Covariant-Contravariant Refinement Modal μ-calculus *

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Abstract

The notion of covariant-contravariant refinement (CC-refinement, for short) is a generalization of the notions of bisimulation, simulation and refinement. This paper introduces CC-refinement modal μ-calculus (CCRMLμ) obtained from the modal μ-calculus system Kμ by adding CC-refinement quantifiers, establishes an axiom system for CCRMLμ and explores the important properties: soundness, completeness and decidability of this axiom system. The language of CCRMLμ may be considered as a specification language for describing the properties of a system referring to reactive and generative actions. It may be used to formalize some interesting problems in the field of formal methods.

Keywords: modal logic, modal μ-calculus, covariant-contravariant refinement, axiom system

1 Introduction

Fixed-points operators make it possible to express most of the properties that are of interest in the study of ongoing behaviours. It is well known that there exists a deep link between behaviour relations and modal logics. Laura Bozelli et al. presented and explored refinement modal logic (RML) and single-agent refinement modal μ-calculus (RMLμ) [6, 7]. RML provides a more abstract perspective of future event logic [14, 15] and arbitrary public announcement logic [3]. In addition to usual modal operators and fixed-point operators, RMLμ contains refinement operator ∃B where B is a set of actions, whose semantics is given in terms of the notion of B-refinement. The notion of refinement (simulation) is often used to describe the refinement relation between reactive systems. To describe the behavioural relation between the systems involving also generative (active) actions (e.g., input/output (I/O) automata) [1] [9, 11], the notion of covariant-contravariant refinement (CC-refinement, for short) is presented

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The notion of CC-refinement captures behavioural preorders between labelled transitions systems (LTSs) and partitions all actions into three sorts: covariant, contravariant and bivariant actions, which represent respectively the passive actions of a system, the generative actions under the control of a system itself and the actions treated as in the usual notion of bisimulation. The transitions labelled with covariant actions in a given specification should be simulated by any correct implementation and the transitions for contravariant actions in an implementation must be allowed by its specification. It is not difficult to see that the notion of CC-refinement generalizes the notions of bisimulation, refinement and simulation considered in [6].

We, inspired by Laura Bozzelli et al’s work, presented covariant-contravariant refinement modal logic (CCRML) based on the notion of CC-refinement, and provided its sound and complete axiom system [12].

This paper considers CC-refinement modal $\mu$-calculus (CCRML$\mu$). We obtain the language $L_{CC}^\mu$ of CCRML$\mu$ from the standard modal $\mu$-calculus language $L_K^\mu$ by adding CC-refinement operator $\exists(A_1,A_2)$, where $A_1$ ($A_2$) is a set of all covariant (contravariant, resp.) actions. The operator $\exists(A_1,A_2)$ acts as a quantifier over the set of all $\exists(A_1,A_2)$-refinements of a given pointed model. Intuitively, the formula $\exists(A_1,A_2)\beta$ represents that we can CC-refine the current model so as to satisfy $\beta$. This operator can be adopted to formalize some interesting problems in the field of formal methods. For instance, if a given specification, expressed by a LTS $N$, refers to the set $A_1$ ($A_2$) of passive (generative, resp.) actions, the problem whether this specification has an implementation which involves some special characterizations and satisfies some given property $\psi$, by applying CC-refinement operators, can be boiled down to the model checking problem: $N \models \exists(A_1,A_2)(\nu q.\varphi \land \Box q \land \psi)$. Here, $\nu q.\varphi \land \Box q$ depicts the characterization that $\varphi$ is true at any time, which is often used in protocol synthesis and verifications.

In this paper, we provide an axiom system for CCRML$\mu$ and explore its important properties: its soundness is established based on the construction of the desired CC-refined models, and its completeness and decidability are obtained through transforming $L_{CC}^\mu$-formulas into $L_K^\mu$-formulas.

We organize this paper as follows. Section 2 recalls the notion of CC-refinement and introduces CC-refinement modal $\mu$-calculus. Section 3 provides a sound and complete axiomatization for CC-refinement modal $\mu$-calculus. Finally, we give a brief discussion in Section 4.

2 CC-refinement modal $\mu$-calculus

In this section, we firstly recall the notion of CC-refinement [10], and then present CC-refinement modal $\mu$-calculus (CCRML$\mu$), which is obtained from CC-refinement modal logic (CCRML) [12] by adding fixed-point operators.

Given a finite set $A$ of actions and a countable infinite set $Atom$ of propositional letters, a model $M$ is a triple $\langle S^M, R^M, V^M \rangle$, where $S^M$ is a non-empty set of states, $R^M$ is an accessibility function from $A$ to $2^{S^M \times S^M}$ assigning to each action $b$ in $A$ a binary relation $R^M_b \subseteq S^M \times S^M$, and $V^M : Atom \to 2^{S^M}$ is a valuation function. In this paper, we also use another valuation function $V^M : S^M \to 2^{Atom}$. A pair $(M, w)$ with $w \in S^M$ is said to be a pointed model. If $q \in Atom$ and $u \in V^M(q)$, $u$ is said to be a $q$-state. For any binary relation
\( R \subseteq S_1 \times S_2, T \subseteq S_1 \) and \( w \in S_1, \ R(w) \triangleq \{ s \mid wR_s \}, \ R(T) \triangleq \bigcup_{w \in T} R(w), \ \pi_1(R) \triangleq \{ s \mid \exists (sR_t) \} \) and \( \pi_2(R) \triangleq \{ t \mid \exists (sR_t) \} \).

As usual, we write \( M \not\models N \) for the disjoint union of two models \( M \) and \( N \) with \( S^M \cap S^N = \emptyset \), which is defined by \( S_1^M \cup S_2^N = S^M \cup S^N, \ R^M_0 \cup R^N_0 \) for each \( b \in A \) and \( V^M(r) \cup V^N(r) \) for each \( r \in \text{Atom} \).

**Definition 2.1 (\( P \)-restricted CC-refinement).** Given \( P \subseteq \text{Atom} \) and \( A_1, A_2 \subseteq A \) with \( A_1 \cap A_2 = \emptyset \), for any model \( M = (S, R, V) \) and \( M' = (S', R', V') \), a binary relation \( Z \subseteq S \times S' \) is a \( P \)-restricted \((A_1, A_2)\)-refinement relation between \( M \) and \( M' \) if, for each pair \( \langle u, u' \rangle \) in \( Z \), the following conditions hold:

- **(forth)** for each \( a \in A - A_1 \) and \( v \in S \), \( uR_av \) implies \( u'R'_av' \) and \( vZv' \) for some \( v' \in S' \);
- **(back)** for each \( a \in A - A_1 \) and \( v' \in S', u'R'_av' \) implies \( uR_av \) and \( vZv' \) for some \( v \in S \).

Here \( A_1 \) (\( A_2 \)) is said to be the **covariant** (contravariant, resp.) set. A pointed model \((M', u')\) is said to be a \( P \)-restricted \((A_1, A_2)\)-refinement of \((M, u)\), in symbols \( M, u \models^P_{(A_1, A_2)} M', u' \), if there exists a \( P \)-restricted \((A_1, A_2)\)-refinement relation between \( M \) and \( M' \) linking \( u \) and \( u' \). We also write \( Z : M, u \models^P_{(A_1, A_2)} M', u' \) to indicate that \( Z \) is a \( P \)-restricted \((A_1, A_2)\)-refinement relation such that \( uZu' \).

If \( P \) is a singleton, say \{ \( q \) \}, we write \( \models^q_{(A_1, A_2)} \) instead of \( \models^P_{(A_1, A_2)} \). If \( P = \emptyset \), we abbreviate the superscript in \( \models^P_{(A_1, A_2)} \). In this case, Definition 2.1 describes indeed the notion of CC-refinement given in \[10\].

The above notion generalizes the notions of bisimulation, simulation and refinement considered in the literature (see, e.g., \[4\]). Formally, a bisimulation relation is exactly an \((\emptyset, \emptyset)\)-refinement, a B-simulation relation a \((B, \emptyset)\)-refinement and a B-refinement relation an \((\emptyset, B)\)-refinement.

We write \( Z : M, s \models^P M', s' \) to indicate that \( Z \) is a \( P \)-restricted bisimulation which witnesses that \((M, s)\) is \( P \)-restricted bisimilar to \((M', s')\). The notation \( \models^P \) also follows the conventions for the notation \( \models^P_{(A_1, A_2)} \) in the above paragraphs.

The motivation behind the notion of CC-refinement lies in the differences between the roles played by different kinds of actions while considering refinement relations between models. The transitions labelled with the actions in \( A_1 \) (covariant actions) need to satisfy \( \text{(forth)} \), i.e., these transitions in a given specification should be simulated by any correct implementation; the transitions labelled with the actions in \( A_2 \) (contravariant actions) need to satisfy \( \text{(back)} \), i.e., these transitions in an implementation must be allowed by its specification; the transitions labelled with the actions in \( A \) - \((A_1 \cup A_2) \) (bivariant actions) satisfy both \( \text{(forth)} \) and \( \text{(back)} \). More descriptions for this can be found in \[12\].

**Proposition 2.2 (\[12\]).**

\begin{enumerate}
\item \( M_1, s_1 \not\models M_2, s_2 \models_{(A_1, A_2)} N_2, t_2 \not\models N_1, t_1 \) implies \( M_1, s_1 \models_{(A_1, A_2)} N_1, t_1 \).
\item If \( M, s \models_{(A_1, A_2)} N, t \) then there exist \((M', s'), (N', t')\) and \( Z \) such that
  \begin{enumerate}
  \item \( M, s \models M', s' \),
  \item \( N, t \models N', t' \),
  \item \( Z : M', s' \models_{(A_1, A_2)} N', t' \), and
  \item \( Z \) is an injective partial function from \( S^{M'} \) to \( S^{N'} \), that is, \( Z \) satisfies
\end{enumerate}
\end{enumerate}
(functionality) \( \forall v \in S^M \forall v_1, v_2 \in S^N (wZv_1 \text{ and } wZv_2 \Rightarrow v_1 = v_2) \);

(injectivity) \( \forall v \in S^N \forall v_1, v_2 \in S^M (wZv \text{ and } wZv \Rightarrow v = v_2) \).

Clearly, the above proposition holds still up to \( \succeq^0_{(A_1, A_2)} \).

**Proposition 2.3** \( \text{[12]} \). Let \( A_1, A_2 \subseteq A \) with \( A_1 \cap A_2 = \emptyset \). Then, for each \( A'_1, A''_1, A'_2 \) and \( A''_2 \) such that \( A'_1 \cup A''_1 = A_1 \) and \( A'_2 \cup A''_2 = A_2 \), it holds that

\[
\succeq(A'_1, A'') \circ \succeq(A'_2, A'') = \succeq(A_1, A_2).
\]

Here, \( \circ \) is used to denote the composition operator of relations.

The above proposition reveals that, through taking compositions, any CC-refinement may be captured by the CC-refinements with singleton covariant and contravariant sets.

**Definition 2.4** (Language \( L^\mu_{CC} \)). Let \( A \) be a finite set of actions and \( \text{Atom} \) a countable infinite set of propositional letters. The language \( L^\mu_{CC} \) of CC-refinement modal \( \mu \)-calculus is generated by the BNF grammar below, where

\[
\varphi ::= r \mid (\neg \varphi) \mid (\varphi \land \varphi) \mid (\odot b \varphi) \mid (\exists (A_1, A_2) \varphi) \mid (\mu q. \varphi)
\]

As usual, the propositional letter \( q \) in \( \varphi \) is required to occur positively in \( \varphi \) (namely occur only in the scope of even number of negations).

The modal operator \( \odot_b \) and propositional connectives \( \bot, \top, \lor, \land \) and \( \leftrightarrow \) are defined in the standard manner. We write \( \forall (A_1, A_2) \varphi \) for \( \neg \exists (A_1, A_2) \neg \varphi \), and the duality of \( \mu q. \varphi \) is defined as \( \nu q. \varphi \equiv \neg \mu q. \neg \varphi \). We also use the symbol \( \eta \) to denote either \( \mu \) or \( \nu \). If both \( A_1 \) and \( A_2 \) are singletons, say \( A_1 = \{a_1\} \) and \( A_2 = \{a_2\} \), we write \( \exists (a_1, a_2) \varphi \) (or \( \forall (a_1, a_2) \varphi \) instead of \( \exists (A_1, A_2) \varphi \) (resp., \( \forall (A_1, A_2) \varphi \)). For the sake of simplicity, this paper supposes that \( A_1 \neq \emptyset \) and \( A_2 \neq \emptyset \). Section \( \text{[14]} \) will discuss how to dispense with this assumption.

The cover operators \( \nabla_b \) (or \( b \in A \)) are adopted in this paper. \( \nabla_b \Phi \) is defined as \((\odot_b \Phi) \land \bigwedge_{\varphi \in \Phi} \odot_b \varphi\), where \( \Phi \) is a finite set of formulas. \( \odot_b \varphi \) and \( \odot_b \varphi \) can be captured by \( \nabla_b \theta \lor \nabla_b \varphi \) and \( \nabla_b \{\varphi, \top\} \) respectively. More information about cover operators may be found in \( \text{[13, 8]} \).

Given a model \( M \), the notion of a formula \( \varphi \in L^\mu_{CC} \) being satisfied in \( M \) at a state \( u \) (in symbols, \( M, u \models \varphi \)) is defined inductively as follows \( \text{[10]} \):

\[
\begin{align*}
M, u = r & \quad \text{iff} \quad u \in V^M (r), \text{ where } r \in \text{Atom} \\
M, u \models \neg \varphi & \quad \text{iff} \quad M, u \not\models \varphi \\
M, u \models \varphi_1 \land \varphi_2 & \quad \text{iff} \quad M, u \models \varphi_1 \text{ and } M, u \models \varphi_2 \\
M, u \models \odot_b \varphi & \quad \text{iff} \quad \text{for all } v \in R^M_b (u), M, v \models \varphi \\
M, u \models \exists (A_1, A_2) \varphi & \quad \text{iff} \quad M, u \models (A_1, A_2) N, v \text{ and } N, v \models \varphi \text{ for some } (N, v) \\
M, u \models \mu q. \varphi & \quad \text{iff} \quad u \in \bigcap \{T \subseteq S^M : \| \varphi \|^M (\varphi \rightarrow T) \subseteq T\}
\end{align*}
\]

Here,

\[
\| \varphi \|^M \triangleq \{ s \in S^M : M, s \models \varphi \}
\]

\( M^{[\varphi \rightarrow T]} \triangleq (S^M, R^M, V) \) with \( V(r) \triangleq \begin{cases} V^M (r) & \text{if } r \neq q \\ T & \text{if } r = q. \end{cases} \)
As usual, for every $\psi \in \mathcal{L}_{CC}^\mu$, $\psi$ is valid, denoted by $\models \psi$, if $M, u \models \psi$ for every pointed model $(M, u)$.

