SYMMETRIC ORNSTEIN-UHLENBECK SEMIGROUPS
AND THEIR GENERATORS

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Abstract. We provide necessary and sufficient conditions for a Hilbert space-valued Ornstein-Uhlenbeck process to be reversible with respect to its invariant measure $\mu$. For a reversible process the domain of its generator in $L^p(\mu)$ is characterized in terms of appropriate Sobolev spaces thus extending the Meyer equivalence of norms to any symmetric Ornstein-Uhlenbeck operator. We provide also a formula for the size of the spectral gap of the generator. Those results are applied to study the Ornstein-Uhlenbeck process in a chaotic environment. Necessary and sufficient conditions for a transition semigroup $(R_t)$ to be compact, Hilbert-Schmidt and strong Feller are given in terms of the coefficients of the Ornstein-Uhlenbeck operator. We show also that the existence of spectral gap implies a smoothing property of $R_t$ and provide an estimate for the (appropriately defined) gradient of $R_t\phi$. Finally, in the Hilbert-Schmidt case, we show that for any $\phi \in L^p(\mu)$ the function $R_t\phi$ is an (almost) classical solution of a version of the Kolmogorov equation.

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1. INTRODUCTION

Consider a linear stochastic differential equation
\[
\begin{aligned}
dZ &= AZdt + \sqrt{Q}dW, \\
Z(0) &= x,
\end{aligned}
\] (1.1)
in a separable real Hilbert space $H$. We assume that $Q$ is a bounded self-adjoint and nonnegative operator on $H$ and $A$ generates on $H$ a strongly continuous semigroup $(S(t))$. The process $W$ is a standard cylindrical Wiener process on $H$. Under appropriate assumptions (see Hypothesis 1.1 below) the solution to (1.1), is given by the formula
\[
Z(t, x) = S(t)x + \int_0^t S(t-s)\sqrt{Q}dW(s), \quad t \geq 0.
\]
The process $Z$, called the Ornstein-Uhlenbeck process, is Gaussian and Markov in $H$ with the transition semigroup
\[
R_t \phi(x) = E\phi(Z(t, x)),
\]
where $\phi$ is a bounded Borel function on $H$. An important class of the Ornstein-Uhlenbeck processes are the so called reversible Ornstein-Uhlenbeck processes which arise in the theory of Interacting Particle Systems and other areas of Mathematical Physics. Let us recall that a probability measure $\mu$ is said to be symmetrizing for the semigroup $(R_t)$, or equivalently the process $Z$ is said to be $\mu$-reversible, if
\[
\int_H \psi(x)R_t\phi(x)\mu(dx) = \int_H \phi(x)R_t\psi(x)\mu(dx),
\] (1.2)
for bounded Borel functions $\phi, \psi$. If such a measure $\mu$ exists then it is necessarily invariant for the semigroup $(R_t)$. In that case $(R_t)$ extends to a strongly continuous semigroup of contractions on $L^p(H, \mu)$ (still denoted by $(R_t)$) for all $p \in [1, \infty)$. Moreover, (1.2) implies that $R_t$ is symmetric in $L^2(H, \mu)$ for each $t \geq 0$.

The aim of this paper is to provide necessary and sufficient conditions for the invariant measure of an arbitrary Ornstein-Uhlenbeck process $Z$ to be symmetrizing and to study some important properties of a symmetric semigroup $(R_t)$ and its generator $L$ in the spaces $L^p(H, \mu)$ under the sole assumption of existence of a nondegenerate invariant measure $\mu$.

The main idea of this paper may be described as follows. Let $H_Q = Q^{1/2}(H)$ with the norm $|x|_Q = |Q^{-1/2}x|$. We will show that the semigroup $(R_t)$ is symmetric in $L^2(H, \mu)$ if and only if $H_Q$ is invariant for $(S(t))$ and its restriction to $H_Q$ defines a $C_0$-semigroup of symmetric contractions. This fact allows us to provide explicit criteria in terms of $A$ and $Q$ for the various interesting properties of the Ornstein-Uhlenbeck semigroup $(R_t)$ and its generator $L$. In particular, we characterize the domain of $L$ in $L^p(H, \mu)$, the spectral gap property, compactness, Hilbert-Schmidt property and the strong Feller property, and finally, the existence of ”almost classical” solutions to the associated Kolmogorov equation. Let us note that the existing
characterizations of those properties for a general Ornstein-Uhlenbeck semigroup are usually not easily applicable, see [13], [5], [6], [7].

Let us emphasise that our starting point is the stochastic differential equation (1.1) and the associated transition semigroup \((R_t)\). For another approach, where the starting point is the Gaussian measure \(\mu = N(0, Q_\infty)\) and the associated Dirichlet form 
\[
E(\phi, \psi) = \frac{1}{2} \int_H \langle Q^{1/2} D\phi(x), Q^{1/2} D\psi(x) \rangle \mu(dx),
\]
see for example [3].

We will describe now the results of this paper in more detail.

In Section 2 we show that the process \(Z\) is reversible if and only if for every \(x \in \text{dom}(A^*)\)
\[
Qx \in \text{dom}(A) \quad \text{and} \quad AQx = QA^*x. \tag{1.3}
\]
This result was proved in [27] for \(Q = I\). Furthermore we show that the operators \(S_Q(t) = Q^{-1/2} S(t) Q^{1/2}\) are bounded in \(H\) and define a \(C_0\)-semigroup of symmetric contractions on \(H\). As a consequence, we find that \(R_t = UT \Gamma(S_Q(t)) U^*\), where \(\Gamma(S_Q(t))\) stands for the second quantization of the operator \(S_Q(t)\) and \(U : L^2(H, \mu) \to L^2(H, \mu)\) is an isometric isomorphism. These characterizations allow us to express various properties of \((R_t)\) in terms of analogous properties of \((S_Q(t))\).

In Section 3 we study the domain of \(L\) in \(L^p(H, \mu)\). The generator \(L\) of the semigroup \((R_t)\) may be easily evaluated on a dense set of cylindrical functions, see [3]:
\[
L\phi(x) = \frac{1}{2} \text{tr} \left( Q D^2 \phi(x) \right) + \langle x, A^* D\phi(x) \rangle, \quad x \in H,
\]
where \(D\) stands for the Fréchet derivative of the function \(\phi : H \to \mathbb{R}\). The problem of an explicit characterization of the domain of \(L\) in \(L^p(H, \mu)\) was an object of intense study for some time, see [25], [13], [5]. We use the results from our recent work [3] to give a complete characterization of the domain \(\text{dom}_p(L)\), \(p \in (1, \infty)\), of the selfadjoint (in \(L^2(H, \mu)\)) generator \(L\) acting in \(L^p(H, \mu)\) in terms of appropriately defined Gauss-Sobolev spaces.

In Section 4 we show that for a symmetric semigroup \((R_t)\), the spectral gap of \(L\), and the compactness of \((R_t)\) are determined by the corresponding properties of the semigroup \((S_Q(t))\) in \(H\). In particular, the spectral gap of \(L\) in \(L^2(H, \mu)\) is the same as the spectral gap of \(A_Q\), the generator of \((S_Q(t))\) in \(H\). Next, we provide necessary and sufficient conditions for the strong Feller property of the semigroup \((R_t)\). Finally, we show that for a bounded function \(\phi\)
\[
\left\| Q^{1/2} DR_t \phi \right\|_\infty \leq \frac{c}{\sqrt{t}} \| \phi \|_\infty,
\]
for an arbitrary symmetric Ornstein-Uhlenbeck semigroup \((R_t)\) with the spectral gap property.
In Section 5 we provide necessary and sufficient conditions for the semigroup \((R_t)\) to be of Hilbert-Schmidt type. Let us note that this class of semigroups includes an important class of strongly Feller semigroups, see \([6]\). We show that \((R_t)\) is of Hilbert-Schmidt type if and only if the semigroup \((S_Q(t))\) is exponentially stable and of Hilbert-Schmidt type. Subsequently we prove that the latter condition is satisfied if \(\mu(H_Q) = 1\). As a consequence we find that the function \(u(t,x) = R_t \phi(x)\) is an (almost) classical solution of the Kolmogorov equation

\[
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t}(t,x) = \frac{1}{2} \text{tr} \left( (D_Q)^2 u(t,x) \right) + \langle Q^{-1/2} x, A_Q D_Q u(t,x) \rangle, \quad t > 0, x \in Q^{1/2}(H), \\
u(0,x) = \phi(x), \quad \phi \in L^p(H, \mu),
\end{array} \right.
\]

where \(D_Q\) denotes the Fréchet derivative in the direction of the subspace \(Q^{1/2}(H)\).

In Section 6 we discuss some examples. In particular, we consider an Ornstein-Uhlenbeck process in a chaotic environment and provide a detailed analysis of its invariant measures and the Spectral Gap Property.

In the last part of this section we formulate the main assumption of the paper. Let

\[ Q_t = \int_0^t S(s)Q S^*(s) ds, \quad t \leq \infty. \quad (1.4) \]

The following hypothesis is a standing assumption for the rest of this paper and the results will be enunciated without further recalling it.

**Hypothesis 1.1.** We assume that

\[ \int_0^\infty \text{tr} \left( S(s)Q S^*(s) \right) ds < \infty, \]

and the operator \(Q_\infty\) is injective.

If Hypothesis 1.1 holds then the solution to (1.1) is well defined and there exists an invariant measure \(\mu\) for the process \(Z\) which is a centered Gaussian measure with the covariance operator \(Q_\infty\), see \([13]\).

2. CHARACTERIZATION OF SYMMETRIC OU SEMIGROUPS

We will study equation (1.1) in a separable real Hilbert space \(H\) with the norm \(|\cdot|\).

2.1. General OU Process. The next two lemmas summarise some basic properties of an arbitrary Ornstein-Uhlenbeck semigroup proved in \([3]\) and \([8]\) which will be useful in the sequel. Lemma 2.3 seems to be new.

**Lemma 2.1.** (a) We have \(S(t)Q^{1/2}_\infty(H) \subset Q^{1/2}_\infty(H)\) for each \(t \geq 0\). The family of operators \(S_0(t) = Q^{-1/2} S(t)Q^{1/2}_\infty, t \geq 0\), defines a \(C_0\)-semigroup \((S_0(t))\) of contractions on \(H\). For each \(t \geq 0\) the adjoint \(S_0^*(t)\) may be identified with the operator \(Q^{1/2}_\infty S^*(t)Q^{-1/2}_\infty\).
(b) Moreover, denoting by $A_0$ the generator of the semigroup $(S_0(t))$, we find that $K = Q_{2\infty}^{1/2}(\text{dom}(A^*))$ is a core for for $A_0^*$ and

$$\langle A_0^* h, h \rangle = -\frac{1}{2} |Vh|^2, \quad \text{for } h \in K,$$

where $V = Q_{2\infty}^{1/2}Q_{2\infty}^{-1/2}$ with the domain $Q_{2\infty}^{1/2}(H)$.

