STABILITY OF N-D TRANSMISSION PROBLEM IN VISCOELASTICITY WITH LOCALIZED KELVIN-VOIGT DAMPING UNDER DIFFERENT TYPES OF GEOMETRIC CONDITIONS

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Abstract. We investigate a multidimensional transmission problem between viscoelastic system with localized Kelvin-Voigt damping and purely elastic system under different types of geometric conditions. The Kelvin-Voigt damping is localized via non smooth coefficient in a suitable subdomain. It was shown that the discontinuity of the material coefficient along the interface elastic/viscoelastic can’t assure an exponential stability of the total system. So, it is natural to hope for a polynomial stability result under certain geometric conditions on the damping region. For this aim, using frequency domain approach combined with a new multiplier technic, we will establish a polynomial energy decay estimate of type $t^{-1}$ for smooth initial data. This result is obtained if either one of the geometric assumptions (A1) or (A2) holds (see below). Also, we establish a general polynomial energy decay estimate on a bounded domain where the geometric conditions on the localized viscoelastic damping are violated and we apply it on a square domain where the damping is localized in a vertical strip. However, the energy of our system decays polynomially of type $t^{-2/5}$ if the strip is localized near the boundary. Else, it’s of type $t^{-1/3}$. The main novelty in this paper is that the geometric situations covered here are richer and less restrictive than those considered in [31], [28], [19] and include in particular an example where the damping region is localized faraway from the boundary. Note that part of the results of this paper was announced in [22].

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1. Introduction. Local viscoelastic damping is a natural phenomena of bodies arising from a solid that have one part made of viscoelastic material, and the other made of elastic material. Let $\Omega \subset \mathbb{R}^N$ be a nonempty bounded open set with Lipschitz boundary $\Gamma$. We consider the wave equation with locally distributed Kelvin-Voigt type damping given in the following equation:

$$
\begin{cases}
\rho(x)u_{tt}(x,t) - \text{div}(a(x)\nabla u + b(x)\nabla u_t) = 0 & \text{in } \Omega \times \mathbb{R}^+,
\quad u(x,t) = 0 & \text{on } \Gamma \times \mathbb{R}^+,
(u(x,0), u_t(x,0)) = (u_0(x), u_1(x)) & \text{in } \Omega,
\end{cases}
$$

(1)

where the coefficient functions $\rho, a, b \in L^\infty(\Omega)$ and we assume that $\rho(x) \geq \rho_0 > 0$, $a(x) \geq a_0 > 0$, $b(x) \geq 0$, $\forall x \in \Omega$.

In 1988, F. Huang proved that when the Kelvin-Voigt damping $\text{div}(b(x)\nabla u_t)$ is globally distributed, i.e. $b(x) \geq b_0 > 0$ for almost every $x$ in $\Omega$, the corresponding semigroup of System (1) is not only exponentially stable, but also is analytic (see [11]). Thus, Kelvin-Voigt damping is stronger than the viscous damping $b(x)u_t$ in this case. Indeed, in [13], it was proved that the semigroup corresponding to the system of wave equations with global viscous damping is exponentially stable but not analytic. However, the exponential stability is still true even if the viscous damping is localized; via a smooth or a non smooth damping coefficient, in a suitable subdomain satisfying the Geometric Control Condition (GCC in short) introduced by C. Bardos, G. Lebeau and J. Rauch in [3] (see also [13] and Definition 3.2 below). Nevertheless, when viscoelastic damping is distributed locally, the situation is more delicate and such comparison between viscous and viscoelastic damping is not valid anymore. In fact, in 1998, K. Liu and Z. Liu considered a one-dimensional wave equation with Kelvin-Voigt damping distributed locally on any subinterval of the region occupied by the beam, where the damping coefficient is the characteristic function of the subinterval. They proved that the semigroup associated with the equation for the transversal motion of the beam is exponentially stable, although the semigroung associated with the equation for the longitudinal motion of the beam is not (see [16]). This shows that Kelvin-Voigt damping does not obey the GCC. This surprising result, due to the discontinuity of the materials and the unboundedness of viscoelastic damping, motivated the study of elastic system with local Kelvin-Voigt damping. Later, in the one-dimensional case, it was found that the smoothness of the damping coefficient at the interface is a critical factor for the stability and the regularity of the solutions (see [9, 17, 18, 20, 21, 29]). However, there are only a small number of publications on the corresponding $N$-dimensional case. In 2006, K. Liu and B. Rao considered this problem in the $N$-dimensional space where the damping region is a neighborhood (in $\Omega$) of the entire boundary $\Gamma$ (see [19]). They proved that the energy of the system goes exponentially to zero as $t$ goes to infinity for all usual initial data by assuming that the damping coefficient $b$ satisfies $b \in C^{1,1}(\overline{\Omega})$, $\Delta b \in L^\infty(\Omega)$ and $|\nabla b(x)|^2 \leq M_0 b(x)$ for almost every $x$ in $\Omega$, where $M_0$ is a positive constant. In 2012, L. Tebou studied the stabilization of the wave equation with Kelvin-Voigt damping (see [28]). He established polynomial energy decay of type $t^{-1}$ provided that the damping region is localized and verifies the Piecewise Multiplier Geometric Condition (PMGC in short) introduced by K. Liu in [15] (see Definition 3.1 below). Also, in 2016, under the same regularity assumptions imposed by K. Liu and B. Rao on the Kelvin-Voigt damping coefficient $b$, S. Nicaise and C. Pignotti established the exponential stability of the wave
equation with local Kelvin-Voigt damping localized around a part of the boundary and an extra boundary damping with time delay where they added an appropriate geometric condition (see Section 3.2 condition (Q4) in [23]). Later on, in 2017, M. Cavalcanti, V. Cavalcanti and L. Tebou showed the exponential decay of the energy of a wave equation with two types of locally distributed mechanisms; a frictional damping and a Kelvin-Voigt type damping where the location of each damping is such that none of them alone is able to exponentially stabilize the system (see [8]). Under the condition that the damping region satisfies the PMGC geometric conditions, they proved that the energy of the system decays polynomially as type $t^{-1}$ in the absence of regularity of the Kelvin-Voigt damping coefficient $b$. However, they established exponential stability when this coefficient is smooth. Recently, in 2018, K. Ammari, F. Hassine and L. Robbianio considered a wave equation with Kelvin-Voigt damping localized in a subdomain $\omega$ faraway from the boundary without geometric conditions (see [1]). They established a logarithmic energy decay rate for smooth initial data. Finally, in 2018, Q. Zhang considered the wave equation with Kelvin-Voigt damping in a nonempty bounded convex domain $\Omega$ with partition $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ where the viscoelastic damping is localized in $\Omega_1$ (see [31]). Under the condition that the damping coefficient $b$ is non smooth, she established a polynomial energy decay rate of type $t^{-1}$ for smooth initial data in the following two cases:

**Case 1.** The damping region $\Omega_1$ is a neighborhood of the entire boundary $\Gamma$ of $\Omega$.

**Case 2.** The domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or $3$), $\partial \Omega_1$ and $\partial \Omega_2$ are either convex curvilinear polygons or curved plane polyhedron, the damping region $\Omega_1$ is a neighborhood of a part $\Gamma_1 \neq \emptyset$ of the boundary $\Gamma$ and $m(x) \cdot \nu_2 \leq 0$ where $m(x) = x - x_0$ for $x_0$ fixed in $\mathbb{R}^N$ ($N = 2, 3$) for all $x \in \Gamma_2 = \Gamma \setminus \Gamma_1$.

In conclusion, several important geometric situations are not covered by all previous cited papers. For example, in the case where the damped region $\{b > 0\}$ satisfies or does not satisfy the GCC condition (see for instance Fig. 1-c, Fig. 2, Fig. 3 and Fig. 4), the problem of the energy decay rate is still open. So, our aim is to answer this open problem.

