Assessing relative volatility/intermittency/energy dissipation

Ole E. Barndorff-Nielsen
The T.N. Thiele Centre for Mathematics in Natural Science
CREATES and Department of Mathematics, Aarhus University
Ny Munkegade 118, 8000 Aarhus C, Denmark
e-mail: oebn@imf.au.dk

Mikko S. Pakkanen*
CREATES and Department of Economics and Business, Aarhus University
Fuglesangs Allé 4, 8210 Aarhus V, Denmark
e-mail: mpakkanen@econ.au.dk

and

Jürgen Schmiegel
The T.N. Thiele Centre for Mathematics in Natural Science
and Department of Engineering, Aarhus University
Finlandsvej 22, 8200 Aarhus N, Denmark
e-mail: schmiegl@imf.au.dk

Abstract: We introduce the notion of relative volatility/intermittency and demonstrate how relative volatility statistics can be used to estimate consistently the temporal variation of volatility/intermittency when the data of interest are generated by a non-semimartingale, or a Brownian semistationary process in particular. This estimation method is motivated by the assessment of relative energy dissipation in empirical data of turbulence, but it is also applicable in other areas. We develop a probabilistic asymptotic theory for realised relative power variations of Brownian semistationary processes, and introduce inference methods based on the theory. We also discuss how to extend the asymptotic theory to other classes of processes exhibiting stochastic volatility/intermittency. As an empirical application, we study relative energy dissipation in data of atmospheric turbulence.

MSC 2010 subject classifications: Primary 62M09; secondary 76F55.
Keywords and phrases: Brownian semistationary process, energy dissipation, intermittency, power variation, turbulence, volatility.

Received February 2014.

Contents

1 Introduction .................................................. 1997
2 Brownian semistationary processes and realised volatility/intermittency 1998

*Corresponding author.
1. Introduction

The concept of volatility expresses the ubiquitous phenomenon that observational fields exhibit more variation than expected; that is, more than the most basic type of random influence\footnote{Often thought of as Gaussian.} envisaged.

Accordingly, volatility is a relative concept, and its meaning depends on the particular setting under investigation. Once that meaning is clarified the question is how to assess the volatility empirically and then to describe it in stochastic terms and incorporate it in a suitable probabilistic model.

The ‘additional’ random fluctuations denoted as volatility/intermittency, generally vary, in time and/or in space, in regard to Intensity (activity rate and duration) and Amplitude. Typically the volatility/intermittency may be further classified into continuous and discrete (i.e., jumps) elements, and long and short term effects.

In turbulence and certain other areas of study the phenomenon is referred to as intermittency (Frisch, 1995, Chapter 8) rather than volatility. Energy dissipation is a key concept in the statistical theory of turbulence, and is in the character of a specific type of intermittency.

In finance the investigation of volatility is well developed and many of the procedures of probabilistic and statistical analysis applied (Barndorff-Nielsen and Shephard, 2010) are similar to those of relevance in turbulence.

In this paper, we introduce the notion of relative volatility/intermittency and the closely related statistics, realised relative power variations. They pave the way for practical applications of some recent advances in the asymptotic theory of power variations of non-semimartingales (see, e.g., Corcera, Nualart and
Woerner (2006) and Barndorff-Nielsen, Corcuera and Podolskij (2011, 2013)) to volatility/intermittency measurements and inference with empirical data.

In the non-semimartingale setting, realised power variations need to be scaled properly, in a way that depends on the smoothness of the process through unknown parameters, to ensure convergence. Realised relative power variations, however, are self-scaling and, moreover, admit a statistically feasible central limit theorem, which can be used, e.g., to construct confidence intervals for the realised relative volatility/intermittency. (Self-scaling statistics have also been recently used by Podolskij and Wasmuth (2013) to construct a goodness-of-fit test for the volatility coefficient of a fractional diffusion.)

This paper is organised as follows. Section 2 presents some results from the theory of Brownian semistationary processes that are pertinent to assessment of volatility/intermittency, and the definitions of relative volatility/intermittency and realised relative power variations are given in Section 3. A stable functional central limit theorem for realised relative power variations of Brownian semistationary processes is presented in Section 4. An application to empirical data on atmospheric turbulence is carried out in Section 5, and Section 6 concludes. Appendices contain a discussion of extending the theory beyond Brownian semistationary processes (Appendix A), an alternative method of assessing the volatility/intermittency of a Brownian semistationary process (Appendix B), and some supporting results (Appendix C).

2. Brownian semistationary processes and realised volatility/intermittency

2.1. Probabilistic setup

Brownian semistationary (BSS) processes, introduced by Barndorff-Nielsen and Schmiegel (2009), may be used as models of timewise development of homogeneous and isotropic turbulent velocity fields. More concretely, a BSS process can be used to describe the velocity at a fixed point in space and in the main direction of the flow in a turbulent field. While the original motivation for BSS processes arose out of a study in turbulence, these processes have since found widespread interest in regard to their theoretical properties and to applications beyond physics, including, e.g., modelling of electricity price dynamics (Barndorff-Nielsen, Benth and Veraart, 2013).

A generic BSS process $Y = \{Y_t\}_{t \geq 0}$ is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$ via the decomposition

$$ Y_t = X_t + A_t, \quad (2.1) $$

where the process

$$ X_t = \int_{-\infty}^{t} g(t-s)\sigma_s dB_s, \quad t \geq 0, \quad (2.2) $$
is constructed from a standard Brownian motion $B = \{B_t\}_{t \in \mathbb{R}}$ and a non-zero\(^2\) càglàd volatility/intermittency process $\sigma = \{\sigma_t\}_{t \in \mathbb{R}}$, both of which are adapted to $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$, and using a square integrable kernel $g : (0, \infty) \to \mathbb{R}$ such that

$$\int_{-\infty}^{t} (t-s)^2 \sigma_s^2 \, ds < \infty \quad \text{a.s.}$$

for all $t \geq 0$. This condition ensures the existence of the stochastic integral in (2.2). In the decomposition (2.1), $A = \{A_t\}_{t \geq 0}$ is a process that allows for skewness in the distribution of $Y_t$. The process $A$ is assumed to fulfill one of two negligibility conditions, viz. (2.7) and (4.3) given below (Appendix C presents more concrete criteria that can be used to check these conditions).

**Example 2.1.** In the context of turbulence, the *gamma kernel*

$$g(t) = ct^\nu e^{-\lambda t}, \quad t > 0, \quad (2.3)$$

where $c > 0$, $\nu > \frac{1}{2}$, and $\lambda > 0$, has a special role. In particular, if $\nu = \frac{5}{6}$ and $\sigma$ is stationary with $\mathbb{E}\{\sigma_0^2\} < \infty$, then the autocorrelation function of $X$ is identical to von Kármán’s autocorrelation function (von Kármán, 1948) for ideal turbulence and also belongs to the Whittle–Matérn family of correlation functions (Guttorp and Gneiting, 2005). The parameter value $\nu = \frac{5}{6}$ agrees with Kolmogorov’s (K41) scaling law of turbulence (Kolmogorov, 1941a,b).

**Example 2.2.** The process $A$ can be specified as

$$A_t = \mu + \int_{-\infty}^{t} q(t-s) \sigma_s^2 \, ds, \quad t \geq 0, \quad (2.4)$$

where the kernel $q$ belongs to $L^1((0, \infty))$, which makes the integral in (2.4) convergent under the assumption $\sup_{t \in \mathbb{R}} \mathbb{E}\{\sigma_t^2\} < \infty$. In particular, $q$ can be chosen to be of the gamma form (2.3). Lemma C.1 in Appendix C provides sufficient conditions for the process $A$ to be negligible in the sense of conditions (2.7) and (4.3) when $q$ is a gamma kernel.

### 2.2. Assessing volatility/intermittency

In relation to the $BSS$ process $Y$, a central question is that of determining the dynamics of volatility/intermittency $\sigma$ from $Y$. If $X$ were a semimartingale and $A$ of finite variation, then the answer would be given by the quadratic variation $[Y]$ of $Y$. In fact, if

$$g(0+) < \infty \quad \text{and} \quad g' \in L^2((0, \infty)), \quad (2.5)$$

then $X$ is a semimartingale with $[X]_t = g(0+) \sigma_t^{2+}$ for any $t \geq 0$, where

$$\sigma_t^{2+} = \int_0^t \sigma_s^2 \, ds$$

\(^2\)More precisely, a.e. sample path is not equal to zero on a set with positive Lebesgue measure.
is the accumulated volatility/intermittency (Barndorff-Nielsen and Schmiegel, 2009). Assuming normalisation \(|g(0+)| = 1\), given a set of equidistant discrete observations of \(Y\) at time points \(0, \delta, \ldots, \lfloor t/\delta \rfloor \delta\), where \(\delta > 0\), the accumulated volatility \(\sigma^2_{t+}\) can then be estimated consistently as the limit in probability for \(\delta \to 0\) of the realised quadratic variation

\[
[Y_{\delta}]_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} (Y_{j\delta} - Y_{(j-1)\delta})^2.
\]

More generally, the volatility/intermittency functional \(\sigma^p_{t+} = \int_0^t |\sigma_s|^p \, ds\) for any \(p > 0\) can be estimated consistently as \(\delta \to 0\) using the realised \(p\)-th order power variation

\[
[Y_{\delta}]^{(p)}_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} |Y_{j\delta} - Y_{(j-1)\delta}|^p
\]

rescaled by \(\frac{\delta^{1-p}}{m_p}\), where \(m_p = \mathbb{E}\{|\xi|^p\}\) with \(\xi \sim N(0, 1)\), see Barndorff-Nielsen et al. (2006).

