Lipschitz Trivial Values of Polynomial Mappings

André Costa¹ · Vincent Grandjean¹ · Maria Michalska¹,²

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Abstract
We prove that a polynomial mapping \( f : \mathbb{K}^n \to \mathbb{K}^p \), where \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), attains a Lipschitz trivial value \( c \) if and only if there exist a polynomial mapping \( g : \mathbb{K}^m \to \mathbb{K}^p \), for which the value \( c \) is a regular value of properness, and a linear surjective projection \( \pi : \mathbb{K}^n \to \mathbb{K}^m \) such that \( f = g \circ \pi \). The integer \( m \) is the \( \mathbb{K} \)-codimension of the accumulation set at infinity of the level \( f^{-1}(c) \) in the hyperplane at infinity. In the complex case, it is equivalent to require the mapping \( g \) be generically finite and dominant. Last, we show this result cannot extend to rational mappings over \( \mathbb{K}^n \).

Keywords Polynomial mapping · Lipschitz fibre bundle · Proper mapping · Bifurcation value

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1 Introduction

Let \( f : \mathbb{K}^n \to \mathbb{K}^p \) be a dominant polynomial mapping over \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). There exists a smallest subset \( \text{Bif}_k(f) \), contained in an algebraic subset of \( \mathbb{K}^p \) of positive codimension, with the following property: the mapping \( f \) induces a locally trivial \( \mathcal{C}^k \)
fibre bundle structure over each connected component of $\mathbb{K}^p \setminus \text{Bif}_k(f)$. The notion of $C^\infty$ triviality is generally finer than that of $C^0$ triviality and over the last fifty years, following the seminal paper [10], a significant literature about the $C^0$ and the $C^\infty$ local triviality has been developed (see for instance [3, 5, 6, 9, 11]).

This paper investigates an intermediate case: when and over which subset of values the mapping induces a locally bi-Lipschitz trivial fibre bundle structure. Our goal is to characterize polynomial mappings admitting \emph{Lipschitz trivial values}, that is over a neighbourhood of which there is a bi-Lipschitz trivialization, problem recently raised in [1]. Our main result Theorem 4.1 when combined with Proposition 3.8 implies the following

**Theorem** Let $f : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial mapping and let $n - 1 - m$ be the dimension of the set of accumulation points at infinity of the fibre $f^{-1}(c)$. The mapping $f$ attains the Lipschitz trivial value $c$ if and only if
\[
f = g \circ \pi,
\]
for a linear surjective projection $\pi : \mathbb{K}^n \to \mathbb{K}^m$ and a polynomial mapping $g : \mathbb{K}^m \to \mathbb{K}^p$ for which $c$ is a regular value of properness.

In the complex case, the statement equivalently requires that $m = p$ and the polynomial mapping $(g : \mathbb{C}^p \to \mathbb{C}^p)$ be dominant and generically finite (see Corollary 5.1). Therefore, either almost all values of a complex polynomial mapping are Lipschitz trivial or there are none. In contrast with the complex case, there exist non-proper real polynomial mappings admitting values of properness.

The main result completely describes the real and complex polynomial mappings admitting Lipschitz trivial values, recovering the case of complex polynomial functions of [1]. Our proof is very different, more concise, and covers both real and complex cases. Moreover, we show that the main theorem cannot extend without further hypotheses to a wider class of rational mappings (even those which are regulous, see [2]).

The article is organized as follows. Section 3 presents properties of Lipschitz trivial values of differentiable mappings. The main result is proved in Sect. 4, while Sect. 5 describes the set of Lipschitz trivial values in the real and complex cases. Section 6 deals with the relation between Lipschitz trivial values of a real mapping and that of its complexification. In Sect. 7, we show that the main result cannot extend to rational mappings.

### 2 Preliminaries

Throughout the paper $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We use the conventions that $\dim \emptyset = -1$ and empty mappings are proper and continuous.

We embed the affine space $\mathbb{K}^n$ in $\mathbb{K}P^n$ as $x \mapsto [x : 1]$. The hyperplane at infinity $H_\infty := \mathbb{K}P^n \setminus \mathbb{K}^n$ consists of the points of $\mathbb{K}P^n$ of the form $[v : 0]$. 

[Springer]
Definition 2.1 The accumulation set at infinity $X^\infty$ of a subset $X$ of $\mathbb{K}^n$ is defined as

$$X^\infty := \overline{X}^{\mathbb{K}^n} \cap H^\infty,$$

where $\overline{X}^{\mathbb{K}^n}$ is the closure of $X$ taken in $\mathbb{K}^n$.

The following notions lie at the heart of our problem.

