Cubic interactions of arbitrary spin fields 
in 3d flat space

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Abstract  
Using light-cone gauge formulation, massive arbitrary spin irreducible fields  
and massless (scalar and spin one-half) fields in three-dimensional flat space  
are considered. Both the integer spin and half-integer spin fields are studied.  
For such fields, we provide classification for cubic interactions and obtain  
explicit expressions for all cubic interaction vertices. We study two forms of  
the cubic interaction vertices which we refer to as first-derivative form and  
higher-derivative form. All cubic interaction vertices are built by using the  
first-derivative form.  

Keywords: massive higher-spin fields, light-cone gauge formalism, cubic interaction vertices.

1. Introduction

Unitary arbitrary integer and half-integer spin irreducible representations of Poincaré algebra  
in three dimensions are associated with the respective bosonic and fermionic massive fields  
propagating in the \( R^{2,1} \) space. Lagrangian description of free bosonic and fermionic arbitrary  
spin massive irreducible fields propagating in the \( R^{2,1} \) space was obtained long ago in reference  
[1]. We recall that, in three dimensions, massless fields with spin equal or greater than \( \frac{3}{2} \) do  
not propagate and these fields are not associated with unitary representations of the Poincaré  
algbera \( iso(2, 1) \). Namely, for massless fields in three dimensions, only scalar, vector and spin  
one-half fields propagate and only these fields are associated with unitary irreps of the Poincaré  
algbera \( iso(2, 1) \). In this paper, we are interested in cubic interactions only for those fields in  
\( R^{2,1} \) which are associated with unitary irreps of the Poincaré algebra \( iso(2, 1) \). Also, we recall,  
that, in light-cone gauge approach, a massless vector field in 3d is treated as a massless scalar field. In view of above-said, we deal with arbitrary integer and half-integer spin massive fields

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and low spin (scalar and spin one-half) massless fields. Our aim is to construct all cubic interaction vertices for such fields. In references [9, 10], for light-cone gauge bosonic and fermionic arbitrary spin massive and massless fields propagating in $R^{d-1,1}$, we studied cubic interactions in flat space for the case of $d \geq 4$. This is to say that, in this paper, we are going to extend results in references [9, 10] to the case of light-cone gauge bosonic and fermionic arbitrary spin massive fields and low spin (scalar and spin one-half) massless fields propagating in $R^{2,1}$ space.

Before proceeding to the main theme in this paper we briefly mention our two long term motivations for our study of massive fields in 3d space. First, in view of simplicity of cubic vertices for light-cone gauge massive fields in $R^{2,1}$ obtained in this paper, we believe that our results may be helpful in the search of yet unknown interesting models of higher-spin massive fields in $R^{2,1}$ and their counterparts in higher dimensions. Second, one expects that arbitrary spin $AdS$ massive fields and low spin massless $AdS$ fields form spectrum of states of $AdS$ superstring. We think then that our results for cubic vertices of light-cone gauge massive fields in $R^{2,1}$ may serve as a good starting point for the study of cubic vertices of light-cone gauge massive fields in $AdS_3$ and hence may find applications in study of superstring in $AdS_3$. Other long term motivations for our study in this paper may be found in conclusions.

This paper is organized as follows.

In section 2, we review the well known light-cone gauge description of arbitrary spin massive bosonic and fermionic and low spin (scalar and half-integer) massless fields propagating in $R^{2,1}$. In section 3, we describe restrictions on $n$-point interaction vertices imposed by kinematical symmetries of the Poincaré algebra $iso(2, 1)$. Section 4 is devoted to equations for cubic vertices. We start with the presentation of restrictions imposed by kinematical and dynamical symmetries of the Poincaré algebra on cubic interaction vertices. After that, we discuss light-cone gauge dynamical principle. Finally we present our complete system of equations which allows us to determine the cubic vertices unambiguously. In section 5, we present our method for solving the complete system of equations for cubic vertices. In section 6, we consider cubic interactions for bosonic arbitrary spin massive fields and massless scalar fields. Using our method, we present all solutions for complete system of equations for cubic vertices obtained in section 4. Section 7 is devoted to Fermi–Bose interactions for two fermionic and one bosonic fields. For such interactions, we present all solutions for cubic vertices. In section 8, we present our conclusions. In appendix A, we outline some technical details of the derivation of cubic vertices. In appendix B, we show how some our vertices can be obtained from vertices of massless fields in $R^{1,1}$ by using dimensional reduction. Also a proposal is made for massive higher-spin theories in $R^{2,1}$.

2. Light-cone gauge formulation of free massive and massless fields in 3d flat space

Poincaré algebra $iso(2, 1)$ in light-cone frame. In reference [16], it has been noted that the problem of finding a light-cone gauge dynamical system amounts to a problem of finding a

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1 For study of various aspects of massless higher-spin dynamics in $R^{2,1}$ and $AdS_3$, spaces, the reader can consult the incomplete list of references [2–8].

2 Lorentz covariant formulation for all light-cone gauge cubic vertices of massless fields in reference [9] was obtained in references [11, 12]. BRST-BV formulation for all light-cone gauge cubic vertices of massive and massless fields in reference [9] was obtained in reference [12]. Recent discussion of this theme and extensive list of references may be found in reference [13].

3 Light-cone gauge superstring action in $AdS_3$ was considered in reference [14]. Interesting use of light-cone gauge approach for studying 3-point function of $AdS$ superstring may be found in reference [15].
light-cone gauge solution for commutators of a space-time symmetry algebra. For theories of
fields propagating in the $R^{2,1}$ space, the space-time symmetries are associated with the Poincaré
algebra $iso(2,1)$. Therefore our aim in this section is to review a realization of the Poincaré
algebra $iso(2,1)$ in the light-cone frame.

In the three-dimensional flat space $R^{2,1}$, the Poincaré algebra $iso(2,1)$ is spanned by the
three translation generators $P^\mu$, the three generators of the $so(2,1)$ Lorentz algebra denoted as
$J^\mu\nu$. We use the following form of commutators of the Poincaré algebra $iso(2,1)$:

\[
[P^\mu, J^{\nu\rho}] = \eta^{\mu\nu} P^\rho - \eta^{\mu\rho} P^\nu, \quad [J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\mu\rho} J^{\nu\sigma} + 3 \text{ terms,}
\]

where $\eta^{\mu\nu}$ stands for the mostly positive Minkowski metric. Starting with the Lorentz
basis coordinates $x^\mu$, $\mu = 0, 1, 2$, we introduce the light-cone basis coordinates $x^\pm$, where
the coordinates $x^\pm$ are defined as

\[
x^\pm \equiv \frac{1}{\sqrt{2}}(x^2 \pm x^0).
\]

From now on, the coordinate $x^+$ is considered as an light-cone time. In the frame of the light-
cone coordinates, a $so(2,1)$ Lorentz algebra vector $X^\mu$ is decomposed as $X^+$, $X^-$, $X^1$. Also we
note that a scalar product of two $so(2,1)$ Lorentz algebra vectors $X^\mu$ and $Y^\nu$ can be decomposed
as

\[
\eta_{\mu\nu} X^\mu Y^\nu = X^+ Y^- + X^- Y^+ + X^1 Y^1.
\]

Relation (2.3) tells us, that in the frame of the light-cone frame coordinates, non-vanishing com-
ponents of the flat metric are given by $\eta_{\pm\pm} = 1$, $\eta_{1\pm} = 1$. Therefore for the covariant
and contravariant components of vector $X^\mu$ we get the relations $X^+ = X_-$, $X^- = X_+$, $X^1 = X_1$.

In light-cone frame, generators of the Poincaré algebra $iso(2,1)$ are separated into the
following two groups:

\[
P^+, \quad P^1, \quad J^{+1}, \quad J^{+-}, \quad \text{kinematical generators;}
\]

\[
P^-, \quad J^{-1}, \quad \text{dynamical generators.}
\]

Note that, in a field-theoretical realization, the kinematical generators (2.4) are quadratic in
fields for $x^+ = 0$, while, the dynamical generators (2.5) consist of quadratic and higher order
terms in fields$^4$.

In order to get commutators of the Poincaré algebra in light-cone frame, we use commutators
in (2.1) and the flat metric $\eta^{\mu\nu}$ which has non-vanishing components given by $\eta^+_- = \eta^+ + = 1,
\eta_{11} = 1$. Hermitian conjugation rules for the generators are assumed to be as follows

\[
P^{\pm\dagger} = P^\pm, \quad P^{1\dagger} = P^1, \quad J^{+-\dagger} = -J^{+-}, \quad J^{\pm1\dagger} = -J^{\pm1}.
\]

We are going to use a field-theoretical realization of generators of the Poincaré algebra on
space of light-cone gauge fields. Therefore we now proceed with a review of light-cone gauge
description of arbitrary spin massive and low spin massless fields.

**Light-cone gauge massive and massless fields.** To study field theories in three dimensions
we use light-cone gauge massive and massless fields which we denote as $\phi_{m,\lambda}(x)$, where an

$^4$Namely, with the exception of $J^{+-}$, all generators (2.4) are quadratic in fields when $x^+ \neq 0$, while the $J^{+-}$ takes the
form $J^{+-} = G_0 + i x^+ P^-$, where $G_0$ is quadratic in fields.
argument \( x \) stands for space time-coordinates \( x^+, x^-, x^1 \), while labels \( m \) and \( \lambda \) denote the respective mass and spin of the field \( \phi_{m,\lambda}(x) \). We note that

\[
\phi_{m,\lambda}(x) \text{ are bosonic fields for } \lambda \in \mathbb{N}_0, \\
\phi_{m,\lambda}(x) \text{ are fermionic fields for } \lambda \in \mathbb{N}_0 + \frac{1}{2},
\]

By definition, all fields given in (2.7) and (2.8), are real-valued

\[
\phi_{m,\lambda}^\dagger(x) = \phi_{m,\lambda}(x).
\]

Using a shortcut \((m,\lambda)\) for the fields \( \phi_{m,\lambda} \) in (2.7) and (2.8), we restrict our study in this paper to the following cases of \( m \) and \( \lambda \)

\[
(m,s), \quad m \neq 0, s \in \mathbb{N}, \quad \text{spin} - s \text{ massive bosonic field};
\]

\[
(m,0), \quad m \neq 0 \quad \text{massive scalar bosonic field};
\]

\[
(0,0), \quad \massless scalar bosonic field;
\]

\[
(m,s + \frac{1}{2}), \quad m \neq 0, s \in \mathbb{N}, \quad \text{spin} - (s + \frac{1}{2}) \text{ massive fermionic field};
\]

\[
(m,\frac{1}{2}), \quad m \neq 0, \quad \massive spin - \frac{1}{2} \text{ fermionic field};
\]

\[
(0,\frac{1}{2}), \quad \massless spin - \frac{1}{2} \text{ fermionic field}.
\]

For fields in (2.10), (2.13) and (2.14), the mass parameter \( m \) is allowed to be positive or negative. Fields in (2.10) and (2.13) with \( m > 0 \) and \( m < 0 \) are referred to as self-dual and anti-self-dual massive fields respectively. The self-dual massive field \((|m|,\lambda)\) and anti-self-dual massive field \((-|m|,\lambda)\) are associated with different irreducible irreps of the Poincaré algebra iso(2,1).

In place of the fields depending on space-time coordinates (2.7) and (2.8), we prefer to deal with the fields obtainable by using the Fourier transform with respect to the spatial coordinates \( x^- \), \( x^1 \),

\[
\phi_{m,\lambda}(x) = \int \frac{d^2\vec{p}}{2\pi} e^{i(3x^- + mx^1)} \phi_{m,\lambda}(x^+ \cdot \vec{p}), \quad d^2\vec{p} \equiv dp\,d\beta,
\]

where the argument \( \vec{p} \) in \( \phi_{m,\lambda}(x^+, \vec{p}) \), is used to indicate the momenta \( \beta, p \). In terms of the field \( \phi_{m,\lambda}(x^+, \vec{p}) \), the hermicity condition shown in (2.9) takes the following form:

\[
\phi_{m,\lambda}^\dagger(\vec{p}) = \phi_{m,\lambda}(-\vec{p}).
\]

Note, that, in (2.16) and below, dependence of the momentum-space fields on the light-cone time \( x^+ \) is implicit.

**Realization of the Poincaré algebra on fields.** We now ready to present the well known field-theoretical realization of the Poincaré algebra on the space of massive and massless fields in three dimensions. This is to say that a realization of the Poincaré algebra (2.1) in terms of differential operators acting on the momentum-space fields \( \phi_{m,\lambda}(\vec{p}) \) (2.16) is given by the following relations:

\[
P_1 = p, \quad P^+ = \beta, \quad P^- = p^-, \quad p^- \equiv -\frac{p^2 + m^2}{2\beta},
\]

where

\[
\phi_{m,\lambda}(x) = \int \frac{d^2\vec{p}}{2\pi} e^{i(3x^- + mx^1)} \phi_{m,\lambda}(x^+ \cdot \vec{p}), \quad d^2\vec{p} \equiv dp\,d\beta,
\]

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\[
P_1 = p, \quad P^+ = \beta, \quad P^- = p^-, \quad p^- \equiv -\frac{p^2 + m^2}{2\beta},
\]
\[ J^+ = i x^+ P^1 + \partial_\beta \beta, \quad J^- = i x^+ P^- + \partial_\beta \beta - \frac{1}{2} e_\lambda, \] (2.19)

\[ J^1 = -\partial_\beta p + \partial_\beta p^- + \frac{1}{\beta} M + \frac{p}{2 \beta} e_\lambda, \quad M = i m \lambda, \] (2.20)

\[ \partial_\beta \equiv \partial / \partial \beta, \quad \partial_\beta \equiv \partial / \partial p, \] (2.21)

\[ e_\lambda = 0 \quad \text{for integer } \lambda, \quad e_\lambda = 1 \quad \text{for half-integer } \lambda. \] (2.22)

We now note that, to quadratic order in the fields \( \phi_{m,\lambda}(\vec{p}) \), the field-theoretical realization of the Poincaré algebra generators (2.1) takes the form

\[ G_{[2]} = \int d^2 \vec{p} \beta^\lambda \beta^\lambda \phi_{m,\lambda}^\dagger G_{\text{diff}} \phi_{m,\lambda}, \] (2.23)

where \( G_{\text{diff}} \) stands for the differential operators given in (2.18)–(2.20), while the notation \( G_{[2]} \) (2.23) is used for the field-theoretical representation for the generators of the Poincaré algebra (2.1).

By definition, the fields \( \phi_{m,\lambda} \) given in (2.7) and (2.8) satisfy the respective Poisson–Dirac equal-time commutation and anti-commutation relations

\[ [\phi_{m,\lambda}(\vec{p}), \phi_{m,\lambda}(\vec{p}')] = \frac{1}{2} \delta^2(\vec{p} + \vec{p}') \delta_{\lambda\lambda'}, \quad \text{for } \lambda \in \mathbb{N}_0, \] (2.24)

\[ \{\phi_{m,\lambda}(\vec{p}), \phi_{m,\lambda}(\vec{p}')\} = \frac{1}{2} \delta^2(\vec{p} + \vec{p}') \delta_{\lambda\lambda'}, \quad \text{for } \lambda \in \mathbb{N}_0 + \frac{1}{2}. \] (2.25)

Taking into account relations above-given it is easy to check the following standard equal-time commutation relations between the fields and the Poincaré algebra generators

\[ [\phi_{m,\lambda}, G_{[2]}] = G_{\text{diff}} \phi_{m,\lambda}. \] (2.26)

3. General structure of \( n \)-point vertices

In this section, we describe restrictions imposed on interacting vertices by the kinematical symmetries of the Poincaré algebra \( \text{iso}(2,1) \).

For interacting fields, dynamical generators the Poincaré algebra receive corrections having higher powers of fields. Namely, the dynamical generators \( G_{\text{dyn}} = P^-, J^- \) can be presented as

\[ G_{\text{dyn}} = \sum_{n=2}^{\infty} G_{[n]}^{\text{dyn}}, \] (3.1)

where \( G_{[n]}^{\text{dyn}} \) stands for a functional that has \( n \) powers of bosonic and fermionic fields. For \( n \geq 3 \), we are going to describe restrictions imposed on the dynamical generators \( G_{[n]}^{\text{dyn}} \) obtained from commutators between \( G_{[n]}^{\text{dyn}} \) and the kinematical generators. Let us discuss the restrictions in turn.

