RIGID SURFACE OPERATOR AND SYMBOL INVARIANT OF PARTITIONS

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Abstract. The symbol is used to describe the Springer correspondence for the classical groups by Lusztig. We refine the explanation that the S-duality maps of the rigid surface operators are symbol preserving maps. And we find that the maps $X_S$ and $Y_S$ used in the construction of S-duality maps are essentially the same. We clear up cause of the mismatch problem of the total number of the rigid surface operators between the $B_n$ and $C_n$ theories. And we construct all the $B_n/C_n$ rigid surface operators which can not have a dual. A classification of the problematic surface operators is made.

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1. Introduction

Surface operators are two-dimensional defects supported on a two-dimensional submanifold of spacetime, which are natural generalisations of the ’t Hooft operators. In [2], Gukov and Witten initiated a study of surface operators in $\mathcal{N} = 4$ super Yang-Mills theories in the ramified case of the Geometric Langlands Program. $S$-duality for certain subclass of surface operators is discussed in [2][3]. The $S$-duality [3] assert that $S : (G, \tau) \rightarrow (G^L, -1/n_B\tau)$ (where $n_B$ is 2 for $F_4$, 3 for $G_2$, and 1 for other semisimple classical groups [2]; $\tau = \theta/2\pi + 4\pi i/g^2$ is usual gauge coupling constant ). This transformation exchanges gauge group $G$ with the Langlands dual group. For example, the Langlands dual groups of Spin$(2n+1)$ are Sp$(2n)/\mathbb{Z}_2$. And the langlands dual groups of SO$(2n)$ are themselves.

In [2], Gukov and Witten extended their earlier analysis [2] of surface operators which are based on the invariants of duality. They identified a subclass of surface operators called ’rigid’
surface operators, which are expected to be closed under $S$-duality. There are two types rigid surface operators: unipotent and semisimple. The rigid semisimple surface operators are labelled by pairs of partitions. And unipotent rigid surface operators arise when one of the partitions is empty. In $\mathcal{E}$, some proposals for the $S$-duality maps related to rigid surface operators were made in the $B_{n}(SO(2n+1))$ and $C_{n}(Sp(2n))$ theories. These proposals involved all unipotent rigid surface operators as well as certain subclasses of rigid semisimple operators.

In $\mathcal{E}$, we analyse and extend the $S$-duality maps proposed by Wyllard, using consistency checks. We propose the $S$-duality for a subclasses of rigid surface operators. The symbol invariant is more convenient than other invariants to study the $S$-duality of surface operators but its calculation is boring. In $\mathcal{E}$, we prove the symbol invariant proposed in $\mathcal{E}$ in 

The $S$ duality maps preserve symbol but not all symbol preserving maps are $S$ duality maps. However more thorough understanding the construction of the $S$ duality of surface operators might lead to progress. A problematic mismatch in the total number of rigid surface operators between the $B_{n}$ and the $C_{n}$ theories was pointed out in $\mathcal{E}$. The discrepancy is clearly a major problem and hamper the attempt to analysis to more general classes of semisimple surface operators. Fortunately, the construction of symbol $\mathcal{E}$ and the classification of symbol preserving maps are helpful to address this problem in $\mathcal{E}$.

In this paper, we attempt to extend the analysis in $\mathcal{E}$, $\mathcal{E}$, and $\mathcal{E}$. Since no noncentral rigid conjugacy classes in the $A_{n}$ theory, we do not discuss surface operators in this case. We also omit the discussion of the exceptional groups, which are more complicated. We will focus on theories with gauge groups $SO(2n)$ and the gauge groups $Sp(2n)$ whose Langlands dual group are $SO(2n + 1)$.

In Section 2 we review the construction of rigid surface operators given in $\mathcal{E}$. We discuss some mathematical results and definitions as preparation. We focus on the symbol invariant of surface operators which are unchanged under the $S$-duality map. In Section 3 we review the symbol invariant proposed in $\mathcal{E}$. We refine the computational rules of symbol found in $\mathcal{E}$. We find the contributions to symbol of a row in the same location of a pairwise rows are the same in the $B_{n}$, $C_{n}$, and $D_{n}$ theories. As applications, the $S$-duality maps proposed in the $\mathcal{E}$, $\mathcal{E}$ can be illustrated more clearly $\mathcal{E}$. We find that the maps $X_{S}$ and $Y_{S}$ are essentially the same map.

The second part of the paper involve the mismatch problem of the total number of the rigid surface operators between the $B_{n}$ and $C_{n}$ theories. We clear up cause of this problem. We give the construction and classification of all the $B_{n}/C_{n}$ rigid surface operators which can not have a dual, revealing some subtle things.

In the appendix, we summarize revelent facts about all rigid surface operators and their associated invariants in the $SO(13)$ and $Sp(12)$ theories as examples.

2. Surface operators in $\mathcal{N} = 4$ Super-Yang-Mills

In this section, we introduce the revelent backgrounds of surface operator. We closely follow paper $\mathcal{E}$ to which we refer the reader for more details.

We consider $\mathcal{N} = 4$ super-Yang-Mills theory on $\mathbb{R}^{4}$ with coordinates $x^{0}, x^{1}, x^{2}, x^{3}$. The most important bosonic fields: a gauge field as 1-form, $A_{\mu}$ ($\mu = 0, 1, 2, 3$), six real scalars, $\phi_{I}$ ($I = 1, \ldots, 6$). All fields take values in the adjoint representation of the gauge group $G$. Surface operators are introduced by prescribing a certain singularity structure of fields near the surface on which the operator is supported. Without loss of generality we can assume the support of the surface operator $D$ to be oriented along the $(x^{0}, x^{1})$ directions. Since the fields satisfy the BPS condition, the combinations $A = A_{2} dx^{2} + A_{3} dx^{3}$ and $\phi = \phi_{2} dx^{2} + \phi_{3} dx^{3}$ must obey Hitchin’s equations $\mathcal{E}$

$$F_{A} - \phi \wedge \phi = 0, \quad d_{A} \phi = 0, \quad d_{A} \star A = 0$$  \tag{2.1}$$

A surface operator is defined as a solution to these equations with a prescribed singularity along the surface $\mathbb{R}^{2}(x^{0}, x^{1})$. 

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For the superconformal surface operator, setting \( x_2 + i x_3 = re^{i\theta} \), the most general possible rotation-invariant Ansatz for \( A \) and \( \phi \) is

\[
A = a(r) \, d\theta, \\
\phi = -c(r) \, d\theta + b(r) \frac{dr}{r}.
\]

On substituting this Ansatz into Hitchin’s equations (2.1) and defining \( s = -\ln r \), equations (2.1) reduces to Nahm’s equations

\[
\frac{da}{ds} = [b, c], \\
\frac{db}{ds} = [c, a], \\
\frac{dc}{ds} = [a, b]
\]

which imply the communication for the constants \( a \), \( b \) and \( c \). Surface operators of this type were discussed in [2].

There is another way to obtain conformally invariant surface operator. Nahm’s equations (2.3) are solved with

\[
a = \frac{t_x}{s + 1/f}, \quad b = \frac{t_y}{s + 1/f}, \quad c = \frac{t_z}{s + 1/f},
\]

where \( t_x, t_y \) and \( t_z \) are elements of the lie algebra \( g \), spanning a representation of \( su(2) \). These \( t_i \)'s are in the adjoint representation of the gauge group. The surface operator is actually conformal invariant if the function \( f \) allowed to fluctuate.

Alternatively, the surface operators can be characterised as the conjugacy class of the monodromy

\[
U = P \exp(\oint A),
\]

where \( A = A + i\phi \). The integration is around a circle near \( r = 0 \). Following from (2.1), one finds that \( F = dA + A \wedge A = 0 \), which means that \( U \) is independent of deformations of the integration contour. For the surface operators (2.4), \( U \) becomes

\[
U = P \exp\left(\frac{2\pi}{s + 1/f} t_+\right),
\]

where \( t_+ \equiv t_x + it_y \) is nilpotent, corresponding to unipotent surface operator.

There are two types of conjugacy classes in a Lie group: unipotent and semisimple. Semisimple classes can also lead to surface operators. With a semisimple element \( S \), one can obtain a surface operator with monodromy \( V = SU \). For a general surface operator, it is constructed by requiring all the fields which are solutions to Nahm’s equations satisfy the following constraint near the surface \( D(4) \)

\[
S\Psi(r, \theta)S^{-1} = \Psi(r, \theta + 2\pi).
\]

From all the surface operators constructed from conjugacy classes, a subclass of surface operators called rigid surface operator is closed on the \( S \)-duality. The rigid surface operators are expected to be superconformal and not to depend on any parameters. A unipotent conjugacy classes is called rigid\(^1\) if its dimension is strictly smaller than that of any nearby orbit. All rigid orbits have been classified \([3][4]\). A semisimple conjugacy classes \( S \) is called rigid if the centraliser of such class is larger than that of any nearby class. Summary, surface operators are called rigid if they based on monodromies of the form \( V = SU \), where \( U \) is unipotent and rigid and \( S \) is semisimple and rigid.

