Tunneling of Dirac particles from accelerating and rotating black holes

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ABSTRACT: Hawking radiation from black holes has been studied as a phenomenon of quantum tunneling of particles through their horizons. We have extended this approach to study the tunneling of Dirac particles from a large class of black holes which includes those with acceleration and rotation as well. We have calculated the tunneling probability of incoming and outgoing particles, and recovered the correct Hawking temperature by this method.
1. Introduction

Hawking’s arguments [1, 2] for thermal radiation and evaporation of black holes were based on quantum field theoretic analysis in curved spacetime. This breakthrough gave rise to a whole new field of investigation where three different areas, the quantum theory, general relativity and thermodynamics, come together. A major development in this direction was to interpret Hawking radiation as a quantum tunneling phenomenon [3-5]. Using this procedure the Hawking temperature was calculated from the tunneling probabilities of incoming and outgoing particles [6, 7]. This generated a lot of interest in studying radiations of different particles in the context of various black hole spacetimes. In this regard different semiclassical approaches were adopted to study radiation by scalar and Dirac particles. In particular, a procedure was set up and tunneling of charged and uncharged Dirac particles was studied for Kerr, Kerr-Newman, Taub-NUT and Gödel spacetimes [8-10]. This approach was applied to other four-dimensional [11-15] and three-dimensional [16, 17] spacetimes as well.

In this paper we study a large class [18-23] of Plebański and Demiański spacetimes for the tunneling procedure. We are basically motivated by two reasons. Firstly, the spacetime we study generalizes many black holes like the Schwarzschild, Kerr and Taub-NUT, and the tunneling probabilities and Hawking temperatures for these black holes can be recovered as special cases of our study. Secondly, the metric we study has very interesting interpretation in itself as accelerating and rotating black holes which have been studied in the literature for various properties. These black holes admit two rotation horizons and two acceleration horizons. In this paper we apply the procedure mentioned above to study tunneling of Dirac particles from these black holes. We calculate their tunneling probability and the Hawking temperature at these horizons.

2. Accelerating and rotating black holes

The metric for accelerating and rotating black holes without the cosmological constant can be written as [18, 19]

\[ ds^2 = \frac{1}{\Omega^2} \left\{ -(\frac{Q}{\rho^2} - \frac{a^2 P \sin^2 \theta}{\rho^2}) dt^2 + \frac{\rho^2}{Q} dr^2 + \frac{\rho^2}{P} d\theta^2 \right. \\
\left. + \left( \frac{P(r^2 + a^2) \sin^2 \theta}{\rho^2} - \frac{Q a^2 \sin^4 \theta}{\rho^2} \right) d\phi^2 - \frac{2a \sin^2 \theta (P(r^2 + a^2) - Q)}{\rho^2 \Omega^2} dt d\phi \right\} \tag{2.1} \]
where

\[ \Omega = 1 - \alpha r \cos \theta, \]  
\[ \rho^2 = r^2 + a^2 \cos^2 \theta, \]  
\[ P = 1 - 2 \alpha M \cos \theta + \alpha^2 a^2 \cos^2 \theta, \]  
\[ Q = (a^2 - 2 Mr + r^2)(1 - \alpha^2 r^2). \]  

Here the parameters \( M, a \) and \( \alpha \) represent the mass, angular momentum per unit mass, and acceleration of the source respectively. Following the notation of Ref. [9] the above metric can be written as

\[ ds^2 = -f(r, \theta)dt^2 + \frac{dr^2}{g(r, \theta)} + \Sigma(r, \theta)d\theta^2 + K(r, \theta)d\phi^2 - 2H(r, \theta)dtd\phi, \]  

where \( f(r, \theta), g(r, \theta), \Sigma(r, \theta), K(r, \theta), H(r, \theta) \) are defined below

\[ f(r, \theta) = \frac{1}{\Omega^2} \left\{ \frac{Q}{\rho^2} - \frac{a^2 P \sin^2 \theta}{\rho^2} \right\}, \]  
\[ g(r, \theta) = \frac{Q \Omega^2}{\rho^2}, \]  
\[ \Sigma(r, \theta) = \frac{\rho^2}{P \Omega^2}, \]  
\[ K(r, \theta) = \frac{P(r^2 + a^2) \sin^2 \theta}{\rho^2 \Omega^2} - \frac{Q a^2 \sin^4 \theta}{\rho^2 \Omega^2}, \]  
\[ H(r, \theta) = \frac{a \sin^2 \theta (P(r^2 + a^2) - Q)}{\rho^2 \Omega^2}. \]  

