COMPLEX OPTIMAL TRANSPORT AND THE PLURIPOTENTIAL THEORY OF KÄHLER-RICCI SOLITONS

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Abstract. Let $(X, L)$ be a (semi-) polarized complex projective variety and $T$ a real torus acting holomorphically on $X$ with moment polytope $P$. Given a probability density $g$ on $P$ we introduce a new type of Monge-Ampère measure $MA_g(\phi)$ on $X$, defined for singular $T$-invariant metrics $\phi$ on the line bundle $L$, generalizing the ordinary Monge-Ampère of global pluripotential theory, which corresponds to the case when $T$ is trivial (or $g = 1$). In the opposite extreme case when $T$ has maximal rank, i.e. $(X, L, T)$ is a toric variety, the solution $\phi$ of the corresponding Monge-Ampère equation $MA_g(\phi) = \mu$ corresponds to the convex Kantorovich potential for the optimal transport map in the Monge-Kantorovich transport problem between $\mu$ and $g$ (with a quadratic cost function). Accordingly, our general setting can be seen as a complex version of optimal transport theory. Our main complex geometric applications concern the pluripotential study of singular (shrinking) Kähler-Ricci solitons. In particular, we establish the uniqueness of such solitons, modulo automorphisms, and explore their relation to a notion of modified K-stability inspired by the work of Tian-Zhu. The quantization of this setup, in the sense of Donaldson, is also studied.

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1. Introduction

1.1. Background and motivation. Ever since the seminal work of Yau [58] and Aubin [2] on Kähler-Einstein metrics on complex manifolds, i.e. Kähler metrics with constant Ricci curvature, complex Monge-Ampère equations have played a central role in complex geometry. Recall that a Riemannian metric on a complex manifold $X$ with complex structure $J$ is said to be Kähler when it can be written as $\omega(\cdot, J\cdot)$ for a closed two-form $\omega$ on $X$, which equivalently means that $\omega$ can be locally written as

$$\omega = \omega_\phi := \frac{i}{2\pi} \partial \bar{\partial} \phi,$$

for a local function $\phi$, which is strictly plurisubharmonic, i.e. the complex Hessian $\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}$ is positive. In the case when $\omega$ has integral periods, i.e. $[\omega] \in H^2(X, \mathbb{Z})$, the
local functions $\phi$ patch to define a Hermitian metric on a positive/ample line bundle $L$ with curvature form $\omega_\phi$, representing the first Chern class $c_1(L)$ in $H^2(X, \mathbb{Z})$. In particular, Yau’s solution of the Calabi conjecture concerning the existence of a Ricci flat Kähler metric $\omega$ on a Calabi-Yau manifold $X$ amounts (in the case when $[\omega] \in H^2(X, \mathbb{Z})$) to the solvability of a complex Monge-Ampère equation, which in local notation may be formulated as

\begin{equation}
\det \left( \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} \right) = f
\end{equation}

for $f$ a given positive smooth density on the $n$–dimensional complex manifold $X$ (similarly, the equation for non-Ricci flat Kähler-Einstein metrics is obtained by replacing $f$ with $e^{\pm \phi}$). In global terms the equation (1.1) thus prescribes the volume form, i.e. the top exterior power $\omega^n_\phi$ of the Kähler metric $\omega_\phi$.

Subsequently Bedford and Taylor [3] developed the local pluripotential theory which, in particular, furnishes a notion of weak solution to the highly non-linear complex Monge-Ampère equation is a (singular) Kähler-Ricci soliton on $X$ if the metric (1.1) by making sense of the the wedge products $(i\partial \bar{\partial} \phi)^n$, as long as $i\partial \bar{\partial} \phi$ is positive in the sense of currents and $\phi$ is locally bounded. The local work of Bedford-Taylor and its extension to compact Kähler manifolds by Kolodziej [33] and Guedj-Zeriahi [30] was generalized to a very general global complex geometric framework in [16]. In particular, using the non-pluripolar product of positive currents defined in [16] this allows one to define the complex Monge-Ampère measure

$$MA(\phi) := \left( \frac{i}{2\pi} \partial \bar{\partial} \phi \right)^n$$

for any (possibly singular) positively curved metric $\phi$ on a big line bundle $L \to X$. As shown in [16] this leads to very general existence and uniqueness results for global complex Monge-Ampère equations of the form (1.1) by using pluripotential capacity techniques to reduce the situation to the original setting of Aubin and Yau.

A direct variational approach to complex Monge-Ampère equations and Kähler-Einstein metrics was recently introduced in [16, 13, 6], which can be seen as a non-linear version of the classical Dirichlet energy variational principle for the Laplace equation on a Riemann surface.

There is also a real version of this story. Indeed, as is well-known, in the case when the plurisubharmonic function $\phi(z)$ is independent of the imaginary part of $z$ - i.e. $\phi(z) = \varphi(x)$ for a convex function $\varphi$ on $\mathbb{R}^n$ - the complex Monge-Ampère measure may be identified with the real Monge-Ampère measure $MA_{\mathbb{R}}(\varphi)$ of the convex function $\varphi$. The latter measure was geometrically defined in the classical works by Alexandrov and Pogorelov by using the multivalued map defined by the subgradient $\partial \varphi$ of $\varphi$ : the mass $MA(\varphi)(E)$ of a Borel set $E$ in $\mathbb{R}^n$ is the Lebesgue volume of the image of $E$ under $\partial \varphi$. This real situation appears naturally in the global complex geometric framework when $(X, L)$ is toric, i.e. when there is an action of the full complex torus $T^c$ on $(X, L)$ - then any metric $\phi$ on $L$ which is invariant under the action of the corresponding real torus $T^n$ is naturally identified with a convex function $\varphi$ on $\mathbb{R}^n$. However, an important flexibility which arises in the setting of the real Monge-Ampère equation is that, for any given (say, continuous) non-negative function $g(p)$ on the space $\mathbb{R}^n$, invariantly viewed as the dual real vector space, the product

\begin{equation}
MA_g(\varphi) : = MA(\varphi)_{\mathbb{R}} g(\nabla \varphi)
\end{equation}
is well-defined as a measure, as long as $\varphi$ is convex (indeed, compared with the previous definition one simply replaces the Lesbegue measure $dp$ on the dual $\mathbb{R}^n$ by $g(p)dp$). In particular, this situation appears naturally in the theory of optimal (mass) transport, originating in the classical works of Monge and Kantorovich. The point is that the corresponding Monge-Ampère equation

$$MA_\varphi(f) = \int f\,dx$$

is equivalent to the mass density $f(x)$ being transported optimally by the $L_\infty$-map defined by gradient $\nabla \varphi$ to the density $g(p)dp$:

$$(\nabla \varphi)_*(f\,dx) = g(p)dp,$$

As shown by Brenier [18] the existence of such a convex function $\varphi$ follows from variational considerations (involving Kantorovich duality), using that $\varphi$ minimizes the Kantorovich cost functional. In fact, as observed in [9] the complex and real variational approaches in [13] and [18], respectively, are essentially equivalent in the toric setting. In particular, the cost in the real setting corresponds, in the complex setting, to energy.

One of the main general aims of the present paper is to consider the general situation where a torus $T$ - not necessarily of maximal rank - acts on $(X, L)$ and define a generalized version of the $g$–Monge-Ampère measure in formula (1.2) for any $T$–invariant, possibly singular, metric $\phi$ on $L$ with positive curvature current. This situation can thus be seen as a hybrid of the complex and the real settings for the Monge-Ampère equation and leads to a complex generalization of the theory of optimal transport (this point will be expanded on elsewhere). Our starting point is the basic observation that for a smooth metric $\phi$ on $L$ there is natural generalization of the gradient map, namely the moment map $m_\phi$ determined by the metric $\phi$ in terms of symplectic geometry, which defines a map from $X$ into the dual of the Lie algebra of $T$ (depending on the local first derivatives of $\phi$ along the torus orbits). Inspired by some ideas originating in some recent work on the connection between filtrations and pluripotential theory on one hand and test configurations and geodesic rays in Kähler geometry on the other [53, 40, 32], we are led to a canonical definition of the generalized $g$–Monge-Ampère measure, which has good continuity properties.

Our main motivation for developing this general framework is to provide a pluripotential theoretic notion of a weak solution to the Kähler-Ricci soliton equation on a, possibly singular, Fano variety $X$, where $L$ is the anti-canonical line bundle $-K_X$, i.e. the top exterior power of the holomorphic tangent bundle of $X$. In the ordinary smooth case a Kähler metric $\omega$ is a Kähler-Ricci soliton precisely when it can be written as the curvature form $\omega_\phi$ of a smooth metric $\phi$ on $-K_X$, locally satisfying the complex Monge-Ampère equation

$$\det\left(\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}\right)e^{f_\phi} = e^{-\phi},$$

where $f_\phi$ is the Hamiltonian function corresponding to the imaginary part of a holomorphic vector field $V$ on $X$, generating an action of a real torus $T$ on $(X, -K_X)$ (i.e. the orbits of the torus $T$ coincide with the closure of the orbits of flow of the imaginary part of $V$). Since the Hamiltonian may be expressed as $f_\phi = \langle m_\phi, \xi \rangle$, where $\xi$ is the element in the Lie algebra of $T$ corresponding to $V$, the right hand side in the equation (1.3) is the density of $MA_{g_V}(\phi)$ for $g_V(\cdot) = \exp \langle \cdot, \xi \rangle$. This
observation will allow us to define the notion of a weak solution $\phi$ of the Kähler-Ricci soliton equation 1.3, a notion which turns out to be very useful for both uniqueness and existence problems, as well as convergence problem. We also develop a general “quantized” (finite dimensional) version of the $g$–Monge-Ampère setting, which generalizes Donaldson’s setting of balanced metrics introduced in [25] and, in particular, leads to a new finite dimensional analog of a Kähler-Ricci soliton.

1.2. Pluripotential theory of moment maps and $g$–Monge-Ampère equations. Let $L$ be a holomorphic line bundle over a compact $n$–dimensional complex manifold $X$ and assume that $(X, L)$ comes with a holomorphic action of a real torus $T$ of rank $m$ (which equivalently means that the $T$–action on $X$ is Hamiltonian). To fix ideas first assume that $L$ is ample. Then any $T$–invariant smooth positively curved metric $\phi$ on $L$ induces, via the symplectic form $\omega_\phi$ defined by the curvature of $\phi$, a moment map

$$m_\phi : X \to \mathbb{R}^m, \quad P := m_\phi(X)$$

for the $T$–action, where we have identified the Lie algebra of $T$ (and its dual) with $\mathbb{R}^m$ in the standard way. As is well-known the image $P$ is compact and independent of $\phi$ - more precisely, $P$ is a convex polytope and thus usually referred to as the moment polytope - and coincides with the support of the corresponding (normalized) Duistermaat-Heckman measure

$$\nu := (m_\phi)_* MA(\phi), \quad MA(\phi) := \frac{1}{c_1(L)^n} \omega_\phi^n$$

defining a probability measure on $\mathbb{R}^m$, which is absolutely continuous with respect to Lebesgue measure and independent of $\phi$ [22].

More generally, this setup applies as long as $L$ is semi-positive and big, i.e. $L$ admits some smooth metric $\phi$ with non-negative curvature form $\omega_\phi$ of positive total volume, which will be assumed henceforth (in particular, by passing to a resolution $X$ may be allowed to be singular). Given a continuous non-negative function $g$ on $\mathbb{R}^n$, or rather on $P$ (usually normalized so that $g\nu$ is a probability measure) we will write

$$MA_g(\phi) := MA(\phi) g(m_\phi), \quad (1.4)$$

which defines, for any smooth and $T$–invariant non-negatively curved metric $\phi$, a measure on $X$ which will be referred to as the $g$–Monge-Ampère measure (or the $g$–modified Monge-Ampère measure). As explained above one of the main points of the present paper is to extend the definition of $MA_g(\phi)$ to the space of all (possibly singular) $T$–invariant metrics on $L$ with positive curvature current and show that it has the same good continuity properties as in the standard case when $g = 1$.

**Theorem 1.1.** Let $L \to X$ be a line bundle with an action of a real torus $T$, as above, and $g$ a continuous function on the corresponding moment polytope $P$. Then there exists a unique extension of the smooth $g$–Monge-Ampère measure $MA_g(\phi)$ defined by formula (1.4) to the space of all $T$–invariant (possibly singular) metrics $\phi$ on $L$ with positive curvature current and show that it has the same good continuity properties as in the standard case when $g = 1$.

- If $\phi_j$ is a sequence of metrics decreasing to a locally bounded metric $\phi$, then the corresponding measures $MA_g(\phi_j)$ converge weakly to $MA_g(\phi)$ on $X$.
- The measure $MA_g(\phi)$ does not charge pluripolar subsets of $X$. 

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The measure $MA_g(\phi)$ is local with respect to the $T$–plurifine topology on $X$.

Moreover, the following properties also hold:

- $\int X MA_g(\phi) \leq \int P g d\nu$ with equality if and only if the metric $\phi$ has full Monge-Ampère mass (i.e. $\int_X MA(\phi) = 1$).
- The convergence statement in the first point above more generally holds for any decreasing sequence of singular metrics converging to a metric $\phi$ of full Monge-Ampère mass.

In particular, the previous theorem implies that for any bounded decreasing sequence $\phi_j$ of smooth positively curved metrics on $L$ the corresponding measures $MA(\phi_j)g(m_{\phi_j})$ have a unique weak limit on $X$, which seems hard to prove directly even if the limiting metric is smooth.

With the previous theorem in hand it is rather straightforward to adapt the variational approach in [13] to prove the following result, generalizing the case when $T$ is trivial considered in [13, 16, 33]:

**Theorem 1.2.** Let $\mu$ be a probability measure on $X$ which is $T$–invariant and assume that $g d\nu$ is a probability measure on the moment polytope $P$. Then $\mu$ does not charge pluripolar sets iff there exists a metric $\phi$ on $L$ with positive curvature current such that

$$MA_g(\phi) = \mu$$

Moreover, the following is equivalent if $g$ is bounded from below by a positive constant:

- The measure $\mu$ has finite (pluricomplex) energy
- The solution $\phi$ has finite (pluricomplex) energy

and in the finite energy case any solution $\phi$ is unique modulo constants. In particular, if $\mu$ has a density $f$ in $L^p(X)$, for some $p > 1$, then the solution $\phi$ is continuous.

The notion of pluricomplex energy is recalled in sections 2.4, 2.6. In the other extreme case, i.e. when $T$ has maximal rank $n$, so that $(X, L)$ is a polarized toric variety, the previous result, concerning the finite energy measures $\mu$, is essentially equivalent to the existence and uniqueness result of Brenier for optimal transport maps [18]. The point is that in this situation finite energy corresponds to finite cost (see [10, 9]). Moreover, the case of a general probability measure $\mu$ can, in the toric situation, be seen as a variant of a result of McCann [36] (who also proves uniqueness).

**1.3. Applications to Kähler-Ricci solitons.** Recall that Hamilton’s Ricci flow emanating from an initial Kähler metric $\omega_0$ on a complex manifold $X$ preserves the Kähler property and can thus (after normalization) be written as the Kähler-Ricci flow

$$\frac{d\omega_t}{dt} := -\text{Ric} \, \omega_t + \omega_t$$

which exists for any positive time $t$ [19] (here $\text{Ric} \, \omega$ denotes, as usual, the Ricci curvature form of a Kähler metric $\omega$). The fixed points of the (normalized) Kähler-Ricci flow are Kähler-Einstein metrics with positive Ricci curvature and more generally, the fixed points of the induced flow on the space of all Kähler metrics modulo automorphisms correspond to (shrinking) Kähler-Ricci solitons on $X$, i.e. a Kähler
metric $\omega$ on $X$ such that there exists a complex holomorphic vector field $V$ on $X$ with the property that

$$\text{Ric} \, \omega = \omega + L_V \omega,$$

where $L_V$ denotes the Lie derivative of $\omega$ with respect to $V$. In particular, the metric $\omega$ is invariant under the flow generated by the imaginary part $\text{Im}V$ of $V$, which equivalently means that $\text{Im}V$ generates a Hamiltonian action of a torus $T$ on $X$ and $L_V \omega = dd^c f$, where $f$ is a Hamiltonian function for the flow of $\text{Im}V$. As a consequence any manifold $X$ carrying a Kähler-Ricci soliton is Fano, i.e. the anti-canonical line bundle $-K_X := \Lambda^nTX$ is positive/ample (since $\text{Ric} \, \omega$ represents the first Chern class $c_1(-K_X)$). According to the Hamilton-Tian conjecture [45, 52] the Kähler-Ricci flow $\omega_t$ on a Fano manifold $X$ defined by $\omega_t$ always (sub)converges in the Gromov-Hausdorff topology to a singular generalization of a Kähler-Ricci soliton $\omega_{\infty}$ defined on a singular normal Fano variety $X_{\infty}$, which is a complex deformation of the original Fano manifold $X$.

Motivated by the Hamilton-Tian conjecture and in particular its relation to K-stability (see below) we will in this paper initiate the pluripotential study of Kähler-Ricci solitons. First note that there is a natural differential geometric definition of a Kähler-Ricci soliton $\omega$ on a singular (normal) Fano variety $X$ (compare [14] for the Kähler-Einstein case): the Kähler metric $\omega$ is defined on the regular part $X_{\text{reg}}$ of $X$, where it solves the equation 1.5 for some holomorphic vector field $V$ on $X_{\text{reg}}$ and moreover the volume of the metric $\omega$ on $X_{\text{reg}}$ is maximal in the sense that it coincides with the global algebraic top intersection number $c_1(-K_X)^n$, i.e. the degree of the Fano variety $X$ (abusing terminology slightly we will also refer to the corresponding pair $(\omega, V)$ as a Kähler-Ricci soliton). Our first result reveals, in particular, that this definition coincides with various a priori stronger definitions previously proposed in the literature [39, 56, 52] (that impose some information on the singularities of $X$ and on $\omega$ along the singular locus of $X$):

**Theorem 1.3.** Let $X$ be a normal Fano variety admitting a Kähler-Ricci soliton $\omega$ such that the imaginary part of the corresponding holomorphic vector field $V$ generates a torus $T$. Then $X$ has log terminal singularities and the Kähler metric $\omega$, originally defined on the regular locus $X_{\text{reg}}$, extends to a unique positive current $\overline{\omega}$ on $X$ in $c_1(-K_X)$ with continuous potentials. More precisely, $\overline{\omega}$ is the curvature current of a continuous $T$-invariant metric $\phi$ on the line bundle $L := -K_X$ satisfying the global $\text{g}_V$–Monge-Ampère equation on $X$ corresponding to the equation 1.5. Conversely, any (singular) $T$–invariant metric $\phi$ on $-K_X$ with positive curvature current and full Monge-Ampère mass which is a weak solution to the equation 1.5 is continuous and has a curvature current which is smooth on $X_{\text{reg}}$, satisfying the equation 1.5 there. In the case that $X$ is a priori assumed to have log terminal singularities the existence of the torus $T$ is automatic.

