Research Article

The $q$-Chlodowsky and $q$-Szasz-Durrmeyer Hybrid Operators on Weighted Spaces

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The main aim of this article is to introduce a new type of $q$-Chlodowsky and $q$-Szasz-Durrmeyer hybrid operators on weighted spaces. To this end, we give approximation properties of the modified new $q$-Hybrid operators. Moreover, in the weighted spaces, we examine the rate of convergence of the modified new $q$-Hybrid operators by means of moduli of continuity. In addition, we derive Voronovskaja’s type asymptotic formula for the related operators.

1. Introduction

Polynomial approach and the classical approximation theory constitute a basic research area in applied mathematics. The development of the approximation theory plays an important role in the numerical solution of partial differential equations, data processing sciences, and many other disciplines. For example, it is widely used in geometric modelling in the aerospace and automotive industries to calculate approximate values with basic functions. Works in this field go back to the 18th century and still continue as a powerful tool in scientific calculations. Furthermore, it is used in civil engineering projects to analyze the energy efficiency and earthquake resistance data of different types of buildings in thermography calculations and earthquake engineering.

Many generalized versions of these polynomials have been studied by many authors. Some of these studies are [1–13]. During the studies of approximation theory, many new operators and their generalizations were introduced with different spaces and variables. For example, on the interval $[0, \infty)$, $[0, 1]$ etc., $q$–analog type, $K$–Petree type, King’s type, and Weighted space type operators are introduced.

In this study, the $q$–analog type operator is defined. The studies regarding the $q$–analog type operator are as follows.

First, the $q$–Bernstein polynomials were produced by Phillips [14]. When $q = 1$ is used, the results are the same as for classical operators. However, new operators with different properties are obtained for $q \neq 1$. Gupta [15] introduced and analyzed the approach characteristics of $q$-Durrmeyer operators. Gupta and Heping [16] identified the $q$-Durrmeyer operators and estimated the rate of convergence for continuous functions with the help of the moduli of continuity. In [2, 17], Mahmudov defined the King-type $q$-Szász operators. He obtained the rate of convergence on weighted spaces and a Voronovskaya-type theorem for these operators. Some other studies based on classical $q$–theory are [17–24].

In light of the above studies, the following result has been obtained. For a real-valued function, $f(u)$ defined on the interval $[0, \infty)$, the operators

$$
\begin{align*}
\sum_{k=0}^{n} P_{n,k,q}(u) & \int_{0}^{\infty} P_{n,k,q}(t)h(t)d_{q}t, \\
\sum_{k=0}^{n} S_{n,k,q}(u) & \int_{0}^{\infty} S_{n,k,q}(t)h(t)d_{q}t, \\
\sum_{k=0}^{n} S_{n,k,q}(u) & \int_{0}^{\infty} P_{n,k,q}(t)h(t)d_{q}t,
\end{align*}
$$

(1)
for the general operator kernels
\[
P_{n,k,q}(x) = \left[ \begin{array}{c} n \\ k \end{array} \right]_q x^k (1 - x)^{n-k} = \left[ \begin{array}{c} n \\ k \end{array} \right] x^k \prod_{\nu=0}^{n-k-1} (1 - q^\nu x),
\]
(2)

\[
S_{n,k,q}(y) = \frac{1}{E_q[|n|_q y]^k} \left[ \begin{array}{c} n \\ k \end{array} \right]_q y^k.
\]
(3)

have been studied by many authors. On the contrary, the operator
\[
\sum_{k=0}^{n} P_{n,k,q}(u) \int_0^{\infty} P_{n,k,q}(t) h(t) d_q t
\]
(4)

has not been studied yet.

