On the robustness of random $k$-cores

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May 5, 2014

Abstract

The $k$-core of a graph is its maximal subgraph with minimum degree at least $k$. In this paper, we address robustness questions about $k$-cores. Given a $k$-core, remove one edge uniformly at random and find its new $k$-core. We are interested in how many vertices are deleted from the original $k$-core to find the new one. This can be seen as a measure of robustness of the original $k$-core. We prove that, if the initial $k$-core is chosen uniformly at random from the $k$-cores with $n$ vertices and $m$ edges, its robustness depends essentially on its average degree $c$. We prove that, if $c \to k$, then the new $k$-core is empty with probability $1 + o(1)$. We define a constant $c'_k$ such that when $k + \varepsilon < c < c'_k - \varepsilon$, the new $k$-core is empty with probability bounded away from zero and, if $c > c'_k + \psi(n)$ with $\psi(n) = \omega(n^{-1/4})$, $\psi(n) > 0$ and $c$ is bounded, then the probability that the new $k$-core has less than $n - h(n)$ vertices goes to zero, for every $h(n) = \omega(\psi(n)^{-1})$.

1 Introduction

The $k$-core of a graph is its maximal subgraph with minimum degree at least $k$. The $k$-core of a graph is unique and it can be obtained by iteratively deleting vertices of degree smaller than $k$. The $k$-core of a graph that already has minimum degree at least $k$ is the graph itself. So we also say that graphs (and multigraphs) with minimum degree at least $k$ are $k$-cores.

The investigation of $k$-cores in random graphs was started by Bollobás [11] in 1984 in connection with $k$-connected subgraphs in random graphs. There has been much success in the use of $k$-cores due to their amenability to analysis. For some earlier results on the $k$-cores of random graphs, see [9, 10, 15].

A seminal result in this area was proved by Pittel, Spencer and Wormald [16]: they determined the threshold $c_k$ for the emergence of a giant $k$-core in $G(n, m)$. Roughly

*The author received an Ontario Graduate Scholarship during this project.
spoking, if the average degree is below this threshold, the $k$-core of $G(n, m)$ is empty with probability going to 1 as $n \to \infty$, and above the threshold the $k$-core has a linear number of vertices with probability going to 1. After this result, many proofs using a variety of techniques were given for the emergence of a giant $k$-core in graphs and hypergraphs; see [3, 5, 2, 8, 6, 7, 18].

We are interested in finding how robust this giant $k$-core of $G(n, m)$ is as a $k$-core. More precisely, if we delete a random edge in the $k$-core of $G(n, m)$ and obtain its new $k$-core, is the new $k$-core much smaller than the original one? This can be seen as a measure of the robustness of the giant $k$-core. We do not restrict ourselves to the $k$-core of $G(n, m)$: we consider a $k$-core chosen uniformly at random with given number of vertices and edges, then we delete an edge from it uniformly at random and obtain the new $k$-core.

We define a constant $c'_k$ and analyse the behaviour of the random $k$-cores with average degree below and above $c'_k$. We work with multigraphs with given degree sequence and then we deduce the desired results for simple graphs. Throughout the paper we use a simple deletion algorithm (and some variants) to find the $k$-core of a graph: the algorithm iteratively removes vertices of degree less than $k$ until all remaining vertices have degree at least $k$. We couple this deletion algorithm with a random walk. For the case with bounded average degree $c > c'_k + \psi(n)$ with $\psi(n) = \omega(n^{-1/4})$ and $\psi(n) > 0$, this strategy works quite well: we prove that the deletion algorithm and the random walk both terminate/die in less than $t(n)$ steps with probability going to 1, for every $t(n) = \omega(\psi(n)^{-1})$. This also implies that, when $2m/n = c_k + \phi(n) > c_k + n^{-\delta}$, where $\delta \in (0, 1/4)$ is a constant, the probability of deleting $\omega(\psi(n)^{-1})$ vertices of the $k$-core of $G(n, m)$ to find its new $k$-core after deleting a single random edge goes to zero.

For the case with average degree $c \leq c'_k - \varepsilon$ where $\varepsilon$ is a positive constant, we use the random walk to show that, for any $h(n) \to \infty$, with probability going to 1, the deletion algorithm deletes $\Theta(n)$ vertices or at most $h(n)$ vertices. When $c \to k$, the probability of deleting $\Theta(n)$ vertices goes to 1. Then we use the differential equation method as described in [19] to show that, if $\Theta(n)$ vertices are deleted, then the deletion algorithm will not stop until the $k$-core has less than $\gamma n$ vertices a.a.s. (where we can choose $\gamma$ as small as we want). Using a result in [6], we prove that in this case the $k$-core must be empty a.a.s. This finishes the proof that, for $k + \varepsilon \leq c \leq c'_k + \varepsilon$ and any $h(n) \to \infty$, the deletion algorithm deletes $n$ vertices or at most $h(n)$ vertices a.a.s.; and that for $c \to k$, we delete $n$ vertices a.a.s. Proving that the probability of deleting all vertices in the case $k + \varepsilon \leq c \leq c'_k - \varepsilon$ is bounded away from zero require some more work: we couple the deletion algorithm for multigraphs and simple graphs for $t(n) \to \infty$ steps. This will then imply that the probability of deleting $h(n)$ vertices for some $h(n) \to \infty$ is bounded away from zero and so we must delete all vertices with probability bounded away from zero.

2 Main results

Let $G = G(k, n, m)$ be a graph sampled uniformly at random from the (simple) $k$-cores with vertex set $[n]$ and $m = m(n)$ edges. For any graph $H$, let $K(H)$ denote the $k$-core of $H$ and let $W(H)$ be $|V(H)| - |V(K(H - e))|$, where $e$ is an edge chosen uniformly at
random from the edges of $H$. That is, $W(H)$ is the number of vertices we delete from $H-e$ to obtain its $k$-core.

For every $k \geq 0$, let
\[
f_k(\lambda) = e^\lambda - \sum_{i=0}^{k-1} \frac{\lambda^i}{i!} \quad \text{and} \quad h_k(\mu) = \frac{e^\mu \mu}{f_{k-1}(\mu)}.
\]

For $k \geq 3$, let $c_k = \inf\{h_k(\mu) : \mu > 0\}$ and let $\mu_{k,c_k}$ be such that $c_k = h_k(\mu_{k,c_k})$. We discuss the existence of $c_k$ and $\mu_{k,c_k}$ later. Let
\[
c'_k = \frac{\mu_{k,c_k} f_{k-1}(\mu_{k,c_k})}{f_k(\mu_{k,c_k})}.
\]

Throughout the text, let $c = 2m/n$. The asymptotics will always be with respect to $n \to \infty$. For a sequence of probability spaces $(\Omega_n, \mathbb{P}_n)_{n \in \mathbb{N}}$, we say that a sequence of events $(E_n)_{n \in \mathbb{N}}$ holds asymptotically almost surely (a.a.s.) if $\mathbb{P}_n(E_n) \to 1$ as $n \to \infty$.

**Theorem 2.1.** Let $k \geq 3$ be a fixed integer. Let $m = m(n)$ and $c = 2m/n$. Then the following hold.

(i) If $c \geq k$ and $c \to k$, then $W(G(k,n,m)) = n$ a.a.s.

(ii) Let $\varepsilon > 0$ be a fixed real. Suppose that $k + \varepsilon \leq c \leq c'_k - \varepsilon$. For any function $h(n) \to \infty$, we have that a.a.s. $W(G(k,n,m)) \leq h(n)$ or $W(G(k,n,m)) = n$. Moreover, $W(G(k,n,m)) = n$ with probability bounded away from zero.

(iii) Let $\psi(n) = \omega(n^{-1/4})$ be a positive function and let $C_0$ be a constant. Suppose that $c'_k + \psi(n) \leq c \leq C_0$. For every $h(n) = \omega(\psi(n)^{-1})$, we have that $\mathbb{P}(W(G(k,n,m)) \geq h(n)) \to 0$.

We apply Theorem 2.1 to study the robustness of the $k$-core of $G(n,m)$, the random graph chosen uniformly at random from all graphs on $[n]$ with $m$ edges.

**Corollary 2.2.** Let $k \geq 3$ be a fixed integer. Let $m = m(n)$ and suppose that $c = 2m/n = c_k + \psi(n) \geq c_k + n^{-\delta}$ and $c \leq C_0$, where $\delta$ is a constant in $(0,1/4)$ and $C_0$ is a constant. Then, for every $h(n) = \omega(\psi(n)^{-1})$, we have that $\mathbb{P}(W(K(G(n,m)))) \geq h(n)) \to 0$.

We remark that there are some known results about the $k$-core of random graphs with given degree sequence under some constraints on the degree sequences (see [6,8,5]). Since the degree sequence of a graph $G$ and the degree sequence of $G-e$ for some edge $e \in E(G)$ are very similar, it is intuitive that one can draw some conclusions about $W(G(k,n,m))$. Indeed, in the case $c \in [c_k+c, C_0]$ one can use [6] to conclude that $W(G(k,n,m)) = o(n)$ a.a.s. We were not able to derive results for the cases (i) and (ii) directly from known results.
2.1 Models of random multigraphs

We use the allocation model restricted to $k$-cores (here we allow multigraphs): let $a : [2m] \rightarrow [n]$ be chosen uniformly at random among the functions such that $|a^{-1}(v)| \geq k$ for any $v \in [n]$; let $G_{\text{multi}} = G_{\text{multi}}(k,n,m)$ be the multigraph on $[n]$ obtained by adding an edge joining $a(i)$ and $a(m+i)$ for every $i \in [m]$. Then every simple $k$-core with $n$ vertices and $m$ edges is generated by $m!2^m$ allocations. This implies that $G_{\text{multi}}(k,n,m)$ conditioned upon simple graphs is a uniform probability space on $k$-cores with vertex set $[n]$ and $m$ edges. Multigraphs do not necessarily have the same probability in $G_{\text{multi}}(k,n,m)$.