Kinematical \( P^1, P^+ \) symmetries. Using commutators between the dynamical generators \( P^-, J^- \) and the kinematical generators \( P^1, P^+ \), we find that, for \( n \geq 3 \), the dynamical generators \( P_{[n]}^- \) and \( J_{[n]}^- \) can be presented as

\[ P_{[n]}^- = \int d\Gamma_{[n]} \Phi_{[n]} P_{[n]}^-, \] (3.2)
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Now turn to studying cubic vertices. Our plan in this section is as follows. First, we represent the generator \( J^{-1} \) in terms of the momenta \( p_a \) and \( \beta_a \). Throughout this paper, the density \( p_{[a]} \) is referred to as an \( n \)-point interaction vertex. For \( n = 3 \), the density \( p_{[3]} \) is referred to as cubic interaction vertex.

**Kinematical \( J^+ \)-symmetry equations.** Commutators between the generators \( P^- \), \( J^{-1} \) and the generator \( J^+ \) lead to following equations for the densities:

\[
\sum_{a=1}^{n} \left( \beta_a \partial_{\beta_a} + \frac{1}{2} \xi_{\beta_a} \right) p_{[a]} = 0, \quad \sum_{a=1}^{n} \left( \beta_a \partial_{\beta_a} + \frac{1}{2} \xi_{\beta_a} \right) j_{[a]}^{-1} = 0. \tag{3.6}
\]

**Kinematical \( J^{1+} \)-symmetry equations.** Commutators between the generators \( P^- \), \( J^{-1} \) and the generator \( J^{1+} \) tell us that the densities \( p_{[a]} \), \( j_{[a]}^{-1} \) depend on the momenta \( p_a \) through new momentum variables \( P_{ab} \),

\[
p_{[a]} = p_{[a]}(P_{ab}, \beta_a), \quad j_{[a]}^{-1} = j_{[a]}^{-1}(P_{ab}, \beta_a), \tag{3.7}
\]

\[
P_{ab} \equiv p_a \beta_b - p_b \beta_a. \tag{3.8}
\]

Relations given in (3.2)–(3.7) provide a complete list restrictions imposed by the kinematical symmetries on the \( n \)-point dynamical generators \( P_{[a]}, J_{[a]}^{-1} \). Below we apply the general discussion above-presented for studying cubic vertices which correspond to the case \( n = 3 \).

**4. Complete system of equations for cubic vertices**

We now turn to studying cubic vertices. Our plan in this section is as follows. First, we represent \( J^{+} \)-symmetry equations given in (3.6) in terms of the momenta \( P_{ab} \) defined in (3.8). Second, we consider restrictions imposed by dynamical symmetries. Third, we formulate our dynamical principle which we refer to as light-cone gauge dynamical principle. Finally, we present the complete system equations which allow us to determine the cubic vertices unambiguously.

**Kinematical symmetries of the cubic densities.** Using the momentum conservation laws

\[
p_1 + p_2 + p_3 = 0, \quad \beta_1 + \beta_2 + \beta_3 = 0, \tag{4.1}
\]

it is easy to check that \( P_{12}, P_{23}, P_{31} \) are expressed in terms of a new momentum \( P \)

\[
P_{12} = P_{23} = P_{31} = P, \tag{4.2}
\]

\[
P \equiv \frac{1}{3} \sum_{a=1,2,3} \beta_a p_a, \quad \beta_a \equiv \beta_{a+1} - \beta_{a+2}, \quad \beta_a \equiv \beta_{a+3}. \tag{4.3}
\]
We now see that our cubic densities \( p_{[3]} \), \( j_{[3]}^{-1} \) are functions of the momenta \( P, \beta_1, \beta_2, \beta_3 \).

\[
p_{[3]} = p_{[3]}(P, \beta_a), \quad j_{[3]}^{-1} = j_{[3]}^{-1}(P, \beta_a).
\]

(4.4)

Thus, the three momenta \( p_a \) enter cubic densities (4.4) through the one momentum \( P \). We note then that it is this feature of the cubic densities that simplifies the study of cubic interactions.

Let us now represent equation (3.6) in terms of the cubic densities given in (4.4).

\[
J^+ = \sum_{a=1,2,3} \left( \beta_a \partial_{\beta_a} + \frac{1}{2} \epsilon_{\lambda} \right).
\]

(4.5)

We now turn to studying restrictions imposed by the dynamical symmetries.

**Dynamical symmetries of the cubic densities.** Restrictions on the interaction vertices obtained from commutators between the dynamical generators of the Poincaré algebra are referred to as dynamical symmetry restrictions in this paper. In our case all that is required is to consider commutator between dynamical generators given in (2.5), \([P^-, J^{-1}_3] = 0\). In the cubic approximation, this commutator leads to the relation

\[
[P_{[2]}, J_{[3]}^{-1}] = [P_{[3]}, J_{[3]}^{-1}] = 0.
\]

(4.7)

Equation (4.7) tells us that the density \( j_{[3]}^{-1} \) can be expressed in terms of the cubic vertex \( p_{[3]} \),

\[
j_{[3]}^{-1} = -(P^+)^{-1} J^{-1}_3 p_{[3]},
\]

(4.8)

where operators \( P^+ \), \( J^{-1}_3 \) are defined by the following relations

\[
P^+ = \frac{p^2}{2\beta} - \sum_{a=1,2,3} \frac{m^2}{2\beta_a}, \quad \beta \equiv \beta_1 \beta_2 \beta_3,
\]

(4.9)

\[
J^{-1}_3 = -\frac{P}{\beta} N_{[3]} - \mathcal{M} + \sum_{a=1,2,3} \frac{\beta_a}{6\beta_a} m^2 \partial_p, \quad N_{[3]} \equiv N_{[\beta]} + \frac{1}{2} \epsilon_{\lambda},
\]

(4.10)

\[
N_{[\beta]} \equiv \frac{1}{3} \sum_{a=1,2,3} \beta_a \partial_{\beta_a}, \quad \epsilon_{\lambda} \equiv \frac{1}{3} \sum_{a=1,2,3} \beta_a \epsilon_{\lambda a}, \quad \mathcal{M} \equiv \sum_{a=1,2,3} \frac{1}{\beta_a} M_a.
\]

(4.11)

Equation (4.8) exhausts restrictions imposed by the dynamical symmetries of the Poincaré algebra on the cubic densities \( p_{[3]} \) and \( j_{[3]}^{-1} \). Kinematical and dynamical symmetry equations given in (4.5) and (4.8) respectively provide complete list of restrictions imposed by Poincaré algebra.

**Light-cone gauge dynamical principle.** Poincaré algebra restrictions given in (4.5), (4.8) do not allow us to fix the cubic vertex \( p_{[3]} \) unambiguously. To fix the cubic vertex \( p_{[3]} \) unambiguously we impose additional restrictions on the densities \( p_{[3]} \) and \( j_{[3]}^{-1} \) which we refer to as light-cone gauge dynamical principle. We formulate the light-cone gauge dynamical principle as follows:
i) The cubic densities $p^{(3)}_i, j^{(3)}_i$ should be polynomial in the momentum $P$ and $\beta$-analytic; 

ii) The cubic vertex $p^{(3)}_i$ should satisfy the following restriction

$$p^{(3)}_i \neq P^* W, \quad W \text{ is polynomial in } P \text{ and } \beta \text{ - analytic}, \quad (4.12)$$

where $P^*$ is given in (4.9). Let us explain restriction (4.12). Upon field redefinitions, the cubic vertex $p^{(3)}_i$ is changed by terms proportional to the quantity $P^*$ (4.9) (for discussion, see, e.g., appendix B in reference [9]). Therefore, ignoring restriction in (4.12) implies that the cubic vertices can be removed by field redefinitions. However our primary interest are the cubic vertices that cannot be removed by field redefinitions. For this reason, we impose the requirement (4.12). We now summarize our discussion of equations and restrictions for the cubic densities.

Complete system of equations for cubic interaction vertex. For the cubic vertex given by

$$p^{(3)}_i = p^{(3)}_i(P, \beta_a) \quad (4.13)$$

the complete system of equations is given by

Poincaré algebra kinematical and dynamical restrictions:

$$J^{+ -} p^{(3)}_i = 0, \quad \text{kinematical } J^{+ -} \text{ symmetry}; \quad (4.14)$$

$$j^{(3)}_i = -(P^*)^{-1} J^{-1} p^{(3)}_i, \quad \text{dynamical } P^-, J^{-1} \text{ symmetries}; \quad (4.15)$$

Light–cone gauge dynamical principle:

$$p^{(3)}_i, j^{(3)}_i \text{ are polynomial in } P \text{ and } \beta \text{ - analytic}; \quad (4.16)$$

$$p^{(3)}_i \neq P^* W, \quad W \text{ is polynomial in } P \text{ and } \beta \text{ - analytic}, \quad (4.17)$$

where operator $J^{+ -}$ is given in (4.6), while the operators $P^*$, $J^{-1}$ are defined in (4.9)–(4.11).

Equations presented in (4.14)–(4.17) constitute the complete system of equations which allow us to find all cubic interaction vertices $p^{(3)}_i$ and the corresponding densities $j^{(3)}_i$ unambiguously.

5. Method for solving complete system of equations for cubic vertices

The most difficult point in the analysis of the equations (4.14)–(4.17) is related to the fact that, in general, the cubic vertex is some complicated polynomial in the momentum $P$. We note however, in light-cone gauge approach in three dimensions, by using field redefinitions, any cubic vertex can be cast into a polynomial of degree-1 in $P$ (see below). Representation for cubic vertex in terms of degree-1 polynomials in $P$ will be referred to as first-derivative form of cubic vertex. Thus our first-derivative vertices are degree-1 polynomials in $P$. In this paper, we find first-derivative form for all cubic vertices. Besides this, for some wide class of first-derivative cubic vertices, we discuss the procedures which allow us, by using field redefinitions, to generate other particular representation for cubic vertices which we refer to as

$\beta$-analytic. If function $f = f(\beta_1, \beta_2, \beta_3)$ takes the form $f = g/h$ where $g, h$ are polynomials in $\beta_1, \beta_2, \beta_3$, then we say that $f$ is $\beta$-analytic. If $f = g/h$, where $g, h, u$ are polynomials in $\beta_1, \beta_2, \beta_3$, then we say that $f$ is $\beta$-nonanalytic. Consider Taylor series expansion $p^{(3)}_i = \sum_{n=0}^N f_n P^n$. If all $f_n$ are $\beta$-analytic, then we say that $p^{(3)}_i$ is $\beta$-analytic. If some $f_n$ are $\beta$-nonanalytic, then we say that $p^{(3)}_i$ is $\beta$-nonanalytic.
higher-derivative form of cubic vertices\(^6\). We start with discussion of first-derivative form for the cubic vertex.

**First-derivative form of cubic vertex.** We find that any cubic vertex \( p^{[3]} \), which involves at least one massive field, can be presented as

\[
p^{[3]} = \frac{1}{2} \left( 1 + \frac{iP}{\kappa} \right) V + \frac{1}{2} \left( 1 - \frac{iP}{\kappa} \right) \bar{V},
\]

where two new vertices \( V, \bar{V} \) do not depend on the momentum \( P \). These two new vertices depend only on the momenta \( \beta_1, \beta_2, \beta_3 \) and satisfy the following decoupled (diagonalized) equations

\[
\kappa \beta \eta V + i \lambda V = 0, \quad \kappa \beta \eta \bar{V} - i \lambda \bar{V} = 0,
\]

where we use notation as in (4.10) and (4.11). Thus, the complete system of equations is reduced to analysis of decoupled equations for the vertices \( V, \bar{V} \) (5.3) and (5.4). Expression for \( j^{-1}_{[3]} \) corresponding to cubic vertex (5.1) is given by

\[
j^{-1}_{[3]} = 2i \kappa \eta \beta V_1, \quad V_1 \equiv \frac{1}{2\kappa} (V - \bar{V}).
\]

Equations (5.3) and (5.4) fix vertices \( V, \bar{V} \) uniquely (up to two coupling constants). If \( \kappa \) (5.2) is \( \beta \)-analytic, then both vertices \( V, \bar{V} \) are also \( \beta \)-analytic, and, in view of (5.1), we obtain two vertices \( p^{[3]} \) (5.1). If \( \kappa \) (5.2) is \( \beta \)-nonanalytic, then both vertices \( V, \bar{V} \) are also \( \beta \)-nonanalytic. For this case, requiring the \( p^{[3]} \) (5.1) to be \( \beta \)-analytic, we get one restriction on two coupling constants. Solving this restriction, we are left with one \( \beta \)-analytic vertex \( p^{[3]} \). Thus, if \( \kappa \) is \( \beta \)-analytic, then there are two vertices, while, if \( \kappa \) is \( \beta \)-nonanalytic, then there is one vertex. We see that number of cubic vertices \( p^{[3]} \) depends only on masses and does not depend on spins.

We now outline our method for derivation of equations given in (5.1)–(5.5). Our method can be described as a sequence of the following steps.

i) Cubic vertex \( p^{[3]} \) is defined by module of field redefinitions. We use field redefinitions to choose the most simple representation for cubic vertex. Upon field redefinitions, the cubic vertex is changed as

\[
p^{[3]} \rightarrow p^{[3]} + P^f, \quad f = f(P, \beta_a),
\]

where \( f(P, \beta_a) \) is polynomial in \( P \) and \( \beta \)-analytic, while \( P^c \) is given in (4.9). By definition the cubic vertex \( p^{[3]} \) is a finite-order polynomial in \( P \) and \( \beta \)-analytic. Taking into account that \( P^c \) is degree-2 polynomial in \( P \) (4.9) it is clear from (5.6) that, by using field redefinition, we

\(^6\)We expect that higher-derivative form of cubic vertices is more convenient as starting point for translation of our light-cone gauge cubic vertices into Lorentz covariant cubic vertices. This is long-term motivation of our interest in studying higher-derivative form of cubic vertices.
can remove in the cubic vertex $p_{[3]}$ all terms $P^q$ with $q \geq 2$. In other words, by using field redefinitions, the cubic vertex $p_{[3]}$ can be made to be degree-1 polynomial in $P$,

$$p_{[3]} = iP V_1 + V_0 \quad V_1 = V_1(\beta_a), \quad V_0 = V_0(\beta_a),$$

(5.7)

where new vertices $V_1, V_0$ do not depend on $P$ and should be $\beta$-analytic. The vertex $p_{[3]}$ (5.7) satisfies requirement (4.17). Thus, by using field redefinitions, we get the simple representation for the vertex (5.7) and respect requirement (4.17). Note that the possibility to cast any cubic vertex into the first-derivative form (5.7) exists only in three dimensions.

ii) Using (5.7), we now consider equation (4.15) and requirement for the density $j_{[3]}^{-1}$ to be polynomial in $P$. Using expression for $J_{[3]}^{-1}$ (4.10), we find the following relation:

$$J_{[3]}^{-1} = -2i P^\beta N^E_\beta V_1 + i \sum_{a=1,2,3} \left( \frac{m_a^2 N^E_\beta}{\beta_a} + \frac{\beta_a m_a^2}{6\beta_a} \right) V_1 - i M V_0$$

(5.8)

Using (5.8), we see that equation (4.15) and requirement for the density $j_{[3]}^{-1}$ to be polynomial in $P$ amount to the following two equations

$$i \sum_{a=1,2,3} \left( \frac{m_a^2 N^E_\beta}{\beta_a} + \frac{\beta_a m_a^2}{6\beta_a} \right) V_1 - i M V_0 = 0,$$

(5.9)

$$\frac{1}{\beta} N^E_\beta V_0 + i M V_1 = 0,$$

(5.10)

and the following representation for the density $j_{[3]}^{-1}$:

$$j_{[3]}^{-1} = 2i N^E_\beta V_1.$$  

(5.11)

iii) Plugging (5.7) into (4.14) we find the following equations for the vertices $V_1, V_0$:

$$J^{+ -} V_0 = 0, \quad (J^{+ -} + 1) V_1 = 0, \quad J^{+ -} \equiv \sum_{a=1,2,3} (\beta_a \partial_{\beta_a} + \frac{1}{2} e_{\lambda_a}).$$

(5.12)

Thus, by using cubic vertex (5.7), we reduced our complete system of equations (4.14)–(4.17) to equations for vertices $V_1, V_0$ in (5.9), (5.10) and (5.12). Equation (5.12) are simple homogeneity equations. It is the equations (5.9) and (5.10) that turn out to be complicated for the analysis.

iv) Vertices $V_1, V_0$ satisfy coupled equations (5.9) and (5.10). Our basic observation is that these equations can be cast into decoupled (diagonalized) form. Namely, in place of vertices $V_1, V_0$, we introduce vertices $V, \bar{V}$ defined by the relations

$$V = V_0 + \kappa V_1, \quad \bar{V} = V_0 - \kappa V_1.$$  

(5.13)

It is the straightforward exercise to show that two equations for the vertices $V_1, V_0$ (5.9) and (5.10) amount to the decoupled (diagonalized) equations for the vertices $V, \bar{V}$ given in (5.3). By definition, the vertices $V_1, V_0$ are $\beta$-analytic. For arbitrary masses $m_a$, the $\kappa$ (5.2) is $\beta$-nonanalytic. Therefore, from (5.13), we see that, in general, the vertices $V, \bar{V}$ are also
\(\beta\)-nonanalytic. This is to say that solution for \(V, \bar{V}\) should be chosen so that to get \(\beta\)-analytic \(V_1, V_0\).

