1 \ General equation for the calculation of amplitudes

It is well known that the high order calculations of the observables within the perturbation theory has the serious difficulties (especially, if we take into account the polarization effects). Method of the direct calculation of amplitudes simplifies these calculations essentially.

There is an even number \(2N\) of fermions in initial and final state for any reaction with Dirac particles. Therefore every diagram contains \(N\) non-closed fermion lines. The expression

\[
M_{if} = \bar{u}_f Qu_i = \text{Tr}(Qu_i\bar{u}_f) \quad (1.1)
\]

corresponds to every line in the amplitude of the process, \(u_i, u_f\) are the Dirac bispinors for free particles and \(Q\) is a matrix operator characterizing the interaction. The operator \(Q\) is expressed as a linear combination of the products of the Dirac \(\gamma\)-matrices (or their contractions with four-vectors) and can have any number of free Lorentz indexes.

We must obtain the expression for the operator

\[
u_i\bar{u}_f \quad (1.2)
\]
in \(1.1\).

Note that Dirac equation

\[
\hat{p}u(p,n) = mu(p,n) , \quad (1.3)
\]

and equation for the axis of the spin projections

\[
\gamma^5\hat{u}u(p,n) = \pm u(p,n) \quad (1.4)
\]
as well as normalization condition for bispinor define \( u(p, n) \) up to the phase factor (\( p \) is four-momentum, \( n \) is the four-vector specifying the axis of the spin projections and \( \hat{a} = \gamma_\mu a^\mu \) for any four-vector \( a \)) \(^1\). Let use this fact to construct the operator \(^2\):

\[
\begin{align*}
\bar{u}_i & \simeq \frac{\bar{u}_i u}{|\bar{u}_i u|} = \frac{\mathcal{P}_i}{\sqrt{\text{Tr}(\mathcal{P}_i)}} u, \quad (1.5) \\
\bar{u}_f & \simeq \frac{\bar{u} u_f}{|\bar{u} u_f|} \bar{u}_f = \bar{u} \frac{\mathcal{P}_f}{\sqrt{\text{Tr}(\mathcal{P}_f)}} , \quad (1.6)
\end{align*}
\]

where \( u \) is an arbitrary bispinor and \( \mathcal{P} \) are projection operators for Dirac particles.

\[
\mathcal{P}(p, n) = u(p, n)\bar{u}(p, n) = \frac{1}{2}(\hat{p} + m)(1 + \gamma_5 \hat{n})
\]

is the projection operator for a particle with mass \( m \),

\[
p^2 = m^2 , \quad n^2 = -1 , \quad pn = 0 , \quad \bar{u} u = 2m .
\]

For massless particle the projection operator has form:

\[
u_\pm(p)\bar{u}_\pm(p) = \frac{1}{2}(1 \pm \gamma_5)\hat{p} = \mathcal{P}_\pm(p)
\]

where:

\[
p^2 = 0 , \quad \bar{u}_\pm \gamma_\mu u_\pm = 2p_\mu
\]

(signs \( \pm \) correspond to the helicity of particle) .

As a result we have

\[
u_i \bar{u}_f \simeq \frac{\mathcal{P}_i \mathcal{P}_f}{\sqrt{\text{Tr}(\mathcal{P}_i) \text{Tr}(\mathcal{P}_f)}} . \quad (1.9)
\]

It follows from \(^1\), \(^2\) that both \( u_i \) and \( \bar{u}_f \) obtain individual phase factors, i.e. direct as well as exchange diagrams multiplied by the same phase factor, and consequently one can ignore it. Therefore we will use the equality sign instead of the symbol \( \simeq \) in formulae for amplitudes.

Finally, we have the general formula for the calculation of amplitudes

\[
\bar{u}_f Qu_i = \frac{\text{Tr}(Q\mathcal{P}_i \mathcal{P}_f)}{\sqrt{\text{Tr}(\mathcal{P}_i) \text{Tr}(\mathcal{P}_f)}} . \quad (1.10)
\]

\(^1\)We use the same metric as in the book \([\Pi]\):

\[
a^\mu = (a_0, \vec{a}), \quad a_\mu = (a_0, -\vec{a}) , \quad ab = a_\mu b^\mu = a_0 b_0 - \vec{a} \vec{b} , \quad \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 .
\]

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This expression enables to calculate the amplitude numerically. The complex values obtained in this way may be used to evaluate the squared amplitude.

Note that this method can be generalized easily to the case of the reaction with antiparticles. To do this it is sufficient in (1.10) to replace the projection operators of particles by the operators of antiparticles.