Since a bisimulation relation is also a CC-refinement relation, due to the equivalence of $\leftrightarrow$ and the transitivity of $\supseteq (A_r, A_s)$ [12, Proposition 2.2], the satisfiability of the formula of the form $\exists \varphi (\varphi \in \mathcal{L}_{CC}^\mu)$ is invariant under bisimulations. Then we can show that $L_{CC}^\mu$-satisfiability is invariant under bisimulations by the proof method of [10, Theorem 2.17].

**Proposition 2.5.** If $M, s \leftrightarrow N, t$ then

$$M, s \models \psi \iff N, t \models \psi \text{ for all formulas } \psi \in \mathcal{L}_{CC}^\mu.$$

In one model, adding copies of the generated submodel of some state will not change this model w.r.t. bisimulation.

**Definition 2.6 (Copy).** A copy of a pointed model $(M, s)$ is a pointed model obtained from $(M, s)$ by renaming every state in $S^M$.

**Proposition 2.7.** A pointed model is bisimilar to its copy.

**Proof.** Obviously.

**Proposition 2.8.** Given a pointed model $(M, s)$, assume that $w$ is a successor of $u \in S^M$ and $(M', w')$ is a copy of the $w$-generated submodel of $M$ such that $M'$ and $M$ are disjoint. Let $N$ be obtained from $M \cup M'$ and defined by

$$S^N \triangleq S^M \cup S^{M'} \quad R_b^N \triangleq R_b^M \cup R_b^{M'} \cup \{(u, w') : uR_b^M w\} \text{ for each } b \in A \quad V^N(r) \triangleq V^M(r) \cup V^{M'}(r) \text{ for each } r \in \text{Atom}.$$

Then $M, s \leftrightarrow N, s$.

**Proof.** By Proposition [2.7] $Z : M, w \leftrightarrow M', w'$ for some $Z$. Set $Z' \subseteq S^M \times S^N$ by $Z' \triangleq \{(v, v') : v \in S^M\} \cup Z$. It is routine to check $Z' : M, s \leftrightarrow N, s$.

### 3 Axiom system

This section will provide a sound and complete axiom system for CCRML$^\mu$, which augments the axiom system for CCRML [12] by adding the axiom schemata and rules for fixed-point operators. Since the uniform substitution rule is not sound in CCRML [12], neither CCRML nor CCRML$^\mu$ is normal.

We use $\mathcal{L}_{K}^\mu$ to denote the set of all $\mathcal{L}_{CC}^\mu$ formulas containing no CC-refinement operators, and $\mathcal{L}_p$ the set of all propositional formulas in $\mathcal{L}_{CC}^\mu$. Obviously, $\mathcal{L}_{K}^\mu$ is indeed the multi-agent modal $\mu$-calculus, which may be axiomatized by the system $K^\mu$ [17].

The axiom schemata and rules for CCRML$^\mu$ are given in Table 1. The axiom schema $F1$ and rule $F2$ are standard [2], which characterize the least fixed-point operators; CCRIn reveals that the operator $\exists_{(a_1, a_2)}$ preserves the inconsistency of $\mathcal{L}_{K}^\mu$-formulas; and $\text{CCR}^\mu$ and $\text{CCR}^\nu$ may be seen as $\exists_{(a_1, a_2)}$-crossing laws. See [12, Page 16-17] for the interpretations of the other axiom schemata and rules.

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1Given a model $M$ with $w \in S^M$, its $w$-generated submodel $N$ is the model $(S, R, V)$ where $S \triangleq M^s(w) \cup \{w\}$ with $R_b^N \triangleq \bigcup_{b \in A} R_b^{M}$, $R_b \triangleq S^2 \cap R_b^{M}$ for each $b \in A$, and $V(r) \triangleq S \cap V^M(r)$ for each $r \in \text{Atom}$ [5].
Table 1: Axiom system of CCRML$^\beta$

Axiom schemata
Here $a_1, a_2, a, b \in A$, $A_1, A_2, B \subseteq A$, $A_1 \cap A_2 = \emptyset$, $r, q \in \text{Atom}$, $\beta \in \mathcal{L}_K^\mu$, $\Gamma \subseteq \mathcal{L}_K^\mu$ and $\Phi, \Phi_b \subseteq \mathcal{L}_C^\mu$.

| Prop  | All propositional tautologies |
|-------|-------------------------------|
| K     | $\Box_a(\varphi \rightarrow \psi) \rightarrow (\Box_a \varphi \rightarrow \Box_a \psi)$ |
| CCR   | $\forall_{(a_1, a_2)}(\varphi \rightarrow \psi) \rightarrow (\forall_{(a_1, a_2)} \varphi \rightarrow \forall_{(a_1, a_2)} \psi)$ |
| CCRp1 | $\forall_{(a_1, a_2)} r \leftrightarrow r$ |
| CCRp2 | $\forall_{(a_1, a_2)} s \leftrightarrow \neg s$ |
| CCRD | $\exists_{(a_1, a_2)} \varphi \leftrightarrow (\exists_{a_1} \cdots \exists_{a_{|A_1 \times A_2|}})_{\varphi}$ where $\{\theta_i\}_{1 \leq i \leq |A_1 \times A_2|}$ is any permutation of elements in $A_1 \times A_2$ |
| CCRKco1 | $\exists_{(a_1, a_2)} \Box_a \varphi \leftrightarrow \Box_{a_1} \Box_{a_2} \varphi$ if $\vdash_{K_{\varphi}} \beta \leftrightarrow \bot$ for some $\beta \in \Gamma$ |
| CCRKco2 | $\exists_{(a_1, a_2)} \Box_a \varphi \leftrightarrow \Box_{a_1} \Box_{a_2} \varphi$ if $\vdash_{K_{\varphi}} \beta \leftrightarrow \bot$ for all $\beta \in \Gamma$ |
| CCRKcontra | $\exists_{(a_1, a_2)} \Box_a \varphi \leftrightarrow \bigvee_{a_1, a_2} \exists_{a_1, a_2} \exists_{(a_1, a_2)} \varphi$ |
| CCRKbis | $\exists_{(a_1, a_2)} \Box_a \varphi \leftrightarrow \Box_b \exists_{(a_1, a_2)} \varphi$ where $b \neq a_1, a_2$ |
| CCRKconj | $\exists_{(a_1, a_2)} \Box_a \varphi \leftrightarrow \bigvee_{a_1, a_2} \exists_{a_1, a_2} \exists_{(a_1, a_2)} \varphi$ |
| F1 | $\varphi[\mu q. \varphi / q] \rightarrow \mu q. \varphi$ |
| CCR$^\mu$ | $\exists_{(a_1, a_2)} \nu q. \beta \leftrightarrow \nu q. \exists_{(a_1, a_2)} \beta$ |
| CCRIn | $\exists_{(a_1, a_2)} \beta \leftrightarrow \bot$ if $\vdash_{K_{\beta}} \beta \leftrightarrow \bot$ |
| CCR$^\mu$ | $\exists_{(a_1, a_2)} \mu q. \beta \leftrightarrow \mu q. \exists_{(a_1, a_2)} \beta$ if $\vdash_{K_{\beta}} \mu q. \beta \leftrightarrow \bot$ |

Rules

| Rule | Symbolic Form |
|------|---------------|
| MP   | $\varphi \rightarrow \psi, \varphi \psi$ |
| NK   | $\Box_a \varphi$ |
| NCCR | $\varphi \rightarrow (\forall_{(a_1, a_2)} \varphi)$ |
| F2   | $\varphi[\psi / q] \rightarrow \psi \mu q. \varphi \rightarrow \psi$ |

As usual, $\vdash \beta$ ($\vdash_{K_{\varphi}} \beta$) means that $\beta$ is a theorem in CCRML$^\beta$ (resp., $K_{\beta}$ [17]).

3.1 Technical preliminary

In this subsection, some technical preparations will be made for establishing the soundness of the axiom system.

Definition 3.1 (Tree-like model). A model $M$ is tree-like whenever the following conditions are satisfied:

(i) there is a unique state $s \in S^M$, called the root, such that $\forall t \in S^M - \{s\}$ ($sR^+_M t$), that is, every $t \in S^M - \{s\}$ is accessible from $s$;
(ii) for each $t \in S^M - \{s\}$, there is a unique $t' \in S^M$ such that $t' R^+_a t$ for some $a \in A$;
(iii) $R^+_a \cap R^+_b = \emptyset$ for all $a, b \in A$ with $a \neq b$;
(iv) $\forall t \in S^M ((t, t) \notin R^+_M)$.

Proposition 3.2. Any rooted model$^2$ is bisimilar to a tree-like model.

As usual, a model is rooted if there is a unique state $s \in S^M$ such that $s$ has no precedent and the other states are accessible from $s$.
Proof. In [5] Proposition 2.15, it was shown that any rooted model is bisimilar to a traditional tree-like model [5] Definition 1.7], i.e., a model satisfying the conditions (i)(ii)(iv) in Definition 3.1. In order to satisfy the condition (iii), we intend to reconstruct the traditional tree-like model given in the proof of [5] Proposition 2.15]. We add agents information to its state names.

Let \( M \) be a model with the root \( u_0 \). Define the model \( M' \) as follows: for each \( b \in A \) and \( v \in \text{Atom} \),

\[
\begin{align*}
S_{M'}' & \triangleq \{ (u_0, a_1, u_1, \ldots, a_n, u_n) : n \geq 0 \text{ and } u_0 R^M_{a_1} u_1 R^M_{a_2} u_2 \cdots R^M_{a_n} u_n \} \\
R^M_b & \triangleq \{ \langle (u_0, a_1, u_1, \ldots, a_n, u_n) \cup \langle u_0, a_1, \ldots, a_n, u_n, b, u_{n+1} \rangle \rangle : n \geq 0 \text{ and } u_0 R^M_{a_1} u_1 R^M_{a_2} u_2 \cdots R^M_{a_n} u_n R^M_n b, u_{n+1} \} \\
V^M(r) & \triangleq \{ (u_0, a_1, u_1, \ldots, a_n, u_n) \in S^M' : n \geq 0 \text{ and } u_n \in V^M(r) \}.
\end{align*}
\]

It is not difficult to see that \( M' \) satisfies the condition (iii) and \( M' \) is a tree-like model. Moreover, it is straightforward to check that \( Z : M, u_0 \leftrightarrow M', u_0 \) where

\[
Z \triangleq \{ (u_n, \langle u_0, a_1, u_1, \ldots, a_n, u_n \rangle) : n \geq 0 \text{ and } u_0 R^M_{a_1} u_1 R^M_{a_2} u_2 \cdots R^M_{a_n} u_n \}. \quad \square
\]

For tree-like models, the results in Proposition 2.2 (2) still hold.

**Proposition 3.3.** Assume that \( (M, s) \) and \( (N, t) \) are both tree-like models. If \( M, s \succeq_{(A_1, A_2)} N, t \) then there exist a tree-like model \( (N', t') \) and \( Z \) such that

1. \( N, t \leftrightarrow N', t' \),
2. \( Z : M, s \succeq_{(A_1, A_2)} N', t' \), and
3. \( Z \) is an injective relation from \( S^M \) to \( S^{N'} \), that is, \( Z \) satisfies

\[
\forall v \in S^N \forall w_1, w_2 \in S^M (w_1 Z v \text{ and } w_2 Z v \Rightarrow w_1 = w_2).
\]

**Proof.** Let \( M, s \succeq_{(A_1, A_2)} N, t \). At first glance, by Proposition 2.8 we may add enough copies of the generated submodel of each non-root state in \( N \).

Since \( (M, s) \) and \( (N, t) \) are both tree-like models, \( Z : M, s \succeq_{(A_1, A_2)} N, t \) for some \( Z \) that \( Z(s) = \{ t \} \) and \( Z^{-1}(t) = \{ s \} \). Let \( u' Z v' \text{ with } Z^{-1}(v') = \{ u' \} \), and let \( v' R^N_b v \) with \( b \in A \) and \( Z^{-1}(v) \neq \emptyset \). Set \( \Delta_v \triangleq Z^{-1}(v) \cap R^N_b(v') \).

For each \( u \in \Delta_v \), the model \( (N_u, v_u) \) is the copy of the \( v \)-generated submodel of \( N \) with each state \( w \) of this submodel renamed \( w_u \). We w.l.o.g. assume that \( N \) and these copies are pairwise disjoint. Let \( N_1 \) be obtained from \( N \uplus \bigcup_{u \in \Delta_v} N_u \) by adding the transitions \( \{ v' \xrightarrow{b} v_u \} \). Clearly, \( (N_1, t) \) is a tree-like model.

If \( |\Delta_v| \geq 1 \), we choose arbitrarily and fix a state \( u_0 \) in \( \Delta_v \) and put \( S \triangleq \{ u_0 \} \), else put \( S \triangleq \emptyset \). To complete this proof, it suffices to show that \( N, t \leftrightarrow N_1, t \) and \( Z_1 : M, s \succeq_{(A_1, A_2)} N_1, t \) where

\[
Z_1 \triangleq (Z - ((S^M - S) \times \{ v \})) \cup \{ \langle u, v_u \rangle \}_{u \in \Delta_v} \cup \{ \langle w, v_u \rangle : w \in R^N(v), w_1 Z w \text{ and } u \in \Delta_v \}.
\]

The former one follows immediately by Proposition 2.8. It is routine to check \( Z_1 : M, s \succeq_{(A_1, A_2)} N_1, t \). Moreover, \( Z_1^{-1}(v) = \{ v_0 \} \) and \( Z_1^{-1}(v_u) = \{ u \} \) for each \( u \in \Delta_v \), as expected. \( \square \)

We next recall the notion of disjunctive formula with fixed points [17].
Definition 3.4 \([df]^{[13, 17]}\). The set of all the disjunctive formulas, \(df\), is the smallest set defined by the following clauses:

(i) \(L_\emptyset \subseteq df\);
(ii) if \(\alpha, \beta \in df\), then \(\alpha \lor \beta \in df\);
(iii) if \(\alpha \in L_\emptyset\) and \(\Phi_b \subseteq df\) for each \(b \subseteq A\), then \(\alpha \land \bigwedge_{b \in B} \nabla_b \Phi_b \in df\);
(iv) if \(q \in Atom\) occurs only positively in \(\alpha\) and not in the context \(q \land \gamma\) for some \(\gamma\), then \(\eta q.\alpha \in df\).

Proposition 3.5 \([13, 17]\). For each \(\psi \in L^\mu_K\), there is a \(df\) formula \(\alpha\) such that \(\vdash_{K^\ast} \psi \iff \alpha\) and \(\models \psi \iff \alpha\).

Proposition 3.15 reveals that, given \(M, s \models \varphi(q)\) where \(q \in Atom\) and \(\eta q.\varphi(q) \in df\), there exists a tree-like model \(N\) with the root \(t\) such that \((N, t)\) is \(q\)-restricted bisimilar to \((M, s)\), \(N, t \models \varphi(q)\) and this satisfiability does not depend on the descendants of the non-root \(q\)-states and the assignments of the propositional letters in \(Atom - \{q\}\) at these \(q\)-states. We intend to show these statements based on operational semantics for the \(L^\mu_K\)-formulas described in [13]. We firstly recall the related notions.