In this paper, we consider the stabilization of the wave equation with Kelvin-Voigt damping in a bounded domain $\Omega \subset \mathbb{R}^N$. The damping is localized in $\Omega$ via non smooth coefficient. The energy of a solution $u$ of System (1) is given by

$$E(u, t) = \frac{1}{2} \int_{\Omega} (\rho(x)|u_t|^2 + a(x)|\nabla u|^2) \, dx.$$  

Then, a straightforward computation gives

$$\frac{d}{dt} E(u, t) = - \int_{\Omega} b(x)|\nabla u_t|^2 \, dx.$$ 

Thus, as $b$ is nonnegative implies that System (1) is dissipative in the sense that its energy is decreasing with respect to time $t$. Assume that $b \geq b_0 > 0$ in a nonempty open subset $\omega$ of $\Omega$ (see condition (LA) below), then the decay is strict and $E(u, t)$ goes to zero as $t$ goes to infinity (see Subsection 2.2). The question is then to know, under the localization condition (LA), what is the energy decay rate under geometric conditions satisfied by $\omega$. First, we establish a polynomial energy decay estimate of type $t^{-1}$ for smooth initial data provided that the damping region $\omega$ satisfies the Geometric Control Condition GCC and $\text{meas}(\overline{\omega} \cup \Gamma) > 0$ (condition...
(A1) below) or \( \omega \) satisfies Strictly Geometric Control Condition SGCC (condition (A2) below). Second, in the case where \( \omega \) does not satisfy the GCC condition, i.e. in the presence of trapped rays that do not meet the damped region \( \omega \), we focus on the 2-dimensional square. We prove that the energy of smooth initial data decays polynomially like \( t^{-2/\ell} \), where \( \ell = 6 \) if condition (LC1) holds and \( \ell = 5 \) if condition (LC2) holds (see below). The frequency domain approach and new multiplier technics are employed. The results of Theorem 3.4 and Theorem 4.1 are new. Indeed, the geometric situations covered by these theorems are richer than that considered in all previous literature (see for instance [31], [28], [19]) and include in particular an example where the damping region is faraway from the boundary and an example where the damping region does not satisfy the GCC condition.

This paper is organized as follows: In Section 2, we study the Well-Posedness and the strong stability of System (1). Section 3 is devoted to study the energy decay rate of System (1) under different types of geometric conditions. We prove that the energy of our system has a polynomial decay rate of type \( t^{-1} \) (see Theorem 3.4).

Finally, in section 4, we study the stabilization of a wave equation with localized Kelvin-Voigt damping in the absence of GCC, and we obtain a general polynomial decay rate of the energy. In particular, we give applications on a bounded square with internal Kelvin-Voigt damping localized in a vertical strip which does not verify any geometric condition. We establish a polynomial decay rate of the energy of type \( t^{-1/3} \) or \( t^{-2/5} \) under conditions (LC1) or (LC2) respectively.

2. Well-Posedness and strong stability. This section is devoted to the study of existence, uniqueness and asymptotic behavior of the solution of System (1).

2.1. Well-Posedness of the problem. In this part, by using semigroup theory, we give the Well-Posedness results for Problem (1). For this aim, we introduce the Hilbert energy space \( \mathcal{H} \) by

\[
\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega),
\]

which is endowed with the usual inner product

\[
(U, \tilde{U})_H = \int_{\Omega} (a \nabla u \cdot \nabla \tilde{u} + \rho \tilde{v} \tilde{v}) \, dx,
\]

where \( U = (u, v) \in \mathcal{H} \) and \( \tilde{U} = (\tilde{u}, \tilde{v}) \in \mathcal{H} \). We use \( ||U||_{\mathcal{H}} \) to denote the corresponding norm. We next define the linear unbounded operator \( A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H} \) by

\[
D(A) = \{(u, v) \in \mathcal{H} \mid v \in H^1_0(\Omega), \text{ div}(a \nabla u + b \nabla v) \in L^2(\Omega)\}
\]

and

\[
A(u, v) = \left( v, \frac{1}{\rho} \text{ div}(a \nabla u + b \nabla v) \right), \quad \forall (u, v) \in D(A).
\]

If \( (u, u_t) \) is a regular solution of System (1), then we transform this system into the following evolution equation

\[
\begin{cases}
U_t = AU, \\
U(0) = U_0,
\end{cases}
\]

where \( U_0 = (u_0, u_1) \in \mathcal{H} \). According to [19], we have the following proposition:

**Proposition 1.** The unbounded linear operator \( A \) is m-dissipative in the energy space \( \mathcal{H} \).
Thanks to Lumer-Philips theorem (see [24]), we deduce that \( \mathcal{A} \) generates a \( C_0 \)-semigroup of contractions \( e^{t\mathcal{A}} \) in \( \mathcal{H} \) and therefore Problem (1) is well-posed. Then we have the following result:

**Theorem 2.1.** For any \( U_0 \in \mathcal{H} \), Problem (2) admits a unique weak solution

\[
U \in C^0(\mathbb{R}^+, \mathcal{H}).
\]

Moreover, if \( U_0 \in D(\mathcal{A}) \), then

\[
U \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).
\]

### 2.2. Strong stability

In this subsection we study the strong stability of System (1) in the sense that its energy \( E(u, t) \) converges to zero when \( t \) goes to infinity for all initial data in \( \mathcal{H} \). For this aim, we assume that there exist a nonempty open set \( \omega \subset \Omega \) and \( b_0 > 0 \) such that

\[
b(x) \geq b_0 \quad \forall x \in \omega.
\]

(LA)

Since the resolvent of \( \mathcal{A} \) is not compact, we have to discuss the full spectrum on the imaginary axis. We use a general criteria of Arendt-Batty in [2] which states that in a reflexive Banach space a \( C_0 \)-semigroup \( e^{t\mathcal{A}} \) is strongly stable if \( e^{t\mathcal{A}} \) is bounded, \( \mathcal{A} \) has no eigenvalues on the imaginary axis and \( \sigma(\mathcal{A}) \cap i\mathbb{R} \) is countable. So, we will prove the following result:

**Theorem 2.2.** (Strong Stability) Assume that \( a \in C^1(\overline{\Omega}) \) and condition (LA) holds.

Then, the semigroup of contractions \( e^{t\mathcal{A}} \) is strongly stable on the energy space \( \mathcal{H} \) in the sense that

\[
\lim_{t \to +\infty} \| e^{t\mathcal{A}} U_0 \|_{\mathcal{H}} = 0, \quad \forall U_0 \in \mathcal{H}.
\]

Before stating the proof of the above theorem, we need to recall the following unique continuation theorem (see [12]) :

**Theorem 2.3.** (Calderón theorem). Let \( \Omega \) be a connected open set in \( \mathbb{R}^n \) and let \( \omega \subset \Omega \) with \( \omega \neq \emptyset \). If \( u \in H^2(\Omega) \) satisfies \( Pu = g(u) \) in \( \Omega \) and \( u(x) = 0 \) in \( \omega \), then \( u \) vanishes in \( \Omega \).

For simplicity, we divide the proof of Theorem 2.2 into two Lemmas

**Lemma 2.4.** Under the same assumptions of Theorem 2.2, we have

\[
\ker(i\beta I - \mathcal{A}) = \{0\}, \quad \forall \beta \in \mathbb{R}.
\]

**Proof.** From Proposition 1, 0 \( \in \rho(\mathcal{A}) \). We still need to show the result for \( \beta \in \mathbb{R}^* \).

