A fidelity treatment of near-resonant states in the atom-optics kicked rotor

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Abstract. We investigate the dynamics of the atom-optics δ-kicked rotor in the vicinity of quantum resonance. Although small deviations from resonant conditions lead to a negligible change in the momentum space probability density, they lead to a significant relative phase change between the different momentum states taking part in the dynamics. By adding a tailored pulse to the kicked rotor pulse sequence, one can measure the overlap between the resonant state and any other state, i.e. perform a fidelity measurement. Using this sequence, we predict a narrow peak around quantum resonance with a width that scales as \(1/N^3\) with \(N\) being the number of pulses in the kicked rotor sequence. This method may be of interest to precision measurements, such as \(\hbar/M\).

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1. Introduction

The δ-kicked rotor is a simple system, yet one that displays a wide range of dynamics. Indeed, it is a paradigmatic system in the study of classical and quantum chaos \[1\]. The atom-optics kicked rotor is a physical realization of the δ-kicked rotor Hamiltonian; it consists of a cloud of laser-cooled atoms subjected to temporally periodic pulses of an off-resonant standing wave of laser light. This yields a time-dependent, sinusoidal potential that transfers momentum to the atoms, and the transferred momentum is typically determined by a time-of-flight measurement. The atom-optics kicked rotor has allowed experimental studies of a wide range of the quantum dynamical phenomena displayed by the kicked rotor system, including ‘dynamical localization’ and ‘quantum resonances’ \[2, 3\]. Furthermore, a number of experiments and theoretical studies have investigated how deviations from the idealized system affect the dynamics. These deviations include nonlinear effects such as particle–particle interactions, additional potentials, noise on the pulse trains and spontaneous emission \[3–6\].

In recent years, the use of a Bose–Einstein condensate as a source of coherent ultra-cold atoms has enabled studies that require accurate control of the initial atomic momentum \[7, 8\], and the use of an atom interferometer for probing the atom-optics kicked rotor has further expanded possible investigations \[9, 10\].

Quantum resonances in the δ-kicked rotor occur for specific combinations of parameters, and are characterized by unbounded ballistic transfer of momentum to the atoms \[3, 11\]. This is in contrast to the generic behavior of the system—dynamical localization—where the transfer of energy to the atoms ceases after the ‘quantum break time’ \[2, 4\]. Quantum resonances occur due to the matter-wave Talbot effect \[8, 12, 13\], where periodic wavefunctions revive/self-image after the Talbot time \[7\], thereby allowing coherent momentum transfer from the optical standing wave to the atom.

Beyond its role in the study of quantum chaos, the dynamics of the atom-optics kicked rotor in a region in parameter space around the quantum resonance is of potential relevance to metrological applications \[9, 14\]. Indeed, the precision with which one can identify the quantum resonance and thereby the Talbot time is of great interest, since a measurement of the Talbot time together with other well-known quantities constitutes a measurement of the fine structure constant α \[9, 15, 16\]; a narrow resonance will enable a precise determination of the Talbot time. In this field, Tonyushkin et al \[9\] reported on the use of quantum resonances to enhance the precision of a time-domain de Broglie interferometer \[17\]. Being a dynamical system, quantum resonances in the δ-kicked rotor are not necessarily limited by the Fourier relation where the width of a resonance scales inversely proportional to the interrogation time \[18, 19\]. In fact, it has been shown that the width of the resonance when the time between pulses is scanned across the Talbot time—as determined by measurements of the energy imparted to the atoms—scales as \(1/N^2\), where \(N\) is the interrogation time in units of time between pulses \[3, 8\].

In this paper, we investigate an alternative to the standard kicked rotor pulse sequence: we propose a method that measures the overlap of a state somewhere in the resonance peak with the state occurring for an exact quantum resonance, thereby performing a type of fidelity measurement. This is achieved by the addition of a single pulse that returns the state resulting from resonant driving to the initial state, followed by a measurement of the population in the initial state. Figure 1 illustrates the standard kicked rotor pulse sequence and the proposed sequence. When the initial state is a plane wave, the proposed sequence shows a resonant peak of height one in the return probability when the time between pulses is scanned across the
Figure 1. (a) The $\delta$-kicked rotor pulse sequence consists of a series of pulses (four pulses in this illustration) of a sinusoidal potential spaced in time by $T$, with the same kicking strength ($\phi_d$) and phase. (b) The proposed fidelity pulse sequence is achieved by adding an extra pulse to the $\delta$-kicked rotor sequence. The extra pulse should have opposite phase (the potential should be shifted by half its spatial period) and a strength of $N\phi_d$.