More details may be found in [2].

2 Example: the calculation of electron-electron scattering amplitudes

As an illustration of application of the method we will consider the lowest order amplitudes of electron-electron scattering. Two Feynman diagrams presented in Fig.1 correspond to this process.

![Figure 1: Diagrams for electron-electron scattering](image)

\[ M_1 = \frac{1}{(p_1 - p_3)^2} \bar{u}_3 \gamma_\rho u_1 \cdot \bar{u}_4 \gamma^\rho u_2 , \]  

\[ M_2 = -\frac{1}{(p_1 - p_4)^2} \bar{u}_4 \gamma_\rho u_1 \cdot \bar{u}_3 \gamma^\rho u_2 . \]  

There are two variants of formula (1.10) for massive Dirac particles:

\[ \bar{u}_f Q u_i = \text{Tr} \left[ (1 - \gamma^5) \hat{q} (\hat{p}_f + m \hat{n}_f + m - \hat{p}_f \hat{n}_f) Q (\hat{p}_i + m \hat{n}_i + m + \hat{p}_i \hat{n}_i) \right] / 8 \sqrt{(qp_i) + m(qn_i) \sqrt{(qp_f) + m(qn_f)}} , \]  

\[ (\bar{u}_f Q u_i)' = \text{Tr} \left[ (1 + \gamma^5) \hat{q} (\hat{p}_f - m \hat{n}_f + m + \hat{p}_f \hat{n}_f) Q (\hat{p}_i - m \hat{n}_i + m - \hat{p}_i \hat{n}_i) \right] / 8 \sqrt{(qp_i) - m(qn_i) \sqrt{(qp_f) - m(qn_f)}} , \]
if $P = \frac{1}{2}(1 + \gamma^5)\hat{q}$ . Relation between (2.3) and (2.4) gives by

$$\bar{u}_f Q u_i = (\bar{u}_f Q u_i)^' \cdot \frac{-\text{Tr} [(1 + \gamma^5)\hat{q}\hat{p}_i\hat{n}_i\hat{q}\hat{p}_f\hat{n}_f]}{8\sqrt{(qp_i)^2 - m^2(qn_i)^2}\sqrt{(qp_f)^2 - m^2(qn_f)^2}} \cdot (2.5)$$

For definiteness we will use (2.3).

Note, to check formula (2.6) it necessary to use identity:

$$(1 \pm \gamma^5)\hat{q}Q(1 \pm \gamma^5)\hat{q} = \text{Tr} \left[(1 \pm \gamma^5)\hat{q}Q\right] (1 \pm \gamma^5)\hat{q} \quad (2.6)$$

where an operator $Q$ is any matrix. (The proof of (2.6) is given in [2].) Besides, we will use well-known formulae of $\gamma$-matrix algebra:

$$\gamma_\rho Q^{2n+1}\gamma^\rho = -2Q^{2n+1}_R \quad (2.7)$$

and

$$(1 \pm \gamma^5)\gamma_\rho Q^{2n}\gamma^\rho = (1 \pm \gamma^5)\text{Tr} \left[(1 \pm \gamma^5)Q^{2n}\right] \quad (2.8)$$

where $Q^{2n+1}$ is a product of odd number of the $\gamma$-matrices; $Q^{2n+1}_R$ is a product of the same $\gamma$-matrices, rewritten in reversal order and $Q^{2n}$ is a product of even number of the $\gamma$-matrices.

Thus, we obtain from (2.3):

$$\bar{u}_3\gamma_\rho u_1 = \frac{\text{Tr} \left[(1 - \gamma^5)\hat{q}(\hat{p}_3 + m\hat{n}_3 + m - \hat{p}_3\hat{n}_3)\gamma_\rho(\hat{p}_1 + m\hat{n}_1 + m + \hat{p}_1\hat{n}_1)\right]}{8\sqrt{(qp_1) + m(qn_1)\sqrt{(qp_3) + m(qn_3)}}} \quad (2.9)$$

$$\bar{u}_4\gamma^\rho u_2 = \frac{\text{Tr} \left[(1 - \gamma^5)\hat{q}(\hat{p}_4 + m\hat{n}_4 + m - \hat{p}_4\hat{n}_4)\gamma^\rho(\hat{p}_2 + m\hat{n}_2 + m + \hat{p}_2\hat{n}_2)\right]}{8\sqrt{(qp_2) + m(qn_2)\sqrt{(qp_4) + m(qn_4)}}} \quad (2.10)$$

