Wilson loops in $\mathcal{N}=4$ SYM theory: rotation in $S^5$

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Abstract

We study Wilson loops in $\mathcal{N}=4$ SYM theory which are non-constant in the scalar ($S^5$) directions and open string solutions associated with them in the context of AdS/CFT correspondence. An interplay between Minkowskian and Euclidean pictures turns out to be non-trivial for time-dependent Wilson loops. We find that in the $S^5$-rotating case there appears to be no direct open-string duals for the Minkowskian Wilson loops, and their expectation values should be obtained by analytic continuation from the Euclidean-space results. In the Euclidean case, we determine the dependence of the “quark – anti-quark” potential on the rotation parameter $\nu$, both at weak coupling (i.e. in the 1-loop perturbative SYM theory) and at strong coupling (i.e. in the classical string theory in $AdS_5 \times S^5$).

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1 Introduction

In the standard picture \[1\] of the AdS/CFT correspondence the closed string states in the bulk of \(AdS_5\) are dual (at large \(N\)) to the (single-trace) operators in the \(\mathcal{N} = 4\) SYM CFT. Recently, there was a progress in understanding details of this correspondence in certain sectors of string states with large (angular momentum) quantum numbers \[2, 3\].

Adding a D3-brane probe located parallel (and close) to the boundary introduces open string sectors – open strings attached to the D3-brane. These are important for the description of Wilson loops as basic non-local observables in the context of AdS/CFT (for a review see \[4\]).

By analogy with the closed-string case of \[2\], one may wonder if considering special open string configurations with non-trivial dependence on angles of \(S^5\) may lead to simplifications and new interesting tests of the AdS/CFT. It is natural to associate such open strings with Wilson loops that depend on scalar fields in a non-trivial way. Here we shall study the dependence of the Wilson loops on a new parameter \(\nu\) corresponding to rotation of the open string configuration along big circle of \(S^5\). While the corresponding Wilson loop will not carry a definite R-charge and thus there will be no direct relation to the closed string case of \[2\], we are motivated by the hope that an investigation of more general scalar Wilson loops may be useful for better understanding of the AdS/CFT duality. In particular, one may be able to identify new cases where certain features of SYM observables can be interpolated from weak to strong ’t Hooft coupling.

The Minkowski-signature Wilson loops depending not only on a contour \(x^m(\tau)\) in Minkowski space \(\mathbb{R}^{1,3}\) but also on a contour \(\theta^i(\tau)\) in \(S^5\) can be related to a propagator of a “ W-boson” \[5, 6, 7\] present in the case of a generic slowly varying scalar field condensate (with \(\theta_i(\tau)\) being a unit 6-vector in the direction of the symmetry breaking condensate)

\[
W(C, \theta) = \frac{1}{\mathcal{N}} \text{tr} P \exp \left[ i \oint_C d\tau \left( A_m(x) \dot{x}^m - \Phi_i(x) \theta^i |\dot{x}| \right) \right]. \tag{1}
\]

Similar generalizations of constant scalar Wilson loops (\(\theta_i = \text{const}\)) were considered early on in the context of AdS/CFT correspondence \[8\] and were recently shown \[9\] to admit new interesting cases in which supersymmetry is partially preserved.

Below we shall study the simplest non-trivial case of “rotation” in \(S^5\)
when the unit 6-vector $\theta_i(\tau)$ is chosen as
\[
\theta^i = (\cos \nu \tau, \sin \nu \tau, 0, 0, 0, 0) \quad \nu = \text{const}.
\] (2)

In the case of a static source, i.e. single straight line $x^m = (\tau, 0, 0, 0)$, that gives
\[
W = \frac{1}{N} \text{tr} \ P \exp \left[ i \int d\tau \left( A_0(\tau, 0) - \Phi_1(\tau, 0) \cos \nu \tau - \Phi_2(\tau, 0) \sin \nu \tau \right) \right].
\] (3)

The expression for the corresponding Wilson loop in the Euclidean-signature case can obtained by the standard continuation $A_0 \to -iA_0, \ x^0 \to ix^0, \ |\dot{x}| \equiv \sqrt{-\dot{x}^2} \to i|\dot{x}|$, giving \[5, 7\]
\[
W(C, \theta) = \frac{1}{N} \text{tr} \ P \exp \left[ \oint_C d\tau \left( iA_m(x)\dot{x}^m + \Phi_i(x)\dot{x}^i|\dot{x}| \right) \right]. \quad (4)
\]

In addition, it seems natural to continue $\tau \to i\tau$. While this does not affect the final result \[4\] for a constant $\theta_i$, this may produce a complex expression when $\theta^i$ is non-trivial and some additional continuation of parameters may be needed. For instance, if we change $\tau$ to $i\tau$ in the Wilson loop (3), we should also substitute $-i\nu$ for $\nu$. The Euclidean counterpart of the Wilson line with rotation in $S^5$ is
\[
W = \frac{1}{N} \text{tr} \ P \exp \left[ \int d\tau \left( iA_0(\tau, 0) + Z(\tau, 0) e^{-i\nu \tau} + \bar{Z}(\tau, 0) e^{i\nu \tau} \right) \right], \quad (5)
\]
where we introduced the complex scalar field $Z = \frac{1}{2}(\Phi_1 + i\Phi_2)$.

