Cooperative Hypothesis Testing by Two Observers with Asymmetric Information

Aneesh Raghavan and John S. Baras

Abstract— We consider the binary hypothesis testing problem with two observers. There are two possible states of nature (or hypotheses). Observations collected by the two observers are statistically related to the true state of nature. The knowledge of joint distribution of the observations collected and the true state of nature is unknown to the observers. There are two problems to be solved by the observers: (i) true state of nature is known: find the distribution of the local information collected; (ii) true state of nature is unknown: collaboratively estimate the same using the distributions found by solving the first problem. We present four algorithms, each having two phases where the two problems are solved, with emphasis on the information exchange between the observers and resulting patterns. We prove different properties of the algorithms including the following: the probability spaces constructed as a consequence of solving the first problem are dependent on the information patterns at the observers; (ii) the rate of decay of probability of error of algorithms while solving the second problem is dependent on the information exchange between the observers. We present a numerical example demonstrating the four algorithms.

I. INTRODUCTION

A. Motivation

Hypothesis testing problems arise in various areas of science, medicine, engineering, and sociology. The standard version of the problem has been studied extensively in the literature. The inherent assumption of the standard problem is that even if there are multiple sensors collecting observations, the observations are transmitted to a single fusion center where the observations are used collectively to arrive at the belief of the true hypothesis. When multiple sensors collect observations, there could be other detection schemes as well. One possible scheme is that, the sensors could send a summary of their observations as finite valued messages to a fusion center where the final decision is made. Such schemes are classified as “Decentralized Detection” [1]. One of the motivations for studying decentralized detection schemes is that, when there are geographically dispersed sensors, or in environments with constrained communication (bandwidth limitations, geometric obstacles), such a scheme could lead to significant reduction in communication cost without compromising much on the detection performance.

In probability theory, the abstract probability space \((\Omega, \mathcal{F}, \mathbb{P})\) (a measure space) is given and always exists. The “observables” or the random variables are measurable functions on the probability space. In a real world experiment, the measurements correspond to these observables.

\[
(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{Y} (\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{D})
\]

Probability World Statistics World

Let \(E\) be an experiment observed by a single observer. Let the outcomes of the experiment be \(O\). The observer observes a function of the outcome of the experiment, \(Y = f(O)\), where \(Y \in \mathcal{X}\). \((\mathcal{X}, \mathcal{B}(\mathcal{X}))\) is a topological space and \(\mathcal{B}(\mathcal{X})\) is the Borel \(\sigma\)-algebra of subsets of \(\mathcal{X}\). Since \(Y\) is a measurable function, \(Y^{-1}(F) \in \mathcal{F}\) for any \(F \in \mathcal{B}(\mathcal{X})\). If the abstract measure, \(\mathbb{P}\), is known then \(\mathbb{D}(F) = \mathbb{P}(Y^{-1}(F)) \forall F \in \mathcal{B}(\mathcal{X})\) and is known as the law of the random variable. The probability space for the observer is \((\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{D})\). \(\mathbb{D}\) is referred to as a distribution when \(\mathcal{X} = \mathbb{R}^n\) and as a measure in a general setting.

Problems in probability theory are often referred to as “forward problems”, i.e., given the probability space what can be said about the random variables on the space. Problems in statistics are referred to as “inverse problems” as often the objective is to infer the probability space given realizations of the random variables, [3]. Consider the scenario where the measure \(\mathbb{P}\) is unknown. The probability space at the observer could be constructed as follows. Given the set \(\mathcal{X}\), let \(\mathcal{F}\) be a semiring of subsets of \(\mathcal{X}\) for which the observer can assign a measure after repeatedly performing the experiment. Assuming each trial of the experiment is independent of other trials, for any \(F \in \mathcal{F}\),

\[
\hat{\mathbb{D}}(F) = \frac{\# \text{ of times } Y \in F}{\# \text{ of trials of experiment}}
\]

\(\hat{\mathbb{D}}(\cdot)\) is a distribution on \(\mathcal{F}\). Since \(\hat{\mathbb{D}}(\cdot)\) is finitely additive and countably monotone, by the Caratheodory - Hahn theorem, the Caratheodory measure \(\mathbb{D}\) induced by \(\hat{\mathbb{D}}\), is an extension of \(\hat{\mathbb{D}}\). Let \(\mathcal{B}(\mathcal{F})\) be the \(\sigma\)-algebra of sets which are measurable with respect to the outer measure induced by \(\hat{\mathbb{D}}(\cdot)\). The probability space constructed by the observer after repeatedly observing the experiment is \((\mathcal{X}, \mathcal{B}(\mathcal{F}), \mathbb{D})\). We note that \(\mathcal{B}(\mathcal{F}) = \mathcal{B}(\mathcal{X})\) if and only if \(\mathcal{F}(\mathcal{X}) \subset \mathcal{F}\), i.e., the semiring \(\mathcal{F}\) contains all open sets. This may not be experimentally feasible.

Suppose each trial of the experiment is observed over time and multiple observations are collected, then the observation space is \(S \times T\), where \(T\) denotes the instances at which the observations are collected. If \(T\) is finite then the probability space construction can be done by following the methodology above. If \(T\) is a countable or uncountable set, then the distributions need to satisfy the Kolmogorov Consistency conditions [4], [5]. Further, the measure obtained by extending the distributions is a measure on the \(\sigma\)-algebra generated by
the cylindrical subsets of $S \times T$. This construction is referred to as the Kolmogorov Construction \cite{4,5}.

Now we consider the scenario where the experiment is observed by two observers, Observer 1 and Observer 2. Observer 1 observes a function of the outcome of the experiment, $Y^1 = f(O)$, while Observer 2 observes a different function $Y^2 = g(O)$ of the outcome of the experiment. Observer 1 (Observer 2) cannot find the distribution of its observation $Y^1 (Y^2)$ from the data. Neither observer can find the joint distribution of $Y^1$, $Y^2$ as Observer 1 and Observer 2 do not know $Y^2$ and $Y^1$ respectively. Even if both of the observers share the same model for the experiment, Observer 1 (Observer 2) cannot find the distribution of $Y^2 (Y^1)$ without knowing the $g (f)$ function. Hence, without sharing information, the observers cannot build the joint distribution of the observations. To build the joint distribution, the observers could send their observations or the functions $f$ and $g$ to a central coordinator.

In conclusion, though the abstract probability space might “exist” and be common to the observers, the probability space at the observers are different as indicated below. When $\mathbb{P}$ is unknown, each observer could construct its own probability space by following the procedure described before. However, events which belong to $\mathcal{B}(\mathcal{X}^1 \times \mathcal{X}^2)$ are not measurable in either measure space.

```
(\Omega, \mathcal{F}, \mathbb{P})

\downarrow

(\mathcal{X}^1, \mathcal{B}(\mathcal{X}^1), \mathbb{I}^{1})

\uparrow

? 

(\mathcal{X}^2, \mathcal{B}(\mathcal{X}^2), \mathbb{I}^{2})
```

The above motivates the following set of questions: (i) Is it even necessary that events in $\mathcal{B}(\mathcal{X}^1 \times \mathcal{X}^2)$ need to be measurable? (ii) What could be a suitable subset of events in $\mathcal{B}(\mathcal{X}^1 \times \mathcal{X}^2)$ that is to be measurable by each observer? (iii) What information should be exchanged by the observers to achieve the same? The questions are tied to the context in which the probability spaces are constructed. Though these problems do not naturally arise in statistics; they are inherent in multi-agent systems. This situation arises in almost all multi-agent decision making problems; though in the literature it is always assumed that $\mathbb{P}$ is known. In this paper, we study the above questions in the context of the hypothesis testing problem (as the simplest decision-making problem) and deviate from the assumption that $\mathbb{P}$ is known. In the following subsection, we present a brief survey of papers which study hypothesis testing from a multi-agent perspective.

\begin{itemize}
  \item \textbf{B. Literature Survey}
  \end{itemize}

In [1], the $M$-ary hypothesis testing problem is considered. A set of sensors collect observations and transmit finite valued messages to the fusion center. At the fusion center, a hypothesis testing problem is considered to arrive at the final decision. For the sensors, to decide what messages they should transmit, the Bayesian and Neyman-Pearson versions of the hypothesis testing problem are considered. The messages transmitted by the sensors are coupled though a common cost function. For both versions of the problem, it is shown that if the observations collected by different sensors conditioned on any hypothesis are independent, then the sensors should decide their messages based on the likelihood ratio test. The results are extended to the cases when the sensor configuration is a tree and when the number of sensors is large. In [2], the binary hypothesis testing problem is considered. The formulation considers two sensors and the joint distribution of the observations collected by the two sensors is known, to both sensors, under either hypothesis. The objective is to find an optimal decision policy for the sensors, based on the observations collected at the sensors locally, with respect to a coupled cost function. Under assumptions on the structure of the cost function, and independence of the observations conditioned on the hypothesis, it is shown that the likelihood ratio test is optimal with thresholds based on the decision rule of the alternate sensor. Conditions under which threshold computations decouple are also presented. In [6], the binary decentralized detection problem over a wireless sensor network is considered. A network of wireless sensors collect measurements and send a summary individually to a fusion center. Based on the information received, the objective of the fusion center is to find the true state of nature. The objective of the study was to find the structure of an optimal sensor configuration with the formulation incorporating constraints on the capacity of the wireless channel over which the sensors are transmitting. For the scenario of detecting deterministic signals in additive Gaussian noise, it is shown that having a set of identical binary sensors is asymptotically optimal. Extensions to other observation distributions are also presented. In [7], sequential problems in decentralized detection are considered. Peripheral sensors make noisy measurements of the hypothesis and send a binary message to a fusion center. Two scenarios are considered. In the first scenario, the fusion center waits for the binary messages (i.e., the decisions) from all the peripheral sensors and then starts collecting observations. In the second scenario, the fusion center collects observations from the beginning and receives binary messages from the peripheral sensors as time progresses. In either scenario, the peripheral sensor and the fusion center need to solve a stopping time problem and declare their decision. Parametric characterization of the optimal policies are obtained and a sequential methodology for finding the optimal policies is presented.

\begin{itemize}
  \item \textbf{C. Problem Description}
  \end{itemize}

We investigate the binary hypothesis testing problem from a novel and fundamental perspective. There are two possible states of nature. There are two observers, Observer 1 and Observer 2. Each observer collects its individual set of observations. The observations collected by the observers are statistically related to the true state of nature. The joint or the marginal distribution of the observations is unknown to the agents. Given the observations, the objective of the two observers is to collaboratively find the true hypothesis. The driving motivation of this paper is to understand this decentralized detection problem from scratch. More specifically the
focus is to understand rigorously and at a fundamental level the construction of the underlying probability spaces, under various data exchange (communication) patterns between the two observers. Surprisingly, this fundamental problem has not been examined carefully, and in detail, in the current literature, with the result being that several a priori assumptions on the underlying probability spaces, widely used in the literature, are in most cases incorrect, or not justified. Further to this fundamental point, at a recent workshop in the 2023 American Control Conference [8], the surprising fact regarding the lack of such serious investigation of the underlying probability spaces and conditioning information models (i.e. σ-algebras) by each of the observers was pointed out and discussed by several authors (V. Borkar, V. Anantharam, J.S. Baras) [8]. We quote from V. Anantharam’s presentation “A problem for the next generation. Develop a meaningful theory of distributed interaction when the agents have different probability assessments on the underlying sample space.” Some related early papers are [9], [10], [11] and a more recent one is [12].

Since, we do not assume that the joint distribution of the observations is known to the observers, i.e., the observers share a common probability space, we emphasize the details of probability space construction by each observer from the data each observer has available. The focus is on understanding more accurately the information exchange (or not exchange) between the observers not only in order to perform collaborative detection, but even more importantly to construct the underlying probability spaces and models employed. One of the authors (Baras) has investigated the need for such detailed and careful constructions, and even the need for new probability models (i.e. von-Neumann-like and not Kolmogorov-like [13]) over the years [14], [15], [16], [17]. The recent PhD thesis of one of the authors (Raghavan) [13] investigated various problems closely related to the problems addressed in this paper.

A similar problem in the study of quantum systems has gained prominence. The scenario of two independent observers conducting measurements on a joint quantum system can be modelled using two approaches: (i) a Hilbert space of tensor product form, each factor associated to one observer. The operators describing the observables are acting only on one tensor factor; (ii) one joint Hilbert space, requiring that all operators associated to different observers commute, i.e. are jointly measurable without causing disturbance. The problem of Tsirelson [19], is to decide whether all quantum correlation functions between two independent observers derived from commuting observables can also be expressed using observables defined on a Hilbert space of tensor product form. Connections between this problem and many other problems, like Connes’ embedding problem [20], Kirchberg’s conjecture [21], etc., have been established.

II. PROBLEM FORMULATION AND CONTRIBUTIONS

In this section, we progress towards formalizing the problem described in the previous subsection, [I-C], and discuss the contributions of the paper.

A. Problem Set Up

1) Both the observers operate on the same time scale; their actions are synchronized. Time is considered to be discrete and is denoted by subscripts. Time instant $n$ is also referred to as iteration $n$.

2) State of nature is the same for both observers. The two states of nature are represented by 0 and 1. The state of nature remains fixed during a given experiment but can change across experiments. The state of nature is denoted by $H$.

3) The observations collected by Observer 1 are denoted by $Y_i, Y_i \in S_1$ where $S_1$ is a finite set of real numbers or real vectors of finite dimension. The observations collected by Observer 2 are denoted by $Z_i, Z_i \in S_2$, where $S_2$ is a finite set of real numbers or real vectors of finite dimension. Let $M = |S_1| \times |S_2|$.

4) The state of nature, $H$, the observations, $\{Y_n, Z_n\}_{n \geq 1}$ (and functions of these observations) are considered to be random variables in an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is unknown.

5) The knowledge of distributions of the observations (joint or marginal) under either hypothesis and the prior distribution of the hypothesis is unknown to both observers.

B. Problems

The information collected by an observer includes observations from nature and the information communicated by the other observer. The sequence of observations collected and information received from the other observer is referred to as the information pattern at an observer. To address the last item in subsection II-A, first we formulate the learning problem for the observers.

Problem 1. The Learning Problem: In this phase, the data collected by the observers comprises of realizations of (i) the true state of nature; (ii) observations which are statistically related to the true state of nature; (iii) information communicated by the other observer which is a function of the observations collected by it. For Observer 1 and 2, the data collected are realizations of the sequences of random variables

\[ \{H_j, Y_{j,1}, \phi^1_1(Z_{j,1}), Y_{j,2}, \phi^2_1(Z_{j,1}, Z_{j,2}), \ldots, Y_{j,n}, \phi^2_n(Z_{j,1}\ldots Z_{j,n})\}, \]

\[ \{H_j, Z_{j,1}, \phi^1_j(Y_{j,1}), Z_{j,2}, \phi^2_j(Y_{j,1}, Y_{j,2}), \ldots, Z_{j,n}, \phi^2_n(Y_{j,1}\ldots Y_{j,n})\}\]

respectively for experiment $j$. The objective of the observers is to utilize the data over the experiments to find the local joint distribution of the random variables.

The information communicated by Observer $i$ during the learning phase is determined by the sequence of functions $\{\phi^i_n(\cdot)\}_{n \geq 1}$. This sequence remains the same across experiments. In the testing problem, given a particular information pattern, possibly the same as the learning phase, the objective is to find the true state of nature.

