Existence of Positive Solution of a Class of Semi-linear Sub-elliptic Equation in the Entire Space $IH^n$

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Abstract In this paper, we study the following problem

$$\begin{aligned}
\Delta_{H^n} u - u + u^p &= 0 \quad \text{in } H^n \\
u &> 0 \quad \text{in } H^n \\
u(x) &\to 0 \quad \rho(x) \to \infty
\end{aligned}$$

where $1 < p < \frac{Q+2}{Q-2}$, $Q$ is the homogeneous dimension of Heisenberg group $H^n$. Our main result is that this problem has at least one positive solution.

Key Words and Phrases: Semilinear subelliptic equation, Heisenberg group.

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1 Introduction

Let $H^n$ be the Heisenberg group, where $\Delta_{H^n} = \sum_{i=1}^{n} (X_i^2 + Y_i^2)$ is its subelliptic Laplacian operator, $\rho(x)$ is the distance function from $x$ to the point 0. Under the real coordinate ($x_1, \cdots, x_n, y_1, \cdots, y_n, t$), the vector field $X_i$ and $Y_i$ are defined by

$$\begin{aligned}
X_i &= \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \\
Y_i &= \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}
\end{aligned}$$

for $i = 1, 2, \cdots, n$.

and the distance function $\rho(x)$ is defined by

$$\rho(x) = \left( \sum_{i=1}^{n} (x_i^2 + y_i^2)^2 + t^2 \right)^{\frac{4}{Q}}.$$
It is well known that \( \{X_i, Y_i\} \) generate the real Lie algebra of Lie group \( H^n \) and
\[
[X_i, Y_i] = 4\delta_{ij} \frac{\partial}{\partial t}, \quad i, j = 1, \ldots, n.
\]
In this Lie group, there is a group of natural dilations defined by
\[
\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0
\]
where \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \). With this group of dilations, the Lie group \( H^n \) is a two step stratified nilpotent Lie group of homogeneous dimension \( Q = 2n + 2 \), and \( \Delta_{H^n} \) is homogeneous partial differential operator of degree 2. In this paper, we deal with the existence of the positive solution to the following semi-linear subelliptic equation
\[
\begin{cases}
\Delta_{H^n} u - u + u^p = 0 & \text{in } H^n \\
u > 0 & \text{in } H^n \\
u(x) \rightarrow 0 & \rho(x) \rightarrow \infty
\end{cases}
\]
where \( 1 < p \leq \frac{Q+2}{Q-2} \)

Equation (1) comes from the CR-Yamabe problem (see [14]) and has been studied by several authors (see [4], [10], [12] and the references therein). In the paper [12], they studied the problem
\[
\begin{cases}
\Delta_{H^n} u + u^p = 0 & \text{in } H^n \\
u > 0 & \text{in } H^n \\
u(x) \rightarrow 0 & \rho(x) \rightarrow \infty
\end{cases}
\]
and showed that if the problem’s solution is cylindrical, then it must be 0. In the works [2] and [4], for \( 1 < p < \frac{Q+2}{Q-2} \) they have gotten some results on the existence of the boundary value problem of equation (1) on the bounded domain and unbounded domain with thin condition. In these condition, the corresponding functional satisfies P.S condition, and the normal variational methods works. In the entire space \( H^n, 1 < p \leq \frac{Q+2}{Q-2} \), the Folland-Stein-Soblev embedding lost compactness. The corresponding functional lost P.S condition. Their methods don’t work. To our knowledge, in these situation, there exists no report of progress on this problem up to now.

In the Euclidean space, the similar problem was studied by many peoples (see [2], [3], [11], and the references therein). In [2], W-Y Ding and W-M Ni gave some beautiful results on the similar semilinear problem in Euclidean. But for our problem, as a consequence of [12], it may not have radical symmetry solution. So our problem is more subtle then them. This is one of the reasons to study the problem (1).
Our main result is the following theorem.