The event horizons of this black hole can be calculated by putting

\[ \frac{1}{g_{11}} = 0, \]  

which implies that

\[ g(r, \theta) = \frac{Q \Omega^2}{\rho^2} = 0, \]  

or

\[ \Omega = 0, Q = 0, \]  

which shows that in addition to the outer and inner horizons corresponding to the Kerr-Newman black hole horizons

\[ r_{\pm} = M \pm \sqrt{M^2 - a^2}, \]
we get two acceleration horizons \([19, 20]\): \( r = \frac{1}{\alpha} \) and \( r = \frac{1}{\alpha \cos \theta} \).

We also define the function \([9]\)

\[
F(r, \theta) = f(r, \theta) + \frac{H^2(r, \theta)}{K(r, \theta)}.
\]  

(2.16)

Putting the values of \( f(r, \theta) \), \( K(r, \theta) \), \( H(r, \theta) \) in this we get

\[
F(r, \theta) = \frac{QP \rho^2}{(P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta) \Omega^2}.
\]  

(2.17)

The angular velocity, for the above metric takes the form \([9]\)

\[
\Omega_H = \frac{H(r_+, \theta)}{K(r_+, \theta)}.
\]  

(2.18)

Substituting \( H(r_+, \theta) \), \( K(r_+, \theta) \) we get

\[
\Omega_H = \frac{a(P(r_+^2 + a^2) - Q(r_+)}{Q(r_+)a^2 \sin^2 \theta + P(r_+^2 + a^2)^2}.
\]  

(2.19)

Using the fact that \( Q(r_+) = 0 \), this can be written as

\[
\Omega_H = \frac{a}{r_+^2 + a^2}.
\]  

(2.20)

3. Tunneling of Dirac particles

In order to study the tunneling of Dirac particles we solve the Dirac equation in the background of accelerating and rotating black holes. The covariant Dirac equation can be written as

\[
i \gamma^\mu (D_\mu) \Psi + \frac{m}{\hbar} \Psi = 0,
\]  

(3.1)

where we have

\[
D_\mu = \partial_\mu + \Omega_\mu,
\]  

(3.2)

\[
\Omega_\mu = -\frac{1}{8} \Gamma^\alpha_\mu [\gamma^\alpha, \gamma^\beta],
\]  

(3.3)

and \([\gamma^\alpha, \gamma^\beta]\) satisfy the commutation relations

\[
[\gamma^\alpha, \gamma^\beta] = -[\gamma^\beta, \gamma^\alpha] \text{ if } \alpha \neq \beta, \quad [\gamma^\alpha, \gamma^\beta] = 0 \text{ if } \alpha = \beta.
\]  

(3.4)
Here $m$ is the mass of the fermions and $\gamma^\mu$ matrices satisfy $[\gamma^\alpha, \gamma^\beta] = 2 g^{\mu\nu} I$, ($I$ is the identity matrix). For fermion tunneling radiation, it is important to choose appropriate $\gamma^\mu$ matrices. We take

$$\gamma^t = \sqrt{\frac{P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta}{PQ\rho^2}} \gamma^0, \gamma^r = \sqrt{\frac{Q\Omega^2}{\rho^2}} \gamma^3, \gamma^\theta = \sqrt{\frac{P\Omega^2}{\rho^2}} \gamma^1,$$

$$\gamma^\phi = \frac{\rho\Omega\gamma^2}{\sin \theta \sqrt{P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta}} + \frac{a(P(r^2 + a^2) - Q)\gamma^0}{\sqrt{F(r, \theta)(P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta)}}. \tag{3.5}$$

Here

$$\gamma^0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \tag{3.7}$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}. \tag{3.8}$$

The $\sigma^i (i = 1, 2, 3)$ are the Pauli sigma matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{3.9}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.10}$$