Our aim will be to compare the weak-coupling (i.e. perturbative SYM) and strong-coupling (i.e. semiclassical string theory in $AdS_5 \times S^5$) predictions for the expectation values of such Wilson loops, generalizing the previously known results for the single straight line and the antiparallel lines \[5\] to the case of $\nu \neq 0$.

While in the $\nu = 0$ case the expressions for the Minkowski and Euclidean signature loops are directly related, we shall find that the two are different in the rotating case. While on the AdS side the Minkowski case may seem natural having a direct open-string energy interpretation, the corresponding perturbative SYM potential turns out to be imaginary (Section 2.1). The latter is found to be a direct analytic continuation $\nu \to i\nu$ of the well-defined
real potential extracted from the Euclidean Wilson loop expectation value (Section 3.1). This may be considered as an indication that non-constant scalar Wilson loops should be associated with Euclidean minimal surfaces on the string side.

For $\nu = 0$, the simplest 1/2 supersymmetric configuration ("straight line") corresponds to a single W-boson at rest, i.e. to the $x^0 = \tau$ line at the boundary of $AdS_5$ which may be thought of as the end-line of the open string stretched from the horizon $z = \infty$ to the boundary $z = 0$ [7, 8]. Below we shall first determine how this simplest classical open-string solution is modified in the rotating case. In the Minkowski case (Section 2.2) we shall find that instead of running all the way to the horizon, the string reaches some maximal height $z_0$ and then returns back to the same point at the boundary. In the Euclidean case (Section 3.2) the classical string solution is still similar to the $\nu = 0$ case: the infinite line along the radial AdS direction. A novel feature is an instability, which leads to shrinking of the minimal surface in $S^5$.

We shall then generalize the minimal surface solution of [8] describing the potential between two W-bosons (i.e. the minimal surface ending on two anti-parallel lines on the boundary separated by distance $L$) to the case with an extra "$S^5$-rotation" characterized by the parameter $\nu$. In general, the potential will have the structure

$$V = \frac{1}{L} \mathcal{V}(l, \lambda) , \quad l \equiv \nu L ,$$

i.e. will be a non-trivial function of the dimensionless parameter $l = \nu L$ and the 't Hooft coupling $\lambda = g_{YM}^2 N = \frac{R^4}{\alpha'}$ (we assume $N \to \infty$). At weak coupling, i.e. in the SYM perturbation theory

$$V = \frac{\lambda}{L} G(l) + O(\lambda^2) .$$

At strong coupling, i.e. in the semiclassical string theory in $AdS_5 \times S^5$ one finds that

$$V = \frac{\sqrt{\lambda}}{L} W(l) + O(1) .$$

The functions $G(l)$ and $W(l)$ will be determined below.

We shall find that in the Minkowski case (Section 2.2) the classical string solution no longer exists if $l$ is larger than certain critical value $l_{\text{max}} \approx 0.678$. 4
In the Euclidean case (Section 3.2), the minimal surface exists for any value of $l = \nu L$ but there is a phase transition at $l = l_c \approx 2.31$, which is similar to the Gross-Ooguri transition \cite{9, 10, 11, 12, 13}, though the mechanism of the transition is different. In our case, the phase transition happens due to the onset of an instability for the minimal surface rotating in $S^5$, while the Gross-Ooguri phase transition is caused by the string breaking. The potential is a non-analytic function of the distance, but it has the same Coulomb-law ($\frac{1}{T}$) asymptotics both at short and at large distances with the same effective charge.

2 Minkowski Wilson loops with $S^5$-rotation

2.1 Perturbative SYM results

2.1.1 Single line

The expectation value of the Wilson loop (3) is given, to the first order in perturbation theory in $\lambda$, by:

$$\ln \langle W(C_T) \rangle = -\frac{\lambda T}{2} \int_0^\infty d\tau \left( 1 - \cos \nu \tau \right) \int \frac{d^4p}{(2\pi)^4} \frac{-ie^{-ip_0\tau}}{-p^2 + i\epsilon}. \quad (9)$$

Here, $T$ is the length of the loop ($-T/2 < x^0 = \tau < T/2$), which serves as an IR cutoff. Computing the integrals over $p_0$ and over $\tau$, we get:

$$\ln \langle W(C_T) \rangle = \frac{\lambda T}{8\pi^2 i} \int_0^\infty dEE \left( \frac{1}{E} - \frac{1}{2} \frac{1}{E + \nu} - \frac{1}{2} \frac{1}{E - \nu - i\epsilon} \right). \quad (10)$$

Now,

$$\frac{1}{E - \nu - i\epsilon} = \varphi \frac{1}{E - \nu} + i\pi \delta(E - \nu)$$

and

$$\varphi \int_0^\infty dEE \left( \frac{1}{E} - \frac{1}{2} \frac{1}{E + \nu} - \frac{1}{2} \frac{1}{E - \nu} \right) = 0.$$ 

The only non-zero contribution comes from the pole at $E = \nu + i\epsilon$:

$$\langle W(C_T) \rangle = \exp \left( -\frac{\lambda}{16\pi} \nu T + O(\lambda^2) \right). \quad (11)$$

\footnote{We use the metric with signature $(- + + +)$ and the standard pole prescription for propagators.}
This result can be obtained from the Euclidean expectation value found in Section 3.1 by the substitution $T \rightarrow iT, \nu \rightarrow -i\nu$.

