ON W-ALGEBRAS ASSOCIATED TO \((2,p)\) MINIMAL MODELS AND THEIR REPRESENTATIONS

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ABSTRACT. For every odd \(p \geq 3\), we investigate representation theory of the vertex algebra \(W_{2,p}\) associated to \((2,p)\) minimal models for the Virasoro algebras. We demonstrate that vertex algebras \(W_{2,p}\) are \(C_2\)-cofinite and irrational. Complete classification of irreducible representations for \(W_{2,3}\) is obtained, while the classification for \(p \geq 5\) is subject to certain constant term identities. These identities can be viewed as "logarithmic deformations" of Dyson and Selberg constant term identities, and are of independent interest.

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1. INTRODUCTION

The Virasoro algebra modules of central charge zero play a prominent role in conformal field theory (CFT). Most recently they also appeared in the physics literature on logarithmic CFT (cf. \[21, 17, 13, 20, 27\], etc.). As with the ordinary CFTs, it is desirable to understand constructions in logarithmic CFT via representations of vertex algebras. While this can be done in some cases, there are examples of central charge zero LCFT which are still poorly understood, at least from algebraic point of view.

In an important paper \[17\] (cf. also \[13\]), certain vertex \(W\)-algebras, denoted by \(W_{q,p}\), associated to central charge \(c_{q,p} = 1 - \frac{6(p-q)^2}{pq}\) Virasoro minimal models were introduced (here \(p\) and \(q\) are co-prime integers). These vertex algebras are extracted from the Felder’s complex via a
pair of screening operators. Similar construction, but with one screening operator, has also been used in the definition of the triplet vertex algebra $W(p)$ [3], a certain extension of the Virasoro vertex algebra $L(c_{1,p},0)$. Compared to the triplet $W(p)$, the vertex algebra $W_{q,p}$ is more difficult to study due to more complicated structure of Virasoro modules and screening operators. Another sharp contrast is that $W_{q,p}$ is no longer simple nor self-dual and in fact combines into a nontrivial extension of the Virasoro vertex algebra $L(c_{q,p},0)$ (the latter is known to be rational [32]). More precisely,

$$0 \rightarrow R_{q,p} \rightarrow W_{q,p} \rightarrow L^{Vir}(c_{q,p},0) \rightarrow 0,$$

where $R_{q,p}$ is the maximal vertex ideal in $W_{q,p}$. The reader should notice that for $q = 2, p = 3$ (i.e. $c = 0$) the above sequence is non-trivial, even though the vertex algebra $L^{Vir}(0,0) \cong \mathbb{C}$ is trivial.

While the paper [17] does provide expected “spectrum” and “generalized characters” of $W_{q,p}$, several results in [17] rely on a conjectural correspondence between the category of $W_{q,p}$-modules and the category of modules of a certain quantum group. Thus, in parallel with the triplet algebra [3], it is not clear how to: (i) obtain a “strong” set of generators of $W_{q,p}$, (ii) prove the $C_2$-cofiniteness of $W_{q,p}$ and (iii) classify irreducible $W_{q,p}$-modules, and finally (iv) construct projective covers. Some partial results in this direction were previously obtained in [17] for all $q$ and $p$, and in [20] for $c = 0$. In addition, in [6] we found an explicit construction of some (but not all!) logarithmic $W_{q,p}$-modules.

Arguably, the simplest minimal models occur within the series $(2, p)$, $p \geq 3$ odd. These models have been linked to classical partition identities (cf. [15]) and their combinatorics and characters are somewhat less complicated compared to the unitary series $(k, k + 1)$. So, at least from this point of view, it seems natural to contemplate $W_{2,p}$ algebras and their representations first.

This paper is meant to provide a very detailed vertex algebraic study of $W$-algebras $W_{2,p}$, following our approach in [3], but supplied with many new ideas. In parallel with [3], we study $W_{2,p}$ as a subalgebra of a rank one lattice vertex algebra $V_L$ such that the conformal vector $\omega$ has central charge $c_{2,p}$ (cf. Section 3). Although we work in full generality of $W_{2,p}$ vertex algebras, our original motivation was to understand the algebra $W_{2,3}$ (cf. [20], [13], [17], etc). A central aim of this work is to study the $C_2$-cofiniteness of the vertex operator algebra $W_{2,p}$. This is an important problem because $C_2$-cofiniteness implies that $W_{2,p}$ admits only finitely many irreducible representations. That is our first result

**Theorem 1.1.** For any $p \geq 3$, the vertex operator algebra $W_{2,p}$ is $C_2$-cofinite.

We should mention that for $p = 3$, Theorem 1.1 was (at least implicitly) conjectured in [20]. The $C_2$-cofiniteness is also important because it allows us to use powerful results from [24] and [22]. In particular, results there endow the category of $W_{2,p}$-modules with a natural braided vertex tensor category structure. The fusion rules for this category are of considerable interest in the physics literature (cf. [20], [33]).

A few words about the proof of Theorem 1.1. As in [11], [12] (see also [41]) we study the singlet vertex algebra $M(1)$ strongly generated by two generators and realized as a subalgebra of $W_{2,p}$. In this way we realize a new family of $W$-algebras with two generators. We also completely determine the associated Zhu’s algebras. It turns out that the structural results on singlet vertex algebra $M(1)$ provide an important step in the proof of Theorem 1.1.
Having established the $C_2$-cofiniteness we can now go on and explore irreducible $\mathcal{W}_{2,p}$-modules. But to achieve this in full generality we have to overcome some difficulties of combinatorial nature (see Section 10). First we only state and prove classification theorem for $p = 3$.

**Theorem 1.2.** The vertex operator algebra $\mathcal{W}_{2,3}$ has precisely 13 irreducible inequivalent modules (explicitly described in the paper). Each irreducible module is a submodule or a subquotient of an irreducible module for the rank one lattice vertex algebra $V_L$ of central charge zero.

The classification of modules for $\mathcal{W}_{2,p}$ is subject to a constant term identity closely related to classical Dyson-Selberg’s identities [9], but with important modifications due to logarithmic factors. We have

**Theorem 1.3.** If Conjecture 10.1 holds, the vertex algebra $\mathcal{W}_{2,p}$ has precisely $4p + \frac{p-1}{2}$ irreducible modules.

As with the triplet vertex algebra it is important to explore the space of generalized characters (cf. [30]) for purposes of modular invariance. We have a partial result in this direction.

**Theorem 1.4.** The space of generalized $\mathcal{W}_{2,p}$-characters is 20-dimensional for $p = 3$. For $p \geq 5$, this space is conjecturally of dimension $\frac{15p-5}{2}$.

There are several directions we plan to pursue at this stage. Many ideas and arguments presented here certainly generalize to other $\mathcal{W}_{q,p}$-algebras and possibly to higher $\mathcal{W}$-algebras [10]. But even before diving deeper into $\mathcal{W}_{q,p}$-algebras, it would be desirable to construct indecomposable $\mathcal{W}_{2,p}$ algebras. For instance, the methods in [6, 19] and this paper can be employed to study logarithmic $\mathcal{W}_{2,p}$-modules. Another direction is to explore possible connections between our construction and generalized twisted modules introduced recently in [23].

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### 2. A Construction of Derivations of Vertex Superalgebras

In this section we shall present a new construction of derivation on a vertex subalgebra realized as the kernel of a screening operator. This construction will be applied to lattice vertex algebras. Further applications will be studied in our future work.

We assume some familiarity with vertex operator algebras and superalgebras. We refer the reader to standard textbooks on this subject (cf. [25] for instance). Assume that $(V, Y, 1, \omega)$ is a $\mathbb{Z}$-graded vertex operator superalgebra with $V = V_0 \oplus V_1$ (parity decomposition) and the vertex operator map $Y(\cdot, x)$, where $Y(u, x) = \sum_{n \in \mathbb{Z}} u_n x^{-n-1}$ and $Y(\omega, x) = \sum_{m \in \mathbb{Z}} L(m)x^{-m-2}$. Let $a \in V$ be odd vector such that

\begin{align*}
\{a_n, a_m\} &= a_n a_m + a_m a_n = 0 \quad \forall n, m \in \mathbb{Z}, \\
L(n)a &= \delta_{n,0}a \quad \forall n \in \mathbb{Z}_{\geq 0},
\end{align*}

so that $a$ is of conformal weight one. Then

$[a_0, L(n)] = 0,$
i.e., $a_0$ is a screening operator. Moreover,

$$ \nabla = \text{Ker}_{\nabla} a_0 $$

is a vertex subalgebra of $\nabla$. We shall now construct a derivation on $\nabla$. Recall that an (even) derivation on a superalgebra $V$ is a linear map $D : V \to V$, such that

$$ D(a_n b) = (Da)_n b + a_n (Db), $$

for all $a, b \in V$ and $n \in \mathbb{Z}$. Define the following operator

$$ G = \sum_{i=1}^{\infty} \frac{1}{i} a_{-i}. $$

Since $G$ commutes with the action of screening operator $a_0$, we have that $G$ is a well-defined operator on $\nabla$.

**Theorem 2.1.** On the vertex superalgebra $\nabla$ we have:

(i) $[L(n), G] = 0$ for every $n \in \mathbb{Z}$, i.e, $G$ is a screening operator which commutes with the action of the Virasoro algebra.

(ii) The operator $G$ is a derivation on the vertex algebra $\nabla$, and in particular, $\exp[G]$ is an automorphism of $\nabla$.

