Research Article

Computing the Hosoya Polynomial of $M$-th Level Wheel and Its Subdivision Graph

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HT_hedetermination of Hosoyapolynomial is the latest scheme, and it provides an excellent and superior role in finding the Weiner and hyper-Wiener index. HT_he application of Weiner index ranges from the introduction of the concept of information theoretic analogues of topological indices to the use as major tool in crystal and polymer studies. In this paper, we will compute the Hosoya polynomial for multirule graph and uniform subdivision of multirule graph. Furthermore, we will derive two well-known topological indices for the abovementioned graphs, first Weiner index, and second hyper-Wiener index.

1. Introduction

Let $G$ be a finite connected graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The distance $d_{u,v}$ between $u, v \in V(G)$ is the length of the shortest path joining $u, v$. The diameter $d(G)$ of $G$ is $\max_{u,v} d_{u,v}$. The terminologies not defined here can be seen in [1, 2]. The Weiner index $W$ was first put forward in chemistry by Harold Weiner to compute the cardinality of the carbon-carbon bonds among all pairs of carbon atoms in alkane. For a molecular graph $G$, it is defined as

$$W(G) = \sum_{u,v \in V} d_{u,v}. \tag{1}$$

To read more about the chemical application of Weiner index, see [3–6], and for its mathematical properties, see [7, 8].

Milan Randic coined the term hyper-Wiener index $WW(G)$ of $G$ as

$$WW(G) = \frac{1}{2} \sum_{u,v \in V} (d_{u,v} + d_{u,v}^2). \tag{2}$$

To read more the properties of hyper-Wiener index, see [9–12]. Hosoya polynomial was first introduced by Hosoya [13] and it received the attention of a lot of researchers. The same notion was independently put forward by Sagan et al. [14] as Weiner polynomial $G$. The Hosoya polynomial $H(G,x)$ of $G$ is defined as

$$H(G,x) = \sum_{u,v \in V} (x^{d_{u,v}}). \tag{3}$$

Let $a(G,k)$ be the number of ordered pair $(u,v)$ in $V$ with $d_{u,v} = k$. Then, the above definition of Hosoya polynomial can be expressed as

$$H(G,x) = \sum_{k=0}^{d(G)} a(G,k) x^k. \tag{4}$$
The Hosoya polynomial has been investigated on polycyclic aromatic hydrocarbons [15], benzenoid chains [16], Fibonacci and Lucas cubes [17], zigzag polyhexanotorus [9], carbon nanotubes [18], Hanoi graphs [19], and circumcoronene series [20]. A significant importance of $H(G, x)$ is that some distance-based topological indices (TIs) such as $W(G)$ and $WW(G)$ of $G$ can be computed from the Hosoya polynomial as

$$W(G) = H'(G; 1), \quad WW(G) = H'(G; 1) + \frac{1}{2}H''(G; 1).$$

(5)

The readers can see the following papers [21–25] for the results on distance-based TIs.

$V(mW_n) = \{c, u_i^j, 0 \leq i \leq n - 1, 1 \leq j \leq m\}$,

$E(mW_n) = \{cu_i^j, 0 \leq i \leq n - 1, 1 \leq j \leq m\} \cup \{u_i^j u_i^{j+1}, 0 \leq i \leq n - 1, 1 \leq j \leq m\}$.  

(6)

Next, the theorem gives the expression for the Hosoya polynomial of $mW_n$.

**Theorem 1.** Let $m, n \geq 1$, then $H(mW_n; x)$ is of the form

$$H(mW_n; x) = (mn + 1) + (2mn)x + \left[ mn(n - 3) + \frac{(n(m - 1))^2}{2}x \right].$$

(7)

Proof. It is easy to observe that the diameter of $mW_n$ is 2. In order to derive the $H(mW_n; x)$, we compute the coefficients $a(mW_n, k)$ for $k = 0, 1, 2$. By definition, we have $a(mW_n, 0) = mn + 1$ and $a(mW_n, 1) = 2mn$. To compute $a(mW_n, 2)$, we use the following notation:

$$\alpha_A = \text{number of pair of vertices in set } A.$$  

(8)

The cardinality of order pairs in $V(mW_n)$ with distance 2 can be characterized by the following two sets:

$$A_1 = \{u_i^j, u_i^{j+1}\}, \quad 1 \leq j \leq m, \quad 0 \leq i \leq n - 1, \quad 2 \leq l \leq n - 2,$$

$$A_2 = \{u_i^j, u_l^h\}, \quad 0 \leq i \leq n - 1, \quad 0 \leq l \leq n - 1, \quad 1 \leq j \leq m - 1, \quad j + 1 \leq h \leq m.$$  

(9)

The cardinality of the above sets is $\alpha_{A_1} = mn(n - 3)$ and $\alpha_{A_2} = (n(m - 1))^2(m)/2$ and hence the coefficient $a(mW_n, 2)$ is equal to $a(mW_n, 2) = \alpha_{A_1} + \alpha_{A_2} = mn(n - 3) + (n(m - 1))^2(m)/2$. Now, using the values of $a(mW_n, 0)$, $a(mW_n, 1)$, and $a(mW_n, 2)$, we get the desired result.  

