4-dimensional Artin-Schelter regular quadratic $\widetilde{H}_4$-algebras

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Abstract

In this paper, quadratic algebras on which $\widetilde{H}_4$, the Heisenberg group of order 64, acts as degree-preserving algebra automorphisms are studied. In particular, we show that if $A$ is a four-dimensional Artin-Schelter regular quadratic $\widetilde{H}_4$-algebra with the degree one part isomorphic to the Schrödinger representation of $\widetilde{H}_4$, then $A$ is (a twist of) a four-dimensional Sklyanin algebra or (a twist of) a quantum Clifford algebra of global dimension 4.

Keywords: Artin-Schelter regular, Sklyanin, Heisenberg group

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1 Introduction

Artin-Schelter (AS) regular algebras were first defined in [1] by Artin and Schelter and later studied by Artin, Tate and Van den Bergh in [2] and [3]. In these papers, they classified all AS-regular algebras of global dimension 3. However, for higher dimensions, a classification is not yet in sight. Important classes of AS-regular algebras of each global dimension were later studied by Artin, Tate and Van den Bergh in [2] and [3]. In these papers, they classified all commutative Koszul algebras, as proven by Tate and Van den Berg in [14]. These classes form noncommutative deformations of the commutative polynomial for each global dimension.

In recent years, interest in four-dimensional Sklyanin algebras has resurfaced, as evidenced in [7], [10] and [15]. The regularity of these algebras was first proved by Smith and Stafford in [13]. In this paper, we will show in the main theorem (theorem 5.1) that almost all four-dimensional Sklyanin algebras, with the exception of 4 quantum algebras as defined in [4].

Most of the algebras studied here occur in the work of Chirvasitu and Smith [7]. In their paper, the algebras \( \mathcal{A}(\alpha_{1,0}, \alpha_{0,1}, \alpha_{1,1}) \) are studied for \((\alpha_{1,0}, \alpha_{0,1}, \alpha_{1,1}) \in \mathbb{A}^3\), which are quotients of the free associative algebra \( \mathbb{C}[w_{0,0}, w_{1,0}, w_{0,1}, w_{1,1}] \) by the following six quadratic relations (set \([x, y]_− := xy - yx\) and \([x, y]_+ := xy + yx\))

\[
\begin{align*}
[w_{0,0}, w_{i,j}]_− + \alpha_{i,j}[w_{j,i+j}, w_{i+j,j}]_+ & = 0, \\
[w_{j,i+j}, w_{i+j,j}]_− + [w_{0,0}, w_{i,j}]_+ & = 0, \\
(i, j) & \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0, 0)\}.
\end{align*}
\]

If \(\alpha_{1,0} \alpha_{0,1} \alpha_{1,1} \neq 0\), then \(\tilde{H}_4\) acts on these algebras as degree-preserving algebra automorphisms by [10] Section 6. If \(\alpha_{1,0} + \alpha_{0,1} + \alpha_{1,1} = 0\), then \(\mathcal{A}(\alpha_{1,0}, \alpha_{0,1}, \alpha_{1,1})\) is a (degenerate) Sklyanin algebra. It is a natural question to ask if there are any other AS-regular \(\tilde{H}_4\)-algebras in this three-dimensional family of algebras. Unfortunately, the answer will be mostly negative.

In section 7 we will show that the Koszul dual of the twisted homogeneous coordinate ring \(\mathcal{O}_\tau(E)\) associated to a Sklyanin algebra \(\mathcal{A}_\tau(E)\) has the following property:

\[
\forall n \in \mathbb{N} : (\mathcal{O}_\tau(E))^n_n \cong \mathcal{H}_4(\mathcal{O}_\tau(E)_n)^*.
\]

In the special case that \(\tau = O\) (the commutative case), that is, \(\mathcal{O}_\tau(E) = \mathcal{O}(E)\), then \(\mathcal{O}(E)\) maps \(\mathcal{H}_4\)-equivariant to a commutative Koszul algebra such that the following diagram of \(\mathcal{H}_4\)-maps is commutative:

\[
\begin{array}{c}
S(V) \xrightarrow{\phi} \mathcal{O}(E) \\
\downarrow \\
S(V)/(v_{0,0}^2, v_{1,0}^2, v_{0,1}^2, v_{1,1}^2)
\end{array}
\]
The Koszul dual of \( S(V)/(v_{0,0}^2, v_{0,1}^2, v_{1,0}^2, v_{1,1}^2) \) is an AS-regular Clifford algebra 

\[
(S(V)/(v_{0,0}^2, v_{0,1}^2, v_{1,0}^2, v_{1,1}^2)) = \mathbb{C}(v_{0,0}, v_{0,1}, v_{1,0}, v_{1,1})/([v_{i,j}, v_{k,l}]_+ \text{ if } (i, j) \neq (k, l)),
\]

which we denote by \( S_{-1}(V^*) \) and the above diagram can be dualized

\[
\begin{array}{ccc}
S(V) & \overset{\pi(E)}{\longrightarrow} & O(E) \\
\downarrow \quad & & \downarrow \\
S(V)/(v_{0,0}^2, v_{0,1}^2, v_{1,0}^2, v_{1,1}^2) & \overset{(\cdot)'}{\longrightarrow} & S_{-1}(V^*)
\end{array}
\]

In addition, it is also true that \( S_{-1}(V^*) \cong \mathcal{H}_4 \). We say that the couple \((S(V), S_{-1}(V^*))\) forms an \( \mathcal{H}_4 \)-duality.

There was hope that this \( \mathcal{H}_4 \)-duality could be extended to the general Sklyanin setting by the existence of an AS regular \( \mathcal{H}_4 \)-algebra \( B(A) \) for each Sklyanin algebra \( A \) such that \( B(A) \cong \mathcal{H}_4 \).

Unfortunately, we will show in this paper that \( B(A) \) does not exist, except in the case that \( \tau \) is of order two (in which case the Sklyanin algebra is a twist of the polynomial ring by an automorphism of order two).

2 Preliminaries

In this paper, we will use the following notations and conventions.

- If \( G \overset{\phi}{\longrightarrow} \text{GL}(V) \) is a representation of \( G \), then \( \overline{\phi} \) is the image of \( G \) in \( \text{PGL}(V) \) under the natural composition \( G \overset{\phi}{\longrightarrow} \text{GL}(V) \longrightarrow \text{PGL}(V) \).
- For \( \phi \in \text{PGL}(V) \), \( \overline{\phi} \) is the image of \( \phi \) in \( \text{PGL}(V) \).
- If \( V \) and \( W \) are two representations of a reductive group \( G \), we denote the fact that \( V \) is isomorphic to \( W \) as a \( G \)-representation by \( V \cong_G W \).
- If \( V \) is a representation of \( G \), then \( \rho_V \) is the associated group morphism \( G \longrightarrow \text{GL}(V) \).
- All the graded algebras \( A \) studied here are connected \((A \cong \mathbb{C})\) and finitely generated in degree 1.
- If \( A \) is a positively graded algebra and \( G \) a reductive group that acts on \( A \) as degree-preserving automorphisms (that is, \( G \subset \text{Aut}_g(A) \)), then we call \( A \) a \( G \)-algebra.
- For two graded \( G \)-algebras \( A \) and \( B \), we denote \( A \cong_G B \) if the following is true:

\[
\forall n \in \mathbb{N} : A_n \cong_G B_n.
\]
• If $G$ is a reductive group, $C \subset G$ a conjugacy class and $A$ a $G$-algebra, then we denote the character series as defined in [6, Chapter 7] by

$$\text{Ch}_A(C,t) := \sum_{n=0}^{\infty} \chi_{A_n}(g)t^n$$

for some $g \in C$.

• If $\lambda \in \mathbb{C}^*$ and $p = [s : t] \in \mathbb{P}^1$, then $\lambda p := [s : \lambda t]$.

• The element $\omega = \exp(\frac{2\pi i}{8}) \in \mathbb{C}$ is a fixed primitive 8th root of unity.

• On $\mathbb{P}^1$, we set $0 := [1 : 0]$ and $\infty := [0 : 1]$.

• For $v, w \in V$ with $V$ any $\mathbb{C}$-vector space, we set $v \approx w$ if $v = \lambda w$ for some $\lambda \neq 0$.

• For $V$ a finite dimensional $\mathbb{C}$-vector space, we set

\begin{align*}
\mathcal{T}(V) &:= \bigoplus_{n=0}^{\infty} V^{\otimes n}, \text{ the free associative algebra generated by the elements of } V. \\
\mathcal{S}(V) &:= \bigoplus_{n=0}^{\infty} S^n(V), \text{ the polynomial algebra generated by the elements of } V. \\
\wedge(V) &:= \bigoplus_{n=0}^{\dim V} \wedge^n V, \text{ the wedge algebra generated by the elements of } V.
\end{align*}

3  The Heisenberg group of order 64

Before we start the classification, we will review the group structure and the representation theory of $\tilde{H}_4$. One can deduce everything from the description of $\tilde{H}_4$ by its natural generators and relations, which is left as an exercise for the reader. Some information can be found in [3, Section 7.1].