**Proof.** For (a) see Proposition 1 and (a) of Proposition 2 in [3].

(b) If $x \in \text{dom}(A^*)$ then $S^*(t)x \in \text{dom}(A^*)$ and $S_0^*(t)Q_{2\infty}^{1/2}x = Q_{2\infty}^{1/2}S^*(t)x$. Hence, $K \subset \text{dom}(A_0^*)$ and $S_0^*(t)K \subset K$. Since $K$ is dense in $H$, we find that $K$ is a core for $A_0^*$ by Theorem 1.9 in [16]. Putting $x = y \in \text{dom}(A^*)$ and $h = Q_{2\infty}^{1/2}x$ in the Liapunov Equation (2.6) below we obtain

$$2 \langle A^*Q_{2\infty}^{-1/2}h, Q_{2\infty}^{1/2}h \rangle = - \langle QQ_{2\infty}^{-1/2}h, Q_{2\infty}^{-1/2}h \rangle,$$

which yields (2.1). \qed

Let us recall that if $T : H \to H$ is a contraction then the second quantization $\Gamma(T) : L^2(H, \mu) \to L^2(H, \mu)$ of the operator $T$ is well defined, see [20] or [3] for details.

**Lemma 2.2.** For each $t \geq 0$ we have $R_t = \Gamma(S_0^*(t))$ and therefore the semigroup $(R_t)$ is symmetric in $L^2(H, \mu)$ if and only if the semigroup $(S_0(t))$ is symmetric in $H$. If $A_0 = A_0^*$ then $V$ is closable and $\text{dom}(V) = \text{dom}(\sqrt{-A_0})$.

**Proof.** Theorem 1 in [3] yields immediately $R_t = \Gamma(S_0^*(t))$. The second statement follows from Lemma 2c) in [3]. To prove the last one, note that $A_0 = A_0^* \leq 0$ and thereby the operator $\sqrt{-A_0}$ is well defined and closed. Moreover,

$$|Vh|^2 = 2 \left| \sqrt{-A_0}h \right|^2, \quad h \in K,$$

by (2.2), with $K$ being a core for $A_0$. Hence, the restricted operator $V|K$ is closable and $\text{dom}(\sqrt{-A_0})$ is the domain of its closure $V|K$. Since $K$ is dense in $Q_{2\infty}^{1/2}(H) = \text{dom}(V)$ in the range norm, we get $V = V|K$ and the lemma follows. \qed

Let us note the following corollary of Hypothesis [1.1].

**Lemma 2.3.** We have $\ker(A_0) = \{0\}$ and $\ker(A^*) = \{0\}$.

**Proof.** To prove that $\ker(A_0) = \{0\}$ we will show first that

$$\lim_{t \to \infty} S(t)Q_{2\infty}^{1/2}x = 0, \quad x \in H. \quad (2.3)$$

Indeed, since by Hypothesis [1.1] the operators $S(s)QS^*(s)$ and $S(s)Q_{2\infty}S^*(s)$ are nonnegative and of trace class, we obtain for any orthonormal base $\{e_i : i \geq 1\}$ in $H$:

$$\int_0^t \text{tr}(S(s)QS^*(s)) \, ds = \sum_{i=1}^\infty \left\langle \int_0^t S(s)QS^*(s)e_i \, ds, e_i \right\rangle$$
\[
\begin{align*}
&= \sum_{i=1}^{\infty} \langle (Q_{\infty} - S(t)Q_{\infty}S^*(t)) e_i, e_i \rangle \\
&= \text{tr} (Q_{\infty}) - \left\| Q_{\infty}^{1/2} S^*(t) \right\|_{HS}^2.
\end{align*}
\]

Hence, taking into account that \( \left\| Q_{\infty}^{1/2} S^*(t) \right\|_{HS} = \left\| S(t) Q_{\infty}^{1/2} \right\|_{HS} \) and in view of Hypothesis 1.1 we find that
\[
\text{tr} (Q_{\infty}) = \lim_{t \to \infty} \int_0^t \text{tr} (S(s)Q_{\infty}S^*(s)) ds
\]
which in particular implies (2.3). Finally, if \( x \in \ker (A_0) \) then \( S_0(t)x = x \), hence \( S(t)Q_{\infty}^{1/2}x = Q_{\infty}^{1/2}x \) for all \( t \geq 0 \). Therefore, by (2.3) \( x \in \ker (Q_{\infty}) = \{0\} \).

Similarly, if \( x \in \ker (A^*) \) then, taking into account that
\[
Q_{\infty} x = Q_{\infty} x - S(t)Q_{\infty}S^*(t)x, \quad x \in H,
\]
we find that
\[
\lim_{t \to \infty} Q_{\infty}^{1/2} S^*(t)x = 0,
\]
which yields \( x = 0 \).

2.2. Characterizations of the Symmetry. The next theorem provides necessary and sufficient conditions for the semigroup \((R_t)\) to be symmetric in \(L^2(H, \mu)\). This problem has been solved in [27] for the case \( Q = I \).

**Theorem 2.4.** The following conditions are equivalent.

(i) The semigroup \((R_t)\) is symmetric in \(L^2(H, \mu)\).

(ii) If \( x \in \text{dom} (A^*) \) then \( Qx \in \text{dom} (A) \) and
\[
AQx = QA^*x.
\]

(iii) \( S(t)Q = QS^*(t) \) for all \( t \geq 0 \).

**Proof.** (i) \(\Rightarrow\) (ii). By Lemma 2.3 \( R_t \) is symmetric if and only if \( A_0^* = A_0 \). Therefore \( S_0^*(t) = S_0(t) \) for all \( t \geq 0 \) or, equivalently,
\[
S(t)Q_{\infty} = Q_{\infty} S^*(t).
\]

If Hypothesis 1.1 holds then
\[
\langle A^*x, Q_{\infty}y \rangle + \langle A^*y, Q_{\infty}x \rangle = -\langle Qx, y \rangle
\]
for \( x, y \in \text{dom} (A^*) \) (see chapter 11.2 of [13]). It follows from (2.6) that if \( x \in \text{dom} (A^*) \) then \( Q_{\infty}x \in \text{dom} (A) \) and thereby, (2.5) yields
\[
AQ_{\infty}x = Q_{\infty}A^*x = -\frac{1}{2} Qx,
\]
which completes the proof. \( \square \)
for all \( x \in \text{dom} \left( (A^*)^2 \right) \). Take \( y \in \text{dom} \left( (A^*)^2 \right) \). Since \( x = A^*y \in \text{dom} \left( A^* \right) \) we conclude from (2.7) that
\[
Q_\infty A^*y \in \text{dom}(A),
\] (2.8)
and
\[
AQ_\infty A^*y = -\frac{1}{2}QA^*y.
\] (2.9)
By (2.7) we also have
\[
Q_\infty A^*y = -\frac{1}{2}Qy,
\] (2.10)
which combined with (2.8) implies that \( Qy \in \text{dom}(A) \) and
\[
AQ_\infty A^*y = -\frac{1}{2}AQy.
\] (2.11)
From (2.9) and (2.11) we obtain
\[
AQx = QA^*x, \quad x \in \text{dom} \left( (A^*)^2 \right).
\] (2.12)
Since \( \text{dom} \left( (A^*)^2 \right) \) is a core for \( A^* \) and the right hand side of (2.12) is well defined for \( x \in \text{dom} \left( A^* \right) \), (ii) follows.

(ii) \( \implies \) (iii) By assumption \((\lambda - A)Qy = Q(\lambda - A^*)y \) for \( \lambda \in \mathbb{R} \) and \( y \in \text{dom} \left( A^* \right) \). Hence, for a certain \( \lambda_0 \) and all \( \lambda > \lambda_0 \)
\[
(\lambda - A)^{-1}Qx = Q(\lambda - A^*)^{-1}x, \quad x \in H.
\]
Then using the formula for the resolvent of generator and properties of the Laplace transform we obtain (iii).

(iii) \( \implies \) (i) From Hypothesis 1.1 and (iii)
\[
S(t)Q_\infty x = \int_0^\infty S(t + s)QS^*(s)xds
\]
\[
= \int_0^\infty S(s)QS^*(t + s)xds = Q_\infty S^*(t)x,
\]
for \( x \in H \), which yields \( S_0(t) = S_0^*(t) \) and (i) follows from Lemma 2.2.

If the Ornstein-Uhlenbeck process (1.1) is diagonal, that is there exists a joint eigenbasis
\[
Ae_k = \alpha_k e_k, \quad Qe_k = q_k e_k, \quad k \geq 1,
\]
for \( A \) and \( Q \), then obviously the corresponding Ornstein-Uhlenbeck semigroup is symmetric provided Hypothesis 1.1 holds. It is easy to see that an Ornstein-Uhlenbeck process with the symmetric transition semigroup need not be diagonal. As the simplest example it is enough to take \( H = \mathbb{R}^2 \),
\[
A = \begin{pmatrix} a & c \\ d & b \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}
\]
with $0 < q \neq 1$. Then Hypothesis 1.1 holds and the corresponding Ornstein-Uhlenbeck semigroup is symmetric if and only if

$$a < 0, \quad \det(A) > 0, \quad d = cq, \quad \text{and} \quad (a - b)^2 + 4c^2q > 0.$$ 

**Corollary 2.5.** Let $(R_t)$ be symmetric. Then the following holds.

(i) $\text{im}(Q^\infty) \subset \text{dom}(A)$ and the operator $AQ^\infty = -\frac{1}{2}Q$ is bounded, symmetric and negative.

(ii) $\text{im}(Q) \subset \text{im}(A)$.

(iii) If $\ker(A) = \{0\}$, then

$$Q^\infty = -\frac{1}{2}A^{-1}Q = -\frac{1}{2}Q(A^*)^{-1}.$$ 

**Proof.** The proof follows immediately from (2.7).

**Remark 2.6.** Note that (2.13) holds if the semigroup $(S(t))$ is stable. In particular, if $H = \mathbb{R}^d$, then (2.13) is a consequence of Hypothesis 1.1. Indeed, by Theorem 2.7(i) below the operator $Q^{-1}$ is bounded in the case of $H = \mathbb{R}^d$ hence,

$$\int_0^\infty \|S(t)\|^2 dt < \infty$$

by Hypothesis 1.1, which implies the exponential stability of the semigroup $(S(t))$. In Section 6 we provide an example in which $(R_t)$ is symmetric and has the Spectral Gap Property but (2.13) is not satisfied.