Problem 2. The Testing Problem: In this phase, the data collected by the observers comprises of realizations of (i)
observations which are statistically related to the true state of nature; (ii) information communicated by the other observer which is a function of the observations collected by it and the information communicated by the former observer. For Observer 1 and 2, the data collected are realizations of the sequences of random variables

\[ \{Y_1, \hat{\phi}_1^1(Z_1), \hat{\phi}_1^2(Z_1, Z_2), \phi_1^2(Z_1, Y_2, Z_1, \ldots, Z_n, \hat{\phi}_1^n(Z_1, \ldots, Z_n)\}, \{Y_2, \hat{\phi}_2^1(Z_1, Y_2), \hat{\phi}_2^2(Z_1, Y_2, Z_1, \ldots, Z_n, \hat{\phi}_2^n(Z_1, \ldots, Z_n)\}, \{Z_1, \hat{\phi}_1^1(Y_1, \hat{\phi}_1^2(Z_1)), Z_2, \hat{\phi}_2^1(Y_1, Y_2, \hat{\phi}_2^2(Z_1, Z_2)) \ldots, Z_n, \hat{\phi}_1^n(Y_1, \ldots, Y_n, \hat{\phi}_1^n(Z_1, \ldots, Z_n)) \} \]

respectively. The objective of the observers is to utilize the data to collaboratively find the true state of nature using the local knowledge, i.e., distributions obtained by solving Problem 7.

The information communicated by Observer 1 during the testing phase is determined by \( \{\phi_1^0(\cdot)\}_{n=1}^\infty, \{\hat{\phi}_1^0(\cdot)\}_{n=1}^\infty \). It is possible that the statistical distribution knowledge to process the information received is not known to the agents, in which case the random variables are treated as \textit{exogenous} random variables, i.e., simply as real numbers without any statistical information.

\section*{C. Contributions: Algorithms}

In Section III we present four different algorithms to solve the problems considered. In each algorithm there are two phases: (a) Learning phase where Problem 1 is solved. The true hypothesis is known, data collected is utilized to build empirical distributions between hypothesis and the observations; (b) Testing phase where Problem 2 is solved. Given a new set of observations, hypothesis testing problems are solved by the observers to find their individual beliefs about the true hypothesis. We make following observation:

- The two observers possess asymmetric information and models, i.e., the distributions obtained from the learning phase are different as the information used to obtain them are different. Hence, their beliefs about the true state of nature is “most likely” different. For them to collaborate and agree upon their beliefs, requires repeated exchange of information leading to a \textit{sequential} approach to solve the problem.

Hence, each of the algorithms involves consensus steps for the observers to agree on their beliefs about the true hypothesis.

In Algorithm-1 (standard), subsection III-A the observations collected by both observers are sent to a central coordinator, the joint distribution between the observations and hypothesis is built and hypothesis testing is done using the collective set of observations. \textit{It should be noted that the joint distribution between the observations collected by the observers is found only for the purpose of comparison between the centralized and decentralized detection schemes. It is not available to observers for processing any information they receive.}

In Algorithm-2, subsection III-B each observer builds its own probability space using local observations. Hypothesis testing problems are formulated for each observer in their respective probability spaces. The observers solve the problems to arrive at their beliefs about the true hypothesis. A consensus algorithm involving exchange of beliefs is presented.

In Algorithm-3, subsection III-C the observers build aggregated probability spaces by building joint distributions between their observations and the alternate observer’s decisions. The decisions transmitted by the observers for probability space construction are the decisions obtained in the second approach. Hypothesis testing problems are formulated for each observer in their new probability spaces. The original decision of the observers is a function of their observations alone. The construction of the aggregated probability space enables an observer to update its information state based on the accuracy of the alternate observer. Based on the updated information state the observer updates its belief about the true hypothesis. A modified consensus algorithm is presented where the observers exchange their decision information twice; the first time they exchange their original beliefs and the second time their updated beliefs.

In Algorithm-4, subsection III-D we assume that the observations collected by the observers are independent conditioned on the hypothesis. In such a case the construction of the aggregated sample space is skipped. An observer receives the accuracy information (to update its information state) from the alternate observer. Hence, the observers exchange real valued information. In this approach the observers also solve the detection problem twice; once with information state obtained from the observations alone and the second time with the information state updated from the accuracy information. The consensus algorithm involves exchange of (i) original decision (ii) accuracy information (iii) updated decision. In our previous work, [22], we presented Algorithms-1, 2 and preliminary results on the analysis of the algorithms.

\section*{D. Contributions: Analysis and Key Ideas of The Paper}

In Section V we analyze properties of the distributions found and the testing algorithms prescribed in Section III. For the learning phase of algorithms, we prove that the estimated distributions equal the true distributions. We analyze the probability space construction in the learning phase of the algorithms. We prove that the probability spaces constructed for the observers in Algorithms-2,3,4 are not \textit{similar} to the probability spaces constructed with full information exchange as in Algorithm-1. Though the sample spaces are the same, the measures are assigned to different \( \sigma \)-algebras which is dictated by the information they possess during the learning phase. Following are the key observations which are used in the proofs:

- At time instant \( n \), the common sample space for both agents is \( \{0,1\} \times S_1^n \times S_2^n \). Since this set is a finite set, any \( \sigma \)-algebra of subsets of it is obtained by union of sets in a partition of the set. A \textit{Partition} of a set, \( E, \bar{E} \), is a finite collection of subsets of \( E, \bar{E} = \{E_1,\ldots,E_n\} \) such that \( \bigcup_{j=1}^n E_j = E \). If we define an \textit{Equivalence Relation} on the set \( E \), the partition generated by it, \( \bar{E} = \{E_1,\ldots,E_n\} \), is such that \( E_i \cap E_j = \emptyset \) and \( \bigcup_{j=1}^n E_j = E \). This implies
that, defining a measure on a σ-algebra of \(\{0,1\} \times S_1^0 \times S_2^0\) generated by a partition from an equivalence relation is equivalent to defining a set function on the partition which sums to 1, \(P : \bar{E} \rightarrow [0,1] : \sum_{j=1}^n P(E_j) = 1\). Thus, to solve Problem 1 it suffices to estimate the distributions on suitable partitions of \(\{0,1\} \times S_1^0 \times S_2^0\).

- Different functions \(\{\phi_1^0(\cdot), \phi_2^0(\cdot)\}_{n \geq 1}\) of the observations for communication have been chosen in the four algorithms in Section III. In Algorithm-2, subsection III-B we demonstrate that for any \(n\) any atomic event (e.g. for \(n = 1\), \(H = h, Y_1 = y_1, Z_1 = z_1\)) of the sample space is not experimentally verifiable. Hence the distributions are defined on partitions whose sets contain more than 1 element of the sample space. In Algorithm-3, subsection III-C the information exchange pattern leads to an equivalence relation on the sample space for both the observers. The corresponding partition at iteration \(n+1\), \(\bar{E}_{n+1}\) (of \(\{0,1\} \times S_1^{n+1} \times S_2^{n+1}\)) is a refinement of the partition at iteration \(n\), \(E_n\) (of \(\{0,1\} \times S_1^n \times S_2^n\)), i.e., set product of every set in \(\bar{E}_n\) with \(S_1 \times S_2\) is obtained by the disjoint union of a unique collection of sets in \(\bar{E}_{n+1}\). The new distribution from iteration \(n\) to \(n+1\) needs to be recomputed due to correlation between the information communicated in \(n+1\) and all the past information. These ideas are further described in subsection IV-B.

- In the Figure 1 an example of the partitions generated with different information exchange has been demonstrated. We consider \(|S_1| = 6\) and \(|S_2| = 4\). As in Algorithm-1, if the observations collected are exchanged during the learning phase, then every member of the sample space is a member of the partition, i.e. \(\bar{E} = \{(h,y,z)\}, h \in \{0,1\}, y \in S_1, z \in S_2\) for \(n = 1\) (top left of the figure). Thus, there exits a unique partition which contains \(2 \times 6 \times 4 = 48\) sets. However, if the information exchanged during the learning phase is only the decisions as in Algorithm-3, the partitions are not unique. One such partition is demonstrated in the figure. Further, for Observer 1, any partition has \(2 \times 6 \times 2 = 24\) sets while for Observer 2 any partition has \(2 \times 4 \times 2 = 16\) sets for \(n = 1\). We note that (subsection IV-B) the equivalence relation is such that it does not depend on \(H\). Hence the partitions are the same under either hypothesis. The same has been captured through the repeating color patterns.

In the testing phase of algorithms, we prove that consensus is achieved in Algorithms-2,3,4 for almost all sample paths. We investigate the benefits of additional data exchange in the testing phase of Algorithm-3 and establish the correlation between Algorithm-3 and Algorithm-4. The performances of these algorithms are compared by comparing the rate of decay of probability of error. We characterize the rate of decay of the probability of error in Algorithm-1 and the rate of decay of the probability of agreement on the wrong hypothesis in Algorithm-2 and prove that the former is upper bounded by the latter. We note that the study of the rate of decay of the probability of error presented in this paper is limited and is to be investigated further with advanced tools from large deviations theory. [23]. With respect to the computation of beliefs in the testing phase of the algorithms, we make the following remark:

- If the joint distributions of the observations under either hypothesis were to be known (i.e., the learning phase of every algorithm could be skipped) while the information pattern in the testing phase was to be retained (Problem 2), the belief could be computed by finding the conditional probability of true state of nature given the information pattern using the joint distributions. This
would require exhaustive search over the sample spaces to identify the observation sequences which lead to the information pattern, which is computationally intensive. However, in the algorithms proposed, the distributions found in the learning phase aid in finding the conditional probability using arithmetic computations, avoiding the exhaustive search. Storing the distributions, especially in Algorithm-3, could be expensive in terms of memory usage.

**Remark 1.** We note that the problem considered in this paper is a decision theoretic, or a parametric hypothesis testing, analogue of the problem considered in [24]. We demonstrate that the information exchanged by the observers in the learning phase dictates the probability space constructed at the observers. We prove that all subsets of the set of possible outcomes need not belong to the \( \sigma \)-algebra of the probability space at either observer. To summarize, the outline of the paper is as follows. In the next section, Section III, we present the four algorithms in complete detail. In Section IV we present the analysis of the algorithms including the probability space construction, rate of decay of probability of error, etc. Simulation results are presented in section V. We conclude and discuss future work in section VI. The proof of the result comparing the rate of decay of the probability of error in the testing phase of Algorithms-1,2 is presented in Section VII.

### III. ALGORITHMS

In this section, we describe different algorithms to solve the problems proposed in subsection IV-B. For each algorithm, there is a learning phase where the distributions are found by solving Problem 1 and subsequently suitable decision making problems are formulated. Following the learning phase, there is a testing phase where the decision making problems are solved followed by consensus steps for collaboration.

#### A. Algorithm-1 Centralized Approach

1) **Algorithm-1 Learning Phase:** In this approach both the observers send the data strings collected by them to a central coordinator. The central coordinator generates new strings by concatenating the observations from Observer 1, observations from Observer 2 and the true hypothesis. Thus, \( \phi_n^1((Y_i)_{i=1}^n) = Y_n \) and \( \phi_n^2((Z_i)_{i=1}^n) = Z_n \). From the data strings, the empirical joint distributions are found. The joint distribution when the true hypothesis is 0 is denoted by \( f_0(y,z) \), and when the true hypothesis is 1 is denoted by \( f_1(y,z) \). We assume, \( 0 < \mathbb{D}_{KL}(f_0||f_1) < \infty \), where \( \mathbb{D}_{KL}(f_0||f_1) \) denotes the Kullback-Leibler (KL) divergence between distributions \( f_0 \) and \( f_1 \) [25]. The prior distribution of the hypothesis is denoted by \( p_h \) for \( h = 0,1 \). Let \( \Omega = \{0,1\} \times S_1 \times S_2 \), \( \omega \in \Omega \), is given by the triple \((h,y,z)\), \( h \in \{0,1\}, y \in S_1 \) and \( z \in S_2 \). Let \( \mathbb{F} = 2^{\Omega} \). Since \( \Omega \) is finite it suffices to define the measure for each element in \( \Omega \). Hence the measure, \( \mathbb{P}^c \) is defined as follows: \( \mathbb{P}^c(\omega) = p_h f_h(y,z) \). The probability space constructed by the central coordinator is \((\Omega, \mathbb{F}, \mathbb{P}^c)\). We assume that the observations received by the observers are i.i.d conditioned on the hypothesis, in which case, the observations are random variables in the product space. The product space is defined as \((\Omega_n, \mathbb{F}_n, \mathbb{P}_n)\), where \( \Omega_n = \{0,1\} \times S_{1n} \times S_{2n} \), \( \mathbb{F}_n = 2^{\Omega_n} \) and \( \mathbb{P}_n(\omega) = p_h \prod_{i=1}^{n} f_h(y_i,z_i) \). The schematic for the centralized approach is shown in Fig. 2. Given an observation sequence \( \{y_i, z_i\}_{i=1}^n \), the objective is to find \( D_n : S_1 \times S_2 \rightarrow \{0,1\} \) such that the following cost is minimized

\[
\mathbb{E}_{P_n} [C_{10} H(1-D_n) + C_{01} (D_n)(1-H)],
\]

where \( H \) denotes the hypothesis random variable, \( C_{10}, C_{01} \) the typical costs for decision errors.

For the purpose of comparison of this algorithm with the remaining algorithms (e.g. subsection IV-B), the joint probability space is extended as follows. A sample space consisting of sequences of the form \((H, (Y_1, Z_1), (Y_2, Z_2), (Y_3, Z_3), \ldots)\) is considered. For \( n \in \mathbb{N} \), let \( B \) be a subset of \( \{(0,1) \times \{S_1 \times S_2\}^n\} \). A cylindrical subset of \( \{(0,1) \times \{S_1 \times S_2\}^n\} \) is,

\[
L_n(B) = \{ \omega \in \{0,1\} \times \{S_1 \times S_2\}^n : (\omega(1), ..., \omega(n+1)) \in B \}.
\]

Let \( \mathbb{F}^c \) be the smallest \( \sigma \)-algebra generated by all cylindrical subsets of the sample space. Since the sequence of product measures \( \mathbb{P}_n^c \) is consistent, i.e.,

\[
\mathbb{P}_n^c(B \times \{S_1 \times S_2\}) = \mathbb{P}_n^c(B) \quad \forall B \in 2^{\{0,1\} \times \{S_1 \times S_2\}^n}, \forall n
\]

by the Kolmogorov extension theorem (subsection VII-A), there exists a measure \( \mathbb{P}_n^{c,*} \) on \( \{(0,1) \times \{S_1 \times S_2\}^n, \mathbb{F}^c \} \), such that,

\[
\mathbb{P}_n^{c,*}(L_n(B)) = \mathbb{P}_n(B) \quad \forall B \in 2^{\{0,1\} \times \{S_1 \times S_2\}^n}, \forall n
\]

Hence the extended probability space at the central coordinator is given by \( \{(0,1) \times \{S_1 \times S_2\}^n, \mathbb{F}^c, \mathbb{P}_n^{c,*}\} \).

2) **Algorithm-1 Testing Phase:** The problem formulated in the above subsection is the standard Bayesian hypothesis testing problem. We let \( \hat{\phi}_n^1(\{Y_i\}_{i=1}^n) = Y_n, \hat{\phi}_n^2(\{Z_i\}_{i=1}^n) = Z_n, \hat{\phi}_n^1(\{Y_i\}_{i=1}^n, \{Z_i\}_{i=1}^n) = \hat{\phi}_n^1(\{Y_i\}_{i=1}^n), \{Z_i\}_{i=1}^n) = \emptyset \).