Definition 2.2 A mapping $\varphi : \mathbb{K}^n \to \mathbb{K}^p$ is proper at the value $c \in \mathbb{K}^p$, if there exists a neighbourhood $V$ of $c$ in $\mathbb{K}^p$ such that the restriction mapping of $\varphi$ to $\varphi^{-1}(V)$ is proper. Let us denote by $J(\varphi)$ the Jelonek set of values at which the mapping $\varphi$ is not proper.

The Jelonek set of a real or complex polynomial mappings is always contained in an algebraic set of dimension at most $n - 1$, see [4, 5].

Definition 2.3 A mapping $\varphi : \mathbb{K}^n \to \mathbb{K}^p$ is topologically trivial at the value $c \in \mathbb{K}^p$, if there exist a neighbourhood $V$ of $c$ in $\mathbb{K}^p$ and a trivializing homeomorphism

$$H : \varphi^{-1}(c) \times V \to \varphi^{-1}(V) \quad (1)$$

which satisfies $(\varphi \circ H)(x, t) = t$.

When $H$ is a $C^\infty$ diffeomorphism, the mapping $\varphi$ is called $C^\infty$ trivial at the value $c$. The complement of the set of values at which the mapping $\varphi$ is $C^\infty$ trivial is the set of bifurcation values $\text{Bif}(\varphi)$, see [10].

Remark 2.4 In particular, the mapping $\varphi$ is locally trivial at any value of the open subset $\mathbb{K}^p \setminus \text{clos}(\text{Im}(\varphi))$, the complement of the closure of the image of $\varphi$.

Example 2.5 Note that the real polynomial function $x \mapsto x^{2021}$ is topologically trivial at each $c \in \mathbb{R}$, but not $C^\infty$ trivial at 0.

Let $K_0(\varphi) := \varphi(\text{crit}(\varphi))$ be the set of critical values of $\varphi$. The mapping $\varphi$ is said to be regular at $c$ once $c \notin K_0(\varphi)$. When a polynomial mapping $\varphi$ is proper and regular at a value $c$, the mapping $\varphi$ is $C^\infty$ trivial at $c$ by Sard’s and Ehresmann’s Theorems. Thus the set $\text{Bif}(\varphi) \setminus K_0(\varphi)$ contains only non-proper values when it is non-empty. Moreover, by [5], when $\varphi$ is polynomial and $n = p$ we get $\text{Bif}(\varphi) = K_0(\varphi) \cup J(\varphi)$.

### 3 Lipschitz Trivial Values

In this section, we present general properties of differentiable mappings with Lipschitz trivial values.

Let again $\varphi : \mathbb{K}^n \to \mathbb{K}^p$ be a mapping.

Any subset $X$ of $\mathbb{K}^n$ inherits a metric structure from restricting the ambient Euclidean distance to $X$, this metric is called the outer metric on $X$. On a product $X \times Y$ of metric spaces we will consider the product metric.
Definition 3.1 A Lipschitz trivial value \( c \in \mathbb{K}^p \) of the mapping \( \varphi \) is a value such that there exists a trivializing homeomorphism as in (1) of Definition 2.3 which is bi-Lipschitz with respect to the outer metrics. Denote by \( L(\varphi) \) the set of Lipschitz trivial values of \( \varphi \).

Remark 3.2 In light of Definition 3.1 and Remark 2.4, the subset \( L(\varphi) \) is open and contains \( \mathbb{K}^p \setminus \text{clos}(\text{Im}(\varphi)) \). In particular, the mapping \( \varphi \) attains a Lipschitz trivial value only if its image is Zariski dense in \( \mathbb{K}^p \).

Example 3.3 The set of Lipschitz trivial values of the polynomial mapping \( \mathbb{K}^3 \to \mathbb{K}^2 \) defined as \( (x, y, z) \mapsto (x, xy + xz) \) is \( \mathbb{K}^2 \setminus 0 \times \mathbb{K} \). There is a single critical value \( (0, 0) \), and none of the values \( (0, b) \) with \( b \neq 0 \) is taken. Moreover, each level \( (a, b) \) with \( a \neq 0 \) is an affine line.

The next result emphasizes the rigid asymptotic behaviour of levels near a Lipschitz trivial value.

Proposition 3.4 Assume that \( c \) is a Lipschitz trivial value of \( \varphi : \mathbb{K}^n \to \mathbb{K}^p \). There exists a neighbourhood \( V \) of \( c \) such that the following properties hold:

(i) the mapping \( \varphi \) is Lipschitz on \( \varphi^{-1}(V) \);

(ii) there exist \( 0 < \delta < \epsilon \) such that

\[
T_{\delta}(\varphi^{-1}(c)) \subset \varphi^{-1}(V) \subset T_{\epsilon}(\varphi^{-1}(c)),
\]

where the open tube \( T_r(S) \) of radius \( r \) around a subset \( S \) of \( \mathbb{K}^n \) is defined as

\[
T_r(S) := \{ x \in \mathbb{K}^n : \text{dist}(x, S) < r \}.
\]