Suppose that there exist a real number \( \beta \neq 0 \) and \( U = (u, v) \in D(\mathcal{A}) \) such that

\[
\mathcal{A}U = i\beta U.
\]

(4)

It follows that

\[
0 = \Re(i\beta \|U\|_{\mathcal{H}}^2) = \Re(\mathcal{A}U, U)_{\mathcal{H}} = -\int_{\Omega} b(x)|\nabla v|^2 \, dx.
\]

(5)

This together with condition (LA) imply that

\[
b(x)\nabla v = 0 \text{ in } \Omega \quad \text{and} \quad \nabla v = 0 \text{ in } \omega.
\]

(6)

Now, writing Equation (4) in a detailed form and inserting (6) into it, we obtain the following system
\[ v = i\beta u \quad \text{in } \Omega, \] (7)
\[ \text{div}(a\nabla u) = i\rho \beta v \quad \text{in } \Omega. \] (8)

Deriving (7) with respect to \( x \) and taking into consideration (6), we get
\[ \nabla u = 0 \quad \text{in } \omega. \] (9)

After inserting (7) in (8), we get the following system
\[ \rho \beta^2 u + \text{div}(a\nabla u) = 0 \quad \text{in } \Omega. \] (10)

Now, let \( x_0 \in \omega \), so there exists \( R > 0 \) such that \( B(x_0, R) \subset \omega \). Consider a cut-off function \( \eta \in C^\infty_c(\Omega) \) such that \( 0 \leq \eta \leq 1 \) and
\[ \eta(x) = \begin{cases} 1 & \text{if } x \in B(x_0, R/2), \\ 0 & \text{if } x \in \Omega \setminus B(x_0, R). \end{cases} \]

Multiply Equation (10) by \( \eta u \) and use integration by parts over \( \Omega \) to obtain
\[ \beta^2 \int_\Omega \rho |\eta u|^2 dx - \int_\Omega a \nabla \eta \cdot \nabla u dx - \int_\Omega a |\nabla u|^2 dx = 0. \]

Using Equation (9) and the fact that \( B(x_0, R) \subset \omega, \beta \neq 0 \) and \( \rho \geq \rho_0 > 0 \) in the above equation, we obtain
\[ u = 0 \quad \text{in } B(x_0, R/2). \] (11)

So, we obtain the following system
\[ \begin{cases} \Delta u = -\frac{\rho a}{\alpha} \beta^2 u - \frac{1}{a} \nabla a \cdot \nabla u & \text{in } \Omega, \\ u = 0 & \text{in } B(x_0, R/2), \\ u = 0 & \text{on } \Gamma. \end{cases} \]

Thus, we define the following elliptic operator \( \mathcal{P} \) by
\[ \mathcal{P} : H^2(\mathcal{V}) \to L^2(\mathcal{V}) \]
\[ u \mapsto \Delta u \]

and the function \( g \) by
\[ g : L^2(\mathcal{V}) \to L^2(\mathcal{V}) \]
\[ u \mapsto -\frac{\rho a}{\alpha} \beta^2 u - \frac{1}{a} \nabla a \cdot \nabla u. \]

Using Theorem 2.3 and taking \( \omega = B(x_0, R/2) \), we get finally that \( u = 0 \) in \( \Omega \) and thus \( U = 0 \) in \( \Omega \). Hence, \( \ker(i\beta I - A) = 0 \) and the proof is thus complete. \( \square \)

**Lemma 2.5.** Under the same assumptions of Theorem 2.2, we have
\[ \text{R}(i\beta I - A) = \mathcal{H}, \quad \forall \beta \in \mathbb{R}. \]

**Proof.** Given \( F = (f, g) \in \mathcal{H} \), we solve equation
\[ (i\beta I - A)U = F. \] (12)

Equivalently, we consider the following system
\[ v = i\beta u - f, \] (13)
\[ \beta^2 u + \frac{1}{\rho} \text{div}(a\nabla u + i\beta b\nabla u) = -(g + i\beta f) + \frac{1}{\rho} \text{div}(b\nabla f). \] (14)

Now, define the linear operator \( \mathcal{L} : H^1_0(\Omega) \to H^{-1}(\Omega) \) by
\[
\mathcal{L}u = -\frac{1}{\rho}\text{div}(a\nabla u + i\beta b\nabla u).
\]

Using Lax-Milgram’s theorem (see [6]), it is easy to show that $\mathcal{L}$ is an isomorphism from $H^1_0(\Omega)$ onto $H^{-1}(\Omega)$. Then we transform (14) into the following form

\[
(\beta^2 I - \mathcal{L})u = -(g + i\beta f) + \frac{1}{\rho}\text{div}(b\nabla f).
\]

Since the operator $\mathcal{L}$ is an isomorphism from $H^1_0(\Omega)$ onto $H^{-1}(\Omega)$ and $I$ is a compact operator from $H^1_0(\Omega)$ onto $H^{-1}(\Omega)$, then using Fredholm’s Alternative theorem, problem (16) admits a unique solution in $H^1_0(\Omega)$ if and only if the operator $\beta^2 I - \mathcal{L}$ is injective. For that purpose, let $u \in \ker(\beta^2 I - \mathcal{L})$. Then, if we set $v = i\beta u$, we deduce that $U = (u, v) \in D(A)$ is a solution of

\[
(i\beta - A)U = 0.
\]

Using Lemma 2.4, we deduce that $u = v = 0$. This implies that equation (16) admits a unique solution $u \in H^1_0(\Omega)$ and $\text{div}(a\nabla u + i\beta b\nabla u - b\nabla f) = -(g + i\beta f) \in L^2(\Omega)$. By setting $v = i\lambda u - f$, we deduce that $U = (u, v) \in D(A)$ is the unique solution of equation (12) and the proof is thus complete. \hfill \Box

**Proof of Theorem 2.2.** Using Lemma 2.4, the operator $A$ has no pure imaginary eigenvalues and by Lemma 2.5, we have $\mathcal{R}(i\beta I - A) = \mathcal{H}$, for all $\beta \in \mathbb{R}$. Therefore, the closed graph theorem implies that $\sigma(A) \cap i\mathbb{R} = \emptyset$. Following Arendt-Batty (see [2]), the $C_0$-semi group of contractions $(e^{tA})_{t \geq 0}$ is strongly stable and the proof is complete.

### 3. Polynomial stability.

In this section, we study the energy decay rate of System (1). Indeed, if the material parameter $b$ is smooth enough at the interface, then the energy of System (1) decays exponentially to zero as $t$ goes to infinity under appropriate geometric conditions imposed on the damped region (see [19], [23], [28], [8]). However, Q. Zhang proved in [30] that the exponential decay fails in any geometry if the damping coefficient $b$ is discontinuous along the interface. Note that, in this case, the lack of exponential stability still holds even in the $1 - d$ case (see [9]). So, it is natural to hope for a polynomial stability result under some geometric considerations not covered previously which represents the main goal of this work. So, we start by recalling the Piecewise Multiplier Geometric condition (PMGC in short) introduced by K. Liu in [15] and the Geometric Control Condition (GCC in short) introduced by J. Rauch and M. Taylor in [26] for manifolds without boundaries and by C. Bardos, G. Lebeau and J. Rauch in [3] (see also [13]) for domains with boundaries. We also introduce a new geometric condition.

**Definition 3.1.** A subset $\omega$ satisfies the Piecewise Multiplier Geometric Condition if there exist $\Omega_j \subset \Omega$ having Lipschitz boundary $\Gamma_j$ and $x_j \in \mathbb{R}^N$, $j = 1, ..., J$ such that $\Omega_j \cap \Omega_i = \emptyset$ for $j \neq i$ and $\omega$ contains a neighborhood in $\Omega$ of the set

\[
\bigcup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \bigcup_{j=1}^J \Omega_j)
\]

where $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ and $\nu_j$ is the outward unit normal vector to $\Gamma_j$.

**Definition 3.2.** For a subset $\omega$ of $\Omega$ and $T > 0$, we shall say that $(\omega, T)$ satisfies the Geometric Control Condition if every geodesic traveling at speed one in $\Omega$ meets $\omega$ in time $t < T$. 

Remark 1. The PMGC is a generalization of the \(\Gamma\)-condition introduced by J-L. Lions in [14] and is much more restrictive than the GCC. For example, in Figure 1, we consider the case where \(\Omega\) is a disk and we draw three different subsets in \(\Omega\). The \(\Gamma\)-condition is only satisfied by \(\omega_0\). The PMGC is satisfied by \(\omega_0\) and \(\omega_1\). However, \(\omega_2\) doesn’t satisfy neither the PMGC nor the \(\Gamma\)-condition. Finally, the GCC is satisfied by the three different subsets of \(\Omega\).