The decision policy is a threshold policy and is a function of the likelihood ratio. The likelihood ratio is defined as, \( \pi_n = \prod_{i=1}^{n} \frac{f_h(y_i,z_i)}{f_1(y_i,z_i)} \). Then the decision is given by

\[
D_n = \begin{cases} 
1, & \text{if, } \pi_n \ge T_c, \\
0, & \text{otherwise},
\end{cases}
\]

where \( T_c = \frac{C_{10}}{C_{01} + C_{10}} \).
B. Algorithm-2 Decentralized Approach

1) Algorithm-2 Learning Phase: In this approach each observer constructs its own probability space. From the data strings collected locally, the observers find their respective empirical distributions. Thus, $\phi_i^1(\{Y_i\}_{i=1}^n) = \phi_i^1(\{Z_i\}_{i=1}^n) = \emptyset$. For Observer 1, the distribution of observations when the true hypothesis is 0 is denoted by $f_1^0(y)$ and when the true hypothesis is 1 is denoted by $f_1^1(y)$. Similarly, Observer 2 finds $f_2^0(z)$ and $f_2^1(z)$. We assume that the prior distribution of the hypothesis remains the same as in the previous approach. We assume, for $i = 1, 2$, $0 < D_{KL}(f_i^0||f_i^1) < \infty$. For consistency we impose:

$$\sum_{y \in S_1} f_h(y, z) = f_h^0(y), \forall y \in S_1, h = 0, 1.$$  
$$\sum_{y \in S_1} f_h(y, z) = f_h^1(y), \forall z \in S_2, h = 0, 1.$$  

Based on these distributions, the probability space constructed by Observer 1 is $(\Omega^1, F_1^0, P_1^0)$. $\Omega^1 = \{0, 1\} \times S_1, F_1^0 = 2^{\Omega^1}$ and $P_1(\omega) = p_h f_h^0(y)$. As in the previous approach, when Observer 1 receives observations which are i.i.d. conditioned on the hypothesis, the observations are treated as random variables in the product space $(\Omega^1, F_1^0, P_1^0)$. Observer 2 the probability space is $(\Omega^2, F_2^0, P_2^0) = \{0, 1\} \times S_2 \times 2^{\Omega^2}$. The product space is denoted $(\Omega^2, F_2^0, P_2^0)$. Given the observation sequences $\{Y_i = y_i\}_{i=1}^n$ and $\{Z_i = z_i\}_{i=1}^n$ for Observer 1 and Observer 2 respectively, the objective is to find $D^n_i : S^n_i \rightarrow \{0, 1\}$ such that following cost is minimized

$$\mathbb{E}_{P^0_{Y_i}}[C_{10} H_i(1 - D^n_i) + C_{01} (D^n_i)(1 - H_i)],$$

where $H_i$ denotes the hypothesis random variable for observers in their respective probability spaces.

Since the decisions of the two observers need not be the same and a sequential approach with random stopping time is to proposed in the following subsection, we extend the probability spaces of the observers as follows. Since the sequences of product measures $(\{P^n_i\}_{i=1}^2, i = 1, 2)$ are consistent, by the Kolmogorov extension theorem (subsection VII-A), for $i=1,2$, there exists measures $P^n_i$ on $(\{0, 1\} \times \{S_i\}^\infty, F^n_i)$, where $F^n_i$ is the $\sigma$-algebra generated by cylindrical sets in $(\{0, 1\} \times \{S_i\}^\infty)$, such that

$$\mathbb{P}_n^i(I^n_i(B)) = P^n_i(B) \forall B \in 2^{(0,1) \times \{S_i\}^\infty}, \forall n$$

where $I^n_i(B) = \{\omega \in \{0, 1\} \times \{S_i\}^\infty : (\omega(1), \ldots, \omega(n+1)) \in B\}$. Thus, the extended probability space for Observer $i$ is $(\{0, 1\} \times \{S_i\}^\infty, P^n_i, F^n_i)$.

Remark 2. Consider the scenario where $f_h(y, z) = f_h^0(y) f_h^1(z)$, $h = 0, 1$. Consider the estimation problem, where $H$ is estimated from $\{Y_1, Z_1, \ldots, Y_n, Z_n\}$. Let $T : S^n_1 \times S^n_2 \rightarrow S^n_1 \times \{0, 1\}$ be the mapping

$$T(Y_1, Z_1, \ldots, Y_n, Z_n) = (Y_1, D^n_1), \ldots, (Y_n, D^n_2).$$

We can consider another Bayesian estimation problem of estimating $H$ from $\{Y_1, D^n_1, \ldots, Y_n, D^n_2\}$. $T$ is a sufficient statistic (Figure 4) for the original estimation problem if and only if

$$\sum_{z_1 \in S_2} \prod_{i=1}^n f_{i,1}^0(z_i) = \sum_{z_1 \in S_2} \prod_{i=1}^n f_{i,1}^1(z_i), \forall z_1 \in S^n_2, \forall S_1.$$  

where $S^n_2$ is a set of sequences in $S^n_2$, which leads to a decision sequence $\{D_1^n = d_1^n, \ldots, D_n^n = d_n^n\}$ and $z_1^n = \{z_1, \ldots, z_n\}$. The above condition is very stringent and might not be true in most cases. Even though the $T$ is not a sufficient statistic, our objective is to design a consensus algorithm based on just the exchange of decision information. The advantage of such a scheme is that, the exchange of information is restricted to 1 bit and the observers do not have to do any other processing on their observations.

2) Algorithm-2 Testing Phase: The information state $\psi^n_i = \mathbb{P}_{P^{\omega_{Y_i}}} (\mathcal{H}_i | \mathcal{F}_n^i), i = 1, 2$, where $\mathcal{F}_n^i$ denotes the $\sigma$-algebra generated by $Y_1, \ldots, Y_n$ and $\mathcal{F}_n^i$ denotes the $\sigma$-algebra generated by $Z_1, \ldots, Z_n$. The decisions are memoryless functions of $\psi^n_i$. More precisely, they are threshold policies. Let $\pi^n_1 = \frac{1}{p_{10} + p_{11}}$ and $\pi^n_2 = \frac{1}{p_{20} + p_{21}}$. Hence, $\psi^n_i = \frac{\pi^n_i}{p_{10} \pi^n_1 + p_{20} \pi^n_2}$. For $0 < T_i < 1, \psi^n_i \geq T_i \Leftrightarrow \pi^n_i \geq \frac{T_i p_{01}}{1 - p_{01}}$. Hence the decision policy for Observer $i$ can be stated as a function of $\pi^n_i$ via:

$$D^n_i = \begin{cases} 1, \text{ if } \pi^n_i \geq T_i, \\ 0, \text{ otherwise.} \end{cases}$$

For the collaboration step, we let

$$\hat{\phi}^1_i(\{Y_i\}_{i=1}^n) = D^n_1, \hat{\phi}^2_i(\{Z_i\}_{i=1}^n) = D^n_2.$$
For an observer, a variable is said to be an exogenous random variable if it is not measurable with respect to the probability space of that observer. When Observer 1 receives the decision of Observer 2 (and vice-versa), it treats that decision as an exogenous random variable as no statistical information is available about the new random variable. Based on this 1 bit information exchange we consider a simple consensus algorithm: Let $n = 1$.

1) Observer 1 collects $Y_n$ while Observer 2 collects $Z_n$.
2) Based on $Y_1, \ldots, Y_n$, $D^1_n$ is computed by Observer 1, while $D^2_n$ is computed by Observer 2 based on $Z_1, \ldots, Z_n$.
3) If $D^1_n = D^2_n$, stop. Else increment $n$ by 1 and return to step 1.

The schematic of this algorithm is depicted in Figure 5.

C. Algorithm-3 Decentralized Approach with Two Bit Exchange

In the previous algorithm, the decision from the alternate observer was considered as an exogenous random variable by the original observer. In this subsection, we propose a scheme where the observers build joint distributions between their own observations and the decision they receive from the alternate observer.

1) Algorithm-3 Learning Phase: We let $\phi^1_n(\{Y_i\}_{i=1}^n) = D^1_n$ and $\phi^2_n(\{Z_i\}_{i=1}^n) = D^2_n$. The probability space construction for Observer 1 is described as follows: Observer 1 collects strings of finite length: $[H, Y_1, D^1_2, Y_2, D^2_3, \ldots, Y_n, D^2_n]$, where $Y_n \in S_1$ and $D^2_n$ is the decision of Observer 2, after repeating the hypothesis testing problem $n$ times. This is done by Observer 1 for every $n \in N$. $Y_1, \ldots, Y_n$ are assumed to be i.i.d. conditioned on the hypothesis and hence can be interpreted in the product space described before (section III-B). The decisions, $D^1_1, \ldots, D^1_n$ are obtained by Observer 2 using the decision policy described in section III-B2. Since $\pi^2_n$ are controlled Markov chains, $D^2_n$ are correlated. From the data strings, Observer 1 finds the empirical joint distribution of $\{H, Y_1, D^2_2\}_{i=1}^n$ denoted as $\mathcal{P}_{1,n}$. For strings of the form $(0, y, D^2_2)_{i=1}^n$ or $(1, y, D^2_2)_{i=1}^n$, which are not observed, measure 0 is assigned. Let $S^2_n = \{(h, (y, D^2_2)_{i=1}^n) : \mathcal{P}_{1,n}(h, (y, D^2_2)_{i=1}^n) > 0\}$. Hence, Observer 1 builds a family of joint distributions, $\mathcal{P}_{1,n}$, on $S^2_n \subset 2^{0(1)} \times S_1 \times \{0, 1\}^n$. We assume that the family of distributions is consistent:

$$\mathcal{P}_{1,n+1}(B \times S_1 \times \{0, 1\}) = \mathcal{P}_{1,n}(B) \forall B \in 2^{S^2_n}, \forall n.$$  

Let $B$ belong to $2^{S^2_n}$. Then a cylindrical subset of $(\{0, 1\} \times S_1 \times \{0, 1\})^n$ is:

$$L_n(B) = \{\omega \in \{0, 1\} \times S_1 \times \{0, 1\}^n : (\omega(1), \ldots, \omega(n+1)) \in B\}.$$  

Let $\mathcal{F}_1$ be the smallest $\sigma$-algebra such that it contains all cylindrical sets, i.e., for all $n$ and all $B$. By the Kolmogorov extension theorem (subsection VII-A), there exists a measure $\mathcal{P}_1$ on $(\{0, 1\} \times S_1 \times \{0, 1\})^\infty$, $\mathcal{F}_1$, such that,

$$\mathcal{P}_1(L_n(B)) = \mathcal{P}_{1,n}(B) \forall B \in 2^{S^2_n}, \forall n,$$

where, $L_n(B)$ is defined as above. Thus, two aggregated probability spaces are constructed. For Observer 1, $(\hat{\Omega}_1, \mathcal{F}_1, \mathcal{P}_1)$ is constructed where $\hat{\Omega}_1 = \{0, 1\} \times \{S_1 \times \{0, 1\}\}^\infty$. For Observer 2, $(\hat{\Omega}_2, \mathcal{F}_2, \mathcal{P}_2)$ is constructed where $\hat{\Omega}_2 = \{0, 1\} \times \{\hat{\Omega}_2 \times \{0, 1\}\}^\infty$. The sequence of measures $\{\mathcal{P}_{1,n}\}_{n \geq 1}$ is a function of the thresholds $T_1$ and $T_2$. Thus, when the thresholds for the decentralized scheme in III-B3 change, the probability space constructed as above also changes.

Based on the new probability space constructed, the observers could find a new pair of decisions. Given the observation sequences $\{Y_i = y_i, D^1_i = d^1_i\}_{i=1}^n$ and $\{Z_i = z_i, D^1_i = d^1_i\}_{i=1}^n$ for Observer 1 and Observer 2 respectively, the objective is to find $O'_n: \{S_1 \times \{0, 1\}\}^n \longrightarrow \{0, 1\}$ such that the following cost is minimized

$$\mathbb{E}_{\mathcal{F}_1}|C_{10}H_1(1 - O'_n) + C_{01}(O'_n)(1 - H_1)|.$$  

2) Algorithm-3 Testing Phase: To solve the problem for Observer 1, we define a new set of information states as:

$$\alpha^1_n = \mathbb{E}_{\mathcal{F}_1}[H_1|Y_n, D^1_n], \alpha^2_n = \mathbb{E}_{\mathcal{F}_1}[H_1|Y_n, D^2_n].$$

$$\alpha^1_n = \mathcal{P}_1(D^1_i = d^1_i|Y_1 = y_1, H_1 = 1) \mathcal{P}_1(Y_1 = y_1, H_1 = 1),$$

and for any $n$,

$$\alpha^1_n = \sum_{j=0,1} \mathcal{P}_1(Y_n = y_n, D^1_n = d^1_n|Y_1 = y_1, H_1 = 1),$$

$$\alpha^2_n = \sum_{j=0,1} \mathcal{P}_1(Y_n = y_n, D^2_n = d^2_n|Y_1 = y_1, H_1 = j),$$

where $\chi$ is the indicator function, $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. The decision policy is:

$$O'_n = \begin{cases} 1, \text{ if } \alpha^1_n \geq \alpha^2_n \\ 0, \text{ otherwise.} \end{cases}$$  

Using a similar procedure, $\{\alpha^2_n\}_{n \geq 1}$ can be defined and $\{O^2_n\}_{n \geq 1}$ can be found by Observer 2. For the collaboration step, we let

$$\phi^1_n(\{Y_i\}_{i=1}^n) = D^1_n, \hat{\phi}^2_n(\{Z_i\}_{i=1}^n) = D^2_n, \hat{\phi}^2_n(\{Y_i\}_{i=1}^n), \hat{\phi}^2_n(\{Z_i\}_{i=1}^n) = O^2_n.$$
The consensus algorithm can be modified from Algorithm-2 (subsection III-B2) as follows. Let $n = 1$.

1. Observer 1 collects $Y_n$ while Observer 2 collects $Z_n$.
2. Based on $Y_n$, $\pi_{n-1}^1$, $D_n^1$ is computed by Observer 1 while $D_n^2$ is computed by Observer 2 based on $Z_n$, $\pi_{n-1}^2$.
3. If $D_n^1 = D_n^2$, stop. Else, $O_n^1$ is computed by Observer 1 using $\alpha_{n-1}^1$, $\{Y_i, D_i^1\}_{i=1}^n$ and $O_n^2$ is computed by Observer 2 using $\alpha_{n-1}^2$, $\{Z_i, D_i^2\}_{i=1}^n$.
4. If $O_n^1 = O_n^2$, stop. Else, increment $n$ by 1 and return to step 1.

Figure 5 captures this approach. We note that while $\{O_n^1\}$ are measurable in the new probability spaces, $\{O_n^2\}$ is not. Hence they are treated as exogenous random variables. The interpretation of the exchange of the additional 1 bit information is presented in subsection IV-D.

D. Algorithm-4 Alternative Decentralized Approach with more than 1 Bit Exchange

Motivated by the interpretation of the information states in the testing phase of Algorithm-3 (subsection III-B2), we present a modification of Algorithm-3 where the construction of the “larger” probability spaces compared to Algorithm-2 can be skipped while the essence of the improved performance of Algorithm-3 (in the testing phase) achieved through additional data exchange can still be achieved through exchange of more than 1 bit.

1. Algorithm-4 Learning Phase: The probability spaces constructed for the observers in learning phase of Algorithm-2, subsection III-B1 are retained. Thus, the probability space for Observer $i$ is $\{(0,1) \times \{S_i\} \times \{\bar{p}_i^1, p_i^1\}\}$. The decision making problems as defined in III-B1 and III-C are also retained.