3.1 Group structure

3.1.1 Definition

The group $\tilde{H}_4$ has generators and relations

$$\tilde{H}_4 = \langle e_1, e_2, z : [e_1, e_2] = z, z \text{ central}, e_1^4 = e_2^4 = z^2, z^4 = 1 \rangle.$$ 

The center of $\tilde{H}_4$ is equal to $\langle z \rangle \cong \mathbb{Z}_4$. As such, we have the exact sequence

$$\mathbb{Z}_4 \longrightarrow \tilde{H}_4 \longrightarrow \mathbb{Z}_4 \times \mathbb{Z}_4,$$

which makes $\tilde{H}_4$ a non-trivial central extension of $\mathbb{Z}_4 \times \mathbb{Z}_4$ with $\mathbb{Z}_4$. There is an isomorphism of commutative groups $\langle e_1^2 z^{-1}, e_2^2 z^{-1}, z^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

The following diagram is commutative, with exact rows and columns, for $Q_8$ the quaternion group of order 8:

\[
\begin{array}{c}
\mathbb{Z}_2 \\
\downarrow \\
\mathbb{Z}_4 \\
\downarrow \\
\mathbb{Z}_2 \\
\downarrow \\
Q_8 \\
\end{array} \xrightarrow{\quad} \begin{array}{c}
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \\
\downarrow \\
\mathbb{Z}_4 \times \mathbb{Z}_4 \\
\downarrow \\
\mathbb{Z}_2 \times \mathbb{Z}_2 \\
\end{array} \xrightarrow{\quad} \\
\end{array}
\]

In $\langle e_1^2 z^{-1}, e_2^2 z^{-1}, z^2 \rangle$, we set $\mathbb{V}_4 := \langle e_1^2 z^{-1}, e_2^2 z^{-1} \rangle$. 

4
3.2 Action of $\text{GL}_2(\mathbb{Z}_4)$

The following three endomorphisms of $\tilde{H}_4$ (which we abbreviate by the images of the generators $e_1$ and $e_2$ of $\tilde{H}_4$) are automorphisms of $\tilde{H}_4$:

$$\begin{array}{ccc}
\psi_S & e_1 e_2 & e_2 \\
\psi_T & e_2 & e_1 \\
\psi_U & e_2 & e_1 \\
\end{array}$$

These three automorphisms are lifts of

$$S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \text{Aut}(\mathbb{Z}_4 \times \mathbb{Z}_4) \cong \text{GL}_2(\mathbb{Z}_4),$$

The automorphisms $\psi_S$, $\psi_T$ and $\psi_U$ together with the subgroup $\text{Inn}(\tilde{H}_4) \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ generate $\text{Aut}(\tilde{H}_4)$ by [5] Proposition 7.12.

From this it follows that the induced action of $\text{GL}_2(\mathbb{Z}_4)$ on $\mathbb{Z}_4 = \langle z \rangle$ is just the determinant, for

$$[e_1^a e_2^b z^m, e_1^c e_2^d z^n] = z^{ad-bc}.$$ 

3.3 Conjugacy classes

This group has 22 conjugacy classes. These can be easily calculated using the presentation of $\tilde{H}_4$ by its natural generators and relations:

- one conjugacy class $\{z^k\}$ for each element $k \in \mathbb{Z}_4$, leading to 4 conjugacy classes with one element,
- one conjugacy class $C_{a,b} = \{e_1^a e_2^b, e_1^c e_2^d z, e_1^e e_2^f z^2, e_1^g e_2^h z^3\}$ for each element $(a,b) \in \mathbb{Z}_4 \times \mathbb{Z}_4 \setminus 2\mathbb{Z}_4 \times 2\mathbb{Z}_4$, leading to 12 conjugacy classes, each with 4 elements of order 8,
- one conjugacy class $C_{a,b}^1 = \{e_1^a e_2^b, e_1^c e_2^d z^2\}$ for each element $(a,b) \in 2\mathbb{Z}_4 \times 2\mathbb{Z}_4 \setminus \{(0,0)\}$, leading to 3 conjugacy classes, each with 2 elements of order 4, and finally
- one conjugacy class $C_{a,b}^2 = \{e_1^a e_2^b z, e_1^c e_2^d z^3\}$ for each element $(a,b) \in 2\mathbb{Z}_4 \times 2\mathbb{Z}_4 \setminus \{(0,0)\}$, leading to 3 conjugacy classes, each with 2 elements of order 2.

3.4 Representation theory

3.4.1 One-dimensional representations

The one-dimensional representations are representations of $\tilde{H}_4/(\langle \tilde{H}_4, \tilde{H}_4 \rangle) = \tilde{H}_4/(\langle z \rangle) \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, consequently there are 16 one-dimensional representations $\chi_{a,b}$:

$$\chi_{a,b}(e_1) = \omega^{2a}, \quad \chi_{a,b}(e_2) = \omega^{2b}.$$ 

3.4.2 Two-dimensional representations

The two-dimensional simple representations of $\tilde{H}_4$ can be deduced from $\tilde{H}_4/(\langle e_1^2 z^{-1}, e_2^2 z^{-1}, z \rangle) = \tilde{H}_4/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \cong Q_8$. From this quotient it follows that there is a two-dimensional simple representation

$$e_1 \mapsto \begin{bmatrix} 0 & \omega^2 \\ \omega^2 & 0 \end{bmatrix}, \quad e_2 \mapsto \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^6 \end{bmatrix}.$$
Call this representation $W_{0,0}$. One checks that

$$W_{0,0} \otimes \chi_{a,b} \cong \mathcal{H}_4 W_{0,0} \Leftrightarrow (a,b) \in 2\mathbb{Z}_4 \times 2\mathbb{Z}_4.$$  

This implies that there are three more two-dimensional simple representations

$$
\begin{align*}
W_{1,0} &= W_{0,0} \otimes \chi_{1,0} \cong \mathcal{H}_4 W_{1+2k,2l}, \\
W_{0,1} &= W_{0,0} \otimes \chi_{0,1} \cong \mathcal{H}_4 W_{2k,1+2l}, \\
W_{1,1} &= W_{0,0} \otimes \chi_{1,1} \cong \mathcal{H}_4 W_{1+2k,1+2l},
\end{align*}
$$

They can also be recognized by the following rule: let $\psi_{a,b}$, $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ be the four one-dimensional representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ defined by

$$
\psi_{a,b}(1,0) = (-1)^a, \quad \psi_{a,b}(0,1) = (-1)^b.
$$

Then one has $W_{a,b} \mid_{\mathcal{V}_4} \cong \mathcal{V}_4 \otimes_{\mathbb{C}} \psi_{a,b}$.  

In addition, one has $W_{a,b} \wedge W_{a,b} \cong \mathcal{H}_4 \chi_{2a,2b}$.

### 3.4.3 Four-dimensional representations

The four-dimensional representations can be defined in the following way: define the following representation $V$ of $\mathcal{H}_4$

$$
e_1 \mapsto \begin{bmatrix} 0 & \omega & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \\ \omega & 0 & 0 & 0 \end{bmatrix}, \quad e_2 \mapsto \begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^3 & 0 & 0 \\ 0 & 0 & \omega^5 & 0 \\ 0 & 0 & 0 & \omega^7 \end{bmatrix}.
$$

Then it is an easy exercise to see that $V$ is simple and that $V \not\cong \mathcal{H}_4 V^*$. Consequently, $V$ and $V^*$ are nonequivalent simple representations.

One can check that $V$ is equivalent to the following representation

$$
e_1 \mapsto \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^3 & 0 & 0 & 0 \\ 0 & \omega^6 & 0 & 0 \end{bmatrix}, \quad e_2 \mapsto \begin{bmatrix} 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \end{bmatrix}
$$

and $V^*$ to

$$
e_1 \mapsto \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^6 & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \end{bmatrix}, \quad e_2 \mapsto \begin{bmatrix} 0 & \omega^6 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \omega^6 & 0 \end{bmatrix}
$$

From this, one deduces that $V \mid_{\mathcal{V}_4} \cong \mathcal{V}_4 V^* \mid_{\mathcal{V}_4} \cong \mathcal{V}_4 \otimes_{\mathbb{C}} \mathbb{C}_{a,b=0} \psi_{a,b}$.