$$\bar{u}_3\gamma_\rho u_1 \cdot \bar{u}_4\gamma^\rho u_2 = \frac{1}{8^2\sqrt{(qp_1) + m(qn_1)\sqrt{(qp_2) + m(qn_2)\sqrt{(qp_3) + m(qn_3)\sqrt{(qp_4) + m(qn_4)}}}}} \cdot \left\{ \text{Tr} \left[(1 - \gamma^5)\hat{q}(\hat{p}_3 + m\hat{n}_3)\gamma_\rho(\hat{p}_1 + m\hat{n}_1)\right] \text{Tr} \left[(1 - \gamma^5)\hat{q}(\hat{p}_4 + m\hat{n}_4)\gamma^\rho(\hat{p}_2 + m\hat{n}_2)\right] \right. $$

$$+ \text{Tr} \left[(1 - \gamma^5)\hat{q}(\hat{p}_3 + m\hat{n}_3)\gamma_\rho(\hat{p}_1 + m\hat{n}_1)\right] \text{Tr} \left[(1 - \gamma^5)\hat{q}(m - \hat{p}_4\hat{n}_4)\gamma^\rho(m + \hat{p}_2\hat{n}_2)\right] $$

$$+ \text{Tr} \left[(1 - \gamma^5)\hat{q}(m - \hat{p}_3\hat{n}_3)\gamma_\rho(m + \hat{p}_1\hat{n}_1)\right] \text{Tr} \left[(1 - \gamma^5)\hat{q}(\hat{p}_4 + m\hat{n}_4)\gamma^\rho(\hat{p}_2 + m\hat{n}_2)\right] $$

$$+ \text{Tr} \left[(1 - \gamma^5)\hat{q}(m - \hat{p}_3\hat{n}_3)\gamma_\rho(m + \hat{p}_1\hat{n}_1)\right] \text{Tr} \left[(1 - \gamma^5)\hat{q}(m - \hat{p}_4\hat{n}_4)\gamma^\rho(m + \hat{p}_2\hat{n}_2)\right] \right\} \quad (2.11)$$
Using (2.6), (2.7), for first term in (2.11) we have:

\[
\text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_3 + m\hat{n}_3) \gamma_\rho(\hat{p}_1 + m\hat{n}_1) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_4 + m\hat{n}_4) \gamma_\rho(\hat{p}_2 + m\hat{n}_2) \right] \\
= 2 \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_3 + m\hat{n}_3) \gamma_\rho(\hat{p}_1 + m\hat{n}_1) \hat{q}(\hat{p}_4 + m\hat{n}_4) \gamma_\rho(\hat{p}_2 + m\hat{n}_2) \right] \\
= -4 \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_3 + m\hat{n}_3)(\hat{p}_4 + m\hat{n}_4) \hat{q}(\hat{p}_1 + m\hat{n}_1)(\hat{p}_2 + m\hat{n}_2) \right].
\] (2.12)

For second term in (2.11) we obtain through (2.6), (2.8)

\[
\text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_3 + m\hat{n}_3) \gamma_\rho(\hat{p}_1 + m\hat{n}_1) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_4\hat{n}_4) \gamma_\rho(m - \hat{p}_2\hat{n}_2) \right] \\
= 2 \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_3 + m\hat{n}_3) \gamma_\rho(\hat{p}_1 + m\hat{n}_1) \hat{q}(m - \hat{p}_4\hat{n}_4) \gamma_\rho(m - \hat{p}_2\hat{n}_2) \right] \\
= 2 \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_3 + m\hat{n}_3)(m + \hat{p}_2\hat{n}_2) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_4\hat{n}_4)(\hat{p}_1 + m\hat{n}_1) \right].
\] (2.13)

In the same way

\[
\text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_3\hat{n}_3) \gamma_\rho(m + \hat{p}_1\hat{n}_1) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_4 + m\hat{n}_4) \gamma_\rho(\hat{p}_2 + m\hat{n}_2) \right] \\
= 2 \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_4 + m\hat{n}_4)(m + \hat{p}_1\hat{n}_1) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_3\hat{n}_3)(\hat{p}_2 + m\hat{n}_2) \right],
\] (2.14)

and

\[
\text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_3\hat{n}_3) \gamma_\rho(m + \hat{p}_1\hat{n}_1) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_4\hat{n}_4) \gamma_\rho(m + \hat{p}_2\hat{n}_2) \right] \\
= -4 \text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_3\hat{n}_3)(m + \hat{p}_4\hat{n}_4) \hat{q}(m - \hat{p}_1\hat{n}_1)(m + \hat{p}_2\hat{n}_2) \right].
\] (2.15)