Whenever the process \(\sigma\) is not identically equal to zero, the condition (2.5) is both sufficient and necessary for \(X\) to be a semimartingale. However, in many interesting situations (2.5) does not hold and thus \(X\) is not a semimartingale. They include the case where \(g\) is a gamma kernel with \(\nu \in (\frac{1}{2}, 1) \cup (1, \frac{3}{2})\), which is of interest for turbulence. Then, in order to determine \(\sigma^2_{t+}\) by a limiting procedure from the realised quadratic variation \(Y_{\delta}\), the latter has to be rescaled by a factor depending on \(\delta\) and the scaling properties of \(X\). Specifically, as shown by Barndorff-Nielsen and Schmiegel (2009), the appropriate scaling can be described using the second-order structure function (or variogram)

\[
R(t) = \mathbb{E}\{(G_t - G_0)^2\}, \quad t \geq 0,
\]

of the Gaussian core \(G\) of \(X\) defined by \(G_t = \int_{-\infty}^t g(t-s) \, dW_s\), \(t \geq 0\).

Let us now recall the general version of the law of large numbers for power variations of \(BSS\) processes, due to Barndorff-Nielsen, Corcuera and Podolskij (2011). To this end, we need to introduce some conditions concerning the kernel \(g\) and the volatility/intermittency process \(\sigma\). Below \(L_f : (0, \infty) \to \mathbb{R}\) stands for a function that is slowly varying at zero, indexed by a given function \(f\). Recall that slow variation at zero requires that \(\lim_{u \to 0^+} L_f(ut)/L_f(t) = 1\) for any \(u > 0\).

**Assumption 2.3.** For some \(\nu \in (\frac{1}{2}, 1) \cup (1, \frac{3}{2})\), the kernel \(g\) and the process \(\sigma\) satisfy:

\(\text{(i) } g(t) = x^{\nu-1}L_g(t)\).

\(\text{(ii) } g'(t) = x^{\nu-2}L_{g'}(t)\) and \(g' \in L^2((\varepsilon, \infty))\) for any \(\varepsilon > 0\). Moreover, \(|g'|\) is non-decreasing on \((a, \infty)\) for some \(a > 0\).

\(\text{(iii) } \int_1^\infty g'(s)^2 \sigma^2_{t-s} \, ds < \infty\) a.s. for any \(t > 0\).
Moreover, the second-order structure function $R$ satisfies:

(iv) $R(t) = t^{2\nu-1} L_R(t)$.
(v) $R''(t) = t^{2\nu-3} L_R''(t)$.
(vi) For some $b \in (0, 1)$,

$$
\limsup_{s \downarrow 0} \sup_{t \in [s, s^b]} \left| \frac{L_R''(t)}{L_R(s)} \right| < \infty.
$$

Example 2.4. If $g$ is the gamma kernel (2.3) with $\nu \in (\frac{1}{2}, 1) \cup (1, \frac{3}{2})$ and $\sup_{t \in \mathbb{R}} \mathbb{E}\{\sigma_t^2\} < \infty$, then Assumption 2.3 is in force, see Barndorff-Nielsen, Corcuera and Podolskij (2011, pp. 1173).

Remark 2.5. Under Assumption 2.3, the process $X$ is not a semimartingale, unless $\sigma$ is identically equal to zero. The parameter $\nu$ describes the smoothness of the process $X$ and is analogous to the Hurst parameter of fractional Brownian motion. In fact, the increments of the Gaussian core $G$ over short time intervals are ‘close’ to increments of fractional Brownian motion with Hurst parameter $\nu - \frac{1}{2}$, see Corcuera et al. (2013, p. 2557).

The following statement is a special case of Theorem 3 of Barndorff-Nielsen, Corcuera and Podolskij (2011) that provides a law of large numbers for multi-power variations of BSS processes.

Theorem 2.6. Let $p > 0$. Suppose that Assumption 2.3 holds and that the process $A$ satisfies the negligibility condition

$$
\frac{\delta}{R(\delta)^{\frac{p}{2}}} [A_\delta]_{t}^{(p)} \xrightarrow{p \delta \rightarrow 0} 0 \quad \text{for any } t \geq 0, \quad (2.7)
$$

where $[A_\delta]_{t}^{(p)}$ is defined analogously to (2.6). Then,

$$
\frac{\delta}{R(\delta)^{\frac{p}{2}}} [Y_\delta]_{t}^{(p)} \xrightarrow{p \delta \rightarrow 0} m_p \sigma_t^{p +} \quad \text{for any } t \geq 0.
$$

Remark 2.7. Assumption 2.3 (iv) implies, by Potter’s bounds for slowly varying functions (Bingham, Goldie and Teugels, 1987, Theorem 1.5.6), that for any $\varepsilon > 0$ and $t_0 \in (0, 1)$ there exist $C, C' > 0$ such that

$$
C t^{2\nu-1+\varepsilon} \leq R(t) \leq C' t^{2\nu-1-\varepsilon} \quad (2.8)
$$

for any $t \in [0, t_0)$. Then, the negligibility condition (2.7) holds if

$$
[A_\delta]_{t}^{(p)} = O_p(\delta^\gamma)
$$

for any $\gamma > p(\nu - \frac{1}{2}) - 1$. Another consequence of (2.8) is that under the assumptions of Theorem 2.6 the ‘raw’ realised quadratic variation $[Y_\delta]_{t}$ satisfies

$$
[Y_\delta]_{t} \xrightarrow{p \delta \rightarrow 0} \begin{cases} 
0, & \nu \in (1, \frac{3}{2}), \\
\infty, & \nu \in (\frac{1}{2}, 1).
\end{cases}
$$
(In the critical case $\nu = 1$ the limit of $[Y_{\delta}]_t$ is indeterminate, unless we have more information on the slowly varying part $L_R$ of the structure function $R$ near zero.)

3. Realised relative volatility/intermittency

3.1. Consistent estimation of relative volatility/intermittency

Using Theorem 2.6 for estimation of the accumulated volatility $\sigma_t^{2+}$ requires knowledge of the scaling factor $\delta/R(\delta)^{2}$. More precisely, the behaviour of the second-order structure function $R$ near zero should be known or determinable from data with sufficient accuracy. We discuss the viability of estimation of the scaling factor in Appendix B.

However, instead of the precise value of $\sigma_t^{2+}$ for fixed $t$, we are often more interested in measuring the dynamics of $\sigma_t^{2+}$, as a function of $t$, in relative terms. That is, for $T > 0$ we are interested in the relative volatility/intermittency process

$$\tilde{\sigma}_{t,T}^{2+} = \frac{\sigma_t^{2+}}{\sigma_T^{2+}}, \quad 0 \leq t \leq T,$$

which captures the variation of $\sigma_t^{2+}$ in $t$ but loses the original scale of measurement. Clearly, under the assumptions of Theorem 2.6, we may estimate $\tilde{\sigma}_{t,T}^{2+}$ consistently using the realised relative quadratic variation of $Y$,

$$[Y_{\delta}]_{t,T} = \frac{[Y_{\delta}]_t}{[Y_{\delta}]_T},$$

that is, $[Y_{\delta}]_{t,T} \xrightarrow{P} \tilde{\sigma}_{t,T}^{2+}$ as $\delta \to 0$. The realised relative quadratic variation $[Y_{\delta}]_{t,T}$ is entirely empirically determined, self-scaling, and its consistency does not require information on the second-order structure function $R$.

More generally, for any $p > 0$, the relative volatility/intermittency functionals

$$\tilde{\sigma}_{t,T}^{p+} = \frac{\sigma_t^{p+}}{\sigma_T^{p+}}, \quad 0 \leq t \leq T,$$

(3.1)

can be estimated consistently using the realised $p$-th order relative power variations

$$[Y_{\delta}]_{t,T}^{(p)} = \frac{[Y_{\delta}]_{t}^{(p)}}{[Y_{\delta}]_{T}^{(p)}}, \quad 0 \leq t \leq T,$$

as outlined in the following result.

**Theorem 3.1.** Let $p > 0$. Suppose that Assumption 2.3 holds and that the process $A$ satisfies (2.7). Then for any $T > 0$,

$$[Y_{\delta}]_{t,T}^{(p)} \xrightarrow{P \delta \to 0} \tilde{\sigma}_{t,T}^{p+}$$

(3.2)

uniformly in $t \in [0,T]$. 

