Parameterized String Equations

Laurent Bulteau
LIGM, CNRS, Université Gustave Eiffel, Marne-la-Vallée France
laurent.bulteau@univ-eiffel.fr

Michael R. Fellows
Department of Informatics University of Bergen Bergen, Norway
michael.fellows@uib.no

Christian Komusiewicz
Fachbereich für Mathematik und Informatik, Philipps-Universität Marburg, Germany
komusiewicz@informatik.uni-marburg.de

Frances Rosamond
Department of Informatics University of Bergen Bergen, Norway
frances.rosamond@uib.no

Abstract
We study systems of String Equations where block variables need to be assigned strings so that their concatenation gives a specified target string. We investigate this problem under a multivariate complexity framework, searching for tractable special cases such as systems of equations with few block variables or few equations. Our main results include a polynomial-time algorithm for size-2 equations, and hardness for size-3 equations, as well as hardness for systems of two equations, even with tight constraints on the block variables. We also study a variant where few deletions are allowed in the target string, and give XP algorithms in this setting when the number of block variables is constant.

2012 ACM Subject Classification Theory of computation → Parameterized complexity and exact algorithms

Keywords and phrases String Equations, String Morphism, Parameterized Algorithms, Parameterized Complexity

Acknowledgements We want to thank Shonan Workshop 045 in which this work was initiated

1 Introduction

String equations are equations in which the variables, called block variables in this work, can be assigned arbitrary strings. We study the case where a string equation can express that the concatenation of a given list of blocks must equal a specific string. We study the problem of finding a solution for a given set of string equations within a multivariate complexity framework, searching for tractable special cases such as systems of equations with few blocks variables or few equations.

Related Work

The single-equation problem can be seen as a String Morphism question, for which many variants have been extensively studied in the literature (see Fernau et al. for an overview of related works). A possible starting point is the pattern discovery problem: given a set of strings, find a pattern (i.e. a string of characters and variables), of minimal length, that may generate each input string by assigning values to each block variable. Deciding if a given pattern generates a string can be seen, in our setting, as a single string equation. Angluin [1] Theorem 3.2.3 proves that this problem is NP-hard, by reduction from SAT. She also identifies the brute-force $O(n^k)$ algorithm (which requires linear space [9]). Fernau et
Parameterized String Equations

\[
\begin{align*}
abcab & \equiv AB \\
abcd\overline{abc}d & \equiv AC\overline{AC} \\
ab & \equiv BC
\end{align*}
\]

\[\sigma(A) := abc\]
\[\sigma(B) := ab\]
\[\sigma(C) := d\]

\[\sigma(d) := \sigma(A)\]

**Figure 1** Left: a string equation system with alphabet \(\Sigma = \{a, b, c, d\}\) and block variables \(B = \{A, B, C\}\). Right: a solution (assignment) \(\sigma\) for this system, with the corresponding substrings highlighted with different colors in the target strings.

al. [7, 8] further prove (among other variants) \(\text{W}[1]\)-hardness when parameterized by the number of block variables. A notable variant is the injective case (where blocks must be assigned distinct strings) which in its simplest version can be formulated as the following NP-hard problem [6]: given a string \(T\) and an integer \(k\), can \(T\) be split into \(k\) distinct factors?

For two equations, if the blocks are used once in each equation, the String Equations formulation can be related to Minimum Common String Partition (where two target strings are given, but the block variables can be permuted in any order). Interestingly, this problem is FPT for the number of blocks [3] but becomes \(\text{W}[1]\)-hard if the permutation of the blocks is fixed (Theorem 7).

**Formal definitions**

Given a string \(S\), we write \(|S|\) for the length of \(S\), and \(S[i]\) for the \(i\)th letter of \(S\) (for \(1 \leq i \leq |S|\)). We use \(\prod_{i=1}^{n} S_i := S_1S_2\ldots S_n\) to denote the concatenation of strings \(S_1\) to \(S_n\). We distinguish *substrings* (obtained by taking consecutive characters from \(s\)) from *subsequences* (obtained by taking not-necessarily consecutive characters from \(s\)).

Given an alphabet \(\Sigma\) and a set \(B\) of block variables (also called blocks for simplification), with \(\Sigma\) and \(B\) being disjoint, a *string equation* over \((\Sigma, B)\) is a pair \((T, X)\), written \(T \equiv X\), where \(T\) (resp. \(X\)) is a non-empty string over \(\Sigma\) (resp. \(B\)). \(T\) is the *target string* of the equation. A *system of equations* is a set of string equations. A block which appears only once in a given system of equations is said to be a joker; and is denoted \(*\). A system of equations is *duplicate-free* if no block appears twice in the same string \(X\).

An *assignment* of the block variables is a function \(\sigma : B \rightarrow \Sigma^*\) that assigns to each block variable a non-empty string over \(\Sigma\). For \(X = X_1X_2\ldots X_c \in B^*\), we write \(\sigma(X) = \prod_{i=1}^{c} \sigma(X_i)\). An equation \(T \equiv X\) is *satisfied* by an assignment \(\sigma\) if \(T = \sigma(X)\), that is, replacing \(X\) by the concatenation of the corresponding strings yields exactly \(T\). A system of equations is satisfied if all equations are satisfied. An example is given in Figure 1.

The String Equations Problem and its Variants

We define the String Equations problem as follows:

**Input:** A system of equations \(E = \{T_1 \equiv X_1, T_2 \equiv X_2, \ldots, T_r \equiv X_r\}\).

**Question:** Does \(E\) admit a satisfying assignment \(\sigma\)?

We consider the *duplicate-free* restriction of String Equations, where no block may appear twice in any \(X_i\). We also consider a more general problem where a number of deletions is allowed, String Equations with Deletions.

**Input:** A system of equations \(E = \{T_1 \equiv X_1, T_2 \equiv X_2, \ldots, T_r \equiv X_r\}\), an integer \(d\)

**Question:** Does there exist \(r\) strings \(T'_1, \ldots, T'_r\) obtained from \(T_1, \ldots, T_r\) by removing a total of at most \(d\) letters, such that the equations \(T'_i \equiv X_i\) admit a satisfying assignment \(\sigma\)?
Figure 2 Running example for reductions from Clique (or Multi-colored Clique): a (3-colored) graph $G = (V,E)$ where $V = \{a, b, c, d\}$ and $E = \{ab, ac, bc, bd, cd\}$. For $\kappa = 3$, $G$ contains two triangles: $\{a, b, c\}$ and $\{b, c, d\}$.

