The Parker-Shearing instability in azimuthaly magnetized discs

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Received June 3; accepted August 18, 1994

Abstract. We describe the effects of both magnetic buoyancy and differential rotation on a disc of isothermal gas embedded in a purely azimuthal magnetic field, in order to study the evolution and interplay of Parker and shearing instabilities.

We perform a linear analysis of the evolution of perturbations in the shearing sheet model. Both instabilities occur on the slow MHD branch of the dispersion relation, and can affect the same waves. We put a stress on the natural polarization properties of the slow MHD waves to get a better understanding of the physics involved. The mechanism of the shearing instability is described in details.

Differential rotation can transiently stabilize slow MHD waves with a vertical wavelength longer than the scale height of the disc, against the Parker instability. Waves with a vertical wavelength shorter than the scale height of the disc are subject to both the Parker and the transient shearing instabilities. They occur in different ranges of radial wavenumbers, i.e. at different times in the shearing evolution; these ranges can overlap or, on the contrary, be separated by a phase of wave-like oscillations, depending on the strength of differential rotation.

These analytical results, obtained in a WKB approximation, are found to be in excellent agreement with numerical solutions of the full set of linearized equations. Our results can be applied to both galactic and accretion discs.

Key words: accretion discs – instabilities – MHD – cosmic rays – Galaxies: ISM – Galaxies: magnetic fields

1. Introduction

The Parker instability (Parker 1966) relies on the natural tendency of the horizontal component of the magnetic flux to escape towards the surface of a vertically stratified atmosphere (Kruskal & Schwarzschild 1954, Tserkovnikov 1960, Newcomb 1961, Yu 1966). This effect of magnetic buoyancy occurs wherever a magnetic field is embedded in a density gradient: its classical manifestations are the sunspots on the surface of the sun (Parker 1955), the escape of the magnetic field lines towards the halo of disc galaxies (Parker 1966, Lachièze-Rey et al. 1980), or the formation of the molecular clouds which appear as “beads on a string” in galaxies (Parker 1966, Mouschovias et al. 1974, Blitz & Shu 1980, Elmegreen 1982b).

The latter application, however, has been criticized (Zweibel & Kulsrud 1975, Parker 1975b, Balbus 1988), and recent observations indicate that the Jeans instability is a major ingredient in this process (Elmegreen et al. 1994). It may be faster than the Parker instability, and self-gravity has the advantage of not relying on the little known topology of the magnetic field lines. The self gravitational collapse may even be helped by the presence of a sheared magnetic field (Lynden-Bell 1966, Elmegreen & Elmegreen 1983, Elmegreen 1987). Some purely gravitational mechanisms have also been advocated (Balbus 1988), as well as thermal instabilities (Elmegreen 1989a,b, 1990). Nevertheless, the Parker mechanism may still play a role (Elmegreen 1991, Hanawa et al. 1992), especially since recent work by Giz & Shu (1993) shows that, with more realistic vertical equilibrium profiles, one should expect a growth rate a factor two higher than in classical calculations. The Parker instability could also be a key ingredient of the dynamo process in stars (Parker 1975a) and galaxies (Ruzmaikin et al. 1988, Hanasz & Lesch 1993 and references therein), if combined with differential rotation.

The recent re-discovery of a powerful shearing instability (Velikhov 1959, Chandrasekhar 1960) in magnetized discs by Balbus & Hawley (1991) has renewed the interest taken in differentially rotating magnetized discs: an axisymmetric instability was found to occur in any magnetic configuration which is not purely azimuthal (Balbus...
The configuration with a purely azimuthal magnetic field is stable to axisymmetric perturbations, but transiently unstable to non-axisymmetric ones. This different instability was very briefly mentioned by Acheson (1978) in the context of stellar interiors, and described in more detail by Foglizzo & Tagger (1991), and Balbus & Hawley (1992b) when applied to galaxies and accretion discs.

One of the most promising results of these shearing instabilities, combined with the Parker instability, could be the description, in a single model, of both the turbulent viscosity and the dynamo action in accretion discs (Vishniac & Diamond 1992, Tout & Pringle 1992). Unfortunately, these attempts are still mainly qualitative, because of the analytical difficulties introduced by differential rotation.

On the other hand, although the case of an initially vertical magnetic field leads to the most powerful shearing instability, its nonlinear development gives birth to a strong azimuthal magnetic component (Zhang et al. 1994). This azimuthal magnetic field is subject to both the Parker and the transient shearing instabilities, and has not been fully studied yet: Balbus & Hawley (1992b) canceled the effects of magnetic buoyancy by studying the shearing instability in the disc midplane. Moreover, they introduced a radial component of the initial magnetic field, thus precluding the possibility of an initial hydrostatic equilibrium.

In this paper we describe in detail the effect of differential rotation on waves in a disc embedded in an azimuthal magnetic field, at the linear stage. This has already been the purpose of an earlier study (Foglizzo & Tagger 1994, hereafter [FT]), where we considered perturbations with a vanishing vertical wavenumber because this is known to be the most favourable configuration to the Parker instability, though it considerably limits the effect of shear. Here, by considering perturbations with a non-vanishing vertical wavenumber, acting on a true hydrostatic equilibrium (with a purely azimuthal initial magnetic field), in a vertical density gradient, we can look at the details of the conjugate effects of both the Parker and the azimuthal shearing instability. We will show that they occur on the same branch (the “slow wave”) of the MHD dispersion relation, at different values of the radial wavenumber. Since shearing motions lead the latter to vary with time, this means that the same wave can evolve through periods of stable oscillations, and shearing or Parker amplification, corresponding to a changing polarization (i.e. the orientation of the perturbed velocities associated with the wave), respectively more radial or more vertical. These two amplification mechanisms are found to act successively or sometimes simultaneously, depending on the strength of differential rotation.

Our paper is organized as follows: in Sect. 2 we first recall our hypotheses, which essentially reduce to the “canonical” ones used in classical works on the Parker instability and to the use of the “shearing sheet” model used in the description of spiral waves in differentially rotating discs. This leads to our set of linearized differential equations. In Sect. 3 we discuss in detail the validity of the WKB approximation, and interpret it in terms of the natural polarization of the slow MHD waves. Throughout the paper we will find that, in addition to giving a clear understanding of the physics involved, the WKB approximation gives results in excellent agreement with exact numerical solutions.

In Sect. 4 we explain the mechanism of the shearing instability. We compare the characteristic features of the shearing and the Parker instabilities, taken separately. We show analytically in Sect. 5 how these two instabilities proceed from the same slow MHD mode, and solve the apparent contradiction with the transient shearing stabilization found in [FT]. We also derive the overall strength of the transient amplification.

The phase of transition between the two instabilities is shown to depend crucially on the strength of differential rotation in Sect. 6. We summarize our main conclusions in Sect. 7, and give the details of our analytical treatment in the appendices.

2. The equations

2.1. Hydrostatic equilibrium

The equilibrium is exactly the same as in [FT], and we recall it here very briefly. In the rotating frame at distance \( r_0 \) from the rotation axis, we define the vectors \( \mathbf{x} \) along the radial direction \( (x = r - r_0) \), \( \mathbf{y} \) along the azimuthal magnetic field \( (B_x = B_y) \), and \( \mathbf{z} \) parallel to the rotation axis \( (\Omega = \Omega z) \).

The vertical gravitational field is taken to be constant \((-\text{sign}(z)g_z \mathbf{z})\). This classical hypothesis allows an easier mathematical formulation.

The magnetic pressure (and the cosmic ray pressure in the case of a galactic disc) are assumed to be proportional to the thermal gas pressure:

\[
P_b = \alpha P_{th} \quad \text{and} \quad P_{cr} = \beta P_{th}.
\]

The gas is convectively neutral \((\gamma = 1)\), and both the sound speed \( a \) and the Alfven speed \( V_A = a(2\alpha)^{1/2} \) are independent of height.
Thus the vertical density profile of the hydrostatic equilibrium is simply:
\[
\rho_o(z) = \rho_o(0) \exp(-|z|/H),
\]
where the scale height is:
\[
H = (1 + \alpha + \beta) \frac{a^2}{g_z}.
\]

In our numerical calculations, we assume that the vertical gravity and rotation frequency are related by:
\[
g_z \sim \frac{H}{r_0} g_r \sim H\Omega^2.
\]
This neglects the gravity of the midplane disc of stars, and consequently would overestimate the scale height \(H\) in the case of a realistic galactic disc.

2.2. Linearized perturbations in the shearing-sheet approximation

We will write our set of equations in terms of the Lagrangian displacement vector defined by:
\[
\frac{\partial}{\partial t} + \mathbf{V}_\phi \cdot \nabla) \xi = \mathbf{v} + (\xi, \nabla)\mathbf{V}_\phi.
\]
The perturbations of the density and magnetic field take the simple form:
\[
\rho = -\nabla \cdot (\rho_o \xi),
\]
\[
b = \nabla \times (\xi \times \mathbf{B}_0).
\]

In the shearing sheet approximation, differential rotation is measured by the Oort constant \(A\):
\[
A = \frac{r_0 \partial \Omega}{2 \partial r}(r_0) < 0.
\]
In the frame rotating at radius \(r_0\) with the rotation frequency \(\Omega = \Omega(r_0)\), the sheared motion is linearized so that the azimuthal speed of the gas at equilibrium is:
\[
V_0 = r_0 \Omega_0 + 2Ax.
\]
The only difference with [FT] is the presence here of a vertical wavenumber \(k_z\), so that perturbed quantities vary as:
\[
\xi(x, y, z, t) = \rho_o \frac{1}{\xi_0} e^{i(k_y y + k_z z - \frac{1}{2} k_x x + k_z z)} e^{i(k_x x - 2Ak_y t)} \xi(k_x, t) dk_x
\]
\(k_x\) was set to zero in [FT] for the sake of simplicity, since classically this is the most unstable Parker mode. In Eq. (10) and hereafter the \((k_y, k_z)\)-dependence of \(\xi(k_x, t)\) is not written explicitly, for the sake of clarity, since linear calculations allow us to keep them constant.