Proof. Pointwise convergence in (3.2) follows immediately from Theorem 2.6. It remains to note that the convergence is uniform since the sample paths of \( \{Y_{t,T}^{(p)} \}_{0 \leq t \leq T} \) are non-decreasing and since \( \{\tilde{\sigma}_{t,T}^{+} \}_{0 \leq t \leq T} \) is a continuous process.

3.2. Connection to relative energy dissipation in turbulence

Let us briefly consider the interpretation of relative volatility/intermittency from the point of view of physics. In the classical theory of turbulence (see, e.g., Frisch, 1995), velocity fields are assumed to be differentiable — that is, in place of a BSS process \( Y \) we would consider a differentiable function \( y : [0, T] \rightarrow \mathbb{R} \) describing the velocity component in the main direction of the flow. Then, for \( t \in [0, T] \), the surrogate energy dissipation of \( y \) at time \( t \) is defined as

\[
\varepsilon(t) = y'(t)^2
\]

and the coarse-grained energy dissipation of \( y \) over the interval \([0, t]\) as

\[
\varepsilon^+(t) = \int_0^t y'(s)^2 \, ds.
\]

Using the mean value theorem, it is easy to show that the realised quadratic variation of \( y \), viz. \([y_{\delta}]_t\), is connected to \( \varepsilon^+(t) \) via the convergence

\[
[y_{\delta}]_t \to_{\delta \to 0} \varepsilon^+(t).
\]

Thus, we find that the realised relative quadratic variation \( \widetilde{[y_{\delta}]}_{t,T} \) satisfies

\[
\widetilde{[y_{\delta}]}_{t,T} \to_{\delta \to 0} \frac{\varepsilon^+(t)}{\varepsilon^+(T)},
\]

where the limit is the relative energy dissipation of \( y \) over the subinterval \([0, t]\) of \([0, T]\). Within the turbulence literature, this definition of the relative energy dissipation is strongly related to the definition of a multiplier in the cascade picture of the transport of energy from large to small scales (see Cleve, Schmiegel and Greiner (2008) and references therein).

Motivated by this discussion, in the turbulence context we interpret \( \tilde{\sigma}_{t,T}^{2+} \) as the relative energy dissipation of \( Y \) over \([0, t] \subset [0, T]\).

4. Central limit theorem for realised relative power variations

4.1. Stable convergence

We are about to derive a stable central limit theorem for realised relative power variations of BSS processes. To this end, recall first that random elements
$U_1, U_2, \ldots$ in some metric space $U$ converge stably (in law) to a random element $U$ in $U$, defined on an extension $(\Omega', \mathcal{F}', P')$ of the underlying probability space $(\Omega, \mathcal{F}, P)$, if

$$E\{f(U_n)V\} \xrightarrow[n \to \infty]{} E'\{f(U)V\}$$

for any bounded, continuous function $f : U \to \mathbb{R}$ and bounded random variable $V$ on $(\Omega, \mathcal{F}, P)$. We write then $U_n \overset{st}{\to} U$. Stable convergence, introduced by Rényi (1963), is stronger than ordinary convergence in law and weaker than convergence in probability. It is essential to note that the limiting random element $U$ is defined on an extension of the original probability space. In fact, when $U$ is $\mathcal{F}$-measurable, the convergence $U_n \overset{st}{\to} U$ is equivalent to $U_n \overset{P}{\to} U$ (Podolskij and Vetter, 2010, Lemma 1).

**Remark 4.1.** The usefulness of stable convergence can be illustrated by the following example that is pertinent to the asymptotic results below. Suppose that $U_n \overset{st}{\to} \theta \xi$ in $\mathbb{R}$, where $\xi \sim N(0,1)$ and $\theta$ is a positive random variable independent of $\xi$. In other words, $U_n$ follows asymptotically a mixed Gaussian law with mean zero and conditional variance $\theta^2$. If $\hat{\theta}_n$ is a positive, consistent estimator of $\theta$, i.e., $\hat{\theta}_n \overset{P}{\to} \theta$, then the stable convergence of $U_n$ allows us to deduce that $U_n/\hat{\theta}_n \overset{d}{\to} N(0,1)$. We refer to Rényi (1963), Aldous and Eagleson (1978), Jacod and Shiryaev (2003, pp. 512–518), and Podolskij and Vetter (2010, pp. 332–334) for more information on the properties of stable convergence.

### 4.2. Stable functional central limit theorem

As a preparation for the stable central limit theorem for realised relative power variations, we recall the stable central limit theorem for realised power variations of BSS processes, due to Barndorff-Nielsen, Corcuera and Podolskij (2011). As usual, the central limit theorem requires somewhat stronger assumptions than the corresponding law of large numbers (Theorem 2.6). In particular, we need to control the Hölder regularity of the volatility/intermittency process $\sigma$ as follows.

**Assumption 4.2.** There exists a constant $\gamma > \frac{1}{2}$ such that for any $q > 0$ and $T > 0$,

$$E\{|\sigma_t - \sigma_s|^q\} \leq C_{q,T}|t-s|^{\gamma q}, \quad s, t \in [0,T],$$

where $C_{q,T} > 0$ is a constant that may depend on $q$ and $T$.

In what follows, we write $D([0,T])$ for the space of càdlàg functions from $[0,T]$ to $\mathbb{R}$, endowed with the usual Skorohod metric (Jacod and Shiryaev, 2003, Chapter V). (Recall, however, that convergence to a continuous function in this metric is equivalent to uniform convergence.) We also introduce a function $\lambda_p : (\frac{1}{2}, \frac{3}{4}) \to (0, \infty)$ given by

$$\lambda_p(\nu) = \sum_{l=2}^{\infty} l a_l^2 \left( 1 + 2 \sum_{j=1}^{\infty} \rho(\nu) j^l \right), \quad (4.1)$$
Assessing relative volatility

where $a_2, a_3, \ldots$ are the coefficients in the expansion of the function $u_p(x) = |x|^p - m_p$, $x \in \mathbb{R}$, in second and higher-order Hermite polynomials $x^2 - 1$, $x^3 - 3x$, $\ldots$, satisfying $\sum_{l=2}^{\infty} l a_l^2 < \infty$ (in the case $p = 2$ we have, clearly, $a_2 = 1$ and $a_l = 0$ for all $l > 2$). The sequence $(\rho_{\nu}(j))_{j=1}^\infty$ is the correlation function of fractional Gaussian noise with Hurst parameter $\nu - \frac{1}{2}$, namely

$$\rho_{\nu}(j) = \frac{1}{2}((j + 1)^{2\nu - 1} - 2j^{2\nu - 1} + (j - 1)^{2\nu - 1}), \quad j \geq 1. \quad (4.2)$$

**Theorem 4.3.** Let $p \geq 1$. Suppose that Assumptions 2.3 and 4.2 hold, $\nu \in (\frac{1}{2}, 1)$, and that $A$ satisfies

$$\sqrt{\frac{\gamma}{R(\delta)}} |A_{\eta}^{(p)}| \xrightarrow{P} 0 \quad \text{for any } t \geq 0. \quad (4.3)$$

Then for any $T > 0$,

$$\delta^{-1/2}\left(\frac{\delta}{R(\delta)} |Y_{\delta,t}^{(p)} - m_p \sigma_\delta^+| \xrightarrow{st} \sqrt{\lambda_p(\nu)} \int_0^t |\sigma_s|^p dW_s \right) \text{ in } D([0,T]),$$

where $\{W_t\}_{t \in [0,T]}$ is a standard Brownian motion, independent of the filtration $\{F_t\}_{t \in \mathbb{R}}$.

**Remark 4.4.** The restriction $p \geq 1$ is not necessary, but we introduce it for the sake of simpler exposition. See Theorem 4 of Barndorff-Nielsen, Corcuera and Podolskij (2011) or Theorem 3.2 of Corcuera et al. (2013) for more general versions of Theorem 4.3.

**Remark 4.5.** Using the bounds (2.8), we deduce that, under Assumption 2.3 (iv), the negligibility condition (4.3) holds if

$$[A_{\delta}]^{(p)}_t = O_P(\delta^\gamma)$$

for any $\gamma > p(\nu - \frac{1}{2}) - \frac{1}{2}$.

Building on Theorem 4.3, we can prove the following stable central limit theorem for realised relative power variations of $Y$.

**Theorem 4.6.** Let $p \geq 1$. Suppose that Assumptions 2.3 and 4.2 hold, $\nu \in (\frac{1}{2}, 1)$, and that $A$ satisfies (4.3). Then for any $T > 0$,

$$\delta^{-1/2}\left(\frac{\delta}{R(\delta)} |\tilde{Y}_{\delta,t}^{(p)} - \tilde{\sigma}_{\delta,T}^+| \xrightarrow{st} \sqrt{\lambda_p(\nu)} \int_0^t |\sigma_s|^p dW_s - \tilde{\sigma}_{\delta,T}^+ \int_0^T |\sigma_s|^p dW_s \right) \quad (4.4)$$

in $D([0,T])$, where $\tilde{\sigma}_{\delta,T}^+$ is given by (3.1) and $W$ is a standard Brownian motion as in Theorem 4.3.

