Threshold odd solutions to the nonlinear Schrödinger equation in one dimension

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Received: 22 March 2022 / Accepted: 7 June 2022 / Published online: 1 July 2022
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Abstract
We consider odd solutions to the Schrödinger equation with the $L^2$-supercritical power type nonlinearity in one dimensional Euclidean space. It is known that the odd solution scatters or blows up if its action is less than twice that of the ground state. In the present paper, we show that odd solutions with action twice that of the ground state scatter or blow up.

Keywords Nonlinear Schrödinger equation · Odd functions · Global dynamics · Threshold

Mathematics Subject Classification 35Q55 · 37K40

1 Introduction

1.1 Background

We consider the following nonlinear Schrödinger equation in one dimensional Euclidean space:

$$\begin{cases}
i \partial_t u + \partial_x^2 u + |u|^{p-1} u = 0, & (t, x) \in I \times \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}$$

(NLS)

where $I$ denotes a time interval and $p > 5$. It is known that the Cauchy problem of (NLS) is locally well-posed in the energy space $H^1(\mathbb{R})$. That is, there exist $T > 0$ and a unique
solution $u \in C((-T, T) : H^1(\mathbb{R}))$ if $u_0 \in H^1(\mathbb{R})$. Moreover, the energy and mass are conserved by the flow, where the energy $E$ and mass $M$ are defined by

$$E(u) := \frac{1}{2} \| \partial_x u \|^2_{L^2} - \frac{1}{p+1} \| u \|_{L^{p+1}}^{p+1}, \quad M(u) := \| u \|_{L^2}^2.$$ 

We also have blow-up alternative. Namely, the $\dot{H}^1$-norm of the solution diverges at the maximal existence time if the time is finite. See [5, 15, 23] for these results. In this paper, we are interested in the global behavior of the solution.

Since the work by Kenig and Merle [22], many researchers have investigated the global dynamics of the nonlinear Schrödinger equations below the ground state. See [10, 17, 18] for 3d cubic NLS. For the one dimensional NLS, i.e. (NLS), Akahori and Nawa [1], where they in fact treated the equations with $L^2$-super and energy-subcritical nonlinearity in all dimensions, determined the global behavior of the solutions below the ground state. The ground state is the solution with the formula

$$Q(x) = \left( \frac{p+1}{2} \right)^{\frac{1}{p-1}} \left\{ \cosh \left( \frac{p-1}{2} x \right) \right\}^{-\frac{2}{p-1}},$$

where the functional is defined by

$$K(u) := \| \partial_x u \|^2_{L^2} - \frac{p-1}{2(p+1)} \| u \|_{L^{p+1}}^{p+1}.$$ 

Precisely, they showed the following: if $K(u_0) \geq 0$, then the solution is global and scatters in both time directions, i.e. there exist $u_{\pm} \in H^1(\mathbb{R})$ such that

$$\| u(t) - e^{it\partial_x^2} u_{\pm} \|_{H^1} \to 0 \text{ as } t \to \pm \infty.$$ 

If $K(u_0) < 0$, then the solution blows up in finite time or grows up at infinite time. In addition, if the variance of the solution is finite, i.e. $\| x u_0 \|_{L^2} < \infty$, then the solution blows up in finite time in both time directions. (See also Fang et al. [14] for the scattering result.)

The global behavior of the solutions with $E(u_0)^{\kappa} M(u_0)^{1-s_c} = E(Q)^{\kappa} M(Q)^{1-s_c}$, which are called threshold solutions, was also investigated by Duyckaerts and Roudenko [13] for 3d cubic NLS. Recently, for one dimensional NLS, Campos et al. [4] studied them, where they treated $L^2$-supercritical NLS, including the energy-critical one, in general dimensions (see also [12]). They showed the following theorems in the one dimensional case:

**Theorem** [4] There exist radial (even) solutions $Q^+$ and $Q^-$ to (NLS) on at least $[0, \infty)$ such that

- $M(Q^+) = M(Q)$ and $E(Q^+) = E(Q)$,
- $Q^\pm$ satisfies

$$\| Q^\pm(t) - e^{itQ} \|_{H^1} \leq C e^{-ct} \text{ for all } t > 0$$

for some constants $C, c > 0$. 

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Theorem [4] Let \( u_0 \) satisfy \( E(u_0)^{sc} M(u_0)^{1-sc} = E(Q)^{sc} M(Q)^{1-sc} \) and \( u \) be the solution to (NLS). Then we have the following:

1. If \( K(u_0) > 0 \), then \( u \) is global and either \( u \) scatters in both time directions or \( u = Q^- \) up to the symmetries of the equation.
2. If \( K(u_0) = 0 \), then \( u = Q \) up to the symmetries of the equation.
3. If \( K(u_0) < 0 \) and \( \|uxu_0\|_{L^2} < \infty \), then either \( u \) blows up in finite both time or \( u = Q^+ \) up to the symmetries of the equation.

The special solutions \( Q^\pm \), which converge to the ground state, appear at the threshold, though we do not have them below the ground state.

Now, let us assume that \( u_0 \) is an odd function. Then the solution \( u \) is also odd. In this case, the second author determined the global behavior of the solutions below the ground state as follows:

Theorem [20, Theorem 1.2] Let \( u_0 \) be odd and \( u \) be the solution to (NLS). Assume that \( u_0 \) satisfies \( E(u_0)^{sc} M(u_0)^{1-sc} < 2E(Q)^{sc} M(Q)^{1-sc} \). Then we have the following.

1. If \( K(u_0) \geq 0 \), then \( u \) is global and scatters in both time directions.
2. If \( K(u_0) < 0 \), then either \( u \) blows up in finite time or grows up at infinite time.

Remark 1 In Theorem 1.2 in [20], the second author gave the condition in terms of the action and the frequency \( \omega \) instead of the energy and mass condition \( E(u_0)^{sc} M(u_0)^{1-sc} < 2E(Q)^{sc} M(Q)^{1-sc} \). As seen in Sect. 2.1 below, the formulation in [20] is equivalent to the energy and mass condition.

Since the ground state \( e^{it} Q \) and the special solutions \( Q^\pm \) are even, the assumption of odd symmetry excludes them. That is why we can determine the global dynamics of odd solutions above the ground state.

In the odd case, the energy and mass condition \( E(u_0)^{sc} M(u_0)^{1-sc} < 2E(Q)^{sc} M(Q)^{1-sc} \) is related to a minimization problem restricted to odd functions just as the relation \( E(u_0)^{sc} M(u_0)^{1-sc} < E(Q)^{sc} M(Q)^{1-sc} \) is. Roughly, the condition comes from a minimizing sequence of odd functions \( \{\psi_n\}_{n\in\mathbb{N}} = \{Q(-n) - Q(-n+1)\}_{n\in\mathbb{N}} \) with \( E(\psi_n)^{sc} M(\psi_n)^{1-sc} \rightarrow 2E(Q)^{sc} M(Q)^{1-sc} \) as \( n \rightarrow \infty \). Moreover, the minimizer is not attained. (See Sect. 2.1 below for more details.) Therefore, we expect that on the threshold \( E(u_0)^{sc} M(u_0)^{1-sc} = 2E(Q)^{sc} M(Q)^{1-sc} \) there are no odd solutions like the ground state and the special solutions. In the present paper, we will show that odd solutions at the threshold scatter or blow up.

Remark 2 Odd solutions for (NLS) can also be regarded as solutions of NLS on half line \([0, \infty)\) with Dirichlet zero boundary condition at the origin. Thus, the result in the paper holds for NLS on half line with this Dirichlet zero boundary. The same goes for NLS on the star graph, which is a metric graph connecting half lines, with Dirichlet zero boundary condition.

1.2 Main result

We use \( H^1_{\text{odd}}(\mathbb{R}) \) to denote the set of odd functions in \( H^1(\mathbb{R}) \). We obtain the following result:
Theorem 1 Let $u_0 \in H^{1}_{\text{odd}}(\mathbb{R})$. Assume $E(u_0)^{\delta} M(u_0)^{1-s_c} = 2E(Q)^{\delta} M(Q)^{1-s_c}$. Then we have the following:

1. If $K(u_0) > 0$, then the solution scatters in both time directions.
2. In addition, we assume $\|xu_0\|_{L^2} < \infty$. If $K(u_0) < 0$, then the solution blows up in finite time in both time directions.

Remark 3 $K(u_0) = 0$ is not possible under the assumption $u_0 \in H^{1}_{\text{odd}}(\mathbb{R})$ and $E(u_0)^{\delta} M(u_0)^{1-s_c} = 2E(Q)^{\delta} M(Q)^{1-s_c}$ (see Sect. 2.1).

Remark 4 The assumption of oddness is essential. Indeed, Nguyen [25] obtained a "two peak" solution $u$ such that

$$
\|u(t) - e^{iy(t)} \sum_{k=1}^{2} Q(\cdot - x_k(t))\|_{H^1} \lesssim t^{-1}
$$

for all $t > 0$, where $x_1 = -x_2$ and $|x_1(t) - x_2(t)| = 2(1 + o(1)) \log t$ as $t \to \infty$. This solution satisfies $E(u_0)^{\delta} M(u_0)^{1-s_c} = 2E(Q)^{\delta} M(Q)^{1-s_c}$. We note that the solitons have the same sign, so $u$ is not odd.

Remark 5 The threshold condition $E(u_0)^{\delta} M(u_0)^{1-s_c} = 2E(Q)^{\delta} M(Q)^{1-s_c}$ is optimal in the sense of scattering and blow-up dichotomy. That is, there exists a non-scattering solution satisfying $E(u_0)^{\delta} M(u_0)^{1-s_c} = 2E(Q)^{\delta} M(Q)^{1-s_c} + \delta$ even if $\delta > 0$ is arbitrarily small. Indeed, by Combet [7], there are odd 2-soliton solutions, which are non-scattering solutions $u^v (v > 0)$ such that $\|u^v(t) - R^v(t)\|_{H^1} \to 0$ as $t \to \infty$, where

$$
R^v(t) := e^{i\gamma t} Q(x - vt)e^{i\left(\frac{\gamma}{2} x - \frac{\gamma^2}{4} t\right)} - e^{i\gamma t} Q(x + vt)e^{i\left(-\frac{\gamma}{2} x - \frac{\gamma^2}{4} t\right)}.
$$

By a direct calculation, these solutions satisfy

$$
M(u^v) = 2M(Q) \quad \text{and} \quad E(u^v) = 2E(Q) + \frac{\gamma^2}{2} M(Q).
$$

By taking $v$ appropriately, the solution satisfies $E(u^v)^{\delta} M(u^v)^{1-s_c} = 2E(Q)^{\delta} M(Q)^{1-s_c} + \delta$.

Remark 6 The blow-up or grow-up result when $K < 0$ is not obtained yet as in the case of [4].

1.3 Idea of proof

Our proof of the scattering result is based on the contradiction argument by Duyckaerts, Landoulsi, and Roudenko [11], where they consider 3d cubic NLS outside an obstacle (see also [2, 24]). If the statement fails, combining a modulation argument and a concentration compactness argument, we construct a critical element at the threshold which has the following compactness property: There exists $x(t)$ such that $u(t, \cdot) - [\psi(t, \cdot - x(t)) - \psi(t, \cdot - x(t))]$ converges to 0 for some $\psi$. If $x(t)$ goes to infinity, then the assumption on the virial functional gives us a contradiction, and thus $x(t)$ must be bounded. However, the boundedness of $x(t)$ implies a contradiction to a modulation argument.

For the blow-up result, we apply the argument by Duyckaerts and Roudenko [13] (see also [4]). If the solution is global, the finite variance and the negativity of the virial functional imply that the modulation parameter $y$ is bounded. However, this gives a contradiction.
The main difficulty appears in the modulation argument. For NLS with potential as in [2, 24], we consider the linearization around the ground state $Q$ and thus we can control the translation parameter $y$ easily. On the other hand, in our setting, we need to consider the linearization around $Q(-y) - Q(+y)$. Therefore, it is more difficult to control the translation parameter $y$. This is similar to [11]. They overcame this difficulty by using the result for NLS on $\mathbb{R}^3$. In our proof, the following plays a very important role:

$$S(Q(-y) - Q(+y)) - 2S(Q) \approx e^{-2y},$$

where $S = E + M$ (see Lemma 30). Since this is positive, we can control the translation parameter $y$. Since $S(Q(-y) + Q(+y)) - 2S(Q) \approx -e^{-2y}$, the oddness is very important.)

By using this, we can control the modulation parameters.

1.4 Notation

We define

$$(f, g) := \text{Re} \int_{\mathbb{R}} f(x) \overline{g(x)} dx.$$

We use following notation: for $y \geq 0$,

$$T_y f(x) := f(x - y),$$

$$R_y f(x) := f(x - y) - f(-x - y) = T_y f(x) - (T_y f)(-x).$$

The function $R_y f$ is an odd function even when $f$ is neither odd nor even. For an even function $f$, we have

$$R_y f(x) = T_y f(x) - T_{-y} f(x) = f(x - y) - f(x + y).$$

We define a smooth and even cut-off function $\chi_R : \mathbb{R} \to [0, 1]$ by

$$\chi_{1}(x) := \begin{cases} 1 & (|x| > 1) \\ 0 & (|x| < 1/2) \end{cases}, \quad \chi_R(x) := \chi_{1} \left( \frac{x}{R} \right)$$

and we use

$$\chi_{R}^{c}(x) := 1 - \chi_R(x), \quad \chi_{R}^{+}(x) := 1_{(0, \infty)}(x) \chi_R(x), \quad \chi_{R}^{-}(x) := 1_{(-\infty, 0)}(x) \chi_R(x),$$

where $1_A$ denotes the characteristic function on a set $A$. We define

$$G_{R,y} f(x) := \chi_{R}^{+}(x) T_y f(x) - \chi_{R}^{-}(x) T_{-y} f(x).$$

We set

$$\mu_{\omega_0}(u) := 2\|\partial_x Q_{\omega_0}\|_{L^2}^2 - \|\partial_x u\|_{L^2}^2,$$

where $Q_{\omega}(x) = \omega^\frac{1}{p-1} Q(\sqrt{\omega} x)$, and we denote $\mu_1$ for simplicity.

We use the notation $A \lesssim B$ if there exists a positive constant $C$ such that $A \leq CB$. The notation $A \approx B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold.
2 Preliminaries

2.1 Variational argument

In this section, we consider the variational structure for odd functions.

2.1.1 Minimizing problem for odd functions

Let $\omega > 0$. We define the action by

$$ S_\omega(f) := E(f) + \frac{\omega}{2} M(f) = \frac{1}{2} \| \partial_x f \|_{L^2}^2 + \frac{\omega}{2} \| f \|_{L^2}^2 - \frac{1}{p+1} \| f \|_{L^{p+1}}^{p+1}. $$

We consider the following minimizing problems:

$$
\begin{align*}
I_\omega &:= \inf\{ S_\omega(f) : f \in H^1(\mathbb{R}) \setminus \{0\}, K(f) = 0 \}, \\
I^\text{odd}_\omega &:= \inf\{ S_\omega(f) : f \in H^1_{\text{odd}}(\mathbb{R}) \setminus \{0\}, K(f) = 0 \}.
\end{align*}
$$

It is known that $I_\omega$ is attained by the ground state $Q_\omega$, i.e. $I_\omega = S_\omega(Q_\omega)$, where we recall $Q_\omega(x) = \omega \frac{1}{\sqrt{1\!+\!1}} Q(\sqrt{\omega} x)$, and $Q_\omega$ is the radial positive solution to the elliptic equation $-\partial_x^2 Q_\omega + \omega Q_\omega - Q_\omega^p = 0$. We note $Q = Q_1$ and set $S := S_1$.

We prove that $I^\text{odd}_\omega = 2I_\omega$ and there is no minimizer.

Lemma 2 We have $I^\text{odd}_\omega = 2I_\omega (= 2S_\omega(Q_\omega))$.

Proof Define a sequence \{\varphi_n\} of odd functions by $\varphi_n(x) := R_n Q_\omega(x).$ Then $\{\varphi_n\}$ satisfies $S_\omega(\varphi_n) \to 2S_\omega(Q_\omega)$ and $K(\varphi_n) \to 0$. Take a parameter $\lambda_n$ such that $K(\lambda_n \varphi_n) = 0$. Then $\lambda_n \to 1$ and thus $S_\omega(\lambda_n \varphi_n) \to 2S(\omega)$. Therefore, we get $I^\text{odd}_\omega \leq 2S_\omega(Q_\omega) = 2I_\omega$.

On the other hand, taking a minimizing sequence \{\varphi_n\} of $I^\text{odd}$ We note that $1_{(0,\infty)} \varphi_n \in H^1(\mathbb{R})$ and $\partial_x (1_{(0,\infty)} \varphi_n) = 1_{(0,\infty)} \partial_x \varphi_n$. Then, it holds that

$$
\begin{align*}
2S(1_{(0,\infty)} \varphi_n) &= S(1_{(0,\infty)} \cdot \varphi_n(\cdot) + 1_{(0,\infty)}(-\cdot) \varphi_n(-\cdot)) = S(\varphi_n) \to I^\text{odd}_\omega, \\
2K(1_{(0,\infty)} \varphi_n) &= K(1_{(0,\infty)} \cdot \varphi_n(\cdot) + 1_{(0,\infty)}(-\cdot) \varphi_n(-\cdot)) = K(\varphi_n) = 0.
\end{align*}
$$

This means that $I^\text{odd}_\omega / 2 \geq I_\omega$. \hfill \Box

Lemma 3 There is no minimizer of $I^\text{odd}_\omega$.

Proof Suppose that $f \in H^1_{\text{odd}}(\mathbb{R})$ is a minimizer, i.e. $S_\omega(f) = I^\text{odd}_\omega$ and $K(f) = 0$. Then we have

$$
\begin{align*}
2S(1_{(0,\infty)} f) &= S(1_{(0,\infty)} \cdot f(\cdot) - 1_{(0,\infty)}(-\cdot) f(-\cdot)) = S(f) = I^\text{odd}_\omega = 2I_\omega, \\
2K(1_{(0,\infty)} f) &= K(1_{(0,\infty)} \cdot f(\cdot) - 1_{(0,\infty)}(-\cdot) f(-\cdot)) = K(f) = 0.
\end{align*}
$$

The uniqueness of the minimizer of $I_\omega$ implies that $1_{(0,\infty)} f = e^{it} Q_\omega(\cdot - y)$. Obviously, this is a contradiction. \hfill \Box

2.1.2 Energy and mass condition

In this section, we prove that the action condition $S_\omega(f) \leq 2S_\omega(Q_\omega)$ is equivalent to the energy and mass condition $E(f)^{\frac{1}{2}} M(f)^{1-x} \leq 2E(Q)^{\frac{1}{2}} M(Q)^{1-x}$ for odd functions $f$.

By properties of the ground state, we have the following:
Lemma 4 (The Pohozaev identity) We have
\[ \frac{1}{p+3} \| Q \|_{L^2}^3 = \frac{1}{p-1} \| \partial_x Q \|_{L^2}^2 = \frac{1}{2(p+1)} \| Q \|_{L^{p+1}}^{p+1} \]
and, in particular,
\[ M(Q) = \frac{2(p+3)}{p-5} E(Q). \]

Lemma 5 (Scaling of the ground state) We have
\[ M(Q_\omega) = \omega^{-\frac{p-5}{2(p-1)}} M(Q), \quad E(Q_\omega) = \omega^{\frac{p+3}{2(p-1)}} E(Q). \]

Proposition 6 For \( f \in H_{odd}^1(\mathbb{R}) \), the following are equivalent:
1. \( (f)^{sc} M(f)^{1-sc} \leq 2 E(Q)^{sc} M(Q)^{1-sc} \).
2. There exists \( \omega > 0 \) such that \( S_\omega(f) \leq 2 S_\omega(Q_\omega) \).

