CATEGORICAL JOINS

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Abstract. We introduce the notion of a categorical join, which can be thought of as a categorification of the classical join of two projective varieties. This notion is in the spirit of homological projective duality, which categorifies classical projective duality. Our main theorem says that the homological projective dual category of the categorical join is naturally equivalent to the categorical join of the homological projective dual categories. This categorifies the classical version of this assertion and has many applications, including a nonlinear version of the main theorem of homological projective duality.

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1. Introduction

The theory of homological projective duality (HPD) introduced in [14] is a powerful tool for understanding the structure of derived categories of algebraic varieties. Given a smooth projective variety $X$ with a morphism to a projective space $\mathbb{P}(V)$ and a special semiorthogonal decomposition of its derived category, HPD associates a smooth projective (but generally “noncommutative” in a sense described below) variety $X^\#$ — called the HPD of $X$ — with a morphism to the dual projective space $\mathbb{P}(V^\vee)$. The terminology stems from the close relation between the construction of $X^\#$ and classical projective duality. The main theorem of HPD describes the derived categories of linear sections of $X$ in terms of those of $X^\#$.

Since many interesting varieties can be expressed as linear sections of “simple” varieties, e.g. many Fano threefolds are linear sections of homogeneous varieties, this gives a potent strategy for studying derived categories:

(1) Obtain an explicit geometric description of the HPD of a simple variety.
We call this the standard Lefschetz structure over \( P \). Such a decomposition is in fact determined by the subcategory \( A \) of the category \( \text{Perf}(X) \) of perfect complexes, namely a semiorthogonal decomposition of the form
\[
\text{Perf}(X) = \langle A_0, A_1 \otimes f^* \mathcal{O}_P(V)(1), \ldots, A_{m-1} \otimes f^* \mathcal{O}_P(V)(m-1) \rangle
\]
where \( 0 = A_0 \subset A_1 \subset \cdots \subset A_1 \subset A_0 \subset \text{Perf}(X) \) is a chain of triangulated subcategories. Such a decomposition is in fact determined by the subcategory \( A_0 \subset \text{Perf}(X) \), which we call the Lefschetz center. The key property of a Lefschetz decomposition — from which its name derives, by analogy with the Lefschetz hyperplane theorem — is that it induces semiorthogonal decompositions of linear sections of \( X \) (Lemma 2.12).

The simplest example to keep in mind is the linear case.

**Example 1.1.** If \( W \subset V \) is a subspace of dimension \( m \), then \( P(W) \) is a Lefschetz variety over \( P(V) \) with Lefschetz decomposition
\[
\text{Perf}(P(W)) = \langle \mathcal{O}_{P(W)}, \mathcal{O}_{P(W)}(1), \ldots, \mathcal{O}_{P(W)}(m-1) \rangle.
\]
We call this the standard Lefschetz structure on \( P(W) \subset P(V) \). The HPD variety of \( P(W) \) is \( P(W^\perp) \subset P(V^\vee) \), where \( W^\perp = \ker(V^\vee \to W^\vee) \) is the orthogonal of \( W \).

In general, given a Lefschetz variety \( X \to P(V) \), the associated HPD variety \( X^2 \to P(V^\vee) \) is noncommutative. This means that \( X^2 \) consists only of the data of a (suitably enhanced) triangulated category \( \text{Perf}(X^2) \) equipped with a \( P(V^\vee) \)-linear structure, i.e. an action of the monoidal category \( \text{Perf}(P(V^\vee)) \). In some cases, \( \text{Perf}(X^2) \) can be identified with the derived category of a variety (as in Example 1.1), or with some other category of geometric origin, like the derived category of sheaves of modules over a finite sheaf of algebras on a variety. In these cases, \( X^2 \) is called “commutative” or “almost commutative”, but in general there is no underlying variety, and the symbol \( X^2 \) just serves as a notational device.

In \([14]\), HPD is developed under the assumption that \( X^2 \) is commutative. This was generalized to the noncommutative case in \([32]\), where both the input \( X \) and output \( X^2 \) are...
allowed to be noncommutative, making the theory completely symmetric. The basic object in this setting is a Lefschetz category over a projective space \( \mathbb{P}(V) \), i.e. a (suitably enhanced) \( \mathbb{P}(V) \)-linear triangulated category with a Lefschetz decomposition. We note that there is a robust framework for linear categories that includes notions of smoothness, properness, and base change, which in the commutative case reduce to their usual meanings. For background and references, see [1.8 and Appendix A. For this introduction, it is enough to know that the objects of a category linear over a scheme \( T \) can be “tensored” with objects of \( \text{Perf}(T) \), and if \( X \) is a scheme over \( T \) then \( \text{Perf}(X) \) — or more generally any admissible subcategory of \( \text{Perf}(X) \) closed under tensoring with pullbacks of objects of \( \text{Perf}(T) \) — is an example of a \( T \)-linear category.

With this in mind, a Lefschetz decomposition of a \( \mathbb{P}(V) \)-linear category \( A \) is a semiorthogonal decomposition of the form

\[
A = \langle A_0, A_1 \otimes \mathcal{O}_{\mathbb{P}(V)}(1), \ldots, A_{m-1} \otimes \mathcal{O}_{\mathbb{P}(V)}(m-1) \rangle
\]

where \( A_0 \supset A_1 \supset \cdots \supset A_{m-1} \). If this decomposition satisfies some technical assumptions (see Definitions 2.1 and 2.6) which are automatic if \( A \) is smooth and proper, then \( A \) is called a Lefschetz category over \( \mathbb{P}(V) \). In this case there is also a “left” analogue of (1.2) that takes the form

\[
A = \langle A_{1-m} \otimes \mathcal{O}_{\mathbb{P}(V)}(-m), \ldots, A_{-1} \otimes \mathcal{O}_{\mathbb{P}(V)}(-1), A_0 \rangle
\]

where \( A_{1-m} \subset \cdots \subset A_{-1} \subset A_0 \). As in the commutative case, both (1.2) and (1.3) are determined by the Lefschetz center \( A_0 \subset A \).

To formulate HPD in this setting, we need two other technical notions. The Lefschetz structure of \( A \) is called right strong if the categories \( A_{i+1} \cap A_i \), \( i \geq 0 \), are admissible in \( A \), and left strong if the categories \( \mathbb{A}_{i-1} \cap A_{i} \), \( i \leq 0 \), are admissible in \( A \). Again, these conditions are automatic if \( A \) is smooth and proper. Further, the Lefschetz structure of \( A \) is called moderate if the length \( m \) of the decomposition (1.2) satisfies \( m < \dim(V) \). This condition is very mild and essentially always satisfied in practice, see Remark 2.10.

In its noncommutative form, HPD

- takes as input a right strong, moderate Lefschetz category \( A \) over \( \mathbb{P}(V) \), and
- gives as output a left strong, moderate Lefschetz category \( \mathbb{A}^2 \) over \( \mathbb{P}(V^\vee) \).

The category \( \mathbb{A}^2 \) is called the HPD category of \( A \).

**Remark 1.2.** As formulated above, the input and output of HPD are not quite symmetric. In fact, there are two versions of HPD: the above is more precisely called right HPD, while left HPD takes a left strong, moderate Lefschetz category \( A \) and gives a right strong, moderate Lefschetz category \( \mathbb{A}^2 \). As \( \mathbb{P}(V^\vee) \)-linear categories \( \mathbb{A}^2 \) and \( \mathbb{A}^2 \) are always equivalent, and if \( A \) is smooth and proper then there is a Lefschetz equivalence \( \mathbb{A}^2 \simeq \mathbb{A}^2 \), i.e. a \( \mathbb{P}(V^\vee) \)-linear equivalence which identifies the Lefschetz decompositions on each side. So in most cases there is no need to distinguish between the two versions of HPD. In this paper, we deal almost exclusively with “right HPD”, which we simply call HPD, but all of the results translate directly to the “left HPD” setting. For more details see Remark 2.20.

The main theorem of HPD describes the base change category \( A_{\mathbb{P}(L)} \) in terms of the base change category \( \mathbb{A}_{\mathbb{P}(L^\perp)} \), where \( L \subset V \) is a vector subspace, \( L^\perp = \ker(V^\vee \rightarrow L^\vee) \) is its orthogonal, and the base changes of the \( \mathbb{P}(V) \)- and \( \mathbb{P}(V^\vee) \)-linear categories \( A \) and \( \mathbb{A}^2 \) are
taken along the natural morphisms $P(L) \to P(V)$ and $P(L^\perp) \to P(V^\perp)$. For the precise formulation of this result, see Theorem 2.24.

In the rest of the introduction, for ease of reading we stick as much as possible to the language of Lefschetz varieties, but in the body of the paper we work in the general setting of Lefschetz categories.

1.2. Categorical joins. Now let us outline the construction of a categorical join. Given two smooth projective varieties $X_1 \subset P(V_1)$ and $X_2 \subset P(V_2)$, their classical join

$$J(X_1, X_2) \subset P(V_1 \oplus V_2)$$

is the union of all the lines between points of $X_1$ and $X_2$ regarded as subvarieties of $P(V_1 \oplus V_2)$. The join is usually very singular (along the union $X_1 \cup X_2$), unless both $X_1$ and $X_2$ are linear subspaces in $P(V_1)$ and $P(V_2)$, i.e. unless

$$X_1 = P(W_1) \subset P(V_1) \quad \text{and} \quad X_2 = P(W_2) \subset P(V_2),$$

in which case $J(X_1, X_2) = P(W_1 \oplus W_2) \subset P(V_1 \oplus V_2)$.

The main problem with running HPD on the classical join is that in general $J(X_1, X_2)$ does not have a natural Lefschetz structure. We would like to construct one from Lefschetz structures on $X_1$ and $X_2$. For this purpose, we pass to the resolved join, defined by

$$\tilde{J}(X_1, X_2) = P_{X_1 \times X_2}(O(-H_1) \oplus O(-H_2))$$

(1.5)

where $H_k$ is the pullback to $X_1 \times X_2$ of the hyperplane class of $P(V_k)$. The resolved join is smooth since it is a $P^1$-bundle over $X_1 \times X_2$, and the canonical embedding

$$O(-H_1) \oplus O(-H_2) \to (V_1 \otimes O) \oplus (V_2 \otimes O) = (V_1 \oplus V_2) \otimes O$$

induces a morphism $\tilde{J}(X_1, X_2) \to P(V_1 \oplus V_2)$ which factors birationally through the classical join. The morphism $\tilde{J}(X_1, X_2) \to J(X_1, X_2)$ blows down the two disjoint divisors

$$\varepsilon_k : E_k(X_1, X_2) = P_{X_1 \times X_2}(O(-H_k)) \to \tilde{J}(X_1, X_2), \quad k = 1, 2,$$

(1.6)

to $X_k \subset J(X_1, X_2)$. Note that both $E_k(X_1, X_2)$ are canonically isomorphic to $X_1 \times X_2$.

There are several advantages of the resolved join over the classical join:

1. Via the morphism $\tilde{J}(X_1, X_2) \to X_1 \times X_2$ the resolved join is more simply related to $X_1$ and $X_2$ than the classical join.

2. The definition (1.5) of the resolved join extends verbatim to the more general situation where $X_1$ and $X_2$ are varieties equipped with morphisms $X_1 \to P(V_1)$ and $X_2 \to P(V_2)$, and with some work even to the case where $X_1$ and $X_2$ are replaced with $P(V_1)$- and $P(V_2)$-linear categories (see Definition 3.2).

3. The resolved join $\tilde{J}(X_1, X_2)$ is smooth and proper if $X_1$ and $X_2$ are.

Points (2) and (3) should be thought of as parallel to the fact that HPD takes as input varieties that are not necessarily embedded in projective space (or even categories), and preserves smoothness and properness.

However, we still face two obstacles to running HPD on $\tilde{J}(X_1, X_2)$ over $P(V_1 \oplus V_2)$: the variety $\tilde{J}(X_1, X_2)$ is “too far” from the classical join, and is still not equipped with a natural Lefschetz structure.

To illustrate these problems, consider the simplest case (1.4) where $X_1$ and $X_2$ are linear subspaces of $P(V_1)$ and $P(V_2)$. Then the classical join $J(X_1, X_2) = P(W_1 \oplus W_2)$ is already smooth and comes with the standard Lefschetz structure (1.1). The resolved join $\tilde{J}(X_1, X_2)$,
being a blowup of $\mathbf{J}(X_1, X_2)$, contains in its derived category a copy of $\text{Perf}(\mathbf{J}(X_1, X_2))$, as well as extra components coming from the blowup centers. These extra components are irrelevant for the geometry of the join, and prevent $\text{Perf}(\tilde{\mathbf{J}}(X_1, X_2))$ from having a natural Lefschetz decomposition. We thus need a general procedure for eliminating these extra components. The solution is the categorical join.

**Definition 1.3.** Let $X_1 \to \mathbf{P}(V_1)$ and $X_2 \to \mathbf{P}(V_2)$ be Lefschetz varieties with Lefschetz centers $A^1_0 \subset \text{Perf}(X_1)$ and $A^2_0 \subset \text{Perf}(X_2)$. The categorical join of $X_1$ and $X_2$ is the full subcategory of $\text{Perf}(\tilde{\mathbf{J}}(X_1, X_2))$ defined by

$$\mathcal{J}(X_1, X_2) = \left\{ C \in \text{Perf}(\tilde{\mathbf{J}}(X_1, X_2)) \mid \begin{array}{l} \varepsilon^1_1(C) \in \text{Perf}(X_1) \otimes A^2_0 \subset \text{Perf}(E_1(X_1, X_2)) \\ \varepsilon^2_2(C) \in A^1_0 \otimes \text{Perf}(X_2) \subset \text{Perf}(E_2(X_1, X_2)) \end{array} \right\}$$

where $\varepsilon_1$ and $\varepsilon_2$ are the morphisms (1.6) and

$$\text{Perf}(X_1) \otimes A^2_0 \subset \text{Perf}(X_1 \times X_2) = \text{Perf}(E_1(X_1, X_2)),$$

$$A^1_0 \otimes \text{Perf}(X_2) \subset \text{Perf}(X_1 \times X_2) = \text{Perf}(E_2(X_1, X_2)),$$

are the subcategories generated by objects of the form $F_1 \otimes F_2$ with $F_1 \in \text{Perf}(X_1)$, $F_2 \in A^2_0$ in the first case, and with $F_1 \in A^1_0$, $F_2 \in \text{Perf}(X_2)$ in the second case.

**Remark 1.4.** In the body of the paper, we generalize the above construction to the noncommutative case, where $X_1$ and $X_2$ are replaced with Lefschetz categories (Definition 3.9). All of the results below are also proved in this context, but for simplicity in the introduction we state them in the case where $X_1$ and $X_2$ are commutative.

The following shows that the categorical join gives a suitable input for HPD.

**Theorem 1.5** (Lemma 3.14 and Theorem 3.21). Let $X_1 \to \mathbf{P}(V_1)$ and $X_2 \to \mathbf{P}(V_2)$ be Lefschetz varieties with Lefschetz centers $A^1_0 \subset \text{Perf}(X_1)$ and $A^2_0 \subset \text{Perf}(X_2)$. Then the categorical join $\mathcal{J}(X_1, X_2)$ has the structure of a Lefschetz category over $\mathbf{P}(V_1 \oplus V_2)$, with Lefschetz center the image of $A^1_0 \otimes A^2_0 \subset \text{Perf}(X_1 \times X_2)$ under pullback along the projection $\tilde{\mathbf{J}}(X_1, X_2) \to X_1 \times X_2$. Moreover, if $X_1$ and $X_2$ are smooth and proper, or right strong, or moderate, then so is $\mathcal{J}(X_1, X_2)$.

As promised, if $X_1$ and $X_2$ are linear subspaces (1.4) with their standard Lefschetz structures (1.1), then there is an equivalence

$$\mathcal{J}(\mathbf{P}(W_1), \mathbf{P}(W_2)) \simeq \text{Perf}(\mathbf{P}(W_1 \oplus W_2))$$

of Lefschetz categories over $\mathbf{P}(V_1 \oplus V_2)$ (Example 3.15).

In general, the categorical join $\mathcal{J}(X_1, X_2)$ is not equivalent to the derived category of a variety, i.e. is not commutative. Rather, $\mathcal{J}(X_1, X_2)$ should be thought of as a noncommutative birational modification of the resolved join $\tilde{\mathbf{J}}(X_1, X_2)$. Indeed, away from the exceptional divisors (1.6), the categorical join coincides with the resolved join, which in turn coincides with the classical join if the morphisms $X_1 \to \mathbf{P}(V_1)$ and $X_2 \to \mathbf{P}(V_2)$ are embeddings. For a precise formulation of this claim, see Proposition 3.17 and Remark 3.18. In particular, if $X_1 \to \mathbf{P}(V_1)$ and $X_2 \to \mathbf{P}(V_2)$ are embeddings of smooth varieties, then $\mathcal{J}(X_1, X_2)$ can be thought of as a noncommutative resolution of singularities of the classical join $\mathbf{J}(X_1, X_2)$.

**Remark 1.6.** There are other notions of noncommutative resolutions in the literature, in particular categorical resolutions in the sense of [17, 22] and noncommutative resolutions in...
the sense of Van den Bergh [36, 35]. In the case where $X_1 \to \mathbf{P}(V_1)$ and $X_2 \to \mathbf{P}(V_2)$ are embeddings of smooth varieties, then using the results of [17] it can be shown that under certain hypotheses, $\mathcal{J}(X_1, X_2)$ is a resolution of $\mathbf{J}(X_1, X_2)$ in these senses. For instance, $\mathcal{J}(X_1, X_2)$ is a categorical resolution if $\mathbf{J}(X_1, X_2)$ has rational singularities.

Our main result describes the HPD of a categorical join. Recall that the HPD of a right strong, moderate Lefschetz category is equipped with a natural Lefschetz structure (see Theorem 2.24[5]).

**Theorem 1.7** (Theorem 4.1). Let $X_1 \to \mathbf{P}(V_1)$ and $X_2 \to \mathbf{P}(V_2)$ be right strong, moderate Lefschetz varieties. Let $X_1^\vee \to \mathbf{P}(V_1^\vee)$ and $X_2^\vee \to \mathbf{P}(V_2^\vee)$ be the HPD varieties. Then $\mathcal{J}(X_1, X_2)$ and $\mathcal{J}(X_1^\vee, X_2^\vee)$ are HPD, i.e. there is an equivalence

$$\mathcal{J}(X_1, X_2)^\vee \simeq \mathcal{J}(X_1^\vee, X_2^\vee)$$

of Lefschetz categories over $\mathbf{P}(V_1^\vee \oplus V_2^\vee)$.

**Remark 1.8.** As discussed in §1.1, the HPD varieties $X_1^\vee$ and $X_2^\vee$ will in general be noncommutative. Nonetheless, as mentioned in Remark 1.4, it is still possible to form their categorical join $\mathcal{J}(X_1^\vee, X_2^\vee)$.

Theorem 1.7 can be thought of as a categorification of the classical result that the operations of classical join and projective duality commute, i.e. for $X_1 \subset \mathbf{P}(V_1)$ and $X_2 \subset \mathbf{P}(V_2)$ we have

$$\mathbf{J}(X_1, X_2)^\vee = \mathbf{J}(X_1^\vee, X_2^\vee) \subset \mathbf{P}(V_1^\vee \oplus V_2^\vee),$$

where $(-)^\vee$ denotes the operation of classical projective duality.

1.3. **The nonlinear HPD theorem.** To explain an important general consequence of Theorem 1.7 we need the following observation. Given two closed subvarieties $X_1 \subset \mathbf{P}(V)$ and $X_2 \subset \mathbf{P}(V)$ of the same projective space, the classical join $\mathbf{J}(X_1, X_2)$ is a subvariety of $\mathbf{P}(V \oplus V)$. Let $W \subset V \oplus V$ be the graph of an isomorphism $\xi : V \to V$ given by scalar multiplication; e.g. $W \subset V \oplus V$ is the diagonal for $\xi = \text{id}$ and the antidiagonal for $\xi = -\text{id}$. Then we have

$$\mathbf{J}(X_1, X_2) \cap \mathbf{P}(W) \cong X_1 \cap X_2.$$

If, more generally, we have morphisms $X_1 \to \mathbf{P}(V_1)$ and $X_2 \to \mathbf{P}(V_2)$ instead of embeddings, then

$$\mathcal{J}(X_1, X_2) \times_{\mathbf{P}(V_1 \oplus V_2)} \mathbf{P}(W) \cong X_1 \times_{\mathbf{P}(V)} X_2.$$

Categorifying this, we show (Proposition 3.17) that if $X_1 \to \mathbf{P}(V_1)$ and $X_2 \to \mathbf{P}(V_2)$ are Lefschetz varieties, then

$$\mathcal{J}(X_1, X_2)_{\mathbf{P}(W)} \cong \text{Perf}(X_1 \times_{\mathbf{P}(V)} X_2)$$

where the left side is the base change of $\mathcal{J}(X_1, X_2)$ along the embedding $\mathbf{P}(W) \to \mathbf{P}(V \oplus V)$.

The orthogonal subspace to the diagonal $V \subset V \oplus V$ is the antidiagonal $V^\vee \subset V^\vee \oplus V^\vee$. Thus combining Theorem 1.7 with the main theorem of HPD, we obtain the following result, that we formulate here loosely; see Theorem 5.5 for the precise statement.

**Theorem 1.9.** Let $X_1 \to \mathbf{P}(V_1)$ and $X_2 \to \mathbf{P}(V_2)$ be right strong, moderate Lefschetz varieties. Let $X_1^\vee \to \mathbf{P}(V_1^\vee)$ and $X_2^\vee \to \mathbf{P}(V_2^\vee)$ be the HPD varieties. Then there are induced semiorthogonal decompositions of

$$\text{Perf}(X_1 \times_{\mathbf{P}(V)} X_2) \quad \text{and} \quad \text{Perf}(X_1^\vee \times_{\mathbf{P}(V^\vee)} X_2^\vee) \quad (1.7)$$
which have a distinguished component in common.

**Remark 1.10.** When the $X_k$ or $X^\natural_k$ are noncommutative, the terms in (1.7) must be interpreted as the relative tensor products

$$\text{Perf}(X_1) \otimes_{\text{Perf}(\mathbb{P}(V))} \text{Perf}(X_2) \quad \text{and} \quad \text{Perf}(X^\natural_1) \otimes_{\text{Perf}(\mathbb{P}(V^\vee))} \text{Perf}(X^\natural_2)$$

of linear categories; see §A.1 and especially Theorem A.2.

**Remark 1.11.** Using [4] or (if the fiber products in (1.7) are assumed Tor-independent) [18], Theorem 1.9 also implies an analogous result at the level of bounded derived categories of coherent sheaves in place of perfect complexes; see Remark 5.7 for details.

**Remark 1.12.** In Theorem 5.11 we prove an iterated version of Theorem 1.9, where we consider a collection of Lefschetz varieties $X_k \to \mathbb{P}(V)$ for $k = 1, 2, \ldots, \ell$. The left side of (1.7) is replaced with $\text{Perf}(X_1 \times_{\mathbb{P}(V)} \cdots \times_{\mathbb{P}(V)} X_\ell)$, but when $\ell > 2$ the right side must be replaced with a certain linear section of the iterated categorical join of the HPD varieties $X^\natural_k \to \mathbb{P}(V^\vee)$, which does not have a straightforward geometric interpretation.

If $X_2 = \mathbb{P}(L) \subset \mathbb{P}(V)$ for a vector subspace $0 \subset L \subset V$, then the HPD variety is the orthogonal space $X^\natural_2 = \mathbb{P}(L^\perp) \subset \mathbb{P}(V^\vee)$. Hence

$$X_1 \times_{\mathbb{P}(V)} X_2 = X_1 \times_{\mathbb{P}(V)} \mathbb{P}(L) \quad \text{and} \quad X^\natural_1 \times_{\mathbb{P}(V^\vee)} X^\natural_2 = X^\natural_1 \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L^\perp)$$

are mutually orthogonal linear sections of $X_1$ and $X^\natural_1$. The result of Theorem 1.9 then reduces to the main theorem of HPD. Accordingly, Theorem 1.9 should be thought of as a nonlinear version of HPD.

### 1.4. Applications.

Theorems 1.7 and 1.9 have many applications. We provide a few of them in §6 of the paper, to show how the theory works.

First, we consider the Grassmannians $\text{Gr}(2,5)$ and $\text{OGGr}^+(5,10)$, which have the special property of being (homologically) projectively self-dual. Given two copies of a Grassmannian of either type, we obtain a pair of varieties by forming their intersection and the intersection of their projective duals. We show in §6.1 that the $\text{Gr}(2,5)$ case gives a new pair of derived equivalent Calabi–Yau threefolds, and in §6.2 that the $\text{OGGr}^+(5,10)$ case gives a new pair of derived equivalent Calabi–Yau fivefolds. These pairs of Calabi–Yaus were recently discussed in [5, 31, 29], where they are shown to be non-birational if the two copies of the Grassmannian are chosen generically.

In §6.3 we also deduce from Theorem 1.7 a relation between the derived category of an Enriques surface and the twisted derived category of a stacky projective plane (Theorem 6.19), which can be regarded as an algebraization of the logarithmic transform.

### 1.5. Categorical cones.

For other applications we develop the notion of a categorical cone. This theory is very similar to, and under mild assumptions turns out to be a special case of, the theory of categorical joins. Assume given an exact sequence of vector spaces

$$0 \to V_0 \to V \to \bar{V} \to 0 \quad (1.8)$$

and a closed subvariety $X \subset \mathbb{P}(\bar{V})$. The classical cone over $X$ with vertex $\mathbb{P}(V_0)$ is the strict transform

$$C_{V_0}(X) \subset \mathbb{P}(V)$$
of $X$ under the linear projection $P(V) \to P(\tilde{V})$ from $P(V_0)$. Let $\tilde{H}$ denote the pullback to $X$ of the hyperplane class of $P(\tilde{V})$. The resolved cone is defined by

$$\tilde{C}_{V_0}(X) = P_X(V_X),$$

where the subbundle $V_X \subset V \otimes O_X$ is defined as the preimage of $O_X(-\tilde{H}) \subset \tilde{V} \otimes O_X$ under the map $V \otimes O_X \to \tilde{V} \otimes O_X$. The embedding of $V_X$ induces a morphism $\tilde{C}_{V_0}(X) \to P(V)$ which factors birationally through the classical cone and blows down the divisor

$$\varepsilon : E(X) = P_X(V_0 \otimes O_X) \hookrightarrow \tilde{C}_{V_0}(X)$$

to $P(V_0) \subset C_{V_0}(X)$. Note that $E(X)$ is canonically isomorphic to $P(V_0) \times X$.

The definition of the resolved cone extends verbatim to the case where $X$ is a variety equipped with a morphism $X \to P(V)$. Following the cue of joins, we use this to categorify cones as follows.

**Definition 1.13.** Let $X \to P(\tilde{V})$ be a Lefschetz variety with Lefschetz center $A_0 \subset \text{Perf}(X)$. The categorical cone of $X$ is the full subcategory of $\text{Perf}(\tilde{C}_{V_0}(X))$ defined by

$$\mathcal{C}_{V_0}(X) = \left\{ C \in \text{Perf}(\tilde{C}_{V_0}(X)) \mid \varepsilon^*(C) \in \text{Perf}(P(V_0)) \otimes A_0 \subset \text{Perf}(E(X)) \right\}.$$

We prove an exact analog of Theorem 1.15 for categorical cones, showing that $\mathcal{C}_{V_0}(X)$ has the structure of a Lefschetz category over $P(V)$, such that if $X$ is smooth and proper, or right strong, or moderate, then so is $\mathcal{C}_{V_0}(X)$ (Theorem 7.20). In particular, categorical cones can be used as inputs for HPD.

We also prove an analog of Theorem 1.7 describing HPD for categorical cones. In fact, we work in a more general setup that simultaneously allows for extensions of the ambient projective space, because this extra generality is useful in applications. Namely, let $V$ be a vector space and assume given a pair of subspaces

$$V_0 \subset V \quad \text{and} \quad V_\infty \subset V^\vee$$

such that $V_0 \subset V_\infty^\perp$, or equivalently $V_\infty \subset V_0^\perp$. Let $\tilde{V} = V_\infty^\perp/V_0$, so that $\tilde{V}^\vee \cong V_0^\perp/V_\infty$. For $V_\infty = 0$ this reduces to the situation (1.8) above. Let $X \to P(\tilde{V})$ be a right strong, moderate Lefschetz variety, with HPD variety $X^2 \to P(\tilde{V}^\vee)$. The categorical cone $\mathcal{C}_{V_0}(X)$ is then a Lefschetz category over $P(V_0^\perp)$. Via the inclusion $P(V_0^\perp) \to P(V)$ we can regard $\mathcal{C}_{V_0}(X)$ as a Lefschetz category over $P(V)$, which we write as $\mathcal{C}_{V_0}(X)/P(V)$ for emphasis. Similarly, we have a Lefschetz category $\mathcal{C}_{V_\infty}(X^2)/P(\tilde{V}^\vee)$ over $P(V^\vee)$.

**Theorem 1.14 (Theorem 8.1).** In the above situation, there is an equivalence

$$(\mathcal{C}_{V_0}(X)/P(V))^2 \cong \mathcal{C}_{V_\infty}(X^2)/P(\tilde{V}^\vee)$$

of Lefschetz categories over $P(V^\vee)$.

Cones can be expressed as joins in most cases. Indeed, assume we have an exact sequence (1.8) with $V_0 \neq 0$. Then for any $X \subset P(\tilde{V})$, a choice of splitting of this sequence induces an isomorphism

$$\mathcal{C}_{V_0}(X) \cong J(P(V_0), X),$$

and an expression of resolved cone $\tilde{C}_{V_0}(X)$ as the blow down of one of the exceptional divisors of the resolved join $J(P(V_0), X)$. At the categorical level, we show that if $X \to P(\tilde{V})$ is a Lefschetz variety, then the choice of a splitting of (1.8) induces a natural equivalence

$$\mathcal{C}_{V_0}(X) \cong J(P(V_0), X)$$
of Lefschetz categories over $\mathbf{P}(V)$ (Proposition 7.24).

Note, however, that for $V_0 = 0$ the above relations do not hold, since then $\mathcal{C}_{V_0}(X) = X$ and $\mathcal{C}_{V_0}(X) \simeq \text{Perf}(X)$, whereas $J(\mathbf{P}(V_0), X) = \emptyset$ and $J(\mathbf{P}(V_0), X) = 0$. Moreover, even if $V_0 \neq 0$, we still need to choose a splitting of (1.8) to be able to form $J(\mathbf{P}(V_0), X)$. When working over a field (as we tacitly do in the introduction) this is not a problem, but it may not be possible when working over a general base scheme, as we do in the body of the paper. This is the main reason that we develop the notion of a categorical cone directly, without the use of categorical joins. On the other hand, to prove Theorem 1.14 we reduce to the analogous Theorem 1.17 for joins.

1.6. HPD for quadrics. We use categorical cones to develop HPD for quadrics. By a quadric, we mean an integral scheme isomorphic to a degree 2 hypersurface in a projective space. Any quadric $Q$ can be expressed as a classical cone $Q = C_K(\overline{Q})$ over a smooth quadric $\overline{Q}$. We define the standard categorical resolution of $Q$ as the categorical cone $Q = \mathcal{C}_K(\overline{Q})$, where $\overline{Q}$ is equipped with a natural Lefschetz decomposition involving spinor bundles, see Lemma 9.9.

We show the class of standard categorical resolutions of quadrics is closed under HPD. Namely, we consider pairs $(Q, f)$ where $Q$ is a quadric and $f: Q \to \mathbf{P}(V)$ is a morphism such that $f^*\mathcal{O}_{\mathbf{P}(V)}(1)$ is the ample line bundle that realizes $Q$ as a quadratic hypersurface in a projective space. We define a generalized duality operation $(Q, f) \mapsto (Q^\sharp, f^\sharp)$ on such pairs, where the target of $f^\sharp: Q^\sharp \to \mathbf{P}(V^\vee)$ is the dual projective space. This generalized duality reduces to classical projective duality when $Q$ has even dimension and $f: Q \to \mathbf{P}(V)$ is an embedding, and involves passing to a double covering or branch divisor in other cases (see Definition 9.14). Our main result for quadrics is as follows.

**Theorem 1.15** (Theorem 9.17). Let $(Q, f)$ and $(Q^\sharp, f^\sharp)$ be a generalized dual pair as above. Then the HPD of the standard categorical resolution of $Q$ over $\mathbf{P}(V)$ is equivalent to the standard categorical resolution of $Q^\sharp$ over $\mathbf{P}(V^\vee)$.

By combining Theorem 1.15 with the nonlinear HPD Theorem 1.9, we obtain a relation between

$$
\text{Perf}(X \times_{\mathbf{P}(V)} Q) \quad \text{and} \quad \text{Perf}(X^\sharp \times_{\mathbf{P}(V^\vee)} Q^\sharp),
$$

(1.9)

where $(Q, f)$ and $(Q^\sharp, f^\sharp)$ is a generalized dual pair, and $X \to \mathbf{P}(V)$ is a right strong, moderate Lefschetz variety with HPD variety $X^\sharp \to \mathbf{P}(V^\vee)$. This gives a quadratic version of the main theorem of HPD, which is very useful for applications. For precise statements, see Theorem 9.18 and Lemma 9.11. As in Remark 1.11 this also implies a similar result for bounded derived categories of coherent sheaves.

We apply the above relation for $X$ being Gr$(2, 5)$ or OGr$^+_+(5, 10)$. In these cases the HPD variety $X^\sharp$ is abstractly isomorphic to $X$. An interesting feature of generalized duality of quadrics is that the dimension of $Q^\sharp$ may be different from the dimension of $Q$ (although the sum of the dimensions is congruent to $\dim(V) \mod 2$). Thus, (1.9) relates the derived categories of varieties of the same type, but different dimensions. Among other things, this proves the duality conjecture for Gushel–Mukai varieties as stated in [24], see §9.5.
1.7. **Homological projective geometry.** Our results suggest the existence of a robust theory of homological projective geometry, of which homological projective duality, categorical joins, and categorical cones are the first instances. In this theory, Lefschetz categories over $\mathbf{P}(V)$ should play the role of projective varieties embedded in $\mathbf{P}(V)$. An interesting feature of the operations known so far is that they preserve smoothness of the objects involved, whereas in classical projective geometry this is far from true. In fact, this principle of homological smoothness guided our constructions.

The vision of homological projective geometry is alluring because the known results are so powerful, and yet they correspond to a small sector of the vast theory of classical projective geometry. For instance, it would be very interesting to categorify secant varieties, and prove an HPD statement for them. The ideas of this paper should be useful for making progress in this area.

1.8. **Noncommutative algebraic geometry framework.** To finish the introduction, we explain the framework of noncommutative algebraic geometry adopted in this paper. As explained above, the idea is to replace varieties with suitable linear categories. Such a framework is absolutely necessary — without one we could not even formulate our main results.

Given a scheme $T$, we need notions of $T$-linear categories and functors, subject to the following desiderata:

- A $T$-linear category should carry an action of the category Perf($T$) of perfect complexes on $T$, and a $T$-linear functor should commute with the Perf($T$)-actions.
- The class of $T$-linear categories should include derived categories of schemes over $T$, with Perf($T$)-action given by pullback followed by tensor product. Moreover, if $\mathcal{C}$ is a $T$-linear category and $\mathcal{A} \subset \mathcal{C}$ is an admissible subcategory which is preserved by the Perf($T$)-action, then $\mathcal{A}$ should be $T$-linear.
- There should be a theory of tensor products, which given $T$-linear categories $\mathcal{C}$ and $\mathcal{D}$ produces (in a suitably functorial way) another $T$-linear category $\mathcal{C} \otimes_{\text{Perf}(T)} \mathcal{D}$. Taking $\mathcal{D}$ to be the derived category of a scheme $T'$ equipped with a morphism $T' \to T$, this defines a base change category $\mathcal{C}_{T'}$, which should carry a natural $T'$-linear structure. If further $\mathcal{C}$ is the derived category of a scheme $X$ over $T$, then the base change $\mathcal{C}_{T'}$ should coincide with the derived category of the geometrically defined base change $X_{T'}$.

Let us briefly describe two alternatives to defining such a class of $T$-linear categories:

1. **Down-to-earth approach:** Define a $T$-linear category to be an admissible subcategory $\mathcal{A}$ of Perf($X$) or $\mathbf{D}_{\text{coh}}^b(X)$ (the bounded derived category of coherent sheaves) which is preserved by the Perf($T$)-action, where $X$ is a proper scheme over $T$, and define a $T$-linear functor between $\mathcal{A} \subset \mathbf{D}_{\text{coh}}^b(X)$ and $\mathcal{B} \subset \mathbf{D}_{\text{coh}}^b(Y)$ to be a functor induced by a Fourier–Mukai functor between $\mathbf{D}_{\text{coh}}^b(X)$ and $\mathbf{D}_{\text{coh}}^b(Y)$ with kernel schematically supported on the fiber product $X \times_T Y$. In this setting, a base change operation (along morphisms satisfying a transversality assumption) with the necessary compatibilities is developed in [18]. A version of HPD in this context is described in [16].

2. **Higher approach:** Define a $T$-linear category to be a small idempotent-complete stable $\infty$-category equipped with a Perf($T$)-module structure, and define a $T$-linear functor to be an exact functor commuting with the Perf($T$)-modules structures. Relying on Lurie’s foundational work [27], this approach is developed in [32] and used to give a version of HPD in this context.
The advantage of (1) is that it avoids the use of higher category theory and derived algebraic geometry. The advantages of (2) are that it includes (1) as a special case, allows us to prove more general results (e.g. over general base schemes and without transversality hypotheses), and there is a complete reference [32] for the results we need in this setting. In particular, [32] proves a version of HPD which allows linear categories as both inputs and outputs.

To fix ideas, in this paper we adopt approach (2). However, the reader who prefers (1) should have no trouble translating everything to that setting. In fact, we encourage the reader who is not already familiar with noncommutative algebraic geometry to assume all noncommutative schemes are “commutative”, i.e. of the form Perf($X$); for intuition, we have explained throughout the paper what our constructions amount to in this situation.

To recapitulate, from now on we use the following definition.

**Definition 1.16.** Let $T$ be a scheme. A $T$-linear category is a small idempotent-complete stable $\infty$-category equipped with a Perf($T$)-module structure, and a $T$-linear functor between $T$-linear categories is an exact functor of Perf($T$)-modules.

In Appendix A we summarize the key facts about linear categories used in this paper.

1.9. **Conventions.** All schemes are assumed to be quasi-compact and separated. Instead of working over a ground field, we work relative to a fixed base scheme $S$ throughout the paper. Namely, all schemes will be defined over $S$ and all categories will be linear over $S$. The only time we make extra assumptions on $S$ is in our discussion of applications in §6 and §9.4 where for simplicity we assume $S$ is the spectrum of a field.

A vector bundle $V$ on a scheme $S$ is a finite locally free $\mathcal{O}_S$-module. Given such a $V$, its projectivization is

$$P(V) = \text{Proj}(\text{Sym}^* (V^\vee)) \to S$$

with $\mathcal{O}_{P(V)}(1)$ normalized so that its pushforward to $S$ is $V^\vee$. Note that we suppress $S$ by writing $P(V)$ instead of $P_S(V)$. A subbundle $W \subset V$ is an inclusion of vector bundles whose cokernel is a vector bundle. Given such a $W \subset V$, the orthogonal subbundle is defined by

$$W^\perp = \ker (V^\vee \to W^\vee). \quad (1.10)$$

We often commit the following convenient abuse of notation: given a line bundle $L$ or a divisor class $D$ on a scheme $T$, we denote still by $L$ or $D$ its pullback to any variety mapping to $T$. Similarly, if $X \to T$ is a morphism and $V$ is a vector bundle on $T$, we sometimes write $V \otimes \mathcal{O}_X$ for the pullback of $V$ to $X$.

Given morphisms of schemes $X \to T$ and $Y \to T$, the symbol $X \times_T Y$ denotes their derived fiber product (see [23] for background on derived algebraic geometry). We note that this agrees with the usual fiber product of schemes whenever the morphisms $X \to T$ and $Y \to T$ are Tor-independent over $T$. To lighten the notation, we write fiber products over our fixed base $S$ as absolute fiber products, i.e. we write

$$X \times Y = X \times_S Y.$$  

If $X$ is a scheme over $T$, we denote by Perf($X$) its category of perfect complexes and by $\text{D}^{\text{b}}_{\text{coh}}(X)$ its bounded derived category of coherent sheaves, which we consider as $T$-linear categories in the sense of Definition 1.16. The Perf($T$)-module structure on these categories is given by the (derived) tensor product with the (derived) pullback of objects from Perf($T$). See Appendix A for background on linear categories.
We always consider derived functors (pullbacks, pushforwards, tensor products, etc.), but write them with underived notation. For example, for a morphism of schemes \( f: X \to Y \) we write \( f^*: \text{Perf}(Y) \to \text{Perf}(X) \) for the derived pullback functor, and similarly for the functors \( f_* \) and \( \otimes \). We always work with functors defined between categories of perfect complexes. Note that in general, \( f_* \) may not preserve perfect complexes, but it does if \( f: X \to Y \) is a perfect (i.e., pseudo-coherent of finite Tor-dimension) proper morphism \([26]\) Example 2.2(a)]. This assumption will be satisfied in all of the cases where we use \( f_* \) in the paper.

Recall that \( f_* \) is right adjoint to \( f^* \). Sometimes, we need other adjoint functors as well. Provided they exist, we denote by \( f^! \) the right adjoint of \( f_*: \text{Perf}(X) \to \text{Perf}(Y) \) and by \( f_! \) the left adjoint of \( f^*: \text{Perf}(Y) \to \text{Perf}(X) \), so that \((f_!, f^*, f_!, f^!)\) is an adjoint sequence. For instance, if \( f: X \to Y \) is a perfect proper morphism and a relative dualizing complex \( \omega_f \) exists and is a perfect complex on \( X \), then \( f^! \) and \( f_! \) exist and are given by

\[
\begin{align*}
 f^!(-) &\simeq f^*(-) \otimes \omega_f \quad \text{and} \quad f_!(-) \simeq f_*(- \otimes \omega_f).
\end{align*}
\] (1.11)

Indeed, the functor \( f_*: \text{D}_{\text{qc}}(X) \to \text{D}_{\text{qc}}(Y) \) between the unbounded derived categories of quasi-coherent sheaves admits a right adjoint \( f^!: \text{D}_{\text{qc}}(Y) \to \text{D}_{\text{qc}}(X) \), the relative dualizing complex of \( f \) is by definition \( \omega_f = f^!(\mathcal{O}_Y) \), and the above formulas for \( f^! \) and \( f_! \) hold on quasi-coherent categories (the first holds by \([26]\) Proposition 2.1] and implies the second). Hence if \( f \) is a perfect proper morphism and \( \omega_f \) is perfect, it follows that all of these functors and adjunctions restrict to categories of perfect complexes. In all of the cases where we need \( f^! \) and \( f_! \) in the paper, the following stronger assumptions will be satisfied.

**Remark 1.17.** Suppose \( f: X \to Y \) is a morphism between schemes which are smooth, projective, and of constant relative dimension over \( S \). Then \( f \) is perfect, projective, and has a relative dualizing complex, which is a shift of a line bundle:

\[
\omega_f = \omega_{X/S} \otimes f^!(\omega_{Y/S})^\vee.
\] (1.12)

In particular, for such an \( f \), all of the functors \( f_!, f^*, f_*, f^! \) are defined and adjoint between categories of perfect complexes.

Given a \( T \)-linear category \( \mathcal{C} \), we write \( \mathcal{C} \otimes F \) for the action of an object \( F \in \text{Perf}(T) \) on an object \( C \in \mathcal{C} \). Given \( T \)-linear categories \( \mathcal{C} \) and \( \mathcal{D} \), we denote by \( \mathcal{C} \otimes_{\text{Perf}(T)} \mathcal{D} \) their \( T \)-linear tensor product, see \([A.1] \). Parallel to our convention for fiber products of schemes, if \( T = S \) is our fixed base scheme, we simplify notation by writing

\[
\mathcal{C} \otimes \mathcal{D} = \mathcal{C} \otimes_{\text{Perf}(S)} \mathcal{D}.
\]

If \( \mathcal{C} \) is a \( T \)-linear category and \( T' \to T \) is a morphism of schemes, we write

\[
\mathcal{C}_{T'} = \mathcal{C} \otimes_{\text{Perf}(T)} \text{Perf}(T')
\]

for the \( T' \)-linear category obtained by base change. By abuse of notation, if \( \mathcal{C} \) is a \( T \)-linear category and \( F: \mathcal{D}_1 \to \mathcal{D}_2 \) is a \( T \)-linear functor, then we frequently write \( F \) for the induced functor

\[
F: \mathcal{C} \otimes_{\text{Perf}(T)} \mathcal{D}_1 \to \mathcal{C} \otimes_{\text{Perf}(T)} \mathcal{D}_2.
\]

Finally, if \( \phi: \mathcal{C} \to \mathcal{D} \) is a \( T \)-linear functor, we write \( \phi^*, \phi^!: \mathcal{D} \to \mathcal{C} \) for its left and right adjoints, if they are defined.
1.10. **Related work.** The results of this paper were first announced in seminar talks in 2015. Since then Jiang, Leung, and Xie have posted a paper \[10\] whose main result is a version of our nonlinear HPD Theorem 1.9. Their result is proved in a less general setting, by a different argument which does not involve joins, and which does not yield an iterated nonlinear HPD theorem as in Remark 1.12. Moreover, they do not prove an HPD statement between two categories as in our main result, Theorem 1.7.

1.11. **Organization of the paper.** In \[2\] we gather preliminaries on HPD, including notably a useful new characterization of the HPD category, on which the proof of our main theorem relies.

In \[3\] we define the categorical join of two Lefschetz categories, show that it is equipped with a canonical Lefschetz structure, and study its behavior under base change.

In \[4\] we prove our main theorem, stated above as Theorem 1.7.

In \[5\] we prove the nonlinear HPD theorem, stated above as Theorem 1.9.

In \[6\] we discuss the applications of the previous two theorems mentioned in \[1.4\].

In \[7\] we define the categorical cone, study it along the lines of \[3\] and show that it can be expressed as a categorical join in the split case.

In \[8\] we prove our theorem on HPD for cones, stated above as Theorem 1.14.

In \[9\] we prove the results on HPD for quadrics discussed in \[1.6\].

In Appendix A we collect some useful results on linear categories.

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2. **Preliminaries on HPD**

In this section, we discuss preliminary material on HPD that will be needed in the rest of the paper. In \[2.1\] we review the notion of a Lefschetz category, and in \[2.2\] we recall the definition of the HPD category and state the main theorem of HPD. In \[2.3\] we prove a useful characterization of the HPD category in terms of the projection functor from the universal hyperplane section. Finally, in \[2.4\] we describe the behavior of HPD when the base projective bundle is replaced with a quotient bundle or with a subbundle.

Recall that we work relative to a general base scheme $S$. In particular, we consider HPD over a projective bundle $\mathbb{P}(V)$, where $V$ is a vector bundle on $S$. This is convenient because it includes various relative versions of HPD (cf. \[14\] Theorem 6.27 and Remark 6.28) into the general framework. We denote by $N$ the rank of $V$ and by $H$ the relative hyperplane class on the projective bundle $\mathbb{P}(V)$ such that $\mathcal{O}(H) = \mathcal{O}_{\mathbb{P}(V)}(1)$.

2.1. **Lefschetz categories.** The fundamental objects of HPD are Lefschetz categories. We summarize the basic definitions following \[32\] \[6\], starting with the notion of a Lefschetz center.