In order to solve the Dirac equation we assume the following ansatz involving arbitrary functions of the coordinates corresponding to the spin-up and spin-down solutions of this equation

$$\Psi_\uparrow(t, r, \theta, \phi) = \begin{pmatrix} A(t, r, \theta, \phi) \xi_\uparrow \\ B(t, r, \theta, \phi) \xi_\uparrow \end{pmatrix} \exp\left[\frac{iI_\uparrow(t, r, \theta, \phi)}{\hbar}\right], \tag{3.11}$$

$$\Psi_\downarrow(t, r, \theta, \phi) = \begin{pmatrix} C(t, r, \theta, \phi) \xi_\downarrow \\ D(t, r, \theta, \phi) \xi_\downarrow \end{pmatrix} \exp\left[\frac{iI_\downarrow(t, r, \theta, \phi)}{\hbar}\right]. \tag{3.12}$$
Here $\xi^\uparrow$ and $\xi^\downarrow$ are the eigenvectors of $\sigma^3$, and $I^\uparrow$ and $I^\downarrow$ denote the actions of the emitted spin-up and spin-down particles, respectively. On using Eq. (3.4) the Dirac equation takes the form

$$(\nu \gamma^t \partial_t + \nu \gamma^r \partial_r + \nu \gamma^\theta \partial_\theta + \nu \gamma^\phi \partial_\phi)\Psi + \frac{m}{\hbar}\Psi = 0.$$  

(3.13)

Now, we substitute the above ansatz into the Dirac equation and compute it term by term. Dividing by the exponential term and neglecting the terms with $\hbar$ we obtain the following four equations

$$0 = -B[\frac{1}{\sqrt{F(r, \theta)}} \partial_t I^\uparrow + \sqrt{\frac{\Omega^2 Q}{\rho^2}} \partial_r I^\uparrow - a(P(r^2 + a^2) - Q) \frac{\partial_\phi I^\uparrow}{\sqrt{F(r, \theta)}(P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta)} + Am],$$  

(3.14)

$$0 = -B[\sqrt{\frac{\Omega^2 P}{\rho^2}} \partial_\theta I^\uparrow + \frac{\nu \rho \Omega}{\sin \theta \sqrt{P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta}} \partial_\phi I^\uparrow],$$  

(3.15)

$$0 = A[\frac{1}{\sqrt{F(r, \theta)}} \partial_t I^\uparrow - \sqrt{\frac{\Omega^2 Q}{\rho^2}} \partial_r I^\uparrow + a(P(r^2 + a^2) - Q) \frac{\partial_\phi I^\uparrow}{\sqrt{F(r, \theta)}(P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta)} + Bm],$$  

(3.16)

$$0 = A[\sqrt{\frac{\Omega^2 P}{\rho^2}} \partial_\theta I^\uparrow + \frac{\nu \rho \Omega}{\sin \theta \sqrt{P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta}} \partial_\phi I^\uparrow].$$  

(3.17)

Taking into account the symmetries of the spacetime we assume the action to be of the form

$$I^\uparrow = -Et + J\phi + W(r, \theta).$$  

(3.18)

Here $E$ and $J$ denote the energy and angular momentum of the emitted particle. We do calculations for the spin-up case; the spin-down case is similar. Inserting this into the above four equations, we get

$$0 = -B[\frac{-E}{\sqrt{F(r, \theta)}} + \sqrt{\frac{\Omega^2 Q}{\rho^2}} W'(r, \theta)]$$

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\[ 0 = B \left[ \sqrt{\frac{\Omega^2 P}{\rho^2}} W_0(r, \theta) + \frac{\nu \rho \Omega}{\sin \theta \sqrt{P(r^2 + a^2)^2 - Q a^2 \sin^2 \theta}} J \right], \tag{3.20} \]

\[ 0 = A \left[ \frac{-E}{\sqrt{F(r, \theta)}} - \sqrt{\frac{\Omega^2 Q}{\rho^2}} W'(r, \theta) \right. \]
\[ + \left. \frac{a(P(r^2 + a^2)^2 - Q)}{\sqrt{F(r, \theta)}(P(r^2 + a^2)^2 - Q a^2 \sin^2 \theta)} J \right] + Bm, \tag{3.21} \]

Expanding Eq. (2.8) in Taylor’s series and neglecting the higher powers we get

\[ g(r, \theta) = g(r_+, \theta) + (r - r_+)g_r(r_+, \theta). \tag{3.23} \]