**Theorem 2.7.** The semigroup $(R_t)$ is symmetric in $L^2(H, \mu)$ if and only if the following conditions are satisfied.

(i) $Q$ is injective.

(ii) We have

$$S(t)Q^{1/2}(H) \subset Q^{1/2}(H), \quad t \geq 0.$$ 

(iii) The family of operators

$$S_Q(t) = Q^{-1/2}S(t)Q^{1/2}, \quad t \geq 0,$$

defines a symmetric $C_0$-semigroup of contractions on $H$.

**Proof.** Let $(R_t)$ be symmetric. Suppose that $Qx = 0$. Then by Hypothesis 1.1 and (iii) of Theorem 2.4

$$Q^\infty x = \int_0^\infty S(2t)Qx dt = 0,$$

and since $Q^\infty$ is injective (i) follows. It follows from (2.9), (2.11) and (i) of Corollary 2.5 that

$$-QA^*x = 2AQ^\infty A^*x = AQx, \quad x \in \text{dom}(A^*),$$ 

(2.15)
and therefore \((2.7)\) implies
\[
\langle -QA^*x, x \rangle \geq 0, \quad x \in \text{dom}(A^*).
\] (2.16)

To prove that \(S(t)Q^{1/2}(H) \subset Q^{1/2}(H)\) we will show that
\[
\left| Q^{1/2}S^*(t)x \right|^2 \leq \left| Q^{1/2}x \right|^2, \quad x \in H. \tag{2.17}
\]

To this end note that for \(x \in \text{dom}(A^*)\) \((2.7)\) implies
\[
Qx = -2 \int_0^\infty S(s)QS^*(s)A^*xds = 2 \int_0^\infty S(s)(-QA^*)S^*(s)xds, \tag{2.18}
\]

and therefore,
\[
\langle S(t)QS^*(t)x, x \rangle = 2 \int_0^\infty \langle S(t+s)(-QA^*)S^*(s)x, x \rangle ds
\]
\[
= 2 \int_0^\infty \langle -QA^*S^*(u)x, S^*(u)x \rangle du \leq \langle Qx, x \rangle,
\]

where the last inequality follows from \((2.16)\) and \((2.18)\). For arbitrary \(x \in H\) \((2.17)\) follows by the density of \(\text{dom}(A^*)\) in \(H\). Finally, by Proposition B.1 of [13], \((2.17)\) yields \(S(t)Q^{1/2}(H) \subset Q^{1/2}(H)\). Hence, the operator \(S_Q(t) = Q^{-1/2}S(t)Q^{1/2}\) is bounded on \(H\) for each \(t \geq 0\). Obviously, operators \(S_Q(t)\) satisfy the semigroup property and moreover,
\[
S_Q'(t) = \frac{Q^{1/2}S^*(t)Q^{-1/2}}{t \geq 0}. \tag{2.19}
\]

Since \((R_t)\) is symmetric, Theorem 2.4 yields \(S(t)Q = QS^*(t)\) and therefore \(S_Q(t)x = S_Q^*(t)x\) for \(x \in Q^{1/2}(H)\) and by the density argument,
\[
S_Q^*(t)x = S_Q(t)x, \quad x \in H. \tag{2.20}
\]

Putting in \((2.17)\) \(x = Q^{-1/2}y\) with \(y \in Q^{1/2}(H)\) we obtain
\[
|S_Q^*(t)y| \leq |y|, \quad y \in Q^{1/2}(H). \tag{2.21}
\]

Since \(Q^{1/2}(H)\) is dense in \(H\), \((2.21)\) holds for all \(y \in H\). The function \(t \rightarrow S_Q^*(t)x\) is weakly continuous at zero for every \(x \in Q^{1/2}(H)\), hence for \(x \in H\), since \(\|S_Q^*(t)\| \leq 1\) for all \(t\). Therefore, the semigroup \(\left(S_Q^*(t)\right)\) is a strongly continuous contraction semigroup in \(H\) and so is \((S_Q(t))\).

Assume now that conditions (i)-(iii) are satisfied and let \(x = Q^{1/2}y\). Then by symmetry of \((S_Q(t))\) we have \(Q^{-1/2}S(t)Qy = Q^{1/2}S^*(t)y\), or equivalently, \(S(t)Qy = QS^*(t)y\), which proves condition (iii) of Theorem 2.4. \(\Box\)
2.3. The First Consequences of Symmetry.

**Corollary 2.8.** Let \((R_t)\) be symmetric. Then the generator \(A_Q\) of the semi-group \((S_Q(t))\) is injective.

**Proof.** If \(x \in \ker(A_Q)\) then \(S_Q(t)x = x\) for each \(t \geq 0\) and hence \(S(t)Q^{1/2}x = Q^{1/2}x\). By Hypothesis 1.1 we find that

\[
\int_0^\infty \left| Q^{1/2}x \right|^2 \, dt = \int_0^\infty \left| S(t)Q^{1/2}x \right|^2 \, dt
\]

\[
\leq |x|^2 \int_0^\infty \left\| S(t)Q^{1/2} \right\|^2_{HS} \, dt = |x|^2 \int_0^\infty \left\| Q^{1/2}S^*(t) \right\|^2_{HS} \, dt < \infty,
\]

which yields \(x \in \ker(Q) = \{0\}\). \(\square\)

**Theorem 2.9.** Assume that \((R_t)\) is symmetric. Then there exists an isometric isomorphism \(U : L^2(H, \mu) \rightarrow L^2(H, \mu)\) such that

\[
R_t = U^{-1} \Gamma (S_Q(t)) U, \quad t \geq 0.
\]

**Proof.** Note first that \(\tilde{K}\) is a core for \(A_0 = A_0^*\). Hence it follows from (2.1) and Lemma 2.2 that \(A_0 = \frac{1}{2}V^* \tilde{V}\). By Lemma 2.3 \(A_0\) is injective, hence so is \(\sqrt{-A_0}\) and since (2.2) extends on \(\text{dom}(\sqrt{-A_0}) = \text{dom}(\tilde{V})\), the operator \(\tilde{V}\) is also injective, in particular \(\tilde{V}^{-1} = \sqrt{-1}\). Moreover, \(\text{im}(\tilde{V}) = H\), because by Theorem 2.7(i) \(\text{im}(Q)\) is dense in \(H\). Therefore, the polar decomposition of \(\tilde{V}\) is given by

\[
\tilde{V} = U \sqrt{-2A_0},
\]

where \(U : H \rightarrow H\) is an isometric isomorphism. Since \(Q_{\infty}^{1/2}(H)\) is invariant for \(S_0^*(t), t \geq 0\), we have for \(h \in Q_{\infty}^{1/2}(H)\)

\[
S_Q^*(t)h = Q^{1/2}Q_{\infty}^{-1/2}S_0^*(t)Q_{\infty}^{1/2}Q^{-1/2}h
\]

\[
= VS_0^*(t)V^{-1}h = U \left| \tilde{V} \right| S_0^*(t) \left| \tilde{V} \right|^{-1} U^{-1}h
\]

\[
= US_0^*(t)U^{-1}h,
\]

where the last equality holds because \(\left| \tilde{V} \right| = \sqrt{-2A_0}\) commutes with \(S_0^*(t) = S_0^*(t)\). By (i) of Theorem 2.7 \(\text{im}(Q^{1/2})\) is dense in \(H\) and therefore (2.23) holds for \(h \in H\). By Lemma 2.4, (2.23) and the properties of the second quantization operator \(\Gamma\) (see 2.6 or 3) we obtain

\[
R_t = \Gamma (S_0(t)) = \Gamma (U^{-1}S_Q(t)U) = \Gamma (U^{-1}) \Gamma (S_Q(t)) \Gamma (U),
\]

and putting \(U = \Gamma (U)\) we complete the proof of the theorem. \(\square\)

**Proposition 2.10.** Assume that \((R_t)\) is symmetric. Then for each \(t > 0\)

\[
Q_{\infty}^{1/2}(H) = Q^{1/2} \left( \text{dom}\sqrt{-A_Q} \right).
\]

(2.24)
Proof. By Corollary B.7 in [13] \( \text{im} \left( Q_t^{1/2} \right) = \text{im}(T) \), where \( T : L^2(0,t;H) \to H \) is given by

\[
Tu = \int_0^t S(t-s)Q^{1/2}u(s)ds.
\]

By Theorem 2.7 \( S(t-s)Q^{1/2}u(s) = Q^{1/2}S_Q(t-s)u(s) \) and therefore

\[
Tu = Q^{1/2} \int_0^t S_Q(t-s)u(s)ds.
\]

By Remarks B8 and A.18 in [13]

\[
\left\{ \int_0^t S_Q(t-s)u(s)ds; \ u \in L^2(0,t;H) \right\} = DA_{AQ} \left( \frac{1}{2}, 2 \right) = \text{dom} \sqrt{-A_Q}
\]

(2.25)

for exponentially stable and symmetric semigroup \((S_Q(t))\). Note that for \( \lambda > 0 \) we have \( \text{dom} (\lambda - A_Q) = \text{dom} (-A_Q) \) and \( \text{dom} \sqrt{\lambda - A_Q} = \text{dom} \sqrt{-A_Q} \) with the equivalent graph norms. Since the interpolation space \( DA_{AQ} \left( \frac{1}{2}, 2 \right) \) is completely determined by the pair of spaces \( (\text{dom} (A_Q), H) \) it follows that

\[
DA_{AQ} \left( \frac{1}{2}, 2 \right) = DA_{AQ-\lambda} \left( \frac{1}{2}, 2 \right).
\]

Finally,

\[
\left\{ \int_0^t e^{-\lambda(t-s)} S_Q(t-s)u(s)ds; u \in L^2(0,t;H) \right\} = \left\{ \int_0^t S_Q(t-s)u(s)ds; u \in L^2(0,t;H) \right\},
\]

and therefore (2.25) still holds if \((S_Q(t))\) is only bounded and symmetric. \( \square \)

Remark 2.11. Let \( H_Q \) denote the space \( Q^{1/2}(H) \) endowed with the norm \( \|x\|_Q = \|Q^{-1/2}x\| \). It follows from Theorem 2.7 that \((R_t)\) is symmetric if and only if the semigroup \((S(t))\) restricted to \( H_Q \) defines a \( C_0 \)-semigroup \((S_r(t))\) of symmetric contractions in this space. Its generator \( A_r \) is the part of \( A \) in \( H_Q \). By Corollary 2.8 \( \ker (A_r) = \{0\} \). Moreover,

\[
Q_t^{1/2}(H) = \text{dom} \left( \sqrt{-A_r} \right).
\]

(2.26)

Indeed, by Proposition 2.10 \( Q_t^{1/2}(H) = Q_t^{1/2}A_Q^{-1/2}(H) \), hence any \( x \in Q_t^{1/2}(H) \) can be written (see for example p. 70 of [22]) in the form

\[
x = Q^{1/2} \int_0^\infty \frac{1}{\sqrt{t}} e^{-\lambda t} S_Q(t)h dt, \quad h \in H,
\]
for a certain $\lambda > 0$. Therefore,

$$x = \int_0^\infty \frac{1}{\sqrt{t}} e^{-\lambda t} S(t) Q^{1/2} h dt, \quad h \in H,$$

(2.27)

with the integral convergent in the norm $|\cdot|_Q$. This, again by the formula on p. 70 of [22], implies that $x \in \text{dom} (\sqrt{-A_r})$ and (2.26) follows.

