The Speed of Gravity Has Not Been Measured From Time Delays

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The recent passage of Jupiter by the quasar QSO J0842+1835 at a separation of 3.7 arcminutes on September 8, 2002, combined with recent advances in interferometric radio timing, has allowed for the first measurement of higher-order post-Newtonian terms in the Shapiro time delay which depend linearly on the velocity of the gravitating body. Claims have been made that these measurements also allow for the measurement of the propagation speed of the gravitational force. This conclusion disagrees with recent calculations done in the parameterized post-Newtonian (PPN) model, which find no dependence of the velocity-dependent terms in the time delay on the “speed of gravity” to the stated order. Here, to test out these claims and counterclaims, we calculate the time delay in the limit of an instantaneous gravitational force, and find that the velocity-dependent terms are in complete agreement with previous PPN calculations, with no dependence on the speed of gravity. We conclude that the speed of gravity cannot be determined by measuring these terms in the Shapiro time delay, and suggest a reason why other groups mistakenly came to the opposite conclusion.

I. HISTORICAL BACKGROUND

Almost 40 years ago, I.I. Shapiro pointed out that the time delay which results from light appearing to slow down as it passes through a gravitational potential could be measured within our solar system [1]. The measurement was done three years later, and provided a new test of the theory of general relativity (GR) [2]. Recent advances have now allowed radio astronomers to measure higher order post-Newtonian (PN) terms in the Shapiro time delay, using Very-Long Baseline Interferometry (VLBI). A recent passage of Jupiter by the position of a bright quasar at a separation of only 3.7 arcminutes has allowed for some of these high-order terms to be measured by Fomalont and Kopeikin, specifically those that have to do with the transverse motion of Jupiter perpendicular to the line of sight to the quasar [3]. In a series of papers, Kopeikin has argued that the terms in the Shapiro time delay which depend upon the velocity of the gravitating body also depend upon the propagation speed of gravity [4,5]. Specifically, in Ref. [4], he derives an equation for the time delay in the form

\[ t_d = -2 \sum_a \frac{G m_a}{c^3} \left( 1 - \frac{\vec{K} \cdot \vec{v}_a}{c_g} \right) \ln\left( r_a - \vec{K} \cdot \vec{r}_a \right) + C, \]  

where \( t_d \) is the time delay, \( m_a \) and \( v_a \) the mass and velocity of the \( a \)'th gravitating body, respectively, \( c_g \) is the propagation speed of gravity, \( \vec{r}_a \) the separation vector between the gravitating body and the observer, \( r_a \) its magnitude, \( C \) is an integration constant, and the vector \( \vec{K} \) (called \( \vec{N} \) by Kopeikin) is given by

\[ \vec{K} = \vec{k} - \frac{1}{c_g} \vec{k} \times (\vec{v} \times \vec{k}), \]  

where \( \vec{k} \) is a unit vector pointing in the direction of the photon’s path. In what follows, for clarity, we will refer to equations in Ref. [4] with the author’s initial, thus our Eqs. 1 and 2 correspond exactly to Eqs. (K22) and (K23).

Recently, Asada [6] has taken issue with the interpretation of the velocity-dependent terms in the time delay for models which assume that \( c_g = c \). Will goes further, evaluating the time delay for models in which \( c_g \neq c \), and concludes that there is an error in Kopeikin’s derivation of the time delay [7]. Using the parameterized post-Newtonian (PPN) framework (see Ref. [8] for more details), he concludes that the proper time delay is given by

\[ t_d = -2 \sum_a \frac{G m_a}{c^3} \left( 1 - \frac{\vec{K} \cdot \vec{v}_a}{c} \right) \ln\left( r_a - \vec{K} \cdot \vec{r}_a \right) + C, \]  

where

\[ \vec{K} = \vec{k} - \frac{1}{c} \vec{k} \times (\vec{v} \times \vec{k}). \]
Using his initials for clarity as well, our Eqs. 3 and 4 correspond exactly with Eqs. (W35) and (W36), where we have set several of the PPN parameters which do not affect the final result equal to the values predicted by GR.

In what follows, we will repeat the calculations of Kopeikin and Will, but in a model where the gravitational force is instantaneous, i.e. \( c_g \to \infty \). We identify where the two methods agree under this assumption, and where they find different mathematical expressions. In short, we find that Will’s result is correct given his assumptions regarding the PPN framework. Kopeikin’s result is inconsistent with his assumptions, revealing that his derivation contains a mathematical error. We identify the likely cause of it in the course of our derivation.

