Weyl asymptotics for Hanoi attractors

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Abstract

The asymptotic behavior of the eigenvalue counting function of Laplacians on Hanoi attractors is determined. To this end, Dirichlet and resistance forms are constructed. Due to the non self-similarity of these sets, the classical construction of the Laplacian for p.c.f. self-similar fractals has to be modified by combining discrete and quantum graph methods.

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1 Introduction

It is a well known fact from the theory of Dirichlet forms (see [10]) that any local and regular Dirichlet form defines a diffusion process on a set. The development of this theory when the set is a fractal started with the construction of Brownian motion on the Sierpiński gasket by Goldstein in [11] and Kusuoka in [25]. Since then, many results concerning both Dirichlet forms and diffusion processes have been obtained for the case of the fractal set being self-similar (see for example [5, 19, 20]). Some particular cases of non strictly self-similarity have been treated in [12, 9].

In this paper, we would like to consider diffusion on a special type of non self-similar sets that we call Hanoi attractors of parameter $\alpha$, for any $\alpha \in (0, 1/3)$. Our interest in them comes from the fact that they have been proved in [2] to be geometrically related to the Sierpiński gasket (see Theorem 1.1 below). These fractals can also be treated as (degenerated) graph directed fractals, introduced in [27], where the contractions associated to some of the edges are not similitudes. An analysis for such objects was first treated in [28] for the special case of the plain Mandala, and it was generalized in [14] for any graph directed fractal. Here the Laplacian is constructed via Dirichlet forms and its spectral asymptotics are calculated. Our work differs from this in that we get to the Dirichlet form by constructing first a resistance form and afterwards choosing a measure that allows us to compute the second term of the spectral asymptotics. The theory of resistance forms was introduced by Kigami in [21] and it has been broadly studied in the context of self-similar and p.c.f. sets, see e.g. [22, 30].

Let us briefly recall the construction of Hanoi attractors: we denote by $\mathcal{H}(\mathbb{R}^2)$ the space of nonempty compact subsets of $\mathbb{R}^2$ and equip it with the Hausdorff distance $h$ given by

$$h(A,B) := \inf \{ \varepsilon > 0 \mid A \subseteq B_{\varepsilon} \text{ and } B \subseteq A_{\varepsilon} \}$$

for $A, B \in \mathcal{H}(\mathbb{R}^2)$, where $A_{\varepsilon} := \{ x \in \mathbb{R}^2 \mid d(x, A) < \varepsilon \}$ denotes the $\varepsilon$–neighborhood of $A$.

This distance defines a metric on $\mathcal{H}(\mathbb{R}^2)$ and $(\mathcal{H}(\mathbb{R}^2), h)$ is a complete metric space.

We consider the points in $\mathbb{R}^2$

$$p_1 := (0, 0), \quad p_2 := \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad p_3 := (1, 0),$$

$$p_4 := \frac{p_2 + p_3}{2}, \quad p_5 := \frac{p_1 + p_3}{2}, \quad p_6 := \frac{p_1 + p_2}{2}.$$
Note that $p_1, p_2, p_3$ are the vertices of an equilateral triangle of side length 1.

For any fixed $\alpha \in (0, 1/3)$ we define the contractions

$$G_{\alpha,i} : \mathbb{R}^2 \to \mathbb{R}^2 \quad x \mapsto A_i(x - p_i) + p_i \quad i = 1, \ldots, 6,$$

where $A_1 = A_2 = A_3 = \frac{1-\alpha}{2}I_2$ and

$$A_4 = \frac{\alpha}{4} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}, \quad A_5 = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_6 = \frac{\alpha}{4} \begin{pmatrix} 1 & \sqrt{3} \\ \frac{1}{\sqrt{3}} & 3 \end{pmatrix}.$$

It follows from [17] that there exists a unique $K_\alpha \in \mathcal{H} (\mathbb{R}^2)$ such that

$$K_\alpha = \bigcup_{i=1}^{6} G_{\alpha,i}(K_\alpha).$$

This set is called the Hanoi attractor of parameter $\alpha$ and it is not self-similar because $G_{\alpha,4}, G_{\alpha,5}$ and $G_{\alpha,6}$ are not similitudes. The parameter $\alpha$ may be understood as the length of the segments joining the copies $G_{\alpha,1}(K_\alpha)$, $G_{\alpha,2}(K_\alpha)$ and $G_{\alpha,3}(K_\alpha)$. The lack of self-similarity carries some difficulties that we discuss later.

Figure 1: The Hanoi attractor of parameter $\alpha = 0.05$.

For the rest of this section, we fix $\alpha \in (0, 1/3)$ and denote by $\mathcal{A}$ the alphabet consisting of the three symbols 1, 2 and 3. For any word $w = w_1 \cdots w_n \in \mathcal{A}^n$ of length $n \in \mathbb{N}$, we define $G_{\alpha,w} : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$G_{\alpha,w} := G_{\alpha,w_1} \circ G_{\alpha,w_2} \circ \cdots \circ G_{\alpha,w_n}$$

and $G_{\alpha,\emptyset} := \text{id}_{\mathbb{R}^2}$ for the empty word $\emptyset$.

We will approximate the Hanoi attractor $K_\alpha$ by a sequence of one-dimensional sets defined as follows:
Firstly, we consider for each $n \in \mathbb{N}_0$ the set
\[
W_{\alpha,n} := \bigcup_{w \in \mathcal{A}^n} G_{\alpha,w}(\{p_1, p_2, p_3\}).
\] (1.1)

Secondly, let $J_{\alpha,0} := \emptyset$ and
\[
J_{\alpha,n} := \bigcup_{m=0}^{n-1} \bigcup_{w \in \mathcal{A}^m} G_{\alpha,w} \left( \bigcup_{i=1}^{3} e_i \right)
\] (1.2)
for each $n \in \mathbb{N}_0$, where $e_i$ denotes the line segment joining $G_{\alpha,j}(p_k)$ and $G_{\alpha,k}(p_j)$ for each triple $\{i, j, k\} = \mathcal{A}$ without its endpoints as shown in Figure 2. Note that $e_i$ has precisely length $\alpha$ for all $i = 1, 2, 3$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{The set $J_{\alpha,1}$.}
\end{figure}

Therefore, $J_{\alpha,n}$ denotes the set of line segments joining the copies of $K_\alpha$ at iteration level $n$.

For each $n \in \mathbb{N}_0$ we finally define the set
\[
V_{\alpha,n} := W_{\alpha,n} \cup J_{\alpha,n}
\] (1.3)
Since the sequence $(V_{\alpha,n})_{n \in \mathbb{N}_0}$ is monotonically increasing as suggested in Figure 3 we can consider the set
\[
V_{\alpha,*} := \bigcup_{n \in \mathbb{N}_0} V_{\alpha,n},
\]
which is dense in $K_\alpha$ with respect to the Euclidean metric (see [1] Lemma 2.1.2 for a proof). We may also say that $V_{\alpha,n}$ is the union of a “discrete part” $W_{\alpha,n}$ and a “continuous part” $J_{\alpha,n}$. Moreover, since $V_{\alpha,0} = \{p_1, p_2, p_3\}$ is independent of $\alpha$, we will denote this set just by $V_0$.

The geometric relationship between Hanoi attractors and the Sierpiński gasket is stated in the following Theorem.
Theorem 1.1. Let $K$ denote the Sierpiński gasket and let $K_\alpha$ be the Hanoi attractor of parameter $\alpha$, $\alpha \in (0, 1/3)$. Then we have:

(i) $h(K_\alpha, K) \to 0$ as $\alpha \downarrow 0$,

(ii) $\dim_H K_\alpha = \frac{\ln 3}{\ln 2 - \ln(1 - \alpha)} =: d$ and $0 < \mathcal{H}^d(K_\alpha) < \infty$. In particular,

$$\dim_H K_\alpha \xrightarrow{\alpha \downarrow 0} \dim_H K.$$

Proof. See [2, Theorem 3.1, Corollary 4.1].

Remark 1.2. Note that part (ii) of this Theorem justifies the condition that $\alpha < 1/3$: If $\alpha \geq 1/3$, then $\dim_H K_\alpha = 1$, reducing the problem to 1-dimensional analysis.

These results awoke the question about what other convergence types could hold and in particular if the spectral dimension would also converge. Since $K_\alpha$ is not self-similar, we could neither define a Dirichlet form for $K_\alpha$ nor calculate its spectral dimension as in the self-similar case treated in [24]. However, $K_\alpha$ still has the good property of being finitely ramified and so we focused on constructing a resistance form $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ on $K_\alpha$. After choosing a suitable Radon measure, this resistance form induced a Dirichlet form and therefore a Laplacian on the fractal.

This paper is organized as follows: Section 2 recalls the construction of the local and regular Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ on $K_\alpha$ introduced in [3] restating some of the results in terms of resistance forms. In particular, we prove that

Theorem 1.3. The pair $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ is a regular resistance form.

In Section 3 we introduce a class of Radon measures in $K_\alpha$ that depend on a parameter $\beta$ and characterize the spectrum of the Laplacian associated with the Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ in the corresponding $L^2$-space. Section 4 analyses the asymptotic behavior of the eigenvalue counting function of this Laplacian by giving the following estimates.
Theorem 1.4. There exist constants $C_{\alpha,1}, C_{\alpha,\beta,1}, C_{\alpha,2}, C_{\alpha,\beta,2} > 0$ and $x_0 > 0$ such that

$$C_{\alpha,1}x^{\frac{\log 3}{\log 5}} + C_{\alpha,\beta,1}x^{1/2} \leq N_D(x) \leq N_N(x) \leq C_{\alpha,2}x^{\frac{\log 3}{\log 5}} + C_{\alpha,\beta,2}x^{1/2}$$

for all $x \geq x_0$.

The eigenvalue counting function $N_N(x)$ (resp. $N_D(x)$) give the number of Neumann (resp. Dirichlet) eigenvalues of the considered Laplacian, counted with multiplicity, lying below $x$. A more precise definition is given at the beginning of Section 4.

From this theorem we can easily deduce that the spectral dimension of $K_\alpha$ equals $\frac{\log 9}{\log 5}$ for all $\alpha \in (0, 1/3)$ and it therefore coincides with the spectral dimension of the Sierpiński gasket. In particular, it turns out that (in contrast with Hausdorff dimension) the spectral dimension of $K_\alpha$ is independent of the parameter $\alpha$. This can be interpreted as the fact that one can “see” this parameter but not “hear” it.

The last section analyses some interesting physical consequences of this result in view of the Einstein relation to be considered for further research.

2 Dirichlet form on the Hanoi attractor $K_\alpha$

This section is devoted to the construction of a resistance form $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ on $K_\alpha$ that will induce the Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$. The main novelty in this procedure consists in the definition of the approximating forms $(\mathcal{E}_{\alpha,n}, \mathcal{D}_{\alpha,n})$, which combines techniques of both discrete and quantum graphs as well as the computation of the corresponding renormalization factors.

Since all results presented in the paper hold for any $\alpha \in (0, 1/3)$, we will not explicitly mention this condition up to this point.

2.1 Approximating forms

The definition of the bilinear form $(\mathcal{E}_{\alpha,n}, \mathcal{D}_{\alpha,n})$ on each of the approximating sets $V_{\alpha,n}$ defined in (1.3) will reflect the fact that $V_{\alpha,n}$ can be decomposed into its “discrete” and “continuous” part, introduced in (1.1) and (1.2).

We start by introducing some useful notations. Concerning to the “discrete part”, we say that any two vertices $x, y \in W_{\alpha,n}$ are $n$-neighbors, and write $x \sim_n y$,

if and only if there exists a word of length $n \in \mathbb{N}_0$, $w \in \mathcal{A}^n$, such that $x, y \in G_{\alpha,w}(V_{\alpha,0})$, i.e., both points are vertices of the same $n$-th level triangle $G_{\alpha,w}(\{p_0, p_1, p_2\})$. Figure 4 illustrates this relation for the level $n = 2$. 

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Figure 4: Examples of 2-neighbors: $x \sim y$ and $z \sim t$.

Concerning to the "discrete" part, we define the set of line segments

$$J_{\alpha,n} := \{ e \mid e \text{ is a connected component of } J_{\alpha,n} \}.$$ 

If necessary, we will specify the endpoints of such a component by writing $e =: (a_e, b_e)$. Note that $a_e, b_e \in W_{\alpha,n}$ for each $e \in J_{\alpha,n}$. Moreover, we denote $H^1(e, dx) := \{ f \circ \varphi_e \mid f \in H^1((0,1), dx) \}$, $H^1((0,1), dx)$ being the classical Sobolev space of functions defined on the unit interval and $\varphi_e$ as defined in (2.3).

**Definition 2.1.** Let $D_0 := \{ u: V_{\alpha,0} \to \mathbb{R} \}$ and

$$D_{\alpha,n} := \{ u: V_{\alpha,n} \to \mathbb{R} \mid u|_e \in H^1(e, dx) \forall e \in J_{\alpha,n} \}$$

for each $n \in \mathbb{N}$. The functional $E_{\alpha,n}: D_{\alpha,n} \to \mathbb{R}$ is defined by

$$E_{\alpha,n}[u] := \sum_{x \sim y} (u(x) - u(y))^2 + \int_{J_{\alpha,n}} |\nabla u|^2 \, dx.$$ 

For each $u \in D_{\alpha,n}$, $E_{\alpha,n}[u]$ is called the energy of $u$ at level $n$.