Defining \(\tilde{\xi}(k_x, t)\) by Eq. (10) permits us to interpret one effect of differential rotation as a linear time-dependence of the radial wavenumber of the sheared perturbation \((K_x(t) = k_x - 2Ak_y t)\). This appears on the differential system satisfied by \(\tilde{\xi}(k_x, t)\):
\[
\frac{\partial^2 \tilde{\xi}}{\partial t^2} = -2\Omega \frac{\partial \tilde{\xi}}{\partial t} - \frac{a^2}{H^2} \mathcal{L}(K_x(t)) \tilde{\xi},
\]
where the dimensionless hermitian matrix \(\mathcal{L}(k_x)\) includes additional terms due to the vertical wavenumber \(k_z\):
\[
\mathcal{L}(k_x) = \begin{pmatrix}
\tilde{k}_x^2 + 2\alpha(\tilde{k}_x^2 + \tilde{k}_y^2 - \tilde{k}_A^2) & \tilde{k}_x \tilde{k}_y & (\frac{1}{2} + \beta - i\tilde{k}_z(1+2\alpha))i\tilde{k}_x \\
\tilde{k}_x \tilde{k}_y & \tilde{k}_y^2 & (\frac{1}{2} + \alpha + \beta - i\tilde{k}_z) \tilde{k}_y \\
2\alpha \tilde{k}_y^2 + (\tilde{k}_z^2 + \frac{1}{4})(1+2\alpha) & \tilde{k}_y & \tilde{k}_y^2 + \tilde{k}_z^2
\end{pmatrix}
\]
We have noted \(\tilde{k} \equiv Hk\) the dimensionless wavenumbers normalized by the scale height of the disc, and crosses denote the complex conjugate of the symmetric terms. The differential force appears through the dimensionless parameter \(\tilde{k}_A\) defined as:
\[
\tilde{k}_A \equiv -\frac{4A\Omega H^2}{V_A^2} > 0.
\]
As stated in [FT], the shearing time-scale \(T_{\text{Shear}} = |A|^{-1}\) occurs both:
(i) as the typical time-scale of the shearing of waves, i.e. the linear growth of the radial wavenumber:
\[
K_x(t) \equiv k_x - 2Ak_y t,
\]
(ii) as a scaling of the radial differential force, which is opposite to the radial magnetic tension. Their ratio is:
\[
-\frac{4A\Omega}{k_y V_A} = \frac{\tilde{k}_A}{k_y} \sim \frac{2\pi}{k_y T_{\text{Parker}}} \frac{T_{\text{Shear}}}{T_{\text{Parker}}}.
\]

3. WKB approximation

3.1. Mathematical transformations

The presence of a vertical wavenumber \(k_z\) still allows the mathematical transformations performed in [FT]: we reduce the sixth order differential system (11) to a second order differential equation, by first performing a Laplace transform:
\[
\tilde{\xi}(k_x, \omega) = e^{-\frac{\pi\omega}{2k_y}} \int_0^{+\infty} e^{i\omega t} \tilde{\xi}(k_x, t) dt.
\]
In Appendix A we derive an ordinary differential equation, of which \(\tilde{\xi}(k_x, \omega)\) is the unique solution bounded at infinity. This equation has a singularity where \(\Delta(\omega^2) = 0,\)
and this polynomial is interpreted in [FT] as the asymptotic dispersion relation at \(|K_x(t)| \to \infty\). Reciprocally, \(\xi(k_x, t)\) is determined by the inverse Laplace transform:

\[
\xi(k_x, t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{i\omega(t - \frac{x}{c_s})} \xi(k_x, \omega) d\omega,
\]

where \(p\) is real and larger than the largest imaginary part of the singularities of \(\xi\).

The solution of the differential equation (A14) can be written in the WKB approximation:

\[
\xi(k_x, t) = \frac{1}{2\pi} \int_{-i\infty}^{i\infty} (\mu_+ e^{i\Psi} - \mu_- e^{-i\Psi}) d\omega.
\]

and the functions \(\mu_{\pm}(\omega)\) vary slowly if a WKB criterion, derived in Appendix B, is satisfied. We show that the major contributions to the integral (17) are then given by the saddle points \(\omega_{\pm}(t)\) defined by:

\[
\frac{\partial \Psi_{\pm}}{\partial \omega}(\omega_{\pm}(t)) = 0.
\]

As in [FT], we can move the integration contour of Eq. (17) so as to pass through these saddle points, and follow the steepest descent path between them. When the WKB criterion is satisfied, this formulation allows us to interpret Eq. (18) as a dispersion relation whose roots are time-dependent. The saddle points are the relevant instantaneous frequencies or growth rates which used to be the poles of the Laplace transform \(\bar{\xi}(\omega)\) in the absence of differential rotation.

3.2. WKB validity and the linear coupling of waves

It was shown in [FT] that if the WKB criterion is satisfied, the evolution is “adiabatic”, i.e. a wave (stable or not) preserves its “identity” throughout its sheared evolution. On the other hand, at low \(|K_x(t)|\), the WKB approximation fails if the differential rotation is realistic, and a linear coupling of waves occurs: an initial pure Parker mode (slow MHD wave) is associated with a complex pattern of motions, due to the interplay of the Coriolis force and magnetic tension with its basic perturbed velocities. As a result it loses its identity and it can emit both magnetosonic (fast MHD) and Alfvénic waves in the disc.

The WKB criterion can be understood as the requirement that the instantaneous frequency and polarization of the waves evolve (because of the time dependence of the radial wavenumber \(K_x(t)\)) sufficiently slowly with time. This criterion is satisfied in two different cases (see Appendix B):

(i) If \(K_x(t)\) varies very slowly with time, the very slow temporal variations of the growth rate and polarization allow a globally adiabatic evolution of the solution. This is made possible if the shearing parameter \(A\) is very small. The effect of the differential force \((4A\Omega\xi_x)\) becomes negligible compared to the radial magnetic tension \((k_y^2V_x^2\xi_x)\) if the magnetic field is not very weak (as in Shu 1974). Nevertheless, the case of a vanishing field allows us to study the effect of the differential force on the Parker instability ([FT] and here, Sect. 5).

(ii) If the polarization and the frequency of the perturbation depend weakly on the radial wavenumber \(K_x(t)\), the WKB analysis is valid even if the shearing parameter is not small. We study in the next section the polarization and the frequency of the slow MHD wave to determine the range of wavenumbers \(K_x, k_z\) that allows this approximation.

3.3. The polarization of the slow MHD mode

The slow MHD mode is characterized by displacements with a strong component along the magnetic field lines, taking advantage of the anisotropy of the magnetic pressure. The amounts of radial and vertical displacements depend crucially on the orientation of the wavevector. Independently of the presence of any instability, the vertical stratification and the rotation of the magnetized plasma are responsible for the following properties:

\[
\frac{\xi_x}{\xi_z} = \mathcal{O} \left( \frac{k_x}{k_z} \right) \to \infty \text{ when } |k_x| \gg k_z, Hk_y^2
\]

\[
\frac{\xi_z}{\xi_x} = \mathcal{O} \left( \frac{k_z}{k_x} \right) \to 0 \text{ when } |k_z| \gg k_x, k_y^2a_s^2/\Omega
\]

\[
\frac{\xi_z}{\xi_x} \sim -\frac{k_x}{k_z} \text{ when } |k_z|, |k_x| \gg Hk_y^2, k_y^2a_s^2/\Omega
\]

The transition between these asymptotic polarizations occurs at \(k_x \sim k_z\) (if \(k_z > Hk_y^2, k_y^2a_s^2/\Omega\)). It should be remembered that in most cases of interest for us, \(Hk_y^2 \sim k_y^2a_s^2/\Omega \sim 1/H\).

These properties are typical of both the Alfvén and slow MHD modes in the classical case of a stratified plasma with uniform rotation, and can be derived in a straightforward way by a normal mode analysis (see Appendix C).

The WKB approximation, which is valid in each of these asymptotic limits, allows us to generalize these properties to the case of differential rotation.

We know from [FT] that differential rotation causes a wave with an initial radial wavenumber \(K_x \sim -\infty\) to evolve linearly with time towards \(K_x \to +\infty\). A slow MHD wave with an initial \(K_x(t = 0) \ll -|k_z|\) is polarized mostly in the vertical plane \((y, z)\) (see Eq. (19)). It rocks naturally towards the horizontal \((x, y)\) plane when, because of the shear, \(|K_x(t)| < k_z\) (see Eq. (20)). It rocks again towards the vertical \((y, z)\) plane when \(K_x(t) > k_z\) (Eq. (19) again). The intrinsic nature of the slow MHD wave is consequently enough to explain that the displacements are successively vertical, horizontal, and vertical again, and thus more prone to buoyancy or Coriolis forces. Figure 4 illustrates...
phase of horizontal polarization shrinks to about one shear
duration of the phase of horizontal polarization scales as:
frequency remains constant. According to Eq. (13), the
polarization is essentially horizontal and the eigen-
value reduces to \( \Delta(\omega^2) = 0 \), which is independent of \( k_x \).
The evolution is then adiabatic and allows a WKB study
of the asymptotic Parker instability (Shu 1974, [FT]).

