Almost primes in generalized Piatetski-Shapiro sequences

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Abstract: We consider a generalization of Piatetski-Shapiro sequences in the sense of Beatty sequences, which is of the form \((\lfloor \alpha n^c + \beta \rfloor)_{n=1}^{\infty}\) with real numbers \(\alpha \geq 1\), \(c > 1\) and \(\beta\). In this paper, we prove that there are infinitely many \(R\)-almost primes in sequences \((\lfloor \alpha n^c + \beta \rfloor)_{n=1}^{\infty}\) if \(c \in (1, c_R)\) and \(c_R\) is an explicit constant depending on \(R\).

Keywords: Piatetski-Shapiro sequences; almost primes; exponent pair
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1. Introduction

For \(1 < c \notin \mathbb{N}\), the Piatetski-Shapiro sequences are sequences of the form

\[ N^{(c)} := (\lfloor n^c \rfloor)_{n=1}^{\infty} \quad (c > 1, \ c \notin \mathbb{N}). \]

Such sequences have been named in honor of Piatetski-Shapiro [5], who published the first paper in this problem. He showed that the counting function

\[ |\{\text{prime } p \leq x : p \in N^{(c)}\}| \sim \frac{x^{\frac{1}{c}}}{\log x} \quad \text{as } x \to \infty, \]

holds for \(1 < c < \frac{12}{11}\). The range for \(c\) of the asymptotic formula of \(\pi^{(c)}(x)\) has been extended many times over the years and is currently known for all \(c \in (1, \frac{243}{205})\) thanks to Rivat and Wu [6]. It is conjectured that there are infinitely many Piatetski-Shapiro primes for \(c \in (1, 2)\). However, we can see that the best known bound for \(c\) in [6] is still far from 2 and it has not been improved for almost 20 years.

Several mathematicians approached this problem in a different direction. For every \(R \geq 1\), we say that a natural number is an \(R\)-almost prime if it has at most \(R\) prime factors, counted with multiplicity. The study of almost primes is an intermediate step to the investigation of primes.
Baker, Banks, Guo and Yeager [1] proved that for any fixed \( c \in (1, \frac{67}{66}) \) there are infinitely many primes of the form \( p = \lfloor n^c \rfloor \), where \( n \) is a natural number with at most eight prime factors. More precisely,

\[
|\{ n \leq x : n \text{ is an 8-almost prime and } \lfloor n^c \rfloor \text{ is prime} \}| \gg \frac{x}{\log^2 x}.
\]

Provided that \( c_R \) is an explicit constant depending on \( R \), for any fixed \( c \in (1, c_R) \), Guo [4] proved that

\[
\left| \{ n \leq x : \lfloor n^c \rfloor \text{ is a } R\text{-almost prime} \} \right| \gg \frac{x}{\log x}
\]

holds for all sufficiently large \( x \).

For fixed real numbers \( \alpha \) and \( \beta \), the associated non-homogeneous Beatty sequence is the sequence of integers defined by

\[
\mathcal{B}_{\alpha, \beta} := (\lceil \alpha n + \beta \rceil)_{n=1}^\infty,
\]

where \( \lfloor t \rfloor \) denotes the integral part of any \( t \in \mathbb{R} \). Such sequences are also called generalized arithmetic progressions. If \( \alpha \) is irrational, it follows from a classical exponential sum estimate of Vinogradov [9] that \( \mathcal{B}_{\alpha, \beta} \) contains infinitely many prime numbers; in fact, one has the asymptotic relation

\[
\#\{ \text{prime } p \leq x : p \in \mathcal{B}_{\alpha, \beta} \} \sim \alpha^{-1} \pi(x) \quad (x \to \infty),
\]

where \( \pi(x) \) is the prime counting function.

It is interesting to generalize the Piatetski-Shapiro sequences in the sense of Beatty sequences, since both Piatetski-Shapiro sequences and Beatty sequences produce infinitely many primes. Let \( \alpha \geq 1 \) and \( \beta \) be real numbers. We investigate the following generalized Piatetski-Shapiro sequences

\[
\mathcal{N}^{(c)}_{\alpha, \beta} := (\lfloor \alpha n + \beta \rfloor)_{n=1}^\infty.
\]

Note that the special case \( \mathcal{N}^{(c)}_{1,0} \) is the normal Piatetski-Shapiro sequences. In this paper, we prove that there are infinitely many almost primes in generalized Piatetski-Shapiro sequences.