The functionals $E^d_{\alpha,n}: D_{\alpha,n} \to \mathbb{R}$ and $E^c_{\alpha,n}: D_{\alpha,n} \to \mathbb{R}$ defined by

$$E^d_{\alpha,n}[u] := \sum_{x \sim y} (u(x) - u(y))^2 \tag{2.1}$$

and

$$E^c_{\alpha,n}[u] := \int_{J_{\alpha,n}} |\nabla u|^2 \, dx \tag{2.2}$$

are called the discrete and resp. continuous part of $E_{\alpha,n}$.

The integral expression in (2.2) has to be understood as follows: for each line segment $(a_e, b_e) \in J_{\alpha,n}$ we consider $\varphi_e: [0,1] \to \mathbb{R}^2$ to be the curve parametrization of $e$, that is

$$\varphi_e(t) := (b_e - a_e) \cdot t + a_e. \tag{2.3}$$
For any function \( u \in \mathcal{D}_{\alpha,n} \),

\[
E^c_{\alpha,n}[u] = \int_{J_{\alpha,n}} |\nabla u|^2 \, dx := \sum_{e \in J_{\alpha,n}} \frac{1}{b_e - a_e} \int_0^1 |(u \circ \varphi_e)'|^2 \, dt.
\]

Applying the polarization identity to this energy functional we obtain the bilinear form \( E_{\alpha,n} : \mathcal{D}_{\alpha,n} \times \mathcal{D}_{\alpha,n} \to \mathbb{R} \),

\[
E_{\alpha,n}(u, v) := \frac{1}{2} (E_{\alpha,n}[u + v] - E_{\alpha,n}[u] - E_{\alpha,n}[v]).
\]

### 2.2 Harmonic extension and renormalization factor

So far we have defined \( E_{\alpha,n} \) just by “gluing” its discrete and continuous part, \( E^d_{\alpha,n} \) and \( E^c_{\alpha,n} \). This means that, until now, both parts of the energy are independent of each other. However, since we want the energy functional \( E_{\alpha,n} \) to be invariant under harmonic extension, we still have to renormalize it. This renormalization is precisely what correlates \( E^d_{\alpha,n} \) and \( E^c_{\alpha,n} \).

#### 2.2.1 Harmonic extension

Although this construction has already been explained in [3, Section 2.2], we include here the most relevant steps for ease of the reading.

**Definition 2.2.** Let \( u \in \mathcal{D}_0 \). Its harmonic extension at level \( n + 1 \) is the function \( \tilde{u} \in \mathcal{D}_{\alpha,n+1} \) satisfying

\[
E_{\alpha,k}[\tilde{u}] = \inf \{ E_{\alpha,k}[v] \mid v \in \mathcal{D}_{\alpha,k} \text{ and } v_{|V_0} \equiv u \}
\]

for all \( 0 \leq k \leq n + 1 \).

Note that the extension is well defined by the next proposition.

**Proposition 2.3.** For any function \( u \in \mathcal{D}_0 \),

\[
\inf \{ E_{\alpha,1}[v] \mid v \in \mathcal{D}_{\alpha,1} \text{ and } v_{|V_0} \equiv u \}
\]

is attained by one and only one function \( \tilde{u} \in \mathcal{D}_{\alpha,1} \) given on \( W_{\alpha,1} \) by

\[
\tilde{u}_1(G_{\alpha,1}(p_j)) = \frac{2 + 3\alpha}{5 + 3\alpha} u(p_i) + \frac{2}{5 + 3\alpha} u(p_j) + \frac{1}{5 + 3\alpha} u(p_k)
\]

for any \( i \in \mathcal{A} \), \( \{i, j, k\} = \mathcal{A} \), and linear interpolation on \( J_{\alpha,1} \).
Proof. Without loss of generality, we may assume that the function \( u_0 \in D_0 \) is given by
\[
  u_0(p_1) = 1, \quad u_0(p_2) = 0 = u_0(p_3).
\]

If we knew the values of the extension \( \tilde{u}_1 \) on \( W_{\alpha,1} \), then we would just have to extend the function \( \tilde{u}_1 \big|_{W_{\alpha,1}} \) linearly to \( J_{\alpha,1} \) by
\[
  \tilde{u}_1|_e(x) := \frac{\tilde{u}_1(b_e) - \tilde{u}_1(a_e)}{b_e - a_e} \cdot x + \frac{\tilde{u}_1(a_e)b_e - \tilde{u}_1(b_e)a_e}{b_e - a_e}
\]
at each \( x \in (a_e, b_e) = (G_{\alpha,i}(p_j), G_{\alpha,j}(p_i)) \subseteq J_{\alpha,1}, \ i \neq j \) (see Figure 5). The integrals of the continuous part of the energy would become
\[
  \int_e |\nabla \tilde{u}_1|^2 \, dx = \frac{(\tilde{u}_1(b_e) - \tilde{u}_1(a_e))^2}{|b_e - a_e|}
\]
and so the total energy \( E_{\alpha,1}[\tilde{u}_1] \) may be written only in terms of \( \tilde{u}_1 \) on \( W_{\alpha,1} \).

Due to the definition of \( u_0 \) and the symmetry of \( V_{\alpha,1} \), the function \( \tilde{u}_1 \) on \( W_{\alpha,1} \) will have the unknown values \( x, y \) and \( z \) as shown in Figure 6.

Let us now define the so-called conductance of an edge \( \{p, q\} \) by
\[
  \nu_{pq}^{\alpha,1} := \begin{cases} 
    1, & \text{if } p \sim_\alpha q, \\
    \alpha^{-1}, & \text{if } e := (p, q) \in J_{\alpha,1}.
  \end{cases}
\]
The energy of the harmonic extension $\tilde{u}_1$ can be thus expressed as the sum

$$E_{\alpha,1}[\tilde{u}_1] = \frac{1}{2} \sum_{p,q \in W_{\alpha,1}} c_{pq}^{\alpha,1}(\tilde{u}_1(p) - \tilde{u}_1(q))^2.$$ 

Solving the minimization problem in (2.4) leads to a linear system of equations whose solution is given by

$$x = \frac{2 + 3\alpha}{5 + 3\alpha}, \quad y = \frac{2}{5 + 3\alpha}, \quad z = \frac{1}{5 + 3\alpha}. \quad (2.6)$$

Because of symmetry and linearity, given an arbitrary function $u_0 : V_0 \to \mathbb{R}$ with $u_0(p_1) = a, u_0(p_2) = b, u_0(p_3) = c, a, b, c \in \mathbb{R}$, the harmonic extension $\tilde{u}_1$ is given by

$$\tilde{u}_1(p) = \frac{2 + 3\alpha}{5 + 3\alpha}a + \frac{2}{5 + 3\alpha}b + \frac{1}{5 + 3\alpha}c$$

for a point $p$ as in Figure 7.

Figure 7: The extension $\tilde{u}_1$ at $p \in W_{\alpha,1}$ for an arbitrary $u_0$.

The uniqueness of the extension is given by the uniqueness of the solution of the linear system corresponding to the minimization problem.

The expression given in (2.5) may be considered as a kind of “extension algorithm”, where $\alpha$ is the length of the segment lines in $J_{\alpha,1}$.

Next proposition generalizes this last argument in order to construct the harmonic extension from any level $n$ to $n + 1$ and finally use this extension iteratively in order to obtain the harmonic extension from zero to any level.

**Proposition 2.4.** Let $d_{\alpha,0} := 0$ and $d_{\alpha,n} := \alpha \left(\frac{1-\alpha}{2}\right)^{n-1}$ for each $n \in \mathbb{N}$.

For any function $u \in D_{\alpha,n}$, the infimum

$$\inf \{E_{\alpha,n+1}[v] \mid v \in D_{\alpha,n+1} \text{ and } v|_{V_{\alpha,n}} = u\}$$
is attained by a unique function $\tilde{u} \in D_{\alpha,n+1}$ which is given at each $p_{wij} := G_{\alpha,wi}(p_j) \in W_{\alpha,n+1}$ by

$$\tilde{u}(p_{wij}) = \frac{2 + 3d_{\alpha,n}}{5 + 3d_{\alpha,n}} u(p_{ui}) + \frac{2}{5} u(p_{wjj}) + \frac{1}{5 + 3d_{\alpha,n}} u(p_{wkk})$$

for $wi \in A^{n+1}$, $\{i,j,k\} = A$, and linear interpolation on $J_{\alpha,n+1} \setminus J_{\alpha,n}$.

**Proof.** We define the conductance of the edges $\{p,q\}$ for $p,q \in W_{\alpha,n}$ by

$$c_{\alpha,n}^{pq} := \begin{cases} 1 & \text{if } p \sim q, \\ d_{\alpha,n}^{-1} & \text{if } e := (p,q) \in J_{\alpha,n} \setminus J_{\alpha,n-1}. \end{cases}$$

The proof works entirely analogous to Proposition 2.3 (see [3, Section 2.2] for details).

### 2.2.2 Renormalization factor

Let $\tilde{u} \in D_{\alpha,n+1}$ denote the harmonic extension of a function $u \in D_{\alpha,n}$. A sequence of bilinear forms $\{B_n : D_{\alpha,n} \times D_{\alpha,n} \to \mathbb{R}\}_{n \in \mathbb{N}_0}$ is said to be invariant under harmonic extension if

$$B_n(u,u) = B_{n+1}(\tilde{u},\tilde{u}) \quad \text{for all } u \in D_{\alpha,n}.$$ 

If we can find a sequence of positive numbers $(\rho_{\alpha,n})_{n \in \mathbb{N}_0}$ such that the sequence of bilinear forms $\{E_{\alpha,n}\}_{n \in \mathbb{N}_0}$ defined by

$$E_{\alpha,n}(u,u) := \rho_{\alpha,n}^{-1} E_{\alpha,n}(u,u)$$

is invariant under harmonic extension, then $\rho_{\alpha,n}$ is called the renormalization factor of $E_{\alpha,n}$ for each $n \in \mathbb{N}_0$. The aim of this section is the computation of this factor. Contrary to the typical self-similar case, this factor will have to be understood as a $2 \times 2$-matrix of the form

$$\rho_{\alpha,n} = \begin{pmatrix} \rho^d_{\alpha,n} & 0 \\ 0 & \rho^c_{\alpha,n} \end{pmatrix}$$

instead of a real number.

We define for each $n \in \mathbb{N}$ the quantities

$$r^d_{\alpha,n} := \frac{15}{(5 + 3d_{\alpha,n})^2} \quad \text{and} \quad r^c_{\alpha,n} := \frac{(1 - \alpha)}{2} r^d_{\alpha,n}, \quad (2.7)$$

where $d_{\alpha,n}$ was defined in Proposition 2.4.

For convenience, we will use in this section the following matrix notation for the energy:

$$E_{\alpha,n}[u] := \begin{pmatrix} E^d_n[u] \\ E^c_n[u] \end{pmatrix}.$$
Lemma 2.5. Let \( u_0 \in D_0 \) and denote by \( \tilde{u}_n \in D_{\alpha,n} \) its harmonic extension to level \( n \in \mathbb{N} \). Then, for any \( n \geq 2 \) and \( w \in A^{n-2} \) it holds that \( \tilde{u}_n \circ G_{\alpha,w} \in D_{\alpha,2} \) and

\[
E_{\alpha,2}[\tilde{u}_n \circ G_{\alpha,w}] = \left( r_{\alpha,n}^d \ 0 \right) \left( 0 \ 1 + r_{\alpha,n}^c \right) E_{\alpha,1}[\tilde{u}_n \circ G_{\alpha,w}|_{V_{\alpha,1}}]. \tag{2.8}
\]

Proof. See [3, Lemma 2.12]. \( \square \)

Proposition 2.6. For each \( n \in \mathbb{N}_0 \) and \( u \in D_{\alpha,n+1} \) it holds that

\[
E_{\alpha,n+1}[u] = \sum_{m=0}^{3} \left( \begin{array}{cc} 1 & 0 \\ 0 & \frac{2}{1-\alpha} \end{array} \right)^m E_{\alpha,n}[u \circ G_{\alpha,i}] + \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) E_{\alpha,1}[u|_{V_{\alpha,1}}].
\]

Proof. See [3, Proposition 2.10]. \( \square \)

Corollary 2.7. Let \( n \in \mathbb{N}_0 \) and \( u \in D_{\alpha,n+1} \). Then,

\[
E_{\alpha,n+1}[u] = \sum_{m=0}^{n} \left( \begin{array}{cc} 1 & 0 \\ 0 & \frac{2}{1-\alpha} \end{array} \right)^m E_{\alpha,n+1-m}[u \circ G_{\alpha,w}]
+ \sum_{k=0}^{m-1} \sum_{w \in A^k} \left( \begin{array}{cc} 0 & 0 \\ 0 & \frac{2}{1-\alpha} \end{array} \right)^k E_{\alpha,1}[u \circ G_{\alpha,w}|_{V_{\alpha,1}}]
\]

holds for \( 0 \leq m \leq n \).

