ON SEMI-MODULAR SUBALGEBRAS OF LIE ALGEBRAS OVER
FIELDS OF ARBITRARY CHARACTERISTIC

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Abstract
This paper is a further contribution to the extensive study by a number of authors of the subalgebra lattice of a Lie algebra. It is shown that, in certain circumstances, including for all solvable algebras, for all Lie algebras over algebraically closed fields of characteristic \( p > 0 \) that have absolute toral rank \( \leq 1 \) or are restricted, and for all Lie algebras having the one-and-a-half generation property, the conditions of modularity and semi-modularity are equivalent, but that the same is not true for all Lie algebras over a perfect field of characteristic three. Semi-modular subalgebras of dimensions one and two are characterised over (perfect, in the case of two-dimensional subalgebras) fields of characteristic different from 2, 3.

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1 Introduction

This paper is a further contribution to the extensive study by a number of authors of the subalgebra lattice of a Lie algebra, and is, in part, inspired by the papers of Varea ([15], [16]). A subalgebra \( U \) of a Lie algebra \( L \) is called

- **modular** in \( L \) if it is a modular element in the lattice of subalgebras of \( L \); that is, if

\[
< U, B > \cap C = < B, U \cap C > \quad \text{for all subalgebras } B \subseteq C,
\]

and

\[
< U, B > \cap C = < B \cap C, U > \quad \text{for all subalgebras } U \subseteq C,
\]

(where, \( < U, B > \) denotes the subalgebra of \( L \) generated by \( U \) and \( B \))
• upper modular in L (um in L) if, whenever B is a subalgebra of L which covers \( U \cap B \) (that is, such that \( U \cap B \) is a maximal subalgebra of B), then \( < U, B > \) covers U;

• lower modular in L (lm in L) if, whenever B is a subalgebra of L such that \( < U, B > \) covers U, then B covers \( U \cap B \);

• semi-modular in L (sm in L) if it is both um and lm in L.

In this paper we extend the study of sm subalgebras started in [12]. In section two we give an example of a Lie algebra over a perfect field of characteristic three which has a sm subalgebra that is not modular. However, it is shown that for all solvable Lie algebras, and for all Lie algebras over an algebraically closed field of characteristic \( p > 0 \) that have absolute toral rank \( \leq 1 \) or are restricted, the conditions of modularity, semi-modularity and being a quasi-ideal are equivalent. The latter extends results of Varea in [16] where the characteristic of the field is restricted to \( p > 7 \). It is then shown that for all Lie algebras having the one-and-a-half generation property the conditions of modularity and semi-modularity are equivalent.

In section three, sm subalgebras of dimension one are studied. These are characterised over fields of characteristic different from 2, 3. This result generalises a result of Varea in [15] concerning modular atoms. In the fourth section we show that, over a perfect field of characteristic different from 2, 3, the only Lie algebra containing a two-dimensional core-free sm subalgebra is \( sl_2(F) \). It is also shown that, over certain fields, every sm subalgebra that is solvable, or that is split and contains the normaliser of each of its non-zero subalgebras, is modular.

Throughout, \( L \) will denote a finite-dimensional Lie algebra over a field \( F \). There will be no assumptions on \( F \) other than those specified in individual results. The symbol ‘\( \oplus \)’ will denote a vector space direct sum. If \( U \) is a subalgebra of \( L \), the core of \( U \), \( U_L \), is the largest ideal of \( L \) contained in \( U \); we say that \( U \) is core-free if \( U_L = 0 \). We denote by \( R(L) \) the solvable radical of \( L \), by \( Z(L) \) the centre of \( L \), and put \( C_L(U) = \{ x \in L : [x, U] = 0 \} \).

2 General results

We shall need the following result from [12].

Lemma 2.1 Let \( U \) be a proper sm subalgebra of a Lie algebra \( L \) over an arbitrary field \( F \). Then \( U \) is maximal and modular in \( < U, x > \) for all \( x \in L \setminus U \).
Proof. We have that $U$ is maximal in $< U, x >$, by Lemma 1.4 of [12], and hence that $U$ is modular in $< U, x >$, by Theorem 2.3 of [12].

In [12] it was shown that, over fields of characteristic zero, $U$ is modular in $L$ if and only if it is sm in $L$. This result does not extend to all fields of characteristic three, as we show next. Recall that a simple Lie algebra is split if it has a splitting Cartan subalgebra $H$; that is, if the characteristic roots of $ad_L h$ are in $F$ for every $h \in H$. Otherwise we say that it is non-split.

