Dually affine Information Geometry modeled on a Banach space

Goffredo Chirco and Giovanni Pistone

Abstract In this chapter, we study Information Geometry from a particular non-parametric or functional point of view. The basic model is a probabilities subset usually specified by regularity conditions. For example, probability measures mutually absolutely continuous or probability densities with a given degree of smoothness. We construct a manifold structure by giving an atlas of charts as mappings from probabilities to a Banach space. The charts we use are quite peculiar in that we consider only instances where the transition mappings are affine. We chose a particular expression of the tangent and cotangent bundles in this affine setting.

Overview

This chapter consists of two parts. In the first part, Sec. 1 and 2 we first present a version of the affine geometry of statistical models based on Weyl’s axioms and notions from Inferential Statistics. Then, we present the fundamental structures of differential geometry in terms of velocity, natural gradient, parallel transport, acceleration. In the second part, Sec. 3 and 4 we present specific cases of model Banach spaces. Finally, we discuss the case of Orlicz spaces of exponential growth in detail.

Our approach covers only the dually flat affine structure of non-parametric Information Geometry. Other chapters treat approaches based on a Riemannian structure or Fréchet spaces as coordinate’s spaces.

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The use of differential geometry in the study of statistical models was first devised by C.R. Rao [42] who recognized that the Fisher Information matrix [25] defines a Riemannian manifold on the model parameters. It was later recognized by B. Efron [21, 22] that the affine geometry of the exponential families provides a more interesting geometric setup that connects with such fundamental topics as entropy, information, and Gibbs-Boltzmann distribution. It was the work of Čenkov [52], and Amari [6], to reconnect the two branches in the topic we call Information Geometry (IG). Pistone and Sempi [41] first presented the nonparametric version we describe here. A further enlargement of the scope of IG to include geometries that depend on the geometry of the sample space is the object of current research and not treated here.

1 Nonparametric affine manifolds

In this chapter, we present a non-parametric version of the Amari-Nagaoka theory of Information Geometry (IG), see [7]. Due to the specific issues of the non-parametric setup, we adopt Bourbaki’s definition of a differentiable manifold. Before starting, we want to make a general caveat. The geometric structure of interest in IG is much more specific than the highly general one used in most Differential Geometry textbooks. First, it is an affine geometry; second, it applies to spaces of probability measures. For this reason, several geometric objects appear in a specific expression. We always use a specially given set of charts and always give a presentation compatible with the specific statistical intuition of IG.

A chart on a set $\mathcal{M}$ is a triple $(s, U, B)$, where $s$ is a 1-to-1 mapping from its domain $U \subset \mathcal{M}$ to an open subset $S(U)$ of a topological vector space $B$. A topological vector space is a vector space endowed with a topology such that all the vector space operations are continuous. We will briefly talk about top-linear mappings to mean continuous linear mappings between topological linear spaces. Normed vector spaces and Banach spaces are instances of topological vector spaces. In most of our applications, the topological vector space of interest will be a Banach space of random variables.

The manifold structure depends on the given atlas of charts. We repeat below the basic definitions to stress that we do not assume the coordinate spaces to be finite-dimensional nor require it to be the same for each chart.

**Definition 1 ($C^k$-atlas)** Let $\mathcal{M}$ be a given set and let $B_\alpha$, $\alpha \in A$, be a family of Banach spaces. For each $\alpha \in A$, $(s_\alpha, U_\alpha, B_\alpha)$ is a chart on $\mathcal{M}$, that is, $s_\alpha: U_\alpha \to B_\alpha$ is 1-to-1 from $U_\alpha \subset \mathcal{M}$ to the open set $s_\alpha(U_\alpha)$. We assume that each couple of charts, say $(s_\alpha, U_\alpha, B_\alpha)$ and $(s_\beta, U_\beta, B_\beta)$, either have disjoint domains, $U_\alpha \cap U_\beta = \emptyset$, or the transition mapping (or, overlap mapping)
is a 1-to-1 $C^k$ mapping between open sets. If such condition holds, the spaces $B_\alpha$ and $B_\beta$ are isomorphic as topological vector spaces and the set of all charts is a $C^k$-atlas $\mathcal{A}$.

Let us stress that there is a diffeomorphism between open sets of the corresponding spaces of coordinates for each couple of overlapping domains, say $B_1$ and $B_2$. The implicit function theorem implies the existence of 1-to-1 continuous linear mapping between $B_1$ and $B_2$. Hence the assumption about the existence of a top-linear isomorphism is unavoidable. This remark is critical in the infinite-dimensional case, and checking it will be a constant concern for us. Moreover, we will see that the specification of a convenient family of such isomorphisms is a way to specify an affine structure.

**Definition 2 (C^k-manifold)** Two atlases are equivalent if their union is again an atlas. A class of equivalent atlases is a manifold.

The two well known examples below present instances of manifold in this sense and are of special interest in IG.

**Example 1 (The stereo-graphic projection of the unit sphere of L^2(μ))** In the probability space $(\Omega, \mathcal{F}, \mu)$, the unit sphere of $L^2(\mu)$ with norm $\|\rho\|_2^2 = \int |\rho|^2 \, d\mu$, is $M = \{\rho \in L^2(\mu) \mid \|\rho\|_\mu = 1\}$. For each $\alpha \in M$, define a chart's domain to be $U_\alpha = M \setminus \{-\alpha\}$ and the coordinates' space to be the Hilbert space of random variables which are orthogonal to $\alpha$, $B_\alpha = \{u \in L^2(\mu) \mid \langle u, \alpha \rangle_\mu = 0\}$. The $\alpha$-chart is the stereo-graphic projection from $-\alpha$ to the hyper-plane $B_\alpha + \alpha$, that is, the vector $(s_\alpha(\beta) + 2\alpha)$ is aligned with the vector $\beta + \alpha$. If $u = s_\alpha(\beta)$, then $u + 2\alpha = \theta(\beta + \alpha)$, were $2 = \theta(\beta, \alpha) + 1$ and $\theta = 1 + \|u/2\|^2$. It follows

$$s_\alpha : \beta \mapsto \frac{2}{1 + \langle \alpha, \beta \rangle} (\beta - \langle \alpha, \beta \rangle \alpha)$$

$$s^{-1}_\alpha : u \mapsto \frac{1}{1 + \|u/2\|^2} (u + (1 - \|u/2\|^2) \alpha)$$

One can check that $s_\alpha$, $\alpha \in M$, is an atlas of the sphere considered as a submanifold of $L^2(\mu)$.

The previous example is related with one of the possible construction of the manifold structure on probability densities. In fact, the sphere is mapped onto the set of probability densities by the square function.