Finally, motivated by the similarities with the Minimum Common String Partition problem, we are interested in the restriction with $r = 2$ duplicate-free equations without joker blocks. In this setting, we consider the variation where blocks may be assigned empty strings (remember that by default, $\sigma$ must be non-erasing, i.e. $\sigma(T)$ is not empty), see Theorem 7.

Since the size of each $\sigma(A)$ can be bounded by the input size, and it is trivial to verify whether an equation is satisfied by a given assignment, String Equations is clearly in NP. Similarly, String Equations with Deletions and all variants mentioned here are also clearly in NP.

Parameterized Complexity

For an introduction to parameterized complexity theory, we refer to the monograph of Downey and Fellows [4]. An instance $(x, \kappa)$ of a parameterized problem consists of a classical problem instance $x$ and a parameter $\kappa \in \mathbb{N}$. A parameterized problem is fixed-parameter tractable, or equivalently is in FPT, if there exists an algorithm that solves every instance $(x, \kappa)$ in $f(\kappa) \cdot |x|^{O(1)}$ time. A parameterized problem has an XP-algorithm, or equivalently is in XP, if there exists an algorithm that solves every instance $(x, \kappa)$ in $|x|^{g(\kappa)}$ time. To show that problems in XP are unlikely to be in FPT, one may use parameterized reductions from W[1]-hard problems, i.e., polynomial-time reduction between parameterized problems such that the parameter of the reduced instance is a function of the original parameter.

We use the following classic W[1]-hard problem in our reduction.

\textbf{Clique}

\textbf{Input:} A graph $G = (V,E)$

\textbf{Question:} Does $G$ contain a clique of size $\kappa$?

In addition, we use a constrained version of the problem, where the vertices are colored so that a clique must use exactly one vertex with each color:

\textbf{Multicolored Clique}

\textbf{Input:} A $\kappa$-partite graph $G = (V = \bigcup_{i=1}^{\kappa} V_i, E)$.

\textbf{Question:} Does $G$ contain a clique of size $\kappa$ in $G$?

Multicolored Clique is W[1]-hard for parameter $\kappa$ [5, 10]. An example instance is given in Figure 2, it will be used as an example for our reductions. By convention we write $n = |V|$, $m = |E|$, $V = \{v_1, \ldots, v_n\}$, $E = \{e_1, \ldots, e_m\}$. For convenience, we assume that edges are size-2 strings over alphabet $V$, that is, $e_j = v_s v_t$ for an edge $\{v_s, v_t\}$ with $s < t$. Moreover, we assume that $E$ is sorted lexicographically, that is, for $e_j = v_s v_t$ and $e_{j'} = v_{s'} v_{t'}$ we have $j < j'$ iff $s < s'$ or $s = s'$ and $t < t'$.

2 Overview of the results

In our complexity study, we consider the following five quantities:

- $t$, the maximum size of any target string
Parameterized String Equations

- \( r \), the number of equations
- \( c \), the maximum number of blocks per equation,
- \( k \), the overall number of blocks
- \( d \), the number of deletions (for String Equations with Deletions)

We only consider \( r, c, k \) and \( d \) as possible parameters (i.e. we always assume \( t \) to be unbounded). Observe that \( k \leq rc \). Consequently, a positive result for parameter \( k \) implies a positive result for \( rc \) and a running time lower bound for \( r + c \) implies a running time lower bound for \( k \). An overview of our results is given in Table 1.

We first show in Section 3 that the problem is polynomial-time solvable when \( k \) is any constant (i.e. \( XP \) for \( k \), Proposition 1) using brute-force enumeration. This raises the possibility of an FPT algorithm for parameter \( k \), which will be ruled out in the next sections, with hardness results proving \( W[1] \)-hardness for parameter \( k \) even in more restricted settings. We also consider the deletions variant: our \( XP \) algorithm extends to parameter \( k + d \) (Proposition 2), but with parameter \( k \) alone, we show that it is already \( NP \)-hard for \( k = 1 \) (Theorem 3). However, it is in \( XP \) for parameter \( r + c \) (Proposition 1).

We then focus, in Section 4, on the number \( r \) of equations. \( W[1] \)-hardness is already known for \( r = 1 \) and parameter \( c \) \([1, 8]\). We prove a similar result in the duplicate-free setting when \( r = 2 \) (\( r = 1 \) is a trivial case for duplicate-free equations). Note that \( W[1] \)-hardness follows for parameter \( k \), since \( k \in O( c ) \) for constant \( r \). We consider two slightly stronger cases: one where both target strings are equal, and the other where block variables may be assigned empty strings.

Finally, Section 5 is devoted to small-size equations (constant \( c \)). On the negative side, String Equations remains \( W[1] \)-hard for parameter \( r \) (and \( k \)) when \( c = 3 \). On the positive side, we give a polynomial-time algorithm for size-2 equations, which generalizes to the case where non-joker blocks must all be prefixes or suffixes of their equations. As a final attempt towards generalizing this algorithm to other cases, we consider equation systems with \( (r - 1) \) size-2 equations together with a single size-\( c \) duplicate-free equation. We show, however, that this special case is \( W[1] \)-hard for parameter \( r + c \) (Theorem 10).

### 3 Constant number of blocks

A straightforward brute-force algorithm that guesses endpoints of each block shows that String Equations is in \( XP \).

**Proposition 1.** String Equations can be solved in \( O^*(t^{2k}) \) time.

**Proof.** For each block \( X \in \mathcal{B} \), pick an equation \( T \equiv X \) where \( X \) appears in \( X \), and branch into all possibilities to choose integers \( 1 \leq i \leq j \leq |T| \leq t \) such that \( \sigma(X) := T[i] \ldots T[j] \). Overall, this creates \( O(t^{2k}) \) branches. For each branch, it remains to check whether the resulting assignment satisfies \( E \), which can be done in linear time. The total running time is thus \( O^*(t^{2k}) \).