Proof We consider the family \( \{C_\omega\}_{\omega > 0} \) of lines such that \( C_\omega := \{(M, E) \in (0, \infty) \times \mathbb{R} : F(M, E, \omega) := E + \frac{\omega}{2} M - 2 S_\omega(Q_\omega) = 0\} \). We have
\[ F(2M(Q_\omega), 2E(Q_\omega), \omega) = 2 \left( E(Q_\omega) + \frac{\omega}{2} M(Q_\omega) \right) - 2 S_\omega(Q_\omega) = 0 \]
and, by Lemma 5, we also get
\[ \partial_\omega F(M, E, \omega) = \frac{1}{2} M - \frac{p+3}{2(p-1)} \omega^{-\frac{p-5}{2(p-1)}} 2S(Q) \]
and thus, by Lemmas 4 and 5,
\[ \partial_\omega F(2M(Q_\omega), 2E(Q_\omega), \omega) = M(Q_\omega) - \frac{p+3}{p-1} \omega^{-\frac{p-5}{2(p-1)}} 2S(Q) \]
\[ = \omega^{-\frac{p-5}{2(p-1)}} M(Q) - \omega^{-\frac{p-5}{2(p-1)}} M(Q) = 0. \]

Therefore, the curve \( \{(2M(Q_\omega), 2E(Q_\omega)) : \omega > 0\} \) is the envelope of the family of lines \( \{C_\omega\} \). This curve is explicitly given by \( E^{sc} M^{1-sc} = 2E(Q)^{sc} M(Q)^{1-sc} \). Indeed, since we have \( M(Q_\omega) = (2M(Q_\omega), 2E(Q_\omega)) = (2\omega^{-sc} M(Q), 2\omega^{1-sc} E(Q)) \) by Lemma 5, by erasing \( \omega \) from it, we get the explicit formula of the curve. It is obvious that the curve is convex. Thus, the tangent line of the curve at \( M = 2M(Q_\omega) \), which is given by \( F(M, E, \omega) = 0 \), is below the curve for any \( \omega > 0 \). This implies the statement. \( \square \)

As a corollary, we see the following:

Corollary 7 For \( f \in H_{odd}^1(\mathbb{R}) \), the following are equivalent:
1. \( (f)^{sc} M(f)^{1-sc} \leq 2 E(Q)^{sc} M(Q)^{1-sc} \).
2. There exists \( \omega > 0 \) such that \( E(f) = 2E(Q_\omega) \) and \( M(f) = 2M(Q_\omega) \).

2.1.3 Reduction by the scaling

As seen in Corollary 7, to prove the main result, we may assume that \( M(u_0) = 2M(Q_\omega) \) and \( E(u_0) = 2E(Q_\omega) \) for some \( \omega > 0 \). By using the scaling invariance of the equation (NLS), we may also assume that \( M(u_0) = 2M(Q) \) and \( E(u_0) = 2E(Q) \). We prove the sufficiency in this section.
Theorem 8  Let \( u_0 \in H^1_{\text{odd}}(\mathbb{R}) \) satisfy \( M(u_0) = 2M(Q) \) and \( E(u_0) = 2E(Q) \). Then we have

1. If \( K(u_0) > 0 \), then the solution scatters in both time directions.
2. In addition, we assume finite variance. If \( K(u_0) < 0 \), then the solution blows up in finite positive and negative time.

If the above Theorem 8 holds, we get the main result, Theorem 1, by a scaling argument.

Proof of Theorem 1 from Theorem 8  Let \( u_0 \in H^1_{\text{odd}}(\mathbb{R}) \) satisfy \( E(u_0)^2 M(u_0)^{1-s_c} = 2E(Q)^2 M(Q)^{1-s_c} \) and \( K(u_0) > 0 \). We have \( M(u_0) > 0 \) by assumption. Thus, there exists \( \omega > 0 \) such that \( M(u_0) = \omega^{\frac{p+5}{2(p-1)}} 2M(Q) = 2M(Q_\omega) \) by Lemma 5. Then since we also have

\[
2E(Q)^2 M(Q)^{1-s_c} = 2(\omega^{\frac{p+3}{2(p-1)}} E(Q_\omega)^2)^{\frac{1}{1-s_c}} \left( \omega^{\frac{p-5}{2(p-1)}} M(Q_\omega)^2 \right)^{\frac{1}{1-s_c}}
\]

we get

\[
E(u_0) = 2E(Q_\omega) = 2\omega^{-\frac{p+3}{2(p-1)}} E(Q).
\]

Therefore, we obtain

\[
\omega^{\frac{p+3}{2(p-1)}} E(u_0) = 2E(Q) \quad \text{and} \quad \omega^{-\frac{p-5}{2(p-1)}} M(u_0) = 2M(Q).
\]

Let \( u_{0,\omega^{-1}}(x) := \omega^{-1/(p-1)} u_0(\omega^{-1/2} x) \). Then we get

\[
E(u_{0,\omega^{-1}}) = 2E(Q) \quad \text{and} \quad M(u_{0,\omega^{-1}}) = 2M(Q).
\]

We also have

\[
K(u_{0,\omega^{-1}}) = \omega^{\frac{p+3}{2(p-1)}} K(u_0) > 0.
\]

By Theorem 8, we find that the solution \( u_{\omega^{-1}} \) with the initial data \( u_{0,\omega^{-1}} \) scatters. Since the equation (NLS) is invariant under this scaling, the behavior of the solution \( u \) with \( u(0) = u_0 \) is same as that of \( u_{\omega^{-1}} \). Thus \( u \) scatters. This argument also works for the case that \( K \) is negative.

Thus, it is enough to consider the case of \( \omega = 1 \).

2.1.4 The virial functional and variational argument

We will show the following in two ways:

Proposition 9  Let \( f \in H^1_{\text{odd}}(\mathbb{R}) \). Assume \( M(f) = 2M(Q) \) and \( E(f) = 2E(Q) \). Then the following are equivalent:

1. \( K(f) > 0 \).
2. \( \| f \|_{L^2}^{2(1-s_c)} \| \partial_x f \|_{L^2}^{2s_c} < 2 \| Q \|_{L^2}^{2(1-s_c)} \| \partial_x Q \|_{L^2}^{2s_c} \).
3. \( \mu(f) > 0 \).

The equivalence of 2 and 3 in Proposition 9 holds by \( M(f) = 2M(Q) \). The equivalence of 1 and 3 follows from \( E(f) = 2E(Q) \):

Lemma 10  If \( E(f) = 2E(Q) \), we have \( K(f) = \frac{p-5}{4} \mu(f) \).
Proof We have
\[
K(f) = \frac{p - 1}{2} E(f) - \frac{p - 5}{4} \| \partial_x f \|_{L^2}^2
= \frac{p - 1}{2} 2E(Q) - \frac{p - 5}{4} \| \partial_x f \|_H^2
= \frac{p - 5}{4} (2 \| \partial_x Q \|_{L^2}^2 - \| \partial_x f \|_{L^2}^2)
= \frac{p - 5}{4} \mu(f),
\]
by the Pohozaev identity
\[
E(Q) = \frac{p - 5}{2} \left( \frac{p - 1}{4} \right) \| \partial_x Q \|_{L^2}^2.
\]
\[\square\]

We also give a direct proof of the equivalence of 1 and 2 in Proposition 9 by using the best constant of the Gagliardo–Nirenberg inequality for odd functions. The best constant will be also used later.

Lemma 11 (The Gagliardo–Nirenberg inequality for odd functions) We have
\[
\| f \|_{L^{p+1}}^{p+1} \leq C_{GN}^{odd} \| f \|_{L^2}^{p+3} \| \partial_x f \|_{L^2}^{p-1}
\]
for any odd functions \( f \in H^1(\mathbb{R}) \) and
\[
C_{GN}^{odd} := \sup \left\{ \frac{\| f \|_{L^{p+1}}^{p+1}}{\| f \|_{L^2}^{p+3} \| \partial_x f \|_{L^2}^{p-1}} : f \in H^1_{odd}(\mathbb{R}) \setminus \{0\} \right\} = 2^{-\frac{p-1}{2}} C_{GN}^{odd}
\]

Proof By the usual Gagliardo–Nirenberg inequality, we have \( C_{GN}^{odd} \leq C_{GN} \). We show \( C_{GN}^{odd} \geq 2^{-\frac{p-1}{2}} C_{GN} \). Applying the Gagliardo–Nirenberg inequality to \( \varphi_n := T_n Q - T_{n} Q \), we get
\[
\| \varphi_n \|_{L^{p+1}}^{p+1} \leq C_{GN}^{odd} \| \varphi_n \|_{L^2}^{p+3} \| \partial_x \varphi_n \|_{L^2}^{p-1}.
\]
Taking the limit \( n \to \infty \), we obtain
\[
\| Q \|_{L^{p+1}}^{p+1} \leq 2^{-\frac{p-1}{2}} C_{GN}^{odd} \| Q \|_{L^2}^{p+3} \| \partial_x Q \|_{L^2}^{p-1}.
\]
Therefore, we have \( C_{GN}^{odd} \geq 2^{-\frac{p-1}{2}} C_{GN} \).

Next, we show \( 2^{-\frac{p-1}{2}} C_{GN} \geq C_{GN}^{odd} \). Let \( \{ f_n \} \subset H^1_{odd} \setminus \{0\} \) satisfy
\[
\frac{\| f_n \|_{L^{p+1}}^{p+1}}{\| f_n \|_{L^2}^{p+3} \| \partial_x f_n \|_{L^2}^{p-1}} \to C_{GN}^{odd}.
\]
We have
\[
(LHS) = 2^{-\frac{p-1}{2}} \frac{\| 1_{(0,\infty)} f_n \|_{L^{p+1}}^{p+1}}{\| 1_{(0,\infty)} f_n \|_{L^2}^{p+3} \| \partial_x (1_{(0,\infty)} f_n) \|_{L^2}^{p-1}} \leq 2^{-\frac{p-1}{2}} C_{GN}.
\]
Since the left hand side goes to \( (C_{GN}^{odd})^{-1} \), we get \( 2^{-\frac{p-1}{2}} C_{GN} \geq C_{GN}^{odd} \). This completes the proof. \( \square \)
Lemma 12 Let \( f \in H^1_{\text{odd}}(\mathbb{R}) \). If \( \| f \|_{L^2}^{2(1-s_c)} \| \partial_x f \|_{L^2}^{2s_c} \leq \| Q \|_{L^2}^{2(1-s_c)} \| \partial_x Q \|_{L^2}^{2s_c} \), then we have \( K(f) \geq 0 \). Moreover, if \( M(f)^{1-s_c} E(f)^{s_c} \leq 2M(Q)^{1-s_c} E(Q)^{s_c} \) and \( K(f) \geq 0 \), then we have
\[
\| f \|_{L^2}^{2(1-s_c)} \| \partial_x f \|_{L^2}^{2s_c} \leq \| Q \|_{L^2}^{2(1-s_c)} \| \partial_x Q \|_{L^2}^{2s_c}.
\]

\textbf{Proof} We assume that \( \| f \|_{L^2}^{2(1-s_c)} \| \partial_x f \|_{L^2}^{2s_c} \leq \| Q \|_{L^2}^{2(1-s_c)} \| \partial_x Q \|_{L^2}^{2s_c} \). This means that
\[
\| f \|_{L^2}^{\frac{p+1}{2}} \| \partial_x f \|_{L^2}^{\frac{p-5}{2}} \leq 2 \| Q \|_{L^2}^{\frac{p+1}{2}} \| \partial_x Q \|_{L^2}^{\frac{p-5}{2}}.
\]
By the Gagliardo–Nirenberg inequality for odd functions and the assumption, we have
\[
K(f) \geq \| \partial_x f \|_{L^2}^2 - \frac{(p-1)C_{G,N}^{\text{odd}}}{2(p+1)} \| f \|_{L^2}^{\frac{p+3}{2}} \| \partial_x f \|_{L^2}^{\frac{p-1}{2}}
\]
\[
= \| \partial_x f \|_{L^2}^2 \left( 1 - \frac{(p-1)C_{G,N}^{\text{odd}}}{2(p+1)} \| f \|_{L^2}^{\frac{p+3}{2}} \| \partial_x f \|_{L^2}^{\frac{p-1}{2}} \right)
\]
\[
\geq \| \partial_x f \|_{L^2}^2 \left( 1 - \frac{(p-1)C_{G,N}^{\text{odd}}}{2(p+1)} \frac{p-5}{2(p-1)} \| f \|_{L^2}^{\frac{p+3}{2}} \| \partial_x Q \|_{L^2}^{\frac{p-5}{2}} \right)
\]
By the Pohozaev identity, we get
\[
\frac{(p-1)C_{G,N}^{\text{odd}}}{2(p+1)} \frac{p-5}{2} \| Q \|_{L^2}^{\frac{p+3}{2}} \| \partial_x Q \|_{L^2}^{\frac{p-5}{2}} = 1.
\]

Therefore, we have \( K(f) \geq 0 \).

Next, assume \( K(f) \geq 0 \). By the Pohozaev identity, Lemma 4, and \( 2M(Q)^{1-s_c} E(Q)^{s_c} \geq M(f)^{1-s_c} E(f)^{s_c} \), we obtain
\[
2M(Q)^{1-s_c} \left( \frac{p-5}{2(p-1)} \| \partial_x Q \|_{L^2}^2 \right)^{s_c} = 2M(Q)^{1-s_c} E(Q)^{s_c} \geq M(f)^{1-s_c} E(f)^{s_c}.
\]
(3)

By the assumption, we have
\[
E(f) \geq E(f) - \frac{2}{p-1} K(f) = \frac{p-5}{2(p-1)} \| \partial_x f \|_{L^2}^2.
\]

Therefore, combining this with (3), we get
\[
2M(Q)^{1-s_c} \left( \frac{p-5}{2(p-1)} \| \partial_x Q \|_{L^2}^2 \right)^{s_c} \geq M(f)^{1-s_c} \left( \frac{p-5}{2(p-1)} \| \partial_x f \|_{L^2}^2 \right)^{s_c}.
\]

This completes the proof. \( \square \)

Lemma 12 includes the equivalence between 1 and 2 of Proposition 9.

The following corollary immediately holds.

Corollary 13 Let \( f \in H^1_{\text{odd}}(\mathbb{R}) \). Assume \( M(f) = 2M(Q) \) and \( E(f) = 2E(Q) \). Then the following are equivalent:

1. \( K(f) < 0 \).
2. \( \| f \|_{L^2}^{2(1-s_c)} \| \partial_x f \|_{L^2}^{2s_c} > 2 \| Q \|_{L^2}^{2(1-s_c)} \| \partial_x Q \|_{L^2}^{2s_c} \).
3. \( \mu(f) < 0 \).
2.1.5 Invariance of the potential-well sets by the flow

**Proposition 14** Let \( u_0 \in H^1_{\text{odd}}(\mathbb{R}) \) satisfy \( M(u_0) = 2M(Q) \) and \( E(u_0) = 2E(Q) \) and \( u(t) \) be a solution to (NLS) with \( u(0) = u_0 \). Then we have the following.

1. If \( K(u_0) > 0 \), then the solution \( u(t) \) exists globally in both time directions and \( K(u(t)) > 0 \) for all \( t \in \mathbb{R} \).
2. If \( K(u_0) < 0 \), then the solution \( u(t) \) satisfies \( K(u(t)) < 0 \) while the solution exists.

**Proof** Since \( u_0 \) is odd, the solution \( u(t) \) is also odd. We consider the case of \( K(u_0) > 0 \). Suppose that there exists a time \( t^* \) such that \( K(u(t^*)) \leq 0 \). Then by the continuity of the flow, there exists a time \( t_0 \) such that \( K(u(t_0)) = 0 \). This and the assumption \( M(u_0) = 2M(Q) \) and \( E(u_0) = 2E(Q) \) mean that \( u(t_0) \), which is not the zero function, is a minimizer of \( l_{\text{odd}} \).

However, this contradicts the non-existence of the minimizer. Therefore we have \( K(u(t)) > 0 \) for all \( t \) in the existence interval. We get an a priori bound by Proposition 9, and the blow-up alternative implies that the solution exists globally in both time directions. In the case of \( K(u_0) < 0 \), we get the result in the same way. \( \square \)

**Remark 7** By Propositions 9 and 14, solutions \( u(t) \) with \( M(u_0) = 2M(Q) \), \( E(u_0) = 2E(Q) \) and \( K(u_0) > 0 \) are uniformly bounded in \( H^1(\mathbb{R}) \).

2.2 Lemmas

We collect some useful lemmas such as the estimate of the ground state, long time perturbation and linear profile decomposition.

2.2.1 Estimate for the ground state

By the explicit formula (2) of the ground state \( Q \), we have \( Q(x) \approx e^{-|x|} \). Thus, the following estimates hold by direct calculations:

**Lemma 15** Let \( \alpha, \beta > 0 \) and \( y > 0 \). We have

\[
\int_{\mathbb{R}} T_y Q(x)^\alpha T_{-y} Q(x)^\beta dx \approx \begin{cases} (1 + y) \exp(-2\alpha y), & \text{if } \alpha = \beta, \\
\exp(-2\min\{\alpha, \beta\}y), & \text{if } \alpha \neq \beta. \end{cases}
\]

**Proof** This follows from \( Q(x) \approx e^{-|x|} \) and direct calculations. \( \square \)

**Lemma 16** Let \( \alpha, \beta \geq 0 \). Let \( y > 0 \). If \( \alpha = 0 \) or \( \beta \neq 0 \), then we have

\[
\int_{0}^{\infty} T_y Q(x)^\alpha T_{-y} Q(x)^\beta dx \approx \begin{cases} (1 + y) \exp(-2\alpha y), & \text{if } \alpha = \beta, \\
\exp\{-\min\{\alpha, \beta\}y\}, & \text{if } \alpha \neq \beta. \end{cases}
\]

2.2.2 Virial identity and its localization

For a solution \( u(t) \), we define

\[
J(u(t)) = J_\infty(u(t)) := \int_{\mathbb{R}} |x|^2 |u(t, x)|^2 dx.
\]

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Then we have
\[ \frac{d}{dt} J(u(t)) = 2 \text{Im} \int_{\mathbb{R}} xu(t, x) \partial_x u(t, x) dx, \]
\[ \frac{d^2}{dt^2} J(u(t)) = 8 K(u(t)). \]

Let \( \varphi \) be an even function in \( C_0^\infty(\mathbb{R}) \) satisfying
\[ \varphi(x) := \begin{cases} x^2, & (|x| < 1), \\ 0, & (|x| > 2). \end{cases} \]
For a solution \( u(t) \), we set
\[ J_R(u(t)) := \int_{\mathbb{R}} R^2 \varphi \left( \frac{x}{R} \right) |u(t, x)|^2 dx. \]
Then, we have
\[ \frac{d}{dt} J_R(u(t)) = 2 \text{Im} \int_{\mathbb{R}} R(\partial_x \varphi) \left( \frac{x}{R} \right) u(t, x) \partial_x u(t, x) dx \]
and
\[ \frac{d^2}{dt^2} J(u(t)) = 4 \int_{\mathbb{R}} (\partial_x^2 \varphi) \left( \frac{x}{R} \right) \left\{ |\partial_x u(t, x)|^2 - \frac{p - 1}{2(p + 1)} |u(t, x)|^{p+1} \right\} dx \]
\[ - \int_{\mathbb{R}} \frac{1}{R^2} (\partial_x^4 \varphi) \left( \frac{x}{R} \right) |u(t, x)|^2 dx \]
\[ = 8 K(u(t)) + A_R(u(t)), \]
where we set
\[ A_R(u(t)) := -4 \int_{|x| > R} \left\{ 2 - (\partial_x^2 \varphi) \left( \frac{x}{R} \right) \right\} \left\{ |\partial_x u(t, x)|^2 - \frac{p - 1}{2(p + 1)} |u(t, x)|^{p+1} \right\} dx \]
\[ - \int_{R < |x| < 2R} \frac{1}{R^2} (\partial_x^4 \varphi) \left( \frac{x}{R} \right) |u(t, x)|^2 dx. \]
For a function \( f \in H^1 \), we set
\[ F_R(f) := 4 \int_{\mathbb{R}} (\partial_x^2 \varphi) \left( \frac{x}{R} \right) \left\{ |\partial_x f|^2 - \frac{p - 1}{2(p + 1)} |f|^{p+1} \right\} dx \]
\[ - \int_{\mathbb{R}} \frac{1}{R^2} (\partial_x^4 \varphi) \left( \frac{x}{R} \right) |f|^2 dx. \]
Then the above equality is written by
\[ A_R(u(t)) = F_R(u(t)) - 8 K(u(t)) = F_R(u(t)) - F_\infty(u(t)), \]
where we note that \( F_\infty(f) = 8 K(f) \).