**Example 2 (Square root embedding)** we can map the unit sphere $M$ into the affine subspace $L^1_1(\mu) = \{p \in L^1(\mu) \mid \int p \, d\mu = 1\}$ by $p \mapsto p^2$, so that the image is the set $P$ of probability densities. Such a mapping is not 1-to-1, but nevertheless it has

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$^3$ See, for example, the pictures in [2, ?]. Both example are related to the $\alpha$-embedding as introduced in [7].
been used to define a geometry on $P$. If $t \mapsto \rho(t)$ is a curve in the set of probability densities, then \( \rho(t) = \sqrt[2]{p(t)} \) is a uniquely defined curve in $M$. We could define $T_\rho(q) = s_{\sqrt[2]{p}}(\sqrt[2]{q}) \in B_{\sqrt[2]{p}}$ and $T_\rho^{-1}(u) = S_{\sqrt[2]{p}}^{-1}(u)$ to get from Eq.s (2) and (3) the mappings

\[
T_\rho : q \mapsto \frac{2}{1 + \int \sqrt{pq} \, d\mu} \left( \sqrt[2]{q} - \int \sqrt{pq} \, d\mu \sqrt[2]{p} \right)
\]

\[
T_\rho^{-1} : u \mapsto \left( \frac{2}{1 + \|u/2\|^2} \left( u - \left(1 - \|u/2\|^2\right) \sqrt[2]{p} \right) \right)^2
\]

The use of $B_{\sqrt[2]{p}} = \{ u \in L^2(\mu) \mid \int u \sqrt[2]{p} \, d\mu = 0 \}$ might seem unnatural. In fact, if $p > 0$, the $u \mapsto p^{-1/2}u$ maps $B_{\sqrt[2]{p}}$ to $\{ v \in L^2(\sqrt[2]{p} \cdot \mu) \mid \int vp \, d\mu = 0 \}$, with the chart

\[
p^{1/2}s_{\sqrt[2]{p}}(\sqrt[2]{q}) = \frac{2}{1 + \int \sqrt{pq} \, d\mu} \left( \sqrt[2]{q} - \int \sqrt{pq} \, p \, d\mu \right)
\]

We do not further discuss this topic here, see the relevant chapter in this Handbook.

1.1 **Affine space**

The manifolds of interest here fall in the particular class of *affine manifolds*. We begin with a general definition and will turn to the specific case of statistical manifolds later on. If the reader is interested in comparing such a general approach to our main applications, she may refer to the example in Sec. 2.1.

The word "affine" above refers to the geometrical construction of vectors associated with displacement according to classical H. Weyl's axioms of an affine space.

Let be given a set $\mathcal{X}$ and a real finite-dimensional vector space $\mathcal{V}$. A *displacement* mapping is a mapping

\[
M \times M \ni (P, Q) \mapsto \overrightarrow{PQ} \in \mathcal{V},
\]

such that

1. for each fixed $P$ the partial mapping $s_P : Q \mapsto \overrightarrow{PQ}$ is 1-to-1 and onto, and
2. the parallelogram law or Chasles rule, $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$, holds true.

The notation $Q = P + \overrightarrow{PQ}$ is also useful to show that the vector space $\mathcal{V}$ acts on the set $M$. From the parallelogram law follows that $\overrightarrow{PP} = 0$ and $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$. The structure $(M, \mathcal{V}, \overrightarrow{)}$ is called *affine space*. The corresponding affine manifold is

\[\text{\cite{51} \S 1.2}. The L. Schwartz' textbook \cite{46} \S III.1] has the same presentation in a slightly different language."
Affine statistical bundles

derived from the atlas of charts \( s_P : M \to V, P \in M \). Notice that the change-of-chart is the choice of a new origin.

Because of our non-parametric perspective, we re-define the object to fit our Def. of a manifold. That is, we consider a generalization of Weyl’s axioms that allows for a family of (possibly infinite-dimensional) top linear spaces instead of a single finite-dimensional vector space.

**Definition 3 (Affine space)** Let \( M \) be a set and let \( B_\mu, \mu \in M \), be a family of toplinear spaces. Let \( (U^v_\nu), v, \mu \in M \) be a family of toplinear isomorphism \( U^v_\nu : B_v \to B_\mu \) satisfying the cocycle condition:

\[ \text{AF0} \quad U^v_\mu = I \text{ and } U^v_\nu U^\nu_\mu = U^v_\mu. \]

\( U^v_\mu \) is the parallel transport from \( B_v \) onto \( B_\mu \). Consider a displacement mapping

\[ \mathbb{S} : (v, \mu) \mapsto s_v(\mu) \in B_v \quad (8) \]

defined on a subset of the product space \( \text{dom}(\mathbb{S}) \subset M \times M \). We assume

\[ \text{AF1} \quad \text{For each fixed } v \text{ the mapping } M_v \ni \mu \mapsto s_v(\mu) = \mathbb{S}(v, \mu) \text{ is injective.} \]

\[ \text{AF2} \quad \mathbb{S}(\mu_1, \mu_2) + U^v_{\mu_2} \mathbb{S}(\mu_2, \mu_3) = \mathbb{S}(\mu_1, \mu_3). \]

We will say that the structure \((M, (B_\mu)_{\mu \in M}, (U^v_\nu)_{\mu, v \in M}, \mathbb{S})\) is an affine space.

When the vector space is constant \( B_\mu = B \) and the parallel transport maps are the identity, we recover Weyl’s definition. Note that in our definition the \( s_\mu \) map is not required to be defined on all of \( M \) and to be surjective.

Def. AF2 with \( \mu_1 = \mu_3 = v \) and \( \mu_2 = \mu \) becomes

\[ \mathbb{S}(v, \mu) + U^v_\mu \mathbb{S}(\mu, v) = 0 . \quad (9) \]

Let us compute, where defined, the change-of-origin map \( s_\mu \circ s_v^{-1} \) in an affine space. At \( \rho = s_v^{-1}(w), w \in B_v \), it holds

\[ s_\mu \circ s_v^{-1}(w) = s_\mu(\rho) = s_\mu(v) + U^v_\mu s_\nu(\rho) = s_\mu(v) + U^v_\mu w . \quad (10) \]

The change-of-origin map extends to a toplinear isomorphism which is an affine map.

### 1.2 Affine manifold

An affine space provides a family of candidates to charts \( s_v : M_v \to B_v, v \in M \), that we could use as an atlas. We want to add a smoothness condition.

\[ ^5 \text{Such a structure supports a full geometrical development, see the Nomizu and Sazaki monograph [33].} \]

\[ ^6 \text{Compare [30 p. 42].} \]
Definition 4 (Affine manifold)

Let \((M, (B_\mu)_{\mu \in M}, (\cup^c V)_{\nu, \nu \in M}, S)\) be an affine space as in Def. 3.

AF3 We assume that for each \(\nu\), the image set \(s_\nu(M_\nu)\) is a neighborhood of 0 in \(B_\mu\).

That is, its interior \(s_\nu(M_\nu)^o\) is an open set containing \(s_\nu(0) = 0\). Define the coordinates domains as \(U_\nu = s^{-1}_\nu (s_\nu(M)^o)\), so that \((s_\nu, U_\nu, B_\nu)\) is a chart on \(M\). Such a chart is said to have origin \(\nu\). It follows, that such charts are compatible and the resulting manifold is by definition the affine manifold associated to the given affine space.

