Composite adaptive backstepping controller design and the energy calculation for active suspension system

Xiaoyu Su¹, Bin Lin¹ and Shuai Liu²
¹School of Electronic and Electrical Engineering, Shanghai University of Engineering Science, Shanghai, China
²Department of Mechatronics Engineering, University of Shanghai for Science and Technology, Shanghai, China

Abstract
The half-car suspension has the coupling of pitch angle and front and rear suspension. Especially when the suspension model has a series of uncertainties, the traditional linear control method is difficult to be applied to the half-car suspension model. At present, there is no systematic method to solve the suspension power. According to the energy storage characteristics of the elastic components of the suspension, the power calculation formula is proposed in this paper. This paper proposes a composite adaptive backstepping control scheme for the half-car active suspension systems. In this method, the correlation information between the output error and the parameter estimation error is used to construct the adaptive law. According to the energy storage characteristics of the elastic components of the suspension, the power calculation formula is introduced. The compound adaptive law and the ordinary adaptive law have good disturbance suppression, both of which can solve the pitching angle problem of the semi-car suspension, but the algorithm of the compound adaptive law is superior in effect. In terms of vehicle comfort, the algorithm of the general adaptive law can achieve stability quickly, but compared with the composite adaptive law, its peak value and jitter are higher, while the algorithm of the composite adaptive law is relatively gentle and has better adaptability to human body. In terms of vehicle handling, both control algorithms can maintain driving safety under road excitation, and the compound adaptive algorithm appears to have more advantages. Compared with the traditional adaptive algorithm, the power consumption of the composite adaptive algorithm is relatively lower than that of the former in the whole process. The simulation results show that the ride comfort, operating stability and safety of the vehicle can be effectively improved by the composite adaptive backstepping

Corresponding author:
Xiaoyu Su, School of Electronic and Electrical Engineering, Shanghai University of Engineering Science, 333 Long Teng Road, Shanghai 201620, China.
Email: su_xy0722@163.com

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controller, and the composite adaptive algorithm is more energy-saving than the conventional adaptive algorithm based on projection operator.

**Keywords**

Automobile active suspension, composite adaptive backstepping control, energy-saving

**Introduction**

The suspension system has a significant influence on the vehicle ride comfort and handling stability. Compared with the traditional automobile suspension system, the active suspension system has great advantages. For example, the vibration between the body and the irregular road surface can be more effectively isolated by the active suspension system. A quarter of the model is very effective in verifying the superiority of the control algorithm. In order to solve the problem of the pitch angle of the suspension and the vertical displacement of the body, the half-car suspension model is adopted in this paper, and the properties of the nonlinear suspension components are taken into account. But the half-car suspension has the coupling of pitch angle and front and rear suspension, which is more difficult than the quarter suspension system in the control algorithm design. Especially, due to the suspension model has a series of uncertainties, the traditional linear control method is difficult to be applied to the half-car suspension model. Therefore, adaptive backstepping control algorithm is adopted to solve the above problems.

The adaptive algorithm has some adaptabilities in the estimation of uncertainties. And according to the change of parameters, this algorithm can automatically maintain the stable operation of the system. Adaptive control is widely used in active suspension system because of its superiority in solving uncertainties. An adaptive control strategy suitable for the CVD skyhook control of the whole vehicle with seven degrees of freedom was proposed by Amin et al., which solved the problems of vehicle ride comfort and safety under different driving modes. Na et al., a new adaptive control algorithm was designed to stabilize the vertical displacement of the vehicle and improve ride comfort. In addition, other suspension requirements related to vehicle safety and mechanical constraints (such as road retention and suspension space limitations) are considered in this paper. The backstepping control is very effective in solving nonlinear problems. this algorithm has strong fusion and can be effectively combined with other algorithms. Chávez-Conde et al., this control algorithm is combined with integral sliding mode control to solve a series of nonlinear characteristics caused by suspension under the road disturbance. In response to the disturbance and uncertainty of suspension on different roads, Basturk proposed an adaptive backstepping control to improve the comfort of the vehicle. But with the promotion of energy conservation, the current adaptive law has been difficult to meet the growing needs of people. In this paper, the conventional adaptive control method with projection algorithm is improved, and a composite adaptive control method is proposed to solve the pitch angle of the half-car suspension and the vertical displacement of the vehicle body. This method is combined with
backstepping control to ensure the travel and safety of suspension. At present, there is no systematic method to calculate the suspension power, and the evaluation index of suspension energy consumption is still in a blank state. According to the energy storage characteristics of the elastic components of the suspension, the power calculation formula is proposed in this paper.

**Problem formation**

In order to solve the problem of the pitch angle of the suspension and the vertical displacement of the body, the half-car suspension model is adopted in this paper.

According to the active suspension structure diagram in Figure 1, the equations of motion of the half-car active suspension model can be described as follows:

\[
\begin{align*}
\text{m}_c \ddot{X}_c + F_{a1} + F_{a2} + F_{b1} + F_{b2} &= u_1 \\
h \ddot{e} + \delta(F_{a1} + F_{b2}) - \gamma(F_{a1} + F_{b1}) &= u_2 \\
\text{m}_a \ddot{x}_1 - F_{a1} - F_{b1} + F_{ar1} + F_{br1} &= -u_a \\
\text{m}_b \ddot{x}_2 - F_{a2} - F_{b2} + F_{ar2} + F_{br2} &= -u_b
\end{align*}
\]

where, \(c_{b1}\) and \(c_{br2}\) are the damping coefficients of the front and rear parts of the suspension, \(t_{ar1}\), \(t_{ar2}\) and \(c_{br1}\), \(c_{br2}\), respectively, represent the stiffness coefficient and damping coefficient of the tire, the spring stiffness coefficients of the front and rear suspension components are \(x_1\) and \(x_2\) respectively, \(m_a\) and \(m_b\) are the unsprung masses of the front and rear wheels respectively, \(u_a\), \(u_b\) are the control forces of the suspension generated by the actuator, \(m_c\) is the moment of inertia of the body mass, and \(h\) is the moment of inertia of pitching motion. \(F_{b1}\), \(F_{b2}\), \(F_{a1}\) and \(F_{a2}\) are the damping forces and spring elastic forces in the front and rear suspension components, \(F_{ar1}\), \(F_{ar2}\), \(F_{br1}\), \(F_{br2}\) are the elastic forces and damping forces generated by the front and rear tires. \(e\) is the pitching angle, \(x_{r1}\), \(x_{r2}\) are inputs of road disturbance displacements.