Proof. The case \( m = 1 \) is Proposition 2.6 and the assertion follows by induction over \( m \). \( \square \)

The renormalization factor for each level \( n \in \mathbb{N}_0 \) is given in the next theorem and since we are using matrix notation, this “factor” becomes a \( 2 \times 2 \) matrix.

Theorem 2.8. Let \( u_0 : V_0 \to \mathbb{R} \) and denote by \( \tilde{u}_n \in D_{\alpha,n} \) its harmonic extension at level \( n \in \mathbb{N} \). Then

(i) if \( n = 0 \),

\[
E_{\alpha,0}[\tilde{u}_1] = \left( \begin{array}{cc} \rho_{\alpha,0}^d & 0 \\ 0 & \rho_{\alpha,0}^c \end{array} \right) E_{\alpha,1}[\tilde{u}_1] \tag{2.9}
\]

for \( \rho_{\alpha,0}^d := \frac{(5 + 3d_{\alpha,1})^2}{15 + 18d_{\alpha,1}} \) and \( \rho_{\alpha,0}^c = 0 \).

(ii) if \( n \in \mathbb{N} \),

\[
E_{\alpha,n}[\tilde{u}_n] = \left( \begin{array}{cc} \rho_{\alpha,n}^d & 0 \\ 0 & \rho_{\alpha,n}^c \end{array} \right) E_{\alpha,1}[\tilde{u}_1], \tag{2.10}
\]

where

\[
\rho_{\alpha,n}^d := \begin{cases} 1, & \text{for } n = 1, \\ \prod_{i=2}^{n} \rho_{\alpha,i}^d, & \text{for } n \geq 2 \end{cases}
\]
and

\[ \rho_{\alpha,n}^c := \begin{cases} 1, & \text{for } n = 1, \\ 1 + \sum_{j=2}^{n} \prod_{i=2}^{j} r_{\alpha,i}^e, & \text{for } n \geq 2, \end{cases} \]

where \( r_{\alpha,i}^d \) and \( r_{\alpha,i}^e \) were defined in (2.7) for each \( i \in \mathbb{N} \).

**Proof.** This is proved by induction (see [3, Lemma 2.13] for details).

We define the renormalized energy at level \( n \) by

\[ E_{\alpha,n}(u,v) := \rho_{\alpha,n}^{-1} E_{\alpha,n}(u,v) = \left( \rho_{\alpha,n}^d \right)^{-1} E_{\alpha,n}^d(u,v) + \left( \rho_{\alpha,n}^c \right)^{-1} E_{\alpha,n}^c(u,v) \]

for \( u, v \in D_{\alpha,n} \).

Note that the sequence \( \rho_{\alpha,n}^c \) converges and for \( \Theta_{\alpha} := \lim_{n \to \infty} \rho_{\alpha,n}^c \) it holds \( 1 < \Theta_{\alpha} < \infty \) (see [3] for details). This quantity will appear in later calculations.

### 2.3 Resistance form and Dirichlet form on \( K_\alpha \)

In this paragraph, we define a resistance form for the whole fractal \( K_\alpha \) which together with a suitable \( L^2 \)-space will induce a Dirichlet form on \( K_\alpha \). We refer to [23] for an outline of the most important results on the theory of resistance forms.

#### 2.3.1 Resistance form

We first recall the definition of resistance form on a locally compact metric space \( (X,d) \).

**Definition 2.9.** The pair \( (\mathcal{E}, \mathcal{F}) \) is called a *resistance form* if the following properties are satisfied:

(R1) \( \mathcal{F} \) is a linear subspace of \( \{ u : X \to \mathbb{R} \} \) that contains constants. Moreover, \( \mathcal{E} \) is a non-negative symmetric bilinear form on \( \mathcal{F} \) and for all \( u \in \mathcal{F} \), \( \mathcal{E}(u,u) = 0 \) if and only if \( u \) is constant.

(R2) For any \( u, v \in \mathcal{F} \), write \( u \sim v \) if and only if \( u - v \) is constant. Then \( (\mathcal{F}/\sim, \mathcal{E}) \) is a complete metric space.

(R3) For any two points \( x, y \in X \), there exists \( u \in \mathcal{F} \) such that \( u(x) \neq u(y) \).

(R4) For any two points \( x, y \in X \),

\[ \sup \left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}(u,u)} \mid u \neq 0, u \in \mathcal{F} \right\} < \infty. \]
(R5) For any \( u \in \mathcal{F}, \pi := 0 \lor u \land 1 \in \mathcal{F} \) and \( \mathcal{E}(\pi, \pi) \leq \mathcal{E}(u, u) \).

For any function \( u : V_{\alpha,*} \to \mathbb{R} \) we define
\[
\mathcal{E}_{\alpha,n}[u] := \mathcal{E}_{\alpha,n}(u|_{V_{\alpha,n}}, u|_{V_{\alpha,n}}).
\]
The sequence \((\mathcal{E}_{\alpha,n}[u])_{n \in \mathbb{N}_0}\) is thus non-decreasing and we can define a (non-trivial) functional \( \mathcal{E}_{K_{\alpha}} : \mathcal{F}_{K_{\alpha}} \to \mathbb{R} \) by
\[
\begin{cases}
\mathcal{F}_{K_{\alpha}} := \{ u : V_{\alpha,*} \to \mathbb{R} \mid \lim_{n \to \infty} \mathcal{E}_{\alpha,n}[u] < \infty \}, \\
\mathcal{E}_{K_{\alpha}}[u] := \lim_{n \to \infty} \mathcal{E}_{\alpha,n}[u].
\end{cases}
\]

Let us now set \( \mathcal{C}(J_{\alpha}) := \{ u : V_{\alpha,*} \to \mathbb{R} \mid u \in \mathcal{C}([a_e, b_e]), \forall e \in J_{\alpha} \} \). We will prove in Proposition 2.11 that any function \( u \in \mathcal{F}_{K_{\alpha}} \cap \mathcal{C}(J_{\alpha}) \) is Hölder – and therefore uniformly – continuous on \( V_{\alpha,*} \). Since \( V_{\alpha,*} \) is dense in \( K_{\alpha} \), \( u \) can be uniquely extended to a continuous function on \( K_{\alpha} \). We denote this extension again by \( u \) and set
\[
\mathcal{F}_{K_{\alpha}} := \{ u : K_{\alpha} \to \mathbb{R} \mid u|_{V_{\alpha,*}} \in \mathcal{F}_{K_{\alpha}} \cap \mathcal{C}(J_{\alpha}), \text{ and } \mathcal{E}_{K_{\alpha}}[u] < \infty \}.
\]

**Theorem 2.10.** \((\mathcal{E}_{K_{\alpha}}, \mathcal{F}_{K_{\alpha}})\) is a resistance form on \( K_{\alpha} \).

Before proving this we need a prior result.

**Proposition 2.11.** Every function in \( \mathcal{F}_{K_{\alpha}} \) is continuous on \( K_{\alpha} \).

**Proof.** Since \( V_{\alpha,*} \) is dense in \( K_{\alpha} \) with respect to the Euclidean norm, it suffices to show continuity on \( V_{\alpha,*} \). Consider \( u \in \mathcal{F}_{K_{\alpha}} \) and \( x, y \in V_{\alpha,*} \).

1. If \( x, y \in W_{\alpha,n} \) are \((\alpha, n)\)-neighbors, then \( |x - y| = \left( \frac{1 - \alpha}{2} \right)^n \) and
\[
\left( \rho_{\alpha,n}^{-d} \right)^{-1} |u(x) - u(y)|^2 \leq \mathcal{E}_{K_{\alpha}}^d[u] \leq \mathcal{E}_{K_{\alpha}}[u],
\]
which implies that
\[
|u(x) - u(y)| \leq \left( \rho_{\alpha,n}^{d} \right)^{1/2} \mathcal{E}_{K_{\alpha}}^{1/2}[u] \leq \mathcal{E}_{K_{\alpha}}^{1/2}[u] |x - y|^{\lambda_{\alpha}},
\]
where \( \lambda_{\alpha} := \frac{\ln 3 - \ln 5}{2 \ln (1 - \alpha) - \ln 2} \).

2. If \( x, y \in W_{\alpha,n} \) are not neighbors we proceed as follows: Consider a chain of points \( x_n, y_{n+1}, x_{n+2}, y_{n+2}, \ldots, x_{n+k-1}, y_{n+k} \in V_{\alpha,*} \) such that \( x_{n+j}, y_{n+j+1} \in W_{\alpha,n+j} \) are \((\alpha, n+j+1)\)-neighbors and \( (y_{n+j+1}, x_{n+j+1}) \in J_{\alpha,n+j} \setminus \mathcal{J}_{\alpha,n+j-1} \) for each \( 0 \leq j \leq k - 1 \) (see Figure 5).
Finally, \( l_\alpha \leq 1/2 \). If there exists some \( k > 1 \) such that \( x := x_n \in W_{\alpha,n} \) and \( y := y_n + k \in W_{\alpha,n+k} \backslash W_{\alpha,n+k-1} \), then, \( |x_{n+j} - y_{n+j+1}| = (\frac{1}{2})^{n+j+1} \) and \( |y_{n+j} - x_{n+j}| = \alpha (\frac{1}{2})^{n+j-1} \) and we get that
\[
|u(x) - u(y)| \leq \sum_{j=0}^{k-1} |u(x_{n+j}) - u(y_{n+j+1})| + \sum_{j=1}^{k-1} |u(y_{n+j}) - u(x_{n+j})| 
\leq \mathcal{E}^{1/2}_{K_\alpha}[u] \sum_{j=0}^{k-1} |x_{n+j} - y_{n+j+1}|^{l_\alpha} + \Theta^{1/2}_{\alpha} \mathcal{E}^{1/2}_{K_\alpha}[u] \sum_{j=1}^{k} |y_{n+j} - x_{n+j}|^{1/2} 
= \mathcal{E}^{1/2}_{K_\alpha}[u] \left( \frac{1 - \alpha}{2} \right)^{(n+1)l_\alpha} \sum_{j=0}^{k-1} \left( \frac{1 - \alpha}{2} \right)^{l_\alpha j} 
+ \Theta^{1/2}_{\alpha} \mathcal{E}^{1/2}_{K_\alpha}[u] \alpha^{1/2} \left( \frac{1 - \alpha}{2} \right)^{\frac{n-1}{2}} \sum_{j=1}^{k} \left( \frac{1 - \alpha}{2} \right)^{j/2}.
\]
Since \( l_\alpha < 1/2 \), \( \alpha (\frac{1-a}{2})^{-1} < 1 \) and \( (\frac{1-a}{2})^{l_\alpha} < 1 \), we get that
\[
|u(x) - u(y)| \leq \mathcal{E}^{1/2}_{K_\alpha}[u] \left( \frac{1 - \alpha}{2} \right)^{n l_\alpha} \sum_{j=0}^{k-1} \left( \frac{1 - \alpha}{2} \right)^{l_\alpha j} 
+ \Theta^{1/2}_{\alpha} \mathcal{E}^{1/2}_{K_\alpha}[u] \left( \frac{1 - \alpha}{2} \right)^{n l_\alpha} \sum_{j=1}^{k} \left( \frac{1 - \alpha}{2} \right)^{l_\alpha j} 
= (1 + \Theta^{1/2}_{\alpha}) \mathcal{E}^{1/2}_{K_\alpha}[u] \left[ 1 - \left( \frac{1 - \alpha}{2} \right)^{l_\alpha} \right]^{-1} \left( \frac{1 - \alpha}{2} \right)^{n l_\alpha}.
\]
Finally, \( (\frac{1-a}{2})^n \leq |x - y| \) because \( y \notin W_{\alpha,n} \) by assumption, hence, if we set \( C := (1 + \Theta^{1/2}_{\alpha}) \left[ 1 - \left( \frac{1-a}{2} \right)^{l_\alpha} \right]^{-1} \), we obtain
\[
|u(x) - u(y)| \leq C \mathcal{E}^{1/2}_{K_\alpha}[u] |x - y|^{l_\alpha}.
\]
In the case \( k = 0 \), i.e. \( x, y \in W_{\alpha,n} \setminus W_{\alpha,n-1} \) are not \((\alpha, n)\)-neighbors, we can join them by at most two such chains, say \( x := x_n, \ldots, y_{n+k} \) and \( y := y'_n, \ldots, y'_{n+k} \) for some \( k \in \mathbb{N} \) and an extra segment \((y_{n+k}, y'_{n+k})\) of length \( \alpha \left(1 - \frac{1}{2}\right)^{n+k-1} \) (in the case that \( y_{n+k} \neq y'_{n+k} \)). The triangular inequality and last calculation leads to

\[
|u(x) - u(y)| \leq 2C\mathcal{E}^{1/2}_{K_\alpha}[u] \left(1 - \frac{\alpha}{2}\right)^{(n+1)l_\alpha} + \Theta^{1/2}_\alpha \mathcal{E}^{1/2}_{K_\alpha}[u] \alpha^{1/2} \left(1 - \frac{\alpha}{2}\right)^{n+k-1}
\]

and by using again the fact that \( l_\alpha < 1/2, \alpha \left(\frac{1}{2}\right)^{-1} < 1, k \geq 1 \) and \( |x - y| > (\frac{1}{2})^{(n+1)} \), we obtain

\[
|u(x) - u(y)| \leq (2C + \Theta^{1/2}_\alpha) \mathcal{E}^{1/2}_{K_\alpha}[u] \left(1 - \frac{\alpha}{2}\right)^{(n+1)l_\alpha}
\]

\[
\leq (2C + \Theta^{1/2}_\alpha) \mathcal{E}^{1/2}_{K_\alpha}[u] |x - y|^{l_\alpha}.
\]