**Theorem 1** For $1 < p < \frac{2Q}{Q-2}$, the problem (1) has a solution $u \in E$.

To proof our theorem, we first give some preliminary definition and Lemmas. For $u \in C_0^\infty(H^n)$ the $C^\infty$ smooth function with compact support, we define norm $\|\|\|\| \text{ by}

$$
\|u\|^2 = \int_{R^n} |\nabla_H u|^2 + u^2
$$

(2)

where $\nabla_H = (\nabla_{X_1}, \cdots, \nabla_{X_2}, \nabla_{Y_1}, \cdots, \nabla_{Y_n})$. Then we define the Folland-Stein-Sobolev space by $E = \text{C}_0^\infty(H^n)$, the completion of $C_0^\infty(H^n)$ under the norm (2). This is a Hilbert space. For $\Omega \subset H^n$, the completion of $C_0^\infty(\Omega)$ in $E$ is denoted by $E(\Omega)$, it is a Hilbert space too. These spaces have embedding theorem like the Sobolev embedding.

**Lemma 1.1** $\forall u \in E, 1 < q \leq \frac{2Q}{Q-2}$, we have

$$
\|u\|_{L^q} \leq C\|u\|
$$

(3)

where $C$ is a constant independent of $u$.

**Lemma 1.2** Let $\Omega \subset H^n$ be bounded smooth domain in $H^n$, the embedding

$$
E(\Omega) \hookrightarrow L^p(\Omega), \quad 1 \leq p < \frac{2Q}{Q-2}
$$

(4)

is compact.

We use two methods to solve the problem (1).

The first method:

In the Folland-Stein-Sobolev space $E$, we define the energy functional

$$
J(u) = \frac{1}{2} \int_{H^n} |\nabla_H u|^2 + u^2 - \frac{1}{p+1} \int u^{p+1}, \quad u \in E
$$

(5)

Let $B_k$ be the ball $B_k = \{x \in H^n | \rho(x) < k\}$. Denoted the completion $C_0^\infty(B_k)$ in $E$ by $E_k$, then

$$
E_k \subset E_{k+1} \subset E
$$

$$
E = \bigcup_{k=1}^{\infty} E_k
$$

Set $J_k = J|_{E_k}$, choose an element $u_0 \in E_1 \subseteq E_1 \subseteq \cdots \subseteq E_k \subseteq E$, such that

$$
J(u_0) < 0, \quad J_k(u_0) < 0
$$

(6)
Let $\Gamma, \Gamma_k$ defined by

$$\Gamma = \{ r : [0, 1] \to E | r(0) = 0, \ r(1) = u_0, \ r \text{ is continuous} \}$$

$$\Gamma_k = \{ r : [0, 1] \to E_k | r(0) = 0, \ r(1) = u_0, \ r \text{ is continuous} \}$$

Define

$$c = \min_{r \in \Gamma} \max_{0 \leq t \leq 1} I(r(t)),$$

$$c_k = \min_{r \in \Gamma_k} \max_{0 \leq t \leq 1} I_k(r(t)).$$

For $\Gamma_k \subseteq \Gamma_{k+1} \subset \Gamma$, we have

$$c_k \geq c_{k+1} \geq c > 0$$

By mountain-path Lemma, we know $c_k$ is a critical value of the functional $I_k$. Let $u_k$ be a critical point of $I_k$ corresponding the critical value, that is $I_k(u_k) = c_k$ and $I_k'(u_k) = 0$. By some complex estimates of $u_k$, we shall proof $c$ is a critical value of $I$, and $u_k \to u$ in $E$, $u$ is a critical point and $I(u) = c$. By the maximum principle we get a positive solution of (1).

the second method:

Define $M = \{ u \in E | \int_{H^n} |u|^{p+1} = 1 \} \subset E$. On the manifold, we define

$$I(u) = \frac{1}{2} \int_{H^n} |\nabla_H u|^2 + u^2$$

(10)

The main idea is, for $I$ have bounded from below, we define

$$c = \inf_{u \in M} I(u)$$

(11)

Then we prove that the critical $c$ can be arrived by $u \in M$. Then by Lagrange multiplier method, we know the problem have a positive solution.