Let $D_k(n,m)$ be the set of $d \in \mathbb{N}^n$ with $\sum_{i=1}^n d_i = 2m$ and $\min_i d_i \geq k$. For every multigraph $H$ with vertex set $[n']$, let $d(H)$ denote the degree sequence of $H$, that is, $(d(H))_i$ is the degree of vertex $i$. For any $d = (d_1, \ldots, d_n) \in \mathbb{N}^n$, let $D_j(d)$ be the number of occurrences of $j$ in $d$ and let $\eta(d) = \sum_{i=1}^n (d_i^2)/m$. We will work with $k$-cores generated using the pairing model with degree sequences in $D_k(n,m)$. Given a degree sequence $d$, let $G_{\text{multi}}(d)$ denote the graph generated using the pairing model: arbitrarily choose a partition of $[2m]$ into sets $S_1, \ldots, S_n$ (which we call bins) such that $|S_i| = d_i$ for every $i$, add a perfect matching uniformly at random on $[2m]$ and contract each $S_i$ to obtain a multigraph. Then $G_{\text{multi}}(k,n,m)$ conditioned upon $d(G_{\text{multi}}(k,n,m)) = d$ has the same distribution as $G_{\text{multi}}(d)$.

It is clear that $d(G_{\text{multi}})$ has multinomial distribution conditioned upon each coordinate being at least $k$, which we denote by $\text{Multi}_{\geq k}(n,m)$. We say that a variable $Y$ taking integer values has truncated Poisson distribution with parameters $(k, \lambda)$ (which we denote by $\text{Po}(k, \lambda)$) if, for every integer $j$,

$$
\P(Y = j) = \begin{cases} 
\frac{\lambda^j}{j!f_k(\lambda)}, & \text{if } j \geq k; \\
0, & \text{otherwise.}
\end{cases}
$$

By straightforward computations, one can show that $\text{Multi}_{\geq k}(n,m)$ has the same distribution as $Y = (Y_1, \ldots, Y_n)$ where the $Y_i$'s are independent truncated Poisson variables with parameters $(k, \lambda)$ conditioned upon the event $\Sigma$ that $\sum_{i=1}^n Y_i = 2m$.

3 Random walks and a deletion procedure

3.1 A deletion procedure

We are given a degree sequence $d \in D_k(n,m)$. Here we describe a procedure for finding the $k$-core of $G_{\text{multi}}(d) - e$, where $e$ is a random edge in $G_{\text{multi}}(d)$. We will sample $G_{\text{multi}}(d)$ using the pairing model by discovering one edge at a time. We start by choosing $e$ by picking two points uniformly at random from the set of all points.

Deletion procedure ($d$)

- Partition $[2m]$ into $n$ bins $S_1, \ldots, S_n$ such that $|S_i| = d_i$ for every $1 \leq i \leq n$. 
• Iteration 0: Choose $e$ by picking distinct points $u$ and $v$ uniformly at random from $[2m]$. Delete $u$ and $v$ and mark all points in bins of size less than $k$.

• Loop: While there is a marked undeleted point, choose one such point $u$ and find the other end $v$ of the edge incident to $u$. Delete $u$ and $v$. If $v$ was in a bin of size exactly $k$ (now of size $k - 1$ because we deleted $v$), mark all the other points in this bin.

After the deletion procedure is over, the $k$-core can be obtained by adding a random matching uniformly at random on the surviving points. Let $Z_0(d)$ denote the number of marked points after the deletion of the edge $e$ chosen in Iteration 0. Note that $Z_0(d) \in \{0, k - 2, k - 1, 2(k - 1)\}$.

Let $Y_j(d)$ be the number of undeleted marked points after the $j$-th iteration of the loop ($Y_0(d) := Z_0(d)$). The procedure stops when $Y_j(d) = 0$. Let $Z_j(d)$ be the number of points that are marked in the $j$-th iteration of the loop. Let $W(d) = \sum_j \left\lceil \frac{Z_j(d)}{k-1} \right\rceil$. Note that $W(d) = W(G_{\text{multi}}(d))$.

We mark new points in an iteration of the loop if $v$ lies in a bin of (current) size $k$. The probability that this happens (denoted by $p_j(d)$) is the ratio of the number of unmarked points in bins of (current) size $k$ and the number of undeleted points other than the one we are exploring. If $v$ is also a marked point, then no new points will be marked and $v$ is deleted. In this case, $Z_j(d) = -1$ and the probability that this happens (denoted by $p'_j(d)$) is the ratio of the marked undeleted points other than $u$ and the number of undeleted points other than $u$. Thus, in the $j$-th iteration of the loop,

$$Z_j(d) = \begin{cases} 
  k-1, & \text{with probability } p_j(d); \\
-1, & \text{with probability } p'_j(d) \\
0, & \text{otherwise.}
\end{cases}$$

The probabilities of $p_j(d)$ and $p'_j(d)$ are analyzed later.

### 3.2 Random walks

Given $c$ and $k$, we will define random walks in $\mathbb{Z}$ that will help us to study the behaviour of the deletion procedure. Let $\lambda_{k,c}$ be the (unique) positive root of $\frac{\lambda f_{k-1}(\lambda)}{f_k(\lambda)} = c$. Such root always exists for $c > k$. For more properties of $\lambda_{k,c}$, see [17]. Let

$$q_{k,c} = \frac{\lambda_{k,c}^{k-1}}{(k-1)!f_{k-1}(\lambda_{k,c})}.$$  

Let $Z(k,c)$ be a random variable such that

$$Z(k,c) = \begin{cases} 
  k-1, & \text{with probability } q_{k,c}; \\
0, & \text{otherwise.}
\end{cases}$$

Let $Y_0 = Z_0(d)$. For $j > 0$, let $Y_j = Y_{j-1} + Z_j - 1$ where $Z_j$ has same distribution as $Z(k,c)$ and the variable $Z_j$ is independent from $Z_1, Z_2, \ldots, Z_{j-1}$. Thus, we defined a
random walk such that the position in iteration $j$ is $Y_j$ and the drift is given by $Z_j - 1$. Similarly, for $\xi = \xi(n) \geq 0$ and $\xi \leq 1 - q_{k,c}$, define the random variable $Z^+(k,c,\xi)$ by

$$Z^+(k,c) = \begin{cases} k - 1, & \text{with probability } q_{k,c} + \xi; \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y_0^+ = Z_0(d)$. For $j > 0$, let $Y_j^+ = Y_{j-1}^+ + Z_j^+ - 1$ where $Z_j^+$ has same distribution as $Z^+(k,c,\xi)$ and the variable $Z_j^+$ is independent from $Z_1^+, Z_2^+, \ldots, Z_{j-1}^+$. Note that $(Y_j)_{j \in \mathbb{N}}$ and $(Y_j^+)_{j \in \mathbb{N}}$ are actually branching processes.

For $\xi = \xi(n) \geq 0$ and $\xi \leq q_{k,c}$, define the random variable and $Z^-(k,c,\xi)$ by

$$Z^-(k,c) = \begin{cases} k - 1, & \text{with probability } q_{k,c} - \xi; \\ -1, & \text{with probability } \xi; \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y_0^- = Z_0(d)$. For $j > 0$, let $Y_j^- = Y_{j-1}^- + Z_j^- - 1$ where $Z_j^-$ has same distribution as $Z^-(k,c,\xi)$ and the variable $Z_j^-$ is independent from $Z_1^-, Z_2^-, \ldots, Z_{j-1}^-$. We say that $Y_j$ is the number of particles alive in iteration $j$ and that $Z_j$ is the number of particles born in iteration $j$ (and similarly for $Y_j^+$, $Z_j^+$, and $Y_j^-$, $Z_j^-$).

The random walk given by $Z^+(k,c,\xi)$ is going to be used to bound the number of marked points in the deletion process by above, while the random walk given by $Z^-(k,c,\xi)$ will bound it from below. Here we will prove some properties of these random walks.

Recall that $h_k(\mu) = \mu e^\mu / f_{k-1}(\mu)$ and $c_k = \inf\{h_k(\mu) : \mu > 0\} = h_k(\mu_{k,c_k})$. Here we justify why the infimum is reached and why it is reached by a unique $\mu$. It is easy to see that $h_k$ is differentiable. Moreover, $h_k(\mu) \to \infty$ when $\mu \to 0$ and when $\mu \to \infty$. The first derivative of $h_k(\mu)$ is

$$\frac{e^\mu}{f_{k-1}(\mu)} \left(1 + \mu - \frac{f_{k-2}(\mu)}{f_{k-1}(\mu)}\right)$$

Using the fact that $f_{k-2}(\mu) = f_{k-1}(\mu) + \mu^{k-2}/(k-2)!$, it is clear that this derivative is at least 0 iff

$$\frac{\mu^{k-1}}{(k-2)!} \leq f_{k-1}(\mu)$$

and the functions on both sides are convex and increasing for $\mu > 0$. Thus, the function $h_k(\mu)$ must reaches its infimum in a unique point $\mu_{k,c_k}$ and the equation $h_k(\mu) = c$ has exactly two roots when $c > c_k$. Let $\mu_{k,c}$ denote the largest root of the equation $h_k(\mu) = c$. Recall that $c_k' = h_k(\mu_{k,c_k})$.

**Proposition 3.1.** The following hold:

(i) $E(Z(k,c))$ is a strictly decreasing function of $c$ for $c > k$ and $E(Z(k,c_k')) = 1$.

(ii) For any $\varepsilon > 0$ with $c_k' - \varepsilon > k$, there exists a positive constant $\alpha$ such that $E(Z(k,c_k' - \varepsilon)) > 1 + \alpha$. 

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(iii) Let \( \psi(n) \) be a nonnegative function with \( \psi(n) \leq C_0 \), where \( C_0 \) is constant. There exists a positive constant \( \beta \) such that \( \mathbb{E}(Z(k, c'_k + \psi(n))) \leq 1 - \beta \psi(n) \).