□
Corollary 1. Let \( m, n \geq 1 \), then \( W(mW_n) \) and \( WW(mW_n) \) are given as
\[
W(mW_n) = (2mn) + 2 \left[ mn(n - 3) + \frac{(n(m - 1))^2}{2} \right].
\]
\[
WW(mW_n) = (2mn) + 3 \left[ mn(n - 3) + \frac{(n(m - 1))^2}{2} \right].
\]

It is easy to observe that order and size of are \( 3mn + 1 \) and \( 4mn \), respectively. In the next theorem, we give the analytic form to derive the \( H(S(mW_n); x) \).

3. Hosoya Polynomial of Subdivision of \( M \)-th Level Wheel Graph

The subdivision graph \( S(mW_n) \) of \( mW_n \) is constructed from \( mW_n \) by adding a vertex into each edge of \( mW_n \). In other words, we replace each edge of \( mW_n \) by a path of length 2. The graph of \( S(mW_n) \) is depicted in Figure 2. If we label the new vertices that we insert in the cycle at the \( j \)-th level by \( x_1^j, x_2^j, \ldots, x_{n-1}^j \) for \( j = 1, 2, \ldots, m \), then the vertex set and edge set of \( S(mW_n) \) can be written as

\[
V(S(mW_n)) = \{c, u'_1, v'_1, x'_1, \ldots, \}
\]
\[
E(S(mW_n)) = \{cv'_1, v'_1u'_1, x'_1u'_1, u'_1x'_{i+1}, \ldots, 0 \leq i \leq n - 1, 1 \leq j \leq m \}. \quad (11)
\]

Theorem 2. Let \( m, n \geq 1 \), then the \( H(S(mW_n); x) \) is of the form
\[
H(S(mW_n); x) = (3mn + 1) + (4mn)x + [5mn + mn(n - 1)]x^2
\]
\[
+ \left[ \frac{(mn)^2(m - 1)}{2} + 3mn + mn(n - 1) \right]x^3 + \left[ mn((m - 1)n)^2 + mn(2n - 5) + mn \right]x^4
\]
\[
+ \left[ mn(n - 4) + \frac{m((m - 1)n)^2}{2} \right]x^5 + \left[ mn(n - 5) + \frac{m((m - 1)n)^2}{2} \right]x^6. \quad (12)
\]

Proof. It is easy to observe that the diameter of \( mW_n \) is 6. In order to derive the \( H(S(mW_n); x) \), we find the coefficients \( a(S(mW_n), k) \) for \( k = 0, 1, 2, \ldots, 6 \). By definition, we have \( a(S(mW_n), 0) = 3mn + 1 \) and \( a(S(mW_n), 1) = 4mn \). To compute \( a(S(mW_n), j) \) for \( j = 2, 3, 4, 5, 6 \), we use the following notation:

\[
\alpha_A = \text{number of pair of vertices in set } A. \quad (13)
\]
The cardinality of order pairs in $V(S(mW_n))$ at distance 2 can be characterized by the following sets:

$B_1 = \{(c, u^i_j), \ 1 \leq j \leq m, 0 \leq i \leq n - 1\},$
$B_2 = \{(v^i_j, v^i_{l+1}), \ 1 \leq j \leq m, 0 \leq i \leq n - 1, 1 \leq l \leq n - 1\},$
$B_3 = \{(v^i_j, x^i_{l+1}), \ 1 \leq j \leq m, 0 \leq i \leq n - 1\},$
$B_4 = \{(v^i_j, x^i), \ 0 \leq i \leq n - 1, 1 \leq j \leq m\},$
$B_5 = \{(u^i_j, u^i_{l+1}), \ 1 \leq j \leq m, 0 \leq i \leq n - 1\},$
$B_6 = \{(x^i_j, x^i_{l+1}), \ 1 \leq j \leq m, 0 \leq i \leq n - 1\}.$

(14)

The cardinality of the above sets is $\alpha_{B_1} = \alpha_{B_2} = \alpha_{B_3} = \alpha_{B_4} = mn, \ \alpha_{B_5} = mn(n - 1)$, and hence

$$\alpha \left( S(mW_n) \right), 2) = \alpha_{B_1} + \alpha_{B_2} + \alpha_{B_3} + \alpha_{B_4} + \alpha_{B_5} + \alpha_{B_6} = 5mn + mn(n - 1).$$

The cardinality of order pairs in $V(S(mW_n))$ at distance 3 can be characterized by the following sets:

$C_1 = \{(c, x^i_j), \ 1 \leq j \leq m, 0 \leq i \leq n - 1\},$
$C_2 = \{(v^i_j, u^i_{l+1}), \ 1 \leq j \leq m, 0 \leq i \leq n - 1, 1 \leq l \leq n - 1\},$
$C_3 = \{(v^i_j, u^i_{l+k}), \ 1 \leq j \leq m, 1 \leq k \leq m - j, 0 \leq i \leq n - 1, 0 \leq l \leq n - 1\},$
$C_4 = \{(u^i_j, x^i_{l+2}), \ 1 \leq j \leq m, 0 \leq i \leq n - 1\},$
$C_5 = \{(x^i_j, u^i_{l+1}), \ 1 \leq j \leq m, 0 \leq i \leq n - 1\}.$