(iii) Assume that $(W, Y_W)$ is any $V$–module and $v \in \nabla$. Then on the submodule $\text{Ker}_W a_0$ we have

$$ Y_W(Gv, z) = [G, Y_W(v, z)]. $$

**Proof.** (i) We start with the formula

$$ [L(n), Y(a, x)] = (x^{n+1} \frac{d}{dx} + (n+1)x^n)Y(a, x). $$

Thus,

$$ [L(n), a_m] = -ma_{m+n} $$

and hence

$$ \sum_{m=1}^{\infty} \frac{1}{m} [L(n), a_{-m}a_m] = \sum_{m=1}^{\infty} a_{-m+n}a_m - \sum_{m=1}^{\infty} a_{-m}a_{m+n}. $$

But the last expression is zero because $a_k$ and $a_l$ anti-commute for any $k$ and $l$, and because $a_0$ acts trivially on $\nabla$.

For (ii) we first observe

$$ [G, Y(u, x)] = \sum_{m=1}^{\infty} \frac{1}{m} [a_m, Y(u, x)] + \sum_{m=1}^{\infty} \frac{1}{m} [a_{-m}, Y(u, x)]a_m. $$

Let us denote the first and second summand on the right hand side in (3) by $A$ and $B$, respectively. Then

$$ A = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} x^{m-n}a_{-m} \binom{m}{n} Y(a_n u, x) + \sum_{n=1}^{\infty} \frac{a_{-n}x^n}{n} Y(a_0 u, x) $$

$$ = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} x^{m-n}a_{-m} \binom{m}{n} Y(a_n u, x), $$

$$ B = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{a_{-n}x^n}{n} Y(a_0 u, x). $$
where we used the property $a_0 a = 0$. Now replace $m - n$ by $m$, so we get

$$A = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} x^m a_{-n-m} \frac{(m+n)}{m} Y(a_n x)$$

$$= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} x^m a_{-n-m} \frac{(m+n-1)}{m} Y(a_n x).$$

(4)

Similarly,

$$B = \sum_{m=1}^{\infty} \frac{1}{m} [a_m, Y(u, x)] a_m$$

$$= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m} (-m) x^{-m-n} Y(a_n x) a_m$$

$$= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m} (-1)^n \frac{(m+n-1)}{n} x^{-m-n} Y(a_n x) a_m.$$  

On the other hand, the iterate formula (see [25]) gives for odd $w$ (for even $w$ just switch the sign):

$$Y(a_m w, x_2) = \text{Res}_{x_1} \left( \frac{1}{(x_1 - x_2)^m} Y(a, x_1) Y(w, x_2) + \frac{1}{(-x_2 + x_1)^m} Y(w, x_2) Y(a, x_1) \right)$$

$$= \sum_{i=0}^{\infty} a_{-m-i} \frac{(-m)}{i} (-1)^i x_2^i Y(w, x_2)$$

$$+ \sum_{i=0}^{\infty} \left( \frac{(-m)}{i} \right) (-1)^{m-i} x_2^{-m-i} Y(w, x_2) a_i$$

$$= \sum_{i=0}^{\infty} a_{-m-i} \frac{(m+i-1)}{i} x_2^i Y(w, x_2)$$

$$+ \sum_{i=0}^{\infty} \left( \frac{m+i-1}{i} \right) (-1)^m x_2^{-m-i} Y(w, x_2) a_i.$$

Now,

$$\sum_{m=1}^{\infty} \frac{1}{m} Y(a_m a w, x_2) =$$

$$\sum_{m=1}^{\infty} \sum_{i=0}^{\infty} \frac{1}{m} a_{-m-i} \left( \frac{m+i-1}{i} \right) x_2^i Y(a_m w, x_2) + \sum_{m=1}^{\infty} \sum_{i=0}^{\infty} \frac{1}{m} \left( \frac{m+i-1}{i} \right) (-1)^m x_2^{-m-i} Y(a_m w, x_2) a_i.$$  

(5)

Denote the first and second sum on the right hand side in (5) by $A'$ and $B'$, respectively.
It is clear that $A = A'$ (just apply $m \to i$ and $n \to m$ in formula (4)). Similarly one shows $B = B'$.

It is clear that all above formulas and calculations hold on $\operatorname{Ker} W_{a_0}$, where $W$ is any $V$-module. The proof follows. □

Assume now that $h \in V$ is an even vector such that

\begin{align}
(6) & \quad L(n)h = \delta_{n,0}h, \quad n \in \mathbb{Z}_{\geq 0}, \\
(7) & \quad [h(n), h(m)] = \gamma \delta_{n+m,0}, \quad \gamma \in \mathbb{Q}, \\
(8) & \quad h(0) \text{ acts semisimply on } V \text{ with eigenvalues in } \frac{1}{2}\mathbb{Z}, \\
(9) & \quad h(n)a = \frac{1}{2} \delta_{n,0}a, \quad n \in \mathbb{Z}_{\geq 0}.
\end{align}

Then $\sigma = \exp[2\pi ih(0)]$ is an automorphism of $V$ of order two.

Let $(W, Y_W)$ be any $V$-module. Then we can construct the $\sigma$-twisted $V$-module $(W^\sigma, Y_W^\sigma)$ as follows (cf. [26]):

\begin{align}
W^\sigma := W \quad \text{as vector space,} \\
Y_W^\sigma(\cdot, x) := Y_W(\Delta(h, x)\cdot, x), \quad \text{where} \\
\Delta(h, x) := x^{h(0)} \exp \left( \sum_{n=1}^{\infty} \frac{h(n)}{n} (-x)^{-n} \right).
\end{align}

The $\Delta$-operator satisfies

\begin{equation}
\Delta(h, x_2)Y_W(u, x_0)\Delta(h, x_2)^{-1} = Y_W(\Delta(h, x_2 + x_0)u, x_0).
\end{equation}

In particular for $u = a$ we get

\begin{equation}
\Delta(h, x_2)Y_W(a, x_0)\Delta(h, x_2)^{-1} = Y_W((x_2 + x_0)^{1/2}a, x_0).
\end{equation}

Let

\begin{equation}
Y_W^\sigma(a, x) = \sum_{m\in \mathbb{Z}} a_{m+1/2}x^{-m-1/2}
\end{equation}

and

\begin{equation}
G_{tw} = \sum_{m=0}^{\infty} \frac{1}{m + 1/2} a_{-m-1/2} a_{m+1/2}.
\end{equation}

Then we have a $\sigma$-twisted version of Theorem 2.1

**Theorem 2.2.** On any $\sigma$-twisted $V$-module $(W^\sigma, Y_W^\sigma)$, we have:

(i) $[L(n), G_{tw}^\sigma] = 0$ for $n \in \mathbb{Z}$, i.e., $G_{tw}^\sigma$ is a screening operator.

(ii) For any $v \in V$ we have

\begin{equation}
[G_{tw}^\sigma, Y_W^\sigma(v, x)] = Y_W^\sigma(Gv, x).
\end{equation}
Proof. The strategy of the proof is as in Theorem 2.1, so we omit unnecessary details. From the twisted commutator formula (cf. [26]) we get

\[ [G_{tw}, Y^\sigma_W(v, x)] = \sum_{m=0}^{\infty} \frac{1}{m + 1/2} a_{m-1/2} a_{m+1/2}, Y^\sigma_W(v, x) + \sum_{m=0}^{\infty} \frac{1}{m + 1/2} a_{m-1/2}, Y^\sigma_W(v, x) a_{m+1/2} \]

\[ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m + 1/2} x^{m+1/2-n} \binom{m + 1/2}{n} a_{m-1/2} Y^\sigma_W(a_n v, x) \]

\[ + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m + 1/2} x^{-m-1/2-n} \binom{-m - 1/2}{n} Y^\sigma_W(a_n v, x) a_{m+1/2} \]

On the other hand, the formula (11) and the (untwisted) iterate formula gives

\[ Y^\sigma_W(Gv, x) = Y^\sigma_W(\Delta(h, x)Gv, x) = \sum_{m=1}^{\infty} \frac{1}{m} Y^\sigma_W(\Delta(h, x)a_{-m} a_m a, x) \]

\[ = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m} \binom{1/2}{n} x^{1/2-n} Y^\sigma_W(a_{-m} \Delta(h, x)a_m v, x) \]

\[ = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{m (-1)^j} \binom{1/2}{n} \binom{n - m}{j} x^{1/2-n+j} a_{n-m-j-1/2} Y^\sigma_W(a_m v, x) + \]

\[ + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{m (-1)^{m+n+j}} \binom{1/2}{n} \binom{n - m}{j} x^{1/2-n-j} a_{n-m-j} Y^\sigma_W(a_m v, x) e_j^{\alpha 1/2}, \]

where as in the previous theorem we assumed that \( v \) is odd.

The rest is an application of Vandermonde convolution for binomial coefficients appearing in (13) and matching appropriate terms in (12) and (13). In the process we also use the fact that \( a_0 \) acts trivially on \( V \).

\[ \square \]

3. Lattice vertex algebras, \( W \)-subalgebras and screening operators

Assume that \( p \) is an odd natural number, \( p \geq 3 \). Let

\[ L = \mathbb{Z} \alpha, \quad \langle \alpha, \alpha \rangle = p. \]

Also, let

\[ V_L = M(1) \otimes \mathbb{C}[L] \]

be the associated vertex superalgebra, where as usual \( M(1) \cong U(\hat{h}_{<0}) \) is the vacuum module for the corresponding Heisenberg algebra \( \hat{h}, h = L \otimes \mathbb{Z} \). Consider \( D = \mathbb{Z}(2\alpha) \). Then \( V_D \) is a vertex subalgebra of \( V_L \).