The algorithm above can be extended to the case where we allow for deletions in the target strings.

**Proposition 2.** String Equations with Deletions is in \( XP \) for parameter \( k + d \).

**Proof.** Branch into the at most \( t^r \) ways of building strings \( T_1', \ldots, T_r' \) out of \( T_1, \ldots, T_r \). Testing each time if the resulting String Equation problem has a solution can be done via Proposition 1 so the overall running time is \( O^*((nr)^d t^{2k}) \).
These results still hold in a stronger setting: Theorem 6 holds for duplicate-free equations with a subsequence of strings \( X \) each.

Compute the starting points of each block in each equation: by the above remark, for \( T \) the starting point of block \( \sigma \) assignment Proposition 4.

\[ k \ r \ \ c \ d \quad \text{Complexity of String Equations} \]

- C * * 0 XP (Proposition 1)
- C * * C XP (Proposition 2)
- C * * P open
- 1 1 P * W[2]-hard (Theorem 3)
- 1 P 1 * W[2]-hard (Theorem 7)
- C C C * XP (Proposition 4)

- P 1 P 0 W[1]-hard (Theorem 3)
- P 2 P 0 \( ^{†} \) W[1]-hard (Theorem 6 and 7)
- P * 2 0 \( ^{†} \) P (Proposition 8)
- P * 2 P open
- P P 0 \( ^{†} \) W[1]-hard (Theorem 8)
- P P 0 \( ^{†} \) W[1]-hard (Theorem 8)

**Table 1** Summary of our results. For each of \( k \) (number of block variables), \( r \) (number of equations), \( c \) (equation size) and \( d \) (number of deletions), we indicate if the result holds for the given integer, for any fixed constant (C, for XP algorithms), when seen as a parameter (P, for FPT or W[1]-hardness) or is unbounded (*).

\( ^{†} \) These results still hold in a stronger setting: Theorem 6 holds for duplicate-free equations with a unique target. Theorem 7 holds for duplicate-free equations allowing empty blocks. Proposition 8 holds for arbitrarily large equations, provided non-border blocks are all jokers. Theorem 10 holds for duplicate-free equations, and all equations except one have size 2.

**Theorem 3.** String Equations with Deletions is NP-hard for \( k = 1 \). Moreover, String Equations with Deletions is W[2]-hard for parameter \( r \) when \( k = c = 1 \) and W[2]-hard for parameter \( c \) when \( k = r = 1 \).

**Proof.** We note that String Equations with Deletions contains Longest Common Subsequence as a sub-problem, using a single block \( X \) and equations \( T_1 \equiv X, \ldots, T_r \equiv X \) to denote the fact that \( \sigma(X) \) should be a common subsequence of each \( T_i \). In this setting, minimizing the number of deletions \( d = \sum_{i=1}^{r} |T_i| - r|\sigma(X)| \) is equivalent to maximizing \( |\sigma(X)| \), i.e. the length of the common subsequence. Since LCS is W[2]-hard for the parameter number of strings \( r \) \[2\], the same result applies to String Equations with Deletions for \( k = c = 1 \).

For the case with a single equation \( (r = 1) \), it suffices to insert a sufficiently long prefix \( P = \sum_{i}^{d} \) before each string, then again \( PT_1PT_2 \ldots PT_r \equiv XX \ldots X \) has a solution reaching the target number of deletions \( d \) iff \( T_1, \ldots, T_r \) have an LCS reaching the corresponding target size \( \frac{1}{r}(\sum_{i=1}^{r} |T_i| - d) \).

**Proposition 4.** String Equations with Deletions is in XP for parameter \( r + c \).

**Proof.** We introduce the following notation: given an equation \( T \equiv X \) satisfied by an assignment \( \sigma \), the starting point for block \( i \) is the index \( j, 1 \leq j \leq |T| \), such that the first character of \( \sigma(X[i]) \) is mapped to \( T[j] \). Thus, if \( j \) is the starting point of block \( i \) and \( j' \) is the starting point of block \( i + 1 \) (or \( j' = |T| + 1 \) if \( i = |X| \)), then \( \sigma(X[i]) \) is a subsequence of \( T[j] \ldots T[j'-1] \).

Consider a solution \( \sigma \) for an instance \( (E, d) \) of String Equations with Deletions. Compute the starting points of each block in each equation: by the above remark, for each \( X \in B \) there exist substrings of \( T_i's \) denoted \( F_1, \ldots, F_h \) such that \( \sigma(X) \) is a common subsequence of strings \( F_1, \ldots, F_h \) (with \( h \leq rc \) corresponding to the number of occurrences
Parameterized String Equations

\[ T = y_0 \ a \ c \ y_1 \ a \ d \ y_2 \ b \ c \ y_3 \ b \ d \ y_4 \ c \ d \ y_5 \]
\[ T \equiv * \ R \ G \ * \ R \ B \ * \ G \ B \ * \]

Figure 3 Reduction of Theorem 5 (using \( r = 1 \) equation) applied to the graph from Figure 2 (using a single string equation, with the target string described first, then its block decomposition). In this and following examples, in order to lighten notations, we use color initials \( R, G, B \) or \( r, g, b \) instead of indices 1, 2, 3 to highlight the fact that we select a vertex of the corresponding color in the graph (or an edge between corresponding colors, i.e. \( R, G, B \) designate \( X_1, X_2, X_3 \) respectively, \( R_b, R_g \) designate \( X_{1,2}, X_{1,3} \), etc). A satisfying assignment is given by colored underlines: \( \sigma(R) = a, \sigma(G) = c, \sigma(B) = d \).

of block \( X \). Note that replacing \( \sigma(X) \) by any longest common subsequence of these strings still yield a valid solution for the same problem, since the number of deletions performed in target strings does not increase.

This leads to the following \( XP \) algorithm: branch into all possibilities to choose a starting point of every block of \( \mathcal{E} \); the number of branches is \( O(t^c) \). For each block \( X \), compute strings \( F_1, \ldots, F_h \) as described above. Set \( \sigma(X) \) to be a longest common subsequence of these strings in time \( O^*(t^h) \): it remains to check whether \( \sigma \) is a solution to the original problem, which can be done in linear time.