2.1. Preliminary

From the above discussions, a classification of unipotent and semisimple conjugacy classes is needed to study surface operators. Here we describe the classification of rigid surface operators in the \( B_n(SO(2n+1)) \), \( C_n(Sp(2n)) \) and \( D_n(SO(2n)) \) theories in detail.

\(^1\) The rigid surface operators here correspond to strongly rigid operators in [1].
The $t_+$ in Eq. (2.6) can be described in block-diagonal basis as follows

$$t_+ = \begin{pmatrix}
  t_{n_1}^n & \cdots & \cdots & \cdots \\
  \cdots & t_{n_k}^n & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & t_{n_l}^n
\end{pmatrix},$$

where $(t_{n_k}^n)$ is the ‘raising’ generator of the $n_k$-dimensional irreducible representation of su(2). For the $B_n$, $C_n$, and $D_n$ theories, there are restrictions on the allowed dimensions of the su(2) irreps since $t_+$ should belong to the relevant gauge group. From the block-decomposition (2.8) we see that unipotent (nilpotent) surface operators are classified by the restricted partitions.

A partition $\lambda$ of the positive integer $n$ is defined by a decomposition $\sum_{i=1}^{l} \lambda_i = n$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$), where the $\lambda_i$ are called parts and $l$ is the length. There is a one-to-one correspondence between partition and Young tableaux. For instance the partition $3^22^31$ corresponds to

$$\lambda = 3^22^31.$$ 

The another representation of partition $\lambda_m \lambda_{m-1} \cdots \lambda_1$ with the length $l = \sum \lambda_i$ as shown in Fig. (1). Young diagrams occur in a number of branches of mathematics and physics. They are also useful to construct the eigenstates of Hamiltonian System [23] [24] [25]. The addition of two partitions $\lambda$ and $\kappa$ is defined by the additions of each part $\lambda_i + \kappa_i$.

We have the following classification of nilpotent orbits in terms of partitions [1]:

- $\mathbf{(B}_n\mathbf{)}$: partitions of $2n+1$, $\sum \lambda_i = 2n+1$, with a constraint that all even integers appear an even number of times;
- $\mathbf{(D}_n\mathbf{)}$: partitions of $2n$, $\sum \lambda_i = 2n$, with a constraint that all even integers appear an even number of times;
- $\mathbf{(C}_n\mathbf{)}$: partitions of $2n$, $\sum \lambda_i = 2n+1$, with a constraint that all odd integers appear an even number of times;

A partition in the $B_n$ or $D_n$($C_n$) theories is called rigid if it satisfies the following conditions,

1. no gaps (i.e. $\lambda_i - \lambda_{i+1} \leq 1$ for all $i$),
2. no odd (even) integer appears exactly twice.

Rigid partitions correspond to rigid surface operators. The following facts are important for studying rigid partitions, which are easy to be proved and omitted here [1].

**Proposition 2.1.** The longest row in a rigid $B_n$ partition always contains an odd number of boxes. And the following two rows of the first row are either both of odd length or both of even length. This pairwise pattern then continues. If the Young tableau has an even number of rows the row of shortest length has to be even.

**Proposition 2.2.** The longest two rows in a rigid $C_n$ partition both contain either an even or an odd number of boxes. This pairwise pattern then continues. If the Young tableau has an even number of rows the row of shortest length has to be even.

**Proposition 2.3.** The longest row in a rigid $D_n$ partition always contains an even number of boxes. And the following two rows are either both of even length or both of odd length. This pairwise pattern then continues. If the Young tableau has an even number of rows the row of the shortest length has to be even.
The rigid semisimple conjugacy classes $S$ in formula correspond to diagonal matrices with elements $-1$ and $1$ along the diagonal in the $B_n$, $C_n$, and $D_n$ theories. The matrices $S$ break the gauge group to its centraliser at the Lie algebra level as follows
\begin{align*}
\text{so}(2n+1) & \rightarrow \text{so}(2k+1) \oplus \text{so}(2n - 2k), \\
\text{so}(2n) & \rightarrow \text{so}(2k) \oplus \text{so}(2n - 2k), \\
\text{sp}(2n) & \rightarrow \text{sp}(2k) \oplus \text{sp}(2n - 2k),
\end{align*}
which imply that the rigid semisimple surface operators correspond to pairs of partitions $(\lambda', \lambda'')$ in the $B_n$, $C_n$, and $D_n$ theories. $\lambda'$ is a rigid $B_k$ partition and $\lambda''$ is a rigid $D_{n-k}$ partition in the $B_n$ case, $\lambda'$ is a rigid $D_k$ partition and $\lambda''$ is a rigid $D_{n-k}$ partition in the $D_n$ case. $\lambda'$ is a rigid $C_k$ partition and $\lambda''$ is a rigid $C_{n-k}$ partition in the $C_n$ case. The rigid unipotent surface operator is a limiting case of rigid semisimple surface operator with $\lambda'' = 0$.

There is a close relationship between the pair of partition $(\lambda', \lambda'')$ and Weyl group. For Weyl groups in the $B_n$, $C_n$, and $D_n$ theories both conjugacy classes and irreducible unitary representations are in one-to-one correspondence with ordered pairs of partitions $[\alpha; \beta]$, where $\alpha$ is a partition of $n_\alpha$ and $\beta$ is a partition of $n_\beta$, with $n_\alpha + n_\beta = n$. Though both the conjugacy classes and unitary representations are parameterised by ordered pair of partitions there is no canonical isomorphism between the two sets.

The Kazhdan-Lusztig map is a map from the unipotent conjugacy classes of a simple group to the set of conjugacy classes of the Weyl group. This map can be extended to the case of rigid semisimple conjugacy classes. The Springer correspondence is a injective map from the unipotent conjugacy classes of a simple group to the set of unitary representations of the Weyl group. For the classical groups the above two maps can be described explicitly by the invariants fingerprint and symbol invariant of partitions, respectively.

Without explanation, we only concern about rigid partition and rigid surface operator in the following sections.

2.2. Invariants of surface operators

Invariants of the surface operators $(\lambda'; \lambda'')$ do not change under the $S$-duality map. The dimension $d$ is the most basic invariant of a rigid surface operator. It is calculated as follows:
\begin{align*}
B_n : & \quad d = 2n^2 + n - \frac{1}{2} \sum_k (s_k')^2 - \frac{1}{2} \sum_k (s_k'')^2 + \frac{1}{2} \sum_k \text{odd} k r_k' + \frac{1}{2} \sum_k \text{odd} k r_k'', \\
D_n : & \quad d = 2n^2 - n - \frac{1}{2} \sum_k (s_k')^2 - \frac{1}{2} \sum_k (s_k'')^2 + \frac{1}{2} \sum_k \text{odd} k r_k' + \frac{1}{2} \sum_k \text{odd} k r_k'', \\
C_n : & \quad d = 2n^2 + n - \frac{1}{2} \sum_k (s_k')^2 - \frac{1}{2} \sum_k (s_k'')^2 - \frac{1}{2} \sum_k \text{odd} k r_k' - \frac{1}{2} \sum_k \text{odd} k r_k'',
\end{align*}
where $s_k'$ denotes the number of parts of $\lambda'$ that are larger than or equal to $k$. And $r_k'$ denotes the number of parts of $\lambda'$ that are equal to $k$. Similarly, $s_k''$ and $r_k''$ correspond to $\lambda''$.

The invariant fingerprint is a pair of partitions $[\alpha; \beta]$ associated with the Weyl group conjugacy class. There is another invariant symbol based on the Springer correspondence, which can be extended to rigid semisimple conjugacy classes. One can construct the symbol of this rigid semisimple surface operator $(\lambda'; \lambda'')$ by calculating the symbols for both $\lambda'$ and $\lambda''$, then add the entries that are “in the same place” of these two partitions. The result symbol is denoted as follows
\begin{equation}
\sigma((\lambda'; \lambda'')) = \sigma(\lambda') + \sigma(\lambda'').
\end{equation}
An example illustrates the addition rule in detail:
\begin{equation}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix} +
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & 1 & 2 & 2 & 2 \\
1 & 2 & 2 & 2 & 2 & 2 & 3
\end{pmatrix}.
\end{equation}

It is checked that the symbol of a rigid surface operator contains the same amount of information as the fingerprint. Compared with the fingerprint invariant, the symbol invariant is much easier to be calculated and more convenient to find the $S$-duality maps of surface operators.

In [4], it was pointed that two discrete quantum numbers 'center' and 'topology' are interchanged under $S$ duality. A surface operator can detect topology then its dual should detect the centre and vice versa. However, there are some puzzles using these discrete quantum numbers to

\footnote{Without confusion, the rigid semisimple surface operators will be called rigid surface operator or surface operator in this study.}
find duality pair \[3\]. There is another problem that the number of rigid surface operators in the \(B_n\) theory is larger than that in the \(C_n\) theory \[3\], which was first observed in the \(B_4/C_4\) theories \[4\]. In this paper, we ignore the first problem for the moment. We focus on the symbol invariant to study the second problem of rigid surface operators between the dual theories. Hopefully, our works will be helpful in making new insight to the surface operator.