Define the Virasoro vector

\[ \omega = \frac{1}{2p} (\alpha(-1)^2 + (p-2)\alpha(-2)) \]
and screening operators

\[ Q = e^0_0, \quad \tilde{Q} = e^0_0. \]

Define the following vertex (super)algebras

\[ \mathcal{V}_L = \text{Ker}_1 V \cap \text{Ker}_1 \tilde{Q} \]

and

\[ \mathcal{W}_{2,p} = \text{Ker}_1 V \cap \text{Ker}_1 \tilde{Q}. \]

Then \( \mathcal{V}_L \) is a \( \mathbb{Z}_{\geq 0} \)–graded vertex superalgebra and \( \mathcal{W}_{2,p} \subset \mathcal{V}_L \) is a vertex operator algebra of central charge \( c_{2,p} = 1 - \frac{3(p-2)^2}{p} \).

In particular, if \( p = 3 \) the central charge is zero; this case is of considerable interest in the paper.

Now we shall apply the general construction from Section 2 in the case of lattice vertex superalgebra \( V_L \). Then we have the following operator

\[ G = \sum_{i=1}^{\infty} \frac{1}{i} e^{-i} e^i. \]

It is convenient to express \( G \) as

\[ \text{Res}_{x_1, x_2} \ln (1 - \frac{x_2}{x_1}) Y(e^0, x_1) Y(e^0, x_2). \]

Then applying Theorem 2.1 we get:

**Theorem 3.1.** On the vertex algebras \( \mathcal{W}_{2,p} \) and \( \mathcal{V}_L \) we have:

(i) \( [L(n), G] = 0 \) for every \( n \in \mathbb{Z} \), i.e., \( G \) is a screening operator.

(ii) \( [Q, G] = [	ilde{Q}, G] = 0 \).

(iii) The operator \( G \) is a derivation on the vertex algebra \( \text{Ker}_1 V \), and in particular on \( \mathcal{V}_L \) and \( \mathcal{W}_{2,p} \).

(iv) Relations (i)-(iii) hold on \( \text{Ker}_1 V_{L+\frac{\ell}{p}} \), where \( \ell \in \{0, \ldots, p-1\} \).

Next we shall consider twisted \( V_L \)–modules. As in [5], we have that

\[ \sigma := \exp\left[ \frac{\pi i}{p} \alpha(0) \right] \]

is a canonical automorphism of order two of the vertex superalgebra \( V_L \). Moreover, \( V_L \) admits \( \sigma \)-twisted modules whose description we briefly recall here [26].

Any irreducible \( \sigma \)-twisted \( V_L \) module \( M \) is isomorphic to

\[ (V_{L+\frac{(\ell+1/2)\alpha}{p}}, Y^\sigma) \quad (0 \leq \ell \leq p-1), \]

where

\[ V_{L+\frac{(\ell+1/2)\alpha}{p}} = V_{L+\frac{\ell\alpha}{p}} \]

as a vector spaces,

and the twisted vertex operator \( Y^\sigma \) is defined

\[ Y^\sigma(\cdot, x) = Y(\Delta(\alpha/2p, x)\cdot, x). \]
Then applying Theorem 2.2 in the case
\[ V = V_L, \ a = e^\alpha, \ h = \frac{\alpha}{2p}, \]
we get the following result:

**Proposition 3.2.**  
(i) The relation
\[ [L(n), G^{tw}] = 0 \text{ for } n \in \mathbb{Z} \]
holds on any \( \sigma \)-twisted \( V_L \)-module.
(ii) The operator \( G^{tw} \) is a \( \sigma \)-twisted derivative of \( \ker V_L Q \), and thus of \( W_{2,p} \). In other words, for any \( v \in \ker V_L Q \) we have
\[ [G^{tw}, v_n] = (Gv)_n \text{ for } n \in \frac{1}{2} \mathbb{Z}. \]

4. **Fusion rules for certain Virasoro modules of central charge \( c_{2,p} \)**

Let us fix some notation. In what follows \( V^{Vir}(c,0) \) denote the universal Virasoro vertex operator algebra of central charge \( c \) (cf. [25] for details). The corresponding simple vertex algebra will be denoted by \( L^{Vir}(c,0) \). It is known (cf. [25], [32]) that any highest weight irreducible module \( L^{Vir}(c,h), h \in \mathbb{C} \) is a \( V^{Vir}(c,0) \)-module. We remark that for \( c = 0 \) the algebra \( L(0,0) \) is 1-dimensional.

The aim of this section is to analyze fusion rules among certain irreducible Virasoro modules of central charge \( c_{2,p} \), in particular \( c = 0 \), viewed as modules for the vertex algebra \( V^{Vir}(c_{2,p},0) \).

Let
\[ h_{r,s} = \frac{(pr - 2s)^2 - (p - 2)^2}{8p}. \]
Here we analyze fusion rules among \( L^{Vir}(c_{2,p}, 3p - 2) \) and some special modules of type \( L^{Vir}(c_{2,p}, n) \), where \( n \in \mathbb{N} \). Related fusion rules for \( p = 3 \) were previously analyzed in [28].

Here is the main result

**Theorem 4.1.** The space of intertwining operators
\[ I \left( \begin{array}{c} L^{Vir}(c_{2,p}, h) \\ L^{Vir}(c_{2,p}, h_{5,1}) \end{array} \right) \]
is nontrivial only if
\[ h \in \{ h_{2n-1,1}, h_{2n+1,1}, h_{2n+3,1}, h_{2n+5,1}, h_{2n+7,1} \}. \]

**Proof.** Proofs of similar results have already appear in the literature so here we do not provide full details (see [29] for instance).

We start from intertwining operator \( \mathcal{Y} \) of type as in (15), and consider highest weight vectors \( u \in L^{Vir}(c_{2,p}, h), v \in L^{Vir}(c_{2,p}, h_{5,1}) \) and \( w \in L(c_{2,p}, h_{2n+3,1}). \) Then the matrix coefficient
\[ \langle w', \mathcal{Y}(u,x)v \rangle \]
is proportional to \( x^{h - h_{2n+3,1} - h_{5,1}} \). It is known (see [15]) that \( L^{Vir}(c_{2,p}, 3p - 2) \) can be obtained as a quotient of the Verma module \( M^{Vir}(c_{2,p}, 3p - 2) \) by the submodule generated by a pair of singular vectors, where one singular vector is of (relative) degree 5. While one can analyze both
vectors explicitly, for our purposes we only analyze the singular vector of degree five. It is not hard to see that

\[ v_{\text{sing}} = \left( L(-1)^5 - 10pL(-2)L(-1)^3 + (21p^2 - 15p)L(-3)L(-1)^2 \\
+ 16p^2L(-2)^2L(-1) + (42p^2 - 18p - 36p^3)L(-4)L(-1) + (-24p^3 + 16p^2)L(-3)L(-2) \\
+ (-66p^3 + 46p^2 + 36p^4 - 12p)L(-5) \right) 1 \in M^{Vir}(c_2, p, 3p - 2) \]

is such a vector (unique up to a constant). This singular vector give rise to a differential equation of degree 5 satisfied by \( x^{h - h_{2n + 3,1} - h_{5,1}} \). This follows from the relations

\[ [L(-n), \mathcal{Y}(u, x)] = (x^{n+1} \frac{d}{dx} - (n - 1)h_{2n+3,1}x^{-n})\mathcal{Y}(u, x), \quad n \geq 1. \]

The differential equation obtained (we omit an explicit formula here) can be now solved and the \( h \)-solutions are given in (16).

**Remark 1.** To prove the “if” part in the theorem we would have to analyze both singular vectors and describe \( A(V^{Vir}(c_2, p, 0)) \)-bimodule \( A(L^{Vir}(c_2, p, 3p - 2)) \) (cf. [29]).

5. **The Vertex Operator Algebra** \( \mathcal{W}_{2,p} \)

We define the following elements in \( V_L \):

\[ a^- = Qe^{-2\alpha}, \quad a^+ = GQe^{-2\alpha}. \]

Since \( Q \) anti-commutes with itself, we have \( Q^2 = 0 \) and hence \( Qa^- = 0 \) and \( Qa^+ = 0 \). It is not hard to see that \( GQe^{-2\alpha} \neq 0 \). Since \( \tilde{Q} \) annihilates \( e^{-2\alpha} \) and commutes with \( Q \) and \( G \), we conclude that \( a^\pm \in \overline{V}_L \). A related vertex operator superalgebra has been encountered in our study of the triplet vertex algebra and is a subject of our forthcoming paper [7] (see also [16]).

Let \( V \) be the subalgebra of \( \overline{V}_L \) generated by 1, \( a^- \), \( a^+ \).

The following lemma is useful for constructing singular vectors in \( \overline{V}_L \):

**Lemma 5.1.** We have:

(i) \( Y(a^-, x)Qe^{-(n+1)\alpha} \in W((x)), \) where

\[ W = U(V^{Vir}).Qe^{-(n+2)\alpha} \cong L^{Vir}(c_2, p, h_{2n+5,1}) \]

(ii) \( G^nQe^{-(n+1)\alpha} \neq 0 \) for \( n \in \mathbb{Z}_{\geq 0} \).

(iii) \( G^nQe^{-(n+1)\alpha} \in V \).