3. IDENTIFICATION OF DOMAINS

Let $K = Q_{\infty}^{1/2} (\text{dom} (A^*))$ and let

$$\mathcal{P}(K) = \text{lin} \{ \phi^n_k : n \geq 0, k \in K \},$$

where $\phi_k(x) = \langle x, Q_{-1/2}^{\infty} k \rangle$. By Lemma 2.1 $K$ is a core for $A_r^*$ and by (15) in [5] $\mathcal{P}(K)$ is a core for $L$. We denote by $W_{1,p}^Q$ the completion of $\mathcal{P}(K)$ in the norm

$$\| \phi \|_{1,p,AQ} = \left( \| \phi \|_p^p + \left\| (AQ)^{1/2} D\phi \right\|_{l^p}^{1/p} \right)^{1/p}.$$ (3.1)

By $W_{2,p}^Q$ we denote the completion of $\mathcal{P}(K)$ with respect to the norm $\| \cdot \|_{1,p,Q}$, where

$$\| \phi \|_{1,p,Q}^p = \| \phi \|_p^p + \int_H \left| Q^{1/2} D\phi(x) \right|^p \mu(dx).$$ (3.2)

The completion of $\mathcal{P}(K)$ with respect to the norm

$$\| \phi \|_{2,p,Q} = \left( \| \phi \|_{1,p,Q}^p + \int_H \left\| Q^{1/2} D^2\phi(x) Q^{1/2} \right\|_{HS}^p \mu(dx) \right)^{1/p}$$ (3.3)

where $\| \cdot \|_{HS}$ denotes the Hilbert-Schmidt norm of an operator, will be denoted by $W_{2,p}^Q$.

**Theorem 3.1.** Assume that $L$ is selfadjoint. Then for every $p \in (1, \infty)$ the spaces $W_{1,p}^Q$, $W_{1,q}^Q$ and $W_{2,p}^Q$ may be identified as subspaces of $L^p(H, \mu)$. Moreover,

$$\text{dom}_p (-L)^{1/2} = W_{1,p}^Q,$$ (3.4)

and

$$\text{dom}_p (L) = W_{2,p}^Q \cap W_{1,p}^Q.$$ (3.5)

In order to prove this theorem we will recall some facts from [8]. We define first the Malliavin gradient

$$D_I = Q_{\infty}^{1/2} D, \quad \text{dom} (D_I) = \mathcal{P}(K).$$

Taking into account that $-A_0$ is nonnegative and selfadjoint we define the gradients

$$D^n_{A_0} = \left( (-A_0)^{1/2} \right)^\otimes n D_I, \quad \text{dom} \left( D^n_{A_0} \right) = \mathcal{P}(K), \quad n = 1, 2.$$
It was shown in [8] that for \( p \in (1, \infty) \) \( D_{A_0} \) is closable in \( L^p(H, \mu) \) and \( D_{A_0}^2 \) is closable in \( \text{dom}_p(D_{A_0}) \) endowed with the graph norm in \( L^p(H, \mu) \). The closed extensions are again denoted by \( D_{A_0}^n, n = 1, 2 \). The next theorem is a special case of Theorem 5.3 in [8].

**Theorem 3.2.** For each \( p \in (1, \infty) \), there exist \( a_p, b_p > 0 \) such that for \( \phi \in \text{dom}_p(\sqrt{-L}) \)

\[
a_p \left( \| \phi \|_p + \| D_{A_0} \phi \|_p \right) \leq \left\| \sqrt{I - L} \phi \right\|_p \leq b_p \left( \| \phi \|_p + \| D_{A_0} \phi \|_p \right),
\]

and for \( \phi \in \text{dom}(L) \)

\[
a_p \left( \| \phi \|_p + \left\| \sqrt{I - A_0 D_{A_0} \phi} \right\|_p + \left\| D_{A_0}^2 \phi \right\|_p \right) \leq \left\| (I - L) \phi \right\|_p
\]

\[
\leq b_p \left( \| \phi \|_p + \left\| \sqrt{I - A_0 D_{A_0} \phi} \right\|_p + \left\| D_{A_0}^2 \phi \right\|_p \right).
\]

We will prove now Theorem 3.1.

**Proof.** Note first, that by (2.16) the operator \( AQ = QA^* \) with \( \text{dom}(AQ) = \text{dom}(A^*) \) may be extended in the sense of Friedrichs to a selfadjoint and nonpositive operator in \( H \). We will use the same notation \( AQ \) for the Friedrichs extension of the operator \( (AQ, \text{dom}(A^*)) \).

In view of Theorem 3.2 it remains to identify the Sobolev norms given in (3.1), (3.2) and (3.3) with the appropriate norms in (3.6) and (3.7). The relationship (3.4) was proved in [25]. We repeat here the argument using our notation for the sake of completeness. For \( \phi \in \mathcal{P}(K) \) we have

\[
|D_{A_0} \phi(x)|^2 = \left| \sqrt{-A_0} Q_\infty^{1/2} D \phi(x) \right|^2
\]

\[
= \langle -Q_\infty^{1/2} A_0 Q_\infty^{1/2} D \phi(x), D \phi(x) \rangle
\]

\[
= \langle AQ_\infty D \phi(x), D \phi(x) \rangle = \frac{1}{2} \left| Q^{1/2} D \phi(x) \right|^2,
\]

by (i) of Corollary 2.3 and (2.7). Hence the operator \( \left( Q^{1/2} D, \mathcal{P}(K) \right) \) is closable in \( L^p(H, \mu) \) and (3.4) follows.

Again, for \( \phi \in \mathcal{P}(K) \) and invoking Corollary 2.3 we have

\[
\left\| D_{A_0}^2 \phi(x) \right\|_{HS}^2 = \left\| (-A_0)^{1/2} Q^{1/2} D^2 \phi(x) Q^{1/2} (-A_0)^{1/2} \right\|_{HS}^2
\]

\[
= \left\| Q^{1/2} A_0 Q^{1/2} D^2 \phi(x) \right\|_{HS}^2 = \left\| AQ_\infty D^2 \phi(x) \right\|_{HS}^2 = \frac{1}{4} \left\| QD^2 \phi(x) \right\|_{HS}^2,
\]

and

\[
\left| A_0 Q^{1/2} D \phi(x) \right|^2 = \langle Q^{1/2} A_0^2 Q^{1/2} D \phi(x), D \phi(x) \rangle.
\]
\[ = \langle AQ_\infty A^* D\phi(x), D\phi(x) \rangle = \frac{1}{2} \langle -QA^* D\phi(x), D\phi(x) \rangle \]
\[ = \frac{1}{2} \left| (-QA^*)^{1/2} D\phi(x) \right|^2, \quad (3.9) \]
where the third equality follows from (2.13). Since the operators \( D^2_{A_0} \) and \( D_{A_0} \) are closable in \( W^{1,p}_Q \) and \( L^p(H,\mu) \) respectively, the above identities yield closability of the operators \( QD^2 \) and \( (-QA^*)^{1/2} D \) in \( W^{1,p}_Q \) and \( L^p(H,\mu) \) respectively, hence the corresponding Sobolev spaces \( W^{1,p}_Q \) and \( W^{2,p}_Q \) are continuously embedded into \( L^p(H,\mu) \). Finally, (3.8), (3.9) and Theorem 3.2 yield (3.5).

The Corollary below extends the result of [19] obtained by a completely different argument.

**Corollary 3.3.** Assume that \((R_t)\) is a symmetric Ornstein-Uhlenbeck semi-group for a process in \( \mathbb{R}^d \). Then for each \( p \in (1,\infty) \)
\[ \text{dom}_p \left( \sqrt{-L} \right) = W^{1,p}_I \quad \text{and} \quad \text{dom}_p(L) = W^{2,p}_I. \]

**Proof.** By Theorem 2.7(i) \( Q \) has bounded inverse and the result follows immediately from Theorem 3.1 and the definition of Sobolev spaces. \( \square \)

The next corollary extends the results of [25], [19], [13] and [11].

**Corollary 3.4.** Assume that \( A = A^* \) and \( Q = (-A)^{-2\alpha} \), where \( \alpha \geq 0 \) and \( \text{tr} (-A)^{-1-2\alpha} < \infty \). Then \( Q_\infty = \frac{1}{2} (-A)^{-1-2\alpha} \) and for \( p \in (1,\infty) \),
\[ \text{dom}_p \left( \sqrt{-L} \right) = W^{1,p}_{(-A)^{-2\alpha}}, \]
and
\[ \text{dom}_p(L) = W^{2,p}_{(-A)^{-2\alpha}} \cap W^{1,p}_{(-A)^{1-2\alpha}}. \]

**Example 3.5.** In the previous Corollary assume that \( H = L^2(0,1) \), and \( A = \frac{\partial^2}{\partial x^2} \) with zero Dirichlet boundary conditions and \( Q = I \). In this case \( \text{dom}(\sqrt{-A}) = H^1_0 \) with equivalent norms. Then \( \phi \in \text{dom}_p(L) \) if and only if \( D\phi(x) \in H^1_0 \) \( \mu \)-a.s.,
\[ \int_H |D\phi(x)|^p_{H^1_0} \mu(dx) < \infty, \]
\( D^2\phi(x) \) is a Hilbert-Schmidt operator on \( H \) \( \mu \)-a.s. and
\[ \int_H \| D^2\phi(x) \|_{HS}^p \mu(dx) < \infty. \]
4. SPECTRAL GAP AND REGULARITY

We start with the result which says that the spectral gap of the operator \( L \) in \( L^2(H, \mu) \) is the same as the spectral gap of \( A_Q \) in \( H \).

