Self-interaction in the Bopp-Podolsky electrodynamics: Can the observable mass of a charged particle depend on its acceleration?

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In this paper we obtain the expression for the self-force in the model with the Lagrangian containing additional terms, quadratic in Maxwell tensor derivatives (so-called Bopp-Podolsky electrodynamics). Features of this force are analyzed for various limiting cases. When a charged particle moves along straight line with a uniform acceleration, an explicit formula is found. In the framework of the considered model, an observable renormalized particle mass is shown to depend on its acceleration. This dependence allows, in principle, to extract experimentally a value of the particle bare mass.

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I. INTRODUCTION

Effects of the quantum field theory in the low-energy limit can be described by the action functional with an effective Lagrangian, which contains additional nonlinear terms and terms with higher derivatives. Furthermore, in recent years the dark energy problem and the accelerated expansion of the Universe have inspired an interest in the various phenomenological models in cosmology, which use Lagrangians of such a type (see, e.g., [1]). Therefore, a reasonable selection of the effective Lagrangian and a search of constraints on its constituent parameters become needed.

A general form of the effective Lagrangian describing the interaction of gravitational and electromagnetic fields, which is composed of invariants containing derivatives up to the fourth order, can be given as

$$\mathcal{L}_{\text{eff}} = \frac{R}{2\kappa} + \frac{1}{16\pi} F_{ik} F^{ik} + c_{MG}^{(1)} R^2 + c_{MG}^{(2)} R_{ik} R^{ik} + c_{NM}^{(1)} R F_{ik} F^{ik} + c_{NM}^{(2)} R F_{ik} F^{ik} F_{km} F^{im} + c_{NM}^{(3)} R R_{ikmn} F_{ij} F^{ij} + c_{NL}^{(1)} (F_{ik} F^{ik})^2 + c_{NL}^{(2)} (F_{ik}^{*} F^{ik})^2 + c_{BP} \nabla_i F^{im} \nabla^k F_{km}. \quad (1)$$

The first two terms in this expression form the Lagrangian of the standard (or minimal) Einstein-Maxwell model. Next two invariants (with phenomenological constants $c_{MG}^{(1)}$, $c_{MG}^{(2)}$) relate to the various modified theories of gravity [2]. Next three cross-terms with coupling parameters $c_{NM}^{(1)}$, $c_{NM}^{(2)}$, and $c_{NM}^{(3)}$ describe the nonminimal interaction of gravity and the electromagnetic field (see [3] for history, references, and the latest results). The last terms with parameters $c_{NL}^{(1)}$, $c_{NL}^{(2)}$ and $c_{BP}$ are associated with the nonlinear [4, 5] and nonlocal self-interaction of the electromagnetic field, respectively. Note that in Eq. (1) we omit terms which can be reduced to already indicated invariants by identical transformations and/or dropping a total 4-divergence.

When we examine the effects relating to particle motion in weak gravitational fields, one can neglect the terms containing the curvature tensor. In this case, the Lagrangian of the theory reduces to

$$\mathcal{L}_{\text{eff}} = \frac{1}{16\pi} F_{ik} F^{ik} + c_{NL}^{(1)} (F_{ik} F^{ik})^2 + c_{NL}^{(2)} (F_{ik}^{*} F^{ik})^2 + c_{BP} \nabla_i F^{im} \nabla^k F_{km}. \quad (2)$$

Moreover, among the remaining terms we focus on the last one only, while the invariants of the fourth order with respect to the Maxwell tensor components leading to the nonlinearity of the equations will not be considered here for simplicity.

The invariant $\nabla_i F^{im} \nabla^k F_{km}$ gives rise to significant physical effects, such as a recoil force caused by the own electromagnetic field of charged particles (i.e., self-force). The electrodynamic model extended by using such terms was first considered independently by Bopp and Podolsky [6, 7]. Unlike the Maxwell electrodynamics, this modification
has two essential features: firstly, an electric field of the point particle possesses a finite energy, secondly, besides the “usual” massless photon, the Bopp-Podolsky (BP) electrodynamics predicts its “heavy” analog, and, in this sense, the BP-model has a close relationship with the Pauli-Villars regularization procedure (see, e.g.,[8]) and Lee-Wick model[9]. A “heavy photon” mass is the only uncertain parameter of this theory.

