Approximation of the Duffing oscillator frequency response function using the FPK equation.

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Abstract. Although a great deal of work has been carried out on structural dynamic systems under random excitation, there has been a comparatively small amount of this work concentrating on the calculation of the quantities commonly measured in structural dynamic tests. Perhaps the most fundamental of these quantities is the Frequency Response Function (FRF). A number of years ago, Yar and Hammond took an interesting approach to estimating the FRF of a Duffing oscillator system which was based on an approximate solution of the Fokker-Planck-Kolmogorov equation. Despite reproducing the general features of the statistical linearization estimate, the approximation failed to show the presence of the poles at odd multiples of the primary resonance which are known to occur experimentally. The current paper simply extends the work of Yar and Hammond to a higher order of approximation and is thus able to show the existence of a third ‘harmonic’ in the FRF.

1. Introduction

Among the many techniques developed for the analysis of nonlinear systems under random excitation, the Fokker-Planck-Kolmogorov (FPK) equation approach is attractive for a number of reasons; among them is the property recognised by Yar and Hammond [1,2], of having considerable potential with regards to obtaining significant response statistics.

When the excitation to a system can be idealised as Gaussian white noise, the solution of the corresponding FPK equations is the transition probability density function. This is considered the most complete statistical description of the process, and when expressed as an expansion of the eigenfunctions of the corresponding FPK operators, can be easily manipulated to give the autocovariance and spectral density functions of the system. Unfortunately, exact solutions to the FPK equations can rarely be found for nonlinear systems, instead methods to approximate the solution must be used.

This paper follows the work of Yar and Hammond [1,2] closely, in using a variational approach to approximate the eigenfunctions of the FPK operators of the Duffing oscillator under Gaussian white noise excitation \( n(t) \),

\[
\ddot{x} + 2c\dot{x} + x + \varepsilon x^3 = n(t)
\]  

(1.1)

where,

\[
E[n(t)] = 0, \quad E[n(t)n(t')] = 2D\delta(t-t')
\]
Here, $c$ represents viscous damping, $\varepsilon$ a cubic spring stiffness, $D$ is a constant which measures the white noise intensity, and $\delta$ is the Dirac delta function; $E[\cdot]$ is the expectation operator.

In this paper, just as in Yar and Hammond’s paper, a first-order approximation is initially employed, which is sufficient to reproduce the general features of the statistical linearisation estimate, but not accurate enough to indicate any presence of the poles at odd multiples of the primary resonance which are known to occur from simulation and experiment [3]. The work is then extended to a third-order approximation which successfully shows the existence of a secondary resonance at three times the natural frequency. The paper follows Yar and Hammond [1] very closely in its description of the basic theory; this is because the paper [1] contains a number of typographical errors which are corrected here. This theory will then form the basis of the extended calculations.

2. The Fokker Planck Kolmogorov (FPK) Equations

The transition probability density $p(x, y, t \mid x_0, y_0, 0)$ for the Duffing oscillator system of equation (1.1) satisfies the backwards and forwards FPK equations,

$$\frac{\partial p}{\partial t} = L(p) \quad \text{(forward)}$$

$$\frac{\partial p}{\partial t} = L^*(p) \quad \text{(backward)} \quad (2.1)$$

with the initial condition: $L_{\tau \rightarrow \tau} p(x, y, t \mid x_\tau, y_\tau) = \delta(x-x_\tau)\delta(y-y_\tau)$.

In the above, $L$ and $L^*$ are adjoint differential operators defined as follows,

$$L(p) = -y \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} (2cy + x + \varepsilon x^3)p + D \frac{\partial^2 p}{\partial y^2}$$

$$L^*(p) = y_\tau \frac{\partial p}{\partial x_\tau} - \frac{\partial}{\partial y_\tau} (2cy_\tau + x_\tau + \varepsilon x_\tau^3)p + D \frac{\partial^2 p}{\partial y_\tau^2} \quad (2.2)$$