**Definition 2.1.** Let $T$ be a scheme over $S$ with a line bundle $\mathcal{L}$. Let $\mathcal{A}$ be a $T$-linear category. An admissible $S$-linear subcategory $\mathcal{A}_0 \subset \mathcal{A}$ is called a *Lefschetz center* of $\mathcal{A}$ with respect to $\mathcal{L}$.
if the subcategories $A_i \subset \mathcal{A}$, $i \in \mathbb{Z}$, determined by
\begin{align}
A_i &= A_{i-1} \cap (A_0 \otimes \mathcal{L}^{-i})^\perp, \quad i \geq 1 \tag{2.1} \\
A_i &= A_{i+1} \cap (A_0 \otimes \mathcal{L}^{-i})^\perp, \quad i \leq -1 \tag{2.2}
\end{align}
are right admissible in $\mathcal{A}$ for $i \geq 1$, left admissible in $\mathcal{A}$ for $i \leq -1$, vanish for all $i$ of sufficiently large absolute value, say for $|i| \geq m$, and provide $S$-linear semiorthogonal decompositions
\begin{align}
\mathcal{A} &= \langle A_0, A_1 \otimes \mathcal{L}, \ldots, A_{m-1} \otimes \mathcal{L}^{m-1} \rangle, \tag{2.3} \\
\mathcal{A} &= \langle A_{1-m} \otimes \mathcal{L}^{1-m}, \ldots, A_{-1} \otimes \mathcal{L}^{-1}, A_0 \rangle. \tag{2.4}
\end{align}

The categories $A_i, i \in \mathbb{Z}$, are called the \textit{Lefschetz components} of the Lefschetz center $A_0 \subset \mathcal{A}$. The semiorthogonal decompositions (2.3) and (2.4) are called \textit{the right Lefschetz decomposition and the left Lefschetz decomposition} of $\mathcal{A}$. The minimal $m$ above is called the \textit{length} of the Lefschetz decompositions.

\textbf{Remark 2.2.} By [32, Lemma 6.3], if the subcategories $A_i \subset \mathcal{A}$ are admissible for all $i \geq 0$ or all $i \leq 0$, then the length $m$ defined above satisfies
\[ m = \min \{ i \geq 0 \mid A_i = 0 \} = \min \{ i \geq 0 \mid A_{-i} = 0 \}. \]

\textbf{Remark 2.3.} If $\mathcal{A}$ is a $T$-linear category equipped with a $T$-linear autoequivalence $\alpha: \mathcal{A} \to \mathcal{A}$, there is a more general notion of a Lefschetz center of $\mathcal{A}$ with respect to $\alpha$, see [32, §6.1]. The case where $\alpha$ is the autoequivalence $- \otimes \mathcal{L}$ for a line bundle $\mathcal{L}$ recovers the above definitions. This notion is also useful for other choices of $\alpha$, see [25, §2].

The following shows that giving a Lefschetz center is equivalent to giving Lefschetz decompositions with suitably admissible components. This is useful in practice for constructing Lefschetz centers.

\textbf{Lemma 2.4.} Let $T$ be a scheme over $S$ with a line bundle $\mathcal{L}$. Let $\mathcal{A}$ be a $T$-linear category with $S$-linear semiorthogonal decompositions
\begin{align}
\mathcal{A} &= \langle A_0, A_1 \otimes \mathcal{L}, \ldots, A_{m-1} \otimes \mathcal{L}^{m-1} \rangle, \quad \text{where } A_0 \supset A_1 \supset \cdots \supset A_{m-1}, \tag{2.5} \\
\mathcal{A} &= \langle A_{1-m} \otimes \mathcal{L}^{1-m}, \ldots, A_{-1} \otimes \mathcal{L}^{-1}, A_0 \rangle, \quad \text{where } A_{1-m} \supset \cdots \supset A_{-1} \supset \cdots \supset A_0. \tag{2.6}
\end{align}
Then the categories $A_i$, $|i| < m$, satisfy (2.1) and (2.2), and the categories defined by (2.1) and (2.2) for $|i| \geq m$ vanish. Hence if $A_i \subset \mathcal{A}$ is right admissible for $i \geq 0$ and left admissible for $i \leq 0$, then $A_0 \subset \mathcal{A}$ is a Lefschetz center.

\textit{Proof.} Note that $A_0$ is left admissible by the first semiorthogonal decomposition and right admissible by the second. The rest follows from [32, Lemma 6.3]. \qed

\textbf{Remark 2.5.} If $\mathcal{A}$ as in Lemma 2.4 is smooth and proper over $S$, then in order for a subcategory $A_0 \subset \mathcal{A}$ to be a Lefschetz center, it is enough to give only one of the semiorthogonal decompositions (2.5) or (2.6). This follows from [32, Lemmas 4.15 and 6.3].

Note that the subcategories $A_i \subset A_0$, $i \in \mathbb{Z}$, associated to a Lefschetz center form two (different in general) chains of admissible subcategories
\[ 0 \subset A_{1-m} \subset \cdots \subset A_{-1} \subset A_0 \supset A_1 \supset \cdots \supset A_{m-1} \supset 0. \tag{2.7} \]
For $i \geq 1$ the \textit{i-th right primitive component} $a_i$ of a Lefschetz center is defined as the right orthogonal to $A_{i+1}$ in $A_i$, i.e.
\[ a_i = A_{i+1}^\perp \cap A_i, \]
so that
\[ A_i = \langle a_i, A_{i+1} \rangle = \langle a_i, a_{i+1}, \ldots, a_{m-1} \rangle. \] (2.8)
Similarly, for \( i \leq -1 \) the \( i \)-th left primitive component \( a_i \) of a Lefschetz center is the left orthogonal to \( A_{i-1} \) in \( A_i \), i.e.
\[ a_i = \perp A_{i-1} \cap A_i, \] so that
\[ A_i = \langle A_{i-1}, a_i \rangle = \langle a_{1-m}, a_{i-1}, a_i \rangle. \] (2.9)
For \( i = 0 \), we have both right and left primitive components, defined by
\[ a_{+0} = A_1^+ \cap A_0 \quad \text{and} \quad a_{-0} = \perp A_{-1} \cap A_0. \]
These are related by the formula \( a_{-0} = a_{+0} \otimes \mathcal{L} \), see [32, Remark 6.4].

To simplify formulas, we sometimes abusively write \( a_0 \) to mean either \( a_{-0} \) or \( a_{+0} \), when it is clear from context which is intended. So for instance the formulas (2.8) and (2.9) make sense for \( i = 0 \), and the right Lefschetz decomposition of \( A \) in terms of primitive categories can be written as
\[ A = \langle a_0, a_{m-1}, a_1 \otimes \mathcal{L}, \ldots, a_{m-1} \otimes \mathcal{L}, \ldots, a_{m-1} \otimes \mathcal{L}^{m-1} \rangle \]
\[ = \langle a_i \otimes \mathcal{L}^t \rangle_{0 \leq t \leq i \leq m-1}, \] (2.10)
while the left Lefschetz decomposition can be written as
\[ A = \langle a_{1-m} \otimes \mathcal{L}^{1-m}, \ldots, a_{1-m} \otimes \mathcal{L}^{-1}, \ldots, a_{-1} \otimes \mathcal{L}^{-1}, a_{m-1}, \ldots, a_0 \rangle \]
\[ = \langle a_i \otimes \mathcal{L}^i \rangle_{1-m \leq i \leq -t \leq 0}. \] (2.11)

**Definition 2.6.** A Lefschetz category \( A \) over \( \mathbf{P}(V) \) is a \( \mathbf{P}(V) \)-linear category equipped with a Lefschetz center \( A_0 \subset A \) with respect to \( \mathcal{O}(H) \). The length of \( A \) is the length of its Lefschetz decompositions, and is denoted by \( \text{length}(A) \).

Given Lefschetz categories \( A \) and \( B \) over \( \mathbf{P}(V) \), an equivalence of Lefschetz categories or a Lefschetz equivalence is a \( \mathbf{P}(V) \)-linear equivalence \( A \simeq B \) which induces an \( S \)-linear equivalence \( A_0 \simeq B_0 \) of centers.

For HPD we will need to consider Lefschetz categories that satisfy certain “strongness” and “moderateness” conditions, defined below.

**Definition 2.7.** A Lefschetz category \( A \) is called right strong if all of its right primitive components \( a_{+0}, a_i, i \geq 1 \), are admissible in \( A \), left strong if all of its left primitive components \( a_{-0}, a_i, i \leq -1 \), are admissible in \( A \), and strong if all of its primitive components are admissible.

**Remark 2.8.** If \( A \) is smooth and proper over \( S \), then any Lefschetz structure on \( A \) is automatically strong, see [32, Remark 6.7].

By [32] Corollary 6.19(1), the length of a Lefschetz category \( A \) over \( \mathbf{P}(V) \) satisfies
\[ \text{length}(A) \leq \text{rank}(V). \] (2.12)

**Definition 2.9.** A Lefschetz category \( A \) over \( \mathbf{P}(V) \) is called moderate if its length satisfies the strict inequality
\[ \text{length}(A) < \text{rank}(V). \]
Example 2.11. Let recall one simple example, and several others are recalled in examples. The categories that arise in practice are moderate; we do not know any interesting immoderate examples.

There are many examples of interesting Lefschetz categories, see [19] for a survey. Here we recall one simple example, and several others are recalled in [6] and [9,4].

Remark 2.10. Moderateness of a Lefschetz category $A$ over $\mathcal{P}(V)$ is a very mild condition. Indeed, we can always embed $V$ into a vector bundle $V'$ of larger rank, e.g. $V' = V \oplus \emptyset$, and then $A$ is a moderate Lefschetz category over $\mathcal{P}(V')$. Moreover, essentially all Lefschetz categories that arise in practice are moderate; we do not know any interesting immoderate examples.

The key property of a Lefschetz category is that its Lefschetz decomposition behaves well under passage to linear sections.

Lemma 2.12. Let $A$ be a Lefschetz category over $\mathcal{P}(V)$ of length $m$. Let $L \subset V$ be a subbundle of corank $s$. Then the functor $A \to A_{\mathcal{P}(L)}$ induced by pullback along $\mathcal{P}(L) \to \mathcal{P}(V)$ is fully faithful on the Lefschetz components $A_i \subset A$ for $|i| \geq s$. Moreover, denoting their images by the same symbols, there are semiorthogonal decompositions

$$A_{\mathcal{P}(L)} = \langle \mathcal{K}_L(A)_i, A_s(H), \ldots, A_{m-1}((m-s)H) \rangle,$$

$$A_{\mathcal{P}(L)} = \langle A_{1-m}((s-m)H), \ldots, A_{-s}(-H), \mathcal{K}_L'(A) \rangle.$$

Proof. This is a special case of [32, Lemmas 6.20 and 6.22(3)].

Remark 2.13. The analogy between Lemma 2.12 and the Lefschetz hyperplane theorem is the source of the terminology “Lefschetz category”. The main theorem of HPD (Theorem 2.24 below) describes the categories $\mathcal{K}_L(A)$ and $\mathcal{K}_L'(A)$ in terms of the HPD category of $A$.

In this paper, we will be concerned with producing certain equivalences of Lefschetz categories. For this, the criteria below will be useful.

Lemma 2.14. Let $\phi: A \to B$ be a $\mathcal{P}(V)$-linear functor between Lefschetz categories $A$ and $B$ over $\mathcal{P}(V)$. Assume:

1. $\phi$ induces an equivalence $A_0 \simeq B_0$.
2. $\phi$ admits a left adjoint $\phi^*: B \to A$.
3. $\phi^*$ induces an equivalence $B_0 \simeq A_0$.

Then $\phi$ is an equivalence of Lefschetz categories.

Remark 2.15. A similar criterion is true if we replace the left adjoint $\phi^*$ with the right adjoint $\phi^!$. 
Proof. First we show $\phi$ is fully faithful. Consider the counit morphism

$$\phi^* \circ \phi \to \text{id}_A.$$  

This is a morphism in the category of $P(V)$-linear functors $\text{Fun}_{\text{Perf}(P(V))}(A, A)$ (note that $\phi^*$ is a $P(V)$-linear functor [32, Lemma 2.11]). Let $\psi: A \to A$ be the $P(V)$-linear functor given by the cone of this morphism. Then the claim that $\phi$ is fully faithful is equivalent to $\psi$ being the zero functor. Our assumptions imply $\psi$ vanishes on the subcategory $A_0 \subset A$. By $P(V)$-linearity it follows that $\psi$ vanishes on $A_0(iH)$, and hence on $A_i(iH) \subset A_0(iH)$, for all $i$. But then the (right or left) Lefschetz decomposition of $A$ implies $\psi$ is the zero functor.

Since $\phi$ is fully faithful, its image is a $P(V)$-linear triangulated subcategory of $B$. By assumption this image contains $B_0$, and hence by $P(V)$-linearity it contains $B_i(iH) \subset B_0(iH)$ for all $i$. But then the (right or left) Lefschetz decomposition of $B$ implies $\phi$ is essentially surjective. 

The second criterion we give for an equivalence of Lefschetz categories is a local one. To formulate it, we note the following. Let $S' \to S$ be a morphism of schemes, and let $V_{S'}$ denote the pullback of $V$ to $S'$. Then if $A$ is a Lefschetz category over $P(V)$, the base change $A_{S'}$ is naturally a Lefschetz category over $P(V_{S'})$ with Lefschetz center given by the base change $(A_0)_{S'} \subset A_{S'}$. This follows from a combination of Lemmas 2.4, A.5, and A.6.

Lemma 2.16. Let $A$ and $B$ be Lefschetz categories over $P(V)$. Let $\phi: A \to B$ be a $P(V)$-linear functor which admits a left or right adjoint. Let $\{U_i \to S\}$ be an fpqc cover of $S$, and let $\phi_{U_i}: A_{U_i} \to B_{U_i}$ denote the induced functor for any $i$. Then $\phi$ is an equivalence of Lefschetz categories over $P(V)$ if and only if $\phi_{U_i}$ is an equivalence of Lefschetz categories over $P(V_{U_i})$ for every $i$.

Proof. Follows from Proposition A.9. 

2.2. The HPD category. Let $H'$ denote the relative hyperplane class on $P(V)\vee$ such that $O(H') = O_{P(V)\vee}(1)$. Let

$$\delta: H(P(V)) \to P(V) \times P(V'\vee).$$

be the natural incidence divisor. We think of $H(P(V))$ as the universal hyperplane in $P(V)$.

If $X$ is a scheme with a morphism $X \to P(V)$, then the universal hyperplane section of $X$ is defined by

$$H(X) = X \times_{P(V)} H(P(V)).$$

This definition extends directly to linear categories as follows.

Definition 2.17. Let $A$ be a $P(V)$-linear category. The universal hyperplane section of $A$ is defined by

$$H(A) = A \otimes_{\text{Perf}(P(V))} \text{Perf}(H(P(V))).$$

The above definition is compatible with the geometric one in the following sense: if $X$ is a scheme over $P(V)$, then by Theorem A.2 there is an equivalence $H(\text{Perf}(X)) \simeq \text{Perf}(H(X))$. We sometimes use the more elaborate notation

$$H(X/P(V)) = H(X) \quad \text{and} \quad H(A/P(V)) = H(A)$$

to emphasize the universal hyperplane section is being taken with respect to $P(V)$.
The natural embedding $\delta$ includes into the following diagram of morphisms

\[
\begin{array}{ccc}
H(P(V)) & \xrightarrow{\delta} & P(V) \\
\downarrow & & \downarrow h \\
P(V) & \leftarrow P(V) \times P(V^\vee) & \xrightarrow{\text{pr}_2} P(V^\vee).
\end{array}
\]

(2.13)

Here we deviate slightly from the notation of [32], where the morphisms $\pi$, $\delta$, and $h$ are instead denoted $p$, $\iota$, and $f$. All schemes in the diagram are smooth and projective over $S$, hence Remark 1.17 applies to all morphisms. For a $P(V)$-linear category $A$, it follows from Theorem A.2 that there are canonical identifications

\[
A \otimes_{\text{Perf}(P(V))} \text{Perf}(P(V) \times P(V^\vee)) \simeq A \otimes_{\text{Perf}(P(V))} \text{Perf}(P(V^\vee)),
\]

by which we will regard the functors induced by morphisms in (2.13) as functors

\[
\delta_* : H(A) \rightarrow A \otimes_{\text{Perf}(P(V^\vee))} \text{Perf}(P(V^\vee)), \quad \pi_* : H(A) \rightarrow A,
\]

and so on.

The next definition differs from the original in [14], but is equivalent to it, as Lemma 2.22 below shows. The advantage of this definition is that it is more symmetric (with respect to the left and the right Lefschetz decompositions of $A$).

**Definition 2.18.** Let $A$ be a Lefschetz category over $P(V)$. Then the **HPD category** $A^\natural$ of $A$ is the full $P(V^\vee)$-linear subcategory of $H(A)$ defined by

\[
A^\natural = \{ C \in H(A) \mid \delta_*(C) \in A_0 \otimes_{\text{Perf}(P(V^\vee))} \}\.
\]

(2.14)

We sometimes use the notation

\[
(A/P(V))^\natural = A^\natural
\]

to emphasize the dependence on the $P(V)$-linear structure.

**Remark 2.19.** The HPD category $A^\natural$ depends on the choice of the Lefschetz center $A_0 \subset A$, although this is suppressed in the notation. For instance, for the “stupid” Lefschetz center $A_0 = A$ we have $A^\natural = H(A)$. The dependence of $A^\natural$ on the $P(V)$-linear structure is addressed by Proposition 2.32 and Corollary 8.3. Under mild hypotheses we can show that there is a Lefschetz equivalence $A^\natural \simeq \nabla A$ [32, Proposition 7.12], but in general we do not know if one exists. In this paper, we will deal almost exclusively with $A^\natural$, and therefore simply refer to it as the HPD category. All of our results can be translated directly to the “left HPD” setting.

**Remark 2.20.** In fact, following [32 §7.1], the category $A^\natural$ should more precisely be called the **right HPD category** of $A$. Indeed, there is also a left HPD category $^\natural A$, which is defined by replacing the right adjoint $\delta_*$ to $\delta^*$ with the left adjoint $\delta_!$ in (2.14). As shown in [32, Lemma 7.2], there is a $P(V^\vee)$-linear equivalence $A^\natural \simeq ^\natural A$. Under mild hypotheses these categories are endowed with natural Lefschetz structures, see [32 §7.2]; for $A^\natural$ this is part of Theorem 2.24 below. Under stronger hypotheses we can show that there is a Lefschetz equivalence $A^\natural \simeq ^\natural A$ [32, Proposition 7.12], but in general we do not know if one exists. In this paper, we will deal almost exclusively with $A^\natural$, and therefore simply refer to it as the HPD category. All of our results can be translated directly to the “left HPD” setting.

**Remark 2.21.** If $A$ is a Lefschetz category over $P(V)$ which is smooth and proper over $S$, then the HPD category $A^\natural$ is also smooth and proper over $S$ ([32, Lemma 7.18]). This is an instance of the homological smoothness principle of homological projective geometry.

Sometimes it is convenient to describe the HPD category $A^\natural$ in terms of the right or left Lefschetz decompositions of $A$ as follows.
Lemma 2.22. Let $\mathcal{A}$ be a Lefschetz category over $\textbf{P}(V)$ of length $m$. Then there are $\textbf{P}(V^\vee)$-linear semiorthogonal decompositions

$$\mathbf{H}(A) = \left\langle A^1, \delta^*(A_1(H) \otimes \text{Perf}(\textbf{P}(V^\vee))), \ldots, \delta^*(A_{m-1}((m-1)H) \otimes \text{Perf}(\textbf{P}(V^\vee))) \right\rangle,$$

$$\mathbf{H}(A) = \left\langle \delta'(A_{1-m}((1-m)H) \otimes \text{Perf}(\textbf{P}(V^\vee))), \ldots, \delta'(A_{-1}(-H) \otimes \text{Perf}(\textbf{P}(V^\vee))), A^2 \right\rangle,$$

where the functors $\delta^*, \delta': \text{Perf}(\textbf{P}(V^\vee)) \to \mathbf{H}(A)$ are fully faithful on the categories to which they are applied. In particular, $A^2$ is an admissible subcategory in $\mathbf{H}(A)$ and its inclusion functor $\gamma: A^2 \to \mathbf{H}(A)$ has both left and right adjoints $\gamma^*, \gamma^! \colon \mathbf{H}(A) \to A^2$.

Proof. This holds by [32, Definition 7.1 and Lemma 7.2]. Note that admissibility is by definition the existence of adjoint functors to the inclusion. □

By [32] Lemma 7.3, if $\mathcal{A}$ is a moderate Lefschetz category the composition

$$\mathcal{A} \xrightarrow{\pi^*} \mathbf{H}(A) \xrightarrow{\gamma^*} A^2$$

is fully faithful on the center $A_0 \subset A$; in this case, we define

$$A^2_0 = \gamma^* \pi^*(A_0).$$

For later use, we note the following.

Lemma 2.23. For a moderate Lefschetz category $\mathcal{A}$ over $\textbf{P}(V)$, the functors $\pi_* \circ \gamma^* : A^2 \to \mathcal{A}$ and $\gamma^* \circ \pi^* : \mathcal{A} \to A^2$ induce mutually inverse equivalences between $A^2_0 \subset A^2$ and $A_0 \subset \mathcal{A}$.

Proof. Since $\gamma^* \circ \pi^*$ is fully faithful on $A_0$ with image $A^2_0$, this follows from the fact that the image of the right adjoint $\pi_* \circ \gamma$ is $A_0$ by [32] Lemma 7.11]. □

The main theorem of HPD, recalled below, shows in particular that $A^2_0 \subset A^2$ is a Lefschetz center under certain hypotheses. This theorem was originally proved in [14] in the “commutative” case. We need the following “noncommutative” version from [32] Theorem 8.7. Recall the definition [110] of the orthogonal of a subbundle.

Theorem 2.24. Let $\mathcal{A}$ be a right strong, moderate Lefschetz category over $\textbf{P}(V)$. Then:

1. $A^2$ is a left strong, moderate Lefschetz category over $\textbf{P}(V^\vee)$ with center $A^2_0 \subset A^2$ and length given by

$$\text{length}(A^2) = \text{rank}(V) - \#\{ i \geq 0 \mid A_i = A_0 \}.$$

2. Let $L \subset V$ be a subbundle and let $L^\perp \subset V^\vee$ be its orthogonal. Set

$$r = \text{rank}(L), \quad s = \text{rank}(L^\perp), \quad m = \text{length}(A), \quad n = \text{length}(A^2).$$

Then there are semiorthogonal decompositions

$$A^1_{\text{P}(L)} = \langle \mathcal{K}_L(A), A_s(H), \ldots, A_{m-1}((m-s)H) \rangle,$$

$$A^1_{\text{P}(L^\perp)} = \langle A_{1-n}^2((r-n)H'), \ldots, A_{-r}((-H'), \mathcal{K}_{L^\perp}(A^2) \rangle,$$

and an $S$-linear equivalence $\mathcal{K}_L(A) \simeq \mathcal{K}_{L^\perp}(A^2)$.

Remark 2.25. The components $A^2_j \subset A^2$ can be expressed explicitly in terms of the right primitive components of $\mathcal{A}$, see [32] §7.2.
Remark 2.26. In the setup of Theorem 2.24, there are equivalences $\overset{\leftrightarrow}{\mathcal{A}} \simeq \mathcal{A}$ and $\overset{\leftrightarrow}{\mathcal{A}} \simeq \mathcal{A}$ of Lefschetz categories over $\mathbf{P}(V)$, where $\overset{\leftrightarrow}{(-)}$ denotes the left HPD operation, see Remark 2.20 and [32] Theorem 8.9. This property justifies HPD being called a “duality”.

We recall the simplest case of HPD. See §6 for others, and [19] for many more.

**Theorem 2.27** ([14] Corollary 8.3). Let $0 \subseteq W \subseteq V$ be a subbundle, and let $W^\perp \subseteq V^\vee$ be the orthogonal bundle. Then there is an equivalence

$$\text{Perf}(\mathbf{P}(W))^\perp \simeq \text{Perf}(\mathbf{P}(W^\perp))$$

of Lefschetz categories over $\mathbf{P}(V^\vee)$.

More precisely, if $\iota: \mathbf{P}(W) \times \mathbf{P}(W^\perp) \to \mathbf{H}(\mathbf{P}(W)/\mathbf{P}(V))$ denotes the natural embedding and $\text{pr}_2: \mathbf{P}(W) \times \mathbf{P}(W^\perp) \to \mathbf{P}(W^\perp)$ denotes the projection, then the above equivalence is induced by the functor

$$\iota_* \circ \text{pr}_2^*: \text{Perf}(\mathbf{P}(W^\perp)) \to \text{Perf}(\mathbf{H}(\mathbf{P}(W)/\mathbf{P}(V))).$$  \hspace{1cm} (2.18)

This result is usually referred to as linear HPD.

### 2.3. Characterization of the HPD category.

Given $\mathcal{A}$ as in Theorem 2.24, we will need later a characterization of the Lefschetz category $\mathcal{A}^\perp$ in terms of the functor $\pi_*: \mathbf{H}(\mathcal{A}) \to \mathcal{A}$.

For this, we must characterize $\mathcal{A}^\perp$ and its Lefschetz center $\mathcal{A}^\perp_0 \subset \mathcal{A}^\perp$ in terms of $\pi_*$. We handle the first in Lemma 2.28 and the second in Proposition 2.31. Recall that given a $T$-linear category $\mathcal{C}$, we write $C \otimes F$ for the action of an object $F \in \text{Perf}(T)$ on an object $C \in \mathcal{C}$.

**Lemma 2.28.** Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(V)$. Then $\mathcal{A}^\perp$ is the full $\mathbf{P}(V^\vee)$-linear subcategory of $\mathbf{H}(\mathcal{A})$ given by

$$\mathcal{A}^\perp = \{ C \in \mathbf{H}(\mathcal{A}) \mid \pi_*(C \otimes h^*F) \in \mathcal{A}_0 \text{ for all } F \in \text{Perf}(\mathbf{P}(V^\vee)) \}.$$

**Proof.** Consider the diagram (2.13) and its base change from $\mathbf{P}(V)$ to $\mathcal{A}$. By [32] Lemma 3.18 the defining property (2.14) of $\mathcal{A}^\perp$ holds for $C \in \mathbf{H}(\mathcal{A})$ if and only if

$$\text{pr}_1^*(\delta_*(C) \otimes \text{pr}_2^*(F)) \in \mathcal{A}_0 \text{ for all } F \in \text{Perf}(\mathbf{P}(V^\vee)).$$

But $\text{pr}_1 \circ \delta = \pi$ and $\text{pr}_2 \circ \delta = h$, hence the result follows from projection formula

$$\text{pr}_1^*(\delta_*(C) \otimes \text{pr}_2^*(F)) \simeq \pi_*(C \otimes h^*(F)). \hspace{1cm} \square$$

The following related result will also be needed later.

**Lemma 2.29.** Let $\mathcal{A}$ be a $\mathbf{P}(V)$-linear category with a $\mathbf{P}(V)$-linear semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{A}', \mathcal{A}'' \rangle$. Then there is a $\mathbf{P}(V^\vee)$-linear semiorthogonal decomposition

$$\mathbf{H}(\mathcal{A}) = \langle \mathbf{H}(\mathcal{A}'), \mathbf{H}(\mathcal{A}'') \rangle,$$

where $\mathbf{H}(\mathcal{A}')$ can be described as the full subcategory of $\mathbf{H}(\mathcal{A})$ given by

$$\mathbf{H}(\mathcal{A}') = \{ C \in \mathbf{H}(\mathcal{A}) \mid \pi_*(C \otimes h^*F) \in \mathcal{A}' \text{ for all } F \in \text{Perf}(\mathbf{P}(V^\vee)) \},$$

and $\mathbf{H}(\mathcal{A}'')$ is given analogously.
Proof. The claimed semiorthogonal decomposition of $\mathbf{H}(\mathcal{A})$ holds by Lemma [A.5] By [32, Lemma 3.18], for $C \in \mathbf{H}(\mathcal{A})$ we have $C \in \mathbf{H}(\mathcal{A}')$ if and only if

$$\pi_*(C \otimes G) \in \mathcal{A}' \text{ for all } G \in \text{Perf}(\mathbf{H}(\mathbf{P}(V))).$$

(2.19)

Since $\delta: \mathbf{H}(\mathbf{P}(V)) \to \mathbf{P}(V) \times \mathbf{P}(V')$ is a closed embedding, Perf$(\mathbf{H}(\mathbf{P}(V)))$ is thickly generated by objects in the image of $\delta^*$. Hence by Lemma [A.1] the category Perf$(\mathbf{H}(\mathbf{P}(V)))$ is thickly generated by the objects $\delta^*(E \boxtimes F)$ for $E \in \text{Perf}(\mathbf{P}(V))$, $F \in \text{Perf}(\mathbf{P}(V'))$. It follows that (2.19) is equivalent to

$$\pi_*(C \otimes \delta^*(E \boxtimes F)) \in \mathcal{A}' \text{ for all } E \in \text{Perf}(\mathbf{P}(V)), F \in \text{Perf}(\mathbf{P}(V')).$$

Note that $\delta^*(E \boxtimes F) \simeq \pi^*(E) \otimes h^*(F)$, hence

$$\pi_*(C \otimes \delta^*(E \boxtimes F)) \simeq \pi_*(C \otimes h^*F) \otimes E.$$

Since $\mathcal{A}'$ is $\mathbf{P}(V)$-linear, the above condition is thus equivalent to

$$\pi_*(C \otimes h^*F) \in \mathcal{A}' \text{ for all } F \in \text{Perf}(\mathbf{P}(V')).$$

To characterize $\mathcal{A}_0^2 \subset \mathcal{A}^2$ we need to introduce some notation. Consider the tautological inclusion

$$\mathcal{O}_{\mathbf{P}(V)}(-H) \to V \otimes \mathcal{O}_{\mathbf{P}(V)}$$
on $\mathbf{P}(V)$, and the tautological surjection

$$V \otimes \mathcal{O}_{\mathbf{P}(V')} \to \mathcal{O}_{\mathbf{P}(V')}(H')$$
on $\mathbf{P}(V')$. By the definition of $\mathbf{H}(\mathbf{P}(V))$, the composition

$$\mathcal{O}_{\mathbf{H}(\mathbf{P}(V))}(-H) \to V \otimes \mathcal{O}_{\mathbf{H}(\mathbf{P}(V))} \to \mathcal{O}_{\mathbf{H}(\mathbf{P}(V'))}(H')$$
of the pullbacks of these morphisms to $\mathbf{H}(\mathbf{P}(V))$ vanishes, and hence can be considered as a complex concentrated in degrees $[-1,1]$. By construction, this complex has cohomology concentrated in degree 0, i.e. it is a monad; we define $M$ as the degree 0 cohomology sheaf,

$$M \simeq \{ \mathcal{O}_{\mathbf{H}(\mathbf{P}(V))}(-H) \to V \otimes \mathcal{O}_{\mathbf{H}(\mathbf{P}(V))} \to \mathcal{O}_{\mathbf{H}(\mathbf{P}(V'))}(H') \},$$

which is a vector bundle of rank $N-2$ on $\mathbf{H}(\mathbf{P}(V))$.

Lemma 2.30. Let $\mathcal{A}$ be a $\mathbf{P}(V)$-linear category. Then there is a semiorthogonal decomposition

$$\mathbf{H}(\mathcal{A}) = \langle \pi^*(\mathcal{A})(-(N-2)H'), \ldots, \pi^*(\mathcal{A})(-H'), \pi^*(\mathcal{A}) \rangle.$$

Moreover, for $0 \leq t \leq N-2$ the projection functor onto the $-tH'$ component (regarded as a functor to $\mathcal{A}$) is given by

$$\eta_t: \mathbf{H}(\mathcal{A}) \to \mathcal{A}, \quad C \mapsto \pi_*(C \otimes \wedge^tM[t]).$$

Proof. Define

$$\mathcal{K} = \ker(V' \otimes \mathcal{O}_{\mathbf{P}(V)} \to \mathcal{O}_{\mathbf{P}(V)}(H')).$$

Then it is easy to see there is an isomorphism $\mathbf{H}(\mathbf{P}(V)) \cong \mathbf{P}(\mathbf{V})/(\mathcal{K})$, under which $H'$ corresponds to the tautological $\mathcal{O}(1)$ line bundle and $M$ corresponds to $\mathcal{O}_{\mathbf{P}(\mathbf{V})}(H')$. Hence we have the standard projective bundle semiorthogonal decomposition

$$\text{Perf}(\mathbf{H}(\mathbf{P}(V))) =$$

$$\langle \pi^*(\text{Perf}(\mathbf{P}(V)))(-(N-2)H'), \ldots, \pi^*(\text{Perf}(\mathbf{P}(V)))(-H'), \pi^*(\text{Perf}(\mathbf{P}(V))) \rangle,$$
whose projection functors for $0 \leq t \leq N - 2$ are given by
\[ \eta_t: \text{Perf}(H(P(V))) \to \text{Perf}(P(V)), \quad F \mapsto \pi_*(F \otimes \wedge^t \mathcal{M}[t]). \]
Now by Lemma A.15 the result follows by base change.

**Proposition 2.31.** Let $A$ be a moderate Lefschetz category over $P(V)$. Then $A^L_0$ is the full subcategory of $A^L$ given by
\[ A^L_0 = \left\{ C \in A^L \mid \pi_*(\gamma(C) \otimes \wedge^t \mathcal{M}) \in \perp A_0 \text{ for all } t \geq 1 \right\}, \]
where $\perp A_0$ is the left orthogonal to the center $A_0 \subset A$.

**Proof.** First note that since $\mathcal{M}$ is a vector bundle of rank $N - 2$, for $C \in A^L$ the condition
\[ \pi_*(\gamma(C) \otimes \wedge^t \mathcal{M}) \in \perp A_0 \]
holds for all $t \geq 1$ if and only if it holds for $1 \leq t \leq N - 2$. This condition is in turn equivalent to
\[ \text{Cone}(\pi^*\pi_\gamma(C) \to \gamma(C)) \in \left\{ \pi^*(A_i(iH)) \otimes \mathcal{O}(-tH') \right\}_{1 \leq t \leq N-2, 1 \leq i \leq m-1}. \tag{2.20} \]
Indeed, this follows from the form of the projection functors for the semiorthogonal decomposition of Lemma 2.30 together with the equality
\[ \perp A_0 = \langle A_1(H), \ldots, A_{m-1}((m-1)H) \rangle \]
where $A_i$ are the components of the right Lefschetz decomposition (2.3) of $A$. It remains to show that (2.20) is equivalent to $C \in A^L_0$.

Suppose (2.20) holds for $C \in A^L$. Then $\gamma^* \pi^*$ kills the left side of (2.20), since by (2.15) all components in the right side are contained in $\perp (A^L)$. Hence $\gamma^* \pi^* \pi_\gamma(C) \simeq \gamma^*(\gamma(C)) \simeq C$. But $\pi_\gamma(C) \in A_0$ by Lemma 2.28 so we conclude $C \in A^L_0$ by the definition (2.17) of $A^L_0$.

Conversely, assume $C \in A^L$ lies in $A^L_0$, i.e. $C = \gamma^* \pi^*(D)$ for some $D \in A_0$. By Lemma 2.23 we have $\pi_\gamma \gamma^* \pi^*(D) \simeq D$. Under this isomorphism, the morphism $\pi^* \pi_\gamma(C) \to \gamma(C)$ is identified with the canonical morphism
\[ \pi^*(D) \to \gamma^* \pi^*(D), \]
whose cone is nothing but $R_{A^L}(\pi^*(D))[1]$, where $R_{A^L}$ is the right mutation functor through the subcategory $A^L \subset H(A)$. But by [32, Lemma 7.8] (or [14, Lemma 5.6] in the commutative case) the object $R_{A^L}(\pi^*(D))$ lies in the subcategory
\[ \left\langle \pi^*(A_i(iH)) \otimes \mathcal{O}(-tH') \right\rangle_{1 \leq t \leq m-1, 1 \leq i \leq m-t} \subset H(A). \]
Note that $m - 1 \leq N - 2$ since $A$ is a moderate Lefschetz category, so this subcategory is contained in the right side of (2.20), and hence (2.20) holds. This completes the proof. \qed

### 2.4. HPD over quotients and subbundles.

Given a surjective morphism $\tilde{V} \to V$ of vector bundles with kernel $K$, we consider the corresponding rational map $P(\tilde{V}) \dashrightarrow P(V)$ and denote by $U = P(\tilde{V}) \setminus P(K) \subset P(\tilde{V})$ the open subset on which it is regular. If $X \to P(\tilde{V})$ is a morphism of schemes which factors through $U$, we thus obtain a morphism $X \to P(V)$. In this situation, we can ask how HPD with respect to the morphism $X \to P(\tilde{V})$ relates to HPD with respect to $X \to P(V)$. Below we answer this more generally when $X$ is replaced by a $P(\tilde{V})$-linear category; then the condition that $X \to P(\tilde{V})$ factors through $U$ is replaced by the condition that $A$ is supported over $U$, in the sense of Definition A.15. Note that the
surjection \( \tilde{V} \to V \) induces an embedding of bundles \( V^\vee \to \tilde{V}^\vee \), so that \( P(V^\vee) \subset P(\tilde{V}^\vee) \). In the following, we use the operation of extending the base scheme of a linear category, see Definition \[A.13\]

**Proposition 2.32.** Let \( A \) be a Lefschetz category over \( P(\tilde{V}) \), with center \( A_0 \). Assume \( \tilde{V} \to V \) is a surjection with kernel \( K \) such that \( A \) is supported over \( P(\tilde{V}) \setminus P(K) \). Then \( A \) has the structure of a Lefschetz category over \( P(V) \) (with the same center \( A_0 \)), and there is a \( \text{Perf}(V^\vee) \)-linear equivalence

\[
(A/P(V))^2 \simeq (A/P(\tilde{V}))^2 \otimes_{\text{Perf}(P(\tilde{V}^\vee))} \text{Perf}(P(V^\vee)).
\]

**Remark 2.33.** The proposition can be generalized to the case where \( A \) is not assumed to be supported over \( P(\tilde{V}) \setminus P(K) \), by working with a suitable “blowup” of \( A \). In the situation where \( A \) is geometric, this is the main result of \[7\]. As we will not need this generalization, we omit the details.

**Proof.** Let \( U = P(\tilde{V}) \setminus P(K) \). Then by the support assumption, \( A \) has a \( U \)-linear structure such that the \( P(\tilde{V}) \)-linear structure is induced by pullback along \( U \to P(\tilde{V}) \). Via the morphism \( U \to P(V) \) given by linear projection, \( A \) also carries a \( \text{Perf}(V) \)-linear structure. Let \( H \) and \( \tilde{H} \) denote the relative hyperplane classes on \( P(V) \) and \( P(\tilde{V}) \). Note that \( \mathcal{O}(H) \) and \( \mathcal{O}(\tilde{H}) \) both pull back to the same object of \( \text{Perf}(U) \), and hence their actions on \( A \) coincide. From this, it follows that the given Lefschetz center \( A_0 \subset A \) is also a Lefschetz center with respect to the \( \text{Perf}(V) \)-linear structure.

Consider the induced embedding \( V^\vee \hookrightarrow \tilde{V}^\vee \). There is a canonical isomorphism

\[
U \times_{P(\tilde{V})} H(P(\tilde{V})) \times_{P(\tilde{V}^\vee)} P(V^\vee) \cong U \times_{P(V)} H(P(V)). \tag{2.21}
\]

Using this, we deduce

\[
H(A/P(\tilde{V})) \otimes_{\text{Perf}(P(\tilde{V}^\vee))} \text{Perf}(P(V^\vee)) = A \otimes_{\text{Perf}(P(\tilde{V}))} \text{Perf}(H(P(\tilde{V}))) \otimes_{\text{Perf}(P(\tilde{V}^\vee))} \text{Perf}(P(V^\vee))
\]

\[
\cong A \otimes_{\text{Perf}(P(\tilde{V}))} \text{Perf}(U) \otimes_{\text{Perf}(P(V^\vee))} \text{Perf}(H(P(\tilde{V}))) \otimes_{\text{Perf}(P(\tilde{V}^\vee))} \text{Perf}(P(V^\vee))
\]

\[
\cong A \otimes_{\text{Perf}(P(\tilde{V}))} \text{Perf} \left( U \times_{P(\tilde{V})} H(P(\tilde{V})) \times_{P(\tilde{V}^\vee)} P(V^\vee) \right)
\]

\[
\cong A \otimes_{\text{Perf}(P(\tilde{V}))} \text{Perf} \left( U \times_{P(V)} H(P(V)) \right)
\]

\[
\cong A \otimes_{\text{Perf}(P(V))} \text{Perf}(H(P(V))) = H(A/P(V)).
\]

Indeed, the second line holds by definition of \( H(A/P(\tilde{V})) \), the third and the sixth follow from the fact that \( A \) is supported over \( U \) (see Lemma \[A.17\]), the fourth holds by Theorem \[A.2\], the fifth holds by \[2.21\], and the last holds by definition. Using the semiorthogonal decomposition \( (2.15) \) from Lemma \[2.22\], it is easy to check that this equivalence induces an equivalence between the subcategories

\[
(A/P(\tilde{V}))^2 \otimes_{\text{Perf}(P(\tilde{V}^\vee))} \text{Perf}(P(V^\vee)) \subset H(A/P(\tilde{V})) \otimes_{\text{Perf}(P(\tilde{V}^\vee))} \text{Perf}(P(V^\vee))
\]

and

\[
(A/P(V))^2 \subset H(A/P(V)). \tag*{\Box}
\]
Remark 2.34. In the situation of Proposition 2.32, note that we have $K = (V')^\perp$ and $\mathcal{A} \otimes_{\text{Perf}(\mathcal{P}(\tilde{V}))} \text{Perf}(\mathcal{P}(K)) = 0$ by the support assumption for $\mathcal{A}$. Assume that $\mathcal{A}$ is right strong and moderate as a Lefschetz category over $\mathcal{P}$. Assume that $\mathcal{A}$ is right strong and moderate as a Lefschetz category over $\mathcal{P}$. Then from Theorem 2.24 we conclude that $K$ is semiorthogonal decomposition where $n = \text{length}(\mathcal{A})$ and $r = \text{rank}(K)$. This provides the left side with a Lefschetz structure of length $n - r$ and center $A^\perp_{n-r}$, with respect to which the equivalence of Proposition 2.32 is a Lefschetz equivalence. We also note that $A^\perp_{n-r} = A^\perp_0$. Indeed, since $\mathcal{A} \simeq \tilde{\mathcal{A}}^\perp$ (Remark 2.26), the left version of Theorem 2.24 gives

$$\text{length}(\mathcal{A}) = \text{rank}(\tilde{V}) - \# \{ i \leq 0 \mid A^\perp_i = A^\perp_0 \},$$

while by moderateness of $\mathcal{A}$ over $\mathcal{P}(V)$ we also have

$$\text{length}(\mathcal{A}) < \text{rank}(V) = \text{rank}(\tilde{V}) - r,$$

and hence $\# \{ i \leq 0 \mid A^\perp_i = A^\perp_0 \} > r$.

There is also a dual result.

Proposition 2.35. Let $\mathcal{A}$ be a right strong, moderate Lefschetz category over $\mathcal{P}(\tilde{V})$. Assume $V \subset \tilde{V}$ is a subbundle such that the HPD category $\mathcal{A}^\perp$ is supported over $\mathcal{P}(\tilde{V}) \setminus \mathcal{P}(V^\perp)$. Further assume that $\mathcal{A}^\perp$ is moderate when regarded as a Lefschetz category over $\mathcal{P}(V^\perp)$ via the linear projection $\mathcal{P}(\tilde{V}) \twoheadrightarrow \mathcal{P}(V^\perp)$. Then the category

$$\mathcal{A}_{\mathcal{P}(V)} = \mathcal{A} \otimes_{\text{Perf}(\mathcal{P}(\tilde{V}))} \text{Perf}(\mathcal{P}(V))$$

has a Lefschetz structure over $\mathcal{P}(V)$, with center the fully faithful image of $\mathcal{A}_0 \subset \mathcal{A}$ under the restriction functor $\mathcal{A} \to \mathcal{A}_{\mathcal{P}(V)}$. Moreover, there is a $\mathcal{P}(V^\perp)$-linear Lefschetz equivalence

$$(\mathcal{A}_{\mathcal{P}(V)})^\perp \simeq \mathcal{A}^\perp / \mathcal{P}(V^\perp).$$

Proof. Applying the left version of Proposition 2.32 and Remark 2.34 to the category $\mathcal{A}^\perp$, we obtain a $\mathcal{P}(V)$-linear Lefschetz equivalence

$$\tilde{\mathcal{A}}^\perp / \mathcal{P}(V^\perp) \simeq \tilde{\mathcal{A}}^\perp \otimes_{\text{Perf}(\mathcal{P}(\tilde{V}))} \text{Perf}(\mathcal{P}(V)).$$

But $\tilde{\mathcal{A}}^\perp \simeq \mathcal{A}$ by Remark 2.26, so taking right HPD categories and using Remark 2.26 again finishes the proof. □

3. Categorical joins

In this section, we introduce the categorical join of two Lefschetz categories, which in the commutative case was briefly described in §1.2. First in §3.1 we define the resolved join of two categories linear over projective bundles, by analogy with the canonical resolution of singularities of the classical join of two projective schemes. In §3.2 we define the categorical join of two Lefschetz categories as a subcategory of the resolved join, and prove some basic properties of this construction. In §3.3 we study base changes of categorical joins, and in particular show that categorical and resolved joins agree away from the “exceptional locus”
of the resolved join. Finally, in §3.4 we construct a canonical Lefschetz structure on the
categorical join of two Lefschetz categories.

We fix nonzero vector bundles $V_1$ and $V_2$ on $S$, and write $H_1, H_2$, and $H$ for the relative
hyperplane classes on $P(V_1), P(V_2)$, and $P(V_1 \oplus V_2)$.

3.1. **Resolved joins.** Let $X_1 \to P(V_1)$ and $X_2 \to P(V_2)$ be morphisms of schemes. The
resolved join of $X_1$ and $X_2$ is defined as the $P^1$-bundle

$$\tilde{J}(X_1, X_2) = P_{X_1 \times X_2}(O(-H_1) \oplus O(-H_2)).$$

The canonical embedding of vector bundles

$$O(-H_1) \oplus O(-H_2) \hookrightarrow (V_1 \otimes O) \oplus (V_2 \otimes O) = (V_1 \oplus V_2) \otimes O$$

induces a morphism

$$\tilde{J}(X_1, X_2) \to P(V_1 \oplus V_2).$$

Recall from §1.2 that if $X_1 \to P(V_1)$ and $X_2 \to P(V_2)$ are embeddings, this morphism factors
birationally through the classical join $J(X_1, X_2) \subset P(V_1 \oplus V_2)$, and provides a resolution of
singularities if $X_1$ and $X_2$ are smooth.

Note that there is an isomorphism

$$\tilde{J}(X_1, X_2) \cong (X_1 \times X_2) \times_{(P(V_1) \times P(V_2))} \tilde{J}(P(V_1), P(V_2)). \quad (3.1)$$

Motivated by this, we call $\tilde{J}(P(V_1), P(V_2))$ the universal resolved join. Denote by

$$p : \tilde{J}(P(V_1), P(V_2)) \to P(V_1) \times P(V_2)$$

the canonical projection morphism, and by

$$f : \tilde{J}(P(V_1), P(V_2)) \to P(V_1 \oplus V_2)$$

the canonical morphism introduced above. Define

$$E_1 = P_{P(V_1) \times P(V_2)}(O(-H_1)) \cong P(V_1) \times P(V_2),$$

$$E_2 = P_{P(V_1) \times P(V_2)}(O(-H_2)) \cong P(V_1) \times P(V_2).$$

These are disjoint divisors in $\tilde{J}(P(V_1), P(V_2))$, whose embeddings we denote by

$$\varepsilon_1 : E_1 \to \tilde{J}(P(V_1), P(V_2))$$

and

$$\varepsilon_2 : E_2 \to \tilde{J}(P(V_1), P(V_2)).$$

We have a commutative diagram

$$\begin{array}{ccc}
E_1 & \xrightarrow{\varepsilon_1} & \tilde{J}(P(V_1), P(V_2)) & \xleftarrow{\varepsilon_2} & E_2 \\
\downarrow{p} & & & & \downarrow{p} \\
P(V_1) \times P(V_2).
\end{array} \quad (3.2)
$$

The next result follows easily from the definitions.