Therefore, we introduced the following operator:
\[
H_{n,q}(h;u) = \left[ \begin{array}{c} n \\ k \end{array} \right] \sum_{k=0}^{n} P_{n,k,q}(\frac{u}{b_n}) q^k \int_0^{\infty} S_{n,k,q}(\frac{t}{b_n}) h(q^k t) d_q t,
\]
(5)

which is \(q\)-Chlodowsky and \(q\)-Szasz-Durrmeyer hybrid operators on weighted spaces. Here, \(P_{n,k,q}(x)\) and \(S_{n,k,q}(y)\) are defined as in (2)-(3), \(u \in [0,b_n]\), \(0 < q < 1\), \(h \in C[0,b_n]\), and \(b_n\) is an increasing and positive sequence with properties \(\lim_{n \to \infty} b_n = \infty\) and \(\lim_{n \to \infty} (b_n/[n]_q) = 0\). In this article, we intend to study the approximation properties of the operator \(H_{n,q}(h; u)\). We produced our study by making use of [25].

The important terms of \(q\)-analysis which are used in this paper are given below, see [26, 27],

Given value of \(q > 0\) and \(n \in \mathbb{N}\), we define the \(q\)-integer \([n]_q\) by
\[
[n]_q = \begin{cases} 1 - q^n & \text{if } q \neq 1, \\ n & \text{if } q = 1. \end{cases}
\]
(6)
The \(q\)-factorial \([n]_q!\) is defined by
\[
[n]_q! = \begin{cases} [n]_q[n-1]_q \ldots [1]_q, & \text{if } n = 1, 2, 3, \ldots, \\ 1, & \text{if } n = 0, \end{cases}
\]
(7)

for \(0 \leq k \leq n\); we define the \(q\)-binomial coefficients \([n \choose k]_q\) by
\[
[n \choose k]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.
\]
(8)
The \(q\)-binomial can be written in the following forms:
\[
[n \choose k]_q = \frac{[n-1]_q + q^k [n-1]_q}{[k]_q + [n-1]_q},
\]
(9)

Exponential function \(e^z\) has two \(q\)-analogs, see [26, 27]:
\[
|z| < \frac{1}{1-q}, |q| < 1,
\]
(10)

\[
\Gamma_q(u) = \int_0^{1/1-q} t^{u-1} E_q(-qt) d_q t,
\]
(14)

\[
\gamma_q^B(u) = \int_0^{\frac{1}{1-q}} t^{u-1} e_q(-t) d_q t.
\]
(15)

For every \(B\) and \(u > 0\), one has
\[
\Gamma_q(u) = K(B; u) \gamma_q^B(u),
\]
(16)

where \(K(B; u) = (1/1 + B)^{u(1 + (1/B))} u^{1/u(1+B)}\). Especially, for any positive integer \(m\),
\[
K(B; m) = q^{m(m-1)/2} \text{ and } \Gamma_q(m) = q^{m(m-1)/2} \gamma_q^B(m),
\]
(17)

see [28].

The present note deals with the study of the \(q\)-Chlodowsky and \(q\)-Szasz-Durrmeyer hybrid operators on weighted spaces. Firstly, we estimate the moments for the \(H_{n,q}(h; u)\) operators.
We also study the rate of convergence for these operators $H_{n,q}(h; u)$. Furthermore, definitions and some properties for weighted spaces are given. We guess the order of approximation by weighted Voronovskaya-type theorem.

\[
\int_0^{\cos^{1-q}} (q^k t)^m S_{n,k,q} \left( \frac{t}{b_n} \right) dt = \int_0^{\cos^{1-q}} (q^k t)^m \frac{1}{E_q([n]_q (t/b_n))} q^{(k-1)/2} \left( \frac{[n]_q (t/b_n)}{[k]_q} \right)^k \frac{1}{[k]_q !} dt
\]

\[
= q^{km} \int_0^{\cos^{1-q}} t^m e_q \left( -[n]_q \frac{t}{b_n} \right) q^{(k-1)/2} \left( \frac{[n]_q t^k}{[k]_q b_n} \right) dt
\]

