A RELATIVE, STRICTLY ERGODIC MODEL THEOREM FOR INFINITE MEASURE-PRESERVING SYSTEMS

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Abstract. Every factor map between given ergodic, measure-preserving transformations on infinite Lebesgue spaces has a strictly ergodic, locally compact Cantor model.

1. Introduction

A Cantor set is a totally disconnected, compact metric space without isolated points. As well known, such topological spaces are all homeomorphic to each other; see for example [10, Theorem 2-97]. A topological dynamical system on a Cantor set is called a Cantor system. A Cantor model of a measure-preserving system Y on a probability space is a Cantor system X which equipped with an invariant probability measure is measure-theoretically isomorphic to the system Y. The Cantor system X is said to be strictly ergodic if it is minimal, i.e. every orbit is dense in X, and in addition, it is uniquely ergodic, i.e. it has a unique, invariant Borel probability measure. Jewett-Krieger Theorem [11, 13] affirms that any ergodic, measure-preserving system on a Lebesgue probability space has a strictly ergodic Cantor model.

B. Weiss provided a relative, strictly ergodic model theorem [17, Theorem 2] by restricting himself to stating some relevant lemmas without detailed proofs, which affirms that any factor map between arbitrary ergodic, measure-preserving systems on Lebesgue probability spaces has a strictly ergodic Cantor model. F. Béguin, S. Crovisier and F. Le Roux [3, Section 1] fully described a proof of the relative model theorem. One of our goals is to prove an infinite counterpart:

Theorem 1.1. If π is a measure-theoretical factor map from an ergodic, measure-preserving system Y on an infinite Lebesgue space to a strictly ergodic, locally compact Cantor system Z, then there exist strictly ergodic, locally compact Cantor model X of the ergodic system Y and open and proper, topological factor map ρ from X to Z for which the diagram in Figure 1 commutes in the category of measure-preserving systems.

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\[ \begin{align*}
Y & \xrightarrow{\pi} X \\
& \downarrow \rho \\
Z & \xrightarrow{q_2} Y_2 \\
& \downarrow \rho \\
& \downarrow q_3 \\
& Y_3
\end{align*} \]

\[ \begin{align*}
Y_1 & \xrightarrow{q_2} Y_2 \\
& \downarrow \rho \\
& \downarrow q_3 \\
Y & \xrightarrow{Y \times Y}
\end{align*} \]

Figure 1. Figure 2. Figure 3.
A locally compact Cantor set [5] is a totally disconnected, locally compact (non-compact) metric space $X$ without isolated points. The one-point compactification of such a topological space is exactly a Cantor set. If $S : X \to X$ is a homeomorphism, then $X := (X, S)$ is referred to as a locally compact Cantor system. The locally compact Cantor system $X$ is said to be strictly ergodic if it is minimal, i.e. the orbit $\text{Orb}_S(x) := \{ S^k x \mid k \in \mathbb{Z} \}$ of any point $x \in X$ is dense in $X$, and in addition, it is uniquely ergodic, i.e. an $S$-invariant Radon measure is unique up to scaling. This definition of strict ergodicity clearly extends that of unique ergodicity for Cantor systems. The system $X$ is called a locally compact Cantor model of a measure-preserving system $Y$ if the system $X$ equipped with an invariant Radon measure is measure-theoretically isomorphic to the system $Y$. F. Béguin, S. Crovisier and F. Le Roux [3, Theorem A.3] also showed that any factor map between ergodic, measure-preserving systems on probability spaces has a “strictly ergodic”, locally compact Cantor model, although they took into account unique ergodicity only within the probability measures.

Combining Theorem 1.1 with [20, Theorem 4.6], we obtain an immediate consequence that if $Y$ and $Z$ are ergodic, measure-preserving systems on infinite Lebesgue spaces and $\pi$ is a factor map from $Y$ to $Z$ then there exist strictly ergodic, locally compact Cantor models $X$ and $W$ of $Y$ and $Z$, respectively, and an open and proper, factor map $\rho$ from $X$ to $W$ so that a square diagram:

$$
\begin{array}{ccc}
Y & \longrightarrow & X \\
\pi \downarrow & & \downarrow \rho \\
Z & \longrightarrow & W
\end{array}
$$

commutes in the category of measure-preserving systems. We then say that $\pi : Y \to Z$ has a strictly ergodic, locally compact Cantor model $\rho : X \to W$.

The above-mentioned consequence has a paraphrase in terms of almost minimal Cantor systems [5]. A Cantor system is said to be almost minimal if it has a unique fixed point and any other point has a dense orbit. An almost minimal Cantor system is regarded as the one-point compactification of a minimal, locally compact Cantor system. If the minimal, locally compact Cantor system is uniquely ergodic, then the associated almost minimal Cantor system is said to be bi-ergodic, as the point mass on a fixed point is always invariant for the almost minimal Cantor system. The bi-ergodicity is dealt with also in [3, Theorem A.3], although their notion is actually different from ours as mentioned above. The paraphrase is now stated as follows. If $Y$ and $Z$ are ergodic, measure-preserving systems on infinite Lebesgue spaces then any factor map $\pi : Y \to Z$ has a bi-ergodic, almost minimal Cantor model $\hat{\rho} : \hat{X} \to \hat{W}$. It is now clear how to describe a paraphrase of Theorem 1.1 in terms of almost minimal Cantor systems.

B. Weiss [17] also gave a sufficient condition for a diagram in the category of ergodic measure-preserving systems on probability spaces to have a strictly ergodic Cantor model. Another goal of ours is to prove its infinite counterpart:

**Theorem 1.2.** In the category of ergodic, measure-preserving systems on infinite Lebesgue spaces, any diagram including no portions with shape of Figure 2 has a strictly ergodic, locally compact Cantor model. Actually, no diagrams including portions with shape of Figure 3 have strictly ergodic, locally compact Cantor models.
We will prove Theorems 1.1 and 1.2 in Sections 5 and 6, respectively, and in particular, accomplish a proof of Theorem 1.1 almost along the line of [3, Sublemma A.1]. In virtue of [20, 21], almost all devices are actually ready to achieve our goals. We will review them in Sections 2.4.

Any relations among measurable sets, any properties of maps between measure spaces, etc. are always understood to hold up to sets of measure zero.

2. Strict ergodicity

Let \((Y, \mathcal{B}, \mu)\) be an infinite Lebesgue space, i.e. a measure space isomorphic to a measure space \(\mathbb{R}\) endowed with Lebesgue measure together with the \(\sigma\)-algebra of Lebesgue measurable subsets. Set \(\mathcal{B}_0 = \{ A \in \mathcal{B} \mid 0 < \mu(A) < \infty \}\). The measure space \((Y, \mathcal{B}, \mu)\) has a countable base \(\mathcal{E} \subset \mathcal{B}_0\), i.e.

- \(\mathcal{E}\) generates the \(\sigma\)-algebra \(\mathcal{B}\), i.e. \(\mathcal{B}\) is the completion of the smallest \(\sigma\)-algebra including \(\mathcal{E}\) with respect to \(\mu\);
- \(\mathcal{E}\) separates points on \(Y\), i.e. only one of any distinct two points in \(Y\) is included in some set belonging to \(\mathcal{E}\).

The author refers the readers to [11, 8] for fundamental properties of a Lebesgue space.

A bi-measurable bijection \(T : Y \to Y\) is said to be measure-preserving if \(\mu(T^{-1}E) = \mu(E)\) for all sets \(E \in \mathcal{B}\). The measure \(\mu\) is then said to be \(T\)-invariant, or simply, invariant. We refer to \(Y := (Y, \mathcal{B}, \mu, T)\) as an infinite measure-preserving system. The system \(Y\) is said to be ergodic if the measure of any invariant set is zero or full, or equivalently, any \(T\)-invariant, measurable function on \(Y\) is constant.

Another infinite measure-preserving system \(Z = (Z, \mathcal{C}, \nu, U)\) is called a factor of the system \(Y\) if there exists a measurable surjection \(\phi : Y \to Z\) for which \(\mu \circ \phi^{-1} = \nu\) and \(\phi \circ T = U \circ \phi\). The map \(\phi\) is then called a factor map from \(Y\) to \(Z\). If in addition \(\phi\) is injective, then \(\phi\) is called an isomorphism and \(Y\) is said to be isomorphic to \(Z\). If the system \(Y\) is ergodic, then the factor \(Z\) is necessarily ergodic, because if a measurable function \(f : Z \to \mathbb{R}\) is \(U\)-invariant then a measurable function \(f \circ \phi\) is \(T\)-invariant.

We say that \(X = (X, S)\) is a topological dynamical system if \(S\) is a homeomorphism on a \(\sigma\)-compact and locally compact metric space \(X\). Given another topological dynamical system \(W = (W, \nu)\), a continuous surjection \(\rho : X \to W\) is called a factor map from \(X\) to \(W\) if \(\rho \circ S = V \circ \rho\). If, in addition, the continuous surjection \(\rho\) is injective, then we say that \(\rho\) is an isomorphism and \(X\) is isomorphic to \(W\). A positive Borel measure \(\lambda\) on the \(\sigma\)-compact and locally compact metric space \(X\) is called a Radon measure if \(\lambda(K) < \infty\) for all compact subsets \(K\) of \(X\), so that any Radon measure is \(\sigma\)-finite. If the homeomorphism \(S\) has a unique, up to scaling, invariant Radon measure, then the system \(X\) is said to be uniquely ergodic. If, in addition, the homeomorphism \(S\) is minimal, then \(X\) is said to be strictly ergodic. If the metric space \(X\) is a Polish space, then an ergodic decomposition (see for example [11, 2.2.9]) shows that the unique ergodicity implies the ergodicity. Actually, if \(\lambda\) is the unique \(S\)-invariant Radon measure, then the ergodic decomposition guarantees that there exist a probability space \((\Omega, m)\) and \(\sigma\)-finite, ergodic Radon measures \(\{ \lambda_\omega \mid \omega \in \Omega \}\) such that for any Borel set \(A \subset X\),

- a function \(\Omega \to \mathbb{R}, \omega \mapsto \lambda_\omega(A)\) is measurable;
- \(\lambda(A) = \int_\Omega \lambda_\omega(A) dm(\omega)\).
Then, for any $\omega \in \Omega$, there exists a constant $c_\omega$ such that $\lambda_\omega = c_\omega \lambda$. Hence, the measure $\lambda$ is ergodic.

Assume that $X = (X, S)$ is a locally compact Cantor system; see Section I. Let $\hat{X}$ denote the one-point compactification $X \cup \{ \omega_X \}$ of the locally compact Cantor set $X$. Define a homeomorphism $\hat{S} : \hat{X} \to \hat{X}$ by

$$\hat{S}x = \begin{cases} Sx & \text{if } x \in X; \\ \omega_X & \text{if } x = \omega_X. \end{cases}$$

We shall refer to $\hat{X} := (\hat{X}, \hat{S})$ as the one-point compactification of the locally compact Cantor system $X$. If the system $X$ is minimal, then the system $\hat{X}$ is almost minimal; see Section I. Clearly, the one-point compactification provides us with a one-to-one correspondence between the class of minimal, locally compact Cantor systems and that of almost minimal Cantor systems.

If $\phi : X \to W$ is a continuous surjection between locally compact Cantor sets, then we let $\hat{\phi}$ denote a surjection from $\hat{X}$ to $\hat{W}$ which is defined in the same way as $\hat{S}$ defined above. If $\hat{\phi}$ is continuous, then $\phi$ must be proper. A continuous map $\phi : X \to W$ is said to be proper if a map $\phi \times \text{id}_Z : X \times Z \to W \times Z$, $(x, z) \mapsto (\phi(x), z)$ is closed for any topological space $Z$; see [4, Definition 1 in §10.1]. This condition is equivalent to each of the following conditions:

- the inverse image $\phi^{-1}(C)$ of any compact subset $C$ of the locally compact Cantor set $W$ is compact in the locally compact Cantor set $X$;
- the map $\phi$ is closed and the inverse image $\phi^{-1}\{w\}$ of any point $w \in W$ is compact.

