MAXWELL’S EQUATIONS, THE EULER INDEX AND MORSE THEORY

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Abstract. We show that the singularities of the Fresnel surface for Maxwell’s equation on an anisotropic material can be accounted for purely topological considerations. The importance of these singularities is that they explain the phenomenon of conical refraction predicted by Hamilton. We show how to desingularise the Fresnel surface, which will allow us to use Morse theory to find lower bounds for the number of critical wave velocities inside the material under consideration. Finally, we propose a program to generalise the results obtained to the general case of hyperbolic differential operators on differentiable bundles.

1. INTRODUCTION

One of the most interesting problems in geometrical optics in the nineteenth century was the phenomenon of double refraction, in which a ray of light entering certain crystals is refracted into two rays. Later on, Hamilton discovered that if the ray of light enters a biaxial crystal in certain directions (known as the optical axis) then the ray of light must be refracted into a cone of rays. This phenomenon, now known as conical refraction, was confirmed a year later by Lloyd, who investigated aragonite at Hamilton’s suggestion.

In mathematical terms, conical refraction is explained by the existence of conical singularities in the Fresnel surface associated to the crystal. The Fresnel surface is constructed as the set of allowed wave speeds imposed by Maxwell’s equations. In this paper we will show that the singularities of the Fresnel surface can be accounted for topological considerations only. Furthermore, we apply Morse theory to establish a connection between the number of these singularities and the critical speeds of wave propagation within the crystal. Inspired by this result, we propose a program to study this problem in the general context of hyperbolic differential equations on manifolds.
Recall that Maxwell’s equations in a medium with dielectric tensor $\epsilon \in S^2 \mathbb{R}^3$ are given by (see [2, pg. 678])

\[
\begin{align*}
\nabla \times H - \frac{\partial}{\partial t}(\epsilon E) &= 0, \\
\nabla \times E + \frac{\partial}{\partial t} H &= 0, \\
\nabla \cdot (\epsilon E) &= 0, \\
\nabla \cdot H &= 0,
\end{align*}
\]

where we have assumed the speed of light to be equal to one. These equations describe the behaviour of the electric and magnetic fields $E$ and $H$, in the medium under consideration. From now on we will always assume that the eigenvalues of $\epsilon$ are all strictly positive real numbers. If for constant vectors $E_0, H_0, \xi \in \mathbb{R}^3$ and $\tau \in \mathbb{R}$ we try to find planar wave solutions of the form

\[
\begin{align*}
E(x,t) &= E_0 \exp(i(\xi \cdot x - \tau t)), \\
H(x,t) &= H_0 \exp(i(\xi \cdot x - \tau t)),
\end{align*}
\]

we are lead to the equations

\[(1.1) \quad Q_\epsilon(\xi) \begin{pmatrix} E_0 \\ H_0 \end{pmatrix} = 0, \quad <\xi, \epsilon E_0> = 0, \quad <\xi, H_0> = 0,
\]

where

\[(1.2) \quad Q_\epsilon(\xi, \tau) = \begin{pmatrix} \epsilon & 0 \\ 0 & I \end{pmatrix} \tau + \begin{pmatrix} 0 & P(\xi) \\ -P(\xi) & 0 \end{pmatrix}
\]

and $P(\xi)$ is the anti-symmetric matrix such that $P(\xi)v = \xi \times v$ for any $v \in \mathbb{R}^3$. For the first of the equations (1.1) to hold we need that

\[(1.3) \quad \det(Q_\epsilon(\xi, \tau)) = 0.
\]

We can assume without loss of generality that the principal axes of $\epsilon$ are aligned with the $x, y$ and $z$ axes in $\mathbb{R}^3$, so that we can write

\[
\epsilon = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}.
\]

A simple calculation shows that if we let $q_\epsilon(\xi, \tau) = \det(Q_\epsilon(\xi, \tau))/\tau^2$ (assuming $\tau \neq 0$) then we have that

\[
q_\epsilon(\xi, \tau) = (\xi_1^2 + \xi_2^2 + \xi_3^2)(\epsilon_1 \xi_1^2 + \epsilon_2 \xi_2^2 + \epsilon_3 \xi_3^2) + \\
+ (\epsilon_1 \epsilon_2 (\xi_1^2 + \xi_2^2) + \epsilon_2 \epsilon_3 (\xi_2^2 + \xi_3^2) + \epsilon_1 \epsilon_3 (\xi_1^2 + \xi_3^2)) \tau^2 + \epsilon_1 \epsilon_2 \epsilon_3 \tau^4
\]

