Time invariants across a fourth-order exceptional point in a parity-time-symmetric qudit

Zhihao Bian$^{1,2}$, Lei Xiao$^{1,3}$, Kunkun Wang$^{1,3}$, Xiang Zhan$^{1,4}$, Franck Assogba
Onanga$^{5}$, Frantisek Ruzicka$^{5,6}$, Wei Yi$^{7,8}$, Yogesh N. Joglekar$^{5}$, Peng Xue$^{1,3,9}$

1 Beijing Computational Science Research Center, Beijing 100084, China
2 School of Science, Jiangnan University, Wuxi 214122, China
3 Department of Physics, Southeast University, Nanjing 211189, China
4 School of Science, Nanjing University of Science and Technology, Nanjing 210094, China
5 Department of Physics, Indiana University Purdue University Indianapolis (IUPUI), Indianapolis, Indiana 46202, USA
6 Institute of Nuclear Physics, Czech Academy of Sciences, Rez 250 68, Czech Republic
7 Key Laboratory of Quantum Information, University of Science and Technology of China, CAS, Hefei 230026, China
8 CAS Center For Excellence in Quantum Information and Quantum Physics and
9 State Key Laboratory of Precision Spectroscopy,
East China Normal University, Shanghai 200062, China

Constants of motion of a closed system, such as its energy or charge, are determined by symmetries of the system. They offer global insights into the system dynamics and were instrumental to advances such as the prediction of neutrinos. In contrast, little is known about time invariants in open systems. Recently, a special class of open systems with parity-time ($\mathcal{PT}$) symmetry has been intensely explored for their remarkable properties. However, a complete characterization and experimental observation of time invariants therein are both still lacking. Here we present an analytical solution for all time invariants of a broad class of $\mathcal{PT}$-symmetric Hamiltonians. Using a single-photon interferometry setup, we confirm our results by simulating the quantum dynamics of a $\mathcal{PT}$-symmetric qudit across a fourth-order exceptional point. We further observe the information flow in the system via the dynamics of qudit entropies. Our results demonstrate conserved quantities in a non-unitary time evolution, and point towards the rich dynamics and enhanced sensitivity of systems with higher-order exceptional points.

I. INTRODUCTION

A $\mathcal{PT}$-symmetric system has balanced gain and loss, and is described by an effective, non-Hermitian Hamiltonian $H_{\mathcal{PT}}$ that is invariant under the combined parity and time-reversal operation $[1][14]$. As the gain-loss strength is increased, the spectrum of $H_{\mathcal{PT}}$ changes from real into complex conjugate pairs, and the corresponding eigenvectors cease to be eigenvectors of the $\mathcal{PT}$ operator. This $\mathcal{PT}$-symmetry-breaking transition occurs at an exceptional point of order $n$ (EP$n$), where $n$ eigenvalues, as well as their corresponding eigenvectors, coalesce $[15][16]$. The $\mathcal{PT}$ transition and the non-unitary time evolution generated by $H_{\mathcal{PT}}$ have been observed in classical optical systems with EP2 $[1][2]$ and EP3 $[10]$. In the quantum domain, $\mathcal{PT}$ transitions across the EP2 have been observed in systems with mode-selective losses comprising single photons $[11]$, ultracold atoms $[12]$, and superconducting qubits $[13]$.

In traditional quantum theory (of closed systems), an observable $O$ is a constant of motion if it commutes with the Hermitian Hamiltonian, where $O = I$ implies the conservation of norm of a quantum state. A $\mathcal{PT}$-symmetric Hamiltonian, on the other hand, describes an open system with balanced gain and loss. What are its conserved quantities, if any? How do they constrain the dynamics, particularly in the $\mathcal{PT}$-symmetry-broken region where exponentially amplifying eigenmodes occur?