\[
\begin{align*}
u_1 &= u_a + u_b \\
u_2 &= \delta u_b - \gamma u_a \\
F_{a1} &= t_{a1} \Delta \dot{t}, \quad F_{a2} = t_{a2} \Delta \dot{p} \\
F_{b1} &= c_{b1} \Delta \dot{t}, \quad F_{b2} = c_{b2} \Delta \dot{p} \\
F_{ar1} &= t_{ar1}(x_1 - x_{r1}), \quad F_{ar2} = t_{ar2}(x_2 - z_{r2}) \\
F_{br1} &= c_{br1}(\dot{x}_1 - \dot{x}_{r1}), \quad F_{br2} = c_{br2}(\dot{x}_2 - \dot{z}_{r2})
\end{align*}
\]

Then, the above equations are constrained as follows:

\[
|F_{a2} + F_{br2}|<F_2 = \frac{(m_c \gamma + m_b (\delta + \gamma))g}{(\delta + \gamma)}
\]

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\[ |F_{ar1} + F_{br1}| < F_1 = (m_c + m_a + m_b)g - F_2 \]  
\[ \Delta \partial_t = X_c + \delta \sin \varepsilon - x_2 \]  
\[ \Delta \partial_p = X_c - \gamma \sin \varepsilon - x_1 \]

In the above formulas, \( \delta \) and \( \gamma \) respectively represent the distances between the front and rear suspension assembly center and the body mass center, \( \Delta \partial_t \) is the travel of the front suspension, \( \Delta \partial_p \) is the travel of the rear suspension. The suspension travel constraints are designed as follows:

\[ |\Delta \partial_t| < \Delta \partial_{\text{max}} \]  
\[ |\Delta \partial_p| < \Delta \partial_{\text{pmax}} \]

where, \( \Delta \partial_{\text{max}} \) and \( \Delta \partial_{\text{pmax}} \) are the maximum value of the suspension displacement.

To proceed further, the following variables are defined:

\[ x_1 = X_c, x_2 = \dot{X}_c, x_3 = \varepsilon, x_4 = \dot{\varepsilon}, x_5 = x_2, x_6 = \dot{x}_2, x_7 = x_1, x_4 = \dot{x}_1, \]

Then,

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{m_c} (-F_{a1} - F_{a2} - F_{b1} - F_{b2} + u_1) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{h} (-\delta(F_{b2} + F_{a2}) + \gamma(F_{b1} + F_{a1}) + u_2) \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= \frac{1}{m_b} (F_{a2} + F_{b2} - F_{ar2} - F_{br2} - u_b) \\
\dot{x}_7 &= x_8 \\
\dot{x}_8 &= \frac{1}{m_a} (F_{ar1} + F_{b1} - F_{ar1} - F_{br1} - u_a)
\end{align*}
\]
Influenced by the original suspension structure and materials, the corresponding \( M \) and \( I \) are in a certain range and conform to the following forms:

\[
m_c \in \Omega_M = \{ m_c : m_{c_{\min}} \leq m_c \leq m_{c_{\max}} \}
\]

\[
\sigma \in \Omega_I = \{ \sigma : \sigma_{\min} \leq \sigma \leq \sigma_{\max} \}
\]

Due to the influence of its structure, the controller will be overloaded in the process of working. This may result in the original control effect is not as expected, so it is necessary to constrain the amplitude of the controller. The amplitude is set as follows:

\[
|u_i(t)| < u_{i_{\max}}, \ i = \alpha, \beta
\]

The next step is to design the controller \( u_i(i = \alpha, \beta) \) to stabilize the suspension system. That is, the travel of the car body’s vertical displacement and angular displacement suspension can meet the following equation conditions:

\[
\lim_{t \to T_r} x_1 = 0
\]

\[
\lim_{t \to T_r} \epsilon = 0
\]

where, \( T_r \) is the pre-designed time.

**Composite adaptive backstepping control**

According to formula (8), the formula of the first subsystem is as follows:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \varphi_1(-F_{a1} - F_{a2} - F_{b1} - F_{b2} + u_1)
\end{align*}
\]

where, \( \varphi_1 = \frac{1}{m_c} \in [\varphi_{1_{\min}} \varphi_{1_{\max}}], \varphi_{1_{\min}} = \frac{1}{m_{c_{\max}}}, \varphi_{1_{\max}} = \frac{1}{m_{c_{\min}}} \)

The tracking error is set as \( e_\alpha = x_1 - x_{d_1} \). According to this formula, we can obtain its derivative as:

\[
\dot{e}_\alpha = x_2 - \dot{x}_{d_1}
\]

And the corresponding error of \( x_2 \) is set as \( e_\beta \), which form is as follows:

\[
e_\beta = x_2 - x_i
\]

where, \( x_i \) is the ideal function controlling the input \( x_2 \) in formula (15).