Talbot time. The width of this peak is narrower than the peak observed in the energy transferred to the atoms around resonance; indeed, the width of the fidelity peak scales as $1/N^3$, whereas the energy peak scales as $1/N^2$. This reveals that important information (in this case, relative phase between momentum components) of near-resonant states is not captured by investigations of the transferred energy. However, our proposed pulse sequence is sensitive to deviations from the resonant state in both relative phase and population. The narrow width and favorable scaling of the fidelity peak make it a better candidate for potential metrological applications than measurements of the transferred energy near resonance. In fact, the fidelity pulse sequence can be interpreted as an atomic interferometer where as in \cite{20, 21}, the measurement precision increases as the square of the number of imparted photon momenta.

We extend our calculations to cover the more realistic case of a Gaussian wavepacket as the initial state. We observe several deviations from the idealized plane wave calculations. These include that the peak height drops as the number of pulses in the sequence increases, and deviations from the $1/N^3$ scaling of the peak width.

Our proposed pulse sequence utilizes a similar principle to the sequence used to reverse quantum dynamics in a classical chaotic system in \cite{22}. It differs from the investigations reported in \cite{23, 24}, in that they analyzed the decay in overlap of states evolved with different kick strengths whereas our technique relies on a perturbation in the time between pulses.

2. Quantum resonances in the $\delta$-kicked rotor

In the atom-optics kicked rotor, ultra-cold atoms are kicked by a pulse sequence of a one dimensional (1D) periodic optical potential formed by two counterpropagating laser beams. The laser light generating the standing wave has wave number $k$ and a frequency tuned sufficiently far from resonances in the atoms so that spontaneous emission can be neglected. If the pulses of the optical potential are sufficiently short, such that the distance an atom travels during the

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pulse is much smaller than the period of the potential—the so-called Raman–Nath regime—then the 1D dynamics of the atoms is well described by the idealized atom-optics kicked rotor Hamiltonian given by [4]

\[
\hat{H}(t) = \frac{\hat{p}^2}{2M} + \hbar \phi_d \cos(2k\hat{x}) \sum_{n=0}^{N-1} \delta(t - nT),
\]

where \(\hat{x}\) and \(\hat{p}\) are, respectively, position and momentum operators, \(t\) the time, \(T\) the time between pulses, \(M\) the mass of the atom, \(k\) the wave number of the light used to form the periodic potential, \(\phi_d\) the kicking strength and \(N\) the number of pulses. An initial state evolves according to

\[
|\psi(t = 0)\rangle = \hat{U}^{N} |\psi(t = 0)\rangle,
\]

where \(\hat{U}\), the one-period evolution operator, is given by

\[
\hat{U} = \exp\left(-\frac{i}{\hbar} \frac{\hat{p}^2}{2M} T\right) \exp(-i\phi_d \cos(2k\hat{x})).
\]

Note that equation (3) factorizes as a product of a kick operator and a free evolution operator due to the \(\delta\)-function time dependence in the Hamiltonian (equation (1)).

Quantum resonances occur for specific combinations of initial atomic momentum and the time between pulses \(T\). We first consider the case where the atom is initially at rest \((|p = 0\rangle \equiv |0\rangle)\) and the ‘first quantum resonance’, for which \(T = T_T = 1/(4\omega_r)\), where \(T_T\) is the Talbot time, and \(\omega_r = \hbar k^2/(2M)\) is the atomic recoil frequency. For the first quantum resonance, the free-space evolution term in equation (3) equals one, so that the kicks add coherently and consecutive pulses serve to increase the amplitude of the phase modulation. Under these conditions, the time-evolved state is given by [25]

\[
|\psi(t = NT_T)\rangle = \exp(-i\phi_dN \cos(2k\hat{x})) |0\rangle = \sum_{n=-\infty}^{\infty} (-i)^n J_n (N\phi_d) |n2\hbar k\rangle,
\]

where \(J_n\) denotes a Bessel function of the first kind of order \(n\).

