A Note on Some Best Proximity Point Theorems Proved under P-Property

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1. Introduction

Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. In this paper, we adopt the following notations and definitions:

$$D(x, B) := \inf \{d(x, y) : y \in B\}, \forall x \in X,$$

$$A_0 := \{x \in A : d(x, y) = \text{dist}(A, B), \text{ for some } y \in B\},$$

$$B_0 := \{y \in B : d(x, y) = \text{dist}(A, B), \text{ for some } x \in A\}.$$

The notion of best proximity point is defined as follows.

Definition 1. Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \to B$ a non-self-mapping. A point $x^* \in A$ is called a best proximity point of $T$ if $d(x^*, Tx^*) = \text{dist}(A, B)$, where

$$\text{dist}(A, B) := \inf \{d(x, y) : (x, y) \in A \times B\}.$$

Similarly, for a multivalued non-self-mapping $T : A \to 2^B$, where $(A, B)$ is a nonempty pair of subsets of a metric space $(X, d)$, a point $x^* \in A$ is a best proximity point of $T$ provided that $D(x^*, Tx^*) = \text{dist}(A, B)$.

Recently, the notion of P-property was introduced in [1] as follows.

Definition 2 (see [1]). Let $(A, B)$ be a pair of nonempty subsets of a metric space $(X, d)$ with $A_0 \neq \emptyset$. The pair $(A, B)$ is said to have P-property if and only if

$$d(x_1, y_1) = \text{dist}(A, B) \quad d(x_2, y_2) = \text{dist}(A, B) \implies d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

By using this notion, some best proximity point results were proved for various classes of non-self-mappings. Here, we state some of them.

Theorem 3 (see [1]). Let $(A, B)$ be a pair of nonempty closed subsets of a complete metric space $X$ such that $A_0$ is nonempty. Let $T : A \to B$ be a weakly contractive non-self-mapping; that is,

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \quad \forall x, y \in A,$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi$ is positive on $(0, \infty)$, $\phi(0) = 0$, and $\lim_{t \to \infty} \phi(t) = \infty$. Assume that the pair $(A, B)$ has the P-property and $T(A_0) \subseteq B_0$. Then, $T$ has a unique best proximity point.
Theorem 4 (see [2]). Let \((A, B)\) be a pair of nonempty closed subsets of a Banach space \(X\) such that \(A\) is compact and \(A_0\) is nonempty. Let \(T : A \to B\) be a nonexpansive mapping; that is,
\[
\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in A.
\] (5)
Assume that the pair \((A, B)\) has the P-property and \(T(A_0) \subseteq B_0\). Then, \(T\) has a best proximity point.

Theorem 5 (see [3]). Let \((A, B)\) be a pair of nonempty closed subsets of a complete metric space \(X\) such that \(A_0 \neq \emptyset\) and \((A, B)\) satisfies the P-property. Let \(T : A \to B\) be a Meir-Keeler non-self-mapping; that is, for all \(x, y \in A\) and for any \(\epsilon > 0\), there exists \(\delta(\epsilon) > 0\) such that
\[
\epsilon \leq d(x, y) < \epsilon + \delta \quad \text{implies} \quad d(Tx, Ty) \leq \epsilon.
\] (6)
Assume that the pair \((A, B)\) has the P-property and \(T(A_0) \subseteq B_0\). Then, \(T\) has a unique best proximity point.

Theorem 6 (see [4]). Let \((A, B)\) be a pair of nonempty closed subsets of a complete metric space \((X, d)\) such that \(A_0 \neq \emptyset\) and \((A, B)\) satisfies the P-property. Let \(T : A \to 2^B\) be a multivalued contraction non-self-mapping; that is,
\[
H(Tx, Ty) \leq \alpha d(x, y),
\] (7)
for some \(\alpha \in (0, 1)\) and for all \(x, y \in A\). If \(T\) is bounded and closed in \(B\) for all \(x \in A\), then \(T\) has a unique best proximity point in \(A\).

Theorem 7 (see [5]). Let \((A, B)\) be a pair of nonempty closed subsets of a complete metric space \((X, d)\) such that \(A_0 \neq \emptyset\) and \((A, B)\) satisfies the P-property. Let \(T : A \to B\) be a Geraghty-contraction non-self-mapping; that is,
\[
d(Tx, Ty) \leq \beta(d(x, y)), \quad \forall x, y \in A,
\] (8)
where \(\beta : [0, \infty) \to [0, 1)\) is a function which satisfies the following condition:
\[
\beta(t_n) \to 1 \iff t_n \to 0.
\] (9)
Suppose that the pair \((A, B)\) has the P-property and \(T(A_0) \subseteq B_0\). Then, \(T\) has a unique best proximity point.