Definition 3.6 (Tableau rules \([13]\)). Let \(\eta q.\varphi(q) \in L^\mu_K\). The system \(S^\varphi\) of tableau rules parameterized by \(\varphi(q)\) is defined as follows:

\[
\begin{array}{ll}
\text{(and)} & \{\alpha, \beta, \Gamma\} \quad \{\alpha \land \beta, \Gamma\} \\
\text{(or)} & \{\alpha, \Gamma\} \quad \{\beta, \Gamma\} \\
\text{(mu)} & \{\mu r.\alpha(r), \Gamma\} \quad \{\alpha(r), \Gamma\} \\
\text{(mod)} & \{\psi\} \quad \{\forall \Psi : \nabla_b \Psi \in \Gamma \land \Psi \neq \Psi\} \quad \text{for each } \nabla_b \Psi \in \Gamma \text{ and } \psi \in \Psi
\end{array}
\]

where \(\alpha, \beta, \psi \in L^\mu_K\), \(\Gamma \subseteq L^\mu_K\) and \(\{\alpha, \Gamma\}\) is used as a shorthand for \(\{\alpha\} \cup \Gamma\).

Definition 3.7 (Tableau \([13]\)). Let \(\eta q.\varphi(q) \in L^\mu_K\). A tableau for \(\varphi(q)\) is a tree \(T = \langle T, L^T \rangle\), where \(T = \langle W^T, E^T \rangle\) is a directed tree in the graph-theoretic sense and \(L^T : W^T \to 2^{L^\mu_K}\) is a labelling function such that:

(i) \(L^T(v_0) = \{\varphi(q)\}\) where \(v_0\) is the root of \(T\);
(ii) the children of a node are created and labeled according to the rules in \(S^\varphi\),
with the rule \(\text{mod}\) applied only when no other rule is applicable.

Leaves and nodes where the rule \(\text{mod}\) is applied are called modal nodes. The root of \(T\) and children of modal nodes are called choice nodes.

We use \(u, v, w\) to range over \(E^T\).

Definition 3.8 (Marking \([13]\)). For a tableau \(T = \langle T, L^T \rangle\), its marking \(w.r.t.\) a pointed model \((M, s_0)\) is a relation \(M \subseteq S^M \times W^T\) satisfying the following conditions:

(i) \(s_0 M v_0\) where \(v_0\) is the root of \(T\).
(ii) If \(s M v\) and a rule other than \(\text{mod}\) is applied in \(v\), then \(s M v'\) for some child \(v'\) of \(v\).
(iii) If \(s M v\) and the rule \(\text{mod}\) is applied at \(v\), then for each \(b \in A\) for which there exists a formula of the form \(\nabla_b \Psi\) in \(L^T(v)\),
(a) for each \(b\)-child \(v'\) of \(v\), there exists \(s' \in R^M_b(s)\) such that \(s' M v'\);
(b) for each \(s' \in R^M_b(s)\), there exists a \(b\)-child \(v'\) of \(v\) such that \(s' M v'\).
Definition 3.9 (Consistent marking 13). Using the notations from Definition 3.8, a marking $M$ of $T$ w.r.t. $(M, s_0)$ is consistent if and only if it satisfies the following conditions:

1. (local consistency) for each modal node $v$ and $s \in S^M$, if $s_Mv$ then $M, s \models \theta$ for each literal $\theta$ occurring in $L^T(v)$;

2. (global consistency) for every path $v_0, v_1, \cdots$ of $T$ with $v_i \in \pi_2(M)$ ($i \geq 0$), the rule (reg) is not applied in $v_i$ for each $i \geq 0$.

Proposition 3.10 (13). Let $\eta_r \varphi(q) \in \mathcal{L}_K^\mu$. So $M, s_0 \models \varphi(q)$ if and only if there exists a consistent marking of a tableau for $\varphi(q)$ w.r.t. $(M, s_0)$.

Proposition 3.11. Let $\eta_r \varphi(q) \in df$ and $T = \langle T, L^T \rangle$ be a tableau for $\varphi(q)$. Then

$$\bigwedge L^T(w) \in df \quad \text{for each } w \in W^T.$$

Proof. Let $w \in W^T$. Proceed by induction on the height of $T$. If $w$ is the root of $T$, then $L^T(w) = \{ \varphi(q) \}$ and the result holds. Assume that $w$ is not the root of $T$ and $\bigwedge L^T(u) \in df$ for any precedent $u$ of $w$. In the following, we distinguish the different cases based on the applied tableau rule to obtain $w$.

Case 1. $\{ \{\alpha, \beta, \Gamma \} \}$. Then $L^T(w) = \{ \alpha, \beta \}$, and $\alpha \land \beta \land \Gamma \in df$ by the induction hypothesis. Immediately, $\bigwedge L^T(w) \in df$.

Case 2. $\{ \{\alpha, \beta, \Gamma \} \}$. Then either $L^T(w) = \{ \alpha, \beta \}$ or $L^T(w) = \{ \beta, \Gamma \}$. Moreover, by the induction hypothesis, $\alpha \lor \beta \land \Gamma \in df$. Further, by definition 3.4 if $\Gamma = \emptyset$, then the result holds clearly, else it must be that $\alpha \lor \beta \in \mathcal{L}_\mathcal{P}$ and $\alpha \lor \beta \land \Gamma \in df$, so $\alpha, \beta \in \mathcal{L}_\mathcal{P}$, and hence $\bigwedge L^T(w) \in df$.

Case 3. $\{ \{\alpha, \beta, \Gamma \} \}$. So $L^T(w) = \{ \alpha, \beta \}$. By the induction hypothesis, $(\mu_r \alpha) \land \alpha \in df$. Then, by definition 3.4 it is not difficult to see $\Gamma = \emptyset$ and $\alpha \in df$. Thus, this result holds.

Case 4. $\{ \{\alpha, \beta, \Gamma \} \}$. Similar to Case 3.

Case 5. $\{ \{\alpha, \beta, \Gamma \} \}$. where $\mu_r \alpha$ is a subformula of $\varphi(q)$. Then we obtain $L^T(w) = \{ \alpha, \beta \}$. Since $\mu_r \alpha$ is a subformula of $\varphi(q)$, we have $\mu_r \alpha \in df$ and $\alpha \in df$. Moreover, $\alpha \land \Gamma \in df$ by the induction hypothesis. Due to these, by definition 3.4 $\Gamma = \emptyset$ follows. Hence, $\bigwedge L^T(w) = \alpha \in df$.

Case 6. $\{ \{\alpha, \beta, \Gamma \} \}$. where $\mu_r \alpha$ is a subformula of $\varphi(q)$. Then we obtain $L^T(w) = \{ \psi \} \cup \{ \psi : \psi \in \Psi \} \in \Gamma \in \pi_2(M)$ and $\psi \in \Psi$. Since $\bigwedge \Gamma \in df$ by the induction hypothesis, the $\psi \in \Psi$ is unique w.r.t. $b$ by definition 3.4 and hence $L^T(w) = \{ \psi \}$. It is clear that $\psi \in df$. \qed

Proposition 3.12. Let $\eta_r \varphi(q) \in df$. Then $M, s_0 \models \varphi(q)$ if and only if there exists a minimal consistent marking of a tableau for $\varphi(q)$ w.r.t. $(M, s_0)$.

Proof. The implication from right to left holds clearly by Proposition 3.10. Now, we check the converse implication. Let $M, s_0 \models \varphi(q)$. By Proposition 3.10 there exists a consistent marking of a tableau $T = \langle T, L^T \rangle$ for $\varphi(q)$ w.r.t. $(M, s_0)$. Below, we show the existence of a minimal consistent marking of $T$ w.r.t. $(M, s_0)$ by Zorn Lemma. Put

$$\Pi \triangleq \{ M : M \text{ is a consistent marking of } T \text{ w.r.t. } (M, s_0) \}.$$
Clearly $\Pi \neq \emptyset$. Then $\langle \Pi, \subseteq \rangle$ is a partially ordered set. Let $\Delta$ be a nonempty chain in $(\Pi, \subseteq)$. It is enough to show that $\bigcap \Delta$ is a consistent marking of $T$ wrt. $(M,s_0)$ by Definition 3.8 and Definition 3.9.

(Definition 3.8 (i)) Since $s_0 M v_0$ ($v_0$ is the root of $T$) for each $M \in \Delta$, we have $(s_0,v_0) \in \bigcap \Delta$.

(Definition 3.8 (ii)) Let $(s,v) \in \bigcap \Delta$. If a rule other than (mod) and (or) is applied in $v$, the proof is trivial. If the rule (or) is applied in $v$, $v$ has two children $v_1$ and $v_2$. Thus, we need to show that either $s M v_1$ for each $M \in \Delta$, or $s M v_2$ for each $M \in \Delta$. On the contrary, assume that for some $M_1, M_2 \in \Delta$, $(s,v_1) \notin M_1$ and $(s,v_2) \notin M_2$. Since the rule (or) is applied in $v$, by Definition 3.8 (ii), we get $(s,v_2) \in M_1$ and $(s,v_1) \in M_2$. Hence, due to $(s,v_1) \notin M_1$ and $(s,v_2) \notin M_2$, it follows that $M_1 \nsubseteq M_2$ and $M_2 \nsubseteq M_1$. This contradict that $M_1, M_2 \in \Delta$ and $\Delta$ is a chain.

(Definition 3.8 (iii)) Assume that $(s,v) \in \bigcap \Delta$ and the rule (mod) is applied in $v$. Let $b \in A$ such that there exists a formula of the form $\forall \Psi$ in $L^T(v)$. Since $\varphi(q) \in df_b$, by Proposition 3.11, the $\forall_b \Psi$ is unique wrt. $b$. Thus, by the rule (mod), it is easy to see that $|E_b^T(v)| = |\Psi|$ where $E_b^T(v)$ is the set of all the $b$-children of $v$. As $\Psi$ is finite, $E_b^T(v)$ is also finite. For each $M \in \Delta$ and $t \in R_b^M(s)$, put

$$W_t^M \triangleq \{w \in W^T : t M w \} \quad \text{and} \quad W^M \triangleq \{W_t^M\}_{t \in R_b^M(s)}.$$ 

Obviously, since $\Delta$ is a chain, $\{W_t^M\}_{M \in \Delta}$ is also a chain, where comparing $W_t^M$ componentwise, that is, $W_t^M \subseteq W_t'^M$ iff $W_t^M \subseteq W_t'^M$ for each $t \in R_b^M(s)$. Further, due to the finiteness of $E_b^T(v)$, the chain $\{W_t^M\}_{M \in \Delta}$ is also finite.

Since $(s,v) \in \bigcap \Delta$ and $M_0 \in \Delta$, we get $s M_0 v$. Thus, on the one hand, by Definition 3.8 (iii)-(a), for each $b$-child $v'$ of $v$, there exists $s' \in R_b^M(s)$ such that $s' M_0 v'$. Further, $(s',v') \in \bigcap \Delta$ due to (3.1.1). Therefore $\bigcap \Delta$ satisfies Definition 3.8 (iii)-(a). On the other hand, by Definition 3.8 (iii)-(b), for each $s' \in R_b^M(s)$, there is a $b$-child $v'$ of $v$ such that $s' M_0 v'$. Then $(s',v') \in \bigcap \Delta$ due to (3.1.1). Hence $\bigcap \Delta$ satisfies Definition 3.8 (iii)-(b).

(consistency) Let $v$ be a modal node and $(s,v) \in \bigcap \Delta$. Assume $M \in \Delta$. So $s M v$. By (local consistency) of $M$, we have $M, s \vdash \theta$ for each literal $\theta$ occurring in $L^T(v)$. Consequently, $\bigcap \Delta$ is local consistent. Let $v_0, v_1, \ldots$ is a path of $T$ with $v_i \in \tau_\Delta(i) \geq 0$. So $v_i \in \tau_\Delta(M)$ for each $i \geq 0$. Further, by (global consistency) of $M$, the rule (reg) is not applied in $v_i$ for each $i \geq 0$. Hence $\bigcap \Delta$ is global consistent.

Notation 3.13. Given a tree-like model $M$ and $T \subseteq S^M$, the model $M \upharpoonright T$ is obtained from $M$ by removing all the descendants of the states in $T$, formally,

$$S^M \upharpoonright T \triangleq S^M - R_M^T(T)$$
$$R_b^M \upharpoonright T \triangleq R_b^M \cap (S^M \upharpoonright T \times S^M \upharpoonright T) \quad \text{for each} \ b \in A$$
$$V^M \upharpoonright T (r) \triangleq V^M(r) \cap S^M \upharpoonright T \quad \text{for each} \ r \in \text{Atom}. $$
Notation 3.14. Given two models $M, N$ and $T \subseteq S^M$, we write $M \models^T N$, if $S^M = S^N$, $R^M = R^N$ and $V^M(u) = V^N(u)$ for any $u \in S^M - T$.

Proposition 3.15. Let $\eta_q \varphi(q) \in df$. If $M, s \models \varphi(q)$, then there exists a tree-like model $N$ with the root $t$ such that

(i) $M, s \models^T N, t \models \varphi(q)$

(ii) $N', t' \models \varphi(q)$ for all $N'$ such that $N' \models (V^N(q) - \{ t \}) =^T N \models (V^N(q) - \{ t \})$

where $T \equiv V^N(q)(V^N(q) - \{ t \})(q) - \{ t \}$.

Proof. The assertion (i) and its proof have been given in the proof of [6] Theorem 40]. Since this proposition is important for our work, we provide its complete proof below.

Let $M, s \models \varphi(q)$. By Proposition 3.2 and Proposition 2.5, we may w.l.o.g. suppose that $M$ is a tree-like model with the root $s$. Moreover, by Proposition 2.8 and Proposition 2.5, we assume that there are enough many copies of the generated submodel of every state in $M$. That is, in a marking of a tableau for $\varphi(q)$ w.r.t. $(M, s)$,

for Definition 3.8 (iii)-(b), the $\nu'$ is unique. \hspace{1cm} (3.1.2)

Due to $M, s \models \varphi(q)$ and $\eta_q \varphi \in df$, by Proposition 3.12 there exists a minimal consistent marking $M$ of a tableau $T = (T, L^T)$ for $\varphi(q)$ w.r.t. $(M, s)$. Let the model $N$ be obtained from $M$ by setting

$V^N(q) \equiv \{ u \in S^M : \exists u \in W^T (u \ominus Mu$ and $q \in L^T(u)) \} \cup (V^M(q) \cap \{ s \})$.

Clearly, $M, s \models^T N, s$. To complete the proof, one crucial claim is needed.

Claim 1 $q \in L^T(u)$ implies $L^T(u) = \{ q \}$.

Let $q \in L^T(u)$. On the contrary, assume $|L^T(u)| > 1$. Firstly, due to $\eta_q \varphi(q) \in df$, by Proposition 3.11 we have

$\bigwedge L^T(w) \in df \quad \text{for each } w \in W^T.$ \hspace{1cm} (3.1.3)

However, since $\eta_q \varphi(q) \in df$, $q \in L^T(u)$ and $|L^T(u)| > 1$, by Definition 3.4 it is easy to see that the formula $\bigwedge L^T(u) \notin df$. A contradiction arises.