Then the formula (15) can be further written as:

\[
\dot{e}_\alpha = e_\beta + x_i - \dot{x}_{d_1}
\]

Lyapunov function is selected as:
\[ V_1 = \frac{1}{2} e_\alpha^2 \]  \hspace{1cm} (18)

In order to stabilize the formula (15), the desired virtual control \( x_t \) is designed as follow:

\[ x_t = \dot{x}_{d1} - \kappa_1 (2 \text{sigmoid}(2e_\alpha) - 1) \]  \hspace{1cm} (19)

The \( \kappa_1 \) is a positive number. Taking the derivative of formula (18) to obtain the following form:

\[ \dot{V}_1 = e_\alpha e_\beta - \kappa_1 e_\alpha (2 \text{sigmoid}(2e_\alpha) - 1) \]  \hspace{1cm} (20)

As we can see from the above formula, if \( e_\beta = 0 \), so \( \dot{V}_1 = -\kappa_1 e_\alpha (2 \text{sigmoid}(2e_\alpha) - 1) \leq 0 \), \( e_\alpha \) will asymptotically to be zero. If this condition to be true, according to formula (16), we can get: as long as \( x_2 = x_t \) is guaranteed to be true, the requirement of this condition to be true can be satisfied. The next step is to design the control law \( u_1 \) to ensure \( e_\beta \to 0 \), so that the state \( x_2 \) tends to the desired virtual control \( x_t \).

The derivative of the error dynamic \( e_\beta = x_2 - x_t \) can be obtained as follow:

\[ \dot{e}_\beta = \varphi_1 (-F_{a1} - F_{a2} - F_{b1} - F_{b2} + u_1) - \ddot{x}_{d1} + \kappa_1 (1 - (2 \text{sigmoid}(2e_\alpha) - 1)^2) \dot{e}_\alpha \]  \hspace{1cm} (21)

The control law \( u_1 \) is designed as follow:

\[ u_1 = \frac{1}{\hat{\varphi}_1} (\ddot{x}_{d1} - \kappa_1 (1 - (2 \text{sigmoid}(2e_\alpha) - 1)^2) \dot{e}_\alpha - \kappa_1 (2 \text{sigmoid}(2e_\beta) - 1) - e_\alpha) \]
\[ + F_{a1} + F_{a2} + F_{b1} + F_{b2} \]  \hspace{1cm} (22)

Where, \( \hat{\varphi}_1 \) is the estimated value of \( \varphi_1 \).

**Design of conventional adaptive law**

The conventional adaptive law is designed as:

\[ \dot{\phi} = \text{Proj}_{\hat{\phi}_1} (\theta \phi) \]  \hspace{1cm} (23)

\[ \phi = (-F_{a1} - F_{a2} - F_{b1} - F_{b2} + u_1) e_\beta = p e_\beta \]  \hspace{1cm} (24)

where, \( p = (-F_{a1} - F_{a2} - F_{b1} - F_{b2} + u_1) \).

\( \theta > 0 \) is the regression factor of the adaptive law, and is the known parameter. \( \text{Proj}_{\hat{\phi}_1} \) is the projection operator of vector, which is defined as follows:

\[ \text{Proj}_{\hat{\phi}_1}(\phi) = [\text{Proj}_{\hat{\phi}_1}(\phi_1), \ldots, \text{Proj}_{\hat{\phi}_1}(\phi_n)]^T \] Among them,
\( \text{Proj}_{\phi_i}(\theta \phi) = \begin{cases} 
0, & \text{when } \hat{\phi}_i = \phi_{i\text{max}} \text{ and } \theta \phi > 0, \\
0, & \text{when } \hat{\phi}_i = \phi_{i\text{min}} \text{ and } \theta \phi < 0, \\
\theta \phi, & \text{Other situations}.
\end{cases} \) (25)

The adaptive law has the following two properties:

1. It is guaranteed that the estimated \( \hat{\phi}_1 \) of this parameter is always within the known bounds:

\[ \phi_{1\text{min}} < \hat{\phi}_1 < \phi_{1\text{max}} \] (26)

2. Guarantee that the following conditions are true:

\[ \hat{\phi}_1 (\theta^{-1} \text{Proj}_{\phi_1}(\theta \phi) - \phi) < 0, \ \forall \phi \] (27)

**Design of compound adaptive law**

In order to make the online estimation \( \hat{\phi}_1 \) of the parameter approximate to its truth value \( \phi_1 \), we use the compound adaptive law instead of the conventional adaptive law. Although the conventional adaptive law is sufficient to meet the control requirements of suspension and has been adopted by a large number of scholars, the composite adaptive law has better convergence than the conventional adaptive law, higher accuracy than the former, and better control effect. The composite adaptive law is as follows:

\[ \hat{\phi}_1 = \text{Proj}_{\phi_1}(\theta \phi - \partial r(P\hat{\phi}_1 - \mu)) - \phi \] (28)

Set \( \sigma_0 = F_{df} + F_r + F_{sf} + F_{sr} - u_z \),

where, \( P(t) = \int_0^t \sigma_0 \sigma_0^T dr \), \( \mu(t) = \int_0^t \sigma_0 u_z dr \), \( \partial \) is an optional positive number. Further analysis of formula (28) shows that:

\[ \dot{\hat{\phi}}_1 = \text{Proj}_{\phi_1}(\theta p \beta - \partial r(\int_0^t \sigma_0 \sigma_0^T dr \hat{\phi}_1 - \int_0^t \sigma_0 u_z dr)) - \phi = \text{Proj}_{\phi_1}(\theta p \beta - \partial r(\int_0^t \sigma_0 \sigma_0^T dr \hat{\phi}_1)) \)

It can be seen that the adaptive law includes not only the system output error information \( p \), but also the term \( \int_0^t \sigma_0 \sigma_0^T dr \hat{\phi}_1 \) related to the parameter estimation error.