**Proof.** First we shall prove assertion (i). Define \( j_0 = -np + p - 2 \). Then we have

\[ a_j^-Qe^{-(n+1)\alpha} = 0 \quad \text{for} \quad j > j_0, \]

\[ a_{j_0}^-Qe^{-(n+1)\alpha} = \mu_nQe^{-(n+2)\alpha} \quad (\mu_n \neq 0). \]

Let us now see that

\[ w_n := GQe^{-(n+1)\alpha} \neq 0. \]
In what follows we shall prove that $w_2 = H = GQe^{-3\alpha} \neq 0$ (cf. Theorem 6.5). Then the relation
\[ G(a^-_2a^-) = a^-_2a^- + a^-_2a^+ = \mu_1 H \]
gives that $w_1 = a^+ \neq 0$. Our Theorem 2.1 implies
\[ e^{(n-1)\alpha}_j w_n = (-1)^{n-1}a^+ \quad (j_1 = (n + 1)(n - 1)p - 1), \]
so we have that $w_n \neq 0$ for every $n \in \mathbb{Z}_{\geq 0}$.

Assume that $j \leq j_0$. Then by using Proposition 4.1 we see that $a^-_j Qe^{-\alpha}(n+1) \in \text{Virasoro-submodule of } M(1) \otimes e^{(n+1)\alpha}$ generated by (co)singular vectors of weights $h_{2n+5,1}$ and $h_{2n+7,1}$. But cosingular vector of weight $h_{2n+7,1}$ is proportional to $GQe^{-(n+3)\alpha} \notin V_L$ (since by (17) we have $w_{n+2} = QGe^{-(n+3)\alpha} \neq 0$). Therefore
\[ a^-_j Qe^{-\alpha}(n+1) \in U(Vir)Qe^{-(n+2)\alpha}. \]

In this way we have proved assertion (i).

We shall prove the assertions (ii) and (iii) by induction on $n \in \mathbb{Z}_{>0}$.

For $n = 1$ we have already proved that $a^+$ is a nontrivial singular vector.

Assume now that assertions (ii)-(iii) hold for certain $n \in \mathbb{Z}_{>0}$. Since $V_L$ is a simple vertex operator algebra we have that
\[ Y(a^+, z)G^nQe^{-(n+1)\alpha} \neq 0, \]
(for the proof see [12]). So there is $k_0 \in \mathbb{Z}$ such that
\[ a^+_{k_0} G^nQe^{-(n+1)\alpha} \neq 0 \quad \text{and} \quad a^+_{j} G^nQe^{-(n+1)\alpha} = 0 \quad \text{for} \quad j > k_0. \]
Since
\[ a^+_{k_0} G^nQe^{-(n+1)\alpha} = \nu_2 G^{n+1}(a^-_{k_0} Qe^{-(n+1)\alpha}), \]
for certain non-zero constant $\nu_2$, then by using assertion (i) and the fact that $G$ is a screening operator we conclude that
\[ a^+_{k_0} G^nQe^{-(n+1)\alpha} \in U(Vir)G^{n+1}Qe^{-(n+2)\alpha}. \]
Therefore $G^{n+1}Qe^{-(n+2)\alpha} \neq 0$ and
\[ G^{n+1}Qe^{-(n+2)\alpha} = \frac{1}{\nu_2 \mu_n} a^+_{j_0} G^nQe^{-(n+1)\alpha} \in V. \]

The proof follows. \[ \square \]

By using Lemma 5.1 and the structure theory of Feigin-Fuchs modules from [14, 15 and 31] one can see that the following theorem holds for every $p \geq 3$:

**Theorem 5.2.** As a Virasoro algebra module, $V_L$ is generated by the family of (non-trivial) singular $\tilde{S}ing$ vectors, two families of subsingular vectors $SSing^{(1)} \cup SSing^{(2)}$, and a family of cosingular vectors...
These vectors satisfy the following conditions:

\[
\begin{align*}
\tilde{\text{Sing}}^{(3)} &= \{w^{(3)}_{(j,n)}, \ | \ n \in \mathbb{Z} > 0, \ 0 \leq j \leq n - 1\}, \\
\tilde{\text{Sing}}^{(2)} &= \{w^{(2)}_{(j,n)}, \ | \ n \in \mathbb{Z} > 0, \ 0 \leq j \leq n - 1\}, \\
\tilde{\text{Sing}}^{(1)} &= \{w^{(1)}_{(j,n)}, \ | \ n \in \mathbb{Z} > 0, \ 0 \leq j \leq n\}.
\end{align*}
\]

Remark 2. The main novelty of our approach is in the fact that we use the screening operator \(G\) for which we have explicit formulae in the context of vertex operator algebras. Screening operators from \([14]\) and \([31]\) are defined by using (complex) contour integrals. In the above case these two approaches are equivalent.

We should mention here that \(G^j e^{-(n+1) \alpha} \in \tilde{\text{Sing}}^{(1)}\) become singular in \(V_L / \text{Ker} \ Q\). But these vectors are not annihilated by \(Q\). So we get

Proposition 5.3. We have

(i) As a Virasoro module, \(V_L\) is generated by the vacuum vector \(1\) and the family of singular vectors

\[
\{G^j Q e^{-(n+1) \alpha} \ | \ j \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{> 0}, \ j \leq n\}.
\]

Moreover,

\[
V_L = V^{vir}(c_{2,p}, 0) \bigoplus_{n=1}^{\infty} \bigoplus_{k=1}^{\infty} (n + 1) L^{vir}(c_{2,p}, \frac{(n + 1)(pn + 2p - 2)}{2}).
\]

(ii) We have \(V = \overline{V}_L\), i.e., the vertex algebra \(V = \overline{V}_L\) is strongly generated by \(1, \omega\) and

\[
a^- = Q e^{-2 \alpha} \quad \text{and} \quad a^+ = G a^-.
\]

Proof. First we notice that

\[
Q e^{-n \alpha} \neq 0, \tilde{Q} e^{-n \alpha} = 0, \quad \tilde{Q} e^{n \alpha} \neq 0, \quad (n \in \mathbb{Z}_{> 0}).
\]

Since the operators \(Q, \tilde{Q}\) and \(G\) mutually commute we have that only singular vectors \(u_{(j,n)}\) are annihilated by both \(Q\) and \(\tilde{Q}\). This proves assertion (i). The assertion (ii) easily follows from Lemma 5.1. \(\square\)
Define the following (nonzero) vectors in the vertex algebra $V_D$:

$$F = Qe^{-3\alpha}, \quad H = GF, \quad E = G^2F.$$ 

Again, it is clear that $F$ (and $H$ and $E$) are inside $\mathcal{W}_{2,p}$.

In the same way as above (see also [3]) we get the following result.

**Proposition 5.4.**

(i) As a Virasoro module, $\mathcal{W}_{2,p}$ is generated by the vacuum vector $1$ and the family of singular vectors

$$\{Ge^{-2(n+1)\alpha} \mid j \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{> 0}, \ j \leq 2n\}.$$ 

Moreover,

$$\mathcal{W}_{2,p} = V^{Vir}(c_{2,p},0) \bigoplus_{n=1}^{\infty} (2n+1)L^{Vir}(c_{2,p},(2n+1)(pm+p-1)).$$

(ii) As a vertex algebra, $\mathcal{W}_{2,p}$ is strongly generated by $1, \omega, E, F, H$

The structure of $\mathcal{W}_{2,p}$ and $V_L$ can be visualized via the embedding diagram for the Virasoro module $V_L$ (cf. [14], [15]):

The left-most "ladder" is $M(1) \otimes e^{-2\alpha} \in V_D$, followed by $M(1) \otimes e^{-\alpha} \in V_{D+\alpha}$, the middle module is $M(1) \in V_D$, $M(1) \otimes e^{\alpha} \in V_{D+\alpha}$ and $M(1) \otimes e^{2\alpha} \in V_D$. Singular vectors in $V_D$ are denoted by $\bullet$, whereas in $V_{D+\alpha}$ singular vectors are denoted by $\square$. These vectors account for $\widetilde{Sing}$ set introduced earlier. Similarly, vectors denoted by $\circ$ in $V_D$ and $\triangle \in V_{D+\alpha}$ give the set $\widetilde{SSing}^{(1)}$. Dotted arrows indicate the action of the $G$-screening.

6. **Singlet vertex algebra $\overline{M(1)}$**

In this section we shall study representation theory of the singlet vertex algebra, which is a subalgebra of $\mathcal{W}_{2,p}$. The results from this section will be used in the proof of $C_2$-cofiniteness of $\mathcal{W}_{2,p}$.

As in [1] and [2] we have the singlet vertex operator algebra

$$\overline{M(1)} = \text{Ker}_M(1)Q \cap \text{Ker}_{\overline{M(1)}}Q,$$
which is a subalgebra of $\mathbf{W}_{2,p}$. The structure of $M(1)$ as a module for the Virasoro algebra can be described by using Theorem 5.2. We have the following result which is analogous to the results obtained in [1] in the case of $(1, p)$ Virasoro modules.

**Proposition 6.1.**
(i) As a Virasoro module, $M(1)$ is generated by the vacuum vector $1$ and the family of singular vectors
\[ \{ G^n Qe^{-(2n+1)\alpha} \mid n \in \mathbb{Z}_{>0} \}. \]
Moreover,
\[ M(1) = V_{Vir}(c_2, p, 0) \bigoplus_{n=1}^\infty L_{Vir}(c_2, p, (2n+1)(pn+p-1)). \]
(ii) As a vertex algebra, $M(1)$ is strongly generated by $1$, $\omega$ and $H$.