Usually, the exponent in the Wilson line vev is interpreted as a response of the vacuum energy to the insertion of an external source. One would then expect an oscillating behavior of the expectation value: $\langle W(C_T) \rangle = \exp(-iE_{\text{ext}}T)$, where $E_{\text{ext}}$ is the energy which the source acquires due to interactions with the Yang-Mills fields. Instead of oscillations, we found an exponential decrease, which, according to the standard lore, is a signal of an instability. One way to understand this instability is to recall that Wilson line with time dependence in $S^5$ describes an infinitely heavy W-boson associated with non-constant Higgs condensate. Time dependence of the condensate allows the Wilson loop to emit on-shell scalars. As a result, the pole of the scalar propagator lies in the region of integration in the one-loop amplitude, and the amplitude acquires an imaginary part from the residue.

2.1.2 Anti-parallel lines

The potential $V(L, \nu)$ is defined by the expectation value of the rectangular loop whose time extent is much larger than the spatial separation ($T \gg L$):

$$\langle W(C_{T \times L}) \rangle = e^{-iVT}.$$  \hspace{1cm} (12)

To the first order in perturbation theory:

$$V = -\frac{i\lambda}{2} \int_{-\infty}^{+\infty} d\tau (1 + \cos \nu \tau) \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip_0\tau + ipL}}{-p^2 + i\varepsilon}.$$  \hspace{1cm} (13)

After angular and energy integrations we get

$$V = -\frac{\lambda}{4\pi^2L} \int_0^\infty dE \sin(EL) \left( \frac{1}{E} - \frac{1}{2} \frac{1}{E + \nu} - \frac{1}{2} \frac{1}{E - \nu - i\varepsilon} \right).$$  \hspace{1cm} (14)

Again, there is a resonant term and the potential picks up an imaginary part from the pole at $E = \nu + i\varepsilon$:

$$V = -\frac{\lambda}{8\pi L}(1 + e^{i\nu L}) + O(\lambda^2).$$  \hspace{1cm} (15)

This expression also turns out to be an analytic continuation in $\nu$ of the corresponding Euclidean potential found in Section 3.1. This suggests that in
general the expectation values of the Minkowski Wilson loops can be obtained by an analytic continuation from the Euclidean space.

As follows from the above expression, the continuation in the leading-order perturbative expression is non-trivial and produces imaginary parts in the expectation values. Almost certainly, continuation from Euclidean space should work to all orders of perturbation theory, though it would be interesting to check this by pushing perturbative calculations to higher orders as it was done for the Wilson loops which are static in $S^5$ \cite{14, 15}. A natural guess then is that well-defined Euclidean Wilson loops are associated with the Euclidean minimal surfaces at strong coupling.

Before turning to the Euclidean case in Section 3, in the remainder of this section we shall consider the Minkowski-signature open strings which end at the boundary of $AdS_5$. It seems that these string solutions are not directly related to Wilson loops, but may be interesting on their own.

### 2.2 Open strings in $AdS_5$ with Minkowski signature

We shall use the $AdS_5 \times S^5$ metric in the following form (Poincare coordinates)

$$ds^2 = \frac{1}{z^2}(dx^m dx^m + dz^p dz^p), \quad z^2 = z^p z^p, \quad p = 1, ..., 6 \quad (16)$$

$$dx^m dx^m = -dx^0 dx^0 + dx^k dx^k, \quad k = 1, 2, 3 \quad (17)$$

Let us first recall the form of the open string solutions in the absence of rotation, i.e. for $\nu = 0$. We shall use the conformal gauge on the world sheet. In the case of single straight line on the boundary along $x^0$, the string is stretched along the radial AdS direction (here we use Minkowski signature in both target space and string world sheet)

$$x^0 = \tau, \quad z = \sigma, \quad 0 < \sigma < \infty \quad (18)$$

The Minkowski version of the two anti-parallel lines configuration \cite{5} is (prime is derivative over $\sigma$)

$$x^0 = \tau, \quad x^1 = x^1(\sigma) \equiv x(\sigma), \quad z = z(\sigma) \quad (19)$$
\( x' = cz^2 \), \( z'^2 = 1 - c^2 z^4 \), \( z\text{max} = \frac{1}{\sqrt{c}} \), \( \left( \frac{dz}{dx} \right)^2 = (c^2 z^4)^{-1} - 1. \) \hfill (20)

c = 0 is the straight line case. Both solutions admit straightforward Euclidean analogs obtained by \( \tau \to i\tau \), \( x^0 \to ix^0 \).