For any matrices \( A \) and \( Q \), the following formula is true:

\[ \hat{\phi}_1 ^T (\theta^{-1} \text{Proj}_{\phi_1}(\theta \phi - \theta \phi_1) - \phi) \leq -\tilde{\theta}_1^T \phi_1 \] (29)

The following Lyapunov function is selected:

\[ V_2 = V_1 + \frac{1}{2} \varepsilon_\beta^2 + \frac{1}{2\theta} \tilde{\phi}_1^T \theta^{-1} \phi_1 \] (30)
The derivative of the above formula can be obtained:

\[ \dot{V}_2 = \dot{V}_1 + e_\beta \dot{\beta} + \phi_1^T \theta^{-1} \dot{\phi} = -\kappa_1 e_\alpha (2 \text{sigmoid}(2e_\alpha) - 1) \]

\[ - \kappa_2 e_\beta (2 \text{sigmoid}(2e_\alpha) - 1) + \phi_1^T \theta^{-1} (\text{Proj}_{\phi_1}(\theta \phi - \partial \theta (P \phi_1 - \mu)) - \phi) \]

(31)

According to lemma 1:

\[ \dot{V}_2 = -\kappa_1 e_\alpha (2 \text{sigmoid}(2e_\alpha) - 1) - \kappa_2 e_\beta (2 \text{sigmoid}(2e_\alpha) - 1) - \partial \phi_1^T P \phi_1 \]

(32)

We know from the definition of \( P \) that \( P(t_0) \) is a positive semidefinite matrix. So if \( P(t_0) \) is not singular, then we have \( \lambda_{\min}(P(t_0)) > 0 \). \( P(t_0) \) is positive definite. For \( t > t_0 \),

\[ P(t) = \int_{t_0}^{t} \sigma_0 \sigma_0^T dr + \int_{t_0}^{t} \sigma_0 \sigma_0^T dr = P(t_0) + \int_{t_0}^{t} \sigma_0 \sigma_0^T dr \]

(33)

Since \( P(t) \), \( P(t_0) \) and \( \int_{t_0}^{t} \sigma_0 \sigma_0^T dr \) are all semidefinite matrices, \( \lambda_{\min}(P(t)) \geq \lambda_{\min}(P(t_0)) \). In summary, if there is a \( t_0 > 0 \) that makes \( P(t_0) \) nonsingular, then there is a positive definite matrix \( P(t) \) for \( t > t_0 \), and \( \lambda_{\min}(P(t)) \geq \lambda_{\min}(P(t_0)) > 0 \). Because \( \partial > 0 \), so \(- \partial \phi_1^T P \phi_1 < 0 \).

\[ \dot{V}_2 \leq -\kappa_1 e_\alpha (2 \text{sigmoid}(2e_\alpha) - 1) - \kappa_2 e_\beta (2 \text{sigmoid}(2e_\alpha) - 1) \leq 0 \]

(34)

The following inequalities can be obtained by integrating both sides of \( A \) from \( 0 \) to \( t \):

\[ V_2 = \int_{0}^{t} \dot{V}_2 d\tau + V_2(0) \leq V_2(0) \]

(35)

Therefore, we can conclude that \( e_\alpha \) and \( e_\beta \) have ranges and fixed upper limits:

\[ |e_\alpha| \leq \sqrt{2V_2(0)} |e_\beta| \leq \sqrt{2V_2(0)} \]

(36)

From the above equation, it can be further obtained that:

\[ |x_1| \leq |x_{d_1}| + \sqrt{2V_2(0)} \leq |x_{d_1}|_\infty + \sqrt{2V_2(0)} \leq |x_{d_1}|_\infty + \sqrt{2V_2(0)} \]

(37)

\[ |x_2| \leq |x_{d_1}| + \kappa_1 |2 \text{sigmoid}(2e_\alpha) - 1| + \sqrt{2V_2(0)} \leq |x_{d_1}|_\infty + (\kappa_1 + 1) \sqrt{2V_2(0)} \]

(38)

The formula (38) and subsequent steps show that all signals of the system are bounded. Therefore, the following formula is established:

\[ -F_{a_1} - F_{a_2} - F_{b_1} - F_{b_2} + u_1 \in L_\infty \]

(39)

So \( \dot{e}_\beta \in L_\infty \), so we get
\[ \dot{V}_2 \leq -\mathcal{N}_1 e_\alpha \left( 1 - \left( 2\text{sigmoid}(e_\alpha) - 1 \right)^2 \right) \dot{e}_\alpha - \mathcal{N}_1 \dot{e}_\alpha (2\text{sigmoid}(e_\alpha) - 1) \]
\[ - \mathcal{N}_2 e_\beta \left( 1 - \left( 2\text{sigmoid}(e_\beta) - 1 \right)^2 \right) \dot{e}_\beta - \mathcal{N}_1 \dot{e}_\beta (2\text{sigmoid}(e_\beta) - 1) \]  
(40)

which is equal to \( C \) is bounded, so \( V_2 \) is uniformly continuous. Furthermore, \( \dot{V}_2 < 0 \). According to Lyapunov theorem, when \( t \to \infty, e_\beta \to 0 \) and \( \dot{\varphi}_1 \to 0 \).

At this point, the design of controller \( u_1 \) and its proof process are finished. Next is the design process of controller \( u_2 \). Since its equation is somewhat similar to the previous formula, only the corresponding design process is given this time. As long as the design of controller \( u_1 \) and \( u_2 \) is completed, we can obtain \( u_\alpha \) and \( u_\beta \) according to the linkage formulas.