3. Contributions to symbol of rows of partition

In this section, we discuss the contributions to symbols of rows of partitions, refining the construction given in \[17\]. As applications, we analyse the \(S\) duality maps proposed in \[5\], with a preparation for the study of the mismatch problem in the \(S\)-duality map between the number of rigid surface operators in the \(B_n\) and \(C_n\) theories in the next section.

3.1. Symbol invariant of partitions

In \[17\], we proposed equivalent definitions of symbols for the partitions in the \(C_n\) and \(D_n\) theories, which are consistent with that in the \(B_n\) theory as much as possible.

Definition 1. \[17\]

- Symbol of a partition \(\lambda\) in the \(B_n\) theory: firstly add \(l - k\) to the \(k\)th part of the partition \(\lambda\). Then arrange the odd parts and the even parts of the sequence \(l + k + \lambda_k\) in increasing sequences \(2f_i + 1\) and \(2g_i\), respectively. Next calculate the terms

\[
\alpha_i = f_i - i + 1, \quad \beta_i = g_i - i + 1.
\]

Finally write the symbol as

\[
\left(\alpha_1 \alpha_2 \alpha_3 \cdots \beta_1 \beta_2 \cdots\right).
\]

- Symbol of a partition \(\lambda\) in the \(C_n\) theory:

1: If the length of partition is even, we compute the symbol as in the \(B_n\) case. And then append an extra 0 on the left of the top row of the symbol.

2: If the length of the partition is odd, we append an extra 0 as the last part of the partition. And then compute the symbol as in the \(B_n\) case. Finally, we delete a 0 in the first entry of the bottom row of the symbol.

- Symbol of a partition \(\lambda\) in the \(D_n\) theory: we append an extra 0 as the last part of the partition and then compute the symbol as in the \(B_n\) case. We delete two 0’s which occupy the first two entries of the bottom row of the symbol.

Remark 3.1. Note that the terms \(\alpha_*\) in formula \[3.14\] are related to \(f_*\) while the terms \(\beta_*\) are related to \(g_*\).

\[\text{Figure 2. Addition of an even row } b\text{ on the left partition leads to a different partition but in the same theory.}\]
Parity of the length of the $i$th row & Parity of $i + t + 1$ & Contribution & $L$

| Parity | Contribution |
|--------|--------------|
| odd    | $\frac{1}{2}(\sum_{k=1}^{m} n_k + 1)$ |
| even   | $\frac{1}{2}(\sum_{k=1}^{m} n_k)$ |
| even   | $\frac{1}{2}(\sum_{k=1}^{m} n_k)$ |
| odd    | $\frac{1}{2}(\sum_{k=1}^{m} n_k - 1)$ |

Table 1. Contribution to symbol of the $i$th row of the partition $\lambda_n^{m} \lambda_{n-1}^{m-1} \cdots \lambda_1^{m-1}$. It depends on the parity of the length of row, the parameter $L$, and the parity of $i + t + 1$ with $t = -1$, $t = 0$, and $t = 1$ for the partitions in the $B_n$, $C_n$, and $D_n$ theories, respectively.

Example: Symbol of the partition $\lambda = 3^2 2^2 1^1$ in the $D_n$ theory,

\begin{equation}
(3.15)
\end{equation}

According to Table 1, the symbol is

\begin{equation}
(3.16)
\end{equation}

\[a_{(3^2 2^2 1^1)} \equiv \left( \begin{array}{c}
1 \\
0
\end{array} \right) + 
\left( \begin{array}{c}
0 \\
1
\end{array} \right) + 
\left( \begin{array}{c}
0 \\
0
\end{array} \right),
\]

where the superscript $D$ indicates that it is a partition in the $D_n$ theory.

Using the Table 1, the calculation of symbol invariant of partition become the combination of blocks. We can further refine the construction of symbol. Firstly, we study the contribution to symbol of each row of a pairwise rows of partitions in different theories. And then we study the contribution to symbol of a pairwise rows of a partition.

For the first step, we study the contributions to symbol of a row with the same location in a pairwise rows of partitions in different theories. The row $a$ in Fig. 3 is the top row of a pairwise rows in the $B_n$, $C_n$, and $D_n$ theories according to Propositions 2.1, 2.2, and 2.3. The row $a$ has the same contribution to symbol in different theories.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{Row $a$ is the top row of a pairwise rows of partitions in the $B_n$, $D_n$, and $C_n$ theories using Propositions 2.1, 2.2 and 2.3. The row $a$ has the same contribution to symbol in different theories.}
\end{figure}

- If the length of the row $a$ is $2n + 1$, according to Table 1, its contributions to symbol is

\[\left( \begin{array}{c}
0 \\
0 \\
\vdots
\end{array} \right) + 
\left( \begin{array}{c}
1 \\
0 \\
\vdots
\end{array} \right) + 
\left( \begin{array}{c}
0 \\
0 \\
\vdots
\end{array} \right),
\]

which are the same in different theories.

- If the length of the row $a$ is $2n$, according to Table 1, its contributions to symbol is

\[\left( \begin{array}{c}
0 \\
0 \\
\vdots
\end{array} \right) + 
\left( \begin{array}{c}
1 \\
0 \\
\vdots
\end{array} \right) + 
\left( \begin{array}{c}
0 \\
0 \\
\vdots
\end{array} \right),
\]
which are the same in different theories.

Similarly, if the row \( a \) is at the bottom of a pairwise rows of a partition, its contribution to symbol is the same in different theories.

Summary, the same location of a row in a pairwise rows partition leads to the same contribution to symbol in different theories.

Figure 4. The first row \( a \) of a partition in the \( B_n \) theory can be regarded as the top row of an odd pairwise rows. With the same location of a pairwise rows, the row \( a \) has the same contribution to symbol in the \( B_n, D_n, \) and \( C_n \) theories.

Secondly, we study the contribution to symbol of the first row of partitions in the \( B_n \) and \( D_n \) theories which do not belong to a pairwise rows according to Propositions 2.1 and 2.3. Let the row \( a \) is the first row of a partition in the \( B_n \) theory with length \( 2n + 1 \) as shown in Fig. (4). According to Table 1, it has a contribution to symbol as follows

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 1 & \ldots & 1 \\
\end{pmatrix}
\]

which is the same as the contribution to symbol of the top row of an odd pairwise rows in the \( B_n, C_n, \) and \( D_n \) theories with length \( 2n + 1 \). Thus we claim that the first row of a partition in the \( B_n \) theory can be regarded as the top row of an odd pairwise rows.

Figure 5. The first row \( a \) in the \( D_n \) theory can be regarded as the top row of an odd pairwise rows. With the same location in a pairwise rows, the row \( a \) have the same contributions to symbols in the \( B_n, D_n, \) and \( C_n \) theories.

Similarly, we find that the first row of a partition in the \( D_n \) theory can be regarded as the top row of an even pairwise rows. As shown in Fig. (5), the row \( a \) with length \( 2n \) has a contribution to symbol in the \( D_n \) theory as follows

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & 1 & \ldots & 1 \\
\ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\end{pmatrix}
\]

which is the same as the contribution to symbol of the top row with the same length of an even pairwise rows in the \( B_n, C_n, \) and \( D_n \) theories according to Table 1.

From the above discussions, we get the following concise proposition.

**Proposition 3.1.** With the same location in a pairwise rows of a partition, one row has the same contribution to symbol for partitions in the \( B_n, D_n, \) and \( C_n \) theories.

The form of the contribution to symbol of a row of a partition is shown in Table 2.
As the applications of Proposition 3.1, we study the contributions to symbol of a pairwise rows of a partition. As shown in Fig. 6, the rows a and b of an odd pairwise rows have the lengths of $2n + 1$ and $2m + 1$, respectively. According to Table 2, the pairwise rows has the contributions to symbol as follows,

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & \\
0 & 0 & \ldots & 0 & 1 & \\
\end{pmatrix},
\]

which are the same in the $B_n$, $D_n$, and $C_n$ theories. Similarly, if the length of a is $2n$ and the length of b is $2m$, they have the contributions to symbol as follows,

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & \\
0 & 0 & \ldots & 0 & 1 & \\
\end{pmatrix},
\]

which are the same in the $B_n$, $D_n$, and $C_n$ theories.

Summary, we get the following lemma.

**Lemma 3.1.** A pairwise rows of partitions in the $B_n$, $D_n$, and $C_n$ theories has the same contributions to symbol.