We now return to the proof of this proposition. Firstly, we have the result:\footnote{\hspace{1cm} This result will be needed in the proof of Proposition 3.16. Its proof is as follows. Let $u \in V^N(q)$. If $u = s$, $u \in V^M(q)$ by the definition of $N$. Assume $u \neq s$. Then there exists $v \in W^T$ such that $u \ominus MV$ and $q \in L^T(v)$. By Claim 1, $L^T(v) = \{ q \}$ follows. So $v$ is a leaf of $T$, and next a modal node of $T$. Thus, due to $u \ominus MV$ and $q \in L^T(v)$, by (local consistency) of $M$ w.r.t. $(M, s)$, we get $u \in V^M(q)$, as desired.}

$V^N(q) \subseteq V^M(q).$ \hspace{1cm} (3.1.4)

In the following, we intend to show $N, s \models \varphi(q)$. By Proposition 3.10 it is enough to check that $M$ is still a consistent marking of $T$ w.r.t. $(N, s)$.

(marking) As $(S^M, R^M) = (S^N, R^N)$, since $M$ is a marking of $T$ w.r.t. $(M, s)$, we have $M \subseteq S^N \times W^T$ and the conditions (i)-(iii) in Definition 3.8 hold also w.r.t. $T$ and $(N, s)$. Then $M$ is a marking of $T$ w.r.t. $(N, s)$.\hspace{1cm}
Let \( v \in W^T \) be a modal node and \( tMv \). Assume that \( \theta \) is a literal occurring in \( L^T(v) \). By (local consistency) of \( \mathcal{M} \) w.r.t. \((M,s)\), we have \( M,t \models \theta \). Consider three cases by \( \theta \) below.

If \( \theta \neq q, \neg q \), it follows from \( M,t \models \theta \) that \( N,t \models \theta \) by the definition of \( N \).

If \( \theta = q \), then by the definition of \( N \), due to \( tMv \) and \( q \in L^T(v) \), we get \( t \in V^N(q) \), i.e., \( N,t \models q \).

If \( \theta = \neg q \), then \( q \notin L^T(w) \) for any \( w \in M(t) \). (Otherwise, \( q \in L^T(w) \) for some \( w' \in M(t) \).) Thus \( L^T(w') = \{ q \} \) by Claim 1, that is, \( w' \) is a leaf of \( T \) and so a modal node of \( T \). Further, due to \( w' \in M(t) \) and \( q \in L^T(w') \), by (local consistency) of \( M \) w.r.t. \((M,s)\), we obtain \( M,t \models q \). Contradict that \( M,t \models \neg q \). Hence, by the definition of \( N \), we get \( t \notin V^N(q) \), i.e., \( N,t \models \neg q \).

Let \( v_0,v_1,\cdots \) be a path of \( T \) where \( v_0 \) is the root of \( T \) and \( v_i \in \pi_2(\mathcal{M}) (i \geq 0) \). By (global consistency) of \( \mathcal{M} \) w.r.t. \((M,s)\), the rule (reg) is not applied in \( v_i \) for each \( i \geq 0 \).

Let \( M' \) be a consistent marking of \( T \) w.r.t. \((N,s)\) and \( M' \subseteq M \). So \( M' \subseteq S^M \times W^T \). Similar to (marking) and (global consistency) above, we can show that \( M' \) is also a global consistent marking of \( T \) w.r.t. \((M,s)\). Below, we check (local consistency) of \( M' \) w.r.t. \((M,s)\). Let \( v \in W^T \) be a modal node and \( tM'v \). So \( tMv \) due to \( M' \subseteq M \). For any literal \( \theta \in L^T(v) \), by (local consistency) of \( M \) w.r.t. \((M,s)\), we have \( M,t \models \theta \). Summarizing, \( M' \) is also a consistent marking of \( T \) w.r.t. \((M,s)\). Since \( M' \subseteq M \) and \( M \) is a minimal consistent marking of \( T \) w.r.t. \((M,s)\), it follows that \( M' = M \).

In the next step, we will show that the descendants of the non-root \( q \)-states and the assignments of the propositional letters in \( \text{Atom} - \{ q \} \) at these states do not affect the satisfiability of \( \varphi(q) \) in \( N \) at the state \( s \). To complete the proof, another crucial claim is needed.

Claim 2  \( uM\mu \) and \( q \in L^T(u) \) imply \( W^{u,u} = \{ u \} \), where \( W^{u,u} \triangleq \{ w \in M(u) : \text{if } w \neq u \, \text{then } u \text{ is not a descendant of } w \} \).

Let \( uM\mu \) and \( q \in L^T(u) \). By Claim 1, \( L^T(u) = \{ q \} \) and so \( u \) is a leaf of \( T \). Moreover, due to (3.1.3), for any \( b \in A \) and \( w \in W^T \), it is clear that

\[
|\{ \nabla_b \Psi : \nabla_b \Psi \in L^T(w) \} | \leq 1. \tag{3.1.5}
\]

On the contrary, suppose \( |W^{u,u}| > 1 \). Since \((M,s)\) is a tree-like model, \( M \) is a marking of the tableau \( T \) w.r.t. \((M,s)\) and (3.1.5), for some \( u_i \in W^{u,u}, u, u_i \), are obtained according to:

- either the rule (or) for some \( v, u', u'_i \in W^T, v \in S^M, \alpha \in L^T(u') \) and \( \beta \in L^T(u'_i) \) such that \( \alpha \lor \beta \in L^T(v) \), \( vMv \), \( vMu' \), \( v'_iM'_i \), \( u' \), \( u'_i \) are children of \( v \), \( u \) is in the \( \nu \)-generated subtree of \( M \), and \( u (u_i) \) is in the subtree with the root \( u' (u'_i, \text{resp.}) \) of \( T \),

- or the rule (mod) for some \( b \in A, v, u', u'_i \in W^T, v, v' \in S^M, \nabla_b \Psi \in L^T(v) \) and \( \psi, \psi_i \in \Psi \) such that \( \psi \in L^T(u'), \psi_i \in L^T(u'_i), v' \in R^L_b(v), vMv, v'Mu' \), \( v'_iMu'_i \), \( u', u'_i \) are children of \( u, v \) is in the \( v' \)-generated subtree of \( M \), and \( u (u_i) \) is in the subtree with the root \( u' (u'_i, \text{resp.}) \) of \( T \) (See Definition (3.8) (iii)).

The former one contradicts the minimality of \( M \), and the latter one contradicts (3.1.2).
We now return to the proof of this proposition. Let \((N', s)\) be a pointed model such that
\[ N' \models (V^{N'}(q) - \{s\}) =^T N \models (V^N(q) - \{s\}) \]
where \(T \triangleq V^{N \#(V^N(q) - \{s\})}(q) - \{s\} = V^{N' \#(V^{N'}(q) - \{s\})}(q) - \{s\}. \) Then
\[
S^{N' \#(V^{N'}(q) - \{s\})} = S^{N \#(V^N(q) - \{s\})} \tag{3.1.6}
\]
\[
R^{N' \#(V^{N'}(q) - \{s\})} = R^{N \#(V^N(q) - \{s\})}. \tag{3.1.7}
\]
Firstly, we get the following assertion:
\[
\mathcal{M} \subseteq S^{N \#(V^N(q) - \{s\})} \times W^T. \tag{3.1.8}
\]
(Its proof is as follows. Let \(u \in V^N(q)\) with \(u \neq s\). So by the definition of \(N\), \(uM_u\) and \(q \in L^T(u)\) for some \(u \in W^T\). Then, by Claim 1, \(L^T(u) = \{q\}\) and \(u\) is a leaf of \(T\), and by Claim 2, \(W^N_u = \{u\}\). Thus, by Definition 3.8 (ii)-(iii), the descendants of \(u\) is impossibly in \(\pi_1(M)\). )

Further, from \([3.1.8]\) and \([3.1.6]\), it follows that
\[
\mathcal{M} \subseteq S^{N' \#(V^{N'}(q) - \{s\})} \times W^T (\subseteq S^N \times W^T). \tag{3.1.9}
\]

Below, we intend to check that \(\mathcal{M}\) is still a consistent marking of \(T\) w.r.t. \((N', s)\), which will imply \(N', s \models \varphi(q)\) by Proposition 3.10 (marking). Since \([3.1.7]\), \([3.1.9]\) and \(\mathcal{M}\) is a marking of \(T\) w.r.t. \((N, s)\), the conditions (i)-(iii) in Definition 3.8 hold also w.r.t. \(T\) and \((N', s)\). Then \(\mathcal{M}\) is also a marking of \(T\) w.r.t. \((N', s)\).

(local consistency) Let \(v \in W^T\) be a modal node and \(tMv\). So \(t \in S^{N' \#(V^{N'}(q) - \{s\})}\) and \(t \in S^{N \#(V^N(q) - \{s\})}\). Assume that \(\theta\) is a literal occurring in \(L^T(v)\). By (local consistency) of \(\mathcal{M}\) w.r.t. \((N, s)\), we have \(N, t \models \theta\).

If \(\theta = q\), then \(t \in T\) due to \(N, t \models \theta\), so \(N', t \models q\) by the definition of \(N'\).

If \(\theta \neq q\), then \(t \notin V^{N'}(q)\). (Otherwise, we get \(t \in T\). By the definition of \(N, q \in L^T(w)\) for some \(w \in M(t)\). Thus \(W^T \subseteq \{w\}\) by Claim 2. Hence, since \(tMv\) and \(W^T \subseteq \{w\}\), \(w\) is a descendant of \(v\), and so \(v\) is not a leaf of \(T\). Further, since \(v\) is a modal node, it is the tableau rule (mod) to be applied at \(v\). Therefore, by \(tMv\) and Definition 3.8 (iii), \(R^T_\varphi(t) \cap \pi_1(M) \neq \emptyset\). However, due to \(t \in T\) and \([3.1.8]\), \(R^T_\varphi(t) \cap \pi_1(M) = \emptyset\). Contradict.) Thus by the definition of \(N'\), due to \(N, t \models \theta\), it holds still that \(N', t \models \theta\).

(global consistency) See (global consistency) above.

Finally, we can conclude that the assertion (ii) in this proposition holds. \(\square\)

The following proposition is used to simplify the construction and verification of a desired CC-refined model in the proofs of Lemma 3.20 and Lemma 3.21.

**Proposition 3.16.** Let \(\eta, \varphi \in \text{df}\). If \(M, s \models \exists_{(A_1, A_2)} \varphi(q)\), then there exists a tree-like model \(N\) with the root \(t\) and an injective relation \(Z\) from \(S^M\) to \(S^N\) such that
(1) \(Z : M, s \models^\eta_{(A_1, A_2)} N, t \models \varphi(q)\)
(2) \(Z^{-1}(V^N(q)) \subseteq V^M(q)\)
(3) \(t \in V^N(q)\) iff \(s \in V^M(q)\)
(4) \(N', t \models \varphi(q)\) for all \(N'\) such that \(N' \models (V^{N'}(q) - \{t\}) =^T N \models (V^N(q) - \{t\})\) where \(T \triangleq V^{N \#(V^N(q) - \{t\})}(q) - \{t\} = V^{N' \#(V^{N'}(q) - \{t\})}(q) - \{t\}.\)
Proof. Let \( M, s \models \exists_{(A_1, A_2)} \varphi(q) \). Then there exists \( (N_1, t_1) \) such that
\[ M, s \models (N_1, t_1) \models \varphi(q). \]

By Proposition 3.3 and Proposition 2.5, we may assume that \( N_1 \) is a tree-like model with the root \( t_1 \). By Proposition 3.3 there exist a tree-like model \( N_1' \) with the root \( t \) and injective relation \( Z \) from \( S^{M} \) to \( S^{N_1} \) such that \( N_1, t_1 \models q \) and \( Z : M, s \models (N_1, t_1) \models \varphi(q) \). Further, by the construction of \( (N_1, t_1) \), we get \( N_1', t \models \varphi(q) \) by Proposition 2.5. Then, due to \( (N_1', t) \models \varphi(q) \), by the construction in the proof of Proposition 3.13 we get the tree-like model \( N \) with the root \( t \) such that \( N_1', t \models \varphi(q) \) and (4) holds w.r.t. \( (N, t) \). Moreover, by (3.1.4), \( V^N(q) \subseteq V^{N_1}(q) \) holds. Further, by the construction of \( (N, t) \), it is evident that \( Z : M, s \models (N_1, t) \models \varphi(q) \) and (2) and (3) still hold w.r.t. \( (N, t) \).

**Proposition 3.17.** If \( M, s \models q_{(A_1, A_2)} N, t \models \eta q \varphi \) then \( M, s \models \exists_{(A_1, A_2)} \eta q \varphi \).

**Proof.** (\( \eta = \mu \)) Let \( M, s \models q_{(A_1, A_2)} N, t \) and \( N, t \models \mu q \varphi \). By Proposition 2.2 there exist \( (M', s') \), \( (N', t') \) and injective partial function \( Z \) from \( S^{M'} \) to \( S^{N'} \) such that \( M, s \models (M', s') \), \( M, s \models (N', t') \) and \( Z : M', s' \models (N', t') \). By Proposition 2.5 we have \( N', t' \models \mu q \varphi \) due to \( N \models q \varphi \) and \( N, t \models \mu q \varphi \).

Set \( N_1 \equiv (N')_{[q \rightarrow Z(V^{M'}(q))]} \).

Obviously, \( V^{N_1}(q) = Z(V^{M'}(q)) \) and \( (N')_{[q \rightarrow T]} = N_{[q \rightarrow T]} \) for any \( T \subseteq S^{N'} \). Further, as \( N', t' \models \mu q \varphi \), by the semantics of \( \mu q \varphi \),
\[ t' \in \bigcap \{ T \subseteq S^{N_1} : \| \varphi \|_{(N')_{[q \rightarrow T]}} \subseteq T \} = \bigcap \{ T \subseteq S^{N_1} : \| \varphi \|_{N_{[q \rightarrow T]}} \subseteq T \}. \]

That is, \( N_1, t' \models \mu q \varphi \).

Below, we show \( Z : M', s' \models (A_1, A_2) N_1, t' \). Since \( Z : M', s' \models q_{(A_1, A_2)} N', t' \), by the definition of \( N_1 \), the conditions (\( \{q\text{-atoms}\} \), (forth) and (back) hold clearly. Next, we check (atoms). Assume \( u Z v \). Let \( u \in V^{M'}(q) \). So we get \( v \in V^{N_1}(q) \) due to \( V^{N_1}(q) = Z(V^{M'}(q)) \). Let \( v \in V^{N_1}(q) \). Then due to \( V^{N_1}(q) = Z(V^{M'}(q)) \), we have \( u' \models Z v \) for some \( u' \in V^{M'}(q) \). Since \( Z \) is an injective partial function from \( S^{M'} \) to \( S^{N'} (= S^{N_1}) \), \( u = u' \) immediately follows from \( u Z v \) and \( u' \models Z v \). Hence \( u \in V^{M'}(q) \) due to \( u' \in V^{M'}(q) \).

