Logarithmic corrections to charged hairy black hole in (2+1) dimensions

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Abstract

We consider a charged black hole with a scalar field that is coupled to gravity in (2 + 1)-dimensions. We compute the logarithmic corrections to the corresponding system using two approaches. In the first method we take advantage of thermodynamic properties. In the second method we use the metric function that is suggested by conformal field theory. Finally, we compare the results of the two approaches.

Keywords: Black hole; Scalar field; Logarithmic correction.

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1 Introduction

Gravity in (2 + 1)-dimensional space-time has been an interesting area of theoretical researches during recent decades. Such studies began in the early 1980 [1-4]. By the discovery of Banados- Teitelboim- Zanelli (BTZ) [5] and Martinez- Teitelboim-Zanelli (MTZ) [6] black holes, it became clear that gravity in (2 + 1) dimensions is much more fascinating in its own place and it is often much easier to analyze black hole solutions in (2 + 1) dimensions than it in other dimensions. Recently, Xu and Zhao analyzed the charged black hole with scalar field in (2 + 1) dimensions, where the scalar field couples to gravity and it couples to itself with...
the self-interacting potential too [7]. Then, similar black hole with a rotational parameter constructed by the Ref. [8] and developed by the Ref. [9]. Also some thermodynamical studies of such kind of black hole may be find in the Refs. [10] and [11]. It is shown that the entropy of large black holes is adequate to its horizon area [12, 13], so while one reduces the size of the black hole, it is important to study what the leading order corrections are, and also proven that when small stable fluctuations around equilibrium are considered, logarithmic corrections to thermodynamic entropy appear in all thermodynamic systems. The stability condition is correspond to being positive of the specific heat, so that the equivalent canonical ensemble is stable [14]. We want to study this logarithmic corrections for the charged black hole with scalar hair in three dimensions and the special cases in section 3. We will study the thermodynamics of these black holes that similarly was done in [11] and use them for compute these corrections. in next section we try thermodynamic entropy as an exact function of the inverse temperature $\beta(S = S(\beta))$. In this part concentrate on the function offered by CFT [15], leads to identical logarithmic corrections like found in the earlier section from thermodynamic considerations. Finally in section 6 we give conclusion.

2 Charged hairy black hole in three dimensions and entropy

The black hole solution in an Einstein-Maxwell scalar gravity that coupled with a nonminimally scalar field in (2 + 1) dimensions was studied. Hairy black hole is the name that black hole solutions in this theories have, and now there is lots of texts about this entry [16-29], and the space-time is not only limited to be (2 + 1)-dimensional [30]. The scalar field may be coupled minimally or nonminimally, to gravity, and it may coupled to itself by a self-interacting potential $U(\varphi)$. In the model that was studied, $\varphi$ couples to gravity in a nonminimal way, and it also couples to itself by a self-potential $V(\varphi)$. The action is given by,

$$s = \int d^{3}x \sqrt{-g} \left( \frac{R}{2} - \frac{1}{2} g^{\alpha\beta} \nabla_{\alpha}\varphi \nabla_{\beta}\varphi - \frac{1}{2} \varepsilon R \varphi^{2} - V(\varphi) - \frac{1}{8} F_{\alpha\beta} F^{\alpha\beta} \right),$$

(1)

where $\varepsilon = \frac{1}{8}$ is a constant shows the coupling power between gravity and the scalar field and the metric function is obtained as the following,

$$f(r) = \frac{r^{2}}{l^{2}} + 3\beta - \frac{Q^{2}}{4} + (2\beta - \frac{Q^{2}}{9}) \frac{B}{r} - Q^{2}(\frac{1}{2} + \frac{B}{3r}) \ln(r),$$

(2)

where $Q$ is the electric charge, $l$ and $\beta$ are integration constants. $l$ related to the cosmological constant by $\Lambda = -\frac{1}{l^{2}}$, it is negative because smooth black hole horizons can be only in presence of a negative cosmological constant in (2 + 1) dimensions, and $r$ explains the radial coordinate. the relation between $B$ and scalar field is as follow,

$$\phi(r) = \pm \sqrt{\frac{8B}{r + B}}.$$

(3)
Also,
\[ \beta = \frac{1}{3} \left( \frac{Q^2}{4} - M \right), \quad (4) \]
is a relation between the black hole charge and mass. So we use them to calculate the related entropy as will see in the next sections.