The work is organized as follows. In Section II we recall some facts relating to the BP-electrodynamics. In Section III the expression for the self-force in BP-model is considered in detail: Subsection III A covers its general properties, while in Subsection III B we focus on the toy-model describing a uniformly accelerated particle. We summarize the results of the paper in Section IV.

Throughout this paper we use the units \( c = \hbar = 1 \) and assume that the spacetime is flat and possesses the metric 
\[
g_{ik} = \text{diag}(1, -1, -1, -1)\]
According to this assumption, all covariant derivatives are replaced below with ordinary partial ones.

II. BOPP-PODOLSKY ELECTRODYNAMICS

A. General formalism

The electromagnetic field in the generalized Bopp-Podolsky electrodynamics is described by the Lagrangian \[6, 7\]
\[
\mathcal{L}_{BP} = \frac{1}{16\pi} F_{ik} F^{ik} - \frac{1}{8\pi \mu^2} \partial_i F^{im} \partial^k F_{km} + A_i j^i. \tag{3}
\]
Here \( A_i \) is the electromagnetic field potential, \( F_{ik} = \partial_i A_k - \partial_k A_i \) is the Maxwell tensor, and \( j^i \) denotes the 4-vector of the current density. Note this model returns to the ordinary Maxwell electrodynamics when \( \mu \) tends to infinity.

The variation of the action functional with the Lagrangian (3) with respect to the potentials \( A_i \) gives the electromagnetic field equation
\[
\partial_m \left( F^{mi} + \mu^{-2} G^{mi} \right) = 4\pi j^i, \]
\[
G_{mn} \equiv \partial_m \partial^k F_{kn} - \partial_n \partial^k F_{km}. \tag{4}
\]
If we take into account the Lorenz gauge condition \( \partial_m A^m = 0 \), it reduces to
\[
(1 + \mu^{-2}\Box) \Box A_n = 4\pi j_n, \quad \Box \equiv \partial_m \partial^m. \tag{5}
\]
As was shown in [8], any solution of (5) can be represented as a difference of two auxiliary field potentials \( A'_n \) and \( A''_n \):
\[
A_n = A'_n - A''_n, \tag{6}
\]
where the first term is a solution to the Maxwell equation, while the second one obeys the Proca equation
\[
\Box A'_n = 4\pi j_n, \quad \Box A''_n + \mu^2 A''_n = 4\pi j_n. \tag{7}
\]
Thus, the fourth-order equation (5) splits into two second-order equations describing, correspondingly, massless and massive vector fields with the same source. In this context, the positive parameter \( \mu \) in the BP-model plays the role of an auxiliary field mass.

The energy-momentum tensor of the electromagnetic field in this model takes the form \[7\]
\[
T_{ik}^{BP} = \frac{1}{4\pi} \left( \frac{1}{4} F_{mn} F^{mn} g_{ik} - F_{im} F_{km} \right) + \mu^{-2} T_{ik}^{(+)} \tag{8}
\]
Here the first component is the standard energy-momentum tensor in the Maxwell theory and the additional term \( T_{ik}^{(+)} \) is defined as follows
\[
T_{ik}^{(+)} = \frac{1}{4\pi} \left( \frac{1}{2} g_{ik} \partial_p F^{pm} \partial^q F_{qm} - \partial^m F_{mi} \partial^n F_{nk} + \frac{1}{2} G_{mn} F^{mn} g_{ik} - G_{im} F_{km} - G_{km} F_{im} \right) \tag{9}
\]
The substitution of (6) into (8) gives that the energy-momentum tensor is also splitting into two parts \[6\]:
\[
T_{ik} = T'_{ik} - T''_{ik}, \tag{10}
\]
where
\[
T'_{ik} = \frac{1}{4\pi} \left( \frac{1}{4} F'_{mn} F'^{mn} g_{ik} - F'_{im} F'^{im} \right),
\]
(11)
\[
F'_{mn} \equiv \partial_m A'_n - \partial_n A'_m,
\]
corresponds to the massless vector field \(A'_i\), and
\[
T''_{ik} = \frac{1}{4\pi} \left( \frac{1}{4} F''_{mn} F''^{mn} g_{ik} - F''_{im} F''^{im} + \mu^2 \left[A''_n A''_m - \frac{1}{2} g_{ik} A''_m A''_m \right] \right),
\]
(12)
\[
F''_{mn} \equiv \partial_m A''_n - \partial_n A''_m,
\]
is the energy-momentum tensor of the Proca-type massive field \(A''_i\). It is necessary to emphasize that this auxiliary field has a negative energy and therefore the massive field \(A''_i\) can be interpreted as a ghost (or phantom) field.

