HIGHEST WEIGHT MODULES OVER
QUANTUM QUEER SUPERA LGEBRA $U_q(q(n))$

DIMITAR GRANTCHAROV$^1$, JI HYE JUNG$^{2,3}$, SEOK-JIN KANG$^2$ AND MYUNGHO KIM$^{2,3}$

Abstract. In this paper, we investigate the structure of highest weight modules over the
quantum queer superalgebra $U_q(q(n))$. The key ingredients are the triangular decomposition
of $U_q(q(n))$ and the classification of finite dimensional irreducible modules over quantum
Clifford superalgebras. The main results we prove are the classical limit theorem and the
complete reducibility theorem for $U_q(q(n))$-modules in the category $O^\geq q$.

Introduction

Since its inception, the representation theory of Lie superalgebras has been known to
be much more complicated than the corresponding theory of Lie algebras. One of the Lie
superalgebra series attracts special attention due to its resemblance of the Lie algebra $gl_n$
on the one hand and because of the unique properties of its structure and representations
on the other. This is the so-called queer (or strange) Lie superalgebra $q(n)$ which consists
of all endomorphisms of $C^{n|n}$ that commute with an odd automorphism $P$ of $C^{n|n}$ such that
$P^2 = Id$. The queer nature of $q(n)$ is partly due to the nonabelian structure of its Cartan
subsuperalgebra $h$ having a nontrivial odd part $h_1$. Another unique property of $q(n)$ is that,
although it has no invariant bilinear form, it admits an invariant odd bilinear form. Because
of the nonabelian structure of $h$, the study of the highest weight modules of $q(n)$ requires some
tools in addition to the standard technique. For example, the highest weight space $v_\lambda$ of an
irreducible highest weight $q(n)$-module $V(\lambda)$ has a Clifford module structure. The case when
$V(\lambda)$ is a tensor module; i.e., a submodule of some tensor power $V^\otimes r$ of the natural $q(n)$-
module $V = C^{n|n}$, was treated first by Sergeev in 1984. In [Se2] Sergeev established several
important results, among which are the complete reducibility of $V^\otimes r$, a character formula of
$V(\lambda)$, and an analog of the fundamental Schur-Weyl duality, often referred as Sergeev duality.
The characters of all simple finite-dimensional $q(n)$-modules have been found by Penkov and
Serganova in 1996 (see [PS2] and [PS3]) via an algorithm using a supergeometric version of the
Borel-Weil-Bott Theorem. In 2004 Brundan, [B], obtained the character formula of Penkov

---

$^1$This research was supported by a UT Arlington REP Grant.
$^2$This research was supported by KRF Grant # 2007-341-C00001.
$^3$This research was supported by BK21 Mathematical Sciences Division.
and Serganova using a different approach and formulated a conjecture for the characters of the irreducible modules in the category $\mathcal{O}$. Important results related to the simplicity of the highest weight $\mathfrak{q}(n)$-modules were obtained recently by Gorelik in [G].

In this paper we initiate the study of highest weight representations of the quantum superalgebra $U_q(\mathfrak{q}(n))$. The aim of this paper is twofold. We want to study highest weight $U_q(\mathfrak{q}(n))$-modules on the one hand, and to build the foundations of the crystal bases theory for the tensor modules of $U_q(\mathfrak{q}(n))$ on the other. The latter problem will be treated in a future work.

A quantum deformation of the universal enveloping algebra of $\mathfrak{q}(n)$ was constructed first by Olshanski in [O]. Olshanski’s construction is a flat deformation of the universal enveloping algebra $U(\mathfrak{q}(n))$ of $\mathfrak{q}(n)$ and is a quantum enveloping superalgebra in the sense of Drinfeld ([Dr], §7). The idea in [O] is to apply a suitable modification of the procedure used by Faddeev, Reshetikhin, and Takhtajan in [RTF] – using an element $S$ in $\text{End}(\mathbb{C}^{n|n})^{\otimes 2}$ that satisfies the quantum Yang-Baxter equation. However, as pointed out by Olshanski, the $r$-matrix $r \in \mathfrak{q}(n)^{\otimes 2}$ does not satisfy the classical Yang-Baxter equation. Thus no quantum analogue of $U(\mathfrak{q}(n))$ can be a quasi-triangular Hopf algebra.

In the present paper, based on the description of Olshanski, we give a presentation of $U_q(\mathfrak{q}(n))$ in terms of generators and relations so that the relations are quantum deformations of the relations of $\mathfrak{q}(n)$ obtained in [LS]. Using this presentation, we find a natural triangular decomposition of $U_q(\mathfrak{q}(n))$, and then introduce the notion of highest weight modules and Weyl modules. Similarly to the case of $\mathfrak{q}(n)$, in order to study highest weight modules, one has to describe the modules over the quantum Clifford superalgebra $\text{Cliff}_q(\lambda)$ for a weight $\lambda$ of $\mathfrak{q}(n)$. These modules, as we show in Section 3, do not have the same structure as the ones over the classical Clifford superalgebra $\text{Cliff}(\lambda)$. For example, the irreducible modules over $\text{Cliff}_q(\lambda)$ are parity invariant for much larger set of weights $\lambda$, compared with the irreducibles over $\text{Cliff}(\lambda)$.

In the last two sections of the paper we focus on the category $\mathcal{O}_{\mathfrak{q}}^{\geq 0}$ of finite dimensional $U_q(\mathfrak{q}(n))$-modules all whose weights are of the form $\lambda_1 \epsilon_1 + \cdots + \lambda_n \epsilon_n$ ($\lambda_i \in \mathbb{Z}_{\geq 0}$). One of our main results is a classical limit theorem for the irreducible modules in $\mathcal{O}_{\mathfrak{q}}^{\geq 0}$. Due to the structure of the quantum Clifford superalgebra, the classical limit theorem is non-standard, as it is not true in general that the classical limit $V^1$ of an irreducible highest weight $U_q(\mathfrak{q}(n))$-module $V^\lambda$ is $V(\lambda)$. In fact, as we show in Section 5 if $\lambda$ has even number of nonzero coordinates $\lambda_1 > \cdots > \lambda_{2k}$, then $\text{ch } V^1 = 2 \text{ch } V(\lambda)$. The “queer” version of the classical limit theorems are Theorem 5.14 and Theorem 5.16. With the aid of the classical limit theorems we obtain another important result in the last section: the category $\mathcal{O}_{\mathfrak{q}}^{\geq 0}$ is semisimple.
The organization of the paper is as follows. In Section 1 we recall some definitions and basic results about \( \mathfrak{q}(n) \). The realization of \( U_q(\mathfrak{q}(n)) \) and its triangular decomposition is provided in Section 2. Section 3 is devoted to the study of the quantum Clifford superalgebra and its modules. In Section 4 we introduce the notion of highest weight modules and Weyl modules. In particular, we show that every Weyl module \( W^\lambda(\lambda) \) has a unique irreducible quotient \( V^\lambda(\lambda) \). The classical limit theorem for the category \( \mathcal{O}_q^{\geq 0} \) is proved in Section 5 and the complete reducibility of \( U_q(\mathfrak{q}(n)) \)-modules in \( \mathcal{O}_q^{\geq 0} \) is established in the last section.

1. The Lie superalgebra \( \mathfrak{q}(n) \) and its representations

The ground field in this section will be \( \mathbb{C} \). By \( \mathbb{Z}_{\geq 0} \) and \( \mathbb{Z}_{>0} \) we denote the nonnegative integers and strictly positive integers, respectively. We set \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \). Every vector space \( V = V_0 \oplus V_1 \) over \( \mathbb{C} \) is \(\mathbb{Z}_2\)-graded with even part \( V_0 \) and odd part \( V_1 \). We will write \( \dim V = m/n \) if \( \dim V_0 = m \) and \( \dim V_1 = n \). By \( \Pi \) we denote the parity change functor; i.e., \( \Pi V \) is a vector space for which \( \Pi V_0 = V_1 \) and \( \Pi V_1 = V_0 \). The direct sum of \( r \) copies of a vector space \( V \) will be written as \( V^{\oplus r} \).

The Lie subsuperalgebra \( \mathfrak{g} = \mathfrak{q}(n) \) of \( \mathfrak{gl}(n|n) \) is defined in matrix form by

\[
\mathfrak{g} = \mathfrak{q}(n) := \left\{ \left( \begin{array}{cc} A & B \\ B & A \end{array} \right) \mid A, B \in \mathfrak{gl}_n \right\}.
\]

By definition, a subsuperalgebra \( \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \) of \( \mathfrak{g} \) is a Cartan subsuperalgebra, if it is a self-normalizing nilpotent subsuperalgebra. Every such \( \mathfrak{h} \) has a nontrivial odd part \( \mathfrak{h}_1 \). We fix \( \mathfrak{h} \) to be the standard Cartan subsuperalgebra, namely the one for which \( \mathfrak{h}_0 \) has a basis \( \{k_1, ..., k_n\} \) and \( \mathfrak{h}_1 \) has a basis \( \{k_1, ..., k_n\} \), where \( k_i := \left( \begin{array}{cc} E_{i,i} & 0 \\ 0 & E_{i,i} \end{array} \right) \), \( k_i := \left( \begin{array}{cc} 0 & E_{i,i} \\ E_{i,i} & 0 \end{array} \right) \) and \( E_{i,j} \) is the \( n \times n \) matrix having 1 in the \( (i, j) \) position and 0 elsewhere. One should note that all Cartan subsuperalgebras of \( \mathfrak{g} \) are conjugate to \( \mathfrak{h} \). Let \( \{\epsilon_1, ..., \epsilon_n\} \) be the basis of \( \mathfrak{h}_0^\circ \) dual to \( \{k_1, ..., k_n\} \). We denote \( k_i - k_{i+1} \) by \( h_i \) for \( i = 1, 2, \cdots, n-1 \). The root system \( \Delta = \Delta_0 \sqcup \Delta_1 \) of \( \mathfrak{g} \) has identical even and odd parts. Namely, \( \Delta_0 = \Delta_1 = \{\epsilon_i - \epsilon_j \mid 1 < i \neq j < n\} \). In particular, the root space decomposition \( \mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \) has the property that \( \mathfrak{g}_\alpha \) has dimension 1|1 for every \( \alpha \in \Delta \). Set \( \alpha_i := \epsilon_i - \epsilon_{i+1} \). Let \( Q = \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \alpha_i \) be the root lattice and \( Q_+ = \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \alpha_i \) be the positive root lattice. The notation \( Q_- = -Q_+ \) will also be used. There is a partial ordering on \( \mathfrak{h}_0^\circ \) defined by \( \lambda \geq \mu \) if and only if \( \lambda - \mu \in Q_+ \) for \( \lambda, \mu \in \mathfrak{h}_0^\circ \). The root space \( \mathfrak{g}_\alpha_i \) is spanned by \( e_i := \left( \begin{array}{cc} E_{i,i+1} & 0 \\ 0 & E_{i,i+1} \end{array} \right) \) and \( e_i := \left( \begin{array}{cc} 0 & E_{i,i+1} \\ E_{i,i+1} & 0 \end{array} \right) \), while \( \mathfrak{g}_{-\alpha_i} \) is spanned.
We modified the relations given in [LS]. More precisely, we replaced the relations

\[
\begin{pmatrix}
E_{i+1,i} & 0 \\
0 & E_{i+1,i}
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
0 & E_{i+1,i} \\
E_{i+1,i} & 0
\end{pmatrix}.
\]

Let \( P := \bigoplus_{i=1}^{n} \mathbb{Z} \varepsilon_{i} \) be the weight lattice of \( \mathfrak{g} \) and denote by \( P^{\vee} := \bigoplus_{i=1}^{n} \mathbb{Z} k_{i} \) the dual weight lattice.

Let \( I := \{1, 2, \ldots, n-1\} \) and \( J := \{1, 2, \ldots, n\} \).

**Proposition 1.1.** [LS] The Lie superalgebra \( \mathfrak{g} \) is generated by the elements \( e_{i}, e_{i}^{\dagger}, f_{i}, f_{i}^{\dagger} \) \( (i \in I) \), \( h_{0} \) and \( k_{l} \) \( (l \in J) \) with the following defining relations:

\[
[h, h'] = 0 \quad \text{for} \ h, h' \in h_{0},
\]

\[
[h, e_{i}] = \alpha_{i}(h)e_{i}, \quad [h, e_{i}^{\dagger}] = \alpha_{i}(h)e_{i}^{\dagger} \quad \text{for} \ h \in h_{0}, \ i \in I,
\]

\[
[h, f_{i}] = -\alpha_{i}(h)f_{i}, \quad [h, f_{i}^{\dagger}] = -\alpha_{i}(h)f_{i}^{\dagger} \quad \text{for} \ h \in h_{0}, \ i \in I,
\]

\[
[h, k_{l}] = 0 \quad \text{for} \ h \in h_{0}, \ l \in J,
\]

\[
[e_{i}, f_{j}] = \delta_{ij}(k_{i} - k_{i+1}), \quad [e_{i}, f_{j}^{\dagger}] = \delta_{ij}(k_{i} - k_{i+1}) \quad \text{for} \ i, j \in I,
\]

\[
[e_{i}, f_{j}] = \delta_{ij}(k_{i} - k_{j+1}), \quad [k_{l}, e_{i}] = \alpha_{i}(k_{l})e_{i} \quad \text{for} \ i, j \in I, \ l \in J,
\]

\[
[k_{l}, f_{j}] = -\alpha_{i}(k_{l})f_{j}, \quad [e_{i}, f_{j}] = \delta_{ij}(k_{i} + k_{i+1}) \quad \text{for} \ i, j \in I, \ l \in J
\]

\[
[k_{l}, e_{i}] = \begin{cases} 
  e_{i} & \text{if} \ l = i, i+1 \\
  0 & \text{otherwise}
\end{cases} \quad \text{for} \ i \in I, \ l \in J,
\]

\[
[k_{l}, f_{i}] = \begin{cases} 
  f_{i} & \text{if} \ l = i, i+1 \\
  0 & \text{otherwise}
\end{cases} \quad \text{for} \ i \in I, \ l \in J
\]

\[
[e_{i}, e_{j}] = [e_{i}, e_{j}] = [f_{i}, f_{j}] = [f_{i}, f_{j}^{\dagger}] = 0 \quad \text{for} \ i, j \in I, \ |i - j| \neq 1,
\]

\[
[e_{i}, e_{j}] = [f_{i}, f_{j}] = 0 \quad \text{for} \ i, j \in I, \ |i - j| > 1,
\]

\[
[e_{i}, e_{i+1}] = [e_{i}, e_{i+1}^{\dagger}], [e_{i}, e_{i+1}] = [e_{i}, e_{i+1}^{\dagger}],
\]

\[
[f_{i+1}, f_{i}] = [f_{i+1}, f_{i}] = [f_{i+1}, f_{i}^{\dagger}] = [f_{i+1}, f_{i}^{\dagger}],
\]

\[
[k_{i}, k_{j}] = \delta_{ij}2k_{i} \quad \text{for} \ i, j \in J,
\]

\[
[e_{i}, [e_{i}, f_{j}]] = [e_{i}, [e_{i}, e_{j}]] = 0 \quad \text{for} \ i, j \in I, \ |i - j| = 1,
\]

\[
[f_{i}, [f_{i}, f_{j}]] = [f_{i}, [f_{i}, f_{j}]] = 0 \quad \text{for} \ i, j \in I, \ |i - j| = 1.
\]

**Remark.** We modified the relations given in [LS]. More precisely, we replaced the relations

\[(1.1)\]

\[
[e_{i}, [e_{i}, e_{j}]] = 0 \quad \text{for} \ i, j \in I, \ |i - j| = 1,
\]

\[
[f_{i}, [f_{i}, f_{j}]] = 0 \quad \text{for} \ i, j \in I, \ |i - j| = 1.
\]
by

\[ [e_i, e_{i+1}] = [e_i, e_{i+1}], [e_i, e_{i+1}] = [e_i, e_{i+1}], \]

\[ [f_i+1, f_i] = [f_i+1, f_i], [f_i+1, f_i] = [f_i+1, f_i]. \]

(1.2)

Since (1.1) can be derived from (1.2) (and other ones), we can easily see that these two presentations are equivalent.

The universal enveloping algebra \( U(\mathfrak{g}) \) is obtained from the tensor algebra \( T(\mathfrak{g}) \) by factoring out by the ideal generated by the elements \( [u, v] - u \otimes v + (-1)^{\alpha \beta} v \otimes u, \) where \( \alpha, \beta \in \mathbb{Z}_2, u \in \mathfrak{g}_\alpha, \ v \in \mathfrak{g}_\beta. \) Let \( U^+ \) (respectively, \( U^0 \) and \( U^- \)) be the subalgebra of \( U(\mathfrak{g}) \) generated by the elements \( e_i, e_i^\dagger \ (i \in I) \) (respectively, by \( k_i, k_i^\dagger \ (i \in J) \) and by \( f_i, f_i^\dagger \ (i \in I) \)). By the Poincaré-Birkhoff-Witt theorem, the universal enveloping algebra has the triangular decomposition:

\[ U(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+. \]

(1.3)

A \( \mathfrak{g} \)-module \( V \) is called a weight module if it admits a weight space decomposition

\[ V = \bigoplus_{\mu \in \mathfrak{h}_0^*} V_\mu, \text{ where } V_\mu = \{ v \in V | hv = \mu(h)v \text{ for all } h \in \mathfrak{h}_0 \}. \]

For a weight \( \mathfrak{g} \)-module \( M \) denote by \( \text{wt}(M) \) the set of weights \( \lambda \in \mathfrak{h}_0^* \) for which \( M_\lambda \neq 0. \) Every submodule of a weight module is also a weight module. If \( \dim_{\mathbb{C}} V_\mu < \infty \) for all \( \mu \in \mathfrak{h}_0^* \), the character of \( V \) is defined to be

\[ \text{ch } V = \sum_{\mu \in \mathfrak{h}_0^*} (\dim_{\mathbb{C}} V_\mu) \ e^\mu, \]

where \( e^\mu \) are formal basis elements of the group algebra \( \mathbb{C}[\mathfrak{h}_0^*] \) with the multiplication given by \( e^\lambda e^\mu = e^{\lambda + \mu} \) for all \( \lambda, \mu \in \mathfrak{h}_0^*. \)

Denote by \( \mathfrak{b}_+ \) the standard Borel subsuperalgebra of \( \mathfrak{g} \) generated by \( k_l, k_l^\dagger \ (l \in J) \) and \( e_i, e_i^\dagger \ (i \in I) \). A weight module \( V \) is called a highest weight module if it is generated over \( \mathfrak{g} \) by a finite dimensional irreducible \( \mathfrak{b}_+ \)-submodule (see [PSI] Definition 4)).

**Proposition 1.2.** [P] Let \( \mathfrak{v} \) be a finite dimensional irreducible \( \mathbb{Z}_2 \)-graded \( \mathfrak{b}_+ \)-module.