As a result we have

\[
\bar{u}_3 \gamma_\rho u_1 \cdot \bar{u}_4 \gamma_\rho u_2 = \frac{1}{32 \sqrt{(qp_1) + m(qn_1)} \sqrt{(qp_2) + m(qn_2)} \sqrt{(qp_3) + m(qn_3)} \sqrt{(qp_4) + m(qn_4)}} \\
\left\{ -2 \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_3 + m\hat{n}_3)(\hat{p}_4 + m\hat{n}_4) \hat{q}(\hat{p}_1 + m\hat{n}_1)(\hat{p}_2 + m\hat{n}_2) \right] \\
+ \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_3 + m\hat{n}_3)(m + \hat{p}_2\hat{n}_2) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_4\hat{n}_4)(\hat{p}_1 + m\hat{n}_1) \right] \\
+ \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_4 + m\hat{n}_4)(m + \hat{p}_1\hat{n}_1) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_3\hat{n}_3)(\hat{p}_2 + m\hat{n}_2) \right] \\
-2 \text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_3\hat{n}_3)(m + \hat{p}_4\hat{n}_4) \hat{q}(m - \hat{p}_1\hat{n}_1)(m + \hat{p}_2\hat{n}_2) \right] \right\}.
\] (2.16)
In principle we may use \((2.16)\) for calculation of amplitude but it possible to simplify essentially the last term in this expression. Really, using consequence of formula \((2.5)\)

\[
\text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_4 \hat{n}_4) \gamma^\rho (m + \hat{p}_2 \hat{n}_2) \right]
= - \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_2 - m \hat{n}_2) \gamma^\rho (\hat{p}_4 - m \hat{n}_4) \right] \cdot \frac{\text{Tr} \left[ (1 - \gamma^5) \hat{q} \hat{p}_4 \hat{n}_4 \hat{q} \hat{p}_2 \hat{n}_2 \right]}{8 ((q p_2) - m (q n_2)) \cdot ([q p_4) - m (q n_4)]},
\]

we have instead of \((2.15)\)

\[
\text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_3 \hat{n}_3) \gamma^\rho (m + \hat{p}_1 \hat{n}_1) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_4 \hat{n}_4) \gamma^\rho (m + \hat{p}_2 \hat{n}_2) \right]
= -2 \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_2 - m \hat{n}_2) (m + \hat{p}_1 \hat{n}_1) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q}(m - \hat{p}_3 \hat{n}_3) (\hat{p}_4 - m \hat{n}_4) \right] \cdot \frac{\text{Tr} \left[ (1 - \gamma^5) \hat{q} \hat{p}_4 \hat{n}_4 \hat{q} \hat{p}_2 \hat{n}_2 \right]}{8 ((q p_2) - m (q n_2)) \cdot ([q p_4) - m (q n_4)]}.\]

Similarly, for first term of \((2.16)\), using the other consequence of formula \((2.5)\)

\[
\text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_4 + m \hat{n}_4) \gamma^\rho (\hat{p}_2 + m \hat{n}_2) \right]
= - \text{Tr} \left[ (1 - \gamma^5) \hat{q}(m + \hat{p}_2 \hat{n}_2) \gamma^\rho (m - \hat{p}_4 \hat{n}_4) \right] \cdot \frac{\text{Tr} \left[ (1 - \gamma^5) \hat{q} \hat{p}_4 \hat{n}_4 \hat{q} \hat{p}_2 \hat{n}_2 \right]}{8 ((q p_2) - m (q n_2)) \cdot ([q p_4) - m (q n_4)]},
\]

we obtain instead of \((2.12)\)

\[
\text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_3 + m \hat{n}_3) \gamma^\rho (\hat{p}_1 + m \hat{n}_1) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_4 + m \hat{n}_4) \gamma^\rho (\hat{p}_2 + m \hat{n}_2) \right]
= -2 \text{Tr} \left[ (1 - \gamma^5) \hat{q}(\hat{p}_3 + m \hat{n}_3) (m - \hat{p}_4 \hat{n}_4) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q}(m + \hat{p}_2 \hat{n}_2) (\hat{p}_1 + m \hat{n}_1) \right] \cdot \frac{\text{Tr} \left[ (1 - \gamma^5) \hat{q} \hat{p}_4 \hat{n}_4 \hat{q} \hat{p}_2 \hat{n}_2 \right]}{8 ((q p_2) - m (q n_2)) \cdot ([q p_4) - m (q n_4)]}.\]