\[
= q^{km} \left[ \frac{b_n^{m+1}}{[n]_q^{m+1} [k]_q^2} q^{(k-1)/2} \int_0^{\cos^{1-q}} \left( \frac{t}{b_n} \right)^{k+m} e_q \left( -[n]_q \frac{t}{b_n} \right) [n]_q dt \right]
\]

\[
= q^{km} \left[ \frac{b_n^{m+1}}{[n]_q^{m+1} [k]_q^2} q^{(k-1)/2} \int_0^{\cos^{1-q}[n]_q} (v)^{k+m} e_q (-v) dv \right]
\]

\[
= q^{km} \left[ \frac{b_n^{m+1}}{[n]_q^{m+1} [k]_q^2} \frac{\Gamma_q(k+m+1)}{q^{(k+m+1)(k+m)/2}} \right] = q^{km} \left[ \frac{b_n^{m+1}}{[n]_q^{m+1} [k]_q^2} \frac{1}{q^{(k+m+1)(k+m)/2}} \right],
\]

where, for $m = 0$,

\[
\int_0^{\cos^{1-q}} (q^k t)^0 S_{n,k,q} \left( \frac{t}{b_n} \right) dt = \frac{b_n}{[n]_q} q^{-k}, \quad (19)
\]

for $m = 1$,

\[
\int_0^{\cos^{1-q}} (q^k t)^1 S_{n,k,q} \left( \frac{t}{b_n} \right) dt = \frac{[k+1]_q b_n^2}{[n]_q} q^{-k-1}, \quad (20)
\]

and for $m = 2$,

\[
\int_0^{\cos^{1-q}} (q^k t)^2 S_{n,k,q} \left( \frac{t}{b_n} \right) dt = \frac{[k+1]_q [k+2]_q b_n^3}{[n]_q^3} q^{-k-3}. \quad (21)
\]

\section{Estimation Moments}

Here, we will prove $H_{n,q}(t^n; u)$ for $m = 0, 1, 2$. By the definition of the $q$–Gamma function $\gamma_q$ in (15), we have

\[
H_{n,q}(1; u) = 1, \quad (22)
\]

\[
H_{n,q}(t; u) = u + \frac{b_n}{q[n]_q}, \quad (23)
\]

\[
H_{n,q}(t^n; u) = u^n + \frac{u^n \left( (q^n + 2q^2 + 1) b_n - q^3 u \right)}{q^n [n]_q} + \frac{(q + 1) b_n^2}{q^n [n]_q}. \quad (24)
\]

\textbf{Lemma 1.} For $u \in \{0, \infty\}$, $0 < q < 1$, and $h \in C[0, \infty)$, we have

\[
H_{n,q}(1; u) = 1, \quad (22)
\]

\[
H_{n,q}(t; u) = u + \frac{b_n}{q[n]_q}, \quad (23)
\]

\[
H_{n,q}(t^n; u) = u^n + \frac{u^n \left( (q^n + 2q^2 + 1) b_n - q^3 u \right)}{q^n [n]_q} + \frac{(q + 1) b_n^2}{q^n [n]_q}. \quad (24)
\]

\textbf{Proof.} Using (19), we obtain

\[
H_{n,q}(1; u) = \frac{[n]_q}{b_n} \sum_{k=0}^n p_{n,k,q} \left( \frac{u}{b_n} \right) q^k \int_0^{\cos^{1-q}} S_{n,k,q} \left( \frac{t}{b_n} \right) dt
\]

\[
= \frac{[n]_q}{b_n} \sum_{k=0}^n p_{n,k,q} \left( \frac{u}{b_n} \right) q^k \left( \frac{b_n}{[n]_q} \right)^{-k} = \sum_{k=0}^n p_{n,k,q} \left( \frac{u}{b_n} \right) = 1. \quad (25)
\]
Using (20) and \([k + 1]_q = q[k]_q + 1\), we obtain