### 3.4.4 Tensor decompositions

Using the restriction of each representation to the center $\{z\}$ and/or to $\mathcal{V}_4$, one finds the following decompositions for the tensor products.

| $\otimes$ | $\chi_{a,b}$ | $W_{a,b}$ | $V$ | $V^*$ |
|------------|--------------|-----------|-----|------|
| $\chi_{a',b'}$ | $\chi_{a+a',b+b'}$ | $W_{a+a',b+b'}$ | $(V^*) \otimes_2 V$ | $(V^*) \otimes_2 V^*$ |
| $W_{a',b'}$ | $W_{a+a',b+b'}$ | $\oplus_{i,j=0}^{3} \mathbb{C}_{2i+(a+a'),2j+(b+b')}$ | $\oplus_{i,j=0}^{3} W_{i,j}$ | $\oplus_{i,j=0}^{3} W_{i,j}$ |
| $V$ | $V$ | $V \otimes_2 V$ | $\oplus_{i,j=0}^{3} \mathbb{C}_{i,j}$ | $\oplus_{i,j=0}^{3} \mathbb{C}_{i,j}$ |
| $V^*$ | $V^*$ | $V^* \otimes_2 V$ | $\oplus_{i,j=0}^{3} \mathbb{C}_{i,j}$ | $\oplus_{i,j=0}^{3} \mathbb{C}_{i,j}$ |
3.4.5 Action of $GL_2(\mathbb{Z}_4)$ on $\widetilde{H}_4 - \text{rep}$

The action of $\text{Aut}(\widetilde{H}_4)$ on $\widetilde{H}_4 - \text{rep}$ induces a right action of $\text{Out}(\widetilde{H}_4) = GL_2(\mathbb{Z}_4)$ on $\widetilde{H}_4 - \text{rep}$ in the following way: if $\mathfrak{A} \in GL_2(\mathbb{Z}_4)$, $A \in \text{Aut}(\widetilde{H}_4)$ is a lift of $\mathfrak{A}$ and $\widetilde{H}_4 \xrightarrow{\rho} GL_n(\mathbb{C})$ is a representation of $\widetilde{H}_4$, then $\rho \cdot A$ is the composition

$$\widetilde{H}_4 \xrightarrow{A} \widetilde{H}_4 \xrightarrow{\rho} GL_n(\mathbb{C}),$$

which is a (possibly non-isomorphic to $\rho$) $\widetilde{H}_4$-representation. This action of $GL_2(\mathbb{Z}_4)$ is well-defined, for the action of $\text{Inn}(\mathbb{Z}_4)$ fixes every $\widetilde{H}_4$-representation up to equivalence. From this, we can describe the $GL_2(\mathbb{Z}_4)$-orbits of all the simple representations.

- The action of $GL_2(\mathbb{Z}_4)$ on the character group of $\widetilde{H}_4$ is the action of $GL_2(\mathbb{Z}_4)$ by right multiplication on row vectors.

  From this it follows that the orbits are $O(\chi_{0,0}) = \{\chi_{0,0}\}$ consisting of 1 element, $O(\chi_{1,0}) = \{\chi_{a,b} : (0,0) \neq (a,b) \in 2\mathbb{Z}_4 \times 2\mathbb{Z}_4\}$ consisting of 3 elements (coming from the group morphism $GL_2(\mathbb{Z}_4) \rightarrow GL_2(\mathbb{Z}_2) \cong S_3$) and the orbit $O(\chi_{1,0}) = \{\chi_{a,b} : (a,b) \not\in 2\mathbb{Z}_4 \times 2\mathbb{Z}_4\}$ consisting of 12 elements.

- By the association $W_{k,l} \wedge W_{k',l'} \cong \chi_{2k,2l}$ for $k,l = 0,1$ and the description of the $SL(\mathbb{Z}_4)$-orbits of the one-dimensional representations, it follows that $O(W_{0,0}) = \{\chi_{0,0}\}$ is an orbit consisting of 1 element and $O(W_{1,0}) = \{W_{1,0}, W_{0,1}, W_{1,0}\}$ is an orbit consisting of 3 elements.

- From the fact that the induced action of $GL_2(\mathbb{Z}_4)$ on $(z)$ is the determinant, it follows that $O(V) = \{V, V^*\}$.

In particular, it follows that the stabilizer of $V$ and $V^*$ is $SL_2(\mathbb{Z}_4)$.

4 Graded $\widetilde{H}_4$-algebras

In [2], Chirvasitu and Smith studied the following three-dimensional family of algebras: for $p = (\alpha_1,0,\alpha_0,1,1,\alpha_1) \in \mathbb{C}^3$, take $A(p)$ to be the graded algebra with quadratic relations $R(p)$ in $\mathbb{C}[w_{0,0}, w_{1,0}, w_{0,1}, w_{1,1}]$ with $R(p)$ generated by:

$$\begin{cases} 
[w_{0,0}, w_{i,j}]_+ + \alpha_{1,i}[w_{j,i+j}, w_{i+j,j}]_+ + [w_{j,i+j}, w_{i+j,j}] - [w_{0,0}, w_{i,j}]_+ , & (i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0,0)\}.
\end{cases}$$

In [10] it was remarked that $\widetilde{H}_4$ acts on $A(p)$ as degree-preserving algebra automorphisms such that $A(p)_1 \cong \widetilde{H}_4 V$ if and only if $\alpha_1,0 \alpha_0,1,1,\alpha_1 \neq 0$. Consequently, this is true for most algebras corresponding to points on the Sklyanin surface $V(\alpha_1,0 + \alpha_0,1 + \alpha_1,1 + \alpha_0,0,1,0,1,1)$, in particular for the Sklyanin algebras themselves. However, $D(\alpha_1,0 \alpha_0,1,1,\alpha_1)$ does not parametrize all quadratic quotients $A = T(V)/(R)$ of $T(V)$ such that $A_2 \cong \widetilde{H}_4 S^2(V)$ or equivalently, all quadratic algebras with relations $R \cong \widetilde{H}_4 V \wedge V$.

From now on, we will look at quadratic quotients of $T(V)$ that lead to $\widetilde{H}_4$-algebras $A$ such that

- $A_1 \cong \widetilde{H}_4 V$, and
• $A_2 \cong \tilde{H}_4 S^2(V)$.

It is clear that such algebras are quotients of $T(V)$ by quadratic relations $R \subset V \otimes V$, $R \cong \tilde{H}_4 V \wedge V$.

4.1 The moduli space

We are looking for all algebras with the following conditions:

• $A = T(V)/(R)$ is an $\tilde{H}_4$-algebra with $R$ quadratic,

• $R \cong \tilde{H}_4 V \wedge V$.

Using the representation theory of $\tilde{H}_4$, we find that

\[ V \wedge V \cong \tilde{H}_4 W_{1,0} \oplus W_{0,1} \oplus W_{1,1}, \]
\[ V \otimes V \cong \tilde{H}_4 W_{0,0}^{\otimes 2} \oplus W_{0,1}^{\otimes 2} \oplus W_{1,1}^{\otimes 2} \oplus W_{0,0}^{\otimes 2}. \]

Consequently, if $R$ is a subspace of $V \otimes V$ isomorphic to $\tilde{H}_4$, then

\[ R \in \{ W \subset V \otimes V : W \cong \tilde{H}_4 V \wedge V \} \cong \mathbb{P}^1_{[s_1 : t_1]} \times \mathbb{P}^1_{[s_0 : t_0]} \times \mathbb{P}^1_{[s_1 : t_1]}, \]

with $(p_{1,0}, p_{0,1}, p_{1,1}) = ([s_1 : t_1], [s_0 : t_0], [s_1 : t_1])$ corresponding to the relations $R := R(p) = W_{1,0}(p) \oplus W_{0,1}(p) \oplus W_{1,1}(p)$, with

\[ W_{i,j}(p) = C_1 r_{i,j} \oplus C_2 r_{i,j}, \]
\[ W_{i,j}(p) = C_1 r_{i,j}^* \oplus C_2 r_{i,j}^*, \]

for $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0, 0)\}$. Consequently, $W_{i,j}(p) \cong \tilde{H}_4 W_{i,j}$. We set $A(p) = T(V)/(R(p))$.

Similarly, we will set $A^*(p)$ to be the quotient of $T(V^*)$ by the quadratic relations

\[ W_{i,j}(p) = C_1 r_{i,j}^* \oplus C_2 r_{i,j}^*, \]

for $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0, 0)\}$.

**Remark 4.1.** One might say that we are studying quadratic $\tilde{H}_4$-algebras with level 4-structure, that is, quadratic $H_4$-algebras $A$ with a fixed basis $\{v_{0,0}, v_{1,0}, v_{0,1}, v_{1,1}\}$ for $A_1$ such that $H_4$ acts on these vectors as

\[ e_1 \cdot v_{0,0} = \omega^2 v_{0,0}, \quad e_1 \cdot v_{1,0} = \omega v_{1,1}, \quad e_1 \cdot v_{0,1} = v_{0,0}, \quad e_1 \cdot v_{1,1} = v_{1,0}, \]
\[ e_2 \cdot v_{0,0} = v_{1,0}, \quad e_2 \cdot v_{1,0} = \omega^2 v_{0,0}, \quad e_2 \cdot v_{0,1} = \omega v_{1,1}, \quad e_2 \cdot v_{1,1} = -v_{0,1}. \]

4.1.1 The action of $SL_2(\mathbb{Z}_4)$

**Lemma 4.2.** The three automorphisms of $\mathbb{P}^1_{[s_0 : t_0]} \times \mathbb{P}^1_{[s_1 : t_1]}$ are

\[ \phi_1(p_{1,0}, p_{0,1}, p_{1,1}) = (-p_{1,0}, p_{0,1}, p_{1,1}), \]
\[ \phi_2(p_{1,0}, p_{0,1}, p_{1,1}) = (-ip_{0,1}, -ip_{1,0}, ip_{1,1}), \]
\[ \phi_3(p_{1,0}, p_{0,1}, p_{1,1}) = (p_{0,1}, p_{1,1}, p_{1,0}). \]

generate a group $H$ isomorphic to $SL_2(\mathbb{Z}_4)$ such that for $p, q \in (\mathbb{P}^1)^3$ the following is true:

\[ p \in O_H(q) \Rightarrow A(p) \cong A(q). \]
Proof. The fact that $H$ has order 48, follows from the exact sequence
\[
1 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow H \rightarrow S_3 \rightarrow 1,
\]
with $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ the group generated by $\phi_1$ and its conjugates under $\phi_3$.