It should be stressed that the regularity statement concerning the solution $\phi$ in the previous theorem is new even when $X$ is smooth. Our next result extends the Tian-Zhu uniqueness theorem for Kähler-Ricci solitons [47, 48] (which in turn generalizes the Bando-Mabuchi uniqueness theorem for Kähler-Einstein metrics) to the singular setting.

**Theorem 1.4.** A Kähler-Ricci soliton $(\omega, V)$ on a Fano variety is unique modulo the action of the group $\text{Aut}(X)_0$ of all holomorphic automorphism of $X$ homotopic...
to the identity. Moreover, a Kähler-Ricci soliton $\omega$ defined with respect to a fixed holomorphic vector field $V$, is unique modulo the subgroup $\text{Aut}(X, V)_0$ of $\text{Aut}(X)_0$ consisting of all automorphism of $X$ commuting with the flow of $V$.

The proof follows closely the proof of the smooth case in [13] extended to the singular Kähler-Einstein setting in [14], which proceeds by connecting two given Kähler-Ricci solitons $\phi_0$ and $\phi_1$ by a weak geodesic curve $\phi_t$. The main issue is that, in the case of a singular Fano variety, the intermediate metrics $V$, holomorphic vector field to the identity. Moreover, a Kähler-Ricci soliton $\omega$ is unique modulo the subgroup $\text{Aut}(X, V)_0$ of $\text{Aut}(X)_0$ consisting of all automorphism of $X$ commuting with the flow of $V$.

In view of the previous theorem it is natural to fix a holomorphic vector field $V$ on the Fano variety $X$, as above, and view the pair $(X, V)$ as the given complex geometric data, denoting by $\text{Aut}(X, V)$ the corresponding automorphism group. If $(X, V)$ admits a Kähler-Ricci soliton in the sense that $X$ admits a Kähler-Ricci soliton with corresponding vector field $V$, then it follows from the previous theorem, just as in the Kähler-Einstein case considered in [21] III, that the group $\text{Aut}(X, V)_0$ is reductive (see Corollary 3.7).

As shown by Tian-Zhu [48] another necessary condition for the existence of a Kähler-Ricci soliton on $(X, V)$, in the smooth case, is the vanishing of the modified Futaki invariant introduced in [48], which can be viewed as a functional $\text{Fut}_V$ on the Lie algebra of $\text{Aut}(X, V)_0$ and algebraically expressed as

$$\text{Fut}_V(W) = -\lim_{k \to \infty} \frac{1}{kN_k} \sum_{i=1}^{N_k} \exp(v_i^{(k)}/k)w_i^{(k)},$$

where $(v_i^{(k)}, w_i^{(k)})$ are the joint eigenvalues for the commuting action of the real parts of the holomorphic vector fields $V$ and $W$ on the $N_k$-dimensional space $H^0(X, -kK_X)$ of all holomorphic section with values in $-kK_X$ (see Proposition 4.7). More generally, in the case $V = 0$ there is a notion of algebro-geometric stability of a Fano manifold $X$ referred to as $K$-stability (or sometimes $K$-polystability) introduced by Tian in [45] saying that $X$ is $K$-polystable if for any $C^r$-equivariant deformation $X'$ of $X$ the Futaki invariant $\text{Fut}_V(X_0)$ of the central fiber $X_0$ (assumed to have log terminal singularities) satisfies $\text{Fut}_V(X_0) \geq 0$ with equality if and only if $X_0$ is biholomorphic to $X$. Similarly, in the general case of a pair $(X, V)$ we will say that $(X, V)$ is $K$-polystable if for any $C^r$-equivariant deformation $(X', V')$ of $(X, V)$ the modified Futaki invariant $\text{Fut}_{V_0}(X_0)$ of the central fiber $X_0$ satisfies $\text{Fut}_{V_0}(X_0) \geq 0$ with equality if and only if $(X_0, V_0)$ is isomorphic to $(X, V)$.

**Theorem 1.5.** Assume that $(X, V)$ admits a Kähler-Ricci soliton. Then $(X, V)$ is $K$-polystable.

This results appears to be new even in the case when $X$ is smooth, where it generalizes the result of Tian-Zhu in [48] concerning product deformations. The proof of the previous theorem builds on [8] where the case $V = 0$ was considered in the setting of singular Fano varieties. In the case $V = 0$ there is also a generalization of Tian’s notion of $K$-stability due to Donaldson [24], which involves more general polarized deformations of $X$ (called test configurations) and which in the end turns out to be equivalent to Tian’s notion. Presumably there is a similar generalization of Donaldson’s notion of $K$-stability in the presence of a non-trivial vector field $V$, as in the setting of extremal Kähler metrics considered in [33], but we will not go further into this here.
According to the fundamental Yau-Tian-Donaldson conjecture recently settled in [21] and [46] a Fano manifold $X$ is K-polystable if and only if $X$ admits a Kähler-Einstein metric (the existence problem in the singular case is still open). It seems natural to conjecture that this correspondence can be extended to the case of pairs $(X, V)$, with $V$ non-trivial, using the notion of K-stability appearing in the previous theorem (a different version of this conjecture involving a notion of geodesic K-stability was recently formulated by He [31]). In this direction we will show the analytic analog of K-polystability does imply the existence of a Kähler-Ricci soliton $\omega$, which moreover can be realized as the large time limit of the Kähler-Ricci flow:

**Theorem 1.6.** Let $X$ be a Fano variety and $V$ a holomorphic vector field on $X$ generating an action on $X$ of a torus $T$. If $(X, V)$ is analytically K-polystable, in the sense that the modified Mabuchi $K$-energy is proper modulo $\text{Aut}(X, V)_0$, then $(X, V)$ admits a Kähler-Ricci soliton. Moreover, the Kähler-Ricci flow $\omega_t$ then converges in the weak topology of currents, modulo the action of the group $\text{Aut}(X, V)_0$, to the Kähler-Ricci soliton $\omega$.

In the case when $X$ is smooth the previous existence result was shown in [20], using a variant of Aubin’s continuity method and the convergence result was shown in [49], using Perelman’s deep estimates [38].

It seems reasonable to expect that the recent powerful techniques developed for the existence problem of Kähler-Einstein metrics (see [21, 46, 44]) - which combine Gromov-Hausdorff convergence theory with $L^2$-estimates for $\overline{\partial}$ - can be extended to prove the missing algebraic counterpart of the previous Theorem by a deformation argument; either by using Tian-Zhu’s modification of Aubin’s continuity path or the Kähler-Ricci flow (as in the three dimensional situation considered in [52]). For some recent results in this direction see [39, 52, 57, 56]. As emphasized in [26, 21], in the ordinary case $V = 0$, the reductivity of the automorphism group of the limiting singular object is an important step in producing a suitable test configuration from the deformation in question. Accordingly, we expect the reductivity established in Corollary 1.7 (resulting from the uniqueness result in Theorem 1.4 above) to play a similar role in the case when $V$ is non-trivial (by replacing the Hilbert scheme used in [21] with its $T_c$-invariant counterpart).

1.4. The quantized setting and balanced metrics. In section 4 we consider the quantization, in the sense of Donaldson [25], of the setup above. In other words, given a line bundle $L \to X$, the infinite dimensional space $\mathcal{H}$ of all metrics $\phi$ on $L$ with (semi-)positive curvature is replaced by the sequence $\mathcal{H}_k$ of finite dimensional symmetric spaces of Hermitian metrics on the complex vector spaces $H^0(X, kL)$ of all holomorphic sections with values in the $k$ th tensor power of $L$, written as $kL$ in additive notation. For example, when $L = -K_X$ a quantized Kähler-Ricci soliton, $\mathcal{H}_k \in \mathcal{H}_k$ may in this framework be defined as a fixed point, modulo $\text{Aut}(X, V)_0$, of Donaldson’s (anti-)canonical iteration on the symmetric space $\mathcal{H}_k$. As conjectured in [25] and confirmed in [14] the latter iteration can be viewed as the quantization of the Kähler-Ricci flow. We recall that the bona fide fixed points in $\mathcal{H}_k$ are called (anti-) canonically balanced metrics and as conjectured by Donaldson [25], and shown in [13], in the case when $X$ admits a Kähler-Einstein $\omega_{KE}$ and $\text{Aut}(X)_0$ is trivial (i.e. $\omega_{KE}$ is unique), such balanced metrics exist for $k$ sufficiently large and the corresponding Bergman metrics $\omega_k$ on $X$ converge, as $k \to \infty$, weakly to the Kähler-Einstein metric on $X$. Here we will introduce a notion of $g$-balanced
metrics which in particular allows us to prove the following generalization of the result in [13]:

**Theorem 1.7.** Let \((X, V)\) be a Fano manifold equipped with a holomorphic vector field \(V\). If \(X\) is strongly analytically K-polystable and all the higher order modified Futaki invariants of \((X, V)\) vanish, then there exist quantized Kähler-Ricci solitons \(H_k\) at any sufficiently large level \(k\), which are unique modulo the action of \(\text{Aut}(X,V)_0\) and as \(k \to \infty\) the corresponding Bergman metrics \(\omega_k\) on \(X\) converge weakly, modulo automorphisms, to a Kähler-Ricci soliton \(\omega\) on \((X, V)\).

The strong analytic K-polystability referred to above simply means that the modified Mabuchi functional is coercive modulo \(\text{Aut}(X,V)_0\), which in the case when \(\text{Aut}(X)_0\) is trivial is well-known to be equivalent to the existence of a unique Kähler-Einstein metric on \(X\) (as well as the properness of the Mabuchi functional). It should be stressed that the previous theorem is new even in the case when \(V = 0\), if \(\text{Aut}(X)_0\) is non-trivial. The condition on the vanishing of the higher order Futaki invariants is then equivalent to the vanishing of Futaki’s higher order invariants \(F_{T_{d(m)}}\) for all integers \(m = 1, .. n\), which is a necessary condition for the existence of balanced metrics \(H_k\) for \(k\) large (and the condition is not implied by the analytic K-polystability assumption even in the case when \(X\) is toric, by the example in [29]). Finally, it should be stressed that even in the simplest case \(\text{Aut}(X)_0 = 0\) the extension of the previous theorem to singular Fano varieties seems challenging. Indeed, the corresponding existence statement may then even turn out to be false as indicated by an example in [37], which reveals that the K-stability of a singular polarized variety \((X, L)\) does not imply the existence of balanced metrics in the (a priori) different sense of [23]. This latter notion of a balanced metric (or critical metric in the terminology of Zhang) is equivalent to the asymptotic Chow polystability of \((X, L)\), in the sense of Geometric Invariant Theory. As shown by Mabuchi [35] and Futaki [28] an analog of the existence part of Theorem 1.7 holds in the latter setting of balanced metrics: more precisely, the existence of a constant scalar curvature metric in \(c_1(L)\) together with the vanishing of the higher order (ordinary) Futaki invariants of \((X, L)\) implies the asymptotic Chow polystability of \((X, L)\), but the convergence problem for the corresponding balanced metrics seems to be open (except in the case case when \(\text{Aut}(X, L)_0 = 0\) originally settled by Donaldson [23]).

**Organization of the paper.** The technical core of the paper is section 2 where the pluripotential theory of \(g\)-Monge-Ampère measure \(MA_g\) is developed. In particular, the continuity properties of \(MA_g\) (stated in Theorem 1.1) as well as the continuity and concavity/convexity properties of the corresponding energy functional \(E_g\) are established. With these results in place the variational approach to complex Monge-Ampère equations developed in [13] is not hard to extend to the present setting and the section is concluded with a proof of Theorem 1.2. Then in section 3 the pluripotential theory developed in the previous section is applied to Kähler-Ricci solitons (by adapting the proofs in [13] concerning singular Kähler-Einstein metrics) and finally extended to the quantized setting in section 4.
2. The pluripotential theory of moment maps and \( g \)-Monge-Ampère equations

2.1. Pluripotential preliminaries. Let \( L \to X \) be a line bundle over a compact complex manifold \( X \) and assume that \( L \) is semi-positive and big, i.e. \( L \) admits a smooth metric \( \phi_0 \) with non-negative curvature form \( \omega_0 \) (representing the first Chern class \( c_1(L) \in H^2(X,\mathbb{Z}) \) of \( L \)) and

\[
c_1(L)^n := \int_X \omega_0^n > 0
\]

We will use additive notation for a metric \( \phi \) on \( L \), i.e. given an open set \( U \subset X \) and a holomorphic trivializing section \( s|_U \) of \( L|_U \) the metric \( \phi \) is represented by the local function \( \phi|_U := \log |s|_\phi^2 \), where \( |s|_\phi \) denotes the length of \( s \) wrt the metric \( \phi \).

Then the (normalized) curvature form of the metric \( \phi \) may be locally represented as

\[
\omega_{\phi} := dd^c \phi|_U, \quad dd^c := \frac{i}{2\pi} \partial \bar{\partial}
\]

(which, abusing notation slightly will occasionally be written as \( \omega_{\phi} = dd^c \phi \)). We will denote by \( PSH(X,L) \) the space of all (possibly singular) metrics \( \phi \) on \( L \) with positive curvature current, i.e. \( \phi \) is locally upper semi-continuous (usc) and integrable and \( \omega_{\phi} \geq 0 \) holds in the sense of currents (equivalently, \( \phi \) is locally plurisubharmonic, or psh for short). By the seminal work of Beford-Taylor the Monge-Ampère (normalized) measure

\[
MA(\phi) := \omega_{\phi}^n/c_1(L)^n
\]

is well-defined for any metric \( \phi \) in \( PSH(X,L) \) which is locally bounded and defines a probability measure on \( X \). More generally, the (non-pluripolar) Monge-Ampère measure \( MA(\phi) \) can be defined for any metric \( \phi \) in \( PSH(X,L) \) by replacing the Bedford-Taylor wedge products used in the previous formula with the non-pluripolar product of positive currents introduced in [16]. This extension is uniquely determined by the properties that (i) \( MA(\phi) \) does not charge pluripolar subset of \( X \) (i.e. sets locally contained in the \( -\infty \)-locus of a psh function) and (ii) \( MA(\phi) \) is local with respect to the plurifine topology (i.e. the coarsest topology making all psh functions continuous). Concretely, this means that,

\[
MA(\phi) := \lim_{k \to \infty} 1_{\{\phi^{(k)} > \phi_0 - k\}} MA(\phi^{(k)}), \quad \phi^{(k)} := \max\{\phi, \phi_0 - k\}
\]

(by the locality of \( MA \) the sequence above is increasing and hence the limit is indeed well-defined).

**Example 2.1.** If the metric \( \phi \) in \( PSH(X,L) \) is locally bounded on a Zariski open subset \( \Omega \) of \( X \), then \( MA(\phi) = 1_\Omega \omega_\phi^n/c_1(L)^n \).

The Monge-Ampère measure defined by formula (2.1) satisfies \( \int_X MA(\phi) \leq 1 \) and accordingly, following [16], a metric \( \phi \) in \( PSH(X,L) \) is said to be of maximal Monge-Ampère mass if \( \int_X MA(\phi) = 1 \). The subspace of all such metrics is denoted by \( \mathcal{E}(X,L) \) and it has the crucial property that the Monge-Ampère operator is continuous wrt decreasing \( \phi_j \), converging to an element \( \phi \in \mathcal{E}(X,L) \). Note however that a metric as in the previous example, with non-empty singularity locus \( X - \Omega \), never has full Monge-Ampère mass. Still we have the following convergence result, which we will have great use for:
Proposition 2.2. Assume that there exists a Zariski open subset $\Omega$ such that the metrics $\phi_j$ and $\phi$ in $\text{PSH}(X, L)$ are locally bounded on $X - \Omega$ and have equivalent singularities (i.e. $\phi_j - \phi$ is bounded, for any fixed $j$). Then $\int \text{MA}(\phi_j) = \int \text{MA}(\phi)$ and if $\phi_j$ decreases to $\phi$, then $\text{MA}(\phi_j)$ converges weakly to $\text{MA}(\phi)$.

Proof. This is well-known and more generally holds as long as $\phi_j$ and $\phi$ have small unbounded locus (in the sense of [16]). For completeness we recall the simple argument. First, the equality of the total masses is a consequence of the fact that $\int \text{MA}(\psi) \leq \int \text{MA}(\phi)$ if $\psi \leq \phi + C$ and $\phi$ and $\psi$ have small unbounded locus (see Theorem 1.16 in [16].) In fact, in our case this follows from a simple max construction interpolating between suitable perturbations of $\phi$ and $\psi$. Finally, since $\text{MA}(\phi_j) \to \text{MA}(\phi)$ in the local weak topology on $\Omega$ (by Bedford-Taylor theory) it then follows from the equalities of the total integrals and basic integration theory that $1_\Omega \text{MA}(\phi_j) \to 1_\Omega \text{MA}(\phi)$, weakly on $X$, as desired. \hfill $\square$

We recall the following generalization (to semi-positive and big line bundles) of Demaillly’s classical regularization result concerning ample line bundles:

Theorem 2.3. (regularization) [27] Any metric in $\text{PSH}(X, L)$ can be written as a decreasing limit of smooth metrics $\phi_j$ in $\text{PSH}(X, L)$ (i.e. $\phi_j \in H(X, L)$).

As explained in the introduction of the paper one of our main aims is to define a modified version of the Monge-Ampère measure, in the presence of a torus action.

2.2. The torus setting and the $g$–Monge-Ampère measure. Let $T$ be an $m$–dimensional real torus acting holomorphically on $(X, L)$. The complexified torus $T_c$ then also acts holomorphically on $(X, L)$ and we denote by $\rho(\tau)$ the automorphism of $(X, L)$ corresponding to $\tau \in T_c$. Writing $\tau := (\tau_1, ..., \tau_m)$ we identify $T_c$ with $\mathbb{C}^m$ in the standard way and set $\tau_j = e^{t_j + i\theta_j}$, where $t_j$ are real coordinates on $(J$ times) the Lie algebra Lie $(T)$, identified with $\mathbb{R}^m$, which we equip with its standard partial order relation $\leq$. Similarly, we identify the dual Lie $(T)^*$ with $\mathbb{R}^m$ with the corresponding dual real coordinates, which will be denote by $\lambda := (\lambda_1, ..., \lambda_m)$.