### 2.2.1 Single line

A simple generalization of the straight line solution (18) is obtained by adding a rotation along the big circle of \( S^5 \), i.e. in the 2-plane \((z_1, z_2)\) in the 6-space

\[ x^0 = \tau \, , \quad \varphi = \nu \tau \, , \quad z = z(\sigma) \, , \quad z_1 + iz_2 = ze^{i\varphi} . \] \hfill (21)

\[ z'^2 = 1 - \nu^2 z^2 \, , \quad \text{i.e.} \quad z = \nu^{-1} \sin \nu \sigma . \] \hfill (22)

For zero angular momentum parameter \( \nu \to 0 \) we get back to the straight line solution (18). For \( \nu \neq 0 \) the radial coordinate \( z \) is periodic in \( \sigma \): for fixed \( \tau \) the string starts at the boundary, reaches the maximal value \( z\text{max} = \frac{1}{\nu} \) and then returns back to the same point \( x_k = 0 \) at the boundary. For changing \( \tau \) it rotates in the \( \varphi \)-direction in \( S^5 \) with speed \( \nu \). As we shall see below, this configuration may be thought of as a limit of a string ending at the two points of the boundary in the case when these points coincide. One may restrict \( \sigma \) to run from 0 to \( \frac{\pi}{\nu} \), so that the string has only one fold (multi-folded strings will have bigger energy).

This open-string configuration is characterized by the values of the conserved charges – the space-time energy and the angular momentum

\[ E = 2\sqrt{\lambda} \int_0^{\frac{\pi}{\nu}} d\sigma \frac{z^{-2} \dot{x}^0}{2\pi} , \quad J = 2\sqrt{\lambda} \int_0^{\frac{\pi}{\nu}} d\sigma \frac{\dot{\varphi}}{2\pi} , \] \hfill (23)

where we added factors of 2 as the integral goes from 0 to the folding point. Not surprisingly, the value of the angular momentum does not depend on \( \nu \) \((J \text{ is dimensionless, } \nu \text{ has a dimension of mass)}\):

\[ J = \frac{\sqrt{\lambda}}{2} . \] \hfill (24)

The expression for the energy is divergent at \( \sigma = 0 \) and can be defined by subtracting, as in the \( \nu = 0 \) case, the value of the infinite straight string
configuration. Note that we have to subtract the straight-line contribution twice as the two segments of the folded string include the singular point $z = 0$. Then the subtracted value of the energy is

$$E = 2\sqrt{\lambda} \left( \int_0^{\pi} \frac{d\sigma}{2\pi} \frac{\nu^2}{\sin^2 \nu \sigma} - \int_0^\infty \frac{d\sigma}{2\pi \sigma^2} \right) = \frac{\sqrt{\lambda}}{\pi} \left( \nu \cot \nu \sigma - \frac{1}{\sigma} \right) \bigg|_{\sigma=0} = 0 .$$

(25)

Thus the subtracted energy vanishes as in the non-rotating case, despite the fact that the $\nu \neq 0$ solution is no longer supersymmetric.

One may wonder if the solution with $\tau$-dependent angle of $S^5$ and no other angles excited is actually stable. This is indeed so in Minkowski signature. Including the two-sphere directions $\psi$ and $\varphi$ of $S^5$ the relevant part of the string action in the conformal gauge is

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[ \frac{1}{z^2} \left( \dot{x}_m^2 + \dot{z}^2 - \dot{x}_m^2 + \dot{z}^2 \right) + \dot{\psi}^2 - \dot{\psi}^2 + \cos^2 \psi \left( \dot{\varphi}^2 - \dot{\varphi}^2 \right) \right] .$$

(26)

For $\varphi = \nu \tau$ the effective potential for $\psi$ is $U = \psi^2 - \nu^2 \cos^2 \psi$. Its minimum $U = -\nu^2$ is indeed at $\psi = 0$, as we were assuming above. The stability of this Minkowski solution can be also confirmed by the direct analysis of the action for small fluctuations as in [16, 17].

The Euclidean analog of the above solution with $\psi = 0$ is obtained by the following continuation: $\tau \to i\tau$, $x^0 \to ix^0$, $\nu \to -i\nu$, i.e.

$$x^0 = \tau , \quad \varphi = \nu \tau , \quad \dot{z}^2 = 1 + \nu^2 z^2 , \quad \text{i.e.} \quad z = \frac{1}{\nu} \sinh \nu \sigma .$$

(27)

Here we get again an infinite open string: $\sigma$ and thus $z$ changes from 0 to infinity. This Euclidean solution is, however, unstable, because its area linearly diverges at the horizon. Its stable counterpart involves non-constant $\psi$-angle and will be described in Section 3.2.

### 2.2.2 Anti-parallel lines

Let us now consider the $\nu \neq 0$ generalization of the two-line solution [13].

This will be an open string with both ends at the boundary separated by

\(^2\)Note that in the Euclidean signature case in Section 3.2 the subtracted value of the Euclidean action will turn out to be non-vanishing.

\(^3\)This generalization was also independently studied by J. Russo.
distance $L$, stretched in $z$ direction and rotating along the angle $\varphi$ in $S^5$. We shall set the angle $\psi = 0$. This solution is similar to the Euclidean solution A in Section 3.2.

