Optimization as a result of the interplay between dynamics and structure

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Abstract

In this work we study the interplay between the dynamics of a model of diffusion governed by a mechanism of imitation and its underlying structure. The dynamics of the model can be quantified by a macroscopic observable which permits the characterization of an optimal regime. We show that dynamics and underlying network cannot be considered as separated ingredients in order to achieve an optimal behavior.

Key words: econophysics; socio-economic evolution; networks
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1 Introduction

In recent times the possibility of using the tools of statistical physics to analyze the rich dynamical behaviors observed in social, technological, biological and economical systems has attracted a lot of attention from the physics community. So far, one of the main contributions to these fields has been the analysis of simple models that capture the basic features of the investigated phenomena. The goal is to identify their relevant ingredients as well as the essential mechanisms governing their dynamics with the hope that this information will help us to understand the physical behavior of real complex systems. A great part of this effort has been devoted to the characterization of real networks, identifying their main features, and understanding how they

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arise since in nature the coupling and interaction between units always has a clear motivation [1,2,3,4]. This approach provides a valuable tool to analyze how the underlying network affects the dynamics of interest. This is a natural step forward along this line of investigation.

For example, studies of an Ising model defined on a small world network [5] showed that for any finite disorder strength the system undergoes a crossover to a mean-field like region for small temperatures in the thermodynamic limit. Also, the analysis of a biological evolution model defined on a small world [6] has shown that even for a small number of quenched long-range jumps in the genotypic space the results are indistinguishable from those obtained by assuming all mutations equiprobable. The dramatic change in some static quantities such as the mean distance between units has been argued as the key mechanism to explain these phenomena. However, there is another set of works pointing out that other elements are necessary in order to observe changes when the underlying structure is modified, something which cannot be directly related to known static properties of the network. Along this line, numerical simulations on a non-equilibrium directed Ising model on a small world [7] showed a non trivial phase diagram. This system exhibits a line of continuous phase transitions below a critical threshold of disorder, above which the transition becomes first order. In agreement to this model other systems display transitions between different regimes but only when a finite number of shortcuts is introduced. In a model for the spreading of rumors [8] this critical value separates two regimes. In the first regime the rumor is bound to a finite neighborhood of an initially infected site while on the other one it reaches a finite fraction of the population. A similar phenomenon was observed in a model for the spreading of diseases [9] where a transition from an endemic situation to an oscillatory one was found.

These are just a few modern examples where significant changes in the properties of models are observed when their underlying structure is modified. Our work fits in this intriguing area. However, we will not focus our attention only on what happens when a structure is replaced by another. There are systems whose behavior can be described in terms of macroscopic observables, which give an idea about how well a system performs in a given scenario. Precisely, in this work the interplay between the dynamics and the underlying network is going to be exploited in order to reach an optimal behavior. Keeping this idea in mind we will show that a clear separation between both ingredients is not possible. In this way given a particular set of features which characterize the system, we wonder for the binomial dynamics/structure that optimizes its performance, measured in terms of the appropriate magnitudes. As we will see the answer is not trivial, even for the simple model of diffusion of innovations [10,11] that will be under consideration in this paper.

The outline of this work is as follows. First we will define the model, and
describe its main features. Then we will analyze the influence of different underlying networks on the dynamical properties of the system. Using this information, we will study the interplay between structure and dynamics. Finally, we will show that both elements need to be taken into account in order to achieve an optimal regime.

2 The model

Our starting point is a model of diffusion of innovations by imitation recently proposed in [11]. In spite of its apparent simplicity it displays a very rich behavior presenting self-organization, subcritical, critical and supercritical regimes and deserves further analysis. The model was originally presented in a computational economics context [4], describes how an external perturbation that affects individuals propagates to the rest of the population by a mechanism of imitation. Such a perturbation, modeled in terms of random updates, is independent of the underlying network and may affect any agent in the system. When an agent changes her state such information will be available only to the agents which are in contact with her. These agents should decide if it is beneficial for them to imitate, or remain in their present state. In the case that any of these agents decides to imitate the information will also propagate to her neighbors. The process ends when everybody is satisfied with their current situation. In this way, the features that characterize the members of the population are diffused as a wave of changes throughout the system, eventually reaching all agents. Clearly, the main features of the whole process will depend on how we model the mechanism of imitation as well as the way in which the agents are connected.

