MATCHINGS AND ENTROPIES OF CYLINDERS

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Abstract. The enumeration of perfect matchings of graphs is equivalent to the dimer problem which has applications in statistical physics. A graph $G$ is said to be $n$-rotation symmetric if the cyclic group of order $n$ is a subgroup of the automorphism group of $G$. Jockusch (Perfect matchings and perfect squares, J. Combin. Theory Ser. A, 67(1994), 100-115) and Kuperberg (An exploration of the permanent-determinant method, Electron. J. Combin., 5(1998), #46) proved independently that if $G$ is a plane bipartite graph of order $N$ with $2n$-rotation symmetry, then the number of perfect matchings of $G$ can be expressed as the product of $n$ determinants of order $N/2n$. In this paper we give this result a new presentation. We use this result to compute the entropy of a bulk plane bipartite lattice with $2n$-rotation symmetry. We obtain explicit expressions for the numbers of perfect matchings and entropies for two types of cylinders. Using the results on the entropy of the torus obtained by Kenyon, Okounkov, and Sheffield (Dimers and amoebae, Ann. Math. 163(2006), 1019–1056) and by Salinas and Nagle (Theory of the phase transition in the layered hydrogen-bonded $SnCl_2 \cdot 2H_2O$ crystal, Phys. Rev. B, 9(1974), 4920–4931), we show that each of the cylinders considered and its corresponding torus have the same entropy. Finally, we pose some problems.

1. Introduction

An automorphism of a graph $G$ is a graph isomorphism with itself, i.e., a bijection from the vertex set of $G$ to itself such that the adjacency relation is preserved. The set of automorphisms defines a permutation group on the vertices of $G$, which is called the automorphism group of the graph $G$. The automorphism group of a graph $G$ characterizes its symmetries, and is therefore very useful for simplifying the computation of some of its invariants. One of the most successful results in this direction is to use the character of the automorphism group of a graph $G$ to compute its characteristic polynomial (see for example Cvetković et al. [5]). Recall that a cyclic group is a group that can be generated by a single element $x$ (the group generator). A cyclic group of order $n$ is denoted by $C_n$, and its generator $x$ satisfies $x^n = 1$, where 1 is the identity element. A graph $G$ is said to be $n$-rotation symmetric if the cyclic group $C_n$ is a subgroup of the automorphism group of $G$. We call $C_n$ the rotational subgroup of an $n$-rotation symmetric graph $G$. We say that a plane graph $G$ is reflective symmetric if it is invariant under the reflection across some straight line.

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A perfect matching of a graph $G$ is a collection of vertex-disjoint edges that are collectively incident to all vertices of $G$. Let $M(G)$ denote the number of perfect matchings of a simple graph $G$. Problems involving enumeration of perfect matchings have been examined extensively not only by mathematicians (see for example [2, 3, 4, 8, 13, 23, 26]) but also by physicists and chemists (see for example [9, 11, 12, 13, 27, 29]). In 1961, Kasteleyn [11] found a formula for the number of perfect matchings of an $m \times n$ quadratic lattice graph. Temperley and Fisher [27] used a different method and arrived at the same result at almost exactly the same time. Both lines of calculation showed that the logarithm of the number of perfect matchings, divided by $mn/2$, converges to $2c/\pi \approx 0.5831$ as $m, n \to \infty$, where $c$ is Catalan’s constant. This limit is called the entropy of the quadratic lattice graph and the corresponding problem was called the dimer problem by the statistical physicists. In 1992, Elkies et al. [6] studied the enumeration of perfect matchings of regions called Aztec diamonds, and showed that the entropy equals $\log 2 \approx 0.35$.

Cohn, Kenyon, and Propp [4] demonstrated that the behavior of random perfect matchings of large regions $R$ was determined by a variational (or entropy maximization) principle, as was conjectured in Section 8 of [6], and they gave an exact formula for the entropy of simply-connected regions of arbitrary shape. Particularly, they showed that computation of the entropy is intimately linked with an understanding of long-range variations in the local statistics of random domino tilings. Kenyon, Okounkov, and Sheffield [14] considered the problem enumerating perfect matchings of the doubly-periodic bipartite graph on a torus, which generalized the results in [4]. They proved that the number of perfect matchings of the doubly-periodic plane bipartite graph $G$ can be expressed in terms of four determinants and they expressed the entropy of $G$ as a double integral.

The dimer problem on hexagonal lattice graphs has also been examined in the past [7, 29, 31]. It can be seen as equivalent to the combinatorial problem of “plane partitions” (see MacMahon [19]). The $a, b, c$ semiregular hexagon is the hexagon whose side lengths are in cyclic order, $a, b, c, a, b, c$. Lozenge tilings of this region are in correspondence with plane partitions with at most $a$ rows, at most $b$ columns, and no part exceeding $c$. MacMahon [19] showed that the number of such plane partitions (i.e., the number of perfect matchings of the corresponding hexagonal lattice graph $G(a, b, c)$) equals

$$
\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i + k + k - 1}{i + j + k - 2}.
$$

Elser [7] obtained the entropy of $G(a, a, a)$ which is approximately 0.2616. The dimer problem on some other types of plane lattices are also considered (see Wu [30] and the references cited therein).

Jockusch [10] first used a combinatorial method to obtained a product formula for the number of perfect matchings of a type of rotational symmetric bipartite graphs. Kuperberg [15] used the representation theory of groups to obtain independently a product formula which can be equivalent to the Jockusch’s one. Ciucu [2], and Yan and Zhang [32, 33, 34] considered the enumeration of perfect matchings of general graphs (lattices) with a certain type of reflective symmetry. Ciucu [2] obtained a basic factorization theorem for the number of perfect matchings of plane bipartite graphs with reflective symmetry. Yan and Zhang [32, 33] extended Ciucu’s result
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The current paper deals with the enumerating problem for perfect matchings of cylinders which can be regarded as the graphs with rotational symmetry. In the next section, we introduce the Pfaffian method for enumerating perfect matchings of plane graphs (see for example \cite{8,11,26,27}). In Section 3, we give a new presentation of the product formula for the number of perfect matchings of plane bipartite graphs with 2\(n\)-rotation symmetry, which was found independently by Jockusch \cite{10} and by Kuperberg \cite{15}. We use this result to compute the entropy of a bulk plane bipartite lattice with 2\(n\)-notation symmetry. In Section 4, we obtain the explicit expressions for the numbers of perfect matchings and entropies for two types of cylinders (bipartite graphs with the cylinder-boundary condition). Using the theorem obtained by Kenyon, Okounkov, and Sheffield \cite{14}, we compute the entropy of a type of toruses (bipartite graphs with the torus-boundary condition). On the other hand, the entropy of another type of toruses was computed by Salinas and Nagle \cite{25}. Our results show that each of the cylinders considered and its corresponding torus have the same entropy (a similar result for the plane quadratic lattices had been obtained by Kasteleyn \cite{11}). Finally, Section 5 poses two open problems.