Proof Clearly, all assumptions of Def. 1 hold true, but the fact that the domains in eq. (1) are both open. For each couple \(\mu, \nu \in M\), we have defined \(U_\mu = s^{-1}_\mu (s_\mu(M)^o)\) and \(U_\nu = s^{-1}_\nu (s_\nu(M)^o)\). Use the change-of-origin equation (10) to see that

\[
s_\nu(U_\nu \cap U_\mu) = s_\nu(M)^o \cap s_\nu \circ s^{-1}_\mu (s_\mu(M)^o) = s_\nu(M)^o \cap (s_\nu(\mu) + U_\nu \circ s_\mu(M)^o) \quad (11)
\]

is open. \(\square\)

Given the affine atlas, we can locally express the displacement mapping \(S\) with respect of any origin \(\sigma \in M\). From the parallelogram law for the points \(\sigma, \nu, \mu\), we write

\[
s_\sigma(\nu) + U_\nu s(\nu, \mu) = s_\sigma(\mu). \quad (12)
\]

Because of the cocycle property Def. 3(AF0), we can write

\[
S(\nu, \mu) = U_\nu s_\nu (s_\mu(\mu) - s_\sigma(\nu)) \quad , \quad (13)
\]

which in turn implies the expression in the chart at \(\sigma\) of the displacement is given by

\[
S_\sigma(u, v) = S(S^{-1}_\sigma(u), S^{-1}_\sigma(v)) = U_\nu \circ s_\sigma(u, v) - u. \quad (14)
\]

The expression \(S_\sigma\) is affine in the second variable, hence the derivative\(^7\)

\[
\frac{d}{dt} S_\sigma(u, v + tk) \bigg|_{t=0} = D_2 S_\sigma(u, v)[k] = U_\nu S^{-1}_\sigma(u) k. \quad (15)
\]

Using (15) and (14), we find that the expression of the displacement solves a differential equation, namely

\[
S_\sigma(u, v) = D_2 S_\sigma(u, v)[v - u]. \quad (16)
\]

The previous equation provides a differential equation for the derivative in the first variable. In fact, the first partial derivative in the direction \(h\) is

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\(^7\) We use the notation \(D_f(u)[k]\) to denote the derivative of the function \(f\) in the direction \(k\). Alternative notations are \(D_0 f(u), D f(u) \cdot k, f'(u) k\). The notation with square brackets is used, for example, in [3].
If Exponential: With the same space strictly positive continuous probability densities) Let $\Omega$ be a compact metric space and let $\mathcal{B}$ be its Borel $\sigma$-algebra. We are given a reference finite measure $m$ on $(\Omega, \mathcal{B})$, $C(\Omega)$ is the Banach space of continuous functions on $\Omega$ with the uniform norm $\|f\|_\infty = \sup_{x \in \Omega} |f(x)|$. Let $M$ be the set of strictly positive continuous probability density $p$, that is, positive functions in $C(\Omega)$ such that $\int f \, dm = 1$. It is easy to see that $M$ an open convex subset of the affine space $B = \{ f \in C(\Omega) \mid \int f \, dm = 1 \}$. We note that the previous set up applies, in particular, to the case where $\Omega$ is finite and $C(\Omega)$ is the space of all real vectors with indeces in $\Omega$.

There are many interesting ways to give to the set $M$ an affine geometry. We introduce here 3 cases to be discussed in detail in the following.

Flat: As $M$ is an open convex subset of an affine space $B$, it inherits the affine geometry of the larger space and the displacement vector is simply $S(p, q) = q - p \in B_1$, where $B_1$ is the vector space parallel to $B$, $B_1 = \{ f \in C(\Omega) \mid \int f \, dm = 0 \}$. That is $s_p(q) = q - p$ and $s_p^U(u) = u + p$ for all $u \in B_1$ such that $u + p > 0$.

Mixture: If $B_p = \{ f \in C(\Omega) \mid \int f \, p \cdot dm = 0 \}$ and $m^U_p u = \frac{d}{d} u$, then we can define $s_p(q) = \frac{d}{d} - 1$ for all $q \in M$. The parallelogram law as follows from

$$
\left( \frac{q}{p} - 1 \right) + \frac{P}{Q} \left( \frac{q}{r} - 1 \right) = \frac{r}{p} - 1 .
$$

(18)

The inverse mapping is $S_p^{-1}(u) = (1 + u) \cdot p$ and it is defined on the open set $B_p = \{ u \in B_p \mid u > -1 \}$. The expression of the displacement is $S_p(u, v) = (1+u)^{-1}(v-u)$ and the partial derivatives are $D_1 S_p(u, v)[h] = -(1+u)^{-2}(1+v)h$ and $D_2 S_p(u, v)[k] = (1+u)^{-1}k$.

Exponential: With the same $B_p$ as above, we define $e^U_p u = u - \int u \cdot q \cdot dm$ and define $s_p(q) = \log \left( \frac{d}{d} \right) - \int \log \left( \frac{d}{d} \right) p \cdot dm$. The parallelogram law follows from

$$
\left( \log \frac{q}{p} - \int \log \frac{q}{p} \cdot dm \right) + \\
\left( \log \frac{r}{q} - \int \log \frac{r}{q} \cdot dm - \int \left( \log \frac{r}{q} - \int \log \frac{r}{q} \cdot dm \right) \cdot dm \right) = \\
\left( \log \frac{r}{p} - \int \log \frac{r}{p} \cdot dm \right)
$$

(19)

The inverse of the chart is easily seen to be

$$
s_p(u) = \exp \left( u - K_p(u) \right) \cdot p , \quad K_p(u) = \log \int e^u \cdot p \, dm , \quad u \in B_p .
$$

(20)

The expression of the displacement is
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\[ S_r(u, v) = (v - u) - \int (v - u)e^{u-K_r(u)} r \ dm . \quad (21) \]

1.3 Affine bundle

The specific form of the atlas defining the affine manifold allows the extension of the same atlas to define an affine bundle.

**Definition 5 (Affine bundle)** Given the affine manifold \( M \) of def. 4 consider the linear bundle

\[ SM = \{ (\mu, v) | \mu \in M, v \in B_\mu \} . \quad (22) \]

The equation

\[ SM \times SM \ni ((v, u), (\mu, v)) \mapsto (s_v(\mu), U^v_\mu v) \in B_v \times B_v \quad (23) \]

defines a displacement on the bundle. For each \( v \) define the chart

\[ s_v : SM \ni (\mu, v) \mapsto (s_v(\mu), U^v_\mu v) \in B_v \times B_v \quad (24) \]

to define the affine bundle \( SM \) as a manifold. Equivalently, we can say that \( SM \) is a linear bundle with trivialization

\[ s_v : (\mu, v) \mapsto (s_v(\mu), U^v_\mu v) . \quad (25) \]

The affine bundle is a convenient expression of the tangent bundle of the affine manifold if we define the velocity as follows.

**Definition 6 (Velocity)** The velocity of the smooth curve \( t \mapsto \gamma(t) \) of the affine manifold is the curve \( t \mapsto (\gamma(t), \dot{\gamma}(t)) \) of the affine bundle whose second component is

\[ \dot{\gamma}(t) = \lim_{h \to 0} h^{-1}(s_{\gamma(t)}(\gamma(t + h))) = \frac{d}{dt} s_{\gamma(t)}(\gamma(t + h)) \bigg|_{h=0} . \quad (26) \]

By (24) and Def. 3(AF2) applied to the points, the expression in the chart centered at \( \gamma(t) \) is

\[ U^{\gamma(t)}_{\gamma(t)} \dot{\gamma}(t) = \lim_{h \to 0} h^{-1}U^{\gamma(t)}_{\gamma(t)} s_{\gamma(t)}(\gamma(t + h)) = \lim_{h \to 0} h^{-1} (s_v(\gamma(t + h)) - s_v(\gamma(t))) = \frac{d}{dt} s_v(\gamma(t)) , \quad (27) \]

and, conversely,

\[ \dot{\gamma}(t) = U^{\gamma(t)}_{\gamma(t)} \frac{d}{dt} s_v(\gamma(t)) . \quad (28) \]

