Soliton Resolution for the Wadati–Konno–Ichikawa Equation with Weighted Sobolev Initial Data

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Abstract. In this work, we employ the $\bar{\partial}$-steepest descent method to investigate the Cauchy problem of the Wadati–Konno–Ichikawa (WKI) equation with initial conditions in weighted Sobolev space $\mathcal{H}(\mathbb{R})$. The long time asymptotic behavior of the solution $q(x, t)$ is derived in a fixed space-time cone $S(y_1, y_2, v_1, v_2) = \{(y, t) \in \mathbb{R}^2 : y = y_0 + vt, \ y_0 \in [y_1, y_2], \ v \in [v_1, v_2]\}$. Based on the resulting asymptotic behavior, we prove the soliton resolution conjecture of the WKI equation which includes the soliton term confirmed by $N(I)$-soliton on discrete spectrum and the $t^{-\frac{3}{2}}$ order term on continuous spectrum with residual error up to $O(t^{-\frac{3}{4}})$.

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1. Introduction

It is well known that the nonlinear Schrödinger (NLS) equation,
\[ iu_t + u_{xx} + 2|u|^2u = 0, \]  
(1.1)
can be adapted to describe the pulse propagation in optical fibers \[1\]. With more in-depth research on NLS equation, it plays an increasingly important role in the field of the optical communication. Motivated by this, more and more scholars devoted to the research on the NLS equation and its extensions \[2–6\]. However, at higher field strength, the optically induced refractive-index change becomes saturated. The saturation effects will give rise to a physical limit of the shortest soliton pulse duration or of the pulse compression by high-order soliton generation. Thus, the NLS equation is not appropriate to describe the propagation of soliton in materials with saturation effects. To study the propagation of soliton in materials with saturation effects, the equation
\[ iA_t + A_{xx} + \frac{|A|^2}{1 + \gamma |A|^2} A = 0, \]  
(1.2)
where \( A \) is the slowly varying amplitude of the field strength and \( \gamma \) is the Kerr parameter, is proposed and investigated \[7,8\]. Regrettably, the model (1.2) is not integrable, which will make it difficult to get its analytical solutions. In order to overcome this difficulty, Wadati, Konno and Ichikawa proposed an integrable model possessing saturation effects \[9\], i.e.,
\[ iq_t + \left( \frac{q}{\sqrt{1 + |q|^2}} \right)_{xx} = 0, \]  
(1.3)
which is later called the Wadati–Konno–Ichikawa (WKI) equation. Furthermore, Wadati, Konno and Ichikawa presented two types of integrable nonlinear evolution equations, and confirmed that the equations have an infinite number of conservation laws \[10\]. The WKI equation (1.3) can also be used to describe nonlinear transverse oscillations of elastic beams under tension \[11,12\]. Since then, there are many significant work about the WKI hierarchy. In 1993,
under the condition of C. Neumann constraint, Qiao gives the proof that the WKI spectrum problem related to the WKI hierarchy is nonlinearized into an Hamilton system [13]. Under the Bargmann constraint between potential and spectral function, Qiao proved that the WKI eigenvalue problem could be nonlinearized into a Liouville completely integrable Hamilton system [14]. In 2005, Qu and Zhang [15] derived the WKI equation (1.3) from the motions of curves in Euclidean geometry \( E^2 \). Because the significant mathematical structures and physical meanings of WKI equation (1.3), many scholars contribute their efforts to the research on the properties of WKI equation. The commutator representation, algebraic structure and the parametric representations of solutions of WKI hierarchy are investigated in [16–18]. In [19], the orbital stability for stationary solutions of the WKI equation (1.3) was given. The algebra-geometric constructions of WKI flows and the existence of global solution for the WKI equation with small initial data were studied in [20, 21]. By using Riemann–Hilbert (RH) method, the soliton solutions of the WKI equation (1.3) related to simple poles and higher-order poles were constructed in [22, 23].

Moreover, through a series of gauge transformation, the WKI equation (1.3) was related to Ablowitz, Kaup, Newell and Segur (AKNS) system [24]. For example, in [24], by adapting dependent and independent variable transformations, the solutions of the WKI equation were constructed based on the solution of the modified Korteweg–de Vries (mKdV) equation. However, it is precisely because of dependent and independent variable transformations that the explicit higher-order solution of the WKI equation cannot be obtained from a solution of the mKdV equation. Therefore, it is meaningful to directly study the explicit solution of the WKI equation (1.3).

In this work, we employ \( \bar{\partial} \)-steepest descent method to investigate the soliton resolution of the WKI equation (1.3) with the initial value condition

\[
q(x, 0) = q_0(x) \in \mathcal{H}(\mathbb{R}),
\]

where

\[
\mathcal{H}(\mathbb{R}) = W^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R}).
\]

The \( W^{2,1}(\mathbb{R}) \) and \( H^{2,2}(\mathbb{R}) \) are defined in (3.1). Compared with the work in [25] that have obtained the long time asymptotic solutions of potential WKI equation by using nonlinear steepest descent method, our work shows that the accuracy of our result can reach \( O(t^{-\frac{3}{4}}) \); this is barely possible for the literature [25].

The study of the long time asymptotic behavior of nonlinear evolution equations can go back to the earlier work of Manakov [26]. Inspired by the previous work, Zakharov and Manakov [27] derived the long time asymptotic solutions of NLS equation with decaying initial value in 1976. In 1993, a nonlinear steepest descent method was developed by Defit and Zhou [28]. This method is able to be adapted to systematically study the long time asymptotic behavior of nonlinear evolution equations. Motivated by the pioneers’ work, later scholars continuously follow their steps. After years of unremitting
research, the nonlinear steepest descent method has been improved and widely applied [29–34]. The authors in [35,36] showed that when the initial value is smooth and decays fast enough, the error term is $O(\log t)$. The work [37] confirmed that if the initial value belongs to the weighted Sobolev space (1.5), the error term could reach $O(t^{-\left(\frac{1}{2}+\epsilon\right)})$ for any $0 < \epsilon < \frac{1}{4}$.

In recent years, in order to study the asymptotic of orthogonal polynomials, McLaughlin and Miller [38,39] presented a $\bar{\partial}$-steepest descent method by combining steepest descent with $\bar{\partial}$-problem. Then, scholars developed this method to investigate defocusing NLS equation with finite mass initial data [40] and with finite density initial data [41]. Compared with the nonlinear steepest descent method, $\bar{\partial}$-steepest descent method has an obvious advantage that the delicate estimates involving $L^p$ estimates of Cauchy projection operators can be avoided during the analysis. In addition, the work [40] showed an improvement that the error term reach $O(t^{-\frac{3}{4}})$ when the initial value belongs to the weighted Sobolev space. Therefore, a series of great work has been done by using $\bar{\partial}$-steepest descent method [42–51].

In [42], M. Borghese, R. Jenkins and K. T. R. McLaughlin have computed the long time asymptotic expansion of the solution $u(x,t)$ of the focusing NLS equation (1.1) by using the $\bar{\partial}$-steepest descent method. Here, we extend above results to study the long time asymptotic behavior of the solution $q(x,t)$ of the WKI equation (1.3). It is worth noting that there are some differences from that on focusing NLS equation (1.1) as mentioned in the following four aspects.

(I) Because the Lax pair (2.1) of the WKI equation possesses two singularities at $z = 0$ and $z = \infty$, the behavior of the solutions of spectral problem (2.1) also need to be studied as spectral parameter $z \to 0$. Then, the solution $q(x,t)$ of the WKI equation (1.3) can be constructed via employing the expansion of the eigenfunction with $t$-part.

(II) When we construct the Riemann–Hilbert problem (RHP) corresponding to the initial value problem for the WKI equation (1.3), an improved transformation needs to be introduced to ensure that the eigenfunctions tend to the identity matrix as the spectral parameter $z \to \infty$. An obvious result is that there exists an exponential term in the solution $q(x,t)$ shown in (3.9).

(III) In order to reconstruct the solution $q(y(x,t),t)$ of (1.3), we need not only the long time asymptotic behavior of error function $E(z)$ but also the long time asymptotic behavior of $E(0)$, see Sect. 7.2.2.

(IV) For the $\bar{\partial}$-RH problem $M^{(3)}(z)$ defined in (7.1), we need to study the asymptotic behavior of $M^{(3)}(0)$ and the long time asymptotic behavior of $M^{(3)}_1(y,z)$. Although the final purpose is to bound the size of $M^{(3)}(z)$, some distinctive scaling techniques need to be taken to investigate the estimates of $M^{(3)}$ including $M^{(3)}(0)$ and $M^{(3)}_1(y,z)$, see Sect. 8.
Main Result and Remark

Our main result displays the long time asymptotic behavior of the solution (1.3) for generic initial value $q_0(x) \in \mathcal{H}(\mathbb{R})$.

**Theorem 1.1.** Suppose that the initial value $q_0(x)$ satisfies Assumption 3.1 and $q_0(x) \in \mathcal{H}(\mathbb{R})$. Let $q(x, t)$ be the solution of WKI equation (1.3). The scattering data is denoted as $\{r, \{z_k, c_k\}_{k=1}^N\}$ which generated from the initial value $q_0(x)$. For fixed $y_1, y_2, v_1, v_2 \in \mathbb{R}$ with $y_1 < y_2, v_1 < v_2$, and $\mathcal{I} = \{z : -\frac{1}{4v_1} < \text{Re} z < -\frac{1}{4v_2}\}$, $z_0 = \frac{y}{4t}$, then as $t \to \infty$ and $(y, t) \in S(y_1, y_2, v_1, v_2)$ which is defined in (B.13), the solution $q(x, t)$ can be expressed as

$$q(x, t) = q(y(x, t), t)e^{-2d}$$

$$= q_{\text{sol}}(y(x, t), t; \hat{\sigma}_d(\mathcal{I}))T^2(0)(1 + T_1) - it^{-\frac{1}{2}}e^{2d} \frac{\partial}{\partial y}f_{12}^\pm + O(t^{-1}), \quad (1.6)$$

$$y(x, t) = x - c_-(x, t, \hat{\sigma}_d(\mathcal{I})) - iT_1^{-1} - it^{-\frac{1}{2}}f_{11}^\pm + O(t^{-1}).$$

Here, $q_{\text{sol}}(x, t; \hat{\sigma}_d(\mathcal{I}))$ is the $N(\mathcal{I})$ soliton solution, $c_-(x, t, \hat{\sigma}_d(\mathcal{I}))$ is defined in (3.6), $T(z)$ is defined in (5.4), $T_1$ is defined in (5.10), $d$ is defined in (2.13) and

$$f_{12}^\pm = \frac{1}{i\sqrt{2}}[M^{(\text{out})}(0)^{-1}(M^{(\text{out})}(z_0)^{-1}M_1^{(\text{pc}, \pm)}(z_0)M^{(\text{out})}(z_0)]_{12},$$

$$f_{11}^\pm = \frac{1}{i\sqrt{2}}[M^{(\text{out})}(0)^{-1}(M^{(\text{out})}(z_0)^{-1}M_1^{(\text{pc}, \pm)}(z_0)M^{(\text{out})}(z_0)]_{11},$$

where $M_1^{(\text{pc}, \pm)}(z)$ can be expressed as

$$M_1^{(\text{pc}, \pm)}(z) = \begin{pmatrix} 0 & -\beta_{12}^\pm(r_0) \\ \beta_{21}^\pm(r_0) & 0 \end{pmatrix},$$

and $M^{(\text{out})}(z)$ is defined in (7.4). Here,

$$\beta_{12}^\pm(r_0) = \beta^\pm(r(z_0)) = \alpha(z_0, \pm)e^{i\frac{y^2}{4t}+iv(z_0)[\log 8|t|]},$$

where

$$|\alpha(z_0, \pm)|^2 = |\nu(z_0)|,$$

$$\arg \alpha(z_0, \pm) = \pm \frac{\pi}{4} \pm \arg \Gamma(i\nu(z_0)) - \arg r(z_0)$$

$$- 4 \sum_{k \in \Delta_{z_0}} \arg(z_0 - z_k) + 2 \int_{-\infty}^{z_0} \log |z_0 - s|d\nu(s),$$

$r(z)$ is defined in (2.22), $\nu(z)$ is defined in (5.3) and $\Gamma$ denotes the gamma function.

**Remark 1.2.** Theorem 1.1 needs the initial value to meet $q_0(x) \in \mathcal{H}(\mathbb{R})$, so that the inverse scattering transform possesses well mapping properties [52]. Indeed, the asymptotic results only depend on the $H^{1,1}(\mathbb{R})$ norm of $r$ in this work. So we restrict the initial potential $q_0(x) \in \mathcal{H}(\mathbb{R})$. Particularly, for any $q_0(x) \in \mathcal{H}(\mathbb{R})$ admitting Assumption 3.1, the process of the long-time analysis and calculations shown in this work is unchanged.
Organization of the Rest of the Work

In Sect. 2, based on the Lax pair of the WKI equation, we introduce two kinds of eigenfunctions to deal with the spectral singularity. Moreover, the analyticity, symmetries and asymptotic properties are analyzed.

In Sect. 3, by using similar ideas to [55], the RHP for \( M(z) \) is constructed for the WKI equation with initial problem.

In Sect. 4, for given initial data \( q_0(x) \in \mathcal{H}(\mathbb{R}) = W^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R}) \), we prove that the reflection coefficient \( r(z) \) belongs to \( H^{1,1}(\mathbb{R}) \).

In Sect. 5, we introduce the matrix function \( T(z) \) to define the new RHP for \( M^{(1)}(z) \). Then, its jump matrix can be decomposed into two triangle matrices near the phase point \( z = z_0 \).

In Sect. 6, we make the continuous extension of the jump matrix off the real axis by introducing a matrix function \( R^{(2)}(z) \) and get a mixed \( \bar{\partial} \)-Riemann–Hilbert (RH) problem.

In Sect. 7, we decompose the mixed \( \bar{\partial} \)-RH problem into two parts that are a model RH problem with \( \bar{\partial}R^{(2)} = 0 \) and a pure \( \bar{\partial} \)-RH problem with \( \bar{\partial}R^{(2)} \neq 0 \), i.e., \( M_R^{(2)} \) and \( M^{(3)} \). Furthermore, we solve the model RH problem \( M_R^{(2)} \) via an outer model \( M^{(out)}(z) \) for the soliton part and local solvable model near the phase point \( z_0 \) which can be solved by matching parabolic cylinder model problem. Besides, the error function \( E(z) \) with a small-norm RH problem is achieved.

In Sect. 8, the pure \( \bar{\partial} \)-RH problem for \( M^{(3)} \) is studied.

Finally, we obtain the soliton resolution and long time asymptotic behavior of the WKI equation.

2. The Spectral Analysis of WKI Equation

In order to study the soliton resolution of the initial value problem (IVP) for the WKI equation via applying \( \bar{\partial} \)-steepest descent method, we first construct a RHP based on the Lax pair of the WKI equation. The WKI equation admits the Lax pair

\[
\psi_x = U \psi, \quad \psi_t = V \psi,
\]

where \( U = -iz \sigma_3 + zQ \),

\[
V = \begin{pmatrix}
-\frac{2iz^2}{\Phi} & \frac{2qz^2}{\Phi} + iz \left( \frac{q}{\Phi} \right)_x \\
-\frac{2\bar{q}z^2}{\Phi} + iz \left( \frac{\bar{q}}{\Phi} \right)x & \frac{2iz^2}{\Phi}
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
0 & q \\
\bar{q} & 0
\end{pmatrix}, \quad \Phi = \sqrt{1 + |q|^2},
\]

and \( \sigma_3 \) is the third Pauli matrix:

\[
\sigma_1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

The \( \bar{q} \) means the complex conjugate of \( q \).

To investigate the IVP of integrable equations, we generally employ the \( x \)-part of the Lax pair to study the long time asymptotic behaviors. The \( t \)-part
of Lax pair is used to control the time evolution of the scattering data based on the inverse scattering transform method. However, the Lax pair (2.1) of the WKI equation possesses two singularities, i.e., \( z = 0 \) and \( z = \infty \). As a result, the behavior of the solutions of spectral problem (2.1) also need to be investigated as spectral parameter \( z \rightarrow 0 \). Then, via employing the \( t \)-part of Lax pair and the expansion of the eigenfunction, the potential function \( q(x,t) \) can be recovered. Therefore, we deal with the two singularities at \( z = 0 \) and \( z = \infty \) using two different transformations in the following analysis.