2. Pfaffians

Let \(B = (b_{ij})_{2n \times 2n}\) be a skew symmetric matrix of order \(2n\). For each partition \(P = \{\{i_1,j_1\}, \{i_2,j_2\}, \ldots, \{i_n,j_n\}\}\) of the set \(\{1,2,\ldots,2n\}\) into pairs, form the expression

\[
b_P = \text{sgn}(i_1j_1i_2j_2\ldots i_nv_n)b_{i_1j_1}b_{i_2j_2}\ldots b_{i_nj_n},
\]

where \(\text{sgn}(i_1j_1i_2j_2\ldots i_nv_n)\) denotes the sign of the permutation \(i_1j_1i_2j_2\ldots i_nv_n\). Note that \(b_P\) depends neither on the order in which the classes of the partition are listed nor on the order of the two elements of a class. So \(b_P\) indeed depends only on the choice of the partition \(P\). The Pfaffian of the skew matrix \(B\) (see \cite{17}), denoted by \(Pf(B)\), is defined as

\[
Pf(B) = \sum_P b_P,
\]

where the summation ranges over all partitions of \(\{1,2,\ldots,2n\}\) which have the form of \(P\).

Theorem 2.1 (The Cayley’s Theorem, \cite{17}). Let \(B = (b_{ij})_{2n \times 2n}\) be a skew symmetric matrix of order of \(2n\). Then

\[
\det(B) = Pf(B)^2.
\]

The Pfaffian method for enumerating perfect matchings of plane graphs was independently discovered by Fisher \cite{8}, Kasteleyn \cite{11}, and Temperley \cite{27}. Given a plane graph \(G\), the method produces a matrix \(A\) such that the number of perfect matchings of \(G\) can be expressed by the determinant of matrix \(A\). By using this method, Fisher \cite{8}, Kasteleyn \cite{11}, and Temperley \cite{27} solved independently a famous problem on enumerating perfect matchings of an \(m \times n\) quadratic lattice graph in statistical physics–Dimer problem. Given a simple graph \(G = (V(G), E(G))\) with vertex set \(V(G) = \{v_1, v_2, \ldots, v_N\}\), let \(G^e\) be an arbitrary orientation. The skew
adjacency matrix of $G^e$, denoted by $A(G^e)$, is defined as follows:

$$A(G^e) = (a_{ij})_{n \times n},$$

where

$$a_{ij} = \begin{cases} 
1 & \text{if } (v_i, v_j) \text{ is an arc of } G^e, \\
-1 & \text{if } (v_j, v_i) \text{ is an arc of } G^e, \\
0 & \text{otherwise.} 
\end{cases}$$

Obvious, $A(G^e)$ is a skew symmetric matrix.

If $D$ is an orientation of a graph $G$ and $C$ is a cycle of even length, we say that $C$ is oddly oriented in $D$ if $C$ contains odd number of edges that are directed in $D$ in the direction of each orientation of $C$. We say that $D$ is a Pfaffian orientation of $G$ if every nice cycle of even length of $G$ is oddly oriented in $D$ (a cycle $C$ of $G$ is called be nice if the induced subgraph $G - C$ of $G$ by $V(G) \setminus V(C)$ has perfect matchings). It is well known that if a graph $G$ contains no subdivision of $K_{3,3}$ then $G$ has a Pfaffian orientation (see Little [16], McCuaig [20], and McCuaig, Robertson et al [21], and Robertson, Seymour et al [24] found a polynomial-time algorithm to show whether a bipartite graph has a Pfaffian orientation. Stembridge [26] proved that the number (or generating function) of nonintersecting $r$-tuples of paths from a set of $r$ vertices to a specified region in an acyclic digraph $D$ can, under favorable circumstances, be expressed as a Pfaffian. See a survey of Pfaffian orientations of graphs in Thomas [28].

Lemma 2.2 (Lovász et al. [17]). If $G^e$ is a Pfaffian orientation of a graph $G$, then

$$M(G) = |Pf(A(G^e))|,$$

where $A(G^e)$ is the skew adjacency matrix of $G^e$.

Lemma 2.3 (Lovász et al. [17]). If $G$ is a plane graph and $G^e$, an orientation of $G$ such that every boundary face–except possibly the infinite face–has an odd number of edges oriented clockwise, then in every cycle the number of edges oriented clockwise is of opposite parity to number of vertices of $G^e$ inside the cycle. Consequently, $G^e$ is a Pfaffian orientation.

Lemma 2.4 ([8] [11] [27]). Every plane graph has a Pfaffian orientation satisfying the condition in Lemma 2.3.

3. A PRODUCT THEOREM

Let $G = (V(G), E(G))$ be a simple connected plane bipartite graph of order $N$ with $2n$-rotation symmetry, which has symmetric axes $\ell_0, \ell_1, \ldots, \ell_{2n-1}$ passing through the rotation center $O$. Figure 1(a) shows an example of a $6$-rotation symmetric graph. If $G$ has some vertices lying on its symmetric axes $\ell_i$’s, then we can rearrange vertices of $G$ such that there are no vertex lying on its new symmetric axes. Hence we can assume that there exists no vertex of $G$ lying on $\ell_0$, $\ell_1$, $\ldots$, $\ell_{2n-1}$. If we delete all the edges intersected by $\ell_0$, $\ell_1$, $\ldots$, and $\ell_{2n-1}$, then $2n$ isomorphic components $G_i = (V(G_i), E(G_i))$ for $0 \leq i \leq 2n - 1$ are obtained, where $V(G_i) = \{v_1^{(i)}, v_2^{(i)}, \ldots, v_{N/2n}^{(i)}\}$ is the vertex set of $G_i$. Note that $G$ is a plane graph. Hence there exists no edges between $G_i$ and $G_j$ for $j \neq i - 1, i + 1$ (mod $2n$). That is, we can arrange all $G_i$’s on a circle clockwise (see Figure 1(a)). Without loss of generality, we assume that the set of edges of $G$ between $G_i$ and
Figure 1. (a) A connected plane bipartite graph $G$ with 6-rotation symmetry, where the local width $w(G)$ of $G$ equals 2. (b) One component $G_0$ of $G$.