When the vertical wavenumber \( |k_z| \gg k_z, H k_y^2 \),
the polarization of the slow MHD wave is essentially vertical and
the eigenfrequency remains constant. The dispersion relation
reduces to \( \Delta(\omega^2) = 0 \), which is independent of \( k_x \).
The evolution is then adiabatic and allows a WKB study
of the asymptotic Parker instability (Shu 1974, [FT]).

When the vertical wavenumber \( |k_z| \gg k_x, H k_y^2 / \Omega \),
the polarization is essentially horizontal and the eigen-
frequency remains constant. According to Eq. (13), the
duration of the phase of horizontal polarization scales as:
\[
T_{\text{horiz}} \sim \frac{k_z}{\Omega k_y}.
\] (22)

Thus the polarization of the slow MHD wave is con-
stantly horizontal on an arbitrary long time if the ver-
tical wavenumber is high enough. The dispersion relation
reduces to \( Q_A(\omega^2) = 0 \), which is independent of \( k_x \) (see Appendix B). The evolution is then adiabatic and allows a WKB approximation.

The strongest effect of the differential force is expected
in this phase of horizontal motions, since this force is radial
and proportional to the radial displacement. Indeed, if the
differential force dominates the radial magnetic tension,
the shearing instability will result, during a time com-
parable to \( T_{\text{horiz}} \) (see Sect. 4-5).

In the opposite case where \( k_z \leq k_y^2 \Omega / \sim 1 / H \), the
phase of horizontal polarization shrinks to about one shear
time, and the evolution may be adiabatic only in the limit
of vanishing differential rotation (see [FT] for \( k_z = 0 \)).

If both the vertical and radial wavenumbers are large
\( K_y \sim k_z \gg H k_y^2, k_y^2 \Omega / \Omega \), the polarization of the slow
MHD wave varies slowly:
\[
\frac{\partial K_z(t)/k_z}{\partial t} \sim -2 A \frac{k_z}{k_z} \ll |A|,
\] (23)
and the WKB approximation is possible again (see Sect. 6).

4. The pure shearing instability

4.1. The pure shearing instability

If a magnetic field line of length \( \Lambda_y = 2\pi / k_y \) is bent
by a radial displacement \( \xi_x \), a magnetic tension
\( F_{\text{m}} \equiv \rho_0 k_y^2 V_A^2 \xi_x \) acts as a restoring force. The differential force
\( F_d \equiv 4 \Omega \rho_0 \xi_x \) is also proportional to the radial displace-
ment, but in the opposite direction. Hence if the following
criterion is satisfied:
\[
4 \Omega + k_y^2 V_A^2 < 0,
\] (24)
a radially bent field line will naturally continue its radial displacement.

This balance is indeed a central feature of the shearing
instability. However, this simple description is not enough
to fully explain it: it ignores the effects of the other forces,
namely the Coriolis force, the pressure force and the other
components of the magnetic force. In particular, the Cori-
olis force is usually stabilizing by transforming any radial
motion into an epicyclic oscillation, as in the case of sound
waves modified by rotation. Moreover, one would like to understand why the criterion
(24) seems to predict an instability in the limit of a vanish-
ing magnetic field, which seems to contradict the stability
ensured by the Rayleigh criterion \( (\kappa^2 = 4 \Omega^2 + 4 \Omega > 0) \)
for a non-magnetized disc.

We now proceed to a detailed description of the mech-
anism, by analysing the linearized system in the “hor-
izontal regime” where \( k_z \gg 1 / H, k_y \), and Eq. (20) is valid.
In particular the stratification and cosmic ray effects dis-
appear here.

An illustrative picture of the mechanism is given in Fig. 1.

It is convenient, for the sake of clarity, to decompose
the magnetic force into two contributions, which have simple
expressions in our approximation:

(i) an anisotropic “pressure” force acting against the
compression of the field lines,
\[
\mathbf{F}_p \equiv \rho_0 V_A^2 \nabla \cdot (\nabla \xi) = -i \rho_0 V_A^2 (K_x \xi_x + k_z \xi_z)
\] (25)
Fig. 2. The shearing instability viewed from above the disc: the rotation of the disc is here anti-clockwise. The amplitude of the perturbation is exaggerated for the sake of clarity. The circles along the perturbed magnetic field line represent the maxima of density. When the displacements of the slow MHD mode are mainly horizontal ($\lambda_x < \lambda_c$), the radial thermal pressure cancels the anisotropic magnetic pressure. Instability occurs if the radial magnetic tension is dominated by the differential force. The azimuthal thermal pressure force acts as a restoring force, which can overcome the stabilizing influence of the Coriolis force.

(ii) a magnetic tension which can be visualized as the result of the “elasticity” of the field lines when they are bent.

$$F_{th} = -i\rho_0 V_A^2 k_y a_s \xi_y = -i\rho_0 V_A^2 k_y^2$$

(i) As a consequence of Eq. (27), the sum of the thermal pressure force and the magnetic force, in the radial direction, is reduced to the single magnetic tension term mentioned above.

(ii) As a consequence of Eq. (28), the azimuthal component of the thermal pressure force can be expressed as a restoring force acting on the azimuthal displacement.

Hence in the limit of small vertical wavelength, the slow MHD mode (the Alfvén mode as well) corresponds to perturbations satisfying the following 2-D system:

$$\begin{cases} \frac{\partial^2 \xi_x}{\partial t^2} = 2\Omega \frac{\partial}{\partial \xi_y} \xi_y - (4A\Omega + k_y^2 V_y^2) \xi_x, \\ \frac{\partial^2 \xi_y}{\partial t^2} = -2\Omega \frac{\partial}{\partial \xi_x} \xi_x - k_y^2 V_c^2 \xi_y. \end{cases}$$

(29)

The only remaining forces are the Coriolis force, the radial magnetic tension modified by the differential force, and an azimuthal restoring force (we have noted $V_c = \alpha_s V_A / (a_s^2 + V_A^2)^{1/2}$ the cusp speed).

We can notice the similarity with the case of the axisymmetric instability of a disc embedded in a magnetic field which is not purely azimuthal: Balbus & Hawley (1992a) obtained an equation similar to Eq. (29), with the following differences:

(i) $k_s$ appears instead of $k_y$ here, because of the different geometry of the equilibrium magnetic field;

(ii) the azimuthal restoring force involves the Alfvén speed instead of the cusp speed here.

They stressed the link with the Hill equations, and gave a simple mechanical analogue: the motion of a particle attached with a spring to a guiding center in orbit in a gravitational potential. The spring is isotropic ($C_x = C_y = k_s^2 V_A^2$). This analogy also holds in our case, by considering an anisotropic spring ($C_x = k_y^2 V_A^2$, $C_y = k_s^2 V_y^2$). One can be easily convinced that the motion of the particle is unstable if the stabilizing effect of the spring is canceled by the differential force ($4A\Omega + C_x < 0$).

The importance of the azimuthal restoring force in the instability mechanism is now clear: the Coriolis force would be able to reduce the shearing instability to a neutral mode if the azimuthal spring constant $C_y$ were to vanish (for example if the sound speed $a_s \to 0$).

This occurrence of the cusp speed may be traced back to the fact that we are here dealing with the destabilization of the slow MHD mode, whereas the axisymmetric instability studied by Balbus and Hawley corresponds to the destabilization of the Alfvén mode.

As shown in Appendix B, the WKB criterion is satisfied in the horizontal regime, so that we can replace time derivatives in Eq. (29) by simple $(-i\omega)$ multiplications (note that radial derivatives have disappeared). The resulting dispersion relation reduces to $Q_A(\omega^2) = 0$:

$$\omega^4 - [\kappa^2 + 2(1 + \alpha)k_y^2 V_c^2] \omega^2 - (4A\Omega + k_y^2 V_A^2) = 0$$

(30)

Note that this does not mean incompressibility: Eq. (26) can be equivalently written in order to express the density perturbation as a function of the azimuthal displacement:

$$\frac{\rho}{\rho_0} = -\frac{i k_y V_A^2}{a_s^2 + V_A^2} \xi_y$$

(28)

Independently of the presence of any instability, this implies in particular that the extrema of the density perturbed by a slow MHD mode occur where the azimuthal displacement vanishes (see Fig. 3).
According to this dispersion relation, instability indeed occurs if the differential force dominates the magnetic tension. The other root of this dispersion relation, corresponding to the Alfvén mode, is always stable and reduces to an epicyclic oscillation (inertial wave) in the case of a vanishing magnetic field.

The missing mode is of course the fast magnetosonic mode, corresponding to fast vertical oscillations, which was excluded from our approximation by Eq. (20).

As the Parker instability, the shearing instability requires a long enough wavelength along the magnetic field lines: in the galactic case, the minimum azimuthal wavelength scales as \( \pi V_A |A\Omega|^{-1/2} \sim 1.5 \text{kpc} \).

The optimal azimuthal wavenumber corresponds to \( k_y^2 V_A^2 = (1 + \alpha) \omega^2_s - 2A\Omega \), where \( \omega_s \) is the optimal growth rate:

\[
\omega_s^2 = \frac{-2A^2}{(1 + 2\alpha) \Omega + 2\alpha + \frac{A}{\Omega}\alpha}.
\]  (31)

\( |\omega_s| \) is a decreasing function of the intensity of the magnetic field \( \alpha \). Thus the highest growth rate corresponds to a vanishing magnetic field, and we obtain in this limit, for an infinite optimal \( k_y \):

\[
\omega_s^2 \rightarrow -A^2.
\]  (32)

This extreme limit is of course never reached, because the diffusion of the magnetic field must be taken into account as soon as the magnetic field is weak: our idealized MHD equations then break down, as shown in details by Acheson (1978).