For $\xi \neq 0$ and $\tau \neq 0$ we can visualise condition (1.3) if we consider the Fresnel surface

\[(1.4) \quad F_\epsilon = \{(\tau \xi/||\xi||^2) \in \mathbb{R}^3 | q_\epsilon(\xi, \tau) = 0 \}.
\]

Physically, the Fresnel surface is the space of the allowable phase velocities. We can visualise $F_\epsilon$ by considering the following cases:

(1) If $\epsilon_1 = \epsilon_2 = \epsilon_3 \neq 0$ then

\[
q_\epsilon(\xi, \tau) = \epsilon_1 (\xi_1^2 + \xi_2^2 + \xi_3^2 - \epsilon_1 \tau^2),
\]

and hence $F_\epsilon$ is a sphere of radius $\epsilon_1^{-1/2}$. 
Figure 1.1. Fresnel surface for $\epsilon_1 = \epsilon_3 = 3$ and $\epsilon_2 = 15$

(2) If exactly two of the $\epsilon_i$’s are equal then $\mathcal{F}_\epsilon$ consists of two smooth surfaces that intersect tangentially (see Figure 1.1).

(3) If all of $\epsilon_i$’s are different from each other, then $\mathcal{F}_\epsilon$ consists two singular surfaces (each having 4 singularities) intersecting at these singular points (see Figure 1.2).

The existence of singularities in case 3 accounts for the phenomena of conical refraction described by Hamilton (see [2, pg. 681]) in which ray of light splits into a cone upon entering at certain directions of a biaxial crystal (e.g aragonite).

2. Maxwell Equations and Symmetric Tensors on the Sphere

In this section we will show that, for the case when the eigenvalues of the dielectric tensor are all different, the singularities of the associated Fresnel surface can be accounted purely from topological considerations.

Throughout this section we will assume $F$ to be an orientable Riemannian real vector bundle of rank 2 over an orientable closed surface $X$, and we will denote the metric in $F$ by $<,>_F$. We will let $S^2F$ stand for the bundle of symmetric endomorphism of $F$, i.e for $p \in X$ we have that $A \in S^2F_p$ iff $A$ is a linear map from $F_p$ to $F_p$ such that

$$<Av, w>_F = <v, Aw>_F \quad \text{for all } v, w \in F_p,$$

For a section $s$ of $S^2F$ we can consider its eigenvalue functions $\lambda_{s,1}, \lambda_{s,2} : X \to \mathbb{R}$, i.e for $p \in X$ we will let $\lambda_{s,1}(p)$ and $\lambda_{s,1}(p)$ be the two eigenvalues of $s(p)$ for all
\[ p \in X. \text{ From now on we will assume that we have chosen } \lambda_{s,1} \text{ and } \lambda_{s,2} \text{ so that } \lambda_{s,1} \leq \lambda_{s,2}. \]

Given a symmetric matrix \( \epsilon \in S^2 \mathbb{R}^3 \) we define \( s_{\epsilon} : S^2 \to S^2(TS^2) \) by letting

\[ <s_{\epsilon}(p)v, w> = <\epsilon v, w> \quad \text{for all } p \in S^{k-1} \text{ and } v, w \in T_pS^2. \]

Observe that we are identifying \( T_pS^2 \) with the subspace of \( \mathbb{R}^3 \) consisting of all the vectors orthogonal to \( p \), and that the metric we are using on \( T_pS^2 \) is that induced from the ambient space \( \mathbb{R}^3 \). To simplify notation we will let

\[ \lambda_{\epsilon, i} = \lambda_{s, i} \quad \text{for } i = 1, 2. \]

The following result expresses the Fresnel surface \( F_\epsilon \) in terms of \( \lambda_{\epsilon,1} \) and \( \lambda_{\epsilon,2} \).

