TAUT SUBMANIFOLDS ARE ALGEBRAIC

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Abstract. We prove that every (compact) taut submanifold in Euclidean space is real algebraic, i.e., is a connected component of a real irreducible algebraic variety in the same ambient space.

1. Introduction

An embedding \( f \) of a compact, connected manifold \( M \) into Euclidean space \( \mathbb{R}^n \) is taut if every nondegenerate (Morse) Euclidean distance function,

\[
L_p : M \to \mathbb{R}, \quad L_p(z) = d(f(z), p)^2, \quad p \in \mathbb{R}^n,
\]

has \( \beta(M, \mathbb{Z}_2) \) critical points on \( M \), where \( \beta(M, \mathbb{Z}_2) \) is the sum of the \( \mathbb{Z}_2 \)-Betti numbers of \( M \). That is, \( L_p \) is a perfect Morse function on \( M \).

A slight variation of Kuiper’s observation in \([7]\) gives that tautness can be rephrased by the property that

\[
(1.1) \quad H_j(M \cap B, \mathbb{Z}_2) \to H_j(M, \mathbb{Z}_2)
\]

is injective for all closed disks \( B \subset \mathbb{R}^n \) and all \( 0 \leq j \leq \dim(M) \). As a result, tautness is a conformal invariant, so that via stereographic projection we can reformulate the notion of tautness in the sphere \( S^n \) using the spherical distance functions. Another immediate consequence is that if \( B_1 \subset B_2 \), then

\[
(1.2) \quad H_j(M \cap B_1) \to H_j(M \cap B_2)
\]

is injective for all \( j \).

Kuiper in \([8]\) raised the question whether all taut submanifolds in \( \mathbb{R}^n \) are real algebraic. We established in \([4]\) that a taut submanifold in \( \mathbb{R}^n \) is real algebraic in the sense that, it is a connected component of a real irreducible algebraic variety in the same ambient space, provided the submanifold is of dimension no greater than 4.

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In this paper, we prove that all taut submanifolds in $\mathbb{R}^n$ are real algebraic in the above sense, so that each is a connected component of a real irreducible algebraic variety in the same ambient space. In particular, any taut hypersurface in $\mathbb{R}^n$ is described as $p(t) = 0$ by a single irreducible polynomial $p(t)$ over $\mathbb{R}^n$. Moreover, since a tube with a small radius of a taut submanifold in $\mathbb{R}^n$ is a taut hypersurface [10], which recovers the taut submanifold along its normals (we will see this in (2.4) below), understanding a taut submanifold, in principle, comes down to understanding the hypersurface case defined by a single algebraic equation.

To achieve the goal, on the one hand we continue to explore the property that certain multiplicity sets are of finite ends as studied in [4]. On the other we employ Morse-Bott theory [3] and further real algebraic geometry in conjunction with Ozawa’s theorem [9] to obtain, in the hypersurface case, a fine structure of the set where the principal multiplicities are not locally constant. As a byproduct, the crucial local finiteness property that is decisive in [4] for establishing that a taut submanifold is algebraic falls out.

It is more convenient to prove that a taut submanifolds in the sphere is real algebraic, though occasionally we will switch back to Euclidean space when it is more convenient for the argument. Since a spherical distance function $d_p(q) = \cos^{-1}(p \cdot q)$ has the same critical points as the Euclidean height function $\ell_p(q) = p \cdot q$, for $p, q \in S^n$, a compact submanifold $M \subset S^n$ is taut if and only if it is tight, i.e., every nondegenerate height function $\ell_p$ has the total Betti number $\beta(M, \mathbb{Z}_2)$ of critical points on $M$. We will use both $d_p$ and $\ell_p$ interchangeably, whichever is more convenient for our argument.

Our proof is based on a fundamental result on taut submanifolds due to Ozawa [9].