**Theorem 1.1.** For any fixed \( c \in (1, c_R) \) we have

\[
\left| \{ n \leq x : \lfloor \alpha n^c + \beta \rfloor \text{ is a } R\text{-almost prime} \} \right| \gg \frac{x}{\log x}
\]

holds for all sufficiently large \( x \). In particular, we have

\[
c_3 := \frac{329}{249} = 1.3319 \ldots, \quad c_4 := \frac{25882}{16071} = 1.6104 \ldots,
\]

and

\[
c_R := 3 - \frac{128}{3(8R-1)} \quad (R \geq 5).
\]
2. Preliminaries

2.1. Notations

We denote by $[t]$ and $\{t\}$ the integer part and the fractional part of $t$, respectively. As is customary, we put $e(t) = e^{2\pi it}$. We make considerable use of the sawtooth function defined by

$$\psi(t) := t - [t] - \frac{1}{2} = \{t\} - \frac{1}{2} \quad (t \in \mathbb{R}).$$

We use notation of the form $m \sim M$ as an abbreviation for $M < m \leq 2M$.

Throughout the paper, implied constants in symbols $O$, $\ll$ and $\gg$ may depend (where obvious) on the parameters $\alpha, \varepsilon$ but are absolute otherwise. For given functions $F$ and $G$, the notations $F \ll G$, $G \gg F$ and $F = O(G)$ are all equivalent to the statement that the inequality $|F| \leq C|G|$ holds with some constant $C > 0$. $F \asymp G$ means that $F \ll G \ll F$.

2.2. Technical lemmas

As we have mentioned the following notion plays a crucial role in our arguments. We specify it to the form that is suited to our applications; it is based on a result of Greaves [2] that relates level of distribution to $R$-almost primality. More precisely, we say that an $N$-element set of integers $A$ has a level of distribution $D$ if for a given multiplicative function $f(d)$ we have

$$\sum_{d \leq D} \max_{\gcd(s,d)=1} \left| \left\{ a \in A, \ a \equiv s \text{ mod } d \right\} - \frac{f(d)}{d} \right| N \leq \frac{N}{\log^2 N}.$$  

As in [2, pp. 174–175] we define the constants

$$\delta_2 := 0.044560, \quad \delta_3 := 0.074267, \quad \delta_4 := 0.103974$$

and

$$\delta_R := 0.124820, \quad R \geq 5.$$ 

We have the following result, which is [2, Chapter 5, Proposition 1].

**Lemma 2.1.** Suppose $A$ is an $N$-element set of positive integers with a level of distribution $D$ and degree $\rho$ in the sense that

$$a < D^\rho \quad (a \in A)$$

holds with some real number $\rho < R - \delta_R$. Then

$$\left| \left\{ a \in A : a \text{ is an } R\text{-almost prime} \right\} \right| \gg \frac{N}{\log N}.$$ 

**Lemma 2.2.** Let $M \geq 1$ and $\lambda$ be positive real numbers and let $H$ be a positive integer. If $f : [1, M] \rightarrow \mathbb{R}$ is a real valued function with three continuous derivatives, which satisfies

$$\lambda \leq |f^{(3)}(x)| \ll \lambda \quad \text{for } 1 \leq x \leq M,$$
then for the sum

\[ S = \frac{1}{H} \sum_{h=H+1}^{2H} \left| \sum_{m=1}^{M_h} \left( \frac{h}{H} f(m) \right) \right|, \]

where the integer \( M_h \) satisfies \( 1 \leq M_h \leq M \) for each \( h \in [H + 1, 2H] \), we have

\[ S \ll M^c \left( M^1/6 H^{-1/9} + M^1/5 + M^{3/4} \right) + \lambda^{-1/3}. \]

Proof. See [7, Theorem 1].

Lemma 2.3. For any \( H \geq 1 \) there are numbers \( a_h, b_h \) such that

\[ \left| \psi(t) - \sum_{0 \leq |h| < H} a_h e(th) \right| \leq \sum_{|h| < H} b_h e(th), \]

where

\[ a_h \ll \frac{1}{|h|}, \quad b_h \ll \frac{1}{H}. \]

Proof. See [8].