When the time between pulses differs from the Talbot time by a small amount \(\epsilon\) or the initial momentum differs from zero by a small amount \(p_0\), equation (4) no longer predicts the time-evolved wavefunction. Since the kick operator is periodic in space, Bloch’s theorem ensures that only momentum states of the form \(|p_0 + n2\hbar k\rangle\) take part in the dynamics, and the wavefunction at \(t = NT\) can be written as

\[
|\psi(t = NT)\rangle = \sum_{n=-\infty}^{\infty} c_n |p_0 + n2\hbar k\rangle,
\]

with the expansion coefficients given by

\[
c_n (\epsilon, p_0) = \langle p_0 + n2\hbar k | \hat{U}^N |p_0\rangle.
\]

Our present goal is to understand the evolution of the time-evolved state in the vicinity of the quantum resonance as a function of the initial momentum and deviation from the Talbot
time. To this end, let us consider the first order Taylor expansion of the $c_n$ coefficients as a function of $\epsilon$ and $p_0$. We begin by evaluating

$$\left. \frac{\partial c_n}{\partial \epsilon} \right|_{\epsilon=0, p_0=0} = \frac{-i}{2M\hbar} \sum_{r=1}^{N} b_{n,r},$$

(7)

where we have defined

$$b_{n,r} \equiv \langle n2\hbar k | e^{-i\phi_d(r-1)\cos(2k\xi)} \hat{p}^2 e^{-i\phi_d(N+1-r)\cos(2k\xi)} | 0 \rangle.$$  

(8)

Using the Bessel function expansion of equation (4) we obtain

$$b_{n,r} = 4\hbar^2 k^2 \sum_{l=-\infty}^{\infty} l^2 J_l((N+1-r)\phi_d) J_l(N\phi_d) (r-1) \frac{(-1)^l}{(-1)^n}.$$  

(9)

Equations (7)–(9) show that when $n$ is even (odd), the partial derivative $\frac{\partial c_n}{\partial \epsilon} \big|_{\epsilon=0, p_0=0}$ is imaginary (real). Furthermore, equation (4) shows that when $n$ is even (odd), $c_n(0,0)$ is real (imaginary). These results mean that, to first order in $\epsilon$, the change can be visualized as a rotation in the complex plane. If we write

$$c_n(\epsilon, p_0) = A_n(\epsilon, p_0) \exp(i\theta_n(\epsilon, p_0)),$$

(10)

with $A_n(\epsilon, p_0)$ and $\theta_n(\epsilon, p_0)$ real functions, then we can express this as $\frac{\partial A_n}{\partial \epsilon} \big|_{\epsilon=0, p_0=0} = 0$. A similar evaluation of $\frac{\partial A_n}{\partial p_0} \big|_{\epsilon=0, p_0=0}$ yields the analogous result $\frac{\partial A_n}{\partial p_0} \big|_{\epsilon=0, p_0=0} = 0$. As such, expanding $A_n(\epsilon, p_0)$ and $\theta_n(\epsilon, p_0)$ to first order in $\epsilon$ and $p_0$ yields

$$c_n(\epsilon, p_0) \simeq J_N(N\phi_d) \exp \left( i \left( \frac{\partial \theta_n}{\partial \epsilon} \big|_{\epsilon=0, p_0=0} \epsilon + \frac{\partial \theta_n}{\partial p_0} \big|_{\epsilon=0, p_0=0} p_0 - \frac{n\pi}{2} \right) \right),$$

(11)

where we have substituted $c_n(0,0)$ from equation (4) and included its complex component $((-i)^n)$ in the argument of the exponential. Evidently, only the phases of the expansion coefficients $c_n$ show first order changes when $\epsilon$ and $p_0$ deviate from resonance.

This is illustrated in figure 2 where in (a), $|c_n|^2$ is plotted against $n$ for $^{85}$Rb atoms initially in $|0\rangle$ after 40 pulses for $\epsilon = 0$ and $1 \times 10^{-8}$ s, and an experimentally realizable value of the kicking strength, $\phi_d = 0.485$. In figure 2(b), the position space densities are plotted for the same two states. The position space densities differ considerably despite close resemblance of the momentum space probability densities, confirming that the main change in the wavefunction when $\epsilon$ deviates from 0 by a small amount is a phase change in $c_n$. Although the position space density could be investigated using optical mask techniques [26], most previous experiments on quantum resonances have investigated the momentum space probability density using time of flight experiments, and have therefore not been sensitive to phase changes in $c_n$ (see for example [5, 7, 8, 19, 27]).