**Theorem 1** (Ozawa). Let $M$ be a taut submanifold in $S^n$, and let $\ell_p, p \in S^n$, be a linear height function on $M$. Let $x \in M$ be a critical point of $\ell_p$, and let $S$ be the connected component of the critical set of $\ell_p$ that contains $x$. Then $S$ is
(a) a smooth compact manifold of dimension equal to the nullity of the Hessian of $\ell_p$ at $x$;
(b) nondegenerate as a critical manifold;
(c) taut in $S^n$.

In particular, $\ell_p$ is perfect Morse-Bott [3]. We call such a connected component of a critical set of $\ell_p$ a critical submanifold of $\ell_p$.

An important consequence of Ozawa’s theorem is the following [5].
Corollary 2. Let $M$ be a taut submanifold in $S^n$. Then given any principal space $T$ of any shape operator $S_\zeta$ at any point $x \in M$, there exists a submanifold $S$ (called a curvature surface) through $x$ whose tangent space at $x$ is $T$. That is, $M$ is Dupin [10].

Let us remark on a few important points in the corollary. It is convenient to work in the ambient Euclidean space $\mathbb{R}^n$. Let $\mu$ be the principal value associated with $T$. Consider the focal point $p = x + \zeta/\mu$. Then the critical submanifold $S$ of the (Euclidean) distance function $L_p$ through $x$ is exactly the desired curvature surface through $x$. The unit vector field

$$\zeta(y) := \mu(p - y)$$

for $y \in S$ extends $\zeta$ at $x$ and is normal to and parallel along $S$. The $(n-1)$-sphere of radius $1/\mu$ centered at $p$ is called the curvature sphere of $Z$.

2. The proof

We do an inductive argument on the following statement:

$S(n)$: All taut submanifolds in $S^n$ are real algebraic.

The statement is true for $n = 1$ since a 0-dimensional taut submanifold is a point. Assuming the statement is true for all $k \leq n - 1$.

We first handle the case when $M$ is a hypersurface. Fix a unit normal field $n$ over $M$ once and for all. We label the principal curvatures of $M$ by $\lambda_1 \leq \cdots \leq \lambda_{n-1}$, which are Lipschitz-continuous functions on $M$ because the principal curvature functions on the linear space $\mathcal{L}$ of all symmetric matrices are Lipschitz-continuous by general matrix theory [1] p. 64], and the Hessian of of $M$ is a smooth function from $M$ into $\mathcal{L}$. Let $\lambda_j = \cot(t_j)$ for $0 < t_j < \pi$. We have the Lipschitz-continuous focal maps

$$f_j(x) = \cos(t_j)x + \sin(t_j)n.$$

In fact, the $l$th focal point $f_l(x)$ along $n$ emanating from $x$ is antipodally symmetric to the $(n - l)$th focal point along $-n$ emanating from $x$. The spherical distance functions $d_{f_l(x)}$ tracing backward following $-n$ thus assumes the same critical point $x$ as the distance function $d_{-f_l(x)}$ tracing backward following $n$; thus we may just consider the former case without loss of generality. Accordingly, we refer to a focal point $p$ as being $f_j(x)$ for some $x$ and $j$.

By the inductive hypothesis, $Z$ must be algebraic since $Z$ lies in its curvature sphere by Corollary [2].
As mentioned earlier, we can regard $M \subset S^n$ as being tight. Suppose $Z$ is a critical submanifold of $M$ cut out by the height function $\ell_p$; assume $\ell_p(Z) = 0$ without loss of generality. Let $W \subset M$ be a tubular neighborhood of $Z$ so small that $\ell_p^{-1}(0)$ is the only critical set of $\ell_p$ in $W$. (We will call such a $W$ a neck around $Z$.)

Let us slightly perturb $\ell_p$ by a linear function $g$ with small coefficients such that $g$ is not a multiple of $\ell_p$ (otherwise $\ell_p + g$ is just $\ell_p$ in essence). Then $\ell_p + g = \ell_q$ for some $q$ close to $p$. $Z$ is not a critical submanifold of $\ell_p + g$, or equivalently, of $g$ since $q \neq p$.