**Proposition 1.** If \( F_\epsilon \) is the Fresnel surface associated with the dielectric tensor \( \epsilon \in S^2\mathbb{R}^3 \), then we have that

\[ F_\epsilon = \bigcup_{i=1}^{2} \{ \lambda_{\epsilon, i}^{-1/2}(\xi)\xi \mid \xi \in S^2 \}. \]

**Proof.** As we have seen, we must have that

\[ Q_\epsilon(\xi, \tau) \begin{pmatrix} E_0 \\ H_0 \end{pmatrix} = 0, \]

where \( Q_\epsilon \) is given by (1.2). The above formula is the same as the formulas

\[ \tau \epsilon E_0 + \xi \times H_0 = 0, \]
\[ \tau H_0 - \xi \times E_0 = 0. \]
From these last two equations we obtain that

\[(\tau/||\xi||)^2 \epsilon E_0 - (\xi/||\xi||) \times ((\xi/||\xi||) \times E_0) = 0.\]

Since \(\xi/||\xi||\) is a unitary vector, we have that the second term in the left hand side of the above equation is just the negative of the projection of \(E_0\) onto the orthogonal space to \(\xi\), and hence

\[(\tau/||\xi||)^2 < \epsilon E_0, v > + < E_0, v > = 0 \text{ for all } v \text{ such that } < v, \xi > = 0.\]

The above equation means that

\[(\sigma_e(\xi/||\xi||)) + (||\xi||/\tau)^2 I)(\pi_\xi E_0) = 0,

where \(\pi_\xi : \mathbb{R}^3 \to T_\xi/||\xi||S^2\) is projection onto the orthogonal complement of \(\xi\), and \(I\) is the identity morphism of the bundle \(S^2(TS^2)\). Hence, we must have that

\[\tau = ||\xi||\lambda_{e,i}^{-1/2}(\xi/||\xi||)\]

where \(\lambda_i : S^2 \to \mathbb{R}\) are the eigenvalue functions of \(\sigma_e\). From this formula and the definition of \(F_e\) (see [1.4]) we obtain the desired result. \(\square\)

**Example 2.** Consider the case were \(\epsilon\) is diagonal with eigenvalues \(\epsilon_1 = 1, \epsilon_2 = 2\) and \(\epsilon_3 = 15\), as shown in Figure [1.2]. The surface at the bottom right corresponds to \(\lambda_{e,1}\) and the one to the left \(\lambda_{e,2}\). Observe that that these surfaces join at the points \(\xi \in S^2\) where \(\lambda_{e,1}(\xi) = \lambda_{e,2}(\xi)\), and exactly at these points are where the singularities of \(F_e\) occur.

The previous example motivates the importance of the following definition.

**Definition 3.** If \(s\) is a section of \(S^2F\) we will say that \(p \in X\) is a multiple point iff \(\lambda_{s,1}(p) = \lambda_{s,2}(p)\), and we will denote the set of such points by \(M_s\).

Let \(S^2_0F\) stand for traceless elements in \(S^2F\). For a given section \(s\) of \(S^2F\) we can construct its traceless part (which is a section of \(S^2_0F\)) by letting

\[s_0 = s - \frac{1}{2}\text{tr}(s)I_F,\]

where \(I_F\) is the identity section in \(S^2F\).

**Proposition 4.** The set \(M_s\) coincides with the set of zeros of the section \(s_0\). Furthermore, if \(s_0\) is transversal to the zero section of \(S^2_0F\) then the eigenvalue function \(\lambda_{s,1}\) and \(\lambda_{s,2}\) have conic singularities on \(M_s\).

**Proof.** It is easy to see that for a section \(s\) of \(S^2F\) we have that

\[\lambda_{s,1}(p) = \frac{1}{2}\text{tr}(s(p)) - ||s_0(p)||\]

\[\lambda_{s,2}(p) = \frac{1}{2}\text{tr}(s(p)) + ||s_0(p)||.\]

where the norm used for an element \(A \in S^2_0F\) is given by \(||A|| = \frac{1}{2}\text{tr}(A^2)\). From this it follows that \(\lambda_{s,1}(p) = \lambda_{s,2}(p)\) iff \(s_0(p) = 0\). If \(s_0\) is transversal to the zero section of \(S^2_0F\) then near a zero of \(s_0\) the norm function behaves like \((x, y) \rightarrow (x^2 + y^2)^{1/2}\), and hence we have conical singularities at these points. \(\square\)

We are now ready to prove that four singularities of \(F_e\) can be accounted from purely topological considerations. We will need the following result first.
Lemma 5. The bundle $S^2_0 F$ is isomorphic to $F \otimes \mathbb{C} F$, as a real vector bundle.