Now we study the rows of partitions which have the same contribution to symbol with different lengths. According to Table 2, the bottom row of an odd pairwise rows has the same contribution to symbol as that of the top row of an even pairwise rows with one more box. Examples are shown in Fig. 7. Without an explanation, the gray boxes denote the box appended and the black boxes denote the boxes omitted in the following sections. The contribution to symbol of the row b with length $2n + 1$ in the $B_n$ theory is

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & \\
0 & 0 & \ldots & 0 & 1 & \\
\end{pmatrix}
\]
Figure 7. Gray boxes are appended at the end of row. The length of $b$ is $l$ and the lengths of $b_1$, $b_2$ and $b_3$ are $l + 1$. The rows $b$, $b_1$, $b_2$, and $b_3$ have the same contributions to symbol.

| Location in a pairwise rows | Length $L$ of row | Contribution |
|-----------------------------|------------------|--------------|
| top                         | $2n + 1$         | $\begin{pmatrix} 0 & 0 \cdots & 0 & 0 \cdots & 0 \\ 0 \cdots & 1 & 1 \cdots & 1 \end{pmatrix}$ |
| bottom                      | $2n$             | $\begin{pmatrix} 0 & 0 \cdots & 0 \cdots & 0 \\ 0 \cdots & 1 & 1 \cdots & 1 \end{pmatrix}$ |

Table 3. Contribution to symbol of the top row of an odd pairwise rows with length $2n + 1$, which is the same as the contribution to symbol of the bottom row of an even pairwise rows with length $2n$.

| Location in a pairwise rows | Length $L$ of row | Contribution |
|-----------------------------|------------------|--------------|
| bottom                      | $2n + 1$         | $\begin{pmatrix} 0 & 0 \cdots & 1 & 1 \cdots & 1 \\ 0 \cdots & 0 & 0 \cdots & 0 \end{pmatrix}$ |
| top                         | $2n$             | $\begin{pmatrix} 0 & 0 \cdots & 1 & 1 \cdots & 1 \\ 0 \cdots & 0 & 0 \cdots & 0 \end{pmatrix}$ |

Table 4. Contribution to symbol of the bottom row of an odd pairwise rows with length $2n + 1$, which is the same as the contribution to symbol of the top row of an even pairwise rows with length $2n$.

which is the same as the contributions of the rows $b_1$, $b_2$, and $b_3$ in the $B_n$, $C_n$, and $D_n$ theories, respectively.

Figure 8. Black boxes are omitted at the end of row. The length of $b$ is $l$ and the lengths of $b_1$, $b_2$, and $b_3$ are $l - 1$. The rows $b$, $b_1$, $b_2$, and $b_3$ have the same contributions to symbol.

According to Table 2, the top row of an even pairwise rows has the same contribution to symbol as that of the bottom row of an odd pairwise rows with one less box. Examples are shown in Fig. 8. The contribution to symbol of the row $b$ with length $2n$ in the $B_n$ theory is

$$
\begin{pmatrix}
0 & 0 \cdots & 1 & 1 \cdots & 1 \\
0 \cdots & 0 & 0 \cdots & 0
\end{pmatrix}
$$

which is the same as that of the rows $b_1$, $b_2$, and $b_3$ in the $B_n$, $C_n$, and $D_n$ theories, respectively.

Summary, we have the following proposition.
Proposition 3.2. The contribution to symbol of the bottom row of an odd pairwise rows with length \(L\) is the same as that of the top row of an even pairwise rows with length \(L + 1\). And the contribution to symbol of the top row of an odd pairwise rows with length \(L\) is the same as that of the bottom row of an even pairwise rows with length \(L - 1\).

This proposition is equivalent to the contents of Tables 3 and 4. Compared with Table 1, the conclusions of Tables 3 and 4 are not limited to certain theory. As shown in Fig. 9, the rows \(a\) and \(b\) form a pairwise rows of the first partition. The second partition is obtained from the first one by omitting the rows under the row \(a\), so it is a partition in the \(C_n\) theory. The third partition is obtained from the first one by omitting the rows under the row \(b\), so it is a partition in the \(B_n\) or \(D_n\) theories, depending on the parity of the length of row \(a\). So partitions can be obtained from partitions in different theories. This picture explain that the row with the same location in a pairwise rows would have the same contribution to symbol in different theories.

According to Propositions 3.1 and 3.2, the contribution to symbol of a row is an invariant. In other words, given the contribution to symbol of a row, we can list out all possible lengths and locations of the row in a pairwise rows. Furthermore, given the symbol invariant, we can list all rigid semisimple surface operators corresponding to the invariant.

3.2. Maps preserving symbol
There are two classes of symbol preserving maps. The first class of maps takes surface operators to surface operators in the same theory. We have made a classification of the first class of maps in [19], with examples shown in Fig.(10). \((\lambda', \lambda'')\) is a rigid semisimple operator in the \(B_n\) theory. Under the map \(S_{e1221}\), the bottom row of a pairwise rows of \(\lambda'\) switches place with the top row of an pairwise row of \(\lambda''\) switch places, which preserves symbol according to Proposition 3.2. Under the map \(D_{e1221}\), the bottom row of a pairwise rows of \(\lambda'\) switches place with the bottom row of an pairwise row of \(\lambda''\) switch places, which preserves symbol according to Proposition 3.1.

The second class of maps takes surface operators to surfaces operator in different theories, for examples, the \(S\) duality maps. Without confusion, the second class of maps will be called the \(S\) duality maps in the following sections. For the construction of the \(S\) duality maps [5], the maps \(X_S\) and \(Y_S\) play significant roles. \(X_S\) map a partition with only odd rows in the \(B_n\) theory to a partition with only even rows in the \(C_n\) theory

\[
X_S : \quad m^{2n_m+1} (m-1)^{2n_{m-1}} (m-2)^{2n_{m-2}} \cdots 2^{2n_2} 1^{2n_1} \mapsto m^{2n_m} (m-1)^{2n_{m-1}+2} (m-2)^{2n_{m-2}+2} \cdots 2^{2n_2+2} 1^{2n_1-2}.
\]  

(3.21)

where \(m\) has to be odd in order for the first object to be a partition in the \(B_n\) theory. As shown in Fig.(11), on the left hand of the map \(X_S\), the two rows in braces form pairwise rows. On the right hand of the map \(X_S\), the black boxes are omitted and the gray boxes are appended. And the two rows in braces belong to different pairwise rows. The bottom row on the left hand side become the top row on the right hand side while the top row on the left hand side become the bottom row on the right hand side.

Using Tables 3 and 4, we can prove the following lemma directly.

**Lemma 3.2.** The map \(X_S\) preserve symbol invariant.

**Proof.** On the left hand side of the map \(X_S\), the 2kth and (2k+1)th rows of the partition in the \(B_n\) theory form a pairwise rows excepting the first row. On the other side, the (2k-1)th and 2kth rows of the partition in the \(C_n\) theory form a pairwise rows. The first row can be regarded as the top of a pairwise rows.

According to Table 3 the contribution to symbol of the 2kth row in the \(B_n\) partition is equal to that of the (2k-1)th row in the \(C_n\) partition. According to Table 4 the contribution to symbol of the (2k+1)th row in the \(B_n\) partition is equal to that of the 2kth row in the \(C_n\) partition. So the symbols on the two sides of the map \(X_S\) are equal. □

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According to Table 3 the contribution to symbol of the 2kth row in the \(B_n\) partition is equal to that of the (2k-1)th row in the \(C_n\) partition. According to Table 4 the contribution to symbol of the (2k+1)th row in the \(B_n\) partition is equal to that of the 2kth row in the \(C_n\) partition. So the symbols on the two sides of the map \(X_S\) are equal. □

![Figure 11](image1.png)  
**Figure 11.** On the left hand of the map \(X_S\), the two rows in braces form a pairwise rows. On the right hand of the map \(X_S\), the black boxes are omitted and the gray boxes are appended. And the two rows in braces belong to different pairwise rows.

![Figure 12](image2.png)  
**Figure 12.** On the left hand of the map \(Y_S\), the two rows in braces form a pairwise rows. On the right hand of the map \(Y_S\), the black boxes are omitted and the gray boxes are appended. And the two rows in braces belong to different pairwise rows.
Next, we introduce the map $Y_S$ which take a rigid partition with only odd rows in the $C_n$ theory to a rigid partition with only even rows in the $D_n$ theory as shown in Fig. (12).

\[
Y_S : m^{2m+1} (m-1)^{2m} (m-2)^{2m-2} \cdots 2^{n} 1^{2n} 
\rightarrow m^{2m} (m-1)^{2m-1+2} (m-2)^{2m-2} \cdots 2^{n} 1^{2n+1}
\]

(3.22)

where $m$ has to be even in order for the first element to be a $C_k$ partition. The bottom row on the left hand side become the top row on the right hand side while the top row on the left hand side become the bottom row on the right hand side. Similarly, we can prove the following lemma.

**Lemma 3.3.** The map $Y_S$ preserve symbol invariant.

Summary, under the map $X_S$, we get a partition $\lambda_{\text{even}}$ with only even rows in the $C_n$ theory from a partition $\rho_{\text{odd}}$ with only odd rows in the $B_n$ theory,

\[
X_S : \rho_{\text{odd}} \rightarrow \rho_{\text{even}}.
\]

Under the map $Y_S$, we get a partition $\lambda_{\text{even}}$ with only even rows in the $D_n$ theory from a partition $\rho_{\text{odd}}$ with only odd rows in the $C_n$ theory,

\[
Y_S : \rho_{\text{odd}} \rightarrow \rho_{\text{even}}.
\]

The common characteristics of the maps $X_S$ and $Y_S$ are to append a box at the end of the bottom row of a pairwise rows and to delete a box at the end of the top row for a partition with only odd rows. Compared Fig. (11) with Fig. (12), the relationship between the map $X_S$ and the map $Y_S$ is

\[
X_S(m \rightarrow m-1) = Y_S.
\]

(3.23)

Thus the map $Y_S$ can be regarded as a special case of the map $X_S$. In fact, the Fig. (11) explain this result.