Since $Z$ is taut by Ozawa’s theorem, in general the height function $g$ cuts $Z$ in several critical submanifolds $Z_1, \ldots, Z_l$; without loss of generality, we assume these critical submanifolds of $Z$ correspond to different critical values of $g$. It suffices to consider $Z_1$, for instance. Assume the codimension of $Z_1$ in $Z$ is $t$ and the dimension of $Z$ is $s$. Let us parametrize $W$ by $v_1, \ldots, v_t, v_{t+1}, \ldots, v_s, u_1, \ldots, u_{n-1-s}$ around 0, where $v_{t+1}, \ldots, v_s$ parametrize $Z_1$, $v_1, \ldots, v_s$ parametrize a neck around $Z_1$ in $Z$, and lastly the variables $v_1, \ldots, v_s, u_1, \ldots, u_{n-1-s}$ parametrize the neck $W$ of dimension $n-1$, which is the dimension of $M$, around $Z$. It is understood that 0 in the coordinate system corresponds to a point on $Z_1$. As in [9], we can assume

\[
\ell_p = \sum_{j=1}^{n-1-s} \alpha_j u_j^2 + O(3),
\]

\[
g = h(u) + \sum_{i=1}^t \beta_i u_i^2 + O(3),
\]

with

\[
h(u) = \sum_{j=1}^{n-1-s} a_i u_i + \sum_{j,k=1}^{n-1-s} b_{jk} u_j u_k
\]

for some small coefficients $a_i$ and $b_{jk}$, where $\alpha_j$ and $\beta_i$ are all nonzero constants. Note that the cross $uv$-terms can always be canceled by an appropriate linear change of coordinates. Moreover, there are no $v_{t+1}, \ldots, v_s$ present in $g$ because when we set the $u$-variables equal to zero, $Z_1$ parametrized by $v_{t+1}, \ldots, v_s$ is a critical submanifold of $g$ over $Z$. Differentiating and setting the derivatives equal to zero, we obtain

\[
0 = \partial(\ell_p + g)/\partial u_j = a_j + 2\alpha_j u_j + 2 \sum_{l=1}^{n-1-s} b_{jl} u_l + O(2) := F_j
\]
for $1 \leq j \leq n - 1 - s$, and
\[ 0 = \partial(\ell_p + g)/\partial v_i = 2\beta_i v_i + O(2) := G_i \]
for $1 \leq i \leq t$.

Since $a_i$ and $b_{jk}$ are small quantities, we know
\[ \partial(F_1, \ldots, F_{k-s})/\partial (u_1, \ldots, u_{n-1-s}) \neq 0 \]
at $u_1 = \cdots = u_{n-1-s} = 0$. Therefore, the implicit function theorem
implies that $u_1, \ldots, u_{n-1-s}$ are all functions of $v_1, \ldots, v_s$. Likewise,
since all $\beta_i$ are nonzero, we can in turn solve $v_1, \ldots, v_t$ in terms of
$v_{t+1}, \ldots, v_s$, the coordinates of $Z_1$. The critical set is thus a graph over
$Z_1$. Hence we have the following.

**Proposition 3.** Consider a neck $W$ around a critical submanifold $Z$
that is cut out by $\ell_p$ in $M$. Let $N_1, \ldots, N_l$ be necks around the critical
submanifolds $Z_1, \ldots, Z_l$ cut out by $g$ in $Z$, respectively. Set up a finite
number of aforementioned coordinate charts and let
\[ \pi : (v_1, \ldots, v_s, u_1, \ldots, u_{n-1-s}) \mapsto (v_1, \ldots, v_s). \]
be the projection. Then the critical set of $\ell_p + g$ in $\pi^{-1}(N_i)$ is a graph
over $Z_i$.