**Proposition 2.2** Let $L$ be a Lie algebra of dimension greater than three over an arbitrary field $F$, and suppose that every two linearly independent elements of $L$ generate a three-dimensional non-split simple Lie algebra. Then there are maximal subalgebras $M_1, M_2$ of $L$ such that $M_1 \cap M_2 = 0$.

Proof: This is proved in Proposition 4 of [8].

**Example**

Let $G$ be the algebra constructed by Gein in Example 2 of [7]. This is a seven-dimensional Lie algebra over a certain perfect field $F$ of characteristic three. In $G$ every linearly independent pair of elements generate a three-dimensional non-split simple Lie algebra. It follows from Proposition 2.2 above that there are two maximal subalgebras $M, N$ in $G$ such that $M \cap N = 0$. Choose any $0 \neq a \in M$. Then $< a, N > \cap M = M$, but $< N \cap M, a > = Fa$, so $Fa$ is not a modular subalgebra of $L$. However, it is easy to see that all atoms of $G$ are sm in $G$.

A subalgebra $Q$ of $L$ is called a quasi-ideal of $L$ if $[Q, V] \subseteq Q + V$ for every subspace $V$ of $L$. It is easy to see that quasi-ideals of $L$ are always semi-modular subalgebras of $L$. When $L$ is solvable the semi-modular subalgebras of $L$ are precisely the quasi-ideals of $L$, as the next result, which is based on Theorem 1.1 of [15], shows.

**Theorem 2.3** Let $L$ be a solvable Lie algebra over an arbitrary field $F$ and let $U$ be a proper subalgebra of $L$. Then the following are equivalent:

(i) $U$ is modular in $L$;

(ii) $U$ is sm in $L$; and

(iii) $U$ is a quasi-ideal of $L$. 

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Proof (i) ⇒ (ii): This is straightforward.

(ii) ⇒ (iii): Let \( L \) be a solvable Lie algebra of smallest dimension containing a subalgebra \( U \) which is sm in \( L \) but is not a quasi-ideal of \( L \). Then \( U \) is maximal and modular in \( L \), by Lemma 2.1 and \( U_L = 0 \). Let \( A \) be a minimal ideal of \( L \). Then \( L = U + A \). Moreover, \( U \cap A \) is an ideal of \( L \), since \( A \) is abelian, whence \( U \cap A = 0 \) and \( L = U \oplus A \). Now \( U \) is covered by \( < U, A > \) so \( A \) covers \( U \cap A = 0 \). This yields that \( \dim(A) = 1 \) and so \( U \) is a quasi-ideal of \( L \), a contradiction.

(iii) ⇒ (i): This is straightforward.

**Corollary 2.4** Let \( L \) be a solvable Lie algebra over an arbitrary field \( F \) and let \( U \) be a core-free sm subalgebra of \( L \). Then \( \dim(U) = 1 \) and \( L \) is almost abelian.

**Proof:** This follows from Theorem 2.3 and Theorem 3.6 of [1].

We now consider the case when \( L \) is not necessarily solvable. First we shall need the following result concerning \( psl_3(F) \).

**Proposition 2.5** Let \( F \) be a field of characteristic 3 and let \( L = psl_3(F) \). Then \( L \) has no maximal sm subalgebra.

**Proof:** Let \( E_{ij} \) be the \( 3 \times 3 \) matrix that has 1 in the \((i, j)\)-position and 0 elsewhere, and denote by \( \overline{E_{ij}} \) the canonical image of \( E_{ij} \) in \( sl_3(F) \) in \( psl_3(F) \).

Put \( e_3 = \overline{E_{23}}, e_2 = \overline{E_{31}}, e_1 = \overline{E_{12}}, e_0 = \overline{E_{11} - E_{22}}, e_1 = \overline{E_{21}}, e_2 = \overline{E_{13}}, e_3 = \overline{E_{32}} \). Then \( e_{-3}, e_{-2}, e_{-1}, e_0, e_1, e_2, e_3 \) is a basis for \( psl_3(F) \) with

\[
[e_0, e_i] = e_i \text{ if } i > 0, \quad [e_0, e_i] = -e_i \text{ if } i < 0, \quad [e_{-i}, e_j] = \delta_{ij} e_0 \text{ if } i, j > 0 \quad \text{and}
\]

\[
[e_i, e_j] = e_{-k} \quad \text{for every cyclic permutation } (i, j, k) \text{ of } (1, 2, 3) \text{ or } (-3, -2, -1).
\]

Put \( B_{i,j} = F e_0 + F e_i + F e_j \) for each non-zero \( i, j \). If \( i, j \) are of opposite sign then \( B_{i,j} \) is a subalgebra, every maximal subalgebra of which is two dimensional.