Noting that at the horizon \( g(r_+, \theta) = 0 \), and its derivative at the horizon is

\[ g_r(r_+, \theta) = \frac{2(1 - \alpha r_+ \cos \theta)^2(r_+ - M)(1 - \alpha^2 r_+^2)}{r_+^2 + a^2 \cos^2 \theta}. \tag{3.24} \]

we obtain

\[ g(r, \theta) = (r - r_+) \frac{2(1 - \alpha r_+ \cos \theta)^2(r_+ - M)(1 - \alpha^2 r_+^2)}{r_+^2 + a^2 \cos^2 \theta}. \tag{3.25} \]

Similarly, noting that at the horizon \( F(r_+, \theta) = 0 \), from Eq. (2.17) we obtain

\[ F(r, \theta) = (r - r_+) \frac{2(r_+^2 + a^2 \cos^2 \theta)(r_+ - M)(1 - \alpha^2 r_+^2)}{(r_+^2 + a^2 \cos^2 \theta)^2 \Omega^2}. \tag{3.26} \]

Now expanding Eqs. (3.19) - (3.22) near the black hole horizon and using Eqs. (3.25) and (3.26) we get

\[ 0 = -B \left[ \frac{-E}{\sqrt{(r - r_+)^2 F(r, \theta)} + \sqrt{(r - r_+)g_r(r_+, \theta)W'(r, \theta)}} \right. \]
\[ + \left. \frac{a(P(r_+^2 + a^2)^2 - Q(r_+))}{\sqrt{(r - r_+)F(r_+, \theta)(P(r_+^2 + a^2)^2 - Q(r_+^2 a^2 \sin^2 \theta)}} J \right] + Am, \tag{3.27} \]
Substituting the values of $F$ in the massless case there exist two possible solutions. We neglect the equations that depend upon $\theta$ and $W$. At the horizon we can further separate (3.29)) which on using Eq. (2.19) take the form

$$0 = -B\left[\sqrt{\frac{\Omega^2(r_+, \theta)P}{r^2(r_+, \theta)}} W_\theta(r, \theta) + \frac{\mu(r_+, \theta)\Omega(r_+, \theta)}{\sin \theta \sqrt{P(r_+^2 + a^2)^2 - Q(r_+)a^2 \sin^2 \theta}} J\right], \quad (3.28)$$

$$0 = A\left[\frac{-E}{\sqrt{(r - r_+)F_r(r_+, \theta)}} - \sqrt{(r - r_+)g_r(r_+, \theta)W'(r, \theta)}\right] - \frac{a(P(r_+^2 + a^2) - Q(r_+))}{\sqrt{(r - r_+)F_r(r_+, \theta)}(P(r_+^2 + a^2)^2 - Q(r_+)a^2 \sin^2 \theta)} J + Bm, \quad (3.29)$$

$$0 = -A\left[\sqrt{\frac{\Omega^2(r_+, \theta)P}{r^2(r_+, \theta)}} W_\theta(r, \theta) + \frac{\mu(r_+, \theta)\Omega(r_+, \theta)}{\sin \theta \sqrt{P(r_+^2 + a^2)^2 - Q(r_+)a^2 \sin^2 \theta}} J\right]. \quad (3.30)$$

We neglect the equations that depend upon $\theta''$, as their contribution to the total tunneling rate is canceled out, and retain only the radial equations (Eqs. (3.27) and (3.29)) which on using Eq. (2.19) take the form

$$0 = -B\left[\frac{-E + \Omega_H J}{\sqrt{(r - r_+)F_r(r_+, \theta)}} + \sqrt{(r - r_+)g_r(r_+, \theta)W'(r, \theta)}\right] + Am, \quad (3.31)$$

$$0 = A\left[\frac{-E + \Omega_H J}{\sqrt{(r - r_+)F_r(r_+, \theta)}} - \sqrt{(r - r_+)g_r(r_+, \theta)W'(r, \theta)}\right] + Bm. \quad (3.32)$$