For $\tau := (t_1, ..., t_n)$ and $\phi \in \text{PSH}(X, L)$ we set

$$\phi_\tau := \rho(\tau)^* \phi,$$

which gives a family of metrics $\phi_\tau$ parametrized by $\mathbb{R}^n$. Next, for $\phi$ smooth we define the corresponding moment map $m_\phi$ by differentiating wrt $t$:

$$m_\phi(x) := \nabla_\tau \phi_\tau(x)|_{\tau=0} : \quad X \to P \subset \mathbb{R}^m,$$

where $P$ is the defined as the image

$$P := m_\phi(X)(\subset \mathbb{R}^m)$$

which is well-known to be independent of $\phi$ (and compact, since it is the image of the compact manifold $X$ under a continuous map). In fact, $P$ is convex and coincides with the support of the corresponding Duistermaat-Heckman measure

$$\nu := (m_\phi)_* \frac{(\omega^\phi)^n}{n!V}$$

on $\mathbb{R}^m$, which is absolutely continuous with respect to Lesbegue measure and independent of $\phi$ (see [22] for the case when $L$ is ample and [1] for the semi-positive...
and taking the inf over all $t$ properties, which follow immediately from its variational definition:

**Lemma 2.5.**
\[
\phi \text{ the constant has been chosen so that } \phi \in \mathbb{E} \text{ and a holomorphic section } \langle 2.2 \rangle \text{ } \Psi \text{ : metric on } L \quad \text{In particular, if } \phi \text{ is locally bounded then } \psi_\lambda \text{ is locally bounded on a Zariski open subset } \Omega \text{ of } X.
\]

**Proof.** By Kiselman’s minimum principle the metric $\psi_\lambda$ is psh (note that $\psi_\lambda$ is automatically upper-semi continuous, since it is the infimum of continuous metrics).

Next, observe that there exists an integer vector $\lambda^{(k)}$ in $P$ such that $\lambda^{(k)}/k \geq \lambda$ and a holomorphic section $s_k$ in $H^0(X,kL)$ (for some large $k$) satisfying $\tau^* s_k = e^{\langle (i\theta + t), \lambda^{(k)} \rangle} s_k$ (as follows from Proposition [14.1]). Set $\phi_k := \frac{1}{k} \log |s_k|^2 - C_k$, where the constant has been chosen so that $\phi \geq \phi_k$. In particular $\tau^* \phi \geq \tau^* \phi_k$ and hence $\phi_t \geq \langle t, \lambda^{(k)}/k \rangle - \langle t, \lambda \rangle + \log |s_k|^2$. Hence, letting $X - \Omega$ be the zero set $\{s_k = 0\}$ and taking the inf over all $t \geq 0$ reveals that $\phi_t(x)$ is locally bounded on $\Omega$. \hfill \Box

The operator $P_\lambda$ on $PSH(X,L)^T$ defined by formula [2.2] has the following basic properties, which follow immediately from its variational definition:

**Lemma 2.6.** In general $P_\lambda \phi \leq \phi$ and

- If $\phi \leq \phi’$ then $P_\lambda \phi \leq P_\lambda \phi$ and $P_\lambda (\phi + C) = P_\lambda (\phi) + C$ for any constant $C \in \mathbb{R}$. 

In particular, if $\phi - \phi’$ is bounded then so is $P_\lambda (\phi) - P_\lambda (\phi’)$. Moreover, if $\phi_j$ decreases to $\phi$ the metric $P_\lambda \phi_j$ decreases to $P_\lambda \phi$.

The relation to the moment map is given by the following

**Lemma 2.7.** Assume that $\phi$ is in $\mathcal{H}(X,L)^T$. Then
\[
\{ m_\phi \geq \lambda \} = \{ P_\lambda \phi = \phi \}
\]

and setting $\chi_\lambda(p) := 1_{\{p \geq \lambda\}}$ on $\mathbb{R}^m$ the following identity holds:

\[
MA_{\chi_\lambda}(\phi) = MA(P_\lambda(\phi))
\]

for almost any $\lambda$ in $P$.

**Proof.** The first relation is an immediate consequence of the convexity of $t \mapsto \phi_t$ on $[0, \infty]^n$ (and the fact that $\phi_0 = \phi$). Next, we observe that $1_{\{\psi_\lambda = \phi\}} MA(\phi) \leq 1_{\{\psi_\lambda = \phi\}} MA(\psi_\lambda)$, as follows immediately from the fact that the set $\{\psi_\lambda = \phi\}$ is, for a.e. $\lambda$, the closure of an open domain of $X$ (by Sard’s theorem). To prove the reversed inequality we apply the comparison principle for the MA-operator to $\psi_\lambda$ and $\phi - \epsilon$ (which indeed applies since $\psi_\lambda$ is more singular than $\phi$; see Remark 2.4 in [16]) to get

\[
\int_{\{\psi_\lambda > \phi - \epsilon\}} MA(\phi) \geq \int_{\{\psi_\lambda > \phi - \epsilon\}} MA(\psi_\lambda).
\]

Letting $\epsilon \to 0$ thus gives $1_{\{\psi_\lambda = \phi\}} MA(\phi) \geq 1_{\{\psi_\lambda \geq \phi\}} MA(\psi_\lambda)$. Finally, the relation [2.3] follows from the fact that $MA(\psi_\lambda) = 0$ on the complement of the set where $\psi_\lambda = \phi$, i.e. on the open set where $\psi_\lambda < \phi$. This follows from general maximality properties of envelopes (compare [11]), but in the present setting it also follows
directly from the fact that $\psi_\lambda$ harmonic along the orbits of $T_c$ in the open subset $\psi_\lambda < \phi$ or from the proof of Proposition 4.4.

2.2.1. The $g$–Monge–Ampère measure. Given a continuous function $g$ on $P$ which will be assumed to be continuous (and usually normalized so that $g\nu$ is a probability measure) we define, for $\phi$ in $\mathcal{H}(X,L)^T$, the measure

\[
MA_g(\phi) := MA(\phi)g(\nu),
\]
on $X$, which will be referred to as the $g$–Monge–Ampère measure (or the $g$–modified Monge–Ampère measure). In order to define $MA_g(\phi)$ for $\phi$ merely locally bounded first consider the (non-continuous) case when $g$ is of the form $g = \chi_\lambda := 1_{\{r \geq \lambda\}}(p)$ and define $MA_\lambda(\phi)$ by the relation (2.4). Next, by imposing linearity wrt $g$ this defines $MA_g(\phi)$ for any step function $g$ (where the “step” is a cube in $\mathbb{R}^m$). In the case of a continuous $g$ we set

\[
MA_g(\phi) := \lim_{j \to \infty} MA_{g_j}(\phi)
\]
where $g_j$ is any sequence of step functions converging uniformly to $g$ on $P$. Finally, for $\phi$ a general (possibly singular) metric in $PSH(X,L)$ we set

\[
MA_g(\phi) := \lim_{k \to \infty} 1_{\{\phi^{(k)} > \phi - k\}}MA_g(\phi^{(k)}), \quad \phi^{(k)} := \max\{\phi, \phi_0 - k\}
\]

Theorem 2.7. The measure $MA_g(\phi)$ is a well-defined measure on $X$, not charging pluripolar subset and

\[
\int_X MA_g(\phi) = \int_P g\nu
\]
if $\phi$ has full Monge–Ampère mass (with an inequality $\leq$ for $\phi$ a general metric in $PSH(X,L)^T$). Moreover, if $\phi_j$ is a sequence of metrics with full Monge–Ampère mass decreasing to $\phi$ with full Monge–Ampère mass, then

\[
MA_g(\phi_j) \to MA_g(\phi)
\]
in the weak topology of measures on $X$ and more generally:

\[
(\psi_j - \phi_0)MA_g(\phi_j) \to (\psi - \phi_0)MA_g(\phi)
\]
if $\psi_j$ decreases to a locally bounded metric $\psi$ in $PSH(X,L)^T$. In particular, the uniqueness statement in Theorem 2.4 also holds.

Proof. Step one: the case of $\phi$ locally bounded.

By the very definition of $MA_g$, for $\phi$ in $\mathcal{H}(X,L)^T$ the equation (2.4) holds for all functions $g$ iff $(\nu \geq \lambda)MA(\phi) = \nu$, where $\nu$ is the canonical D-H measure attached to $T$. As pointed out above the latter push-forward relation indeed holds for smooth (a proof in the spirit of the present paper is explained in the remark below which also applies to metrics $\phi$ which are merely locally bounded). In particular, if $g_j$ and $g_j'$ are two sequences of step functions converging to the same continuous function $g$, then $\int |MA_{g_j}(\phi) - MA_{g_j'}(\phi)|$ tends to zero when $j \to \infty$ which shows that $MA_g(\phi)$ is well-defined for $\phi$ locally bounded. To prove the continuity statement when $\phi$ is locally bounded first observe that the continuity with respect to decreasing sequences holds for $g$ of the form $g = \chi_\lambda$ and hence, by linearity as long as $g$ is a step function. Indeed, if $\phi_j$ decreases to $\phi$, then $P_\lambda \phi_j$ decreases to $P_\lambda \phi$ and
$P_h\phi - P_h\phi_j$ is bounded (by Lemma 2.5). But then the desired continuity follows immediately from Prop 2.2. To handle the case of a general $g$ we take two sequences of step functions $g^\pm_k$ satisfying $g^\pm_k \leq g \leq g^\pm_k$ and converging uniformly to $g$. Fixing a smooth positive function $u$ on $X$ we then have

$$\int MA_{g^-}(\phi_j)u \leq \int MA_g(\phi_j)u \leq \int MA_{g^+}(\phi_j)u$$

for $j$ and $k$ fixed. Letting $j \to \infty$ for $k$ fixed and using the continuity property of $MA_{g_k}$ established above thus gives

$$\int MA_{g^-}(\phi)u \leq \lim_{j \to \infty} \int MA_g(\phi_j)u \leq \int MA_{g^+}(\phi)u$$

Finally, letting $k \to \infty$ and using the definition of $MA_g(\phi)$ reveals that the sequences $\int MA_{g^-}(\phi)u$ and $\int MA_{g^+}(\phi)u$ both converge to $\int MA_g(\phi)u$, which concludes the proof in the locally bounded case.

**Step two: the case of $\phi$ with full Monge-Ampère mass**

By the definition of $MA_g(\phi)$ and the previous case it will be enough to show that, for any continuous function $h$ on $X$,

$$\int_X h \left( MA_g(\psi^{(k)}) - MA_g(\psi) \right)$$

tends to zero uniformly for all $\psi \geq \phi$. But the absolute value of the integral above may be estimated by a constant $C$ times

$$\int_{\{\psi \leq \phi_0 - k\}} \left( MA(\psi^{(k)}) + MA(\psi) \right),$$

where the constant $C$ only depends on $X$ and $\sup_X |h|$ and $\sup_P |g|$. But, as shown in the proof of Theorem 2.17 in [10] the latter integral tends to zero if $\phi$ has full Monge-Ampère mass, which thus proves the general continuity statement. In particular, regularizing $\phi$ and applying the previous step also gives equality in equation 2.10 for $\phi$ of full Monge-Ampère mass (and inequality in general). The proof of the last convergence statement is proved in a similar manner. Finally, the fact that the limit $2.10$ is well-defined for any $\phi \in PSH(X, L)\cap T$ follows from the $T-$pluripolar locality of $MA_g$ (acting on locally bounded $\phi$; see section 2.2.2 below). Indeed, setting $E_k := \{\phi^{(k)} > \phi - \phi_0\}$ gives, by locality, that $1\{\phi^{(k)} > \phi - \phi_0\}MA_g(\phi^{(k)}) = 1_{E_k}MA_g(\phi|_{E_k})$, which is increasing in $k$, with uniformly bounded mass, which ensure the existence of the limit. Finally, since $MA(\phi)$ does not charge pluripolar subsets so doesn’t $MA_g(\phi)$ and this argument also proves the uniqueness statement in Theorem 1.1 (since $\{\phi = \infty\}$ is pluripolar).

**Remark 2.8.** The arguments above can be used to give a (non-standard) proof of the independence of the measure $\nu$. The point is that if $|P_h(\phi_1) - P_h(\phi_2)| \leq C$ then $|P_h(\phi_1) - P_h(\phi_2)| \leq C$ (compare Lemma 2.5) and hence, by Prop 2.2, the total Monge-Ampère mass $\int MA(P_h\phi)$ is the same for any locally bounded metric $\phi$. But then it follows (just as above) that, for any $g$, $\int MA_g(\phi)(= \int_P g\nu_\phi)$ is independent of $\phi$,mas desired.

An immediate consequence of the previous theorem, combined with the regularization result in Theorem 2.3 together with the compactness of $P$, is the following
Corollary 2.9. The following inequalities hold:

\[ \inf_P g_X MA(\phi) \leq MA_g(\phi) \leq \sup_P g_X MA(\phi) \]

2.2. Fine locality and consequences. By definition, the \( T \)-plurifine topology on \( X \) is the coarsest topology making all sets of the form \( O := \{ \phi < \psi \} \) open, for \( \phi \) and \( \psi \) metrics in \( PSH(X,L)^T \) (which by a max construction may be assumed locally bounded).

Proposition 2.10. The operator \( \phi \mapsto MA_g(\phi) \) on \( PSH(X,L)^T \) is local with respect to the \( T \)-plurifine topology, i.e. if \( \phi = \phi' \) on a \( T \)-plurifine open set \( O \) then \( 1_O MA_g(\phi) = 1_O MA_g(\phi') \). In particular,

\[ 1_{\{\phi > \psi\}} MA_g(\phi) = 1_{\{\phi > \psi\}} MA_g(\max(\phi, \psi)) \]

Proof. By the definition of \( MA_g \) we may as well assume that \( \phi \) and \( \phi' \) are locally bounded. Regularizing \( \phi \) and \( \phi' \) and invoking the convergence result in Theorem 2.7, the proof is reduced - precisely as in the classical case when \( g = 1 \) [4] - to the case when \( \phi \) and \( \phi' \) are smooth and \( O \) is open with respect to the standard topology. But then the locality in question follows trivially from formula 2.4. \( \square \)

Just as in the case \( g = 1 \) formula 2.7 implies that the comparison principle holds (also using that the total mass of \( MA_g(\phi) \) is independent of \( \phi \) when \( \phi \) has maximal Monge-Ampère mass).

Proposition 2.11. (the comparison principle). Assume that \( \phi \) and \( \psi \) are in \( PSH(X,L)^T \) and of full Monge-Ampère mass. Then

\[ \int_{\{\psi > \phi\}} MA_g(\psi) \leq \int_{\{\psi > \phi\}} MA_g(\phi) \]

Another useful consequence of the formula 2.7 is the following inequality (compare [10] for the case \( g = 1 \)):

Proposition 2.12. Assume that \( \phi \) and \( \psi \) are in \( PSH(X,L)^T \) and that there exists a positive measure \( \mu \) such that \( MA_g(\phi) \geq \mu \) and \( MA_g(\psi) \geq \mu \). Then

\[ MA_g(\max(\phi, \psi)) \geq \mu \]

2.3. Vector fields generating torus actions on singular varieties. In this section we assume that \( L \) is the pull-back of an ample line bundle \( A \) on a (possible singular) projective variety \( Y \). Let \( V \) be a complex holomorphic vector field on \( X \) (i.e. \( V \in H^0(X,T^{1,0}) \)) with a fixed holomorphic lift to \( L \). Denote by \( H(X,L)^V \) the space of all smooth metrics on \( L \) with semi-positive curvature form such that \( \phi \) is invariant under the flow \( \exp(t \text{Im}V) \) defined by the imaginary part of \( V \) (equivalently: \( L_{\text{Im}V} \phi = 0 \), where \( L_{\text{Im}V} \) denotes the Lie derivative along the imaginary part of \( V \) ) and define \( PSH(X,L)^V \) etc in a similar manner.

Lemma 2.13. Let \( V \) be a holomorphic vector field on \( X \) with admits a holomorphic lift to a line bundle \( L \to X \), where \( L \) is assumed to be the pull-back of an ample line bundle \( A \) on a (possible singular) projective variety \( Y \). If there exists a metric \( \phi \) on \( L \) with positive curvature current which is invariant under the flow of the imaginary part of \( V \) and such that \( \phi \) is locally bounded, then there exists a complex torus \( T_c \) acting holomorphically on \( (X,L) \) such that the imaginary (as well as the
real) part of $V$ may be identified (in the standard way) with an element $\xi$ in the Lie algebra of the corresponding real torus $T \subset T_e$ and $PSH(X,L)^V = PSH(X,L)^T$. Moreover, in the case when $A = -K_Y$ for $Y$ a Fano variety with log terminal singularities the boundedness assumption on $\phi$ may be replaced by the assumption that $\phi$ has full Monge-Ampère mass.

**Proof.** Let $K$ be the subgroup of $Aut(X,L)_0$ fixing the metric $\phi$ and denote by $G$ the one-parameter subgroup $G$ of $K$ defined by the flow of the imaginary part of $V$. First observe that $K$ is a compact Lie group. Indeed, induces a $K$–invariant Hermitian norm on $H^0(X,kL)$. In the case that $\phi$ is locally bounded the norm may be defined by $\|s_k\|^2 := \int |s_k|^2 e^{-k\phi} MA(\phi)$ and in the case $A = -K_Y$ we can replace $MA(\phi)$ with the pull-back to $X$ of the measure $\mu_\phi$ on $Y$ with local density $e^{-\phi}$ defined by the metric $\phi$ (compare formula 3.1). Since $\phi$ is assumed to have full MA-mass it has no Lelong numbers and hence $\|s_k\|^2 := \int_Y |s_k|^2 e^{-(k+1)\phi} < \infty$ for any $k > 0$ (see the appendix in [14]). The $K$–invariant norm induces a Kodaira embedding of $X$ into $\mathcal{P}H^0(X,kL)$ and hence $K$ may be identified with the subgroup of $U(N+1,\mathbb{C})$ preserving the image of $X,$ which is clearly a compact Lie group. Next, denote by $G$ the topological closure of $G$ in $K,$ which defines a topological compact and connected abelian subgroup of the compact Lie group $K.$ But it is well-known that any such subgroup is a Lie subgroup and hence it follows from the standard classification of such Lie groups that $G$ is a real torus. Finally, by general principles (analytic continuation) any holomorphic action of a real torus on $(X,L)$ can be extended holomorphically to give an action of the corresponding complex torus, which concludes the proof. □

Now, given a metric $\phi$ in $H(X,L)^V$ we define the following smooth function on $X$ $$f_\phi := \langle m_\phi, \zeta \rangle,$$
which in (in the case when $\omega_\phi > 0$) is a **Hamiltonian function** for the $\omega^\phi$–symplectic real vector field $\text{Im} V$ and

$$\omega^\phi(\text{Im} V , \cdot ) = dh_\phi,$$
or equivalently:

$$\nabla^\phi h_\phi = \text{Re} V$$
where $\nabla^\phi$ is the gradient defined with respect to $\omega^\phi$ (the previous relation holds where it is well-defined, i.e. where $\omega^\phi > 0$). In particular, it follows from the compactness of $P \subset \mathbb{R}^n$ that there exists a universal constant $C$ such that

$$\sup_X |f_\phi| \leq C$$
(in the case when $L$ is ample this was shown by a different argument in [55]). Now given a continuous function $G(v)$ on $\mathbb{R}$ we set $g(\lambda) := G(\langle \lambda, \zeta \rangle)$ and obtain, by the construction in section 2.2.1, a modified Monge-Ampère measure

$$MA_{(G,V)}(\phi) := MA_g(\phi),$$
defined for any metric $\phi$ in $PSH(X,L)^V$ with the property that

$$MA_{(G,V)}(\phi) := MA(\phi)G(h_\phi)$$
for $\phi$ smooth. In particular, in the setting of Kähler-Ricci solitons considered in section 3, we will take $G(v) = e^v/C$ and write $g_V$ for the corresponding function on
the Lie algebra of $T$. Here the normalizing constant $C$ is chosen so that $g_V \nu$ is a probability measure.