(3) If \( x, y \) belong to the same component \( e \in J_{\alpha,n} \) for some \( n \in \mathbb{N} \), \( u \) is in particular continuous on \( e \) so we get by Cauchy-Schwartz that

\[
|u(x) - u(y)|^2 = \int_x^y \nabla u \, dx \leq \int |\nabla u|^2 \, dx \cdot |x - y|,
\]

and therefore

\[
\left(\rho_{\alpha,n}^c\right)^{-1} |u(x) - u(y)|^2 \leq |x - y| \left(\rho_{\alpha,n}^c\right)^{-1} \int_{J_{\alpha,n}} |\nabla u|^2 \, dx
\]

\[
\leq \mathcal{E}_{K_\alpha}[u] |x - y|,
\]

which leads to

\[
|u(x) - u(y)| \leq \left(\rho_{\alpha,n}^c\right)^{1/2} \mathcal{E}^{1/2}_{K_\alpha}[u] |x - y|^{1/2} \leq \Theta^{1/2}_\alpha \mathcal{E}^{1/2}_{K_\alpha}[u] |x - y|^{1/2}.
\]

The same calculations apply if \( x \in e \in J_{\alpha,n} \) and \( y \in W_{\alpha,n} \) is one of its endpoints.

(4) If \( x, y \in J_{\alpha,n} \setminus J_{\alpha,n-1} \) do not belong to the same line segment, then there exists \( e_1, e_2 \in J_{\alpha,n} \) such that \( x \in e_1, y \in e_2 \). Now we can join both points as follows: consider \( x' \in W_{\alpha,n} \) the nearest endpoint of \( e_1 \) to \( x \), and \( y' \in W_{\alpha,n} \) the nearest in \( e_2 \) to \( y \). Then, by an analogous calculation as the previous case and the applying the triangular inequality we have

\[
|u(x) - u(y)| \leq (2C + 3\Theta^{1/2}_\alpha) \mathcal{E}^{1/2}_{K_\alpha}[u] |x - y|^{l_\alpha}.
\]

Now, choosing \( \tilde{C} := 2C + 3\Theta^{1/2}_\alpha \), it follows from cases (3) and (4) that

\[
|u(x) - u(y)| \leq \tilde{C} \mathcal{E}^{1/2}_{K_\alpha}[u] |x - y|^{l_\alpha}
\]

for all \( x, y \in W_{\alpha,n} \), hence \( u \) is uniformly Hölder-continuous.
Figure 9: Chain with $x \in e_1$, $y \in e_2$.

(5) The case when $x \in J_{\alpha,n}$ and $y \in W_{\alpha,n}$ follows by combining the two last cases.

\[ \Box \]

Proof of Theorem 2.10

We check the properties of a resistance form given in Definition 2.9.

(RF1) $F_{K_{\alpha}}$ is clearly a linear subspace of $\{ u : K_{\alpha} \to \mathbb{R} \}$ and $E_{K_{\alpha}}$ is a non-negative quadratic form on $F_{K_{\alpha}}$. Moreover, it follows from the definition of $E_{\alpha,n}$ that $0 = E_{K_{\alpha}}[u] = \lim_{n \to \infty} E_{\alpha,n}[u]$ if and only if $u \equiv \text{const}$, which implies that $F_{K_{\alpha}}$ contains constants.

(RF2) Define the equivalence relation on $F_{K_{\alpha}}$ by $u \sim v \iff u - v \equiv \text{const}$ and consider the space $(F_{K_{\alpha}}/\sim, E_{K_{\alpha}})$. We prove now that this is a Hilbert space.

All properties of $E_{K_{\alpha}}$ for being an inner product are satisfied by definition except that $E_{K_{\alpha}}[u] = 0 \iff u \equiv 0$. This follows from the fact that $E_{K_{\alpha}}[u] = 0$ if and only if $u \equiv \text{const}$ on $K_{\alpha}$ and constants are the zero class in $F_{K_{\alpha}}/\sim$.

In order to prove that $(F_{K_{\alpha}}/\sim, E_{K_{\alpha}})$ is complete, we identify the set $F_{K_{\alpha}}/\sim$ with the set $\mathcal{R}_{\alpha} := \{ u \in F_{K_{\alpha}} | u(p_1) = 0 \}$.

Let $(u_m)_{m \in \mathbb{N}_0}$ be a Cauchy sequence in $\mathcal{R}_{\alpha}$. For all $x \in W_{\alpha,*}$, $(u_n(x))_{n \in \mathbb{N}_0}$ is a Cauchy sequence on $\mathbb{R}$ and therefore convergent, so its limit $\bar{u}$ exists.

On the other hand, we know that for each $e \in J_{\alpha}$, $u_{n|e} \in H^1(e, dx)$ and

\[ \left\| \nabla u_{n|e} - \nabla u_{m|e} \right\|_2 \leq E_{K_{\alpha}}^{1/2}[u_n - u_m] \xrightarrow{m,n \to \infty} 0. \tag{2.11} \]

Hence $(\nabla u_{n|e})_{n \in \mathbb{N}_0}$ is a Cauchy sequence in $L^2(e, dx)$ and there exists $v^e \in L^2(e, dx)$ such that $\left\| \nabla u_{n|e} - v^e \right\|_2 \xrightarrow{n \to \infty} 0$. We also may choose $v^e$ to be continuous.
Since $a_e \in W_{\alpha, s}$, we can set $u(a_e) := \lim_{n \to \infty} u_n(a_e)$ and define

$$u^c(x) := \int_{a_e}^x v^c \, dx + u(a_e) \quad \forall x \in (a_e, b_e).$$

A straightforward calculation proves that

$$\lim_{n \to \infty} \mathcal{E}_{K_{\alpha}}[u_n - u] = 0$$

for

$$u(x) := \begin{cases} \tilde{u}(x) = \lim_{n \to \infty} u_n(x), & x \in W_{\alpha, s}, \\ u^c(x), & x \in e \in J_{\alpha}. \end{cases} \tag{2.12}$$

Note that $u_n \in \mathcal{R}_{\alpha}$ implies $u_n$ is continuous, hence $u \in C(V_{\alpha, s})$. Moreover, $u(p_1) = \lim_{n \to \infty} u_n(p_1) = 0$, hence $u \in \mathcal{R}_{\alpha}$ and we are done.

(RF3) If $x, y \in K_{\alpha}$ are such that $x \neq y$, we can consider $x, y \in V_{\alpha, n}$ for some $n \in N_0$ and take $B_\varepsilon(x)$ such that $y \in V_{\alpha, n} \setminus B_\varepsilon(x)$ ($=: V_\varepsilon$). Define $u_n : V_{\alpha, n} \to \mathbb{R}$ by $u_n(x) := 1$, $u_n|_{V_\varepsilon} \equiv 0$. (If $x \in e \in J_{\alpha, n}$, define $u_n$ on $e = (a_e, b_e)$ to be some smooth function with $u_n(a_e) = 0 = u_n(b_e)$ and $u(x) = 1$.)

Clearly, $u_n \in D_{\alpha, n}$ because $\mathcal{E}_{\alpha, n}[u_n] < \infty$ and by defining $u : K_{\alpha} \to \mathbb{R}$ as the harmonic extension of $u_n$, the we get that $\mathcal{E}_{K_{\alpha}}[u] = \mathcal{E}_{\alpha, n}[u_n] < \infty$, hence $u \in \mathcal{F}_{K_{\alpha}}$ and $u(x) \neq u(y)$ as desired.

(RF4) From Proposition 2.11 there exists $c > 0$ such that

$$\sup \left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}_{K_{\alpha}}[u]} \big| u \in \mathcal{F}_{K_{\alpha}}, \mathcal{E}_{K_{\alpha}}[u] < \infty \right\} \leq c |x - y|^{2\beta} < \infty. \tag{2.13}$$

(RF5) $\mathcal{E}_{K_{\alpha}}$ satisfies the Markov property: $\mathcal{E}_{\alpha, n}$ fulfills by definition the Markov property, hence

$$\mathcal{E}_{K_{\alpha}}[0 \vee u \wedge 1] = \lim_{n \to \infty} \mathcal{E}_{\alpha, n}[0 \vee u|_{V_{\alpha, n}} \wedge 1] \leq \lim_{n \to \infty} \mathcal{E}_{\alpha, n}[u] = \mathcal{E}_{K_{\alpha}}[u] < \infty.$$ 

In particular $0 \vee u \wedge 1 \in \mathcal{F}_{K_{\alpha}}$ and we are done.

\[ \square \]

The mapping $R : K_{\alpha} \times K_{\alpha} \to [0, \infty)$ given by the supremum in (R4) defines metric on $K_{\alpha}$, the so-called resistance metric (see [21] Theorem 2.3.4 for a proof). This metric satisfies the following property, whose immediate consequence is crucial for the regularity of the form.
Lemma 2.12. The topology induced by the resistance metric $R$ associated with $E_{K_\alpha, \mathcal{D}_{K_\alpha}}$ coincides with the original topology of $(K_\alpha, |\cdot|)$.

Proof. The proof works analogous to [4, Proposition 7.18].

Corollary 2.13. $K_\alpha$ is a compact set with respect to the resistance metric.

Corollary 2.14. The resistance form $(E_{K_\alpha}, \mathcal{F}_{K_\alpha})$ is regular.

Proof. This follows from [23, Corollary 6.4].

Finally, we state a kind of scaling property of this form.

Proposition 2.15. For any $u,v \in \mathcal{D}_{K_\alpha}$ it holds that

$$E_{K_\alpha}(u,v) = \sum_{i=1}^{3} \left( \frac{5}{3} E_{d_{K_\alpha}}(u_i, v_i) + \frac{2}{1-\alpha} E_{c_{K_\alpha}}(u_i, v_i) \right) + \Theta_\alpha^{-1} E_{c_{K_\alpha}}(u|_{V_\alpha}, v|_{V_\alpha}),$$

where $u_i := u \circ G_{\alpha,i}$ for any $u \in \mathcal{D}_{K_\alpha}$, $i \in A$, and $\Theta_\alpha := \lim_{n \to \infty} \rho_{c_{\alpha,n}}$.

Proof. By Proposition 2.6 we have that

$$E_{\alpha,n+1}(u,v) = \frac{\rho_{d_{\alpha,n}}}{\rho_{d_{\alpha,n+1}}} \sum_{i=1}^{3} E_{d_{K_\alpha}}(u_i, v_i) + \frac{\rho_{c_{\alpha,n}}}{\rho_{c_{\alpha,n+1}}} \sum_{i=1}^{3} \frac{2}{1-\alpha} E_{c_{K_\alpha}}(u_i, v_i)$$

$$+ \left( \frac{\rho_{c_{\alpha,n}}}{\rho_{c_{\alpha,n+1}}} \right)^{-1} E_{c_{K_\alpha}}(u,v).$$

Now, note that

$$\frac{\rho_{d_{\alpha,n}}}{\rho_{d_{\alpha,n+1}}} = \prod_{i=1}^{n} \frac{r_{d_{\alpha,i}}}{r_{d_{\alpha,i+1}}} = \frac{1}{\prod_{i=1}^{n+1} r_{d_{\alpha,i}}} = \frac{\left( 5 + 3\alpha \left( \frac{1}{2} - \frac{\alpha}{2} \right) n \right)^2}{15} \xrightarrow{n \to \infty} \frac{5}{3},$$

and since $0 < \Theta_\alpha < \infty$, we have that $\frac{\rho_{c_{\alpha,n}}}{\rho_{c_{\alpha,n+1}}} x \to \infty$.

Letting $n \to \infty$ in both sides of the equality (2.15), we obtain (2.14).

