Efficiency fluctuations in steady-state machines

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Abstract

We study the statistics of the efficiency in a class of isothermal cyclic machines with realistic coupling between the internal degrees of freedom. We derive, under fairly general assumptions, the probability distribution function for the efficiency. We find that the macroscopic efficiency is always equal to the most likely efficiency, and it lies in an interval whose boundaries are universal as they only depend on the input and output thermodynamic forces, and not on the details of the machine. The machine achieves the upper boundary of such an interval only in the limit of tight coupling. Furthermore, we find that the tight coupling limit is a necessary, yet not sufficient, condition for the engine to perform close to the reversible efficiency. The reversible efficiency is the least likely regardless of the coupling strength, in agreement with previous studies. By using a large deviation formalism for the energy currents we derive a fluctuation relation for the efficiency which holds for any number of internal degrees of freedom in the system.

Keywords: stochastic thermodynamics, fluctuation relations, microscopic machines

(Some figures may appear in colour only in the online journal)

1. Introduction

Since the dawn of thermodynamics, the capacity of a machine to convert available sources of energy into useful work has been an ubiquitous subject of investigation. As a matter of fact, the second law of thermodynamics sets a limit on a thermal machine’s performance. In particular, the maximum efficiency of an isothermal machine, as given by the ratio of work performed by the machine to the energy used, is $1$. The lossless limit in which energy conversion into work is performed with efficiency $1$ is nonetheless attained in the reversible quasi-static limit, in which the machine operates infinitely slowly. A machine in this reversible quasi-static
regime delivers zero output power, and so it is useless for practical purposes. Accordingly, many efforts have been devoted to the study of the condition for finite non-zero, possibly maximal, power production. One of the first discussions on this topic is attributed to Moritz von Jacobi already around 1840 [1].

The blossoming of experimental techniques aimed at investigating the fluctuations of thermodynamic quantities in microscopic systems [2] has paved the way to the extension of the laws of thermodynamics to address the stochastic properties of quantities such as work, heat or entropy production [3]. According to this revised version of thermodynamics, the efficiency itself is a fluctuating quantity [4] as it is given by the ratio of two fluctuating quantities: the entropy production rates $\sigma_i$ associated to the output ($i = 2$) and input ($i = 1$) currents along a single stochastic trajectory

$$\eta = -\frac{\sigma_2}{\sigma_1}. \quad (1)$$

As such, the trajectory dependent efficiency of a microscopic machine performing at the energy scale of the thermal fluctuations ($k_B T$) [4] can indeed surpass the reversible limit (or the Carnot limit for thermal motors). Furthermore, collective effects such as synchronization in arrays of $N$ interacting microscopic motors can decrease the energy dissipation [5] and possibly increase the thermodynamic efficiency with respect to the single motor case [6–11], and even beat the Carnot limit at finite entropy production rate [12]. The study of the statistical properties of the stochastic efficiency is thus of crucial importance in order to characterize the performance of microscopic machines operating in out-of-equilibrium conditions.

In this paper we derive via stochastic thermodynamics the statistics of the efficiency for a class of cyclic isothermal energy transducers [13, 14], whose internal degrees of freedom are mechanically coupled with realistic physical interactions described by a many-body potential. Starting from the simplest case of a machine consisting of two degrees of freedom and concluding with the $N$-particle system, we are able to derive the full probability density function (PDF) of the efficiency under fairly general assumptions. The efficiency PDF is known to exhibit power law long tails [15, 16], and as such finite moments of any order cannot be calculated. However, our approach allows us to identify the macroscopic efficiency with the most likely value, i.e. the maximum of the efficiency PDF. Thanks to the mechanistic description used here we can also derive the exact expression of the machine response as a function of the force intensity: this in turn allows us to accurately study the weak and tight coupling limits, and the large input/output force regime, so as our investigation is not limited to the linear regime. Furthermore, in the linear regime the response matrix coefficients turn out to depend explicitly on the interaction strength. With respect to other works (e.g. [4, 16]), this approach has thus the advantage that in the linear regime the transition from the weak to the tight coupling regime is controlled directly and in a transparent way by the interaction potential strength.

As far as the least likely efficiency is concerned, we find that it corresponds to the reversible efficiency, in accordance with the findings of [4]. In that reference the fluctuation theorem for the entropy production [3, 17, 18] was used to prove this result on the least likely efficiency. Here we take one step further, and show that the fluctuation theorem for the energy currents [19–22] implies a fluctuation relation for the efficiency itself: the PDF of $\eta$ turns out to show a symmetry which resembles those obtained previously for, e.g. the work or the heat PDFs [3, 18, 23–26]. While we initially assume that the input and output energy currents are Gaussian distributed as, e.g. in [16, 27], we provide solid evidence that the fluctuation relation for the efficiency holds beyond the linear regime, and for a general interaction potential.
The paper is organized in the following manner. In section 2 we review a few useful results on the Brownian particle in a tilted periodic potential, which will be used in the following discussion in the paper. In section 3 we consider the minimal model for an isothermal cyclic energy transducer, namely a system with two degrees of freedom and a periodic interaction potential. We then derive the efficiency PDF, discuss its extremal points, and introduce the fluctuation relation for $\eta$. In section 4 we generalize our results to the case of a machine with $N$ degrees of freedom. In section 5 we summarize our results.

2. Single oscillator

A Brownian particle in a one-dimensional periodic ring potential $U_0(y)$ and driven by a force $f$ is the minimal model for the study of isothermal systems driven into a non-equilibrium steady state [28–33]. Furthermore, its properties are relevant for the study of a system with many degrees of freedom, interacting through periodic potentials, as we argue in the following sections. We thus review some of its features and include a few novel results as well in this section.

The trajectory $y(t)$ of an overdamped Brownian particle in a periodic potential $U_0(y)$ with period $L$ and subject to a constant drift force $f$ is generated by the Langevin equation

$$\dot{y} = f - U_0'(y) + \zeta(t),\quad (2)$$

where the friction coefficient is set to unity $\Gamma = 1$, and a dot and a prime indicate time and space derivatives, respectively. The quantity $\zeta(t)$ is a stochastic force with a Gaussian distribution and correlations given by the fluctuation-dissipation relation

$$\langle \zeta(t) \zeta(t') \rangle = 2T \delta(t - t'),\quad (3)$$

that accounts for thermal fluctuations due to energy exchange between the system and the surrounding medium at temperature $T$. The Boltzmann constant $k_B$ is set to unity throughout this paper.

Furthermore, in the following the quantity $k$ will express the typical amplitude of the periodic potential corrugations, the simplest example being $U_0(y) = -k \cos y$. We will not assume any specific for $U_0(y)$, unless differently stated.

The equation for the time evolution of the probability distribution function (PDF) of the phase $y$ reads

$$\partial_t P(y, t) = \mathcal{L}_y P(y, t),\quad (4)$$

where $\mathcal{L}$ is the Fokker–Planck (FP) differential operator

$$\mathcal{L}_y = -\partial_y (f - U_0'(y) - T \partial_y).\quad (5)$$

The PDF in the steady state is thus [9, 34, 35]

$$P(y) = \frac{\mathcal{N}}{T} e^{-U_0(y) + fy}/T \left[ I(L)/\left[ 1 - \exp (-Lf/T) \right] - I(y) \right],\quad (6)$$

where $I(y) = \int_0^y dy' \exp \left[ -(U_0(y') + fy')/T \right]$, and $\mathcal{N}$ is a normalization constant that depends implicitly on $T$, $k$ and $f$, and which is fixed by the normalization condition

$$\int_0^L P(y) dy = 1.\quad (7)$$
In order to obtain the solution for $P(y)$, equation (6), one requires that the distribution is periodic with period $L$ [34]. The steady-state velocity of the dynamical variable $y$ reads [9, 30]

$$\bar{v}_y(k, f) = L N,$$

(8)

in which the dependency on the temperature is implicit. This is an exact result that holds for any potential strength $k$. The velocity $\bar{v}_y$ depends in particular on the form of the potential $U_0(y)$. However, the asymptotic behaviors can be predicted by using some physical arguments: (a) in the limit of large corrugation amplitude ($k \gg T, fL$) the particle is effectively trapped in a potential well, and so $\bar{v}_y \to 0$; (b) in the opposite limit ($k \ll T, fL$) the potential is flattened by the tilting force, hence $\bar{v}_y \to f$. We can thus express the steady-state velocity in terms of a function $c(k, f)$,

$$\bar{v}_y(k, f) = f[1 - c(k, f)],$$

(9)

such that $0 \leq c(k, f) \leq 1$, $c(0, f) = 0$ and $c(\infty, f) = 1$. Finally, we notice that the integrals contained in the expression for the normalization constant equations (6) and (7) typically do not have an analytic solution, though the steady-state velocity can be expanded in power series of $k$ [9].

2.1. Stochastic work

The total work done on the particle along individual trajectory is defined by the functional [36, 37]

$$w_y[y(\tau)] = \int_0^t f\dot{y}(\tau) \, d\tau = f \cdot (Y_t - Y_0).$$

(10)

Here we have introduced a second coordinate $Y$ to account for the total traveled distance: such coordinate is unbounded ($-\infty < Y < \infty$) in contrast to the bounded periodic coordinate $y$. The stochastic processes for $y$ and $Y$ (and hence $w_y$) are characterized by the same Langevin equation equation (2), the only difference being that the former coordinate is periodic while the latter is unbounded. In particular both coordinates have the same velocity in the steady-state $\langle \dot{Y} \rangle = \langle \dot{y} \rangle$.

