How the Phase Slips in a Current-Biased Narrow Superconducting Stripe?

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The theory of current transport in a narrow superconducting channel accounting for thermal fluctuations is revisited. The value of voltage appearing in the sample is found as the function of temperature (close to transition temperature $T - T_c \ll T_c$) and bias current $J < J_c$ ($J_c$ is a value of critical current calculated in the framework of the BCS approximation, neglecting thermal fluctuations). It is shown that the careful analysis of vortex crossing of the stripe results in considerable increase of the activation energy.

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I. INTRODUCTION

The fundamental property of dissipationless current flow through the superconducting components underlies operation of numerous nano-electronic devices. One of such components is a narrow superconducting stripe (NSS) and the role of thermal and quantum fluctuations in formation of the resistive state in it is a problem of great importance. Various models have been proposed to explain the appearance of non-zero resistance of NSS and its temperature dependence in the region of low temperatures (for review see Ref.[5]).

The role of thermal fluctuations in the energy dissipation in the process of current flow through the one-dimensional superconductor for the first time was considered in the paper of Langer and Ambegaokar[6] almost fifty years ago. Publication of this paper has strongly influenced all further research in this field, it became classical, and corresponding results were included in many monographs and handbooks on superconductivity.[7,8]

It is necessary to mention that the “one-dimensional superconductor” de facto often is a narrow stripe with finite width $W$, much less than the Ginzburg-Landau coherence length $\xi_{GL}(T) = \sqrt{\pi D / 16 T \tau}$ ($\tau = 1 - T / T_c$ is the reduced temperature, $D$ is the electron diffusion coefficient). The energy dissipation in it is related to the phase slip processes consisting in the crossing of the stripe by the vortices. It is clear, that such events cannot be realized remaining in the frameworks of the one-dimensional model. Indeed, the solution found in Ref.[2] shows that even when the density of the flowing through the one-dimensional superconductor current reaches its critical value $J_c$, the minimal value of the order parameter is $(2/3)^{1/2} \Delta_{BCS}$, while in order to perform the phase slip event it should turn zero at least at one point.

In this work we will resolve the mentioned paradox, describing the true mechanism of phase slip events in NSS and determining the corresponding value of the activation energy. We will demonstrate that the saddle point solution of Ginzburg-Landau (GL) equation for the order parameter $\tilde{\Delta}$ in presence of a fixed current $J$, possessing at least one vortex, exists only for weak enough currents $J < J_{c1} = \eta (L / \xi) J_c$, ($J_c$ is the critical current of the stripe, and $\eta = 0.0312$ is the small number which will be found below). Under the definition “saddle point solution” $\tilde{\Delta}(x, y, J, r_1, .. r_i)$ we understand the solution of GL equation, which depends not only on the coordinates $x, y$, and current $J$, but also on some set of parameters $\{r_i\}$ satisfying the extremes conditions for the GL functional $\mathcal{F}_s$:

$$\frac{\partial}{\partial r_i} \mathcal{F}_s \left[ \tilde{\Delta}(x, y, J, r_1, .. r_i), J \right] = 0. \tag{1}$$

In the case under consideration, when one or several vortices penetrate through the stripe, such parameters can be chosen as the vortex centers coordinates (zeros of the order parameter function).

When the current $J$ exceeds the value $J_{c1}$ the saddle point solutions leading to the phase slip events do not exist more, the described above scenario is exhausted, but another mechanism comes in play. In order to explain it let us recall that the minimum of the GL equation solution is reached for the ground state, corresponding to the solution with the space independent modulus $|\Delta_{gs}(x, y)| = \Delta_0$. When $J < J_{c1}$ the saddle point solutions of GL equations, including vortices, exist with energies higher than that one of the ground state. The transition from the ground state to the saddle point solution one can imagine as the motion of the order parameter “vector” in Hilbert space, accompanying the motion of the “point” $\{r_i\}$ in the finite-dimensional space of the parameters.