3. Quantum resonance fidelity pulse sequence

In order to access the phase information contained in the wavefunction and thereby resolve the quantum resonance with greater precision, we now consider an alternative to the $N$-pulse $\delta$-kicked rotor pulse sequence: we add to the end of the sequence a pulse with kick strength $N\phi_d$.
Figure 2. (a) Momentum space probability density after a kicked rotor pulse sequence of $N = 40$, $\phi_d = 0.485$ and $\epsilon = 0$ (red ◦) and $\epsilon = 1 \times 10^{-8}$ s (blue x) (lines to guide the eye). The two distributions closely resemble each other. (b) Unnormalized position space densities of the same two wavefunctions (red dashed line: $\epsilon = 0$).

and with the spatial phase of the optical standing wave shifted by $\pi$, such that the 1D dynamics of the atoms is governed by the following Hamiltonian:

$$\hat{H}(t) = \frac{\hat{p}^2}{2M} + \hbar \phi_d \cos(2k\hat{x}) \sum_{n=0}^{N-1} \delta(t - nT) - \hbar N \phi_d \cos(2k\hat{x}) \delta(t - NT).$$  \hfill (12)

The phase shift of the final pulse can be realized using conventional techniques, where the phases of the two laser beams forming the periodic potential may be individually controlled, such as those described in [8, 28]. Instead of measuring the momentum distribution (from which one can deduce the kinetic energy imparted to the atoms), we propose to measure the proportion of atoms whose net momentum change after evolution by equation (12) is zero. In order to be able to resolve this, we require the initial momentum spread of the atoms to be smaller than the photon recoil momentum. Sub-recoil momentum distributions can be realized experimentally by using a Bose–Einstein condensate or by velocity selection of a sample of laser-cooled atoms [8, 15, 21].

The probability that an atom initially in a momentum eigenstate $|p_0\rangle$ will be found in this state after the sequence is given by

$$F(\epsilon, p_0) = |\langle p_0 | e^{(iN\phi_d \cos(2k\hat{x}))} (e^{-i\hat{p}^2/(2\hbar)}e^{i(T\phi_d + \epsilon)}) e^{(-i\phi_d \cos(2k\hat{x}))})N | p_0 \rangle|^2. \hfill (13)$$

At resonance, the wavefunction for atoms initially at rest is given by $\exp(-iN\phi_d \cos(2k\hat{x}))|p = 0\rangle$; therefore $F(\epsilon, p_0)$ measures the projection of the state after an $N$-pulse $\delta$-kicked rotor sequence on this state, i.e. it describes a fidelity measurement. As such, $F(\epsilon, p_0)$ will be sensitive to the phase deviations from the resonant value of $c_n$ that time-of-flight investigations have missed.
3.1. Approximate analytic description

Using equations (4) and (11), we can obtain an approximate expression for the fidelity that is valid in the vicinity of the resonance:

\[ F(\epsilon, p_0) \approx \left| \sum_n J_n^2 (N \Phi_d) \exp(-i\Theta_n) \right|^2 , \]

where

\[ \Theta_n = \left. \frac{\partial \theta_n}{\partial \epsilon} \right|_{\epsilon=0, p_0=0} \epsilon + \left. \frac{\partial \theta_n}{\partial p_0} \right|_{\epsilon=0, p_0=0} p_0 . \]