**Lemma 17** We have \( K(e^{i\theta} T_y Q) = 0 \) and thus \( 8^{-1} F_\infty(e^{i\theta} R_y Q) = K(e^{i\theta} R_y Q) = O((1 + y)e^{-2y}) \) for any \( \theta \in \mathbb{R} \) and \( y > 0 \).

**Proof** \( K(e^{i\theta} T_y Q) = 0 \) follows from the Pohozaev identity. The latter statement follows from the first statement and the estimate of cross terms by Lemma 15. \( \square \)
Lemma 18 We have \( F_R(e^{i\theta T_y Q}) = 0 \) and thus \( F_R(e^{i\theta R_y Q}) = O((1 + y)e^{-2y}) \) for any \( \theta \in \mathbb{R} \) and \( y > 0 \).

Proof The first statement follows from [24, Lemma 2.9]. The latter statement holds as in Lemma 17.

2.2.3 Strichartz estimates and long time perturbations

We use the following Lebesgue exponents:

\[
\begin{align*}
  r &:= p + 1, \\
  a &:= \frac{2(p - 1)(p + 1)}{p + 3}, \quad \text{and} \\
  b &:= \frac{2(p - 1)(p + 1)}{(p - 1)^2 - (p - 1) - 4}.
\end{align*}
\]

We have the following Strichartz estimates (see [5, 6, 14]):

\[
\begin{align*}
  \| e^{it\partial_x^2} f \|_{L^r_x L^a_t} &\lesssim \| f \|_{H^1}, \\
  \| \int_0^t e^{i(t-s)\partial_x^2} g(s) ds \|_{L^r_x L^b_t} &\lesssim \| g \|_{L^a_t L^r_x}.
\end{align*}
\]

In the following result we have a sufficient condition for scattering.

Proposition 19 Let \( u_0 \in H^1(\mathbb{R}) \) and \( u \) be the corresponding solution to (NLS) with initial data \( u(0) = u_0 \). If \( u \in L^a_t L^r_x((0, \infty) \times \mathbb{R}) \), then \( u \) scatters in the positive time direction.

It is known that the Cauchy problem is well-posed. We also see that the final value problem is well-posed, that is there exists a wave operator.

Proposition 20 [26, Theorem 3] Let \( \psi \in H^1(\mathbb{R}) \). There exists \( T \in \mathbb{R} \) and the solution \( u(t) \) to (NLS) on \((T, \infty)\) such that

\[
\| u(t) - e^{it\partial_x^2} \psi \|_{H^1} \to 0 \quad \text{as} \quad t \to \infty.
\]

We will use this long time perturbation result (see [1, 14] for the proof).

Lemma 21 (Long time perturbation) For any \( M > 0 \), there exist \( \epsilon = \epsilon(M) > 0 \) and a positive constant \( C = C(M) \) such that the following occurs. Let \( v : I \times \mathbb{R} \to \mathbb{C} \) be a solution of the integral equation with source term \( e \):

\[
v(t) = e^{it\partial_x^2} \varphi + i \int_0^t e^{i(t-s)\partial_x^2} (|v(s)|^{p-1} v(s)) ds + e(t)
\]

with \( \| v \|_{L^a_t L^r_x(I \times \mathbb{R})} < M \) and \( \| e \|_{L^a_t L^r_x(I \times \mathbb{R})} < \epsilon \). Assume moreover that \( u_0 \in H^1(\mathbb{R}) \) is such that \( \| u_0 - \varphi \|_{H^1} < \epsilon \), then the solution \( u : I \times \mathbb{R} \to \mathbb{C} \) to (NLS) with initial data \( u_0 \):

\[
u(t) = e^{it\partial_x^2} u_0 + i \int_0^t e^{i(t-s)\partial_x^2} (|u(s)|^{p-1} u(s)) ds,
\]

satisfies \( u \in L^a_t L^r_x(I \times \mathbb{R}) \) and moreover \( \| u - v \|_{L^a_t L^r_x(I \times \mathbb{R})} < C \epsilon \).

2.2.4 Linear profile decomposition

We have a linear profile decomposition for odd functions.
Proposition 22. (Linear profile decomposition for odd functions) Let \( \{ \varphi_n \}_{n \in \mathbb{N}} \) be a bounded sequence in \( H^1_{\text{odd}}(\mathbb{R}) \). Then, up to subsequence, we can write

\[
\varphi_n = \sum_{j=1}^{J} e^{-it_j \partial_x^2} \mathcal{R}_{x_j} \psi_j + R^J_n, \quad \forall J \in \mathbb{N},
\]

where \( t_j^n \in \mathbb{R}, x_j^n \geq 0, \psi_j \in H^1(\mathbb{R}), R^J_n \in H^1_{\text{odd}}(\mathbb{R}), \) and the following hold:

- for any fixed \( j \), we have:
  
  either \( t_j^n = 0 \) for any \( n \in \mathbb{N} \), or \( t_j^n \to \pm \infty \) as \( n \to \infty \),
  
  either \( x_j^n = 0 \) for any \( n \in \mathbb{N} \), or \( x_j^n \to \infty \) as \( n \to \infty \).

- orthogonality of the parameters:
  
  \[ |t_j^n - t_k^n| + |x_j^n - x_k^n| \to \infty \text{ as } n \to \infty, \quad \forall j \neq k. \]

- smallness of the remainder:
  
  \[ \forall \varepsilon > 0, \exists J = J(\varepsilon) \in \mathbb{N} \text{ such that } \limsup_{n \to \infty} \| e^{it\partial_x^2} R^J_n \|_{L^{\infty}_{t,x}} < \varepsilon. \]

- orthogonality in norms: for any \( J \in \mathbb{N} \)
  
  \[
  \| \varphi_n \|^2_{L^2} = \sum_{j=1}^{J} \| \mathcal{R}_{x_j} \psi_j \|^2_{L^2} + \| R^J_n \|^2_{L^2} + o_n(1),
  \]
  
  \[
  \| \partial_x \varphi_n \|^2_{L^2} = \sum_{j=1}^{J} \| \partial_x \mathcal{R}_{x_j} \psi_j \|^2_{L^2} + \| \partial_x R^J_n \|^2_{L^2} + o_n(1).
  \]

Moreover, we have

\[
\| \varphi_n \|^q_{L^q} = \sum_{j=1}^{J} \| e^{-it_j \partial_x^2} \mathcal{R}_{x_j} \psi_j \|^q_{L^q} + \| R^J_n \|^q_{L^q} + o_n(1), \quad q \in (2, \infty), \quad \forall J \in \mathbb{N}.
\]

Proof We give a sketch of the proof. See also [19, Theorem 3.5]. By the linear profile decomposition, e.g. [14, Theorem 5.1.], we have

\[
\varphi_n = \sum_{j=1}^{J} e^{-it_j \partial_x^2} T_{x_j} \tilde{\psi}^j + \tilde{R}^J_n.
\]

Since \( \varphi_n \) is odd, we have

\[
\varphi_n(x) = \frac{\varphi_n(x) + \varphi_n(-x)}{2} = \sum_{j=1}^{J} e^{-it_j \partial_x^2} \frac{(T_{x_j} \tilde{\psi}^j)(x) + (T_{x_j} \tilde{\psi}^j)(-x)}{2} + \frac{\tilde{R}^J_n(x) + \tilde{R}^J_n(-x)}{2}.
\]

Setting \( \psi^j := \tilde{\psi}^j/2 \) and \( \tilde{R}^J_n := (\tilde{R}^J_n(x) + \tilde{R}^J_n(-x))/2 \), we get (4) and the other statements follow from the linear profile decomposition. See [20] for a direct proof.

Remark 8 The \( L^q_t L^r_x \)-norm of \( e^{it\partial_x^2} R^J_n \) is controlled by its \( L^\infty_{t,x} \)-norm. See [2, Remark 2.3].
3 Coercivity of the linearized operator

In this section, we give the coercivity of a linearized operator, which is used in the following modulation argument. Before stating our coercivity, we recall the coercivity statement of [4] for the linearized equation.

We set

$$\Phi(f, g) = \text{Re} \int_{\mathbb{R}} \partial_x f(x) \overline{\partial_x g(x)} + f(x) g(x) dx - |Q|^{p-1} (p f_1(x) g_1(x) + f_2(x) g_2(x)) dx,$$

where \( f = f_1 + i f_2 \) and \( g = g_1 + i g_2 \) and we also set \( \Phi(f) := \Phi(f, f) \).

Lemma 23 [4, Lemma 3.5] Let \( h \in H^1(\mathbb{R}) \) satisfy the following orthogonality conditions:

$$\text{Im} \int h^T y Q dx = \text{Re} \int h \partial_x (T_y Q) dx = \text{Re} \int h (T_y Q)^p dx = 0.$$

Then, there exists a positive constant \( c \) such that

$$\Phi(T_y h) \geq c \| h \|^2_{H^1}.$$

See also [13, 27].

Using Lemma 23, we obtain the following coercivity statement:

Lemma 24 (Coercivity) Let \( h \in H^1_{\text{odd}}(\mathbb{R}) \) satisfy the orthogonality conditions:

$$\text{Im} \int h^T y Q dx = \text{Re} \int h \partial_x (\chi_R^+ T_y Q) dx = \text{Re} \int h (\chi_R^+ T_y Q)^p dx = 0.$$

Then, there exists a positive constant \( c \) such that

$$\Phi(T_y (\chi_R^+ h)) \geq c \| \chi_R h \|^2_{H^1} - \frac{1}{R} \| h \|^2_{H^1}$$

for \( R > 1 \).

Proof We set

$$\mathcal{A} := \{ f \in H^1_{\text{odd}}(\mathbb{R}) : \text{Re} \int f Q^p dx = \text{Im} \int f Q dx = \text{Re} \int f \partial_x Q dx = 0 \},$$

$$\mathcal{B} := \text{span}_{\mathbb{R}} \{ i Q, Q^p, \partial_x Q \}.$$

Then, we write \( T_y (\chi_R^+ h) = T_y \tilde{h} + T_y r \) such that \( T_y \tilde{h} \in \mathcal{A} \) and \( T_y r \in \mathcal{B} \). Since \( T_y r \in \mathcal{B} \), we write

$$T_y r = \alpha \partial_x Q + \beta i Q + \gamma Q^p,$$

where \( \alpha, \beta, \gamma \in \mathbb{R} \). By multiplying (5) by \(-i Q\) and taking real part and integral, we get

$$\beta = \frac{1}{\| Q \|_{L^2}^2} \langle T_y r, i Q \rangle = \frac{1}{\| Q \|_{L^2}^2} \langle T_y (\chi_R^+ h) - T_y \tilde{h}, i Q \rangle.$$

We have \( \langle T_y \tilde{h}, i Q \rangle = 0 \) since \( T_y \tilde{h} \in \mathcal{A} \) and we also have \( \langle T_y (\chi_R^+ h), i Q \rangle = 0 \) by the orthogonality assumption. Thus, \( \beta = 0 \). By multiplying (5) by \( Q^p \) and taking integral and
real part, we get \( \gamma = 0 \) in a similar way. By multiplying (5) by \( \partial_x Q \) and taking integral and imaginary part, we get

\[
\alpha = \frac{1}{\| \partial_x Q \|^2_{L^2}} \langle T_{-y} r, \partial_x Q \rangle = \frac{1}{\| \partial_x Q \|^2_{L^2}} \langle T_{-y} (\chi_R^+ h), \partial_x Q \rangle
\]

since \( T_{-y} \tilde{h} \in \mathcal{A} \). Now, we have

\[
\langle T_{-y} (\chi_R^+ h), \partial_x Q \rangle = \text{Re} \int \chi_R^+ h \partial_x T_y Q \, dx = \text{Re} \int \chi_R^+ h \partial_x T_y Q \, dx - \text{Re} \int h \partial_x (\chi_R^+ T_y Q) \, dx,
\]

where we use \( \text{Re} \int h \partial_x (\chi_R^+ T_y Q) \, dx = 0 \) by the orthogonality assumption. By a direct calculation, it holds that

\[
\left| \text{Re} \int \chi_R^+ h \partial_x T_y Q \, dx - \text{Re} \int h \partial_x (\chi_R^+ T_y Q) \, dx \right| \lesssim R^{-1} \| h \|_{L^2}.
\]

Therefore, we obtain

\[
\| r \|_{H^1} = \| T_{-y} r \|_{H^1} = |\alpha| \| \partial_x Q \|_{H^1} \lesssim \frac{1}{R} \| h \|_{L^2}
\]

and thus

\[
| \Phi(T_{-y} r) | \lesssim \frac{1}{R^2} \| h \|^2_{L^2}.
\]

(6)

Now, since \( \Phi \) is bilinear, we get

\[
\Phi(T_{-y} (\chi_R^+ h)) = \Phi(T_{-y} \tilde{h} + T_{-y} r) = \Phi(T_{-y} \tilde{h}) + \Phi(T_{-y} r) + 2 \Phi(T_{-y} \tilde{h}, T_{-y} r).
\]

(7)

Then, by Lemma 23, we have

\[
\Phi(T_{-y} \tilde{h}) \gtrsim \| T_{-y} \tilde{h} \|^2_{H^1}.
\]

(8)

Moreover, we have

\[
\| T_{-y} \tilde{h} \|^2_{H^1} \gtrsim \| T_{-y} (\chi_R^+ h) \|^2_{H^1} - \| T_{-y} r \|^2_{H^1} \gtrsim \| \chi_R^+ h \|^2_{H^1} - \frac{1}{R^2} \| h \|^2_{H^1}.
\]

(9)

Now, we have

\[
| \Phi(T_{-y} \tilde{h}, T_{-y} r) | \lesssim \frac{1}{R} \| \tilde{h} \||H^1| \| h \|_{H^1} \lesssim \frac{1}{R} \| h \|^2_{H^1}.
\]

(10)

Combining (6)–(10), we obtain

\[
\Phi(T_{-y} (\chi_R^+ h)) \gtrsim \| \chi_R^+ h \|^2_{H^1} - \frac{1}{R} \| h \|^2_{H^1}.
\]

By the symmetry, we have \( 2 \| \chi_R^+ h \|^2_{H^1} = \| \chi_R h \|^2_{H^1} \). This completes the proof. \( \square \)
4 Modulation

We start with a simple version of modulation.

**Lemma 25** There exists $\mu_0 > 0$ and a function $\varepsilon : (0, \mu_0) \to (0, \infty)$ with $\varepsilon(\mu) \to 0$ as $\mu \to 0$ such that the following holds. For any $\mu < \mu_0$ and for all $f \in H_{\text{odd}}^1(\mathbb{R})$ satisfying $E(f) = 2E(Q)$, $M(f) = 2M(Q)$ and $\mu(f) < \mu$, there exist $(\theta, y) \in \mathbb{R} \times [0, \infty)$ such that

$$
\|f - e^{i\theta} R_y Q\|_{H^1} \leq \varepsilon(\mu). 
$$

**Proof** We use a contradiction argument. We suppose that the statement fails. Then there exists $\mu_0 > 0$ such that for any $\mu < \mu_0$ and for all $f \in H_{\text{odd}}^1(\mathbb{R})$ satisfying $E(f) = 2E(Q)$, $M(f) = 2M(Q)$ and $\mu(f) < \mu$, there exist $(\theta, y) \in \mathbb{R} \times [0, \infty)$ such that

$$
\inf_{\theta \in \mathbb{R}} \inf_{y \in [0, \infty)} \|f - e^{i\theta} R_y Q\|_{H^1} > \varepsilon_0. 
$$

Since $f$ is odd, we have $\mathbb{I}_{(0, \infty)} f \in H^1(\mathbb{R})$ and it holds that

$$
S(\mathbb{I}_{(0, \infty)} f_n) \to S(Q) \text{ and } K(\mathbb{I}_{(0, \infty)} f_n) \to K(Q) = 0. 
$$

Therefore, $\{\mathbb{I}_{(0, \infty)} f_n\}$ is a minimizing sequence, and we obtain $(\theta, y) \in \mathbb{R}^2$ such that $\mathbb{I}_{(0, \infty)} f_n \to e^{i\theta} T_y Q$ in $H^1$ by the characterization of the ground state. By the symmetry, we also have $\mathbb{I}_{(-\infty, 0)} f_n \to -e^{i\theta} T_{-y} Q$ in $H^1$. Therefore it holds that

$$
\|f_n - e^{i\theta} R_y Q\|_{H^1} \to 0.
$$

This is a contradiction. $\square$

We have the following modulation with orthogonality conditions.

**Lemma 26** (Modulation) Let $R > 0$ be sufficiently large. There exist $\mu_0 > 0$ and a function $\varepsilon : (0, \mu_0) \to (0, \infty)$ with $\varepsilon(\mu) \to 0$ as $\mu \to 0$ such that the following holds. For any $\mu < \mu_0$ and for all $f \in H_{\text{odd}}^1(\mathbb{R})$ satisfying $E(f) = 2E(Q)$, $M(f) = 2M(Q)$ and $\mu(f) < \mu$, there exist $(\tilde{\theta}, y) \in \mathbb{R} \times (R, \infty)$ such that

$$
\|e^{-i\tilde{\theta}} f - R_y Q\|_{H^1} < \varepsilon(\mu)
$$

and

$$
\begin{align*}
\text{Im} \int_{\mathbb{R}} g \chi_R^+ T_y Q dx &= 0, \\
\text{Re} \int_{\mathbb{R}} g \partial_x (\chi_R^+ T_y Q) dx &= 0,
\end{align*}
$$

where $g = e^{-i\tilde{\theta}} f - R_y Q$.