The coordinate $Y$ represents a time integrated current for the Brownian particle, and the study of its fluctuations is propaedeutic to the subsequent study of the efficiency fluctuations. In particular, we notice that the time evolution of its PDF is governed by the analogous evolution operator to that for the variable $y$ equation (5):

$$\partial_t P(Y, t) = \hat{L}_y P(Y, t).$$

However, the knowledge of the steady state PDF for $y$ (equation (6)) does not allow one to evaluate the time dependent PDF $P(Y, t)$, not even in the long time regime.

2.2. Fluctuations of $Y$

In view of studying the fluctuations of the variable $Y$, it is convenient to introduce the evolution operator $\hat{L}$ for the joint probability $P(y, Y, t)$ that reads [20, 21, 38–40]

$$\hat{L} = -\partial_y(f - U'_0(y)) - \partial_y(f - U'_0(y)) + T(\partial^2_y + \partial^2_Y + 2\partial_y \partial_Y).$$

(11)

Because of the specific symmetry exhibited by the Fokker–Planck operator (11) [20, 21], the PDF $P(Y, t)$ exhibits a fluctuation relation

$$P(Y, t) = P(-Y, t)e^{\Delta f/T}.$$

(12)
As a consequence, the scaling cumulant generating function defined as \[ \mu_0(\lambda) \equiv \lim_{t \to \infty} \frac{\ln(e^{\lambda Y_t})}{t} \] (13) that corresponds to the largest eigenvalues of the operator (11), exhibits the following symmetry \[ \mu_0(\lambda) = \mu_0(-\lambda - f/T). \] (14)

We next introduce the generating function
\[ \Psi(y, \lambda, t) = \int_{-\infty}^{+\infty} dY \exp(\lambda Y)P(y, Y, t), \] (15)
whose time evolution \[ \partial_t \Psi = \mathbf{L}_\lambda \Psi \] is governed by the differential operator
\[ \mathbf{L}_\lambda = -\partial_y (f - U'_0(y) - T\partial_y) + (f - U'_0(y))\lambda + T\lambda^2 - 2T\lambda \partial_y, \] (16)
which is a simplified version of the operator (11) as discussed in \[42\]. Considering the separation ansatz for \[\Psi(y, \lambda, t)\] as given by equation (16), we express the operator \(\mathbf{L}_\lambda\) in matrix form, hence we need a complete and orthonormal basis, \(\langle j|f \rangle = \delta_{j}^{f}\). Because of the periodic nature of the system, a suitable choice for the basis is \[40\]
\[ \langle j|y \rangle = \frac{e^{-i2\pi y/L}}{\sqrt{L}}, \quad \langle y|j \rangle = \frac{e^{i2\pi y/L}}{\sqrt{L}}. \] (19)

Expanding the eigenfunctions \(\varphi_n(y, \lambda)\) into the chosen basis
\[ \varphi_n(y, \lambda) = \langle y|\varphi_n(\lambda)\rangle = \langle y|\sum_{l} c_l^{(n)}(\lambda)|l\rangle = \sum_{l} c_l^{(n)}(\lambda)\langle y|l\rangle, \] (20)
equation (18) for the eigenvalue \(n\) reads
\[ \sum_{j} L_{lj} c_j^{(n)}(\lambda) = \mu_n(\lambda) c_l^{(n)}(\lambda), \] (21)

with \(L_{lj} \equiv \langle l|\mathbf{L}_\lambda|j \rangle = \int_{0}^{L} dy \langle l|y\rangle \mathbf{L}_\lambda \langle y|j \rangle\).

Considering a cosine potential \(U_0(y) = -k \cos y\) the matrix turns out to be tridiagonal with elements
\[ L_{jj} = -T (j + 1)^2 - if(j + 1\lambda), \quad \text{if } j = l; \] (22)
\[ L_{lj \pm 1} = \mp \frac{k}{2} (j + 1\lambda), \quad \text{if } j - l = \pm 1; \] (23)
\[ L_{jl} = 0, \quad \text{if } j \neq l \text{ and } j - l \neq \pm 1. \] (24)

The practicality of dealing with the joint PDF of \(y\) and \(Y P(y, Y, t)\) lies in the fact that the eigenfunctions of the operator \(\mathbf{L}_\lambda\), as given by equation (16), can be expanded in the periodic
basis equations (19) and (20), and thus one can obtain a matrix representation of \( \hat{L}_\lambda \), equations (22)–(24). Had one considered the time evolution of \( P(Y, t) \) alone, the corresponding matrix representation would not be available, given that the eigenfunctions (and thus the eigenvalues) of the corresponding differential operator are unknown, even for the cosine potential.

2.3. Perturbative approach

In its matrix form, the operator \( \hat{L}_\lambda \) equation (16) is an infinite matrix, whose size can be truncated to some finite value in order to solve the linear system equation (21). We write \( \hat{L}_\lambda \) as the sum of a diagonal matrix and another one including the upper and the lower diagonals, \( \hat{L}_\lambda = \hat{L}_\lambda^{(0)} + k \hat{L}_\lambda^{(1)} \). The latter turns out to be proportional to the potential strength \( k \), and therefore a perturbation theory can be used to obtain the eigenvalues as series expansions in terms of \( k \): \( \mu_n(\lambda) = \mu_n(\lambda) + k \mu_n^{(1)}(\lambda) + k^2 \mu_n^{(2)}(\lambda) + k^3 \mu_n^{(3)}(\lambda) + k^4 \mu_n^{(4)}(\lambda) + O(k^5) \).

However, the perturbation theory employed in quantum mechanics to solve, e.g. the Schrödinger equation, cannot be used here since neither the operator \( \hat{L}_\lambda \) of the Fokker–Planck equation nor the unperturbed operator \( \hat{L}_\lambda^{(0)} \) are Hermitian. Therefore, there is no set of functions to form a complete orthonormal basis for \( \hat{L}_\lambda^{(0)} \), and so the corrections to the eigenvectors cannot be calculated.

One possible way to proceed in order to avoid this limitation is to recast the equation for the characteristic polynomial into the following form

\[
\det [\hat{L}_\lambda - \mu(\lambda) \mathbb{I}] = 0 \Rightarrow \det [M] \det \left[ \left( \mathbb{I} + k M^{-1} \hat{L}_\lambda^{(1)} \right) \right] = 0, \tag{25}
\]

where \( \hat{L}_\lambda^{(0)} \) is a diagonal matrix with entries \( L_{jj} \), \( M = \hat{L}_\lambda^{(0)} - \mu(\lambda) \mathbb{I} \), and where we have employed the property that the determinant for the product of matrices is the product of their determinants. The matrix \( M \) is diagonal and thus its determinant reduces to the product of its entries: it depends on \( \mu(\lambda) \) and can thus be easily expanded in powers of \( k \). The expansion of the second determinant in equation (25) requires additional analysis that is included in appendix A.

2.4. Largest eigenvalue

According to equation (17), the long-time limit behavior of the generating function \( \Psi(y, \lambda, t) \) is dominated by the largest eigenvalue of the operator \( \hat{L}_\lambda \), which corresponds to the cumulant generating function \( \mu_0(\lambda) \) introduced in equation (13), and so we can write the generating function for \( Y \) as

\[
\psi(\lambda, t) = \int_0^L dy \Psi(y, \lambda, t) \sim \exp[\mu_0(\lambda) t]. \tag{26}
\]

Thus, starting from the zeroth order, we apply the perturbative approach described above for the largest eigenvalue (labeled by \( n = 0 \)) to obtain the expansion for the largest eigenvalue up to fourth order,

\[
\mu_0(\lambda) \approx \lambda (f + T\lambda) \left[ 1 - \frac{k^2}{2 T^2} + \frac{1}{(f + 2T\lambda)^2} \left( \frac{k^4 - 3f^2T\lambda + f^2T^2(4 + \lambda^2) + fT^3(9 + 8\lambda^2) + T^4(5 + 9\lambda^2 + 4\lambda^4)}{(f^2 + (f + 2T\lambda)^2)^4(4f^2 + (f + 2T\lambda)^2)} \right) \right] + O[k^6]. \tag{27}
\]
We recall that this last result is specific for the cosine potential \( U_0(y) = -k \cos y \).

It is worth noting that by substituting the leading contribution of the generating function equations (26) into (15), deriving with respect to \( \lambda \), and evaluating the result at \( \lambda = 0 \), we obtain the following identity
\[
\partial_\lambda \mu_0(\lambda)|_{\lambda=0} = \langle Y \rangle/T = \bar{v}_y.
\] (28)

The last equation, together with equation (27), provides thus an expansion of the steady-state velocity \( \bar{v}_y \) in powers of \( k \). An identical result for \( \bar{v}_y \) is obtained by considering equation (8) and expanding the normalization constant, as given by equations (6) and (7), in powers of \( k \).

Further, the diffusion coefficient can be computed from \( \mu_0(\lambda) \),
\[
D = \lim_{t \to \infty} \frac{\langle Y^2 \rangle - \langle Y \rangle^2}{2t} = \frac{1}{2} \partial_\lambda^2 \mu_0(\lambda)|_{\lambda=0}.
\] (29)

For small \( k \), figure 1, there is a good agreement between the diffusion coefficient as given by equations (27) and (29), and the analytic result calculated for the overdamped Brownian motion in a tilted periodic potential as obtained in [43].