For the second method of subcritical case to overcome the difficult that the functional $I$ lost P.S condition, we give some Lions’ version concentration-compactness Lemmas. This is one of bones in this work.

For the case $1 < p < \frac{Q+2}{Q-2}$, by our proof we know, for every smooth bounded domain $\Omega$, the Dirichlet problem

$$\begin{cases}
\varepsilon^2 \Delta_{H^n} u - u + u^p = 0 & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

(12)

have a least energy solution $u_\varepsilon$. Let $x_\varepsilon \in \Omega, u(x_\varepsilon) = \max_{x \in \Omega} u(x)$, like J.Wei in the paper [13], we want to know what is the lim $\varepsilon \to 0$ dist($x_\varepsilon, \partial \Omega$). In one of our preparing
works ([1]), we shall proof that
\[
\text{dist}(x, \partial \Omega) \to \max_{x \in \Omega} d(x, \partial \Omega), \quad \varepsilon \to 0
\]
and we shall publish this result elsewhere. In the following, as the Euclidean case, we shall study the effect of topology of the unbounded \( \Omega \).

2 The Proofs of Main Theorem

2.1 The first method

In this subsection, we shall use the mountain-path lemma and domain extension method to proof the Theorem in the subcritical exponent case \( 1 < p < \frac{Q+2}{Q-2} \). And more, we get that the problem have a least energy solution, and proof that
\[
c = \inf_{r \in \Gamma} \max_{0 \leq t \leq 1} J(rt)
\]
can be arrived by a path \( r_0 \in \Gamma \). This is the foundation of our paper [1].

For the Folland-Stein-Sobolev embedding \( E_k \hookrightarrow L^{p+1}, 1 < p < \frac{Q+2}{Q-2} \) is compact ([5]), by the standard method we have the following lemma.

Lemma 2.1 For \( k \in \mathbb{N} \), the functional \( J_k \) defined in the Hilbert \( E_k \) satisfies P.S condition.

For an element \( e \in E_1 \subset E_k \subset E, \|e\| = 1, \forall k \in \mathbb{N} \), we have
\[
J_k(te) = \frac{t^2}{2} - \frac{t^{p+1}}{p+1} \int |e|^{p+1} dx
\]  
(13)
For \( p + 1 > 2 \), we have the following Lemma 2.2.

Lemma 2.2 For any \( k \in \mathbb{N} \), there exists an element \( u \in \bigcap_{k=1}^{\infty} E_k \cap E \), such that
\[
I_k(u_0) < 0, \quad J(u_0) < 0
\]  
(14)
For \( \|u\| = 1 \), we have
\[
J(tu) = \frac{t^2}{2} - \frac{t^{p+1}}{p+1} \int_{H^n} |u|^{p+1}
\]  
(15)
By the Lemma 2.1, there is a positive constant \( C > 0 \) independent of \( u \), such that
\[
\int_{H^n} |u|^{p+1} \leq C
\]  
(16)
Combine the inequality (16) and the formula (15) we have

\[ J(tu) \geq \frac{t^2}{2} - \frac{tp^+}{p+1}C \]  

(17)

Since \( p + 1 > 2 \), we have the following Lemma 2.3.

**Lemma 2.3** There is a neighborhood \( U_k \) of 0 respectively in \( E_k \), and a neighborhood \( U \) of 0 in \( E \), such that

\[ J_k(u) \geq \alpha, \quad J(u) \geq \alpha \]  

(18)

for all \( u \in U_k \) or \( u \in U \) respectively, where \( \alpha > 0 \) is a positive constant.

From mountain path lemma and the above Lemma, we have the following Lemma 2.4.

