Six-dimensional \((2,0)\) theory on tori

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Abstract. The six-dimensional \((2,0)\) theories are a comparatively new and rather abstract type of quantum theory with important relations both to supersymmetric Yang-Mills theory in lower dimensions and to string- and \(M\)-theory in higher dimensions. After a short introduction to these theories, we focus on the case when they are considered on flat tori \([1][2]\). In particular, we give an example of how their ground state degeneracies can be computed, and also briefly discuss the spectrum of BPS-states. Finally, we comment on the automorphic transformation properties of the partition function of such a theory under the mapping class group of a six-torus.

The maximal dimension of a space(-time) which admits superconformal symmetry is \(d = 1+5\). The symmetry algebra is then

\[ osp(2,6|2n) = so(2,6) \oplus sp(2n) \oplus \text{odd generators} \]

for some \(n = 1,2,\ldots\), where the first two terms are the conformal algebra in six-dimensions and the \(R\)-symmetry algebra \([3]\).

Indications for the existence of a quantum theory with such symmetry (for \(n = 2\)) follows by considering type IIB string theory on a \((1+9)\)-dimensional space-time with a codimension four singularity of some \(ADE\)-type. A self-consistent six-dimensional theory without dynamical gravity on the locus of the singularity then decouples from the bulk theory \([4]\). These so called \((2,0)\) theories are highly unique: Apart from their \(ADE\)-type, they have no other discrete or continuous parameters.

Some reasons to study the \((2,0)\) theories:

- They are quite different from other theories we know of, and still rather mysterious. Understanding them is likely to lead to much new mathematics and physics.
- They give a good opportunity to learn about important aspects of string theory without having to deal with quantum gravity.
- They are related to Yang-Mills theory in lower dimensions. In particular, they give a geometric understanding of \(S\)-duality of \(N = 4\) super Yang-Mills theory in four dimensions.
- The study of \((2,0)\) theory might be our best way towards a rigorous definition of quantum theory with infinitely many degrees of freedom.

A way to introduce a parameter in a \((2,0)\) theory is to consider it on a space-time of the form \(M^{1.5} = M^{1.4} \times S^1\) with a compact direction of radius \(R\). At longer distances, the effective
theory is then given by maximally supersymmetric Yang-Mills theory on $M^{1,4}$ with coupling constant $g = R^{1/2}$ and action

$$S = \frac{1}{R} \int_{M^{1,4}} \text{Tr}(F \wedge *F + \ldots).$$

The gauge group is of the form $G/C$, where $G$ is simply connected with center subgroup $C$:

| type  | $G$     | $C$       |
|-------|---------|-----------|
| $A_{n-1}$ | $SU(n)$ | $\mathbb{Z}_n$ |
| $D_{2k}$    | $Spin(4k)$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $D_{2k+1}$  | $Spin(4k+2)$ | $\mathbb{Z}_4$ |
| $E_6$       | $E_6$    | $\mathbb{Z}_3$ |
| $E_7$       | $E_7$    | $\mathbb{Z}_2$ |
| $E_8$       | $E_8$    | $1$       |

In this way, $(2,0)$ theory can be seen as providing an ultra-violet completion of the Yang-Mills theory. The negative power of $R$ indicates that $(2,0)$ theory has no Lagrangian description [5].

Let $M^{1,4} = \mathbb{R} \times M^4$, where the first factor denotes time. Quantum states of the Yang-Mills theory on this space are characterized by their magnetic 't Hooft flux $m \in H^2(M^4, C)$ which determines the topological class of the gauge bundle over $M^4$, and their electric 't Hooft flux $e \in \text{Hom}(H^1(M^4, C), U(1)) \simeq H^3(M^4, C)$ which determines the transformation properties under 'large' gauge transformations [6]. From the perspective of $(2,0)$ theory on $M^{1,5} = M^{1,4} \times S^1$, we instead have a self-dual 't Hooft flux

$$f = e + m \in H^3(M^4 \times S^1, C).$$

A particularly interesting case is to consider $M^{1,5} = \mathbb{R} \times M^4 \times S^1$ with $M^4 = T^4$ a flat four-torus, since this preserves 16 supersymmetries. We can think of this as Yang-Mills theory on $\mathbb{R} \times T^4$ or as $(2,0)$ theory on $\mathbb{R} \times T^5$ with

$$T^5 = T^4 \times S^1.$$ 

The quantum states are characterized by their

- self-dual 't Hooft flux $f \in H^3(T^5, C)$
- energy $E \in \mathbb{R}^+$
- spatial momentum $p \in H^1(T^5, \mathbb{R})$
  (with the fifth component given by the Yang-Mills instanton number over $T^4$)
- $sp(4)$ $R$-symmetry representation $R$.

Supersymmetry implies the energy bound

$$E \geq |p|.$$ 

There are three classes of states:

- Vacuum states have $E = p = 0$ and are annihilated by all supercharges.
- BPS states have $E = |p| > 0$ and are annihilated by half of the supercharges.
• non-BPS states have $E > |p|$.

The spectra of vacua and BPS states are invariant under smooth deformations of the geometry of $T^5$, and can thus be followed from weak to strong coupling in the Yang-Mills perspective. We will discuss the computation of the spectrum of vacua from the Yang-Mills perspective. (Similar reasoning may be applied to the BPS-states, although we do not yet have any non-trivial checks on the results.)