The next table shows the new variables $w_{0,0}$, $w_{1,0}$, $w_{0,1}$ and $w_{1,1}$ one has to take in order to check whether $\mathcal{A}(p)$ and $\mathcal{A}(\phi_1(p))$, $\mathcal{A}(\phi_2(p))$ and $\mathcal{A}(\phi_3(p))$ are isomorphic.

|      | $\phi_1$ | $\phi_2$ | $\phi_3$ |
|------|---------|---------|--------|
| $w_{0,0}$ | $v_{0,0}$ | $v_{0,0}$ | $v_{0,0}$ |
| $w_{1,0}$ | $v_{1,0}$ | $v_{1,0}$ | $v_{1,0}$ |
| $w_{0,1}$ | $\omega^2 v_{0,1}$ | $v_{0,1}$ | $v_{0,1}$ |
| $w_{1,1}$ | $\omega^2 v_{1,1}$ | $v_{1,1}$ | $v_{1,0}$ |

The fact that $H$ is isomorphic to $SL_2(\mathbb{Z}_4)$ follows from $\text{Stab}_{GL_2(\mathbb{Z}_4)}(V) = SL_2(\mathbb{Z}_4)$, or equivalently, as $\rho_V$ is faithful, $N_{PGL_4(\mathbb{C})}(H_4)/H_4 \cong SL_2(\mathbb{Z}_4)$.

From this one deduces that $A^3_{(\alpha_{0,0},\alpha_{0,1},\alpha_{1,1})} = D(s_{1,0}s_{0,1}s_{1,1})/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$. However, one has to be careful: the algebra corresponding to $(\alpha_{1,0},\alpha_{0,1},\alpha_{1,1}) = (0,0,0)$ is not the commutative polynomial ring, in fact $H_4$ does not even act on this algebra.

From the relation $\alpha_{1,0} + \alpha_{0,1} + \alpha_{1,1} + \alpha_{1,0}\alpha_{0,1}\alpha_{1,1}$ describing the (Zariski-closure) of the surface parametrizing Sklyanin algebras, one deduces

**Proposition 4.3.** The Sklyanin surface (the Zariski closure in $\mathbb{P}^1_{[s_{1,0}:t_{1,0}]} \times \mathbb{P}^1_{[s_{0,1}:t_{0,1}]} \times \mathbb{P}^1_{[s_{1,1}:t_{1,1}]}$ of the variety parametrizing Sklyanin algebras) is the surface

\[
S = V((t_{1,0}s_{0,1}s_{1,1})^2 + (s_{1,0}s_{0,1}s_{1,1})^2 + (s_{1,0}s_{0,1}t_{1,1})^2 + (t_{1,0}s_{0,1}t_{1,1})^2).
\]

Looking at the 8 standard open subsets of $\mathbb{P}^1_{[s_{1,0}:t_{1,0}]} \times \mathbb{P}^1_{[s_{0,1}:t_{0,1}]} \times \mathbb{P}^1_{[s_{1,1}:t_{1,1}]}$, we see that $S$ has singularities in the points $(0,0,0)$, $(0,\infty,\infty)$, $(\infty,0,\infty)$ and $(\infty,\infty,0)$. The point $(0,0,0)$ corresponds to $S(V)$, the other 3 points correspond to Zhang twists of $S(V)$ with elements of $\mathbb{P}_4$, which will follow from the next part.

### 4.1.2 Zhang twists

Recall the following from [16]: if $A$ is graded algebra and $\phi$ is an automorphism of $A$ that is degree-preserving, then the Zhang twist of $A$ with $\phi$ is the algebra $A^\phi$ with underlying graded vector space $A$ and multiplication rule for homogeneous elements

\[
a *_\phi b = a\phi^n(b) \text{ if } \deg(a) = n,
\]

and distributively extended for general elements. In the case that $A$ is an AS-regular algebra, then $A^\phi$ is also AS-regular.

**Theorem 4.4.** If $G$ is a reductive group that acts degree-preserving on a graded algebra $A = T(V)/(\mathcal{R})$ and $\phi$ is a degree-preserving algebra automorphism of $A$ such that

\[
\forall g \in G \exists \lambda \in \mathbb{C}^* : [\phi, g] = \lambda I,
\]

or equivalently, if $\overline{\phi}$ commutes with $\overline{g}$ in $\text{PGL}(V)$, then $G$ also acts on $A^\phi$ as degree-preserving algebra automorphisms.
Consequently, there is a rational action of \( \omega \) respectively corresponding to algebras which are twist equivalent.

so that \( \tilde{w} \), showing that \( w \) is a relation of \( A^\phi \) if and only if for every \( g \in G \), \( g \cdot w \) is a relation of \( A^\phi \).

While this shows that \( G \) acts on \( A^\phi \), this does not imply that \( A \cong_A A^\phi \) in general, as the following example shows.

**Example 4.5.** Let \( G = D_4 = \langle s, t : s^4 = t^2 = 1, tst = s^3 \rangle \) and \( A = S(V) \) with \( V \) the unique two-dimensional simple representation of \( D_4 \). Then \( \phi = s \) has the property that \( \mathcal{S} \) in \( \text{PGL}_2(\mathbb{C}) \) commutes with \( G \). In \( A^\phi \), one checks that for \( x \) and \( y \) two eigenvectors of \( s \) with eigenvalues \( \omega^2 \), respectively \( \omega^3 \), that

\[
    x \ast \phi y + y \ast \phi x = \omega^3 xy - \omega^2 yx = 0,
\]

so that \( A^\phi = \mathbb{C}(x, y)/(xy + yx) \). However, \( \mathbb{C}(xy + yx) \not\cong_{D_4} \mathbb{C}(xy - yx) \), so that \( A^\phi \not\cong_{D_4} A \).

**Proposition 4.6.** Twisting the algebra \( A(p) \) with one of the non-trivial elements of \( \mathcal{V}_4 \) gives the following algebras:

\[
\begin{array}{c|c|c}
    e_1^2z^{-1} & A(p_{1,0}^{-1}, -p_{0,1}, -p_{1,1}^{-1}) \\
    e_2^2z^{-1} & A(-p_{1,0}, -p_{0,1}, p_{1,1}) \\
    e_3^2e_2^2z^2 & A(-p_{1,0}, p_{0,1}, -p_{1,1}) \\
\end{array}
\]

Consequently, there is a rational action of \( \mathcal{V}_4 \) on \((\mathbb{P}^1)^3\) with elements in the same orbit corresponding to algebras which are twist equivalent.

**Proof.** It is enough to show this for \( e_1^2z^{-1} \) and \( e_2^2z^{-1} \), for the twist with \( e_3^2e_2^2z^2 \) is the composition.

Set \( [x, y]_\pm = x \ast y \pm y \ast x \).

For \( \phi = e_1^2z^{-1} \), we find

\[
\begin{align*}
    t_{1.0}[v_{0,0}, v_{1,0}]^+ + s_{1.0}[v_{0,1}, v_{1,1}]^+ &= -t_{1.0}[v_{0,0}, v_{1,0}]^+ - s_{1.0}[v_{0,1}, v_{1,1}]^- = 0, \\
    s_{0.1}[v_{0,0}, v_{1,0}]^- - t_{0.1}[v_{1,1}, v_{0,0}]^+ &= s_{0.1}[v_{0,0}, v_{1,0}]^- + t_{0.1}[v_{1,1}, v_{0,0}]^+ = 0, \\
    t_{1.1}[v_{0,0}, v_{1,1}]^- - s_{1.1}[v_{0,1}, v_{1,0}]^+ &= -t_{1.1}[v_{0,0}, v_{1,1}]^- - s_{1.1}[v_{0,1}, v_{1,0}]^+ = 0,
\end{align*}
\]

which is sufficient by Theorem 4.3, as we know that \( \tilde{H}_4 \) has to act on this Zheng twist.