3. Two coupled oscillators

A minimal model for a machine with an input and an output energy current consists of two overdamped Brownian particles with coordinates \( x_1 \) and \( x_2 \) coupled through a periodic potential \( U_0(x_1 - x_2) \) of strength \( k \) [5, 30, 44]. Each particle is subject to a constant tilting force of opposite sign, so that one injects energy into the system \( f_1 > 0 \), whereas the other extracts energy, \( f_2 < 0 \). The two particles will be termed the producer and the user, respectively. In the limit of weak coupling (\( k \) small) the two particles tend to move independently, each one at its own ‘natural frequency’ \( f \), while strengthening \( k \) will increasingly synchronize their motion.

The dynamic equations for the two coupled oscillators read,
\[
\dot{x}_1 = f_1 - \partial_{x_1} U_0(x_1 - x_2) + \zeta_1(t),
\] (30)
\[
\dot{x}_2 = f_2 - \partial_{x_2} U_0(x_2 - x_1) + \zeta_2(t).
\] (31)

We assume uncorrelated Gaussian white noises, \( \langle \zeta_i(t) \zeta_j(t') \rangle = 2T \delta_{ij} \delta(t-t') \), \( i,j = 1,2 \). The efficiency equation (1) for a single trajectory of this isothermal engine is the rate between the work equation (10) extracted by the user along an individual trajectory and the work injected by the producer along the same trajectory
\[
\eta = -\frac{w_2}{w_1} = \frac{-f_2 X_2}{f_1 X_1}.
\] (32)

We employ the same notation as in section 2.1 to distinguish the unbounded coordinates, \( X_i \), from the bounded periodic ones \( x_i \). The unbounded coordinates \( X_i \) obey the same set of Langevin equations as the periodic ones (30) and (31). Since the efficiency depends on the unbounded coordinates \( X_i \) (equation (32)), in the following we will focus on their dynamics, rather than on the periodic coordinates’ dynamics.

The PDF of the efficiency is then [45],
\[
P(\eta, t) = \int_{-\infty}^{+\infty} dX_1 \int_{-\infty}^{+\infty} dX_2 P(X_1, X_2, t) \delta \left( \eta + \frac{f_2 X_2}{f_1 X_1} \right). 
\] (33)
We need thus to evaluate the joint PDF $P(X_1, X_2, t)$ for the unbounded coordinates, that amounts to find the steady state solution to the FP equation corresponding to the Langevin equations (30) and (31). To simplify the question, we introduce a coordinate transformation to recast the problem into the center of mass (CM) and the relative coordinate motion $X \equiv \frac{X_1 + X_2}{2}$, $Y \equiv \frac{X_1 - X_2}{2}$, (34)

so that the dynamic equations (30) and (31) decouple into

$\dot{X} = f_x + \zeta_x(t),$ (35)

$\dot{Y} = f_y - U_0'(2Y)/2 + \zeta_y(t), \quad \text{(36)}$

where $f_x \equiv (f_1 + f_2)/2$, $f_y \equiv (f_1 - f_2)/2$, and $\langle \zeta_\alpha(t) \zeta_\beta(t') \rangle = T \delta_{\alpha\beta} \delta(t - t').$

The Langevin equation for the CM equation (35) describes unidimensional overdamped Brownian motion subject to a constant force $f_x$. Its PDF is hence a Gaussian centered at $f_x t$ (equation (5.20) [35])

$p(X,t) = \frac{1}{\sqrt{2\pi T t}} \exp \left( -\frac{(X - f_x t)^2}{2Tt} \right), \quad \text{(37)}$

where for consistency with our previous notation we have introduced the coordinate $X$ to remark that it is an unbounded degree of freedom. On the other hand, the Langevin equation for the relative coordinate equation (36) is analogous to equation (2) introduced in section 1 to discuss the single oscillator. Thus we will exploit the results contained in that section to evaluate the PDF for the unbounded relative coordinate $P(Y, t)$. Unless otherwise
indicated, we will assume the long-time limit, so that \( y \) and \( Y \) are uncorrelated. In this regime we can use the separation ansatz equation (17) and integrate over the bounded coordinate \( y \), so as to obtain the long time behaviour of \( P(Y, t) \). In the following we will discuss different scenarios and approximations to obtain \( P(Y, t) \).

However, before proceeding to analyze the PDF \( P(Y, t) \), we consider the PDF of the efficiency equation (33), and discuss a few simplifications. According to the transformation equation (34), we have that \( P(X_1, X_2, t) \propto P(Y, t)P(Y, t) \) up to a constant given by the Jacobian of the coordinate transformation, hence the PDF of the efficiency equation (33) reads

\[
P(\eta, t) = \frac{1}{2} \int_{-\infty}^{+\infty} dX \int_{-\infty}^{+\infty} dY P(X, t) P(Y, t) \delta \left( \eta + \frac{f_2(X - Y)}{f_1(X + Y)} \right).
\]

Let us introduce the rescaled trajectory dependent efficiency \( \tilde{\eta} = -f_1 \eta / f_2 \), and the new variable

\[
\xi = \frac{X}{Y} = \frac{1 + \tilde{\eta}}{1 - \tilde{\eta}}.
\]

Its PDF \( \Phi(\xi, t) \) is such that the following general relation between the two probability distributions holds,

\[
P(\eta, t) = \left| \frac{d\xi}{d\eta} \right| \Phi(\xi, t) = \left| \frac{f_1}{f_2} \right| \frac{2}{(1 - \tilde{\eta})^2} \Phi(\xi, t).
\]

Equation (40) is a central result as it indicates that any symmetry exhibited by \( P(\eta, t) \) will arise from a corresponding symmetry of \( \Phi(\xi, t) \). The PDF of the new variable \( \xi \) reads thus

\[
\Phi(\xi, t) = \int dY d\delta \left( \xi - \frac{X}{Y} \right) P(X, t) P(Y, t) = \int dY \frac{|Y|}{\sqrt{2\pi T}} e^{-\frac{|Y|^2}{2T}} P(Y, t),
\]

where the rhs of the equation is obtained by substituting the PDF for \( X \) equation (37), and by introducing the rescaled variable \( J_Y \equiv Y / t \), corresponding to the current associated to \( Y \).

### 3.1. Efficiency distribution with Gaussian approximation for the variable \( Y \)

The first case we consider is when \( P(Y, t) \) is a Gaussian distribution. We notice that there exists an ample literature on the statistic of the ratio of two normal random variables [46]. The assumption that the fluxes are normally distributed is commonly made when studying the thermodynamic properties of microscopic devices in the linear regime [4, 16, 27, 47]. As argued below, the Gaussian approximation is accurate in the limit of small \( f_s \). Given the definition of \( f_s = (f_1 - f_2)/2 \), this translates into the requirement that the system is close to equilibrium, since the two forces \( f_1 \) and \( f_2 \) need to have opposite sign in order to constitute a duo of input/output power sources.

Taking \( P(Y, t) \) to be Gaussian corresponds to truncate the cumulant generating function \( \mu_0(\lambda) \) to second order in \( \lambda \). While the first order coefficient is fixed by the average value of the velocity, equation (28), the second order coefficient is dictated by the symmetry imposed by the fluctuation relation \( \mu_0(\lambda) = \mu_0(-\lambda - f_s/T_s) \) equation (14), where \( T_s = T/2 \) is the temperature appearing in the fluctuation-dissipation relation for the noise on the rhs of equation (36). The cumulant generating function of \( Y \) for the Gaussian approximation thus reads

\[
\mu_0(\lambda) = \nu_s(k, f_s) \lambda (1 + \lambda T_s / f_s),
\]
where \( \bar{v}_y(k_f) = \langle \bar{Y} \rangle \) is given by equation (8). The expression (42) for \( \mu_0(\lambda) \) sets a constraint on the diffusion coefficient in the Gaussian distribution \( P(Y, t) \) that reads

\[
P(Y, t) = \exp \left[ -\frac{f_y(Y - \bar{v}_y t)^2}{2T\bar{v}_y} \right] \frac{1}{\sqrt{2\pi T\bar{v}_y/|f_y|}},
\]

(43)

where we have dropped the dependency of \( \bar{v}_y \) on \( k \) and \( f_y \) to simplify the notation. The same result is obtained if one imposes directly the fluctuation relation equation (12) on a Gaussian distribution with average \( \bar{v}_y t \). In order to check the accuracy of the Gaussian approximation equation (43), we compare in figure 2 the diffusion coefficient as given by our approximation for the PDF equation (43),

\[
D = \frac{\sigma_y^2}{2T} = \frac{T\bar{v}_y(k_f)}{2f_y},
\]

(44)

with the exact expression for the effective diffusion coefficient obtained in [43] for the overdamped Brownian motion in a tilted periodic potential and with its approximation [48]. The diffusion coefficient as a function of the tilt force as given by equation (44) exhibits the qualitative behavior predicted in [43]. However our result equation (44) agrees quantitatively with the exact result of [43] only in the limit of small force \( f_y \), as highlighted by the insets in figure 2. The motion along the coordinate \( Y \) when the force \( f_y \) is weak can be described as a random diffusion along the periodic potential, slightly biased by the force \( f_y \). In this regime the Gaussian approximation entailed in equation (44) captures the essential features of the dynamics. However, equation (44) fails to display the giant acceleration of free diffusion observed in [43] near the threshold tilt, beyond which the periodic potential is overturned by the external constant force \( f_y \), and the solution of the Langevin equation (36) is dominated by the force itself. On the other hand, it should be noted that the degree of agreement is improved in the weak coupling regime, notice the different scales employed in the vertical axes of the inset plots in figure 2.