2.3.1. The case of a singular Fano variety. By normality it is enough to define the holomorphic vector field $V$, appearing in Lemma 2.13, on the regular part of a singular normal variety $Y$. In particular, if $Y$ is a normal Fano variety, i.e. $-K_Y$ is ample (see section 3), then any holomorphic vector field $V_0$ defined on the regular locus $Y_0$ of $Y$ admits a canonical lift to $-K_{Y_0}$ (since $V_0$ naturally acts on the tangent bundle of $Y_0$). Hence, if $\pi : X \to Y$ is a resolution such that $\pi$ is an isomorphism over $Y_0$ then, by normality, the vector field $V_0$ admits a unique extension $V_X$ to $X$ such that $V_X$ lifts to the line bundle $L := \pi^*(-K_X)$. As a consequence the corresponding $g-$Monge-Ampère measure $MA_{g_V}(\phi)$ can be defined on the singular variety $Y$, by passing to the resolution $X$. The measure $MA_{g_V}(\phi)$ thus defined is independent of the resolution $X$, since the $g-$Monge-Ampère measure on $X$ does not charge pluripolar sets and in particular it does not charge the exceptional locus of $\pi$.

2.4. Energy type functionals and psh projections. As is well-known the classical Monge-Ampère measure $MA$ viewed as a one-form on the convex space $\mathcal{H}(X, L)$ is closed and hence exact. In particular, $MA$ admits a primitive, i.e. a functional $\mathcal{E}$ on $\mathcal{H}(X, L)$ such that $d\mathcal{E}|_\phi = MA(\phi)$, where $d\mathcal{E}$ denotes the differential of $\mathcal{E}$, which is uniquely determined by the normalization condition $\mathcal{E}(\phi_0) = 0$. Integrating along affine segments in $\mathcal{H}(X, L)$ one arrives at the following explicit energy type expression:

$$\mathcal{E}(\phi) := \frac{1}{(n+1)c_1(L)^n} \int_X (\phi - \phi_0) \sum_{j=0}^n (dd^c \phi)^{n-j} \wedge (dd^c \phi_0)^j$$  \hspace{1cm} (2.8)

Following [10] the functional $\mathcal{E}(\phi)$ admits a unique increasing and use extension to all of $PSH(X, L)$ given by

$$\mathcal{E}(\phi) := \inf_{\psi \geq \phi} \mathcal{E}(\psi),$$

where $\psi$ ranges over all elements of $\mathcal{H}(X, L)$ bounded from below by $\phi$. A metric $\phi$ is said to have finite (pluricomplex) energy if $\mathcal{E}(\phi) > -\infty$ and the space of all finite energy metrics is denoted by $\mathcal{E}^1(X, L)$. As shown in [10] any finite energy metric has full Monge-Ampère mass, i.e.

$$\mathcal{H}(X, L) \subset \mathcal{E}^1(X, L) \subset \mathcal{E}(X, L)$$

Let us also recall the definition of the psh projection operator $P$ from the space $C(X, L)$ of all continuous metrics on $L$ to the space of all continuous and psh (i.e. positively curved) metrics defined by the following point-wise upper envelope:

$$P\phi := \sup \{ \psi : \psi \leq \phi, \psi \in PSH(X, L) \}$$

Next, turning to the present torus setting we start by observing that the one-form $MA_{g}$ on $\mathcal{H}(X, L)^T$ defined by the $g-$Monge-Ampère measure is also closed and hence exact (this observation is essentially due to Zhu [55]):

Lemma 2.14. There exists a functional $\mathcal{E}_g$ on $\mathcal{H}(X, L)^T$ such that $(d\mathcal{E}_g)|_\phi = MA_g(\phi)$. The functional $\mathcal{E}_g$ is uniquely determined by the normalization condition $\mathcal{E}_g(\phi_0) = 0$. 

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Proof. This was essentially shown by Zhu [55] in the vector field setting for $g(p)$ the exponential function (compare section 2.3) and the proof in the general case is essentially the same (see also [15]). Alternatively, the existence of the functional $\mathcal{E}_g$ also follows from the existence of its quantized version $\mathcal{L}_g$ (see Prop 4.5). □

Proceeding as before we may now extend $\mathcal{E}_g$ to all of $PSH(X,L)^T$ by setting

$$
\mathcal{E}_g(\phi) := \inf_{\psi \geq \phi} \mathcal{E}_g(\psi),
$$

where $\psi$ ranges over all elements of $\mathcal{H}(X,L)^T$ bounded from below by $\phi$. It follows immediately from the definition of $\mathcal{E}_g(\phi)$ and Cor 2.9 that there exists a constant $C$ such that

$$
\mathcal{E}_g(\phi) \geq C \mathcal{E}(\phi)
$$

under the normalizing condition $\sup_X (\phi - \phi_0) = 0$. In particular, $\mathcal{E}_g(\phi)$ is finite for any metric $\phi$ of finite energy.

**Proposition 2.15.** The functional $\mathcal{E}_g(\phi)$ on $PSH(X,L)^T$ has the following properties:

- $\mathcal{E}_g$ is increasing and satisfies $\mathcal{E}_g(\phi + c) = \mathcal{E}_g(\phi) + c$ for any constant $c \in \mathbb{R}$ iff $gd\nu$ is a probability measure.
- $\mathcal{E}_g$ is upper semi-continuous with respect to the $L^1$-topology on $PSH(X,L)^T$.
- $\mathcal{E}_g$ is continuous along decreasing sequences in $PSH(X,L)^T$.

Proof. The first point follows immediately from the fact that $MA_g(\phi)$ is a probability measure when $\phi \in H(X,L)^T$. In the case when $\phi$ is locally bounded the continuity of $\mathcal{E}_g$ wrt decreasing sequences follows from the convergence in Theorem 2.7, by writing $\mathcal{E}_g(\phi) = \int_{X \times [0,1]} (\phi - \phi_0) MA_g(t\phi + (1 - t)\phi_0) dt$. By a simple max construction this implies the general upper-semicontinuity in the second point and hence, since $\mathcal{E}_g$ is increasing, it also implies the general continuity wrt decreasing sequences in the third point (just as in the proof of Prop 2.10 in [16]). □

We also have the following generalization of the differentiability theorem in [11] to the torus setting, which plays a key role in the variational approach to complex Monge-Ampère equations [13]:

**Proposition 2.16.** The composed functional $\mathcal{E}_g \circ P$ is Gateaux differentiable on the affine space $C(X,L)^T$ of all $T$-invariant continuous metrics on $L$ and

$$
(2.9) \quad d(\mathcal{E}_g \circ P)_{|_\phi} = MA_g(P\phi)
$$

More generally, if $\phi$ is in $\mathcal{E}^1(X,L)^T$ and $u$ is a continuous function on $X$, then

$$
\frac{d (\mathcal{E}_g(P(\phi + tu)))}{dt} \bigg|_{t=0} = \int MA_g(P\phi) u
$$

Proof. First observe that the following “orthogonality relation” holds for $\phi$ a metric in $C(X,L)^T$:

$$
\int MA_g(P\phi)(\phi - P\phi) = 0
$$

Indeed, for $g = 1$ this is well-known (compare [11]) and the general case then follows immediately from the upper bound in Cor 2.9. Combining the orthogonality relation above with the comparison principle in Prop 2.11 the proof of formula (2.9) then follows using the corresponding comparison principle (Prop 2.11), just as in
Finally, the proof of the last formula in the proposition is reduced to the previous case by approximation, just as in the proof of Lemma 4.2 in \cite{13}.

2.5. Energy along $T$--invariant bounded geodesics. As noted by Mabuchi, Semmes and Donaldson, independently, there is a natural Riemannian metric on the infinite dimensional space $\mathcal{H}_\infty$ of all smooth positively curved metrics on a given line bundle $L$, such that corresponding geodesics $\phi_t$ - when they exist - are solutions to a complex Monge-Ampère equation. Here we will only use the pluripotential version of this setup where a notion of “weak geodesic” or more precisely “bounded geodesic” is defined directly without any reference to a Riemannian structure \cite{12,13}.

Given two elements $\phi_0$ and $\phi_1$ in $PSH(X,L) \cap L^\infty_{loc}$ there is a unique “bounded geodesic” $\phi_t$ defining a curve in $PSH(X,L) \cap L^\infty_{loc}$ connecting $\phi_0$ and $\phi_1$. The curve $\phi_t$ may be defined as follows: first we set $t := \log |\tau|$, for $\tau \in \mathbb{C}$ and identify $\phi_t$ with an $S^1$--metric $\Phi$ on $\pi^*L \rightarrow X \times A$, where $A$ is the annulus $\{\tau \in \mathbb{C} : 0 \leq \log |\tau| \leq 1\}$ in $\mathbb{C}$ equipped with its standard $S^1$--action and $\pi$ is the standard projection from $X \times A$ to $X$. The restriction $\Phi|_{\partial(X \times A)}$ to the boundary of $X \times A$ is determined by the given metrics $\phi_0$ and $\phi_1$ and the extension to all of $X \times A$ of $\Phi$ is given by the following envelope:

$$\Phi := \sup_{\Psi} \left\{ \Psi : \Psi|_{\partial(X \times A)} \leq \Phi|_{\partial(X \times A)} \right\}$$

where $\Psi$ ranges of all $S^1$--invariant bounded metrics on $\pi^*L \rightarrow X \times A$ with positive curvature current (the corresponding curve $\psi_t$ will be called a bounded subgeodesic).

As observed in \cite{15} a simple barrier argument reveals that $\Phi$ is continuous at the boundary in the sense that $\phi_t \rightarrow \phi_0$ uniformly as $t \rightarrow 0$ and similarly for $t = 1$.

Indeed, this follows directly from the fact that the continuous metric

$$\Psi := \max\{\Phi_0 - A \log |\tau|, \Phi_1 - A(1 - \log |\tau|)\}$$

is, for $A$ sufficiently large a candidate for the sup defining $\Phi$. Moreover, by standard properties of free envelopes

$$(dd^c \Phi)^n = 0$$

on the interior of $X \times A$. In other words, the extension $\Phi$ is a solution of the corresponding Dirichlet problem for the Monge-Ampère operator on the domain $X \times A$ (as shown in \cite{12} $\Phi$ is continuous on all of $X \times A$ if the given boundary data is). It follows immediately from the envelope construction above that if $\phi_0$ and $\phi_1$ are $T$--invariant, then so is $\phi_t$ for any $t \in [0,1]$.

**Proposition 2.17.** Let $\phi_t$ be a curve in $\mathcal{E}^1(X,L)^T$ defined for $t \in [0,1]$ and set $f(t) := \mathcal{E}_g(\phi_t)$

- If $\phi_t$ is a bounded subgeodesic, then the function $f(t)$ is convex on $[0,1]$.
- If $\phi_t$ is a bounded geodesic then $f(t)$ is affine on $[0,1]$ (and continuous up to the boundary)
- If $\phi_t$ is affine wrt $t$, then $f(t)$ is concave, i.e. the functional $\mathcal{E}_g$ is concave on the space $\mathcal{E}^1(X,L)^T$ equipped with its affine structure.

**Proof.** A direct calculation reveals that, if the corresponding metric $\Phi$ on $\pi^*L$ is smooth, then the function $F(\tau) := \mathcal{E}_g(\Phi(\cdot, \tau))$ on $A$ satisfies

$$dd^c F = \int_X MA(\Phi)g(m_{\Phi(\cdot, \tau)}),$$

\cite{11}.
where \(\int_X \cdot\) denotes the push-forward to \(X\) (see the appendix in \cite{15} for the special case when \(g\) is a function on \(\mathbb{R}\) of the form \(g(v) = e^{av}\) for \(a \in \mathbb{R}\); the general case reduces to this special case by expressing \(g\) in terms of its Fourier transform).

Note that the integrand may be written as the \(g\)-Monge-Amp\`ere measure \(MA_g(\Phi)\) on \(X \times A\), defined with respect to the induced \(T \times S^1\)-action on \((X \times A, \pi^*L)\), identifying \(g\) with a function on \(\mathbb{R}^{m+1}\). Let now \(U\) be a \(S^1\)-invariant open set whose closure is contained in the interior of \(A\). Then it follows from a variant of Theorem \ref{thm:2.3} (or the extension argument in the claim below) that \(\phi_t\) can be written as a decreasing sequence \(\phi_t^{(j)}\) of smooth subgeodesics defined for \(t \in U\). By Prop \ref{prop:2.14}, \(E_g(\phi_t^{(j)}) \to E_g(\phi_t)\), as \(j \to \infty\) and since \(E_g(\phi_t^{(j)})\) is convex wrt \(t\) (by formula \ref{eq:2.11}) so is \(E_g(\phi_t)\), which proves the first point.

To prove the second point we will use the following claim which furnishes approximations with good extension properties: the restriction \(\Phi|_U\) of the given bounded \(T \times S^1\)-invariant metric on \(\pi^*L \to X \times A\) defining the geodesic \(\phi_t\) extends to a metric \(T \times S^1\)-invariant metric \(\tilde{\Phi}\) with positive curvature current on the line bundle \(L_m := \pi^*L \otimes O(m)\) over \(X \times P^1\), for some positive integer \(m\). Accepting this for the moment we can, by regularization, take smooth \(\tilde{\Phi}\) such metrics on \(L_m\) decreasing to \(\tilde{\Phi}\). In particular, by Theorem \ref{thm:2.7}

\[
(2.12) \quad \int_{X \times P^1} MA_g(\tilde{\Phi}) u \to \int_{X \times P^1} MA_g(\tilde{\Phi}) u
\]

for any smooth function \(u\) on \(X \times P^1\). Now, setting \(F_j(\tau) := \tilde{E}_g(\tilde{\Phi}_j(\cdot, \tau))\) we have, as before, that \(F_j(\tau) \to F(\tau)\) for any fixed \(\tau\) and since \(dd^c F_j \geq 0\) this shows that \(dd^c F(\tau) \geq 0\), i.e. \(f(t)\) is convex on \([0, 1]\) as desired. Moreover, since \(\Phi\) defines a geodesic we have \(MA(\Phi) = 0\) in the interior of \(X \times A\) and in particular \(MA(\tilde{\Phi}) = 0\) on \(X \times U\). Hence, applying the convergence result in formula \ref{eq:2.12} to a function \(u\) which is the pull-back to \(X \times U\) of a smooth function compactly supported in \(U\) reveals that \(\int_A dd^c F_j \to \int_{X \times U} MA_g(\tilde{\Phi}) u\), which thus vanishes (using Cor \ref{cor:2.9}). But then it follows that \(f(t)\) is affine on \([0, 1]\), as desired. Finally, the continuity up to the boundary on \([0, 1]\) follows from the corresponding continuity property of the curve \(\phi_t\) together with Prop \ref{prop:2.13}. This concludes the proof of the second point up to the claimed extension property of \(\Phi|_U\), which is obtained by setting

\[
\Phi := \max\{\Psi + B \rho + 1/B, \Phi|_U\}
\]

for \(B\) sufficiently large, \(\Psi\) is the metric defined by formula \ref{eq:2.10} and \(\rho\) is the psh function \(\rho := \max\{\log |\tau|, (\log |\tau| - 1)\}\) on \(X \times A\), which has the property that \(\rho \leq 0\) and \(\rho < 1\) in the interior. Then \(\Phi\) is a positively curved extension of \(\Phi\) from \(U\) coinciding with \(\Psi + B \rho + 1/B\) close to the boundary of \(X \times A\) and since \(2 \log |\tau|\) extends to define a metric on \(O(1) \to P^1\) it follows from the definition \ref{eq:2.10} of \(\Psi\) that \(\Psi\) extends to a metric on \(L_m\) for \(m\) a sufficiently large integer (depending on \(A\)), which concludes the proof of the claim.

Finally, to prove the last point it is, by approximation, enough to consider the case when \(\phi_t\) is an affine curve in \(H(X, L)^T\). But then the concavity follows immediately from the fact that \(MA(\Phi) \leq 0\) for \(\Phi\) corresponding to an affine curve in \(H(X, L)^T\) (and using that \(g\) is non-negative).

\[\square\]

2.6. Monge-Ampère equations and variational principles. Recall that the pluricomplex energy of a measure \(\mu\), that we shall denote by \(E(\mu)\), may be defined
by

\[ E(\mu) := \sup_{\phi \in PSH(X,L)} -J_\mu(\phi) \]

where \( J_\mu \) is the following \( \mathbb{R} \)-invariant functional

\[(2.13) \quad J_\mu(\phi) = -\mathcal{E}(\phi) + \mathcal{L}_\mu(\phi), \quad \mathcal{L}_\mu(\phi) := \int_X (\phi - \phi_0)\mu \]

(see [13]). Similarly, in the presence of a torus \( T \) we define, for any given function \( g \) on the corresponding moment polytope \( P \), the \( g \)-modified energy \( E_g(\mu) \) be replacing the functional \( \mathcal{E} \) with \( \mathcal{E}_g \) and the space \( PSH(X,L) \) with its \( T \)-invariant subspace \( PSH(X,L)^T \).

**Theorem 2.18.** Let \( \mu \) be a probability measure on \( X \) which is \( T \)-invariant and assume that \( g \mu \) is a probability measure on the moment polytope \( P \) such that \( 1/C \leq g \leq C \) for some positive constant \( C \). Then there exists a finite energy \( T \)-invariant solution \( \phi \) to the equation

\[(2.14) \quad MA_g(\phi) = \mu \]

iff \( \mu \) has finite (pluricomplex) energy. Moreover, the solution is unique modulo constants and can be characterized as the unique (mod \( \mathbb{R} \)) minimizer of the functional \( J_{\mu,g} \).