2. Algorithm-4 Testing Phase: We define,

$$D_n^1 = \frac{P_n^1(D_n^1 = d_1^1 | \{D_j^1 = d_j^1\}_{j=1}^{n-1}, H_i = 0)}{P_n^2(D_n^2 = d_1^2 | \{D_j^2 = d_j^2\}_{j=1}^{n-1}, H_i = 1)},$$

as the estimate of accuracy of Observer $i$ by the alternate observer (subsection IV-D), and, the modified information state recursively as,

$$\alpha_{n}^1 = \frac{\psi_1}{(1 - \beta_1^2)\psi_1 + \beta_1^2}, \alpha_{n}^2 = \frac{\psi_2}{(1 - \beta_1^2)\psi_2 + \beta_1^2},$$

Then the following algorithm is executed. Let $n = 1$,

1. Observer 1 collects $Y_n$ while Observer 2 collects $Z_n$.
2. Based on $Y_n$, $\pi_{n-1}^1$, $\pi_{n-1}^2$ is found by Observer 1. Using $\pi_{n-1}^1$, $D_n^1$ is found by Observer 1. Based on $Z_n$, $\pi_{n-1}^2$, $\pi_{n-2}^2$ is found by Observer 2. Using $\pi_{n-2}^2$, $D_n^2$ is found by Observer 2, as in subsection III-B2.

3. The observers exchange their decisions. $D_n^1$ is treated as an exogenous random variable by Observer 2 while $D_n^2$ is treated as an exogenous random variable by Observer 1. If $D_n^1 = D_n^2$, stop. Else $\beta_{n}^1$ is sent by Observer 1 to Observer 2 while $\beta_{n}^2$ is sent by Observer 2 to Observer 1.

4. Using $Y_n$, $\alpha_{n-1}^1$ and $\beta_{n-1}^2$, $\tilde{\alpha}_{n}^1$ is computed by Observer 1 while using $Z_n$, $\alpha_{n-1}^2$, and $\beta_{n-1}^1$, $\tilde{\alpha}_{n}^2$ is computed by Observer 2. Using $\tilde{\alpha}_{n}^1$, $O_n^1$ is computed by Observer 1 while using $\tilde{\alpha}_{n}^2$, $O_n^2$ is computed by Observer 2 as in subsection III-C.

5. The observers exchange their new decisions. $O_n^1$ is treated as an exogenous random variable by Observer 2 while $O_n^2$ is treated as an exogenous random variable by Observer 1. If $O_n^1 = O_n^2$, then stop. Else increment $n$ by 1 and return to step 1.

Figure 6 captures the above modified algorithm. The advantage of this scheme is that the construction of the aggregated probability space is not needed. The scheme can be executed even when conditions on the joint distribution of the observations and decisions from the alternate observer do not hold, though it might not be useful.

IV. ANALYSIS OF THE ALGORITHMS

In this section, we analyze the structural, qualitative, and quantitative properties of the algorithms presented in the previous section. First, in subsection IV-A, we prove that convergence of the distributions estimated in the learning phase to the true distributions. In subsection IV-B, we discuss the probability space construction in the learning phase. In subsection IV-C, we prove the convergence of the consensus step in the testing phase of the algorithms. In subsection IV-D, we compare the rate of decay of the probability of error in Algorithm-1 and of the probability of agreement on the wrong belief in Algorithm-2.
A. Algorithm-1 and Algorithm-3 Learning Phase: Convergence of Distributions

1) Algorithm-1: Consider the abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is unknown. Let $\{H_j, Y_j, Z_j\}_{j \in \mathbb{N}} \subset L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the sequence of random variables used to estimate the distribution, $\mathbb{P}^\varepsilon(\cdot)$, where $j$ denotes the experiment number. We assume that this sequence of random variables is i.i.d, i.e.,

$$
P(\{H_1, Y_1, Z_1\} \in E_1 \cap \{H_2, Y_2, Z_2\} \in E_2 \cap \ldots \cap \{H_j, Y_j, Z_j\} \in E_j) = \prod_{p=1}^j P(\{H_{p}, Y_{p,k}, Z_{p,k}\} \in E_{p})
$$

for any $j \in \mathbb{N}$. $E \subset \{0,1\} \times \{S_1 \times S_2\}$, i.e., that the random variables have the same distribution across experiments. The estimated distributions are defined as,

$$
\mathcal{P}_1, n(\{h, y, d_k^{D}_{k}\}_{k=1}^n) = \lim_{j \rightarrow \infty} \sum_{j=1}^j P(\{y_{1:k}\} \in \{H_{y_{1:k}}, D_{y_{1:k}}\}_{k=1}^n),
$$

Theorem IV.2. The estimated distributions equal the true distribution, i.e.,

$$
P(x_{y_{1:k}}(\{H_{y_{1:k}}, D_{y_{1:k}}\}_{k=1}^n)) = P(x_{h_{1:k}}(\{H_{h_{1:k}}, D_{h_{1:k}}\}_{k=1}^n)) = P(x_{z_{1:k}}(\{H_{z_{1:k}}, D_{z_{1:k}}\}_{k=1}^n))
$$

Proof. Follows from the strong law of large numbers. \[ \square \]

The same proof can be extended to prove that the estimated measures in Algorithm-2 equal the true distributions,

$$
P_1(h, y) = \lim_{j \rightarrow \infty} \sum_{j=1}^j P(\{y_{1:k}\} \in \{H_{y_{1:k}}, D_{y_{1:k}}\}_{k=1}^n),
$$

Proof. Follows from the strong law of large numbers. \[ \square \]

2) Algorithm-3: Consider the abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{H_j, Y_j, Z_j\}_{j \in \mathbb{N}} \subset L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the sequence of random variables used to estimate the distributions, $\mathbb{P}_1(\cdot), \mathcal{P}_2(\cdot)$, $\mathcal{P}_3(\cdot)$, $\mathcal{P}_4(\cdot), \mathcal{P}_5(\cdot)$. Index $j$ denotes the experiment number, while index $k$ denotes the iteration; for example the random variable $Y_{j,k}$ corresponds to the observation of Observer 1 in the $k$th iteration of experiment $j$. It is assumed that,

$$
\{H_{j,1}, Y_{j,1}, Z_{j,1}\}_{k \in \mathbb{N}}, \{H_{j,2}, Y_{j,2}, Z_{j,2}\}_{k \in \mathbb{N}}, \{H_{j,3}, Y_{j,3}, Z_{j,3}\}_{k \in \mathbb{N}}, \ldots
$$

are independent for any $n \in \mathbb{N}$ and $\{j_1, \ldots, j_n\} \subset \mathbb{N}$, i.e., the random variables are independent across the experiments. Thus,

$$
P(\{H_{j,1}, Y_{j,1}, Z_{j,1}\}_{k=1}^n \in E_1 \cap \{H_{j,2}, Y_{j,2}, Z_{j,2}\}_{k=1}^n \in E_2 \cap \ldots \cap \{H_{j,j}, Y_{j,j}, Z_{j,j}\}_{k=1}^n \in E_j)
$$

for any $j \in \mathbb{N}$, $\{h_1, \ldots, h_j\} \subset \mathbb{N}$, $\{n_1, \ldots, n_j\} \subset \mathbb{N}$, and $E_p \subset \{0,1\} \times \{S_1 \times S_2\}$, $p = 1, \ldots, j$. The following assumption is made on the distribution.

$$
P(\{H_{j,1}, Y_{j,1}, Z_{j,1}\}_{k=1}^n \in E) = P(\{H_{j,2}, Y_{j,2}, Z_{j,2}\}_{k=1}^n \in E)
$$

for any $j_1, j_2, n \in \mathbb{N}$, $E \subset \{0,1\} \times \{S_1 \times S_2\}$, i.e., that the random variables have the same distribution across experiments.
B. Algorithm-3 Learning Phase: Probability Space Construction

Our objective is to compare the distributions \( \mathcal{P}_1, \mathcal{P}_2 \) (subsection III-C) and \( \mathbb{P}^{c,s} \) (subsection III-A). Two distributions on the same \( \sigma \)-algebra can be compared, if the distributions are equal on every sample space. Even if we find a common sample space to redefine the measures, the relationship between the corresponding \( \sigma \)-algebras is not clear. We investigate the same in this subsection. Comparison of distributions on two different \( \sigma \)-algebras is not obvious. Though two \( \sigma \)-algebras could have different subsets of the same sample space, if they have the “same” number of sets, then the distributions defined on them could be compared as below.

**Definition 1.** Two probability spaces are said to be similar if there exists a bijection between the \( \sigma \)-algebras and the measures are equal up to this bijection. \((\Gamma_1,F_1,P_1) \simeq (\Gamma_2,F_2,P_2)\).

If \( \exists \phi : F_1 \to F_2 \) such that
\[
\phi(E_1) = \phi(\hat{E}_1) \Rightarrow E_1 = \hat{E}_1
\]
\[
\forall E_2 \in F_2, \exists E_1 \in F_1 \exists \phi^{-1}(E_2) = E_1
\]
\[
P_1(E_1) = P_2(\phi(E_1)) \text{ and } P_1(\phi^{-1}(E_2)) = P_2(E_2)
\]

Since \( \phi \) also preserves the \( \sigma \)-algebra structure, it can be considered to be an isomorphism between the two probability spaces. We use the notation \( \simeq \) to denote non-similarity of probability spaces.

We now present an alternate construction of the probability space for Observer 1 in Algorithm-3 so that its sample space is the same as that of the central coordinator in Algorithm-1. Consider, a “modified” observation space at sample number, \( (0,1) \times S_1 \times S_2 \). Two sequences \( \{h,\{y_i,z_i\}_{i=1}^n\} \) and \( \{h,\{y_i,z_i\}_{i=1}^n\} \) are said to be related, denoted by \( \{h,\{y_i,z_i\}_{i=1}^n\} \sim \{h,\{y_i,z_i\}_{i=1}^n\} \), if \( \{z_i\}_{i=1}^n \) and \( \{\bar{z}_i\}_{i=1}^n \) lead to the same decision sequence, \( \{d_i\}_{i=1}^n \). The relation \( \sim \) is:
- reflexive: \( \{h,\{y_i,z_i\}_{i=1}^n\} \sim \{h,\{y_i,z_i\}_{i=1}^n\} \)
- symmetric: \( \{h,\{y_i,z_i\}_{i=1}^n\} \sim \{h,\{y_i,z_i\}_{i=1}^n\} \Rightarrow \{h,\{y_i,\bar{z}_i\}_{i=1}^n\} \sim \{h,\{y_i,\bar{z}_i\}_{i=1}^n\} \)
- transitive: \( \{h,\{y_i,z_i\}_{i=1}^n\} \sim \{h,\{y_i,\bar{z}_i\}_{i=1}^n\}, \{h,\{y_i,\bar{z}_i\}_{i=1}^n\} \sim \{h,\{y_i,\bar{z}_i\}_{i=1}^n\} \Rightarrow \{h,\{y_i,z_i\}_{i=1}^n\} \sim \{h,\{y_i,\bar{z}_i\}_{i=1}^n\} \)

Hence \( \sim \) is an equivalence relation. Let \( E_{n}^1 = \{0,1\} \times S_1 \times S_2 \) \( \sim \) be the quotient set. Hence \( E_{n}^1 \) consists of all equivalent classes where each class contains all sequences which are equivalent to each other. Let \( \Sigma_n^1 \) be the \( \sigma \)-algebra generated by the classes in \( E_{n}^1 \). Since an equivalence relation partitions the set, any pair of classes in \( E_{n}^1 \) are mutually exclusive. Thus, \( \Sigma_n^1 \) is obtained by taking finite unions of classes in \( E_{n}^1 \). Every sequence in \( \Sigma_n^1 \) corresponds to a unique class in \( E_{n}^1 \). For every class in \( E_{n}^1 \), there exists a sequence in \( \Sigma_n^1 \). Hence there is a bijection, \( \phi_n^1 \) from \( \Sigma_n^1 \) to \( E_{n}^1 \). The measure on \( (E_{n}^1,\Sigma_n^1) \) can be defined as,
\[
\mathcal{P}_n^1(E) = \mathcal{P}_n^1(\phi_n^{-1}(E)), \forall E \in \Sigma_n^1
\]

Thus by construction,
\[
(E_{n}^1,\Sigma_n^1,\mathcal{P}_n^1) \simeq (E_{n}^1,\Sigma_n^1,\mathcal{P}_n^1), \forall n.
\]

From the consistency of \( \mathcal{P}_n^1 \), it follows that
\[
\mathcal{P}_{n+1}(B \times S_1 \times S_2) = \mathcal{P}_{n+1}(B) \forall B \in \Sigma_n^1, \forall n.
\]

Let \( B \) belong to \( \Sigma_n^1 \). Then a cylindrical subset of \( (\{0,1\} \times \{S_1 \times S_2\})^\omega \) is:
\[
I_n(B) = \{ \omega \in \{0,1\} \times \{S_1 \times S_2\}^\omega : (\omega(1),...\omega(n+1)) \in B \}
\]

Let \( G_1 \) be the smallest \( \sigma \)-algebra such that it contains all cylindrical sets, i.e., for all \( n \) and all \( B \). By the Kolmogorov extension theorem (subsection VII-A), there exists a measure \( \mathcal{P}_n^1 \) on \( (\{0,1\} \times \{S_1 \times S_2\})^\omega, G_1 \) such that,
\[
\mathcal{P}_n(I_n(B)) = \mathcal{P}_{n+1}(B) \forall B \in \Sigma_n^1, \forall n
\]

where, \( I_n(B) \) is defined just above. Define the mapping \( \phi^1 : \mathcal{F}_1 \to G_1 \) as:
\[
\phi^1(I_n(B)) = I_n(\phi_n(B)), I_n(B) \in \mathcal{F}_1,
\]
\[
\phi^1\left(\bigcup_{i=1}^\infty I_n(B_i)\right) = \bigcup_{i=1}^\infty I_n(\phi_n(B_i)),
\]

where \( I_n(B) \) is a cylindrical set. It is straightforward to show that \( \phi^1 \) is a bijection by using the following properties:
\[
(I_n(B))^\circ = I_n(B^\circ), I_n(B_1) = I_n(B_2) \iff B_1 = B_2.
\]

For Observer 2, an equivalence relation like above can be defined and \( \Sigma_n^2 \) can be found. Let \( G_2 \) be the smallest \( \sigma \)-algebra which contains all the cylindrical sets constructed from \( \{\Sigma_n^2\}^\omega_{n=1} \). For Observer 2, the probability space constructed is \( (\{0,1\} \times \{S_1 \times S_2\})^\omega, G_2, \mathcal{P}_2 \), where \( \mathcal{P}_2 \) is the measure obtained from Kolmogorov extension theorem (subsection VII-A). A bijection \( \phi^2 : \mathcal{F}_2 \to G_2 \) can also be defined. Now let us consider the central coordinator (mentioned in section II.B).

We recall that \( \mathbb{P}^{c,s} \) is the smallest \( \sigma \)-algebra which contains all the cylindrical sets constructed from \( \{2^{\{0,1\} \times \{S_1 \times S_2\}}\}^\omega_{n=1} \) and the extended probability space associated with central coordinator is \( (\{0,1\} \times \{S_1 \times S_2\})^\omega, \mathbb{P}^{c,s}, \mathcal{P}^{c,s} \).