2.1. Eigenfunction Expansion of the FPK

Following the work of Johnson and Scott [4], the transition probability density function can be expressed as an expansion of the eigenfunctions of the FPK operators $L$ and $L^*$ as follows

$$p(x, y, t \mid x_0, y_0, 0) = \sum_{i=1}^{\infty} e^{-\lambda_i t} u_i(x, y) u^*_i(x_0, y_0) \quad (2.3)$$

where,

$$Lu_i + \lambda_i u_i = 0 \quad (2.4a)$$

$$L^* u^*_i + \lambda_i u^*_i = 0 \quad (2.4b)$$

Since $L$ and $L^*$ are adjoint, their corresponding eigenvalues $\lambda_i$ are the same. This leads to the following biorthogonality relation which will prove useful later,

$$\int \int u_i u^*_j \, dx \, dy = \delta_{ij} \quad (2.5)$$

Where $\delta_{ij}$ is the Kronecker delta which is unity when $i = j$, and zero else.
2.2. The stationary probability density

An important solution to (2.4a) occurs when $\lambda_0 = 0$, i.e. $L(u_0) = 0$. $u_0$ or $p_s$ is called the stationary probability density and can be used to find the lower-order statistical moments of the process, such as the displacement variance $\sigma_x^2$. For the Duffing oscillator

$$u_0 = p_s(x, y) = Ae^{-\frac{c}{D}x^2+y^2}$$

(2.6)

where the normalisation constant $A$ is given by,

$$A^{-1} = \sqrt{\frac{\pi D}{c}} \Gamma\left(\frac{1}{2}\right) U\left(0, \sqrt{\frac{c}{\epsilon D}}\right)$$

and $U(a,b)$ is the parabolic cylindrical function.

2.3 The Operator $G$

Once the stationary probability density function has been found, as in the work of Caughey [5], Yar and Hammond [1,2] and Johnson and Scott [4], it is found to be advantageous to consider the operator $G$ which is related to $L$ by the following relation,

$$G(w_i) = \frac{L(p_s(x, y)w_i)}{p_s}$$

(2.7)

From which $G$ is found to be,

$$G(w) = -y \frac{\partial w}{\partial x} + (2cy + \epsilon x^3) \frac{\partial w}{\partial y} + D \frac{\partial^2 w}{\partial y^2}$$

(2.8a)

where,

$$G(w) + \lambda w = 0$$

(2.8b)

Now, since,

$$G(w) + \lambda w = 0 \Rightarrow \frac{L(p_s,w)}{p_s} + \lambda w = 0 \Rightarrow L(p_s,w) + \lambda p_s w = 0$$

it follows that if $w_i(x, y)$ is an eigenfunction of $G$ then $w_i(x, y)p_s(x, y)$ is an eigenfunction of $L$. It is also clear here that $G$ may be obtained from $L^*$ on simply replacing $y$ by $-y$. Therefore, if $w_i(x, y)$ is an eigenfunction of $G$, $w_i(x, -y)$ is an eigenfunction of $L^*$. Using these two observations equation (2.3) may now be re-expressed as,

$$p(x, y; t \mid x_0, y_0, 0) = p_s(x, y) \sum_{i=1}^{\infty} e^{-\lambda t} w_i(x, y)w_i(x_0, -y_0)$$

(2.9)

and the relation (2.5) can be re-expressed as,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_i(x, y)w_i(x, y)w_j(x, -y)dx\,dy = E[w_i(x, y)w_j(x, -y)] = \delta_{ij}
\]  
(2.10)

2.4 Autocovariance and Spectral Density Functions.

Expressions for the autocovariance function and spectral density function in terms of the eigenfunctions \(w_i(x, y)\) may now be easily derived.

The autocovariance function, \(\phi_{xx}(\tau)\) is defined as,

\[
\phi_{xx}(t_1 - t_0) = E[x(t_0)x(t_1)] = E[x_0x_1]
\]

In an obvious notation. In full,

\[
\phi_{xx}(t_1 - t_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_j(x_0, y_0, t_0; x_1, y_1, t_1) x_0 x_1 dx_0 dy_0 dt_0 dt_1.
\]  
(2.11)

where \(p_j(x_0, y_0, t_0; x_1, y_1, t_1)\) is the complete joint probability density function of \((x_0, y_0)\) and \((x_1, y_1)\). When a stationary solution of the FPK exists, the joint probability density function can be related to the transition probability density by,

\[
p_j(x, y, t; x', y', t') = p(x', y', t - t' | x, y, 0) p_j(x, y)
\]  
(2.12)

(which is essentially the definition of a conditional probability [6]).