**Proof.** Existence: Let \( PSH(X,L)_0 \) be the subspace of all metrics \( \phi \) such that \( \sup_X (\phi - \phi_0) = 0 \), which is compact in the \( L^1 \)-topology. In particular, \( -\mathcal{E}(\phi) \geq 0 \) on \( PSH(X,L)_0 \). Let us first assume that \( \mu \) has finite energy, i.e. \( J_\mu \) is bounded from below. Then, as shown in [13], the functional \( J_\mu \) is even coercive in the sense

\[ J_\mu(\phi) \geq A(-\mathcal{E}(\phi)) - B \]

for some positive constants \( A \) and \( B \). In particular, by the assumption on \( g \) the corresponding \( g \)-modified functional \( J_{\mu,g} \) is also coercive. Hence, taking a sequence \( \phi_j \) in \( PSH(X,L)_0 \) such that \( J_{\mu,g}(\phi_j) \to \inf J_{\mu,g} \) the coercivity implies that \( -\mathcal{E}(\phi)(\phi_j) \leq C \) for some constant \( C \). Accordingly, after passing to a subsequence, \( \phi_j \) converges in \( L^1 \) to a metric \( \phi_* \) in \( \{ -\mathcal{E} \leq C \} \) (using that \( \mathcal{E} \) is usc, i.e. the previous sublevel sets are compact). Next, as shown in [13] \( L_\mu \) is finite and even continuous on the compact sublevel sets \( \{ -\mathcal{E} \leq C \} \) and since \( \mathcal{E}_g(\phi) \) is also upper semi-continuous (by Prop 2.15) it follows that

\[ J_{\mu,g}(\phi_*) = \lim_{j \to \infty} J_{\mu,g}(\phi_j) := \inf J_{\mu,g} \]

i.e. \( \phi_* \) is a minimizer of \( J_{\mu,g} \). Finally, to show that \( \phi_* \) satisfies the equation [2.14] we set \( f(t) := -\mathcal{E}_g(P(\phi + tu)) + L_\mu(\phi) \) for a given smooth function \( u \) on \( X \), which defines a function on \( \mathbb{R} \) with an absolute minimum at \( t = 0 \) (using that \( P \) is decreasing) and which is differentiable (by Prop 2.16). In particular, the derivative of \( f \) vanishes at \( t = 0 \) which by Prop 2.16 implies (since \( u \) was arbitrary) that the equation [2.14] holds, as desired.

Conversely, if \( \phi \) is a finite energy solution of the equation [2.14] then it follows from the affine concavity of \( \mathcal{E}_g \) (Prop 2.17) that \( \phi \) is a global minimizer of the convex functional \( J_{\mu,g} \).

**Uniqueness:** Let now \( \psi_0 \) and \( \psi_1 \) be two finite energy solutions to the equation [2.14] in the convex space \( V \) of all finite energy metrics \( \psi \) normalized so that \( L_\mu(\psi) = 0 \). By the affine concavity of \( \mathcal{E}_g \) on \( V \) this means that \( \psi_1 \) are minimizers of the functional \( -\mathcal{E}_g \) on \( V \). Let now \( \psi_t \) be the affine curve connecting \( \psi_0 \) and \( \psi_1 \) and set
terminal singularities with the property that the $X$ consisting of a Fano variety for some fixed positive constants $A$.

3.1. Setup. Our basic complex geometric data in this section will be a pair $(X, V)$ consisting of a Fano variety $X$ (i.e. $X$ is a projective normal variety with log terminal singularities with the property that the $\mathbb{Q}$-line bundle $-K_X$ is ample) and $V$ a complex holomorphic vector field on the regular locus $X_{reg}$ (see [14] and references therein for further background on Fano varieties). We will denote by $\text{Aut}(X, V)$ the group of all biholomorphisms of $X$ which commute with the flow of $V$ (or equivalently: all automorphisms $F$ such that $F_* V = V$) and by $\text{Aut}(X, V)_0$ the connected component of the identity.

Any given metric $\phi \in PSH(X, -K_X)$ induces a measure $\mu_{\phi}$ on $X$, which may be defined as follows: if $U$ is a coordinate chart in $X_{reg}$ with local holomorphic coordinates $z_1, \ldots, z_n$ we let $\phi_U$ be the representation of $\phi$ with respect to the local trivialization of $-K_X$ which is dual to $dz_1 \wedge \cdots \wedge dz_n$. Then we define the restriction of $\mu_{\phi}$ to $U$ as

$$
\mu_{\phi} = e^{-\phi_U} i^n dz_1 \wedge d\bar{z}_1 \cdots \wedge dz_n \wedge d\bar{z}_n
$$

We claim that there exists a positive constant $a$ such that

$$
\frac{d^2}{dt^2} f_g \geq a \frac{d^2}{dt^2} f \geq 0
$$

weakly on $[0, 1]$. Accepting this for the moment it follows that $\frac{d}{dt} f_g(0) = \frac{d}{dt} f_g(1) = 0$ and hence $f(t)$ is also affine on $[0, 1]$. But this means that $\frac{d}{dt} f(0) = \frac{d}{dt} f(1) = 0$, i.e. $\int (\psi_0 - \psi_1) MA(\psi_0) = \int (\psi_0 - \psi_1) MA(\psi_1) = 0$. In particular, $I(\phi_0, \phi_1) := \int (\psi_0 - \psi_1)(MA(\psi_1) - MA(\psi_0)) = 0$. But as is well-known this implies by an argument essentially due to Blocki (see [13, 16] and references therein) that $\psi_0 - \psi_1$ is constant and thus zero, by our normalization assumption. Finally, the claimed inequality (2.15) follows from formula (2.11) which gives that for $\psi_i$ smooth

$$
dd^c f_g = -\int_X MA_g(\Psi), \quad dd^c f = -\int_X MA(\Psi)
$$

Moreover, for any affine path $\psi$, we have (by a direct calculation) that $MA(\Psi) \leq 0$ and hence, by definition, $-MA_g(\Psi) \geq -aMA(\Psi)$ proving the inequality (2.15) in the smooth case. The general case then follows by regularization using that $f_g^{(j)} \to f_g$ etc as in the proof of Prop (2.17).

2.6.1. Completion of the proof of Theorem 2.18. First observe that since, by assumption $g \geq 1/C > 0$, Cor (2.9) implies that any solution $\phi$ satisfies $MA(\phi) \leq C\mu$. In particular, if $\mu$ has a density in $L^p_{loc}$ for some $p > 1$ then so has $MA(\phi)$ and thus it follows from the generalizations of Kolodziej’s results in [27] that $\phi$ is continuous. More generally, this is the case as long as $\mu \leq A\text{Cap}^p$ for some $p > 1$, where $\text{Cap}$ denotes the set functional defined by the global Bedford-Taylor capacity on $X$. Finally, to prove the existence of a solution $\phi$ for any $T$-invariant pluripolar probability measure $\mu$ one uses, just as in [13], a decomposition argument to reduce the problem (using the continuity properties for $MA_g$ established in section above) to the case when $\mu$ has finite energy, or more precisely to the special case when $\mu$ is in the weakly compact subset $\mathcal{A}$ of all probability measures satisfying $\mu \leq A\text{Cap}^p$ for some fixed positive constants $A$ and $p$ such that $p > 1$.
This expression is readily verified to be independent of the local coordinates $z$ and hence defines a measure $\mu_\phi$ on $X_{\text{reg}}$ which we then extend by zero to all of $X$. The Fano variety $X$ has log terminal (klt) singularities precisely when the total mass of $\mu_\phi$ is finite for some $\phi \in \text{PSH}(X, L)$ (or equivalently: the finiteness holds for any locally bounded metric $\phi$). More precisely, if $X$ has log terminal singularities then the the pull-back of $\mu_\phi$ to any given resolution $X'$ of $X$ has has local densities in $L^p_{\text{loc}}$ for some $p > 1$ (see [14] for the equivalence with the usual algebraic definition involving discrepancies on $X'$).

We will say that a Kähler metric $\omega$ defined on the regular part $X_{\text{reg}}$ of a normal Fano variety $X$ is a (singular) Kähler-Ricci soliton on $X$ if the metric $\omega$ solves the equation

$$\text{Ric} \omega = \omega + L_V \omega,$$

for some complex holomorphic vector field $V$ on $X_{\text{reg}}$ (where $L_V$ denotes the Lie derivative along $V$) and moreover the volume of the metric $\omega$ on $X_{\text{reg}}$ is maximal in the sense that it coincides with the global algebraic top intersection number $c_1(-K_X)^n$ (then the pair $(\omega, V)$ will also be called a Kähler-Ricci soliton). In the case when $X$ is smooth it is well-known that $(\omega, V)$ is a Kähler-Ricci soliton precisely when $\omega = \omega_\phi$ for a unique smooth metric $\phi$ in $\mathcal{H}(-K_X)$, which is invariant under $\text{Im} V$, satisfying the equation

$$MA_{g_V}(\phi) = \mu_\phi$$

where $g_V$ is the normalized exponential function $g_V$ on the Lie algebra of the real torus $T$ acting on $(X, -K_X)$ generated by $V$, i.e. $g_V(m_\phi) = e^{f_\phi}/C$, where $f_\phi$ is the Hamiltonian function on $X$ determined by the canonical lift of $V$ to $-K_X$ (compare section [23]). Indeed, denoting by $\psi_\omega$ the metric on $-K_X$ determined by the volume form of $\omega$ we have, by definition, that $\text{Ric} \omega_\phi$ is the curvature form of $\psi_\omega$ and since $L_V \omega_\phi = dd^c f_\phi$ the equation (3.2) holds iff $\psi_\omega - (\phi + h_\omega)$ is constant, i.e. iff the equation (3.3) holds.

As a final matter of notation we will say that a (singular) metric $\phi$ in $\text{PSH}(-K_X)^T$ is a weak Kähler-Ricci soliton (wrt $V$ generating an action of $T$) if it has full Monge-Ampère mass and the equation (3.3) holds on $X$.

3.2. Tian-Zhu’s modified functionals and Futaki invariant. Recall that in the smooth case the Mabuchi K-energy functional $\mathcal{M}$ of a Fano manifold $X$ equipped with its standard polarization $L = -K_X$ is, up to an additive constant, defined by the property that its differential on $\mathcal{H}(X, -K_X)$ is given by

$$d\mathcal{M}_\phi = -(\text{Ric} \omega_\phi - \omega_\phi) \wedge \omega_\phi^n.$$

In particular its critical points are Kähler-Einstein metrics on $X$. Similarly, in the presence of a vector field $V$ the modified K-energy functional of Tian-Zhu [43], that we shall denote by $\mathcal{M}_V$, is obtained by replacing $\text{Ric} \omega_\phi$ with the modified Ricci curvature $\text{Ric} \omega_\phi - L_V \omega_\phi$ (compare [43]). In order to deal with singular metrics varieties we will modify the singular setup for Kähler-Einstein metrics introduced in [44], by defining the modified Mabuchi functional $\mathcal{M}_V(\phi)$ on the space $\mathcal{E}(-K_X)^T$ by the formula

$$\mathcal{M}_V(\phi) := F_V(\mathcal{M}(\phi)),$$
where $F_V$ is the modified free energy functional on the space of all $T$-invariant probability measure with finite energy, defined by

$$F_V(\mu) := -E_V(\mu) + H(\mu, \mu_{\phi_0}),$$

where $E_V(\mu) := E_{g_V}(\mu)$ is the $g$-energy defined in the beginning of section 2.6 and $H(\mu, \mu_{\phi_0})$ is the entropy of $\mu$ relative to $\mu_{\phi_0}$, i.e.

$$H(\mu, \mu') = \int_X \log(\frac{d\mu}{d\mu'}) d\mu$$

if $\mu$ is absolutely continuous wrt $\mu'$ and $H(\mu, \mu') = \infty$ otherwise. Next, we define the modified Ding functional $D_V$ by

$$D_V(\phi) = -\mathcal{E}_V(\phi) + \mathcal{L}(\phi), \quad \mathcal{L}(\phi) = -\log \int_X e^{-\phi}$$

(compare [55], [47] for the smooth case). We will have great use for the convexity properties of the functional $\mathcal{L}(\phi)$, originating in the work of Berndtsson [15], which combined with Prop 2.17 gives the following

**Proposition 3.1.** Let $\phi_t$ be a bounded geodesic. Then the function $t \mapsto \mathcal{L}(\phi_t)$ is convex on $[0,1]$. Moreover, if the function is affine, then there exists a family of automorphisms $F_t$ in $\text{Aut}(X)_0$ such that $F_t^* \phi_t = \phi_t$. As a consequence, by Prop 2.17, the function $t \mapsto D_V(\phi_t)$ is also convex with the same necessary conditions to be affine.

Next, using the thermodynamic formalism in [6] we also have the following

**Proposition 3.2.** The functional $D_V$ is bounded iff the functional $M_V$ is bounded and their infima coincide. Moreover, in general, $M_V(\phi) \geq D_V(\phi)$ with equality iff $\phi$ is a Kähler-Ricci soliton.

**Proof.** This is shown precisely as in [6] using that, by definition, $D_V = -E^*_V + H^*$ where the upper star denotes the Legendre transform between convex functionals on the space $\mathcal{P}$ of all probability measures on $X$ and concave functionals on the space of all continuous metrics on $L$ (using the pairing $\langle \phi, \mu \rangle := -\int (\phi - \phi_0) d\mu$ between the latter spaces).

**Theorem 3.3.** Let $X$ be a Fano variety with log terminal singularities. Then the following is equivalent for a metric $\phi$ in $E^1(-K_X)$:

- $\phi$ minimizes the modified Ding functional $D_V$
- $\phi$ minimizes the modified Mabuchi functional $M_V$
- $\phi$ is a weak Kähler-Ricci soliton for $(X,V)$.

**Proof.** If $\phi$ is a weak Kähler-Ricci soliton for $(X,V)$, then, by the convexity in the previous theorem it minimizes $D_V$. Conversely, if $\phi \in E^1(-K_X)$ minimizes $D_V$, then we deduce that it satisfies the Kähler-Ricci soliton equation, by repeating the projection argument used in the proof of the existence part in Theorem 2.18 (replacing the functional $L_\mu$ used there with the functional $\mathcal{L}$). The proof is now concluded by invoking the previous proposition. \[\square\]
3.2.1. The modified Futaki invariant. Recall that, in the smooth case, the modified Futaki invariant of Tian-Zhu, attached to $(X, V)$ [48], may be defined as the following real valued function on the space of all holomorphic vector fields $W$ generating a $\mathbb{C}^∗$—action

$$\text{Fut}_V(W) := \frac{M(\phi^W_t)}{dt}, \quad \phi^W_t := \exp(tW)^∗φ,$$

where $φ$ is any metric in $\mathcal{H}(X, −K_X)$ which is invariant under the $S^1$—action generated by the imaginary part of $W$. Strictly speaking the original definition in [48] may, in our notation, be written as

$$\text{Fut}_V(W) = c_n \int W(f^W_{φ_0} − h_{φ_0})ω^n/\phi_0,$$

where $f^W_{φ_0}$ is the Hamiltonian function corresponding to $W$ (compare the Lemma below) and $h_{φ_0}$ is the normalized Ricci potential of $ω_{φ_0}$. The equivalence with the definition (3.4) follows from standard integration by parts. Anyway, here it will be convenient to use the definition (3.4) which applies verbatim in the general singular setting and which is in line with the notion of modified K-stability introduced below. The modified Futaki invariant is independent of $φ$ and always finite, as follows, for example, from the following alternative formula:

Lemma 3.4. Let $X$ be a Fano variety with log terminal singularities. Then

$$\text{Fut}_V(W) = -\frac{dE_V(φ^W_t)}{dt} = -\int_X f^W_{φ_0} \exp(f^V_φ)MA(φ_0)$$

which is independent of $φ_0$, where $f^W_{φ_0}$ and $f^V_φ$ denote the Hamiltonian functions determined by the canonical lift to $−K_X$ of the vector fields $V$ and $W$, respectively.

Proof. By definition, $M_V(φ) − D_V(φ) = C \int h_φ e^{h_φ}ω^n/φ$ where $C$ is a constant and $h_φ$ is the normalized Ricci potential of $φ$, i.e. $h_φ = \log MA(φ) − (φ − L(φ)).$ But the latter integral is invariant under $φ → F^∗φ$ for any automorphism $F$ of $X$ and in particular for $F = \exp(tW)^∗φ$. In other words, $M_V(φ^W_t) − D_V(φ^W_t)$ is independent of $t$ and in particular the derivatives wrt $t$ vanish, which proves the first identity in the Lemma. The second identity follows directly from the very definition of the objects involved and the independence wrt $φ_0$ then follows from the independence wrt $φ_0$ of the D-H measure attached to the torus $S^1 × T$ generated by $(W, V)$. □

In section 4.4 a “quantized” version of the previous lemma will be given which provides an algebraic (i.e. metric free) formulation of the modified Futaki invariant.

Proposition 3.5. Assume that $(X, V)$ is such that $M_V$ (or equivalently $D_V$) is bounded from below. Then Fut$_V(W) = 0$ for any vector field $W$. In particular, this is the case if $(X, V)$ admits a Kähler-Ricci soliton.

Proof. Assuming that $D_V(φ^W_t)$ is bounded from below and using that $L(φ^W_t)$ is constant wrt $t$ it follows that $E_V(φ^W_t)$ is bounded from below, forcing $−\frac{dE_V(φ^W_t)}{dt} ≥ 0$. Finally, replacing $W$ with $−W$ forces the converse inequality $\frac{dE_V(φ^W_t)}{dt} ≥ 0$. □
3.3. Regularity: Proof of Theorem 1.3. Global continuity: Given a smooth Kähler-Ricci soliton ω it extends, by normality to a unique positive current ω in $c_1(-K_X)$. In particular, $\omega = \omega_\psi$ for a metric $\phi$ in $PSH(X, -K_X)$. Set $\psi := -\log MA_\phi(\phi)$ which defines a smooth metric on $-K_X \to X_{reg}$ whose curvature form $\omega_\psi$ coincides with $\omega_\phi$ on $X_{reg}$ (by the Kähler-Ricci soliton equation). In particular, $\psi$ extends to a unique element in $PSH(X, -K_X)$, whose curvature current will still be denoted by $\omega_\psi$. But then $\omega_\phi = \omega_\psi$ on $X_{reg}$ implies, by normality, that $\phi = \psi + C$ on $X$ for a constant $C$ and hence restricting to $X_{reg}$ reveals that the equation 3.3 holds for $\phi$ holds on $X_{reg}$, up to replacing $\phi$ by $\phi - C$. Moreover, since $MA_\phi(\phi)$ does not charge pluripolar sets the equation 3.3 holds globally on $X$. In particular, the mass of the measure $\mu_\phi$ with density $e^{-\phi}$ is bounded from above by $\int_X MA_\phi(\phi)$ for $\phi \in PSH(X, -K_X)$ and hence finite, which shows that $X$ has log terminal singularities. Moreover, by the volume assumption, $\int_X MA(\phi) := \int_{X_{reg}} \omega_\phi^\alpha/e^\beta(-K_X) = 1$ and hence the singular metric $\phi$ in $PSH(X, -K_X)$ has maximal Monge-Ampère mass. But then $\phi$ has no Lelong numbers and the measure corresponding to $\mu_\phi$ on any smooth resolution $X'$ of $X$ has a density in $L^p_{loc}$ for any $p > 1$ (see the appendix in [14]). Finally, combining Cor 2.9 with the global equality $MA_\phi(\phi) = \mu_\phi$ gives $MA(\phi) \leq C \mu_\phi$ and hence, by [27], the $L^p$-property of $\mu_\phi$ implies that $\phi$ is continuous.