**Lemma 3.1.** The following hold:

1. The morphism $f : \tilde{J}(P(V_1), P(V_2)) \to P(V_1 \oplus V_2)$ is the blowup of $P(V_1 \oplus V_2)$ in the disjoint
   union $P(V_1) \sqcup P(V_2)$, with exceptional divisor $E_1 \sqcup E_2$.

2. The $O(1)$ line bundle for the $P^1$-bundle $p : \tilde{J}(P(V_1), P(V_2)) \to P(V_1) \times P(V_2)$ is $O(H)$. 

(3) We have the following equalities of divisors modulo linear equivalence:
\[ E_1 = H - H_2, \quad H|_{E_1} = H_1, \]
\[ E_2 = H - H_1, \quad H|_{E_2} = H_2. \]

(4) The relative dualizing complex of the morphism \( p \) is given by
\[ \omega_p = \mathcal{O}(H_1 + H_2 - 2H)[1]. \]

Part (1) of the lemma can be summarized by the blowup diagram
\[
\begin{array}{ccc}
E_1 & \xrightarrow{\epsilon_1} & \mathcal{J}(\mathcal{P}(V_1), \mathcal{P}(V_2)) & \xleftarrow{\epsilon_2} & E_2 \\
\downarrow & & \downarrow f & & \downarrow \\
\mathcal{P}(V_1) & \rightarrow & \mathcal{P}(V_1 \oplus V_2) & \leftarrow & \mathcal{P}(V_2)
\end{array}
\] (3.3)

All schemes in the diagram are smooth and projective over \( S \), hence Remark 1.17 applies to all morphisms.

Following [3.1] we define the resolved join of categories linear over \( \mathcal{P}(V_1) \) and \( \mathcal{P}(V_2) \) by base change from the universal resolved join.

**Definition 3.2.** Let \( \mathcal{A}^1 \) be a \( \mathcal{P}(V_1) \)-linear category and \( \mathcal{A}^2 \) a \( \mathcal{P}(V_2) \)-linear category. The **resolved join** of \( \mathcal{A}^1 \) and \( \mathcal{A}^2 \) is the category
\[
\mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) = (\mathcal{A}^1 \otimes \mathcal{A}^2) \otimes_{\mathcal{P}(V_1) \times \mathcal{P}(V_2)} \mathcal{D}(\mathcal{P}(V_1), \mathcal{P}(V_2)).
\]

Further, for \( k = 1, 2 \), we define
\[
E_k(\mathcal{A}^1, \mathcal{A}^2) = (\mathcal{A}^1 \otimes \mathcal{A}^2) \otimes_{\mathcal{P}(V_1) \times \mathcal{P}(V_2)} \mathcal{D}(E_k).
\]

**Remark 3.3.** The isomorphism \( E_k \cong \mathcal{P}(V_1) \times \mathcal{P}(V_2) \) induces a canonical equivalence
\[
E_k(\mathcal{A}^1, \mathcal{A}^2) \simeq \mathcal{A}^1 \otimes \mathcal{A}^2.
\]

We identify these categories via this equivalence; in particular, below we will regard subcategories of the right side as subcategories of the left.

**Remark 3.4.** If \( X_1 \rightarrow \mathcal{P}(V_1) \) and \( X_2 \rightarrow \mathcal{P}(V_2) \) are morphisms of schemes, then by the isomorphism [3.1] and Theorem A.2 the resolved join satisfies
\[
\mathcal{J}(\mathcal{D}(X_1), \mathcal{D}(X_2)) \simeq \mathcal{D}(\mathcal{J}(X_1, X_2)).
\]

If \( \mathcal{A}^1 \) and \( \mathcal{A}^2 \) are linear over \( \mathcal{P}(V_1) \) and \( \mathcal{P}(V_2) \), we sometimes abbreviate notation by writing
\[
\mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) = \mathcal{J}(\mathcal{D}(X_1), \mathcal{A}^2), \quad \mathcal{J}(\mathcal{A}^1, X_2) = \mathcal{J}(\mathcal{A}^1, \mathcal{D}(X_2)),
\]
but the notation \( \mathcal{J}(X_1, X_2) \) will always refer to the resolved join as a scheme.

Below we gather some elementary lemmas about resolved joins.

Let \( \gamma_k : \mathcal{A}^k \rightarrow \mathcal{B}^k \) be \( \mathcal{P}(V_k) \)-linear functors. Then we have a \( \mathcal{P}(V_1) \times \mathcal{P}(V_2) \)-linear functor
\[
\gamma_1 \otimes \gamma_2 : \mathcal{A}^1 \otimes \mathcal{A}^2 \rightarrow \mathcal{B}^1 \otimes \mathcal{B}^2,
\]
and by base change along the morphism \( p \) we obtain a \( \mathcal{J}(\mathcal{P}(V_1), \mathcal{P}(V_2)) \)-linear functor
\[
\mathcal{J}(\gamma_1, \gamma_2) : \mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) \rightarrow \mathcal{J}(\mathcal{B}^1, \mathcal{B}^2).
\] (3.4)
Lemma 3.5. Let \( \gamma_k : A^k \to B^k \) be \( P(V_k) \)-linear functors. There is a commutative diagram

\[
\begin{array}{ccc}
\check{J}(A^1, A^2) & \to & \check{J}(B^1, B^2) \\
p^* & \downarrow & \downarrow p^* \\
A^1 \otimes A^2 & \to & B^1 \otimes B^2
\end{array}
\]

Moreover, if \( \gamma_1 \) and \( \gamma_2 \) both admit left or right adjoints, then so does \( \check{J}(\gamma_1, \gamma_2) \). If further \( \gamma_1 \) and \( \gamma_2 \) are fully faithful or equivalences, then so is \( \check{J}(\gamma_1, \gamma_2) \).

Proof. The formalism of base change for linear categories gives the claimed commutative diagram. The rest follows from Lemma \([A.4]\) \( \square \)

Lemma 3.6. Let \( A^1 \) be a \( P(V_1) \)-linear category and \( A^2 \) a \( P(V_2) \)-linear category. Then for any \( P(V_1) \)-linear semiorthogonal decomposition \( A^1 = \langle A', A'' \rangle \), there is a \( \check{J}(P(V_1), P(V_2)) \)-linear semiorthogonal decomposition

\[
\check{J}(A^1, A^2) = \langle \check{J}(A', A^2), \check{J}(A'', A^2) \rangle.
\]

A semiorthogonal decomposition of \( A^2 \) induces an analogous decomposition of \( \check{J}(A^1, A^2) \).

Proof. Follows from the definition of the resolved join and Lemma \([A.5]\) \( \square \)

Lemma 3.7. Let \( A^1 \) be a \( P(V_1) \)-linear category and \( A^2 \) a \( P(V_2) \)-linear category. Then the functor

\[
p^* : A^1 \otimes A^2 \to \check{J}(A^1, A^2)
\]

is fully faithful, and there is a semiorthogonal decomposition with admissible components

\[
\check{J}(A^1, A^2) = \langle p^*(A^1 \otimes A^2), p^*(A^1 \otimes A^2)(H) \rangle.
\]

Proof. By virtue of the \( P^1 \)-bundle structure \( p : \check{J}(P(V_1), P(V_2)) \to P(V_1) \times P(V_2) \), we have a semiorthogonal decomposition

\[
\text{Perf}(\check{J}(P(V_1), P(V_2))) = \langle p^*\text{Perf}(P(V_1) \times P(V_2)), p^*\text{Perf}(P(V_1) \times P(V_2))(H) \rangle.
\]

Now by Lemmas \([A.5] \ [A.6]\) and \([A.4]\) the result follows by base change. \( \square \)

Lemma 3.8. Let \( A^1 \) and \( A^2 \) be categories linear over \( P(V_1) \) and \( P(V_2) \) which are smooth and proper over \( S \). Then the resolved join \( \check{J}(A^1, A^2) \) is smooth and proper over \( S \).

Proof. By \([32] \) Lemma 4.8] combined with (the proof of) \([32] \) Chapter I.1, Corollary 9.5.4], the category \( A^1 \otimes A^2 \) is smooth and proper over \( S \). Moreover, \( \check{J}(A^1, A^2) \) is obtained from \( A^1 \otimes A^2 \) by base change along the smooth and proper morphism \( \check{J}(P(V_1), P(V_2)) \to P(V_1) \times P(V_2) \). Hence the result follows from \([32] \) Lemma 4.11]. \( \square \)

3.2. Categorical joins. We define the categorical join of Lefschetz categories over \( P(V_1) \) and \( P(V_2) \) as a certain subcategory of the resolved join.

Definition 3.9. Let \( A^1 \) and \( A^2 \) be Lefschetz categories over \( P(V_1) \) and \( P(V_2) \). The categorical join \( \tilde{J}(A^1, A^2) \) of \( A^1 \) and \( A^2 \) is defined by

\[
\tilde{J}(A^1, A^2) = \left\{ C \in \check{J}(A^1, A^2) \mid \begin{array}{c}
\varepsilon_1^*(C) \in A^1 \otimes A^2_0 \subset E_1(A^1, A^2) \smallsetminus \varepsilon_2^*(C) \in A^0_0 \otimes A^2 \subset E_2(A^1, A^2)
\end{array} \right\}.
\]
Here, we have used the identifications of Remark 3.3. If \( \mathcal{A}^1 = \text{Perf}(X_1) \) for a scheme \( X_1 \) over \( \mathbf{P}(V_1) \) or \( \mathcal{A}^2 = \text{Perf}(X_2) \) for a scheme \( X_2 \) over \( \mathbf{P}(V_2) \), we abbreviate notation by writing

\[
\mathcal{J}(X_1, \mathcal{A}^2) = \mathcal{J}(\text{Perf}(X_1), \mathcal{A}^2), \quad \mathcal{J}(\mathcal{A}^1, X_2) = \mathcal{J}(\mathcal{A}^1, \text{Perf}(X_2)).
\]

**Remark 3.10.** The categorical join depends on the choice of Lefschetz centers for \( \mathcal{A}^1 \) and \( \mathcal{A}^2 \), although this is suppressed in the notation. For instance, for the “stupid” Lefschetz centers \( A^1_0 = \mathcal{A}^1 \) and \( A^2_0 = \mathcal{A}^2 \), the condition in the definition is void, so \( \mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) = \mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) \).

To show that \( \mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) \) is an admissible subcategory of \( \mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) \) and to describe its orthogonal category, we need the following noncommutative version of [17, Proposition 4.1], whose proof translates directly to our setting. Recall that \( \varepsilon_1 \) denotes the left adjoint of the pullback functor \( \varepsilon^*: \text{Perf}(Z) \to \text{Perf}(E) \), see (1.11).

**Proposition 3.11.** Let \( Y \) be a scheme over a base scheme \( T \). Let \( \varepsilon: E \to Y \) be the embedding of a Cartier divisor in \( Y \) over \( T \), and let \( \mathcal{L} = \mathcal{O}_E(-E) \) be the conormal bundle of \( E \subset Y \). Let \( \mathcal{A} \) be a \( T \)-linear category and set

\[
\mathcal{A}_Y = \mathcal{A} \otimes_{\text{Perf}(T)} \text{Perf}(Y) \quad \text{and} \quad \mathcal{A}_E = \mathcal{A} \otimes_{\text{Perf}(T)} \text{Perf}(E).
\]

Assume \( \mathcal{A}_E \) is a Lefschetz category with respect to \( \mathcal{L} \) with Lefschetz center \( \mathcal{A}_{E,0} \) and Lefschetz components \( \mathcal{A}_{E,i}, i \in \mathbb{Z} \). Set \( m = \text{length}(\mathcal{A}_E) \). Then:

1. The full subcategory of \( \mathcal{A}_Y \) defined by
   \[
   \mathcal{B} = \{ C \in \mathcal{A}_Y \mid \varepsilon^*(C) \in \mathcal{A}_{E,0} \}
   \]
   is admissible.
2. The functor \( \varepsilon_1: \mathcal{A}_E \to \mathcal{A}_Y \) is fully faithful on the subcategories \( \mathcal{A}_{E,i} \otimes \mathcal{L}^i \) for \( i \geq 1 \), and there is a semiorthogonal decomposition
   \[
   \mathcal{A}_Y = (\mathcal{B}, \varepsilon_1(\mathcal{A}_{E,1} \otimes \mathcal{L}), \varepsilon_1(\mathcal{A}_{E,2} \otimes \mathcal{L}^2), \ldots, \varepsilon_1(\mathcal{A}_{E,m-1} \otimes \mathcal{L}^{m-1})).
   \]
3. The functor \( \varepsilon_*: \mathcal{A}_E \to \mathcal{A}_Y \) is fully faithful on the subcategories \( \mathcal{A}_{E,i} \otimes \mathcal{L}^i \) for \( i \leq -1 \), and there is a semiorthogonal decomposition
   \[
   \mathcal{A}_Y = (\varepsilon_*(\mathcal{A}_{E,1-m} \otimes \mathcal{L}^{1-m}), \ldots, \varepsilon_*(\mathcal{A}_{E,-2} \otimes \mathcal{L}^{-2}), \varepsilon_*(\mathcal{A}_{E,-1} \otimes \mathcal{L}^{-1}), \mathcal{B}).
   \]

In the next lemma we apply the above proposition to the resolved join.

**Lemma 3.12.** For \( k = 1, 2 \), let \( \mathcal{A}^k \) be a Lefschetz category over \( \mathbf{P}(V_k) \) of length \( m_k \). Then the categorical join \( \mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) \) is an admissible \( \mathbf{P}(V_1 \oplus V_2) \)-linear subcategory of \( \mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) \), and there are \( \mathbf{P}(V_1 \oplus V_2) \)-linear semiorthogonal decompositions

\[
\tilde{\mathcal{J}}(\mathcal{A}^1, \mathcal{A}^2) = \left\langle \mathcal{J}(\mathcal{A}^1, \mathcal{A}^2), \varepsilon_1!(\mathcal{A}^1 \otimes \mathcal{A}^2_1(H_2)), \ldots, \varepsilon_1!(\mathcal{A}^1 \otimes \mathcal{A}^2_{m_2-1}((m_2-1)H_2)), \right\rangle
\]
\[
\tilde{\mathcal{J}}(\mathcal{A}^1, \mathcal{A}^2) = \left\langle \mathcal{J}(\mathcal{A}^1, \mathcal{A}^2), \varepsilon_2!(\mathcal{A}^1_1(H_1) \otimes \mathcal{A}^2), \ldots, \varepsilon_2!(\mathcal{A}^1_{m_1-1}((m_1-1)H_1) \otimes \mathcal{A}^2) \right\rangle,
\]
\[
(3.5)
\]
\[
\mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) = \left\langle \mathcal{J}(\mathcal{A}^1, \mathcal{A}^2), \varepsilon_1!*\left(\mathcal{A}^1 \otimes \mathcal{A}^2_{-m_2}((1-m_2)H_2)\right), \ldots, \varepsilon_1!*\left(\mathcal{A}^1 \otimes \mathcal{A}^2_{-1}(-H_2)\right) \right\rangle
\]
\[
\varepsilon_2!*\left(\mathcal{A}^1_{1-m_1}((1-m_1)H_1) \otimes \mathcal{A}^2\right), \ldots, \varepsilon_2!*\left(\mathcal{A}^1_{-1}(-H_1) \otimes \mathcal{A}^2\right), \right\rangle
\]
\[
(3.6)
\]
**Proof.** We apply Proposition 3.11 in the following setup:

\[ T = \mathbf{P}(V_1) \times \mathbf{P}(V_2), \quad Y = \tilde{J}(\mathbf{P}(V_1), \mathbf{P}(V_2)), \quad E = E_1 \sqcup E_2, \quad \text{and} \quad A = A^1 \otimes A^2. \]

Then \( \mathcal{A}_Y = \tilde{J}(A^1, A^2) \) and \( \mathcal{A}_E = E_1(A^1, A^2) \oplus E_2(A^1, A^2) \). We claim that

\[ \mathcal{A}_{E,0} = (A^1 \otimes A^2_0) \oplus (A^1_0 \otimes A^2) \]

is a Lefschetz center of \( \mathcal{A}_E \) with respect to \( \mathcal{L} = \mathcal{O}_E(-E) \), with Lefschetz components

\[ \mathcal{A}_{E,i} = (A^1 \otimes A^2_i) \oplus (A^1_i \otimes A^2). \]

Indeed, by Lemma 3.1 we have

\[ \mathcal{L}|_{E_1} = \mathcal{O}_{E_1}(-E_1) = \mathcal{O}_{E_1}(H_2 - H_1) \quad \text{and} \quad \mathcal{L}|_{E_2} = \mathcal{O}_{E_2}(-E_2) = \mathcal{O}_{E_2}(H_1 - H_2), \]

from which the claim follows easily.

In the above setup, the category \( \mathcal{B} \) of Proposition 3.11 coincides with the definition of the categorical join \( \tilde{J}(A^1, A^2) \). Hence the proposition shows \( \tilde{J}(A^1, A^2) \) is an admissible subcategory of \( \tilde{J}(A^1, A^2) \), and gives the semiorthogonal decompositions (3.5) and (3.6).

It remains to show the categorical join and the decompositions are \( \mathbf{P}(V_1 \oplus V_2) \)-linear. Since the categorical join is the orthogonal of the other components in the decompositions, it is enough to check that every other component is \( \mathbf{P}(V_1 \oplus V_2) \)-linear. By diagram (3.3) the morphism \( E_1 \rightarrow \mathbf{P}(V_1 \oplus V_2) \) factors through the projection \( E_1 \cong \mathbf{P}(V_1) \times \mathbf{P}(V_2) \rightarrow \mathbf{P}(V_1) \). Thus, since the subcategory \( A^1 \otimes A^2_1(iH_2) \subset E_1(A^1, A^2) \) is \( \mathbf{P}(V_1) \)-linear (because \( A^1 \) is), it is also \( \mathbf{P}(V_1 \oplus V_2) \)-linear. Since \( \varepsilon_1 \) is a morphism over \( \mathbf{P}(V_1 \oplus V_2) \), it follows that \( \varepsilon_1! (A^1 \otimes A^2_1(iH_2)) \) is also \( \mathbf{P}(V_1 \oplus V_2) \)-linear for any \( i \geq 1 \). The same argument works for the other components in (3.5) and (3.6), which finishes the proof.

**Remark 3.13.** The last two rows in (3.5) and the first two rows in (3.6) are completely orthogonal since \( E_1 \) and \( E_2 \) are disjoint.

Categorical joins preserve smoothness and properness:

**Lemma 3.14.** Let \( A^1 \) and \( A^2 \) be Lefschetz categories over \( \mathbf{P}(V_1) \) and \( \mathbf{P}(V_2) \) which are smooth and proper over \( S \). Then the categorical join \( \tilde{J}(A^1, A^2) \) is smooth and proper over \( S \).

**Proof.** Follows from Lemma 3.8, Lemma 3.12, and [32, Lemma 4.15].

**Example 3.15.** As an example, we consider the categorical join of two projective bundles. Let \( W_1 \subset V_1 \) and \( W_2 \subset V_2 \) be subbundles, so that \( \mathbf{P}(W_1) \subset \mathbf{P}(V_1) \) and \( \mathbf{P}(W_2) \subset \mathbf{P}(V_2) \). The classical join of these spaces is given by \( J(\mathbf{P}(W_1), \mathbf{P}(W_2)) = \mathbf{P}(W_1 \oplus W_2) \). Consider the Lefschetz structures of \( \mathbf{P}(W_1) \) and \( \mathbf{P}(W_2) \) defined in Example 2.11. Then the pullback functor

\[ f^* : \text{Perf}(\mathbf{P}(W_1 \oplus W_2)) \rightarrow \text{Perf}(\tilde{J}(\mathbf{P}(W_1), \mathbf{P}(W_2))) \]

induces a \( \mathbf{P}(W_1 \oplus W_2) \)-linear equivalence

\[ \text{Perf}(\mathbf{P}(W_1 \oplus W_2)) \simeq \tilde{J}(\mathbf{P}(W_1), \mathbf{P}(W_2)). \]

Indeed, this follows easily from Lemma 3.1, Orlov’s blowup formula, and the definitions. Moreover, Theorem 3.21 below equips \( \tilde{J}(\mathbf{P}(W_1), \mathbf{P}(W_2)) \) with a canonical Lefschetz structure. It is easy to check that the above equivalence is a Lefschetz equivalence.
3.3. **Base change of categorical joins.** Let $T \to \mathbf{P}(V_1 \oplus V_2)$ be a morphism of schemes. The base change of diagram (3.3) along this morphism gives a diagram

\[
\begin{array}{ccc}
E_1T & \longrightarrow & \mathbf{J}(\mathbf{P}(V_1), \mathbf{P}(V_2))T \\
\downarrow & & \downarrow \\
\mathbf{P}(V_1)T & \longrightarrow & T & \longrightarrow & \mathbf{P}(V_2)T \\
\end{array}
\]  \hspace{1cm} (3.7)

with cartesian squares. Note that the isomorphisms $E_k \cong \mathbf{P}(V_1) \times \mathbf{P}(V_2)$, $k = 1, 2$, induce isomorphisms

\[
E_1T \cong \mathbf{P}(V_1)T \times \mathbf{P}(V_2)T, \quad E_2T \cong \mathbf{P}(V_1) \times \mathbf{P}(V_2)T.
\]

If $\mathcal{A}^k$ is $\mathbf{P}(V_k)$-linear for $k = 1, 2$, then by the definition of $E_k(\mathcal{A}^1, \mathcal{A}^2)$ we have

\[
E_k(\mathcal{A}^1, \mathcal{A}^2)_T \cong (\mathcal{A}^1 \otimes \mathcal{A}^2) \otimes_{\text{Perf}(\mathbf{P}(V_1) \times \mathbf{P}(V_2))} \text{Perf}(E_kT).
\]

Hence by the above isomorphisms and Lemma A.7 we have equivalences

\[
E_1(\mathcal{A}^1, \mathcal{A}^2)_T \cong \mathcal{A}^1_{\mathbf{P}(V_1)T} \otimes \mathcal{A}^2, \quad (3.8)
\]

\[
E_2(\mathcal{A}^1, \mathcal{A}^2)_T \cong \mathcal{A}^1 \otimes \mathcal{A}^2_{\mathbf{P}(V_2)T}. \quad (3.9)
\]

We identify these categories via these equivalences.

**Lemma 3.16.** For $k = 1, 2$, let $\mathcal{A}^k$ be a Lefschetz category over $\mathbf{P}(V_k)$ of length $m_k$. Let $T \to \mathbf{P}(V_1 \oplus V_2)$ be a morphism of schemes. Then there is a $T$-linear semiorthogonal decomposition

\[
\tilde{\mathbf{J}}(\mathcal{A}^1, \mathcal{A}^2)_T = \langle \mathbf{J}(\mathcal{A}^1, \mathcal{A}^2)_T, \epsilon_{1!}(\mathcal{A}^1_{\mathbf{P}(V_1)_T} \otimes \mathcal{A}^2_{\mathbf{P}(V_2)_T}(H_2)), \ldots, \epsilon_{2!}(\mathcal{A}^1_{\mathbf{P}(V_1)_T} \otimes \mathcal{A}^2_{\mathbf{P}(V_2)_T}(m_2 - 1)(m_2 - 1)H_2) \rangle,
\]

\[
\epsilon_{2!}(\mathcal{A}^1_{\mathbf{P}(V_1)_T} \otimes \mathcal{A}^2_{\mathbf{P}(V_2)_T}(H_1)), \ldots, \epsilon_{2!}(\mathcal{A}^1_{\mathbf{P}(V_1)_T} \otimes \mathcal{A}^2_{\mathbf{P}(V_2)_T}(m_1 - 1)H_1) \rangle.
\]

**Proof.** This is the base change of (3.5) with the identifications (3.8) and (3.9) taken into account. \(\square\)

In the next proposition we use the notion of a linear category being supported over a closed subset, see Definition A.15.

**Proposition 3.17.** For $k = 1, 2$, let $\mathcal{A}^k$ be a Lefschetz category over $\mathbf{P}(V_k)$. Assume $\mathcal{A}^k$ is supported over a closed subset $Z_k \subset \mathbf{P}(V_k)$. Assume $T \to \mathbf{P}(V_1 \oplus V_2)$ is a morphism which factors through the complement of $Z_1 \cup Z_2$ in $\mathbf{P}(V_1 \oplus V_2)$. Then there is a $T$-linear equivalence

\[
\mathbf{J}(\mathcal{A}^1, \mathcal{A}^2)_T \cong \tilde{\mathbf{J}}(\mathcal{A}^1, \mathcal{A}^2)_T. \quad (3.10)
\]

If further $T \to \mathbf{P}(V_1 \oplus V_2)$ factors through the complement of $\mathbf{P}(V_1) \cup \mathbf{P}(V_2)$ in $\mathbf{P}(V_1 \oplus V_2)$, then there is an equivalence

\[
\tilde{\mathbf{J}}(\mathcal{A}^1, \mathcal{A}^2)_T \cong \mathcal{A}^1_T \otimes_{\text{Perf}(T)} \mathcal{A}^2_T, \quad (3.11)
\]

where the factors in the tensor product are the base changes of $\mathcal{A}^1$ and $\mathcal{A}^2$ along the morphisms $T \to \mathbf{P}(V_1)$ and $T \to \mathbf{P}(V_2)$ obtained by composing $T \to \mathbf{P}(V_1 \oplus V_2)$ with the linear projections of $\mathbf{P}(V_1 \oplus V_2)$ from $\mathbf{P}(V_2)$ and $\mathbf{P}(V_1)$.
Proof. The assumption on $T \rightarrow P(V_1 \oplus V_2)$ implies that $P(V_k)_T \rightarrow P(V_k)$ factors through the open subset $P(V_k) \setminus Z_k$. Thus we have $A_k^{i,T} \simeq 0$ and the equivalence \((3.10)\) follows from Lemma 3.16.

By the definition of $\tilde{J}(A^1, A^2)$ we have an equivalence

$$\tilde{J}(A^1, A^2) \simeq (A^1 \otimes A^2) \otimes_{\text{Perf}(P(V_1) \times P(V_2))} \text{Perf}(\tilde{J}(P(V_1), P(V_2)))_T.$$

By Lemma 3.3 the morphism $f: \tilde{J}(P(V_1), P(V_2)) \rightarrow P(V_1 \oplus V_2)$ is an isomorphism over the complement of $P(V_1) \cup P(V_2)$. Hence if $T \rightarrow P(V_1 \oplus V_2)$ factors through this complement, we have an isomorphism $\tilde{J}(P(V_1), P(V_2))_T \cong T$. Combining this isomorphism and the above equivalence with Corollary A.8 proves (3.11).

Remark 3.18. If $X_1 \subset P(V_1)$ and $X_2 \subset P(V_2)$ are closed subschemes, then the morphism $\tilde{J}(X_1, X_2) \rightarrow J(X_1, X_2)$ is an isomorphism over $U = P(V_1 \oplus V_2) \setminus (X_1 \sqcup X_2)$, so the pullback functor $\text{Perf}(\tilde{J}(X_1, X_2)) \rightarrow \text{Perf}(\tilde{J}(X_1, X_2))$ becomes an equivalence after base change to $U$. Hence if $\text{Perf}(X_1)$ and $\text{Perf}(X_2)$ are equipped with Lefschetz structures, by Proposition 3.17 the categorical join $\tilde{J}(X_1, X_2)$ becomes equivalent to $\text{Perf}(\tilde{J}(X_1, X_2))$ after base change to $U$.

3.4. The Lefschetz structure of a categorical join. Our next goal is to show that given Lefschetz categories over $P(V_1)$ and $P(V_2)$, their categorical join has a canonical Lefschetz structure. Recall that $p: \tilde{J}(P(V_1), P(V_2)) \rightarrow P(V_1) \times P(V_2)$ denotes the projection.

Lemma 3.19. Let $A^1$ and $A^2$ be Lefschetz categories over $P(V_1)$ and $P(V_2)$. Then the image of the subcategory

$$A_0^1 \otimes A_0^2 \subset A^1 \otimes A^2$$

under the functor $p^*: A^1 \otimes A^2 \rightarrow \tilde{J}(A^1, A^2)$ is contained in the categorical join $\tilde{J}(A^1, A^2)$ as an admissible subcategory.

Proof. By Lemma 3.7 the functor $p^*: A^1 \otimes A^2 \rightarrow \tilde{J}(A^1, A^2)$ is fully faithful with admissible image. By Lemma A.6 the subcategory $A_0^1 \otimes A_0^2 \subset A^1 \otimes A^2$ is admissible, so its image under $p^*$ is admissible. Finally, it follows from Definition 3.9 that this image is contained in the categorical join $\tilde{J}(A^1, A^2)$.

Definition 3.20. For Lefschetz categories $A^1$ and $A^2$ over $P(V_1)$ and $P(V_2)$, we define

$$\tilde{J}(A^1, A^2)_0 = p^*(A_0^1 \otimes A_0^2) \subset \tilde{J}(A^1, A^2). \quad (3.12)$$

Note that the containment $\tilde{J}(A^1, A^2)_0 \subset \tilde{J}(A^1, A^2)$ holds by Lemma 3.19.

Theorem 3.21. Let $A^1$ and $A^2$ be Lefschetz categories over $P(V_1)$ and $P(V_2)$. Then the categorical join $\tilde{J}(A^1, A^2)$ has the structure of a Lefschetz category over $P(V_1 \oplus V_2)$ with center $\tilde{J}(A^1, A^2)_0$ given by (3.12), and Lefschetz components given by (3.15) and (3.16) below. If $A^1$ and $A^2$ are both either right or left strong, then so is $\tilde{J}(A^1, A^2)$. Moreover, we have

$$\text{length}(\tilde{J}(A^1, A^2)) = \text{length}(A^1) + \text{length}(A^2),$$

and $\tilde{J}(A^1, A^2)$ is moderate if and only if one of $A^1$ or $A^2$ is moderate.

The proof of Theorem 3.21 takes the rest of this section. We let $A^1$ and $A^2$ be as in the theorem. Further, we let

$$m_1 = \text{length}(A^1), \quad m_2 = \text{length}(A^2), \quad m = m_1 + m_2,$$
and set
\[ \mathcal{J}_0 = \mathfrak{J}(A^1, A^2)_0. \]

Note that by Lemma 3.12, the categorical join \( \mathfrak{J}(A^1, A^2) \) is naturally \( \mathbf{P}(V_1 \oplus V_2) \)-linear. To prove the theorem, we will explicitly construct the required Lefschetz decompositions of \( \mathfrak{J}(A^1, A^2) \) and apply Lemma 2.4.

For \( k = 1, 2 \), let \( a^k_i, 0 \neq i \in \mathbb{Z} \), and \( a^k_{0+i}, a^k_{0-j} \), be the primitive components of the Lefschetz category \( A^k \) as defined in §2.1. We define
\[ j_i = \bigoplus_{i_1 + i_2 = i + 1} p^*(a^1_{i_1} \otimes a^2_{i_2}), \quad i \leq 0, \tag{3.13} \]
where in the formula \( a^k_i \) denotes \( a^k_{0+i} \) for \( k = 1, 2 \). Note that \( j_0 = 0 \).

Similarly, we define
\[ j_i = \bigoplus_{i_1 + i_2 = i + 1} p^*(a^1_{i_1} \otimes a^2_{i_2}), \quad i \leq 0, \tag{3.14} \]
where in the formula \( a^k_i \) denotes \( a^k_{0+i} \) for \( k = 1, 2 \). Note that \( j_0 = 0 \) with this definition, which is consistent with (3.13) for \( i = 0 \).

**Lemma 3.22.** We have semiorthogonal decompositions
\[ \mathcal{J}_0 = \langle j_0, j_1, \ldots, j_{m-1} \rangle \quad \text{and} \quad \mathcal{J}_0 = \langle j_{-m}, \ldots, j_{-2}, j_{-1}, j_0 \rangle. \]

**Proof.** Applying Lemma A.5 to \( A^1_0 = \langle a^1_0, a^1_1, \ldots, a^1_{m-1} \rangle \) and \( A^2_0 = \langle a^2_0, a^2_1, \ldots, a^2_{m-1} \rangle \), we obtain a semiorthogonal decomposition
\[ A^1_0 \otimes A^2_0 = \langle a^1_{i_1} \otimes a^2_{i_2} \rangle_{0 \leq i_1 \leq m_1-1, 0 \leq i_2 \leq m_2-1} \]
with components \( a^1_{i_1} \otimes a^2_{i_2} \) and \( a^1_{i_1} \otimes a^2_{i_2} \) semiorthogonal if \( i_1 < j_1 \) or \( i_2 < j_2 \). Since \( p^* \) defines an equivalence between \( A^1_0 \otimes A^2_0 \) and \( \mathcal{J}_0 \), we obtain a semiorthogonal decomposition
\[ \mathcal{J}_0 = \langle p^*(a^1_{i_1} \otimes a^2_{i_2}) \rangle_{0 \leq i_1 \leq m_1-1, 0 \leq i_2 \leq m_2-1} \]
with the same semiorthogonalities between the components. It follows that the summands in the right hand side of (3.13) are completely orthogonal as subcategories of \( \mathcal{J}_0 \), so that we indeed have inclusions \( j_i \subset \mathcal{J}_0 \) for \( i \geq 0 \), which give the first claimed semiorthogonal decomposition of \( \mathcal{J}_0 \). This is illustrated in Figure 1 below.

![Figure 1](image-url)

**Figure 1.** The semiorthogonal decomposition of \( \mathcal{J}_0 \) into the components \( j_i \) for \( m_1 = 3 \) and \( m_2 = 6 \). For simplicity, \( p^* \) is omitted from \( p^*(a^1_{i_1} \otimes a^2_{i_2}) \).
The second claimed semiorthogonal decomposition of $\mathcal{J}_0$ is constructed analogously.

We define two descending chains of subcategories of $\mathcal{J}_0$ by

\[
\mathcal{J}_i = \langle j_i, j_{i+1}, \ldots, j_{m-1} \rangle, \quad 0 \leq i \leq m-1,
\]

\[
\mathcal{J}_i = \langle j_{m-i}, \ldots, j_1, j_0 \rangle, \quad 1 - m \leq i \leq 0.
\]

(3.15) (3.16)

Note that $j_0 = 0$ implies $\mathcal{J}_{-1} = \mathcal{J}_0 = \mathcal{J}_1$.

**Lemma 3.23.** The subcategories $\mathcal{J}_i \subset \mathcal{J}(A^1, A^2)$ are right admissible for $i \geq 0$ and left admissible for $i \leq 0$. Further, if $A^1$ and $A^2$ are both right strong (or left strong), then the subcategory $\mathcal{J}_i \subset \mathcal{J}(A^1, A^2)$ is admissible for $i \geq 0$ (or $i \leq 0$).

**Proof.** For $i = 0$ we know $\mathcal{J}_0 \subset \mathcal{J}(A^1, A^2)$ is admissible by Lemma 3.19. If $i > 0$, then by definition we have a semiorthogonal decomposition

\[
\mathcal{J}_i = \langle j_0, \ldots, j_{i-1}, j_i \rangle.
\]

It follows that $\mathcal{J}_i$ is right admissible in $\mathcal{J}_0$, and hence also in $\mathcal{J}(A^1, A^2)$ as $\mathcal{J}_0$ is. Similarly, $\mathcal{J}_i$ is left admissible in $\mathcal{J}(A^1, A^2)$ for $i < 0$.

Finally, assume $A^1$ and $A^2$ are both right strong. Then Lemma A.6 implies the subcategory $A^1_i \otimes A^2_i \subset A^1_0 \otimes A^2_0$ is admissible for $i \geq 0$, and hence by Lemma 3.19 the summands defining $j_i$, $i \geq 0$, in (3.13) are admissible in $\mathcal{J}(A^1, A^2)$. So by [32] Lemma 3.10 we conclude that $j_i$, $i \geq 0$, is admissible in $\mathcal{J}(A^1, A^2)$. A similar argument applies if $A^1$ and $A^2$ are left strong.

The following alternative expressions for the categories $\mathcal{J}_i$ are sometimes useful, and can be proved by unwinding the definitions.

**Lemma 3.24.** For $i \neq 0$ we have $\mathcal{J}_i = p^*(\mathcal{J}_i)$, where $\mathcal{J}_i \subset A^1 \otimes A^2$ is the subcategory defined by

\[
\mathcal{J}_i = \begin{cases} 
\langle A^1_{i-m} \otimes A^2_0, A^1_{i-2} \otimes A^2_1, \ldots, A^1_{i-m} \otimes A^2_{i-2} \rangle & \text{if } 1 \leq i \leq m-2, \\
\langle A^1_{i-m} \otimes A^2_0, A^1_{i-2} \otimes A^2_1, \ldots, A^1_{i-m} \otimes A^2_{i-2} \rangle & \text{if } m-1 \leq i \leq m-1,
\end{cases}
\]

\[
\mathcal{J}_i = \begin{cases} 
\langle A^1_{i+1} \otimes A^2_{i+2}, A^1_{i+2} \otimes A^2_{i+1}, A^1_{i+2} \otimes A^2_{i+1} \rangle & \text{if } -m \leq i \leq -1, \\
\langle A^1_{i+1} \otimes A^2_{i+2}, A^1_{i+2} \otimes A^2_{i+1} \rangle & \text{if } -m \leq i \leq -m-1,
\end{cases}
\]

where for $k = 1, 2$, the symbol $a^k_0$ denotes $a^k_{i+0}$ in the first two equalities and $a^k_{i+0}$ in the last two equalities.

To prove Theorem 3.21 we will show that we have semiorthogonal decompositions

\[
\mathcal{J}(A^1, A^2) = \langle \mathcal{J}_0, \mathcal{J}_1(H), \ldots, \mathcal{J}_{m-1}((m-1)H) \rangle,
\]

(3.17)

\[
\mathcal{J}(A^1, A^2) = \langle \mathcal{J}_{1-m}((-1)(m)H), \ldots, \mathcal{J}_{1}(-H), \mathcal{J}_0 \rangle,
\]

(3.18)

and then apply Lemma 2.4. We focus on proving (3.17) below; an analogous argument proves (3.18).
Lemma 3.25. The sequence of subcategories

\[ \mathcal{J}_0, \mathcal{J}_1(H), \ldots, \mathcal{J}_{m-1}((m-1)H), \]  

in (3.17) is semiorthogonal.

Proof. By the definitions (3.15) and (3.13), it is enough to check that for any integer \( t \) such that \( 1 \leq t \leq j_1 + j_2 + 1 \), the subcategories

\[ p^*(a^1_j \otimes a^2_{j_2}), \quad p^*(a^1_{j_1} \otimes a^1_{j_2})(tH), \]

of \( \mathcal{J}(A^1, A^2) \) are semiorthogonal. Let \( C_1 \in a^1_{i_1}, C_2 \in a^2_{i_2} \) and \( D_1 \in a^1_{j_1}, D_2 \in a^2_{j_2} \). Since \( a^1_{i_1} \otimes a^2_{i_2} \) and \( a^1_{j_1} \otimes a^2_{j_2} \) are thickly generated by objects of the form \( C_1 \boxtimes C_2 \) and \( D_1 \boxtimes D_2 \) respectively, we must show that

\[ p^*(D_1 \boxtimes D_2)(tH) \in p^*(C_1 \boxtimes C_2). \]

Recall that \( p_t \) denotes the left adjoint functor of \( p^* \), see (1.11). By adjunction and the projection formula, this is equivalent to

\[ (D_1 \boxtimes D_2) \otimes p_t(0_{J_0}(p(V_1), p(V_2)))(tH) \in p^*(C_1 \boxtimes C_2). \]

Using the formula of Lemma 3.24 for the dualizing complex of \( p_t \), for \( t \geq 1 \) we obtain

\[ p_t(0_{J_0}(p(V_1), p(V_2)))(tH) \simeq \bigoplus_{t_1 + t_2 = t, t_1, t_2 \geq 1} 0_{p(V_1) \times p(V_2)}(t_1 H_1 + t_2 H_2)[1]. \]  

So, it is enough to show that

\[ D_1(t_1 H_1) \boxtimes D_2(t_2 H_2) \in p^*(C_1 \boxtimes C_2) \]

for all \( t_1, t_2 \geq 1 \) such that \( t_1 + t_2 = t \). The left side is contained in \( a^1_{j_1}(t_1 H_1) \otimes a^2_{j_2}(t_2 H_2) \), while \( C_1 \boxtimes C_2 \in a^1_{i_1} \otimes a^2_{i_2} \), so it suffices to show the pair \( (a^1_{j_1}(t_1 H_1) \otimes a^2_{j_2}(t_2 H_2), a^1_{i_1} \otimes a^2_{i_2}) \) is semiorthogonal. By Lemma A.3 and (2.10), this holds for \( 1 \leq t_1 \leq j_1 \) because of the semiorthogonality of the first factors, and for \( 1 \leq t_2 \leq j_2 \) because of the semiorthogonality of the second factors. Since the assumption \( t_1 + t_2 = t \leq j_1 + j_2 + 1 \) implies either \( t_1 \leq j_1 \) or \( t_2 \leq j_2 \), this finishes the proof. \( \square \)
To show that the categories in (3.19) generate \( \mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) \), we consider the idempotent-complete triangulated subcategory \( \mathcal{P} \) of the resolved join \( \mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) \) generated by the following subcategories:

\[
\begin{align*}
p^*(a_{i_1}^1 \otimes a_{i_2}^2)(tH), & \quad 0 \leq i_1 \leq m_1 - 1, 0 \leq i_2 \leq m_2 - 1, 0 \leq t \leq i_1 + i_2 + 1, \\
\varepsilon_{1*}(A_{i_1}^1 \otimes a_{i_2}^2(s_2H_2)), & \quad 0 \leq i_2 \leq m_2 - 1, 0 \leq s_2 \leq i_2 - 1, \\
\varepsilon_{2*}(a_{i_1}^1(s_1H_1) \otimes \mathcal{A}^2), & \quad 0 \leq i_1 \leq m_1 - 1, 0 \leq s_1 \leq i_1 - 1.
\end{align*}
\]

(3.21) (3.22) (3.23)

It follows from the definitions (3.13) and (3.15) that (3.19) and (3.21) generate the same line of the semiorthogonal decomposition (3.5) of \( \tilde{\mathcal{J}} \). Hence to establish (3.17), it suffices to show

\[
\varepsilon_1(C) \simeq \varepsilon_{1*}(C(H_1 - H_2)[-1]),
\]

so it follows from \( \mathcal{P}(V_1) \)-linearity of \( \mathcal{A}^1 \) and the definitions that (3.22) generates the second line of the semiorthogonal decomposition (3.5) of \( \tilde{\mathcal{J}}(\mathcal{A}^1, \mathcal{A}^2) \). Similarly, (3.23) generates the third line of (3.5). Hence to establish (3.17), it suffices to show \( \mathcal{P} = \tilde{\mathcal{J}}(\mathcal{A}^1, \mathcal{A}^2) \). For this, we will need the following lemma.

**Lemma 3.26.** For all integers \( i_1, i_2, s_1, s_2, t \), such that

\[
0 \leq s_1 \leq i_1 \leq m_1 - 1, 0 \leq s_2 \leq i_2 \leq m_2 - 1, 0 \leq t \leq i_1 + i_2 - (s_1 + s_2) + 1,
\]

the subcategory

\[
p^*(a_{i_1}^1 \otimes a_{i_2}^2)(s_1H_1 + s_2H_2 + tH) \subset \tilde{\mathcal{J}}(\mathcal{A}^1, \mathcal{A}^2)
\]

is contained in \( \mathcal{P} \).

**Proof.** We argue by induction on \( s = s_1 + s_2 \). The base case \( s_1 = s_2 = 0 \) holds by (3.21) in the definition of \( \mathcal{P} \). Now assume \( s > 0 \) and the result holds for \( s - 1 \). Either \( s_1 > 0 \) or \( s_2 > 0 \). Assume \( s_1 > 0 \). Consider the exact sequence

\[
0 \rightarrow \mathcal{O}(-E_2) \rightarrow \mathcal{O} \rightarrow \varepsilon_{2*}\mathcal{O}_{E_2} \rightarrow 0
\]

(3.24)

on \( \tilde{\mathcal{J}}(\mathcal{P}(V_1), \mathcal{P}(V_2)) \). By Lemma 3.1[3] we have \(-E_2 = H_1 - H \) and \( H|_{E_2} = H_2 \), so twisting this sequence by \( \mathcal{O}((s_1 - 1)H_1 + s_2H_2 + (t + 1)H) \) gives

\[
0 \rightarrow \mathcal{O}(s_1H_1 + s_2H_2 + tH) \rightarrow \mathcal{O}((s_1 - 1)H_1 + s_2H_2 + (t + 1)H) \\
\rightarrow \varepsilon_{2*}\mathcal{O}_{E_2}((s_1 - 1)H_1 + (s_2 + t + 1)H_2) \rightarrow 0.
\]

For \( C_1 \in a_{i_1}^1, C_2 \in a_{i_2}^2 \), tensoring this sequence with \( p^*(C_1 \boxtimes C_2) \) gives an exact triangle

\[
p^*(C_1 \boxtimes C_2)(s_1H_1 + s_2H_2 + tH) \rightarrow p^*(C_1 \boxtimes C_2)((s_1 - 1)H_1 + s_2H_2 + (t + 1)H) \\
\rightarrow \varepsilon_{2*}(C_1((s_1 - 1)H_1) \boxtimes C_2((s_2 + t + 1)H_2)),
\]

where we have used the projection formula and diagram (3.2) to rewrite the third term. The second term of this triangle is in \( \mathcal{P} \) by the induction hypothesis, and the third term is in \( \mathcal{P} \) by (3.23) since \( s_1 - 1 \leq i_1 - 1 \) by the assumption of the lemma. Hence the first term is also in \( \mathcal{P} \). By Lemma A.1, the objects \( p^*(C_1 \boxtimes C_2) \) for \( C_1 \in a_{i_1}^1, C_2 \in a_{i_2}^2 \), thickly generate \( p^*(a_{i_1}^1 \otimes a_{i_2}^2) \), so we deduce the required containment

\[
p^*(a_{i_1}^1 \otimes a_{i_2}^2)(s_1H_1 + s_2H_2 + tH) \subset \mathcal{P}.
\]

The case \( s_2 > 0 \) follows by the same argument (with \( E_2 \) replaced by \( E_1 \)). This completes the induction. \( \square \)
Proof of Theorem 3.21. Let us show $\mathcal{P} = \mathcal{J}(A^1, A^2)$, which as observed above will complete the proof of the semiorthogonal decomposition (3.17). By Lemma 3.7 we have a semiorthogonal decomposition

$$\mathcal{J}(A^1, A^2) = \langle p^*(A^1 \otimes A^2), p^*(A^1 \otimes A^2)(H) \rangle,$$

so it suffices to show $\mathcal{P}$ contains both components of this decomposition. But tensoring the decompositions (2.10) for $A^1$ and $A^2$, we see that $A^1 \otimes A^2$ is generated by the categories

$$(a_1^1 \otimes a_2^1)(s_1H_1 + s_2H_2), \quad 0 \leq s_1 \leq i_1 \leq m_1 - 1, \quad 0 \leq s_2 \leq i_2 \leq m_2 - 1.$$ 

Hence taking $t = 0$ in Lemma 3.26 shows $\mathcal{P}$ contains $p^*(A^1 \otimes A^2)$, and taking $t = 1$ shows $\mathcal{P}$ contains $p^*(A^1 \otimes A^2)(H)$, as required.