B. Point electric charge field

1. Charged particle at rest

Let us consider the field of a point charged particle. When the particle is at rest, its current density 4-vector takes the form (hereafter we apply bold letters to denote spatial vectors)
\[
j^0 = q \delta (\mathbf{r}), \quad j = 0,
\]
(13)
where \(q\) is a charge of the particle, \(\delta (x)\) is the Dirac delta function. Solving Eqs.(7), we obtain the field potential for the static case in the BP-electrodynamics [6, 7, 10] as follows
\[
A_0 = \frac{q}{r} (1 - e^{-\mu r}).
\]
(14)
For \(r \gg 1/\mu\) the formula (14) turns into the well-known expression for the Coulomb electrostatic potential, \(A_0 = \frac{q}{r}\); while at the origin it remains nonsingular, \(\lim_{r \to 0} A_0 = \mu q\). Notice that the same formula for the potential can be derived in the framework of the Maxwell electrodynamics, if one supposes the electric charge has a spatial distribution with density \(j^0 = \frac{\mu^2 q}{4\pi r} e^{-\mu r}\). In that case, the quantity \(1/\mu\) plays the role of a particle effective “radius”. The energy \(m_{\text{em}}^{(0)}\) of the electrostatic field (14) in the BP-model is also finite and expressed by the formula
\[
m_{\text{em}}^{(0)} = \int dV T_{00} = \frac{1}{2} \mu q^2.
\]
(15)
The obtained value can be identified as an electromagnetic component of the mass of the rest charged particle, or, for simplicity, its “electromagnetic” mass.

2. Moving charged particle

Let us proceed to a field configuration produced by a point charge moving along a given trajectory \(\mathbf{r} = \mathbf{r}_0(t)\) with a speed \(v_0(t)\). In this case, the components of the current density 4-vector take the form
\[
j^0 (r, t) = q \delta (\mathbf{r} - \mathbf{r}_0 (t)), \quad j (r, t) = q v_0 (t) \delta (\mathbf{r} - \mathbf{r}_0 (t)).
\]
(16)
As was shown by Landé and Thomas [10], the solution to Eqs.(17) is given by the formula
\[
A_i = \mu q \int_0^{+\infty} d\xi J_1 (\mu \xi) \frac{1}{\xi} \frac{\partial R_i}{\partial \xi},
\]
(17)
where \(R^i\) denotes a position vector of the particle, \(R^i = x^i - x^i_0 = (t - \tau, \mathbf{r} - \mathbf{r}_0 (\tau))\), the nonnegative parameter \(\xi\) is defined by the relation \(\xi^2 = R_p R^p\), \(\tau\) is a retarded time which is determined implicitly by the equation \(\tau = \ldots\)
The self-force phenomenon for charged particles moving in the flat spacetime was elaborated and described in detail (see, e.g., [16, 17] and references therein). The origin of the particle self-force is related with the inertial properties of the electromagnetic field. In other words, the self-force is the reaction of radiation. The equation of motion for a charged particle, which is under the influence of an external force \( F^i_{\text{ext}} \), taking into account self-force effects yields

\[
m \omega^i = F^i_{\text{ext}} + f^i_{\text{sf}},
\]

where \( m \) is the particle mass, \( \omega^i \) denotes 4-vector of its acceleration, and \( f^i_{\text{sf}} \) is the self-force.

In contrast to the Maxwell electrodynamics, in the BP-model the expression of the point charge field potential (14) is free from a singularity at the origin. Therefore, to derive a formula for \( f^i_{\text{sf}} \), we can omit a renormalization procedure and write down immediately

\[
f^i_{\text{sf}} = q F_{ik} u^k,
\]

where \( F^{ik} \) is the Maxwell tensor of the electromagnetic field produced by the moving particle, and \( u^k \) is the 4-vector of its velocity, and both quantities have to be calculated at the present location of the charged particle. The substitution of the formula for the strength tensor (18) into (22) yields

\[
f^i_{\text{sf}} = \mu q^2 \int_0^{+\infty} d\xi \, J_1(\mu \xi) \frac{1}{\xi} \frac{\partial}{\partial \xi} \left[ R^i_k \frac{1}{\xi} \frac{\partial R^i_k}{\partial \xi} - R^i_k \frac{1}{\xi} \frac{\partial}{\partial \xi} \right].
\]
The structure of this expression points to essential nonlocality of the self-force, because the whole path traversed by the particle up to the present time contributes to $f_{st}^i$ ($\xi^2 = R_\mu R^\mu$ runs from 0 to $+\infty$).