1. The maximal nilpotent subsuperalgebra \( \mathfrak{n} \) of \( \mathfrak{b}_+ \) acts on \( \mathfrak{v} \) trivially.
2. For any weight \( \mu \in \mathfrak{h}_0^* \), consider the symmetric bilinear form \( F_\mu(u, v) := \mu([u, v]) \) on \( \mathfrak{h}_1 \) and let \( \text{Cliff}(\mu) \) be the Clifford superalgebra of the quadratic space \( (\mathfrak{h}_1, F_\mu). \) Then there exists a unique weight \( \lambda \in \mathfrak{h}_0^* \) such that \( \mathfrak{v} \) is endowed with a canonical \( \mathbb{Z}_2 \)-graded \( \text{Cliff}(\lambda) \)-module structure and \( \mathfrak{v} \) is determined by \( \lambda \) up to \( \Pi. \)
3. \( \mathfrak{h}_0 \) acts on \( \mathfrak{v} \) by the weight \( \lambda \) determined in (2).
From the above proposition, we know that the dimension of the highest weight space of a highest weight \(g\)-module with highest weight \(\lambda\) is the same as the dimension of an irreducible \(\text{Cliff}(\lambda)\)-module. On the other hand all irreducible \(\text{Cliff}(\lambda)\)-modules have the same dimension (see, for example, [ABS, Table 2]). Thus the dimension of the highest weight space is constant for all highest weight modules with highest weight \(\lambda\).

**Definition 1.3.** Let \(v(\lambda)\) be the irreducible \(b_+\)-module determined by \(\lambda\) up to \(\Pi\). The Weyl module \(W(\lambda)\) of \(g\) with highest weight \(\lambda\) is defined to be

\[ W(\lambda) := U(g) \otimes U(b_+) v(\lambda). \]

Note that the structure of \(W(\lambda)\) is determined by \(\lambda\) up to \(\Pi\).

**Remark.** One may define the Verma module corresponding to \(\lambda\) by \(M(\lambda) := U(g) \otimes U(b_+) \text{Cliff}(\lambda)\). Since the Verma modules are not highest weight modules, they will not be considered in this paper.

We will denote by \(\Lambda_0^+\) and \(\Lambda^+\) the set of \(\mathfrak{gl}_n\)-dominant integral weights and the set of \(g\)-dominant integral weights, respectively. These are given by

\[ \Lambda_0^+ := \{ \lambda_1 \epsilon_1 + \cdots + \lambda_n \epsilon_n \in b_0^* \mid \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I \} \]

\[ \Lambda^+ := \{ \lambda_1 \epsilon_1 + \cdots + \lambda_n \epsilon_n \in \Lambda_0^+ \mid \lambda_i = \lambda_{i+1} \Rightarrow \lambda_i = \lambda_{i+1} = 0 \text{ for all } i \in I \}. \]

**Proposition 1.4.** [P]

1. For any weight \(\lambda\), \(W(\lambda)\) has a unique maximal submodule \(N(\lambda)\).
2. For each finite dimensional irreducible \(g\)-module \(V\), there exists a unique weight \(\lambda \in \Lambda_0^+\) such that \(V\) is a homomorphic image of \(W(\lambda)\).
3. \(V(\lambda) := W(\lambda)/N(\lambda)\) is finite dimensional if and only if \(\lambda \in \Lambda^+\).

Now we restrict our attention to the following subcategory of the category of finite dimensional \(g\)-modules.

**Definition 1.5.** Set \(P_{\geq 0} := \{ \lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_n \epsilon_n \in P \mid \lambda_j \geq 0 \text{ for all } j = 1, \cdots, n \} \). The category \(\mathcal{O}_{\geq 0}\) consists of finite dimensional \(U(g)\)-modules \(M\) with weight space decomposition \(M = \bigoplus_{\lambda \in P} M_\lambda\) such that \(\text{wt}(M) \subset P_{\geq 0}\).

Clearly, \(\mathcal{O}_{\geq 0}\) is closed under finite direct sum, tensor product and taking submodules and quotient modules. Because a \(q(n)\)-module in \(\mathcal{O}_{\geq 0}\) can be decomposed into a direct sum of irreducible highest weight \(\mathfrak{gl}_n\)-modules, one can easily prove the following proposition (see, for example, [HK, Theorem 7.2.3]).
Proposition 1.6. For each $\lambda \in \Lambda^+ \cap P_{\geq 0}$, $V(\lambda)$ is an irreducible $U(\mathfrak{g})$-module in the category $\mathcal{O}^{\geq 0}$. Conversely, every irreducible $U(\mathfrak{g})$-module in the category $\mathcal{O}^{\geq 0}$ has the form $V(\lambda)$ for some $\lambda \in \Lambda^+ \cap P_{\geq 0}$.

In [Sc1], Sergeev has presented an explicit set of generators of $Z = Z(U(\mathfrak{g}))$, the center of $U(\mathfrak{g})$, and showed that each Weyl module $W(\lambda)$ ($\lambda \in h^*_0$) admits a central character. Let $\chi_\lambda \in \text{Hom}_C(Z, \mathbb{C})$ be the central character afforded by $W(\lambda)$; i.e., every element $z \in Z$ acts on $W(\lambda)$ as scalar multiplication by $\chi_\lambda(z)$. Following [B] (2.12)], to each weight $\lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_n \epsilon_n \in P$, one can assign a formal symbol

$$\delta(\lambda) := \delta_{\lambda_1} + \cdots + \delta_{\lambda_n}$$

such that $\delta_0 = 0$ and $\delta_{-i} = -\delta_i$.

Proposition 1.7. [B] Theorem 4.19, [PS2] Proposition 1.1] For $\lambda, \mu \in P$, $\chi_\lambda = \chi_\mu$ if and only if $\delta(\lambda) = \delta(\mu)$.

The following proposition will be very useful in Section 5.

Proposition 1.8. Let $V$ be a finite dimensional highest weight module over $\mathfrak{g}$ with highest weight $\lambda \in \Lambda^+ \cap P_{\geq 0}$. Then $V$ is isomorphic to an irreducible highest weight module $V(\lambda)$.

Proof. If $V$ is reducible, since it is finite dimensional, it contains a nonzero proper irreducible submodule $W$. Then $W$ is isomorphic to an irreducible highest weight module $V(\mu)$ for some weight $\mu \in \Lambda^+ \cap P_{\geq 0}$ by Proposition 1.4. We know that $\mu \preceq_+ \lambda$ and $\chi_\lambda = \chi_\mu$. But, by Proposition 1.7, $\delta(\lambda) = \delta(\mu)$. Since $\lambda, \mu \in \Lambda^+ \cap P_{\geq 0}$, we have $\lambda = \mu$, which is a contradiction. Thus $V$ is irreducible and by Proposition 1.4 it must be isomorphic to the irreducible highest weight module $V(\lambda)$ up to $\Pi$.

The next proposition gives a sufficient condition for the finite dimensionality of a highest weight $\mathfrak{g}$-module.

Proposition 1.9. Let $V$ be a highest weight module over $\mathfrak{g}$ with highest weight $\lambda \in \Lambda^+$. If $f_i^{\lambda(h_i)+1}v = 0$ for all $v \in V_\lambda$ and $i \in I$, then $V$ is finite dimensional.

Proof. Let $\{x_1, x_2, \ldots, x_r\}$ and $\{y_1, y_2, \ldots, y_r\}$ be bases of $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively. Then by the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{g})$ has a basis consisting of elements of the form $y_1^{\epsilon_1}y_2^{\epsilon_2} \cdots y_r^{\epsilon_r}x_1^{n_1}x_2^{n_2} \cdots x_r^{n_r}$ where $\epsilon_j = 0$ or 1 and $n_j \in \mathbb{N} \cup \{0\}$. Because $\{y_1^{\epsilon_1}y_2^{\epsilon_2} \cdots y_r^{\epsilon_r} | \epsilon_j = 0, 1\}$ is a finite set, it is enough to show that $U(\mathfrak{g}_0)V_\lambda$ is finite dimensional. For any $v \in V_\lambda$, we know that $U(\mathfrak{g}_0)v$ is a highest weight module over $\mathfrak{g}_0$ with highest weight $\lambda$ satisfying $f_i^{\lambda(h_i)+1}v = 0$ for all $i \in I$. Thus it is finite dimensional. Since $U(\mathfrak{g}_0)V_\lambda \subset \sum_{v \in V_\lambda} U(\mathfrak{g}_0)v$, we have the desired result. \qed
We say that a weight \( \lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_n \epsilon_n \in \mathfrak{h}_0^* \) is \( \alpha \)-typical if \( \alpha = \epsilon_i - \epsilon_j \) and \( \lambda_i + \lambda_j \neq 0 \).

In [Sc2], Sergeev proved the following character formula for \( V(\lambda) \) (\( \lambda \in \Lambda^+ \cap P_{\geq 0} \)):

\[
\text{ch} V(\lambda) = \frac{\dim \mathfrak{v}_\lambda}{D} \sum_{w \in W} \text{sgn} \, w \left( e^{\lambda + \rho_0} \prod_{\alpha \in \Delta_0^+} (1 + e^{-\alpha}) \right),
\]

where \( \mathfrak{v}_\lambda \) is an irreducible Clifford(\( \lambda \))-module, \( W \) is the Weyl group of \( \mathfrak{g}_0 = \mathfrak{gl}_n \), \( \rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha \) and \( D = \sum_{w \in W} \text{sgn} \, w \, e^{w(\rho_0)} \) is the Weyl denominator. In [PS2], the formula (1.4) is called the generic character formula and an explicit algorithm for computing the character of an arbitrary finite dimensional irreducible \( \mathfrak{g} \)-module is presented.

2. The Quantum Superalgebra \( U_q(\mathfrak{q}(n)) \)

In [O], Olshanski constructed the quantum deformation \( U_q(\mathfrak{q}(n)) \) of the universal enveloping algebra of \( \mathfrak{q}(n) \). The quantum superalgebra \( U_q(\mathfrak{q}(n)) \) is defined to be the associative algebra over \( \mathbb{C}(q) \) generated by \( L_{ij}, \ i \leq j \), with defining relations

\[
L_{ii} L_{-i,-i} = L_{-i,-i} L_{ii} = 1, \\
(-1)^{p(i,j)p(k,l)} q^{\varphi(j,i)} L_{ij} L_{kl} + \{k \leq j < l\} \theta(i, j, k) (q - q^{-1}) L_{il} L_{kj} \\
+ \{i \leq -l < j \leq -k\} \theta(-i, -j, k) (q - q^{-1}) L_{i-l} L_{kJ} \\
= q^{\varphi(i,k)} L_{kl} L_{ij} + \{k < i \leq l\} \theta(i, j, k) (q - q^{-1}) L_{il} L_{kj} \\
+ \{-l \leq i < -k \leq j\} \theta(-i, -j, k) (q - q^{-1}) L_{i-l} L_{k-j} ,
\]

where \( \varphi(i, j) = \delta_{|i|,|j|} \text{sgn}(j) \), \( \theta(i, j, k) = \text{sgn}(\text{sgn}(i) + \text{sgn}(j) + \text{sgn}(k)) \), \( p(i, j) = \begin{cases} 0 & \text{if } ij > 0 \\ 1 & \text{if } ij < 0 \end{cases} \)

for any indices \( i \leq j, k \leq l \) in \( \{ \pm 1, \cdots, \pm n \} \) and the symbol \( \{ \cdots \} \) (the dots stand for some inequalities) is equal to 1 if all of these inequalities are fulfilled and 0 otherwise.

Following [O, Remark 7.3], we consider the set of generators of \( U_q(\mathfrak{g}) = U_q(\mathfrak{q}(n)) \) as follows:

\[
q^{k_i} := L_{i,i}, \ q^{-k_i} := L_{-i,-i}, \ e_i := -\frac{1}{q - q^{-1}} L_{-i,-i}, \ f_i := \frac{1}{q - q^{-1}} L_{i,i+1}, \\
e_i := -\frac{1}{q - q^{-1}} L_{i-1,i}, \ f_i := -\frac{1}{q - q^{-1}} L_{i,i+1}, \ k_i := -\frac{1}{q - q^{-1}} L_{i,i}.
\]

Our first main result is the following presentation of \( U_q(\mathfrak{g}) \).

**Theorem 2.1.** The quantum superalgebra \( U_q(\mathfrak{g}) \) is isomorphic to the unital associative algebra over \( \mathbb{C}(q) \) generated by the elements \( e_i, f_i, e_i, f_i \) (\( i = 1, \cdots, n-1 \)), \( k_l \) (\( l = 1, \cdots, n \)), and \( q^h \) (\( h \in P^\vee \)), satisfying the following relations

\[
q^0 = 1, q^{h_1 + h_2} = q^{h_1} q^{h_2} \text{ for } h_1, h_2 \in P^\vee ,
\]
\[ q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \text{ for } h \in P^\vee \]
\[ q^h k_i q^{-h} = k_i, \quad e_i q^{-h} q^h = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \text{ for } h \in P^\vee \]
\[ e_i f_i - f_i e_i = \frac{1}{q - q^{-1}} \left( q^{k_i-k_{i+1}} - q^{-k_{i}+k_{i+1}} \right), \]

\[ q e_{i+1} f_i - f_i e_{i+1} = e_i f_{i+1} - q f_{i+1} e_i = e_i f_j - f_j e_i = 0 \text{ if } |i-j| > 1, \]
\[ e_i f_i - f_i e_i = q^{-k_{i+1}} k_i - k_{i+1} q^{-k_i}, \]
\[ q e_{i+1} f_i - f_i e_{i+1} = e_i f_{i+1} - q f_{i+1} e_i = e_i f_j - f_j e_i = 0 \text{ if } |i-j| > 1, \]
\[ e_i f_i - f_i e_i = q^{k_{i+1}} k_i - k_{i+1} q^{k_i}, \]
\[ q e_{i+1} f_i - f_i e_{i+1} = e_i f_{i+1} - q f_{i+1} e_i = e_i f_j - f_j e_i = 0 \text{ if } |i-j| > 1, \]
\[ k_i e_i - q e_i k_i = e_i q^{-k_i}, \quad q k_i e_i - e_i k_i = -q^{-k_i} e_i, \]
\[ k_i e_j - e_j k_i = 0 \text{ for } j \neq i \text{ and } j \neq i - 1, \]
\[ k_i f_i - q f_i k_i = -q f_i k_i, \quad q k_i f_i - f_i k_i = q k_i f_i, \]
\[ k_i f_j - f_j k_i = 0 \text{ for } j \neq i \text{ and } j \neq i - 1, \]
(2.3) \[ k_i^2 = \frac{q^{2k_i} - q^{-2k_i}}{q^2 - q^{-2}}, \quad k_i k_j = -k_j k_i \text{ for } i \neq j, \]
\[ e_i f_i + f_i e_i = \frac{q^{k_i+k_{i+1}} - q^{-k_i-k_{i+1}}}{q - q^{-1}} + (q - q^{-1}) k_i k_{i+1}, \]
\[ q e_{i+1} f_i + f_i e_{i+1} = e_i f_{i+1} + q f_{i+1} e_i = e_i f_j + f_j e_i = 0 \text{ if } |i-j| > 1, \]
\[ k_i e_i + q e_i k_i = e_i q^{-k_i}, \quad q k_i e_i + e_i k_i = q^{-k_i} e_i, \]
\[ k_i e_j + e_j k_i = 0 \text{ for } j \neq i \text{ and } j \neq i - 1, \]
\[ k_i f_i + q f_i k_i = f_i q^{k_i}, \quad q k_i f_i + f_i k_i = q^{k_i} f_i, \]
\[ k_i f_j + f_j k_i = 0 \text{ for } j \neq i \text{ and } j \neq i - 1, \]
\[ e_i^2 = \frac{q - q^{-1}}{q + q^{-1}} e_i^2, \quad f_i^2 = \frac{q - q^{-1}}{q + q^{-1}} f_i^2, \]
\[ e_i e_j - e_j e_i = f_i f_j - f_j f_i = e_i e_j + e_j e_i = f_i f_j + f_j f_i = 0 \text{ if } |i-j| > 1, \]
\[ e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0 \text{ if } |i-j| \neq 1, \]
\[ e_i e_{i+1} - e_{i+1} e_i = e_i e_{i+1} + e_{i+1} e_i, \quad f_i f_{i+1} = f_i f_{i+1} + f_{i+1} f_i, \]
\[ e_i e_{i+1} - e_{i+1} e_i = e_i e_{i+1} - e_{i+1} e_i, \quad f_i f_{i+1} = f_i f_{i+1} - f_{i+1} f_i, \]
\[ q e_i^2 e_{i+1} - (q + q^{-1}) e_i e_{i+1} e_i + q^{-1} e_i e_{i+1} e_i^2 = 0, \]
\[ q f_i f_{i+1} - (q + q^{-1}) f_i f_{i+1} f_i + q^{-1} f_i f_{i+1} f_i^2 = 0, \]
\[ qe_i e_{i+1}^2 - (q + q^{-1})e_{i+1} e_i + q^{-1} e_{i+1}^2 e_i = 0, \]
\[ qf_i f_{i+1}^2 - (q + q^{-1})f_{i+1} f_i + q^{-1} f_{i+1}^2 f_i = 0, \]
\[ qe_i^2 e_{i+1} - (q + q^{-1})e_i e_{i+1} e_i + q^{-1} e_i e_{i+1}^2 = 0, \]
\[ qf_i^2 f_{i+1} - (q + q^{-1})f_i f_{i+1} f_i + q^{-1} f_i f_{i+1}^2 = 0, \]
\[ qe_i e_{i+1} - (q + q^{-1})e_i e_{i+1} e_i + q^{-1} e_i^2 e_{i+1} = 0, \]
\[ qf_i f_{i+1} - (q + q^{-1})f_i f_{i+1} f_i + q^{-1} f_i^2 f_{i+1} = 0. \]

Proof. Let \( U \) be the unital associative algebra over \( \mathbb{C}(q) \) generated by the elements \( e_i, f_i, e_i, f_i \) \((i = 1, \ldots, n - 1), k_l (l = 1, \ldots, n), \) and \( q^h \) \((h \in P^+) \) with defining relations given in \((2.3)\). Using \((2.1)\) and \((2.2)\), the relations in \((2.3)\) can be derived easily. Thus there is a well-defined algebra homomorphism \( \phi : U \rightarrow U_q(\mathfrak{g}) \).

From the relation \((2.1)\), we obtain

\[
L_{i, i+j} = (q - q^{-1}) q^{-\sum_{h=1}^{j-1} k_{i+h}} \prod_{h=1}^{j-1} \text{ad} f_{i+h}(f_i),
\]
\[
L_{-i, i+j} = -(q - q^{-1}) q^{-\sum_{h=1}^{j-1} k_{i+h}} \prod_{h=1}^{j-1} \text{ad} f_{i+h}(f_i),
\]
\[
L_{-i, j-i} = (-1)^j(q - q^{-1}) q^{\sum_{h=1}^{j-1} k_{i+h}} \prod_{h=1}^{j-1} \text{ad} e_{i+h}(e_i),
\]
\[
L_{-i, j-i} = (-1)^j(q - q^{-1}) q^{\sum_{h=1}^{j-1} k_{i+h}} \prod_{h=1}^{j-1} \text{ad} e_{i+h}(e_i),
\]

where \( \text{ad} b_i(b_j) := b_i b_j - b_j b_i, \prod_{h=1}^{j-1} \text{ad} b_{i+h}(b_i) := \text{ad} b_{i+j} \cdots \text{ad} b_{i+1}(b_i) \) and \( \prod_{h=1}^{0} \text{ad} b_{i+h}(b_i) = b_i \) for \( b_i = e_i, f_i, f_i (i = 1, \cdots, n - 1, j > 0). \) It follows that the homomorphism \( \phi \) must be surjective.