Theorem 4.6 follows from Theorem 4.3 by invoking the following simple result concerning the stable convergence of a process that has been normalised by its terminal value.
Lemma 4.7. Let $T > 0$ be fixed and suppose that:

- $Z^n = \{Z^n_i\}_{0 \leq i \leq T}$, for any $n \in \mathbb{N}$, is a process defined on $(\Omega, \mathcal{F}, P)$ with non-decreasing sample paths in $D([0, T])$ such that $Z^n_T \neq 0$ a.s.,
- $Z = \{Z_i\}_{0 \leq i \leq T}$ is a process defined on $(\Omega, \mathcal{F}, P)$ with non-decreasing sample paths in $C([0, T])$ such that $Z_T \neq 0$ a.s.,
- $\xi = \{\xi_t\}_{0 \leq t \leq T}$ is a process defined on an extension $(\Omega', \mathcal{F}', P')$ of $(\Omega, \mathcal{F}, P)$ with sample paths in $C([0, T])$.

If
\[ \sqrt{n}(Z^n_i - Z_i) \xrightarrow{\text{st}}_{n \to \infty} \xi_t \quad \text{in} \quad D([0, T]), \]  
then
\[ \sqrt{n}\left(\frac{Z^n_i - Z_i}{Z^n_T - Z_T}\right) \xrightarrow{\text{st}}_{n \to \infty} \frac{1}{Z_T} (\xi_t - \frac{Z_i}{Z_T} \xi_t) \quad \text{in} \quad D([0, T]). \]

Proof. Since $Z^n$ and $Z$ have non-decreasing sample paths and the sample paths of $Z$ are continuous, we have
\[ \sup_{0 \leq t \leq T} \left| \frac{Z^n_t - Z_t}{Z^n_T - Z_T} \right| \leq \frac{2}{|Z_T|} \sup_{0 \leq t \leq T} |Z^n_t - Z_t| \xrightarrow{p} 0 \]
by (4.5). Due to the properties of stable convergence, we obtain then
\[ \left(\sqrt{n}(Z^n_i - Z_t), \frac{Z^n_i}{Z^n_T} \right) \xrightarrow{\text{st}}_{n \to \infty} \left(\xi_t, \frac{Z_t}{Z_T} \right) \quad \text{in} \quad D([0, T])^2. \]  

Let us now consider the decomposition
\[ \sqrt{n}\left(\frac{Z^n_i - Z_t}{Z^n_T - Z_T}\right) = \frac{1}{Z_T} \left(\sqrt{n}(Z^n_i - Z_i) - \sqrt{n}(Z^n_T - Z_T) \frac{Z^n_i}{Z^n_T}\right). \]

Using again the fact that convergence to a continuous function in $D([0, T])$ is equivalent to uniform convergence, it follows that the map $(x, y) \mapsto x - x(T)y$ from $D([0, T])$ to $D([0, T])$ is continuous on $C([0, T])^2$. Since $\xi$ and $Z$ have continuous sample paths, the assertion follows from (4.6) and the properties of stable convergence. \qed

For practical applications, we need a statistically feasible version of Theorem 4.6. Conditional on $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$, the limiting process on the right-hand side of (4.4) is a Gaussian bridge. In particular, its (unconditional) marginal law at time $t \in [0, T]$ is mixed Gaussian with mean zero and conditional variance
\[ \frac{\lambda_p(\nu)}{(m_p \sigma_p^T)^2} \left( (1 - \tilde{\sigma}_p^T)^2 \sigma_p^T + (\tilde{\sigma}_p^T)^2 (\sigma_p^T - \tilde{\sigma}_p^T)^2 \right). \]  

To be able to estimate the conditional variance (4.7), we need a consistent estimator of the factor $\lambda_p(\nu)$ that depends on the smoothness parameter $\nu$. To this end, the following fact is crucial.
Lemma 4.8. The function \( \nu \mapsto \lambda_p(\nu) \) is continuous.

Proof. It suffices to show that \( \nu \mapsto \lambda_p(\nu) \) is continuous on \((\frac{1}{2}, \bar{\nu})\) for any \( \bar{\nu} \in (\frac{1}{2}, \frac{3}{2}) \). Applying the mean value theorem twice to (4.2), we can show that there is a constant \( C > 0 \) such that \( |\rho_j(j)| \leq C j^{2\nu-3} \) for any \( j \geq 1 \) and \( \nu \in (\frac{1}{2}, \bar{\nu}) \). Thus for any \( l \geq 2 \) the function \( \nu \mapsto \sum_{j=1}^{\infty} \rho_j(j)^l \) is continuous on \((\frac{1}{2}, \bar{\nu})\), by Lebesgue’s dominated convergence theorem. Moreover, since \( |\rho_j(j)| \leq 1 \) and \( 6 - 4\nu > 1 \), we have for any \( \nu \in (\frac{1}{2}, \bar{\nu}) \) and \( l \geq 2 \),

\[
\left| \sum_{j=1}^{\infty} \rho_j(j)^l \right| \leq \sum_{j=1}^{\infty} \rho_j(j)^2 \leq C^2 \sum_{j=1}^{\infty} \frac{1}{j^{6-4\nu}} < \infty.
\]

The continuity of \( \lambda_p \) follows then by applying Lebesgue’s dominated convergence theorem to the outer sum in (4.1) (recall that \( \sum_{l=2}^{\infty} \delta \alpha_l^2 < \infty \)).

Barndorff-Nielsen, Corcuera and Podolskij (2011, 2013) and Corcuera et al. (2013) have developed estimators \( \hat{\nu}_\delta \) of \( \nu \), based on the observations \( Y_0, Y_\delta, \ldots, Y_{(T/\delta)\delta} \), that are consistent as \( \delta \to 0 \). Using such an estimator, Lemma 4.8, and the properties of stable convergence, we obtain a feasible central limit theorem for realised relative power variations.

Proposition 4.9. Suppose that \( \hat{\nu}_\delta \overset{p}{\to} \nu \) as \( \delta \to 0 \). Then under the assumptions of Theorem 4.6, we have for any \( T > 0 \) and \( t \in (0, T) \),

\[
\frac{\delta^{-1/2} \left( \tilde{Y}_\delta^{(p)}_{t,T} - \tilde{\sigma}_{t,T}^{p+} \right)}{\sqrt{V_t(T)}} \overset{d}{\to} N(0, 1),
\]

where

\[
V_t(T) = \frac{\lambda_p(\hat{\nu}_\delta)}{\delta m_{2p}(\tilde{Y}_\delta^{(p)}_{T,T})^2} \left( \left(1 - \frac{\tilde{Y}_\delta^{(p)}_{t,T}^2}{\tilde{Y}_\delta^{(2p)}_{t,T}}\right)^2 + \left(\frac{\tilde{Y}_\delta^{(p)}_{t,T}}{\tilde{Y}_\delta^{(2p)}_{t,T}}\right)^2 \left(\frac{\tilde{Y}_\delta^{(2p)}_{t,T}}{\tilde{Y}_\delta^{(2p)}_{t,T}} - \tilde{Y}_\delta^{(2p)}_{t,T}\right) \right).
\]

4.3. Inference on relative volatility/intermittency

Proposition 4.9 can be used to construct approximative, pointwise confidence intervals for the relative volatility/intermittency \( \tilde{\sigma}_{t,T}^{p+} \). Since, by construction, \( \tilde{\sigma}_{t,T}^{p+} \) assumes values in \([0, 1]\), it is natural to constrain the confidence interval to be a subset of \([0, 1]\). Thus, we define for any \( a \in (0, 1) \) the corresponding \((1 - a) \cdot 100\%\) confidence interval as

\[
\left[ \max \left\{ \tilde{Y}_\delta^{(p)}_{t,T} - z_{1-a/2} \sqrt{V_t(T)}, 0 \right\}, \min \left\{ \tilde{Y}_\delta^{(p)}_{t,T} + z_{1-a/2} \sqrt{V_t(T)}, 1 \right\} \right],
\]

where \( z_{1-a/2} > 0 \) is the \( 1 - \frac{a}{2} \)-quantile of the standard Gaussian distribution.
Another application of the central limit theory is a non-parametric homoskedasticity test that is similar in nature to the classical Kolmogorov–Smirnov and Cramér–von Mises goodness-of-fit tests for empirical distribution functions. This extends the homoskedasticity tests proposed by Dette, Podolskij and Vetter (2006) and Dette and Podolskij (2008) to a non-semimartingale setting. Another extension of these tests to non-semimartingales, namely fractional diffusions, is given by Podolskij and Wasmuth (2013). The approach is also similar to the cumulative sum of squares test (Brown, Durbin and Evans, 1975) of structural breaks studied in time series analysis literature. To formulate our test, we introduce the hypotheses

\[
\begin{align*}
H_0 &: \sigma_t = \sigma_0 \text{ for all } t \in [0, T], \\
H_1 &: \sigma_t \neq \sigma_0 \text{ for some } t \in [0, T].
\end{align*}
\]