Using this relation, equation (2.11) becomes,

\[
\phi_{xx}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y, \tau | x_0, y_0, 0) p_j(x_0, y_0) x_0 x_1 dx_0 dy_0 dt_0 dt_1.
\]

which, on substituting equation (2.3), eventually yields,

\[
\phi_{xx}(\tau) = \sum_{i=1}^{\infty} e^{-\lambda_i \tau} \gamma_i \eta_i
\]  
(2.13)

where,

\[
\gamma_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x w_i(x, y) p_i(x, y) dx\,dy = E[x w_i(x, y)]
\]

\[
\eta_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x w_i(x, -y) p_i(x, y) dx\,dy = E[x w_i(x, -y)]
\]

The spectral density function is obtained by taking the Fourier transform of the autocovariance function and is straightforwardly found to be,

\[
S_{xx}(\omega) = 2 \sum_{i=1}^{\infty} \gamma_i \eta_i \frac{\lambda_i}{(\lambda_i^2 + \omega^2)}
\]  
(2.14)

where \(\gamma_i\) and \(\eta_i\) are as defined above. One can then trivially express the FRF (magnitude) as,
$$|H(\omega)| = \sqrt{\frac{S_{xx}(\omega)}{2D}}$$

(2.15)

3. Approximating the eigenfunctions of the FPK operators

Since the eigenfunctions of the FPK equations for the Duffing system can not be found exactly, an approximation technique must be used. A variational method based on the Rayleigh Ritz method is used here as employed by Yar and Hammond [2] and Atkinson [7].

As shown in references [2,5,7], approximate eigenfunctions \( w^*(x,y) \) of the operator \( G \) can be expressed in the form,

$$w^*(x,y) = \sum_{i=1}^{N} C_i z_i(x,y)$$

(3.1)

where \( \{z_i(x,y), i=1,\ldots,N\} \) is a set of independent trial functions orthonormal with respect to the stationary density \( p_s(x,y) \), and \( C_i \) are constants to be found. The constant \( N \) is fixed by the degree of approximation required. Now, upon noting that,

$$p_s(x,y) = a e^{-\frac{c}{b}x^2} b e^{-\frac{c}{b}(x^2+y^4)} = p_s(x)p_s(y)$$

(3.2)

where \( a \) and \( b \) are appropriate normalisations, one can see immediately that the trial functions will factorise as,

$$w^*(x,y) = \phi(x)H_j(y)$$

(3.3)

where the \( \{\phi(x)\} \) will be a set of functions orthonormal with respect to \( p_s(x) \), and the \( \{H_j(y)\} \) will be a set of functions orthonormal with respect to \( p_s(y) \). The approximation is further simplified by the observation that only eigenfunctions that are odd polynomials in \( x \) and \( y \) will contribute to the autocovariance and spectral density functions [2]. A set of basis functions \( \{z_j\} \) consisting of products of the odd polynomials of \( \{\phi(x), H_j(y)\} \) can therefore be used in the place of the full set in the approximation (3.1).

Once the basis functions \( \{z_j\} \) have been determined, the constants \( C_i \), along with the approximate eigenvalues for the original eigenvalue problem (2.8b), are found by solving the matrix eigenvalue problem [1],

$$(P + \lambda I)C = 0$$

(3.4)

where the \( ij \)th element of the matrix \( P \) is \( E[z_i(z_j)z_j ] \).

3.1 Orthonormal trial functions

To construct the required basis functions, the set \( \{\phi(x)\} \) of functions orthonormal with respect to \( p_s(x) \), and the set \( \{H_j(y)\} \) orthonormal with respect to \( P_s(y) \) must first be found. Since \( y \) is normally distributed, the \( \{H_j(y)\} \) take the form of Hermite polynomials. The required orthonormal set is given by,
where
\[ \{h_j(y)\} = \left(\frac{\nu}{\pi}\right)^{\frac{j}{2}} J^{(1)}_j\left(y\frac{\nu}{\pi}\right)^{\frac{1}{2}} \] and \( \{h_j^{(1)}(y)\} \) are the standard Hermite polynomials orthogonal with respect to the normal probability distribution with density \( \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \).