As in [1] and [3] we have the following useful proposition:

**Proposition 6.2.**
(i) $V_{Vir}(c_2, p, 0) \cong \text{Ker}_{M(1)} G$.
(ii) $H_i H \in U(Vir) \cdot 1$ for every $i \geq -3p -1$.
(iii) $H_{-3p-2} H = CG^2 Q e^{-5\alpha} + U(Vir) \cdot 1$ ($C \neq 0$).

Now we want to classify all irreducible $M(1)$–modules. We shall use similar approach as in [1]. So we need to evaluate certain singular vectors for the Virasoro algebra on top components of $M(1)$–modules $M(1, \lambda)$. It turns out that this case involve more complicated combinatorics.

We shall start with the general following theorem which gives non-triviality of action of singular vectors for the Virasoro algebra on top components of $M(1)$–modules.

**Theorem 6.3.** Assume that $u \in M(1)$ is a singular vector for the Virasoro algebra. Let $u(0) := u_{\deg u-1}$. Assume that $M(1, \lambda)$ is a $M(1)$–module, which is irreducible as a module for the Virasoro algebra with central charge $c_2, p$ (In other words $M(1, \lambda)$ is an irreducible Feigin-Fuchs module). Then
\[ u(0)v_\lambda = \nu v_\lambda \text{ for certain } \nu \neq 0. \]

**Proof.** Since $M(1, \lambda)$ is an irreducible module for the simple vertex algebra $M(1)$ we have that
\[ Y(u, z)v_\lambda \neq 0, \]
(for the proof see [12]). So there is $j_0 \in \mathbb{Z}$ such that
\[ u_{j_0} v_\lambda \neq 0 \text{ and } u_j v_\lambda = 0 \text{ for } j > j_0. \]
Since $u$ and $v_\lambda$ are singular vectors for the Virasoro algebra we have that
\[ L(n) u_{j_0} v_\lambda = 0 \text{ for } n \geq 1. \]
Therefore $u_{j_0} v_\lambda$ is a non-trivial singular vector in the irreducible Feigin-Fuchs module $M(1, \lambda)$ and conclude that it is proportional to $v_\lambda$. Therefore $j_0 = \deg u - 1$ and $u(0)v_\lambda = \nu v_\lambda$ for certain non-zero constant $\nu$.

By using expression (14) for the operator $G$ and similar calculation to that in [1], [2], [3], [5] we get:

**Proposition 6.4.**
(1) Assume that \( v_\lambda \) is a lowest weight vector in \( M(1) \)-module \( M(1, \lambda) \) and \( t = \lambda(\alpha) \). Then we have:

\[
H(0)v_\lambda = -(\text{Res}_{x_1, x_2, x_3} \ln(1 - x_2/x_1)(x_1 x_2 x_3)^{-3\rho}(x_1 - x_2)^\rho(x_2 - x_3)^\rho(1 + x_1)^t(1 + x_2)^t(1 + x_3)^t) v_\lambda.
\]

(2) Let \( v = G^2 Q e^{-5\alpha} \). Then

\[
o(v)v_\lambda = \tilde{G}_p(t)v_\lambda \neq 0
\]

where

\[
\tilde{G}_p(t) = \text{Res}_{x_1, x_2, x_3, x_4, x_5} (\ln(1 - x_2/x_1)\ln(1 - x_4/x_3)(x_1 x_2 x_3 x_4 x_5)^{-5p} \\
\Delta(x_1, x_2, x_3, x_4, x_5)^p (1 + x_1)^t(1 + x_2)^t(1 + x_3)^t(1 + x_4)^t(1 + x_5)^t).
\]

(Here \( \Delta(x_1, x_2, \ldots, x_n)^p \) denotes the \( p \)-th power of the Vandermonde determinant.)

The next result appears to be fairly difficult to prove.

**Theorem 6.5.** Assume that \( v_\lambda \) is a lowest weight vector in \( M(1) \)-module \( M(1, \lambda) \) and \( t = \lambda(\alpha) \). Then we have:

\[
H(0)v_\lambda = A_p \left( \frac{t + p}{2p - 1} \right) \left( \frac{t}{2p - 1} \right) \left( \frac{t + p/2}{2p - 1} \right) v_\lambda \quad (A_p \neq 0).
\]

This theorem will be proven in the appendix, where we also compute the constant \( A_p \) (this is not really needed because \( A_p \neq 0 \) is everything we need).

Now we are in the position to determine the Zhu’s algebra of \( \overline{M(1)} \) and therefore classify their irreducible \( \mathbb{Z}_{\geq 0} \)-graded modules. By using Proposition 6.2, Theorem 6.5 and similar arguments to that of Theorem 6.1 of [1] we get:

**Theorem 6.6.** Zhu’s algebra of the singlet vertex algebra \( A(\overline{M(1)}) \) is isomorphic to the commutative associative algebra

\[
A(\overline{M(1)}) \cong \frac{\mathbb{C}[x, y]}{(P(x, y))}
\]

where

\[
P(x, y) = y^2 - C_p \prod_{i=0}^{2p-2} \left( x - \frac{(2p - 2 - i)(p - i)}{2p} \right)^{2p-2} \prod_{i=0}^{2p-2} \left( x - \frac{(3p - 4 - 2i)(p - 2i)}{8p} \right) \quad (C_p \neq 0).
\]

In particular, for \( p = 3 \) the polynomial \( P(x, y) \) is given by

\[
y^2 - C_3(x + \frac{1}{24})(x - \frac{5}{8})^2(x - \frac{1}{8})^2 x^4(x - 1)^2(x - 2)^2(x - \frac{1}{3})^2
\]

where \( C_3 \) is a constant.
7. THE $C_2$-COFINITENESS OF VERTEX ALGEBRAS $\mathcal{W}_{2,p}, p \geq 3$

As usual, for a vertex operator algebra $V$ we let

$$C_2(V) = \{a_{-2}b : a, b \in V\}. $$

It is a fairly standard fact (cf. [34]) that $V/C_2(V)$ has a Poisson algebra structure with the multiplication

$$\overline{a} \cdot \overline{b} = \overline{a_{-1}b},$$

where $- \cdot$ denotes the natural projection from $V$ to $V/C_2(V)$. If $\dim(V/C_2(V))$ is finite-dimensional we say that $V$ is $C_2$-cofinite.

**Lemma 7.1.** In $\mathcal{W}_{2,p}$ we have the following relations

$$E_{-1}F + F_{-1}E + 2H_{-1}H = 0, \tag{19}$$

$$E_{-1}E = F_{-1}F = 0. \tag{20}$$

**Proof.** Relation $F_{-1}F = 0$ follows from definition. Since $E_{-1}E$ is proportional to $G^4(F_{-1}F)$ we get (20). Then we have that

$$E_{-1}F + F_{-1}E + 2H_{-1}H = G^2(F_{-1}F) = 0.$$  

The proof follows. \(\square\)

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$$C_2(V) = \{a_{-2}b : a, b \in V\}. $$

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$$\overline{a} \cdot \overline{b} = \overline{a_{-1}b},$$

where $- \cdot$ denotes the natural projection from $V$ to $V/C_2(V)$. If $\dim(V/C_2(V))$ is finite-dimensional we say that $V$ is $C_2$-cofinite.

The aim of this section is the following result

**Theorem 7.2.** The vertex operator algebra $\mathcal{W}_{2,p}$ is $C_2$-cofinite.

**Proof.** First we notice $\mathcal{W}_{2,p}/C_2(\mathcal{W}_{2,p})$ is generated by the set $\{\overline{E}, \overline{F}, \overline{H}, \overline{\omega}\}$. By using Lemma 7.1 and commutativity of $\mathcal{W}_{2,p}/C_2(\mathcal{W}_{2,p})$, we obtain that

$$\overline{E}^2 = \overline{F}^2 = 0,$$

and

$$\overline{H}^2 = -\overline{E}\overline{F}$$

which implies that

$$\overline{H}^4 = 0.$$

Moreover, the description of Zhu’s algebra from Theorem 6.6 implies that

$$\overline{H}^2 = C\overline{\omega}^{6p-3}, \quad (C \neq 0).$$

Since $\overline{H}^4 = 0$, we conclude that $\overline{\omega}^{12p-6} = 0$. Therefore, every generator of the commutative algebra $\mathcal{W}_{2,p}/C_2(\mathcal{W}_{2,p})$ is nilpotent and therefore $\mathcal{W}_{2,p}/C_2(\mathcal{W}_{2,p})$ is finite-dimensional. \(\square\)
8. Classification of Irreducible $\mathcal{W}_{2,3}$-Modules

In this section we shall consider the case $p = 3$. We shall construct and classify all irreducible modules for $\mathcal{W}_{2,3}$. It turns out that the proof of classification result is analogous to the case of triplet vertex algebra $\mathcal{W}(p)$ from [3].