Proof Denote by \( d_c \) the outer metric on \( \varphi^{-1}(c) \) and by \( U := \varphi^{-1}(V) \). Since \( \varphi \) has a Lipschitz trivial value at \( c \), there exist an open ball \( V = B^p(c, r) \) of \( \mathbb{K}^p \) and a bi-Lipschitz homeomorphism

\[
G = (\varphi, \psi) : U \mapsto V \times \varphi^{-1}(c).
\]

Therefore, there exists \( L > 1 \) such that for any \( x, x' \in U \) we have

\[
\frac{1}{L} \| x - x' \| \leq \| G(x) - G(x') \| \leq L \| x - x' \|.
\]

Point (i) follows from inequalities (2) since for all \( x, x' \in U \) we find

\[
\| \varphi(x) - \varphi(x') \| \leq \| \varphi(x) - \varphi(x') \| + d_c(\psi(x), \psi(x')) \leq L \| x - x' \|.
\]

To prove (ii), define the following radii

\[
\delta := \frac{r}{L} \quad \text{and} \quad \epsilon := Lr.
\]
Point (i) yields the first inclusion $T_\delta(\varphi^{-1}(c)) \subset \mathcal{U}$. Note that for $c, t \in V$ and $x' \in \varphi^{-1}(t)$, there exists $x \in \varphi^{-1}(c)$ such that $\psi(x) = \psi(x')$. Therefore, estimates (2) provide

$$\frac{1}{L} \|x - x'\| \leq \|G(x) - G(x')\| = \|c - t\| + \text{dist}(\psi(x), \psi(x')) = \|c - t\|. $$

Thus, we obtain $\mathcal{U} \subset T_\epsilon(\varphi^{-1}(c))$. \qed

**Remark 3.5** Point (i) of Proposition 3.4 implies that each first-order partial derivative of each component of the mapping $\varphi$ is bounded over $\varphi^{-1}(V)$.

**Property 3.6** Let $\tau : K^m \to K^p$ be a mapping Lipschitz trivial at the value $c$ and $\pi : K^n \to K^m$ be a linear surjective projection. Then the mapping $\tau \circ \pi : K^n \to K^p$ is Lipschitz trivial at the value $c$.

**Proof** Let $\varphi := \tau \circ \pi$. Up to a $K$-linear change of coordinates in $K^n$, we can assume that for any subset $V$ of $K^p$ the following holds true

$$\varphi^{-1}(V) = \tau^{-1}(V) \times K^{n-m}. $$

Denote $(u, v) \in K^m \times K^{n-m} = K^n$. If the bi-Lipschitz homeomorphism $G : \tau^{-1}(V) \mapsto V \times \tau^{-1}(c)$ provides trivialization of $\tau$ over a neighbourhood $V$ of $c$, then the mapping

$$H : \varphi^{-1}(V) \to V \times \varphi^{-1}(c), \quad (u, v) \mapsto (G(u), v)$$

is a bi-Lipschitz homeomorphism trivializing $\varphi$ over $V$. \qed

**Property 3.7** Let $\varphi : K^n \to K^p$ be a smooth mapping with a nowhere dense set of critical values. Any regular value of $\varphi$ that is also a value of properness is a Lipschitz trivial value.

**Proof** Let $V$ be a non-empty open subset of $K^p$ such that $\varphi$ is proper over $V$. We can further assume that $\text{clos}(V)$ does not intersect with $K_0(\varphi)$. Therefore, the mapping $\varphi$ is $C^\infty$ locally trivial over $V$ by Ehresmann’s Theorem. The restriction of any $C^\infty$ trivialization of $\varphi$ over $V$ to any open subset $\mathcal{U}$ relatively compact in $V$ is necessarily bi-Lipschitz over $\mathcal{U}$. \qed

**Proposition 3.8** Let $\varphi : R^n \to R^p$ be a $C^k$ mapping with $k \geq \max(n - p + 1, 1)$. Any Lipschitz trivial value of $\varphi$ is a regular value.

The proof will follow from the next result.

**Lemma 3.9** Let $f : \mathcal{U} \to \mathcal{V}$ be a $C^1$ mapping, where $\mathcal{U}$ and $\mathcal{V}$ are open subsets of $R^n$ and $R^p$, respectively. If there exists a bi-Lipschitz homeomorphism

$$(f, \psi) : \mathcal{U} \to \mathcal{V} \times F$$

where $F$ is a $C^1$ sub-manifold of some $R^q$, then $f$ has no critical points in $\mathcal{U}$. \qed

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Proof Let $H : \mathcal{V} \times F \to U$ be the inverse of the bi-Lipschitz homeomorphism $(f, \psi)$. Since $H$ is bi-Lipschitz, it is differentiable almost everywhere. At a point $y_0$ where $H$ is differentiable, its differential $D_{y_0}H$ has rank $n$. Moreover, having $H$ bi-Lipschitz implies that any limit of the form $D = \lim_{y_n \to y} D_{y_n}H$ at the given point $y$ of $\mathcal{V} \times F$ has also rank $n$. Therefore the differential of $f$ has rank $p$ at each point of $U$. \qed

Proof of Proposition 3.8 By hypothesis $\varphi$ satisfies Sard’s Theorem. Let $c$ be a Lipschitz trivial value of $\varphi$.