The construction of \( \{\phi_i(x)\} \) is approached using the Gram-Schmidt method. This method takes a finite set of linearly independent functions \( \{U_n\} \) and generates an orthogonal set \( \{\psi_n\} \) that spans the same subspace. A set of orthonormalised functions \( \{\phi_n\} \) can then be obtained from the orthogonal sequence. Here \( \{U_n\} = \{1, x, x^2, ..., x^n\} \).

### 3.2 Second and higher-order moments of x and y

The constructed orthonormal functions will involve the displacement variance \( E[x^2] \), and higher-order moments such as \( E[x^4] \). While the stationary probability density can be used to find the second order moments \( E[x^2] \) and \( E[y^2] \), recursion relations must be used to express any higher-order moments. If one considers \( E[x^m] \) first, one observes that,

\[
E[x^m] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^m p_s(x, y) \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^m A e^{-\frac{(y^2 + x^2)}{2\epsilon}} \, dx \, dy \tag{3.6}
\]

Integrating by parts, one establishes the following recursion relation,

\[
E[x^{m-2}] + \epsilon E[x^m] = \frac{(m-3)DE[x^{m-4}]}{2\epsilon} \tag{3.7}
\]

Using this relation one can find all the higher-order moments providing the displacement variance \( E[x^2] \) is known. Directly from the work of To [8], one finds the displacement variance to be

\[
E[x^2] = \sqrt{\frac{D}{4\epsilon \epsilon}} U \left[ 1, \sqrt{\frac{c}{D\epsilon}}, 0, \frac{c}{D\epsilon} \right]^{-1} \tag{3.8}
\]

where \( U[a, b] \) is again the parabolic cylindrical function.

It remains now to find an expression for \( E[y^2] \), and the higher-order moments involving y. As before, one begins with,
\[ E[y^m] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^m p_x(x, y)dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^m Ae^{\frac{c}{2}(y^2 + x^2 + \xi^4)} dx dy \quad (3.9) \]

and integrates by parts; the following recursion relation is found,

\[ E[y^m] = \frac{(m-1)DE[y^{m-2}]}{2c} \quad (3.10) \]

Furthermore, it follows trivially from this recurrence that,

\[ E[y^2] = \frac{D}{2c} \]

Everything is now in place to use (3.1) to approximate the eigenfunctions of the system.

### 3.3 First order approximation, N=1.

Yar and Hammond [2] used the approximation in (3.1) with N=2, which amounts to a first-order approximation in \(x\) and \(y\). In this case the basis functions are \( z_i(x, y) = \phi_i(x), z_2(x, y) = H_i(y) \), where,

\[ \phi_i = \frac{x}{\|\psi_i\|} \quad \text{with} \quad \|\psi_i\| = \sqrt{E[x^2]} = \sigma_x \]

\[ H_i = \frac{y}{\|h_i\|} \quad \text{with} \quad \|h_i\| = \sqrt{E[y^2]} = \sigma_y = \frac{D}{2c} \]

and the matrix \( P \) of equation (3.4) takes the form,

\[
\begin{bmatrix}
E[G(\phi_1)\phi_1] & E[G(\phi_1)H_1] \\
E[G(H_1)\phi_1] & E[G(H_1)H_1]
\end{bmatrix} = \begin{bmatrix}
0 & -\sigma_y \\
\frac{\sigma_x}{\sigma_y} & \frac{\sigma_x}{2c}
\end{bmatrix}
\]

which yields eigenvalues completely consistent with a statistical linearization approach, as observed by Yar and Hammond.