Finally, \( M, s : \exists_{(A_1, A_3)} \mu q \varphi \) holds due to \( M, s \models (M', s')_{(A_1, A_2)} N_1, t' \) and \( N_1, t' \models \mu q \varphi \), as desired.

(\( \eta = \nu \)) Similarly.

By Proposition 3.17, to show that \( M, s : \exists_{(A_1, A_2)} \eta q \varphi \), it is sufficient to find a model which \( q \text{-restricted} \ (A_1, A_2) \text{-refines} \ (M, s) \) and satisfies \( \eta q \varphi \).

To simplify the proofs of Lemma 3.20 and Lemma 3.21, we give the following model construction in a uniform fashion.
Definition 3.18. Let \((M, s)\) with \(W \subseteq S^M\) be a tree-like model with the root \(s\) and \((N_u, v_u)\) \((u \in W)\) be a tree-like model with the root \(v_u\) such that \(\{M\} \cup \{N_u\}_{u \in W}\) are pairwise disjoint. The model \(N\) is obtained from \((M \upharpoonright W) \uplus \biguplus_{u \in W} N_u\) and defined by, for each \(b \in A\) and \(r \in \text{Atom}\),

\[
\begin{align*}
S^N &\triangleq S^{M \upharpoonright W} \cup \bigcup_{u \in W} (S^{N_u} - \{v_u\}) \\
R^N_b &\triangleq (S^N)^2 \cap \left( R^{M \upharpoonright W}_b \cup \bigcup_{u \in W} R^{N_u}_b \right) \cup \{\langle u, w \rangle : u \in W \text{ and } v_u R^N_b w\} \\
V^N(r) &\triangleq (S^N - W) \cap \left( V^{M \upharpoonright W}(r) \cup \bigcup_{u \in W} V^{N_u}(r) \right) \cup \{u \in W : v_u \in V^{N_u}(r)\}
\end{align*}
\]

The model \(N\) is denoted by \((M, s) \oplus W \{\langle N_u, v_u \rangle\}_{u \in W}\). An illustration for this construction is given in Figure 1 with the hollow dots and dashed arrows removed.

![Figure 1](image-url)

Figure 1: The model \((M, s) \oplus W \{\langle N_u, v_u \rangle\}_{u \in W}\), where \(a, b, c, d \in A\), \(W = \{u, w\}\), \(P_1 = V^{N_u}(v_u)\), \(P_2 = V^{N_w}(v_w)\) and \(M'\) is obtained from \(M \upharpoonright W\) by setting \(V^{M'}(u) = P_1\) and \(V^{M'}(w) = P_2\).

### 3.2 Soundness

This subsection intends to establish the soundness of the axiom system CCRML\(^\mu\). Firstly we will give several validities concerned with fixed points.

**Lemma 3.19.** \(\models \exists (A_1, A_2) \eta q. \varphi \rightarrow \eta q. \exists (A_1, A_2) \varphi\).

**Proof.** By the reflexivity of the relation \(\geq_{(A_1, A_2)}\), we have

\(\models \psi \rightarrow \exists (A_1, A_2) \psi\).
Moreover, it is obvious that
\[ \models \eta q. \varphi \leftrightarrow \varphi[\eta q. \varphi/q], \quad \text{and} \quad (3.2.1) \]
\[ \models \eta q. \exists(A_1, A_2) \varphi \leftrightarrow \exists(A_1, A_2) \varphi[\eta q. \exists(A_1, A_2) \varphi/q]. \quad (3.2.2) \]

Due to (3.2.1), we also have
\[ \models \exists(A_1, A_2) \eta q. \varphi \leftrightarrow \exists(A_1, A_2) \varphi[\eta q. \varphi/q]. \quad (3.2.3) \]

To complete the proof, we prove the claim below

**Claim 1.** \[ \models \eta q. \varphi \rightarrow \eta q. \exists(A_1, A_2) \varphi \]

Let \((M, s)\) be a pointed model.

(\(\eta = q\)) Due to the inductive characterization idea of the greatest fixed point of monotone functions (see, e.g., [2]), it holds that
\[ M, s \models \nu q. \beta \iff s \in \bigcap_{\tau < \tau^*} \|\beta\|_{\tau} \]
where \(\tau^*\) is a limit ordinal and \(\|\beta\|_{\tau}\) is defined by
\[ \|\beta\|_{0} \triangleq S^M, \quad \text{and} \quad \|\beta\|_{\tau} \triangleq \|\beta\|_{M^{(q \rightarrow \tau, r, \leq q, \leq \|\beta\|_{\tau^*})}}. \]
Since \[ \models \varphi \rightarrow \exists(A_1, A_2) \varphi, \|\varphi\|_{\tau} \subseteq \|\exists(A_1, A_2) \varphi\|_{\tau} \quad \text{for any } \tau < \tau^*. \]
\[ \bigcap_{\tau < \tau^*} \|\varphi\|_{\tau} \subseteq \bigcap_{\tau < \tau^*} \|\exists(A_1, A_2) \varphi\|_{\tau}, \]
which implies \[ M, s \models \nu q. \varphi \rightarrow \nu q. \exists(A_1, A_2) \varphi. \]

(\(q = \mu\)) As \[ \models \varphi \rightarrow \exists(A_1, A_2) \varphi, \] it is clear that
\[ \{ T \subseteq S^M : \|\varphi\|_{M^{(q \rightarrow \tau, r, \leq q, \leq \|\varphi\|_{\tau})}} \leq T \} \subseteq \{ T \subseteq S^M : \|\exists(A_1, A_2) \varphi\|_{M^{(q \rightarrow \tau, r, \leq \|\varphi\|_{\tau})}} \leq T \}, \]
and hence,
\[ \bigcap \{ T \subseteq S^M : \|\varphi\|_{M^{(q \rightarrow \tau, r, \leq q, \leq \|\varphi\|_{\tau})}} \leq T \} \subseteq \bigcap \{ T \subseteq S^M : \|\exists(A_1, A_2) \varphi\|_{M^{(q \rightarrow \tau, r, \leq \|\varphi\|_{\tau})}} \leq T \}. \]

Thus, \[ M, s \models \mu q. \varphi \rightarrow \mu q. \exists(A_1, A_2) \varphi. \] Because \((M, s)\) is arbitrary, it holds that
\[ \models \eta q. \varphi \rightarrow \eta q. \exists(A_1, A_2) \varphi. \]

Now we return to the proof. By Claim 1, it follows that
\[ \models \varphi[\eta q. \varphi/q] \rightarrow \varphi[\eta q. \exists(A_1, A_2) \varphi/q]. \]

Further,
\[ \models \exists(A_1, A_2) \varphi[\eta q. \varphi/q] \rightarrow \exists(A_1, A_2) \varphi[\eta q. \exists(A_1, A_2) \varphi/q]. \quad (3.2.4) \]

Thus, due to (3.2.4), (3.2.3) and (3.2.2),
\[ \models \exists(A_1, A_2) \eta q. \varphi \rightarrow \eta q. \exists(A_1, A_2) \varphi. \quad \square \]

In the proof of Lemma 3.20 given \(M, s \models \nu q. \exists(A_1, A_2) \varphi\) with \(\nu q. \varphi \in \text{df}\), we intend to show that \(M, s \models \exists(A_1, A_2) \nu q. \varphi\). To this end, by Proposition 3.17, it is enough to construct a pointed model \((N, t)\) such that
\begin{align*}
\text{C1:} & \quad M, s \subseteq_{(A_1, A_2)} N, t, \quad \text{and} \\
\text{C2:} & \quad N, t \models \nu q. \varphi.
\end{align*}
Referring to the method used in [5, Theorem 40], we decorate and verify this construction. In the following, we will explain the idea behind this construction. At first glance, in order to satisfy \( C_2 \), by the semantics of the greatest fixed point, it suffices to realize that \( V^N(q) \) is a postfixed point of the function \( \lambda X. \| \varphi(q) \|^N_{\langle a \rightarrow X \rangle} \) and \( t \in V^N(q) \), i.e.,

\[
\text{C3: } N, u \models \varphi \text{ for each } u \in V^N(q), \text{ and } t \in V^N(q).
\]

Inspired by this observation, we next consider how to satisfy \( C_1 \) whenever \( \text{C3} \) is satisfied. Since \( M, s \models \nu q. \exists (A_1, A_2) \varphi, \) for some \( T \subseteq S^M \), we have \( s \in T \) and

\[
M[|q\rightarrow T], w \models q \rightarrow \exists (A_1, A_2) \varphi \quad \text{for all } w \in S^M.
\]

This brings us that \( M[|q\rightarrow T], s \models q \) and \( M[|q\rightarrow T], s \models \exists (A_1, A_2) \varphi. \) Thus, there exists a \( q \)-restricted CC-refined pointed model of \( (M, s) \) satisfying \( \varphi \land q. \) Unfortunately, it is not always guaranteed that all the \( q \)-states satisfy \( \varphi \) in this CC-refined model (that is, \( \text{C3} \) is not necessarily satisfied).

However, by Proposition 3.15 and Proposition 3.16 in a model, the descendents of the non-root \( q \)-states do not affect the satisfiability of the df-formula \( \varphi(q) \) at the root. Since \( M[|q\rightarrow T], s \models \exists (A_1, A_2) \varphi(q) \), by Proposition 3.16 we can obtain a \( q \)-restricted CC-refined tree-like model, say \( (N_0, t_0) \), of \( (M, s) \) such that \( N_0, t_0 \models \varphi \land q. \) Thus, we intend to realize \( \text{C3} \) by changing the descendents of some non-root \( q \)-states of \( (N_0, t_0) \) safely (that is, \( \text{C1} \) is still satisfied).

To this end, we define two sets \( T_0 \) and \( W_0 \):

\[
T_0 \triangleq V^{N_0}(q) - \{ t_0 \} \quad \text{and} \quad W_0 \triangleq S_{N_0 \upharpoonright T_0} \cap T_0.
\]

Obviously, \( V^{N_0 \upharpoonright T_0}(q) = \{ t_0 \} \cup W_0. \) For each \( u \in W_0 \), we can, based on [3.2.5], find an available tree-like model \( (N_u, v_u) \) which \( q \)-restricted CC-refines \( (M, s) \) and satisfies \( \varphi \land q. \) In the model \( N_0 \upharpoonright T_0, \) we add the \( b \)-transition \( \langle u, v \rangle \) for each \( b \)-successor \( (b \in A) v \) of \( v_u \) in \( N_u \), and change the assignments at \( u \) according to the ones at \( v_u. \) Thus, in the obtained pointed model, say \( (N_1, t_1) \), \( u \) satisfies \( \varphi \land q. \) It is not difficult to see that \( \text{C1} \) is still satisfied. But, regrettably, the \( q \)-states in the added parts \( (S^{N_1} - S^{N_0}) \) do not necessarily satisfy \( \varphi. \)

Going on like the above, we can inductively construct a sequence of tree-like pointed models \( \{ (N_i, t_i) \}_{i < \omega}. \) For each \( i < \omega, \) \( N_{i+1} \) is obtained from \( N_i \) by changing the generated submodels of the states in \( W_i, \) and the two sets \( T_{i+1} \) and \( W_{i+1} \) are respectively defined as the set of all the \( q \)-states in \( S^{N_{i+1}} - S^{N_i}, \) and \( W_{i+1} \triangleq S^{N_{i+1}} \upharpoonright T_{i+1} \cap T_{i+1}. \) Then it holds that

\[
V^{N_{i+1 \upharpoonright T_{i+1}}}(q) = \{ t_0 \} \cup \bigcup_{0 \leq j \leq i+1} W_j.
\]

Moreover, \( \text{C1} \) can be guaranteed in this process. A sketchy illustration for the sequence \( \{ (N_i, t_i) \}_{i < \omega} \) is given in Figure 2.

Finally, \( N_\omega \triangleq \left( \lim_{i < \omega} S^{N_i}, \left\{ \lim_{i < \omega} R^N_b \right\}_{b \in A}, \left\{ \lim_{i < \omega} V^{N_i}(r) \right\}_{r \in \text{Atom}} \right) \) and \( T_\omega \triangleq 0, \) which can be verified to be the desired model.

\( ^5\)As usual, given a sequence of sets \( \{ Z_n \}_{n < \omega}, \) the limit \( \lim_{n < \omega} Z_n \) is defined as,

\[
\lim_{n < \omega} Z_n \triangleq \{ x : \exists m < \omega \forall k (k \geq m \Rightarrow x \in Z_k) \},
\]

i.e.,

\[
\lim_{n < \omega} Z_n = \{ x : \exists m < \omega (x \in \bigcap_{k \geq m} Z_k) \}.
\]
Figure 2: \(\{ (N'_i, t_i) \}_{i<\omega} \), where \(N'_i\) is obtained from \(N_i \uparrow W_i\) by for each \(u \in W_i\), setting \(V^{N'_i}(u) = V^{N_i}(u)\)

Lemma 3.20. Let \(\nu q. \varphi \in df\). Then \(\models \nu q. \exists (A_1,A_2) \varphi \rightarrow \exists (A_1,A_2) \nu q. \varphi\).

Proof. Let \(M, s \models \nu q. \exists (A_1,A_2) \varphi\). So \(s \in \bigcup \{ T \subseteq S^M : T \subseteq \| \exists (A_1,A_2) \varphi \|^{|M^{\varphi+T}|} \}\). Then, for some \(T \subseteq S^M\), we have that \(s \in T\) and

\[
M^{\varphi+T}, w \models q \rightarrow \exists (A_1,A_2) \varphi \quad \text{for all } w \in S^M.
\] (3.2.6)

Clearly, \(M^{\varphi+T}, s \models q \land \exists (A_1,A_2) \varphi\). We will intend to construct a model \((N_\omega, t)\) such that \(M, s \geq (A_1,A_2) N_\omega, t \models \nu q. \varphi\), which by Proposition 3.17 will imply \(M, s \models \exists (A_1,A_2) \nu q. \varphi\), as desired. Firstly, \(\varphi \in df\) as \(\nu q. \varphi \in df\).