\[
H_{n,q}(t, u) = \frac{[n]_q}{b_n} \sum_{k=0}^{n} P_{\ast,k,q} \left( \frac{u}{b_n} \right) \frac{q^k}{[n]_q} \int_{0}^{\frac{t}{b_n}} q^{k-1} \, dq \frac{(q^k t)}{[n]_q} S_{n,k,q} \left( \frac{t}{b_n} \right)
\]

\[
= \frac{[n]_q}{b_n} \sum_{k=0}^{n} P_{\ast,k,q} \left( \frac{u}{b_n} \right) q^k \left( \frac{[k + 1]_q b_n^3}{[n]_q} q^{-k-3} \right) = \sum_{k=0}^{n} P_{\ast,k,q} \left( \frac{u}{b_n} \right) \left( \frac{[k + 1]_q b_n^3}{[n]_q} q^{-k-3} \right) \tag{26}
\]

Using \((20)\) and \([k + 1]_q [k + 2]_q = q^3 [k]_q^2 + (2q^2 + 1)[k]_q + q + 1\), we obtain

\[
H_{n,q}(t^2, u) = \frac{[n]_q}{b_n} \sum_{k=0}^{n} P_{\ast,k,q} \left( \frac{u}{b_n} \right) q^k \left( \frac{[k + 1]_q [k + 2]_q b_n^3}{[n]_q} q^{-k-3} \right) = \sum_{k=0}^{n} P_{\ast,k,q} \left( \frac{u}{b_n} \right) \left( \frac{[k + 1]_q [k + 2]_q b_n^3}{[n]_q} q^{-k-3} \right) \tag{27}
\]

which completes the proof. \(\square\)

**Lemma 2.** Let \(u \in [0, \infty)\), \(0 < q < 1\), and \(h \in C[0, \infty)\), and we have

\[
H_{n,q}(t - u; u) = \frac{b_n}{q[n]_q},
\]

\[
H_{n,q}(t - u^2; u) = u \frac{(q^3 + 1)b_n - q^3 u}{q^3 [n]_q^3} + \frac{(q + 1)b_n^2}{q^3 [n]_q^3} \tag{28}
\]
where $H_{n,q}$ obtained the following result in [25]:

$$H_{n,q}(t; u) = H_{n,q}(t^2; u) = 2uH_{n,q}(t; u) + u^2H_{n,q}(1; u)$$

$$= u^2 + u\left(\frac{(q^3 + 2q^2 + 1)b_n - q^3u}{q^3[n]_q}\right) + \frac{(q + 1)b_n^2}{q^3[n]_q} - 2u\left(u + \frac{b_n}{q[n]_q}\right) + u^2$$

which completes the proof.

By simple calculations, we obtain

$$H_{n,q}(t; u) \leq \frac{b_n^2}{q^3[n]_q} + \frac{2b_n^2}{q^3[n]_q}$$

where $H_{n,q}(t; u)$ appears to be constrained in the above inequality, and this result is available in [25]. Likewise, A. İzgi obtained the following result in [25]:

$$H_{n,q}(t; u) \leq \frac{144b_n^2}{n^2}.$$

## 3. Approximation of $H_{n,q}(h; u)$ in Weighted Spaces

In this section, we use Gadjev’s Korovkin-type theorems on the weighted spaces [4, 29]. Let $B_p$ be the set of all functions $h$ over the real line, $\rho(u) = 1 + u^2$, where $u \in (-\infty, \infty) = \mathbb{R}$ such that

$$|h(u)| \leq M_{h,p}(u),$$

where $M_h$ is a positive constant depending on the function $h$. Now, let

$$C_p^k(\mathbb{R}) = \left\{ h \in C_p(\mathbb{R}), h \text{ is continuous}\right\},$$

$$C_p^k(\mathbb{R}) = \left\{ h \in C_p(\mathbb{R}), \lim_{|x| \to \infty} \frac{|h(u)|}{\rho(u)} = K_h < \infty\right\}.$$  