If $c$ does not lie in $\operatorname{Im}(\varphi)$, the image of $\varphi$, then it belongs to $\mathbb{R}^p \setminus \operatorname{clos}(\operatorname{Im}(\varphi))$, thus is a regular value.

Assume $c$ is a value taken by $\varphi$. Let $\mathcal{V}$ be an open neighbourhood of $c$ over which $\varphi$ is Lipschitz trivial. Thus, by Lemma 3.9, the mapping $\varphi$ has no critical point in $\varphi^{-1}(\mathcal{V})$, therefore $c$ is a regular value. \qed

Property 3.10 Let $\varphi : \mathbb{K}^n \to \mathbb{K}^p$ be a mapping locally Lipschitz trivial over the connected open subset $\mathcal{V} \subset \mathbb{K}^p$. Then the family $(\varphi^{-1}(t))_\infty$ of accumulation sets at infinity of the levels of $\varphi$ is constant, i.e.

$$\varphi^{-1}(t)_\infty = \varphi^{-1}(\mathcal{V})_\infty, \quad \forall t \in \mathcal{V}.$$ 

Proof Consider two sequences $(x_k)_k$ and $(x'_k)_k$ of $\mathbb{K}^n$ satisfying the following property: there exists a positive constant $A$ such that

$$|x_k - x'_k| \leq A \quad \text{for } k \gg 1.$$ 

If furthermore $|x_k|$ goes to $\infty$ and does so such that $[x_k : 1] \to [\lambda : 0] \in H_\infty$ as $k \to \infty$, then we deduce that $|x'_k|$ goes to $\infty$ and $[x'_k : 1] \to [\lambda : 0]$ as $k$ goes to $\infty$.

Let $c \in \mathcal{V}$ be a Lipschitz trivial value of $\varphi$. Up to taking a smaller $\mathcal{V}$ containing $c$, point (ii) of Proposition 3.4 states that $\varphi^{-1}(\mathcal{V})$ is contained in the open tube $T_\varepsilon(\varphi^{-1}(c))$ for some positive radius $\varepsilon$. The first part of the proof then gives the result. \qed

4 Nonproper Polynomial Mappings with Lipschitz Trivial Values Depend on Fewer Variables

Let $f : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial mapping. The level $f^{-1}(t)$ is denoted by $F_t$.

Theorem 4.1 Let $f : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial mapping with $\dim F_c^\infty = n - 1 - m$ for a value $c \in \mathbb{K}^p$. If the mapping $f$ attains $c$ as a Lipschitz trivial value, then there exist a polynomial mapping $g : \mathbb{K}^m \to \mathbb{K}^p$ which is proper at $c$ and a linear surjective projection $\pi : \mathbb{K}^n \to \mathbb{K}^m$ such that

$$f = g \circ \pi.$$ 

We start with the following key result.
Lemma 4.2 Let \( f : \mathbb{K}^n \to \mathbb{K}^p \) be a polynomial mapping and assume that the point \([1 : 0 : \cdots : 0]\) lies in \( F^\infty_c \). If there exists a neighbourhood \( \mathcal{V} \) of \( c \) such that \( f \) satisfies points (i) and (ii) of Proposition 3.4, then the mapping \( f \) does not depend on the coordinate \( x_1 \).

**Proof** Since \([1 : 0 : \cdots : 0] \in F^\infty_c \), there exists an arc \( \gamma : I \to F_c \) parametrized as

\[
\gamma(t) = (t^d, p(t) + A_0(1/t)) \in \mathbb{K} \times \mathbb{K}^{n-1},
\]

where \( I \) is a connected component of the complement of an Euclidean ball of \( \mathbb{K} \), the mapping \( p : \mathbb{K} \to \mathbb{K}^{n-1} \) is polynomial of degree \( \leq d - 1 \), and \( A_0 \) is a \( \mathbb{K} \)-analytic map germ \((\mathbb{K}, 0) \to (\mathbb{K}^{n-1}, 0)\).