The semiorthogonal decomposition (3.18) holds by a similar argument. Thus by Lemma 2.4 and Lemma 3.23 we deduce that $j_0 \subseteq \mathcal{J}(A_1, A_2)$ is a Lefschetz center with $j_i, i \in \mathbb{Z}$, the corresponding Lefschetz components. The strongness claims follow from the definitions and Lemma 3.23, and the claims about the length and moderateness of $\mathcal{J}(A^1, A^2)$ follow from the definitions and (2.12).

4. HPD for Categorical Joins

In this section we prove our main theorem, which says that (under suitable hypotheses) the formation of categorical joins commutes with HPD.

Theorem 4.1. Let $A^1$ and $A^2$ be right strong, moderate Lefschetz categories over $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$. Then there is an equivalence

$$\mathcal{J}(A^1, A^2)^\diamond \simeq \mathcal{J}((A^1)^\diamond, (A^2)^\diamond)$$

of Lefschetz categories over $\mathbb{P}(V_1^\vee \oplus V_2^\vee)$.

Remark 4.2. The Lefschetz structures on the categories $\mathcal{J}(A^1, A^2)^\diamond$ and $\mathcal{J}((A^1)^\diamond, (A^2)^\diamond)$ in Theorem 4.1 are the ones obtained by combining Theorems 2.27 and 3.21.

The key object in the proof of Theorem 4.1 is a certain fiber product of resolved joins, which we call a double resolved join. We discuss this construction in §4.1 and then use it in §4.2 to define a functor $\gamma_j : \mathcal{J}((A^1)^\diamond, (A^2)^\diamond) \to \mathbf{H}(\mathcal{J}(A^1, A^2))$. In §4.2 we also prove various properties of $\gamma_j$, which we use in §4.3 to show $\gamma_j$ induces the equivalence of Theorem 4.1.

4.1. Double resolved joins. For $k = 1, 2$, let $V_k$ be a vector bundle on $S$ and denote by $H_k$ and $H'_k$ the relative hyperplane classes on $\mathbb{P}(V_k)$ and $\mathbb{P}(V_k^\vee)$.

In this section sometimes we will consider pairs of $\mathbb{P}(V_k)$-linear categories, so we can form their resolved join over $\mathbb{P}(V_1 \oplus V_2)$, and sometimes we will consider pairs of $\mathbb{P}(V_k^\vee)$-linear categories, so we can form their resolved join over $\mathbb{P}(V_1^\vee \oplus V_2^\vee)$. Moreover, sometimes we will consider pairs of $\mathbb{P}(V_k) \times \mathbb{P}(V_k^\vee)$-linear categories, so that we can form both types of joins for them. To distinguish notationally between the two types of joins we will write

$$\mathcal{J}(Y_1, Y_2) = \mathbb{P}_{Y_1 \times Y_2}(0(-H_1) \oplus 0(-H_2)),$$

$$\mathcal{J}(B^1, B^2) = (B^1 \otimes B^2) \otimes_{\text{Perf}(\mathbb{P}(V_1) \times \mathbb{P}(V_2))} \text{Perf}(\mathcal{J}(\mathbb{P}(V_1), \mathbb{P}(V_2)))$$
if $Y_k$ are schemes over $P(V_k)$ and $B^k$ are $P(V_k)$-linear categories, and
\[
\tilde{J}^\vee(Y_1, Y_2) = P_{Y_1 \times Y_2}(O(-H_1') \oplus O(-H_2'))
\]
\[
\tilde{J}^\vee(B^1, B^2) = (B^1 \otimes B^2) \otimes_{\text{Perf}(P(V_1) \times P(V_2))} \text{Perf}(\tilde{J}^\vee(P(V_1'), P(V_2')))
\]
if $Y_k$ are schemes over $P(V_k')$ and $B^k$ are $P(V_k')$-linear categories. We will also use this convention for schemes over $P(V_k) \times P(V_k')$ and for $P(V_k) \times P(V_k')$-linear categories. Note, however, that we do not extend this convention to categorical joins, or to resolved joins of functors.

Let $Y_1$ and $Y_2$ be $S$-schemes equipped with morphisms
\[
Y_1 \to P(V_1) \times P(V_1'), \quad Y_2 \to P(V_2) \times P(V_2').
\]
We define the double resolved join of $Y_1$ and $Y_2$ as the fiber product
\[
\tilde{J}J(Y_1, Y_2) = \tilde{J}(Y_1, Y_2) \times_{(Y_1 \times Y_2)} \tilde{J}^\vee(Y_1, Y_2).
\]
In particular, we can consider the universal double resolved join with its natural projection
\[
\tilde{J}J((P(V_1) \times P(V_1')), (P(V_2) \times P(V_2'))) \to (P(V_1) \times P(V_1')) \times (P(V_2) \times P(V_2')).
\]
Now, given for $k = 1, 2$, a category $B^k$ which has a $P(V_k) \times P(V_k')$-linear structure, the double resolved join $\tilde{J}J(B^1, B^2)$ of $B^1$ and $B^2$ is defined as
\[
(B^1 \otimes B^2) \otimes_{\text{Perf}(P(V_1) \times P(V_1'))) \text{Perf}(\tilde{J}J((P(V_1) \times P(V_1')), (P(V_2) \times P(V_2')))),
\]
that is the base change of $B^1 \otimes B^2$ along $B^k$.

For us, the key case of a double resolved join is when $Y_1$ and $Y_2$ are the universal spaces of hyperplanes in $P(V_1)$ and $P(V_2)$, which we denote by
\[
H_1 = H(P(V_1)) \quad \text{and} \quad H_2 = H(P(V_2)).
\]
Note that for $k = 1, 2$, the space $H_k$ indeed comes with natural maps to $P(V_k)$ and $P(V_k')$, hence we can form the double resolved join of $H_1$ and $H_2$. The following commutative diagram summarizes the spaces involved and names the relevant morphisms:

\[
\begin{array}{ccc}
\tilde{J}J(H_1, H_2) & \xrightarrow{p} & \tilde{J}J^\vee(H_1, H_2) \\
\downarrow{\bar{q}} & & \downarrow{h_1 \times h_2} \\
\tilde{J}J^\vee(P(V_1', P(V_2')) & \xrightarrow{q} & H_1 \times H_2 \\
\downarrow{g} & & \downarrow{\pi_1 \times \pi_2} \\
\{P(V_1' \oplus V_2') & \xrightarrow{\pi} & H_1 \times H_2 \\
\downarrow{h_1 \times h_2} & & \downarrow{\pi_1 \times \pi_2} \\
\{P(V_1' \oplus V_2') & \xrightarrow{f} & P(V_1') \times P(V_2') \\
\end{array}
\]

All of the squares in this diagram are cartesian.

Since $\tilde{J}J(P(V_1), P(V_2))$ maps to $P(V_1 \oplus V_2)$, we can form the corresponding universal hyperplane section, which sits as a divisor in the product
\[
H(\tilde{J}J(P(V_1), P(V_2))) \subset \tilde{J}J(P(V_1), P(V_2)) \times P(V_1' \oplus V_2').
\]
Lemma 4.3. We have a commutative diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{J}}(H_1, H_2) & \xrightarrow{\alpha} & \mathcal{H}(\mathcal{J}(P(V_1), P(V_2))) \\
\tilde{\mathcal{J}}^\vee(H_1, H_2) & \xrightarrow{\tilde{p}^*} & P(V_1) \times P(V_2) \times P(V_1^\vee + V_2^\vee)
\end{array}
\]

(4.4)

All schemes in the diagram are smooth and projective over \( S \); hence Remark 1.17 applies to all morphisms.

Proof. From diagram (4.2), we see that \( \tilde{\mathcal{J}}^\vee(H_1, H_2) \) maps to \( P(V_1) \times P(V_2) \) and \( P(V_1^\vee + V_2^\vee) \), giving the lower left arrow in the diagram. Furthermore, its composition with the morphism \( \tilde{p} \) from (4.2) is easily seen to factor through the embedding (4.3) via a morphism which we denote \( \alpha : \tilde{\mathcal{J}}(H_1, H_2) \to H(\mathcal{J}(P(V_1), P(V_2))) \). This gives (4.4). The second claim of the lemma is evident. \( \square \)

4.2. The HPD functor for categorical joins. Let \( A^1 \) and \( A^2 \) be Lefschetz categories over \( P(V_1) \) and \( P(V_2) \). For \( k = 1, 2 \), we denote by

\[
\gamma_k : (A^k)^\natural \to H(A^k)
\]

the canonical \( P(V_k^\vee) \)-linear inclusion functor. Then (4.4) defines a \( P(V_1^\vee + V_2^\vee) \)-linear functor

\[
\tilde{\mathcal{J}}(\gamma_1, \gamma_2) : \tilde{\mathcal{J}}^\vee((A^1)^\natural, (A^2)^\natural) \to \tilde{\mathcal{J}}^\vee(H(A^1), H(A^2)).
\]

By base changing the \( P(V_1) \times P(V_2) \)-linear category \( A^1 \otimes A^2 \) along the top of diagram (4.4), we obtain a diagram of functors

\[
\begin{array}{ccc}
\tilde{\mathcal{J}}(H(A^1), H(A^2)) & \xrightarrow{\tilde{p}^*} & \tilde{\mathcal{J}}^\vee((A^1)^\natural, (A^2)^\natural) \\
\tilde{\mathcal{J}}^\vee((A^1)^\natural, (A^2)^\natural) & \xrightarrow{\tilde{\mathcal{J}}(\gamma_1, \gamma_2)} & \tilde{\mathcal{J}}^\vee(H(A^1), H(A^2)) \\
\end{array}
\]

(4.4)

\[
\begin{array}{ccc}
\tilde{\mathcal{J}}^\vee(H(A^1), H(A^2)) & \xrightarrow{\alpha_*} & H(\tilde{\mathcal{J}}(A^1, A^2)) \end{array}
\]

Since the diagram (4.4) is over \( P(V_1^\vee + V_2^\vee) \), all of the above functors are \( P(V_1^\vee + V_2^\vee) \)-linear.

By composing the functors in the diagram, we obtain a \( P(V_1^\vee + V_2^\vee) \)-linear functor

\[
\gamma_{\tilde{\mathcal{J}}} = \alpha_* \circ \tilde{p}^* \circ \tilde{\mathcal{J}}(\gamma_1, \gamma_2) : \tilde{\mathcal{J}}^\vee((A^1)^\natural, (A^2)^\natural) \to H(\tilde{\mathcal{J}}(A^1, A^2)).
\]

(4.5)

Our goal is to show that \( \gamma_{\tilde{\mathcal{J}}} \) induces the desired equivalence \( \tilde{\mathcal{J}}((A^1)^\natural, (A^2)^\natural) \simeq \tilde{\mathcal{J}}(A^1, A^2)^\natural \) when \( A^1 \) and \( A^2 \) satisfy the assumptions of Theorem 4.1. In this subsection, we focus on proving some compatibility properties between the functor \( \gamma_{\tilde{\mathcal{J}}} \) and the functors \( \gamma_1 \) and \( \gamma_2 \), which will be used in §4.3 to prove Theorem 4.1. We start with the following observation, which will also be needed later.

Lemma 4.4. The functor \( \gamma_{\tilde{\mathcal{J}}} \) has both left and right adjoints.

Proof. The functors \( \gamma_k \) have both left and right adjoints by Lemma 2.22. Therefore, \( \tilde{\mathcal{J}}(\gamma_1, \gamma_2) \) has both left and right adjoints by Lemma 3.5. On the other hand, the functors \( \alpha_* \) and \( \tilde{p}^* \) have both left and right adjoints by Lemma 4.3 and Remark 1.17. \( \square \)
Let $H$ and $H'$ denote the relative hyperplane classes on $\mathbf{P}(V_1 \oplus V_2)$ and $\mathbf{P}(V_1^\vee \oplus V_2^\vee)$. As in [2,3], let $\mathcal{M}$ be the cohomology sheaf of the monad

$$\mathcal{M} \simeq \{ \mathcal{O}(-H) \to (V_1 \oplus V_2) \otimes \mathcal{O} \to \mathcal{O}(H') \}$$

(4.6) on $\mathbf{H}(\mathbf{P}(V_1 \oplus V_2))$. Similarly, for $k = 1, 2$, we let $\mathcal{M}_k$ be the cohomology sheaf of the monad

$$\mathcal{M}_k \simeq \{ \mathcal{O}(-H_k) \to V_k \otimes \mathcal{O} \to \mathcal{O}(H'_k) \}.$$  

(4.7) on $\mathbf{H}_k$. Pushforward along the morphisms

$$\pi_{\tilde{J}} : \mathbf{H}(\tilde{J}(\mathbf{P}(V_1), \mathbf{P}(V_2))) \to \tilde{J}(\mathbf{P}(V_1), \mathbf{P}(V_2))$$

$$\pi_k : \mathbf{H}_k \to \mathbf{P}(V_k), \ k = 1, 2,$$

induces functors

$$\pi_{\tilde{J}*} : \mathbf{H}(\tilde{J}(\mathbf{A}^1, \mathbf{A}^2)) \to \tilde{J}(\mathbf{A}^1, \mathbf{A}^2),$$

$$\pi_{k*} : \mathbf{H}(\mathbf{A}^k) \to \mathbf{A}^k, \ k = 1, 2.$$  

For $t \geq 0$, we aim to relate the composition

$$\pi_{\tilde{J}*} \circ (\otimes \wedge^t \mathcal{M}) \circ \gamma_{\tilde{J}} : \tilde{J}^\vee((\mathbf{A}^1)^\wedge, (\mathbf{A}^2)^\wedge) \to \tilde{J}(\mathbf{A}^1, \mathbf{A}^2)$$

(4.8) to the analogous compositions

$$\pi_{k*} \circ (\otimes \wedge^t \mathcal{M}_k) \circ \gamma_k : (\mathbf{A}^k)^\wedge \to \mathbf{A}^k, \ k = 1, 2.$$  

(4.9)

Combined with the results of [2,3], this will be the key ingredient in our proof that $\gamma_{\tilde{J}}$ induces the equivalence of Theorem 4.1. The following result handles the case $t = 0$.

**Proposition 4.5.** There is an isomorphism

$$\pi_{\tilde{J}*} \circ \gamma_{\tilde{J}} \simeq p^* \circ ((\pi_{1*} \circ \gamma_1) \otimes (\pi_{2*} \circ \gamma_2)) \circ q_*$$

of functors $\tilde{J}^\vee((\mathbf{A}^1)^\wedge, (\mathbf{A}^2)^\wedge) \to \tilde{J}(\mathbf{A}^1, \mathbf{A}^2)$.

**Proof.** Consider the commutative diagram

$$\begin{array}{ccc}
\tilde{J}^\vee(H_1, H_2) & \xrightarrow{\tilde{p}} & \tilde{J}^\vee(H_1, H_2) \\
\downarrow \tilde{q} & & \downarrow \alpha \\
\tilde{J}(\mathbf{P}(V_1), \mathbf{P}(V_2)) & \xrightarrow{\tilde{p}_H} & \mathbf{H}(\tilde{J}(\mathbf{P}(V_1), \mathbf{P}(V_2))) \\
\downarrow \pi_{1\times\pi_2} & & \downarrow \pi_3 \\
\mathbf{P}(V_1) \times \mathbf{P}(V_2) & \xrightarrow{p} & \mathbf{P}(V_1) \times \mathbf{P}(V_2)
\end{array}$$

(4.10)

where the squares marked by $\star$ are cartesian and Tor-independent since $p$, being a $\mathbf{P}^1$-bundle, is flat. Therefore, we have a chain of isomorphisms

$$\pi_{\tilde{J}} \circ \alpha_* \circ \tilde{p}^* \simeq (\pi_{1\times\pi_2})_* \circ \tilde{q}_* \circ \tilde{p}^*$$

$$\simeq (\pi_{1\times\pi_2})_* \circ p_{H*}^* \circ q_{H*}$$

$$\simeq p^* \circ (\pi_{1\times\pi_2})_* \circ q_{H*}$$

as required.
of functors $\text{Perf}(\tilde{J}^\vee(H_1, H_2)) \to \text{Perf}(\tilde{J}(P(V_1), P(V_2)))$. After base change from $P(V_1)$ to $A^1$ and from $P(V_2)$ to $A^2$ and composition with the functor $\tilde{J}(\gamma_1, \gamma_2)$, we obtain an equivalence

$$\pi_{\tilde{J}*} \circ \alpha_* \circ \tilde{p}^* \circ \tilde{J}(\gamma_1, \gamma_2) \simeq p^* \circ (\pi_1 \times \pi_2)_* \circ q_{H*} \circ \tilde{J}(\gamma_1, \gamma_2)$$

of functors $\tilde{J}^\vee((A^1)^{\circ}, (A^2)^{\circ}) \to \tilde{J}(A^1, A^2)$. The left hand side is $\pi_{\tilde{J}*} \circ \gamma_{\tilde{J}}$. On the other hand, from the commutative diagram of Lemma 3.5 we obtain an isomorphism

$$q_{H*} \circ \tilde{J}(\gamma_1, \gamma_2) \simeq (\gamma_1 \otimes \gamma_2) \circ q_*,$$

that allows us to rewrite the right hand side as

$$p^* \circ (\pi_1 \times \pi_2)_* \circ (\gamma_1 \otimes \gamma_2) \circ q_*,$$

which is equivalent to the right hand side in the statement of the proposition. \qed

To relate the functor (4.8) to the functors (4.9) for arbitrary $t \geq 0$, we will need the following lemma. By definition $\tilde{J}(H_1, H_2)$ admits projections to $H_1$ and $H_2$, and also maps to $H(P(V_1 \oplus V_2))$ via the composition

$$\tilde{J}(H_1, H_2) \xrightarrow{\alpha} H(J(P(V_1), P(V_2))) \xrightarrow{H(f)} H(P(V_1 \oplus V_2)).$$

We denote by $\tilde{M}, \tilde{M}_1, \tilde{M}_2$, the pullbacks to $\tilde{J}(H_1, H_2)$ of the sheaves $M, M_1, M_2$, defined by (4.6) and (4.7) above. Note that by Lemma 3.1(4) we have the formula

$$\omega_{\tilde{p}} = \mathcal{O}(H_1 + H_2 - 2H)[1]$$

for the relative dualizing complex of the morphism $\tilde{p}: \tilde{J}(H_1, H_2) \to \tilde{J}^\vee(H_1, H_2)$.

We write

$$u = q_{H} \circ \tilde{p} = p_{H} \circ \tilde{q}: \tilde{J}(H_1, H_2) \to H_1 \times H_2$$

for the canonical $P^1 \times P^1$-bundle, see (4.10). For $k = 1, 2$, we write

$$\text{pr}_k: H_1 \times H_2 \to H_k$$

for the projection.

**Lemma 4.6.** There is an isomorphism

$$u_* (\omega_{\tilde{p}} \otimes \wedge^t \tilde{M}) \simeq \wedge^t (\text{pr}_1^* M_1 \oplus \text{pr}_2^* M_2).$$

**Proof.** The pullback to $\tilde{J}(H_1, H_2)$ of the monad (4.6) and the direct sum of the pullbacks to $\tilde{J}(H_1, H_2)$ of the monads (4.7) fit into the following bicomplex on $\tilde{J}(H_1, H_2)$:

$$
\begin{array}{ccc}
\mathcal{O}(H_1') \oplus \mathcal{O}(H_2') & \longrightarrow & \mathcal{O}(H') \\
\uparrow & & \uparrow \\
\mathcal{O}(-H) & \longrightarrow & (V_1 \oplus V_2) \otimes \mathcal{O} \longrightarrow \mathcal{O}(H') \\
\uparrow & & \uparrow \\
\mathcal{O}(-H) & \longrightarrow & \mathcal{O}(-H_1) \oplus \mathcal{O}(-H_2)
\end{array}
$$

The nontrivial cohomology sheaves with respect to the horizontal differential are given by $\mathcal{O}(H - H_1 - H_2)$, $\tilde{M}$, $\mathcal{O}(H_1' + H_2' - H')$ in degrees $(0, -1), (0, 0), (0, 1)$. The only nontrivial
cohomology sheaf with respect to the vertical differential is $\tilde{M}_1 \oplus \tilde{M}_2$ in degree $(0,0)$. It follows that there is a filtration of $\tilde{M}$ whose associated graded is

$$\mathcal{O}(H - H_1 - H_2) \oplus (\tilde{M}_1 \oplus \tilde{M}_2) \oplus \mathcal{O}(H'_1 + H'_2 - H').$$

Hence $\wedge^t \tilde{M}$ has a filtration whose associated graded is

$$\left(\wedge^{t-1}(\tilde{M}_1 \oplus \tilde{M}_2) \otimes \mathcal{O}(H - H_1 - H_2)\right)$$

$$\oplus \left(\wedge^t (\tilde{M}_1 \oplus \tilde{M}_2)\right)$$

$$\oplus \left(\wedge^{t-2}(\tilde{M}_1 \oplus \tilde{M}_2) \otimes \mathcal{O}(H - H_1 - H_2 + H'_1 + H'_2 - H')\right)$$

$$\oplus \left(\wedge^{t-1}(\tilde{M}_1 \oplus \tilde{M}_2) \otimes \mathcal{O}(H'_1 + H'_2 - H')\right).$$

Since $u: \tilde{JJ}(H_1, H_2) \to H_1 \times H_2$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle (whose relative hyperplane classes are $H$ and $H'$), we have

$$u_*(\mathcal{O}(aH + a'H')) = 0$$

if either $a = -1$ or $a' = -1$.

Moreover, since $u = q_{\mathbb{H}} \circ \tilde{p}$ where $q_{\mathbb{H}}$ and $\tilde{p}$ are $\mathbb{P}^1$-bundles, we have

$$u_*(\omega_{\tilde{p}}) \simeq \mathcal{O}.$$

Hence by the formula (4.11) for $\omega_{\tilde{p}}$ and the above description of the associated graded of the filtration of $\wedge^t \tilde{M}$, we find

$$u_*(\omega_{\tilde{p}} \otimes \wedge^t \tilde{M}) \simeq \wedge^t (\text{pr}^*_1 \tilde{M}_1 \oplus \text{pr}^*_2 \tilde{M}_2)$$

as desired. \hfill \square

Recall that $p_!$ denotes the left adjoint of $p^*: \text{Perf}(\mathbb{P}(V_1) \times \mathbb{P}(V_2)) \to \text{Perf}(\tilde{JJ}(\mathbb{P}(V_1), \mathbb{P}(V_2))).$

**Proposition 4.7.** There is an isomorphism

$$p_! \circ \pi_{J*} \circ (\ominus \otimes \wedge^t \tilde{M}) \circ \gamma_J \circ q^* \simeq \bigoplus_{t_1 + t_2 = t} (\pi_{1*} \circ (\ominus \otimes \wedge^t \tilde{M}_1) \circ \gamma_1) \otimes (\pi_{2*} \circ (\ominus \otimes \wedge^t \tilde{M}_2) \circ \gamma_2)$$

of functors $(\mathcal{A}_1^1)^\otimes \otimes (\mathcal{A}_2^2)^\otimes \to \mathcal{A}_1^1 \otimes \mathcal{A}_2^2$.

**Proof.** The proof is similar to that of Proposition 4.5. First, using the commutative diagram in Lemma 3.5, we obtain an isomorphism

$$\tilde{J}(\gamma_1, \gamma_2) \circ q^* \simeq q_{\mathbb{H}}^* \circ (\gamma_1 \otimes \gamma_2).$$

Using the definition of $\gamma_J$, this allows us to rewrite the left hand side in the statement of the proposition as

$$p_! \circ \pi_{J*} \circ (\ominus \otimes \wedge^t \tilde{M}) \circ \alpha_* \circ \tilde{p}^* \circ q_{\mathbb{H}}^* \circ (\gamma_1 \otimes \gamma_2).$$

By the projection formula, we can rewrite this as

$$p_! \circ \pi_{J*} \circ \alpha_* \circ (\ominus \otimes \wedge^t \tilde{M}) \circ \tilde{p}^* \circ q_{\mathbb{H}}^* \circ (\gamma_1 \otimes \gamma_2).$$

Note that $p_! = p_* \circ (\ominus \otimes \omega_{\tilde{p}})$ by (1.11). Further, the pullback of $\omega_{\tilde{p}}$ to $\tilde{JJ}(H_1, H_2)$ is $\omega_{\tilde{p}}$, since $\tilde{p}$ is a base change of $p$. Hence again by the projection formula, we can rewrite the above composition as

$$p_* \circ \pi_{J*} \circ \alpha_* \circ (\ominus \otimes \omega_{\tilde{p}} \otimes \wedge^t \tilde{M}) \circ \tilde{p}^* \circ q_{\mathbb{H}}^* \circ (\gamma_1 \otimes \gamma_2).$$
By commutativity of the diagram (4.10) we have \( p \circ \pi_j \circ \alpha = (\pi_1 \times \pi_2) \circ u \), where recall \( u = \gamma H \circ \tilde{p} \).

Hence we can rewrite the above composition as

\[
(\pi_1 \times \pi_2)_* \circ u_* \circ (- \otimes \omega_{\tilde{p}} \otimes \wedge^t \tilde{M}) \circ u^* \circ (\gamma_1 \otimes \gamma_2).
\]

By the projection formula, the composition \( u_* \circ (- \otimes \omega_{\tilde{p}} \otimes \wedge^t \tilde{M}) \circ u^* \) is equivalent to the functor given by tensoring with the object \( u_*(\omega_{\tilde{p}} \otimes \wedge^t \tilde{M}) \). But by Lemma 4.6 this object is isomorphic to \( \wedge^t (pr_1^* M_1 \oplus pr_2^* M_2) \). Therefore, the functor we are interested in is equivalent to the direct sum of the functors

\[
(\pi_1 \times \pi_2)_* \circ (- \otimes (\wedge^t pr_1^* M_1 \otimes \wedge^t pr_2^* M_2)) \circ (\gamma_1 \otimes \gamma_2),
\]

over all \( t_1 + t_2 = t \). It remains to note that each summand is isomorphic to the corresponding summand in the right hand side of the statement of the proposition.

Remark 4.8. The functor \( \gamma_J \) can be described in terms of Fourier–Mukai kernels. For simplicity, in this remark we restrict ourselves to the geometric case as in [14], but the same description works in general using the formalism of Fourier–Mukai kernels from [32, \S 5]. Namely, assume for \( k = 1, 2 \), that \( \mathcal{A}^k = \text{Perf}(X_k) \) for a \( \mathcal{P}(V_k) \)-scheme \( X_k \), and there is a \( \mathcal{P}(V_k) \)-scheme \( Y_k \) and a \( \mathcal{P}(V_k) \)-linear equivalence

\[
\phi_k : \text{Perf}(Y_k) \simto \text{Perf}(X_k)^2
\]

which is induced by a Fourier–Mukai functor

\[
\Phi_{\mathcal{E}_k} : \text{Perf}(Y_k) \to \text{Perf}(H(X_k)),
\]

where \( \mathcal{E}_k \in \text{Perf}(H(X_k) \times \mathcal{P}(V_k) Y_k) \). Via the identifications

\[
\text{Perf}(J^\vee(Y_1, Y_2)) \simeq J^\vee(\text{Perf}(Y_1), \text{Perf}(Y_2)),
\]

\[
\text{Perf}(H(J(X_1, X_2))) \simeq H(J(\text{Perf}(X_1), \text{Perf}(X_2))),
\]

we can consider \( \gamma_J \circ J(\phi_1, \phi_2) \) as a \( \mathcal{P}(V_1^\vee \oplus V_2^\vee) \)-linear functor

\[
\gamma_J \circ J(\phi_1, \phi_2) : \text{Perf}(J^\vee(Y_1, Y_2)) \to \text{Perf}(H(J(X_1, X_2))).
\]

This functor can be described as follows. We have a commutative diagram

\[
\begin{array}{ccc}
\tilde{J}J(H_1, H_2) & \xrightarrow{u} & H_1 \times H_2 \\
\downarrow v & & \downarrow \\
H(J(P_1), P_2) \times_{\mathcal{P}(V_1^\vee \oplus V_2^\vee)} J^\vee(P_1^\vee, P_2^\vee) & \to & \mathcal{P}(V_1) \times \mathcal{P}(V_2) \times \mathcal{P}(V_1^\vee) \times \mathcal{P}(V_2^\vee)
\end{array}
\]

Base changing from \( \mathcal{P}(V_1), \mathcal{P}(V_2), \mathcal{P}(V_1^\vee), \mathcal{P}(V_2^\vee) \), to \( X_1, X_2, Y_1, Y_2 \), gives a commutative diagram

\[
\begin{array}{ccc}
\tilde{J}J(H(X_1), H(X_2)) & \xrightarrow{u} & (H(X_1) \times_{\mathcal{P}(V_1^\vee)} Y_1) \times (H(X_2) \times_{\mathcal{P}(V_2^\vee)} Y_2) \\
\downarrow v & & \downarrow \\
H(J(X_1, X_2)) \times_{\mathcal{P}(V_1^\vee \oplus V_2^\vee)} J^\vee(Y_1, Y_2) & \to & X_1 \times X_2 \times Y_1 \times Y_2
\end{array}
\]
where we abusively still denote the top and left maps by \( u \) and \( v \). The object

\[
\mathcal{E} = v_* u^*(\mathcal{E}_1 \boxtimes \mathcal{E}_2)
\]

(4.12)

in \( \text{Perf}\left( H(J(X_1, X_2)) \times_{\mathcal{P}(V_1 \oplus V_2)} J^\vee(Y_1, Y_2) \right) \) is a Fourier–Mukai kernel for a \( \mathcal{P}(V_1 \oplus V_2) \)-linear functor

\[
\Phi_\mathcal{E} : \text{Perf}(J^\vee(Y_1, Y_2)) \to \text{Perf}(H(J(X_1, X_2))).
\]

This is the desired functor: it is straightforward to verify there is an isomorphism

\[
\Phi_\mathcal{E} \simeq \gamma_J \circ \tilde{J}(\phi_1, \phi_2).
\]

4.3. Proof of Theorem 4.1. The categorical join \( \mathcal{J}(\mathcal{A}_1, \mathcal{A}_2) \) fits into a \( \mathcal{P}(\mathcal{V}_1 \oplus \mathcal{V}_2) \)-linear semiorthogonal decomposition (3.5), which we can write in a simplified form as

\[
\tilde{J}(\mathcal{A}_1, \mathcal{A}_2) = \langle \mathcal{J}(\mathcal{A}_1, \mathcal{A}_2), \perp \mathcal{J}(\mathcal{A}_1, \mathcal{A}_2) \rangle.
\]

By Lemma 2.29 we have a semiorthogonal decomposition

\[
H(\tilde{J}(\mathcal{A}_1, \mathcal{A}_2)) = \langle H(\mathcal{J}(\mathcal{A}_1, \mathcal{A}_2)), H(\perp \mathcal{J}(\mathcal{A}_1, \mathcal{A}_2)) \rangle.
\]

The HPD category \( \mathcal{J}(\mathcal{A}_1, \mathcal{A}_2) \) is a \( \mathcal{P}(\mathcal{V}_1 \oplus \mathcal{V}_2) \)-linear subcategory of \( H(\mathcal{J}(\mathcal{A}_1, \mathcal{A}_2)) \), and hence also of \( H(\tilde{J}(\mathcal{A}_1, \mathcal{A}_2)) \).

We will prove the following more precise version of Theorem 4.1.

**Theorem 4.9.** Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be right strong, moderate Lefschetz categories over \( \mathcal{P}(\mathcal{V}_1) \) and \( \mathcal{P}(\mathcal{V}_2) \). The functor

\[
\gamma_J : \tilde{J}^\vee((\mathcal{A}_1)^\natural, (\mathcal{A}_2)^\natural) \to H(\tilde{J}(\mathcal{A}_1, \mathcal{A}_2))
\]

defined by (4.5) induces a Lefschetz equivalence between the subcategories

\[
\mathcal{J}((\mathcal{A}_1)^\natural, (\mathcal{A}_2)^\natural) \subset \tilde{J}^\vee((\mathcal{A}_1)^\natural, (\mathcal{A}_2)^\natural) \quad \text{and} \quad \mathcal{J}(\mathcal{A}_1, \mathcal{A}_2) \subset H(\tilde{J}(\mathcal{A}_1, \mathcal{A}_2)).
\]

The following lemma guarantees that \( \gamma_J \) does indeed induce a functor between the categories of interest.

**Lemma 4.10.** The image of the functor \( \gamma_J : \tilde{J}^\vee((\mathcal{A}_1)^\natural, (\mathcal{A}_2)^\natural) \to H(\tilde{J}(\mathcal{A}_1, \mathcal{A}_2)) \) is contained in the HPD category \( \mathcal{J}(\mathcal{A}_1, \mathcal{A}_2) \subset H(\tilde{J}(\mathcal{A}_1, \mathcal{A}_2)) \).

**Proof.** By Lemmas 2.28 and 2.29 together with \( \mathcal{P}(\mathcal{V}_1 \oplus \mathcal{V}_2) \)-linearity of the functor \( \gamma_J \), it suffices to show that the image of \( \pi_{J_{A^k}} \circ \gamma_J \) is contained in the Lefschetz center

\[
\mathcal{J}(\mathcal{A}_1, \mathcal{A}_2) \subset \mathcal{J}(\mathcal{A}_1, \mathcal{A}_2) \subset \tilde{J}(\mathcal{A}_1, \mathcal{A}_2).
\]

By Proposition 4.5 we have

\[
\pi_{J_{A^k}} \circ \gamma_J \simeq p^* \circ ((\pi_{A^k} \circ \gamma_1) \otimes (\pi_{A^k} \circ \gamma_2)) \circ q_*.
\]

By Lemma 2.28 the image of \( \pi_{k_*} \circ \gamma_k \) is \( \mathcal{A}_k \subset \mathcal{A}_k \), hence the image of \( \pi_{J_{A^k}} \circ \gamma_J \) is contained in \( p^*(\mathcal{A}_0 \boxtimes \mathcal{A}_0) \), which is nothing but \( \mathcal{J}(\mathcal{A}_1, \mathcal{A}_2) \) by definition (3.12). \( \Box \)

By Lemma 4.10, the restriction of the functor \( \gamma_J \) to the \( \mathcal{P}(\mathcal{V}_1 \oplus \mathcal{V}_2) \)-linear subcategory \( \mathcal{J}((\mathcal{A}_1)^\natural, (\mathcal{A}_2)^\natural) \subset \tilde{J}^\vee((\mathcal{A}_1)^\natural, (\mathcal{A}_2)^\natural) \) induces a \( \mathcal{P}(\mathcal{V}_1 \oplus \mathcal{V}_2) \)-linear functor

\[
\phi : \mathcal{J}((\mathcal{A}_1)^\natural, (\mathcal{A}_2)^\natural) \to \mathcal{J}(\mathcal{A}_1, \mathcal{A}_2).
\]
which fits into a commutative diagram

\[
\begin{array}{ccc}
\mathcal{J}((A^1)^\natural, (A^2)^\natural) & \overset{\gamma_1}{\longrightarrow} & \mathbf{H}(\mathcal{J}(A^1, A^2)) \\
j \downarrow & & \downarrow \gamma \\
\mathcal{J}((A^1)^\natural, (A^2)^\natural) & \overset{\phi}{\longrightarrow} & \mathcal{J}(A^1, A^2)^\natural
\end{array}
\]

(4.13)

where \(j\) and \(\gamma\) are the inclusions. Our goal is to show \(\phi\) is an equivalence of Lefschetz categories. We will prove this by verifying the criteria of Lemma 2.14.

**Lemma 4.11.** The functor \(\phi\) takes the Lefschetz center \(\mathcal{J}((A^1)^\natural, (A^2)^\natural)_0 \subset \mathcal{J}((A^1)^\natural, (A^2)^\natural)\) to the Lefschetz center \(\mathcal{J}(A^1, A^2)^\natural_0 \subset \mathcal{J}(A^1, A^2)^\natural\).

**Proof.** By Proposition 2.31 we must show that for any \(C \in \mathcal{J}((A^1)^\natural, (A^2)^\natural)_0\) and \(t \geq 1\), we have

\[\pi_{A^1}(\gamma_J(C) \otimes \wedge^1 \mathcal{M}) \in \mathcal{J}(A^1, A^2)_0.\]

Since \(\mathcal{J}(A^1, A^2)_0 = p^* (A^0_0 \otimes A^0_0)\), by adjunction the desired conclusion is equivalent to

\[\pi_{A^1}(\gamma_J(C) \otimes \wedge^1 \mathcal{M}) \in \mathcal{J}(A^1, A^2)_0.\]

Since \(\mathcal{J}((A^1)^\natural, (A^2)^\natural)_0 = q^* (A^1_0 \otimes A^2_0)\), by Lemma A.1 we may assume that \(C\) is of the form \(C = q^* (C_1 \boxtimes C_2)\) for \(C_k \in (A^k)^\natural_0\). Then by Proposition 4.7, we have

\[\pi_{A^1}(\gamma_J(q^* (C_1 \boxtimes C_2)) \otimes \wedge^1 \mathcal{M}) \simeq \bigoplus_{t_1 + t_2 = t} (\pi_{A^1}(\gamma_1(C_1) \otimes \wedge^{l_1} \mathcal{M}_1)) \otimes (\pi_{A^2}(\gamma_2(C_2) \otimes \wedge^{l_2} \mathcal{M}_2)).\]

By Proposition 2.31 again, we have

\[\pi_{A^1}(\gamma_1(C_1) \otimes \wedge^{l_1} \mathcal{M}_1) \in \mathcal{J}(A^1, A^2)_0\] if \(t_1 \geq 1\),

\[\pi_{A^2}(\gamma_2(C_2) \otimes \wedge^{l_2} \mathcal{M}_2) \in \mathcal{J}(A^1, A^2)_0\] if \(t_2 \geq 1\).

Since \(t \geq 1\), \(t_1 + t_2 = t\) then either \(t_1 \geq 1\) or \(t_2 \geq 1\). It follows that if \(t_1 \geq 1\) then the \((t_1, t_2)\) summand in the above expression lies in \((A^1_0 \otimes A^2_0)\), and if \(t_2 \geq 1\) then it lies in \((A^1_0 \otimes A^2_0)\). We conclude by Lemma A.5 that every summand in the above expression lies in \((A^1_0 \otimes A^2_0)\), and hence so does their sum \(\pi_{A^1}(\gamma_J(q^* (C_1 \boxtimes C_2)) \otimes \wedge^1 \mathcal{M}).\)

**Lemma 4.12.** The functor \(\phi\) induces an equivalence \(\mathcal{J}((A^1)^\natural, (A^2)^\natural)_0 \simeq \mathcal{J}(A^1, A^2)_0\).

**Proof.** By Lemma 4.11, \(\phi\) induces a functor \(\mathcal{J}((A^1)^\natural, (A^2)^\natural)_0 \to \mathcal{J}(A^1, A^2)_0\), and by Lemma 2.23 the functor \(\pi_J \circ \gamma: \mathcal{J}(A^1, A^2)_0 \to \mathcal{J}(A^1, A^2)^\natural_0 \simeq \mathcal{J}(A^1, A^2)_0\). Hence it suffices to show the composition \(\pi_J \circ \gamma \circ \phi \simeq \pi_J \circ (\pi \circ (\pi^* \circ \gamma_1) \otimes (\pi^* \circ \gamma_2)) \circ q_*\).

By the definitions of \(\mathcal{J}(A^1, A^2)_0\) and \(\mathcal{J}(A^1, A^2)_0\), the functor \(q_*\) induces an equivalence \(\mathcal{J}((A^1)^\natural, (A^2)^\natural)_0 \simeq (A^0_0 \otimes A^0_0)\), and \(\pi^*\) induces an equivalence \(A^0_0 \otimes A^0_0 \simeq \mathcal{J}(A^1, A^2)_0\). Hence it remains to observe \(\pi_k \circ \gamma_k\) induces an equivalence \((A^k)^\natural_0 \simeq A^k\) for \(k = 1, 2\), again by Lemma 2.23.

**Lemma 4.13.** The functor \(\phi\) admits a left adjoint \(\phi^*: \mathcal{J}(A^1, A^2)^\natural \to \mathcal{J}((A^1)^\natural, (A^2)^\natural)\).
Lemma 4.14. The functor $\phi^*$ induces an equivalence $\mathcal{J}(A^1, A^2)_0 \simeq \mathcal{J}((A^1)^2, (A^2)^2)_0$.

Proof. By Lemma 2.23 the functor $\gamma^* \circ \pi_j^*: \mathcal{J}(A^1, A^2) \to \mathcal{J}((A^1)^2, (A^2)^2)_0$ induces an equivalence $\mathcal{J}(A^1, A^2)_0 \simeq \mathcal{J}((A^1)^2, (A^2)^2)_0$. So, it suffices to show that the composition $\phi^* \circ \gamma^* \circ \pi_j^*$ induces an equivalence $\mathcal{J}(A^1, A^2)_0 \simeq \mathcal{J}((A^1)^2, (A^2)^2)_0$.

On the one hand, by taking left adjoints in the diagram (4.13) and composing with $\pi_j^*$ we obtain

$$\phi^* \circ \gamma^* \circ \pi_j^* \simeq j^* \circ \gamma_j^* \circ \pi_j^*.$$ 

On the other hand, taking left adjoints in Proposition 4.5 gives

$$\gamma_j^* \circ \pi_j^* \simeq q^* \circ ((\gamma_1^* \circ \pi_1^*) \otimes (\gamma_2^* \circ \pi_2^*)) \circ p_1.$$ 

So it suffices to show the right side induces an equivalence $\mathcal{J}(A^1, A^2)_0 \simeq \mathcal{J}((A^1)^2, (A^2)^2)_0$. By the definitions of $\mathcal{J}(A^1, A^2)_0$ and $\mathcal{J}((A^1)^2, (A^2)^2)_0$, the functor $p_1$ induces an equivalence $\mathcal{J}(A^1, A^2)_0 \simeq A_0^1 \otimes A_0^2$ (note that $p \circ p^* \simeq \text{id}$ as $p$ is a $\mathbb{P}^1$-bundle) and $q^*$ induces an equivalence $(A_0^1)^2 \otimes (A_0^2)^2 \simeq \mathcal{J}((A^1)^2, (A^2)^2)_0$. Hence it remains to observe $\gamma_k^* \circ \pi_k^*$ induces an equivalence $A_0^k \simeq (A^k)^2_0$ for $k = 1, 2$, again by Lemma 2.23.

Proof of Theorems 4.9 and 4.1. Lemmas 4.12, 4.13, and 4.14 verify the criteria of Lemma 2.14 for

$$\phi: \mathcal{J}((A^1)^2, (A^2)^2) \to \mathcal{J}(A^1, A^2)^2$$

to be an equivalence of Lefschetz categories. This completes the proof of Theorem 4.9 and hence also of Theorem 4.1. 

5. Nonlinear HPD theorems

The main theorem of HPD (Theorem 2.24) relates the linear sections of a Lefschetz category to the orthogonal linear sections of the HPD category. In this section, we show that Theorem 4.1 leads to a nonlinear version of this result, where the role of a linear subspace and its orthogonal is replaced by a Lefschetz category and its HPD category. This is the content of 5.1. We give an extension of this result in 5.2 which describes the tensor product of an arbitrary number of Lefschetz categories over a projective bundle in terms of a linear section of the categorical join of their HPD categories.

5.1. The nonlinear HPD theorem. Recall that if $A^1$ and $A^2$ are Lefschetz categories over $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$, then Theorem 3.21 provides their categorical join $\mathcal{J}(A^1, A^2)$ with the structure of a Lefschetz category over $\mathbb{P}(V_1 \oplus V_2)$. We denote by $\mathcal{J}_i$, $i \in \mathbb{Z}$, the Lefschetz components of the categorical join $\mathcal{J}(A^1, A^2)$, defined by (3.15) and (3.16).

Lemma 5.1. Let $A^1$ and $A^2$ be Lefschetz categories over $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$. For $i \in \mathbb{Z}$ let $\mathcal{J}_i$ be the Lefschetz components of the categorical join $\mathcal{J}(A^1, A^2)$. Set

$$m = \text{length}(\mathcal{J}(A^1, A^2)) = \text{length}(A^1) + \text{length}(A^2).$$
Let $W \subset V_1 \oplus V_2$ be a subbundle of corank $s$, and denote by $H$ the relative hyperplane class on $\mathbf{P}(W)$. Then the functor $\mathcal{J}(A^1, A^2) \to \mathcal{J}(A^1, A^2)\mathbf{P}(W)$ induced by pullback along the embedding $\mathbf{P}(W) \to \mathbf{P}(V_1 \oplus V_2)$ is fully faithful on $\mathcal{J}_i$ for $|i| \geq s$, and there are semiorthogonal decompositions

$$\mathcal{J}(A^1, A^2)\mathbf{P}(W) = \langle \mathcal{K}_W(A^1, A^2), \mathcal{J}_s(H), \ldots, \mathcal{J}_{m-1}((m-s)H) \rangle, = \langle \mathcal{J}_{1-m}((s-m)H), \ldots, \mathcal{J}_{-s}(-H), \mathcal{K}_W'(A^1, A^2) \rangle.$$  

Proof. Follows by combining Theorem 3.21 and Lemma 2.12. □

Now assume that the compositions $W \to V_1 \oplus V_2 \to V_1$ and $W \to V_1 \oplus V_2 \to V_2$ are both inclusions, that is $\mathbf{P}(W) \subset \mathbf{P}(V_1 \oplus V_2)$ is contained in the complement of $\mathbf{P}(V_1) \sqcup \mathbf{P}(V_2)$. Then Proposition 3.17 gives an equivalence

$$\mathcal{J}(A^1, A^2)\mathbf{P}(W) \simeq A^1_{\mathbf{P}(W)} \otimes_{\text{Perf}(\mathbf{P}(W))} A^2_{\mathbf{P}(W)}. \quad (5.1)$$

If $s$ denotes the corank of $W \subset V_1 \oplus V_2$, then by this equivalence and Lemma 5.1 for $|i| \geq s$ we may consider the Lefschetz component $\mathcal{J}_i$ of the categorical join $\mathcal{J}(A^1, A^2)$ as a subcategory of $A^1_{\mathbf{P}(W)} \otimes_{\text{Perf}(\mathbf{P}(W))} A^2_{\mathbf{P}(W)}$.

Remark 5.2. Let us directly describe $\mathcal{J}_i$, $|i| \geq s$, as a subcategory of $A^1_{\mathbf{P}(W)} \otimes_{\text{Perf}(\mathbf{P}(W))} A^2_{\mathbf{P}(W)}$ without reference to categorical joins. First note that $\mathcal{J}_i$ as a subcategory of $\mathcal{J}(A^1, A^2)$ is the image under the functor $p^*$ of the subcategory $\mathcal{J}_i \subset A^1 \otimes A^2$ described explicitly in Lemma 3.21. Hence for $|i| \geq s$, the subcategory

$$\mathcal{J}_i \subset A^1_{\mathbf{P}(W)} \otimes_{\text{Perf}(\mathbf{P}(W))} A^2_{\mathbf{P}(W)}$$

is the fully faithful image of $\mathcal{J}_i$ under the composition

$$A^1 \otimes A^2 \overset{p^*}{\longrightarrow} \mathcal{J}(A^1, A^2) \to \mathcal{J}(A^1, A^2)\mathbf{P}(W) \overset{\sim}{\longrightarrow} A^1_{\mathbf{P}(W)} \otimes_{\text{Perf}(\mathbf{P}(W))} A^2_{\mathbf{P}(W)}, \quad (5.2)$$

where the second functor is given by base change along the inclusion $\mathbf{P}(W) \to \mathbf{P}(V_1 \oplus V_2)$. It remains to describe this composition without reference to categorical joins. The inclusions $W \to V_1$ and $W \to V_2$ induce a morphism $\mathbf{P}(W) \to \mathbf{P}(V_1) \times \mathbf{P}(V_2)$. Base changing the $\mathbf{P}(V_1) \times \mathbf{P}(V_2)$-linear category $A^1 \otimes A^2$ along this morphism gives a functor

$$A^1 \otimes A^2 \to (A^1 \otimes A^2) \otimes_{\text{Perf}(\mathbf{P}(V_1) \times \mathbf{P}(V_2))} \text{Perf}(\mathbf{P}(W)) \simeq A^1_{\mathbf{P}(W)} \otimes_{\text{Perf}(\mathbf{P}(W))} A^2_{\mathbf{P}(W)}, \quad (5.3)$$

where the equivalence is given by Lemma 4.18. Alternatively, (5.3) is given by the tensor product of the restriction functors $A^1 \to A^1_{\mathbf{P}(W)}$ and $A^2 \to A^2_{\mathbf{P}(W)}$. Unwinding the definitions shows the functors (5.2) and (5.3) are isomorphic.