We also introduce the following geometric condition.

Definition 3.3. For a subset \(\omega\) of \(\Omega\), we shall say that \(\omega\) satisfies Strictly the Geometric Control Condition (SGCC in short) if there exists an open subset \(\tilde{\omega}\) included strictly in \(\omega\) (i.e. \(\tilde{\omega} \subset \omega\)) and satisfying the GCC.

Remark 2. It is easy to see that, if \(\omega\) verifies the SGCC, then it verifies the GCC. The converse of this implication is false (see Fig. 1-c).

Remark 3. It is easy to see that, Figure 4 does not satisfy any geometry.

For the study of the energy decay rate we need the following geometric assumptions:

(A1) the open subset \(\omega\) verifies the GCC and \(\text{meas}(\tilde{\omega} \cap \Gamma) > 0\),

(A2) the open subset \(\omega\) verifies the SGCC.

There are several geometries that verify the previous assumptions. For example:

Fig. 1: Elastic-viscoelastic waves interaction models satisfying the assumption (A1)

(a) \(\omega_0\)        (b) \(\omega_1\)        (c) \(\omega_2\)

Fig. 2: A model satisfying assumption (A2)

Fig. 3: A model satisfying both (A1) and (A2)
We are now in position to state our main result of this part.

**Theorem 3.4.** Let \( a, \rho \in C^2(\Omega) \) and assume that the boundary \( \Gamma \) is of class \( C^3 \). Assume that condition \((LA)\) holds. Assume also assumption \((A1)\) or assumption \((A2)\) holds. Then there exists a constant \( C > 0 \) such that for all initial data \( U_0 \in D(A) \), the energy of System \((1)\) satisfies the following estimation

\[
E(U,t) \leq C t \| U_0 \|^2_{D(A)}, \quad \forall t > 0.
\]

\( (17) \)

**Proof.** Following Borichev and Tomilov [5] (see also [20], [4]), a \( C_0 \)-semigroup of contractions \( (e^{tA})_{t \geq 0} \) on a Hilbert space \( \mathcal{H} \) verifies \((17)\) if

\[
i \mathbb{R} \subset \rho(A) \quad \text{(S1)}
\]

and

\[
\limsup_{|\lambda| \to \infty} \frac{1}{|\lambda|^2} \|(i\lambda - A)^{-1}\|_{L(\mathcal{H})} < \infty \quad \text{(S2)}.
\]

Since the resolvent of the operator \( A \) is not compact in the energy space \( \mathcal{H} \) (see [19]) and \( 0 \in \rho(A) \), then to prove \( i \mathbb{R} \subset \rho(A) \) is equivalent to prove that \( i \beta I - A \) is bijective in the energy space \( \mathcal{H} \) for all \( \beta \in \mathbb{R}^* \). This last is proven in Subsection 2.2 according to a unique continuation theorem and Fredholm’s alternative. Then, we still need to prove condition \((S2)\). This is checked by using a contradiction argument. Indeed, suppose there exists \( \{ (\lambda_n, U_n := (u_n, v_n)) \}_{n \geq 1} \subset \mathbb{R}^*_+ \times D(A) \), such that

\[
\begin{align*}
\lambda_n & \to +\infty, \quad \|U_n\|_{\mathcal{H}} = 1, \\
\lambda_n^2 (i\lambda_n I - A)U_n &= (f_n, g_n) \to 0 \text{ in } \mathcal{H}.
\end{align*}
\]

\( (18) \)

\( (19) \)

Our aim is to show that \( \|(u_n, v_n)\|_{\mathcal{H}} \to 0 \). This condition permits to conclude a contradiction with \((18)\). By detailing equation \((19)\), we get the following system

\[
\begin{align*}
i \lambda_n u_n - v_n &= \lambda_n^{-2} f_n \quad \text{in } H^1_0(\Omega), \\
i \rho \lambda_n v_n - \text{div}(a \nabla u_n + b \nabla v_n) &= \lambda_n^{-2} g_n \quad \text{in } L^2(\Omega).
\end{align*}
\]

\( (20) \)

\( (21) \)

The proof of our theorem is divided into several lemmas.
Lemma 3.5. Assume that assumption (LA) holds. Then the solution \((u_n, v_n) \in D(\mathcal{A})\) of Equations (20)-(21) satisfies the following asymptotic behavior estimations

\[
||u_n||_{L^2(\Omega)} = O(\lambda_n^{-1}), \quad (22)
\]

\[
||\nabla v_n||_{L^2(\omega)} = o(\lambda_n^{-1}). \quad (23)
\]

Proof. Using Equations (18) and (20), we deduce directly the first estimation (22). Now, taking the inner product of (19) with \(U_n = (u_n, v_n)\) in \(H\), then using the fact that \(U_n\) is uniformly bounded in \(H\), we get

\[
\int_\Omega b(x)|\nabla v_n|^2dx = \Re \langle (i\lambda_n I - \mathcal{A})U_n, U_n \rangle_{\mathcal{H}} = o(\lambda_n^{-2}).
\]

It follows from the localization condition (LA) that

\[
\int_\omega |\nabla v_n|^2 = o(\lambda_n^{-2}).
\]

\[\square\]

Lemma 3.6. Assume that assumptions (LA) and (A1) hold. Then the solution \((u_n, v_n) \in D(\mathcal{A})\) of Equations (20)-(21) satisfies the following asymptotic behavior estimation

\[
\int_\omega |u_n|^2 = o(\lambda_n^{-2}). \quad (24)
\]

Proof. Since assumption (A1) holds, then using Poincaré’s inequality and Equation (23), we obtain

\[
\lambda_n^2 \int_\omega |v_n|^2dx = o(1). \quad (25)
\]

Multiplying Equation (20) by \(-i\lambda_n \bar{v}_n\) and integrating over \(\omega\), we get

\[
\lambda_n^2 \int_\omega |u_n|^2dx + i\lambda_n \int_\omega v_n \bar{v}_ndx = -i\lambda_n^{-1} \int_\omega \bar{v}_n f_n dx.
\]

Using estimations (22), (25) and the fact that \(f_n\) converges to zero in \(L^2(\Omega)\), we deduce from the above equation that

\[
\lambda_n^2 \int_\omega |u_n|^2dx = o(1).
\]

The desired estimation (24) is established. \[\square\]

Lemma 3.7. Assume that assumptions (LA) and (A2) hold. Then the solution \((u_n, v_n) \in D(\mathcal{A})\) of Equations (20)-(21) satisfies the following asymptotic behavior estimation

\[
\int_\omega |u_n|^2 = o(\lambda_n^{-2}). \quad (26)
\]