See [2, Appendix C]. This equivalence allows us to verify that if a continuous surjection $\phi : X \to W$ is proper then the surjection $\hat{\phi} : \hat{X} \to \hat{W}$ is continuous.

If the continuous surjection $\phi : X \to W$ is a factor map between minimal, locally compact Cantor systems, then it is natural to assume that $\phi$ is proper as well. See also [14].

**Lemma 2.1** ([20, a portion of Proposition 3.3]). Assume that a locally compact Cantor system $X = (X, S)$ has a compact open set $K \subset X$ whose orbit $\bigcup_{n \in \mathbb{Z}} S^n(K)$ coincides with $X$. Then,

- $X$ has an invariant Radon measure;
- $X$ is uniquely ergodic if and only if the following two conditions hold:
  - $\#(\text{Orb}_S(x) \cap K) = \infty$ for all $x \in X$;
  - to any compact open set $A \subset X$ and $\epsilon > 0$ correspond $c \geq 0$ and $m \in \mathbb{N}$ such that for all $x \in K$,
    $$S_n1_K(x) \geq m \Rightarrow \left| \frac{S_n1_A(x)}{S_n1_K(x)} - c \right| < \epsilon,$$
    where $1_A$ is the indicator function of $A$ and for a function $f$ on $X$,
    $$S_n f = \sum_{i=-n}^{n-1} f \circ S^i.$$ 

**Remark 2.2.** Let $X$ be as in the hypothesis of Lemma 2.1. If $X$ is uniquely ergodic under a unique, up to scaling, invariant Radon measure $\lambda$, then its minimality is equivalent to saying that the support $\text{supp}(\lambda)$ of $\lambda$ coincides with $X$, which is the smallest closed subset of $X$ whose complement has measure zero with respect to $\lambda$. Compare this fact with [16, Theorem 6.17].
There exist bi-ergodic, for whose definition see Section 1 almost minimal subshifts over finite alphabets 19, which arise from non-primitive substitutions.

**Corollary 2.3.** If $\phi$ is a proper factor map from a strictly ergodic, locally compact Cantor system $X = (X, S)$ to a locally compact Cantor system $W = (W, V)$, then the system $W$ is strictly ergodic.

**Proof.** Any point $w \in W$ equals $\phi(x)$ for some point $x \in X$. Since $\phi$ is continuous,

$$W = \phi(X) = \phi(\text{Orb}_S(x)) \subset \phi(\text{Orb}_S(x)) = \text{Orb}_V(w),$$

and hence $W$ is minimal.

Fix a nonempty compact open subset $L$ of $W$. Since $\phi$ is proper and continuous, an inverse image $\phi^{-1}(L)$ is compact and open. Since $W$ is minimal, it holds that $\bigcup_{i \in \mathbb{Z}} V^i L = W$ and hence $\bigcup_{i \in \mathbb{Z}} S^i \phi^{-1}(L) = X$. Let $\epsilon > 0$ and compact open subset $B$ of $W$ be arbitrary. Since $X$ is uniquely ergodic, in virtue of Lemma 2.1, there exist $c \geq 0$ and $m \in \mathbb{N}$ such that for all $x \in \phi^{-1}(L)$,

$$S_n 1_{\phi^{-1}(L)}(x) \geq m \Rightarrow \left| \frac{S_n 1_{\phi^{-1}(B)}(x)}{S_n 1_{\phi^{-1}(L)}(x)} - c \right| < \epsilon,$$

which implies that for all $y \in L$,

$$V_n 1_L(y) \geq m \Rightarrow \left| \frac{V_n 1_B(y)}{V_n 1_L(y)} - c \right| < \epsilon,$$

because $S_n 1_{\phi^{-1}(B)} = (V_n 1_B) \circ \phi$. Finally, Lemma 2.1 again guarantees the unique ergodicity of the system $W$. □

3. **Symbolic factors associated with partitions**

Let $Y = (Y, B, \mu, T)$ be an ergodic, infinite measure-preserving system. Simply by a partition of $Y$, we mean an ordered, finite family $\alpha = \{ A_1, A_2, \ldots, A_m \}$ of nonempty, measurable subsets of $Y$ satisfying that

- $m \geq 2$;
- $\alpha$ is a partition of $Y$ in the usual sense;
- $\mu(A_i) = \infty$ if and only if $i = 1$.

Each member of $\alpha$ is called an atom. In particular, a single atom $A_1$ and the other atoms $A_2, \ldots, A_m$ are called infinite atom and finite atoms, respectively. Set $K_\alpha = Y \setminus A_1$, which is called a finite support of $\alpha$. We let $\mathcal{A}_\alpha$ denote the set of subscripts of atoms of the partition $\alpha$, i.e. $\mathcal{A}_\alpha = \{ 1, 2, \ldots, m \}$.

Let $\beta = \{ B_1, B_2, \ldots, B_n \}$ be another partition of $Y$. If each atom of the partition $\beta$ is included in an atom of the partition $\alpha$, then the partition $\beta$ is called a refinement of the partition $\alpha$. It is then designated by $\beta \succ \alpha$. If, in addition, the partitions $\alpha$ and $\beta$ have their finite supports in common, i.e. $K_\beta = K_\alpha$, then the refinement is designated by $\beta \succ \alpha$. If $m = n$, then the partitions $\alpha$ and $\beta$ have a distance defined by

$$d(\alpha, \beta) = \sum_{i \neq 1} \mu(A_i \triangle B_i),$$

which is a complete metric in the sense that if $(\alpha_i)_{i \in \mathbb{N}}$ is a Cauchy sequence of partitions in the distance $d$ then there exists a partition $\alpha$ to which the sequence $(\alpha_i)_{i \in \mathbb{N}}$ converges in the distance $d$. 
A join $\alpha \lor \beta$ of the partitions $\alpha$ and $\beta$ is defined to be a partition:

$$\{ A \cap B \mid A \in \alpha, B \in \beta \}.$$  

Clearly, the notion of join can be extended to any finite number of partitions. Given partitions $\alpha_1, \alpha_2, \ldots, \alpha_k$ of $Y$, it is natural to regard $\mathcal{A}_{\alpha_1 \lor \alpha_2 \lor \cdots \lor \alpha_k}$ as a subset of a product set $\prod_{i=1}^k \mathcal{A}_{\alpha_i}$. For example, an infinite atom of $(\mathcal{A}_{\alpha_1 \lor \alpha_2 \lor \cdots \lor \alpha_k})$ has subscript $(1,1,\ldots,1)$. We let $\mathcal{F}_\alpha$ denote the algebra generated by $\bigcup_{k=1}^\infty \alpha_{k-1}$, where

$$\alpha_k^k = T^{-\ell} \alpha \lor T^{-(\ell+1)} \alpha \lor \cdots \lor T^{-k} \alpha$$

for integers $k, \ell$ with $\ell \leq k$. Also, set

$$\tag{3.1} \alpha^T = \left\{ \bigcup_i T^{k_i} A_{\ell_i} \mid k_i \in \mathbb{Z}, A_{\ell_i} \in \alpha \cap \mathcal{B}_0, \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j \right\}.$$  

Remark that $\#(\alpha \cap \mathcal{B}_0) < \infty$. Given a set $E \in \mathcal{B}$, we use notation $E^\mathbb{C} \in \alpha^T$ to mean that $\mu(E \triangle F) \leq \epsilon$ for some set $F \in \alpha^T$.

**Remark 3.1.** Assume that $\#\alpha = \#\beta$ in other words $\mathcal{A}_\alpha = \mathcal{A}_\beta$. Since

$$\mu \left( \bigcap_{j=-k+1}^{k-1} T^{-j} A_{i_j} \triangle \bigcap_{j=-k+1}^{k-1} T^{-j} B_{i_j} \right) \leq \sum_{j=-k+1}^{k-1} \mu(A_{i_j} \triangle B_{i_j}) \leq (2k-1)d(\alpha, \beta)$$

for arbitrary $k \in \mathbb{N}$ and $\{ i_j \in \{1,2,\ldots,n\} \mid -k+1 \leq j \leq k-1 \}$, it follows that if $\mathcal{A}_{\alpha_{k-1}} = \mathcal{A}_{\beta_{k-1}}$, viewed as subsets of $\prod_{k=-k+1}^{k-1} \mathcal{A}_\alpha$ and $\prod_{k=-k+1}^{k-1} \mathcal{A}_\beta$ respectively, then $d(\alpha_{k-1}, \beta_{k-1}) \leq (2k-1) \cdot \#(\alpha_{k-1}) \cdot d(\alpha, \beta)$. This implies that $\alpha_{k-1} \to \alpha_{k-1} \to \alpha_{k-1}$ in $d$ as $i \to \infty$ if $\mathcal{A}(\alpha_{i_{k-1}}) = \mathcal{A}(\alpha_{i_{k-1}})$ for every $i \in \mathbb{N}$ and if $\alpha_i \to \alpha$ in $d$ as $i \to \infty$.

Again, let $\alpha = \{ A_1, A_2, \ldots, A_m \}$ be a partition of $Y$. The set $\mathcal{A}_\alpha$ is now regarded as a (finite) alphabet, and so, its elements are sometimes called letters. Let $\Lambda$ and $(\mathcal{A}_\alpha)^*$ denote the empty word and the set of (possibly empty) words over the alphabet $\mathcal{A}_\alpha$, respectively. An infinite product space $(\mathcal{A}_\alpha)^\mathbb{C}$ is a Cantor set under the product topology of discrete topologies. The family of cylinder subsets is a base for the topology. A cylinder subset $[u,v]$ of $(\mathcal{A}_\alpha)^\mathbb{C}$ is associated with words $u, v \in (\mathcal{A}_\alpha)^*$ in such a way that

$$[u,v] = \left\{ x = (x_i)_{i \in \mathbb{Z}} \in (\mathcal{A}_\alpha)^\mathbb{C} \mid x_{[-|u|,|v|]} := x_0 \ldots x_{|u|} x_{-|u|+1} \ldots x_{|v|-1} = uv \right\},$$

where $|u|$ is the length of the word $u$. A cylinder subset $[\lambda, v]$ is abbreviated to $[v]$. Consider a homeomorphism $S_\alpha : (\mathcal{A}_\alpha)^\mathbb{C} \to (\mathcal{A}_\alpha)^\mathbb{C}$, $(x_i)_{i \in \mathbb{Z}} \mapsto (x_{i+1})_{i \in \mathbb{Z}}$, which is the so-called left shift on $(\mathcal{A}_\alpha)^\mathbb{C}$. Define a map $\phi_\alpha : Y \to (\mathcal{A}_\alpha)^\mathbb{C}$ by $T^i y \in A_{\phi_\alpha(Y)}$, for every $i \in \mathbb{Z}$. The map $\phi_\alpha$ is measurable. Since for any $i \in \mathbb{Z}$ and $y \in Y$, a point $T^{i+1} y$ belongs to both of atoms $A_{\phi_\alpha(Ty)}$ and $A_{\phi_\alpha(y)_{i+1}}$, we obtain that $\phi_\alpha(Ty) = S_\alpha \phi_\alpha(y)$ for all $y \in Y$. Set $\lambda_\alpha = \mu \circ \phi_\alpha^{-1}$, which is an $S_\alpha$-invariant, infinite Borel measure on $(\mathcal{A}_\alpha)^\mathbb{C}$. Set $\hat{\lambda}_\alpha = \text{supp}(\lambda_\alpha)$. Since $\hat{\lambda}_\alpha$ is $S_\alpha$-invariant, so
is $\hat{X}_\alpha$. Set