We also need the method of exponent pair. A detailed definition of exponent pair can be found in [3, p. 31]. For an exponent pair \((k, l)\), we denote

\[ A(k, l) = \left( \frac{k}{2k+2}, \frac{k+l+1}{2k+2} \right) \]

and

\[ B(k, l) = \left( l - \frac{1}{2}, k + \frac{1}{2} \right), \]

the A-process and B-process of the exponent pair, respectively.

3. Proof of Theorem 1.1

Now we prove our main results. The set we sieve is

\[ \mathcal{A} = \{ m \leq x^c : m = [an^c + \beta] \text{ for integer } n \}. \]

For any \( d \leq D \), where \( D \) is a fixed power of \( x \), we estimate

\[ \mathcal{A}_d := \{ m \in \mathcal{A} : d \mid m \}. \]

We know that \( rd \in \mathcal{A} \) if and only if

\[ rd \leq an^c + \beta < rd + 1 \quad \text{and} \quad rd \leq x^c. \]

Within \( O(1) \) the cardinality of \( \mathcal{A}_d \) is equal to the number of integers \( n \leq x \) for which the interval \( ((an^c + \beta - 1)d^{-1}, (an^c + \beta)d^{-1}] \) contains a natural number. Hence

\[ |\mathcal{A}_d| = \sum_{n \leq x} \left( \left\lfloor (an^c + \beta)d^{-1} \right\rfloor - \left\lfloor (an^c + \beta - 1)d^{-1} \right\rfloor \right) + O(1) \]
\[= Xd^{-1} + \sum_{n \leq x} \left( \psi((an^c + \beta - 1)d^{-1}) - \psi((an^c + \beta)d^{-1}) \right) + O(1),\]

where

\[X = \sum_{n \leq x} 1 = x.\]

By Lemma 2.1 we need to show that for any sufficiently small \( \varepsilon > 0, \)

\[\sum_{d \leq D} \left| |A_d| - Xd^{-1} \right| \leq Xx^{-\frac{\varepsilon}{4}},\]

for sufficiently large \( x. \) Splitting the range of \( d \) into dyadic subintervals, it is sufficient to prove that

\[\sum_{d \sim D} \left| \sum_{N \leq n \leq N_1} \left( \psi((an^c + \beta - 1)d^{-1}) - \psi((an^c + \beta)d^{-1}) \right) \right| \ll x^{1-\frac{\varepsilon}{2}}, \tag{3.1}\]

holds uniformly for \( D_1 \leq D, N \leq x, N_1 \sim N. \) Our aim is to establish (3.1) with \( D \) as large as possible.

We define

\[S := \sum_{N \leq n \leq N_1} \left( \psi((an^c + \beta - 1)d^{-1}) - \psi((an^c + \beta)d^{-1}) \right). \tag{3.2}\]

By Lemma 2.3 and taking \( H = Dx^c, \) we have

\[S = S_1 + O(S_2),\]

where

\[S_1 := \sum_{N \leq n \leq N_1} \sum_{0 < |h| < H} a_h \left( e(h(an^c + \beta - 1)d^{-1}) - e(h(an^c + \beta)d^{-1}) \right)\]

and

\[S_2 := \sum_{N \leq n \leq N_1} \sum_{|h| \leq H} b_h \left( e(h(an^c + \beta - 1)d^{-1}) + e(h(an^c + \beta)d^{-1}) \right).\]

We split \( S_1 \) into two parts

\[S_1 = S_1^{(1)} + S_1^{(2)}, \tag{3.3}\]

where

\[S_1^{(1)} := \sum_{N \leq n \leq N_1} \sum_{0 < |h| < H} a_h \left( e(h(an^c + \beta - 1)d^{-1}) - e(han^c d^{-1}) \right),\]

and

\[S_1^{(2)} := \sum_{N \leq n \leq N_1} \sum_{0 < |h| < H} a_h \left( e(han^c d^{-1}) - e(h(an^c + \beta)d^{-1}) \right).\]

We consider \( S_1^{(1)} \). Writing that

\[\phi_h := e(h(\beta - 1)d^{-1}) - 1 \ll 1.\]

Using the exponent pair \((k, l),\) we obtain that

\[S_1^{(1)} = \sum_{N \leq n \leq N_1} \sum_{0 < |h| < H} a_h \phi_h e(han^c d^{-1})\]
\[
\ll \sum_{0<h<H} h^{-1} \sum_{N<n<N_1} e(ha_n d^{-1})| \\
\ll \sum_{0<h<H} h^{-1} \left( (hd^{-1}N^{c-1})^k N^l + (hd^{-1})^{-1} N^{1-c} \right) \\
\ll H^k d^{-k}N^{kc-k+l} + H^{-1}dN^{1-c}. \tag{3.4}
\]