Nevertheless, this calculation confirms the study by Balbus & Hawley (1992a) suggesting that the Oort constant \( A \) is the maximum growth rate for such shearing instabilities.

The instability still exists when the magnetic field is strong, but the growth rate decreases as \( \alpha^{-1/2} \). This instability may play an important role, not only in weakly magnetized accretion discs, but also in the gaseous disc of galaxies, where the magnetic pressure is comparable to the thermal pressure: if \( \alpha \sim 1 \) and \( A/\Omega = -0.5 \), the e-folding time of the shearing instability is about one half of the rotation period for a \( m = 1 \) perturbation at \( R \sim 10 \text{kpc} \).

We can notice here an important difference with the axisymmetric instability of a disc embedded in vertical magnetic field: in this latter case, the instability criterion again compares the differential force to the magnetic tension: \( 4A\Omega + k_y^2 V_A^2 < 0 \). In particular, this condition implies that the instability cannot exist if the magnetic energy is comparable to or larger than the thermal energy. Indeed, the vertical wavelength of the perturbation must be shorter than the scaleheight of the disc, so that axisymmetric instability requires (Balbus & Hawley 1991):

\[
V_A < \frac{1}{\pi} |A\Omega|^2 H \sim \text{Max}(a_s, V_A).
\]  (33)

By contrast, the azimuthal wavelength is only bounded by the circumference of the disc, so that the non-axisymmetric instability we are studying occurs as long as:

\[
V_A < \frac{1}{\pi} |A\Omega|^2 R \sim \text{Max}(a_s, V_A) \frac{R}{H}.
\]  (34)

The shearing instability might therefore be efficient for magnetic fields exceeding the equipartition value, in the outer parts of the disc. This might be an important property of the instability, especially if it were related to a dynamo mechanism.

4.2. Comparison of the growth rates

It is interesting at this point to compare the characteristic length scales and growth rates of the optimal shearing and Parker instabilities. Both occur at large azimuthal scales, i.e. at low \( k_y \). The Parker instability requires \( k_y < k_P \) at \( |k_z| \rightarrow \infty \) (where \( k_P \) is defined in Eq. (A8)), whereas the shearing instability requires \( k_y < k_A \) at \( |k_z| \rightarrow \infty \) (where \( k_A \) is defined in Eq. (12)).

As an illustration, Fig. 3 (top) shows the shearing instability occurring at an azimuthal wavelength which is stable against the Parker instability (\( k_A > k_y > k_P \)). It was obtained by integrating numerically the differential system (11). The bottom graph shows the corresponding slow-wave solutions of the WKB dispersion relation. The two solutions (corresponding to slow waves propagating in opposite directions) are not exactly symmetric with respect to \( \omega \) axis, showing an effect of rotation which was already noted by Shu (1974). The two solutions merge at \( K_x \approx -k_z \) where they become unstable. The numerical solution shows that, as a result, when one starts with a “pure” solution on the top branch, one gets at positive \( K_x \) a mixture of the two waves. This illustrates the linear coupling, which occurs because the WKB criterion is not strictly satisfied where the two slow MHD waves merge: the change of polarization and frequency between the oscillation phase and the growing phase is too fast (the shear, corresponding to a flat rotation curve, is strong: \( -A/\Omega = .5 \)). Figure 3 illustrates this rapid change.

We have plotted in Fig. 4 the range of magnetic pressures and rotation profiles for which, for any azimuthal wavenumber, the shearing growth rate is always higher than the Parker growth rate (contour levels > 1), and conversely (contour levels < 1). For instance, along the contour level labeled “2”, the shearing growth rate is twice as large as the optimal Parker growth rate, at the wavenumber \( k_y \) which is optimal for the Parker instability.
A “realistic” galactic disc ($\alpha \sim \beta \sim 1, -A/\Omega \sim .5$ to .75) would lie in the zone where the strengths of the two instabilities are comparable. On the basis of the comparison, in the linear regime, of the growth rates of these two instabilities, it is not possible to favour one particular vertical wavenumber. This contrasts with the case of a uniformly rotating disc, where the maximization of the growth rate of the Parker instability over all possible wavenumbers would select the mode with $k_z = 0$ (Parker 1966, Zweibel & Kulsrud 1975).

Non-linear effects will play a major role in selecting the vertical wavelength of the most unstable perturbations, but this is beyond the scope of the present paper.

In an accretion disc with Keplerian rotation ($-A/\Omega \sim .75$), the growth rate of the shearing instability is higher than the Parker growth rate at any $\alpha < 2$. In such a disc, the fastest growing perturbations, in the linear regime, have a short vertical wavelength and are essentially horizontal.

We must bear in mind that this comparison is quantitatively dependent on the relationship assumed between the scale height $H$ and the rotation frequency $\Omega$ (Eq. (4)). Figure 4 is, however, qualitatively relevant. A lower scale height would slightly favour the Parker instability. On the other hand, if the scaleheight of the magnetic pressure were higher than the scale height of the gas, the Parker instability would be weakened, while the shearing instability would remain unchanged.

**Fig. 3.** With $k_A > k_y > k_P$, the perturbation is stable against the Parker instability at $|K_z(t)| > |k_z|$, and unstable to the shearing instability at $|K_z(t)| < |k_z|$. The evolution of the displacement vector (top figure), is in very good agreement with the calculation of the slow MHD root of the WKB-dispersion relation (bottom figure). Parameters are $-A/\Omega = .5, k_z = 10, \alpha = 0.1, \beta = 0$. The three components $(x, y, z)$ of the displacement vector $\xi$ are displayed. Wavenumbers are in units of $H^{-1}$; frequencies in units of $V_A/H$.

In the transition between these two regions, the choice of the azimuthal wavenumber determines which instability is faster. Introducing a cosmic-ray gas favours the Parker instability only: the pure-shearing instability does not depend on the cosmic-ray pressure, because the displacements involved are essentially horizontal and the differential force does not act on the massless cosmic-ray gas (see Eq. (30)).

**Fig. 4.** Comparison of the growth rates of the shearing and Parker instabilities, depending on the strengths of the magnetic field and differential rotation. The contour levels give a conservative estimation of the ratio $\omega_S/\omega_P$: it is minimized (resp. maximized) over every $k_y$ when $\omega_S/\omega_P > 1$ (resp. $\omega_S/\omega_P < 1$).
This comparison of the two isolated optimal instabilities is of course only a first estimate of their relative importance. A refined study would require taking into account the temporal evolution of the radial wavenumber, and noting how the instabilities may be simultaneous or mutually exclusive.

We already know that the Parker instability is favoured by large radial wavenumbers and vanishing vertical ones, whereas the shearing instability favours large vertical wavenumbers and limited radial ones. The next sections are devoted to a further study of this point.

5. The Parker-Shearing instability

5.1. The simplified dispersion relation for slow MHD waves

As mentioned by Balbus & Hawley (1994), the shearing instability is the consequence of the differential force rather than of the shearing of the flow. In particular, this local instability would not exist in a shear flow between two parallel planes, as can be easily checked by canceling the rotation terms (Coriolis and differential forces) and keeping the shearing of waves (time-dependent wavenumber).

Nevertheless, the sheared motion is responsible for the transience of the instability, as well as a linear coupling of the waves at low $|K_x(t)|$, as was shown in [FT]. We concentrate here on the nature of the Parker and shearing instabilities, and avoid the complications due to this coupling by considering the case of low shear ($A/\Omega \to 0$).

In order to study the effects of the differential force, we keep it comparable with the magnetic tension, i.e.

$$k_y^2 V_A^2 \sim 4A\Omega.$$  

This leads us to consider the case of a vanishing magnetic field ($\alpha \to 0$) so as to keep:

$$k_A \sim k_y,$$  

$$\tilde{\Omega}^2 \sim 1/\alpha \gg 1.$$  

Hence we use the same WKB-approximation as in [FT], but include here a non-vanishing vertical wavenumber $\bar{k}_z$. Assuming that $\bar{\omega} = \mathcal{O}(1/\bar{\Omega})$, we can write the leading terms of the WKB-dispersion relation (B11) and obtain:

$$\left(\bar{k}_z^2 + \frac{1}{4}\right) (\bar{\omega}\bar{\Omega})^2 + 2\bar{K}_y \bar{k}_y \bar{k}_z (\bar{\omega}\bar{\Omega}) - \bar{k}_y^2 R = 0,$$  

with:

$$R = \bar{k}_y^2 (\bar{k}_y^2 - \bar{k}_A^2) + \bar{k}_y^2 (\bar{k}_y^2 - \bar{k}_Q^2) + \bar{k}_A^2 \bar{k}_z^2 / 4 - \bar{k}_A^2 (\bar{k}_y^2 - \bar{k}_y^2).$$  

When $k_A = 0$, we recover the classical criterion for Parker instability at $K_x = 0$:

$$k_y^2 + k_z^2 < k_Q^2.$$  

This indicates that the dispersion relation [38] gives, for finite values of $K_x(t)$, the behaviour of the two slow MHD frequencies.

Now we study the stabilizing or destabilizing influence of the differential force on the slow MHD branch.

5.2. Stabilization of the Parker instability by differential rotation

When $\bar{k}_z^2 < 1/4$, a strong differential force ($k_A > k_y$) impedes the Parker instability at low $|K_x(t)|$ even though $k_y^2 + k_z^2 < k_Q^2$. This stabilization was found in [FT] for $\bar{k}_z = 0$, and is illustrated in Fig. 5 for $\bar{k}_z = 0.25$. Nevertheless, this stabilization lasts a few shear times only.