4 Constant number of equations

For the sake of completeness (and as a warm-up for following, more complex reductions), we give a \( W[1] \)-hardness proof for the single-equation case (see also Fernau et al. [8]).

\textbf{Theorem 5. String Equations with one equation (that is, \( r = 1 \)) is \( W[1] \)-hard for \( c \).}

\textbf{Proof.} See Figure 3. Consider an instance \((G = (V, E), \kappa)\) of \textsc{Clique}. Introduce \( m + 1 \) separators (i.e., new characters) denoted \( y_0, \ldots, y_m \), let \( \Sigma = V \cup \{y_0, \ldots, y_m\} \). Introduce \( \kappa \) non-joker blocks \( \{X_1, \ldots, X_\kappa\} \) and \((\binom{\kappa}{2} + 1)\) joker blocks (for a total of \( k = \mathcal{O}(\kappa^2) \) blocks). Let

\[ \mathcal{E} := \{T \equiv X\} \text{ where } T := y_0 \prod_{i=1}^{m}(e_i y_j) \text{ and } \mathcal{X} := * \prod_{1 \leq i, j \leq \kappa} (X_i X_j^*) \]

This concludes the construction of the instance. We now show the correctness of the reduction.

\( (\Rightarrow) \) Assume that \( G \) has a \( \kappa \)-clique \( \{u_1, \ldots, u_\kappa\} \). Set \( X_i := u_i \). Then each \( X_i X_j \) with \( 1 \leq i < j \leq \kappa \) corresponds to an edge in \( G \), hence it appears as a substring in \( T \), in lexicographical order (recall that we assume that the edges of \( G \) are ordered). Finally, joker blocks can be matched to the gaps around selected edges, which are non-empty (they contain at least one separator), so this assignment satisfies \( \mathcal{E} \).

\( (\Leftarrow) \) Assume that \( \mathcal{E} \) has a satisfying assignment \( \sigma \). First, note that \( \sigma(X_i) \) may contain only characters from \( V \) (since \( X_i \) is repeated \( \kappa - 1 \) times and separators only once), and at most one character (otherwise, \( \sigma(X_i X_j) \) contains three characters, so by definition of \( T \) at least one separator). Thus \( X_i \in V \). Furthermore, \( X_i X_j \) is an edge of \( E \) for each \( 1 \leq i < j \leq \kappa \), so \( \{X_1, \ldots, X_\kappa\} \) is a \( \kappa \)-clique.

Only with one equation the problem is already hard, but this definitely needs duplications (since an instance having 1 duplicate-free equation is trivial). Hence we look at instances
This concludes the construction of the instance. We now show the correctness of the reduction.

Theorem 6. String Equations with \( r = 2 \) duplicate-free equations, a unique target and parameter \( c \) is \( W[1] \)-hard.

Proof. Consider an instance \( G = (V, E, \kappa) \) of Clique. Introduce \( n + m + 3 \) separators (i.e., new characters) denoted \( x_0, \ldots, x_n, y_0, \ldots, y_m \) and \( z \). Let \( \Sigma \) consist of \( V \) and the set of separators. Introduce \( 2\kappa(\kappa - 1) \) coding blocks \( \{X_{i,j}, X'_{i,j}\} \) for \( i \neq j \), a starting block \( Z \) and \( \kappa + \binom{\kappa}{2} + 2 \) gap blocks \( A_0, \ldots, A_\kappa, B_0 \) and \( B_{i,j} \) for \( 1 \leq i < j \leq \kappa \). Define the following string equations (we decompose the strings into vertex and edge sections for ease of presentation):

\[
\mathcal{E} := \{ T \equiv X, T \equiv X' \}
\]

where

\[
T := z \prod_{i=1}^{n} (e_i^{k-1} x_i) \quad y_0 \prod_{j=1}^{m} (e_j y_j)
\]

\[
X := Z A_0 \prod_{1 \leq i < j \neq \kappa} (X_{i,j} A_{i,j}) \quad Z' B_0 \prod_{1 \leq i < \kappa} \prod_{1 \leq i < j \leq \kappa} (X'_{i,j} X'_{j,i} B_{i,j})
\]

\[
X' := Z' A_0 \prod_{1 \leq i < \kappa} \prod_{1 \leq i < j \neq \kappa} (X'_{i,j} A_{i,j}) \quad Z B_0 \prod_{1 \leq i < \kappa} \prod_{1 \leq i < j \leq \kappa} (X_{i,j} X_{j,i} B_{i,j})
\]

This concludes the construction of the instance. We now show the correctness of the reduction.

(⇒) If \( G \) has a \( \kappa \)-clique \( K = \{w_1, \ldots, w_\kappa\} \). Set \( \sigma(Z) = \sigma(Z') = z \) and \( \sigma(X_{i,j}) = \sigma(X'_{i,j}) = w_i \) for all \( j \neq i \). Then each \( \sigma(\prod_{j \neq i} X_{i,j}) \) equals \( x_i^{k-1} \), and these substrings appear in the same order as in \( X \) in the vertex section of \( T \), and \( \sigma(X'_{i,j} X'_{j,i}) \), with \( i < j \), correspond to edges of \( G \) which appear in this order in the edge section of \( T \). Pick \( \sigma(A_i) \) to be the gaps around the selected vertices and \( \sigma(B_{i,j}) \) to be the gaps around selected edges. We match \( X' \) in exactly the same way with \( T \) (exchanging the roles of \( X_i \) and \( X'_i \)), so that the gaps are identical in both matchings. Thus this assignment satisfies \( \mathcal{E} \).