It remains to prove \( \phi \) is injective. For this purpose, we will show that the relations in \((2.1)\) can be derived from the ones in \((2.3)\). The proof of our assertion is quite lengthy and tedious. But the basic idea is just the case-by-case check-up.

We define the sets

\[ \Lambda = \{(i, j) \in \mathbb{Z}/\{0\} \times \mathbb{Z}/\{0\} \mid -n \leq i \leq j \leq n\}, \quad \Lambda_1 = \{(i, j) \in \Lambda \mid i > 0, j > 0 \text{ and } i < j\}, \]
\[ \Lambda_2 = \{(i, j) \in \Lambda \mid i < 0, j > 0 \text{ and } |i| < |j|\}, \quad \Lambda_3 = \{(i, j) \in \Lambda \mid i < 0, j > 0 \text{ and } |i| > |j|\}, \]
\[ \Lambda_4 = \{(i, j) \in \Lambda \mid i < 0, j < 0 \text{ and } |i| > |j|\}, \quad \Lambda_5 = \{(i, j) \in \Lambda \mid i < 0, j > 0 \text{ and } |i| = |j|\}. \]
For \(((i, j), (k, l)) \in \Lambda \times \Lambda\), let \(a = \min\{|i|, |j|\}\), \(b = \max\{|i|, |j|\}\), \(c = \min\{|k|, |l|\}\), \(d = \max\{|k|, |l|\}\). We list all possible subsets of \(\Lambda \times \Lambda\):

\[
C_1 = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid c < d < a < b\}, \quad C_2 = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid c < d = a < b\},
\]

\[
C_3 = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid c < a < d < b\}, \quad C_4 = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid c < a < d = b\},
\]

\[
C_5 = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid c < a < b < d\}, \quad C_6 = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid c = a < d < b\},
\]

\[
C_7 = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid c = a < d < b\}, \quad C_8 = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid c = a < b < d\},
\]

\[
C_9 = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid a < c < d < b\}, \quad C_{10} = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid a < c < d = b\},
\]

\[
C_{11} = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid a < c < b < d\}, \quad C_{12} = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid a < b = c < d\},
\]

\[
C_{13} = \{((i, j), (k, l)) \in \Lambda \times \Lambda \mid a < b < c < d\}, \quad D_1 = \{((i, j), (k, l)) \in \Lambda_5 \times \Lambda \mid |i| < c < d\},
\]

\[
D_2 = \{((i, j), (k, l)) \in \Lambda_5 \times \Lambda \mid |i| = c < d\}, \quad D_3 = \{((i, j), (k, l)) \in \Lambda_5 \times \Lambda \mid c < |i| < d\},
\]

\[
D_4 = \{((i, j), (k, l)) \in \Lambda_5 \times \Lambda \mid c < |i| = d\}, \quad D_5 = \{((i, j), (k, l)) \in \Lambda_5 \times \Lambda \mid c < d < |i|\},
\]

\[
D_6 = \{((i, j), (k, l)) \in \Lambda_5 \times \Lambda \mid |k| < a < b\}, \quad D_7 = \{((i, j), (k, l)) \in \Lambda_5 \times \Lambda \mid |k| = a < b\},
\]

\[
D_8 = \{((i, j), (k, l)) \in \Lambda_5 \times \Lambda \mid a < |k| < b\}, \quad D_9 = \{((i, j), (k, l)) \in \Lambda_5 \times \Lambda \mid a < b = |k|\},
\]

\[
D_{10} = \{((i, j), (k, l)) \in \Lambda_5 \times \Lambda \mid a < b < |k|\}.
\]

We consider all cases for \(\Lambda_s \times \Lambda_t \cap C_i\) \((1 \leq s, t \leq 4, 1 \leq i \leq 13)\) and \(\Lambda_s \times \Lambda_t \cap D_i\) \((s = 5, 1 \leq t \leq 4\) or \(1 \leq s \leq 4, t = 5\) and \(1 \leq i \leq 10)\). Since the remaining cases can be checked similarly, we just prove:

\[
L_{i,j} L_{k,l} L_{i,j}^{-1} = q^{\varphi(l,i) - \varphi(k,i)} L_{k,l} \quad \text{if} \quad (k, l) \in \Lambda_1 \cup \Lambda_2,
\]

\[
L_{i,j} L_{k,l} - L_{k,l} L_{i,j} = 0 \quad \text{if} \quad ((i, j), (k, l)) \in \Lambda_1 \times \Lambda_1 \cap C_1,
\]

\[
L_{i,j} L_{k,l} - L_{k,l} L_{i,j} = (q - q^{-1}) L_{i,j} L_{k,l} \quad \text{if} \quad ((i, j), (k, l)) \in \Lambda_1 \times \Lambda_1 \cap C_2,
\]

\[
(L_{i,j})^2 = \frac{q - q^{-1}}{q + q^{-1}} (L_{-i,j})^2 \quad \text{if} \quad (i, j) \in \Lambda_2.
\]

From (2.4), we obtain

\[
L_{i,j} = \frac{L_{i,j}^{-1} j^{-1} j^{-1} (L_{j-1,j} L_{i,j-1} - L_{i,j-1} L_{j-1,j})}{q - q^{-1}} \quad \text{if} \quad (i, j) \in \Lambda_1 \cup \Lambda_2,
\]

\[
L_{i,j} = \frac{L_{i,j}^{-1} j^{-1} j^{-1} (L_{i,j+1} L_{i+1,j} - L_{i+1,j} L_{i,j+1})}{q - q^{-1}} \quad \text{if} \quad (i, j) \in \Lambda_3 \cup \Lambda_4.
\]

To prove (2.5), we use induction on \(l - k\):

\[
L_{i,j} L_{k,l} L_{i,j}^{-1} = \frac{L_{i,j}^{-1} j^{-1} j^{-1} (L_{l-1,j} L_{k,l-1} - L_{k,l-1} L_{l-1,j}) L_{i,j}^{-1}}{q - q^{-1}}
\]
From (2.33), we know that $f_i f_j - f_j f_i = 0$ if $|i - j| > 1$. By using induction on $j - i$ and (2.35), one can show that $L_{i,j} L_{k,k+1} - L_{k,k+1} L_{i,j} = 0$ when $((i,j), (k,k+1)) \in \Lambda_1 \times \Lambda_1 \cap C_1$. Similarly, one can prove $L_{i,j} L_{k,l} - L_{k,l} L_{i,j} = 0$ by induction on $l - k$. The proof of (2.7) is analogous (we use induction on $l - k$ and (2.5), (2.6)):

\[ L_{i,j} L_{k,l} = \frac{L_{i,j} - L_{i,j} L_{k,l} - L_{k,l} L_{i,j}}{q - q^{-1}} L_{k,l} = \frac{L_{i,j} - L_{i,j} L_{k,l} - L_{k,l} L_{i,j}}{q - q^{-1}} (L_{k,l} - L_{k,l} L_{i,j}) \]

To verify the relation (2.8), it suffices to show that

\[ (L_{j-1,j} L_{i,j-1} - L_{i,j-1} L_{j-1,j})^2 = \frac{q - q^{-1}}{q + q^{-1}} (L_{j-1,j} L_{i,j-1} - L_{i,j-1} L_{j-1,j})^2. \]

For this purpose, we need the following formulas for $(i,j) \in \Lambda_2$ which can be derived using induction:

\[ L_{j-1,j} L_{i,j-1} L_{j-1,j} = \frac{1}{q + q^{-1}} (q L_{i,j-1} L_{j-1,j}^2 + q^{-1} L_{j-1,j}^2 L_{i,j-1}), \]

\[ q L_{i,j-1} L_{j-1,j} - (q + q^{-1}) L_{j-1,j} L_{i,j-1} L_{i,j-1} + q^{-1} L_{j-1,j}^2 L_{i,j-1} = 0. \]

Using these formulae, we can verify the desired relations

\[ (L_{j-1,j} L_{i,j-1} - L_{i,j-1} L_{j-1,j})^2 \]

\[ = (L_{j-1,j} L_{i,j-1} L_{j-1,j}) L_{i,j-1} - \frac{q - q^{-1}}{q + q^{-1}} L_{j-1,j} L_{i,j-1} L_{j-1,j} - L_{i,j-1} L_{j-1,j} L_{i,j-1} \]

\[ + L_{i,j-1} (L_{j-1,j} L_{i,j-1} L_{j-1,j}) \]

\[ = \frac{q - q^{-1}}{q + q^{-1}} \left( \frac{q^{-1}}{q + q^{-1}} L_{j-1,j}^2 L_{i,j-1} + \frac{q}{q + q^{-1}} L_{i,j-1}^2 L_{j-1,j} - L_{j-1,j} L_{i,j-1} L_{j-1,j} \right) \]

\[ = \frac{q - q^{-1}}{q + q^{-1}} \left( (L_{j-1,j} L_{i,j-1} L_{j-1,j} - \frac{q}{q + q^{-1}} L_{i,j-1} L_{j-1,j}) L_{i,j-1} \right) \]
\[ + L_{-i,j-1}(L_{j-1,j}L_{-i,j-1}L_{j-1,j} - \frac{q^{-1}}{q+q^{-1}}L_{j-1,j}^2L_{-i,j-1}) - L_{j-1,i}L_{-i,j-1} \]

\[ = \frac{q-q^{-1}}{q+q^{-1}} (L_{j-1,j}L_{-i,j-1} - L_{-i,j-1}L_{j-1,j})^2. \]

\[ \square \]

Set \( \deg f_i = \deg f_i = -\alpha_i \), \( \deg q^h = \deg k_i = 0 \), \( \deg e_i = \deg e_i = \alpha_i \). Since all the defining relations of the quantum superalgebra \( U_q(\mathfrak{g}) \) are homogeneous, it has a root space decomposition

\[ U_q(\mathfrak{g}) = \bigoplus_{\alpha \in \mathbb{Q}} (U_q)_\alpha, \]

where \( (U_q)_\alpha = \{ u \in U_q(\mathfrak{g}) \mid q^h u q^{-h} = q^{\alpha(h)} u \text{ for all } h \in P^\vee \} \).

**Remark.** If we define

\( F_i = f_i q^{-k_{i+1}}, \ E_i = q^{k_{i+1}} e_i, \)

one can see that the relations involving \( E_i, F_i \) and \( q^h \) are the same as the standard relations for \( U_q(\mathfrak{gl}_n) \) (see, for example, [HK, Definition 7.1.1]). Hence \( U_q(\mathfrak{gl}_n) \) is a subalgebra of \( U_q(\mathfrak{g}) \).

The comultiplication \( \Delta \) of \( U_q(\mathfrak{g}) \) is given by the formula

\[ (2.9) \quad \Delta(L_{i,j}) = \sum_{k=i}^j L_{i,k} \otimes L_{k,j}, \]

(see §4 in [O]). In terms of the new generators we have:

\[ \Delta(q^h) = q^h \otimes q^h \text{ for every } h \in P^\vee, \]

\[ \Delta(e_i) = q^{-k_{i+1}} \otimes e_i + e_i \otimes q^{-k_i}, \]

\[ \Delta(f_i) = q^{k_i} \otimes f_i + f_i \otimes q^{k_{i+1}}, \]

\[ \Delta(e_i) = q^{-k_{i+1}} \otimes e_i - (q-q^{-1}) e_i \otimes k_i \]

\[ + (q-q^{-1}) \left( \sum_{j=1}^{i-1} (-1)^{j+1} q^{\sum_{h=1}^{j} k_{i-j+h}} \prod_{h=1}^{j} \text{ad } e_{i-j+h}(e_{i-j}) \right) \]

\[ \otimes q^{-\sum_{h=1}^{j-1} k_{i-j+h}} \prod_{h=1}^{j-1} \text{ad } f_{i-j+h}(f_{i-j}) \right) \]

\[ + (q-q^{-1}) \left( \sum_{j=1}^{i-1} (-1)^{j} q^{\sum_{h=1}^{j} k_{i-j+h}} \prod_{h=1}^{j} \text{ad } e_{i-j+h}(e_{i-j}) \right) \]
\[ \otimes q^{-\sum_{i=1}^{n-1}k_{i-j+h}} \prod_{h=1}^{j-1} \text{ad } f_{i-j+h}(f_{i-j}) + e_i \otimes q^{k_i}, \]

\[ \Delta(f_i) = q^{-k_i} \otimes f_i \]

\[ + (q - q^{-1}) \left( \sum_{j=1}^{i-1} (-1)^j q^{\sum_{h=1}^{j-1}k_{i-j+h}} \prod_{h=1}^{j-1} \text{ad } e_{i-j+h}(e_{i-j}) \right) \]

\[ \otimes q^{-\sum_{i=1}^{n-1}k_{i-j+h}} \prod_{h=1}^{j-1} \text{ad } f_{i-j+h}(f_{i-j}) \]

\[ + (q - q^{-1}) \left( \sum_{j=1}^{i-1} (-1)^j q^{\sum_{h=1}^{j-1}k_{i-j+h}} \prod_{h=1}^{j-1} \text{ad } e_{i-j+h}(e_{i-j}) \right) \]

\[ \otimes q^{-\sum_{i=1}^{n-1}k_{i-j+h}} \prod_{h=1}^{j-1} \text{ad } f_{i-j+h}(f_{i-j}) \]

\[ + (q - q^{-1}) k_i \otimes f_i + f_i \otimes q^{k_i+1}, \]

\[ \Delta(k_i) = q^{-k_i} \otimes k_i \]

\[ + (q - q^{-1}) \left( \sum_{j=1}^{i-1} (-1)^j q^{\sum_{h=1}^{j-1}k_{i-j+h}} \prod_{h=1}^{j-1} \text{ad } e_{i-j+h}(e_{i-j}) \right) \]

\[ \otimes q^{-\sum_{i=1}^{n-1}k_{i-j+h}} \prod_{h=1}^{j-1} \text{ad } f_{i-j+h}(f_{i-j}) \]

\[ + (q - q^{-1}) \left( \sum_{j=1}^{i-1} (-1)^j q^{\sum_{h=1}^{j-1}k_{i-j+h}} \prod_{h=1}^{j-1} \text{ad } e_{i-j+h}(e_{i-j}) \right) \]

\[ \otimes q^{-\sum_{i=1}^{n-1}k_{i-j+h}} \prod_{h=1}^{j-1} \text{ad } f_{i-j+h}(f_{i-j}) \]

\[ + k_i \otimes q^{k_i}. \]

Let $U_q^+$ (respectively, $U_q^-$) be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements $e_i, e_i^-$ (respectively, $f_i, f_i^-$) for $i = 1, \ldots, n-1$, and let $U_q^0$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $q^h$ ($h \in P^\vee$) and $k_l$ for $l = 1, \ldots, n$. In addition, let $U_q^{\geq 0}$ (respectively, $U_q^{\leq 0}$) be the subalgebra of $U_q(\mathfrak{g})$ generated by $U_q^+$ and $U_q^0$ (respectively, by $U_q^-$ and $U_q^0$). We will show that the quantum superalgebra $U_q(\mathfrak{g})$ has a triangular decomposition. For this purpose, we need the following lemma.

**Lemma 2.2.**

\[ U_q^{\geq 0} \cong U_q^0 \otimes U_q^+, \quad U_q^{\leq 0} \cong U_q^- \otimes U_q^0. \]
Proof. We will prove the second part. Let \( \{ f_\zeta \mid \zeta \in \Omega \} \) be a basis of \( U_q^- \) consisting of monomials in \( f_i \) and \( f_i \)'s (\( i \in I \)). Consider a set \( \Omega' = \{(a_1, \ldots, a_n) \mid a_i = 0 \text{ or } 1 \text{ for all } i \in J \} \). Then \( \{ q^h k_\eta \mid h \in P^\vee, \eta \in \Omega' \} \) is a basis of \( U_q^0 \), where \( k_\eta = k_\eta^{a_1} \cdots k_\eta^{a_n} \) for \( \eta = (a_1, \ldots, a_n) \) by [O Theorem 6.2]. By the defining relations of \( U_q \), it is easy to see that the elements \( f_\zeta q^h k_\eta \ (\zeta \in \Omega, \ h \in P^\vee, \ \eta \in \Omega') \) span \( U_q^{\leq 0} \). Thus there is a surjective \( \mathbb{C}(q) \)-linear map \( U_q^{-} \otimes U_q^{0} \rightarrow U_q^{\leq 0} \) given by \( f_\zeta \otimes q^h k_\eta \rightarrow f_\zeta q^h k_\eta \). To show that this map is injective, it suffices to show that the elements \( f_\zeta q^h k_\eta \ (\zeta \in \Omega, \ h \in P^\vee, \ \eta \in \Omega') \) are linearly independent over \( \mathbb{C}(q) \).