2.1. The Singularity at \( z = 0 \)

Based on the initial condition that \( q_0 \in \mathcal{H}(\mathbb{R}) \), letting \( x \rightarrow \pm \infty \), we can obtain the asymptotic scattering problem and construct the two Jost solutions, i.e.,

\[
\psi_\pm \sim e^{-i(2x+2iz^2)t}\sigma_3, \ x \rightarrow \pm \infty. \tag{2.3}
\]

Then, we make a gauge transformation

\[
\psi(x,t;z) = \mu^0(x,t;z)e^{-i(2x+2iz^2)t}\sigma_3. \tag{2.4}
\]

As a result, we obtain \( \mu^0(z) \sim \mathbb{I}(x \rightarrow \pm \infty) \). In addition, the equivalent Lax pair of \( \mu^0(z) \) can be written as

\[
\begin{align*}
\mu_0^2 + iz[\sigma_3, \mu^0] &= U_1 \mu^0, \\
\mu_t^0 + 2iz^2[\sigma_3, \mu^0] &= V_1 \mu^0,
\end{align*} \tag{2.5}
\]

where

\[
U_1 = zQ, \ V_1 = \begin{pmatrix} 2iz^2(1 - \frac{1}{\Phi}) & 2qz^2 + iz\left(\frac{q}{\Phi}\right)_x \\
-2qz^2 + iz\left(\frac{q}{\Phi}\right)_x & -2iz^2(1 - \frac{1}{\Phi}) \end{pmatrix},
\]

and \( \mu^0 = \mu^0(x,t;z) \). Besides, \([A,B]\) means \( AB - BA \) where \( A \) and \( B \) are \( 2 \times 2 \) matrices. Next, we rewrite the Lax pair (2.5) in full derivative form, i.e.,

\[
d(e^{i(2x+2z^2)t}\sigma_3 \mu_0^0) = e^{i(2x+2z^2)t}\sigma_3(U_1dx + V_1dt)\mu^0_0, \tag{2.6}
\]

where \( e^{\sigma_3 A} = e^{\sigma_3} Ae^{-\sigma_3} \). Then, the solutions of (2.6) can be derived as Volterra integrals

\[
\mu_\pm^0(x,t;z) = \mathbb{I} + \int_{-\infty}^{\infty} e^{-iz(x-y)}\sigma_3 U_1(y,t;z)\mu^0_\pm(y,t;z)dy, \tag{2.7}
\]

from which we can derive the analytical properties of \( \mu^0_\pm \).

**Proposition 2.1.** It is assumed that \( q(x) - q_0 \in H^{1,1}(\mathbb{R}) \). Then, \( \mu^0_{-,1}, \mu^0_{+,2} \) are analytic in \( \mathbb{C}^+ \) and \( \mu^0_{-,2}, \mu^0_{+,1} \) are analytic in \( \mathbb{C}^- \). The \( \mu^0_{\pm,j} (j = 1,2) \) mean the \( j \)-th column of \( \mu^0_{\pm} \), and \( \mathbb{C}^\pm \) mean the upper and lower complex \( z \)-plane, respectively.

Furthermore, we study the asymptotic property of \( \mu^0_\pm \) as \( z \rightarrow 0 \). Substituting the following asymptotic expansions

\[
\mu^0_\pm = \mu^0_\pm^{(0)} + \mu^0_\pm^{(1)} z + O(z^2), \ z \rightarrow 0,
\]
into the Lax pair (2.5) and comparing the same power coefficients of $z$, we obtain the expressions of $\mu_{\pm}^{0}(0)$ and $\mu_{\pm}^{0}(1)$. It should be pointed out that $\mu_{\pm}^{0}(j)(j = 1, 2, \ldots)$ are independent of $z$.

**Proposition 2.2.** The functions $\mu_{\pm}^{0}(x,t; z)$ admit the following asymptotic property as $z \to 0$,

$$
\mu_{\pm}^{0}(x,t; z) = 1 + \int_{-\infty}^{x} zQ \, dx + O(z^2). \quad (2.8)
$$

### 2.2. The Singularity at $z = \infty$

In this part, due to the singularity at $z = \infty$, our first purpose is to control the asymptotic behavior of the Lax pair (2.1) as $z \to \infty$.

However, it is difficult to directly analyze the Lax pair (2.1) of the WKI equation (1.3). This difficulty was successfully overcome by Boutet de Monvel and Shepelsky through introducing a transformation to modify the Lax pair of the Camassa–Holm equation and short wave equations [53–55]. Following the idea [53], we introduce the transformation

$$
\psi(x,t;z) = G(x,t)\phi(x,t;z), \quad (2.9)
$$

where

$$
G(x,t) = \sqrt{\frac{\Phi + 1}{2\Phi}} \left( \frac{1}{i(1-\Phi)} \right) \left( \frac{i(1-\Phi)}{q(x,t)} \right),
$$

then, the Lax pair related to $\phi(x,t; z)$ can be derived as

$$
\phi_{x} + iz\Phi \sigma_{3} \phi = U_{2} \phi, \quad \phi_{t} + \left( 2iz^{2} + z \frac{q\bar{q}x - q_{x}}{2\Phi^{2}} \right) \sigma_{3} \phi = V_{2} \phi, \quad (2.10)
$$

where

$$
U_{2} = \begin{pmatrix}
-\frac{\bar{q}_{x} - q_{x}}{4\Phi(1+\Phi)} & -i\frac{q(\bar{q}_{x} - q_{x}) - |q|_{x}^{2}}{4\Phi^{2}(\Phi^{2} - 1)} \\
\frac{i\bar{q}[q(\bar{q}_{x} - q_{x}) + |q|_{x}^{2}]}{4\Phi^{2}(\Phi^{2} - 1)} & -\frac{q_{x}^{2} - q_{x}}{4\Phi(1+\Phi)}
\end{pmatrix},
$$

$$
V_{2} = \begin{pmatrix}
v_{2,11} & v_{2,12} \\
v_{2,21} & -v_{2,11}
\end{pmatrix}
$$

with

$$
v_{2,11} = -\frac{q\bar{q}_{t} - q_{t}\bar{q}}{4\Phi(1+\Phi)},
$$

$$
v_{2,12} = \frac{i\bar{q}[q(\bar{q}_{x} - q_{x}) - 2q_{x}(\Phi - 1)]}{2\Phi^{3}(\Phi - 1)\bar{q}} z - \frac{i\bar{q}[q(\bar{q}_{t} - q_{t}) - 2\bar{q}_{t}(\Phi - 1)]}{4\Phi(1+\Phi)\bar{q}},
$$

$$
v_{2,21} = \frac{i\bar{q}[q(\bar{q}_{x} - q_{x}) + 2q_{x}(\Phi - 1)]}{2\Phi^{3}(\Phi - 1)\bar{q}} z + \frac{i\bar{q}[q(\bar{q}_{t} - q_{t}) + 2q_{t}(\Phi - 1)]}{4\Phi^{2}(\Phi - 1)\bar{q}}.
$$

Define $p_{x}(x,t; z) = iz\Phi \sigma_{3}$ and $p_{t}(x,t; z) = (2iz^{2} + z \frac{q\bar{q}x - q_{x}}{2\Phi^{2}})\sigma_{3}$, then $p_{x}$ and $p_{t}$ are compatible, i.e., $p_{xt} = p_{tx}$. We rewrite this relation as

$$
\begin{pmatrix}
q\bar{q}_{x} - q_{x}\bar{q} \\
\frac{q_{x}^{2} - q_{x}}{2\Phi^{2}}
\end{pmatrix}_{x} = 0.
$$
which is the conservation law [10] of the WKI equation. Therefore, we can define \( p(x, t; z) \) as
\[
p(x, t; z) = iz \left( x - \int_{x}^{-\infty} (\Phi(y) - 1) \, dy \right) + 2iz^2 t. \tag{2.11}
\]
Furthermore, we define \( \varphi = \phi \sigma_3 \), we can obtain the equivalent Lax pair
\[
\begin{align*}
\varphi_x(x, t; z) + p(x, t; z) \varphi(\sigma_3, \varphi(x, t; z)) &= U_2 \varphi(x, t; z), \\
\varphi_t(x, t; z) + p_t(x, t; z) \varphi(\sigma_3, \varphi(x, t; z)) &= V_2 \varphi(x, t; z).
\end{align*}
\]
Because the potential \( q(x, t) \) is complex valued, the diagonal elements of the matrix \( U_2 \) do not equal to zero which leads to the solutions of spectral problem do not approximate the identity matrix as \( z \to \infty \). Thus, we introduce an improved transformation
\[
\psi(x, t; z) = G(x, t) e^{d_+ \sigma_3} \mu(x, t; z) e^{-d_- \sigma_3}, \tag{2.12}
\]
where
\[
\begin{align*}
d_+ &= \int_{x}^{\infty} \frac{q_x - q_x \bar{q}}{4\Phi(\Phi + 1)} (s, t) \, ds, \\
d_- &= \int_{-\infty}^{x} \frac{q_x - q_x \bar{q}}{4\Phi(\Phi + 1)} (s, t) \, ds, \\
d &= d_+ + d_- = \int_{-\infty}^{\infty} \frac{q_x - q_x \bar{q}}{4\Phi(\Phi + 1)} (s, t) \, ds. \tag{2.13}
\end{align*}
\]
Then, the equivalent Lax pair of \( \psi(x, t; z) \) (2.1) can be written as
\[
\begin{align*}
\mu_x + p_x [\sigma_3, \mu] &= -e^{-d_+ \sigma_3} U_3 \mu, \\
\mu_t + p_t [\sigma_3, \mu] &= -e^{-d_+ \sigma_3} V_3 \mu, \tag{2.14}
\end{align*}
\]
where
\[
\begin{align*}
U_3 &= U_2 + \frac{q_x - q_x \bar{q}}{4\Phi(\Phi + 1)} \sigma_3, \\
V_3 &= V_2 + \frac{q_x - q_x \bar{q}}{4\Phi(\Phi + 1)} \sigma_3.
\end{align*}
\]
Furthermore, (2.14) can be written as
\[
d(e^{-p(x, t; z) \sigma_3} \mu) = e^{-p(x, t; z) \sigma_3} e^{-d_+ \sigma_3} (U_3 dx + V_3 dt) \mu, \tag{2.15}
\]
from which we derive Volterra type integral equations
\[
\mu_-(x, t; z) = I + \int_{\pm \infty}^{x} e^{[p(x, t; z) - p(s, t; z)] \sigma_3} e^{-d_+ \sigma_3} U_3(s, t; z) \mu_+(s, t; z) \, ds. \tag{2.16}
\]
Then, according to the definition of \( \mu(x, t; z) \) and the above integrals (2.16), the properties of the eigenfunctions \( \mu_\pm(x, t; z) \) can be easily derived.

**Proposition 2.3.** (Analytic property) It is assumed that \( q(x) - q_0 \in H^{1,1}(\mathbb{R}) \). Then, \( \mu_{-1}, \mu_{+2} \) are analytic in \( \mathbb{C}^+ \) and \( \mu_{-2}, \mu_{+1} \) are analytic in \( \mathbb{C}^- \). The \( \mu_{\pm,j} \) \((j = 1, 2)\) mean the \( j \)-th column of \( \mu_\pm \).
Proposition 2.4. (Symmetry property) The eigenfunctions $\mu_{\pm}(x, t; z)$ satisfy the following symmetry relation

$$
\bar{\mu}_{\pm}(x, t; \bar{z}) = -\sigma_2 \mu_{\pm}(x, t; z) \sigma_2.
$$

(2.17)

Proposition 2.5. (Asymptotic property for $z \to \infty$) The eigenfunctions $\mu_{\pm}(x, t; z)$ satisfy the following asymptotic behavior

$$
\mu_{\pm}(x, t; z) = I + O(z^{-1}), \ z \to \infty.
$$

(2.18)

2.3. The Scattering Matrix

For $z \in \mathbb{R}$ both eigenfunctions $\mu_{+}(x, t; z)$ and $\mu_{-}(x, t; z)$ are the fundamental matrix solutions of Eq. (2.14), there exists a matrix $S(z)$, the scattering matrix, satisfying that

$$
\mu_{+}(x, t; z) = \mu_{-}(x, t; z) e^{-p(x, t; z)\hat{\sigma}_3} S(z), \quad z \in \mathbb{R},
$$

(2.19)

where $S(z) = (s_{ij}(z))$ ($i, j = 1, 2$) is independent of the variable $x$ and $t$. The coefficients $s_{11}(z)$ and $s_{22}(z)$ can be expressed as

$$
s_{11}(z) = \det(\mu_{+,1}, \mu_{-,2}), \quad s_{22}(z) = \det(\mu_{-,1}, \mu_{+,2}).
$$

Then, on the basis of the above propositions and the definition of the scattering matrix $S(z)$, we obtain the following standard results. The similar proofs can be found in many literatures [see, e.g., [56]].

Proposition 2.6. The scattering matrix $S(z)$ possesses the properties:

- (Analytic property) The scattering coefficients $s_{11}$ and $s_{22}$ are, respectively, analytic in $\mathbb{C}^-$ and $\mathbb{C}^+$.
- (Symmetry property) The scattering coefficients $s_{ij}(z)$ ($i, j = 1, 2$) possess the following relations:

$$
s_{11}(z) = \overline{s_{22}(\bar{z})}, \quad s_{12}(z) = -\overline{s_{21}(\bar{z})}.
$$

(2.20)

- (Asymptotic property for $z \to \infty$)

$$
S(z) = I + O(z^{-1}), \ z \to \infty.
$$

(2.21)

Additionally, we define the reflection coefficient as

$$
r(z) = \frac{s_{12}(z)}{s_{22}(z)},
$$

(2.22)

then, it follows from (2.20) that

$$
\frac{s_{12}(z)}{s_{22}(z)} = -\frac{\overline{s_{21}(\bar{z})}}{\overline{s_{11}(\bar{z})}} = -\overline{r(\bar{z})} = -\overline{\bar{r}(z)}
$$

for $z \in \mathbb{R}$. 
2.4. The Connection Between $\mu_{\pm}(x, t; z)$ and $\mu^0_{\pm}(x, t; z)$

In next part, we can use the eigenfunctions $\mu_{\pm}(x, t; z)$ to construct the matrix $M(x, t; z)$ and further formulate a RHP. While in order to obtain the reconstruction formula between the solution $q(x, t)$ and the RHP, the asymptotic behavior of $\mu_{\pm}$ as $z \to 0$ is needed. Thus, we need to establish the connection between $\mu_{\pm}(x, t; z)$ and $\mu^0_{\pm}(x, t; z)$.

Referring to the transformation (2.4) and (2.12), we assume that the eigenfunctions $\mu_{\pm}(x, t; z)$ and $\mu^0_{\pm}(x, t; z)$ related to each other as

$$
\mu_{\pm}(x, t; z) = e^{-d_{\pm}^s \sigma_3} G^{-1}(x, t) \mu_{\pm}^0(x, t; z) e^{-i(zx + 2z^2 t) \sigma_3} C_{\pm}(z) e^{p(x, t; z) \sigma_3} e^{d_{\pm} \sigma_3},
$$

(2.23)

where $C_{\pm}(z)$ are independent of $x$ and $t$. Taking $x \to \infty$, (2.23) gives

$$
C_-(z) = \mathbb{I}, \quad C_+(z) = e^{-d_{\pm} \sigma_3} e^{-izc \sigma_3},
$$

where $c = \int_{-\infty}^{+\infty} (\Phi(s) - 1) ds$ is a quantity conserved under the dynamics governed by (1.3). As a result, we obtain

$$
\mu_-(x, t; z) = e^{-d_+ \sigma_3} G^{-1}(x, t) \mu_-^0(x, t; z) e^{-iz \int_{-\infty}^{+\infty} (\Phi(s) - 1) ds \sigma_3} e^{d_+ \sigma_3},
\mu_+(x, t; z) = e^{-d_- \sigma_3} G^{-1}(x, t) \mu_+^0(x, t; z) e^{-iz \int_{-\infty}^{+\infty} (\Phi(s) - 1) ds \sigma_3}.
$$

(2.24)

3. The Riemann–Hilbert Problem for WKI Equation

In order to avoid dealing with many possible pathologies in the following part, we first make some assumptions.