Proof. Since $F$ has a metric, we can consider it as an $\text{SO}(2)$ bundle. For

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in \text{SO}(2)$$

and

$$A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in S^2_0 \mathbb{R}^2$$

we have that

$$RAR^T = \begin{pmatrix} p & q \\ q & -p \end{pmatrix},$$

where

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Hence, the transition functions of the bundle $S^2_0 F$ are the same as those of the bundle $F \otimes \mathbb{C} F$. □

Proposition 6. If $s$ is a section of $S^2 F$ such that $s_0$ is transversal to the zero section of $S^2 \mathbb{R}^2$, then the cardinality of $M_s$ is at least $2 \left| \hat{e}(F) \right|$.

Proof. From Proposition 5 we have that $M_s = Z_0$, where $Z_0$ is the set of zeros of $s_0$. We have that

$$\int_X e(S^2_0 F) = \sum_{p \in Z_0} i_{s_0}(p),$$

where $i_{s_0}(p)$ is the index of $s_0$ at $p$. From the transversality assumption we have that $i(p) = \pm 1$, and from Lemma 5 we have that $e(S^2_0 F) = e(F \otimes \mathbb{C} F) = 2e(F)$. We conclude that

$$|Z_0| \geq 2 \left| \int_X e(F) \right|,$$

where $|Z_0|$ is the cardinality of $Z_0$. □

Corollary 7. If $X$ has genus $g$ and $s : X \to S^2(TX)$ is such that $s_0$ is transversal to the zero section of $S^2_0(TX)$, then cardinality of $M_s$ is at least $4 - 4g$.

Proof. Apply the above proposition and the formula

$$\int_X e(TX) = 2 - 2g.$$ □

We know that the singularities of $F_\epsilon$ occur at the multiplicity set $M_\epsilon$ of the section $s_\epsilon$. And the from the above Corollary we know that the cardinality of $M_\epsilon$ must be greater or equal than 4. This is accordance with the known fact that $F_\epsilon$ has exactly four singularities, but our argument has been purely topological. In this discussion we have implicitly assumed that $(s_\epsilon)_0$ is transversal to the zero section of $S^2(TS^2)$. We can explicitly find the points in $M_\epsilon$ and verify that this is actually the case.

Proposition 8. If the the eigenvalues of $\epsilon \in S^2 \mathbb{R}^3$ are all different from each other, then the traceless part of $s_\epsilon$ is transversal to the zero section of $S^2_0(TS^2)$. 


Proof. We can assume that \( \epsilon \) is diagonal with diagonal elements \( 0 < \epsilon_1 < \epsilon_2 < \epsilon_3 \). The zero set of \((s_\epsilon)_0\) does not contain neither \((0,0,1)\) nor \((0,0,-1)\). We can use polar coordinates

\[
p(\theta, \varphi) = (\cos(\theta) \cos(\varphi), \sin(\theta) \cos(\varphi), \sin(\varphi)) \quad \text{for} \ 0 \leq \theta < 2\pi, -\pi/2 < \varphi < \pi/2,
\]

and let

\[
u(\theta, \varphi) = (-\sin(\theta), \cos(\theta), 0)
v(\theta, \varphi) = (-\cos(\theta) \sin(\varphi), -\sin(\theta) \sin(\varphi), \cos(\varphi)).
\]

The triple \((p,u,v)\) forms and orthonormal frame, so that locally we can write

\[
s_\epsilon = \begin{pmatrix}
< \epsilon u, u > & < \epsilon u, v > \\
< \epsilon v, u > & < \epsilon v, v >
\end{pmatrix}
\]

If we identify \(S^2_0 \mathbb{R}^2\) with \(\mathbb{R}^2\) by letting

\[(a,b) \sim \begin{pmatrix} a \\ b \\ -a \end{pmatrix}\]

then we can write

\[
(s_\epsilon)_0 = \frac{1}{2} \begin{pmatrix} < \epsilon u, u > & -< \epsilon v, v >, 2 < \epsilon u, v > \end{pmatrix}
\]

where

\[
2 < \epsilon u, v > = 2(\epsilon_1 - \epsilon_2) \cos(\theta) \sin(\varphi),
\]
\[
< \epsilon u, u > - < \epsilon v, v > = (\epsilon_2 - \epsilon_1 \sin^2(\varphi)) \cos^2(\theta) +
\]
\[
+ (\epsilon_1 - \epsilon_2 \sin^2(\varphi)) \sin^2(\theta) - \epsilon_3 \cos^2(\varphi).
\]

From this formula it is easy to see that the zeros of \((s_\epsilon)_0\) occur at the points \((0, \varphi_m), (0, \varphi_m + \pi/2), (\pi, \varphi_m)\) and \((0, \varphi_m + \pi/2)\), where

\[
\varphi_m = \arctan \left( \frac{\epsilon_3 - \epsilon_2}{\epsilon_2 - \epsilon_1} \right)^{\frac{1}{2}}.
\]

