KPZ Equation in One Dimension and Line Ensembles

Herbert Spohn
Physik Department and Zentrum Mathematik, Technische Universität München,
D-85747 Garching, Germany
spohn@ma.tum.de

Abstract. For suitably discretized versions of the Kardar-Parisi-Zhang equation in one space dimension exact scaling functions are available, amongst them the stationary two-point function. We explain one central piece from the technology through which such results are obtained, namely the method of line ensembles with purely entropic repulsion.

Keywords. exact solution, two-point function, probability density functions

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1. Introduction

Since STAPHYS 21 our understanding of the statistical properties of the one-dimensional KPZ equation (Kadar, Parisi, and Zhang [1]) for surface growth has improved substantially. Thus an obvious scheme would be to sketch the main results and to list the relevant articles. While the latter will be done anyhow, it seems to me that a more interesting strategy is to explain one central method through which these advances became possible, namely the mapping to line ensembles. Given the appropriate line ensemble, to obtain properties of physical interest still requires a tough asymptotic analysis for which I refer to the articles [2–10]. The link to multi-matrix models is illustrated in our contribution to the Proceedings of the ICMP 14 [11].

It seems difficult to handle directly the continuum KPZ equation and one has to rely on suitable discretizations. In one dimension the slope is locally conserved. Thus a natural discretization is to assume that the slope takes only two values and that the dynamics is defined through a suitable stochastic exchange rule. Thereby one arrives at the totally asymmetric simple exclusion process (TASEP), which is also a standard model for driven lattice gases in one dimension [12]. We briefly recall the definition. At each site $j \in \mathbb{Z}$ there is at most one particle. Thus the occupation variables $\eta_j(t)$ at time $t$ take only the values 0 (site $j$ empty) and 1 (site $j$ occupied). Under the stochastic dynamics a particle with an empty right neighbor site jumps independently to that site after an exponentially distributed waiting time with rate 1. The master equation for the Markov chain thus reads

$$\frac{d}{dt} f_t(\eta) = L f_t(\eta), \quad t \geq 0,$$

(1)
with generator

\[ Lf(\eta) = \sum_{j \in \mathbb{Z}} \eta_j (1 - \eta_{j+1}) (f(\eta^{j+1}) - f(\eta)) \]  

as acting on functions \( f \) over configuration space. Here \( \eta^{j+1} \) denotes the configuration \( \eta \) with the occupations at sites \( j \) and \( j + 1 \) interchanged.

In the interpretation of surface growth, the height function \( h_j(t) \) at time \( t \) has the height differences \( h_{j+1}(t) - h_j(t) = 1 - 2\eta_{j+1}(t) \) with the convention that \( h_0(t) = 2N_i, N_i \) being the number of jumps through the bond from 0 to 1 in the time span \([0, t]\). Then our dynamical rule simply means that local minima of \( h_j(t) \) are filled independently after an exponentially distributed waiting time. The stationary measures of the TASEP are Bernoulli, \( \text{Prob}_\rho(\eta_j = 1) = \rho \). Therefore \( \langle \eta_i \eta_j \rangle_\rho = \rho^2 = \delta_{ij} \rho(1 - \rho) \) and the stationary current is \( j(\rho) = \rho(1 - \rho) \). Since \( j''(\rho) \neq 0 \), the TASEP is in the KPZ universality class for every density \( \rho, \rho \neq 0, 1 \). More details can be found in [13].

For the initial conditions there are three prototypical choices. In fact, while the dynamical scaling exponent is always \( z = 3/2 \), the scaling functions do depend on the initial conditions.

- **flat initial data**, which is the standard choice for numerical simulations. For the TASEP it means \( \eta_j(0) = (1 + (-1)^j)/2 \). Then \( \langle h_0(t) \rangle = \frac{1}{2} t \). From a statistical mechanics point of view the object of interest are random fluctuations. Their behavior is only partially understood. Assuming universality, there is strong evidence for

\[ h_0(t) \approx \frac{1}{2} t - \xi_{\text{GOE}} t^{1/3} \quad \text{for } t \to \infty \]  