To be more precise, let us explicitly enumerate the dynamical rules governing the behavior of the individuals. Each site (or agent) $i$ is characterized by a real variable $a_i$. In a general way we can consider this quantity as a characteristic of a given individual that other agents might want to imitate. When an agent has adopted a new characteristic, her neighbors become aware of the change and balance their interest (quantified as $a_i - a_j$) with their resistance to change $C$ to decide if they would like to imitate this change. In this way $C$ controls the mechanism of imitation. This parameter is constant and the same for all the agents in the system.

The dynamics can be summarized as follows [11]:

(1) The system is asynchronously updated, that is, at each time step a randomly selected agent $a_i$ updates her state

$$a_i \Rightarrow a_i + \Delta_i,$$

(1)
where $\Delta_i$ is a random variable exponentially distributed with mean $\lambda$. This driving process accounts for the external pressure that may lead an individual to spontaneously adopt a new characteristic.

(2) All agents $j \in \Gamma(i)$, where $\Gamma(i)$ is the set of neighbors of agent $i$, decide whether they want to upgrade or not, according to the following rule

$$a_i - a_j \geq C \Rightarrow a_j = a_i,$$

(2)

(3) If any $j \in \Gamma(i)$ has decided to imitate, we also let her neighbors decide whether they want to imitate this behavior or not. In this way the information of an update may spread beyond the first neighbors of the originally perturbated site. This procedure is repeated until no one else wants to change, concluding an avalanche of imitation events. In this way we have assumed that the time scale of the imitation process is much shorter than the one corresponding to the external driving.

Regardless of the particular connectivity pattern between agents, there are some common trends that helps us to understand intuitively the collective behavior of the system. Hence, it is easy to understand the role of the parameter $C$ and see how different dynamical regimes come out in a natural way. For small $C$ there is almost no resistance to change, and the information spreads easily through the system. Avalanches involve a large number of units and therefore agents are continually updating their state, although by small amounts. On the other hand, for large $C$, since the resistance to change is high, the number of units involved in the diffusion process is rather small which implies a localized region of propagation for most of the updates. As a consequence, distant agents can present very different characteristics, and only in few occasions they are able to find out about their differences. When this happens the changes (variation of $a_i$) adopted will be large. For intermediate values of $C$ the behavior is rather complex and the most interesting from a collective standpoint.

A desirable situation would be one in which most of the time the agents adopt changes that lead to large advances. In this context the following question arises: what is the resistance to change that permits the agents to reach a given average level with a minimum number of upgrades? By measuring the mean rate of advance $\rho$ it has been shown that there exists a unique value of $C$ for which this optimal regime can be achieved [11]. This quantity is simply defined as the ratio between the total advancement over the total number of upgrades, that is

$$\rho = \lim_{T \to \infty} \rho(T) = \lim_{T \to \infty} \frac{\sum_{t=0}^{T} A(t)}{\sum_{t=0}^{T} s(t)}$$

(3)

where $s(t)$ is the number of sites that have changed (i.e. the avalanche size).
at time $t$, and $A(t) = \sum_i (a_i(t) - a_i(t - 1))$ is the advance triggered by the avalanche. The optimal growth regime is characterized by the presence of a peak, $\rho_{\text{max}}$, in the mean rate of advance for a given value of the parameter $C$. This peak scales with system size as a power law $\rho_{\text{max}} \sim N^\alpha$. This property is very convenient in certain frameworks. For example, from an economic point of view the model displays scale effects, i.e., large economies grow faster.