2. Main Result

In this section, we show that the existence of a best proximity point in the main theorems of [1–5] can be obtained from the existence of the fixed point for a self-map. We begin our argument with the following lemmas.

Lemma 8 (see [6]). Let \((A, B)\) be a pair of nonempty closed subsets of a complete metric space \((X, d)\) such that \(A_0 \neq \emptyset\) and \((A, B)\) has the P-property. Then, \((A_0, B_0)\) is a closed pair of subsets of \(X\).

Lemma 9. Let \((A, B)\) be a pair of nonempty closed subsets of a metric space \((X, d)\) such that \(A_0 \neq \emptyset\) is nonempty. Assume that the pair \((A, B)\) has the P-property. Then there exists a bijective isometry \(g : A_0 \to B_0\) such that \(d(x, gx) = \text{dist}(A, B)\).

Proof. Let \(x \in A_0\); then there exists an element \(y \in B_0\) such that
\[
d(x, y) = \text{dist}(A, B).
\] (10)
Assume that there exists another point \(\hat{y} \in B_0\) such that
\[
d(x, \hat{y}) = \text{dist}(A, B).
\] (11)
By the fact that \((A, B)\) has the P-property, we conclude that \(y = \hat{y}\). Consider the non-self-mapping \(g : A_0 \to B_0\) such that \(d(x, gx) = \text{dist}(A, B)\). Clearly, \(g\) is well defined. Moreover, \(g\) is an isometry. Indeed, if \(x_1, x_2 \in A_0\), then
\[
d(x_1, gx_1) = \text{dist}(A, B), \quad d(x_2, gx_2) = \text{dist}(A, B).
\] (12)
Again, since \((A, B)\) has the P-property,
\[
d(x_1, x_2) = d(gx_1, gx_2);
\] (13)
that is, \(g\) is an isometry.

Here, we prove that the existence and uniqueness of the best proximity point in Theorem 3 are a sample result of the existence of fixed point for a weakly contractive self-mapping.

Theorem 10. Let \((A, B)\) be a pair of nonempty closed subsets of a complete metric space \(X\) such that \(A_0 \neq \emptyset\). Let \(T : A \to B\) be a weakly contractive mapping. Assume that the pair \((A, B)\) has the P-property and \(T(A_0) \subseteq B_0\). Then, \(T\) has a unique best proximity point.

Proof. Consider the bijective isometry \(g : A_0 \to B_0\) as in Lemma 9. Since \(T(A_0) \subseteq B_0\), for the self-mapping \(g^{-1}T : A_0 \to A_0\), we have
\[
d\left(g^{-1}(Tx), g^{-1}(Ty)\right) = d(Tx, Ty) \leq \varphi(d(x, y)),
\] (14)
for all \(x, y \in A_0\) which implies that the self-mapping \(g^{-1}T\) is weakly contractive. Note that \(A_0\) is closed by Lemma 8. Thus, \(g^{-1}T\) has a unique fixed point [7]. Suppose that \(x^* \in A_0\) is a unique fixed point of the self-mapping \(g^{-1}T\); that is, \(g^{-1}T(x^*) = x^*\). So, \(Tx^* = gx^*\), and then
\[
d(x^*, Tx^*) = d(x^*, gx^*) = \text{dist}(A, B),
\] (15)
from which it follows that \(x^* \in A_0\) is a unique best proximity point of the non-self weakly contractive mapping \(T\).

Remark 11. By a similar argument, using the fact that every nonexpansive self-mapping defined on a nonempty compact and convex subset of a Banach space has a fixed point, we conclude Theorem 4. Also, the existence and uniqueness of best proximity point for Meir-Keeler non-self-mapping \(T\) (Theorem 5) follow from the Meir-Keeler’s fixed point theorem (8)). Moreover, in Theorem 6, Nadler’s fixed point theorem (9) ensures the existence of a best proximity point for multivalued non-self mapping \(T\). Finally, Theorem 7 due to Caballero et al., is obtained from Geraghty’s fixed point theorem (10)).
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