Corollary 2.16. For any $m \in \mathbb{N}_0$ and $u,v \in \mathcal{D}_{K_\alpha}$ we have that

$$E_{K_\alpha}(u,v) = \sum_{w \in \mathcal{A}_m} \left( \frac{5}{3} \right)^m E_{d_{\alpha,n}}(u \circ G_{\alpha,w}, v \circ G_{\alpha,w})$$

$$+ \sum_{w \in \mathcal{A}_m} \left( \frac{2}{1-\alpha} \right)^m E_{c_{\alpha,n}}(u \circ G_{\alpha,w}, v \circ G_{\alpha,w})$$

$$+ \Theta_\alpha^{-1} \sum_{k=0}^{m-1} \sum_{w \in \mathcal{A}_k} E_{c_{\alpha,1}}(u \circ G_{\alpha,w|_{V_\alpha}}, v \circ G_{\alpha,w|_{V_\alpha}}).$$
2.3.2 Dirichlet form

In order to obtain a Dirichlet form from the resistance form, we need a locally finite regular measure $\mu_\alpha$ on $K_\alpha$. Due to the non self-similarity of $K_\alpha$, there is no "canonical" choice of such measure. Hence we will not specify it until the next section, when it becomes necessary for the study of the associated Laplacian.

Let $\mu_\alpha$ be an arbitrary finite Radon measure on $K_\alpha$ and let $L^2(K_\alpha, \mu_\alpha)$ be the associated Hilbert space. From Proposition 2.11 it follows that $F_{K_\alpha} \subseteq L^2(K_\alpha, \mu_\alpha)$ so we can define

$$E_{K_\alpha, 1}(u,v) := \mathcal{E}_{K_\alpha}(u,v) + \int_{K_\alpha} uv \, d\mu_\alpha \quad u, v \in F_{K_\alpha}. \quad (2.16)$$

This turns out to be an inner product in $F_{K_\alpha}$ and thus we can consider the norm $\|\cdot\|_{E_{K_\alpha, 1}} := \mathcal{E}_{K_\alpha}^{1/2}$.

Let $\mathcal{D}_{K_\alpha}$ denote the closure of $C_0(K_\alpha) \cap F_{K_\alpha}$ with respect to the norm $\|\cdot\|_{E_{K_\alpha, 1}}$, where $C_0(K_\alpha)$ denotes the set of compactly supported continuous functions in $K_\alpha$ (in fact $C(K_\alpha)$). On the one hand, it follows from Corollary 2.14 that $\mathcal{D}_{K_\alpha}$ is dense in $C(K_\alpha)$. On the other hand, it is a well known result from classical analysis that $C_0(K_\alpha)$ is dense in $L^2(K_\alpha, \mu_\alpha)$.

Thus $\mathcal{D}_{K_\alpha}$ is dense in $L^2(K_\alpha, \mu_\alpha)$ too and the pair $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ is called the Dirichlet form derived from the resistance form $(\mathcal{E}_{K_\alpha}, F_{K_\alpha})$. Moreover, by Corollary 2.13 $K_\alpha$ is $R$-compact, thus $\mathcal{D}_{K_\alpha} = F_{K_\alpha}$.

**Theorem 2.17.** The Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ on $L^2(K_\alpha, \mu_\alpha)$ is local and regular.

**Proof.** By Corollary 2.14 $(\mathcal{E}_{K_\alpha}, F_{K_\alpha})$ is a regular resistance form, hence by [23, Theorem 9.4] its associated Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ is a regular Dirichlet form.

If we consider $u, v \in \mathcal{D}_{K_\alpha}$ such that supp$(u) \cap$ supp$(v) = \emptyset$, since supp$(u)$ and supp$(v)$ are compact sets, there exists some $n \in \mathbb{N}$ such that for all $w \in \mathcal{A}^n$, either supp$(u) \cap G_{\alpha,w}(K_\alpha) = \emptyset$ or supp$(v) \cap G_{\alpha,w}(K_\alpha) = \emptyset$. By Corollary 2.16 we get that $\mathcal{E}_{K_\alpha}(u,v) = 0$, hence the form is local. \[\square\]

3 Measure and Laplacian on $K_\alpha$

Since the definition of $\Delta_{\mu_\alpha}$ strongly depends on the choice of the measure on $K_\alpha$, we need to fix one up to this point. In general, there is no canonical choice of it and the one constructed here has been chosen in this particular manner for technical reasons.
3.1 Measure on $K_\alpha$

The following result gives a decomposition of $K_\alpha$ that will be very useful in the definition of the measure $\mu_\alpha$.

**Lemma 3.1.** Let $F_\alpha$ be the unique nonempty compact subset of $\mathbb{R}^2$ satisfying $F_\alpha = \bigcup_{i=1}^3 G_{\alpha,i}(F_\alpha)$ and define $J_\alpha := \bigcup_{n \in \mathbb{N}_0} J_{\alpha,n}$. Then,

$$K_\alpha = F_\alpha \cup J_\alpha.$$ 

**Proof.** See [1, Lemma 2.1.1] \hfill \square

Now, let $\lambda$ denote the $1$–dimensional Hausdorff measure and consider $\beta$ any positive number satisfying

$$0 < \beta < \left(\frac{2}{3(1-\alpha)}\right)^2. \tag{3.1}$$

On the one hand, we define the self-similar measure on $F_\alpha$ given by

$$\mu_\alpha^d(A) := \frac{1}{2^d \mathcal{H}^d(F_\alpha)} \mathcal{H}^d(A) \quad \text{for } A \subseteq \mathbb{R}^2 \text{ Borel},$$

where $d_\alpha := \dim_H K_\alpha = \frac{\ln 3}{\ln 2 - \ln(1-\alpha)}$ and $\mathcal{H}^d$ denotes the $d$–dimensional Hausdorff measure.

On the other hand, we define the Radon measure on $\mathbb{R}^2$ given by

$$\mu_\alpha^c(A) := \mu_\alpha^c(A) := \frac{1}{2^e \tilde{\mu}_{\alpha,e}(J_\alpha)} \tilde{\mu}_{\alpha,e}(A) \quad \text{for } A \subseteq \mathbb{R}^2 \text{ Borel},$$

where

$$\tilde{\mu}_{\alpha,e}(A) := \sum_{e \in J_\alpha} \beta_e \lambda(A \cap e),$$

and $\beta_e := \beta^k$ if $e \in J_{\alpha,k+1} \setminus J_{\alpha,k}$, $\beta$ being the constant chosen in (3.1). Note that $\text{supp}(\mu_\alpha^c) = J_\alpha$.

In view of Lemma 3.1 we may define a finite Radon measure on $\mathbb{R}^2$ as the sum

$$\mu_\alpha(A) := \mu_\alpha^c(A) := \mu_\alpha^d(A \cap F_\alpha) + \mu_\alpha^c(A \cap J_\alpha) \quad \text{for } A \subseteq \mathbb{R}^2 \text{ Borel}.$$ 

Note that $\text{supp}(\mu_\alpha) = K_\alpha$ and $\mu_\alpha(K_\alpha) = 1$.

**Remark 3.2.**...
(1) The choice of $\beta$ in the definition of $\tilde{\mu}_c^\alpha$ ensures that $\tilde{\mu}_c^\alpha(J_\alpha) < \infty$.

(2) Let $\mathcal{A}^* := \bigcup_{n \in \mathbb{N}} \mathcal{A}^n$ be the set of all words on the alphabet $\mathcal{A}$ of finite length. The measure $\mu_\alpha$ is a Borel probability measure and belongs to the set

$$
\mathcal{M}^1(K_\alpha) := \left\{ \mu \mid \mu \text{ is a probability measure on } K_\alpha, \mu(\{x\}) = 0 \forall x \in K_\alpha, \mu(G_{\alpha,w}(K_\alpha)) > 0 \text{ and } \mu(G_{\alpha,w}(V_0)) = 0 \text{ for any } w \in \mathcal{A}^*, i \in \mathcal{A} \right\}.
$$

(3) It will follow directly from Lemma 4.5 that $\mu_\alpha$ is an elliptic measure, i.e. there exists $\gamma \in (0, \infty)$ such that $\mu_\alpha(G_{\alpha,wi}(K_\alpha)) \geq \gamma \mu_\alpha(G_{\alpha,w}(K_\alpha))$ for all $w \in \mathcal{A}^*$, $i \in \mathcal{A}$.

(4) For each $w \in \mathcal{A}^*$, define $\mu_w^\alpha := \frac{1}{\mu_\alpha(G_{\alpha,w}(K_\alpha))} \mu \circ G_{\alpha,w}$. We have that $\mu_w^\alpha \in \mathcal{M}^1(K_\alpha)$ and for any Borel measurable function $u: K_\alpha \to \mathbb{R}$,

$$
\int_{K_\alpha} u \circ G_{\alpha,w} \, d\mu_w^\alpha(x) = \frac{1}{\mu_\alpha(G_{\alpha,w}(K_\alpha))} \int_{G_{\alpha,w}(K_\alpha)} u \, d\mu_\alpha. \quad (3.2)
$$

3.2 Laplacian

As we know from the theory of Dirichlet forms (see e.g. [10]) $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ defines a Laplacian on $K_\alpha$ in the weak sense as the unique non-positive, self-adjoint and densely defined operator $\Delta_{\mu_\alpha}: \mathcal{D}(\Delta_{\mu_\alpha}) \to L^2(K_\alpha, \mu_\alpha)$ such that for any $u \in \mathcal{D}(\Delta_{\mu_\alpha})$

$$
\mathcal{E}_{K_\alpha}(u, v) = (-\Delta_{\mu_\alpha} u, v)_{\mu_\alpha} \quad \forall v \in \mathcal{D}_{K_\alpha}.
$$

Moreover, it can be proved following the same arguments as in [22] that all functions in the domain $\mathcal{D}(\Delta_{\mu_\alpha})$ have normal derivative (in the sense of [29, Definition 2.3.1]) equal zero on the boundary $V_0$. Hence we can say that the functions in $\mathcal{D}(\Delta_{\mu_\alpha})$ satisfy homogeneous Neumann boundary conditions. From now on we will thus adopt the notation $\Delta_{\mu_\alpha}^N$ for the Neumann Laplacian associated to $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$.

The Laplacian $\Delta_{\mu_\alpha}^D$ subject to Dirichlet boundary conditions is defined analogously by modifying the domain $\mathcal{D}_{K_\alpha}$ to $\mathcal{D}_{K_\alpha}^0 := \{ u \in \mathcal{D}_{K_\alpha} \mid u|_{V_{\alpha,0}} \equiv 0 \}$.

**Theorem 3.3.** The operator $-\Delta_{\mu_\alpha}^N$ has pure point spectrum consisting of countable many non-negative eigenvalues with finite multiplicity and only accumulation point at $+\infty$. The same holds for the operator $-\Delta_{\mu_\alpha}^D$.

**Proof.** Lemma 2.12 together with [23] Theorem 10.4 imply that the Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ has a jointly continuous kernel. Hence the semigroup $e^{-\Delta_{\mu_\alpha} t}$ is ultracontractive by [6] Lemma 2.1.2 and the claim follows from [6] Theorem 2.1.4. \[\square\]
4 Spectral dimension

This last result about the spectrum of the operators $\Delta^N_{\mu_\alpha}$ and $\Delta^D_{\mu_\alpha}$ allows us to study the asymptotic behavior of the eigenvalue counting function associated with each of them. In the following, whenever a statement holds for both operators, we will use the notation $\Delta^{N/D}_{\mu_\alpha}$.

**Definition 4.1.** The eigenvalue counting function of $-\Delta^N_{\mu_\alpha}$ is defined for each $x \geq 0$ as

$$N_{N/D}(x) := \#\{\kappa \mid \kappa \text{ eigenvalue of } -\Delta^N_{\mu_\alpha} \text{ and } \kappa \leq x\},$$
counted with multiplicity.

**Remark 4.2.** Given a Dirichlet form $(\mathcal{E}, \mathcal{D})$ on a Hilbert space $H$, we say that $\kappa \in \mathbb{R}$ is an eigenvalue of $\mathcal{E}$ if and only if there exists $u \in \mathcal{D}$, $u \neq 0$, such that $\mathcal{E}(u, v) = \kappa(u, v)$ for all $v \in \mathcal{D}$.

The eigenvalue counting function can thus be defined for a Dirichlet form $(\mathcal{E}, \mathcal{D})$ on a Hilbert space $H$ at any $x > 0$ by

$$N(x; \mathcal{E}, \mathcal{D}) := \#\{\kappa \mid \kappa \text{ eigenvalue of } \mathcal{E} \text{ and } \kappa \leq x\},$$
counted with multiplicity.

Furthermore, it follows from [26, Proposition 4.2] that $N_N(x) = N(x; \mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ and $N_D(x) = N(x; \mathcal{E}_{K_\alpha}^0, \mathcal{D}_{K_\alpha}^0)$.

Given two functions $f, g: \mathbb{R} \to \mathbb{R}$, let us write $f(x) \asymp g(x)$ if there exist constants $C_1, C_2 > 0$ such that $C_1f(x) \leq g(x) \leq C_2f(x)$.

The spectral dimension of $K_\alpha$ describes the asymptotic behavior of both eigenvalue counting functions and it is defined as the number $d_S(K_\alpha) > 0$ (in case it exists) such that

$$N_{N/D}(x) \asymp x^{d_S} \quad \text{as } x \to \infty.$$  

The following estimate of the eigenvalue counting function is therefore crucial to determine $d_S K_\alpha$.