Smoothness: To conclude the proof we need to show that any continuous metric $\phi$ satisfying the equation 3.3 is in fact smooth on the regular locus of $X$. To this end we let $\phi_t$ be the Kähler-Ricci flow starting at the continuous metric $\phi$(constructed by Song-Tian [42]; compare the proof of Theorem 3.13 below). By [12] $\phi_t$ is smooth on $X_{reg}$ for any $t > 0$. Next, we note that $\psi_t := \exp(tV)^* \phi_t$ evolves according to the modified Kähler-Ricci flow and in particular $D_V(\psi_t)$ is decreasing in $t$ (as in the proof of Theorem 3.13). Since $\psi_0$ minimizes $D_V$ (by Theorem 3.3), it then follows that so does $\psi_t$ for any $t > 0$ and hence invoking Theorem 3.3 again gives that $\psi_t$ is a Kähler-Ricci soliton for $(X, V)$ and hence a stationary point for the modified Kähler-Ricci flow, i.e. $\psi_t$ is independent of $t > 0$. But then letting $t \to 0$ we conclude that $\psi_t = \psi_0 = \phi$ for any $t > 0$, which shows that $\phi$ is smooth on $X_{reg}$, as desired.

3.4. Uniqueness and reductivity.

Theorem 3.6. Let $X$ be a Fano variety and $(\omega_0, V_0)$ and $(\omega_1, V_1)$ be two Kähler-Ricci solutions. Then there exists $F \in Aut(X)_0$ such that $F^* \omega_1 = \omega_0$ and $F^* V_1 = V_0$. In particular, if $V_0 = V_1$ then $F \in Aut(X, V_0)_0$. More precisely, in the general case $F$ can be taken as the time one flow map defined by a real vector field $Y$ on $X$ such that $JY$ is in the compact isometry group $Iso(X, \omega_1)$.

Proof. Let us first fix a vector field $V$ and let $\phi_0$ and $\phi_1$ be two weak Kähler-Ricci solitons, defined with respect to $V$ and in particular $T$-invariant. By Theorem 1.20 $\phi_0$ and $\phi_1$ are locally bounded. Denote by $\phi_t$ the corresponding bounded geodesic curve in $PSH(-K_X)^T$. Consider the function $f(t) := D_V(\phi_t)$. By the minimization property in Theorem 3.3 and the fact that $f(t)$ is convex (by Prop 3.3) it follows that $f(t)$ is affine on $[0, 1]$. In particular, by Prop 3.3 $L(\phi_t)$ is affine. But then Prop 3.3 implies that there exists $F_t$ in $Aut(X)_0$ such that $\phi_t = F_t^* \phi_0$. Next, we note that $F_t$ preserves $V$, i.e. $F_t$ is in $Aut(X, V)_0$. Indeed, $\phi_t$ minimizes $D_V$ for any $t$ and hence by Theorem 3.3 $(\phi_t, V)$ is a weak Kähler-Ricci soliton. But since $\phi_t = F_t^* \phi_0$ it then follows that $L_Z \omega_0 = 0$ for $Z := F_t^* V - V$. But then $dd^c f = 0$.
for $f$ the Hamiltonian function of $F_t^* V - V$ defined on $X_{reg}$, which, by normality, forces $f = 0$, i.e. $F_t^* V = V$, as desired. To see that there exists a vector field $W$ as in the theorem, we recall that the proof of the convexity properties of $D_Y$ in \cite{15,14} realizes $F_t$ by integrating a family of holomorphic vector fields $V_t$ where $V_t$ has Hamiltonian function $h_t := d\phi_t/dt$ and hence $W_t := \text{Re}V_t = J\text{Im}V_t$, where $\text{Im}V_t$ preserves $\omega_t$. Moreover, as pointed out in the end of the exposition of the proof in the appendix of \cite{21} (III) it follows from the relation $\phi_t = F_t^* \phi_0$ combined with the fact that $\phi_t$ is a smooth geodesic on $X_{reg}$ that $F_t$ is, in fact, a one-parameter group generated by the vector field $Y := Y_0$.

Finally, the case of different vector fields $V_0$ and $V_1$ can be reduced to the previous case, by the arguments in \cite{48}. For completeness we recall the elegant argument. First, by Iwasawa's theorem, we may, after perhaps replacing $V_1$ with $F_* V_1$, for some $F \in \text{Aut}(X)_0$, assume that the one-parameter isometry groups generated by the imaginary parts of $V_0$ and $V_1$, respectively, are contained in the same compact subgroup $K$ of isometries as $(X,\omega_0)$. In fact, since $[V, W] = 0$ (by the argument in the beginning of the proof of the next corollary) we may even assume that they are contained in Lie algebra of the same subtorus $T'$ of $K$. Now consider the functional on the Lie algebra of $T'$, defined by $\mathcal{F}(V) := \int_X \exp(f^V_{\phi_0})\omega_{\phi_0}^n$ (which is independent of $\phi_0$). Its differential is given by $(d\mathcal{F}(V))(W) := \int_X f^W_{\phi_0} \exp(f^V_{\phi_0})\omega_{\phi_0}^n$ and hence if $V$ is defined by a Kähler-Ricci soliton, then it is a critical point of $V$ (compare Prop 3.5). But the functional $\mathcal{F}(V)$ is strictly convex and hence $V$ is uniquely determined, as desired.

**Corollary 3.7.** Let $X$ be a Fano variety and $V$ a holomorphic vector field on $X$. If $(X, V)$ admits a Kähler-Ricci soliton $\omega$, then $\text{Aut}(X, V)_0$ is reductive. More precisely, the group $\text{Aut}(X, V)_0$ may be identified with the complexification of the compact group $\text{Iso}(X, \omega)$ consisting of all isometries of $(X, \omega)$, i.e. all elements $F$ in $\text{Aut}(X)_0$ such that $F^* \omega = \omega$.

**Proof.** First observe that $\text{Iso}(X, \omega) \subset \text{Aut}(X, V)_0$. Indeed, by assumption, $F_* \omega = \omega$ and hence applying $F^*$ to both sides in the Kähler-Ricci soliton equation forces $L_{F_* V - V} \omega = 0$. But then it follows as in the proof of the previous theorem that $F_* V - V = 0$. Hence, $F$ is in $\text{Aut}(X, V)_0$, as desired. Moreover, the group $K := \text{Iso}(X, \omega)$ is a compact Lie group (compare the proof of Lemma \cite{13,14}) and we denote by $K_c$ its complexification. Given the previous theorem the rest of the argument proceeds exactly as in the elegant argument for the Kähler-Einstein case in \cite{21}: let $g$ be an element in the Lie group $G := \text{Aut}(X, V)_0$. Then $g^* \omega$ is also a Kähler-Ricci soliton wrt $V$ and hence, by the previous theorem, there exists a one parameter group $F_t$ generated by a vector field $Y$ in the Lie algebra of $K_c$ and $F^*_t g^* \omega = \omega$. But then $g \circ (F_1)^{-1}$ is in $K$ and since $F_1$ is in $K_c$ we conclude (since $K \circ K_c \subset K_c$) that so is $g$.

**3.5. K-stability.** Recall that a special test configuration for a Fano variety $X$ is defined by a variety $\mathcal{X}$ equipped with a $\mathbb{C}^*$-action $\rho$ and an equivariant morphism $\pi$ to $\mathbb{C}$ (with its standard $\mathbb{C}^*$-action) such that the (scheme theoretic) fibers are Fano varieties with log terminal singularities and $\pi^{-1}\{1\} = X$. We will denote by $\mathcal{W}$ the holomorphic vector field on $\mathcal{X}$ generating the $\mathbb{C}^*$ action, which restricts to a holomorphic vector field $W_0$ on the central fiber $X_0$. More generally, we will say that $(\mathcal{X}, \rho_\mathcal{W}, V)$ is a special test configuration for $(X, V)$, if the vector field $V$ on the
generic fiber $X$ is the restriction of a holomorphic vector field $\mathcal{V}$ on $\mathcal{X}$ preserving the fibers of $\mathcal{X}$ and commuting with $\mathcal{W}$.

**Definition 3.8.** The modified Futaki invariant $\text{Fut}(\mathcal{X}, \rho_{\mathcal{W}}, \mathcal{V})$ of a test configuration $(\mathcal{X}, \rho_{\mathcal{W}}, \mathcal{V})$ is defined as the modified Futaki invariant $F_{\mathcal{V}_0}(W_0)$ of the induced holomorphic vector field $W_0$ on the central fiber $X_0$.

More concretely, realizing $X$ as subvariety of a projective space $\mathbb{P}^N$ in such a way that a multiple of $-K_X$ gets identified with $\mathcal{O}(1)|_X$ it is enough to consider test configurations $(\mathcal{X}, \rho, \mathcal{V})$ such that $\mathcal{X} \subset \mathbb{P}^N \times \mathbb{C}$ with $\rho$ the restriction to $\mathcal{X}$ of a $\mathbb{C}^*$--action on $\mathbb{P}^N$ preserving $\mathcal{X}$ and $\mathcal{V}$ the restriction to $\mathcal{X}$ of a holomorphic vector field on $\mathbb{P}^N$ commuting with $\rho$ and preserving the fibers of $\mathcal{X}$ and coinciding on the generic fiber $X$ with the torus action generated by $V$.

**Definition 3.9.** A pair $(X, V)$ consisting of a Fano variety $X$ with log terminal singularities and a holomorphic vector field $V$ on $X$ is said to be $K$-polystable if the modified Futaki invariant of any special test configuration $(\mathcal{X}, \rho_{\mathcal{W}}, \mathcal{V})$ for $(X, V)$ is non-negative and zero iff $(X_0, V_0)$ is isomorphic to $(X, V)$.

3.5.1. **Proof of Theorem 1.5.** First recall that a test configuration $(\mathcal{X}, \rho_{\mathcal{W}})$ for a Fano variety $X$ together with a continuous metric $\phi_0$ in $PSH(-K_X)$ induces a geodesic ray $\phi_t$ of continuous metrics in $PSH(-K_X)$. The curve $\phi_t$ may, using the $\mathbb{C}^*$ action $\rho_{\mathcal{W}}$, be identified with an $S^1$--invariant continuous metric $\Phi$ in $PSH(M, -K_{M/\Delta})$ where $M := \mathcal{X}|_{\Delta}$ and $\Delta$ is the unit-disc in $\mathbb{C}$ (see [13] and references therein) and $\Phi$ satisfies $(dd^c \Phi)^{n+1} = 0$ in the interior of $M$ and may be identified with $\phi_0$ on $\partial M$. In the case when $(\mathcal{X}, \rho_{\mathcal{W}}, \mathcal{V})$ is a test configuration for $(X, V)$ and $\phi_0$ is invariant under the real torus $T$ induced by $V$ it follows (from the uniqueness of solutions to the Dirichlet problem above) that $\Phi$ is invariant under the corresponding real torus action on $\mathcal{X}$ and hence $\phi_t$ is a geodesic ray in $PSH(-K_X)^T$.

Now, if $(X, V)$ admits a Kähler-Ricci soliton then we can take $\phi_0$ as the corresponding metric on $-K_X$. Consider the functional $f(t) := \mathcal{D}_V(t)$. By the convexity of $f(t)$ (see Prop 3.1) we then get

\begin{equation}
0 \leq \lim_{t \to \infty} \frac{df(t)}{dt} \bigg|_{t=0} = \lim_{t \to \infty} \frac{d\mathcal{E}_V(\phi_t)}{dt} \bigg|_{t=0} + 0,
\end{equation}

using in the last equality that

\[
\lim_{t \to \infty} \frac{d\mathcal{L}(\phi_t)}{dt} = 0
\]

for a special test configuration, as shown in [13] (we recall that the key point is the vanishing of the Lelong number of the $L^2$--metric on the direct image of $-K_{X/\Delta}$). Hence, by combining the following proposition with Lemma 2.13 it follows that $0 \leq \text{Fut}(\mathcal{X}, \rho_{\mathcal{W}}, \mathcal{V})$.

**Proposition 3.10.** Let $(\mathcal{X}, \rho_{\mathcal{W}}, \mathcal{V})$ be special test configuration for $(X, V)$, $\phi_0$ a continuous metric in $PSH(-K_X)^T$ and denote by $\phi_t$ the corresponding geodesic ray. Then

\begin{equation}
\frac{d\mathcal{E}_V(\phi_t)}{dt} = \frac{d\mathcal{E}_{V_0}(\phi_t^W)}{dt}
\end{equation}
More generally, the equality holds when $t \to \infty$ if $\phi_t$ is replaced by the subgeodesics defined by a locally bounded metric $\Phi$ on $-K_X/\Delta$.

**Proof.** We start by noting that the last statement about subgeodesics follows from the first one about geodesics. Indeed, if $\phi_t$ and $\psi_t$ are such geodesics and subgeodesics, respectively, then $|\phi_t - \psi_t| \leq C$ for a constant independent of $t$ and hence $f(t) := dE_V(\psi_t) - E_V(\phi_t)$ is a bounded convex function. In particular, its derivative tends to zero as $t \to \infty$. Next, we first assume that $X$ is smooth. By Prop 2.17 $dE_V(\phi_t)/dt$ is constant and equal to the total integral of $\partial_{\phi_t}MA_{gv}(\phi_t)$. In other words, $dE_V(\phi_t)/dt$ is equal to the first moment $\int_R w\gamma_{\phi_t}(w)$ of the probability measure

$$\gamma_{\phi_t} := (d\phi_t/dt)_{\ast}MA_{gv}(\phi_t)$$
onumber

on $\mathbb{R}$. The previous proposition can now be obtained as a special case of the following convergence of probability measures on $\mathbb{R}$ (that holds for any $t \in [0,\infty[$):

$$\gamma_{\phi_t} = \gamma_{(V_0,W_0)} := h^W_0(MA_{gv0}(\phi_t^{W_0})) = \lim_{k \to \infty} \frac{1}{N_k} \sum_{l=1}^{N_k} \exp(v^{(k)}_l/k) \delta_{u_l^{(k)}/k}$$

where $(v^{(k)},u^{(k)})$ are the joint eigenvalues on $H^0(X_0, -kK_{X_0})$ of the real parts of $V_0$ and $W_0$, respectively (compare Prop 4.7). This result is obtained by adapting the proof of the main result in [32] (concerning the case when there is no torus action, i.e. $g = 1$) to the present torus setting. The point is that the commuting pair $(V,W)$ of vector fields on $X$ generate an $m + 1$-dimensional torus $S^1 \times T$ on $X$. In particular, integrating formula 3.7 against the linear test function $w$ gives

$$\frac{dE_V(\phi_t)}{dt} = \int_R w\gamma_{\phi_t}(w) = \lim_{k \to \infty} \frac{1}{N_k} \sum_{l=1}^{N_k} \exp(v^{(k)}_l/k)u_l^{(k)}$$

and invoking Prop 4.7 thus concludes the proof of formula 3.6. □

Next, in the case when $0 = \text{Fut}(X, \rho_W, V)$ we must have equalities in the inequality in formula 3.5. In particular, $D(\phi_t)$ is affine and hence, by Prop 3.1, there exists a family of holomorphic vector fields $Y_\tau$ on $X$ inducing biholomorphisms $F_\tau$ from $X$ to $X_\tau$ such that $F_\tau^\ast \phi_\tau = \phi$. Finally, using that $\Phi$ is continuous on all of $X$ one obtains, just as in the [8] that $F_\tau$ converges as $\tau \to 0$ to a biholomorphism between $X$ and $X_0$. The proof is concluded by noting that $F_\tau$ commutes with the flow of $V$. Indeed, as observed above, minimizes $D_V$ and is hence a Kähler-Ricci soliton wrt $V$. But $F_\tau^\ast \phi_\tau$ is also a Kähler-Ricci soliton wrt $F_\tau^\ast V$ and hence (just as in the in the proof of Theorem 3.6) it follows that $F_\tau^\ast V = V$, as desired.

### 3.6. Analytic K-stability and the Kähler-Ricci flow

Given a pair $(X, V)$ with $X$ a Fano variety (with log terminal singularities) and $V$ a holomorphic vector field generating an action of a torus $T$, we will say that a functional $F$ on $H(-K_X)^T$ is **proper modulo $\text{Aut}(X,V)$** if it is proper with respect to the $\text{Aut}(X,V)$–invariant exhaustion function

$$\bar{J}(\phi) := \inf_{F \in \text{Aut}(X,V)} J(F^\ast \phi)$$

i.e. if $D_V(\phi) \leq C$ implies that $\bar{J}(\phi) \leq C'$, where $C'$ only depends on $C$. We will also say that a functional $F$ on $H(-K_X)^T$ is **strongly proper (or coercive)**
modulo $\text{Aut}(X, V)_0$ if there exist positive constants $A$ and $B$ such that, for any $\phi \in \mathcal{H}(-K_X)^T$ there exists $F \in \text{Aut}(X, V)_0$ satisfying
\[ F(\phi) \geq A \tilde{J} - B \]

We will say that $X$ is analytically $K$-polystable if the modified Mabuchi functional $\mathcal{M}_V$ on $\mathcal{H}(-K_X)^T$ is proper modulo $\text{Aut}(X, V)_0$ and analytically strongly $K$-polystable if $\mathcal{M}_V$ is coercive modulo $\text{Aut}(X, V)_0$.

**Theorem 3.11.** Let $(X, V)$ be a Fano variety with a holomorphic vector field. Assume that either the modified Ding functional $\mathcal{D}_V$ or the modified Mabuchi functional $\mathcal{M}_V$ is proper modulo $\text{Aut}(X, V)_0$. Then $(X, V)$ admits a Kähler-Ricci soliton.

*Proof.* First assume that $\mathcal{D}_V$ is proper on $\mathcal{H}(-K_X)^T$ modulo $\text{Aut}(X, V)_0$ and in particular bounded from below (strictly speaking, by Prop 3.2 it is enough to consider the case when $\mathcal{D}_V$ is proper on $\mathcal{H}(-K_X)^T$ modulo $\text{Aut}(X, V)_0$, but for future reference we start by considering the case of $\mathcal{D}_V$ separately). Then $\mathcal{D}_V$ has to be invariant under the action of $\text{Aut}(X, V)_0$. Indeed, $\mathcal{D}_V(\phi^W)$ is bounded from below with respect to $t \in \mathbb{R}$ and $W$ a holomorphic vector field commuting with $V$, i.e., an element in the Lie algebra of $\text{Aut}(X, V)_0$ and hence letting $t \to \pm \infty$ gives $d\mathcal{D}_V(\phi^W)/dt = \lim_{t \to \infty} d\mathcal{D}_V(\phi^W)/dt = 0$, as desired. By the assumed properness this means that we can take a sequence $\phi_j$ in $\mathcal{H}(-K_X)^T$ such that
\[ \lim_{j \to \infty} \mathcal{D}_V(\phi_j) = \inf_{\mathcal{H}(-K_X)} \mathcal{D}_V = \inf_X \mathcal{D}_V, \sup(\phi_j - \phi_0) = 0, -\mathcal{E}(\phi_j) \leq C \]

Hence, taking a sequence $\phi_j$ in $PSH(X, L)_0$ such that $J_{\mu, g}(\phi_j) \to \inf J_{\mu, g}$ the coercivity implies that $-\mathcal{E}(\phi_j) \leq C$ for some constant $C$. Accordingly, by the upper semicontinuity of $\mathcal{E}_V$ we may assume, perhaps passing to a subsequence, that $\phi_j$ converges in $L^1$ to a metric $\phi_\infty$ in $\mathcal{E}(-K_X)$. Moreover, as shown in [12] the functional $\mathcal{L}$ is continuous on the closed subset $-\mathcal{E}(\phi) \leq C$ and hence if follows that the limit $\phi_\infty$ realizes the infimum of $\mathcal{D}_V$. But then $(\phi_\infty, V)$ is a Kähler-Ricci soliton (by Theorem 3.3).

