Quantum spin liquids are low-temperature phases of matter with fractionalized excitations and emergent gauge fields [1–5]. Efforts at identifying possible spin liquids has led to hundreds of candidates due to the various possible symmetries present in lattice systems. However, a broader view of the nature of the fractionalized excitations and gauge field leads to only a few prominent types [2], some of which are realized in exactly solvable models [6–14]. Here, we show that coupling a spin liquid to an environment can lead to a qualitatively new kind of phase which cannot occur in any closed system.

Dissipative systems can display unusual phenomenology not seen in closed systems. These range from unusual phase transitions and critical phases [15–17] to new topological phases [18–21]. One prominent class of phenomena can be understood in regimes where a non-Hermitian description [18, 19, 22–27] of the system is appropriate. This allows for the appearance of exceptional points (EPs) in the spectrum when two eigenvectors coincide [28–31]. In non-interacting systems, band crossings with such exceptional points result in an unconventional square-root dispersion at low energies as opposed to a typical Dirac dispersion as seen in graphene. These band crossings in 2D systems are generic unlike the accidental symmetry-protected crossings in graphene [18, 21]. The conventional bulk-boundary correspondence is also shown to be broken due to an exotic non-Hermitian skin effect [32–38], which results in localization of all eigenstates at the boundary. This results in an exponential sensitivity of the system to boundary conditions. Based on work of free systems, interest is now drawn to understanding effects in interacting systems [16, 39–43]. It is then natural to ask how the emergent phenomena in strongly correlated spin liquids looks like when the system is described by such an effective non-Hermitian Hamiltonian.

In this Letter we show that these phenomena can be realized in an interacting spin model giving rise to a qualitatively new kind of spin liquid. We illustrate this by coupling the Kitaev honeycomb model [7] to an environment (Fig. 1, left panel). In certain regimes, the two Dirac points generically split into four exceptional points (Fig. 1, right panel). The four exceptional points are paired up with each pair connected through

![FIG. 1. Left: The lattice of the Kitaev honeycomb model. The coupling to the environment is described by jump operators $L^\alpha$ ($\alpha = x, y, z$). Right: the 3D spectrum diagram for the non-Hermitian Kitaev honeycomb model at $G_x = 2, G_y = 1, G_z = 2.5 \exp(i\pi/3)$. The Fermi arc (Re$E = 0$) is labelled with the red line and the green line indicates the Im$E = 0$ curve.](https://example.com/fig1.png)
and real space of spin-1/2:

\[ H_0 = - \sum_{\langle jk \rangle \alpha} J_{\alpha} \sigma_j^{\alpha} \sigma_k^{\alpha}, \tag{1} \]

where \( \langle jk \rangle_\alpha \) labels the lattice (Fig. 1) and \( \alpha = x, y, z \) labelling the three types of links of a hexagonal lattice with \( \sigma_\alpha \) the corresponding Pauli matrices.

We consider an open system where the Kitaev Hamiltonian is coupled to an environment. The resulting open system is described by a Lindblad master equation [45] for the density matrix

\[ \frac{d}{dt} \rho = -i[H_0, \rho] + \gamma \sum_{n, \alpha} \left( L_n^{\alpha} \rho L_n^{\alpha \dagger} - \frac{1}{2} \{ L_n^{\alpha \dagger} L_n^{\alpha}, \rho \} \right), \tag{2} \]

where \( L_n^{\alpha} \) are jump operators describing how the system is coupled to the bath and we have set \( \hbar = 1 \). The dynamics can be interpreted in terms of deterministic evolution of a trajectory (wavefunction) described by an effective non-Hermitian Hamiltonian \( H_{\text{NH}} = H_0 - (\gamma/2) \sum_n L_n^{\alpha} L_n^{\alpha \dagger} \), interspersed with quantum jumps to different states through the \( L_n^{\alpha \dagger} L_n^{\alpha} \) term [46–49]. Thus when we are measuring at times before the first jump, the dynamics is governed by the non-Hermitian Hamiltonian \( H_{\text{NH}} \).