We now proceed to calculate the PDF \( P(\eta, t) \). Plugging the expression for \( P(Y, t) \) equations (43) into (41), integrating over \( Y \), and inverting the change of variables equation (39), we obtain

\[
P(\eta, t) = e^{-\frac{f_y^2\eta^2}{2f_y^2}} f_y \pi \sqrt{f_y^2(\eta - 1)^2|f_y - f_y|} \left[ 1 + \sqrt{\pi f_y(\eta)} e^{\frac{\eta f_y^2}{2f_y^2}} \text{erf} \left( \sqrt{\pi f_y(\eta)} \right) \right].
\]

(45)

where \( h(\eta) = (f_y^2 - f_y^2)(\eta - 1)\sqrt{\pi f_y(\eta - 1)^2|f_y - f_y|} \), see appendix B for the details. Finally, in the long time limit the leading terms of the efficiency’s PDF are

\[
P(\eta, t) = e^{-\frac{f_y^2\eta^2}{2f_y^2}} f_y \pi \sqrt{f_y^2(\eta - 1)^2|f_y - f_y|} \left[ 1 - \frac{h(\eta)}{|h(\eta)|} + \sqrt{\pi f_y(\eta)} e^{\frac{\eta f_y^2}{2f_y^2}} \right].
\]

(46)

and they exhibit a good agreement with the exact expression equation (45), see figure 3. We employ a logarithmic scale on vertical axes in order to show the minimum of the PDF \( P(\eta, t) \) to be further discussed, and we include a zoom in linear scale around the maximum so as to better display the agreement between the exact expression equation (45) and the long time approximation for \( P(\eta, t) \) equation (46). The long time approximation equation (46) exhibits a discontinuity at \( \eta = 1 \) that fades when \( \sqrt{\pi f_y} \) is large. As such it can be observed in the tight coupling regime, i.e. when \( \bar{v}_y \) is small, even at a large time, see figure 3 rightmost panel.
The efficiency distribution in figure 3 exhibits a maximum and a minimum, which correspond to the large deviation function’s extremal points, as detailed below. The super-Carnot local maximum found in [16] belongs to a subdominant decay mode, and thus it does not appear in the plot ranges of figure 3 as it is displaced towards infinity in the long time limit. For the set of parameters chosen in figure 3 the local maxima occur at very large value of $\eta$, and are not displayed here. From equation (46) one obtains the large deviation function of the PDF of the efficiency

$$J(\eta) \equiv \lim_{t \to \infty} \frac{1}{t} \ln P(\eta, t),$$

that has two extremal points

$$\eta_+ = \frac{f_2 (f_1 - f_2) \cdot c (k f_1 - f_2) + 2 f_2}{f_1 (f_1 - f_2) \cdot c (k f_1 - f_2) - 2 f_1}; \quad \eta_- = 1,$$

which are a maximum and a minimum, respectively; where we have written the velocity $\bar{v}_y$ in the form of equation (9) in section 2. It is worth noting that the maximum and the minimum equation (48) of the large deviation function $J(\eta)$ do match the extremal points of the efficiency distribution as shown in figure 3.

The minimum $\eta_- = 1$, corresponding to the reversible efficiency, is in accordance with the findings of [4] on the least likely efficiency in stochastic machines, and is a direct consequence of the fluctuation relation equation (12). Furthermore, we find that the most likely value of the efficiency $\eta_+$ is always equal to the macroscopic efficiency $\bar{\eta}$

$$\eta_+ = \bar{\eta} = \frac{f_2 \langle X_2 \rangle}{f_1 \langle X_1 \rangle},$$

which, differently from the minimum, depends on the coupling strength $k$ and on the forces. We will now consider the limiting cases of weak and tight coupling and of large forces.
In this analysis we will avail ourselves of the results on the single oscillator velocity discussed in section 2. For $k = 0$, we have $\bar{v}_y(0, f_y) = f_y$ and thus

$$\lim_{k \to 0} \eta_+ = -\left(\frac{f_2}{f_1}\right)^2.$$  (50)

The same result is obtained in the limit $f_y \to \infty$; indeed, in section 2 we have found

$$\lim_{f_y \to \infty} \bar{v}_y(k, f_y) = f_y, \forall k < \infty,$$  (51)

and recalling that $f_y = (f_1 - f_2)/2$, one finds

$$\lim_{f_y \to \infty} \eta_+ = \lim_{f_1 \to +\infty} \eta_+ = \lim_{f_2 \to -\infty} \eta_+ = -\left(\frac{f_2}{f_1}\right)^2.$$  (52)

Thus we find that a large applied force renormalizes the interaction potential, leading to a non-interacting system with negative macroscopic efficiency. From equation (49) one finds negative macroscopic efficiency $\bar{\eta}$ both in the limit of weak coupling and of large forces. In these regimes the two oscillators are effectively decoupled, meaning that the energy transferred between them can be neglected and hence their motion is only due to the external forces applied on each of them, with the velocity of particle 2 being negative ($f_2 < 0$). Thus the system can be no longer considered a motor (work-to-work transducer), and therefore a negative efficiency has no major physical implications.

In the limit of tight coupling $k \to \infty$ the variable $Y$ becomes confined, and one has

$$\lim_{k \to \infty} \bar{v}_y(k, f_y) = 0, \forall f_y < \infty,$$  (53)

thus

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure3.pdf}
\caption{PDF of the efficiency for weak coupling ($k = 0.25$, left panel), moderate coupling ($k = 1$, middle panel), and tight coupling ($k = 4$, right panel). Red symbols: exact expression for $P(\eta, t)$ equation (45). Blue line: long time approximation for $P(\eta, t)$ equation (46). Vertical black lines: extremal points of $P(\eta, t)$ equation (48). Insets: zoom of the corresponding main plot around the maximum. Parameters choice: $T = 1, f_x = 0.05, f_y = 0.1, t = 10^5$. The average velocity $\bar{v}_y$ appearing in equations (45) and (46) is calculated through the exact expression (8), for the interaction potential $U_0(y) = -2k \cos y$.}
\end{figure}
\[
\lim_{k \to \infty} \eta^+ = -\frac{f_2}{f_1}.
\]  

(54)

We now argue that the values given in equations (50) and (54) are respectively the lower and the upper bounds for the most likely and thus for the macroscopic efficiency equation (49). Indeed, one finds that

\[
\partial_k \eta^+ = \frac{2f_2 f_\xi}{f_1 (f_\xi + \tilde{\nu}_r (k, f_\xi))} \partial_k \tilde{\nu}_r (k, f_\xi) > 0,
\]

(55)

where we have used the fact that \( \partial_k \tilde{\nu}_r (k, f_\xi) < 0 \) for \( f_\xi > 0 \), and we have assumed that \( f_1 > 0, f_2 < 0, \) with \( |f_1| > |f_2| \). Furthermore, \( -(f_2/f_1)^2 < -f_2/f_1 \), and so \( \eta^+ \) is restricted in this interval of values. Thus we conclude that (i) the optimal maximal/macroscopic efficiency is always obtained in the limit of tight coupling, where the relative coordinate \( Y \) and its fluctuations are suppressed [9], and (ii) the only way that the maximal/macroscopic efficiency can reach the reversible value 1 is in the limit \( f_2 \to -f_1 \) for which the total entropy production in the environment vanishes. The latter result is relevant in connection with the argument raised in [47], where it was argued that a machine at diverging power output can achieve the reversible efficiency limit. On the one hand, our results show clearly that, for finite coupling strength \( k \), taking diverging \( f_1 \) gives the lower bound equation (52). On the other hand, by taking first the tight coupling limit, \( k \to \infty \), and then \( f_1 \) and \( f_2 \) large but with \( f_1 L \ll k \) (where \( L \) is the period of the periodic potential \( U_0(x_1 - x_2) \)), the machine can achieve a large power output \( (f_2 \tilde{\chi}_2) \), but at the expenses of a large power input \( (f_1 \tilde{\chi}_1) \), given that \( \tilde{\nu}_r (k \to \infty, f_1) \to 0 \) (equation (53)) and thus \( \langle \tilde{\chi}_2 \rangle \to \langle \tilde{\chi}_1 \rangle \). Therefore, the macroscopic efficiency 1 can only be achieved close to the stall condition \( f_2 \to -f_1 \), making the machine a dud.

