A method of constructing 2-resolvable $t$-designs for $t = 3, 4$

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Abstract
The paper introduces a method for constructing 2-resolvable $t$-designs for $t = 3, 4$. The main idea is based on the assumption that there exists a partition of a $t$-design into Steiner 2-designs. A remarkable property of the method is that it enables the construction of 2-resolvable $t$-designs with a large variety of block sizes. For $t = 4$, it is required that the Steiner 2-designs of the partition are projective planes and this case would also lead to a construction of 3-resolvable 5-designs. For instance, we show the existence of an infinite series of 3-resolvable 5-designs having $N = 5$ resolution classes with parameters $5-(14+8m, 7, 10(9+8m)(1+m))$ for any $m \geq 0$ as a byproduct. Moreover, it turns out that the method is very effective, as it yields infinitely many 2-resolvable 3-designs. However, the question of simplicity of the constructed designs has not been yet investigated.

Keywords $s$-resolvable $t$-design · Steiner 2-design · Projective plane

Mathematics Subject Classification 05B05

1 Introduction

A $t-(v, k, \lambda)$ design is called $s$-resolvable if it can be partitioned into $s-(v, k, \delta)$ designs with $s < t$. The interesting case is $s \geq 2$. Especially, the $s$-resolvability of the complete $k-(v, k, 1)$ design is known in the literature as a large set of an $s-(v, k, \delta)$ design. Large sets are an essential element in proving the existence of simple $t$-designs for arbitrarily large $t$ which have been intensively studied over three decades, see for instance [1, 10–14]. By contrast, very little is known about $s$-resolvability of non-trivial $t$-designs, when $s > 1$, see [4, 15, 17–19]. We are interested in non-trivial $t$-designs having $s$-resolutions. By focussing on $s = 2$ we introduce a method of constructing 2-resolvable $t$-designs, for $t = 3, 4$. In essence, the method is based on the assumption that there exists a $t$-design which can be partitioned...
into Steiner 2-designs, and for \( t = 4 \) it is further required that the Steiner 2-designs must be projective planes. Some examples among others satisfying the assumption can be found in large sets of 2-(\( v, 3, 1 \)) Steiner triple systems for \( v \equiv 1, 3 \text{ mod } 6, v \neq 7 \), in partition of certain infinite classes of 3-(\( v, 4, 1 \)) Steiner quadruple systems into 2-(\( v, 4, 1 \)) designs, for \( v = 2^{2m}, m \geq 2, \) \([4]\), and \( v = 2p^n + 2, p \in \{7, 31, 127\} \) \([15]\), or in large sets of the projective planes of order 3, i.e. a symmetric 2-(13, 4, 1) design. \([6, 8]\) It appears that the method is very effective, actually, when starting with examples above, it will provide a huge number of 2-resolvable 3-designs for a large variety of block sizes. Moreover, with suitable parameters for \( t = 4 \), we can also construct 4-(2\( k + 1, k, \lambda \)) designs having 2-resolutions and therefore they can be extended to 3-resolvable 5-(2\( k + 2, k + 1, \Lambda \)) designs. For instance, the case corresponding to the projective plane of order 3 yields a 3-resolvable 5-(14, 7, 90) design, which in turn leads to the existence of an infinite series of 3-resolvable 5-designs having \( N = 5 \) resolution classes with parameters 5-(14 + 8\( m \), 7, 10(9 + 8\( m \))(1 + \( m \))) for any \( m \geq 0 \) as a byproduct.

We recall a few basic definitions. A \( t \)-design, denoted by \( t-(v, k, \lambda) \), is a pair \((X, B)\), where \( X \) is a \( v \)-set of \emph{points} and \( B \) is a collection of \( k \)-subsets of \( X \), called \emph{blocks}, such that every \( k \)-subset of \( X \) is a subset of exactly \( \lambda \) blocks of \( B \). A \( t \)-design is called \emph{simple} if no two blocks are identical, otherwise, it is called \emph{non-simple}. A \( t-(v, k, 1) \) design is called a \emph{Steiner \( t \)-design}. It can be shown by simple counting that a \( t-(v, k, \lambda) \) design is an \( s-(v, k, \lambda_s) \) design for \( 0 \leq s \leq t \), where \( \lambda_s = \lambda(v-s\choose t-s)\lambda(s\choose \lambda) \)\( = \lambda(v-s\choose \lambda)\lambda(s\choose \lambda) \). Since \( \lambda_s \) is an integer, necessary conditions for the parameters of a \( t \)-design are \( (v-s\choose t-s)\lambda(s\choose \lambda) \) for \( 0 \leq s \leq t \). The smallest positive integer \( \lambda \) for which these necessary conditions are satisfied is denoted by \( \lambda_{\text{min}}(t, v, k) \) or simply \( \lambda_{\text{min}} \). If \( B \) is the set of all \( k \)-subsets of \( X \), then \((X, B)\) is a \( t-(v, k, \lambda_{\text{max}}) \) design, called the \emph{complete} design, where \( \lambda_{\text{max}} = \lambda(v\choose k) \). If we take \( \delta \) copies of the complete design, we obtain a \( t-(v, k, \delta(v-s\choose k-s)) \) design, to which we refer as a \emph{trivial \( t \)-design}. Again a \( t-(v, k, \lambda) \) design \((X, B)\) is said to be \emph{s-resolvable}, for \( 0 < s < t \), if its block set \( B \) can be partitioned into \( N \geq 2 \) classes \( A_1, \ldots, A_N \) such that each \((X, A_i)\) is an \( s-(v, k, \delta) \) design for \( i = 1, \ldots, N \). Each \( A_i \) is called an \emph{s-resolution class} or simply a resolution class and the set of \( N \) classes is called an \emph{s-resolution of \((X, B)\)}. If the complete \( k-(v, k, 1) \) design is \( t \)-resolvable, i.e. it can be partitioned into \( N \) disjoint \( t-(v, k, \lambda) \) designs, where \( k > t \), then we say that there exists a \emph{large set} of size \( N \) of \( t \)-designs denoted by \( LS(N)(t, k, v) \) or by \( LS_{\lambda}(t, k, v) \) to emphasize the value \( \lambda \).

For more information about \( s \)-resolvable \( t \)-designs with \( 1 < s < t \), see for instance \([16–19]\). It should be remarked that \( s \)-resolvable \( t \)-designs have been used in the construction of \( t \)-designs \([16]\).

2 Description of the method

The details of the method are described in this section. Here, two elements are required.

1. Let \((X, B)\) be a \( 2 \)-resolvable \( t-(v, k, \lambda) \) design, where each class is a \( 2-(v, k, 1) \) design. Thus, there are \( N = \lambda(v-s\choose k-s) \) resolution classes. Let \( B_1, \ldots, B_N \) denote the resolution classes of \((X, B)\), so each \((X, B_i)\) is a \( 2-(v, k, 1) \) design. We call \((X, B)\) the \emph{outer design}.