A simple calculations shows that

\[
D(s_\epsilon)_0(0, \varphi_m) = (\epsilon_2 - \epsilon_1)^{3/2}(\epsilon_3 - \epsilon_2)(\epsilon_3 - \epsilon_1)^{-1/2} > 0,
\]

and hence \((s_\epsilon)_0\) is transversal to the zero section of \(S^2(TS^2)\) at that point. Due to the symmetry of the problem, the same condition holds at the other points. \(\square\)

3. Desingularising the Fresnel Surface and Morse Theory

We are interested in finding the critical points of the functions \(\lambda_{\epsilon,1}\) and \(\lambda_{\epsilon,2}\) described in the previous section. One reason why this problem is interesting is that the maximum and minimum are critical points of these functions, and they correspond to directions of maximal and minimal wave velocities. We want use a topological argument to find lower bounds to the number and types of critical points. To do this, we can apply Morse theory. Recall that if \(f : X \to \mathbb{R}\) is a smooth function and \(p \in X\) is a critical point of \(f\) (i.e \(df(p) = 0\)) then we can define its hessian \(d^2f(p)\), which is symmetric bilinear form in \(T_pX\). A critical point \(p \in X\) of \(f\) is said to be non-degenerate iff \(d^2f(p)\) is non-degenerate, and in this case we define the index of \(p\) as the dimension of the maximal subspace of \(T_pX\).
at which $d^2f(p)$ is negative definite. Morse inequalities assert that if $f$ has only non-degenerate critical points the we have that (see [9, pg. 29])

$$C_i(f) \geq \dim(H^i(X, \mathbb{R}))$$

$$\sum_{i=0}^{\dim(X)} (-1)^i \dim(H^i(X, \mathbb{R})) = \sum_{i=0}^{\dim(X)} (-1)^i C_i(f),$$

where $C_i(f)$ are the number of critical points of $f$ of index $i$, and $H^i(X, \mathbb{R})$ is the cohomology group of $X$ with real coefficients. This formulas allow us to estimate the number of critical points of a given index in terms of topological invariants of $X$.

We have seen in the previous section both $\lambda_{e,1}$ and $\lambda_{e,2}$ are non-smooth at $\mathcal{M}_e$, so we cannot apply Morse Theory directly to them. To solve this problem we desingularise these functions as follows. For a section $s : X \to S^2F$ we define

$$\mathcal{E}_s = \bigcup_{p \in X} \{l_x \in PF | p \text{ is spanned by an eigenvector of } s(p)\},$$

where $PF$ is the projectivisation of $F$. We will refer to $\mathcal{E}_s$ as the space of eigenlines of $s$. Over this space we can define the eigenvalue function of $s$ by

$$\lambda_s(l_p) = \text{eigenvalue of } s(p) \text{ corresponding to } l_p.$$

It turns out that if $s_0$ is transversal to the zero section of $S^2F$ then we have that $\mathcal{E}_s$ is a smooth submanifold of $PF$, and $\lambda_s$ is a smooth function (see [11, Theorem 5]). Furthermore, the space $\mathcal{E}_s$ is topologically equivalent to the space obtained by the following procedure (see [11, Proposition 6])

1. Let $X_0$ be obtained by removing small open disks, centred at the points of $\mathcal{M}_s$, from $X$.
2. The space $\mathcal{E}_s$ is homeomorphic to the space obtained by joining two disjoint copies of $X_0$ along cylinders of the form $S^1 \times (-\delta, \delta)$ (see Figure 3.1).

If we restrict the projection map $\pi : PF \to X$ to $\mathcal{E}_s$ we obtain a projection map $\pi_s : \mathcal{E}_s \to X$ with the following properties (see [11, Theorem 5])

1. We have that $\pi_s^{-1}(X - \mathcal{M}_s)$ consists of two connected component $X_1$ and $X_2$ such that $\pi_s : X_i \to X - \mathcal{M}_s$ is a diffeomorphism and $\lambda_{s,i} \circ \pi_s = \lambda_s$.
2. For any $p \in \mathcal{M}_s$ we have that $\pi_s^{-1}(p)$ is diffeomorphic to a circle and $\lambda_s$ can have 0, 1 or 2 critical points at $\pi_s^{-1}(p)$. Furthermore, if any of these critical points is non-degenerate it must have index equal to one.