For \( \phi = e_2^2z^{-1} \), we find

\[
\begin{align*}
    s_{1.0}[v_{0,0}, v_{1,0}]^- - t_{1.0}[v_{0,1}, v_{1,1}]^+ &= s_{1.0}[v_{0,0}, v_{1,0}]^- + t_{1.0}[v_{0,1}, v_{1,1}]^+ = 0, \\
    t_{0.1}[v_{0,0}, v_{1,0}]^- - s_{0.1}[v_{1,1}, v_{0,0}]^+ &= -t_{0.1}[v_{0,0}, v_{1,0}]^- - s_{0.1}[v_{1,1}, v_{0,0}]^+ = 0, \\
    t_{1.1}[v_{0,0}, v_{1,1}]^- + s_{1.1}[v_{1,0}, v_{0,1}]^+ &= -t_{1.1}[v_{0,0}, v_{1,1}]^- + s_{1.1}[v_{1,0}, v_{0,1}]^+ = 0,
\end{align*}
\]

finishing the proof. \( \square \)
This action of $\mathbb{V}$ on $(\mathbb{P}^1)^3$ divides the classical $2^3 = 8$ open subsets of this projective variety in two sets:

- the open subsets $D(s_{0,1}t_{0,1}s_{1,1}), D(s_{1,0}t_{0,1}t_{1,1}), D(t_{1,0}t_{0,1}s_{1,1})$, and $D(t_{1,0}t_{0,1}t_{1,1})$.
- the open subsets $D(t_{1,0}t_{0,1}t_{1,1}), D(t_{1,0}s_{0,1}s_{1,1}), D(s_{1,0}t_{0,1}s_{1,1})$, and $D(s_{1,0}t_{0,1}t_{1,1})$.

Therefore, it is enough to find all the points in $U_s = D(s_{1,0}s_{0,1}s_{1,1})$ and $U_t = D(t_{1,0}t_{0,1}t_{1,1})$ that correspond to Artin-Schelter regular algebras, for the other points can be found using the twisting action of $\mathbb{V}$, which preserves Artin-Schelter regularity by [16]. However, we can even do better:

**Proposition 4.7.** For every $p \in (\mathbb{P}^1)^3$, $p \neq (\infty, \infty, \infty), (\infty, 0, 0), (0, \infty, 0)$ or $(0, 0, \infty)$, one has $\mathcal{O}_U(p) \cap U_s \neq \emptyset$.

**Proof.** If $p = (p_{1,0}, p_{0,1}, p_{1,1})$ with none of the coordinates equal to 0 or $\infty$, then the proposition is trivially true. Assume that for example $p_{1,0} = \infty$, then either $e_1^2z^{-1} \cdot p$ or $e_1^2e_2^2 \cdot p$ lies in $U_s$, except if $p_{0,1} = p_{1,1} = 0$. If this is the case, then $e_2^2z^{-1} \cdot p = (\infty, \infty, \infty)$. By symmetry, the other cases are similar.

Consequently, if we classify all the points $p \in U_s$ such that $\mathcal{A}(p)$ has the correct Hilbert series up to degree 3, then we have found all possible AS-regular algebras by adding the orbits under $\mathbb{V}$ and adding $\mathcal{O}_{U_s}(p)$, as the algebra $\mathcal{A}(\infty, \infty, \infty)$ is indeed regular.

Therefore, we may assume that we are working on $U_s$ and set $s_{1,0} = s_{0,1} = s_{1,1} = 1$.

### 4.2 Special quadratic relations of $\mathcal{A}(p)$

**Proposition 4.8.** Let $R_{i,j} \cong H_4$ be a $H_4$-subrepresentation of $V \otimes V$ and take $p = (t_{1,0}, t_{0,1}, t_{1,1}) \in U_s$. Set

$$
\begin{align*}
1w_{i,j} &= 2t_{i,j}v_{0,0}v_{i,j} + (t_{i,j}^2 + 1)v_{j,i+j}v_{i,j} + (t_{i,j}^2 - 1)v_{j,i+j}v_{i,j+i}, \\
2w_{i,j} &= 2t_{i,j}v_{i,j}v_{0,0} - (t_{i,j}^2 - 1)v_{j,i+j}v_{i,j} - (t_{i,j}^2 + 1)v_{j,i+j}v_{i,j+i}.
\end{align*}
$$

Then the following three statements are equivalent:

- $R_{i,j} = W_{i,j}(p)$,
- $1w_{i,j} \in R_{i,j}$,
- $2w_{i,j} \in R_{i,j}$.

**Proof.** By a standard computation, we find that

$$
\begin{align*}
1w_{i,j} &= t_{i,j}r_{i,j} + 2r_{i,j}, \\
2w_{i,j} &= t_{i,j}r_{i,j} + 2r_{i,j},
\end{align*}
$$

proving that $1w_{i,j}$ and $2w_{i,j}$ are elements of $W_{i,j}(p)$. But then

$$
\mathbb{C}(H_4) \cdot 1w_{i,j} = \mathbb{C}(H_4) \cdot 2w_{i,j} = W_{i,j}(p),
$$

as $W_{i,j}(p)$ is simple.

The elements $1w_{i,j}$ and $2w_{i,j}$ can be characterized as the only elements (up to scalar multiplication) in $W_{i,j}(p)$ fulfilling

$$
V \otimes \mathbb{C}v_{0,0} \cap \mathbb{C}1w_{i,j} = \mathbb{C}v_{0,0} \otimes V \cap \mathbb{C}2w_{i,j} = 0.
$$
5 The main theorem

Theorem 5.1. Let $\mathcal{A}(p) = T(V)/\langle \mathcal{R}(p) \rangle$ be an Artin-Schelter regular algebra. Then one of the following is true:

- $p \in S$, that is, $\mathcal{A}(p)$ is a Skylanin algebra or a degeneration of Skylanin algebras, or
- $p = (\infty, \infty, \infty)$, $(\infty, 0, 0)$, $(0, \infty, 0)$ or $(0, 0, \infty)$. In this case $\mathcal{A}(p)$ is a twist of $\mathcal{A}(\infty, \infty, \infty)$ with an element of $\mathcal{V}_4$.

Proof. By proposition 1.6 and the discussion thereafter, it is enough to prove this for the points in $U_s = D(s_1,0, s_0,0, s_1,1)$.

Assume that $\mathcal{A}(p)$ is a quadratic AS-regular algebra. Then

$$\dim \mathcal{R}(p) \otimes V \cap V \otimes \mathcal{R}(p) = \dim(V \cap V) \otimes V \cap V \otimes (V \cap V) = 4$$

and $\mathcal{R}(p) \otimes V \cap V \otimes \mathcal{R}(p)$ is an $\widetilde{H}_4$-subrepresentation of $V^\otimes 3$. By the structure of the representations of $\widetilde{H}_4$, it follows that $V^\otimes 3 \cong H_4 (V^*)^\otimes 16$. Consequently,

$$\mathcal{R}(p) \otimes V \cap V \otimes \mathcal{R}(p) \cong \widetilde{H}_4 V^*.$$

For any algebra $\mathcal{A}(p)$, the same reasoning shows that

$$\mathcal{R}(p) \otimes V \cap V \otimes \mathcal{R}(p) \cong \widetilde{H}_4 (V^*)^\otimes e$$

for some $e \in \mathbb{N}$. As $V^*|_{\mathcal{V}_4} \cong \mathcal{V}_4 \oplus_{a,b=0}^1 \psi_{a,b}$, it follows that

$$e = \dim(\mathcal{R}(p) \otimes V \cap V \otimes \mathcal{R}(p))^\mathcal{V}_4.$$ 

Assume now that $w \in (\mathcal{R}(p) \otimes V \cap V \otimes \mathcal{R}(p))^\mathcal{V}_4$. It is easy to see from the fact that $W_{i,j}|_{\mathcal{V}_4} \cong \mathcal{V}_4 \psi_{i,j}^\otimes 2$ that

$$W_{i,j}(p) \otimes V^\mathcal{V}_4 = W_{i,j}(p) \otimes \mathcal{V}_4.$$ 

From this it follows that

$$\mathcal{C}w \cap \mathcal{C}_{v_0,0} \otimes V \cap V = V \otimes V \otimes \mathcal{C}_{v_0,0} \cap \mathcal{C}w = 0.$$

Assume now that $e \neq 0$, which is certainly true for AS-regular algebras. Then there exists $0 \neq (\gamma_{1,0}, \gamma_{0,1}, \gamma_{1,1}) \in \mathbb{A}^3$ such that

$$w = \gamma_{1,0} v_{1,0} v_{1,0} + \gamma_{0,1} v_{0,1} v_{0,1} + \gamma_{1,1} v_{1,1} v_{1,1},$$

or in matrix form,

$$w = \begin{bmatrix} v_{1,0} & v_{0,1} & v_{1,1} \end{bmatrix} \begin{bmatrix} 2t_{1,0} \gamma_{1,0} v_{0,0} & (t_{1,0}^2 - 1) \gamma_{1,0} v_{1,1} & (1 + t_{1,0}^2) \gamma_{1,0} v_{0,1} \frac{2}{2} \end{bmatrix} \begin{bmatrix} v_{1,0} \frac{2}{2} \end{bmatrix} \begin{bmatrix} v_{0,1} \frac{2}{2} \end{bmatrix}$$

The condition that $w \in \mathcal{R}(p) \otimes V$ is then equivalent to

$$2t_{i,j} \gamma_{i,j} v_{i,j} v_{i,j} v_{i,j} + \frac{2}{2} (1 + t_{i,j}^2) \gamma_{i,j} v_{i,j} v_{i,j} v_{i,j} + \frac{2}{2} t_{i,j}^2 - 1) \gamma_{i,j} v_{i,j} v_{i,j} v_{i,j} \in \mathcal{C}w_{i,j},$$