So far, we have considered general aspects of the model. Since the main goal of this paper is to analyze the interplay between underlying structure and dynamics, let us put some attention in the influence of the connectivity pattern on the collective properties of the system and, in particular, in macroscopic observables such as $\rho$. There are two simple cases which are systematically analyzed: a fully connected network and a regular lattice. In the globally coupled case the information referent to any change elicited in an arbitrary position of the network is immediately available to every other agent. As a consequence, the state of all the agents $a_i$ is bound in a gap of width $C$. This limits the advance of any agent to a maximum of $C$. On the other hand, when the system is defined on a 1D ring this limitation only applies to the nearest neighbors. If the information spreads beyond these neighbors, the advance achieved can exceed $C$. In this extreme cases different qualitative behaviors are observed, which can be quantified by the exponent $\alpha$. When one considers the dynamics of the model on a ring $\alpha = 0.20(2)$ while mean field calculations and numerical simulations of the dynamics of the model on a fully connected network show that $\alpha = 0.50$ [11].

Summarizing, we have presented a model that presents an optimal regime which can be characterized by the mean rate of advance $\rho$. The scaling properties of this magnitude can be used to quantitatively distinguish among different dynamical behaviors. These dynamical behaviors have been observed in two particular underlying networks. In the next section we will consider more general structures and analyze their effects on the dynamical properties of the model.

### 3 Scaling analysis

The two particular cases considered so far are usually chosen either for their numerical simplicity, as in the ring, or because they allow for simple mean field calculations as in the fully connected network. Clearly, the structure of both cases is far from the much more complex pattern of interactions observed in realistic systems. However, they appear as paradigms of two opposite generic situations either a scenario where the propagation of information can be constrained to a local neighborhood, or one where it may reach the whole population in just one step. One wonders which will be the dominant features
It is well known that, starting from a ring, the random addition of a few links produces changes in the properties of the network, such as a rapid drop in the average distance between nodes, maintaining the local structure [13]. It has been shown that this feature can be related to significant changes in the dynamical properties in some systems [5,6]. In order to study what effects may be present in our model, we will consider the dynamics on a ring lattice with $N$ vertices and $k = 2$ edges per vertex, adding a new link at random with probability $p$ per edge [14]. In general, when one modifies the underlying structure, a quantitative variation of the numerical values of the dynamical magnitudes that characterize the system is observed. However, we will focus our attention on whether the scaling properties are modified, since they are an indication of qualitative changes in these properties. In order to quantify the changes in the scaling behavior of the system we have computed the exponent $\alpha$ for different values of $p$, as shown in Fig. 3.

The main conclusion that can be extracted from the figure is that the addition of new links does not modify substantially the scaling properties of the system. For small $p$ we observe a small decrease in the value of $\alpha$, effect that seems to be correlated with the rapid drop in the average distance between agents. As $p$ increases to $p = 1$ a slight growth of $\alpha$ is reported. It is important to understand why the qualitative behavior of the system is similar to the one characteristic of the ring. In this case, an avalanche propagates through steps in which the information reaches the nearest neighbors of a modified site. To generate a large event or simply to reach agents far away from the initial updated unit a large number of steps is required. In this sense, the information propagates by a local process. The addition of new links allows the information to reach agents through short cuts, but it does not change the
mean mechanism of diffusion, i.e. in order to proceed further, an avalanche still requires a large number of steps, it is still dominated by a local process. One cannot under-stress the fact that for the imitation strategy described in this paper, in contrast to what is observed in other models, the characteristic behavior of the ring is dominant even when a more general structure, such as a small world network, is considered.

The local character of the diffusion process typical of low connectivity networks is lost when considering densely connected networks. In this situation the information about the state of a given agent is available to any other agent in just a few steps. The small world construction is far away from this limit, even when $p = 1$ the mean number of links per site, i.e. the connectivity of the system, has increased very slightly. In order to reach a highly connected state we added links randomly to the ring up to a fraction $f$ of the total possible links in the system. This means adding $f[N(N-1)/2] - N$ connections to the ring. This recipe allows us to interpolate between the ring ($f = 0$) and the fully connected model ($f = 1$). For finite values of $f$ the system corresponds to a ring with a superimposed random graph. For this construction the connectivity of the system will be $fN$, and thus, for a fixed value of $f$, the connectivity will increase as $N$ grows. If this plays a significant role in the dynamical behavior of the system, we expect that it will be reflected in an important quantitative change in $\alpha$. In Fig. 3 we present the behavior of the peak of the mean rate of advance $\rho_{\text{max}}$ as a function of system size $N$ when $f = 0.005$. For low values of $N$ the behavior resembles the one observed in the small world case. As $N$ grows there is a crossover to the fully connected behavior and $\rho_{\text{max}} \sim N^{0.5}$.