For $\varphi = \nu \tau$ and $x_1 = x(\sigma)$ one finds (cf. (20))

$$x' = c z^2, \quad z'^2 = 1 - \nu^2 z^2 - c^2 z^4,$$

i.e.

$$z'^2 = (1 - b^2 z^2)[1 + (b^2 - \nu^2)z^2], \quad c = b\sqrt{b^2 - \nu^2}. \quad (28)$$

We introduced, instead of the integration constant $c$, the parameter $b \geq \nu$, so that $b^{-1}$ is the maximal value of $z$. Then

$$L = \int dx = 2\sqrt{1 - v^2} \int_0^1 \frac{w^2 dw}{\sqrt{(1 - w^2)[1 + (1 - v^2)w^2]}}, \quad \quad \quad (30)$$

$$w = bz, \quad v = \frac{\nu}{b}. \quad \quad \quad (31)$$

Here $0 \leq v \leq 1$. In the case of $\nu = 0$ (i.e. $v = 0$)

$$L = \int dx = b^{-1}k_0, \quad k_0 = \frac{(2\pi)^{3/2}}{[\Gamma(\frac{1}{4})]^2} \approx 1.198. \quad \quad \quad (32)$$

Eq. (31) determines the maximal value $z_{\text{max}} = b^{-1}$ in terms of $L$ and $\nu$. Equivalently, it gives a relation between the two dimensionless parameters $l = \nu L$ and $v = \frac{\nu}{b}$:

$$l = 2v\sqrt{1 - v^2} \int_0^1 \frac{w^2 dw}{\sqrt{(1 - w^2)[1 + (1 - v^2)w^2]}}. \quad \quad \quad (33)$$

The dimensionless distance, as a function of $v$, $l(v)$, vanishes at $v = 0, 1$ and, consequently, has maximum at some $0 < v_0 < 1$ (numerically, $v_0 \approx 0.75$). As a result, the string solution is possible only for $l = L^\nu$ smaller than $l_{\text{max}} = l(v_0) \approx 0.678$. For fixed $\nu$ we cannot make $L$ arbitrarily large, or, equivalently, for fixed $L$ there exists a maximal value of $\nu$ above which the solution does not exist. As we change the parameter $v$ from zero to one, $l$ first grows, then reaches its maximal value and then returns back to zero. To summarize, the solution cannot be continued past the maximal distance
Figure 1: The energy \((U(l), \text{bold line})\) and the angular momentum \((F(l))\) as functions of \(l = \nu L\). Both curves go from bottom to top as \(v\) goes from 0 to 1.

between the lines, and the classical string world sheet cannot connect lines on the boundary which are separated by a larger distance.

The energy and the angular momentum of the string are given by (sub-
tracting an infinite constant from the energy):

\[
E = 2\sqrt{\lambda} \int \frac{d\sigma}{z^2} - E_0
\]

\[
= \frac{\sqrt{\lambda} l}{\pi L v} \left\{ \int_0^1 \frac{dw}{w^2} \left[ \frac{1}{\sqrt{(1 - w^2)[1 + (1 - v^2)w^2]}} - 1 \right] - 1 \right\}
\]

\[
= -\frac{\sqrt{\lambda} l^2 \sqrt{1 - v^2}}{L} \frac{1}{2\pi v^2}
\]

\[
\equiv \frac{\sqrt{\lambda}}{L} U(l), \quad (34)
\]

\[
J = 2\nu\sqrt{\lambda} \int d\sigma = \frac{\nu\sqrt{\lambda}}{\pi} \sigma_{\text{max}}(v) = \frac{\sqrt{\lambda}}{\pi} \nu \int_0^1 \frac{dw}{\sqrt{(1 - w^2)[1 + (1 - v^2)w^2]}}
\]

\[
\equiv \sqrt{\lambda} F(l). \quad (35)
\]

The functions \(U\) and \(F\) determining the dependence of \(E\) and \(J\) on \(\nu\) are shown in fig. It should be noted that very similar expressions for the energy and the angular momentum arise also for a time-independent string solution, where the string is stretched along the big circle of \(S^5\) [5].

The angular momentum \(J\) grows from 0 at \(v = 0\) to its maximal value \(\frac{\sqrt{\lambda}}{2}\) at \(v = 1\), which corresponds to the folded single-line solution discussed above. The potential vanishes at the point where \(J\) has its maximum value. In contrast to the closed string case in [2], here the angular momentum \(J\) does not have an intrinsic meaning on the gauge-theory side so it does not seem natural to express \(E\) as a function of \(J\).

We conclude that there is no similarity between the perturbative SYM and the semiclassical string-theory expressions for the Minkowski-signature Wilson loop expressions in the \(\nu \neq 0\) case. We shall therefore turn now to the study of the Euclidean signature case where the results will be more consistent with the AdS/CFT duality.

\(^4\) We use the notation \(U\) instead of \(W\) in (8) to distinguish the Minkowski case from the Euclidean one.
3 Euclidean Wilson loops with rotation in $S^5$

3.1 Perturbative SYM results

3.1.1 Single line

We start with the Euclidean Wilson line (5) which rotates in $S^5$. The expectation value of this operator is an even function of the rotation parameter $\nu$, which has the dimension of mass. Actually, $\nu$ is the only dimensionful parameter in the problem, except for an IR cutoff $T$. On dimensional grounds, $\nu$ must enter in the combination $|\nu|T$. We expect the expectation value to have an exponential form:

$$\langle W(C) \rangle = e^{-f(\lambda)|\nu|T} \quad \text{for } T \to \infty ,$$

(36)

where $f(\lambda)$ is some function of the 't Hooft coupling. Because the expectation value depends on $|\nu|$, it is not analytic in $\nu$ at $\nu \to 0$. This turns out to be a generic feature of Euclidean Wilson loops which are non-constant in $S^5$. Below, we assume that $\nu > 0$ and thus write $\nu$ instead of $|\nu|$.