3 Logarithmic correction with thermodynamic properties

There is the exact entropy in any values of \( \beta \) as,
\[ S(\beta) = \ln Z(\beta) + \beta E, \quad (5) \]
where \( Z(\beta) \) is the partition function in which \( k\beta = 1 \) is used. By expanding \( S(\beta) \) around equilibrium temperature \( (\beta = \beta_0) \) one can obtain,
\[ S(\beta) = S_0 + \frac{1}{2} (\beta - \beta_0)^2 S''_0 + ..., \quad (6) \]
where \( S''(\beta_0) = \left( \frac{\partial S(\beta)}{\partial \beta} \right)_{(\beta=\beta_0)} = 0, \beta_0 = \frac{1}{T_0}, \) and \( S(\beta_0) = S_0 \). Using \( S = \ln \rho(E) \) we have \([31]\),
\[ S = S_0 - \frac{1}{2} \ln S''_0 + ... \quad (7) \]
From the entropy at any temperature \( S(\beta) \) there is the right canonical entropy at equilibrium temperature \( (7) \). Second derivative of \( (5) \) gives the following relation,
\[ S''(\beta) = \frac{1}{Z} \frac{\partial^2 S(\beta)}{(\partial \beta)^2} - \frac{1}{Z^2} \frac{\partial S(\beta)}{(\partial \beta)^2}, \quad (8) \]
where we used \( <E> = -\left( \frac{\partial S(\beta)}{\partial \beta} \right)_{(\beta=\beta_0)}. \)
We have \( S''(\beta) \) as a fluctuation square of energy. As \( C = (\frac{\partial E}{\partial T})_{(T=T_0)} \) the following equation will be obtained,
\[ S''(\beta) = CT^2 \quad (9) \]
By putting \( (9) \) in \((7)\) the corrected entropy will be,
\[ S = S_0 - \frac{1}{2} \ln CT^2 + ..., \quad (10) \]
We will use this formula for some black holes in next section. For black holes we have \( T_0 = T_H \). If we consider \( 16\pi G_N = 1 \), therefore,
\[ S_0 = S_{BH} = 4\pi r_+ \quad (11) \]
We need some thermodynamic properties of black holes like,
\[ T = \frac{dM}{dS}, \]  
(12)
and,
\[ C = \frac{dM}{dT} = T \frac{dS}{dT}, \]  
(13)
where \( M \) is the mass of black hole, by replacing them to (10) the entropy will be obtained.

### 3.1 Charged BTZ black hole

In the equation (2) if \( B = 0 \) we have a black hole without coupling with scalar field, the metric function becomes,
\[ f(r) = \frac{r^2}{l^2} - M - \frac{Q^2}{2} \ln (r), \]  
(14)
therefore the black hole mass establish as the following,
\[ M = \frac{r_+^2}{l^2} - \frac{Q^2}{2} \ln (r_+). \]  
(15)
By setting \( r_+ \) in terms of \( S_{BH} \), we have,
\[ M = -\frac{Q^2}{2} \ln \frac{S_0}{4\pi} + \frac{S_0^2}{16\pi^2 l^2}. \]  
(16)
Using equation (15) the temperature is given by,
\[ T = -\frac{Q^2}{2S_0} + \frac{S_0}{8\pi^2 l^2}. \]  
(17)
From (13) and (17) we can calculate \( C \), putting on (10), we obtain,
\[ S = S_0 + \frac{1}{2} \ln S_0 + \frac{1}{2} \ln(S_0^2 + 4\pi^2 l^2 Q^2) - \frac{3}{2} \ln(S_0^2 - 4\pi^2 l^2 Q^2) + \ln 8\pi^2 l^2. \]  
(18)
If we set \( Q = 0 \) we have BTZ black hole and the logarithmic correction as the following,
\[ S = S_0 - \frac{1}{2} \ln S_0 + \ln 8\pi^2 l^2 \]  
(19)
This is the same result as we saw in [31].
### 3.2 Uncharged hairy AdS black hole

In this case we have,

\[
\begin{align*}
f(r) &= -M\left(1 + \frac{2B}{3r}\right) + \frac{r^2}{l^2}, \\
M &= \frac{r_+^2}{l^2\left(1 + \frac{2B}{3r_+}\right)}. 
\end{align*}
\]

(20)  

and therefore,

\[
\begin{align*}
M &= \frac{r_+^2}{l^2\left(1 + \frac{2B}{3r_+}\right)}. 
\end{align*}
\]

(21)

So the following term will be the temperature,

\[
T = \frac{9S_0^2(S_0 + 4\pi B)}{8\pi^2l^2(3S_0 + 8\pi B)^2}.
\]

(22)

The correct entropy can be,

\[
S = S_0 - \frac{5}{2} \ln S_0 + \frac{3}{2} \ln \frac{3S_0 + 8\pi B}{S_0 + 4\pi B} + \frac{1}{2} \ln(3S_0^2 + 24\pi BS_0 + 64\pi^2B^2) + \ln \frac{8\pi^2l^2}{9}.
\]

(23)

Therefore if the coupling of scalar field be ignored we have,

\[
S = S_0 - \frac{3}{2} \ln S_0 + \ln 8\pi^2l^2.
\]

(24)

Again the leading order correction is logarithmic.