**Assumption 3.1.** For the Cauchy problem of WKI equation (1.3), the initial value $q_0$ generates generic scattering data in the sense that:

- For $z \in \mathbb{R}$, no spectral singularities exist, i.e., $s_{22}(z) \neq 0 \ (z \in \mathbb{R})$;
- Suppose that $s_{22}(z)$ possesses $N$ zero points, denoted as $\mathcal{Z} = \{(z_j, Im z_j > 0) | j = 1, \ldots, N\}$.
- The discrete spectrum is simple, i.e., if $z_0$ is the zero of $s_{22}(z)$, then $s'_{22}(z_0) \neq 0$.

Define weighted Sobolev spaces

$$
W^{k,p}(\mathbb{R}) = \{ f(x) \in L^p(\mathbb{R}) : \partial^j f(x) \in L^p(\mathbb{R}), j = 1, 2, \ldots, k \},
$$

$$
H^{k,2}(\mathbb{R}) = \{ f(x) \in L^p(\mathbb{R}) : x^2 \partial^j f(x) \in L^p(\mathbb{R}), j = 1, 2, \ldots, k \},
$$

(3.1)

$$
\mathcal{H}(\mathbb{R}) = W^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R}),
$$

then we can further show that

**Proposition 3.2.** *If the initial data $q_0(x) \in \mathcal{H}(\mathbb{R})$, then $r(z) \in H^{1,1}(\mathbb{R})$ and the map $q_0(x) \to r(z)$ is Lipschitz continuous from $\mathcal{H}(\mathbb{R})$ into $H^{1,1}(\mathbb{R})$.***
Next, we define a sectionally meromorphic matrices
\[
\tilde{M}(x, t; z) = \begin{cases} 
\tilde{M}^+(x, t; z) = \left( \mu_{-1}(x, t; z), \frac{\mu_{+2}(x, t; z)}{s_{22}(z)} \right), & z \in \mathbb{C}^+,
\tilde{M}^-(x, t; z) = \left( \frac{\mu_{+1}(x, t; z)}{s_{11}(z)}, \mu_{-2}(x, t; z) \right), & z \in \mathbb{C}^-,
\end{cases}
\]
where \(\tilde{M}^\pm(x, t; z) = \lim_{\varepsilon \to 0^\pm} \tilde{M}(x, t; z \pm i\varepsilon), \varepsilon \in \mathbb{R}\).

For the initial data that admits Assumption 3.1, the matrix function \(\tilde{M}(x, t; z)\) solves the following matrix RHP.

**Riemann-Hilbert Problem 3.3.** Find an analysis function \(\tilde{M}(x, t; z)\) with the following properties:

- \(\tilde{M}(x, t; z)\) is meromorphic in \(\mathbb{C} \setminus \mathbb{R}\);
- \(\tilde{M}^+(x, t; z) = \tilde{M}^-(x, t; z)\tilde{V}(x, t; z), z \in \mathbb{R}\), where
  \[
  \tilde{V}(x, t; z) = \begin{pmatrix} 1 & r(z)e^{-2zp(x, t, z)} \\ -\bar{r}(z)e^{2zp(x, t, z)} & 1 + |r(z)|^2 \end{pmatrix};
  \]  \hspace{1cm} (3.3)
- \(\tilde{M}(x, t; z) = I + O(z^{-1})\) as \(z \to \infty\).

Referring to (2.19), there exist norming constants \(b_j\) such that
\[
\mu_{+2}(z_j) = b_j e^{-2p(z_j)} \mu_{-1}(z_j).
\]
Then, the residue condition of \(\tilde{M}(x, t; z)\) can be shown as
\[
\begin{align*}
Res_{z=z_j} \tilde{M} &= \lim_{z \to z_j} \tilde{M} \begin{pmatrix} 0 & c_j e^{-2p(z_j)} \\ 0 & 0 \end{pmatrix}, \\
Res_{z=z_j} \tilde{M} &= \lim_{z \to z_j} \tilde{M} \begin{pmatrix} 0 & 0 \\ -\bar{c}_j e^{2p(z)} & 0 \end{pmatrix},
\end{align*}
\]
where \(c_j = \frac{b_j}{s_{22}(z_j)}\).

**Remark 3.4.** On the basis of the Zhou’s vanishing lemma, the existence of the solutions of RHP 3.3 for \((x, t) \in \mathbb{R}^2\) is guaranteed. According to the results of Liouville’s theorem, we know that if a solution exists, it is unique.

Next, our purpose is to reconstruct the solution \(q(x, t)\). So we need to study the asymptotic behavior of \(\tilde{M}(x, t; z)\) as \(z \to 0\), i.e.,
\[
\tilde{M}(x, t; z) = e^{-d_+ \sigma_3} G^{-1}(x, t) \left[ \mathbb{I} + z \int_{\pm \infty}^x \left( \begin{array}{cc} 0 & q \\ -\bar{q} & 0 \end{array} \right) dx - ic_- \sigma_3 \right] + O(z^2)
\]
\times e^{d\sigma_3}, \quad z \to 0,
\]  \hspace{1cm} (3.5)
where \(c_-(x, t) = \int_x^{\pm \infty} (\Phi(s, t) - 1) ds\). However, because \(p(x, t; z)\) that appears in jump matrix (3.3) is not clear, it is quite difficult to reconstruct the solution \(q(x, t)\) from (3.5). Boutet de Monvel and Shepelsky have overcome this problem.
by changing the spatial variable of the Camassa–Holm equation and short wave equations [57, 58]. Therefore, following the idea in [57], we introduce a new scale
\[ y(x, t) = x - \int_x^{-\infty} (\Phi(s, t) - 1) ds = x - c_-(x, t), \] (3.6)
which leads to the jump matrix can be expressed explicitly. But the solution \( q(x, t) \) can be expressed only in implicit form: It will be given in terms of functions in the new scale, whereas the original scale will also be given in terms of functions in the new scale. Based on the definition of \( y(x, t) \), we further define that
\[ \tilde{M}(x, t; z) = M(y(x, t), t; z), \]
then, the \( M(y(x, t), t; z) \) satisfies the following matrix RHP.

**Riemann-Hilbert Problem 3.5.** Find an analysis function \( M(y, t; z) \) with the following properties:
- \( M(y, t; z) \) is meromorphic in \( \mathbb{C} \setminus \mathbb{R} \);
- \( M^+(y, t; z) = M^-(y, t; z)V(y, t; z) \), \( z \in \mathbb{R} \), where
  \[
  V(y, t; z) = e^{-i(zy + 2z^2 t)} \begin{pmatrix} 1 & r(z) \\ -\frac{1}{r(z)} - |r(z)|^2 \end{pmatrix}; \]
- \( M(y, t; z) = I + O(z^{-1}) \) as \( z \to \infty \);
- \( M(y, t; z) \) possesses simple poles at each point in \( Z \cup \bar{Z} \) with:
  \[
  \text{Res}_{z=z_j} M(z) = \lim_{z \to z_j} M(z) \begin{pmatrix} 0 & c_j e^{-2i(z_j y + 2z_j^2 t)} \\ 0 & 0 \end{pmatrix},
  \text{Res}_{\bar{z}=\bar{z}_j} M(z) = \lim_{z \to \bar{z}_j} M(z) \begin{pmatrix} 0 & 0 \\ -\bar{c}_j e^{2i(\bar{z}_j y + 2\bar{z}_j^2 t)} & 0 \end{pmatrix}. \]

**Proposition 3.6.** If \( M(y, t; z) \) satisfies the above conditions, then the RHP 3.5 possesses a unique solution. Additionally, on the basis of the solution of RHP 3.5, the solution \( q(x, t) \) of the initial value problem (1.3) and (1.4) can be derived in parametric form, i.e., \( q(x, t) = q(y(x, t), t) \) where
\[
q(y, t) = e^{2d} \lim_{z \to 0} \frac{\partial}{\partial y} \left( \frac{M^{-1}(y, t; 0)M(y, t; z)}{z} \right)_{12},
\]
\[
x(y, t) = y + \lim_{z \to 0} \left( \frac{M^{-1}(y, t; 0)M(y, t; z)}{z} \right)_{11} - 1. \]

**Proof.** A fact that the jump matrix \( V(y, t; z) \) is a Hermitian matrix gives rise to that the RHP 3.5 indeed has a solution. Moreover, due to the normalize condition, i.e., \( M(y, t; z) = I + O(z^{-1}) \) as \( z \to \infty \), the RHP 3.5 possesses only one solution.

Referring to the asymptotic formula (3.5), the statements of the solution \( q(x, t) \) can be derived.
4. The Scattering Maps

In this section, our purpose is to give the proof of the correctness of Proposition 3.2.

In the following part, we only take the $x$-part of Lax pair into consideration to prove Proposition 3.2. In fact, by considering the $t$-part of Lax pair (2.14) and carrying out the standard direct scattering transform, we can derive the linear time evolution of the reflection coefficient $r(z)$, i.e., $r(z,t) = e^{2iz^2t}r(z,0)$. Next, in order to give the proof of Proposition 3.2, we first give some notations.

- If $I$ is an interval on the real axis $\mathbb{R}$, and $X$ is a Banach space, we denote $C^0(I,X)$ as a space of continuous functions on $I$ taking values in $X$. The norm of $f(x) \in C^0(I,X)$ is denoted as
  \[ \|f(x)\|_{C^0(I,X)} = \sup_{x \in I} \|f(x)\|_X. \]

- We denote $C^0_B(X)$ as a space of bounded continuous functions on $X$.

- If $f = (f_1,f_2)^T \in X$, the norm of vector function $f \in X$ is denoted as
  \[ \|f(x)\|_X \triangleq \|f_1\|_X + \|f_1\|_X. \]

Then, taking $t = 0$, we consider the case of singularity at $z = \infty$. According to the above analyses, the matrix function $G(x,t)$ can be rewritten as
\[
G(x) = \sqrt{\frac{\Phi(x) + 1}{2\Phi(x)}} \left( \frac{1}{i(1-\Phi(x))} \frac{i(1-\Phi(x))}{q(x)} \right),
\]

\[ p(x) = iz \left( x - \int_x^\infty (\Phi(s) - 1)ds \right). \]

Then, we make the following transformation
\[
\psi_{\pm}(x,z) = G(x)e^{d_+\bar{\sigma}_3}\mu_{\pm}(x,z)e^{-d_-\sigma_3e^{-p(x)}\sigma_3}. \tag{4.1}
\]

The equivalent Lax pair of (2.14) is obtained as
\[
\mu_{\pm,x} + p_x[\sigma_3,\mu_{\pm}] = e^{-d_+\bar{\sigma}_3}U_3\mu_{\pm}. \tag{4.2}
\]

It should be pointed out that $\mu_{\pm}$ satisfy that
\[
\mu_{\pm} \sim I, \quad x \to \pm \infty. \tag{4.3}
\]

Then, we derive the following Volterra integral equations,
\[
\mu_{\pm}(x,z) = I + \int_x^{\pm\infty} e^{(p(x) - p(y))\bar{\sigma}_3}e^{-d_+\bar{\sigma}_3}U_3\mu_{\pm}(y,z)dy. \tag{4.4}
\]

Next, in order to approach our aim, it is necessary to estimate the $L^2$-integral property of $\mu_{\pm}(z)$ and their derivatives. We first introduce some results of functional analysis that would be used in the following analysis [44].

**Lemma 4.1.** $F$ is a two factorial square matrix and $g$ is a column vector. Then $|Fg| \leq |F||g|$. 

Lemma 4.2. For $\psi(\eta) \in L^2(\mathbb{R})$, $f(x) \in L^{2,1/2}(\mathbb{R})$, then

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}} f(y)e^{-2i\eta(p(x) - p(y))}\psi(\eta)d\eta dy \right| = \left| \int_{\mathbb{R}} f(y)\psi(2(p(x) - p(y)))dy \right|$$

$$\lesssim \left( \int_{\mathbb{R}} |f(y)|^2 dy \right)^{1/2} \| \psi \|_2;$$

$$\int_0^\infty \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)e^{2i\eta(p(x) - p(y))}dy \right|^2 d\eta dx \lesssim \| f \|_{2,1/2}^2.$$

The proof of the above lemmas are trivial, so we omit it.

Next, observing a fact that the properties of $\mu_{\pm,2}$ can be obtained immediately by applying Proposition 2.4, we only need to consider the properties of $\mu_{\pm,1}$. For convenience, we define

$$[\mu_{\pm}]_1(x, z) - e_1 \triangleq n_{\pm}(x, z) = \begin{pmatrix} n_{\pm}^{(1)} \\ n_{\pm}^{(2)} \end{pmatrix},$$

(4.5)

where $e_1$ is a column vector $(1 \ 0)^T$. Next, we introduce the integral operators $P_{\pm}$ satisfying

$$P_{\pm}(f)(x, z) = \int_{\mathbb{R}} K_{\pm}(x, y, z)f(y)dy,$$

(4.6)

where

$$K_{\pm}(x, y, z) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2(p(x) - p(y))} \end{pmatrix} \tilde{U}_3(y),$$

(4.7)

with

$$\tilde{U}_3 = -\begin{pmatrix} 0 & -i\frac{q\Phi(q\tilde{q}_x - q_x \tilde{q}) - |q|_2^2}{4\Phi^2(\Phi^2 - 1)} e^{-2d_z} \\ \frac{iq\Phi(q\tilde{q}_x - q_x \tilde{q}) + |q|_2^2}{4\Phi^2(\Phi^2 - 1)} e^{2d_z} & 0 \end{pmatrix}.$$

Then, the Volterra integral equations (4.4) are transformed into

$$n_{\pm} = P_{\pm}(\mu_{\pm,1}) = P_{\pm}(e_1) + P_{\pm}(n_{\pm}).$$

(4.8)

Then, taking derivation of (4.8) with respect to $z$ yields

$$[n_{\pm}]_z = [P_{\pm}]_z(e_1) + [P_{\pm}]_z(n_{\pm}) + P_{\pm}([n_{\pm}]_z).$$

(4.9)

Moreover, $[P_{\pm}]_z$ also are integral operators with integral kernel $[K_{\pm}]_z(x, y, z)$, i.e.,

$$[K_{\pm}]_z(x, y, z) = -2(\bar{p}(x) - \bar{p}(y)) \begin{pmatrix} 0 & 0 \\ 0 & e^{-2(p(x) - p(y))} \end{pmatrix} \tilde{U}_3(y),$$

(4.10)

where $\bar{p}(x) = i(x - \int_x^\infty (\Phi(s) - 1)ds)$.

Then, the properties of the integral operators $[T_{\pm}]$ and $[T_{\pm}]_z$ can be confirmed by the following lemma. To make the analysis clearer, we use $C^0_B$, $C^0$ and $L^2_{xz}$, respectively, to represent $C^0_B(\mathbb{R}_x^+ \times \mathbb{R}_z)$, $C^0(\mathbb{R}_x^+, L^2(\mathbb{R}_z))$ and $L^2(\mathbb{R}_x^+ \times \mathbb{R}_z)$. 
Lemma 4.3. If $P_{\pm}$ and $[P_{\pm}]_z$ are integral operators defined above, then $P_{\pm}(e_1)$ $(x, z)$ and $[P_{\pm}]_z(e_1)(x, z)$ belong to $C^0_B \cap C^0 \cap L^2_{xz}$.

