NOT ALL PURE STATES ON $\mathcal{B}(H)$ ARE DIAGONALIZABLE

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Abstract. Assuming the continuum hypothesis, we prove that $\mathcal{B}(H)$ has a pure state whose restriction to any masa is not pure. This resolves negatively an old conjecture of Anderson.

Let $H$ be a separable infinite-dimensional Hilbert space and let $\mathcal{B}(H)$ be the algebra of bounded operators on $H$. Anderson [4] conjectured that every pure state on $\mathcal{B}(H)$ is diagonalizable, i.e., of the form $f(A) = \lim U \langle Ae_n, e_n \rangle$ for some orthonormal basis $(e_n)$ and some ultrafilter $U$ over $\mathbb{N}$.

A masa of $\mathcal{B}(H)$ is a maximal abelian self-adjoint subalgebra, and an atomic masa is the set of all operators which are diagonalized with respect to some given orthonormal basis of $H$. Anderson’s conjecture is related to a fundamental problem in C*-algebra, the Kadison-Singer problem [6], which asks whether every pure state on an atomic masa of $\mathcal{B}(H)$ has a unique extension to a pure state on $\mathcal{B}(H)$. If $(e_n)$ is an orthonormal basis of $H$, then every pure state $f_0$ on the corresponding atomic masa $\mathcal{M}$ has the form $f_0(A) = \lim U \langle Ae_n, e_n \rangle$ for some ultrafilter $U$ over $\mathbb{N}$ and all $A \in \mathcal{M}$, and Anderson [3] showed that the same formula, now for $A \in \mathcal{B}(H)$, defines a pure state $f$ on $\mathcal{B}(H)$. Thus, a positive solution to the Kadison-Singer problem would say that $f$ is the only pure state on $\mathcal{B}(H)$ which extends $f_0$.

In the presence of a positive solution to the Kadison-Singer problem, Anderson’s conjecture is equivalent to the weaker statement that every pure state on $\mathcal{B}(H)$ restricts to a pure state on some atomic masa. However, assuming the continuum hypothesis, we show that this weaker statement is false: in fact, there exist pure states on $\mathcal{B}(H)$ whose restriction to any masa is not pure. It follows that there are pure states on $\mathcal{B}(H)$ that are not diagonalizable. It seems likely that the statement “every pure state on $\mathcal{B}(H)$ restricts to a pure state on some atomic masa” is also consistent with standard set theory. This together with a positive solution to the Kadison-Singer problem would imply the consistency of a positive answer to Anderson’s conjecture.

The key lemma we need is the following. Let $\mathcal{K}(H)$ be the algebra of compact operators on $H$, let $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ be the Calkin algebra, and let $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$ be the natural quotient map. We also write $\dot{a}$ for $\pi(a)$, for any $a \in \mathcal{B}(H)$.

**Lemma 0.1.** Let $\mathcal{A}$ be a separable C*-subalgebra of $\mathcal{B}(H)$ which contains $\mathcal{K}(H)$, let $f$ be a pure state on $\mathcal{A}$ that annihilates $\mathcal{K}(H)$, and let $\mathcal{M}$ be a masa of $\mathcal{B}(H)$. Then there is a pure state $g$ on $\mathcal{B}(H)$ that extends $f$ and whose restriction to $\mathcal{M}$ is not pure.

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Proof. By Proposition 6 of [2] we can find an infinite-rank projection \( p \in \mathcal{B}(H) \) such that
\[
\hat{p}\hat{p} = f(a)\hat{p}
\]
for all \( a \in \mathcal{A} \).

Lemma 1.4 and Theorem 2.1 of [5] imply that \( \pi(M) \) is a masa of \( \mathcal{C}(H) \). It follows that there is a projection \( q \in M \) such that \( \hat{q} \) neither contains nor is orthogonal to \( \hat{p} \). Otherwise \( \hat{p} \) would be in the commutant of \( \pi(M) \), and hence would belong to \( \pi(M) \) by maximality. But this would mean \( \hat{p} \) is minimal in \( \pi(M) \) because any nonzero projection below \( \hat{p} \) neither contains nor is orthogonal to \( \hat{p} \), and \( \pi(M) \) has no minimal projections.

Let \( \phi : \mathcal{C}(H) \to \mathcal{B}(K) \) be an irreducible representation of the Calkin algebra. It is faithful because \( \mathcal{C}(H) \) is simple. Therefore \( \phi(\hat{q}) \) neither contains nor is orthogonal to \( \phi(\hat{p}) \), so we can find a unit vector \( v \in K \) in the range of \( \phi(\hat{p}) \) which is neither contained in nor orthogonal to the range of \( \phi(\hat{q}) \). Finally, define \( g(a) = \langle \phi(a)v, v \rangle \) for all \( a \in \mathcal{B}(H) \). This is a pure state on \( \mathcal{B}(H) \) because \( \phi \circ \pi \) is an irreducible representation of \( \mathcal{B}(H) \). It extends \( f \) because, using (1),
\[
g(a) = \langle \phi(a)v, v \rangle = \langle \phi(\hat{a})\phi(\hat{p})v, \phi(\hat{p})v \rangle = \langle \phi(\hat{p}\hat{a}\hat{p})v, v \rangle = (f(a)\phi(\hat{p})v, v) = f(a)
\]
for all \( a \in \mathcal{A} \). Finally, its restriction to \( M \) is not pure because the projection \( q \in M \) has the property that
\[
g(q) = \langle \phi(\hat{q})v, v \rangle
\]
is strictly between 0 and 1, since \( v \) is neither contained in nor orthogonal to the range of \( \phi(\hat{q}) \).