We note that, for the case of three massless fields, \(\kappa = 0\). Therefore transformation (5.13) is not invertible. For analysis of this particular case, we will use basis of vertices \(V_0\) and \(V_1\).

**Higher-derivative form of cubic vertex.** We start with the definition of higher-derivative vertices we use in this paper. The first-derivative vertex is degree-1 polynomial in \(P\). Applying field redefinitions to the first derivative vertex, we obtain a general vertex which is degree-\(n\), \(n \geq 2\), polynomial in \(P\). Consider the general vertex for spin \(s_1, s_2, s_3\) bosonic fields and let us introduce quantities \(L_a, L^c_{\text{crit}, a}, a = 1, 2, 3\), defined below in (5.17) and (6.25). If the general vertex depends on \(P\) through expressions \(L_a, L^c_{\text{crit}, a}, \epsilon_a = 1\), then we refer to such general vertex as higher-derivative vertex\(^7\). Fermi–Bose higher-derivative vertices involve some additional factor which depends linearly on \(P\) (see factor \(K\) in (5.16)). The first-derivative vertices do not impose any constraints on spins, while, as we demonstrate below, some higher-derivative vertices impose certain constraints on spins. This is to say that, first-derivative vertices provide us the full list of vertices, while higher-derivative vertices we find in this paper provide us the particular list of vertices.

Some first-derivative vertices can be used to generate their higher-derivative counterparts in a rather straightforward way\(^8\). Namely, some higher-derivative cubic vertices are obtained by replacement \(\kappa = iP\) in expression for \(p_{\text{crit}}(5.1)\) (for some details, see appendix A). Doing so, we get for some Bose and Fermi–Bose cubic vertices the following higher-derivative representation:

\[
p_{\text{HD}}^3 = V_{\text{HD}}, \quad V_{\text{HD}} = \big|_{\kappa = iP}, \quad (5.14)
\]

\[
V_{\text{HD}} = V_b(L_a), \quad \text{for Bose vertices,} \quad (5.15)
\]

\[
V_{\text{HD}} = KV_{\text{FB}}(L_a), \quad \text{for Fermi–Bose vertices,} \quad (5.16)
\]

\[
L_a \equiv \frac{\beta_a}{2\beta_a} m_a + \frac{m_a^2 + a = m_a^2}{2m_a^2 + 2}, \quad K \equiv \frac{1}{\beta_1 \beta_2} \left(iP + m_1 \beta_2 - m_2 \beta_1\right), \quad (5.17)
\]

where for Fermi–Bose vertices, the external line indices \(a = 1, 2\) stand for two fermionic fields entering the cubic vertex. Note that \(L_a\) appear in (5.15) and (5.16) only if \(m_a \neq 0\). Expression for \(j_{\text{HD}}(5.1)\) corresponding to cubic vertex (5.14) is given by

\[
\begin{align*}
\tilde{j}_{\text{HD}}^1\ &= \sum_{a=1,2,3} \frac{2i\beta_a}{3\beta_a} \partial_a p_{\text{crit}}^3, \quad \text{for Bose vertices,} \quad (5.18) \\
\tilde{j}_{\text{HD}}^{-1} \ &= \left(\frac{i\beta_3}{3\beta_1 \beta_2} - K \sum_{a=1,2,3} \frac{2i\beta_a}{3\beta_a} \partial_a\right) V_{\text{FB}}. \quad \text{for Fermi–Bose vertices.} \quad (5.19)
\end{align*}
\]

We emphasize that expressions for \(p_{\text{HD}}^3\) (5.15), (5.16) and for \(j_{\text{HD}}^1\) in (5.18) and (5.19) are valid for some (not all) cubic vertices we discuss below. For the remaining cubic vertices, the higher-derivative form of \(p_{\text{HD}}^3\) depends not only on \(L_1, L_2, L_3\) but also on \(\beta_1, \beta_2, \beta_3\). Besides this, we

\(\text{We use the expressions } L^c_{\text{crit}} \text{ because we expect interrelations between such expressions and linearized curvatures in Lorentz covariant formulations. We use the expressions } L^c_{\text{crit}} \text{ because such expressions appear upon dimensional reduction from } 4\text{d massless fields to } 3\text{d massive fields (see appendix B).}

\(\text{In general, the higher-derivative vertices are degree-}n, n \geq 2, \text{ polynomials in } P. \text{ For fields with particular values of spins, some higher-derivative vertices are degree-1 polynomials in } P. \text{ For these particular cases, expressions for higher-derivative and first-derivative cubic vertices coincide.}
find also a wide class of higher-derivative vertices that are expressed entirely in terms of $L_{\text{crit},a}$. For some such cubic vertices, the procedure for generating higher-derivative vertices $P_{[3]}$ is clarified in appendix A, while the expressions $P_{[3]}$ and $J_{[3]}$ are presented explicitly in sections 6 and 7. We now apply our result above-presented for discussion of all cubic vertices in turn.

6. Cubic interaction vertices for bosonic fields

Classification of Bose vertices. Cubic vertices describing interaction of three bosonic fields we refer to as Bose vertices. Consider Bose vertex for three fields having masses $m_1, m_2, m_3$. To develop classification of Bose vertices we introduce quantities $D, P_{cm}$ defined by the relations

$$D \equiv m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_2^2m_3^2 - 2m_3^2m_1^2,$$

$$P_{cm} \equiv \sum_{\mu=1,2,3} \epsilon_{\mu}m_\mu, \quad \epsilon_1^2 = 1, \quad \epsilon_2^2 = 1, \quad \epsilon_3^2 = 1,$$

$$D = (m_1 + m_2 + m_3)(m_1 + m_2 - m_3)(m_1 - m_2 + m_3)(m_1 - m_2 - m_3),$$

where, in (6.3), we present helpful alternative representation for the quantity $D$ defined in (6.1). From (6.2) and (6.3), it is seen that $D \neq 0$ if $P_{cm} \neq 0$ for all admitted values of $\epsilon_1, \epsilon_2, \epsilon_3$. Also, from (6.2) and (6.3), we see that, if $P_{cm} = 0$ for some values of $\epsilon_1, \epsilon_2, \epsilon_3$, then $D = 0$.

Depending on the masses $m_1, m_2, m_3$, we split cubic vertices in the following four groups:

- **Ia)** $m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad D \neq 0$; \hspace{1cm} (6.4)
- **Ib)** $m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad P_{cm} = 0, \quad D = 0$; \hspace{1cm} (6.5)
- **IIa)** $m_1 \neq 0, \quad m_2 \neq 0, \quad |m_1| \neq |m_2|, \quad m_3 = 0$; \hspace{1cm} (6.6)
- **IIb)** $m_1 \neq 0, \quad m_2 \neq 0, \quad |m_1| = |m_2|, \quad m_3 = 0$; \hspace{1cm} (6.7)
- **III)** $m_1 = 0, \quad m_2 = 0, \quad m_3 \neq 0$; \hspace{1cm} (6.8)
- **IV)** $m_1 = 0, \quad m_2 = 0, \quad m_3 = 0$. \hspace{1cm} (6.9)

Now, using our classification in (6.4)–(6.9), we discuss the respective cubic vertices in turn.

**Ia)** Cubic vertex for three arbitrary spin massive fields with masses $D \neq 0$. Using notation as in (2.10)–(2.12), we consider a cubic vertex for three fields with the following masses and spins:

$$(m_1, s_1) - (m_2, s_2) - (m_3, s_3), \quad s_1, s_2, s_3 \in \mathbb{N}_0,$$

$$m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0; \quad D \neq 0,$$

where $D$ is defined in (6.1). Solution for vertices $V, \bar{V}$ entering cubic vertex $P_{[3]}$ (5.1) is given by

$$V = CV_\kappa, \quad \bar{V} = CV_{-\kappa},$$

$$V_\kappa \equiv L_{s_1,1}^sL_{s_2,2}^sL_{s_3,3}^s, \quad \bar{L}_{s,a} \equiv \frac{\kappa}{\beta_a} + \frac{\beta_a}{2\beta_a}a + \frac{a^2}{2\beta_a^{a+1}} - \frac{m_{a+1}^2}{2\beta_a},$$

where $C$ is coupling constant and $\kappa$ is defined in (5.2). Plugging $V$ and $\bar{V}$ (6.11) into (5.1), we get first-derivative cubic vertex $P_{[3]}$. In this paper, unless otherwise specified, all coupling constants are real-valued. Note also that, in general, our coupling constants might depend on
masses and spins of fields entering cubic vertex. Thus, there is only one type of first-derivative cubic vertex.

**Higher-derivative form.** Higher-derivative form of the vertex in (6.10) is obtained by plugging \( \kappa = \frac{\epsilon}{P} \) into expressions for \( V, V \) (6.11) and \( p_{[3]} \) (5.1). Doing so, we get one higher-derivative cubic vertex

\[
p_{[3]} = CL_1^{[a]} L_2^{[b]} P_3^{[c]}, \quad L_a \equiv \frac{\epsilon P}{\beta_a^2} + \frac{\beta_a^2 - m_a^2}{2m_a^2}, \quad (6.13)
\]

where the coupling constant \( C \) (6.13) coincides with \( C \) in (6.11). Expression for \( j_{[3]}^{-1} \) can be obtained by plugging \( p_{[3]} \) (6.13) into (5.18).

**Ib) Cubic vertices for three arbitrary spin massive fields with masses \( P_{em} = 0 \).** Using notation in (2.10)–(2.12), we consider a cubic vertices for three fields with the following expressions for \( \epsilon \)

\[
\epsilon \equiv \sum_{a=1,2,3} \beta_a s_a \epsilon_a, \quad S_i \in \mathbb{Z},
\]

(6.15)

\[
p_{[3]}^{-1} = C_{\epsilon_1 \epsilon_2 \epsilon_3} (i P + P_{em}) P_{em}^{\frac{1}{6} S_1 - 1} \prod_{a=1,2,3} \beta_a^{S_a}, \quad S_i \in \mathbb{Z},
\]

(6.16)

\[
P_{[3]} = C_{\epsilon_1 \epsilon_2 \epsilon_3} (i P - P_{em}) P_{em}^{\frac{1}{6} S_1 - 1} \prod_{a=1,2,3} \beta_a^{S_a}, \quad S_i \in \mathbb{Z},
\]

(6.17)

\[
P_{em} \equiv \sum_{a=1,2,3} \beta_a \epsilon_a m_a, \quad P_{em} \equiv \sum_{a=1,2,3} \epsilon_a m_a, \quad \epsilon_1 = 1, \quad \epsilon_2 = 1, \quad \epsilon_3 = 1,
\]

(6.18)

where coupling constants \( C_{\epsilon_1 \epsilon_2 \epsilon_3} \) can depend on masses, spins, and the parameters \( \epsilon_1, \epsilon_2, \epsilon_3 \). Vertex (6.16) is obtained from (6.15) by the replacements \( \epsilon_a \to \epsilon_a \), i.e., vertices (6.15) and (6.16) describe overcomplete basis of vertices. To avoid the overcounting of the vertices we can impose, for example, the restriction \( \epsilon_1 \epsilon_2 \epsilon_3 = 1 \). Using (5.7) and (5.11), we find that expressions for \( j_{[3]}^{-1} \) corresponding to vertices (6.15) and (6.16) take the form

\[
j_{[3]}^{-1} = -2iC_{\epsilon_1 \epsilon_2 \epsilon_3} \left( S_i P_{em}^{\frac{1}{6} (S_1 - 1) P_{em}} \right) \prod_{a=1,2,3} \beta_a^{S_a}, \quad (6.19)
\]

\[
j_{[3]}^{-1} = 2iC_{\epsilon_1 \epsilon_2 \epsilon_3} \left( S_i P_{em}^{\frac{1}{6} (S_1 + 1) P_{em}} \right) \prod_{a=1,2,3} \beta_a^{S_a}, \quad (6.20)
\]

**Higher-derivative form.** For \( S_i > 0 \) and \( S_i < 0 \), the first-derivative cubic vertices (6.15) and (6.16) admit the following higher-derivative form:

\[
p_{[3]}^{-1} = C_{\epsilon_1 \epsilon_2 \epsilon_3}^{HD} \left( i P + P_{em} \right) S_1 \prod_{a=1,2,3} \beta_a^{S_a}, \quad S_i > 0, \quad C_{\epsilon_1 \epsilon_2 \epsilon_3}^{HD} \equiv 2^{1-S_i} C_{\epsilon_1 \epsilon_2 \epsilon_3}, \quad (6.21)
\]
\[
\tilde{p}_{[3]} = \tilde{C}_{\ell_1\ell_2\ell_3}^{\text{HD}}(i\mathbb{P} - \mathbb{P}_m)^S \prod_{a=1,2,3} \beta_a^{\omega_a}, \quad S_i < 0, \quad \tilde{C}_{\ell_1\ell_2\ell_3}^{\text{HD}} \equiv (2)^{1+S} \tilde{C}_{\ell_1\ell_2\ell_3},
\]

(6.22)

where we use notation (6.17), while \( C_{\ell_1\ell_2\ell_3}, \tilde{C}_{\ell_1\ell_2\ell_3} \) (6.21) and (6.22) coincide with coupling constants in (6.15) and (6.16). To find \( j_{[3]}^{-1} \) we use general relation (4.15). Doing so and using notation (6.17) and (6.18), we get \( j_{[3]}^{-1} \) corresponding to \( \tilde{p}_{[3]} \) (6.21) and (6.22),

\[
j_{[3]}^{-1} = -2iC_{\ell_1\ell_2\ell_3}^{\text{HD}} S_i(i\mathbb{P} + \mathbb{P}_m)^{S_i-1} \prod_{a=1,2,3} \beta_a^{\omega_a}, \quad S_i > 0,
\]

(6.23)

\[
j_{[3]}^{-1} = 2iC_{\ell_1\ell_2\ell_3}^{\text{HD}} S_i(i\mathbb{P} - \mathbb{P}_m)^{S_i-1} \prod_{a=1,2,3} \beta_a^{\omega_a}, \quad S_i < 0.
\]

(6.24)

Vertices (6.21) and (6.22) turn out to be building blocks for vertices obtained via dimensional reduction from cubic vertices of massless fields in \( R^{3,1} \). For details of derivation and our proposal for higher-spin theory of massive fields in \( R^{3,1} \), see appendix B. Note that restrictions on \( S_i \) in (6.21)–(6.24) can be ignored only for the case when \( s_1 = s_2 = s_3 = 0 \).

Note that vertices (6.15) and (6.16) are defined for all \( S_i \in \mathbb{Z} \), while in (6.21) and (6.22) the values of \( S_i \) are restricted. In other words, first-derivative vertices (6.15) and (6.16) provide us the full list of vertices, while higher-derivative vertices (6.21) and (6.22) provide us the particular list of vertices.

As a remark, we note that vertices (6.21) and (6.22) can be represented in terms of quantities \( L_{\text{crit},1}, L_{\text{crit},2} \) defined by

\[
L_{\text{crit},1} \equiv \frac{1}{\beta_1} (i\mathbb{P} + \mathbb{P}_m), \quad L_{\text{crit},2} \equiv \frac{1}{\beta_2} (i\mathbb{P} - \mathbb{P}_m).
\]

(6.25)

For example, vertex (6.21) can be represented as \( C_{\ell_1\ell_2\ell_3}^{\text{HD}} L_{\text{crit},1}^{\ell_1} L_{\text{crit},2}^{\ell_2} L_{\text{crit},3}^{\ell_3} \).