A Maple routine [10] was used at this point to solve the matrix eigenvalue problem for a low level of excitation, where one would expect to observe the FRF of the underlying linear system; the parameters: \( D=0.005, c=0.05 \) and \( \varepsilon=0.05 \) were adopted (in order to make contact with the results obtained in [1]). With these choices, the approximate eigenvalues of the system were found to be,

\[-0.05 \pm 1.002i.\]

Looking at the eigenvalues gives a quick and simple validation as to whether the approximation is working, as the imaginary parts of the eigenvalues should represent the resonances of the system.
this case the imaginary parts of the eigenvalues are indicating that the primary resonance occurs at
around unit frequency, which is expected.

From this approximation the autocorrelation function and FRF magnitude were computed using
equations (2.14) and (2.16) and are shown below. The usual forms associated with the single-degree-
of-freedom linear oscillator are observed.

\begin{align*}
\text{Figure 1. Autocorrelation function} \\
\text{Figure 2. FRF magnitude.}
\end{align*}

3.3 Third order approximation, \( N=6 \).

Equation (3.1) was then used with \( N=6 \) to compute the third-order approximation of the FPK solution.
The required basis functions in this case are \( \{ \phi(x), H_1(y), \phi(x)H_2(y), \phi_2(x)H_1(y), \phi_3(x), H_3(x) \} \),
and \( P \) therefore becomes a 6x6 matrix.

\[
P = \begin{bmatrix}
E[G(\phi)\phi] & E[G(\phi)H_1] & E[G(\phi)\phi H_2] & E[G(\phi)\phi H_1] & E[G(\phi)H_3] & E[G(\phi)H_4] \\
E[G(H_1)\phi] & E[G(H_1)H_1] & E[G(H_1)\phi H_2] & E[G(H_1)\phi H_1] & E[G(H_1)H_3] & E[G(H_1)H_4] \\
E[G(\phi H_2)\phi] & E[G(\phi H_2)H_1] & E[G(\phi H_2)\phi H_2] & E[G(\phi H_2)\phi H_1] & E[G(\phi H_2)H_3] & E[G(\phi H_2)H_4] \\
E[G(\phi H_1)\phi] & E[G(\phi H_1)H_1] & E[G(\phi H_1)\phi H_2] & E[G(\phi H_2)\phi H_1] & E[G(\phi H_1)H_3] & E[G(\phi H_1)H_4] \\
E[G(\phi)\phi] & E[G(\phi)H_1] & E[G(\phi)\phi H_2] & E[G(\phi)\phi H_1] & E[G(\phi)H_3] & E[G(\phi)H_4] \\
E[G(H_3)\phi] & E[G(H_3)H_1] & E[G(H_3)\phi H_2] & E[G(H_3)\phi H_1] & E[G(H_3)H_3] & E[G(H_3)H_4]
\end{bmatrix}
\]

A Maple routine was once again used to solve the 6x6 matrix eigenvalue problem (3.4). FRFs and
spectral density functions are yet to be computed for the third degree approximation; instead, for a
quick validation of the approximation one can once again consider the approximate eigenvalues. For a
higher level of excitation which is expected to force marked nonlinear behaviour: for \( D=1, c=0.05 \) and
\( \varepsilon=0.001 \) (again, following [1]), the approximate eigenvalues of the system are found to be,

\[
\begin{align*}
-0.016+1.027i & \quad -0.016-1.027i \\
-0.070+3.081i & \quad -0.070-3.081i \\
-0.048+1.012i & \quad -0.048-1.012i
\end{align*}
\]

As expected, one pair of the eigenvalues have imaginary part close unity, representing the frequency
concerning the primary resonance. The second pair have imaginary parts close to three, which
confirms that the approximation is recreating the secondary resonance at three times the natural
frequency. The third pair of eigenvalues suggest the presence of a double pole at around unit
frequency, which is a conclusion consistent with a parallel study carried out using the Volterra series
[9].
4. Conclusions

Although the autocorrelation and spectral density functions of the third-order approximation have not yet been computed, one can conclude from the approximate eigenvalues that the third-order approximation is sufficient to indicate the presence of the secondary resonance at three times the natural frequency. It remains now to compute these functions for the third-order approximation and assess their accuracy by comparison with numerical simulation. Further study could naturally extend to a higher-order approximation to see if further harmonics at odd multiples of the natural frequency can be recreated.

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