**Lemma 2.4** The value \( c_k \) is a critical value of functional \( I_k \), and more we have

\[ c_k \geq c_{k+} \geq c > \alpha > 0 \]  

(19)

Suppose \( u_k \) is a critical point of \( J_k \) corresponding the critical value \( c_k \). Then we have

\[ J'(u_k)u_k = \|u_k\|^2 - \int_{H^n} |u|^{p+1} \]  

(20)

\[ J(u_k) = \|u_k\|^2 - \frac{1}{p+1} \int_{H^n} |u_k|^{p+1} = c_k > \alpha \]  

(21)

From (20) and (21), we have

\[ c_1 \geq \frac{p}{p+1} \|u_k\|^2 = c_k > \alpha \]  

(22)

That is to say \( \{u_k\} \) is a bounded point set in \( E \). So there is a subset of \( \{u_k\} \), we still denote it by \( \{u_k\} \), and a point \( \overline{u} \in E \), such that

\[ u_k \rightharpoonup \overline{u}, \]  

(23)

and \( \overline{u} \geq 0 \) is a weak solution of

\[ \Delta_{H^n} u - u + u^p = 0 \]  

in \( H^n \).

By the method of Ding and Ni(see [2]), If we can prove \( \overline{u} \neq 0 \), then \( \overline{u} \) is a critical of functional \( J \), and

\[ J(\overline{u}) = c. \]
Then by maximum principal, we know $u$ is a positive solution of problem (1), and it is a positive least energy solution of it. So if we can prove $u \not\equiv 0$, our theorem is proved. Next we locus on this problem.

For $u_k \in E$ is a solution of

$$
\Delta_{H^n} u - u + u^p = 0 \quad \text{in } H^n,
$$

we have

$$
\int |\nabla_{H^n} u_k|^2 + u_k^2 - \int u_k^{p+1} = 0.
$$

Then we have

$$
\int u_k^2 (u_k^{p-1} - 1) = \int |\nabla_{H^n} u_k|^2 \geq 0.
$$

Since $u_k \not\equiv 0$, there must be exists $\xi_k \in H^n$, such that

$$
u_k(\xi_k) = \max_{H^n} u_k \geq 1 \quad (24)
$$

We claim that $\{\xi_k\}$ is a bounded subset of $H^n$. This is our next lemma.

**Lemma 2.5** The subset $\{\xi_k\}$ defined by (24) is a bounded subset of $H^n$.

**Proof.** For $\{u_k\}$ is bounded subset of $E$, by some standard estimates and the Folland-Stein-Sobolev embedding theorem, there is a positive constant $\alpha$, such that

$$
\sup_{H^n} u_k \leq \alpha.
$$

So there is a large enough $\beta > 0$ such that

$$
- \Delta_{H^n} u_k + \beta u_k = u_k^p - (\beta - 1)u_k \leq 0 \quad (25)
$$

Define function $v = ce^{-\delta \rho(x)}$, on $H^n$ where $c$ and $\delta$ are positive number which shall be determined.

For $\Delta_{H^n}$ is a 2 order operator. So $\Delta_{H^n} v$ is a -1 order function. Then there are large positive numbers $R_0$, and $\delta > 0$, such that for all $\xi, \rho(\xi) > R_0$, and large positive number $\beta'$ such that

$$
- \Delta_{H^n} v(\xi) + \beta' v(\xi) \geq 0. \quad (26)
$$

Choose large positive number $R_0$, for all $\xi, \rho(\xi) = R_0$, we have

$$
(v - u)(\xi) \geq 0 \quad (27)
$$

Set $\beta'' = \max\{\beta, \beta'\}$, then by (25, 24, 27) we have
\[
\begin{cases}
-\Delta_{H^\alpha}(v-u) + \beta''(v-u) \geq 0 & \text{in } H^n \backslash B_{R_0}(0), \\
v - u \geq 0, & \text{on } \partial(H^n \backslash B_{R_0}(0))
\end{cases}
\]

By the maximum principle, this implies that for all $\xi > R_0$, for any $k$,

\[
u_k \leq ce^{-\delta_p(\xi)}
\]

(28)

The inequality implies that $\xi_k$ is bounded.