$$\mathcal{L}(\alpha) = \left\{ u \in (\mathfrak{A}_\alpha)^* \mid \bigcap_{i=1}^{\lfloor |u| / 2 \rfloor} T^{-i} A_{u_i} \neq \emptyset \right\};$$

$$\mathcal{L}(\alpha)' = \left\{ u \in \mathcal{L}(\alpha) \mid u \neq 1^{\lfloor |u| / 2 \rfloor} := 11\ldots1 \right\};$$

$$\mathcal{L}_n(\alpha)' = \left\{ u \in \mathcal{L}(\alpha)' \mid |u| = n \right\},$$

where we have to recall that “a measurable set is nonempty” should be interpreted as “a measurable set has a positive measure”. Since it follows from definition of $\text{supp}(\hat{\lambda}_\alpha)$ that given a sequence $x = (x_i)_{i \in \mathbb{Z}} \in (\mathfrak{A}_\alpha)^\mathbb{Z}$,

$$x \in \hat{X}_\alpha \iff x_{[-n,n]} \in \mathcal{L}(\alpha) \text{ for all } n \in \mathbb{N},$$

we see that a sequence $1^\infty$, which consists of the letter 1 only, belongs to $X_\alpha$. Since $T$ is ergodic, we obtain that $\hat{\lambda}_\alpha(\{1^\infty\}) = \mu\left(\bigcap_{i \in \mathbb{Z}} T^{-i} A_1\right) = 0$. In addition, it is easy to see that $0 < \hat{\lambda}_\alpha([u,v]X_\alpha) < \infty$ whenever $uv \in \mathcal{L}(\alpha)'$, where $[u,v]X_\alpha = [u,v] \cap X_\alpha$. Hence, $\hat{\lambda}_\alpha$ is $\sigma$-finite. Let $\hat{X}_\alpha$ denote a subshift $(\hat{X}_\alpha, \hat{S}_\alpha)$, where $\hat{S}_\alpha$ is the restriction of the left shift $S_\alpha$ to $\hat{X}_\alpha$. The map $\phi_\alpha$ is a factor map from the system $\mathbf{Y}$ to an ergodic, infinite measure-preserving system $(\hat{X}_\alpha, \hat{\lambda}_\alpha, \hat{S}_\alpha)$. To see that $\hat{\lambda}_\alpha$ is non-atomic, let $x \in \hat{X}_\alpha$ be arbitrary. We may assume that $x \in \hat{X}_\alpha \setminus \{1^\infty\}$, so that $\hat{\lambda}_\alpha(\{x\}) < \infty$. If $x$ is shift-periodic, i.e. $\hat{S}_\alpha^p x = x$ for some $p \in \mathbb{N}$, then $\bigcup_{i=0}^{p-1} T^i \phi_\alpha^{-1}\{x\}$ is a $T$-invariant set, because $T^p \phi_\alpha^{-1}\{x\} = \phi_\alpha^{-1}\{x\}$, and hence, $0 \leq \hat{\lambda}_\alpha(\{x\}) = \mu(\phi_\alpha^{-1}\{x\}) \leq \mu\left(\bigcup_{i=0}^{p-1} T^i \phi_\alpha^{-1}\{x\}\right) = 0$ since $T$ is ergodic. If $x$ is aperiodic, then $W := \phi_\alpha^{-1}\{x\}$ is a wandering set for $T$, i.e. $T^i W \cap T^j W = \emptyset$ if $i \neq j$. Since $T$ is conservative [1 Proposition 1.2.1], we have $\hat{\lambda}_\alpha(\{x\}) = \mu(W) = 0$.

Suppose that partitions $\alpha = \{A_1, \ldots, A_m\}$ and $\beta = \{B_1, \ldots, B_n\}$ have a relation $\beta \succ \alpha$. Define a map:

$$f_{\alpha,\beta} : \mathfrak{A}_\beta \to \mathfrak{A}_\alpha$$

by $B_j \subset A_{f_{\alpha,\beta}(j)}$ for each $j \in \mathfrak{A}_\beta$. Define a map $\phi_{\alpha,\beta} : \hat{X}_\beta \to \hat{X}_\alpha, (x_i)_{i \in \mathbb{Z}} \mapsto (f_{\alpha,\beta}(x_i))_{i \in \mathbb{Z}}$, which is a factor map from the subshift $\hat{X}_\beta$ to the subshift $\hat{X}_\alpha$. It is readily verified that $\phi_{\alpha,\beta}(1^\infty) = 1^\infty$ and that $\hat{\lambda}_\beta \circ \phi_{\alpha,\beta}^{-1} = \hat{\lambda}_\alpha$ because $\phi_{\alpha,\beta} = \phi_{\alpha,\beta} \circ \phi_{\alpha,\beta}$. It is clear from definition that $\phi_{\alpha,\beta}$ maps each cylinder subset of $X_\beta$ onto a cylinder subset of $X_\alpha$, so that $\phi_{\alpha,\beta}$ is an open map.

**Definition 3.2** ([20 Definition 4.1]). A set $A \in \mathcal{B}_0$ is said to be uniform relative to a set $K \in \mathcal{B}_0$ if to any number $\epsilon > 0$ corresponds $m \in \mathbb{N}$ such that for any point $y \in K$,

$$T_n 1_K(y) \geq m \Rightarrow \left| \frac{T_n 1_A(y)}{T_n 1_K(y)} - \frac{\mu(A)}{\mu(K)} \right| < \epsilon.$$

A partition $\alpha$ of $Y$ is said to be uniform if every set belonging to a family $\mathcal{B}_0 \cap \bigcup_{n=1}^\infty \alpha_n^{-1}$ is uniform relative to the finite support $K_\alpha$ of the partition $\alpha$.

If a uniform partition $\alpha$ of $Y$ is such that for all $x \in \hat{X}_\alpha \setminus \{1^\infty\}$,

$$\#(\text{Orb}_{S_\alpha}(x) \cap K_{\phi_\alpha(x)}) = \infty,$$

(3.5)
where $K_{\varphi_{\alpha}}(\alpha) = \bigcup_{i \neq 1} \varphi_{\alpha}(A_i)$, then the partition $\alpha$ is said to be *strictly uniform*.

It is easy to see that a partition $\alpha$ is uniform if and only if there exists a sequence $1 \leq n_1 < n_2 < \ldots$ of integers for which every set belonging to a family $B_0 \cap \bigcup_{k=1}^{\infty} \alpha_{n_k}^{-1}$ is uniform relative to $K_{\alpha}$.

**Lemma 3.3** ([20] Lemma 4.2). Let $\alpha$ be a partition of $Y$. Then, the partition $\alpha$ is strictly uniform if and only if the subshift $X_{\alpha}$ is almost minimal and bi-ergodic.

## 4. Towers

Let $Y = (Y, \mathcal{B}, \mu, T)$ be again an ergodic, infinite measure-preserving system. A partition $t = \{ T^j B_i \mid 1 \leq i \leq k, 0 \leq j < h_i \}$ of $Y$ is called a tower with *base* $B(t) := \bigcup_{i=1}^{k} B_i$. An atom of the tower $t$ is called a *level*. Since the tower $t$ is required to be a partition of $Y$, there necessarily exists a unique $i_0$ for which $\mu(B_{i_0}) = \infty$. Then, it is necessary that $h_{i_0} = 1$. We may always assume that $i_0 = 1$. The unique level $B_1$ of infinite measure is referred to as an *infinite level* of $t$. Another tower $t'$ is called a refinement of the tower $t$ if $B(t') \subset B(t)$ and $t' \succ t$.

Each subfamily $c_i := \{ T^j B_i \mid 0 \leq j < h_i \}$ is called a column of the tower $t$. We refer to $B_i$ and $h_i$ as the *base* and *height* of the column $c_i$, respectively. With abuse of terminology, we also say that $\bigcup_{j=0}^{h_i-1} T^j B_i$ is a column of $t$. The column $c_i$ is said to be principal if $i \neq 1$. Let $h(t)$ denote the least value of heights of principal columns of $t$.

A set $\{ T^j y \mid 0 \leq j < h_i \}$ with $y \in B_i$ and $1 \leq i \leq k$ is called a fiber of the tower $t$. Any set of the form $\{ T^j y \mid m \leq i \leq n \}$ with $m \leq n$ is called a section of the orbit $\text{Or}B_T(y)$, so that any fiber is a section. The word $\varphi_{\alpha}(y)[m, n]$ over the alphabet $\mathcal{A}_n$ is called the $\alpha$-name of the section. Sections $\{ T^j y \mid m \leq i \leq n \}$ and $\{ T^j y' \mid m' \leq i \leq n' \}$ are said to be consecutive if $T(T^m y) = T^{m'} y'$.

Let $K \in \mathcal{B}_0$. We say that a tower $t$ is $K$-standard [20] Definition 2.1] if

1. $K$ has a nonempty levels intersection with each principal column of $t$;
2. $K$ is a union of levels included in the principal columns of $t$.

These conditions imply that the number of those points in a fiber which belong to the set $K$ is positive and constant on each column of the tower $t$ and that the number is zero for all the fibers on the infinite level of the tower $t$. Let $h_K(t)$ and $H_K(t)$ denote the least and greatest values of such numbers over fibers on principal columns of $t$, respectively.

**Lemma 4.1** ([20] Proposition 2.5]). If $C$ is a measurable subset of $K$, $0 < \epsilon < \mu(K)$ and $M \in \mathbb{N}$, then there exists $N \in \mathbb{N}$ such that if a $K$-standard tower $t$ satisfies $h(t) > N$ then the union of those fibers $\{ T^j y \mid 0 \leq j < h_y \}$ of the tower $t$, for which

$$\left| \frac{\sum_{j=0}^{h_y-1} 1_C(T^j y)}{\sum_{j=0}^{h_y-1} 1_K(T^j y)} - \frac{\mu(C)}{\mu(K)} \right| < \epsilon \quad \text{and} \quad \sum_{j=0}^{h_y-1} 1_K(T^j y) \geq M,$$

covers at least $\mu(K) - \epsilon$ of $K$ in measure.

Apart from measure-preserving systems, we then discuss towers for an almost minimal Cantor system $Z = (\hat{Z}, \mathcal{U})$, which is the one-point compactification of a minimal, locally compact Cantor system $Z = (Z, U)$. Put $\hat{Z} = Z \cup \{ \omega_Z \}$ as done in Section [2] which is a Cantor set. Take a clopen neighborhood $C$ of $\omega_Z$ so that
C ≠ ♯. Since ω_Z is an accumulation point of the forward orbit \{ U^k z \mid k ∈ N \} of any point z ∈ ♯ [9 Theorem 1.1], the return time function:

\[ r_C : C → N, z ↦ \min \{ k ∈ N \mid U^k z ∈ C \} \]

is well-defined. For each k ∈ N, an inverse image:

\[ r_C^{-1} \{ k \} = (C \cap ♯ U^{-k} C) \setminus \bigcup_{i=1}^{k-1} ♯ U^{-i} C \]

is open as the right hand side looks. Since the clopen set C is compact, the function \( r_C \) is bounded. Then, put \( r_C(C) = \{ h_1 < h_2 < \cdots < h_c \} \). Since C includes a fixed point \( ω_Z \) of ♯, it is necessary that \( h_1 = 1 \). Since ♯ \setminus C ≠ ∅, the almost minimality of ♯ implies that \( C \setminus ♯ U^{-1}(C) ≠ ∅ \), so that we must have \( c ≥ 2 \). It is readily verified that

\[ (4.1) \quad \mathcal{P} = \left\{ ♯ U^j C_i \mid 0 ≤ j < h_i, 1 ≤ i ≤ c \right\}, \]

where \( C_i = r_C^{-1} \{ h_i \} \), is a family of mutually disjoint, nonempty, clopen subsets of ♯. It is a partition of ♯ in the usual sense, because a clopen subset \( \bigcup_{A ∈ \mathcal{P}} A \) of ♯ is ♯-invariant. A partition of the form (4.1) is called a Kakutani-Rohlin partition [9] (abbreviated to K-R partition) of the almost minimal Cantor system ♯. The above discussion guarantees that every almost minimal Cantor system possesses a K-R partition.