**Proof** We define

$$
J(\tilde{\theta}, y, v) = \begin{pmatrix} J_1(\tilde{\theta}, y, v) \\ J_2(\tilde{\theta}, y, v) \end{pmatrix} := \begin{pmatrix} \text{Im} \int_{\mathbb{R}} (e^{-i\tilde{\theta}} v - R_y Q) \chi_R^+ T_y Q dx \\ \text{Re} \int_{\mathbb{R}} (e^{-i\tilde{\theta}} v - R_y Q) \partial_x (\chi_R^+ T_y Q) dx \end{pmatrix}
$$
for $\tilde{\theta} \in \mathbb{R}$, $y \gg R, v \in H^1_{\text{odd}}$. Then $J(0, y, \mathcal{R}_y Q) = 0$. We have
\[
\frac{\partial J_1}{\partial \tilde{\theta}}(0, y, \mathcal{R}_y Q) = -\int_{\mathbb{R}} \mathcal{R}_y Q \chi_R^+ T_y Q dx
= -\int_{\mathbb{R}} |T_y Q|^2 dx + O(e^{-2y})
= -\|Q\|_{L^2}^2 + O(e^{-2y})
\]
and
\[
\frac{\partial J_2}{\partial y}(0, y, \mathcal{R}_y Q) = -\int_{\mathbb{R}} \partial_y (T_y Q + T_{-y} Q) \partial_x (\chi_R^+ T_y Q) dx
= \|\partial_x Q\|_{L^2}^2 + O(R^{-1} + e^{-2y}).
\]
Therefore we have
\[
\frac{\partial J(\tilde{\theta}, y, v)}{\partial (\tilde{\theta}, y)}(0, y, \mathcal{R}_y Q)
= \left(\begin{array}{cc}
-\int_{\mathbb{R}} \mathcal{R}_y Q \chi_R^+ T_y Q dx & 0 \\
0 & -\int_{\mathbb{R}} \partial_x (T_y Q + T_{-y} Q) \partial_x (\chi_R^+ T_y Q) dx
\end{array}\right)
= \left(\begin{array}{cc}
-\|Q\|_{L^2}^2 + O(e^{-2y}) & 0 \\
0 & \|\partial_x Q\|_{L^2}^2 + O(R^{-1} + e^{-2y})
\end{array}\right).
\]
This is invertible for large $R$ and $y$. By the implicit function theorem, we get a function $(\tilde{\theta}, y) : H^1_{\text{odd}} \rightarrow \mathbb{R} \times \mathbb{R}$ such that
\[
\text{Im} \int_{\mathbb{R}} (e^{-i\tilde{\theta}(v)} - \mathcal{R}_y(v) Q) \chi_R^+ T_y(v) Q dx
= \text{Re} \int_{\mathbb{R}} (e^{-i\tilde{\theta}(v)} - \mathcal{R}_y(v) Q) \partial_x (\chi_R^+ T_y(v) Q) dx
= 0.
\]
This completes the proof. \(\square\)

Let $u$ be an odd solution satisfying
\[
M(u(t)) = 2M(Q) \quad \text{and} \quad E(u(t)) = 2E(Q).
\] (13)
We set $I_{\mu_0} := \{ t \in I_{\text{max}} : \mu(t) < \mu_0 \}$, where $I_{\text{max}}$ denotes the maximal existence time interval of the solution. By Lemma 26, we have $C^1$ functions $\tilde{\theta} = \tilde{\theta}(t)$ and $y = y(t)$ for $t \in I_{\mu_0}$. We set $\theta := \tilde{\theta} - 1$. We also have orthogonality conditions (12). We set
\[
u(t, x) = e^{i\tilde{\theta}(t) + it} (\mathcal{R}_y(t) Q(x) + g(t, x))
= e^{i\tilde{\theta}(t) + it} (\mathcal{R}_y(t) Q(x) + \rho(t) \mathcal{G}_{R, y(t)}(Q(x) + h(t, x)),
\] (14)
where
\[
\rho(t) := \frac{\text{Re} \int g \chi_R^+(T_y(t) Q)^p dx}{\int (\chi_R^+)^2 (T_y(t) Q)^{p+1} dx}.
\] (15)
Then it follows from (12), (14) and (15) that
\[
\text{Im} \int_{\mathbb{R}} h \chi_R^+ T_y(t) Q dx = \text{Re} \int_{\mathbb{R}} h \partial_x (\chi_R^+ T_y(t) Q) dx = \text{Re} \int_{\mathbb{R}} h \chi_R^+(T_y(t) Q \omega)^p dx = 0.
\] (16)
4.1 Estimate of the parameters

We give estimates of the parameters.

**Lemma 27** We have

\[
|\rho| \lesssim \|g\|_{L^2}, \\
\|g\|_{H^1} \lesssim |\rho| + \|h\|_{H^1}, \\
\|h\|_{H^1} \lesssim |\rho| + \|g\|_{H^1} \lesssim \|g\|_{H^1}.
\]

**Proof** These follow from \( g = \rho \mathcal{G}_{R,y} Q + h \) and the definition of \( \rho \). \( \square \)

By the above lemma and \( \|g\|_{H^1} \leq \mu_0 \), taking \( \mu_0 \) sufficiently small, we may assume that \( \|h\|_{H^1} \) and \( |\rho| \) are sufficiently small.

**Lemma 28** We have

\[
|\rho| \lesssim \|h\|_{H^1} + (1 + y)e^{-2y}.
\]

**Proof** We have

\[
M(u(t)) = M(\mathcal{R}_y Q + \rho \mathcal{G}_{R,y} Q + h) \\
= 2\|Q\|_{L^2}^2 - 2 \int T_y Q T_{-y} Q dx \\
+ 2 \text{Re} \int \mathcal{R}_y Q (\rho \mathcal{G}_{R,y} Q + h) dx + \|\rho \mathcal{G}_{R,y} Q + h\|_{L^2}^2.
\]

It holds from \( M(u(t)) = 2M(Q) \) that

\[
-2 \int T_y Q T_{-y} Q dx + 2 \text{Re} \int \mathcal{R}_y Q (\rho \mathcal{G}_{R,y} Q + h) dx + \|\rho \mathcal{G}_{R,y} Q + h\|_{L^2}^2 = 0.
\]

and thus

\[
2\rho \text{Re} \int \mathcal{R}_y Q \mathcal{G}_{R,y} Q dx = 2 \int T_y Q T_{-y} Q dx + 2 \text{Re} \int \mathcal{R}_y Q h dx + \|\rho \mathcal{G}_{R,y} Q + h\|_{L^2}^2.
\]

(17)

Since \( |\text{Re} \int \mathcal{R}_y Q \mathcal{G}_{R,y} Q dx| > C \), where \( C \) is independent of \( y \), we get

\[
|\rho| \lesssim (1 + y)e^{-2y} + \|h\|_{H^1} + O(|\rho|^2 + \|h\|_{H^1}^2)
\]

by Lemma 15. Since \( |\rho| \) and \( \|h\|_{H^1} \) is small, we get the statement. \( \square \)

**Lemma 29** We have

\[
|\rho| \approx |\mu(t)| + O(e^{-2y} + \|h\|_{H^1}^2).
\]

**Proof** By a direct calculation, we have

\[
\mu(t) = 2 \int \partial_x T_y Q \partial_x T_{-y} Q dx - 2\rho \text{Re} \int \partial_x (\mathcal{R}_y Q) \partial_x (\mathcal{G}_{R,y} Q) dx \\
- 2 \text{Re} \int \partial_x (\mathcal{R}_y Q) \partial_x h dx - \|\partial_x (\rho \mathcal{G}_{R,y} Q + h)\|_{L^2}^2.
\]

(18)
We note that
\[ -\partial_x^2 (\mathcal{R}_y Q) = -\mathcal{R}_y Q + (T_y Q)^p - (T_{-y} Q)^p \]  
(19)
since \( Q \) satisfies the elliptic equation. It holds from integration by parts and (19) that
\[
\text{Re} \int \partial_x (\mathcal{R}_y Q) \partial_x h dx = -\text{Re} \int \mathcal{R}_y Q h dx + \text{Re} \int \{(T_y Q)^p - (T_{-y} Q)^p\} h dx.
\]
(20)
The second term of (20) is estimated as follows.
\[
\left| \text{Re} \int \{(T_y Q)^p - (T_{-y} Q)^p\} h dx \right| \lesssim e^{-py} \|h\|_{H^1}.
\]
(21)
Indeed, we have
\[
\text{Re} \int \{(1 - \chi_R^+(T_y Q)^p - (1 - \chi_R^-)(T_{-y} Q)^p\} h dx
\]
\[
= \text{Re} \int \{(1 - \chi_R^+(T_y Q)^p - (T_{-y} Q)^p\} h dx
\]
\[
+ \text{Re} \int \{(1 - \chi_R^+(T_y Q)^p - (1 - \chi_R^-)(T_{-y} Q)^p\} h dx.
\]
(22)
The first term in the right hand side is 0 by (16) and the symmetry. It follows from Lemma 16
that the second term is estimated by
\[
\left| \text{Re} \int \{(1 - \chi_R^+(T_y Q)^p - (1 - \chi_R^-)(T_{-y} Q)^p\} h dx \right| \lesssim_R e^{-py} \|h\|_{H^1}.
\]
(23)
Thus we get
\[
\text{Re} \int \partial_x (\mathcal{R}_y Q) \partial_x h dx = -\text{Re} \int \mathcal{R}_y Q h dx + O(e^{-py} \|h\|_{H^1}).
\]
(24)
Now we recall (17). We have
\[
-2 \text{Re} \int \mathcal{R}_y Q h dx = -2 \int T_y Q T_{-y} Q dx
\]
\[
+ 2\rho \text{Re} \int \mathcal{R}_y Q \mathcal{G}_{R,y} Q dx + \|\rho \mathcal{G}_{R,y} Q + h\|_{L^2}^2.
\]
(25)
Combining (18), (24) and (25), we get
\[
\mu(t) = 2 \int \partial_x T_y Q \partial_x T_{-y} Q dx - 2\rho \text{Re} \int \partial_x (\mathcal{R}_y Q) \partial_x (\mathcal{G}_{R,y} Q) dx
\]
\[
+ 2 \int T_y Q T_{-y} Q dx - 2\rho \text{Re} \int \mathcal{R}_y Q \mathcal{G}_{R,y} Q dx
\]
\[
+ O(|\rho|^2 + \|h\|_{H^1}^2 + e^{-py} \|h\|_{H^1}).
\]
where we used \( \|\rho \mathcal{G}_{R,y} Q + h\|_{H^1}^2 \lesssim |\rho|^2 + \|h\|_{H^1}^2 \). By (1), we have
\[
\int \partial_x T_y Q \partial_x T_{-y} Q dx + \int T_y Q T_{-y} Q dx = \int (T_y Q)^p T_{-y} Q \approx e^{-2y}.
\]
Thus we get

\[
2 \rho \left( \int \partial_x (R_y Q) \partial_x (G_{R,y} Q) dx + \text{Re} \int R_y Q G_{R,y} Q dx \right)
\approx -\mu(t) + e^{-2y} + O(|\rho|^2 + \|h\|_{H^1}^2 + e^{-py} \|h\|_{H^1}).
\]

By integration by parts and (19) again, we get

\[
\int \partial_x (R_y Q) \partial_x (G_{R,y} Q) dx + \text{Re} \int R_y Q G_{R,y} Q dx = \int \{(T_y Q)^p - (T_{-y} Q)^p\} G_{R,y} Q dx
\]

\[
= 2 \int \chi_R^+(T_y Q)^{p+1} dx + O(e^{-2y})
\]

\[
= 2 \int (T_y Q)^{p+1} dx + O(e^{-2y}),
\]

where we used Lemma 16. Therefore we get

\[
|\rho| \approx |\mu(t)| + e^{-2y} + O(|\rho|^2 + \|h\|_{H^1}^2 + e^{-py} \|h\|_{H^1}).
\]

This means that

\[
|\rho| \approx \mu(t) + O(e^{-2y} + \|h\|_{H^1}^2).
\]

This completes the proof. \(\square\)

Next we consider the estimate of \(\mu\) by using \(S(u(t)) = 2S(Q)\). The following is a key in our proof.

**Lemma 30** Let \(y\) be a positive number. For sufficiently large \(y > 0\), it holds that

\[
S(R_y Q) - 2S(Q) \approx e^{-2y}.
\]

**Proof** By a direct calculation, we have

\[
S(R_y Q) - 2S(Q) = - \int \partial_x T_y Q \partial_x T_{-y} Q dx - \int T_y Q T_{-y} Q dx - \frac{1}{p+1} \left( \|R_y Q\|_{L^{p+1}}^{p+1} - 2\|Q\|_{L^{p+1}}^{p+1} \right).
\]

By (1), we get

\[
- \int \partial_x T_y Q \partial_x T_{-y} Q dx - \int T_y Q T_{-y} Q dx = - \int (T_y Q)^p T_{-y} Q dx
\]

\[
= - \int_0^\infty (T_y Q)^p T_{-y} Q dx + O(e^{-py}),
\]

where we used Lemma 16. Since \(|R_y Q|^{p+1}\) is even, it holds from Lemma 16 that

\[
\|R_y Q\|_{L^{p+1}}^{p+1} - 2\|Q\|_{L^{p+1}}^{p+1} = 2 \int_0^\infty (|R_y Q|^{p+1} - |T_y Q|^{p+1}) dx + O(e^{-(p+1)y}).
\]
Thus we get

\[ \int_0^\infty (|\mathcal{R}_y Q|^{p+1} - |T_y Q|^{p+1}) dx \]

\[ = - \int_0^\infty (p + 1)(T_y Q)^p T_{-y} Q + p(p + 1)(T_y Q)^{p-1} (T_{-y} Q)^2 + O((T_{-y} Q)^3) dx \]

\[ = - \int_0^\infty (p + 1)(T_y Q)^p T_{-y} Q dx + O(e^{-3y}). \]

Therefore, we obtain

\[ S(\mathcal{R}_y Q) - 2S(Q) = \int_0^\infty (T_y Q)^p T_{-y} Q dx + O(e^{-3y}) \]

\[ \approx e^{-2y} + O(e^{-3y}). \]

For large \( y \), we obtain the result. \( \square \)

Since \( y(t) > R \), the assumption in the above lemma is satisfied by taking \( R \) large enough.

**Lemma 31** We have

\[ e^{-2y} + \|h\|_{H^1}^2 \lesssim |\rho|^2. \]

**Proof** We have

\[ 0 = S(u(t)) - 2S(Q) = S(\mathcal{R}_y Q + g) - S(\mathcal{R}_y Q) + S(\mathcal{R}_y Q) - 2S(Q). \]

By Lemma 30, the second term is estimated by

\[ S(\mathcal{R}_y Q) - 2S(Q) \gtrsim e^{-2y}. \]

By Taylor expansion, we have

\[ S(\mathcal{R}_y Q + g) - S(\mathcal{R}_y Q) = (S'(\mathcal{R}_y Q), g) + \frac{1}{2} (S''(\mathcal{R}_y Q)g, g) + o(\|g\|^2_{H^1}), \]

where we note that

\[ S'(\mathcal{R}_y Q) = -\partial_x^2 \mathcal{R}_y Q + \mathcal{R}_y Q - |\mathcal{R}_y Q|^{p-1} \mathcal{R}_y Q, \]

\[ (S''(\mathcal{R}_y Q)\phi, \psi) = \text{Re} \int \nabla \phi \nabla \psi + \omega \phi \psi dx - \text{Re} \int |\mathcal{R}_y Q|^{p-1}(\phi \psi_1 + i\phi_2)(\mathcal{R}_y Q dx \]

for \( \phi = \phi_1 + i\phi_2, \psi \in H^1 \).

First we give an estimate of \( (S'(\mathcal{R}_y Q), g) \). By (19), we have

\[ (S'(\mathcal{R}_y Q), g) = \langle (T_y Q)^p - (T_{-y} Q)^p - |\mathcal{R}_y Q|^{p-1} \mathcal{R}_y Q, g \rangle. \]

By a nonlinear estimate, we obtain

\[ |(T_y Q)^p - (T_{-y} Q)^p - |\mathcal{R}_y Q|^{p-1} \mathcal{R}_y Q| \lesssim \|(T_y Q)^{p-2} + (T_y Q)^{p-2}\) T_y Q T_{-y} Q. \]

Thus we get

\[ \int |(T_y Q)^p - (T_{-y} Q)^p - |\mathcal{R}_y Q|^{p-1} \mathcal{R}_y Q| g | dx \]

\[ \lesssim \int \|(T_y Q)^{p-2} + (T_y Q)^{p-2}\) T_y Q T_{-y} Q| g | dx \]

\[ \lesssim e^{-2y} \|g\|_{L^2}. \]

(26)
By the Young inequality and \( \|g\|^2_{L^2} \lesssim |\rho|^2 + \|h\|^2_{H^1}, \) we get
\[
|\langle S'(\mathcal{R}_y Q), g \rangle| \lesssim e^{-2y}\|g\|_{L^2} \lesssim C e^{-4y} + |\rho|^2 + \varepsilon \|h\|^2_{H^1}
\]
for arbitrary small \( \varepsilon > 0. \) Next we consider \( \langle S''(\mathcal{R}_y Q)g, g \rangle. \) We set
\[
B(\phi, \psi) := \langle S''(\mathcal{R}_y Q)\phi, \psi \rangle.
\]
Since \( B \) is bilinear and \( g = \rho \mathcal{G}_{R,y} Q + h, \) we have
\[
B(g, g) = |\rho|^2 B(\mathcal{G}_{R,y} Q, \mathcal{G}_{R,y} Q) + 2\rho B(\mathcal{G}_{R,y} Q, h) + B(h, h).
\]
It is obvious that
\[
|\rho|^2 |B(\mathcal{G}_{R,y} Q, \mathcal{G}_{R,y} Q)| \lesssim |\rho|^2.
\]
Now we have
\[
B(\mathcal{G}_{R,y} Q, h) = B(\mathcal{R}_y Q, h) + B(\mathcal{G}_{R,y} Q - \mathcal{R}_y Q, h). \tag{27}
\]
The second term of (27) is estimated by
\[
|B(\mathcal{G}_{R,y} Q - \mathcal{R}_y Q, h)| \lesssim e^{-y}\|h\|_{H^1} + R^{-1}\|h\|_{H^1} \tag{28}
\]
since we have
\[
\int |\mathcal{G}_{R,y} Q - \mathcal{R}_y Q| |h| dx = \int |\chi^-_R - 1| |T_y Q| |h| + |1 - \chi^-_R| |T_{-y} Q| |h| dx \lesssim e^{-y}\|h\|_{L^2}
\]
and
\[
\int |\partial_x (\mathcal{G}_{R,y} Q - \mathcal{R}_y Q)||\partial_x h| dx = \int |\chi^-_R - 1||\partial_x T_y Q||\partial_x h| + |1 - \chi^-_R||\partial_x T_{-y} Q||\partial_x h| dx + O(R^{-1}\|h\|_{L^2})
\]
\[
\lesssim e^{-y}\|\partial_x h\|_{L^2} + O(R^{-1}\|h\|_{L^2}).
\]
The first term of (27) is estimated as follows. By (19) and the fact that \( \mathcal{R}_y Q \) is real-valued, we get
\[
B(\mathcal{R}_y Q, h) = \int_{\mathbb{R}} \{(T_y Q)^p - (T_{-y} Q)^p - p|\mathcal{R}_y Q|^{p-1} \mathcal{R}_y Q\} h dx
\]
\[
= p \int_{\mathbb{R}} \{(T_y Q)^p - (T_{-y} Q)^p - |\mathcal{R}_y Q|^{p-1} \mathcal{R}_y Q\} h dx
\]
\[
+ (1-p) \int_{\mathbb{R}} \{(T_y Q)^p - (T_{-y} Q)^p\} h dx.
\]
This first term is estimated by
\[
\left| \int \{(T_y Q)^p - (T_{-y} Q)^p - |\mathcal{R}_y Q|^{p-1} \mathcal{R}_y Q\} h dx \right| \lesssim e^{-2y}\|h\|_{H^1}
\]
as in (26). We recall (21) and thus the second term is estimated by
\[
\left| \int \{(T_y Q)^p - (T_{-y} Q)^p\} h dx \right| \lesssim e^{-py}\|h\|_{H^1}.
\]
Thus we have
\[ |B(\mathcal{R}_{\gamma} Q, h)| \lesssim e^{-2\gamma} \| h \|_{H^1}. \quad (29) \]

Combining (27), (28) and (29), we obtain
\[
|\rho| |B(G_{R_{\gamma} Q}, h)| \lesssim e^{-\gamma} |\rho| \| h \|_{H^1} + R^{-1} |\rho| \| h \|_{H^1} \\
\lesssim |\rho|^2 + e^{-2\gamma} \| h \|_{H^1}^2 + R^{-2} \| h \|_{H^1}^2
\]

by the Young inequality. At last, we calculate \( B(h, h) \). We divide it into three terms as follows.