A simple one-loop calculation gives (cf. (11))

$$f(\lambda) = \frac{\lambda}{8\pi^2} \int_0^{+\infty} d\tau \frac{1 - \cos(\nu \tau)}{\tau^2} + O(\lambda^2) = \frac{\lambda}{16\pi} + O(\lambda^2) .$$

(37)

3.1.2 Antiparallel lines

Next, let us consider the potential between two heavy sources, each of which involves rotation in $S^5$ with the same frequency. This corresponds to the anti-parallel Wilson lines separated by the distance $L$. To determine the potential, we should subtract the self-energy:

$$\langle W(C_{L\times T}) \rangle \langle W(C_T) \rangle^2 = e^{-V(L)T} \quad \text{for } T \gg L .$$

(38)

At weak coupling one finds (cf. (13))

$$V(L) = -\frac{\lambda}{8\pi^2} \int_{-\infty}^{+\infty} d\tau \frac{1 + \cos(\nu \tau)}{\tau^2 + L^2} + O(\lambda^2) = -\frac{\lambda}{8\pi} \frac{1 + e^{-\nu L}}{L} + O(\lambda^2) .$$

(39)

This determines the function $G(l)$ in (7).
3.2 Open string minimal surface

3.2.1 Single line

The AdS dual of the Wilson loop \((5)\) is a straight string which starts near the boundary of \(\text{AdS}_5\) and rotates in \(S^5\). Since the rotation at infinity leads to linearly divergent area, the string will slide down the sphere as it moves towards the interior of \(\text{AdS}_5\). Such a solution corresponds to the ansatz

\[
x^0 = x^0(\tau), \quad \varphi = \varphi(\tau), \quad z = z(\sigma), \quad \psi = \psi(\sigma).
\] (40)

The string action is the Euclidean analog of \((7)\)

\[
S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \left[ (\partial_a z)^2 + \frac{(\partial_a x^m)^2}{z^2} + (\partial_a \varphi)^2 + \cos^2 \psi (\partial_a \varphi)^2 \right].
\] (41)

The minimal-area solution can be found by satisfying the conformal-gauge constraint separately on the \(\text{AdS}_5\) and \(S^5\) parts. The \(\text{AdS}_5\) part of the solution is then the same as for \(\nu = 0\):

\[
x^0 = \tau, \quad z = \sigma.
\] (42)

For the rotating string, we also have

\[
\varphi = \nu \tau.
\] (43)

The remaining constraint (i.e. the equation of motion for the azimuthal angle \(\psi\)) can then be easily solved to give:

\[
\sin \psi = \tanh(\nu \sigma).
\] (44)

The metric induced on the minimal surface is that of the \(\text{AdS}_2+ S^2/\mathbb{Z}_2\):

\[
ds^2 = \frac{d\tau^2 + d\sigma^2}{\sigma^2} + \nu^2 \frac{d\tau^2 + d\sigma^2}{\cosh^2(\nu \sigma)}.
\] (45)

The area of \(\text{AdS}_2\) is subtracted by the regularization, while the area of the hemi-sphere is

\[
A = \nu T .
\] (46)

In general, the single-line vev depends non-trivially only on the ’t Hooft coupling, while its dependence on \(\nu\) is determined by the conformal symmetry.
Including the string tension factor, the strong-coupling prediction for the function $f$ in (36) is thus

$$f(\lambda) = \frac{\sqrt{\lambda}}{2\pi} + O(1).$$

(47)

Hence, the function $f(\lambda)$ is expected to smoothly interpolate between $\sim \lambda$ at weak coupling and $\sim \sqrt{\lambda}$ at strong coupling, similarly to many other observables in $\mathcal{N} = 4$ SYM theory.

### 3.2.2 Anti-parallel lines

As was already mentioned above, on dimensional grounds, the potential between antiparallel lines is a function of $l = \nu L$, i.e. is given by (8) where the function $W(l)$ is proportional to the area per unit length for the surface that connects two anti-parallel lines, with the area of the two disconnected surfaces (46) subtracted.

The ansatz for the minimal surface is a generalization of (40)

$$x^0 = \tau, \quad \varphi = \nu \tau, \quad z = z(\sigma), \quad x^1 = x(\sigma), \quad \psi = \psi(\sigma).$$

(48)

The action is the same as in (41). The resulting equations of motion can be integrated:

$$
\begin{align*}
x' &= c \nu^2 z^2, \\
z' &= \pm \sqrt{1 + a \nu^2 z^2 - c^2 \nu^4 z^4}, \\
\psi' &= \pm \nu \sqrt{\cos^2 \psi - \cos^2 \psi_0},
\end{align*}
$$

(49)

where $a$, $c$ and $\cos^2 \psi_0$ are constants of integration. The constraint,

$$\frac{z'^2 + x'^2 - 1}{z^2} + \psi'^2 - \nu^2 \cos^2 \psi = 0,$$

(50)

reduces to the condition

$$a = \cos^2 \psi_0.$$

(51)