Next, assuming instead that $\mathcal{M}_V$ is proper modulo $\text{Aut}(X, V)_0$ we deduce just as above that there exists a sequence $\phi_j$ converging in $L^1$ to $\phi_\infty$ such that
\[ \lim_{j \to \infty} \mathcal{M}_V(\phi_j) = \inf_{\mathcal{H}(-K_X)} \mathcal{M}_V = \inf_X \mathcal{M}_V, \sup(\phi_j - \phi_0) = 0, -\mathcal{E}(\phi_j) \leq C \]

In particular, by the assumed properness, setting $\mu_j := MA(\phi_j)$ the entropies $H(\mu_j)$ are uniformly bounded from above. But then it follows from Theorem 2.17 in [13] that $\phi_j \to \phi$ in energy (i.e., in the so-called strong topology introduced in [13]) and in particular $E(\mu_j) \to E(MA(\phi_\infty))$. By the lower semi-continuity of the entropy wrt the weak topology this shows that $\phi_\ast$ minimizes $\mathcal{M}_V$ and hence, invoking Theorem 3.3, we conclude that $(\phi_\infty, V)$ is a Kähler-Ricci soliton. □

**Remark 3.12.** Combining the previous theorem with Theorem [13] reveals that analytic $K$-polystability of $(X, V)$ implies $K$-polystability of $(X, V)$. It seems natural to conjecture that the converse also holds, as well as the equivalence between analytic $K$-polystability and strong analytic $K$-polystability.

**Theorem 3.13.** Assume that $(X, V)$ is analytically $K$-polystable. Then the Kähler-Ricci flow $\omega_t$ converges in the weak topology of currents, modulo the action of the group $\text{Aut}(X, V)_0$, to a Kähler-Ricci soliton $\omega$ for $(X, V)$. 

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Proof. Let us denote by $\phi_t$ the Kähler-Ricci flow on the level of metrics on $-K_X$, defined by

$$
\frac{d\phi_t}{dt} = \log \frac{MA(\phi_t)}{\mu_{\phi_t}}, \quad \phi_{t=0} = \phi_0
$$

(3.8)

where the initial data $\phi_0$ is assumed to be a continuous metric in $PSH(X - K_X)^T$. In the singular case one demands that the metric $\phi_t(x)$ be smooth on $]0, \infty[ \times X_{\text{reg}}$ and continuous on $X$; the existence and uniqueness of such a flow was established in [12] (see also the exposition in [17]). Next, setting $\psi_t := \exp(tV)^* \phi_t$ gives a solution to the modified Kähler-Ricci flow $\psi_t$ obtained by replacing $MA$ with $MA_{\theta_V}$. As is well-known, at least in the smooth case, the modified Ding functional $\Delta$ is decreasing along the modified Kähler-Ricci flow. In the singular case this is shown by regularization, just as in the case $V = 0$ considered in [14] (using, in the case $V \neq 0$, the continuity properties of $\mathcal{E}_V$ in Prop 2.13). In particular, by the assumed properness mod Aut$(X,V)_0$ of $D_V$, there exist $F_t$ in Aut$(X,V)_0$ such that $J(F_t(\psi_t)) \leq C$ and $D_V(F_t(\psi_t)) = D_V(\psi_t)$. The proof in the case $V = 0$ in [14] can now be repeated, mutatis mutandis, to get that any subsequence $\psi_{t_j}$ converges in energy to a minimizer $\psi_*$ of $D_V$ and hence (by Theorem 3.3) to a Kähler-Ricci soliton $\psi_*$ (wrt $V$). However, a priori $\psi_*$ depends on the subsequence. To get around this issue we observe that $F_t$ above may be taken so that $F_t \phi_0$ minimizes the functional $J$ on the orbit Aut$(X,V)_0 \phi_0$ and, in particular, $J(F_t^* \phi_t) \leq J(F^* \phi_t)$ for any $F$ in Aut$(X,V)_0$. But then, letting $t \to \infty$ reveals that any limit point $\psi_*$ minimizes the functional $J$ on the space $\mathcal{H}_{KHS}$ of all KR-solitons on $(X,V)$, which by Theorem 2.13 may be identified with quotient Aut$(X,V)_0 \psi_* / K$, where $K$ is the stabilizer of $\psi_0$. Finally, since $J$ is strictly convex on Aut$(X,V)_0 \psi_* / K$ equipped with its natural Riemannian structure (where the geodesics are one parameter subgroups) there is a unique such minimizer, which must hence correspond to $\psi_*$. Hence, the whole curve $\psi_t$ converges (up to normalization) to $\psi_*$ in energy, which concludes the proof.  

4. The quantized setting and semi-classical asymptotics

We continue with the setup in sections 2.2 involving the holomorphic action of a real torus $T$ (and its complexification $T_c$) on $(X,L)$, where the line bundle $L$ is assumed semi-positive and big, i.e. it admits a smooth and semi-positively curved metric $\phi_0$ whose curvature is strictly positive at some point. We will fix such a $T$–invariant reference metric $\phi_0$. Moreover, we will denote by $V$ a fixed holomorphic vector field on $X$ generating the action of $T$ in the sense of Lemma 2.13. In order to be consistent with the setup in Section 3 we do allow $X$ to be singular (with log terminal singularities), but in the proofs we may as well assume that $X$ is smooth by passing to a resolution.

4.1. Convergence of the spectral measure towards the Duistermaat-Heckman measure. Let $P_k := \{ \lambda^{(k)}_i \} \subset \mathbb{Z}^m$ be the set of all weights for the action of the complex torus $T_c$ on the $N_k$–dimensional vector space $H^0(X,kL)$ of all holomorphic sections on $X$ with values in $kL$, i.e. there is a decomposition

$$
H^0(X,kL) = \bigoplus_{\lambda^{(k)}_i \in P_k} E_{\lambda^{(k)}_i}, \quad s \in E_{\lambda^{(k)}_i} \iff \rho(\tau)^* s = \tau_1^{\lambda^{(k)}_1} \cdots \tau_m^{\lambda^{(k)}_m} s
$$
Equivalently, we can view $\lambda_{i}^{(k)}$ as the joint eigenvalues, counted with multiplicity (=the dimension of $E_{\lambda_{i}^{(k)}}$), of the commuting infinitesimal actions of the real parts of the holomorphic vector fields $V_i$ generating the action of $T_{\phi}$. Here the real part of a vector field $W$ on $X$ (with a fixed lift to $L$) acts on $H^{0}(X,kL)$ by $(\text{Re}V)s := \frac{1}{s!} \exp(i\text{Re}V)s$. We let

$$\psi_{k} := \frac{1}{N_{k}} \sum_{i=1}^{N_{k}} \delta_{\lambda_{i}^{(k)}/k}$$

be the corresponding normalized spectral measure on $\mathbb{R}^{m}$, supported on the joint spectrum $P_{k}$. Similarly, we denote by $\psi_{k}^{V}$ the spectral measure on $\mathbb{R}$ attached to the infinitesimal action of the real part of $V$ on $H^{0}(X,kL)$:

$$\psi_{k}^{V} = \frac{1}{N_{k}} \sum_{i=1}^{N_{k}} \delta_{\lambda_{i}^{(k)}/k}$$

Hence, identifying $\text{Re}V$ with the corresponding element $\xi$ in $\mathbb{R}^{m}$, i.e. writing $\text{Re}V = \sum_{i=1}^{m} \xi_{i} \text{Re}V_{i}$, we have $\psi_{i}^{(k)} = \left< \lambda_{i}^{(k)}, \xi \right>$. 

**Proposition 4.1.** Assume that $L$ is semi-positive and big. Then the spectral measures $\psi_{k}$ of the infinitesimal action on $H^{0}(X,kL)$ of the real torus $T$ converge weakly, as $k \to \infty$, to the D-H measure $\nu^{T} := (m_{\phi})_{*} MA(\phi)$. In particular, if the torus $T$ is generated by a vector field $V$ corresponding to the element $\xi$ in $\mathbb{R}^{m}$, then the corresponding spectral measures $\psi_{k}^{V}$ converge weakly to $\nu^{V} = f_{\phi}^{V}_{*} MA(\phi)$, where $f_{\phi}^{V}$ is the Hamiltonian function determined by $V$.

**Proof.** For $L$ ample and $T = S^{1}$ this was shown in [53] (but it also follows from general results on torus actions in symplectic geometry). The case of a torus clearly reduces, using that the generators commute, to the case of $T = S^{1}$. In order to deal with the case of $L$ semi-positive and big we fix a $T-$invariant metric $\phi$ which is smooth and of non-negative curvature and note that, following the argument in [53], the problem is reduced to establishing the corresponding spectral asymptotics for a Toeplitz operator with smooth symbol $f$ (in this case equal to $f_{\phi}^{V}$). In the case of $L$ semi-positive and big the asymptotics in question were obtained in [53].

**Corollary 4.2.** For $\lambda$ an interior point in $P$ denote by $N_{k}(\lambda)$ the number of eigensections in the $N_{k}-$ dimensional space $H^{0}(X,kL)$ with joint eigenvalues $\lambda^{(k)}$ such that $\lambda^{(k)}/k \geq \lambda$ as vectors in $\mathbb{R}^{m}$. Then, for almost any $\lambda$,

$$\lim_{k \to \infty} \frac{N_{k}(\lambda)}{N_{k}} = \int_{X} MA(P_{\lambda} \phi)$$

where $\phi$ is any locally bounded $T-$invariant metric on $L$.

**Proof.** For $\phi$ smooth and of non-negative curvature this follows from combing the previous proposition with formula [22] using $\chi_{\lambda}$ as a test function. But, by Prop 2.24 the mass of $MA(P_{\lambda} \phi)$ is the same if $\phi$ is replaced with a locally bounded metric (since the action of the operator $P_{\lambda}$ only changes with a bounded term). 

**Remark 4.3.** Conversely, if one takes the latter corollary as granted then it implies the previous proposition by a standard measure theory argument (compare the approach used in [32]).
4.2. The quantized energy functionals and their asymptotics. Recall that for any positive integer \( k \) the quantization at level \( k \) of the space \( \mathcal{H}(L) \) is the space \( \mathcal{H}_k \) of all Hermitian metrics \( H \) on the \( N_k \)-dimensional complex vector space \( H^0(X, k L) \) (see [25]). Fixing a reference element \( H_0 \) the space \( \mathcal{H}_k \) may be identified with the symmetric space \( GL(N_k, \mathbb{C})/U(N_k) \). We will equip \( \mathcal{H}_k \) with the corresponding symmetric Riemannian metric and denote by \( \mathcal{H}^T_k \) the corresponding T-invariant subspace. This means that the geodesics in \( \mathcal{H}_k \) corresponds to one-parameter subgroups in \( GL(N_k, \mathbb{C}) \). Recall also that there is a map, the “Fubini-Study map”

\[
FS_k : \mathcal{H}_k \to \mathcal{H}(L), \quad FS_k(H) := \sup_{s \in H_0(X, k L)} \log \left( \frac{|s|^2}{H(s, s)} \right)
\]

which is compatible with the torus action. Next, recall that \( E_{\lambda(k)} \) denotes the joint eigenspace in \( H^0(X, k L) \) corresponding to the joint eigenvalue \( \lambda(k) \) in \( \mathbb{R}^m \) attached to the torus \( T \). To any triple \( (\phi, \mu, g) \) consisting of a \( T \)-invariant smooth metric \( \phi \) on \( L \) with semi-positive curvature, a \( T \)-invariant probability measure \( \mu \) on \( X \) (of finite energy) and a bounded function \( g \) on \( P \), we attach a \( T \)-invariant Hilbert norm (metric) denoted by \( \text{Hilb}(\phi, \mu, g) \), defined by

\[
\text{Hilb}(\phi, \mu, g)(s_i, s_i) = g(\lambda_i(k)/k)^{-1} \int_X |s_i|^2 e^{-k\phi} d\mu \quad \text{for } s_i \in E_{\lambda(k)} \subset H^0(X, k L)
\]

and declaring that the different subspace \( E_{\lambda(k)} \) are mutually orthogonal wrt \( \text{Hilb}(\phi, \mu, g) \). For \( g \) strictly positive this defines a Hilbert norm on \( H^0(X, k L) \), i.e. an element in \( \mathcal{H}_k \) and in general it defines a Hilbert norm on the subspace of \( H^0(X, k L) \) spanned by all eigensections such that \( g(\lambda_i(k)) > 0 \).

4.2.1. The \( g \)-Bergman measure. To a triple \( \text{Hilb}(\phi, \mu, g) \) we associate the following Bergman type function at level \( k \) :

\[
B_{k\phi, \mu, g}(s_i, s_i) := \sup_{s \in H_0(X, k L)} \frac{|s|^2_{k\phi}}{\text{Hilb}(k\phi, \mu, g)(s, s)}
\]

coinciding with the classical Bergman function for \( g = 1 \), also called the density of states (see [31]). In the case when \( \mu = dV \) for a fixed volume form \( dV \) on \( X \) we will drop the explicit dependence of \( \mu \) from the notation and simply write \( B_{k\phi, g} := B_{k\phi, \mu, g} \) which can thus be written as

\[
B_{k\phi, g} := \sum_{\lambda_i(k) \in P_k} g(\lambda_i(k)/k) B_{k\phi, \lambda(k)}
\]

where \( B_{k\phi, \lambda(k)} \) is the ordinary Bergman function of the subspace \( E_{\lambda(k)} \) of \( H^0(X, k L) \).

According to the following proposition one can view the \( g \)-Bergman measure \( B_{k\phi, g} dV/N_k \) as the quantization, at level \( k \), of the \( g \)-Monge-Ampère measure:

**Proposition 4.4.** Assume that \( \phi \) is in \( \mathcal{H}(L)^T \). Then the following convergence holds in the weak topology of measures on \( X \)

\[
\frac{B_{k\phi, g} dV}{N_k} \to MA_g(\phi)
\]

Moreover, if \( L \) is ample, \( g \) is smooth and \( \phi \) has positive curvature, then the convergence above holds in the uniform topology (on the level of densities).
Proof. Weak convergence: Proceeding as in the proof of Theorem 2.7 it is enough to prove the case when \( g = \chi \lambda \), so that \( B_{k \phi \chi} \) is the Bergman function of the subspace generated by all eigensections with joint eigenvalues \( \lambda^{(k)} \) such that \( \lambda^{(k)}/k \geq \lambda \). First, by the local holomorphic Morse inequalities in [5] the following point-wise upper bound holds:

\[
\limsup_{k \to \infty} \frac{B_{k \phi \chi \chi} dV}{N_k} \leq 1_{\{dd^c \phi \geq 0\}} MA(\phi)
\]

together with the uniform bound \( \frac{B_{k \phi \chi \chi} dV}{N_k} \leq C \). Moreover, there exists a constant \( C \) such that

\[
(4.1) \quad \frac{B_{k \phi \chi \chi} dV}{N_k} \leq C \exp(-k(\phi - P_\lambda \phi)).
\]

Indeed, by the uniform bound above \( \phi_k := \phi + \frac{1}{k} \log B_{k \phi \chi} - \log N_k - \log C \leq \phi \). But then it follows from the definition of the envelope \( P_\lambda \phi \) that \( \phi_k \leq P_\lambda \phi \) (compare the proof of Lemma 2.4), which proves the inequality (4.1). All in all this means that

\[
\limsup_{k \to \infty} \frac{B_{k \phi \chi \chi} dV}{N_k} \leq 1_{\{dd^c \phi \geq 0\} \cap \{P_\lambda \phi = \phi\}} MA(\phi)
\]

Finally, by Cor 4.2 \( \int \frac{B_{k \phi \chi \chi} dV}{N_k} \) converges to \( \int MA(P_\lambda \phi) \) which by formula 2.3 coincides with \( 1_{\{P_\lambda \phi = \phi\}} MA(\phi) \) and hence, by basic integration theory, we conclude that the desired weak convergence holds (in fact, one even gets the \( L^1 \)–convergence of the densities as in [5]).

Uniform convergence: As in the the proof of Prop 4.1 it is enough to consider the case when the rank of \( T \) is one. We denote by \( \xi_k \) the linear operator on \( H^0(X, kL) \) corresponding to \( 1/k \) times differentiation wrt the real part of the generator \( V \) of \( T \). Next observe that \( B_{k \phi \gamma} \) can be written as the scalar product

\[
B_{k \phi \gamma}(x) = \left\langle g(\xi_k) K_x^{(k)}, K_x^{(k)} \right\rangle,
\]

where \( K_x^{(k)}(y) = K^{(k)}(x, y) \) is the Bergman kernel of \( H^0(X, kL) \), i.e. the integral kernel of the orthogonal projection \( \Pi_k \) from the space \( L^2(X, kL) \) of all square integrable sections of \( kL \) (equipped with the \( L^2 \)–norm defined by \( dV, k\phi \)). Moreover, we may as well assume that the function \( g \) is a polynomial (by a simple approximation argument using the uniform bound \( B_{k \phi}/k^n \leq C \)). By well-known results (see the review [54] and references therein) \( K^{(k)}(x, y) \) admits, in the case when \( L \) is ample and \( \phi \) has positive curvature a local asymptotic expansion of the form

\[
K^{(k)}(x, y)/k^n \sim e^{k \psi(x, y)} (b_0(x, y) + b_1(x, y) k^{-1} + \ldots)
\]

in the \( C^\infty \)–topology, where \( \psi \) is a certain local smooth function. Hence, fixing local coordinates centered at \( x \) and expressing the vector field \( V \) in terms of the local coordinates reveals that \( g(\xi_k) K_x^{(k)}(y) \) admits a local asymptotic expansion of a similar form. Finally, setting \( y = x \) we deduce in particular that \( B_{k \phi \gamma}(x) \) converges in the uniform topology to some limiting function, which by the weak convergence above must coincide with the density of \( MA_g(\phi) \).