Suppose
\[
\sum_{\zeta \in \Omega, \ h \in P^\vee, \ \eta \in \Omega'} C_{\zeta, h, \eta} f_\zeta q^h k_\eta = 0 \quad \text{for some} \quad C_{\zeta, h, \eta} \in \mathbb{C}(q).
\]

We may write
\[
\sum_{\beta \in Q_+} \left( \sum_{\deg f_\zeta = -\beta, \ h \in P^\vee, \ \eta \in \Omega'} C_{\zeta, h, \eta} f_\zeta q^h k_\eta \right) = 0 \quad \text{for some} \quad C_{\zeta, h, \eta} \in \mathbb{C}(q).
\]

Since \( U_q(\mathfrak{g}) = \bigoplus_{\beta \in Q} (U_q)_\beta \), we have
\[
\sum_{\deg f_\zeta = -\beta, \ h \in P^\vee, \ \eta \in \Omega'} C_{\zeta, h, \eta} f_\zeta q^h k_\eta = 0 \quad \text{for each} \quad \beta \in Q_+.
\]

Write \( \beta = -\sum_{i=1}^{n-1} m_i \alpha_i \) \( (m_i \in \mathbb{Z}_{\geq 0}) \), and let \( h_\beta = \sum_{i=1}^{n-1} m_i k_{i+1} \). Since \( f_\zeta \) is a monomial in \( f_i \) and \( f_i \)'s, the term of degree \( (-\beta, 0) \) in \( \Delta(f_\zeta) \) is \( f_\zeta \otimes q^{h_\beta} \). We consider the terms of degree \( (0,0) \) in \( \Delta(k_\eta) \) where \( \eta = (a_1, \cdots, a_n) \). Then the terms of degree \( (0,0) \) in \( \Delta(k_\eta) \) can be written as
\[
(q^{-k_i} \otimes k_1 + k_i \otimes q^{k_i})^{a_1} \cdots (q^{-k_n} \otimes k_n + k_n \otimes q^{k_n})^{a_n} = \prod_{i=1}^{n} \left( \sum_{j=0}^{a_i} q^{-(a_i-j)k_i} k_j^{a_i-j} \right)
\]
\[
= \sum_{(j_1, \ldots, j_n) \in \Omega'} \prod_{i=1}^{n} \left( q^{-(a_i-j_i)k_i} k_i^{a_i-j_i} \otimes q^{j_i k_i} k_i^{a_i-j_i} \right).
\]

Since the terms of degree \( (-\beta, 0) \) of \( \sum C_{\zeta, h, \eta} \Delta(f_\zeta q^h k_\eta) \) must sum to zero, we have
\[
(2.10) \quad 0 = \sum_{\eta = (a_1, \cdots, a_n) \in \Omega'} \left( \sum_{\deg f_\zeta = -\beta, \ h \in P^\vee, \ \eta \in \Omega'} C_{\zeta, h, \eta} f_\zeta q^h \left( \prod_{i=1}^{n} q^{-(a_i-j_i)k_i} k_i^{a_i-j_i} \right) \otimes q^{h_\beta + h} \left( \prod_{i=1}^{n} q^{j_i k_i} k_i^{a_i-j_i} \right) \right).
\]
For all \((a_1 - j_1, \ldots, a_n - j_n) \in \Omega'\) and \(h \in P^\vee\), the elements \(q^h \left( \prod_{i=1}^n k_i^{a_i - j_i} \right)\) are linearly independent. Set \(\eta_1 := (1, \ldots, 1)\). Since there is only one pair of \((a_1, \ldots, a_n)\) and \((j_1, \ldots, j_n)\) such that \(\prod_{i=1}^n k_i^{a_i - j_i} = k_{\eta_1}\) in the above sum, we obtain

\[
0 = \sum_{h \in P^\vee} \sum_{\deg f_\zeta = -\beta} C_{\zeta, h, \eta_1} f_\zeta q^{h - \sum_{i=1}^n k_i} \otimes q^{h_\beta + h k_{\eta_1}}
\]

\[
+ \sum_{\eta = (a_1, \ldots, a_n), (j_1, \ldots, j_n), (a_1 - j_1, \ldots, a_n - j_n) \neq \eta_1} \sum_{h \in P^\vee, \deg f_\zeta = -\beta} C_{\zeta, h, \eta} f_\zeta q^{h \left( \prod_{i=1}^n q^{-(a_i - j_i)k_i k_i^{j_i}} \right)} \otimes q^{h_\beta + h \left( \prod_{i=1}^n q^{j_i k_i a_i - j_i} \right)}.
\]

Thus we have

\[
\sum_{\deg f_\zeta = -\beta} C_{\zeta, h, \eta_1} f_\zeta q^{h - \sum_{i=1}^n k_i} = 0 \text{ for all } h \in P^\vee.
\]

Multiplying by \(q^{h_\beta + \sum_{i=1}^n k_i}\) from the right we obtain

\[
\sum_{\deg f_\zeta = -\beta} C_{\zeta, h, \eta_1} f_\zeta = 0 \text{ for all } h \in P^\vee.
\]

Using the linear independence of \(f_\zeta\), we conclude all \(C_{\zeta, h, \eta_1} = 0\) for all \(\zeta \in \Omega, h \in P^\vee\). Now consider general \(\eta = (a_1, \ldots, a_n) \in \Omega'\). Assume that for all \(\eta' = (a'_1, \ldots, a'_n)\) such that \(a'_i \geq a_i\) for all \(i \in J\) and \(\eta' \neq \eta\), \(C_{\zeta, h, \eta'} = 0\) for all \(\zeta \in \Omega, h \in P^\vee\). Then there is only one pair of \((a_1, \ldots, a_n)\) and \((j_1, \ldots, j_n)\) such that \((a_1 - j_1, \ldots, a_n - j_n) = \eta\) in (2.10). Repeating the above argument, we conclude \(C_{\zeta, h, \eta} = 0\) for all \(\zeta \in \Omega, h \in P^\vee\).

For example, consider \(\eta_2 = (0, 1, \ldots, 1)\). Since \(C_{\zeta, h, \eta_1} = 0\), there is only one pair of \((a_1, \ldots, a_n)\) and \((j_1, \ldots, j_n)\) such that \((a_1 - j_1, \ldots, a_n - j_n) = (0, 1, \ldots, 1)\) in (2.10). Thus we have

\[
\sum_{\deg f_\zeta = -\beta} C_{\zeta, h, \eta_2} f_\zeta q^{h - \sum_{i=1}^n k_i} = 0 \text{ for all } h \in P^\vee.
\]

Multiplying \(q^{h_\beta + \sum_{i=1}^n k_i}\) and using the linear independence of \(f_\zeta\), we obtain \(C_{\zeta, h, \eta_2} = 0\) for all \(\zeta \in \Omega, h \in P^\vee\). \(\square\)

We are now ready to prove the triangular decomposition for \(U_q(\mathfrak{g})\).

**Theorem 2.3.** There is a \(\mathbb{C}(q)\)-linear isomorphism

\[
U_q(\mathfrak{g}) \cong U_q^- \otimes U_q^0 \otimes U_q^+.
\]

**Proof.** Let \(\{ f_\zeta \mid \zeta \in \Omega \}, \{ q^h k_\eta \mid h \in P^\vee, \eta \in \Omega' \}\), and \(\{ e_\tau \mid \tau \in \Omega \}\) be monomial bases of \(U_q^-, U_q^0\) and \(U_q^+\) respectively, where \(\Omega\) and \(\Omega'\) are the index sets as in the proof for Lemma
It suffices to show that the elements $f_{\zeta} q^h k_\eta e_\tau$ ($\zeta, \tau \in \Omega$, $h \in P^\vee, \eta \in \Omega'$) are linearly independent over $\mathbb{C}(q)$.

Suppose
\[ \sum_{\zeta, h, \eta, \tau} C_{\zeta, h, \eta, \tau} f_{\zeta} q^h k_\eta e_\tau = 0 \quad \text{for some } C_{\zeta, h, \eta, \tau} \in \mathbb{C}(q). \]

The root space decomposition of $U_q(\mathfrak{g})$ yields
\[ \sum_{\deg f_{\zeta} + \deg e_\tau = \gamma} C_{\zeta, h, \eta, \tau} f_{\zeta} q^h k_\eta e_\tau = 0 \quad \text{for all } \gamma \in Q. \]

Using the partial ordering on $\mathfrak{h}_0^*$, we can choose $\alpha = \deg f_{\zeta}$ and $\beta = \deg e_\tau$, which are minimal and maximal, respectively, among those for which $\alpha + \beta = \gamma$ and $C_{\zeta, h, \eta, \tau}$ is nonzero. If $\alpha = -\sum m_i \alpha_i$, set $h_\alpha = \sum m_i k_i + 1$, and if $\beta = \sum n_i \alpha_i$, set $h_\beta = \sum n_i k_i + 1$. The term of degree $(0, \beta)$ in $\Delta(e_\tau)$ is $q^{-h_\beta} \otimes e_\tau$ and the term of degree $(\alpha, 0)$ of $\Delta(f_{\zeta})$ is $f_{\zeta} \otimes q^{h_\alpha}$.

Since the terms of degree $(\alpha, \beta)$ of $\sum C_{\zeta, h, \eta, \tau} \Delta(f_{\zeta} q^h k_\eta e_\tau)$ must sum to zero, we have
\[ \sum_{\deg f_{\zeta} = \alpha, \deg e_\tau = \beta} \sum_{(j_1, \ldots, j_n) \in \mathbb{R}^D, \sum j_i = \alpha} C_{\zeta, h, \eta, \tau} f_{\zeta} q^h \left( \prod_{i=1}^n q^{-(a_i-j_i)k_i - h_\beta j_i} k_i^{j_i} \right) \otimes q^{h_\alpha + h} \left( \prod_{i=1}^n q^{j_i k_i a_i - j_i} \right) e_\tau = 0. \]

The elements $f_{\zeta} q^h \left( \prod_{i=1}^n k_i^{j_i} \right)$ are linearly independent for $\zeta \in \Omega$, $h \in P^\vee$, $(j_1, \ldots, j_n) \in \Omega'$ by Lemma 2.2. By the similar argument in the proof for Lemma 2.2, we obtain
\[ \sum_{\deg e_\tau = \beta} C_{\zeta, h, \eta, \tau} e_\tau = 0 \quad \text{for all } h \in P^\vee, \zeta \in \Omega, \text{ and } \eta \in \Omega'. \]

Using the linear independence of $e_\tau$, we conclude that $C_{\zeta, h, \eta, \tau} = 0$ for all $\zeta \in \Omega, h \in P^\vee, \eta \in \Omega'$, and $\tau \in \Omega$, as desired. \qed

3. The Quantum Clifford Superalgebra $\text{Cliff}_q(\lambda)$

We first introduce some notation that will be used in this section only. Let $\mathbb{K}$ be a field of zero characteristic and $A$ be an associative $\mathbb{K}$-algebra. Denote by Mat$_n(A)$ the associative $\mathbb{K}$-algebra of $n \times n$ matrices with entries in $A$. If $A$ is a superalgebra, then Mat$_n(A)$ is a superalgebra as well by setting Mat$_n(A)_{\overline{1}} = \text{Mat}_n(A_{\overline{1}})$. By sMat$_{n|n}(\mathbb{K})$ we denote the associative superalgebra of $2n \times 2n$ matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A$, $B$, $C$, and $D$ are in Mat$_n(\mathbb{K})$ and
\[ \text{sMat}_{n|n}(\mathbb{K})_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad \text{sMat}_{n|n}(\mathbb{K})_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}. \]
Let $Q_n(\mathbb{K})$ be the subsuperalgebra of $s\text{Mat}_{n|n}(\mathbb{K})$ with elements $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$. In particular, $Q_n(\mathbb{K})_0 = Q_n(\mathbb{K})_\bar{1} = \text{Mat}_n(\mathbb{K})$. There are $\mathbb{K}$-superalgebra isomorphisms

\[ \text{Mat}_r(\text{sMat}_{|1}(\mathbb{K})) \cong \text{sMat}_{r|\bar{r}}(\mathbb{K}), \text{Mat}_r(Q_{1}(\mathbb{K})) \cong Q_r(\mathbb{K}). \]

Note that if $\mathbb{K} = \mathbb{C}$, then the superalgebra $Q_n(\mathbb{C})$ coincides with $\mathfrak{g}$ as a complex vector space. Another example of a $\mathbb{K}$-superalgebra is any extension $\mathbb{K}(\alpha)$ of $\mathbb{K}$ of degree $2$ considering $\alpha$ as an odd element. If $\alpha^2 = \beta \in \mathbb{K}$ we will denote $\mathbb{K}(\alpha)$ by $\mathbb{K}(\sqrt{\beta})$.

In this section, we set $\mathbb{F} = \mathbb{C}(q)$. For every $\lambda \in P$ we define $I^q(\lambda)$ to be the left ideal of $U_q^0$ generated by $q^h - q^{\lambda(h)}1$, $h \in P^\vee$. Set $\text{Cliff}_q(\lambda) := U_q^0/I^q(\lambda)$. We may consider $\text{Cliff}_q(\lambda)$ as the associative $\mathbb{F}$-algebra generated by the identity $1 = 1 + I^q(\lambda)$ and $t_i := k_i + I^q(\lambda)$ satisfying the relations

\[ t_it_j + t_jt_i = \delta_{ij} \frac{2(q^{2\lambda_i} - q^{-2\lambda_i})}{q^2 - q^{-2}} 1, \quad i, j = 1, \ldots, n. \]

Furthermore, $\text{Cliff}_q(\lambda)$ has an obvious $\mathbb{Z}_2$-grading (and thus a superalgebra structure) by assuming that $t_i$ are odd. More precisely, $\text{Cliff}_q(\lambda)_0$ is spanned by $1$ and the monomials $t_{i_1} \cdots t_{i_{2k}}$ of even degree, while $\text{Cliff}_q(\lambda)_{\bar{1}}$ is spanned by those of odd degree. In this section we will describe the structure of $\text{Cliff}_q(\lambda)$ and will classify its irreducible modules. Because of its superalgebra structure, $\text{Cliff}_q(\lambda)$ has both $\mathbb{Z}_2$-graded and nongraded modules and both cases will be addressed.

The results in this section may be derived from more general statements about quadratic forms and Clifford superalgebras over arbitrary fields (see, for example, [Lam] and [Sh]). For the sake of completeness we will give an outline of the proofs. The results and the proofs in this section will also help us to describe explicitly the action of $U_q^0$ on the highest weight vectors of an irreducible highest weight module over $U_q(\mathfrak{g})$. This is demonstrated in Example 3.10 for the case $n = 3$ and $\lambda = (4, 2, 1)$.

In this section, we fix $V := \bigoplus_{i=1}^n \mathbb{F}t_i$ and $\Lambda := (\Lambda_1, \ldots, \Lambda_n) \in \mathbb{F}^n$ and denote by $B_\Lambda : V \times V \to \mathbb{F}$ the symmetric bilinear form defined by $B_\Lambda(t_i, t_j) = \delta_{ij}\Lambda_i$. Let $\text{Cliff}_q(\Lambda)$ be the unique up to isomorphism Clifford algebra associated to $V$ and $B_\Lambda$. If $\Lambda_i = \frac{q^{2\lambda_i} - q^{-2\lambda_i}}{q^2 - q^{-2}}$, then we have $\text{Cliff}_q(\Lambda) \simeq \text{Cliff}_q(\lambda)$.

Define $V(\Lambda) := V/\ker B_\Lambda$, where $\ker B_\Lambda := \{v \in V \mid B_\Lambda(v, u) = 0, \text{ for every } u \in V\}$ and denote by $\beta_\Lambda$ the restriction of $B_\Lambda$ on $V(\Lambda)$. Let $N_\Lambda = \{i \mid \Lambda_i \neq 0\}$, $Z_\Lambda = \{j \mid \Lambda_j = 0\}$, and $|\Lambda| = \#N_\Lambda$. Set $\Lambda_N := (\Lambda_{i_1}, \ldots, \Lambda_{i_{|\Lambda|}})$, $0_Z := (\Lambda_{j_1}, \ldots, \Lambda_{j_{n-|\Lambda|}}) = (0, \ldots, 0)$, where $N_\Lambda = \{i_1, \ldots, i_{|\Lambda|}\}$, $Z_\Lambda = \{j_1, \ldots, j_{n-|\Lambda|}\}$, and $i_1 < \ldots < i_{|\Lambda|}$. It is clear that $\ker B_\Lambda = \bigoplus_{j \in Z_\Lambda} \mathbb{F}t_j$ and)}
that $\text{Cliff}_q(\Lambda_N) = \bigoplus_{i \in \Lambda} \mathbb{F} t_i$ is the Clifford algebra corresponding to $(V(\Lambda), \beta_\Lambda)$. Furthermore,

$$\text{Cliff}_q(\Lambda) \simeq \text{Cliff}_q(\Lambda_N) \otimes_{\mathbb{F}} \text{Cliff}_q(0_Z) \simeq \text{Cliff}_q(\Lambda_N) \otimes_{\mathbb{F}} \bigwedge \ker B_\Lambda.$$  

Here $\bigwedge W$ denotes the exterior algebra of the vector space $W$. Thanks to the above isomorphisms every $\text{Cliff}_q(\Lambda)$-module can be considered as a $\text{Cliff}_q(\Lambda_N)$-module under the embedding $\text{Cliff}_q(\Lambda_N) = \text{Cliff}_q(\Lambda_N) \otimes_{\mathbb{F}} 1 \to \text{Cliff}_q(\Lambda_N) \otimes_{\mathbb{F}} \text{Cliff}_q(0_Z)$. The class $\overline{\Delta(\Lambda)}$ of $\Delta(\Lambda) = \Pi_{i \in \Lambda} \Lambda_i$ in $\mathbb{F}/\mathbb{F}^2$ is called the discriminant of $(V, B_\Lambda)$.

The following lemma is standard and the proof is left to the reader.

**Lemma 3.1.** Let $M$ be an irreducible $\text{Cliff}_q(\Lambda)$-module. Then $M$ is an irreducible $\text{Cliff}_q(\Lambda_N)$-module and $t_i^* v = 0$ for every $i \in \Lambda$. Conversely, if $M_0$ is an irreducible $\text{Cliff}_q(\Lambda_N)$-module then $M_0$ considered as a $\text{Cliff}_q(\Lambda)$-module with trivial action of $\text{Cliff}_q(0_Z)$ is irreducible as well.

Since our goal in this section is to classify the irreducible representations of $\text{Cliff}_q(\Lambda)$, thanks to the above lemma, we may assume that $\Lambda_i$ are nonzero. So, for simplicity we fix $Z_\Lambda = \emptyset$, and thus $B_\Lambda = \beta_\Lambda$ and $V(\Lambda) = V$, in all statements preceding Corollary 3.7.

Recall that a vector $v$ in $V$ is called $\beta_\Lambda$-isotropic (or simply isotropic) if $\beta_\Lambda(v, v) = 0$. A subspace $W$ of $V$ is $\beta_\Lambda$-isotropic subspace if $\beta_\Lambda(u, w) = 0$ for every $u$ and $w$ in $W$. A subspace $W$ of $V$ is anisotropic if it contains no nonzero $\beta_\Lambda$-isotropic vector. An isotropic subspace $W$ of $V$ is maximal isotropic if there is no larger $\beta_\Lambda$-isotropic subspace containing $W$.

**Lemma 3.2.** Let $W$ be an isotropic subspace of $V$. Then there exists an isotropic subspace $W^*$ and a subspace $Z$ of $V$ such that

$$V = Z \oplus W \oplus W^*, \quad \dim W = \dim W^*,$$

$$\beta_\Lambda(z, w) = \beta_\Lambda(z, w^*) = 0 \quad \text{for every } z \in Z, w \in W, w^* \in W^*.$$  

Moreover, there exist bases $\{w_1, \ldots, w_m\}$ and $\{w_1^*, \ldots, w_m^*\}$ of $W$ and $W^*$, respectively, such that $\beta_\Lambda(w_i, w_j^*) = \delta_{ij}$.

**Proof.** The lemma follows by induction on $\dim W$. If $\dim W = 1$, then $W^*$ is spanned by $w_1^* = x - \frac{1}{2} \beta_\Lambda(x, x) w_1$, where $x \in V$ is arbitrarily chosen so that $\beta_\Lambda(w_1, x) = 1$. Then we define $Z$ to be

$$Z = \{z \in V \mid \beta_\Lambda(z, w_1) = \beta_\Lambda(z, w_1^*) = 0\}.$$  

For the complete proof, see [SH, Lemma 1.3].

The decomposition $V = Z \oplus W \oplus W^*$ in Lemma 3.2 is called a weak Witt decomposition of $V$. For any weak Witt decomposition $V = Z \oplus W \oplus W^*$, we denote by $\text{Cliff}(\Lambda_Z)$ the Clifford
algebra corresponding to \((Z, \beta_\Lambda|Z)\). If \(V = Z \oplus W \oplus W^*\) is a weak Witt decomposition for which \(Z\) is anisotropic (or, equivalently, \(W\) is maximal isotropic) we call it a Witt decomposition. We may identify \(W^*\) with the dual space of \(W\) via the nondegenerate form \(\beta_\Lambda\). If \(V = Z \oplus W \oplus W^*\) is a weak Witt decomposition for which \(Z\) is anisotropic (or, equivalently, \(W\) is maximal isotropic) we call it a Witt decomposition. We say that the Witt index is maximal if \(\dim Z \leq 1\). Recall that if the ground field is \(\mathbb{C}\), the Witt index is always maximal. In the case of arbitrary \(F\) though, the Witt index is generally not maximal as we verify in Lemma 3.6. In order to find a Witt decomposition and the Witt index of \((V, \beta_\Lambda)\) we need some preparatory statements.