(2.17)

(2.18)

(2.19)

(2.20)
The final equation is
\[
\bar{u}_3 \gamma_\rho u_1 \cdot \bar{u}_4 \gamma_\rho u_2 = \frac{1}{32 \sqrt{(qp_1) + m(qn_1)} \sqrt{(qp_2) + m(qn_2)} \sqrt{(qp_3) + m(qn_3)} \sqrt{(qp_4) + m(qn_4)}} \cdot \left\{ \text{Tr} \left[ (1 - \gamma^5) \hat{q} (\hat{p}_3 + m\hat{n}_3) (m + \hat{p}_2 \hat{n}_2) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q} (m - \hat{p}_4 \hat{n}_4) (\hat{p}_1 + m\hat{n}_1) \right] \\
+ \text{Tr} \left[ (1 - \gamma^5) \hat{q} (\hat{p}_4 + m\hat{n}_4) (m + \hat{p}_1 \hat{n}_1) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q} (m - \hat{p}_3 \hat{n}_3) (\hat{p}_2 + m\hat{n}_2) \right] \\
- \left\{ \text{Tr} \left[ (1 - \gamma^5) \hat{q} (\hat{p}_3 + m\hat{n}_3) (m - \hat{p}_4 \hat{n}_4) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q} (m + \hat{p}_2 \hat{n}_2) (\hat{p}_1 + m\hat{n}_1) \right] \\
+ \text{Tr} \left[ (1 - \gamma^5) \hat{q} (\hat{p}_2 - m\hat{n}_2) (m + \hat{p}_1 \hat{n}_1) \right] \text{Tr} \left[ (1 - \gamma^5) \hat{q} (m - \hat{p}_3 \hat{n}_3) (\hat{p}_4 - m\hat{n}_4) \right] \right\} \\
\cdot \frac{\text{Tr} \left[ (1 - \gamma^5) \hat{q} \hat{p}_4 \hat{n}_4 \hat{p}_2 \hat{n}_2 \right]}{8 [(qp_2) - m(qn_2)] \cdot [(qp_4) - m(qn_4)]} \right\}.
\]

The expression for
\[
\bar{u}_4 \gamma_\rho u_1 \cdot \bar{u}_3 \gamma_\rho u_2
\]
is obtained from (2.21) by replacing the subscripts 3 ↔ 4.

The obtained formulae are simple enough and may be used for numerical calculation of the amplitudes.

If the denominators in (2.21) become zero for some values of the vectors of problem, then one needs to change the values of arbitrary four-vector \( q \) in all formulae.

### 3 Construction of polarization vectors of photons from the vectors of the problem

If photons participate in reaction, we must to construct their polarization vectors from other vectors coming in amplitude calculation. It is necessary to use Gram – Schmidt orthonormalization process to do this (see [3]).

As it shown in [3], in the Minkowski space, this process give

1. Let \( p \) is an arbitrary four-momentum, such that
\[
\sum p^2 = m^2 \neq 0,
\]

\( a, b \) and \( c \) are arbitrary vectors. Then four vectors \( l_0, l_1, l_2, l_3 \) form an orthonormal basis:
\[
l_0 = \frac{p}{m},
\]  
(3.1)
\[(l_1)_\rho = \frac{G\left(\frac{p}{p} a\right)}{m\left[-G\left(\frac{p}{p} a\right)\right]^{1/2}} = \frac{m^2a_\rho - (pa)p_\rho}{m\sqrt{(pa)^2 - m^2a^2}}, \quad (3.2)\]

\[(l_2)_\rho = -\frac{G\left(\frac{p}{p} a b\right)}{\sqrt{(pa)^2 - m^2a^2\sqrt{2(pa)(pb)(ab) + m^2a^2b^2 - m^2(ab)^2 - a^2(pb)^2 - b^2(pa)^2}}} = \frac{[a^2(pb) - (pa)(ab)]p_\rho + [m^2(ab) - (pa)(pb)]a_\rho + [(pa)^2 - m^2a^2]b_\rho}{\sqrt{(pa)^2 - m^2a^2\sqrt{2(pa)(pb)(ab) + m^2a^2b^2 - m^2(ab)^2 - a^2(pb)^2 - b^2(pa)^2}}}, \quad (3.3)\]

\[(l_3)_\rho = \frac{G\left(\frac{p}{p} a b c\right)}{\left[-G\left(\frac{p}{p} a b\right)G\left(\frac{p}{p} a b\right)^{1/2}} = \frac{\varepsilon_{\rho\alpha\beta\lambda}p^\rho a^\alpha b^\beta c^\lambda}{\sqrt{2(pa)(pb)(ab) + m^2a^2b^2 - m^2(ab)^2 - a^2(pb)^2 - b^2(pa)^2}}\right], \quad (3.4)\]

Note that in the final formulae the vector \(c\) is absent.