We have
\[ B(h, h) = B(\chi_R h, \chi_R h) + 2B(\chi_R h, \chi_R^c h) + B(\chi_R^c h, \chi_R^c h). \]

We consider the middle term \( B(\chi_R h, \chi_R^c h) \). We have
\[
B(\chi_R h, \chi_R^c h) = \langle \partial_x (\chi_R h), \partial_x (\chi_R^c h) \rangle + \langle \chi_R h, \chi_R^c h \rangle \\
- p \langle |\mathcal{R}_{\gamma} Q|^{p-1} \chi_R h_1, \chi_R^c h_1 \rangle - \langle |\mathcal{R}_{\gamma} Q|^{p-1} \chi_R h_2, \chi_R^c h_2 \rangle.
\]

It holds that
\[
\langle \partial_x (\chi_R h), \partial_x (\chi_R^c h) \rangle = \int \chi_R \chi_R^c \| \partial_x h \|^2 dx + O(R^{-1} \| h \|_{H^1}^2)
\]

and
\[
\langle \chi_R h, \chi_R^c h \rangle = \int \chi_R \chi_R^c \| h \|^2 dx.
\]

Moreover, by Lemma 16, we have
\[
|p \langle |\mathcal{R}_{\gamma} Q|^{p-1} \chi_R h_1, \chi_R^c h_1 \rangle - \langle |\mathcal{R}_{\gamma} Q|^{p-1} \chi_R h_2, \chi_R^c h_2 \rangle| \\
\lesssim \int |\mathcal{R}_{\gamma} Q|^{p-1} \chi_R \chi_R^c \| h \|^2 dx \\
\lesssim e^{-(p-1)\gamma} \| h \|_{L^2}^2.
\]

Thus, we can estimate the middle term as follows.
\[
B(\chi_R h, \chi_R^c h) \approx \int \chi_R \chi_R^c (|h|^2 + |\partial_x h|^2) dx + O(R^{-1} \| h \|_{H^1}^2 + e^{-(p-1)\gamma} \| h \|_{H^1}^2).
\]

In the similar way, we can estimate the third term \( B(\chi_R^c h, \chi_R^c h) \) as follows.
\[
B(\chi_R^c h, \chi_R^c h) \approx \int (\chi_R^c)^2 (|h|^2 + |\partial_x h|^2) dx + O(R^{-1} \| h \|_{H^1}^2 + e^{-(p-1)\gamma} \| h \|_{H^1}^2).
\]

Next we consider the first term \( B(\chi_R h, \chi_R h) \). By the symmetry, we have
\[
B(\chi_R h, \chi_R h) \\
= 2 \left[ \int_0^\infty |\partial_x (\chi_R^+ h)|^2 + |\chi_R^+ h|^2 dx - \int_0^\infty |\mathcal{R}_{\gamma} Q|^{p-1} \{ p(\chi_R^+ h_1)^2 + (\chi_R^+ h_2)^2 \} dx \right] \\
= 2 \Phi(T_{-\gamma}(\chi_R^+ h)) + 2 \int_0^\infty (|T_{\gamma} Q|^{p-1} - |\mathcal{R}_{\gamma} Q|^{p-1}) \{ p(\chi_R h_1)^2 + (\chi_R h_2)^2 \} dx.
\]
Since it holds that
\[
\left| \int_0^\infty (|T_y Q|^{p-1} - |R_y Q|^{p-1}) (p(\chi_R h_1)^2 + (\chi_R h_2)^2) \right| \leq \int_0^\infty (|T_y Q|^{p-2} T_{-y} Q + (T_{-y} Q)^{p-1}) |h|^2 dx \leq e^{-2y} \| h \|_{H^1}^2.
\]

For \( \Phi \), by Lemma 24, we have
\[
\Phi(T_{-y}(\chi_R h)) \geq \| \chi_R h \|_{H^1}^2 - R^{-1} \| h \|_{H^1}^2.
\]

Therefore, combining all estimates above, we get
\[
e^{-2y} + \| h \|_{H^1}^2 \lesssim |\rho|^2 + R^{-1} \| h \|_{H^1}^2 + e^{-2y} \| h \|_{H^1}^2 + C_\varepsilon e^{-4y} + \varepsilon \| h \|_{H^1}^2.
\]

where we used
\[
\| \chi_R h \|_{H^1}^2 + \int \chi_R \chi_R^c(|h|^2 + |\partial_x h|^2) dx + \int (\chi_R^c)^2(|h|^2 + |\partial_x h|^2) dx \geq \| h \|_{H^1}^2 - R^{-1} \| h \|_{H^1}^2.
\]

This implies
\[
e^{-2y} + \| h \|_{H^1}^2 \lesssim |\rho|^2
\]

by taking small \( \varepsilon > 0 \) and sufficiently large \( R \). \( \square \)

**Corollary 32** We have
\[
e^{-2y} + \| h \|_{H^1}^2 \lesssim \mu(t)^2.
\]

**Proof** By Lemma 29, we have
\[
|\rho|^2 \lesssim \mu(t)^2 + e^{-4y} + \| h \|_{H^1}^4.
\]

This implies the result by taking sufficiently small \( \mu_0 \), which ensures the smallness of \( \| h \|_{H^1} \). \( \square \)

**Corollary 33** We have
\[
(1 + y)e^{-2y} \lesssim \mu(t)^{2-\delta} \lesssim \mu_0^{1-\delta} \mu(t),
\]

where \( \delta > 0 \) is arbitrary small.

**Proof** It follows from Corollary 32 and \( \mu(t) < \mu_0 \) that
\[
(1 + y)e^{-2y} \lesssim e^{-(2-\delta)y} \lesssim \mu(t)^{2-\delta} \lesssim \mu_0^{1-\delta} \mu(t)
\]

for any small \( \delta \). \( \square \)
4.2 Estimate of the derivatives of the parameters

The derivatives of the parameters are estimated as follows.

**Lemma 34** We have

\[ |\theta'(t)| + |\rho'(t)| + |y'(t)| \lesssim \mu(t). \]

**Proof** Since we have

\[ h = e^{-i\theta(t) - it} u(t) - (R_{y(t)} Q + \rho(t) G_{R,y(t)} Q), \]

we get

\[ h' = -i(\theta'(t) + 1) e^{-i\theta(t) - it} u(t) + e^{-i\theta(t) - it} u'(t) \]
\[ - (y'(t) R_{y(t)}^+ \partial_x Q + \rho'(t) G_{R,y(t)} Q - \rho(t) y'(t) G_{R,y(t)}^+ \partial_x Q), \]

where we define

\[ R_{y(t)}^+ f(x) := T_y f(x) + T_{-y} f(x), \]
\[ G_{R,y(t)}^+ f(x) := \chi_R^+(x) T_y f(x) + \chi_R^-(x) T_{-y} f(x). \]

From this, we have

\[ ih' + \partial_x^2 h = \theta'(t)(R_{y(t)} Q + g) + h \]
\[ - |R_{y(t)} Q + g|^p - (R_{y(t)} Q + g) + |R_{y(t)} Q|^p \]
\[ - |R_{y(t)} Q|^p + |T_y Q|^p - |T_{-y} Q|^p \]
\[ - i(-y'(t) R_{y(t)}^+ \partial_x Q + \rho'(t) G_{R,y(t)} Q - \rho(t) y'(t) G_{R,y(t)}^+ \partial_x Q) \]
\[ - \rho(t) \{ (\partial_x^2 \chi_R^+) T_y Q - (\partial_x^2 \chi_R^-) T_{-y} Q \} \]
\[ - \rho(t) \{ (\partial_x \chi_R^+) \partial_x T_y Q - (\partial_x \chi_R^-) \partial_x T_{-y} Q \} \]
\[ + \rho(t) G_{R,y(t)}(Q^p). \]

where we use (1) (or (19)).

Since we have

\[ \text{Im} \int h' \chi_R^+ T_y Q dx = 0 \]

by the orthogonality condition (16), multiplying \( \chi_R^+ T_y Q \) into this equation and taking the integral and the real part, we obtain

\[ \theta'(t) = O(||h||_{H^1} + ||g||_{H^1} + ||\rho|| + e^{-2y}). \]

Similarly, since we have

\[ \text{Re} \int h' \chi_R^+ (T_y Q)^p dx = 0 \]

by the orthogonality condition (16), multiplying \( \chi_R^+ (T_y Q)^p \) into this equation and taking the integral and the imaginary part, we obtain

\[ \rho'(t) = O(||h||_{H^1} + ||\theta'(t)|| ||g||_{H^1} + ||g||_{H^1} + e^{-2y} + e^{-2y} |y'(t)|). \]
In the same way, since we have
\[ \text{Re} \int h' \partial_x (\chi_R^- T_y Q) \, dx = 0 \]
by the orthogonality condition (16), multiplying \( \partial_x (\chi_R^- T_y Q) \) into this equation and taking the integral and the imaginary part, we obtain
\[ y'(t) = O(\|h\|_{H^1} + |\theta'(t)|\|g\|_{H^1} + e^{-2y}|\rho(t)| + \|g\|_{H^1} + |\rho| + e^{-2y}). \]
These estimates imply the result. \( \square \)

5 Proof of scattering

5.1 Compactness of a critical element

Suppose that Proposition 14 (1) fails. Then there exists an odd solution \( u \) with
\[ E(u) = 2E(Q), \quad M(u) = 2M(Q) \text{ and } K(u(t)) > 0, \]
where the solution is global by the variational argument, such that
\[ \|u\|_{L^6_t L^\infty_x (\mathbb{R} \times \mathbb{R})} = \infty. \]
We call the solution a critical element. We may consider only the positive time direction by time reversibility. Thus we may suppose that
\[ \|u\|_{L^6_t L^\infty_x ((0, \infty) \times \mathbb{R})} = \infty. \]

Proposition 35 (Compactness of a critical element) Let \( u \) be an odd solution with
\[ E(u) = 2E(Q), \quad M(u) = 2M(Q) \text{ and } K(u(t)) > 0, \] (30)
such that
\[ \|u\|_{L^6_t L^\infty_x ((0, \infty) \times \mathbb{R})} = \infty. \]
Then the solution \( u \) satisfies the following compactness property: There exists a function \( x : [0, \infty) \to [0, \infty) \) such that for any \( \varepsilon > 0 \) there exists \( R = R(\varepsilon) > 0 \) such that
\[ \int_{\{|x-x(t)| > R\} \cap \{|x+x(t)| > R\}} |\partial_x u(t,x)|^2 + |u(t,x)|^2 \, dx < \varepsilon \]
for all \( t \in [0, \infty) \).

Proof Let \( \{\tau_n\} \) be an arbitrary time sequence such that \( \tau_n \to \infty \). The sequence \( \{u(\tau_n)\} \) is bounded in \( H^1(\mathbb{R}) \) by (30) and Proposition 9. By the linear profile decomposition (Lemma 22), we have, up to subsequence,
\[ u(\tau_n) = \sum_{j=1}^J e^{-i\xi_j^2 \frac{\tau_n}{\varepsilon}} R_{x_n} \psi_j^\tau \]
and the properties in the statement hold. We set \( \psi_n^\tau := e^{-i\xi_j^2 \frac{\tau_n}{\varepsilon}} R_{x_n} \psi_j^\tau \) for simplicity.

It is easy to show that \( J = 0 \) does not occur. Indeed, by the linear profile decomposition, if \( J = 0 \), then \( \|e^{i\xi_j^2 \frac{\tau_n}{\varepsilon}} u(\tau_n)\|_{L^6_t L^\infty_x ((0, \infty) \times \mathbb{R})} \to 0 \) as \( n \to \infty \) (see Remark 8). Then, by the
long time perturbation (Lemma 21), we get \( \|u\|_{L_t^2 L_x^4((\tau_n, \infty) \times \mathbb{R})} \lesssim 1 \) for large \( n \in \mathbb{N} \). This contradicts that \( u \) does not scatter.

Thus we have \( J \geq 1 \). We will prove that \( J = 1 \) by contradiction. Suppose that \( J \geq 2 \) and thus we have at least two non-zero functions \( \psi^1 \) and \( \psi^2 \). First, by the linear profile decomposition, we have

\[
\lim_{n \to \infty} \left( \sum_{j=1}^{J} M(\psi^j_n) + M(R^j_n) \right) = \lim_{n \to \infty} M(u(\tau_n)) = M(u_0) = 2M(Q),
\]

\[
\lim_{n \to \infty} \left( \sum_{j=1}^{J} \|\partial_x \psi^j_n\|_{L^2}^2 + \|\partial_x R^j_n\|_{L^2}^2 \right) = \lim_{n \to \infty} \|u(\tau_n)\|_{L^2}^2 \leq 2\|\partial_x Q\|_{L^2}^2.
\]

It holds that, for any \( j \) and for large \( n \),

\[
\|\psi^j_n\|_{L^2}^{1-\varsigma_c} \|\partial_x \psi^j_n\|_{L^2}^{\varsigma_c} < 2^{1/2} \|Q\|_{L^2}^{1-\varsigma_c} \|\partial_x Q\|_{L^2}^{\varsigma_c},
\]

\[
\|R^j_n\|_{L^2}^{1-\varsigma_c} \|\partial_x R^j_n\|_{L^2}^{\varsigma_c} < 2^{1/2} \|Q\|_{L^2}^{1-\varsigma_c} \|\partial_x Q\|_{L^2}^{\varsigma_c}.
\]

From these and Lemma 12, we get \( K(\psi^j_n), K(R^j_n) > 0 \). It follows from \( E \geq \frac{p-1}{2} K \) that \( E(\psi^j_n), E(R^j_n) \geq 0 \). Now, since we also have

\[
\lim_{n \to \infty} \left( \sum_{j=1}^{J} E(\psi^j_n) + E(R^j_n) \right) = \lim_{n \to \infty} E(u(\tau_n)) = E(u_0) = 2E(Q),
\]

we obtain \( E(\psi^j_n), E(R^j_n) \leq 2E(Q) \).

Since \( J \geq 2 \), there exists \( \delta > 0 \) such that \( M(\psi^j_n)^{1-\varsigma_c} E(\psi^j_n)^{\varsigma_c} < 2M(Q)^{1-\varsigma_c} E(Q)^{\varsigma_c} - \delta \) for all \( j \). By reordering, we may choose \( J_1, \ldots, J_4 \) such that

\[
1 \leq j \leq J_1 \Rightarrow t^j_n = 0 \ (\forall n \in \mathbb{N}) \text{ and } x^j_n = 0 \ (\forall n \in \mathbb{N}), \quad \text{(Case 1)}
\]

\[
J_1 + 1 \leq j \leq J_2 \Rightarrow t^j_n = 0 \ (\forall n \in \mathbb{N}) \text{ and } x^j_n \to \infty \ (n \to \infty), \quad \text{(Case 2)}
\]

\[
J_2 + 1 \leq j \leq J_3 \Rightarrow |t^j_n| \to \infty \ (n \to \infty) \text{ and } x^j_n = 0 \ (\forall n \in \mathbb{N}), \quad \text{(Case 3)}
\]

\[
J_3 + 1 \leq j \leq J_4 \Rightarrow |t^j_n| \to \infty \ (n \to \infty) \text{ and } x^j_n \to \infty \ (n \to \infty), \quad \text{(Case 4)}
\]

where we are assuming that there is no \( j \) such that \( a \leq j \leq b \) if \( a > b \). We will define nonlinear profiles associated with \( \psi^j_n \). If there is no \( j \) such that \( J_k + 1 \leq j \leq J_{k+1} \) for some \( k \in \{0, 1, 2, 3\} \), where \( J_0 = 0 \), then skip the construction of nonlinear profiles in the following steps.

**Case 1.** We first consider the case of \( 1 \leq j \leq J_1 \). By the orthogonality of the parameter \( t^j_n \) and \( x^j_n \), we note that \( J_1 = 0 \) or 1. (Skip this step if \( J_1 = 0 \).) We define a solution \( N \) to (NLS) with the initial data \( N(0, x) = R_0 \psi^1(x) = \psi^1(x) - \psi^1(-x) \). Then, the solution \( N \) is global and satisfies \( \|N\|_{L_t^2 L_x^4(\mathbb{R} \times \mathbb{R})} \lesssim 1 \) by \( [20] \), where the global dynamics for below the threshold for odd solutions is obtained, since \( M(R_0 \psi^1)^{1-\varsigma_c} E(R_0 \psi^1)^{\varsigma_c} < 2M(Q)^{1-\varsigma_c} E(Q)^{\varsigma_c} - \delta \) and \( K(R_0 \psi^1) \geq 0 \). 

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Case 2. We consider the case of $J_1 + 1 \leq j \leq J_2$. We define a solution $U^j$ to (NLS) with the initial data $ψ^j$. Now, we have
$$
\lim_{n \to \infty} M(ψ^j_n) = 2M(ψ^j) \quad \text{and} \quad \lim_{n \to \infty} E(ψ^j_n) = 2E(ψ^j).
$$
Thus we get $M(ψ^j)^{1−s_c}E(ψ^j)^{s_c} < M(Q)^{1−s_c}E(Q)^{s_c}−δ/2$ and we also obtain $K(ψ^j) ≥ 0$. It holds that $∥U^j∥_{L^s_tL^r_x(\mathbb{R}×\mathbb{R})} ≤ 1$ by [1, 14]. We set $U^j(t, x) := \mathcal{R}_{x_n}U^j(t)$. Then we also have $∥U^j∥_{L^s_tL^r_x(\mathbb{R}×\mathbb{R})} ≤ 2∥U^j∥_{L^s_tL^r_x(\mathbb{R}×\mathbb{R})} ≤ 1$.