In the case when $J > J_{c1}$ the saddle point solutions of GL equations possessing vortices do not exist more. In this region the minimal activation energy is reached at some function $\Delta_v(x, y, r_1)$ corresponding to the state with the single vortex. We choose such a gauge (i.e. the form of vector potential $A$) where the phase of the order parameter is determined only by the vortex position and the boundary conditions at the stripe edges. The modulus of $\Delta_v(x, y, r_1)$ in this case is the even function of the
longitudinal coordinate $y$. In order to find the modulus of the function $\Delta_{\nu}(x,y,r_1)$ we will use the variational principle with three free parameters. The first one is the distance $r_1$ from the edge of the stripe to the center of vortex, the physical sense of two other will be explained below.

II. GENERALITIES

In order to calculate the value of activation energy $\delta F$ for the current-biased NSS we start from the free energy functional $\mathcal{F}_s$ including both GL and the current-field interaction terms:

$$\mathcal{F}_s = \nu \int d^3r \left[ -\tau |\Delta(r)|^2 + \frac{\pi D}{8T} |\partial_\nu \Delta(r)|^2 + \frac{7\zeta(3)}{16\pi^2T^2} |\Delta(r)|^4 \right] + \frac{1}{c} \int d^3r (\mathbf{A} - (c/2e) \nabla \phi) \cdot \mathbf{j}_\infty.$$

Here $\nu = mp_F/(2\pi^2)$ is the density of states ($p_F$ is the electron Fermi momentum), $\partial_\nu = \partial/\partial r - 2ie\mathbf{A}/c$, $\zeta(x)$ is the Riemann zeta-function, $J_\infty = J/S$, $S$ is the cross-section of the stripe, $c$ is the speed of light. We use the system of units where $\hbar = 1$ and $\hbar = 1$.

Close to zero current value. Let us start from the simplest case of zero current, $J_\infty = 0$. In this case the infinite number of the saddle point solutions exist (see Fig.1). When the saddle point solution has only one zero (corresponds to the one vortex state) the phase and modulus of the order parameter are determined as:

$$[\partial \varphi(x,y) / \partial r]^2 = \frac{\pi^2}{I^2} \frac{1}{\sin^2(\pi x/L) + \sin^2(\pi y/L)},$$

$$|\Delta| = \frac{\pi\Delta_0}{L \cosh \frac{2\pi y}{L}} \frac{1}{\pi} \tanh \frac{y_0}{2\xi_{GL}}.$$

where $y_0$ is the free parameter. Substitution of Eqs. (3) and (4) to Eq. (3) gives the value of the activation energy versus $y_0$:

$$\delta F^{(1)}(y_0) = 4\nu \Delta_0^2 \tau S \xi_{GL} \left[ \frac{2}{3} - \frac{\pi y_0}{8\xi_{GL}} + \frac{L}{4\pi\xi_{GL}} \left( \frac{8\pi^2(y_0/L)^2 + 4\zeta(2) - 1}{6} + \frac{\pi y_0^2}{2L^2} I_0 \left( \frac{\pi y_0}{L} \right) \right) \right].$$

Substitution of this value to $\delta F^{(1)}(y_0)$ results in

$$\delta F^{(1)} = 4\nu \Delta_0^2 \tau S \xi_{GL} \left[ \frac{2}{3} + 0.058 \frac{L}{\xi_{GL}} \right].$$

The analogous consideration of the two vortices case gives for $\delta F^{(2)}$ the answer similar to Eq. (5) with the twice smaller second term in parenthesis. Further increase of the number of zeros of order parameter results in decrease of the second term in $\delta F^{(n)}$ by factor $n$ with respect to $\delta F^{(1)}$. In the limit $n \to \infty$ the latter reaches the value

$$\delta F^{(\infty)} = 8\nu \Delta_0^2 \tau S \xi_{GL} / 3$$

first obtained in Ref.[2] in the frameworks of one-dimensional model.