Proof. Let \( g(c) = \mathbb{E}(Z(k, c)) \). Note that \( g(c) = (k - 1)q_{k,c} \). By the definition of \( c'_k \), we have that (1) holds with equality for \( \mu = \mu_{k,c_k} \). This clearly implies \( g(c'_k) = 1 \). We have that \( \lambda_{k,c} \) is a strictly increasing function of \( c \) and vice-versa (see the derivative computation in [17, Lemma 1]). If \( c' > k \), then \( \lambda_{k,c} > 0 \). Thus, by considering \( c = c(\lambda) = \lambda f_{k-1}(\lambda)/f_k(\lambda) \) and differentiating with respect to \( \lambda \), we get

\[
\frac{d}{d\lambda} q_{k,c} = \frac{\lambda^{k-2}(k - 1 - \mathbb{E}(\text{Po}(k - 1, \lambda))))}{(k - 2)!f_k(\lambda)} < 0
\]

since \( \mathbb{E}(\text{Po}(k - 1, \lambda))) > k - 1 \). Thus, \( g(c) \) is strictly decreasing for \( c > k \).

It is easy to see that \( c(\lambda) \) is a smooth function on \( \lambda \in [\varepsilon', \infty) \) for any \( \varepsilon' > 0 \). By the Inverse Function Theorem, this implies that \( \lambda_{k,c} \) is a smooth function on \( c \in [c(\varepsilon'), C_0] \) and so \( g(c) \) is a smooth function on \( c \). Thus, the supremum \( \sup\{g'(c) : c'_k \leq c \leq C_0\} \) and the infimum \( \inf\{g'(c) : c'_k \leq c \leq C_0\} \) are both achieved and are both negative constants since \( g(c) \) is strictly decreasing. By the Mean Value Theorem, there are positive constants \( \alpha \) and \( \beta \) such that \( g(c) \geq 1 + \alpha|c - c'_k| \) for \( c(\varepsilon') < c < c'_k \) and \( g(c) \leq 1 - \beta|c - c'_k| \) for \( c'_k < c < C_0 \).

Proposition 3.2. Let \( k, c, \xi \) be such that \( \mathbb{E}(Z^-(k, c, \xi)) > 1 + \varepsilon \), for some constant \( \varepsilon > 0 \). Then \( \mathbb{P}(Y_j^- > 0, \forall j \geq 0) \) is bounded away from 0 and, for any function \( h(n) \to \infty \),

\[
\mathbb{P}(Y_j^- > 0, \forall j \geq h(n)) = 1 + o(1).
\]

Proof. The first part follows from the fact that \( (Y_j^-)_{j \geq 0} \) is a random walk in \( \mathbb{R} \) with positive expected drift (see e.g. [4, p. 366]). The second part is a straightforward application of the method of bounded differences since the variables \( Z_j^- \) are independent random variables with range \([-1, k - 1]\) (see [13]).

4 The case \( c > c'_k + \omega(n^{-1/4}) \)

Here we prove Theorem 2.1(iii). We start by proving a version of Theorem 2.1(iii) for random multigraphs with given degree sequence.

Theorem 4.1. Let \( \psi(n) = \omega(n^{-1/4}) \) be a positive function and let \( C_0 \) be a constant. Suppose that \( m = m(n) \) is such that \( c = 2m/n \) satisfies \( c'_k + \psi(n) \leq c \leq C_0 \). Let \( d \in D_k(n,m) \) be such that \( |D_k(d) - \mathbb{E}(D_k(Y))| \leq n\phi(n) \) for \( \phi(n) = o(\psi(n)) \), where \( Y = (Y_1, \ldots, Y_n) \) and the \( Y_i \)'s are independent truncated Poisson variables with parameters \( (k, \lambda_{k,c_k}) \). For every \( h(n) = \omega(\psi(n)^{-1}) \), we have that \( \mathbb{P}(W(G_{\text{multi}}(d)) \geq h(n)) = o(1) \).

Using Theorem 4.1 we can deduce a result about multigraphs with given number of vertices and edges, which is then used to prove Theorem 2.1(iii).
Corollary 4.2. Let $\psi(n) = \omega(n^{-1/4})$ be a positive function and let $C_0$ be a constant. Suppose that $c = 2m/n$ is such that $c^1 + \psi(n) \leq c \leq C_0$. For every $h(n) = \omega(\psi(n)^{-1})$, we have that $\mathbb{P}(W(G_{\text{multi}}(k, n, m)) \geq h(n)) = o(1)$.

Now we prove Theorem 4.1. We will choose $\xi$ big enough so that $Z_j(d)$ is stochastically bounded from above by $Z_j^+$ for $j \leq t(n)$ steps, where $Z_j^+$ has the same distribution as $Z^+(k, c, \xi)$. Recall we start the deletion process with $n$ bins with $d_i$ points inside each bin $i$. Let $p$ denote the initial ratio between the number of points in bins of size $k$ and the total number of points. Note that $p = kD_k(d)/2m = q_{k,c}(1 + \phi_1(n))$, for some function $\phi_1(n)$ such that $\phi_1(n) = O(\phi(n))$. Choose $t(n) = \psi(n)^{-1}n^\alpha$, where $\alpha$ is constant in $(0, 1/2)$. Then, for $1 \leq j \leq t(n)$,

$$\frac{kD_k(d) - (j + 2)(k - 1)}{2m - 2j - 2} \leq p_j(d) \leq \frac{kD_k(d)}{2m - 2j - 2}$$

and so $p_j(d) = p + O(t(n)/n)$. We can assume $h(n) \leq t(n)$. Proposition 3.1 implies that $q_{k,c} \leq 1/(k - 1)$. Since $t(n)/n = o(\psi(n))$, we can choose $\xi > 0$ such that $\xi = o(\psi(n))$ and $\xi < 1 - q_{k,c}$ and $Z_j^+ \geq Z_j$ for all $j \leq t$. Now $\mathbb{E}(Z^+(k, c, \xi)) = 1 - \psi(n) + (k - 1)\xi$ according to Proposition 4.1 for some positive constant $\beta$. Since $\xi = o(\psi)$, we have $\mathbb{E}(Z^+(k, c, \xi)) \leq 1 - \beta^2\psi(n)$ for some positive constant $\beta$. Thus, we have that $\mathbb{E}(Y_i^+) = O((1 - \beta^2\psi(n))^t(n)) = O(\exp(-t(n)\beta^2\psi(n))) = o(1)$ because $t(n) = n^\alpha/\psi(n)$ with $\alpha > 0$. This implies that the deletion procedure stops before $t(n)$ steps a.a.s., which proves Lemma 4.1.

4.1 Proof of Corollary 4.2 and Theorem 2.1(iii)

Let $h(n) = \omega(\psi(n)^{-1})$. Choose $\phi(n)$ such that $\phi(n) = o(\psi(n))$ and $\phi(n) = \omega(n^{-1/4})$. First we will prove Corollary 4.2. We will show that the degree sequences that satisfy the hypotheses in Lemma 4.1 are the ‘typical’ degree sequences for $G_{\text{multi}}(k, n, m)$. Let $\tilde{D}_k(n, m)$ be the set of degree sequences $d$ satisfying $|D_k(d) - \mathbb{E}(D_k(Y))| \leq n\phi(n)$. Recall that $d(G_{\text{multi}}(k, n, m))$ has the same distribution as $Y = (Y_1, \ldots, Y_n)$ such the $Y_i$’s are independent truncated Poisson variables with parameters $(k, \lambda_{k,c})$ and conditioned to the event $\Sigma$ that $\sum_i Y_i = 2m$. Using Chebyshev’s inequality,

$$\mathbb{P}(|D_k(Y) - \mathbb{E}(D_k(Y))| \geq \phi(n)n) \leq \frac{n}{n^2\phi(n)^2}. $$

By [17] Theorem 4(a)], it is easy to see that the probability of $\Sigma$ is $\Omega(1/\sqrt{n})$. Thus,

$$\mathbb{P}(d(G_{\text{multi}}(k, n, m)) \notin \tilde{D}_k(n, m)) \leq \frac{\mathbb{P}(Y \notin \tilde{D}_k(n, m))}{\mathbb{P}(\Sigma)} = O\left(\frac{n\sqrt{n}}{n^2\phi(n)^2}\right) = o(1). \quad (2)$$

For every $n \in \mathbb{N}$, since the set $\tilde{D}_k(n, m)$ is finite, there exists a degree sequence $d^*(n)$ such that $\mathbb{P}(W(G_{\text{multi}}(d^*(n))) \geq h(n)) = \max \{\mathbb{P}(W(G_{\text{multi}}(d)) \geq h(n)) : d \in \tilde{D}_k(n, m)\}$. Set $r(n) = \mathbb{P}(W(G_{\text{multi}}(d^*(n))) \geq h(n))$. Theorem 4.1 implies that $r(n) = o(1)$. Thus, for any sequence $(d(n))_{n \in \mathbb{N}}$ such that $d(n) \in \tilde{D}_k(n, m)$ for every $n \in \mathbb{N}$, we have that $\mathbb{P}(W(G_{\text{multi}}(d(n))) \geq h(n)) \leq r(n) = o(1)$. This is usually expressed by saying that
\[ P(W(G_{\text{multi}}(d(n))) \geq h(n)) \rightarrow 0 \text{ uniformly for } d \in D_k(n, m). \] Together with (2), this implies that \( P(W(G_{\text{multi}}(k, n, m)) \geq h(n)) = o(1), \) proving Corollary 4.2.