Although the general phenomenology of what follows is largely independent of the form of the jump operators, we consider here jump operators \( L_{\alpha - k}^{\alpha} = \sigma_{\alpha}^{\alpha} + \sigma_{\alpha}^{\gamma} \) along each \( \alpha \)-type link \( j - k \) for illustration. Similar results can be obtained by considering the effect of dephasing noise [50]. This results in an effective non-Hermitian description of the the form of Eq. (1) but with the coupling constants being complex and henceforth labeled by \( G_\alpha \). This model is solved through a Majorana representation of the spin operators. Introducing four Majorana fermions \( (c_j, b_j^x, b_j^y, b_j^z) \) at each site, the spin is represented as \( \sigma_{\alpha}^{\beta} = ic_j b_j^\alpha \). Defining bond operators \( u_{\langle jk \rangle \alpha} = ib_j^\alpha b_k^\beta \), the effective model is

\[ H_{\text{NH}} = - \sum_{\langle jk \rangle \alpha} G_\alpha \sigma_j^{\alpha} \sigma_k^{\alpha} + i \sum_{\alpha \langle jk \rangle \alpha} G_\alpha u_{\langle jk \rangle \alpha} c_j c_k. \tag{3} \]

The Hamiltonian has an extensive set of conserved quantities, \( [W_p, H] = 0 \), defined by a product of the spin components around a plaquette \( W_p = \sigma_i^x \sigma_j^y \sigma_k^z \sigma_l^\gamma \) where \( \gamma \) is a gauge transformation operator. Since \( D^2 = 1 \) and it commutes with the Hamiltonian, this is a \( \mathbb{Z}_2 \) gauge theory. The product \( u_{jk} = ib_j^x b_k^x \) \( (j, k \) on an \( \alpha \)-type link) is a constant of motion in the enlarged space. It can be viewed as the \( \mathbb{Z}_2 \) gauge field. The plaquette operator now takes the form \( W_p = \prod_{j - k \in p} u_{jk} \). The sector with all \( W_p = 1 \) is viewed as vortex free and \( W_p = -1 \) means a \( \mathbb{Z}_2 \)-vortex at \( p \).

The eigenstates of the model can be decomposed into different \( \mathbb{Z}_2 \)-flux sectors as in the original Hermitian model, where the zero-flux is the relevant one at low (real) energies due to Lieb’s theorem [7, 51]. In fact, the zero-flux sector is still relevant for the open systems in appropriate regimes where all the phenomenology we discuss is realized. To illustrate this, consider the Hermitian model at temperatures much lower than the veson gap. If we consider multiplying it by an overall complex number, there is a parametric separation in lifetimes of states between different flux sectors and the zero-flux sector corresponds to states with the longest lifetimes. If we now add a generic perturbation, the emergent Dirac points will split up into exceptional points as indicated below.

The gauge field can be chosen as \( u_{jk} = 1 \) for \( j \) on one sublattice and \( k \) on the other. The Hamiltonian becomes a tight binding Majorana model, taking a simple form in momentum space:

\[ \tilde{H} = \sum_{\mathbf{q}} \begin{pmatrix} c_{-\mathbf{q}, 1} & c_{-\mathbf{q}, 2} \end{pmatrix} \begin{pmatrix} 0 & iA(\mathbf{q}) \\ -iA(\mathbf{q}) & 0 \end{pmatrix} \begin{pmatrix} c_{\mathbf{q}, 1} \\ c_{\mathbf{q}, 2} \end{pmatrix}, \tag{4} \]