So, using the three vectors (one of which is a four-momentum of a massive particle) and a total antisymmetric Levi-Civita tensor, an orthonormal basis can be always constructed in the Minkowski space.

2. Let us consider now the construction of the basis for reaction with massless particles.

Let \(k_1\) and \(k_2\) are four-momentums of the problem, such that

\[k_1^2 = k_2^2 = 0.\]

We consider a particular form of the basis \([3.1] - [3.4]\) at

\[p = k_1 + k_2, \quad m = \sqrt{2(k_1k_2)}, \quad a = k_2:\]
It is easy to check that this vector satisfies standard conditions for polarization vector:

\[
l_0 = \frac{k_1 + k_2}{\sqrt{2(k_1k_2)}} , \tag{3.5}
\]

\[
l_1 = \frac{-k_1 + k_2}{\sqrt{2(k_1k_2)}} , \tag{3.6}
\]

\[
(l_2)_\rho = -\frac{(k_2b)(k_1)_\rho - (k_1b)(k_2)_\rho + (k_1k_2)b_\rho}{\sqrt{2(k_1k_2)(k_1b)(k_2b)}} , \tag{3.7}
\]

\[
(l_3)_\rho = \frac{\varepsilon_{\rho\alpha\beta\lambda}k_1^\alpha q_2^\beta k_2^\lambda b_\lambda}{\sqrt{2(k_1k_2)(k_1b)(k_2b)}} . \tag{3.8}
\]

If \( k_1 \) is four-momentum of photon, polarizations vectors for it may be constructed from (3.7), (3.8), where it is appropriate to choose \( b = q \), \( [q] \) is the same massless vector which contained in expressions for amplitudes (2.3), (2.4).

\[
(l_2)_\rho = -\frac{(k_2q)k_1_\rho + (k_1k_2)q_\rho - (k_1q)k_2_\rho}{\sqrt{2(k_1k_2)(k_1q)(k_2q)}} ,
\]

\[
(l_3)_\rho = -\frac{\varepsilon_{\rho\alpha\beta\lambda}q_2^\beta k_2^\lambda}{\sqrt{2(k_1k_2)(k_1q)(k_2q)}} ,
\]

\[
e^\pm_\rho(k_1) = -\frac{(l_2)_\rho \pm i(l_3)_\rho}{\sqrt{2}} = \frac{\text{Tr}[(1 \pm \gamma^5)\gamma_\rho \hat{k}_1 \hat{q} \hat{k}_2]}{8 \sqrt{(k_1k_2)(k_1q)(k_2q)}} . \tag{3.9}
\]

It is easy to check that this vector satisfies standard conditions for polarization vector:

\[
(k_1 e^\pm) = (e^\pm)^2 = 0, \quad (e^\pm e^\mp) = -1 . \tag{3.10}
\]

Then, using formulae of Fierz transformation

\[
[(1 \mp \gamma^5)\gamma^\rho]_{kl} [(1 \pm \gamma^5)\gamma_\rho]_{ij} = 2[1 \mp \gamma^5]_{k[i} [1 \pm \gamma^5]_{j]} ,
\]

\[
[(1 \pm \gamma^5)\gamma^\rho]_{kl} [(1 \pm \gamma^5)\gamma_\rho]_{ij} = -[(1 \pm \gamma^5)\gamma_\rho]_{kj} [(1 \pm \gamma^5)\gamma_\rho]_{il} ,
\]

\((i, j, k, l\) are indices that label the components of \(4 \times 4\)-matrices), we obtain:

\[
(1 \mp \gamma^5)\gamma^\rho \text{Tr}[(1 \pm \gamma^5)\gamma_\rho \hat{k}_1 \hat{q} \hat{k}_2] = 2(1 \mp \gamma^5)\hat{k}_1 \hat{q} \hat{k}_2(1 \pm \gamma^5) = 4(1 \mp \gamma^5)\hat{k}_1 \hat{q} \hat{k}_2 ,
\]

\[
(1 \pm \gamma^5)\gamma^\rho \text{Tr}[(1 \pm \gamma^5)\gamma_\rho \hat{k}_1 \hat{q} \hat{k}_2] = -(1 \pm \gamma^5)\gamma_\rho \hat{k}_1 \hat{q} \hat{k}_2(1 \mp \gamma^5)\gamma^\rho =
\]