5.3. Destabilization of the slow MHD mode by differential rotation

When $\bar{k}_y^2 > 1/4$, the discriminant of Eq. (38) is negative if $K_x(t)^2 < \bar{K}_x^2$, with:

$$\bar{K}_x^2 \equiv (4\bar{k}_z^2 + 1) \frac{\bar{k}_y^2 (\bar{k}_y^2 - \bar{k}_Q^2 - \bar{k}_A^2) + \bar{k}_A^2 (\bar{k}_y^2 - 1/4)}{4\bar{k}_z^2 (\bar{k}_Q^2 + 1/2) + \bar{k}_y^2 - 1/2}.$$  

The differential term $k_A^2$ clearly contributes to destabilize the slow MHD mode, resulting in the shearing instability.

When $\bar{k}_z \gg \bar{k}_y, \bar{K}_x, 1/2$, we recover the pure shearing instability studied in Sect. 4:

$$\bar{\omega}_S^2 \sim -\frac{k_y^2}{\bar{\Omega}} (\bar{k}_A^2 - \bar{k}_y^2),$$  

with the usual criterion for instability:

$$k_A > k_y \iff 4A\Omega + k_y^2 V_A^2 < 0.$$  

For $\bar{k}_y^2 = \bar{k}_A^2 / 2$ we recover the optimal growth rate:

$$|\bar{\omega}| \sim |A|.$$  

5.4. Efficiency of the shearing instability at $\bar{k}_x > 1/2$

Eq. (38) is useful in determining the overall strength of the transient shearing amplification. When the discriminant of Eq. (38) is negative, the complex root is:

$$\bar{\Omega}_{\bar{\omega}} = -\frac{\bar{k}_y \bar{k}_z \bar{K}_x}{\bar{k}_z^2 + 1/4} \left[ 1 \pm i \left( 1 + \left( \bar{k}_z^2 + \frac{1}{4} \right) \frac{R}{\bar{k}_z^2 \bar{K}_z} \right)^{\frac{1}{2}} \right].$$  

We define the complex amplification factor $\varphi$ so that the amplitude of the wave after the transient amplification is multiplied by $\exp(\varphi)$:

$$\varphi \equiv \int_{-K_x}^{+K_x} i\bar{\omega}(\bar{K}_x) d\bar{K}_x.$$  

This comparison of the two isolated optimal instabilities is of course only a first estimate of their relative importance. A refined study would require taking into account the temporal evolution of the radial wavenumber, and noting how the instabilities may be simultaneous or mutually exclusive.
5.5. Comment on the transient character of the Shearing instability

We wish to emphasize that the shearing instability, although transient and occurring preferentially at low \( k_x \), is essentially local and does grow on a sufficiently short time scale that its transient nature poses no physical limit to its growth: this contrasts with e.g. spiral density waves, which travel in the disc and can experience amplification only for a limited time (as they are reflected from corotation), unless the boundary conditions at the disc center allow them to be reflected back toward corotation.

Indeed the time required for the growing perturbation to travel, vertically or radially, along a distance comparable to the scaleheight of the disc is much longer than the typical e-folding time for realistic values of differential rotation. This can be checked by derivating the dispersion relation (B14) with respect to \( k_x \) (or \( k_z \)) to obtain the radial (or vertical) group velocity:

\[
\frac{v_{gx}}{H} = \frac{1}{H} \frac{\partial \omega}{\partial k_x} \sim \frac{\Omega}{H k_z} \ll |A|
\]

Thus perturbations can be considered to grow where they were created.

The transient character of any instability may be an obstacle to its development only if the instability does not have enough e-folding time to produce a significant effect: we have shown that the shearing instability does not suffer this limitation if we allow for a short enough vertical wavelength.

Incidentally, it has been shown in the context of non-magnetized discs that any local instability is stabilized by the effects of viscosity in a differentially rotating disc (see Korycansky (1992) or Dubrulle & Knobloch (1992)). This stabilization arises when the sheared radial wavenumber increases towards infinity, i.e. when the radial wavelength is smaller than the typical scale for dissipative processes. Nevertheless, such a mathematical limitation of the linear regime has no physical implication if the “transient” instability has had enough e-folding time to produce perturbations which lead to the non linear regime.

As an illustration, we know that the Parker instability is stabilized by ambipolar diffusion for scales \(< 1\) pc in the galaxy (see Cesarsky 1980): this leaves more than 100 e−folding time for the Parker instability to occur on larger radial scales.

Similarly, the shearing instability is limited by the minimum vertical scale at which the ideal MHD equations are valid. It can be estimated, in a galactic disc, as the same critical scale used in the radial direction for the Parker instability. Although transient by definition, the shearing
instability is not *physically* more transient than the Parker instability.

**Fig. 6.** Shearing instability at $|K_x| < K_s \sim 20$, and Parker instability at $|K_x| > K_p \sim 40$, separated by an oscillating phase. Here the shearing growth rate is lower than the Parker one. Parameters are $\alpha = 1$, $\beta = 0$, $-A/\Omega = .25$, and $k_y = .57$ is optimal for the Parker instability $k_z = 20$. We have then $k_A = k_P = 1$, and $k_Q = .87$

6. Transition between the Parker and the shearing instability

6.1. General evolution of the unstable slow MHD wave

The dispersion relation (38) demonstrates that both the shearing and the Parker instabilities belong to the same slow MHD branch, thus confirming the qualitative argument of Sect. 3.3.: the Parker instability occurs when the polarization of the slow MHD wave is vertical ($K_x(t) \ll -|k_z|$), and is stabilized when the polarization rocks towards the plane of the disc ($|K_x(t)| \sim k_z$). The differential force then acts, in the horizontal plane, on the radially displaced gas in the same destabilizing way as did the gradient of magnetic pressure, in the vertical plane, on the vertically displaced gas. While the shearing goes on, the polarization rocks back towards the vertical plane, leading to the Parker instability again at $K_x(t) \gg k_z$. This behaviour is illustrated in Figs.(5–8).

The relative growth rates of the two successive instabilities (Parker/shearing) simply depend on the strengths
of the magnetic field and differential rotation, as studied in Sect. 4. The two instabilities can follow one another without any interruption of growth as $K_s(t)$ varies (Fig. 3), or they can be separated by a phase where the slow MHD mode is oscillating as in Figs.(6-7).

6.2. The case of weak differential rotation

Even without differential rotation, the Parker instability is naturally stabilized at low $k_x$ when the vertical wavenumber is high. This appears on the classical criterion for the Parker instability $k_{Q}^2 + k_{z}^2 < k_{Q}^2$ at $k_x = 0$. This defines a radial wavenumber $K_p$ such that stabilization occurs at $|k_x| < K_p$. It can be calculated from the classical dispersion relation without differential rotation (B12). An easy estimate of $K_p$ is possible in the absence of rotation ($\Omega = \bar{k}_A = 0$), for large values of $\bar{k}_z$:

$$K_p \sim \left(\frac{\bar{k}_z^2}{k_p^2 - k_y^2}\right)^{1/2} \bar{k}_z. \quad (50)$$

The shearing instability occurs at low $|K_x(t)|$ only, and is stabilized for $|K_x(t)| > K_s$ even when the Parker instability is not present (see Balbus & Hawley 1992b). The value of $K_s$ depends on the intensity of the differential rotation as in Eq. (41): $K_s^2$ is negative if $k_A < k_y$, and it naturally increases when the shear parameter increases. We obtain, for $\bar{k}_z \gg \bar{k}_y$:

$$K_s \sim \left(\frac{\bar{k}_A^2 - \bar{k}_y^2}{k_y^2}\right)^{1/2} \bar{k}_z. \quad (51)$$

A weak differential rotation is therefore likely to imply $K_s < K_p$.

6.3. The case of realistic differential rotation

We can derive a criterion for the presence of an oscillating phase of transition between the two instabilities, for large values of $\bar{k}_z$. According to our previous estimate, such a transition occurs at $\bar{K}_x \sim \bar{k}_z$. The WKB-approximation is then possible even if the differential rotation is strong (see Appendix B).