where in the primed summation \( \sum'_{\mathbf{q}} \) we count the pair \( q \), \(-q\) only once. This is because \( c_{-\mathbf{q}} = c_{\mathbf{q}}^\dagger \). The off-diagonal element is \( A(\mathbf{q}) = 2 \left(G_{\mathbf{q}} e^{-i(\mathbf{q} \cdot \mathbf{r}_1 + \mathbf{q} \cdot \mathbf{r}_2 + \mathbf{G}_2)} \right) \) and the subscripts 1, 2 label the two sublattices of the honeycomb system. The momentum-space operators satisfy \( \{c_{\mathbf{q}, \lambda}, c_{-\mathbf{q}', \gamma}^\dagger\} = \delta_{\mathbf{q}, \mathbf{q}'}, \delta_{\lambda, \gamma} \). Depending on whether \( A(\mathbf{q}) \) can go zero, the system exhibits a gapped or a gapless phases. In the Hermitian model, the gapped phase is equivalent to a toric code spin liquid [6] while the gapless phase can possess non-Abelian statistics in the presence of magnetic field [7]. The gapless condition is given that the lengths \( |G_x|, |G_y|, |G_z| \) admit a triangle:

\[ |G_x| \leq |G_y| + |G_z|, \quad |G_y| \leq |G_x| + |G_z|, \quad |G_z| \leq |G_x| + |G_y|. \tag{5} \]

Inside the gapless region, there are two Dirac points for the Hermitian model. When they come closer and fuse, the system transits into the gapped phase. Notice that in the above equation, the Majoranas are agnostic to whether the coupling is ferromagnetic or antiferromagnetic, and only depends on the modulus \( |G_\alpha| \). However, the \( G_\alpha = 0 \) point is equivalent to an array of 1D gapless chains.

**Exceptional points and Fermi arcs** - The spectrum is obtained by the eigenvectors of the matrix (4) \( E^2(\mathbf{q}) = A(\mathbf{q})A(-\mathbf{q}) \). So the existence of (complex) band touching point is still given by Eq. (5) with complex-valued \( G_\alpha \). However, the band touching points are no longer Dirac points. As we will calculate explicitly, they have a square-root dispersion and are exceptional points.

For convenience of computation here, we can extract out the phase of \( G_{\mathbf{q}} \) as an overall phase of \( H \), so the Hamiltonian is now parameterized by \( \tilde{\phi}_x = \phi_x - \phi_z, \tilde{\phi}_y = \phi_y - \phi_z \). In the Fourier-space Brillouin zone, it is more convenient to parametrize as \( \mathbf{q} = q_1 (2\pi) + q_2 (2\pi) \), where \( q_1 \) and \( q_2 \) are the reciprocal lattice vectors. The values \( q_1, q_2 \) uniquely
A cut at \( G_x = 2, G_y = 1 \) for different complex \( G_z \) is shown in Fig. 2, where we also plot the absolute energy \( |E| \) and its real part, \( \text{Re} E \), at different \( G_z \). One can see the splitting of each Hermitian band-touching point into two non-Hermitian exceptional points. At the phase boundary, the branch-cut for \( E \) disappears. In this situation, the band-touching point is not protected and thus gets gapped out when crossing the phase boundary. This illustrates how, similar to Weyl points in 3D, the exceptional points can only be gapped out when combined pairwise, in glaring contrast to the 2D Dirac points that are inherently symmetry protected.

Notably, in the non-Hermitian situation, we can go from an anti-ferromagnetic Kitaev spin liquid \( J_\alpha < 0 \) to an ferromagnetic Kitaev spin liquid \( J_\alpha > 0 \) by splitting and reconnecting exceptional points. While in the Hermitian case, one has to go through the critical regime \( J_\alpha = 0 \) exhibiting nodal lines. The non-Hermitian coupling thus provides paths circumventing this critical point as illustrated in the phase diagram in the left panel of Fig. 2.

**Skin effects** - Boundary conditions do not affect the spectrum for Hermitian systems in the thermodynamic limit except for additional edge states. In contrast, non-Hermitian systems display a strong sensitivity to boundary conditions. To illustrate this, we consider Eq. (3) with two parallel zigzag boundaries. We place the open boundary condition (OBC) perpendicular to the \( y \)-type link, which is along the \( x \)-direction. The \( x \)-direction is still chosen to be periodic with \( N_x \) unit cells. For convenience of calculation and comparison with PBC in both directions, the number of layers sandwiched by the two boundaries is taken to be even \( M = 4M' \). The states are labeled by the momentum \( q_x \) and the layer index \( m \). The Hamiltonian takes the form:

\[
\tilde{H} = \sum_{m,m',q_x} A_{m,m'}^{Q}(q_x) c_{-q_x,m} c_{q_x,m'},
\]

with the constraint \( G_x \sin(q_1 + \bar{q}_x) = -G_y \sin(q_2 + \bar{q}_y) \) to fix the \( \pm \) signs above. These equations admit at most two solutions which we denote as \( q_c, q'_c \). In the Hermitian situation, as \( \bar{q}_x = \bar{q}_y = 0 \), one has \( q_c = -q'_c \). Since \( A^*(q) = A(-q) \) there are no further zero energy solutions and linearizing \( A \) it directly follows that we have Dirac points with a conventional linear dispersion away from the degeneracies.

For complex, i.e., non-Hermitian, parameters, we have \( A^*(q) \neq A(-q) \) and instead find \( A(-q) = 0 \) at \( q_c = -q'_c - 2(\bar{q}_x, \bar{q}_y) \) implying four \( E = 0 \) exceptional points \( \pm q_c, \pm q'_c \), at which the Hamiltonian matrix becomes non-diagonalizable and the two eigenvectors coincide. The dispersion near the exceptional points takes a square-root form instead of the conic form since \( A(q) \) and \( A(-q) \) generically do not vanish simultaneously. The \( \text{Re} E = 0 \) branch cuts associated with the square-roots have a natural interpretation as bulk Fermi arcs, and the exceptional points are connected by these Fermi arcs and their imaginary counterparts \( \text{Im} E = 0 \) (cf. Fig. 1, right panel).

When the coupling constants are tuned out of the triangle regime Eq. (5), the four exceptional points fuse into two exceptional points and then disappears. A cut at \( G_x = 2, G_y = 1 \) for different complex \( G_z \) is shown in Fig. 2, where we also
where \( c_{q,m} = \sum_n (1/\sqrt{2N_x}) e^{i q x} c_{n,m} \). The matrix \( A^O_{m,n}(q_x) \) is quasi-diagonal:

\[
A^O_{m,n} = i \delta_{m+1,n} r(q_x) + t + \frac{(-1)^m}{2} [r(q_x) - \bar{t}] \frac{r'(q_x) + t' + \frac{(-1)^m}{2} [r'(q_x) - t']}{\delta_{m+n+1}} \tag{9}
\]

where \( r(q_x) = r(-q_x) = 2 [G_x e^{i q x / 2} + G_y e^{-i q x / 2}] \) and \( t = t' = -2G_z \). The boundary conditions require the wave function to satisfy \( \psi(0) = 0 \) and \( \psi(M+1) = 0 \). The question can be solved by a transfer matrix method \([52, 53]\). We group the wave functions into a doublet \( \Psi(m) = (\psi(2m), \psi(2m+1)) \). Then the equation of motion takes the form \( \Psi(m) = T \Psi(m-1) \), where \( T \) is the transfer matrix:

\[
T = \frac{1}{2 \pi i} \begin{pmatrix} tt' - itE & -itE \\ -itE & t - t' \end{pmatrix} \tag{10}
\]

Each eigenstate \( \Psi(m) \) can be therefore constructed as a superposition of the two eigenvectors of \( T \), \( \Psi(m) = \alpha s_1 \Psi_1 + \beta s_2 \Psi_2 \) with \( \Psi_{1,2} \) the eigenvectors of \( T \) and \( s_{1,2} \) the corresponding eigenvalues, such that it satisfies \( \psi(0) = \psi(M+1) = 0 \). If we consider Hermitian couplings, then \( |s_1| = |s_2| = 1 \) and the eigenstates propagate within the bulk. However in the non-Hermitian situation, \( s_1 \approx s_2 \) are either both larger or smaller than one and the states are piled up against one of the boundaries. The is known as the non-Hermitian skin effects \([32, 33]\). The general criterion for the skin effect is when \( s_1 s_2 \) is given by

\[
| \det T | = \left| \frac{r'(q_x)}{r(q_x)} \right| = \left| \frac{G_x e^{i q x / 2} + G_y}{G_x e^{-i q x / 2} + G_y} \right| \neq 1. \tag{11}
\]

In this case, all eigenstates are exponentially localized to the boundary \( \psi(m) \sim \exp(-m/l) \), with \( l = (\ln | \det T |)/2 \). We see that this occurs when the relative phase \( \phi_x - \phi_y \) is non-zero. By rotation symmetry, we can draw the conclusion that for two parallel zigzag open boundaries perpendicular to \( \alpha \)-type links, the skin effects can be turned on by giving a non-trivial relative phase \( \phi_{\beta} - \phi_{\gamma} \), where \( \beta, \gamma \neq \alpha \).