**Theorem 8.1.** In Zhu’s algebra $A(\mathcal{W}_{2,3})$ we have the following relation

$$f([\omega]) = 0,$$

where

$$f(x) = x^3 \left( (x - 1)(x - 2)(x - \frac{5}{3})(x - \frac{7}{3})(x - \frac{1}{3}) \right)^2$$

$$\times (x - 5)(x - 7)(x - \frac{10}{3})(x + \frac{1}{3})(x - \frac{33}{8})(x - \frac{41}{8})(x - \frac{35}{24});$$

(21)

**Proof.** First we notice that $Qe^{-5\alpha} \in O(\mathcal{W}_{2,p})$, which implies that

$$v = G^2Qe^{-5\alpha} \in O(\mathcal{W}_{2,p}) \quad \text{and} \quad [v] = 0 \in A(\mathcal{W}_{2,p}).$$

By using structure of $\mathcal{W}_{2,p}$ and similar arguments as in [3] we conclude that

$$[v] = f([\omega])$$

for certain polynomials of degree 20. This polynomials can be determined by evaluating $o(v)v_\lambda$. Direct calculation (note that in the case $p=3$ Conjecture 10.1 holds) and Proposition 6.4 give explicit formula (21).

Define the ideal:

$$\mathcal{R}_{2,3} = \mathcal{W}_{2,3}.\omega.$$

**Proposition 8.2.** We have:

(i) $\mathcal{R}_{2,3}$ is a non-trivial ideal in $\mathcal{W}_{2,3}$ and $\mathcal{W}_{2,3}/\mathcal{R}_{2,3} = \mathbb{C}$.  

(ii) $\mathcal{R}_{2,3}$ is an irreducible $\mathcal{W}_{2,3}$-module.

**Proof.** First we notice that

$$L(n + 1)\omega = X(n)\omega = 0 \quad \text{for} \quad X \in \{E, F, H\}, \quad n \in \mathbb{Z}_{\geq 0}.$$ 

(Here $X(n) := X_{14+n}$.) This easily implies that $1 \notin \mathcal{R}_{2,3}$. So, $\mathcal{R}_{2,3} \neq \mathcal{W}_{2,3}$. Clearly, $E, F, H \in \mathcal{R}_{2,3}$, which implies that

$$G^jQe^{-(2n+1)\alpha} \in \mathcal{R}_{2,3} \quad \text{for} \quad n \in \mathbb{Z}_{>0}, \quad j \leq 2n.$$ 

This proves assertion (i). Assume now that $N \subset \mathcal{R}_{2,3}$ is a non-trivial submodule. Then $N$ is a $\mathbb{Z}_{\geq 0}$-graded, and top component $N(0)$ must have conformal weight which is greater than 2. Now Theorem 8.1 implies that top component $N(0)$ of $N$ must have conformal weight 5 or 7. This is a contradiction, since every vector in $N(0)$ is a singular vector for the Virasoro algebra and since there are no singular vectors of conformal weights 5 and 7 in $\mathcal{R}_{2,3}$. The proof follows.

Define the set

$$\mathcal{S}_{2,3} = \mathcal{S}_{2,3}^{(1)} \cup \mathcal{S}_{2,3}^{(2)},$$

where

$$\mathcal{S}_{2,3}^{(1)} = \{0, 1, 2, \frac{1}{3}, \frac{1}{8}, \frac{3}{8}, \frac{-1}{24}\} \quad \mathcal{S}_{2,3}^{(2)} = \{5, 7, \frac{10}{3}, \frac{21}{8}, \frac{33}{8}, \frac{35}{24}\}.$$
Define the following $W_{2,3}$–modules:

$$W(2) = x_{1,1}^+ = R_{2,3} = W_{2,3} \cdot \omega \subset W_{2,3},$$
$$W(0) = W_{2,3}/x_{1,1}^+,$$
$$W(7) = x_{1,1}^- = W_{2,3} \cdot (e^{-2\alpha} + \text{Ker}_{V_L} Q) \subset V_L/\text{Ker}_{V_L} Q,$$
$$W(1) = x_{2,1}^+ = W_{2,3} \cdot e^\alpha \subset V_L,$$
$$W(5) = x_{2,1}^- = W_{2,3} \cdot (e^{2\alpha} + V_L) \subset V_L/\overline{V_L},$$
$$W(-\tfrac{1}{24}) = x_{3,2}^+ = W_{2,3} \cdot e^{\alpha/6} \subset V_{L+\alpha/6},$$
$$W(\tfrac{1}{5}) = x_{2,2}^+ = W_{2,3} \cdot e^{\alpha/2} \subset V_{L+\alpha/2},$$
$$W(\tfrac{4}{3}) = x_{3,1}^+ = W_{2,3} \cdot e^{2\alpha/3} \subset V_{L+2\alpha/3},$$
$$W(\tfrac{5}{8}) = x_{1,2}^+ = W_{2,3} \cdot e^{5\alpha/6} \subset V_{L+5\alpha/6},$$
$$W(\tfrac{35}{24}) = x_{5,2}^- = W_{2,3} \cdot e^{7\alpha/6} \subset V_{L+7\alpha/6},$$
$$W(\tfrac{11}{8}) = x_{2,2}^- = W_{2,3} \cdot e^{3\alpha/2} \subset V_{L+3\alpha/2},$$
$$W(\tfrac{10}{3}) = x_{3,1}^- = W_{2,3} \cdot e^{5\alpha/3} \subset V_{L+5\alpha/3},$$
$$W(\tfrac{33}{8}) = x_{1,2}^- = W_{2,3} \cdot e^{11\alpha/6} \subset V_{L+11\alpha/6}.$$

We also indicate the $x$-parametrization for the modules following [17]. In the parenthesis next to a module we denote the lowest conformal weight.

**Theorem 8.3.** For every $h \in S_{2,3}$, $W(h)$ is a $\mathbb{N}$-graded irreducible $W_{2,3}$–module whose top component $W(h)(0)$ has lowest conformal weight $h$. Moreover, $W(h)(0)$ is 1-dimensional if $h \in S_{2,3}^{(1)}$ and 2-dimensional if $h \in S_{2,3}^{(2)}$.

**Proof.** The proof is similar to that of the irreducibility result obtain in Proposition 8.2 for $R_{2,3}$ and to that of Theorem 3.12 of [3]. So we omit some technical details. First we see that each $W(h)$ is a $\mathbb{Z}_{>0}$–graded $W_{2,3}$–module whose top component is irreducible module for the Zhu’s algebra $A(W_{2,3})$. Then by using Theorem 5.1 and simple analysis of weights of modules $W(h)$, we see that they are irreducible $W_{2,3}$–modules. □

As in [3] we have the following result on the structure of Zhu’s algebra $A(W_{2,3}).$

**Proposition 8.4.** The Zhu’s associative algebra $A(W_{2,3})$ is generated by $[1], [\omega], [E], [F]$ and $[H]$. We have the following relations:

\begin{align*}
(22) & \quad [H] \ast [F] - [F] \ast [H] = -2q([\omega]) [F], \\
(23) & \quad [H] \ast [E] - [E] \ast [H] = 2q([\omega]) [E], \\
(24) & \quad [E] \ast [F] - [F] \ast [E] = -2q([\omega]) [H].
\end{align*}

where $q$ is a certain polynomial.
Theorem 8.5. The set 
\[ \{W(h), \ h \in S_{2,3}\} \]
provides, up to isomorphism, all irreducible modules for the vertex operator algebra \( W_{2,3} \).

Proof. The proof is also similar to that of the classification result in [1]. It is enough to see that
\( W(h)(0), \ h \in S_{2,3} \) gives all irreducible modules for Zhu’s algebra \( A(W_{2,3}) \). Assume that \( U \) is an irreducible \( A(W_{2,3}) \)-module. Relation \( f([\omega]) = 0 \) in \( A(W_{2,3}) \) implies that
\[ L(0)|U = h \text{ Id}, \ \text{for} \ \ h \in S_{2,3}. \]

Assume first that \( h \in S_{2,3}^{(2)} \). By combining Proposition 8.4 and Theorem 8.3 we have that \( q(h) \neq 0 \). Define
\[ e = \frac{1}{\sqrt{2q(h)}} E, \ \ f = -\frac{1}{\sqrt{2q(h)}} F, \ \ h = \frac{1}{q(h)} H. \]

Therefore \( U \) carries the structure of an irreducible, \( sl_2 \)-module with the property that \( e^2 = f^2 = 0 \) and \( h \neq 0 \) on \( U \). This easily implies that \( U \) is 2-dimensional irreducible \( sl_2 \)-module. Moreover, as an \( A(W_{2,3}) \)-module \( U \) is isomorphic to \( W(h)(0) \).

Assume next that \( h \in S_{2,3}^{(1)} \). If \( q(h) \neq 0 \), as above we conclude that \( U \) is an irreducible 1-dimensional \( sl_2 \)-module. Therefore \( U \cong W(h)(0) \).

If \( q(h) = 0 \) from Proposition 8.4 we have that the action of generators of \( A(W_{2,3}) \) commute on \( U \). Irreducibility of \( U \) implies that \( U \) is 1-dimensional. Since \([H], [E]^2, [F]^2\) must act trivially on \( U \), we conclude that \([H], [E], [F]\) also act trivially on \( U \). Therefore \( U \cong W(h)(0) \). \( \square \)

9. DESCRIPTION OF THE SPACE OF GENERALIZED CHARACTERS FOR \( W_{2,3} \)

By using Proposition 6.2, Theorem 8.1 and the proof similar to that of Corollary 3.6 in [1] we obtain the following useful consequence.

Proposition 9.1. Inside the Poisson algebra \( V/C_2(V) \), we have the relation \( \omega^{20} = 0 \).

Next result is essentially from [17]

Lemma 9.2. The \( SL(2, Z) \)-closure of the space of irreducible characters is 20-dimensional.

Proposition 9.1, Lemma 9.2, and the results from [8] now give

Theorem 9.3. Any generalized \( W_{2,3} \) character satisfies a unique modular differential equation of order 20. Consequently, the space of generalized characters is precisely 20-dimensional and coincides with the space in Lemma 9.2.