We note that

\[ \frac{\partial \theta_n}{\partial \epsilon} \bigg|_{\epsilon=0, p_0=0} = \frac{\partial \phi_n}{\partial \epsilon} \bigg|_{\epsilon=0, p_0=0} \frac{i \hbar}{M} , \]

so by equations (7) and (9) (and similar expressions for \( \frac{\partial \phi_n}{\partial p_0} \bigg|_{\epsilon=0, p_0=0} \)), \( \Theta_n \) can be evaluated. To find an alternative expression for \( \Theta_n \), we make use of the following expressions:

\[ b_{n,r} = (2\hbar k)^2 \phi_d^2 (N + 1 - r)^2 \langle n2\hbar k | e^{-i\phi_d N \cos(2k\hat{x})} \sin^2(2k\hat{x}) | 0 \rangle \]
\[ \times \langle n2\hbar k | e^{-i\phi_d N \cos(2k\hat{x})} \cos(2k\hat{x}) | 0 \rangle , \]

\[ \langle n2\hbar k | e^{-i\phi_d N \cos(2k\hat{x})} \cos(2k\hat{x}) | 0 \rangle = \frac{1}{2} \left[ (-i)^{n-1} J_{n-1} (\phi_d N) + (-i)^{n+1} J_{n+1} (\phi_d N) \right] , \]

\[ \langle n2\hbar k | e^{-i\phi_d N \cos(2k\hat{x})} \sin^2(2k\hat{x}) | 0 \rangle = \frac{1}{2} ((-i)^{n} J_{n} (\phi_d N) - (-i)^{n-2} J_{n-2} (\phi_d N) \]
\[ - \frac{1}{2} ((-i)^{n+2} J_{n+2} (\phi_d N) . \]

Using these expressions, equation (16) becomes

\[ \frac{\partial \theta_n}{\partial \epsilon} \bigg|_{\epsilon=0, p_0=0} = 4k^2 \hbar \frac{\epsilon \epsilon'}{2M} \left[ \frac{1}{6} \left( N - \frac{1}{N} \right) n - \frac{1}{6} \phi_d \left( N^2 - 1 \right) \frac{J_{n-1}(N \phi_d)}{J_n(N \phi_d)} - \left( \frac{1}{3} N + \frac{1}{2} + \frac{1}{6} N \right) n^2 \right] . \]

In figure 3 we show \( \Theta_n = \frac{\partial \theta_n}{\partial \epsilon} \bigg|_{\epsilon=0, p_0=0} \) obtained from equation (19) together with the phase difference relative to exact quantum resonance of the different momentum eigenstates (computed using equation (1)). A similar treatment of deviations from resonance in \( p_0 \) yields

\[ \frac{\partial \theta_n}{\partial p_0} \bigg|_{\epsilon=0, p_0=0} = \frac{\epsilon}{M} \left( N + 1 \right) n , \]

where we have omitted a constant term (i.e. independent of \( n \)) that will not affect the calculation of the fidelity in equation (14). Combining equations (14), (15), (19) and (20), allows for quick computation of the fidelity; that is, the probability that an atom has zero net momentum change after the pulse sequence.
Figure 3. Momentum state phase difference between exact resonance and $\epsilon = 5 \times 10^{-9}$ s for $^{85}$Rb atoms with $N = 40$, $\phi_d = 0.485$ and $p_0 = 0$. Green ×: full quantum calculation based on the Hamiltonian equation (1); blue +: $\frac{\partial \theta}{\partial \epsilon} |_{\epsilon=0, p_0=0} \epsilon$ evaluated using equation (19); solid red line: $\frac{\partial \theta}{\partial \epsilon} |_{\epsilon=0, p_0=0} \epsilon$ evaluated using equation (21).

3.2. Asymptotic behavior of the fidelity

To provide insight, we now present derivations of simple expressions for the behavior of the fidelity for asymptotic values of relevant parameters.

3.2.1. $p_0 = 0$. We first consider how the width of the fidelity peak in $\epsilon$ depends on the parameters $N$ and $\phi_d$ for the case $p_0 = 0$. To obtain an analytic result for the general behavior of the fidelity, we make two approximations: firstly, we keep only the dominant term in $\Theta_n$ from equation (19), and secondly we substitute the quantum momentum probability distribution with its classical counterpart.