(34)

It is clear that $C_p^k(\mathbb{R}) \subset C_p(\mathbb{R}) \subset B_p(\mathbb{R})$, where $B_p(\mathbb{R})$ is the linear normed space with the norm

$$\|h\|_p = \sup_{u \in (-\infty, \infty)} \frac{|h(u)|}{\rho(u)} \quad h \in B_p.$$  

(35)

Let $b_n$ be an increasing and positive sequence with features

$$\lim_{n \to \infty} q_n = 1,$$

$$\lim_{n \to \infty} b_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n}{n_q} = 0.$$  

(36)

**Lemma 3.** $H_{n,q}(h; u)$ defined in (5) is a sequence of positive and linear operators that move $C_p[0,b_n]$ to $B_p[0,b_n]$. So, there is

$$\lim_{n \to \infty} H_{n,q}(\rho(t); u) = \rho(u),$$

(37)

for $\rho(u) \in C_p[0,b_n]$ on $[0,b_n]$.

**Proof.** Taking advantage of equations (22) and (24), we have

$$H_{n,q}(\rho(t); u) = \rho(u) + u\left(\frac{(q^3 + 2q^2 + 1)b_n - q^3u}{q^3[n]_q}\right) + \frac{(q + 1)b_n^2}{q^3[n]_q}$$

$$+ \frac{(q + 1)b_n^3}{q^3[n]_q} = 0.$$  

(38)

Therefore, $\|H_{n,q}(h; u)\|_{p,[0,b_n]}$ is uniformly bounded on $[0,b_n]$, since

$$\lim_{n \to \infty} \sup_{u \in [0,b_n]} \left(\frac{(q^3 + 2q^2 + 1)b_n - q^3u}{q^3[n]_q}\right) + \frac{(q + 1)b_n^2}{q^3[n]_q} = 0,$$

(39)

under the condition in (36) that completes the proof. 

**Theorem 1.** Let $h \in C_p^k[0,b_n]$. Then,

$$\lim_{n \to \infty} \left\|H_{n,q} h - h\right\|_{p,[0,b_n]} = 0.$$  

(40)
Proof. From Lemma 1, we have

\[
\lim_{n \to \infty} \left\| H_{n,q}(1; u) - 1 \right\|_{\mathcal{P}[0,b_n]} = 0,
\]

\[
\lim_{n \to \infty} \left\| H_{n,q}(t; u) - u \right\|_{\mathcal{P}[0,b_n]} = \lim_{n \to \infty} \sup_{u \in \mathcal{P}[0,b_n]} \left| \frac{b_n}{q[n]_q} \frac{1}{1 + u^2} \right| = \lim_{n \to \infty} \frac{b_n}{q[n]_q} = 0,
\]

\[
\lim_{n \to \infty} \left\| H_{n,q}(t^2; u) - u^2 \right\|_{\mathcal{P}[0,b_n]} = \lim_{n \to \infty} \sup_{u \in \mathcal{P}[0,b_n]} \left| \left( \frac{(q^2 + 2q^2 + 1)b_n - q^3 u}{q^3[n]_q} + (q + 1)b_n^2 \frac{1}{1 + u^2} \right) \frac{1}{1 + u^2} \right|
\]

\[
\leq \lim_{n \to \infty} \frac{2q^2 + 1}{4q^3[n]_q} b_n^2 + \frac{(q + 1)b_n^2}{q^3[n]_q} = 0,
\]

according to (36), which completes the proof. \hfill \Box

4. Main Results

Here, we estimate the rate of approximation of the \( H_{n,q}(h; u) \) hybrid operators. The following theorem gives the rate of approximation of the sequence of \( H_{n,q}(h; u) \) operators in terms of moduli of continuity of a function \( h \in \mathcal{C}_p^k[0, b_n] \). For \( h \in \mathcal{C}_p^k[0, b_n] \), the moduli of continuity is defined as follows:

\[
\omega(h, \delta) = \sup \{|h(t) - h(u)|; t, u \in [a, b], |t - u| \leq \delta\},
\]

(42)

where \( \delta \to 0 \).