As mentioned above, Theorem 4.6 implies that under \( H_0 \),

\[
\delta^{-1/2} \left( \frac{1}{T} \sum_{t=0}^{T} \left( \frac{W_t}{T} - \frac{t}{T} W_T \right) \right) \xrightarrow{\delta \to 0} \text{dist} \left( \mathcal{W}_{\nu}, \mathcal{W}_{\nu(\delta)} \right). \tag{4.8}
\]

The distance between the realised relative power variation and the linear function \( t \mapsto \frac{t}{T} \) can be measured using various norms and metrics. Here, we consider the typical sup and \( L^2 \) norms that correspond to the Kolmogorov–Smirnov and Cramér–von Mises test statistics, respectively. More precisely, we define the statistics

\[
\begin{align*}
S_{\delta}^{\text{KS}} &= m_p \sqrt{T} \sup_{k=1,\ldots,(T/\delta)-1} \left| \frac{1}{\delta} \sum_{t=k\delta}^{(k+1)\delta-1} \left( \frac{Y_{t\delta}}{\sqrt{p}} - \frac{k}{T/\delta} \right) \right|, \\
S_{\delta}^{\text{CvM}} &= m_p^2 \frac{[T/\delta]-1}{\Lambda_p(\nu_\delta)} \sum_{k=1}^{[T/\delta]-1} \left( \frac{1}{\sqrt{p}} \sum_{t=k\delta}^{(k+1)\delta-1} \left( \frac{Y_{t\delta}}{\sqrt{p}} - \frac{k}{T/\delta} \right) \right)^2,
\end{align*}
\]

where \( \nu_\delta \) is any consistent estimator of \( \nu \). Under \( H_0 \), these statistics have the classical Kolmogorov–Smirnov and Cramér–von Mises limiting distributions, respectively, as outlined in the following result.

**Proposition 4.10.** Suppose that the assumptions of Theorem 4.6 hold. Then, under \( H_0 \),

\[
\begin{align*}
S_{\delta}^{\text{KS}} \xrightarrow{\delta \to 0} \sup_{0 \leq s \leq 1} \left| \mathcal{W}_s \right|, \tag{4.9} \\
S_{\delta}^{\text{CvM}} \xrightarrow{\delta \to 0} \int_0^1 \mathcal{W}_s^2 \, ds. \tag{4.10}
\end{align*}
\]

where \( \{ \mathcal{W}_t \}_{t \in [0,1]} \) is a standard Brownian bridge, independent of the filtration \( \{ \mathcal{F}_t \}_{t \in \mathbb{R}} \). Moreover, under \( H_1 \), both \( S_{\delta}^{\text{KS}} \) and \( S_{\delta}^{\text{CvM}} \) diverge to infinity as \( \delta \to 0 \).
Proof. Under $H_0$, we have

$$S_{\delta}^{KS} = \frac{m_p \sqrt{T}}{\delta \lambda_p(\hat{\nu}_\delta)} \sup_{0 \leq t \leq T} \left| \tilde{Y}_{\delta}^{(p)}(t, T) - \frac{t}{T} \right| + O_p(\delta^{1/2}) \sup_{0 \leq s \leq 1} |\tilde{\rho}|,$$

$$S_{\delta}^{CvM} = \frac{m^2_p}{\delta \lambda_p(\hat{\nu}_\delta)} \int_0^T \left( \frac{\tilde{Y}_{\delta}^{(p)}}{t} - \frac{t}{T} \right)^2 dt + O_p(\delta^{1/2}) \int_0^1 \tilde{\rho}^2 ds,$$

by (4.8), Lemma 4.8, and the scaling properties of Brownian motion. The divergence of $S_{\delta}^{KS}$ and $S_{\delta}^{CvM}$ as $\delta \to 0$ under $H_1$ is a straightforward consequence of Theorem 3.1.

Remark 4.11. Well-known series expansions for the cumulative distribution functions of the limiting functionals in (4.9) and (4.10) can be found, e.g., in Lehmann and Romano (2005, p. 585) and Anderson and Darling (1952, p. 202), respectively.

Remark 4.12. The finite-sample performance of the test statistics $S_{\delta}^{KS}$ and $S_{\delta}^{CvM}$ is explored in a separate paper (Bennedsen, Lunde and Pakkanen, 2014a).

5. Application to turbulence data

We apply the methodology developed above to empirical data of turbulence. The data consist of a time series of the main component of a turbulent velocity vector, measured at a fixed position in the atmospheric boundary layer using a hotwire anemometer, during an approximately 66 minutes long observation period at sampling frequency of 5 kHz (i.e. 5000 observations per second). The measurements were made at Brookhaven National Laboratory (Long Island, NY), and a comprehensive account of the data has been given by Drhuva (2000).

As a first illustration, we study the observations up to time horizon $T = 800$ milliseconds. Using the smallest possible lag, $\delta = 0.2$ ms, this amounts to 4000 observations. Figure 1(a) displays the squared increments corresponding to these observations. As a comparison, the same time horizon is captured in Figure 1(b) but with lag $\delta = 0.8$ ms. Figure 1(c) compares the associated accumulated realised relative energy dissipations/quadratic variations. The graphs for these two lags show very similar behaviour, exhibiting how the total time interval is divided into a sequence of intervals over which the slope of the energy dissipation is roughly constant. On the other hand, the amplitudes of the volatility/intermittency are of the same order in the whole observation interval.

To be able to draw inference on relative volatility/intermittency using the data, we need to address two issues. Firstly, for this time series, the lags $\delta = 0.2$ ms and $\delta = 0.8$ ms are below the so-called inertial range of turbulence, where a BSS process with a gamma kernel, a model of ideal turbulence, provides an accurate description of the data—see Corcuera et al. (2013), where the same data are analysed. Secondly, the data were digitised using a 12-bit analog-to-digital converter. Thus, the measurements can assume at most $2^{12} = 4096$ different values, and due to the resulting discretisation error, a non-negligible number
of increments are in fact equal to zero (roughly 20% of all increments). These discretisation errors are bound to bias the estimation of the parameter $\nu$, which is needed for the inference methods. We mitigate these issues by subsampling, namely, we apply the inference methods using a considerably longer lag, $\delta = 80$ ms, which is near the lower bound of the inertial range for this time series (Corcuera et al., 2013, Figure 1).

We divide the time series into 66 non-overlapping one-minute-long subperiods, testing the constancy of $\sigma$, i.e., the null hypothesis $H_0$, within each subperiod. Figure 2(a) displays the estimates of $\nu$ for each subperiod using the change-of-frequency method (Barndorff-Nielsen, Corcuera and Podolskij, 2013; Corcuera et al., 2013). All of the estimates belong to the interval $(\frac{5}{6}, 1)$ and they are scattered around the value $\nu = \frac{5}{6}$ predicted by Kolmogorov’s (K41) scaling.
Fig 2. Brookhaven turbulence data: (a) Estimates of $\nu$, using the change-of-frequency method and lag $\delta = 80$ ms, for each one-minute subperiod and the value predicted by Kolmogorov’s (K41) scaling law. (b) and (c) Kolmogorov–Smirnov and Cramér–von Mises-type test statistics and the corresponding critical values for the constancy of $\sigma$ for each subperiod. The red bars indicate the 27th and 40th subperiods that are analysed in more detail in Figure 3.

To understand what kind of intermittency the tests are detecting in the data, we look into two extremal cases, the 27th and 40th subperiods (the red bars in Figure 2(b) and (c)). To this end, we plot the realised relative energy dissipations, with $\delta = 80$ ms, during the 27th and 40th subperiods in Figure 3(a) and (b), respectively. We also include the pointwise confidence intervals, the p-values
of the homoskedasticity tests, and as a reference, the realised relative quadratic variations using the smallest possible lag $\delta = 0.2$ ms. While the realised relative quadratic variations exhibit a slight discrepancy between the lags $\delta = 80$ ms and $\delta = 0.2$ ms, it is clear that 40th subperiod indeed contains significant intermittency, whereas during the 27th subperiod, the (accumulated) realised relative energy dissipation grows nearly linearly.

6. Conclusion

We have introduced the concept of relative volatility/intermittency and we have shown how relative volatility/intermittency can be assessed using realised relative quadratic variations in the context of non-semimartingale Brownian semi-stationary (BSS) processes. (Straightforward extensions of the methodology beyond BSS processes are discussed in Appendix A.)

Realised relative quadratic variations are parameter-free statistics that provide estimates of the relative volatility/intermittency in subintervals of the full observation range, by relating the realised quadratic variation over each subinterval to the total realised quadratic variation for the entire range. They provide robust estimates of the relative accumulated volatility/intermittency as this develops over time and are intimately connected to the concept of relative energy dissipation in the statistical theory of turbulence. An extension to vector valued processes is an issue of interest, in particular in relation to the definition of the energy dissipation in three-dimensional turbulent fields.