**Definition 7 (Integral curve and flow of a section)** Let \( F \) be a section of the affine bundle, that is, \((\mu, F(\mu)) \in SM \). An integral curve of the section \( F \) is a curve
Especially, for all $u_i \in \mathbb{R}$

**Definition 8 (Auto-parallel curve)** A curve $I: t \mapsto \gamma(t)$ is auto-parallel in the affine bundle if

$$\hat{\gamma}(t) = \mathbb{U}_{\gamma(s)}^\gamma(t) \gamma(s) \quad s, t \in I .$$

**Proposition 1** The following conditions are equivalent.

1. The curve $\gamma$ is autoparallel.
2. The expression of the curve in each chart is affine.
3. For all $s, t$

$$\gamma(t) = S_{\gamma(s)}^{-1}((t-s)\gamma(s)) .$$

**Proof** From eq. (30) we see that the velocity is constant in each chart, $\mathbb{U}_{\gamma(t)}^\gamma(t) \gamma(t) = \mathbb{U}_{\gamma(s)}^\gamma(t) \gamma(s)$, and, by (27), we have that $\frac{d}{d\xi} s_{\gamma}(\gamma(t)) = \frac{d}{d\xi} s_{\gamma}(\gamma(s))$. Hence $t \mapsto s_{\gamma}(t)$ is an affine curve, $s_{\gamma}(\gamma(t)) - s_{\gamma}(\gamma(s)) = (t-s) \frac{d}{d\xi} s_{\gamma}(\gamma(s)) = (t-s) \mathbb{U}_{\gamma(s)}^\gamma(t) \gamma(s)$. Now put $v = \gamma(t)$ to get $s_{\gamma(s)}(\gamma(t)) = (t-s) \gamma(s)$ hence (31). And conversely. □

**Proposition 2** The affine bundle is an affine manifold for the displacement mapping

$$((\nu, v), (\mu, w)) \mapsto (\mathbb{S}(\mu, v), \mathbb{U}_\mu w - v) \in B_\nu \times B_\nu,$$

and the transports $\mathbb{U}_\mu^\nu \times \mathbb{U}_\nu^\mu$.

**Proof** Check all the properties AF0–3. □

**Definition 9 (Acceleration)** Consider the curve $t \mapsto \mu(t)$ with velocity $t \mapsto \dot{\mu}(t)$. The acceleration $t \mapsto \ddot{\mu}(t)$ is defined as the velocity $t \mapsto (\mu(t), \ddot{\mu}(t))$.

$$\ddot{\mu}(t) = \lim_{h \to 0} h^{-1} \mathbb{U}_{\mu(t), \ddot{\mu}(t)}^\mu (\mu(t+h), \dot{\mu}(t+h)) .$$

Especially, for all $\mu \in M$,

$$\ddot{\mu}(t) = \mathbb{U}_\mu^\mu \frac{d}{dt} \mu(t) .$$

From this equation, it follows that

**Proposition 3** A curve with 0 acceleration is auto-parallel.
Example 4 (Running example: follows from Example 3) If \( t \mapsto p(t) \) is a curve, let us compute the velocity by implicitly assuming enough smoothness to justify all the computations. The velocity in the mixture manifold is

\[
\dot{p}(t) = \lim_{h \to 0} h^{-1} \frac{1}{p(t)} \int d\mu^{(t)}_{p(t+h)} \mathcal{S}(p(t), p(t+h)) = \lim_{h \to 0} h^{-1} \left( \frac{p(t)}{p(t+h)} \left( \frac{p(t+h)}{p(t)} - 1 \right) \right) = \frac{\dot{p}(t)}{p(t)}. \tag{35}
\]

The velocity in the exponential manifold is

\[
\dot{p}(t) = \lim_{h \to 0} h^{-1} \frac{1}{p(t)} \int d\mu^{(t)}_{p(t+h)} \mathcal{S}(p(t), p(t+h)) = \lim_{h \to 0} h^{-1} \left( \log \frac{p(t+h)}{p(t)} - \int \log \frac{p(t+h)}{p(t)} \ dm(p(t)) \right) = \frac{\dot{p}(t)}{p(t)}. \tag{36}
\]

It is remarkable that the expression of the velocity is the same in both cases. In the statistical literature the quantity \( \dot{p}(t) \) is called the Fisher’s score of the 1-dimensional model \( p(t) \). The exponential velocity for a curve of the form of a Gibbs model \( p(t) \propto e^{a(t)U} \cdot p \), that is \( p(t) = e^{a(t)U} \cdot \psi(t) \cdot p \), is

\[
\dot{p}(t) = \frac{d}{dt} (a(t)U - \psi(t)) = \dot{a}(t)U - \dot{\psi}(t) = \dot{a}(t) \left( U - \int U \cdot p(t) \ dm \right). \tag{37}
\]

In this case, the quantity \( \dot{p}(t) \) is seen as \( \dot{a}(t) \) times the fluctuation \( U - \int U \cdot p(t) \ dm \).

Let us compute the acceleration in both cases. In the mixture case,

\[
\ddot{p}(t) = \frac{d}{dt} \frac{\dot{p}(t)}{p(t)} = \frac{d}{dt} \left( \frac{\dot{p}(t)}{p(t)} \right) = \frac{\dot{p}(t) \cdot \dot{p}(t)}{p(t)} = \frac{\ddot{p}(t)}{p(t)}. \tag{38}
\]

In the exponential case,

\[
\ddot{p}(t) = \frac{d}{dt} \frac{\dot{p}(t)}{p(t)} = \frac{d}{dt} \left( \frac{\dot{p}(t)}{p(t)} \right) = \frac{\dot{p}(t) \cdot \dot{p}(t)}{p(t)} = \frac{\ddot{p}(t)}{p(t)} \tag{39}
\]

For the Gibbs model above, the exponential acceleration is proportional to the velocity, namely

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8 This notion is due to R. Fisher. See, for example, the textbook [23].
9 See any textbook of Statistical Physics, for example [29].
The auto-parallel curves in the mixture geometry are of the form \( \gamma(t) = \gamma(0) + \dot{\gamma}(0)t = (1 + \mathbf{\gamma}(0))\gamma(0) = (1 - t)\gamma(0) + t\gamma(1) \). The last expression explains the name. In the exponential geometry, the form of the auto-parallel curve is derived from Eq. (31): \( \gamma(t) = S_{\gamma(0)}^{-1}(t\gamma(0)) = e^{\mathbf{\gamma}(0) - \mathbf{K}_{\gamma(0)}(\mathbf{\gamma}(0))} \cdot \gamma(0) \), that is, it is an exponential family. The auto-parallel interval is \( \gamma(t) \propto \gamma(0)^{1 - t} \gamma(1)^t \).