The thermodynamic uncertainty relation [49, 50] sets an upper bound for the thermodynamic efficiency, which reads [51],

\[
\bar{\eta} \leq \frac{1}{1 + 2 \langle \tilde{\nu}_2 \rangle \bar{T}/\Delta_2},
\]

(56)

where \( \langle \tilde{\nu}_2 \rangle \) is the average output power and \( \Delta_2 \) its fluctuations,

\[
\langle \tilde{\nu}_2 \rangle = \frac{-f_2 \langle X_2 \rangle}{t}, \quad \Delta_2 = \lim_{t \to \infty} \langle (\tilde{\nu}_2 (t) - \langle \tilde{\nu}_2 \rangle)^2 \rangle t.
\]

(57)

By virtue of the coordinate transformation equation (34) and given the distribution of \( X \) equation (37), the mean power and its fluctuations read,

\[
\langle \tilde{\nu}_2 \rangle = -f_2 \left( \frac{f_1 + f_2}{2} - \tilde{\nu}_r \right), \quad \Delta_2 = \lim_{t \to \infty} \frac{f_2^2}{t} (T + \Delta_Y).
\]

(58)

The steady-state mean velocity \( \tilde{\nu}_r \) has already been discussed in section 2 (equation (9)), whereas the fluctuations of \( Y \) decrease from \( \Delta_Y = t \bar{T} \) in the weak coupling limit \( (k \to 0) \) to \( \Delta_Y = 0 \) in the tight coupling limit \( (k \to \infty) \), when the two oscillators are fully coupled and so the relative coordinate vanishes. We therefore find that, for decreasing \( k \),

\[
0 \leq \Delta_Y \leq t \bar{T}.
\]

(59)

According to this argument, the upper bound for the macroscopic efficiency \( (f_1 \neq -f_2) \), as given by equation (56), takes the values

\[
\bar{\eta} \leq \infty, \quad \text{for} \quad k \to 0.
\]

(60)
\[ \eta \leq \frac{-f_2}{f_1}, \quad \text{for} \quad k \to \infty. \quad (61) \]

Comparing these results with the asymptotic behaviors of the most likely efficiency equations (50) and (54), we notice that the upper bound equation (56) turns out to overestimate by far the macroscopic efficiency in the weak coupling, whereas for in the tight coupling we prove that the upper bound corresponds to the actual value for the macroscopic efficiency.

### 3.2. Saddle-point approach

We now calculate the probability distribution \( \Phi(\xi, t) \), equation (41), and the most and the least probable value of the efficiency without making any assumption on the relative coordinate distribution. Recalling that \( \mu_0(\lambda) \) introduced in equation (13) is the cumulant generating function of \( Y \), we have that in the long time limit

\[ P(Y, t) \sim \int d\lambda e^{[\mu_0(\lambda) - \lambda J_Y]} \sim e^{[\mu_0(\lambda^*) - \lambda^* J_Y]}, \quad (62) \]

where \( \lambda^* \) is implicitly defined by the saddle-point condition

\[ \partial_\lambda \mu_0(\lambda)|_{\lambda^*} = J_Y. \quad (63) \]

The integral in equation (41) is dominated by the saddle point \( J_Y^{**} \) defined implicitly by

\[ \lambda^*(J_Y^{**}) = -\frac{\xi}{T}(J_Y^{**} \xi - f_x). \quad (64) \]

where equation (63) is exploited to simplify the last expression. Thus one obtains

\[ \Phi(\xi, t) \sim e^{G(\xi)} = e^{-\frac{1}{2}(J_Y^{**} \xi - f_x)^2/(2T - (\mu_0(\lambda^{**}) - \lambda^{**} J_Y^{**}))}, \quad (65) \]

with \( \lambda^{**} = \lambda^*(J_Y^{**}) \). Let us now find the stationary points of \( G(\xi(\eta)) \). We first notice that

\[ \frac{\partial G}{\partial \eta} = \frac{\partial G}{\partial \xi} \frac{\partial \xi}{\partial \eta} = -\frac{f_1}{f_2} \frac{1}{T} \frac{1}{(1 + \xi)^2} \frac{\partial G}{\partial \xi}. \quad (66) \]

Exploiting equations (63) and (64), a straightforward calculation leads to the expression for the stationary points

\[ \frac{\partial G}{\partial \xi} = \frac{\lambda^{**} J_Y^{**}}{\xi} = 0. \quad (67) \]

Thus, we are left with the two equations,

\[ \lambda^{**} = 0; \quad J_Y^{**} = 0. \quad (68) \]

The first equation together with equation (64) gives

\[ 0 = \xi (J_Y^{**} \xi - f_x). \quad (69) \]

The solution \( \xi = 0 \) must be discarded, because \( \xi \) appears in the denominator of equation (67). so we have

\[ \xi_+ = f_x / \bar{v}_y \Rightarrow \eta_+ = \frac{f_2 (\bar{v}_y - f_x)}{f_1 (f_x + \bar{v}_y)}, \quad (70) \]

where we have used equation (63) and the fact that when \( \lambda^{**} = 0 \),
\[ J_Y^{**} = \partial_\lambda \mu_0(\lambda)|_{\lambda^{**} = 0} = \bar{v}_y. \]  

(71)

The other solution of equation (67), \( J_Y^{**} = 0 \), implies
\[ \partial_\lambda \mu_0(\lambda)|_{\lambda^{**} = 0} = 0, \]

(72)

so \( \lambda^{**} \) in this case is the minimum of \( \mu_0(\lambda) \), which is a convex function. Since the fluctuation relation for the FP equation with operator given by equation (16) implies \( \mu_0(\lambda) = \mu_0(-\lambda - 2f_y/T) \) [20], and thus \( \mu_0(\lambda) \) is symmetric around \( \lambda = -f_y/T \), we have that the minimum is exactly at this symmetry point. Hence, \( \lambda^{**} = -f_y/T \), and thus exploiting equation (63) the least likely \( \xi \) and \( \eta \) are
\[ \xi = -f_y/f_x \Rightarrow \eta = 1. \]

(73)

Therefore, the solutions for \( \eta_+ \), \( \eta_- \) are the same as those in equation (48), obtained with the Gaussian approximation for the current \( J_Y \).

### 3.3. Linear regime and singular coupling

After the usual coordinate transformation into the CM and the relative coordinate equation (34), the average velocities for the two oscillators read
\[ \bar{v}_1 = \langle \dot{x}_1 \rangle = \langle \dot{X} \rangle + \langle \dot{Y} \rangle = f_x + \bar{v}_y (f_y, k), \]

(74)

\[ \bar{v}_2 = \langle \dot{x}_2 \rangle = \langle \dot{X} \rangle - \langle \dot{Y} \rangle = f_x - \bar{v}_y (f_y, k), \]

(75)

with \( f_{x_0} = (f_1 \pm f_2)/2 \), and where we notice that in order for the machine to extract work from the input source of power, the two forces must be of opposite sign as discussed above.

We consider the linear regime between fluxes (particle velocities) and thermodynamic forces \( f_1 \) and \( f_2 \),
\[ \bar{v}_y = L_{\alpha f_y}, \]

(76)

where the linear response matrix is
\[ L = \frac{1}{2} \begin{bmatrix} 1 + b(k) & 1 - b(k) \\ 1 - b(k) & 1 + b(k) \end{bmatrix}, \]

and
\[ b(k) = \frac{\partial \bar{v}_y}{\partial f_y} \bigg|_{f_y = 0} = I_0^{-2}(k), \]

(77)

is the partial derivative of the steady state velocity equation (8) calculated at \( f_y = 0 \); \( I_0(k) = \frac{1}{2\pi} \int_0^{2\pi} dy \exp \left( \frac{1}{2} k \cos y \right) \) is the zeroth order modified Bessel function of the first kind. In deriving the expression for \( L \), we have assumed that since \( f_1 \) and \( f_2 \) are both small, their difference \( 2f_0 \) is small as well, and the interacting potential is \( U_0(y) = -2k \cos y \).

The function \( b(k) \) is monotonically decreasing from \( b(0) = 1 \) to \( b(k \rightarrow \infty) \rightarrow 0 \). Thus, in the linear regime the macroscopic efficiency \( \eta \) achieves the reversible limit 1 only in the limit of tight coupling \( k \rightarrow \infty \). This result is in agreement with the analysis we discussed in section 3.1 for the general case of arbitrary forces \( f_1 \) and \( f_2 \) in the tight coupling limit, as summarized by equation (54). In the same section we derived the range of values for the macroscopic efficiency by using a general argument. Obviously, the values of the macroscopic
efficiency are limited in that range in the linear regime too. It is however interesting to investigate whether in the linear regime one can attain the condition called singular coupling in [16], where the reversible efficiency can be achieved when the linear response matrix tends to the inverse of a degenerate matrix. The entries of the inverse matrix of \( L \) are

\[
L^{-1}_{ii} = \frac{1}{b(k)} L_{ii}; \tag{78}
\]

\[
L^{-1}_{ij} = -\frac{1}{b(k)} L_{ij}, \quad i \neq j. \tag{79}
\]

Such a matrix becomes degenerate in the limit \( b(k) \to \infty \), which is not a physically meaningful limit: the response of a current (in our case the derivative of \( \bar{v}_r \), i.e. the particle current) cannot be infinite for any finite value of the corresponding thermodynamic force (in our case \( f_i \)). Therefore, when one considers a physical model for an engine, with realistic physical interaction between the thermodynamic forces, and thus between the corresponding energy currents, the necessary (but not sufficient) condition for the engine to operate at a macroscopic efficiency near the reversible (Carnot efficiency) is that the coupling between the input and output currents is tight.

3.4. Fluctuation theorem for the efficiency PDF

The PDF of the position of the single particle described by equation (2) exhibits the long time fluctuation relation as given by equation (43). We now explore whether the PDF of the efficiency near the reversible (Carnot efficiency) is that the coupling between the input and output currents is tight.

\[
\Phi(\xi, t) = e^{-R(\xi)} \Phi(g(\xi), t) = e^{-R(\xi)} \int dY |Y|/(\sqrt{2\pi T}) \exp \left[ -\frac{t(J_Y g(\xi) - f_i)^2}{2T} \right] P(Y, t), \tag{80}
\]

where we have set again \( J_Y \equiv Y/t \) in (80). In order to discard the trivial solution \( g(\xi) = \xi \), \( R(\xi) = 0 \), we introduce a scaling function \( \alpha(\xi) \) for the current \( J_Y = \alpha(\xi) J_Y \), amounting to a change of integration variable in equation (80). Assuming a Gaussian distribution for \( P(Y, t) \) (equation (43)) and setting

\[
g(\xi) = \frac{2f_i f_x - \xi (f_i \bar{v}_r - f_i^2)}{f_i \bar{v}_r - f_i^2 + 2\xi f_i \bar{v}_r}; \tag{81}
\]

\[
\alpha(\xi) = \left| \frac{-f_i \bar{v}_r + f_i^2 - 2\xi f_i \bar{v}_r}{f_i \bar{v}_r + f_i^2} \right|, \tag{82}
\]

we find that the following fluctuation relation for the stochastic variable \( \xi \) holds

\[
\Phi(\xi, t) = e^{-R(\xi)} \Phi(g(\xi), t), \tag{83}
\]

with \( R(\xi) = \ln \alpha^2(\xi) \).