Moreover, we have applied our estimation and inference methods to assess relative intermittency/energy dissipation in empirical data of atmospheric turbulence. In ongoing work (Bennedsen, Lunde and Pakkanen, 2014b), these meth-
ods are also being applied to volatility estimation with electricity price data, which exhibit non-negligible correlations in returns that can be successfully captured by models based on BSS processes (Barndorff-Nielsen, Benth and Veraart, 2013).

Acknowledgements

The authors would like to thank Mikkel Bennedsen and Mark Podolskij for valuable comments. O. E. Barndorff-Nielsen and M.S. Pakkanen were supported by CREATES (DNRF78), funded by the Danish National Research Foundation. Moreover, M.S. Pakkanen acknowledges support from from the Aarhus University Research Foundation regarding the project “Stochastic and Econometric Analysis of Commodity Markets” and from the Academy of Finland (project 258042).

Appendix A: Relative volatility/intermittency in the context of fractional processes and beyond

We have introduced relative volatility/intermittency in the context of BSS processes, but the concept has much wider applicability. The key asymptotic results for realised relative power variations, Theorems 3.1 and 4.6, can easily be generalised to other classes of processes. Indeed, Lemma 4.7 can take any stable\textsuperscript{3} functional central limit theorem for power variations of some process (provided that the limiting process is continuous) as an ‘input’ to produce a ‘relative’ version of the result. As an example, we consider now briefly a generalisation to another class of non-semimartingales, namely fractional processes that are defined as integrals with respect to fractional Brownian motion. We also list below a number of other possible generalisations.

More concretely, let us consider a process \( Y^\prime = \{Y^\prime_t\}_{t \geq 0} \) given by

\[
Y^\prime_t = \int_0^t u_s \Delta Z^H_s, \tag{A.1}
\]

where \( Z^H = \{Z^H_t\}_{t \geq 0} \) is a fractional Brownian motion with Hurst parameter \( H \in (0, 1) \) and \( u = \{u_t\}_{t \geq 0} \) is a volatility/intermittency process with finite \( r \)-variation for some \( r < \frac{1}{1-H} \) (we refer to Corcuera, Nualart and Woerner (2006) for the definition of \( r \)-variation). The integral in (A.1) is defined pathwise, in particular, it is not necessary to assume that \( u \) is adapted to the natural filtration of \( Z^H \). We could also add to \( Y^\prime \) a skewness term analogous to \( A_t \) of (2.1), but for simplicity it is eschewed here.

Corcuera, Nualart and Woerner (2006, Theorem 1) show that for any \( p > 0 \) and \( t \geq 0 \), the \( p \)-th power variation of \( Y^\prime \) satisfies

\[
\delta^{1-pH} \left[ Y^\prime_t \right]^{(p)}_{\delta \to 0} \to m_p u_t^{p+},
\]

\textsuperscript{3}Stable convergence is crucial for the validity of Lemma 4.7.
where $u_t^{p+} = \int_0^t |u_s|^p ds$. Thus, analogously to Theorem 3.1, we find that for any $T > 0$,

$$\tilde{Y}_\delta^{(p)}|_{t,T} \overset{p}{\Rightarrow} \tilde{u}_t^{p+},$$

uniformly in $t \in [0, T]$, where

$$\tilde{Y}_\delta^{(p)}|_{t,T} = \frac{[Y_\delta^{(p)}|_{t,T}]}{[Y_\delta^{(p)}]} = \frac{u_t^{p+}}{u_T^{p+}} - \tilde{u}_t^{p+},$$

Further, when $p \geq 1$, $H \in (0, \frac{3}{4})$, and the sample paths of $u$ are $\gamma$-Hölder continuous with $\gamma > \frac{1}{2}$, it holds that (Corcuera, Nualart and Woerner, 2006, Theorem 4) for any $T > 0$,

$$\delta^{-\frac{1}{2}}(\delta^{-pH}[Y_\delta^{(p)}|_{t,T}] - m_p u_t^{p+}) \overset{st}{\Rightarrow} \sqrt{\lambda_p \left( H + \frac{1}{2} \right)} \int_0^t |u_s|^p dW_s \quad \text{in } D([0, T]),$$

where $W$ is a standard Brownian motion independent of the natural filtration of $Z^H$. Using Lemma 4.7, we can then conclude that

$$\delta^{-\frac{1}{2}}(\tilde{Y}_\delta^{(p)}|_{t,T} - \tilde{u}_t^{p+}) \overset{st}{\Rightarrow} \sqrt{\frac{\lambda_p \left( H + \frac{1}{2} \right)}{m_p u_T^{p+}}} \left( \int_0^t |u_s|^p dW_s - \tilde{u}_t^{p+} \int_0^T |u_s|^p dW_s \right)$$

in $D([0, T])$.

In addition to $BSS$ and fractional processes, relative volatility/intermittency statistics could be used in a similar vein at least in the following settings:

- Power and multipower variations of continuous Itô semimartingales, based on the asymptotic theory developed by Barndorff-Nielsen et al. (2006). Also, the consistency of realised relative power variations of certain multifractal processes (Duvernet, 2010; Duvernet, Robert and Rosenbaum, 2010; Ludeña and Soulier, 2014), which are non-Itô semimartingales, could be shown.
- Power variations of stochastic integrals with respect to symmetric $\alpha$-stable Lévy processes (Corcuera and Farkas, 2010).
- Power variations of $BSS$ processes using higher-order increments (Barndorff-Nielsen, Corcuera and Podolskij 2013; Corcuera et al., 2013). With second or higher order increments, the restriction $\nu < 1$ in Theorem 4.3 (and in its applications) can be lifted.
- Power variations of two-parameter ambit fields driven by white noise, observed on a line segment (Barndorff-Nielsen and Graversen, 2011) or on a square lattice (Pakkanen, 2014). However, in these settings only consistency of realised relative power variations can be established using the currently available asymptotic theory.
Appendix B: Estimating the scaling factor of realised quadratic variation

As seen in Sections 2 and 4, the asymptotic theory for power variations of the BSS process $Y$ requires a suitable scaling of the realised power variation by a factor that depends on the second-order structure function $R$. We will now discuss whether the scaling factor can be estimated from the observed data, which would be an alternative to using relative volatility/intermittency statistics. For simplicity, we focus on quadratic variations, which are the most relevant in practical applications.

Assumption 2.3 postulates that $R(\delta)$ behaves like $\delta^{2\nu-1}$ as $\delta \to 0$, apart from a slowly varying factor $L_R(\delta)$. If $L_R(\delta)$ is ‘well-behaved’ and normalised in the sense that $\lim_{\delta \to 0} L_R(\delta) = 1$, then in Theorem 2.6 for the case $p = 2$ the scaling factor $\frac{\delta^2 R(\delta)}{R(\delta)}$ can be replaced with $\delta^{2-2\nu}$, to wit,

$$\delta^{2-2\nu}[Y_\delta]_t \xrightarrow{P, \delta \to 0} \sigma_t^2$$  \hspace{1cm} (B.1)

for any $t \geq 0$. The condition $\lim_{\delta \to 0} L_R(\delta) = 1$ holds, e.g., when $g$ is the gamma kernel (2.3) with $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and $c$ is chosen in a suitable way (Barndorff-Nielsen, Corcuera and Podolskij, 2011, p. 1173). If, additionally, $L_R(\delta) = 1 + o(\delta^{1/2})$ as $\delta \to 0$, which is again true in the aforementioned situation with $g$ of the gamma form, the convergence in the central limit theorem (Theorem 4.3) in the case $p = 2$ can be simplified to

$$\delta^{1/2}(\delta^{2-2\nu}[Y_\delta]_t - \sigma_t^2) \xrightarrow{\text{st, } \delta \to 0} \sqrt{2} \int_0^t \sigma_s^2 dW_s \quad \text{in } D([0,T]).$$ \hspace{1cm} (B.2)

As shown by Barndorff-Nielsen, Corcuera and Podolskij (2013) and Corcuera et al. (2013), the smoothness parameter $\nu$ can be estimated consistently in the infill asymptotic setting with an estimator $\hat{\nu}_\delta$ with the usual rate of convergence $\delta^{-1/2}$. Then it is natural to ask, whether we can simply substitute $\nu$ with $\hat{\nu}_\delta$ in (B.1) and (B.2) without affecting the asymptotic behaviour of the scaled realised quadratic variation. From the following result we learn that $[Y_\delta]_t$ with the estimated scaling $\delta^{2-2\hat{\nu}_\delta}$ indeed attains consistency. However, the second-order behaviour is affected by the estimated scaling: the rate of convergence becomes slower and the asymptotic distribution is non-standard, due to the estimation error of $\nu$. Similar results have been shown (under constant volatility) by Coeurjolly (2001, Proposition 4) in the context of fractional Brownian motion and by Brouste and Iacus (2013, Theorem 1) in the context of fractional Ornstein–Uhlenbeck processes.