Proof. According to the definition of the operators $P_{\pm}$, we have

$$P_{\pm}(e_1)(x, z) = \int_x^{+\infty} \left( \begin{array}{cc} 1 & 0 \\ 0 & e^{-z(p(x) - p(y))} \end{array} \right) \tilde{U}_3(y)e_1 dy. \quad (4.11)$$

Then, the following result is derived immediately

$$\|P_{\pm}(e_1)(x, z)\|_{\infty} \lesssim \|q_x\|_1. \quad (4.12)$$

Moreover, by referring to Lemma 4.2, we obtain

$$\|P_{\pm}(e_1)(x, z)\|_{C^0} \lesssim \|q_x\|_2, \quad (4.13)$$

$$\|P_{\pm}(e_1)(x, z)\|_{L^2_{xz}} \lesssim \|q_x\|_{2, \frac{1}{2}}. \quad (4.14)$$

On the basis of (4.10), the operator $[P_{\pm}]_z(e_1)(x, z)$ is expressed as

$$[P_{\pm}]_z(e_1)(x, z) = \int_x^{+\infty} -2(\tilde{p}(x) - \tilde{p}(y)) \left( \begin{array}{cc} 0 & 0 \\ 0 & e^{2(p(x) - p(y))} \end{array} \right) \tilde{U}_3(y)e_1 dy. \quad (4.15)$$

Observing a fact that $|\tilde{p}(x) - \tilde{p}(y)| \leq |x - y| + \|q\|_1$, the following results are derived similarly,

$$\|P_{\pm}]_z(e_1)(x, z)\|_{L^2_{xz}} \lesssim \|q_x\|_{1, 1, 1} + \|q_x\|_{1, 1} \|q\|_1, \quad (4.16)$$

$$\|P_{\pm}]_z(e_1)(x, z)\|_{C^0} \lesssim \|q_x\|_{2, 1, 2} + \|q_x\|_2 \|q\|_1, \quad (4.17)$$

$$\|P_{\pm}]_z(e_1)(x, z)\|_{L^2_{xz}} \lesssim \|q_x\|_{2, \frac{1}{2}} + \|q_x\|_{2, \frac{1}{2}} \|q\|_1. \quad (4.18)$$

$\square$

Lemma 4.4. The integral operators $P_{\pm}$ and $[P_{\pm}]_z$ map $C^0_B \cap C^0 \cap L^2_{xz}$ to itself. Additionally, $(I - P_{\pm})^{-1}$ exist as bounded operators on $C^0_B \cap C^0 \cap L^2_{xz}$.

Proof. Referring to the definition of integral kernel $K_{\pm}(x, y, z)$, i.e.,

$$K_{\pm}(x, y, z) = \left( \begin{array}{cc} 1 & 0 \\ 0 & e^{z(p(x) - p(y))} \end{array} \right) \tilde{U}_3(y), \quad (4.19)$$

we obtain the result

$$|K_{\pm}(x, y, z)| = |\tilde{U}_3(y)|. \quad (4.20)$$

Then, for any vector function $f(x, z) \in C^0_B \cap C^0 \cap L^2_{xz}$, by adapting Lemma 4.1, we have

$$|P_{\pm}(f)(x, z)| \leq \int_x^{+\infty} |\tilde{U}_3(y)|dy \|f\|_{C^0_B}. \quad (4.21)$$
Besides, we further give the result
\[
\left(\int_{\mathbb{R}} |P_\pm(f)(x,z)|^2 dz\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}} |K_\pm(x,y,z)f(y,z)|^2 dy dz\right)^{\frac{1}{2}}
\leq \left|\int_x^{\pm\infty} \left(\int_{\mathbb{R}} |K_\pm(x,y,z)|^2 |f(y,z)|^2 dy\right)^{\frac{1}{2}} dz\right|
\leq \int_x^{\pm\infty} |\tilde{U}_3(y)| \|f(y,z)\|_{L^2_x} dy \leq \|\tilde{U}_3\|_1 \|f\|_{C^0},
\] (4.22)
which implies
\[
\left(\int_{\mathbb{R}} \int_{\mathbb{R}} |P_\pm(f)(x,z)|^2 dz dx\right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}} \left(\int_x^{\pm\infty} |\tilde{U}_3(y)| \|f(y,z)\|_{L^2_x} dy\right)^2 dx\right)^{\frac{1}{2}}
\leq \left(\int_{\mathbb{R}} \left(\int_0^y |\tilde{U}_3(y)|^2 \int_{\mathbb{R}} |f(y,z)|^2 dz dx\right) dy\right)^{\frac{1}{2}}
\leq \|\tilde{U}_3\|_{2,\frac{1}{2}} \|f\|_{L^2_{x,z}}.
\] (4.23)
The integral kernel of Volterra operator $[P_\pm]^n$ are, respectively, denoted as $K^n_\pm$. Then, $K^n_\pm$ can be expressed as
\[
K^n_\pm(x,y_n,z) = \int_x^{y_n} \cdots \int_x^{y_2} K_\pm(x,y_1,z)K_\pm(y_1,y_2,z) \cdots K_\pm(y_{n-1},y_n,z) dy_1 \cdots dy_{n-1}.
\] (4.24)
and satisfy
\[
|K^n_\pm(x,y,z)| \leq \frac{1}{(n-1)!} \left(\int_x^{\pm\infty} |\tilde{U}_3(y)| dy\right)^{n-1} |\tilde{U}_3(y)|.
\] (4.25)
Furthermore, similar to the above analysis, we show that
\[
\| [P_\pm]^n \|_{B(\mathcal{C}^\alpha_\beta)} \leq \frac{\|\tilde{U}_3\|_1^n}{(n-1)!},
\]
\[
\| [P_\pm]^n \|_{B(\mathcal{C}^0)} \leq \frac{\|\tilde{U}_3\|_1^n}{(n-1)!},
\]
\[
\| [P_\pm]^n \|_{B(L^2_{x,z})} \leq \frac{\|\tilde{U}_3\|_1^{n-1}}{(n-1)!} \|\tilde{U}_3\|_2,\frac{1}{2}.
\]
Then, on the basis of the standard Volterra theory, the following operator norm can be obtained,
\[
\| (\mathbb{I} - P_\pm)^{-1} \|_{B(\mathcal{C}^\alpha_\beta)} \leq e^{\|\tilde{U}_3\|_1} \|\tilde{U}_3\|_1,
\]
\[
\| (\mathbb{I} - P_\pm)^{-1} \|_{B(\mathcal{C}^0)} \leq e^{\|\tilde{U}_3\|_1} \|\tilde{U}_3\|_1,
\]
\[
\| (\mathbb{I} - P_\pm)^{-1} \|_{B(L^2_{x,z})} \leq e^{\|\tilde{U}_3\|_1} \|\tilde{U}_3\|_2,\frac{1}{2}.
\]
Next, for the integral operator $[P_\pm]_z$, by using a fact that
\[
|[K_\pm]_z| \leq |\tilde{p}(x) - \tilde{p}(y)||\tilde{U}_3(y)|,
\] (4.26)
we obtain
\[
\| [P_\pm]_z \|_{S(C^0_B)} & \leq \| \tilde{U}_3 \|_{L^1_{1,1}} + \| \tilde{U}_3 \|_{L^1_{2,2}} \| q \|_{L^1} \\
\| [P_\pm]_z \|_{S(C^0)} & \leq \| \tilde{U}_3 \|_{L^1_{1,1}} + \| \tilde{U}_3 \|_{L^1_{2,2}} \| q \|_{L^1} \\
\| [P_\pm]_z \|_{S(L^2_{x,z})} & \leq \| \tilde{U}_3 \|_{L^2_{1,\frac{3}{2}}} + \| \tilde{U}_3 \|_{L^2_{2,\frac{3}{2}}} \| q \|_{L^1} .
\]

Then, according to the above lemmas, we know that \([P_\pm]_z(n_{\pm})\) belong to \(C^0_B \cap C^0 \cap L^2_{x,z}\). Therefore, we can conclude that \(([P_\pm]_z(e_1) + [P_\pm]_z(n_{\pm}))\) belong to \(C^0_B \cap C^0 \cap L^2_{x,z}\). Due to the existence of the operator \((I - P_\pm)^{-1}\), the equations (4.8) and (4.9) can be transformed as
\[
n_{\pm}(x,z) = (I - P_\pm)^{-1}(P_\pm(e_1)(x,z)),
\]
\[
[n_{\pm}(x,z)]_z = (I - P_\pm)^{-1}([P_\pm]_z(e_1)(x,z) + [P_\pm]_z(n_{\pm})(x,z)).
\]
Then, based on the above lemmas and the definition of \(n_{\pm}(x,z)\) (4.5), we give the following proposition to show the property of \(\mu_{\pm}(x,z)\).

**Proposition 4.5.** If \(u \in W^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R})\), then \(\mu_{\pm}(0,z) - I\) and its \(z\)-derivative \([\mu_{\pm}(0,z)]_z\) belong to \(C^0_B(\mathbb{R}) \cap L^2(\mathbb{R})\).

Next, we give the proof of Proposition 3.2. According to (3.3), we only need to confirm that \(r(z), r'(z)\) and \(zr(z)\) belong to \(L^2(\mathbb{R})\), i.e.,
\[
r(z) = \frac{s_{12}(z)}{s_{22}(z)}, \quad r'(z) = \frac{s'_{12}(z) - s_{12}(z)s'_{22}(z)}{s_{22}(z)} - \frac{s_{12}(z)s'_{22}(z)}{s_{22}(z)} , \quad zr(z) = \frac{zs_{12}(z)}{s_{22}(z)} \in L^2(\mathbb{R}).
\]

According to (2.19), (4.5) and Proposition 2.4, we rewrite the scattering coefficients \(s_{12}(z)\) and \(s_{22}(z)\) as
\[
s_{22}(z) = -\mu_{-11}(0,z)\tilde{\mu}_{+,11}(0,\tilde{z}) - \mu_{-,21}(0,z)\tilde{\mu}_{+,21}(0,\tilde{z}) \quad = - (n^{(1)}_{-11}(0,z) + 1)(n^{(2)}_{+,11}(0,\tilde{z}) + 1) - n^{(2)}_{-,21}(0,z)n^{(2)}_{+,21}(0,\tilde{z}),
\]
\[
s_{12}(z) = -\mu_{-11}(0,z)\mu_{+,21}(0,z) - \mu_{-,21}(0,\tilde{z})\mu_{+,11}(0,\tilde{z}) \quad = - (n^{(1)}_{+1}(0,\tilde{z}) + 1)n^{(2)}_{-21}(0,\tilde{z}) + n^{(2)}_{-,21}(0,\tilde{z})(n^{(1)}_{+1}(0,\tilde{z}) + 1).
\]

Then, the boundedness of \(s_{11}(z), s'_{11}(z), s_{21}(z), s'_{21}(z)\) and the \(L^2\)-integrability of \(s_{21}(z), s'_{21}(z)\) are confirmed by Proposition 4.5. Next, we explain that \(zs_{21}(z)\) belongs to \(L^2(\mathbb{R})\).

Referring to (4.4), we obtain
\[
\mu_{1,1}(0,z) - e_1 = (n^{(1)}_{-11}, n^{(2)}_{-,21})^T
\]
\[
= \int_{0}^{\pm\infty} \begin{pmatrix} 1 & 0 \\ e^{-2(p(x)-p(y))} & 0 \end{pmatrix} \tilde{U}_3(y)(n_{\pm} - e_1)dy.
\]

Following the ideas in [44], it is easy to verify the \(L^2\)-integrability of \(zs_{21}(z)\) on \(\mathbb{R}\). In summary, we obtain the result shown in Proposition 3.2.
5. Conjugation

In this section, our purpose is to re-normalize the Riemann–Hilbert problem 3.5 so that the RHP is well behaved as $t \to \infty$ along an arbitrary characteristic. Therefore, we apply a function to establish the transformation $M \mapsto M^{(1)}$.

Recall that the jump matrix in RHP 3.5 is

$$V(y, t; z) = e^{-i(zy + 2z^2t)}\hat{\sigma}_3 \begin{pmatrix} \frac{1}{\bar{r}(z)} & r(z) \bar{r}(z) + |r(z)|^2 \end{pmatrix}.$$ 

Then, the oscillation term can be rewritten as $e^{-2i\theta(z)} \left( \theta(z) = \left( \frac{z^2}{t} + 2z^2 \right) \right)$ from which we obtain a phase point

$$z_0 = -\left( \frac{y}{4t} \right).$$

Therefore, we can rewrite $\theta(z)$ as

$$\theta(z) = 2(z - z_0)^2 - 2z_0^2.$$  \hfill (5.1)

By evaluating the real part of $2it\theta(z)$, i.e., $\text{Re}(2i\theta) = -4(\text{Re} z - z_0)\text{Im} z$, we can derive the decaying domains of the oscillation term, see Fig. 1.

To make the following analyses more clear, we introduce some notations.

$$\triangle^-_{z_0} = \{ k \in \{1, \ldots, N\} | \text{Re}(z_k) < z_0 \},$$

$$\triangle^+_{z_0} = \{ k \in \{1, \ldots, N\} | \text{Re}(z_k) > z_0 \}.$$  \hfill (5.2)

For $I = [a, b]$, define

$$\mathcal{Z}(I) = \{ z_k \in \mathcal{Z} : \text{Re}z_k \in I \},$$

$$\mathcal{Z}^-(I) = \{ z_k \in \mathcal{Z} : \text{Re}z_k < a \},$$

$$\mathcal{Z}^+(I) = \{ z_k \in \mathcal{Z} : \text{Re}z_k > b \}.$$
For $z_0 \in \mathcal{I}$, define
\[
\triangle^-_{z_0}(\mathcal{I}) = \{ k \in \{1, \ldots, N\} : a \leq \text{Re} z_k < z_0 \},
\]
\[
\triangle^+_{z_0}(\mathcal{I}) = \{ k \in \{1, \ldots, N\} : z_0 < \text{Re} z_k \leq b \}.
\]

In the following part, we mainly focus on the case that $t \to -\infty$, and the analysis of the case $t \to +\infty$ is essentially the same.

Next, in order to re-normalize the Riemann–Hilbert problem such that it is well behaved for $t \to -\infty$ with fixed phrase point $z_0$, we first introduce the function
\[
\delta(z) = \exp \left[ i \int_{-\infty}^{z_0} \frac{\nu(s)}{s - z} ds \right],
\]
\[
\nu(s) = -\frac{1}{2\pi} \log(1 + |r(s)|^2),
\]
and
\[
T(z) = T(z, z_0) = \prod_{k \in \triangle^-_{z_0}} \frac{z - \bar{z}_k}{z - z_k} \delta(z).
\]
Moreover, the trace formula of the scattering coefficient $s_{22}(z)$ can be easily derived
\[
s_{22}(z) = \prod_{k=1}^{N} \frac{z - \bar{z}_k}{z - z_k} \exp \left( i \int_{-\infty}^{+\infty} \frac{\nu(s)}{s - z} ds \right).
\]

Comparing (5.4) with (5.5), we can easily find that the function $T(z, z_0)$ approaches the scattering coefficient $1/s_{22}(z)$ as $z_0 \to \infty$. Besides, the function $T(z, z_0)$ possesses well properties.

**Proposition 5.1.** The function $T(z, z_0)$ satisfies that
\[(a)\] $T$ is meromorphic in $C \setminus (-\infty, z_0]$. $T(z, z_0)$ possesses simple pole at $z_k (k \in \triangle^-_{z_0})$ and simple zero at $\bar{z}_k (k \in \triangle^-_{z_0})$.
\[(b)\] For $z \in C \setminus (-\infty, z_0]$, $\tilde{T}(z) = \frac{1}{T(z)}$.
\[(c)\] For $z \in (-\infty, z_0]$, the boundary values $T_\pm$ satisfy that
\[
\frac{T_+}{T_-}(z) = 1 + |r(z)|^2, z \in (-\infty, z_0).
\]
\[(d)\] As $|z| \to \infty$ with $|\text{arg}(z)| \leq c < \pi$,
\[
T(z) = 1 + \frac{i}{z} \left[ 2 \sum_{k \in \triangle^-_{z_0}} \text{Im} z_k - \int_{-\infty}^{z_0} \nu(s) ds \right] + O(z^{-2}).
\]
\[(e)\] As $z \to z_0$ along any ray $z_0 + e^{i\phi} R_+$ with $|\phi| \leq c < \pi$
\[
|T(z, z_0) - T_0(z_0)(z - z_0)^{i\nu(z_0)}| \leq C \| r \|_{H^1(R)} |z - z_0|^{\frac{1}{2}},
\]
where $T_0(z_0)$ is the complex unit
\[
T_0(z_0) = \prod_{k \in \Delta^{-}_{z_0}} \left( \frac{z_0 - \bar{z}_k}{z_0 - z_k} \right) e^{i\beta(z_0, z_0)},
\]
(5.9)
\[
\beta(z, z_0) = -\nu(z_0) \log(z - z_0 + 1) + \int_{-\infty}^{z_0} \frac{\nu(s) - \chi(s)\nu(z_0)}{s - z} ds,
\]
with $\chi(s) = 1$ as $s \in (z_0 - 1, z_0)$, and $\chi(s) = 0$ as $s \in (-\infty, z_0 - 1]$.