$G_{i+1}$ is exactly \{$v_{r_k}^{(i)}v_{s_k}^{(i+1)} | k = 1, 2, \ldots, p$\} (see Figure 1(a)), where $G_{2n} = G_0$, $v_{s_k}^{(2n)} = v_{s_k}^{(0)}$, and $v_{r_k}^{(2n)} = v_{r_k}^{(0)}$, and the generator $x$ of the rotation subgroup $C_{2n}$ of the automorphism group of $G$ maps $v_{r_k}^{(i)}$ (resp. $v_{s_k}^{(i)}$) to $v_{r_k}^{(i+1)}$ (resp. $v_{s_k}^{(i+1)}$) for $1 \leq k \leq p$, respectively. The local width of a 2n-rotation symmetric graph $G$ with 2n isomorphic components $G_i$’s, denoted $w(G)$, is defined to be the number of edges in $G_i$ lying on the boundary face of $G$ which contains the rotation center $O$. For the graph $G$ illustrated in Figure 1(a), $w(G) = 2$. For the underlying graph $G$ (which is a 6-rotation symmetric graph) of the digraph illustrated in Figure 4(a), $w(G) = 1$.

Note that $G_0$ is a plane graph. By Lemma 2.4, there exists a Pfaffian orientation $G_0^\circ$ of $G_0$ satisfying the condition in Lemma 2.3. For the corresponding graph $G_0$ illustrated in Figure 1(b), the orientation $G_0^\circ$ shown in Figure 2(a) is a Pfaffian orientation satisfying the condition in Lemma 2.3. Now we define a Pfaffian orientation $G^\circ$ from $G_0^\circ$ as follows. If we reverse the orientation of each arc of $G_0^\circ$, then we can obtain an orientation of $G_0$ satisfying conditions in Lemma 2.3, denoted by $G_0^{-\circ}$. Hence, $G_0^{-\circ}$ is the converse of $G_0^\circ$. Note that $G_0$ is bipartite. Therefore, $G_0^{-\circ}$ is a Pfaffian orientation of $G_0$ satisfying the condition in Lemma 2.3. Since all $G_i$’s are isomorphic, we define the orientations of all $G_{2i}$’s to be the same as $G_0^{-\circ}$ and the orientations of all $G_{2i+1}$’s to be same as $G_0^{-\circ}$ for $0 \leq i \leq n - 1$, respectively. In order to obtain an orientation $G^\circ$ of $G$, we need to orient all edges with the form of $v_{r_k}^{(i)}v_{s_k}^{(i+1)}$ in $G$. Note that the subgraph $G_{0,1}$ of $G$ induced by $V(G_0) \cup V(G_1)$ is a plane graph. Define the orientations of $G_0$ and $G_1$ in $G_{0,1}$ to be $G_0^\circ$ and $G_0^{-\circ}$,
respectively, and orient edges between $G_0$ and $G_1$ one by one such that every face between $G_0$ and $G_1$ has an odd number of edges oriented clockwise. Hence we have obtained an orientation $G_{0,1}^r$ of $G_{0,1}$ satisfying the condition in Lemma 2.3 such that the two induced orientations of $G_0$ and $G_1$ in $G_{0,1}$ are exactly $G_0^r$ and $G_0^{-r}$.

For the graph $G$ illustrated in Figure 1(a) and the orientation $G_0^r$ shown in Figure 2(a), the corresponding orientation $G_{0,1}^r$ is pictured in Figure 2(b).

Hence we have oriented the edges $v_{r_k}^{(0)}v_{s_k}^{(1)}$ for $1 \leq k \leq p$ in $G$. Define the orientations of edges $v_{r_k}^{(1)}v_{s_k}^{(1)}$ for $1 \leq i \leq 2n - 2$ to be from $v_{r_k}^{(i)}$ to $v_{s_k}^{(i+1)}$ if the orientation of $v_{r_k}^{(0)}v_{s_k}^{(1)}$ is from $v_{r_k}^{(0)}$ to $v_{s_k}^{(1)}$, and the orientations of edges $v_{r_k}^{(i)}v_{s_k}^{(i+1)}$ to be from $v_{r_k}^{(i+1)}$ to $v_{r_k}^{(i)}$ otherwise. So far, we have oriented all edges of $G$ except the edges $v_{r_k}^{(2n-1)}v_{s_k}^{(0)}$ for $1 \leq k \leq p$. In order to orient the edges $v_{r_k}^{(2n-1)}v_{s_k}^{(0)}$ such that $G^c$ is a Pfaffian orientation, we must distinguish the following four cases:

Case (i). both $n$ and $w(G)$ (the local width of $G$) are even;
Case (ii). $n$ is odd and $w(G)$ is even;
Case (iii). $n$ is even and $w(G)$ is odd;
Case (iv). both $n$ and $w(G)$ are odd.

For Cases (i), (ii), and (iii), define the orientations of edges $v_{r_k}^{(2n-1)}v_{s_k}^{(0)}$ to be from $v_{r_k}^{(2n-1)}$ to $v_{s_k}^{(0)}$ if the orientation of $v_{r_k}^{(0)}v_{s_k}^{(1)}$ is from $v_{r_k}^{(0)}$ to $v_{s_k}^{(1)}$, and the orientations of edges $v_{r_k}^{(2n-1)}v_{s_k}^{(0)}$ to be from $v_{s_k}^{(0)}$ to $v_{r_k}^{(2n-1)}$ otherwise (see Figure 3(a)).

For Case (iv), define the orientations of edges $v_{r_k}^{(2n-1)}v_{s_k}^{(0)}$ to be from $v_{r_k}^{(2n-1)}$ to $v_{s_k}^{(0)}$ if the orientation of $v_{r_k}^{(0)}v_{s_k}^{(1)}$ is from $v_{r_k}^{(0)}$ to $v_{s_k}^{(1)}$, and the orientations of edges $v_{r_k}^{(2n-1)}v_{s_k}^{(0)}$ to be from $v_{s_k}^{(0)}$ to $v_{r_k}^{(2n-1)}$ otherwise (see Figure 4(a)).