At the horizon we can further separate $W(r, \theta)$ as

$$W(r, \theta) = W(r) + \Theta(\theta) \quad (3.33)$$

In the massless case there exist two possible solutions

$$B = 0, W'(r) = W'_+(r) = \frac{(E - \Omega_H J)}{\sqrt{(r - r_+)F_r(r_+, \theta)}\sqrt{(r - r_+)g_r(r_+, \theta)}}, \quad (3.34)$$

and

$$A = 0, W'(r) = W'_-(r) = \frac{(-E + \Omega_H J)}{\sqrt{(r - r_+)F_r(r_+, \theta)}\sqrt{(r - r_+)g_r(r_+, \theta)}}. \quad (3.35)$$

Substituting the values of $F_r(r_+, \theta)$ and $g_r(r_+, \theta)$ the above equations become

$$W'_+(r) = \frac{(E - \Omega_H J)(r_+^2 + a^2)}{2(r - r_+)(r_+ - M)(1 - \alpha^2 r_+^2)}, \quad (3.36)$$

$$W'_-(r) = \frac{(-E + \Omega_H J)(r_+^2 + a^2)}{2(r - r_+)(r_+ - M)(1 - \alpha^2 r_+^2)}. \quad (3.37)$$
Here prime denotes the derivative with respect to $r$ and $+/-$ correspond to the outgoing/incoming solutions. For finding the value of $W(r)$ we integrate the above result

$$W_+(r) = \int \frac{(E - \Omega_H J)(r_+^2 + a^2)}{2(r - r_+)(r_+ - M)(1 - \alpha^2 r_+^2)}.$$ (3.38)

Integrating around the pole $r = r_+$ this gives

$$W_+(r) = \pi i \frac{(E - \Omega_H J)(r_+^2 + a^2)}{2(r_+ - M)(1 - \alpha^2 r_+^2)}. \tag{3.39}$$

Dropping the subscript we write

$$W(r) = \pi i \frac{(E - \Omega_H J)(r_+^2 + a^2)}{2(r_+ - M)(1 - \alpha^2 r_+^2)}, \tag{3.40}$$

$$Im W = \pi \frac{(E - \Omega_H J)(r_+^2 + a^2)}{2(r_+ - M)(1 - \alpha^2 r_+^2)}. \tag{3.41}$$

So the tunneling probabilities of fermions are

$$P_{\text{emission}} = \exp[-2Im I] = \exp[-2(Im W_+ + Im \Theta)], \tag{3.42}$$

$$P_{\text{absorption}} = \exp[-2Im I] = \exp[-2(Im W_- + Im \Theta)]. \tag{3.43}$$

Since $Im W_+ = -Im W_-$, we see that the total probability that the particle tunnels from inside the event horizon to outside is

$$\Gamma \sim \frac{P_{\text{emission}}}{P_{\text{absorption}}} = \exp[-4Im W_+], \tag{3.44}$$

or

$$\Gamma = \exp\left[-\frac{2\pi (E - \Omega_H J)(r_+^2 + a^2)}{(r_+ - M)(1 - \alpha^2 r_+^2)}\right]. \tag{3.45}$$

Comparing this with $\Gamma = \exp[-\beta E]$ where $\beta = 1/T_H$ we find that the Hawking temperature [6, 7] is given by

$$T_H = \frac{(r_+ - M)}{2\pi} \frac{(1 - \alpha^2 r_+^2)}{(r_+^2 + a^2)}, \tag{3.46}$$

where $r_+$ is given by Eq. (2.15). If we put acceleration equal to zero in formulae (3.45) and (3.46), they reduce to the tunneling probability and temperature of the Kerr black hole [8, 23]. Similarly, setting rotation equal to zero will recover expressions
for the Schwarzschild black hole. Comparing with the Kerr black hole, we note that the effect of acceleration is that it increases the temperature.

From Eqs. (3.42)-(3.45) it appears that for some value of \( E \) and \( J \) the probabilities may become larger than 1 and hence violate unitarity. However, this is not the case because apart from the spatial contribution there is also a contribution to the imaginary part \( \text{Im}(E \Delta t) \), from the temporal part of the action [24, 25]. This shifts the time by an imaginary amount which contributes both to \( P_{\text{emission}} \) and \( P_{\text{absorption}} \) and results in a correct value of \( \Gamma \). If we do not take this contribution into account, we will obtain the Hawking temperature twice as large as the actual value [26, 27].