(⇐) If \( \mathcal{E} \) has a satisfying assignment \( \sigma \). First, note that \( \sigma(Z) = z \): The first block in \( X \) is \( Z \), so \( Z \) must start with \( z \). Moreover, \( z \) is followed by different characters in \( X \) and \( X' \) \((x_0 \text{ and } y_0)\). Similarly, \( \sigma(Z') = z \). Thus, the vertex (edge) sections of \( X \) and \( X' \) are matched only to the vertex (edge) section of \( T \). Hence, \( \sigma(X_{i,j}) \) is a substring of both sections of \( T \), which may only be single characters corresponding to some character \( w_{i,j} \). In the vertex section, characters \( \sigma(X_{i,j}) \) for \( j \neq i \) appear consecutively in \( T \) so all vertices \( w_{i,j} \) are equal (and denoted \( w_i, 1 \leq i \leq \kappa \)). In the edge section, each string \( \sigma(X'_{i,j} X'_{j,i}) = w_j \) for \( 1 \leq i < j \leq \kappa \) is a substring of \( T \) so it is an edge of \( G \), i.e. \( \{w_1, \ldots, w_\kappa\} \) is a clique in \( G \).
Parameterized String Equations

\[ T = \gamma \phi_1 \phi_2 \gamma x_0 \phi_1 \phi_2 \gamma x_1 \phi_1 \phi_2 \gamma x_2 \phi_1 \phi_2 \gamma x_3 \phi_1 \phi_2 \gamma x_4 \ldots \]

\[ T' = \gamma \phi_1 \phi_2 \gamma x_0 \phi_1 \phi_2 \gamma x_1 \phi_1 \phi_2 \gamma x_2 \phi_1 \phi_2 \gamma x_3 \phi_1 \phi_2 \gamma x_4 \ldots \]

\[ T \equiv \Gamma_0 \Gamma_0 \ldots \Gamma_{1,3} \Phi_1 \Phi_2 Z \Gamma_0 A_0 \Gamma_{0,1}^{\prime} \Gamma_{1,1} A_1 \Gamma_{0,2} G \Gamma_{1,2} A_2 \Gamma_{0,3} B \Gamma_{1,3} A_3 \ldots \]

\[ T \equiv \Gamma_0' \Gamma_0' \ldots \Gamma_{1,3} \Phi_2 \Phi_1 Z' \Gamma_0 A_0 \Gamma_{0,1}^{\prime} \Gamma_{1,1} A_1 \Gamma_{0,2} G \Gamma_{1,2} A_2 \Gamma_{0,3} B \Gamma_{1,3} A_3 \ldots \]

**Figure 5** Prefix and vertex sections in the reduction of Theorem 7 (with \( r = 2 \) equations and empty blocks) applied to the graph from Figure 2.

Building on the result above, we prove that STRING EQUATIONS is \( \mathsf{W[1]} \)-hard even if blocks are allowed to be assigned empty string sections (in this case, we need two distinct target empty blocks: having a single target would have trivial solutions by assigning empty strings to all but one blocks).

**Theorem 7.** STRING EQUATIONS allowing empty blocks with \( r = 2 \) duplicate-free equations and parameter \( c \) is \( \mathsf{W[1]} \)-hard.

**Proof.** See Figure 5. We build on the proof of Theorem 6 and create a similar instance of STRING EQUATIONS with more separating gadgets, mainly to enforce that strings \( \sigma(Z) \) and \( \sigma(X) \) may not be empty. We start from \( T, X' \) as defined in the previous proof. We introduce characters \( \gamma, \phi_1, \phi_2 \), as well as blocks \( \Gamma_0, \Gamma_0', \Gamma_{h, i}, \Gamma_{h, i} \) (with \( h \in \{0, 1\} \) and \( 1 \leq i \leq \kappa \), \( \Phi_1 \) and \( \Phi_2 \).

We build the new equation system \( \mathcal{E} \) as follows (differences to the construction in the proof of Theorem 6 are highlighted):

\[ \mathcal{E} := \{ T \equiv X', T' \equiv X' \}, \]

where:

\[ T := \gamma^{2\kappa+1} \phi_1 \phi_2 \gamma x_0 \prod_{i=1}^{n} (\gamma x_i^{k-1}) \gamma x_1 \gamma x_2 \ldots \gamma x_m \prod_{j=1}^{m} (e_j y_j) \]

\[ T' := \gamma^{2\kappa+1} \phi_2 \phi_1 \gamma x_0 \prod_{i=1}^{n} (\gamma x_i^{k-1}) \gamma x_1 \gamma x_2 \ldots \gamma x_m \prod_{j=1}^{m} (e_j y_j) \]

\[ X := \Gamma_0 \prod_{i=1}^{\kappa} \Gamma_{0, i} \Gamma_{1, i} A_2 Z \Gamma_0 A_0 \prod_{i=1}^{\kappa} (\Gamma_{0, i} \prod_{j \neq i} (X_{i, j} X_{j, i} B_{i, j})) \]

\[ X' := \Gamma_0' \prod_{i=1}^{\kappa} \Gamma_{0, i} \Gamma_{1, i} A_2 Z' \Gamma_0 A_0 \prod_{i=1}^{\kappa} (\Gamma_{0, i} \prod_{j \neq i} (X_{i, j} X_{j, i} A_{i, j})) \]

This concludes the construction of the instance. We now show the correctness of the reduction.

\( \rightarrow \) Assume that \( G \) has a \( \kappa \)-clique. We set \( \gamma = \sigma(\Gamma_0) = \sigma(\Gamma_0') = \sigma(\Gamma_{h, i}) = \sigma(\Gamma_{h, i}') \) for \( h \in \{0, 1\} \) and \( 1 \leq i \leq \kappa \). We set \( \phi_i = \sigma(\Phi_i), i \in \{1, 2\} \), and keep the assignments to other
blocks from Theorem 6 (inserting character $\gamma$ within separators $G_{i,j}$ and $G'_{i,j}$ since only those around blocks $X_{i,j}$ are covered by $\Gamma_{h,i}$ and $\Gamma'_{h,i}$). The new prefix section is correctly covered by these new blocks, as are the occurrences of $\gamma$ in the vertex sections (those around selected vertices are covered by $\Gamma_{h,i}$ and $\Gamma'_{h,i}$; others are inserted in separators $G_{i,j}$ and $G'_{i,j}$).