(f) As $z \to 0$, $T(z)$ can be expressed as
\[
T(z) = T(0)(1 + zT_1) + O(z^2),
\]
(5.10)
where
\[
T_1 = 2 \sum_{k \in \Delta^{-}_{z_0, 1}} \frac{\text{Im} \frac{z_k}{z_k}}{z_k} - \int_{-\infty}^{z_0} \frac{\nu(s)}{s^2} ds.
\]

Proof. The above properties of $T(z)$ can be proved by a direct calculation, for details, see [42,43,49].

Next, applying the partial scattering coefficient $T(z, z_0)$, we define an unknown matrix-value function $M^{(1)}(z)$
\[
M^{(1)}(z) = M(z)T(z)^{\sigma_3}.
\]
(5.11)
Then, $M^{(1)}(z)$ satisfies the following matrix RHP.

Riemann-Hilbert Problem 5.2. Find a function $M^{(1)}$ with the following properties:

- $M^{(1)}$ is meromorphic in $C \setminus \mathbb{R}$;
- $M^{(1)}(z) = \mathbb{I} + O(z^{-1})$ as $z \to \infty$;
- For $z \in \mathbb{R}$, the boundary values $M^{(1)}_{\pm}(z)$ satisfy the jump relationship
  \[
  M^{(1)}_{+}(z) = M^{(1)}_{-}(z)V^{(1)}(z),
  \]
  (5.12)

- $M^{(1)}(z)$ has simple poles at each $z_k \in \mathbb{Z}$ and $\bar{z}_k \in \bar{\mathbb{Z}}$ at which

\[
\text{Res } M^{(1)} = \begin{cases}
\lim_{z \to z_k} M^{(1)} \left( e^{-1} \left( \frac{1}{T'}(\bar{z}_k) \right) -2 e^{2it\theta(\bar{z}_k)} \right), k \in \Delta^{-}_{z_0}, \\
\lim_{z \to \bar{z}_k} M^{(1)} \left( 0 \right), k \in \Delta^{-}_{\bar{z}_0},
\end{cases}
\]
(5.13)
\[
\text{Res } M^{(1)} = \begin{cases}
\lim_{z \to z_k} M^{(1)} \left( 0 \right), k \in \Delta^{-}_{z_0}, \\
\lim_{z \to \bar{z}_k} M^{(1)} \left( 0 \right), k \in \Delta^{-}_{\bar{z}_0},
\end{cases}
\]
Proof. On the basis of the definition of $M^{(1)}$, Proposition 5.1 and the properties of $M(y, t; z)$, the analyticity, jump matrix and asymptotic behavior of $M^{(1)}$ can be obtained directly. Then, for the residue conditions of $M^{(1)}$, when $k \in \Delta_{z_0}^+$, $T(z)$ is analytic at the points $z_k, \bar{z}_k$. Thus, at these points, the residue conditions can be obtained directly from (3.8) and (5.11). When $k \in \Delta_{z_0}^-$, $z_k$ is the pole of $T(z)$. As a result, we have

$$\text{Res}_{z = z_k} M^{(1)}_0 = \text{Res}_{z = z_k} (M_2 T^{-1}) = 0,$$

$$M^{(1)}_2(z_k) = \lim_{z \to z_k} (M_2(z) T^{-1}(z)) = c_k e^{-2it\theta(z_k)} M_1(z_k) \left(\frac{1}{T}\right)'(z_k),$$

$$\text{Res}_{z = z_k} M^{(1)}_1 = \text{Res}_{z = z_k} (M_1 T) = M_1(z_k) \lim_{z \to z_k} T(z)(z - z_k)$$

$$= c_k^{-1} e^{2it\theta(z_k)} M^{(1)}_2(z_k) \left(\frac{1}{T}\right)'(z_k)^{-1}.$$

Now, a direct calculation gives the first formula shown in (5.13). Similarly, we can obtain the residue condition at $\bar{z}_k (k \in \Delta_{z_0}^-)$. □

6. Continuous Extension to a Mixed $\bar{\partial}$-RH Problem

Following the ideas in [38–42], we make the continuous extensions of the jump matrix off the real axis. Applying these extensions, the oscillatory jump can be deformed onto new contours along which the jumps are decaying. In order to achieve this goal, we define the contours

$$\Sigma_j = -\frac{x}{4t} + e^{\frac{(2j-1)\pi i}{4}} R_+^-, \quad j = 1, 2, 3, 4,$$

$$\Sigma^2 = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4.$$  \hspace{1cm} (6.1)

Then, the complex plane $\mathbb{C}$ is separated into six open sectors $\Omega_j (j = 1, \ldots, 6)$ by $\mathbb{R} \cup \Sigma^2$, see Fig. 2. Moreover, define

$$\rho = \frac{1}{2} \min_{\lambda, \zeta \in Z \cup \bar{Z}, \lambda \neq \mu} |\lambda - \zeta|,$$  \hspace{1cm} (6.2)

and $\chi_Z \in C_0^\infty(C, [0, 1])$ which is supported near the discrete spectrum such that

$$\chi_Z(z) = \begin{cases} 1, & \text{dist}(z, Z \cup \bar{Z}) < \rho/3, \\ 0, & \text{dist}(z, Z \cup \bar{Z}) > 2\rho/3. \end{cases}$$  \hspace{1cm} (6.3)

The formula (6.2) implies that $\text{dist}(Z \cup \bar{Z}, \mathbb{R}) > \rho, k = 1, 2, \ldots, N$. Then, we define extensions of the jump matrices of (5.12) in the following proposition.

Proposition 6.1. There exist functions $R_j : \bar{\Omega}_j \to C, j = 1, 3, 4, 6$ with boundary values such that

$$R_1(z) = \begin{cases} r(z) T^{-2}(z), & z \in (z_0, \infty), \\ r(z_0) T_0^{-2}(z_0) (z - z_0)^{-2it(z_0)} (1 - \chi_Z(z)), & z \in \Sigma_1, \end{cases}$$
Riemann–Hilbert Problem 6.2. Find a matrix value function $M^{(2)}$ satisfying

- $M^{(2)}(y, t, z)$ is continuous in $\mathbb{C}\setminus(\Sigma^2 \cup \mathcal{Z} \cup \tilde{\mathcal{Z}})$.
- $M^+_t(y, t, z) = M^-_t(y, t, z)V^{(2)}(y, t, z)$, $z \in \Sigma^2$, where the jump matrix $V^{(2)}(x, t, z)$ satisfies

  \[ V^{(2)}(z) = \mathbb{I} + (1 - \chi_Z(z))\tilde{V}^{(2)}, \]
Figure 2. Definition of $R^{(2)}$ in different domains

with

$$
\bar{V}^{(2)}(z) = \begin{cases} 
(0 \ r(z_0) T_0(z_0)^{-2} (z - z_0)^{-2i\nu(z_0)} e^{-2it\theta(z_0)}), & z \in \Sigma_1, \\
0 & z \in \Sigma_2, \\
\frac{\bar{r}(z_0) T_0(z_0)^2}{1+|\bar{r}(z_0)|^2} (z - z_0)^{-2i\nu(z_0)} e^{2it\theta(z_0)}, & z \in \Sigma_3, \\
(0 \ 0), & z \in \Sigma_4.
\end{cases}
$$

(6.8)

• $M^{(2)}(x, t, z) \to I, \quad z \to \infty.$

• For $\mathbb{C}\setminus(\Sigma^2 \cup \Sigma \cup \bar{\Sigma}), \ \bar{\partial}M^{(2)} = M^{(2)} \bar{\partial}R^{(2)}(z),$ where

$$
\bar{\partial}R^{(2)} = \begin{cases} 
(0 -\bar{\partial}R_1 e^{-2it\theta}), & z \in \Omega_1, \\
(0 \ 0), & z \in \Omega_2, \\
\bar{\partial}R_3 e^{2it\theta}, & z \in \Omega_3, \\
0 & z \in \Omega_4, \\
\bar{\partial}R_4 e^{-2it\theta}, & z \in \Omega_5, \\
\bar{\partial}R_6 e^{2it\theta}, & z \in \Omega_6, \\
(0 0), & z \in \Omega_7.
\end{cases}
$$

(6.9)
$R(2) = U_R^{-1}$  \hspace{1cm} $R(2) = W_R^{-1}$  
\hspace{1cm} $\Omega_1$  \hspace{1cm} $\Omega_2$  
\hspace{1cm} $\Omega_3$ \hspace{1cm} $\Omega_4$  
\hspace{1cm} $\Sigma_1$  
\hspace{1cm} $\Sigma_2$  
$\Sigma_3$ \hspace{1cm} $\Sigma_4$  
\hspace{1cm} $z_k$  
\hspace{1cm} $\bar{z}_k$  
\hspace{1cm} $Rez$  
\hspace{1cm} $R(2) = U_L$  \hspace{1cm} $R(2) = W_L$  

**Figure 3.** Jump matrix $V(2)$, pink parts support $\bar{\partial}$ derivative: $\bar{\partial}R(2) \neq 0$. White parts do not support $\bar{\partial}$ derivative: $\bar{\partial}R(2) = 0$

- $M(2)$ admits the residue conditions at poles $z_k \in \mathcal{Z}$ and $\bar{z}_k \in \bar{\mathcal{Z}}$, i.e. (Fig. 3),

$$\text{Res}_{z = z_k} M(2) = \begin{cases} 
\lim_{z \to z_k} M(2) \left( c_k^{-1} \left( \frac{(T')'(z_k)}{T'}(z_k) \right)^{-2} e^{2i\theta(z_k)} \right), & k \in \bigtriangleup_{-z_0}, \\
\lim_{z \to z_k} M(2) \left( 0 \ c_k T(z_k)^{-2} e^{-2i\theta(z_k)} \ 0 \right), & k \in \bigtriangleup_{+z_0},
\end{cases}$$

$$\text{Res}_{z = \bar{z}_k} M(2) = \begin{cases} 
\lim_{z \to \bar{z}_k} M(2) \left( 0 \ (-\bar{c}_k)^{-1} T'(\bar{z}_k)^{-2} e^{-2i\theta(\bar{z}_k)} \right), & k \in \bigtriangleup_{-\bar{z}_0}, \\
\lim_{z \to \bar{z}_k} M(2) \left( 0 \ -c_k T(\bar{z}_k)^2 e^{2i\theta(\bar{z}_k)} \ 0 \right), & k \in \bigtriangleup_{+\bar{z}_0}.
\end{cases}$$

**7. Dropping the RH Component of the Solution**

In order to construct the solution $M(2)$, we decompose the mixed $\bar{\partial}$-RH problem, i.e., RHP 6.2, into two parts, including a model RH problem with $\bar{\partial}R(2) = 0$ and a pure $\bar{\partial}$-RH problem with $\bar{\partial}R(2) \neq 0$. For the first step, we construct a solution $M_R^{(2)}$ to the model RH problem with $\bar{\partial}R(2) = 0$.

**7.1. The Decomposition of the Mixed $\bar{\partial}$-RH Problem for $M^{(2)}$**

**Riemann-Hilbert Problem 7.1.** Find a matrix value function $M_R^{(2)}$, admitting

- $M_R^{(2)}$ is analytical in $\mathbb{C} \setminus (\Sigma^2 \cup \mathcal{Z} \cup \bar{\mathcal{Z}})$;
- $M_{R,+}^{(2)}(y,t,z) = M_{R,-}^{(2)}(y,t,z)V(2)(y,t,z)$, $z \in \Sigma^2$, where $V(2)(y,t,z)$ is the same with the jump matrix appearing in RHP 6.2;
- As $z \to \infty$, $M_R^{(2)}(y,t,z) = I + o(z^{-1})$;
- $M_R^{(2)}$ possesses the same residue condition with $M^{(2)}$. 

In this section, the following part will give the proof of the existence and asymptotic of $M_R^{(2)}$. It is meaningful to point out that if $M_R^{(2)}$ exists, a pure $\bar{\partial}$-RH problem can be generated from RHP 6.2 by using $M_R^{(2)}$ to establish a transformation.

Assuming that $M_R^{(2)}$ is a solution of the RHP 7.1, defining
\[
M^{(3)}(z) = M^{(2)}(z)M_R^{(2)}(z)^{-1}, \tag{7.1}
\]
then, $M^{(3)}(z)$ satisfying the following pure $\bar{\partial}$-RH problem.

**Riemann-Hilbert Problem 7.2.** Find a matrix value function $M^{(3)}$ admitting
- $M^{(3)}$ is continuous with sectionally continuous first partial derivatives in $\mathbb{C}\setminus(\Sigma^2 \cup \mathcal{Z} \cup \mathbb{Z})$;
- For $z \in \mathbb{C}$, we obtain $\bar{\partial}M^{(3)}(z) = M^{(3)}(z)W(z)$, where
\[
W^{(3)} = M_R^{(2)}(z)\bar{\partial}R^{(2)}M_R^{(2)}(z)^{-1}, \tag{7.2}
\]
- As $z \to \infty$,
\[
M^{(3)}(z) = I + o(z^{-1}). \tag{7.3}
\]

**Proof.** Referring to the properties of the $M_R^{(2)}$ in RHP 7.1 and $M^{(2)}$ in 6.2, the analytic and asymptotic properties of $M^{(3)}$ can be derived easily. According to the construction of the $M_R^{(2)}$, we know that $M_R^{(2)}$ possesses the same jump matrix with $M^{(2)}$. As a result, we obtain that
\[
M^{(3)}_-(z)^{-1}M^{(3)}_+(z) = M_R^{(2)}(-z)M^{(2)}_-(z)M^{(2)}_+(z)^{-1}
\]
\[
= M_R^{(2)}(-z)V(z)(M^{(2)}_-(z)V(z)^{-1} = I,
\]
which implies that $M^{(3)}$ has no jump. Next, we give the proof that $M^{(3)}$ only has removable singularities at each $z_k$. We use $N_k$ to denote nilpotent matrix which appears in the left side of the residue condition of RHP 6.2 and RHP 7.1. Then, we obtain the Laurent expansions
\[
M^{(2)}(z) = C(z_k) \left[ \frac{N_k}{z - z_k} + I \right] + O(z - z_k),
\]
\[
M_R^{(2)}(z) = \hat{C}(z_k) \left[ \frac{N_k}{z - z_k} + I \right] + O(z - z_k),
\]
where $C(z_k)$ and $\hat{C}(z_k)$ are constant terms. Then, we can derive that
\[
M^{(2)}(z)M_R^{(2)}(z)^{-1} = O(1),
\]
which infers to that $M^{(3)}$ has removable singularities at $z_k$. Finally, on the basis of the definition of $M^{(3)}$, the $\bar{\partial}$-derivative of $M^{(3)}$ is derived as
\[
\bar{\partial}M^{(3)}(z) = \bar{\partial}(M^{(2)}(z)M_R^{(2)}(z)^{-1}) = \bar{\partial}M^{(2)}(z)M_R^{(2)}(z)^{-1} = M^{(2)}(z)\bar{\partial}R^{(2)}(z)
\times M_R^{(2)}(z)^{-1}
\]
\[
= M^{(2)}(z)M_R^{(2)}(z)^{-1}(M^{(2)}_R(z)\bar{\partial}R^{(2)}(z)M^{(2)}_R(z)^{-1}) = M^{(3)}(z)W^{(3)}(z).\]
\[\square\]
7.2. Constructing the Solution $M_R^{(2)}(z)$

Firstly, we define an open neighborhood

$$U_{z_0} = \{ z : |z - z_0| < \rho/2 \},$$

on which $M_R^{(2)}(z)$ is pole free. Then, in order to construct the solution $M_R^{(2)}(z)$, we decompose $M_R^{(2)}$ into two parts

$$M_R^{(2)}(z) = \begin{cases} E(z)M^{(out)}(z), & z \in \mathbb{C}\setminus U_{z_0}, \\ E(z)M^{(out)}(z)M^{(pc)}(z_0, r_0), & z \in U_{z_0}, \end{cases} \quad (7.4)$$

where $M^{(out)}$ solves a model RHP, $M^{(pc)}$ is a known parabolic cylinder model and $E(z)$, an error function, is the solution of a small-norm Riemann–Hilbert problem.