This symmetry turns out to be analogous to the fluctuation relations of the work or the heat PDFs [3, 18, 23–26]. One finds that for any value of \( f_x, f_i \) and \( \bar{v}_r \), \( \partial g(\xi)/\partial \xi < 0 \), \( \forall \xi \neq f_i^2/(2f_i \bar{v}_r) \) where the function \( g(\xi) \) has a vertical asymptote. The variable transformation function \( g(\xi) \) has also a horizontal asymptote at \( g(\xi \to \pm \infty) = (f_i^2 - f_i \bar{v}_r)/(2f_i \bar{v}_r) \), and therefore it is bijective:
some text...
\[ \dot{x}_i = f_i - \sum_j k_{ij} \partial_x u_0(x_i - x_j) + \zeta_i(t). \]  

(88)

We assume uncorrelated Gaussian white noises, \( \langle \zeta_i(t) \zeta_j(t') \rangle = 2T \delta_{ij} \delta(t - t'), i, j = 1, \ldots, N. \) The case in which \( k_{ij} = k, \forall i, j, \) and \( u_0(x) = -\cos(x) \) was first introduced by Sakaguchi [52] as an extension of the Kuramoto model [53–57]. However, in the following we will not make any assumption on the specific form of the potential \( u_0. \) As in the previous section, section 3, depending on the force sign, each oscillator can be considered either an energy producer \((f_i > 0)\) or an energy user \((f_i < 0)\). The single trajectory efficiency equation (1) of this isothermal engine is then the rate between the work extracted by the users \((u)\) and the work injected by the producers \((p)\) along a single trajectory,

\[ \eta = -\frac{\sum_i^u f_i x_i}{\sum_i^p f_i x_i}, \]  

(89)

where we retain the notation as in the previous sections, and the capital letters indicate the unbounded coordinates. The superscripts appearing in the sum at the numerator and denominator of equation (89) indicate that the sum is restricted to the users or producers, respectively. Accordingly, the PDF of the efficiency reads

\[ P(\eta, t) = \int dX_1 \cdots dX_N \delta \left( \eta + \frac{\sum_i^u f_i X_i}{\sum_i^p f_i X_i} \right) P(X_1, \ldots, X_N, t), \]  

(90)
where the PDF $P(X_1, \ldots, X_N, t)$ depends implicitly on the forces $f_1, \ldots, f_N$. Analogously to the unidimensional case equation (26), we introduce the multidimensional version of the generating function

$$\psi(\lambda, t) = \int dX_1 \cdots dX_N e^{\lambda \cdot X} P(X_1, \ldots, X_N, t),$$

(91)

where underlined symbols represent vectors, and Einstein convention for the summation of repeated indexes is adopted. The generating function is dominated by the largest eigenvalue $\mu_0(\lambda)$ of the multidimensional FP operator corresponding to the single coordinate operator equation (16),

$$\psi(\lambda, t) \sim e^{\mu_0(\lambda) t}.$$  

(92)

The fluctuation relation for the multidimensional PDF $P(X_1, \ldots, X_N, t)$ implies the symmetry for the largest eigenvalue [20],

$$\mu_0(\lambda) = \mu_0(-\lambda_i - f_i/T), \quad \forall \lambda.$$  

(93)

In the following subsections we will explore the statistical properties of the efficiency for two different choices of the constant forces applied to the particles.

4.1. Two terminals

In the first case that we consider the system has two terminals, one where energy is injected and the other where it is extracted. We consider $f_1 > 0$ the input force and $f_N < 0$ the output force, with $f_i > -f_N, \ f_i = 0, \ i \neq 1,N$. Then, the system of Langevin equation (88) reads...
\[ \dot{x}_i = (\delta_{i1} + \delta_{iN}) f_i - \sum_j k_{ij} \partial_x u_0 (x_i - x_j) + \zeta_i (t). \]  

(94)

Thus, the stochastic efficiency equation (89) reduces to

\[ \eta = -\frac{f_N X_N}{f_1 X_1}; \]

(95)

we introduce the rescaled efficiency

\[ \hat{\eta} = -\frac{f_1}{f_N} \frac{\eta}{X_1} = \frac{X_N}{X_1}, \]

(96)

whose PDF reads

\[ P(\hat{\eta}) = \int \int dX_1 dX_N \delta \left( \hat{\eta} - \frac{X_N}{X_1} \right) \tilde{P}(X_1, X_N, t), \]

(97)

and \( P(X_1, X_N, \cdots, X_{N-1}, X_N, t) \) is the solution of the FP equation associated to the Langevin equation (94).

In the long time limit, the dominant term of \( \tilde{P}(X_1, X_N, t) \) can be obtained trough saddle-point integration of the generating function,

\[ P(X, t) \sim \int d\lambda e^{\lambda X} e^{\mu_0 (\lambda)} \]

(98)

Hence

\[ P(X_1, X_N, t) \propto e^{(\bar{\mu}_0 (\lambda^*_1, \lambda^*_N) - J_1 \lambda^*_1 - J_N \lambda^*_N)}, \]

(99)

with the saddle points implicitly defined by

\[ \partial_{\lambda_1} \bar{\mu}_0 (\lambda_1, \lambda_N) \bigg|_{\lambda^*_1, \lambda^*_N} = J_1, \]

(100)

\[ \partial_{\lambda_N} \bar{\mu}_0 (\lambda_1, \lambda_N) \bigg|_{\lambda^*_1, \lambda^*_N} = J_N, \]

(101)

and where \( J_1 \equiv X_1/t, J_N \equiv X_N/t, \) and \( \bar{\mu}_0 (\lambda_1, \lambda_N) \) is the cumulant generating function of \( P(X_1, X_N, t). \)

\[ \bar{\mu}_0 (\lambda_1, \lambda_N) = \mu_0 (\lambda_1, 0, \ldots, 0, \lambda_N). \]

(102)

The details of the calculations are given in appendix C. Plugging equations (99) into (97), and rearranging the terms in the integral, the PDF for the rescaled efficiency reads

\[ P(\hat{\eta}) \sim \int \int dX_1 dX_N \delta (X_N - X_1 \hat{\eta}) |X_1| \ e^{(\bar{\mu}_0 (\lambda^*_1, \lambda^*_N) - J_1 \lambda^*_1 - J_N \lambda^*_N)}. \]

(103)

4.1.1. Extremal points. The extremal points of the efficiency’s PDF correspond to the most likely efficiency,

\[ \eta_+ = -\frac{f_N (J_N)}{f_1 (J_1)} \]

(104)
and the least likely efficiency,

\[ \eta_- = 1, \]

see appendix C.

Thus we conclude that the study of the extremal points for the PDF of the efficiency of this machine made of \( N \) all-to-all interacting oscillators with two terminals of input and output energy leads to the same efficiency features as for the two coupled oscillators machine studied in section 3, namely the most likely efficiency is the macroscopic efficiency, whereas the least likely corresponds to the efficiency of the machine performing reversibly.

4.1.2. Gaussian assumption. In order to obtain an expression for the PDF of the efficiency equation (103) we need to assume a certain distribution for the variables \( X_1 \) and \( X_N \). We assume thus a Gaussian distribution, that according to equation (42) implies the following expression for the cumulant generating function,

\[ \bar{\mu}_0(\lambda_1, \lambda_N) = \bar{\nu}_1 \lambda_1 \left( 1 + \lambda_1 T/f_1 \right) + \bar{\nu}_N \lambda_N \left( 1 + \lambda_N T/f_N \right), \]

so as to fulfill the fluctuation relation equation (93), and where \( \bar{\nu}_i = \langle \dot{X}_i \rangle, (i = 1, N) \). Hence, \( P(\eta, \tau) \) reads

\[ P(\eta, \tau) = \frac{e^{-\tau/4}}{\pi a(\eta) \sqrt{|C|}} \left( 1 + \sqrt{\pi \tau} h(\eta) e^{\tau h^2(\eta)} \text{erf}(\sqrt{\tau} h(\eta)) \right), \]

with

\[ \tau = t f_N \bar{\nu}_N + f_1 \bar{\nu}_1, \]

\[ a(\eta) = (1 - \eta)^2 + \frac{1}{|C|} \left( \eta - \bar{\eta} \right)^2, \]

\[ h(\eta) = \frac{1 - \eta}{2 \sqrt{a(\eta)}}, \]

\[ C = \frac{1}{f_N \bar{\nu}_N + f_1 \bar{\nu}_1} \begin{bmatrix} f_N \bar{\nu}_N & 0 \\ 0 & f_1 \bar{\nu}_1 \end{bmatrix}, \]

and \( \bar{\eta} = -f_N \bar{\nu}_N / (f_1 \bar{\nu}_1) \). The expression for \( P(\eta, \tau) \) is the analogous of the one obtained in [16] for two coupled currents in the linear regime.