IIa) Cubic vertex for two arbitrary spin massive fields with masses \( |m_1| \neq |m_2| \) and one scalar massless field. Using notation as in (2.10)–(2.12), we consider a cubic vertex for three fields with the following masses and spins:

\[
(m_1, s_1) - (m_2, s_2) - (0, 0), \quad s_1, s_2 \in \mathbb{N}_0,
\]

\[
m_1 \neq 0, \quad m_2 \neq 0, \quad |m_1| \neq |m_2|.
\]

(6.26)

Solution for vertices \( V, \bar{V} \) entering cubic vertex \( \tilde{p}_{[3]} \) (5.1) is given by

\[
V = CV, \quad \bar{V} = CV, \quad \kappa \equiv -\beta_2 \beta_3 m_1^2 - \beta_1 \beta_3 m_2^2,
\]

(6.27)

\[
V_{\kappa} \equiv L_{n_{\ell_1}}^{\kappa} L_{n_{\ell_2}}^{\kappa},
\]

(6.28)

\[
L_{n_{\ell_1}} \equiv \frac{\kappa}{\beta_1} + \frac{\beta_1}{2\beta_{11}} m_1 + \frac{m_1^2}{2m_1}, \quad L_{n_{\ell_2}} \equiv \frac{\kappa}{\beta_2} + \frac{\beta_2}{2\beta_{22}} m_2 - \frac{m_2^2}{2m_2},
\]

(6.29)

where \( C \) is a coupling constant. Plugging \( V \) and \( \bar{V} \) (6.27) into (5.1), we get first-derivative form of the cubic vertex \( \tilde{p}_{[3]} \). Thus, for masses and spins in (6.26), there is one vertex.

Higher-derivative form. Higher-derivative form of vertex (6.26) is obtained by plugging \( \kappa = i\mathbb{P} \) into expressions for \( V, \bar{V} \) (6.27) and \( \tilde{p}_{[3]} \) (5.1). Doing so, we get

\[
\tilde{p}_{[3]} = C L_{1}^{\ell_1} L_{2}^{\ell_2}, \quad L_1 \equiv \frac{i\mathbb{P}}{\beta_1} + \frac{\beta_1}{2\beta_{11}} m_1 + \frac{m_1^2}{2m_1}, \quad L_2 \equiv \frac{i\mathbb{P}}{\beta_2} + \frac{\beta_2}{2\beta_{22}} m_2 - \frac{m_2^2}{2m_2}.
\]

(6.30)
where the coupling constant $C$ in (6.30) is the same as the one in (6.27). Expression for $j_{[3]}^{-1}$ can be obtained by plugging $p_{[3]}$ (6.30) into (5.18).

**IIb) Cubic vertices for two arbitrary spin massive fields with masses $|m_1| = |m_2|$ and one scalar massless field.** Using notation as in (2.10)–(2.12), we consider a cubic vertex for three fields with the following masses and spins:

$$(m_1, s_1) - (m_2, s_2) - (m_3, 0), \quad s_1, s_2 \in \mathbb{N}_0,$$

$$m_1 \neq 0, \quad m_2 \neq 0, \quad |m_1| = |m_2|, \quad m_3 = 0.$$  \hfill (6.31)

The constraint $|m_1| = |m_2|$ leads to the consideration of two different cases: $m_1 = m_2$ and $m_1 = -m_2$. For these two cases, we find four vertices which we label as **IIb1–IIb4**. First-derivative form of these four vertices is given by

**IIb1)** \hspace{1cm} $p_{[3]}^- = C_+ \left(1 + \frac{i\beta_2}{m_1 \beta_3} \right) \left(\frac{\beta_3}{\beta_1} \right)^{\epsilon_1} \left(\frac{\beta_3}{\beta_2} \right)^{\epsilon_2}, \quad m_1 = m_2$;  \hfill (6.32)

**IIb2)** \hspace{1cm} $p_{[3]}^- = \bar{C}_+ \left(1 - \frac{i\beta_2}{m_1 \beta_3} \right) \left(\frac{\beta_3}{\beta_1} \right)^{\epsilon_1} \left(\frac{\beta_3}{\beta_2} \right)^{\epsilon_2}, \quad m_1 = m_2$;  \hfill (6.33)

**IIb3)** \hspace{1cm} $p_{[3]}^- = C_- \left(1 + \frac{i\beta_2}{m_1 \beta_3} \right) \left(\frac{\beta_3}{\beta_1} \right)^{\epsilon_1} \left(\frac{\beta_3}{\beta_2} \right)^{\epsilon_2}, \quad m_1 = -m_2$;  \hfill (6.34)

**IIb4)** \hspace{1cm} $p_{[3]}^- = \bar{C}_- \left(1 - \frac{i\beta_2}{m_1 \beta_3} \right) \left(\frac{\beta_3}{\beta_1} \right)^{\epsilon_1} \left(\frac{\beta_3}{\beta_2} \right)^{\epsilon_2}, \quad m_1 = -m_2$.  \hfill (6.35)

We see that for masses $m_1 = m_2$ there are two cubic vertices given in (6.32) and (6.33), while, for masses $m_1 = -m_2$, two cubic vertices are given in (6.34) and (6.35). Expressions for $j_{[3]}^{-1}$ corresponding to cubic vertices (6.32)–(6.35) are obtained by using general relations (5.7) and (5.11).

As a remark, we note that all vertices (6.32)–(6.35) can be presented on an equal footing as

$$p_{[3]} = C_{1,1} \left(1 + \frac{i\epsilon_2}{m_2 \beta_3} \right) \left(\frac{\beta_3}{\beta_1} \right)^{\epsilon_1} \left(\frac{\beta_3}{\beta_2} \right)^{\epsilon_2}, \quad \epsilon_1 m_1 + \epsilon_2 m_2 = 0, \quad \epsilon_1 = 1, \quad \epsilon_2 = 1,$$

$$C_{1,1} = C_+, \quad C_{1,-1} = \bar{C}_+, \quad C_{-1,1} = C_-, \quad C_{-1,-1} = \bar{C}-,$$  \hfill (6.36)

where, in (6.37), we identify coupling constants in (6.32)–(6.35) and (6.36).

**Higher-derivative form.** For the cases **IIb1, IIb2, IIb4**, we find the following higher-derivative representation,

**IIb1)** \hspace{1cm} $p_{[3]}^{HD} = 2C_+ \left(\frac{\beta_3}{\beta_1} \right)^{\epsilon_1} L_2^{\epsilon_2}, \quad C_+ = \frac{2C_+}{(2m_2)^{\epsilon_1}}, \quad m_1 = m_2$;  \hfill (6.38)

**IIb2)** \hspace{1cm} $p_{[3]}^{HD} = 2\bar{C}_+ L_1^{\epsilon_1} \left(\frac{\beta_3}{\beta_2} \right)^{-\epsilon_2}, \quad \bar{C}_+ = \frac{2\bar{C}_+}{(-2m_2)^{\epsilon_1}}, \quad m_1 = m_2$;  \hfill (6.39)

**IIb4)** \hspace{1cm} $p_{[3]}^{HD} = 2\bar{C}_- L_1^{\epsilon_1} L_2^{\epsilon_2}, \quad \bar{C}_- = \frac{2\bar{C}_-}{(2m_2)^{\epsilon_1}}, \quad m_1 = -m_2$;  \hfill (6.40)

$$L_1 = \frac{1}{\beta_1} (i\bar{P} - m_1 \beta_3), \quad L_2 = \frac{1}{\beta_2} (iP + m_2 \beta_3),$$  \hfill (6.41)
where, in (6.38)–(6.40), we also show interrelations between coupling constants of higher-derivative vertices and coupling constants of first-derivative vertices (6.32)–(6.35). Expression for \( j_{[3]} \) corresponding to \( P_{[3]} \) (6.40) can be obtained by plugging \( P_{[3]} \) (6.40) into (5.18), while expressions for \( j_{[3]} \) corresponding to \( p_{[3]} \) in (6.38) and (6.39) are given by

\[
\begin{align*}
\text{IIb1:} & \quad j_{[3]}^1 = C_{+}^{HD} \left( 2ix_{1} - \frac{2i\tilde{\beta}_{2}}{3\beta_{2} s_{1}} \right) \left( \frac{\beta_{3}}{\beta_{1}} \right)^{s_{2} - 1} L_{2}^{-1}, \quad m_{1} = m_{2}, \quad s_{2} \geq 1; \\
\text{IIb2:} & \quad j_{[3]}^1 = -C_{+}^{HD} \left( 2ix_{2} + \frac{2i\tilde{\beta}_{1}}{3\beta_{1} s_{1}} \right) \left( \frac{\beta_{3}}{\beta_{2}} \right)^{s_{1} - 1} L_{1}^{-1}, \quad m_{1} = m_{2}, \quad s_{1} \geq 1.
\end{align*}
\]

Restrictions \( s_{2} \geq 1 \) (6.42) and \( s_{1} \geq 1 \) (6.43) are obtained by requiring that the \( j_{[3]}^1 \) given in (6.42) and (6.43) be polynomial in \( L_{2} \) and \( L_{1} \) respectively. These restrictions are redundant for \( j_{[3]} \) corresponding to all first-derivative vertices in (6.32)–(6.35). Thus, we see that first-derivative vertices (6.32)–(6.35) provide us the full list of vertices, while higher-derivative vertices (6.38)–(6.40) provide us the particular list of vertices.

As a remark, we note that, by using \( L_{\text{crit}} \) (6.25), we find alternative higher-derivative form for some first-derivative vertices (6.32)–(6.35) given by

\[
p_{[3]} = C_{+}^{HD} (i\bar{p} + \bar{p} m_{2}) \beta_{1}^{s_{1}} \beta_{2}^{s_{2}}, \quad \bar{p} m_{2} = \epsilon_{2} m_{2} \beta_{2}, \quad C_{+}^{HD} = \frac{2C_{+}^{HD}}{(2\epsilon_{2} m_{2})^{N}}, \quad (6.44)
\]

where \( \epsilon_{2}^{2} = s_{2} = 1, \beta_{1} = \epsilon_{1} s_{1} + \epsilon_{2} s_{2} \). Vertices (6.44) and corresponding \( j_{[3]}^1 \) are obtained by setting \( m_{1} = 0, s_{3} = 0 \) in (6.21) and (6.23). This implies the restriction \( S_{1} > 0 \) in (6.44) which is redundant for first-derivative vertices (6.36). Thus, higher-derivative vertices (6.44) provide us the particular vertices, while first-derivative vertices (6.32)–(6.35) provide us the full list of vertices.

**III. Cubic vertex for two scalar massless fields and one arbitrary spin massive field.**

Using notation as in (2.10)–(2.12), we consider a cubic vertex for three fields with the following masses and spins:

\[
(0, 0) \rightarrow (0, 0) - (m_{1}, s_{1}), \quad m_{3} \neq 0, \quad s_{3} \in \mathbb{N}_{0}.
\]

Solution for vertices \( V, V \) entering cubic vertex \( p_{[3]} \) (5.1) is given by

\[
V = CV_{\kappa}, \quad \bar{V} = CV_{-\kappa},
\]

\[
V_{\kappa} \equiv L_{\kappa}, \quad L_{\kappa,3} \equiv \frac{\kappa}{\beta_{3}} + \frac{\beta_{3}}{2\beta_{3}} m_{3}, \quad \kappa^{2} = -\beta_{1} \beta_{2} m_{3}^{2},
\]

where \( C \) is a coupling constant. Plugging \( V, \bar{V} \) (6.46) into (5.1), we get first-derivative form of the cubic vertex \( p_{[3]} \).

**Higher-derivative form.** Higher-derivative form of the cubic vertex in (6.45) is obtained by plugging \( \kappa = i\bar{p} \) into expressions for \( V, \bar{V} \) (6.46) and \( p_{[3]} \) (5.1). Doing so, we get

\[
p_{[3]} = CL_{3}^{s_{3}}, \quad L_{3} \equiv i\bar{p} \beta_{3} + \frac{\beta_{3}}{2\beta_{3}} m_{3},
\]

where the coupling constant \( C \) in (6.48) is the same as the one in (6.46). Expression for \( j_{[3]} \) corresponding to \( p_{[3]} \) (6.48) can be obtained by plugging \( p_{[3]} \) (6.48) into (5.18).
IV) Cubic vertices for three scalar massless fields. Using notation as in (2.10)–(2.12), we consider a cubic vertex for three massless scalar fields:

\[(0,0) - (0,0) - (0,0)\].

(6.49)

For this case \(\kappa = 0\). Transformation (5.13) is not invertible and we use the vertices \(V_0, V_1\) which should satisfy equation (5.12) and equations obtained by setting \(m_1 = m_2 = m_3 = 0\) in (5.9) and (5.10). For \(m_1 = m_2 = m_3 = 0\), equation (5.9) is satisfied automatically. Thus, for vertices \(V_0\) and \(V_1\), we get equation (5.12) and equation \(N_3V_0 = 0\). Solution to these equations is given by

\[V_0 = C_0, \quad V_1 = \frac{1}{\beta_3} v \left( \frac{\beta_1}{\beta_2} \right)\].

(6.50)

where \(C_0\) is coupling constant, while \(v\) is \(\beta\)-analytic in \(\beta_1/\beta_2\).

We recall that, for \(\kappa \equiv 0\), we dealt with one or two vertices. We now see that, for three massless fields, \(\kappa \equiv 0\), we have infinite number of light-cone gauge vertices (6.50) parametrized by the constant \(C_0\) and by the function \(v\). The vertex \(V_0 = C_0\) is associated with the standard \(\phi^3\) self-interaction of scalar field in Lorentz covariant approach, while vertex \(V_1\) is associated with the Yang–Mills cubic interaction, when \(v = \text{const}\). To our knowledge, Lorentz covariant vertices associated with our light-cone gauge vertex \(V_1\), when \(v \neq \text{const}\), are not available in the literature. In other words, we encounter a mismatch between classifications of light-cone gauge and Lorentz covariant cubic vertices. For massless field in 4d, the mismatch between classifications of light-cone gauge vertices and Lorentz covariant cubic vertices is well known (see, e.g., reference [17]).

7. Cubic interaction vertices for fermionic and bosonic fields

Classification of Fermi–Bose vertices. Cubic vertices describing interaction of two fermionic fields and one bosonic field we refer to as Fermi–Bose vertices. Our conventions for fields in the cubic vertex are as follows. In the Bose–Fermi vertices, two fermionic fields carry external line indices \(a = 1, 2\), while one bosonic field corresponds to \(a = 3\). Namely, using notation as in (2.10)–(2.15), in the Bose–Fermi vertices, two fermionic fields and one bosonic field are identified as

\[(m_1, s_1 + \frac{1}{2}), (m_2, s_2 + \frac{1}{2})\] – two fermionic fields, \( (m_3, s_3)\) – one bosonic field.

(7.1)

As before, to develop classification of Fermi–Bose vertices we use the quantities \(D, P_{\text{em}}\) defined in (6.1) and (6.2). Namely, depending on the masses of two fermionic fields \(m_1, m_2\), and mass of one bosonic field \(m_3\), we split vertices in the following four groups:

\begin{align*}
\text{Ia)} & \quad m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad D \neq 0; \quad (7.2) \\
\text{Ib)} & \quad m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad P_{\text{em}} = 0, \quad D = 0; \quad (7.3) \\
\text{IIa1)} & \quad m_1 \neq 0, \quad m_2 \neq 0, \quad |m_1| \neq |m_2|, \quad m_3 = 0; \quad (7.4) \\
\text{IIa2)} & \quad m_1 = 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad |m_2| \neq |m_3|; \quad (7.5) \\
\text{IIb1)} & \quad m_1 \neq 0, \quad m_2 \neq 0, \quad |m_1| = |m_2|, \quad m_3 = 0; \quad (7.6)
\end{align*}
IIb) \( m_1 = 0, \ m_2 \neq 0, \ m_3 \neq 0 \quad |m_2| = |m_3|; \quad (7.7) \\
IIIa) \quad m_1 = 0, \ m_2 = 0, \ m_3 \neq 0; \quad (7.8) \\
IIIb) \quad m_1 \neq 0, \ m_2 = 0, \ m_3 = 0; \quad (7.9) \\
IV) \quad m_1 = 0, \ m_2 = 0, \ m_3 = 0. \quad (7.10)

Now, using our classification in (7.2)–(7.10), we discuss the respective cubic vertices in turn.