Further, we note that \( P_{\text{absorption}} \) will actually be equal to 1, because the trajectories of incoming particles do not face any barrier. This is, in fact, taken care of if the temporal contribution is also taken into account [24, 25], and we obtain the correct value of the tunneling probability.

For the massive case Eqs. (3.27) and (3.29) do not decouple. We eliminate the function \( W' \) from these two equations by multiplying Eq. (3.29) by \( B \) and Eq. (3.27) by \( A \) and subtracting to yield

\[
\frac{A}{B} = -(E - J \Omega_H) \pm \frac{\sqrt{(E - J \Omega_H)^2 + m^2 F_r(r_+ \theta)(r - r_+)} - \sqrt{g_r(r_+ \theta)(r - r_+)W'(r)}}{m \sqrt{F_r(r_+ \theta)(r - r_+)}}. \tag{3.47}
\]

In the limit \( r \to r_+ \) the two roots give either \( \frac{A}{B} \to 0 \) or \( \frac{A}{B} \to -\infty \), i.e. either \( A \to 0 \) or \( B \to 0 \). For \( A \to 0 \) we find the value of \( m \) from Eq. (3.32)

\[
m = \frac{-A}{B} \left( \frac{-E + J \Omega_H}{\sqrt{F_r(r_+ \theta)(r - r_+)}} - \sqrt{g_r(r_+ \theta)(r - r_+)W'(r)}. \right) \tag{3.48}
\]

Putting in Eq. (3.31) and simplifying we get

\[
W_r(r, \theta) = W'_+(r) = \frac{(E - J \Omega_H)(1 + A^2/B^2)}{\sqrt{F_r(r_+ \theta)g_r(r_+ \theta)(r - r_+)(1 - A^2/B^2)}}. \tag{3.49}
\]

Integrating with respect to \( r \) as done before we finally get

\[
W(r) = \pi t \frac{(E - \Omega_H J)(r_+^2 + a^2)}{2(r_+ - M)(1 - \alpha^2 r_+^2)}, \tag{3.50}
\]

\[
\text{Im} W = \pi \frac{(E - \Omega_H J)(r_+^2 + a^2)}{2(r_+ - M)(1 - \alpha^2 r_+^2)}. \tag{3.51}
\]

For \( B \to 0 \) we simply get

\[
-10-
\]
\[ W_r(r, \theta) = W_\theta(r) = \frac{\pi}{2} \frac{-(E - \Omega_H J)(r_+^2 + a^2)}{(r_+^2 - M)(1 - \alpha^2 r_+^2)}. \]  

We note that we obtain the same tunneling probabilities as before, and hence the same temperature. This is because near the black hole horizon massive particles behave as massless.

4. The acceleration horizon

As mentioned earlier the black holes under consideration have an acceleration horizon at \( r_\alpha = 1/\alpha \) also, apart from the rotation horizons. The functions \( F_r(r, \theta) \) and \( g_r(r, \theta) \) in this case will be

\[
F_r(r_\alpha, \theta) = \frac{(r_\alpha^2 + a^2 \cos^2 \theta)(a^2 - 2Mr_\alpha + r_\alpha^2)(-2r_\alpha^2)}{(r_\alpha^2 + a^2)^2 \Omega^2(r_\alpha, \theta)}, \tag{4.1}
\]

\[
g_r(r_\alpha, \theta) = \frac{(1 - \cos \theta)^2(\alpha^2 a^2 - 2M\alpha + 1)(-2\alpha)}{(1 + \alpha^2 a^2 \cos^2 \theta)}, \tag{4.2}
\]

and from Eqs. (3.31) and (3.32) we see that the massless case gives rise to two possible solutions

\[
B = 0, W'(r) = W'_+(r) = \frac{-(-E + \Omega_H J)}{\sqrt{(r - r_\alpha)F_r(r_\alpha, \theta)\sqrt{(r - r_\alpha)g_r(r_\alpha, \theta)}}}, \tag{4.3}
\]

and

\[
A = 0, W'(r) = W'_-(r) = \frac{(-E + \Omega_H J)}{\sqrt{(r - r_\alpha)F_r(r_\alpha, \theta)\sqrt{(r - r_\alpha)g_r(r_\alpha, \theta)}}}, \tag{4.4}
\]