Retracing the steps in section 3.4, we exploit the fluctuation symmetry for \( \mu_0(\lambda) \) equation (93) and find that the transformations

\[ g(\eta) = -\frac{\eta + (\eta - 2) \bar{\eta}}{1 - 2\eta + \bar{\eta}}, \]

\[ \alpha(\eta) = \pm \frac{1 - 2\eta + \bar{\eta}}{-1 + \bar{\eta}}, \]

give the fluctuation relation

\[ P(\eta, \tau) = P(g(\eta), \tau) e^{-R(\eta)}, \]

with \( R(\eta) = \ln \alpha^2(\eta) \). This relation is graphically checked in figure 6, where we also show the extremal points equations (104) and (105) and the behavior around the peak of the PDF in linear scale, for a particular choice of the parameters.
The fluctuation relation for the efficiency has been derived for an isothermal motor, with two energy currents coupled through a general potential. However, our results remain valid for other types of systems, for example the heat engine considered in [4], as long as the energy currents obey a fluctuation relation of the same type as equation (93), where the term $f_i/T$ is replaced by the corresponding generalized thermodynamic force associated with the current $J_i$.

4.2. Distribution of forces

As a second case we consider a machine in which every oscillator is subject to a biasing force, so that we have a certain quenched distribution of input ($f_i > 0$) and output forces ($f_i < 0$). The system of Langevin equations is the same as in equation (88). Analogously to what we did in section 3.1 for the relative coordinate, we assume a $N$-dimensional Gaussian PDF for $P(X_1, \ldots, X_N, t)$, the cumulant generating function reads

$$\mu_0(\lambda) = \lambda_i \bar{v}_i + \alpha_{ij} \lambda_i \lambda_j, \quad (112)$$

where $\bar{v}_i \equiv \langle \dot{x}_i \rangle$, and $\alpha_{ij} = \delta_{ij} \bar{v}_i T/f_i$ due to the symmetry imposed by the fluctuation relation equation (93). As discussed in section 3.1 we expect such a Gaussian approximation to hold in the limit of small forces.

We define the input and output stochastic work as $W_{\text{out}} = \sum_i f_i X_i$, $W_{\text{in}} = \sum_i f_i X_i$.

Accordingly, the PDF of the efficiency equation (90) reads

$$P(\eta, t) = \int \int dW_{\text{out}} dW_{\text{in}} \delta \left( \eta + \frac{W_{\text{out}}}{W_{\text{in}}} \right) P(W_{\text{out}}, W_{\text{in}}, t). \tag{113}$$

Given equation (112) one can easily check that the joint PDF on the right hand side of the last equation factorizes,

$$P(W_{\text{out}}(\text{in}), t) = \int d\lambda e^{t \lambda} \left[ P_{\text{out}}(\text{in}) - (1 - \lambda T) \bar{P}_{\text{out}}(\text{in}) \right], \tag{114}$$

and where $P_{\text{out}}(\text{in}) = \sum_i \rho(P)f_i \bar{v}_i$ are the output (input) power, averaged over the force distribution. One thus obtains the bidimensional Gaussian distribution

$$P(W_{\text{out}}, W_{\text{in}}, t) = \frac{1}{4\pi T} \left( \bar{P}_{\text{out}} \bar{P}_{\text{in}} \right)^{-1/2} \exp \left[ -\frac{t}{4T} \left( \frac{(P_{\text{out}} - \bar{P}_{\text{out}})^2}{\bar{P}_{\text{out}}} + \frac{(P_{\text{in}} - \bar{P}_{\text{in}})^2}{\bar{P}_{\text{in}}} \right) \right]. \tag{115}$$

Therefore, the efficiency PDF equation (113) will be analogous to equations (107) and (108), with

$$\tau = \frac{t(\bar{P}_{\text{out}} + \bar{P}_{\text{in}})}{T},$$

$$C = \frac{1}{\bar{P}_{\text{out}} + \bar{P}_{\text{in}}} \begin{bmatrix} \bar{P}_{\text{out}} & 0 \\ 0 & \bar{P}_{\text{in}} \end{bmatrix}, \tag{116}$$

and $\dot{\eta} = -\bar{P}_{\text{out}}/\bar{P}_{\text{in}}$. Accordingly, the fluctuation relation for the PDF of the efficiency is given by equation (111), with the transformation equations (109) and (110).

The statistical features of the efficiency of an isothermal engine made up of $N$-coupled oscillators, that are either producers or users according to a given distribution, are thus analogous to the statistical features of the efficiency in a device that couples two thermodynamic currents that fluctuate with normal law [16]. However, differently from [16], our system is not linear, the features of the non-linear interacting potential being hidden in the average velocities $\bar{v}_i$ that appear in equation (112).
We end up this section by studying the extremal points of the PDF $\eta^{\pm}$. They can be obtained by requiring that the transformation equation (109) maps each of them into itself, such that

$$g(\eta^{\pm}) = \eta^{\pm},$$

with $\alpha(\eta^{\pm}) = 1$. According to this condition, the extremal points of the PDF equation (107) are $\eta^{+} = \bar{\eta}$ and $\eta^{-} = 1$, which correspond again to the macroscopic efficiency and the reversible efficiency, respectively.

However, we do not obtain the second maximum in the super-Carnot efficiency region $\eta \geq \eta^{-}$ obtained in [16] in the intermediate time regime. This is due to the fact that the long-time limit is already implicit in the derivation of the PDF $P(W_{\text{out}}, W_{\text{in}}, t)$ equation (107).

5. Summary

We have studied the statistics of the efficiency in isothermal cyclic machines with realistic interactions between the internal degrees of freedom. Such a realistic potential interaction has the advantage that we can consider explicitly the weak and the tight coupling limits as well as the small and large force limits.

We first investigate a minimal model consisting of two coupled degrees of freedom. By separating the center of mass and the relative coordinate motion, we are able to express the PDF of the efficiency as an integral of a closed form. Besides, we derive an analytic solution for the efficiency PDF in the limit of weak coupling and small forces.
The study of the extremal points of the efficiency PDF reveals that the most likely efficiency is always the macroscopic efficiency, whereas the least likely is the reversible efficiency. The macroscopic efficiency, which depends on the interaction strength, is bounded between a minimal value obtained for weak coupling or strong forces, and a maximal value achieved in the tight coupling limit. These boundaries turn out to be universal in the sense that they depend only on the thermodynamic forces, and not on the details of the interaction potential.

We investigate the condition under which the machine operates close to the macroscopic reversible efficiency, and we conclude that the tight coupling limit between the input and output currents is a necessary, yet not sufficient, condition for achieving the lossless limit. As a matter of fact, given the realistic physical interaction between the thermodynamic forces, the reversible macroscopic efficiency is attained in the tight coupling limit and close to the stall condition, in which the difference between the input and the output forces vanishes, thus making the machine useless.

Assuming a normal distribution for the relative coordinate current, the long time fluctuation relation for the input and output currents implies a fluctuation relation for the efficiency, that resembles the long time relations previously obtained for other stochastic thermodynamic quantities. Even though this relation is derived under the conjecture of Gaussian distributed currents, whose range of validity is limited to the range of small forces and weak coupling, we provide numerical evidence that it holds for a wide range of forces, and hence beyond the linear regime.

We finally explore the case where the machine consists of $N$ degrees of freedom, and show that the efficiency fluctuations can be studied by focusing on the input and the output energy currents alone, i.e. mapping the $N$ body model into a model with two coupled fluctuating currents. Thus we find that the results obtained for the minimal model hold true for an arbitrary number of degrees of freedom.

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Appendix A. Determinant near identity

The expansion of the second determinant in equation (25) is

$$
\det \left[ \left( 1 + k M^{-1} \hat{L}_\lambda^{(1)} \right) \right] = \left[ 1 + k f_1 ( M^{-1} \hat{L}_\lambda^{(1)} ) + k^2 f_2 ( M^{-1} \hat{L}_\lambda^{(1)} ) + k^3 f_3 ( M^{-1} \hat{L}_\lambda^{(1)} ) + k^4 f_4 ( M^{-1} \hat{L}_\lambda^{(1)} ) + O(k^5) \right],
$$

(A.1)

where the series expansion terms of the determinant near identity can be derived from the Jacobi’s formula [55, 56], and by setting $A \equiv M^{-1} \hat{L}_\lambda^{(1)}$ we have

$$
f_1 (A) = \text{Tr}[A],
$$

$$
f_2 (A) = \frac{\text{Tr}^2[A] - \text{Tr}[A^2]}{2},
$$

$$
f_3 (A) = \frac{1}{3!} (\text{Tr}^3 (A) - 3 \text{Tr}(A) \text{Tr}(A^2) + 2 \text{Tr}(A^3)),
$$

$$
f_4 (A) = \frac{1}{4!} \text{Tr}^4 (A) - \frac{1}{4} \text{Tr}(A^4) + \frac{1}{8} \text{Tr}^2 (A^2) + \frac{1}{3} \text{Tr}(A) \text{Tr}(A^3) - \frac{1}{4} \text{Tr}^2 (A) \text{Tr}(A^2).
$$
The matrix $\mathbf{M}^{-1}$ depends on $\mu(\lambda)$ too, and thus the terms $f_i(\mathbf{M}^{-1} \hat{\mathbf{L}}_1^{(1)})$ have to be expanded in powers of $k$ as well, so as to take into account all the contributions for each in $k$.