Proof. Since assumption (A2) holds, then there exists a nonempty open subset \(\tilde{\omega}\) of \(\omega\) such that \(\tilde{\omega} \subset \omega\). Hence, \(\tilde{\omega} \cap (\Omega \setminus \omega) = \emptyset\), and thus we define the function \(\eta \in C_\infty(\mathbb{R}^N)\) by \(0 \leq \eta(x) \leq 1\) and

\[
\eta(x) = \begin{cases} 
0, & \text{if } x \in \Omega \setminus \omega, \\
1, & \text{if } x \in \tilde{\omega}.
\end{cases}
\]
Multiplying Equation (21) by \(\eta \widetilde{\varphi}_n\) and integrating over \(\Omega\), we obtain
\[
i\lambda_n \int_\Omega \rho |v_n|^2 \, dx + \int_\Omega (a\nabla u_n + b \nabla v_n) \cdot (\eta \nabla \varphi_n + \nabla \eta) \, dx = \lambda_n^{-2} \int_\Omega \rho \eta g_n \varphi_n \, dx.
\]
Using estimations (22), (23), then taking into consideration that \(\text{Supp} \eta \subset \omega\) and the fact that \(v_n, \nabla u_n\) are uniformly bounded in \(L^2(\Omega)\) and \(g_n\) converges to zero in \(L^2(\Omega)\), we deduce that
\[
\begin{align*}
&\int_\Omega a \eta \nabla u_n \cdot \nabla \varphi_n \, dx \leq \|a\|_\infty \left( \int_\Omega \eta |\nabla u_n|^2 \, dx \right)^{1/2} \left( \int_\Omega \eta |\nabla v_n|^2 \, dx \right)^{1/2} = o \left( \lambda_n^{-1} \right), \\
&\int_\Omega b \eta |\nabla v_n|^2 \, dx = \leq \|\eta\|_\infty \int_\Omega b |\nabla v_n|^2 \, dx = o \left( \lambda_n^{-2} \right), \\
&\int_\Omega b \nabla u_n \cdot \nabla \eta \, dx \leq \|\nabla \eta\|_\infty \left( \int_\Omega |\nabla u_n|^2 \, dx \right)^{1/2} \left( \int_\Omega |\nabla v_n|^2 \, dx \right)^{1/2} = o \left( \lambda_n^{-1} \right), \\
&\int_\Omega \rho \eta g_n \varphi_n \, dx \leq \|\rho\|_\infty \|\eta\|_\infty \left( \int_\Omega |g_n|^2 \, dx \right)^{1/2} \left( \int_\Omega |v_n|^2 \, dx \right)^{1/2} = o(1).
\end{align*}
\]
It follows that
\[
i\lambda_n \int_\Omega \rho |v_n|^2 \, dx + \int_\Omega a \nabla u_n \cdot \nabla \eta \, dx = o \left( \lambda_n^{-1} \right). \quad (27)
\]
Now, taking the imaginary part and applying Young’s inequality on the second term of (27), we deduce that
\[
\lambda_n \int_\Omega \rho |v_n|^2 \, dx \leq \frac{\|a\nabla \eta\|_\infty^2}{2} \int_\Omega |\nabla u_n|^2 \, dx + \frac{\|a\nabla \eta\|_\infty^2}{2} \int_\Omega |v_n|^2 \, dx + o \left( \lambda_n^{-1} \right).
\]
Using the fact that \((u_n, v_n)\) is uniformly bounded in \(H\) and \(\rho(x) \geq \rho_0 > 0\), we get
\[
\int_\Omega \eta |v_n|^2 \, dx = o(1).
\]
Using the definition of \(\eta\), we deduce that
\[
\int_\Omega |v_n|^2 \, dx = o(1).
\]
Finally, multiply Equation (20) by \(i\eta \lambda_n \varphi_n\) to get the desired estimation (26).

**Lemma 3.8.** Let \(\mathcal{U}\) be a nonempty open subset of \(\Omega\) satisfying the GCC such that \(\mathcal{U} \subseteq \omega\). Then, for any \(\lambda_n \in \mathbb{R}_+^*\), the solution \(\varphi_n \in H^2(\Omega) \cap H^1_0(\Omega)\) of system
\[
\begin{align*}
\left\{ \begin{array}{l}
\rho \lambda_n^2 \varphi_n + \text{div}(a \nabla \varphi_n) - i \lambda_n (1_{\mathcal{U}} b)(x) \varphi_n = u_n \quad \text{in} \quad \Omega, \\
\varphi_n = 0 \quad \text{on} \quad \Gamma,
\end{array} \right.
\end{align*}
\]
satisfies the following estimation
\[
||\lambda_n \varphi_n||_{L^2(\Omega)} + ||\nabla \varphi_n||_{L^2(\Omega)} \leq M ||u_n||_{L^2(\Omega)} \quad (29)
\]
where \(M\) is positive constant independent of \(n\).

**Proof.** We consider the following auxiliary problem, namely the wave equation with local viscous damping:
\[
\begin{align*}
\left\{ \begin{array}{l}
\rho(x) \varphi_{tt}(x,t) - \text{div}(a(x) \nabla \varphi) + (1_{\mathcal{U}} b)(x) \varphi_t = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\
\varphi = 0 \quad \text{on} \quad \Gamma,
\end{array} \right.
\end{align*}
\]
where \(a, \rho \in C^2(\overline{\Omega})\). Since \(\mathcal{U}\) satisfies the GCC, then the wave equation with local viscous damping (30) is exponentially stable (see [7]) in the associated energy space.
$\mathcal{H}_{1,a} = H_0^1(\Omega) \times L^2(\Omega)$. So, following Huang [10] and Pruss [25], the resolvent set of its associated operator $A_a$ defined by

$$D(A_a) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega),$$

$$A_a(\varphi, \psi) = \left(\psi, \frac{1}{\rho}(\text{div}(a\nabla \varphi) - (\mathbb{1}_u b)(x)\psi)\right)$$

contains $i\mathbb{R}$ and the resolvent $(i\lambda_n I - A_a)^{-1}$ of $A_a$ is uniformly bounded on the imaginary axis. Consequently, there exists a positive constant $M > 0$ independent of $n$ such that

$$\sup_{\lambda_n \in \mathbb{R}} ||(i\lambda_n I - A_a)^{-1}||_{\mathcal{L}(\mathcal{H}_{1,a})} \leq M. \quad (31)$$

Now, since $u_n \in L^2(\Omega)$, then there exists a unique $(\varphi_n, \psi_n) \in D(A_a)$ solution of

$$(i\lambda_n I - A_a)(\varphi_n, \psi_n) = \left(0, -\frac{1}{\rho}u_n\right),$$

equivalently

$$i\lambda_n \varphi_n - \psi_n = 0,$$

$$i\lambda_n \rho \psi_n - \text{div}(a\nabla \varphi_n) + (\mathbb{1}_u b)(x)\psi_n = -u_n.$$

Finally, from (31), we deduce that

$$|| (\varphi_n, \psi_n) ||_{\mathcal{H}_{1,a}} \leq M ||u_n||_{L^2(\Omega)}$$

which gives the desired estimation. \hfill \square

Lemma 3.9. Assume that assumption (LA) and assumptions (A1) or (A2) hold. Then the solution $(u_n, v_n) \in D(A)$ of (20)–(21) satisfies the following asymptotic behavior estimation

$$\int_{\Omega} |\lambda_n u_n|^2 dx = o(1). \quad (32)$$

Proof. Case 1. Under assumptions (LA) and (A1), $\omega$ satisfies the GCC. So, we consider System (28) where $U \equiv \omega$ and $\varphi_n$ is its solution. Multiplying Equation (20) by $i\lambda_n^3 \rho \varphi_n$ and Equation (21) by $\lambda_n^3 \varphi_n$ and using Green’s formula to obtain

$$- \lambda_n^4 \int_{\Omega} \rho u_n \varphi_n dx + i\lambda_n^3 \int_{\Omega} \rho u_n \varphi_n dx = i\lambda_n \int_{\Omega} \rho \varphi_n f_n dx \quad (33)$$

and

$$i\lambda_n^3 \int_{\Omega} \rho u_n \varphi_n dx + \lambda_n^2 \int_{\Omega} (a\nabla u_n + b\nabla v_n) \cdot \nabla \varphi_n dx = \int_{\Omega} \rho \varphi_n g_n dx. \quad (34)$$

By using (22), (29) and the fact that $f_n \to 0$, $g_n \to 0$ in $L^2(\Omega)$, we get

$$i\lambda_n \int_{\Omega} \rho \varphi_n f_n dx = o(1) \quad \text{and} \quad \int_{\Omega} \rho \varphi_n g_n dx = o(1). \quad (35)$$

Now, taking the second term of the left hand side of (34), we obtain

$$\lambda_n^2 \int_{\Omega} (a\nabla u_n + b\nabla v_n) \cdot \nabla \varphi_n dx = \lambda_n^2 \int_{\Omega} a\nabla u_n \cdot \nabla \varphi_n dx + \lambda_n^2 \int_{\Omega} b\nabla v_n \cdot \nabla \varphi_n dx. \quad (36)$$