In Fig. 3a-3c, we show the OBC spectra for different constants. The average localization of the wave function, \( \bar{m} = \sum |\psi(m)|^2 \), is indicated with the color plot. For a non-vanishing \( \phi_x - \phi_y \), in addition to the zero-energy boundary state, the bulk states are also piling up the \( m = 1 \) boundary, exhibiting the skin effects. The localization shift from one boundary to the other at \( q_x = 0, \pi \), in accordance with Eq. (11). The spectrum is strikingly different from the PBC spectrum. For a vanishing \( \phi_x - \phi_y \) and non-vanishing \( \phi_z \), we can however see in Fig. 3b that there is no skin effect despite the presence of bulk exceptional points and the OBC spectrum coincides with the PBC spectrum. Fig. 3c shows a Hermitian example contrasting the novel non-Hermitian behaviour.

Discussion - We have shown in this Letter that genuinely non-Hermitian phenomenology, exceptional points and skin effects, intriguingly conspire with fractionalization in the interacting Kitaev honeycomb model in dissipative environments. This results in a qualitatively new type of non-equilibrium matter which we call exceptional spin liquids, which is by its dissipative nature, lies beyond earlier classification schemes and potentially displays new dynamics beyond current spin liquids \([54–57]\). Remarkably, this new phase is generic in the sense that it does not rely on any underlying symmetries—this may in fact greatly facilitate the prospects for observing gapless spin liquids in synthetic setups.

The exceptional points can naturally arise in many of the proposed realizations of the Kitaev honeycomb model \([58–63]\). For example, spontaneous emission is an inherent part of ultracold atoms in optical lattices and is treated as a noise to be minimized by techniques such as a large detuning or blue-detuned lattices \([58]\). A continuous observation of the system then results in a measurement backaction \([48, 49, 64–66]\) which can be incorporated as a non-Hermitian perturbation to the Kitaev model in absence of decay. From this perspective, any amount of such noise will generically result in exceptional points and larger amounts of noise result in a bigger separation of the exceptional points.

Systems of cold-atoms in optical lattices are also currently restricted to small systems which has limited the scope of ob-
serving interesting many-body physics. In particular, signatures of gapless Dirac cones in the Hermitian Kitaev model would be hard to see given avoided level crossings in finite systems. However, the physics of exceptional points is also visible in finite size systems [28, 64], and even small non-Hermitian systems exhibit the non-Hermitian skin effect.

The exquisite control and ubiquitous presence of dissipation in the suggested synthetic implementations of our ideas might even open the door for novel technological applications such ultra sensitive sensing devices based harnessing the non-Hermitian skin effect [67] by judiciously manipulating the boundary conditions.

As strongly correlated many-body states exhibit a rich variety of emergent phenomena, such as non-Abelian statistics, the study of their interplay with genuinely non-Hermitian effects as advanced here is likely to provide fertile ground for new fundamental discoveries.