We begin by finding the dominant term in $\Theta_n$. When $N\phi_d$ is large, the dominant terms in the sum in equation (14) are those for which $n \sim N\phi_d$. That is, the majority of atoms are diffracted into the wings of the momentum distribution, as can be seen in figure 4. Additionally, if $N$ is large then the dominant term in equation (19) will be

$$\frac{\partial \theta_n}{\partial \epsilon} |_{\epsilon=0, p_0=0} = -4k^2 \frac{\hbar}{2M} \frac{1}{3} Nn^2.$$ (21)

From equation (4), the quantum mechanical momentum space probability density for an atom initially at rest and exposed to a single, short (Raman–Nath) pulse of kick strength $N\phi_d$, is given by $P_Q(n2\hbar k) = J_n^2(N\phi_d)$. The classical analogue to this quantum mechanical momentum distribution is the one we would get from an ensemble of classical atoms that is initially at rest, distributed evenly over a period of the standing wave potential, and exposed to the same pulse (also neglecting motion during the pulse). This classical momentum space probability density
Figure 4. Quantum and classical momentum probability functions. The quantum case (blue crosses, lines to guide the eye) is given by $P_Q(n2\hbar k) = J_2^2(N\phi_d)$, and the classical equivalent is given by equation (22) (red dashed line). Parameter values: $N = 40$ and $\phi_d = 0.485$.

(with $n$ the momentum in units of $2\hbar k$) is given by

$$P_C(n) \, dn = \frac{1}{\pi N \phi_d} \sqrt{\frac{1}{1 - (n/N\phi_d)^2}} \, dn,$$

for $n \in (-N\phi_d, N\phi_d)$ and 0 elsewhere. Figure 4 shows $P_C$ and the corresponding quantum distribution $P_Q$. For $F(\epsilon, p_0 = 0)$ to deviate significantly from 0, $\Theta_n$ must vary slowly with $n$. The oscillatory behavior of $P_Q$ can therefore be ignored so that the quantum momentum space probability density in equation (14) can be replaced with its classical counterpart $P_C$. Additionally, the sum over discrete momentum states may be replaced by an integral, resulting in

$$F(\epsilon, p_0 = 0) \simeq \left| \frac{1}{\pi N \phi_d} \int_{-N\phi_d}^{N\phi_d} \sqrt{\frac{1}{1 - (n/N\phi_d)^2}} \exp\left(-i2k^2 \frac{\hbar}{M} \frac{1}{3} N\epsilon n^2\right) \, dn \right|^2,$$

which can be evaluated to give

$$F(\epsilon, p_0 = 0) \simeq J_0^2 \left( \frac{\hbar k^2}{3M} N^3 \phi_d^2 \epsilon \right).$$

Figure 5 shows equation (24) together with $F(\epsilon, p_0 = 0)$ obtained from a numerical evaluation of equation (13); we see that equation (24) accurately predicts the width of the fidelity peak for the parameters used. From figure 5, it is evident that the transferred energy about quantum resonance is much broader than the fidelity peak, thereby confirming that the first order deviations from the resonant wavefunction occur because of the phase change in $c_n$, rather than a change in magnitude.

Equation (24) shows that the width of the fidelity peak scales as $1/N^3\phi_d^2$, in contrast to the $1/N^2\phi_d$ scaling of quantum resonances predicted in [3]. When measuring the Talbot time using quantum resonances or the fidelity pulse sequence, the width of the peak determines
Figure 5. Red dashed line: energy of atoms initially in $|p = 0\rangle$ after a pulse sequence with $N = 40$ pulses and $\phi_d = 0.485$ normalized to the on-resonance energy, as a function of the time between pulses $T$ for $^{85}\text{Rb}$; blue solid line: numerical evaluation of equation (13). Blue dash-dot line: equation (24).

the precision of the measurement, such that a small width gives rise to a more precise measurement. The above scaling therefore indicates that both quantum resonances and the fidelity measurements show a sub-Fourier resonance peak, given that the precision of a Fourier limited measurement—for example Rabi spectroscopy [29]—would show a peak with width scaling as $1/N$, with $N$ being the interrogation time. However, the fidelity peak shows a more favorable scaling and a narrower resonance peak compared to the transferred energy about quantum resonance.

The reason for the sub-Fourier scaling in both cases is that the momentum of the atoms changes during the pulse sequence. More specifically, the standard kicked rotor part of the sequence (i.e. the pulse sequence shown in figure 1(a)), leads to a momentum spread $\Delta p$ that grows linearly in $N\phi_d$ at resonance [8]. In a measurement of the energy transferred about quantum resonance, the width of the peak therefore scales as $1/\Delta p \times 1/N$, where the second term derives from the standard Fourier scaling of a resonance with interrogation time. In the fidelity pulse sequence, the width of the peak scales as $1/(\Delta p)^2 \times 1/N$, indicating optimal sensitivity to the quadratic phase accumulation in the free-evolution Hamiltonian $\hat{p}^2/2M$. This is analogous to the interferometers in [20, 21, 30] and is achieved through the final pulse which ensures quantum interference between all momentum states populated by the kicked rotor part of the sequence.