The flow of any finite current through the stripe results in the finiteness of the number of the saddle point solutions. Their number rapidly decreases with the current growth and already at so small current as $J_\infty = N_\xi |J_c$ the only saddle point solution with one vortex remains. At higher currents the saddle point solutions do not exist more, the critical points appear instead of them.

One can see, that at zero current the GL equation solution found in Ref.[2] actually is the limit point for the multiple-vortex solutions obtained above. In result we can state that the point $J = 0$ is the singular point in the
current dependence of the activation energy. Thery the dependence of \( \delta F^{(\infty)}(J) \), obtained in Ref.\(^2\) for small currents as the linear, in fact turns to be essentially more complicated. The fine structure of this dependence can be investigated experimentally. It is worth to mention that the multiplicity of saddle point solutions in the domain of weak currents results in increase of the possibilities of the phase slip events, i.e. to the noticeable increase of the pre-exponential factor.

III. WEAK CURRENTS

Now we consider the simplest one-vortex state in the region of weak currents \( J \leq J_{c1} \). Here the saddle point solution with the vortex center shifted with respect to the stripe central axis exists. Denoting the distance between the axis and vortex center \((x = 0)\), \( \delta \), we look for the solution of the Ginzburg-Landau equation in the form

\[
|\Delta| = \Delta_0(\Gamma) Z^{1/2} \left( \sqrt{y^2 + y_0^2} \right) \Phi(x, y, \delta),
\]

Substitution of this ansatz to the GL equations gives the explicit value for \( \mathcal{L}(\Gamma) \):

\[
\mathcal{L} = \frac{1}{3} + \frac{2}{3} \sin \left( \frac{\pi}{6} + \frac{2}{3} \arcsin \sqrt{1 - \Gamma^2} \right).
\]

The phase of the order parameter satisfies the equation

\[
\left( \frac{\partial \varphi(x, y)}{\partial x} \right)^2 = \left( \frac{\pi}{\mathcal{L}} \right)^2 \frac{\cos^2 \frac{\pi \delta}{\mathcal{L}}}{\cosh^2 \left( \frac{\pi y}{\mathcal{L}} \right)} \Phi^2(x, y, \delta).
\]

Substitution of the Eqs. (7), (8), and (12) into Eq. (2) leads to the expression for free energy.

\[
\delta F_{\delta}(y_0) = 4\nu \Delta_0^2 \tau S \xi_{GL} \left\{ \frac{2}{3} - \frac{\pi y_0}{8 \xi_{GL}} - \frac{16\pi \xi_{GL} \Gamma^2}{27L} \left( \frac{\pi}{4a} + \frac{L}{8y_0} + I_1(a) \right) + \frac{L}{4\pi \xi_{GL}} \left[ \frac{4a^2}{3} + \frac{2}{3} \zeta(2) - \frac{1}{6} + \frac{a^2}{2} I_0(a) \right] \right\},
\]

where

\[
a^2 = \frac{\pi^2 y_0^2}{L^2} + \frac{32\pi^2 \Gamma^2}{27L^2} \xi_{GL}^2
\]

and

\[
I_1(a) = \int_0^\infty dx \frac{1}{\cosh x} \frac{1}{x^2 + 4a^2}.
\]
The Eq. (15) determines $\delta$ as the function of $\Gamma$

$$\sin \frac{\pi \delta}{L} = \left( \frac{2}{3} \right)^{3/2} \frac{\pi^2 \xi_{\text{GL}} \Gamma}{L [2a^2/3 + \zeta(2)/3 + a^2 I_0 (a)]}. \quad (16)$$

Finally, Eqs. (14) and (15) determine the value $y_0$. Corresponding equation is very cumbersome and we do not present it here. Important, that it has the solution only in the very narrow currents interval

$$\Gamma^2 \leq \Gamma_{c1}^2 = 0.009 \frac{L^2}{\xi_{\text{GL}}}. \quad (17)$$

Corresponding value of the free energy $\delta F_{\delta}$ in the critical point $J = J_{c1}$ is

$$\delta F_{\delta} (\tau, J_{c1}) = 4 \nu \Delta_0^2 (\Gamma) \tau S \xi_{\text{GL}} \left( \frac{2}{3} + 0.054 \frac{L}{\xi_{\text{GL}}} \right). \quad (18)$$