To simplify notation we define

$$\mathcal{E}_\epsilon = \mathcal{E}_{s_\epsilon}, \pi_\epsilon = \pi_{s_\epsilon} \text{ and } \lambda_\epsilon = \lambda_{s_\epsilon}.$$ 

As an application of the above results, we have the following.

**Proposition 9.** If $\epsilon \in S^2\mathbb{R}^3$ has eigenvalues $0 < \epsilon_1 < \epsilon_2 < \epsilon_3$ then then space $\mathcal{E}_\epsilon$ is a smooth surface of genus 3 and $\lambda_\epsilon : \mathcal{E}_\epsilon \to \mathbb{R}$ is a smooth function having no critical points at $\pi_\epsilon^{-1}(\mathcal{M}_\epsilon)$.

**Proof.** The spaces $\mathcal{E}_\epsilon$ is obtained joining two disjoint copies of $S^2 - D_1 \cup D_2 \cup D_3 \cup D_4$ along cylinders of the form $S^1 \times (-\delta, \delta)$, where the $D_i$'s are small open disks centred at the four points in $\mathcal{M}_\epsilon$ (see Figure 3.1). From this it follows that $\mathcal{E}_\epsilon$ is a surface of genus 3. To check that $\lambda_\epsilon$ has no singularities in $\pi_\epsilon^{-1}(\mathcal{M}_\epsilon)$ we write a explicit
formula for \( \lambda_\epsilon \) and check that this is actually the case for \( 0 < \epsilon_1 < \epsilon_2 < \epsilon_3 \). Consider polar coordinates on \( S^2 - \{(0,0,1),(0,0,-1)\} \) given by

\[
p(\theta, \varphi) = (\cos(\theta) \cos(\varphi), \sin(\theta) \cos(\varphi), \sin(\varphi))
\]

and the orthonormal frame \( \{u(\theta, \varphi), v(\theta, \varphi)\} \) in \( T(S^2 - \{(0,0,1),(0,0,-1)\}) \) given by

\[
u(\theta, \varphi) = (-\cos(\theta) \sin(\varphi), -\sin(\theta) \sin(\varphi), \cos(\varphi)).
\]

If we let \( \alpha \) be the angle of a line \( l \in P(TS^2) \) with respect to \( u(\theta, \varphi) \) then we can use \((\theta, \varphi, \alpha)\) as coordinates for \( P(TS^2) \). In these coordinates we have that (see [11] proof of Theorem 5)

\[
E_\epsilon = f_\epsilon^{-1}(0)
\]

\[
\lambda_\epsilon = g_\epsilon|E_\epsilon|
\]
where

\[
\begin{align*}
f_s(\theta, \varphi, \alpha) &= (\epsilon_2 - \epsilon_1) \cos(2\alpha) \cos(\theta) \sin(\theta) \sin(\varphi) \\
&+ \frac{1}{2} \sin(2\alpha) \cos^2(\theta) (\epsilon_2 - \epsilon_1 \sin^2(\varphi)) \\
&+ \frac{1}{2} \sin(2\alpha) \sin^2(\theta) (\epsilon_1 - \epsilon_2 \sin^2(\varphi) - \epsilon_3 \cos^2(\varphi)) \\
g_s(\theta, \varphi, \alpha) &= -\sin(2\alpha) \sin(\theta) \sin(\varphi) \\
&+ \frac{1}{2} (\epsilon_1 \sin^2(\theta) + \epsilon_2 \cos^2(\theta) + \epsilon_3 \cos^2(\varphi)) \\
&+ \frac{1}{2} (\epsilon_1 \cos^2(\theta) + \epsilon_2 \sin^2(\theta)) \sin(\varphi) \\
&+ \frac{1}{2} \cos(2\alpha) (\epsilon_2 - \epsilon_1 \sin^2(\varphi)) \\
&+ \frac{2}{3} \cos(2\alpha) (\epsilon_1 - \epsilon_2 \sin^2(\varphi) - \epsilon_3 \cos^2(\varphi))
\end{align*}
\]