### 3.3 Conformally dressed black hole

Another black hole which gives \(\beta = -\frac{B^2}{l^2}\), has the following metric function,

\[
\begin{align*}
f(r) &= -3\frac{B^2}{l^2} - \frac{2B^3}{l^2r} + \frac{r^2}{l^2}, \\
M &= \frac{3S_0^2}{64\pi^2l^2}.
\end{align*}
\]

(25)  

Because of \(\beta = -\frac{M}{3}\) the mass will be,

\[
M = \frac{3S_0^2}{64\pi^2l^2}.
\]

(26)

So the temperature and specific heat are,

\[
T = \frac{3S_0}{32\pi^2l^2},
\]

(27)

and

\[
C = S_0.
\]

(28)

Therefore,

\[
S = S_0 - \frac{3}{2} \ln S_0 + \ln \frac{32\pi^2l^2}{3}.
\]

(29)

As it can be seen, this is the logarithmic that we predicted before, now we want to study the charged black hole which coupled to scalar field.
3.4 Charged hairy black hole in three dimensional

By using the explanation of $\beta$ in (4), the following metric function obtained,

$$f(r) = \frac{r^2}{l^2} - M - \frac{2MB}{3r} - \frac{Q^2B}{18r} - Q^2\left(\frac{1}{2} + \frac{B}{3r}\right)\ln(r).$$  (30)

If we set $f(r_+) = 0$ the mass will be reduced to the following equation,

$$M = \frac{9S_0^3 + 32\pi^3l^2Q^2B - 72\pi^2l^2Q^2S_0\ln\left(\frac{S_0}{4\pi}\right) - 192\pi^3l^2Q^2B\ln\frac{S_0}{4\pi}}{48\pi^2l^2(3S_0 + 8\pi B)}.  \quad (31)$$

Applying the equations (12) and (13) to the equation (31) we yield to the following temperature,

$$T = \frac{9S_0^4 + 36\pi BS_0^3 - 36\pi^2l^2Q^2S_0^2 - 208\pi^3l^2Q^2BS_0 - 256\pi^4l^2Q^2B^2}{8\pi^2l^2(3S_0 + 8\pi B)^2}.  \quad (32)$$

The related leading order correction for this black hole get

$$S = S_0 + \ln(S_0) + 2\ln(3S_0 + 8\pi B) + \frac{1}{2}\ln(27S_0^3 + 72\pi BS_0^2 - 36\pi^2l^2Q^2S_0 + \frac{256\pi^4l^2Q^2B^2}{S_0})$$

$$- \frac{6}{3S_0 + 8\pi B}\left(9S_0^4 + 36\pi BS_0^3 - 3\pi^2l^2Q^2S_0^2 - 208\pi^3l^2Q^2BS_0 - 256\pi^4l^2Q^2B^2\right)$$

$$- \frac{3}{2}\ln(9S_0^4 + 36\pi BS_0^3 - 36\pi^2l^2Q^2S_0^2 - 208\pi^3l^2Q^2S_0 - 256\pi^4l^2Q^2B^2)$$

$$+ \ln(8\pi^2l^2).  \quad (33)$$

If the scalar field be discarded ($B = 0$), we have the charged BTZ black hole and we see the same entropy as equation (19).

In the equation (32) when we neglect charge ($Q = 0$) we have the similar result of (19).

We apply $Q = 0$ and $B = 0$ on (33) to obtain,

$$S = S_0 - \frac{3}{2}\ln S_0 + \ln 8\pi^2l^2  \quad (34)$$

In next step we show the exact of these special black holes by the suggestion form of CFT.

