Abstract. Non-relativistic conformal ("Schrödinger") symmetry is derived in a Kaluza-Klein type framework. Reduction of the massless Dirac equation from 5D Minkowski space yields Lévy-Leblond’s non-relativistic equation for a spin 1/2 particle. Combining with the osp(1,1) SUSY found before provides us with a super-Schrödinger symmetry.

1. Introduction.
Twenty years ago Niederer, Hagen, and Jackiw, [1] found that the maximal invariance group of the Schrödinger equation of a free, non-relativistic particle was larger than just the Galilei group and contains in fact two more ‘conformal’ generators, namely

\[ D = tH - \frac{1}{4}\{p, r\} \quad \text{and} \quad K = t^2 H - 2tD + \frac{m^2}{2}r^2 \]

(1.1) dilatations expansions

These operators generate, with the Hamiltonian \( H = p^2/2m \), an \( o(2,1) \) symmetry algebra. Adding the Galilei group yields the Schrödinger group (2.4).

That the Schrödinger equation has a symmetry larger than the Galilei group is in contrast with the relativistic case, where the free Klein-Gordon equation admits the Poincaré group as symmetry, and the conformal group \( o(4,2) \) only occurs for massless particles.

The \( o(2,1) \) in Eq. (1.1) is also a symmetry for a charged, spin 0 particle in a Dirac monopole field [2] and even for the Pauli Hamiltonian

\[ H = \frac{1}{2m} \left( \pi^2 - q \frac{r_3}{r^2} \right), \quad \pi = -i\nabla - qA_D \]

(1.2) of a non-relativistic spin \( \frac{1}{2} \) particle in the field of a Dirac monopole [3]. Eq. (1.2) has, furthermore, a superconformal symmetry: the fermionic charges

\[ Q = \frac{1}{\sqrt{2m}} \pi.\sigma \quad \text{and} \quad S = \sqrt{\frac{m}{2}} r.\sigma - tQ \]

(1.3) close into the symmetry algebra \( osp(1,1)_{superconf} \) (see eqn. (3.6) below).

It was argued [4-5] that the proper arena for non-relativistic physics is extended ‘Bargmann’ spacetime \( M \), endowed with a metric \( g \) with signature \((-,+,+,+,+))\), and with a covariantly-constant null vector \( \xi \). Classical motions are massless

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(1) In Proc. XXIth Int. Conf. on Diff. Geom. Meths. in Phys., Nankai’92. Editors C. N. Yang, M. L. Ge, X. W. Zhou. Int. J. Mod. Phys. A (Proc. Suppl.) 3A:339-342 (1993). Singapore: World Scientific.
geodesics in $M$ and hence *conformally invariant*. Our construction differs from Kaluza-Klein theory in that the extra dimension is null, rather than space-like.

2. Spinless particles.

For a spinless, free particle the extended space is $M = \{t, r, s\} = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$, endowed with the flat metric $g_{\mu\nu}dx^\mu dx^\nu = dr^2 + 2dt ds$, and with the null Killing vector $\xi = \partial_s$. Viewing the wave function as an equivariant function on extended space, $\partial_s \Phi = im\Phi$, the free Schrödinger equation $-\Delta \varphi = 2mi\partial_t \varphi$ (where $\Delta$ is the Laplacian on ordinary 3-space) can be written as

$$
\Delta_g \Phi = 0.
$$

$g$ being the Laplacian on $M$ and $\Phi = e^{ims}\varphi$. Consider now those $\xi$-preserving conformal isometries $C$ of $M$,

$$
C^* g = \Omega^2 g, \quad C^* \xi = \xi.
$$

called *non-relativistic conformal transformations*. For a free particle for example, the conformal diffeomorphisms form $O(5, 2)$ which is reduced by the $\xi$-constraint to a 13 dimensional subgroup. Its infinitesimal action on extended spacetime $\{t, r, s\}$ is

$$
(\kappa t^2 + \delta t + \epsilon, \omega \times r + (\frac{\delta}{2} + \kappa t)r + \beta t + \gamma, -\frac{1}{2}kr^2 + \beta r + \eta),
$$

where $\omega \in so(3)$, $\beta, \gamma \in \mathbb{R}^3$, $\epsilon, \delta, \kappa, \eta \in \mathbb{R}$. This is the extended Schrödinger algebra, with $\omega$ representing rotations, $\beta$ Galilei boosts, $\gamma$ space-translations, $\epsilon$ time-translations, $\delta$ dilatations, $\kappa$ expansions, and $\eta$ translations in the vertical direction.

The associated conserved quantities $L, b, p, H, D, K$ and $m$ satisfy the commutation relations of the extended Schrödinger algebra,

$$
[L, L] = iL \quad [L, b] = ib \quad [L, p] = ip \quad [L, H] = 0
$$

$$
[b, H] = ip \quad [b, b] = 0 \quad [b, p] = im \quad [p, p] = [p, H] = 0
$$

$$
[H, D] = iH \quad [H, K] = 2iD \quad [D, K] = iK
$$

$$
[p, D] = \frac{i}{2}p \quad [b, D] = -\frac{i}{2}b \quad [p, K] = -ib \quad [b, K] = 0
$$

$$
[L, K] = 0 \quad [L, D] = 0 \quad [b, K] = 0
$$

and all generators commute with $m$. The action of conformal transformations on a wavefunction can be deduced from that of $O(2, 5)$ on $M$, yielding a representation of the extended Schrödinger group.