**Proposition B.1.** Let $\delta \in (0, 1)$ and let $\hat{\nu}_\delta$ be an estimator of the smoothness parameter $\nu$ such that

$$\delta^{1/2}(\hat{\nu}_\delta - \nu) \xrightarrow{\text{st, } \delta \to 0} \xi,$$ \hspace{1cm} (B.3)

where $\xi$ is an a.s. finite random variable.
(a) If the assumptions of Theorem 2.6 hold and \( \lim_{\delta \to 0} L_R(\delta) = 1 \), then for any \( t \geq 0 \),
\[
\delta^{2-2\nu_\delta} |Y_\delta|_t \xrightarrow{\delta \to 0} \sigma_t^{2+}.
\]

(b) If the assumptions of Theorem 4.3 hold and \( L_R(\delta) = 1 + o(\delta^{\frac{1}{2}}) \) as \( \delta \to 0 \), then
\[
\frac{\delta^{-\frac{1}{2}}}{\log(\delta^{-1})}(\delta^{2-2\nu_\delta} |Y_\delta|_t - \sigma_t^{2+}) \xrightarrow{\delta \to 0} 2\xi \sigma_t^{2+} \text{ in } D([0, T]).
\]

Proof. (a) Let us write
\[
\delta^{2-2\nu_\delta} |Y_\delta|_t = \delta^{-2(\nu_\delta - \nu)} \delta^{2-2\nu} |Y_\delta|_t = e^{Q_\delta} \delta^{2-2\nu} |Y_\delta|_t,
\]
where \( Q_\delta = 2\log(\delta^{-1})(\nu_\delta - \nu) \). By the condition (B.3), we find that
\[
Q_\delta = 2\delta^{\frac{1}{2}} \log(\delta^{-1}) \delta^{-\frac{1}{2}} (\nu_\delta - \nu) \xrightarrow{\delta \to 0} 0. \tag{B.4}
\]
Thus, \( e^{Q_\delta} \xrightarrow{\delta \to 0} 1 \) as \( \delta \to 0 \), and the assertion follows then from (B.1).

(b) Let us consider the decomposition
\[
\frac{\delta^{-\frac{1}{2}}}{\log(\delta^{-1})}(\delta^{2-2\nu} |Y_\delta|_t - \sigma_t^{2+}) = U_\delta \delta^{2-2\nu} |Y_\delta|_t + \frac{\delta^{-\frac{1}{2}}}{\log(\delta^{-1})}(\delta^{2-2\nu} |Y_\delta|_t - \sigma_t^{2+}),
\]
where
\[
U_\delta = \frac{\delta^{-\frac{1}{2}}}{\log(\delta^{-1})}(\delta^{-2(\nu_\delta - \nu)} - 1) = \frac{\delta^{-\frac{1}{2}}}{\log(\delta^{-1})}(e^{Q_\delta} - 1).
\]
By (B.2), we have clearly
\[
\frac{\delta^{-\frac{1}{2}}}{\log(\delta^{-1})}(\delta^{2-2\nu} |Y_\delta|_t - \sigma_t^{2+}) \xrightarrow{\delta \to 0} 0 \text{ in } D([0, T]).
\]
Due to (B.1) and the properties of stable convergence, it suffices now to show that \( U_\delta \xrightarrow{\delta \to 0} 2\xi \) as \( \delta \to 0 \). To this end, define \( u(x) = e^x - 1 - x, \ x \in \mathbb{R} \). Observe that
\[
U_\delta = 2\delta^{-\frac{1}{2}} (\nu_\delta - \nu) + \frac{u(Q_\delta)}{\delta^{\frac{1}{2}} \log(\delta^{-1})}, \tag{B.5}
\]
and in view of the condition (B.3) it remains to show that the second term on right-hand side of (B.5) converges to zero in probability as \( \delta \to 0 \). To this end, let \( \eta > 0 \) and consider
\[
P\left( \left| \frac{u(Q_\delta)}{\delta^{\frac{1}{2}} \log(\delta^{-1})} \right| > \eta \right) \leq P\left( \left| \frac{u(Q_\delta)}{\delta^{\frac{1}{2}} \log(\delta^{-1})} \right| > \eta, |Q_\delta| \leq 1 \right) + P\{|Q_\delta| > 1\},
\]
where \( \lim_{\delta \to 0} P\{|Q_\delta| > 1\} = 0 \) by (B.4). Using the elementary inequality \( |u(x)| \leq 3x^2 \), valid when \( |x| \leq 1 \), we finally deduce that
\[
P\left( \left| \frac{u(Q_\delta)}{\delta^{\frac{1}{2}} \log(\delta^{-1})} \right| > \eta, |Q_\delta| \leq 1 \right) \leq P\left( \left| \frac{3Q_\delta^2}{\delta^{\frac{1}{2}} \log(\delta^{-1})} \right| > \eta \right) \xrightarrow{\delta \to 0} 0,
\]
since
\[
\frac{3Q_3^2}{\delta^2 \log(\delta^{-1})} = 12 \delta \frac{1}{2} \log(\delta^{-1}) \left( \delta^{-\frac{1}{2}} (\rho - \nu) \right)^2 \rightarrow 0, \quad \delta \to 0,
\]
which in turn is a simple consequence of the condition (B.3).

\[\square\]

**Appendix C: Sufficient conditions for the negligibility of the skewness term**

This appendix provides some methods of checking the negligibility conditions (2.7) and (4.3) with some concrete specifications of the process \( A = \{A_t\}_{t \geq 0} \).

Suppose first that the process \( A \) is given by
\[
A_t = \mu + \int_0^t a_s ds,
\]
where \( \mu \in \mathbb{R} \) is a constant and the process \( \{a_t\}_{t \geq 0} \) is measurable and locally bounded. Then we can establish rather simple conditions for its negligibility in the asymptotic results for power variations. By Jensen’s inequality, we have for any \( p \geq 1 \), \( s \geq 0 \), and \( t \geq 0 \),
\[
|A_s - A_t|^p \leq C_a \cdot |s - t|^p,
\]
where \( C_a > 0 \) is a random variable that depends locally on the path of \( a \). Thus, we find that for any \( t \geq 0 \),
\[
[A^\delta]_t^{(p)} = O_{\text{a.s.}}(\delta^{p-1})
\]
as \( \delta \to 0 \). Then, in view of Remarks 2.7 and 4.5 and the restriction \( \nu < \frac{3}{2} \), the condition (2.7) holds always and (4.3) holds provided that \( p > \frac{1}{3 - 2\nu} \) (which is always true if \( p \geq 1 \)).

Suppose now, instead, that \( A \) follows
\[
A_t = \mu + \int_{-\infty}^t q(t - s) a_s ds,
\]
where \( q \) is the gamma kernel
\[
q(t) = c' t^{\eta-1} e^{-\rho t}
\]
for some \( c' > 0 \), \( \eta > 0 \), and \( \rho > 0 \). We assume that the process \( \{a_t\}_{t \in \mathbb{R}} \) is measurable, locally bounded, and satisfies
\[
A_t^\ast = \sup_{0 \leq u \leq t} \int_{-\infty}^u q(u - s) |a_s| ds < \infty \quad \text{a.s.} \quad \text{(C.2)}
\]
for any \( t \geq 0 \), which is true, e.g., when the auxiliary process \( \int_{-\infty}^u q(u - s) |a_s| ds \), \( u \geq 0 \), has a càdlàg or continuous modification.
Lemma C.1. If $A$ is given by (C.1), and (C.2) holds, then for any $p > 0$ and $t \geq 0$,

$$[A_{\delta}]_{t}^{(p)} = O_{a.s.}(\delta^{p \min\{\eta, 1\} - 1})$$

(C.3)

as $\delta \to 0$. Thus the condition (2.7) holds if $\min\{\eta, 1\} > \nu - \frac{1}{2p}$ and (4.3) holds if $\min\{\eta, 1\} > \nu - \frac{p - 1}{2p}$.

Proof. Let us first look into the properties of $q$. For the sake of simpler notation, we make the innocuous assumption that $c' = 1$. Since

$$q'(t) = \left(\frac{\eta - 1}{t} - \rho\right)q(t),$$

(C.4)

we find that $q$ is decreasing when $\eta \leq 1$. When $\eta > 1$, $q$ is increasing on $\left(0, \frac{\eta - 1}{\rho}\right)$ and decreasing on $\left(\frac{\eta - 1}{\rho}, \infty\right)$.