4 Exact Entropy Function

In previous stages we have seen that the logarithmic corrections can be obtained from thermodynamics properties like temperature and specific heat of black holes. we will try to make over these corrections by considering an exact entropy function $S(\beta)$, which follows from $CFT$ and other quantum theories of gravity. If we return to the equation (7) for using it to study entropy of some black holes we need $S''(\beta_0)$. By considering the special form of function of entropy that is approximated by the parabolic form (6). The special form is,

$$S(\beta) = x\beta + \frac{y}{\beta}.  \quad (35)$$
A general form of entropy function was assumed to check necessity and influenced on results, that is,

\[ S(\beta) = x\beta^m + \frac{y}{\beta^n}, \tag{36} \]

where the case \( m = n = 1 \) is commanded by CFT. By finding the extremum of \( S(\beta) \) and getting the second derivative of \( S(\beta) \) then by some calculation similar to the Ref. [31] we have the following form of entropy,

\[ S = S_0 - \frac{1}{2} \ln S_0 T^2. \tag{37} \]

Now we use the above entropy and apply it to previous black holes.

### 4.1 Charged BTZ black hole

In that case we obtain,

\[ S = S_0 + \frac{1}{2} \ln S_0 - \ln S_0^2 - 4\pi^2 l^2 Q^2 + \ln 8\pi^2 l^2. \tag{38} \]

We see that this is same as (18) by one different expression that is,

\[ \frac{1}{2} \ln \frac{S_0^2 + 4\pi^2 l^2 Q^2}{S_0^2 - 4\pi^2 l^2 Q^2}, \tag{39} \]

so if \( Q = 0 \) the logarithmic correction of BTZ black hole will obtained.

### 4.2 Uncharged hairy AdS black hole

By using (37) for this black hole the entropy will be,

\[ S = S_0 - \frac{5}{2} \ln S_0 + \ln \frac{(3S_0 + 8\pi B)^2}{S_0 + 4\pi B} + \ln \frac{8\pi^2 l^2}{9}. \tag{40} \]

There is one expression that makes it different from (23), this is,

\[ \frac{1}{2} \ln \frac{3S_0^2 + 24\pi B S_0 + 64\pi^2 B^2}{3S_0 + 8\pi B S_0 + 4\pi B}, \tag{41} \]

if we set \( B = 0 \), we see the same relation as (19).

### 4.3 Conformally dressed black hole

For this part the answer is the same as (29) as we predicted.
4.4 Charged hairy black hole

In that case we calculate,

\[ S = S_0 + \frac{1}{2} \ln S_0 + 2 \ln(3S_0 + 8\pi B) - \ln(9S_0^4 + 36\pi BS_0^3 - 36\pi^2 l^2 Q^2 S_0^2 - 208\pi^3 l^2 Q^2 BS_0 - 256\pi^4 l^2 Q^2 B^2) + \ln 8\pi^2 l^2. \]  

(42)

This is the leading order correction of (33) without two expressions that is,

\[ \frac{1}{2} \left[ \ln(27(S_0)^4 + 72\pi BS_0^3 - 36\pi^2 l^2 Q^2 S_0^2 + 256\pi^4 l^2 Q^2 B^2) - \frac{6S_0}{3S_0 + 8\pi B} (9S_0^4 + 36\pi BS_0^3 - 36\pi^2 l^2 Q^2 S_0^2 - 208\pi^3 l^2 Q^2 BS_0 - 256\pi^4 l^2 Q^2 B^2) - \ln(9S_0^4 + 36\pi BS_0^3 - 36\pi^2 l^2 Q^2 S_0^2 - 208\pi^3 l^2 Q^2 BS_0 - 256\pi^4 l^2 Q^2 B^2) \right], \]  

(43)

and if the scalar field and charge are ignored we will derive the same shape of (29). We saw that both of these ways for getting the corrections have the same result as shows above.

5 Conclusion

In this paper, we studied some special types of charged black hole in three dimensions coupled to the scalar fields. We calculated the logarithmic correction for these black holes with two approaches. The first approach is the thermodynamics properties and the second one is the entropy function, which is suggested by CFT area. We obtained corrections of the entropy as the natural logarithm of the cosmological constant for four different kinds of black hole and found good agreement between them. This means that the presence of the cosmological constant is necessary to have logarithmic corrections. The completed form of the black hole led to (38) using the first method and (53) using the second method, where logarithmic corrections were illustrated. It is obvious that the logarithmic corrections increase entropy and yield more radiation from the black hole. These logarithmic corrections are due to thermal fluctuations of the black hole around its equilibrium. In this paper we achieved the logarithmic form of entropy. We compared the results and demonstrated that by two approaches. Here we have seen that the results for the two approaches are the same without the scalar fields and the electric charge. In this case the corresponding black hole will be a BTZ black hole.

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