A mathematical treatment of the bump structure of particle-laden flows with particle features

Kaname Matsue\textsuperscript{1,2} · Kyoko Tomoeda\textsuperscript{3}

Received: 15 May 2021 / Revised: 7 June 2022 / Accepted: 27 June 2022 / Published online: 30 July 2022 © The Author(s) 2022

Abstract
In this study, we consider particle-laden flows on an inclined plane under the effect of gravity. Previous experimental works have noted that a particle-rich ridge is generated near the contact line. We investigate the bump structure observed in particle-rich ridges in terms of Lax’s shock waves in the mathematical theory of conservation laws, explicitly considering the effect of particles with nontrivial radii on morphology of particle-laden flows. We also extract the dependence of radius and concentration of particles on the bump structure.

Keywords Particle-laden flows · Conservation laws · Shock wave

Mathematics Subject Classification 35L65 · 37M20 · 76A20

1 Introduction

Particle-laden flows comprise fluids including large concentrations of particles playing dominant and considerable roles in dynamics, such as debris flows, slurry transport, and mixed pharmaceuticals (cf. Zhou et al. [14]). In [14], experimental results on the behavior of particle-laden inclined film flows were presented, in which particle-laden flows of polydisperse glass beads (diameter 250–425 $\mu$m) were directed down an acrylic plate with fixed, adjustable inclination angles $\alpha$.
It was been observed in [14] that particles may behave in three different manners depending on the inclination angle and particle concentration of the flow. (a) At low inclination angles and the particle concentration, particles settle to the substrate and the clear silicone oil flows over the particle bed. (b) At high inclination angles and the particle concentration, particles accumulate near the contact line, forming a particle-rich ridge. (c) At intermediate inclination angles and particle concentrations, particles remain well-mixed in the fluid.

A lubrication model was also derived in [14] to capture the behavior of the particle-rich ridge regime as observed in the ridged regime (b), in which the governing equations of mixture motion was considered. The governing equations are given by the following system of partial differential equations for the particle concentration \( \phi(x,t) \) \((0 \leq \phi(x,t) \leq \phi_m)\) and the velocity of particle laden \( u = (u,v) \).

\[
\begin{align*}
-\nabla \cdot \left( -pI + \mu(\phi)(\nabla u + \nabla u^T) \right) &= \rho(\phi)g, \\
\nabla \cdot u &= 0, \quad 0 < y < h(x,t), \quad t \geq 0,
\end{align*}
\]

where \( h(x,t) \) is the total thickness of particle laden, \( g = g(\sin \alpha, -\cos \alpha) \) is the acceleration of gravity, \( p \) is pressure and \( I \) is the 2-dimensional identity matrix. \( \mu(\phi) \) is the effective viscosity given by \( \mu(\phi) = \phi_m^2/(\phi_m - \phi)^2 \) with the maximum packing fraction \( \phi_m \approx 0.67 \). \( \rho(\phi) \) is the effective density given by \( \rho(\phi) = (\rho_f \phi + \rho_p(1 - \phi)) \), where \( \rho_f \) and \( \rho_p \) are the densities of the fluid and particulate phases, respectively. The velocity \( u \) is the average of the two velocities of the fluid and the particle phases, defined as

\[
u = (1 - \phi)u_f + \phi u_p \quad \text{in} \quad 0 < y < h(x,t), \quad t \geq 0,
\]

where \( u_f \) and \( u_p \) are the velocities of the fluid and particulate phases, respectively. At the free surface, the normal stress balance and the tangential stress balance are imposed as

\[
\begin{align*}
n \cdot \left( -pI + \mu(\phi)(\nabla u + \nabla u^T) \right) \cdot n &= \gamma \kappa, \\
t \cdot \left( -pI + \mu(\phi)(\nabla u + \nabla u^T) \right) \cdot n &= 0, \quad \text{at} \quad y = h(x,t),
\end{align*}
\]

where \( n \) and \( t \) are the unit outward normal and unit tangential vectors to the free surface, \( \gamma \) is the surface tension of the mixture and independent of particle concentration, and \( \kappa \) is the curvature of the free surface. The boundary condition at the wall is

\[
u = 0 \quad \text{at} \quad y = 0.
\]

Note that the governing Eqs. (1)–(4) assume that \( \phi \) is independent of \( y \), which prohibits particles from settling to the substrate. Therefore, these governing equations cannot explain the settling behaviors that occur in the settled regime (a). Cook [1] proposed an equilibrium model based on the balance of the hindered settling and shear-induced migration, which exhibited good agreement with the experimental data in [14] and captured the transition between the well-mixed regime (c) and the
settled regime (a). Other experimental results were presented and similar models derived in e.g. [6–8, 12, 13].

Cook et al. [2] revisited a lubrication model in [14] with more complete explanations. The lubrication model derived from the governing Eqs. (1)–(4) comprises the following system of conservation laws (see [2, 14]).

\[
\begin{align*}
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left( h^3 \frac{\rho(\phi)}{\mu(\phi)} \right) &= 0, \\
\frac{\partial}{\partial t}(\phi h) + \frac{\partial}{\partial x} \left( h^3 \frac{\rho(\phi)}{\mu(\phi)} + v_s h \phi (1 - \phi) F(\phi) W(h) \right) &= 0,
\end{align*}
\]

(5)

where \( F(\phi) = (1 - \phi)^{5.1} \) and \( v_s = \frac{2}{3} \left( \frac{\rho_p - \rho_f}{\rho_f} \right) \left( \frac{a}{h_0} \right)^2 \), \( a \) is the distance from the center of the particle to the wall and

\[
W(h) = \frac{1}{\sqrt{1 + \left[ \frac{1}{18} \left( \frac{h}{a} \right)^2 \right]^2}}.
\]

(6)

The term \( v_s h \phi (1 - \phi) F(\phi) W(h) \) represents the relative velocity of particles to the fluid incorporating the interference among particles themselves, as well as among particles and the wall. The former is described by \( F(\phi) \), which is known as Richardson-Zaki correction and is valid for high concentration of particles (cf. [8, 9, 14]), whereas the latter is approximately described by \( W(h) \) in (6). The function \( W(h) \) is introduced in [2, 14] and has the asymptotics

\[
W(h) \approx \begin{cases} 0 & h \ll a, \\ 1 & h \gg a. \end{cases}
\]