Hence we have obtained an orientation $G^c$ of $G$. For the graph $G$ in Figure 1(a) and the orientation $G_0^r$ of $G_0$ in Figure 2(a), by the above definition, the corresponding orientation $G^c$ of $G$ is shown in Figure 3(a). For the underlying graph $G$ of the digraph illustrated in Figure 4(a), the corresponding orientation $G^c$ is shown in Figure 4(a). Note that $G$ is a bipartite graph. Every boundary face of the $2n$ $G_i$'s in $G^c$ is oddly oriented clockwise. Moreover, it is not difficult to see that every boundary face in $G^c$ intersected by the rotation axes $\ell_1, \ell_2, \ldots, \ell_{2n-1}$—except the face containing the rotation center $O$—is oddly oriented clockwise. For Cases (i), (ii), and (iii), the number of edges oddly oriented clockwise in the face containing the rotation center $O$ equals $w(G) \times n + 2n - 1 \equiv 1 \pmod{2}$ (For the
Figure 3. (a) A regular orientation $G^e$ of $G$ with local width $w(G) = 2$. (b) The weighted digraph $D_j$.

Figure 4. (a) A regular orientation $G^e$ of $G$ with the local width $w(G) = 1$. (b) The corresponding weighted digraph $D_j$. 
orientation illustrated in Figure 3(a), the number of edges oddly oriented clockwise in the face containing the rotation center \( O \) equals \( 2 \times 3 + 5 = 11 \). For Case \((iv)\), the number of edges oddly oriented clockwise in the face containing the rotation center \( O \) equals \( w(G) \times n + 2n = 1 \) (mod 2) (For the orientation shown in Figure 4(a), the number of edges oddly oriented clockwise in the face containing the rotation center \( O \) equals \( 1 \times 3 + 6 = 9 \)). Thus the orientation \( G^c \) of \( G \) satisfies the condition in Lemma 2.3. Consequently, \( G^c \) is a Pfaffian orientation of \( G \). We call \( G^c \) a regular orientation of \( G \).

For \( j = 0, 1, \ldots, n - 1 \), define \( \alpha_j(G) \) as

\[
\alpha_j(G) = \begin{cases} 
\cos \frac{2j\pi}{2n} + i \sin \frac{2j\pi}{2n} & \text{if } n = 1 \text{ (mod 2)}, \ w(G) = 0 \text{ (mod 2)}; \\
\cos \frac{(2j+1)\pi}{2n} + i \sin \frac{(2j+1)\pi}{2n} & \text{otherwise}.
\end{cases}
\]

(1)

For convenience, set \( \alpha_j = \alpha_j(G) \) for \( 0 \leq j \leq n - 1 \).

Now we construct \( n \) weighted digraphs \( D_0, D_1, \ldots, D_{n-1} \) of order \( N/2n \) from \( G^c \), which may contain both loops and multiple arcs, as follows. Note that \( G_0^c \) is a digraph containing neither loops nor multiple arcs and the vertex set of \( G_0^c \) is \( V(G_0^c) = \{v_1^{(0)}, v_2^{(0)}, \ldots, v_{N/2n}^{(0)}\} \). Let \( G_0^c \) be the weighted digraph obtained from \( G^c \) by replacing each arc \( (v_i^{(0)}, v_j^{(0)}) \) with two arcs \( (v_i^{(0)}, v_j^{(0)}) \) and \( (v_j^{(0)}, v_i^{(0)}) \) with weights 1 and \(-1\), respectively. For each \( j = 0, 1, 2, \ldots, n - 1 \), define \( D_j \) to be the weighted digraph obtained from \( G_0^c \) by the following procedures:

1. For each \( i = 1, 2, \ldots, p \), if \( v_i^{(0)} = v_r^{(0)} \), then add a loop at vertex \( v_i^{(0)} \) with weight \( \alpha_j + \alpha_j^{-1} \) if \( (v_i^{(0)}, v_i^{(1)}) \) is an arc in \( G^c \) and with weight \( -\alpha_j - \alpha_j^{-1} \) if \( (v_i^{(1)}, v_i^{(0)}) \) is an arc in \( G^c \).

2. For each \( i = 1, 2, \ldots, p \), if \( v_i^{(0)} \neq v_r^{(0)} \), then add two arcs \( (v_i^{(0)}, v_i^{(1)}) \) and \( (v_i^{(1)}, v_i^{(0)}) \) with weights \( \alpha_j \) and \( \alpha_j^{-1} \) if \( (v_i^{(0)}, v_i^{(1)}) \) is an arc in \( G^c \) and with weights \( -\alpha_j \) and \( -\alpha_j^{-1} \) if \( (v_i^{(1)}, v_i^{(0)}) \) is an arc in \( G^c \).

For the graph \( G \) illustrated in Figure 1(a), the corresponding weighted digraph \( D_j \) is shown in Figure 3(b), where each arc \( (v_s^{(0)}, v_t^{(0)}) \) of \( D_j \), which is also an arc in \( G_0^c \), must be regarded as two arcs \( (v_s^{(0)}, v_t^{(0)}) \) and \( (v_t^{(0)}, v_s^{(0)}) \) with weights 1 and \(-1\), respectively. For the orientation \( G^c \) shown in Figure 4(a), the corresponding digraph \( D_j \) is illustrated in Figure 4(b).

By a suitable labelling of vertices of the regular orientation \( G^c \) of \( G \), for Cases \((i)-(iii)\), the skew adjacency matrix \( A_1(G^c) \) of \( G^c \) has the following form:

\[
A_1(G^c) = \begin{pmatrix}
A & R & 0 & \cdots & 0 & R^T \\
-R^T & -A & R & \cdots & 0 & 0 \\
0 & -R^T & A & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A & R \\
-R & 0 & 0 & \cdots & -R^T & -A \\
\end{pmatrix}_{2n \times 2n}
\]
and for Case (iv), the skew adjacency matrix $A_2(G^c)$ of $G^c$ has the following form:

$$
A_2(G^c) = \begin{pmatrix}
A & R & 0 & \cdots & 0 & -R^T \\
-R^T & -A & R & \cdots & 0 & 0 \\
0 & -R^T & A & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A & R \\
R & 0 & 0 & \cdots & -R^T & -A
\end{pmatrix}_{2n \times 2n},
$$

where $A$ is the skew adjacency matrix of $G_0^c$, and $R$ is the adjacent relation in $G_{0,1}^c$ between $G_0$ and $G_1$.

By the definitions of $\alpha_j$’s and $D_j$’s, it is not difficult to see that the adjacency matrix $A_j$ of weighted digraph $D_j$ defined above equals exactly $A + \omega^j R + \omega^{-j} R^T$ if $n = 1 \pmod{2}$ and $A + \omega_j R + \omega_j^{-1} R^T$ otherwise, where $\omega = \cos \frac{2\pi}{2n} + i \sin \frac{2\pi}{2n}$ and $\omega_j = \cos \frac{(2j+1)\pi}{2n} + i \sin \frac{(2j+1)\pi}{2n}$.