For \(S_1^{(2)}\), by a similar argument with \(\phi_h\) replaced by \(\varphi_h\) defined by

\[
\varphi_h := 1 - e(h\beta d^{-1}) \ll 1.
\]

One can derive that

\[
S_1^{(2)} \ll H^k d^{-k}N^{kc-k+l} + H^{-1}dN^{1-c}. \tag{3.5}
\]

Now we consider \(S_2\). The contribution of \(S_2\) from \(h = 0\) is

\[
\sum_{N<n<N_1} b_h \ll NH^{-1}. \tag{3.6}
\]

By a similar arguments of \(S_1\) with a shift of \(n\), the contribution of \(S_2\) from \(h \neq 0\) is

\[
\ll \sum_{N<n<N_1} \sum_{0<h<H} b_h \left( e(h(a_n d^{-1} + \beta - 1)d^{-1}) + e(h(a_n d^{-1} + \beta)d^{-1}) \right) \\
\ll \sum_{N<n<N_1} \sum_{0<h<H} b_h \phi_h e(ha_n d^{-1}) + \sum_{N<n<N_1} \sum_{0<h<H} b_h \varphi_h e(ha_n d^{-1}) \\
\ll \sum_{0<h<H} H^{-1} \left| \sum_{N<n<N_1} e(ha_n d^{-1}) \right| \\
\ll \sum_{0<h<H} H^{-1} (h^k d^{-k}N^{kc-k+l} + h^{-1}dN^{1-c}) \\
\ll H^k d^{-k}N^{kc-k+l} + H^{-1}dN^{1-c} \log H. \tag{3.7}
\]

Substituting (3.4) and (3.5) to (3.3), and combining (3.6) and (3.7), the left hand side of (3.1) is

\[
\ll \sum_{d=D_1} (H^k d^{-k}N^{kc-k+l} + H^{-1}dN^{1-c} \\
+ H^{-1}dN^{1-c} \log H + NH^{-1}) \ll Dx^{kc-k+l+ke} + Dx^{1-c+e}.
\]

Therefore, to make (3.1) to be true, we need that

\[
D x^{ke-k+l+ke} \ll x^{1-\frac{c}{2}}, \tag{3.8}
\]

and

\[
D x^{1-c+e} \ll x^{1-\frac{c}{2}}. \tag{3.9}
\]

Combining (3.8) and (3.9), we obtain that

\[
D \ll \min \left( x^{-\frac{c}{2}}, x^{1-ke-k-l-e} \right). \tag{3.10}
\]
3.1. Exponent pair estimation for $R = 3$

By Lemma 2.1, $A$ contains $\gg x/\log x$ $R$-almost primes. We apply the weighted sieve with the choice