To complete the analysis we have performed a finite size study of the crossover between both regimes. The results are presented in Fig. 3. As one can expect, and in agreement with the previous discussion, the jump from the ring-like scaling behavior to the highly connected network scaling behavior becomes
Fig. 3. Exponent $\alpha$ vs $f$. Finite size effects are clearly present. The maximum system sizes used in the fits are $N = 256$ (circles), $N = 512$ (squares) and $N = 1024$ (triangles). The lines are a guide to the eye.

sharper as $N$ grows being the crossover located at $f \approx \frac{1}{N}$. Numerical simulations support this result suggesting that in the thermodynamic limit a transition to $\alpha = 0.5$ should be present for any non-vanishing value of $f$, i.e. provided the connectivity of the system grows with the system size. Therefore, by using this construction we have found that there are two dominant behaviors, one for networks with a number of links scaling with the size of the system, $O(N)$, and another where the connectivity grows with the size of the system, i.e., the number of links go as $O(N^2)$. It will be very interesting to analyze the situation for other complex situations such as scale free networks. This is currently under study and will be published elsewhere.

We have remarked the fact that the behaviors observed in the ring and the fully connected network were generic regarding the way the information propagates through the system. We found that these dynamical behaviors are dominant in more complex structures. In a small world network the dynamics is dominated by local processes. In order to reach the qualitative behavior observed in a fully connected network a higher connectivity is necessary. In fact, we have shown that by introducing a mechanism which takes this effect into account we were able to reach this qualitative behavior. The identification of the dominant behaviors of our model in more general networks is an interesting issue in itself, but it is also an essential step in exploiting the interplay between dynamics and structure to reach an optimal regime. Precisely, to consider both elements in a unified framework is the main purpose of this work. The following section is devoted to the investigation of our model under this new exciting point of view.
4 Identification of an optimal behavior

Up to now, we have analyzed different mechanisms for the spreading of the information. On one hand, $C$ appears as a parameter that can be tuned in order to reach an optimal regime for a fixed underlying network. On the other hand, we have also seen that diverse structures lead to different ways in which the information spreads. This line of reasoning naturally leads to the question of which is the structure that gives the optimal regime for a fixed value of $C$. For small $C$ there is almost no resistance to change and the information easily spreads. In fact we expect that if $C \to 0$ then $\rho$ will also decrease independently of the underlying network. On the other extreme, for $C \to \infty$ the behavior of the system will resemble a random deposition process, and again the behavior of $\rho$ is expected to decrease when considering any general structure. To analyze what happens for intermediate values of $C$ we should proceed with care as we will see immediately.

In order to do this we consider a system with a certain distribution of couplings between agents and a fixed $C$. In general, the value of $\rho$ will be smaller than $\rho_{\text{max}}$. Now, suppose that we can modify the connectivity pattern in order to reach the optimal behavior. What should we do? Is it better to increase the number of links and tend to a densely connected network or, on the contrary, we should go towards a regular structure with a small number of connections per site? The question cannot be answered properly without looking at the parameter that control the dynamics, i.e., the strategy to follow depends precisely on $C$. To optimize the behavior of the system one cannot split the problem in two independent parts; dynamics and the underlying structure must be considered as a whole.