For the Folland-Stein-Sobolve spaces have similar embedding theorems with the Sobolev embedding and the Sub-Laplacian operator have similar characters with the Laplacian operator (see [5]), so by the method of Noussair, Ezzat S. and Swanson, Charles A (see [13]), we have the following lemma.

**Lemma 2.6.** There is a subsequence of $\{u_k\}$ we still denote it by $\{u_k\}$, such that for any bounded domain $\Omega$, $u_k \to \overline{u}$ in $C^{2+\alpha}(\Omega)$, where $\alpha$ is a positive number. That is $u_k \to \overline{u}$ in $C^{2+\alpha}_{loc}(H^n)$.

From Lemma 2.5 and Lemma 2.6, we have the following Lemma.

**Lemma 2.7.** The functional defined by (23)

\[
\overline{u} \neq 0.
\]

Proof. For $\{\xi_k\}$ is bounded in $H^n$, so we may assume that there is a $\xi_0 \in H^k$, such that $\xi_k \to \xi_0$. So we have

\[
u_k(\xi_k) \to \overline{u}(\xi_0).
\]

By the inequality (24), we have $u(\xi_0) \geq 1$. That is to say $u \neq 0$. #

### 2.2 the second method

In this subsection, we shall use the constraint functional method to study the problem (1). First we defined the manifold

\[
M = \{u \in E \mid \int |u|^{p+1} dx = 1\}
\]

(29)

On this manifold, define a functional

\[
I(u) = \frac{1}{2} \int |\nabla_{H}u|^2 + u^2, \ \forall u \in M
\]

(30)

It is obviously that the functional $I$ is bounded from below. We shall study whether the functional defined by (30) arrive its minimum on the manifold $M$. That is we want to find a $u_0 \in M$, such that

\[
I(u_0) = \min_{u \in M} I(u) = \alpha
\]

(31)
For the embedding $E \hookrightarrow L^{p+1}(H^n)$ lost compactness, so the functional $I$ does not satisfy P.S condition. To overcome this difficult, we first transplant the Lions’ concentration-compactness Lemma([6,7,11]) to Heisenberg group case.

Lemma 2.2.1 Let $(\rho_m)_{m \geq 1}$ be a sequence in $L^1(H^n)$ satisfying:

$$\rho_m \geq 0 \text{ in } H^n, \quad \int_{H^n} \rho_m = 1 \quad (32)$$

Then there exists a sequence $(\rho_{m_k})_{k \geq 1}$ satisfying one the following three possibilities:

(i) (Compactness) There exists a sequence $z_k \in H^n$ such that $\rho_{m_k}(z)$ is tight, i.e

$$\forall \varepsilon > 0, \exists R < \infty, \quad \int_{z_k + B_R} \rho_{m_k}(z) dz \geq 1 - \varepsilon; \quad (33)$$

(ii) (Vanishing) $\lim_{k \to \infty} \sup_{y \in B_R} \int \rho_{m_k}(z) dz = 0$, for all $R < \infty$;

(iii) (Dichotomy) There exists $\alpha \in (0,1)$ such that for all $\varepsilon > 0$, there $k_0 \geq 1$ and $\rho^1_k, \rho^2_k \in L^1_+(H^n)$ satisfying for $k \geq k_0$,

$$\|\rho_{m_k} - (\rho^1_k + \rho^2_k)\|_{L^1} \leq \varepsilon$$

$$|\int_{H^n} \rho^1_k dz - \alpha| \leq \varepsilon$$

and $\text{dist}(\text{supp}\rho^1_k, \text{supp}\rho^2_k) \to +\infty, k \to +\infty$, where $dz = dx dy dt$.

For the measure $dx dy dt$ on $H^n$, it has translation invariant and it is a homogeneous on dilations $\delta_\lambda$ like the measure on $R^{2n+1}$. That is for $u \in L^1(H^n), z_0 \in H^n$,

$$\int_{H^n} u(z) dz = \int_{H^n} u(z \cdot z_0^{-1}) dz$$

$$\int_{H^n} u(\lambda z) dz = \lambda^{-Q} \int_{H^n} u(z) dz$$

Where $\lambda > 0$. So, just like P.L.Lions[6,7], we can prove this lemma. We omit its proof here.