On the other hand, at a point $x \in Z$ away from $Z_1, \ldots, Z_l$, we
we can still parametrize $W$ around $x$ by $v_1, \ldots, v_s, u_1, \ldots, u_{n-1-s}$ where
$v_1, \ldots, v_s$ parametrize $Z$ around $x$ identified with 0. Then slightly
different from the earlier expression we have
\[ \ell_p = \sum_{j=1}^{n-1-s} \alpha_j u_j^2 + O(3), \]
\[ g = h(u) + \sum_{i=1}^{s} \gamma_i v_i + \sum_{i=1}^{s} \delta_i v_i^2 + O(3) \]
where at least one of $\gamma_i$ is nonzero since $p$ is a nondegenerate point of
$g$ on $Z$. Once more by setting $\partial(\ell_p + g)/\partial u_j$ equal to zero we see that
$u_1, \ldots, u_{n-1-s}$ are all functions of $v_1, \ldots, v_s$. On the other hand, we
may assume none of the $\delta_i$ are zero. For, suppose $\gamma_1 \neq 0$ and some
$\delta_j = 0$. Then replacing $v_1$ by $v_1 + v_j^2$ and keeping all other variables
unchanged will result in a nonzero coefficient for $v_j^2$ with all other $\delta_i$
unchanged. Then setting $\partial(\ell_p + g)/\partial v_i$ equal to zero yields
\[ 0 = \gamma_i + 2\delta_i v_i + O(2) =: H_i, \quad 1 \leq i \leq s. \]

As before, we see
\[ \partial(H_1, \ldots, H_s)/\partial (v_1, \ldots, v_s) \neq 0. \]
Therefore, the implicit function theorem implies that there is only a single point solution, which is a nondegenerate critical point of $\ell_p + g$ in a small neighborhood of $x$ in $W$, which we can thus ignore.

Recall the local finiteness property in [4] that holds the key for proving that a taut hypersurface is real algebraic. We denote by $G$ the subset of $M$ where the multiplicities of principal values are locally constant, and by $G^c$ its complement in $M$.

**Definition 4.** A connected Dupin hypersurface $M$ of $S^n$ has the local finiteness property if there is a subset $S \subset G^c$, closed in $M$, such that $S$ disconnects $M$ into only a finite number of connected components, and for each point $x \in G^c \setminus S$, there is an open neighborhood $O$ of $x$ in $M$ such that $O \cap G$ contains a finite number of connected open sets whose union is dense in $O$.

It suffices to establish that $G$ satisfies the local finiteness property for $M$ to be real algebraic [4, Theorem 8]. We begin with a convenient lemma. Recall the global minimum or maximum level set of a height function on $M$ is called a top set. Setting $j = 0$ in (1.1) we see a top set is always connected.

**Lemma 5.** Let $T_i$ be a sequence of top sets of dimension $l$ at $q_i$ in the taut hypersurface $M$. Suppose $T_i$ converge to a top set $T$ of dimension $m$ at $p$. Then $H_l(T) \neq 0$.

**Proof.** First off, the top-dimensional homology of a top set of $M$ is nonzero. This follows from the Poincare duality (with $\mathbb{Z}_2$ coefficients) and that a top set is connected.

Now let $W$ be a tubular neighborhood of the top set $T$ at $p$ so small that $T$ is the only critical set in it. Let $j$ be so large that a tubular neighborhood $W_j$ of the top set $T_j$, containing only $T_j$, is brought to lie inside $W$. Then by (1.2)

$$H_k(T_j) \to H_k(T)$$

is an injection for all $k$. It follows that $H_l(T)$ is nonzero by what is said in the preceding paragraph. \qed