**Theorem 4.1.** Assume that the semigroup \((R_t)\) is symmetric. Then
\[
|S_Q(t)h| \leq e^{-\beta t}|h|, \quad h \in H,
\]
if and only if
\[
\left\| R_t \phi - \int_H \phi d\mu \right\|_2 \leq e^{-\beta t} \|\phi\|_2, \quad \phi \in L^2(H, \mu).
\]

**Proof.** Assume that (4.1) holds and let \( \Pi_0 \phi = \int_H \phi d\mu \). By (2.22) and the properties of the second quantization operator (Lemma 1c of [5]) we have
\[
\|R_t - \Pi_0\|_2 = \|S_Q(t)\| \leq e^{-\beta t},
\]
hence (4.2) is satisfied. The converse statement follows from (4.3) in the same way.

**Theorem 4.2.** Assume that \((R_t)\) is symmetric. Then the following conditions are equivalent.

(i) \( \|S_Q(t)\| = e^{-\beta t} \).

(ii) \( \|S_0(t)\| = e^{-\beta t} \).

(iii) \( \text{im} \left( Q_t^{1/2} \right) = \text{im} \left( Q_{\infty}^{1/2} \right) \) for \( t > 0 \).

(iv) \( \text{im} \left( Q_{\infty}^{1/2} \right) \subset \text{im} \left( Q^{1/2} \right) \).

(v) The generator \( L \) of \((R_t)\) satisfies the Logarithmic Sobolev Inequality:
\[
\int_H |\phi(x)|^2 \log |\phi(x)| \mu(dx) \leq \frac{2}{\beta} \langle -L\phi, \phi \rangle + \|\phi\|^2 \log \|\phi\|
\]

(vi) \((R_t)\) is hypercontractive from \( L^p(H, \mu) \) to \( L^q(H, \mu) \) for all \( p, q \) such that
\[
1 < p < q \leq 1 + (p - 1)e^{2\beta t}.
\]

**Proof.** (i) \( \Leftrightarrow \) (ii). This follows immediately from (2.23). (ii) \( \Leftrightarrow \) (iii) It is enough to recall that by Proposition 2b) in [3] (iii) is equivalent to \( \|S_0(t)\| < 1 \) for \( t > 0 \). Then the symmetry of the semigroup \((S_0(t))\) implies that for a certain \( \beta > 0 \) we have \( \|S_0(t)\| \leq e^{-\beta t} \) for all \( t \geq 0 \). The same result in [3] shows that (ii) implies (iii).

(ii) \( \Leftrightarrow \) (iv) By Proposition B.1 in [3] (iv) holds if and only if there exists \( a > 0 \) such that
\[
|Vx| \geq a|x|, \quad x \in Q_{\infty}^{1/2}(H).
\]

By Lemma 2.1 (4.4) is equivalent to the condition
\[
\langle A_0^* x, x \rangle \leq -\frac{1}{2}a^2|x|^2, \quad x \in \text{dom}(A_0^*),
\]
and the last inequality is equivalent to (ii).

(iii) ⇔ (v) By Theorem 2 in [3] (iii) and (vi) are equivalent and by [18], see also [24], (v) and (vi) are equivalent for symmetric semigroups.

Remark 4.3. Let us recall that for a finite Borel measure ν we denote by $\|\nu\|_{\text{var}}$ the variation norm of ν and the measure $\nu_Rt$ is defined by the formula

$$\nu_Rt(B) = \int_H R_t I_B(x) \nu(dx).$$

Assume that $(R_t)$ is symmetric. Then (1.2) (or any of the conditions of Theorem 4.2) holds if and only if for each probability measure ν on H such that $\nu \ll \mu$ and $\frac{d\nu}{d\mu} \in L^2(H,\mu)$ there exists $C_\nu < \infty$ such that

$$\|\nu_Rt - \mu\|_{\text{var}} \leq C_\nu e^{-\beta t}, \quad t \geq 0.$$ 

This fact follows immediately from Theorem 1.1 and the result in [20], see also [23].

Remark 4.4. Let $(R_t)$ be a symmetric Ornstein-Uhlenbeck semigroup. 
(a) If $H = \mathbb{R}^d$ then by Theorem 2.7(i) $Q$ is boundedly invertible, hence (4.1) holds if and only if $(S(t))$ is exponentially stable. Then by Theorem 4.1 condition (4.2) is equivalent to the exponential stability of $(S(t))$. In $H = \mathbb{R}^d$ the latter follows from Hypothesis 1.1 (see Remark 2.6), hence (4.2) always holds.
(b) If dim($H$) = $\infty$, then properties from (a) are not true in general. In Example 1 of Section 6 the Ornstein-Uhlenbeck semigroup is symmetric but it does not satisfy (4.2). In Example 2 the semigroup $(S(t))$ is not stable but (4.2) still holds.

The next corollary provides characterization of the Reproducing Kernel Hilbert Space $Q_{1/2}^{1/2}(H)$ of the invariant measure $\mu$. It follows immediately from Proposition 2.10 and Theorem 4.2. Let us recall, that in the general nonsymmetric case the space $Q_{1/2}^{1/2}(H)$ is explicitly characterized as an interpolation space $D_A(\frac{1}{2}, 2)$ only if $(S(t))$ is analytic and $Q$ is boundedly invertible, see Remark B.8 in [13].

Corollary 4.5. Assume that $(R_t)$ is symmetric and $(S_Q(t))$ is exponentially stable. Then

$$Q_{1/2}^{1/2}(H) = Q^{1/2} \left( \text{dom} \sqrt{-A_Q} \right).$$

Theorem 4.6. Assume that $(R_t)$ is symmetric. Then $(R_t)$ is a compact semigroup in $L^2(H,\mu)$ if and only if $(S_Q(t))$ is a compact and exponentially stable semigroup in $H$.

Proof. If $(R_t)$ is compact then by Proposition 2 in [5] and Theorem 2.4 for each $t > 0 \|S_Q(t)\| < 1$ and the semigroup $(S_Q(t))$ is compact. By the symmetry of $(S_Q(t))$ the former implies that $(S_Q(t))$ is exponentially stable.
If \((S_Q(t))\) is compact and exponentially stable then \(\|S_Q(t)\| < 1\), hence \((R_t)\) is compact by the result in [5].

**Remark 4.7.** If \((R_t)\) is symmetric and compact then by Theorem [3,1] the embedding of \(W_{Q}^{1,p}\) into \(L^p(H,\mu)\) is compact for each \(p \in (1,\infty)\).

The next result extends some results of Da Prato, see for example [10], where the estimate (4.5) is proved for the case \(A = A^*\). In the theorem below we use the notation

\[\|\phi\|_\infty = \text{ess sup} |\phi(x)|,\]

for \(\phi \in L^\infty(H,\mu)\).

**Theorem 4.8.** Assume that \((R_t)\) is symmetric and \((S_Q(t))\) is exponentially stable. Then \(R_t\) is a bounded operator from \(L^p(H,\mu)\) into \(W_{Q}^{1,p}\) for each \(p \in (1,\infty)\) and \(t > 0\) and there exists \(c(p) < \infty\) such that

\[\left\|Q^{1/2}DR_t\phi\right\|_p \leq \frac{c(p)}{\sqrt{t}} \|\phi\|_p.\]  

Moreover, (4.3) still holds for \(p = \infty\) for all bounded Borel functions \(\phi\) and with the operator \(Q^{1/2}D, W_{Q}^{1,p}\) taken for an arbitrary \(p \in (1,\infty)\).

**Proof.** For \(p \in (1,\infty)\) the estimate (4.5) follows immediately from (3.4) and properties of analytic semigroups but we need another argument for the case \(p = \infty\). Note that \(V^* = Q_{\infty}^{-1/2}Q^{1/2}\) and \(\text{dom} (V^*) = \left\{ x \in H : Q^{1/2}x \in Q_{\infty}^{1/2}(H) \right\}\) is dense in \(H\) since \(V\) is closable. For \(x \in \text{dom}(V^*)\)

\[S(t)Q^{1/2}x = Q_{\infty}^{1/2}S_0(t)V^*x \in \text{im} \left( Q_t^{1/2} \right),\]

since \(\text{im} \left( Q_t^{1/2} \right) = \text{im} \left( Q_{\infty}^{1/2} \right)\) by Theorem 4.2. Hence the operator

\[Q_t^{-1/2}S(t)Q^{1/2} = Q_{\infty}^{-1/2}Q_t^{1/2}S_0(t)V^*,\]

with the domain \(\text{dom} (V^*)\) is densely defined and since \(V^* = \sqrt{-2A_0U^*}\) (see the proof of Theorem 2.3) and \(Q_t^{-1/2}Q_{\infty}^{1/2}\) is bounded, it extends to a bounded operator on \(H\). We will show that

\[\left\|Q_t^{-1/2}S(t)Q^{1/2}\right\| \leq \frac{1}{\sqrt{t}}.\]  

(4.6)

To this end note that for \(h \in H\)

\[Q_t h = \int_0^t S(s)Q S^*(s)h ds = Q^{1/2} \int_0^t S_Q(2s)ds Q^{1/2}h,\]

hence for \(x \in Q^{1/2}(H)\)

\[Q^{-1/2}Q_t Q^{-1/2}x = -\frac{1}{2} A_Q^{-1} (I - S_Q(2t)) x.\]  

(4.7)
It is easy to see from (2.19) that \((S_Q^*(t))\) defines a \(C_0\)-semigroup in \(H_Q\) (see Remark 2.11) and since \((S_Q(t))\) is symmetric we obtain from (4.7) for \(x \in \text{dom}(A_Q|H_Q)\), the domain of the part of \(A_Q\) in \(H_Q\),

\[
Q^{1/2}Q_t^{-1}Q^{1/2}x = -2A_Q(I - S_Q(2t))^{-1}x.
\]

Since \(Q_t^{-1/2}S(t)Q^{1/2} = Q_t^{-1/2}Q^{1/2}S_Q(t)\) we obtain

\[
\left|Q_t^{-1/2}S(t)Q^{1/2}h\right|^2 = \left|Q_t^{-1/2}Q^{1/2}S_Q(t)h\right|^2
\]

\[
= 2 \left|\sqrt{-A_Q(I - S_Q(2t))^{-1}}S_Q(2t)h\right|^2.
\]

By the Functional Calculus for selfadjoint operators

\[
\left\|A_Q(I - S_Q(2t))^{-1}S_Q(2t)\right\| \leq \sup_{\lambda > 0} \left(\frac{\lambda e^{-2\lambda t}}{1 - e^{-2\lambda t}}\right) \leq \frac{1}{2t},
\]

hence (4.6) holds. Fix \(p \in (1, \infty)\) and let \(D_t\) denote the closure in \(L^p(H, \mu)\) of the Malliavin gradient (see below (3.5)). By (ii) of Theorem 4.2 \(A_Q\) is boundedly invertible and by the argument in the proof of Theorem 2.9 so is \(\bar{V}\), hence the operator \(\bar{V}D_t\) with its maximal domain is closed in \(L^p(H, \mu)\). Since \(\bar{V}D_t\phi = Q^{1/2}D\phi\) for \(\phi \in \mathcal{P}(K)\) we conclude that \(\bar{V}D_t \supset Q^{1/2}D\). Let \(\phi\) be bounded. By the first part of the theorem \(R_t\phi \in \mathcal{W}^{1,p}_Q\), hence

\[
Q^{1/2}DR_t\phi(x) = \bar{V}D_tR_t\phi(x), \quad \mu - \text{a.s.} \quad (4.8)
\]