Putting values of the functions \( F_r(r_+, \theta) \) and \( g_r(r_+, \theta) \) in the above equations and integrating as before we obtain

\[
W_+(r) = \frac{\pi i (E - \Omega_H J)(1 + \alpha^2 a^2)}{2\alpha (\alpha^2 a^2 - 2M\alpha + 1)}. \tag{4.5}
\]

Similarly

\[
W_-(r) = \frac{-\pi i (E - \Omega_H J)(1 + \alpha^2 a^2)}{2\alpha (\alpha^2 a^2 - 2M\alpha + 1)}. \tag{4.6}
\]

Proceeding as before to find the tunneling probability, \( \Gamma = \exp[-\beta E] \), the resulting Hawking temperature at the acceleration horizon comes out to be

\[
T_H = \frac{\alpha (\alpha^2 a^2 - 2M\alpha + 1)}{2\pi (1 + \alpha^2 a^2)}. \tag{4.7}
\]
5. Calculation of the action

Here we will work out the action explicitly. We have already seen that \( r \)- and \( \theta \)-dependence decouples in Eqs. (3.27) - (3.30) near the black hole horizon. This allows us to separate the action as

\[
W(r, \theta) = R(r) + \Theta(\theta).
\]  
(5.1)

Using this separation in Eqs. (3.31) and (3.32) we get

\[
0 = -B\left(\frac{-E + \Omega_HJ}{\sqrt{(r-r_+)F_r(r_+, \theta)}}\right) + \sqrt{(r-r_+)g_r(r_+, \theta)R'(r)} + Am, \quad (5.2)
\]

\[
0 = A\left(\frac{E - \Omega_HJ}{\sqrt{(r-r_+)F_r(r_+, \theta)}}\right) - \sqrt{(r-r_+)g_r(r_+, \theta)R'(r)} + Bm. \quad (5.3)
\]

If \( m = 0 \), from Eq. (5.2) we get

\[
B = 0, R'(r) = R'_+(r) = \frac{E - \Omega_HJ}{\sqrt{(r-r_+)F_r(r_+, \theta)}} \sqrt{(r-r_+)g_r(r_+, \theta)}. \quad (5.4)
\]

Similarly from Eq. (5.3) we get

\[
A = 0, R'(r) = R'_-(r) = \frac{-E + \Omega_HJ}{\sqrt{(r-r_+)F_r(r_+, \theta)}} \sqrt{(r-r_+)g_r(r_+, \theta)}. \quad (5.5)
\]

Substituting the values of \( F_r(r_+, \theta) \) and \( g_r(r_+, \theta) \) in these equations gives

\[
R'_+(r) = \frac{(E - \Omega_HJ)(r_+^2 + a^2)}{2(r-r_+)(r_+ - M)(1 - \alpha^2r_+^2)}, \quad (5.6)
\]

\[
R'_-(r) = \frac{(E - \Omega_HJ)(r_+^2 + a^2)}{2(r-r_+)(r_+ - M)(1 - \alpha^2r_+^2)}, \quad (5.7)
\]

Integrating Eqs. (5.6) and (5.7) we get

\[
R_+(r) = \frac{(E - \Omega_HJ)(r_+^2 + a^2)}{2(r-r_+)(r_+ - M)(1 - \alpha^2r_+^2)} \ln(r - r_+). \quad (5.8)
\]

\[
R_-(r) = \frac{(E - \Omega_HJ)(r_+^2 + a^2)}{2(r-r_+)(r_+ - M)(1 - \alpha^2r_+^2)} \ln(r - r_+). \quad (5.9)
\]

For the massive case from Eq. (5.2) we get

\[
R'(r) = \frac{Am}{B \sqrt{(r-r_+)g_r(r_+, \theta)}} + \frac{E - \Omega_HJ}{(r-r_+) \sqrt{g_r(r_+, \theta)F_r(r_+, \theta)}}, \quad (5.10)
\]
where $A$ and $B$ are functions of $(t, r, \theta, \phi)$. After integrating it with respect to $r$ and using the values of $g_r(r_+, \theta)$ and $F_r(r_+, \theta)$ we get for the outgoing particle

$$R_+(r) = \int \frac{Am}{B \sqrt{(r-r_+)} g_r(r_+, \theta)} dr + \frac{(E - \Omega_H J)(r_+^2 + a^2)}{2(r_+ - M)(1 - \alpha^2 r_+^2)} \ln (r - r_+).$$  \hspace{1cm} (5.11)