3.2.2. $\epsilon = 0$. We will now analyze the case where the time between pulses is exactly the Talbot time but let the initial momentum deviate from $p = 0$. It should be noted that the approach found in [13] can be extended to treat the fidelity pulse sequence as well.

Inserting equation (20) into equation (14) and evaluating using Graf’s identity [28] yields

$$F (\epsilon = 0, p_0) = J_0 \left( \phi_d N \sqrt{2 + 2 \cos \left( \frac{T_T}{M} (N + 1) p_0 + \pi \right)} \right).$$

(25)
Taking only the lowest order in $p_0$ of the argument of the Bessel function gives

$$F (\epsilon = 0, p_0) = J^2_0 \left( k \frac{T_T}{M} \phi_d N (N + 1) p_0 \right).$$

Equation (26) shows that the width of the fidelity peak in $p_0$ scales as $1/\phi_d N(N + 1)$, which again gives a more velocity sensitive sequence than the quantum resonances where the width of the resonances scale as $1/N$ [3, 8]. Our pulse sequence can therefore—like the one proposed in [22]—be used as a narrow velocity filter.

### 3.3. Finite momentum spread

In the previous sections we considered the initial state of the atoms to be a momentum eigenstate. In an experiment this will never be exact due to the finite size of the atomic sample, or its finite temperature. We therefore test the outcome of the fidelity pulse sequence when the initial state is a Gaussian wavepacket with standard deviation 100 $\mu$m. Figure 6 shows a numerical calculation of the fidelity peak based on equation (12) together with the prediction of equations (14), (15), (19) and (20). We see that our approximate analytical model captures the behavior of the Gaussian wavepacket as successfully as the plane waves previously investigated. In contrast to the case of plane waves, the peak height is no longer one at resonance for the parameters used. This is due to the fact that the momentum spread of the initial Gaussian wavepacket is larger than the momentum width of the resonance. The width of the fidelity peak for Gaussian wavepackets is also shown in figure 7 as a function of $N$. Here we also observe deviations from the single momentum calculation. First, we note deviations from the $1/N^3$ scaling for ‘large’ numbers of pulses ($N > 20$), where the narrowing of the peak slows down. Next, we observe that for small and intermediate numbers of pulses the half-width of the fidelity peak is slightly smaller than for the momentum eigenstate calculation. This can be understood from equation (15), since a nonzero initial momentum component will have nonzero $\Theta_n$, even

![Figure 6](http://www.njp.org/)
when $\epsilon = 0$. For low $N$, this will not cause a significant drop in the peak value of the fidelity (see equation (26)). However, given that the contribution to $\Theta_n$ from nonzero $p_0$ is added to the contribution from nonzero $\epsilon$, it will take a smaller $\epsilon$ to cause the fidelity of this component to drop, thereby leading to a narrower peak.

4. Summary

In the vicinity of the quantum resonance in the atom-optics $\delta$-kicked rotor, small deviations from resonant conditions cause significant relative phase changes between the different momentum states taking part in the dynamics, but negligible change in the momentum space probability density. By adding a strong, phase-inverted pulse to the kicked rotor pulse sequence, one can measure the overlap between the resonant state and any other state and thereby be sensitive to the relative phase changes of the different momentum states as well as changes in the momentum space probability density. We have derived approximate analytic predictions for the outcome of the proposed pulse sequence. For the case of a plane wave as the initial state, we found a narrow peak around quantum resonance with a width that scales as $1/N^3$ where $N$ is the number of pulses in the kicked rotor sequence. We subsequently considered a Gaussian wavepacket as our initial state and found that for a low number of pulses, the Gaussian wavepacket gave the narrowest fidelity peak whereas the opposite was true for large $N$. This narrow peak provides a method to measure the Talbot time that may be used in a measurement of the fine structure constant.

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