Let $t \geq 0$ be fixed, $\delta \in (0, 1)$, and let $j \geq 1$ be such that $j\delta \leq t$. Below, all big $O$ estimates hold uniformly in such $j$. We consider the decomposition

$$A_{j\delta} - A_{(j-1)\delta} = \int_{(j-1)\delta}^{j\delta} q(j\delta - s)a_s ds$$

$$+ \int_{(j-2)\delta}^{(j-1)\delta} (q(j\delta - s) - q((j - 1)\delta - s))a_s ds$$

$$+ \int_{s^*}^{(j-2)\delta} (q(j\delta - s) - q((j - 1)\delta - s))a_s ds$$

$$+ \int_{-\infty}^{s^*} (q(j\delta - s) - q((j - 1)\delta - s))a_s ds$$

$$= I_1^\delta + I_2^\delta + I_3^\delta + I_4^\delta,$$

where

$$s^* = - \max \left\{ \frac{\eta - 1}{\rho}, 1 \right\}.$$}

When $\eta \geq 1$, $q$ is bounded and we have $|I_1^\delta + I_2^\delta| = a_t^* O(\delta)$, where

$$a_t^* = \sup_{s^* \leq s \leq t} |a_s| < \infty \text{ a.s.},$$

and when $\eta < 1$, we find that

$$|I_1^\delta + I_2^\delta| \leq 2a_t^* \int_0^\delta q(s) ds = a_t^* O(\delta^\eta).$$

Next, we want to show that

$$|I_3^\delta| = a_t^* O(\delta^{\min\{\eta, 1\}}).$$

(C.5)
In the case $\eta \geq 2$ the derivative $q'$ is bounded and (C.5) is immediate. Suppose that $\eta < 2$. Then, $|q'(t)| \leq C t^{\eta-2}$ on any finite interval, where $C > 0$ depends on the interval. Using the mean value theorem, we obtain

$$|I^3_\delta| \leq C a^*_s \delta \int_{s_*}^{(j-2)\delta} ((j-1)\delta - s)^{\eta-2} ds,$$

which implies (C.5). To bound $|I^4_\delta|$, note that, by (C.4), $|q'(t)| \leq C' q(t)$ for all $t \geq -s^*$, where $C' > 0$ is a constant. For any $s < s^*$, we have $(j-1)\delta - s > \frac{\eta-1}{\rho}$. Thus, by the mean value theorem,

$$\left|(q(j\delta - s) - q((j-1)\delta - s))\right| \leq C' q((j-1)\delta - s) \delta$$

and, consequently,

$$|I^4_\delta| \leq C' \delta \int_{-\infty}^{(j-1)\delta} q((j-1)\delta - s)|a_s| ds = A^*_s O(\delta).$$

Collecting the estimates, we have

$$|A^j_\delta - A^j_{(j-1)\delta}| = \max \{a^*_s, A^*_s\} O(\delta^{\min\{\eta, 1\}})$$

uniformly in $j$, whence (C.3) follows. Checking the sufficiency of the asserted criteria for (2.7) and (4.3) is now a straightforward task (based on Remarks 2.7 and 4.5).

References

Aldous, D. J. and Eagleson, G. K. (1978). On mixing and stability of limit theorems. Ann. Probab. 6 325–331. MR0517416

Anderson, T. W. and Darling, D. A. (1952). Asymptotic theory of certain “goodness of fit” criteria based on stochastic processes. Ann. Math. Statistics 23 193–212. MR0050238

Barndorff-Nielsen, O. E., Benth, F. E. and Veraart, A. E. D. (2013). Modelling energy spot prices by volatility modulated Lévy-driven Volterra processes. Bernoulli 19 803–845. MR3131315

Barndorff-Nielsen, O. E., Corcuera, J. M. and Podolskij, M. (2011). Multipower variation for Brownian semi-stationary processes. Bernoulli 17 1159–1194. MR2854786

Barndorff-Nielsen, O. E., Corcuera, J. M. and Podolskij, M. (2013). Limit theorems for functionals of higher order differences of Brownian semi-stationary processes. In Prokhorov and Contemporary Probability Theory (A. N. Shiryaev, S. R. S. Varadhan and E. Presman, eds.) 69–96. Springer, Berlin.

Barndorff-Nielsen, O. E. and Graeversen, S. E. (2011). Volatility determination in an ambit process setting. J. Appl. Probab. 48A 263–275. MR2865631

Barndorff-Nielsen, O. E. and Schmiegel, J. (2009). Brownian semi-stationary processes and volatility/intermittency. In Advanced Financial
Modelling, (H. Albrecher, W. Rungaldier and W. Schachermayer, eds.). Radon Ser. Comput. Appl. Math. 8 1–25. Walter de Gruyter, Berlin. MR2648456

Barndorff-Nielsen, O. E. and Shephard, N. (2010). Volatility. In Encyclopedia of Quantitative Finance (R. Cont, ed.) 1898–1901. Wiley, Chicester.

Barndorff-Nielsen, O. E., Graversen, S. E., Jacod, J., Podolskij, M. and Shephard, N. (2006). A central limit theorem for realised power and bipower variations of continuous semimartingales. In From Stochastic Calculus to Mathematical Finance (Y. Kabanov, R. Liptser and J. Stoyanov, eds.) 33–68. Springer, Berlin. MR2233534

Bennedsen, M., Lunde, A. and Pakkanen, M. S. (2014a). Discretization of Lévy semistationary processes with application to estimation. Available at http://arxiv.org/abs/1407.2754.

Bennedsen, M., Lunde, A. and Pakkanen, M. S. (2014b). Modelling energy prices by Brownian semistationary processes. In preparation.

Bingham, N. H., Goldie, C. M. and Teugels, J. L. (1987). Regular Variation. Cambridge University Press, Cambridge. MR898871

Brouste, A. and Iacus, S. M. (2013). Parameter estimation for the discretely observed fractional Ornstein-Uhlenbeck process and the Yuima R package. Comput. Statist. 28 1529–1547. MR3120827

Brown, R. L., Durbin, J. and Evans, J. M. (1975). Techniques for testing the constancy of regression relationships over time. J. Roy. Statist. Soc. Ser. B 37 149–192. MR0378310

Cleve, J., Schmiegel, J. and Greiner, M. (2008). Apparent scale correlations in a random multiplicative process. Eur. Phys. J. B 63 109–116.

Coeurjolly, J.-F. (2001). Estimating the parameters of a fractional Brownian motion by discrete variations of its sample paths. Stat. Inference Stoch. Process. 4 199–227. MR1856174

Corcuera, J. M. and Farkas, G. (2010). Power variation for Itô integrals with respect to α-stable processes. Stat. Neerl. 64 276–289. MR2683461

Corcuera, J. M., Nualart, D. and Woerner, J. H. C. (2006). Power variation of some integral fractional processes. Bernoulli 12 713–735. MR2248234

Corcuera, J. M., Hedevang, E., Pakkanen, M. S. and Podolski, M. (2013). Asymptotic theory for Brownian semi-stationary processes with application to turbulence. Stochastic Process. Appl. 123 2552–2574. MR3054536

Dette, H., Podolskij, M. and Vetter, M. (2006). Estimation of integrated volatility in continuous-time financial models with applications to goodness-of-fit testing. Scand. J. Statist. 33 259–278. MR2279642

Dette, H. and Podolski, M. (2008). Testing the parametric form of the volatility in continuous time diffusion models—a stochastic process approach. J. Econometrics 143 56–73. MR2384433

Drhuva, B. R. (2000). An experimental study of high Reynolds number turbulence in the atmosphere. PhD thesis, Yale University.

Duvernet, L. (2010). Convergence of the structure function of a multifractal random walk in a mixed asymptotic setting. Stoch. Anal. Appl. 28 763–792. MR2739317
Assessing relative volatility

DUVERNET, L., ROBERT, C. Y. and ROSENBAUM, M. (2010). Testing the type of a semi-martingale: Itô against multifractal. *Electron. J. Stat.* 4 1300–1323. MR2738534

FRISCH, U. (1995). *Turbulence: The Legacy of A. N. Kolmogorov*. Cambridge University Press, Cambridge. MR1428905

GUTTORP, P. and GNEITING, T. (2005). On the Whittle–Matérn correlation family. Technical Report No. 080, The National Research Center for Statistics and the Environment, University of Washington.

JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, Second ed., Springer, Berlin. MR1943877

KOLMOGOROV, A. N. (1941a). Dissipation of energy in locally isotropic turbulence. *Dokl. Akad. Nauk SSSR* 32 19–21.

KOLMOGOROV, A. N. (1941b). The local structure of turbulence in incompressible viscous fluids. *Dokl. Akad. Nauk SSSR* 30 301–305.

LEHMAN, E. L. and ROMANO, J. P. (2005). *Testing Statistical Hypotheses*, Third ed., Springer, New York. MR2135927

LUDEÑA, C. and SOULIER, P. (2014). Estimating the scaling function of multifractal measures and multifractal random walks using ratios. *Bernoulli* 20 334–376. MR3160585

PAKKANEN, M. S. (2014). Limit theorems for power variations of ambit fields driven by white noise. *Stochastic Process. Appl.* 124 1942–1973. MR3170230

PODOLSKII, M. and VETTER, M. (2010). Understanding limit theorems for semimartingales: A short survey. *Stat. Neerl.* 64 329–351. MR2683464

PODOLSKII, M. and WASMUTH, K. (2013). Goodness-of-fit testing for fractional diffusions. *Stat. Inference Stoch. Process.* 16 147–159. MR3071885

RÉNYI, A. (1963). On stable sequences of events. *Sankhyā Ser. A* 25 293 302. MR0170385

VON KÁRMÁN, T. (1948). Progress in the statistical theory of turbulence. *J. Marine Research* 7 252–264. MR0028750