Let $\{u_m\} \subset M, I(u) \to \min_{u \in M} I(u) = \alpha, m \to \infty$. By the Folland-Stein-Sobolev and there exists a constant $c > 0$, such that

$$\|u\|_p \leq c\|u\|, \quad \forall u \in E \quad (36)$$

So $\alpha = \min_{u \in M} I(u) > 0$.

Lemma 2.2.2 For the sequence $\{u_m\}$, there is a positive number $\{R_m\}$, for the function

$$\nu_m(z) = \frac{1}{R_m^q} u_m(\delta_{\frac{z}{R_m}}(z))$$

\(9\)
such that
\[
\sup_{z \in H^n} \int_{B_1(z)} |\nu_m|^q(w)dw = \frac{1}{2} = \int_{B_1(0)} |\nu_m|^qdw
\] (38)

Proof. For \(u_m \in \{u_m\}, r > 0, z^r_m \in H^n\), we define
\[
u_m(z) = R^{-2/n}u_m(\delta_{1/m}(z \cdot z^r_m))
\] (39)

From (35), we have
\[
\int_{H^n} \int_{H^n} |u_m(\delta_{1/m}(zz^r_m))|^q = \int_{H^n} |u_m|^q = 1
\] (40)

So there exists a \(R_m\), for every \(z^r_m \in H^n\),
\[
\int_{B_1(z^r_m)} |u_m^r|^q = \int_{B_{R_m}(0)} |u_m|^q = \frac{1}{2}
\] (41)

Define \(\nu_m(z) = R^{-2/n}u_m(\delta_{1/m}(z))\). From the formula (11), we have
\[
\sup_{z \in H^n} \int_{B_1(z)} |\nu_m|^qdx = \int_{B_1(0)} |\nu_m|^qdx = \frac{1}{2}
\] #.

Let \(\rho_m = |\nu_m|^q\), then \(\rho_m \in L^1(H^n)\), and \(\int_{H^n} \rho_m = 1\). From Lemma 2.2.2, we know case (ii) in Lemma 2.2.1 can’t occurs. We declare that the case (iii) can’t also. That is our following lemma.

**Lemma 2.2.3** For the function \(\rho_m \in L^1(H^n)\) defined above, there is \(z_m \in H^n\), such that \(\rho_m(z \cdot z^{-1}_m)\) is tight, i.e. there exists a number \(R > 0\) large enough, such that
\[
\int_{z_m \cdot B_R(0)} \rho_m(z)dz \geq 1 - \varepsilon
\] (42)

Proof. By the Lemma 2.2.1 and Lemma 2.2.2, we only need prove the case (iii) in Lemma 2.2.1 doesn’t occur. On contrary, there is a number \(\beta \in (0, \lambda)\) such that for all \(\varepsilon > 0\), there exist \(m_0 \geq 1\) and \(\rho_m^1, \rho_m^2 \in L^1(H^n)\) satisfies for \(m > m_0\),
\[
\|\rho_m - (\rho_m^1 + \rho_m^2)\|_m \leq \varepsilon
\]
\[
|\int_{H^n} \rho_m^1dz - \beta| \leq \varepsilon
\]
\[
|\int_{H^n} \rho_m^2dz - (1 - \beta)| \leq \varepsilon
\] (43)

and \(\text{dist}(\text{supp}\rho_m^1, \text{supp}\rho_m^2) \to +\infty\).
Choose \( r_m > 0 \), such that \( \text{supp} \rho_m^1 \subset B_{r_m}(0) \), \( \text{supp} \rho_m^2 \subset H^n \setminus B_{r_m}(0) \), and \( r_m \to +\infty \) as \( m \to +\infty \). Set \( \varphi \in C_0^\infty(B_2(0)) \) such that \( \varphi \equiv 1 \) in \( B_1(0) \), \( 1 \leq \varphi \leq 1 \) and let \( \varphi_m(\frac{x}{r_m}) \). Decompose

\[
\nu_m = \varphi_m \nu_m + (1 - \varphi_m) \nu_m
\]