We will now prove Theorem 2.1(iii). To deduce the result for simple graphs, we impose further conditions on the degree sequences: let \( \tilde{D}_k(n, m) \) be the set of degree sequences in \( D_k(n, m) \) that satisfy the conditions that \( \max_i d_i \leq n^\varepsilon \) for some \( \varepsilon \in (0, 0.25) \) and that \( |\eta(d) - \mathbb{E}(\eta(Y))| \leq \phi(n). \) One can easily prove that \( \text{Var}(Y_i(Y_i - 1)) = O(1) \) and so uniformly for \( n \) and \( m \) with \( c < C_0, \) by Chebyshev's inequality,

\[
P \left( |\eta(Y) - \mathbb{E}(\eta(Y))| \geq \phi(n) \right) = O \left( \frac{1}{n\phi(n)^2} \right).
\]

For \( j_0 > 2c\lambda_{k,c}, \) we have that \( P(Y_1 > j_0) = O(\exp(-j_0/2)). \) This holds because the ratio \( P(Y_1 = j + 1)/P(Y_1 = j) \) is less than \( 1/e \) for \( j \geq j_0/2 \) (This is the same equation as [17] Equation (27)). Thus, \( P(\max_j Y_j \geq n^\varepsilon) = O(n \exp(-n^\varepsilon/2)). \) This implies that \( P(d(G_{\text{multi}}(k, n, m)) \in \tilde{D}_k(n, m)) \) is also \( 1 + o(1). \) For \( d \in \tilde{D}_k(n, m), \) the probability of that \( G_{\text{multi}}(d) \) is simple is already known (see [13][14]):

\[
P(G_{\text{multi}}(d) \text{ simple}) = \exp \left( -\frac{\eta(d)}{2} - \frac{\eta(d)^2}{4} + O \left( \frac{\max_i d_i^4}{n} \right) \right)
\sim \exp \left( -\frac{\bar{\eta}_c}{2} - \frac{(\bar{\eta}_c)^2}{4} + O \left( \frac{\max_i d_i^4}{n} \right) \right),
\]

where \( \bar{\eta}_c := \lambda_{k,c}f_{k-2}(\lambda_{k,c})/f_{k-1}(\lambda_{k,c}). \) We can apply the same argument on the uniformity of the bound for \( P(W(G_{\text{multi}}(d)) \geq h(n)) = o(1) \) as above to \( P(G_{\text{multi}}(d) \text{ is simple}) - \exp (-\eta_c/2 - \eta_c^2/4) \) and conclude that

\[
P(G_{\text{multi}}(k, n, m) \text{ is simple}) = \exp (-\eta_c/2 - \eta_c^2/4) + o(1) = \Omega(1)
\]

and so

\[
P \left( W(G(k, n, m)) \geq h(n) \right)
\triangleq P(W(G_{\text{multi}}(k, n, m)) \geq h(n) \mid G_{\text{multi}}(k, n, m) \text{ is simple})
\leq \frac{P(W(G_{\text{multi}}(k, n, m)) \geq h(n))}{P(G_{\text{multi}}(k, n, m) \text{ is simple})}
= o(1).
\]

This finishes the proof of Theorem 2.1(iii).

4.2 The \( k \)-core of \( G(n, m) \)

In this section we prove Corollary 2.2. We will use [16] Theorem 2. Although this result does not state the number of edges in the \( k \)-core, it can be obtained from its proof with the main steps in [16] Equations (6.18),(6.34) and [16] Corollary 1 applied to \( J_1. \) We restate [16] Theorem 2 with the number of edges here:
Theorem 4.3 ([16] Theorem 2]). Suppose \( c > c_k + n^{-\delta} \), \( \delta \in (0,1/2) \) being fixed. Fix \( \sigma \in (3/4,1 - \delta/2) \) and \( \tilde{\zeta} = \min\{2\sigma - 3/2,1/6\} \). Then with probability \( \geq 1 + O(\exp(-n^\delta)) \) (\( \forall \zeta < \tilde{\zeta} \)), the random graph \( G(n,m = cn/2) \) contains a giant \( k \)-core with \( e^{-\mu_k \cdot f_k(m_k,c)n + O(n^\sigma)} \) vertices and \( (1/2)\mu_k e^{-\mu_k \cdot f_k-1(m_k,c)n + O(n^\sigma)} \) edges.

We are now ready to prove Corollary 5.2. Recall that \( c \geq c_k + n^{-\delta} \), where \( \delta \in (0,1/4) \).

So \( \delta = 1/4 - \varepsilon \), where \( \varepsilon \) is a constant in \( (0,1/4) \). Let \( \varepsilon' < \varepsilon \) be a constant such that \( \varepsilon' < 1/4 - \delta/2 \). Fix \( \sigma = 3/4 + \varepsilon' \). Thus, the average degree of the \( k \)-core is

\[
\frac{\mu_k \cdot f_k^{-1}(\mu_k, c)}{f_k(\mu_k, c)} (1 + O(n^{-1/4 + \varepsilon'})).
\]

Recall that \( h'(\mu_k, c) = 0 \) and \( h'(\mu) > 0 \) for \( \mu > \mu_k, c \). This implies that \( \mu_k, c = \mu_k, c_k + \Omega(c - c_k) \). Moreover, the function \( x \mapsto x f_k^{-1}(x)/f_k(x) \) is smooth. Thus, the average degree of the \( k \)-core of \( G(n,m) \) is \( (\varepsilon' + \Theta(c - c_k))(1 + O(n^{-1/4 + \varepsilon'})) \). Since \( c - c' > n^{-\delta} = n^{-1/4 + \varepsilon} \) with \( \varepsilon > \varepsilon' \), the average degree of the \( k \)-core is \( \epsilon' + \Theta(c - c_k) \). We can now apply Theorem 2.1(iii) to obtain the desired result.

5 The case \( k \leq c \leq c'_k - \varepsilon \): deleting \( \Theta(n) \) vertices

The following result is an intermediate step for the proof of Theorem 2.1(i) and (ii).

Theorem 5.1. Let \( \varepsilon > 0 \) be a fixed real. Suppose that \( k \leq c \leq c'_k - \varepsilon \). Let \( \phi(n) = o(1) \).

Let \( d \) be such that \( D_k(d) \geq E(D_k(Y))(1 - \phi(n)) \), where \( Y = (Y_1,\ldots,Y_n) \) and the \( Y_i \)'s are independent truncated Poisson variables with parameters \( (k,\lambda_{k,c}) \).

Then there exists a constant \( \varepsilon' > 0 \) (depending on \( \varepsilon \)) such that, for every function \( h(n) \rightarrow \infty \), we have that a.a.s. \( W(G_{\text{multi}}(d)) \leq h(n) \) or \( W(G_{\text{multi}}(d)) \geq \varepsilon'n \). Moreover, \( W(G_{\text{multi}}(d)) \geq \varepsilon'n \) with probability bounded away from zero.

The proof of the following corollary is very similar to the proof in Section 4.1 and so we omit it.

Corollary 5.2. Let \( \varepsilon > 0 \) be a fixed real. Suppose that \( k \leq c \leq c'_k - \varepsilon \). Then there exists a constant \( \varepsilon' > 0 \) (depending on \( \varepsilon \)) such that, for every function \( h(n) \rightarrow \infty \), we have that a.a.s. \( W(G\text{multi}(k,n,m)) \leq h(n) \) or \( W(G\text{multi}(k,n,m)) \geq \varepsilon'n \). Moreover, \( W(G\text{multi}(k,n,m)) \geq \varepsilon'n \) with probability bounded away from zero.

For the case \( c \rightarrow k \), Theorem 5.1 implies a stronger result because there is a function \( h(n) \rightarrow \infty \) such that \( W(G\text{multi}(k,n,m)) \geq h(n) \) steps a.a.s. From this one can deduce the following result.

Corollary 5.3. If \( c \geq k \) and \( c \rightarrow k \), then there exists a constant \( \varepsilon' > 0 \) such that \( W(G\text{multi}(k,n,m)) \geq \varepsilon'n \) a.a.s.

Now we prove Theorem 5.1. We will choose \( \xi \) so that \( Z_j(d) \) is stochastically bounded from below by \( Z^-_j \), where \( Z^-_j \) has the same distribution as \( Z^-(k,c,\xi) \), and so that \( E(Z^-(k,c,\xi)) \) is bounded away from 1 from above. For \( j \geq 1 \), we have already seen that
$p_j(d) = (1 + O(j/n) + O(\phi(n)))q_{k,c}$. Moreover, for $j \geq 1$, the probability that $Z_j(d) = -1$ is at most $(k-1)(j+2)/(2m-2j-2)$.

Proposition 3.1 implies that $q_{k,c} > 1/(k-1)$ and that $E(Z^-(k,c,\xi)) \geq 1 + \alpha' - (k-1)\xi$ for some constant $\alpha' > 0$. Choose $\xi \in (0,\alpha'/(k-1))$. Thus, we have $E(Z^-(k,c,\xi)) \geq 1 + \alpha$ for some $\alpha > 0$. We can now choose $\varepsilon'' > 0$ small enough so that $p_j(d) \geq q_{k,c} - \xi$ and $\mathbb{P}(Z_j(d) = -1) \leq \xi$ for all $j \leq \varepsilon''n$. Thus, we can couple the processes for at least $t(n) = \varepsilon''n$ steps.

By Proposition 3.2, a.a.s. either $Y_j^- \leq 0$ for some $j \leq h(n)$ or $Y_j^- > 0$ for all $j$. Moreover, the latter occurs with probability bounded away from zero. Since the coupling holds for $t(n)$ steps with $Z_j^- \leq Z_j(d)$, a.a.s. either $Y_j(d) = 0$ for some $j \leq h(n)$ or $Y_j(d) > 0$ for $1 \leq j \leq \varepsilon''n$. Thus, a.a.s. either $W(G_{\text{multi}}(d)) \leq h(n) + 2$ or $W(G_{\text{multi}}(k,n,m)) \geq \varepsilon''n/(k-1)$. This completes the proof of Theorem 5.1.

6 The case $k \leq c \leq c_k' - \varepsilon$

In this section we will prove Theorem 2.1(i). We will also prove Theorem 2.1(ii) except for the claim that $W(G(k,n,m)) = n$ with probability bounded away from zero, which is handled in Section 7. We use the differential equation method as described in [19] Theorem 6.1] with stopping times. We will also use some results from [2].

We will use the pairing-allocation model $P(M,L,V,k)$ as described in [2]: given a set $M$ of points together with a perfect matching $E_M$ on $M$ and two disjoint set $L,V$ let $h$ be chosen uniformly at random from the functions mapping $M$ to $L \cup V$ such that $|h^{-1}(v)| \geq k$ for all $v \in V$ and $|h^{-1}(v)| = 1$ for all $v \in L$. Let $G_P = G_P(M,L,V,k)$ be the multigraph obtained by adding edges joining $h(a)$ and $h(b)$ for every $ab \in E_M$ and $h(a), h(b) \in V$. Note that $G_{\text{multi}}(k,n,m) = G_P([2m],\emptyset,[n],k)$ with $E_M = \{ \{i,m+i\} : i \in [m] \}$.