We do not require in general that \( h_1, \ldots, h_c \) are mutually distinct, but always require that \( h_1 = 1 \) (and hence \( ω_Z ∈ C_1 \)). A K-R partition is topological counterpart of a tower for a measure-preserving system, as the clopen set \( C_1 \) corresponds to the infinite level of a tower. It is possible to naturally import terminology and notation defined in the category of measure-theoretical systems into that of almost minimal Cantor systems. For example, each subfamily \( \{ ♯ U^j C_i \mid 0 ≤ j < h_i \} \) is called a column, and the clopen set \( C_1 \) is called an infinite level.

We then discuss symbolic factors of the almost minimal Cantor system ♯. Suppose that \( α = \{ A_1, A_2, \ldots, A_m \} \) with \( m ≥ 2 \) is a partition of ♯ into nonempty, clopen subsets such that \( ω_Z ∈ A_1 \). Actually, this is a definition of a partition of ♯. The atom \( A_1 \) corresponds to the infinite atom of a partition of a measure space, which is discussed in Section 3. So, we let \( K_α \) denote the finite support \( \bigcup_{i=2}^m A_i \) of the partition \( α \). Define \( \mathcal{L}(α) \) by literally interpreting (3.2) after replacing \( T \) with ♯. With abuse of notation, define a subshift:

\[ (4.2) \quad X_α = \{ x = (x_i)_{i∈Z} ∈ (2^A_α)^Z \mid x_{[-n,n)} ∈ \mathcal{L}(α) \text{ for all } n ∈ N \}, \]

where \( A_α = \{ 1, 2, \ldots, m \} \) again. Define a factor map \( φ_α : ♯ → X_α \) by \( ♯ U^i z ∈ A_{φ_α(z)} \) for all \( i ∈ Z \). We can also define a factor map \( φ_α,β : X_β → X_α \) as in Section 3 if \( β \succ α \). The subshift \( X_α := (X_α, S_α) \) is almost minimal; recall the proof of Corollary [8, 3].

We then discuss K-R partitions of the subshift \( X_α \) which are associated with return words; see [6, 7] for the minimal subshifts and [19, 24] for the almost minimal subshifts. They induce K-R partitions of the almost minimal Cantor system ♯. Choose \( u, v ∈ \mathcal{L}(α) \) so that \( uv ∈ \mathcal{L}(α) \setminus \{ ∅ \} \). A word \( w ∈ \mathcal{L}(α) \) is called a return
word to \( uv \) if \( uvw \in \mathcal{L}(\alpha) \) and in addition \( uv \) occurs in \( uvw \) exactly twice as prefix and suffix. Let \( \mathcal{R}_n(\alpha) \) denote the set of return words to \( 1^n.1^n \). It is clear that \( 1 \in \mathcal{R}_n(\alpha) \). In virtue of [10] Section 3] and [21] Lemma 5.2, we know that

- \( \mathcal{R}_n(\alpha) \) is a finite set for all \( n \in \mathbb{N} \);
- for every \( n \in \mathbb{N} \), each word \( w \in \mathcal{R}_{n+1}(\alpha) \) has a unique decomposition into words belonging to \( \mathcal{R}_n(\alpha) \);
- if \( w \in \mathcal{R}_n(\alpha) \setminus \{1\} \), then \( |w| > 2n, w_{[1,n]} = 1^n, w_{n+1} \neq 1 \) and \( w_{[w-(n-1),w]} = 1^n, w_{[w-n]} \neq 1 \).

\begin{equation}
\lim_{n \to \infty} \min \{ |w|_{-1} \mid w \in \mathcal{R}(\alpha) \setminus \{1\} \} = +\infty,
\end{equation}

where \( |w|_{-1} = \# \{ 1 \leq i \leq |w| \mid w_i \in \mathcal{A} \setminus \{1\} \} \).

To see (4.3), assume that it is not the case. Find a sequence \( x \in \hat{X}_\alpha \) of the form \( 1^\infty.w1^\infty \) with a word \( w \) whose first and last letter is neither 1, where the dot means a separation between the nonnegative and negative coordinates. Since the only accumulation point of Orb \( \hat{\alpha} \) is \( 1^\infty \), we obtain that \( \hat{X}_\alpha = \text{Orb}_\alpha(x) \cup \{1^\infty\} \), because \( \hat{X}_\alpha \) is almost minimal. This leads to a contradiction that \( \hat{Z} \) has a nonempty open wandering set.

Now, it is readily verified that for each \( n \in \mathbb{N} \),

\[ P_n(\alpha) := \left\{ S^j_\alpha([1^n.w1^n]_{\hat{X}_\alpha}) \mid w \in \mathcal{R}(\alpha), 0 \leq j < |w| \right\} \]

is a K-R partition of \( \hat{X}_\alpha \) with base \( B(P_n(\alpha)) = [1^n.1^n]_{\hat{X}_\alpha} \). Set

\[ \hat{t}_n(\alpha) = \phi_\alpha^{-1}P_n(\alpha) := \{ \phi_\alpha^{-1}(A) \mid A \in P_n(\alpha) \} \]

\begin{equation}
= \left\{ \hat{U}^j\phi_\alpha^{-1}([1^n.w1^n]_{\hat{X}_\alpha}) \mid w \in \mathcal{R}_n(\alpha), 0 \leq j < |w| \right\},
\end{equation}

which is a K-R partition of \( \hat{Z} \). Observe that

\begin{equation}
\phi_\alpha^{-1}([1^n.w1^n]_{\hat{X}_\alpha}) = \bigcap_{i=-n}^{-1} \hat{U}^{-i}A_{1^n.w1^n}^{i+n+1} \text{ and } B(\hat{t}_n(\alpha)) = \bigcap_{j=-n}^{n-1} \hat{U}^{-j}A_1.
\end{equation}

For every \( n \in \mathbb{N} \), \( \hat{t}_{n+1}(\alpha) \) is a refinement of \( \hat{t}_n(\alpha) \), because

- \( B(\hat{t}_{n+1}(\alpha)) = \phi_\alpha^{-1}([1^n.1^{n+1}]_{\hat{X}_\alpha}) \subset \phi_\alpha^{-1}([1^n.1^n]_{\hat{X}_\alpha}) = B(\hat{t}_n(\alpha)) \);
- letting \( w = u_1u_2\ldots u_k \) (\( u_i \in \mathcal{R}(\alpha) \)) be a unique decomposition of a given word \( w \in \mathcal{R}_{n+1}(\alpha) \), we have that if \( |u_1 \ldots u_{i-1}| \leq j < |u_1 \ldots u_i| \) for \( 1 \leq i \leq k \) then \( S^j_\alpha([1^n.1^{n+1}]_{\hat{X}_\alpha}) \subset S^j_\alpha([1^n.1^n]_{\hat{X}_\alpha}) \) and hence \( \hat{U}^j\phi_\alpha^{-1}([1^n.1^{n+1}]_{\hat{X}_\alpha}) \subset \hat{U}^j\phi_\alpha^{-1}([1^n.1^n]_{\hat{X}_\alpha}) \), where we use a convention that \( |u_1 \ldots u_0| = 0 \).

**Lemma 4.2.** The refining sequence \( \{ \hat{t}_n(\alpha) \}_{n \in \mathbb{N}} \) of \( K_\alpha \)-standard K-R partitions of the almost minimal Cantor system \( \hat{Z} \) satisfies that

(1) \( h_{K_\alpha}(\hat{t}_n(\alpha)) \to +\infty \) as \( n \to \infty \);

(2) for each \( n \in \mathbb{N} \), the \( \alpha \)-names of fibers of \( \hat{t}_n(\alpha) \) are constant on each column.

In addition, if a partition \( \beta \) of \( \hat{Z} \) implements a relation that \( \beta \succ \alpha \), then for each \( n \in \mathbb{N} \), \( \hat{t}_n(\beta) \) is a refinement of \( \hat{t}_n(\alpha) \).
Proof. Property (1) follows from (4.3). Property (2) follows from (4.4) and the first half of (4.5). Actually, each word $w \in \mathcal{R}_n(\alpha)$ is the $\alpha$-name of fibers on the column with base $\phi_{\alpha}^{-1}(1^n \cdot w1^n\hat{x}_\alpha)$.

Then, let $\beta$ be as in the hypothesis. Since $f_{\alpha,\beta}(1) = 1$, we have that $[1^n \cdot 1^n]\hat{x}_\beta \subset \phi_{\alpha,\beta}^{-1}(1^n \cdot 1^n \hat{x}_\alpha)$, so that $B(\hat{t}_n(\beta)) = \phi_{\alpha}^{-1}(1^n \cdot 1^n \hat{x}_\alpha) \subset \phi_{\alpha}^{-1}(1^n \cdot 1^n \hat{x}_\alpha) = B(\hat{t}_n(\alpha))$, since $\phi_{\alpha} = \phi_{\alpha,\beta} \circ \phi_{\beta}$. What remains to be shown is that each level of $\mathcal{P}_n(\beta)$ is included in some level of $\phi_{\alpha,\beta}^{-1}\mathcal{P}_n(\alpha)$, since $\phi_{\alpha} = \phi_{\alpha,\beta} \circ \phi_{\beta}$. Since $f_{\alpha,\beta}(1) = 1$, for any word $w \in \mathcal{R}_n(\beta)$, there exist unique words $u_1, u_2, \ldots, u_k \in \mathcal{R}_n(\alpha)$ so that

- $|w| = |u_1 u_2 \ldots u_k|$;
- $f_{\alpha,\beta}(w) = (u_1 u_2 \ldots u_k)$, for every integer $i$ with $1 \leq i \leq |w|$.

It follows from the conditions that

$[1^n \cdot w1^n]\hat{x}_\beta \subset \phi_{\alpha,\beta}^{-1}(1^n \cdot u_1 u_2 \ldots u_k 1^n \hat{x}_\alpha),$

and hence,

$S_j \left( [1^n \cdot w1^n]\hat{x}_\beta \right) \subset S_j \left( 1^n \cdot u_1 \ldots u_{i-1} 1^n \hat{x}_\alpha \right)$

for any integer $j$ with $0 \leq j < |w|$ if $|u_1 \ldots u_{i-1}| \leq j < |u_1 \ldots u_i|$ for $1 \leq i \leq k$. This completes the proof.

□

Remark 4.3. In the proof of Lemma 4.2, it is necessary that $k = 1$ if $\beta \succ \alpha$.

Next, take a countable base $\{D_1, D_2, \ldots\}$ for the topology of the locally compact Cantor set $Z$ which consists of compact open subsets of $Z$. Form a sequence $\{\alpha_i\}_{i \in \mathbb{N}}$ of partitions of $\hat{Z}$ by setting $\alpha_i = \left\{ \hat{Z} \setminus D_1, D_1 \right\} \vee \left\{ \hat{Z} \setminus D_1, D_2 \right\} \vee \cdots \vee \left\{ \hat{Z} \setminus D_i, D_i \right\}$. Clearly, for each $i \in \mathbb{N}$, the partition $\alpha_{i+1}$ refines the partition $\alpha_i$, i.e. $\alpha_{i+1} \succ \alpha_i$. Set $\hat{t}_i = \phi_{\alpha_i}^{-1}\mathcal{P}_1(\alpha_i)$. In view of Lemma 4.2, we have a refining sequence $\{\hat{t}_i\}_{i \in \mathbb{N}}$ of K-R partitions of the almost minimal Cantor system $\hat{Z}$. The construction shows that

- for all $i \in \mathbb{N}$ and integer $j$ with $j \geq i$, the compact open subset $D_i$ of $Z$ is a union of levels in the principal columns of $\hat{t}_j$;
- each principal column of $\hat{t}_i$ has a nonempty intersection with $D_i$, because it is a refinement of $\hat{t}_i$.