Now we can give a new presentation of the product theorem obtained independently by Jockusch [10] and by Kuperberg [15] as follows.

**Theorem 3.1 (Product Theorem).** Let $G$ be a simple connected plane bipartite graph of order $N$ with $2n$-rotation symmetry and $G_i$’s for $0 \leq i \leq 2n - 1$ be the $2n$ graphs defined above. Suppose there are $w(G)$ edges in $G_i$, lying on the boundary face of $G$ which contains the rotation center $O$ and there exists no vertex of $G$ lying on the rotation axes $l_0, l_1, \ldots, l_{2n-1}$ passing $O$. Let $G^c$ be a regular orientation and $D_j$’s be the $n$ weighted digraphs defined above. Then the number of perfect matchings of $G$ can be expressed by

$$
M(G) = \prod_{j=0}^{n-1} |\det(A_j)|,
$$

where $A_j = (a^{(j)}_{st})$ is the adjacency matrix of $D_j$ each entry $a^{(j)}_{st}$ of which equals the sum of weights of all arcs from vertices $v_s^{(0)}$ to $v_t^{(0)}$ in $D_j$, and $\alpha_j = \alpha_j(G)$ satisfies (1).

**Corollary 3.2.** Let $G$ be a bulk plane bipartite lattice of order $N$ with $2n$-rotation symmetry (where $N \to \infty$). If $N' = \frac{N}{2n}$ is a constant, then the entropy of $G$

$$
\lim_{N' \to \infty} \frac{2}{N'} \log M(G) = \frac{1}{N'\pi} \int_0^\pi \log |\det(D)| dx,
$$

where $D$ is the matrix of order $N'$ obtained from $A_j$ by replacing each $\alpha_j$ in entries of $A_j$ with $\cos x + i \sin x$, $i^2 = -1$.

**Proof.** By Theorem 3.1, we have

$$
M(G) = \prod_{j=0}^{n-1} |\det(A_j)|,
$$

where $A_j$ is the adjacency matrix of $D_j$. Hence

$$
\lim_{N \to \infty} \frac{2}{N} \log M(G) = \lim_{n \to \infty} \frac{2}{2nN'} \sum_{j=0}^{n-1} \log |\det(A_j)| = \lim_{n \to \infty} \frac{1}{nN'} \sum_{j=0}^{n-1} \log |\det(A_j)|.
$$
By the definition of $A_j$'s, it is not difficult to see that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |\det(A_j)| = \frac{1}{\pi} \int_0^{\pi} \log |\det(D)| \, dx
\]
which implies the corollary.

Similarly, we have the following

**Corollary 3.3.** Let $G$ be a bulk plane bipartite lattice of order $N$ with $2n$-notation symmetry, where $N' = \frac{N}{2n} \to \infty$. Then the entropy of $G$
\[
\lim_{N \to \infty} \frac{2}{N} \log M(G) = \frac{1}{\pi} \int_0^{\pi} \left[ \lim_{N' \to \infty} \log |\det(D)| \right] \, dx,
\]
where $D$ is the same as in Corollary 3.5.

## 4. TWO TYPES OF TILINGS OF CYLINDERS

Two bulk lattice graphs, denoted by $G_1^*(m, 2n)$ and $G_2^*(m, 2n)$, are illustrated in Figure 5(a) and Figure 5(b), respectively, where $G_1^*(m, 2n)$ is a finite subgraph of an edge-to-edge tilings of the plane with two types of vertices—8.8.6 and 8.8.4 vertices, and $G_2^*(m, 2n)$ is a finite subgraphs of 8.8.4 tilings in the Euclidean plane which has been used to describe phase transitions in the layered hydrogen-bonded $SnCl_2 \cdot 2H_2O$ crystal [25] in physical systems [1, 22, 25]. The bulk lattice graph $G_1^*(m, 2n)$ is composed of $2mn$ hexagons whose fundamental part is a hexagon. Similarly, the bulk lattice graph $G_2^*(m, 2n)$ is composed of $2mn$ quadrangles whose fundamental part is a quadrangle. Physicists call each of this kind of bulk graphs “an $(m, 2n)$-bipartite graphs with the free-boundary condition whose fundamental domain is $G^*$” (see [13]).

![Figure 5.](image)

**Figure 5.** (a) The lattice graph $G_1^*(m, 2n)$. (b) The lattice graph $G_2^*(m, 2n)$.

If we add edges $(a_i, a_i^*)$, $(b_i, b_i^*)$ for $1 \leq i \leq m$ and $(c_i, c_i^*)$ for $1 \leq i \leq 2n$ in $G_1^*(m, 2n)$, we obtain an $(m, 2n)$-bipartite graph with the doubly-periodic condition...
on a torus (see the definition in [14]), denoted by \( G_1^t(m, 2n) \). Similarly, if we add edges \((a_i, a_i^*)\) for \(1 \leq m\) and \((b_i, b_i^*)\) for \(1 \leq i \leq 2n\) in \( G_2^t(m, 2n) \), then an \((m, 2n)\)-bipartite graph with the doubly-periodic condition on a torus, denoted by \( G_2^t(m, 2n) \), is obtained. For some related work on the plane bipartite graph with the doubly-periodic condition, see Kenyon, Okounkov, and Sheffield [14] and Cohn, Kenyon, and Propp [4]. Salinas and Nagle [25] showed that the entropy of \( G_2^t(m, 2n) \), denoted by \( \lim_{n,m \to \infty} \frac{2}{8mn} \log(M(G_2^t(m, 2n))) \), equals

\[
\frac{1}{2\pi} \int_0^{\pi/2} \log \left[ \frac{5 + \sqrt{25 - 16 \cos^2 \theta}}{2} \right] \, d\theta \approx 0.3770. \tag{2}
\]

In this section, as applications of the product theorem in Section 3, we enumerate perfect matchings of two types of subgraphs of tilings of cylinders—\( G_1(m, 2n) \) and \( G_2(m, 2n) \), where \( G_1(m, 2n) \) (resp. \( G_2(m, 2n) \)) is obtained from \( G_1^t(m, 2n) \) (resp. \( G_2^t(m, 2n) \)) by adding extra edges \((a_i, a_i^*), (b_i, b_i^*)\) for \(1 \leq i \leq m\) (resp. \((a_i, a_i^*), (b_i, b_i^*)\) for \(1 \leq i \leq m, 1 \leq j \leq 2n\)) between each pair of opposite vertices of both sides of them. We call each of \( G_1(m, 2n) \) and \( G_2(m, 2n) \) “an \((m, 2n)\)-bipartite graph with the cylinder-boundary condition” (simply cylinder). We also obtain the exact solutions for the entropies of two types of corresponding tilings of cylinders. We observe that both the Archimedean 8.8.4 tilings of the Euclidean plane and the corresponding tilings of the cylinder have the same entropy which is approximately 0.3770. Based on the result obtained in [14], we also show that both \( G_1(m, 2n) \) and \( G_1^t(m, 2n) \) have the same entropy which approximately 0.3344. These are reasonable conclusions from the physical intuition.