10. ON CLASSIFICATION OF IRREDUCIBLE \( W_{2,p} \)-MODULES, \( p \geq 5 \)

In this section we briefly discuss the classification of irreducible modules for the vertex operator algebra \( W_{2,p} \). One can apply the similar methods as in Section 8. The main technical problem is in the determination of the polynomial \( \tilde{G}_p(t) \) from Proposition 6.4.

We have the following conjecture about polynomial \( \tilde{G}_p(t) \):

\[ \text{Conjecture: } \]
Conjecture 10.1.

\[ \widetilde{G}_p(t) = B_p \left( \frac{t}{3p-1} \right) \left( \frac{t + p/2}{3p-1} \right) \left( \frac{t + p}{3p-1} \right) \left( \frac{t + 3p/2}{3p-1} \right) \left( \frac{t + 2p}{3p-1} \right) \]  

(B_p \neq 0).

Remark 3. We verified Conjecture 10.1 for \( p = 3 \) and \( p = 5 \) by using Mathematica package. Since polynomial \( \widetilde{G}_p(t) \) is important in the representation theory of the vertex algebra \( \mathcal{W}_{2,p} \), we plan to return to this Conjecture in our future work.

We shall now assume that Conjecture 10.1 holds. Since \( v = G^2 Q e^{-5\alpha} \in O(\mathcal{W}_{2,p}) \) and \( o(v) v_\lambda = \widetilde{G}_p(t) v_\lambda \), we get the following important relation in Zhu’s algebra \( A(\mathcal{W}_{2,p}) \):

Theorem 10.2. Assume that Conjecture 10.1 holds. Then in Zhu’s algebra \( A(\mathcal{W}_{2,p}) \) we have:

\[ f_p([\omega]) = 0, \]

where

\[
\begin{align*}
    f_p(x) &= \left( \prod_{i=1}^{3p-1} (x - h_{1,i}) \right) \left( \prod_{i=1}^{3p-1} (x - h_{1,2p-i}) \right) \left( \prod_{i=1}^{3p-1} (x - h_{2,i}) \right) \\
    &= \left( \prod_{i=1}^{\frac{p-1}{2}} (x - h_{1,i}) \right) ^3 \left( \prod_{i=p}^{2p-1} (x - h_{1,i}) \right) ^2 \left( \prod_{i=1}^{p-1} (x - h_{2,i}) \right) ^2 \\
    &\cdot (x - h_{p,2}) \left( \prod_{i=2p}^{3p-1} (x - h_{1,i}) \right) \left( \prod_{i=2p}^{3p-1} (x - h_{2,i}) \right) .
\end{align*}
\]

(25)

By using same methods as in Section 8 one can conclude that \( \mathcal{W}_{2,p} \) has \( 4p + \frac{p-1}{2} \) irreducible modules. The list of irreducible modules includes \( \frac{p-1}{2} \) minimal models for the Virasoro algebra:

\[ L(c_{2,p}, h_{1,i}) , \quad (1 \leq i \leq \frac{p-1}{2}), \]

and \( 4p \) modules \( W_p(h) \), parameterized by lowest weights

\[ h \in \{ h_{1,i}, h_{2,j}, h_{2,k} \mid p \leq i \leq 3p - 1, \ 1 \leq j \leq p, \ 2p \leq k \leq 3p - 1 \}. \]

Remark 4. The relation \( f_p([\omega]) = 0 \) also indicates on the existence of \( \mathbb{Z}_{\geq 0} \)-graded logarithmic \( \mathcal{W}_{2,p} \)-modules whose top components have generalized conformal weights \( h_{1,i} \) (\( 1 \leq i \leq 2p \)) or \( h_{2,i} \) (\( 1 \leq i \leq p - 1 \)) with Jordan blocks, with respect to \( L(0) \), of size two or three. Some of these logarithmic modules for \( \mathcal{W}_{2,p} \) were constructed explicitly in [6].

Let \( k \geq 0 \) and also, let \( p \geq 3 \). Consider

\[ o(G^k Q e^{-2k-1}) v_\lambda = (-1)^k E_{k,p}(t) v_\lambda . \]

Then

\[ E_{k,p}(t) = \text{Res}_{x_1,\ldots,x_{2k+1}} \frac{\Delta(x_1,\ldots,x_{2k+1})^p}{(x_1 \cdots x_{2k+1})^{(2k+1)p}} \prod_{i=1}^{k} \ln(1 - x_{2i}/x_{2i-1}) \prod_{i=1}^{2k+1} (1 + x_i)^k . \]
Conjecture 10.3. Let \( k \geq 0 \), and let also \( p \geq 1 \) be odd. Then

\[
\text{Res}_{x_1, \ldots, x_{2k+1}} \frac{\Delta(x_1, \ldots, x_{2k+1})^p}{(x_1 \cdots x_{2k+1})^{(2k+1)p}} \prod_{i=1}^{k} \ln \left(1 - \frac{x_{2i}}{x_{2i-1}}\right) \prod_{i=1}^{2k} \left(1 + x_i\right)^t
\]

\[
= \lambda_{k,p} \prod_{i=0}^{2k} \left( \frac{t + \frac{mi}{2}}{(k+1)p - 1} \right),
\]

where \( \lambda_{k,p} \neq 0 \) is a constant not depending on \( t \).

For \( k = 0 \), we clearly have

\[
o(Q^{-a})v_\lambda = \left( \frac{t}{p-1} \right)^{v_\lambda}.
\]

The \( k = 1 \) case of the conjecture was verified in Theorem 11.1 below. The \( k = 2 \) case is equivalent to Conjecture 10.1.

11. Appendix: Some Constant Term Identities

In this part we prove an interesting (constant term) identity by combining combinatorial and representation theoretic methods. We often provide two different proofs for the same result to make this part accessible both to combinatorists and algebraists.

Theorem 11.1. Let \( p \geq 1 \) be odd. Then

\[
\text{Res}_{x_1, x_2, x_3} \frac{1}{(x_1 x_2 x_3)^{3p}} \ln \left(1 - \frac{x_2}{x_1}\right) (x_1 - x_2)^p (x_1 - x_3)^p (x_2 - x_3)^p (1 + x_1)^t (1 + x_2)^t (1 + x_3)^t
\]

\[
= A_p \left( \frac{t}{2p - 1} \right) \left( \frac{t + p}{2p - 1} \right) \left( \frac{t + p/2}{2p - 1} \right),
\]

where

\[
A_p = \frac{(-1)^{(p-1)/2}}{6} \frac{(3p)! (\frac{p-1}{2})^3}{(3p-1)(\frac{3p-1}{2})! (p!)^3 (\frac{3p-1}{2})!}.
\]

We denote the triple residue in the theorem by \( F(p, t) \) and the product of three binomial coefficients \( f(p, t) \) (we omit the constant \( A_p \) at this point). It is easy to see that \( F \) and \( f \) are polynomials of degree at most \( 6p - 3 \), by expanding in Laurent series. So it is sufficient to show that these polynomials agree at \( 6p - 2 \) values (including derivatives). More precisely, we will show the following:

\[
F(p, t) = 0, \quad -p \leq t \leq 2p - 2,
\]

(26)

\[
F'(p, t) = 0, \quad 0 \leq t \leq p - 2
\]

(27)

\[
F(p, t) = 0, \quad \frac{-p}{2} \leq t \leq \frac{3p}{2} - 2 \quad \text{and}
\]

(28)

\[
F(p, 2p - 1) = f(p, 2p - 1) \neq 0.
\]

(29)

All these properties can be easily verified by \( f(p, t) \).
We start with a combinatorial lemma which also has a representation theoretic version (see Proposition 11.4 below). We include the proof here because it is needed in Lemma 11.9 below.

**Lemma 11.2.** We have $F(p, t) = 0$ for $0 \leq t \leq 2p - 2$.

**Proof.** We expand $F(p, t)$ by using binomial expansion. This leads to summation over 7 variables. By using residue condition we trim down to four summation variables:

$$
F(p, t) = \sum_{i,j,k,m} (-1)^{i+j+k} \frac{1}{m} \binom{p}{i} \binom{p}{j} \binom{p}{k} \binom{t}{m+3p-1-i-j} \binom{t}{2p-m-1+i-k} \binom{t}{p-1+j+k}.
$$

Observe that $\binom{m}{n} = 0$, for $n < m$. We argue that for $0 \leq t \leq 2p - 2$ at least one of $t$-binomial coefficients is zero. If not then $m + 3p - 1 - i - j \leq 2p - 2$, $2p - m - 1 + i - k \leq 2p - 2$ and $p - 1 + j + k \leq 2p - 2$. But we get contradiction because $(m + 3p - 1 - i - j) + (2p - m - 1 + i - k) + (p - 1 + j + k) = 6p - 3$.

**Lemma 11.3.** We have $F'(p, t) = 0$ for $t = 0, \ldots, p - 2$.

**Proof.** If we differentiate $F(p, t)$ with respect to the $t$ variable we obtain

$$
F'(p, t) = F(p, t)(\ln(1 + x_1) + \ln(1 + x_2) + \ln(1 + x_3)).
$$

Now, for $t = 0, \ldots, p - 2$ we clearly have

$$Res_{x_3} F(p, t)(\ln(1 + x_1) + \ln(1 + x_2)) = 0.
$$

But we also have

$$Res_{x_1} (\ln(1 + x_3)F(p, t) = 0.
$$

The proof follows. □

It is interesting that Lemma 11.2 has a representation theoretic version. We include it here as an illustration of the method.