2. Let \((Y, C)\) be a \( t-(v-1\choose k-1, \ell, \mu) \) design. We call \((Y, C)\) the \emph{inner design}.
Consider a fixed resolution class \((X, B_i)\). Let \(Y = \{1, \ldots, \frac{v-1}{k-1}\}\) be the point set of the inner design. For a point \(x \in X\), let \(\mathcal{Y}_{i,x} = \{B^1_{i,x}, \ldots, B^{|Y|-1}_{i,x}\}\) denote the set of \(\frac{v-1}{k-1}\) blocks through \(x\) of \(B_i\), i.e. \(B^j_{i,x} \in B_i\) with \(x \in B^j_{i,x}, 1 \leq j \leq |Y|\). For a block \(C \in \mathcal{C}\), define
\[
D^C_{i,x} = \bigcup_{j \in C} B^j_{i,x},
\]
and
\[
\mathcal{D}_{i,x} = \{D^C_{i,x} \mid C \in \mathcal{C}\}.
\]
That is, block \(D^C_{i,x}\) is formed by the union of blocks in \(\mathcal{Y}_{i,x}\) indexed by \(C\), and \(\mathcal{D}_{i,x}\) is the set of \(\mu_0\) such blocks \(D^C_{i,x}\). Further, define
\[
\mathcal{D}_i = \bigcup_{x \in X} \mathcal{D}_{i,x},
\]
and
\[
\mathcal{D} = \bigcup_{i=1}^N \mathcal{D}_i.
\]
Similarly, define
\[
D^{*C}_{i,x} = \bigcup_{j \in C} B^j_{i,x} \setminus \{x\}, \quad \mathcal{D}^{*}_{i,x} = \{D^{*C}_{i,x} \mid C \in \mathcal{C}\},
\]
\[
\mathcal{D}^*_i = \bigcup_{x \in X} \mathcal{D}^{*}_{i,x},
\]
and
\[
\mathcal{D}^* = \bigcup_{i=1}^N \mathcal{D}^{*}_{i}.
\]
If \((X, \mathcal{D})\) or \((X, \mathcal{D}^*)\) forms a \(t\)-design, we call it the constructed design. For \(t = 3\), we show that \((X, \mathcal{D})\) and \((X, \mathcal{D}^*)\) are 3-designs. For \(t = 4\), if each resolution class of the outer design is a symmetric 2-(\(v, k, 1\)) design, i.e. a projective plane of order \((k-1)\) with \(v = q^2 + q + 1, k = q + 1\), we prove that \((X, \mathcal{D})\) and \((X, \mathcal{D}^*)\) will form 4-designs. Further, it is shown that \((X, \mathcal{D}_i)\) and \((X, \mathcal{D}^*_i)\) are 2-designs. Obviously, the construction method makes clear that the constructed designs \((X, \mathcal{D})\) and \((X, \mathcal{D}^*)\) are 2-resolvable, as they are the union of designs \((X, \mathcal{D}_i)\) and \((X, \mathcal{D}^*_i)\), respectively. In case \(t = 4\) and for suitable parameters of the outer design, the constructed design can be extended to a 3-resolvable 5-design, as shown in the subsequent section. A further investigation shows that if the inner design is also 2-resolvable with \(L\) resolution classes, then the constructed design is 2-resolvable with \(NL\) resolution classes. A major advantage of the method is the fact that it enables us to construct 2-resolvable \(t\)-designs with a large variety of block sizes, because there is no restriction on the parameters of the inner designs.
3 2-Resolvable 3-designs

In this section we deal with the case \( t = 3 \). We prove that \((X, D)\) and \((X, D_i)\) are 3-(\(v, \ell(k - 1) + 1, \Lambda\)) and 2-(\(v, \ell(k - 1) + 1, \delta\)) designs, respectively. Similarly, \((X, D^*)\) and \((X, D^*_i)\) are 3-(\(v, \ell(k - 1), \Lambda^*\)) and 2-(\(v, \ell(k - 1), \delta^*\)) designs. Thus, we need to determine \(\Lambda, \delta, \Lambda^*, \delta^*\). Recall that we consider the complete 2-(\(v, 2, 1\)) design as a \(t-(v, 2, 0)\) design for \(t \geq 3\).

3.1 \((X, D)\) and \((X, D_i)\) designs

We use the notation as described in the construction method. In the first step we show that \((X, D_i)\) is a 2-\((v, \ell(k - 1) + 1, \delta)\) design, and in the next step \((X, D)\) is a 3-\((v, \ell(k - 1) + 1, \Lambda)\) design.

Step 1 \((X, D_i)\) is a 2-\((v, \ell(k - 1) + 1, \delta)\) design.

Recall that \((X, B_i)\) is a 2-(\(v, k, 1\)) design and \((Y, C)\) is a 3-\((\frac{v-1}{k-1}, \ell, \mu)\) design with \(Y = \{1, \ldots, \frac{v-1}{k-1}\}\). As usual \(\mu_1\) (resp. \(\mu_2\)) denote the number of blocks of \((Y, C)\) containing a point (resp. two points). For a given point \(x \in X\), there are \(|Y|\) blocks of \(B_i\), say \(B^1_{i,x}, \ldots, B^{|Y|}_{i,x}\) containing \(x\). Let \(C = \{j_1, \ldots, j_\ell\} \subseteq Y\) be a block of \(C\). Then block \(D^C_{i,x} \subseteq D_{i,x}\) is defined by \(D^C_{i,x} = B^1_{i,x} \cup \cdots \cup B^{|Y|}_{i,x}\). Now let \(a, b \in X, a \neq b\). Let \(B\) be the unique block of \(B_i\) containing \([a, b]\). We distinguish two types of points of \(X\), namely points \(x \in B\) and points \(x \in X \setminus B\). If \(x \in B\), then \(B\) is one of the blocks \(B^1_{i,x}, \ldots, B^{|Y|}_{i,x}\), thus by forming the blocks of \(D_{i,x}\) we see that block \(B\) is contained in \(\mu_1\) blocks of \(D_{i,x}\), consequently \([a, b]\) appears in \(\mu_1\) blocks \(D\) of \(D_{i,x}\). Thus \(k\) points of \(B\) contribute \(k\mu_1\) blocks \(D \supseteq [a, b]\). If \(x \in X \setminus B\), then \([a, x]\) and \([b, x]\) determine two distinct blocks \([a, x, \ldots]\) and \([b, x, \ldots]\) of \(B^1_{i,x}, \ldots, B^{|Y|}_{i,x}\). All the blocks \(D \in D_{i,x}\) containing \([a, x, \ldots]\) and \([b, x, \ldots]\) will contain \([a, b]\). So, there are \(\mu_2\) blocks \(D\) containing \([a, x, \ldots]\) and \([b, x, \ldots]\). Thus \([a, b]\) appears in \(\mu_2\) blocks \(D\) of \(D_{i,x}\). Hence, \((v - k)\) points of \(x \in X \setminus B\) contribute \((v - k)\mu_2\) blocks \(D \supseteq [a, b]\). Altogether it gives

\[
\delta = k\mu_1 + (v - k)\mu_2.
\]

Hence, \((X, D_i)\) is a 2-(\(v, \ell(k - 1) + 1, \delta)\) design.

Step 2 \((X, D)\) is a 3-(\(v, \ell(k - 1) + 1, \Lambda)\) design.

Let \(T = \{a, b, c\} \subseteq X\). Note that among the \(N\) resolution classes \(B_1, \ldots, B_N\) of \((X, B)\) there are \(\lambda\) classes, say, \(B_1, \ldots, B_\lambda\) having the property that each has a unique block containing \(T\).

(i) We first focus on blocks \(D\) containing \(T\) constructed from classes \(B_1, \ldots, B_\lambda\). Consider \(B_1\). Let \(B\) be its unique block containing \(T\). Each point of \(B\) gives \(\mu_1\) blocks \(D\) containing \(T\). Whereas, each point of \(X \setminus B\) gives \(\mu\) blocks \(D\) containing \(T\). Thus class \(B_1\) contributes \(k\mu_1 + (v - k)\mu\) blocks \(D \supseteq T\). It follows that the classes \(B_1, \ldots, B_\lambda\) together give \(\lambda(k\mu_1 + (v - k)\mu)\) blocks \(D \supseteq T\).

(ii) The remaining \(N - \lambda = \lambda \frac{v-k}{k-2}\) classes \(B_{\lambda+1}, \ldots, B_N\) of \((X, B)\) have the property that \(|B \cap T| \leq 2\), for any block \(B \in B_i, i = \lambda + 1, \ldots, N\). Consider \(B_{\lambda+1}\). Let \(B_{ab} = \{a, b, x_2, \ldots, x_k\}, B_{ac} = \{a, c, y_2, \ldots, y_k\}, \) and \(B_{bc} = \{b, c, z_2, \ldots, z_k\}\) be three unique blocks in \(B_{\lambda+1}\) containing \([a, b], [a, c], [b, c]\), respectively. Two types of points of \(X\) need to be distinguished.
(I) \(3(k-1)\) points of \(B_{ab} \cup B_{ac} \cup B_{bc}\).

(II) \((v-3(k-1))\) points of \(X \setminus B_{ab} \cup B_{ac} \cup B_{bc}\).

Each point of type (I) gives \(\mu_2\) blocks \(D \supseteq T\). Hence points of type (I) contribute \(3(k-1)\mu_2\) blocks \(D \supseteq T\).