**Definition 10 (Duality)** Let be given two affine manifolds on the same base set \( M \), \( \mathcal{M}_i = (M, (B^i_\mu)_{\mu \in M}, (\langle{}^{(i)}U^\mu_\nu^v \rangle)_{\mu, v \in M, i}, i = 1, 2 \), and let be given for each \( \mu \in M \) a duality pairing

\[
B^1_\mu \times B^2_\mu \ni (u_1, u_2) \mapsto \langle u_1, u_2 \rangle_\mu .
\]

(41)

The affine manifolds \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are in duality if for all \( \mu, \nu \in M, u \in B^1_\mu, v \in B^2_\nu \), it holds

\[
\langle u, {}^{\langle}1U^\mu_\nu^v \rangle \rangle_\mu = \langle {}^{\langle}1U^\nu_\mu^v, v \rangle \rangle_\nu .
\]

(42)

**Example 5 (Duality. Follows from Examples 3 and 4)** In the present case, the mixture and the exponential fibers are equal, \( mB_p = eB_p = B_p \), and there is a separating pairing \( \langle u, v \rangle_p = \int uv \, dm \). The mixture affine manifold and the exponential affine manifold are actually dual. Let us check this. For \( u \in B_p \) and \( v \in B_q \)

\[
\langle m^{\langle}U^p_q u, v \rangle \rangle_q = \int \frac{p}{q} uv q \, dm = \int uv \, p \, dm = \int u (v - \int vp \, dm) \, p \, dm = \langle \gamma, e^{\langle}U^p_q v \rangle \rangle_p .
\]

(43)

**Definition 11 (Gradient)** Consider a \( M \) which is base of two dual affine manifolds \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \). A real function \( \phi \) on \( \mathcal{M}_1 \) has a gradient \( \nabla \phi \) if \( \nabla \phi \) is a section of the affine bundle \( S\mathcal{M}_2 \) and for all smooth curve \( t \mapsto \gamma(t) \in M \) it holds

\[
\frac{d}{dt} \phi(\gamma(t)) = \left( \nabla \phi(\gamma(t)), \dot{\gamma}(t) \right)_{\gamma(t)} .
\]

(44)

The gradient as defined above is related, but does not coincide, with the *natural gradient* of S.I. Amari. Let us express the gradient \( \nabla \phi \) in a chart with origin \( \sigma \) with the ordinary gradient \( \nabla_{\sigma} \). In the 1-chart, it holds \( \phi \circ \gamma(t) = \left( \phi \circ 1S_{\sigma}^{-1} \right) \circ (1s_\sigma \circ \gamma(t)) = \phi_{\sigma} \circ (1s_\sigma \circ \gamma(t)) \), where \( \phi_{\sigma}: U_{\sigma} \to \mathbb{R} \) is the expression of \( \phi \), so that

\[
\frac{d}{dt} \phi \circ \gamma(t) = d\phi_{\sigma} \left[ \frac{d}{dt} (1s_\sigma \circ \gamma(t)) \right] = d\phi_{\sigma} \left[ {}^{\langle}1U^{\gamma(t)}_{\gamma(t)} \dot{\gamma}(t) \right] = \left( \nabla_{\sigma} \phi_{\sigma}, {}^{\langle}1U^{\gamma(t)}_{\gamma(t)} \dot{\gamma}(t) \right)_{\gamma(t)} ,
\]

(45)
where we have used (45) and \( \nabla_{\phi} \) denotes the gradient computed in the duality of \( {\mathcal{B}}_{\phi} \) with \( \partial_{\phi} \). In conclusion, the gradient of \( \phi: M \) equals the gradient of \( \phi_{\phi}: {\mathcal{B}}_{\phi}, \)

\[
\nabla \phi (\mu) = \nabla_{\phi} \phi_{\phi} (\mu) = \nabla_{\mu} \phi_{\phi}(\mu). \tag{46}
\]

Example 6 (Gradient of the entropy. Follows from Examples 3, 4, and 5) The entropy is \( \mathcal{H}(q) = -\int q \log q \, dm \). The expression of the entropy of \( q \) at \( p \) is \( \mathcal{H}_p(v) = -\int (1 + v)p \log((1 + v)p) \, dm, \, v \in B_p \). Let \( \iota \mapsto v(t) \) be a smooth curve in \( B_q \) with \( v(0) = 0 \). We have

\[
\frac{d}{dt} \mathcal{H}_p(v(t))\bigg|_{t=0} = \int (1 + \log((1 + v(t))p)v(t) \, p \, dm \bigg|_{t=0} = -\log((1 + v(t))p) + \int \log((1 + v(t))p) \, dm, \, v(t) \bigg|_{t=0} = -\log q - \mathcal{H}(q), \, \dot{v}(0) \) \tag{47}
\]

In conclusion \( \nabla_m \mathcal{H}(q) = -\log q + \mathcal{H}(q) \). The same result holds for \( \nabla_q \).

In the exponential geometry, as \( \gamma(t) = \frac{d}{dt} \log \gamma(t) \), the gradient flow equation for the entropy \( \dot{\gamma}(t) = -\nabla \mathcal{H}(\gamma(t)) \) becomes

\[
\frac{d}{dt} \log \gamma(t) = \log \gamma(t) - \int \gamma(t) \log \gamma(t) \, dm \tag{48}
\]

If we compute the acceleration, we find the remarkable result \( \ddot{\gamma}(t) = \ddot{\gamma}(t) \). The curve \( \gamma(t) = e^{\gamma(t)} - K_\gamma(v) \cdot p \) has \( \dot{\gamma}(t) = \dot{\gamma}(v - \int v \, \gamma(t) \, dm) = \frac{\dot{a}(t)}{a(t)} \dot{\gamma}(t) \). We want \( \dot{a}(t) = \ddot{a}(t) \), that is \( a(t) = ce^t + b \).

Example 7 (Differentiable densities) In this example, we discuss how to construct an affine manifold for differentiable densities. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) and let \( m \) denote the Lebesgue measure on \( \Omega \). Let \( C^m_b(\Omega) \) be the Banach space of functions which are continuous on \( \overline{\Omega} \) and \( n \)-times differentiable on \( \Omega \), with bounded partial derivatives. The norm is \( \|f\|_v = \|f\|_\infty + \sum_{|\alpha|=n} x \|\partial_\alpha f\|_\infty \). Let \( M_1 \) be the affine subspace where \( \int f \, dm = 1 \). Convex subset \( M \) of positive functions is an open subset of \( M_1 \). It is the set of positive differentiable density functions on \( \Omega \).

A displacement map\(^{10}\) based on derivatives is given by

\[
\mathcal{S}(p, q) = \nabla \log \frac{q}{p} = \nabla q \frac{p}{q} - \frac{\nabla p}{p} \tag{49}
\]

Let us define the fibers as

\[
B_p = \left\{ u \in C^{n-1}_b(\Omega; \mathbb{R}^d) \mid u = \nabla U, \, U \in C^n_b(\Omega), \int U \, p \, dm = 0 \right\}. \tag{50}
\]

\(^{10}\) This displacement appears in [27]
All the fibers are equal and we can assume trivial transports. The semi-group property is clear. Let us compute the inverse of a chart. If \( B_p = \{ U \in C^n_p(\Omega) \mid \int U \cdot p \, dm = 0 \} \) then \( s_p^{-1}(u) = e^{U - K_p(U)} \cdot p \), \( u = \nabla U. \) It follows that, in the notations of Example 3, we can write

\[
\frac{1}{s_p^{-1}(u)} = e^{U - K_p(U)} \cdot p, \quad u = \nabla U.
\] (51)

The bundle on \( M \) with fibers \( B_p \) has an interesting inner product,

\[
\langle u, v \rangle_p = \int \nabla U \cdot \nabla V \, p \, dm.
\] (52)

We do not further follow this example, but refer to the relevant chapter in this Handbook.