By Theorem 1 in [4], condition (iii) of Theorem 4.2 implies that for a bounded Borel \(\phi\) and \(x \in H\), \(D_tR_t\phi(x)\) exists as a Fréchet derivative in the direction \(Q_{\infty}^{1/2}(H)\) and

\[
\langle D_tR_t\phi(x), h \rangle = \int_H \langle \Lambda^*(t)S_0(t)h, Q_t^{-1/2}y \rangle \phi(S(t)x + y) \mu_t(dy), \quad (4.9)
\]

where \(\Lambda^*(t)S_0(t) = Q_t^{-1/2}S(t)Q_{\infty}^{1/2}\). Fix \(x \in H\), such that (4.8) holds. Then for \(h \in \text{dom}(V^*)\) (4.3) yields

\[
\langle Q^{1/2}DR_t\phi(x), h \rangle = \langle D_tR_t\phi(x), V^*h \rangle
\]

\[
= \int_H \langle Q_t^{-1/2}S(t)Q^{1/2}h, Q_t^{-1/2}y \rangle \phi(S(t)x + y) \mu_t(dy).
\]

Therefore by (4.6)

\[
\left|\langle Q^{1/2}DR_t\phi(x), h \rangle\right| \leq \sqrt{\frac{2}{\pi}} \left\|Q_t^{-1/2}S(t)Q^{1/2}\phi\right\|_\infty |h|
\]

\[
\leq \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{t}} \left\|\phi\right\|_\infty |h|, \quad h \in \text{dom}(V^*).
\]

Since \(\text{dom}(V^*)\) is dense in \(H\) we obtain (4.3) for \(p = \infty\). \(\square\)
Remark 4.9. It follows from Remark 2.11 and Theorem 4.1 that (4.2) holds if and only
\[ \|S_{r}(t)\| \leq e^{-\beta t}. \] (4.10)
Moreover, \((R_{t})\) is compact in \(L^{p}(H,\mu)\), \(p \in (1,\infty)\) if and only if (4.10) holds and \((S_{r}(t))\) is compact in \(H_{Q}\). Finally, if (4.10) holds then
\[ Q_{1/2}^{1/2}(H) = \text{dom} \left( \sqrt{-A_{r}} \right), \]
by Remark 2.11 and Theorem 4.2.

5. HILBERT-SCHMIDT CASE

We start with necessary and sufficient conditions for the semigroup \((R_{t})\) to be Hilbert-Schmidt. This case was studied in [6], where conditions for the Hilbert-Schmidt property were given in terms of the semigroup \((S_{0}(t))\), hence rather difficult to apply in special cases.

**Theorem 5.1.** Assume that \((R_{t})\) is symmetric.
(a) The following conditions are equivalent.
(i) \((R_{t})\) is a Hilbert-Schmidt semigroup in \(L^{2}(H,\mu)\).
(ii) \((S_{Q}(t))\) is a Hilbert-Schmidt and exponentially stable semigroup in \(H\).
(b) Moreover, \(\mu(Q_{1/2}^{1/2}(H)) = 1\) if and only if \((S_{Q}(t))\) is a Hilbert-Schmidt semigroup and
\[ \int_{0}^{\infty} \|S_{Q}(t)\|_{HS}^{2} dt < \infty. \] (5.1)

**Proof.** (a) In view of Proposition 2 in [6] the proof is completely analogous to the proof of Theorem 4.6 and therefore omitted.
(b) Assume that (5.1) holds. Let \(H_{Q}\) denote the space defined in Remark 2.11 and similarly, let \(H_{0} = Q_{1/2}^{1/2}(H)\) be endowed with the norm \(|x|_{0} = |Q_{1/2}^{1/2}x|\). By Theorem 1.2 \(Q_{1/2}^{1/2}(H) \subset Q_{1/2}^{1/2}(H)\), hence the corresponding imbedding \(i : H_{0} \to H_{Q}\) is continuous. It is easy to check, that \(\mu(Q_{1/2}^{1/2}(H)) = 1\) if and only if \(i\) is a Hilbert-Schmidt operator, see also pp. 48-50 of [3]. Let \(\{e_{k} : k \geq 1\}\) be a CONS in \(H\). Then \(\{Q_{1/2}^{1/2}e_{k} : k \geq 1\}\) is a CONS in \(H_{0}\) and
\[ \sum_{k=1}^{\infty} |iQ_{1/2}^{1/2}e_{k}|_{Q}^{2} = \sum_{k=1}^{\infty} |Q^{-1/2}Q_{1/2}^{1/2}e_{k}|^{2}. \]
Hence it is enough to show that the operator \(Q^{-1/2}Q_{1/2}^{1/2}\) is Hilbert-Schmidt. For \(x \in Q_{1/2}^{1/2}(H)\) we have
\[ Q_{1/2}^{1/2}Q^{-1/2}x = \int_{0}^{\infty} S(s)QS^{*}(s)Q^{-1/2}xds = Q^{1/2} \int_{0}^{\infty} S_{Q}(2s)xds. \]
\[ = -\frac{1}{2} Q_{1/2}^{1/2}A_{r}^{-1}x. \] (5.2)
Hence
\[ Q^{-1/2}Q_\infty Q^{-1/2}x = -\frac{1}{2}A_Q^{-1}x, \quad x \in Q^{1/2}(H) \]
and thereby
\[ Q^{-1/2}Q_\infty^{1/2}Q^{-1/2} = -\frac{1}{2}A_Q^{-1}. \tag{5.3} \]
Since \( A_Q^{-1} \) is nuclear, (5.3) yields the Hilbert-Schmidt property of \( Q^{-1/2}Q_\infty^{1/2} \) and thereby \( \mu \left( Q^{1/2}(H) \right) = 1. \)

Conversely, assume that \( \mu \left( Q^{1/2}(H) \right) = 1. \) Then it follows from the properties of Gaussian measures (see for example Theorem 2.5.8 in [2]) that \( Q_\infty^{1/2}(H) \subset Q^{1/2}(H), \) hence by Theorem 4.2 \( (S_Q(t)) \) is exponentially stable. Consequently, \( A_Q^{-1} \) is bounded and (5.2) holds for \( x \in Q^{1/2}(H), \) which implies (5.3). Since by (iii), \( Q^{-1/2}Q_\infty^{1/2} \) is a Hilbert-Schmidt operator, it follows from (5.3) that \( A_Q^{-1} \) is a nuclear operator. Hence, \( S_Q(t) \) is Hilbert-Schmidt for each \( t > 0 \) and
\[ \int_0^\infty \|S_Q(t)\|_{HS}^2 \, ds = \frac{1}{2} \text{tr} \left( -A_Q^{-1} \right) < \infty. \]

\[ \square \]

**Lemma 5.2.** Let \( Z \) be a solution to (1.1) and assume that the corresponding Ornstein-Uhlenbeck semigroup is symmetric. Moreover, assume that (5.1) holds. Let \( \tilde{Z}(\cdot,x) \) denote a solution to the equation
\[ \begin{cases} d\tilde{Z} = A_Q\tilde{Z}dt + dW, \\ \tilde{Z}(0,x) = x \in H. \end{cases} \tag{5.4} \]
Then \( Q^{-1/2}Z(t,x) = \tilde{Z} \left( t, Q^{-1/2}x \right) \) for each \( x \in Q^{1/2}(H). \)

**Proof.** By assumption the stochastic integral
\[ \int_0^t S_Q(t-s) dW(s) \]
is well defined. Moreover, since \( Q^{-1/2} \) is closed in \( H \) it is enough to note that for \( x \in Q^{1/2}(H) \)
\[ Q^{-1/2}Z(t,x) = Q^{-1/2}S(t)x + Q^{-1/2} \int_0^t S(t-s)Q^{1/2}dW(s) \]
\[ = S_Q(t)Q^{-1/2}x + \int_0^t S_Q(t-s)dW(s) \]
\[ \square \]
Let \( u(t, x) = E\phi(Z(t, x)) \) with \( \phi \in L^p(H, \mu) \). We will show that for each \( \phi \in L^p(H, \mu) \) the function \( u \) satisfies almost everywhere an appropriate version of the following Backward Kolmogorov Equation

\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) &= \frac{1}{2} \text{tr} \left( Q D^2 u(t, x) \right) + \langle x, A^* Du(t, x) \rangle,
\end{align*}
\]

(5.5)

For \( \psi : H \to K \), where \( K \) is a Banach space, let \( D^Q \psi(x) \) denote the Fréchet derivative in the direction of the space \( Q^{1/2}(H) \) of \( \psi \) at the point \( x \in H \). This means that \( D^Q \psi(x) \) is a unique element of \( \mathcal{L}(H, K) \) (note that \( H = H^* \)) such that

\[
\lim_{H \ni h \to 0} \frac{\psi(x + Q^{1/2}h) - \psi(x) - D^Q \psi(x)h}{|h|} = 0.
\]

We write \((D^Q)^2 \psi(x) = D^Q(D^Q \psi)(x)\).

**Theorem 5.3.** Assume that \((R_t)\) is symmetric and the semigroup \((S_Q(t))\) satisfies (5.4). Let \( v(t, x) = u(t, Q^{1/2}x) \). Then the following holds.

(a) \( v \in C^{1,2}((0, \infty) \times H, \mathbb{R}) \).

(b) The functions \((t, x) \to \langle x, A_Q D^Q u(t, Q^{1/2}x) \rangle \) and \((t, x) \to \text{tr} \left( (D^Q)^2 u(t, Q^{1/2}x) \right)\) are well defined and continuous on \((0, \infty) \times H\).