Returning to establishing the local finiteness property, let $S \subset G^c$ be the set of points where the principal multiplicities are $(1, \dim (M) - 1)$ or $(\dim (M) - 1, 1)$. The set is closed; or else a boundary point of which would assume the single principal multiplicity $(\dim (M))$ so that $M$ would be a sphere. $S$ must be a subset of $G^c$. This is because if multiplicities $(1, \dim (M) - 1)$ exist on an connected open set $O \subset G$, let $p_i \in O$ be a sequence which converges to $p$ on the boundary of $O$. The multiplicities at $p$ must remain to be $(1, \dim (M))$, or else it
would drop to the single multiplicity \((\dim(M))\). On the other hand, there must be a sequence \(q_i\) of points converging to \(p\) with fixed multiplicities \((\cdots, l)\) where \(l < \dim(M) - 1\). Therefore, on the one hand, the curvature surface \(S_i\) at \(p_i\) with principal multiplicity \(\dim(M) - 1\), which is a top set sphere of dimension \(\dim(M) - 1\), converges to the top set curvature sphere at \(p\), which is also a sphere \(S_p\) of dimension \(\dim(M) - 1\). This is because each \(S_i\) is cut out from its curvature sphere by a unique hyperplane \(L_i\) in the ambient Euclidean space, so that the limiting hyperplane also cuts out a sphere, which is \(S_p\), from the limiting curvature sphere. On the other hand, at \(q_i\) the curvature surface \(T_i\) with principal multiplicity \(l\) is a top set as well, and so by Lemma 5 the \(l\)-dimensional homology in \(S_p\) is nontrivial, which is absurd.

We next show that \(S\) disconnects \(M\) into only finitely many components. Recall the following definition in [4].

**Definition 6.** For each natural number \(m\) we define \((U^*_m)^+\) to be the collection of all \(x \in M\) for which there is a \(t > 0\) such that \((x, t)\) is a regular point of the normal exponential map

\[
E : (x, t) \mapsto \cos(t)x + \sin(t)n
\]

and such that the spherical distance function \(d_y\), where \(y = E(x, t)\), has index \(m\) at \(x\).

We showed in Corollary 20 of [4] that \((U^*_m)^+\) has a finite number of connected components for all \(m\).

**Remark 7.** The + sign in \((U^*_m)^+\) is merely to indicate that we traverse in the positive \(n\) direction, which we have agreed to undertake earlier.

Consider \(A_m : = (U^*_m)^+\) for \(m = 1, \cdots, \dim(M) - 1\). Let \(B : = \bigcup_{m=2}^{\dim(M)-1} A_m\) and \(A : = A_1\). Then it is readily checked that \(M = A \cup B\) and furthermore \(C : = A \cap B\) is exactly \(A\) with points of multiplicities \((1, \dim(M) - 1)\) removed. Therefore, the Mayer-Vietoris sequence

\[
0 \to H^0(M) \to H^0(A) \oplus H^0(B) \to H^0(C) \to H^1(M) \to \cdots
\]

establishes that \(C\) has finitely many components, which is what we are after.

Now let \(x \in G^c \setminus S\) and let \(Z\) through \(x\) be a critical submanifold with focal point \(p\). By the nature of \(S\) we know that

\[
\dim(Z) \leq \dim(M) - 2;
\]

in particular, \(Z\) does not disconnect \(M\). From this point onward we diversify into two cases.
Case 1. None of the curvature spheres of $\ell_p + g$ contain $Z$.

This means that $Z$ is not a level set of $g$ so that $g$ cuts $Z$ in proper taut submanifolds. Let $I$ be the index range such that
\begin{equation}
(2.1) \quad p = f_a(x), \forall a \in I.
\end{equation}
Let $W$ be a neck of $Z$. Let $O \subset W$ around $x$ be an open ball. The set
$$
\mathcal{F}_O := \cup_{a \in I} f_a(O)
$$
is a connected set of focal points around the focal point $p$.

We pick the open ball $O$ so small that any critical submanifold of $\ell_p + g = \ell_q$, for focal points $q \in \mathcal{F}_O$, lies completely in $W$ when its intersection with $O$ is not empty. (From now on we identify an element $q$ in $\mathcal{F}_O$ with the corresponding $g$ interchangeably.) Proposition 3 ensures that these critical submanifolds of $\ell_q$ on $W$ are all graphs over the corresponding critical submanifolds $Z_g$ that $g$ cut out on $Z$.