7.2.1. Outer Model RH Problem: $M^{(out)}$. Recall that $\theta(z) = 2(z - z_0)^2 - 2z_0^2$. Then, for the jump matrix $V^{(2)}$, by applying the spectral bound (6.2), we have the following estimate.

$$||V^{(2)} - I||_{L^\infty(\Sigma^2)} = \begin{cases} o(e^{-4|t|z - z_0|^2}), & z \in \Sigma^2\setminus U_{z_0}, \\ o(|z - z_0|^{-1}t^{-\frac{1}{2}}), & z \in \Sigma^2 \cap U_{z_0}. \end{cases} \quad (7.5)$$

We can obviously see that the estimate (7.5) decays exponentially in $\Sigma^2 \setminus U_{z_0}$. Therefore, it is reasonable to construct a model solution outside $U_{z_0}$ when we omit the jump completely.

So, we next establish a model RH problem and prove that its solution can be approximated by a finite sum of soliton solutions.

Riemann–Hilbert Problem 7.3. Find a matrix value function $M^{(out)}$ satisfying

- $M^{(out)}(y, t; z)$ is analytic in $\mathbb{C}\setminus(\Sigma^2 \cup Z \cup \bar{Z})$;
- As $z \to \infty$,
  $$M^{(out)}(y, t; z) = I + o(z^{-1}); \quad (7.6)$$
- $M^{(out)}(y, t; z)$ has simple poles at each point in $Z \cup \bar{Z}$ and admits the same residue condition in RHP 6.2 with $M^{(out)}(y, t; z)$ replacing $M^{(2)}(y, t; z)$.

Proposition 7.4. There exists unique solution $M^{(out)}$ of RHP 7.3. Particularly,

$$M^{(out)}(z) = M^{\Delta_{z_0}}(z|\sigma_{d}^{out}), \quad (7.7)$$

where $M^{\Delta_{z_0}}(z)$ is the solution of RHP B.3 with $\Delta = \Delta_{z_0}$ and $\sigma_{d}^{out} = \{(z_k, \tilde{c}_k(z_0))\}_{k=1}^{N}$ with

$$\tilde{c}_k(z_0) = c_k e^{\frac{i}{\pi} \int_{-\infty}^{z_0} \frac{\log(1 + |r(s)|^2)}{s - z_k} ds}. \quad (7.8)$$

Moreover,

$$\|M^{(out)}(z)\|_{L^\infty(\mathbb{C}\setminus(Z \cup \bar{Z}))} \lesssim 1. \quad (7.9)$$
In addition, for \( t \to \infty \),
\[
q_{\text{sol}}(y, t; \sigma_{d}^{\text{out}}) = e^{2d} \lim_{z \to 0} \frac{\partial}{\partial y} \left( \frac{M^{(\text{out})}(0)^{-1} M^{(\text{out})}(z)}{z} \right)_{12}
\]
(7.10)
where \( q_{\text{sol}}(y, t; \sigma_{d}^{\text{out}}) \) is the \( N \)-soliton solution of Eq. (1.3) corresponding the scattering data \( \sigma_{d}^{\text{out}} \), \( \mu(I) \) and \( \sigma_{d}(I) \) are, respectively, defined in (B.12) and (B.15).

Proof. Note that when \( \Delta = \Delta_{-}^{\lambda} \) and \( \sigma_{d}^{\text{out}} = \sigma_{d}^{\text{out}} \) in RHP B.3, \( M^{(\text{out})}(z) \) is the same as defining \( M^{\Delta}(z) \) in RHP B.3. The existence and uniqueness of solutions to RHR B.3 are guaranteed by Proposition B.2. The inequality (7.9) can be obtained by substituting (7.7) into (B.5). Finally, on the basis of (7.7), (B.11) and Proposition B.4, (7.10) can be obtained by a simple calculation. \( \square \)

7.2.2. Local Solvable Model Near Phase Point. For \( z \in \mathcal{U}_{0} \), the (7.5) implies that \( V^{(2)} - I \) does not have a uniform estimate for large time. Therefore, by introducing a model \( M^{(\text{out})}(z) M^{(\text{pc})}(z_{0}, r_{0}) \) to match the jumps of \( M^{(2)}_{R} \) on \( \Sigma^{2} \cap \mathcal{U}_{z_{0}} \), we establish a local solvable model for the error function \( E(z) \). Recall the definition of \( \theta(z) \) (5.1) and introduce the transformation
\[
\lambda = \lambda(z) = 2i \sqrt{2t}(z - z_{0}).
\]
(7.11)
Then, we can derive that
\[
2t \theta = -\frac{1}{2} \lambda^{2} - 4t z_{0}^{2},
\]
(7.12)
from which we know that \( \mathcal{U}_{z_{0}} \) is mapped into an expanding neighborhood of \( \lambda = 0 \). If we let
\[
r_{0}(z_{0}) = r(z_{0}) T_{0}^{-2}(z_{0}) e^{2i(\nu(z_{0}) \log(2i \sqrt{2t}))} e^{-4it z_{0}^{2}},
\]
(7.13)
and consider the fact that \( 1 - \chi_{Z} = 1 \) as \( z \in \mathcal{U}_{z_{0}} \), the jump of \( M^{(2)}_{R} \) in \( \mathcal{U}_{z_{0}} \) is translated into
\[
V^{(2)}(z) \mid_{z \in \mathcal{U}_{z_{0}}} = \begin{cases} 
\lambda(z)^{-i\nu \sigma_{3}} e^{i \lambda(z)^{2}/4 \sigma_{3}} \begin{pmatrix} 1 & r_{0}(z_{0}) \\
0 & 1 \end{pmatrix}, & z \in \Sigma_{1}, \\
\lambda(z)^{-i\nu \sigma_{3}} e^{i \lambda(z)^{2}/4 \sigma_{3}} \begin{pmatrix} 1 & 0 \\
1/r_{0}(z_{0}) & 1 \end{pmatrix}, & z \in \Sigma_{2}, \\
\lambda(z)^{-i\nu \sigma_{3}} e^{i \lambda(z)^{2}/4 \sigma_{3}} \begin{pmatrix} 1 & r_{0}(z_{0}) \\
0 & 1 \end{pmatrix}, & z \in \Sigma_{3}, \\
\lambda(z)^{-i\nu \sigma_{3}} e^{i \lambda(z)^{2}/4 \sigma_{3}} \begin{pmatrix} 1 & 0 \\
1/r_{0}(z_{0}) & 1 \end{pmatrix}, & z \in \Sigma_{4}.
\end{cases}
\]
(7.14)
It is obvious that the jump \( V^{(2)}(z) \mid_{z \in \mathcal{U}_{z_{0}}} \) in (7.14) is equivalent to the jump of the parabolic cylinder model problem (A.5) whose solutions is shown in “Appendix A.” Moreover, on the basis of a fact that \( M^{(\text{out})}(z) \) is an analytic
and bounded function in $\mathcal{U}_{z_0}$ and referring to the definition of $M_R^{(2)}(z)$, i.e., $M_R^{(2)}(z) = M^{(out)}(z)M^{(pc)}(z_0, r_0)(z \in \mathcal{U}_{z_0})$, a direct calculation shows that $M^{(out)}(z)M^{(pc)}(z_0, r_0)$ satisfies the jump $V^{(2)}(z)$ of $M_R^{(2)}(z)$.

7.2.3. The Small Norm RHP for $E(z)$. On the basis of Proposition 7.4 and $M^{(out)}(z)M^{(pc)}(z_0, r_0)$ analyzed in the above subsection, the unknown error function $E(z)$ defined in (7.4) can be shown as

$$E(z) = \begin{cases} M_R^{(2)}(z)M^{(out)}(z)^{-1}, & z \in \mathbb{C}\setminus \mathcal{U}_{z_0}, \\ M_R^{(2)}(z)M^{(pc)}(z_0, r_0)^{-1}M^{(out)}(z)^{-1}, & z \in \mathcal{U}_{z_0}. \end{cases} \tag{7.15}$$

It is obvious that $E(z)$ is analytic in $\mathbb{C}\setminus \Sigma^{(E)}$ where

$$\Sigma^{(E)} = \mathcal{U}_{z_0} \cup (\Sigma^{(2)}\setminus \mathcal{U}_{z_0}), \tag{7.16}$$

with clockwise direction for $\partial \mathcal{U}_{z_0}$. Then, we can show that $E(z)$ satisfies the Riemann–Hilbert problem.

**Riemann-Hilbert Problem 7.5.** Find a matrix-valued function $E(z)$ satisfies that (Fig. 4)

- $E$ is analytic in $\mathbb{C}\setminus \Sigma^{(E)}$;
- $E(z) = 1 + O(z^{-1})$, $z \to \infty$;
- $E_{+}(z) = E_{-}(z)V^{(E)}(z)$, $z \in \Sigma^{(E)}$, where

$$V^{(E)}(z) = \begin{cases} M^{(out)}(z)V^{(2)}(z)M^{(out)}(z)^{-1}, & z \in \Sigma^{(2)}\setminus \mathcal{U}_{z_0}, \\ M^{(out)}(z)M^{(pc)}(\xi, r_0)M^{(out)}(z)^{-1}, & z \in \partial \mathcal{U}_{z_0}. \end{cases} \tag{7.17}$$
Referring to (7.5), the boundedness of \(M^{(\text{out})} (\mathbf{7.9})\) and (A.7), the following estimates can be immediately obtained.

\[
|V^{(E)}(z) - \mathbb{I}| = \begin{cases} \mathcal{O}(e^{-t\rho^2}) & z \in \Sigma^2 \setminus \mathcal{U}_{z_0}, \\ \mathcal{O}(t^{-1/2}) & z \in \partial \mathcal{U}_{z_0}, \end{cases} \tag{7.18}
\]

where the constant \(\rho\) is defined in (6.2). Then, we obtain

\[
\left\| (z - z_0)^k (V^{(E)} - \mathbb{I}) \right\|_{L^p(\Sigma_E)} = o(t^{-1/2}), \quad p \in [1, +\infty), \quad k \geq 0. \tag{7.19}
\]

The estimates (7.18) imply that the bound on \(V^{(E)}(z) - \mathbb{I}\) decay uniformly. Therefore, RHP 7.5 is a small-norm Riemann–Hilbert problem whose existence and uniqueness have been guaranteed by [36,37]. Furthermore, based on the Beals–Coifman theory, the solution of RHP 7.5 is obtained as

\[
E(z) = \mathbb{I} + \frac{1}{2\pi i} \int_{\Sigma^{(E)}} \frac{\mu_E(s)(V^{(E)}(s) - \mathbb{I})}{s - z} \, ds, \tag{7.20}
\]

where \(\mu_E \in L^2(\Sigma^{(E)})\) satisfies

\[
(1 - C_{\omega_E})\mu_E = \mathbb{I}. \tag{7.21}
\]

The integral operator \(C_{\omega_E}\) is defined by

\[
C_{\omega_E} f = C_-(f(V^{(E)} - \mathbb{I})),
\]

\[
C_-(f(z) \lim_{z \to \Sigma^{(E)}} \int_{\Sigma^{(E)}} \frac{f(s)}{s - z} \, ds,
\]

where \(C_-\) is the Cauchy projection operator. Based on the properties of the Cauchy projection operator \(C_-\) and (7.19), we obtain that

\[
\|C_{\omega_E}\|_{L^2(\Sigma^{(E)})} \lesssim \|C_-\|_{L^2(\Sigma^{(E)}) \to L^2(\Sigma^{(E)})} \|V^{(E)} - \mathbb{I}\|_{L^\infty(\Sigma^{(E)})} \lesssim \mathcal{O}(t^{-1/2}), \tag{7.22}
\]

from which we know that \(1 - C_{\omega_E}\) is invertible. As a result, the existence and uniqueness of \(\mu_E\) and \(E(z)\) are guaranteed. This explains that it is reasonable to define \(M^{(2)}_R\) in (7.4). In turn, we can solve (7.1) to the unknown \(M^{(3)}\) which admits the Riemann–Hilbert Problem 7.2.

Furthermore, to reconstruct the solutions of \(q(y,t)\), it is necessary to study the asymptotic behavior of \(E(z)\) as \(z \to 0\) and large time asymptotic behavior of \(E(0)\). By observing the estimate (7.18), for \(t \to -\infty\), we just need to consider the calculation on \(\partial \mathcal{U}_{z_0}\) because it approaches to zero exponentially on other boundary. Firstly, as \(z \to 0\), we show that

\[
E(z) = E(0) + E_1 z + O(z^2), \tag{7.23}
\]

where

\[
E(0) = \mathbb{I} + \frac{1}{2\pi i} \int_{\Sigma^{(E)}} \frac{(\mathbb{I} + \mu_E(s))(V^{(E)}(s) - I)}{s} \, ds, \tag{7.24}
\]

\[
E_1 = -\frac{1}{2\pi i} \int_{\Sigma^{(E)}} \frac{(\mathbb{I} + \mu_E(s))(V^{(E)}(s) - I)}{s^2} \, ds. \tag{7.25}
\]
Then, as $t \to -\infty$, the asymptotic behavior of $E(0)$ and $E_1$ can be calculated as

$$E(0) = \mathbb{I} + \frac{1}{2i\pi} \int_{\partial \mathbb{U}_{z_0}} (V^{(E)}(s) - \mathbb{I}) ds + o(|t|^{-1})$$

$$E_1 = -\frac{1}{i2\sqrt{2t}z_0} M^{(out)}(z_0)^{-1} M^{(pc,-)}(z_0) M^{(out)}(z_0),$$

(7.26)

$$E_1 = -\frac{1}{i2\sqrt{2t}z_0^2} M^{(out)}(z_0)^{-1} M^{(pc,-)}(z_0) M^{(out)}(z_0),$$

(7.27)

where

$$M^{(pc,-)}_1 = \left( \begin{array}{cc} 0 & -\beta_{12}(r_0) \\ \beta_{21}(r_0) & 0 \end{array} \right).$$

By using (A.6) and (7.13), we obtain

$$\beta_{12}(r(z_0)) = \beta^{-}(r(z_0)) = \alpha(z_0, -)e^{i\frac{x^2}{4t} + i\nu(z_0)\log 8|t|},$$

where

$$|\alpha(z_0, -)|^2 = |\nu(z_0)|,$$

$$\arg \alpha(z_0, -) = -\frac{\pi}{4} - \arg \Gamma(i\nu(z_0)) - \arg r(z_0)$$

$$-4 \sum_{k \in \triangle z_0} \arg (z_0 - z_k) - 2 \int_{-\infty}^{z_0} \log |z_0 - s| d\nu(s).$$

Moreover, from (7.26), a direct calculation shows that

$$E(0)^{-1} = \mathbb{I} + O(t^{-1/2}).$$

(7.28)

8. Pure $\bar{\partial}$-Problem

In this section, our purpose is to study the existence and asymptotic behavior of the remaining $\bar{\partial}$-problem $M^{(3)}(z)$. The $\bar{\partial}$-RH problem 7.2 for $M^{(3)}(z)$ is equivalent to the following integral equation

$$M^{(3)}(z) = \mathbb{I} - \frac{1}{\pi} \int \frac{M^{(3)}W^{(3)}}{s - z} dA(s),$$

(8.1)

where $dA(s)$ is Lebesgue measure. Furthermore, the integral equation (8.1) can be written in operator form, i.e.,

$$(\mathbb{I} - S)M^{(3)}(z) = \mathbb{I},$$

(8.2)

where $S$ is Cauchy operator

$$S[f](z) = -\frac{1}{\pi} \int \frac{f(s)W^{(3)}(s)}{s - z} dA(s).$$

(8.3)