Comparing the Eqs. (17) and (5) one can see that the state with $J = J_{c1}$, when the only saddle point solution with one vortex remains, energetically differs from that one with $J = 0$ by very small quantity $0.004 \left( \frac{L}{\xi_{\text{GL}}} \right) 4 \nu \Delta_0^2 (\Gamma) \tau S \xi_{\text{GL}}$.

### IV. “Strong” Currents

Let us pass to discussion of the mechanism of energy dissipation in the wide range of currents $J_{c1} \ll J < J_c$, when the GL equations do not have more any saddle point solution. Let us suppose that through the edge of the stripe penetrates a single vortex and assume that its center is located at some small distance $r_1 (r_1 \ll L)$ from the edge, i.e. the vortex center coordinates are: $(-L/2 + r_1, 0)$. Our goal is to obtain the maximal possible value of the “penetration length” $r_1$ at which the requirement of existence of the conditional extremum of the functional (2) is still satisfied. In order to do this we look for the phase and the modulus of the order parameter in the form containing three free parameters: $r_1, y_0, \gamma$

$$\left( \frac{\partial \varphi (x, y)}{\partial r} \right)^2 = \left( \frac{\varphi (x, y)}{L} \right)^2 \frac{\sin^2 \left( \frac{\pi r_1}{L} \right)}{\cosh^2 \left( \frac{\pi r_1}{2L} \right)} |Q(x, y)|^{-1}, \quad (19)$$

and

$$|\Delta (x, y)| = \Delta_0 (\Gamma) \frac{\ln \left[ 1 + \frac{\pi L^2 Q^{1/2}(x, y)}{2 \nu^2 \xi_{\text{GL}}^2} \right]}{\ln \left[ \frac{2 \nu^2 \xi_{\text{GL}}^2}{\pi L^2} \right]} Z^2 (\Gamma, y + y_0). \quad (20)$$

The function

$$Q(x, y, r_1) = \left\{ 4 \sinh^2 \left( \frac{\pi y}{L} \right) \left[ \cosh^2 \left( \frac{\pi y}{L} \right) + \cos \frac{\pi r_1}{L} \sin \left( \frac{\pi x}{L} \right) \right] + \left[ \sin \left( \frac{\pi x}{L} \right) + \cos \frac{\pi r_1}{L} \right]^2 \right\} \left[ \cosh \left( \frac{\pi y}{2L} \right) \right]^{-4}, \quad (21)$$

is chosen in the way to satisfy the boundary conditions for the order parameter and its derivatives at the edge of the stripe and at infinity.

The current conservation law leads to the next expression for the essential part of the vector potential $A_y$

$$A_y^{(0)} (y) = \frac{A_x \Delta_0^2 (\Gamma) \tau S \xi_{\text{GL}}}{(\Delta (x, y)^2)_x}, \quad (22)$$

where the value of the vector potential at infinity is determined by the current density

$$\frac{J}{S} = - \frac{\nu e^2 D}{T} \Delta_0^2 (\Gamma) A_x. \quad (23)$$

Replacing the expressions (18), (19), and (20) to the GL functional (2) and calculating the integral one can find the value of excess free energy $\delta F_s$ of the stripe with the penetrated vortex with respect to its ground state with the fixed current. The requirement of existence of the conditional extremum determines the value $y_0$

$$y_0 = 2 \xi_{\text{GL}} Y_{\text{L}} / \sqrt{3 \xi_{\text{GL}} - 2}, \quad (24)$$

$$\tanh Y_{\text{L}} = 2 \left[ \frac{(1 - \xi_{\text{GL}})}{4 - 3 \xi_{\text{GL}} + \sqrt{(16 - 15 \xi_{\text{GL}})}} \right]^{1/2}. \quad (25)$$