**Lemma 3.3.** Let \(V = Z \oplus W \oplus W^*\) be a weak Witt decomposition and let \(m = 2^{\dim W}\). Then \(\text{Cliff}_q(\Lambda) \cong \text{Mat}_m(\text{Cliff}_q(\Lambda|Z))\). Moreover, we have

\[
\text{Cliff}_q(\Lambda)|_0 \cong \begin{cases} 
\text{Mat}_m(\text{Cliff}_q(\Lambda|Z)) & \text{if } Z \neq 0, \\
\text{Mat}_{m/2}(F) \oplus \text{Mat}_{m/2}(F) & \text{if } Z = 0.
\end{cases}
\]

**Proof.** For the complete proof, see [Sh, Theorem 2.6]. The proof follows by induction on \(\dim W\). We sketch the proof for \(\dim W = 1\). In this case there is an isomorphism \(\Psi : \text{Cliff}_q(\Lambda) \rightarrow \text{Mat}_m(\text{Cliff}_q(\Lambda|Z))\) defined by its restriction \(\Psi|_V\) on \(V\):

\[
z + rw_1 + sw_1^* \mapsto \begin{pmatrix} z & r \\ s & -z \end{pmatrix}.
\]

Notice that if \(Z \neq 0\), \(\Psi\) is not necessarily parity preserving. In such a case we choose the isomorphism \(\Theta : \text{Cliff}_q(\Lambda) \rightarrow \text{Mat}_m(\text{Cliff}_q(\Lambda|Z))\) defined by \(\Theta(\alpha) = D^{-1}\Psi(\alpha)D\), where \(D = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\) and any \(g \in Z\) with \(\beta_\Lambda(g, g) \neq 0\). \(\square\)

**Lemma 3.4.** The nondegenerate Legendre’s equation always has a nontrivial solution in \(F\): for every nonzero \(A, B, C\) in \(F\), there exist \(X, Y, Z \in F\) with \((X, Y, Z) \neq (0, 0, 0)\) such that \(AX^2 + BY^2 + CZ^2 = 0\).

**Proof.** We modify the proof of the classical Legendre’s Theorem (see, for example, [IR, §17.3]). We first assume that \(A, B, C, X, Y, Z\) are polynomials in \(\mathbb{C}[q]\), where \(A, B, C\) are square free. We may fix \(C = -1\), since if \((X, Y, Z)\) is a solution of \(ACX^2 + BCY^2 = Z^2\) then \((X, Y, \sqrt{-1}Z)\) is a solution of \(AX^2 + BY^2 + CZ^2 = 0\). We prove that \(AX^2 + BY^2 = Z^2\) has a nontrivial solution by induction on \(N := \max\{\deg A, \deg B\}\).

If \(N = 0\); i.e., \(A\) and \(B\) are constant polynomials, then \(AX^2 + BY^2 = Z^2\) has a solution (constant polynomials). Assume that \(\deg B \leq \deg A\) and \(\deg A \geq 1\). Recall that every polynomial \(R \in \mathbb{C}[q]\) is a quadratic residue modulo any square free polynomial \(S\). Indeed, if
$S$ is constant, our assertion is obvious. Otherwise, let $S(q) = \Pi_{i=1}^r (q - z_i)$ with $z_i \neq z_j$, and let $y_i \in \mathbb{C}$ be such that $y_i^2 = R(z_i)$. Then $y_i^2 \equiv R \pmod{(q - z_i)}$. Using the Chinese Remainder Theorem, we find $y \in \mathbb{C}[q]$ for which $y \equiv y_i \pmod{(q - z_i)}$. But then $y^2 \equiv R \pmod{(q - z_i)}$ and thus $y^2 \equiv R \pmod{S}$.

We fix $C_1$ with $\deg C_1 < \deg A$ such that $C_1^2 \equiv B \pmod{A}$. Then $C_1^2 - B = AT = AA_1 M^2$ for some square free polynomial $A_1$. Since $\deg A + \deg A_1 \leq \deg (AA_1 M^2) = \deg (C_1^2 - B) < 2 \deg A$, we have $0 \leq \deg A_1 < \deg A$. Now we observe that if $(X_1, Y_1, Z_1)$ is a solution of $A_1 X^2 + B Y^2 = Z^2$, then $(A_1 X_1 M, C_1 Y_1 + Z_1, Z_1 C_1 + B Y_1)$ is a solution of $AX^2 + BY^2 = Z^2$.

Using the induction hypothesis, we complete the proof.

**Remark.** Lemma 3.3 may be proved with a standard algebro-geometric argument using dimensions, see, for example, [Har, Exercise 11.6]. The lemma is also a particular case of the following Theorem of Tsen-Lang: if $K$ is a field of transcendence degree $n$ over an algebraically closed field $k$, then any quadratic form over $K$ of dimension bigger than $2^n$ is isotropic. For details, see [Lam, Chapter XI].

In what follows, we assume $\Lambda_i = \frac{q^{2 \lambda_i} - q^{-2 \lambda_i}}{q^2 - q^{-2}}$. For simplicity, we will write $\beta_\lambda, |\lambda|$, and $\Delta(\lambda)$ for $\beta_\lambda, |\Lambda|$, and $\Delta(\Lambda)$, respectively. The following technical lemma can be easily verified.

**Lemma 3.5.** Define an equivalence relation $\sim$ in \( \{ \lambda_i \mid i = 1, \ldots, n \} \) by $\lambda_i \sim \lambda_j$ if $\lambda_i^2 = \lambda_j^2$ and denote by $o(\lambda_i)$ the orbit of $\lambda_i$ relative to $\sim$. Then $\overline{\Delta(\lambda)} = \overline{1}$ (or, equivalently, $\Delta(\lambda)$ is a square in $\mathbb{F}$) if and only if the orbit $o(\lambda_i)$ of every $\lambda_i \neq \pm 1$ contains even number of elements.

**Lemma 3.6.** The space $V$ is anisotropic if and only if $\dim V = 1$ or $\dim V = 2$ and $\overline{\Delta(\lambda)} \neq \overline{1}$. If $V$ is isotropic, there is a Witt decomposition $V = Z \oplus W \oplus W^*$ of $V$ such that

1. $\dim W = k$ if $\dim V = 2k + 1$, $k \geq 1$ (maximal Witt index);
2. $\dim W = k - 1$ if $\dim V = 2k$ and $\overline{\Delta(\lambda)} \neq \overline{1}$;
3. $\dim W = k$ if $\dim V = 2k$ and $\overline{\Delta(\lambda)} = \overline{1}$ (maximal Witt index).

In particular, if $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$, then $\dim W = \left\lceil \frac{n-1}{2} \right\rceil$.

**Proof.** The proof consists of several steps.

Step 1: The case $\dim V = 1$. This case is straightforward.

Step 2: The case $\dim V = 2$. In this case, $v = a_1 t_1 + a_2 t_2$ is $\beta_\lambda$-isotropic if and only if $a_1^2 A_1 + a_2^2 A_2 = 0$. The latter equation has a solution for $a_1$ and $a_2$ if and only if $\frac{A_1}{A_2}$ is a square (or equivalently, $\Lambda_1 A_2$ is a square).

Step 3: If $\dim V \geq 3$, then $V \cong Fw \oplus F w^* \oplus F v_3 \oplus \ldots \oplus F v_n$, where
where 
\[
\beta_\lambda(w, w) = \beta_\lambda(w^*, w^*) = \beta_\lambda(w, v_i) = \beta_\lambda(w^*, v_i) = 0 \quad \text{for } i \geq 3,
\]
\[
\beta_\lambda(w, w^*) = 1, \quad \beta_\lambda(v_3, v_3) = \Lambda_1\Lambda_2\Lambda_3, \quad \beta_\lambda(v_i, v_i) = \Lambda_i \quad \text{if } i \geq 4.
\]

Let us first consider the case \( \dim V = 3 \). We use Lemma 3.4 to find \( w = x_1t_1 + x_2t_2 + x_3t_3 \) such that \( \beta_\lambda(w, w) = 0 \). Applying Lemma 3.2 to \( W = \mathbb{F}w \), we find \( w^* = y_1t_1 + y_2t_2 + y_3t_3 \) and \( z = z_1t_1 + z_2t_2 + z_3t_3 \) such that 
\[
\beta_\lambda(w^*, w^*) = \beta_\lambda(w^*, z) = \beta_\lambda(w, z) = 0, \quad \beta_\lambda(w, w^*) = 1.
\]
The choice of \( z \) is unique up to a multiplication by a nonzero constant in \( \mathbb{F} \). A simple calculation shows that \( z_i \) may be chosen as follows 
\[
z_1 = \sqrt{-1}\Lambda_2\Lambda_3(x_2y_3 - x_3y_2),
\]
\[
z_2 = \sqrt{-1}\Lambda_1\Lambda_3(x_3y_1 - x_1y_3),
\]
\[
z_3 = \sqrt{-1}\Lambda_1\Lambda_2(x_1y_2 - x_2y_1).
\]

Then one can easily verify that \( \beta_\lambda(z, z) = \Lambda_1\Lambda_2\Lambda_3 \).

In the case \( \dim V > 3 \), write \( V = \mathbb{F}t_1 \oplus \mathbb{F}t_2 \oplus \mathbb{F}t_3 \oplus \left( \bigoplus_{i \geq 4} \mathbb{F}t_i \right) \). Fix \( w, w^*, z \in \mathbb{F}t_1 \oplus \mathbb{F}t_2 \oplus \mathbb{F}t_3 \) as above, and set \( v_3 = z \) and \( v_i = t_i \) for \( i \geq 4 \).

**Step 4:** If \( \dim V \geq 3 \), then \( V \) has a Witt decomposition 
\[
V \cong Z \oplus W \oplus W^*,
\]

where 
\[
\dim Z = \begin{cases} 
0 & \text{if } \dim V \text{ is even and } \Lambda_1\Lambda_2...\Lambda_n \text{ is a square}, \\
1 & \text{if } \dim V \text{ is odd}, \\
2 & \text{if } \dim V \text{ is even and } \Lambda_1\Lambda_2...\Lambda_n \text{ is not a square}.
\end{cases}
\]

This follows from an inductive argument using Step 1, Step 2, and Step 3. \( \square \)

**Lemma 3.7.**

(1) Assume that \( \dim V = 1 \). Then 
\[
\text{Cliff}_q(\lambda) \cong \begin{cases} 
Q_1(\mathbb{F}) & \text{if } \Delta(\lambda) = 1 \text{ (equivalently, } \Lambda_1 \text{ is a square in } \mathbb{F}), \\
\mathbb{F}(\sqrt{\Lambda_1}) & \text{if } \Delta(\lambda) \neq 1 \text{ (equivalently, } \Lambda_1 \text{ is not a square in } \mathbb{F}).
\end{cases}
\]

(2) Assume that \( \dim V = 2 \). Then \( \text{Cliff}_q(\lambda) \cong \text{Mat}_2(\mathbb{F}) \text{ as (nongraded) algebras and } \text{Cliff}_q(\lambda)_0 \cong \text{Cliff}_q(\Lambda_1\Lambda_2) \).
We now apply Lemma 3.7 (1), (2) and prove (1).

We are now ready to describe the superalgebra structure of Cliff_q(λ).

Proposition 3.8.

1. If n is even, then Cliff_q(λ) ⊃ Mat_r(Λ(q, 1)) and r = 2^{n-1}.
   Furthermore, Cliff_q(λ) ⊃ Mat_{2r}(F) as (nongraded) algebras and
   \[ \text{Cliff}_q(\lambda)_0 \cong \begin{cases} \text{Mat}_r(F) \oplus \text{Mat}_r(F) & \text{if } \Delta(\lambda) = \overline{1}, \\ \text{Mat}_r(F(\sqrt{\Delta(\lambda)}) & \text{if } \Delta(\lambda) \neq \overline{1}. \end{cases} \]

2. If n is odd, then Cliff_q(λ) ⊃ Mat_r(B), where B = Cliff_q(\Delta(λ)) and r = 2^{n-1}.
   Furthermore,
   \[ \begin{cases} \text{Cliff}_q(\lambda) \cong Q_r(F), & \text{Cliff}_q(\lambda)_0 \cong \text{Mat}_r(F) & \text{if } \Delta(\lambda) = \overline{1}, \\ \text{Cliff}_q(\lambda) \cong \text{Mat}_r(F(\sqrt{\Delta(\lambda)}) & \text{Cliff}_q(\lambda)_0 \cong \text{Mat}_r(F) & \text{if } \Delta(\lambda) \neq \overline{1}. \end{cases} \]

In particular, Cliff_q(λ) is a simple superalgebra which is isomorphic to
- a direct sum of two isomorphic simple algebras if n is odd and \( \Delta(\lambda) = \overline{1} \);
- a simple algebra otherwise.

Proof. We first consider the case when n is even and let r = 2^{n-1}. If A_1...A_n is a square in F, then (1) is proved by Lemma 3.6 (3) and Lemma 3.3. Now if \( \Delta(\lambda) \neq \overline{1} \), by Lemma 3.3 and Step 3 in the proof of Lemma 3.6 we have Cliff_q(λ) ⊃ Mat_r(Λ(q, 1)) and A = Cliff_q(A_1...A_{n-1}, A_n).

We now apply Lemma 3.7 (1), (2) and prove (1).

Next, assume that n is odd and let r = 2^{n-1}. By Lemma 3.3 and Step 3 in the proof of Lemma 3.6 we have Cliff_q(λ) ⊃ Mat_r(B), where B is the 2-dimensional Clifford superalgebra Cliff_q(A_1...A_n). We use Lemma 3.7 (1) to complete the proof. □

In the statement of the following corollary we allow \( \lambda_i \) to be zero for some i. Recall that \( |\lambda| \) is the number of nonzero \( \lambda_i \). We also set \( \lambda_N := (\lambda_{i_1}, ..., \lambda_{i_{|\lambda|}}) \) where \( N_\lambda = \{i_1, ..., i_{|\lambda|}\} \) and \( i_1 < ... < i_{|\lambda|} \).

Corollary 3.9. Every \( \mathbb{Z}_2 \)-graded Cliff_q(\lambda_N)-module is completely reducible. Furthermore, the superalgebra Cliff_q(λ) has up to isomorphism
(1) two simple modules $E^q(\lambda)$ and $\Pi(E^q(\lambda))$ of dimension $2^{k-1}|2^{k-1}$ if $|\lambda| = 2k$ and $\Delta(\lambda) = 1$;
(2) one simple module $E^q(\lambda) \cong \Pi(E^q(\lambda))$ of dimension $2^k|2^k$ if $|\lambda| = 2k$ and $\Delta(\lambda) \neq 1$ (in particular, if $\lambda_1 > \ldots > \lambda_{2k} > 0$);
(3) one simple module $E^q(\lambda) \cong \Pi(E^q(\lambda))$ of dimension $2^k|2^k$ if $|\lambda| = 2k + 1$.

Proof. Thanks to Lemma 3.1, we may assume that $\lambda_i \neq 0$; i.e., $|\lambda| = n$. The category of all $\mathbb{Z}_2$-graded $\text{Cliff}_q(\lambda)$-modules is equivalent to the category of all nongraded $\text{Cliff}_q(\lambda)_0$-modules. Indeed, the reverse correspondence is obtained by

$$V_0 \mapsto \text{Cliff}_q(\lambda) \otimes_{\text{Cliff}_q(\lambda)_0} V_0.$$ 

The corollary follows from Proposition 3.8 and the characterization of the simple and indecomposable (nongraded) modules of $\text{Mat}_r(\mathbb{F}) \oplus \text{Mat}_r(\mathbb{F})$, $\text{Mat}_r(\mathbb{F})$, and $\text{Mat}_r(\mathbb{F}(\sqrt{\Delta(\lambda)}))$.

(This characterization may be found, for example, in [Lang, Chapter XVII].) \hfill \square

Example 3.10. Let $n = 3$ and $\lambda = (4, 2, 1)$. We describe the action of $t_i$ ($i = 1, 2, 3$) on $E^q(\lambda)$. We have

$$\Lambda_1 = (q^2 + q^{-2})(q^4 + q^{-4}), \quad \Lambda_2 = q^2 + q^{-2}, \quad \Lambda_3 = 1.$$ 

For simplicity, let $t = q^2 + q^{-2}$. We first find a solution of Legendre’s equation

$$(3.1) \quad \Lambda_1 X^2 + \Lambda_2 Y^2 + \Lambda_3 Z^2 = 0$$

We follow the proof of Lemma 3.4. Let $z = t z'$ and $y = \sqrt{-1} y'$. In order to solve the equation $(t^2 - 2)X^2 + t Z'^2 = Y'^2$ we find $C_1 \in \mathbb{C}[t]$ for which $C_1^2 - t$ is a multiple of $t^2 - 2$. Using the Chinese Remainder Theorem, we choose

$$C_1 = \frac{\sqrt{3}}{4}(1 - \sqrt{-1})t + \frac{\sqrt{2}}{2}(1 + \sqrt{-1}).$$

Then we solve the equation $A_1 X_1^2 + B Z_1^2 = Y_1^2$ for $A_1 = -\frac{\sqrt{2}}{4} \sqrt{-1}$ and $B = t$. A solution for this is

$$(X_1, Y_1, Z_1) = \left(1, \frac{\sqrt{2}}{4}(1 - \sqrt{-1}), 0 \right).$$

Then (3.1) has a solution

$$(A_1 X_1, \sqrt{-1}(Y_1 C_1 + B Z_1), t(C_1 Z_1 + Y_1)) = \left(-\frac{\sqrt{2}}{4} \sqrt{-1}, \frac{\sqrt{2}}{4} t + \frac{1}{2} \sqrt{-1}, \frac{\sqrt{8}}{4} (1 - \sqrt{-1})t \right).$$

Multiplying by an appropriate constant and changing signs, we fix the following solution of (3.1)

$$w = (X, Y, Z) = (1, \sqrt{-1} t - \sqrt{2}, \sqrt{2}(1 + \sqrt{-1}) t).$$
We consider \( w \) as an element in \( V \) relative to the basis \( \{t_1, t_2, t_3\} \). We use Lemma 3.2 to find a Witt decomposition \( V = \mathbb{F} w \oplus \mathbb{F} w^* \oplus \mathbb{F} z \). As mentioned in the proof of Lemma 3.2, we find

\[
w^* = c(X, Y, -Z),
\]

where \( c = \sqrt{\frac{1}{4v^2}} t^{-2} \) such that \( \beta_\lambda(w, w^*) = 1 \) and \( \beta_\lambda(w^*, w^*) = 0 \). Then, as pointed out in Step 3 of the proof of Lemma 3.6, we can find

\[
z = c(tYZ, -t(t^2 - 2)XZ, 0)
\]

such that \( \beta_\lambda(z, w) = \beta_\lambda(z, w^*) = 0 \) and \( \beta_\lambda(z, z) = -\frac{1}{4}\Lambda_1\Lambda_2\Lambda_3 = -\frac{1}{4}t^2(t^2 - 2) \). Set \( \alpha = \sqrt{\Delta(\lambda)} = \sqrt{t^2(t^2 - 2)} \). Using Lemma 3.3, we define an isomorphism \( \Theta : \text{Cliff}_q(\lambda) \rightarrow \text{Mat}_2(\mathbb{F}(\alpha)) \) by

\[
w \mapsto \begin{pmatrix} 0 & \alpha^{-1} \\ 0 & 0 \end{pmatrix}, \quad w^* \mapsto \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}.
\]