From (8.2), we know that if the inverse operator $(\mathbb{I} - S)^{-1}$ exists, the solution $M^{(3)}(z)$ also exists. In order to prove the operator $\mathbb{I} - S$ is reversible, we give the following proposition.
Proposition 8.1. For large $t$, there exists a constant $c$ that enables the operator (8.3) to admit the following relation

$$
|||S|||_{L^\infty \rightarrow L^\infty} \leq c|t|^{-1/4}.
$$

Proof. We mainly focus on the case that the matrix function supported in the region $\Omega_1$, the other case can be proved similarly. Assume that $f \in L^\infty(\Omega_1)$ and $s = p + iq$. Then, based on (6.9) and (7.2), we can obtain the following inequality

$$
||S[f](z)|| \leq \frac{1}{\pi} \iint_{\Omega_1} \frac{|f(s)M_R^{(2)}(s)\bar{\partial}R_1(s)M_R^{(2)}(s)^{-1}|}{|s - z|} df(s)
$$

$$
\leq c \frac{1}{\pi} \iint_{\Omega_1} \frac{||\bar{\partial}R_1(s)|| e^{4tq(p-z_0)}}{|s - z|} df(s),
$$

where $c$ is a constant. Then, on the basis of (6.4) and the estimates demonstrated in “Appendix C,” we obtain the following norm estimate.

$$
|||S|||_{L^\infty \rightarrow L^\infty} \leq c(I_1 + I_2 + I_3) \leq ct^{-1/4},
$$

where

$$
I_1 = \iint_{\Omega_1} \frac{||\partial_X z(s)|| e^{4tq(p-z_0)}}{|s - z|} df(s),
$$

$$
I_2 = \iint_{\Omega_1} \frac{|p'(p)|| e^{4tq(p-z_0)}}{|s - z|} df(s),
$$

and

$$
I_3 = \iint_{\Omega_1} \frac{|s - z_0|^{-\frac{1}{2}}| e^{4tq(p-z_0)}}{|s - z|} df(s).
$$

Next, our ultimate goal is to reconstruct the potential $q(x, t)$ as $t \rightarrow \infty$. To approach this goal, according to (3.9), it is necessary to study the long time asymptotic behaviors of $M^{(3)}(0)$ and $M^{(3)}_1(y, t)$. They are defined in the asymptotic expansion of $M^{(3)}(z)$ as $z \rightarrow 0$, i.e.,

$$
M^{(3)}(z) = M^{(3)}(0) + M^{(3)}_1(y, t)z + O(z^2), \ z \rightarrow 0,
$$
where
\[ M^{(3)}(0) = \mathbb{I} - \frac{1}{\pi} \int \int_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{s} dA(s), \]
\[ M^{(3)}_1(y, t) = \frac{1}{\pi} \int \int_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{s^2} dA(s). \]

The \( M^{(3)}(0) \) and \( M^{(3)}_1(y, t) \) satisfy the following proposition.

**Lemma 8.2.** For \( t \to -\infty \), \( M^{(3)}(0) \) and \( M^{(3)}_1(y, t) \) satisfy the following inequality
\[
\| M^{(3)}(0) - \mathbb{I} \|_{L^\infty} \lesssim |t|^{-\frac{3}{4}}, \quad (8.9)
\]
\[
M^{(3)}_1(y, t) \lesssim |t|^{-1}. \quad (8.10)
\]

The proof of this proposition is similar to the process shown in “Appendix C.”

9. Soliton Resolution for the WKI Equation

Now, we are ready to give the proof of Theorem 1.1 as \( t \to -\infty \). Inverting a series of transformation including (5.11), (6.5), (7.1) and (7.4), i.e.,
\[
M(z) \rightleftharpoons M^{(1)}(z) \rightleftharpoons M^{(2)}(z) \rightleftharpoons M^{(3)}(z) \rightleftharpoons E(z),
\]
we then obtain
\[
M(z) = M^{(3)}(z)E(z)M^{(out)}(z)R^{(2)}(z)T^{-\sigma_3}(z), \quad z \in \mathbb{C} \setminus \mathbb{U}_0.
\]

In order to recover the potential \( q(x, t) \), we take \( z \to 0 \) along the imaginary axis that means \( z \in \Omega_2 \) or \( z \in \Omega_5 \), as a result \( R^{(2)}(z) = \mathbb{I} \). Then, we obtain
\[
M(0) = M^{(3)}(0)E(0)M^{(out)}(0)T^{-\sigma_3}(0),
\]
\[
M = \left( M^{(3)}(0) + M^{(3)}_1 z + \cdots \right) \left( E(0) + E_1 z + \cdots \right) \left( M^{(out)}(z) \right)
\times \left( T^{-\sigma_3}(0) + \hat{T}_1^{-\sigma_3} z + \cdots \right).
\]

Then, by simple calculation, we immediately obtain
\[
M(0)^{-1} M(z) = T^{\sigma_3}(0)M^{(out)}(0)^{-1} M^{(out)}(z)T^{-\sigma_3}(0)z
\]
\[
+ T^{\sigma_3}(0)M^{(out)}(0)^{-1} E_1 M^{(out)}(z)T^{-\sigma_3}(0)z
\]
\[
+ T^{\sigma_3}(0)M^{(out)}(0)^{-1} M^{(out)}(z)T^{-\sigma_3}(0)z + O(t^{-\frac{3}{4}}).
\]

Then, according to the reconstruction formula (3.9), (7.10) and (7.27), as \( t \to -\infty \), we obtain that
\[
q(x, t) = q(y(x, t), t)
\]
\[
= q_{sol}(y(x, t), t; \hat{\sigma}_d(I))T^2(0)(1 + T_1) - it^{-\frac{1}{2}} e^{2d \frac{\partial}{\partial y}} f_{12}^* + O(t^{-\frac{3}{4}}), \quad (9.1)
\]
where
\[ y(x, t) = x - c_-(x, t, \hat{\sigma}_d(I)) - iT_1^{-1} - it^{\frac{1}{2}} f_{11}^- + O(t^{-\frac{3}{4}}), \]
\[ f_{12}^- = \frac{1}{iz_0^2 2\sqrt{2}} [M^{(out)}(0)^{-1}(M^{(out)}(z_0)^{-1}M^{(pc,-)}_1(z_0)M^{(out)}(z_0))]_{12}, \]
\[ f_{11}^- = \frac{1}{iz_0^2 2\sqrt{2}} [M^{(out)}(0)^{-1}(M^{(out)}(z_0)^{-1}M^{(pc,-)}_1(z_0)M^{(out)}(z_0))]_{11}, \]
where \( M^{(pc,-)}_1 \) is defined in Sect. 7.2.3.

For the initial value problem of the WKI equation, i.e., \( q_0(x) \in H(\mathbb{R}) \), the long time asymptotic behavior (9.1) gives the soliton resolution which contains the soliton term confirmed by \( N(I) \)-soliton on discrete spectrum and the \( t^{-\frac{1}{2}} \) order term on continuous spectrum with residual error up to \( O(t^{-\frac{3}{4}}) \). Additionally, our results reveal that the soliton solutions of WKI equation are asymptotic stable.

Remark 9.1. The steps in the steepest descent analysis of RHP 3.5 for \( t \to +\infty \) are similar to the case \( t \to -\infty \) which has been presented in Sects. 5, 6, 7, 8 and 9. When we consider \( t \to +\infty \), the main difference can be traced back to the fact that the regions of growth and decay of the exponential factors \( e^{2it\theta} \) are reversed, see Fig. 1. Here, we leave the detailed calculations to the interested reader.

Finally, we can give the results shown in Theorem 1.1.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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Appendix A: The Parabolic Cylinder Model Problem

Here, we describe the solution of parabolic cylinder model problem [59,60]. Define the contours \( \Sigma^{pc} = \bigcup_{j=1}^4 \Sigma_j^{pc} \) where
\[
\lambda^{-i\nu\sigma_3} e^{\frac{i\lambda^2}{4}\sigma_3} \left( \begin{array}{cc} \frac{1}{1+|r_0|^2} & 0 \\ 0 & \frac{1}{1+|r_0|^2} \end{array} \right) \Omega_2 \\
\lambda^{-i\nu\sigma_3} e^{\frac{i\lambda^2}{4}\sigma_3} \left( \begin{array}{cc} 1 & r_0 \\ 0 & 1 \end{array} \right) \Omega_4 \\
\lambda^{-i\nu\sigma_3} e^{\frac{i\lambda^2}{4}\sigma_3} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \Omega_5
\]

\[
\lambda^{-i\nu\sigma_3} e^{\frac{i\lambda^2}{4}\sigma_3} \left( \begin{array}{cc} \frac{r_0}{1+|r_0|^2} & 0 \\ 0 & \frac{r_0}{1+|r_0|^2} \end{array} \right) \Omega_3 \\
\lambda^{-i\nu\sigma_3} e^{\frac{i\lambda^2}{4}\sigma_3} \left( \begin{array}{cc} 1 & \bar{r}_0 \\ 0 & 1 \end{array} \right) \Omega_6
\]

**Figure 5. Jump matrix \( V^{(pc)} \)**

\[
\Sigma_j^{pc} = \left\{ \lambda \in \mathbb{C} | \arg \lambda = \frac{2j - 1}{4} \pi \right\}. \quad (A.1)
\]

For \( r_0 \in \mathbb{C} \), let \( \nu(r) = -\frac{1}{2\pi} \log(1 + |r_0|^2) \), consider the following parabolic cylinder model Riemann–Hilbert problem.

**Riemann-Hilbert Problem 9.2.** Find a matrix-valued function \( M^{(pc)}(\lambda) \) such that (Fig. 5)

- \( M^{(pc)}(\lambda) \) is analytic in \( \mathbb{C} \setminus \Sigma^{pc} \), \quad (A.2)
- \( M^{(pc)}_+(\lambda) = M^{(pc)}_-(\lambda) V^{(pc)}(\lambda), \quad \lambda \in \Sigma^{pc} \), \quad (A.3)
- \( M^{(pc)}(\lambda) = \mathbb{I} + \frac{M_1}{\lambda} + O(\lambda^2), \quad \lambda \to \infty \), \quad (A.4)

where

\[
V^{(pc)}(\lambda) = \left\{ \begin{array}{c}
\lambda^{-i\nu\sigma_3} e^{\frac{i\lambda^2}{4}\sigma_3} \left( \begin{array}{cc} 1 & r_0 \\ 0 & 1 \end{array} \right), \quad \lambda \in \Sigma_1^{pc}, \\
\lambda^{-i\nu\sigma_3} e^{\frac{i\lambda^2}{4}\sigma_3} \left( \begin{array}{cc} 1 & 0 \\ \frac{r_0}{1+|r_0|^2} & 1 \end{array} \right), \quad \lambda \in \Sigma_2^{pc}, \\
\lambda^{-i\nu\sigma_3} e^{\frac{i\lambda^2}{4}\sigma_3} \left( \begin{array}{cc} 1 & 0 \\ 0 & \frac{1}{1+|r_0|^2} \end{array} \right), \quad \lambda \in \Sigma_3^{pc}, \\
\lambda^{-i\nu\sigma_3} e^{\frac{i\lambda^2}{4}\sigma_3} \left( \begin{array}{cc} 1 & 0 \\ \bar{r}_0 & 1 \end{array} \right), \quad \lambda \in \Sigma_4^{pc}. 
\end{array} \right. \quad (A.5)
\]

We have the parabolic cylinder equation expressed as [61]

\[
\left( \frac{\partial^2}{\partial z^2} + \left( \frac{1}{2} - \frac{z^2}{2} + a \right) \right) D_a = 0.
\]

As shown in the literature [28, 62], we obtain the explicit solution \( M^{(pc)}(\lambda, r_0) \):

\[
M^{(pc)}(\lambda, r_0) = \Phi(\lambda, r_0) \mathcal{P}(\lambda, r_0) e^{\frac{i}{2} \lambda^2 \sigma_3} \lambda^{-i\nu\sigma_3},
\]
where

\[ P(\lambda, r_0) = \begin{cases} 
(1 - r_0, 1), & \lambda \in \Omega_1, \\
(1, 0), & \lambda \in \Omega_2, \\
1 + |r_0|^2, & \lambda \in \Omega_3, \\
(1 - r_0, 1), & \lambda \in \Omega_4, \\
0 + |r_0|^2, & \lambda \in \Omega_5, \\
I, & \lambda \in \Omega_6,
\end{cases} \]

and

\[ \Phi(\lambda, r_0) = \begin{cases} 
\frac{e^{-\frac{3}{4}\pi\nu} D_{\nu'} \left(e^{-\frac{3}{4}\pi\lambda}\right)}{i\beta_{21} e^{-\frac{3}{2}\nu\nu'} D_{\nu'-1} \left(e^{-\frac{3}{2}\pi\nu'}\lambda\right)} + i\beta_{12} e^{-\frac{3}{2}\nu\nu'} D_{\nu'-1} \left(e^{-\frac{3}{2}\pi\nu'}\lambda\right), & \lambda \in \mathbb{C}^+, \\
\frac{e^{\frac{3}{2}\nu\nu'} D_{\nu'} \left(e^{\frac{3}{2}\pi\nu'}\lambda\right)}{i\beta_{21} e^{-\frac{3}{2}\nu\nu'} D_{\nu'-1} \left(e^{\frac{3}{2}\pi\nu'}\lambda\right)} + i\beta_{12} e^{-\frac{3}{2}\nu\nu'} D_{\nu'-1} \left(e^{\frac{3}{2}\pi\nu'}\lambda\right), & \lambda \in \mathbb{C}^-,
\end{cases} \]

with

\[ \beta_{21} = \frac{\sqrt{2\pi} e^{i\pi/4} e^{-\pi\nu'/2}}{r_0 \Gamma(-i\nu)}, \]
\[ \beta_{12} = \frac{-\sqrt{2\pi} e^{-i\pi/4} e^{-\pi\nu'/2}}{r_0 \Gamma(i\nu)} = \frac{\nu}{\beta_{21}}. \] (A.6)

Then, it is not hard to obtain the asymptotic behavior of the solution by using the well-known asymptotic behavior of \( D_a(z) \),

\[ M^{(pc)}(r_0, \lambda) = I + \frac{M_{1}^{(pc)}}{i\lambda} + O(\lambda^{-2}), \] (A.7)

where

\[ M_{1}^{(pc)} = \begin{pmatrix} 0 & -\beta_{12} \\
\beta_{21} & 0 \end{pmatrix}. \] (A.8)

**Appendix B: Meromorphic Solutions of the WKI Riemann–Hilbert Problem**

Here, we study RHP 3.5 for the reflectionless case, i.e., \( r(z) = 0 \). Under this condition, we know that \( M(y, t; z) \) has no jump across the real axis. Then, for given scattering data \( \sigma_d = \{(z_k, c_k), z_k \in \mathbb{Z}\}_{k=1}^{N} \) satisfying \( z_k \neq z_j \) for \( k \neq j \), we obtain the following Riemann–Hilbert problem from RHP 3.5.

**Riemann–Hilbert Problem B.1** Find a matrix value function \( M(y, t; z|\sigma_d) \) satisfying
• \( M(y,t; z|\sigma_d) \) is analytic in \( \mathbb{C}\backslash(\mathcal{Z} \cup \bar{\mathcal{Z}}) \);
• \( M(y,t; z|\sigma_d) = I + O(z^{-1}) \), \( z \to \infty \);
• \( M(y,t; z|\sigma_d) \) satisfies the following residue conditions at simple poles \( z_k \in \mathcal{Z} \) and \( \bar{z}_k \in \bar{\mathcal{Z}} \):
\[
\text{Res}_{z=z_k} M(y,t; z|\sigma_d) = \lim_{z \to z_k} M(y,t; z|\sigma_d)N_k, \\
\text{Res}_{z=\bar{z}_k} M(y,t; z|\sigma_d) = \lim_{z \to \bar{z}_k} M(y,t; z|\sigma_d)\bar{N}_k\sigma_2, \tag{B.1}
\]
where
\[
N_k = \left( \begin{array}{cc} 0 & \gamma_k(x,t) \\ 0 & 0 \end{array} \right), \quad \gamma_k(x,t) = c_ke^{-2it\theta(z_k)}, \tag{B.2}
\]
\[
\theta(z_k) = 2\bar{z}_k^2 + \frac{y}{t}z_k. \tag{B.3}
\]

Then, based on the Liouville’s theorem, the uniqueness of the solution is a direct result. Referring to the symmetry shown in Proposition 2.4, we obtain
\[
M(y,t; z|\sigma_d) = -\sigma_2 M(y,t; \bar{z}|\sigma_d)\sigma_2, \tag{B.4}
\]
from which we can derive the following expansion, i.e.,
\[
M(y,t; z|\sigma_d) = I + \sum_{k=1}^{N} \left[ \frac{1}{z - z_k} \left( \begin{array}{cc} 0 & \zeta_k(x,t) \\ 0 & \eta_k(x,t) \end{array} \right) + \frac{1}{z - \bar{z}_k} \left( \begin{array}{cc} \eta_k(x,t) & 0 \\ -\zeta_k(x,t) & 0 \end{array} \right) \right], \tag{B.5}
\]
where \( \zeta_k(x,t) \) and \( \eta_k(x,t) \) are unknown coefficients to be determined. Next, in the similar way shown in [42], we obtain the following proposition.