Appendix B. PDF of the efficiency for the Gaussian approximation

We assume a normal distribution for the relative coordinate $Y$, $P(Y,t)$ equation (43). After integrating equation (41), the PDF of $\xi$ reads

$$
\Phi(\xi, t) = \exp \left[ -t \left( \frac{f_x^2 + f_y^2}{2T} \right) \right] \frac{\sqrt{f_x f_y}}{\pi (f_x + v_x \xi^2)} \left( 1 + e^{\frac{h(\xi)^2}{\sqrt{\pi T h(\xi)}}} \right),
$$

(B.1)

where $\hat{h}(\xi) = (f_y + f_x \xi) (2T(f_x + v_x \xi^2)/\hat{v})^{-1/2}$. In the long time limit the error function can be expanded $\text{erf} (\sqrt{\pi h(\xi)}) \sim 1 - e^{-h(\xi)^2}/(\sqrt{\pi T h(\xi)})$ \[57\]; taking into account that the $\text{erf} (\sqrt{\pi h(\xi)})$ change sign at $h(\xi) = 0$, the long time limit of $\Phi(\xi, t)$ equation (B.1) reads

$$
\Phi(\xi, t) = \exp \left[ -t \left( \frac{f_x^2 + f_y^2}{2T} \right) \right] \frac{\sqrt{f_x f_y}}{\pi (f_x + v_x \xi^2)} \left( 1 - \frac{\hat{h}(\xi)}{|h(\xi)|} + \sqrt{\pi T h(\xi)} e^{h(\xi)^2} \right).
$$

(B.2)

We obtain the PDF of $\eta$ after inverting the change of variables equation (39),

$$
P(\eta, t) = \exp \left[ -t \left( \frac{f_x^2 + f_y^2}{2T} \right) \right] \frac{4f_x T h(\eta)^2}{(f_x + f_y)^2 f_x f_y \sqrt{\pi h(\eta) - 1}^2 |h(\eta)|} \left( 1 + \sqrt{\pi T h(\eta)} e^{h(\eta)^2} \text{erf} \left( \sqrt{\pi T h(\eta)} \right) \right),
$$

(B.3)

where $h(\eta) = (f_x^2 - f_y^2)(\eta - 1)/\sqrt{\pi} \left( (f_x(\eta - 1) + f_y(\eta + 1)) \sqrt{2T \left( f_x + \frac{\hat{h}(\eta)(\eta - 1) + \hat{h}(\eta)(\eta + 1)}{f_x(\eta - 1) + f_x(\eta + 1)} \right)} \right)^{-1}$.

Applying the former expansion for the error function, the long time limit PDF of the efficiency reads

$$
P(\eta, t) = \exp \left[ -t \left( \frac{f_x^2 + f_y^2}{2T} \right) \right] \frac{4f_x T h(\eta)^2}{(f_x + f_y)^2 f_x f_y \sqrt{\pi h(\eta) - 1}^2 |h(\eta)|} \left( 1 - \frac{h(\eta)}{h(\eta)} + \sqrt{\pi T h(\eta)} e^{h(\eta)^2} \right).
$$

(B.4)

Appendix C. PDF of the efficiency for two terminals and its extremal points

The leading term of the integral in equation (103) is

$$
g(\hat{\eta}) = \int \int dx \, dX \, e^{i \mu_0(\lambda_1^* \lambda_2^*) - J_1^* \lambda_1^* \lambda_2^* + \hat{h}(J_2 - J_1 \hat{\eta})},
$$

(C.1)

where we have used the integral expression for the Dirac delta,

$$
\delta (X_N - X_1 \hat{\eta}) = \frac{1}{2\pi} \int dx \, e^{i \mu(\lambda_1^* \lambda_2^*) - J_1^* \lambda_1^* \lambda_2^* + \hat{h}(J_2 - J_1 \hat{\eta})}.
$$

(C.2)

Integrating over $X_1$ and $X_N$ by the saddle-point approximation we obtain

$$
g(\hat{\eta}) = \int dx \, e^{i \mu(\lambda_1^* \lambda_2^*) - J_1^* \lambda_1^* \lambda_2^* + \hat{h}(J_2 - J_1 \hat{\eta})},
$$

(C.3)

with $J_1^*$ and $J_1^*$ implicitly defined by the equations

$$
\partial_{\hat{\eta}} \left[ \mu_0(\lambda_1^* \lambda_2^*) - \lambda_1^* J_1 - \lambda_2^* J_2 + \hat{h}(J_2 - J_1 \hat{\eta}) \right] = 0,
$$

(C.4)
\[
\partial_{J_N} \left[ \mu_0 (\lambda_1^*, \lambda_N^* - \lambda_1^* J_1 - \lambda_N^* J_N + \hat{\xi} (J_N - J_1 \hat{\eta})) \right] \bigg|_{J_1^*=J_N^*} = 0. \tag{C.5}
\]

By employing the conditions that define \( \lambda_1^* \) equation (100) and \( \lambda_N^* \) equation (101), and labeling \( \lambda_1^{**} \equiv \lambda_1^* (J_1^{**}, J_N^{**}) \) and \( \lambda_N^{**} \equiv \lambda_N^* (J_1^{**}, J_N^{**}) \), the equations that define \( J_1^{**} \) and \( J_N^{**} \) are

\[
- \lambda_1^{**} = - \lambda_N^{**} = 0, \tag{C.6}
\]

\[
- \lambda_1^{**} + \hat{\xi} \hat{\eta} = 0, \tag{C.7}
\]

The integral over \( s \) in equation (C.3) can be solved by a saddle-point approximation as well

\[
g(\hat{\eta}) = e^{\mu_0 (\lambda_1^{***}, \lambda_N^{***}) - J_1^{***} \lambda_1^{***} - J_N^{***} \lambda_N^{***} + \frac{1}{2} \xi s J_N^{***} + \hat{\xi} \xi \lambda_1^{***} - J_1^{***} \hat{\eta} \xi)}, \tag{C.8}
\]

\[
\partial_{\hat{\eta}} \left[ \mu_0 (\lambda_1^{***}, \lambda_N^{***} - \lambda_1^{***} J_1^{***} - \lambda_N^{***} J_N^{***} + \frac{1}{2} \xi s (J_N^{***} - J_1^{***} \hat{\eta})) \right] \bigg|_{\hat{\eta}}^{****} = 0. \tag{C.9}
\]

Taking into account equations (100), (101), (C.6) and (C.7), the condition for \( s^{***} \) can be rewritten after some algebraic manipulation as

\[
J_N^{***} - J_1^{***} \hat{\eta} = 0. \tag{C.10}
\]

The extremal points of the efficiency’s PDF will be given by those of \( g(\hat{\eta}) \) equation (C.8), that is, the solution of

\[
\partial_{\hat{\eta}} \left[ \mu_0 (\lambda_1^{***}, \lambda_N^{***} - \lambda_1^{***} J_1^{***} - \lambda_N^{***} J_N^{***} + \frac{1}{2} \xi s (J_N^{***} - J_1^{***} \hat{\eta})) \right] \bigg|_{\hat{\eta}}^{****} = 0, \tag{C.11}
\]

that simplifies into

\[
s^{***} J_1^{***} = 0, \tag{C.12}
\]

after applying equations (100), (101), (C.6), (C.7) and (C.10). The two solutions of equation (C.12) are

\[
s^{***} = 0, J_1^{***} = 0. \tag{C.13}
\]

When \( s^{***} = 0 \), then \( \lambda_1^{***} = \lambda_N^{***} = 0 \) equations (C.6) and (C.7). Plugging equations (92) into (91), deriving with respect to \( \lambda_1 \) and evaluating at \( \lambda_1 = \lambda_N = 0 \), we obtain the identity (analogous to equation (28))

\[
\frac{\partial_\lambda \mu_0 (\lambda)}{\lambda_i = \lambda_N = 0} = \frac{\langle X_i \rangle}{\bar{f}} = \langle J_i \rangle. \tag{C.14}
\]

Exploiting equation (C.14), we can compute \( J_1^{***} \) and \( J_N^{***} \) appearing in equation (C.10) from equations (100) and (101). Then we can solve equation (C.10) for \( \hat{\eta} \) and we find that the most likely efficiency is,

\[
\hat{\eta}_+ = \frac{\langle J_N \rangle}{\langle J_1 \rangle} \Rightarrow \eta_+ = - \frac{f_N \langle J_N \rangle}{f_1 \langle J_1 \rangle}, \tag{C.15}
\]

where the transformation in equation (96) has been taken into account.

Considering the second solution \( J_1^{***} = 0 \), then \( J_N^{***} = 0 \) because of equation (C.10). Bearing in mind that the largest eigenvalue is a convex function, then \( (\lambda_1^{***}, \lambda_N^{***}) \) are the coordinates of its minimum, for \( J_1^{***} = 0 \) implies that \( \partial_\lambda \mu_0 (\lambda_1, \lambda_N) \bigg|_{\lambda_1^{***}, \lambda_N^{***}} = 0 \) according to equations (100) and (101). Then the symmetry imposed by the fluctuation relation
equation (93) is such that the symmetry point, i.e. the minimum, is located at \((-f_1/2T, -f_N/2T)\). Thus the least likely efficiency is,
\[
\eta_- = 1;
\]
(C.16)
where the definitions of \(J_1^\ast\) equations (C.6) and \(J_N^\ast\) (C.7) have been employed, together with equation (96).

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