The weighted moduli of continuity is defined as follows:

\[
\Lambda_n(h, \delta) = \sup_{|h| \leq \delta} \frac{|h(u + a) - h(u)|}{(1 + u^2)(1 + a^2)}\left(1 + \frac{|t - u|}{\delta}\right) \Lambda_n(h, \delta),
\]

(43)

It is seen that they provide the characteristics of the continuity module. In what follows,

\[
\lim_{h \to 0} \Lambda_n(h, \delta) = 0,
\]

\[
|h(t) - h(u)| \leq \left(1 + (t - u)^2\right)\left(1 + u^2\right)\left(1 + \frac{|t - u|}{\delta}\right) \Lambda_n(h, \delta),
\]

(44)

\[
\left| H_{n,q}(t; u) - h(u) \right| \leq H_{n,q}(\omega(h(t) - h(u)))\left(1 + (t - u)^2\right)\left(1 + \frac{|t - u|}{\delta}\right);
\]

\[
\leq \Lambda_n(h, \delta) \left(1 + u^2\right) H_{n,q}\left(1 + (t - u)^2\right)\left(1 + \frac{|t - u|}{\delta}\right); u)
\]

\[
\leq \Lambda_n(h, \delta) \left(1 + u^2\right)\left[H_{n,q}\left(1 + (t - u)^2\right)u\right] + H_{n,q}\left(1 + (t - u)^2\right)\left(\frac{|t - u|}{\delta}; u\right),
\]

(47)
from Lemmas 1-2, (31), and (32), we obtain
\[
H_{n,q}(1 + (t - u)^2) ; u = H_{n,q}(1; u) + H_{n,q}(t - u)^2 ; u \\
\leq 1 + \frac{b_n^2}{q^2[n]_q} + \frac{2b_n^2}{q^2[n]_q} \leq \mu_1(b_n, n, q),
\]
(48)

As a result
\[
H_{n,q}(1 + (t - u)^2) \left(1 + \frac{|t - u|}{\delta}\right); u \leq H_{n,q}(1 + (t - u)^2); u \leq H_{n,q}\left(\left(\frac{|t - u|}{\delta}\right)^2 ; u\right)^{1/2}. 
\]
(49)

We can write by (32)-(33) that
\[
H_{n,q}(1 + (t - u)^2) = H_{n,q}(1; u) + 2H_{n,q}(t - u)^2; u + H_{n,q}(t - u)^4; u \leq 1 + 2\mu_1 + \frac{144b_n^4}{n^2} \leq \mu_2(b_n, n, q),
\]
(50)

where \(\delta = \frac{b_n^2}{q^3[n]_q}\).

As a result
\[
\|H_{n,q}(h; u) - h(u)\|_{p; [0, b_n]} \leq \mu(b_n, n, q)\Lambda_n\left(h, \frac{b_n^2}{q^3[n]_q}\right),
\]
(51)

which completes the proof.

Now, we prove a Voronovskaja-type result for the \(H_{n,q}(h(t); u)\) operators.

\[
\mathcal{O}(t; u) = \begin{cases} 
\frac{h(t) - h(u) - (t - u)_qD_qh(u) - 1/[2_q(t - u)_q]D_q^2h(u)}{(t - u)_q^2}, & t \neq u, \\
0, & t = u.
\end{cases}
\]
(55)

By implementing \(q\)-L’Hospital’s rule twice,
\[
\lim_{t \rightarrow u} \mathcal{O}(t; u) = \lim_{t \rightarrow u} \frac{D_q^2h(t) - D_q^2h(u)}{2} = 0.
\]
(56)