Consider the incidence space $\mathcal{I} \subset \mathcal{F}_O \times W \subset S^n \times S^n$ given by
$$
\mathcal{I} := \{(g, z) : z \in \text{a critical submanifolds of } \ell_p + g \text{ in } W,
\text{and } \dim (Z_h) \text{ is not locally constant for } h \text{ around } g\}.
$$

Let
$$
\Pi : S^n \times S^n \to S^n
$$
be the standard projection onto the second factor. Then
\begin{equation}
(2.2) \quad W \cap (\mathcal{G}^c \setminus Z) = \Pi(\mathcal{I}).
\end{equation}
Note that $\Pi|_{\mathcal{I}}$ is an open finite (hence proper) map; the finiteness is because through each point in $M$ there are only at most $\dim (M)$ worth of critical submanifolds, while the openness follows from that of $\Pi$.

The following lemma, based on our inductive hypothesis, makes the structure of $\mathcal{I}$ clear.

**Lemma 8.** $\mathcal{I}$ is a piecewise smooth simplicial complex of dimension at most $\dim (M) - 1$.

**Proof.** Since $Z \subset S^n$ is algebraic by the inductive hypothesis, the set $\cup \mathcal{N}^{\circ}$ of unit normals $\xi$ of $Z$ at which the shape operator $S_\xi$ has multiplicity change is semialgebraic. This can be seen as follows. Let $\dim (M) = s$ and let $(y, \xi) \in Z \times S^{n-s-1}$ parametrize the unit normal bundle of $Z$. The characteristic polynomial of $S_\xi$ is of the form
$$
\lambda^s + a_{s-1}\lambda^{s-1} + \cdots + a_1\lambda + a_0,
$$
where $a_1, \cdots, a_{s-1}$ are polynomials in the zero jet of $\xi$ and the second jets of $y$; hence they are Nash functions. By the slicing theorem [2, p. 30], $Z \times S^{n-t-1}$ is decomposed into finitely many disjoint semialgebraic
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sets $A_1, \ldots, A_m$, where each $A_i$ is equipped with semialgebraic functions
$f_{i1} < \cdots < f_{ii}$ that solve the characteristic polynomial. Where
multiplicities are not locally constant occurs at some $A_1, \ldots, A_m$ whose
dimensions are lower than $n - 1$, the dimension of $Z \times S^{n-1-s}$.

Now in view of Corollary 2, for a unit normal $\xi$ to $Z$, we let $q_1^\xi, q_2^\xi, \ldots,$
and $q_{dim(\xi)}$ be the focal point of the curvature surface through the base
point of $\xi$ corresponding to the principal curvature function
$\lambda^1(\xi), \ldots, \lambda^{dim(\xi)}(\xi)$ of $S_\xi$, respectively. The remark following Corollary 2
gives the focal maps $g_1, g_2, \ldots, g_{dim(\xi)}$ that send $\xi$ to the respective focal
points; by the algebraic nature of $Z$, all these maps are semialgebraic.

Consider the semialgebraic set $X \subset UN^o \times S^n \times S^n$ defined by

$$X := \{(\xi, q, r) : q = g_j^\xi(\xi) \text{ for some } j; r \text{ belongs a critical set of } Z \text{ of the}
\text{height function } \ell_q \text{ centered at } q\}.$$  

Due to the nature of all these defining functions, $X$ is semialgebraic.
(For instance, critical submanifolds are obtained by setting the first
derivative of the height function equal to zero on $Z$, which is a semialgebraic process.) Let $pr : UN^o \times S^n \times S^n \to S^n \times S^n$ be the standard
projection, and let $J := pr(X)$. The set $J$ is also semialgebraic.