We have

$$\lambda_n^2 \int_{\Omega} b\nabla v_n \cdot \nabla \varphi_n dx \leq \lambda_n \left( \int_{\Omega} b|\nabla v_n|^2 dx \right)^{1/2} \lambda_n \sqrt{\|b\|_\infty} \left( \int_{\Omega} |\nabla \varphi_n|^2 dx \right)^{1/2}. $$
We continue by multiplying Estimation (29) by \( \lambda_n \) and taking into consideration Estimation (22), we obtain
\[
\| \varphi_n \|_{L^2(\Omega)} = O(\lambda_n^{-2}) \quad \text{and} \quad \| \nabla \varphi_n \|_{L^2(\Omega)} = O(\lambda_n^{-1}).
\] (37)

Using Estimations (23) and (37), we conclude that
\[
\| \varphi_n \|_{L^2(\Omega)} = O(\lambda_n^{-2} n) \quad \text{and} \quad \| \nabla \varphi_n \|_{L^2(\Omega)} = O(\lambda_n^{-1} n).
\] (37)

Applying Green’s formula on the first term of the right hand side of Equation (36) and taking into consideration (38), we get
\[
\lambda_n^2 \int_{\Omega} (a \nabla u_n + b \nabla v_n) \cdot \nabla \varphi_n \, dx = -\lambda_n^2 \int_{\Omega} \text{div}(a \nabla \varphi_n) u_n \, dx + o(1).
\] (39)

Now, adding (33) and (34) and using (35) and (39), we obtain
\[
\lambda_n^4 \int_{\Omega} \rho u_n \varphi_n \, dx + \lambda_n^2 \int_{\Omega} \text{div}(a \nabla \varphi_n) u_n \, dx = o(1).
\] (40)

By inserting the first equation of (28) in (40), we get
\[
i \lambda_n^3 \int_{\Omega} (\mathbb{1}_{\omega} b)(x) \varphi_n u_n \, dx = o(1).
\] (41)

We have
\[
\left| \lambda_n^3 \int_{\Omega} (\mathbb{1}_{\omega} b)(x) \varphi_n u_n \, dx \right| \leq \lambda_n^2 \sqrt{\| b \|_\infty} \left( \int_{\Omega} |\varphi_n|^2 \, dx \right)^{1/2} \lambda_n \left( \int_{\Omega} (\mathbb{1}_{\omega} b)(x) |u_n|^2 \, dx \right)^{1/2}.
\]

So, from (24) (or (26) for assumption (A2)) and (37), we deduce that
\[
i \lambda_n^3 \int_{\Omega} (\mathbb{1}_{\omega} b)(x) \varphi_n u_n \, dx = o(1)
\]
which together with (41) give the desired Estimation (32).

**Case 2.** Under assumptions (LA) and (A2), there exists \( \hat{\omega} \) contained strictly in \( \omega \) and satisfying the GCC. Similarly, we consider System (28) where \( \mathcal{U} \equiv \hat{\omega} \) and \( \varphi_n \) is its solution. Following the same technique as in Case 1, we get
\[
i \lambda_n^3 \int_{\Omega} (\mathbb{1}_{\hat{\omega}} b)(x) \varphi_n u_n \, dx = o(1)
\]
to continue in the same way and get Estimation (32). \( \square \)

**Lemma 3.10.** Assume that assumption (LA) and assumptions (A1) or (A2) hold. Then the solution \((u_n, v_n) \in D(A)\) of (20)-(21) satisfies the following asymptotic behavior estimation
\[
\int_{\Omega} |\nabla u_n|^2 \, dx = o(1).
\] (42)

**Proof.** Multiplying Equation (21) by \( \varphi_n \) and applying Green’s formula, we obtain
\[
i \lambda_n \int_{\Omega} \rho u_n \varphi_n \, dx + \int_{\Omega} (a \nabla u_n + b \nabla v_n) \cdot \nabla \varphi_n \, dx = \lambda_n^{-2} \int_{\Omega} \rho \varphi_n g_n \, dx.
\]
Taking into consideration (18), (23), (32) and the fact that $g_n$ converges to zero in $L^2(\Omega)$, we deduce that
\[ \int_{\Omega} a |\nabla u_n|^2 \, dx = o(1) \]
which together with the fact that $a(x) \geq a_0 > 0$ for almost every $x \in \Omega$ give the desired estimation (42).

**Proof of Theorem 3.4** It follows from (32) and (42) that $||U_n||_{\mathcal{H}} = o(1)$ which contradicts (18). Consequently, condition (S2) holds and the proof is thus complete.

**Remark 4.** i) The main result of Section 3 covers the stabilization over several geometries not taken into consideration in previous works; for example, Fig. 1-c, Fig. 2, Fig. 3, Fig. 4 and Fig. 6. In fact, the result of Theorem 3.4 generalizes that of [19], [28] and [31]. Indeed, the geometric situations covered by this theorem are richer than that considered in the previous references. For instance, in these literature mentioned, the damping is a neighborhood of the entire boundary, localized via smooth damping coefficient (see [19]) or a part of the boundary satisfying the PMGC and localized via smooth or nonsmooth coefficient (see [28]). Also, in [31], the author considered $\Omega$ to be a bounded convex domain, such that the damping region is either a neighborhood of the whole boundary or near a part of the boundary. However, in the second case, the partitions of $\Omega$: $\Omega_1$ and $\Omega_2$, are either convex curvilinear polygons or curved plane polyhedron. Also, the damping region $\Omega_1$ is a neighborhood of a part $\Gamma_1 \neq \emptyset$ of the boundary $\Gamma$ and satisfies the $\Gamma$- condition. The difference is that we obtain our result when the damping region is localized internally faraway from the boundary (see assumption (A2)) or internally having an intercept with the boundary (see assumption (A1)) and it obeys the GCC which is less restrictive than the PMGC or the $\Gamma$- condition. Also, in [31], it was necessary to have more viscoelastic wave than the elastic wave in order for the condition $(m \cdot \nu_2)|_{\partial \Omega} \leq 0$; where $m(x) = x - x_0$ such that $x_0$ is a fixed point in $\mathbb{R}^n$, $(n = 2, 3)$, to apply (see Fig. 5). In our case, we do not care about the previous condition and we get the result despite the quantity of the viscoelastic wave, whether it’s more or less than the elastic part (see Fig. 6). In addition, unlike the result of Theorem 4.1 in [31], our result holds for all $n \geq 2$ and for non-convex domains.

(ii) There is a relation between the geometric condition affecting the damping region and the smoothness of the boundary and regularity of the density and metric coefficients ($\rho$, $a$). Namely, if the $\Gamma$- condition applies, then it is enough to have Lipschitz boundary conditions and coefficients of class $C^1$. Although, knowing that the GCC is an optimal condition, so it costs more and thus needs a minimal regularity of $C^2$ coefficients and $C^3$ boundary (see [7]). Remark that Burq-Dehman-Le Rousseau have dropped the previous condition to a boundary smoothness of class $C^2$ and coefficients of class $C^1$ (Burq-Dehman-Le Rousseau; Control for wave equation with rough coefficients, oral communication).