It is convenient to introduce dimensionless variables

$$\zeta = \nu z, \quad \xi = \nu x, \quad s = \nu \sigma.$$
They satisfy the following equations:

\[
\begin{align*}
\xi' &= c\zeta^2, \\
\zeta' &= \pm\sqrt{1 + \cos^2\psi_0\zeta^2 - c^2\zeta^4}, \\
\psi' &= \pm\sqrt[4]{\cos^2\psi - \cos^2\psi_0}.
\end{align*}
\] (52)

The solution is symmetric under the reflection of \( s \). The upper signs should be chosen to the left of the turning point \( s_0 \) and the lower signs to the right. The derivatives of \( \zeta \) and \( \psi \) must vanish at \( s = s_0 \). Consequently, \( \psi(s_0) = \psi_0 \). The AdS radius at the turning point, \( \zeta(s_0) = \zeta_0 \), is the root of the equation:

\[
1 + \cos^2\psi_0\zeta_0^2 - c^2\zeta_0^4 = 0.
\] (53)

It is convenient to consider one half of the solution to the left of the turning point and then continue it by symmetry.

There are two possible solutions which we shall call ( A) and ( B). One of them does not depend on the azimuthal angle: \( \psi \equiv 0 \), or \( \psi_0 = 0 \). The boundary conditions fix the remaining constant of integration:

\[
\frac{l}{2} = c \int_0^{\zeta_0} \frac{d\zeta \zeta^2}{\sqrt{1 + \zeta^2 - c^2\zeta^4}}, \quad \psi_0 = 0 \quad (A) \quad (54)
\]

In the other solution, the azimuthal angle is non-trivial, and the boundary conditions reduce to

\[
\frac{l}{2} = c \int_0^{\zeta_0} \frac{d\zeta \zeta^2}{\sqrt{1 + \cos^2\psi_0\zeta^2 - c^2\zeta^4}} \quad (B) \quad (55)
\]

\[
\int_0^{\psi_0} \frac{d\psi}{\sqrt{\cos^2\psi - \cos^2\psi_0}} = \int_0^{\zeta_0} \frac{d\zeta}{\sqrt{1 + \cos^2\psi_0\zeta^2 - c^2\zeta^4}} \quad (B) \quad (56)
\]

The straightforward calculations of the induced metric and the area, i.e. the function \( W(l) \) in (8), for the two solutions give:

\[
W(l) = 2l \left[ \int_0^{\zeta_0} \frac{d\zeta}{\zeta^2 \left( \frac{1}{\sqrt{1 + \zeta^2 - c^2\zeta^4}} - 1 \right) - \frac{1}{\zeta_0} - 1} + \int_0^{\zeta_0} \frac{d\zeta}{\sqrt{1 + \zeta^2 - c^2\zeta^4}} \right]
\]

\[
= l \left[ -cl - 2 + 2 \int_0^{\zeta_0} \frac{d\zeta}{\sqrt{1 + \zeta^2 - c^2\zeta^4}} \right] \quad (A) \quad (57)
\]
\begin{align*}
W(l) &= 2l \left[ \int_0^{\zeta_0} \frac{d\zeta}{\zeta^2} \left( \frac{1}{\sqrt{1 + \cos^2 \psi_0 \zeta^2 - c^2 \zeta^4}} - 1 \right) - \frac{1}{\zeta_0} - 1 ight] \\
&\quad + \int_0^{\psi_0} d\psi \frac{\cos^2 \psi}{\sqrt{\cos^2 \psi - \cos^2 \psi_0}} \\
&= l \left[ -cl - 2 + 2 \int_0^{\psi_0} d\psi \frac{\cos^2 \psi}{\sqrt{\cos^2 \psi - \cos^2 \psi_0}} \right]
\end{align*}

(B) (58)

Here we have subtracted the area of $AdS_2$ ($\int \frac{d\zeta}{\zeta^2}$) for the sake of regularization as well as the self-energy (46) from the bare action calculated on the solutions of (52).

![Figure 2: $W(l)$ as a function of $l$ (bold curve). The area of the unstable solution $A$ (thin curve). The dashed line corresponds to pure Coulomb potential at $\nu = 0$.](image)

We have computed the function $W(l)$ in the potential (8) numerically. The results are shown in fig. 2.

### 3.2.3 Phase transition

The rotation in $S^5$ effectively gives mass to the azimuthal fluctuations of the string: $(\partial \varphi)^2 = \nu^2$ in (11). Since the mass-squared is negative, the solution with $\psi = 0$ is clearly unstable. However, for a sufficiently short string, this instability has no time to develop.
To see this, let us analyze the equation of motion for the azimuthal angle:

\[
\frac{d^2 \psi}{ds^2} + \cos \psi \sin \psi = 0, \quad s = \nu \sigma .
\]  

(59)

This is the Newton’s equation for a pendulum. As the string propagates from one Wilson line to another, the azimuthal angle changes from \( \psi = 0 \) to \( \psi = \psi_0 \), and then bounces back to \( \psi = 0 \), so the pendulum makes one half of the oscillation. The total period of oscillation is equal to \( 4s_0 \), twice the internal length of the string. The period depends on the energy: it is bigger for larger energy. The total energy is equal to the potential at the turning point, \( -\cos^2 \psi_0 \). Hence, the smaller \( \cos \psi_0 \), the larger is the period. When \( \cos \psi_0 \) reaches zero, the period goes to infinity. But the period cannot be arbitrarily small. It approaches a finite limit as \( \cos \psi_0 \) goes to 1. The smallest period is equal to the period of harmonic oscillations around the minimum of the potential. Consequently, sufficiently short string, whose length is smaller than half-period of the harmonic oscillations, cannot move in the azimuthal direction.