Each point of type (II) gives \(\mu\) blocks \(D \supseteq T\). Hence points of type (II) contribute \((v-3(k-1))\mu\) blocks \(D \supseteq T\).

It follows that all \(N-\lambda\) classes \(B_{\lambda+1}, \ldots, B_N\) contribute

\[(N-\lambda)(3(k-1)\mu_2 + (v-3(k-1))\mu)\]

blocks \(D \supseteq T\).

Hence, Cases (i) and (ii) together show that

\[\Lambda = \lambda(k\mu_1 + (v-k)\mu) + (N-\lambda)(3(k-1)\mu_2 + (v-3(k-1))\mu).\]

Thus \((X, D)\) is a 3-design.

To compute the values of \(\Lambda\) and \(\delta\) in terms of \(v, k, \lambda, \ell, \mu\) we have to separate two cases: \(\ell = 2\) and \(\ell = 3\).

\(\ell = 2:\)

In this case the inner design is the \(2-(\frac{v-1}{k-1}, 2, 1)\) design, which is considered as a degenerated 3-design with \(\mu = 0\), \(\mu_2 = 1\) and \(\mu_1 = \frac{v-k}{k-1}\). Therefore

\[\delta = k\mu_1 + (v-k)\mu_2 = k\frac{v-k}{k-1} + (v-k) = (v-k)\frac{(2k-1)}{(k-1)},\]

and

\[\Lambda = \lambda(k\mu_1 + (v-k)\mu) + (N-\lambda)(3(k-1)\mu_2 + (v-3(k-1))\mu) = \lambda\left(k\frac{v-k}{k-1} + \frac{v-k}{k-2}\right)(3(k-1)) = \lambda(v-k)\frac{(2k-1)(2k-3)}{(k-1)(k-2)}.\]

\(\ell \geq 3:\)

The inner design with parameters \(3-(\frac{v-1}{k-1}, \ell, \mu)\) will give \(\mu_2 = \mu\frac{(v-2k+1)}{(k-1)(\ell-2)}\) and \(\mu_1 = \mu\frac{(v-k)(v-2k+1)}{(k-1)^2(\ell-1)(\ell-2)}\). Replacing \(\mu_2\) and \(\mu_1\) by their values in the formulas for \(\delta\) and \(\Lambda\) and so simplifying we obtain

\[\delta = k\mu_1 + (v-k)\mu_2 = k\mu\frac{(v-k)(v-2k+1)}{(k-1)^2(\ell-1)(\ell-2)} + (v-k)\mu\frac{(v-2k+1)}{(k-1)(\ell-2)} = \frac{(v-k)(v-2k+1)}{(k-1)^2(\ell-1)(\ell-2)}\mu(k\ell - \ell + 1),\]

and

\[\Lambda = \lambda(k\mu_1 + (v-k)\mu) + \lambda\frac{(v-k)}{(k-2)}(3(k-1)\mu_2 + (v-3(k-1))\mu)\]
Theorem 3.1

Assume that the following designs exist.

in Step 1 is the union of $L$ with $(v-2k+1)$ therefore it will be omitted. The results show that

We further study the resolvability of the constructed designs when the inner designs are 2-resolvable. Suppose that the inner $3-(v-1, \ell, \mu)$ design $(Y, C)$ is 2-resolvable with $L$ resolution classes. Let $C_1, \ldots, C_L$ be the $L$ classes of $(Y, C)$. Then

where each $(Y, C_i)$ is a 2-$(v-1, \ell, \mu)$ design, and $\mu_2 = \frac{v-2k+1}{(k-1)(\ell-2)}$. It follows that

This is because the 2-$(v, \ell(k-1)+1, \delta)$ design $(X_i, D_i)$ constructed from $(X, B_i)$ and $(Y, C_i)$ in Step 1 is the union of $L$ disjoint 2-$(v, \ell(k-1)+1, \frac{\delta}{L})$ designs $(X, E_{i,j})$, $j = 1, \ldots, L.$ Each $(X, E_{i,j})$ is the 2-design constructed from $(X, B_i)$ and $(Y, C_j)$.

As a result, the constructed design $(X, D)$ is 2-resolvable with $NL$ resolution classes, and each class is a 2-$(v, \ell(k-1)+1, \frac{\delta}{L})$ design.

3.1.1 The case with 2-resolvable inner designs

To show that $(X, D^\ast)$ and $(X, D_i^\ast)$ are designs, a very similar proof as above is to be employed, therefore it will be omitted. The results show that $(X, D_i^\ast)$ is a 2-$(v, \ell(k-1), \delta^\ast)$ design with

and $(X, D^\ast)$ is a 3-$(v, \ell(k-1), \Lambda^\ast)$ design with

Putting the explicit values of $\mu_1, \mu_2$, both $\delta^\ast$ and $\Lambda^\ast$ are expressed in terms of $v, k, \lambda, \ell, \mu$ as shown in the next theorem.

The resolvability of $(X, D^\ast)$ and $(X, D_i^\ast)$ is the same as that of $(X, D)$ and $(X, D_i)$.

We summarize the results in the following theorem.

Theorem 3.1

Assume that the following designs exist.

(i) A 2-resolvable 3-$(v, k, \lambda)$ design $(X, B)$ having $N = \lambda \frac{v-2}{k-2}$ resolution classes and each class is a 2-$(v, k, 1)$ design.

(ii) A 3-$(\frac{v-1}{k-1}, \ell, \mu)$ design $(Y, C)$.

Then there exist 2-resolvable 3-$(v, (k-1)\ell+1, \Lambda)$ and 3-$(v, (k-1)\ell, \Lambda^\ast)$ designs $(X, D)$ and $(X, D^\ast)$, with $N$ resolution classes, where each class is a 2-$(v, (k-1)\ell+1, \delta)$ and 2-$(v, (k-1)\ell, \delta^\ast)$ design, respectively.
(i) For $\ell = 2$,
\[
\Lambda = \lambda(v - k) \frac{(2k - 1)(2k - 3)}{(k - 1)(k - 2)}, \quad \delta = (v - k) \frac{(2k - 1)}{(k - 1)},
\]
\[
\Lambda^* = 2\lambda(v - k) \frac{(2k - 3)}{(k - 1)}, \quad \delta^* = (v - k) \frac{(2k - 3)}{(k - 1)}.
\]

(ii) For $\ell \geq 3$,
\[
\Lambda = \frac{(v - k)(v - 2k + 1)}{(k - 1)^2(k - 2)(\ell - 1)(\ell - 2)} \lambda \mu ((k - 1)^2 \ell^2 - 1),
\]
\[
\delta = \frac{(v - k)(v - 2k + 1)}{(k - 1)^2(\ell - 1)(\ell - 2)} \mu (k\ell - \ell + 1),
\]
\[
\Lambda^* = \frac{(v - k)(v - 2k + 1)}{(k - 1)^2(\ell - 1)(\ell - 2)} \lambda \mu ((\ell(k - 1) - 1)(\ell(k - 1) - 2)),
\]
\[
\delta^* = \frac{(v - k)(v - 2k + 1)}{(k - 1)^2(\ell - 1)(\ell - 2)} \mu (k\ell - \ell - 1).
\]

Further, if $(Y, C)$ is 2-resolvable with $L$ resolution classes, then $(X, D)$ and $(X, D^*)$ are 2-resolvable with $NL$ resolution classes and each class is a 2-$(v, (k - 1)\ell + 1, \frac{\delta}{\mu})$ and 2-$(v, (k - 1)\ell, \frac{\delta^*}{\mu})$ design, respectively.

To illustrate the effectiveness of Theorem 3.1 we show a concrete example. For the outer design take a 3-$(64, 4, 1)$ design [4] which is partitioned into $N = 31$ Steiner 2-$(64, 4, 1)$ designs. The inner design can be chosen from all possible 3-$(21, \ell, \mu)$ designs with the following parameters.

1. 2-(21, 2, 1),
2. 3-(21, 3, 1),
3. 3-(21, 4, m6), $1 \leq m \leq 3$
4. 3-(21, 5, m3), $1 \leq m \leq 51$
5. 3-(21, 6, m4), $1 \leq m \leq 204$
6. 3-(21, 7, m15), $1 \leq m \leq 204$
7. 3-(21, 8, m84), $1 \leq m \leq 102$
8. 3-(21, 9, m42), $1 \leq m \leq 442$
9. 3-(21, 10, m72), $1 \leq m \leq 442$.