2 Non-parametric statistical affine manifolds

Let us focus now on affine manifolds whose base set is the vector space of signed measures \( M \) on a given measurable space \( (\Omega, \mathcal{F}) \). We recall that the set of finite measures on a measurable space \( (X, \mathcal{X}) \) is a lattice and a convex pointed cone. A signed measure is the numerical difference of two finite measures, \( \mu = \mu_1 - \mu_2 \). There exists a unique minimal decomposition \( \mu = \mu_+ - \mu_- \), the Jordan decomposition, where the positive part \( \mu_+ \) and the negative part \( \mu_- \) have disjoint supports, that is, \( \mu_- \wedge \mu_+ = 0 \). The measure \( |\mu| = \mu_+ + \mu_- \) is the absolute value of \( \mu \) and \( \mu \mapsto \|\mu\|_{TM} = \int d |\mu| \) is the total variation norm. The affine subspace of signed measures with total mass \( \tau \) is denoted \( \mathcal{M}_\tau \). In particular, we are interested with the affine subspace \( \mathcal{M}_1 \) and in the vector subspace \( \mathcal{M}_0 \). The affine subspace \( \mathcal{M}_1 \) contains the closed convex set of probability measures \( P \).

The integral induces a natural pairing on \( M \times L^\infty \), that is, \( \langle \mu, f \rangle \mapsto \int f \, d\mu \). The space \( \mathcal{M}_0 \) is closed in the weak topology. We have \( \langle \mu, f \rangle \leq \|f\|_\infty \|\mu\|_{TM} \), and the weak convergence induced on \( L^\infty \) implies the point-wise convergence.

For each curve \( t \mapsto \mu(t) \) in \( \mathcal{M}_1 \) and for any topology on \( \mathcal{M} \) compatible with the operations of vector space such that \( \mathcal{M}_0 \) is closed, it holds \( \frac{d}{dt} \mu(t) = \dot{\mu}(t) \in \mathcal{M}_0 \) provided the curve is differentiable at \( t \), that is

\[
\lim_{h \to 0} \left( h^{-1}(\mu(t + h) - \mu(t)) - \dot{\mu}(t) \right) = 0
\] (53).

---

11 The first reference is [34].
12 We refer to [9] for a full treatment of Information Geometry in \( M \). See the relevant chapter in this Handbook. We refer to [43] for basic Functional Analysis and Measure theory. More advanced textbooks are [8] and [11].
13 See [5] Ch. 10-11.
The following special case is of high interest. Assume that the curve stays in \( \mathbb{P} \). In such a case, \( \mu_t(A) = 0 \) implies that \( s \) is a minimum point of \( t \mapsto \mu_t(A) \), hence \( \mu_s(A) = 0 \) so that the absolute continuity \( \mu_t \ll \mu_s \) holds. The Fisher’s score \( \dot{\mu}_t = \frac{d\mu_t}{d\mu_s} \in L^1(\mu_t) \) is defined for smooth statistical models. Notice that \( \int \dot{\mu}_t \, d\mu_t = \dot{\mu}_t(X) = 0 \).

For any topological vector space on \( \mathbb{M} \) such the the mapping \( \mu \mapsto \int f \, d\mu = \langle f, \mu \rangle \) is continuous, the Fisher-Rao equation holds,

\[
\frac{d}{dt} \int f \, d\mu_t = \int f \, d\dot{\mu}_t = \int \left( f - \int f \, d\mu_t \right) \frac{d\mu_t}{d\mu_s} \, d\mu_t = \int \left( f - \int f \, d\mu_t \right) \frac{d\mu_t}{d\mu_s} \, d\mu_t = \left\langle f - \int f \, d\mu_t, \frac{d\mu_t}{d\mu_s} \right\rangle_{\mu_t}. \tag{54}
\]

Notice that \( \dot{\mu}_t \in L^1(\mu_t) \), that is, possibly a different space for each \( t \).

The existence of a common dominating measure is an option to be considered, that is, \( \mu(t) = p_t \cdot \mu \). Assume moreover \( p(t) > 0 \) \( \mu \)-almost surely. In such a case,

\[
\dot{\mu}(t) = \frac{\dot{p}_t \cdot \mu}{p_t \cdot \mu} = \frac{\dot{p}_t}{p_t} = \frac{d}{dt} \log p_t \quad \mu\text{-a.s.} \tag{55}
\]

This basic scheme has several variations in the various instances of statistical affine spaces. A few schemes of examples, non further developed in this chapter, are listed below. However, we will focus on the cases described in the following sections.

**Example 8 (\( \mathbb{P} \) on a measurable space with the total variation norm)** Let the base manifold be \( \mathbb{M} = \mathbb{M}_1 \) and let the fibers be \( B_\mu = \mathbb{M}_0 \) for all \( \mu \in \mathbb{P} \). In this case, the affine structure is simply the affine structure of the affine space \( \mathbb{M}_1 \). Recall that \( \mathbb{M} \) is a Banach space for the total variation norm. This space has even more structure. It is a *Dedekind complete Banach lattice* for the total variation norm.\(^\text{14}\) That is, any bounded above set (respectively, below) in the natural order of signed measures has an upper (respectively, a lower) limit. In particular, \( \mu \lor \nu \) and \( \mu \land \nu \) exist, and open intervals are open sets. We define a displacement by \( \mathcal{S}(\nu, \mu) = \mu - \nu \), that is, \( s_\nu : \mathbb{P} \ni \mu \mapsto \mu - \nu \in \mathbb{M}_0 \). For each \( \nu \) the mapping \( \mu \mapsto \mathcal{S}(\nu, \mu) \) is clearly 1-to-1 and the image is the set of all \( \xi \in \mathbb{M}_0 \). The dual space is the space of bounded measurable functions with the topology of bounded convergence. This space supports many special structures, particularly the geometry of scores and an embedded Riemannian geometry.\(^\text{15}\)

**Example 9 (Positive probability densities with the \( L^1(m) \) topology)** Let us take \( M = \{ \rho \in L^1(m) \mid \rho \geq 0, \int \rho \, dm = 1 \} \) and \( B = L_0^1(m) \). These are subsets of case in Ex.\(^\text{8}\)

\(^{14}\) See [8] Ch. 8–9.

\(^{15}\) This construction was done in detail in the monograph [9].
The mapping \( s_\eta : M \ni \rho \mapsto \rho - \eta \in B \) is 1-to-1. Consider that the image of \( s_\eta \) has empty interior in the \( L^1(m) \) topology if the sample space is not finite. This, in fact, is a counter-example showing that no trivial construction is feasible in infinite dimension.

**Example 10 (Continuous probability densities with respect to a Borel probability measure on a compact space)** This has already been used in the previous section as an introductory example. We assume \( \Omega \) is metric and compact.\(^{[6]} \) We consider a reference Radon measure \( m \), that is, a positive, hence continuous, linear functional on \( C(\Omega) \).

We construct an affine space with base \( M = \{ p \in C(\Omega) \mid p > 0, \int p \, dm = 1 \} \) by setting \( p q = q - p \). In this case, \( s_p : M \to U_p = \{ u \in C(\Omega) \mid u + p > 0, \int u \, dm = 0 \} \), the space of vectors is \( B = \{ u \in C(\Omega) \mid \int u \, dm = 0 \} \). We want to show that \( U_p \) is open in \( B \). In fact, if \( u \in U_p \), there is an \( \epsilon > 0 \) such that \( u - \epsilon > p, \epsilon = \min(u + p) \). If we define \( B_p = \{ u \in C(\Omega) \mid \int u \, dm = 0 \} \), then the velocity is \( \dot{p}_t = \dot{p}_t/p_t \in B_p \).