$(\Leftarrow)$ If $\mathcal{E}$ has an assignment, possibly with empty blocks, we focus on proving that blocks $Z$ and $X_{i,j}$ are not empty. First consider characters $\phi_i$: they must each be in their own block ($T$ and $T'$ have no other common substring containing these). Write $A$ and $B$ respectively for the blocks assigned $\phi_1$ and $\phi_2$, then $AB$ is a substring of $\mathcal{X}$ and $BA$ is a substring of $\mathcal{X}'$. It can be verified that $\Phi_1$, $\Phi_2$ are the only blocks satisfying this property, so $\sigma(\Phi_1) = \phi_i$ for $i = 1, 2$. Thus, from the prefix section of $T$ and $\mathcal{X}$, we have $\sigma(\Gamma_0 \prod_{i=1}^{c}(\Gamma_{0,i},\Gamma_{1,i})) = \gamma^{2^k+1}$, so each $\sigma(\Gamma_0)$, $\sigma(\Gamma_{h,i})$ only contains characters $\gamma$, and may not contain more than one (since there is no $\gamma^\gamma$ outside the vertex section, where the corresponding block is matched in $T' \equiv \mathcal{X}'$), so it must contain exactly $\gamma$. Similarly, $\sigma(\Gamma_0') = \sigma(\Gamma'_{h,i}) = \gamma$. It follows that $\sigma(Z)$ is not empty and starts with $z$, which gives the separation into vertex and edge sections as in the proof of Theorem 6. For each $i$, string $\sigma(\prod_{j \neq i} X_{i,j})$ only contains characters from vertex and edge sections (so only of the form $v_\nu$, not $\gamma$ or separators), and $T$ contains $\gamma \sigma(\prod_{j \neq i} X_{i,j}) \gamma$ as a substring, so again $X_{i,j} = w_i$ for some $i$ and every $j \neq i$. Finally, $w_i w_j$ must be an edge of $G$ for each $i$ and $j$, so $\{w_1, \ldots, w_n\}$ forms a clique of $G$.

We note that the above reduction requires two distinct target strings (otherwise, allowing empty blocks leads to trivial solutions where a single block is non-empty). However, the two strings differ by a single character inversion, that force each and every other block to be non-empty.

## 5 Constant Equation Sizes

We now consider the special case where the right hand side of each equation in the system has only a constant number of blocks. In other words, the case when $c$ is constant. We show that the case where $c = 2$ can be solved in polynomial time. In fact, we show that a more general case is polynomial-time solvable. To describe this special case, we introduce the following notation.

A block $X_p$ is a border block of an equation if it appears as first or as last block in the right hand side of that equation. A block is a border block of a system of equations $\mathcal{E}$ if it is a border block of at least one equation of $\mathcal{E}$. We say that $\mathcal{E}$ has only border blocks if non-border blocks are all jokers. In particular, in such a setting, a border block of $\mathcal{E}$ is a border block of any equation in which it occurs (otherwise, it would both appear twice and be a joker: a contradiction).

**Proposition 8.** String Equations with only border blocks is polynomial-time solvable. In particular, String Equations with equation size $c = 2$ can be solved in polynomial time.

**Proof.** Consider an equation $T_i \equiv \mathcal{X}_i$. We say that the string $T_i$ starts with block $X_p$ if $\mathcal{X}_i[1] = X_p$. Similarly, $T_i$ ends with $X_p$ if $\mathcal{X}_i[|\mathcal{X}_i|] = X_p$. Some length $\ell \in \mathbb{N}$ is valid for block $X_p$ if the length-$\ell$ prefixes of all strings starting with $X_p$ and the length-$\ell$ suffixes of all strings ending with $X_p$ are equal (then we denote this common substring $\sigma_\ell(X_p)$).

We let $\{X_1, \ldots, X_k\} \subseteq \mathcal{B}$ denote the border blocks of the instance. Recall that, since the instance has only border blocks, all other blocks are jokers, and border blocks are border blocks in every equation that contains them. Build a $2$SAT formula as follows. For each border block $X_p$ and each integer $0 \leq \ell \leq t$, introduce a boolean variable denoted $X_{p, \ell} \leq \ell$
10 Parameterized String Equations

(which corresponds to assigning a string with length at most \( \ell \) to \( X_p \), its negation is denoted \( \overline{X_p} \)). Create formula \( \Phi_E \) using the following 1- and 2-clauses.

1. For each \( X_p \), add clause \( X_p \lor \overline{X_p} \).
2. For each \( X_p \) and \( 0 \leq \ell < \ell' \leq n \), add clause \( X_p \lor \overline{X_p} \lor X_{p+\ell'} \).
3. For each \( X_p \) and \( 1 \leq \ell \leq n \). If \( \ell \) is not a valid length for \( X_p \), add clause \( X_p \lor \overline{X_p} \lor X_{p+\ell} \).
4. For each equation \( T_i \equiv X_i \) where \( X_i = X_p \), add clause \( X_p \lor \overline{X_p} \lor X_{|T_i|} \).
5. For each equation \( T_i \equiv X_i \) where \( X_i = X_pX_q \), and \( 0 \leq \ell \leq |T_i| \), add clause \( X_p \lor \overline{X_p} \lor X_{p+\ell} \).
6. For each equation \( T_i \equiv X_i \) where \( X_i \) starts with \( X_p \) and ends with \( X_q \), for each \( 0 \leq \ell \leq |T_i| \), add clause \( X_p \lor \overline{X_p} \lor X_{p+\ell} \).

We then prove the following claim, which completes the proof that \( E \) can be solved in polynomial time:

\[ E \text{ admits a satisfying assignment } \iff \Phi_E \text{ is satisfiable.} \]

\((\Rightarrow)\) Consider a satisfying assignment \( \sigma \) of \( E \), let \( \ell_p \) be the length of \( \sigma(X_p) \), and set all variables \( X_p \leq \ell \) to true iff \( \ell \geq \ell_p \). We now verify each clause (each time we use the notations from the clause definition, and for clauses of the form \( A \rightarrow B \), we assume that \( A \) is true and directly prove \( B \)).