The existence of 3-$(21, \ell, \mu)$ designs above for $4 \leq \ell \leq 10$ can be found in [7]. For each value of $m$ for which a 3-$(21, \ell, \mu)$ design exists, the parameters of 2-resolvable 3-$(64, 3\ell + 1, \Lambda)$ and 3-$(64, 3\ell, \Lambda^*)$ designs for $\ell = 2, \ldots, 10$, and their corresponding 2-designs in the resolution constructed from Theorem 3.1 are as follows.

(i) 3-(64, 7, $70 \times 5$), 2-(64, 7, $70 \times 2$),
\quad 3-(64, 6, $200 \times 10$), 2-(64, 6, $200 \times 5$),
(ii) 3-(64, 10, $380 \times 20$), 2-(64, 10, $200 \times 5$),
\quad 3-(64, 9, $190 \times 28$), 2-(64, 9, $10 \times 8$),
(iii) 3-(64, 13, $95m \times 286$), 2-(64, 13, $95m \times 52$),
\quad 3-(64, 12, $380m \times 55$), 2-(64, 12, $380m \times 11$), $1 \leq m \leq 3$,
(iv) 3-(64, 16, $304m \times 35$), 2-(64, 16, $304m \times 5$),
\quad 3-(64, 15, $133m \times 65$), 2-(64, 15, $133m \times 10$), $1 \leq m \leq 51$,
(v) 3-(64, 19, $38m \times 323$), 2-(64, 19, $38m \times 38$),
\quad 3-(64, 18, $76m \times 136$), 2-(64, 18, $76m \times 17$), $1 \leq m \leq 204$. 
Theorem 3.1 we have the following result.

Let Corollary 3.2 $L^2$-resolvable with $(v,\Lambda)$.

(iii) $(3, 24, 95m \times 2300)$, $2-(64, 25, 95m \times 200)$, $1 \leq m \leq 102$,

(viii) $3-(64, 28, 2660m \times 39)$, $2-(64, 28, 2660m \times 3)$, $3-(64, 27, 95m \times 975)$, $2-(64, 27, 95m \times 78)$, $1 \leq m \leq 442$,

(ix) $3-(64, 31, 1178m \times 145)$, $2-(64, 31, 38m \times 310)$, $3-(64, 30, 76m \times 2030)$, $2-(64, 30, 76m \times 145)$, $1 \leq m \leq 442$.

**Remarks 3.1** 1. Observe that all values of $\Lambda$ and $\Lambda^*$ of the constructed designs above are really small. For example, by taking a $3-(21, 9, 42)$ design as the inner design, the parameters of the $3-(64, 27, \Lambda^*)$ constructed design become $3-(64, 27, 95 \times 975)$, compared with its general parameters $3-(64, 27, m^* \times 975)$, where $1 \leq m^* \leq 60961764003119$. Even if the complete $3-(21, 9, 442 \times 42)$ design is used, the corresponding constructed design will be of parameters $3-(64, 27, 41990 \times 975)$, showing that $m^* = 41990 \ll 60961764003119$ is still quite small.

2. The constructed $3-(64, 10, 380 \times 20)$ and $3-(64, 9, 190 \times 28)$ designs under (ii) are $2$-resolvable with $NL = 31.19 = 589$ resolution classes each, this is because the $3-(21, 3, 1)$ inner design can be partitioned into $L = 19$ Steiner $2-(21, 3, 1)$ designs. Other examples are the $3-(64, 13, 95m \times 286)$ and $3-(64, 12, 380m \times 55)$ designs with $m = 3$ from (iii). Here, the inner design is the complete $3-(21, 4, 3 \times 6)$ design, which again can be partitioned into $L = 19$ disjoint $2-(21, 4, 9)$ designs. Thus, both $3-(64, 13, 95 \times 3 \times 286)$ and $3-(64, 12, 380 \times 3 \times 55)$ designs are $2$-resolvable with $NL = 589$ resolution classes.

In general, when the inner design is the complete $3-(21, \ell, (\ell \div 2))$ design, we may employ the knowledge of large sets $LS_L(2, \ell, 21)$ to obtain further refinement of the resolution for the constructed design. For instance, there are $LS_{17}(2, \ell, 21)$ for $\ell = 5, 6, 7, 8$, thus the constructed designs under (i), (ii), (iii) have $NL = 31.17 = 527$ resolution classes.

The following corollaries show some applications of Theorem 3.1. It is a well-known result that there exists an $LS_{\text{min}}(2, 3, v)$ for $v \neq 7$. In particular, if $v \equiv 1, 3$ (mod 6), then $v_{\text{min}} = 1$, i.e. the $3-(v, 3, 1)$ design can be partitioned into $N = (v - 2)$ disjoint $2-(v, 3, 1)$ designs. Take the $3-(v, 3, 1)$ design as the outer design. Take the $2-(\frac{v-1}{2}, 2, 1)$ and $3-(\frac{v-1}{2}, 3, 1)$ design as the inner design. Again, in the second case the $3-(\frac{v-1}{2}, 3, 1)$ design is $2$-resolvable with $L = \frac{v-5}{2v_{\text{min}}}$ resolution classes, each class is a $2-(\frac{v-1}{2}, 3, v_{\text{min}})$. Now applying Theorem 3.1 we have the following result.

**Corollary 3.2** Let $v_{\text{min}} = v_{\text{min}}(2, 3, \frac{v-1}{2})$, where $v$ is an integer such that $v \equiv 1, 3$ (mod 6), $v \neq 7$. Let $N = (v - 2)$ and $L = \frac{v-5}{2v_{\text{min}}}$. Then

(i) There exists a $2$-resolvable $3-(v, 5, \frac{15}{2}(v - 3))$ design having $N = (v - 2)$ resolution classes, each class is a $2-(v, 5, \frac{5}{2}(v - 3))$ design.

(ii) There exists a $2$-resolvable $3-(v, 7, \frac{35}{8}(v - 3)(v - 5))$ design having $NL$ resolution classes, each class is a $2-(v, 7, \frac{7}{4}v_{\text{min}}(v - 3))$ design.

(iii) There exists a $2$-resolvable $3-(v, 6, \frac{5}{2}(v - 3)(v - 5))$ design having $NL$ resolution classes, each class is a $2-(v, 6, \frac{5}{4}v_{\text{min}}(v - 3))$ design.

For $n \geq 2$ there is a $2$-resolvable $3-(2^n, 4, 1)$ design with $N = 2^{2n-1} - 1$ resolution classes and each class is a $2-(2^n, 4, 1)$ design, see [4]. Take this design as the outer design.
Now any 3-(\(\frac{2^2n-1}{3}\), \(\ell\), \(\mu\)) design can be used as the inner design. Thus it produces innumerable 2-resolvable 3-designs with a large variety of block sizes. As an example, the next corollary shows the results for the first two cases with \(\ell = 2, 3\), i.e. the inner design is the 2-(\(\frac{2^2n-1}{3}\), 2, 1) and 3-(\(\frac{2^2n-1}{3}\), 3, 1) design. Again, note that the 3-(\(\frac{2^2n-1}{3}\), 3, 1) design can be partitioned into \(L = \frac{2^{2n}-7}{3\nu_{\min}}\) classes of 2-(\(\frac{2^2n-1}{3}\), 3, \(\nu_{\min}\)) designs.

**Corollary 3.3** Let \(\nu_{\min} = \nu_{\min}(2, 3, \frac{2^2n-1}{3})\), \(n \geq 2\). Let \(N = (2^2n-1)\) and \(L = \frac{2^{2n}-7}{3\nu_{\min}}\) Then

(i) There exists a 2-resolvable 3-(\(2^2n\), 7, \(\frac{35}{6}\)(\(2^2n-4\))) design having \(N\) resolution classes, each class is a 2-(\(2^2n\), 7, \(\frac{7}{2}\)(\(2^2n-4\))) design.