**Theorem IV.3.** Given the above constructions,
\[
(\tilde{\mathcal{D}}_n, \tilde{\mathcal{F}}_n, \tilde{\mathcal{P}}_n) \simeq \{(0,1) \times \{S_1 \times S_2\}^\omega, G_1, \tilde{\mathcal{P}}_1\}
\]

If \( |S_1| > 2 \) and \( |S_2| > 2 \), then \( \{(0,1) \times \{S_1 \times S_2\}^\omega, G_1, \tilde{\mathcal{P}}_1\} \sim \{(0,1) \times \{S_1 \times S_2\}^\omega, \mathbb{P}^{c,s}\} \sim \{(0,1) \times \{S_1 \times S_2\}^\omega, \mathbb{P}^{c,s}\} \sim \{(0,1) \times \{S_1 \times S_2\}^\omega, \mathbb{P}^{c,s}\}\)

**Proof.** The first similarity in Equation (1) follows from the constructions and the existence of the bijections \( \phi^1 \). First, we note that the sample space for the two observers and the central coordinator are the same. The associated \( \sigma \)-algebras are different. Since the equivalence relation partitions the observation space and each equivalent class contains multiple sequences, \( \Sigma_n^1, \Sigma_n^2 \subset \{2^{\{0,1\} \times \{S_1 \times S_2\}}\}^\omega_{n=1} \). We assume \( |S_1| > 2 \) and \( |S_2| > 2 \) to avoid pathological situations where the equality of the algebras could possibly hold. For \( n \in \mathbb{N} \), the number of sequences in the observation space is \( 2 \times |S_1|^n \times |S_2|^n \). For Observer 1, the maximum number of equivalent classes is \( 2 \times |S_1|^n \times 2^n \). Since \( 2 \times |S_1|^n \times 2^n < 2 \times |S_1|^n \times |S_2|^n \), by the Pigeon hole principle [28], there is at least one class which contains
multiple sequences from the observation space. Let this class be denoted by $C_n$. The cylindrical set corresponding to $C_n$ is

$$I_n(C_n) = \{ \omega \in \{0, 1\} \times \{S_1 \times S_2\}^{\infty} : (\omega(1), ..., \omega(n + 1)) \in C_n \},$$

and it belongs to $G_1$ and $\mathbb{F}^\circ$. For any $\{h, y_i, z_i\}^{n}_{i=1} \in C_n$, the cylindrical set,

$$I_n(\{h, y_i, z_i\}^{n}_{i=1}) = \{ \omega \in \{0, 1\} \times \{S_1 \times S_2\}^{\infty} : (\omega(1), ..., \omega(n + 1)) \in \{h, y_i, z_i\}^{n}_{i=1} \}$$

(3)

belongs to $\mathbb{F}^\circ$, but does not belong to $G_1$. The reasoning for the same is as follows. This cylindrical set cannot be obtained from $I_n(C_n)$ as the set $C_n \setminus \{h, y_i, z_i\}^{n}_{i=1} \notin \Sigma^n$. We note that the partition from the equivalence relation at $n + 1$ is a refinement of the partition from the equivalence relation at $n$, in the following sense,

$$\{C^n_i : C^n_i \cap C^n_j = \emptyset, \cup C^n_i = \{0, 1\} \times S^n_i \times S^n_j, C^n_i \times S^n_j \in P \}$$

Thus any cylindrical set $I_{n+1}(B_{n+1}), B_{n+1} \in \Sigma^{n+1}$ is a subset of some cylindrical set $I_n(B_n), B_n \in \Sigma^n$. From the above relations, we note that the cylindrical set for a sequence like the set in Equation (3), cannot be obtained from the union of cylindrical sets corresponding to equivalence classes at $m, m > n$ on the observation space, $\{0, 1\} \times S^m \times S^m$. The finite union of cylindrical sets corresponding to equivalent classes at $m, m > n$, will only result in cylindrical sets corresponding to equivalent classes at $n$. Hence, the set in Equation (3) is neither an atomic element of the $\sigma$-algebra nor can it be obtained through the unions and intersections of other subsets in the $\sigma$-algebra. For each one of the sets, $I_n(\{h, y_i, z_i\}^{n}_{i=1}), \{h, y_i, z_i\}^{n}_{i=1} \in C_n$ one cannot find a unique element to which it gets mapped in $I_n(E^n_i : E^n_i \in 2^{\mathbb{S}^n})$. However one can find a surjection from $I_n(E^n_i : E^n_i \in 2^{\mathbb{S}^n})$ to $I_n(E^n_i : E^n_i \in 2^{\mathbb{S}^{n+1}})$. Hence the set of all cylindrical subsets for Observer 1 (and Observer 2) is a strict subset of the set of all cylindrical subsets for the central coordinator, which implies that $G_1 \subset \mathbb{F}^\circ$ and $G_2 \subset \mathbb{F}^\circ$, which in turn implies Equation (2). □

If the aggregated probability spaces of the two observers are similar then $(\mathbb{S}^1_n, 2\mathbb{S}^1_n, \mathcal{P}_{1,n}) \simeq (\mathbb{S}^2_n, 2\mathbb{S}^2_n, \mathcal{P}_{2,n})$, $\forall n$. Such a condition implies $|S_1| = |S_2|$, and that the distribution $\mathcal{P}_{1,n}$ is a permutation of $\mathcal{P}_{2,n}$ (and vice-versa) for all $n$. Except for this pathological case it is safe to conclude that $|\{0, 1\} \times \{S_1 \times S_2\}^{\infty}, G_1, \mathcal{P}_1| = |\{0, 1\} \times \{S_1 \times S_2\}^{\infty}, G_2, \mathcal{P}_2|$. Thus in the approach mentioned in section III-C1, a probability measure is not assigned to every subset of the observation space, but is assigned to those subsets which correspond to an observable outcome. The true $\sigma$-algebra is a coarse $\sigma$-algebra compared to the original $\sigma$-algebra. The same concept has been emphasized in □ 23, i.e., models often require coarse event $\sigma$-algebras. Through examples, it is shown that in certain experiments it might not be possible to assign measure to the Borel $\sigma$-algebra.

C. Algorithm-2 Testing Phase: Convergence to Consensus

**Theorem IV.4.** In Algorithm-2, in the testing phase, for almost all the sequence paths $\omega^1 \in \{0, 1\} \times \{S_1\}^{\infty}$ and $\omega^2 \in \{0, 1\} \times \{S_2\}^{\infty}$, there exists $n(\omega^1, \omega^2) \in \mathbb{N}$ such that $D_n(\omega^1, \omega^2) = D_n(\omega^1, \omega^2) \in (0, 1]$, i.e., the observers eventually agree upon their decisions about the true state of nature.

**Proof.** $\{\psi^n_\alpha, \mathcal{J}_n^n\}_{n \geq 1}$ are martingales in $\{\{0, 1\} \times \{S_1\}^{\infty}, \mathbb{F}_i^n, \mathbb{P}_i^n\}$, i.e.,

$$\mathbb{E}_{\mathbb{F}_i^n}[\psi_{n+1}^\alpha | \mathcal{J}_n^n] = \mathbb{E}_{\mathbb{F}_i^n}[H | \mathcal{J}_n^n],$$

Hence by Doob’s theorem □ 26, it follows that

$$\lim_{n \rightarrow \infty} \psi^n_\alpha = H_i, \mathbb{P}_i^n \text{ a.s.}$$

Hence for almost all sequence paths $\omega^1 \in \{0, 1\} \times \{S_1\}^{\infty}$ there exist $N(\omega^1) \in \mathbb{N}$ such that $D_{N(\omega^1)} = H_i \forall n \geq N(\omega^1)$. For observer $i$, for almost all sample paths (or any sequence of observations), $\omega^1$, there exists a finite natural number $N(\omega^1)$ such that the decision after collecting $N(\omega^1)$ observations, or more, will always be the true hypothesis. Hence, after both observers collect max($N(\omega^1), N(\omega^2)$) number of samples, both their decisions will be the true hypothesis. Hence the observers achieve consensus for almost all sample paths. □

D. Algorithm 3 Testing Phase: Interpretation of Additional Data Exchange

The information states used to solve the hypothesis testing problem in subsection III-C1 can be interpreted as follows. For Observer 1, $n = 1, \alpha_1^1 = \mathbb{E}_{\mathbb{P}_1}[H_1 | Y_1, D_1^2]$ which can be expanded as follows,

$$\alpha_1^1 = \frac{\mathcal{P}_1[D_1^2 = d_1^2 | Y_1 = y_1, H_1 = 1]}{\sum_{i=0,1} \mathcal{P}_1[D_1^2 = d_1^2 | Y_1 = y_1, H_1 = i]} \mathcal{P}_1(Y_1 = y_1, H_1 = i),$$

where,

$$\beta_1^2 = \frac{\mathcal{P}_1[D_1^2 = d_1^2 | Y_1 = y_1, H_1 = 0]}{\mathcal{P}_1[D_1^2 = d_1^2 | Y_1 = y_1, H_1 = 1]},$$

The decision by Observer 1 after finding $\alpha_1^1 = 0 \leq 1 = 1$ if $\alpha_1^1 \geq 0$. Hence, $O_1^1 = 0$. $O_1^1$ is sent to Observer 2 which treats it as an exogenous random variable. $O_1^1$ is found by Observer 2 and sent to Observer 1 which treats it as an exogenous random variable. Suppose $\beta_2^2 = 1 + x$, then $\alpha_1^1 = \frac{w^1}{1 + x(1 - w^1)}$. Consider the case where $D_1^2 = 0$ and $D_1^2 = 1$. If $\beta_2^2 > 1$, i.e., $x > 0$, then $\alpha_1^1 < \psi_1^1, \alpha_1^1$ could be less than the threshold, which implies $O_1^1 = 0$. If $\beta_2^2 > 1$ then consensus is achieved. If $\beta_2^2 < 1$, i.e., $x < 0$, then $\alpha_1^1 > \psi_1^1, \alpha_1^1$ remains greater than the threshold, which implies $O_1^1 = 1$. Hence $\beta_2^2$ could be interpreted as an estimate of the accuracy of Observer 2 by Observer 1.
The above expansion and interpretation of $\alpha_1$, motivates us to consider the expansion of the information state, $\alpha_n$, for all $n$ and understand that it can be defined by alternative means without construction of the extended probability space as in subsection III-C. Indeed, it can be achieved through the following assumption and subsequent expansion. Suppose for Observer 1 the observations collected are independent of the decisions received from Observer 2 conditioned on either hypothesis, i.e., for $j = 0, 1$,

$$\mathcal{P}_1(\{Y_i = y_i | H_1 = j\}) = \mathcal{P}_1(\{Y_i = y_i | H_1 = j\}) \mathcal{P}_1(\{D_i^n = d_i^n | H_1 = j\}) = \left[ \prod_{i=1}^n \mathcal{P}_1(Y_i = y_i | H_1 = j) \right] \mathcal{P}_1(\{D_i^n = d_i^n | H_1 = j\}).$$

Similarly for Observer 2, for $j = 0, 1$,

$$\mathcal{P}_2(\{Z_i = z_i, D_i = d_i | H_2 = j\}) = \left[ \prod_{i=1}^n \mathcal{P}_2(Z_i = z_i | H_2 = j) \right] \mathcal{P}_2(\{D_i = d_i | H_2 = j\}).$$

A sufficient condition for the above is that the observations collected by Observer 1 and Observer 2 are independent conditioned on the hypothesis. Then, $\alpha_n$ can be computed as follows:

$$\alpha_n = \frac{\mathcal{P}_1(\{D_2^n = d_2^n | H_1 = 1\}) \sum_{j=0,1} \prod_{i=1}^n \mathcal{P}_1(Y_i = y_i | H_1 = j) \mathcal{P}_1(\{D_i^n = d_i^n | H_1 = j\})}{\mathcal{P}_1(\{D_2^n = d_2^n | H_1 = 1\}) \sum_{j=0,1} \prod_{i=1}^n \mathcal{P}_1(Y_i = y_i | H_1 = j) \mathcal{P}_1(\{D_i^n = d_i^n | H_1 = j\}).}

$$

In the above, the main component needed for finding $\alpha_n$ is,

$$\beta_n^2 = \frac{\mathcal{P}_1(D_2^n = d_2^n | H_1 = 0) \mathcal{P}_1(D_2^n = d_2^n | H_1 = 1)}{\mathcal{P}_1(D_2^n = d_2^n | H_1 = 0) \mathcal{P}_1(D_2^n = d_2^n | H_1 = 1)} = \frac{\mathcal{P}_2(D_2^n = d_2^n | \{D_i^n = d_i^n | H_1 = 0\}, H_2 = 0)}{\mathcal{P}_2(D_2^n = d_2^n | \{D_i^n = d_i^n | H_1 = 1\}, H_2 = 1)}$$

$\beta_n^2$ can be computed by Observer 2 as above using the distributions estimated in the learning phase of Algorithm-2 and the following expansion. Given $\{d_i^n\}_{i=1}^n$,

$$\sum_{\{z_i \in \Sigma_2: \chi_1(d_i^n) (\sigma_2^n \geq t_2) + \chi_0(d_i^n) (\sigma_2^n < t_2)\}} \sum_{\{z_i \in \Sigma_2: \chi_1(d_i^n) (\sigma_2^n \geq t_2) + \chi_0(d_i^n) (\sigma_2^n < t_2)\} \chi_1(1) \alpha_{n-1} + \chi_0(1) (1 - \alpha_{n-1})} \mathcal{P}_2(Y_n = y_n | H_1 = 1) \alpha_{n-1} + \mathcal{P}_1(Y_n = y_n | H_1 = 0) (1 - \alpha_{n-1} - \beta_n^2).$$

$$R_c^* \geq \min_{\eta_0, \eta_0 \geq 0} \left[ \mathbb{D}_{KL}(Q^{h}_0 || f_0), \mathbb{D}_{KL}(Q^{h}_f || f_1) \right].$$
2) Algorithm 2: To compare the rate of decay of the probability of error in the second approach to that in the first approach, we consider that in the second approach there is a hypothetical central coordinator where the joint distribution was built. Let,
\[ \mathcal{B}_n^1 = \{(Y_i, Z_i)_{i=1}^n \in S_1 \times S_2 | D_n^1 = 1 \} \]  \hspace{0.5cm} (6)
\[ \mathcal{B}_n^2 = \{(Y_i, Z_i)_{i=1}^n \in S_1 \times S_2 | D_n^1 = 0 \} \]  \hspace{0.5cm} (7)
For the probability space \((\Omega_n, \mathcal{F}_n, \mathbb{P}_n)\), the algebra \(\mathcal{F}_n\) contains all possible subsets of the product space. Hence \(\mathcal{B}_n^1\) and \(\mathcal{B}_n^2\) are measurable sets. Note that, the decision regions \(\mathcal{B}_n^1\) and \(\mathcal{B}_n^2\) depend on thresholds \(T_1\) and \(T_2\) respectively; by changing the thresholds different decision regions can be generated. Given a fixed number of samples, \(n\), to both the observers, let \(D_n^1\) and \(D_n^2\) denote their decisions. The probability that the two observers agree on the wrong belief is, \(\rho_n\),
\[ \rho_n = \mathbb{P}_n(D_c \neq H) = p_0 \mu_n + p_1 v_n, \] where \(D_c = D_n^1 = D_n^2\). The rate of decay of the probability of agreement on the wrong belief for the decentralized approach is defined as:
\[ R_d = \lim_{n \to \infty} - \frac{1}{n} \log_2 (\rho_n). \] The optimal rate of decay of the probability of agreement on the wrong belief for the decentralized approach is defined by optimizing over thresholds:
\[ R_d^* = \lim_{n \to \infty} - \frac{1}{n} \log_2 \left( \min_{\mathcal{A}_n^1, \mathcal{A}_n^2 \subseteq S_1 \times S_2} \rho_n \right). \]
Define, the following probability distributions: for \(h = 0, 1\), \(\lambda_h \geq 0, \sigma_0 \geq 0\)
\[ \mathcal{Q}_{\lambda_h, \sigma_0}^h(y, z) = \frac{K_h f_0(y, z) (f_0^1(y))^{\lambda_h} (f_0^2(z))^{\sigma_0}}{(f_1^1(y))^{\lambda_h} (f_1^2(z))^{\sigma_0}} \]  \hspace{0.5cm} (8)
\[ K_h = \left[ \sum_{y,z} f_0(y, z) (f_0^1(y))^{\lambda_h} (f_0^2(z))^{\sigma_0} \right]^{-1} \]
where \(s(h) = 1\) if \(h = 1\) and \(s(h) = -1\) if \(h = 0\). Then,
\[ R_d^* = \max_{\lambda_h \geq 0, \sigma_0 \geq 0, |h| = 1} \min \left[ \mathbb{D}_{KL}(\mathcal{Q}_{\lambda_h, \sigma_0}^0 || f_0), \mathbb{D}_{KL}(\mathcal{Q}_{\lambda_h, \sigma_0}^1 || f_1) \right]. \] \hspace{0.5cm} (9)

Theorem IV.5. Given the above definitions, and if \(f_0(y, z) = f_0^1(y) f_0^2(z)\) and \(f_1(y, z) = f_1^1(y) f_1^2(z)\), then
\[ R_d^* \geq R^*. \] \hspace{0.5cm} (10)

For the proof of Equations (4), (5), (8), (9), and the above result we refer to the Appendix.