The obtained results allow us to write down the expression for the activation energy in the whole range of “strong currents” $J_{c1} \ll J < J_c$

$$
\delta F (\tau, J) = 4 \nu \Delta_0^2 (\Gamma) \tau S \xi_{\text{GL}} \sqrt{(3 \xi_{\text{GL}} - 2)} \left\{ \frac{1 - \tanh Y_{\text{L}}}{6 \xi_{\text{GL}}} \left[ 4 + (3 \xi_{\text{GL}} - 2) \tanh Y_{\text{L}} \right] \left( 1 + \tanh Y_{\text{L}} \right) \right\}.
\quad (26)
$$

It is seen that the difference between Eq. (26) and the expression for the activation energy of the one-dimensional superconducting channel carrying current $J$ (the main result of Ref [4]) consists of the contribution occurring due
to the nonzero value of the parameter $Y_L$, i.e. due to existence of the conditional extremum of the free energy functional at the distance $y_0 \neq 0$. Let us stress that the activation energy $\delta F$ depends on the geometry of a sample, which here is assumed as a stripe. The increase of the energy barrier in the Arrenius law with respect to the result of Ref.\cite{2} is related to the necessity of the vortex penetration in a sample at the moment of the phase slip event.

The expression for activation energy $\delta F^{(LA)}(\tau, J)$ of the one-dimensional superconducting channel found in Ref.\cite{2} can be easily reproduced from Eq. (23) just putting $Y_L = 0$ (what follows from Eq. (22)). One can compare the result of our careful consideration of the vortex penetration mechanisms with the latter:

$$\frac{\delta F(\tau, J) - \delta F^{(LA)}(\tau, J)}{\delta F^{(LA)}(\tau, J)} = \left[ \frac{3L - 2 \tanh Y_L}{3L} \cosh^2 Y_L - \frac{2 \tanh Y_L}{3L} \right] + \sqrt{\frac{2(1 - L^2)}{3L^2 - 2}} \arctan \left( \frac{3L - 2 \tanh Y_L}{3L^2 - 2} \right).$$  \hspace{1cm} (24)

The most interesting limit is $J \to J_c(\Gamma \to 1)$, when $\tanh Y_L \to 1/\sqrt{3}$, $\cosh^2 Y_L \to 3/2$; $1 + 2^{-1} \cosh^{-2} Y_L - L \tanh^2 Y_L/[2(1 - L^2)] \to 1$; $3L - 2 \to 2^{3/2} \sqrt{1 - \Gamma} / \sqrt{3}$. In this case the relative difference Eq. (24) diverges:

$$\frac{\delta F(\tau, J) - \delta F^{(LA)}(\tau, J)}{\delta F^{(LA)}(\tau, J)} \bigg|_{J \to J_c} = \frac{5J_c}{6\sqrt{J_c^2 - J^2}}. \hspace{1cm} (25)$$

Indeed, the discrepancy between the theoretical prediction of the Ref.\cite{2} and the experimental findings has been known many years\cite{3} and we believe that our result explains this long standing enigma. The behavior of Eq. (24) as the function of $\Gamma$ in the interval $[0, 0.9]$ is presented in Fig. 2.

V. PRE-EXPONENTIAL FACTOR

In order to obtain the exact value of the pre-exponential factor $\Omega$ for phase slip events one should have in possession the expression for the effective action of superconducting stripe containing vortices. In the Ref.\cite{2} was proposed a general procedure which, in the regime of thermal fluctuations, is reduced to solution of the spectral problem for linear operator corresponding to the action at its saddle point. The difficulty of the problem under consideration consists in the fact that nor microscopic action operator is known nor saddle point (for currents $J_{c1} < J$) exists. Nevertheless, the knowledge of action would allow one to get the precise value of $\Omega$ at least for weak currents $J < J_{c1}$, while for strong currents one could believe that change of the saddle point to a singular point would not strongly effect on the value of pre-exponential factor. In light of the above said the evaluations of $\Omega$ in both papers\cite{26} seem doubtful: use of the time-dependent GL equation below $T_c$ as today is well known cannot be justified.