II. CALCULATING THE TIME DELAY

For the following calculations, we make use of the following assumptions. A photon travels along the x-axis of our coordinate system. It passes a body of mass \( m_a \) moving with velocity \( \vec{v}_a = (v_\parallel, v_\perp, 0) \) along a straight line. The origin of the time coordinate is defined such that the instantaneous separation vector between the gravitating body and the photon is perpendicular to the photon’s path at \( t = 0 \). This does not correspond exactly to the moment of closest approach between the photon and the gravitating body, which can easily be calculated to be first-order in \( v_a/c \). Using these velocities, we find that the position of the of the gravitating body is given by \( \vec{x}_a(t) = (v_\parallel t, v_\perp t + y_0, z_0) \) and that of the photon by \( \vec{x}_p(t) = (ct, 0, 0) \). Our observer is placed far from the gravitating body at some distance \( x_o \) along the photon’s path, at position \( \vec{x}_o = (x_o, 0, 0) \). As these calculations are traditionally done by placing an observer at the barycenter of the system, we should technically include at least one other gravitating body, but we see immediately that it will have no effect to lowest order on the measured time delay.

To calculate the time delay for the photon passage, we note that since we can make the gravitating mass sufficiently small, we can ignore all effects associated with the deflected path of the photon, which are of higher order in \( v_a/c \). The equations of motion for the photon’s path as a function of its wave vector \( k^\mu = dx^\mu/dt = (1, k^i) \) are completely determined from a description of the spacetime metric, and the constraint equations for a null geodesic. These constraint equations are given by both Eq. (W11) and Eq. (K14), in complete agreement, in the form

\[
\frac{dx^i}{dt} + k^\mu k^\nu (\Gamma^i_{\mu\nu} - k^i \Gamma^0_{\mu\nu}) = 0
\]

(5)

\[
\left(\frac{ds}{dt}\right)^2 = g_{00} + 2g_{0i}k^i + g_{ij}k^ik^j = 0.
\]

(6)

While Kopeikin and Will agree on the geodesic equation for the photon, they disagree on the form of the metric. Will, using the standard parameterized post-Newtonian (PPN) formalism, assumes that the metric, Eq. (W9), takes the form

\[
g_{00} = -1 + 2U
\]

(7)

\[
g_{0i} = -4V_i
\]

(8)

\[
g_{ij} = (1 + 2U)\delta_{ij}
\]

(9)

where we have set the PPN parameters \( \gamma \) and \( \alpha_1 \) equal to the GR values \( \gamma = 1, \alpha_1 = 0 \), and the retarded Liénard-Wiechert mass and momentum potentials are given by Eq. (W10) as

\[
U(t, \vec{x}) = \frac{Gm_a}{c^2r_e(t, \vec{x})}
\]

(10)

\[
V_i(t, \vec{x}) = \frac{Gm_a\vec{v}_a}{c^3r_e(t, \vec{x})},
\]

(11)

where we define \( r_e \) to be the “effective” distance of the retarded potential, such that \( r_e(t, \vec{x}) = |\vec{x} - \vec{x}_a(t')| \), where the “emission” time of the gravitational force must satisfy \( t - t' = r_e/c_g \), where \( c_g \) is the speed of gravity.

Kopeikin’s metric is slightly different, in ways that deserve some clarification. He introduces a parameter \( \tau \), which is used to describe all quantities involving gravitation in his framework. It is defined such that \( c_g\tau = ct \). Unfortunately, the gravitational equations treat this parameter as a physical time for gravitation. Thus, the wave equation for linear metric perturbations, Eq. (K5), reads

\[
\left(-\frac{1}{c_g^2} \frac{\partial^2}{\partial \tau^2} + \nabla^2\right) \bar{\gamma}^{\mu\nu}(\tau, \vec{x}) = -\frac{16\pi G}{c^4} \Theta^{\mu\nu}(\tau, \vec{x}),
\]