We now estimate the dimension of $J$. Consider the the map

$$PR := \Pi|_J.$$  

It is readily seen that $PR : J \to Z$. For a fixed $z$ in the image of $PR$, the
preimage $PR^{-1}(z)$ consists of the focal points that come from the
$\xi \in UN^o$ where the base point of $\xi$ is $z$. At $z$, the eigenvalue problem
is an algebraic one; therefore, the set $S$ of $\xi$ based at $z$ where principal
multiplicities is not locally constant is a subvariety of the unit normal
sphere at $z$ of dimension at most $n - dim(Z) - 2$. Each $\xi$ in $S$ gives
rise to at most $dim(Z)$ worth of taut submanifolds through $z$, and vice versa, whose focal points are the ones in $PR^{-1}(z)$. Therefore,

$$dim(PR^{-1}(z)) \leq n - dim(Z) - 2.$$  

As a result, as $z$ varies in $Z$

$$dim(J) \leq n - dim(Z) - 2 + dim(Z) = dim(M) - 1.$$  

Since a semialgebraic set assumes a triangulation of semialgebraic simplicial complexes [2 p. 217], the structure of $J$ is clear. Consider the map

$$F : I \to J \text{ given by}$$

$$F : (g, z) \to (g, \pi(z)),$$

where $\pi$ is given in Proposition 3. The preimage of each point is finite
with cardinality at most $\beta(M, Z_2)$ between the two spaces with the
naturally induced metrics. Hence, $F$ is a finite covering map, since for a fixed $g$ the map $\pi$ maps a critical manifold of $\ell_p + g$ to $Z_g$ diffeomorphically. As a consequence $I$ inherits from $J$ a piecewise smooth triangulation of dimension $\dim(M) - 1$ sitting in $S^n \times S^n$. In fact we can work our way down the skeletons of the simplicial complex dimension by dimension. Each open face of the skeleton is defined by a finite set of polynomial functions $H < 0$, so that the pullback maps $H \circ F < 0$ define the corresponding open face for $I$.

Since the natural projection $\Pi : S^n \times S^n \to S^n$ into the second slot is an open finite map when restricted to $I$ as mentioned earlier, we see that at $x \in Z$ with preimages $x_1, \ldots, x_k \in I$, the projection $\Pi$ sends $k$ disjoint piecewise smooth (local) finite simplicial complexes $C_1, \ldots, C_k$ (of dimension at most $\dim(M) - 1$) around $x_1, \ldots, x_k$, respectively, to $x \in S^n$. Over each $C_j$, the differential $d\Pi$ is not defined over the skeletons of dimension $\leq \dim(M) - 2$; call this set $K_j$, which is a rectifiable set [6, p. 251]. Hence by the general area-coarea formula [6, p. 258]

$$\mathcal{H} \cdot \dim(\Pi(K_j)) \leq \dim(M) - 2$$

since $\Pi_I$ is a finite map; here $\mathcal{H} \cdot \dim$ denotes the Hausdorff dimension. On the other hand, $d\Pi$ is defined over the $(\dim(M) - 1)$-dimensional open faces $F_{jl}$ of $C_j$. By Federer’s version of Sard’s theorem [6, p. 316], the critical value set $\Theta_{jl}$ of $\Pi$ over $F_{jl}$ satisfies

$$\mathcal{H}^{\dim(M) - 1}(\Theta_{jl}) = 0,$$

where $\mathcal{H}^\nu$ denotes the Hausdorff $\nu$-dimensional measure. Therefore,

$$\mathcal{H}^{\dim(M) - 1}(\Pi(K_j \cup_l F_{jl})) = 0,$$

which implies that $\Pi(K_j \cup_l F_{jl})$ does not disconnect $M$ [11, p. 269].

Case 2. There are some $g$ such that $\ell_p + g$ contain $Z$.