In order to analyze the general formula (28) for an arbitrary type of motion, let us consider the power expansion of $f_{st}^i$ in terms of the parameter $\mu$, using the asymptotic formula from [18]

$$\int_0^{+\infty} \frac{J_1(\mu x)}{x} x y(x) \, dx = y(0) + \frac{1}{\mu} y'(0) + \frac{1}{2\mu^2} y''(0) - \frac{1}{8\mu^4} y^{(4)}(0) + \ldots.$$  \hspace{1cm} (24)

If $\xi = 0$ we have either $\tau = t$, or $\frac{r_0(t) - r_0(\tau)}{t - \tau} = 1$. But the last relation is impossible, because massive particles cannot move with the speed being more or equal to the speed of light. Hence, $\lim R^i = 0$.

Using this fact, let us obtain the expansion of the position vector $R^i$ with respect to $\xi$. The particle trajectory will be considered to be parameterized by a natural parameter $s$. At the present point $s = s_0$, while the retarded moment $\tau$ corresponds to $s = s_0 - \Delta s$, where $\Delta s > 0$. It is easy to see that the expansion of $R^i$ in terms of $\Delta s$ takes the form

$$R^i = x^i(s_0) - x^i(s_0 - \Delta s) = u^i \Delta s - \frac{1}{2} \omega^i \Delta s^2 + \frac{1}{6} \omega^i \Delta s^3 - \frac{1}{24} \omega^i \Delta s^4 + \ldots,$$ \hspace{1cm} (25)

where $\omega^i = du^i/ds$ is the particle acceleration, the dot denotes the derivative with respect to $s$, and the quantities $u^i$, $\omega^i$, etc are calculated at the present point of the particle, i.e., at $s = s_0$. From the condition $R_i R^i = \xi^2$ we find the relation, which connects the parameters $\Delta s$ and $\xi$

$$\xi^2 = \Delta s^2 - \frac{1}{12} \omega_i \omega^i \Delta s^4 + \frac{1}{12} \omega_i \omega^i \Delta s^5 + \ldots.$$ \hspace{1cm} (26)

Solving this equation with respect to $\Delta s$ and taking into account that $\Delta s > 0$, we arrive at

$$\Delta s = \xi + \frac{1}{24} \omega_i \omega^i \xi^3 \frac{1}{24} \omega_i \omega^i \xi^4 + \ldots.$$ \hspace{1cm} (27)

The substitution of this relation into (28) yields

$$R^i = u^i \xi - \frac{1}{2} \omega^i \xi^2 + \frac{1}{6} (\omega^i + \frac{3}{4} u^i \omega^k \omega_k) \xi^3 - \frac{1}{24} (\omega^i + u^i \omega^k \omega_k + \omega_i \omega^k \omega_k) \xi^4 + \ldots.$$ \hspace{1cm} (28)

Hence, using (28) and (29), we obtain at last that

$$f_{st}^i = -\frac{\mu q^2}{2} \omega^i + \frac{2q^2}{3} (\omega^i + u^i \omega^k \omega_k) - \frac{3q^2}{8\mu} (\omega^i + 3u^i \omega^k \omega_k + \frac{3}{2} u^i \omega^k \omega_k) + \ldots.$$ \hspace{1cm} (29)

Note that this formula was found earlier by McManus [19], who was guided by quite different motivation.

The expansion (29) is just a formal one, because it is unknown whether this series converges. Nevertheless, basing on (29), one can discover a number of specific properties of the self-force $f_{st}^i$.

When $\mu \to \infty$, dropping the first infinite term, which corresponds to the classical renormalization of the mass, the expression (29) transforms into the well-known formula for the Lorentz-Dirac force

$$f_{LD}^i = \frac{2q^2}{3} (\omega^i + u^i \omega^k \omega_k).$$ \hspace{1cm} (30)

But, in contrast to the Maxwell electrodynamics, the first term $\frac{4q^2}{3} \omega^i$ in (29) being proportional to the acceleration of the moving particle is finite. The factor is equal to the electromagnetic field energy of the pointlike particle at rest (see [13]), and this feature is in agreement with Frenkel’s work [20], where this problem was considered in the nonrelativistic approximation.