with the random amplitude \( \xi_{\text{GOE}} \) distributed as the largest eigenvalue of a \( N \times N \) GOE random matrix in the limit \( N \to \infty \) [14,6]. This distribution is known as Tracy-Widom at \( \beta = 1 \) [15]. Note that instead of centering we subtracted the asymptotic mean. In particular, it turns out that \( \langle \xi_{\text{GOE}} \rangle < 0 \). As a stronger statistical property, under diffusive scaling with scale parameter \( t^{2/3} \) and in the limit \( t \to \infty \), the process

\[ x \mapsto t^{-1/3} \left( h_{\lfloor t^{2/3} x \rfloor}(t) - \frac{1}{2} t \right), \quad \lfloor \cdot \rfloor \text{ integer part}, \]  

should have the same fluctuations as the top line in Dyson’s Brownian motion at \( \beta = 1 \) in the limit \( N \to \infty \), see [6] for a complete discussion.

- **droplet geometry**, which takes \( \eta_j(0) = 1 \) for \( j \leq 0 \) and \( \eta_j(0) = 0 \) for \( j \geq 0 \). In terms of the height, \( h_j(0) = |j| \). The asymptotic shape of the droplet is \( h_{\lfloor t \rfloor}(t) \approx t(x^2 + 1)/2, |x| \leq 1 \), which has nonzero curvature in contrast to the flat initial conditions. The droplet geometry occurs naturally in the growth of an Eden cluster and other growth processes in the plane starting from a point seed. In our context the droplet geometry derives its fame from the work of K. Johansson [2], who proved for the height at the origin

\[ h_0(t) \approx \frac{1}{2} t - \xi_{\text{GUE}} t^{1/3} \quad \text{for } t \to \infty \]  

with the random amplitude \( \xi_{\text{GUE}} \) distributed as the largest eigenvalue of a \( N \times N \) GUE random matrix in the limit \( N \to \infty \). For the droplet geometry even stronger property (4) is
available. The limit is the stationary Airy process \([16,3]\). (For the sake of notational clarity we omit numerical multiplicative factors. They are provided in the references given.)

- stationary initial conditions. This means that at time \(t = 0\) the \(\eta_j(0)\) are independent with \(\text{Prob}_\rho(\eta_j(0) = 1) = \rho\). The stochastic process \(\eta_j(t)\) is stationary in both variables and can be viewed as a two-dimensional field theory in infinite volume. The natural statistical object is then the two-point function

\[
S(j, t) = \langle \eta_j(t) \eta_0(0) \rangle_\rho - \rho^2.
\]

It has the scaling form, introducing the compressibility \(\chi(\rho) = \rho(1 - \rho)\),

\[
S(j, t) \cong \chi(4\chi^{1/3}t^{2/3})^{-1} \frac{1}{8} g''((4\chi^{1/3}t^{2/3})^{-1}(j - (1 - 2\rho)t))
\]

for \(j - (1 - 2\rho)t = O(t^{2/3})\) and \(t \to \infty\). A plot of the scaling function \(g''\) can be found in \([17]\). For the TASEP a proof based on line ensembles is under construction \([18]\).

### 2. Nonintersecting Line Ensembles

Line ensembles are familiar from bosonic and fermionic paths integrals. In our context the relevant example are free fermions on the one-dimensional lattice with Hamiltonian

\[
H = \sum_j (a^*(j)a(j + 1) + a^*(j + 1)a(j)).
\]

Here the Fermi field \(a(j)\) satisfies the anticommutation relations \(\{a(i), a(j)\} = 0,\ \{a(i), a(j)^*\} = \delta_{ij}\). Let us denote by \(x_j(s), j = 1, \ldots, N\), a collection of independent symmetric nearest neighbor random walks on \(\mathbb{Z}\) with jump rate 1. We condition them not to intersect, indicated by NC for non-crossing constraint. Then the \(N\)-particle transition amplitude is expressed as

\[
\langle x_1, \ldots, x_N|e^{-sH}|y_1, \ldots, y_N \rangle = \text{Prob}^{\text{NC}}(x_1(s) = x_1, \ldots, x_N(s) = x_N|x_1(0) = y_1, \ldots, x_N(0) = y_N),
\]

i.e. the random walks start at \(y_1 < \ldots < y_N\) at time 0 and end up at \(x_1 < \ldots < x_N\) at time \(s\) constrained not to intersect. Since the constraint is purely geometric, it is also called entropic repulsion. The transition probability of the TASEP, \(e^{Lt}\), cannot be written as a line ensemble with purely entropic interaction. Nevertheless, in a much more hidden way there is such a line ensemble, but it depends in a delicate way on the initial conditions as will be explained in the following section.