(7)

In [2, 14], solutions of (5) with initial jumps in height from the upstream film thickness \( h_L \) to the precursor thickness \( b \) and given concentrations on tips and ends, that is

\[
h(x, 0) = \begin{cases} h_L & \text{for } x \leq 0, \\ b & \text{for } x > 0, \end{cases} \quad \phi(x, 0) = \begin{cases} \phi_{0L} & \text{for } x \leq 0, \\ \phi_{0R} & \text{for } x > 0, \end{cases}
\]

(8)

are treated only in the asymptotic case \( W(h) \approx 1 \), where \( h_L, b, \phi_{0L}, \) and \( \phi_{0R} \) are constants with specified values. In the initial value problem (5) and (8), namely the Riemann problem of (5), numerical solutions have been found with a 1-shock wave from the upstream state \( (h_L, \phi_{0L}, h_L) \) to an intermediate state \( (h_I, \phi_I, h_I) \) with height \( h_I \) and concentration \( \phi_I \) slightly larger than \( h_L \) and \( \phi_{0L} \), respectively, and a 2-shock wave from this intermediate state to the precursor \( (b, \phi_{0R}, b) \). According to their results, the double-shock wave (1-shock and 2-shock) can generate a particle-rich ridge, which forms “a bump”, as observed in the ridged regime (b) (see Fig. 1). Despite good agreement with the results of experiments and a simple description of structures, the explicit dependence of several particle characteristics such as radius is not clearly
mentioned in these results, which can lead to misunderstandings in interpreting phenomena relating to concrete particle-laden flows.

Our main aim here is to explicate one of these unclear points in detail. In particular, the effect of particles with nontrivial radii on the morphology of particle-laden flows is explicitly considered, which includes the asymptotics (7) in the Riemann problem for (5) with the initial condition (8). We then elucidate the influence of interaction between particles of nontrivial size and the wall on the formation of the particle-rich ridge through the shock structure. The remainder of this study is organized as follows. In Sect. 2, we briefly review a theory of shock waves for the Riemann problem of the system of conservation laws (5) with (8). It is mathematically known that the general $m \times m$ system of the hyperbolic conservation laws

$$
\partial_t U + \partial_x (F(U)) = 0
$$

can admit discontinuous solutions such as shock waves and smooth solutions such as rarefaction waves, where $U = (U_1, \cdots, U_m)^T \in \mathbb{R}^m$, $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, $\mathbb{R}_+ \equiv [0, +\infty)$ and $F(U) = (F_1(U), \cdots, F_m(U))^T$ is a vector-valued function which is $C^2$ in some open subset $D \subset \mathbb{R}^m$ (see [5, 11]). To construct a solution with shock waves of the system (5), we consider the case where solutions have discontinuities, and hence we deal with weak solutions mentioned below. Weak solutions consisting of shock
waves are constructed by applying mathematical theories established in [5, 11] to the system (5). In Sect. 3, families of weak solutions of (5) are numerically studied. In particular, we focus on the particle-rich ridge from a mathematical perspective. To this end, solutions with bump structure are constructed by means of physically relevant shock waves. We also provide an associated definition of bump structure in terms of shocks. We especially investigate characteristics of the family of solutions with the required structure, such as their range of existence, height, and propagation speed. In the proposed model (5), solutions can depend on the radius \(a\), the explicit contributions of which are overlooked and overestimated in previous studies such as [2, 14], and the concentration \(\phi\) of particles in flows. This work mathematically investigates the qualitative nature of the particle-rich ridge considering nontrivial particle features.

2 Preliminaries from the theory of conservation laws

Here, we apply terminology from the theory of conservation laws to (5) with a brief review. Setting \(n = \phi h\), the system (5) is reformulated to the system of conservation laws for \((h, n)\) as follows.

\[
\begin{align*}
\partial_t h + \partial_x \left( h^3 f \left( \frac{n}{h} \right) \right) &= 0, \\
\partial_t n + \partial_x \left( h^2 n f \left( \frac{n}{h} \right) + h g \left( \frac{n}{h} \right) W(h) \right) &= 0,
\end{align*}
\]  

(9)

where

\[
\begin{align*}
f \left( \frac{n}{h} \right) &= \left( 1 + \frac{\rho_p - \rho_f}{\rho_f} \left( \frac{n}{h} \right) \right) \cdot \frac{\left( \phi_m - \phi \right)^2}{\phi_m^2}, \\
g \left( \frac{n}{h} \right) &= \frac{2}{3} \frac{\rho_p - \rho_f}{\rho_f} \frac{a^2}{h_0^2} \left( \frac{n}{h} \right) \phi (1 - \phi)^{6.1}.
\end{align*}
\]

Let

\[
U = \left( \frac{h}{n} \right), \quad F(U) = \left( \begin{array}{c}
h^3 f \left( \frac{n}{h} \right) \\
h^2 n f \left( \frac{n}{h} \right) + h g \left( \frac{n}{h} \right) W(h)
\end{array} \right).
\]

Then, the system (9) can be rewritten in the form

\[
\partial_t U + \partial_x (F(U)) = 0.
\]

(10)

It is well-known that a solution to conservation laws (10) can be discontinuous even if the initial profile is smooth. We therefore consider weak solutions, as defined below.
Definition 1 (e.g. [11]) A bounded measurable function $U(x, t)$ on $\mathbb{R} \times \mathbb{R}_+$ is called a weak solution of the initial-value problem for (10) with bounded and measurable initial data $U(x, 0)$, provided that

$$
\int_0^\infty \int_{\mathbb{R}} (U\psi_t + F(U)\psi_x)dxdt + \int_{\mathbb{R}} U(x, 0)\psi(x, 0)dx = 0
$$

holds for all $\psi \in C^1_0(\mathbb{R} \times \mathbb{R}_+)$, where $C^1_0(X)$ is the space of continuously differentiable functions defined on a topological space $X$ with compact supports in $X$.