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First we identify the nature of the band touching point. We focus on non-zero $G_q$. The energy is given by $E(q) = \pm \sqrt{A(q)A(-q)}$. In the Hermitian case, $A(-q) = A^*(q)$, the dispersion must be at least linear around the band touching point. In the non-Hermitian case, $A(-q) \neq A^*(q)$ in general. The dispersion is usually square-root dependent around the band-touching point. In fact, if $A(-q) = A^*(q) = 0$, then we can find that $G_x \sin(\tilde{q}_1) = -G_y \sin(\tilde{q}_2)$. This requires either $\phi_x = \phi_y$.
(mod $\pi$) or the band-touching point at the inversion-invariant point ($\hat{q}_1, \hat{q}_2 = 0, \pi$). In the first situation, for simplicity we can absorb the phases $\phi_x$ and $\phi_y$ into $\phi_z$. Using $\text{Im}A(q) = \text{Im}A(-q) = 0$, we obtain $G_x \sin(\hat{q}_1) + G_y \sin(\hat{q}_2) + |G_z| \sin \phi_z = 0$ and $-G_x \sin(\hat{q}_1) - G_y \sin(\hat{q}_2) + |G_z| \sin \phi_z = 0$. So $\phi_z = 0$ (mod $\pi$), all $G_x$ have to share the same phase and the system is simply obtained by the Hermitian one times an overall phase. In the second situation where the band-touching points are at the inversion-invariant points, i.e. $\hat{q}_1, \hat{q}_2 = 0, \pi$, we as before absorb the phase of $G_z$ into the overall phase of the Hamiltonian. The band-touching condition is:

$$|G_x| \cos(\hat{q}_1 + \phi_x) + |G_y| \cos(\hat{q}_2 + \phi_y) + G_z = 0, \quad |G_x| \sin(\hat{q}_1 + \phi_x) + |G_y| \sin(\hat{q}_2 + \phi_y) = 0. \quad (S1)$$

From these two equations, one finds that

$$|G_x| = \frac{\sin(\tilde{\phi}_y + \hat{q}_2)}{\sin(\phi_x + \hat{q}_1 - \phi_y - \hat{q}_2)}|G_z|, \quad |G_y| = \frac{\sin(\tilde{\phi}_x + \hat{q}_1)}{\sin(\phi_y + \hat{q}_2 - \phi_x - \hat{q}_1)}|G_z|. \quad (S2)$$

This means we need a fine tuning to have the band-touching point located at the inversion-invariant point. Therefore, we conclude that the band-touching point is Dirac-like only for Hermitian Hamiltonians up to an overall complex phase or fine tuning where the band-touching point takes place at the inversion-invariant point in the Brillouin zone. Otherwise, the band-touching points possess a defect Hamiltonian and are exceptional points.

Then we consider the boundary between the gapless phase $B$ and the gapped phase $A$ when the Hamiltonian is non-Hermitian. We rewrite the band-touching condition as

$$\cos(\hat{q}_1 + \tilde{\phi}_x) = \frac{|G_y|^2 - |G_z|^2 - |G_x|^2}{2|G_x||G_z|}, \quad \cos(\hat{q}_2 + \tilde{\phi}_y) = \frac{|G_x|^2 - |G_z|^2 - |G_y|^2}{2|G_y||G_z|}, \quad (S3)$$

$$|G_x| \sin(\hat{q}_1 + \phi_x) + |G_y| \sin(\hat{q}_2 + \phi_y) = 0. \quad (S4)$$

The boundary is reached when the solution to Eq. (S3) is at its extremes. That is, $|\cos(\hat{q}_1 + \tilde{\phi}_x)| = 1$ or $|\cos(\hat{q}_2 + \tilde{\phi}_y)| = 1$ in Eq. (S3). According to Eq. (S4), we actually have $|\cos(\hat{q}_1 + \tilde{\phi}_x)| = |\cos(\hat{q}_2 + \tilde{\phi}_y)| = 1$. In this situation, there is only one solution to $A(q) = 0$. For the Hermitian situation, this means the two Dirac points appearing in pairs $q_1, -q_1$ are fusing into one. For general non-Hermitian case, according to the discussion in the above paragraph, the band-touching point $q_c$ is usually not inversion invariant. So we obtain two exceptional points at the phase boundary $q_c, -q_c$. We conclude that in general when the non-Hermitian system is evolving from the gapless to gapped phase, the four exceptional points fuse into two and then get gapped out.