With an approach inspired by the early work on pseudo-Hermicity $[17][19]$, we conclusively address these questions. For a broad class of $\mathcal{PT}$-symmetric Hamiltonians encompassing all finite-dimensional discrete models, we analytically construct a maximal set of linearly independent intertwining operators with time-invariant expectation values. We then experimentally confirm our construction in a minimal $\mathcal{PT}$-symmetric Hamiltonian with an EP4, by encoding a qudit ($d = 4$) in the path and polarization of a single photon $[20]$ and simulating its non-unitary dynamics. We track all four linearly independent time invariants in the $\mathcal{PT}$-symmetric phase, at the $\mathcal{PT}$ transition point (EP4), and in the $\mathcal{PT}$-broken phase for different initial states. We also demonstrate the enhanced sensitivity of the system’s spectrum near the EP4, and further characterize the information-flow dynamics through the time evolution of qudit entropies.
II. CONSTRUCTING TIME-INVARIANTS FOR $H_{\mathcal{PT}}$

Consider an open system described by a $d$-dimensional Hamiltonian $H_{\mathcal{PT}}$ with an energy scale $J$ ($\hbar = 1$). A Hermitian operator $\hat{\eta}$ is called an intertwining operator \cite{18, 19} for $H_{\mathcal{PT}}$ if it satisfies

$$\hat{\eta}H_{\mathcal{PT}} = H_{\mathcal{PT}}^{\dagger}\hat{\eta}. \quad (1)$$

It follows that the non-unitary time evolution operator $G(t) = \exp(-iH_{\mathcal{PT}}t)$ keeps $\hat{\eta}$ unchanged, i.e. $G(t)^{\dagger}\hat{\eta}G(t) = \hat{\eta}$. The expectation value of $\hat{\eta}$ in an arbitrary quantum state is therefore a time invariant. Thus, in principle, all time invariants can be obtained by solving a set of $d^2$ linear equations \cite{1}, an approach used for $\mathcal{PT}$-symmetric dimer and trimer cases \cite{21}.

Instead, here we derive analytical expressions for all intertwining operators of an arbitrary Hamiltonian with simultaneous transpose and $\mathcal{PT}$ symmetries, i.e. $H_{\mathcal{PT}} = H_{\mathcal{PT}}^{\dagger}$. This broad class includes all experimentally investigated $\mathcal{PT}$-symmetric systems \cite{11, 13} and most of theoretically studied lattice models \cite{22, 23}. The transpose symmetry of the Hamiltonian implies $TH_{\mathcal{PT}} = H_{\mathcal{PT}}^{\dagger}$, where the time-reversal operator $T$ is given by complex conjugation. It follows from the $\mathcal{PT}$ symmetry of the Hamiltonian that the parity operator $\mathcal{P}$ is an intertwining operator, i.e. $\hat{\eta}_1 = \mathcal{P}$. We then construct a sequence of dimensionless, linearly independent operators by the recursion relation $\hat{\eta}_k = \hat{\eta}_{k-1}H_{\mathcal{PT}}/J$ ($k = 2, \cdots, d$). The intertwining nature of $\hat{\eta}_{k-1}$ implies that $\hat{\eta}_k$ is also an intertwining operator. This sequence terminates with $\hat{\eta}_d$ because of the characteristic equation for $H_{\mathcal{PT}}$ is a polynomial of order $d$. Thus $\mathcal{P}(H_{\mathcal{PT}}/J)^d$ is a linear combination of lower-order intertwining operators. By expressing $\hat{\eta}$ and $H_{\mathcal{PT}}$ in the bi-orthogonal eigenbasis of $H_{\mathcal{PT}}$ \cite{19}, it is easy to show that there are no additional linearly-independent intertwining operators as long as the spectrum of $H_{\mathcal{PT}}$ is non-degenerate. The recursive procedure thus yields a maximal set of $d$ linearly independent constants of motion for a $d$-dimensional Hamiltonian.