(ii) There exists a 2-resolvable 3-(\(2^2n\), 6, \(\frac{10}{3}\)(\(2^2n-4\))) design having \(N\) resolution classes, each class is a 2-(\(2^2n\), 6, \(\frac{5}{3}\)(\(2^2n-4\))) design.

(iii) There exists a 2-resolvable 3-(\(2^2n\), 10, \(\frac{20}{9}\)(\(2^2n-4\))(\(2^2n-7\)))(\(2^2n-7\)) design having \(NL\) resolution classes, each class is a 2-(\(2^2n\), 10, \(\frac{5}{3}\nu_{\min}(2^2n-4))\) design.

(iv) There exists a 2-resolvable 3-(\(2^2n\), 9, \(\frac{14}{9}\)(\(2^2n-4\))(\(2^2n-7\)))(\(2^2n-7\)) design having \(NL\) resolution classes, each class is a 2-(\(2^2n\), 9, \(\frac{4}{3}\nu_{\min}(2^2n-4))\) design.

### 4 2-Resolvable 4-designs

This section deals with the case, where the designs in the resolution of the outer design are symmetric 2-(\(v, k, 1\)) designs, i.e. each resolution class is a projective plane of parameters 2-(\(q^2 + q + 1, q + 1, 1\)). Obviously, \((X, D_l)\) and \((X, D^*)\) are 2-designs, as shown in the previous section. We prove that \((X, D)\) and \((X, D^*)\) are 4-designs.

#### 4.1 4-(\(v, \ell(k-1) + 1\), \(\Lambda\)) design \((X, D)\)

Again use the notation as described in the construction method. We omit the proof that \((X, D_l)\) is a 2-(\(v, \ell(k-1) + 1, \delta\)) design, as it is the same as that in the previous section. Here, we focus on the proof in the main step that \((X, D)\) is a 4-(\(v, \ell(k-1) + 1, \Lambda\)) design. **Main step** \((X, D)\) is a 4-(\(v, \ell(k-1) + 1, \Lambda\)) design.

To simplify the writing we temporarily keep the parameters 2-(\(v, k, 1\)) for the symmetric design of the resolution, and will replace them with 2-(\(q^2 + q + 1, q + 1, 1\)) at the end of the proof.

Let \(T = \{a, b, c, d\} \subseteq X\). With respect to \(T\), there are three types of resolution classes:

(i) Classes having a unique block \(B\) containing \(T\).

(ii) Classes having a unique block \(B\) with \(|B \cap T| = 3\).

(iii) Classes having only blocks \(B\) with \(|B \cap T| \leq 2\).

The number of classes of type (i) is \(\lambda\), of type (ii) \(4(\lambda \frac{v-3}{k-3} - \lambda) = 4\lambda \frac{v-3}{k-3}\). The remaining \(N - (4\lambda \frac{v-3}{k-3} + \lambda)\) classes are of type (iii). So, w.l.o.g., we may assume that \(B_1, \ldots, B_3\) are classes of type (i) and \(B_4, \ldots, B_j, B_{\lambda+4\lambda \frac{v-3}{k-3}}\) classes of type (ii).

(i) Consider class \(B_1\) of type (i). Let \(B\) be its unique block containing \(T\). Each point of \(B\) gives \(\mu_1\) blocks \(D\) containing \(T\). Whereas, each point of \(X \setminus B\) gives \(\mu\) blocks \(D\) containing \(T\). Thus class \(B_1\) produces \(k\mu_1 + (v - k)\mu\) blocks \(D\). It follows that the classes \(B_1, \ldots, B_j\) together give \(\lambda(k\mu_1 + (v - k)\mu)\) blocks \(D \supseteq T\).

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(ii) Each of $B_{k+1}, \ldots, B_{\frac{k+4v-1}{3}}$ classes of type (ii) has a unique block $B$ with $|B \cap T| = 3$. Consider class $B_{k+1}$. There are four 3-subsets of $T$. So, w.l.o.g., we assume that $B \cap T = \{a, b, c\}$. Let $B := B_{abc} = \{a, b, c, u_3, \ldots, u_k\}$, $B_{da} = \{d, a, x_2, \ldots, x_k\}$, $B_{db} = \{d, b, y_2, \ldots, y_k\}$, and $B_{dc} = \{d, c, z_2, \ldots, z_k\}$ be the four unique blocks in $B_{k+1}$ containing $\{a, b, c\}, \{d, a\}, \{d, b\}$ and $\{d, c\}$, respectively. In $B_{k+1}$, the contribution to blocks $D \supseteq T$ depends on three distinct point types of $X$, that are the following.

(I) $k$ points of $B_{abc}$. These points produce $k \mu_2$ blocks $D \supseteq T$.

(II) $1 + (3(k - 2)) = 3k - 5$ points of $B_{da}, B_{db}$ and $B_{dc}$ different from $a, b, c$. These points give $(3k - 5) \mu_3$ blocks $D \supseteq T$.

(III) $(v - 4k + 5)$ points of $X \setminus B_{abc} \cup B_{da} \cup B_{db} \cup B_{dc}$. These points produce $(v - 4k + 5) \mu$ blocks $D \supseteq T$.

So, class $B_{k+1}$ gives $(k \mu_2 + (3k - 5) \mu_3 + (v - 4k + 5) \mu)$ blocks $D \supseteq T$. It follows that for all four 3-subsets $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$ of $T$, the $4 \lambda \frac{v - k}{k - 3}$ classes of type (ii) produce $4 \lambda \frac{v - k}{k - 3} (k \mu_2 + (3k - 5) \mu_3 + (v - 4k + 5) \mu)$ blocks $D \supseteq T$ in total.

(iii) Consider the remaining $N - (4 \lambda \frac{v - k}{k - 3} + \lambda)$ classes of type (iii). Let $B_j$ be such a class. Since $(X, B_j)$ is a 2-$(v, k, 1)$ projective plane, and $|B \cap T| \leq 2$ for any block $B \in B_j$, the 6 pairs of points of $T = \{a, b, c, d\}$ are on 6 unique blocks.

$$B_{ab} = \{a, b, x_3, x_4, \ldots, x_k\},$$
$$B_{cd} = \{c, d, x_3, y_4, \ldots, y_k\},$$
$$B_{ad} = \{a, d, x'_3, x'_4, \ldots, x'_k\},$$
$$B_{bc} = \{b, c, x'_3, y'_4, \ldots, y'_k\},$$
$$B_{ac} = \{a, c, x''_3, x''_4, \ldots, x''_k\},$$
$$B_{bd} = \{b, d, x''_3, y''_4, \ldots, y''_k\}.$$

These blocks partition the points of $X$ in 3 types.

(I) $(6k - 14)$ points:

$$a, b, c, d, x_4, \ldots, x_k, y_4, \ldots, y_k, x'_4, \ldots, x'_k, y'_4, \ldots, y'_k, x''_4, \ldots, x''_k.$$

These points give $(6k - 14) \mu_3$ blocks $D \supseteq T$.

(II) $3$ points: $x_3, x'_3, x''_3$. These points give $3 \mu_2$ blocks $D \supseteq T$.

(III) $(v - 6k + 11)$ points of $X \setminus (B_{ab} \cup B_{cd} \cup B_{ad} \cup B_{bc} \cup B_{ac} \cup B_{bd})$. These points produce $(v - 6k + 11) \mu$ blocks $D \supseteq T$.

Altogether $B_j$ has $3 \mu_2 + (6k - 14) \mu_3 + (v - 6k + 11) \mu$ blocks $D \supseteq T$. Hence the

$$N - (4 \lambda \frac{v - k}{k - 3} + \lambda) \text{ classes of type (iii) produce}$$

$$(N - (4 \lambda \frac{v - k}{k - 3} + \lambda)) (3 \mu_2 + (6k - 14) \mu_3 + (v - 6k + 11) \mu)$$

blocks $D \supseteq T$.