We inductively construct a sequence of tree-like pointed models \(\{ (N_n, t_n) \}_{n<\omega} \) and sets \(T_n \subseteq V^{N_n}(q)\) and \(W_n = S^N_n \uparrow T_n \cap T_n\) \((n < \omega)\), which satisfy the following properties: for each \(n < \omega\),

(i) \(Z_n : M^{[\varphi+T]}, s \models q \rightarrow \exists (A_1,A_2) N_n, t_n\) that is an injective relation from \(S^M\) to \(S^N_n\)

(ii) \(Z_n^{-1}(W_n) \subseteq T\)

(iii) \(\forall j < n \ (W_n \subseteq R_{N_n}^\downarrow (W_j))\)

(iv) \(\forall j < n \ (N_n \uparrow W_j = W_j) \uparrow W_j\)

(v) \(V^{N_n \uparrow W_j}(q) = \{ t_0 \} \cup \bigcup_{j \leq i} W_j\)

(vi) \(\forall j < n \ \forall u \in W_j \ (N_{j+1}, u \models \varphi \Rightarrow \forall N' \ ((N'_n \uparrow W_{j+1} = W_{j+1} \uparrow W_{j+1} \\
\text{and } W_{j+1} \subseteq V^{N'}(q) \Rightarrow N', u \models \varphi))\)
Proposition 2.5, due to Zarbitrarily and fix a tree-like model $N_0$ with the root $t_0$ and injective relation $Z_0$ from $S^M$ to $SN_0$ such that

$$Z_0 : M^{[q,T]}, s = q(A_1,A_2)N_0, t_0 \models q \land \varphi$$

Further, it is evident that the properties (i) state

Then, by Proposition 3.16, we may choose and fix a tree-like injective $w$ to $S_n$ such that $v$ satisfying (i) and $w$ from $Z_0(q)$.

For $u \in Z_i$, due to $Z_i^{-1}(W_i) \subseteq T$, we can choose arbitrarily and fix a state $w_u \in T$ such that $w_u Z_i u$. Further, $M^{[q,T]}, w_u \models q \land \exists(A_1,A_2)\varphi$ by (3.2.8). Then, by Proposition 3.16, we may choose and fix a tree-like model $N_u$ with the root $v_u$ and injective relation $Z_u$ from $S^M$ to $S^N_n$ such that

$$Z_u : M^{[q,T]}, w_u \geq q(A_1,A_2)N_u, v_u \models q \land \varphi$$

For $u \notin \pi_2(Z_i)$, we obtain the model $(N_u, v_u)$ from $(N_0, t_0)$ by renaming $w$ to $w_u$ for each $u \in S^N_0$. Set $Z_u \triangleq \{(v, w_u) : v \in Z_u \cap W_i\}$. It is easy to see that $Z_u$ is an injective relation from $S^M$ to $S^N_n$ and

$$Z_u : M^{[q,T]}, s \geq q(A_1,A_2)N_u, v_u \models q \land \varphi$$

W.l.o.g., we suppose that all the models in $\{N_i\} \cup \{N_u\}_{u \in W_i}$ are pairwise disjoint. The pointed model $(N_{i+1}, t_{i+1})$ is defined by:

$$N_{i+1} \triangleq (N_i, t_i) \oplus_{W_i} (\{N_u, v_u\})_{u \in W_i}$$

$$t_{i+1} \triangleq t_i$$

Since $W_i \subseteq V^{N_i}(q)$ by the induction hypothesis, $w_u Z_i u (u \in W_i \cap \pi_2(Z_i))$ and (3.2.9), it is not difficult to see that $V^{N_i}(u) = V^{N_u}(v_u) (u \in W_i \cap \pi_2(Z_i))$. Hence, by the construction of $N_{i+1}$ (See Definition 3.18), it holds that

$$V^{N_i}(u) = V^{N_{i+1}}(u)$$

for each $u \in W_i \cap \pi_2(Z_i)$.

Moreover, by the above construction, we easily see that, $N_{i+1}$ is still a tree-like model with the root $t_{i+1}$; and for each $u \in W_i$, $N_{i+1}, u \models N_0, v_u$, so by Proposition 2.5, due to $N_u, v_u \models \varphi$,

$$N_{i+1} \models \varphi.$$  

(3.2.10)
Put
\[ T_{i+1} \triangleq \bigcup_{u \in W_i} (V_{N_1}^*(q) - \{v_u\}) \]
\[ W_{i+1} \triangleq S_{N_i+q \uparrow T_{i+1}} \cap T_{i+1} \]
\[ Z_{i+1} \triangleq (Z_i \cup \bigcup_{u \in W_i} Z_u) \cap (S^M \times S_{N_{i+1}}) \]

So
\[ W_{i+1} \subseteq T_{i+1} \subseteq R_{N_{i+1}}^+(W_i) \]
\[ N_i \uparrow W_i = W_i \uparrow N_i \]
\[ V_{N_i+q \uparrow W_{i+1}}(q) = V_{N_i+q \uparrow W_i}(q) \cup W_{i+1} \]

Intuitively, \( T_{i+1} \) is indeed the set of all the \( q \)-states in \( (S_{N_{i+1}} - S_{N_i}) \).

Below, we check that \((N_{i+1}, t_{i+1})\) satisfies the properties (i)-(vii).

(i). Let \( v_1 Z_{i+1} w \) and \( v_2 Z_{i+1} w \). Then \( w \in S_{N_{i+1}} \). Because all the models in \( \{N_i\} \cup \{N_u\}_{u \in W_i} \) are pairwise disjoint, by the definition of \( N_{i+1} \), either \( w \in S_{N_i} \) or \( w \in S_{N_{i+1}} \) for some \( u \in W_i \), and only one of the two holds. Assume \( w \in S_{N_i} \). The other case is similar. Thus, by the definition of \( Z_{i+1} \), \( v_1 Z_i w \) and \( v_2 Z_i w \). By induction hypothesis, \( Z_i \) is an injective relation from \( S^M \) to \( S_{N_i} \). So \( v_1 = v_2 \).

Hence, \( Z_{i+1} \) is an injective relation from \( S^M \) to \( S_{N_{i+1}} \).

In the following, we check that \( Z_{i+1} : M^{(q \rightarrow T_i), s \geq q} \rightarrow N_{i+1}, t_{i+1} \) (i.e., \((N_{i+1}, t_{i+1})\)). Clearly \( s Z_{i+1} t_{i+1} \) since \( s Z_i t_i \). Let \( w Z_{i+1} w' \). The condition \( \{\{q\}\text{-atoms}\} \) holds trivially. Now we check (forth) and (back).

(forth) Let \( w R_{w}^{N_i} \) and \( b \in A - A_2 \). Due to \( w Z_{i+1} w' \), by the definition of \( Z_{i+1} \), \( w Z_i w' \) or \( w Z_u w' \) for some \( u \in W_i \).

For the former one, we have \( w' \in S_{N_i} \). So by the definition of \( N_{i+1} \), there are two alternatives.

If \( w' \notin W_i \), then \( w' \in S_{N_i+q \uparrow T_i} - W_i \). Since \( w Z_i w' \) and \( w R_{w}^{N_i} \), there exists \( w'_1 \in S_{N_i} \) such that \( w' R_{w}^{N_i} w'_1 \) and \( w Z_i w'_1 \). By the definitions of \( N_{i+1} \) and \( Z_{i+1} \), we have \( w'_1 \in S_{N_i+q \uparrow T_i} \subseteq S_{N_{i+1}} \), \( w' R_{w}^{N_{i+1}} w'_1 \) and \( w Z_i w'_1 \), as desired.

If \( w' \in W_i \), we have \( w' w Z_i w' \) and \( w w Z_i w' \). Since \( Z_i \) is an injective relation from \( S^M \) to \( S_{N_i} \), \( w = w' \) due to \( w Z_i w' \), \( w w Z_i w' \). Thus, as \( w w Z_i w' \), \( w Z_i w' \), \( w R_{w}^{N_i} w'_1 \) and \( w Z_i w'_1 \) for some \( w'_1 \in S_{N_i} \). Further, by the definitions of \( N_{i+1} \) and \( Z_{i+1} \), we get \( w R_{N_{i+1}}^{N_{i+1}} w'_1 \) due to \( v_1 Z_i w'_1 \) and \( w Z_i w'_1 \). For the latter one, from \( w Z_i w' \) and \( w R_{w}^{N_i} w'_1 \), it follows that \( w' R_{w}^{N_i} w'_1 \) and \( w Z_i w'_1 \) for some \( w'_1 \in S_{N_{i+1}} \). Clearly, \( w'_1 \in S_{N_{i+1}} \), \( w' R_{w}^{N_{i+1}} w'_1 \) and \( w Z_i w'_1 \) by the definitions of \( N_{i+1} \) and \( Z_{i+1} \).

(back) Let \( w' R_{w}^{N_{i+1}} w'_1 \) and \( b \in A - A_1 \). So \( w', w'_1 \in S_{N_{i+1}} \). We below deal with the different cases based on \( w' \).

For \( w' \notin W_i \), by the above construction of \( N_{i+1} \), due to \( w' R_{w}^{N_{i+1}} w'_1 \), we have \( w w' R_{w}^{N_i} w'_1 \), \( w w Z_i w' \) and \( w w Z_i w' \). Since \( Z_i \) is an injective relation from \( S^M \) to \( S_{N_i} \), \( w = w' \) immediately follows from \( w Z_i w' \) and \( w Z_i w' \). Further, it follows from \( w w Z_i w' \) and \( w R_{w}^{N_i} w'_1 \) and \( w R_{w}^{N_i} w'_1 \) that \( w w R_{w}^{N_i} w'_1 \) (i.e., \( w R_{w}^{N_i} w'_1 \) and \( w Z_i w' \)). Moreover, \( w Z_i w'_1 \) due to \( w', w'_1 \in S_{N_{i+1}} \).

For \( w' \notin W_i \), there are two alternatives by the definition of \( N_{i+1} \).

If \( w' \in S_1 \), then \( w' \in S_{N_i+q \uparrow T_i} - W_i \). By the definitions of \( N_{i+1} \) and \( Z_{i+1} \), we have \( w'_1 \in S_{N_i+q \uparrow T_i} \), \( w R_{w}^{N_i} w'_1 \) and \( w Z_i w' \). Hence, \( w R_{w}^{N_i} w'_1 \) and \( w Z_i w'_1 \) (also \( w Z_i w'_1 \)) for some \( w'_1 \in S^M \).
If \( w' \notin S_i \), then \( w' \in S^{N_{i+1}} \setminus S^N_i \). By the definition of \( N_{i+1} \), we have \( w' \in S^{N_i} \) and \( w'R^N_i w'_1 \) for some \( u \in W_i \). Next \( wZ_u w' \) by the definition of \( Z_{i+1} \). Thus, due to \( wZ_u w' \) and \( w'R^N_u w'_1 \), there exists \( w_1 \in S^M \) such that \( wR^M_i w_1 \) and \( w_1 Z_u w'_1 \). Obviously, \( wZ_{i+1} w' \).

(ii). By the above construction, \( Z_{i+1}^{-1}(V^N(q)) \subseteq T \) for each \( u \in W_i \). So by the definition of \( Z_{i+1} \) and \( T_{i+1} \), \( Z_{i+1}^{-1}(T_{i+1}) \subseteq T \). Further, \( Z_{i+1}^{-1}(W_{i+1}) \subseteq T \) due to (3.2.11).

(iii). Let \( j < i + 1 \). For \( j = i \), it holds by (3.2.11). Consider \( j < i \). By the induction hypothesis, \( W_j \subseteq R^{N_i}(W_j) \). By the above construction, \( N_i \uparrow W_i \) is remained in \( N_{i+1} \). So \( W_i \subseteq R^{N_{i+1}}(W_j) \). This, together with (3.2.11), implies \( W_{i+1} \subseteq R^{N_{i+1}}(W_j) \).

(iv). Let \( j < i + 1 \). For \( j = i \), it holds by (3.2.12). For \( j < i \), by the induction hypothesis, \( W_j \subseteq R^{N_i}(W_j) \), which together with (3.2.12), implies \( N_{i+1} \uparrow W_j = N_i \uparrow W_j \). Hence, due to \( N_i \uparrow W_j = W_j \), \( N_i \uparrow W_j \) by the induction hypothesis, we have \( N_{i+1} \uparrow W_j = W_j \), \( N_i \uparrow W_j \).

(v). By the induction hypothesis, we have \( V^{N_i}(q) = \{t_0\} \cup \bigcup_{q \leq i} W_j \). Then, it follows from (3.2.13) that \( V^{N_{i+1}}(W_{i+1}) = \{t_0\} \cup \bigcup_{q \leq i} W_j \).

(vi). Let \( j < i + 1 \). For \( j = i \), it holds by the induction hypothesis. Consider \( j = i \). Let \( u \in W_i \). We have \( N_{i+1}, u \models \varphi \) by (3.2.10). Let \( N' \) be a model such that \( N' \uparrow W_{i+1} = W_{i+1}, N_{i+1} \uparrow W_{i+1} \subseteq V^{N_i}(q) \). Moreover, \( W_{i+1} \subseteq V^{N_{i+1}}(q) \) by (v) and \( W_{i+1} \subseteq R^{N_{i+1}}(W_i) \) by (iii). Thus, the conditions holds in Proposition (3.16) (4) w.r.t. the \( u \)-generated submodels of \( N' \) and \( N_{i+1} \). So, \( N', u \models \varphi \) by Proposition (3.16) (4).

(vii). For each \( u \in W_i \), it holds by (3.2.10). For all \( j < i \) and \( u \in W_j \), from \( N_{i+1}, u \models \varphi \) by the induction hypothesis, \( W_{i+1} \subseteq V^{N_{i+1}}(q) \) by (v) and \( N_{i+1} \uparrow W_j = W_{i+1} \), \( N_{i+1} \uparrow W_j \) by (iv), it follows that \( N_{i+1}, u \models \varphi \) due to (vii).

Now we are ready to construct the model \( N_\omega \).

(\( N_\omega \)) We define

\[
N_\omega \triangleq \left\langle \lim_{i < \omega} S^{N_i}, \left\{ \lim_{r < \omega} R^N_i \right\}_{b \in A}, \left\{ \lim_{i < \omega} V^{N_i}(r) \right\}_{r \in \text{Atom}} \right\}
\]

\[
T_\omega \triangleq 0
\]

\[
Z_\omega \triangleq \lim_{i < \omega} Z_i \quad \text{where} \quad Z_\omega \subseteq S^M \times S^{N_\omega}.
\]

In the following, we verify \( (N_\omega, t_0) \) is exactly the desired model, i.e., we prove that \( M, s \models (A_1, A_2) \) \( N_\omega, t_0 \models \varphi \). From the above inductive construction and the properties (i)-(vii), we can find the following interesting statements:

(a) For any \( j < i < \omega \), \( S^{N_j}(W_i) \subseteq S_i^{N_j} \) and \( S^{N_j}(W_j) \subseteq S_i^{N_j} \).

(b) For any \( j < i < \omega \) and \( b \in A \), \( R^{N_j}(W_i) \subseteq R^N_i \) and \( R^{N_j}(W_j) \subseteq R^N_i \).

(c) For any \( j < i < \omega \), if \( wZ_j w' \) and \( w' \in S^{N_j}(W_i) \) then \( \langle w, w' \rangle \) will be remained in \( Z_i \) and \( Z_\omega \).

(d) \( V^{N_i}(q) = \{t_0\} \cup \bigcup_{i < \omega} W_i \).

(e) For any \( i < \omega \), \( N_i \) and \( N_\omega \) are tree-like models with the root \( t_0 \) and \( N_\omega \uparrow W_i = W_i, N_i \uparrow W_i \).