III. IMPLEMENTING $\mathcal{PT}$-SYMMETRIC QUDIT WITH AN EP

For experimental demonstrations, we use the Hamiltonian

$$H_{\mathcal{PT}}(\gamma) = \frac{1}{2} \begin{pmatrix} 3i\gamma & -\sqrt{3}J & 0 & 0 \\ -\sqrt{3}J & i\gamma & -2J & 0 \\ 0 & -2J & -i\gamma & -\sqrt{3}J \\ 0 & 0 & -\sqrt{3}J & -3i\gamma \end{pmatrix}. \quad (2)$$

Equation (2) is compactly written as $H_{\mathcal{PT}}(\gamma) = -JS_x + i\gamma S_z$, where $S_x$ and $S_z$ are spin-3/2 representations of the SU(2) group. It represents a $\mathcal{PT}$-symmetric qudit with an anti-diagonal parity operator $\mathcal{P} = \text{antidiag}(1,1,1,1)$. The four equally spaced eigenvalues are given by $\lambda_k = \{-3/2,-1/2,1/2,3/2\} \sqrt{J^2 - \gamma^2}$, which give rise to an EP4 at the $\mathcal{PT}$-breaking threshold $\gamma = J$.

The advantage of choosing Hamiltonian (2) is that it can be easily generalized to an arbitrary dimensional system where it still remains analytically solvable and has an exceptional point with the order equal to the system dimension $10$, $24$. The single energy gap $\sqrt{J^2 - \gamma^2}$ in the spectrum of (2) and the fermionic nature of spin-3/2 representation lead to anti-periodic ($G(T) = -I$) dynamics with the period $T(\gamma) = 2\pi/\sqrt{J^2 - \gamma^2}$ for $\gamma \leq J$. Further, for $\gamma = 0$, occupations $P_k(t) = |\langle k|\psi(t)\rangle|^2$ ($k = 1, 2, 3, 4$) of the four modes obey a shifted mirror symmetry, with $P_k(t) = P_{d-k}(t + T/2)$, which indicates a perfect state transfer.

We encode the four modes of the qudit in the spatial and polarization degrees of freedom of a single photon, and label them as $|1\rangle = |UV\rangle$, $|2\rangle = |UV\rangle$, $|3\rangle = |DH\rangle$, $|4\rangle = |DV\rangle$. Here, $\{|H\rangle, |V\rangle\}$ are the horizontal and vertical polarizations, and $\{|U\rangle, |D\rangle\}$ denote the upper and lower paths, which undergo gain and loss respectively (Fig. 1A). Mapping the $\mathcal{PT}$-symmetric Hamiltonian $H_{\mathcal{PT}}$ into a passive $\mathcal{PT}$-symmetric one with mode-selective losses $H_{\mathcal{L}}(\gamma) = H_{\mathcal{PT}}(\gamma) - 3i\gamma 1/2$, we implement the $4 \times 4$ non-unitary matrix $G_{\mathcal{L}}(t) = \exp(-iH_{\mathcal{L}}t)$ via a lossy linear optical circuit, which is related to $G(t)$ through $G(t) = G_{\mathcal{L}}(t)\exp(3\gamma t/2)$ \cite{11} (Supplementary Information).

We experimentally measure the occupation $P_k(t)$ by projecting the time-evolved state onto $|k\rangle$ (Supplementary Information). The perfect state transfer for $\gamma = 0$ is confirmed by the transfer of occupation from the first mode to the fourth mode (Fig. 1B). In the $\mathcal{PT}$-symmetric phase with a finite $\gamma$, whereas the occupations are periodic in time with a period $T(\gamma)$, there is no perfect state transfer at time $T(\gamma)/2$ due to the non-unitary dynamics (Fig. 1C).
At the exceptional point $\gamma = J$, the observed time-dependent norm $P(t) = \sum_k P_k(t)$ grows algebraically with time as $t^6$ (Fig. 1a). Such a scaling is dictated by the order of the exceptional point. At the EP4, $H_{\mathcal{PT}}^4 = 0$ and the power-series expansion of $G(t)$ terminates at the third order, giving rise to the $t^6$ dependence for the norm. When a general $\mathcal{PT}$-symmetric Hamiltonian is perturbed from the exceptional point by a small detuning $\delta$, the complex eigenvalues in the vicinity of EP$n$ are given by a Puiseux series in $\delta^{1/n}$ [24], indicating enhanced classical sensitivity proportional to the order of the exceptional point [10, 25]. To demonstrate this, we simulate the dynamics of a perturbed Hamiltonian $H_{\delta}(\gamma) = H_{\mathcal{PT}}(\gamma = J) - \delta \gamma |1\rangle \langle 1|$ and probe its eigenvalues via interference-based measurements (Fig. 1c; Supplementary Information). The real and imaginary parts of perturbed eigenvalues $\lambda_k$ as functions of distance $\delta$ from the EP4 (Fig. 1d) show $\text{Re}(\lambda_k) \sim \delta^{1/4}$ and $\text{Im}(\lambda_k) \sim \delta^{1/4}$ and provide an independent evidence of the fourth-order exceptional point. We note that this is a first direct measurement of eigenenergies in single-photon dynamics.