We say that the vertices in $V$ are heavy vertices and the vertices in $L$ are light vertices. We will also say that point $i \in M$ is in $v$ if $h(i) = v$.

Cain and Wormald [2] analyse a deletion procedure for obtaining the $k$-core. Here we will use a similar procedure with the only modifications being in the first step. The procedure receives as input $h : [2m] \to [n]$ such that $|h^{-1}(v)| \geq k$ for all $v \in [n]$.

Deletion procedure – pairing-allocation ($h$):

- Let $M = [2m]$, $L = \emptyset$ and $V = [n]$.
- Iteration 0: Choose $i \in [m]$ uniformly at random. Find $v = h(i)$ and $u = h(m+i)$. Delete $i$ and $m+i$ from $M$. If $u \neq v$ and $|h^{-1}(v)| = k$, then delete $v$ from $V$, add $k-1$ new elements to $L$ and redefine the action of $h$ on $h^{-1}(v) \setminus \{i\}$ as a bijection to the new elements. Similarly to $u$, if $u \neq v$ and $|h^{-1}(u)| = k$, then delete $u$ from $V$, add $k-1$ new elements to $L$ and redefine the action of $h$ on $h^{-1}(u) \setminus \{m+i\}$ as a bijection to the new elements. If $u = v$ and $|h^{-1}(v)| \leq k+1$, then delete $v$ from $V$, add $|h^{-1}(v)|-2$ new elements to $L$ and redefine the action of $h$ on $h^{-1}(v) \setminus \{i,m+i\}$ as a bijection to the new elements.
- Loop: While $L \neq \emptyset$, choose $j \in h^{-1}(L)$ uniformly at random. Delete $j$ and $m+j$ of $M$ and delete $h(j)$ from $L$. Find $v = h(m+j)$. If $v \in L$, delete $v$ from $L$. If
Let $h_0,M_0,L_0,V_0$ be the values of $h,M,L,V$, resp., after Iteration 0. Let $h_i,M_i,L_i,V_i$ be the values of $h,M,L,V$, resp., after the $i$-th iteration of the loop. Then the proof of \cite[Lemma 6]{2} gives us the same conclusion as \cite[Lemma 6]{2}:

**Lemma 6.1.** Starting with $h = \mathcal{P}(2m,\emptyset,[n],k)$ and conditioning upon the values of $M_i,L_i$ and $V_i$, we have that $h_i$ has same distribution as $\mathcal{P}(M_i,V_i,L_i,k)$.

### 6.1 The case $c \to k$

Here we prove Theorem \ref{2}–(i). We assume that $c = 2m/n = k + \phi(n)$, where $\phi(n) = o(1)$ and $\phi(n) \geq 0$. Let $S_i$ denote the number of points in heavy vertices just after the $i$-th iteration of the loop. Let $S_0$ denote the number of points in heavy vertices after Iteration 0. We will use $x$ as $i/n$ and $y(i/n)$ to approximate $S_i/n$.

Define $D_\gamma = \{(x,y): -\gamma < 2x < k - \gamma, \gamma < y < k + \gamma\}$. Note that $D_\gamma$ is bounded, connected and open. We choose $\gamma < \min\{\gamma_0/3,k\}$ so that the $k$-core cannot not empty and smaller than $\gamma_0n$ a.a.s. ($\gamma_0$ is given by Lemma \ref{2}). Moreover, we work with $n$ big enough so that $\phi(n) < \gamma$. After the first step there are at most $2(k - 1)$ points in $L_0$ and all the other vertices in $V_0$ and so $S_0 \geq 2m - 2(k - 1)$. Then it is clear that $S_0/n \leq k + \phi + k + \gamma$ and $S_0/n > \gamma$.

Let $T_D = \min\{i: (i/n,S_i/n) \notin D\}$. Let $W_i$ denote the number of light vertices after iteration $i$ is performed. We also use the stopping time $T = \min\{i: W_i = 0\}$. That is, there are no light vertices to be deleted and the deletion process has actually ended. We need to check the boundedness hypothesis, trend hypothesis and Lipschitz hypothesis (see \cite{19} for more details). The boundedness hypothesis is trivially true: $|S_i - S_{i+1}| \leq k$ always.

Now we check the trend hypothesis. Let $f(x,y) = -ky/(k - 2x)$. Let $H_i$ denote the history of the process at iteration $i \geq 1$. We need to show that $\xi_1 := |E(S_{i+1} - S_i|H_i) - f(i/n,S_i/n)| = o(1)$ while the $i < T$ and $i < T_D$. We have that $S_i - S_{i+1}$ if $j$ is matched to a light vertex, is $-1$ if $j$ is matched to a heavy vertex with degree $k$ and is $-k$ if $j$ is matched to a point in a heavy vertex with degree exactly $k$. The probability that $j$ is matched to a point in a heavy vertex is $S_i/(2m - 2i - 2)$. The probability that such a heavy vertex has degree $k$ is at least $1 - \sum \{d_i : d_i > k\}/S_i$ where $d$ is the degree sequence (we do not sample the degree sequence, we just decide if the vertex had degree $k$ or not). But for every possible degree sequence $1 - \sum \{d_i : d_i > k\}/S_i \geq 1 + n\phi(n)/S_i = 1 + o(1)$ whenever $S_i \geq \gamma n$. Then

$$E(S_{i+1} - S_i|H_i) = \frac{-k|S_i|}{2m - 2i - 2} + o(1),$$

and so the trend hypothesis holds. It is easy to see that the Lipschitz hypothesis also holds in $D_\gamma \cap \{(x,y) : x \geq 0\}$.

According to \cite{19} Theorem 6.1, the $y'(x) = f(x,y)$ has a unique solution in $D_\gamma$, say $y^*$, with $y(0) = k$ and a unique solution in $D_\gamma$, say $y^{**}$, with $y(0) = S_0/n$. Note that $y^*$
is a fixed function while \( y^{**} \) is a random variable because \( S_0 \) is a random variable. The Lipschitz condition implies that, for any \( x \) with both \( (x, y^*(x)) \) and \( (x, y^{**}(x)) \) in \( D_\gamma \), we have that \( |y^*(x) - y^{**}(x)| = |k - S_0/n| R =: \xi_3 \), where \( R \) is some big constant and so \( \xi_3 = o(1) \). Let \( \xi_2 = o(1) \) and \( \xi_2 > \xi_1 \) and \( \xi_2 > \xi_3 \). By [19, Theorem 6.1], there is a constant \( C \) and a function \( \xi \to 0 \), such that, a.a.s. at each step \( i < \min\{T, n/\sigma\} \) we have that

\[
|S_i - ny^*(i/n)| \leq \xi n,
\]

where \( \sigma \) denotes the supremum of \( x \) such that \( (x', y^*(x')) \) and \( (x', y^{**}(x')) \) are at \( \ell^\infty \)-distance at most \( C\xi_2 \) of the boundary of \( D_\gamma \) for all \( 0 \leq x' \leq x \).

Let \( \varepsilon' \) be given by Corollary 5.3. For \( \varepsilon' < x < (k - \gamma)/2 \), we have that

\[
(k - 2x) - y^*(x) = (k - 2x) \left( 1 - \left( \frac{k - 2x}{k} \right)^{k/2-1} \right) \geq \frac{2\gamma \varepsilon'}{k}.
\]

This implies that, if \( (3) \) holds at \( i \) where \( \varepsilon'n < 2i < (k - \gamma)n \), then \( W_i = 2m - 2i - 2 - S_i = \Omega(n) \). Thus, if \( (3) \) holds for some step \( i \in (\varepsilon'n, \sigma n] \) with \( T > i \), then \( T > i + 1 \) because there are still \( \Omega(n) \) points to be deleted. This implies that, conditioning upon \( T > \varepsilon'n \), we have that \( T > \sigma n \) a.a.s.

For any constant \( \alpha \in (0, \gamma) \), using the fact that \( \xi_3 = o(1) \), there exists \( x \) such that \( x \leq \sigma n \) and \( (x, y^*(x)) \) and \( (x, y^{**}(x)) \) are at \( \ell^\infty \)-distance \( C\xi_2, \alpha \) of the boundary of \( D_\gamma \). For such an \( x \) we have \( T > x \) a.a.s. because \( T > \sigma n \) a.a.s. Thus, \( (3) \) holds a.a.s. Since \( x \) is at \( \ell^\infty \)-distance at most \( \alpha \) of the boundary of \( D_\gamma \), either \( 2x \geq k - \gamma - \alpha \) or \( y^*(x) \leq \gamma + \alpha \). We excluded \( y^*(x) \leq k + \gamma - \alpha \) because \( y^*(0) = k \) and \( y^* \) decreases as \( x \) increases. For \( n \) sufficiently large so that \( |\xi(n)| < \gamma \), the equation \( (3) \) for at either \( 2x \geq k - \gamma - \alpha \) or \( y^*(x) \leq \gamma + \alpha \) shows that \( S_i \leq n \gamma_0 \) a.a.s.

Since \( T > \varepsilon'n \) a.a.s. by Corollary 5.3 the \( k \)-core would have to be smaller than \( \gamma_0 n \) and so it must be empty a.a.s. (see Section 6.3). We conclude that \( W(G_{\text{multi}}(k, n, m)) = n \) a.a.s. Since the probability that \( G_{\text{multi}}(k, n, m) \) is simple is \( \Omega(1) \) (see [13, 14]), we have that \( W(G(k, n, m)) = n \) a.a.s.

### 6.2 The case \( c \in [k + \varepsilon, c + k' - \varepsilon] \)

We prove Theorem 2.1(ii) except for the claim that \( W(G(k, n, m)) = n \) with probability bounded away from zero, which is addressed in Section 7. Let \( h(n) \to \infty \). Let \( \varepsilon' \) be given by Corollary 5.2. Assume that \( c \to C \in [k + \varepsilon, c + k' - \varepsilon] \), where \( C \) is a constant. We will explain later how to drop this constraint.