Let \( M \) be a maximal sm subalgebra of \( L \). For each \( i, j \) of opposite sign, if \( B_{i,j} \not\subseteq M \) then \( M \cap B_{i,j} \) is two dimensional. Since \( M \) is at most five-dimensional, by considering the intersection with each of \( B_1, B_2, B_3 \) it is easy to see that \( e_0 \in M \). But then, considering \( B_1 \) again, we have either \( e_1 \in M \) or \( e_{-1} \in M \). Suppose the former holds. Taking the intersection of \( M \) with \( B_{2,-3} \) shows that \( e_{-3} \in M \); then with \( B_{2,-1} \) gives \( e_2 \in M \); next with \( B_{3,-2} \) gives \( e_{-2} \in M \); finally with \( B_{3,-1} \) yields \( e_3 \in M \).
But then $M = L$, a contradiction. A similar contradiction is easily obtained if we assume that $e_{-1} \in M$.

Let $(L_p, [p], i)$ be any finite-dimensional $p$-envelope of $L$. If $S$ is a subalgebra of $L$ we denote by $S_p$ the restricted subalgebra of $L_p$ generated by $i(S)$. Then the (absolute) toral rank of $S$ in $L$, $TR(S, L)$, is defined by

$$TR(S, L) = \max\{\dim(T) : T \text{ is a torus of } (S_p + Z(L_p))/Z(L_p)\}.$$ 

This definition is independent of the $p$-envelope chosen (see [11]). We write $TR(L, L) = TR(L)$. Then, following the same line of proof, we have an extension of Lemma 2.1 of [16].

**Lemma 2.6** Let $L$ be a Lie algebra over an algebraically closed field of characteristic $p > 0$ such that $TR(L) \leq 1$. Then the following are equivalent:

(i) $U$ is modular in $L$;

(ii) $U$ is $sm$ in $L$; and

(iii) $U$ is a quasi-ideal of $L$.

**Proof:** We need only show that (ii) $\Rightarrow$ (iii). Let $U$ be a $sm$ subalgebra of $L$ that is not a quasi-ideal of $L$. Then there is an $x \in L$ such that $< U, x > \neq U + Fx$. We have that $U$ is maximal and modular in $< U, x >$, by Lemma 2.1 and $< U, x >$ is not solvable, by Theorem 2.3. Furthermore $TR(< U, x >) \leq TR(L) \leq 1$, by Proposition 2.2 of [11], and $< U, x >$ is not nilpotent so $TR(< U, x >) \neq 0$, by Theorem 4.1 of [11], which yields $TR(< U, x >) = 1$. We may therefore suppose that $U$ is maximal and modular in $< U, x >$, of codimension greater than one in $L$, and that $TR(L) = 1$.

Put $L^\infty = \bigcap_{n \geq 1} L^n$. Suppose first that $R(L^\infty) \not\subseteq U$. Then $U \cap R(L^\infty)$ is maximal and modular in the solvable subalgebra $R(L^\infty)$, so $U \cap R(L^\infty)$ has codimension one in $R(L^\infty)$. Since $U$ is maximal in $L$ we have $L = U + R(L^\infty)$ and so $\dim(L/U) = 1$, which is a contradiction. This yields that $R(L^\infty) \subseteq U$. Moreover, $L^\infty \not\subseteq U$, since this would imply that $U/L^\infty$ is maximal in the nilpotent algebra $L/L^\infty$, giving $\dim(L/U) = 1$, a contradiction again. It follows that $(U \cap L^\infty)/R(L^\infty)$ is modular and maximal in $L^\infty/R(L^\infty)$. But now $L^\infty/R(L^\infty)$ is simple, by Theorem 2.3 of [17], and $1 = TR(L) \geq TR(L^\infty, L) \geq TR(L^\infty/R(L^\infty))$ by section 2 of [11], so $TR(L^\infty/R(L^\infty)) = 1$. This implies that

$$p \neq 2, \quad L^\infty/R(L^\infty) \in \{sl_2(F), W(1:1), H(2:1)^{(1)}\} \text{ if } p > 3$$
and $L^\infty/R(L^\infty) \in \{\text{sl}_2(F), \text{psl}_3(F)\}$ if $p = 3$, by [9] and [10].