From Proposition 3.8 and Corollary 3.9 we find that \( E^q(\lambda) = \mathbb{F}(\alpha)^{\oplus 2} \). Let \( v_1 \) and \( v_2 \) be the standard basis vectors of the \( \mathbb{F}(\alpha) \)-vector space \( E^q(\lambda) \), and let \( \bar{v}_i = \alpha v_i \) (\( i = 1, 2 \)). The action of \( \text{Cliff}_q(\lambda) \) on \( E^q(\lambda) \) is given by

\[
\begin{align*}
z(v_1) &= \bar{v}_1, z(v_2) = -\bar{v}_2, z(\bar{v}_1) = t^2(t^2 - 2)v_1, z(\bar{v}_2) = -t^2(t^2 - 2)v_2 \\
w(v_1) &= 0, w(v_2) = (t^2(t^2 - 2))^{-1}v_1, w(\bar{v}_1) = 0, w(\bar{v}_2) = v_1 \\
w^*(v_1) &= \bar{v}_2, w^*(v_2) = 0, w^*(\bar{v}_1) = t^2(t^2 - 2)v_2, w^*(\bar{v}_2) = 0
\end{align*}
\]

In order to determine the action of \( t_i \) (\( i = 1, 2, 3 \)) on \( E^q(\lambda) \), we need to express \( t_1, t_2, t_3 \) in terms of \( z, w, w^* \). With simple computations we find:

\[
\begin{align*}
t_1 &= \frac{\sqrt{-1}}{4\sqrt{2}} t^2 - 2 t \quad w + t(t^2 - 2)w^* + \frac{\sqrt{8}(1 - \sqrt{-1})}{2} \frac{\sqrt{-1}t - \sqrt{2}}{t} z, \\
t_2 &= \frac{\sqrt{-1}}{4\sqrt{2}} \frac{\sqrt{-1}t - \sqrt{2}}{t} \quad w + (\sqrt{-1}t^2 - \sqrt{2}t)w^* + \frac{\sqrt{8}(1 - \sqrt{-1})}{2} \frac{1}{t} z, \\
t_3 &= \frac{1 - \sqrt{-1}}{4\sqrt{2}} \quad w + \frac{\sqrt{2}(1 + \sqrt{-1})}{\sqrt{2}} t \quad w^*.
\end{align*}
\]

4. **Highest Weight Representation Theory of \( U_q(\mathfrak{g}) \)**

A \( U_q(\mathfrak{g}) \)-module \( V^q \) is called a **weight module** if it admits a **weight space decomposition**

\[
V^q = \bigoplus_{\mu \in \Pi} V^q_{\mu}, \quad \text{where } V^q_{\mu} = \{ v \in V^q \mid q^h v = q^{\mu(h)} v \text{ for all } h \in \Pi^V \}.
\]
For a weight $U_q(\mathfrak{g})$-module $V^q$, we set $\text{wt} V^q = \{ \lambda \in P \mid V^q_\lambda \neq 0 \}$. By the same argument as in [HK, Ch.3], it can be verified that every submodule of a weight $U_q(\mathfrak{g})$-module is also a weight module. If $\dim_{\mathbb{C}(q)} V^q_\mu < \infty$ for all $\mu \in P$, then the character of $V^q$ is defined to be

$$\text{ch } V^q = \sum_{\mu \in P} (\dim_{\mathbb{C}(q)} V^q_\mu) \, e^\mu,$$

where $e^\mu$ are formal basis elements of the group algebra $\mathbb{C}(q)[P]$ with the multiplication given by $e^\lambda e^\mu = e^{\lambda+\mu}$ for all $\lambda, \mu \in P$.

A weight module $V^q$ is called a \textit{highest weight module} if it is generated over $U_q(\mathfrak{g})$ by a finite dimensional irreducible $U^q_{\geq 0}$-module $\mathfrak{v}^q$. Note that $\mathfrak{v}^q$ also admits a weight space decomposition. We call a vector in $\mathfrak{v}^q$ a \textit{highest weight vector} of $V^q$. Combining Lemma 2.2 and the triangular decomposition of $U_q(\mathfrak{g})$ (Theorem 2.3), we obtain $V^q = U_q^{-} \mathfrak{v}^q$.

**Proposition 4.1.** If $\mathfrak{v}^q$ is a finite dimensional irreducible $U^q_{\geq 0}$-module with a weight space decomposition $\mathfrak{v}^q = \bigoplus_{\mu \in P} \mathfrak{v}^q_\mu$, then $\mathfrak{v}^q$ is irreducible as a $U^q_0$-module and $\mathfrak{v}^q = \mathfrak{v}^q_\lambda$ for some $\lambda \in P$. Conversely, if $\mathfrak{v}^q$ is an irreducible $U^q_0$-module on which the even part of $U^q_0$ acts by a weight $\lambda$, then $\mathfrak{v}^q$ can be endowed with the structure of an irreducible $U^q_{\geq 0}$-module by letting $U^q_+$ act trivially on $\mathfrak{v}^q$.

**Proof.** Because $\mathfrak{v}^q$ is finite dimensional, there exists a weight $\lambda \in P$ such that $\mathfrak{v}^q_\lambda \neq 0$ and $\mathfrak{v}^q_{\lambda+i\alpha_i} = 0$ for all $i \in I$. Then we have $U^q_+ \mathfrak{v}^q_\lambda = \mathfrak{v}^q_\lambda$ and $U^q_0 \mathfrak{v}^q_\lambda = \mathfrak{v}^q_\lambda$. Thus $\mathfrak{v}^q_\lambda$ is a $U^q_{\geq 0}$-submodule of $\mathfrak{v}^q$ and hence $\mathfrak{v}^q_\lambda = \mathfrak{v}^q$. The other direction is obvious from the defining relations of $U_q(\mathfrak{g})$ in Theorem 2.1. □

**Remark.** If $\mathfrak{v}^q$ is a finite dimensional irreducible $U^q_{\geq 0}$-module which generates a highest weight module $V^q$ of highest weight $\lambda$, then, by Proposition 4.1, we know that $\mathfrak{v}^q$ is an irreducible $U^q_0$-module of weight $\lambda$. Thus $\mathfrak{v}^q$ is a finite dimensional irreducible module over $\text{Cliff}_q(\lambda) = U^q_0/I^q(\lambda)$. Conversely, if $E^q$ is a finite dimensional irreducible $\text{Cliff}_q(\lambda)$-module, then it is clear that $E^q$ is an irreducible $U^q_0$-module of weight $\lambda$.

By Corollary 3.9 we know that, up to isomorphism, $\text{Cliff}_q(\lambda)$ has at most two simple modules: $E^q(\lambda)$ and $\Pi(E^q(\lambda))$. The $U_q(\mathfrak{g})$-module $W^q(\lambda) = U_q(\mathfrak{g}) \otimes_{U^q_{\geq 0}} E^q(\lambda)$ is called the \textit{Weyl module} of $U_q(\mathfrak{g})$ corresponding to $\lambda$ (defined up $\Pi$).

**Proposition 4.2.**

1. $W^q(\lambda)$ is a free $U_q^{-}$-module of rank $\dim E^q(\lambda)$.
2. Every highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda$ is a homomorphic image of $W^q(\lambda)$.
3. Every Weyl module $W^q(\lambda)$ has a unique maximal submodule $N^q(\lambda)$. 
Proof. (1) This is clear from the definition.

(2) Let \( V^q \) be a highest weight module with highest weight \( \lambda \) generated by the irreducible \( U_q^{\geq 0} \)-module \( v^q \). Because \( v^q \) is irreducible over \( \text{Cliff}_q(\lambda) \), it is isomorphic to \( E^q(\lambda) \) up to \( \Pi \). Thus the map \( \phi : W^q(\lambda) \rightarrow V^q \) induced by \( E^q(\lambda) \rightarrow v^q \) is a surjective \( U_q(\mathfrak{g}) \)-module homomorphism.

(3) Since \( E^q(\lambda) \) is an irreducible \( \text{Cliff}_q(\lambda) \)-module, any proper submodule \( N^q \) of \( W^q(\lambda) \) does not contain highest weight vectors (the vectors in \( U^q(\lambda) \)-module \( q \)). Thus the sum of two proper submodules is again a proper submodule of \( W^q(\lambda) \). Then the sum \( N^q(\lambda) \) of all proper submodules of \( W^q(\lambda) \) is the unique maximal submodule of \( W^q(\lambda) \). \( \square \)

For \( \lambda \in P \), the unique irreducible quotient \( V^q(\lambda) := W^q(\lambda)/N^q(\lambda) \) is called the irreducible highest weight module over \( U_q(\mathfrak{g}) \) with highest weight \( \lambda \) (defined up to \( \Pi \)).

We introduce the notation
\[
[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}},
\]
which is called a \( q \)-integer. We also define \([0]_q! := 1\) and \([n]_q! := [n]_q \cdot [n-1]_q \cdots [1]_q \). We define the divided powers of \( e_i \) and \( f_i \) as follows:
\[
e_i^{(k)} := \frac{e_i^k}{[k]_q!}, \quad f_i^{(k)} := \frac{f_i^k}{[k]_q!}.
\]

By a straightforward induction argument, we can prove the following lemma.

Lemma 4.3. For all \( i \in I \) and \( k \in \mathbb{Z}_{\geq 0} \), we have
\[
e_i f_i^{(k)} = f_i^{(k)} e_i + f_i^{(k-1)} q^{h_i} q^{-k+1} - q^{-h_i} q^{k-1}.
\]

Proposition 4.4. Let \( \lambda \in \Lambda^+ \) and \( V^q(\lambda) \) be the irreducible highest weight \( U_q(\mathfrak{g}) \)-module generated by an irreducible finite dimensional \( U_q^{\geq 0} \)-module \( v^q \). Then \( f_i^{\lambda(h_i)+1} v = 0 \) for all \( v \in v^q \) and \( i \in I \).

Proof. Lemma 4.3 implies
\[
e_i f_i^{(k)} v = [\lambda(h_i) - k + 1]_q f_i^{(k-1)} v \quad \text{for all } v \in v^q.
\]

If \( k = \lambda(h_i) + 1 \), we see that \( e_i f_i^{\lambda(h_i)+1} v = 0 \). Moreover, for \( j \neq i \), we already know \( e_j f_i^{\lambda(h_i)+1} v = 0 \) and \( e_j f_j^{\lambda(h_j)+1} v = 0 \), since \( V^q(\lambda) = \bigoplus_{\mu \leq \lambda} V^q(\mu) \).

Suppose that \( e_i f_i^{\lambda(h_i)+1} v \neq 0 \). We have
\[
e_i(e_i f_i^{\lambda(h_i)+1} v) = e_i(e_i f_i^{\lambda(h_i)+1} v) = 0,
\]
\[
e_i(e_i f_i^{\lambda(h_i)+1} v) = \frac{q - q^{-1}}{q + q^{-1}} e_i^2 f_i^{\lambda(h_i)+1} v = 0.
\]
Also, \( e_j(e_i f_i^{\lambda(h_i) + 1} v) = e_j(e_i f_i^{\lambda(h_i) + 1} v) = 0 \) for \( j \neq i \), since \( V^q(\lambda) = \bigoplus_{\mu \leq \lambda} V^q_\mu \).

If \( \lambda(h_i) \geq 1 \), then \( \text{wt}(e_i f_i^{\lambda(h_i) + 1} v) = \lambda - \lambda(h_i) \alpha_i < \lambda \). Thus \( e_i f_i^{\lambda(h_i) + 1} v \) would generate a nontrivial proper submodule of \( V^q(\lambda) \), which contradicts the irreducibility of \( V^q(\lambda) \).

If \( \lambda(h_i) = 0 \), then we have \( \lambda_i = \lambda_{i+1} = 0 \) so that \( k_i v = k_{i+1} v = 0 \) by Lemma 3.1. From the defining relation of \( U_q(\mathfrak{g}) \), we know

\[
e_i f_i v = f_i e_i v + (q^{k_i+1} k_i - q^{k_i} k_{i+1}) v = 0.
\]

Therefore, in any case, \( e_i f_i^{\lambda(h_i) + 1} v = 0 \) for all \( v \in \mathfrak{v}^q \).

Similarly, if \( f_i^{\lambda(h_i) + 1} \neq 0 \), it would generate a nontrivial proper submodule of \( V^q(\lambda) \). Hence we conclude \( f_i^{\lambda(h_i) + 1} v = 0 \) for all \( v \in \mathfrak{v}^q \). \( \square \)

5. Classical limits

Let \( A_1 := \{ f/g \in \mathbb{C}(q) \mid f, g \in \mathbb{C}[q], g(1) \neq 0 \} \). For an integer \( n \in \mathbb{Z} \), we formally define

\[
[y; n]_x := \frac{yx^n - y^{-1}x^{-n}}{x - x^{-1}}, \quad (y; n)_x := \frac{yx^n - 1}{x - 1}.
\]

For example,

\[
[q^h; 0]_q = \frac{q^h - q^{-h}}{q - q^{-1}}, \quad (q^h; 0)_q = \frac{q^h - 1}{q - 1}.
\]

**Definition 5.1.** We define the \( A_1 \)-form \( U_{A_1} \) of the quantum superalgebra \( U_q(\mathfrak{g}) \) to be the \( A_1 \)-subalgebra of \( U_q(\mathfrak{g}) \) with 1 generated by the elements \( e_i, e_i, f_i, f_i, q^h, k_i \) and \( (q^h; 0)_q \) \( (i \in I, l \in J, h \in P^\vee) \).

We denote by \( U_{A_1}^+ \) (respectively, \( U_{A_1}^- \)) the \( A_1 \)-subalgebra of \( U_q(\mathfrak{g}) \) with 1 generated by \( e_i, e_i \) (respectively, \( f_i, f_i \)) for \( i \in I \), and by \( U_{A_1}^0 \) the \( A_1 \)-subalgebra of \( U_q(\mathfrak{g}) \) with 1 generated by \( q^h, k_i \) and \( (q^h; 0)_q \) for \( l \in J, h \in P^\vee \).

**Lemma 5.2.**

1. \( (q^h; n)_q \in U_{A_1}^0 \) for all \( n \in \mathbb{Z} \) and \( h \in P^\vee \).
2. \( [q^h; 0]_q \in U_{A_1}^0 \) for all \( n \in \mathbb{Z} \) and \( h \in P^\vee \).

**Proof.** Our assertions follow immediately from the following identities:

\[
(q^h; n)_q = q^n (q^h; 0)_q + \frac{q^n - 1}{q - 1},
\]

\[
[q^h; 0]_q = \frac{q - 1}{q^2 - 1} (1 + q^{-h})(q^h; 0)_q.
\]

\( \square \)
Note that
\[ k_i^2 = [q^{2k_i}; 0]q^2 = q^2 \frac{q^2 - 1}{q^2 - 1} (1 + q^{-2k_i}) \frac{1}{q + 1} (q^{2k_i}; 0)_q. \]

**Proposition 5.3.** We have the triangular decomposition of the algebra \( U_{A_1} \). Namely,
\[ U_{A_1} \cong U^-_{A_1} \otimes U^0_{A_1} \otimes U^+_{A_1} \]
as \( A_1 \)-modules.

**Proof.** Recall the canonical isomorphism \( U_q(\mathfrak{g}) \xrightarrow{\sim} U_q^- \otimes U_q^0 \otimes U_q^+ \) given by Theorem 2.3.
The following commutation relations hold:
\[
\begin{align*}
e_i(q^h; 0)_q &= (q^h; -\alpha_i(h))_q e_i, & e_i(q^h; 0)_q &= (q^h; -\alpha_i(h))_q e_i, \\
(q^h; 0)_q f_i &= f_i(q^h; -\alpha_i(h))_q, & (q^h; 0)_q f_i &= f_i(q^h; -\alpha_i(h))_q, \\
e_if_i &= f_ie_i + [q^{k_i-k_{i+1}}; 0]_q, & e_if_i &= f_ie_i + [q^{k_i-k_{i+1}}; 0]_q, \\
e_{i+1}f_i &= q^{-1}f_ie_{i+1}, & e_{i+1}f_i &= qf_{i+1}e_i, & e_if_j - f je_i &= 0 \text{ for } |i - j| > 1, \\
e_if_i &= -f_i e_i + [q^{k_i+k_{i+1}}; 0]_q + (q - q^{-1})h_i h_{i+1}, & e_if_i &= -f_i e_i + [q^{k_i+k_{i+1}}; 0]_q + (q - q^{-1})h_i h_{i+1}, \\
e_jf_i &= -q^{-1}f_{i+1}e_i, & e_jf_i &= -qf_{i+1}e_i, & e_if_j &= -f_je_i = 0 \text{ for } |i - j| > 1.
\end{align*}
\]

Together with Lemma 5.2, one can show that the image of the canonical isomorphism lies inside \( U^-_{A_1} \otimes U^0_{A_1} \otimes U^+_{A_1} \), when restricted to \( U_{A_1} \). Its inverse map is given by multiplication. Hence the two spaces are isomorphic as \( A_1 \)-modules. \( \square \)

In what follows, \( V^q \) is a highest weight module over \( U_q(\mathfrak{g}) \) with highest weight \( \lambda \in P \) generated by a finite dimensional irreducible \( U_q \)-submodule \( \mathfrak{v}^q \). Then \( \mathfrak{v}^q \) is a finite dimensional irreducible \( \text{Cliff}_q(\lambda) \)-module. Since it is irreducible, it is generated by a nonzero vector \( v \in (\mathfrak{v}^q)_0 \); i.e., \( \mathfrak{v}^q = \text{Cliff}_q(\lambda)v \). Note that
\[
\frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}} = q^{2n-2} + q^{2n-6} + \cdots + q^{-2n+6} + q^{-2n+2} \in A_1 \quad \text{for } n \in \mathbb{Z}_{>0}.
\]

We denote by \( \text{Cliff}_{A_1}(\lambda) \) the \( A_1 \)-subalgebra of \( \text{Cliff}_q(\lambda) \) generated by \( \{t_i \mid i \in J\} \).

**Definition 5.4.** Let \( V^q \) be a highest weight \( U_q(\mathfrak{g}) \)-module generated by a finite dimensional irreducible \( U_q \)-module \( \mathfrak{v}^q \) and let \( E^{A_1}(\lambda) \) be the \( \text{Cliff}_{A_1}(\lambda) \)-submodule of \( \mathfrak{v}^q \cong E^q(\lambda) \) generated by a nonzero element \( v \in (\mathfrak{v}^q)_0 \). The \( A_1 \)-form of \( V^q \) is defined to be the \( U_{A_1} \)-submodule \( V_{A_1} \) of \( V^q \) generated by \( E^{A_1}(\lambda) \).