If an oscillating transition occurs, the two slow MHD frequencies $\hat{\omega}(\bar{K}_x)$ are purely real, and satisfy the following dispersion relation:

$$Q_A + \left(\frac{\bar{K}_x}{\bar{k}_z}\right)^2 \Delta + 2\bar{\Omega}(1 + \alpha + \beta)\bar{k}_y \left(\frac{\bar{K}_x}{\bar{k}_z}\right) = 0. \quad (52)$$

The two frequencies $\hat{\omega}(\bar{K}_x)$ pass through an extremum (see Fig. 1 and Fig. 2) at $\hat{\omega}_\pm(K_x)$ such that:

$$\frac{\partial \hat{\omega}}{\partial k_x} = 0 \iff \frac{\bar{K}_x}{\bar{k}_z} = \frac{\hat{\omega}_\pm(1 + \alpha + \beta)\bar{k}_y}{\Delta}. \quad (53)$$

In the limit of $\bar{k}_z \gg 1$, the two extremal frequencies $\hat{\omega}_\pm$ are thus solutions of the polynomial of degree 4 in $\hat{\omega}^2$:

$$\Delta Q_A = \bar{\Omega}^2\hat{\omega}^2(1 + \alpha + \beta)^2 \bar{k}_y^2. \quad (54)$$

The limiting case of a vanishing period of oscillations corresponds to $\hat{\omega}_+ = \hat{\omega}_-$ and $K_+ = K_-$. This condition means that the polynomial (54) has a double root. It can be translated into a condition on its coefficients, depending on $\bar{k}_y, \alpha, \beta, A/\Omega$. It is solved numerically and displayed

Fig. 8. Parker-Shearing instabilities occurring successively: we can clearly see the two instabilities occurring at the same time while $K_s > |K_x(t)| > K_p$, with $K_s \sim 50$ and $K_p \sim 20$; this appears as a distinct bump in the instant growth rate (bottom figure). The shearing growth rate is slightly larger than the Parker one. Parameters are $\alpha = 1, \beta = 0, -A/\Omega = .75, \bar{k}_z = 20$, and $\bar{k}_y = .57$ is optimal for the Parker instability. Then $\bar{k}_A = 1.73, \bar{k}_P = 1.,$ and $\bar{k}_Q = .87$.
on Fig. 9. We can stress the following properties, which are independent of \((\alpha, \beta)\):

Fig. 9. Range of azimuthal wavenumbers leading to an oscillating transition between the instabilities \((K_s < K_p)\), or a phase of simultaneous instabilities \((K_s > K_p)\), depending on the strengths of the magnetic field and differential rotation. The dotted line corresponds to \(K_s = K_p\). In the top figure, \(\beta = 0\) and \(\alpha = 1\). At \(-A/\Omega = 0.5\) (bottom figure), the two instabilities are never simultaneous. The two dotted curves correspond to the two critical wavenumbers \(k_y\) leading, for any strength \(\alpha\) of the magnetic field, to \(K_s = K_p\). Any other \(k_y\) implies \(K_s < K_p\).

(i) If the rotation curve decreases more slowly than the flat rotation curve \((-A/\Omega < 0.5)\), the two instabilities will always be separated by an oscillating transition \((K_s \leq K_p)\). If \(-A/\Omega\) is close enough to the critical value \(0.5\), the duration of the transition can be arbitrarily short: there is a unique azimuthal wavenumber \(k_y\) leading to \(K_s = K_p\).

Even in this case, the transition is slow because the growth rate still vanishes between the two instabilities (see Fig. 10 as an illustration).

Fig. 10. Optimal transition between the Parker and the shearing instabilities when \(-A/\Omega < 0.5\): the growth rate vanishes between the two instabilities. Here, \(\alpha = 1\), \(\beta = 0\), \(-A/\Omega = 0.4\), \(k_s = 20\), and we choose \(k_y = 0.48\) according to Fig. 9 so that \(K_s = K_p\) \((\sim 35)\). Then \(k_A = 1.26\), \(k_P = 1.0\), and \(k_Q = 0.87\).

(ii) If the rotation curve decreases faster than the flat rotation curve \((-A/\Omega > 0.5)\), the two instabilities can follow each other continuously, and even be simultaneous: \(K_s > K_p\) as in Fig. 8 where the rotation is Keplerian \((-A/\Omega = 0.75)\).

(iii) If the rotation curve is exactly flat \((-A/\Omega = 0.5)\), Fig. 8 (bottom) shows that there are two critical values of \(k_y\) allowing \(K_s = K_p\), for any strength of the magnetic
field. This demonstrates that the lowest value of \(-A/\Omega\) permitting \(K_s > K_p\) is precisely \(-A/\Omega = .5\), for any \(\alpha\).

7. Discussion and conclusions

Our study reveals that the WKB approximation is a very powerful tool to study analytically the effect of differential rotation on the slow MHD waves. Our numerical computations are in very good agreement with the WKB dispersion relation, even when the WKB criterion is not strictly satisfied. We summarize here our main conclusions:

(i) The shearing instability comes from the non-axisymmetric destabilization of the slow MHD wave by the differential force. However, a parallel can be drawn with the axisymmetric shearing instability of the Alfvén mode in a disc embedded in a vertical magnetic field.

(ii) The Parker instability and the shearing instability both correspond to the destabilization of the same slow MHD mode. The occurrence of each instability can be understood by looking at the natural polarization of the slow MHD mode, depending on whether their radial wavelength is longer or shorter than their vertical wavelength.

(iii) Although transient, the shearing instability may last an arbitrarily long time if the vertical wavelength is short enough. In this respect it is a true local instability. Moreover, it still exists for a magnetic field strength exceeding the equipartition value.

(iv) The action of differential rotation is opposite depending on whether the vertical wavelength of the slow MHD wave is longer or shorter than the scale height of the disc. Perturbations with a long vertical wavelength \((k_z < 1/2H)\) are transiently stable against the Parker instability, during a few shear times. On the contrary, slow MHD waves with a short vertical wavelength are destabilized by differential rotation almost as long as their polarization is in the plane of the disc, i.e. while \(|K_z(t)| < k_z\). The Parker instability naturally occurs when the polarization of the slow MHD wave is vertical, i.e. when \(|K_z(t)| > k_z\).

(v) A refined look at the transition between the Parker and shearing instabilities shows that the intensity of the differential rotation is the crucial parameter. If the rotation curve decreases more slowly than the flat rotation curve \((-A/\Omega < 1/2)\), the growth rate will always vanish between the two instabilities. If the rotation curve decreases faster than the flat rotation curve \((-A/\Omega > 1/2)\), the two instabilities can occur simultaneously: in discs with a strong differential rotation, the most unstable perturbations may not be the ones with \(k_z = 0\), because of this shearing instability.

Astrophysical and observational consequences of these results are beyond the scope of this paper, which leaves many questions open: observing shearing instabilities might prove even more difficult than looking for the signature of Parker instabilities, because here one is dealing with small vertical wavelengths, on which the gas disc and the field are hardly homogeneous. This might actually be seen as an obstacle to the development of the instability, or as a consequence of it (see the discussion in Zweibel & Kulsrud 1975, Parker 1975b).

Furthermore, since we are limited to linearized analysis, we cannot answer the question of the ultimate fate of these waves. If they reach very large amplitudes before the shear takes them to very large \(K_z\) they might more easily, through the non-linear effect of self-gravity on the resulting clumps of gas, lead to Jeans collapse of these clumps. On the other hand if they reach large \(K_z\) their ultimate fate would be more likely a cascade towards small-scale turbulence.

One should also remember, in such a discussion, that in galaxies the gas disc is swept and shocked by the spiral arms on a time scale of the order of a rotation time, i.e. at best a few amplification times, so that the coherent growth of a Parker or Shearing instability on a longer time scale is rather dubious. The conclusions would obviously depend on assumptions on the initial state, i.e. the amplitude of fluctuations generated in the disc and ready to be amplified by these instability mechanisms.

On the other hand self-gravity might play a quite different role, which we consider as most promising for generating strong Parker-like perturbations in the disc: here and in [FT] we have shown that differential rotation results in a linear coupling of waves, i.e. a wave on a given branch of the dispersion relation evolves into a mixture of the three MHD waves (slow and fast magnetosonic, and Alfvén waves). If we had included self-gravity in this discussion, the fast magnetosonic wave would have appeared as the classical spiral density waves of galactic discs, modified by inclusion of the magnetic field (as was done, in the case of a vertical field, by Tagger et al. 1990). But one can conclude that a spiral wave does, by this mechanism of linear coupling, generate Alfvén and slow magnetosonic waves, i.e. Parker rather than shearing instabilities since this occurs at \(k_z = 0\). We know that the spiral wave is strong (the density contrast is of the order of 100% in the gas), and the coupling is also strong since the rotation curve has \(-A/\Omega \sim 5\). This means that the resulting Parker perturbation will also have a very large amplitude, even before it has started to grow by Parker’s mechanism.

We consider this as the most promising way of generating strong Parker perturbations in the disc, and will discuss it in future work.

Acknowledgements. We thank Prof. R. Pudritz for his useful comments. T.F. wishes to acknowledge fruitful discussions with Dr. H. Spruit and Dr. F. Meyer.
A. Appendix: Laplace Transform of the solution with shear

We define the dimensionless rotation parameter and epicyclic frequency as:

\[ \dot{\Omega}^2 \equiv \frac{4H^2\Omega^2}{V_A^2}, \quad (A1) \]

\[ \tilde{\kappa}^2 \equiv \frac{\kappa^2 H^2}{V_A^2} = \Omega^2 - \tilde{k}_A^2. \quad (A2) \]

In order to reduce the degree of the differential system\(^1\), we introduce the convenient function:

\[ \tilde{\xi}_w = -\frac{1}{2\alpha} \left\{ K_x \tilde{\xi}_x + \frac{\tilde{k}_y \tilde{\xi}_y}{1 + 2\alpha} + \left( \tilde{k}_x + \frac{i}{2 + 4\alpha} \right) \tilde{\xi}_z \right\}. \quad (A3) \]

The only time-dependent parameter, \( K_x \), now multiplies the functions \( \tilde{\xi}_x \) and \( \tilde{\xi}_y \) only. Thus we can write two equations without any \( K_x \)-multiplication, and two other equations expressing \( K_x \tilde{\xi}_x \) and \( K_x \tilde{\xi}_w \) in terms of \( \tilde{\xi}_w, \tilde{\xi}_x, \tilde{\xi}_y, \tilde{\xi}_z \). These latter are written:

\[ \dot{K}_x \tilde{\xi}_w = \left( \frac{H^2}{V_A^2} \frac{\partial^2}{\partial t^2} + \tilde{k}_y^2 + \tilde{k}_A^2 \right) \tilde{\xi}_x - \dot{\Omega} \frac{H}{V_A} \frac{\partial}{\partial t} \tilde{\xi}_x, \quad (A4) \]

\[ \dot{K}_x \tilde{\xi}_w = -\frac{2\alpha}{1 + 2\alpha} \tilde{k}_x \tilde{\xi}_w - \frac{\tilde{k}_y}{1 + 2\alpha} \tilde{\xi}_y - \left( \tilde{k}_x + \frac{i}{2 + 4\alpha} \right) \tilde{\xi}_z. \quad (A5) \]

Performing at this point the Laplace transform\(^2\) changes the \( K_x \)-multiplications into \( \omega \)-derivatives, and time-derivatives change to simple \( \omega \)-multiplications, with some additional \( \omega \)-polynomials whose coefficients depend linearly on the initial conditions \( \tilde{\xi}(k_x, t = 0) \) and \( \frac{\partial}{\partial \omega} \tilde{\xi}(k_x, t = 0) \).