If the weak solution $U(x, t)$ exhibits a discontinuity along a curve $x = x(t)$, the solution $U$ and the curve $x = x(t)$ must satisfy the Rankine-Hugoniot relation (jump condition)

$$
s(U_L - U_R) = F(U_L) - F(U_R),
$$

where $U_L = U(x(t) - 0, t)$ is the limit of $U$ approaching $(x, t)$ from the left and $U_R = U(x(t) + 0, t)$ is the limit of $U$ approaching $(x, t)$ from the right, and $s = \frac{dx}{dt}$ is the propagation speed of $x(t)$. The discontinuity is referred to as a shock wave, or simply a shock, and $s$ is called the shock speed.

We consider the Riemann problem for the conservation laws (9), namely the initial value problem with the following form of initial conditions called the Riemann data.

$$
U(x, 0) = \begin{cases}
U_L & x < 0 \\
U_R & x > 0
\end{cases}.
$$

The Jacobian matrix of $F$ in (10) at $U$ is

$$
DF(U) = \begin{pmatrix}
F_{11}(U) & F_{12}(U) \\
F_{21}(U) & F_{22}(U)
\end{pmatrix},
$$

where

$$
F_{11}(U) = 3h^2f - nh(d_\phi f),
$$
$$
F_{12}(U) = h^2(d_\phi f),
$$
$$
F_{21}(U) = 2nhf - n^2(d_\phi f) + \left(-\frac{n}{h}(d_\phi g) + g\right)W + hg(d_h W),
$$
$$
F_{22}(U) = h^2f + nh(d_\phi f) + (d_\phi g)W,
$$

and their eigenvalues, which are called characteristic fields, of $DF(U)$ are

$$
\lambda_1(U) = \frac{P - \sqrt{Q}}{2}, \quad \lambda_2(U) = \frac{P + \sqrt{Q}}{2},
$$

where

$$
P = F_{11}(U) + F_{22}(U), \quad Q = (F_{11}(U) - F_{22}(U))^2 + 4F_{12}(U)F_{21}(U).
$$
Assuming that $Q > 0$, eigenvalues $\lambda_j(U) (j = 1, 2)$ are real-valued and $\lambda_1(U) < \lambda_2(U)$ holds for any $U \in \Omega$, where $\Omega = \{(h, n) \in \mathbb{R}^2 : 0 < h, \ 0 \leq n < \phi_m h\}$. The system (9) is called strictly hyperbolic in $\Omega$ if this condition holds. The right eigenvectors associated with the eigenvalues $\lambda_j(U)$ are

$$r_1(U) = \left( \frac{-2F_{12}(U)}{Z + \sqrt{Q}} \right), \quad r_2(U) = \left( \frac{-2F_{12}(U)}{Z - \sqrt{Q}} \right),$$

where $Z = F_{11}(U) - F_{22}(U)$. To characterize characteristic fields further, the quantities

$$\nabla \lambda_j(U) \cdot r_j(U), \quad j = 1, 2$$

are calculated for each $U \in \Omega$. If $\nabla \lambda_j(U) \cdot r_j(U) \neq 0$ holds in a subset $N \subset \Omega$, the $j$-th characteristic field $\lambda_j(U)$ is called genuinely nonlinear in $N$, which ensures the existence of solutions of the Riemann problem with the left state $U_L$ in $N$ and right states close to $U_L$. We omit the details of the required calculations owing to their length. We remark instead that the genuine nonlinearity of both characteristic fields in regions of our interest in the following arguments may be numerically confirmed.

In this case, weak solutions of the Riemann problem consist of at most three constant states $U_L, U_I, U_R$ followed by either shock or rarefaction waves (e.g. [5, 11]).

In the present work, our interest is restricted to solutions of (9) consisting of discontinuities, namely shocks. As mentioned, discontinuities arising in weak solutions of (10) have constraints determined by the Rankine-Hugoniot relation (12). To calculate such discontinuities for (9), we fix the reference point $U_L = (h_L, n_L)$ and consider right states $U_R = U = (h, n)$ solving (12). If a weak solution of (9) includes a jump discontinuity between the left state $U_L$ and the right state $U$, then $U$ must satisfy

$$\begin{cases}
    s(h - h_L) = F_1(h, n) - F_1(h_L, n_L), \\
    s(n - n_L) = F_2(h, n) - F_2(h_L, n_L),
\end{cases}$$

where

$$F_1(h, n) = h^3 f \left( \frac{n}{h} \right),$$
$$F_2(h, n) = h^2 n f \left( \frac{n}{h} \right) + h g \left( \frac{n}{h} \right) W(h).$$

Eliminating $s$ from these equations, we obtain

$$\frac{n - n_L}{h - h_L} = \frac{F_2(h, n) - F_2(h_L, n_L)}{F_1(h, n) - F_1(h_L, n_L)} = 0,$$

the graph of which is referred to as the Hugoniot locus. Among solution families in the Hugoniot locus, physically relevant discontinuities additionally require the following $k$-entropy inequalities ($k = 1, 2$) in the sense of Lax (e.g. [11]).

$$s < \lambda_1(U_L), \quad \lambda_1(U) < s < \lambda_2(U), \quad (1 - \text{entropy inequality}),$$
\[ \lambda_1(U_L) < s < \lambda_2(U_L), \quad \lambda_2(U) < s, \quad (2 - \text{entropy inequality}), \quad (18) \]

where \( s \) is the shock speed

\[
\begin{align*}
    s &= \frac{F_1(h, n) - F_1(h_L, n_L)}{h - h_L} = \frac{F_2(h, n) - F_2(h_L, n_L)}{n - n_L}. 
\end{align*}
\]

If \( U \) satisfies (16) and (17), then \( U \) can be connected to \( U_L \) from the right followed by a 1-shock wave. Similarly, \( U \) can be connected to \( U_L \) from the right followed by a 2-shock wave, provided \( U \) satisfies (16) and (18).

### 3 Dependence of radius and concentration of particles on bump generation

Our primary concern is the investigation of “bump” structure described by the composite waves of shocks in the following sense.

**Definition 2** We say that a weak solution of the Riemann problem (9) with the Riemann data (8) admits a bump structure if the solution consists of three states \( \{(h_L, \phi_L), (h_I, \phi_I), (h_R, \phi_R)\} \) such that

(B1) the leftmost state \((h_L, \phi_L)\) is connected to an intermediate state \((h_I, \phi_I)\) with \( h_L < h_I \) and \( \phi_L < \phi_I \) followed by a 1-shock with the shock speed \( s_1 \), and

(B2) the state \((h_I, \phi_I)\) is connected to the rightmost state \((h_R, \phi_R)\) with \( h_R < h_I \) and \( \phi_R < \phi_I \) followed by a 2-shock with the shock speed \( s_2 > s_1 \).