\[
= -2(1 \pm \gamma^5)\gamma^\rho \hat{k}_1 \hat{q} \hat{k}_2 \gamma_\rho = 4(1 \pm \gamma^5)\hat{k}_2 \hat{q} \hat{k}_1 .
\]
Using identity (2.6) we have
\[
\gamma^\mu \text{Tr} \left[ (1 \pm \gamma^5) \gamma_\mu \hat{k}_1 \hat{q} \hat{k}_2 \right] = 2(1 \mp \gamma^5) \hat{k}_1 \hat{q} \hat{k}_2 + 2(1 \pm \gamma^5) \hat{k}_2 \hat{q} \hat{k}_1
\]
and we have
\[
\hat{e}^\pm (k_1) = \frac{(1 \pm \gamma^5) \hat{k}_2 \hat{q} \hat{k}_1 + (1 \mp \gamma^5) \hat{k}_1 \hat{q} \hat{k}_2}{4 \sqrt{(k_1 k_2)(k_1 q)(k_2 q)}}. \tag{3.11}
\]

Note that \( k_2 \) is an arbitrary massless vector of problem. It follows from this that for photon with four-momentum \( k_2 \) polarization vectors may be chosen in form (3.9) too (see also [5]):
\[
\hat{e}_\rho^\pm (k_1) = \frac{\text{Tr} \left[ (1 \pm \gamma^5) \gamma_\rho \hat{k}_1 \hat{q} \hat{k}_2 \right]}{8 \sqrt{(k_1 k_2)(k_1 q)(k_2 q)}} = \hat{e}_\rho^\mp (k_2). \tag{3.12}
\]

This circumstance simplifies calculations for two-photon processes essentially. For example, take place expression
\[
\hat{e}_\mu^\pm (k_1) \hat{e}_\nu^\pm (k_2) = \frac{\text{Tr} \left[ (1 \pm \gamma^5) \gamma_\mu \hat{k}_1 \gamma_\nu \hat{k}_2 \right]}{8 (k_1 k_2)}, \tag{3.13}
\]
et al.

In conclusion of this section we consider one more problem. In reality, polarization vectors for the photon with four-momentum \( k_1 \), satisfied (3.10), may be constructed with help of any vectors \( a \) and \( b \):
\[
\hat{e}_\rho^\pm (k_1; a, b) = \frac{\text{Tr} \left[ (1 \pm \gamma^5) \gamma_\rho \hat{k}_1 \hat{a} \hat{b} \right]}{4 \sqrt{2G \left( k_1 \begin{array}{c} a \\ b \end{array} \right)}}. \tag{3.14}
\]

Using identity (2.6) we have
\[
\text{Tr}[(1 \pm \gamma^5) \gamma_\rho \hat{k}_1 \hat{a} \hat{b}] \text{Tr}[(1 \mp \gamma^5) \hat{k}_1 \hat{d} \hat{c} \hat{k}_1 \hat{c}] = \text{Tr}[(1 \pm \gamma^5) \gamma_\rho \hat{k}_1 \hat{a} \hat{b}] 8G \left( k_1 \begin{array}{c} c \\ d \end{array} \right) = \]
\[
= \text{Tr}[(1 \mp \gamma^5) \hat{k}_1 \hat{a} \hat{b} \gamma_\rho (1 \mp \gamma^5) \hat{k}_1 \hat{d} \hat{c} \hat{k}_1 \hat{c} \hat{d}] = \text{Tr}[\hat{k}_1 \hat{a} \hat{b} (1 \mp \gamma^5) \gamma_\rho \hat{k}_1 \hat{d} \hat{c} (1 \pm \gamma^5) \hat{k}_1 \hat{c} \hat{d}] = \]
\[
= 4 k_1 \rho \text{Tr}[(1 \mp \gamma^5) \hat{k}_1 \hat{a} \hat{b} \hat{d} \hat{c} \hat{k}_1 \hat{c} \hat{d}] - \text{Tr}[\hat{k}_1 \hat{a} \hat{b} (1 \pm \gamma^5) \hat{k}_1 \gamma_\rho \hat{d} \hat{c} (1 \pm \gamma^5) \hat{k}_1 \hat{c} \hat{d}] = \]
\[
= 4 k_1 \rho \text{Tr}[(1 \mp \gamma^5) \hat{k}_1 \hat{a} \hat{b} \hat{d} \hat{c} \hat{k}_1 \hat{c} \hat{d}] - \text{Tr}[(1 \mp \gamma^5) \gamma_\rho \hat{k}_1 \hat{c} \hat{d}] \text{Tr}[(1 \pm \gamma^5) \hat{k}_1 \hat{c} \hat{d} \hat{a} \hat{b}] .
\]
Thus
\[ e^\pm(k_1; a, b) = -e^\pm(k_1; c, d) \frac{\text{Tr}[(1 \pm \gamma^5)\hat{k}_1 \hat{d} \hat{k}_1 \hat{a} \hat{b}]}{8\sqrt{G\left(\begin{array}{cc} k_1 & a \\ k_1 & b \end{array}\right) G\left(\begin{array}{cc} k_1 & c \\ k_1 & d \end{array}\right)}} + \]
\[ + k_1 \rho \frac{\text{Tr}[(1 \mp \gamma^5)\hat{k}_1 \hat{d} \hat{k}_1 \hat{a} \hat{b}]}{\sqrt{2G\left(\begin{array}{cc} k_1 & a \\ k_1 & b \end{array}\right) \cdot 8G\left(\begin{array}{cc} k_1 & c \\ k_1 & d \end{array}\right)}}. \]