Ia) Cubic vertex for two fermionic arbitrary spin massive fields and one bosonic arbitrary spin massive field with masses \( D \neq 0 \). Using notation as in (2.10)–(2.15), we consider a cubic vertex for three fields with the following masses and spins:

\[
\left( m_1, s_1 + \frac{1}{2} \right) - \left( m_2, s_2 + \frac{1}{2} \right) - (m_3, s_3), \quad s_1, s_2, s_3 \in \mathbb{N}_0,
\]

\[
m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0; \quad D \neq 0, \quad (7.11)
\]

where \( D \) is defined in (6.1). Solution for vertices \( V, \bar{V} \) entering cubic vertex \( p_{[3]}^{(1)} (5.1) \) is given by

\[
V = CV, \quad \bar{V} = CV, \quad (7.12)
\]

\[
V_\kappa = K_i L_{i,1} L_{i,2} L_{i,3}, \quad K_\kappa \equiv \frac{1}{\kappa \beta_1 \beta_2} (\kappa + m_1 \beta_2 - m_2 \beta_1), \quad (7.13)
\]

\[
L_{\kappa, a} \equiv \frac{\kappa}{\beta_a} + \frac{\beta_a}{2 \beta_a} m_a + \frac{m_{a+1}^2 - m_a^2}{2 m_a}, \quad \kappa^2 = -\beta \sum_{a=1,2,3} \frac{m_a^2}{\beta_a}, \quad (7.14)
\]

where \( C \) is coupling constant. Plugging \( V, \bar{V} \) (7.12) into (5.1), we get first-derivative form of the cubic vertex \( p_{[3]}^{(1)} \). We recall that throughout this paper all coupling constants are real valued and, in general, our coupling constants might depend on masses and spins of fields entering cubic vertex.

**Higher-derivative form.** Higher-derivative form of the vertex in (7.11) is obtained by plugging \( \kappa = i \beta \) into expressions for \( V, \bar{V} \) (7.12) and \( p_{[3]}^{(1)} \) (5.1). Doing so, we get

\[
p_{[3]}^{(1)} = CK L_{i,1} L_{i,2} L_{i,3}, \quad K \equiv \frac{1}{\beta_1 \beta_2} (i \beta + m_1 \beta_2 - m_2 \beta_1), \quad (7.15)
\]

\[
L_a \equiv \frac{i}{2 \beta_a} + \frac{\beta_a}{2 \beta_a} m_a + \frac{m_{a+1}^2 - m_a^2}{2 m_a}, \quad (7.16)
\]

where the coupling constant \( C \) in (7.15) is the same as the one in (7.12). Expression for \( p_{[3]}^{-1} \) corresponding to \( p_{[3]}^{(1)} \) (7.15) can be obtained by using \( p_{[3]}^{(1)} \) (7.15) and relations in (5.14), (5.16) and (5.19).

Ib) Cubic vertices for two fermionic arbitrary spin massive fields and one bosonic arbitrary spin massive field with masses \( P_{em} = 0 \). Using notation as in (2.10)–(2.15), we consider a cubic vertex for three fields with the following masses and spins:

\[
\left( m_1, s_1 + \frac{1}{2} \right) - \left( m_2, s_2 + \frac{1}{2} \right) - (m_3, s_3), \quad s_1, s_2, s_3 \in \mathbb{N}_0,
\]

\[
m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad P_{em} = 0; \quad (7.17)
\]
where \( P_{\text{em}} \) is defined in (6.2). First-derivative form of two cubic vertices describing interactions of fields in (7.17) is given by

\[
P_{[3]} = C_{c_{123}}(iP + P_{\text{em}}) P^{-1}_{a} \beta_{1}^{\frac{1}{2}} \beta_{2}^{\frac{1}{2}} \prod_{a=1,2,3} \beta_{a}^{-\epsilon_{a}}, \quad S_{f}^{i} \in \mathbb{Z},
\]

(7.18)

\[
P_{[3]} = \bar{C}_{c_{123}}(iP - P_{\text{em}}) P^{-1}_{a} \beta_{1}^{\frac{1}{2}} \beta_{2}^{\frac{1}{2}} \prod_{a=1,2,3} \beta_{a}^{\epsilon_{a}}, \quad S_{f}^{i} \in \mathbb{Z},
\]

(7.19)

\[
S_{f}^{i} = \sum_{a=1,2,3} s_{a}^{i}, \quad \epsilon_{a}^{i} = \epsilon_{a}^{s_{a}^{i}}, \quad s_{1}^{i} = s_{1} + \frac{1}{2}, \quad s_{2}^{i} = s_{2} + \frac{1}{2}, \quad s_{3}^{i} = s_{3},
\]

(7.20)

where \( P_{\text{em}} \) and \( \epsilon_{a} \) are defined in (6.18). Coupling constants \( C_{c_{123}}, \bar{C}_{c_{123}} \) can depend on masses, spins, and the parameters \( \epsilon_{a} \). Vertex (7.19) is obtained from (7.18) by the replacements \( \epsilon_{a} \to -\epsilon_{a} \), i.e., vertices (7.18) and (7.19) describe overcomplete basis of vertices. Expressions for \( f_{[3]}^{i} \) corresponding to vertices (7.18) (7.19) are obtained from (6.19) (6.20) by the replacement \( \{ s_{a} \} \to \{ s_{a}^{i} \} \) and by multiplication on the factor \( (\beta_{1} \beta_{2})^{-1/2} \).

**Higher-derivative form.** For \( S_{f}^{i} > 0 \) and \( S_{f}^{i} < 0 \), the cubic vertices (7.18) (7.19) admit the following alternative higher-derivative representation:

\[
P_{[3]} = C_{c_{123}}^{\text{HD}}(iP + P_{\text{em}}) S_{f}^{1/2} \beta_{1}^{\frac{1}{2}} \beta_{2}^{\frac{1}{2}} \prod_{a=1,2,3} \beta_{a}^{-\epsilon_{a}}, \quad S_{f}^{i} > 0,
\]

(7.21)

\[
C_{c_{123}}^{\text{HD}} = 2^{1-S_{f}} C_{c_{123}},
\]

(7.22)

\[
P_{[3]} = \bar{C}_{c_{123}}^{\text{HD}}(iP - P_{\text{em}}) S_{f}^{1/2} \beta_{1}^{\frac{1}{2}} \beta_{2}^{\frac{1}{2}} \prod_{a=1,2,3} \beta_{a}^{\epsilon_{a}}, \quad S_{f}^{i} < 0,
\]

(7.23)

\[
\bar{C}_{c_{123}}^{\text{HD}} = (-2)^{1+S_{f}} \bar{C}_{c_{123}},
\]

where, in (7.21) and (7.22), \( C_{c_{123}}^{\text{HD}}, \bar{C}_{c_{123}}^{\text{HD}} \) are the coupling constants entering first-derivative vertices (7.18) and (7.19). Expressions for \( f_{[3]}^{i} \) corresponding to \( p_{[3]}^{i} \) (7.21) and (7.22) are given by

\[
f_{[3]}^{1} = -2i C_{c_{123}}^{\text{HD}} S_{f}^{i} (iP + P_{\text{em}}) S_{f}^{1/2} \beta_{1}^{\frac{1}{2}} \beta_{2}^{\frac{1}{2}} \prod_{a=1,2,3} \beta_{a}^{-\epsilon_{a}},
\]

(7.23)

\[
f_{[3]}^{2} = 2i C_{c_{123}}^{\text{HD}} S_{f}^{i} (iP - P_{\text{em}}) S_{f}^{1/2} \beta_{1}^{\frac{1}{2}} \beta_{2}^{\frac{1}{2}} \prod_{a=1,2,3} \beta_{a}^{\epsilon_{a}},
\]

(7.24)

where \( S_{f}^{i} = \sum_{a=1,2,3} \beta_{a}^{\epsilon_{a}} \) and we use the notation as in (7.20).

**IIa1) Cubic vertex for two fermionic arbitrary spin massive fields with masses \( \{ m_{1}, m_{2} \} \) and one scalar massless field.** Using notation as in (2.10)–(2.15), we consider a cubic vertex for three fields with the following masses and spins:

\[
\left( m_{1}, s_{1} + \frac{1}{2} \right) - \left( m_{2}, s_{2} + \frac{1}{2} \right) - (m_{3}, 0), \quad s_{1}, s_{2} \in \mathbb{N}_{0},
\]
\[ m_1 \neq 0, \quad m_2 \neq 0, \quad |m_1| \neq |m_2|, \quad m_3 = 0. \] (7.25)

Solution for vertices \( V, \tilde{V} \) entering cubic vertex \( p_{(3)}^- \) (5.1) is given by
\[
V = CV_c, \quad \tilde{V} = CV_{-c}, \quad (7.26)
\]
\[
V_c \equiv K_c L_{a1}^1 L_{a2}^2, \quad K_c \equiv \frac{1}{\beta_1 \beta_2} (\kappa + m_1 \beta_2 - m_2 \beta_1), \quad (7.27)
\]
\[
L_{a1} \equiv \frac{\kappa}{\beta_1} + \frac{\beta_1}{2 \beta_1} m_1 + \frac{m_3^2}{2 m_1}, \quad L_{a2} \equiv \frac{\kappa}{\beta_2} + \frac{\beta_2}{2 \beta_2} m_2 - \frac{m_1^2}{2 m_2}, \quad (7.28)
\]
\[
\kappa^2 = -\beta_2 \beta_1 m_1^2 - \beta_1 \beta_2 m_2^2. \quad (7.29)
\]

where \( C \) is a coupling constant. Plugging \( V \) and \( \tilde{V} \) (7.26) into (5.1), we get first-derivative form of the cubic vertex \( p_{(3)}^- \).

**Higher-derivative form.** Higher-derivative form of the vertex in (7.25) is obtained by plugging \( \kappa = i\bar{p} \) into expressions for \( V, \tilde{V} \) (7.26) and \( p_{(3)}^- \) (5.1). Doing so, we get
\[
p_{(3)}^- = CKL_1^1 L_2^2, \quad K \equiv \frac{1}{\beta_1 \beta_2} (i\bar{p} + m_1 \beta_2 - m_2 \beta_1), \quad (7.30)
\]
\[
L_1 \equiv \frac{i \bar{p}}{\beta_1} + \frac{\beta_1}{2 \beta_1} m_1 + \frac{m_3^2}{2 m_1}, \quad L_2 \equiv \frac{i \bar{p}}{\beta_2} + \frac{\beta_2}{2 \beta_2} m_2 - \frac{m_1^2}{2 m_2}. \quad (7.31)
\]

where the coupling constant \( C \) in (7.30) is the same as the one in (7.26). Expression for \( p_{(3)}^- \) corresponding to \( p_{(3)}^- \) (7.30) are obtainable by using \( p_{(3)}^- \) (7.30) and general relations in (5.14), (5.16) and (5.19).

IIa2) Cubic vertex for one fermionic massless field, one fermionic arbitrary spin massive field and one bosonic arbitrary spin massive field with masses \( |m_2| \neq |m_3| \). Using notation as in (2.10)–(2.15), we consider a cubic vertex for three fields with the following masses and spins:

\[
\left( m_1, \frac{1}{2} \right) - \left( m_2, s_2 + \frac{1}{2} \right) - \left( m_3, s_3 + \frac{1}{2} \right), \quad s_2, s_3 \in \mathbb{N}_0,
\]
\[
m_1 = 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad |m_2| \neq |m_3| \]. (7.32)

Solution for vertices \( V, \tilde{V} \) entering cubic vertex \( p_{(3)}^- \) (5.1) is given by
\[
V = CV_c, \quad \tilde{V} = CV_{-c}, \quad (7.33)
\]
\[
V_c \equiv K_c L_{a1}^1 L_{a2}^3, \quad K_c \equiv \frac{1}{\beta_1 \beta_2} (\kappa - m_2 \beta_1), \quad (7.34)
\]
\[
L_{a2} \equiv \frac{\kappa}{\beta_2} + \frac{\beta_2}{2 \beta_2} m_2 + \frac{m_3^2}{2 m_2}, \quad L_{a3} \equiv \frac{\kappa}{\beta_3} + \frac{\beta_3}{2 \beta_3} m_3 - \frac{m_1^2}{2 m_3}, \quad (7.35)
\]
\[
\kappa^2 = -\beta_1 \beta_2 m_1^2 - \beta_1 \beta_3 m_2^2. \quad (7.36)
\]

where \( C \) is a coupling constant. Plugging \( V, \tilde{V} \) (7.33) into (5.1), we get first-derivative form of the cubic vertex \( p_{(3)}^- \).
Higher-derivative form. Higher-derivative form of the vertex in (7.32) is obtained by plugging \( \kappa = \imath p \) into expressions for \( V, \bar{V} \) (7.33) and \( \bar{p}_3 \) (5.1). Doing so, we get

\[
\bar{p}_3 = CKL_2^2 \bar{L}_3^n, \quad K \equiv \frac{1}{\beta_1 \beta_2} (\imath p - m_2 \beta_1),
\]

\[
L_2 \equiv \frac{\imath p}{\beta_2} + \frac{\beta_2}{2 \beta_2} m_2 + \frac{m_2^2}{2 \bar{m}_2}, \quad L_3 \equiv \frac{\imath p}{\beta_3} + \frac{\beta_3}{2 \beta_3} m_3 - \frac{m_3^2}{2 m_3},
\]

where the coupling constant \( C \) in (7.37) is the same as the one in (7.33). Expression for \( \bar{p}_3^{-1} \) corresponding to \( \bar{p}_3 \) (7.37) can be obtained by using \( \bar{p}_3^{-1} \) and general relations in (5.14), (5.16) and (5.19).

Iib1) Cubic vertices for two fermionic arbitrary spin massive fields with masses \( |m_1| = |m_2| \) and one scalar massless field. Using notation as in (2.10)–(2.15), we consider a cubic vertex for three fields with the following masses and spins:

\[
\left( m_1, s_1 + \frac{1}{2} \right) - \left( m_2, s_2 + \frac{1}{2} \right) - (m_3, 0), \quad s_1, s_2 \in \mathbb{N}_0,
\]

\[
m_1 \neq 0, \quad m_2 \neq 0, \quad |m_1| = |m_2|, \quad m_3 = 0.
\]

The constraint \( |m_1| = |m_2| \) leads to the consideration of two different cases: \( m_1 = m_2 \) and \( m_1 = -m_2 \). For these two cases, we find four vertices which we label to as Iib1.1–Iib1.4.

First-derivative form of these four vertices is given by

Iib1.1) \( \bar{p}_3 = C_+ \frac{1}{\beta_2} \left( 1 + \frac{\imath p}{m_1 \beta_3} \right) \left( \frac{\beta_3}{\beta_1} \right)^{-s_1} \left( \frac{\beta_3}{\beta_2} \right)^{s_2}, \quad m_1 = m_2; \)

Iib1.2) \( \bar{p}_3 = C_+ \frac{1}{\beta_1} \left( 1 - \frac{\imath p}{m_2 \beta_3} \right) \left( \frac{\beta_3}{\beta_1} \right)^{s_1} \left( \frac{\beta_3}{\beta_2} \right)^{-s_2}, \quad m_1 = m_2; \)

Iib1.3) \( \bar{p}_3 = C_+ \frac{1}{\beta_3} \left( 1 + \frac{\imath p}{m_1 \beta_2} \right) \left( \frac{\beta_2}{\beta_1} \right)^{s_1} \left( \frac{\beta_2}{\beta_3} \right)^{-s_2}, \quad m_1 = -m_2; \)

Iib1.4) \( \bar{p}_3 = C_+ \frac{1}{\beta_1 \beta_2} \left( 1 - \frac{\imath p}{m_1 \beta_3} \right) \left( \frac{\beta_3}{\beta_1} \right)^{s_1} \left( \frac{\beta_3}{\beta_2} \right)^{s_2}, \quad m_1 = -m_2. \)

Thus, for masses \( m_1 = m_2 \), we find two cubic vertices given in (7.40) and (7.41), while, for masses \( m_1 = -m_2 \), two cubic vertices are given in (7.42) and (7.43).

As a remark, we note that all vertices (7.40)–(7.43) can be presented on an equal footing as

\[
\bar{p}_3 = \frac{C_{112}}{\beta_3} \left( 1 + \frac{i \epsilon_2 \beta_2}{m_2 \beta_3} \right) \left( \frac{\beta_3}{\beta_1} \right)^{s_1} \left( \frac{\beta_3}{\beta_2} \right)^{s_2}, \quad \epsilon_1 m_1 + \epsilon_2 m_2 = 0,
\]

\[
C_{-11} = C_+, \quad C_{1-1} = \bar{C}_+, \quad C_{-1-1} = C_-, \quad C_{11} = \bar{C}_-,
\]

where \( \epsilon^2_1 = 1, \epsilon^2_2 = 1 \). In (7.45), we match coupling constants in (7.40)–(7.43) and (7.44).