**Example 11 (Arens-Eells)** Let \( (\Omega, d) \) be a metric space with Borel measurable space \((\Omega, \mathcal{B})\). Let \( B \) be the vector space of all signed measures \( \xi \) of the form

\[
\xi = \int (\delta_x - \delta_y) \, a(dx, dy) \quad \text{with} \ a \ \text{a signed measure on} \ \Omega \times \Omega. \tag{56}
\]

That is, \( f \in L^m(\mathcal{B}) \),

\[
\int f \, d\xi = \int (f(x) - f(y)) \, a(dx, dy), \tag{57}
\]

in particular, \( \xi(\Omega) = 0 \). The Arens-Eells norm is

\[
\xi \mapsto \sup \left\{ \int d(x, y) a(dx, dy) \mid \xi = \int (\delta_x - \delta_y) \, a(dx, dy) \right\} \tag{58}
\]

is its Arens-Eells norm. Take \( M \) to be a maximal set of probability measures such that \( \nu, \mu \in M \) implies \( \mu - \nu \in B \).\(^{[7]} \)

### 2.1 Exponential affine manifold

In this section, we set the exponential affine geometry already described in Examples \(^{[3]}\) to \(^{[5]} \) in a larger framework. There are many feasible choices for the Banach spaces acting as coordinate spaces. However, not all settings will work, and we consider it important to spell out general requirements. The following sections will describe two specific choices of Banach spaces.

---

\(^{[6]} \) This assumption is satisfied in the finite state space case. A large part of the literature in non-parametric Information Geometry is actually based on this assumption. The tutorial \(^{[39]} \) is a presentation along such lines.

\(^{[7]} \) This is clear on a finite state space, otherwise see \(^{[3]} \) Ch. 15] and \(^{[50]} \) § 3.1].
Let the base set $M$ be the set of all probability measures equivalent to a reference $\sigma$-finite measure $m$. That is, $\mu = p \cdot m$ and $p > 0$ m-a.s. The equivalent set of all positive $m$-desities is the maximal possible base set. Many models we will define below apply to a smaller set of densities.

As all the measures $p \cdot m$, $p \in M$, are equivalent, the vector spaces of $p \cdot m$-equivalent classes of real random variables are equal, $L^0(p \cdot m) = L^0(m)$. $L^0(m)$ is a topological vector space with the convergence in $m$-measure. However, the bundle $M \times L^0(m)$ seems too big to support an exponential geometric structure because the random variable $U$ and $e^U$ will not always be $m$-integrable unless the state space is finite.

Let us assume that $M$ is a set of positive $m$-probability densities possibly smaller than the maximal one. Each probability density $\mu \in M$ defines its own Banach space of integrable random variables $F_1(\mu) = F_1(p \cdot m)$. In general, the spaces are not equal for different densities. A sufficient condition for $F_1(\mu) = F_1(\nu)$ for all of $\mu, \nu \in M$, is that the density ratio is bounded above and below for all couples, $k \leq q/p \leq K$ for some $0 < k \leq K$. In fact,

$$\int |f| \ q \ dm = \int |f| \ p \ dm \leq K \int |f| \ p \ dm.$$  \hfill (59)

We want a set of random variables that are integrable for all densities in our base set $M$. That is, we look for a topological vector space $B$ of $m$-classes of random variables such that

$$B \ni \cap_{p \in M} L^1(p) \mapsto L^0(m).$$ \hfill (60)

Example 12 (Compact sample space, continuous densities) If $\Omega$ is compact and $p, q$ are assumed to be continuous, then a bound in Eq. (59) always exists.

Example 13 (Bounded random variables) Clearly, that $\cap_{p \in M} L^1(p) = L^\infty(m)$, so that one could restrict the attention to bounded random variable only. This does not seem to produce a model with sufficient applicability. Frequently, useful random variables are unbounded.

Given a vector space $B$ satisfying Eq. (60), we define the family of spaces

$$B_p = \left\{ u \in B \mid \int u \ p \ dm = 0 \right\}$$ \hfill (61)

together with the transports

$$e^{U_p}: B_p \ni u \mapsto u - \mathbb{E}_q [u] \in B_q.$$ \hfill (62)

Notice that the transports compose correctly as a cocycle,

$$e^{U_p} e^{U_q} = e^{U_{pq}} e^{U_p}.$$ \hfill (63)

Moreover, we assume $M$ is such that for each couple $p, q \in M$ the log-ratio is well defined in $B$, $\log \frac{q}{p} \in B$. An affine space is now defined by the displacement
Affine statistical bundles

mapping

\[ \mathcal{S} : (p, q) \mapsto \log \frac{q}{p} - \mathbb{E}_p \left[ \log \frac{q}{p} \right] \in B_p \subset L^1_0(p) . \]  

(64)

In fact, the parallelogram law holds,

\[
\left( \log \frac{q}{p} - \mathbb{E}_p \left[ \log \frac{q}{p} \right] \right) + ^e^u \left[ \log \frac{r}{q} - \mathbb{E}_q \left[ \log \frac{r}{q} \right] \right] = \\
\log \frac{q}{p} - \mathbb{E}_p \left[ \log \frac{q}{p} \right] + \log \frac{r}{p} - \mathbb{E}_p \left[ \log \frac{r}{p} - \log \frac{q}{p} \right] \\
\log \frac{r}{p} - \mathbb{E}_p \left[ \log \frac{r}{p} \right] 
\]

(65)

Finally, we want to check that the charts

\[ s_p : M \ni q \mapsto \log \frac{q}{p} - \mathbb{E}_p \left[ \log \frac{q}{p} \right] \]  

(66)

with

\[ s_p^{-1} : u \mapsto e^{u - K_p(u)} \cdot p , \; K_p(u) = \mathbb{E}_p \left[ e^u \right] . \]  

(67)

are such that the image of each \( s_p \) is a neighborhood of 0 in \( B_p \).

The velocity \( (26) \) of the curve

\[ t \mapsto p(t) = e^{u(t) - K_p(u(t))} \cdot p \]  

(68)

is

\[ p(t) = \frac{d}{dt} \log p(t) = (u(t))_t - \frac{d}{dt} K(u(t)) = ^e^u_p (u(t))_t , \]  

(69)

where the last equality follows from a well-known property of the derivative of the cumulant generating function\[^{10}\]

The auto-parallel curves are the exponential families

\[ t \mapsto e^{i u - K_p(u)} \cdot p \; \; u \in B_p . \]  

(70)

Let us compute the acceleration with Eq. (34)

\[^{10}\text{A classical reference for exponential families is [12]}\]
Example 14 We have seen that $\dot{\dot{p}}(t) = 0$ implies $p(t) = e^{t u - K_{p(t)}} \cdot p, p(0) = p, u \in B_{p}$. In a different time scale, $q(t) = \exp(a(t) v - K_{p}(a(t)v) \cdot p, a > 0, v \in B_{p}$.

We have

$$q(t) = \dot{a}(t) \left( v - \int v q(t) \, dm \right) \quad \dot{q}(t) = \ddot{a}(t) \left( v - \int v q(t) \, dm \right)$$

so that

$$\ddot{q}(t) = \dddot{a}(t) \dddot{q}(t) = \frac{d}{dt} \log \dot{a}(t) q(t).$$

In particular, $a(t) = -1/t$, yields the equation $t \dddot{q}(t) + 2\dot{q}(t) = 0$.