In summary, cases (i), (ii), (iii) together yield

$$\Lambda = \lambda (k \mu_1 + (v - k) \mu) + 4 \lambda \frac{v - k}{k - 3} (k \mu_2 + (3k - 5) \mu_3 + (v - 4k + 5) \mu)$$

$$+ (N - (4 \lambda \frac{v - k}{k - 3} + \lambda)) (3 \mu_2 + (6k - 14) \mu_3 + (v - 6k + 11) \mu).$$
we find that

(1) for \(\ell = 2\),

\[
\Lambda = \frac{2\lambda q}{(q-2)(4q^2-1)}, \quad \delta = q(2q+1).
\]

(2) for \(\ell = 3\),

\[
\Lambda = \frac{\lambda q}{2(q-2)(9q^2-1)(3q-2)}, \quad \delta = \frac{q(q-1)}{2}(3q+1),
\]

(3) for \(\ell \geq 4\),

\[
\Lambda = \frac{\lambda \mu q}{(\ell - 1)(\ell - 2)(\ell - 3)}(q^2 \ell^2 - 1)(q \ell - 2), \quad \delta = \frac{q(q-1)(q-2)(q \ell + 1)}{(\ell - 1)(\ell - 2)(\ell - 3)} \mu.
\]

The 4-design \((X, D)\) is 2-resolvable with \(N\) resolutions classes, because it is the union of 2-designs \((X, D_i)\)’s. Further, if the inner design \((Y, C)\) is also 2-resolvable with \(L\) resolution classes, then the same argument as above shows that \((X, D)\) is 2-resolvable with \(NL\) resolution classes.

4.2 4-(\(v\), \(\ell(k-1)\), \(\Lambda^*\)) design \((X, D^*)\)

Again, this case may be handled in a similar manner as that of \((X, D)\), and therefore we will omit the proof, despite the fact that several tiresome calculations for \(\Lambda^*\) have to be carefully carried out.

We record the results for both cases in the following theorem.

**Theorem 4.1** Assume that the following designs exist.

(1) A 4-(\(q^2 + q + 1\), \(q + 1\), \(\lambda\)) design \((X, B)\) that can be partitioned into \(N = \lambda \frac{(q^2 + q - 1)(q^2 + q - 2)}{(q-1)(q-2)}\) symmetric 2-(\(q^2 + q + 1\), \(q + 1\), 1) designs, i.e. projective planes.

(2) A 4-(\(q + 1\), 1, \(\ell\), \(\mu\)) design \((Y, C)\).

Then there exist 2-resolvable 4-(\(q^2 + q + 1\), \(q \ell + 1\), \(\Lambda\)) and 4-(\(q^2 + q + 1\), \(q \ell\), \(\Lambda^*\)) designs \((X, D)\) and \((X, D^*)\) with \(N\) resolution classes, where each class is a 2-(\(q^2 + q + 1\), \(q \ell + 1\), \(\delta\)) and a 2-(\(q^2 + q + 1\), \(q \ell\), \(\delta^*\)) design, respectively.

(i) For \(\ell = 2\),

\[
\Lambda = \frac{2\lambda q}{(q - 2)}(4q^2 - 1), \quad \delta = q(2q + 1),
\]

\[
\Lambda^* = \frac{2\lambda q}{(q - 2)}(2q - 1)(2q - 3), \quad \delta^* = q(2q - 1).
\]

(ii) For \(\ell = 3\),

\[
\Lambda = \frac{\lambda q}{2(q-2)}(9q^2 - 1)(3q - 2), \quad \delta = \frac{q(q-1)}{2}(3q + 1),
\]

\[
\Lambda^* = \frac{3\lambda q}{2(q-2)}(3q - 1)(3q - 2)(q - 1), \quad \delta^* = \frac{q(q-1)}{2}(3q - 1).
\]
(iii) For $\ell \geq 4$, 
\[
\Lambda = \frac{\lambda \mu q}{(\ell - 1)(\ell - 2)(\ell - 3)} (q^2 \ell^2 - 1)(q \ell - 2),
\]
\[
\delta = \frac{q(q - 1)(q - 2)(q \ell + 1)}{(\ell - 1)(\ell - 2)(\ell - 3)} \mu,
\]
\[
\Lambda^* = \frac{\lambda \mu q}{(\ell - 1)(\ell - 2)(\ell - 3)} (q \ell - 1)(q \ell - 2)(q \ell - 3),
\]
\[
\delta^* = \frac{q(q - 1)(q - 2)(q \ell + 1)}{(\ell - 1)(\ell - 2)(\ell - 3)} \mu.
\]

Further, if $(Y, C)$ is 2-resolvable with $L$ resolution classes, then $(X, \mathcal{D})$ and $(X, \mathcal{D}^*)$ are 2-resolvable with $NL$ resolution classes and each class is a 2-$(q^2 + q + 1, q \ell + 1, \frac{\delta}{L})$ and a 2-$(q^2 + q + 1, q \ell, \frac{\delta^*}{L})$ design, respectively.

We illustrate Theorem 4.1 by showing the following examples. Let $q = 2^m$, $m \geq 5$ odd. Consider two infinite classes of 4-designs with parameters $4-(q + 1, 5, 5)$ and $4-(q + 1, 6, 10)$. The first one can be found in [2] and the second in [5]. All these designs are 3-resolvable with $L = \frac{(q - 2)}{6}$ resolution classes. Each resolution class of the 4-$(q + 1, 5, 5)$ designs is a 3-$(q + 1, 5, 15)$ design, which is also a 2-$(q + 1, 5, 5(q - 1))$ design. Further, each resolution class of the 4-$(q + 1, 6, 10)$ designs is a 3-$(q + 1, 6, 20)$ design, which is also a 2-$(q + 1, 6, 5(q - 1))$ design. Taking these 4-$(q + 1, 5, 5)$ and 4-$(q + 1, 6, 10)$ designs as the inner design $(Y, C)$ and applying Theorem 4.1 we obtain the following result.

**Corollary 4.2** Let $q = 2^m$, $m \geq 5$ odd and let $L = \frac{(q - 2)}{6}$. Assume that there exists a 4-$(q^2 + q + 1, q \ell + 1, \lambda)$ design that can be partitioned into $N = \frac{\lambda(q^2 + q - 1)(q^2 + q - 2)}{(q - 1)(q - 2)}$ projective planes of order $q$. Then there exist 2-resolvable 4-$(q^2 + q + 1, q \ell + 1, \Lambda)$ and 4-$(q^2 + q + 1, q \ell, \Lambda^*)$ designs $(X, \mathcal{D})$ and $(X, \mathcal{D}^*)$ with $NL = \frac{\lambda(q^2 + q - 1)(q^2 + q)}{6}$ resolution classes, where classes are 2-$(q^2 + q + 1, q \ell + 1, \delta)$ and 2-$(q^2 + q + 1, q \ell, \delta^*)$ designs $(X, \mathcal{E}_i)$ and $(X, \mathcal{E}_i^*)$, respectively.

(i) $(X, \mathcal{D})$: 4-$(q^2 + q + 1, 5q + 1, \Lambda)$, 
\[
\Lambda = \frac{5q}{24} (5q + 1)(5q - 1)(5q - 2),
\]
\[
(X, \mathcal{E}_i): 2-(q^2 + q + 1, 5q + 1, \delta), \quad \delta = \frac{5}{4} q(q - 1)(5q + 1).
\]
(ii) $(X, \mathcal{D}^*)$: 4-$(q^2 + q + 1, 5q, \Lambda^*)$, 
\[
\Lambda^* = \frac{5q}{24} (5q - 1)(5q - 2)(5q - 3),
\]
\[
(X, \mathcal{E}_i^*): 2-(q^2 + q + 1, 5q, \delta^*), \quad \delta^* = \delta = \frac{5}{4} q(q - 1)(5q - 1).
\]
(iii) $(X, \mathcal{D})$: 4-$(q^2 + q + 1, 6q + 1, \Lambda)$, 
\[
\Lambda = \frac{5q}{6} (6q + 1)(6q - 1)(6q - 2),
\]
\[
(X, \mathcal{E}_i): 2-(q^2 + q + 1, 6q + 1, \delta), \quad \delta = \frac{5}{6} q(q - 1)(6q + 1),
\]
(iv) $(X, \mathcal{D}^*)$: 4-$(q^2 + q + 1, 6q, \Lambda^*)$, 
\[
\Lambda^* = \frac{5q}{6} (6q - 1)(6q - 2)(6q - 3),
\]
\[
(X, \mathcal{E}_i^*): 2-(q^2 + q + 1, 6q, \delta^*), \quad \delta^* = \delta = q(q - 1)(6q - 1).
\]

Under the condition of Corollary 4.2 we may find more infinite classes of 2-resolvable 4-designs by using the inner design $(Y, C)$ as 3-resolvable 4-$(q + 1, k, \lambda)$ designs for $k = 8, 9$ in [5, 17].