Another application is provided by the harmonic oscillator. The Bargmann space is again $M = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$, endowed with the metric $g = dx^2 + 2ds dt - \omega^2 r^2 dt^2$. Now the transformation $f : (t, x, s) \mapsto (\tau, \xi, \sigma)$ where

$$
\tau = \frac{1}{\omega} \tan \omega t, \quad \xi = \frac{x}{\cos \omega t}, \quad \sigma = s - \frac{\omega r^2}{2} \tan \omega t
$$
satisfies \( f^*(d\xi^2 + 2d\sigma d\tau) = (\cos \omega t)^{-2}g \), so that (every half period of) the harmonic oscillator is mapped onto a free motion. The two systems have therefore the same symmetries [6].

Electromagnetic fields can be included as external fields. We only consider the theory obtained by minimal coupling, \( \partial_t \mapsto \partial_t + ieV, \ p \mapsto \pi = p - eA \). Those conformal space-time symmetries which preserve the electromagnetic fields will still act as symmetries.

For example, the Bargmann space of a Dirac monopole is \( \{(t, r, s)\} = \mathbb{R} \times \mathbb{R}^3 \setminus \{0\} \times \mathbb{R} \), with the flat metric above. Its conformal symmetries form the subgroup of the Schrödinger group which preserves the origin and are readily identified with \( SO(3) \times SL(2, \mathbb{R}) \), generated by the angular momentum \( L = r \times \pi - q\hat{r} \), and by \( H = \pi^2/2m \), and \( D \) and \( K \) in (1.1). Since they also preserve the electromagnetic field, they are symmetries.

3. Spinning particles and Supersymmetry.

Chosing Dirac matrices to satisfy \( \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \) on 5-dimensional extended space, spin \( \frac{1}{2} \) particles are described by the massless minimally coupled Dirac equation,

\[
\hat{D}\Psi \equiv \gamma^\mu D_\mu \Psi = 0,
\]

Using the particular form of the Bargmann metric, one gets the Lévy-Leblond [7] equation

\[
\sigma \cdot \pi \varphi + 2m\chi = 0
\]

\[
(i\partial_t - eV)\varphi - \sigma \cdot \pi \chi = 0
\]

Since

\[
\hat{D}^2 = \left[ 2m(i\partial_t - eV) - \pi^2 - e\sigma \cdot B \right],
\]

each component satisfies the Pauli equation.

Since we are working with the massless Dirac equations, all \( \xi \)-preserving conformal transformations of extended space are symmetries. For a free particle, we get an irreducible representation of the centrally extended Schrödinger group [8].

For the Dirac monopole, the origin-preserving subgroup \( SO(3) \times SL(2, \mathbb{R}) \) of the Schrödinger group yields the bosonic symmetry algebra for the Lévy-Leblond equation and thus also for its square, the Pauli equation.

For an arbitrary static magnetic field \( A \), the helicity operator

\[
Q = \frac{1}{\sqrt{2m}} \sigma \cdot \pi
\]

is conserved, \([Q, \hat{D}] = 0\). Commuting the free helicity operator \( Q = \sigma \cdot p \) with the generators of the Schrödinger group yields two new supercharges, namely

\[
\Sigma = i[Q, b] = \sqrt{\frac{m}{2}} \sigma \quad \text{and} \quad S = -i[K, Q] = \frac{1}{\sqrt{2m}} \sigma \cdot b.
\]
The commutation relations are deduced from those of the extended Schrödinger group. This yields, for a free article with spin 1/2, the ‘super-Schrödinger algebra’ [9] with commutation relations (2.4) (with the total angular momentum, $J = L + \frac{1}{2} \sigma$, replacing $L$), supplemented with

\[
\begin{align*}
\{Q, D\} &= \frac{i}{2} Q \\
\{Q, J\} &= 0 \\
\{Q, b\} &= -i \Sigma \\
\{Q, p\} &= 0 \\
\{Q, H\} &= 0 \\
\\end{align*}
\]

(3.6)

\[
\begin{align*}
\{S, J\} &= 0 \\
\{\Sigma, J\} &= i \Sigma \\
\{S, p\} &= i \Sigma \\
\{S, K\} &= 0 \\
\{Q, Q\} &= 2 H \\
\{Q, S\} &= -2 D \\
\{S, S\} &= 2 K \\
\{\Sigma, \Sigma\} &= m \\
\{\Sigma, Q\} &= p \\
\{\Sigma, S\} &= b \\
\end{align*}
\]

Having $Q$ act on the o(2,1) subalgebra yields an osp(1,1) sub-superalgebra spanned by $H$, $D$ and $K$ and by the odd charges $Q$ and $S$. These are the symmetries which remain unbroken when a Dirac monopole is added [3].

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