Then

\[
\int_{H^n} |\nabla_H \nu_m|^2 + |\nu_m|^2 = \int_{H^n} |\nabla_H (\varphi_m \nu_m)|^2 + \int_{H^n} (\varphi_m \nu_m)^2 + \int_{H^n} |\nabla_H (1 - \varphi_m) \nu_m)|^2
\]

Next we estimate the last two terms in formula (44) respectively.

\[
\int_{H^n} \nabla_H (\varphi_m \nu_m) \cdot \nabla_H ((1 - \varphi_m) \nu)
\geq - \int_{H^n} |\nabla_H (\varphi_m \nu_m) \cdot \nabla_H (1 - \varphi_m) \nu_m)|
\geq - \int_{H^n} |\nabla_H (\varphi_m \nu_m)| |\nabla_H ((1 - \varphi_m) \nu_m)|
= - \int_{B_2^m(0) \setminus B_{r_m}(0)} |\nabla_H (\varphi_m \nu_m)| |\nabla_H ((1 - \varphi_m) \nu_m)|
\geq \frac{1}{2} \int_{B_2^m(0) \setminus B_{r_m}(0)} |\nabla_H (\varphi_m \nu_m)|^2 + \int_{B_2^m(0) \setminus B_{r_m}(0)} |\nabla_H ((1 - \varphi_m) \nu_m)|^2
\geq - \frac{1}{2} \left\{ |\nabla_H (\varphi_m|^2 \nu_m^2 + 2 \nabla_H \varphi_m \cdot \nabla_H \nu_m \cdot \varphi_m \nu_m + \varphi_m^2 |\nabla m \nu_m|^2 \right\}
+ |\nabla H \varphi_m|^2 \nu_m^2 - 2 \nabla_H \varphi_m \cdot \nabla_H \nu_m \cdot \varphi_m \nu_m + (1 - \varphi_m)^2 |\nabla m \nu_m|^2
\geq - c \int_{B_2^m(0) \setminus B_{r_m}(0)} \nu_m^2 + |\nabla_H \nu_m|^2
\geq c \int_{B_2^m(0) \setminus B_{r_m}(0)} |\nu_m|^{p+1}
\]

Then we have

\[
\int_{H^n} \nabla_H (\varphi_m \nu_m) \cdot ((1 - \varphi_m) \nu_m) + \int \varphi_m \nu_m (1 - \varphi_m) \nu_m \geq - c \int_{B_2^m(0) \setminus B_{r_m}(0)} |\nu_m|^{p+1} \tag{45}
\]

From the proof of Lemma 2.2.1, we have \( \forall \varepsilon > 0, \exists m_0 \), such that for \( m > m_0 \),

\[
\int_{B_{r_m}(0)} |\nu_m|^{p+1} \leq \beta
\]

\[
\int_{H^n \setminus B_{r_m}(0)} |\nu_m|^{p+1} \leq 1 - \beta + \varepsilon \tag{46}
\]
So by the inequalities of (43) and (46), we have

\[
\int_{B_{2r_m}(0) \setminus B_{r_m}(0)} |\nu_m|^{p+1} \leq c \left[ \int_{H^n} |\nu_m|^{p+1} - \int_{H^n} (\rho_m^1 + \rho_m + m^2) \right] + \varepsilon
\]  

(47)

The inequality (47) means that

\[
\int_{B_{2r_m}(0) \setminus B_{r_m}(0)} |\nu_m|^{p+1} = o(1)
\]  

(48)

where \(o(1) \to 0, m \to +\infty\).