A schematic illustration of a physical realization of weak solutions with a bump structure is shown in Fig. 2. As mentioned in Introduction, \( h_L \) represents the upstream film flow thickness, and \( h_R \) denotes the precursor thickness.\(^1\) The requirement for distributions of three individual states includes the existence of bump structure\(^2\) in the particle-laden flow generated by shock waves. The additional requirement for shock speeds indicates that the bump structure persists in the sense that two shocks do not interact during their evolution over time; otherwise two shocks would collide with each other to create a single shock structure (cf. [11]). For simplicity, we refer to the above structure as a bump structure generated by Lax’s shocks. We do not require any relationships between \( h_L \) and \( h_R \), and \( \phi_L \) and \( \phi_R \), respectively.

Analytic treatments of this problem with concrete data are relatively difficult owing to the complexity of the nonlinearity and eigensystems \( \{\lambda_j(U), r_j(U)\}_{j=1}^2 \). Instead, we

---

\(^1\) Distributed flows from the tips of bumps whose height can be lower than \( h_R \) are not considered in the present model, which was developed in terms of the general initial value problem of (5) or the unreduced model (1).

\(^2\) Herein, the bump structure in solutions to (5) refers to a pair of step-up and step-down functions as described in Fig. 2.
investigate the structure of weak solutions numerically, particularly shocks. Our objective is then reduced to numerical computations of Hugoniot loci, i.e., solutions of (15), through the fixed left states \((h_L, \phi_L)\), and possibly intermediate states \((h_I, \phi_I)\). As mentioned in the previous section, the pair of functions \((h, n) = (h, \phi h)\) is considered instead of \((h, \phi)\) itself to allow the mathematical theory of conservation laws to be applied directly. (15) includes three unknowns \((h, n, s)\), whereas the system (5) comprises two equations. To ensure unique solvability, we regard one of unknowns as a parameter and compute the loci as 1-parameter families of solutions. In this study, we regard \(h\) as a parameter and the remaining unknowns \((s, n)\) as functions of \(h\), which induce the following parameter-dependent zero-finding problem.

\[
G(x; p) = 0, \quad G : \mathbb{R}^2 \times \mathbb{R}^k \rightarrow \mathbb{R}^2, \quad x = (s, n)^T, \quad p = (p, \mu)^T,
\]

where \(x\) is a phase variable vector and \(p = (p, \mu)\) is a parameter vector consisting of a varying parameter \(p \in \mathbb{R}\) and a fixed parameter vector \(\mu \in \mathbb{R}^{k-1}\). The integer \(k\) denotes the number of parameters, which is given in advance. In the present case, we set

\[
p = h, \quad \mu = (h_p, n_p, a, \rho_p, \rho_f, \phi_m), \quad k = 7.
\]

This represents a given left state \((h_p, n_p)\) and the environmental parameters of the system. The nonlinearity \(G\) becomes

\[
G(x; p) = \begin{cases} h^3 f \left( \frac{n}{h} \right) - h^3 f \left( \frac{n_p}{h_p} \right) - s(h - h_p) \\ h^2 n f \left( \frac{n}{h} \right) - h^2 n f \left( \frac{n_p}{h_p} \right) - s(h - h_p) \\ h^2 n f \left( \frac{n}{h} \right) - h^2 n f \left( \frac{n_p}{h_p} \right) - s(h - h_p) \\ + \left\{ h g \left( \frac{n}{h} \right) - h p g \left( \frac{n_p}{h_p} \right) \right\} - s(n - n_p) \end{cases}
\]

(20)
We then apply the pseudo-arclength continuation method well-established in numerical bifurcation theory (e.g., [3, 4]) to (19) to obtain Hugoniot loci connected to \((h_p, n_p)\). From the definition, the triple \(\{(h_p, n_p), (h, n), s\}\) solving (19) generates a shock with speed \(s\).

To ensure that the triple \(\{(h_p, n_p), (h, n), s\}\) generates a shock in the sense of Lax, we additionally check the entropy conditions (17) and (18). If either (17) or (18) is satisfied, we can conclude that the triple \(\{(h_p, n_p), (h, n), s\}\) generates a 1-shock if (17) is satisfied, and generates a 2-shock if (18) is satisfied, as long as the original system (5) satisfies the strict hyperbolicity in the domain including the chain of loci of interest. See Remark 4 below on the strict hyperbolicity of this problem.

The operations mentioned here are summarized as follows.

1. Fix \((h_p, n_p) = (h_L, n_L)\) and solve (19).
2. For each solution \((h, n, s)\), verify the entropy conditions (17) and (18).
3. Fix a point \((h, n)\) on the obtained loci as a new left state \(\{(h_p, n_p)\}\) corresponding to an intermediate state \((h_I, n_I)\), and return to 1.

Following our purpose, we restrict operation 3 to the case that the state \(\{(h_p, n_p), (h, n), s\}\) satisfies the 1-entropy condition (17) and the first our requirement between \((h_L, n_L)\) and \((h, n)\), namely \(h_L < h\) and \(\phi_L = n_L/h_L < n/h = \phi\).

**Remark 1** When we compute Hugoniot loci for the system of conservation laws, typical unknown variables corresponding to \(\mathbf{x}\) in (19) are the state variables \((h, n) \equiv U\) in (15), and the shock speed \(s\) is regarded as a component of parameters. However, \(s\) is determined by the left state \((h_p, n_p)\) and the right state \((h, n)\) and is relatively difficult to predict a priori. \(s\) can be eliminated through the Rankine-Hugoniot condition like (16), which induces extra calculations to construct the corresponding zero-finding problem and reduce unnecessary complexity of the system. In contrast, the dependence of states \((h, n)\) in our interest around \((h_L, n_L)\) is very clear. For example, the initial guess of \(h\) should be sufficiently close to \(h_L\). Therefore it is easy to control solutions \((n, s)\) depending on the parameter \(h\), which yields the present methodology.