Again we use the differential equation method as in [19] and the deletion procedure described in the beginning of the section. For each \( i \), let \( S_i \) denote the number of points in heavy vertices just after iteration \( i \), let \( T_i \) denote the number of heavy vertices just after iteration \( i \) and let \( W_i \) denote the number of points in light vertices just after iteration \( i \). We will use the differential equation method to approximate \( S_i \) and \( T_i \). Note that \( W_i = 2m - 2i - 2 - S_i \). We will use \( y(i/n) \) to approximate \( S_i/n \) and \( z(i/n) \) to approximate \( T_i/n \).
Let $\gamma$ be a positive constant with $\gamma < \min\{1, C - k\}$ to be chosen later. Define

$$D_{\gamma} = \{(x,y,z) : \gamma < z < 1 + \gamma, -\gamma < x < C - \gamma, \gamma < y < C + \gamma, y > (k + \gamma)z\}.$$  

Then $D_{\gamma}$ is bounded, connected and open. We have $T_0 \in \{n, n-1, n-2\}$ and $S_0 \in [2m - 2 - 2(k-1), 2m - 2]$. Thus, $T_0/n = 1 + o(1/n)$ and $S_0/n = C + o(1)$. Then $D_{\gamma}$ contains the closure of the points $(0,y,z)$ such that $\mathbb{P}(S_i = yn \text{ and } T_i = zn) \neq 0$ for some $n$.

We use the stopping time $T = \min \{i : W_i = 0\}$ again. We have to check the boundedness hypothesis, the trend hypothesis and the Lipschitz hypothesis. The boundedness hypothesis is again easy: $|S_{i+1} - S_i| \leq k$ and $|T_{i+1} - T_i| \leq 1$ always.

The trend hypothesis is exactly like in [2] with $\mathbb{E}(T_{i+1} - T_i|H_i) - f_z(i/n) = \xi_z = o(1)$ and $|\mathbb{E}(S_{i+1} - S_i|H_i) - f_y(i/n)| = \xi_y = o(1)$ with

$$f_z(x) = -\frac{y}{C - 2x} \left(1 - \frac{\mu z}{y}\right) \quad \text{and} \quad f_y(x) = -\frac{y}{C - 2x} \left(k - (k-1)\frac{\mu z}{y}\right),$$

where $\mu = \lambda_{k,y,z}$. The Lipschitz hypothesis is straightforward to check.

According to [19, Theorem 6.1], the $y'(x) = f_y(x)$ and $z'(x) = f_z(x)$ has unique solutions $(y^*, z^*)$ and $(y^{**}, z^{**})$, with initial conditions $y(0) = C$ and $z(0) = 1$, and $y(0) = S_0/n$ and $z(0) = T_0$, resp. The Lipschitz hypothesis implies that, there exists a constant $R$ such that, for any $x$ with both $(x, y^*(x), z^*(x))$ and $(x, y^{**}(x), z^{**}(x))$ in $D_{\gamma}$, we have $\max\{|y^*(x) - y^{**}(x)|, |z^*(x) - z^{**}(x)|\} \leq x|k - S_0/n| R =: \xi_3$. Note that $\xi_3 = o(1)$. Let $\xi_2 > \xi_z, \xi_2 > \xi_y, \xi_2 > \xi_3, \xi_2 = o(1)$. Thus, by [19, Theorem 6.1], there is a constant $C_0$ and a function $\xi \to 0$, such that, a.a.s. at each step $i < \min\{T, n\sigma\}$ we have that

$$\max\{|S_i - ny^*(i/n)|, |T_i - nz^*(i/n)|\} \leq \xi_n, \tag{4}$$

where $\sigma$ denotes the supremum of $x$ such that, for all $0 \leq x' \leq x$, we have $(x', y^*(x'), z^*(x'))$ and $(x', y^{**}(x'), z^{**}(x'))$ are at $\ell^\infty$-distance at least $C_0 \xi_2$ of the boundary of $D_{\gamma}$.

According to [2], we have that $\mu^2/(C - 2x)$ and $(z\varepsilon'')/f_k(\mu)$ are constants. With initial conditions $y(0) = C$ and $z(0) = 1$, we get $\mu^2/(C - 2x) = \lambda_{k,C}^2/C$ and $z\varepsilon''/f_k(\mu) = e^{\lambda_{k,C}/f_k(\lambda_{k,C})}$, which can be used to deduce that

$$y^* = (C - 2x)\frac{h_k(\lambda_{k,C})}{h_k(\mu)}. \tag{5}$$

For $x \geq \varepsilon'/2$, we must have $\mu(x) \leq \lambda_{k,C} \sqrt{1 - \varepsilon'/C}$ and so $h_k(\mu) \geq (1 + \varepsilon'')h_k(\lambda_{k,C})$, for some $\varepsilon'' > 0$. Thus, for every $x$ such that $\varepsilon' \leq 2x \leq C - \gamma$ using (5),

$$C - 2x - (C - 2x)\frac{h_k(\lambda_{k,C})}{h_k(\mu)} \geq \gamma\varepsilon''.$$ 

This implies that, if (3) holds at $i$ with $\varepsilon'n < 2i < (C - \gamma)n$, then $W_i = 2m - 2i - 2 - S_i = \Omega(n)$. Thus, if (3) holds for some step $i \in (\varepsilon'n, \sigma n)$ with $T > i$, then $T > i + 1$ because there are still $\Omega(n)$ points to be deleted. This implies that, conditioning upon $T > \varepsilon'n$, we have that $T > \sigma n$ a.a.s.
For any constant $\alpha \in (0, \gamma)$, using the fact that $\xi_3 = o(1)$, there exists $x$ such that $x \leq \sigma n$ and $(x, y^*(x), z^*(x))$ and $(x, y'^*(x), z'^*(x))$ are at $\ell^\infty$-distance $(C_0\xi_2, \alpha)$ of the boundary of $D_\gamma$. For such an $x$ we have $T > x$ a.a.s. because $T > \sigma n$ a.a.s. Thus, (4) holds a.a.s. Since $x$ is at $\ell^\infty$-distance at most $\alpha$ of the boundary of $D_\gamma$, either $z^*(x) \leq \gamma + \alpha$ or $2x \geq C - \gamma - \alpha$ or $y^*(x) \leq \gamma + \alpha$ or $y^*(x)/z^*(x) \leq k + \gamma + \alpha$. We excluded $y^*(x) \geq C + \gamma - \alpha$ and $z^*(x) \geq 1 + \gamma - \alpha$ because $y^*(0) = C$ and $z^*(0) = 1$ and $f_y$ and $f_z$ are decreasing.

If $2x \geq C - \gamma - \alpha$, then, using that $\mu^2/(C - 2x)$ remains constant and $\mu(0) = \lambda_{k,C}$ with $C < \ell_k'$, we have that $h_k(\mu) \geq h_k(\lambda_{k,C})$ and so $y^*(x) \leq C - 2x \leq \gamma + \alpha$. For $n$ sufficiently large so that $|\xi(n)| < \gamma$, it is easy to see that the equation (3) for at $y^*(x) \leq \gamma + \alpha$ or $z^*(x) \leq \gamma + \alpha$ implies that $S_i \leq 3\gamma n$ a.a.s. We still have to check what happens when $y^*(x)/z^*(x) \leq k + \gamma + \alpha$. In this case, $\mu = O(\gamma + \alpha)$. This holds because the function $g_k(x) = x f_{k-1}(x)/f_k(x)$ with domain $(0, 2\gamma)$ is strictly increasing and the limit of its one-sided derivative with $x \to 0$ is $1/(k + 1)$. This limit can be computed by the derivative of this function which is $(1/x) g_k(x) (1 + g_{k-1}(x) - g_k(x))$ and then using Taylor’s approximation for $g_k(x)$ and $g_{k-1}(x)$ around $x \to 0$. For more details on this computation see [17] Lemma 1 and its proof. Using that $\mu^2/(C - 2x)$ during the process, we then have $C - 2x = O(\gamma^2)$, we can then conclude that the $S_i = O(\gamma n)$ a.a.s.

Thus, conditioned upon $T > \varepsilon' n$ the $k$-core has at most $O(\gamma n)$ vertices a.a.s. Let $\gamma_0$ be the constant given by Lemma 6.2. By choosing $\gamma$ small enough, we can conclude that, conditioned upon $T > \varepsilon' n$, the $k$-core has less than $\gamma_0 n$ vertices a.a.s. and which implies, by Lemma 6.2 that the $k$-core must be empty a.a.s. By Corollary 5.2 we have that $W(G_{\text{multi}}(k, n, m)) \leq h(n)$ or $W(G_{\text{multi}}(k, n, m)) = n$ with probability $1 + o(1)$ conditioned upon $T > \varepsilon' n$ (where the convergence depends on $c$).

Recall that we assumed $c \to C$. We show how to drop this assumption here. Let $(c_i)_{i \in \mathbb{N}}$ such that every $c_i \in [k + \varepsilon, c + k' - \varepsilon]$. Let $r(n)$ be the probability that neither $W(G_{\text{multi}}(k, n, m)) \leq h(n)$ nor $W(G_{\text{multi}}(k, n, m)) = n$. Then every subsequence of $(c_i)_{i \in \mathbb{N}}$ has a subsequence that converges to some constant $C_0$ and in that subsequence $r(n) \to 0$. So by the subsequence principle $r(n) \to 0$. Since the probability that $G_{\text{multi}}(k, n, m)$ is simple is $\Omega(1)$, we have that $W(G(k, n, m)) \leq h(n)$ or $W(G(k, n, m)) = n$ a.a.s.

### 6.3 No small $k$-cores

**Lemma 6.2.** Let $C_0$ be a constant. Suppose that $m = m(n)$ satisfies $kn \leq 2m \leq C_0 n$. Then there exists a constant $\gamma$ such that a.a.s. the graph obtained from $G_{\text{multi}}(k, n, m)$ by deleting an edge chosen uniformly at random either has a $k$-core of size at least $\gamma n$ or its $k$-core is empty.