Set
\[ e_\gamma = x_3 - x_{d3} \]  
(41)
\[ e_\delta = x_4 - x_{i2} \]  
(42)

If \( x_{d3} \) and \( u_2 \) are defined as follows:
\[ x_{d3} = \dot{x}_{d3} - \mathcal{N}_3 (2\text{sigmoid}(2e_\gamma) - 1) \]  
(43)
\[ u_2 = \frac{1}{\dot{\varphi}_2} \left( \dot{x}_{d3} - \mathcal{N}_3 \left( 1 - \left( 2\text{sigmoid}(2e_\gamma) - 1 \right)^2 \right) \dot{e}_\alpha \right) + F_\varphi \]  
(44)
\[ \dot{\varphi} = \text{Proj}_{\dot{\varphi}_2} \left( \theta_2 \dot{\varphi}_2 - \partial \theta_2 (P \hat{\varphi}_2 - \mu) \right) - \phi_2 \]  
(45)

Then the error dynamic equation of pitching motion is stable. The \( \mathcal{N}_3 \) and \( \mathcal{N}_4 \) are controllable parameters. \( F_\varphi \) and \( \phi_2 \) are defined as follows:
\[ F_\varphi = \delta (F_{b2} + F_{a2}) - \gamma (F_{b1} + F_{a1}) \]  
(46)
\[ \phi_2 = (u_2 - F_\varphi) e_\delta \]  
(47)

where, \( \theta_2 \) is the adjustable parameter, the following stability analysis can be obtained:
\[ |x_3| \leq |x_{d3}|_\infty + \sqrt{2V_4(0)} \]  
(48)
\[ |x_2| \leq |x_{d3}|_\infty + (\mathcal{N}_3 + 1) \sqrt{2V_4(0)} \]  
(49)

Finally, according to \( u_1 \) and \( u_2 \):
\[ u_\alpha = \frac{\gamma u_1 + u_2}{\delta + \gamma} \]  
(50)
\[ u_\beta = \frac{\delta u_1 - u_2}{\delta + \gamma} \]  

(51)

The next part is the proof of zero dynamic stability.

**Stability proof**

The above-mentioned discussion devotes to controller design for a fourth-order error system, while the whole is an eight-order system, a zero-dynamic analysis is required. According to the requirement of zero dynamics, we set \( e_\alpha = e_\gamma \), that is, \( e_\beta = e_\delta \), so the following formulas can be obtained:

\[ u_1 = m_c \ddot{x}_{d1} + F_{b1} + F_{a1} + F_{b2} + F_{a2} \]  

(52)

\[ u_2 = h \ddot{x}_{d3} + \delta(F_{b2} + F_{a2}) - \gamma(F_{b1} + F_{a1}) \]  

(53)

Through the above controller form, we can obtain:

\[ u_\alpha = \frac{\gamma M}{\delta + \gamma} \ddot{x}_{d1} + \frac{h}{\delta + \gamma} \ddot{x}_{d3} + F_{b2} + F_{a2} \]  

(54)

\[ u_\beta = \frac{\gamma M}{\delta + \gamma} \ddot{x}_{d1} - \frac{h}{\delta + \gamma} \ddot{x}_{d3} + F_{b1} + F_{a1} \]  

(55)

According to (54) and (55), the zero dynamic equation can be obtained:

\[ \dot{x} = Ax + BW + B_a x_r \]  

(56)

where, \( x = \begin{bmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \), \( B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma a}{m_a} & \frac{\gamma a b}{m_a} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{\gamma a}{m_a} & \frac{\gamma a b}{m_a} \end{bmatrix} \), \( A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma a}{m_a} & \frac{\gamma a b}{m_a} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{\gamma a}{m_a} & -\frac{\gamma a b}{m_a} \end{bmatrix} \), \( 

W = \begin{bmatrix} W_1 \\ \hat{W}_1 \\ W_2 \end{bmatrix}, B_a = \begin{bmatrix} -\frac{\gamma m_c}{m_a(\delta + \gamma)} & -\frac{\gamma m_c}{m_a(\delta + \gamma)} & 0 & 0 \\ 0 & 0 & \frac{h}{m_a(\delta + \gamma)} & \frac{h}{m_a(\delta + \gamma)} \end{bmatrix}, x_r = \begin{bmatrix} \ddot{x}_{d1} \\ \ddot{x}_{d3} \end{bmatrix} \)

The Lyapunov function is defined as follows:

\[ V = x^T P x \]  

(57)

Then,

\[ \dot{V} = \dot{x}^T P x + x^T \dot{P} x = x^T (A^T P + AP) x + 2x^T PBW + 2x^T PB_a x_r \]  

(58)
Since the eigenvalues of $A$ are all negative real parts, $A^TP + AP = -Q < 0$, and

$$2x^TPBW \leq \frac{1}{\gamma_1}x^TPBB^TPx + \gamma_1 W^TW \tag{59}$$

$$2x^TPB_a x_r \leq \frac{1}{\gamma_2}x^TPB_a B_a^TPx + \gamma_2 x_r^T x_r \tag{60}$$

where $\gamma_1$ and $\gamma_2$ are adjustable positive parameters, then,

$$\dot{V} \leq -x^T Q x + \frac{1}{\gamma_1}x^T PBB^TPx + \gamma_1 W^TW + \frac{1}{\gamma_2}x^T PBB^TPx + \gamma_2 x_r^T x_r$$

$$\leq \left[ -\lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}) + \frac{1}{\gamma_1}\lambda_{\max}(P^{-\frac{1}{2}}BB^TP^{-\frac{1}{2}}) + \frac{1}{\gamma_2}\lambda_{\max}(P^2B_aB_a^TP^{-\frac{1}{2}}) \right] V + \gamma_1 W^TW + \gamma_2 x_r^T x_r \tag{61}$$

Select appropriate parameters to obtain:

$$\lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}) - \frac{1}{\gamma_1}\lambda_{\max}(P^{-\frac{1}{2}}BB^TP^{-\frac{1}{2}}) - \frac{1}{\gamma_2}\lambda_{\max}(P^2B_aB_a^TP^{-\frac{1}{2}}) \geq \exists_1 \geq 0 \tag{62}$$