We give the form of the Hamiltonian near the band-touching point. The Hamiltonian can be parameterized by the Pauli matrix as

$$H = \frac{A(q) + A(-q)}{2} \sigma_x + \frac{A(q) - A(-q)}{2} i\sigma_y. \quad (S5)$$

For an Hermitian Hamiltonian, $A(-q) = A^\dagger(q)$. At the Dirac point $q_a$, we have $A(q) \simeq g_a \delta q_a$, where $g_a$ is a complex vector and $\delta q = q - q_a$. So the Hamiltonian takes the form $H = \sigma_x \delta q_a \text{Re} g_a - \sigma_y \delta q_a \text{Im} g_a$. In general Re $g_a$ and Im $g_a$ are linearly independent. So the dispersion of the energy is linear $E \simeq \sqrt{g_a g_a^\dagger \delta q_a \delta q_a}$. At the Hermitian phase boundary, using $q_a = -q_a$, one can deduce that Re $g_a = 0$ and $g_a g_a^\dagger = 0$. Then the Hamiltonian becomes $H = ig_a \delta \sigma_y$ and $E \sim \delta q_a^2$ for $\delta q$ perpendicular to Im $g_a$. In the non-Hermitian situation, at the band touching point only one of $A(q), A(-q)$ vanishes. Let's take $A(q_c) = 0$ and $A(q) \simeq g_a \delta q_a$. Then the Hamiltonian and the energy are expressed as

$$H \simeq \begin{pmatrix} 0 & g_a \delta q_a \\ A(-q_c) & 0 \end{pmatrix}, \quad E \simeq \sqrt{A(-q_c) g_a \delta q_a} \quad (S6)$$

As before, the real and the imaginary part of $g_a$ are usually linearly independent. The energy is square-root dependent on $\delta q$. At the gap-gapless boundary, by examining the structure of Eq. (S1) as done in the last paragraph, the vector $g_a$ is real up to a complex phase. In that case, along the orthogonal direction to $g_a$, the dispersion $E(q)$ is linear in $\delta q$.

Below we list how the exceptional points and the Fermi arcs evolves when $\phi_z$ going from 0 to $\pi$ at fixed $G_x = 2, G_y = 1$ in Fig. S1. From ferromagnetic $G_x$ coupling to anti-ferromagnetic $G_z$ coupling, one can observe that there are two recombination processes of the Fermi arc. Two Fermi arcs appear in splitting each Dirac point to two exceptional points. Between $\phi_z = \pi/4$ and $\phi_z = 5\pi/12$, the two Fermi arcs collide for the first time. How they connect the four exceptional points into two pairs changes. Each Fermi arc together with their imaginary counterpart winds over the torus hole, forming a closed path that can not shrink into a point. Between $\phi_z = 7\pi/12$ to $\phi_z = 5\pi/4$, the Fermi arcs collide again. This time the Fermi arcs again connect the exceptional points in the same way as for small $\phi_z$. And they eventually diminish when $\phi_z$ is approaching $\pi$.  


\[ \phi_z = \frac{\pi}{12} \]
\[ \phi_z = \frac{\pi}{4} \]
\[ \phi_z = \frac{5\pi}{12} \]
\[ \phi_z = \frac{7\pi}{12} \]
\[ \phi_z = \frac{3\pi}{4} \]
\[ \phi_z = \frac{11\pi}{12} \]

**FIG. S1.** (a)-(f) The evolution of the band-touching points, Fermi arcs (red) and the Im\(E = 0\) curve (green) for \(G_x = 2, G_y = 1\) and \(G_z = 2.5 \exp(i\phi_z)\) from \(\phi_z = \pi/12\) to \(\phi_z = 11\pi/12\).