**Proof.** This is an application of a result by Luczak and Janson [6, Lemma 5.1]: if a degree sequence $(d_n)_{n \in \mathbb{N}}$ satisfies $\sum d_i \leq Rn$ for constants $\alpha$ and $R$, then there is a constant $\gamma$ such that a.a.s. no subgraph of $G_{\text{multi}}(d)$ with less than $\gamma n$ vertices has average degree at least $k$. We set $\alpha < 1/3$ and we will choose $R$ later. Let $D_k(n, m) \subseteq D_k(n)$ be the set of degree sequences $d$ such that $\sum d_i \leq Rn$. It suffices to show that the degree sequence $d = d(G_{\text{multi}}(k, n, m))$ is in $D_k(n, m)$ a.a.s.
Let $Y = (Y_1, \ldots, Y_n)$ be such that the $Y_i$’s are independent random variables with distribution $\text{Po}(k, \lambda_{k,c})$. As already mentioned before, $d$ has the same distribution of $Y$ conditioned upon the event that $\sum_i Y_i = 2m$. Using [17] Theorem 4, one can prove that $P(\sum_i Y_i = 2m) = \Omega(1/\sqrt{n})$.

For $J_0$ big enough (depending only on $C_0$), we have that $\lambda_{k,c}/J_0 \leq e^{-1}$, which implies $\lambda_{k,c}/J_0 \leq e^{-1}$. Clearly, $\sum_{j \leq J_0} e^{\alpha j} D_j(Y) \leq e^{\alpha J_0} n$. Let $J_1 = J_0 + (1 + \beta) \log n$ with $\beta \in (1/2, (2\alpha)^{-1} - 1)$. Let $p = \lambda_{k,c}^{-1}/((J_0 - 1)!f_k(\lambda_{k,c}))$. Then

$$P(\exists j > J_1 \text{ with } D_j(Y) > 0) \leq np \sum_{i \geq 0} e^{(1+\beta)\log n+i} \frac{1}{1-e^{-1}} = O(n^{-\beta}).$$

Using the fact that $P(\sum_i Y_i = 2m) = \Omega(1/\sqrt{n})$ and $\beta \in (1/2, (2\alpha)^{-1} - 1)$, we conclude that $P(\max_i d_i > J_1) = o(1)$. And so $\sum_{j > J_1} e^{\alpha j} D_j(Y) = 0$ a.a.s.

Now we consider $j \in (J_0, J_1)$. By Hoeffding’s inequality and using the fact that $P(\sum_i Y_i = 2m) = \Omega(1/\sqrt{n})$, we have that $P(|D_j(Y) - p^{(j)} n| \geq a\sqrt{n}) = O(\sqrt{n})e^{-a^2}$, where $p^{(j)} = \lambda_{k,c}^{-1}/(j!f_k(\lambda_{k,c}))$. Thus,

$$P\left(|D_j(Y) - p^{(j)} n| \geq a\sqrt{n} \text{ for some } j \in (J_0, J_1)\right) = (1 + \beta) \log n O(\sqrt{n})e^{-a^2} = O(n^{-\beta}),$$

for $a = \sqrt{(1+\beta')\log n}$ with $\beta' > 0$. Thus, a.a.s.

$$\sum_{j = J_0+1}^{J_1} e^{\alpha j} D_j(Y) \leq e^{\alpha J_0} \sum_{j = 1}^{(1+\beta)\log n} e^{\alpha j} \left( p \frac{1}{e^j n + a\sqrt{n}} \right) \leq e^{\alpha J_0} \left( np \sum_{j = 1}^{(1+\beta)\log n} e^{-2j/3} + e^{\alpha(J_1-J_0)}(J_1-J_0)a\sqrt{n} \right) \leq e^{\alpha J_0} \left( p \frac{1}{e^{2/3}(1-e^{-2/3})} + \frac{n^{(1+\beta)\alpha+1/2}(\log n)^{3/2}}{n\sqrt{1+\beta'(1+\beta)}} \right) n.$$

Using that $1 + \beta > (2\alpha)^{-1}$, we can set $R = e^{\alpha J_0}(1 + p/(1-e^{-2/3}) + \sqrt{1+\beta'(1+\beta)})$. □

7 Working with simple graphs

Lemma 7.1. Let $\varepsilon > 0$ be a fixed real. Suppose that $c = 2m/n$ satisfies $k+\varepsilon \leq c \leq c' - \varepsilon$. Then there exists a function $h(n) \to \infty$ such that $P(W(G(k,n,m)) \geq h(n)) = \Omega(1)$.

Together with Section 6.2, this lemma implies that $P(W(G(k,n,m)) = n) = \Omega(1)$, which completes the proof of Theorem 2.1(ii).

We now prove Lemma 7.1. We will work with degree sequences. Let $G(d)$ be chosen uniformly at random from all (simple) $k$-cores with degree sequence $d$. Let $\phi(n) = o(1)$
with \( \phi(n) = \omega(n^{-1/4}) \). Let \( Y = (Y_1, \ldots, Y_n) \) be such that the \( Y_i \)'s are independent truncated Poisson variables with parameters \((k, \lambda_{k,c})\). Let \( \tilde{D}_k(n, m) \) be the degree sequences \( d \) such that \( |D_k(d) - E(D_k(Y))| \leq n\phi(n) \) and \( \max_i d_i \leq n^\beta \) for some \( \beta \in (0, 0.25) \) and \( |\eta(d) - E(\eta(Y))| \leq \phi(n) \). Similarly to the proof in Section 4.1, one can prove that \( d(G(k, n, m)) \in \tilde{D}_k(n, m) \) a.a.s. Thus, it suffices to show that, there exists \( h(n) \to \infty \) such that

\[
\mathbb{P}(W(G(d)) \geq h(n)) = \Omega(1),
\]

for \( d \in \tilde{D}_k(n, m) \).

For \( d \in \tilde{D}_k(n, m) \), we will couple deletion algorithms for \( G(d) \) and \( G_{\text{multi}}(d) \) so that they coincide for \( t(n) \to \infty \) steps. We use a deletion algorithm that is essentially the same as the one we used in the other sections. The only difference is that we explore a whole vertex at a time (instead of an edge at a time) and mark the vertices that have to be deleted.

**Deletion procedure by vertex:**

- Iteration 0: Choose an edge \( uv \) uniformly at random, delete \( uv \) and mark the vertices with degree less than \( k \).
- Loop: While there is an undeleted marked vertex, say \( w \), find its neighbours, delete \( w \) and the edges incident to it, and then mark all neighbours of \( w \) that now have degree less than \( k \).

If we can do such a coupling for \( t(n) \to \infty \) iterations of the loop, then we can choose \( h(n) \to \infty \) such \( h(n) \leq \min\{t(n), \varepsilon' n\} \) with \( \varepsilon' \) as in Theorem 5.1 so that \( \mathbb{P}(W(G_{\text{multi}}(d)) \geq h(n)) = \Omega(1) \). This would imply that the deletion algorithm did not stop for at least \( h(n) \) steps and so \( \mathbb{P}(W(G(d)) \geq h(n)) \geq \Omega(1) \).

In the rest of this section, we show that there exists \( t(n) \to \infty \) such that we can couple the deletion algorithms for \( G(d) \) and \( G_{\text{multi}}(d) \) so that they coincide for \( t(n) \) iterations of the loop. For now assume that \( t(n) \to \infty \) with \( t(n) \leq \log n \). Later we add more restrictions on the growth of \( t(n) \). We show that the probabilities that a certain edge \( uv \) is chosen in the first step are asymptotically equivalent for \( G(d) \) and \( G_{\text{multi}}(d) \) and so the first step can be coupled. For the other steps \( i \leq t(n) \), we show that the probabilities that the set of neighbours of the vertex \( w \) is some specific set are again asymptotically equivalent for \( G(d) \) and \( G_{\text{multi}}(d) \) with some error \( \xi(n) = o(1) \). So we can couple the deletion algorithms for \( t(n) \) steps, where \( t(n) \) will depend on \( \xi(n) \). In the computations in this section, we will use \( \mathbb{P}_{\text{multi}} \) to denote the probabilities in the deletion procedure for \( G_{\text{multi}}(d) \) and we will use \( \mathbb{P} \) to denote the probabilities in the deletion procedure for \( G(d) \). First we analyse the procedure for multigraphs. Let \( uv \in \binom{V}{2} \). Then

\[
\mathbb{P}_{\text{multi}}(uv \text{ is chosen in the first step}) = \mathbb{P}_{\text{multi}}(uv \in E(G_{\text{multi}}(d)))
\]

\[
= \frac{d_u d_v}{m (2m - 1) !} (2m - 2)! 2^m m!
\]

\[
= \frac{d_u d_v}{m (2m - 1) !} (2m - 2)! 2^m (m - 1)!
\]

\[
= \frac{d_u d_v}{m (2m - 1) !} = \frac{d_u d_v}{m (2m) !} (1 + \xi_1(n)),
\]

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simple graph exists. In our case, Erdős-Gallai Theorem will be enough to ensure such a graph.

Let $L$ be a graph on $[n]$ with at most $kt(n)$ edges. Let $G$ be a supergraph of $H$ with at most $k$ edges more than $H$ such that there is a simple graph $G$ with degree sequence $d$ such that $G \cap L = H$. Then

$$\Pr(L \subseteq G(d) \mid H \subseteq G(d)) = \frac{\prod_{i=1}^{n} [d_i - h_i]_{\tilde{\ell}}}{2^{\ell_{E(J)}|E(J)|}} (1 + \xi_2(n)),$$

where $\xi_2(n) = o(1)$ because $m \geq m - kt(n) \geq m - k \log n$. Now we have to compute estimates for the probabilities in the deletion algorithm for simple graphs. The following lemma is an application of [12 Theorem 10].

**Lemma 7.2.** Let $d \in D_k(n, m)$ be such that $\max_i d_i \leq n^{0.25}$. Let $H$ be a graph on $[n]$ with at most $kt(n)$ edges. Let $L$ be a supergraph of $H$ with at most $k$ edges more than $H$ such that there is a simple graph $G$ with degree sequence $d$ such that $G \cap L = H$. Then

$$\Pr(L \subseteq G(d) \mid H \subseteq G(d)) = \frac{\prod_{i=1}^{n} [d_i - h_i]_{\tilde{\ell}}}{2^{\ell_{E(J)}|E(J)|}} (1 + v(n)),$$

where $h$ is the degree sequence of $H$, $J = L - E(H)$, $j$ is the degree sequence of $J$, and $v(n) = o(1)$.