Set $\gamma_i = \hat{t}_i$. Let $K_{\gamma_i}$ be as above, i.e. the union of principal columns of $\hat{t}_i$. For each integer $i$ with $i > 1$, set $\gamma_i = \left\{ \hat{Z} \setminus K_{\gamma_i}, C \mid C \in \hat{t}_i, C \subset K_{\gamma_i} \right\}$.

Lemma 4.4. The sequence $\{\gamma_i\}_{i \in \mathbb{N}}$ of partitions of $\hat{Z}$ into clopen subsets satisfies that

- $\gamma_1 \preceq \gamma_2 \preceq \cdots$;
- $\bigcup_{i \in \mathbb{N}}(\gamma_i)^{j-1}$ is a base for the topology of $\hat{Z}$.

Proof. The first property is clear. The second property follows from a fact that each principal column of $\hat{t}_i$ has a nonempty intersection with $K_{\gamma_i}$.

□

5. PROOF OF THEOREM 1.1

Let $Y = (Y, \mathcal{B}, \mu, T)$ be as in Theorem 1.1.
Definition 5.1. Let $\alpha$ and $\beta = \{ B_1, B_2, \ldots, B_n \}$ be partitions of $Y$ with $\#\alpha = \#\beta = n$. The $\alpha$ $(2k - 1)$-block empirical distribution over a section with $\alpha$-name $w$ is said to be within $\epsilon$ of the $\beta^{k-1}$ distribution [20, Definition 4.3] if for every word $v \in L_{2k-1}(\alpha)'$,

$$\left| \frac{\# \{ 1 \leq j < |w| \mid w_{(j-k+j+k)} = v \}}{\# \{ 1 \leq j < |w| \mid w_j \neq 1 \}} - \frac{\mu(\cap_{i=k}^{k+1} T_i B_{\nu+i})}{\mu(K_\beta)} \right| < \epsilon,$$

where $w_{(j-k+j+k)} = \begin{cases} w_{[1,j+k)} & \text{if } j - k < 0; \\ w_{(j-k,w]} & \text{if } j + k > n + 1. \end{cases}$

We say that the partition $\alpha$ has $(H, k, \epsilon)$-uniformity if the $\alpha (2k - 1)$-block distribution over every section having at least $H$ points in $K_\alpha$ is within $\epsilon$ of the $\alpha^{k-1}$ distribution.

Let $Z = (Z, U)$ and $\pi$ be as in Theorem [1.1] so that there exists a unique, up to scaling, infinite, $U$-invariant Radon measure $\nu$ satisfying that $\mu \circ \pi^{-1} = \nu$. Let $C$ be the completion of the Borel $\sigma$-algebra with respect to $\nu$. Let $\hat{Z} = (\hat{Z}, \hat{U})$ be as posterior to Lemma [4.1]. Extend the measure $\nu$ naturally to a unique, non-atomic measure $\hat{\nu}$ onto the Borel $\sigma$-algebra of $\hat{Z}$. Complete the Borel $\sigma$-algebra with respect to $\hat{\nu}$, which is denoted by $\hat{C}$. The map $\pi : Y \to Z$ is naturally regarded as a factor map from $Y$ to $(\hat{Z}, \hat{C}, \hat{\nu}, \hat{U})$.

Let $\{ \gamma_i \}_{i \in \mathbb{N}}$ be a refining sequence of partitions of $\hat{Z}$ as in Lemma [4.3]. In virtue of Corollary [2.3], for each $i \in \mathbb{N}$, the subshift $X_{\gamma_i}$ is a bi-ergodic, almost minimal Cantor system. For each $i \in \mathbb{N}$, set $\beta_i = \pi^{-1} \gamma_i$. It follows that $\{ \beta_i \}_{i \in \mathbb{N}}$ is a sequence of partitions of $Y$ satisfying that $\beta_1 \preceq \beta_2 \preceq \ldots$. Let $K$ denote their common finite support, which equals $\pi^{-1}(K_{\gamma_i})$ for all $i \in \mathbb{N}$. Since the minimality of $U$ implies that any nonempty open subset of $Z$ has positive measure with respect to $\nu$, it follows from (3.3) and (1.2) that $X_{\beta_i} = X_{\gamma_i}$ for each $i \in \mathbb{N}$. This together with Lemma [3.3] implies that each $\beta_i$ is strictly uniform. Also, observe that $\phi_{\gamma_i} \circ \pi = \phi_{\beta_i}$ for each $i \in \mathbb{N}$. Choose a countable base $E = \{ E_i \}_{i \in \mathbb{N}} \subset B_0$ for the measure space $(Y, B, \mu)$ so that each member appears in $E$ infinitely often. Setting $K_0 = K \cap E_1$ and

$$K_i = (K \cap T^{-1}E_1) \setminus \bigcup_{j=1}^{i-1} T^{-j}E_1$$

for $i \in \mathbb{N}$, we have a disjoint union $\bigcup_{i=0}^{\infty} T^i K_i = E_1$, because $Y = \bigcup_{j=1}^{\infty} T^{-j}E_1$ by the ergodicity. This allows us to find a partition $\tau_1$ with finite support $K$ implementing an approximation:

$$E_1 = \left( \tau_1 \right)^{1/2},$$

by means of a finite union of mutually disjoint translates of mutually disjoint atoms; recall (3.1). Fix $n_1 \in \mathbb{N}$ and $\delta_1 > 0$ so that

$$\frac{1}{2^{n_1}} < \frac{1}{3} \min \left\{ \frac{\mu(A)}{\mu(K)} \mid A \in (\tau_1 \cup \beta_1) \cap B_0 \right\};$$

and

$$\frac{1}{4 \cdot 2^{n_1}} < \frac{\mu(K)}{4 \cdot 2^{n_1}} < \frac{1}{2^3}.$$
We regard $\mathbf{A}_{\tau_1 \vee \beta_1}$ as a subset of a product set $\mathbf{A}_{\tau_1} \times \mathbf{A}_{\beta_1}$, so that in particular an infinite atom of $\tau_1 \vee \beta_1$ has a subscript (1, 1).

**Remark 5.2.** Since every tower which we will encounter in this proof is $K$-
standard, we will suppress the term ‘$K$-standard’ for the sake of simplicity. Also, every partition will have the set $K$ as its finite support.

We appeal to a series of lemmas. Recall that given a partition $\alpha$ of $Y$, $\mathcal{F}_\alpha$ denotes the algebra generated by $\bigcup_{k=1}^\infty \alpha^{k^{-1}}$.

**Lemma 5.3** (Step 1). There exist partition $\alpha_{1,1}$, $\mathcal{F}_{\beta_1}$-measurable tower $t_1$ and $N_1 \in \mathbb{N}$ such that

- $\alpha_{1,1} \cong \beta_1$;
- $E_1^{1/2^2} \subseteq (\alpha_{1,1})^T$;
- if a partition $\alpha$ of $Y$ is such that the $\alpha$-name of any fiber of $t_1$ coincides with the $\alpha_{1,1}$-name of some fiber of $t_1$, then the partition $\alpha$ has the \( (N_1,1,1/2^n) \)-uniformity.

**Proof.** Applying Lemma 4.1 to the finite atoms of a partition $\tau_1 \vee \beta_1$, we obtain $H_1 \in \mathbb{N}$ such that if $t$ is a tower with $h(t) \geq H_1$ then the good fibers of $t$, i.e. the $\tau_1 \vee \beta_1$ 1-block empirical distributions over the fibers are within $\delta_1$ of the $\tau_1 \vee \beta_1$ distribution, covers at least $\mu(K) - \delta_1$ of $K$ in measure. Lemma 4.2 allows us to find an $\mathcal{F}_{\beta_1}$-measurable tower $t'_1$ with $h(t'_1) \geq H_1$ such that the $\beta_1$-names of fibers are constant on each column of $t'_1$. The tower $t'_1$ has the form $\pi^{-1}t_\alpha(\gamma_1)$. Within each principal column of $t'_1$, copy the $(\tau_1 \vee \beta_1)$-name of a good fiber into bad fibers, provided that good fibers exist in the column. This yields a new partition $\alpha_{1,1}$ with $\mathbf{A}_{\alpha_{1,1}} \subset \mathbf{A}_{\tau_1 \vee \beta_1}$. In virtue of the above-mentioned property of the tower $t'_1$, the copying procedure never changes $\beta_1$-names although it may change $\tau_1$-names. Since it follows from (5.2) and (5.3) that $\delta_1 < \frac{\mu(K)}{4 \cdot 2^{n_1}}$, we obtain that $\mathbf{A}_{\alpha_{1,1}} = \mathbf{A}_{\tau_1 \vee \beta_1}$. Since a given point has the $\tau_1$-name 1 if and only if it has the $\beta_1$-name 1, we see that $K_{\alpha_{1,1}} = K_{\tau_1 \vee \beta_1}$. In order to see that $\alpha_{1,1} \cong \beta_1$, again in virtue of the above-mentioned property of $t'_1$, it is sufficient to observe that any enlargement or reduction of each finite atom of $\tau_1 \vee \beta_1$ caused by the copying procedure is carried out within a unique atom of $\beta_1$. Moreover, for the same reason, the refinement $\alpha_{1,1} \cong \beta_1$ is implemented so that $f_{\beta_1,\alpha_{1,1}} = f_{\beta_1,\tau_1 \vee \beta_1}$; recall (3.4). Since $d(\alpha_{1,1}, \tau_1 \vee \beta_1) \leq \delta_1 < 1/2^3$, it follows from (5.1) that $E_1^{1/2^2} \subseteq (\alpha_{1,1})^T$.

Let $R_1$ denote the intersection of the set $K$ with the union of those principal columns of $t'_1$ which do not include any good fibers. As a consequence of the copying procedure, we know that if a fiber on a principal column of $t'_1$ is disjoint from $R_1$ then the fiber has $\alpha_{1,1}$-1-block empirical distribution within $\delta_1$ of the $\tau_1 \vee \beta_1$ distribution. Let us verify that this leads to the existence of $M_1 \in \mathbb{N}$ such that for any point $y \in K$,

\[ A \in \alpha_{1,1} \cap \mathcal{B}_0 \text{ and } T_n 1_{K \setminus R_1}(y) \geq M_1 \Rightarrow \left| \frac{T_n 1_{A \cap R_1}(y)}{T_n 1_{K \setminus R_1}(y)} - \frac{\mu(B)}{\mu(K)} \right| < \delta_1, \] (5.4)
where $B$ is a finite atom of $\tau_1 \lor \beta_1$ which shares a subcript with the finite atom $A$ of $\alpha_{1,1}$. Let $\mathcal{F}$ denote the family of good fibers of $\tau'_1$. For each integer $m$ with $m \geq 3$, put

$$\epsilon_m = \frac{4}{m - 2} \cdot \frac{\max_{F \in \mathcal{F}} \#(F \cap K)}{\min_{F \in \mathcal{F}} \#(F \cap K)}.$$ 

Fix an integer $m_0$ with $m_0 \geq 3$ so that

$$\epsilon_{m_0} < \delta_1 - \max \left\{ \left| \frac{\#(F \cap A)}{\#(F \cap K)} - \frac{\mu(B)}{\mu(K)} \right| \mid A \in \alpha_{1,1} \cap B_0, B \in (\tau_1 \lor \beta_1) \cap B_0, F \in \mathcal{F}, \right. \left. B \text{ shares a subcript with } A. \right\}.$$