Define $W^TW \leq W_{\max}$, $x_r^T x_r \leq x_{r_{\max}}$, then:

$$\gamma_1 W^TW + \gamma_2 x_r^T x_r \leq \exists_2 = \gamma_1 W_{\max} + \gamma_2 x_{r_{\max}} \tag{63}$$

$$\dot{V} \leq \exists_1 V + \exists_2 \tag{64}$$

The above expressions indicate that the function $V$ is bounded, namely:

$$V(t) \leq V(0)e^{-\exists_1 t} + \exists_2(1 - e^{-\exists_1 t}) \leq \max \left[ V(0)\exists_2|x_k| \leq \frac{\sqrt{\max|V(0)\exists_2|}}{\lambda_{\min}(P)}, k = 5, 6, 7, 8 \right] \tag{65}$$

The next part is to estimate the bounds of suspension parameters:

$$|\Delta \delta_l| \leq |x_1| + \delta|\sin(x_3)| + |x_5| \leq |x_1| + \delta|x_3| + |x_5|$$

$$\leq |x_{d1}| + \sqrt{2V_2(0)} + \delta|x_3| + \delta \sqrt{2V_2(0)} + \frac{\sqrt{\max|V(0)\exists_2|}}{\lambda_{\min}(P)} = \Delta \delta_{lbd} \leq \Delta \delta_{lmax} \tag{66}$$

$$|\Delta \delta_p| \leq |x_1| + b|\sin(x_3)| + |x_5| \leq |x_1| + b|x_3| + |x_5|$$

$$\leq |x_{d1}| + \sqrt{2V_2(0)} + b|x_3| + b \sqrt{2V_4(0)} + \frac{\sqrt{\max|V(0)\exists_2|}}{\lambda_{\min}(P)} = \Delta \delta_{pdb} \leq \Delta \delta_{pmax} \tag{67}$$
\[ |F_{ar2} + F_{br2}| \leq |t_{ar2}(x_5 - x_{r2}) + c_{br2}(x_6 - \dot{x}_{r2})| \leq t_{ar2}|x_5| + t_{ar2}|x_{r2}| + c_{br2}|x_6| + c_{br2}|\dot{x}_{r2}| \\
\leq (t_{ar2} + c_{br2})\sqrt{\max[V(0) \geq 0]} + t_{ar2}\|x_{r2}\|_{\infty} + c_{br2}\|\dot{x}_{r2}\|_{\infty} = F_{fbd} \]  
\hspace{1cm} (68)

\[ |F_{ar1} + F_{br1}| \leq |c_{br1}(x_7 - x_{r1}) + c_{br1}(x_8 - \dot{x}_{r1})| \leq t_{ar1}|x_7| + t_{ar1}|x_{r1}| + c_{br1}|x_8| + c_{br1}|\dot{x}_{r1}| \\
\leq (t_{ar1} + c_{br1})\sqrt{\max[V(0) \geq 0]} + t_{ar1}\|x_{r1}\|_{\infty} + c_{br1}\|\dot{x}_{r1}\|_{\infty} = F_{rbd} \]  
\hspace{1cm} (69)

After adjustment:

\[ F_{fbd} \leq F_f \]  
\hspace{1cm} (70)

\[ F_{rbd} \leq F_r \]  
\hspace{1cm} (71)

then,

\[ |F_{ar2} + F_{br2}| \leq F_f \]  
\hspace{1cm} (72)

\[ |F_{ar1} + F_{br1}| \leq F_r \]  
\hspace{1cm} (73)

Similarly, it can be concluded that:

\[ |u_1| \leq \frac{1}{\varphi_m} (||\dot{x}_{d1}\|_{\infty} + \kappa_1|\dot{e}_1| + \kappa_2|e_\beta| + |e_\alpha|) + |F_{a1}| + |F_{a2}| + |F_{b1}| + |F_{b2}| \leq u_{1bd} \]  
\hspace{1cm} (74)

\[ |u_2| \leq \frac{1}{\varphi_m} (||\dot{x}_{d3}\|_{\infty} + \kappa_3|e_\gamma| + \kappa_4|e_6| + |e_\gamma|) + a(|F_{a2}| + |F_{b2}|) + b(|F_{a1}| + |F_{b1}|) \leq u_{2bd} \]  
\hspace{1cm} (75)

\[ |u_\alpha| \leq \frac{\gamma u_{1bd} + u_{2bd}}{\delta + \gamma} \]  
\hspace{1cm} (76)

\[ |u_2| \leq \frac{\delta u_{1bd} + u_{2bd}}{\delta + \gamma} \]  
\hspace{1cm} (77)

The adjustable gains can be adjusted to:

\[ \frac{\gamma u_{1bd} + u_{2bd}}{\delta + \gamma} \leq u_{a_{max}} \]  
\hspace{1cm} (78)

\[ \frac{\delta u_{1bd} + u_{2bd}}{\delta + \gamma} \leq u_{b_{max}} \]  
\hspace{1cm} (79)

So,
Solution of active suspension power

Nowadays, the performance evaluation of automobile suspension has been quite mature.\textsuperscript{14} Compared with ordinary suspension, it has great advantages in improving passenger riding comfort, driving maneuverability and driving stability.\textsuperscript{15} But in the process of driving, compared with other suspension systems, the active suspension system consumes more energy. At present, most literatures focus on the research of energy-fed suspension and adopt different control methods for the energy saving of the suspension. In the aspect of energy saving, there are some researchers on the study. Kou et al.\textsuperscript{16} proposed a coordinated control strategy using linear motor to control active suspension energy management, which improved the efficiency of energy regeneration, reduced the energy consumption of the suspension, and improved the performance of the suspension to a certain extent. Zhang et al.\textsuperscript{17} designed a new suspension mechanism, which is a hydro-electric power regeneration structure of automobile. It can ensure the recovery and utilization of energy in the process of driving on steep roads, and it has been proved in the experiment that the modified device has strong ability in energy recovery. However, there is no systematic method to solve the suspension power. For example, the evaluation index of suspension energy consumption is still in a blank state. How to develop a new energy consumption evaluation method is of great practical significance for the promotion of active suspension.