Combining Lemma 5.1 with the equivalence (5.1) we arrive at the following nonlinear analogue of Lemma 2.12 which describes linear sections of Lefschetz categories. Note that although by Remark 5.2 this result can be stated without categorical joins, the proof uses them.

Corollary 5.3. In the setup of Lemma 5.1, assume that the subbundle $W \subset V_1 \oplus V_2$ is such that the compositions $W \to V_1 \oplus V_2 \to V_1$ and $W \to V_1 \oplus V_2 \to V_2$ are both inclusions. Then there are semiorthogonal decompositions

$$A^1_{\mathbf{P}(W)} \otimes_{\text{Perf}(\mathbf{P}(W))} A^2_{\mathbf{P}(W)} = \langle \mathcal{K}_W(A^1, A^2), \mathcal{J}_s(H), \ldots, \mathcal{J}_{m-1}((m-s)H) \rangle, = \langle \mathcal{J}_{1-m}((s-m)H), \ldots, \mathcal{J}_{-s}(-H), \mathcal{K}_W'(A^1, A^2) \rangle.$$
Remark 5.4. If \( W = V_1 = V_2 \) and \( A^2 = \text{Perf}(P(L)) \) for a subbundle \( L \subset V_2 \), then Corollary 5.3 reduces to Lemma 2.12.

Now we arrive at the nonlinear HPD theorem. Like Corollary 5.3, the statement can be explained without appealing to categorical joins, but the proof uses them.

Theorem 5.5. Let \( A^1 \) and \( A^2 \) be right strong, moderate Lefschetz categories over projective bundles \( P(V_1) \) and \( P(V_2) \). For \( i, j \in \mathbb{Z} \) let \( J_i \) and \( J_j^\natural \) be the Lefschetz components of the categorical joins \( J(A^1, A^2) \) and \( J((A^1)^\natural, (A^2)^\natural) \) respectively. Assume \( V_1 \) and \( V_2 \) have the same rank, and set

\[
N = \text{rank}(V_1) = \text{rank}(V_2).
\]

Let \( W \) be a vector bundle on \( S \) equipped with isomorphisms

\[
\xi_k: W \xrightarrow{\sim} V_k, \quad k = 1, 2,
\]

and let

\[
(\xi_k^{-1})^\vee: W^\vee \xrightarrow{\sim} V_k^\vee, \quad k = 1, 2,
\]

be the inverse dual isomorphisms. Set

\[
m = \text{length}(A^1) + \text{length}(A^2) \quad \text{and} \quad n = \text{length}((A^1)^\natural) + \text{length}((A^2)^\natural).
\]

Denote by \( H \) and \( H' \) the relative hyperplane classes on \( P(W) \) and \( P(W^\vee) \). Then there are semiorthogonal decompositions

\[
A_P^1(W) \otimes_{\text{Perf}(P(W))} A_P^2(W) = \left\langle H_W(A^1, A^2), J_N(H), \ldots, J_{m-1}((m-N)H) \right\rangle,
\]

\[
(A^1)^\natural_P(W^\vee) \otimes_{\text{Perf}(P(W^\vee))}(A^2)^\natural_P(W^\vee) = \left\langle J_1^\natural((N-n)H'), \ldots, J_k^\natural(-H'), K_W((A^1)^\natural, (A^2)^\natural) \right\rangle,
\]

where we consider \( J_i \) and \( J_j^\natural \) as subcategories in the left sides as explained in Remark 5.2. Furthermore, we have an \( S \)-linear equivalence

\[
K_W((A^1, A^2) \simeq K_W((A^1)^\natural, (A^2)^\natural).
\]

Proof. Consider the inclusion of vector bundles

\[
(\xi_1, \xi_2): W \to V_1 \oplus V_2 = V.
\]

The orthogonal subbundle is given by the inclusion of vector bundles

\[
((\xi_1^{-1}, -(\xi_2^{-1})^{-1}): W^\vee \to V_1^\vee \oplus V_2^\vee = V^\vee.
\]

Now the semiorthogonal decompositions follow from Corollary 5.3 and the equivalence of categories follows from a combination of (5.1) with Theorem 4.1 and Theorem 2.24(2), noting that \(-(\xi_2^{-1})^{-1} \) and \((\xi_2^{-1})^{-1}\) induce the same morphism \( P(W^\vee) \to P(V_2^\vee) \).

Remark 5.6. Theorem 5.5 can be regarded as a nonlinear version of the main theorem of HPD, i.e. Theorem 2.24(2). Indeed, consider the “linear” case where \( A^2 = \text{Perf}(P(L)) \) for a subbundle \( 0 \subsetneq L \subsetneq V_2 \). Then by Theorem 2.27 the HPD category is \((A^2)^\natural \simeq \text{Perf}(P(L^\perp))\).

Now if \( W = V_1 = V_2 \) and \( \xi_1 = \xi_2 = \text{id} \), then Theorem 5.5 reduces to Theorem 2.24(2).

In the following remark, we explain how to deduce results for bounded derived categories of coherent sheaves in place of perfect complexes.
Remark 5.7. Given a proper $T$-linear category $\mathcal{A}$, where $T$ is noetherian over a field of characteristic 0, in [32 Definition 4.27] a bounded coherent category $\mathcal{A}^{\text{coh}}$ is defined. In case $X \to T$ is a proper morphism of finite presentation, where $X$ is possibly a derived scheme, then $\text{Perf}(X)^{\text{coh}}$ recovers $D^b_{\text{coh}}(X)$. By [32 Proposition 4.28], a semiorthogonal decomposition of $\mathcal{A}$ (with all components except possibly the first or last admissible) induces a semiorthogonal decomposition of $\mathcal{A}^{\text{coh}}$. This gives rise to a bounded coherent version of Theorem 5.5. Namely, assume the categories $\mathcal{J}_i$ and $\mathcal{J}_j$ appearing in the semiorthogonal decompositions of

$$\mathcal{C} = \mathcal{A}^1_{\mathbb{P}(W)} \otimes_{\text{Perf}(\mathbb{P}(W))} \mathcal{A}^2_{\mathbb{P}(W)} \quad \text{and} \quad \mathcal{D} = (\mathcal{A}^1)^{\sharp}_{\mathbb{P}(W)} \otimes_{\text{Perf}(\mathbb{P}(W))} (\mathcal{A}^2)^{\sharp}_{\mathbb{P}(W)}$$

are admissible. For instance, this is automatic if $\mathcal{A}^1$ and $\mathcal{A}^2$ are smooth and proper over $S$, by Lemma 3.14 combined with [32 Lemmas 4.15 and 4.13]. Then there are semiorthogonal decompositions

$$\mathcal{C}^{\text{coh}} = \left( (\mathcal{K}_W(\mathcal{A}^1, \mathcal{A}^2))^{\text{coh}}, (\mathcal{J}_N)^{\text{coh}}(H), \ldots, (\mathcal{J}_{m-1})^{\text{coh}}((m-N)H) \right), \quad (5.4)$$

$$\mathcal{D}^{\text{coh}} = \left( (\mathcal{J}_1^{\sharp})^{\text{coh}}((-N-n)H'), \ldots, (\mathcal{J}_l^{\sharp})^{\text{coh}}((-H'), (\mathcal{K}_W'((\mathcal{A}^1)^{\sharp}, (\mathcal{A}^2)^{\sharp}))^{\text{coh}} \right), \quad (5.5)$$

and an $S$-linear equivalence

$$(\mathcal{K}_W(\mathcal{A}^1, \mathcal{A}^2))^{\text{coh}} \simeq (\mathcal{K}_{W'}((\mathcal{A}^1)^{\sharp}, (\mathcal{A}^2)^{\sharp}))^{\text{coh}}.$$

Note that if $\mathcal{A}^k = \text{Perf}(X_k)$ for a $\mathbb{P}(V_k)$-scheme $X_k$, then there is an equivalence

$$\mathcal{C} \simeq \text{Perf}(X_1 \times_{\mathbb{P}(W)} X_2)$$

where $X_k$ is regarded as a $\mathbb{P}(W)$-scheme via the isomorphism $\xi_k: \mathbb{P}(W) \xrightarrow{\sim} \mathbb{P}(V_k)$. Hence if our base $S$ is noetherian over a field of characteristic 0 and $X_k \to \mathbb{P}(V_k)$ is a proper morphism, then there is an equivalence

$$\mathcal{C}^{\text{coh}} \simeq D^b_{\text{coh}}(X_1 \times_{\mathbb{P}(W)} X_2).$$

5.2. An iterated nonlinear HPD theorem. Theorem 5.5 describes the tensor product of two Lefschetz categories over a projective bundle in terms of their HPD categories. This generalizes to a description of the tensor product of an arbitrary number of Lefschetz categories over a projective bundle. The key point is to consider iterated categorical joins of a collection of Lefschetz categories.

Definition 5.8. For $k = 1, 2, \ldots, \ell$, let $\mathcal{A}^k$ be a Lefschetz category over $\mathbb{P}(V_k)$. The categorical join of $\mathcal{A}^1, \ldots, \mathcal{A}^\ell$ is the Lefschetz category over $\mathbb{P}(V_1 \oplus \cdots \oplus V_\ell)$ defined inductively by the formula

$$\mathcal{J}(\mathcal{A}^1, \ldots, \mathcal{A}^\ell) = \mathcal{J}(\mathcal{J}(\mathcal{A}^1, \ldots, \mathcal{A}^{\ell-1}), \mathcal{A}^{\ell}).$$

As an immediate consequence of Theorem 4.11 we obtain the following.

Theorem 5.9. For $k = 1, 2, \ldots, \ell$, let $\mathcal{A}^k$ be right strong, moderate Lefschetz category over $\mathbb{P}(V_k)$. Then there is an equivalence

$$\mathcal{J}(\mathcal{A}^1, \ldots, \mathcal{A}^\ell)^{\sharp} \simeq \mathcal{J}((\mathcal{A}^1)^{\sharp}, \ldots, (\mathcal{A}^\ell)^{\sharp})$$

of Lefschetz categories over $\mathbb{P}(V_1^\vee \oplus \cdots \oplus V_\ell^\vee)$.
For $k = 1, 2, \ldots, \ell$, let $A^k$ be a Lefschetz category over $P(V_k)$. Let $J_i$, $i \in \mathbb{Z}$, be the Lefschetz components of the categorical join $J(A^1, \ldots, A^\ell)$. If $W \subset V_1 \oplus \cdots \oplus V_\ell$ is a subbundle of corank $s$, then by Lemma 2.12 the functor $J_i : J(A^1, \ldots, A^\ell) \to J(A^1, \ldots, A^\ell)_{P(W)}$ is fully faithful for $|i| \geq s$, so we can consider $J_i$ as a subcategory of $J(A^1, \ldots, A^\ell)_{P(W)}$. If moreover the composition $W \to V_1 \oplus \cdots \oplus V_\ell \to V_k$ is an inclusion of vector bundles for each $k$, then Proposition 3.17 gives an equivalence

$$J(A^1, \ldots, A^\ell)_{P(W)} \simeq A^1_{P(W)} \otimes_{\text{Perf}(P(W))} \cdots \otimes_{\text{Perf}(P(W))} A^\ell_{P(W)},$$

so in this case we can consider $J_i$ as a subcategory of $A^1_{P(W)} \otimes_{\text{Perf}(P(W))} \cdots \otimes_{\text{Perf}(P(W))} A^\ell_{P(W)}$ as soon as $|i| \geq s$. Finally, this subcategory can be described as in Remark 5.2 without appealing to categorical joins, as the image of an explicit subcategory of $A^1 \otimes \cdots \otimes A^\ell$ (defined along the lines of Lemma 3.24) under the functor given by base change along the induced morphism $P(W) \to \prod_{k=1}^\ell P(V_k)$. Combining Lemma 2.12 with the equivalence (5.6), we obtain the following iterated version of Corollary 5.3.

**Proposition 5.10.** For $k = 1, 2, \ldots, \ell$, let $A^k$ be a Lefschetz category over $P(V_k)$. For $i \in \mathbb{Z}$ let $J_i$ be the Lefschetz components of the categorical join $J(A^1, \ldots, A^\ell)$. Let $W$ be a vector bundle on $S$ equipped with inclusions of vector bundles $W \to V_k$ for all $k$. Set

$$m = \sum_k \text{length}(A^k) \quad \text{and} \quad s = \sum_k \text{rank}(V_k) - \text{rank}(W).$$

Denote by $H$ the relative hyperplane class on $P(W)$. Then there are semiorthogonal decompositions

$$A^1_{P(W)} \otimes_{\text{Perf}(P(W))} \cdots \otimes_{\text{Perf}(P(W))} A^\ell_{P(W)} = \left\langle J_1(W(A^1, \ldots, A^\ell), J_{s}(H), \ldots, J_{m-1}(m-s)H) \right\rangle,$n

$$= \left\langle J_{1-m}(s-m)H), \ldots, J_{-s}(-H), K_W(A^1, \ldots, A^\ell) \right\rangle.$$

Now we state the iterated nonlinear HPD theorem, which describes $K_W(A^1, \ldots, A^\ell)$ in terms of the HPD categories $(A^k)^{\mathbb{Z}}$ and reduces to Theorem 5.5 when $\ell = 2$. When $\ell > 2$, the description is in terms of the categorical join of the categories $(A^k)^{\mathbb{Z}}$, and cannot be expressed in terms of a tensor product of the $(A^k)^{\mathbb{Z}}$ over a projective bundle.

**Theorem 5.11.** For $k = 1, 2, \ldots, \ell$, let $A^k$ be a right strong, moderate Lefschetz category over $P(V_k)$. For $i, j \in \mathbb{Z}$ let $J_i$ and $J_j$ be the Lefschetz components of the categorical joins $J(A^1, \ldots, A^\ell)$ and $J((A^1)^{\mathbb{Z}}, \ldots, (A^\ell)^{\mathbb{Z}})$. Let $W$ be a vector bundle on $S$ equipped with inclusions of vector bundles

$$\xi_k : W \to V_k, \ k = 1, \ldots, \ell,$$

and let

$$W^\perp = \{ (\theta_1, \ldots, \theta_\ell) \in V_1^\vee \oplus \cdots \oplus V_\ell^\vee \mid \sum_k \theta_k \circ \xi_k = 0 \in W^\vee \} \subset V_1^\vee \oplus \cdots \oplus V_\ell^\vee$$

be the orthogonal to the induced inclusion $W \to V_1 \oplus \cdots \oplus V_\ell$. Let $H$ and $H'$ denote the relative hyperplane classes on $P(W)$ and $P(W^\perp)$, and set

$$r = \text{rank}(W), \ s = \text{rank}(W^\perp), \ m = \sum_k \text{length}(A^k), \ n = \sum_k \text{length}((A^k)^{\mathbb{Z}}).$$
Then there are semiorthogonal decompositions

\[ A_{\mathbf{P}(W)}^1 \otimes \text{Perf}(\mathbf{P}(W)) \cdots \otimes \text{Perf}(\mathbf{P}(W)) A_{\mathbf{P}(W)}^\ell = \left\langle \mathcal{K}_W(A^1, \ldots, A^\ell), \mathcal{J}_s(H), \ldots, \mathcal{J}_{m-1}((m-s)H) \right\rangle, \]

\[ \mathcal{J}((A^1)^\natural, \ldots, (A^\ell)^\natural)_{\mathbf{P}(W)} = \left\langle \mathcal{J}^\natural_{1-n}((r-n)H'), \ldots, \mathcal{J}^\natural_{-r}(-H'), \mathcal{K}_{W'}^\perp(\mathcal{J}((A^1)^\natural, \ldots, (A^\ell)^\natural)) \right\rangle, \]

and an \( S \)-linear equivalence

\[ \mathcal{K}_W(A^1, \ldots, A^\ell) \cong \mathcal{K}_{W'}^\perp(\mathcal{J}((A^1)^\natural, \ldots, (A^\ell)^\natural)). \]

Proof. The argument of Theorem 5.5 works, using Proposition 5.10 the equivalence (5.6), and Theorem 5.9 in place of the corresponding results for \( \ell = 2 \).

Remark 5.12. In Theorem 5.11 we do not require \( \xi_k : W \rightarrow V_k \) to be an isomorphism, as in Theorem 5.5. The reason is that this assumption does not lead to a simplification in the statement of the conclusion when \( \ell > 2 \).

6. Applications

In this section, we discuss some applications of HPD for categorical joins (Theorem 4.1) and the nonlinear HPD theorem (Theorem 5.5). For simplicity, we assume the base scheme \( S \) is the spectrum of an algebraically closed field \( k \) of characteristic 0.

6.1. Intersections of two Grassmannians. Let \( V_5 \) be a 5-dimensional vector space and let \( \text{Gr}(2, V_5) \) be the Grassmannian of 2-dimensional vector subspaces of \( V_5 \). The Plücker embedding

\[ \text{Gr}(2, V_5) \rightarrow \mathbf{P}(\wedge^2 V_5) \cong \mathbf{P}^9 \]

endows \( \text{Perf} \left( \text{Gr}(2, V_5) \right) \) with a \( \mathbf{P}(\wedge^2 V_5) \)-linear structure. We will need the following description of HPD for the Grassmannian \( \text{Gr}(2, V_5) \), which is a categorical version of the fact that \( \text{Gr}(2, V_5) \) is projectively self-dual.

Theorem 6.1 ([13 Section 6.1 and Theorem 1.2]). Let \( \mathcal{U} \) and \( \mathcal{U}' \) be the tautological rank 2 subbundles on \( \text{Gr}(2, V_5) \) and \( \text{Gr}(2, V_5') \). Then \( \text{Perf}(\text{Gr}(2, V_5)) \) and \( \text{Perf}(\text{Gr}(2, V_5')) \) have the structure of strong, moderate Lefschetz categories over \( \mathbf{P}(\wedge^2 V_5) \) and \( \mathbf{P}(\wedge^2 V_5') \) of length 5, with Lefschetz components given by

\[ A_i = \langle \mathcal{O}, \mathcal{U}^\vee \rangle \quad \text{and} \quad A_i' = \langle \mathcal{U}', \mathcal{O} \rangle \]

for \( |i| \leq 4 \). Moreover, there is an equivalence

\[ \text{Perf}(\text{Gr}(2, V_5))^\natural \cong \text{Perf}(\text{Gr}(2, V_5')) \]

of Lefschetz categories over \( \mathbf{P}(\wedge^2 V_5') \).

Remark 6.2. Theorem 6.1 has a conjectural generalization to \( \text{Gr}(2, V) \) for an arbitrary vector space \( V \) (see [19 Conjecture 5.4]), which is proved if \( \dim(V) \leq 7 \) in [12] and if \( \dim(V) \) is any odd integer in [33].

Using the nonlinear HPD theorem we obtain the following.

Theorem 6.3. Let \( V_5 \) be a 5-dimensional vector space. Let \( W \) be a vector space equipped with two isomorphisms

\[ \xi_1 : W \sim \wedge^2 V_5, \quad \xi_2 : W \sim \wedge^2 V_5, \]

and let

\[ \xi_1^\vee : \wedge^2 V_5^\vee \sim W^\vee, \quad \xi_2^\vee : \wedge^2 V_5^\vee \sim W^\vee, \]
be the dual isomorphisms. Define
\[ X = \xi_1^{-1}(\text{Gr}(2, V_5)) \times_{\mathbb{P}(W)} \xi_2^{-1}(\text{Gr}(2, V_5)), \]
\[ Y = \xi_1'(\text{Gr}(2, V_5')) \times_{\mathbb{P}(W')} \xi_2'(\text{Gr}(2, V_5')). \]

Then there are equivalences of categories
\[ \text{Perf}(X) \simeq \text{Perf}(Y) \quad \text{and} \quad D^b_{\text{coh}}(X) \simeq D^b_{\text{coh}}(Y). \]

Proof. The first equivalence follows by combining Theorem 5.5 with Theorem 6.1, and the second follows by Remark 5.7. \(\square\)

Remark 6.4. When smooth of the expected dimension 3, the varieties \(X\) and \(Y\) in Theorem 6.3 are Calabi–Yau threefolds. For a generic choice of the isomorphisms \(\xi_1\) and \(\xi_2\), this pair of varieties was recently shown to give the first example of deformation equivalent, derived equivalent, but non-birational Calabi–Yau threefolds, and as a consequence the first counterexample to the birational Torelli problem for Calabi–Yau threefolds [5, 31].

6.2. Intersections of two orthogonal Grassmannians. For a vector space \(V\) of even dimension \(2n\) with a nondegenerate quadratic form \(q \in \text{Sym}^2 V\), the Grassmannian of \(n\)-dimensional isotropic subspaces of \(V\) has two connected components, which are abstractly isomorphic. We denote by \(\text{OGr}_+(n, V)\) one of these components and by \(\text{OGr}_-(n, V)\) the other. The Plücker embedding \(\text{OGr}_+(n, V) \to \mathbb{P}(\wedge^n V)\) is given by the square of the generator of \(\text{Pic}(\text{OGr}_+(n, V))\); the generator itself gives an embedding
\[ \text{OGr}_+(n, V) \to \mathbb{P}(S_{2n-1}), \]
where \(S_{2n-1}\) is a \(2^{n-1}\)-dimensional half-spinor representation of \(\text{Spin}(V)\).

In the case of a 10-dimensional vector space \(V_{10}\), the orthogonal Grassmannian in its spinor embedding \(\text{OGr}_+(5, V_{10}) \subset \mathbb{P}(S_{16})\) shares a very special property with the Grassmannian \(\text{Gr}(2, V_5) \subset \mathbb{P}(\wedge^2 V_5)\): both are projectively self-dual, and even homologically projectively self-dual. To be more precise, the classical projective dual of \(\text{OGr}_+(5, V_{10}) \subset \mathbb{P}(S_{16})\) is given by the spinor embedding \(\text{OGr}_-(5, V_{10}) \subset \mathbb{P}(S'_{16})\). This lifts to the homological level as follows.

Theorem 6.5 ([13, Section 6.2 and Theorem 1.2]). Let \(\mathcal{U}\) and \(\mathcal{U}'\) be the tautological rank 5 sub-bundles on \(\text{OGr}_+(5, V_{10})\) and \(\text{OGr}_-(5, V_{10})\). Then \(\text{Perf}(\text{OGr}_+(5, V_{10}))\) and \(\text{Perf}(\text{OGr}_-(5, V_{10}))\) have the structure of strong, moderate Lefschetz categories over the spinor spaces \(\mathbb{P}(S_{16})\) and \(\mathbb{P}(S'_{16})\) of length 8, with Lefschetz components given by
\[ A_i = (\mathcal{O}, \mathcal{U}) \quad \text{and} \quad A'_i = (\mathcal{U}', \mathcal{O}) \]
for \(|i| \leq 7\). Moreover, there is an equivalence
\[ \text{Perf}(\text{OGr}_+(5, V_{10}))^3 \simeq \text{Perf}(\text{OGr}_-(5, V_{10})) \]
of Lefschetz categories over \(\mathbb{P}(S_{16})\).

Using Theorem 6.5 in place of Theorem 6.1 we obtain the following spin analogue of Theorem 6.3.

Theorem 6.6. Let \(V_{10}\) be a 10-dimensional vector space, and let \(S_{16}\) be a half-spinor representation of \(\text{Spin}(V_{10})\). Let \(W\) be a vector space equipped with two isomorphisms
\[ \xi_1: W \xrightarrow{\sim} S_{16}, \quad \xi_2: W \xrightarrow{\sim} S_{16}, \]
and let 
\[ \xi_1^*: S_{16}^v \simto W^v, \quad \xi_2^*: S_{16}^v \simto W^v, \]
be the dual isomorphisms. Define
\[ X = \xi_1^{-1}(OGr_+(5, V_{10})) \times_{P(W)} \xi_2^{-1}(OGr_+(5, V_{10})), \]
\[ Y = \xi_1^{-1}(OGr_-(5, V_{10})) \times_{P(W)} \xi_2^{-1}(OGr_-(5, V_{10})). \]
Then there are equivalences of categories
\[ \text{Perf}(X) \simeq \text{Perf}(Y) \quad \text{and} \quad D^b_{\text{coh}}(X) \simeq D^b_{\text{coh}}(Y). \]

**Remark 6.7.** When smooth of the expected dimension 5, the varieties \( X \) and \( Y \) in Theorem 6.6 are Calabi–Yau fivefolds, which were recently studied in [29]. There, following [5, 31] it is shown that for generic \( \xi_1 \) and \( \xi_2 \), these varieties are non-birational.

### 6.3. Enriques surfaces

The goal of this subsection is to prove that for a general Enriques surface \( \Sigma \), there is a stacky projective plane \( \mathcal{P} \) (with stack structure along the union of two cubic curves), such that the subcategory \((\mathcal{O}_\Sigma)^{-} \subset \text{Perf}(\Sigma)\) is equivalent to the orthogonal of an exceptional object in the twisted derived category of \( \mathcal{P} \). The precise statement is Theorem 6.19 below. As we will see, the result falls out naturally by considering the categorical join of two Veronese surfaces.

Let \( W \) be a 3-dimensional vector space, and let \( V = \text{Sym}^2 W \). The double Veronese embedding \( P(W) \to P(V) \), given by the linear system \([\mathcal{O}_{P(W)}(2)]\), endows \( \text{Perf}(P(W)) \) with a \( P(V) \)-linear structure. Below we will consider \( \text{Perf}(P(W)) \) as a Lefschetz category over \( P(V) \) of length 2, with right Lefschetz components given by
\[
\mathcal{L}_i = \left\{ \langle \mathcal{O}_{P(W)}, \mathcal{O}_{P(W)}(1) \rangle \quad \text{for} \ i = 0, \right. \\
\left. \langle \mathcal{O}_{P(W)} \rangle \quad \text{for} \ i = 1. \right\}
\]
We call this the **double Veronese Lefschetz structure** on \( P(W) \), to distinguish it from the standard Lefschetz structure on a projective space from Example 2.11.

We need the description of the HFPD of \( \text{Perf}(P(W)) \) from [15]. The universal family of conics in \( P(W) \) is a conic fibration over \( P(V^\vee) \). Associated to this fibration are the sheaves \( \text{Cliff}_0 \) and \( \text{Cliff}_1 \) of even and odd parts of the corresponding Clifford algebra on \( P(V^\vee) \), which as sheaves of \( \mathcal{O}_{P(V^\vee)} \)-modules are given by
\[
\text{Cliff}_0 = \mathcal{O}_{P(V^\vee)} \oplus ( \wedge^2 W \otimes \mathcal{O}_{P(V^\vee)}(-1) ), \quad (6.1)
\]
\[
\text{Cliff}_1 = ( W \otimes \mathcal{O}_{P(V^\vee)} ) \oplus ( \wedge^3 W \otimes \mathcal{O}_{P(V^\vee)}(-1) ). \quad (6.2)
\]
Note that \( \text{Cliff}_0 \) is a sheaf of \( \mathcal{O}_{P(V^\vee)} \)-algebras via Clifford multiplication, and \( \text{Cliff}_1 \) is a module over \( \text{Cliff}_0 \). Before continuing, we need a brief digression on the noncommutative scheme associated to a sheaf of algebras.

**Notation 6.8.** Suppose \( X \) is a scheme (or stack) equipped with a sheaf \( \mathcal{R} \) of \( \mathcal{O}_X \)-algebras, such that \( \mathcal{R} \) is finite locally free over \( \mathcal{O}_X \). We denote by \( \text{D}_{qC}(X, \mathcal{R}) \) the unbounded derived category of quasi-coherent sheaves of \( \mathcal{R} \)-modules, and by \( \text{Perf}(X, \mathcal{R}) \subset \text{D}_{qC}(X, \mathcal{R}) \) the full subcategory of objects which are perfect as complexes of \( \mathcal{R} \)-modules.

**Remark 6.9.** In the above situation, \( \text{Perf}(X, \mathcal{R}) \) naturally has the structure of an \( X \)-linear category. Moreover, there is a geometric description of tensor products of categories of this form. Namely, let \((X_1, \mathcal{R}_1)\) and \((X_2, \mathcal{R}_2)\) be two pairs as in Notation 6.8 and assume \( X_1 \) and
and \(X_2\) are defined over a common scheme \(T\), so that both \(\text{Perf}(X_1, \mathcal{R}_1)\) and \(\text{Perf}(X_2, \mathcal{R}_2)\)
can be considered as \(T\)-linear categories. Assume \(X_1\) and \(X_2\) are perfect stacks in the sense of [3]. Then there is an equivalence
\[
\text{Perf}(X_1, \mathcal{R}_1) \otimes_{\text{Perf}(T)} \text{Perf}(X_2, \mathcal{R}_2) \simeq \text{Perf}(X_1 \times_T X_2, \mathcal{R}_1 \boxtimes \mathcal{R}_2).
\]
This is a mild generalization of the case where \(\mathcal{R}_1 = \mathcal{O}_{X_1}\) and \(\mathcal{R}_2 = \mathcal{O}_{X_2}\) proved in [3], and follows by the same argument. See also [13] for the case where \(\mathcal{R}_1\) and \(\mathcal{R}_2\) are Azumaya algebras.

We consider the \(\mathbf{P}(V)^\vee\)-linear category \(\text{Perf}(\mathbf{P}(V)^\vee, \text{Cliff}_0)\). From \(\text{Cliff}_0\) and \(\text{Cliff}_1\) we obtain exceptional objects \(\text{Cliff}_i \in \text{Perf}(\mathbf{P}(V)^\vee, \text{Cliff}_0)\) for all \(i \in \mathbb{Z}\) by the prescription
\[
\text{Cliff}_{i+2} = \text{Cliff}_i \otimes \mathcal{O}_{\mathbf{P}(V)^\vee}(1).
\]

**Theorem 6.10 ([15] Theorem 5.4).** The category \(\text{Perf}(\mathbf{P}(V)^\vee, \text{Cliff}_0)\) has a Lefschetz structure over \(\mathbf{P}(V)^\vee\) of length 5, with left Lefschetz components given by
\[
B_i = \begin{cases} 
\langle \text{Cliff}_0 \rangle & \text{for } i = -4, \\
\langle \text{Cliff}_1, \text{Cliff}_0 \rangle & \text{for } -3 \leq i \leq 0.
\end{cases}
\]
Moreover, there is an equivalence
\[
\text{Perf}(\mathbf{P}(W))^2 \simeq \text{Perf}(\mathbf{P}(V)^\vee, \text{Cliff}_0)
\]
of Lefschetz categories over \(\mathbf{P}(V)^\vee\), where \(\mathbf{P}(W)\) is considered with its double Veronese Lefschetz structure.

**Remark 6.11.** In [15], an HPD theorem is proved more generally for the double Veronese embedding of \(\mathbf{P}(W)\) where \(W\) is of arbitrary dimension.

Now we consider the categorical join of two copies of the above data. Namely, for \(k = 1, 2\), let \(W_k\) be a 3-dimensional vector space, let \(V_k = \text{Sym}^2 W_k\), and let \(\text{Cliff}_k^i, i \in \mathbb{Z}\), denote the Clifford sheaves on \(\mathbf{P}(V_k^\vee)\) from above. We set
\[
A_k = \text{Perf}(\mathbf{P}(W_k))
\]
with the double Veronese Lefschetz category structure over \(\mathbf{P}(V_k)\), and
\[
B_k = \text{Perf}(\mathbf{P}(V_k^\vee, \text{Cliff}_0^k))
\]
with the Lefschetz category structure over \(\mathbf{P}(V_k^\vee)\) from above.

By Theorem 3.21, the categorical join \(J(A^1, A^2)\) is a Lefschetz category over \(\mathbf{P}(V_1 \oplus V_2)\) of length 4, and \(J(B^1, B^2)\) is a Lefschetz category over \(\mathbf{P}(V_1^\vee \oplus V_2^\vee)\) of length 10. Moreover, it follows from Lemma 3.24 that the right Lefschetz components of \(J(A^1, A^2)\) are given by
\[
J(A^1, A^2)_i = \begin{cases} 
\langle 0, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1) \rangle & \text{if } i = 0, 1, \\
\langle 0, \mathcal{O}(1, 0), \mathcal{O}(0, 1) \rangle & \text{if } i = 2, \\
\langle 0 \rangle & \text{if } i = 3,
\end{cases}
\]
and the left Lefschetz components of \(J(B^1, B^2)\) are given by
\[
J(B^1, B^2)_i = \begin{cases} 
\langle \text{Cliff}_0, \text{Cliff}_0 \rangle & \text{if } i = -9, \\
\langle \text{Cliff}_{-1, 0}, \text{Cliff}_{0, -1}, \text{Cliff}_{0, 0} \rangle & \text{if } i = -8, \\
\langle \text{Cliff}_{-1, -1}, \text{Cliff}_{-1, 0}, \text{Cliff}_{0, -1}, \text{Cliff}_{0, 0} \rangle & \text{if } -7 \leq i \leq 0,
\end{cases}
\]
where we write $\text{Cliff}_{1,2} = \text{Cliff}_{1} \boxtimes \text{Cliff}_{2}$. Here, the objects generating the Lefschetz components abusively denote the pullbacks of the corresponding objects of $\text{Perf}(P(W_1) \times P(W_2))$ and $\text{Perf}(P(V_1^\vee) \times P(V_2^\vee), \text{Cliff}_{1} \boxtimes \text{Cliff}_{2})$. By Theorem 4.1 combined with Theorem 6.10 there is an equivalence

$$\mathcal{J}(A^1, A^2) \simeq \mathcal{J}(B^1, B^2)$$

of Lefschetz categories over $P(V_1^\vee \oplus V_2^\vee)$.

Now let $L \subset V_1 \oplus V_2$ be a vector subspace of codimension 3. Then combining the above with Theorem 2.24(2), we obtain the following.

**Corollary 6.12.** There are semiorthogonal decompositions

$$\mathcal{J}(A^1, A^2)_{P(L)} = \langle K_L, \emptyset \rangle,$$
$$\mathcal{J}(B^1, B^2)_{P(L^\perp)} = \langle \text{Cliff}_{0} \boxtimes \text{Cliff}_{2}, K'_L \rangle,$$

and an equivalence $K_L \simeq K'_L$.

Our goal below is to describe geometrically the linear section categories appearing in Corollary 6.12 for generic $L$.

First we consider the category $\mathcal{J}(A^1, A^2)_{P(L)} = \mathcal{J}(P(W_1), P(W_2))_{P(L)}$. The resolved join $\mathcal{J}(P(W_1), P(W_2))$ of $P(W_1)$ and $P(W_2)$ with respect to their double Veronese embeddings is a $P(V_1 \oplus V_2)$-scheme. We define $\Sigma_L$ as its base change along $P(L) \subset P(V_1 \oplus V_2)$, i.e.

$$\Sigma_L = \mathcal{J}(P(W_1), P(W_2))_{P(L)}.$$

**Lemma 6.13.** Assume $P(L)$ does not intersect $P(W_1) \sqcup P(W_2)$ in $P(V_1 \oplus V_2)$. Then there is an equivalence

$$\mathcal{J}(A^1, A^2)_{P(L)} \simeq \text{Perf}(\Sigma_L).$$

**Proof.** Follows from Proposition 3.17. □

The assumption of Lemma 6.13 holds generically. In this case, $\Sigma_L$ is a familiar variety:

**Lemma 6.14 ([21] Lemma 3).** For generic $L \subset V_1 \oplus V_2$ of codimension 3, the scheme $\Sigma_L$ is an Enriques surface. Moreover, a general Enriques surface is obtained in this way.

Now we turn to $\mathcal{J}(B^1, B^2)_{P(L^\perp)}$. Note that $\dim(L^\perp) = 3$, so $P(L^\perp) \cong P^2$.

**Lemma 6.15.** For $k = 1, 2$, assume the composition $L^\perp \to V_1^\vee \oplus V_2^\vee \to V_k^\vee$ is an inclusion. Let $\text{Cliff}_{0} \mid P(L^\perp)$ denote the pullback of $\text{Cliff}_{0}$ along the induced embedding $P(L^\perp) \to P(V_k^\vee)$. Then there are equivalences

$$\mathcal{J}(B^1, B^2)_{P(L^\perp)} \simeq \text{Perf} \left( P(L^\perp), \text{Cliff}_{0} \mid P(L^\perp) \right) \otimes_{\text{Perf}(P(L^\perp))} \text{Perf} \left( P(L^\perp), \text{Cliff}_{0} \mid P(L^\perp) \right) \quad (6.3)$$
$$\simeq \text{Perf} \left( P(L^\perp), (\text{Cliff}_{0} \boxtimes \text{Cliff}_{2}) \mid P(L^\perp) \right). \quad (6.4)$$

**Proof.** The first equivalence follows from Proposition 3.17 combined with Remark 6.9 and the second follows by applying Remark 6.9 again. □

The assumption of Lemma 6.15 is equivalent to $P(L^\perp)$ not meeting $P(V_1^\vee)$ or $P(V_2^\vee)$ in $P(V_1^\vee \oplus V_2^\vee)$, and holds generically for dimension reasons. In this case, we will give a more geometric description of $\mathcal{J}(B^1, B^2)_{P(L^\perp)}$ by rewriting the factors in the tensor product in (6.3).
Being the base of the universal family of conics $\mathcal{X}_k \to \mathbb{P}(V_k^\vee)$ in $\mathbb{P}(W_k)$, the space $\mathbb{P}(V_k^\vee)$ has a stratification

$$\mathbb{P}(W_k^\vee) \subset D_k \subset \mathbb{P}(V_k^\vee),$$

where $D_k$ is the discriminant locus parameterizing degenerate conics and $\mathbb{P}(W_k^\vee) \subset \mathbb{P}(V_k^\vee)$ is the double Veronese embedding, which parameterizes non-reduced conics (double lines). Note that $D_k \subset \mathbb{P}(V_k^\vee)$ is a cubic hypersurface, with singular locus $\mathbb{P}(W_k^\vee)$.

Under the assumption of Lemma 6.15, for $k = 1, 2$, we have an embedding

$$\xi_k: \mathbb{P}(L^\perp) \to \mathbb{P}(V_k^\vee).$$

The stratification of $\mathbb{P}(L^\perp)$ associated to the pullback family of conics $(\mathcal{X}_k)_{\mathbb{P}(L^\perp)} \to \mathbb{P}(L^\perp)$ is the preimage of the stratification of $\mathbb{P}(V_k^\vee)$, i.e.

$$\xi_k^{-1}(\mathbb{P}(W_k^\vee)) \subset \xi_k^{-1}(D_k) \subset \mathbb{P}(L^\perp).$$

We write $C_k = \xi_k^{-1}(D_k)$ for the discriminant locus. Note that for $L \subset V_1 \oplus V_2$ generic, the locus $\xi_k^{-1}(\mathbb{P}(W_k^\vee))$ is empty, $C_k$ is a smooth cubic curve in the projective plane $\mathbb{P}(L^\perp)$, and the curves $C_1$ and $C_2$ intersect transversally. We define

$$\mathcal{P}_k = \mathbb{P}(L^\perp)(\sqrt{C_k})$$

as the square root stack (see [6, §2.2] or [1, Appendix B]) of the divisor $C_k \subset \mathbb{P}(L^\perp)$. Note that $\mathcal{P}_k$ is a Deligne–Mumford stack with coarse moduli space

$$\rho_k: \mathcal{P}_k \to \mathbb{P}(L^\perp),$$

where $\rho_k$ is an isomorphism over $\mathbb{P}(L^\perp) \setminus C_k$ and a $\mathbb{Z}/2$-gerbe over $C_k$.

**Lemma 6.16.** For $k = 1, 2$, assume the composition $L^\perp \to V_1^\vee \oplus V_2^\vee \to V_k^\vee$ is an inclusion and $C_k \neq \mathbb{P}(L^\perp)$. Then there is a finite locally free sheaf of algebras $\mathcal{R}_k$ on $\mathcal{P}_k$ such that $\rho_k^* \mathcal{R}_k \cong \text{Cliff}^5 \big|_{\mathbb{P}(L^\perp)}$ and the induced functor

$$\rho_k^*: \text{Perf}(\mathcal{P}_k, \mathcal{R}_k) \xrightarrow{\sim} \text{Perf} \left( \mathbb{P}(L^\perp), \text{Cliff}^5 \big|_{\mathbb{P}(L^\perp)} \right)$$

is an equivalence. Moreover, $\mathcal{R}_k$ is Azumaya over the complement of $\xi_k^{-1}(\mathbb{P}(W_k^\vee))$ in $\mathbb{P}(L^\perp)$.

**Proof.** Follows from [15, §3.6], cf. [2, Proposition 1.20].

In the situation of the lemma, we define

$$\mathcal{P} = \mathcal{P}_1 \times_{\mathbb{P}(L^\perp)} \mathcal{P}_2.$$ 

This space carries a finite locally free sheaf of algebras given by

$$\mathcal{R} = \mathcal{R}_1 \boxtimes \mathcal{R}_2.$$ 

**Lemma 6.17.** Under the assumption of Lemma 6.16, there is an equivalence

$$\mathcal{R}(\mathbb{B}^1, \mathbb{B}^2)_{\mathbb{P}(L^\perp)} \cong \text{Perf}(\mathcal{P}, \mathcal{R}),$$

where $\mathcal{R}$ is Azumaya over the complement of $\xi_1^{-1}(\mathbb{P}(W_1^\vee)) \cup \xi_2^{-1}(\mathbb{P}(W_2^\vee))$ in $\mathbb{P}(L^\perp)$.

**Proof.** Follows from Lemma 6.15, Lemma 6.16 and Remark 6.9. 

\[\square\]
Remark 6.18. The space $\mathcal{P}$ is a stacky projective plane. More precisely, consider the stratification

$$C_1 \cap C_2 \subset C_1 \cup C_2 \subset \mathbb{P}(L^\perp).$$

Then the canonical morphism $\rho: \mathcal{P} \to \mathbb{P}(L^\perp)$ can be described over the open strata as follows:

- $\rho$ is an isomorphism over $\mathbb{P}(L^\perp) \setminus (C_1 \cup C_2)$.
- $\rho$ is a $\mathbb{Z}/2$-gerbe over $(C_1 \cup C_2) \setminus (C_1 \cap C_2)$.
- $\rho$ is a $\mathbb{Z}/2 \times \mathbb{Z}/2$-gerbe over $C_1 \cap C_2$.

When $\xi_1^1(\mathbb{P}(W_1^\vee))$ and $\xi_2^1(\mathbb{P}(W_2^\vee))$ are empty — which holds for generic $L \subset V_1 \oplus V_2$ — Lemma 6.17 thus gives a satisfactory geometric interpretation of $\mathcal{J}(B_1, B_2) \mathbb{P}(L^\perp)$. Indeed, $\mathcal{R}$ is Azumaya by Lemma 6.17, and an étale cover over which it becomes a matrix algebra will do.

The following result summarizes our work above. Note that the structure sheaf of an Enriques surface is an exceptional object of its derived category.

Theorem 6.19. Let $\Sigma$ be a general Enriques surface. Then there exist (non-canonical) smooth, transverse cubic plane curves $C_1$ and $C_2$ in $\mathbb{P}^2$ and Azumaya algebras $\mathcal{R}_1$ and $\mathcal{R}_2$ on the square root stacks $\mathbb{P}^2(\sqrt{C_1})$ and $\mathbb{P}^2(\sqrt{C_2})$, such that if

$$\mathcal{P} = \mathbb{P}^2(\sqrt{C_1}) \times_{\mathbb{P}^2} \mathbb{P}^2(\sqrt{C_2}),$$

then $\mathcal{R} = \mathcal{R}_1 \boxtimes \mathcal{R}_2 \in \text{Perf}(\mathcal{P}, \mathcal{R})$ is an exceptional object and there is an equivalence between the subcategories

$$\langle O_{\Sigma} \rangle^\perp \subset \text{Perf}(\Sigma) \quad \text{and} \quad \langle \mathcal{R} \rangle^\perp \subset \text{Perf}(\mathcal{P}, \mathcal{R}).$$

Remark 6.20. Theorem 6.19 can be thought of as an algebraization of the logarithmic transform, that creates an Enriques surface from a rational elliptic surface with two marked fibers.

### 7. Categorical cones

In this section, we introduce the operation of taking the categorical cone of a Lefschetz category. The constructions, arguments, and organization of this section closely parallel those of §3. First in §7.1 we define the resolved cone of a category linear over a projective bundle, by analogy with the canonical resolution of singularities of the classical cone over a projective scheme. In §7.2 we define the categorical cone of a Lefschetz category as a subcategory of the resolved cone, and prove some basic properties of this construction. In §7.3 we study base changes of categorical cones, and in particular show that categorical and resolved cones agree away from the vertex. In §7.4 we construct a canonical Lefschetz structure on the categorical cone of a Lefschetz category. Finally, in §7.5 we show that taking the categorical join with a projective bundle can be described as a categorical cone.

We fix an exact sequence

$$0 \to V_0 \to V \to \tilde{V} \to 0 \quad (7.1)$$

of vector bundles on $S$. We write $H_0$, $H$, and $\tilde{H}$ for the relative hyperplane classes on the projective bundles $\mathbb{P}(V_0)$, $\mathbb{P}(V)$, and $\mathbb{P}(\tilde{V})$, and denote by $N_0$ the rank of $V_0$. 
7.1. **Resolved cones.** Denote by \( \text{pr}: \mathbb{P}(\bar{V}) \to S \) the projection. Let \( \mathcal{V} \) be the vector bundle on \( \mathbb{P}(\bar{V}) \) defined as the preimage of \( \mathcal{O}(-\bar{H}) \subset \text{pr}^*\bar{V} \) under \( \text{pr}^*\mathcal{V} \to \text{pr}^*\bar{V} \), so that on \( \mathbb{P}(\bar{V}) \) we have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{pr}^*V_0 & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{O}(-\bar{H}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{pr}^*V_0 & \longrightarrow & \text{pr}^*V & \longrightarrow & \text{pr}^*\bar{V} & \longrightarrow & 0
\end{array}
\]

(7.2)

with exact rows.

Now let \( X \to \mathbb{P}(\bar{V}) \) be a morphism of schemes. Then the **resolved cone** over \( X \) with vertex \( \mathbb{P}(V_0) \) is defined as the projective bundle

\[
\tilde{C}_{V_0}(X) = \mathbb{P}_X(\mathcal{V}_X),
\]

(7.3)

where \( \mathcal{V}_X \) denotes the pullback of \( \mathcal{V} \) to \( X \). The embedding \( \mathcal{V}_X \hookrightarrow V \otimes \mathcal{O}_X \) induces a morphism \( \tilde{C}_{V_0}(X) \to \mathbb{P}(V) \).

Recall from §1.5 that if \( X \to \mathbb{P}(\bar{V}) \) is an embedding, then this morphism factors birationally through the classical cone \( C_{V_0}(X) \subset \mathbb{P}(V) \), and provides a resolution of singularities if \( X \) is smooth.