On the other hand, besides terms, which are known from the usual Lorentz-Dirac theory, in the formula (29) there exist additional terms depending on the higher derivatives of the acceleration $\omega^i$. They vanish in the limit $\mu \to \infty$, but these terms can exert essential influence on the behavior of the self-force at the finite values of $\mu$. For example, one can believe that due to these corrections there are no self-accelerated, or runaway solutions in the BP-model (see [21, 22]).
B. Self-interaction in the case of uniformly accelerated motion

For the sake of simplicity, let a charged particle move along a straight line under the influence of a constant external force, being collinear to particle’s motion. Then the motion equation of this particle takes the form

\[ t(s) = \frac{1}{w} \sinh ws, \quad x(s) = \frac{1}{w} \cosh ws, \quad y(s) = z(s) = 0. \]  

(31)

Here we assume that the trajectory lies along the axis Ox and the natural parameter \( s \) runs from \(-\infty\) to \(+\infty\). The constant \( w \) is a magnitude of the particle acceleration, \( -\omega \omega^i \).

In this case, the velocity \( u^i \), the acceleration \( \omega^i \), and the position vector \( R^i = x^i(s) - x^i(s - \Delta s) \) are of the form

\[ u^i = \cosh ws \delta_0^i + \sinh ws \delta_1^i, \]

(32)

\[ \omega^i = w \left( \sinh ws \delta_0^i + \cosh ws \delta_1^i \right), \]

(33)

\[ R^i = \frac{1}{w} \left[ (\sinh ws - \sinh w(s - \Delta s)) \delta_0^i + (\cosh ws - \cosh w(s - \Delta s)) \delta_1^i \right]. \]

(34)

From the condition \( R_{\mu} R^\mu = \xi^2 \) we obtain that the parameters \( \xi \) and \( \Delta s \) are connected by the following relations

\[ \cosh w \Delta s = 1 + \frac{w^2 \xi^2}{2}, \]

(35)

\[ \sinh w \Delta s = w \xi \sqrt{1 + \frac{w^2 \xi^2}{4}}. \]

(36)

Hence we arrive to the expression for the vector \( R^i \) expressed in terms of \( \xi \)

\[ R^i = \xi \sqrt{1 + \frac{w^2 \xi^2}{4}} u^i - \frac{\omega^i \xi^2}{2}. \]

(37)

Substituting now (37) into the general formula for the self-force (28), we find that in the case of uniformly accelerated motion the vector \( f_{\text{sf}}^i \) is aligned along the particle acceleration \( \omega^i \)

\[ f_{\text{sf}}^i = -m_{\text{em}}(w) \omega^i, \]

(38)

where the factor \( m_{\text{em}}(w) \) is given by the relation

\[ m_{\text{em}}(w) = \frac{\mu q^2}{2} \int_0^{+\infty} \frac{d\xi}{\xi} \frac{J_1(\mu \xi)}{\left(1 + \frac{w^2 \xi^2}{4}\right)^{3/2}} = \frac{\mu^2 q^2}{w} I_1\left(\frac{\mu}{w}\right) K_1\left(\frac{\mu}{w}\right). \]

(39)

Here \( I_1(x) \) and \( K_1(x) \) are the modified Bessel functions of the first and second kind, respectively. As it is easy to see, the quantity \( m_{\text{em}} \), which can be identified as a “electromagnetic” mass of a moving particle, depends on the magnitude of its acceleration \( w \). If \( w \to 0 \), then obviously \( m_{\text{em}} \to m_{\text{em}}^{(0)} = \frac{\mu q^2}{w} \). Otherwise, the function \( m_{\text{em}}(w) \) approaches to zero when \( w \to \infty \). To illustrate the behavior of this function in the mentioned cases, we present the plot (see Fig. 1) and two asymptotic expansions (\( \gamma \) denotes Euler’s constant)

\[ \frac{m_{\text{em}}}{m_{\text{em}}^{(0)}} = 1 - \frac{3}{8} \left(\frac{w}{\mu}\right)^2 + \ldots, \quad \text{if} \quad \frac{w}{\mu} \ll 1, \]

(40)

\[ \frac{m_{\text{em}}}{m_{\text{em}}^{(0)}} = \frac{2\mu}{w} + \left[ \frac{1}{2} \ln \frac{\mu}{2w} + \frac{\gamma}{2} + \frac{1}{8} \right] \left(\frac{\mu}{w}\right)^2 + \ldots, \quad \text{if} \quad \frac{w}{\mu} \gg 1. \]