Corollary IV.6. If the inequality is strict in Equation (10), then \( \exists N \) such that
\[ \min_{\mathcal{A}_n^1, \mathcal{A}_n^2 \subseteq S_1 \times S_2} \rho_n < \min_{\lambda_h \geq 0, \sigma_0 \geq 0} \gamma_n, \ \forall n \geq N. \]
of observer 2 is 0 ($B_n^1$). The observers can be wrong only in regions $B_n^1$ and $B_n^2$ depending on the true hypothesis. Since the measure of regions $B_n'_{1}$ and $B_n'_{2}$ are likely going to be less than the measure of the regions $A_n$ or $A_n'$ the probability of the observers agreeing and being wrong in the second approach is going to be likely less than the probability of error of the central coordinator.

**Remark 3.** The consensus algorithm presented in section III-(B2) translates to considering sets of the form $\{(Y_i, Z_i)^n_{i=1} \in S_k^1 \times S_k^2 \} \ni \{D^1_1 \neq D^2_1 \}_{j=1}^{n-1}, D^1_n = D^2_n = 1$ and $\{(Y_i, Z_i)^n_{i=1} \in S_k^1 \times S_k^2 \} \ni \{D^1_1 \neq D^2_1 \}_{j=1}^{n-1}, D^1_n = D^2_n = 0$ in section IV-(E2). It is essential that these sets can be equivalently captured by a set of distributions, we consider a superset of the sets described in (6) and (7). Thus, we are able to only obtain an upper bound for the probability of error in section IV-(E3).

**Remark 4.** There could be other possible schemes for decentralized detection. For example each observer could individually solve a stopping time problem. The times at which they stop are functions of the probability of error they want to achieve. Hence the observers stop at random times and send their decision information when they stop. The same consensus protocol could be used, i.e., the observers stop only when they both arrive at the same decision. In this scheme the probability of error of the decentralized scheme is upper bounded by the max of the probability of error of the individual observers.

4) **Remark on the Rate of Convergence of Algorithm-3:**

Even though the two observers do not share a common probability space, to compare the probability error we consider the same joint distribution as in Algorithm-1. The probability of error is given by:

$$P_{e,n} = \sum_{\{y^*, z^* : (\alpha_0^1 \geq T_3 \cap \alpha_0^2 \geq T_4)\}} f_0(y^*, z^*) + \sum_{\{y^*, z^* : (\alpha_0^1 < T_3 \cap \alpha_0^2 < T_4)\}} f_1(y^*, z^*)$$

where $T_4 = \frac{C_{G}}{C_{G} + C_{G_1}}$. In this scenario, it is difficult to characterize the error rate. In the previous subsection, for Algorithm-2, the method of types was used to find the error rate. The sets used to characterize the error rate would now depend on the decision sequence from the alternate observer. For a particular type, there could be multiple decision sequences. Hence, the same approach cannot be extended. The advantage of this algorithm is that it has faster rate of convergence due to step 4 of the consensus algorithm. However, the characterization of the rate of decay in Algorithm-3 and Algorithm-4 remains an open problem.

**V. Simulation Results**

Simulations were performed to evaluate the performance of the algorithms. The setting is described as follows. The cardinality of the sets of observations collected by Observers 1 and 2 are 3 and 4 respectively. The joint distribution of the observations under either hypothesis is given in Table 1. Note that under either hypothesis, the observations received by the two observers are independent. The prior distribution of the hypothesis was considered to be $p_0 = 0.4$ and $p_1 = 0.6$. $D_{KL}(f_1||f_0) = 0.7986$ and $D_{KL}(f_0||f_1) = 0.7057$. The empirical probability of error achieved by using the centralized scheme as $n$ increases has been plotted in Figure 9 (Algo-1). The empirical probability of the observers agreeing on the wrong belief conditioned on the observers agreeing in the decentralized scheme (III-B2) has been plotted in Figure 8 (Algo-2). In order to construct the aggregated sample space, the joint distribution of the observations and decision was found by the frequentist approach. 2 x $10^7$ samples were used to construct the aggregated sample space. The empirical probability of error achieved by the centralized sequential hypothesis testing scheme (using sequential probability ratio test), by the decentralized scheme in section III-B2, by the decentralized scheme in section III-C2, by the decentralized scheme in section III-D, has been plotted against the expected stopping time in Figure 9. Algo-1, Algo-2, Algo-3, and Algo-4 respectively. It is clear that the centralized sequential scheme performs the best among the four schemes. 13 aggregated probability sample spaces were constructed by varying $T_1$ and $T_2$. The pairs of $T_1$ and $T_2$ which were considered are $(1, 1), (2, 2), (3, 3), \ldots, (n, n), (\frac{3}{2}, n), (\frac{7}{2}, n), \ldots, (7, \frac{7}{2}), (\frac{9}{2}, 7)$. By varying $T_3$ and $T_4$ and choosing the best pair of expected stopping time and probability of error, the graphs Algo-3 and Algo-4 were obtained in Figure 9. The construction of the aggregated probability space (III-C1) is helpful, since for a given expected stopping time the probability of error achieved by the second decentralized scheme (III-C2) is lower than the probability of error achieved by the first decentralized scheme (III-B3). As discussed in section III-D, the performance of the decentralized scheme with greater than 1 bit exchange (Figure

**Table 1. Joint distribution of observations under either hypothesis**

| Y = 1 | Y = 2 | Y = 3 | Y = 4 | Z = 1 | Z = 2 | Z = 3 | Z = 4 |
|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.02  | 0.03  | 0.05  | 0.07  | 0.06  | 0.09  | 0.09  | 0.09  |
| 0.03  | 0.075 | 0.105 | 0.09  | 0.095 | 0.09  | 0.09  | 0.09  |
| 0.05  | 0.125 | 0.175 | 0.15  | 0.175 | 0.15  | 0.15  | 0.15  |
| 0.18  | 0.135 | 0.09  | 0.095 | 0.09  | 0.09  | 0.09  | 0.09  |
| 0.21  | 0.18  | 0.175 | 0.16  | 0.175 | 0.16  | 0.16  | 0.16  |
| 0.32  | 0.26  | 0.22  | 0.2  | 0.22  | 0.2  | 0.2  | 0.2  |
| 0.12  | 0.09  | 0.06  | 0.03  | 0.06  | 0.03  | 0.03  | 0.03  |
When Observer 1 needs to receive the joint distribution of the observations and decision pairs \((Y_1, D_1^1), ..., (Y_n, D_n^2)\) and then use the joint distribution with the appropriate sequences to find the conditional probability. This is not an efficient approach as computation time increases exponentially with increase in the number of samples. An alternate approach would be to store the sequences found at stage \(n\) and then use them to find the sequences at stage \(n+1\). In this approach the memory used for storage increases exponentially. Hence even upon knowing the joint distribution of the observations, the computation of \(\alpha_n^2\) is intensive. For the fourth approach, Observer \(i\) needs to compute \(\beta_n^i\) which requires the joint distribution of \(D_1^i, ..., D_n^i,\) and \(H\). Again, each observer needs to search over its observation space for finding the observation sequences which lead to that particular decision sequence. Since this approach is computationally intensive, the joint distribution of the decisions was estimated by the frequentist approach. For each observer, \(2 \times 2^7 = 256\) decision sequences are possible. From \(2 \times 10^7\) samples, the joint distribution of the decision sequence and hypothesis is estimated.

We considered another setup, where the cardinality of the sets of observations collected by Observers 1 and 2 are 2 and 3 respectively. The joint distribution of the observations under either hypothesis is given in Table 2. Under either hypothesis, the observations received by the two observers are not independent. The prior distribution of the hypothesis was considered to be \(p_0 = 0.4\) and \(p_1 = 0.6\). \(D_{KL}(f_1 || f_0) = 0.0627\) and \(D_{KL}(f_0 || f_1) = 0.0649\). The empirical probability of error achieved by using the centralized scheme as \(n\) increases has been plotted in Figure 10 (Algo-1). The empirical probability of the observers agreeing on the wrong belief conditioned on the observers agreeing in the decentralized scheme has been plotted in Figure 10 (Algo-2). \(2 \times 10^7\) samples were used to construct the aggregated probability space, while the maximum number of possible sequences is \(2 \times 2^7 \times 3^7 = 559872\). The empirical probability of error achieved by using the centralized sequential hypothesis testing scheme (using the sequential probability ratio test), by the decentralized scheme in section III-B2 by the decentralized scheme in section III-C2 by the decentralized scheme in section III-D have been plotted against the expected stopping time in Figure 11 (Algo-1, Algo-2, Algo-3, and Algo-4 respectively). There is a significant difference between performance of the centralized and the decentralized schemes. One possible reason is that the marginal distributions are closer, i.e., \(D_{KL}(f_1^i || f_0^i) = 0.0290\) and \(D_{KL}(f_0^i || f_1^i) = 0.0244\). The performance of the first decentralized scheme (III-B2) and the second decentralized scheme are almost similar. Hence the construction of the aggregated probability space is not helpful in this example.

VI. CONCLUSION AND FUTURE WORK

To conclude, we considered the problem of collaborative binary hypothesis testing with no prior joint distributions knowledge available to the observers. We presented different algorithms to solve the problem with emphasis on information exchange between the observers. The algorithms were analyzed and compared from different perspectives. We proved that the probability space constructed at the observers were a function of information patterns at the observers. We compared the performance of the algorithms by comparing the rate of

![Fig. 8. Probability of error / conditional probability of agreement on wrong belief] vs number of samples

![Fig. 9. Probability of error vs expected stopping time]
The methods of Grunwald and Topsoe in [25], [29], [30], and [49] developed game theoretic approaches to this problem following operative game with two observers. In future work, we plan to and asymmetric information can also be studied as a co-decay of the probability of error and proved it is a function of the information exchanged.

The binary hypothesis testing problem with two observers and asymmetric information can also be studied as a cooperative game with two observers. In future work, we plan to develop game theoretic approaches to this problem following the methods of Grunwald and Topsoe in [25], [29], [30], and [31].

**REFERENCES**

[1] J. N. Tsitsiklis et al., “Decentralized Detection,” Advances in Statistical Signal Processing, vol. 2, no. 2, pp. 297–344, 1993.

[2] R. R. Tenney and N. R. Sandell, “Detection with Distributed Sensors,” IEEE Transactions on Aerospace and Electronic Systems, no. 4, pp. 501–512, 1981.

[3] V. N. Vapnik, V. Vapnik, et al., Statistical Learning Theory. Wiley New York, 1998.

[4] T. M. Liggett, Continuous Time Markov Processes: An Introduction. American Mathematical Society, Providence, RI, 2010.

[5] R. Durrett, Probability: Theory and Examples, 5th Edition. Cambridge University Press, Cambridge, 2019.

[6] J.-F. Chamberland and V. V. Veeravalli, "Decentralized Detection in Sensor Networks," IEEE Transactions on Signal Processing, vol. 51, no. 2, pp. 407–416, 2003.

[7] A. Nayyar and D. Teneketzis, "Sequential Problems in Decentralized Detection with Communication," IEEE Transactions on Information Theory, vol. 57, no. 8, pp. 5410–5415, 2011.

[8] "Foundations of Systems and Control for the 21st Century in Honor and Memory of Pravin Varaiya," vol. One Day Workshop at the American Control Conference (ACC) 2023. San Diego, California, USA: American Automatic Control Council, May 30 2023, https://sites.google.com/tamu.edu/varaiya-workshop-2023/home.

[9] D. Teneketzis and P. Varaiya, “Consensus in Distributed Estimation With Inconsistent Beliefs,” Systems and Control Letters, vol. 4, pp. 217–221, 1984.

[10] D. Teneketzis and D. Castanon, “Informational Aspects of a Class of Subjective Games of Incomplete Information: Static Case,” Journal of Optimization Theory and Applications, vol. 54, no. 2, pp. 413–422, 1987.

[11] D. Castanon and D. Teneketzis, “Further Results on the Asymptotic Agreement Problem,” IEEE Transactions on Automatic Control, vol. 33, no. 6, pp. 515–523, 1988.

[12] S. Sántas, S. Gezici, and S. Yüksel, “Hypothesis Testing Under Subjective Priors and Costs as a Signaling Game,” IEEE Transactions on Signal Processing, vol. 67, no. 19, pp. 5169–5183, 2019.

[13] J. Hinzikka, “Quantum Logic as a Fragment of Independence-friendly Logic,” Journal of Philosophical Logic, vol. 31, no. 3, pp. 197–209, 2002.

[14] J. S. Baras, “Noncommutative Probability Models in Quantum Communication and Multi-agent Stochastic Control,” Ricerche di Automatica, vol. 10, no. 2, pp. 216–265, 1979.

[15] ——, “Distributed Asynchronous Detection: General Models,” in Proceedings 26th IEEE Conference on Decision and Control, 1987, pp. 1832–1835.

[16] ——, “Multi-agent Stochastic Control: Models Inspired From Quantum Physics,” in Proceedings Physics and Control International Conference, (PhysCon), 2003, pp. 747–758.

[17] ——, “Multi-agent Collaborative Decision Making: Constrained Event Algebras, New Logics and Their Game Theoretic Semantics,” in Workshop on Information, Decisions and Networks, in Honor of D. Teneketzis 65th Birthday, 2016, invited address.

[18] A. Raghavan, The Role of Information in Multi-agent Decision Making. University of Maryland, College Park, 2019, PhD Thesis.

[19] V. B. Scholz and R. F. Werner, “Tsirelson’s problem,” arXiv preprint arXiv:0812.4305, 2008.

[20] M. Junge, M. Navascues, C. Palazuelos, D. Perez-Garcia, V. B. Scholz, and R. F. Werner, “Connes’ embedding problem and tsirelson’s problem,” Journal of Mathematical Physics, vol. 52, no. 1, 2011.

[21] T. Fritz, “Tsirelson’s problem and kirchberg’s conjecture,” Reviews in Mathematical Physics, vol. 24, no. 05, p. 1250012, 2012.

[22] A. Raghavan and J. S. Baras, “Binary Hypothesis Testing by Two Collaborating Observers: A Fresh Look,” in 2019 27th Mediterranean Conference on Control and Automation (MED). IEEE, 2019, pp. 244–249.

[23] H. Touchette, “A Basic Introduction to Large Deviations: Theory, Applications, Simulations,” arXiv preprint arXiv:1106.4146, 2011.

[24] J. C. Willems, “Open Stochastic Systems,” IEEE Transactions on Automatic Control, vol. 58, no. 2, pp. 406–421, 2013.

[25] P. D. Grünwald and A. P. Dawid, “Game Theory, Maximum Entropy, Minimum Discrepancy and Robust Bayesian Decision Theory,” Annals of Statistics, pp. 1367–1433, 2004.

[26] L. Korolov and Y. G. Sinai, Theory of Probability and Random Processes. Springer Science & Business Media, 2007.