**IV. OBSERVING TIME INVARIANTS**

For a four-dimensional system, the recursive procedure gives four intertwining operators $\hat{\eta}_k = \mathcal{P}(H_{\mathcal{PT}}/J)^{k-1}$ ($k = 1, 2, 3, 4$). To obtain their time-dependent expectation values, we carry out quantum-state tomography on the time-evolved states (Fig. 2a). Figure 2b shows the measured qudit density matrix $\rho(t)$ for a system with $\gamma = 0.2J$ and an initial density matrix $\rho(0) = |1\rangle \langle 1|/3 + |2\rangle \langle 2|/4$ at times $t/T(\gamma) = (0, 1, 0.4, 0.7, 1)$. In our experiment, we also reconstruct the time-evolved states $|\psi_k(t)\rangle$ across multiple time scales and over a wide range of $\gamma$ for four different initial states, given by...
FIG. 2. Time invariants across the EP4. (a) Schematic of quantum-state tomography required to reconstruct the time-evolved state $|\psi(t)\rangle$ or the $4 \times 4$, complex, density matrix $\rho(t)$. (b) Reconstructed time-dependent density matrix of the $\mathcal{P}\mathcal{T}$-symmetric qudit with $\gamma = 0.2J$ and initial value $\rho(0) = |1\rangle \langle 1| + 2|4\rangle \langle 4|/3$, measured at times $t/T(\gamma) = \{0.1, 0.4, 0.7, 1\}$, shows a periodic time evolution consistent with the $\mathcal{P}\mathcal{T}$-symmetric phase. (c)-(f) For a symmetric initial state $|\psi_2\rangle$, the measured expectation values $\eta_k(t)$ of the four, dimensionless intertwining operators depend on $\gamma$, but remain time invariant. (g) At the EP4 ($\gamma = J$), $\eta_3(t)$ remains constant over two time-scales that differ by a factor of 15, but depends on the initial state. (h) Deep in the $\mathcal{P}\mathcal{T}$-broken region ($\gamma = 1.2J$), measured $\eta_3(t)$ remain time invariant although the norm of each state $|\psi_a(t)\rangle$ exponentially grows with time.

$|\psi_1\rangle = |1\rangle$, $|\psi_2\rangle = (|1\rangle + |2\rangle + |3\rangle + |4\rangle)/2$, $|\psi_3\rangle = (|1\rangle + \sqrt{2}|4\rangle)/\sqrt{3}$, $|\psi_4\rangle = (|1\rangle + |4\rangle)/\sqrt{2}$, to obtain time-dependent expectation values $\eta_k(t) = \langle \psi(t) | \tilde{\eta}_k | \psi(t) \rangle$ across the EP4.