From this, we obtain that

\[
\nabla f_s(0, \varphi_m, \alpha) = \left(\frac{(\epsilon_1 - \epsilon_2)(\epsilon_3 - \epsilon_2)^{1/2}}{(\epsilon_3 - \epsilon_1)^{1/2}} \cos(2\alpha), (\epsilon_2 - \epsilon_1)^{1/2}(\epsilon_3 - \epsilon_2)^{1/2} \sin(2\alpha), 0\right)
\]

\[
\nabla g_s(0, \varphi_m, \alpha) = \left(\frac{(\epsilon_3 - \epsilon_2)^{1/2}}{(\epsilon_3 - \epsilon_1)^{1/2}} \sin(2\alpha), (\epsilon_2 - \epsilon_1)^{1/2}(\epsilon_3 - \epsilon_2)^{1/2}(1 - \cos(2\alpha), 0)\right)
\]

where \((0, \varphi_m)\) corresponds to one of the zeros of \((s_\epsilon)_0\) as in the proof of Proposition 8 (by symmetry the other zeros of \((s_\epsilon)_0\) are dealt in the same way). The critical points of \(\lambda_\epsilon\) correspond to the \(\alpha\)'s at which \(\nabla f_s(0, \varphi_m, \alpha) = \nabla g_s(0, \varphi_m, \alpha)\). By using the above formulas we have that this equality can not hold if all the \(\epsilon_i\)'s are different from each other. \(\square\)

**Corollary 10.** If \(\epsilon \in S^2 \mathbb{R}^3\) has eigenvalues \(0 < \epsilon_1 < \epsilon_2 < \epsilon_3\) then we have that \(C_i(\lambda_\epsilon) = C_i(\lambda_{\epsilon,1}) + C_i(\lambda_{\epsilon,2})\) for \(i = 0, 1, 2\).

**Proof.** By the previous proposition we have that \(\lambda_\epsilon\) has no critical points on \(\pi_\epsilon^{-1}(\mathcal{M}_\epsilon)\).

But we know that on each of the two connected components of \(\mathcal{E}_\epsilon - \pi_\epsilon^{-1}(\mathcal{M}_\epsilon)\) the function \(\lambda_\epsilon\) coincides (up to diffeomorphism) with \(\lambda_{\epsilon,1}\) or \(\lambda_{\epsilon,2}\). \(\square\)

Applying Morse inequalities to \(\lambda_\epsilon\) and using the above results we obtain that

\[
\begin{align*}
C_0(\lambda_{\epsilon,1}) + C_0(\lambda_{\epsilon,2}) &\geq 1 \\
C_1(\lambda_{\epsilon,1}) + C_1(\lambda_{\epsilon,2}) &\geq 3 \\
C_2(\lambda_{\epsilon,1}) + C_2(\lambda_{\epsilon,2}) &\geq 1
\end{align*}
\]

\[
\sum_{i=0}^{2} C_i(\lambda_{\epsilon,1}) + C_0(\lambda_{\epsilon,2}) = -4
\]

The above holds under the assumption that \(\lambda_\epsilon\) is a Morse function, but it can easily be seen that this is the case if the eigenvalues of \(\epsilon\) are all different from each other.

**4. Generalisations - Future Work**

In this section we propose a program to generalise our results to general hyperbolic differential operators on bundles.
Let $E$ and $F$ be vector bundles of rank $m$ and $k$ over a Riemannian manifold $X$ of dimension $n$. Consider a linear differential operator of degree $d \in \mathbb{Z}^+$ of the form

$$(4.1)\quad L = \sum_{i=0}^{d} \left( L_i \circ \frac{\partial^{d-i}}{\partial t^{d-i}} \right),$$

where $L_i : C^\infty(E) \to C^\infty(F)$ is a differential operator of order $i$. Observe that $L$ acts on time dependent sections of $E$. We are interested in high frequency solutions of the equation $Lu = 0$, i.e. in solutions to the asymptotic partial differential equation

$$L (u_0 e^{is \varphi}) = 0 \quad \text{as} \quad s \to \infty,$$

where $u_0$ is a section of $E$ and $\varphi$ is a smooth real valued function in $X \times \mathbb{R}$ (known in the physics literature as the phase function). The above asymptotic equation leads to the equation (see [4, pg. 31])

$$(4.2)\quad \left( \sum_{i=0}^{d} \sigma_i(d_x \varphi(x,t)) \left( \frac{\partial \varphi}{\partial t}(x,t) \right)^{d-i} \right) u_0(x) = 0,$$

where $\sigma_i : T^*X \to \text{Hom}(E, F)$ is the principal symbol of $L_i$. If we choose local coordinates for $X$ and local trivialisation of $E$ and $F$ over an open set $U \subset X$, we can write

$$L_i = \sum_{|\alpha| \leq i} A_{i,\alpha} \frac{\partial^\alpha}{\partial x^\alpha}$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_n), |\alpha| = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{(\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}}$$

and for every $\alpha$ and $i$ we have that $A_{i,\alpha}$ is a function from $U$ to the $k \times m$ matrices with coefficients in $\mathbb{R}$. The local expression for $\sigma_i$ is then given by

$$\sigma_i(x, \xi) = \sum_{|\alpha| = i} A_{i,\alpha}(x) \xi^\alpha \quad \text{where} \quad \xi = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}.$$ 