Higher-derivative form. For the vertices in (7.40), (7.41) and (7.43), we find the following higher-derivative representation,

Iib1.1) \( \bar{p}_3 = C^{ho} \frac{1}{\beta_2} \left( \frac{\beta_3}{\beta_1} \right)^{s_1} \bar{L}_2^n, \quad C^{ho} = \frac{2 C_+}{(2 m_2)^n}, \quad m_1 = m_2; \)
where \( C^H_+ \), \( C^H_- \) are coupling constants entering higher-derivative form of cubic vertices. In (7.46)–(7.48), we show how these constants are related to the coupling constants entering first-derivative form of cubic vertices (7.40)–(7.43). Expressions for the \( j^{[1]}_{[3]} \) corresponding to \( p^{[3]}_a \) in (7.46)–(7.48) are given by

\[
\text{IIb1.1) } j^{[1]}_{[3]} = C^H_+ \left( 2i \left( s_1 - \frac{\beta_3}{2 \beta_2} \right) - 2i \beta_3 \right) \frac{1}{\beta_1} \left( \frac{\beta_3}{\beta_1} \right)^{s_2} L^{L_{a1} - 1}_2, \quad m_1 = m_2, \quad s_2 \geq 1;
\]

\[
\text{IIb1.2) } j^{[1]}_{[3]} = -C^H_+ \left( 2i \left( s_2 - \frac{\beta_1}{2 \beta_3} \right) + 2i \beta_1 \right) \frac{1}{\beta_2} \left( \frac{\beta_1}{\beta_2} \right)^{s_1} L^{L_{a2} - 1}_2, \quad m_1 = m_2, \quad s_1 \geq 1;
\]

\[
\text{IIb1.4) } j^{[1]}_{[3]} = \frac{i \beta_3 \beta_2 C^H_+}{\beta^2_1 \beta^2_2} (i \mathbb{P} - m_1 \beta_3) L^{L_{a1} - 1}_2 - \sum_{a=1,2} \frac{2i \beta_a}{\beta_a} \partial_a p^{[3]}_a, \quad m_1 = -m_2,
\]

where, in (7.52), \( s_1 + s_2 \geq 1 \). Note that, in (7.52), \( \beta_1 L_1 = \beta_2 L_2 \). We see that first-derivative vertices (7.40)–(7.43) are valid for arbitrary \( s_1, s_2 \) in (7.39), while, for the higher-derivative vertices, we should use restrictions on \( s_1, s_2 \) in (7.50), (7.51) and the restriction \( s_1 + s_2 \geq 1 \) in (7.52).

As a remark, we note that, by using \( L_{c1a} \) (6.25), we find alternative higher-derivative form for some first-derivative vertices (7.40)–(7.43) given by

\[
p^{[3]}_a = C^{H}_{c1a} \left( i \mathbb{P} + \mathbb{P} \right) \frac{s_{c1}}{\beta_a} \left( \frac{1 - e^{-s_{c1} + i}}{2} \right) - \frac{1}{2} \beta_a^{s_{c1} + i} \frac{1}{2} C^{H}_{c1a} = 2(2e_2 m_2)^{8} C^{H}_{c1a}, \quad m_1 = m_2, \quad s_1 \geq 1.
\]

Vertices (7.53) and corresponding \( j^{[1]}_{[3]} \) are obtained by setting \( m_3 = 0, s_3 = 0 \) in (7.21) and (7.23) respectively. This gives the restriction \( S^I_{c1} > 0 \) in (7.53) which is redundant for first-derivative vertices (7.40)–(7.43). Thus, higher-derivative vertices (7.53) provide us the particular vertices, while first-derivative vertices (7.40)–(7.43) provide us the full list of vertices.

**IIb2) Cubic vertices for one fermionic massless field, one fermionic arbitrary spin massive fields and one bosonic arbitrary spin massive field with masses \( |m_2| = |m_3| \).** Using notation as in (2.10)–(2.15), we consider a cubic vertex for three fields with the following masses and spins:

\[
(m_1, \frac{1}{2}) - (m_2, s_2 + \frac{1}{2}) - (m_3, s_3), \quad s_2, s_3 \in \mathbb{N}_0,
\]
Higher-derivative representation, of cubic vertices (7.56)–(7.59). Expressions for we show how these constants are related to the coupling constants entering first-derivative form of these four vertices is given by

**First-derivative form of these four vertices is given by**

\[
P_{[3]} = C_+ \left( 1 + \frac{i \vec{p}}{m_3 \beta_1} \right) \left( \frac{\beta_1}{\beta_2} \right)^{s_3} \left( \frac{\beta_1}{\beta_3} \right)^{s_3}, \quad m_2 = m_3; \quad (7.56)
\]

\[
P_{[3]} = C_+ \left( 1 - \frac{i \vec{p}}{m_3 \beta_1} \right) \left( \frac{\beta_1}{\beta_2} \right)^{s_3} \left( \frac{\beta_1}{\beta_3} \right)^{s_3}, \quad m_2 = m_3; \quad (7.57)
\]

\[
P_{[3]} = C \left( 1 + \frac{i \vec{p}}{m_3 \beta_1} \right) \left( \frac{\beta_1}{\beta_2} \right)^{s_3} \left( \frac{\beta_1}{\beta_3} \right)^{s_3}, \quad m_2 = -m_3; \quad (7.58)
\]

\[
P_{[3]} = \tilde{C} \left( 1 - \frac{i \vec{p}}{m_3 \beta_1} \right) \left( \frac{\beta_1}{\beta_2} \right)^{s_3} \left( \frac{\beta_1}{\beta_3} \right)^{s_3}, \quad m_2 = -m_3. \quad (7.59)
\]

Thus, for masses \(m_2 = m_3\), we find two cubic vertices given in (7.56) and (7.57), while, for masses \(m_2 = -m_3\), two cubic vertices are given in (7.58) and (7.59).

As a remark, we note that all vertices (7.56)–(7.59) can be presented on an equal footing as

\[
P_{[3]} = C_{[3]10} \left( 1 + \frac{i \vec{p} \beta_1}{m_3 \beta_1} \right) \left( \frac{\beta_1}{\beta_2} \right)^{s_2} \left( \frac{\beta_1}{\beta_3} \right)^{s_3}, \quad \epsilon_2 m_2 + \epsilon_3 m_3 = 0; \quad (7.60)
\]

\[
C_{-1} = C_+, \quad C_{1-1} = \tilde{C}_+, \quad C_{11} = C_-, \quad C_{-1-1} = \tilde{C}_-. \quad (7.61)
\]

where, \(\epsilon_2^2 = \epsilon_3^2 = 1\), and, in (7.61), we identify coupling constants in (7.56)–(7.59) and (7.60).

**Higher-derivative form.** For the vertices in (7.56), (7.57) and (7.59), we find the following higher-derivative representation,

\[
P_{[3]} = C^{HD}_{+} \left( \frac{\beta_1}{\beta_2} \right)^{s_2} \left( \frac{\beta_1}{\beta_3} \right)^{s_3}, \quad C^{HD}_{+} = \frac{2 C_+}{(2m_3)^3}, \quad m_2 = m_3; \quad (7.62)
\]

\[
P_{[3]} = \tilde{C}^{HD}_{+} \left( \frac{\beta_1}{\beta_2} \right)^{s_2} \left( \frac{\beta_1}{\beta_3} \right)^{s_3}, \quad \tilde{C}^{HD}_{+} = \frac{2 \tilde{C}_+}{(-2m_3)^3}, \quad m_2 = m_3; \quad (7.63)
\]

\[
P_{[3]} = \tilde{C}^{HD}_{-} \left( \frac{\beta_1}{\beta_2} \right)^{s_2} \left( \frac{\beta_1}{\beta_3} \right)^{s_3}, \quad \tilde{C}^{HD}_{-} = \frac{2 \tilde{C}_-}{(2m_3)^3}, \quad m_2 = -m_3; \quad (7.64)
\]

\[
L_2 = \frac{1}{\beta_2} \left( \vec{p} m_2 \beta_1 \right), \quad L_3 = \frac{1}{\beta_3} \left( \vec{p} m_3 \beta_1 \right); \quad (7.65)
\]

where \(C^{HD}_{+}, \tilde{C}^{HD}_{+}, C^{HD}_{-}\) are coupling constants of higher-derivative cubic vertices. In (7.62)–(7.64), we show how these constants are related to the coupling constants entering first-derivative form of cubic vertices (7.56)–(7.59). Expressions for \(f_{[3]1}^{-1}\) corresponding to \(p_{[3]}\) in (7.62)–(7.64) are given by

\[
f_{[3]}^{-1} = C^{HD}_{+} \left( 2i(s_2 + \frac{1}{2}) - \frac{2i \beta_3}{s \beta_3} \right) \left( \frac{\beta_1}{\beta_2} \right)^{s_2} L_3^{s-1}, \quad m_2 = m_3, \quad s_3 \geq 1; \quad (7.66)
\]
IIb.2) \[ j_{[3]} = -iC_{[3]}^{HD}\left(2ix(s + \frac{1}{2}m_2 + \frac{1}{3}m_3)\right) - \frac{1}{\beta_2}L_2^{s-1} - \frac{1}{\beta_3}L_3^{s-1}, \quad m_2 = m_3, \quad s_2 \geq 1; \]

(7.67)

IIb.3) \[ j_{[3]} = -iC_{[3]}^{HD}\left(2ix(s + \frac{1}{2}m_2 + \frac{1}{3}m_3)\right) - \frac{1}{\beta_2}L_2^{s-1} - \frac{1}{\beta_3}L_3^{s-1}, \quad m_2 = -m_3; \]

(7.68)

where, in (7.68), we assume the restriction \( s_2 + s_3 \geq 1 \). Note that, in (7.68), \( \beta_2 L_2 = \beta_3 L_3 \).

As a remark, we note that, by using \( L_{crit/a} \) (6.25), we find alternative higher-derivative form for some first-derivative vertices (7.56)–(7.59) given by

\[ p_{[3]} = C_{[3]}^{HD}(iP + P_{ren})\left(\frac{1}{\beta_1} - \frac{1}{\beta_2} - \frac{1}{\beta_3}\right), \quad C_{[3]}^{HD} = 2(2\epsilon_3m_3)^{-3}C_{[3]}^{HD}, \]

(7.69)

\[ p_{ren} = c_0m_3\beta_1, \quad S_i = \frac{1}{2}c_1 + c_2(s_2 + \frac{1}{2}) + c_3s_3, \quad c_1 = c_2 = c_3 = 1. \]

(7.70)

Vertices \( p_{[3]} \) (7.69) and corresponding \( j_{[3]} \) are obtained by setting \( m_1 = 0, m_2 = 0 \) in (7.21) and (7.23) respectively. This implies the restriction \( S_3 \geq 0 \) in (7.69) which is redundant for first-derivative vertices (7.56)–(7.59). Thus, higher-derivative vertices (7.69) provide us the particular list of vertices, while first-derivative vertices (7.56)–(7.59) provide us the full list of vertices.

IIIa) Cubic vertex for two fermionic massless fields and one arbitrary spin massive field. Using notation as in (2.10)–(2.15), we consider a cubic vertex for three fields with the following masses and spins:

\[ \left(0, \frac{1}{2}\right) - \left(0, \frac{1}{2}\right) = \left(m_3, s_3\right), \quad m_3 \neq 0, \quad s_3 \in \mathbb{N}_0. \]

(7.71)

Solution for vertices \( V, V \) entering cubic vertex \( p_{[3]} \) (5.1) is given by

\[ V = CV_+, \quad V = CV_-. \]

(7.72)

\[ V_+ \equiv \frac{1}{\kappa}L_{[3]}^{[3]}, \quad L_{[3]} \equiv \frac{\beta_3}{\beta_2}m_3, \quad \kappa^2 = -\beta_1\beta_2m_3^2, \]

(7.73)

where \( C \) is coupling constant. Plugging \( V \) and \( V \) (7.72) into (5.1), we get first-derivative form of the cubic vertex \( p_{[3]} \).

**Higher-derivative form.** Higher-derivative form of the cubic vertex in (7.71) is obtained by using expressions for \( V, V \) (7.72) and \( p_{[3]} \) (5.1) (for some details, see appendix A). Doing so, we get

\[ p_{[3]} = C_{[3]}^{HD}K_{[3]}^{[3]}, \quad K \equiv \frac{i\beta_3}{\beta_1\beta_2}, \quad L_3 \equiv i\beta_3 - \frac{\beta_3}{\beta_2}m_3, \quad C_{[3]}^{HD} = \frac{1}{m_3}C, \]

(7.74)

where the coupling constant \( C \) appearing in (7.74) is the same as the one in (7.72). Expression for \( j_{[3]} \) can be obtained by using \( p_{[3]} \) (7.74) and general relations in (5.14), (5.16) and (5.19).

IIIb) Cubic vertex for one fermionic arbitrary spin massive field, one fermionic massless field, and one scalar massless field. Using notation as in (2.10)–(2.15), we consider a
cubic vertex for three fields with the following masses and spins:
\[
\left( m_1, s_1 + \frac{1}{2} \right) - \left( 0, \frac{1}{2} \right) - (0, 0), \quad m_1 \neq 0, \quad s_1 \in \mathbb{N}_0.
\]  
(7.75)

Solution for vertices \( V, \bar{V} \) entering cubic vertex \( p^{[3]} \) (5.1) is given by
\[
V = CV_\kappa, \quad \bar{V} = CV_{-\kappa}, \quad V_\kappa \equiv K_\kappa L_{\kappa,1}^\kappa,
\]  
(7.76)

where \( C \) is coupling constant. Plugging \( V \) and \( \bar{V} \) (7.76) in (5.1), we get first-derivative form of the cubic vertex \( p^{[3]} \).

**Higher-derivative form.** Higher-derivative form of the cubic vertex in (7.75) is obtained by plugging \( \kappa = i \beta \) into expressions for \( V, \bar{V} \) (7.76) and \( p^{[3]} \) (5.1). Doing so, we get
\[
p^{[3]} = CKL_1^\kappa, \quad K \equiv \frac{1}{\beta_1 \beta_2} (\kappa + m_1 \beta_2), \quad L_1^\kappa \equiv \frac{\kappa}{\beta_1} + \frac{\beta_1}{2\beta_2} m_1, \quad \kappa^2 = -\beta_2 \beta_3 m_1^2,
\]  
(7.77)

where the coupling constant \( C \) in (7.78) is the same as the one in (7.76). Expression for \( f^{[1]} \) can be obtained by using \( p^{[3]} \) (7.78) and general relations in (5.14), (5.16) and (5.19).

**IV) Cubic vertices for two fermionic massless fields and one scalar massless field.** Using notation as in (2.10)–(2.15), we consider a cubic vertex for two fermionic massless fields and one massless scalar field:
\[
\left( 0, \frac{1}{2} \right) - \left( 0, \frac{1}{2} \right) - (0, 0).
\]  
(7.79)

For this case \( \kappa = 0 \) and we use basis of cubic vertices \( V_0, V_1 \) with equation (5.12) and equations obtained by setting \( m_1 = m_2 = m_3 = 0 \) in (5.9) and (5.10). For vertices \( V_0 \) and \( V_1 \), we get equation (5.12) and equation \( N_0^{[3]} V_0 = 0 \). The \( \beta \)-analytic solution to these equations is given by
\[
V_0 = 0, \quad V_1 = \frac{1}{\beta_1 \beta_2} \nu \left( \frac{\beta_1}{\beta_2} \right),
\]  
(7.80)

where \( \nu \) is \( \beta \)-analytic. We see that, for three massless fields, we have an infinite number of light-cone gauge vertices (7.80) parametrized by the function \( \nu \). For \( \nu = \text{const} \), the vertex \( V_1 \) is associated with the \( \bar{\psi} \psi \phi \) Yukawa interaction in Lorentz covariant approach. For \( \nu = \beta_1 / \beta_3 \), i.e., \( \nu = 1 - \frac{\beta_1}{\beta_3} / 1 + \frac{\beta_1}{\beta_3} \), the vertex \( V_1 \) (7.80) is associated with the cubic vertex for coupling of two spin one-half massless fields to one spin-1 massless field in Lorentz covariant approach.