There is a second example which is formulated rather differently, but close to \((8)\) in fact. Following Dyson \([19]\), we consider the stochastic process \(s \mapsto A(s)\) which takes values in the \(N \times N\) Hermitian matrices and is governed by the linear Langevin equation

\[
\frac{d}{ds}A(s) = -\frac{1}{N}A(s) + W(s).
\]

Here \(W(s)\) is an \(N \times N\) Hermitian matrix whose matrix elements are independent Brownian motions. We assume \(A(s)\) to be stationary in time. \(A(s)\) has the real eigenvalues
\( \lambda_{-N+1}(s) < \ldots < \lambda_0(s) \). They never cross. At a fixed time \( s \), \( A(s) \) has the Gaussian distribution \( Z^{-1} \exp [-\frac{1}{4} \text{tr} A^2] dA \), known as GUE of random matrices [20]. The eigenvalues \( \lambda_j(s), j = -N + 1, \ldots, 0 \), form a line ensemble with purely entropic repulsion. In particular, the 2-dimensional random field

\[
\phi_N(x, s) = \sum_{j=-N+1}^{0} \delta(\lambda_j(s) - x)
\]

has determinantal moments constructed out of the extended Hermite kernel [21].

As to be detailed, very roughly, the height function in the growth process has the same statistics as the top eigenvalue in Dyson’s Brownian motion (10), where space \( x \) corresponds to the Brownian motion time \( s \) and the growth time \( t \) corresponds to the matrix size \( N \). This correspondence explains why the universal distribution functions for the KPZ equation are expressed through quantities appearing also in the edge scaling of random matrices.

3. Line Ensemble for the TASEP

We pick the case of step initial conditions, which is the easiest one to explain and well illustrates the principle. Thus initially \( \eta_j = 1 \) for \( j \leq 0 \) and \( \eta_j = 0 \) for \( j > 0 \). Let us label the particle initially at site \( j \) by the index \(-j + 1\). We define \( w(i, j) \) as the \( i \)-th jump time of particle number \( j \). Thus \( w(i, j), i, j = 1, 2, \ldots, \) is an infinite array of independent exponentially distributed random variables, \( \text{Prob}(w(i, j) \in [y, y + dy]) = e^{-y}dy, y \geq 0 \). Given the \( w(i, j)'s \) one constructs from them a growth process by filling the square \((i, j)\) after the random time \( w(i, j) \) starting from the moment when the two adjacent squares \((i - 1, j)\) and \((i, j - 1)\) are already filled. Let us denote by \( G(m, n) \) the time when the square \((m, n)\) is first filled. Then the cluster grown at time \( t, \{(m, n) | G(m, n) \leq t\} \), has as its border the height function \( h_j(t) \) up to a rotation by \( \pi/4 \).

\( G(m, n) \) can be considered as a sort of dual growth process. It satisfies the random recursion relation

\[
G(m, n) = \max \{G(m - 1, n), G(m, n - 1)\} + w(m, n),
\]

(12)
\( G(0, n) = 0 = G(m, 0) \). \( G(m, n) \) has also the interpretation of the optimal time for a last passage directed percolation, which can be seen by introducing the directed polymer, \( \omega \). It is an up/right path on \((\mathbb{Z}_+)^2\) starting at \((1, 1)\). The “length” of a polymer is given by

\[
\tau(\omega) = \sum_{(i,j)\in\omega} w(i, j).
\]

(13)

\( G(m, n) \) is the length of the optimal path at fixed endpoints,

\[
G(m, n) = \max_{\omega: (1,1)\rightarrow(m,n)} \tau(\omega).
\]

(14)

Since \( h_j(t) \) is increasing linearly in \( t \), so is \( G(n + k, n - k) \) in \( n \), and the scaling behavior of \( h_j(t) \) for large \( t \) is determined by the scaling behavior of \( G(n + k, n - k) \) for large \( n \).
For example, $G(n, n)$ is the time it takes for $h_0(t)$ to reach the level $2n$. Thus the assertion (5) follows from a corresponding one for $G(n, n)$ in the limit of large $n$.