Notice that to use this lemma one has to check the existence of a simple graph $G$ with certain properties. In our case, Erdős-Gallai Theorem will be enough to ensure such a simple graph exists.

**Lemma 7.3.** Let $n$ be sufficiently large so that $n - n^{0.25} - k \log n > \sqrt{n}$. Let $n' \geq n - \log n$. Let $g$ be a sequence on $[n']$ such that $g_1 \geq g_2 \geq \cdots \geq g_{n'}$. Let $g_i$ is even, $g_1 \leq n^{0.25}$, $|\{j : g_j = 0\}| \leq k \log n$. Then there exists a simple graph with degree sequence $g$.

The proofs for these lemmas are presented in Section 7.1. Now we can analyse the deletion algorithm for simple graphs. Let $uv \in \binom{V}{2}$. Then

$$\Pr(uv \text{ is chosen in the first step}) = \Pr(uv \in E(G(d))) \frac{1}{m}.$$

We need to compute $\Pr(uv \in E(G(d)))$. Note that this is the same as $\Pr(L \subseteq G(d) \mid H \subseteq G(d))$ with $L = ([n], \{uv\})$ and $H = ([n], \emptyset)$. In order to use Lemma 7.2 we need to check if there is a simple graph $G$ with $G \cap L = H$ with degree sequence $d$. This is the same as saying that there exists a simple graph $G$ with degree sequence $d$ such that $uv \notin E(G)$. It suffices to show that, for every set of vertices $S \subseteq [n] \setminus \{u, v\}$ of size $d_v$, there is a simple graph with degree sequence $d'$, where $d'$ is obtained from $d$ by deleting $v$ and decreasing the degree of every vertex in $S$ by 1 (that is, $S$ can be the set of neighbours of $v$ and it does not include $u$). Note that $\sum_i d'_i$ is even because $\sum_j d_j$ is even. Moreover,
Lemma 7.3, there is a simple graph with degree sequence $d$ and so we can use part (a) of [12, Theorem 2.10] to obtain that

$$
\mathbb{P}(uv \in G) = \frac{d_u d_v}{2m}(1 + \xi_3(n)),
$$

where $\xi_3(n) = o(1)$. Now suppose we are in the $i$-th iteration of the loop and deleting a vertex $w$. Let $\tilde{n}$ be the number of undeleted vertices in the beginning of iteration $i$ and let $\tilde{m}$ be the number of undeleted edges at the beginning of iteration $i$. Let $\tilde{d}$ be the current degree sequence (that is, $\tilde{d}_u$ is the number neighbours $u$ has among the undeleted vertices). At each iteration we delete at most $k$ edges and only one vertex. So $\tilde{n} \geq n - t(n)$ and $\tilde{m} \geq m - kt(n)$. Let $\ell := \tilde{d}_w$ and $\{u_1, \ldots, u_{\ell}\}$ be a set with $\ell$ (undeleted) vertices. Let $U$ be the neighbours of $w$ discovered in iteration $i$. We want to compute the probability that $U = \{u_1, \ldots, u_{\ell}\}$. In order to use Lemma 7.2, we have to check if there exists a simple graph $G$ with degree sequence $\tilde{d}$ such that $G \cap L = H$, where $H$ is the graph discovered so far (which includes the deleted vertices) and $L = ([n], E(H) \cup \{wu_1, \ldots, wu_{\ell}\})$, which is the same as checking if it is possible to get a simple graph such that $w$ as no neighbours in $\{u_1, \ldots, u_{\ell}\}$. Let $U'$ be a set of $\ell$ undeleted vertices such that $U' \cap \{u_1, \ldots, u_{\ell}\} = \emptyset$. There are plenty of choices for $U'$ since $t(n) \leq \log(n)$. Let $\tilde{d}'$ be the degree sequence on $\tilde{n} - 1$ obtained from $\tilde{d}$ by deleting $w$ and decreasing the degree of each vertex in $U'$ by 1. Then $\tilde{n} \geq n - \log(n)$, $\max_i \tilde{d}'_i \leq n^{0.25}$ and $|\{j: \tilde{d}'_j = 0\}| \leq kt(n) \leq k \log n$. Using Lemma 7.3 there is a simple graph with degree sequence $\tilde{d}'$ and so, by Lemma 7.2

$$
\mathbb{P}(U = \{u_1, \ldots, u_{\ell}\}) = \frac{\ell! \prod_{i=1}^{\ell} \tilde{d}_{u_i}}{2^\ell \tilde{m}^\ell}(1 + \xi_4(n)),
$$

where $\xi_4(n) = o(1)$. Thus, there exists a function $\xi(n) = o(1)$ such that

$$
\mathbb{P}(U = \{u_1, \ldots, u_{\ell}\}) = \mathbb{P}_{\text{multi}}(U = \{u_1, \ldots, u_{\ell}\})(1 + \xi(n))
$$

and, for every $uv \in \binom{V}{2}$,

$$
\mathbb{P}(uv \text{ is chosen in the first step}) \sim \mathbb{P}_{\text{multi}}(uv \text{ is chosen in the first step}).
$$

We conclude that the deletion algorithms can be coupled for $t(n)$ steps as long as $(1+\xi)^t = 1 + o(1)$. Thus, it suffices to choose $t = o(1/\xi)$.

### 7.1 Proofs of Lemma 7.2 and Lemma 7.3

**Proof of Lemma 7.2** Let $\Delta_L$ be the maximum degree in $L$ and let $\Delta$ be the maximum degree in $D$. Note that $\Delta_L \leq |E(H)| + k \leq k \log n + k$ and $\Delta \leq n^{0.25}$. Then

$$
|E(G(d))| - |E(H)| - |E(J)| \geq m - kt(n) - k \geq n^{0.25}(2n^{0.25})
$$

$$
\geq \Delta(\Delta + \Delta_L) =: D.
$$

So we can use part (a) of [12, Theorem 2.10] to obtain that

$$
\mathbb{P}(L \subseteq G(d) | H \subseteq G(d)) \leq \frac{\prod_{i=1}^{\nu_1(n)} \left(d_i - h_i \right) j_i}{2 |E(J)| \left[ m - |E(H)| - D \right] |E(J)|}
$$

$$
= \frac{\prod_{i=1}^{\nu_1(n)} \left(d_i - h_i \right) j_i}{2 |E(J)| \left[ m \right] |E(J)|}(1 + \nu_1(n))
$$

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with \( \nu_1(n) = o(1) \) because \( |E(J)| \leq k \) and \( m - |E(H)| - D \geq m - k \log n - 2\sqrt{n} \). Now we will use part (b) of Theorem 2.10 from \cite{12}. We have that
\[
|E(G(d))| - |E(H)| - |E(J)| \geq m - kt(n) - k \geq n^{0.25}(n^{0.25} + 1)
\geq \Delta(\Delta + \Delta_L + 2) + \Delta(\Delta_L + 1),
\]
so we can apply \cite{12}[Theorem 2.10(b)]. We have to bound some errors given by \cite{12}[Theorem 2.10(b)]. We have that
\[
0 \leq \frac{\Delta(\Delta_L + 1)}{m - |E(H)| - |E(J)| - \Delta(\Delta + \Delta_L + 2)} \\
\leq \frac{n^{0.25}(n^{0.25} + 1)}{n - k \log n - n^{0.25}(2n^{0.25} + 2)} =: \nu_2(n),
\]
with \( \nu_2(n) = o(1) \), and
\[
0 \leq \frac{\Delta^2}{2(|E(G)| - |E(H)| - D - (1 - 1/e)|E(J)|)} \\
\leq \frac{\sqrt{n}}{2(n - k \log n - n^{0.25}(2n^{0.25}) - (1 - 1/e)k)} =: \nu_3(n),
\]
with \( \nu_3(n) = o(1) \). Then \cite{12}[Theorem 2.10(b)] implies that
\[
\mathbb{P}(L \subseteq G(d) \mid H \subseteq G(d)) \geq \frac{\prod_{i=1}^{n}[d_i - h_i]_{j_i}}{2^{|E(J)|}|m||E(J)|}(1 + \nu_4(n)) \left( \frac{1 + \nu_2(n)}{1 + \nu_3(n)} \right)^{|E(J)|}.
\]
with \( \nu_4(n) = o(1) \). Since \( \nu_i(n) = o(1) \) for \( i = 1, 2, 3, 4 \), we can conclude that
\[
\mathbb{P}(L \subseteq G(d) \mid H \subseteq G(d)) = \frac{\prod_{i=1}^{n}[d_i - h_i]_{j_i}}{2^{|E(J)|}|m||E(J)|}(1 + \nu(n)),
\]
where \( \nu = o(1) \).

**Proof of Lemma 7.3.** We will use Erdős-Gallai Theorem: \( g \) is the degree sequence of a simple graph iff, for every \( 1 \leq \ell \leq n' \),
\[
\sum_{i=1}^{\ell} g_i \leq \ell(\ell - 1) + \sum_{j=\ell+1}^{n'} \min\{\ell, g_j\}.
\]
If \( \ell \geq n^{0.25} + 1 \), then \( \sum_{i=1}^{\ell} g_i \leq \ell g_1 \leq \ell(\ell - 1) \). If \( \ell \leq n^{0.25} \),
\[
\sum_{i=1}^{\ell} g_i \leq \ell n^{0.25} \leq \sqrt{n} \leq n - n^{0.25} - k \log n \leq n' - \ell - |\{j : g_j = 0\}|
\leq \sum_{j=\ell+1}^{n'} 1 - |\{j : g_j = 0\}| \leq \sum_{j=\ell+1}^{n'} \min\{\ell, g_j\}.
\]
\( \square \)
Acknowledgments

I would like to thank my advisor Nicholas Wormald for his supervision during this project.

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