We include a further application of Theorem 4.1. In [12] Teirlinck proves that an $LS_{\nu_{\min}}(3, 4, n)$ exists if $n \equiv 0 \pmod{3}$. Let $q$ be a prime power such that $q \equiv 2 \pmod{3}$. Take the 4-$(q + 1, 4, 1)$ design as the inner design, which is the union of $L$ disjoint 3-$(q + 1, 4, \nu_{\min})$ designs. Thus $L = \frac{q^2 - 2}{\nu_{\min}}$. Notice that a 3-$(q + 1, 4, \nu_{\min})$ design is also a 2-$(q + 1, 4, \nu_{\min} \frac{q - 1}{2})$ design. Now applying Theorem 4.1 gives the following result.
Proof Let \( q \) be an odd prime power such that \( q \equiv 2 \mod 3 \). Let \( v_{\min} = v_{\min}(3, 4, q + 1) \) and let \( L = \frac{q - 2}{v_{\min}} \). Assume that there exists a 4-(\( q^2 + q + 1 \), \( q + 1 \), \( \lambda \)) design that can be partitioned into \( N = \lambda \frac{(q^2 + q - 1)(q^2 + q - 2)}{(q - 1)(q - 2)} \) projective planes of order \( q \). Then there exist 2-resolvable 4-(\( q^2 + q + 1 \), \( 4q + 1 \), \( \Lambda \)) and 4-(\( q^2 + q + 1 \), 4\( q \), \( \Lambda^* \)) designs \((X, D)\) and \((X, D^*)\) with \( NL = \lambda \frac{(q^2 + q - 1)(q^2 + q - 2)}{(q - 1)(q - 2)} \) resolution classes, where classes are 2-(\( q^2 + q + 1 \), \( 4q + 1 \), \( \frac{q}{L} \)) and 2-(\( q^2 + q + 1 \), \( 4q \), \( \frac{q}{L} \)) designs \((X, E_i)\) and \((X, E^*_i)\), respectively.

(i) \((X, D): 4-(q^2 + q + 1, 4q + 1, \Lambda), \Lambda = \frac{2q}{6}(4q - 1)(4q + 1)(4q - 2), \)  
\((X, E_i): 2-(q^2 + q + 1, 4q + 1, \frac{\delta}{L}), \delta = v_{\min} \frac{q(q-1)(q+1)}{6}, \)

(ii) \((X, D^*): 4-(q^2 + q + 1, 4q, \Lambda^*), \Lambda^* = \frac{2q}{6}(4q - 1)(4q - 2)(4q - 3), \)  
\((X, E^*_i): 2-(q^2 + q + 1, 4q, \frac{\delta^*}{L}), \delta^* = v_{\min} \frac{q(q-1)(q+1)}{6}. \)

5 5-Designs

Let us take a close look at the constructed design \((X, D^*)\) with parameters 4-(\( q^2 + q + 1 \), \( q \ell \), \( \Lambda^* \)) in Theorem 4.1, when \( q \) is odd. Observe that if the inner design \((Y, C)\) is a 4-(\( q + 1 \), \( \frac{q+1}{\mu} \), \( \Lambda^* \)) design, then the parameters of \((X, D^*)\) become 4-(\( q^2 + q + 1 \), \( q \), \( \frac{q+1}{\mu} \), \( \Lambda^* \)).

In this case, \((X, D^*)\) can be extended to a 5-(\( q^2 + q + 2 \), \( \frac{q(q+1)}{2} \), \( + 1 \), \( \Lambda^* \)) design, by a theorem of Alltop [2, 3], which is described as follows.

Let \((X, B)\) be a \( t \)-(\( 2k + 1 \), \( k \), \( \lambda \)) design with \( t \) even, and let \( \infty \notin X \). Define
\[
B^+ = \{B \cup \{\infty\} \mid B \in B\},
\]
\[
B^- = \{X \setminus B \mid B \in B\}.
\]
Then \((X \cup \{\infty\}, B^+ \cup B^-)\) is a \((t + 1)-(2k + 2, k + 1, \lambda)\) design.

We prove the following lemma.

Lemma 5.1 Let \((X, B)\) be a \( t \)-(\( 2k + 1 \), \( k \), \( \lambda \)) design with \( t \) even. Let \((X \cup \{\infty\}, B^+ \cup B^-)\) be its \((t + 1)-(2k + 2, k + 1, \lambda)\) extending design. Assume that \((X, B)\) is \( s \)-resolvable with \( N \) resolution classes; each class is an \( s \)-(\( 2k + 1 \), \( k \), \( \delta \)) design.

(i) If \( s \) is even, then the extending design is \((s + 1)\)-resolvable with \( N \) resolution classes, each class is an \( s \)-(\( 2k + 1 \), \( 2k + 2, \), \( \delta \)) design.

(ii) If \( s \) is odd, then the extending design is \( s \)-resolvable with \( N \) resolution classes, each class is an \( s \)-(\( 2k + 2, \), \( k + 1, \), \( \delta \)) design.

Proof Let \( B_1, \ldots, B_N \) be the \( N \) resolution classes of \((X, B)\), where each \((X, B_i)\) is an \( s \)-(\( 2k + 1 \), \( k \), \( \delta \)) design and \( \delta = \frac{2q}{N} \).

(i) \( s \) even. Applying the Alltop theorem, we find
\[
B^+ = B^+_1 \cup \cdots \cup B^+_N, \quad B^- = B^-_1 \cup \cdots \cup B^-_N.
\]
Hence
\[
B^+ \cup B^- = (B^+_1 \cup B^-_1) \cup \cdots \cup (B^+_N \cup B^-_N).
\]
Each \((X \cup \{\infty\}, B^+_i \cup B^-_i)\) is an \((s + 1)-(2k + 2, k + 1, \delta)\) design, for \( i = 1, \ldots, N \). Thus, \((X \cup \{\infty\}, B^+ \cup B^-)\) is \((s + 1)\)-resolvable.
(ii) $s$ odd. Each class $(X, B_i)$ is an $s$-$(2k + 1, k, \delta)$ design. Thus, $(X, B_i)$ may be considered as an $(s - 1)$-$(2k + 1, k, \delta_{s-1})$ design with $(s - 1)$ even and $\delta_{s-1} = \delta \frac{2k+1-(s-1)}{k-(s-1)}$. Again, applying the Alltop theorem shows that the extending design $(X \cup \{\infty\}, B^+ \cup B^-)$ is $s$-resolvable, and each resolution class is an $s$-$(2k + 2, k + 1, \delta_2 \frac{2k+2-s}{k+1-s})$ design.

Thus, starting with an inner design $(Y, C)$ of parameters $4$-$(q + 1, \frac{q+1}{2}, \mu)$ for $q$ odd and applying Lemma 5.1 we find that the constructed design $(X, D^*)$ in Theorem 4.1 is extended to a $3$-resolvable $5$-design $(X \cup \{\infty\}, D^{*+} \cup D^{*-})$.

We state the result in the following theorem.

**Theorem 5.2** Let $q$ be an odd positive integer. Assume that there is a $2$-resolvable $4$-$(q^2 + q + 1, q + 1, \lambda)$ design with $N = \frac{\lambda (q^2+q-1)(q^2+q-2)}{(q-1)(q-2)}$ resolution classes, each class is a symmetric $2$-$(q^2 + q + 1, q + 1, 1)$ design. Assume that there is also a $4$-$(q + 1, \frac{q+1}{2}, \mu)$ design. Then there is a $3$-resolvable $5$-$(q^2 + q + 2, \frac{q(q+1)}{2} + 1, \Lambda^*)$ design with $N$ resolution classes; each class is a $3$-$(q^2 + q + 2, \frac{q(q+1)}{2} + 1, \delta^*)$ design, where $\Lambda^*$ and $\delta^*$ are as follows.