We are now in the position to construct out of $G(m, n)$ an ensemble of lines with purely entropic repulsion. We fix $\tau$ and introduce the lines $h_{\ell}(j), \ell = 0, \ldots, -\tau + 1, h_{\ell}(j) \in \mathbb{R}_+$ and $h_{\ell}(2j) - h_{\ell}(2j - 1) \geq 0$ while $h_{\ell}(2j + 1) - h_{\ell}(2j) \leq 0$. Furthermore $h_{\ell}(j) = 0$ for $|j| \geq 2\tau - 2\ell - 1$, see Figure 1. Each jump of absolute size $\delta$ carries the weight $e^{-\delta/2}$, $\delta \geq 0$. The weight of the whole line ensemble is then the product of the weights of all jumps. Finally one has to impose the non-crossing constraint. It is most easily formulated through extending $j \mapsto h_{\ell}(j)$ to piecewise constant lines in the plane as

$$h_{\ell}(s) = h_{\ell}(j) \quad \text{for} \quad j - \frac{1}{2} \leq s \leq j + \frac{1}{2}. \quad (15)$$

Note that for each constant piece both endpoints are included. We then require

$$(NC) \quad h_{\ell-1}(s) < h_{\ell}(s) \quad (16)$$

for all $\ell$, all $s$, and all endpoints, compare with Figure 1.

The idea of the line ensemble is that the top line, $h_{0}(j)$, records the statistics of $G(m, n)$, while the others are introduced only for book-keeping purposes. This can be seen by generating the line ensemble through the Robinson-Schensted-Knuth algorithm [22] (it is the Knuth of TeX). $\tau$ is regarded as growth time parameter and we set $w(1, 1) = h_{0}(0, \tau = 1)$, see Figure 2. The up-step (at $s = -\frac{1}{2}$) is shifted one unit to the left and the down-step (at $s = \frac{1}{2}$) is shifted one unit to the right, yielding $\tilde{h}_{0}(j, \tau = 1.5), j = -1, 0, 1$. Then we add mass according to $h_{0}(-1, \tau = 1.5) = \tilde{h}_{0}(-1, \tau = 1.5) + w(1, 2), h_{0}(0, \tau = 1.5) = \tilde{h}_{0}(0, \tau = 1.5), h_{0}(1, \tau = 1.5) = \tilde{h}_{0}(1, \tau = 1.5) + w(2, 1), etc.. In this way the top line $h_{0}(s, \tau), s \in \mathbb{R}, \tau = 1, 2, \ldots,$ satisfies with the identities

$$G(\tau + j, \tau - j) = h_{0}(2j, \tau), \quad G(\tau + j, \tau - j - 1) = h_{0}(2j + 1, \tau). \quad (17)$$

The line $h_{-1}$ is generated from $h_{0}$ through a fully deterministic rule: When moving the up-steps and down-steps of $h_{0}$ to $\tilde{h}_{0}$ there will be excess mass at coalescing steps. This
excess mass is recorded as nucleation for $h_{-1}$. The up-steps and down-steps of $h_{-1}$ move deterministically in the same fashion as explained for $h_0$. The nucleation for the line $h_{-2}$ is generated by the excess mass of $h_{-1}$, etc. Note that the only random input is to the top line $h_0$. Of course, it requires an argument, see [10,18], that at growth time $\tau$ the lines have exactly the probability distribution as claimed, namely weighing the jumps exponentially and imposing the non-crossing constraint.

We have to issue a word of caution. For the multi-line growth process probabilities referring to two different growth times are meaningful. In particular we may want to enquire on $h_0(s, \tau)$ for two different $\tau$'s. This information is no retraceable from the static, fixed $\tau$, line ensemble. In case of the TASEP, line ensembles can be used to analyze the statistics of $h_j(t)$, $j \in \mathbb{Z}$, jointly with prescribed initial conditions but not for, say, $h_0(t_1)$ and $h_0(t_2)$ jointly.