1. \( \ell_p > 0 \) so \( X_p \lor \overline{X_p} \).
2. \( \ell_p \) is a valid length, so \( \ell \neq \ell_p \), if \( X_p \leq \ell \), then \( \ell_p < \ell \) and \( X_p \lor \overline{X_p} \).
3. \( \ell_p \) is a valid length, so \( \ell \neq \ell_p \), if \( X_p \leq \ell \), then \( \ell_p < \ell \) and \( X_p \lor \overline{X_p} \).
4. We have \( |T_i| = |\sigma(X_i)| = \ell_p \), so \( \ell_p \lor |T_i| - 1 \) and \( X_p \lor |T_i| - 1 \).
5. We have \( |T_i| = |\sigma(X_i)| = \ell_p \lor \ell_q \), and \( \ell \geq \ell_p \lor \ell_q \geq |T_i| - \ell \) and \( X_q \lor |T_i| - 1 - \ell \).
6. We have \( T_i = \sigma(X_i) \), so \( |T_i| \geq \ell_p \lor \ell_q + |X_i| - 2 \) (blocks of \( X_i \) have length at most 1, except the first and last that have length \( \ell_p \) and \( \ell_q \)). If \( X_p \lor \overline{X_p} \), then \( \ell_p \lor \ell_q \), and \( \ell_q \leq |T_i| - 1 - |X_i| + 2 \leq |T_i| - 1 - |X_i| + 1 \).

\((\Leftarrow)\) For each \( p \), let \( \ell_p \) be the smallest value of \( \ell \) such that \( X_p \leq \ell \) is true. In particular, \( \ell_p > 0 \) (since \( X_p \leq 0 \) is false, Clause 1), and \( \ell_p \) is a valid length for \( X_p \) (Clause otherwise, we would also have \( X_p \leq \ell_p - 1 \) set to true). Set \( \sigma(X_p) = \sigma_{\ell_p}(X_p) \). Consider an equation \( T_i \equiv X_i \). By Clause 3, if \( X_i \) starts with \( X_p \) and ends with \( X_q \), then \( \sigma(X_p) \) is a prefix of \( T_i \) and \( \sigma(X_q) \) is a suffix of \( T_i \). In particular, \( \ell_p \lor \ell_q \leq |T_i| \). If \( |X_i| = 2 \), then clause 2 \( \ell_p \lor |T_i| \lor |X_i| + 2 \) is a joker block, we have \( \ell_p \lor \ell_q \lor |X_i| - 2 \leq |T_i| \) (indeed, \( X_p \lor \ell_p \lor |X_p| \), so \( X_q \lor |T_i| \lor |X_i| - (\ell_p - 1) \) by Clause hence \( \ell_q \leq |T_i| - |X_i| - \ell_p + 2 \), which gives the result.

\[ \textbf{Theorem 9. STRING EQUATIONS with duplicate-free equations of size } c = 3 \text{ and parameter } r \text{ is W[1]-hard.} \]

\textbf{Proof.} See Figure 6. Consider an instance \( (G = (V, E, \kappa)) \) of Multi-Colored Clique. Introduce three separators denoted \( x, y, z \), let \( \Sigma = V \cup \{x, y, z\} \). Introduce \( \kappa + 3 |c| \) non-joker blocks \( \{X_i \mid 1 \leq i \leq \kappa\} \) and \( \{E_{i,j}, A_{i,j}, B_{i,j} \mid 1 \leq i < j \leq \kappa\} \), and \( 2\kappa + 4 |c| \) joker blocks (for a total of \( k = O(k^2) \) blocks). The equation system is defined as follows.

\[ E := \{T_1 \equiv X_{1,i} \mid 1 \leq i \leq \kappa\} \]
\[ \cup \{T_2 \equiv X_{2,i,j} \mid 1 \leq i < j \leq \kappa\} \]
\[ \cup \{T_3 \equiv X_{3,i,j} \mid 1 \leq i < j \leq \kappa\} \]
Theorem 10. **String Equations** with $r - 1$ size-2 equations and one size-$c$ duplicate-free equation is W[1]-hard for parameter $r + c$.

**Proof.** See Figure 7. Consider an instance $(G = (V, E), \kappa)$ of Clique. Introduce additional characters $x$ and $y$, and let $\Sigma = V \cup \{x, y\}$. Given a vertex $v_i \in V$, define the (non-empty) strings \text{pre}(v_i) := x \prod_{i=1}^{\kappa-1} v_i$ and \text{suffix}(v_i) := \prod_{i=1}^{\kappa} v_i$. We first prove the following property on these functions:
Parameterized String Equations

\[ T_v := x \ ab \ cd \]

\[ T_v \equiv RR' \]
\[ T_v \equiv BG' \]
\[ T_v \equiv BR' \]
\[ T_i := y \ x \ d \ x \ d \ y \ x \ ac \ d \ y \ x \ a b \ d \ y \]

\[ T_i \equiv \ast R_b G_r' \ast R_b B_r' \ast G_b B_r' \ast \]

\[ \text{Figure } 7 \] Reduction of Theorem 10 (using \( r - 1 \) size-2 equations and one size-c duplicate-free equation) applied to the graph from Figure 2.

\( \triangleright \) Claim 11. If \( v_i, v_j, v_{i'}, v_{j'} \) are vertices with \( i < j \) and \( \text{pre}(v_i)\text{suf}(v_j) = \text{pre}(v_{i'})\text{suf}(v_{j'}) \), then \( i = i' \) and \( j = j' \).

**Proof.** Character \( v_i \) does not appear in \( \text{pre}(v_i)\text{suf}(v_j) \), so \( i' \leq i < j' \). Furthermore, \( v_{i-1} \) (or \( x \) if \( i = 0 \)) appears in \( \text{pre}(v_i)\text{suf}(v_j) \) but may not appear in \( \text{suf}(v_{j'}) \), so it appears in \( \text{pre}(v_{i'}) \) and \( i = i' \). Removing the common prefix yields \( \text{suf}(v_j) = \text{suf}(v_{j'}) \), and \( j = j' \).