(iii) It is unknown whether the polynomial decay rate obtained in (17) is optimal in the sense that, for any $\varepsilon > 0$, we can not expect the decay rate of type $t^{-1-\varepsilon}$ for all initial data $U_0 \in D(\mathcal{A})$. From our point of view, the energy decay rate (17) is not optimal, and we conjecture an optimal decay of type $t^{-2}$.
4. Polynomial stability in the absence of GCC. As we have seen in Section 3, the GCC condition (also the PMGC and the SGCC) is sufficient to obtain a polynomial energy decay rate. Thus, in this section, we are interested in studying the nature of the energy decay rate whenever the geometric conditions are violated. We consider the following auxiliary problem, namely the wave equation on a nonempty open bounded domain $\Omega$ of Lipschitz boundary $\Gamma$ with local viscous damping:

\[
\begin{cases}
\rho(x)\varphi_{tt}(x,t) - \text{div}(a\nabla \varphi) + (1_\omega b)(x)\varphi_t = 0 & \text{in } \Omega \times \mathbb{R}^+, \\
\varphi = 0 & \text{on } \Gamma,
\end{cases}
\]

where $\omega$ is an open subset of $\Omega$ not satisfying any geometry. Assume that the energy of the above wave equation with local viscous damping (43) decays polynomially as $t^{-2/\beta}$; $\beta > 0$, for smooth initial data, i.e. there exists $C > 0$ such that

\[
E(t) \leq \frac{C}{t^{2/\beta}}||U_0||_{D(A_0)}^2, \quad \forall t > 0.
\]

Now we are in position to state our main result.

**Theorem 4.1.** Assume that assumptions (LA) and (LE) hold. Then there exists a constant $C > 0$ such that for all initial data $U_0 \in D(A)$, the energy of System (1) satisfies the following estimation

\[
E(t) \leq \frac{C}{t^{1/\beta + 1}}||U_0||_{D(A)}^2, \quad \beta > 0, \quad \forall t > 0.
\]

As mentioned in the proof of Theorem 3.4, it is enough for the proof of Theorem 4.1, according to Borichev and Tomilov, to show that

\[
\sup_{\lambda \in \mathbb{R}} \| (i\lambda I - A)^{-1} \|_{\mathcal{L}(H)} = O \left( |\lambda|^{\frac{\ell}{2}} \right), \quad \text{where } \ell = 2\beta + 2.
\]

This is checked by using a contradiction argument. Indeed, suppose there exists $\{ (\lambda_n, U_n := (u_n, v_n)) \}_{n \geq 1} \subset \mathbb{R}^*_+ \times D(A)$, such that

\[
\lambda_n \to +\infty, \quad ||U_n||_H = 1,
\]

\[
\lambda_n^\ell (i\lambda_n I - A) U_n = (f_n, g_n) \to 0 \text{ in } H.
\]

Our aim is to show that $||(u_n, v_n)||_H \to 0$. This condition permits to conclude a contradiction with (46). By detailing equation (47), we get the following system

\[
\begin{align*}
i\lambda_n u_n - v_n &= \lambda_n^{-\ell} f_n \quad \text{in } H_0^1(\Omega), \\
i\rho \lambda_n v_n - \text{div}(a\nabla u_n + b\nabla v_n) &= \lambda_n^{-\ell} \rho g_n \quad \text{in } L^2(\Omega).
\end{align*}
\]

The proof of Theorem (4.1) is divided into several lemmas.
Lemma 4.2. Assume that conditions (LA) and (LE) hold. Then the solution \((u_n, v_n) \in D(A)\) of Equations (48)-(49) satisfies the following asymptotic behavior estimations

\[
\begin{align*}
\|u_n\|_{L^2(\Omega)} &= O\left(\lambda_n^{-1}\right), \\
\|\nabla v_n\|_{L^2(\omega)} &= o\left(\lambda_n^{-\ell/2}\right).
\end{align*}
\]

**Proof.** Using Equations (46) and (48), we deduce directly the first estimation (50).

Now, taking the inner product of (47) with \(U_n = (u_n, v_n)\) in \(\mathcal{H}\), then using the fact that \(U_n\) is uniformly bounded in \(\mathcal{H}\), we get

\[
\int_\Omega b(x)|\nabla v_n|^2\,dx = \Re \langle (i\lambda_n I - A)U_n, U_n \rangle_\mathcal{H} = o\left(\lambda_n^{-\ell}\right).
\]

It follows from the localization condition (LA) that

\[
\int_\omega |\nabla v_n|^2 = o\left(\lambda_n^{-\ell}\right).
\]

\[\square\]

Lemma 4.3. Assume that conditions (LA) and (LE) hold. Then the solution \((u_n, v_n) \in D(A)\) of Equations (48)-(49) satisfies the following asymptotic behavior estimation

\[
\int_\omega |u_n|^2 = o\left(\lambda_n^{-\ell-2}\right).
\]

**Proof.** Using Poincaré’s inequality and Equation (51), we obtain

\[
\lambda_n^\ell \int_\omega |v_n|^2\,dx = o(1).
\]

Multiplying Equation (48) by \(-ib(x)\pi_n\), and using estimation (53) and the fact that \(f_n\) converges to zero in \(L^2(\Omega)\), we deduce from the above equation that

\[
\lambda_n^{\ell+2} \int_\omega |u_n|^2\,dx = o(1).
\]

The desired estimation (52) is established.

\[\square\]

Lemma 4.4. Assume that conditions (LA) and (LE) hold. Then, for any \(\lambda_n \in \mathbb{R}_+^*\), the solution \(\varphi_n \in H^2(\Omega) \cap H^1_0(\Omega)\) of system

\[
\begin{cases}
\rho \lambda_n^2 \varphi_n + \text{div}(\alpha \nabla \varphi_n) - i\lambda_n(1\omega b)(x)\varphi_n = u_n & \text{in } \Omega, \\
\varphi_n = 0 & \text{on } \Gamma,
\end{cases}
\]

satisfies the following estimation

\[
|\lambda_n||\varphi_n|_{L^2(\Omega)} + ||\nabla \varphi_n||_{L^2(\Omega)} \leq M|\lambda_n|^\beta ||u_n||_{L^2(\Omega)}
\]

where \(M\) is positive constant independent of \(n\).

**Proof.** As stated in the beginning of Section 4, the energy of System (43) decays polynomially as \(t^{-2/\beta}\) with \(\beta > 0\) (see condition (LE)). The associated energy space is given by \(\mathcal{H}_a = H^1_0(\Omega) \times L^2(\Omega)\). So, following Borichev and Tomilov in [5], the resolvent set of its associated operator \(A_a\) defined by

\[
D(A_a) = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega),
\]
\[ \mathcal{A}(\varphi, \psi) = \left( \psi, \frac{1}{\rho} \left( \text{div}(a \nabla \varphi) - (1_\omega b)(x)\psi \right) \right) \]

contains \(i\mathbb{R}\) and \(|\lambda_n|^{-\beta}((i\lambda_n I - A_n)^{-1})\) is uniformly bounded on the imaginary axis. Consequently, there exists a positive constant \(M > 0\) independent of \(n\) such that
\[
\sup_{\lambda_n \in \mathbb{R}} |\lambda_n|^{-\beta}||((i\lambda_n I - A_n)^{-1})||_{L(H_\omega)} \leq M. \tag{56}
\]

Now, since \(u_n \in L^2(\Omega)\), then there exists a unique \((\varphi_n, \psi_n) \in D(A_n)\) solution of
\[
(i\lambda_n I - A_n)(\varphi_n, \psi_n) = \left(0, -\frac{1}{\rho}u_n\right).
\]

Equivalently, we have
\[
\begin{align*}
  i\lambda_n \varphi_n - \psi_n &= 0, \\
i\lambda_n \rho \psi_n - \text{div}(a \nabla \varphi_n) + (1_\omega b)(x) \psi &= -u_n.
\end{align*}
\]

Finally, from (56), we deduce that
\[
|\lambda_n|^{-\beta}||((\varphi_n, \psi_n)||_{H_\omega} \leq M||u_n||_{L^2(\Omega)}
\]
which gives the desired estimation. \(\square\)

**Lemma 4.5.** Assume that conditions (LA) and (LE) hold. Then if \(\ell \geq 2\beta + 2\), the solution \((u_n, v_n) \in D(A)\) of (48)-(49) satisfies the following asymptotic behavior estimate
\[
\int_{\Omega} |\lambda_n u_n|^2 dx = o(1). \tag{57}
\]