As in (11) for Dyson's Brownian motion the line ensemble $h_\ell(j, \tau)$ determines the random field $\phi_\tau(x, j)$ through

$$
\phi_\tau(x, j) = \sum_{\ell=\tau+1}^0 \delta(h_\ell(j, \tau) - x).
$$

(18)

In analogy to femionic path integrals it is convenient to think of $x \in \mathbb{R}_+$ as space, $j \in \mathbb{Z}$, $|j| \leq \tau$, as time, and $\tau$ as the number of growth steps. As already anticipated $\phi_\tau$ has determinantal moments. This means that for a time-ordered sequence $u_1 \leq \ldots \leq u_m$ and arbitrary space points $x_1, \ldots, x_m$ one has

$$
\langle \prod_{j=1}^m \phi_\tau(x_j, u_j) \rangle = \det \{ K^{(2\tau)}(x_i, u_i; x_j, u_j) \}_{1 \leq i,j \leq m}.
$$

(19)

To write down a formula for the kernel $K^{(2\tau)}$ we introduce the operators $T_\pm$ on $L^2(\mathbb{R})$ as given by the integral kernels, $x, y \in \mathbb{R}$,

$$
T_+(x, y) = \begin{cases} 
    e^{-(x-y)/2}, & \text{if } x \geq y, \\
    0, & \text{if } x < y,
\end{cases}
$$

$$
T_-(x, y) = \begin{cases} 
    0, & \text{if } x > y, \\
    e^{-(y-x)/2}, & \text{if } x \leq y.
\end{cases}
$$

(20)
We also introduce the projection $P_-$ onto the half line $(-\infty, 0]$. Then, for $\tau \geq 1$, $x, y > 0$,

$$K^{(2\tau)}(x, 2u; y, 2v) = (T_+ T_-)^u (T_+ T_-^{-1})^\tau P_-(T_+^{-1} T_-)^\tau (T_+ T_-)^{-v}(x, y)$$

in case $-\tau < u \leq v < \tau$ and

$$K^{(2\tau)}(x, 2u; y, 2v) = (T_+ T_-)^u ((T_+ T_-^{-1})^\tau P_-(T_+^{-1} T_-)^\tau - 1)(T_+ T_-)^{-v}(x, y)$$

in case $-\tau < v < u < \tau$. There are similar formula for odd arguments.

As a particular case one may consider equal times $u, v = 0$, i.e. the moments of $\phi_{2\tau}(x, 0)$. Specializing in (21) results in the equality

$$K^{(2\tau)}(x, 0; y, 0) = \sum_{j=0}^{\tau-1} L_j(x) L_j(y) e^{-x/2} e^{-y/2},$$

$x, y > 0$, where $L_j$ are the order 0 Laguerre polynomials normalized as

$$\int_0^\infty dx e^{-x} L_i(x) L_j(x) = \delta_{ij}.$$  

Such a kernel is known from the theory of random matrices [23]. There one considers $N \times N$ complex matrices $A$ where the matrix elements $A_{ij}$ are independent Gaussian random variables normalized as $\langle |A_{ij}|^2 \rangle = 1$. Then the eigenvalues $A^* A$ have determinantal moments with kernel (23). Thus the eigenvalues of $A^* A$ have the same joint distribution as $h_{\ell}^0(0, \tau), \ell = 0, \ldots, -\tau + 1$.

It follows from (19) that the joint distribution function for the top line at times $u_1, \ldots, u_m$ is expressed through the determinant of an operator acting on $L^2(\mathbb{R}) \otimes \mathbb{C}^m$, i.e. acting on $m$-spinors over $\mathbb{R}$. We choose $u_1 < \ldots < u_m$ and define the integral kernel

$$K(x, i; y, j) = K^{(2\tau)}(x, u_i; y, u_j).$$

Let $G$ be multiplication by the indicator function $\chi_{[s_i, \infty)}(v)$, $s_j > 0$. Then

$$\text{Prob}(h_0(u_1, \tau) \leq s_1, \ldots, h_0(u_m, \tau) \leq s_m) = \det(1 - G^{1/2} K G^{1/2}).$$

To obtain the asymptotic statistics of $h_0(j, \tau)$ thus requires an understanding of the asymptotics of the kernel $K^{(2\tau)}$ as $\tau \to \infty$ at suitable rescaling of the arguments $(x, u; y, v)$.