At present, the expressions for solving power are generally divided into the following three situations: (1) Consider the internal structure of the actuator structure and the influence of internal liquid flow. (2) Consider power analysis in the time domain. (3) The structure of the actuator is not considered in the calculation process, and the loss of the suspension’s average working energy during the whole movement is mainly considered. However, the above three methods have different limitations according to different situations. If the internal structure of the actuator and the influence of internal liquid flow are considered, the power expression is:

$$P = \frac{P_{sus} \cdot S}{2T_{sim}} \int_{0}^{T_{sim}} |V|dt$$

(81)

where, $S$ is the area of the structure, $T_{sim}$ is the simulation time, $V$ is the moving speed, and $P_{sus}$ is the pressure. When considering the influence of liquid and simulation time, in order to ensure the accuracy of the data, we only measured half of its value, so the denominator is correspondingly added with 2. However, it is worth pointing out that this method has some defects, that is, it ignores the power loss caused by other electrical structures.

If the power problem is analyzed in the time domain and the average velocity of the liquid is obtained by standard deviation, the power formula can be written in the following form:
\[ P = \frac{P_{sus}}{2T} 0.799\sigma_1 \quad (82) \]

This formula has the same meaning as the letters above, except that \( \sigma_1 \) is the standard deviation of the liquid velocity. This formula is suitable for the analysis of power in the time domain. However, when considering the standard deviation, there are some defects due to the statistical error.

In the process of analyzing the suspension in this paper, since the structure of the actuator is not considered in the study, in the process of vibration analysis, we pay more attention to the average value method to calculate the power.

According to the characteristics of the active suspension and the energy storage property of the elastic element, the following parameters are introduced as follows:

\[ F_1 = F_{b_2} + u_\alpha, \quad F_2 = F_{b_1} + u_\beta \quad (83) \]

where, \( F_1 \) and \( F_2 \) are the front and rear suspension forces.

Then the front and rear power of the suspension are:

\[ P_1 = (F_{b_2} + u_\alpha)(\dot{X}_c - \dot{x}_2) \quad (84) \]
\[ P_2 = (F_{b_1} + u_\beta)(\dot{X}_c - \dot{x}_1) \quad (85) \]

The change of energy consumed in \( dt \) time is:

\[ dE_1 = P_1 dt \quad (86) \]
\[ dE_2 = P_2 dt \quad (87) \]

Since \( P = \frac{\xi}{T} \), the average power in \([0\sim t] \) is:

\[ P_{1f} = \lim_{T \to \infty} \frac{1}{2} \int_0^T (F_{b_2} + u_\alpha)(X_c - x_2)dt \quad (88) \]
\[ P_{2f} = \lim_{T \to \infty} \frac{1}{2} \int_0^T (F_{b_1} + u_\beta)(X_c - x_1)dt \quad (89) \]

\[ x_{jd}(t) = \begin{cases} a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4, & t < T_{jr} \\ 0, & t \geq T_{jr} \end{cases} \quad (90) \]

**Simulation**

The parameters of the half active suspension model are selected as follows:

- \( m_c = 1100\text{kg} \), \( m_\alpha = m_\beta = 80\text{kg} \), \( h = 580\text{kgm}^2 \), \( t_{a_1} = t_{a_2} = 16000\text{N/m} \),
- \( t_{ar_2} = 190000\text{N/m} \), \( c_{br_2} = 1650\text{N}\cdot\text{s/m} \), \( c_{br_1} = 1550\text{N}\cdot\text{s/m} \), \( c_{b_1} = 1600\text{N}\cdot\text{s/m} \), \( \delta = 1.6, \gamma = 1.7 \).

The road surface disturbance is set as:
$W = 0.03\sin(10\pi t)$

The simulation structures are shown in Figures 2 to 16. Figure 3 shows that both the composite adaptive algorithm and the ordinary adaptive algorithm have achieved the effect of controlling the vertical displacement response. However, the fluctuation of the passive suspension in the whole process is too large.
The existence of pitch angle will cause the coupling of front and rear suspension, increase the influence of uncertainty and nonlinearity on the whole system, and then affect the performance of semi-car suspension. Figures 4 and 5 respectively show that both the compound adaptive law and the ordinary adaptive law have good disturbance suppression, both of which can solve the pitching angle problem of the semi-car suspension, but the algorithm of the compound adaptive law is superior in effect. However, the relatively passive suspension is difficult to achieve fast stability in the whole process. In Figure 4, stability is achieved after 3 s, while
in Figure 5, stability is achieved after 2.3 s, which has obvious disadvantages compared with the two adaptive control methods. Figure 5 shows that both algorithms can improve the ride comfort of vehicles. The algorithm of the general adaptive law can achieve stability quickly, but compared with the composite adaptive law, its peak value and jitter are higher, while the algorithm of the composite adaptive law is relatively gentle and has better adaptability to human body. Compared with the above two algorithms, the vibration of the passive suspension in the whole process is relatively large, and it tends to be gentle at 4.3 s, and the comfort is relatively
poor. Figures 6 and 7 show the response of unsprung mass displacement. In contrast, the peak value of the compound adaptive law algorithm in the stabilization process is relatively low.

The dynamic travel of suspension affects the life of automobile suspension. One of the objectives of the controller is to stabilize the value of the data within its maximum limit, so as to avoid irreversible mechanical damage caused by exceeding the value. The suspension travel response has a direct impact on the above safety problems. Figures 8 and 9 show the forward and backward travel of the suspension.

Figure 8. Front suspension travel.

Figure 9. Rear suspension travel.
Compared with the ordinary adaptive algorithm, the composite adaptive algorithm can quickly achieve stability, and its peak value and jitter degree are relatively low and relatively gentle, which can reduce the probability of component damage. Figures 10 and 11 show the control input, indicating that both algorithms can achieve their corresponding control expectations, while the composite adaptive algorithm has a lower peak value and more advantages in terms of energy consumption.