Case 3. We consider the case of $J_2 + 1 \leq j \leq J_3$. If $j$ satisfies $t_n^j → −∞$, then we define a solution $V^j$ to (NLS) that scatters to $R_0ψ^j$ as $t → +∞$. Now, we have
$$
∥ψ^j_n∥_{L^{p+1}}^p = ∥e^{−ıt_n^jδ^2}ψ^j∥_{L^{p+1}}^p → 0
$$
by the dispersive estimate. Since we have
$$
\lim_{n \to \infty} M(ψ^j_n)^{1−s_c}E(ψ^j_n)^{s_c} = M(ψ^j)^{1−s_c}E(ψ^j)^{s_c},
$$
we obtain $M(V^j)^{1−s_c}E(V^j)^{s_c} = M(Q)^{1−s_c}E(Q)^{s_c}−δ/2$. We also have $K(V^j) ≥ 0$ since $lim_{n \to \infty}(K(V^j(t_n^j)) − K(ψ^j_n)) = 0$ and $lim_{n \to \infty} K(ψ^j_n) = ∥ψ^j_n∥^2_{H^1} > 0$. Therefore, by [20], we find that $V^j$ is global and satisfies $∥V^j∥_{L^s_tL^r_x(\mathbb{R}×\mathbb{R})} ≤ 1$. If $j$ satisfies $t_n^j → +∞$, then we define a solution $W^j$ to (NLS) that scatters to $R_0ψ^j$ as $t → +∞$. In the same way, we find that $W^j$ is global in both time directions and $∥W^j∥_{L^s_tL^r_x(\mathbb{R}×\mathbb{R})} ≤ 1$. We set $W^j(t, x) := W^j(t − t_n^j, x)$.

Case 4. We consider the case of $J_3 + 1 \leq j \leq J_4$. If $j$ satisfies $t_n^j → -∞$, then we define a solution $V^j$ to (NLS) that scatters to $ψ^j$ as $t → +∞$. Now, we have
$$
∥ψ^j_n∥_{L^{p+1}}^p = ∥e^{−ıt_n^jδ^2}R_{x_n}ψ^j∥_{L^{p+1}}^p ≤ 2∥e^{−ıt_n^jδ^2}ψ^j∥_{L^{p+1}}^p → 0
$$
by the dispersive estimate. Since we have
$$
\lim_{n \to \infty} M(ψ^j_n)^{1−s_c}E(ψ^j_n)^{s_c} = (2M(ψ^j))^{1−s_c} \left(2·∥ψ^j∥_{H^1}^2\right)^{s_c} = 2M(V^j)^{1−s_c}E(V^j)^{s_c},
$$
we obtain $M(V^j)^{1−s_c}E(V^j)^{s_c} = M(Q)^{1−s_c}E(Q)^{s_c}−δ/2$. We also have $K(V^j) ≥ 0$ since $lim_{n \to \infty}(K(V^j(t_n^j)) − K(e^{−ıt_n^jδ^2}ψ^j)) = 0$ and $lim_{n \to \infty} K(e^{−ıt_n^jδ^2}ψ^j) = ∥ψ^j∥^2_{H^1} > 0$. Therefore, by [1, 14], we find that $V^j$ is global and satisfies $∥V^j∥_{L^s_tL^r_x(\mathbb{R}×\mathbb{R})} ≤ 1$. If $j$ satisfies $t_n^j → +∞$, then we define a solution $V^j$ to (NLS) that scatters to $ψ^j$ as $t → +∞$. Then, in the same way, $V^j$ is global and $∥V^j∥_{L^s_tL^r_x(\mathbb{R}×\mathbb{R})} ≤ 1$. We set $V^j(t, x) := \mathcal{R}_{x_n}V^j(t − t_n^j, x)$. Then we have $∥V^j_n∥_{L^s_tL^r_x(\mathbb{R}×\mathbb{R})} ≤ 2∥V^j∥_{L^s_tL^r_x(\mathbb{R}×\mathbb{R})} ≤ 1$.

We denote all functions $N, U^j_n, W^j_n, V^j_n$ by $v^j_n$ and we define a nonlinear profile by $Z^j_n := \sum_{j=1}^J v^j_n$. We have
$$
Z^j_n = e^{ıt_n^jδ^2}u(τ_n) + i\int_0^t e^{ıt(s−τ_n^j)δ^2}Z^j_n|^{p−1}Z^j_n ds − e^{ıt_n^jδ^2}R^j_n + s_n^j
$$
with $s_n^j∥_{L^s_tL^r_x} → 0$ as $n → \infty$ and $lim sup_{n → \infty} ∥e^{ıt_n^jδ^2}R^j_n∥_{L^s_tL^r_x} < ε$ for large $J$. Moreover, $lim sup_{n → \infty} ∥Z^j_n∥_{L^s_tL^r_x}$ is bounded independently on $J$. By the long time perturbation, we
obtain \( \|u(\tau_n)\|_{L^2_tL^1_x} \lesssim 1 \). This is a contradiction. Therefore, we get \( J = 1 \). Thus, we get a parameter \((t_n, x_n)\) such that

\[
u(\tau_n) = e^{-it_n\partial_x^2} \mathcal{R}_{x_n} \psi + R_n
\]

and \( \lim_{n \to \infty} \|R_n\|_{H^1} = 0 \). Assuming \( |t_n| \to \infty \), we derive a contradiction to the non-scattering of \( u \) by the same argument as in Case 3 and 4. Therefore, we have

\[
u(\tau_n) = \mathcal{R}_{x_n} \psi + R_n.
\]

By Proposition 36 below, we get the statement.

**Proposition 36** Let \( u \) be an odd solution to \((\text{NLS})\) such that \( M(u_0) = 2M(Q) \), \( E(u_0) = 2E(Q) \) and \( K(u_0) > 0 \). Assume that for any time sequence \( \{t_n\} \subset [0, \infty) \) satisfying \( t_n \to \infty \) as \( n \to \infty \), there exist a subsequence of \( \{t_n\} \), which is also denoted by \( \{t_n\} \), \( \{x_n\} \subset \mathbb{R} \) and \( \psi \in H^1(\mathbb{R}) \setminus \{0\} \) such that

\[
\lim_{n \to \infty} \|u(t_n) - \mathcal{R}_{x_n} \psi\|_{H^1} = 0.
\]

Then, there exists a function \( x : [0, \infty) \to [0, \infty) \) such that for any \( \varepsilon > 0 \) there exists \( R = R(\varepsilon) > 0 \) such that

\[
\int_{|x-x(t)| > R \cap |x+x(t)| > R} |\partial_x u(t, x)|^2 + |u(t, x)|^2 dx < \varepsilon
\]

for all \( t \in [0, \infty) \).

**Lemma 37** Let \( u(t) \) satisfy the assumption in Proposition 36. Then there exists \( c > 0 \) such that

\[
\sup_{x_0 \geq 0} \|u(t)\|_{L^2(\{\|x-x_0\| < 1\} \cup \{\|x+x_0\| < 1\})} \geq c
\]

for all \( t \in \mathbb{R} \).

**Proof** Let \( B_y := \{|x-y| < 1\} \) and \( A_y := B_y \cup B_{-y} \) for \( y \geq 0 \). By the symmetry, it is enough to show that there exists \( c > 0 \) such that \( \sup_{x_0 \in \mathbb{R}} \|u(t)\|_{L^2(A_{x_0})} \geq c \) for all \( t \). If not, there exists \( \{t_n\} \) satisfying \( t_n \to \infty \) such that

\[
\sup_{x_0 \in \mathbb{R}} \|u(t_n)\|_{L^2(A_{x_0})} \to 0.
\]

By the assumption, we get \( \{x_n\} \) and \( \psi \) such that \( u(t_n) - \mathcal{R}_{x_n} \psi \to 0 \) strongly in \( H^1 \). If \( \{x_n\} \) is bounded, we may assume that \( \{x_n\} \) converges to \( x_\infty \in [0, \infty) \) by taking a subsequence and thus we may take \( x_n \equiv 0 \) by replacing \( \mathcal{R}_{x_n} \psi \) by \( \psi \). If \( \{x_n\} \) is unbounded, we may assume that \( x_n \to \infty \) as \( n \to \infty \) by taking a subsequence. If \( x_n \equiv 0 \) (i.e. \( u(t_n) \to \psi \) in \( H^1 \) strongly), then we have

\[
\|\psi\|_{L^2(A_{x_0})} \leq \|u(t_n) - \psi\|_{L^2(A_{x_0})} + \sup_{x_0 \in \mathbb{R}} \|u(t_n)\|_{L^2(A_{x_0})}
\]

for any \( x_0 \in \mathbb{R} \). The right hand side goes to zero. This means that \( \psi \equiv 0 \). This contradicts \( \psi \neq 0 \).
Next we consider the case of $x_n \to \infty$. Letting $\psi^+(x) = \psi(-x)$, we have
\[
\| \mathcal{R}_{x_n} \psi \|^2_{L^2(A_{x_0+x_n})} = \| \psi(x-x_n) - \psi^+(x+x_n) \|^2_{L^2(A_{x_0+x_n})}
\]
\[
= \int_{A_{x_0+x_n}} |\psi(x-x_n)|^2 + |\psi^+(x+x_n)|^2 \, dx
\]
\[
- \int_{A_{x_0+x_n}} 2 \text{Re} \, \psi(x-x_n)\psi^+(x+x_n) \, dx
\]
\[
= \int_{|x-x_0|<1} |\psi(x)|^2 \, dx
\]
\[
+ \int_{|x-x_0-2x_n|<1} |\psi^+(x)|^2 \, dx
\]
\[
- 2 \text{Re} \int_{A_{x_0+x_n}} \psi(x-x_n)\psi^+(x+x_n) \, dx.
\]
When $x_n \to \infty$, we have
\[
\int_{A_{x_0+x_n}} \psi(x-x_n)\psi^+(x+x_n) \, dx \to 0,
\]
\[
\int_{|x-x_0+2x_n|<1} |\psi(x)|^2 \, dx + \int_{|x-x_0-2x_n|<1} |\psi^+(x)|^2 \, dx \to 0.
\]
Therefore, we obtain
\[
\| \mathcal{R}_{x_n} \psi \|^2_{L^2(A_{x_0+x_n})} \to \| \psi \|^2_{L^2(B_{x_0})} + \| \psi^+ \|^2_{L^2(B_{-x_0})} = 2\| \psi \|^2_{L^2(B_{x_0})}.
\]
On the other hand, we have
\[
\| \mathcal{R}_{x_n} \psi \|^2_{L^2(A_{x_0+x_n})} \leq \| u(t_n) - \mathcal{R}_{x_n} \psi \|^2_{L^2(A_{x_0+x_n})} + \| u(t_n) \|^2_{L^2(A_{x_0+x_n})}
\]
\[
\leq \| u(t_n) - \mathcal{R}_{x_n} \psi \|^2_{L^2(\mathbb{R})} + \sup_{x_0 \in \mathbb{R}} \| u(t_n) \|^2_{L^2(A_{x_0})}
\]
for any $n$. This implies that $\| \mathcal{R}_{x_n} \psi \|^2_{L^2(A_{x_0+x_n})} \to 0$ as $n \to \infty$. These estimates mean that $\| \psi \|^2_{L^2(B_{x_0})} = 0$ for any $x_0 \in \mathbb{R}$ and thus $\psi(x) = 0$. This is a contradiction.

**Proof of Proposition 36** By Lemma 37, there exists a function $x : [0, \infty) \to [0, \infty)$ satisfying
\[
\| u(t) \|^2_{L^2(|x-x(t)|<1) \cup |x+x(t)|<1)} \geq c
\]
for all $t \in \mathbb{R}$. Now we have the following.

**Claim.** $|x(t_n) - x_n|$ is bounded.

**Proof of Claim** Since, for any sequence $\{t_n\}$, there exists $\{x_n\}$ and $\psi$ such that
\[
u(t_n) = \mathcal{R}_{x_n} \psi + o_n(1)
\]
by the assumption and
\[
\| u(t_n) \|^2_{L^2(|x-x(t_n)|<1) \cup |x+x(t_n)|<1)} \geq c
\]
by (31), we obtain
\[
\| \mathcal{R}_{x_n} \psi \|^2_{L^2(|x-x(t_n)|<1) \cup |x+x(t_n)|<1)} \geq c/2
\]
for sufficiently large $n$. If $|x(t_n) - x_n|$ is unbounded, a simple calculation gives us $\| \mathcal{R}_{x_n} \psi \|^2_{L^2(|x-x(t_n)|<1) \cup |x+x(t_n)|<1)} \to 0$ as $n \to \infty$. This is a contradiction.
We suppose that the result fails. Then there exists $\varepsilon > 0$ and a time sequence $\{t_n\}$ such that
\[
\int_{\{|x-x(t_n)|>n\} \cap \{|x+x(t_n)|>n\}} |\partial_x u(t_n, x)|^2 + |u(t_n, x)|^2 dx \geq \varepsilon.
\]
By the assumption, there exist $\{x_n\}$ and $\psi$ such that
\[
u(t_n) = \mathcal{R}_{x_n} \psi + o_n(1).
\]
Thus we also have
\[
\int_{\{|x-x(t_n)|>n\} \cap \{|x+x(t_n)|>n\}} |\partial_x \mathcal{R}_{x_n} \psi(x)|^2 + |\mathcal{R}_{x_n} \psi(x)|^2 dx \geq \frac{\varepsilon}{2}
\]
for large $n$. This is a contradiction since $|x(t_n) - x_n|$ is bounded.

Next we will show that we can replace $x(t)$ given in Proposition 35 by the modulation parameter $y(t)$ for $t \in I_{\mu_0}$.

**Lemma 38** Let $\mu_0$ be sufficiently small. Then there exists $C > 0$ such that
\[
|x(t) - y(t)| < C
\]
for $t \in I_{\mu_0}$.

**Proof** If not, there exists a sequence $\{t_n\} \subset I_{\mu_0}$ such that
\[
|x(t_n) - y(t_n)| \to \infty.
\]
Since $t_n \in I_{\mu_0}$, by the modulation, we write $u(t_n) = \mathcal{R}_{y(t_n)} Q + g$. Then
\[
c \leq \int_{\{|x-x(t_n)|<1\} \cup \{|x+x(t_n)|<1\}} |u(t_n, x)|^2 dx
\]
\[
\lesssim \int_{\{|x-x(t_n)|<1\} \cup \{|x+x(t_n)|<1\}} |\mathcal{R}_{y(t_n)} Q|^2 dx + \int_\mathbb{R} |g(t_n)|^2 dx.
\]
Since the first term goes to zero as $n \to \infty$, we have
\[
c \lesssim \|g(t_n)\|^2_{L^2}
\]
for sufficiently large $n$. On the other hand, we have $\|g(t_n)\|^2_{L^2} \lesssim \mu(t_n) < \mu_0$. Thus, we get a contradiction since $\mu_0 \ll 1$. \qed

We define new function
\[
X(t) := \begin{cases} x(t), & t \in [0, \infty) \setminus I_{\mu_0}, \\ y(t), & t \in I_{\mu_0}. \end{cases} \quad (32)
\]

Then it is easy to check that for any $\varepsilon > 0$, there exists $R > 0$ such that
\[
\int_{\{|x-X(t)|>R\} \cap \{|x+X(t)|>R\}} |\partial_x u(t, x)|^2 + |u(t, x)|^2 dx < \varepsilon.
\]

**Lemma 39** Let $\{t_n\} \subset [0, \infty)$ be an arbitrary time sequence. If $|X(t_n)|$ is unbounded, taking a subsequence of $\{t_n\}$, which is also denoted by $\{t_n\}$, we have $\psi \in H^1$ such that
\[
u(t_n) - \mathcal{R}_{X(t_n)} \psi \to 0.
If \(|X(t_n)|\) is bounded, taking a subsequence of \([t_n]\), which is also denoted by \([t_n]\), we have \(\psi \in H^1\) such that

\[u(t_n) \to \psi.\]

**Proof** If \(|X(t_n)|\) is unbounded, we may assume that \(X(t_n) \to \infty\) by taking a subsequence. By Proposition 35, we have \(\{x_n\}\) and \(\psi \in H^1\) such that

\[u(t_n) - R_{x_n} \psi \to 0.\]

Since \(|x(t_n) - X(t_n)|\) and \(|x_n - x(t_n)|\) are bounded (see Lemma 38 and the claim in the proof of Proposition 36), taking a subsequence, we may assume that \(x(t_n) - X(t_n) \to x'\) and \(x_n - x(t_n) \to x''\). Then we have

\[u(t_n) - R_{X(t_n)} \psi = u(t_n) - R_{x_n} \psi + R_{x_n} \psi - R_{x(t_n)} \psi + R_{x(t_n)} \psi - R_{X(t_n)} \psi \to R_{x'} \psi + R_{x''} \psi =: \tilde{\psi}.

Then, if \(\tilde{\psi} \not= 0\), by the long time perturbation, we get a contradiction to non-scattering of \(u\).

If \(|X(t_n)|\) is bounded, we may assume that \(X(t_n)\) converges to \(X_\infty\) by taking a subsequence. By Proposition 35, we have \(\{x_n\}\) and \(\psi \in H^1\) such that

\[u(t_n) - R_{x_n} \psi \to 0.\]

Since \(|x(t_n) - X(t_n)|\) is bounded, \(|x(t_n)|\) is bounded and thus \(\{x_n\}\) is bounded by boundedness of \(|x_n - x(t_n)|\). Thus we can take \(x_n = 0\) and \(u(t_n) \to R_0 \psi\).

\(\square\)

### 5.2 Extinction of the critical element

We derive a contradiction. First we show that \(X\) must be bounded. Second we will prove that the boundedness of \(X\) implies a contradiction. In this subsection, we set \(J_R(t) := J_R(u(t))\) and thus \(J_R'(t)\) denotes the time derivative of \(J_R(u(t))\).

#### 5.2.1 \(X(t)\) is bounded

We show \(X(t)\) is bounded by contradiction.

**Proposition 40** Let \(u\) be an odd solution with \(E(u) = 2E(Q), M(u) = 2M(Q)\) and \(K(u(t)) \geq 0\) such that any \(\varepsilon > 0\) there exists \(R = R(\varepsilon) > 0\) such that

\[
\int_{\{|x-X(t)|>R\} \cap \{|x+X(t)|>R\}} |\partial_x u(t,x)|^2 + |u(t,x)|^2 \, dx < \varepsilon
\]

for all \(t \in [0, \infty)\), where \(X\) is defined in (32). Then \(X(t)\) is bounded.

To prove this, we prepare some lemmas.

**Lemma 41** There exists \(C_\varepsilon > 0\) such that

\[
\int_{t_1}^{t_2} \mu(t) \, dt \lesssim C_\varepsilon (1 + \sup_{t \in [t_1, t_2]} |X(t)|)(\mu(t_1) + \mu(t_2))
\]

for any \(t_1, t_2 > 0\)

**Proof** We give estimates of \(J_R'\) and \(A_R\).
Thus, in any cases, we have

$$|J'_{R}(t)| \lesssim R \int |u||\partial_x u|dx \lesssim R \|u\|_{L^\infty H^1}^2 \lesssim R \frac{\mu(t)}{\mu_0}.$$  

When $\mu(t) \leq \mu_0$, since $\mathcal{R}(t)Q$ is real-valued, we have

$$|J'_{R}(t)| = 2R \text{Im} \int \left| \overline{u} \partial_x u(t, x) - e^{i\theta(t)+it} \mathcal{R}(t)Q \partial_x (e^{i\theta(t)+it} \mathcal{R}(t)Q) \right| \partial_x \varphi \left( \frac{x}{R} \right) dx \lesssim R \|u\|_{H^1} \|\mathcal{R}(t)Q\|_{H^1} \|u - e^{i\theta(t)+it} \mathcal{R}(t)Q\|_{H^1} \lesssim R \mu(t).$$

Thus, in any cases, we have $|J'_{R}(t)| \lesssim R \mu(t).$ \hfill \(\square\)

**Claim 2.** There exists sufficiently small $\tilde{\delta} > 0$ such that $|A_R| \leq \tilde{\delta} \mu(t)$ for all $t > 0$.