So, the solution A with \( \cos \psi = 1 \) is stable for sufficiently small \( l \), when the string is too short and the solution B does not exist. At sufficiently large \( l \), the string becomes long enough to support the motion in the azimuthal direction and then the non-trivial solution takes over. As a consequence, the area and the potential are continuous, but non-analytic functions of the distance. There is a transition from one solution to another which leads to non-analyticity of the potential. In a similar context, the non-analyticity (Gross-Ooguri phase transition \[9, 10, 11, 12, 13\]) occurs due to the string breaking. Here the mechanism is different. The order of the transition is also different. In the present case, the potential undergoes, as will become clear shortly, a second order transition, while string breaking lead to the first order transition.

The solution B ceases to exist at the critical separation between the lines \( l = l_c \), at which the left-hand side of (56) reaches its minimal value \( \frac{\pi}{2} \):

\[
\frac{\pi}{2} = \int_0^{\zeta_0} \frac{d\zeta}{\sqrt{1 + \zeta^2 - c_c^2 \zeta^4}}.
\]  

(60)

The first constraint (55) then determines the critical distance

\[
\frac{l_c}{2} = c_c \int_0^{\zeta_0} \frac{d\zeta \zeta^2}{\sqrt{1 + \zeta^2 - c_c^2 \zeta^4}}.
\]  

(61)
Numerically, \( l_c \approx 2.31 \). The areas of the solutions A and B are equal at the critical point, so the transition is of the second order here: the area is continuous together with its first derivative.

It is easy to analyze the potential in the two limiting cases of small and large distances:

**Short distances**: \( L \to 0 \)

The area (57) and the equation for \( c \), (55), differ from the results at \( \nu = 0 \) by the last term in (57) and by the extra \( \zeta^2 \) in the argument of the square root. These differences disappear as \( l \to 0 \), since then \( c \to \infty \) and the region of integration with \( \zeta \sim c^{-1/2} \) gives the dominant contribution. To the leading order in \( 1/l \), the computation of the potential repeats the calculation of [3], so

\[
V(L) = -\frac{4\pi^2}{[\Gamma(1/4)]^4} \cdot \frac{\sqrt{\lambda}}{L} + \ldots \quad (L \to 0).
\]  

**Long distances**: \( L \to \infty \)

The solution at large distances is very different, but, surprisingly, we get the same potential, although for a different reason: at \( l \to \infty \) the length of the string becomes very large, and the period of oscillation in the azimuthal direction must be large too. In other words, \( \cos \psi_0 \) rapidly converges to zero as \( l \) goes to infinity. Then the last two terms in (58) almost cancel each other with an exponentially small leftover. Also, the quadratic piece in \( \zeta \) in the argument of the square root gets multiplied by an exponentially small factor and can be neglected. What remains after such approximations is the same formulas as in the calculation of [3]. As a result, we get the same expression for the potential, up to exponentially small corrections (cf. (39)):

\[
V(L) = -\frac{4\pi^2}{[\Gamma(1/4)]^4} \cdot \frac{\sqrt{\lambda}}{L} + \ldots \quad (L \to \infty).
\]

4 **Discussion**

We have studied open string solutions which involve rotation in \( S^5 \) with a constant frequency. In the SYM theory, these solutions are naturally associated with Wilson loops whose coupling to scalars corresponds to rotation
in a plane in internal 6-dimensional space. Unexpectedly, we found that Minkowskian and Euclidean solutions are absolutely different and cannot be obtained from each other by analytic continuation.

On the other hand, for the perturbative SYM expectation values of the Wilson loops the analytic continuation does work, though time dependence of the scalars ($S^5$ coordinates) makes Wick rotation quite subtle. After continuation from Euclidean to Minkowski space, vev’s of rotating Wilson loops become complex-valued. This presents a serious obstacle for their stringy interpretation directly in Minkowski-signature $AdS_5 \times S^5$ space. It seems likely that the duality between Wilson loops and open strings is well-defined only in the Euclidean version of the AdS/CFT correspondence.

The role of the classical solutions for the open strings in the Minkowski case described in Section 2.2 is unclear to us. It would be very interesting to clarify their meaning and to understand what do they correspond to on the gauge theory side of the AdS/CFT duality.

One of the lessons of this work is that there seems to be no direct relation between the open and closed string states with rotation in $S^5$. This is probably related to the fact there is no sense in which a Wilson loop can carry a definite R-charge. Indeed, a generic Wilson loop is a mixture of states with various R-charges – its local expansion contains operators of different R-charges [18]. Perhaps, a study of the OPE coefficients [19, 20] for operators with large R-charge may help to establish a contact between the semiclassical string picture of [2, 3] and the Wilson loops.

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