Now $H(2 : 1)^{(1)}$ has no modular and maximal subalgebras, by Corollary 3.5 of [15]; likewise $\text{psl}_3(F)$ by Proposition 2.5. It follows that $L^\infty/R(L^\infty)$ is isomorphic to $W(1 : 1)$, which has just one proper modular subalgebra and this has codimension one, by Proposition 2.3 of [15], or to $\text{sl}_2(F)$ in which the proper modular subalgebras clearly have codimension one. Hence $\dim(L^\infty/(U \cap L^\infty)) = 1$. Since $L = U + L^\infty$ we conclude that $\dim(L/U) = \dim(L^\infty/(U \cap L^\infty)) = 1$. This contradiction gives the claimed result.

We then have the following extension of Theorem 2.2 of [16]. The proof is virtually as given in [16], but as the restriction to characteristic $> 7$ has been removed the details need to be checked carefully. The proof is therefore included for the convenience of the reader.

**Theorem 2.7** Let $L$ be a restricted Lie algebra over an algebraically closed field $F$ of characteristic $p > 0$, and let $U$ be a proper subalgebra of $L$. Then the following are equivalent:

(i) $U$ is modular in $L$;

(ii) $U$ is sm in $L$; and

(iii) $U$ is a quasi-ideal of $L$.

**Proof:** As before it suffices show that (ii) $\Rightarrow$ (iii). Let $U$ be a sm subalgebra of $L$ that is not a quasi-ideal of $L$. Then there is an $x \in L$ such that $<U, x> \not= U + Fx$. First note that $<U, x>$ is a restricted subalgebra of $L$. For, suppose not and pick $z < U, x>_p$ such that $z \not\in <U, x>$. Since $<U, x>$ is an ideal of $<U, x>_p$ we have that $[z, U] \leq <U, x> \cap <U, z>$. But $U$ is maximal in $<U, z>$, by Lemma 2.1 and so $<U, x> \cap <U, z> = U$, giving $[z, U] \leq U$. But $U$ is self-idealizing, by Lemma 1.5 of [12], so $z \in U$. This contradiction proves the claim. So we may as well assume that $L = <U, x>$. Moreover, $U$ is restricted since it is self-idealizing, whence $(U_L)_p \leq U$. As $(U_L)_p$ is an ideal of $L$ we have that $U_L = (U_L)_p$. It follows that $L/U_L$ is also restricted. We may therefore assume that $U$ is a core-free modular and maximal subalgebra of $L$ of codimension greater than one in $L$.

Now $L$ is spanned by the centralizers of tori of maximal dimension, by Corollary 3.11 of [17], so there is such a torus $T$ with $C_L(T) \not\subseteq U$. Let $L = C_L(T) \oplus \sum L_\alpha(T)$ be the decomposition of $L$ into eigenspaces with
respect to $T$. We have that $C_L(T)$ is a Cartan subalgebra of $L$, by Theorem 2.14 of [17]. It follows from the nilpotency of $C_L(T)$ and the modularity of $U$ that $U \cap C_L(T)$ has codimension one in $C_L(T)$.

Now let $L^{(\alpha)} = \sum_{i \in P} L_{i\alpha}(T)$, where $P$ is the prime field of $F$. From the modularity of $U$ we see that $U \cap L^{(\alpha)}$ is a modular and maximal subalgebra of $L^{(\alpha)}$. Since $U$ is core-free and self-idealizing, $Z(L) = 0$. But then $TR(T,L) = TR(L)$, since $T$ is a maximal torus, whence $TR(L^{(\alpha)}) \leq 1$, by Theorem 2.6 of [11]. It follows from Lemma 2.6 that $M \cap L^{(\alpha)}$ is a quasi-ideal of $L^{(\alpha)}$.

As $U \cap L^{(\alpha)}$ is maximal in $L^{(\alpha)}$, we have that $\dim(L^{(\alpha)}/(U \cap L^{(\alpha)})) \leq 1$ and $L^{(\alpha)} = U \cap L^{(\alpha)} + C_L(T)$. This yields that $L = U + C_L(T)$ and hence that $\dim(L/U) = \dim(C_L(T)/(U \cap C_L(T))) = 1$, a contradiction. The result follows.

We shall say that the Lie algebra $L$ has the one-and-a-half generation property if, given any $0 \neq x \in L$, there is an element $y \in L$ such that $\langle x, y \rangle = L$. Then we have the following result.

**Theorem 2.8** Let $L$ be a Lie algebra, over any field $F$, which has the one-and-a-half generation property. Then every sm subalgebra of $L$ is a modular maximal subalgebra of $L$.