4.1. THE CYLINDER \( G_1(m, 2n) \). Note that the cylinder \( G_1(m, 2n) \) can be regarded as a \(2n\)-rotation symmetric plane bipartite graph. There exists a regular orientation \( G_1(m, 2n)^c \) of \( G_1(m, 2n) \) as stated in Section 3, which is illustrated in Figure 6(a). For \( G_1(m, 2n)^c \), all hexagons in the first column have the same orientation, all hexagons in the second column have the inverse of the orientation of hexagons in the first column, and so on.

**Theorem 4.1.** For the cylinder \( G_1(m, 2n) \), the number of perfect matchings of \( G_1(m, 2n) \) can be expressed by

\[
M(G_1(m, 2n)) = \frac{1}{2^m} \prod_{j=0}^{n-1} \frac{1}{\sqrt{4 + \beta_j^2}} \left[ \left( \sqrt{4 + \beta_j^2} + \beta_j \right)^{2m+1} + \left( \sqrt{4 + \beta_j^2} - \beta_j \right)^{2m+1} \right],
\]

and the entropy of \( G_1(m, 2n) \), i.e.,

\[
\lim_{m,n \to \infty} \frac{2}{12mn} \log M(G_1(m, 2n)) \text{, equals}
\]

\[
\frac{2}{3\pi} \int_0^{\pi/2} \log(\cos x + \sqrt{4 + \cos^2 x}) \, dx \approx 0.3344,
\]

where \( \beta_j = \cos \frac{4\pi \alpha}{n} \) if \( \alpha \) is odd and \( \beta_j = \cos \frac{(2j+1)\pi}{2n} \) otherwise.

**Proof.** For the regular orientation \( G_1(m, 2n)^c \) of the cylinder \( G_1(m, 2n) \) shown in Figure 6(a), by the definition of \( D_j \) for \(0 \leq j \leq n-1\) defined above, the corresponding digraphs \( D_j \)'s have the form illustrated in Figure 7(a), where \( \alpha_j \)'s satisfy (1), and each arc \((v_i, v_j)\) in \( D_j \) whose weight is neither \( \alpha_j \) nor \( \alpha_j^{-1} \) must be regarded
Figure 6. (a) The regular orientation $G_1(m, 2n)^e$ of $G_1(m, 2n)$. (b) The regular orientation $G_2(m, 2n)^e$ of $G_2(m, 2n)$.

Figure 7. (a) The digraphs $D_j$'s corresponding to the regular orientation $G_1(m, 2n)^e$ in Figure 6(a). (b) The digraph $D_j'$ obtained from $D_j$ by deleting vertex $c_i^*$ shown in Figure 7(a). (c) The digraphs $D_j$'s corresponding to the regular orientation $G_2(m, 2n)^e$ in Figure 6(b). (d) The digraph $D_j'$ obtained from $D_j$ by deleting vertex $b_i^*$ shown in Figure 7(c).
as two arcs \((v_i, v_j)\) and \((v_j, v_i)\) with weights 1 and \(-1\), respectively. Let \(D'_j\) be the
digraph obtained from \(D_j\) by deleting vertex \(c_i^*\) (see Figure 7(b)).

For \(j = 0, 1, \ldots, n - 1\), set

\[
L_m(j) = \det(A_j), \quad L'_m(j) = \det(A'_j),
\]

where \(A_j\) (resp. \(A'_j\)) is the adjacency matrix of the digraph \(D_j\) (resp. \(D'_j\)). It
is not difficult to prove that \(\{L_m(j)\}_{m \geq 0}\) and \(\{L'_m(j)\}_{m \geq 0}\) satisfy the following
recurrences:

\[
\begin{align*}
L_m(j) &= (4 + 4\beta^2_j)L_{m-1}(j) + 4\beta_jL'_{m-1}(j) \quad \text{for } m \geq 1, \\
L'_m(j) &= 4\beta_jL_{m-1}(j) + 4L'_{m-1}(j) \quad \text{for } m \geq 1, \\
L_0(j) &= 1, \quad L'_0(j) = 0.
\end{align*}
\]

Hence we have the following:

\[
\begin{align*}
L_m(j) &= (8 + 4\beta^2_j)L_{m-1}(j) - 16L_{m-2}(j) \quad \text{for } m \geq 2, \\
L_0(j) &= 1, \quad L_1(j) = 4 + 4\beta^2_j,
\end{align*}
\]

which implies the following:

\[
L_m(j) = \frac{1}{2} \frac{1}{\sqrt{4 + \beta^2_j}} \left[ \left( \sqrt{4 + \beta^2_j + \beta_j} \right)^{2m+1} + \left( \sqrt{4 + \beta^2_j - \beta_j} \right)^{2m+1} \right].
\]

Hence the equality (3) follows from the product theorem (Theorem 3.1) and (4).

So the entropy of \(G_1(m, 2n)\)

\[
\lim_{m, n \to \infty} \frac{2}{12mn} \log M(G_1(m, 2n)) = \lim_{m, n \to \infty} \frac{1}{6mn} \times
\]

\[
\left\{ -n \log 2 - \frac{1}{2} \sum_{j=0}^{n-1} \log(4 + \beta^2_j) + \sum_{j=0}^{n-1} \log \left[ \left( \sqrt{4 + \beta^2_j + \beta_j} \right)^{2m+1} + \left( \sqrt{4 + \beta^2_j - \beta_j} \right)^{2m+1} \right] \right\}
\]

\[
= \lim_{m, n \to \infty} \frac{1}{6mn} \sum_{j=0}^{n-1} \log \left[ \left( \sqrt{4 + \beta^2_j + \beta_j} \right)^{2m+1} + \left( \sqrt{4 + \beta^2_j - \beta_j} \right)^{2m+1} \right]
\]

\[
= \lim_{m, n \to \infty} \frac{1}{3mn} \sum_{j=0}^{n-1} \log \left( \sqrt{4 + \beta^2_j + \beta_j} \right)^{2m+1} + \left( \sqrt{4 + \beta^2_j - \beta_j} \right)^{2m+1}
\]

\[
= \lim_{m, n \to \infty} \frac{1}{3mn} \sum_{j=0}^{n-1} \log(\sqrt{4 + \beta^2_j + \beta_j})
\]

\[
= \frac{2}{\pi} \int_0^{\pi/2} \log(\cos x + \sqrt{4 + \cos^2 x}) \, dx \approx 0.3344
\]

and the theorem thus follows. \(\square\)