3.3. $S$-duality maps for rigid surface operators

Combined the addition rules (2.12) the maps $X_S$ and $Y_S$ can be used to construct the $S$ duality maps of surface operators. The $S$ duality maps have the following form

\[
S : (\lambda, \rho)_{GL} \rightarrow (\lambda', \rho'')_{GL}.
\]

which preserve symbol. In [6], Wyllard made explicit proposals for how the $S$-duality map should act on unipotent surface operators and certain subclasses of semisimple surface operators, which passe all consistency checks. In [7], we made new proposals for certain subclasses of semisimple surface operators.

These $S$-duality maps can be explained naturally as the symbol preserving maps using Propositions (3.1) and (3.2) drawing the conclusion directly and avoid complicated derivation.

**For rigid unipotent operators ($\lambda, \emptyset$) in the $B_n$ theory**

The $S$-duality map is

\[
(3.25) \quad WB : (\lambda, \emptyset)_B \rightarrow (\lambda_{\text{odd}} + \lambda_{\text{even}}, \emptyset) \rightarrow (X_S \lambda_{\text{odd}}, \lambda_{\text{even}})_C.
\]

Start by splitting the Young tableau $\lambda$ into tableau $\lambda_{\text{even}}$ constructed from even rows only and tableau $\lambda_{\text{odd}}$ constructed from the odd rows only. Next the map $X_S$ turns $\lambda_{\text{odd}}$ to a partition with only even rows while $\lambda_{\text{even}}$ is left unchanged. Finally, the duality operator corresponding to $(\lambda, \emptyset)$ in the $C_n$ theory is $(X_S \lambda_{\text{odd}}, \lambda_{\text{even}})$. According to Proposition (3.1) and Lemma (3.2) the map $WB$ preserve the symbol. An example illustrates the procedure.

**Example:** For the $B_{16}$ partition, $\lambda = 5 4^2 3^3 2^4 1^3$, applying the map $WB$, we find

\[
(3.26) \quad WB : \quad \rightarrow \quad \left( \begin{array}{cccccccccccc} 
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array} \right)
\]

which leads to the semisimple $C_{16}$ surface operator $(2^4 1^8, 2^6 1^4)$.

**For rigid unipotent operators ($\lambda, \emptyset$) in the $C_n$ theory**

Similarly, the following $S$-duality map preserve symbol,

\[
(3.27) \quad WC : (\lambda, \emptyset)_C \rightarrow (\lambda_{\text{odd}} + \lambda_{\text{even}}, \emptyset) \rightarrow (X_S^{-1} \lambda_{\text{even}}, Y_S \lambda_{\text{odd}})_B.
\]

The unipotent conjugacy classes (nilpotent orbits) are related to the partitions by Kazhdan-Lusztig map. It would be interesting to study the inspiration of the relationship on the nilpotent orbits.
For semisimple surface operators \((\rho; \rho)\) in the \(C_n\) theory

The \(S\)-duality map is

\[(3.28) \quad W_{CC} : (\rho; \rho)_{C} \rightarrow (\rho_{\text{even}} + \rho_{\text{odd}}; \rho_{\text{odd}} + \rho_{\text{even}}) \rightarrow (\rho_{\text{even}} + X_{S}^{-1} \rho_{\text{even}}; \rho_{\text{odd}} + Y_{S} \rho_{\text{odd}})_{B}.\]

Firstly, split two equal tableaux into even-row tableaux \(\rho_{\text{even}}\) and odd-row tableaux \(\rho_{\text{odd}}\). Then apply the map \(X_{S}\) to one of the odd-row tableaux and apply the map \(Y_{S}^{-1}\) to the even-row tableau in the other semisimple factor. Next add the altered and unaltered even-row tableaux to form one of the two partitions in a semisimple \(B_n\) operator. Finally, do the same to the odd-row tableaux and lead to a semisimple operator in the \(B_n\) theory.

\((\rho_{\text{even}} + X_{S}^{-1} \rho_{\text{even}}; \rho_{\text{odd}} + Y_{S} \rho_{\text{odd}})_{B}\) is a rigid surface operator. An illustration is made through an example as shown in Fig. 13. A pairwise rows of \(\rho_{\text{even}}\) are placed between the bottom and the top row of a a pairwise rows of \(X_{S}^{-1} \rho_{\text{even}}\), not violating the rigid conditions. A pairwise rows of \(\rho_{\text{odd}}\) are placed between the bottom and the top row of a pairwise rows of \(Y_{S} \rho_{\text{odd}}\), not violating the rigid conditions. According to Proposition 3.1, the partitions \(\rho_{\text{even}}\) and \(\rho_{\text{odd}}\) have the same contributions to symbol on the two sides of the map \(W_{CC}\). According to Proposition 3.2, the partitions \(X_{S}^{-1} \rho_{\text{even}}\) and \(Y_{S} \rho_{\text{odd}}\) have the same contributions to symbol on the two sides of the map.

For semisimple surface operators \((\lambda_{\text{even}}; \rho_{\text{odd}})\) in the \(C_n\) theory

In [6], we propose a \(S\)-duality map in the sense of symbol invariant as follows,

\[(3.29) \quad C_{B_{co}} : (\lambda_{\text{even}}; \rho_{\text{odd}})_{C} \rightarrow (X_{S}^{-1} \lambda_{\text{even}}; Y_{S} \rho_{\text{odd}})_{B},\]

which preserves symbol according to Proposition 3.1. One example of this duality is shown in the eighteenth example in the appendix.

3.4. Discussions

The \(S\) duality maps preserve symbol invariant and other invariants of partitions. Compared to other invariants, the symbol is more easier to be calculated and more convenient to find the \(S\)-duality maps. Through not all symbol preserving maps are \(S\) duality maps, a more thorough understanding them might lead to progress. Propositions 3.1 and 3.2 make the the contribution to symbol of rows visualization. Proposition 3.1 implies the symbol preserving operations that moving a row of a partition to another partition with the same location in a pairwise rows. One example is that leaving \(\lambda_{\text{even}}\) unchanged in the \(S\) duality map \(WB\). Proposition 3.2 implies the symbol preserving operations such as the maps \(X_{S}, Y_{S}\) and their inverse maps. We also find the important maps \(X_{S}\) and \(Y_{S}\) are essentially the same map.
In fact, the contribution to symbol of a row in a partition is also an invariant. It do not change under the first class of maps and second one. Fig. 9 explain this result.

With these principles in mind, we will discuss the constructions of the rigid operators in the $B_n$ theory from the $C_n$ theory and vice versa in next section, where the operations in Propositions 3.1 and 3.2 will be used frequently as well as the maps $X_S$, $Y_S$.

4. Mismatch of the rigid semisimple surface operator between dual theories

There is a discrepancy of the number of rigid surface operators between the $B_n$ and $C_n$ theories, which was first observed in the $B_4/C_4$ theories in [4]. Using the generating function for the total number of rigid surface operators(both unipotent and semisimple), Wyllard found that the difference of number of operators between the $B_n$ and $C_n$ theory is

$$q^9 + 2q^{11} + 4q^{13} + 5q^{15} + 9q^{17} + 12q^{19} + 17q^{21} + 23q^{23} + \cdots$$

where the degree corresponds to the rank $n$ of Lie algebra.

The discrepancy issue is clearly a major problem. Wyllard gave examples and made a preliminary analysis of the problematic surface operators in [4]. As shown in the appendix, it seems that there are two types of mismatches of rigid surface operators between the $B_n$ theory and $C_n$ theory. The first one is that certain surface operators in $B_n/C_n$ theory do not have duals. And the second one is that the number of surface operators with certain invariants in $B_n$ theory is more than that in the $C_n$ theory.

In this section, we analyse the mismatch problem based on constructions of symbol presented in previous sections. We find that the discrepancy issue originates from the rigid conditions of rigid partitions.

4.1. Changes of the first row of a partition under $S$ duality

According to Tables 3 and 4, the contribution to the symbol of each row of a partition will not change under the symbol preserving map, which means the contribution to symbol of a row is an invariant. So the longest row of the two factors of a rigid surface operator will still be the longest row on the other side of the $S$-duality map. According to Propositions 2.1, 2.2, and 2.3, the first two rows of the $C_n$ partitions form a pairwise rows, while the first row of partitions in the $B_n$ theory do not belongs to a pairwise rows. With these facts in mind, there are two choices for the movements of the longest row in the second class of the symbol preserving maps ($S$-duality maps).