**Proof**: Let $U$ be a sm subalgebra of $L$ and let $0 \neq u \in U$. Then there is an element $x \in L$ such that $L = \langle u, x \rangle = \langle U, x \rangle$. It follows from Lemma 2.1 that $U$ is modular and maximal in $L$.

**Corollary 2.9** Let $L$ be a Lie algebra over an infinite field $F$ of characteristic different from $2, 3$ which is a form of a classical simple Lie algebra. Then every sm subalgebra of $L$ is a modular maximal subalgebra of $L$.

**Proof**: Under the given hypotheses $L$ has the one-and-a-half generation property, by Theorem 2.2.3 and section 1.2.2 of [3], or by [5].

We also have the following analogue of a result of Varea from [15].

**Corollary 2.10** Let $F$ be an infinite perfect field of characteristic $p > 2$, and assume that $p^n \neq 3$. Then the subalgebra $W(1 : n)_0$ is the unique sm subalgebra of $W(1 : n)$.

**Proof**: Let $L = W(1 : n)$ and let $\Omega$ be the algebraic closure of $F$. Then $L \otimes_F \Omega$ is simple and has the one-and-a-half generation property, by Theorem
4.4.8 of [3]. It follows that \( L \) has the one-and-a-half generation property (see section 1.2.2 of [3]). Let \( U \) be a \( \mathfrak{m} \) subalgebra of \( L \). Then \( U \) is modular and maximal in \( L \) by Theorem 2.8. Suppose that \( U \neq L_0 \). Then \( L = U + L_0 \) and \( U \cap L_0 \) is maximal in \( L_0 \). But \( L_0 \) is supersolvable (see Lemma 2.1 of [13] for instance) so \( \dim(L_0/(L_0 \cap U)) = 1 \). It follows that \( \dim(L/U) = \dim(L_0/(L_0 \cap U)) = 1 \), whence \( U = L_0 \), which is a contradiction.

### 3 Semi-modular atoms

We say that \( L \) is almost abelian if \( L = L^2 \oplus Fx \) with \( \text{ad} \ x \) acting as the identity map on the abelian ideal \( L^2 \). A \( \mu \)-algebra is a non-solvable Lie algebra in which every proper subalgebra is one dimensional. A subalgebra \( U \) of a Lie algebra \( L \) is a strong ideal (respectively, strong quasi-ideal) of \( L \) if every one-dimensional subalgebra of \( U \) is an ideal (respectively, quasi-ideal) of \( L \); it is modular* in \( L \) if it satisfies a dualised version of the modularity conditions, namely

\[
\langle U, B \rangle \cap C = \langle B, U \cap C \rangle \quad \text{for all subalgebras } B \subseteq C,
\]

and

\[
\langle U \cap B, C \rangle = \langle B, C \rangle \cap U \quad \text{for all subalgebras } C \subseteq U.
\]

**Example**

Let \( K \) be the three-dimensional Lie algebra with basis \( a, b, c \) and multiplication \( [a, b] = c \), \( [b, c] = b \), \( [a, c] = a \) over a field of characteristic two. Then \( K \) has a unique one-dimensional quasi-ideal, namely \( Fc \). Thus for each \( 0 \neq u \in Fc \) and \( k \in K \setminus Fc \) we have that \( \langle u, k \rangle \) is two dimensional. However \( K \) is not almost abelian. In fact \( K \) is simple, \( Fc \) is core-free and is the Frattini subalgebra of \( K \), and so any two linearly independent elements not in \( Fc \) generate \( K \).

We shall need a result from [4]. However, because of the above example, there is a (slight) error in three results in this paper. The error comes from an incorrect use of Theorem 3.6 of [1]. The three corrected results are as follows:

**Lemma 3.1** (Lemma 2.2 of [3]) If \( Q \) is a strong quasi-ideal of \( L \), then \( Q \) is a strong ideal of \( L \), or \( L \) is almost abelian, or \( F \) has characteristic two, \( L = K \) and \( Q = Fc \).
Proof. Assume that \( Q \) is a strong quasi-ideal and that there exists \( q \in Q \) such that \( Fq \) is not an ideal of \( L \). Then Theorem 3.6 of [1] gives that \( L \) is almost abelian, or \( F \) has characteristic two, \( L = K \) and \( Q = Fc \). The result follows.

The proof of the following result is the same as the original.

**Proposition 3.2** (Proposition 2.3 of [4]) Let \( Q \) be a proper quasi-ideal of a Lie algebra \( L \) which is modular* in \( L \). Then \( Q \) is a strong quasi-ideal and so is given by Lemma 3.1.