Note that there is an isomorphism

\[
\tilde{C}_{V_0}(X) \cong X \times_{\mathbb{P}(\bar{V})} \tilde{C}_{V_0}(\mathbb{P}(\bar{V})).
\]

(7.4)

Motivated by this, we call \( \tilde{C}_{V_0}(\mathbb{P}(\bar{V})) = \mathbb{P}_{\mathbb{P}(\bar{V})}(\mathcal{V}) \) the **universal resolved cone** with vertex \( \mathbb{P}(V_0) \). Denote by

\[
\bar{\mu}: \tilde{C}_{V_0}(\mathbb{P}(\bar{V})) \to \mathbb{P}(\bar{V})
\]

the canonical projection morphism. Note that the rank of \( \mathcal{V} \) is \( N_0 + 1 \), so \( \bar{\mu} \) is a \( \mathbb{P}^{N_0} \)-bundle.

Further, denote by

\[
f: \tilde{C}_{V_0}(\mathbb{P}(\bar{V})) \to \mathbb{P}(V)
\]

the morphism induced by the canonical embedding \( \mathcal{V} \hookrightarrow \text{pr}^*V \). Define

\[
E = \mathbb{P}_{\mathbb{P}(\bar{V})}(\text{pr}^*V_0) \cong \mathbb{P}(V_0) \times \mathbb{P}(\bar{V})
\]

and let

\[
\varepsilon: E \to \tilde{C}_{V_0}(\mathbb{P}(\bar{V}))
\]

be the canonical divisorial embedding. We have a commutative diagram

\[
\begin{array}{cccccc}
E & \xrightarrow{\varepsilon} & \tilde{C}_{V_0}(\mathbb{P}(\bar{V})) & \xrightarrow{\bar{\mu}} & \mathbb{P}(\bar{V}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}(V_0) & \longrightarrow & \mathbb{P}(V)
\end{array}
\]

(7.5)

The isomorphism \( E \cong \mathbb{P}(V_0) \times \mathbb{P}(\bar{V}) \) is induced by the product of the vertical arrow and \( \bar{\mu} \circ \varepsilon \).

The next result follows easily from the definitions.

**Lemma 7.1.** The following hold:

1. The morphism \( f: \tilde{C}_{V_0}(\mathbb{P}(\bar{V})) \to \mathbb{P}(V) \) is the blowup of \( \mathbb{P}(V) \) in \( \mathbb{P}(V_0) \), with exceptional divisor \( E \).
2. The \( \mathcal{O}(1) \) line bundle for the projective bundle \( \bar{\mu}: \tilde{C}_{V_0}(\mathbb{P}(\bar{V})) \to \mathbb{P}(\bar{V}) \) is \( \mathcal{O}(\bar{H}) \).
(3) We have the following equality of divisors modulo linear equivalence:
\[ E = H - \overline{H}, \quad H|_E = H_0. \]

(4) The relative dualizing complex of the morphism \( \bar{p} \) is given by
\[ \omega_{\bar{p}} \cong \det(\bar{p}^* V^\vee)(-(N_0 + 1)H)[N_0]. \]

Following (7.4) we define the resolved cone of a category linear over \( P(\bar{V}) \) by base change from the universal resolved cone.

**Definition 7.2.** Let \( A \) be a \( P(\bar{V}) \)-linear category. The resolved cone over \( A \) with vertex \( P(V_0) \) is the category
\[ \tilde{C}_{V_0}(A) = A \otimes_{\text{Perf}(P(\bar{V}))} \text{Perf}(\tilde{C}_{V_0}(P(\bar{V}))). \]

Further, we define
\[ E(A) = A \otimes_{\text{Perf}(P(\bar{V}))} \text{Perf}(E). \]

**Remark 7.3.** The isomorphism \( E \cong P(V_0) \times P(\bar{V}) \) induces a canonical equivalence
\[ E(A) \simeq \text{Perf}(P(V_0)) \otimes A. \]

We identify these categories via this equivalence; in particular, below we will regard subcategories of the right side as subcategories of the left.

**Remark 7.4.** If \( X \) is a scheme over \( P(\bar{V}) \), then by the isomorphism (7.4) and Theorem A.2 the resolved cone satisfies
\[ \tilde{C}_{V_0}(\text{Perf}(X)) \simeq \text{Perf}(\tilde{C}_{V_0}(X)). \]

Below we gather some elementary lemmas about resolved cones.

**Lemma 7.5.** Let \( \gamma: A \to B \) be a \( P(\bar{V}) \)-linear functor. There is a commutative diagram
\[
\begin{array}{ccc}
\tilde{C}_{V_0}(A) & \xrightarrow{\bar{p}^*} & \tilde{C}_{V_0}(B) \\
\gamma^* \downarrow & & \downarrow \gamma^* \\
A & \xrightarrow{\bar{p}^*} & B \\
\end{array}
\]

Moreover, if \( \gamma \) admits a left or right adjoint, then so does \( \tilde{C}_{V_0}(\gamma) \). If further \( \gamma \) is fully faithful or an equivalence, then so is \( \tilde{C}_{V_0}(\gamma) \).

**Proof.** The proof of Lemma 3.5 applies. \(\square\)

**Lemma 7.6.** Let \( \gamma: A \to B \) be a \( P(\bar{V}) \)-linear functor. There is a commutative diagram
\[
\begin{array}{ccc}
\tilde{C}_{V_0}(A) & \xrightarrow{\bar{p}^*} & \tilde{C}_{V_0}(B) \\
\gamma^* \downarrow & & \downarrow \gamma^* \\
A & \xrightarrow{\bar{p}^*} & B \\
\end{array}
\]

Moreover, if \( \gamma \) admits a left or right adjoint, then so does \( \tilde{C}_{V_0}(\gamma) \). If further \( \gamma \) is fully faithful or an equivalence, then so is \( \tilde{C}_{V_0}(\gamma) \).

**Proof.** Follows from the definition of the resolved cone and Lemma A.5. \(\square\)
Lemma 7.7. Let $\mathcal{A}$ be a $\mathbf{P}(\bar{V})$-linear category. Then the functor 

$$\bar{p}^*: \mathcal{A} \to \check{C}_{V_0}(\mathcal{A})$$

is fully faithful, and there is a semiorthogonal decomposition with admissible components

$$\check{C}_{V_0}(\mathcal{A}) = \langle \bar{p}^*(\mathcal{A}), \bar{p}^*(\mathcal{A})(H), \ldots, \bar{p}^*(\mathcal{A})(N_0 H) \rangle,$$

where $N_0$ is the rank of $V_0$.

Proof. This is similar to the proof of Lemma 3.7, using the $\mathbf{P}^{N_0}$-bundle $\bar{p}$ instead of the $\mathbf{P}^1$-bundle $p$. □

Lemma 7.8. Let $\mathcal{A}$ be a $\mathbf{P}(\bar{V})$-linear category which is smooth and proper over $S$. Then the resolved cone $\check{C}_{V_0}(\mathcal{A})$ is smooth and proper over $S$.

Proof. Similar to the proof of Lemma 3.8, using that the morphism $\bar{p}: \check{C}_{V_0}(\mathbf{P}(\bar{V})) \to \mathbf{P}(\bar{V})$ is smooth and proper. □

7.2. Categorical cones. We define the categorical cone of a Lefschetz category over $\mathbf{P}(\bar{V})$ as a certain subcategory of the resolved cone.

Definition 7.9. Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. The categorical cone $\mathcal{C}_{V_0}(\mathcal{A})$ over $\mathcal{A}$ with vertex $\mathbf{P}(V_0)$ is the subcategory of $\check{C}_{V_0}(\mathcal{A})$ defined by

$$\mathcal{C}_{V_0}(\mathcal{A}) = \left\{ C \in \check{C}_{V_0}(\mathcal{A}) \mid \varepsilon^*(C) \in \text{Perf}(\mathbf{P}(V_0)) \otimes A_0 \subset E(\mathcal{A}) \right\}.$$

Here, we have used the identification of Remark 7.3. If $\mathcal{A} = \text{Perf}(X)$ for a scheme $X$ over $\mathbf{P}(\bar{V})$, we abbreviate notation by writing

$$\mathcal{C}_{V_0}(X) = \mathcal{C}_{V_0}(\text{Perf}(X)).$$

Remark 7.10. The categorical cone depends on the choice of a Lefschetz center for $\mathcal{A}$, although this is suppressed in the notation. For instance, for the “stupid” Lefschetz center $A_0 = \mathcal{A}$, the condition in the definition is void, so $\mathcal{C}_{V_0}(\mathcal{A}) = \check{C}_{V_0}(\mathcal{A})$.

We note that if $V_0 = 0$, then taking the categorical cone does nothing:

Lemma 7.11. Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. If $V_0 = 0$ then $\mathcal{C}_{V_0}(\mathcal{A}) \simeq \mathcal{A}$.

Proof. If $V_0 = 0$ then $\mathcal{V} \cong 0_{\mathbf{P}(\bar{V})}(-\bar{H})$ by (7.2), hence $\check{C}_{V_0}(\mathbf{P}(\bar{V})) = \mathbf{P}(\mathbf{P}(\bar{V})) \cong \mathbf{P}(\bar{V})$ and $\check{C}_{V_0}(\mathcal{A}) \simeq \mathcal{A}$. Furthermore, the divisor $E$ is empty in this case, hence the defining condition of $\mathcal{C}_{V_0}(\mathcal{A}) \subset \check{C}_{V_0}(\mathcal{A})$ is void and $\mathcal{C}_{V_0}(\mathcal{A}) = \check{C}_{V_0}(\mathcal{A})$. □

Analogous to Lemma 3.12 by applying Proposition 3.11 we obtain the following. Recall that $\varepsilon_!$ denotes the left adjoint of $\varepsilon^*$.

Lemma 7.12. Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$ of length $m$. Then the categorical cone $\mathcal{C}_{V_0}(\mathcal{A})$ is an admissible $\mathbf{P}(V)$-linear subcategory of $\check{C}_{V_0}(\mathcal{A})$, and there are $\mathbf{P}(V)$-linear
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semiorthogonal decompositions

\[ \tilde{C}_V(A) = \left\langle \varepsilon_1(\text{Perf}(P(V_0)) \otimes A_1(H)), \ldots, \varepsilon_1(\text{Perf}(P(V_0)) \otimes A_{m-1}((m-1)H)) \right\rangle, \quad (7.7) \]

\[ \tilde{C}_V(A) = \left\langle \varepsilon_*(\text{Perf}(P(V_0)) \otimes A_{1-m}((1-m)H)), \ldots, \varepsilon_*(\text{Perf}(P(V_0)) \otimes A_{-1}(-H)) \right\rangle, \quad (7.8) \]

Proof. Apply Proposition 3.11 with \( T = P(\bar{V}), Y = \tilde{C}_V(P(\bar{V})), \) and \( E = E. \) Then we have \( A_Y = \tilde{C}_V(A) \) and \( A_E = E(A) = \text{Perf}(P(V_0)) \otimes A, \) and the result follows by the same argument as in the proof of Lemma 3.12. \qed

Categorical cones preserve smoothness and properness:

**Lemma 7.13.** Let \( A \) be a Lefschetz category over \( P(\bar{V}) \) which is smooth and proper over \( S. \) Then the categorical cone \( C_{V_0}(A) \) is smooth and proper over \( S. \)

**Proof.** Follows from Lemma 7.8, Lemma 7.12, and [32, Lemma 4.15]. \qed

Before going further, we consider an example of a categorical cone, similar to Example 3.15.

**Example 7.14.** Let \( \bar{W} \subset \bar{V} \) be a subbundle, so that \( P(\bar{W}) \subset P(\bar{V}). \) The classical cone over \( P(\bar{W}) \) with vertex \( P(V_0) \) is given by \( C_{V_0}(P(\bar{W})) = P(W), \) where \( W \subset V \) is the preimage of \( \bar{W} \) under \( V \to \bar{V}. \) Consider the Lefschetz structure of \( P(W) \) defined in Example 2.11. Then the pullback functor

\[ f^*: \text{Perf}(P(W)) \to \text{Perf}(\tilde{C}_V(P(\bar{W}))) \]

induces a \( P(W) \)-linear equivalence

\[ \text{Perf}(P(W)) \simeq C_{V_0}(P(\bar{W})). \]

Moreover, Theorem 3.21 below equips \( C_{V_0}(P(\bar{W})) \) with a canonical Lefschetz structure. It is easy to check that the above equivalence is a Lefschetz equivalence.

**7.3. Base change of categorical cones.** Let \( T \to P(V) \) be a morphism of schemes. The base change of diagram (7.5) along this morphism gives a fiber square

\[ \begin{array}{ccc} E_T & \longrightarrow & \tilde{C}_V(P(\bar{V}))_T \\ \downarrow & & \downarrow \\ P(V_0)_T & \longrightarrow & T \end{array} \]

Note that the isomorphism \( E \cong P(V_0) \times P(\bar{V}) \) induces an isomorphism \( E_T \cong P(V_0)_T \times P(\bar{V}), \) which gives rise to an identification

\[ E(A)_T \cong \text{Perf}(P(V_0)_T) \otimes A \quad (7.9) \]

for any \( P(\bar{V}) \)-linear category \( A. \)
Lemma 7.15. Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$ of length $m$. Let $T \to \mathbf{P}(V)$ be a morphism of schemes. Then there is a $T$-linear semiorthogonal decomposition

$$\tilde{\mathcal{C}}_{V_0}(\mathcal{A})_T = \langle \mathcal{C}_{V_0}(\mathcal{A})_T,\varepsilon_1(\text{Perf}(\mathbf{P}(V_0)_T \otimes A_1(\bar{H})),\ldots,\varepsilon_1(\text{Perf}(\mathbf{P}(V_0)_T \otimes A_{m-1}((m-1)\bar{H})) \rangle.$$

Proof. This is the base change of (7.7) with the identification (7.9) taken into account. □

Proposition 7.16. Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. Let $T \to \mathbf{P}(V)$ be a morphism of schemes which factors through the complement of $\mathbf{P}(V_0)$ in $\mathbf{P}(V)$. Then there are $T$-linear equivalences

$$\mathcal{C}_{V_0}(\mathcal{A})_T \simeq \tilde{\mathcal{C}}_{V_0}(\mathcal{A})_T \simeq \mathcal{A}_T,$$

where the base change of $\mathcal{A}$ is taken along the morphism $T \to \mathbf{P}(\bar{V})$ obtained by composing $T \to \mathbf{P}(V)$ with the linear projection from $\mathbf{P}(V_0) \subset \mathbf{P}(V)$.

Proof. Similar to the proof of Proposition 3.17. □

For future use we fix the following immediate corollary of the proposition.

Corollary 7.17. Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. Let $T \to \mathbf{P}(V)$ be a morphism of schemes which factors through the complement of $\mathbf{P}(V_0)$ in $\mathbf{P}(V)$, and such that the composition $T \to \mathbf{P}(\bar{V})$ is an isomorphism. Then there is an equivalence

$$\mathcal{C}_{V_0}(\mathcal{A})_T \simeq \mathcal{A}.$$

7.4. The Lefschetz structure of a categorical cone. Our goal in this subsection is to equip any categorical cone with a canonical Lefschetz structure.

Lemma 7.18. Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. Then the image of $A_0$ under the functor $\tilde{p}^*: \mathcal{A} \to \tilde{\mathcal{C}}_{V_0}(\mathcal{A})$ is contained in the categorical cone $\mathcal{C}_{V_0}(\mathcal{A})$. Moreover, if $A' \subset A_0$ is a (left or right) admissible subcategory, then so is its image $\tilde{p}^*(A') \subset \mathcal{C}_{V_0}(\mathcal{A})$. In particular, if $A_i$ are the Lefschetz components of $\mathcal{A}$, then $\tilde{p}^*(A_i) \subset \mathcal{C}_{V_0}(\mathcal{A})$ is left admissible for $i < 0$, admissible for $i = 0$, and right admissible for $i > 0$.

Proof. Similar to the proof of Lemma 3.19. □

Definition 7.19. Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. For $i \in \mathbb{Z}$, we define a subcategory $\mathcal{C}_{V_0}(\mathcal{A})_i \subset \mathcal{C}_{V_0}(\mathcal{A})$ by

$$\mathcal{C}_{V_0}(\mathcal{A})_i = \begin{cases} \tilde{p}^*(A_{i+N_0}) & \text{if } i \leq -N_0, \\ \tilde{p}^*(A_0) & \text{if } -N_0 \leq i \leq N_0, \\ \tilde{p}^*(A_{i-N_0}) & \text{if } i \geq N_0, \end{cases} \quad (7.10)$$

where $N_0$ is the rank of $V_0$.

Note that the containment $\mathcal{C}_{V_0}(\mathcal{A})_i \subset \mathcal{C}_{V_0}(\mathcal{A})$ holds by Lemma 7.18.

Theorem 7.20. Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. Then the categorical cone $\mathcal{C}_{V_0}(\mathcal{A})$ has the structure of a Lefschetz category over $\mathbf{P}(V)$ with Lefschetz components $\mathcal{C}_{V_0}(\mathcal{A})_i$ given by (7.10). If $\mathcal{A}$ is either right or left strong, then so is $\mathcal{C}_{V_0}(\mathcal{A})$. Moreover, we have

$$\text{length}(\mathcal{C}_{V_0}(\mathcal{A})) = \text{length}(\mathcal{A}) + N_0,$$

and $\mathcal{C}_{V_0}(\mathcal{A})$ is moderate if and only if $\mathcal{A}$ is moderate.
The proof of Theorem 7.20 takes the rest of this subsection. We let \( \mathcal{A} \) be as in the theorem, let \( m = \text{length}(\mathcal{A}) \), and set
\[
\mathcal{C}_i = \mathcal{C}_{V_0}(\mathcal{A}).
\]
Note that by Lemma 7.12, the categorical cone \( \mathcal{C}_{V_0}(\mathcal{A}) \) is naturally \( \mathbb{P}(V) \)-linear. To prove the theorem, we will show that we have semiorthogonal decompositions
\[
\mathcal{C}_{V_0}(\mathcal{A}) = \langle \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{m+N_0-1}((m+N_0-1)H) \rangle, \quad (7.11)
\]
\[
\mathcal{C}_{V_0}(\mathcal{A}) = \langle \mathcal{C}_1-m-N_0((1-m-N_0)H), \ldots, \mathcal{C}_{-1}(-H), \mathcal{C}_0 \rangle, \quad (7.12)
\]
and then apply Lemma 2.4. We focus on proving (7.11) below; an analogous argument proves (7.12).

**Lemma 7.21.** The sequence of subcategories
\[
\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{m+N_0-1}((m+N_0-1)H) \quad (7.13)
\]
of \( \mathcal{C}_{V_0}(\mathcal{A}) \) is semiorthogonal.

**Proof.** This is similar to the proof of Lemma 3.25: from Lemma 7.14 we get
\[
\pi(O(tH)) \simeq \begin{cases} 0, & \text{if } 1 \leq t \leq N_0, \\ \det(V^\vee) \otimes \text{Sym}^{t-N_0-1}(V^\vee)[N_0], & \text{if } t \geq N_0 + 1. \end{cases}
\]
Using this in place of (3.20) together with the observation that
\[
\det(V^\vee) \otimes \text{Sym}^{t-N_0-1}(V^\vee) \in \langle \text{Perf}(S)\bar{H}, \text{Perf}(S)(2\bar{H}), \ldots, \text{Perf}(S)((t-N_0)\bar{H}) \rangle
\]
by (7.2), we deduce the claim. \( \square \)

To show that the categories (7.13) generate \( \mathcal{C}_{V_0}(\mathcal{A}) \), we consider the idempotent-complete triangulated subcategory \( \mathcal{P} \) of the resolved cone \( \hat{\mathcal{C}}_{V_0}(\mathcal{A}) \) generated by the categories in (7.13) and
\[
\varepsilon_*(\text{Perf}(\mathcal{P}(V_0))) \otimes \mathcal{A}_i((i-1)\bar{H})), \quad 1 \leq i \leq m-1. \quad (7.14)
\]
By (1.11) and Lemma 7.13 we have
\[
\varepsilon_!(C) \simeq \varepsilon_*(C(\bar{H_0}-\bar{H})[-1]),
\]
so by Lemma 7.12 the categories (7.14) generate the left orthogonal to \( \mathcal{C}_{V_0}(\mathcal{A}) \) in \( \hat{\mathcal{C}}_{V_0}(\mathcal{A}) \). Hence to establish (7.11), it suffices to show \( \mathcal{P} = \hat{\mathcal{C}}_{V_0}(\mathcal{A}) \). For this, we will need the following lemma.

**Lemma 7.22.** For all integers \( i, s, t \), such that \( 0 \leq s \leq i \leq m-1 \) and \( 0 \leq t \leq i-s+N_0 \) the subcategory
\[
p^*(A_i(s\bar{H}))(tH) \subset \hat{\mathcal{C}}_{V_0}(\mathcal{A})
\]
is contained in \( \mathcal{P} \).

**Proof.** Similar to the proof of Lemma 3.26, using
\[
0 \to O(-E) \to \emptyset \to \varepsilon_*O_E \to 0
\]
in place of (3.24). \( \square \)
Proof of Theorem 7.20. Let us show $\mathcal{P} = \check{C}_{V_0}(A)$, which as observed above will complete the proof of the semiorthogonal decomposition (7.11). By Lemma 7.7 we have a semiorthogonal decomposition
\[
\check{C}_{V_0}(A) = \langle \tilde{p}^*(A), \tilde{p}^*(A)(H), \ldots, \tilde{p}^*(A)(N_0H) \rangle,
\]
so it suffices to show $\mathcal{P}$ contains all the components of this decomposition. But $A$ is generated by the categories $A_i(iH)$, $0 \leq i \leq m - 1$, so taking $s = i$ in Lemma 7.22 shows $\mathcal{P}$ contains $\tilde{p}^*(A)(tH)$ for $0 \leq t \leq N_0$, as required.

The semiorthogonal decomposition (7.12) holds by a similar argument. Thus by Lemma 2.4 and Lemma 7.18, we deduce that $\check{C}_{V_0}(A)$ is a Lefschetz center with $\check{C}_i$, $i \in \mathbb{Z}$, the corresponding Lefschetz components. The strongness claims follow from the definitions and Lemma 7.18, and the claims about the length and moderateness of $\check{C}_{V_0}(A)$ follow from the definitions and (2.12).

7.5. Relation to categorical joins. In this subsection, we assume $V_0 \neq 0$ and we are given a splitting of (7.1):
\[
V = V_0 \oplus \bar{V}.
\]
Under these assumptions, we relate the cone operations discussed above (classical, resolved, and categorical) to taking a join (in the corresponding senses) with $\mathbf{P}(V_0)$.

The relation between the classical operations is easy: if $X \subset \mathbf{P}(\bar{V})$ is a closed subscheme, then the classical join of $X$ with $\mathbf{P}(V_0)$ coincides with the cone over $X$ with vertex $\mathbf{P}(V_0)$, i.e.
\[
\mathbf{J}(\mathbf{P}(V_0), X) = \check{C}_{V_0}(X) \subset \mathbf{P}(V).
\]
Note that the assumption $V_0 \neq 0$ is necessary for this equality; if $V_0 = 0$ then $\mathbf{P}(V_0) = \emptyset$ and hence $\mathbf{J}(\mathbf{P}(V_0), X) = \emptyset$, while $\check{C}_{V_0}(X) = X$.

To describe the relation between the categorical operations, we first compare the universal resolved join
\[
\tilde{\mathbf{J}}(\mathbf{P}(V_0), \mathbf{P}(\bar{V})) = \mathbf{P}(\mathbf{P}(V_0) \times \mathbf{P}(\bar{V})) (\mathcal{O}(-H_0) \oplus \mathcal{O}(-\bar{H}))
\]
to the universal resolved cone
\[
\check{C}_{V_0}(\mathbf{P}(\bar{V})) = \mathbf{P}(\mathbf{P}(\bar{V})) (\text{pr}^*V_0 \oplus \mathcal{O}(-\bar{H}))
\]
where $\text{pr}: \mathbf{P}(\bar{V}) \to S$ is the projection. The natural embedding $\mathcal{O}(-H_0) \hookrightarrow \text{pr}^*V_0$ induces a morphism
\[
\beta: \tilde{\mathbf{J}}(\mathbf{P}(V_0), \mathbf{P}(\bar{V})) \to \check{C}_{V_0}(\mathbf{P}(\bar{V})).
\]
Denoting $Z = \mathbf{P}(\mathbf{P}(\bar{V}))(\mathcal{O}(-\bar{H})) \cong \mathbf{P}(\bar{V})$, the diagram (3.3) (with $V_1 = V_0$ and $V_2 = \bar{V}$) and the diagram (7.5) merge to a commutative diagram
\[
\begin{array}{cccccc}
\mathbf{E}_1 & \longrightarrow & \tilde{\mathbf{J}}(\mathbf{P}(V_0), \mathbf{P}(\bar{V})) & \longrightarrow & \mathbf{E}_2 \\
\downarrow \cong & \downarrow \beta & & \downarrow & \\
\mathbf{E} & \longrightarrow & \check{C}_{V_0}(\mathbf{P}(\bar{V})) & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{P}(V_0) & \longrightarrow & \mathbf{P}(V_0 \oplus \bar{V}) & \longrightarrow & \mathbf{P}(\bar{V})
\end{array}
\]
where under the isomorphisms $\mathbf{E}_2 \cong \mathbf{P}(V_0) \times \mathbf{P}(\bar{V})$ and $Z \cong \mathbf{P}(\bar{V})$, the map $\mathbf{E}_2 \to Z$ is identified with the projection.
Lemma 7.23. The morphism $\beta: \tilde{J}(P(V_0), P(\tilde{V})) \to \tilde{C}_{V_0}(P(\tilde{V}))$ is the blowup of $\tilde{C}_{V_0}(P(\tilde{V}))$ in $Z$, with exceptional divisor $E_2$.

Proof. Follows from Lemmas [3.1][1] and [7.1][1]. □

Using this, we can compare categorical joins and cones.

Proposition 7.24. Let $A$ be a Lefschetz category over $P(\tilde{V})$, and let $V_0$ be a nonzero vector bundle on $S$. Consider the Lefschetz structure on $P(V_0)$ defined in Example [2.11]. Then there is an equivalence

$$\mathcal{C}_{V_0}(A) \simeq \tilde{J}(P(V_0), A)$$

of Lefschetz categories over $P(V_0 \oplus \tilde{V})$. More precisely, pullback and pushforward along the morphism $\beta: \tilde{J}(P(V_0), P(\tilde{V})) \to \tilde{C}_{V_0}(P(\tilde{V}))$ give functors

$$\beta^*: \tilde{C}_{V_0}(A) \to \tilde{J}(P(V_0), A),$$

$$\beta_*: \tilde{J}(P(V_0), A) \to \tilde{C}_{V_0}(A),$$

which induce mutually inverse Lefschetz equivalences between the subcategories

$$\mathcal{C}_{V_0}(A) \subset \tilde{C}_{V_0}(A) \quad \text{and} \quad \tilde{J}(P(V_0), A) \subset \tilde{J}(P(V_0), A).$$

Moreover, for any $P(\tilde{V})$-linear functor $\gamma: A \to B$ there is a commutative diagram

$$\begin{array}{ccc}
\tilde{J}(P(V_0), A) & \xrightarrow{\tilde{J}(\text{id}, \gamma)} & \tilde{J}(P(V_0), B) \\
\beta_* \downarrow \quad & & \beta_* \downarrow \\
\tilde{C}_{V_0}(A) & \xrightarrow{\gamma} & \tilde{C}_{V_0}(B). \\
\end{array}$$

(7.16)

Proof. Diagram (7.16) is obtained from the functor $\gamma$ by base change along the morphism $\beta$. Lemma [7.23] together with Orlov’s blowup formula implies $\beta^*: \tilde{C}_{V_0}(A) \to \tilde{J}(P(V_0), A)$ is fully faithful and gives an equivalence onto the subcategory

$$\beta^*(\tilde{C}_{V_0}(A)) = \left\{ C \in \tilde{J}(P(V_0), A) \mid \varepsilon_2^*(C) \in \text{Perf}(S) \otimes A \subset E_2(\text{Perf}(P(V_0)), A) \right\}.$$

Since $\beta$ maps $E_1$ isomorphically onto $E$, it thus follows from Definition [7.9] that $\beta^*$ induces an equivalence from $\tilde{C}_{V_0}(A)$ onto the subcategory

$$\beta^*(\tilde{C}_{V_0}(A)) = \left\{ C \in \tilde{J}(P(V_0), A) \mid \varepsilon_1^*(C) \in \text{Perf}(P(V_0)) \otimes A_0 \subset E_1(\text{Perf}(P(V_0)), A), \varepsilon_2^*(C) \in \text{Perf}(S) \otimes A \subset E_2(\text{Perf}(P(V_0)), A) \right\},$$

with the inverse equivalence induced by $\beta_*$. But by Definition [3.9] this subcategory coincides with $\tilde{J}(P(V_0), A)$ since $\text{Perf}(S)$ is the Lefschetz center of $\text{Perf}(P(V_0))$.

It remains to see that $\beta^*$ identifies the Lefschetz centers of $\tilde{C}_{V_0}(A)$ and $\tilde{J}(P(V_0), A)$. This follows directly from the definition of these Lefschetz centers (Definitions [3.20] and [7.19]) and the commutative diagram

$$\begin{array}{ccc}
\tilde{J}(P(V_0), P(\tilde{V})) & \xrightarrow{p} & P(V_0) \times P(\tilde{V}) \\
\beta \downarrow & & \beta \downarrow \\
\tilde{C}_{V_0}(P(\tilde{V})) & \xrightarrow{\tilde{\beta}} & P(\tilde{V}).
\end{array}$$

□
Remark 7.25. Proposition 7.24 does not apply if $V_0 = 0$. Indeed, if $V_0 = 0$ then $P(V_0) = \emptyset$ and hence $\mathfrak{J}(P(V_0), A) = 0$, while $\mathfrak{C}_{V_0}(A) \simeq A$ by Lemma 7.11.

Remark 7.26. Let $A^1$ and $A^2$ be Lefschetz categories over $P(V_1)$ and $P(V_2)$, where $V_1$ and $V_2$ are nonzero. Then one can prove there is a $P(V_1 \oplus V_2)$-linear equivalence

$$\mathfrak{J}(A^1, A^2) \simeq \mathfrak{C}_{V_1}(A^2) \otimes_{\text{Perf}(P(V_1 \oplus V_2))} \mathfrak{C}_{V_2}(A^1),$$

but we omit the proof as we shall not need this. The right side can be endowed with a semiorthogonal decomposition by an application of Corollary 5.3, which can be shown to be a Lefschetz decomposition compatible with the Lefschetz structure of the left side. Note also that the equivalence of Proposition 7.24 is a special case of this. Indeed, just take $A^1 = P(V_1)$ and use the fact that $\mathfrak{C}_{V_2}(P(V_1)) \simeq \text{Perf}(P(V_1 \oplus V_2))$ by Example 7.14.

8. HPD for categorical cones

In this section we show that (under suitable hypotheses) the formation of categorical cones commutes with HPD. We formulate the theorem in a way that allows for extensions of the base projective bundle (in the sense of Definition A.13), because this extra generality is useful in applications.

Theorem 8.1. Let $V$ be a vector bundle on $S$, let

$$V_0 \subset V \quad \text{and} \quad V_\infty \subset V^\vee$$

be subbundles such that the natural pairing $V \otimes V^\vee \to \mathcal{O}_S$ is zero on $V_0 \otimes V_\infty$, so that we have a pair of filtrations

$$0 \subset V_0 \subset V_\infty^\perp \subset V \quad \text{and} \quad 0 \subset V_\infty \subset V_0^\perp \subset V^\vee. \quad (8.1)$$

Set

$$\bar{V} = V_\infty^\perp/V_0, \quad \text{so that} \quad V_0^\perp/V_\infty \cong \bar{V}^\vee. \quad (8.2)$$

Let $A$ be a right strong, moderate Lefschetz category over $P(\bar{V})$. Then there is an equivalence

$$(\mathfrak{C}_{V_0}(A)/P(V))^\sharp \simeq \mathfrak{C}_{V_\infty}(A^\sharp)/P(V^\vee)$$

of Lefschetz categories over $P(V^\vee)$.

Remark 8.2. Let us explain the structure of the categories appearing in Theorem 8.1. By Theorem 7.20 the categorical join $\mathfrak{C}_{V_0}(A)$ is a right strong, moderate Lefschetz category over $P(V_0^\perp)$. By extending the base along the inclusion $P(V_\infty^\perp) \to P(V)$, we obtain by Remark A.14 a right strong, moderate Lefschetz category $\mathfrak{C}_{V_0}(A)/P(V)$ over $P(V)$. Hence by Theorem 2.24(1), the HPD category $(\mathfrak{C}_{V_0}(A)/P(V))^\sharp$ has the structure of a Lefschetz category over $P(V^\vee)$. The structure of $\mathfrak{C}_{V_\infty}(A^\sharp)/P(V^\vee)$ as a Lefschetz category over $P(V^\vee)$ is similarly obtained by a combination of Theorem 2.24(1), Theorem 7.20 and base extension.

The case $V_0 = 0$ gives the following (we take $V_\infty = W^\perp \subset V^\vee$).

Corollary 8.3. Let $W \subset V$ be an inclusion of vector bundles on $S$. Let $A$ be a right strong, moderate Lefschetz category over $P(W)$. Then there is an equivalence

$$(A/P(V))^\sharp \simeq \mathfrak{C}_{W^\perp}(A^\sharp)$$

of Lefschetz categories over $P(V^\vee)$. 
One could prove Theorem 8.1 by mimicking the proof of Theorem 4.1. Instead, we use several steps to bootstrap the result from Theorem 4.1. First, we define a functor
\[ \gamma_C : C_{V_\infty}(A^2)/P(V^\vee) \to H(C_{V_0}(A)/P(V)) \]
via a double cone construction, an analogue of the double join construction of §4.1. Next, we use Theorem 4.1 and a local-to-global argument to show that this functor gives the required equivalence when both \( V_0 \) and \( V_\infty \) are nonzero. Then we use Proposition 2.32 to remove the assumption that \( V_\infty \) is nonzero, and duality to remove the assumption that \( V_0 \) is nonzero.

8.1. Double resolved cones and the HPD functor for categorical cones. Throughout this section we fix filtrations (8.1), and use (8.2) to identify their quotients with \((V_0, \bar{V}, V^\vee)\) and \((V_\infty, \bar{V}, V^\vee)\) respectively.

Let \( Y \) be a scheme equipped with a morphism \( Y \to P(\bar{V}) \times P(\bar{V}^\vee) \). In this situation, we can form two categorical cones, \( \tilde{C}_{V_0}(Y) \) and \( \tilde{C}_{V_\infty}(Y) \). Parallel to §4.1, we define the double resolved cone over \( Y \) as
\[ \tilde{C}_{V_0, V_\infty}(Y) = \tilde{C}_{V_0}(Y) \times_Y \tilde{C}_{V_\infty}(Y). \]
In particular, we can consider the universal double resolved cone with its natural projection
\[ \tilde{C}_{V_0, V_\infty}(P(\bar{V}) \times P(\bar{V}^\vee)) \to P(\bar{V}) \times P(\bar{V}^\vee). \] (8.3)

Now, given a category \( B \) which has a \( P(\bar{V}) \times P(\bar{V}^\vee) \)-linear structure, we define the double resolved cone \( \tilde{C}_{V_0, V_\infty}(B) \) over \( B \) as
\[ \tilde{C}_{V_0, V_\infty}(B) = B \otimes_{\text{Perf}(P(\bar{V}) \times P(\bar{V}^\vee))} \text{Perf}(\tilde{C}_{V_0, V_\infty}(P(\bar{V}) \times P(\bar{V}^\vee))), \]
that is the base change of \( B \) along (8.3).

As in the proof of Theorem 4.1, the key case for us is when \( Y \) is the universal space of hyperplanes in \( P(\bar{V}) \), which we denote by
\[ H = H(P(\bar{V})). \]
Note that \( \tilde{H} \) indeed naturally maps to \( P(\bar{V}) \times P(\bar{V}^\vee) \), hence we can form the double resolved cone over \( \tilde{H} \). We write \( H(C_{V_0}(P(\bar{V}))/P(V)) \) for the universal hyperplane section of \( C_{V_0}(P(\bar{V})) \) with respect to the morphism \( C_{V_0}(P(\bar{V})) \to P(V_\infty) \to P(V) \). The following analogue of Lemma 4.3 holds by the same argument.

**Lemma 8.4.** We have a commutative diagram
\[ \begin{array}{ccc}
\tilde{C}_{V_0, V_\infty}(\tilde{H}) & \xrightarrow{\alpha} & H(C_{V_0}(P(\bar{V}))/P(V)) \\
\tilde{C}_V(\tilde{H}) \downarrow & & H(C_{V_0}(P(\bar{V}))/P(V)) \\
\tilde{P}(\bar{V}) \times \tilde{P}(V^\vee) & \xrightarrow{\tilde{p}} & \tilde{H} \\
\end{array} \] (8.4)

All schemes in the diagram are smooth and projective over \( S \), hence Remark 1.17 applies to all morphisms.
Let $\mathcal{A}$ be a $\mathbf{P}(\check{V})$-linear category. We denote by
\[
\gamma: \mathcal{A}^\natural \to \mathbf{H}(\mathcal{A})
\]
the canonical $\mathbf{P}(\check{V}^\natural)$-linear inclusion functor. By Lemma \ref{lem:7.5}, it induces a $\mathbf{P}(V^\perp)$-linear functor
\[
\check{C}_{V,\infty}(\gamma): \check{C}_{V,\infty}(\mathcal{A}^\natural) \to \check{C}_{V,\infty}(\mathbf{H}(\mathcal{A})),
\]
which can be regarded as a $\mathbf{P}(V^\natural)$-linear functor
\[
\check{C}_{V,\infty}(\gamma): \check{C}_{V,\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\natural) \to \check{C}_{V,\infty}(\mathbf{H}(\mathcal{A}))/\mathbf{P}(V^\natural).
\]
Here, we have written $\check{C}_{V,\infty}(\mathbf{H}(\mathcal{A}))/\mathbf{P}(V^\natural)$ to emphasize that we regard $\check{C}_{V,\infty}(\mathbf{H}(\mathcal{A}))$ as a $\mathbf{P}(V^\natural)$-linear category, via the inclusion $\mathbf{P}(V_0^\perp) \subset \mathbf{P}(V^\natural)$.

Similar to the case of joins, by base change from diagram \ref{diag:8.4} we obtain a diagram of $\mathbf{P}(V^\natural)$-linear functors
\[
\begin{array}{ccc}
\check{C}_{V,\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\natural) & \xrightarrow{\check{C}_{V,\infty}(\gamma)} & \check{C}_{V,\infty}(\mathbf{H}(\mathcal{A}))/\mathbf{P}(V^\natural) \\
\check{C}_{V,\infty}(\mathbf{H}(\mathcal{A}))/\mathbf{P}(V^\natural) & \xrightarrow{\mathbf{H}((\check{C}_{V,\infty}(\mathcal{A}))/\mathbf{P}(V))} & H(\check{C}_{V,\infty}(\mathcal{A}))/\mathbf{P}(V))
\end{array}
\]

We define a $\mathbf{P}(V^\natural)$-linear functor as the composition
\[
\gamma_{\hat{C}} = \alpha_* \circ \check{p}^* \circ \check{C}_{V,\infty}(\gamma): \check{C}_{V,\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\natural) \to \mathbf{H}((\check{C}_{V,\infty}(\mathcal{A}))/\mathbf{P}(V)). \tag{8.5}
\]

The following analogue of Lemma \ref{lem:4.4} holds by a similar proof and will be used later.

**Lemma 8.5.** The functor $\gamma_{\hat{C}}$ has both left and right adjoints.

**Remark 8.6.** The functor $\gamma_{\hat{C}}$ can also be described in terms of Fourier–Mukai kernels, similarly to Remark \ref{rem:4.8}. We leave this as an exercise.

As in the case of joins, note that the HPD category $(\check{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural$ is a $\mathbf{P}(V^\natural)$-linear subcategory of $\mathbf{H}((\check{C}_{V_\infty}(\mathcal{A}))/\mathbf{P}(V))$.

### 8.2. The nonzero case.

The goal of this subsection is to prove the following more precise version of Theorem \ref{thm:8.1} when $V_0$ and $V_\infty$ are nonzero. In \ref{thm:8.3} we will use this to deduce the Theorem \ref{thm:8.1} in general, without any restrictions on $V_0$ or $V_\infty$. In fact, by a careful analysis of the argument in \ref{thm:8.3} it is also possible to prove the following result without any restrictions on $V_0$ or $V_\infty$, but we will not need this.

**Proposition 8.7.** Let $\mathcal{A}$ be a right strong, moderate Lefschetz category over $\mathbf{P}(\check{V})$. Assume $V_0$ and $V_\infty$ are nonzero. Then the functor
\[
\gamma_{\hat{C}}: \check{C}_{V,\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\natural) \to \mathbf{H}((\check{C}_{V_\infty}(\mathcal{A}))/\mathbf{P}(V))
\]
induces a Lefschetz equivalence between the subcategories
\[
\check{C}_{V,\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\natural) \subset \check{C}_{V,\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\natural) \quad \text{and} \quad (\check{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural \subset \mathbf{H}((\check{C}_{V_\infty}(\mathcal{A}))/\mathbf{P}(V)).
\]

The proof takes the rest of the subsection. Let us outline the strategy.

By Lemma \ref{lem:8.5} the functor $\gamma_{\hat{C}}$ has adjoints. Therefore, by Lemma \ref{lem:2.16} the claim of Proposition \ref{prop:8.7} is fpqc-local, so it is enough to prove it over a fpqc cover of the base scheme $S$. Passing to such a cover we may assume that the filtrations \ref{eq:8.1} split, so that
\[
V = V_0 \oplus \check{V} \oplus V_\infty^\vee.
\]
Then the orthogonal to \( V \) in the beginning of § the analogous diagram of resolved joins and cones. Below we resume the convention explained for the rest of this subsection, we fix such a splitting. Using this, we will reduce Proposition 8.7 to Theorem 4.1 (in its more precise form Theorem 4.9). We set

\[
V_1 = V_0 \oplus V_\infty^\vee.
\]

Then the orthogonal to \( V_0 \subset V_1 \) is \( V_\infty \subset V_1^\vee \), so by Theorem 2.27 there is an equivalence

\[
\text{Perf}(\mathbf{P}(V_0))^2 \cong \text{Perf}(\mathbf{P}(V_\infty))
\]  

(8.6)
of Lefschetz categories over \( \mathbf{P}(V_1^\vee) \). Hence we have a commutative diagram of equivalences of Lefschetz categories over \( \mathbf{P}(V^\vee) \):

\[
\begin{array}{ccc}
\mathcal{C}(\mathbf{P}(V_\infty), \mathcal{A}^2) / \mathcal{P}(V^\vee) & \xrightarrow{\gamma} & \mathcal{C}(\mathbf{P}(V_0), \mathcal{A}) / \mathcal{P}(V) \\
\xi\downarrow & & \downarrow \\
\mathcal{C}_V(\mathcal{A}^2) / \mathcal{P}(V^\vee) & \xrightarrow{\gamma} & \mathcal{C}_V(\mathcal{A}) / \mathcal{P}(V)
\end{array}
\]

(8.7)

where the vertical equivalences are consequences of Proposition 7.24 the top equivalence is given by Theorem 4.9 (note that \( V = V_1 \oplus \bar{V} \)) combined with (8.6), and the bottom equivalence is the composition of the other three. To prove Proposition 8.7, we check that the bottom equivalence is in fact induced by the functor \( \gamma_C \).

**Remark 8.8.** The above argument (even without checking the bottom arrow is induced by \( \gamma_C \)) already proves Theorem 8.1 under the assumptions that \( V_0 \) and \( V_\infty \) are nonzero and the filtrations (8.1) are split. However, for the local-to-global argument above by which we reduced to the split case, it is essential that we verify the equivalence is given by a globally defined functor.

To check that the bottom equivalence in (8.7) is induced by \( \gamma_C \), we prove commutativity of the analogous diagram of resolved joins and cones. Below we resume the convention explained in the beginning of § 4.1 of using \( J \) and \( J^\gamma \) in our notation for resolved joins.

**Proposition 8.9.** There is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}(\mathbf{P}(V_\infty), \mathcal{A}^2) / \mathcal{P}(V^\vee) & \xrightarrow{\gamma_\infty} & \mathcal{H}(\mathcal{C}(\mathbf{P}(V_0), \mathcal{A}) / \mathcal{P}(V)) \\
\beta_\infty\downarrow & & \downarrow \beta_0 \\
\mathcal{C}_V(\mathcal{A}^2) / \mathcal{P}(V^\vee) & \xrightarrow{\gamma} & \mathcal{H}(\mathcal{C}_V(\mathcal{A}) / \mathcal{P}(V)),
\end{array}
\]

(8.8)

where \( \beta_0 \) and \( \beta_\infty \) are the blowup morphisms from Lemma 7.23 of the cones with vertices \( \mathbf{P}(V_0) \) and \( \mathbf{P}(V_\infty) \), and \( \gamma_\infty \) is the composition

\[
\gamma_\infty : \mathcal{C}(\mathbf{P}(V_\infty), \mathcal{A}^2) / \mathcal{P}(V^\vee) \xrightarrow{\sim} \mathcal{C}(\text{Perf}(\mathbf{P}(V_0))^2, \mathcal{A}^2) / \mathcal{P}(V^\vee) \xrightarrow{\gamma} \mathcal{H}(\mathcal{J}(\mathbf{P}(V_0), \mathcal{A}) / \mathcal{P}(V))
\]

where the equivalence is induced by (8.6) and \( \gamma_\infty \) is defined by (4.15) (with \( V_2 = \bar{V} \)).

**Proof.** Recall from Theorem 2.27 that the functor

\[
t_\ast \circ \text{pr}_2^*: \text{Perf}(\mathbf{P}(V_\infty)) \to \text{Perf}(\mathbf{H}(\mathbf{P}(V_0)))
\]

induces the HPD between \( \mathbf{P}(V_0) \) and \( \mathbf{P}(V_\infty) \), where \( \mathbf{H}(\mathbf{P}(V_0)) = \mathbf{H}(\mathbf{P}(V_0) / \mathbf{P}(V_1)) \) is the universal hyperplane section of the morphism \( \mathbf{P}(V_0) \to \mathbf{P}(V_1) \), : \( \mathbf{P}(V_0) \times \mathbf{P}(V_\infty) \to \mathbf{H}(\mathbf{P}(V_0)) \).
Lemma 7.23 thus combine to give a morphism

\[ \beta \circ \iota \circ \tilde{\gamma} \]

where the morphisms

\[ \tilde{\gamma}(H(P(V_0)), H) \xrightarrow{\delta_0} \tilde{\gamma}(H(P(V_0)), H) \xrightarrow{\alpha_0} H(\tilde{\gamma}(P(V_0), P(V))/P(V)) \]

are the base change along \( P(V_0) \to P(V_1) \) of the top of diagram (4.4) (with \( V_2 = \bar{V} \)). Further, note that we can write \( \tilde{\gamma} \) as a composition

\[ \tilde{\gamma}(P(V_0), H) \xrightarrow{J(id, \gamma)} \tilde{\gamma}(P(V_0), H(A)) \xrightarrow{J(id, \gamma)} \tilde{\gamma}(P(V_0) \times P(V), H(A)) \]

and hence

\[ \tilde{\gamma}(H(P(V_0)), H(A)) \xrightarrow{\gamma} \tilde{\gamma}(H(P(V_0)), H(A)) \]

(8.9)

By definition, \( \gamma \) is a composition of three functors analogous to \( \tilde{\gamma}(id, \gamma) \), \( \delta_0 \), and \( \alpha_0 \) in (8.9). To prove the proposition, we will relate the analogous functors appearing in these compositions, using the blowup morphisms \( \beta_\infty \) and \( \beta_0 \) and the morphisms \( \iota \) and \( pr_2 \).