(16)

The cardinality of the above sets is $\alpha_{C_1} = \alpha_{C_2} = \alpha_{C_5} = mn, \alpha_{C_3} = mn(n - 1), \alpha_{C_4} = (mn)^2 (m - 1)/2$, and hence
\[
\alpha(S(mW_n), 3) = \alpha_{C_1} + \alpha_{C_2} + \alpha_{C_3} + \alpha_{C_4} + \alpha_{C_5}
\]
\[
= \frac{(mn)^2 (m - 1)}{2} + 3mn + mn(n - 1).
\]

(17)

The cardinality of order pairs in \( V(S(mW_n)) \) at distance 4 can be characterized by the following sets:
\[
D_1 = \{(v'_i, x'_{i+l}), \ 1 \leq j \leq m, 0 \leq i \leq n - 1, 2 \leq l \leq n - 1\},
\]
\[
D_2 = \{(v'_i, x'_{i+k}), \ 1 \leq j \leq m - 1, 1 \leq k \leq m - j, 0 \leq i \leq n - 1, 0 \leq l \leq n - 1\},
\]
\[
D_3 = \{(u'_i, u'_{i+l}), \ 1 \leq j \leq m, 0 \leq i \leq n - 1, 2 \leq l \leq n - 2\},
\]
\[
D_4 = \{(u'_i, u'_{i+k}), \ 1 \leq j \leq m - 1, 1 \leq k \leq m - j, 0 \leq i \leq n - 1, 0 \leq l \leq n - 1\},
\]
\[
D_5 = \{(x'_i, x'_{i+l}), \ 1 \leq j \leq m, 0 \leq i \leq n - 1\}.
\]

(18)

The cardinality of the above sets is \( \alpha_{D_1} = mn(n - 2), \alpha_{D_2} = m((m - 1)n)^2/2, \alpha_{D_3} = mn(n - 3), \alpha_{D_4} = m((m - 1)n)^2/2, \alpha_{D_5} = mn \), and hence
\[
\alpha(S(mW_n), 4) = \alpha_{D_1} + \alpha_{D_2} + \alpha_{D_3} + \alpha_{D_4} + \alpha_{D_5}
\]
\[
= m((m - 1)n)^2 + mn(2n - 4).
\]

(19)

The cardinality of order pairs in \( V(S(mW_n)) \) at distance 5 can be characterized by the following sets:
\[
E_1 = \{(u'_i, x'_{i+l}), \ 1 \leq j \leq m, 0 \leq i \leq n - 1, 3 \leq l \leq n - 2\},
\]
\[
E_2 = \{(u'_i, x'_{i+k}), \ 1 \leq j \leq m - 1, 1 \leq k \leq m - j, 0 \leq i \leq n - 1, 0 \leq l \leq n - 1\}.
\]

(20)

The cardinality of the above sets is \( \alpha_{E_1} = mn(n - 4), \alpha_{E_2} = m((m - 1)n)^2/2, \), and hence
\[
\alpha(S(mW_n), 5) = \alpha_{E_1} + \alpha_{E_2}
\]
\[
= mn(n - 4) + \frac{m((m - 1)n)^2}{2}.
\]

(21)

The cardinality of order pairs in \( V(S(mW_n)) \) at distance 6 can be characterized by the following sets:
\[
F_1 = \{(x'_i, x'_{i+l}), \ 1 \leq j \leq m, 0 \leq i \leq n - 1, 3 \leq l \leq n - 3\},
\]
\[
F_2 = \{(x'_i, x'_{i+k}), \ 1 \leq j \leq m - 1, 1 \leq k \leq m - j, 0 \leq i \leq n - 1, 0 \leq l \leq n - 1\}.
\]

(22)

The cardinality of the above sets is \( \alpha_{F_1} = mn(n - 5), \alpha_{F_2} = m((m - 1)n)^2/2, \), and hence
\[
\alpha(S(mW_n), 6) = \alpha_{F_1} + \alpha_{F_2}
\]
\[
= mn(n - 5) + \frac{m((m - 1)n)^2}{2}.
\]

(23)

Now, using the values of \( \alpha(S(mW_n), 0), \alpha(S(mW_n), 1), \alpha(S(mW_n), 2), \alpha(S(mW_n), 3), \alpha(S(mW_n), 4), \alpha(S(mW_n), 5), \) and \( \alpha(S(mW_n), 6) \), we get the desired result. □

**Corollary 2.** Let \( m, n \geq 1 \), then the \( W(mS_n) \) and \( WW(mS_n) \) are
4. Conclusion

We examined the Hosoya polynomial and two vastly studied TIs $W(G)$ and $WW(G)$ for multiwheel graph $mW_n$ and subdivision of multiwheel graph $mS_n$.

Data Availability

No data were used for this study.

Disclosure

Mathematics subject classification: 05C09, 05C92, 92E10.

Conflicts of Interest

The authors hereby declare that there are no conflicts of interest regarding the publication of this paper.

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