**Lemma 3.3** (Lemma 4.1 of [4]) Let \( L \) be a Lie algebra over an arbitrary field \( F \). Let \( U \) be a core-free subalgebra of \( L \) such that \(< u, z >\) is either two dimensional or a \( \mu \)-algebra for every \( 0 \neq u \in U \) and \( z \in L \setminus U \). Then one of the following holds:

(i) \( L \) is almost abelian;
(ii) \(< u, z >\) is a \( \mu \)-algebra for every \( 0 \neq u \in U \) and \( z \in L \setminus U \);
(iii) \( F \) has characteristic two, \( L = K \) and \( Fu = Fc \).

*Proof: This is the same as the original proof except that the following should be inserted at the end of sentence six: “or \( \text{char}F = 2 \) and \( L = K \)”.

Using the above we now have the following result.

**Lemma 3.4** Suppose that \( Fu \) is sm in \( L \) but not an ideal of \( L \). Then one of the following holds:

(i) \( L \) is almost abelian;
(ii) \(< u, x >\) is a \( \mu \)-algebra for every \( x \in L \setminus Fu \);
(iii) \( F \) has characteristic two, \( L = K \) and \( Fu = Fc \).

*Proof: Pick any \( x \in L \setminus Fu \). Then \( Fu \) is maximal in \(< u, x >\), by Lemma 2.1. Now let \( M \) be a maximal subalgebra of \(< u, x >\). If \( u \in M \) then \( M = Fu \). So suppose that \( u \notin M \). Then \( Fu \) is a maximal subalgebra of \(< u, x >\)\(< u, M >\), whence \( Fu \cap M = 0 \) is maximal in \( M \), since \( Fu \) is lm in \( L \). It follows that every maximal subalgebra of \(< u, x >\) is one dimensional. The claimed result now follows from Lemma 3.3.*

We shall need the following result concerning ‘one-and-a-half generation’ of rank one simple Lie algebras over infinite fields of characteristic \( \neq 2, 3 \).
Theorem 3.5 Let $L$ be a rank one simple Lie algebra over an infinite field $F$ of characteristic $\neq 2, 3$ and let $Fx$ be a Cartan subalgebra of $L$. Then there is an element $y \in L$ such that $<x, y> = L$.

Proof: Since $L$ is rank one simple it is central simple. Let $\Omega$ be the algebraic closure of $F$ and put $L_\Omega = L \otimes_F \Omega$, and so on. Then $L_\Omega$ is simple and $\Omega x$ is a Cartan subalgebra of $L_\Omega$. Let

$$L_\Omega = \Omega x \oplus \sum_{\alpha \in \Phi} (L_\Omega)_\alpha$$

be the decomposition of $L_\Omega$ into its root spaces relative to $\Omega x$. Then, with the given restrictions on the characteristic of the field, every root space $(L_\Omega)_\alpha$ is one dimensional (see [2]).

Let $M$ be a maximal subalgebra of $L$ containing $x$. Then $M_\Omega$ is a subalgebra of $L_\Omega$ and $\Omega x \subseteq M_\Omega$. So, $M_\Omega$ decomposes into root spaces relative to $\Omega x$,

$$M_\Omega = \Omega x \oplus \sum_{\alpha \in \Delta} (M_\Omega)_\alpha.$$

We have that $\Delta \subseteq \Phi$ and $(M_\Omega)_\alpha \subseteq (L_\Omega)_\alpha$ for all $\alpha \in \Delta$. As $(L_\Omega)_\alpha$ is one dimensional for every $\alpha \in \Phi$, we have $(M_\Omega)_\alpha = (L_\Omega)_\alpha$ for every $\alpha \in \Delta$. Hence there are only finitely many maximal subalgebras of $L$ containing $x$: $M_1, \ldots, M_r$ say. Since $F$ is infinite, $\bigcup_{i=1}^r M_i \neq L$, so there is an element $y \in L$ such that $y \not\in M_i$ for all $1 \leq i \leq r$. But now $<x, y> = L$, as claimed.

If $U$ is a subalgebra of $L$, then the normaliser of $U$ in $L$ is the set

$$N_L(U) = \{x \in L : [x, U] \subseteq U\}.$$

We can now give the following characterisation of one-dimensional semi-modular subalgebras of Lie algebras over fields of characteristic $\neq 2, 3$.