Since last term may be neglected because of gauge invariance, we obtain that replacement of the vectors \(a, b\) in polarization vector by another vectors \(c, d\) leads to the phase factor
\[ -\frac{\text{Tr}[(1 \pm \gamma^5)\hat{k}_1 \hat{c} \hat{k}_1 \hat{a} \hat{b}]}{8\sqrt{G\left(\begin{array}{cc} k_1 & a \\ k_1 & b \end{array}\right) G\left(\begin{array}{cc} k_1 & c \\ k_1 & d \end{array}\right)}} = (e^\pm(k_1; a, b)e^{\mp}(k_1; c, d)). \]

4 Some remarks about other methods of amplitude calculation

Formula (1.10) is a particular case of general equation:
\[ \bar{u}_f Qu_i \simeq \frac{\bar{u}_i Zu_f}{\sqrt{\text{Tr}(\bar{Z} u_i Zu_f \bar{u}_f)}} = \frac{\text{Tr}(Qu_i \bar{u}_i Zu_f \bar{u}_f)}{\text{Tr}(Zu_i \bar{u}_i Zu_f \bar{u}_f)} \]
\[ = \frac{\text{Tr}[(\hat{p}_f + m_f)(1 + \gamma^5 \hat{n}_f)Q(\hat{p}_i + m_i)(1 + \gamma^5 \hat{n}_i)]}{4\sqrt{[m_i m_f + (p_i p_f)][1 - (n_i n_f)] + (p_i n_f)(p_f n_i)}}. \]  \hspace{1cm} (4.1)

where \(Z\) is matrix operator, \(\bar{Z} = \gamma^0 Z^+ \gamma^0\). Formula (1.1) describes all methods of amplitude calculation where \(Z\) may be \(1, \gamma^5, \gamma^0, 1 + \gamma^0\) et al. Detailed classification of different methods is given in [2]. Many methods are analyzed in the book [6] also.

The most of the methods give more simple expressions for amplitudes in comparison with (1.10). For example, if \(Z = 1\) (see [7]), we have:
\[ \bar{u}_f Qu_i \simeq \frac{\text{Tr}[(\hat{p}_f + m_f)(1 + \gamma^5 \hat{n}_f)Q(\hat{p}_i + m_i)(1 + \gamma^5 \hat{n}_i)]}{4\sqrt{[m_i m_f + (p_i p_f)][1 - (n_i n_f)] + (p_i n_f)(p_f n_i)}}. \]  \hspace{1cm} (4.2)

However, the scheme considered here has two essential disadvantages in general case:

1. There is denominator in (1.1), hence the ambiguity of the type \(0/0\) can appear during the calculations. For example, in (4.2) the denominator is equal to zero for
\[ n_f = -n_i + \frac{(p_f n_i)}{m_i m_f + (p_i p_f)}(p_i + \frac{m_i}{m_f p_f}). \]
2. Each of the fermion lines obtain phase factor [see (4.1)]:

\[
\frac{\bar{u}_i Z u_f}{|\bar{u}_i Z u_f|}.
\]

If we have the interference diagrams, this circumstance leads to the fact that expressions for the amplitudes corresponding to different channels obtain different phase factors in general case. In this situation the additional calculations are necessary: calculation of the relative phase for interference diagrams, the Fierz transformations et al. (see [2]).

At the same time, the presented method, as mentioned above, has no these disadvantages.

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