**Proof** By the compactness, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\int_{\{|x-X(t)|>C_\varepsilon\} \cap \{|x+X(t)|>C_\varepsilon\}} |\partial_x u(t, x)|^2 + |u(t, x)|^2 dx < \varepsilon.$$  

We choose

$$R = C_\varepsilon + \sup_{t \in [t_1, t_2]} |X(t)|$$  

so that it holds that

$$\{|x| > R\} \subset \{|x-X(t)| > C_\varepsilon\} \cap \{|x+X(t)| > C_\varepsilon\}$$

for $t \in [t_1, t_2]$. By the error estimate, we have

$$|A_R(t)| \lesssim \varepsilon.$$  

If $\mu(t) > \mu_0$, we get

$$|A_R(t)| \lesssim \varepsilon \frac{\mu(t)}{\mu_0}.$$  

By Lemma 17, we get

$$A_R(t) = F_R(u(t)) - F_\infty(u(t)) \quad - F_R(e^{i\theta+iot}Q) + F_\infty(e^{i\theta+iot}Q) + O((1+y)e^{-2\sqrt{\mu}}).$$

Now we have

$$|F_R(u(t)) - F_\infty(u(t)) - F_R(e^{i\theta+iot}Q) + F_\infty(e^{i\theta+iot}Q)| \lesssim (\|u(t)\|_{H^1(|x|>R)} + \|\mathcal{R}(t)Q\|_{H^1(|x|>R)}) \|u(t) - e^{i\theta+iot}Q\|_{H^1}.$$  

By the compactness, $R = C_\varepsilon + \sup |X(t)|$ and $X(t) = y(t)$ for $t \in I_{\mu_0}$, we get

$$|A_R(u(t))| \lesssim \varepsilon \mu(t) + \mu_0^{1-\delta} \mu(t).$$
In both cases, we have

$$|A_R(u(t))| \lesssim \min \{\varepsilon + \mu_0^{1-\delta}, \varepsilon \mu_0^{-1}\} \mu(t).$$

Since $\mu_0$ is sufficiently small, taking $\varepsilon > 0$ small, we get $|A_R(u(t))| \leq \tilde{\delta} \mu(t)$.

By Claim 2, we obtain

$$J''_R(t) = 8K(u(t)) + A_R(u(t)) \gtrsim \mu(t) - \tilde{\delta}\mu(t) \gtrsim \mu(t).$$

Integrating this inequality on $[t_1, t_2]$, we get

$$J'_R(t_2) - J'_R(t_1) \gtrsim \int_{t_1}^{t_2} \mu(t)dt.$$

This completes the proof.

Lemma 42 Let $\{t_n\}$ be a time sequence satisfying $t_n \to \infty$ as $n \to \infty$. Then we have

$$X(t_n) \to \infty \iff \mu(t_n) \to 0.$$  

Proof If $\mu(t_n) \to 0$, then we have $X(t_n) = y(t_n)$ for large $n$. Thus we have

$$e^{-2y(t_n)} \lesssim \mu(t_n) \to 0$$

and this implies $X(t_n) = y(t_n) \to \infty$.

Next we consider the case of $X(t_n) \to \infty$. We use a contradiction argument and we suppose that there exists $\{t_n\}$ satisfying $t_n \to \infty$ and $\delta > 0$ such that $X(t_n) \to \infty$ and $\mu(t_n) \geq \delta > 0$. By the compactness, there exist a subsequence, which is also denoted by $\{t_n\}$, and $\psi \in H^1(\mathbb{R})$ such that

$$\|u(t_n) - R_{X(t_n)}\psi\|_{H^1} \to 0$$

as $n \to \infty$. Since $X(t_n) \to \infty$, we have

$$2M(Q) = M(u(t_n)) \to 2M(\psi),$$

$$2E(Q) = E(u(t_n)) \to 2E(\psi),$$

and

$$-\delta \geq \|\partial_x u(t_n)\|_{L^2}^2 - 2\|\partial_x Q\|_{L^2}^2 \to 2\|\partial_x \psi\|_{L^2}^2 - 2\|\partial_x Q\|_{L^2}^2$$

and thus we get

$$M(Q) = M(\psi), \quad E(Q) = E(\psi), \quad \|\partial_x \psi\|_{L^2}^2 < \|\partial_x Q\|_{L^2}^2.$$
By the result of [4], the solution $v(t)$ to (NLS) with $v(0) = \psi$ scatters in positive or negative time direction. We define the approximate solution $\tilde{v}_n(t, x) = v(t, x - X(t_n)) - v(t, -x - X(t_n))$, which satisfies

$$i \partial_t \tilde{v}_n + \partial^2_x \tilde{v}_n + |\tilde{v}|^{p-1} \tilde{v}_n = e,$$

where $e = |\tilde{v}|^{p-1} \tilde{v}_n - (|\mathcal{T}_{X(t_n)} v|)^{p-1} \mathcal{T}_{X(t_n)} v - |\mathcal{T}_{-X(t_n)} v|^{p-1} \mathcal{T}_{-X(t_n)} v$. We note that $\|u(t_n) - \tilde{v}_n(0)\|_{H^1} \to 0$ as $n \to \infty$.

In the case that $v$ scatters in positive time direction, we have $\|\tilde{v}_n\|_{L^p_t L^r_x(0, \infty)} \lesssim \|v\|_{L^p_t L^r_x(0, \infty)} < \infty$. By the long time perturbation, we get

$$\|u\|_{L^p_t L^r_x(t_n, \infty)} = \|u(t_n + t)\|_{L^p_t L^r_x(0, \infty)} \lesssim C + \|\tilde{v}_n\|_{L^p_t L^r_x(0, \infty)} \lesssim C$$

for large $n$. This contradicts non-scattering of $u$.

In the case that $v$ scatters in negative direction, we have $\|\tilde{v}_n\|_{L^p_t L^r_x(-\infty, 0)} \lesssim \|v\|_{L^p_t L^r_x(-\infty, 0)} < \infty$. By the long time perturbation, we get

$$\|u\|_{L^p_t L^r_x(-\infty, t_n)} = \|u(t_n + t)\|_{L^p_t L^r_x(-\infty, 0)} \lesssim C + \|\tilde{v}_n\|_{L^p_t L^r_x(-\infty, 0)} \lesssim C$$

for large $n$. As $n \to \infty$, this also contradicts non-scattering of $u$. Therefore, we get $\mu(t_n) \to 0$.

\[\Box\]

**Lemma 43** There exists $C > 0$ such that

$$|X(t) - X(s)| \leq C$$

for all $t, s \in [0, \infty)$ with $|t - s| \leq 1$.

**Proof** We prove by contradiction that there exist $\eta > 0$ and $C_0 > 0$ such that

$$|X(t) - X(s)| \leq C_0$$

for all $t, s \in [0, \infty)$ with $|t - s| \leq \eta$. We suppose that for any $\eta > 0$, there exist $t_n$ and $s_n$ such that

$$|t_n - s_n| \leq \eta \text{ and } |X(t_n) - X(s_n)| \to \infty$$

as $n \to \infty$.

It follows from the Duhamel formula and the Sobolev embedding that

$$\|u(t) - u(s)\|_{L^2} \leq \int_s^t \|e^{i(t-\tau)\partial^2_x} |u|^{p-1} u\|_{L^2} d\tau \leq \|t - s\| \|u\|_{L^\infty_t H^1}^p \lesssim |t - s|$$

for $s < t$. By the compactness, for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\sup_{t \geq 0} \int_{|x - X(t)| > R \cap [|x + X(t)| > R]} |u(t, x)|^2 dx < \varepsilon.$$

Let $B(t) := \{|x - X(t)| \leq R\} \cup \{|x + X(t)| \leq R\}$. Then we have

$$\|1_{B(t)} u(t) - 1_{B(s)} u(s)\|_{L^2} \leq \|1_{B(t)} u(t) - u(t)\|_{L^2} + \|u(t) - u(s)\|_{L^2} + \|u(s) - 1_{B(s)} u(s)\|_{L^2}$$

$$\lesssim \varepsilon + |t - s|$$

(33)
for all \( s < t \). On the other hand, since \(|X(t_n) - X(s_n)| \to \infty\) and thus \(B(t_n) \cap B(s_n) = \emptyset\) for large \( n \), we have

\[
\| \mathbb{I}_{B(t_n)} u(t_n) - \mathbb{I}_{B(s_n)} u(s_n) \|_{L^2} \geq 2(M(u) - \varepsilon) = 2(2M(Q) - \varepsilon)
\]

(34)

for large \( n \). Taking \( \varepsilon > 0 \) and \( \eta > 0 \) sufficiently small, we get a contradiction combining (33) and (34). When \( \eta \geq 1 \), the statement holds obviously. If \( \eta < 1 \), then for \( s < t \) with \(|t - s| < 1\), take \( s = t_0 < t_1 < \cdots < t_l = t\) such that \(|t_j - t_{j+1}| < \eta\) for \( j = 0, \ldots, l - 1 \). Then \( l \leq \eta^{-1} \) and thus we get

\[
|X(t) - X(s)| \leq \sum_{j=0}^{l-1} |X(t_j) - X(t_{j+1})| \leq lC_0 \leq C_0 \eta^{-1}.
\]

In any way, we obtain the statement by setting \( C := \max\{C_0, C_0\eta^{-1}\} \). \( \square \)

**Lemma 44** There exists a constant \( C \) such that

\[
|X(\tau_1) - X(\tau_2)| \leq C \int_{\tau_1}^{\tau_2} \mu(t) \, dt
\]

for any \( \tau_1, \tau_2 \) satisfying \( \tau_1 + 1 \leq \tau_2 \).

**Proof** We start with the following claim.

**Claim 1.** There exists \( \mu_1 \) such that

\[
\inf_{t \in [\tau_1, \tau_1 + 1]} \mu(t) \geq \mu_1 \text{ or } \sup_{t \in [\tau_1, \tau_1 + 2]} \mu(t) < \mu_0
\]

for any \( \tau_1 > 0 \).

**Proof of Claim 1** If not, for any \( n \) there exists \( t_n \) such that

\[
\inf_{t \in [t_n, t_n + 2]} \mu(t) < n^{-1} \text{ and } \sup_{t \in [t_n, t_n + 2]} \mu(t) \geq \mu_0
\]

and thus there exists \( t_n', t_n'' \in [t_n, t_n + 2] \) such that

\[
\mu(t_n') < n^{-1} \text{ and } \mu(t_n'') \geq \mu_0.
\]

Thus we have

\[
\mu(t_n') \to 0, \quad \mu(t_n'') \geq \mu_0 \quad \text{and} \quad |t_n' - t_n''| \leq 2.
\]

We may assume that \( t_n' - t_n'' \to \tau \in [-2, 2] \) as \( n \to \infty \) extracting a subsequence if necessary.

Then we may assume that \( X(t_n') \) converges. Indeed, if \( \{t_n''\} \) is bounded, \( X(t_n'') \) is bounded by Lemma 43. When \( X(t_n'') \) is bounded, we may \( X(t_n'') \) converges by taking a subsequence. If \( t_n'' \) is unbounded, we may assume that \( t_n'' \to \infty \) along a subsequence. If \( X(t_n'') \to \infty \), then we must have \( \mu(t_n'') \to 0 \) by Lemma 42. However, this contradicts the definition of \( t_n'' \).

By Lemma 39, we have

\[
u(t_n'') \to \psi \text{ strongly in } H^1.
\]

Therefore we have

\[
M(\psi) = 2M(Q_\omega), \quad E(\psi) = 2E(Q_\omega), \quad \|\partial_x(\psi)\|_{L^2}^2 < 2\|\partial_x Q_\omega\|_{L^2}^2 - \mu_0.
\]
Then the solution $v$ to (NLS) with $v(0) = \psi$ is global and $\tilde{\mu}(v(t)) < 0$ for all $t$. By the continuity of the flow, we have
\[
\|u(t_n'' + \tau) - u(\tau)\|_{\dot{H}^1} \to 0
\]
since the initial data satisfy $\|u(t_n'' - v(0))\|_{\dot{H}^1} \to 0$. Thus we also have $\mu(u(t_n'' + \tau)) \to \mu(v(\tau)) > 0$ as $n \to \infty$ and thus $\mu(u(t_n'' + \tau)) > c_0 > 0$ for large $n$ and some constant $c_0$.

On the other hand, we have
\[
\|u(t_n'') - u(t_n' + \tau)\|_{\dot{H}^1} \to 0
\]
which is proved later. Then $\mu(u(t_n'' + \tau)) \to 0$ since $\mu(u(t_n')) \to 0$ and $\|u(t_n'') - u(t_n' + \tau)\|_{\dot{H}^1} \to 0$. This is a contradiction. We will prove (35). Let $I_n = [t_n', t_n'' + \tau]$ (or $I_n = (I_n' + \tau, t_n')$). Now we have
\[
\|u(t_n') - u(t_n'' + \tau)\|_{\dot{H}^1} = \| \int_{I_n} e^{i(t-s)\frac{\alpha^2}{2}} |u(s)|^{p-1} u(s) ds \|_{\dot{H}^1}
\leq \int_{I_n} \|u(s)|^{p-1} u(s)\|_{\dot{H}^1} ds
\leq \int_{I_n} \|u(s)|^{p-1} u(s)\|_{L^2(p-1)} ds
\leq |I_n| \|u\|_{L^\infty_t H^1} \to 0
\]
since $|I_n| \to 0$ by $t_n' - (t_n'' + \tau) \to 0$. 

**Claim 2.** There exists a constant $C$ such that
\[
|X(\tau_1) - X(\tau_2)| \leq C \int_{\tau_1}^{\tau_2} \mu(t) dt
\]
for any $\tau_1, \tau_2$ satisfying $\tau_1 + 1 \leq \tau_2 \leq \tau_1 + 2$.

**Proof of Claim 2** In the first case of Claim 1, we have
\[
\int_{\tau_1}^{\tau_2} \mu(t) dt \geq 2 \inf_{t \in [\tau_1, \tau_1 + 2]} \mu(t) \geq \mu_1
\]
for $\tau_1 + 1 < \tau_2 < \tau_1 + 2$. By Lemma 43, we have
\[
|X(\tau_1) - X(\tau_2)| \leq |X(\tau_1) - X(\tau_1 + 1)| + |X(\tau_1 + 1) - X(\tau_2)|
\leq 2C \frac{2}{\mu_1} \int_{\tau_1}^{\tau_2} \mu(t) dt.
\]

In the second case of Claim 1, we have $[\tau_1, \tau_1 + 2] \subset I_{\mu_0}$ and thus $X(t) = \gamma(t)$ for $t \in [\tau_1, \tau_1 + 2]$. We have
\[
|X'(t)| = |\gamma'(t)| \leq \mu(t)
\]
for $t \in (\tau_1, \tau_1 + 2)$. Integrating this on $[\tau_1, \tau_2]$, we get
\[
|X(\tau_2) - X(\tau_1)| = \int_{\tau_1}^{\tau_2} X'(t) dt \leq \int_{\tau_1}^{\tau_2} \mu(t) dt.
\]
We get Claim 2. 

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At last, we show the statement. For arbitrary \( \tau_1, \tau_2 \) satisfying \( \tau_1 + 1 < \tau_2 \), we divide the interval \([\tau_1, \tau_2]\) into the intervals \([\sigma_0, \sigma_1] = [\tau_1, \tau_1], [\sigma_1, \sigma_2], \ldots [\sigma_{k-1}, \tau_2] = [\sigma_{k-1}, \sigma_k]\), whose length are larger than 1 and less than 2. By Claim 2, we have
\[
|X(\sigma_j) - X(\sigma_{j+1})| \leq C \int_{\sigma_j}^{\sigma_{j+1}} \mu(t) dt.
\]
Therefore we obtain
\[
|X(\tau_2) - X(\tau_1)| = |X(\sigma_0) - X(\sigma_k)| \\
\leq \sum_{j=0}^{k-1} |X(\sigma_j) - X(\sigma_{j+1})| \\
\leq C \sum_{j=0}^{k-1} \int_{\sigma_j}^{\sigma_{j+1}} \mu(t) dt \\
= C \int_{\tau_1}^{\tau_2} \mu(t) dt.
\]
This completes the proof of Lemma 44.

**Proof of Proposition 40** We prove the statement by contradiction. Suppose that \( |X(t)| \) is unbounded. Then we take a sequence \( \{t_n\} \) such that \( |X(t_n)| \to \infty \) and \( \sup_{t \in [0, t_n]} |X(t)| = |X(t_n)| \). By Lemma 42, we have \( \mu(t_n) \to 0 \). Let \( N \) such that \( C \mu(t_n) < 100^{-1} \) for all \( n > N \). By Lemma and , for \( n \gg N \), which ensures \( t_n > t_N + 1 \) we have
\[
X(t_n) - X(t_N) \leq C \int_{t_N}^{t_n} \mu(t) dt \\
\leq C (1 + |X(t_n)|) (\mu(t_N) + \mu(t_n)) \\
\leq \frac{2}{100} (1 + X(t_n))
\]
and thus
\[
X(t_n) \lesssim 1 + X(t_N) < \infty.
\]
This is a contradiction.

5.2.2 If \( X(t) \) is bounded, contradiction

By the previous section, we see that \( X(t) \) is bounded. In this section, we get a contradiction from the boundedness.

Since \( X(t) \) is bounded, for any \( \varepsilon > 0 \), there exists \( R > 0 \) such that
\[
\int_{|x| > R} |\partial_x u(t, x)|^2 + |u(t, x)|^2 dx < \varepsilon.
\]

**Lemma 45** It holds that
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mu(t) dt = 0.
\]
Proof We consider the localized virial identity $J_R$. By a direct calculation, we have $|J'_R(t)| \lesssim R$. We also have $J''_R(t) = 8\mu(t) + A_R(u(t))$. We can estimate the error term by compactness as follows:

$$|A_R(u(t))| \lesssim \int_{|x|>R} |\partial_x u|^2 + |u|^{p+1} + \frac{1}{R^2} |u|^2 dx \leq \varepsilon$$

for large $R > 0$. We have

$$\int_0^T 8\mu(t) + A_R(u(t)) dt = \int_0^T J'_R(t) dt \leq |J'_R(t)| + |J'_R(0)| \lesssim R.$$

Therefore we have

$$\int_0^T \mu(t) dt \lesssim R + \varepsilon T$$

and thus we get

$$\frac{1}{T} \int_0^T \mu(t) dt \to 0$$

as $T \to \infty$.

Corollary 46 There exists a time sequence $\{t_n\}$ such that

$$\lim_{n \to \infty} \mu(t_n) = 0.$$

Proof If not, there exists $\delta > 0$ such that $\mu(t) > \delta$ for all $t$. This implies

$$\frac{1}{T} \int_0^T \mu(t) dt \geq \delta.$$

This is a contradiction.