Theorem 3.6 Let $L$ be a Lie algebra over a field $F$, of characteristic $\neq 2, 3$ if $F$ is infinite. Then $Fu$ is sm in $L$ if and only if one of the following holds:

(i) $Fu$ is an ideal of $L$;

(ii) $L$ is almost abelian and $ad u$ acts as a non-zero scalar on $L^2$;

(iii) $L$ is a $\mu$-algebra.
Proof. It is easy to check that if (i), (ii), or (iii) hold then \( Fu \) is sm in \( L \). So suppose that \( Fu \) is sm in \( L \), but that (i), (ii) do not hold. First we claim that \( L \) is simple.

Suppose not, and let \( A \) be a minimal ideal of \( L \). If \( u \in A \), choose any \( b \in L \setminus A \). Then \( \langle u, b \rangle \cap A \) is an ideal of \( \langle u, b \rangle \). Since \( 0 \neq u \in \langle u, b \rangle \cap A \) and \( b \notin A \), \( \langle u, b \rangle \) cannot be a \( \mu \)-algebra. But then \( L \) is almost abelian, by Lemma 3.4 a contradiction. So \( u \notin A \). By Lemma 3.3 of [12], \( ua = \lambda a \) for all \( a \in A \) and some \( \lambda \in F \). But now \( Fu + Fa \) is a two-dimensional subalgebra of \( \langle u, a \rangle \), a \( \mu \)-algebra, which is impossible. Hence \( L \) is simple.

Now \( Fu \) is um in \( L \) and not an ideal of \( L \), so \( N_L(Fu) = Fu \), by Lemma 1.5 of [12]. Hence \( Fu \) is a Cartan subalgebra of \( L \), and \( L \) is rank one simple. Now \( F \) cannot be finite, since there are no \( \mu \)-algebras over finite fields, by Corollary 3.2 of [6]. Hence \( F \) is infinite. But then there is an element \( y \in L \) such that \( \langle u, y \rangle = L \), by Theorem 3.5, and \( L \) is a \( \mu \)-algebra. The result is established.

As a corollary to this we have a result of Varea, namely Corollary 2.3 of [14].

**Corollary 3.7** (Varea) Let \( L \) be a Lie algebra over a perfect field \( F \), of characteristic \( \neq 2, 3 \) if \( F \) is infinite. If \( Fu \) is modular in \( L \) but not an ideal of \( L \) then \( L \) is either almost abelian or three-dimensional non-split simple.

Proof: This follows from Theorem 3.6 and the fact that with the stated restrictions on \( F \) the only \( \mu \)-algebras are three-dimensional non-split simple (Proposition 1 of [7]).

4 **Semi-modular subalgebras of higher dimension**

First we consider two-dimensional semi-modular subalgebras. We have the following analogue of Theorem 1.6 of [15].

**Theorem 4.1** Let \( L \) be a Lie algebra over a perfect field \( F \) of characteristic different from 2, 3, and let \( U \) be a two-dimensional core-free sm subalgebra of \( L \). Then \( L \cong sl_2(F) \).

Proof: If \( U \) is modular then the result follows from Theorem 1.6 of [15], so we can assume that \( U \) is not a quasi-ideal of \( L \). Thus, there is an element \( x \in L \) such that \( \langle U, x \rangle \neq U + Fx \). Put \( V = \langle U, x \rangle \). Then \( U_V = U \) implies that \( \langle U, x \rangle = U + Fx \), a contradiction; if \( U_V = 0 \) then \( V \cong sl_2(F) \) by Lemma
\[2.1\] and Theorem 1.6 of [15], and \(< U, x > = U + Fx\), a contradiction. It follows that \(\dim(U_V) = 1\). Put \(U_V = Fu\). Now \(\dim(U/U_V) = 1\) and \(V/U_V\) is three-dimensional non-split simple, by Theorem 3.6 and Proposition 1 of [7]. Thus \(V = Fu \oplus S\), where \(S\) is three-dimensional non-split simple, by Lemma 1.4 of [15], and \(Fu, S\) are ideals of \(V\).

Now we claim that \(0 \neq Z(< U, y >) \subseteq U\) for every \(y \in L \setminus U\). We have shown this above if \(< U, y > \neq U + Fy\). So suppose that \(< U, y > = U + Fy\). Then \(< U, y >\) is three-dimensional and not simple (since \(U\) is two dimensional and abelian), and so solvable. Then, by using Corollary 2.4, we have that \(U\) contains a one-dimensional ideal \(K\) of \(U + Fy\) such that \((U + Fy)/K\) is two-dimensional non-abelian, and \(K = Z(< U, y >)\).