The relation between \( \tilde{\gamma}(id, \gamma) \) and \( \tilde{\gamma}_V(\gamma) \) is provided by the commutative diagram (7.16), that in our case takes the form

\[ \tilde{\gamma}(P(V_0), A^2) \xrightarrow{\tilde{\gamma}(id, \gamma)} \tilde{\gamma}(P(V_0), H(A)) \]

(8.10)

To relate the other functors, we write down diagrams of schemes that induce diagrams of functors by base change. First note that we have a fiber square

\[ \tilde{\gamma}(P(V_0) \times P(V), H) \]

(8.11)

where \( \tilde{\gamma}(\iota, id) \) denotes the morphism between the double resolved joins induced by the morphisms \( \iota: P(V_0) \times P(V) \to H(P(V_0)) \) and \( id: H \to H \). Next observe that

\[ \tilde{\gamma}(P(V_0) \times P(V), H) = \tilde{\gamma}(P(V_0) \times P(V), H) \times_{(P(V_0) \times P(V)) \times H} \tilde{\gamma}(P(V_0) \times P(V), H) \]

\[ \cong \tilde{\gamma}(P(V_0), H) \times_H \tilde{\gamma}(P(V), H), \]

where the first equality holds by definition. We also have by definition

\[ \tilde{\gamma}(P(V_0), H) = \tilde{\gamma}_V(H) \times_H \tilde{\gamma}_V(H). \]

The blowup morphisms \( \beta_0: \tilde{\gamma}(P(V_0), H) \to \tilde{\gamma}_V(H) \) and \( \beta_\infty: \tilde{\gamma}(P(V_0), H) \to \tilde{\gamma}_V(H) \) from Lemma 7.23 thus combine to give a morphism

\[ \beta_0 \circ \tilde{\gamma}(P(V_0), H) \to \tilde{\gamma}(P(V_0) \times P(V), H). \]
It is easy see that the morphism $\beta_{0\infty}$ makes the diagrams
\[
\begin{align*}
\tilde{\mathcal{J}}(\mathbf{P}(V_0), \bar{\mathbf{H}}) & \xleftarrow{\tilde{\mathcal{J}}(pr_2, id)} \tilde{\mathcal{J}}(\mathbf{P}(V_0) \times \mathbf{P}(V_\infty), \bar{\mathbf{H}}) & \xleftarrow{\tilde{p}_{0\infty}} \tilde{\mathcal{J}}(\mathbf{P}(V_0) \times \mathbf{P}(V_\infty), \bar{\mathbf{H}}) \\
\mathcal{C}_\infty(\bar{\mathbf{H}}) & \xleftarrow{\beta_{0\infty}} \mathcal{C}_\infty(\bar{\mathbf{H}})
\end{align*}
\] (8.12)
and
\[
\begin{align*}
\tilde{\mathcal{J}}(\mathbf{P}(V_0) \times \mathbf{P}(V_\infty), \bar{\mathbf{H}}) & \xrightarrow{\tilde{\mathcal{J}}(\iota, id)} \tilde{\mathcal{J}}(\mathbf{P}(V_0)), \bar{\mathbf{H}}) & \xrightarrow{\alpha_0} H(\tilde{\mathcal{J}}(\mathbf{P}(V_0), \mathbf{P}(V))/\mathbf{P}(V)) \\
\mathcal{C}_\infty(\bar{\mathbf{H}}) & \xrightarrow{\alpha} H(\mathcal{C}_\infty(\mathbf{P}(V))/\mathbf{P}(V))
\end{align*}
\] (8.13)
commutative, where in \[ \text{[8.13]} \] we abuse write $\beta_0$ for the morphism induced by the blowup $\beta_0 : \tilde{\mathcal{J}}(\mathbf{P}(V_0), \mathbf{P}(V)) \to \mathcal{C}_\infty(\mathbf{P}(V))$. Note also that since $\beta_{0\infty}$ is a product of two blowup morphisms, we have an isomorphism of functors
\[
\beta_{0\infty} \circ \beta_{0\infty}^* \simeq \text{id}. \quad \text{(8.14)}
\]

Finally, combining the above ingredients we obtain
\[
\begin{align*}
\beta_{0*} \circ \gamma_{\mathcal{C}} & \circ \beta_{\infty}^* \simeq \beta_{0*} \circ \alpha_{0*} \circ \tilde{p}_0^* \circ \tilde{\mathcal{J}}(t_*, id) \circ \tilde{\mathcal{J}}(pr_2, id) \circ \tilde{\mathcal{J}}(id, \gamma) \circ \beta_{\infty}^* \quad \text{(8.9)} \\
& \simeq \beta_{0*} \circ \alpha_{0*} \circ \tilde{\mathcal{J}}(t_*, id) \circ \tilde{\mathcal{J}}(pr_2, id) \circ \beta_{\infty}^* \circ \mathcal{C}_\infty(\gamma) \quad \text{(8.10)} \\
& \simeq \beta_{0*} \circ \alpha_{0*} \circ \tilde{\mathcal{J}}(t_*, id) \circ \beta_{0\infty}^* \circ \tilde{\mathcal{J}}(pr_2, id) \circ \beta_{\infty}^* \circ \mathcal{C}_\infty(\gamma) \quad \text{(8.11)} \\
& \simeq \beta_{0*} \circ \alpha_{0*} \circ \tilde{\mathcal{J}}(t_*, id) \circ \beta_{0\infty}^* \circ \tilde{p}_0^* \circ \tilde{\mathcal{J}}(pr_2, id) \circ \beta_{\infty}^* \circ \mathcal{C}_\infty(\gamma) \quad \text{(8.12)} \\
& \simeq \alpha_{0*} \circ \beta_{0\infty}^* \circ \beta_{\infty}^* \circ \tilde{p}_0^* \circ \tilde{\mathcal{J}}(pr_2, id) \circ \beta_{\infty}^* \circ \mathcal{C}_\infty(\gamma) \quad \text{(8.13)} \\
& \simeq \alpha_{0*} \circ \tilde{p}_0^* \circ \tilde{\mathcal{J}}(pr_2, id) \circ \beta_{\infty}^* \circ \mathcal{C}_\infty(\gamma) \quad \text{(8.14)} \\
& = \gamma_{\mathcal{C}} \quad \text{(8.5)}
\end{align*}
\]
which completes the proof.

\[ \square \]

**Proof of Proposition 8.7.** As explained above, we first consider the case where the filtrations \[ \text{[8.1]} \] split. Then we have a commutative diagram \[ \text{[8.8]} \], whose vertical arrows and top horizontal arrow induce the corresponding arrows of \[ \text{[8.7]} \]. Hence by commutativity of these diagrams, the functor $\gamma_{\mathcal{C}}$ induces the Lefschetz equivalence given by the bottom horizontal arrow of \[ \text{[8.7]} \].

In the nonsplit case we use Proposition \[ \text{[A.9]} \] with $\mathcal{C} = \mathcal{C}_\infty(\mathcal{A}^2)$, $\mathcal{D} = H(\mathcal{C}_\infty(\mathcal{A})/\mathbf{P}(V))$, $\phi = \gamma_{\mathcal{C}}$, $\mathcal{A} = \mathcal{C}_\infty(\mathcal{A}^2)$, and $\mathcal{B} = (\mathcal{C}_\infty(\mathcal{A})/\mathbf{P}(V))^2$. We note that the assumptions of the proposition are satisfied by Lemmas \[ \text{[7.12]} \] and \[ \text{[8.5]} \]. We take an fpqc cover of our base scheme $S$ over which the filtrations \[ \text{[8.1]} \] split. By the argument above the functor $\gamma_{\mathcal{C}}$ induces the desired Lefschetz equivalence after base change to any element of the cover. Hence by Proposition \[ \text{[A.9]} \] the functor $\gamma_{\mathcal{C}}$ induces an equivalence between $\mathcal{C}_\infty(\mathcal{A}^2)/\mathbf{P}(V')$ and $(\mathcal{C}_\infty(\mathcal{A})/\mathbf{P}(V))^2$, which is in fact a Lefschetz equivalence by Lemma \[ \text{[2.16]} \].

\[ \square \]
8.3. The general case. In this subsection we bootstrap from Proposition 8.7 to the general case of Theorem 8.1.

Lemma 8.10. The claim of Theorem 8.1 holds if \( V_0 \) is nonzero.

Proof. By Proposition 8.7 we only need to consider the case where \( V_\infty = 0 \). Take an auxiliary nonzero vector bundle \( V_\infty \) on \( S \), and set \( \tilde{V} = V \oplus \nabla \nabla \). Then \( V_0 \subset \tilde{V} \) and \( \nabla \subset \nabla \) are such that the pairing \( \tilde{V} \otimes \nabla \rightarrow \mathcal{O}_S \) is zero on \( V_0 \otimes \nabla \). Hence Proposition 8.7 gives an equivalence

\[
\mathcal{C}_{\nabla}(A^2)/P(\nabla) \simeq (\mathcal{C}_{V_0}(A)/P(\nabla))^2 \tag{8.15}
\]

of Lefschetz categories over \( P(\nabla) \). By base change along the embedding \( P(\nabla) \rightarrow P(\nabla) \) we obtain a \( P(\nabla) \)-linear equivalence

\[
\left( \mathcal{C}_{\nabla}(A^2)/P(\nabla) \right) \otimes_{\text{Perf}(P(\nabla))} \text{Perf}(P(\nabla)) \simeq (\mathcal{C}_{V_0}(A)/P(\nabla))^2 \otimes_{\text{Perf}(P(\nabla))} \text{Perf}(P(\nabla)),
\tag{8.16}
\]

On the one hand, we have \( P(\nabla) \)-linear equivalences

\[
\left( \mathcal{C}_{\nabla}(A^2)/P(\nabla) \right) \otimes_{\text{Perf}(P(\nabla))} \text{Perf}(P(\nabla)) \simeq A^2/P(\nabla) \simeq \mathcal{C}_{\nabla}(A^2)/P(\nabla),
\tag{8.17}
\]

where the first holds by Corollary 7.17 and the second by Lemma 7.11 since \( V_\infty = 0 \). On the other hand, note that \( \mathcal{C}_{V_0}(A)/P(\nabla) \) is supported over the open \( \text{Perf}(V) \setminus \nabla(\nabla) \) in the sense of Definition A.15, since this category’s \( P(\nabla) \)-linear structure is induced from a \( P(\nabla) \)-linear structure via the morphism \( P(\nabla) \rightarrow P(\nabla) \). Hence by Proposition 2.32 we have a \( P(\nabla) \)-linear equivalence

\[
(\mathcal{C}_{V_0}(A)/P(\nabla))^2 \otimes_{\text{Perf}(P(\nabla))} \text{Perf}(P(\nabla)) \simeq (\mathcal{C}_{V_0}(A)/P(\nabla))^2.
\tag{8.18}
\]

Combining the above equivalences, we thus obtain a \( P(\nabla) \)-linear equivalence

\[
\mathcal{C}_{\nabla}(A^2)/P(\nabla) \simeq (\mathcal{C}_{V_0}(A)/P(V))^2.
\tag{8.17}
\]

Finally, tracing through the equivalences (8.16) and (8.18) and using Remark 2.34 and the fact that (8.15) is a Lefschetz equivalence, one verifies that (8.18) identifies the Lefschetz centers on each side.

Now we can handle the general case.

Proof of Theorem 8.1. By Lemma 8.10 it remains to consider the case where \( V_0 = 0 \). We may assume \( V_\infty \neq 0 \), otherwise there is nothing to prove. Taking the left HPD (see Remark 2.20), by Remark 2.26 the claim of Theorem 8.1 is equivalent to the existence of a Lefschetz equivalence

\[
\mathcal{C}_{V_0}(A)/P(V) \simeq \left( \mathcal{C}_{\nabla}(A^2)/P(\nabla) \right).
\]

Since \( V_\infty \neq 0 \) we can apply (the left version of) Lemma 8.10 to obtain a Lefschetz equivalence

\[
\left( \mathcal{C}_{\nabla}(A^2)/P(\nabla) \right) \simeq \mathcal{C}_{V_0}(A^2)/P(V).
\]

But again by Remark 2.26 we have Lefschetz equivalence \( \left( A^2 \right) \simeq A \), so the result follows.

Remark 8.11. We could also use Proposition 2.35 in place of Proposition 2.32 to prove the case where \( V_0 = 0 \) analogously to Lemma 8.10.
9. HPD for quadrics and its applications

In this section, we use categorical cones to describe HPD for quadrics. For simplicity, we assume the base scheme $S$ is the spectrum of an algebraically closed field $k$ of characteristic 0. We study the following class of morphisms from a quadric to a projective space.

**Definition 9.1.** Let $Q$ be a quadric, i.e. an integral scheme over $k$ which admits a closed immersion into a projective space as a quadric hypersurface. We denote by $\mathcal{O}_Q(1)$ the restriction of the line bundle $\mathcal{O}(1)$ from this ambient space. A morphism $f : Q \to \mathbb{P}(V)$ is standard if there is an isomorphism

$$f^*\mathcal{O}_{\mathbb{P}(V)}(1) \cong \mathcal{O}_Q(1).$$

Note that $Q$ is not required to be smooth. In the preliminary §9.1, we recall that if $Q$ is smooth then it has a natural Lefschetz structure, and if $f : Q \to \mathbb{P}(V)$ is a divisorial embedding or a double covering then the HPD category can be described in terms of classical projective duality. In §9.2 we construct for a general standard morphism $f : Q \to \mathbb{P}(V)$ a Lefschetz category $Q$ over $\mathbb{P}(V)$ — called the standard categorical resolution of $Q$ — which is smooth and proper over $k$ and agrees with Perf($Q$) over the complement of $f(Sing(Q)) \subset \mathbb{P}(V)$.

In §9.3 we introduce a “generalized duality” operation that associates to a standard morphism $f : Q \to \mathbb{P}(V)$ of a quadric another such morphism $f^\vee : Q^\vee \to \mathbb{P}(V^\vee)$. We prove that this notation is compatible with the notation for the HPD category, i.e. that the HPD of the standard categorical resolution of $Q$ is Lefschetz equivalent to the standard categorical resolution of the generalized dual $Q^\vee$ (Theorem 9.17). By combining this with the nonlinear HPD theorem, we prove a quadratic HPD theorem (Theorem 9.18). As an application, we prove the duality conjecture for Gushel–Mukai varieties from [24], as well as a “spin” version of this result.

9.1. HPD for smooth quadrics. In this subsection, we consider HPD for smooth quadrics. The Lefschetz structure on such a quadric will be defined in terms of spinor bundles. We follow [30] for our conventions on spinor bundles, and recall some of the key facts here.

Let $Q$ be a smooth quadric of even dimension $2d$, and write $H$ for the hyperplane class so that $\mathcal{O}(H) = \mathcal{O}_Q(1)$. Let Spin($Q$) be the universal covering of the special orthogonal group $SO(Q)$ associated with the quadric $Q$. Then $Q$ carries a pair of Spin($Q$)-equivariant vector bundles $S_+$ and $S_-$ of rank $2d−1$, called the spinor bundles. If $d = 1$ then $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $S_+ = \mathcal{O}(-1,0)$ and $S_- = \mathcal{O}(0,-1)$; if $d \geq 2$ then $S_+$ and $S_-$ both have determinant $\mathcal{O}_Q(-2d^{-2})$. Denoting by $S_\pm$ the $2d$-dimensional half-spinor representations of Spin($Q$), there are canonical exact sequences

$$0 \to S_+ \to S_+ \otimes \mathcal{O}_Q \to S_-(H) \to 0, \quad 0 \to S_- \to S_- \otimes \mathcal{O}_Q \to S_+(H) \to 0.$$  \hspace{1cm} (9.1)

Moreover, if $f : Q \to \mathbb{P}(V)$ is an embedding of $Q$ as a quadric hypersurface these sequences can be glued to short exact sequences on $\mathbb{P}(V)$:

$$0 \to S_- \otimes \mathcal{O}_{\mathbb{P}(V)}(-H) \to S_+ \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to f_*(S_-(H)) \to 0,$$

$$0 \to S_+ \otimes \mathcal{O}_{\mathbb{P}(V)}(-H) \to S_- \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to f_*(S_+(H)) \to 0.$$  \hspace{1cm} (9.2)

Another nice property of spinor bundles is their self-duality up to a twist:

$$S_\pm(H) \cong \begin{cases} S_\pm^\vee, & \text{if } d \text{ is even}, \\ S_\pm^\vee, & \text{if } d \text{ is odd}. \end{cases}$$  \hspace{1cm} (9.3)
Similarly, if $Q$ is a smooth quadric of odd dimension $2d - 1$, it carries one spinor bundle $S$ of rank $2^{d-1}$ that fits into an exact sequence
\[ 0 \to S \to S \otimes \mathcal{O}_Q \to S(H) \to 0, \tag{9.4} \]
where $S$ is the spinor representation of $\text{Spin}(Q)$, and such that
\[ S(H) \cong S^\vee. \tag{9.5} \]
Moreover, if $Q$ is represented as a hyperplane section of a smooth quadric $Q'$ of even dimension, then $S$ is isomorphic to the restriction of either of the spinor bundles on $Q'$.

In what follows, when $Q$ is a smooth quadric of arbitrary dimension, we will denote by $S$ a chosen spinor bundle — the only one in the odd-dimensional case, or one of the two in the even-dimensional case. With this convention, we have the following result.

**Lemma 9.2.** Let $f: Q \to \mathbf{P}(V)$ be a standard morphism of a smooth quadric $Q$. Let $S$ denote a spinor bundle on $Q$. Then $\text{Perf}(Q)$ is smooth and proper over $k$, and has the structure of a strong, moderate Lefschetz category over $\mathbf{P}(V)$ with Lefschetz center
\[ Q_0 = \langle S, \mathcal{O} \rangle \subset \text{Perf}(Q) \]
and length $\dim(Q)$. Further, if $p \in \{0, 1\}$ is the parity of $\dim(Q)$, i.e. $p = \dim(Q) \pmod{2}$, then the nonzero Lefschetz components of $\text{Perf}(Q)$ are given by
\[ Q_i = \begin{cases} \langle S, \mathcal{O} \rangle & \text{for } |i| \leq 1 - p, \\ \langle \mathcal{O} \rangle & \text{for } 1 - p < |i| \leq \dim(Q) - 1. \end{cases} \]

**Proof.** By Kapranov’s semiorthogonal decomposition of the derived category of a quadric [11] together with Lemma 2.4 and (9.1), it follows that the $Q_i \subset \text{Perf}(Q)$ are the components of a Lefschetz structure. Since $Q$ is smooth and proper over $k$, so is $\text{Perf}(Q)$ by [32, Lemma 4.9]. The Lefschetz structure is thus strong by Remark 2.8, and moderate by definition. □

**Remark 9.3.** Let $f: Q \to \mathbf{P}(V)$ be a standard morphism of a smooth quadric $Q$. Then we always use the center given by Lemma 9.2 when we regard $\text{Perf}(Q)$ as a Lefschetz category over $\mathbf{P}(V)$. If $\dim(Q) = 2d$ is even there are two spinor bundles $S_+$ and $S_-$, so there is an apparent choice involved in the structure of $\text{Perf}(Q)$ as a Lefschetz category. However, there exists an (noncanonical) automorphism $a$ of $Q$ over $\mathbf{P}(V)$ such that $a^*(S_{\pm}) \cong S_{\mp}$ (corresponding to the automorphism of the Dynkin diagram of type $D_{d+1}$). The resulting autoequivalence $a^*$ of $\text{Perf}(Q)$ identifies the Lefschetz center of Lemma 9.2 defined by $S = S_+$ with that defined by $S = S_-$. Hence, if $\dim(Q)$ is even, the structure of $\text{Perf}(Q)$ as a Lefschetz category over $\mathbf{P}(V)$ is still uniquely determined, up to noncanonical equivalence.

**Remark 9.4.** The Lefschetz center $Q_0$ of $\text{Perf}(Q)$ can be also written as
\[ Q_0 = \langle \mathcal{O}, S' \rangle \]
where $S' = S$ if $\dim(Q)$ is not divisible by 4, and the other spinor bundle otherwise. This follows from the exact sequences (9.1) and (9.4) and the dualities (9.3) and (9.5). The nonzero primitive Lefschetz components of $\text{Perf}(Q)$ are given by
\[ q_i = \begin{cases} \langle \mathcal{O} \rangle & \text{if } i = -(\dim(Q) - 1), \\ \langle S' \rangle & \text{if } i = -(1 - p), \\ \langle S \rangle & \text{if } i = 1 - p, \\ \langle \mathcal{O} \rangle & \text{if } i = \dim(Q) - 1. \end{cases} \]
The following result describes HPD for standard morphisms of smooth quadrics that are either divisorial embeddings or double coverings. This will be generalized to arbitrary standard morphisms of quadrics in Theorem 9.17. Recall that the classical projective dual of a smooth quadric hypersurface \( Q \subset P(V) \) is itself a smooth quadric hypersurface \( Q^\vee \subset P(V^\vee) \).

**Theorem 9.5 ([19, Theorem 5.2]).** Let \( f: Q \to P(V) \) be a standard morphism of a smooth quadric \( Q \), which is either a divisorial embedding or a double covering branched along a smooth quadric. Then there is an equivalence

\[
\text{Perf}(Q)^\natural \simeq \text{Perf}(Q^\vee)
\]

of Lefschetz categories over \( P(V^\vee) \), where:

1. If \( f \) is a divisorial embedding and \( \dim(Q) \) is even, then \( Q^\natural \to P(V^\vee) \) is the projective dual of \( Q \).
2. If \( f \) is a divisorial embedding and \( \dim(Q) \) is odd, then \( Q^\natural \to P(V^\vee) \) is the double cover branched along the projective dual of \( Q \).
3. If \( f \) is a double covering and \( \dim(Q) \) is even, then \( Q^\natural \to P(V^\vee) \) is the projective dual of the branch locus of \( f \).
4. If \( f \) is a double covering and \( \dim(Q) \) is odd, then \( Q^\natural \to P(V^\vee) \) is the double cover branched along the projective dual of the branch locus of \( f \).

There is no proof of Theorem 9.5 in the literature, so we supply one. The proof is quite long and takes the rest of this subsection. We divide it into a number of steps, which we explain here. Most of the proof concerns case \( \text{[1]} \), when \( \dim(Q) = 2d \) and \( f: Q \to P(V) \) is a divisorial embedding. In this case we aim to prove a Lefschetz equivalence

\[
\text{Perf}(Q)^\natural \simeq \text{Perf}(Q^\vee).
\]

Let \( H \) and \( H' \) denote the hyperplane classes on \( P(V) \) and \( P(V^\vee) \). We consider the universal hyperplane section \( H(Q) \) as a family of quadrics \( h_Q: H(Q) \to P(V^\vee) \) of dimension \( 2d - 1 \). In Step 1 we use this fibration to isolate a semiorthogonal component of \( \text{Perf}(H(Q)) \) equivalent to the derived category of modules over the corresponding sheaf of Clifford algebras. In Step 2 we use central reduction to rewrite this category as the \( Z/2 \)-equivariant category of the derived category of modules over an Azumaya algebra on the double covering \( Z \) of \( P(V^\vee) \) branched over the quadric \( Q^\vee \). In Step 3 we show that this Azumaya algebra is Morita trivial, and identify the above category with the \( Z/2 \)-equivariant derived category of \( Z \). In Step 4 we further decompose this category into two components, the derived category of \( Q^\vee \) and that of \( P(V^\vee) \). In Steps 5 and 6 we rewrite the embedding functor of \( \text{Perf}(P(V^\vee)) \) into \( \text{Perf}(H(Q)) \), and check that together with the other components of the semiorthogonal decomposition discussed in Step 1, it provides the Lefschetz part of \( \text{Perf}(H(Q)) \), and thus the remaining component \( \text{Perf}(Q^\vee) \) identifies with the HPD category \( \text{Perf}(Q)^\natural \). In Step 7 we describe the Fourier–Mukai kernel for the embedding \( \text{Perf}(Q^\vee) \to \text{Perf}(H(Q)) \) constructed above. In Step 8 we rewrite this description in terms of spinor bundles on \( Q \) and \( Q^\vee \). In Step 9 we use this to check that the equivalence given by this functor is a Lefschetz equivalence, finishing the proof of case \( \text{[1]} \). Finally, in Step 10 we deduce the remaining cases \( \text{[2]} - \text{[4]} \) by applying Proposition 2.32.

For Steps 1–9 below, we assume as above that \( \dim(Q) = 2d \) and \( f: Q \to P(V) \) is a divisorial embedding. Further, we assume that the Lefschetz structure of \( \text{Perf}(Q) \) is chosen so that \( Q_0 = (S_+, 0) \).
Step 1: A decomposition of $\text{Perf}(\mathbf{H}(Q))$. The vector bundle on $\mathbf{P}(V^\vee)$ corresponding to the family $h_Q: \mathbf{H}(Q) \to \mathbf{P}(V^\vee)$ of quadrics is

$$W = \ker(V \otimes \mathcal{O} \to \mathcal{O}(H')) \cong \Omega_{\mathbf{P}(V^\vee)}(H'),$$

and the corresponding family of quadratic forms is given by the composition

$$\mathcal{O} \to \text{Sym}^2 V^\vee \otimes \mathcal{O} \to \text{Sym}^2 W^\vee,$$

where the first morphism is given by the equation of $H$, and the second is the tautological surjection. By \cite[Theorem 4.2]{15} we have

$$\text{Perf}(\mathbf{H}(Q)) = \langle \text{Perf}(\mathbf{P}(V^\vee), \text{Cliff}_0(W)), h_Q^*(\text{Perf}(\mathbf{P}(V^\vee)))(H), \ldots, h_Q^*(\text{Perf}(\mathbf{P}(V^\vee)))((2d-1)H) \rangle,$$

where $\text{Cliff}_0(W)$ is the sheaf of even parts of the universal Clifford algebra on $\mathbf{P}(V^\vee)$ for this family of quadrics, and $\text{Perf}(\mathbf{P}(V^\vee), \text{Cliff}_0(W))$ is the category of perfect complexes of $\text{Cliff}_0(W)$-modules on $\mathbf{P}(V^\vee)$ as in Notation 6.8. The embedding of this category into $\text{Perf}(\mathbf{H}(Q))$ is given by the functor

$$\gamma: \text{Perf}(\mathbf{P}(V^\vee), \text{Cliff}_0(W)) \to \text{Perf}(\mathbf{H}(Q)), \quad \mathcal{F} \mapsto h_Q^*\mathcal{F} \otimes_{\text{Cliff}_0(W)} \mathcal{E},$$

where $\mathcal{E}$ is the sheaf of $\text{Cliff}_0(W)$-modules on $\mathbf{H}(Q)$ defined by the exact sequence

$$0 \to \mathcal{O}(-H) \otimes \text{Cliff}_0(W) \to \mathcal{O} \otimes \text{Cliff}_1(W) \to i_*\mathcal{E} \to 0$$

on $\mathbf{H} = \mathbf{P}(V^\vee)(W)$, where $i: \mathbf{H}(Q) \to \mathbf{H}$ is the natural embedding, $\text{Cliff}_1(W)$ is the sheaf of odd parts of the Clifford algebra, and the first morphism is induced by the Clifford multiplication. Note that the space $\mathbf{H}$ is simply the universal hyperplane in $\mathbf{P}(V)$.

Step 2: Central reduction. Next, we use the argument of \cite[§3.6]{15} to describe the category $\text{Perf}(\mathbf{P}(V^\vee), \text{Cliff}_0(W))$ in more detail. We denote by $\text{Cliff}(V)$ the Clifford algebra of the quadric $Q$. Since the source bundle for the family of quadratic forms (9.6) is trivial (hence is a square), there is a sheaf of Clifford algebras $\text{Cliff}(W)$ on $\mathbf{P}(V^\vee)$ for this family of quadratic forms, that combines the even and the odd parts of the Clifford algebra; as a sheaf of $\mathcal{O}$-modules it has rank $2^{2d+1}$ and can be written as

$$\text{Cliff}(W) = \text{Cliff}_0(W) \oplus \text{Cliff}_1(W) \cong \mathcal{O} \oplus W \oplus \wedge^2 W \oplus \cdots \oplus \wedge^{2d+1} W \subset \text{Cliff}(V) \otimes \mathcal{O},$$

with the algebra structure induced by that of $\text{Cliff}(V)$. The subalgebra

$$\mathfrak{Z} = \mathfrak{Z}_0 \oplus \mathfrak{Z}_1 = \mathcal{O} \oplus \wedge^{2d+1} W \subset \text{Cliff}(W)$$

is central (and moreover $\text{Cliff}(W)$ is the centralizer of $\mathfrak{Z}$ in $\text{Cliff}(V)$), and the morphism

$$\zeta: Z = \text{Spec}_{\mathbf{P}(V^\vee)}(\mathfrak{Z}) \to \mathbf{P}(V^\vee)$$

is the double covering branched over the projective dual quadric $Q^\vee \subset \mathbf{P}(V^\vee)$. Note that $Z$ is a smooth quadric of dimension $2d + 1$. We consider the $\mathbb{Z}/2$-action on $Z$ generated by the involution of the double covering. Note that it is induced by the natural $\mathbb{Z}/2$-grading of $\mathfrak{Z}$. The sheaf of algebras $\text{Cliff}(W)$ is a module over $\mathfrak{Z}$, hence there is a sheaf of algebras $\mathcal{R}$ of rank $2^{2d}$ on $Z$ such that

$$\text{Cliff}(W) \cong \zeta_*\mathcal{R}.$$
Furthermore, the direct sum decomposition $\text{Cliff}(W) = \text{Cliff}_0(W) \oplus \text{Cliff}_1(W)$ provides $\text{Cliff}(W)$ with the structure of a $\mathbb{Z}/2$-graded $\mathfrak{z}$-module, hence provides $\mathcal{R}$ with a $\mathbb{Z}/2$-equivariant structure. By definition the invariant part of $\zeta_*\mathcal{R}$ is
\[(\zeta_*\mathcal{R})^{\mathbb{Z}/2} \cong \text{Cliff}_0(W),\]
hence there is an equivalence of categories
\[\phi: \text{Perf}(Z, \mathcal{R})^{\mathbb{Z}/2} \xrightarrow{\sim} \text{Perf}(\mathbb{P}(V^\vee), \text{Cliff}_0(W)), \quad \mathcal{F} \mapsto (\zeta_*\mathcal{F})^{\mathbb{Z}/2}, \quad (9.10)\]

**Step 3: Morita triviality of $\mathcal{R}$.** The sheaf of algebras $\mathcal{R}$ is an Azumaya algebra by [15, Proposition 3.16]. We claim it is in fact Morita trivial. Indeed, let $S_+$ and $S_-$ be the two $2^d$-dimensional half-spinor modules for $\text{Cliff}_0(V)$ (which appeared earlier as the half-spinor representations of $\text{Spin}(Q)$). Then the sum
\[S = S_+ \oplus S_- \quad (9.11)\]
is naturally a $\text{Cliff}(V)$-module, and hence by restriction a $\text{Cliff}(W)$-module as well. In particular, it is a $\mathfrak{z}$-module, hence gives a vector bundle on $Z$. Moreover, thinking of the direct sum decomposition $\mathcal{Z}$ as a $\mathbb{Z}/2$-grading, i.e. as a $\mathbb{Z}/2$-equivariant structure, we see that there is an object $S_Z$ of $\text{Perf}(Z, \mathcal{R})^{\mathbb{Z}/2}$ such that
\[\zeta_* S^\vee_Z \cong (S_+ \oplus S_-) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)} \quad \text{and} \quad (\zeta_* S^\vee_Z)^{\mathbb{Z}/2} = S_+ \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}. \quad (9.12)\]

Actually, $S^\vee_Z$ is the dual spinor bundle (of rank $2^d$) on the smooth odd-dimensional quadric $Z$ (of dimension $2d + 1$), hence the notation.

Since $S^\vee_Z$ is an equivariant $\mathcal{R}$-module on $Z$, we have a natural equivariant morphism $\mathcal{R} \rightarrow \text{End}(S^\vee_Z)$, which is fiberwise injective because $\mathcal{R}$ is an Azumaya algebra, and hence is an isomorphism because the ranks of the source and the target are both equal to $2^{2d}$. Consequently, we have an equivalence
\[\mu: \text{Perf}(Z)^{\mathbb{Z}/2} \xrightarrow{\sim} \text{Perf}(Z, \mathcal{R})^{\mathbb{Z}/2}, \quad \mathcal{F} \mapsto \mathcal{F} \otimes S^\vee_Z. \quad (9.13)\]

**Step 4: Root stack decomposition.** Finally, the equivariant category $\text{Perf}(Z)^{\mathbb{Z}/2}$ can be considered as the derived category of the quotient stack $[Z/(\mathbb{Z}/2)]$, i.e. of the root stack of $\mathbb{P}(V^\vee)$ along $Q^\vee$, and consequently has a semiorthogonal decomposition
\[\text{Perf}(Z)^{\mathbb{Z}/2} = \langle \text{Perf}(Q^\vee), \text{Perf}(\mathbb{P}(V^\vee)) \rangle, \quad (9.14)\]
with the embedding functors given by
\[\alpha_Q: \text{Perf}(Q^\vee) \rightarrow \text{Perf}(Z)^{\mathbb{Z}/2}, \quad \mathcal{F} \mapsto j_* \mathcal{F} \otimes \chi, \quad \text{where} \quad j: Q^\vee \rightarrow Z \text{ is the embedding of the ramification divisor and } \chi \text{ is the nontrivial character of } \mathbb{Z}/2, \quad \text{and by} \]
\[\alpha_P: \text{Perf}(\mathbb{P}(V^\vee)) \rightarrow \text{Perf}(Z)^{\mathbb{Z}/2}, \quad \mathcal{F} \mapsto \zeta^* \mathcal{F}, \quad \text{where } \zeta^* \mathcal{F} \text{ is given the natural equivariant structure (23, Theorem 4.1)}.

Combining (9.7) with (9.10), (9.13), and (9.14), we obtain a $\mathbb{P}(V^\vee)$-linear semiorthogonal decomposition
\[\text{Perf}(\mathcal{H}(Q)) = \langle \Phi(\text{Perf}(Q^\vee)), \Psi(\text{Perf}(\mathbb{P}(V^\vee))) \rangle, \quad h^*_Q(\text{Perf}(\mathbb{P}(V^\vee)))(H), \ldots, h^*_Q(\text{Perf}(\mathbb{P}(V^\vee)))(2d - 1)H). \quad (9.15)\]
The embedding functors $\Phi$ and $\Psi$ of the first two components are discussed below.

**Step 5: Rewriting the functor $\Psi$.** According to the construction in Steps 1–4 above, the second component of (9.15) is embedded by the functor

$$
\Psi = \gamma \circ f \circ \phi \circ \mu \circ \alpha_P : \text{Perf}(P(V^\vee)) \to \text{Perf}(H(Q)),
$$

which is given by

$$
\mathcal{F} \mapsto h^*_Q \left( \zeta^* (\zeta^* \mathcal{F} \otimes S^\vee_Z) \otimes \text{Cliff}_0(W) \right) \mathcal{E}
$$

$$
\simeq h^*_Q \left( \mathcal{F} \otimes (\zeta ^* S^\vee_Z) \otimes \text{Cliff}_0(W) \right) \mathcal{E}
$$

$$
\simeq (h^*_Q \mathcal{F} \otimes S_+) \otimes \text{Cliff}_0(W) \mathcal{E},
$$

where we used the projection formula for the first isomorphism and (9.12) for the second. This can be simplified further as follows. We consider the commutative diagram

$$
\begin{array}{ccc}
H(Q) & \xrightarrow{i} & H & \xrightarrow{h} & P(V^\vee) \\
\pi_Q \downarrow & & \downarrow \pi & & \\
Q & \xrightarrow{f} & P(V)
\end{array}
$$

with cartesian square, where $h_Q = h \circ i$ by definition. Using the resolution (9.9), we obtain on $H = P_{P(V^\vee)}(W)$ a distinguished triangle

$$
(h^* \mathcal{F} \otimes S_+) \otimes \text{Cliff}_0(W) (\mathcal{O}(-H) \otimes \text{Cliff}_0(W)) \to (h^* \mathcal{F} \otimes S_+) \otimes \text{Cliff}_0(W) \mathcal{O}(\text{Cliff}_1(W)) \to i^*(\Psi(\mathcal{F}))
$$

(9.17)

with the first map induced by the Clifford multiplication. The first term is evidently isomorphic to $h^* \mathcal{F} \otimes S_+ \mathcal{O}(-H)$. For the second note that

$$
S_+ \otimes \text{Cliff}_0(W) \mathcal{Cliff}_1(W) \cong S_+ \otimes \text{Cliff}_0(W) (\mathcal{Cliff}_0(W) \otimes \text{Cliff}_0(V) \mathcal{Cliff}_1(V))
$$

$$
\cong S_+ \otimes \text{Cliff}_0(V) \mathcal{Cliff}_1(V)
$$

$$
\cong S_-
$$

(9.18)

where the last isomorphism follows from the standard isomorphisms

$$
\text{Cliff}_0(V) \cong \text{End}(S_+) \oplus \text{End}(S_-) \quad \text{and} \quad \text{Cliff}_1(V) \cong \text{Hom}(S_+, S_-) \oplus \text{Hom}(S_-, S_+).
$$

Hence the second term in (9.17) is isomorphic to $h^* \mathcal{F} \otimes S_-$. Thus, we can rewrite (9.17) as

$$
h^* \mathcal{F} \otimes S_+ \mathcal{O}(-H) \to h^* \mathcal{F} \otimes S_- \to i^*(\Psi(\mathcal{F}))
$$

(9.19)

with the first map induced by the Clifford multiplication.

On the other hand, on $P(V)$ we have exact sequences (9.2). Pulling the second of them back via $\pi : H \to P(V)$, using the base change isomorphism for the square in diagram (9.16), and tensoring by $h^* \mathcal{F}$, we obtain a distinguished triangle

$$
h^* \mathcal{F} \otimes S_+ \mathcal{O}(-H) \to h^* \mathcal{F} \otimes S_- \to i^*(h^*_Q \mathcal{F} \otimes \pi_Q^*(S_+(H)))
$$

with the first map induced by the Clifford multiplication. It follows from the above that this triangle agrees with (9.19). In summary, we conclude that the second component of the decomposition (9.15) is embedded via the functor

$$
\Psi : \text{Perf}(P(V^\vee)) \to \text{Perf}(H(Q)), \quad \mathcal{F} \mapsto h^*_Q \mathcal{F} \otimes \pi_Q^*(S_+(H)).
$$

(9.20)
Step 6: An equivalence between $\text{Perf}(Q')$ and $\text{Perf}(Q)^3$. By Lemma 2.2.2, we have a $P(V')$-linear semiorthogonal decomposition

$$\text{Perf}(H(Q)) = \langle \text{Perf}(Q)^5, Q_1(H) \otimes \text{Perf}(P(V')) \rangle,$$

$$Q_2(2H) \otimes \text{Perf}(P(V')) \otimes \ldots \otimes Q_{2d-1}((2d-1)H) \otimes \text{Perf}(P(V')) \rangle,$$

where $Q_1 = \langle s_+, 0 \rangle$ and $Q_2 = \cdots = Q_{2d-1} = \langle 0 \rangle$ are the Lefschetz components given by Lemma 9.2. Comparing with the decomposition (9.15) and using (9.20), we conclude there is a $P(V')$-linear equivalence

$$\text{Perf}(Q') \cong \text{Perf}(Q)^3.$$

This equivalence is induced by the functor

$$\Phi = \gamma \circ \phi \circ \mu \circ \alpha_Q : \text{Perf}(Q') \to \text{Perf}(H(Q)),$$

which is given by

$$\mathcal{F} \mapsto h^*_Q \left( (j_+ \mathcal{F} \otimes \chi) \otimes S^\vee_Z \right)^{Z/2} \otimes_{\text{Cliff(W)}} \mathcal{E} \cong h^*_Q g_* \left( \mathcal{F} \otimes (j^* S^\vee_Z \otimes \chi)^{Z/2} \right) \otimes_{\text{Cliff(W)}} \mathcal{E},$$

(9.21)

where $g : Q' \to P(V')$ is the inclusion of the branch divisor of $\zeta : Z \to P(V')$. It remains to check that $\Phi$ is a Lefschetz equivalence.

Step 7: Fourier–Mukai kernel for the functor $\Phi$. The kernel for the $P(V')$-linear functor $\Phi$ will be an object in the derived category of the variety $H(Q, Q')$ defined by the cartesian square

$$\begin{array}{ccc}
H(Q, Q') & \xrightarrow{\hat{h}} & Q' \\
\downarrow \hat{g} & & \downarrow g \\
H(Q) & \xrightarrow{h_Q} & P(V')
\end{array}$$

Note that by definition the bundle $j^* S^\vee_Z$ from (9.21) is the cokernel of the natural map

$$(s_+ \oplus s_-) \otimes g^* Z_1 \to (s_+ \oplus s_-) \otimes 0$$

induced by the action of $Z_1 \subset \text{Cliff}_1(W) \subset \text{Cliff}_1(V) \otimes 0$ on $(s_+ \oplus s_-) \otimes 0$. Since this action swaps the grading and $Z_1 \cong \text{det}(W) \cong \mathcal{O}(-H')$, it follows that on $P(V')$ we have two exact sequences

$$0 \to s_- \otimes \mathcal{O}(-H') \to s_+ \otimes \mathcal{O} \to g_*(j^* S^\vee_Z)^{Z/2} \to 0,$$

$$0 \to s_+ \otimes \mathcal{O}(-H') \to s_- \otimes \mathcal{O} \to g_*(j^* S^\vee_Z \otimes \chi)^{Z/2} \to 0,$$

with the first morphisms given by the Clifford multiplication. Comparing these sequences with (9.2), for the spinor bundles $s'_+ and s'_-$ on $Q' \subset P(V')$, we obtain isomorphisms

$$(j^* S^\vee_Z)^{Z/2} \cong s'_-(H'),$$

$$j^* S^\vee_Z \otimes \chi)^{Z/2} \cong s'_+(H').$$

Combining this with the formula (9.21) for $\Phi$, we find that $\Phi$ can be rewritten as follows:

$$\Phi(\mathcal{F}) \cong h^*_Q g_* \left( \mathcal{F} \otimes s'_+(H') \right) \otimes_{\text{Cliff}_0(W)} \mathcal{E} \cong \tilde{g}_* \left( \hat{h}^* \mathcal{F} \otimes (\hat{h}^* s'_+(H') \otimes_{\text{Cliff}_0(W)} \tilde{g}^* \mathcal{E}) \right).$$

Therefore, the Fourier–Mukai kernel for this functor is given by the object

$$\mathcal{K} = \hat{h}^* s'_+(H') \otimes_{\text{Cliff}_0(W)} \tilde{g}^* \mathcal{E} \in \text{Perf}(H(Q, Q')).$$
Step 8: Rewriting the kernel $K$. Consider the commutative diagram

$$
\begin{array}{ccc}
H(Q, Q^\vee) & \xrightarrow{\tilde{i}} & H(P(V), Q^\vee) \\
\downarrow{\tilde{g}} & & \downarrow{g_H} \\
H(Q) & \xrightarrow{i} & H
\end{array}
$$

with cartesian square. Since $\tilde{h} = pr_2 \circ \tilde{i}$, using the projection formula and base change we compute

$$\tilde{i}_* K \cong pr_2^* (S'_+(H')) \otimes_{\text{Cliff}(W)} \tilde{g}_* \tilde{g}^* E \cong pr_2^* (S'_+(H')) \otimes_{\text{Cliff}(W)} g_H^* (i_* E).$$

Using the resolution (9.9) of $i_* E$ and taking into account that $S'_+ \otimes_{\text{Cliff}(W)} \text{Cliff}(W) \cong S'_-$ (which follows from (9.18) and the resolutions (9.2) for $S'_+$), we obtain an exact sequence

$$0 \to pr_1^* \mathcal{O}_{P(V)}(-H) \otimes pr_2^* S'_+(H') \to pr_1^* \mathcal{O}_{P(V)} \otimes pr_2^* S'_-(H') \to \tilde{i}_* K \to 0$$
on $H(P(V), Q^\vee)$, where $pr_1$ stands for the projection to $P(V)$, and the first map is induced by the Clifford multiplication. By (9.2) we have a surjection $S_- \otimes \mathcal{O}_{Q^\vee} \to S'_+(H')$, and the composition

$$S_- \otimes pr_1^* \mathcal{O}_{Q}(-H) \otimes pr_2^* \mathcal{O}_{Q^\vee} \to pr_1^* \mathcal{O}_{Q}(-H) \otimes pr_2^* S'_+(H') \to pr_1^* \mathcal{O}_{Q} \otimes pr_2^* S'_-(H')$$
can be rewritten as

$$S_- \otimes pr_1^* \mathcal{O}_{Q}(-H) \otimes pr_2^* \mathcal{O}_{Q^\vee} \to S_+ \otimes pr_1^* \mathcal{O}_{Q} \otimes pr_2^* \mathcal{O}_{Q^\vee} \to pr_1^* \mathcal{O}_{Q} \otimes pr_2^* S'_-(H'),$$

where the first map is induced by the Clifford multiplication $S_- \otimes \mathcal{O}_{Q}(-H) \to S_+ \otimes \mathcal{O}_{Q}$. By (9.1) the image of this map is the spinor bundle $S_+$ on $Q$, hence we obtain an exact sequence

$$pr_1^* S_+ \otimes pr_2^* \mathcal{O}_{Q^\vee} \to pr_1^* \mathcal{O}_{Q} \otimes pr_2^* S'_-(H') \to K \to 0.$$n

Therefore, we have on $Q \times Q^\vee$ an exact sequence

$$0 \to S_+ \otimes \mathcal{O}_{Q^\vee} \to \mathcal{O}_{Q} \otimes S'_-(H') \to \delta_* K \to 0,$$

where $\delta: H(Q, Q^\vee) \to Q \times Q^\vee$ is the natural embedding.

Step 9: Lefschetz equivalence. Finally, we use the criterion of [32 Remark 8.3] to prove

$$\Phi: \text{Perf}(Q^\vee) \to \text{Perf}(H(Q))$$

induces a Lefschetz equivalence $\text{Perf}(Q^\vee) \simeq \text{Perf}(Q)$. By our work above, we must check that $\Phi^* \circ \pi_Q^*: \text{Perf}(Q) \to \text{Perf}(Q^\vee)$, where $\pi_Q: H(Q) \to Q$ is the projection, is fully faithful on the right twisted primitive components of $\text{Perf}(Q)$ (as defined in [32 (6.7)]), and takes them to the left primitive components of $\text{Perf}(Q^\vee)$. For this, note that $\pi_Q^* \circ \Phi^*: \text{Perf}(Q^\vee) \to \text{Perf}(Q)$ is the Fourier–Mukai functor given by the kernel $\delta_* K \in \text{Perf}(Q \times Q^\vee)$, so the left adjoint $\Phi^* \circ \pi_Q^*$ is given by the kernel $(\delta_* K)^\vee \otimes pr_1^* \omega_Q$, where $pr_1: Q \times Q^\vee \to Q$ is the projection. By (9.22) we have a distinguished triangle

$$(\delta_* K)^\vee \otimes pr_1^* \omega_Q \to \omega_Q \otimes S'_-(H') \to (S'_+ \otimes \omega_Q) \otimes \mathcal{O}_{Q^\vee}.$$
Using this, it follows that $\Phi^+ \circ \pi_Q^+$ takes the generators $\mathcal{O}_Q$ and $R_{\mathcal{O}_Q}(S_+)$ of the Lefschetz center of $\text{Perf}(Q)$ to the generators $S_Q^\vee(-H')$ and $\mathcal{O}_{Q'}$ (up to shift) of an appropriate Lefschetz center of $\text{Perf}(Q')$. Moreover, it also follows that the generators $R_{\mathcal{O}_Q}(S_+)$ and $R_{\mathcal{O}_Q}(S_+)(\mathcal{O}_Q)$ of the twisted right primitive components of $\text{Perf}(Q)$ go to the generators $\mathcal{O}_{Q'}$ and $S_Q^\vee$ (up to shift again) of the left primitive components of this Lefschetz center. This completes the proof of case \([1]\) of Theorem \([9.5]\).