Set

$$M_1 = m_0 \max_{F \in \mathcal{F}} \#(F \cap K),$$

which is independent of the choice of the finite atom $A$. If $T_n\mathbb{1}_{K \setminus R_1}(y) \geq M_1$ then there are integer $m_1$ with $m_1 \geq m_0$, nonempty subsets $C_{i_1}, C_{i_{m_1}}$ of good fibers of $\tau'_1$, whose intersections with $K$ are nonempty, and fibers $C_{i_2}, \ldots, C_{i_{m_1 - 1}} \in \mathcal{F}$ for which $T_n\mathbb{1}_{K \setminus R_1}(y) = \sum_{j=1}^{m_1} \#(C_{i_j} \cap K)$. Assume that $T_n\mathbb{1}_{K \setminus R_1}(y) \geq M_1$ and $A \in \alpha_{1,1} \cap B_0$ is arbitrary. Since

$$\frac{T_n\mathbb{1}_{A \setminus R_1}(y)}{T_n\mathbb{1}_{K \setminus R_1}(y)} = \frac{\#(C_{i_1} \cap A) + \sum_{j=2}^{m_1 - 1} \#(C_{i_j} \cap A) + \#(C_{i_{m_1}} \cap A)}{\#(C_{i_1} \cap K) + \sum_{j=2}^{m_1 - 1} \#(C_{i_j} \cap K) + \#(C_{i_{m_1}} \cap K)}$$

$$= \frac{\#(C_{i_1} \cap A) + \sum_{j=2}^{m_1 - 1} \#(C_{i_j} \cap A)}{\#(C_{i_1} \cap K) + \sum_{j=2}^{m_1 - 1} \#(C_{i_j} \cap K)} + \frac{\#(C_{i_{m_1}} \cap A)}{\#(C_{i_{m_1}} \cap K)} + 1$$

and since $\frac{\sum_{j=2}^{m_1 - 1} \#(C_{i_j} \cap A)}{\sum_{j=2}^{m_1 - 1} \#(C_{i_j} \cap K)} \leq 1$, we obtain that

$$\frac{\#(C_{i_1} \cap A)}{\#(C_{i_1} \cap K)} - \frac{\sum_{j=2}^{m_1 - 1} \#(C_{i_j} \cap A)}{\sum_{j=2}^{m_1 - 1} \#(C_{i_j} \cap K)} \leq \epsilon_{m_1}.$$ 

In view of an inequality:

$$\min_{1 \leq i \leq k} \frac{a_i}{b_i} \leq \sum_{i=1}^{k} \frac{a_i}{b_i} \leq \max_{1 \leq i \leq k} \frac{a_i}{b_i} (a_i, b_i > 0, k \in \mathbb{N}),$$
By using Lemma 4.2, take an \( F \) this allows us to obtain that
\[
\frac{|T_n 1_{A \setminus R_1}(y) - \mu(B)|}{T_n 1_{K \setminus R_1}(y)} \leq \max_{F \in \mathcal{F}} \left| \frac{(F \cap A)}{(F \cap K)} - \frac{\mu(B)}{\mu(K)} \right| + \epsilon_{m_1}
\]
\[
\leq \max_{F \in \mathcal{F}, A \in \alpha_1, R_0, B \in (\tau_1 \cup \beta_1) \cap R_0} \left| \frac{(F \cap A)}{(F \cap K)} - \frac{\mu(B)}{\mu(K)} \right| + \epsilon_{m_0} < \delta_1,
\]
which shows \((5.4)\). Observe that in order to obtain \((5.4)\) we are not required to consider which good fiber each fiber \( C_i \) is. This is one of facts which guarantee the last property of the lemma.

On the other hand, since \( R_1 \) is \( \mathcal{F}_{\beta_1} \)-measurable, and hence, uniform relative to \( K \), and since
\[
\mu(R_1) \leq \delta_1 < \frac{\mu(K)}{4 \cdot 2^{n_1}},
\]
there exists \( N'_1 \in \mathbb{N} \) with
\[
N'_1 \geq \left(1 - \frac{1}{4 \cdot 2^{n_1}}\right)^{-1} M_1
\]
such that for any point \( y \in K \),
\[
T_n 1_K(y) \geq N'_1 \Rightarrow \frac{T_n 1_{R_1}(y)}{T_n 1_K(y)} < \frac{1}{4 \cdot 2^{n_1}},
\]
which implies further that
\[
T_n 1_{K \setminus R_1}(y) > \left(1 - \frac{1}{4 \cdot 2^{n_1}}\right) \cdot T_n 1_K(y) \geq M_1.
\]
It follows from the above argument that for any point \( y \in K \), if \( T_n 1_K(y) \geq N'_1 \) then for every finite atom \( A \) of \( \alpha_{1,1} \),
\[
\frac{|T_n 1_A(y) - \mu(A)|}{T_n 1_K(y)} \leq \left| \frac{T_n 1_{A \setminus R_1}(y)}{T_n 1_K(y)} - \frac{\mu(B)}{\mu(K)} \right| + \left| \frac{T_n 1_{R_1}(y)}{T_n 1_K(y)} - \frac{\mu(A)}{\mu(K)} \right| + \frac{\mu(B)}{\mu(K)} \leq \frac{1}{2 \cdot 2^{n_1}}
\]
\[
< \delta_1 + \frac{3}{4 \cdot 2^{n_1}} < \frac{1}{2^{n_1}}.
\]
where \( B \) is a finite atom of \( \tau_1 \cup \beta_1 \) which shares a subscript with the atom \( A \) of \( \alpha_{1,1} \). By using Lemma 4.2 take an \( \mathcal{F}_{\beta_1} \)-measurable refinement \( t_1 \) of \( \kappa_1 \) with \( h_K(t_1) \geq N'_1 \). The tower \( t_1 \) has the form \( \pi^{-1} t_1(\gamma_1) \). By argument similar to deducing \((5.4)\), we can find an integer \( N_1 \) with \( N_1 \geq N'_1 \) for which the last property of the lemma is valid. \( \square \)
In the same manner as (5.1), find a partition \( \tau_2 \) with finite support \( K \) implementing an approximation:

\[
E_2^{1/2^4} \in (\tau_2)^T.
\]

The alphabet \( \mathcal{A}_{\alpha_{1,1} \lor \tau_2 \lor \beta_2} \) is regarded as a subset of a product set \( \mathcal{A}_{\alpha_{1,1}} \times \mathcal{A}_{\tau_2} \times \mathcal{A}_{\beta_2} \). Put

\[
r_2 = \#(\alpha_{1,1} \lor \tau_2 \lor \beta_2).
\]

Fix integer \( n_2 \) with \( n_2 > n_1 \) and real number \( \delta_2 > 0 \) so that

\[
\frac{1}{2^{n_2}} < \frac{1}{3} \min \left\{ \frac{\mu(A)}{\mu(K)} \left| A \in (\alpha_{1,1} \lor \tau_2 \lor \beta_2)^{1}_{-1} \cap B_0 \right\}; \delta_2 < \min \left\{ \frac{1}{3 \cdot 4 \cdot 2^{n_2} \tau_2}, \frac{\mu(K)}{3 \cdot 4 \cdot 2^{n_2} \tau_2^2}, \frac{1}{2^4} \right\}.
\]

It is an immediate consequence that \( n_2 \geq 2 \).

**Lemma 5.4** (Step 2). There exist partitions \( \alpha_{2,2} \) and \( \alpha_{2,1} \), \( \mathcal{F}_{\beta_2} \)-measurable refinement \( t_2 \) of \( t_1 \) and integer \( N_2 \) with \( N_2 > N_1 \) such that

- \( \alpha_{2,2} \succ \beta_2 \), \( \alpha_{2,1} \succ \beta_1 \) and \( \alpha_{2,2} \nprec \alpha_{2,1} \);
- \( E_1^{1/2^3+1/2^3} \in (\alpha_{2,1})^T \) and \( E_2^{1/2^3} \in (\alpha_{2,2})^T \);
- \( \#\alpha_{2,2} = r_2 \);
- \( d(\alpha_{2,1}, \alpha_{1,1}) \leq \delta_2 < 2^{-n_2} \);
- the \( \alpha_{2,1} \)-name of any fiber of \( t_1 \) coincides with the \( \alpha_{1,1} \)-name of some fiber of \( t_2 \);
- if a partition \( \alpha \) is such that the \( \alpha \)-name of any fiber of \( t_2 \) coincides with the \( \alpha_{2,2} \)-name of some fiber of \( t_2 \) then \( \alpha \) has the \( (N_2, 2, 1/(2^{n_2} \tau_2^4)) \)-uniformity.

**Proof.** Since \( t_1 \) has the form \( \pi^{-1} \widehat{t}_n(\gamma_1) \), Lemmas 4.1 and 4.2 allow us to obtain an \( \mathcal{F}_{\beta_2} \)-measurable refinement \( t'_2 \) of \( t_2 \) such that

- the \( \beta_2 \)-names of fibers of \( t'_2 \) are constant on each column;
- good fibers of \( t'_2 \), i.e. the \( \alpha_{1,1} \lor \tau_2 \lor \beta_2 \) 3-block empirical distributions on the fibers are within \( \delta_2 \) of the \( (\alpha_{1,1} \lor \tau_2 \lor \beta_2)^{1}_{-1} \) distribution, covers at least \( \mu(K) - \delta_2 \) of \( K \) in measure.

In fact, when applying Lemma 4.1, we may have to consider translates \( T(A) \) or \( T^{-1}(A) \) instead of finite atoms \( A \in (\alpha_{1,1} \lor \tau_2 \lor \beta_2)^{1}_{-1} \) themselves. The tower \( t'_2 \) has the form \( \pi^{-1} \widehat{t}_w(\gamma_2) \). Since \( t'_2 \) is a refinement of \( t_1 \), the \( \alpha_{1,1} \)-name of any fiber of \( t'_2 \) is a concatenation of \( \alpha_{1,1} \)-names of fibers of \( t_1 \).

We obtain a new partition \( \alpha_{2,2} \) by copying the \( (\alpha_{1,1} \lor \tau_2 \lor \beta_2) \)-name of a good fiber in each column of \( t'_2 \) into bad fibers in the same column, provided that good fibers exist in the column. The \( \beta_2 \)-part (or, \( \beta_2 \)-coordinate) of any \( (\alpha_{1,1} \lor \tau_2 \lor \beta_2) \)-name never changes under the copying procedure although the other parts may change. In particular, the copying procedure preserves \( (\alpha_{1,1} \lor \tau_2 \lor \beta_2) \)-name \((1, 1, 1)\). Recall that \( K_{\beta_2} = K_{\alpha_{1,1}} = K_{\tau_2} = K \). These facts guarantee that \( K_{\alpha_{2,2}} = K \). Since

\[
d(\alpha_{1,1} \lor \tau_2 \lor \beta_2, \alpha_{2,2}) \leq \delta_2 < \frac{1}{2^4} \]

it follows from (5.3) that \( E_2^{1/2^3} \in (\alpha_{2,2})^T \). Since \( \delta_2 < \mu(A) \) for all finite atoms \( A \) of \( (\alpha_{1,1} \lor \tau_2 \lor \beta_2)^{1}_{-1} \), we see that \( \mathcal{A}_{(\alpha_{2,2})^{1}_{-1}} = \mathcal{A}_{(\alpha_{1,1} \lor \tau_2 \lor \beta_2)^{1}_{-1}} \) as well as \( \mathcal{A}_{\alpha_{2,2}} = \mathcal{A}_{(\alpha_{1,1} \lor \tau_2 \lor \beta_2)^{1}_{-1}} \). Since, as mentioned above, the \( \beta_2 \)-part of any \( (\alpha_{1,1} \lor \tau_2 \lor \beta_2) \)-name
never changes under the copying procedure, we see that \( \alpha_{2,2} \succeq \beta_2 \) and moreover \( f_{\beta_2,\alpha_{2,2}} = f_{\beta_2,\alpha_{1,1}\vee \tau_2 \vee \beta_2}. \) In virtue of Remark B.1, we obtain that

\[
d((\alpha_{1,1} \vee \tau_2 \vee \beta_2)_{1,-1}, (\alpha_{2,2})_{1,-1}) < 3r_2^3 \delta_2 < \frac{1}{4 \cdot 2^{n_2} r_2^4}.
\]

The intersection \( R_2 \) of the set \( K \) with the union of principal columns of \( t_2 \) which include no good fibers is uniform relative to \( K \), because it is \( \mathcal{F}_{\beta_2} \)-measurable. Applying the same argument as we developed above in order to find \( N'_1 \) in the proof of Lemma 5.3, we obtain an integer \( N'_2 \) with \( N'_2 > N_1 \) such that for any point \( y \in K \),

\[
A \in (\alpha_{2,2})_{1,-1} \cap B_0 \text{ and } T_n 1_K(y) \geq N'_2 \Rightarrow \left\vert \frac{T_n 1_A(y)}{T_n 1_K(y)} - \frac{\mu(A)}{\mu(K)} \right\vert < \frac{1}{2^{n_2} r_2^4}.
\]

Lemma 4.2 allows us to have an \( \mathcal{F}_{\beta_2} \)-measurable refinement \( t'_2 \) of \( t_2 \) with \( h_K(t_2) \geq N'_2 \). We then obtain an integer \( N_2 \) with \( N_2 \geq N'_2 \) for which the last property of the lemma is valid.