1. For the first choice, the longest row moves from one factor of the rigid semisimple surface operator to the other factor, which will be studied in Sections 4.2, 4.3.

2. For the second one, the longest row stays in the same factor, which will be studied in Sections 4.4.

These two choices correspond to two strategies to construct the $S$-duality maps.

The first class of the symbol maps which is the maps between the rigid semisimple surface operator have been classified in [4]. They are one to one correspondence on the two side of the $S$-duality map, which will be illustrated in Section 4.3.

4.2. Generating $B_n$ rigid semisimple surface operators from the $C_n$ theory

In this subsection, we propose algorithms to generate $B_n$ rigid semisimple surface operators from that of the $C_n$ theory. The two factors of $C_n$ rigid semisimple surface operators are partitions in the $C_n$ theory. The first two rows of a rigid $C_n$ partition form a pairwise rows according to Proposition 2.2. And thus the parities of the length of the first two rows of the factors of the $C_n$ rigid semisimple surface operator have the same parity or different.

Firstly, consider the case that the first two rows of both factors of the $C_n$ rigid semisimple surface operator have the same parities. And there are two cases according to the parity of the length of the first row.

- The first two rows of both factors of a rigid surface operator are even. The algorithm $EE$ is defined in Fig. 13. Without lose of generality, we assume the first row of the partition $C_2$ is the longest row of the partitions $C_1$ and $C_2$. Take the longest row from one factor to another one and append a gray box at the end of it. The partition $C_1$ become the partition $B_1$ and the partition $C_2$ become the partition $D_2$. 
The partitions $C_1$ and $C_2$ are in the $C_n$ theory, with first two rows even. And the partitions $B_1$ and $D_2$ are in the $B_n$ and $D_n$ theories, respectively.

• The first two rows of both factors of a rigid surface operator are odd. The algorithm $OO$ is defined in Fig. 15. Without lose of generality, we assume the first row of the partition $C_2$ is the longest row of the partitions $C_1$ and $C_2$. Take the longest row from one factor to another one and append a gray box at the end of it. The partition $C_1$ become the partition $D_2$ and the partition $C_2$ become the partition $B_1$.

According to Tables 3 and 4, we have the following proposition.

Proposition 4.1. The algorithms $EE$ and $OO$ preserve symbol. These algorithms also preserve the rigid conditions.

Proposition 4.2. The algorithms $EE$ and $OO$ preserve rigid conditions of partitions.

Proof. We prove the proposition for the algorithm $EE$. As shown in Fig. 14, there are no gaps appearing in the $B_n$ rigid semisimple surface operator $(B_1, D_2)$. And the even integers in the partitions $C_1, C_2$ become the odd integers in the partitions $B_1, D_2$. Since no even integer appears exactly twice in the symplectic ($C_n$) partitions $C_1, C_2$, no odd integer appears exactly twice in the orthogonal $D_n$ partitions $D_2$ and no odd integer ($\geq 3$) appears exactly twice in the orthogonal $B_n$ partitions $B_1$. Since the difference of lengths between the longest row appended a gray box and the second row of the partition $B_1$ is odd, the part '1' would not appear twice in the partition $B_1$.

Similarly, we can prove the algorithms $OO$ preserve the rigid conditions of partitions. □

Secondly, consider the case that the first two rows of factors of the $C_n$ rigid semisimple surface operator are of different parities. According to the parity of the length of the longest row, there are two cases.

• The length of the longest row of two factors is even. If the first row of $C_2$ is the longest and the length even, we propose an algorithm $CE$ to get a $B_n$ rigid semisimple surface operator from the $C_n$ one as shown in Fig. 16. We add the longest row to $C_1$ and append a gray box, leading to a $B_n$ partition $B_1$ and a $D_n$ partition $D_2$. The $D_n$ partition $D_2$ satisfy the rigid conditions as Proposition 4.2.
The first row of $C_2$ is the longest of the two partitions on the left hand side of $CE$. Add it to $C_1$ and append a gray box as the last part of the longest row.

The first row of $C_2$ is the longest of the two partitions on the left hand side of $CO$. Add it to $C_1$ and append a gray box as the last part of the longest row.

- The length of the longest row of two factors is odd. If the first row of $C_2$ is the longest and the length is odd, we propose an algorithm $CO$ as shown in Fig. (17). We add the longest row to $C_1$ and append a gray box, leading to a $D_n$ partition $D_2$ and a $B_n$ partition $B_1$.
  
  The $B_n$ partition $B_1$ satisfy the rigid conditions as Proposition 4.2.

It is easy to prove the following proposition according to Tables 3 and 4.

**Proposition 4.3.** The algorithms $CE$ and $CO$ preserve symbol.

However, under the algorithms $CE$ and $CO$, the partitions $B_1$ and $D_2$ do not always preserve the rigid condition.

**IC type problematic surface operators:** $L(C_1)$ and $L(C_2)$ denote the lengths of the partitions of $C_1$ and $C_2$, respectively.

- If $L(C_1) = L(C_2) - 1$, the part ‘1’ appear twice in the $B_n$ partition $B_1$ under the algorithm $CE$, violating rigid condition (2) in Section 2.1.
- If $L(C_1) = L(C_2) - 1$, the part ‘1’ appear twice in the $D_n$ partition $D_2$ under the algorithm $CO$, violating rigid condition (2) in Section 2.1.

For these problematic operators, we may try to add the shorter row of the first rows of the factors of the $C_n$ rigid semisimple surface operator from one factor to the other one. However, these procedures do not lead to rigid surface operators, violating the rigid condition $\lambda_i - \lambda_{i+1} \leq 1$ as shown in Figs. (15) and (16).

- If $L(C_1) = L(C_2) - 1$, and then $\lambda_{i-1} - \lambda_i = 2$ in the $D_n$ partition $D_2$ under the algorithm $COS$, violating rigid condition (1) in Section 2.1.
- If $L(C_1) = L(C_2) - 1$, and then $\lambda_{i-1} - \lambda_i = 2$ in the $B_n$ partition $B_1$ under the algorithm $CES$, violating rigid condition (1) in Section 2.1.

To dispel the obstruction of the algorithm $CE$, we may try to map the $C_n$ operator to another $C_n$ operator with the same symbol as shown in Fig. (20) before taking the algorithm $CE$. We swap the row $a$ of the partition $C_1$ with the row $b$ of the partition $C_2$ by deleting the last box of the row $b$ and appending a box at the end of the row $a$. However the first two rows of the new factor $C_2$ would have the same lengths, violating the rigid condition (1). We can get the same conclusion for the operator $(C_1, C_2)$ before taking the algorithm $CO$. 

Summary, the $C_n$ rigid semisimple surface operators $(C_1, C_2)$ with $|L(C_1) - L(C_2)| = 1$ can not have rigid $B_n$ duals. These problematic surface operators are denoted as the $IC$ type.

For one class of the special rigid semisimple surface operator $(\lambda_{\text{even}}, \lambda_{\text{odd}})_C$, there is another strategy to construct the $S$-duality maps. We will come back this problem in Section 4.7.

4.3. Generating $C_n$ rigid semisimple surface operators from the $B_n$ theory

The construction of rigid semisimple surface operators in the $C_n$ theory from that in the $B_n$ theory is roughly parallel to the discussions in the last subsection. According to Propositions 2.1 and 2.2, the first row of the partitions in $B_n$ theory is odd and the first row of the partitions in $D_n$ theory is even.

There are two cases according to the location of the longest row of the factors of the $B_n$ rigid semisimple surface operators.

- The longest row of the rigid semisimple surface operator is the first row of the $B_n$ partition $B1$. We suggest the algorithm $BO$ as shown in Fig. (21): delete the last box of the longest row and then add it to the $D_n$ partition $D2$. Then the first two rows of the $C_n$ partitions $C2$ are even. And the partition $C1$ satisfies the rigid condition naturally.

- The longest row of the rigid semisimple surface operator is the first row of the $D_n$ partition $D2$. We suggest the algorithm $BE$ as shown in Fig. (22): delete the last box of the longest row and then add it to the $B_n$ partition $B1$. Then the first two rows of the $C_n$ partitions $C1$ are odd. And the partition $C2$ satisfies the rigid condition naturally.
Partitions $B_1$ and $D_2$ are in the $B_n$ and $D_n$ theories, respectively. Partitions $C_1$ and $C_2$ are in the $C_n$ theory. Algorithm $BO$ maps $B_n$ rigid semisimple surface operators to $C_n$ rigid semisimple surface operators.

Partitions $B_1$ and $D_2$ are in the $B_n$ and $D_n$ theories, respectively. Partitions $C_1$ and $C_2$ are in the $C_n$ theory. Algorithm $BE$ maps $B_n$ rigid semisimple surface operators to $C_n$ rigid surface operators.

However, the partitions $C_2$ under the algorithms $BO$ and $C_1$ under the algorithms $BE$ do not always preserve the rigid conditions.

**IB type problematic surface operators:** $L(B_1)$ and $L(D_2)$ denote the lengths of the partitions $B_1$ and $D_2$, respectively.