12
for \((i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_2, (i, j) \neq (0, 0)\). Equivalently, the following three matrices have rank 1:

\[
\begin{bmatrix}
2t_{1,0} \gamma_{1,0} & (1 + t_{1,1}^2) \gamma_{0,1} & (t_{1,1}^2 - 1) \gamma_{1,1} \\
2t_{1,0} & 1 - t_{1,1}^2 & -(1 + t_{1,1}^2) \\
2t_{0,1} \gamma_{0,1} & (1 + t_{1,1}^2) \gamma_{1,1} & (t_{1,1}^2 - 1) \gamma_{0,1} \\
2t_{0,1} & 1 - t_{0,1}^2 & -(1 + t_{0,1}^2) \\
2t_{1,1} \gamma_{1,1} & (1 + t_{1,1}^2) \gamma_{1,1} & (t_{1,1}^2 - 1) \gamma_{0,1} \\
2t_{1,1} & 1 - t_{1,1}^2 & -(1 + t_{1,1}^2)
\end{bmatrix}.
\]

This leads to 9 equations, linear in \((\gamma_{1,0}, \gamma_{0,1}, \gamma_{1,1})\), with a non-zero solution:

\[
\begin{align*}
2t_{1,0}((t_{1,1}^2 - 1) \gamma_{1,0} + (t_{0,1}^2 + 1) \gamma_{0,1}) &= 0, \\
2t_{0,1}((t_{1,1}^2 - 1) \gamma_{0,1} + (t_{1,1}^2 + 1) \gamma_{1,0}) &= 0, \\
2t_{1,1}((t_{1,1}^2 - 1) \gamma_{1,1} + (t_{1,1}^2 + 1) \gamma_{0,1}) &= 0, \\
2t_{1,0}((t_{1,1}^2 - 1) \gamma_{1,0} + (t_{0,1}^2 + 1) \gamma_{1,1}) &= 0, \\
2t_{0,1}((t_{0,1}^2 + 1) \gamma_{0,1} + (t_{1,1}^2 - 1) \gamma_{0,1}) &= 0, \\
2t_{1,1}((t_{1,1}^2 + 1) \gamma_{1,1} + (t_{0,1}^2 - 1) \gamma_{0,1}) &= 0,
\end{align*}
\]

These 9 equations are equivalent to the matrix equation

\[
\begin{bmatrix}
2t_{1,0}(t_{1,1}^2 - 1) & 2t_{1,0}(t_{0,1}^2 + 1) & 0 \\
0 & 2t_{0,1}(t_{0,1}^2 - 1) & 2t_{1,0}(t_{1,1}^2 + 1) \\
2t_{1,1}(t_{1,1}^2 + 1) & 0 & 2t_{1,1}(t_{1,1}^2 - 1) \\
2t_{1,0}(t_{1,1}^2 + 1) & 0 & 2t_{1,1}(t_{1,1}^2 - 1) \\
2t_{0,1}(t_{1,1}^2 + 1) & 0 & 2t_{1,1}(t_{1,1}^2 - 1) \\
0 & 2t_{1,1}(t_{0,1}^2 - 1) & 2t_{1,1}(t_{1,1}^2 + 1) \\
-(t_{0,1}^2 - 1)(t_{1,1}^2 - 1) & 0 & -(t_{0,1}^2 + 1)(t_{1,1}^2 - 1) \\
0 & (t_{1,1}^2 - 1)(t_{0,1}^2 - 1) & -(t_{1,1}^2 + 1)(t_{0,1}^2 + 1) \\
(t_{1,1}^2 + 1)(t_{1,1}^2 + 1) & -(t_{1,1}^2 + 1)(t_{0,1}^2 - 1) & 0
\end{bmatrix}
\begin{bmatrix}
\gamma_{1,0} \\
\gamma_{0,1} \\
\gamma_{1,1}
\end{bmatrix} = 0.
\]

Let \(M(p)\) be this \(9 \times 3\)-matrix with entries in the polynomial ring \(\mathbb{C}[t_{1,0}, t_{0,1}, t_{1,1}]\). Consequently, if \(A(p)\) is AS-regular, then \(p \in \{x \in U_A : \text{rank } M(x) \leq 2\}\). For any choice of pairwise distinct indices \(1 \leq i < j < k \leq 9\), let \(M(p)_{i,j,k}\) be the \(3 \times 3\)-submatrix of \(M(p)\) with rows \(i, j, k\) and columns \(i, j, k\). Assume that \(p \not\in S\).

- From the first three rows, we find

\[
\det(M(p)_{1,2,3}) \approx t_{1,0} t_{0,1} t_{1,1} \left( \prod_{(i,j) \in \mathcal{V}, (i,j) \neq (0,0)} (t_{i,j}^2 - 1) + \prod_{(i,j) \in \mathcal{V}, (i,j) \neq (0,0)} (t_{i,j}^2 + 1) \right)
\]

\[
\approx t_{1,0} t_{0,1} t_{1,1} (t_{1,0}^2 + t_{0,1}^2 + t_{1,1}^2 + (t_{1,0} t_{0,1} t_{1,1})^2).
\]

This implies that, if \(p \not\in S\), then one of the affine coordinates of \(p\) is 0. By symmetry, we may assume \(t_{1,0} = 0\).
• From the last three rows, we find
  \[ \det(M(p)) \approx \prod_{(i,j) \in V_4, (i,j) \neq (0,0)} (t_{i,j}^2 + 1)^2 - \prod_{(i,j) \in V_4, (i,j) \neq (0,0)} (t_{i,j}^2 - 1)^2 \]
  \[ \approx (t_{1,0}^2 + t_{0,1}^2 + t_{1,1}^2 + (t_{0,1}t_{1,1})^2)^2 + (t_{0,1}t_{1,1})^2 + (t_{0,1}t_{1,1})^2 + 1. \]
  Together with the assumption \( t_{1,0} = 0 \) and \( p \not\in S \), this implies that \( (t_{0,1}t_{1,1})^2 + 1 = 0. \)

• Taking now row 2,3 and 7, we find
  \[ \det(M(p)) \approx t_{0,1}t_{1,1}((t_{0,1}^2 - 1)(t_{1,1}^2 - 1) - (t_{0,1}^2 + 1)(t_{1,1}^2 + 1)) \]
  \[ \approx t_{0,1}t_{1,1}(t_{0,1}^2 + t_{1,1}^2). \]

In short, we have to solve the following two equations

\[ \begin{cases} (t_{0,1}t_{1,1})^2 + 1 = 0, \\ t_{0,1}^2 + t_{1,1}^2 = 0. \end{cases} \]

But these equations imply that \( t_{0,1}^2 = \pm 1 \), \( t_{1,1}^2 = \mp 1 \), which implies that \( p \in S \), which is a contradiction.

**Corollary 5.2.** If \( A(p) \) is AS-regular, then \( A(p) \cong \mathcal{H} \text{ } S(V) \).

If \( p \in S \), then \( A(p) \cong \mathcal{H} \text{ } A(0,0,0) \) by \([8, Corollary 2.1.9]\), which only leaves us the four exceptional cases

\[ p \in V = \{ (\infty, \infty, \infty), (\infty, 0, 0), (0, \infty, 0), (0, 0, \infty) \}. \]

For these points, it is easier to find the decomposition of \( A(p) \) and show that \( A(p)^! \cong \mathcal{H} \text{ } V^* \), as this completely determines the decomposition of \( A(p) \) by the formula for character series of Koszul algebras and their duals

\[ \text{Ch}_{A(p)}(C,t) \text{Ch}_{A(p)^!}(C,-t) = 1. \]

It is easy to see that

\[ \wedge V^* \cong \mathcal{H} \chi_{0,0} \oplus V^* \oplus (W_{1,0} \oplus W_{0,1} \oplus W_{1,1}) \oplus V \oplus \chi_{0,0}. \]

Using the representation theory of \( \mathcal{H} \) and the Hilbert series of \( A(p)^! \), one sees that

\[ A^! \cong \mathcal{H} \chi_{0,0} \oplus V^* \oplus (W_{1,0} \oplus W_{0,1} \oplus W_{1,1}) \oplus V \oplus \chi_{1,j}. \]

We have to show that \( (i,j) = (0,0) \). As \( A(p) \) is a quantum algebra, the degree four part is as \( \mathbb{C} \)-vector space generated by either \( e_{0,0}e_{1,0}e_{0,1}e_{1,1} \) or \( e_{0,0}e_{1,0}e_{1,0}e_{1,1} \). In addition, by the relations of \( A(p) \), we have

\[ [v_{0,0}^*, v_{1,0}^*] = 0 \iff [v_{0,1}^*, v_{1,1}^*] = 0, \]
\[ [v_{0,1}^*, v_{0,0}^*] = 0 \iff [v_{1,0}^*, v_{1,1}^*] = 0. \]

From this, one deduces that

\[ e_1 \cdot v_{0,0}v_{0,1}v_{1,0}v_{1,1} = v_{0,0}v_{0,1}v_{1,0}v_{1,1}, \quad e_2 \cdot v_{0,0}v_{1,0}v_{0,1}v_{1,1} = v_{0,0}v_{1,0}v_{0,1}v_{1,1}, \]

showing that \( (i,j) = (0,0) \), as required.
6 Artin-Schelter regular $\widetilde{H}_4$-algebras

The main theorem has the following corollary:

**Corollary 6.1.** If $A$ is a quadratic Artin-Schelter regular $\widetilde{H}_4$-algebra with Hilbert series $(1-t)^{-4}$, then one of the following is true:

- $A$ is (a twist of) a Sklyanin algebra,
- $A$ is (a twist of) a degeneration of Sklyanin algebras, or
- $A$ is (a twist of) $C_{-1}[v_{0,0}, v_{1,0}, v_{0,1}, v_{1,1}]$.