Introduce \( 2\kappa + 2\binom{\kappa}{2} \) non-joker blocks \( \{ X_i \mid 1 \leq i \leq \kappa \} \) and \( \{ X_{i,j}, X_{i,j}' \mid 1 \leq i < j \leq \kappa \} \), and \( \binom{\kappa}{2} + 1 \) joker blocs (for a total of \( k = O(\kappa^2) \) blocks). The equation system is defined as follows.

\[ E := \{ T_v \equiv X_i \mid 1 \leq i \leq \kappa \} \]
\[ \cup \{ T_v \equiv X_{1,i,j}, T_v \equiv X_{2,i,j} \mid 1 \leq i < j \leq \kappa \} \]
\[ \cup \{ T_e \equiv X_e \} \]

where:

\[ T_v := x \prod_{i=1}^{n} v_i \]
\[ T_e := y \prod_{u \in E} (\text{pre}(u)\text{suf}(v) \ y) \]
\[ \mathcal{X}_i := X_i \]
\[ X_{i,j} := X_{i,j}' \]
\[ X_{1,i,j} := X_{i,j} X_{i,j}' \]
\[ X_{2,i,j} := X_j X_{i,j}' \]

If \( G \) contains a clique \( K = \{ w_1, \ldots, w_n \} \), let \( \sigma(X_i) := \text{pre}(w_i) \) and \( \sigma(X'_i) := \text{suf}(w_i) \). For each \( i < j \), \( \sigma(X_{i,j}) = X_i \) and \( \sigma(X'_{i,j}) = X_j' \). Since, for each \( w_i \), we have \( T_v = \text{pre}(w_i)\text{suf}(w_i) \), the equations \( T_v \equiv X_i \), \( T_v \equiv X_{1,i,j} \) and \( T_v \equiv X_{2,i,j} \) are satisfied. Furthermore, since \( K \) is a clique, for any \( i < j \) we have \( \sigma(X_{i,j} X'_{i,j}) = \text{pre}(u)\text{suf}(v) \) for some edge \( uv \) of \( G \), so strings \( \sigma(X_{i,j} X'_{i,j}) \) are indeed substrings of \( T_e \), in the same order as in \( X_e \), so \( T_e \equiv X_e \) is satisfied as well.

Assume now that \( E \) admits a satisfying assignment \( \sigma \). Note that by \( T_s \equiv X_i \), \( \sigma(X_i) = \text{pre}(w_i) \) and \( \sigma(X'_i) = \text{suf}(w_i) \) for some vertex \( w_i \in V \). Using other equations with \( T_s \), we get \( \sigma(X_{i,j}) = \sigma(X_i) \) and \( \sigma(X'_{i,j}) = \sigma(X_j') \) for each \( i < j \). For each \( i < j \), write \( E_{i,j} = \sigma(X_{i,j} X'_{i,j}) \). Then \( E_{i,j} \) is a substring of \( T_e \), starting with \( x \), ending with \( v_n \), and not containing any occurrence of \( y \), so \( E_{i,j} = \text{pre}(u)\text{suf}(v) \) for some edge \( uv \) of \( G \). By Claim 11.
L. Bulteau, M. R. Fellow, C. Komusiewicz and F. A. Rosamond

since $E_{i,j} = \text{pre}(w_i)\text{suf}(w_j)$, we have $u = w_i$ and $v = w_j$, so $w_i w_j$ is an edge for each $i < j$: \{w_1, \ldots, w_k\} is a clique of $G$.

\section{Open Questions}

We identified and focused on parameters $k$, $r$, $c$ and $d$ for our multivariate study of the String Equations and String Equations with Deletions problems. Two questions remain open, both in the deletions setting. Can the XP-algorithm for String Equations with Deletions parameterized by $k + d$ (Proposition \ref{prop:deletionsXP}) be improved to an algorithm with running time $f(d) \cdot t^{f(k)}$, that is, to an FPT-algorithm for $d$ when $k$ is constant? Similarly, is there an $f(d)\text{O}(1)$-time algorithm for String Equations with Deletions with size-2 equations (or, even better, equations with only border blocks of arbitrary size)?

We reserve the study of other parameters for future works: possible candidates could be the alphabet size $|\Sigma|$, the size of the target string $t$, or the maximum length of a string assigned to a block $|\sigma(X_i)|$. However, most combinations bring hardness for the related String Morphism problem \cite{Fernau2016}.

Finally, another variant would allow undefined characters (question marks) in the target strings. Does this introduce an additional difficulty, and can our algorithms be extended to this setting?

\section*{References}

1. Dana Angluin. Finding patterns common to a set of strings. In Proceedings of the eleventh annual ACM Symposium on Theory of Computing, pages 130–141, 1979.
2. Hans L. Bodlaender, Rodney G. Downey, Michael R. Fellows, and Harold T. Wareham. The parameterized complexity of sequence alignment and consensus. Theor. Comput. Sci., 147(1&2):31–54, 1995.
3. Laurent Bulteau and Christian Komusiewicz. Minimum common string partition parameterized by partition size is fixed-parameter tractable. In Chandra Chekuri, editor, Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014, pages 102–121. SIAM, 2014.
4. Rodney G. Downey and Michael R. Fellows. Fundamentals of Parameterized Complexity. Texts in Computer Science. Springer, 2013.
5. Michael R. Fellows, Danny Hermelin, Frances A. Rosamond, and Stéphane Vialette. On the parameterized complexity of multiple-interval graph problems. Theor. Comput. Sci., 410(1):53–61, 2009.
6. Henning Fernau, Florin Manea, Robert Mercas, and Markus L. Schmid. Pattern matching with variables: Efficient algorithms and complexity results. ACM Trans. Comput. Theory, 12(1):6:1–6:37, 2020.
7. Henning Fernau and Markus L. Schmid. Pattern matching with variables: A multivariate complexity analysis. Inf. Comput., 242:287–305, 2015.
8. Henning Fernau, Markus L. Schmid, and Yngve Villanger. On the parameterised complexity of string morphism problems. Theory Comput. Syst., 59(1):24–51, 2016.
9. Oscar H. Ibarra, Ting-Chuen Pong, and Stephen M. Sohn. A note on parsing pattern languages. Pattern Recognition Letters, 16(2):179–182, 1995. URL: \url{https://www.sciencedirect.com/science/article/pii/016786559400091G}, \url{doi:https://doi.org/10.1016/0167-8655(94)00091-G}.
10. Krzysztof Pietrzak. On the parameterized complexity of the fixed alphabet shortest common supersequence and longest common subsequence problems. J. Comput. Syst. Sci., 67(4):757–771, 2003.