Consider the following dominant polynomial mapping

\[
\Gamma : \mathbb{K} \times \mathbb{K}^{n-1} \to \mathbb{K}^n, \quad (t, \varepsilon) \mapsto (t^d, p(t) + \varepsilon).
\]

Let \( \gamma_\varepsilon : \mathbb{K} \to \mathbb{K}^n \) be the polynomial arc

\[
\gamma_\varepsilon : t \mapsto \gamma_\varepsilon(t) := \Gamma(t, \varepsilon).
\]

Since \( A_0(1/t) \to 0 \) as \( |t| \to \infty \), we conclude that

\[
\|\gamma_\varepsilon(t) - \gamma(t)\| = \|\varepsilon - A_0(1/t)\| \to \|\varepsilon\|.
\]

Take a neighbourhood \( \mathcal{V} \) of \( c \) in \( \mathbb{K}^n \) such that \( f \) satisfies points (i) and (ii) of Proposition 3.4. Point (ii) of Proposition 3.4 and the definition of \( \Gamma \) guarantee the existence of constants \( \delta > 0 \) and \( R > 0 \) such that

\[
\gamma_\varepsilon(t) \in f^{-1}(\mathcal{V}) \quad (3)
\]

for any \( \|\varepsilon\| < \delta \) and \( t \in I \) such that \( |t| > R \). Since \( f \) is Lipschitz on \( f^{-1}(\mathcal{V}) \), we get

\[
\|f(\gamma_\varepsilon(t)) - c\| \leq L \cdot \|\varepsilon - A_0(1/t)\| \to L \cdot \|\varepsilon\|
\]

as \( |t| \to \infty \). Therefore for each \( \|\varepsilon\| < \delta \), the polynomial mapping \( t \to f \circ \gamma_\varepsilon(t) \) is bounded, thus constant. Writing \( x = (x_1, y) \), we deduce

\[
0 \equiv \frac{d}{dt}(f \circ \gamma_\varepsilon)(t) = d \cdot t^{d-1} \cdot \partial_{x_1}f(\gamma_\varepsilon(t)) + \partial_yf(\gamma_\varepsilon(t)) \cdot p'(t).
\]

Since \( p \) has degree \( \leq d - 1 \) and the first-order partial derivatives of \( f \) are bounded along \( \gamma_\varepsilon \) by Remark 3.5, for \( \|\varepsilon\| < \delta \) we conclude that

\[
(\partial_{x_1}f) \circ \gamma_\varepsilon \equiv 0.
\]

Since the subset \( \mathbb{K} \times \{ \varepsilon : \|\varepsilon\| < \delta \} \subset \mathbb{K}^n \) is open and non-empty and the mapping \( \Gamma \) is dominant, we conclude that the mapping \( \partial_{x_1}f \) is identically null. \( \square \)
Proof of Theorem 4.1} Note that $f$ satisfies the claim of Proposition 3.4. If $\dim F_c^\infty = -1$, then the fibre $F_c$ is compact and from point (ii) of Proposition 3.4 the subset $f^{-1}(\mathcal{V})$ is compact for a small compact neighbourhood $\mathcal{V}$ of $c$. Thus $f$ is proper at $c$ and taking $\pi$ as the identity mapping of $\mathbb{K}^n$ yields the claim.

Assume $F_c^\infty$ is of dimension $n - 1 - m \geq 0$. Thus there exist $n - m$ points $[v_1 : 0], \ldots, [v_{n-m} : 0]$ of $F_c^\infty$ such that the vectors $v_1, \ldots, v_{n-m}$ are $\mathbb{K}$-linearly independent in $\mathbb{K}^n$. Take a $\mathbb{K}$-linear change of coordinates $\ell : \mathbb{K}^n \to \mathbb{K}^n$ such that $(\ell(e_j) = v_j)$ for $j = 1, \ldots, n - m$, where $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis of $\mathbb{K}^n$. Applying Lemma 4.2 we conclude that the polynomial mapping $f \circ \ell$ depends only on $u := (x_{n-m+1}, \ldots, x_n)$. Let $g$ be the polynomial mapping restriction of $f \circ \ell$ to $\mathbb{K}^m$, the subspace of $\mathbb{K}^n$ generated by $e_{n-m+1}, \ldots, e_n$. Let $\pi_0 : \mathbb{K}^n \to \mathbb{K}^m$ be the orthogonal projection of $\mathbb{K}^n$ onto the subspace $\mathbb{K}^m$. Therefore, we find

$$f = g \circ \pi_0 \circ \ell^{-1}.$$ 

Note that $(f \circ \ell)^{-1}(t) = \mathbb{K}^{n-m} \times g^{-1}(t)$. Since

$$n - 1 - m = \dim F_c^\infty = n - m + \dim g^{-1}(c)^\infty,$$

we deduce that $g^{-1}(c)$ is compact. From Proposition 3.4 applied to the levels of $f \circ \ell$ over $\mathcal{V}$ we get that $g^{-1}(\mathcal{V})$ is bounded. Thus $c$ is a value of properness of $g$. \qed

5 On the Set of Lipschitz Trivial Values of Real and Complex Mappings

This section presents some consequences of Theorem 4.1. In particular, complex polynomial mappings admitting Lipschitz trivial values have a very rigid structure, while the real setting allows for more variety.