[27] J. R. Kumar and P. Varaiya, Stochastic Systems: Estimation, Identification, and Adaptive Control. SIAM-Society for Industrial and Applied Mathematics, 2015.

[28] I. N. Herstein, Topics in Algebra. Blaisdell Publishing Company, 1964.

[29] F. Topsoe, “Basic Concepts, Identities and Inequalities-The toolkit of Information Theory,” Entropy, vol. 3, no. 3, pp. 162–190, 2001.

[30] P. Harremoes and F. Topsoe, “Unified Approach to Optimization Techniques in Shannon Theory,” in Information Theory, 2002. Proceedings. 2002 IEEE International Symposium on. IEEE, 2002, p. 238.

[31] F. Topsoe, “Game Theoretical Optimization Inspired by Information Theory,” Journal of Global Optimization, vol. 43, no. 4, p. 553, 2009.

[32] P. Billingsley, Probability and Measure. John Wiley & Sons, 2017.

[33] T. M. Cover and J. A. Thomas, Elements of Information Theory. John Wiley & Sons, 2012.

[34] I. Csiszár, “The Method of Types [information theory],” IEEE Transactions on Information Theory, vol. 44, no. 6, pp. 2505–2523, 1998.

**VII. APPENDIX**

A. Kolmogorov Consistency Theorem

This theorem has been invoked in subsections, [III-A1][II-B1][II-C1][IV-B] and has been mentioned in subsection [IV-A]. We mention the statement of the theorem for the completeness of the paper and for proof we refer to [26], [32].
Let $E$ be a parameter set (usually $E = \mathbb{N}, \mathbb{R}_+$) and $(\mathcal{X}, \mathcal{F}_X)$ be a topological space (usually $\mathcal{F} = \mathbb{R}, \mathbb{R}^n$ or a finite set). Let $\Omega = \{\omega \in E \mapsto \mathcal{X}\}$. Let $\mathcal{B}(\mathcal{F}_n)$ be a $\sigma$-algebra of subsets of $\mathcal{F}_n$. Let $\{\tau_1, \ldots, \tau_n\} \subset E$ and $B \in \mathcal{B}(\mathcal{F}_n)$. A cylindrical subset of $\Omega$ is

$$I_{[\tau_1, \ldots, \tau_n]}(B) = \{\omega \in \Omega : (\omega(t_1), \ldots, \omega(t_n)) \in B\}.$$ 

The collection of all cylindrical subsets for which $\{\tau_k\}_{k=1}^n$ is fixed and $B$ is allowed to vary is a $\sigma$-algebra denoted by $\mathcal{F}_{[\tau_1, \ldots, \tau_n]}$. Let $\mathcal{F}(\Omega)$ be the smallest $\sigma$-algebra containing $\mathcal{F}_{[\tau_1, \ldots, \tau_n]}$ for all $n$, $\{\tau_k\}_{k=1}^n$. A collection of probability measures, $\mathcal{P}_{[\tau_1, \ldots, \tau_n]}$, on $\mathcal{F}_{[\tau_1, \ldots, \tau_n]}$; satisfy the consistency conditions if

- For every permutation, $\pi$, for every $\{\tau_1, \ldots, \tau_n\}$, and $B \in \mathcal{B}(\mathcal{F}_n)$,

$$\mathcal{P}_{[\tau_1, \ldots, \tau_n]}(I_{[\tau_1, \ldots, \tau_n]}(B)) = \mathcal{P}_{[\pi(\tau_1), \ldots, \pi(\tau_n)]}(I_{[\pi(\tau_1), \ldots, \pi(\tau_n)]}(B)).$$

- For every $\{\tau_1, \ldots, \tau_n\} \subset E$, $B \in \mathcal{B}(\mathcal{F}_n)$,

$$\mathcal{P}_{[\tau_1, \ldots, \tau_n]}(I_{[\tau_1, \ldots, \tau_n]}(B)) = \mathcal{P}_{[\tau_1, \ldots, \tau_n]}(B \times \mathcal{F}).$$

The Kolmogorov Consistency Theorem or Kolmogorov Existence Theorem is stated as,

**Theorem VII.1.** Assume that there exists a family of probability measures, $\{\mathcal{P}_{[\tau_1, \ldots, \tau_n]}\}$, which satisfy the consistency conditions. Then, there exists a unique $\sigma$-additive measure $\mathcal{P}$ on $\mathcal{F}(\Omega)$ whose restriction to any $\mathcal{F}_{[\tau_1, \ldots, \tau_n]}$ coincides with $\mathcal{P}_{[\tau_1, \ldots, \tau_n]}$.

In this paper, $E = \mathbb{N}$ and $\mathcal{X}$ is a finite set. The distributions found by the observers are for $E_n = \{1, \ldots, n\}, n \in \mathbb{N}$. Hence the distributions are available for only one permutation of the time stamps, specifically, $t_1 < t_2 < \ldots < t_n$. Other permutations are not found as they are not needed for the problem considered. Hence, the first condition of consistency is skipped when the theorem is invoked.

### B. Method of Types

In this subsection, we recall one result from the method of types [33], [34], which will be used in subsequent proofs. Let $(Y^n, Z^n) = [(Y_1, Z_1), \ldots, (Y_n, Z_n)]$. For an observation sequence $(Y^n, Z^n = y^n, z^n)$, the type associated with it is:

$$Q_{y^n, z^n}(y, z) = \frac{1}{n} \sum_{i=1}^n \chi_{\{y, z\}}(y_i, z_i), \forall (y, z) \in S_1 \times S_2.$$

With the above definition, when $(Y_1, Z_1), \ldots, (Y_n, Z_n)$ are i.i.d. conditioned on the hypothesis, for $h = 0, 1$,

$$\mathbb{P}_n(Y^n, Z^n = y^n, z^n | H = h) = 2^{-nH(Q_{y^n, z^n} \parallel Q_{y^n, z^n}^h \parallel f_0)}.$$ 

Let $T_0$ be $\max \log \frac{f_1(y, z)}{f_0(y, z)}$ and $T_1$ be $\min \log \frac{f_1(y, z)}{f_0(y, z)}$. For threshold $T$ such that $T_0 < log2T < T_1$ the likelihood ratio test can be equivalently written as:

$$D_{KL}(Q_{y^n, z^n}^h \parallel f_0) - D_{KL}(Q_{y^n, z^n} \parallel f_1) \geq \frac{1}{n} \log \frac{2}{T}.$$

### C. Proof of Equations (4) and (5)

We present the proof for Equations (4) and (5) in subsection [V-E].

**Proof.** Let $\mathcal{F}$ denote the set of probability distributions on $S_1 \times S_2$. For a vector $Q \in \mathcal{F}$, $Q = [Q(1), Q(2), \ldots, Q(|S_1| \times |S_2|)]$, the element $Q(i)$ corresponds to the joint probability of observing $y_i$ and $z_i$, where $i = i_1 |(S_1|, k = i_2 |(S_2|$. If $i - i_1 |(S_1| \times |S_2| = 0$, then $k = |S_2|$. $Q(i)$ and $Q(y, z)$ are used interchangeably. For a set $S$, let $int(S)$ denote the interior of the set and $\overline{S}$ denote the closure set. Let,

$$V = \left[\log \frac{f_1(y_1, z_1)}{f_0(y_1, z_1)}, \ldots, \log \frac{f_1(y_{|S_1|}, z_{|S_1|})}{f_0(y_{|S_1|}, z_{|S_1|})}\right].$$

For the given threshold $T$, the objective is to find the rate of decay of the probability of error. The set of distributions for which the decision in the centralized case is 1 is

$$S_1 = \{Q \in \mathcal{F} \mid D_{KL}(Q \parallel f_0) - D_{KL}(Q \parallel f_1) \geq \log_2 T\}.$$ 

Let $e_i(e_{zy})$, $1 \leq i \leq |S_1| \times |S_2|$ represent the canonical basis of $\mathbb{R}(|S_1| \times |S_2|)$. The set $S_1$ can also be described as:

$$S_1 = \{Q \in \mathcal{F} \mid : -V^T Q + \log_2 T \leq 0, \sum_{y, z} Q(y, z) = 1, -e_i Q \leq 0, 1 \leq i \leq |S_1| \times |S_2|\}$$

Since $T_0 < \log_2 T < T_1$, $int(S_1) \neq \emptyset$ and $int(\overline{S_1}) \neq \emptyset$. Since $S_1$ and $\overline{S_1}$ are closed, connected sets with nonempty interiors they are regular closed sets i.e., $S_1 = \overline{int(S_1)}$ and $\overline{S_1} = \overline{int(\overline{S_1})}$. Thus, by Sanov’s theorem [33], it follows that

$$\lim_{n \to \infty} -\frac{1}{n} \log_2(\kappa_n) = D_{KL}(Q_{0}^n \parallel f_0),$$

$$\lim_{n \to \infty} -\frac{1}{n} \log_2(\xi_n) = D_{KL}(Q_{1}^n \parallel f_1),$$

$$\kappa_0 = \arg \min_{Q \in S_1} D_{KL}(Q \parallel f_0), \kappa_1 = \arg \min_{Q \in S_1} D_{KL}(Q \parallel f_0).$$

Since the optimization problems are convex, to solve them the Lagrangian can be setup as follows:

$$K_h(Q(y, z), \tau_h, \nu_h, \epsilon_h) = \sum_{y, z} Q(y, z) \log_2 \left( \frac{Q(y, z)}{f_0(y, z)} \right) + s(h) \tau_h \sum_{y, z} \nu_h(y, z, y) \log_2 \left( \frac{f_1(y, z)}{f_0(y, z)} \right) - \log_2 T - \sum_{y, z} \nu_h(y, z, y) Q(y, z) - \epsilon_h \sum_{y, z} Q(y, z) - 1.$$

where $s(h) = -1$ if $h = 0$ and $s(h) = 1$ if $h = 1$. Setting $\partial_{\nu_h}(K_h(Q, \tau_h, \nu_h, \epsilon_h)) = 0$, for $(y, z) \in S_1 \times S_2$ we get

$$\log_2 \left( \frac{Q(y, z)}{f_0(y, z)} \right) - s_h \tau_h \log_2 \left( \frac{f_1(y, z)}{f_0(y, z)} \right) + \epsilon_h - \nu_h(y, z) = -1.$$

Hence, Equation (4) in subsection [V-E] follows. The dual functions for the above optimization problems are:

$$J_h(\tau_h, \nu_h, \epsilon_h) = K_h(Q_{\tau_h}^n, \tau_h, \nu_h, \epsilon_h),$$
and the dual optimization problems are:

$$\Delta^*_h = \max_{\tau_h \in \mathbb{R}, \nu_h \in \mathbb{R}^{S_1 \times |S_2|}} J_h(\tau_h, \nu_h, \epsilon_h)$$

s.t. $-\tau_h \leq 0, -\epsilon_h \nu_h \leq 0, 1 \leq i \leq |S_1| \times |S_2|$

Since the interior of the sets $S_1$ and $S_1^\epsilon$ are not empty, Slater’s condition holds and hence strong duality holds. Suppose $\tau_h^*$ is such that:

$$\frac{d}{d \tau_h} \left[ \sum_{y \in S_1} Q^{h^*}(y, z) \log_2 \left( \frac{Q^{h^*}(y, z)}{f_h(y, z)} \right) + s(h) \tau_h \right] = \left( \sum_{y \in S_1} Q^{h^*}(y, z) \log_2 \left( \frac{f_h(y, z)}{f_0(y, z)} \right) \right)_{\tau_h = \tau_h^*} = s(h) \log_2 T. \quad (11)$$

Then, since strong duality holds,

$$\lim_{n \to \infty} -\frac{1}{n} \log_2(\kappa_n) = \Delta_h^*, \lim_{n \to \infty} -\frac{1}{n} \log_2(\bar{\xi}_n) = \Delta^*_h,$$

Thus, for the given threshold $T$, the rate of decay of probability of error is:

$$\lim_{n \to \infty} -\frac{1}{n} \log_2(p_n) = \min \left[ D_{KL}(Q^{h^*}_0 || f_0), D_{KL}(Q^{h^*_1} || f_1) \right].$$

By changing the threshold $T$ (or equivalently $\tau_0$ and $\tau_1$) different decay rates can be achieved. Thus the optimal rate of decay is achieved by searching over pairs $(\tau_0, \tau_1)$ such that $\tau_0 \geq 0$ and $\tau_1 \geq 0$. Further if $R^*_h$ is achieved by the pair $\tau_0, \tau_1$, i.e.,

$$R^*_h = \min \left[ D_{KL}(Q^{h^*_0}_0 || f_0), D_{KL}(Q^{h^*_1} || f_1) \right],$$

then $R^*_h = D_{KL}(Q^{h^*_0}_0 || f_0)$ or $R^*_h = D_{KL}(Q^{h^*_1} || f_1)$. The threshold which achieves the optimal decay rate is found by evaluating the L.H.S of Equation (11) at the appropriate $\tau_h$ (i.e., the one that achieves $R^*_h$).

**D. Proof of Equations (9) and (10)**

In the decentralized scenario, the observation sequence $(Y^n, Z^n = y^n, z^n)$ induces a type on $S_1$ and $S_2$:

$$Q^n_{1*}(y) = \frac{1}{n} \sum_{i=1}^n \chi_{i}(y_i) = \sum_{y \in S_1} Q^n_{y*} n_{y*}^*(y, z), \forall y \in S_1,$$

$$Q^n_{2*}(z) = \frac{1}{n} \sum_{i=1}^n \chi_{i}(z_i) = \sum_{y \in S_1} Q^n_{y*} n_{y*}^*(y, z), \forall z \in S_2.$$

Let,

$$T^*_1 = \max_{y \in S_1} \log_2 f_1(y) f_0^*(y), T^*_2 = \max_{z \in S_2} \log_2 f_2^*(z) f_0(z),$$

$$\tilde{T}^*_1 = \min_{y \in S_1} \log_2 f_2(y) f_0^*(y), \tilde{T}^*_2 = \min_{z \in S_2} \log_2 f_2^*(z) f_0(z).$$

Let $T_1$ and $T_2$ be such that $T^*_1 < \log_2 T_1 < T^*_2$ and $T^*_2 < \log_2 T_2 < T^*_1$. The individual likelihood ratio tests for the observers with thresholds $T_1$ and $T_2$ are:

$$D_{KL}(Q^n_{1*} || f_0) - D_{KL}(Q^n_{2*} || f_1) \geq \frac{1}{n} \log_2 T_1,$$

$$D_{KL}(Q^n_{2*} || f_0^*) - D_{KL}(Q^n_{2*} || f_1^*) \geq \frac{1}{n} \log_2 T_2.$$

Now, we present the proof for Equation (9) in subsection IV-E2.