Combine the formula (48), (44) and the inequality (45) we have

\[
\int_{H^n} |\nabla H \nu_m|^2 + |\nu_m|^2 = \|\varphi_m \nu_m\|^2 + \|(1 - \varphi_m) \nu_m\|^2 + o(1)
\]  

(49)

By the Folland-Stein-Sobolev embedding and formula (49), we have

\[
\|\nu_m\|^2 = \|\varphi_m \nu_m\|^2 + \|(1 - \varphi_m) \nu_m\|^2 + o(1)
\]  

\[
\geq S(\|\varphi_m \nu_m\|_{L^p}^{p+1} + \|(1 - \varphi_m) \nu_m\|_{L^p}^{p+1}) + o(1)
\]  

(50)

\[
\geq S\left( \int_{H^n} \rho_m^1 \right)^{\frac{2}{p+1}} + \left( \int_{H^n} \rho_m^2 \right)^{\frac{2}{p+1}} + o(1)
\]  

\[
\geq S(\beta^{\frac{2}{p+1}} + (1 - \beta)^{\frac{2}{p+1}}) + o(1)
\]

By the define of \(\alpha\) and the independence of domain of the best Folland-Stein-Sobolev constant we know \(S = \alpha\). And by the define \(\nu_m\) we have \(\|\nu_m\|^2 \to \alpha(m \to \infty)\). So by the inequality (50) we have

\[
\alpha \geq \alpha(\beta^{\frac{2}{p+1}} + (1 - \beta)^{\frac{2}{p+1}})
\]  

(51)

For \(\frac{2}{p+1} < 1, 0 < \beta < 1\), we get \(\alpha > \alpha\), that is a contradiction. So the case (iii) of Lemma 2.2.1 can’t occur.

**Theorem 2.2.1** There is a subsequence of \(\{\nu_m\}\), we still denote it by \(\{\nu_m\}\); there exists a point \(u_0 \in M\), such that \(\nu_m \to \nu_0\) in \(E\), and

\[
I(\nu_0) = \alpha
\]

Proof: For \(\{\nu_m\}\) is bounded in \(E\), we have a subsequence of it, and we still denote it by \(\{\nu_m\}\), and there exists a \(\varepsilon, \nu_0 \in E\) such that \(\nu_m \to \nu_0\). For \(\varepsilon < \frac{1}{2}\), and Lemma 2.1.2, we get \((z_m \cdot B_R(0)) \cap B_1(0) \neq \emptyset\), so the points sequence \(\{z_m\}\) is a bounded
set. That is implies that there is a subsequence of \( \{z_m\} \), we still denote it by \( \{z_m\} \) and a point \( z_0 \in H^n \), such that \( z_m \to z_0, (m \to \infty) \). Then we have

\[
\int_{z_0 \cdot B_{1+2R}(0)} |\nu_m|^{p+1} > 1 - \varepsilon
\]  

(52)

From the Folland-Stein-Sobolev embedding, we know there is a subsequence \( \{\nu_m\} \) we still denote it by \( \{\nu_m\} \), such that \( \nu_m \to \nu_0, m \to \infty \) in \( H^{1,2}(B_{1+2R}(0)) \) and

\[
\int_{z_0 \cdot B_{1+2R}(0)} |\nu_0|^{p+1} > 1 - \varepsilon
\]  

(53)

From the Fatou Lemma we know

\[
\int_{H^n} |\nu_0|^{p+1} \leq \lim_{m \to \infty} \int_{H^n} |\nu_m|^{p+1} = 1
\]  

(54)

Combine the inequalities of (53) and (54), we have

\[
\int_{H^n} |\nu_0|^{p+1} = 1
\]  

(55)

This implies that \( \nu_m \to \nu_0, m \to \infty \) in \( E \). So we have

\[
I(\nu_0) = \lim_{m \to \infty} I(\nu_m) = \alpha
\]

By the Lagrange multiplier, there is a positive number, such that

\[
\Delta_{H^n} \nu_0 - \nu_0 + \lambda \nu_0^p = 0
\]

Set \( u_0 = \lambda^{p-1} u \), then we have

\[
\Delta_{H^n} u_0 - u_0 + u_0^p = 0
\]

By the maximum principle, we get our main theorem 1.1.

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