**Proposition 11.4.** Recall $H(0)v_\lambda = F(p, t)v_\lambda$, where $\langle \lambda, \alpha \rangle = t$. Then the polynomial $F(p, t)$ is divisible with $t(t-1)\cdots(t-2p+2)$.

**Proof.** We recall $H = GF = GQe^{-3\alpha}$. It is sufficient to show that

$$H(0)e^{\frac{i\alpha}{2p}} = H_{6p-4}e^{\frac{i\alpha}{2p}} = 0$$

for $i = 0, 2, \ldots, 4p - 4$.

It is easy to see that

$$Qe^{\frac{i\alpha}{2p}} = G e^{\frac{i\alpha}{2p}} = 0$$

for all $i$ as above.

Also, observe the relation

$$(GF)_{6p-4} = GF_{6p-4} - F_{6p-4}G$$
and
\[(Qe^{-3\alpha})_{6p-4} = Qe^{-3\alpha} - e^{-3\alpha}Q.\]

Therefore
\[H(0)e^{\frac{\alpha}{2p}} = GQe^{-3\alpha}e^{\frac{\alpha}{2p}}.\]

From
\[e^{-3\alpha}e^{\frac{\alpha}{2p}} = \text{Coeff}_{x-6p+3}Y(e^{-3\alpha}, x)e^{\frac{\alpha}{2p}}.\]

and
\[\langle -3\alpha, \frac{i\alpha}{2p} \rangle = -\frac{3}{2}i.\]

it follows that
\[e^{-3\alpha}e^{\frac{\alpha}{2p}} = 0\]
for all \(i\) even from 0 to \(4p-4\).

\[\square\]

**Proposition 11.5.** The polynomial \(F\) satisfies \(F(p, t) = -F(p, p - t - 2)\). Consequently, \(F(p, t) = 0, -p \leq t \leq -1.\)

**Proof.** Denote by \(M(1, \lambda)^\circ\) the contragradient module of \(M(1, \lambda)\).

Then
\[\langle H(0)w', w \rangle = -\langle w', H(0)w \rangle.\]

But \(M(1, \lambda)^\circ \cong M(1, p - \lambda - 2)\) (see [2]), so the proof follows. The second statement follows now form Lemma [11.2].

For half-integer roots more we shall make use of twisted \(V_L\)-modules and operator \(G_{tw}\) introduced earlier.

**Lemma 11.6.** We have
\[G_{tw}e^{\frac{2\ell+1}{2p}\alpha} = 0, \text{ for } \ell \geq -\frac{p + 1}{2}.\]

**Proof.** The proof follows from the fact
\[\text{wt}(e^{\frac{2\ell+1}{2p}\alpha}) < \text{wt}(e^{2\alpha + \frac{2\ell+1}{2p}\alpha}),\]

for \(\ell\) in this range. \(\square\)

**Lemma 11.7.** Assume that \(\ell \leq \frac{3p-1}{2}\), then
\[F(0)e^{\frac{2\ell+1}{2p}\alpha} = 0.\]
Proof. By using the definition of twisted $V_L$–modules, we have that the vector
\[ F(0)e^{2\frac{\ell+1}{2}p}\alpha \]
can be expressed as a linear combination of vectors of the form
\[ (e_\alpha^n e^{-3\alpha})_{6p-4} e^\frac{\ell}{2}p\alpha, \quad (n \in \mathbb{Z}). \] (31)

But every vector (31) belongs to $M(1) \otimes e^\frac{\ell}{2}p\alpha - 2\alpha$. Since
\[ wt(e^\frac{\ell}{2}p\alpha) < wt(e^{\frac{\ell}{2}p\alpha-2\alpha}) \text{ if } \ell \leq \frac{3p-3}{2}, \]
we get the assertion. \qed

**Proposition 11.8.** $F(p, t) = 0$ for $t \in \{-\frac{p}{2}, -\frac{p}{2} + 1, \ldots, -\frac{p}{2} + 2p - 2\}$.

**Proof.** The proof follows from Lemma 11.6, Lemma 11.7 and relation
\[ H(0)e^{2\frac{\ell+1}{2}p}\alpha = G^{tw} F(0)e^{2\frac{\ell+1}{2}p}\alpha - F(0)G^{tw} e^{\frac{\ell}{2}p\alpha}. \] \qed

**Lemma 11.9.** We have
\[ F(p, 2p - 1) = \sum_{m=1}^{p} \sum_{k=0}^{p} \frac{(-1)^{m+k}}{m} \binom{p}{k} \left( \binom{p}{m+k} \right). \]

**Proof.** The main point for choosing $t = 2p - 1$ comes from equation (30). Observe that three $t$-binomial coefficients in (30) will be nonzero only if $i, j, k, m$ satisfy
\[ (m + 3p - 1 - i - j) = 2p - 1, \]
\[ (2p - m - 1 + i - k) = 2p - 1, \]
and
\[ (p - 1 + j + k) = 2p - 1. \]
This gives $i = m + k$ and $j = p - k$ where $k$ is the free variable. Therefore
\[ F(p, 2p - 1) = \sum_{m=1}^{p} \sum_{k=0}^{p} \frac{(-1)^{m+k}}{m} \binom{p}{k} \left( \frac{p}{m+k} \right). \] \qed

The next result will give exact evaluation for the sum $F(2, 2p - 1)$. Similar formulas were obtained earlier

**Proposition 11.10.** We have
\[ \sum_{m=1}^{p} \sum_{k=0}^{p} \frac{(-1)^{m+k}}{m} \left( \binom{p}{k} \right)^2 \left( \frac{p}{m+k} \right) = \frac{(-1)^{(p+1)/2} (3p)!}{3 p!!^3}. \]
Proof. We define the \( n \)-th harmonic number

\[
H_n = \sum_{i=1}^{n} \frac{1}{i},
\]

and consider the sums

\[
C_k = \sum_{m=1}^{p} (-1)^m \frac{1}{m} \binom{p}{m+k}.
\]

It is not hard to see (simply by using basic properties of binomial coefficients) that

\[
C_0 = -H_p.
\]

Similarly, by using induction and the formula

\[
\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1},
\]

we prove the relation

\[
C_k = -\binom{p}{k} (H_n - H_k), \quad k \geq 1.
\]

Putting together we obtain

\[
\sum_{m=1}^{p} \sum_{k=0}^{p} \frac{(-1)^{m+k}}{m} \binom{p}{k} 2 \binom{p}{m+k} = -\sum_{k=0}^{p} (-1)^k \binom{p}{k}^3 (H_p - H_k) = 0 + \sum_{k=0}^{p} (-1)^k \binom{p}{k}^3 H_k
\]

Fortunately the last sum is a difficult, albeit known, binomial-harmonic sum computed by W. Chu and M. Fu (cf. Example 1 on page 3 in [11]). Their identity reads

\[
\sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k}^3 H_k = \frac{(-1)^{m+1}(6m + 3)!(m!)^3}{6 \cdot (1 + 3m)!(1 + 2m)!^3}.
\]

The rest follows easily if we let \( p = 2m + 1 \) and observe an easy identity

\[
(-1)^{(p+1)/2} \frac{(3p)!^3}{3(p!)^6} = \frac{(-1)^{m+1}(6m + 3)!(m!)^3}{6 \cdot (1 + 3m)!(1 + 2m)!^3}.
\]

\( \square \)

The previous identity can be also reformulated as a constant term identity similar with three variables, but now “disturbed” logarithmically!

**Corollary 11.11.** Let \( p \geq 1 \) be an odd integer. Then

\[
\text{CT}_{x_1,x_2,x_3} \ln \left( 1 - \frac{x_2}{x_1} \right) \left( 1 - \frac{x_2}{x_1} \right)^p \left( 1 - \frac{x_3}{x_2} \right)^p \left( 1 - \frac{x_1}{x_3} \right)^p = \frac{(-1)^{(p+1)/2} (3p)!^3}{3p!^3}.
\]

The previous corollary brings us to the realm of Dyson’s constant term identities [9], which state that for any positive \( k \in \mathbb{N} \) and \( n \geq 2 \) the constant term

\[
\text{CT}_{x_1,\ldots,x_k} \frac{1}{(x_1 \cdots x_k)^{m(k-1)}} \prod_{1 \leq i<j \leq k} (x_i - x_j)^{2m}
\]
is equal (up to a sign) to \[
\frac{(km)!}{(m!)^k}.
\]
Notice that the power of the Vandermonde factor is always even (if the power is odd the constant term is trivially zero!). That is why any identity involving \(\prod_{i<j} (x_i - x_j)^{2m+1}\) has to include additional factors.

Motivated by Corollary [11.11] we present a very precise conjecture in this direction.

**Conjecture 11.12.** Let \(k\) and \(m\) be positive integers. Then (up to a sign)
\[
CT_{x_1,\ldots,x_{2k+1}} \left(\frac{1}{(x_1 \cdot \cdots \cdot x_{2k+1})^{(2m+1)k}} \prod_{i=1}^{k} \ln \left(1 - \frac{x_{2i}}{x_{2i-1}}\right) \prod_{1 \leq i < j \leq 2k+1} (x_i - x_j)^{2m+1}\right)
\]
equals \[
\frac{((2k+1)(2m+1))!!}{(2k+1)!!(2m+1)!!^{2k+1}}.
\]

We verified this conjecture for small \(k\) and \(m\) by using computer.

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