Let us analyze some features associated to the mechanism of imitation and consider the difference between the highest ($a_{\text{max}}$) and the lowest ($a_{\text{min}}$) characteristic values in the system. For sufficiently high connectivity this gap will be limited to $a_{\text{max}} - a_{\text{min}} \leq C$ and, as a consequence, for these systems the value of $\rho$ cannot exceed $C$. On the other hand, when the propagation of the information is constrained to advance through local processes, a more heterogeneous profile of characteristic values can be formed, allowing for a larger gap between $a_{\text{max}}$ and $a_{\text{min}}$. In this case, avalanche events consisting of a large number of steps will produce a large advance, allowing the value of $\rho$ to exceed $C$. In fact, by using the probability distribution of avalanches $P(s)$ and advances $P(H)$ obtained using numerical simulations in [11] this can be easily verified analytically. In this situation a sparsely connected network will necessarily have a greater $\rho$ than a highly connected one. For increasing values of $C$, avalanche events will easily get blocked in a few steps. In this context, large advances in a sparsely connected network will become very rare. More frequent advances will be observed in a highly connected network since many agents
Fig. 4. Mean rate of advance $\rho$ vs. $C$ for a fixed system size $N = 512$ and for $f = 0.00$ and $f = 0.0025$. The inset presents the behavior of $\rho$ vs. $f$ for $C = 2.0$ and $C = 4.0$. 

are permitted to find out about an update in any step. This situation offers the possibility for a higher $\rho$ to be observed in highly connected structures.

Following this analysis we have considered a general system of fixed size $N$. The evolution of the mean rate of advance $\rho$ vs $C$ has been studied for two different situations: the ring and another structure where the number of links (measured in terms of $f$) is large enough to observe fully connected behavior. The results are illustrated in Fig. 4. Note that as $f$ is varied the qualitative shape of the $\rho$ curve is similar. However, the position of the peak corresponds to different values of $C$. For increasing values of $f$ the peak corresponds to larger values of $C$, eventually reaching the curve corresponding to the globally coupled case. It is important to stress that, as $C$ is varied, the optimal network may change from a highly connected to a sparsely connected one. This behavior is clearly reflected in the inset, where we present the behavior of $\rho$ vs. $f$ for two different values of $C$. For $C = 2$ a decrease in $\rho$ is observed as $f$ grows. The addition of links is harmful for the system. On the other hand, for $C = 4.0$ the opposite behavior is observed, and $\rho$ increases its value as $f$ grows. Clearly, in this case, the addition of links is beneficial.

These results show that in order to optimize the behavior of the system a non trivial combination of dynamical rules and underlying structure should be considered. When the interplay between both allows for $\rho/C > 1$, a sparsely connected structure performs better than a highly connected one. On the other hand, when $\rho/C < 1$ the opposite is true. Note that when $\rho/C \sim 1$ the behavior should be independent of the underlying network. In fact, Fig 4 shows that when $\rho/C \sim 1$ both curves intersect. Numerical simulations for different $N$ and $f$ also show these qualitative behaviors.
5 Conclusions

The search for the essential mechanisms that govern the dynamics of real complex systems has led recent efforts to highlight the role of the complex underlying structures present in these systems. A large amount of work has been devoted to the characterization and identification of their main features and the role they play on the dynamical properties. This scenario suggests that a more general framework, devised to address these issues, should consider the interplay between the dynamics and the underlying structure as a single ingredient.

The main aim of this work has been to take a first step into this direction. With this picture in mind we analyzed a model of diffusion by imitation which presents a very interesting property, the possibility of defining an optimal regime by tuning a parameter. This regime can be quantified by a macroscopic observable. In order to be able to consider the interplay between dynamics and structure we have studied the behavior of the model on general networks and characterized the dominant behaviors in these structures. An analysis of how the optimal regime can be achieved has led to the conclusion that the dynamical rules and the underlying structure cannot be considered as separate ingredients.

The rich dynamical behaviors observed in the more general framework makes a detailed analysis of the model necessary. In particular, we are focusing our attention on the behavior when $f \to 0$, in order to characterize a possible transition. Since the connectivity plays an important role in the properties of the system, a natural extension to more general structures, such as scale-free networks, is under consideration. Finally, a very interesting new point of view which we are also analyzing concerns the behavior of the model on dynamical networks. Although we cannot underestimate the results that will follow from these studies we believe that the main results presented in this work will not be affected in their generality.

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