(c) For every \( t > 0 \) and \( y \in Q^{1/2}(H) \) the function \( u \) satisfies the following version of (5.3):

\[
\begin{align*}
\frac{\partial u}{\partial t}(t, y) &= \frac{1}{2} \text{tr} \left( (D^Q)^2 u(t, y) \right) + \langle Q^{-1/2}y, A_Q D^Q u(t, y) \rangle,
\end{align*}
\]

(5.6)

**Proof.** Let

\[
\tilde{R}_t \phi(x) = E\phi(\tilde{Z}(t, x)).
\]

Then we have for any \( \phi \in L^p(H, \mu) \)

\[
v(t, x) = R_t \phi \left( Q^{1/2}x \right) = E\phi \left( Q^{1/2} \tilde{Z}(t, x) \right)
= E\tilde{\phi} \left( \tilde{Z}(t, x) \right) = \tilde{R}_t \tilde{\phi}(x),
\]

where \( \tilde{\phi}(x) = \phi \left( Q^{1/2}x \right) \). Clearly, \( \tilde{\phi} \in L^p(H, \mu) \). Moreover, by Corollary 9.22 in [13] (or by (5.9) below) \( \tilde{R}_t \) is strong Feller. Hence, by Theorem 5 in [6] the function \( v(t, x) = \tilde{R}_t \tilde{\phi}(x) \) satisfies the following conditions.

(i) \( v \in C^{1,2}((0, \infty) \times H, \mathbb{R}) \).

(ii) The functions \((t, x) \to A_Q Dv(t, x) \) and \((t, x) \to \text{tr} \left( D^2 v(t, x) \right)\) are well defined and continuous on \((0, \infty) \times H\).

(iii) For every \( t > 0 \) and \( x \in H \)

\[
\frac{\partial v}{\partial t}(t, x) = \frac{1}{2} \text{tr} \left( D^2 v(t, x) \right) + \langle x, A_Q Dv(t, x) \rangle.
\]

(5.7)
Since $v$ is Fréchet differentiable for each $(t, x) \in (0, \infty) \times H$, the very definition of $v$ implies that
\[
\lim_{h \to 0} \frac{v(t, x + h) - v(t, x) - Dv(t, x)h}{|h|} = \lim_{h \to 0} \frac{u(t, Q^{1/2}x + Q^{1/2}h) - u(t, Q^{1/2}x) - Dv(t, x)h}{|h|} = 0.
\]
Hence, there exists $D^Q u(t, Q^{1/2}x) = Dv(t, x)$. Analogously, $(D^Q)^2 u(t, Q^{1/2}x) = D^2v(t, x)$, $(t, x) \in (0, \infty) \times H$.

Therefore, (b) follows from (ii) and (5.7) yields
\[
\frac{\partial u}{\partial t}(t, Q^{1/2}x) = \frac{1}{2} \text{tr} \left( (D^Q)^2 \left( t, Q^{1/2}x \right) \right) + \langle x, A_Q D^Q \left( t, Q^{1/2}x \right) \rangle,
\]
for $t > 0$ and $x \in H$. Putting $y = Q^{1/2}x$ in (5.8) we obtain (5.6).

**Proof.** By [12] (see also [13]) $(R_t)$ is strongly Feller if and only if
\[
\text{im} (S(t)) \subset Q^{1/2} \left( \text{dom} \left( \sqrt{-A_Q} \right) \right).
\]
If $(R_t)$ is strongly Feller then $(S_Q(t))$ is exponentially stable and of Hilbert-Schmidt type.

**Corollary 5.4.** Assume that $(R_t)$ is symmetric. Then $(R_t)$ is strong Feller if and only if for each $t > 0$
\[
\text{im} (S(t)) \subset Q^{1/2} \left( \text{dom} \left( \sqrt{-A_Q} \right) \right),
\]
for $t > 0$.

By Proposition 2.10, $(R_t)$ is strongly Feller if and only if
\[
\text{im} (Q_t^{1/2}) \subset \text{dom} \left( \sqrt{-A_Q} \right),
\]
and consequently, (5.9) is equivalent to the strong Feller property of $(R_t)$.

Let $(R_t)$ be strongly Feller. Then by [13] (see also Proposition 3 in [3])
\[
\text{im} (Q_t^{1/2}) = \text{im} (Q_{\infty}^{1/2}),
\]
and by Theorem 4.2 $(S_Q(t))$ is exponentially stable and by (5.10) $Q_{\infty}^{-1/2} S(t)$ is bounded for $t > 0$. This implies that for $t > 0$ the operator $S_0(t) = \left( Q_{\infty}^{-1/2} S(t) \right) Q_{\infty}^{1/2}$ is of Hilbert-Schmidt type and by (2.21) so is $S_Q(t)$ for all $t > 0$.

**Remark 5.5.** Remark 4.5 and Corollary 5.4 imply that the semigroup $(R_t)$ is strongly Feller if and only if
\[
\text{im} (S(t)) \subset \text{dom} \left( \sqrt{-A_r} \right),
\]
for $t > 0$. 

\[
\text{im} (Q_t^{1/2}) = \text{dom} \left( \sqrt{-A_Q} \right),
\]
and consequently, (5.9) is equivalent to the strong Feller property of $(R_t)$.
6. EXAMPLES

6.1. Example 1. The example below was introduced in [17] and later studied in a more general framework in [21]. Let \( \{e_k : k \geq 1\} \) be a CONS in \( H \) and let
\[
Qe_k = \frac{1}{k^3} e_k \quad \text{and} \quad Ae_k = -\frac{1}{k} e_k.
\]
Then \( S(t) = e^{tA} \) and \( \|S(t)\| = 1 \) for all \( t \geq 0 \). We have also
\[
Q_t = \frac{1}{2} A^2 \left( I - e^{2tA} \right), \quad Q_\infty = \frac{1}{2} A^2, \quad \text{and} \quad A Q = A.
\]
We shall show that \( \text{im} \left( Q_{1/2}^t \right) \) is constant for all \( t > 0 \) but \( \text{im} \left( Q_\infty^{1/2} \right) \) \( \neq \) \( \text{im} \left( Q_{1/2}^{1/2} \right) \). Indeed, Proposition 2.10 yields
\[
\text{im} \left( Q_{1/2}^t \right) = Q_{1/2}^{1/2}(H) = A^{3/2}(H), \quad t > 0,
\]
while \( Q_{1/2}^{1/2}(H) = A(H) \). It follows from Theorem 1.2 that \( R_t \) is not hypercontractive for any \( t > 0 \) and the generator \( L \) of \( (R_t) \) has no spectral gap.

Using Theorem 1a from [3] we find that there exists a bounded Borel function \( \phi, x \in H \) and \( h \in Q_{\infty}^{1/2}(H) \) such that the function \( t \to R_t \phi(x + th) \) is not continuous.

Let us recall that for noninteger \( \alpha \) the space \( W_{Q_\infty}^{\alpha, 2}(H) \) is defined by interpolation (for details, see [5]). It follows from Theorem 4c in [3] that \( R_t \left( L^2(H, \mu) \right) \) is not contained in \( W_{Q_\infty}^{\alpha, 2}(H) \) for any \( \alpha > 0 \). Hence \( \text{dom} \left( L \right) \) is not contained in \( W_{Q_\infty}^{\alpha, 2}(H) \) for any \( \alpha > 0 \). This fact can be also directly deduced from Theorem 3.1 which yields
\[
\text{dom}_2(L) = W_{-A^2}^{2, 2} \cap W_{-A^4}^{1, 2}.
\]

6.2. Example 2. The stochastic heat equation \((6.4)\) in a weighted space \( L^2(\mathbb{R}, \rho(\zeta) d\zeta) \) was considered in [14] as an example of the Ornstein-Uhlenbeck process in a chaotic environment. Here we investigate some properties of the transition semigroup associated to \((6.4)\), using different methods. In particular, we improve some results from [14].

Let \( H_\kappa = L^2(\mathbb{R}, \rho_\kappa(\zeta) d\zeta) \), where \( \rho_\kappa(\zeta) = e^{-\kappa|\zeta|} \) with \( \kappa \geq 0 \). In particular \( H^0 = L^2(\mathbb{R}) \). The scalar product and the norm in \( H_\kappa \) will be denoted by \( \langle \cdot, \cdot \rangle_\kappa \) and \( | \cdot |_\kappa \) respectively. Fix \( m > 0 \) and let \( A^{(0)} = \Delta - mI \), where \( \Delta \) is the Laplacian in \( H^0 \) and let \( S^{(0)}(t) \) denote the semigroup on \( H^0 \) generated by \( A^{(0)} \). Then \( A^{(0)} \) is selfadjoint in \( L^2(\mathbb{R}) \) and \( \text{dom} \sqrt{-A^{(0)}} = H^{1, 2}(\mathbb{R}) \). The semigroup \( (S^{(0)}(t)) \) generated by \( A^{(0)} \) has the property
\[
\left\| S^{(0)}(t) \right\| = e^{-mt}. \tag{6.1}
\]
Let \( H^{1,2}_\kappa \) denote the space of functions \( x \in H_\kappa \) such that the distributional derivative \( x' \in H_\kappa \) and

\[
|x|_{\kappa,1}^2 = |x'|^2_\kappa + m|x|^2_\kappa < \infty.
\]

**Lemma 6.1.** For any \( \kappa, m \geq 0 \) there exist \( \alpha > 0 \) and \( \omega \in \mathbb{R} \) such that

\[
\langle -A^{(0)} x, x \rangle_\kappa \geq \alpha |x|_{\kappa,1}^2 + \omega |x|_{\kappa}^2, \quad x \in C_0^\infty(\mathbb{R}).
\]

In particular, \( A^{(0)} \) extends to a generator \( A^{(\kappa)} \) of an analytic semigroup \( (S^{(\kappa)}(t)) \) in \( H^\kappa \) and

\[
\|S^{(\kappa)}(t)\| \leq e^{-\omega t}.
\]

Moreover, if \( \frac{1}{4}\kappa^2 < m \) then \( \omega > 0 \).

**Proof.** For \( x \in C_0^\infty(\mathbb{R}) \) we have

\[
\langle -A^{(0)} x, x \rangle_\kappa = \langle -A^{(0)} x, \rho_\kappa x \rangle_0 = \langle x', \rho_\kappa x \rangle_0 + m|x|_{\kappa}^2
\]

\[
= |x'|^2_\kappa + \langle x', \rho_\kappa \rangle_0 + m|x|_{\kappa}^2.
\]

Since \( |\rho'_\kappa(\zeta)| = k\rho_\kappa(\zeta) \) for \( \zeta \neq 0 \) we obtain for any \( \epsilon > 0 \):

\[
\langle -A^{(0)} x, x \rangle_\kappa \geq |x'|^2_\kappa + m|x|_{\kappa}^2 - \frac{k\epsilon}{2} |x'|_{\kappa}^2 - \frac{k}{2\epsilon} |x|_{\kappa}^2
\]

\[
= \left( 1 - \frac{k\epsilon}{2} \right) \left( |x'|_{\kappa}^2 + m|x|_{\kappa}^2 \right) + \left( \frac{mk\epsilon}{2} - \frac{k}{2\epsilon} \right) |x|_{\kappa}^2.
\]

Hence, (6.2) follows provided \( k\epsilon < 2 \). The remaining part of the lemma follows easily from the Theorem of Lions (see p. 389 of [13]). \( \square \)

We will consider equation (1.1) written in a slightly different form

\[
dZ = A^{(\kappa)} Z dt + JdW,
\]

where \( W \) is standard cylindrical Wiener process on \( H^{(0)} \) and \( J : H^{(0)} \to H^{(\kappa)} \) is an embedding: \( Jx = x \). Then \( Q = JJ^* \) and it is easy to check that

\[
J^*x = \rho_\kappa x = Qx, \quad Q^{1/2}x = \rho^{1/2}_\kappa x.
\]

It was proved in [14] that for any \( \kappa > 0 \) and \( m > 0 \) the solution (6.4) is well defined in \( H^\kappa \) and it admits an invariant measure \( \mu = N(0, Q_\kappa) \). Let \( (R_t) \) be the Ornstein-Uhlenbeck semigroup corresponding to (6.4).