Their proofs are given as follows.
(a)-(b). Let \( j < i < \omega \), \( u \in S^{N_i \upharpoonright W_i} \) and \( v R_b^{N_j \upharpoonright W_j} w \). By (iv), it follows that \( N_i \upharpoonright W_j - W_j \), \( N_j \upharpoonright W_j \). Immediately \( u \in S^{N_i} \) and \( v R_b^{N_i} w \). Further, as \( i \) is arbitrary, by the definition of \( N_\omega \), we get \( u \in S^{N_\omega} \) and \( v R_b^{N_\omega} w \).

(c). Let \( j < \omega \). Suppose \( w Z_i w' \) and \( w' \in S^{N_j \upharpoonright W_j} \). We show below \( w Z_i w' \) by induction on \( i \geq j \). The base case is trivial. For the induction step \( i = k + 1 \), assume \( w Z_k w' \). Due to \( w' \in S^{N_j \upharpoonright W_j} \) and (a), we have \( w' \in S^{N_i} \) and \( w' \in S^{N_{k+1}} \). So, by the definition of \( Z_{k+1} \), \( w Z_{k+1} w' \) follows from \( w Z_k w' \).

Moreover, as \( i \geq j \) is arbitrary, \( w Z_j w' \) holds by the definition of \( Z_\omega \).

(d). By (v) and the definition of \( V^{N_\omega} \), it is clear that \( \{t_0\} \cup \bigcup_{i<\omega} W_i \subseteq V^{N_\omega}(q) \). Let \( w \in V^{N_\omega}(q) \). By the definition of \( V^{N_\omega} \), there is \( k < \omega \) such that \( w \in V^{N_k}(q) \) for all \( k' \geq k \). So \( w \in V^{N_k}(q) \subseteq S^{N_k} \) and \( w \in V^{N_{k+1}}(q) \subseteq S^{N_{k+1}} \). Then, by the definition of \( N_{k+1} \), \( w \in S^{N_k \upharpoonright W_k} \). Further, \( w \in V^{N_k \upharpoonright W_k}(q) \) due to \( w \in V^{N_\omega}(q) \). Hence, by (v), we get \( w \in \{t_0\} \cup \bigcup_{i<\omega} W_i \).

(e). Let \( i < \omega \). By the above inductive construction, we easily see that \( N_i \) is a tree-like model with the root \( t_0 \) and \( t_0 \in S^{N_\omega} \). In the following, we check that \( N_\omega \) is a tree-like model with the root \( t_0 \) by Definition 3.1 (Definition 3.1 (iii)) Let \( w \in S^{N_\omega} - \{t_0\} \). By the definition of \( S^{N_\omega} \), there is \( k < \omega \) such that \( w \in S^{N_{k'}} \) for all \( k' \geq k \). So \( w \in S^{N_k} \) and \( w \in S^{N_{k+1}} \). Then, \( w \in S^{N_k \upharpoonright W_k} \) by the definition of \( N_{k+1} \). Since \( N_k \upharpoonright W_k \) is a tree-like model with the root \( t_0 \), we get \( t_0 R_b^{N_k \upharpoonright W_k} w \), and so \( t_0 R_b^{N_\omega} w \) by (b), i.e., each non-\( t_0 \) state is accessible from \( t_0 \) in \( N_\omega \).

Now we check the uniqueness of \( t_0 \). On the contrary, there is a state \( t'_0 \in S^{N_\omega} \) such that \( t'_0 \not= t_0 \) and each non-\( t'_0 \) state is accessible from \( t'_0 \) in \( N_\omega \). Since each non-\( t_0 \) state is accessible from \( t_0 \) in \( N_\omega \), it holds that \( t_0 R_b^{N_\omega} t_0 \). By the definition of \( R^{N_\omega} \), there is \( k_1 < \omega \) such that \( t_0 R_b^{N_{k_1}} t_0 \) for all \( k_1 \geq k_1 \). So \( t_0 R_b^{N_{k_1}} t_0 \). This contradicts that \( N_{k_1} \) is a tree-like model.

(Definition 3.1 (ii)-(iv)) Similar to the proof for Definition 3.1 (i).

Below we check \( N_\omega \upharpoonright W_i = W_i \). Let \( b \in A \) and \( r \in \text{Atom} \). Firstly, by (iv), we have

\[
N_k \upharpoonright W_i = W_i \quad \text{for all } k > i \tag{3.2.14}
\]

On the one hand, \( S^{N_k \upharpoonright W_i} \subseteq S^{N_\omega} \) by (a) and \( R_b^{N_k \upharpoonright W_i} \subseteq R_b^{N_\omega} \) by (b). Moreover, by (3.2.14), \( V^{N_k \upharpoonright W_i}(r) - W_i \subseteq V^{N_\omega}(r) \) for all \( k > i \), and so \( V^{N_k \upharpoonright W_i}(r) - W_i \subseteq V^{N_\omega}(r) \) by the definition of \( V^{N_\omega} \). Further, as all these models are tree-like, it is easy to see that \( S^{N_k \upharpoonright W_i} \subseteq S^{N_\omega \upharpoonright W_i} \), \( R_b^{N_k \upharpoonright W_i} \subseteq R_b^{N_\omega \upharpoonright W_i} \), and \( V^{N_k \upharpoonright W_i}(r) - W_i \subseteq V^{N_\omega \upharpoonright W_i}(r) \). On the other hand, by the definition of \( N_\omega \), there is \( i < k_2 < \omega \) such that \( S^{N_k \upharpoonright W_i} \subseteq S^{N_{k_2} \upharpoonright W_i} \), \( R_b^{N_k \upharpoonright W_i} \subseteq R_b^{N_{k_2} \upharpoonright W_i} \) and \( V^{N_k \upharpoonright W_i}(r) \subseteq V^{N_{k_2} \upharpoonright W_i}(r) \) for all \( k' \geq k_2 \). Since all these models are tree-like, \( S^{N_k \upharpoonright W_i} \subseteq S^{N_{k_2} \upharpoonright W_i} \), \( R_b^{N_k \upharpoonright W_i} \subseteq R_b^{N_{k_2} \upharpoonright W_i} \) and \( V^{N_k \upharpoonright W_i}(r) \subseteq V^{N_{k_2} \upharpoonright W_i}(r) \). Hence, due to (3.2.14), \( S^{N_\omega \upharpoonright W_i} \subseteq S^{N_\omega \upharpoonright W_i} \), \( R_b^{N_\omega \upharpoonright W_i} \subseteq R_b^{N_\omega \upharpoonright W_i} \) and \( V^{N_\omega \upharpoonright W_i}(r) - W_i \subseteq V^{N_\omega \upharpoonright W_i}(r) \). Summarizing, it follows that \( N_\omega \upharpoonright W_i = W_i \).}

Below, we show \( N_\omega, t_0 \models \nu q. \varphi \). Let \( u \in V^{N_\omega}(q) \). So by (d), either \( u = t_0 \) or \( u \in W_j \) for some \( j < \omega \). For the former, \( W_i \subseteq R_b^{N_\omega}(W_0) \) by (iii) and \( N_\omega \upharpoonright W_i = W_i \). \( N_i \upharpoonright W_i \) by (e) for each \( i \geq 1 \). Hence \( \bigcup_{i \geq 1} W_i \subseteq \bigcup_{i \geq 1} W_i \).
Then by (d),
\[ W_0 = V^{N, \omega}(v^{N, \omega}(q) - \{t_0\})(q) - \{t_0\}. \]
(3.2.15)
Due to \( N_\omega \upharpoonright W_0 = w_0 \) \( N_\omega \upharpoonright W_0 \) by (e), (3.2.7), (3.2.8) and (3.2.15), we get
\[ N_\omega \upharpoonright (V^{N, \omega}(q) - \{t_0\}) = w_0 \] \( N_\omega \upharpoonright (V^{N_0}(q) - \{t_0\}) \).

Further, by Proposition 3.16 (4), it follows from \( N_0, t_0 \models \varphi \) that \( N_\omega, t_0 \models \varphi \).
For the latter, \( N_{j+1}, u \models \varphi \) by (vii), \( N_\omega \upharpoonright W_{j+1} = W_{j+1} \) \( N_{j+1} \upharpoonright W_{j+1} \) by (e)
and \( W_{j+1} \subseteq V^{N_\omega}(q) \) by (d). These imply \( N_\omega, u \models \varphi \) due to (vi). Thus
\[ V^{N_\omega}(q) \subseteq \| \varphi \|^{N_\omega} = \| \varphi \|^{N_{0, \varphi + V^{N_\omega}(q)}}. \]

Hence,
\[ t_0 \in V^{N_\omega}(q) \in \{ T \subseteq S^{N_\omega} : T \subseteq \| \varphi \|^{N_{0, \varphi + T}} \}. \]

By the semantics of \( \nu q, \varphi, N_\omega, t_0 \models \nu q, \varphi \).
Finally, to complete the proof, it remains to show:

**Claim 1.** \( Z_\omega : M^{[q < \omega \iota T]} \models q \subseteq_{(A_1, A_2)} N_\omega, t_0 \).

Firstly, it follows from \( sZ_\omega t_0 \) and \( t_0 \in S^{N_\omega \upharpoonright T_\omega} \) that \( sZ_\omega t_0 \) due to (c). Let \( vZ_\omega v' \). By the definition of \( Z_\omega \), \( \exists i < \omega \forall j \geq i (vZ_\omega v') \). So \( \forall j \geq i (v'_i \in S^{N_\omega}) \). The condition\( \{\varphi\}_\{\omega\}-\{\omega\} \)-\{\varphi\}_\{\omega\}-\{\omega\} holds trivially. We below check (forth) and (back).

**(forth)** Let \( v'R^{M}_{b} w' \) and \( b \in A - A_2 \). Since \( vZ_{i+1} v' \) and \( vR^{M}_{b} w' \), we get \( v'R^{N_{i+2}}_{b} w' \) and \( wZ_{i+1} w' \) for some \( w' \in S^{N_{i+1}} \).
In the following, we analyze three cases based on \( v' \in S^{N_{i+1}} \).

**Case 1.** \( v' \in S^{N_{i} \upharpoonright T_{i}} - W_{i} \). By the definition of \( N_{i+1} \) and \( Z_{i+1} \), we have \( w' \in S^{N_{i+1} \upharpoonright T_{i+1}} \), \( v'R^{N_{i}}_{b} w' \) and \( wZ_{i+1} w' \). Then, \( v'R^{N_{i+1}}_{b} w' \) due to \( v'R^{N_{i+1} \upharpoonright T_{i+1}}_{b} w' \) and (b), and \( wZ_{i+1} w' \) follows from \( w' \in S^{N_{i} \upharpoonright T_{i}} \), \( wZ_{i+1} w' \) and (c), as desired.

**Case 2.** \( v' \in W_{i} \). By (3.2.11), \( T_{i+1} \subseteq R^{+}_{N_{i+1}}(W_{i}) \). Since \( v' \in W_{i} \), \( v'R^{N_{i+1} \upharpoonright T_{i+1}}_{b} w' \) and \( N_{i+1} \) is tree-like, it is evident that \( w' \notin R^{N_{i+1} \upharpoonright T_{i+1}}_{i+1} \). So, \( w' \in S^{N_{i+1} \upharpoonright T_{i+1}} \) and \( v'R^{N_{i+1} \upharpoonright T_{i+1}}_{b} w' \). Further, due to \( v'R^{N_{i+1} \upharpoonright T_{i+1}}_{b} w' \) and (b), we get \( v'R^{N_{i+1}}_{b} w' \), and due to \( w' \in S^{N_{i+1} \upharpoonright T_{i+1}} \) and \( wZ_{i+1} w' \), by (c), we have \( wZ_{i+1} w' \).

**Case 3.** \( v' \in S^{N_{i+1}} - S^{N_{i}} \). If \( w' \in S^{N_{i+1} \upharpoonright T_{i+1}} \), then \( v'R^{N_{i+1}}_{b} w' \) which implies \( v'R^{N_{i+1}}_{b} w' \) by (b), and \( wZ_{i+1} w' \) due to \( wZ_{i+1} w' \) by (c).
Otherwise, we get \( w' \notin S^{N_{i+2}} \). Since \( N_{i+1} \) is tree-like, from \( v'R^{N_{i+1} \upharpoonright T_{i+1}}_{b} \) and \( v' \in S^{N_{i+2}} \), it follows that \( v' \in W_{i+1} \). Consequently, the transition \( v' \rightarrow w' \) will be removed for \( N_{i+2} \) so that we have to search for another \( b \)-labelled transition outgoing from \( v' \) in \( N^{\omega} \) in order to match the transition \( v' \rightarrow w \) w.r.t. \( Z_\omega \). Since \( vZ_{i+2} v' \) and \( vR^{M}_{b} w' \), we have that \( v'R^{N_{i+2}}_{b} w' \) and \( wZ_{i+2} w' \). Similar to the analysis for Case 2, due to \( v' \in W_{i+1} \) and \( v'R^{N_{i+2}}_{b} w' \), we have \( w' \in S^{N_{i+2} \upharpoonright T_{i+2}} \) and \( v'R^{N_{i+2} \upharpoonright T_{i+2}}_{b} w' \). Thus \( v'R^{N_{i+2}}_{b} w' \) by (b). Moreover, \( w' \in S^{N_{i+2} \upharpoonright T_{i+2}} \) and \( wZ_{i+2} w' \), it follows by (c) that \( wZ_{i+2} w' \), as desired.

**(back)** Let \( v'R^{N_{i}}_{b} w' \) and \( b \in A - A_1 \). By the definition of \( R^{N_{i}} \), \( \exists k < \omega \forall j \geq k (v'R^{N_{i}}_{b} w') \). Let \( b = \max\{i, k\} \). So \( v'R^{N_{i}}_{b} w' \) and \( v'R^{N_{i+1}}_{b} w' \). Thus, we get \( v'R^{N_{i+1} \upharpoonright T_{i+1}}_{b} w' \) and \( w' \in S^{N_{i+1} \upharpoonright T_{i+1}} \) by the construction of \( N_{i+1} \). As \( v'R^{N_{i}}_{b} w' \) and \( vZ_{i} w' \), we have \( vR^{M}_{b} w' \) and \( wZ_{i} w' \) for some \( w \in S^{M} \). Moreover, \( wZ_{i} w' \) due to \( wZ_{i} w' \), \( w' \in S^{N_{i} \upharpoonright T_{i+1}} \) and (c).
Lemma 3.21. \( \models \mu q.\exists_{(A_1,A_2)} \varphi \rightarrow \exists_{(A_1,A_2)} \mu q.\varphi \) whenever \( \mu q.\varphi \in df \) is satisfiable.