- If $L(B_1) = L(D_2) + 1$, then $\lambda_{l-1} - \lambda_l = 2$ in the partition $C_2$ under the algorithm $BO$, violating the rigid condition.
- If $L(B_1) = L(D_2) - 1$, then $\lambda_{l-1} - \lambda_l = 2$ in the partition $C_1$ under the algorithm $BE$, violating the rigid condition.

To dispel the obstruction of the algorithm $BO$ with $L(B_1) = L(D_2) + 1$, we may try to take the $B_n$ operator to another $B_n$ operator by symbol preserving map as shown in Fig. (23)(a). We swap the row $a$ with row $b$, deleting the last box of the row $b$ and appending a box at the end of the row $a$. However this operation will not lead to a rigid surface operator since the integer '$1$' would appear twice in the $B_n$ partition $B_1$, violating the rigid condition. We may swap the even row $b$ with even row $c$ as shown in Fig. (23)(b). From the condition $L(B_1) = L(D_2) + 1$, we have $L(b) \geq L(a)$. So this operation will not lead to a rigid surface operator $B_1$ in the end.

Similarly, the above operations will not improve the algorithm $BE$ to get a rigid semisimple surface operator under the condition $L(B_1) = L(D_2) - 1$.

Summary, the $B_n$ rigid semisimple surface operators $(B_1, D_2)$ with $|L(B_1) - L(D_2)| = 1$ cannot have rigid $C_n$ duals. These problematic surface operators are denoted as the $IB$ type.

For the class of the special rigid surface operators $(\lambda_{\text{odd}}, \lambda_{\text{even}})_B$, there is another strategy to construct the $S$-duality maps. We will come back to this problem in Section 4.7.

### 4.4. One to one correspondence of maps preserving symbol

The second class of symbol preserving maps is also called $S$-duality maps, which take rigid semisimple surface operator to another rigid semisimple surface operator in the dual theory. For examples, the algorithms proposed in the last two subsections. We find the following relationship between the symbol preserving maps on the two side of these algorithms.
Proposition 4.4. For the algorithms EE, OO, CO, CE, BO, and BE preserving symbol and the rigid conditions, there are one to one correspondence of the first class of symbol preserving maps on the two side of these algorithms.

Proof. We prove the proposition for the algorithm EE as shown in Fig. (24). According to the discussions in Section 4.1 for generating rigid semisimple surface operator in the $B_n$ theory from that in the $C_n$ theory, the change of the longest row is fixed. The changes are one to one correspondence between the blue parts on the two sides of algorithm EE.

Similarly, we can prove the proposition for the algorithms OO, CO, CE, BO, and BE. □

Remark 4.1. The algorithms EE, OO, CO, CE, BO, and BE can be regarded as functors between dual theories, since they not only map the operators in one theory to that of the dual theory but also map the changes on one side of the algorithms to that of the other side.

We illustrate this proposition by two examples as shown in Fig. (25) and Fig. (26). The algorithm EE map the $C_n$ surface operators to $B_n$ surface operators. The rows $c_{11}$, $c_{12}$, $c_{21}$, and $c_{22}$ have the same parities.

For the first example as shown in Fig. (25), the operation that the rows $c_{11}$ and $c_{21}$ swap places is denoted by down arrow on the left hand side of the algorithms EE, which leads to a new rigid semisimple surface operator in the $C_n$ theory. According to Proposition 3.3, this operation preserves symbol and corresponds to the operation swapping $c_{11}$ with $c_{21}$ denoted by down arrow on the right hand side of the algorithms EE.
Figure 25. Algorithm $EE$ take the map preserving symbol of the $C_n$ rigid surface operator to that of the $B_n$ rigid surface operator.

For the second example as shown in Fig. (26), the row $c21$ of $C2$ is inserted into $C1$. The row $c21$ and rows above it of the partition $C2$ would change parities as well as the rows above the $c11$ of the partition $C1$. This operation is denoted by down arrow on the left hand side of the algorithms $EE$, leading to a new semisimple rigid semisimple surface operator in the same theory. According to Proposition 3.2, this operation preserve symbol and corresponds to operation denoted by down arrow on the right hand side of the algorithms $EE$.

Figure 26. Algorithm $EE$ take the map preserving symbol of the $C_n$ rigid surface operator to that of the $B_n$ rigid surface operator.

As an application, Proposition 4.4 ensure the equality of the number of rigid surface operators on two sides of these algorithms ($S$ duality maps).

4.5. $II$ type problematic surface operators

Besides the $IC$ and $IB$ problematic surface operators, there is another kind of problematic surface operators: the number of surface operators of one theory is more than that of the dual theory
Figure 27. Partition $C_1$ with only even rows and the partition $C_2$ with odd rows are in the $C_n$ theory. The partitions $B_1$ with only odd rows and $D_2$ with only even rows are in the $B_n$ and $D_n$ theories, respectively. Algorithm $OE$ take the surface operators in the $C_n$ theory to that in the $B_n$ theory.

with the same symbol invariant. The number of surface operators in the $B_n$ theory is one more than that in the $C_n$ theory as shown in the 18th and 19th examples in the appendix.

This kind of problematic surface operators appear in the second strategy for the construction of the $S$ duality maps. $(\lambda_{even}, \rho_{odd})_C$ is a surface operator in the $C_n$ theory, and $\lambda_{even}$ and $\rho_{odd}$ are partitions with even rows and odd rows only, respectively. We take the following algorithm $OE$ to get the $B_n$ rigid semisimple surface operators from that of the $C_n$ theory as shown in Fig. 27.

$$OE : (\lambda_{even}, \rho_{odd})_C \rightarrow (X_S^{-1}\lambda_{even}, Y_S\rho_{odd})_B \rightarrow (\lambda'_{odd}, \rho'_{even})_B.$$ 

Figure 28. A pairwise rows $r_2$ and $r_3$ of the partition $C_1$ are inserted into the partition $C_2$. And A pairwise rows $r_2$ and $r_3$ of the partition $B_1$ are inserted into the partition $D_2$. These two operations are one to one correspondence under the algorithm $OE$.

On the other hand, the algorithm $OE$ as a functor map the symbol preserving changes of $C_n$ surface operator $(\lambda_{even}, \rho_{odd})_C$ to that of the $B_n$ one as shown in Fig. 28. However not all the changes on the right hand side of $OE$ could be realized on the left hand side. As shown in Fig. 29, the even row $r_1$ is the top row of a pairwise rows of the partition $C_1$, and the odd row $r_2$ is the bottom row of a pairwise rows of the partition $C_2$. The length of $r_1$ is shorter than that of the row $r_2$. Under the algorithm $OE$, to preserve the symbol, the row $r_1$ becomes odd and becomes the bottom row of a pairwise rows of $B_1$, and the row $r_2$ becomes even and becomes the top row of a pairwise rows of $D_2$. Now we take the $B_n$ rigid semisimple surface operator $(\lambda'_{even}, \rho'_{odd})$ to another $B_n$ rigid semisimple surface operator under the down arrow on the right hand side of $OE$. We put the $r_1$ and the parts above it above $r_2$ of $D_2$. This change of the $B_n$ rigid semisimple surface operator $(X_S^{-1}\lambda_{even}, Y_S\rho_{odd})$ can not be realized in the $C_n$ rigid
semisimple surface operator $(\lambda_{\text{even}}, \rho_{\text{odd}})$. Assume $r_1$ is putted above $r_2$ on the left hand side of $OE$, corresponding to the down arrow on the left hand side of $OE$ as shown in Fig. (29), then they form a pairwise rows with different parities, which is a contradiction.

**Figure 29.** Algorithm $OE$ take the $C_n$ rigid semisimple surface operator $(\lambda_{\text{even}}, \rho_{\text{odd}})$ to the $B_n$ rigid semisimple surface operator $(X_{S}^{-1}\lambda_{\text{even}}, Y_{S}\rho_{\text{odd}})$. The row $r_1$ and the rows above $r_1$ of $B_1$ are placed upon the row $r_2$ of $D_2$ under the algorithm $OE$. The operation corresponding to the down arrow on the right hand of $OE$ fail to be realized on the left hand side of $OE$.

The algorithms in Figs. (28) and (29) are particularly revealing. For the $C_n$ operators $(\lambda_{\text{even}}, \rho_{\text{odd}})$, the algorithm $OE$ will work when the algorithm in Sections 4.2 fail to preserve rigid conditions. Since not all the symbol preserving maps of surface operators on the right side of $OE$ can be realized on the left side, the number of rigid $B_n$ surface operators is more than that of the $C_n$ surface operators with the symbol invariant of the rigid semisimple surface operator in Fig.(29). We denote them as the $IIC$ type problematic surface operators.

Similarly, we can propose an algorithm $OE$ to get $C_n$ rigid semisimple surface operators from that of the $B_n$ theory as follows

$$EO : (\lambda^{\prime}_{\text{odd}}, \rho^{\prime}_{\text{even}})B \rightarrow (X_{S}\lambda^{\prime}_{\text{even}}, Y_{S}^{-1}\rho^{\prime}_{\text{odd}})C \rightarrow (\lambda_{\text{even}}, \rho_{\text{odd}})C.$$

And we come to the conclusion that the number of rigid $C_n$ surface operators is more than that of the $B_n$ surface operators for certain symbol invariant. We denote them as the $IIB$ type problematic surface operators.