### 4. Extensions

We discussed the line ensemble for step initial conditions of the TASEP. For large $\tau$, the dual growth process $h_0(j, \tau)$ has a definite shape, which is bent downwards and has a jump of macroscopic size at both endpoints. Thus under edge scaling $h_0(j, \tau)$ is governed by the stationary Airy process, which has joint distributions of a structure very similar to (26); only the kernel $K$ is replaced by the extended Airy kernel [3,16].
By the same method also general step initial conditions can be handled, to say: we pick the left density $\rho_-$ and the right density $\rho_+$ and require that under the initial conditions for the TASEP the $\eta_j$’s are independent with $\text{Prob}(\eta_j = 1) = \rho_-$ for $j \leq 0$ and $\text{Prob}(\eta_j = 1) = \rho_+$ for $j > 0$. In the mapping to the directed polymer the $w(i, j)$’s have to be extended by the row $w(j, 0), j \geq 0$, and the column $w(0, j), j \geq 0$. The directed polymer starts at $(0, 0)$. More precisely the $w(i, j)$ are independent exponentials with the following means,

$$
\langle w(i, j) \rangle = 1 \text{ for } i, j \geq 1, \quad w(0, 0) = 0,
$$

$$
\langle w(j, 0) \rangle = (1 - \rho_+)^{-1}, \quad \langle w(0, j) \rangle = \rho_-^{-1} \text{ for } j > 0. \quad (27)
$$

The construction explained in the previous section can be carried through only if the $w(i, j)$’s are independent exponentials with $\text{Prob}(w(i, j) \in [y, y + dy]) = (\alpha_i + \beta_j) \exp[-(\alpha_i + \beta_j)y]dy, y > 0, \alpha_i, \beta_j > 0$. Starting from the left, the weight for an up-step of size $\delta$ is then $\exp[-\alpha \delta]$ and for a down-step of size $\delta$ it is $\exp[-\beta \delta]$. $\delta > 0$. For the step initial conditions, $\rho_- = 1, \rho_+ = 0$, clearly $\alpha_\delta = (\frac{1}{2}, \frac{1}{2}, \ldots)$ and $\beta_\delta = (\rho_- - \frac{1}{2}, \frac{1}{2}, \ldots)$, which requires $0 \leq \rho_+ < \frac{1}{2}, \frac{1}{2} < \rho_- \leq 1$. Up to an error of order $1$ the TASEP height is then represented by the corresponding height of the dual growth process.

The restrictions on $\rho_+, \rho_-$ are somewhat surprising, since the directed polymer with random waiting times according to (27) is well defined for any choice of $\rho_+, \rho_-$. To cover the left out domain in densities one has to analytically continue in $\rho_+, \rho_-$. The details tend to be lengthy and we refer to [10] for such a continuation in case of the discrete time TASEP. Here we briefly explain stationary initial conditions, $\rho_+ = \rho = \rho_-$, as the one of most physical interest. In this case $\alpha_0 + \beta_0 = 0$, formally, which means that the weight of the very first deposition is $1$ independent of its height. Such a weight cannot be normalized to $1$, but averages of physical quantities can still be computed through a suitable limit procedure. As a net result one finds that for joint distribution functions

$$
\text{Prob}_\rho(h_0(u_1, \tau) \leq s_1, \ldots, h_0(u_m, \tau) \leq s_m) = \frac{d}{ds} \det(1 - G_s^{1/2} \overline{K} G_s^{1/2})_{s=0} + \mathcal{O}(1), \quad (28)
$$

compare with (26). Here $\text{Prob}_\rho$ refers to the stationary TASEP with initial Bernoulli measure of density $\rho$. $h_0(u, \tau)$ is the top line of the corresponding line ensemble after $\tau$ growth steps. $G_s$ is the multiplication by the indicator function of the interval $[s_j + s, \infty)$ and $\overline{K}$ is the extended Laguerre kernel (21) perturbed by a rank $1$ operator which contains the information on the density $\rho$. Such a structure seems to be novel.