**Remark 2** In typical (and mathematical) treatments of the Riemann problem, in addition to the left state, the right state \((h_R, n_R)\) is also fixed as the initial state (cf. [2]). In contrast, the right state is regarded as an unknown in (19) because our interest here is the possible distribution of states generating shock structures of our interests, whereas the Riemann problem with given end states as the initial state focuses on all possibilities of solution structures, including rarefactions.

**Remark 3** In these computations, (19) is solved by Newton’s method with initial points close to the left states mentioned above. Continuations of solution families were found to end successfully as far as we computed. Entropy conditions were verified by simply checking inequalities (17) and (18) for given left states and computing the right states. If one of these inequalities is satisfied, the solution is labeled as
A mathematical treatment of the bump structure...

A mathematical treatment of the bump structure...

...a 1- or 2-shock according to inequalities. Note that the strict hyperbolicity indicates that only a single inequality can be satisfied at most.

Even if neither entropy conditions are satisfied, computation of zeros of (19) continues, because overcompressive or undercompressive shocks may still be present (cf. [10]), although these shocks are beyond the scope of the present work.

In these computations, the strict hyperbolicity of the system (5) and genuine nonlinearity for characteristic fields \{\lambda_1(U), \lambda_2(U)\} was also calculated independently. Note again that the former requires the separation of the solution structure among different type of elementary waves, including shocks of Lax’s type, and the latter guarantees the existence of shock chains rigorously, at least for right states \((h_R, n_R)\) sufficiently close to the left state \((h_L, n_L)\).

Here, we focus on the shock structure of the system with particles under negative buoyancy. The corresponding situation determines the parameter \(\rho_p\). Unless otherwise noted, we fix \(\rho_p = 2500\) for arguments in this section, while the remaining parameter values \((\rho_f, \phi_m) = (970, 0.67)\) are fixed in the entire set of computations. The parameter \(\phi_m\) represents the maximal concentration of particles in flow, which is required for all solutions of (5). All of the computed solutions shown below satisfy \(\phi < \phi_m\).

We should also mention the treatment of the ratio \((a/h_0)^2\) given below (5), which is a primary focus of this work. In previous works such as [1, 14], the ratio \((a/h_0)^2\) was not clearly described. In contrast, \((a/h_0)^2\) was set around 0.01 in subsequent works, e.g., [8]. In this study, we selected the ratio \((a/h_0)^2\) arbitrarily, and hence the solution structure for several sample values of \((a/h_0)^2\) was available to evaluate the qualitative similarity and difference, although the validity of the model itself for relatively large \((a/h_0)^2\) is beyond the scope of this work.

Table 1  The function \(W(1)\) with fixed \(a\)

| \(a\)     | \(W(1)\)     | \(W'(1)\)          |
|----------|--------------|---------------------|
| 0.01     | 0.999998     | 6.51247 \times 10^{-6} |
| 0.02     | 0.999974     | 1.04192 \times 10^{-4} |
| 0.05     | 0.998985     | 4.05793 \times 10^{-5} |
| 0.1      | 0.998985     | 6.20717 \times 10^{-2} |
| 0.2      | 0.810979     | 0.555772             |
| 0.5      | 0.216517     | 0.413147             |
| 0.99     | 0.0564799    | 0.112712             |
| 0.999    | 0.05547      | 0.110709             |
| 1.1      | 0.0460486    | 0.0917185            |

3 These values are quoted from Table I in [12].
4 This value can relate to the sensitive dependence of experimental data on precursor film, according to [14].
In association with the ratio, we review the actual distribution of the function $W(h)$ in (6). The sample values $W(1) ≡ W(h)|_{h=1}$ itself and its derivative for fixed values of the particle radius parameter $a$ are shown in Table 1. Indeed, for sufficiently small $a$ such as 0.01, the asymptotic behavior (7) is valid, whereas the differential $W'(1) ≡ (dW/dh)(1)$ provides a nontrivial contribution to various functions involving the system even for relatively small $a$.

**Remark 4** As shown in figures mentioned below, strict hyperbolicity and genuine nonlinearity are satisfied in the domain of $(h, \phi)$ where the shock structure is considered, as far as we have computed. Therefore, the failure of these properties during the following arguments is irrelevant.

### 3.1 Effect of particle radius variation on shock structure

Here, we consider the effect of $a$ on the distribution of shock structure. To this end, we fix the left state as representatives

$$h_0 = h_L = 1, \quad \phi_L = 0.3$$
and solve (19). Examples of Hugoniot loci representing right states centered at the left state \((h_L, \phi_L)\) followed by shocks are drawn in Fig. 3. We typically find two branches of Hugoniot loci through \((h_L, \phi_L)\). One branch comprises right states followed by 1-shocks in the direction where \(h\) is increasing and pieces of states in which Lax’s entropy conditions are not satisfied, while another comprises segments of right states followed by 2-shocks in the direction where \(h\) is decreasing and segments of states which Lax’s entropy conditions are not satisfied. When right states admitting 1-shocks are distributed in the region \(\{h > h_L, \phi > \phi_L\}\), Hugoniot loci regarding a point on the locus in \(\{h > h_L, \phi > \phi_L\}\) as the new left state \((h_I, \phi_I)\) are additionally computed (Fig. 3b). From our purpose, only the loci admitting 2-shocks are computed there. We see that the loci through various \((h_I, \phi_I)\) have a similar structure to that through \((h_L, \phi_L)\). Our computations also showed (in Fig. 3c) collections of right states \((h, \phi)\) connected from \((h_I, \phi_I)\) followed by 2-shocks whose shock speeds \(s_2\) were much higher than the speeds \(s_1\) of the 1-shocks connecting \((h_L, \phi_L)\) and \((h_I, \phi_I)\). Consequently, the triple
\[
\{(h_L, \phi_L), (h_I, \phi_I), (h_R, \phi_R)\}
\]
is a candidate of states generating the bump structure generated by Lax’s shocks, where \((h_L, \phi_L)\) is fixed, \((h_I, \phi_I)\) is on the locus through \((h_L, \phi_L)\) followed by a 1-shock satisfying \(h_I > h_L\) and \(\phi_I > \phi_L\), and \((h_R, \phi_R)\) is on the locus through \((h_I, \phi_I)\) followed by a 2-shock (cf. Fig. 2).

Changing \(a\) during computation of Hugoniot loci with the required properties, we observe (at least) three interesting features, which may be observed in Fig. 4.