**Theorem 4.3.** There exist constants $C_{\alpha,1}, C_{\alpha,\beta,1}, C_{\alpha,2}, C_{\alpha,\beta,2} > 0$ and $x_0 > 0$ such that

$$C_{\alpha,1} x^{\frac{\log 3}{\log 5}} + C_{\alpha,\beta,1} x^{\frac{1}{2}} \leq N_D(x) \leq N_N(x) \leq C_{\alpha,2} x^{\frac{\log 3}{\log 5}} + C_{\alpha,\beta,2} x^{\frac{1}{2}} \quad (4.1)$$

for all $x \geq x_0$.

The proof of this result will be divided into several lemmas and it mainly follows the ideas of Kajino in [18], based on the minimax principle for the eigenvalues of non-negative self-adjoint operators. Details about this principle can be found in [7, Chapter 4].
4.1 Preliminaries

In this paragraph we prove some technical results that will be used in the lemmas leading to Theorem 4.3. As usual, we work with the alphabet \( \mathcal{A} = \{1, 2, 3\} \) and the set \( \mathcal{A}^* := \bigcup_{n \in \mathbb{N}} \mathcal{A}^n \). Moreover, given any word \( w \in \mathcal{A}^* \), we write \( K_{\alpha,w} := G_{\alpha,w}(K_\alpha) \).

**Lemma 4.4.** For any \( m \in \mathbb{N}_0 \) and \( w \in \mathcal{A}^m \) it holds that

\[
\mu_\alpha(K_{\alpha,w}) = \frac{1}{3^m} \left( \beta \frac{1 - \alpha}{2} \right)^m.
\]

**Proof.** Fix \( m \in \mathbb{N}_0 \) and \( w \in \mathcal{A}^m \). By the definition of \( \mu^{d}_\alpha \) and \( (\frac{1 - \alpha}{2})^{d_\alpha} = \frac{1}{3} \) we have

\[
\mu^{d}_\alpha(G_{\alpha,w}(F_\alpha)) = \left( \frac{1}{3} \right)^m \mu^{d}_\alpha(F_\alpha).
\]

On the other hand, by definition of \( \mu^{c}_\alpha \) and since the length of the largest edges in \( G_{\alpha,w}(J_\alpha) \) is \( \alpha \left( \frac{1 - \alpha}{2} \right)^m \), we get that

\[
\tilde{\mu}^{c}_\alpha(G_{\alpha,w}(J_\alpha)) = \left( \frac{1 - \alpha}{2} \right)^m \beta^m \tilde{\mu}^{c}_\alpha(J_\alpha),
\]

hence \( \mu^{c}_\alpha(G_{\alpha,w}(J_\alpha)) = \left( \frac{1 - \alpha}{2} \right)^m \beta^m \). Equality (4.3) and (4.4) finally lead to (4.2). \( \square \)

**Lemma 4.5.** The measure \( \mu_\alpha \) is elliptic.

**Proof.** Choosing \( \gamma := \beta \frac{1 - \alpha}{2} \in (0, 1) \) we have that \( \mu_\alpha(K_{\alpha,w}) \geq \gamma \mu_\alpha(K_{\alpha}) \) for any \( w \in \mathcal{A}^* \) and \( i \in \mathcal{A} \). \( \square \)

We finish this paragraph with a definition and a remark that connect directly with the beginning of the proof of Theorem 4.3.

**Definition 4.6.** For any non-empty set \( U \subseteq K_{\alpha} \), we define

\[
C_{U} := \{ u \in \mathcal{D}_{K_{\alpha}} \mid \text{supp}(u) \subseteq U \}, \quad \mathcal{D}_{U} := \overline{C}_{U},
\]

where the closure is taken with respect to \( ||\cdot||_{\mathcal{E}_{K_{\alpha},1}} \), and write \( \mathcal{E}_{U} := \mathcal{E}_{K_{\alpha}}|_{\mathcal{D}_{U} \times \mathcal{D}_{U}} \). The pair \( (\mathcal{E}_{U}, \mathcal{D}_U) \) is called the *part of the Dirichlet form* \( (\mathcal{E}_{K_{\alpha}}, \mathcal{D}_{K_{\alpha}}) \) on \( U \).

**Remark 4.7.** Since \( u \equiv 0 \) \( \mu_{\alpha} \)-a.e. on \( K_{\alpha} \setminus U \) for any \( u \in \mathcal{D}_{U} \), we can regard \( \mathcal{D}_{U} \) as a subspace of \( L^2(U, \mu_{\alpha}|_{U}) \). In the case when \( U \subseteq K_{\alpha} \) is open, we know from [10, Theorem 4.4.3] that \( (\mathcal{E}_{U}, \mathcal{D}_U) \) is a Dirichlet form on \( L^2(U, \mu_{\alpha}|_{U}) \). We denote by \( H_U \) the non-negative self-adjoint operator on \( L^2(U, \mu_{\alpha}|_{U}) \) associated to \( (\mathcal{E}_{U}, \mathcal{D}_U) \).
4.2 Spectral asymptotics of the Laplacian

This section is devoted to the proof of Theorem 4.3, that we divide into several lemmas.

First of all, note that for any \( m \in \mathbb{N} \), \( J_{\alpha,m} \) is an open set, hence \((\mathcal{E}_{J_{\alpha,m}}, \mathcal{D}_{J_{\alpha,m}})\) is a Dirichlet form on \( L^2(J_{\alpha,m}, \mu_{\alpha|J_{\alpha,m}}) \). Moreover, since \( J_{\alpha,m} \) is just the finite union of 1-dimensional open intervals and due to the definition of \( \mu_{\alpha} \) we may identify \( \mathcal{D}_{J_{\alpha,m}} \) with the Sobolev space \( \bigoplus_{e \in J_{\alpha,m}} H^1_0(e, dx) \).

Lemma 4.8. For each \( m \in \mathbb{N} \), the non-negative self-adjoint operator \( H_{J_{\alpha,m}} \) associated with the Dirichlet form \((\mathcal{E}_{J_{\alpha,m}}, \mathcal{D}_{J_{\alpha,m}})\) on \( L^2(J_{\alpha,m}, \mu_{\alpha|J_{\alpha,m}}) \) has compact resolvent. Further, there exist constants \( C_{\alpha,\beta,1}, C_{\alpha,\beta,2} > 0 \) depending on \( \alpha \) and \( \beta \), and \( x_0 > 0 \) such that

\[
C_{\alpha,\beta,1} x^{1/2} \leq N_{J_{\alpha,m}}(x) \leq C_{\alpha,\beta,2} x^{1/2}
\]

for all \( x \in [x_0, \infty) \).

Proof. Note that the operator \( H_{J_{\alpha,m}} \) is nothing but the classical one-dimensional Laplacian \( \Delta \) restricted to the set \( J_{\alpha,m} \) and hence has compact resolvent.

Let us now prove the inequality (4.5). Let \( u \in \mathcal{D}_{J_{\alpha,m}} \) be an eigenfunction of \((\mathcal{E}_{J_{\alpha,m}}, \mathcal{D}_{J_{\alpha,m}})\) with eigenvalue \( \kappa \) and for any \( e \in J_{\alpha,m} \) and \( h \in H^1_0(e, dx) \) define

\[
\tilde{h}(x) := \begin{cases} h(x), & \text{if } x \in e, \\ 0, & \text{if } x \in J_{\alpha,m} \setminus e. \end{cases}
\]

Then, \( \tilde{h} \in \mathcal{D}_{J_{\alpha,m}} \) and

\[
\int_e \nabla u \nabla h \, dx = \Theta_\alpha \lim_{n \to \infty} (\rho_{\alpha,n})^{-1} \sum_{e \in J_{\alpha,m}} \int_e \nabla u \nabla \tilde{h} \, dx
\]

\[
= \Theta_\alpha \kappa \int_e uh \, d\mu_\alpha = \Theta_\alpha \kappa \beta_e \int_e uh \, dx,
\]

where \( \beta_e = \beta^n \) for \( e \in J_{\alpha,n+1} \setminus J_{\alpha,n} \) and \( \Theta_\alpha = \lim_{n \to \infty} \rho_{\alpha,n} \).

Thus,

\[
\int_e \nabla u \nabla h \, dx = \kappa \Theta_\alpha \beta_e \int_e uh \, dx \quad \forall h \in H^1_0(e, dx),
\]

which implies that \( \kappa \Theta_\alpha \beta_e \) is an eigenvalue of the classical Laplacian \( -\Delta \) on \( L^2(e, dx) \) subject to Dirichlet boundary conditions with eigenfunction \( u|_e \).

Conversely, it is easy to see that if for any \( e \in J_\alpha \), \( \kappa \Theta_\alpha \beta_e \) is an eigenvalue of the classical Laplacian \( -\Delta \) on \( L^2(e, dx) \) subject to Dirichlet boundary conditions with eigenfunction \( u \in H^1_0(e, dx) \), then \( \kappa \) is an eigenvalue of \((\mathcal{E}_{J_{\alpha,m}}, \mathcal{D}_{J_{\alpha,m}})\) with eigenfunction

\[
\tilde{u}(x) := \begin{cases} u(x), & x \in e, \\ 0, & x \in J_{\alpha,m} \setminus e. \end{cases}
\]

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Hence, if we denote by $N_e(x)$ the eigenvalue counting function of the classical $-\Delta_e$ subject to Dirichlet boundary conditions we have

$$N_{J_{\alpha,m}}(x) = \sum_{e \in J_{\alpha,m}} N_e(\beta_e \Theta_\alpha x). \quad (4.6)$$

Since all components $e \in J_{\alpha,m}$ are 1-dimensional sets, we know from Weyl’s theorem [31] that

$$N_e(x) \asymp \frac{\lambda(e)}{\pi} x^{1/2} + o(x^{1/2}) \quad \text{as } x \to \infty$$

for each $e \in J_{\alpha,m}$ and thus there exist constants $\tilde{c}_1, \tilde{c}_2 > 0$ and $x_0 > 0$ such that

$$\tilde{c}_1 \Theta_\alpha^{1/2} \sum_{e \in J_{\alpha,m}} \beta_e^{1/2} \lambda(e) \leq N_{J_{\alpha,m}}(x) \leq \tilde{c}_2 \Theta_\alpha^{1/2} x^{1/2} \sum_{e \in J_{\alpha,m}} \beta_e^{1/2} \lambda(e).$$

Since $\beta_e$ was chosen in (3.1) so that $\sum_{e \in J_{\alpha,m}} \beta_e^{1/2} \lambda(e) < \infty$, by setting

$$C_{\alpha,\beta,1} := \frac{\tilde{c}_1 \Theta_\alpha^{1/2}}{\pi} \quad \text{and} \quad C_{\alpha,\beta,2} := \frac{\tilde{c}_1 \Theta_\alpha^{1/2}}{\pi} \sum_{e \in J_{\alpha,m}} \beta_e^{1/2} \lambda(e),$$

the assertion is proved.

Upper bound

Let us now define for each $m \in \mathbb{N}$ the set $K_{\alpha,m} := \bigcup_{w \in A^m} K_{\alpha,w}$ and consider the pair $\left(\mathcal{E}_{K_{\alpha,m}}, \mathcal{D}_{K_{\alpha,m}}\right)$ given by

$$\left\{\begin{array}{l}
\mathcal{D}_{K_{\alpha,m}} := (\mathcal{D}_{J_{\alpha,m}})^{\perp}, \\
\mathcal{E}_{K_{\alpha,m}} := \mathcal{E}_{K_{\alpha,m}} |_{\mathcal{D}_{K_{\alpha,m}} \times \mathcal{D}_{K_{\alpha,m}}},
\end{array}\right.$$

where $(\mathcal{D}_{J_{\alpha,m}})^{\perp}$ denotes the orthogonal complement of $\mathcal{D}_{J_{\alpha,m}}$ in $\mathcal{D}_{K_{\alpha}}$ with respect to the inner product $\mathcal{E}_{K_{\alpha,1}}$ defined in (2.16). By definition, $u \equiv 0 \mu_{\alpha}-\text{a.e.}$ on $K_{\alpha} \setminus K_{\alpha,m}(= J_{\alpha,m})$ for all $u \in \mathcal{D}_{K_{\alpha,m}}$, hence $\mathcal{D}_{K_{\alpha,m}}$ can be regarded as a subspace of $L^2(K_{\alpha,m}, \mu_{\alpha}|_{K_{\alpha,m}})$.

**Lemma 4.9.** The pair $\left(\mathcal{E}_{K_{\alpha,m}}, \mathcal{D}_{K_{\alpha,m}}\right)$ defined in (4.7) is a Dirichlet form on $L^2(K_{\alpha,m}, \mu_{\alpha}|_{K_{\alpha,m}})$.

**Proof.** Any function $u \in L^2(K_{\alpha,m}, \mu_{\alpha}|_{K_{\alpha,m}})$ may be extended by zero to a function $\tilde{u} \in L^2(K_{\alpha}, \mu_{\alpha})$ that can be approximated in the $L^2-$norm by a
sequence \((\tilde{u}_n)_{n \in \mathbb{N}} \subseteq D_K\) such that \(\tilde{u}_n = \tilde{v}_n + \tilde{w}_n\), where \(\tilde{u}_n \in D_{K,a,m}\) and \(\tilde{w}_n \in D_{J,a,m}\) for each \(n \in \mathbb{N}\).