**8. Conclusions**

In this paper, we exploited light-cone gauge approach for studying interacting arbitrary integer and half-integer spin massive fields and scalar and one-half massless spin fields propagating flat 3d space. For such fields we build all cubic interaction vertices. As it happens in any other approaches, the light-cone gauge interaction vertices have freedom related to field redefinitions. Using this freedom we worked out two equivalent forms of light-cone gauge vertices which we refer to as first-derivative form and higher-derivative form. We expect that our results could have a lot of very interesting generalizations and applications which we now discuss.
In this paper, we built cubic interactions vertices for both the bosonic and fermionic massive (massless) fields in $R^{2,1}$. We expect therefore that our results for cubic interaction vertices provide good starting point for studying supersymmetric interacting massive (massless) fields in $R^{2,1}$. Light-cone gauge approach turns out to be convenient for studying massless higher-spin theories in $R^{3,1}$. Using this approach, all cubic interaction vertices for massless arbitrary spin $N$-extended supermultiplets in $R^{3,1}$ have been found in references [18, 19]. As the light-cone gauge formulation of supersymmetric massive fields in $R^{2,1}$ is similar to the light-cone gauge formulation of massless fields in $R^{3,1}$ we think that studying of supersymmetric massive field theory in $R^{2,1}$ by using light-cone gauge approach will be fruitful. In the framework of Lorentz covariant approach, the study of free massive fields in 3d may be found in references [22–24], while the supersymmetric free massive field theories are investigated in references [25–29]. The gravitational interaction of higher-spin massive fields in 3d is considered in reference [32].

In the above-presented study, we restricted our attention to interacting arbitrary spin massive and low spin massless fields propagating in $R^{2,1}$. It would be very interesting to generalize our study to massive (massless) fields propagating in $AdS_3$. Such generalization would be important in view of potentially interesting applications to superstring theory in $AdS_3$ space because one expects that it is massive arbitrary spin fields and low spin massless fields that enter spectrum of states of a superstring in $AdS$ background. Light-cone gauge formulation of free arbitrary spin massive fields propagating in $AdS_3$ space was obtained long ago in reference [33, 34], while method for studying interacting light-cone fields in $AdS$ space was developed in reference [35]. We expect therefore that method and results in references [34, 35] will allow us to study interacting massive fields in $AdS_3$. We note also, that results for cubic vertices of massless fields in $AdS_3$ in reference [35] and procedure of dimensions reduction (digression) in $AdS$ space developed in references [34, 37] open interesting possibility for studying cubic vertices of massive fields in $AdS_3$. We think also that knowledge of light-cone gauge formulation of interacting massive fields in $AdS_3$ will give new opportunities for study of $AdS/CFT$ correspondence along the lines in references [38–40].

Conformal higher-spin theories in 3d space have attracted some interest in the recent time (see, e.g., references [41–45]). In this respect we note that the light-cone gauge formulation of ordinary-derivative free and interacting totally symmetric conformal fields propagating in $R^{3−1,1}$ space with even $d$ was developed in reference [46], while the ordinary-derivative Lorentz covariant formulation of conformal fields in 3d space is available from reference [47]. Method in reference [46] provides the possibility to re-cast the Lorentz covariant ordinary-derivative formulation in reference [47] into the light-cone gauge formulation. As the light-cone gauge formulation considerably simplifies the whole analysis we expect then that the light-cone gauge formulation of conformal fields in 3d space will be useful for better understanding conformal higher-spin fields in 3d space.
iv) Recently, the light-cone gauge approach has fruitfully been used for studying loop corrections in higher-spin theories. In references [48, 49], it was shown that the massless higher-spin chiral theory [50] is free of one-loop divergencies and some arguments were given for cancellation of all loop divergencies. We note that, as discussed in references [51, 52], upon use of a proper regularization scheme, the quantum properties of some low spin supersymmetric field theories and their dimensionally reduced counterparts (with all massive KK modes included) are equivalent. In this respect, it would be interesting to investigate quantum properties of massless higher-spin chiral theory by using its dimensionally reduced counterpart which is realized as a massive higher-spin chiral theory in 3d. For the description of massive higher-spin chiral theory in 3d, see appendix B.

v) As we noted, the light-cone formulations of massive fields in $R^{2,1}$ and massless fields in $R^{3,1}$ share some features. The light-cone gauge approach was used in references [53, 54] for studying quartic interaction vertices of massless fields in 4d. We expect then that some considerations in references [53, 54] can in relatively straightforward way be generalized to quartic vertices for massive fields in 3d. We think also that study of quartic vertices for massive fields along the lines in reference [55] could be of some interest. Interesting recent discussion of $n$-point vertices of higher-spin fields may be found in reference [56]. See also general discussion by using light-cone gauge approach in reference [57].

vi) For analysis of various hypergravity theories in 3d, BRST technique was used in reference [58], while in reference [59] BRST approach was used for studying massless fields in 4d. Note that it is use of twistor-like variables that considerably simplifies BRST analysis in reference [59]. Interesting use of twistor-like variables for studying string model in AdS space may be found in reference [60]. Extension of ideas and approaches in references [58–60] to the case of massive fields in 4d could of some interest.

vii) In this paper, we studied integer and half-integer spin fields. Poincaré algebra admits continuous-spin fields. Cubic vertices for light-cone gauge continuous-spin massive and massless fields in $R^{d−1,1}$ with $d \geq 4$ were studied in references [62, 63]. We think that extension of our studies to the case of continuous spin field could be very interesting. For the reader convenience, we note that Lorentz covariant cubic vertex for coupling of massless and massive continuous-spin field to two massive scalar fields were studied in reference [64]. BRST approach for studying continuous-spin free field was applied in references [65–67], while various Lagrangian formulations of supersymmetric continuous-spin free field were considered in reference [68]. For AdS space, light-cone gauge Lagrangian for continuous-spin free field was obtained in references [69, 70], while metric-like and frame-like covariant Lagrangian formulations were studied in the respective references [71, 72] and reference [73].

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Appendix A. Derivation of cubic vertices

Properties of $L_{\kappa,a}$. Quantities $L_{\kappa,a}$ (6.12) and $L_{a}$ (5.17), $L_{\text{crit},a}$, $\bar{L}_{\text{crit},a}$ (6.25) are building blocks for our first-derivative and higher-derivative cubic vertices respectively. Basic properties of $L_{\kappa,a}$, $L_{a}$ and $L_{\text{crit},a}$, are given by the following differential relations

$$\frac{\kappa}{\beta} \frac{\partial}{\partial \beta} L_{\kappa,a} = \frac{m_{a}}{\beta} L_{\kappa,a},$$

(A.1)
We now respect the requirement for \( \mu \beta \) transformations from the vertices \( V \) for vertex (6.11). Therefore to avoid the repetitions, below we present explicit form of \( \beta \) equations (A.10). The \( L_{\text{crit}} \) is obtained from \( L_a \) by the replacements \( m_b \to \epsilon_b m_b \) for all \( b = 1, 2, 3 \) and by using the constraint \( P_{\mu} = 0 \). The \( L_{\text{crit}} \) is obtained from \( L_{\text{crit}} \) by the replacements \( \epsilon_b \to -\epsilon_b \) for all \( b = 1, 2, 3 \). We note the algebraic relations

\[
L_{\alpha,\mu} L_{\nu,\nu} = \frac{1}{4m_a^2} D,
\]

(4.14)

\[
L_{\alpha,1} L_{\nu,2} = \frac{m_{231} m_{312}}{4\beta_1 \beta_2 m_1 m_2} K_{12},
\]

(5.1)

\[
L_{\alpha,1} L_{\nu,3} = -\frac{m_{123} m_{312}}{8\beta_1 m_2 m_3} K_{12} K_{23} K_{31},
\]

(5.12)

\[
\bar{K}_{\alpha} \equiv \kappa + m_a \beta_b - m_b \beta_a + m_{abc} \equiv m_a + m_b - m_c,
\]

(5.12)

\[
L_{\text{crit}} L_{\text{crit}} = -\frac{2 \beta^2}{p_{\text{crit}}} P_{\mu},
\]

(5.12)

where \( P_{\mu} \), \( \beta \), and \( D \) are given in (4.9) and (6.1).

We now outline the derivation of our cubic vertices in turn.

**Cubic vertex** (6.11). In place of vertices \( V, \bar{V} \), we introduce new vertices \( V^{(1)}, \bar{V}^{(1)} \),

\[
V = L_{\alpha,1}^{(1)} L_{\nu,2}^{(1)} L_{\nu,3}^{(1)} V^{(1)}, \quad \bar{V} = L_{\alpha,1}^{(1)} L_{\nu,2}^{(1)} L_{\nu,3}^{(1)} \bar{V}^{(1)}.
\]

(6.11)

In terms of new vertices, equations for \( V, \bar{V} \) (5.3) and (5.4) take the form

\[
N_{\beta} V^{(1)} = 0, \quad N_{\beta} \bar{V}^{(1)} = 0, \quad \sum_{a=1,2,3} \beta_a \partial_{\beta_a} V^{(1)} = 0, \quad \sum_{a=1,2,3} \beta_a \partial_{\beta_a} \bar{V}^{(1)} = 0.
\]

(6.10)

Equations (A.10) tell us that the vertices \( V^{(1)}, \bar{V}^{(1)} \) do not depend on the momenta \( \beta_1, \beta_2, \beta_3 \),

\[
V^{(1)} = C, \quad \bar{V}^{(1)} = \bar{C}.
\]

(6.11)

We now should respect the requirement for \( p_{[3]} \) (5.1) to be \( \beta \)-analytic. From (5.7), we see that \( p_{[3]} \) is \( \beta \)-analytic iff \( V_1, V_0 \) are also \( \beta \)-analytic. Using (5.13), we get

\[
V_0 = \frac{1}{2} (V + \bar{V}), \quad V_1 = \frac{1}{2\kappa} (V - \bar{V}).
\]

(6.12)

Vertices \( V, \bar{V} \) in (A.9) are polynomial in \( \kappa \). From (A.9), (A.12) and \( \kappa^2 \) (5.2), we learn that \( V_0 \) is \( \beta \)-analytic provided \( V_0 \) is polynomial in \( \kappa^2 \). From (A.12), we see that this happens for \( C = \bar{C} \). For such choice of \( C, \bar{C} \), we see that \( V_1 \) (A.12) is also \( \beta \)-analytic.

Procedure of the derivation of all remaining vertices is the same we have just described for vertex (6.11). Therefore to avoid the repetitions, below we present explicit form of transformations from the vertices \( V, \bar{V} \) to vertices \( V^{(1)}, \bar{V}^{(1)} \) which do not depend on \( \beta_1, \beta_2, \beta_3 \).
Solution for forms. The \( \kappa (5.2) \) is chosen to be \( \kappa = \rho_{cm} \). Using notation in (6.17) and (6.18), we note the transformations and helpful relation,

\[
V = \rho_{cm} \prod_{a=1,2,3} \beta_a^{-s_a} V^{(1)}, \quad \bar{V} = \rho_{cm} \prod_{a=1,2,3} \beta_a^{m_a} V^{(1)},
\]

\( S \rho_{cm} + \frac{\beta}{3} S \rho_{cm} = \beta \mathcal{M}_{mm} \), \( \mathcal{M}_{mm} \equiv \sum_{a=1,2,3} S_{a}\bar{m}_{a} \beta_{a} \).

Cubic vertex (6.27).

\[
V = L_{n_1}^{3} L_{n_2}^{3} V^{(1)}, \quad \bar{V} = L_{-n_1}^{3} L_{-n_2}^{3} \bar{V}^{(1)}.
\]

Cubic vertices (6.32)–(6.35). For illustration, consider vertices (6.32) and (6.33). The \( \kappa (5.2) \) is chosen to be \( \kappa = m_1 \beta_3 \). The transformations and equation (5.3) take the respective forms

\[
V = \left( \frac{\beta_3}{\beta_1} \right)^{s_1} \beta_2^{s_2} V^{(1)}, \quad \bar{V} = \left( \frac{\beta_3}{\beta_1} \right)^{s_1} \beta_2^{s_2} \bar{V}^{(1)},
\]

\( \bar{V} = (s_1 \beta_2 + s_2 \beta_1) V \), \( \bar{V} = -(s_1 \beta_2 + s_2 \beta_1) \bar{V} \).

Solution for \( V \) and \( \bar{V} \) turns out to be \( \beta \)-analytic without any constraints on \( V^{(1)} \) and \( \bar{V}^{(1)} \). The solution for \( V, \bar{V} \) leads to the respective two vertices (6.32) and (6.33).

Cubic vertex (6.46).

\[
V = L_{n_1}^{3} V^{(1)}, \quad \bar{V} = L_{-n_1}^{3} \bar{V}^{(1)}.
\]

Cubic vertex (7.12). Helpful relation is given by (A.5). The transformations take the form

\[
V = \beta_1^{-1} \beta_2^{-1} L_{n_1}^{3} L_{n_2}^{3} V^{(1)}, \quad \bar{V} = \beta_1^{-1} \beta_2^{-1} \bar{V}^{(1)}.
\]

Cubic vertices (7.18) and (7.19). The \( \kappa (5.2) \) is chosen to be \( \kappa = \rho_{cm} \). The transformations take the form

\[
V = \beta_1^{-1} \beta_2^{-1} \rho_{cm} \prod_{a=1,2,3} \beta_a^{-s_a} V^{(1)}, \quad \bar{V} = \beta_1^{-1} \beta_2^{-1} \rho_{cm} \prod_{a=1,2,3} \beta_a^{m_a} V^{(1)}.
\]

Cubic vertex (7.26). The helpful relation is obtained by setting \( m_3 = 0 \) in (A.5). The transformations take the form

\[
V = \beta_1^{-1} \beta_2^{-1} L_{n_1}^{3} L_{n_2}^{3} V^{(1)}, \quad \bar{V} = \beta_1^{-1} \beta_2^{-1} \beta_3^{-1} L_{n_1}^{3} L_{n_2}^{3} V^{(1)}.
\]

Cubic vertex (7.32). The helpful relation and the transformations take the form

\[
\frac{1}{\beta_1 \beta_2} L_{n_1}^{2} = -\frac{1}{2m_2} K_2^{2}.
\]

Cubic vertices (7.40)–(7.43). For illustration, consider vertices (7.40) and (7.41). The \( \kappa (5.2) \) is chosen to be \( \kappa = m_1 \beta_3 \). The transformations and equations for \( V, \bar{V} (5.3) \) take the respective forms

\[
V = \frac{1}{\beta_2} \left( \frac{\beta_3}{\beta_1} \right)^{s_1} \beta_2^{s_2} V^{(1)}, \quad \bar{V} = \frac{1}{\beta_2} \left( \frac{\beta_3}{\beta_1} \right)^{s_1} \beta_2^{s_2} \bar{V}^{(1)}.
\]
\[ N^2_{\beta} \dot{V} = \left( (s_1 + \frac{1}{2}) \beta_2 + (s_2 + \frac{1}{2}) \beta_1 \right) V, \quad N^2_{\beta} \ddot{V} = - \left( (s_1 + \frac{1}{2}) \beta_2 + (s_2 + \frac{1}{2}) \beta_1 \right) \ddot{V}, \quad (A.24) \]

where \( N^2_{\beta} \equiv N_3 - \frac{1}{\epsilon} \beta_3 \), while \( N_3 \) is given in (4.11). Solution for \( V \) and \( \dot{V} \) turns out to be \( \beta \)-analytic without any constraints on \( V^{(i)} \) and \( \ddot{V}^{(i)} \). The solution for vertices \( V \), \( \dot{V} \) leads to the respective two vertices (7.40) and (7.41).

Cubic vertices (7.56)–(7.59). For illustration, consider vertices (7.56) and (7.57). The \( \kappa \) (5.2) is chosen to be \( \kappa = m_1 \beta_1 \). The transformation and equation (5.3) take the respective forms

\[ V = \frac{1}{\beta_1} \left( \frac{\beta_1}{\beta_2} \right)^{s_2} \left( \frac{\beta_1}{\beta_3} \right)^{s_3} V^{(1)}, \quad \dot{V} = \frac{1}{\beta_2} \left( \frac{\beta_1}{\beta_2} \right)^{s_2} \left( \frac{\beta_1}{\beta_3} \right)^{s_3} \dot{V}^{(1)}, \quad (A.25) \]

\[ N^2_{\beta} V = \left( s_2 + \frac{1}{2} \right) \beta_3 + s_3 \beta_2 \right) V, \quad N^2_{\beta} \dot{V} = - \left( s_2 + \frac{1}{2} \right) \beta_3 + s_3 \beta_2 \right) \dot{V}. \quad (A.26) \]

Solution for \( V \) and \( \dot{V} \) turns out to be \( \beta \)-analytic without any restrictions on \( V^{(1)} \) and \( \dot{V}^{(1)} \). The solution for vertices \( V, \dot{V} \) leads to the respective two vertices (7.56) and (7.57).