Figures 2c-2f show measured expectation values $\eta_k(t)$ for the symmetric initial state $|\psi_2\rangle$. While generically dependent on $\gamma$, the expectation values $\eta_k(t)$ remain time invariant regardless of whether the system is Hermitian ($\gamma = 0$), in the $\mathcal{P}\mathcal{T}$-symmetric phase ($\gamma = 0.2J$), at the EP4 ($\gamma = J$), or deep in the $\mathcal{P}\mathcal{T}$-symmetry broken region ($\gamma = 1.2J$). Interestingly, the time invariant $\eta_3(t)$ measured at the transition point EP4 and over two different time scales shows that it is not positive-definite and can be tuned by an appropriate choice of the initial state (Fig. 2g). The same is true deep in the $\mathcal{P}\mathcal{T}$-symmetry broken regime (Fig. 2h). The persistence of time invariants in these last two cases leads to phase locking in the time-evolved states (Supplementary Information), while their norms increase with time algebraically or exponentially [23]. Results in Fig. 2 clearly demonstrate that despite their non-unitary evolution, open systems governed by $\mathcal{P}\mathcal{T}$-symmetric Hamiltonians support conserved quantities.

V. OBSERVING INFORMATION DYNAMICS

A crucial aspect of dynamics is the information flow between the system and environment, or between gain and loss sectors within the system. This is reflected in the qudit entropy $S(t) = -\text{Tr} [\tilde{\rho}(t) \log_2 \tilde{\rho}(t)]$, where $\tilde{\rho}(t) = \rho(t)/\text{Tr} [\rho(t)]$ is the instantaneously normalized density matrix, and
The gain- and loss-sector entropies, $S_{\text{Gain}}(t)$ and $S_{\text{Loss}}(t)$, respectively. These are obtained from the gain- and loss-sector reduced density matrices $\rho_{\text{Gain}}(t) = \text{Tr}_{3,4}[\rho(t)]$ and $\rho_{\text{Loss}}(t) = \text{Tr}_{1,2}[\rho(t)]$. A full knowledge of the time-dependent state through the quantum-state tomography allows us to experimentally explore the information flow. Here, we focus on the quantum dynamics in the $PT$-symmetric region. At the EP4 or in the $PT$-symmetry broken region, due to the diverging state norm, the normalized density matrix $\tilde{\rho}(t)$ and various entropies approach a steady state (Supplementary Information) [26, 27].

Starting with a mixed initial state $\rho_{2}(0) = 0.9|\psi_{1}\rangle\langle\psi_{1}| + 0.1|\psi_{2}\rangle\langle\psi_{2}|$, the time-dependent qudit entropy $S(t)$ is constant in the Hermitian limit and shows periodic oscillations in the $PT$-symmetric region (Fig. 3). This demonstrates an exchange of quantum information between the system and its environment in the $PT$-symmetric region. By contrast, $S_{\text{Gain}}(t)$ and $S_{\text{Loss}}(t)$ oscillate in the Hermitian limit and show a qualitatively similar behavior throughout the $PT$-symmetric region. The reduced density matrices $\tilde{\rho}_{\text{Gain}}(t)$ and $\tilde{\rho}_{\text{Loss}}(t)$, on the other hand, trace out distinct trajectories on the Bloch sphere (Figs. 3b and 3c). Figure 3d shows the entropy dynamics for a family of mixed initial states $\rho_{\alpha}(0)$. Dynamics of the entanglement entropy $S_{E}(t)$ with different non-Hermiticities $\gamma$ (e) and different pure initial states (f).

Finally, we explore the dynamics of the entanglement entropy by choosing pure initial states. For
a symmetric initial state $|\psi_2\rangle$, $S_E(t) = S_{\text{Gain}}(t) = S_{\text{Loss}}(t)$ shows little difference between the Hermitian case and the $\mathcal{PT}$-symmetric case (Fig. 3). On the other hand, the periodic results for $S_E(t)$ in Fig. 3 show that its behavior, maxima, and minima can be engineered by the choice of the initial state.

VI. DISCUSSION

By analytically tabulating and directly observing the time invariants, we have evinced the mathematical structure and consequences of conserved quantities in a wide variety of $\mathcal{PT}$-symmetric systems. The observed enhanced sensitivity at the fourth-order exceptional point, phase locking, and quantum information flow signal the rich dynamics of $\mathcal{PT}$-symmetric systems with higher-order EPs that is potentially useful in developing ultrasensitive detection devices [10, 25].