(12)
where $\gamma_{\mu\nu}$ is the standard trace-reversed linear metric perturbation, and $\Theta_{\mu\nu}$ is his modified stress-energy tensor. We note that for this equation to give a traveling wave solution moving at speed $c_p$, we must interpret $\tau$ as the physical time, not $t$. We believe it is confusion between these two quantities which leads to errors in his final conclusions. In any case, Kopeikin modifies not only the stress-energy tensor but the metric perturbation as well, yielding a metric that agrees with Will’s (if we view $\tau$ as the physical time-variable) but for a different momentum potential, given by Eq. (K10)

$$V_i(t, \vec{x}) = \frac{G m_a(\vec{v}_a)_{\parallel}}{c^2 c_p r_e(t, \vec{x})}.$$  

In the calculation that follows we will see that the differences in the two metric formulations lead to different values for the gravitomagnetic drag experienced by a photon passing by a moving body. We will also see that Will’s calculation is completely consistent, whereas Kopeikin’s final answer is inconsistent with the photon propagation equation.

We start our calculation by deriving the magnitude of the photon’s wavevector in our metric, Eq. 8, noting that

$$\vec{k}^2 = (k^{\parallel}, 0, 0)$$

and the momentum potential $V_i$ by $k$ and $V$, respectively. The x-component of the momentum potential is denoted $V^i_{\parallel}$. From Eqs. 6–9, we find

$$(-1 + 2U) - 8V^i_{\parallel}k + (1 + 2U)k^2 = 0 \rightarrow k \equiv \frac{dx_p}{dt} = \frac{4V^i_{\parallel} + \sqrt{1 - 4U^2 + 16V^2_{\parallel}}}{1 + 2U}. \tag{14}$$

Working only to 1.5PN order, we throw out all terms involving $U^2$ or $V^2$, and find

$$k = \frac{dx_p}{dt} = \frac{1 + 4V^i_{\parallel}}{1 + 2U} \sim (1 - 2U + 4V^i_{\parallel}). \tag{15}$$

To find the time delay, we integrate this expression over the path length, finding for the time delay

$$t_d = (\int dt) - t_0 = \left[\int \frac{dx_p}{c} \frac{1}{1 - 2U + 4V^i_{\parallel}}\right] - c\delta x \sim \int \frac{dx_p}{c}(2U - 4V^i_{\parallel}), \tag{16}$$

where $t_0$ is the time required to travel a distance $\delta x$ in the absence of gravitating bodies. The two terms here have entirely different meanings, but both are easily understood. The first, $\infty \int Udx_p$, which appears for static gravitational sources as well, is the simple geometric time delay which results from photon’s traveling through any gravitational potential. The second term, $\infty \int V_i dx_p$ is the gravitomagnetic contribution to the time delay. In simplest terms, it can be thought of as a “frame-dragging” term, whereby the moving object pulls the photon along it’s path. At this point, it is useful to compare our result with those of Will and Kopeikin. To do so, we make use of the ratio between the momentum potential and the mass potential, which is constant throughout space and time in both formalisms so long as the gravitating body moves at constant velocity. From Eqs. 10 and 11, we see that in Will’s formalism that

$$V_i = U \frac{d}{dx}$$

whereas from Eqs. 10 and 13, Kopeikin’s method yields $V_i = U \frac{d}{dx}$. We find, respectively,

$$t_d \sim 2 \left(1 - \frac{2K \cdot \vec{v}_a}{c}\right) \int \frac{dx_p}{c} U \quad \text{(Will)} \tag{17}$$

$$\sim 2 \left(1 - \frac{2K \cdot \vec{v}_a}{c_g}\right) \int \frac{dx_p}{c} U \quad \text{(Kopeikin)}, \tag{18}$$

which correspond with Eq. (W16), and Eq. (K21).

Calculating the time-delay integral is straightforward, so long as we calculate the effective distance correctly for our chosen value of $c_p$. To simplify matters, and emphasize the difference between the frameworks used by Will and Kopeikin, we will evaluate the time delay for a model with instantaneous gravitational propagation, i.e., $c_g \rightarrow \infty$. In Kopeikin’s notation, this represents the limit $\epsilon \rightarrow 0$, which he refers to as the Newtonian limit. Given the trajectories of the photon and gravitating body stated above, we find that the effective radius is given as a function of time as