At weak coupling, vacuum states are localized at orbifold singularities of the moduli space of flat connections over $T^4$. The low energy theory is given by maximally supersymmetric matrix quantum mechanics based on the subgroup $S$ of the gauge group $G/C$ left unbroken by the configuration at the singularity. This quantum mechanical model has a number $n_S$, depending on $S$, of normalizable ground states. Summing over the orbifold singularities gives the complete spectrum of vacua, which may be decomposed according to their electric and magnetic ’t Hooft fluxes $e$ and $m$.

Covariance under the $SL_4(\mathbb{Z})$ mapping class group of $T^4$ is manifest in the Yang-Mills formulation, but $(2,0)$ theory indicates covariance under the $SL_5(\mathbb{Z})$ mapping class group of $T^5 = T^4 \times S^1$. This leads to predictions that appear quite non-trivial from the Yang-Mills point of view.

As an example, we consider the $D_{2k+1}(2,0)$ theories. There are $6$ $SL_5(\mathbb{Z})$ orbits of self-dual ’t Hooft flux $f \in H^3(T^5, \mathbb{Z}_4)$. But a single orbit may be realized in different ways in the corresponding $Spin(4k + 2)/\mathbb{Z}_4$ Yang-Mills theory. In this way we get alternative expressions for the generating functions

$$N_f(q) = \sum_{k=0}^{\infty} N_f(D_{2k+1}) q^{4k+2}$$

of the number $N_f(D_{2k+1})$ of vacua with ’t Hooft flux $f$. (Here $q$ is a formal parameter.) E.g. for a certain $SL_5(\mathbb{Z})$ orbit of $f$, we have three alternative expressions (modulo $q^{4k}$-terms) for $N_f(q)$:

$$N_f(q) = \frac{1}{8} (P_{even}^8(q) + P_{odd}^8(q))$$
$$= \frac{1}{4} P_{even}^4(q) P_{odd}^4(q)$$
$$= Q^4(q) \left(P_{odd}^4(q^2) P_{even}^0(q^2) + 3 P_{odd}^5(q^2) P_{even}^7(q^2) + 3 P_{odd}^7(q^2) P_{even}^5(q^2) + P_{odd}^9(q^2) P_{odd}^3(q^2)\right)$$
$$= q^6 + 10q^{10} + 67q^{14} + 350q^{18} + \ldots.$$

Here

$$P_{even}(q) = \frac{1}{2} \prod_{k=1}^{\infty} \left(1 + q^{2k-1}\right) + \frac{1}{2} \prod_{k=1}^{\infty} \left(1 - q^{2k-1}\right)$$
$$P_{odd}(q) = \frac{1}{2} \prod_{k=1}^{\infty} \left(1 + q^{2k-1}\right) - \frac{1}{2} \prod_{k=1}^{\infty} \left(1 - q^{2k-1}\right)$$
$$Q(q) = \prod_{k=1}^{\infty} \left(1 + q^{2k}\right).$$

A promising approach to understand the complete spectrum of states is to consider the partition functions

$$Z_f = \text{Tr}_{\mathcal{H}_f} \exp(-tE + ix \cdot P + iAR),$$
where $H_f$ is the Hilbert space of states with self-dual ’t Hooft flux $f \in H^3(T^5, C)$, and $t$, $x$, and $A$ are some formal parameters. After continuation to Euclidean time, these partition functions can be seen as pertaining to a particular decomposition of a flat six-torus $T^6 = S^1 \times T^5$ defined by the $T^5$ geometry together with $t$ and $x$, where the first factor denotes the 'time' direction. Twisting by $R$-symmetry in the spatial directions determines, together with the parameters $A$, a flat $sp(4)$ connection over this $T^6$.

The set of partition functions $Z_f$ for $f \in H^3(T^5, C)$ should have automorphic properties under the $SL_6(Z)$ mapping class group of $T^6$. Indeed, these partition function can be regarded as components of an element $Z$ of a certain vector space $V$.

We then have a basis $E_f$, $f \in H^3(T^5, C)$ of $V$. This is such that

\[ E_f = \Phi_f E_0 \]

The $Z_f$ are the components of $Z$ relative to this basis.

E.g. under a continuous shift of the 'time' cycle of $T^6$ by an integer linear combination $\beta$ of the spatial cycles, $Z_f$ is multiplied by an $f$-dependent phase factor:

\[ Z_f \mapsto Z_v \exp \left( \pi i \int_{T^5} f \wedge f[\beta] \right). \]

The Hamiltonian interpretation is that the spatial momentum $p \in H^1(T^5, \mathbb{R})$ obeys the shifted quantization law

\[ p - f \cdot f \in H^1(T^5, \mathbb{Z}), \]

where $f \cdot f \in H^1(T^5, \mathbb{R}/\mathbb{Z})$. The best known example of this phenomenon is the possible non-integrality of the fifth component of $p$ (i.e. the instanton number over $T^4$) for a non-trivial $G/C$ bundle (i.e. of non-trivial magnetic 't Hooft flux $m$). Another example is the possible non-integrality of the four spatial components of $p$ in situations where both the electric and magnetic 't Hooft fluxes $e$ and $m$ are non-trivial.

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