4.2.2. Quantized energy functionals. Next, following [25] [13], we consider the “quantizations” on \( H^0_k \) of the functionals \( \mathcal{E} \) and \( J \) that we shall denote as follows:

\[
(4.2) \quad \mathcal{E}^{(k)}(H) := -\frac{1}{kN_k} \log \det(H), \quad J^{(k)}(H) = -\mathcal{E}^{(k)}(H) + \mathcal{L}_{\mu_0}(FS(H)),
\]

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where we have identified $H$ with an Hermitian positive definite matrix, using the reference element $H_0 := \text{Hilb}(k \phi_0, dV)$ (writing $\mu_0 = dV$) and the functional $\mathcal{L}_{\mu_0}$ was defined in formula \[2.13\] More generally, $g$--analogs of these functional may be defined by setting

$$
\mathcal{E}^{(k)}_g(H) := \sum_{\lambda_i^{(k)} \in \mu_k} g(\lambda_i^{(k)} / k) \mathcal{E}^{(k)}_{E_{\lambda_i^{(k)}}}(H), \quad \mathcal{E}^{(k)}_{E_{\lambda_i^{(k)}}}(H) := -\frac{1}{kN_k} \log \det H_{E_{\lambda_i^{(k)}}}
$$

Concretely, picking a base $s_i^{(k)}$ in $H^0(X, kL)$ which is $H_0$--orthonormal and $H$--orthogonal and writing $H(s_i^{(k)}, s_i^{(k)}) = e^{-\mu_i^{(k)}} H_0(s_i^{(k)}, s_i^{(k)})$ we can express

$$
\mathcal{E}^{(k)}_g(H) = \frac{1}{kN_k} \sum_{i=1}^{N_k} g(\lambda_i^{(k)}/k) \mu_i^{(k)}
$$

This expression reveals that $\mathcal{E}^{(k)}_g(H)$ is affine along geodesics in $\mathcal{H}_T$ (since the corresponding geodesics $H_t$ are defined by the scaled eigenvalues $t \mu_i^{(k)}$). Next, we introduce a $g$--analog of Donaldson’s $L$--functional on the space $\mathcal{H}(L)^T$:

$$
\mathcal{L}^{(k)}_{(\mu, g)}(\phi) := \mathcal{E}^{(k)}_g(\text{Hilb}(k \phi, \mu)) = \mathcal{E}^{(k)}_{(\mu, g)}(\text{Hilb}(k \phi, \mu, g))
$$

**Proposition 4.5.** The differential of $\phi \mapsto \mathcal{L}^{(k)}_{(\mu, g)}(\phi)$ is naturally identified with the corresponding Bergman measure:

$$
\text{d} \mathcal{L}^{(k)}_{(\mu, g)}|_\phi = \frac{1}{N_k} B_{(k \phi, \mu, g)} \mu
$$

and if $\mu$ is a volume form (i.e. $\mu = dV$), then $\mathcal{L}^{(k)}_{(\mu, g)}(\phi)$ converges to $\mathcal{E}^g(\phi)$, as $k \to \infty$, for any $\phi \in \mathcal{H}(X, L)^T$. Moreover, in general, the functional $\mathcal{L}^{(k)}_{(\mu, g)}(\phi)$ is concave along affine curves in $\mathcal{H}(L)^T$.

**4.3.** $(\mu, g)$--balanced metrics. Given a pair $(\mu, g)$ such that $\mu$ and $g \nu$ are probability measures on $X$ and $P$, respectively, with $\mu$ of finite energy and $g$ continuous, we introduce the map

$$
\mathcal{T}_{(g, \mu)} := \text{Hilb}_{(g, \mu)} \circ \text{FS} : \mathcal{H}_k \to \mathcal{H}_k
$$

A metric in $\mathcal{H}_k$ will be said to be $(\mu, g)$--balanced (at level $k$) if it is a fixed point of $\mathcal{T}_{(g, \mu)}$. This is thus the $g$--analog of the ordinary notion of a balanced metric, defined by a measure $\mu$ \[2.23\]. Moreover, iterating the map $\mathcal{T}_{(g, \mu)}$ gives the $g$--analog of Donaldson’s iteration \[2.25\]:

$$
H_m^{(k)} := (\mathcal{T}_{(g, \mu)})^m H_0
$$

where $m$ is a non-negative integer (the discrete time parameter). Just as in the ordinary case $g = 1$ a metric is $(\mu, g)$--balanced metric iff it is a critical point of the following functional on $\mathcal{H}_k^T$

$$
J^{(k)}_{(\mu, g)}(H) = -\mathcal{E}^g(H) + \mathcal{L}_\mu(\text{FS}(H)),
$$

The following result is the quantization of Theorem \[2.18\]

**Theorem 4.6.** Let $(X, L)$ be a polarised manifold and $T$ a real torus acting holomorphically on $(X, L)$ with moment polytope $P$. Given a pair $(\mu, g)$ such that $\mu$ and $g \nu$ are probability measures on $X$ and $P$, respectively, with $\mu$ of finite energy and
concludes the proof.

\[ \square \]

\[ \text{Proof.} \] This is shown by adapting the proof of Theorem 7.1 in [13] to our setting and reducing the problem to Theorem 2.18 (or rather its proof). Since a very similar argument will be carried out in the course of the proof of Theorem 1.7 the details are omitted. \( \square \)

4.4. The quantized Kähler-Ricci soliton setting.

4.4.1. Quantized modified Futaki invariants. Let us start by introducing the following quantized analog of the modified Futaki invariant defined on the Lie algebra of \( \text{Aut}(X,V)_0 \), to be referred to as the quantized modified Futaki invariant at level \( k \):

\[ \text{Fut}_{V,k}(W) := - \sum_{l=1}^{N_k} \exp(v_l^{(k)}/k)w_l^{(k)}, \]

where \( (v_l^{(k)}, w_l^{(k)}) \) are the joint eigenvalues for the commuting action of the real parts of the holomorphic vector fields \( V \) and \( W \) on \( H^0(X, -kK_X) \) (using the canonical lifts to \( -K_X \) of the vector fields \( V \) and \( W \)).

**Proposition 4.7.** Given a pair \( (X,V) \), consisting of a Fano variety equipped with a holomorphic vector field \( V \), let \( W \) be a holomorphic vector field on \( X \) generating a \( \mathbb{C}^* \)-action and commuting with \( V \). Then

\[ \text{Fut}_V(W) = \lim_{k \to \infty} \frac{1}{kN_k} \text{Fut}_{V,k}(W). \]

Moreover, the sequence in the rhs above coincides with the time derivative of the function \( t \mapsto -L_{\nu(V)}^k(H^W_t) \), where \( H^W_t = \exp(tW)^* H_0 \).

**Proof.** Consider the commuting pair \( (W,V) \) inducing an action of \( S^1 \times T \). According to Lemma 3.4 \( \text{Fut}(X, \rho_W, V) = - \int_{\mathbb{R}^2} v e^w \nu(V,W) \), where \( \nu(V,W) \) denotes the corresponding DH-measure on \( \mathbb{R}^2 \). But then the proposition follows from Prop 4.3 applied to the torus \( S^1 \times T \), which gives that \( \sum_{l=1}^{N_k} \delta_{(v_l^{(k)}, w_l^{(k)}/k)} \) converges to \( \nu(V,W) \). Indeed integrating the latter convergence against the function \( ve^w \) on \( \mathbb{R}^2 \) concludes the proof. \( \square \)

We also note that in the case when \( X \) is smooth there exists, for \( k \) large, a polynomial expansion

\[ \text{Fut}_{V,k}(W) = k^{n+1} \text{Fut}_V^{(0)}(W) + k^n \text{Fut}_V^{(1)}(W) + \cdots + \text{Fut}_V^{(n)}(W), \]

where the invariant \( \text{Fut}_V^{(m)}(W) \) defined by the coefficients in the expansion above will be called the \( m \)th order modified Futaki invariant of \( W \) (by the previous proposition \( \text{Fut}_V^{(0)}(W) \) is proportional to \( \text{Fut}_V(W) \)). The previous expansion may be obtained by writing

\[ \text{Fut}_{V,k}(W) = - \frac{d}{dt}_{|t=0} \text{Tr} (e^{V+ tW})|_{H^0(X,-kK_X)}, \]

and evaluating the rhs using the equivariant Riemann-Roch theorem.
Remark 4.8. For $V = 0$ the vanishing of $\text{Fut}^{(m)}(W)$ for all integers $m$ in $[0, n]$ is equivalent to the vanishing of Futaki’s higher invariants $\mathcal{F}_{T_d(\phi)}(W)$ for all integers $p$ in $[1, n]$, which in turn is known to be equivalent to the vanishing of Mabuchi’s obstruction to asymptotic Chow semi-stability (see [28], [31], [35]). To see this we first note that by the equivariant Riemann-Roch theorem $\text{Fut}^{(m)}(W)$ is the coefficient corresponding to $t^k \alpha^{n+1-m}$ in the expansion of

\begin{equation}
- \int_X e^{k(\omega_\phi + f_\phi)} \wedge T_d(\Theta + t L_W),
\end{equation}

where $T_d$ is the Todd polynomial and $\Theta$ is the End $(TX)$-valued Chern curvature form of the metric $\omega_\phi$ and $f_\phi$ denotes, as before, the Hamiltonian function for $W$ determined by the canonical lift of $W$ to $L := -K_X$. Now, $u_\phi := f_\phi - \int f_\phi \omega_\phi^n$ defines another Hamiltonian for $W$ satisfying the normalization condition $\int u_\phi \omega_\phi^n = 0$ used by Futaki [28]. As explained in [28] the vanishing of $\mathcal{F}_{T_d(\phi)}(W)$ for all integers $m$ in $[1, n]$ is equivalent to the vanishing of all the coefficients in the expansion obtained by replacing $f_\phi$ with $u_\phi$ in formula (1.6) and it implies that the ordinary Futaki invariant Fut $(W)$ vanishes (since $\text{Fut}(W) = c \mathcal{F}_{T_d(\phi)}(W)$). Moreover, the vanishing of $\text{Fut}^{(m)}(W)$ for $[0, n]$ also implies that $\text{Fut}(W) = 0$ (since $\text{Fut}(W) = \text{Fut}(0)(W)$).

Finally, observing that $- \int f_\phi \omega_\phi^n = \text{Fut}(W)$ (by Lemma 3.4) we conclude that the vanishing of $\text{Fut}^{(m)}(W)$ for all integers $m$ in $[0, n]$ is indeed equivalent to the vanishing of Futaki’s invariants $\mathcal{F}_{T_d(\phi)}(W)$ for all integers $p$ in $[1, n]$.

4.4.2. Quantized Kähler-Ricci solitons and balanced metrics. Let us next recall the definition of Donaldson’s (anti-)canonical map $\mathcal{T}_k$ on $\mathcal{H}_k$ in the “anti-canonical” case $L = -K_X$. First, we define the anti-canonical Hilb map by

$$\text{Hilb}(k\phi) := \text{Hilb}(k\phi, \mu_\phi),$$

where $\mu_\phi$ is the canonical measure on $X$ determined by $\phi$ (see formula [3.1]). Then the map $\mathcal{T}_k$ on $\mathcal{H}_k$ may be defined as

$$\mathcal{T}_k := \text{Hilb} \circ F S$$

and a metric $H$ in $\mathcal{H}_k$ is said to be anti-canonically balanced (at level $k$) if it is fixed by $\mathcal{T}_k$. As conjectured by Donaldson and shown in [7] the corresponding iteration $\mathcal{H}_{k,m}$ on $\mathcal{H}_k$ converges, in the double scaling limit where $m/k \to t$, to the (normalized) Kähler-Ricci flow on $\mathcal{H}(-K_X)$. Accordingly, we will say that a metric $H$ in $\mathcal{H}_k$ is a quantized Kähler-Ricci soliton with respect to a holomorphic vector field $V$ on $X$ if

$$\mathcal{T}_k H \Rightarrow \exp(V)^* H$$

where $\exp(V)^*$ denotes the automorphism of $\mathcal{H}_k$ induced by the pull-back along the time-one flow of $V$ (a similar notion of quantized extremal metrics was introduced in [11] for any ample line bundle $L$, but defined with respect to a different definition of the Hilb map, obtained by replacing $\mu_\phi$ with $MA(\phi)$, as in [24]). Equivalently, replacing Hilb with Hilb$_V$ defined by Hilb$_V(k\phi) := \text{Hilb}(k\phi, \mu_\phi, g_V)$ and $\mathcal{T}_k$ with $\mathcal{T}_k := \text{Hilb}_V \circ F S$ a metric $H$ is a quantized Kähler-Ricci soliton wrt $V$ iff $H$ is fixed by the map $\mathcal{T}_{k,g_V}$, i.e. if it is “anti-canonically balanced with respect to $g_V$”.

In this setting we define the quantization of the modified Ding functional by

$$\mathcal{D}_V^{(k)}(H) := \mathcal{D}_{g_V}^{(k)}(H) := -\mathcal{E}_{g_V}^{(k)}(H) + \mathcal{L}(FS(H)), \quad \mathcal{L}(\phi) = -\log \int_X e^{-\phi},$$

whose critical points are quantized Kähler-Ricci solitons wrt $V$.  

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Proposition 4.9. Let $(X, V)$ be a Fano variety equipped with a holomorphic vector field $V$. Then the following is equivalent:

- There exists a quantized Kähler-Ricci soliton at level $k$
- The functional $D^{(k)}_\nu$ is invariant under $\text{Aut} (X, V)_0$ and proper on $\mathcal{H}_k / \text{Aut} (X, V)_0$.
- The functional $D^{(k)}_\nu$ is coercive modulo $\text{Aut} (X, V)_0$, i.e. of at least linear growth along geodesics in $\mathcal{H}_k / \text{Aut} (X, V)_0$.

Proof. By basic properties of convex functions on finite dimensional spaces it will be enough to show that $D^{(k)}_\nu (H) = \mathcal{H}_k / \text{Aut} (X, V)_0$. But, as explained above, the functional $E^{(k)}_\nu (H)$ is affine along geodesics in $\mathcal{H}_k$ and hence the result follows from the well-known convexity properties of $H \mapsto \mathcal{L}(FS(H))$ (see [13] and reference therein).

4.4.3. Proof of Theorem [1,7] We will adapt the proof of Theorem 7.1 in [13] to our setting. The starting point is the following comparison inequality:

Lemma 4.10. There exists a sequence $\delta_k \to 0$ of positive numbers such that

$$J_\nu (\phi_k) \leq (1 + \delta_k) J^{(k)}_\nu (H_k) + \delta_k, \quad \phi_k := FS(H_k)$$

Proof. This is the $g$–analog of Lemma 7.7 in [13] and since the proofs are similar we just outline the argument. First one connects $H_k$ with a geodesic $H^t_k$ to the reference metric $H_0 := \text{Hilb} (k \phi_0, dV)$ in $\mathcal{H}_k$ and introduces the function $f_k (t) := E_\nu (FS(H^t_k)) - E^{(k)}_\nu (H_k)$. The latter function is convex since $E^{(k)}_\nu (H_k)$ is affine on $\mathcal{H}_k$, while $E_\nu (FS(H_k))$ is convex. The desired inequality is obtained by using the convexity of $f$ and the error terms $\delta_k$ come from estimating the value of $f_k (0)$ and the derivative $f_k’ (0)$, using the uniform convergence in Prop 4.4.

Now, given $H_k$ in $\mathcal{H}_k^T$ there exists, by the properness assumption on $D_\nu$, an element $F \in \text{Aut} (X, V)_0$ such that

$$J_\nu (FS(F^* H_k)) (1 - \delta) + (\mathcal{L} - \mathcal{L}_{\mu_0}) (FS(F^* H_k)) \geq -C$$

(where we have used that $\mu_0$ is bounded on $P$ and $D_\nu$ is invariant under the action of $\text{Aut} (X, V)_0$, since the Futaki invariants automatically vanish; compare section 3). Hence, using the inequality (4.7) we get, for $k$ large, that

$$J^{(k)}_\nu (F^* H_k) (1 - \delta) + (\mathcal{L} - \mathcal{L}_{\mu_0}) (FS(F^* H_k)) \geq -2C,$$

i.e.

$$D^{(k)}_\nu (F^* H_k) \geq \frac{\delta}{2} J^{(k)}_\nu (F^* H_k) - 2C \geq \frac{\delta}{2C} J^{(k)}_\nu (F^* H_k) - 2C$$

(using that $g$ is bounded in the last inequality). Now, assuming that the quantized modified Futaki invariants $\text{Fut}_V(k)(W)$ vanish for any $W \in \text{aut} (X, V)_0$ (which by the expansion 1.5 is equivalent to the vanishing of the $n+1$ higher order modified Futaki invariants of $(X, V)$) the functional $E^{(k)}_\nu (H_k)$ and hence $D^{(k)}_\nu$ is also invariant under the action of $\text{Aut} (X, V)_0$ (by the last statement in Prop 4.7). Hence, minimizing over $\text{Aut} (X, V)_0$ in the inequalities above gives

$$D^{(k)}_\nu (H_k) \geq \delta’ \inf_{F \in \text{Aut} (X, V)_0} J^{(k)}_\nu (F^* H_k) - C’$$

\footnote{For the latter smooth uniform convergence to hold we need $X$ to be smooth.}
In particular, since $J^{(k)}$ is an exhaustion function for the space $\mathcal{H}_T^k/\mathbb{R}$ we conclude that $D_g^{(k)}(H_k)$ is proper on $\mathcal{H}_T^k/\mathbb{R}$ mod Aut $(X,V)_0$ and hence, by Prop 4.9 admits a minimizer $H_k$ which is unique mod Aut $(X,V)_0$. The minimizer $H_k$ is uniquely determined (mod $\mathbb{R}$) by the normalization condition that the corresponding metric $\phi := FS(H_k)$ minimizes $J$ on the corresponding Aut $(X,V)_0$-orbit (using the convexity properties of $J$ as in the proof of Theorem 3.13). Moreover, by the minimizing property of $H_k$ we have $D_{g,k}(H_k) \leq D_{g,k}(\text{Hilb}(k\psi))$ for any fixed smooth metric $\psi$ with positive curvature and hence letting $k \to \infty$ and using the convergence in Prop 4.5 gives

\begin{equation}
\mathcal{D}_g^{(k)}(H_k) \leq \inf_{\psi \in \mathcal{H}^{-K_X}} \mathcal{D}_g = \mathcal{D}_g(\phi_{KRS}),
\end{equation}

where $\phi_{KRS}$ is the unique Kähler-Ricci soliton on $X$ normalized as above. Finally, by the inequality (4.7) (and using that $g$ is bounded)

\[ D_g(\phi_k) \leq D_g^{(k)}(H_k)(\phi_k) + \delta_k J^{(k)}(H_k) + \delta_k \]

and since $J^{(k)}(H_k)$ is uniformly bounded (by the inequalities 4.8 and 4.9) we conclude that $\phi_k$ is an asymptotically minimizing sequence for $D_g$. Hence it follows, just as in the proof of Theorem 3.11 that $\phi_k$ converges in $L^1$ (and even in energy) to a minimizer of $\mathcal{D}_g$, which thus concludes the proof.

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