$$r_\epsilon(t) = [(c - v^2)2t^2 + (v^2 t + v_0^2)^2 + z_0^2]^{0.5} \tag{19}$$

$$= [(c^2 - 2cv^2 + v_0^2)t^2 + 2y_0v_\perp t + y_0^2 + z_0^2]^{0.5}.$$
To simplify the expression, we note that at the moment of closest approach, at \( t_{\text{min}} = -y_0 v_{\perp} t / [(c^2 - 2c v_{\parallel} + v_{\perp}^2) t^2] \), the distance reaches its minimum value of \( r_{\text{min}} = r_c(t_{\text{min}}) \). At all other times, the effective distance is given by

\[
r_c(t) = \sqrt{r_{\text{min}}^2 + |(\ddot{u}c(t - t_{\text{min}}))|^2},
\]

where \( \ddot{u} = \tilde{k} - \vec{v}_a / c \), \( \tilde{k} \) is the unit 3-vector pointing along the photon’s path, and we know that \( \vec{r}_{\text{min}} \cdot \ddot{u} = 0 \). Note that the appearance of \( c \) in these expressions has nothing to do with assumptions about the speed of gravity, it merely represents the speed of the photon when we calculate relative distances and velocities.

The calculation of the time delay will allow for a direct test of Will and Kopeikin’s results, since both derive the same integral for the time delay, albeit with different prefactors, as we found in Eqs. 17 and 18. The time delay integral is analytic, since we can integrate along the unperturbed path of the photon. Changing variables to \( dt = dx_p / c \), shifting the time axis to a new time variable offset by \( t_{\text{min}} \), and defining \( u = |\ddot{u}| \), we find

\[
t_d \sim \int \frac{dt}{U(t)} \sim Gm_a \int \frac{dt}{r_c(t)} \sim Gm_a \int \frac{dt}{\sqrt{r_{\text{min}}^2 + u^2 c^2 t^2}}
\]

\[
\sim Gm_a \left( \frac{1}{u c} \right) \ln[\sqrt{r_{\text{min}}^2 + u^2 c^2 t^2 - u c t_f}],
\]

where \( t_f \) is the arrival time of the photon. Note that in deriving this expression, we made no assumptions about the geometry of the system at the moment of closest approach, which Kopeikin discusses in his appendix. In what follows, we will only use the fact that vectorial velocity difference between the photon and the gravitating body is perpendicular to the separation vector at closest approach.

We start with an analysis of the prefactor \( 1/c \). Calculating this to first order in \( \vec{w}_a = \vec{v}_a / c \) shows us that

\[
\frac{1}{uc} = \frac{1}{c|\vec{k} - \vec{w}_a|} = \frac{1}{c \sqrt{\vec{k} \cdot \vec{v}_a + \vec{w}_a \cdot \vec{w}_a}} = \frac{1}{c \sqrt{1 - 2v_{\perp}^2 + v_{\parallel}^2}}
\]

\[
\sim \frac{1}{c(1 - w_{\parallel})} \sim \frac{1}{c(1 + w_{\parallel})} = \frac{1}{c(1 + v_{\parallel}/c)} = \frac{1}{c(1 + \vec{k} \cdot \vec{v}_a / c)}.
\]

This multiplicative factor is derived directly from the integral of the gravitational potential along the line of sight, and has absolutely nothing to do with the speed of gravity, which we are assuming here to be instantaneous. It is essentially a Doppler correction, representing the increased (decreased) time that the photon spends in the deepest part of the gravitational potential because of the gravitating body’s parallel (anti-parallel) velocity component. Comparing to Eq. 17, we see that the “Doppler” term is half the magnitude and opposite in sign from the gravitomagnetic correction term. On the other hand, this result is at odds with the gravitomagnetic term derived by Kopeikin in Eq. 18, which contains \( c_g \) in the denominator. We will discuss this error below.

Moving on to the logarithmic term in the integral, we define the separation vector \( \vec{r}(t) \equiv \vec{x}_p(t) - \vec{x}_a(t) = \vec{r}_{\text{min}} + c t \ddot{u} \) whose length is \( r_c(t) = \sqrt{r_{\text{min}}^2 + u^2 c^2 t^2} \). Additionally, we note that

\[
\vec{k} \cdot \vec{r}(t) = \vec{k} \cdot \vec{r}_{\text{min}} + c t (\vec{k} \cdot \vec{v}_a) = \vec{k} \cdot \vec{r}_{\text{min}} + c t (1 - \vec{k} \cdot \vec{v}_a / c).
\]