– The \(\phi\)-component of the right states followed by 2-shocks remains almost constant, that is, the corresponding right states create plateau regions where the ratio \(a/h_R\) is approximately less than 1. More precisely, \(\phi\) behaves as a monotonously increasing function of \(h\) on the plateau region.
– For different choices of \(a\), the length of curves of right states followed by 1-shocks varies nontrivially. At a minimum, the length does not change monotonously.
– The speeds of 1-shocks \(s_1\) and 2-shocks \(s_2\) always satisfy \(s_1 < s_2\) for all mentioned states observed in Fig. 4. In particular, our requirement on the speeds of shocks stated in (B2) is always satisfied for bumps generated by Lax’s shocks, as far as we computed in this study.

The first observation reflects the validity of the model from a physical perspective. The ratio \(a/h\) measures the physical characteristic of films in finite-volume flows. To ensure that particle-laden flows behave as colloids, in particular continua, \((a/h_0)^2 \ll 1\) is required (cf. [8] \(^5\)). Interestingly, the assumption \((a/h_0)^2 \ll 1\) approximately corresponds to the existence of plateau regions on 2-shock curves as far as

\(^5\) For reference, we note that \((a/h_0)^2 = 0(10^{-2})\) in the end of early-stage transient evolution of particle-laden flows; i.e., a well-mixed state is a physically different situation from the present argument.
Fig. 4 Solution families of (19) with \((h_L, \phi_L) = (1.0, 0.3)\) and various \(a\). The basic description of the figures is the same as that of previous figures. In all figures, the horizontal axis is \(h\) with the log-scale plotting, whereas the vertical represents \(\phi\). a \(a = 0.001\). b \(a = 0.01\). c \(a = 0.05\). d \(a = 0.1\). e \(a = 0.2\). f \(a = 0.5\). The length of 1-shock curves depends on \(a\). Moreover, we see that, for small \(a\) such as \(a \leq 0.1\), the 2-shock curves are almost flat for \(h > a\)
A mathematical treatment of the bump structure...

$a/h < 1$ holds for the right state. That is, we observed that loci of states in $(h, \phi)$-plane admitting 2-shocks are strongly curved if $a/h \geq 1$ holds or $a$ is relatively large, which can correspond to a violation of the continuum hypothesis. Taking the treatment of particle-laden flows as continua into account, which is essential to derive the present model, the plateau regions demonstrate the physical reliability of arguments. Moreover, the property of $\phi$ on the locus admitting 2-shocks being an increasing function of $h$ is important in the following arguments.

The second observation strongly relates to generations of the bump structure by Lax’s shocks in various choices of particles in mixture fluids. Indeed, Fig. 4 shows a collection of composite Hugoniot loci, from which we can choose triples $\{(h_L, \phi_L), (h_I, \phi_I), (h_R, \phi_R)\}$ satisfying requirements (B1) and (B2). It should be noted that the last comment in the first observation is considered in concluding (B2). It follows from the definition of state variables that the length of curves of right states followed by 1-shocks determines the upper bounds of the bump height. Moreover, the property of $\phi$ on the locus admitting 2-shocks being an increasing function of $h$ is important in the following arguments.

We remark on the relationship of speeds $s_1, s_2$ of shocks observed in Fig. 4 as the third observation.

Here, we compare our results with preceding works such as [2, 14]. In all studies, results with $(h_L, \phi_L) = (1.0, 0.3)$ and $\rho_p = 2500$ were obtained. According to [14], the bump structure is obtained in this setting by means of shocks\(^6\) as the triple of constant states $(U_L, U_I, U_R)$ with

\[
U_L \equiv (h_L, \phi_L) = (1.0, 0.3), \quad U_I \equiv (h_I, \phi_I) \approx (1.1619, 0.3620), \quad U_R \equiv (h_R, \phi_R) \approx (0.1, 0.3)
\]

such that $(U_L, U_I)$ is followed by the 1-shock with the speed $s_1 \approx 0.4556$ and that $(U_I, U_R)$ is followed by the 2-shock with the speed $s_2 \approx 0.4956$. In contrast, according to our computations with $(a/h_0)^2 = 0.0001$, the bump structure (in the sense of Lax) can be constructed when $h_I \leq 1.1210$ and $\phi_I \leq 0.3903$ for 1-shocks whose corresponding shock speed is $s_1 \approx 0.44927$. In addition, according to our computations, the shock speed $s_2$ following the states $(h_I, \phi_I) = (1.1210, 0.3903)$ and $(h_R, \phi_R) \approx (0.1, 0.3)$ is $s_2 \approx 0.4930$, which approximately agrees with the results in preceding studies up to the small corrections of values. Of note, the present model (9) was originally obtained in [2], and differs slightly from [14]. In [2], the entropy conditions used to characterize Lax’s shock are incorrect, whereas our entropy conditions (17)-(18) follow from the general theory of shocks (cf. [11]). This difference can induce small but nontrivial variations in the range of constant states admitting Lax’s shocks.

Finally we mention results in [2] where shock structure with given right states $(h_R, \phi_R)$ were studied. In particular, significantly small $\phi_R$ such as 0.002 and 0.0005 were also considered. Incorporating our present results with the preceding results,

\(^6\) In [14], the coefficient $1/18a^2$ appearing in (6) is replaced by $1/2$, corresponding to $a = 1/3$. 

 Springer
the shock structure in the sense of Lax may be considered physically meaningful only if \( a < h \) holds, which indicates that an excessively small \( \phi_R \) can be meaningless in describing Lax’s shocks when the effect of particle-wall interaction \( W(h) \) is explicitly considered and the radius \( a \) is inappropriately selected, while it is not explicitly mentioned in [2]. It should be also mentioned that the choice of small \( \phi_R \) does not deny the possibility of solutions of (5) with the right state \((h_R, \phi_R)\) by means of other type of (elementary) waves, such as rarefactions or singular shocks which cannot be characterized by means of Hugoniot loci.