### 4.6. Generating $D_n$ rigid semisimple surface operator from the $D_n$ theory

Since the langlands dual groups of $SO(2n)$ are themselves, the $D_n$ theory is self duality. The first class of symbol preserving maps is the same with the second class of symbol preserving map. Thus the $S$-duality pairs can be realized by the first class of symbol preserving maps, which do not lead to semisimple surface operators violating rigid conditions. For the certain symbol invariant with only one rigid semisimple surface operator, we suggest the following $S$ duality map

$$1 : (\lambda, \rho)_{D} \rightarrow (\lambda, \rho)_{D},$$

which map a rigid surface operator to itself.

### 4.7. Classification of problematic surface operators and discussions

We find new type of of problematic operators excepting all the ones given in [5]. Even more, the algorithms proposed give all the problematic rigid surface operators. The classification of the problematic surface operators in the previous sections is given by Table 5. The discussions in Section 4.4 and Proposition 4.4 ensure it is a completeness classification.
Table 5. Classification of problematic surface operators. The subscripts odd and even mean partitions with only odd and even rows, respectively.

We find two types of problematic surface operators in this study which are denoted as I and II. For the non-problematic surface operators, we have the commutation relation as shown in Fig. (30). The I type surface operators exists only in one theory which mean the map \( f \) can not work in Fig. (30). For example, the algorithm \( BO \) can not map the operators in \( B_n \) theory to that in \( C_n \) theory with restriction conditions given in Table 5. And the II type one are surface operators which do not have the same number of operators in dual theories with certain symbol. It means that the map \( g \) can be only realized in one theory as shown in Fig. (29). The origin of both types of problematic surface operators are the rigid conditions.

We can learn much from Table 5. When the algorithms \( CE \) and \( CO \) work, they would realize all the \( S \) duality pairs with certain symbol. When the algorithms \( CE \) and \( CO \) fail to realize the \( S \) duality pairs, the algorithm \( OE \) is the only choice to work, which is an evidence of the \( S \) duality map \( CB_{eo} \) (3.29).

From formula (4.30), one gets further insight into the mismatch problem. The coefficient is positive, which imply that the number of rigid surface operators in the \( B_n \) theory is larger than that in the \( C_n \) theory. A naive guess would be that there are more \( B_n \) surface operators than \( C_n \) surface operators with the given symbol. In fact, in Fig. 30, it only point out that the number of the \( B_n \) surface operators is more than the number of the \( C_n \) surface operators. They do not find that there are rigid surface operators in the \( C_n \) theory which do not have candidate duals in the \( B_n \) theory. However, according to Table 5, the IC type \( C_n \) problematic surface operators can not have duals in the \( B_n \) theory under the algorithms \( BE \) and \( BO \). They also did not find the IIC type problematic surface operators in the \( C_n \) theory.

The number of the rigid surface operators which do not have candidate duals in the \( C_n \) theory do increase with the rank \( n \) from the discussion in Section 4.2. Fortunately, the excess number of states divided by the total number appears to approach zero as \( n \to \infty \). So one hopes that only a minor modification is needed to make the numbers match, which is consistent with the fact that most rigid surface operators do seem to have candidate duals.

The physical reason for the discrepancy is still unknown. Throughout this paper we will only consider strongly rigid operators which we refer to as rigid surface operator. From the discussions, we should also take account of the larger class including the weakly rigid surface operators discussed in [3] or the quantum effect to resolve the mismatch in the total number of rigid surface operators. Clearly more work is required.
Furthermore, the construction of symbol invariant can be used to study the $S$ duality of the rigid surface operators in other Langlands dual groups such as exception Lie algebra, and the research problems related to the Springer correspondence.

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Appendix A. Rigid semisimple surface operators in $SO(13)$ and $Sp(12)$

The first column is the type of the duality maps listed in [6]. The second and third columns list pairs of partitions corresponding to the surface operators in the $B_n$ and $C_n$ theories. The other columns are the dimension, symbol invariant, and fingerprint invariant of the surface operator, respectively. Even the mismatch in the total number of rigid surface operators in the $B_n$ and $C_n$ theories can be explained. The 18th and 19th pairs of rigid semisimple surface operators belong to the $II$ type mismatch. The 20th, 23th, and 24th pairs of rigid semisimple surface operators belong to the $I$ type mismatch.
| Num | $Sp(12)$ | $SO(13)$ | Dim | Symbol | Fingerprint |
|-----|----------|----------|-----|--------|-------------|
| 1   | $(1^{12}; 0)$ | $(1^{13}; 0)$ | 0   | $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ | $[1^6; 0]$ |
| 2   | $(2^{10}; 0)$ | $(1; 1^{12})$ | 12  | $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | $[1^5; 1]$ |
| 3   | $(1^{10}; 1^{2})$ | $(2^{2}; 1^{9}; 0)$ | 20  | $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$ | $[2^{14}; 0]$ |
| 4   | $(2^{2}; 0)$ | $(1; 2^{2}; 1^{8})$ | 30  | $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ | $[1^{3}; 1^{3}]$ |
| 5   | $(2^{18}; 1^{2})$ | $(1^{3}; 1^{10})$ | 30  | $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ | $[1^{3}; 1^{3}]$ |
| 6   | $(1^{8}; 1^{4})$ | $(2^{4}; 1^{5}; 0)$ | 32  | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 \end{pmatrix}$ | $[2^{2}; 1^{2}; 0]$ |
| 7   | $(2^{4}; 1^{4}; 0)$ | $(3^{2}; 1^{6}; 0)$ | 36  | $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ | $[1^{2}; 1^{4}]$ |
| 8   | $(1^{8}; 2^{1^{2}})$ | $(1^{9}; 1^{4})$ | 36  | $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ | $[1^{2}; 1^{4}]$ |
| 9   | $(1^{6}; 1^{6})$ | $(2^{6}; 1; 0)$ | 36  | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix}$ | $[2^{2}; 0]$ |
| 10  | $(2^{2}; 1^{2}; 0)$ | $(1; 2^{4}; 1^{4})$ | 40  | $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ | $[1^{1}; 1^{5}]$ |
| 11  | $(2^{1^{6}}; 1^{4})$ | $(1^{5}; 1^{8})$ | 40  | $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ | $[1^{1}; 1^{5}]$ |
| 12  | $(1^{6}; 2^{1^{4}})$ | $(1^{7}; 1^{6})$ | 42  | $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ | $[0; 1^{6}]$ |
| 13  | $(3^{2}; 2^{1^{4}}; 0)$ | $(1^{3}; 2^{2}; 1^{6})$ | 44  | $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ | $[3^{2}; 1^{2}]$ |
| 14  | $(2^{3}; 1^{4}; 1^{2})$ | $(2^{2}; 1; 1^{8})$ | 44  | $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ | $[3^{2}; 1^{2}]$ |
| 15  | $(2^{1}; 2^{1}; 0)$ | $(1; 3^{2}; 2^{1^{5}})$ | 44  | $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ | $[2^{1}; 2^{2}]$ |
| 16  | $(2^{4}; 1^{2}; 1^{2})$ | $(2^{2}; 1^{5}; 1^{4})$ | 48  | $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ | $[3^{1}; 2^{2}]$ |
| 17  | $(2^{1}; 2^{1})$ | $(1; 3^{2}; 2^{1})$ | 48  | $\begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ | $[2^{2}; 2]$ |
| 18  | $(2^{3}; 1^{2}; 1^{4})$ | $(1^{5}; 2^{2}; 1^{4})$ | 50  | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 \end{pmatrix}$ | $[3; 1^{3}]$ |
| 19  | $-$ | $(2^{2}; 1^{3}; 1^{6})$ | 50  | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 \end{pmatrix}$ | $[3; 1^{3}]$ |
| 20  | $-$ | $(2^{4}; 1^{1}; 4)$ | 52  | $\begin{pmatrix} 0 & 1 & 1 \\ 2 & 2 \end{pmatrix}$ | $[3^{2}; 0]$ |
| 21  | $(2^{3}; 2^{1}; 2^{1})$ | $(1^{3}; 3^{2}; 2^{1^{3}})$ | 54  | $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ | $[3^{1}; 2]$ |
| 22 $^*$ | $(3^{2}; 2^{1}; 1^{2})$ | $(2^{2}; 1; 2^{2}; 1^{4})$ | 54  | $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix}$ | $[3; 2^{1}]$ |
| 23  | $-$ | $(1^{5}; 3^{2}; 2^{1})$ | 56  | $\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 \end{pmatrix}$ | $[3; 2^{1}]$ |
| 24  | $-$ | $(2^{2}; 1; 3^{2}; 2^{1})$ | 60  | $\begin{pmatrix} 2 & 2 \\ 2 \end{pmatrix}$ | $[0; 2^{3}]$ |

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