In order to prove the following corollary, we need to introduce a formula of the
entropy for an \((n, n)\)-bipartite graphs with the doubly-period condition obtained by
Kenyon, Okounkov, and Sheffield \cite{14}. Let \(G\) be a \(Z^2\)-period bipartite graph which
is embedded in the plane so that translations in the plane act by color-preserving
isomorphisms of \(G\)—isomorphisms which map black vertices to black vertices and
white to white. Let \(G_n\) be the quotient of \(G\) by the action of \(nZ^2\). Then \(G_n\) is a
bipartite graph with the doubly-period condition. Let \(P(z, w)\) be the characteristic
polynomial of \(G\) (see the definition in page 1029 in \cite{14}). Authors in \cite{14} showed
that the entropy of \(G_n\)

\[
\lim_{n \to \infty} \frac{2}{n^2 |G_1|} \log M(G_n) = \frac{2}{|G_1|(2\pi i)^2} \int_D \log |P(z, w)| \frac{dz}{z} \frac{dw}{w},
\]

(5)
where $D = \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\}$ and $i^2 = -1$.

**Corollary 4.2.** Both $G_1(m, 2n)$ and $G'_1(m, 2n)$ have the same entropy, that is,
\[
\lim_{m, n \to \infty} \frac{2}{12mn} \log(M(G_1(m, 2n))) = \lim_{m, n \to \infty} \frac{2}{12mn} \log(M(G'_1(m, 2n))) \approx 0.3344.
\]

**Proof.** Note that by the definition in [14] the fundamental domain of $G'_1(m, 2n)$ is composed of two hexagons (see Figure 8). Otherwise, if we use a hexagon as the fundamental domain, then it does not satisfy the condition “color-preserving isomorphisms”. It is not difficult to show that the characteristic polynomial of $G'_1(m, 2n)$
\[
P(z, w) = 10 - 4(z + z^{-1}) - (w + w^{-1}).
\]

Hence, by (5), we have
\[
\lim_{m, n \to \infty} \frac{2}{12mn} \log(M(G'_1(m, 2n))) = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \log(10 - 8\cos x - 2\cos y) dx dy.
\]

Let $F(y) = \int_0^{2\pi} \log(10 - 8\cos x - 2\cos y) dx = 2 \int_0^{\pi} \log(10 - 8\cos x - 2\cos y) dx$.

Then
\[
F'(y) = 4\sin y \int_0^{\pi} \frac{dx}{10 - 8\cos x - 2\cos y} = \frac{2\pi \sin y}{\sqrt{\cos^2 y - 10\cos y + 9}}.
\]

Hence we have
\[
F(y) = 2\pi \log(5 + \cos y + \sqrt{\cos^2 y - 10\cos y + 9})
\]

implying
\[
\frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log(10 - 8\cos x - 2\cos y) dx dy = \frac{1}{12\pi} \int_0^{2\pi} \log(5 + \cos x + \sqrt{\cos^2 x - 10\cos x + 9}) dx.
\]

So, by Theorem 4.1, it suffices to prove the following:
\[
4 \int_0^{\frac{\pi}{2}} \log(\cos x + \sqrt{4 + \cos^2 x}) dx = \int_0^{\pi} \log(5 + \cos x + \sqrt{\cos^2 x - 10\cos x + 9}) dx.
\]

Note that
\[
4 \int_0^{\frac{\pi}{2}} \log(\cos x + \sqrt{4 + \cos^2 x}) dx = 2 \int_0^{\frac{\pi}{2}} \ln(2 \cos^2 x + 4 + \sqrt{16 \cos^2 x + 4 \cos^4 x}) dx
\]
\[
= \int_0^{\pi} \log(2 \cos^2 x + 4\sqrt{16 \cos^2 x + 4 \cos^4 x}) dx = \int_0^{\pi} \log(5 + \cos x + \sqrt{\cos^2 x + 10\cos x + 9}) dx
\]
in Figure 6(c) (resp. Figure 6(d)). It is not difficult to prove that

\[
\begin{align*}
A & \text{ where } \\
& = \int_0^n \log(5 - \cos x + \sqrt{\cos^2 x - 10 \cos x + 9}) \, dx.
\end{align*}
\]

The corollary has thus been proved. \qed

4.2. THE CYLINDER \(G_2(m, 2n)\). Note that the cylinder \(G_1(m, 2n)\) can be embedded into the plane such that it is a 2n-rotation symmetric plane bipartite graph. There exists a regular orientation \(G_2(m, 2n)^c\) of \(G_2(m, 2n)\) as stated in Section 3, which is illustrated in Figure 6(b).

**Theorem 4.3.** For the cylinder \(G_2(m, 2n)\), the number of perfect matchings of \(G_2(m, 2n)\) can be expressed by

\[
M(G_2(m, 2n)) = \prod_{j=0}^{n-1} \left[ \frac{\sqrt{9 + 16\beta_j^2} + 3}{2\sqrt{9 + 16\beta_j^2}} \left( \frac{5 + \sqrt{9 + 16\beta_j^2}}{2} \right)^m + \frac{\sqrt{9 + 16\beta_j^2} - 3}{2\sqrt{9 + 16\beta_j^2}} \left( \frac{5 - \sqrt{9 + 16\beta_j^2}}{2} \right)^m \right],
\]

and the entropy, i.e.,

\[
\lim_{m,n \to \infty} \frac{1}{2mn} \log M(G_2(m, 2n)),
\]

equals

\[
\frac{1}{2\pi} \int_0^{\pi/2} \log \left[ \frac{5 + \sqrt{25 - 16 \cos^2 \theta}}{2} \right] \, d\theta \approx 0.3770,
\]

where \(\beta_j = \cos \frac{2\pi j}{n}\) if \(n\) is odd and \(\beta_j = \cos \frac{(2j+1)\pi}{2n}\) otherwise.