We see that over the fibre variables of $T^*X$ the symbol $\sigma_i$ is an homogeneous polynomial of degree $i$ whose coefficients are real $k \times m$ matrices.

We define the Fresnel hypersurface associated to $L$ by

$$(4.3)\quad \mathcal{F}_L = \left\{ (x, -\tau \xi) \in T^*X \mid (x, \xi) \in S(T^*X) \quad \text{and ker} \left( \sum_{i=0}^{d} \sigma_i(x, \xi) x^{d-i} \right) \neq 0 \right\},$$

We will assume that the following conditions hold

1. We have that $F$ is a Riemannian real vector bundle and $E = F$. For example, if we were modelling small oscillations of an elastic surface $X$ then we could assume that $E = F = TX$.

2. For all $(x, \xi) \in T^*X$ and $1 \leq i \leq d$ we have that $\sigma_i(x, \xi)$ is a symmetric operator with respect to a metric in $F$, i.e.

$$< \sigma_i(x, \xi)v, w>_F = < v, \sigma_i(x, \xi)w>_F \quad \text{for all} \quad v, w \in F_x.$$
It turns out that two conditions above hold in many cases of physical interest, and in particular those physical problems that arise from variational principles. Under the above assumptions we can write
\[ F_L = \{(x, -\tau \xi) \in S(T^*X)| p_L(x, \xi, \tau) = 0\}, \]
where \( S(T^*X) \) are the unit vectors in \( T^*X \) and
\[ p_L(x, \xi, \tau) = \det \left( \sum_{i=0}^{d} \sigma_i(x, \xi) \tau^{d-i} \right). \]

**Definition 11.** A polynomial \( p = p(\tau) \) is said to be *hyperbolic* if it has as many real roots as its degree (counting multiplicities) and it is said to be *strictly hyperbolic* if all these roots are different from each other.

For a fixed \((x, \xi) \in S(T^*X)\) let
\[ p_{L,x,\xi}(\tau) = p_L(x, \xi, \tau). \]

**Definition 12.** The operator \( L \) is said to be hyperbolic if \( p_{L,x,\xi} \) is hyperbolic for all \((x, \xi) \neq 0\) in \( T^*X \).

If \( L \) is hyperbolic we have functions \( \lambda_i : T^*X \to \mathbb{R} \) for \( 1 \leq i \leq \dim(F) \cdot \deg(L) \) given as as the roots of \( p_{L,x,\xi} \), and we can write
\[ F_L = \bigcup_{i=1}^{d} \{-\lambda_i(\xi)\xi | \xi \in S(T^*X)\}. \]

We define \( M_L \) as the set of points in \( S(T^*X) \) at which two or more or the \( \lambda_i \)'s coincide. The singularities of \( F_L \) occur at points on the set \( M_L \). Inspired by our work on Maxwell’s equations, we pose the following problems.

1. **Find topological obstructions to the condition \( M_L = \emptyset \).** This will provide obstructions to the existence of strictly hyperbolic differential operators on \( F \). It is expected that the characteristic classes of \( F \) should be involved in the answer to this problem.

2. **Desingularisation of \( F_L \).** As in the case of Maxwell’s equation, we would like to find a smooth space \( \mathcal{E}_L \) and a smooth function \( \lambda_L : \mathcal{E}_L \to \mathbb{R} \) that contains all the information of the \( \lambda_{L,i} \)'s. We would then expect to be able to apply Morse theory to \( \lambda_L \) to obtain similar results as in the case of Maxwell’s equations.

In relation with Problem 1 above we mention the work in the articles [3, 5, 6, 7, 8, 10], which contain some local and global results related to the multiplicity of eigenvalues of symbols of differential and pseudo-differential operators. Regarding Problem 2, we mention that the non-smoothness of \( \lambda_{L,i} \)'s at \( M_L \) explain phenomena like wave transformation (see [1, pg. 223]) in which longitudinal waves transforms into transversal ones (or viceversa).

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