Since we have a time sequence $\{t_n\}$ such that $\lim_{n \to \infty} \mu(t_n) = 0$ by Corollary 46, Lemma 42 implies that $X(t_n)$ must tend to $\infty$. This contradicts that $X$ is bounded. As a conclusion, we get Proposition 14 (1).

6 Proof of blow-up

In this section, we prove the blow-up result, Theorem 1 (2) through Theorem 8 (2).

Lemma 47 Let $\varphi \in C^1(\mathbb{R}; \mathbb{R})$ be even and $f \in H_{\text{odd}}^1(\mathbb{R})$. Assume that they satisfy $\int_{\mathbb{R}} |\partial_x \varphi|^2 |f| dx < \infty$, $M(f) = 2M(Q)$, and $E(f) = 2E(Q)$. Then we have the following inequality.

$$|\text{Im} \int_{\mathbb{R}} \partial_x \varphi(x) \partial_x f(x) \overline{f(x)} dx| \lesssim \mu(f)^2 \int_{\mathbb{R}} |\partial_x \varphi(x)|^2 |f(x)|^2 dx.$$

Proof Let $\lambda \in \mathbb{R}$. By the Gagliardo–Nirenberg inequality (Lemma 11), we have

$$\|f\|_{L^{p+1}}^{p+1} = \|e^{i\lambda \varphi} f\|_{L^{p+1}}^{p+1} \leq C_{\text{odd}}^G \|f\|_{L^2}^{p+3} (\|e^{i\lambda \varphi} f\|_{L^2}^{p+1} \frac{p+3}{p+1}).$$
Now, we have
\[
\| (e^{i\lambda \varphi} f)' \|_{L^2}^2 = \lambda^2 \| \varphi' f \|_{L^2}^2 + 2\lambda \Im \int_{\mathbb{R}} \varphi' f' \overline{f} \, dx + \| f' \|_{L^2}^2.
\]
Therefore, we obtain
\[
\| f \|_{L^{p+1}_{L^{p+1}}} \leq C_{\mathrm{GN}}^{\mathrm{odd}} \| f \|_{L^{\frac{p+3}{2}}}^{\frac{p+3}{2}} \left( \lambda^2 \| \varphi' f \|_{L^2}^2 + 2\lambda \Im \int_{\mathbb{R}} \varphi' f' \overline{f} \, dx + \| f' \|_{L^2}^2 \right)^{\frac{p-1}{p}}.
\]
This means that
\[
\lambda^2 \| \varphi' f \|_{L^2}^2 + 2\lambda \Im \int_{\mathbb{R}} \varphi' f' \overline{f} \, dx + \| f' \|_{L^2}^2 - \left( \frac{\| f \|_{L^{p+1}}^{p+1}}{C_{\mathrm{GN}}^{\mathrm{odd}} \| f \|_{L^2}^{p+3}} \right)^{\frac{4}{p-1}} \geq 0.
\]
This is the inequality related to the quadratic equation for $\lambda$. Thus, we obtain
\[
\left| \Im \int_{\mathbb{R}} \varphi' f' \overline{f} \, dx \right|^2 \leq \| \varphi' f \|_{L^2}^2 \left\{ \| f' \|_{L^2}^2 - \left( \frac{\| f \|_{L^{p+1}}^{p+1}}{C_{\mathrm{GN}}^{\mathrm{odd}} \| f \|_{L^2}^{p+3}} \right)^{\frac{4}{p-1}} \right\}.
\]
Now, we have $\| f' \|_{L^2}^2 = 2\| \partial_x Q \|_{L^2}^2 - \mu(f)$ and it holds from $E(f) = 2E(Q)$ that $\| f \|_{L^{p+1}} = 2\| Q \|_{L^{p+1}} - \frac{p+1}{2} \mu(f)$. Therefore, we obtain
\[
\| f' \|_{L^2}^2 - \left( \frac{\| f \|_{L^{p+1}}^{p+1}}{C_{\mathrm{GN}}^{\mathrm{odd}} \| f \|_{L^2}^{p+3}} \right)^{\frac{4}{p-1}} = 2\| \partial_x Q \|_{L^2}^2 - \mu(f) - \left( \frac{\| f \|_{L^{p+1}}^{p+1}}{C_{\mathrm{GN}}^{\mathrm{odd}} \| f \|_{L^2}^{p+3}} \right)^{\frac{4}{p-1}} (C_{\mathrm{GN}} 2^{-\frac{p-5}{4}} \| Q \|_{L^2}^{p+3})^{-\frac{4}{p-1}},
\]
where we also used $\| f \|_{L^2}^2 = 2\| Q \|_{L^2}^2$ and $C_{\mathrm{GN}}^{\mathrm{odd}} = 2^{-\frac{p-1}{4}} C_{\mathrm{GN}}$. Then, by the Taylor expansion and the Pohozaev identity
\[
(C_{\mathrm{GN}} \| Q \|_{L^2}^{p+3})^{-\frac{4}{p-1}} = \frac{p-1}{2(p+1)} \| Q \|_{L^{p+1}}^{\frac{(p-5)(p+1)}{p-1}},
\]
we obtain
\[
2\| \partial_x Q \|_{L^2}^2 - \mu(f) - \left( \frac{\| f \|_{L^{p+1}}^{p+1}}{C_{\mathrm{GN}}^{\mathrm{odd}} \| f \|_{L^2}^{p+3}} \right)^{\frac{4}{p-1}} (C_{\mathrm{GN}} 2^{-\frac{p-5}{4}} \| Q \|_{L^2}^{p+3})^{-\frac{4}{p-1}}
\]
\[
\leq 2\| \partial_x Q \|_{L^2}^2 - \mu(f)
\]
\[
- \left( \frac{2}{p-1} \| Q \|_{L^{p+1}}^{\frac{(p+1)(p+3)}{p-1}} - \frac{2(p+1)}{p-1} \| Q \|_{L^{p+1}}^{\frac{(p-5)(p+1)}{p-1}} \mu(f) - C |\mu(f)|^2 \right)
\]
\[
\times (C_{\mathrm{GN}} 2^{-\frac{p-5}{4}} \| Q \|_{L^2}^{p+3})^{-\frac{4}{p-1}}
\]
\[
= C' |\mu(f)|^2.
\]
Combining these estimates, we get
\[
\left| \text{Im} \int_{\mathbb{R}} \phi' f' \overline{f'} dx \right|^2 \leq \|\phi' f\|_{L^2}^2 \left\{ \|f\|_H^2 - \left( \frac{\|f\|_{L^{p+1}}^{p+1}}{C_{G\text{N}} C_{\text{odd}} \|f\|_{L^2}^{p+1}} \right)^{\frac{4}{p+1}} \right\} \\
\lesssim \|\phi' f\|_{L^2}^2 |\mu(f)|^2.
\]
This completes the proof. \(\square\)

**Corollary 48** Under the assumption of Lemma 47 with \(K(f) < 0\), we have
\[
\left| \text{Im} \int_{\mathbb{R}} \partial_x \phi(x) \partial_x f(x) \overline{f(x)} dx \right| \lesssim |K(f)|^2 \int_{\mathbb{R}} |\partial_x \phi(x)|^2 |f(x)|^2 dx.
\]
**Proof** This follows from Lemmas 10 and 47. \(\square\)

**Proposition 49** We assume the same assumption in Proposition 14 (2). Suppose that \(u\) is global in positive time direction. Then we have
\[
\text{Im} \int_{\mathbb{R}} x \partial_t u(t, x) \overline{u(t, x)} dx > 0
\]
for all existence time \(t\). Moreover, there exists \(c > 0\) such that
\[
\int_t^\infty |\mu(s)| ds \lesssim e^{-ct}
\]
for any \(t > 0\).

**Proof** We set \(J(t) := J(u(t)) = \int_{\mathbb{R}} x^2 |u(t, x)|^2 dx\). We will first show that \(J'(t) > 0\) for all existence time. If not, there exists \(t_1\) such that \(J'(t_1) \leq 0\). Since \(y'' < 0\) for all existence time, we have
\[
J'(t_2) - J'(t_1) = \int_{t_1}^{t_2} J''(s) ds = c_2 \int_{t_1}^{t_2} K(u(s)) ds < 0
\]
for \(t_2 > t_1\). Thus, we have \(J'(t_2) < J'(t_1)\). By the similar argument as above, it holds that \(J'(t) \leq J'(t_2) < 0\) for any \(t > t_2\). This means there exists \(t^*\) such that \(J(t^*) = 0\). This is a contradiction to that \(u\) is a non-zero forward global solution.

Next, we will show that \(J'(t) \lesssim e^{-ct}\) for \(t \geq 0\). By Corollary 48 as \(\phi(x) = x^2\) and \(f(x) = u(t, x)\), we obtain
\[
|J'(t)|^2 \lesssim (J''(t))^2 J(t)
\]
for all existence time \(t\). Since \(J > 0\), \(J' > 0\), and \(J'' < 0\), we obtain
\[
\frac{J'(t)}{\sqrt{J(t)}} \lesssim -J''(t).
\]
Integrating this on \((0, t)\), we get
\[
\sqrt{J(t)} - \sqrt{J(0)} \lesssim -J'(t) + J'(0) \lesssim J'(0).
\]
This means that $y$ is bounded on $(0, \infty)$. Using this boundedness and (36) again, we have $J'(t) \lesssim -J''(t)$. This implies $J'(t) \lesssim e^{-ct}$ for $t \geq 0$. We obtain

$$0 \leq -\int_{t}^{\infty} \mu(s)ds \lesssim -\int_{t}^{\infty} K(u(s))ds \approx -\int_{t}^{\infty} J''(s)ds = -[J'(s)]_{s=t}^{\infty} = J'(t) \lesssim e^{-ct}.$$

This completes the proof. \hfill \Box

**Corollary 50** We assume the assumption of Proposition 14 (2) and that the solution is global in positive time, then $u$ blows up in negative time.

**Proof** Suppose that $u$ is global in negative time. Set $v(t, x) = \overline{u(-t, x)}$. Then, $v$ is a solution of (NLS) satisfying the above assumption. Thus, it holds that

$$\text{Im} \int_{\mathbb{R}} x \partial_{x} v(t, x) \overline{v(t, x)}dx > 0$$

for all $t$. We get

$$0 < \text{Im} \int_{\mathbb{R}} x \partial_{x} v(-t, x) \overline{v(-t, x)}dx = \text{Im} \int_{\mathbb{R}} x \partial_{x} u(t, x)u(t, x)dx$$

$$= -\text{Im} \int_{\mathbb{R}} x \partial_{x} u(t, x)u(t, x)dx < 0.$$

This is a contradiction. \hfill \Box

**Proof of Proposition 14 (2)** Suppose that $u$ is global in positive time direction. Then, by Corollary 50, the solution blows up in negative time. By Proposition 49, we have $\lim_{t \to -\infty} \mu(t) = 0$. Thus, there exists a sequence $\{t_n\}$ such that $t_n \to -\infty$ and $\mu(t_n) \to 0$ as $n \to \infty$. We will prove that $\mu(t) \to 0$ as $t \to \infty$. If not, there exists $\epsilon_1 \in (0, \mu_0)$ and $\{t'_n\}$ such that $-\mu(t'_n) > \epsilon_1$. We can take a sequence $\{t''_n\}$ such that

$$t_n < t''_n, \quad -\mu(t''_n) = \epsilon_1, \quad -\mu(t) < \epsilon_1 \text{ for all } t \in [t_n, t''_n).$$

On the interval $[t_n, t''_n]$, the parameter $\rho$ is well defined. By the estimate of the modulation parameter Lemma 34, we have

$$|\rho(t''_n) - \rho(t_n)| \leq \int_{t_n}^{t''_n} |\rho'(t)|dt \lesssim e^{-ct_n} \to 0$$

as $n \to \infty$. By the definition of $t_n$, we have $|\rho(t_n)| \sim |\mu(t_n)| \to 0$. However, we have $|\rho(t''_n)| \sim |\mu(t''_n)| = \epsilon_1 > 0$ by the estimate of the modulation parameter (see Lemma 29 and Corollary 32) and the definition of $t''_n$. This is a contradiction. This means that $\mu(t) \to 0$ as $t \to \infty$.

Therefore, it follows from the estimate of the modulation parameter Lemma 34 that

$$|y(t_2) - y(t_1)| = \int_{t_1}^{t_2} |y'(t)|dt \lesssim \int_{t_1}^{t_2} |\mu(t)|dt \lesssim e^{-ct_1}$$

for large $t_2 > t_1$. This implies that $y(t)$ converges to $y_\infty \in \mathbb{R}$ as $t \to \infty$. However, this means that

$$e^{-2y(t)} \to e^{-2y_\infty} > 0.$$
This contradicts $e^{-2y(t)} \lesssim |\mu(t)| \to 0$ as $t \to \infty$. As a consequence, the solution is not global in positive time direction. 

**Acknowledgements** The second author deeply appreciates the support by JSPS Overseas Research Fellowship. He also thanks OU and UBC for giving him the chance to study abroad. Research of the first author is partially supported by an NSERC Discovery Grant.

**Data availability** There are no data because this study is purely mathematical.

**Declarations**

**Conflict of interest** The authors declare no conflicts of interest associated with this paper.

**Appendix: Cut of odd functions**

**Lemma 51** Let $f \in H^1_{\text{odd}}(\mathbb{R})$. Then $F^+(x) := \mathbb{1}_{(0,\infty)}(x) f(x)$ also belongs to $H^1(\mathbb{R})$. Moreover, we have

$$\frac{d}{dx} F^+(x) = \mathbb{1}_{(0,\infty)}(x) \frac{d}{dx} f(x).$$

**Proof** We have

$$\|F^+\|_{L^2(\mathbb{R})} = \|f\|_{L^2((0,\infty))} \leq \|f\|_{L^2(\mathbb{R})} < \infty.$$ 

The weak derivative of $F^+$ is

$$\frac{d}{dx} F^+(x) = \mathbb{1}_{(0,\infty)}(x) \frac{df}{dx}(x).$$

Indeed, for any $\varphi \in C_0^\infty(\mathbb{R})$, we have

$$\int_\mathbb{R} F^+(x) \frac{d}{dx} \varphi(x) dx = \int_0^\infty f(x) \frac{d}{dx} \varphi(x) dx$$

$$= - \int_0^\infty \frac{d}{dx} f(x) \varphi(x) dx$$

$$= - \int_\mathbb{R} \mathbb{1}_{(0,\infty)} \frac{d}{dx} f(x) \varphi(x) dx$$

by the integration by parts and $f(0) = 0$. Moreover,

$$\left\| \frac{d}{dx} F^+ \right\|_{L^2(\mathbb{R})} = \left\| \frac{d}{dx} f \right\|_{L^2((0,\infty))} \leq \left\| \frac{d}{dx} f \right\|_{L^2(\mathbb{R})} < \infty.$$ 

Thus, we get $F^+ \in H^1(\mathbb{R})$. 

**References**

1. Akahori, T., Nawa, H.: Blowup and scattering problems for the nonlinear Schrödinger equations. Kyoto J. Math. **53**(3), 629–672 (2013)
2. Ardila, A., Inui, T.: Threshold scattering for the focusing NLS with a repulsive Dirac delta potential. J. Differ. Equ. **313**, 54–84 (2022)
3. Arora, A.K., Dodson, B., Murphy, J.: Scattering below the ground state for the 2d radial nonlinear Schrödinger equation. Proc. Am. Math. Soc. 148(4), 1653–1663 (2020)

4. Campos, L., Farah, L. G., Roudenko, S.: Threshold solutions for the nonlinear Schrödinger equation. Preprint arXiv:2010.14434

5. Cazenave, T.: Semilinear Schrödinger Equations. Courant Lecture Notes in Mathematics, vol. 10. American Mathematical Society, Providence, RI (2003)

6. Cazenave, T., Weissler, F.B.: Rapidly decaying solutions of the nonlinear Schrödinger equation. Commun. Math. Phys. 147(1), 75–100 (1992)

7. Combet, V.: Multi-existence of multi-solitons for the supercritical nonlinear Schrödinger equation in one dimension. Discrete Contin. Dyn. Syst. 34(5), 1961–1993 (2014)

8. Dodson, B., Murphy, J.: A new proof of scattering below the ground state for the 3D radial focusing cubic NLS. Proc. Am. Math. Soc. 145(11), 4859–4867 (2017)

9. Dodson, B., Murphy, J.: A new proof of scattering below the ground state for the non-radial focusing NLS. Math. Res. Lett. 25(6), 1805–1825 (2018)

10. Duyckaerts, T., Holmer, J., Roudenko, S.: Scattering for the non-radial 3D cubic nonlinear Schrödinger equation. Math. Res. Lett. 15(6), 1233–1250 (2008)

11. Duyckaerts, T., Landoulsi, O., Roudenko, S.: Threshold solutions in the focusing 3D cubic NLS equation outside a strictly convex obstacle. J. Funct. Anal. 282, no. 5, Paper No. 109326 (2022)

12. Duyckaerts, T., Merle, F.: Dynamic of threshold solutions for energy-critical NLS. Geom. Funct. Anal. 18(6), 1787–1840 (2009)

13. Duyckaerts, T., Roudenko, S.: Threshold solutions for the focusing 3D cubic Schrödinger equation. Rev. Mat. Iberoam. 26(1), 1–56 (2010)

14. Fang, D.Y., Xie, J., Cazenave, T.: Scattering for the focusing energy-subcritical nonlinear Schrödinger equation. Sci. China Math. 54(10), 2037–2062 (2011)

15. Ginibre, J., Velo, G.: On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case. J. Funct. Anal. 32, 1–32 (1979)

16. Goodman, R.H., Holmes, P.J., Weinstein, M.I.: Strong NLS soliton-defect interactions. Physica D 192, 215–248 (2004)

17. Holmer, J., Roudenko, S.: A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation. Commun. Math. Phys. 282(2), 435–467 (2008)

18. Holmer, J., Roudenko, S.: Divergence of infinite-variance nonradial solutions to the 3D NLS equation. Commun. Partial Differ. Equ. 35(5), 878–905 (2010)

19. Ikeda, M., Inui, T.: Global dynamics below the standing waves for the focusing semilinear Schrödinger equation with a repulsive Dirac delta potential. Anal. PDE 10(2), 481–512 (2017)

20. Inui, T.: Global dynamics of solutions with group invariance for the nonlinear Schrödinger equation. Commun. Pure Appl. Anal. 16(2), 557–590 (2017)

21. Inui, T.: Remarks on the global dynamics for solutions with an infinite group invariance to the nonlinear Schrödinger equation. Commun. Pure Appl. Anal. 16, 557–590 (2018)

22. Kenig, C.E., Merle, F.: Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. Invent. Math. 166(3), 645–675 (2006)

23. Linares, F., Ponce, G.: Introduction to Nonlinear Dispersive Equations. Universitext, 2nd edn. Springer, New York (2015)

24. Miao, C., Murphy, J., Zheng, J.: Threshold scattering for the focusing NLS with a repulsive potential. Preprint arXiv:2102.07163

25. Nguyễn, T.V.: Existence of multi-solitary waves with logarithmic relative distances for the NLS equation. C. R. Math. Acad. Sci. Paris 357(1), 13–58 (2019)

26. Strauss, W.A.: Nonlinear scattering theory at low energy. J. Funct. Anal. 41(1), 110–133 (1981)

27. Weinstein, M.: The nonlinear Schrödinger equation—a singularity formation, stability and dispersion. Contemp. Math. 99, 213–232 (1989)