**Proof.** Let \(\varphi_n\) be the solution of System (43). Multiplying Equation (48) by \(i\lambda_n^3 \rho \varphi_n\) and Equation (49) by \(\lambda_n^3 \varphi_n\) and using Green’s formula, we obtain
\[
\begin{align*}
  -\lambda_n^4 \int_{\Omega} \rho v_n \varphi_n dx - i\lambda_n^3 \int_{\Omega} \rho v_n \varphi_n dx &= i\lambda_n^3 \int_{\Omega} \rho \varphi_n f_n dx \\
i\lambda_n^3 \int_{\Omega} \rho v_n \varphi_n dx + \lambda_n^2 \int_{\Omega} (a \nabla u_n + b \nabla v_n) \cdot \nabla \varphi_n dx &= \lambda_n^2 \int_{\Omega} \rho \varphi_n g_n dx.
\end{align*}
\]

Multiply Estimation (55) by \(\lambda_n\) to get after using (50)
\[
\|\varphi_n\|_{L^2(\Omega)} = O(\lambda_n^{2-\beta}) \quad \text{and} \quad \|\nabla \varphi_n\|_{L^2(\Omega)} = O(\lambda_n^{2-1}). \tag{60}
\]

Now, taking into consideration Estimation (60), and the fact that \(f_n \to 0\) in \(H_0^1(\Omega), g_n \to 0\) in \(L^2(\Omega)\), we get
\[
\begin{align*}
i \int_{\Omega} \rho \varphi_n f_n dx &= o(\lambda_n^{-\ell + \beta + 1}) \quad \text{and} \quad i \int_{\Omega} \rho \varphi_n g_n dx = o(\lambda_n^{-\ell + \beta}). \tag{61}
\end{align*}
\]

We continue under conditions (LA) and (LE), following the same technique used to get Estimation (39) and taking into consideration Estimations (51) and (60) on the second term of the left hand side of Equation (59), we get
\[
\lambda_n^2 \int_{\Omega} (a \nabla u_n + b \nabla v_n) \cdot \nabla \varphi_n dx = \lambda_n^2 \int_{\Omega} a \nabla u_n \cdot \nabla \varphi_n dx + o(\lambda_n^{-\ell/2 + \beta + 1}).
\]

Applying Green’s formula on the above equation gives
\[
\lambda_n^2 \int_{\Omega} (a \nabla u_n + b \nabla v_n) \cdot \nabla \varphi_n dx = -\lambda_n^2 \int_{\Omega} \text{div}(a \nabla \varphi_n) u_n dx + o(\lambda_n^{-\ell/2 + \beta + 1}). \tag{62}
\]
So, adding (58) and (59) and using (61) and (62), we obtain

\[ \lambda^4 \int_\Omega \rho u_n \nabla^2 u_n |dx| + \lambda^2 \int_\Omega \text{div}(a \nabla \nabla u_n) u_n |dx| = o(\lambda^{-2+\beta}). \tag{63} \]

Inserting the first equation of (43) in (63), we get

\[ \lambda^2 \int_\Omega |u_n|^2 |dx| - i \lambda^3 \int_\Omega (\mathbb{1}_\omega b)(x) \nabla u_n u_n |dx| = o(\lambda^{-\ell+\beta+1}). \tag{64} \]

We have

\[ \left| \lambda^3 \int_\Omega (\mathbb{1}_\omega b)(x) \nabla u_n u_n |dx| \right| \leq \lambda^3 \sqrt{\|b\|_\infty} \left( \int_\Omega |\varphi_n|^2 |dx| \right)^{1/2} \left( \int_\Omega (\mathbb{1}_\omega b)(x) |u_n|^2 |dx| \right)^{1/2}. \]

Using Estimations (52) and (61), we deduce from the above inequality that

\[ i \lambda^3 \int_\Omega (\mathbb{1}_\omega b) \nabla u_n u_n |dx| = o(\lambda^{-\ell+\beta}) \]

which together with (64) give

\[ \lambda^2 \int_\Omega |u_n|^2 |dx| = o(\lambda^{-\ell+\beta+1}). \]

Thus, taking \( \ell \geq 2\beta + 2 \) gives the desired estimation (57). The proof of Lemma 4.3 is thus complete.

\[ \square \]

**Lemma 4.6.** Assume that conditions (LA) and (LE) hold. Then the solution \((u_n, v_n) \in D(\mathcal{A})\) of (48)-(49) satisfies the following asymptotic behavior estimation

\[ \int_\Omega |\nabla u_n|^2 |dx| = o(1). \tag{65} \]

**Proof.** Multiplying Equation (49) by \( \nabla_n \) and applying Green’s formula, we obtain

\[ i \lambda_n \int_\Omega \rho u_n \nabla_n |dx| + \int_\Omega (a \nabla u_n + b \nabla v_n) \cdot \nabla \nabla_n |dx| = \lambda_n^\ell \int_\Omega \rho \nabla_n g_n |dx|. \]

Taking into consideration (46), (51), (57) and the fact that \( g_n \) converges to zero in \( L^2(\Omega) \), we deduce that

\[ \int_\Omega a |\nabla u_n|^2 |dx| = o(1) \]

which together with the fact that \( a(x) \geq a_0 > 0 \) for almost every \( x \in \Omega \) give the desired estimation (65).

\[ \square \]

**Proof of Theorem 4.1** Taking \( \ell = 2\beta + 2 \), it follows from (57) and (65) that \( ||U_n||_H = o(1) \) which contradicts (46). Consequently, condition (45) holds. The proof is thus complete.

### 4.1. Applications. Example 1.

Consider System (1) with constant density and metric coefficients on a square domain with viscoelastic damping localized in a sub-domain which contains a vertical strip. Thus, we construct the following geometry:

\[ \omega \supset \omega_0 = \{ (x_1, x_2) \in \mathbb{R}^2 : \epsilon_1 < x_1 < \epsilon_2, \ 0 < x_2 < L \}. \tag{LC1} \]

Setting \( a, \rho \) to be any strictly positive constants, it was proven that under conditions (LA) and (LC1), the energy of the wave equation (43) with local viscous damping decays polynomially as \( t^{-1} \) for smooth initial data (see Example 3 in [20]). Hence,
taking $\beta = 2$ in Estimation (44) gives a polynomial decay rate of the energy of System (1) of type $t^{-3}$, i.e. there exists $C > 0$ such that
\[ E(t) \leq \frac{C}{t^{1/3}} ||U_0||^2_{D(A)}, \quad \forall t > 0. \]

**Example 2.** Under the same conditions applied on System (1) in Example 1, we construct a different geometry:
\[ \omega \supset \omega_0 = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < \epsilon_1, \quad 0 < x_2 < L\}. \quad \text{(LC2)} \]

Also, taking $a, \rho$ to be any strictly positive constants in System (43), it was proven in [27] that the the energy of the wave equation (43) decays polynomially as $t^{-4/3}$ for smooth initial data. Similarly, taking $\beta = 3/2$ in Estimation (44) improves the polynomial decay rate of the energy of System (1) to be of type $t^{-2/5}$ i.e. there exists $C > 0$ such that
\[ E(t) \leq \frac{C}{t^{2/5}} ||U_0||^2_{D(A)}, \quad \forall t > 0. \]

5. **Conclusion and open problems.** In this paper, we have studied the stabilization of a wave equation with local internal Kelvin-Voigt damping under different geometries not studied in previous literature. We obtained polynomial decays of the energy of systems with different rates, regarding that the damping region satisfies or does not satisfy the GCC. However, the optimality of these decay rates is still an open problem where we conjecture $t^{-2}$ to be the optimal one. In fact, in order to get the optimality, we suggest to take $\Omega$ to be a disk where the viscoelastic part is near the boundary. Also, in [1], they studied problem (1) with constant coefficients and on $C^\infty$ domains where the damping region doesn’t obey any geometry. So, it would be interesting to study the stabilization of System (1) on bounded domains with boundary of class $C^2$ in $\mathbb{R}^n$, $n \geq 2$, whenever the damping region is localized internally and faraway from the boundary without any geometric conditions. However, the coefficients of System (1) can be taken of class $C^1(\overline{\Omega})$.

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