Cutting Cakes and Kissing Circles

ALEXANDER MÜLLER-HERMES

Dividing a cake among a given number of people is a relevant problem at many birthday parties. Mathematically, this problem has often been studied in the context of “fair division,” going back to Hugo Steinhaus [8]. Here, every cake-eating participant should receive a share that they themselves consider to be fair. Many protocols for fair division have been studied (see, for instance, [5, 9, 10]), but they do not seem to be practical at birthday parties with more than two participants, because the partygoers might lack the enthusiasm to perform complicated protocols or be unable to accurately compare the sizes of different pieces of cake. For example, the author himself finds it difficult even to cut a cake into two pieces from which the larger piece cannot be selected immediately. In this article, we present precise geometric constructions for how to cut circular cakes into equal-sized pieces. Our constructions are based on the following assumptions:

1. Cakes are perfect circles.
2. Straight lines can be carved into the cake (or the table) using a knife.

It is our firm opinion that geometric considerations of cake-cutting should use only a knife, which is considered equivalent to a straightedge (i.e., an unmarked ruler) in Euclidean geometry. Noncircular cakes are beyond the scope of this article, but we challenge the reader to find techniques to cut whatever shape of baked good they might encounter into pieces of equal size.

History of Cake Cutting: The Poncelet–Steiner Theorem

With the assumptions stated above, we can cut cakes by geometric constructions following the Euclidean axioms but without using the compass. To substitute for the compass, the cakes themselves may be used as preexisting circles. The study of such constructions has a long history, and we will start with the following theorem proved by Jakob Steiner [7] in 1833 after being conjectured by Victor Poncelet. We recommend [4] for a well-written exposition of its proof.

**Theorem 1** (Poncelet–Steiner) Every straightedge-and-compass construction in Euclidean geometry can be performed using a straightedge alone, provided that a circle with its center is given.

By the Poncelet–Steiner theorem, it is possible to cut a cake with marked center into \( n \) pieces of equal size whenever the regular \( n \)-gon is constructible in Euclidean geometry, i.e., whenever \( n \) is a product of distinct Fermat primes and a power of 2.

Many cakes encountered in the wild have their center marked by some kind of decoration, but cakes with unmarked or inaccurately marked center are also quite common. Unfortunately, the conclusion of the Poncelet–Steiner theorem is no longer true when the center of the given circle is not known. This observation is due to David Hilbert, as noted in [2], and an exposition of his argument can be found in [6]. As a consequence, it is impossible to cut a cake in half exactly when the center is not known and only a knife may be used. Luckily, having more cakes available saves the day, as shown by Detlef Cauer [2] in 1912. Focusing on situations most relevant for cutting cakes, it is possible to construct the center of a circle using a straightedge alone if another circle touching \( c \) from the outside is given or if two additional circles are given in any position such that none of the three circles lies inside another. By the Poncelet–Steiner theorem, this implies the following theorem.

**Theorem 2** (Cauer) Every straightedge-and-compass construction in Euclidean geometry can be performed
using a straightedge alone, provided that either two circles are given touching from the outside, or three circles are given in any position such that none of the three circles lies inside another.

Since cakes, unlike circles drawn on a piece of paper, are movable objects, we can push two cakes together so that they touch at a point. By Cauer's theorem, two cakes are then enough to perform general Euclidean constructions using a knife alone. Although we provide all ingredients for cutting pairs of touching cakes in this article, there is an important caveat: in the constructions known to the author (see, for instance, Figure 1 for a bisection), auxiliary lines have to be carved into the table rather than into the surface of the cake. While we have not excluded this possibility in the assumptions stated above, it might create conflict in practice, and the aspiring cake-cutter might no longer be invited to parties with cakes to cut. In the following, we will present constructions that use only the surface of the cakes but require at least three cakes or a marked center. Specifically, we present such constructions for cutting a cake into \( n \) pieces of equal size where \( n = 2, 3, 4, \) and \( 6. \)

**Bisecting an Odd Number of Cakes**

Our method for bisecting cakes is based on an elementary construction using kissing circles. We say that a pair of circles \( c_1, c_2 \) are kissing at a point \( S \) if the following conditions are satisfied:

1. The intersection satisfies \( c_1 \cap c_2 = \{S\}. \)
2. The circles \( c_1 \) and \( c_2 \) lie on opposite sides of their common tangent in \( S. \)

See Figure 2 for an illustration. Given a pair of kissing circles and a point \( P \) on one of the circles, we can construct a point \( P' \) on the other circle by passing \( P \) through the kissing point, as defined below.

**Definition 3** (Passing through the kissing point) Consider circles \( c_1 \) and \( c_2 \) kissing at a point \( S. \) For a point \( P \) on \( c_1, \) we construct a point \( P' \) on \( c_2 \) as follows:

1. If \( P = S, \) then we set \( P' = S. \)
2. If \( P \neq S, \) then we set \( P' \) to be the intersection point different from \( S \) of the line \( PS \) with \( c_2. \)

We will say that \( P' \) is obtained by passing \( P \) through the kissing point \( S. \)

Suppose now that we are given an odd number of cakes. To bisect one of them, we first push them together such that they form a closed chain of kissing circles, as in Figure 3, i.e., circles \( c_1, \ldots, c_{2n+1} \) such that the pair \( (c_i, c_{i+1}) \) are kissing at the point \( S_i \) and the pair \( (c_{2n+1}, c_1) \) are kissing at the point \( S_{2n+1}. \) Starting with a point \( P_1 \) on \( c_1, \) we construct a point \( P_2 \) on \( c_2 \) by passing \( P_1 \) through the kissing point \( S_1. \) Repeating this successively for the other circles leads to points \( P_2, \ldots, P_{2n+1} \) on \( c_2, \ldots, c_{2n+1}, \) as in Figure 3. Finally, we construct a point \( Q \) on \( c_1 \) by passing \( P_{2n+1} \) through the kissing point \( S_{2n+1}. \) We claim that the chord \( QP \) is a diameter and hence bisects the first cake.