Then, \(\mathcal{O}(u; u) = 0\) and \(\mathcal{O}(\ldots; u) \in C_{pq}[0, b_n]\). As a result, we can write

\[
\lim_{t \rightarrow u} \mathcal{O}(t; u) = 0, \\
\mathcal{O}(\ldots; u) = 0,
\]

and using Cauchy–Schwartz inequality

\[
\text{Theorem 3. Let } h, D_qh(u), D_q^2h(u) \in C_{pq}[0, b_n], \text{ and } u \in [0, b_n] \text{ be fixed, and we have}
\]
(53)

\[
\lim_{n \rightarrow \infty} \frac{[n]_q}{b_n}_q\|H_{n,q}(h(u) - h(u))\| = D_q^2h(u).
\]
(54)

\[
\text{Proof. By } q\text{-Taylor formula for } h \in C_{pq}[0, b_n] \text{ and } 0 < q < 1,
\]
(53)

\[
h(t) = h(u) + (t - u)_qD_qh(u)
\]

\[
+ \frac{1}{[2_q(t - u)_q]D_q^2h(u) + \mathcal{O}(t; u)(t - u)_q^2},
\]
(54)

where \((t - u)_q^2 = (t - u)(t - qu)\) and
Now, let us use the Cauchy–Schwarz inequality in the last term of the last equation:

\[ H_{n,q}(\bar{\sigma}(t;u)(t-u)_q^2;u) \leq \left\{ H_{n,q}(t-u)_q^4;u \right\}^{1/2} \left\{ H_{n,q}(\bar{\sigma}^2(t;u);u) \right\}^{1/2}. \]  

(58)

Since \( \bar{\sigma}(t.;u) \in C^4_\rho [0,b_n] \) and \( \bar{\sigma}(t;u) \rightarrow \bar{\sigma}(u;u) \rightarrow 0 \) as \( t \rightarrow u \), applying Lemma 2 and (31) and (32),

\[
\begin{align*}
\lim_{n \to \infty} H_{n,q}(\bar{\sigma}^2(t;u);u) &= \bar{\sigma}^2(u;u) = 0, \\
\lim_{n \to \infty} H_{n,q}(t-u)_q^4;u &< \infty.
\end{align*}
\]

(59)

As a result

\[
H_{n,q}(h;u) - h(u) = H_{n,q}( (t-u)_q;u) D_q^2 h(u) + \frac{1}{[2]_q} H_{n,q}( (t-u)_q^2;u) D_q^4 h(u)
\]

\[
= D_q^2 h(u) \frac{b_n}{q[n]_q} + \frac{1}{[2]_q} D_q^4 h(u) \left( u \left( \frac{q^4+1}{q} b_n - q^3 u \right) \frac{[n]_q q^2}{[n]_q q^4} + \frac{(q+1)b_n^2}{q^4[n]_q^2} \right),
\]

(60)

from (33), and we have

\[
\lim_{n \to \infty} \frac{[n]_q}{b_n} \left\{ H_{n,q}(h;u) - h(u) \right\} = D_q^2 h(u) + uD_q^4 h(u),
\]

(61)

which completes the proof. \( \Box \)

5. Conclusion

In this paper, the approximation properties and rate of convergence of \( q \)-Chlodowsky and \( q \)-Szász-Durrmeyer hybrid operators in weighted spaces are investigated.

For further research in this topic, it would be interesting to study whether the quality of the approximation pythagorean fuzzy set operators, \( q \)-statistical convergence operators, and \( q \)-complex operators directly influence the quality of the approximation of the characteristics. Some studies can be used for future research, such as [30–34].

Data Availability

All data generated or analyzed during this study are included in this published article. They are cited at relevant places within the text as references.

Conflicts of Interest

The authors do not have any conflicts of interest.

Authors’ Contributions

All authors contributed equally to this paper.

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