$$R = 3, \quad \delta_3 = 0.074267$$

and choose

$$\Lambda_R = 3 - \frac{3}{40} = \frac{117}{40} < R - \delta_R.$$  

By (3.10) we require that

$$1 - kc + k - l > \frac{40}{117} \quad \text{and} \quad c > \frac{40}{117},$$  

then

$$c < \frac{77 - 117l}{117k} + 1.$$

Taking the exponent pair

$$BAAAAAB(0, 1) = \left(\frac{19}{42}, \frac{32}{63}\right),$$

we have

$$c < \frac{329}{247} = 1.3319 \ldots.$$  

3.2. Exponent pair estimation for $R = 4$

Similarly, we apply the weighted sieve with the choice

$$R = 4, \quad \delta_4 = 0.103974$$

and choose

$$\Lambda_R = 4 - \frac{13}{125} = \frac{487}{125} < R - \delta_R.$$  

By taking the exponent pair

$$BABABAABAB(0, 1) = \left(\frac{33}{128}, \frac{75}{128}\right),$$

we can get

$$c < \frac{362 - 487l}{487k} + 1 = \frac{25882}{16071} = 1.6104 \ldots.$$  

3.3. The bound of $c$ for $R \geq 5$

For $R \geq 5$, we estimate (3.2) by Lemma 2.2. From (3.4) we have

$$S_1(1) \ll \log H \max_{1 \leq T \leq H} S(T, N),$$

where

$$S(T, N) = \frac{1}{T} \sum_{h \sim T} \sum_{n \sim N} e(hd^{-1}n^r).$$
By Lemma 2.2 with \( f(n) = Td^{-1}(n + N)^c \) and
\[
\lambda = c(c - 1)(c - 2)Td^{-1}N^{c-3},
\]
it follows that
\[
S(T, N) \ll N^{1+\epsilon} (Td^{-1}N^{c-3})^{\frac{d}{2}} T^{-\frac{1}{2}} + N^{1+\epsilon} (Td^{-1}N^{c-3})^{\frac{d}{2}} + N^{\frac{d}{2}+\epsilon} + (Td^{-1}N^{c-3})^{-\frac{d}{4}}
\]
\[
\ll T^{\frac{d}{2}} d^{-\frac{1}{2}} N^{\frac{d}{2}+\epsilon} + T^{\frac{d}{2}} d^{-\frac{1}{2}} N^{\frac{d}{2}+\epsilon} + N^{\frac{d}{2}+\epsilon} + T^{-\frac{d}{4}} d^{\frac{1}{2}} N^{1-\frac{d}{2}}.
\]
Hence
\[
S^{(1)} \ll H^{\frac{d}{2}} d^{-\frac{1}{2}} N^{\frac{d}{2}+\epsilon} + H^{\frac{d}{2}} d^{-\frac{1}{2}} N^{\frac{d}{2}+\epsilon} + N^{\frac{d}{2}+\epsilon} + d^{\frac{1}{2}} N^{1-\frac{d}{2}}.
\]
Similarly, we can get the estimation of \( S^{(2)} \). The contribution of \( S_2 \) from \( h \neq 0 \) can be estimated by the same method and achieve the same upper bound. Together with the contribution of \( S_2 \) from \( h = 0 \), by (3.6) we obtain that the left-hand side of (3.1) is
\[
\sum_{d-D_1} |S| \ll \sum_{d-D_1} \left| H^{\frac{d}{2}} d^{-\frac{1}{2}} N^{\frac{d}{2}+\epsilon} + H^{\frac{d}{2}} d^{-\frac{1}{2}} N^{\frac{d}{2}+\epsilon} + N^{\frac{d}{2}+\epsilon} + d^{\frac{1}{2}} N^{1-\frac{d}{2}} \right|
\]
\[
\ll H^{\frac{d}{2}} D^{\frac{d}{2}} N^{\frac{d}{2}+\epsilon} + H^{\frac{d}{2}} D^{\frac{d}{2}} N^{\frac{d}{2}+\epsilon} + DN^{\frac{d}{2}} \log H + D^{\frac{d}{2}} N^{1-\frac{d}{2}}
\]
\[
\ll D^{\frac{d}{2}} x^{\frac{d}{2}+\epsilon} + D^{\frac{d}{2}} x^{\frac{d}{2}+\epsilon} + D^{\frac{d}{2}} x^{\frac{d}{2}+\epsilon} + D^{\frac{d}{2}} x^{1-\frac{d}{2}}.
\]
To ensure the left-hand side of (3.1) is \( \ll x^{1-\epsilon/2} \), we require that
\[
D \ll \min \left( x^{\frac{d}{2}+\epsilon}, x^{\frac{d}{2}+\epsilon}, x^{\frac{d}{2}+\epsilon}, x^{\frac{d}{2}+\epsilon} \right).
\] (3.12)
We apply the weighted sieve with the choice
\[
\delta_R = 0.124820 \quad (R \geq 5)
\]
and choose
\[
\Lambda_R = R - \frac{1}{8} < R - \delta_R.
\]
To apply Lemma 2.1, by (3.12) we need that
\[
\min \left( \frac{9}{16} - \frac{3c}{16}, \frac{3c}{16} - \frac{c}{5}, \frac{c}{5}, \frac{1}{4}, \frac{1}{4} \right) > \frac{1}{R - \frac{1}{8}},
\]
which gives that
\[
c < 3 - \frac{128}{3(8R - 1)}.
\]
4. Conclusions

In this paper, we investigate the following generalized Piatetski-Shapiro sequences

\[ N^{(c)}_{\alpha, \beta} = (\lfloor an^c + \beta \rfloor)_{n=1}^\infty. \]

We prove that there are infinitely many $R$-almost primes in generalized Piatetski-Shapiro sequences by the Van der Corput’s method of exponential sums and exponent pairs.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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