Before we prove this corollary, we need a lemma and a corollary.

**Lemma 6.2.** Let $A = T(V)/(R)$ be a quadratic $\widetilde{H}_4$-algebra and let $W \subseteq R$ be a two-dimensional subrepresentation of $R$, $W \cong_{\widetilde{H}_4} W_{i,j}$. Then $W^{e_1} e_2 \cong W_{i+l,j+k}$ in $R^{e_1} e_2$.

**Proof.** From proposition 1.1 we know that $A^\phi$ is again an $\widetilde{H}_4$-algebra. Secondly, if $W' \cong_{\widetilde{H}_4} W'$ are two two-dimensional subrepresentations of $V \otimes V$, then $W'^\phi \cong_{\widetilde{H}_4} W'^\phi$ for any $\phi \in \widetilde{H}_4$. This defines an action of $\widetilde{H}_4/([\widetilde{H}_4, \widetilde{H}_4]) \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ on the set of two-dimensional simple representations of $\widetilde{H}_4$. Consequently, this gives a homomorphism $\mathbb{Z}_4 \times \mathbb{Z}_4 \rightarrow S_4$. The only commutative 2-subgroups of $S_4$ are $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2$. We will show that the image is $\mathbb{Z}_2 \times \mathbb{Z}_2$, embedded as the unique normal subgroup of order 4 in $S_4$.

To show this, it is enough to take any element in $w \in W^{2}_{0,0} \subset V \otimes V$, calculate $w^{e_1}$, $w^{e_2}$, $w^{e_1}$, $w^{e_2}$ and determine in which $W_{i,j}^{(e_1,e_2)} \subset V \otimes V$ these twists of $w$ lie. For simplicity, take $w = v^{2,0}_{0,0}$. Then find

$$(v^{2,0}_{0,0})^{e_1} \approx v_{0,0}v_{0,1} \in W^{2}_{0,1}, \quad (v^{2,0}_{0,0})^{e_2} \approx v_{0,0}v_{1,0} \in W^{2}_{1,0},$$

showing that the image of $\mathbb{Z}_4 \times \mathbb{Z}_4$ in $S_4$ is a group of order 4, generated by two elements of order 2. As the orbit of $W^{2}_{0,0}$ under this action of $\mathbb{Z}_4 \times \mathbb{Z}_4$ is of order at least 3 (as it contains $W^{2}_{0,0}$, $W^{2}_{1,0}$ and $W^{2}_{0,1}$), it is of order 4 and therefore the image is the unique normal subgroup of order 4 in $S_4$. □

**Corollary 6.3.** Let $A(p) = T(V)/(R(p))$ be a quadratic Artin-Schelter regular $\widetilde{H}_4$-algebra with Hilbert series $(1-t)^{-4}$. Then $R(p)$ is multiplicity-free as an $\widetilde{H}_4$-representation.

**Proof.** If $R(p)$ is not multiplicity free, then $W_{i,j}^{2 \phi} \subseteq R(p)$ for some $(i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. By the previous lemma, we can therefore twist $A$ with an element $\phi \in \widetilde{H}_4$ such that $W_{0,0}^{2 \phi} \subseteq R(p)^\phi$.

But then $v_{0,0}^{2 \phi} \in W_{0,0}^{2 \phi} \subseteq R(p)^\phi$, which implies that $A^\phi$ and therefore $A$ is not a domain. By [3, Theorem 3.9], this is impossible. □

**Proof of Corollary 6.1.** If $A = T(V)/(R)$ is an AS-regular $\widetilde{H}_4$-algebra with Hilbert series $(1-t)^{-4}$, then by the previous corollary its relations decompose as $W_{i,j} \oplus W_{k,j} \oplus W_{m,n}$, the direct sum of three pairwise non-isomorphic two-dimensional simple $\widetilde{H}_4$-representations. If $\text{Hom}_{\widetilde{H}_4}(W_{0,0}, R) = 0$, then we are in the case of theorem 5.1. If $\text{Hom}_{\widetilde{H}_4}(W_{0,0}, R) = 1$, then we can twist $A$ with an element $\phi \in \widetilde{H}_4$ such that $\text{Hom}_{\widetilde{H}_4}(W_{0,0}, R^\phi) = 0$, which implies that we are again in the case of theorem 5.1. □
Corollary 6.4. Let \( \mathcal{O}_\tau(E) \) be the twisted homogeneous coordinate ring of \( E \) associated to \( \tau \in E \). Then \( \mathcal{O}_\tau(E) \) is the quotient of four \( \mathcal{A} \) regular \( \overline{H}_4 \)-algebras. One of these is a Sklyanin algebra \( \mathcal{A}_\tau(E) \), the other three are twists with \( e_1, e_2 \) and \( e_1e_2 \) of three Sklyanin algebras \( \mathcal{A}_{\tau_0,0}(E) \), \( \mathcal{A}_{\tau_1,0}(E) \) and \( \mathcal{A}_{\tau_1,1}(E) \) with the property
\[
\tau - \tau_{1,0}, \tau - \tau_{0,1}, \tau - \tau_{1,1} \in E[4].
\]

Proof. If \( \mathcal{A}_\tau(E) \) is a Sklyanin algebra, then it maps to \( \mathcal{O}_\tau(E) \). But then the twist \( \mathcal{A}_\tau(E)^\phi \) maps to \( \mathcal{O}_\tau(E)^\phi \) (if \( \phi \) descends to an automorphism of \( \mathcal{O}_\tau(E) \), which is the case). If \( \phi \) of order 8 in \( \overline{H}_4 \), then \( \bar{\phi} \) is of order 4, which implies that \( \mathcal{O}_\tau(E)^\phi = \mathcal{O}_{P+\tau}(E) \) for some \( P \in E[4] \), proving the claim.

7 Duality for twisted homogeneous coordinate rings

In this section, we prove the following theorem.

Theorem 7.1. Let \( \mathcal{O}_\tau(E) = \mathcal{A}(p)/(1_{r_0,0}, 2_{r_0,0}) \) be a twisted homogeneous coordinate ring of an elliptic curve \( E \) and a point \( \tau \in E \) (\( \tau \) may be 0), written as the quotient of a Sklyanin algebra \( \mathcal{A}(p) \) by 2 central elements of degree two as in for example [13]. Then \( \mathcal{O}_\tau(E) \) is a Koszul \( \overline{H}_4 \)-algebra and for each \( n \in \mathbb{N} \), we have
\[
(\mathcal{O}_\tau(E))^n \cong \mathcal{H}_4^n (\mathcal{O}_\tau(E))^n.
\]

Proof. As \( E \) is an \( \overline{H}_4 \)-variety, it follows that the ideal of \( \mathcal{A}(p) \) that annihilates all the point modules corresponding to points of \( E \) is an \( \overline{H}_4 \)-ideal. This ideal is equal to \((1_{r_0,0}, 2_{r_0,0})\), showing that \( C_{1r_0,0} \oplus C_{2r_0,0} \) is an \( \overline{H}_4 \)-subrepresentation of \( V \otimes V \). Using for example [10] or [13], we can show that \( C_{1r_0,0} \oplus C_{2r_0,0} \cong \psi_{W_0,0}^{O_4} \) and therefore \( C_{1r_0,0} \oplus C_{2r_0,0} \cong W_{0,0} \). By Hilbert series considerations, the sequence of \( \mathcal{A}(p) \)-modules
\[
0 \longrightarrow \mathcal{A}(p)[-4] \longrightarrow \mathcal{A}(p)[-2]^{\oplus 2} \longrightarrow \mathcal{A}(p) \longrightarrow \mathcal{O}_\tau(E) \longrightarrow 0
\]
is exact.