**Step 10: Other types of quadrics.** We deduce the other cases of Theorem \([9.5]\) by using Proposition \([2.32]\). Let $Q \subset \mathbb{P}(V)$ be a smooth even-dimensional quadric hypersurface, so that by case \([1]\) of Theorem \([9.5]\) proved in Steps 1–9 there is a Lefschetz equivalence

$$\text{(Perf}(Q)/\mathbb{P}(\tilde{V}))^\natural \simeq \text{Perf}(Q').$$

Choose a point in $\mathbb{P}(\tilde{V}) \setminus Q$, let $K \subset \tilde{V}$ be the corresponding one-dimensional subspace, and set $V = \tilde{V}/K$. Then the composition of the embedding $Q \hookrightarrow \mathbb{P}(\tilde{V})$ with the linear projection $\mathbb{P}(\tilde{V}) \dashrightarrow \mathbb{P}(V)$ is a standard map of the quadric $Q$ which is a double covering (and any such standard map can be realized in this way). By Proposition \([2.32]\) we conclude that

$$\text{(Perf}(Q)/\mathbb{P}(V))^\natural \simeq (\text{Perf}(Q)/\mathbb{P}(\tilde{V}))^\natural \otimes_{\text{Perf}(\mathbb{P}(\tilde{V}^\vee))} \text{Perf}(\mathbb{P}(V'))$$

$$\simeq \text{Perf}(Q') \otimes_{\text{Perf}(\mathbb{P}(\tilde{V}^\vee))} \text{Perf}(\mathbb{P}(V'))$$

$$\simeq \text{Perf}(Q' \times_{\mathbb{P}(\tilde{V}^\vee)} \mathbb{P}(V')).$$

Note that $Q' \times_{\mathbb{P}(\tilde{V}^\vee)} \mathbb{P}(V')$ is projectively dual to the branch divisor of $Q \rightarrow \mathbb{P}(V)$. Moreover, the above equivalence is a Lefschetz equivalence. Indeed, this follows from Remark \([2.34]\) and the fact that the restriction of a spinor bundle on an even-dimensional quadric to a hyperplane section is again a spinor bundle. This proves case \([3]\) of Theorem \([9.5]\) and by duality also case \([2]\).

Similarly, let $Q \subset \mathbb{P}(\tilde{V})$ be a smooth odd-dimensional quadric hypersurface, so that by case \([2]\) of Theorem \([9.5]\) proved above there is a Lefschetz equivalence

$$\text{(Perf}(Q)/\mathbb{P}(\tilde{V}))^\natural \simeq \text{Perf}((Q')_{\text{cov}})$$

where $(Q')_{\text{cov}} \rightarrow \mathbb{P}(\tilde{V}^\vee)$ is the double cover branched along the projective dual $Q' \subset \mathbb{P}(\tilde{V}^\vee)$. Choose a point in $\mathbb{P}(\tilde{V}) \setminus Q$, let $K \subset \tilde{V}$ be the corresponding one-dimensional subspace, and set $V = \tilde{V}/K$. Then the composition of the embedding $Q \hookrightarrow \mathbb{P}(\tilde{V})$ with the linear projection $\mathbb{P}(\tilde{V}) \dashrightarrow \mathbb{P}(V)$ is a standard map of the quadric $Q$ which is a double covering. By Proposition \([2.32]\) we conclude that

$$\text{(Perf}(Q)/\mathbb{P}(V))^\natural \simeq (\text{Perf}(Q)/\mathbb{P}(\tilde{V}))^\natural \otimes_{\text{Perf}(\mathbb{P}(\tilde{V}^\vee))} \text{Perf}(\mathbb{P}(V'))$$

$$\simeq \text{Perf}((Q')_{\text{cov}} \otimes_{\text{Perf}(\mathbb{P}(\tilde{V}^\vee))} \text{Perf}(\mathbb{P}(V'))$$

$$\simeq \text{Perf}((Q')_{\text{cov}} \times_{\mathbb{P}(\tilde{V}^\vee)} \mathbb{P}(V')).$$

Note that $(Q')_{\text{cov}} \times_{\mathbb{P}(\tilde{V}^\vee)} \mathbb{P}(V')$ is projectively dual to the branch divisor of $Q \rightarrow \mathbb{P}(V)$, and $(Q')_{\text{cov}} \times_{\mathbb{P}(\tilde{V}^\vee)} \mathbb{P}(V') \rightarrow \mathbb{P}(V')$ is the double cover branched along $(Q')_{\text{cov}} \times_{\mathbb{P}(\tilde{V}^\vee)} \mathbb{P}(V')$. Again, using Remark \([2.34]\) we see that the above equivalence is a Lefschetz equivalence. This proves case \([4]\) of Theorem \([9.5]\). \(\square\)
9.2. Standard categorical resolutions of quadrics. In this subsection, we will obtain a categorical resolution of a singular quadric by expressing it as a cone over a smooth quadric, and then taking a categorical cone.

More precisely, let \( f : Q \to \mathbf{P}(V) \) be a standard morphism of a quadric \( Q \) which is not necessarily smooth. We denote by \( \langle Q \rangle \) the linear span of \( f(Q) \) in \( \mathbf{P}(V) \), and by \( i : \langle Q \rangle \hookrightarrow \mathbf{P}(V) \) its inclusion. Then \( f \) factors as \( i \circ f_0 \), where \( f_0 : Q \to \langle Q \rangle \) is either an embedding of \( Q \) as a quadric hypersurface, or a double cover branched along a quadric hypersurface in \( \langle Q \rangle \). We say \( f \) is of embedding type in the first case, and of covering type in the second.

The morphism \( f_0 : Q \to \langle Q \rangle \) can be described more explicitly as follows. Write \( \langle Q \rangle = \mathbf{P}(W) \) for a linear subspace \( W \subset V \), and write \( Q \subset \mathbf{P}(\hat{W}) \) as a quadric hypersurface. Then the morphism \( f_0 \) corresponds to a surjective linear map \( \hat{f} : \hat{W} \to W \), such that:

1. If \( f \) is of embedding type, then \( \hat{f} \) is an isomorphism.
2. If \( f \) is of covering type, then \( \ker(\hat{f}) \) is 1-dimensional.

The map \( \hat{f} \) induces a rational map \( \mathbf{P}(\hat{W}) \dashrightarrow \mathbf{P}(W) \) (which is actually a regular isomorphism in case \([1]\)), and the morphism \( f_0 \) is the composition

\[
f_0 : Q \to \mathbf{P}(\hat{W}) \dashrightarrow \mathbf{P}(W).
\]

We can express \( Q \) in terms of a cone as follows. The embedding \( Q \subset \mathbf{P}(\hat{W}) \) corresponds to a quadric form on \( \hat{W} \) (defined up to nonzero scalars), whose kernel we denote by \( \bar{K} = \ker(Q) \). There is an induced quadratic form on \( \hat{W}/\bar{K} \), which defines a smooth quadric

\[
\bar{Q} \subset \mathbf{P}(\hat{W}/\bar{K})
\]

that we call the base quadric of \( Q \). Then \( Q \) is the cone over \( \bar{Q} \) with vertex \( \mathbf{P}(\bar{K}) \), i.e.

\[
Q = \mathbf{C}(\bar{Q}) \subset \mathbf{P}(\hat{W}). \tag{9.23}
\]

Note that \( \hat{f} : \hat{W} \to W \) is injective on \( \bar{K} \); we denote its image by \( K \) and define a space \( \hat{W} \) by the exact sequence

\[
0 \to K \to W \to \hat{W} \to 0.
\]

The map \( \hat{f} : \hat{W} \to W \) induces a surjection \( \hat{W}/\bar{K} \to \hat{W} \), whose kernel is identified with \( \ker(\hat{f}) \).

There is a standard morphism \( \bar{Q} \to \mathbf{P}(W) \) given as the composition

\[
\bar{Q} \hookrightarrow \mathbf{P}(\hat{W}/\bar{K}) \dashrightarrow \mathbf{P}(\hat{W}),
\]

which has the same type (embedding or covering) as \( f \).

**Definition 9.6.** Let \( f : Q \to \mathbf{P}(V) \) be a standard morphism of a quadric. Using the above notation, the standard categorical resolution of \( Q \) over \( \mathbf{P}(V) \) is the Lefschetz category \( \mathcal{Q} \) over \( \mathbf{P}(V) \) defined as the categorical cone over the base quadric \( \bar{Q} \):

\[\mathcal{Q} = \mathcal{C}(\bar{Q})/\mathbf{P}(V).\]

Below we will explicitly describe the Lefschetz components of \( Q \). First we define some useful numerical invariants of a standard morphism of a quadric.

**Definition 9.7.** Let \( f : Q \to \mathbf{P}(V) \) be a standard morphism of a quadric. Then:

- \( r(Q) \) denotes the rank of \( Q \), i.e. the rank of the quadratic form corresponding to \( Q \).
- \( p(Q) \in \{0, 1\} \) denotes the parity of \( r(Q) \), i.e. \( p(Q) = r(Q) \pmod{2} \).
- \( k(Q) = \dim \ker(Q) \).
• $c(f)$ denotes the codimension of $(Q) \subset \mathbf{P}(V)$.
• $t(f) \in \{0, 1\}$ denotes the type of $Q$, defined by

$$t(f) = \begin{cases} 
0 & \text{if } f \text{ is of embedding type,} \\
1 & \text{if } f \text{ is of covering type.}
\end{cases}$$

Note that our convention that $Q$ is integral is equivalent to $r(Q) \geq 3$.

**Remark 9.8.** As indicated by the notation, $r(Q)$, $p(Q)$, and $k(Q)$ depend only on $Q$, while $c(f)$ and $t(f)$ are invariants of the morphism $f$. Moreover, if $Q$ is the base quadric of $Q$, we have $r(Q) = r(Q)$ and $p(Q) = p(Q)$. We also note the relations

$$\dim(Q) = r(Q) + k(Q) - 2, \quad \text{(9.24)}$$

$$\dim(V) = r(Q) + k(Q) + c(f) - t(f). \quad \text{(9.25)}$$

**Lemma 9.9.** Let $f : Q \to \mathbf{P}(V)$ be a standard morphism of a quadric. Let $S$ be a spinor bundle on the base quadric of $Q$. Then the standard categorical resolution $\mathbf{Q}$ of $Q$ over $\mathbf{P}(V)$ is smooth and proper over $\mathbf{k}$, with a strong, moderate Lefschetz structure of length $\dim(Q)$. If $k = k(Q)$ and $p = p(Q)$, then its nonzero Lefschetz components are given by

$$Q_i = \begin{cases} 
\langle S, 0 \rangle & \text{for } |i| \leq k + 1 - p, \\
\langle 0 \rangle & \text{for } k + 1 - p < |i| \leq \dim(Q) - 1,
\end{cases}$$

and its nonzero primitive Lefschetz components are given by

$$q_i = \begin{cases} 
\langle 0 \rangle & \text{if } i = -(\dim(Q) - 1), \\
\langle S' \rangle & \text{if } i = -(k + 1 - p), \\
\langle S \rangle & \text{if } i = k + 1 - p, \\
\langle 0 \rangle & \text{if } i = \dim(Q) - 1,
\end{cases}$$

where $S'$ is described in Remark 9.4.

**Proof.** Combine Theorem 7.20, Lemma 7.13, Lemma 9.2, and the formula (9.24). □

**Remark 9.10.** In Lemma 9.9, we have tacitly identified the objects $\mathcal{O}$, $S$, and $S'$ on the base quadric of $Q$ with their pullbacks to $Q$.

The following lemma relates standard categorical resolutions of quadrics to geometry. Note that the singular locus of a quadric $Q$ is given by $\text{Sing}(Q) = \mathbf{P}(\ker(Q))$.

**Lemma 9.11.** Let $f : Q \to \mathbf{P}(V)$ be a standard morphism of a quadric. Let $\mathbf{Q}$ be the standard categorical resolution of $Q$ over $\mathbf{P}(V)$. Let $U = \mathbf{P}(V) \setminus f(\text{Sing}(Q))$.

1. There is a $\mathbf{P}(V)$-linear functor $\text{Perf}(Q) \to \mathbf{Q}$ whose base change to $U$ gives an equivalence $\mathbf{Q}_U \simeq \text{Perf}(\mathbf{Q}_U)$.

2. Let $\mathcal{A}$ be a $\mathbf{P}(V)$-linear category supported over $U$ (see Definition A.15). Then there is an equivalence

$$\mathcal{A} \otimes_{\text{Perf}(\mathbf{P}(V))} \mathbf{Q} \simeq \mathcal{A} \otimes_{\text{Perf}(\mathbf{P}(V))} \text{Perf}(Q).$$

In particular, if $\mathcal{A} = \text{Perf}(X)$ for a scheme $X$ over $\mathbf{P}(V)$ supported over $U$, then

$$\mathcal{A} \otimes_{\text{Perf}(\mathbf{P}(V))} \mathbf{Q} \simeq \text{Perf}(X \times_{\mathbf{P}(V)} Q).$$
Proof. Part (1) follows from Proposition 7.16 once we note that the morphism $\tilde{C}_K(\bar{Q}) \to \mathbb{P}(V)$ factorizes as

$$\tilde{C}_K(\bar{Q}) \to C_K(\bar{Q}) = Q \overset{f}{\to} \mathbb{P}(V),$$

where the first map is the blowup in $\mathbb{P}(K) = \text{Sing}(Q)$. Part (2) follows from Lemma A.17 and part (1). □

9.3. Generalized quadratic duality and HPD. Our goal is to define a geometric duality operation on standard morphisms of quadrics, which after passing to standard categorical resolutions corresponds to the operation of taking the HPD category.

The desired duality operation will be defined using a combination of the following three operations.

**Definition 9.12.** Let $f: Q \to \mathbb{P}(V)$ be a standard morphism of a quadric.

- If $f: Q \to \mathbb{P}(V)$ is of embedding type, we denote by
  $$f^\vee: Q^\vee \to \mathbb{P}(V^\vee)$$
  the embedding of the classical projective dual of $Q \subset \mathbb{P}(V)$.

- If $f$ is of embedding type, we define
  $$f_{\text{cov}}: Q_{\text{cov}} \to \mathbb{P}(V)$$
  as the composition of the double cover $Q_{\text{cov}} \to \langle Q \rangle$ branched along $Q \subset \langle Q \rangle$ with the embedding $\langle Q \rangle \hookrightarrow \mathbb{P}(V)$.

- If $f$ is of covering type, we define
  $$f_{\text{br}}: Q_{\text{br}} \to \mathbb{P}(V)$$
  as the composition of the inclusion $Q_{\text{br}} \hookrightarrow \langle Q \rangle$ of the branch divisor of the double cover $Q \to \langle Q \rangle$ with the embedding $\langle Q \rangle \hookrightarrow \mathbb{P}(V)$.

**Remark 9.13.** All of the above operations preserve the integrality of $Q$, except for the branch divisor operation in case $r(Q) = 3$ and $f$ is a morphism of covering type. Indeed, this follows from the formulas:

$$r(Q^\vee) = r(Q), \quad r(Q_{\text{cov}}) = r(Q) + 1, \quad r(Q_{\text{br}}) = r(Q) - 1.$$

Note, however, that the above operations are defined even for non-integral quadrics.

The next definition is modeled on the cases considered in Theorem 9.5.

**Definition 9.14.** Let $f: Q \to \mathbb{P}(V)$ be a standard morphism of a quadric. The *generalized dual* of $f$ is the standard morphism

$$f^\natural: Q^\natural \to \mathbb{P}(V^\natural)$$

of the quadric $Q^\natural$ defined as follows:

- If $f: Q \to \mathbb{P}(V)$ is of embedding type, then:
  - If $r(Q)$ is even, we set $Q^\natural = Q^\vee$ and $f^\natural = f^\vee: Q^\natural \to \mathbb{P}(V^\natural)$.
  - If $r(Q)$ is odd, we set $Q^\natural = (Q^\vee)_{\text{cov}}$ and $f^\natural = (f^\vee)_{\text{cov}}: Q^\natural \to \mathbb{P}(V^\natural)$.

- If $f: Q \to \mathbb{P}(V)$ is of covering type, then:
  - If $r(Q)$ is even, we set $Q^\natural = (Q_{\text{br}})^\vee$ and $f^\natural = (f_{\text{br}})^\vee: Q^\natural \to \mathbb{P}(V^\natural)$.
  - If $r(Q)$ is odd, we set $Q^\natural = ((Q_{\text{br}})^\vee)_{\text{cov}}$ and $f^\natural = ((f_{\text{br}})^\vee)_{\text{cov}}: Q^\natural \to \mathbb{P}(V^\natural)$.
In other words, we first pass to a morphism of the embedding type (by taking the branch divisor if necessary), then apply projective duality, and then if necessary go to the double covering.

**Remark 9.15.** Generalized duality affects the numerical invariants of \( f \) as follows:
\[
r(Q^2) = r(Q) + p(Q) - t(f), \quad p(Q^2) = t(f), \quad k(Q^2) = c(f), \quad c(f^2) = k(Q), \quad t(f^2) = p(Q).
\]
In particular, note that generalized duality preserves the integrality of \( Q \). Note also that by [9.24] we have
\[
dim(Q^2) = r(Q) + p(Q) + c(f) - t(f) - 2.
\]

**Remark 9.16.** By [9.24], [9.26], and [9.25] we have
\[
dim(Q^2) + dim(Q) = (r(Q) + k(Q) + c(f) - t(f)) + (r(Q) + p(Q) - 4)
\]
\[
= dim(V) + (r(Q) + p(Q) - 4),
\]
which is congruent to \( \dim(V) \) mod 2 since by definition \( p(Q) \equiv r(Q) \) mod 2. This means that if \( \dim(V) \) is even, then the parities of the dimensions of \( Q^2 \) and \( Q \) are the same, and if \( \dim(V) \) is odd, then the parities are opposite.

Now we can bootstrap from Theorem 9.5 to a result for arbitrary standard morphisms.

**Theorem 9.17.** Let \( f: Q \to P(V) \) be a standard morphism of a quadric. Let \( Q \) be the standard categorical resolution of \( Q \) over \( P(V) \). Then the HPD category \( Q^2 \) is Lefschetz equivalent to the standard categorical resolution of \( Q^2 \) over \( P(V^\vee) \).

**Proof.** Follows from Theorem 9.5, Theorem 8.1, and the definitions. \( \square \)

### 9.4. A quadratic HPD theorem.
Combining our results on HPD for quadrics with the nonlinear HPD theorem (Theorem 5.3), we obtain the following.

**Theorem 9.18.** Let \( A \) be a right strong, moderate Lefschetz category over \( P(V) \). Let
\[
f: Q \to P(V) \quad \text{and} \quad f^2: Q^2 \to P(V^\vee)
\]
be a standard map of a quadric and its generalized dual. Let \( Q \) be the standard categorical resolution of \( Q \) over \( P(V) \), and let \( Q^2 \) be the standard categorical resolution of \( Q^2 \) over \( P(V^\vee) \).

Let \( S \in Q \) and \( S^2 \in Q^2 \) be the pullbacks of spinor bundles on the base quadrics of \( Q \) and \( Q^2 \).

Let \( H \) and \( H' \) denote the hyperplane classes on \( P(V) \) and \( P(V^\vee) \). Let
\[
N = \dim(V), \quad m = \text{length}(A), \quad n = \text{length}(A^2), \quad d = \dim(Q), \quad d^2 = \dim(Q^2).
\]
Then there are semiorthogonal decompositions
\[
A \otimes_{\text{Perf}(P(V))} Q = \left\langle X_Q(A), \quad A_{d^2}(H) \otimes \langle S \rangle, \ldots, A_{m-1}(m - d^2)H \otimes \langle S \rangle, \quad A_{N-d}(H) \otimes \langle 0 \rangle, \ldots, A_{m-1}(m + d - N)H \otimes \langle 0 \rangle \right\rangle.
\]
\[
A^2 \otimes_{\text{Perf}(P(V^\vee))} Q^2 = \left\langle A_{d^2-1-n}((N - d^2 - n)H') \otimes \langle 0 \rangle, \ldots, A_{d^2-n}(-H') \otimes \langle 0 \rangle, \quad A_{d^2-n}((d - n)H') \otimes \langle (S^2)^\vee \rangle, \ldots, A_{d^2-n}(-H') \otimes \langle (S^2)^\vee \rangle, \quad X'_{Q^2}(A^2) \right\rangle.
\]
Proof. Let $\mathcal{J}_i = \mathcal{J}(A^i, Q)_i$ and $\mathcal{J}^j = \mathcal{J}(A^j, Q^j)_j$ be the Lefschetz components of the jointed joins, which have lengths equal to

$$\text{length}(A) + \text{length}(Q) = m + d \quad \text{and} \quad \text{length}(A^j) + \text{length}(Q^j) = n + d^j$$

respectively. Then Theorem 5.5 gives semiorthogonal decompositions of the form

$$\mathcal{A} \otimes \text{Perf}(\mathcal{P}(V')) = \langle \mathcal{K}_Q(A), \mathcal{J}_N(H), \ldots, \mathcal{J}_{m+d-1}((m + d - N)H) \rangle,$$

and an equivalence $\mathcal{K}_Q(A) \simeq \mathcal{K}'_Q(A^2)$. By Lemmas 3.24 and 9.1 for $i \geq N$ we have

$$\mathcal{J}_i = \langle A_{i-k+p-2} \otimes \langle S \rangle, A_{i-d} \otimes \langle 0 \rangle \rangle \subset \mathcal{A} \otimes \text{Perf}(\mathcal{P}(V')) Q.$$

Combined with the observation that $k - p + 2 = N - d^j$ by (9.25) and (9.26), it follows that the semiorthogonal decomposition (9.27) takes the claimed form. Using the expression for the numerical invariants of $Q^j$ in terms of those of $Q$ (Remark 9.15), it follows similarly that the semiorthogonal decomposition (9.28) takes the claimed form.

It is natural to combine Theorem 9.18 with the result of Lemma 9.11(2) that provides the left hand sides of the semiorthogonal decompositions with a clear geometric meaning. We give some sample applications below.

9.5. Duality of Gushel–Mukai varieties. We will prove [24, Conjecture 3.7] on the duality of Gushel–Mukai varieties. We refer to [24] for the context of this conjecture, and in particular its relation to rationality questions.

Using the terminology of Definition 9.1, the definition of this class of varieties from [8] can be rephrased as follows. Recall that $V^5$ denotes a 5-dimensional vector space over $k$.

**Definition 9.19.** A smooth Gushel–Mukai (GM) variety is a smooth dimensionally transverse fiber product

$$X = \text{Gr}(2, V^5) \times_{\text{P}(\wedge^2 V^5)} Q,$$

where $V^5$ is a 5-dimensional vector space, $\text{Gr}(2, V^5) \rightarrow \text{P}(\wedge^2 V^5)$ is the Plücker embedding of the Grassmannian of 2-dimensional subspaces of $V^5$, and $Q \rightarrow \text{P}(\wedge^2 V^5)$ is a standard morphism of a quadric.

It is thus natural to apply Theorem 9.18 to GM varieties. Recall that the Grassmannian $\text{Gr}(2, V^5)$ is homologically projectively self-dual by Theorem 6.1 whose notation we use below.

**Theorem 9.20.** Let

$$X = \text{Gr}(2, V^5) \times_{\text{P}(\wedge^2 V^5)} Q \quad \text{and} \quad Y = \text{Gr}(2, V^5) \times_{\text{P}(\wedge^2 V^5)} Q^j$$

be smooth GM varieties of dimensions $d_X \geq 2$ and $d_Y \geq 2$, where $Q \rightarrow \text{P}(\wedge^2 V^5)$ is a standard morphism of a quadric and $Q^j \rightarrow \text{P}(\wedge^2 V^5)$ is its generalized dual. Let $U_X$ and $U_Y$ denote the pullbacks of $U$ and $U'$ to $X$ and $Y$, and let $O_X(1)$ and $O_Y(1)$ denote the pullbacks of the $O(1)$ line bundles on $\text{P}(\wedge^2 V^5)$ and $\text{P}(\wedge^2 V^5)$.

Then there are semiorthogonal decompositions

$$\text{Perf}(X) = \langle \mathcal{K}(X), O_X(1), U_X(1), \ldots, O_X(d_X - 2), U_X(d_X - 2) \rangle,$$

and an equivalence $\mathcal{K}(X) \simeq \mathcal{K}'(Y)$. 

$$\text{Perf}(Y) = \langle U_Y(2 - d_Y), O_Y(2 - d_Y), \ldots, U_Y(-1), O_Y(-1), \mathcal{K}'(Y) \rangle,$$
Proof. This is a combination of Theorem 9.18, Theorem 6.1, and Lemma 9.11(2). Indeed, the smoothness of $X$ and $Y$ implies that the Grassmannians in (9.29) do not intersect the singular loci of the quadrics, so by Lemma 9.11(2) we have

$$\text{Perf} (\text{Gr}(2, V_5)) \otimes_{\text{Perf}(P(\wedge^2 V_5))} Q \simeq \text{Perf}(X),$$
$$\text{Perf} (\text{Gr}(2, V_5')) \otimes_{\text{Perf}(P(\wedge^2 V_5'))} Q^2 \simeq \text{Perf}(Y).$$

We just need to show the semiorthogonal decompositions of Theorem 9.18 take the prescribed form.

The codimension of $\text{Gr}(2, V_5)$ in $P(\wedge^2 V_5)$ is 3, so by dimensional transversality

$$\text{dim}(Q^2) = d_X + 3 \geq 5.$$

Since the length of the Lefschetz decomposition of $\text{Perf}(\text{Gr}(2, V_5'))$ is $m = 5$, it follows that $\mathcal{S}$ does not show up in the semiorthogonal decomposition of $\text{Perf}(X)$. The same argument shows that $(\mathcal{S}^\vee)$ does not show up in the decomposition of $\text{Perf}(Y)$. Similarly,

$$\text{dim}(\wedge^2 V_5) - \text{dim}(Q) = 10 - (d_X + 3) = 7 - d_X.$$

Since the nonzero Lefschetz components of $\text{Perf}(\text{Gr}(2, V_5'))$ are given by $A_i = \langle O, U_i^\vee \rangle$ for $|i| \leq 4$, it follows that $\langle O_X, U_i^\vee \rangle$ appears $d_X - 2$ times in the decomposition of $\text{Perf}(X)$. The same argument shows that $\langle U_Y, O_Y \rangle$ appears $d_Y - 2$ times in the decomposition of $\text{Perf}(Y)$. Hence the semiorthogonal decompositions of Theorem 9.18 take the prescribed form.

Corollary 9.21 ([24, Conjecture 3.7]). Smooth generalized dual GM varieties have equivalent GM categories.

Proof. Any pair of smooth generalized dual GM varieties $X$ and $Y$ can be obtained as in Theorem 9.20 from an appropriate pair of generalized dual quadrics, cf. [8, Proposition 3.28]. Twisting the decomposition (9.30) by $O_X(-1)$ shows that $\mathcal{K}(X)$ is equivalent to the GM category of $X$, as defined in [24, Definition 2.5]. On the other hand, twisting the decomposition (9.31) by $O_Y(1)$ and using [24, (2.20) and (2.21)] shows that $\mathcal{K}(Y)$ is equivalent to the GM category of $Y$.

9.6. Duality of spin GM varieties. The self-duality of $\text{Gr}(2, V_5)$ is at the root of much of the rich geometry of GM varieties. Recall from §6.2 that if $V_{10}$ denotes a 10-dimensional vector space over $k$, then the orthogonal Grassmannian $\text{OGr}_+(5, V_{10})$ is embedded into the projectivization $P(S_{16})$ of the half-spinor space $S_{16}$ and is projectively dual to $\text{OGr}_-(5, V_{10})$. Motivated by this, we define a smooth spin GM variety to be a smooth dimensionally transverse fiber product

$$X = \text{OGr}_+(5, V_{10}) \times_{P(S_{16})} Q,$$

where $Q \to P(S_{16})$ is a standard morphism of a quadric. The argument of Theorem 9.22 proves the following spin analogue. We use the notation of Theorem 6.5.

Theorem 9.22. Let

$$X = \text{OGr}_+(5, V_{10}) \times_{P(S_{16})} Q \quad \text{and} \quad Y = \text{OGr}_-(5, V_{10}) \times_{P(S_{16})} Q^2$$

be smooth spin GM varieties of dimensions $d_X \geq 3$ and $d_Y \geq 3$, where $Q \to P(S_{16})$ is a standard morphism of a quadric and $Q^2 \to P(S_{16})$ is its generalized dual. Let $U_X$ and $U_Y$
denote the pullbacks of $U$ and $U'$ to $X$ and $Y$, and let $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ denote the pullbacks of the $\mathcal{O}(1)$ line bundles on $\mathbb{P}(S_{16})$ and $\mathbb{P}(S'_{16})$. Then there are semiorthogonal decompositions

$$\text{Perf}(X) = \langle \mathcal{K}(X), \mathcal{O}_X(1), \mathcal{U}_X(1), \ldots, \mathcal{O}_X(dx - 3), \mathcal{U}_X(dx - 3) \rangle,$$

$$\text{Perf}(Y) = \langle \mathcal{U}_Y(3 - dy), \mathcal{O}_Y(3 - dy), \ldots, \mathcal{O}_Y(-1), \mathcal{O}_Y(-1), \mathcal{K}'(Y) \rangle,$$

and an equivalence $\mathcal{K}(X) \simeq \mathcal{K}'(Y)$.

**Remark 9.23.** We call the category $\mathcal{K}(X)$ occurring in (9.32) a spin GM category. Spin GM categories should be thought of as 3-dimensional counterparts of GM categories. Indeed, whereas a GM category is always (fractional) Calabi–Yau of dimension 2, a spin GM category is (fractional) Calabi–Yau of dimension 3 by [20, Remark 4.9]. It would be interesting to investigate the rationality question for spin GM varieties in relation to Theorem 9.22, following the GM case discussed in [24, §3]. The results of [8] should extend to the spin GM setting, and would be useful for this problem.

### Appendix A. Linear categories

In this appendix, we collect some results on linear categories that are needed in the body of the text. As in [32, Part I], in this section we will be considering general linear categories, as opposed to Lefschetz categories or categories linear over a projective bundle. To emphasize this we tend to denote categories with the letters $C$ or $D$ as opposed to $A$ or $B$.

**A.1. Tensor products of linear categories.** Here we quickly recall some salient features of tensor products of linear categories from [32], and formulate a couple extra lemmas that are used in the paper.

First recall that if $T$ is a scheme, then by Definition 1.16 a $T$-linear category is a small idempotent-complete stable $\infty$-category equipped with a $\text{Perf}(T)$-module structure. The basic example of such a category is $\mathcal{C} = \text{Perf}(X)$ where $X$ is a scheme over $T$; in this case, the action functor $\mathcal{C} \times \text{Perf}(T) \to \mathcal{C}$ is given by $(C, F) \mapsto C \otimes \pi^*(F)$, where $\pi: X \to T$ is the structure morphism.

Given $T$-linear categories $\mathcal{C}$ and $\mathcal{D}$, we can form their tensor product

$$\mathcal{C} \otimes_{\text{Perf}(T)} \mathcal{D},$$

which is a $T$-linear category characterized by the property that for any $T$-linear category $\mathcal{E}$, the $T$-linear functors $\mathcal{C} \otimes_{\text{Perf}(T)} \mathcal{D} \to \mathcal{E}$ classify bilinear maps $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ (suitably defined). In particular, there is a canonical functor

$$\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes_{\text{Perf}(T)} \mathcal{D},$$

whose action on objects we denote by $(C, D) \mapsto C \boxtimes D$.

**Lemma A.1** ([32, Lemma 2.7]). The category $\mathcal{C} \otimes_{\text{Perf}(T)} \mathcal{D}$ is thickly generated by objects of the form $C \boxtimes D$ for $C \in \mathcal{C}, D \in \mathcal{D}$, i.e. the smallest idempotent-complete triangulated subcategory containing all of these objects is $\mathcal{C} \otimes_{\text{Perf}(T)} \mathcal{D}$ itself.

A fundamental result is that in the geometric case, tensor products of linear categories correspond to fiber products of schemes.

**Theorem A.2** ([3, Theorem 1.2]). Let $X \to T$ and $Y \to T$ be morphisms of derived schemes. Then there is a canonical equivalence

$$\text{Perf}(X \times_T Y) \simeq \text{Perf}(X) \otimes_{\text{Perf}(T)} \text{Perf}(Y),$$
where \( X \times_T Y \) is the derived fiber product.

**Remark A.3.** In [3], the theorem is formulated for \( X_1, X_2, \) and \( T \) being so-called perfect stacks. Any quasi-compact, separated derived scheme is a perfect stack [3, Proposition 3.19], so with our conventions from [17], any derived scheme is perfect.

Let \( \phi_1 : C_1 \to D_1 \) and \( \phi_2 : C_2 \to D_2 \) be \( T \)-linear functors. They induce a functor between the tensor product categories \( C_1 \otimes_{\text{Perf}(T)} C_2 \to D_1 \otimes_{\text{Perf}(T)} D_2 \), which we denote by \( \phi_1 \otimes \phi_2 \). This operation is compatible with adjunctions.

**Lemma A.4 ([32, Lemmas 2.12]).** Let \( \phi_1 : C_1 \to D_1 \) and \( \phi_2 : C_2 \to D_2 \) be \( T \)-linear functors.

1. If \( \phi_1 \) and \( \phi_2 \) both admit left adjoints \( \phi_1^* \) and \( \phi_2^* \) (or right adjoints \( \phi_1^! \) and \( \phi_2^! \)), then the functor \( \phi_1 \otimes \phi_2 : C_1 \otimes_{\text{Perf}(T)} C_2 \to D_1 \otimes_{\text{Perf}(T)} D_2 \) has a left adjoint given by \( \phi_1^* \otimes \phi_2^* \) (or right adjoint given by \( \phi_1^! \otimes \phi_2^! \)).

2. If \( \phi_1 \) and \( \phi_2 \) both admit left or right adjoints and are fully faithful, then so is \( \phi_1 \otimes \phi_2 \).

3. If \( \phi_1 \) and \( \phi_2 \) are both equivalences, then so is \( \phi_1 \otimes \phi_2 \).

Semiorthogonal decompositions and admissible subcategories of linear categories are defined as in the usual triangulated case [32, Definitions 3.1 and 3.5]. We will frequently need to use that they behave well under tensor products.

**Lemma A.5 ([32, Lemma 3.15]).** Let \( C = \langle A_1, \ldots, A_n \rangle \) and \( D = \langle B_1, \ldots, B_n \rangle \) be \( T \)-linear semiorthogonal decompositions. Then the tensor product of the embedding functors

\[
A_i \otimes_{\text{Perf}(T)} B_j \to C \otimes_{\text{Perf}(T)} D
\]

is fully faithful for all \( i, j \). Moreover, there is a semiorthogonal decomposition

\[
C \otimes_{\text{Perf}(T)} D = \langle A_i \otimes_{\text{Perf}(T)} B_j \rangle_{1 \leq i \leq m, 1 \leq j \leq n}
\]

where the ordering on the set \( \{ (i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n \} \) is any one which extends the coordinate-wise partial order. The projection functor onto the \((i, j)\)-component of this decomposition is given by

\[
\text{pr}_{A_i} \otimes \text{pr}_{B_j} : C \otimes_{\text{Perf}(T)} D \to A_i \otimes_{\text{Perf}(T)} B_j,
\]

where \( \text{pr}_{A_i} : C \to A_i \) and \( \text{pr}_{B_j} : D \to B_j \) are the projection functors for the given decompositions.

**Lemma A.6 ([32, Lemma 3.17]).** Let \( C \) and \( D \) be \( T \)-linear categories. If \( A \subset C \) is a left (or right) admissible \( T \)-linear subcategory, then so is \( A \otimes_{\text{Perf}(T)} D \subset C \otimes_{\text{Perf}(T)} D \).

Finally, in the paper we need a couple formal tensor product identities. Before stating them, we note the following. Let \( T_1, T_2, \) and \( X \) be schemes, and for \( k = 1, 2 \), let \( C_k \) be a \( T_k \times X \)-linear category. Then the tensor product

\[
C_1 \otimes_{\text{Perf}(X)} C_2
\]

is naturally a \( T_1 \times T_2 \)-linear category via the equivalence \( \text{Perf}(T_1 \times T_2) \simeq \text{Perf}(T_1) \otimes \text{Perf}(T_2) \).

**Lemma A.7.** Let \( T_1, T_2, X, \) and \( Y \) be schemes, and for \( k = 1, 2 \), let \( C_k \) be a \( T_k \times X \)-linear category and let \( D_k \) be a \( T_k \times Y \)-linear category. Then there is an equivalence

\[
(C_1 \otimes_{\text{Perf}(X)} C_2) \otimes_{\text{Perf}(T_1 \times T_2)} (D_1 \otimes_{\text{Perf}(Y)} D_2) \simeq (C_1 \otimes_{\text{Perf}(T_1)} D_1) \otimes_{\text{Perf}(X \times Y)} (C_2 \otimes_{\text{Perf}(T_2)} D_2).
\]

**Proof.** The equivalence is induced by the transposition of the middle two factors. \( \square \)
Recall that if \( \mathcal{C} \) is a \( T \)-linear category and \( T' \to T \) is a morphism, then we denote by \( \mathcal{C}_{T'} = \mathcal{C} \otimes_{\text{Perf}(T)} \text{Perf}(T') \) the \( T' \)-linear base change category.

**Corollary A.8.** For \( k = 1, 2 \), let \( T_k \) be a scheme and let \( \mathcal{C}_k \) be a \( T_k \)-linear category. Let \( Y \) be a scheme with a morphism \( Y \to T_1 \times T_2 \) corresponding to morphisms \( Y \to T_1 \) and \( Y \to T_2 \). Then there is an equivalence

\[
(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes_{\text{Perf}(T_1 \times T_2)} \text{Perf}(Y) \simeq (\mathcal{C}_1)_{Y} \otimes_{\text{Perf}(Y)} (\mathcal{C}_2)_{Y}
\]

**Proof.** There is a canonical equivalence \( \text{Perf}(Y) \simeq \text{Perf}(Y) \otimes_{\text{Perf}(Y)} \text{Perf}(Y) \). Now the result follows from Lemma [A.7] by taking \( X = S \) to be our base scheme and \( \mathcal{D}_1 = \mathcal{D}_2 = \text{Perf}(Y) \). □

### A.2. Local properties of linear functors

The main result of this subsection is the following.

**Proposition A.9.** Let \( \phi: \mathcal{C} \to \mathcal{D} \) be a \( T \)-linear functor with a left (resp. right) adjoint. Let \( A \subset \mathcal{C} \) be a left (resp. right) \( T \)-linear admissible subcategory, and let \( \mathcal{B} \subset \mathcal{D} \) be a \( T \)-linear subcategory which is either left or right admissible. Let \( \{ U_i \to T \} \) be an fpqc cover, and let \( \phi_{U_i}: \mathcal{C}_{U_i} \to \mathcal{D}_{U_i} \) denote the induced functor for any \( i \). Then \( \phi \) induces an equivalence \( A \simeq \mathcal{B} \) if and only if \( \phi_{U_i} \) induces an equivalence \( A_{U_i} \simeq \mathcal{B}_{U_i} \) for all \( i \).

We build up some preliminary results before giving the proof. If \( \mathcal{C} \) is a \( T \)-linear category and \( T' \to T \) is a morphism, we write \( C|_{T'} \) for the image of \( C \in \mathcal{C} \) under the canonical functor \( \mathcal{C} \to \mathcal{C}_{T'} \) induced by pullback, i.e. \( C|_{T'} = C \boxtimes \mathcal{O}_{T'} \).

**Lemma A.10.** Let \( \mathcal{C} \) be a \( T \)-linear category, and let \( C \in \mathcal{C} \). Let \( \{ U_i \to T \} \) be an fpqc cover. Then \( C \simeq 0 \) if and only if \( C|_{U_i} \simeq 0 \) for all \( i \).

**Proof.** The forward implication is obvious. Conversely, by the Künneth formula in the form of [32, Lemma 2.10], we have

\[
\hom_T(C, C)|_{U_i} \simeq \hom_{U_i}|C|_{U_i}, C|_{U_i},
\]

where \( \hom_T(C, C) \in \mathcal{D}_{\text{qc}}(T) \) is the mapping object defined in [32, §2.3.1]. Hence if \( C|_{U_i} \simeq 0 \) for all \( i \), then \( \hom_T(C, C) \simeq 0 \) and so \( C \simeq 0 \). □

**Lemma A.11.** Let \( \phi: \mathcal{C} \to \mathcal{D} \) be a \( T \)-linear functor. Let \( \{ U_i \to T \} \) be an fpqc cover. Then \( \phi \simeq 0 \) if and only if \( \phi_{U_i} \simeq 0 \) for all \( i \).

**Proof.** The forward implication is obvious. Conversely, we must show that \( \phi(C) \simeq 0 \) for all \( C \in \mathcal{C} \) if \( \phi_{U_i} \simeq 0 \) for all \( i \). For this, just note that \( \phi(C)|_{U_i} \simeq \phi_{U_i}(C|_{U_i}) \) and apply Lemma [A.10] □

**Lemma A.12.** Let \( \phi: \mathcal{C} \to \mathcal{D} \) be a \( T \)-linear functor. Let \( \mathcal{B} \subset \mathcal{D} \) be a \( T \)-linear subcategory which is left or right admissible. Let \( \{ U_i \to T \} \) be an fpqc cover. Then \( \phi \) factors through the inclusion \( \mathcal{B} \subset \mathcal{D} \) if and only if \( \phi_{U_i} \) factors through the inclusion \( \mathcal{B}_{U_i} \subset \mathcal{D}_{U_i} \) for all \( i \).

**Proof.** We consider the case where \( \mathcal{B} \) is left admissible; the right admissible case is similar. Let \( j^i: \mathcal{B} \to \mathcal{D} \) be the inclusion of the left orthogonal to \( \mathcal{B} \). This functor admits a right adjoint \( j_i^! \) since \( \mathcal{B} \subset \mathcal{D} \) is left admissible. Moreover, \( \phi \) factors through \( \mathcal{B} \subset \mathcal{D} \) if and only if the composition \( j^i \circ \phi \) vanishes. By Lemma [A.11], this composition vanishes if and only if its base change to \( U_i \) vanishes for all \( i \). But this base change identifies with \( j_{U_i}^! \circ \phi_{U_i} \), where \( j_{U_i}^! \) is the right adjoint to the inclusion \( \mathcal{B}_{U_i} \subset \mathcal{D}_{U_i} \), and hence vanishes if and only if \( \phi_{U_i} \) factors through \( \mathcal{B}_{U_i} \subset \mathcal{D}_{U_i} \). □
Proof of Proposition A.9. We consider the left adjoints case of the proposition; the right adjoints case is similar. First assume $A = \mathcal{C}$ and $B = \mathcal{D}$. Note that a functor with a left adjoint is an equivalence if and only if the cones of the unit and counit of the adjunction vanish. If $\psi$ denotes the cone of the unit or counit for the adjoint pair $(\phi, \phi^\ast)$, then $\psi_{|U_i}$ is the cone of the unit or counit for the adjoint pair $(\phi_{U_i}, \phi_{U_i}^\ast)$, cf. Lemma A.4. Hence applying Lemma A.11 proves the lemma in this case.

Now consider the case of general $A$ and $B$. Denote by $\alpha: A \to \mathcal{C}$ and $\beta: B \to \mathcal{D}$ the inclusions. If $\phi_{U_i}$ induces an equivalence $A_{U_i} \simeq B_{U_i}$ for all $i$, then by Lemma A.12 there is a functor $\phi_A: A \to B$ such that $\phi \circ \alpha = \beta \circ \phi_A$. We want to show $\phi_A$ is an equivalence. But $\phi_A$ admits a left adjoint, namely $\alpha^\ast \circ \phi^\ast \circ \beta$, and $(\phi_A)_{U_i}: A_{U_i} \to B_{U_i}$ is an equivalence for all $i$, so we conclude by the case handled above.

A.3. Support of linear categories. Let $T \to T'$ be a morphism of schemes. If $X$ is a $T$-scheme, then we can regard $X$ as a $T'$-scheme via composition of the structure morphism $T \to T'$. We introduce notation for the analogous operation on linear categories.

Definition A.13. Let $\mathcal{C}$ be a $T$-linear category, and let $T \to T'$ be a morphism of schemes. We write $\mathcal{C}/T'$ for $\mathcal{C}$ regarded as a $T'$-linear category via the pullback functor $\text{Perf}(T') \to \text{Perf}(T)$, and say $\mathcal{C}/T'$ is obtained from $\mathcal{C}$ by extending the base scheme along $T \to T'$.

Remark A.14. If $A$ is a Lefschetz category over $\mathcal{P}(V)$ and $V \to V'$ is an embedding of vector bundles, then the category $A/\mathcal{P}(V')$ is naturally a Lefschetz category over $\mathcal{P}(V')$, with the same center. Moreover, this operation preserves (right or left) strongness and moderateness of Lefschetz categories.

The “support” of a $T$-linear category $\mathcal{C}$ over $T$ should be thought of as the locus of points in $T$ over which $\mathcal{C}$ is nonzero. Instead of fully developing this notion, we make ad hoc definitions that capture the idea of the support being contained in a closed or open subset of $T$, which are sufficient for our purposes.

Definition A.15. Let $\mathcal{C}$ be a $T$-linear category.

1. If $Z \subset T$ is a closed subset, we say $\mathcal{C}$ is supported over $Z$ if $\mathcal{C}_U \simeq 0$, where $U = T \setminus Z$.
2. If $U \subset T$ is an open subset, we say $\mathcal{C}$ is supported over $U$ if the canonical $T$-linear functor

$$\mathcal{C} \to \mathcal{C}_{U}/T$$

induced by the restriction functor $\text{Perf}(T) \to \text{Perf}(U)$ is an equivalence.

Remark A.16. Let $X \to T$ be a morphism of schemes. Then the support of the $T$-linear category $\text{Perf}(X)$ should be thought of as the image of $X$ in $T$. Correspondingly, we have:

1. If $Z \subset T$ is a closed subset containing the image of $X$, then $\text{Perf}(X)$ is supported over $Z$.
2. If $U \subset T$ is an open subset containing the image of $X$, then $\text{Perf}(X)$ is supported over $U$.

The following is a formal consequence of the definitions.

Lemma A.17. Let $T$ be a scheme and let $U \subset T$ be an open subscheme. Let $\mathcal{C}$ be a $T$-linear category which is supported over $U$. Then for any $T$-linear category $\mathcal{D}$, there is a canonical $T$-linear equivalence

$$\mathcal{C} \otimes_{\text{Perf}(T)} \mathcal{D} \simeq \mathcal{C}_U \otimes_{\text{Perf}(T)} \mathcal{D}_U.$$

Proof. We have equivalences

$$\mathcal{C} \otimes_{\text{Perf}(T)} \mathcal{D} \simeq \mathcal{C}_U \otimes_{\text{Perf}(T)} \mathcal{D} \simeq \mathcal{C} \otimes_{\text{Perf}(T)} \text{Perf}(U) \otimes_{\text{Perf}(T)} \mathcal{D} \simeq \mathcal{C} \otimes_{\text{Perf}(T)} \mathcal{D}_U.$$
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