**Proof.** We can prove easily the statement in the theorem on the entropy from (5). Hence it suffices to prove that (5) holds. For the regular orientation \(G_2(m, 2n)^c\) of the cylinder \(G_2(m, 2n)\) shown in Figure 6(b), by the definition of \(D_j\) for \(0 \leq j \leq n-1\) defined above, the corresponding digraphs \(D_j\)'s have the form illustrated in Figure 7(c), where \(\alpha_i\)’s satisfy (1), and each arc \((v_i, v_j)\) in \(D_j\) whose weight is neither \(\alpha_j\) (or \(-\alpha_j\)) nor \(\alpha_j^{-1}\) (or \(-\alpha_j^{-1}\)) must be regarded as two arcs \((v_i, v_j)\) and \((v_j, v_i)\) with weights 1 and \(-1\), respectively. Let \(D_j^\prime\) be the digraph obtained from \(D_j\) by deleting vertex \(b_i^\prime\) (see Figure 7(d)).

For \(j = 0, 1, \ldots, n-1\), set

\[
P_m(j) = \det(A_j), \quad P'_m(j) = \det(A'_j),
\]

where \(A_j\) (resp. \(A'_j\)) is the adjacency matrix of the digraph \(D_j\) (resp. \(D'_j\)) illustrated in Figure 6(c) (resp. Figure 6(d)). It is not difficult to prove that \(\{P_m(j)\}_{m \geq 0}\) and \(\{P'_m(j)\}_{m \geq 0}\) satisfy the following recurrences:

\[
\left\{
\begin{array}{ll}
P_{2m+1}(j) = 4P_{2m}(j) + 2\beta_j P'_{2m}(j), & P_{2m+1}(j) = -2\beta_j P_{2m}(j) - P'_{2m}(j) \quad m \geq 0, \\
P_{2m}(j) = 4P_{2m-1}(j) - 2\beta_j P'_{2m-1}(j), & P'_{2m}(j) = 2\beta_j P_{2m-1}(j) - P'_{2m-1}(j) \quad m \geq 1,
\end{array}
\right.
\]

\[
P_0(j) = 1, \quad P'_0(j) = 0, \quad P_1(j) = 4, \quad P'_1(j) = -2\beta_j.
\]

Hence from the recurrences above we have the following:

\[
\left( \begin{array}{c} P_{2m+1}(j) \\ P'_{2m+1}(j) \end{array} \right) = \left( \begin{array}{cc} 16 + 4\beta_j^2 & -10\beta_j \\ -10\beta_j & 4\beta_j^2 + 1 \end{array} \right) \left( \begin{array}{c} P_{2m-1}(j) \\ P'_{2m-1}(j) \end{array} \right)
\]

and

\[
\left( \begin{array}{c} P_{2m}(j) \\ P'_{2m}(j) \end{array} \right) = \left( \begin{array}{cc} 16 + 4\beta_j^2 & 10\beta_j \\ 10\beta_j & 4\beta_j^2 + 1 \end{array} \right) \left( \begin{array}{c} P_{2(m-1)}(j) \\ P'_{2(m-1)}(j) \end{array} \right).
\]
Let \( a_m = P_{2m+1}(j) \) and \( b_m = P_{2m}(j) \) for \( j \geq 0 \). Hence we have

\[
\begin{align*}
  a_m &= (18\beta_j^2 + 17)a_{m-1} - 16(1 - \beta_j^2)^2a_{m-2} & \text{for } m \geq 2, \\
  b_m &= (18\beta_j^2 + 17)b_{m-1} - 16(1 - \beta_j^2)^2b_{m-2} & \text{for } m \geq 2, \\
  a_0 &= 4, a_1 = 64 + 36\beta_j^2, b_0 = 1, b_1 = 16 + 4\beta_j^2.
\end{align*}
\]

By solving the recurrences in (9), then we have

\[
\begin{align*}
  a_m &= \prod_{j=0}^{n-1} \left[ \frac{9 + 16\beta_j^2 + 3}{2\sqrt{9 + 16\beta_j^2}} \left( \frac{5 + \sqrt{9 + 16\beta_j^2}}{2} \right)^{2m+1} + \frac{9 + 16\beta_j^2 - 3}{2\sqrt{9 + 16\beta_j^2}} \left( \frac{5 - \sqrt{9 + 16\beta_j^2}}{2} \right)^{2m+1} \right], \\
  b_m &= \prod_{j=0}^{n-1} \left[ \frac{9 + 16\beta_j^2 + 3}{2\sqrt{9 + 16\beta_j^2}} \left( \frac{5 + \sqrt{9 + 16\beta_j^2}}{2} \right)^{2m} + \frac{9 + 16\beta_j^2 - 3}{2\sqrt{9 + 16\beta_j^2}} \left( \frac{5 - \sqrt{9 + 16\beta_j^2}}{2} \right)^{2m} \right].
\end{align*}
\]

Hence (6) has been proved and the theorem thus follows. \( \square \)

From (2) and Theorem 4.2, both the 8.8.4 tilings of the Euclidean plane and the corresponding tilings of the cylinder have the same entropy which is approximately 0.3770.

5. CONCLUDING REMARKS

In the classical work on the dimer problem for plane quadratic lattices, Kasteleyn [11] proved that the \( m \times n \) quadratic lattice with the free-boundary condition, the \( m \times n \) quadratic lattice with the doubly-period condition (i.e., the \( m \times n \) torus), and the \( m \times n \) quadratic lattice with the cylinder-boundary condition (i.e., the \( m \times n \) cylinder) have the same entropy (= \( 2c/\pi \approx 0.5831 \), where \( c \) is Catalan’s constant). In this paper we have investigated the problem on enumeration of perfect matchings of plane bipartite graphs with \( 2n \)-rotation symmetry. We compute the entropy of a bulk plane bipartite lattice with \( 2n \)-rotation symmetry. We obtain the explicit expressions for the numbers of perfect matchings and entropies for two types of tilings of (the surface of) cylinders and we showed that both the bipartite graph \( G_1(m, 2n) \) with the cylinder-period condition and the bipartite graph \( G_t_1(m, 2n) \) with the doubly-period condition have the same entropy and that both the bipartite graph \( G_2(m, 2n) \) with the cylinder-period condition and the bipartite graph \( G_t_2(m, 2n) \) with the doubly-period condition have the same entropy. A natural problem is how to enumerate perfect matchings of the plane bipartite graph with \( (2n + 1) \)-rotation symmetry. A more general problem is to enumerate perfect matchings of the plane graph (not necessary to be bipartite) with \( n \)-rotation symmetry.

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