This means $Z$ is contained in a level set for such $g$. Suppose $Z$ is contained in a critical submanifold of $g$. Then by Corollary 2 the height functions $\ell_p$ and $\ell_p + g = \ell_q$ share the same center of the curvature sphere through $Z$, so that it must be that $p = q$, which is not the case. Therefore, all points of $Z$ are regular points of $g$. Similar to the equations following Proposition 3 we have
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\[ \ell_p = \sum_{j=1}^{n-1-s} \alpha_j u_j^2 + O(3), \]

\[ g = h(u) + O(3); \]

That \( g \) has no \( v \) terms is because \( g(Z) \) is a constant. Analogous analysis as before shows that \( u_1, \ldots, u_{n-1-s} \) are functions of \( v_1, \ldots, v_s \), so that the critical manifolds of \( \ell + g \) are graphs over \( Z \).

In fact, we can understand all these \( g \) explicitly. Let \( S^l \) be the smallest sphere containing \( Z \). It is more convenient to view what goes on in \( \mathbb{R}^n \) when we place the pole of the stereographic projection on the \( S^l \) containing \( Z \). Then we are looking at an \( \mathbb{R}^l \), which we may assume is the standard one contained in \( \mathbb{R}^n \), in which \( Z \) sits. Let \( E \simeq \mathbb{R}^{n-l} \) be the orthogonal complement of the \( \mathbb{R}^l \). Any \( \mathbb{R}^{n-l-1} \) in \( E \) gives rise to an \( \mathbb{R}^{n-1} \) containing \( Z \), and vice versa. Back on the sphere, this means that we have an \( (n-l-1) \)-parameter family of \( S^{n-1} \) containing \( Z \).

The focal points of these \( S^{n-1} \) is an \( S^{n-l-1} \) on the equator. Now the critical sets of this \( (n-l-1) \)-parameter family of distance functions centered at the focal sphere \( S^{n-l-1} \) are all graphs over \( Z \) by the analysis following (2.3). It follows that we have a manifold structure \( Z \times S^{n-l-1} \) of dimension

\[ \dim(Z) + n - l - 1 \leq n - 2 = \dim(M) - 1 \]

if \( \dim(Z) < l \), in which case, the set of all these critical submanifolds locally disconnects \( M \) in at most two components. If on the other hand \( \dim(Z) = l \), then \( Z = S^l \). The manifold structure \( Z \times S^{n-l-1} \) of dimension \( n \) then fills up \( M \), which means there is no multiplicity change around \( Z \) so that \( Z \) can be ignored.

In summary, we have established the local finiteness property, and so \( M \) is algebraic when it is a hypersurface.

We now handle the case when \( M \) is a taut submanifold. It is more convenient to work in \( \mathbb{R}^n \). Let \( M_\epsilon \) be a tube over \( M \) of sufficiently small radius that \( M_\epsilon \) is an embedded hypersurface in \( \mathbb{R}^n \). Then \( M_\epsilon \) is a taut hypersurface [10], so that by the above \( M_\epsilon \) is algebraic. Consider the focal map \( F_\epsilon : M_\epsilon \to M \subset \mathbb{R}^n \) given by

\[ F_\epsilon(x) = x - \epsilon \xi, \]

where \( \xi \) is the outward field of unit normals to the tube \( M_\epsilon \). Any point of \( M_\epsilon \) has an open neighborhood \( U \) parametrized by an analytic algebraic map. The first derivatives of this parametrization are also analytic algebraic [2 p. 54], and thus the Gram-Schmidt process applied to these first derivatives and some constant non-tangential vector
produces the vector field $\xi$ and shows that $\xi$ is analytic algebraic on $U$. Hence $F_\varepsilon$ is analytic algebraic on $U$ and so the image $F_\varepsilon(U) \subset M$ is a semialgebraic subset of $\mathbb{R}^n$. Covering $M_\varepsilon$ by finitely many sets of this form $U$, we see that $M$, being the union of their images under $F_\varepsilon$, is a semialgebraic subset of $\mathbb{R}^n$. Then the Zariski closure $\overline{M}_{\text{zar}}$ of $M$ is an irreducible algebraic variety of the same dimension as $M$ and contains $M$.

The inductive procedure is thus completed.

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