This expression is similar to the term \( utc \) which appears in logarithmic term of the time delay, but not equal to it at the required order because of the appearance of a term \( \vec{k} \cdot \vec{r}_{\text{min}} \). However, we see that

\[
\vec{r}_{\text{min}} \cdot \ddot{u} = 0 \Rightarrow \vec{r}_{\text{min}} \cdot \vec{k} - \vec{r}_{\text{min}} \cdot \vec{w}_a = 0.
\]

Denoting by \( r_{\parallel} \) and \( r_{\perp} \) the components of \( \vec{r}_{\text{min}} \) parallel to \( \vec{v}_a \) and \( \vec{v}_a \), respectively, we find, to lowest order,

\[
r_{\perp} w_{\perp} = \vec{r}_{\parallel} (1 - w_{\parallel}) \sim \vec{r}_{\parallel} \equiv \vec{r}_{\text{min}} \cdot \vec{k}.
\]

Thus, we can subtract off the term \( \vec{r}_{\text{min}} \cdot \vec{k} \) in Eq. 24 by taking the product of the transverse velocity of the gravitating body with the minimum separation vector. It is trivial to deduce that \( \vec{k} \times (\vec{v}_a \times \vec{k}) = \vec{v}_a \). We see though, that to first order in \( v_{\parallel} / c \),

\[
\vec{w}_{\perp} \cdot \vec{r}(t) = \vec{w}_{\perp} \cdot \vec{r}_{\text{min}} + c t (\vec{w}_{\perp} \cdot \vec{k} - \vec{w}_{\perp} \cdot \vec{w}_a)
\]

\[
\sim w_{\perp} r_{\perp},
\]
where the second term vanishes from orthogonality, and the last because it is second-order in $v_a/c$. Combining these results, we conclude

$$\ln[\sqrt{r^2_{\min} + u^2c^2t_f^2} - uc t_f] \sim \vec{r}(t) - ct \left( 1 - \frac{\vec{k} \cdot \vec{v}}{c} \right)$$

(29)

$$\sim r(t) - \left( \vec{k} - \frac{1}{c} \vec{k} \times (\vec{v}_a \times \vec{k}) \right) \cdot \vec{r}(t).$$

(30)

Plugging Eqs. 21, 23, and 30 into Eq. 17 we recover Eqs. (W35) and (W36). Plugging the same three equations into Eq. 18 does not recover Kopeikin’s Eqs. (K22) and (K23), which have the speed of gravity $c_g$ where we have the speed of light $c$. We conclude that his calculation is erroneous, since the time-delay integral we calculated had nothing to do with the speed of gravity, which we took to be instantaneous. It is also independent of all metric quantities other than the gravitational potential which appears in $g_{00}$. Why then does he find different numbers than ours?

A clue is provided by the fact that our results (as well as those of Will, whose results ours reproduce) can be brought into agreement with Kopeikin’s if we replace $v_a$, the velocity of the gravitating body, with $v_a c/c_g$. In his notation, this is the transformation $v_a \rightarrow \epsilon v_a$. We believe the error he makes is to use $\tau = \epsilon t$ as the physical time for the gravitating object, declaring that its physical velocity is given by $d\vec{x}_a/d\tau = \vec{v}_a$, while keeping $t$ as the time variable used to describe the photon’s motion, as in Eq. (K13). If we convert the velocity he uses for the gravitating object from $\tau$-based coordinates to $t$-based coordinates, we find immediately that $\vec{v}_a(t) = d\vec{x}_a/dt = d\vec{x}_a/d\tau (d\tau/dt) = \epsilon \vec{v}_a$, which corresponds to the unphysical and erroneous velocity found in his calculations.

**III. CONCLUSIONS**

We have repeated the calculations performed by Will and Kopeikin, in a model where the speed of gravity is instantaneous. We find that first-order terms in the time delay resulting from the motions of gravitating bodies are independent of the speed of gravity, and follow the form written down by Will in every respect. We find in addition that Kopeikin’s equation for the time delay, which asymptotically approaches the time-delay for static bodies in the limit of an infinite speed of gravity, is wrong. We conclude that since the first-order velocity terms in the time delay are independent of the speed of gravity, Fomalont and Kopeikin’s high-precision measurements of the time delay of light from quasar QSO J0842+1835 passing by the edge of Jupiter on September 8, 2002 [3], while a truly impressive observational feat in radio astronomy, provided a measure of the speed of light only, not the speed of gravity as was claimed.

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