The main contribution to the average time between two subsequent phase-slip events is related to the existence of the “zero-mode” (see Ref.\cite{27}). In the case under consideration the size of the vortex is determined by the transversal size $L$ of the stripe. The vortices which slip on the distances larger than $L$ can be considered as independent. It is why the main factor determining the pre-exponential one is the ratio of the transversal size $L$ to the stripe length $L$: $L / L$. Another coefficient which forms the pre-exponential factor is the characteristic “crossing time” $\Delta t_{cross} = L / v_{cross}$ of the stripe by the vortex, moving with the velocity $v_{cross}$

$$v_{cross} = \frac{cJ \sqrt{\tau}}{4SH_{c2} \sigma_n}, \hspace{1cm} (26)$$

where $\sigma_n$ is the conductivity of the stripe in its normal phase. Finally, accounting for Arrenius factor, one finds the characteristic time $\Delta t$ between two phase slipping
\[ \Delta t = \frac{L}{2} \left( \frac{L}{v_{\text{cross}}} \right) \exp \left( \frac{\delta F}{k_B T} \right). \]  

(27)

The average voltage \( V \) at the stripe is related to the average time interval \( \Delta t \) between the voltage jumps by the Josephson relation \(^9\):

\[ V = \frac{\pi \hbar}{e} \left( \frac{1}{\Delta t} \right). \]

Corresponding resistance of the stripe is

\[ R = \frac{\pi \hbar c \sqrt{\tau}}{4 e H_c^2 L^2} \exp \left[ -\frac{\delta F(\tau, \Gamma)}{k_B T} \right], \]

where \( R_0 = \frac{L}{(\sigma_n S)} \) is the normal resistance of the stripe. It is necessary to mention that the used above approximation of the independent phase slips is valid only (according to Eq. (27)) when \( \delta F > k_B T \).

VI. CONCLUSIONS

We have demonstrated that just the approach to the saddle point which corresponds to the independent on the transverse coordinate order parameter (see Ref.\(^2\)) in the case of a narrow superconducting stripe carrying current \( J \neq 0 \) is not sufficient to describe correctly the properties of its resistive state. Namely, the value of activation energy in the Arrenius law for the resistance of the narrow superconducting channel differs from the simple difference of the free energies of such a saddle point and the ground state. The mechanism of the phase slip events turns out to be much more sophisticated than that one described in Ref.\(^2\).

At weak currents the subsequence of saddle points, characterized by the number \( n \) of the order parameter zeros along the transverse coordinate, appears. The energy of such state equals with that one found by Langer and Ambegaokar\(^2\) only in the limit \( n \to \infty \), when the flowing current is absent. I.e. the point \( J = 0 \) is singular.

When current increases the number \( n \) of the saddle points rapidly decreases. The number \( n \) reaches 1 when the current achieves the value \( J_{c1} = 0.0312 \left( L/\xi \right) J_c \); at this point the only stationary state remains. When the \( J \geq J_{c1} \) no stationary solutions of the GL equations with the fixed current and a vortex in the stripe exist. Instead of them one should look for the critical point, corresponding to the existence of the specific conditional extremum of the GL functional. These conditions are: the fixed current \( J \) and the maximal possible distance between the vortex center and the stripe edge. The energy of such a state turns out to be higher than the activation energy \( \delta F^{(LA)}(\tau, J) \) obtained in Ref.\(^2\). The normalized difference \( 25\) increases with growth of the current and when the latter approaches to the critical value \( J_c \), the former diverges (see Fig. 2).

Very probably that namely the described phenomenon is responsible for the considerable discrepancy between the experimental data and the theoretical prediction observed in the vicinity of the critical current\(^3\).

VII. ACKNOWLEDGMENTS

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