As \( \mathcal{A}(p) \) is a deformation of \( S(V) \) by \( \overline{H}_4 \)-algebras (that is, both \( p \) and \((0,0,0)\) lie on \( S \)), it follows from [5, Corollary 2.1.9] that \( \mathcal{A}(p) \cong \mathcal{H}_4 S(V) \). From the action of \( \overline{H}_4 \) on \( V \), we therefore find the following character series for \( \mathcal{A}(p) \).

\[
\begin{align*}
\text{Ch}_{\mathcal{A}(p)}(\{z^k\}, t) &= (1 - \omega^{2k}t)^{-4}, \\
\text{Ch}_{\mathcal{A}(p)}(C_{a,b}, t) &= (1 + t^2)^{-1}, \\
\text{Ch}_{\mathcal{A}(p)}(C_{a,b}^1, t) &= (1 + t^2)^{-2}, \\
\text{Ch}_{\mathcal{A}(p)}(C_{a,b}^2, t) &= (1 - t^2)^{-2}.
\end{align*}
\]

From the fact that \( C_{1r_0,0} \oplus C_{2r_0,0} \cong \mathcal{H}_4 W_{0,0} \) and that these relations are of degree two, we can deduce the character series of \( \mathcal{O}_\tau(E) \) from the character series of \( \mathcal{A}(p) \).

\[
\begin{align*}
\text{Ch}_{\mathcal{O}_\tau(E)}(\{z^k\}, t) &= (1 + \omega^{2k}t)^2(1 - \omega^{2k}t)^{-2}, \\
\text{Ch}_{\mathcal{O}_\tau(E)}(C_{a,b}, t) &= (1 - \omega^2t)(1 + \omega^2t)(1 + t^4)^{-1} = 1, \\
\text{Ch}_{\mathcal{O}_\tau(E)}(C_{a,b}^1, t) &= (1 + t^2)(1 + t^2)(1 + t^2)^{-2} = 1, \\
\text{Ch}_{\mathcal{O}_\tau(E)}(C_{a,b}^2, t) &= (1 - t^2)(1 - t^2)(1 - t^2)^{-2} = 1.
\end{align*}
\]
By Koszul duality of $\mathcal{O}_X(E)$ (which follows from [12, Section 3]), it follows that
\[
\text{Ch}_{\mathcal{O}_X(E)}(\{z^k\}, t) = \text{Ch}_{\mathcal{O}_X(E)}(\{z^k\}, -t)^{-1} = (1 + \omega^{-2k}t)^2(1 - \omega^{-2k}t)^{-2},
\]
\[
\text{Ch}_{\mathcal{O}_X(E)}(C_{a,b}, t) = \text{Ch}_{\mathcal{O}_X(E)}(C_{a,b}, -t)^{-1} = 1,
\]
\[
\text{Ch}_{\mathcal{O}_X(E)}(C_{a,b}, t) = \text{Ch}_{\mathcal{O}_X(E)}(C_{a,b}, -t)^{-1} = 1,
\]
\[
\text{Ch}_{\mathcal{O}_X(E)}(C_{a,b}, t) = \text{Ch}_{\mathcal{O}_X(E)}(C_{a,b}, -t)^{-1} = 1,
\]
from which the claim of the theorem follows. □

8 $\tilde{H}_4$-duality

Recall from the introduction that in the commutative case there is a duality
\[
\begin{array}{ccc}
S(V) & \xrightarrow{\text{ev}} & \mathcal{O}(E) \\
S(V)/(t_0^a_0, s^1_0, \cdots, s^1_1) & \xrightarrow{(\cdot)^*} & \mathcal{O}(E)^* \\
S_1(V^*) & \xrightarrow{-} & S_1(V^*)
\end{array}
\]

We say that the couple $(S(V), S_1(V^*))$ forms an $\tilde{H}_4$-duality.

Let $\mathcal{A}(p)$ now be a Sklyanin algebra and assume that there exists an $\tilde{H}_4$-AS regular algebra $B(p) := B(\mathcal{A}(p))$ such that $(\mathcal{A}(p), B(p))$ forms an $\tilde{H}_4$-duality. Then by construction $B(p) = \mathcal{A}^*(-p_1^0, -p_0^1, -p_1^0)$.

**Theorem 8.1.** Let $S \subset (\mathbb{P}^1)^3$ be the Sklyanin surface and $X = S \cup V \subset (\mathbb{P}^1)^3$ with $V = \{(\infty, \infty, \infty), (\infty, 0, 0), (0, \infty, 0), (0, 0, \infty)\}$. Let $p \in X$ be a point such that $\mathcal{A}^*(-p_1^0, -p_0^1, -p_1^0)$ is regular. Then $p$ is one of the following exceptional points:

$(0, 0, 0), (0, \infty, 0), (\infty, 0, 0), (\infty, \infty, 0), (\infty, \infty, 0), (0, 0, 0)$. 

**Proof.** Assume that the couple $(\mathcal{A}(p_1^0, -p_0^1, p_1^0), \mathcal{A}^*(-p_1^1, -p_0^1, -p_1^1))$ forms an $\tilde{H}_4$-duality. Then both $(p_1^0, -p_0^1, p_1^0)$ and $(-p_1^1, -p_0^1, -p_1^1)$ must lie on $X$. It is clear that this is true if $p$ is one of the 8 exceptional points. If $p$ is not one of the 8 exceptional points, then we have

$(p_1^0, -p_0^1, p_1^0), (-p_1^1, -p_0^1, -p_1^1) \in S$.

Consequently, we are looking at the solution set $Y$ of the equations

$$
\begin{cases}
(t_1, 0, 0, 1 s_1, 1)^2 + (s_1, 0, 0, 1 s_1, 1)^2 + (s_1, 0, 0, 1 t_1, 1)^2 + (t_1, 0, 0, 1 t_1, 1)^2 = 0, \\
(s_1, 0, 0, 1 t_1, 1)^2 + (t_1, 0, 0, 1 s_1, 1)^2 + (s_1, 0, 0, 1 s_1, 1)^2 + (s_1, 0, 0, 1 t_1, 1)^2 = 0
\end{cases}
$$

in $(\mathbb{P}^1)^3$. For any standard open subset $\mathcal{A}^1_{2, y, z} = U \subset (\mathbb{P}^1)^3$, we have $Y \cap U = V(x^2 + y^2 + z^2 + (xy)z^2, (yz)^2 + (xz)^2 + (xy)^2 + 1)$. These two equations are then equivalent to

$$
\begin{cases}
(x^2 + y^2 + 1 + (xy)^2)(1 + z^2) = 0, \\
(x^2 + y^2 - 1 - (xy)^2)(1 - z^2) = 0
\end{cases}
$$

There are three cases to consider:
If \( z^2 \neq \pm 1 \), then it follows that \( x^2 + y^2 = 1 + (xy)^2 = 0 \), from which it follows that \( x^2 = \pm 1, y^2 = \mp 1 \).

If \( z^2 = 1 \), then \( (x^2 + y^2 + (xy)^2) = (x^2 + 1)(y^2 + 1) = 0 \), so that either \( x^2 = -1 \) or \( y^2 = -1 \).

If \( z^2 = -1 \), then \( (x^2 + y^2 - 1 - (xy)^2) = -(x^2 - 1)(y^2 - 1) = 0 \), so that either \( x^2 = 1 \) or \( y^2 = 1 \).

In any case, it follows that \( p \) lies in the \( SL_2(Z_4) \)-orbit of \((1, \omega^2, p_{1,1})\) for some \( p_{1,1} \in \mathbb{P}^1 \). But these algebras are not regular by [13, Corollary 1.3], as there are zero divisors.

9 Further remarks and future work

Theorem 5.1 of this paper determines the possible Artin-Schelter regular quadratic \( \tilde{H}_4 \)-algebras in the moduli space \( \{ \mathcal{R} \subset V \otimes V : \mathcal{R} \cong \tilde{H}_4 V \wedge V \} = (\mathbb{P}^1)^3 \) of quadratic \( \tilde{H}_4 \)-algebras with level-4 structure.

However, in order to classify all \( \tilde{H}_4 \)-AS regular quadratic algebras in \((\mathbb{P}^1)^3\), one still has to study the \( \tilde{H}_4 \)-degenerations of the Sklyanin algebras and determine which of these algebras are regular. In particular, it would be nice if the 24 lines found in the proof of theorem 8.1 parametrize the only non-regular algebras corresponding to points lying on the Sklyanin surface.

Another interesting direction to take is to generalise these results to higher dimensions, that is, for the analogue group \( \tilde{H}_n \) and its associated Schrödinger representation \( V \), find all Artin-Schelter regular quadratic \( \tilde{H}_n \)-algebras which are quotients of \( T(V) \). This problem quickly becomes very complex due to the group structure and the representation theory of \( \tilde{H}_n \). For example, for \( n = 5 \) there are at least 57 surfaces in \( \text{Grass}(2,5) = \{ W \subset V \otimes V : W \cong \tilde{H}_5 V \wedge V \} \) that contain an open subset that parametrize AS-regular algebras:

- 25 surfaces parametrize (possibly trivial) Zhang twists of Clifford algebras studied in [9],
- 2 surfaces parametrize Sklyanin algebras and their analogues in [II], and
- 30 surfaces parametrize algebras isomorphic to (Zhang twists of) \( \tilde{H}_5 \)-quantum algebras (as studied in for example [4]).

In addition, the fact that \( \text{Out}(\tilde{H}_5) \cong A_5 \) greatly complicates calculations. The case \( n = 6 \) should be easier, as in this case \( \tilde{H}_6 \cong Q_8 \times \tilde{H}_3 \), which makes describing the simple representations of \( \tilde{H}_6 \) and their tensor products more manageable.

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