![Figure 1. Bisection of a cake. The red line indicates the final cut bisecting the larger cake.](image-url)
The dilations form a group under composition. In particular, the composition of two central dilations with distinct centers $S_1$ and $S_2$ and scale factors $k_1, k_2$ is a translation when $k_1 \cdot k_2 \neq 1$. The composition of a translation and a central dilation with scale factor $k$ is again a central dilation with the same scale factor $k$ but different center, unless the translation is the identity.

Given two circles $c_1$ and $c_2$ kissing at a point $S$, our operation “passing through the kissing point” introduced above and transforming $c_1$ into $c_2$ is a central dilation with center $S$ and negative scale factor $k = -r_2/r_1$, where $r_1$ and $r_2$ denote the radii of $c_1$ and $c_2$, respectively. Consider a closed chain of kissing circles $c_1, \ldots, c_n$ kissing at points $S_1, \ldots, S_n$, as in Theorem 4, and let $f_1, \ldots, f_n$ denote the central dilations such that $f_i$ passes points of $c_i$ through the kissing point $S_i$. Consider now the composition $g = f_n \circ \cdots \circ f_1$ of these central dilations. We know that $g$ is a dilation, and since $g(c_1) = c_1$, we conclude that $g$ is a central dilation with center $Z$ coinciding with the center of the circle $c_1$ and scale factor $k$ either $+1$ or $-1$. Since each $f_i$ has a negative scaling factor, we see (by the composition rules outlined above) that $k = -1$ when $n$ is odd. In this case, the central dilation $g$ is the point reflection at the center $Z$, and $Q = g(P_1)$ is the point diametrically opposite $P_1$ on $c_1$. On the other hand, if $n$ is even, then we see that $k = +1$, and the central dilation $g$ is the identity and $Q = g(P_1) = P_1$. This proves Theorem 4.

**Cutting Cakes into More Pieces**

So far, we have discussed how to find a diameter of a cake when there are at least three cakes available. Repeating the construction after rotating the cake gives another diameter intersecting the first one at the center of the cake. When cutting three or more cakes, we may therefore assume that the center is marked on each of them. By the Poncelet–Steiner theorem mentioned above, it is then possible to cut each of the cakes into three, four, or six pieces of equal size, since the corresponding regular $n$-gons are constructible in Euclidean geometry. In the following, we will show how to do this in practice.

Our constructions are based on two tricks developed by Jacob Steiner in [7], which can also be found in [4]. The first trick constructs a line parallel to a given line $PQ$ through points $P$ and $Q$ when the midpoint $Z$ of the segment $PQ$ is given. The second trick, closely related to the first, constructs the midpoint $Z$ of a segment $PQ$ when a parallel line to the segment is given. Both statements follow directly from Ceva’s theorem [3, p 220], and we leave their proofs to the reader.

**Theorem 5** (Two tricks by Steiner) Consider points $P$ and $Q$.

1. Let $Z$ be the point bisecting the segment $PQ$ and consider a point $R$ not on the line $PQ$, and a point $A$ on the segment $PR$. We construct a point $B$ as the intersection of $QA$ and $ZR$. Let $C$ denote the intersection of the line $PB$ with the line $QR$. Then the line $AC$ is parallel to the line $PQ$.
2. Let points $A$ and $C$ be given such that the lines $AC$ and $PQ$ are parallel. We construct a point $R$ as the intersection of $PA$ and $QC$, and a point $B$ as the...
intersection of PC and QA. Then the line RB intersects PQ at the point Z bisecting the segment PQ. See Figure 4 for an illustration.

We will now show how to cut a cake with marked center Z into four pieces of equal size. Let P and Q denote a pair of antipodal points on the rim of the cake, i.e., such that the chord PQ is a diameter. To cut the cake into four pieces of equal size, we need to construct a diameter perpendicular to PQ. Note that the center Z bisects the diameter PQ. By choosing a point R on the rim of the cake and a point A on the segment PR, we can use the construction from the first case of Theorem 5 to find a point C such that AC is parallel to PQ. Intersecting the line AC with the circle yields points A’ and C’ such that the quadrilateral QC’A’P is an isosceles trapezoid. Finally, the intersection point B of the diagonals PC’ and QA’ lies on a diameter perpendicular to PQ, and by cutting along the lines PQ and ZB’, we divide the cake into four pieces of equal size.

Next, we present a construction for cutting a cake into three pieces of equal size. Again, we assume that a cake with marked center is given, and using the previous construction, we find four points P, Q, P’, Q’ on its rim such that the quadrilateral QP’PQ’ is a square, as in Figure 6. We first note that the segments PP’ and PQ are parallel. After choosing a point R on the rim of the cake, we apply the construction from the second case of Theorem 5 to find a point X1 bisecting the segment PQ. Repeating this construction for the parallel segments QQ’ and PP’ (and another point M on the rim of the cake), we obtain the point X2 bisecting the segment QQ’. The line X1X2 intersects the rim at the points W and Z such that P’WZ is an
equilateral triangle. Finally, we cut from each point $P_0$, $W$, and $Z$ to the center of the cake and obtain three pieces of equal size. Note that the previous construction also allows the cake to be cut into six pieces of equal size by extending the red segments in Figure 6.

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