In the following, we use the dimensionless variable \( \tilde{\omega} \equiv H\omega/V_A \).

The first couple of equations, independent of \( K_x \), relate \((\xi_x, \xi_z)\) to \((\tilde{\xi}_w, \tilde{\xi}_x)\) in a simple algebraic manner:

\[ \Delta \left| \begin{array}{cc} \xi_x' \xi_z' & \xi_x'' \xi_z'' \\ \xi_x' \xi_z' & \xi_x'' \xi_z'' \end{array} \right| = \left| \begin{array}{cc} q_a & q_b \\ q_c & q_d \end{array} \right| \left| \begin{array}{cc} \tilde{\xi}_w' \tilde{\xi}_x' \tilde{\xi}_y' \tilde{\xi}_z' \\ \tilde{\xi}_w'' \tilde{\xi}_x'' \tilde{\xi}_y'' \tilde{\xi}_z'' \end{array} \right| + \frac{i}{\nu} \left| \begin{array}{cc} \tilde{\xi}_x' \tilde{\xi}_z' \tilde{\xi}_y' \tilde{\xi}_z'' \\ \tilde{\xi}_x'' \tilde{\xi}_z'' \tilde{\xi}_y'' \tilde{\xi}_z'' \end{array} \right| \left| \begin{array}{cc} Q_y & Q_z \\ \nabla \xi_x'' \nabla \xi_z'' \end{array} \right|. \quad (A6) \]

The polynomial \( \Delta \), already introduced in [FT], identifies with the asymptotic dispersion relation at \( k_x \to \infty \):

\[ \Delta \equiv (1 + 2\alpha)\tilde{\omega}^4 - 2 \left( (1 + \alpha)\tilde{k}_y^2 + (1 + \alpha + \beta) \frac{\alpha - \beta}{4\alpha} \right) \tilde{\omega}^2 + \tilde{k}_y^2 (\tilde{k}_y^2 - \tilde{k}_A^2). \quad (A7) \]

The maximum azimuthal wavenumber \( k_y \) allowing the Parker instability at \( k_x \to \infty \) is defined as:

\[ \tilde{k}_y^2 \equiv (1 + \alpha + \beta) \frac{\alpha + \beta}{2\alpha} > \frac{1}{2}. \quad (A8) \]

Equations (A4)-(A5) may be expressed as a differential system of second order on the functions \((\xi_w, \xi_x)\), using Eq. (A6).

\[ \frac{\tilde{k}_y}{\tilde{\Omega}} \frac{\partial}{\partial \omega} \tilde{\xi}_w = \left( \frac{p_a}{p_c} \frac{p_b}{p_d} \right) \tilde{\xi}_w + \lambda \left( \frac{P_w}{P_x} \right), \quad (A9) \]

where \((p_a, p_b, p_c, p_d)\) are simple polynomials of \( \tilde{\omega} \) whose coefficients are independent of \( \tilde{k}_y \).

\( P_w \) and \( P_x \) are two polynomials of \( \tilde{\omega} \) of respective degree 5 and 3, whose coefficients depend linearly on the six initial conditions

\[ \left( \tilde{\xi}_x, \tilde{\xi}_y, \tilde{\xi}_z, \frac{\partial}{\partial t} \tilde{\xi}_x, \frac{\partial}{\partial t} \tilde{\xi}_y, \frac{\partial}{\partial t} \tilde{\xi}_z \right) (k_x, t = 0), \quad (A10) \]

independently from the \( \tilde{k}_x \) and \( \tilde{k}_A \) parameters.

The coefficients of the matrix in Eq. (A9) are given by:

\[ p_a = \tilde{\Omega} \tilde{k}_y \tilde{\omega} \left( \tilde{\omega}^2 \tilde{k}_x^2 - (1 + \alpha + \beta) \left( \frac{\beta}{2\alpha} + i\tilde{k}_z \right) \right) \]

\[ p_b = -i \left( \tilde{\omega}^2 - \tilde{k}_y^2 - \tilde{k}_A^2 \right) \Delta (\tilde{\omega}^2) - \tilde{\Omega}^2 \tilde{k}_y^2 \left( \tilde{\omega}^2 - \tilde{k}_y^2 + \tilde{k}_A^2 \right) \]

\[ p_c = -i Q_2 (\tilde{\omega}^2) - i \tilde{k}_y^2 (1 + 2\alpha) \tilde{\omega}^2 \]

\[ p_d = -\tilde{\Omega} \tilde{k}_y \tilde{\omega} \left( \tilde{\omega}^2 - \tilde{k}_y^2 + (1 + \alpha + \beta) \left( \frac{\beta}{2\alpha} - i\tilde{k}_z \right) \right) \]

The determinant of (A9) is \( p_a p_d - p_b p_c = \Delta (Q_3 - \tilde{k}_y^2 Q_A) \).

Defining \( \tilde{k}_y^2 \equiv \tilde{k}_y^2 - 1/4 \) leads us to write the polynomials \( Q_2, Q_3 \) and \( Q_A \) as:

\[ Q_2 \equiv 2\alpha \tilde{\omega}^4 - (1 + 2\alpha) \left( \tilde{k}_y^2 + \frac{1}{4} \right) \tilde{\omega}^2 + \tilde{k}_y^2 (\tilde{k}_y^2 - \tilde{k}_A^2) \]

\[ Q_3 \equiv (\tilde{\omega}^2 - \tilde{k}_y^2 - \tilde{k}_A^2) Q_2 (\tilde{\omega}^2) - \tilde{k}_y^2 \tilde{\Omega}^2 (\tilde{\omega}^2 + \tilde{k}_3^2 - \tilde{k}_y^2) \]

\[ Q_A \equiv (1 + 2\alpha) \tilde{\omega}^4 - \tilde{\omega}^2 (2 (1 + \alpha) \tilde{k}_y^2 + (1 + 2\alpha) \tilde{k}_A^2) \]

\[ + \tilde{k}_y^2 (\tilde{k}_y^2 - \tilde{k}_A^2). \quad (A12) \]

The differential system (A9) can easily be transformed, by substitution, into the second order differential equation:

\[ \left\{ \frac{\partial^2}{\partial \omega^2} + \frac{\partial \log f(\tilde{\omega})}{\partial \tilde{\omega}} \frac{\partial}{\partial \tilde{\omega}} + g(\tilde{\omega}) \right\} \tilde{\xi}(k_x, \tilde{\omega}) = h(k_x, \tilde{\omega}), \quad (A13) \]
where \( h \) depends linearly on the initial conditions, and \((f, g, h)\) are singular where \( \Delta(\omega^2) = 0 \). We derive for \( \xi_x \):

\[
\begin{align*}
\frac{\partial \log f_x}{\partial \omega} &= \frac{\partial \log \Delta/pc}{\partial \omega} - \frac{\hat{\Omega}}{k_A^2 k_y^2} \frac{p_a + p_d}{\Delta}, \\
g_x(\omega^2) &= \frac{\hat{\Omega}^2}{k_A^2 k_y^2} \left\{ \frac{Q_A - k_y^2 Q_A}{\Delta} - \frac{k_y^2 k_y p_c \partial p_d}{\hat{\Omega} \Delta \partial e \omega} \right\}.
\end{align*}
\]

(A15)

(A16)

B. Appendix: WKB approximation

We concentrate here on the function \( \xi_x \), without specifying the subscript \( x \) for the sake of clarity. The homogeneous differential equation associated to Eq. (A14) may be written in its canonical form:

\[
\left\{ \frac{\partial^2}{\partial \omega^2} + W(\omega^2) \right\} (f^{1/2}\xi) = 0,
\]

(B1)

where \( W(\omega^2) \) is a fractional function of \( \omega^2 \) defined as:

\[
W(\omega^2) \equiv g(\omega^2) - \frac{1}{2} \frac{\partial^2 \log f(\omega^2)}{\partial \omega^2} - \frac{1}{4} \left( \frac{\partial \log f(\omega^2)}{\partial \omega} \right)^2.
\]

(B2)

We have shown in [FT] that if the WKB criterion is satisfied,

\[
\left| \frac{1}{4} \frac{\partial^2 \log W}{\partial \omega^2} - \left( \frac{1}{4} \frac{\partial \log W}{\partial \omega} \right)^2 \right| \ll g,
\]

(B3)

the solutions of the differential equation (B1) can be approximated by a linear combination of the two independent functions:

\[
\bar{\xi}_\pm(\omega) \equiv \frac{1}{(f^2W)^{1/4}} \exp \left( \pm i \int_{\omega_0}^{\omega} W^{1/2}(\omega^2) d\omega' \right).
\]

(B4)