(i) For $q = 3$,

$$
\Lambda^* = \frac{2\lambda q}{(q-2)(2q-1)(2q-3)} = 90,
\delta^* = \frac{q(2q-1)}{(2q-3)} = 15.
$$

(ii) For $q = 5$,

$$
\Lambda^* = \frac{3\lambda q}{2(q-2)} (3q-1)(3q-2)(q-1) = \lambda 1820,
\delta^* = \frac{q(q-1)}{2} (3q-1) = 140.
$$

(iii) For $q \geq 7$,

$$
\Lambda^* = \frac{\lambda \mu q}{(q-3)(q-5)} (q+2)(q^2+q-4)(q^2+q-6),
\delta^* = \frac{4q(q-1)(q^2-4)}{(q-3)(q-5)} \mu.
$$

Further, if the $4$-$(q + 1, \frac{q+1}{2}, \mu)$ design is $2$-resolvable with $L$ resolution classes, then the $5$-$(q^2 + q + 2, \frac{q(q+1)}{2} + 1, \Lambda^*)$ design is $3$-resolvable with $NL$ resolution classes and each class is a $3$-$(q^2 + q + 2, \frac{q(q+1)}{2} + 1, \delta^*)$ design.

In 1978, Magliveras conjectured that there will exist a large set of projective planes of order $q$ for $q \geq 3$, provided $q$ is the order of a projective plane. This conjecture is still an unsettled problem, except for $q = 3$, [8]. The main assumption of Theorems 4.1 and 5.2 is the existence of a $2$-resolvable $4$-$(q^2 + q + 1, q + 1, \lambda)$ design as the outer design, whose resolution classes are projective planes of order $q$. In particular, if we take the complete $4$-$(q^2 + q + 1, q + 1, (q^2+q-3)(q-3))$ design as the outer design, then the assumption is equivalent to the existence of a large set of projective planes of order $q$. To further clarify Theorems 4.1 and 5.2 we focus on this special case.

Consider case (i) with $q = 3$ of Theorem 5.2. The outer design becomes the $4$-$(13, 4, 1)$ design, which can be partitioned into $N = 55$ symmetric $2$-$(13, 4, 1)$ designs by [6] and [8].
Applying Theorem 4.1 with the 2-(4, 2, 1) inner design yields a 2-resolvable 4-(13, 6, 90) design with \( N = 55 \) resolution classes, where each class is a 2-(13, 6, 15) design. By Theorem 5.2, this 4-design is extendable to a 3-resolvable 5-(14, 7, 90) design with the same number of resolution classes and each class is a 3-(14, 7, 15) design. Note that both 4-(13, 6, 90) and 5-(14, 7, 90) designs are not simple, since the complete 4-(13, 6, \( \lambda_{\text{max}} \)) and 5-(14, 7, \( \lambda_{\text{max}} \)) design will have \( \lambda_{\text{max}} = 36 \). However, they are also non-trivial, since 90 is not a multiple of 36. It should be remarked that the designs in both resolutions are simple. This is an interesting fact that we want to record in the following corollary.

**Corollary 5.3**

(i) There is a non-trivial 2-resolvable 4-(13, 6, 90) design with repeated blocks having \( N = 55 \) resolution classes, where each class is a simple 2-(13, 6, 15) design.

(ii) There is a non-trivial 3-resolvable 5-(14, 7, 90) design with repeated blocks having \( N = 55 \) resolution classes, where each class is a simple 3-(14, 7, 15) design.

Case (ii) with \( q = 5 \) displays another feature of Theorem 5.2. Assume that there is a partition of a 4-(31, 6, \( \lambda \)) outer design into projective planes of order 5. If \( \lambda = \lambda_{\text{max}} = 117 \times 3 \), the constructed design will have parameters 5-(32, 16, 16380 \times 39). Note that the index of this 5-design is much less than that of its corresponding complete 5-(32, 16, 334305 \times 39) design. By contrast, if \( \lambda = \lambda_{\text{min}} = 3 \), the index of the corresponding 5-(32, 16, \( \Lambda^* \)) constructed design would be drastically reduced to \( \Lambda^* = 140 \times 39 \). Further, since the 3-(6, 3, 1) inner design is 2-resolvable with \( L = 2 \) resolution classes, the number of 3-resolution classes of the constructed design is \( NL = \frac{2}{3}406 \).

For some small values of \( q \), for example \( q = 7, 9, 11 \), we may use the large sets \( LS_5(2, 4, 8) \), \( LS_{14}(2, 5, 10) \), \( LS_{42}(2, 6, 12) \) for the inner designs. Thus, if there would exist a partition of 4-(9, 11, \( \lambda \)) design into projective planes of order \( q = 7, 9, 11 \), then Theorems 5.2 would yield 3-resolvable 5-designs having parameters 5-(58, 29, \( \lambda_{63} \times 325 \)), 5-(92, 46, \( \lambda_{198} \times 903 \)), 5-(134, 67, \( \frac{2}{3}2002 \times 2016 \)) with \( NL = \lambda_{495}, \lambda_{1958}, \frac{2}{3}23842 \) resolution classes, respectively.

### 6 An infinite series of 3-resolvable 5-designs derived from the 5-(14, 7, 90) design

This short excursus we will focus on the 3-resolvable 5-(14, 7, 90) design in Corollary 5.3 and explain how to create an infinite series of 3-resolvable 5-designs from this single design. For the reader’s convenience we include here a result in a recent paper by the author [19].

**Corollary 6.1**

(Corollary 3.4 [19]) Suppose that there exists an \( s \)-resolvable \( t \)-(v, k, \( \lambda \)) design with \( N \) resolution classes such that \( z = \frac{\lambda}{(k-\lambda)} = \frac{Nu}{n} \), where \( u, n \) are positive integers. If there exists an \( LS[n](k - 2, k - 1, v - 1) \), then there exists an \( s \)-resolvable \( t \)-(v + m(v - k + 1), k, z \(^{(v-t+1)n(v-k+1)} \)) design with \( N \) resolution classes for any \( m \geq 0 \).

Observe the main fact of Corollary 6.1: it states that one can construct an infinite series of \( s \)-resolvable \( t \)-designs from a single \( t \)-design and a single large set. Now we will apply this recursive construction to the 5-(14, 7, 90) design in Corollary 5.3. As the design is 3-resolvable with 55 resolution classes, it is especially 3-resolvable with \( N = 5 \) resolution classes. The expression \( z = \frac{\lambda}{(k-\lambda)} = \frac{Nu}{n} \) becomes \( z = \frac{5}{2} \), which implies that \( n = 2 \). Further, since an \( LS[2](5, 6, 13) \) exists [9], there exists a 3-resolvable 5-(14 + 8m, 7, 10(9 +
design having $N = 5$ resolution classes for any $m \geq 0$ by Corollary 6.1. This design is obviously nonsimple, since the $5-(14 + 8m, 7, \lambda_{\text{max}})$ design will have $\lambda_{\text{max}} = 4(9 + 8m)(1 + m)$, however it is nontrivial, since $10(9 + 8m)(1 + m)$ is not a multiple of $\lambda_{\text{max}}$. We record the result in the following theorem.

**Theorem 6.2** There exists a 3-resolvable nonsimple and nontrivial $5-(14 + 8m, 7, 10(9 + 8m)(1 + m))$ design having $N = 5$ resolution classes for any $m \geq 0$.

Moreover, it should be noted that there are at least two non-isomorphic series of 3-resolvable $5-(14 + 8m, 7, 10(9 + 8m)(1 + m))$ designs in Theorem 6.2 due to the existence of two non-isomorphic large sets $L5[55][2, 4, 13]$ as proven by Kolotoğlu and Magliveras [8].

### 7 Conclusion

The paper presents a method for constructing 2-resolvable $t$-designs for $t = 3, 4$ based on the assumption that there exists a partition of a $t$-design into Steiner 2-designs. The case $t = 4$ corresponds to partitioning a 4-design into projective planes. Especially, if the order of the projective planes is odd, it also enables to construct 3-resolvable 5-designs with a largest possible block size. In general, the method appears to be very effective, as it yields infinitely many 2-resolvable 3-designs with a large variety of blocks sizes. A study of simplicity of the constructed designs remains a challenging problem.

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