The stability of handling directly affects the safety of driving, which to some extent reflects the contact between automobile tires and road surface. This
indicator represents the maneuverability of the vehicle. Figures 12 and 13 show tire load. According to the above graph, both control algorithms can maintain driving safety under road excitation, and the compound adaptive algorithm appears to have more advantages. Figures 14 and 15 show the tracking error, indicating that the dynamic equation can converge in a finite time. Figure 16 shows the instantaneous power consumption. Because spring components have the characteristics of energy storage, they generate energy during the release process, so the power will generate negative values. Compared with the traditional adaptive algorithm, the
power consumption of the composite adaptive algorithm is relatively lower than that of the former in the whole process. In addition, when releasing energy, the process of releasing energy of this algorithm is relatively gentle, without the drastic energy release of ordinary adaptive control algorithm. Therefore, compared with the former compound adaptive algorithm, it has more advantages in energy saving.

**Figure 14.** The tracking error $e_{\alpha}$.

**Figure 15.** The tracking error $e_{\beta}$.
Conclusion

This paper proposes a composite adaptive backstepping control scheme for the half-car active suspension systems. In order to improve the convergence of the adaptive algorithm, the compound adaptive law is introduced. And the corresponding adaptive backstepping controller is designed. The simulation results show that the ride comfort, operating stability and safety of the vehicle can be effectively improved by the composite adaptive backstepping controller. According to the energy storage characteristics of the elastic components of the suspension, the power calculation formula is introduced. The simulation results show that the composite adaptive algorithm is more energy-saving than the conventional adaptive algorithm based on projection operator.

This algorithm improves the disadvantage of the traditional adaptive rate and lays a foundation for the design of energy saving controller in the future. How to design a control algorithm focusing on energy saving is my future research direction.

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References

1. Cao F, Sun H, Li Y, et al. Fuzzy adaptive fault-tolerant control for a class of active suspension systems with time delay. *Int J Fuzzy Syst* 2019; 21(7): 2054–2065.
2. Pang H, Zhang X, Yang J, et al. Adaptive backstepping-based control design for uncertain nonlinear active suspension system with input delay. *Int J Robust Nonlinear Control* 2019; 29(1): 5781–5800.
3. Wang W, Song Y, Chen J, et al. A novel optimal fuzzy integrated control method of active suspension system. *J Braz Soc Mech Sci Eng* 2018; 40(1): 29.
4. Yao J, Wang M, Li Z, et al. Research on model predictive control for automobile active tilt based on active suspension. *Energies* 2021; 14(3): 671.
5. Gu C, Yin J, Chen X, et al. Robust control and optimization of a rocker-pushrod electromagnetic active suspension. *Automot Eng* 2018; 40(1): 34–40.
6. Soh M, Jang H, Park J, et al. Development of preview active suspension control system and performance limit analysis by trajectory optimization. *Int J Automot Technol* 2018; 19(6): 1001–1012.
7. Amin M, Hudha K, Kadir Z A, et al. Skyhook control for 7 DOF ride model of armored vehicle due to road disturbance. In: *2015 10th Asian Control Conference (ASCC)*. IEEE, 2015.
8. Na J, Huang Y, Wu X, et al. Active adaptive estimation and control for vehicle suspensions with prescribed performance. *IEEE Trans Control Syst Technol* 2018; 26: 2063–2077.
9. Chávez-Conde E, Beltrán-Carbajal F, Garcaí-Rodríguez C, et al. Sliding mode based differential flatness control and state estimation of vehicle active suspensions. In: *6th International conference on electrical engineering, computing science and automatic control*, Toluca, Mexico, 10–13 January 2009, pp.1–6. New York, NY: IEEE.
10. Basturk H. A backstepping approach for an active suspension system. In: *2016 American Control Conference (ACC)*, Boston, MA, 6–8 July 2016.
11. Li H, Jing X and Karimi HR. Output-feedback-based $H_\infty$ control for vehicle suspension systems with control delay. *IEEE Trans Ind Electron* 2014; 61(1): 436–446.
12. Chen M and Tao G. Adaptive fault-tolerant control of uncertain nonlinear large-scale systems with unknown dead zone. *IEEE Trans Cybern* 2016; 46(8): 1851–1862.
13. Zhu Q, Li L, Chen CJ, et al. Low-cost lateral active suspension system of high-speed train for ride quality based on resonant control method. *IEEE Trans Ind Electron* 2017; 65(5): 4187–4196.
14. Wang R, Jing H, Karimi HR, et al. Robust fault-tolerant $H_\infty$ control of active suspension systems with finite-frequency constraint. *Mech Syst Sig Process* 2015; 62–63: 341–355.
15. Bououden S, Chadli M and Karimi HR. A robust predictive control design for nonlinear active suspension systems. *Asian J Control* 2016; 18(1): 122–132.
16. Kou F, Chen L and Zhang D. Energy management strategy for active suspension with linear motor. In: *2016 IEEE advanced Information Management, Communicates,
Electronic and Automation Control Conference (IMCEC), Xi’an, China, 3–5 October 2016, pp. 419–423. New York, NY: IEEE.

17. Zhang H, Guo X, Hu S, et al. Simulation analysis on hydraulic-electrical energy regenerative semi-active suspension control characteristic and energy recovery validation test. Trans Chin Soc Agric Eng 2017; 33(16): 64–71.

Author biographies

Xiaoyu Su graduated from Harbin Engineering University on 2014 and got the doctor degree. Now she is a lecturer at Shanghai University of Engineering Science. Her main research areas are about the roll attitude control of the ship and control analysis of the active suspension systems.

Bin Lin is currently an undergraduate student in Electronic and Electrical Engineering from Shanghai University of Engineering Science. His main research area is adaptive control.

Shuai Liu is currently an undergraduate student in Mechanical electronic engineering from University of Shanghai for Science and Technology. His main research area is backstepping control.