Cubic vertex (7.72).

\[ V = \frac{1}{\kappa} L^{\kappa}_{\kappa,3} V^{(1)}, \quad \ddot{V} = - \frac{1}{\kappa} L^{3}_{\kappa,3} \ddot{V}^{(1)}. \quad (A.27) \]

Cubic vertex (7.76). The transformation and helpful relation take the form

\[ V = \beta_1^{-\frac{1}{2}} \beta_2^{-\frac{1}{2}} \left( L_{\kappa,3}^{31} \right)^{s_2} V^{(1)}, \quad \dot{V} = \beta_1^{-\frac{1}{2}} \beta_2^{-\frac{1}{2}} \left( L_{\kappa,3}^{31} \right)^{s_3} \dot{V}^{(1)}, \quad \frac{1}{\beta_1} L_{\kappa,3} = \frac{1}{2m_1} \kappa^2. \quad (A.28) \]

Interrelation between first-derivative and higher-derivative cubic vertices. We now clarify our procedure of derivation of higher-derivative cubic vertex (5.14) from first-derivative vertex (5.1). This procedure is straightforward when the first-derivative vertex is polynomial in \( \kappa \). Namely, taking into account definition of \( \kappa^2 \) (5.2), we represent relation (4.9) as

\[ \kappa^2 = -P^2 + 2\beta P^\perp. \quad (A.29) \]

From (A.29), we see that, if first-derivative vertex \( p_{13}(\kappa) \) (5.1) is polynomial in \( \kappa \), then the following relation holds true

\[ p_{13}(\kappa) \big|_{\kappa^2 = -P^2 + 2\beta P^\perp} = p_{13}(iP^\perp) + P^\perp f, \quad (A.30) \]

where \( f \) is polynomial in \( P \). The \( P^\perp f \)-term in (A.30) can be removed by using field redefinitions (see (5.6)). This implies that higher-derivative vertex \( V_{1D} \equiv p_{13}(iP^\perp) \) (5.14) is obtainable from first-derivative vertex \( p_{13}(\kappa) \) (5.1) by using field redefinitions.

The substitution \( \kappa = iP^\perp \) is straightforward when the first-derivative vertex is expressed in terms of quantities \( L_{\kappa,3} \) (6.12). However some first-derivative vertices are not expressed in terms of \( L_{\kappa,3} \) (see Bose vertices (6.15), (6.16), (6.32)–(6.35) and Fermi–Bose vertices (7.18), (7.19), (7.40)–(7.43), (7.56)–(7.59)). To illustrate our method for these cases, we consider vertices (6.15) and (6.32).
First-derivative vertices (6.15), (6.16) and higher-derivative vertices (6.21) and (6.22). For this case, the $\kappa$ (5.2) is chosen to be $\kappa = P_{cm}$. By using (A.29), we get then the relation

$$P_{cm}^2 = -P^2 + 2\beta P^-.$$  

(A.31)

Upon substitution (A.31) in cubic vertex, we can ignore $P^-$-terms because all contributions proportional to $P^-$ can be removed by field redefinitions. Thus, we have the equivalence relation $P_{cm}^2 \sim -P^2$ for the cubic vertices. Using this equivalence relation, we get

$$(iP + P_{cm})^2 \sim 2(iP + P_{cm})P_{cm}, \quad (iP - P_{cm})^2 \sim -2(iP - P_{cm})P_{cm}. \quad \text{(A.32)}$$

Using (A.32), we find that higher-derivative vertices (6.21) and (6.22) amount to first-derivative vertices (6.15) and (6.16).

First-derivative vertex (6.32) and higher-derivative vertex (6.38). For masses (6.31), the $\kappa^2$ (5.2) takes the form $\kappa^2 = m_1^2 \beta_2^2$. Relation (A.29) implies then the following equivalence relation $P^2 \sim -m_1^2 \beta_2^2$. Using this equivalence relation, we find the desired equivalence relation

$$(iP + m_2 \beta_3)^{12} \sim 2^{n-1}m_2^{12}\left(\frac{\beta_3}{\beta_2}\right)^{12}\left(1 + iP\right). \quad \text{(A.33)}$$

Using (A.33) for first-derivative vertex (6.32) gives higher-derivative vertex (6.38).

First-derivative vertex (7.72) and higher-derivative vertex (7.74). Finally we make comment on vertex (7.72). This vertex depends on $L_{m,3}$ however is not polynomial in $\kappa$ (it appears in the denominator in (7.73)). For this case we represent $p_{[3]}$ as $p_{[3]}(\kappa) = \frac{1}{\kappa}p_{[3]}(\kappa)$, where $p_{[3]}(\kappa)$ is polynomial in $\kappa^2$. Then higher-derivative vertex (7.74) is obtained simply as $p_{[3]}^- = \frac{1}{\kappa^2}p_{[3]}^-(iP)$.

Appendix B. Dimensional reduction from massless fields in $R^{3,1}$ to massive fields in $R^{2,1}$

We now outline how our Bose vertices (6.21) and (6.22) can be obtained via dimensional reduction from vertices for massless fields in reference [74]. Generalization of our consideration to the Fermi–Bose vertices is straightforward. In helicity basis, bosonic spin-$s$, $s \geqslant 1$, massless field in $R^{3,1}$ is described by two complex-valued fields $\phi_{\pm,s}(p^i, x^\beta, \beta)$, $\phi_s(p^i, x^\beta, \beta)$, $\phi_{-s}(p^i, x^\beta, \beta)$, where $p^i$, $s$, $\beta$ are momenta and dependence on light-cone time $x^\beta$ is implicit. To proceed with dimensional reduction we make Fourier transform with respect to the momentum $p^i$ and introduce fields $\phi_{\pm,s}(p^i, x^\beta, \beta)$. Now making compactification on a circle $x^\beta \rightarrow \varphi$, $\varphi \in (0, 2\pi)$, we get fields $\phi_{\pm,s}(p^i, \varphi, \beta)$. For such fields, we use the following Fourier expansion into infinite set of fields in $R^{2,1}$,

$$\phi_{\lambda}(p^i, \varphi, \beta) = \sum_{n = -\infty}^{\infty} e^{i\mu n \varphi} \phi_{\lambda,n}(p^i, \beta), \quad \mu_{n} \equiv n, \quad \lambda = cs, \quad \epsilon = \pm 1, \quad \text{(B.1)}$$

where $\phi_{\lambda,n,\pm,s}$ are two complex-valued fields. Hermicity conditions for fields (B.1) take the form

$$\phi_{\lambda,n}(p^i, \varphi, \beta) = \phi_{-\lambda,n}(p^i, -\varphi, -\beta), \quad \phi_{\lambda,n}(p^i, \beta) = \phi_{-\lambda,n}(-p^i, -\beta). \quad \text{(B.2)}$$

Footnote 13 Fourier transform and its inverse are chosen to be as follows $f(x) = \int_{-\infty}^{\infty} e^{i\mu n x} \hat{f}(p) dp$, $\hat{f}(p) = \int_{-\infty}^{\infty} e^{-i\mu n x} f(x)$. 

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The second relation in (B.2) implies that two complex-valued fields $\phi_{m_{\alpha}, s}(p^i, \beta)$ can be represented in terms of two fields (which are real-valued in x-space)

$$
\phi_{m_{\alpha}, s} = \phi_{m_{\alpha}, s}^{(1)} + i\phi_{m_{\alpha}, s}^{(2)}, \quad \phi_{m_{\alpha}, -s} = \phi_{m_{\alpha}, s}^{(1)} - i\phi_{m_{\alpha}, s}^{(2)},
$$

where two fields $\phi_{m_{\alpha}, s}^{(1,2)}$, satisfy the hermicity condition as in (2.17). Now we show that $\phi_{m_{\alpha}, s}^{(1,2)}$ can be identified with the massive fields in (2.7). To this end we prove that, under action of the Poincaré algebra $iso(2, 1)$, the fields $\phi_{m_{\alpha}, s}^{(1,2)}$ transform in the same way as fields (2.7). With the exception of $J^{-1}$-transformations, matching all transformations in (2.18)–(2.20) is obvious. To match $J^{-1}$-transformations we note that $J^{-1}$-transformations of fields $\phi_{\lambda}$ (B.1) are realized by the following differential operator$^{14}$:

$$
J^{-1} = -\partial_0 p^1 + \partial_0 p^1 + \frac{i\lambda p^2}{\beta} + \frac{p^2}{2\beta} e_\lambda, \quad p^- = -\frac{p^1 + p^2}{2\beta}, \quad p^2 = -i\partial_\beta,
$$

where $e_\lambda$ is relevant only for half-integer fields. Acting with $J^{-1}$ on $\phi_{\lambda}$ (B.1) and using $\lambda = \epsilon x$, $p \equiv p^1$, we find that action of $J^{-1}$ on fields $\phi_{m_{\alpha}, s}^{(1,2)}$ (B.3) is realized by the differential operator

$$
J^{-1} = -\partial_0 p^1 + \partial_0 p^+ + \frac{ism_p}{\beta} + \frac{p^2}{2\beta} e_\lambda, \quad p^- = -\frac{p^1 + m^2}{2\beta}.
$$

Comparing (B.5) with (2.20), we see that each field $\phi_{m_{\alpha}, s}^{(1,2)}$ is indeed realized as spin-$s$ and mass $m_\alpha$ field, while a basis of the fields $\phi_{m_{\alpha}, s}$ (B.3) can be interpreted as some kind of helicity basis for massive fields in $R^{3,1}$. Relation (B.1) tells us that massless spin-$s$ field in $R^{3,1}$ described by the fields $\phi_{m_{\alpha}, s}$ amounts to infinite set of Kaluza–Klein massive spin-$s$ fields $\phi_{m_{\alpha}, s}^{(1,2)}$ in $R^{3,1}$ with the spectrum of masses $m_\alpha = n, n = \pm 1, \ldots, \pm \infty$ (plus two massless scalar fields in $R^{3,1}$). For $\lambda = 0$ in (B.1), we can use a scalar field $\phi_0$, which is real valued in x-space, and fix $\epsilon = 1$.

We now turn to cubic vertices. All that is required is to plug (B.1) into cubic vertices for massless fields in $R^{3,1}$ obtained in reference [74]. Let us present those vertices by using the conventions in reference [19]. In terms of the momentum–space fields, the cubic vertex for three massless fields having helicities $\lambda_1, \lambda_2, \lambda_3$ takes the form

$$
P_{[3];\lambda_1, \lambda_2, \lambda_3}^\alpha = \int d\Gamma_{[3]}(\vec{p}_1)(\vec{p}_2)\phi_{\lambda_1}^{(1)}(\vec{p}_1, \vec{p}_2)\phi_{\lambda_2}^{(2)}(\vec{p}_2, \vec{p}_3)\phi_{\lambda_3}^{(3)}(\vec{p}_3)C^{\lambda_1\lambda_2\lambda_3}\frac{\beta^{\lambda_1}2^{\lambda_2}3^{\lambda_3}}{\beta_12_23_3} + \text{h.c.},
$$

where coupling constants $C^{\lambda_1\lambda_2\lambda_3}$ are complex-valued in general. The $P^1$, $P^2$ are defined as in (4.3), and we use $d\Gamma_{[3]} = (2\pi)^3\delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)\prod_{a=1,2,3}^3 \delta_\beta_3^a/2\pi\beta_3^a$. To proceed we should transform fields in (B.6) from coordinates $p_1^a, p_2^a, \beta_a$ to coordinates $p_0^a, \varphi_a, \beta_a$. Doing so, we get fields as in (B.1). Using fields (B.1) in (B.6) and integrating out angle variables $\varphi_1, \varphi_2, \varphi_3$, we get

$$
P_{[3];\lambda_1, \lambda_2, \lambda_3}^\alpha = \sum_{n_1, n_2, n_3 = -\infty}^\infty \delta_{P_{m_0}} \int d\Gamma_{[3]}P_{m_{\alpha}} P_{m_{\alpha}} P_{m_{\alpha}} \prod_{a=1,2,3} \phi_{m_{\alpha}, \alpha}(\vec{p}_a) + \text{h.c.},
$$

$^{14}$ To get (B.4), we use $J^{-1} = (J^{-1} + J^{-1})/\sqrt{2}$ and $J^{-1}, J^+$ given in (2.26) and (2.27) in reference [19].
$P_{m_1,m_2,m_3} = \frac{\kappa C^{\lambda_1 \lambda_2 \lambda_3}}{(i\sqrt{2})^{\lambda_1 + \lambda_2 + \lambda_3}} \left( \mathbb{P}_m + \mathbb{P}_c \right) \mathbb{P} \delta_{\lambda_1 + \lambda_2 + \lambda_3}^{\lambda_1 \lambda_2 \lambda_3}, \quad \lambda_0 = \epsilon \mu \delta_0, \quad \text{(B.8)}$

$\epsilon_1^2 = \epsilon_2^2 = \epsilon_3^2 = 1, m_{a_1} = n_{a_2}, k = 2\pi, \text{ where } \delta_{a_1} = 1 \text{ for } k = n \text{ and } \delta_{a_1} = 0 \text{ for } k \neq n. \text{ The } \Delta \Gamma_{[3]}^{(B.7)} \text{ is obtained from (3.4) by equating } n = 3. \text{ In (B.7) and (B.8), we use } \mathbb{P}_0 = p_0, \beta_0 \text{ and } p_0 = \mathbb{P}_0. \mathbb{P} \equiv \mathbb{P}^1. \text{ The } \mathbb{P}_{em} \text{ and } \mathbb{P}_{cm} \text{ are obtained from (6.2) and (6.18) by the substitution for masses } m_a \to n_a. \text{ In view of } \delta \mathbb{P}_{em}, \text{ we note the condition } \mathbb{P}_{em} = 0 \text{ for Kaluza–Klein masses}^{15}.$

In references [53, 54], we studied massless higher-spin theory with the following cubic interactions:

$P_{(3)[\lambda_1 \lambda_2 \lambda_3]} = \sum_{\lambda_1, \lambda_2, \lambda_3 = -\infty}^{\infty} P_{(3)[\lambda_1 \lambda_2 \lambda_3]} \text{,} \quad \text{(B.9)}$

where $P_{(3)[\lambda_1 \lambda_2 \lambda_3]}$ is given in (B.6). Some solution for coupling constants $C^{\lambda_1 \lambda_2 \lambda_3}$ in references [53, 54] was found. Using such solution and ignoring Hermitian conjugated part in (B.6), the massless higher-spin chiral theory was proposed in reference [50]. By using the solution for coupling constants $C^{\lambda_1 \lambda_2 \lambda_3}$ in references [53, 54], it seems then naturally to suggest massive higher-spin chiral theory by using (B.7), (B.8) and removing Hermitian conjugated part in (B.7). Finally, we note that massive higher-spin theory with full (Hermitian) Hamiltonian (B.7) and (B.8) is also worth studying.

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15 For $\epsilon_1 = \epsilon_2 = \epsilon_3$, the condition $P_m = 0$ takes the form $m_1 + m_2 + m_3 = 0$. The condition $m_1 + m_2 + m_3 = 0$ have also been encountered in reference [75] when studying totally symmetric fields in arbitrary dimensions. In our study all conditions $P_m = 0$ are realized. We cordially thank M Tsulaia for drawing our attention to reference [75]. In the exponent in (B.1), we can set $\epsilon = 1$ for all $\lambda = 0, \pm 1, \ldots, \pm \infty$. This implies the Hermitian rule $\phi^{[\alpha \beta]}(\beta) = \phi^{[\beta \alpha]}(-\beta)$. The convention $\epsilon = 1$ for all $\lambda$ implies also setting $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ in $P_{em}, P_{cm}$ and using $\lambda_0 = 0, \pm 1, \ldots, \pm \infty$ in (B.7) and (B.8). The convention $\epsilon = 1$ for all $\lambda$ might be more convenient for some practical computations.
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