Appendix A: Data analysis

Time-dependent occupation numbers for the four modes are defined as $P_k(t) = |\langle k|\psi(t)\rangle|^2$ $(k=1,2,3,4)$, where $|\psi(t)\rangle = G(t)|\psi(0)\rangle$. In our experiment, instead of $G(t)$, we realize the lossy evolution operator $G_L(t)$ at any given time $t$ and access time-evolved states by enforcing $G_L(t)$ on the initial state. We define a correction factor $C = \exp(3\gamma t/2)$ to map experimental realizations of $G_L(t)$ into $G(t)$ with gain and loss. We perform projective measurement on the time-evolved states, and obtain the number of the photons $N_k$ in each mode $|k\rangle$ and the raw data for the occupation numbers $P_k^\text{exp}(t) = N_k(t)/\sum_k N_k(t) + N_{\text{Loss}}(t)$, where $N_{\text{Loss}}(t)$ is the number of the photons which are lost from the optical circuit, as required by passive $\mathcal{PT}$-symmetric dynamics. After the correction, we obtain the experimental results of the occupation numbers for each mode $P_k^\text{exp}(t) = |C|^2 P_k^\text{exp}(t)$. Similar to the time-dependent norm $P(t) = |\langle \psi(t)|\psi(t)\rangle|$, we obtain the raw data for the norm $P^\text{exp}(t) = \sum_k P_k^\text{exp}(t)$ via projective measurement. After the correction, the experimental result for the norm is $P^\text{exp}(t) = |C|^2 P(t)$.

For the eigenenergies $\lambda_k$ of the perturbed Hamiltonian $H_\delta = H_{\mathcal{PT}}(\gamma = J) - i\delta\gamma |1\rangle\langle 1|$ at the EP4, we simulate passive non-unitary dynamics governed by a perturbed Hamiltonian $H_\delta' = H_\delta - 3i\gamma/2$. The eigenvalues $\xi_k$ of non-unitary time evolution operator $G_k(t) = \exp(-iH_\delta't)$ can be obtained via interference-based measurement [25]. The evolution time is fixed to $Jt = 0.5$ for this case. The eigenenergies $\lambda_k^\text{exp}$ are obtained from the experimental data, i.e. $\lambda_k^\text{exp} = 2t \ln \xi_k + 3i\gamma/2$.

For time invariants, we reconstruct the normalized density matrix $\rho^\text{exp}(t)$ of the time-evolved states via quantum-state tomography. After the correction, the density matrix of the states evolving under $H_L$ is given by $\rho'(t) = \rho^\text{exp}(t) \sum_i N_i(t)/\sum_i N_i(t) + N_{\text{Loss}}(t)$, where $i = U,D$ are the spatial modes of photons. Raw time invariants are obtained as $\eta_k^\text{exp} = \text{Tr}[\rho^\text{exp}(t)]$. After the correction, the experimental results for time invariants are obtained as $\eta_k^\text{exp} = |C|^2 \eta_k^\text{exp}(t)$. The entropy calculations use instantaneously normalized density matrices and are carried out using $\rho^\text{exp}(t)$.

Acknowledgments This work has been supported by the Natural Science Foundation of China (Grant No. 11674056), the Natural Science Foundation of Jiangsu Province (Grant No. BK20160024), the National Key R&D Program (Grant Nos. 2016YFA0301700, 2017YFA0304100) and NSF DMR-1054020. YJ thanks Andrew Harper and Joshua Feinberg for discussions.