Step initial conditions lead to boundary terms for the directed polymer. A further natural modification is to retain the independent exponentially distributed $w(i, j)$ of mean $1$ but to alter along the diagonal to $\text{Prob}(w(i, j) \in [y, y + dy]) = \gamma e^{-\gamma y}dy, y > 0, \gamma > 0$. This results in two well-studied, but unresolved problems:

(i) For the TASEP the rate for jumps from $0$ to $1$ is changed from $1$ to $\gamma$. For $\gamma \ll 1$, particles pile up behind the slow bond and the average current is reduced. It is conjectured that at $\gamma_c = 1$ the current reaches its maximal value of $\frac{1}{4}$ [24]. A more intricate scenario is proposed in [25].

(ii) On the other hand the directed polymer can be interpreted as an interface in a random
potential with a pinning potential along the diagonal \( \{i = j\} \). For \( \gamma < \gamma_c \) the interface is pinned and it unbinds as \( \gamma \to \gamma_c \). For \( \gamma > \gamma_c \) the directed polymer has \( n^{2/3} \) transverse fluctuations [26]. It is conjectured that the unbinding transition occurs at \( \gamma_c = 1 \) [27, 28].

The exponential distribution of the \( w(i, j) \)'s derives from the exponential waiting times in the jumps of the TASEP. On the other hand, from the point of view of the directed polymer any other distribution looks fine, perhaps imposing some moment conditions. Unfortunately the construction of purely entropic line ensembles requires that the \( w(i, j) \)'s are allowed to be independent with the geometric distribution \( \text{Prob}(w(i, j) = n) = (1 - a_{ij})(a_{ij})^n, \quad n = 0, 1, \ldots \), where the \( a_{ij} \) must be of product form, \( 0 < a_{ij} < 1 \) and \( a_{ij} = b_i b_j \). This includes the exponential limit, discussed previously. In the opposite limit, the Poisson limit, the \( w(i, j) \)'s are mostly 0 and once in while take the value 1. Thereby one arrives at a Poisson point process in the quadrant \( \{x_1 \geq 0, x_2 \geq 0\} \) with an average density, \( \rho \), of product form, \( \rho(x_1, x_2) = \rho_1(x_1)\rho_2(x_2) \). The directed polymer lives then in the continuum. It moves from Poisson point to Poisson point, linearly interpolating, restricted to be increasing in both coordinates and starting from \((0, 0)\) reaching some prescribed endpoint \((x_1, x_2)\) in the positive quadrant. The corresponding growth process is the polynuclear growth (PNG). It has a continuous growth time, \( t \), and the height function \( h(x, t) \) is over the real line. \( x \mapsto h(x, t) \) is piecewise constant and takes integer values only. The dynamical rules can be read off the directed polymer. Accordingly, an up-step moves with speed one to the left and a down-step with speed one to the right. Whenever two steps meet they annihilate. In addition, randomly and uniformly in space-time, a spike of height one is nucleated on the top of the current height profile. The spike, then immediately widens under the deterministic motion. The line ensemble for the PNG is constructed as before, namely by using the mass annihilated at line \( h_{-\ell}(x, t) \) as nucleation in the lower lying line \( h_{-\ell-1}(x, t) \). The droplet geometry can be imposed by requiring that initially \( h(x, 0) = 0 \) for \( x \neq 0 \) and \( h(0, 0) = 1 \) and that nucleation is allowed only on the top of the first layer. As for the TASEP, the resulting random field has determinantal moments.

In general, however, line ensembles do not have to be determinantal. A point in case is the PNG half droplet with an extra source of rate \( \gamma \) at the origin [5]. In the corresponding line ensemble the lines are pinned at \( x = t \), i.e. \( h_{-\ell}(t, t) = -\ell \), while at \( x = 0 \) they satisfy a “pairing rule”. In particular, the point process \( \{h_{-\ell}(0, t), \ell = 0, 1, \ldots\} \) has its moments given through a suitable Pfaffian. The top line at \( x = 0 \) has in the scaling limit for \( \gamma = 0 \) the same fluctuations as the largest eigenvalue of a GOE random matrix and for \( \gamma = 1 \) the one of a GSE random matrix. A further point in case is the PNG model with flat initial conditions [6]. While in space-time the process is not determinantal, the point process \( \{h_{-\ell}(0, t), \ell = 0, 1, \ldots\} \) has moments given through a Pfaffian.

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