### 3.2 Effect of particle concentration on shock structure

Next, we study the effect of the initial concentration \( \phi_L \) of particles on the shock structure. Here, we fix \( a \) as 0.01 and vary \( \phi_L \), leaving all remaining parameters unchanged. Figure 5 shows the Hugoniot loci with left states \((h_L, \phi_L) = (1.0, 0.2), (1.0, 0.3) \) and \((1.0, 0.4)\), respectively. In any cases, two Hugoniot loci through \((h_L, \phi_L)\) are computed, either side of which consists of right states connected to \((h_L, \phi_L)\) followed by 1- or 2-shocks. Interestingly, a collection of right states \((h, \phi)\) connected from \((h_L, \phi_L)\) followed by a 1-shock satisfying \( h_L < h \) and \( \phi_L < \phi \) may be observed when \((h_L, \phi_L) = (1.0, 0.2) \) and \((1.0, 0.3)\), whereas it is not evident when \((h_L, \phi_L) = (1.0, 0.4)\).

The robustness of Lax’s shock structure ensures that the above observation persists under small perturbations of \((h_L, \phi_L)\) with an appropriate choice of \((h_I, \phi_I)\), which can be confirmed at least in the numerical sense.\(^7\) Then, our next interest is the limit of \((h_L, \phi_L)\) where the bump structure generated by Lax’s shocks persists. In this study, we fix \( h_L = 1 \) throughout and vary \( \phi_L \). Once again, \( a \) is fixed as 0.01.

\(^7\) In the present system, the regularity of the Jacobian matrix of \( G \) in (20) with respect to \((s, n)\), the continuity of eigenvalues \( \lambda_1, \lambda_2 \), Lax’s entropy condition, and the genuine nonlinearity of characteristic fields as well as the Implicit Function Theorem for \( G(x, p) = 0 \) ensure robustness.
A mathematical treatment of the bump structure...

Numerical investigations based on the bisection method yield the following observation, which indicates the limitation of generating bump structure.

(Limit) When we fix $h_L = 1.0$ and $a = 0.01$, there exists an intermediate state $(h_I, \phi_I)$ with $h_I \leq 1.4$ followed by a 1-shock if and only if $\phi_L \leq \phi_{L,\ast}$, where $\phi_{L,\ast} \approx 0.3314575$.

### 3.3 2-Parameter dependence on shock structure

The observation (Limit) implies that the initial concentration $\phi_L$ of particles therefore affect the generation of bumps, in particular the limitation of bump structure generation. In contrast, observations in Fig. 4 indicate that the generation of bump structure by means of Lax’s shocks depends not only on $\phi_L$ but also $a$. It is then worth investigating the distribution of parameters $(\phi_L, a)$ where the bump structure can be generated. According to Fig. 3, collections of right states $(h_R, \phi_R)$ from $(h_I, \phi_I)$ followed by 2-shocks satisfying (B2) are typically generated, and hence investigation of the existence of intermediate states satisfying (B1) provides a necessary, and possibly sufficient environment to generate the bump structure. To this end, we fix $h_L = 1.0$ as before and vary both $\phi_L$ and $a$. Hugoniot loci including states satisfying (B1) with various $\phi_L$ and $a$ are mainly considered.

First, distribution of $(a, h)$ with various $\phi_L$ such that 1-shocks are admitted is summarized in Fig. 6, where the range of $h$ is restricted to $h > 1$. From the results for fixed $\phi_L$, we first observe that the upper bound $h_{\text{max}}$ of $h > h_I$ admitting 1-shocks converged to a certain value as $a \to 0$ when $\phi_L$ was relatively small, say $\phi_L \approx 0.3$. For example, when $\phi_L = 0.3$, the upper bound was $h_{\text{max}} \approx 1.121185$, while it was $h_{\text{max}} \approx 1.6519053$ when $\phi_L = 0.2$ and 1-shocks were not generated for small $a$ when $\phi_L = 0.4$. This observation indicates that the bump generation

![Fig. 6](image-url)
mechanism is robust among different particles with sufficiently small radii if $\phi_L$ is relatively small. For $\phi_L = 0.2$ (Fig. 6a), 1-shocks are admitted for a large range of $a$, while the range of $h$ admitting 1-shocks varies significantly as $a$ increases. When $a$ becomes larger, the range of $h$ admitting 1-shocks also increases, which is significant in the middle particle size range, say $a \in [0.3, 0.5]$. When $a$ becomes much larger, say $a \geq 0.54$, the region is separated into two subregions. In particular, only small and considerably large bumps can be generated, while the region of large $h$ is admitted in a thin range of $a$. The qualitatively similar feature can be also seen in environments with a slightly larger $\phi_L$, say 0.3 (Fig. 6b). In contrast, when $\phi_L = 0.3$, we also see that only low-height bumps can be generated whose height decreases monotonously in $a$ with $a \leq 0.6$. Furthermore, for much larger
A mathematical treatment of the bump structure...  

A mathematical treatment of the bump structure...  

\[ \phi_L, \text{ say } 0.4 (\text{Fig. 6c}), \text{ bump structure is generated only for the middle particle size range, say } a \in [0.2, 0.4]. \text{ This result is compatible with the observation (Limit).} \]

Comparing all figures in Fig. 6, the region admitting 1-shocks appears increasingly small as \( \phi_L \) increases monotonously, while it is indeed nontrivial because contributions of \( a \) and \( \phi_L \) in the problem (19) is nonlinear. We then studied the distribution of \((\phi_L, h)\) with various \( a \) admitting 1-shocks, and the results are summarized in Fig. 7, where the range of \( h \) is restricted to \( h > 1 \) like Fig. 6. As may be observed from Fig. 7, the region where 1-shocks are admitted monotonously shrinks as \( \phi_L \) becomes large, for small \( a \) such as 0.01 and 0.1. When \( a = 0.01 \), the upper bound of \( \phi_L \) where 1-shocks are admitted is \( \phi_L^* \) as computed before (see the observation (Limit)). The monotonously decreasing behavior may also be observed in the middle-sized particles, on the order of \( a = 0.3 \), but the region of \( h \) admitting 1-shocks itself is significantly large for all \( \phi_L \) under consideration. Compared with Fig. 6, this feature reflects the significant increase of the region for \( a \) around 0.3. When \( a \) is considerably large, say \( a = 0.5 \), the region of the presence of 1-shocks depends on \( \phi_L \) in a nontrivial manner. As in the previous observation, the region is separated at \( \phi_L \approx 0.23397 \) and the boundary of the region in \((\phi_L, h)\)-plane becomes nonsmooth at \( \phi_L \approx 0.2642 \). As \( \phi_L \) becomes much larger, say \( \phi_L \geq 0.32 \) with \( a = 0.5 \), no 1-shocks of interest can be generated, as may be observed from Fig. 6.