Define \( \alpha_{2,1} \) to be a partition whose atom with a subscript \( i \in \mathfrak{A}_{\alpha_{1,1}} \) is the union of atoms of \( \alpha_{2,2} \) with subscripts of the form \((i, *, *) \in \mathfrak{A}_{\alpha_{1,1} \vee \tau_2 \vee \beta_2} = \mathfrak{A}_{\alpha_{2,2}} \). Since the copying procedure changes names on a set of measure \( \delta_2 \) at most, we see that \( d(\alpha_{2,1}, \alpha_{1,1}) \leq \delta_2 \), which together with Lemma 5.3 implies that

\[
E_1^{1/2^3 + 1/2^3} \in (\alpha_{2,1})^T.
\]

Also, it immediately follows from definition of \( \alpha_{2,1} \) that \( \alpha_{2,2} \succeq \alpha_{2,1} \). Since \( f_{\beta_2,\alpha_{2,2}} = f_{\beta_2,\alpha_{1,1}\vee \tau_2 \vee \beta_2}, \beta_2 \succeq \beta_1, \alpha_{2,2} \succeq \beta_2 \) and \( \alpha_{1,1} \succeq \beta_1 \), it is easy to see that

\[
\begin{align*}
f_{\beta_1,\alpha_{2,2}} &= f_{\beta_1,\beta_2} \circ f_{\beta_2,\alpha_{2,2}} = f_{\beta_1,\beta_2} \circ f_{\beta_2,\alpha_{1,1}\vee \tau_2 \vee \beta_2} = f_{\beta_1,\alpha_{1,1}\vee \tau_2 \vee \beta_2} = f_{\beta_1,\alpha_{1,1}} \circ f_{\alpha_{1,1},\alpha_{1,1}\vee \tau_2 \vee \beta_2} \end{align*}
\]

It follows therefore that for all \((i, j, k), (i, j', k') \in \mathfrak{A}_{\alpha_{2,2}} \),

\[
f_{\beta_1,\alpha_{2,2}}(i, j, k) = f_{\beta_1,\alpha_{1,1}}(i) = f_{\beta_1,\alpha_{2,2}}(i, j', k'),
\]

which shows that \( \alpha_{2,1} \succeq \beta_1 \).

Since \( t'_2 \) is a refinement of \( t_1 \), the copying procedure, which is carried out on \( t'_2 \), guarantees that the \( \alpha_{1,1} \)-part of the \( \alpha_{2,2} \)-name of a given fiber of \( t_1 \) coincides with the \( \alpha_{1,1} \)-name of some fiber on a column where the given fiber lies. It follows therefore from definition of \( \alpha_{2,1} \) that the \( \alpha_{2,1} \)-name of any fiber of \( t_1 \) coincides with the \( \alpha_{1,1} \)-name of some fiber of \( t_1 \).

Hence, the partition \( \alpha_{2,1} \) has the \((N_1, 1, 1/2^{n_1})\)-uniformity in virtue of the last property of Lemma 5.3. As a consequence of the last property of Lemma 5.4, we know that the partition \( \alpha_{2,1} \) has the \((N_2, 1, 1/2^{n_2})\)- and \((N_2, 2, 1/2^{n_2})\)-uniformity, because every finite atom of \( \alpha_{2,1} \) (resp. \( (\alpha_{2,1})_{1,-1} \)) is a union of at most \( \#(\alpha_{1,1} \vee \tau_2 \vee \beta_2) \times \#\tau_2 \times \#\beta_2 \) (resp. \( \#(\tau_2 \vee \beta_2) \))^2 \) finite atoms of \( (\alpha_{2,2})_{1,-1} \). Similarly, the partition \( \alpha_{2,2} \) has the \((N_2, 1, 1/2^{n_2})\)-uniformity.

Continuing the inductive steps, we will find triangular array:

\[
\{ \alpha_{i,j} \mid i \in \mathbb{N}, 1 \leq j \leq i \}
\]
of partitions of \( Y \), sequences \( 1 \leq n_1 < n_2 < \ldots \) and \( 1 < N_1 < N_2 < \ldots \) of integers, and refining sequence \( \{ t_i \mid i \in \mathbb{N} \} \) of towers so that for each \( i \in \mathbb{N} \),
(i) \( \alpha_{i,1} \leq \alpha_{i,2} \leq \cdots \leq \alpha_{i,i} \);
(ii) \( \beta_j \leq \alpha_{i,j} \) for each integer \( j \) with \( 1 \leq j \leq i \);
(iii) \( E_j \leq (\alpha_{i,j})^T \) if \( 1 \leq j \leq i \), where \( \epsilon_{i,j} = \sum_{k=j}^{i} 2^{-(k+1)} = 2^{-j} - 2^{-(i+1)} \);
(iv) \( d(\alpha_{i,j}, \alpha_{i,j+1}) < 2^{-n_i + 1} \) for each integer \( j \) with \( 1 \leq j \leq i \);
(v) \( t_i \) is \( \mathcal{F}_{\beta_i} \)-measurable;
(vi) for each integer \( j \) with \( 1 \leq j \leq i \), the \( \alpha_{i,j} \)-name of any fiber of \( t_j \) coincides with the \( \alpha_{j,j} \)-name of some fiber of \( t_j \),
(vii) if a partition \( \alpha \) of \( Y \) satisfies that the \( \alpha \)-name of any fiber of \( t_i \) coincides with the \( \alpha_{i,i} \)-name of some fiber of \( t_i \), then the partition \( \alpha \) has the \((N_i, i, 1/(2^n r_i^2))\)-uniformity, where

\[
\begin{cases}
\# \alpha_{i,i} & \text{if } i \neq 1; \\
1 & \text{if } i = 1.
\end{cases}
\]

Lemma 5.5 (Consequence of the inductive steps). There exist a sequence \( \alpha_1 \leq \alpha_2 \leq \cdots \) of strictly uniform partitions of \( Y \) satisfying that \( \beta_j \leq \alpha_j \) and \( E_j^{1/2^j} \in (\alpha_j)^T \) for every \( j \in \mathbb{N} \).

Proof. It follows from (iv) that for each \( j \in \mathbb{N} \), the sequence \( \{ \alpha_i \}_{i \in \mathbb{N}} \) is a Cauchy sequence in \( d \). For each \( j \in \mathbb{N} \), let \( \alpha_j \) denote a partition of \( Y \) for which

\[
\lim_{i \to \infty} d(\alpha_j, \alpha_{i,j}) = 0.
\]

It follows from (i) and (ii) that for each \( j \in \mathbb{N} \), \( \alpha_j \leq \alpha_{j+1} \) and \( \beta_j \leq \alpha_j \), respectively. It follows from (5.6) and (vi) that for each \( j \in \mathbb{N} \), the \( \alpha_j \)-name of any fiber of \( t_j \) coincides with the \( \alpha_{j,j} \)-name of some fiber of \( t_j \). This together with (vii) implies that each \( \alpha_j \) has the \((N_j, j, 1/(2^n r_j^2))\)-uniformity.

Now, fix \( j \in \mathbb{N} \). Remark that \( \# \alpha_i = r_i \) for every integer \( i \) with \( i \geq 2 \). Let \( \epsilon > 0 \) and \( k \in \mathbb{N} \). There exists \( i \in \mathbb{N} \) with \( i \geq j \lor k \) for which \( 1/2^i < \epsilon \). Since every finite atom of \((\alpha_j)_{k+1}^{-1} \) is the union of at most \( r_i 2^{-i} \) finite atoms of \((\alpha_i)_{i+1}^{-1} \), the partition \( \alpha_j \) has the \((N_i, k, \epsilon)\)-uniformity. Thus, the partition \( \alpha_j \) is uniform since \( k \) and \( \epsilon \) are arbitrary. Moreover, the partition \( \alpha_j \) is strictly uniform, because \( \beta_j \leq \alpha_j \) and \( \beta_j \) satisfies (3.1).

The second property of the lemma is a consequence of (iii) and (5.6). \( \square \)

Let \( \alpha_1, \alpha_2, \ldots \) be strictly uniform partitions of \( Y \) as in Lemma 5.5. Let \( \hat{X} = (\hat{X}, \hat{S}) \) denote an inverse limit of an inverse system \( (\hat{X}_{\alpha_i}, \hat{S}_{\alpha_i}, \phi_{\alpha_i, \alpha_{i+1}}) \) for every \( i \in \mathbb{N} \), i.e.

\[
\hat{X} = \left\{ \left( x_i \right)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} \hat{X}_{\alpha_i} \mid x_i = \phi_{\alpha_i, \alpha_{i+1}}(x_{i+1}) \text{ for all } i \in \mathbb{N} \right\},
\]

which is a Cantor set under the relative topology induced by the product topology on \( \prod_{i \in \mathbb{N}} \hat{X}_{\alpha_i} \), and \( \hat{S} \) is a homeomorphism on \( \hat{X} \) defined by \( (x_i)_{i \in \mathbb{N}} \mapsto (\hat{S}_{\alpha_i}(x_i))_{i \in \mathbb{N}} \). Since it follows from Lemma 3.3 that the subshift \((\hat{X}_{\alpha_i}, \hat{S}_{\alpha_i})\) is almost minimal for every \( i \in \mathbb{N} \), it is readily verified that the system \( \hat{X} \) is almost minimal; see for more details [20, Lemma 3.1]. Let \( X = (X, S) \) denote a minimal, locally compact Cantor system whose one-point compactification is the almost minimal system \( \hat{X} \).

Since \( \lambda_{\alpha_{i+1}} \circ (\phi_{\alpha_i, \alpha_{i+1}})^{-1} = \lambda_{\alpha_i} \) for each \( i \in \mathbb{N} \), Kolmogorov’s extension theorem [15, 18] for infinite measures allows us to have a unique, \( \sigma \)-finite measure \( \hat{\lambda} \) on the
Borel σ-algebra of $X$ satisfying a condition that $\lambda \circ p_{\alpha_i}^{-1} = \lambda_{\alpha_i}$ for all $i \in \mathbb{N}$, where $p_{\alpha_i}$ is the projection from $X$ to $X_{\alpha_i}$. The condition implies that $\lambda$ is $\hat{S}$-invariant. The restriction $\lambda$ of $\hat{\lambda}$ to the Borel σ-algebra of $X$ is a Radon measure, because a family:

$$
\bigcup_{i \in \mathbb{N}} \{ p_{\alpha_i}^{-1}(E) \mid E \text{ is a compact and open subset of } X_{\alpha_i} \}
$$

is a base for the topology of $X$ and $\hat{\lambda}(p_{\alpha_i}^{-1}(E)) = \lambda_{\alpha_i}(E) < \infty$ for all compact and open subset $E$ of $X_{\alpha_i}$ and $i \in \mathbb{N}$. Since it follows from Lemma 5.3 again that for each $i \in \mathbb{N}$, the almost minimal system $(X_{\alpha_i}, \hat{S}_{\alpha_i})$ is bi-ergodic, it is readily verified that so is $(\hat{X}, \hat{S})$; see for more details [20, Lemma 3.2]. Let $\hat{A}$ denote the completion of the Borel σ-algebra of $X$ with respect to $\hat{\lambda}$.