**Proposition 6.2.** For any \( \kappa > 0 \) and \( m > 0 \) the following holds.

(i) \( \ker (Q_\kappa) = \{0\} \) in \( H^\kappa \).

(ii) \( R_t = R^*_t \) in \( L^2(H^\kappa, \mu) \).

(iii) The semigroup \( (R_t) \) satisfies all the statements of Theorem 4.2 with \( \beta = m \).
Proof. (i) Note that if (6.4) has an invariant measure \( \mu = N(0, Q_{\infty}) \) then
\[
\ker (Q_{\infty}) \subset \ker (Q).
\]
Indeed, if \( x \in \ker (Q_{\infty}) \) then
\[
0 = \langle Q_{\infty}x, x \rangle = \int_0^\infty \left| Q^{1/2}S^*(t)x \right|^2 dt.
\]
Hence, for a.a. \( t \geq 0, Q^{1/2}S^*(t)x = 0 \) and by continuity \( Q^{1/2}x = 0 \). Thus
\[(i) \text{ follows from (6.3) and (6.6).}
\]
(ii) For \( x \in H^0 \) and \( y \in H^\kappa \)
\[
\langle S^{(\kappa)}(t)x, y \rangle = \langle S^{(0)}(t)x, \rho_\kappa y \rangle
\]
and thereby
\[
\left( S^{(\kappa)}(t) \right)^* y = \rho_\kappa^{-1}S^{(0)}(t) (\rho_\kappa y),
\]
and by (3.3)
\[
Q \left( S^{(\kappa)}(t) \right)^* y = S^{(0)}(t) (\rho_\kappa y) = S^{(\kappa)}(t)Qy.
\]
Therefore, (ii) holds by Theorem 2.4.
(iii) By Theorem 2.7 and (6.5) we find that for \( x \in H^\kappa \)
\[
\left| S^{(\kappa)}(t)x \right|^2 \kappa = \int_{\mathbb{R}} \left( \rho_\kappa^{-1/2}S^{(\kappa)}(t) (\rho_\kappa x) (\zeta) \right)^2 \rho_\kappa(\zeta) d\zeta
\]
\[
= \left| S^{(0)}(t) \left( \rho_\kappa^{1/2}x \right) \right|^2_0.
\]
Then by (6.1)
\[
\left\| S^{(\kappa)}(t) \right\|_\kappa = e^{-mt},
\]
and (iii) follows from Theorem (1.2).

Corollary 6.3. For any \( \kappa > 0 \) and \( m > 0 \) the following holds.
(i) For \( \phi \in L^2(H^\kappa, \mu) \)
\[
\left\| R_t \phi - \int_{H^\kappa} \phi d\mu \right\|_2 = e^{-mt} \| \phi \|_2.
\]
(ii) \( R_t \) is not strong Feller on \( H^\kappa \).
(iii)
\[
Q^{1/2}_{\infty}(H^\kappa) = \text{dom} \left( \sqrt{-A^{(0)}} \right) = H^{1,2}(\mathbb{R}) = Q^{1/2}_t(H^\kappa).
\]
Proof. Part (i) follows from (6.7) and Theorem 4.4. Note that $H_Q = H^0$ and $(S^{(\kappa)}(t))$ restricted to $H_Q$ is isometrically isomorphic to $(S^{(0)}(t))$. Since $(S^{(0)}(t))$ is not compact, the semigroup $(R_t)$ is not compact by Theorem 4.4 and Remark 4.7. Hence (ii) follows. Similarly, we obtain (iii) from Remark 5.5.

If $\frac{1}{4} \kappa^2 < m$ then by (6.3) the semigroup $(S^{(\kappa)}(t))$ is exponentially stable, hence $\mu$ is a unique invariant measure for (6.4). It was shown in [14] that for $0 < m < \frac{1}{4} \kappa^2$ there are infinitely many invariant measures for (6.4). Below we improve this result.

**Proposition 6.4.** If $m \geq \frac{1}{4} \kappa^2$ then there exists a unique invariant measure.

For $0 < m < \frac{1}{4} \kappa^2$ there exists a family $\{\mu_{\lambda} : \lambda \in \mathbb{R}\}$ of Borel probability measures on $H^\kappa$ such that for any $\lambda \in \mathbb{R}$ the following holds.

(i) The measure $\mu_{\lambda}$ is symmetrizing for $(R_t)$.

(ii) The Logarithmic Sobolev Inequality holds in $L^2(H^\kappa, \mu_{\lambda})$:

$$\int_{H^\kappa} \phi^2(x) \log |(\phi(x)|^2 \mu(dx) \leq \frac{2}{m} \langle -L\phi, \phi \rangle_\kappa + ||\phi||^{2}_\kappa \log ||\phi||^{2}_\kappa.$$  

(iii) If $\lambda_1 \neq \lambda_2$ then $\mu_{\lambda_1}$ and $\mu_{\lambda_2}$ are singular and in particular, $\mu_{\lambda_1} \perp \mu = \mu_0$ if $\lambda \neq 0$.

**Proof.** (i) If $m > \frac{1}{4} \kappa^2$ then the uniqueness of the invariant measure follows from (6.3). For $m = \frac{1}{4} \kappa^2$, assume that for a certain $x \in \text{dom}(A)$ the equation $A^{(\kappa)}x = 0$ has a solution. Then $\Delta x = mx$, and therefore $x \in \text{dom}((-\Delta)^n)$ for all $n \geq 1$. Hence $x \in C^2(\mathbb{R})$ and $x'' = mx$, which is impossible for $x \in H^\kappa$.

(iii) Following [14], let $g(\zeta) = e^{\sqrt{mc}}$. Then

$$A^{(\kappa)}g = 0.$$  

Therefore, $\mu_{\lambda} = N(\lambda g, Q_\infty)$, $\lambda \in \mathbb{R}$, is also invariant for (6.4), see [14] for details. By (iii) of Corollary 6.3, $Q_\infty^{1/2}(H^\kappa) \subset H^0$. Since for $\lambda \neq 0$, $\lambda g \notin H^0$ we obtain (iii) from the Feldman-Hayek Theorem.

(i) Since $\mu_{\lambda}$ is invariant for (6.4), $(R_t)$ is a $C_0$-semigroup of contractions on $L^p(H^\kappa, \mu_{\lambda})$ for $p \in [1, \infty)$. For $a \in H^\kappa$ let $T_a$ denote the shift operator on $H^\kappa$: $T_a x = a + x$. Note that for a bounded Borel $\phi \geq 0$ and $\mu_a = N(a, Q_\infty)$ we have

$$\int_{H^\kappa} \phi(T_a x) \mu(dx) = \int_{H^\kappa} \phi(x) \mu_a(dx),$$

hence the map

$$\phi \to \phi \circ T_a : L^p(H^\kappa, \mu_a) \to L^p(H^\kappa, \mu),$$

is an isometric isomorphism. For $a = \lambda g$ (6.8) yields

$$Z(t,a + x) = S^{(\kappa)}(t)a + Z(t,x) = a + Z(t, x),$$

for
which implies that
\[ R_t \phi (T_a x) = R_t (\phi \circ T_a) (x), \quad \phi \in L^1 (H^\kappa, \mu_\lambda). \] 
(6.11)
Taking into account that \((R_t)\) is symmetric in \(L^2 (H^\kappa, \mu)\), by (6.9) and (6.11) we have for \(\phi, \psi \in L^2 (H^\kappa, \mu_\lambda)\)
\[ \int_{H^\kappa} \psi (x) R_t \phi (x) \mu_\lambda (dx) = \int_{H^\kappa} \phi \circ T_a (x) R_t (\psi \circ T_a) (x) \mu (dx) \]
\[ = \int_{H^\kappa} \phi (x) R_t \psi (x) \mu_\lambda (dx), \]
which proves (i).

(ii) It follows from (iii) of Proposition 6.2, (6.10) and (6.11) that the semi-group \((R_t)\) is hypercontractive in each \(L^p (H^\kappa, \mu_\lambda)\) and therefore (ii) follows from (i).

Remark 6.5. If \(m < \frac{1}{4} \kappa^2\) then the asymptotic behaviour of the semigroups \((S_Q^{(\kappa)} (t))\) and \((S^{(\kappa)} (t))\) on \(H^\kappa\) is different: while the former is exponentially stable, the latter is not. Actually, for \(m < \frac{1}{4} \kappa^2\)
\[ \left\| S_Q^{(\kappa)} (t) \right\| \leq e^{-mt}, \quad \lim_{t \to \infty} \left\| S^{(\kappa)} (t) \right\| = \infty. \]
Indeed, let \(x_\alpha (\zeta) = e^{\alpha \zeta}\). Then \(x_\alpha \in H^\kappa\) (but not to \(H^0\)), provided \(|\alpha| < \frac{1}{2} \kappa\)
and \((\Delta - m I) x_\alpha = (\alpha^2 - m) x_\alpha\), hence \(S^{(\kappa)} (t) x_\alpha = e^{t (\alpha^2 - m)} x_\alpha\). Thereby, for \(m < \alpha^2 < \frac{1}{4} \kappa^2\) we obtain \(|S^{(\kappa)} (t) x_\alpha|_\kappa = e^{t (\alpha^2 - m)} |x_\alpha|_\kappa \to \infty\) for \(t \to \infty\).

Remark 6.6. We show in part (iii) of Proposition 6.4 that the invariant measures \(\mu_\lambda\) must be mutually singular. Since the generator \(L\) of \((R_t)\) in \(L^2 (H^\kappa, \mu_\lambda)\) can be associated to an irreducible symmetric Dirichlet form, this fact can be also deduced from \([1]\). For similar results for processes which are not associated to Dirichlet forms see \([4]\).

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