The transfer matrix solution to open boundary condition

Now we solve the OBC problem. The Hamiltonian is diagonal in momentum \(q_z\). Remember that it takes the form \(\tilde{H} = \sum_{m,m',q_z} A^{O}_{m,n} (q_z) c_{-q_z,m} c_{q_z,m'}\), where the matrix \(A^O\) is given by

\[
A^O = \begin{pmatrix}
0 & -i r(q_x) \\
ir'(q_x) & it \\
-ir'(q_x) & 0 \\
-it' & 0 & ir(q_x) \\
& & it \\
& & & ... & ...
\end{pmatrix}. \tag{S7}
\]

We use the transfer matrix to analyze the eigenstates \(\psi(m)\) of the above matrix. Like in the main text, we rewrite them as a doublet \(\Psi(m) = (\psi(2m), \psi(2m + 1))\). The eigenstate equation becomes

\[
T_+ \Psi(m) + T_- \Psi(m - 1) = 0, \quad \Rightarrow \Psi(m) = T \Psi(m - 1) = 0, \tag{S8}
\]

where those transfer matrices are given by

\[
T_+ = \begin{pmatrix}
ir & 0 \\
-E & it
\end{pmatrix}, \quad T_- = \begin{pmatrix}
-it' & -E \\
0 & -ir'
\end{pmatrix}, \quad T = -T_-^{-1} T_+ = \frac{1}{tr} \begin{pmatrix}
-tt' & -itE \\
-rr' & -E^2
\end{pmatrix}. \tag{S9}
\]

In order to find the solution for OPB, we need to compute \(T^{M/2}\), which is convenient by transforming \(T\) into a diagonal matrix through a similarity transformation:

\[
T = P \left( \begin{array}{cc}
s_1 & 0 \\
0 & s_2
\end{array} \right) P^{-1}, \quad s_1, s_2 = \frac{1}{2tr} \left[ rr' + tt' - E^2 \mp \sqrt{(rr' - tt' - E^2)^2 - 4tt'E^2} \right]. \tag{S10}
\]
FIG. S2. The spectra for OBC (colored) and PBC (light gray) and the corresponding localisation variance of the state. The calculation is performed on a $M = 80$-row lattice

(a) The skin effect situation. (b) No skin effect exhibits when only $G_x$ is tuned complex. The zero-mode edge state is performed on a slightly smaller OBC lattice. (c) The Hermitian result, also with the zero-mode edge computed on a slightly smaller OPC lattice. (d)-(f) The corresponding logarithmic distribution of the wave functions $\ln |\psi(j)|^2$ on the lattice at $q_x = 2\pi/3$.

where $P$ is formed by the columns of the right eigenvectors of $T$ and $P^{-1}$ is formed by the rows of the left eigenvectors of $T$. The boundary condition now can be written as:

$$P \begin{pmatrix} s_1^{M/2} & 0 \\ 0 & s_2^{M/2} \end{pmatrix} P^{-1} \begin{pmatrix} 0 \\ \psi(1) \end{pmatrix} = \begin{pmatrix} \psi(M) \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} A \left( s_1^{M/2} - s_1^{M/2} \right) \\ Bs_1^{M/2} + (1-B)s_2^{M/2} \end{pmatrix} \psi(1) = \begin{pmatrix} \psi(M) \\ 0 \end{pmatrix},$$ (S11)

where the values $A, B$ are given by

$$A = \frac{it}{\sqrt{(rr' - tt' - E^2)^2 - 4tt'E^2}}, \quad B = \frac{1}{2} - \frac{rr' - tt' - E^2}{2\sqrt{(rr' - tt' - E^2)^2 - 4tt'E^2}}.$$ (S12)

The energy is solved by the implicit equation $Bs_1^{M/2} + (1-B)s_2^{M/2} = 0$. In the large-$M$ limit, $|s_1/s_2| = |(1-B)/B|^{2/M}$. This value goes to 1 if neither $B$ nor $1-B$ vanishes, which requires $E \neq 0$. Therefore for a state with finite $|E|$, we can deduce $|s_1| \simeq |s_2|$. The distribution of the bulk state is thus derived as in the main text as $|s_1| \simeq |s_2| \simeq \sqrt{|\det T|}$. In Ref. [52], the authors show that in order to obtain a dense spectrum in the $M \to \infty$ limit, one should have $|s_1| = |s_2|$. States with $|s_1| \neq |s_2|$ are exceptional and treated as boundary states.

In figure S2, we plot the average square position $\Delta m^2 = \sum |m - (M + 1)/2|^2 |\psi(m)|^2$ and the distribution of the wave function on the real-space lattice.