Define a measurable map $\theta : Y \to \hat{X}, y \mapsto (\phi_{\alpha_i}(y))_{i \in \mathbb{N}}$. Since $\phi_{\alpha_i} = p_{\alpha_i} \circ \theta$ for all $i \in \mathbb{N}$, we obtain that $(\mu \circ \theta^{-1}) \circ p_{\alpha_i}^{-1} = \mu \circ \phi_{\alpha_i}^{-1} = \lambda_{\alpha_i}$ for all $i \in \mathbb{N}$, so that $\mu \circ \theta^{-1} = \hat{\lambda}$ in virtue of the uniqueness of Kolmogorov’s extension theorem, i.e. the map $\theta$ is measure-preserving. The map $\theta$ is injective, because Lemma 5.5 shows that $E_i \in (\alpha_i)^T$ for every $i \in \mathbb{N}$ and because each member of the base $E$ appears infinitely often in $E$. Since in view of (3.2), the image of each $\phi_{\alpha_i}$ is written as the intersection of a decreasing sequence of those finite unions of cylinder subsets which have full measure with respect to $\hat{\lambda}_{\alpha_i}$, the image is measurable and has full measure with respect to $\hat{\lambda}_{\alpha_i}$. This implies that the image of the map $\theta$ has full measure with respect to $\hat{\lambda}$, so that the map $\theta$ is surjective. At last, we know that $\theta$ is an isomorphism between $(Y, B, \mu, T)$ and $(\hat{X}, \hat{A}, \hat{\lambda}, \hat{S})$, because for any point $y \in Y$,

$$
\theta \circ T(y) = (\phi_{\alpha_i}(T(y)))_{i \in \mathbb{N}} = (\hat{S}_{\alpha_i}, \phi_{\alpha_i}(y))_{i \in \mathbb{N}} = \hat{S} \circ \theta(y).
$$

In a similar way as above, we can see that the inverse limit $\hat{W} = (\hat{W}, \hat{V})$ of an inverse system $\big\{ (X_{\beta_i}, \hat{S}_{\beta_i}, \phi_{\beta_i, \beta_{i+1}}) \big\}_{i \in \mathbb{N}}$ is a bi-ergodic, almost minimal Cantor system. Since $\hat{X}_{\beta_i} = X_{\gamma_i}$ and $\phi_{\beta_i, \beta_{i+1}} = \phi_{\gamma_i, \gamma_{i+1}}$ for all $i \in \mathbb{N}$, the system $\hat{W}$ is identical with the inverse limit of an inverse system $\big\{ (X_{\gamma_i}, \hat{S}_{\gamma_i}, \phi_{\gamma_i, \gamma_{i+1}}) \big\}_{i \in \mathbb{N}}$. Let $\hat{\xi}$ denote a unique, $\sigma$-finite measure $\hat{\xi}$ satisfying that $\hat{\nu} \circ p_{\beta_i}^{-1} = \hat{\lambda}_{\gamma_i}$ for all $i \in \mathbb{N}$. The measure $\hat{\xi}$ is $\hat{V}$-invariant. Remark that for all $i \in \mathbb{N}$, $\hat{\lambda}_{\beta_i} = \hat{\lambda}_{\gamma_i}$, in virtue of a fact that $\pi$ is measure-preserving and $p_{\beta_i} = p_{\gamma_i}$ by definition. Define a map $\iota : \hat{Z} \to \hat{W}, z \mapsto (\phi_{\gamma_i}(z))_{i \in \mathbb{N}}$, which is continuous because $\iota^{-1}(p_{\gamma_i}^{-1}([u, v]_{X_{\gamma_i}})) = \phi_{\gamma_i}^{-1}([u, v]_{X_{\gamma_i}})$ is open for all words $u$ and $v$ satisfying that $uv \in \mathcal{L}(\gamma_i)$. The map $\iota$ is injective in virtue of the second property of Lemma 4.14 and surjective because the inverse image $\iota^{-1}\{ (x_i)_{i \in \mathbb{N}} \}$ of a given point $(x_i)_{i \in \mathbb{N}} \in \hat{W}$ is written as the intersection of decreasing sequence $\{ \phi_{\gamma_i}^{-1}(x_i) \}_{i \in \mathbb{N}}$ of nonempty, compact subsets of $\hat{Z}$. It follows that $\iota$ is a homeomorphism. It is now readily verified that $\iota$ is an isomorphism between bi-ergodic, almost minimal systems $\hat{Z}$ and $\hat{W}$. We obtain that $\hat{\nu} \circ \iota^{-1} = \hat{\xi}$ because $(\hat{\nu} \circ \iota^{-1}) \circ p_{\alpha_i}^{-1} = \hat{\nu} \circ \phi_{\alpha_i}^{-1} = \hat{\lambda}_{\alpha_i}$ for all $i \in \mathbb{N}$.

Define a continuous map $\kappa : \hat{X} \to \hat{W}, (x_i)_{i \in \mathbb{N}} \mapsto (\phi_{\beta_i, \alpha_i}(x_i))_{i \in \mathbb{N}}$, which is verified to be surjective in virtue of a fact that for all $i \in \mathbb{N}$,

$$
\phi_{\beta_i, \beta_{i+1}} \circ \phi_{\beta_{i+1}, \alpha_i} = \phi_{\beta_i, \alpha_i} \circ \phi_{\alpha_i, \alpha_{i+1}}.
$$
By using a fact that for all \( i \in \mathbb{N} \),
\[
\phi_{\beta_i, \alpha_i} \circ S_{\kappa_i} = S_{\beta_i} \circ \phi_{\beta_i, \alpha_i},
\]
we see that \( \kappa \) is a factor map from \( \hat{X} \) to \( \hat{W} \). The map \( \kappa \) is measure-preserving because for all \( i \in \mathbb{N} \),
\[
(\lambda \circ \kappa^{-1}) \circ p_{\beta_i}^{-1} = \hat{\lambda} \circ p_{\alpha_i}^{-1} \circ (\phi_{\beta_i, \alpha_i})^{-1} = \hat{\lambda}_{\alpha_i} \circ (\phi_{\beta_i, \alpha_i})^{-1} = \hat{\lambda}_{\beta_i}.
\]

We obtain that for any point \( y \in Y \),
\[
\kappa \circ \theta(y) = (\phi_{\beta_i, \alpha_i}(\phi_{\alpha_i}(y)))_{i \in \mathbb{N}} = (\phi_{\beta_i}(y))_{i \in \mathbb{N}} = (\phi_{\gamma_i}(\pi(y)))_{i \in \mathbb{N}} = \iota(\pi(y)).
\]

Putting \( \rho = (\iota^{-1} \circ \kappa)|_X \) completes the proof of Theorem 1.1.

6. PROOF OF THEOREM 1.2

In view of Theorem 1.1, it is sufficient to prove the last assertion of Theorem 1.2. We shall now start the proof with the following lemma.

**Lemma 6.1.** Suppose that \( X_i = (X_i, S_i) \) is a locally compact Cantor system for each \( i \in \{1, 2, 3\} \). Suppose that \( p_i \) is a proper factor map from \( X_i \) to \( X_i \) for each \( i \in \{2, 3\} \). If \( X_1 \) is strictly ergodic and \( p_2^{-1}(U_2) \cap p_3^{-1}(U_3) \neq \emptyset \) for all nonempty open sets \( U_2 \subset X_2 \) and \( U_3 \subset X_3 \), then a locally compact Cantor system \( X_2 \times X_3 := (X_2 \times X_3, S_2 \times S_3) \) is strictly ergodic.

**Proof.** For each \( i \in \{1, 2, 3\} \), let \( \hat{X}_i = (\hat{X}_i, \hat{S}_i) \) be the one-point compactification of \( X_i \) and \( \hat{X}_i = X_i \cup \{ \omega_i \} \). We have another one-point compactification \( \hat{X}_2 \times \hat{X}_3 = (X_2 \times X_3) \cup \{ (\omega_2, \omega_3) \} \) of a locally compact Cantor set \( X_2 \times X_3 \), which gives a topological dynamical system \( \hat{X}_2 \times \hat{X}_3 = (X_2 \times X_3, S_2 \times S_3) \). Define a map \( \hat{p}_2 \times \hat{p}_3 : \hat{X}_1 \to \hat{X}_2 \times \hat{X}_3 \) by
\[
\hat{p}_2 \times \hat{p}_3(x) = \begin{cases} (p_2(x), p_3(x)) & \text{if } x \in X_1; \\ (\omega_2, \omega_3) & \text{if } x = \omega_1. \end{cases}
\]

Since the maps \( X_1 \to X_1 \times X_1, x \mapsto (x, x) \) and \( X_1 \times X_1 \to X_2 \times X_3, (x, x') \mapsto (p_2(x), p_3(x')) \) are both proper, so is their composition \( p_2 \times p_3 : X_1 \to \hat{X}_2 \times \hat{X}_3, x \mapsto (p_2(x), p_3(x)). \) Hence, the map \( p_2 \times p_3 \) is continuous. If \( p_2 \times p_3 \) is not surjective, then there exist a nonempty open subset \( U \subset \hat{X}_2 \times \hat{X}_3 \) which is disjoint from the point \( (\omega_2, \omega_3) \), such that \( \hat{p}_2 \times \hat{p}_3^{-1}(U) = \emptyset \). However, it is clearly impossible in view of the hypothesis of the lemma. Hence, the map \( p_2 \times p_3 \) is a factor map from \( \hat{X}_1 \) to \( \hat{X}_2 \times \hat{X}_3 \). Corollary 2.3 completes the proof.

Assume that \( Y = (Y, \mathcal{B}, \mu, T) \) is an ergodic, infinite measure-preserving system whose cartesian product \( Y \times Y \) with itself is ergodic; see [12] for the existence of such systems. In Figure 4, let \( Y_2 = Y_3 = Y, Y_1 = Y_2 \times Y_3 \) and \( q_i : Y_1 \to Y_i \), the projection for each \( i \in \{2, 3\} \). Let \( \text{gr}(\mu, \text{id}) \) denote the diagonal measure on \( Y \times Y \), i.e. \( \text{gr}(\mu, \text{id})(A \times B) = \mu(A \cap B) \) for all sets \( A, B \in \mathcal{B} \).

Assume now that the diagram in Figure 4 has a strictly ergodic, locally compact Cantor model:

```
    X1
   /   \   \\
  X2----X3
    \   /   \
      p3  p2
```

For each $i \in \{1, 2, 3\}$, let $\sigma_i$ denote the relevant isomorphism from $Y_i$ to $X_i := (X_i, S_i)$. Suppose that $U_i \subset X_i$ is nonempty and open for each $i \in \{2, 3\}$. Since $\mu(\sigma_i^{-1}(U_i)) > 0$ for each $i \in \{2, 3\}$, we know that

$$(\mu \times \mu)(q_2^{-1}q_2^{-1}(U_2) \cap q_3^{-1}q_3^{-1}(U_3)) > 0,$$

and hence, $p_2^{-1}(U_2) \cap p_3^{-1}(U_3) \neq \emptyset$. It follows from Lemma [6.1] that $X_2 \times X_3$ is strictly ergodic. This leads to a contradiction that $\mu \times \mu$ coincides, up to a positive constant multiple, with the diagonal measure $\text{gr}((\mu, \text{id}))$. This completes the proof of Theorem [1.2].

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