**Proof.**

$$\nu = [1, 1, \ldots, 1] \in \mathbb{R}^{|S_2|}, v_1 = [1, 1, \ldots, 1] \in \mathbb{R}^{|S_1| \times |S_2|}$$

$$u = \left[ \log_2 f_1(z_1), \log_2 f_1(z_2), \ldots, \log_2 f_1(z_{|S_2|}) \right] \in \mathbb{R}^{|S_2|},$$

$$v_2 = \left[ \log_2 f_1(y_1), \log_2 f_1(y_2), \log_2 f_1(y_3), \ldots, \log_2 f_1(y_{|S_1|}) \right] \in \mathbb{R}^{|S_1| \times |S_2|}, v_3 = [u, u, \ldots, u] \in \mathbb{R}^{|S_1| \times |S_2|}, ||Q||_2 =$$

$$\max_{Q} T_2^*.$$ For a feasible $Q$, the objective is to find the rate of decay of the probability of false alarm and of the probability of miss detection. We first focus on the rate of decay of the probability of false alarm. The set of distributions for which the decisions of both observers is 1 is

$$\mathcal{S}_1 = \{ Q \in \mathcal{S} : D_{KL}(Q || f_0^*) - D_{KL}(Q || f_1^*) \geq \log_2 T_2^* \},$$

where $Q_1$ and $Q_2$ are induced by $Q$ on $S_1$ and $S_2$ respectively. The set $\mathcal{S}_1$ can also be described as:

$$\mathcal{S}_1 = \{ Q \in \mathbb{R}^{|S_1| \times |S_2|} : -v_1^T Q + \log_2 T_1 \leq 0, -v_2^T Q + \log_2 T_2 \leq 0, -e_1 Q \leq 0, 1 \leq i \leq |S_1| \times |S_2| \}$$

The first objective is to find threshold pairs $T_1, T_2$ for which $\mathcal{S}_1$ is non empty. Note that,

$$\max_{Q \in \mathcal{S}_1} v_1^T Q = \max_{y \in S_1} \log_2 f_1(y) f_0^*(y), \max_{Q \in \mathcal{S}_1} v_2^T Q = \max_{z \in S_2} \log_2 f_2^*(z) f_0(z),$$

$$\min_{Q \in \mathcal{S}_1} v_1^T Q = \min_{y \in S_1} \log_2 f_1(y) f_0^*(y), \min_{Q \in \mathcal{S}_1} v_2^T Q = \min_{z \in S_2} \log_2 f_2^*(z) f_0(z).$$

Since $T_1^* < \log_2 T_2 < T_2^*$, and $g(Q) = v_2^T Q$ is continuous, $\exists Q_0 \in \mathcal{S}_1$ such that $v_1^T Q_0 = \log_2 T_2$. For a feasible $T_2$, we would like to find the set of feasible $T_1$ so that that the set $\mathcal{S}_1$ is nonempty. Consider:

$$\Psi(T_2) = \max_{Q \in \mathbb{R}^{|S_1| \times |S_2|}} v_2^T Q$$

s.t. $-v_1^T Q + \log_2 T_2 \leq 0, v_1^T Q = 1, -e_1 Q \leq 0, 1 \leq i \leq |S_1| \times |S_2|$

$$\Phi(T_2) = \min_{Q \in \mathbb{R}^{|S_1| \times |S_2|}} v_1^T Q$$

s.t. $-v_1^T Q + \log_2 T_2 \leq 0, v_1^T Q = 1, -e_1 Q \leq 0, 1 \leq i \leq |S_1| \times |S_2|$

Since the above optimization problems are linear programs for every $T_2$, the maximum and the minimum occur at one of the vertices of the convex polygon, $\mathcal{S}_1 = \mathcal{S} \cap \{ Q : -v_1^T Q - \log_2 T_2 \leq 0 \}$. Let int$(S)$ denote the interior of a set $S$. Let $Q$ be a boundary point of the set $S$. Let $C(Q, S) = \{ h : h \geq 0, \epsilon \in \text{int}(S), \forall \epsilon \in [0, \epsilon] \}$. Since the set $\mathcal{S}$ is convex, for any point $Q_0$, in the interior of the set $Q$ and on its boundary, the vector $Q_0 - Q$ belongs to...
$C(Q, \mathcal{M})$. For a given $T_1, T_2$, if $\Phi(T_2) < \log_2 T_1 < \Psi(T_2)$, then the pair is a feasible pair. If not, we choose an alternative $T_1$ which satisfies the above inequalities. Further we choose $T_1$ to be such that $\Phi(T_2) < \log_2 T_1 < \log_2 T < \Psi(T_2)$. Since the function $f(Q) = v_3^3 Q$ is continuous, $\exists Q_o \in \mathcal{F}_2$ such that $f(Q_o) = \log_2 T$. Hence $Q_o \in \mathcal{F}_2$ is such that $v_3^3 Q_o < \log_2 T_1$ and $v_3^3 Q_o \geq \log_2 T_2$, which implies that the set $\mathcal{F}_1$ is nonempty.

If $Q_o$ is an interior point of $\mathcal{F}_2$ then it is an interior point for $\mathcal{F}_1$. Suppose $Q_o$ is a boundary point of $\mathcal{F}_2$, such that $v_3^3 Q_o = \log_2 T_2$ and $Q_o(i) > 0$ for all $i$. There exists a direction $h$ such that $v_3^3 h > 0$ and for epislon small enough, $(Q_o + \epsilon h)$ belongs to the interior of $\mathcal{F}_2$. Suppose $Q_o$ is a boundary point of $\mathcal{F}_2$, such that $Q_o(i) = 0$ for some $i$. Then, set $C(Q_o, \mathcal{F}) \cap \{h : v_3^3 h \geq 0\}$ is nonempty.

Indeed, if the set is empty then $C(Q_o, \mathcal{F}) \subseteq \{h : v_3^3 h < 0\}$ which implies that $v_3^3 Q < \log_2 T_2 \forall Q \in \operatorname{int}(\mathcal{F})$, which is a contradiction as $\log_2 T_2 < T_3^3$. This can be proven by the following argument. Let $Q_e$ be such that $v_3^3 Q_e = T_3^3$. Note that $Q_e$ is a boundary point of $\mathcal{F}$. Let $\epsilon = \frac{T_3^3 - \log_2 T_2}{\epsilon_3}$. By continuity of $v_3^3 Q$, there exists $\delta > 0$ such that $\|Q - Q_e\| < \delta$ implies $|v_3^3 Q - v_3^3 Q_e| < \epsilon$. This implies for every $Q$ such that $\|Q - Q_e\| < \delta$, $v_3^3 Q > T_3^3 - \epsilon > \log_2 T_2$. Since $Q_o$ is a boundary point of $\mathcal{F}_2$, there exists at least one interior point of $\mathcal{F}$ in the ball, $\|Q - Q_e\| < \delta$. Hence there exists an interior point, $Q_d$ such that $v_3^3 Q_d > \log_2 T_2$, which contradicts our conclusion that $v_3^3 Q < \log_2 T_2 \forall Q \in \operatorname{int}(\mathcal{F})$.

Thus, there exits $Q_o$ a boundary point of $\mathcal{F}$, such that $b_n(i) > 0 \forall i, v_3^3 Q_o > \log_2 T_2, |Q_o - Q_o|_m < \epsilon$ and

$$v_3^3 Q_o = v_3^3 Q_o + v_3^3 Q_o - v_3^3 Q_o$$

$$\geq \log_2 T - |Q_o - Q_o|_m \times M_1$$

$$\geq \log_2 T - \epsilon \times M_1.$$ 

We choose $\epsilon$ such that $\epsilon < \frac{\log_2 T - \log_2 T_1}{\epsilon_3 \times M_1}$. Then, $v_3^3 Q_o > \log_2 T_2$. Hence $Q_o$ is an interior point of $\mathcal{F}_1$. Thus, for the $T_1, T_2$ pair, there exists $Q \in \mathcal{F}$ such that $Q(i) > 0 \forall i, v_3^3 Q > \log_2 T_2, v_3^3 Q > \log_2 T_2$. Hence the interior of the set $\mathcal{F}_1$ is also nonempty. Clearly, $\mathcal{F}_2$ is a convex set and has strong duality holds. Hence, 

$$\lim_{n \to \infty} - \frac{1}{n} \log_2 (\mu_n) = d^*.$$ 

Suppose $\lambda^*_o$ and $\sigma^*_o$ are such that:

$$\frac{\partial}{\partial \lambda^*_o} \left[ \sum_{y, z} Q^{0}_{\lambda^*_o, \sigma^*_o}(y, z) \log_2 \left( \frac{Q^{0}_{\lambda^*_o, \sigma^*_o}(y, z)}{f_0(y, z)} \right) \right] = 0$$

$$\frac{\partial}{\partial \sigma^*_o} \left[ \sum_{y, z} Q^{0}_{\lambda^*_o, \sigma^*_o}(y, z) \log_2 \left( \frac{f_1(y)}{f_0(y)} \right) \right] = 0$$

By solving the above equations, the optimizers $\lambda^*_o$ and $\sigma^*_o$ can be found as functions of $T_1$ and $T_2$ and the distribution which achieves the optimal rate for this pair of thresholds is $Q^{0}_{\lambda^*_o, \sigma^*_o}$. To study the rate of decay of the probability of miss detection.
we consider the set of distributions for which the decision of both observers is 0, \( \mathcal{S}_3 \),

\[
\mathcal{S}_3 = \{ Q \in \mathbb{R}^{[S_1] \times [S_2]} : v_1^T Q - \log_2 T_1 \leq 0, v_2^T Q = 1, \quad v_1^T Q - \log_2 T_2 \leq 0, \quad -v_1 Q \leq 0, \quad 1 \leq |S_1| \times |S_2| \}. 
\]

It is clear that \( \mathcal{S}_3 \) is closed, convex and has nonempty interior (as \( T_2^* < T_2 \) and \( \Phi(T_2) < \log_2 T_1 \)). Again by Sanov’s theorem,

\[
\lim_{n \to \infty} -\frac{1}{n} \log_2(p_h) = D_{KL}(Q_{h}^1 || f_1),
\]

where,

\[
Q_{h}^1 = \arg \min_{Q \in \mathcal{S}_3} D_{KL}(Q || f_1). 
\]

The optimization problem can be solved to show that \( Q_{h}^1 \) satisfies Equation (8) for \( h = 1 \). The dual problem can be solved to find \( \lambda_{h}^* \) and \( \sigma_{h}^* \). Thus for the given thresholds (and hence decision policy), the error rate is

\[
\lim_{n \to \infty} -\frac{1}{n} \log_2(p_h) = \min_{Q \in \mathcal{S}_3} \left[ D_{KL}(Q_{h}^0 || f_0), D_{KL}(Q_{h}^1 || f_1) \right],
\]

since the exponential rate is determined by the worst exponent. By changing the thresholds (and hence \( \lambda_{h}, \sigma_{h}, h = 0, 1 \)), different error rates can be obtained. Thus the best error rate is obtained by taking the maximum over \( \lambda_{h} > 0 \) and \( \sigma_{h} > 0, h = 0, 1 \). Thus, Equation (9), in section IV-E2, follows. Suppose the above maximum is achieved at \( (\tilde{\lambda}_{0}, \tilde{\sigma}_{0}) \) and \( (\tilde{\lambda}_{1}, \tilde{\sigma}_{1}) \). Then \( R_d^e = D_{KL}(Q_{\lambda_{0}, \sigma_{0}} || f_0) \) or \( R_d^e = D_{KL}(Q_{\lambda_{1}, \sigma_{1}} || f_1) \). Suppose \( R_d^e = D_{KL}(Q_{\lambda_{0}, \sigma_{0}} || f_0) \). Then, the thresholds which achieve the optimal rate of decay can be found by evaluating the L.H.S. of Equation (12) at \( (\tilde{\lambda}_{0}, \tilde{\sigma}_{0}) \). For the other case, the thresholds can be found from equations analogous to Equation (12), which arise from the dual optimization problem obtained while finding the rate of decay of the probability of miss detection.

E. Proof of Equation (10)

Suppose the observation collected by Observer 1 is independent of the observation collected by Observer 2 under either hypothesis, i.e.,

\[
f_0(y, z) = f_0^1(y) f_0^0(z), \quad f_1(y, z) = f_1^1(y) f_1^0(z). 
\]

Let \( C_1 \) be a subset of the positive cone, \( C_1 = \{(\lambda_0, \sigma_0, \lambda_1, \sigma_1) \in \mathbb{R}^4 : \lambda_0 > 0, \sigma_0 > 0, \lambda_1 > 0, \sigma_1 > 0 \} \). For such quadruplets,

\[
Q_{h}^h |_{\lambda_h = \sigma_h = \tau_h} = Q_{h}^h, 
\]

Thus,

\[
R_d^e = \max_{\lambda_0 \geq 0, \sigma_0 \geq 0} \min_{\lambda_1 \geq 0, \sigma_1 \geq 0} \left[ D_{KL}(Q_{0,0}^0 || f_0), D_{KL}(Q_{1,1}^1 || f_1) \right] 
\]

\[
\geq \max_{\lambda_0 \geq 0, \sigma_0, \lambda_1, \sigma_1 \in C_1} \min_{\lambda_0 \geq 0, \sigma_0, \lambda_1, \sigma_1} \left[ D_{KL}(Q_{0,0}^0 || f_0), D_{KL}(Q_{1,1}^1 || f_1) \right] 
\]

\[
= \max_{\sigma_0, \lambda_1 \geq 0} \min_{\lambda_0 \geq 0} \left[ D_{KL}(Q_{0,0}^0 || f_0), D_{KL}(Q_{1,1}^1 || f_1) \right] = R_d^c 
\]

The above result can be understood as follows: in the centralized case, the probability simplex is divided into two regions by a hyperplane, while in the decentralized case the simplex is divide into four regions by two hyperplanes. Hence, the minimum of the Kullback-Liebler divergence between the decision regions (in the probability simplex) and the observation distributions in the centralized scenario is likely to be lower than in the decentralized case as the sets are “larger” in the centralized scenario (Figure 12). 

Anees Raghavan received the Bachelor of Technology degree in Instrumentation Engineering from the Indian Institute of Technology, Kharagpur in 2013 and Ph.D. degree in Electrical Engineering from the University of Maryland, College Park, in 2020. Currently, he is a Post-Doc at the Division of Decision and Control Systems, KTH, Royal Institute of Technology, Stockholm. His research interests include collaborative inference, active learning, and, neuro-symbolic reasoning and learning.

John S. Baras received the M.S. and Ph.D. degrees in Applied Mathematics from Harvard University, Cambridge, MA, USA, in 1971 and 1973, respectively. Since 1973, he has been with the Department of Electrical and Computer Engineering, University of Maryland at College Park, MD, USA, where he is currently a Distinguished University Professor, holding a Permanent Joint Appointment with the Institute for Systems Research (ISR) and the Lockheed Martin endowed Chair in Systems Engineering. He is also a Faculty member of the Applied Mathematics, Statistics and Scientific Computation Program, and an Affiliate Professor in the Departments of Computer Science, Fischell Bioengineering, Mechanical Engineering, Aerospace Engineering, Decision, Operations and Information Technologies of the Robert H. Smith School of Business. From 1985 to 1991, he was the Founding Director of the ISR (one of the first six National Science Foundation Engineering Research Centers). Since 1992, he has been the Director of the Maryland Center for Hybrid Networks (HYNET), which he co-founded.

He is an IEEE Life Fellow, and Fellow of SIAM, AAAS, NAI, IFAC, AMS, AIAA, Member of the National Academy of Inventors (NAI) and a Foreign Member of the Royal Swedish Academy of Engineering Sciences (IVA). Major honors and awards include the 1980 George Axelby Award from the IEEE Control Systems Society, the 2006 Leonard Abraham Prize from the IEEE Communications Society, the 2014 Tage Erlander Guest Professorship from the Swedish Research Council, and a three year (2014-2017) Senior Hans Fischer Fellowship from the Institute for Advanced Study of the Technical University of Munich, Germany. In 2016 he was inducted in the University of Maryland A. J. Clark School of Engineering Innovation Hall of Fame. He was awarded the 2017 Institute for Electrical and Electronics Engineers (IEEE) Simon Ramo Medal, the 2017 American Automatic Control Council (AACC) Richard E. Bellman Control Heritage Award, and the 2018 American Institute for Aeronautics and Astronautics (AIAA) Aerospace Communications Award. In 2018 he was awarded a Doctorate Honoris Causa by the National Technical University of Athens, Greece. He has educated 102 doctoral students, 160 MS students and has mentored 70 postdoctoral fellows. He has given many plenary and keynote addresses in major international conferences worldwide. He has been awarded nineteen patents and has been honored worldwide with many awards as innovator and leader of economic development.