**Proposition B.2.** For the given scattering data \( \sigma_d = \{(z_k, c_k), z_k \in \mathcal{Z}\}_{k=1}^{N} \) such that \( z_k \neq z_j \) for \( k \neq j \), the solution of RHP B.1 is unique for each \( (x,t) \in \mathbb{R}^2 \).
Moreover, the solution satisfies
\[
\|M(y,t; z|\sigma_d)\|_{L^\infty(\mathbb{C}\backslash(\mathcal{Z} \cup \bar{\mathcal{Z}}))} \lesssim 1. \tag{B.6}
\]

**B.1 Renormalization of the RHP for Reflectionless Case**

For the reflectionless case, following from the trace formula (5.5), we obtain
\[
s_{22}(z) = \prod_{k=1}^{N} \left( \frac{z - z_k}{z - \bar{z}_k} \right). \tag{B.7}
\]
Following from the ideas in [42], we define \( \Delta \subseteq \{1, 2, \ldots, N\} \), \( \nabla \subseteq \{1, 2, \ldots, N\} \) \( \backslash \Delta \), and
\[
s_{22,\Delta} = \prod_{k \in \Delta} \frac{z - z_k}{z - \bar{z}_k}, \\
s_{22,\nabla} = \frac{s_{11}}{s_{11,\Delta}} = \prod_{k \in \nabla} \frac{z - z_k}{z - \bar{z}_k}. \tag{B.8}
\]

Then, the normalized transformation
\[
M^\Delta(y,t; z|\sigma_d^\Delta) = M(y,t; z|\sigma_d)s_{22,\Delta}(z)^{-\sigma_3}, \tag{B.9}
\]
splits the poles between the columns of \( M(y,t; z|\sigma_d) \) based on the selection of different \( \Delta \). Then, we can get the modified Riemann–Hilbert problem.
Riemann–Hilbert Problem B.3 Given scattering data \( \sigma_d = \{(z_k, c_k)\}_{k=1}^{N} \) and \( \Delta \subseteq \{1, 2, \cdots, N\} \), find a matrix value function \( M^\Delta \) satisfying

- \( M^\Delta(y, t; z|\sigma_d^\Delta) \) is analytic in \( \mathbb{C} \setminus (\mathcal{Z} \cup \tilde{\mathcal{Z}}) \);
- \( M^\Delta(y, t; z|\sigma_d^\Delta) = I + O(z^{-1}), \quad z \to \infty \);
- \( M^\Delta(y, t; z|\sigma_d^\Delta) \) satisfies the following residue conditions at simple poles \( z_k \in \mathcal{Z} \) and \( \tilde{z}_k \in \tilde{\mathcal{Z}} \):
  \[
  \text{Res} M^\Delta(y, t; z|\sigma_d^\Delta) = \lim_{z \to z_k} M^\Delta(y, t; z|\sigma_d^\Delta) N_k^\Delta,
  \]
  \[
  \text{Res} M^\Delta(y, t; z|\sigma_d^\Delta) = \lim_{z \to \tilde{z}_k} M^\Delta(y, t; z|\sigma_d^\Delta) \sigma_2 N_k^\Delta \sigma_2^\Delta,
  \]

where

\[
N_k^\Delta = \begin{cases} 
  \begin{pmatrix} 0 & \gamma_k^\Delta \\
 0 & 0 \end{pmatrix}, & k \in \nabla, \\
  \begin{pmatrix} \gamma_k^\Delta & 0 \\
 0 & 0 \end{pmatrix}, & k \in \Delta,
\end{cases}
\]

\[
\gamma_k^\Delta = \begin{cases} 
  c_k(s_{22,\Delta}(z_k))^2 e^{-2it\theta(z_k)}, & k \in \nabla, \\
  c_k^{-1}(s'_{22,\Delta}(z_k))^{-2} e^{2it\theta(z_k)}, & k \in \Delta,
\end{cases}
\]

\[
\theta(z_k) = 2z_k^2 + \frac{y}{t} z_k.
\]

Because \( M^\Delta(y, t; z|\sigma_d^\Delta) \) is directly transformed from \( M(y, t; z|\sigma_d) \), it is obvious to find out that RHP B.3 has a unique solution.

For given scattering data \( \sigma_d^\Delta \), using \( q_{sol}(y, t) = q_{sol}(y, t; \sigma_d^\Delta) \) to denote the unique \( N \)-soliton solution of the WKI equation (1.3), by applying (B.8), we can derive that

\[
q_{sol}(y, t; \sigma_d^\Delta) = e^{2d} \lim_{z \to 0} \frac{\partial}{\partial y} \left( \frac{M(0; y, t|\sigma_d^\Delta)^{-1}M(z; y, t|\sigma_d^\Delta)}{z} \right)_{12}.
\]

This indicates that each normalization encodes \( q_{sol}(y, t) \) in the same way. When the scattering coefficient \( s_{22}(z) \) only possesses one zero point \( z_1 \), the one soliton solution can be derived. Taking \( z_1 = \xi + i\eta, \xi > 0, \eta > 0 \), the one soliton solution of the WKI equation (1.3) is derived as [23]

\[
q(x, t) = q(y(x, t), t) = \frac{2\eta(\xi - \eta i)[\xi \sinh(2\phi) + i\eta \cosh(2\varphi)] e^{2d-2i\varphi}}{\eta[(\xi^2 + \eta^2) \cosh(2\phi) - 2\eta^2]},
\]

\[
x = y - \frac{2\eta}{\eta^2(1 + e^{4\phi})};
\]

where \( \phi = \phi(y, t) \) and \( \varphi = \varphi(y, t) \) are, respectively, defined as

\[
\varphi(y, t) = \xi y + 2(\xi^2 - \eta^2)t - \frac{1}{2} \text{arg}(c_1),
\]

\[
\phi(y, t) = 4\xi \eta t - \eta y - \frac{1}{2} \text{log}(|c_1|).
\]

The constant \( c_1 \) is the norming constant and \( d \) is defined in (2.13). However, when the scattering coefficient \( s_{22}(z) \) possesses multiple zero point, the exact formula of the solution is too complicated to derive, we do not give them here. In fact, after the elastic collisions, the \( N \)-soliton asymptotically separate into \( N \) single-soliton solutions as \( t \to \infty \). Of course, the non-generic case, for
example, two points of scattering data lie on a vertical line, is an exception. Next, we study the asymptotic behavior of the soliton solutions.

**B.2 Long-Time Behavior of Soliton Solutions**

Define a distance

$$\mu(I) = \min_{z_k \in \mathbb{Z} \setminus \mathbb{Z}(I)} \{Im(z_k)dist(Rez_k, I)\}, \quad (B.12)$$

and a space-time cone

$$S(y_1, y_2, v_1, v_2) = \{(y, t), y = y_0 + vt \text{ with } y_0 \in [y_1, y_2], v \in [v_1, v_2]\}, \quad (B.13)$$

where $v_1 \leq v_2 \in \mathbb{R}$ are given velocities (Fig. 6).

**Proposition B.4.** Given scattering data $\sigma_d^{\Delta z_0} = \{(z_k, c_k)\}$, fix $y_1, y_2, v_1, v_2 \in \mathbb{R}$ and $y_1 < y_2$, $v_1 < v_2$. Let $I = [-\frac{v_2}{4}, -\frac{v_1}{4}]$. Then as $t \to \infty$ and $(y, t) \in S(y_1, y_2, v_1, v_2)$, we have

$$M^{\Delta I_0}(z|\sigma_d^{\Delta z_0}) = (I + O(e^{-8\mu|t|}))M^{\Delta I}(z|\hat{\sigma}_d(I)), \quad (B.14)$$

where $M^{\Delta I}(z|\hat{\sigma}_d(I))$ is $N(I) = |\mathbb{Z}(I)|$-soliton solutions corresponding to scattering data (Fig. 7)

$$\hat{\sigma}_d(I) = \{(z_k, c_k(I)), z_k \in \mathbb{Z}(I)\},$$

$$c_k(I) = c_k \prod_{z_j \in \mathbb{Z} \setminus \mathbb{Z}(I)} \left(\frac{z_k - z_j}{z_k - \bar{z_j}}\right)^2. \quad (B.15)$$
Figure 7. For fixed $v_1 < v_2$, $\mathcal{I} = \left[ -\frac{v_2}{4}, -\frac{v_1}{4} \right]$

Proof. We first consider the case of $M_{\Delta z_0}^\triangle (z)\sigma_d^{\Delta z_0}$. Define

$$\triangle^- (\mathcal{I}) = \{ k : \text{Re} z_k < -\frac{v_2}{4} \},$$

$$\triangle^+ (\mathcal{I}) = \{ k : \text{Re} z_k > -\frac{v_1}{4} \}.$$

Then, if we choose $\triangle = \triangle^- (\mathcal{I})$ in RHP B.3, it is easy to check that

$$||N_k^{\Delta^- (\mathcal{I})}|| = \begin{cases} o(1) & k \in \mathcal{Z}(\mathcal{I}), \\ o(e^{-8\mu(\mathcal{I})|t|}) & k \in \mathcal{Z}\setminus\mathcal{Z}(\mathcal{I}), \end{cases} t \to -\infty,$$

which implies that the residues with $z_k \in \mathcal{Z}\setminus\mathcal{Z}(\mathcal{I})$ have little contribution to the solution $M_{\Delta z_0}^\pm$.

For each discrete spectrum point $z_k \in \mathcal{Z}\setminus\mathcal{Z}(\mathcal{I})$, we make a small disk $D_k$ corresponding to each spectrum point $z_k$. And the radius of the disk $D_k$ is sufficiently small to guarantee that they are non-overlapping. Denote $\partial D_k$ as the boundary of $D_k$. Then, we introduce that

$$\Xi(z) = \begin{cases} I - \frac{N_k^{\Delta^- (\mathcal{I})}}{z - z_k} & z \in D_k, \\ I - \frac{\sigma_2(N_k^{\Delta^- (\mathcal{I})})\sigma_2}{z - \bar{z}_k} & z \in \overline{D_k}, \\ I, & \text{elsewhere.} \end{cases}$$

(B.17)

By introducing a transformation that $\hat{M}_{\Delta z_0}^\triangle (z) = M_{\Delta z_0}^\triangle (z)\Xi(z)$, we can derive that $\hat{M}_{\Delta z_0}^\triangle (z)$ has a new jump in $\partial D_k$. Then, $\hat{M}_{\Delta z_0}^\triangle (z)$ satisfied the following jump relationship

$$\hat{M}_{\Delta z_0}^\triangle (z) = \hat{M}_{\Delta z_0}^\triangle (z)\hat{V}, \quad z \in \partial D_k \cup \partial \overline{D_k}.$$
By using the estimate (B.16), the jump matrix $\hat{V}$ satisfies that
\[
\|\hat{V} - I\| = O(e^{-8\mu|t|}), \quad z \in \partial D_k \cup \overline{\partial D_k}, \quad t \to -\infty. \tag{B.19}
\]
Observing a fact that $\hat{M}^{\Delta_0^+}(z|\sigma_d)$ and $M^{\Delta_0^+}(z|\dot{\sigma}_d(\mathcal{I}))$ possess the same poles and residue conditions. Therefore, we can show that
\[
\hat{M}^{\Delta_0^+}(z|\sigma_d)[M^{\Delta_0^+}(z|\dot{\sigma}_d(\mathcal{I}))]^{-1} \triangleq \varepsilon(z) \tag{B.20}
\]
has no poles. And, its jumps across the $\partial D_k \cup \overline{\partial D_k}$ satisfy the same estimates with (B.19). Then, with the application of the theory of small-norm Riemann–Hilbert problems, one can easily derive that
\[
\varepsilon(z) = \mathbb{I} + O(e^{-8\mu|t|}), \quad t \to \infty,
\]
which together with $\hat{M}^{\Delta_0^+}(z) = M^{\Delta_0^+}(z)\Xi(z)$ gives the formula (B.14). The other case of $M^{\Delta_0^+}(z|\sigma_d^{\Delta_0^+})$ can be proved similarly.

\[\square\]

**Appendix C: Detailed Calculations for the Pure $\bar{\partial}$-Problem**

**Proposition C.1.** For large $t$, there exist constants $c_j (j = 1, 2, 3)$ such that $I_j (j = 1, 2, 3)$ defined in (8.7) and (8.8) possess the following estimate
\[
I_j \leq c_j t^{-\frac{4}{3}}, \quad j = 1, 2, 3. \tag{C.1}
\]

**Proof.** Let $s = p + iq$ and $z = \xi + i\eta$. Considering the fact that
\[
\left\| \frac{1}{s - z} \right\|_{L^2(q + z_0)} = \left( \int_{q + z_0}^{+\infty} \frac{1}{|s - z|^2} dp \right)^{\frac{1}{2}} \leq \frac{\pi}{q - \eta},
\]
we can derive that
\[
|I_1| \leq \int_0^{+\infty} \int_{q + z_0}^{+\infty} \left| \frac{\partial \chi (s)}{s - z} \right| e^{-4|t||q(p - z_0)|} dp dq
\]
\[
\leq \int_0^{+\infty} e^{-4|t||q|^2} \left| \frac{\partial \chi (s)}{s - z} \right|_{L^2(q + z_0)} \left\| \frac{1}{s - z} \right\|_{L^2(q + z_0)} dq \tag{C.2}
\]
\[
\leq c_1 \int_0^{+\infty} e^{-4|t||q|^2} \sqrt{|q - \eta|} dq \leq c_1 |t|^{-\frac{1}{3}}.
\]
Similarly, considering that $r \in H^{1,1}(\mathbb{R})$, we obtain the estimate
\[
|I_2| \leq \int_0^{+\infty} \int_{q + z_0}^{+\infty} \left| r'(p) \right| e^{-4|t||q|^2} \left| \frac{1}{s - z} \right| dq dp \leq c_2 |t|^{-\frac{1}{3}}. \tag{C.3}
\]
To obtain the estimate of $I_3$, we consider the following $L^k (k > 2)$ norm
\[
\left\| \frac{1}{\sqrt{|s - z_0|}} \right\|_{L^k} \leq \left( \int_{q + z_0}^{+\infty} \frac{1}{|p - z_0 + iq|^\frac{1}{2}} dp \right)^{\frac{1}{k}} \leq c q^{\frac{1}{2} - \frac{1}{2k}}. \tag{C.4}
\]
Similarly, we can derive that
\[ \left\| \frac{1}{s-z} \right\|_{L^k} \leq c|q - \eta|^{\frac{1}{2}} - \frac{1}{2}. \]  
(C.5)

By applying (C.4) and (C.5), it is not hard to check that
\[ |I_3| \leq \int_0^{\infty} \int_q^{\infty} |z - z_0|^{-\frac{1}{2}} e^{-4|t|q(p-z_0)} \frac{|s-z|}{|s-z_0|} dpdq \]
\[ \leq \int_0^{\infty} e^{-4|t|q^2} \left\| \frac{1}{\sqrt{|s-z_0|}} \right\|_{L^k} \left\| \frac{1}{|s-z|} \right\|_{L^k} dq \leq c_3 t^{-\frac{1}{2}}. \]  
(C.6)

Now, we complete the estimates of \( I_j (j = 1, 2, 3) \).

\[ \square \]

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