Since \(\text{supp}(\tilde{u}), \text{supp}(\tilde{v}_n) \subseteq K_{a,m}\) and \(\text{supp}(\tilde{w}_n) \subseteq J_{a,m}\), we have that

\[
\| \tilde{u} - \tilde{u}_n \|_{L^2(K_{a,m})} = \| \tilde{u} - \tilde{u}_n \|_{L^2(K_{a,m})}^2 - \int_{J_{a,m}} |\tilde{w}_n|^2 \, d\mu_{a,m} \xrightarrow{n \to \infty} 0,
\]

and thus \(D_{K,a,m}\) is a dense subspace of \(L^2(K_{a,m}, \mu_{a,K_{a,m}})\).

Finally, \((D_{K,a,m}, E_{K,a,m})\) is a Hilbert space because \(D_{K,a,m}\) is a closed subspace of \(D_K\) and the Markov property is inherited from the form \(E_{K,a}\).

Although the following lemma may not seem very deep, it is in fact essential for the proof of Theorem 4.3: here we use the idea of decomposing \(K\) (and therefore the domain of the Laplacian) into two suitable pieces where we have a better control of the eigenvalues.

**Lemma 4.10.** For any \(m \in \mathbb{N}_0\), let \(H_{J,a,m}\) be the non-negative self-adjoint operator on \(L^2(J_{a,m}, \mu_{a,J_{a,m}})\) associated with the Dirichlet form \((E_{J,a,m}, D_{J,a,m})\) and let \(H_{K,a,m}\) be the non-negative self-adjoint operator on \(L^2(K_{a,m}, \mu_{a|K,a,m})\) associated with the Dirichlet form \((E_{K,a,m}, D_{K,a,m})\). Then, \(H_{J,a,m}\) and \(H_{K,a,m}\) have both compact resolvent and for \(N_{J,a,m}(x) := N(x; E_{J,a,m}, D_{J,a,m}) = N_{H_{J,a,m}}(x)\) and \(N_{K,a,m}(x) := N(x; E_{K,a,m}, D_{K,a,m}) = N_{H_{K,a,m}}(x)\), we have that

\[
N_N(x) \leq N_{K,a,m}(x) + N_{J,a,m}(x)
\]

holds for any \(x \geq 0\).

**Proof.** The statements about compactness of the resolvent are proved in Lemma 4.8 and Lemma 4.13 respectively.

On one hand, we have that \(L^2(K_{a,m}, \mu_{a}) = L^2(K_{a,m}, \mu_{a|K,a,m}) \oplus L^2(J_{a,m}, \mu_{a|J,a,m})\) because \(K_{a,m} \cap J_{a,m} = \emptyset\) and by definition of \((E_{K,a,m}, D_{K,a,m})\) we have that \(E_K = E_{K,a,m} \oplus E_{J,a,m}\).

On the other hand, \(D_K \subseteq D_{K,a,m} \oplus D_{J,a,m}\) and it follows from [26, Proposition 4.2, Lemma 4.2] that

\[
N_N(x) \leq N(x; E_K, D_{K,a,m} \oplus D_{J,a,m}) = N_{K,a,m}(x) + N_{J,a,m}(x),
\]

as we wanted to prove.

We recall now the following result from spectral theory of self-adjoint operators.
Lemma 4.11. Let \((E, D)\) be a Dirichlet form on a Hilbert space \(H\) and let \(A\) be the non-negative self-adjoint operator on \(H\) associated with it. Further, define
\[
\kappa(L) := \sup \{ E(u, u) \mid u \in L, \|u\|_H = 1 \}, \quad L \subseteq D \text{ subspace},
\]
and
\[
\kappa_n := \inf \{ \kappa(L) \mid L \text{ subspace of } D, \dim L = n \}.
\]
If the sequence \(\{\kappa_n\}_{n=1}^\infty\) is unbounded, then the operator \(A\) has compact resolvent.

Proof. This follows from \([7, \text{Theorem } 4.5.3]\) and the converse of \([7, \text{Theorem } 4.5.2]\). \(\square\)

The proof of the next lemma will make use of the following so-called uniform Poincaré inequality.

Definition 4.12. A Dirichlet form \((E, D)\) on \(L^2(K_\alpha, \mu_\alpha)\) is said to satisfy the uniform Poincaré inequality if and only if there exists a constant \(C_P > 0\) such that for any \(w \in A^*\) and all \(u \in \{ u \in L^2(K_\alpha, \mu_\alpha^w) \mid \exists v \in D \cap C(K_\alpha), \ u \equiv v \circ G_{\alpha,w} \}\)
\[
E(u, u) \geq C_P \int_{K_\alpha} \left| u - \bar{\mu}_\alpha^w \right|^2 \, d\mu_\alpha^w,
\]
where \(\mu_\alpha^w\) is the measure defined in Remark 3.2 (4) and \(\bar{\mu}_\alpha^w := \int_{K_\alpha} u \, d\mu_\alpha^w\).

In our case, the uniform Poincaré inequality holds for the Dirichlet form \((E_{K_\alpha, D_{K_\alpha}})\) by \([18, \text{Proposition } 4.4]\) because \(F_{K_\alpha} = D_{K_\alpha} \subseteq C(K_\alpha)\) and \((E_{K_\alpha}, F_{K_\alpha})\) is a resistance form whose associated resistance metric is compatible with the original (Euclidean) topology of \(K_\alpha\) by Lemma 2.12.

Lemma 4.13. Let \(m \in \mathbb{N}_0\) and define
\[
\kappa(L) := \sup \left\{ E_{K_\alpha,m}[u] \mid u \in L, \int_{K_{\alpha,m}} |u|^2 = 1 \right\}, \quad L \subseteq D_{K_\alpha,m} \text{ subspace},
\]
and
\[
\kappa_n := \inf \{ \kappa(L) \mid L \text{ subspace of } D_{K_\alpha,m}, \dim L = n \}.
\]
Then, there exists a constant \(C_P > 0\) such that
\[
\kappa_{3m+1} \geq 5^m C_P.
\]
In particular, the non-negative self-adjoint operator on \(L^2(K_\alpha,m, \mu_{\alpha|_{K_\alpha,m}})\) associated with \((E_{K_\alpha,m}, D_{K_\alpha,m})\) has compact resolvent.
The last assertion follows from Lemma 4.11 in view of inequality (4.8). The proof of this inequality uses the same argumentation as [18, Lemma 4.5] but we include the details for completeness.

Let us consider $L_0 := \{ \sum_{w \in A^m} a_w 1_{K_{\alpha,m}} | a_w \in \mathbb{R} \}$, which is a $3^m$-dimensional subspace of $D_{K_{\alpha,m}}$. Note that $E_{K_{\alpha,m}}|_{L_0 \times L_0} \equiv 0$. Now, consider a $(3^m + 1)$-dimensional subspace $L \subseteq D_{K_{\alpha,m}}$ and set $\tilde{L} := L_0 + L$. The bilinear form $E_{K_{\alpha,m}}$ on $\tilde{L}$ is associated with a non-negative self-adjoint operator $A$ satisfying $E_{K_{\alpha,m}}(u,v) = \int_{K_{\alpha,m}} (Au)v d\mu_\alpha$ for all $u,v \in \tilde{L}$.

By the theory of finite-dimensional real symmetric matrices, the $(3^m + 1)$-th smallest eigenvalue of $A$ is given by

\[
\kappa_A := \inf \{ \kappa(L') \mid L' \text{ is a subspace of } \tilde{L}, \dim L' = 3^m + 1 \}.
\]

Let $u_A \in \tilde{L}$ be the eigenfunction corresponding to the eigenvalue $\kappa_A$ and normalize it so that $\int_{K_{\alpha,m}} |u_A|^2 d\mu_\alpha = 1$. Note that this function is orthogonal to $L_0$, so we can apply the Poincaré inequality to it. Now, since $\frac{3}{5} < \frac{2}{1-\alpha}$ and $3^m \mu_\alpha(K_{\alpha,w}) < 1$ for all $w \in A^m$, we have that

\[
\kappa(L) \geq \kappa_A = \kappa_A \int_{K_{\alpha,m}} |u_A|^2 d\mu_\alpha = E_{K_{\alpha,m}}[u_A]
\]

\[
\geq \sum_{w \in A^m} \left( \frac{5}{3} \right)^m E_{K_{\alpha}}[u_A \circ G_{\alpha,w}]
\]

\[
\geq \left( \frac{5}{3} \right)^m \sum_{w \in A^m} C_P \int_{K_{\alpha}} |u_A \circ G_{\alpha,w}|^2 d\mu_\alpha^w
\]

\[
\geq \left( \frac{5}{3} \right)^m \frac{C_P}{\max_{w \in A^m} \{ \mu_\alpha(K_{\alpha,w}) \}} \int_{K_{\alpha,w}} |u_A|^2 d\mu_\alpha
\]

\[
\geq 5^m C_P.
\]

It follows that $\kappa_{3^m+1} \geq 5^m C_P$, as we wanted to prove.

\[\square\]

**Proposition 4.14.** There exist $C_{\alpha,2}, C_{\alpha,\beta,2} > 0$ depending on $\alpha$ and $\beta$, and $x_0 > 0$ such that

\[
N_N(x) \leq C_{\alpha,2} x^{\frac{\ln 3}{\ln 5}} + C_{\alpha,\beta,2} x^{1/2}
\]

(4.9) for all $x \geq x_0$.

**Proof.** Let $x_0 > C_P$ and $x \geq x_0$. Then we can choose $m \in \mathbb{N}$ such that $C_P 5^{m-1} \leq x < C_P 5^m$. From Lemma 4.13 we know that

\[
\kappa_{3^m+1} \geq 5^m C_P > x,
\]

hence $N_{K_{\alpha,m}}(x) \leq 3^{m} \leq C_{\alpha,2} x^{\frac{\ln 3}{\ln 5}}$, where $C_{\alpha,2} := 3C_P^{\frac{\ln 3}{\ln 5}}$. Finally Lemma 4.10 and Lemma 4.8 lead to (4.9). \[\square\]
Lower bound

Let us write $K_\alpha^0 := K_\alpha \setminus V_0$ and $K_{\alpha,w}^0 := G_{\alpha,w}(K_\alpha^0)$ for any $w \in \mathcal{A}^*$ and set $K_{\alpha,m}^0 := \bigcup_{w \in \mathcal{A}^m} K_{\alpha,w}^0$.

Lemma 4.15. Let $m \in \mathbb{N}$ and consider the partition $\mathcal{A}^m$. For any $w \in \mathcal{A}^m$, the operators $H_{K^0_{\alpha,w}}$ and $H_{K^0_{\alpha,m}\cup J_{\alpha,m}}$ have compact resolvent and for any $x > 0$ we have that

$$\sum_{w \in \mathcal{A}^m} N_{K_{\alpha,w}^0}(x) + N_{J_{\alpha,m}}(x) = N_{K_{\alpha,m}^0\cup J_{\alpha,m}}(x) \leq N_D(x).$$

(4.10)

Proof. Note that by definition, $D_U \subseteq D_{K^0_{\alpha}}$ and $\mathcal{E}_U = \mathcal{E}_{K^0_{\alpha}}|_{U \times U}$ for both $U \in \{D_{K^0_{\alpha,m}}, D_{K^0_{\alpha,m}\cup J_{\alpha,m}}\}$ and any $m \in \mathbb{N}$. Since $H_{K^0_{\alpha}}$ has compact resolvent by Theorem 3.3, the minimax principle implies that the operators $H_{K^0_{\alpha,w}}$ and $H_{K^0_{\alpha,m}\cup J_{\alpha,m}}$ also have compact resolvent and the inequality in (4.10) holds.

The equality

$$N_{K_{\alpha,m}^0\cup J_{\alpha,m}}(x) = N_{J_{\alpha,m}}(x) + N_{K_{\alpha,w}^0}(x).$$

(4.11)

follows by the same argumentation as in [18] Lemma 4.8: Let $u \in D_{J_{\alpha,m}}$. Since $K_\alpha \setminus J_{\alpha,m} \subseteq K_{\alpha,m}$, $L_\alpha := \operatorname{supp}_{K_\alpha}(u) \cap K_\alpha \subseteq J_{\alpha,m}$ and therefore $u \cdot 1_{J_{\alpha,m}} \in C(K_\alpha)$ and $\operatorname{supp}_{K_\alpha}(u \cdot 1_{J_{\alpha,m}}) \subseteq J_{\alpha,m}$. Since $L_\alpha$ is compact and $J_{\alpha,m}$ is open, we know by [10] Exercise 1.4.1 that we can find a function $\varphi \in \mathcal{D}_K$ such that $\varphi \geq 0$, $\varphi|_{L_\alpha} \equiv 1$ and $\varphi|_{K_{\alpha,m}} \equiv 0$. Then, $u \cdot 1_{J_{\alpha,m}} = w \psi \in \mathcal{D}_{J_{\alpha,m}}$ and $u \cdot 1_{J_{\alpha,m}} \in C_{J_{\alpha,m}}$ (recall Definition 4.6).