According to the definition (A15) of the function \( f \), we can write:

\[
\bar{\xi}_\pm(\omega) \equiv \frac{p_c^{1/2}}{\Delta^{1/2} W^{1/4}} \exp \int_{\omega_0}^{\omega} \left( \frac{\hat{\Omega}}{k_y^2 k_A^2} \frac{p_a + p_d}{2i\Delta} \pm W^{1/2} \right) d\omega.
\]

(B5)

Let us define the phase \( \Psi_\pm(\omega, t) \) in a dimensionless form as:

\[
\Psi_\pm \equiv -\frac{i}{2} \int_{\omega_0}^{\omega} \left( \frac{\hat{\Omega}}{k_y^2 k_A^2} \frac{p_a + p_d}{2i\Delta} \pm W^{1/2} \right) d\omega.
\]

(B6)

The solution of the differential equation (A14) is then written as:

\[
\bar{\xi}(k_x, t) = \frac{1}{2\pi} \int_{\mu_+ - e^{i\omega} - \mu_- e^{-i\omega}}^{\mu_+ + e^{i\omega} + \mu_- e^{-i\omega}} d\omega,
\]

(B7)

where the two functions \( \mu_\pm(\omega) \) vary slowly if the WKB criterion is satisfied. The dispersion relation is therefore:

\[
W(\omega^2) = \frac{\hat{\Omega}^2}{k_A^2 k_y^2} \left( \bar{K}_x(t) - \frac{p_a + p_d}{2i\Delta} \right)^2.
\]

(B8)

The function \( W(\omega^2) \) defined in (B2) can be written:

\[
W = -\frac{\hat{\Omega}^2}{k_y^2 k_A^2 \Delta^2} \left\{ \frac{p_c p_b + \left( \frac{p_a - p_d}{2} \right)^2}{\Delta} - \frac{1}{2} \frac{\partial^2 \log \Delta}{\partial pc} \right\}
\]

\[
+ \frac{\hat{\Omega} p_c}{2k_y^2 k_A^2 \Delta} \frac{\partial p_a - \partial p_d}{\frac{2}{pc}} - \frac{1}{4} \left( \frac{\partial \log \Delta}{\partial p_c} \right)^2.
\]

(B9)

The general dispersion relation becomes:

\[
(p_a - i\bar{K}_x \Delta)(p_d - i\bar{K}_x \Delta) - p_c p_b + \frac{\hat{\Omega} k_y^2 \Delta}{2\hat{\Omega}^2} \frac{\partial p_a - \partial p_d}{\partial \omega} + \frac{\hat{\Omega}^2}{4\Omega^2} \left\{ \frac{2}{2} \frac{\partial^2 \log \Delta}{\partial pc} + \left( \frac{\partial \log \Delta}{\partial pc} \right)^2 \right\} = 0.
\]

(B10)

We recognize the first two terms as the determinant of the system (A5). Some additional terms must be taken into account, especially in the limit of both vanishing shear and magnetic field (Sect. 5).

It can be divided by \( \Delta \) to obtain:

\[
Q_3 - Q_A k_y^2 - \Delta \bar{k}_x \Delta + 2\hat{\Omega}(1 + \alpha + \beta) k_y k_x k_z
\]

\[
+ \frac{k_y^2 k_A^2}{2\hat{\Omega}} \left\{ 3\omega^2 - k_y^2 + (1 + \alpha + \beta) \right\} \frac{\partial \log p_c}{\partial \omega^2} = 0.
\]

(B11)

Both the WKB criterion (B3) and the dispersion relation (B11) become much simpler in the two cases we investigate in Sect. (4-6):

(i) If the shear vanishes totally, we will obtain the usual dispersion relation:

\[
Q_3 - Q_A k_y^2 = 0.
\]

(ii) If, as in Sect. 4 and Sect. 6, \( k_z \to \infty \), we simply obtain:

\[
W(\omega^2) \sim -\frac{\hat{\Omega}^2}{k_y^2 k_A^2 \Delta^2} \left( -ip_b(\bar{k}_y^2 - (1 + 2\alpha) \bar{k}_x \Delta) \right).
\]

(B13)

Thus the criterion (B3) is satisfied in the limit of large vertical wavenumbers. In Sect. 4, the radial wavenumber is bounded \(|K_x(t)| \ll |k_z|\), and the dispersion relation (B11) becomes \( Q_A(\bar{\omega}^2) = 0 \).

In Sect. 6, \( |K_x(t)| \approx |k_z| > 1/H \), and the dispersion relation (B11) becomes:

\[
Q_A + \left( \frac{\bar{K}_x}{k_z} \right)^2 \Delta - 2\hat{\Omega}(1 + \alpha + \beta) k_y \left( \frac{\bar{K}_x}{k_z} \right) = \frac{1}{k_z^2}.
\]

(B15)
(iii) If, as in Sect. 5, $\alpha \to 0$ and $\alpha \tilde{\Omega}^2 \sim cte$, then:

$$W(\tilde{\omega}^2) \sim g(\tilde{\omega}^2) \sim O(\tilde{\Omega}^4),$$

(B16)

and the WKB criterion is again satisfied. The simple dispersion relation \( [B15] \) describing the slow MHD frequencies for finite values of $k_z$, $K_z$ comes from the leading terms, when $\tilde{\omega} \tilde{\Omega} \sim 1$, of:

$$Q_3 - Q_A k_x^2 - \Delta \tilde{k}_x^2 + 2\tilde{\Omega}^2(1 + \alpha + \beta)k_y k_z$$

$$+ \tilde{k}_y^2 k_A^2 \left[ 3\tilde{\omega}^2 - \tilde{k}_y^2 (1 + 2\alpha + \beta) \right] = 0.$$  

(B17)

C. Appendix: polarization of a slow MHD wave

Here we recall the $\mathbf{(k_x, k_z)}$-dependence of the polarization of a slow MHD wave in absence of shear. Strictly speaking, the parameter $\tilde{k}_z$ ought to be zero in what follows. If $\tilde{\omega}$ is determined by the dispersion relation \( [B12] \), the equations \( [C9] \) and \( [A6] \) will imply:

$$\xi_x = \frac{k_x Q_A - \tilde{\Omega} \tilde{\omega} (1 + \alpha + \beta) k_y k_z + iR_1}{k_x \Delta - \tilde{\Omega} \tilde{\omega} (1 + \alpha + \beta) k_y k_z + iR_2},$$

(C1)

where $R_1, R_2$ are independent of $(\tilde{k}_x, \tilde{k}_z)$:

$$R_1(\tilde{\omega}) = \left( 1 + \beta \right) (\tilde{k}_x^2 - \tilde{\omega}^2) (\tilde{k}_y^2 - \tilde{\omega}^2) + \frac{k_x^2}{\beta^2} (\tilde{k}_y^2 - \tilde{\omega}^2).$$

$$R_2(\tilde{\omega}) = \tilde{\Omega} \tilde{\omega} k_y \left[ \tilde{\omega}^2 - \tilde{k}_y^2 (1 + \alpha + \beta) \right].$$

(C3)

Let us study the polarization of the Alfven mode: we have seen in Sect. 4 that its eigenfrequency is a root of $Q_A(\tilde{\omega}^2) = 0$ when the vertical wavenumber $k_z \to \infty$, and we know from \([FT]\) that it is a root of $\Delta(\tilde{\omega}^2) = 0$ when $k_z \to \infty$. Unlike the magnetosonic eigenfrequency (fast MHD wave), it remains finite in both cases. Thus this calculation is valid for both the Alfven and the slow MHD mode.

In the limit of $\tilde{k}_x \to \infty$, the dispersion relation \( [B12] \) shows that:

$$k_x \Delta \sim 2\tilde{\Omega} \tilde{\omega} (1 + \alpha + \beta) k_y k_z.$$  

(C4)

The polarization \( [C1] \) is then:

$$\xi_x \sim \frac{-\tilde{\Omega} \tilde{\omega} (1 + \alpha + \beta) k_y}{\tilde{k}_y k_x \sim \tilde{\Omega} \tilde{\omega} (1 + \alpha + \beta) k_y k_z + iR_2(\tilde{\omega}_A)} \to \infty,$$

(C5)

corresponding to a vertical mode.

In the limit of $\tilde{k}_z \to \infty$, the dispersion relation \( [B12] \) implies:

$$\tilde{k}_z Q_A \sim 2\tilde{\Omega} \tilde{\omega} (1 + \alpha + \beta) k_y k_z.$$  

(C6)

The polarization \( [C1] \) becomes:

$$\xi_x \sim \frac{\tilde{\Omega} \tilde{\omega} (1 + \alpha + \beta) k_y k_x + iR_1(\tilde{\omega}_A)}{-\tilde{\Omega} \tilde{\omega} (1 + \alpha + \beta) k_y} \frac{1}{k_z} \to 0,$$

(C7)

and corresponds to a horizontal mode.

If both $k_x, k_z \gg k_y$, the polarization \( [C1] \) is simply:

$$\xi_x \sim \frac{\tilde{k}_y}{k_z}.$$  

(C8)

Thus the plane of polarization of the slow MHD (and Alfven) mode is naturally vertical or horizontal, depending on the ratio $k_x/k_z$.

These properties are specific of the slow magnetosonic mode \textit{in a stratified rotating disc}. In a uniform plasma ($H \to \infty$ and $\Omega \to 0$), the polarization of the slow magnetosonic mode is indeed completely different, and is exactly:

$$\xi_x = \frac{k_z}{k_x}.$$  

(C9)

It can easily be shown that:

(i) the property \( [C8] \) is an effect of stratification, and is valid only if $k_z \gg H k_x$,

(ii) the property \( [C7] \) is an effect of rotation, and is valid only if $k_z \gg k_y^2 \alpha^2 / \Omega$.

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