Proof. By the inductive characterization idea of the least fixed point of monotone functions (see, e.g., [2]), it holds that

\[ M, s \models \mu q.\exists_{(A_1,A_2)} \varphi \text{ iff } s \in \| \exists_{(A_1,A_2)} \varphi \|_\tau \text{ for some ordinal } \tau, \]

where \( \| \exists_{(A_1,A_2)} \varphi \|_\tau \) is defined by

\[ \| \exists_{(A_1,A_2)} \varphi \|_0 \triangleq \emptyset, \text{ and } \]

\[ \| \exists_{(A_1,A_2)} \varphi \|_{\tau+1} \triangleq \| \exists_{(A_1,A_2)} \varphi \|_{\tau} \cup \{ \mu q.\varphi \} \]

Assume that \( M, s \models \mu q.\exists_{(A_1,A_2)} \varphi \) and \( \mu q.\varphi \in df \) is satisfiable. So \( s \in \| \exists_{(A_1,A_2)} \varphi \|_\tau \)

for some least ordinal \( \tau \). Clearly, \( \tau \neq 0 \). Next, it suffices to show that

\[ s \in \| \exists_{(A_1,A_2)} \varphi \|_\tau \text{ implies } M, s \models \exists_{(A_1,A_2)} \mu q.\varphi. \]

We show this inductively over \( \tau > 0 \). By Proposition 3.17, we intend to construct a pointed model which \( q \)-restricted \( (A_1,A_2) \)-refines \( (M,s) \) and satisfies \( \mu q.\varphi \).

**Base case** \( \tau = 1 \) Assume that \( s \in \| \exists_{(A_1,A_2)} \varphi \|_1 \). Then \( s \in \| \exists_{(A_1,A_2)} \varphi \|_{M^{[q \rightarrow 0]}} \).

Namely, \( M^{[q \rightarrow 0]}, s \models \exists_{(A_1,A_2)} \varphi \). By Proposition 3.16, for some tree-like model \( N \) with the root \( t \) and \( Z_1 \), we have that \( Z_1 : M^{[q \rightarrow 0]}, s \models_{(A_1,A_2)} N \models \varphi \),

\[ Z_1^{-1}(V^N(q)) \subseteq V^{M^{[q \rightarrow 0]}}(q) = \emptyset, \quad t \notin V^N(q) \text{ due to } s \notin V^{M^{[q \rightarrow 0]}}(q), \text{ and } \]

\[ N \models V^N(q), t \models \varphi. \]

Let \( W \triangleq S^N \cap V^N(q) \cap V^N(q) \). Then \( N \models V^N(q) = N \uparrow W, \)

\[ V^{N \uparrow W}(q) = W \text{ and } Z_1^{-1}(W) = \emptyset. \]

Furthermore, due to \( N \models V^N(q), t \models \varphi \) and \( N \models V^N(q) = N \uparrow W, t \models \varphi \).

For each \( u \in W, \) as \( \mu q.\varphi \) is satisfiable, we may choose arbitrarily and fix a tree-like model \( N_u \) with the root \( v_u \) such that \( N_u, v_u \models \mu q.\varphi \).

W.l.o.g., we assume that all the models in \( \{ N \} \cup \{ N_u \}_{u \in W} \) are pairwise disjoint. Let \( N_1 \triangleq (N, t) \cup W \{ (N_u, v_u) \}_{u \in W} \) (See Definition 3.18). Clearly,

\[ W \subseteq \| \mu q.\varphi \|_1. \quad (3.2.16) \]

Although, for each \( u \in W, \) we do not know whether the assignments of the propositional letters \( \text{Atom} - \{ q \} \) in \( N_u \) at \( v_u \) agree with the ones in \( N \) at \( u \) or not, it is fortunate that except this, \( N_1^{[q \rightarrow \uparrow W]} \models W \) and \( N \uparrow W \) coincide, so that by Proposition 3.16 (4), due to \( V^{N \uparrow W}(q) = W, t \notin V^N(q) \) and \( N \uparrow W, t \models \varphi \), it still holds that

\[ N_1^{[q \rightarrow \uparrow W]}, t \models \varphi. \]

So by the monotonicity of \( \lambda X.\| \varphi \|_{N_1^{[q \rightarrow \uparrow X]}}, \) it follows from 3.2.16 that

\[ N_1^{[q \rightarrow \uparrow \mu q.\varphi]}, t \models \varphi. \]

This, together with \( \models \varphi[\mu q.\varphi/q] \rightarrow \mu q.\varphi \), implies \( N_1, t \models \mu q.\varphi \).

Since \( Z_1^{-1}(W) = \emptyset, \) it should be evident that

\[ Z_1 \cap (S^M \times S^{N_1}) : M^{[q \rightarrow 0]}, s \models_{(A_1,A_2)} N_1, t. \]

Then \( M, s \models_{(A_1,A_2)} N_1, t. \) Hence \( M, s \models \exists_{(A_1,A_2)} \mu q.\varphi \) by Proposition 3.17.
Let \( \varphi \). Thus, by Proposition \( 3.16 \) we get a tree-like model \( N \) with the root \( t \) and an \emph{injective} relation \( S \) from \( S^N \) to \( S^N \) such that \( \exists \langle A_1, A_2 \rangle \varphi \). Therefore, by the same reasoning as in \( 3.16 \), we can get an \emph{injective} relation from \( S^N \) to \( S^N \), and \( \exists \langle A_1, A_2 \rangle \varphi \) is valid. Due to Proposition \( 3.16 \), we can get \( N \models \exists \langle A_1, A_2 \rangle \varphi \).

For each \( u \in W \cap \pi_2(Z_r) \), as \( \exists \langle A_1, A_2 \rangle \varphi \) is valid. For all \( \psi \in \mathcal{L}_{CC}^\mu \), \( \models \psi \) implies \( \models \psi \).

\textbf{Proof.} As usual, it suffices to prove that all the axiom schemata are valid and all the rules are sound. The axiom schemata and rules from the axiom system for CCRML in \( 12 \), do not focus on fixed point operators. Hence by the same proof as in \( 12 \), these axiom schemata are valid and these rules sound. The axiom schema \( F1 \) is valid by the semantics of fixed points, and the rule \( F2 \) is sound by the semantics of the least fixed points. It is trivial to check that the axiom schema \( CCR_{\text{in}} \) is valid. Due to Lemma \( 3.19 \) and Lemma \( 3.20 \) the axiom schema \( CCR_{\text{in}} \) is valid. From Lemma \( 3.19 \) and Lemma \( 3.21 \) it follows that the axiom schema \( CCR_{\text{in}} \) is valid.
3.3 Completeness

This subsection devotes itself to establishing the completeness of the axiom system CCRML\(^\mu\). This follows by the same method as in [6]. By this method, the completeness of CCRML has been gotten in [12]. We will prove that every \(L_{CC}^\mu\)-formula is provably equivalent to a \(K^\mu\)-formula, which brings the completeness of CCRML\(^\mu\) based on the completeness of \(K^\mu\).

We start with several general statements as the preparations for the reduction argument.

**Proposition 3.23.** Let \(\varphi_1, \varphi_2, \psi \in L_{CC}^\mu\) and \(p \in \text{Atom}\) such that there is no sub-formula of the form \(\eta p.\alpha\) in \(\psi\). Then

\[\vdash \varphi_1 \iff \varphi_2 \implies \vdash \psi[\varphi_1/p] \iff \psi[\varphi_2/p].\]

**Proof.** Proceed by induction on \(\psi\).

**Proposition 3.24.**

1. \(\vdash \forall (a_1, a_2) (\varphi \land \psi) \iff \forall (a_1, a_2) \varphi \land \forall (a_1, a_2) \psi.\)
2. \(\vdash \exists (a_1, a_2) (\varphi \lor \psi) \iff \exists (a_1, a_2) \varphi \lor \exists (a_1, a_2) \psi.\)
3. \(\vdash \forall (a_1, a_2) (\varphi \land \psi) \implies \forall (a_1, a_2) (\varphi \lor \psi).\)
4. \(\vdash \exists (a_1, a_2) (\varphi \land \psi) \implies \exists (a_1, a_2) \varphi \land \exists (a_1, a_2) \psi.\)

**Proof.** Trivially.

**Proposition 3.25 ([12]).** For any \(\alpha \in L_p\), we have

1. \(\vdash \forall (a_1, a_2) \alpha \iff \alpha\)
2. \(\vdash \exists (a_1, a_2) \alpha \iff \alpha\)

**Proposition 3.26.** Let \(\alpha \in L_p\) and \(\varphi \in L_{CC}^\mu\). We have

\[\vdash \exists (a_1, a_2) (\alpha \land \varphi) \iff (\alpha \land \exists (a_1, a_2) \varphi).\]

**Proof.** See [12, Proposition 4.13].

At this point, we can show that any formula of the form \(\exists (a_1, a_2) \alpha\) with \(\alpha \in L_K^\mu\) can be provably reduced to an \(L_K^\mu\)-formula.

**Proposition 3.27.** Let \(\alpha \in L_K^\mu\). Then

\[\vdash \exists (a_1, a_2) \alpha \iff \xi \text{ for some } \xi \in L_K^\mu.\]

**Proof.** By Proposition 3.5 we may w.l.o.g. assume that \(\alpha\) is a df formula. Then proceed by induction on the structure of \(\alpha\).

For \(\alpha \in L_p\), it follows by proposition 3.25 (2).

For \(\alpha \equiv \alpha_1 \lor \alpha_2\), by Proposition 3.24 (2), we have

\[\vdash \exists (a_1, a_2) \alpha \iff \exists (a_1, a_2) \alpha_1 \lor \exists (a_1, a_2) \alpha_2.\]

So, by the induction hypothesis and Proposition 3.23, we get

\[\vdash \exists (a_1, a_2) \alpha \iff \xi_1 \lor \xi_2 \text{ for some } \xi_1, \xi_2 \in L_K^\mu.\]

For \(\alpha \equiv \nu q \varphi\), by the axiom schema CCR\(^\mu\), we get

\[\vdash \exists (a_1, a_2) \alpha \iff \nu q \exists (a_1, a_2) \varphi.\]
Then, by the induction hypothesis and Proposition 3.23 it holds that
\[ \vdash \exists (a_1, a_2) \alpha \leftrightarrow \nu q. \xi \quad \text{for some } \xi \in L^K_{\mu}. \]

For \( \alpha \equiv \mu q. \varphi \), if \( \vdash K^\mu \alpha \leftrightarrow \bot \) by CCRIn, else the analysis is similar as in the case with \( \alpha \equiv \nu q. \varphi \) by CCR\( ^\mu \).

For \( \alpha \equiv a_0 \land \bigwedge_{b \in B} \nabla_b \Phi_b \) with \( a_0 \in L_p \) and \( \Phi_b \subseteq df \) for each \( b \in B \), by Proposition 3.26 it follows that
\[ \vdash \exists (a_1, a_2) \alpha \leftrightarrow \alpha_0 \land \bigwedge_{b \in B} \exists (a_1, a_2) \nabla_b \Phi_b. \]

Further, by the axiom schema CCRKconj and Proposition 3.23 we obtain
\[ \vdash \exists (a_1, a_2) \alpha \leftrightarrow \alpha_0 \land \bigwedge_{b \in B} \exists (a_1, a_2) \nabla_b \Phi_b. \]

Thus the proof is completed by checking that, for each \( b \in B \),
\[ \vdash \exists (a_1, a_2) \nabla_b \Phi_b \leftrightarrow \xi_b \quad \text{for some } \xi_b \in L^K_{\mu}. \quad (3.3.1) \]

Since \( \Phi_b \subseteq df \subseteq L^K_{\mu} \), applying the axiom schemata CCRKco1, CCRKco2, CCRKcontra and CCRKbis, we get \( \vdash \exists (a_1, a_2) \nabla_b \Phi_b \leftrightarrow \gamma \) for some \( \gamma \) in which the operators \( \exists (a_1, a_2) \) are over only the formulas in \( \Phi_b \). Consequently, by the induction hypothesis and Proposition 3.23 the claim (3.3.1) holds.

Now we can prove that all \( L_{\mu CC} \)-formulas can be provably reduced to \( L^K_{\mu} \)-formulas. This is the crucial step in establishing the completeness of CCRML\( ^\mu \).

**Proposition 3.28.** For each \( \psi \in L_{\mu CC}^{\mu} \),
\[ \vdash \psi \leftrightarrow \xi \quad \text{for some } \xi \in L^K_{\mu}. \]

**Proof.** Proceed by induction on the number of the occurrences of the CC-refinement quantifiers in \( \psi \). See [12, Proposition 4.16].

**Proposition 3.29.** Let \( \psi \in L_{\mu CC}^{\mu} \) and \( \alpha \in L^K_{\mu} \) such that \( \vdash \psi \leftrightarrow \alpha \). If \( \alpha \) is a theorem in \( K^\mu \), then so is \( \psi \) in CCRML\( ^\mu \).

**Proof.** Since the axiom system \( K^\mu \) is contained in CCRML\( ^\mu \), \( \vdash \alpha \) due to \( \vdash K^\mu \alpha \). Hence, \( \vdash \psi \) follows immediately from \( \vdash \psi \leftrightarrow \alpha \).

**Theorem 3.30** (Completeness). For all \( \psi \in L_{\mu CC}^{\mu} \), \( \models \psi \) implies \( \vdash \psi \).

**Proof.** Let \( \psi \in L_{\mu CC}^{\mu} \) and \( \models \psi \). By Proposition 3.28 \( \vdash \psi \leftrightarrow \xi \) for some \( \xi \in L^K_{\mu} \). So by Theorem 3.22 we obtain \( \models \psi \leftrightarrow \xi \), which implies \( \models \xi \) due to \( \models \psi \). Then \( \vdash K^\mu \xi \) due to the completeness of \( K^\mu \). Hence, \( \vdash \psi \) holds by Proposition 3.29.

From our proof that every \( L_{\mu CC}^{\mu} \)-formula can be provably reduced to an \( L^K_{\mu} \)-formula, it follows easily that there is an algorithm for transforming every \( L_{\mu CC}^{\mu} \)-formula \( \psi \) into an \( L^K_{\mu} \)-formula \( \alpha \) such that \( \vdash \psi \iff \vdash K^\mu \alpha \). Then, due to the decidability of the system \( K^\mu \), we get

**Theorem 3.31** (Decidability). CCRML\( ^\mu \) is decidable.
4 Discussion

The notion of CC-refinement generalizes the notions of bisimulation, simulation and refinement. This paper presents a sound and complete axiom system for the CC-refinement operators $\exists(A_1, A_2)$ and fixed-point operators under the assumption that neither $A_1$ nor $A_2$ is empty. This assumption is not particularly restrictive. Analogizing $\exists(A_1, A_2)$, based on Proposition 2.3, the operators $\exists(\emptyset, A_2)$ (or, $\exists(A_1, \emptyset)$) can be reduced to the operators $\exists(\emptyset, a)$ (or, $\exists(a, \emptyset)$, resp.). Further, from the axiom system in Table 1, through modifying it slightly, we can obtain the axiom system for $\exists(\emptyset, A_2)$ (or, $\exists(A_1, \emptyset)$) and fixed-point operators. The former axiom system obtained, with $|A| = 1$, is indeed the one provided in [6]. The reader may be referred to [12, Section 5] for these modifications. Furthermore, the relative proofs and constructions with minor modifications still work.

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