In what follows, \( V^q \) will denote a a highest weight \( U_q(\mathfrak{g}) \)-module.

**Proposition 5.5.** \( V_{A_1} = U^-_{A_1} E^{A_1}(\lambda) \).
Proof. In view of Proposition 5.3, it suffices to show that $U_{\mathbb{A}_1}^E E^{A_1}(\lambda) = E^{A_1}(\lambda)$ and $U_{\mathbb{A}_1}^0 E^{A_1}(\lambda) = E^{A_1}(\lambda)$. The first assertion is clear by the definition of highest weight modules. For the second assertion, we observe that

$$q^h w = q^{h(\lambda)} w,$$

$$(q^h, 0)_q w = \frac{q^{h(\lambda)} - 1}{q - 1} w \quad \text{for all } w \in E^{A_1}(\lambda).$$

Hence we obtain $V_{\mathbb{A}_1} = U_{\mathbb{A}_1} E^{A_1}(\lambda) = U_{\mathbb{A}_1}^0 E^{A_1}(\lambda)$. \qed

For each $\mu \in P$, let us denote by $(V_{\mathbb{A}_1})_{\mu}$ the space $V_{\mathbb{A}_1} \cap V_{\mu}^q$. The following assertion can be proved using the same arguments as in [HK, Proposition 3.3.6].

**Proposition 5.6.** $V_{\mathbb{A}_1}$ has the weight space decomposition $V_{\mathbb{A}_1} = \bigoplus_{\mu \leq \lambda} (V_{\mathbb{A}_1})_{\mu}$.

**Proposition 5.7.** For each $\mu \in P$, the weight space $(V_{\mathbb{A}_1})_{\mu}$ is a free $A_1$-module with rank$_{A_1}(V_{\mathbb{A}_1})_{\mu} = \dim_{\mathbb{C}(q)} V_{\mu}^q$. In particular, rank$_{A_1} E^{A_1}(\lambda) = \dim_{\mathbb{C}(q)} E^q(\lambda)$.

Proof. Because $A_1$ is a principal ideal domain, every finitely generated torsion free module over $A_1$ is free. Furthermore, since $\mathbb{C}(q)$ is the field of quotients of the integral domain $A_1$, a finite subset of a $\mathbb{C}(q)$-vector space is linearly independent over $\mathbb{C}(q)$ if and only if it is linearly independent over $A_1$. Thus it is enough to show that each $V_{\mu}^q$ has a $\mathbb{C}(q)$-basis which is also contained in $(V_{\mathbb{A}_1})_{\mu}$. The highest weight space $V^q = E^q(\lambda)$ has a linearly independent subset of $\{t_{1_1}^{\epsilon_1} t_{2_1}^{\epsilon_2} \cdots t_{n_1}^{\epsilon_n} v \mid \epsilon_j = 0 \text{ or } 1\}$ which generates $E^q(\lambda)$ over $\mathbb{C}(q)$, since $E^q(\lambda) = \text{Cliff}_q(\lambda)_v$. By definition, this subset is contained in $E^{A_1}(\lambda)$. For $V_{\mu}^q$, it is easy to show that there is a basis of $V_{\mu}^q$ whose elements are of the form $f_1^{\epsilon_1} f_2^{\epsilon_2} \cdots t_{n_1}^{\epsilon_n} v$, where $f_\zeta$ are monomials in $f_i$ and $f_j$. This basis is also contained in $(V_{\mathbb{A}_1})_{\mu}$, which proves the proposition. \qed

**Corollary 5.8.** The map $\phi : \mathbb{C}(q) \otimes_{A_1} V_{\mathbb{A}_1} \longrightarrow V^q$ given by $f \otimes v \longmapsto fv$ ($f \in \mathbb{C}(q), v \in V_{\mathbb{A}_1}$) is a $\mathbb{C}(q)$-linear isomorphism.

Let $J_1$ be the ideal of $A_1$ generated by $q - 1$. Then there is a canonical isomorphism of fields

$$A_1/J_1 \xrightarrow{\sim} \mathbb{C} \quad \text{given by } f(q) + J_1 \longmapsto f(1).$$

Define the $\mathbb{C}$-linear vector spaces

$$U_1 = (A_1/J_1) \otimes_{A_1} U_{\mathbb{A}_1},$$

$$V^1 = (A_1/J_1) \otimes_{A_1} V_{\mathbb{A}_1}.$$  

Then $V^1$ is naturally a $U_1$-module. Note that $U_1 \cong U_{\mathbb{A}_1}/J_1 U_{\mathbb{A}_1}$ and $V^1 \cong V_{\mathbb{A}_1}/J_1 V_{\mathbb{A}_1}$. We use the bar notation for the images under these maps. The passage under these maps is referred to as taking the classical limit.
Since \( V_{A_1} = U_{A_1}E^{A_1}(\lambda) \), we have:

\[
V^1 \cong V_{A_1}/J_1V_{A_1} = U_{A_1}E^{A_1}(\lambda)/J_1U_{A_1}E^{A_1}(\lambda) = (U_{A_1}/J_1U_{A_1}) \cdot (E^{A_1}(\lambda)/J_1E^{A_1}(\lambda)).
\]

Hence \( V^1 \) is generated by \( E^{A_1}(\lambda)/J_1E^{A_1}(\lambda) \) over \( U^1 \). For each \( \mu \in P \), denote by \( V_{\mu}^1 \) the space \((A_1/J_1) \otimes_{A_1} (V_{A_1})_{\mu} \cong (V_{A_1})_{\mu}/J_1(V_{A_1})_{\mu}\).

**Proposition 5.9.**

1. \( V^1 = \bigoplus_{\mu \leq \lambda} V_{\mu}^1 \)
2. For each \( \mu \in P \), \( \dim_{\mathbb{C}} V_{\mu}^1 = \text{rank}_{A_1}(V_{A_1})_{\mu} \).

**Proof.** The first assertion follows from Proposition 5.6. Using the same argument as in [HK] Lemma 3.4.1, we can prove the second assertion. \( \square \)

Let \( \bar{h} \in U_1 \) be the classical limit of \((q^h; 0)_q \in U_{A_1}\). Using [HK] Lemma 3.4.3, we have:

**Lemma 5.10.**

1. For all \( h \in P^\vee \), we have \( \bar{h}^q = 1 \).
2. For any \( h, h' \in P^\vee \), \( \bar{h} + h' = \bar{h} + h' \).

**Theorem 5.11.**

1. The elements \( \bar{e}_i, \bar{f}_i, \bar{f}_i', \bar{f}_i'' \) (\( i \in I \)), \( \bar{c}_i \) (\( l \in J \)) and \( \bar{h} \ (h \in P^\vee) \) satisfy the defining relations of \( U(\mathfrak{g}) \). Hence there exists a surjective \( \mathbb{C} \)-algebra homomorphism \( \psi : U(\mathfrak{g}) \twoheadrightarrow U_1 \) and the \( U_1 \)-module \( V^1 \) has a \( U(\mathfrak{g}) \)-module structure.
2. For each \( \mu \in P \) and \( h \in P^\vee \), the element \( \bar{h} \) acts on \( V_{\mu}^1 \) as scalar multiplication by \( \mu(h) \). So \( V_{\mu}^1 \) is the \( \mu \)-weight space of the \( U(\mathfrak{g}) \)-module \( V^1 \).
3. There is an isomorphism \( \text{Cliff}(\lambda) \twoheadrightarrow \text{Cliff}_1(\lambda) := \text{Cliff}_{A_1}(\lambda)/J_1 \text{Cliff}_{A_1}(\lambda) \).
4. As a \( U(\mathfrak{g}) \)-module, \( V^1 \) is a highest weight module or the sum of two highest weight modules with highest weight \( \lambda \in P \).

**Proof.** (1) The first relation for \( U(\mathfrak{g}) \) is trivial. Since

\[
(q^h; 0)_q e_i - e_i(q^h; 0)_q = e_i(q^h; \alpha_i(h)) - e_i(q^h; 0)_q = q^{\alpha_i(h) - 1} e_i q^h,
\]

we obtain \( \bar{[h, e_i]} = \alpha_i(h) \bar{e_i} \) by letting \( q \rightarrow 1 \). Similarly,

\[
\bar{[h, e_i]} = \alpha_i(h) \bar{e_i}, \quad \bar{[h, f_i]} = -\alpha_i(h) \bar{f_i}, \quad \bar{[h, f_i']} = -\alpha_i(h) \bar{f_i'} \quad \text{and} \quad \bar{[h, \bar{c}_i]} = 0.
\]

We have \( e_i f_i - f_i e_i = [q^{h_i}; 0]_q = \frac{q}{q + 1} (1 + q^{-h_i})(q^{h_i}; 0)_q \).
Taking the classical limit to both sides above leads to \( \bar{e}_i f_i - f_i \bar{e}_i = \frac{1}{2} \bar{h}_i = \bar{h}_i. \)

Also
\[
k^2_i = [q^{2k_i}; 0]_q^2 = \frac{q^2 q^2 - 1}{q^2 - 1} (1 + q^{-2k_i}) \frac{1}{q + 1} (q^{2k_i}; 0)_q.
\]

When we take \( q \to 1 \), we obtain \( \bar{k}^2_i = \bar{k}_i. \)

Since we can obtain the following relations in \( U(\mathfrak{g}) \) by the Jacobi identity,
\[
[e_i, [e_i, e_j]] = 0, \quad [e_i, [e_i, e_j]] = [e_i, [e_i, e_j]], \quad \text{for } |i - j| = 1,
\]
in order to prove the corresponding relations in \( U_1 \), it suffices to show that \( [\bar{e}_i, [\bar{e}_i, \bar{e}_j]] = 0 \). The latter relation can be checked easily by letting \( q \to 1 \). The rest of the relations can be derived in a similar manner.

Therefore, there exists a surjective algebra homomorphism \( \psi : U(\mathfrak{g}) \to U_1 \) defined by \( e_i \mapsto \bar{e}_i, e_i \mapsto \bar{e}_i, f_i \mapsto \bar{f}_i, f_i \mapsto \bar{f}_i, h \mapsto \bar{h}, k_l \mapsto \bar{k}_l \) \( (i \in I, l \in J) \), which can be used to define a \( U(\mathfrak{g}) \)-module structure on \( V_1. \)

(2) For \( v \in (V_{A_1})_\mu \) and \( h \in P^\vee \), we have
\[
(q^h; 0)_q v = \frac{q^{\mu(h)} - 1}{q - 1} v.
\]

Taking the classical limit of both sides yields our assertion.

(3) Note that \( \bar{t}_i t_j + \bar{t}_j t_i = 2 \delta_{ij} \lambda_i \) in \( \text{Cliff}_1(\lambda) \) and \( \text{Cliff}(\lambda) \) is the associative \( \mathbb{C} \)-algebra with 1 generated by \( \{ \bar{k}_i \mid i \in J \} \) with defining relations \( \bar{k}_i \bar{k}_j + \bar{k}_j \bar{k}_i = 2 \delta_{ij} \lambda_i \). Thus we have a surjective \( \mathbb{C} \)-algebra homomorphism \( \text{Cliff}(\lambda) \to \text{Cliff}_1(\lambda) \). Observe that
\[
\dim_\mathbb{C} \text{Cliff}_1(\lambda) = \text{rank}_{A_1} \text{Cliff}_{A_1}(\lambda)
= \dim_\mathbb{C} \text{Cliff}_{(q)}(\lambda)
= \dim_\mathbb{C} \text{Cliff}(\lambda).
\]

The first two equalities follow by using the same reasoning as in Proposition 5.9 and Proposition 5.7 respectively. It is well known that the dimension of the Clifford algebra associated with a symmetric bilinear form on a vector space of dimension \( k \) is \( 2^k \). This result holds for any base field of characteristic different from 2. Thus we proved the last equality.

(4) \( V^q \) is generated by a finite dimensional irreducible \( U_{\bar{q}}^{\geq 0} \)-submodule \( \mathfrak{v}^q \cong E^q(\lambda) \) up to \( \Pi \). By Corollary 3.9
\[
\dim E^q(\lambda) = \begin{cases} 
2^k & \text{if } |\lambda| = 2k \text{ and } \overline{\Delta(\lambda)} = 1, \\
2^{k+1} & \text{if } |\lambda| = 2k \text{ and } \overline{\Delta(\lambda)} \neq 1, \\
2^{k+1} & \text{if } |\lambda| = 2k + 1.
\end{cases}
\]
It is well known that the dimension of the $\mathbb{Z}_2$-graded irreducible $\text{Cliff}(\lambda)$-modules is $2^{|\lambda|-1}$ (see, for example, [ABS]). With this in mind we deduce that $E^{A_1}(\lambda)/J_1E^{A_1}(\lambda)$ is an irreducible $\text{Cliff}(\lambda)$-module when $|\lambda| = 2k + 1$ or $|\lambda| = 2k$ and $\Delta(\lambda) = 1$, and the direct sum of two irreducible $\text{Cliff}(\lambda)$-modules otherwise. Since $E^q(\lambda)$ is a parity invariant module over $\text{Cliff}_q(\lambda)$ for $|\lambda| = 2k$ and $\Delta(\lambda) \neq 1$, $E^{A_1}(\lambda)/J_1E^{A_1}(\lambda)$ is a parity invariant $\text{Cliff}(\lambda)$-module as well. Hence $E^{A_1}(\lambda)/J_1E^{A_1}(\lambda) = V(\lambda) \oplus \Pi V(\lambda)$ for some irreducible $\text{Cliff}(\lambda)$-module $V(\lambda)$. By definition, $V^1$ is a highest weight $U(\mathfrak{g})$-module generated by $E^{A_1}(\lambda)/J_1E^{A_1}(\lambda)$ or the sum of two highest weight modules generated by $V(\lambda)$ and $\Pi V(\lambda)$ for some irreducible $\text{Cliff}(\lambda)$-module $V(\lambda)$.

By Propositions 5.7 and 5.9 and Theorem 5.11 we obtain the following identity between the characters of a highest weight $U(\mathfrak{g})$-module and a highest weight $\text{U}_q(\mathfrak{g})$-module.

**Proposition 5.12.** $\text{ch } V^1 = \text{ch } V^q$.

**Corollary 5.13.** $V^q(\lambda)$ is finite dimensional if and only if $\lambda \in \Lambda^+$.

**Proof.** Let $V^q = V^q(\lambda)$. If $\lambda \in \Lambda^+$, then we have $f^{\lambda(h_i)+1}_i v = 0$ for all $v \in V^q(\lambda)$ by Proposition 4.4. Taking the classical limit, we have $f^{\lambda(h_i)+1}_i \tilde{v} = 0$ for all $\tilde{v} \in V^q(\lambda)$. Because $V^1$ is a highest weight module or the sum of two highest weight modules, it is finite dimensional by Proposition 4.9, and hence $V^q$ is finite dimensional by Proposition 5.12. Conversely, assume that $\lambda$ is not in $\Lambda^+$. Then $V^1$ has a submodule which is a highest weight module and whose irreducible quotient is isomorphic to an irreducible highest weight module with highest weight $\lambda$. It is not finite dimensional by (2) of Proposition 4.4. Again by Proposition 5.12 $V^q$ cannot be finite dimensional. 

**Theorem 5.14.** If $\lambda \in \Lambda^+ \cap P_{\geq 0}$ and $V^q$ is the irreducible highest weight $U_q(\mathfrak{g})$-module $V^q(\lambda)$ with highest weight $\lambda$, then $V^1$ is isomorphic to

1. $V(\lambda)$ or $\Pi V(\lambda)$ if $|\lambda| = 2k$ and $\Delta(\lambda) = 1$,
2. $V(\lambda) \oplus \Pi V(\lambda)$ if $|\lambda| = 2k$ and $\Delta(\lambda) \neq 1$ (in particular, if $\lambda_1 > \ldots > \lambda_{2k} > 0$),
3. $V(\lambda) \cong \Pi V(\lambda)$ if $|\lambda| = 2k + 1$.

Hence, $\text{ch } V^q(\lambda) = \begin{cases} \text{ch } V(\lambda) & \text{if } |\lambda| = 2k \text{ and } \Delta(\lambda) = 1, \\ 2 \text{ch } V(\lambda) & \text{if } |\lambda| = 2k \text{ and } \Delta(\lambda) \neq 1, \\ \text{ch } V(\lambda) & \text{if } |\lambda| = 2k + 1. \end{cases}$
Proof. By Theorem 5.11 (4), $V^1$ is a highest weight module or the sum of two highest weight modules over $U(g)$ with highest weight $\lambda$. By Proposition 1.8, we have

$$V^1 \cong \begin{cases} V(\lambda) & \text{if } |\lambda| = 2k \text{ and } \Delta(\lambda) = \bar{1}, \\ V(\lambda) \oplus \Pi V(\lambda) & \text{if } |\lambda| = 2k \text{ and } \Delta(\lambda) \neq \bar{1}, \\ V(\lambda) \cong \Pi V(\lambda) & \text{if } |\lambda| = 2k + 1. \end{cases}$$

The second assertion follows from Proposition 5.12.

Remark. The main reason we restrict our attention in Theorem 5.14 to the dominant set of weights $\Lambda^+ \cap P_{\geq 0}$ is the statement of Proposition 1.8. We still believe that the theorem holds in more general setting and conjecture that it is true for any weight $\lambda \in \Lambda^+$ for which the generic character formula (1.4) holds.

Corollary 5.15. If $V_q$ is a finite dimensional highest weight module over $U_q(g)$ with highest weight $\lambda \in \Lambda^+ \cap P_{\geq 0}$, then $V_q$ is isomorphic to $V^q(\lambda)$ up to $\Pi$.

Proof. Note that $V^1$ is a highest weight module or the sum of two highest weight modules over $U(g)$ with highest weight $\lambda$ and it is finite dimensional by Proposition 5.12. From Proposition 1.8, we know that $V^1$ is an irreducible module or the direct sum of two irreducible modules. Thus we get $\text{ch } V^q = \text{ch } V^1 = \text{ch } V^q(\lambda)$ by Theorem 5.14 and hence $V^q \cong V^q(\lambda)$.

Define the subalgebras $U^\pm_i := A_1/J_i \otimes A_1$ and $U^0_i := A_1/J_i \otimes A_1$ of $U_1$.

Theorem 5.16. The classical limit $U_1$ of $U_q(g)$ is isomorphic to the universal enveloping algebra $U(g)$.

Proof. By Theorem 5.11 (1), there exists a surjective algebra homomorphism $\psi : U(g) \rightarrow U_1$ defined by $e_i \mapsto r_i$, $f_i \mapsto t_i$, $h \mapsto I$, $\bar{t}_i \mapsto f_i$ for $i \in I$, $h \in P^\vee$ and $l \in J$. From (1.3), $U(g) \cong U^- \otimes U^0 \otimes U^+$.