Similarly, if $u \in \mathcal{D}_{K_{\alpha,m}^0}$ and $\tilde{L}_\alpha := \operatorname{supp}_{K_\alpha}(u) \cap K_{\alpha,m} \subseteq K_{\alpha,m}$, we can find $\psi \in \mathcal{D}_{K_\alpha}$ such that $\psi \geq 0$, $\psi|_{\tilde{L}_\alpha} \equiv 1$ and $\psi|_{J_{\alpha,m}} \equiv 0$. Thus $u \cdot 1_{K_{\alpha,m}} = w \psi \in \mathcal{D}_{K_{\alpha,m}^0}$ and we have that $\mathcal{E}_{K_{\alpha,m}^0\cup J_{\alpha,m}} = \mathcal{E}_{K_{\alpha,m}} \oplus \mathcal{E}_{J_{\alpha,m}}$, both spaces being orthogonal to each other with respect to $\mathcal{E}_{K_\alpha}$ and the inner product of $L^2(K_\alpha, \mu_\alpha)$. Taking the closure with respect to $\mathcal{E}_{K_\alpha,1}$ we get that $\mathcal{D}_{K_{\alpha,m}^0\cup J_{\alpha,m}} = \mathcal{D}_{K_{\alpha,m}^0} \oplus \mathcal{D}_{J_{\alpha,m}}$, where both spaces keep being orthogonal to each other. Hence (4.11) follows.

The equality

$$N_{K_{\alpha,m}^0}(x) = \sum_{w \in \mathcal{A}^m} N_{K_{\alpha,w}^0}(x)$$

follows by an analogous argument and the inequality (4.10) is therefore proved.

For the proof of the next lemma we need to introduce the following identification mapping. Consider $\{\mathbb{R}^2; S_i, i = 1, 2, 3\}$ the IFS associated with the Sierpiński gasket $K$ and the set $V_0 = \bigcup_{n \in \mathbb{N}_0} \bigcup_{w \in \mathcal{A}^n} S_w(V_0)$. 30
Moreover, recall the IFS \( \{ \mathbb{R}^2; G_{\alpha,i}, i = 1, \ldots, 6 \} \) associated with \( K_\alpha \) and consider the set \( W_{\alpha,*} := \bigcup_{n \in \mathbb{N}_0} \bigcup_{w \in \mathcal{A}^n} G_{\alpha,w}(V_0) \). We know that for any \( x \in W_{\alpha,*} \), there exists a word \( w^x \in \mathcal{A}^* \) such that \( x = G_{\alpha,w^x}(p_i) \) for some \( p_i \in V_0 \), so we can define
\[
I: W_{\alpha,*} \rightarrow V_*,
\]
\[
x \mapsto S_{w^x}(p_i).
\]
This mapping allows us to construct functions in \( \mathcal{D}_{K_\alpha} \) from functions in the domain of the classical Dirichlet form \( (\mathcal{E}_K, \mathcal{D}_K) \) on \( K \) (see e.g. [24] for definitions and details about this form).

For any function \( u \in \mathcal{D}_K \), we define the function \( u_\alpha : V_{\alpha,*} \rightarrow \mathbb{R} \) by
\[
u_{\alpha}(x) := \begin{cases}
   u \circ I(x), & x \in W_{\alpha,*}, \\
   u \circ I(a_e), & x \in [a_e, b_e], e \in J_\alpha,
\end{cases}
\] (4.12)
which is well defined since \( I(a_e) = I(b_e) \) for all \( e \in J_\alpha \).

Then
\[
\mathcal{E}_{K_\alpha}[u_\alpha] = \lim_{n \rightarrow \infty} \frac{3^n}{5^n \rho_{\alpha,n}^d} \left( \frac{3}{5} \right)^{-n} E_{\alpha,n}^d[u],
\]
and since
\[
L := \lim_{n \rightarrow \infty} \frac{3^n}{5^n \rho_{\alpha,n}^d} \leq \lim_{n \rightarrow \infty} \left( 1 + \frac{\alpha}{5} \left( \frac{1 - \alpha}{2} \right)^{n-1} \right) 2^n < \infty,
\]
we get that
\[
\mathcal{E}_{K_\alpha}[u_\alpha] = E_{K_\alpha}^d[u_\alpha] = L \cdot \mathcal{E}_K[u] < \infty,
\]
hence \( u_\alpha \in \mathcal{D}_{K_\alpha} \).

**Lemma 4.16.** Let \( m \in \mathbb{N} \). There exists \( C_D \geq 0 \) such that for all \( w \in \mathcal{A}^m \)
\[
\kappa_1(K_{\alpha,w}^0) := \inf_{u \in \mathcal{C}_{K_{\alpha,w}^0}^{\alpha}} \left\{ \frac{\mathcal{E}_{K_\alpha}[u]}{\|u\|_{L^2(K_{\alpha,w}^0)}} \right\} \leq 5^m C_D.
\] (4.13)

**Proof.** Let \( v \in \mathcal{A}^* \) such that \( S_v(K) \subseteq K \setminus V_0 \) and consider \( u \in \mathcal{D}_K^0 \) a function such that \( \text{supp}_K(u) \subseteq K \setminus V_0 \) and \( u \equiv 1 \) on \( S_v(K) \) (such a function exists by [10], Exercise 1.4.1) because \( S_v(K) \) is compact and \( K \setminus V_0 \) is open). The function \( u_\alpha \in \mathcal{D}_{K_\alpha}^0 \) defined as in (4.12) has by construction the property that \( u_\alpha \equiv 1 \) on \( K_{\alpha,w}^0 \). Now, for any \( w \in \mathcal{A}^m \) we define the function
\[
w^w(x) := \begin{cases}
   u_\alpha \circ G^{-1}_{\alpha,w}(x), & x \in K_{\alpha,w}^0, \\
   0, & x \in K_\alpha \setminus K_{\alpha,w}^0.
\end{cases}
\]
Then, \( u^w \in \mathcal{C}_{K_{\alpha,w}} \), and by Corollary 2.16 we have that

\[
E_{K_{\alpha}}[u^w] = \left(\frac{5}{3}\right)^m \sum_{w' \in A^m} E_{K_{\alpha}}^d [u^w \circ G_{\alpha,w'}] + \left(\frac{2}{1 - \alpha}\right)^m \sum_{w' \in A^m} E_{K_{\alpha}}^c [u^w \circ G_{\alpha,w'}]
\]

\[
+ \Theta^{-1} \sum_{k=0}^{m-1} \left(\frac{2}{1 - \alpha}\right)^k \sum_{w' \in A^k} E_{\alpha,1}^c [u^w \circ G_{\alpha,w'}]
\]

\[
= \left(\frac{5}{3}\right)^m E_{K_{\alpha}}^d [u^w \circ G_{\alpha,w}] = \left(\frac{5}{3}\right)^m L E_K[u].
\]

(4.14)

On the other hand,

\[
\int_{K_{\alpha}} |u^w(x)|^2 d\mu_\alpha(x) \geq \int_{K_{\alpha,w}} d\mu_\alpha(G_{\alpha,w}(y)) = \mu_\alpha(K_{\alpha,w}) \geq \gamma |v| \mu_\alpha(K_{\alpha,w})
\]

(4.15)

for some \( \gamma \in (0, \infty) \) as in Lemma 4.5. From inequalities (4.14) and (4.15) and the fact that \( 3^m \mu_\alpha(K_{\alpha,w}) > \frac{1}{2} \), we obtain

\[
\inf_{u \in \mathcal{C}_{K_{\alpha,w}}^0, u \neq 0} \left\{ \frac{E_{K_{\alpha}}[u]}{\|u\|_{L^2(K_{\alpha,0})}^2 d\mu_\alpha} \right\} \leq \frac{\frac{\left(\frac{5}{3}\right)^m L E_K[u]}{\gamma |v| \mu_\alpha(K_{\alpha,w})}}{\int_{K_{\alpha,w}} |u|^{2} d\mu_\alpha} \leq \frac{\frac{\left(\frac{5}{3}\right)^m L E_K[u]}{\gamma |v| \mu_\alpha(K_{\alpha,w})}}{\int_{K_{\alpha,w}} |u|^{2} d\mu_\alpha} \leq C_D 5^m,
\]

where \( C_D := \frac{2LE_K[u]}{\gamma |v|} \) is independent of \( w \). Thus, inequality (4.13) follows for all \( w \in A^m \).

Now we can prove the lower bound of Theorem 4.3

**Proposition 4.17.** There exist \( C_{\alpha,1}, C_{\alpha,\beta,1} > 0 \) depending on \( \alpha \) and \( \beta \), and \( x_0 > 0 \) such that

\[
C_{\alpha,1} x^{\ln 2}_{m} + C_{\alpha,\beta,1} x^{1/2} \leq N_D(x)
\]

for all \( x \geq x_0 \).

**Proof.** For \( x \geq C_D \), choose \( m \in \mathbb{N}_0 \) such that \( C_D 5^m \leq x < C_D 5^{m+1} \). We know from Lemma 4.10 that

\[
\kappa_1(K_{\alpha,w}^0) \leq C_D 5^m \quad \forall w \in A^m,
\]

which implies that \( N_{K_{\alpha,w}^0}(x) \geq 1 \) for all \( w \in A^m \). 32
By Lemmas 4.15 and 4.8 we get that

\[ N_D(x) \geq \sum_{w \in A^m} N_{K_0,w}^0(x) + N_{J_{a,m}}(x) \geq C_{\alpha,1} x^{\frac{\ln 3}{\ln 5}} + C_{\alpha,\beta,1} x^{1/2}, \]

where \( C_{\alpha,1} := \frac{1}{3} C_D^{-\frac{\ln 3}{\ln 5}} \).

Finally, we are ready to prove Theorem 4.3

**Proof of Theorem 4.3.** First note that the Dirichlet form \((E_{K_0}, D_{K_0})\) corresponds to \((E_{K_0}, D_{K_0})\) in the notation of Definition 4.6. By Theorem 3.3, its associated non-negative self-adjoint operator on \(L^2(K_0, \mu_0)\) has compact resolvent and since \(D_{K_0} \subseteq D_{K_}\) and \(E_{K_0} = E_{K_0}^{D_{K_0} \times D_{K_0}}\), it follows from the minimax principle that \(N_D(x) \leq N_N(x)\) for any \(x \geq 0\).

Finally, consider \(x_0 > \max\{C_P, C_D\}\). Then Propositions 4.14 and 4.17 provide the first and third inequality and the theorem is proved.

**Corollary 4.18.** For any \(\alpha \in (0, 1/3)\), it holds that

\[ d_S(K_\alpha) = \frac{2 \log 3}{\log 5} = d_S(K). \]

**Proof.** Looking to the definition of spectral dimension in (4), the assertion follows from Theorem 4.3 and the previous Remark.

### 5 Conclusions and open problems

As pointed out in the introduction, any local regular Dirichlet form as \((E_{K_\alpha}, D_{K_\alpha})\) has an associated diffusion process \((X_t)_{t \geq 0}\). The space-time relation of this process is given by the so-called walk dimension. If one has Li-Yau type sub-Gaussian estimates for the heat kernel,

\[ p(t, x, y) \geq \frac{C_1}{\mu(B_d(x, t^{1/\delta}))} \exp \left( -C_2 \left( \frac{d(x, y)^\delta}{t} \right)^{1/(\delta-1)} \right), \]

then the walk dimension coincides with the parameter \(\delta\) of the estimate (see [16, Example 3.2] for the case of the Sierpiński gasket).

Spectral dimension and walk dimension are in general related by the so-called Einstein relation

\[ d_Sd_w = 2d_H, \]

33
where $d_H$ denotes the Hausdorff dimension of the set. This relation shows the connection between three fundamental points of view on a set, namely analysis, probability theory and geometry.

The Einstein relation has not yet been proven to hold in general but it is known to be truth in the case of the Sierpiński gasket (see e.g. [8]). The case of Hanoi attractors seems to be quite interesting because of the fact that

$$d_H(K_\alpha) < d_S(K_\alpha) \quad \forall \alpha \in \left(1 - \frac{2}{\sqrt{5}}, \frac{1}{3}\right).$$

In case this relation holds, then we get $d_w(K_\alpha) < 2$ for $\alpha \in (1 - \frac{2}{\sqrt{5}}, \frac{1}{3})$. This would mean that the diffusion process associated with the Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ for these $\alpha$’s moves faster than two-dimensional Brownian motion. Of course, this superdiffusive behavior is brought by the properties of the measure $\mu_{\alpha,\beta}$ giving high conductance to the very small wires in the set. However, the process is still a diffusion and has no jumps. This apparent contradiction with the by now established models for fractal networks arises many interesting questions that should be investigated. Answering these questions may have applications in the design of “superconductors”.

We would also like to note that the resulting process can perhaps be understood as asymptotically lower dimensional (ALD). Such processes were first treated in the context of abc-gaskets in [15], and studied later on Hambly and Kumagai in [13] for some particular nested fractals.

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