Since \(U\) is maximal in \(< U, x >\) we have \(< U, x > \neq L\). Pick \(y \in L \setminus < U, x >\). Then \(0 \neq Z(< U, x + y >) \subseteq U\) by the above. Suppose \(Z(< U, x >) \neq Z(< U, y >)\). Then \(U = Z(< U, x >) \oplus Z(< U, y >)\). Let \(0 \neq z \in Z(< U, x + y >)\) and write \(z = z_1 + z_2\) where \(z_1 \in Z(< U, x >), z_2 \in Z(< U, y >)\). Then \(0 = [z, (x + y)] = [z_2, x] + [z_1, y], \) so \([z_2, x] = -[z_1, y]\). Now, if \(z_1 = 0\), then \([z_2, x] = 0\), whence \(z_2 \in Z(< U, x >) \cap Z(< U, y >)\), a contradiction. Similarly, if \(z_2 = 0\), then \([z_1, y] = 0\), whence \(z_1 \in Z(< U, x >) \cap Z(< U, y >)\), a contradiction again. Hence \(z_1, z_2 \neq 0\).

Since \(z_1, z_2 \in U\) we deduce that \([z_1, y] = -[z_2, x] \in < U, x > \cap < U, y > = U\). Thus \(y \in N_L(U) = U\), a contradiction. It follows that \(Z(< U, x >) = Z(< U, y >)\) for all \(y \in L\), whence \([L, Z(< U, x >)] = 0\) and \(Z(< U, x >)\) is an ideal of \(L\), contradicting the fact that \(U\) is core-free.

Next we establish analogues of two results of Varea from [15].

**Theorem 4.2** Let \(L\) be a Lie algebra over an algebraically closed field \(F\) of characteristic \(p > 5\). If \(U\) is a sm subalgebra of \(L\) such that \(U/U_L\) is solvable and \(\dim(U/U_L) > 1\), then \(U\) is modular in \(L\), and hence \(L/U_L\) is isomorphic to \(sl_2(F)\) or to a Zassenhaus algebra.

**Proof** Let \(L\) be a Lie algebra of minimal dimension having a sm subalgebra \(U\) which is not modular in \(L\), and such that \(U/U_L\) is solvable and \(\dim(U/U_L) > 1\). Then \(U_L = 0\) and \(U\) is solvable. Since \(U\) is not a quasi-ideal there is an element \(x \in L \setminus U\) such that \(S = < U, x > \neq U + Fx\). Let \(K = U_S\). If \(\dim(U/K) = 1\) then \(S/K\) is almost abelian, by Theorem 3.6, whence \(U\) is a quasi-ideal of \(S\), a contradiction. It follows that \(\dim(U/K) > 1\). If \(U/K\) is modular in \(S/K\) then \(\dim(S/U) = 1\), by Theorem 2.4 of [15], a contradiction. The minimality of \(L\) then implies that \(S = L\). This yields that \(U\) is modular in \(L\), by Lemma 2.1. This contradiction establishes the result.
We say that the subalgebra $U$ of $L$ is split if $\text{ad}_L x$ is split for all $x \in U$; that is, if $\text{ad}_L x$ has a Jordan decomposition into semisimple and nilpotent parts for all $x \in U$.

**Theorem 4.3** Let $L$ be a Lie algebra over a perfect field $F$ of characteristic $p$ different from 2. If $U$ is a sm subalgebra of $L$ which is split and which contains the normaliser of each of its non-zero subalgebras, then $U$ is modular, and one of the following holds:

(i) $L$ is almost abelian and $\text{dim}(U) = 1$;

(ii) $L \cong \mathfrak{sl}_2(F)$ and $\text{dim}(U) = 2$;

(iii) $L$ is a Zassenhaus algebra and $U$ is its unique subalgebra of codimension one in $L$.

**Proof:** Let $L$ be a Lie algebra of minimal dimension having a sm subalgebra $U$ which is split and which contains the normaliser of each of its non-zero subalgebras, but which is not modular in $L$. Since $U$ is not a quasi-ideal there is an element $x \in L \setminus U$ such that $S = \langle U, x \rangle \neq U + Fx$. If $S \neq L$ then $U$ is modular in $S$, by the minimality of $L$. It follows from Theorem 2.7 of [15] that $U$ is a quasi-ideal of $S$, a contradiction. Hence $S = L$. Once again we see that $U$ is modular in $L$, by Lemma 2.1. This contradiction establishes that $U$ is modular in $L$. The result now follows from Theorem 2.7 of [15].

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