[1] C. E. Rüter et al. Observation of parity-time symmetry in optics. Nat. Phys. 6, 192-195 (2010).
[2] A. Regensburger et al. Parity-time symmetric synthetic photonic lattices, Nature 488, 167-171 (2012).
[3] L. Feng et al. Single-mode laser by parity-time symmetry breaking, Science 346, 972-975 (2014).
[4] B. Peng et al. Parity-time-symmetric whispering-gallery microcavities. Nat. Phys. 10, 394-398 (2014).
[5] L. Chang et al. Parity-time symmetry and variable optical isolation in active-passive-coupled microresonators. Nat. Photon. 8, 524-529 (2014).
[6] M. Brandstetter et al. Reversing the pump dependence of a laser at an exceptional point. Nat. Commun. 5, 4034 (2014).
[7] S. Weimann et al. Topologically protected bound states in photonic parity-time-symmetric crystals. Nat. Mater. 16 433-438 (2016).
[8] S. Assawaworrarit, X. Yu, and S. Fan. Robust wireless power transfer using a nonlinear parity-time-symmetric circuit. Science 546, 387 (2017).
[9] L. Feng, R. El-Ganainy, and L. Ge. Non-Hermitian photons based on parity-time symmetry. Nat. Photon. 11, 752-762 (2017).
[10] H. Hodaei et al. Enhanced sensitivity at higher-order exceptional points. Nature 548, 187-191 (2017).
[11] L. Xiao et al. Observation of topological edge states in parity-time-symmetric quantum walks.
Nat. Phys. 13, 1117-1123 (2017).

[12] J. Li et al. Observation of parity-time symmetry breaking transitions in a dissipative Floquet system of ultracold atoms. Nat. Commun. 10, 855 (2019).

[13] M. Naghiloo, M. Abbas, Y. N. Joglekar, and K. W. Murch. Quantum state tomography across the exceptional point in a single dissipative qubit. arXiv:1901.07968.

[14] C. M. Bender and S. Boettcher. Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry. Phys. Rev. Lett. 80, 5243 (1998).

[15] T. Kato, Perturbation Theory of Linear Operators (Springer, New York, 1980).

[16] W. D. Heiss. The physics of exceptional points. J. Phys. A 45, 444016 (2012).

[17] C. M. Bender, D. C. Brody, and H. F. Jones. Complex extension of quantum mechanics. Phys. Rev. Lett. 89, 270401 (2002).

[18] A. Mostafazadeh. Pseudo-Hermiticity versus $\mathcal{PT}$ symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian. J. Math Phys. 43, 205 (2002).

[19] A. Mostafazadeh. Pseudo-Hermitian representation of quantum mechanics. Int. J. Geom. Methods Mod. Phys. 07, 1191-1306 (2010).

[20] J. Carolan, et al. Universal Linear Optics. Science 349, 711 (2015).

[21] Y. Choi, C. Hahn, J. W. Yoon, and S. H. Song. Observation of an anti-$\mathcal{PT}$-symmetric exceptional point and energy-difference conserving dynamics in electrical circuit resonators. Nat. Commun. 9, 2182 (2018).

[22] L. Jin and Z. Song. Solutions of $\mathcal{PT}$-symmetric tight-binding chain and its equivalent Hermitian counterpart. Phys. Rev. A 80, 052107 (2009).

[23] Y. N. Joglekar, F. A. Onanga, and A. K. Harter. Time-invariant $\mathcal{PT}$ product and phase locking in $\mathcal{PT}$-symmetric lattice models. Phys. Rev. A 97, 012128 (2018).

[24] E. M. Graefe, U. Günther, H. J. Korsch, and A. E. Niederle. A non-Hermitian $\mathcal{PT}$-symmetric Bose-Hubbard model: eigenvalue rings from unfolding higher-order exceptional points. J. Phys. A 41, 255206 (2008).

[25] W. Chen, Ş. K. Özdemir, G. Zhao, J. Wiersig, and L. Yang. Exceptional points enhance sensing in an optical microcavity. Nature 548, 192-196 (2017).

[26] K. Kawabata, Y. Ashida, and M. Ueda. Information retrieval and criticality in parity-time-symmetric systems. Phys. Rev. Lett. 119, 190401 (2017).

[27] L. Xiao, et al. Observation of critical phenomena in parity-time-symmetric quantum dynamics. arXiv:1812.01213.

[28] K. K. Wang, et al. Simulating dynamic quantum phase transitions in photonic quantum walks. Phys. Rev. Lett. 122, 020501 (2019).