Corollary 5.1 A complex polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^p$ attains a Lipschitz trivial value if and only if there exist a dominant polynomial mapping $g : \mathbb{C}^p \to \mathbb{C}^p$ and a linear surjective projection $\pi : \mathbb{C}^n \to \mathbb{C}^p$ such that

$$f = g \circ \pi.$$ 

In such a case we get

$$L(f) = \mathbb{C}^p \setminus \text{Bif}(g).$$

In particular, the set of regular Lipschitz trivial values is either empty or the complement of an algebraic hypersurface.

Proof Assume there exists a dominant polynomial mapping $g : \mathbb{C}^p \to \mathbb{C}^p$ such that $f = g \circ \pi$ for some linear surjective projection $\pi : \mathbb{C}^n \to \mathbb{C}^p$. Therefore $g$ is generically finite and by Properties 3.6 and 3.7, the set $L(f)$ of Lipschitz trivial values is not empty. For the converse statement, note that $f$ is dominant and $n - p = \dim F_c = 1 + \dim F_c^\infty$ for a generic level $c$ of $f$, so Theorem 4.1 gives the claim.
For the second part of the assertion, observe that the subset $L(g) \cap J(g) \cup K_0(g)$ is empty by Property 3.6 and Proposition 3.8, since $g$ is generically finite. We recall that $J(g) \cup K_0(g) = \text{Bif}(g)$ and if non-empty, it is an algebraic hypersurface by [5].

As a consequence, and with a different proof, we recover the main result of [1].

**Corollary 5.2** [1, Theorem 10] A complex polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ admits a Lipschitz trivial value if and only if it depends on a single variable, i.e. there exist $(n - 1)$ linearly independent vectors $v_2, \ldots, v_n$, of $\mathbb{C}^n$ such that $\partial_{v_i} f \equiv 0$ for all $i \geq 2$.

The real case is more nuanced than the complex one and Lipschitz trivial values admit a richer structure as we can see below.

**Corollary 5.3** Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial mapping admitting a Lipschitz trivial value. There exists a polynomial mapping $g : \mathbb{R}^m \to \mathbb{R}^p$ and linear surjective projection $\pi : \mathbb{R}^n \to \mathbb{R}^m$ such that $f = g \circ \pi$ and

$$L(f) = \mathbb{R}^p \setminus (J(g) \cup K_0(g)).$$

Moreover, the mapping $g$ is unique up to linear changes of coordinates.

**Proof** The demonstration is similar to proof of Corollary 5.1. Moreover, up to a linear change of coordinates in $\mathbb{R}^n$, we have $Df = Dg \oplus 0 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^p$. Thus using Property 3.6, Proposition 3.8 and Theorem 4.1 we get

$$L(f) = L(g).$$

To show uniqueness take $g : \mathbb{R}^m \to \mathbb{R}^p$ proper at $c$ such that $f = g \circ \pi$ for a linear surjective mapping $\pi : \mathbb{R}^n \to \mathbb{R}^m$. For any polynomial mapping $h : \mathbb{R}^k \to \mathbb{R}^p$ and a linear surjective projection $\sigma : \mathbb{R}^n \to \mathbb{R}^k$ such that $f = h \circ \sigma$, we get, up to a linear change of coordinates $\ell : \mathbb{R}^k \to \mathbb{R}^k$, that

$$(h \circ \ell)^{-1}(t) = g^{-1}(t) \times \mathbb{R}^{k-m}$$

for $t \in \mathbb{R}^p$, since at least the level $c$ of $g$ is compact. Thus either $m < k$ and $h$ does not attain a proper value, or $m = k$ and $h \circ \ell = g$. \qed

**Example 5.4** Let $f : \mathbb{R}^3 \hookrightarrow \mathbb{R}$ be the suspension at infinity of the Motzkin polynomial given by

$$f(x, y, z) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1.$$  

We have $J(f) = [1, \infty)$ and $K_0(f) = [0, 1]$. Moreover,

$$L(f) = (-\infty, 1) \setminus \{0\}.$$
Indeed, the values of $[1, +\infty)$ are not Lipschitz trivial values of $f$, since $f$ does not satisfy the necessary condition (ii) of Proposition 3.4 (the distance between any two levels in $(1, +\infty)$ is zero).

Example 5.4 illustrates that the set of Lipschitz trivial values of a real polynomial mapping can be open and not dense in the image, whereas for complex mappings Lipschitz trivial values follow a local-global principle as stated in Corollary 5.1.