We first show that $U^0$ is isomorphic to $U^0_1$. Consider the restriction $\psi_0$ of $\psi$ to $U^0$. Note that $\text{Cliff}_{A_1}(\lambda)$ is a $U^0_{A_1}$-module. Indeed, as in the proof of Proposition 5.5, we know that

$$q^h w = q^{\lambda(h)} w, \quad (q^h) w = \frac{q^{\lambda(h)} - 1}{q - 1} w \quad \text{for all } w \in \text{Cliff}_{A_1}(\lambda).$$

In particular, the action of $k_i$ is just the left multiplication by $t_i$. Let $g \in \ker \psi_0$. By the Poincaré-Birkhoff-Witt theorem, we can write $g = \sum_{i=1}^{2n} g_i k_i$, where $k_i = k_i^{a_i} \cdots k_1^{a_1}$, $0 \leq a_j \leq 1$ for all $j \in J$ and each $g_i$ is a polynomial in $k_1, \ldots, k_n$. For each $\lambda \in P$ we have

$$0 = \psi_0(g) \cdot T = \sum_{i=1}^{2n} \lambda(g_i) \bar{t}_i \in \text{Cliff}_1(\lambda).$$
where $\lambda(g_i)$ denotes the polynomial in $\lambda_j$ corresponding to $g_i$. Since $\{\ell_1, \ldots, \ell_h\}$ is a linearly independent subset of $\text{Cliff}_1(\lambda) \cong \text{Cliff}(\lambda)$, we have $\lambda(g_i) = 0$ for all $i = 1, \ldots, 2n$. Since we may take any integer value for $\lambda_j$, $g_i$ must be zero for all $i = 1, \ldots, 2n$ and hence $g$ is identically zero. Thus $\psi_0$ is injective.

Next we show that the restriction of $\psi_-$ of $\psi$ to $U^-$ is an isomorphism of $U^-$ onto $U_1^-$. Suppose $\ker \psi_- \neq 0$ and $u = \sum b_i f_i \in \ker \psi_-$, where $b_i \in \mathbb{C}$ and $f_i$ are monomials in $f_i$ and $f_i$’s. Let $N$ be the maximal length of the monomials $f_i$ in the expression of $u$, and choose $\lambda \in \Lambda^+ \cap P_{\geq 0}$ satisfying $\lambda(h_i) > N$ and $|\lambda| = 2k$ and $\Delta(\lambda) = 1$ or $|\lambda| = 2k + 1$ for all $i \in I$. By Theorem 5.14, the classical limit $V^1$ of $V^q(\lambda)$ is isomorphic to the irreducible $U(\mathfrak{g})$-module $V(\lambda)$ when $|\lambda| = 2k$ and $\Delta(\lambda) = 1$, or $|\lambda| = 2k + 1$. Set $r = 2\lfloor \frac{|\lambda|}{2} \rfloor$. Consider the map $\phi : (U^-)^{\oplus r} \to V^1$, given by $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^r \psi(x_i) \cdot v_i$ for a basis $(v_i \mid i = 1, \ldots, r)$ of $V^1$. Then by Proposition 4.13 and Proposition 1.9 $\ker \phi$ is the left ideal of $(U^-)^{\oplus r}$ generated by $(J^i_{(h_i)^{+1}}, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, f_i^{(h_i)^{+1}})$ for $i \in I$. In particular, $(u, 0, \ldots, 0) = (\sum b\xi f_i, 0, \ldots, 0) \not\in \ker \phi$. That is $\psi_-(u) v_1 \neq 0$, which is a contradiction. So $\ker \psi_- = 0$ and $U^-$ is isomorphic to $U_1^-$. Similarly, we can show that $U^+ \cong U_1^+$. By the triangular decomposition we have

$$U(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+ \cong U_1^- \otimes U_1^0 \otimes U_1^+ \cong U_1.$$}

It can be checked easily that this isomorphism is an algebra isomorphism. \hfill \square

**Theorem 5.17.** Let $\lambda \in P$. If $V^q$ is the Weyl module $W^q(\lambda)$ over $U_q(\mathfrak{g})$ with highest weight $\lambda$, then its classical limit $V^1$ is isomorphic to

1. $W(\lambda)$ or $\Pi W(\lambda)$ if $|\lambda| = 2k$ and $\Delta(\lambda) = 1$,
2. $W(\lambda) \oplus \Pi W(\lambda)$ if $|\lambda| = 2k$ and $\Delta(\lambda) \neq 1$ (in particular, if $\lambda_1 > \ldots > \lambda_2k > 0$),
3. $W(\lambda) \cong \Pi W(\lambda)$ if $|\lambda| = 2k + 1$.

**Proof.** Let $\mathfrak{v}(\lambda)$ be a finite dimensional irreducible $\mathfrak{b}_+$-module of weight $\lambda$ which generates $W(\lambda)$. Since $U^- \cong U_1^-$ and $E^{A_1}(\lambda)/J_1 E^{A_1}(\lambda)$ is isomorphic to $\mathfrak{v}(\lambda)$ or $\mathfrak{v}(\lambda) \oplus \Pi \mathfrak{v}(\lambda)$ as a $\text{Cliff}(\lambda)$-module, it suffices to show that $V^1$ is a free $U_1^-$-module whose rank is $\dim_\mathbb{C} \mathfrak{v}(\lambda)$ or $2 \dim_\mathbb{C} \mathfrak{v}(\lambda)$.

By Proposition 3.2 we know that $W^q(\lambda)$ is a free $U_q^-$-module generated by $E^q(\lambda)$. Since $V_{A_1}$ is a subspace of $V^q$, taking Proposition 5.7 into account, $V_{A_1}$ is a free $U_{A_1}^-$-module generated by $E^{A_1}(\lambda)$. Taking the classical limit, we see that $V^1 = U_1^- \cdot \left( E^{A_1}(\lambda)/J_1 E^{A_1}(\lambda) \right)$ and

$$\dim_\mathbb{C} E^{A_1}(\lambda)/J_1 E^{A_1}(\lambda) = \dim_\mathbb{C} E^q(\lambda) = \dim_\mathbb{C} \mathfrak{v}(\lambda) or 2 \dim_\mathbb{C} \mathfrak{v}(\lambda).$$
By a similar argument as in [HK] Proposition 3.4.10, we can show that $V^1$ is a free $U_1$-module. When $|\lambda| = 2k$ and $\Delta(\lambda) \neq \bar{1}$, $E^q(\lambda)$ is parity invariant. Hence we have

$$V^1 \cong \begin{cases} W(\lambda) \text{ or } \Pi W(\lambda) & \text{if } |\lambda| = 2k \text{ and } \Delta(\lambda) = \bar{1}, \\ W(\lambda) \oplus \Pi W(\lambda) & \text{if } |\lambda| = 2k \text{ and } \Delta(\lambda) \neq \bar{1}, \\ W(\lambda) \cong \Pi W(\lambda) & \text{if } |\lambda| = 2k + 1. \end{cases}$$

\square

6. Complete reducibility of the category $O_q^{\geq 0}$

In this section, we prove the complete reducibility theorem for $U_q(\mathfrak{g})$-modules in the category $O_q^{\geq 0}$.

**Definition 6.1.** The category $O_q^{\geq 0}$ consists of finite dimensional $U_q(\mathfrak{g})$-modules $M$ with a weight space decomposition $M = \bigoplus_{\lambda \in P} M_{\lambda}$ such that $\text{wt}(M) \subset P_{\geq 0}$.

**Remark.** The complete reducibility theorem for $O_q^{\geq 0}$, which we establish at the end of this section, implies that $O_q^{\geq 0}$ is isomorphic to the category $T_q$ of tensor modules; i.e., submodules of a tensor power of the natural representation $\mathbb{C}(q)^n$. Indeed, using the description of $T_q$ provided by Olshanski and Sergeev we first check that every simple object of $O_q^{\geq 0}$ is a tensor module. Then, by the complete reducibility result for $T_q$, obtained again by Sergeev and Olshanski, we conclude that the two categories are isomorphic.

One can easily prove the following proposition (see, for example, [HK] Theorem 7.2.3).

**Proposition 6.2.** For each $\lambda \in \Lambda^+ \cap P_{\geq 0}$, $V_q(\lambda)$ is an irreducible $U_q(\mathfrak{g})$-module in the category $O_q^{\geq 0}$. Conversely, every finite dimensional irreducible $U_q(\mathfrak{g})$-module in the category $O_q^{\geq 0}$ has the form $V_q(\lambda)$ for some $\lambda \in \Lambda^+ \cap P_{\geq 0}$.

Let $S$ be the antipode on $U_q(\mathfrak{g})$ defined in [O] Section 4. We have $S(q^h) = q^{-h}$ for all $h \in P_\vee$. Because $S$ is an anti-automorphism on $U_q(\mathfrak{g})$, one can define two $U_q(\mathfrak{g})$-module structures on the dual vector space of a $U_q(\mathfrak{g})$-module $V \in O_q^{\geq 0}$ by

$$\langle x \cdot \phi, v \rangle := \langle \phi, S(x) \cdot v \rangle$$

and

$$\langle x \cdot \phi, v \rangle := \langle \phi, S^{-1}(x) \cdot v \rangle$$

for each $x \in U_q(\mathfrak{g})$ and linear functional $\phi$ on $V$. We denote these modules by $V^*$ and $V'$, respectively. As vector spaces both modules are just $\bigoplus_{\mu \in P} V^*_\mu$, where $V^*_\mu = \text{Hom}_{\mathbb{C}(q)}(V_\mu, \mathbb{C}(q))$. The following lemma is an immediate consequence of the definitions.

**Lemma 6.3.** Suppose that $V$ is a $U_q(\mathfrak{g})$-module in the category $O_q^{\geq 0}$.
(1) There exist canonical \( U_q(\mathfrak{g}) \)-module isomorphisms \( (V^*)' \cong V \cong (V')^* \).

(2) The space \( V^*_\mu \) is a weight space of weight \(-\mu\).

Since \( q^h S(e_i)q^{-h} = q^{\alpha_i(h)} S(e_i) \), we have \( S(e_i)V_\mu \subset V_{\mu + \alpha_i} \), which implies \( e_i V^*_\mu \subset V^*_\mu - \alpha_i \). By Lemma 6.3, we get \( e_i (V^*)_\mu \subset (V^*)_\mu + \alpha_i \). Similarly, we also have \( e_i (V^*)_\mu \subset (V^*)_\mu + \alpha_i \), \( f_i (V^*)_\mu \subset (V^*)_\mu - \alpha_i \), \( f_i (V^*)_\mu \subset (V^*)_\mu - \alpha_i \) for all \( i \in I \) and \( k_i (V^*)_\mu \subset (V^*)_\mu - \mu \) for all \( i \in J \). A weight module \( M \) is called a lowest weight module with lowest weight \( \lambda \in P \) if it is generated over \( U_q(\mathfrak{g}) \) by an irreducible finite dimensional \( U^\leq_0 \)-module. By a similar argument as in Proposition 4.1, one can show that \( \left( V^q(\lambda) \right)^* \) is an irreducible \( U^\leq_0 \)-module so that \( V^q(\lambda)^* \) and \( V^q(\lambda)' \) are lowest weight modules of lowest weight \(-\lambda\).

Suppose that \( V \) is a \( U_q(\mathfrak{g}) \)-module in the category \( \mathcal{O}^\geq_q \). Because \( V \) is finite dimensional, we may choose a maximal weight \( \lambda \in \text{wt}(V) \) with the property that \( \lambda + \alpha_i \) is not a weight of \( V \) for any \( i \in I \). Then the weight space \( V_\lambda \) is a \( U^\geq_q \)-module. Fix an irreducible \( U^\geq_q \)-submodule \( \mathfrak{v} \) of \( \mathfrak{V}_\lambda \) and set \( L = U_q(\mathfrak{g}) \mathfrak{v} \). Then \( L \) is a highest weight \( U_q(\mathfrak{g}) \)-module with highest weight \( \lambda \). By the assumption, \( \lambda \in \Lambda^+ \cap P_{\geq 0} \) and from Corollary 5.15, we know \( L \cong V^q(\lambda) \) up to \( \Pi \).

Now consider \( \bar{\mathfrak{v}} = \text{Hom}_{\mathbb{C}(q)}(\mathfrak{v}, \mathbb{C}(q)) \subset V^* \), and set

\[
\bar{L} = U_q(\mathfrak{g}) \bar{\mathfrak{v}} \subset V^*.
\]

It is easy to show that \( \bar{\mathfrak{v}} \) is an irreducible \( U^\leq_q(\mathfrak{g}) \)-module and \( \bar{L} \) is a lowest weight module with lowest weight \(-\lambda\). Translating Corollary 5.15 to the case of lowest weight modules, we get the following lemma.

**Lemma 6.4.** The \( U_q(\mathfrak{g}) \)-module \( \bar{L} \) is isomorphic to the irreducible lowest weight module \( V^q(\lambda)^* \) with lowest weight \(-\lambda\) and lowest weight space \( \bar{\mathfrak{v}} \).

Now we can prove the completely reducibility theorem for \( U_q(\mathfrak{g}) \)-modules in the category \( \mathcal{O}^\geq_q \).

**Theorem 6.5.** Every \( U_q(\mathfrak{g}) \)-module \( V \) in the category \( \mathcal{O}^\geq_q \) is completely reducible.

**Proof.** Take a maximal weight \( \lambda \) and consider a submodule of \( V \), say \( L \), generated by an irreducible \( U^\geq_q \)-submodule of \( V_\lambda \). We want to show \( V \cong L \oplus V/L \). Taking dual with respect to \( S^{-1} \) of the inclusion \( \bar{L} \to V^* \), we obtain a \( U_q(\mathfrak{g}) \)-module homomorphism \( V \cong (V^*)' \to (\bar{L})' \). Thus we have a map:

\[
\psi : L \hookrightarrow V \to (\bar{L})'.
\]

It is easy to check that \( \psi \) is a nontrivial homomorphism. Since both \( L \) and \( (\bar{L})' \) are irreducible, \( \psi \) is an isomorphism by Schur’s lemma and we see that the following short exact sequence splits:

\[
0 \to L \to V \to V/L \to 0.
\]

Since \( V/L \in \mathcal{O}^\geq_q \), using induction on the dimension of \( V \), we complete the proof. \( \square \)
Corollary 6.6. The tensor product of a finite number of $U_q(\mathfrak{g})$-modules in the category $O_{\geq 0}^q$ is completely reducible.

Remark. The same argument can be applied to prove the completely reducibility of $O_{\geq 0}$. In that case, the antipode is given by $S(x) = -x$ for all $x \in \mathfrak{g}$ (see [N, Section 4]) and Proposition [LS] plays the same role as Proposition [5.15].

Acknowledgements. We would like to thank Ivan Penkov and Vera Serganova for the stimulating discussions. D.G. gratefully acknowledges the hospitality and excellent working conditions at the Seoul National University where most of this work was completed.

References

[ABS] M.F. Atiyah, R. Bott, A. Shapiro, Clifford modules, Topology 3 (1964), 3–38.

[B] J. Brundan, Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{q}(n)$, Adv. Math. 182 (2004), 28–77.

[BKM] G. Benkart, S.-J. Kang, D. Melville, Quantized enveloping algebras for Borcherds superalgebras, Trans. Amer. Math. Soc. 350 (1998), 3297–3319.

[Dr] V. Drinfel’d, Quantum groups, Proceedings of the International Congress of Mathematicians, Vol. 1 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[G] M. Gorelik, Shapovalov determinants of $Q$-type Lie superalgebras, Int. Math. Res. Pap., Article ID 96895 (2006), 1–71.

[Har] J. Harris, Algebraic Geometry, A first course. Corrected reprint of the 1992 original. Graduate Texts in Mathematics 133 Springer-Verlag, New York, 1995.

[HK] J. Hong, S.-J. Kang, Introduction to Quantum Groups and Crystal Bases, Graduate Studies in Mathematics 42, American Mathematical Society, 2002.

[IR] K. Ireland, M. Rosen, A Classical Introduction to Modern Number Theory, 2nd ed., Graduate Texts in Mathematics 84, Springer-Verlag, New York, 1990.

[K] V. Kac, Lie superalgebras, Adv. Math. 26 (1977), 8–96.

[Lam] T. Y. Lam, Introduction to Quadratic Forms over Fields, Graduate Studies in Mathematics 67, American Mathematical Society, Providence, RI, 2005.

[Lang] S. Lang, Algebra, Revised third edition, Graduate Texts in Mathematics 211, Springer-Verlag, New York, 2002.

[LS] D. Leites, V. Serganova, Defining relations for classical Lie superalgebras I. Superalgebras with Cartan matrix or Dynkin-type diagram, Proc. Topological and Geometrical Methods in Field Theory (Eds. J. Mickelson, et al), World Sci., Singapore, 1992, 194–201.

[N] M. Nazarov, Capelli identities for Lie superalgebras, Ann. Sci. Ecole Norm. Sup. (4) 30, 6 (1997), 847–872.

[O] G. Olshanski, Quantized universal enveloping superalgebra of type $Q$ and a super-extension of the Hecke algebra, Lett. Math. Phys. 24 (1992), 93–102.

[P] I. Penkov, Characters of typical irreducible finite-dimensional $\mathfrak{q}(n)$-modules, Funct. Anal. Appl. 20 (1986), 30–37.

[PSI] I. Penkov, V. Serganova, Generic irreducible representations of finite-dimensional Lie superalgebras, International Journal of Mathematics 5 (1994), 389–419.
[PS2] I. Penkov, V. Serganova, Characters of irreducible $G$-modules and cohomology of $G/P$ for the Lie supergroup $G = Q(N)$. *J. Math. Sci. (New York)* **84** (1997), 1382–1412.

[PS3] I. Penkov, V. Serganova, Characters of finite-dimensional irreducible $q(n)$-modules, *Lett. Math. Phys.* **40** (1997), 147–158.

[RTF] N. Reshetikhin, L. Takhtadzhyan, L. Faddeev, Quantization of Lie groups and Lie algebras. (Russian) *Algebra i Analiz* **1** (1989), 178–206; translation in *Leningrad Math. J.* **1** (1990), 193–225

[Se1] A. Sergeev, The centre of enveloping algebra for Lie superalgebra $Q(n, C)$, *Lett. Math. Phys.* **7** (1983), 177–179.

[Se2] A. Sergeev, Tensor algebra of the identity representation as a module over the Lie superalgebras $\text{Gl}(n, m)$ and $Q(n)$ (Russian), *Mat. Sb. (N.S.)* **123(165)** (1984), 422–430.

[Sh] G. Shimura, Arithmetic and Analytic Theories of Quadratic Forms and Clifford Groups, *Mathematical Surveys and Monographs* **109**, American Mathematical Society, Providence, RI, 2004.

**Department of Mathematics, University of Texas at Arlington**, Arlington, TX 76021, USA

*E-mail address: grandim@uta.edu*

**Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University**, San 56-1 Sillim-dong, Gwanak-gu, Seoul 151-747, Korea

*E-mail address: jhjung@math.snu.ac.kr*

**Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University**, San 56-1 Sillim-dong, Gwanak-gu, Seoul 151-747, Korea

*E-mail address: sjkang@math.snu.ac.kr*

**Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University**, San 56-1 Sillim-dong, Gwanak-gu, Seoul 151-747, Korea

*E-mail address: mkim@math.snu.ac.kr*