Similarly from Eq. (5.3), for the incoming particle we get

$$R_-(r) = \int \frac{Bm}{A \sqrt{(r-r_+)} g_r(r_+, \theta)} dr - \frac{(E - \Omega_H J)(r_+^2 + a^2)}{2(r_+ - M)(1 - \alpha^2 r_+^2)} \ln (r - r_+).$$  \hspace{1cm} (5.12)

Now, we come to Eqs. (3.28) and (3.30) which take the form

$$0 = -B \sqrt{\Omega^2 P \rho^2 \Theta} + \frac{\iota p \Omega}{\sin \theta(\sqrt{P(r^2 + a^2)^2} - Qa^2 \sin^2 \theta)} J, \hspace{1cm} (5.13)$$

$$0 = -A \sqrt{\Omega^2 P \rho^2 \Theta} + \frac{\iota p \Omega}{\sin \theta(\sqrt{P(r^2 + a^2)^2} - Qa^2 \sin^2 \theta)^2} J. \hspace{1cm} (5.14)$$

Note that both the equations give

$$\Theta = \frac{-\iota p^2 J}{\sin \theta \sqrt{P(r^2 + a^2)^2} - Qa^2 \sin^2 \theta}, \hspace{1cm} (5.15)$$

or at horizon

$$\Theta = \frac{-\iota p^2 (r_+, \theta) J}{(r_+^2 + a^2) P \sin \theta}. \hspace{1cm} (5.16)$$

Substituting the values of $\rho$ and $P$, and integrating we get

$$\Theta(\theta) = \frac{-\iota J}{(r_+^2 + a^2)} \int \frac{(r_+^2 + a^2 \cos^2 \theta) d\theta}{\sin \theta[1 - 2\alpha M \cos \theta + \alpha^2 a^2 \cos^2 \theta]} \hspace{1cm} (5.17)$$

Using the method of partial fractions we evaluate the integral and finally get

$$\Theta(\theta) = L_1 \ln(1 + \cos \theta) - L_2 \ln(1 - \cos \theta) + L_3 \ln(1 - 2\alpha M \cos \theta + \alpha^2 a^2 \cos^2 \theta)$$

$$+ L_4 \ln\left(\frac{\alpha a^2 \cos \theta - M - \sqrt{M^2 - a^2}}{\alpha a^2 \cos \theta - M + \sqrt{M^2 - a^2}}\right) - L_5 \ln\left(\frac{\alpha a^2 \cos \theta - M - \sqrt{M^2 - a^2}}{\alpha a^2 \cos \theta - M + \sqrt{M^2 - a^2}}\right).$$

where $L_i$ are
\begin{align*}
L_1 &= \frac{iJ}{2(1 + 2\alpha M + \alpha^2 a^2)}, \\
L_2 &= \frac{iJ}{2(1 - 2\alpha M + \alpha^2 a^2)}, \\
L_3 &= \frac{iJ\alpha M}{(1 - 2\alpha M + \alpha^2 a^2)(1 + 2\alpha M + \alpha^2 a^2)}, \\
L_4 &= \frac{iJ\alpha^2 [a^2 (1 + \alpha^2 a^2) - 2M^2]}{2\alpha \sqrt{M^2 - a^2(1 - 2\alpha M + \alpha^2 a^2)(1 + 2\alpha M + \alpha^2 a^2)}}, \\
L_5 &= \frac{iJa^2}{2\alpha \sqrt{M^2 - a^2 (r_+^2 + a^2)}}.
\end{align*}

This determines the action completely.

6. Conclusion

For studying Hawking radiations from black holes different types of approaches have been adopted in the literature. This has been done by using the Newman-Penrose formalism and the so-called Hamilton-Jacobi method. In particular, tunneling of Dirac particles has been studied for the Kerr and Schwarzschild black holes. We have extended this semi-classical approach to study a large class of black holes which include those with acceleration and rotation. We obtained the tunneling probability for the incoming and outgoing particles and correctly recover the Hawking temperature. We have explicitly calculated the action as well. An indication of the generality of our results is that, in appropriate limits, they reduce to those for the Kerr and Schwarzschild black holes.

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