For polynomial mappings Property 3.10 is refined as the following necessary condition on the fibres.

**Property 5.5** Let $f : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial mapping. There exists a $\mathbb{K}$-linear subspace $A$ of $\mathbb{K}^n$ of positive codimension such that

$$\hat{F}_t^\infty = A, \quad \text{for all } t \in L(f) \cap \text{Im}(f),$$

where $\hat{F}_t^\infty$ is the $\mathbb{K}$-cone of $\mathbb{K}^n$ over $F_t^\infty$ with vertex at the origin, and where the cone over the empty set is defined as the null subspace.

**Example 5.6** The polynomial mapping $f : \mathbb{K}^3 \to \mathbb{K}^2$, defined as $(x, y, z) \mapsto (x, xy + z)$, is surjective and $\mathcal{C}^\infty$ trivial at each $c \in \mathbb{K}^2$. Each level $F_t$ is an affine line, like the mapping in Example 3.3. Yet, the family of accumulation sets at infinity $(\hat{F}_t^\infty)_{t \in \mathbb{K}^2}$ is nowhere locally constant. Therefore Property 5.5 implies that this mapping cannot admit any Lipschitz trivial value.

## 6 Lipschitz Trivial Values of Real Mappings and Their Complexifications

Throughout this section let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a real polynomial mapping and let $f_C$ be its complexification. We will show that the set of real Lipschitz trivial values of $f_C$ is either empty or equal to the set of Lipschitz trivial values of $f$, up to a set of positive codimension.

**Proposition 6.1** If $f_C$ admits a Lipschitz trivial value, then $L(f)$ is a semi-algebraic dense open subset of $\mathbb{R}^p$. More precisely,

$$L(f_C) \cap \mathbb{R}^p \subset L(f).$$

**Proof** Denote $F_{C,t} := f_C^{-1}(t)$. If $L(f_C) \cap \text{Im}(f_C)$ is not empty, then $f_C$ is dominant, and thus the image of $f$ is Zariski dense. Corollary 5.1 and Property 3.6 imply the existence of a $\mathbb{C}$-linear subspace $A_C$ of $\mathbb{C}^n$ of dimension $n - p$ such that for each $t \in \mathbb{C}^p \setminus \text{Bif}(f_C)$, the level $F_{C,t}$ is a disjoint union of finitely many affine subspaces of $\mathbb{C}$-dimension $n - p$, parallel to $A_C$. Since $f(\mathbb{R}^n)$ is Zariski dense, there exists an open set $V$ in $\mathbb{R}^p$ such that the level $F_t$ is of dimension $n - p$ for $t \in V$. As the intersection of the complex fibre $F_{C,t}$ with $\mathbb{R}^n$, the fibre $F_t$ is necessarily a disjoint union of parallel real affine subspaces. Therefore, $\dim f^{-1}(V)^\infty = n - 1 - p$. Note that $f$, as the restriction of $f_C$ to $\mathbb{R}^n$, satisfies assumptions (i) and (ii) of Lemma 4.2.
Using Theorem 4.1 we get that \( f = g \circ \pi \) for some real linear surjective projection \( \pi : \mathbb{R}^n \to \mathbb{R}^p \) and real polynomial mapping \( g : \mathbb{R}^P \to \mathbb{R}^P \). Thus \( f_C = g_C \circ \pi_C \) and necessarily \( g_C \) is generically finite. Since Corollaries 5.1 and 5.3 yield

\[
L(f_C) = \mathbb{C}^P \setminus \text{Bif}(g_C), \quad \text{and} \quad L(f) = \mathbb{R}^P \setminus \text{Bif}(g).
\]

We get the claim since \( \text{Bif}(g) = J(g) \cup K_0(g) \) is semi-algebraic of positive codimension by [5].

\[\square\]

**Proposition 6.2** Following the notations of the proof of Proposition 6.1, the subset \( B := \text{Bif}(g_C) \cap \mathbb{R}^P \setminus \text{Bif}(g) \) is semi-algebraic of dimension \( \leq p - 1 \). Either

\[
L(f) \setminus B = L(f_C) \cap \mathbb{R}^P \quad \text{when} \quad \dim F_c^\infty = n - 1 - p \quad \text{for some} \ c \in L(f),
\]

or \( L(f_C) = \emptyset \) otherwise.