Theorem 0.2. Assume the continuum hypothesis. Then there is a pure state on \( \mathcal{B}(H) \) whose restriction to any masa is not pure.

Proof. Let \((a_{\alpha})\), \( \alpha < \aleph_1 \), enumerate the elements of \( \mathcal{B}(H) \). Since every von Neumann subalgebra of \( \mathcal{B}(H) \) is countably generated, a simple cardinality argument shows that there are only \( \aleph_1 \) such subalgebras. Hence \( \mathcal{B}(H) \) has only \( \aleph_1 \) masas. Let \((M_{\alpha})\), \( \alpha < \aleph_1 \), enumerate the masas of \( \mathcal{B}(H) \).

We now inductively construct a nested transfinite sequence of unital separable \( C^* \)-subalgebras \( A_{\alpha} \) of \( \mathcal{B}(H) \) together with pure states \( f_{\alpha} \) on \( A_{\alpha} \) such that for all \( \alpha < \aleph_1 \)
\[
(1) a_{\alpha} \in A_{\alpha+1}
(2) \text{if } \beta < \alpha \text{ then } f_{\alpha} \text{ restricted to } A_{\beta} \text{ equals } f_{\beta}
(3) A_{\alpha+1} \text{ contains a projection } q_{\alpha} \in M_{\alpha} \text{ such that } 0 < f_{\alpha+1}(q_{\alpha}) < 1.
\]
Begin by letting \( A_0 \) be any separable \( C^* \)-subalgebra of \( \mathcal{B}(H) \) that is unital and contains \( \mathcal{K}(H) \) and let \( f_0 \) be any pure state on \( A_0 \) that annihilates \( \mathcal{K}(H) \). At successor stages, use the lemma to find a projection \( q_{\alpha} \in M_{\alpha} \) and a pure state \( g \) on \( \mathcal{B}(H) \) such that \( g|_{A_{\alpha}} = f_{\alpha} \) and \( 0 < g(q_{\alpha}) < 1 \). By (1), Lemma 4) there is a separable \( C^* \)-algebra \( A_{\alpha+1} \subseteq \mathcal{B}(H) \) which contains \( A_{\alpha}, a_{\alpha}, \) and \( q_{\alpha} \) such that the restriction \( f_{\alpha+1} \) of \( g \) to \( A_{\alpha+1} \) is pure. Thus the construction may proceed. At limit ordinals \( \alpha \), let \( A_{\alpha} \) be the closure of \( \bigcup_{\beta < \alpha} A_{\beta} \). The state \( f_{\alpha} \) is determined by the condition \( f_{\alpha}|_{A_{\beta}} = f_{\beta} \), and it is easy to see that \( f_{\alpha} \) must be pure. (If \( g_1 \) and \( g_2 \) are states on \( A_{\alpha} \) such that \( f_{\alpha} = (g_1 + g_2)/2 \), then for all \( \beta < \alpha \) purity of \( f_{\beta} \) implies that \( g_1 \) and \( g_2 \) agree when restricted to \( A_{\beta} \); thus \( g_1 = g_2 \).) This completes the description of the construction.
Now define a state $f$ on $B(H)$ by letting $f|_{A_\alpha} = f_\alpha$. By the reasoning used immediately above, $f$ is pure, and since $0 < f(q_\alpha) < 1$ for all $\alpha$, the restriction of $f$ to any masa is not pure.

It is interesting to contrast Theorem 0.2 with Theorem 9 of [2], which states that (assuming the continuum hypothesis) any state on $C(H)$ restricts to a pure state on some masa of $C(H)$. This does not conflict with our result because there are many masas of $C(H)$ which do not come from masas of $B(H)$ (regardless of the truth of the continuum hypothesis). Indeed, $B(H)$ has $2^{\aleph_0}$ masas but $C(H)$ has $2^{2^{\aleph_0}}$ masas. This can be seen by first finding $2^{\aleph_0}$ mutually orthogonal nonzero projections $p_\alpha$ in $C(H)$ [7], then finding projections $q^{1}_\alpha, q^{2}_\alpha < p_\alpha$ such that $q^{1}_\alpha q^{2}_\alpha \neq q^{2}_\alpha q^{1}_\alpha$ for each $\alpha$, and finally for each set $S \subseteq 2^{\aleph_0}$ choosing a masa of $C(H)$ that contains $\{q^{1}_\alpha : \alpha \in S\}$ and $\{q^{2}_\alpha : \alpha \notin S\}$. It is easy to see that this produces $2^{2^{\aleph_0}}$ distinct masas.

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