(41)

As the result, the equation of charged particle motion under influence of the constant external force \( F_{\text{ext}}^i \) and the self-force (38) is written as

\[ [m + m_{\text{em}}(w)] \omega^i = F_{\text{ext}}^i. \]

(42)

It is easy to see that the quantity \( m_{\text{obs}} = m + m_{\text{em}}(w) \) plays the role of an observable mass of the particle. If the particle acceleration vanishes, then the observable mass \( m_{\text{obs}} = m + \frac{\mu q^2}{w} \) consists as expected of the proper, or “bare”, mass of the particle \( m \) and the term associated with the electrostatic field energy (15). However, when the particle is in uniformly accelerated motion, its observable mass decreases and tends to \( m \), if \( w \to \infty \). This phenomenon opens up a possibility to reveal, in principle, the bare mass of charged particles.
Figure 1: Plot of the factor $m_{em}$ normalized to the electromagnetic mass of the rest particle against the dimensionless parameter $w/\mu$. At the origin, i.e., when the acceleration $w$ vanishes, $m_{em}/m_{em}^{(0)} = 1$. The function decreases monotonically and tends to zero at infinity.

IV. CONCLUSION

The Bopp-Podolsky electrodynamics, which is the simplest high-order modification of the Maxwell theory, was proposed in the 1940s, but it has been remaining topical (see, e.g., recent papers [13–15, 23]). We can point out three reasons to explain this fact:

1. This model is free of divergences, since the potential and the electrostatic field energy are finite.
2. In addition to the phenomenological way, the emergence of high-order terms in the Lagrangian can be predicted in the framework of the quantum field theory.
3. This version of the theory of electromagnetism is elaborated insufficiently and offers the possibilities of new physics.

In the present paper we focus on one of such new phenomena. From the toy-model discussed above it follows that in the Bopp-Podolsky electrodynamics the interaction between a charged particle and its electromagnetic field gives rise to the self-force directed against the acceleration of the particle. A related phenomenon known as a virtual mass effect occurs in hydrodynamics, when a solid moves with an acceleration through the liquid column (see, e.g., [24]), but for our case the factor $m_{em}$ depends on the particle acceleration.

It is important to note that the equations given in Subsection III B have been obtained for a uniform movement of the particle provided $t \in (-\infty; +\infty)$. Nevertheless, the effect described above takes place even for arbitrary motion. In order to explain this statement we consider the expansion (29) of the self-force $f_{st}$. Since it contains terms like $\omega_i \omega_k \omega_k$, one can conclude that the factor in front of the acceleration 4-vector, $m_{em}$, will depend on the particle acceleration for the general case. Moreover, we can say that the observable mass depends not only on the magnitude of the acceleration as in the formula (38), but also on its direction and derivatives $\dot{\omega}_i$, $\ddot{\omega}_i$, etc. Thus, for a more realistic example, we can consider particle motion within a plate-parallel capacitor, and obtain dependence between the accelerating force and the particle acceleration by experiments. As for the model considered in Subsection III B, it represents the simplest example, for which we can calculate this effect explicitly.

Dependence between the observable mass and the particle acceleration gives us a hypothetical possibility to obtain experimentally the model parameter $\mu$ as well as the bare mass of the charged particle, which is customary believed to be unobservable. Experimental testing of the obtained laws allows to define or put restrictions on parameter values in the effective Lagrangian and, as a consequence, validity of one or another of the approaches to produce it.

It is worth recalling that in our investigation we ignored the terms quartic in the Maxwell tensor. This is legitimate provided the magnitude of the neglected terms is much less than the rest. For a field produced by a rest charge at $r = 0$ we have

$$F_{ik}F^{ik} \sim \mu^4 q^2, \quad \frac{1}{\mu^2} \partial^i F_{ik} \partial_m F^{mk} \sim \mu^4 q^2, \quad (F_{ik} F^{ik})^2 \sim \mu^8 q^4.$$  \hspace{1cm} (43)

Hence we obtain that the disregard for the terms indicated by constants $c_{NL}^{(1)}$ and $c_{NL}^{(2)}$ in (1) is justified for small values of $\mu$, namely,

$$\mu^4 \ll \frac{1}{q^2 c_{NL}^{(1,2)}}.$$  \hspace{1cm} (44)
Otherwise, for big values of the parameter $\mu$ it is necessary to take into account the quartic terms. This work will be done by us in future papers.

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