**Proof** In view of Proposition 6.1 we only need to show the inclusion

\[
L(f) \setminus B \subset L(f_C).
\]

If \( L(f) \neq \emptyset \), then Property 5.5 holds true and let \( m := \dim A \). From Corollary 5.1, the condition \( m = n - p \) is necessary to have \( L(f_C) \neq \emptyset \). When \( m = n - p \), there exist a polynomial mapping \( g : \mathbb{R}^P \to \mathbb{R}^P \) with Zariski dense image and a linear surjective projection \( \pi : \mathbb{R}^n \to \mathbb{R}^P \) such that \( f = g \circ \pi \). Thus \( f_C = g_C \circ \pi_C \), and \( g_C \) is dominant. We have \( \text{Bif}(g) \cup B = \text{Bif}(g_C) \cap \mathbb{R}^P \) and the claim follows from Corollaries 5.1 and 5.3.

\[\square\]

### 7 Rational Functions and Lipschitz Trivial Values

It is natural to ask whether we can extend the category of mappings that satisfy the claim of our main result. We answer in the negative, as Property 7.3 demonstrates that Theorem 4.1 is sharp in the sense that it does not hold for rational but non-polynomial mappings.

Let \( f : \mathbb{K}^n \to \mathbb{K} \) be a rational function, \( n \geq 2 \). Its indeterminacy locus \( I(f) \) is the subset of \( \mathbb{K}^n \) where denominator and numerator vanish simultaneously (for all representations of \( f \) as a fraction). Let \( \overline{\mathbb{K}} \) be the compactification of \( \mathbb{K} \) defined as follows

\[
\overline{\mathbb{K}} := \mathbb{C} \cup \{ \infty \} \quad \text{and} \quad \overline{\mathbb{R}} := \mathbb{R} \cup \{ \pm \infty \}.
\]

**Property 7.1** Assume that the rational function \( f : \mathbb{K}^n \to \mathbb{K} \) does not extend continuously through the point \( x_0 \in \mathbb{K}^n \), i.e. the subset \( J := \{ \lim_{x \to x_0} f(x) \} \subset \overline{\mathbb{K}} \) of accumulation values of \( f \) at \( x_0 \) does not reduce to a single value in \( \mathbb{K} \). Then \( L(f) \cap J = \emptyset \).
Proof If $\mathbb{K} = \mathbb{C}$ then $J = \overline{\mathbb{C}}$. When $\mathbb{K} = \mathbb{R}$, the set $J$ is closed, semi-algebraic and has non-empty interior. Suppose $J \cap L(f)$ is non-empty, thus it is open. In such a case, there exists an open subset $V$ of $J \cap L(f)$ such that $f$ is Lipschitz trivial over $V$. Then any trivializing bi-Lipschitz homeomorphism satisfies Estimates (2), contradicting the fact that $x_0$ lies in the closure of any level $f^{-1}(t)$ when $t \in V$. \qed

Corollary 7.2 A complex rational function with Lipschitz trivial values has empty indeterminacy locus.

On the other hand, the real setting is more flexible. Real rational functions may extend continuously (or even smoothly) through their indeterminacy locus onto $\mathbb{R}^n$, in such a case they are called regulous.

Proposition 7.3 There exist rational functions $f : \mathbb{K}^n \to \mathbb{K}$ with empty indeterminacy locus that admit Lipschitz trivial values which are not values of properness, and are never of the form $g \circ \pi$ with $g : \mathbb{K}^m \to \mathbb{K}$ a rational function and $\pi : \mathbb{K}^n \to \mathbb{K}^m$ a linear surjective projection with $n > m$.

Proof Let $h : \mathbb{R}^{n-1} \to \mathbb{R}$ be the non-constant function $x \mapsto h(x) := 1 + \sum_{i=1}^{n-1} x_i^2$ and consider the rational $C^\infty$ function $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x, y) = y - \frac{1}{h(x)}.$$

We have $I(f) = \emptyset$ and $f$ has no critical point. Observe that the partial derivatives of $f$ are uniformly bounded over $\mathbb{R}^n$, thus $f$ is a Lipschitz function over $\mathbb{R}^n$.

For $c \in \mathbb{R}$ define the following mapping

$$G : \mathbb{R}^n \to \mathbb{R} \times f^{-1}(c), \quad (x, y) \mapsto \left( f(x, y), \left( x, c + \frac{1}{h(x)} \right) \right).$$

It is a Lipschitz homeomorphism with inverse

$$G^{-1} \left( t, \left( x, c + \frac{1}{h(x)} \right) \right) = \left( x, t + \frac{1}{h(x)} \right).$$

The inverse $G^{-1}$ is also Lipschitz, thus each value $c$ of $\mathbb{R}$ is Lipschitz trivial for $f$. Since any level of $f$ is a graph over $\mathbb{R}^n$, the function $f$ cannot be proper at $c$. Last, there exists no vector $v$ of $\mathbb{R}^n \setminus \{0\}$ such that $\partial_v f \equiv 0$. \qed

Remark 7.4 The function $f$ defined in the proof of Proposition 7.3 is regulous [2, 7, 8].
Declarations

Conflict of interest  The authors have no financial or proprietary interests in any material discussed in this article.

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