**Remark 5** We present Figs. 6 and 7 as a collection of sample points \((a, h)\) in Fig. 6 and \((\phi_L, h)\) in Fig. 7, respectively. More precisely, we numerically investigated the existence of 1-shocks through the 1-entropy condition (17) at \( a = i \Delta a \) and \( \phi_L = 0.2 + i \Delta \phi_L \), respectively, for \( i \geq 0 \). In the present investigation, we fixed \( \Delta a = 1/500 \) and \( \Delta \phi_L = 1/1500 \). At one fixed \( a \) or \( \phi_L \), \( h \) is continued starting at \( h = 1 \) through the pseudo-arclength continuation as far as \( h \leq 4 \); otherwise \( i \) is replaced by \( i + 1 \) and the continuation is operated from \( h = 1 \) again. The index \( i \) has run over \( \mathbb{Z}_{\geq 0} \) while \( a \leq 0.6 \) and \( \phi_L \leq 0.4 \), respectively, is satisfied.

4 Concluding remark

In this study, we have explored the bump structure of particle-laden flows through a simplified model of conservation laws. We have mainly investigated the dependence of the radius and initial concentration of particles in films on a ridge, considering particle-particle and particle-wall interactions, although the latter is intrinsically considered only in an asymptotic sense in prior works. The bump structure is considered by means of composite waves consisting of Lax’s shocks. We have observed the nonlinear dependence of radius and concentration of particles on bump structure of shocks. For relatively small radii of particles enabling the treatment of particle-laden flows as continua is valid, the bump structure is generically admitted with a non-trivial but limited range of heights which depend on the initial concentration of particles. As the radius increased, the bump formation mechanism becomes complex and significantly high bumps could be also generated, whereas bumps with moderate height were prevented. For fixed particle radii, the range of bump heights monotonically decreased as the initial concentration of particles increased when the
radius was relatively small. When the radius became relatively large, the mechanism of bump generation behaved in a nonlinear manner as a function of initial concentration. In contrast to prior works such as [2, 14], we extracted the genuine contribution of particle characteristics to the morphology of particle-laden flows, possibly with their physical reliability in terms of mathematical arguments based on the theory of conservation and objects such as Hugoniot loci.

In contrast, in Murisic et al. [8], a modified model including the effect of shear-induced migration is proposed, where the importance of the particle size considerations we have also mentioned was emphasized in their experiments, especially in the transient (well-mixed) regime. Several numerical simulations in e.g. [13] have shown the existence of the bump (double-shock) and singular shock structure with relatively high initial particle concentrations $\phi_L = 0.5$. The reduced model used to characterize their targeting objects by means of shocks differs from the model considered in this study, and comparison of solution structures with those in the present model remain as directions for future research on extracting the similarity and difference of results towards the study of the intrinsic contribution of, e.g., shear-induced migration.

Acknowledgements This work was supported by JSPS Grant-in-Aid for Scientific Research (C) (No. JP18K03437). KM was partially supported by Program for Promoting the reform of national universities (Kyushu University), Ministry of Education, Culture, Sports, Science and Technology (MEXT), Japan, World Premier International Research Center Initiative (WPI), MEXT, Japan and JSPS Grant-in-Aid for Young Scientists (B) (No. JP17K14235). The authors would also like to thank Dr. Akane Kawaharada, who helped us with experimental observation of particle-laden flows. Finally, we would like to thank Editage (www.editage.com) for English language editing in our paper.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Cook, B.: Theory for particle settling and shear-induced migration in thin-film liquid flow. Phys. Rev. E 78, 045303 (2008)
2. Cook, B., Bertozzi, A., Hosoi, A.: Shock solutions for particle-laden thin films. SIAM J. Appl. Math. 68, 760–783 (2008)
3. Doedel, E., Keller, H.B., Kernevez, J.P.: Numerical analysis and control of bifurcation problems (I): bifurcation in finite dimensions. Int. J. Bifurcat. chaos 1(03), 493–520 (1991)
4. Kuznetsov, Y.A.: Elements of Applied Bifurcation Theory, vol. 112. Springer, New Yrk (2013)
5. Lax, P.D.: Hyperbolic system of conservation laws II. Commun. Pure Appl. Math. 10, 537–566 (1957)
6. Mavromoustaki, A., Bertozzi, A.L.: Hyperbolic systems of conservation laws in gravity-driven, particle-laden thin-film flows. J. Eng. Math. 88, 29–48 (2014)
7. Murisic, N., Ho, J., Hu, V., Latterman, P., Koch, T., Lin, K., Mata, M., Bertozzi, A.: Particle-laden viscous thin-film flows on an incline: experiments compared with a theory based on shear-induced migration and particle settling. Physica D 240, 1661–1673 (2011)
8. Murisic, N., Pausader, B., Peschka, D., Bertozzi, A.L.: Dynamics of particle settling and resuspension in viscous liquids. J. Fluid Mech. 717, 203–231 (2013)
9. Richardson, J.F., Zaki, W.N.: The sedimentation of a suspension of uniform spheres under conditions of viscous flow. Chem. Eng. Sci. 3, 65–73 (1954)
10. Schecter, S., Marchesin, D., Plohr, B.J.: Structurally stable Riemann solutions. J. Differ. Equ. 126(2), 303–354 (1996)
11. Smoller, J.: Shock waves and reaction–diffusion equations, 258 of Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), 2nd edn. Springer-Verlag, New York (1994)
12. Wang, L., Bertozzi, A.L.: Shock solutions for high concentration particle laden thin films. SIAM J. Appl. Math. 74(2), 322–344 (2014)
13. Wang, L., Mavromoustaki, A., Bertozzi, A.L., Urdaneta, G., Huang, K.: Rarefaction-singular shock dynamics for conserved volume gravity driven particle-laden thin film. Phys. Fluids 27(3), 033301 (2015)
14. Zhou, J., Dupuy, B., Bertozzi, A.L., Hosoi, A.E.: Theory for shock dynamics in particle-laden thin films. Phys. Rev. Lett. 94, 117803 (2005)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.