3 Banach spaces of random variables as coordinate spaces

In our definition of affine statistical manifold we consider a set of probability measures $M$ and, for each $v \in M$, a mapping $s_{v}: M \rightarrow B_{v}$, where $B_{v}$ is a topological vector space, a top-linear space. The specific needs of the modeling dictate the choice of the displacement map, the only restriction being the parallelogram law.

The choice of the family of top-linear spaces $B_{v}$ and the family of parallel transport $U_{v}^{\mu}, v, \mu \in M$, could be challenging. There are two topological requirements:

1. $U_{v}^{\mu}$ is an isomorphism of $B_{v}$ onto $B_{\mu}$, and
2. The set image of $M$ with $s_{v}$, that is the set of all coordinates, is open in $B_{v}$.

The previous requirements are strict, but nevertheless there is a wide choice of possible set-ups. The number of possible variations is too large to provide an exhaustive list. As it is in other fields, Functional Analysis provides a toolbox of methods to adapt to each specific case. For convenience of the reader and for future reference, we offer in Table 1 of the displacement maps we have introduced in the previous sections.

We will discuss two instances of the base set $M$.

---

19 A classical reference for Boltzmann-Gibbs is [29]
Table 1 Cases of affine charts

| chart        | $s_p(q)$                                      | $S^{-1}(u)$                                      |
|--------------|-----------------------------------------------|-------------------------------------------------|
| mixture      | $u = \frac{q}{p^{-1}}$                        | $q = (1 + u) \cdot p$                           |
| exponential  | $u = \log \frac{q}{p} - \int \log \frac{q}{p} \cdot p \, dm$ | $q = e^{u - K_p(u)} \cdot p$                     |
| Hyvärinen    | $u = \nabla \log \frac{q}{p}$                | $q = e^{U - K_p(U)} \cdot p$ $\quad u = \nabla U$ |

### §3.1 Continuous and continuously differentiable probability functions on an open real domain.

### §3.2 Densities in Orlicz spaces.

#### 3.1 Continuous and continuously differentiable density functions on a bounded domain of $\mathbb{R}^d$.

This is actually the simplest case and we have already presented it in Examples 3, 4, 5. In applications where one can assume bounded densities for a standard reference measure, one can focus on specific classes of bounded functions, such as bounded measurable functions or continuous functions on a compact metric space. However, we note that such a setup is not suitable in many statistical applications. For example, Gaussian probability measures on $\mathbb{R}^d$ do not transform via bounded factors. However, there is a considerable scope of application in cases where the sample space is naturally bounded, for example, in some applications of Statistical Physics or Data Science. Specifically, we will restrict our discussion to real bounded domains and continuous densities or continuously differentiable densities.

Let $\Omega$ be an open bounded domain of $\mathbb{R}^d$. Let $m$ be a probability measure on $\Omega$ such that $\int f \, dm = 0$ implies $f = 0$ for all continuous non-negative $f$. For example, the uniform probability measure $\int f \, dm = \int_{\Omega} f(x) \, dx / \int_{\Omega} \, dx$ will work, while a finite mixture of Dirac measures will not.

Let us take as model space the vector space $B_1 = \{ u \in C^k_{b}(\Omega) \mid \int u \, f \, dm = 0 \}$, where $k$ is a given non-negative integer. As $B_1$ is a closed vector subspace of $C^k_{b}(\Omega)$, it is a Banach space for the restriction of the uniform norm of $C^k_{b}(\Omega)$. We can define the mixture and the exponential displacement of table 1. The base set is the set of $C^k_{b}(\Omega)$ and positive densities, that is,

$$M = \left\{ p \in C^k_{b}(\Omega) \mid p(x) > 0, x \in \overline{\Omega}, \int p \, dm = 1 \right\}.$$  

$(74)$
The fibers are $B_p = \{ u \in C^k_b(\Omega) \mid \int u \; p \; dm = 0 \}$. All the check of the properties of affine manifold are easy and some depend on the fact every $f \in C^1_b(\Omega)$, $f > 0$, is bounded below by a positive constant.

Let us consider the Hyvärinen chart. Consider the case $k = 2$. For each couple $q, p$ of $C^2_b(\Omega)$ and positive densities, the vector random variable $u = \nabla \log \frac{q}{p}$ belongs to $C^1_b(\Omega; \mathbb{R}^d)$. If $\nabla U = u$ and $\int U \; p \; dm = 0$, then $q = e^{U-K_U} \cdot p$.

### 3.2 Probability densities in Orlicz spaces

In the theory of Lebesgue spaces, one evaluates the integrated value of a function $f$ in some power scale $|f|^{\alpha}$, $1 \leq \alpha < \infty$. Then, one computes the norm as $\|f\|_\alpha = (\int |f|^{\alpha})^{1/\alpha}$. In Orlicz spaces, we use a more general scale $\Phi(f)$, for example, $\Phi(f) = \cosh f - 1$. The presentation below is not the most general possible, but nevertheless it covers all the examples in our scope.

If $\phi \in C[0, +\infty[$ satisfies:

1. $\phi(0) = 0$,
2. $\phi$ is strictly increasing, and
3. $\lim_{u \to +\infty} \phi(u) = +\infty$,

its primitive function

$$
\Phi(x) = \int_0^x \phi(u) \; du, \quad x \geq 0, \tag{75}
$$

is strictly convex and a diffeomorphism of $]0, \infty[$. The function $\Phi$ is extended to $\mathbb{R}$ by symmetry, $\Phi(x) = \Phi(|x|)$ and it is called Young function.

The inverse function $\psi = \phi^{-1}$ has the same properties 1) to 3) as $\phi$, so that its primitive

$$
\Psi(y) = \int_0^y \psi(v) \; dv, \quad y \geq 0, \tag{76}
$$

is again a Young function. The couple $(\Phi, \Psi)$, is a couple of conjugate Young functions. The relation is symmetric and we write both $\Psi = \Phi^*$ and $\Phi = \Psi^*$.

The following properties are easy to check. The Young inequality holds true,

$$
\Phi(x) + \Psi(y) \geq xy, \quad x, y \geq 0, \tag{77}
$$

and the Legendre equality holds true,

$$
\Phi(x) + \Psi(\phi(x)) = x\phi(x), \quad x \geq 0, \tag{78}
$$

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20 The monograph by Adams and Fournier [4, Ch. 8] and the Musielak monograph [32, Ch. I-II] provide basic references to this topic.

21 Notice that $|x|^\alpha$, $\alpha > 1$ is included in our definition, while $|x|$ is not. If more generality is needed, see [4] §8.2 and [32] §7.1.
Table 2 Examples of Young functions. In the table first line $\alpha, \beta > 1$.

| $\Phi = \Psi_*$ | $\Psi = \Phi_*$ |
|----------------|-----------------|
| $x^{\alpha}/\alpha$ | $\int_0^y y^{1/(\alpha-1)} \, dv = y^\beta/\beta$, \quad $1/\alpha + 1/\beta = 1$ |
| $\exp_2(x) = e^x - 1 - x$ | $(\exp_2)_+(y) = \int_0^y \log(1 + v) \, dv = (1 + y) \log(1 + y) - y$ |
| $\cosh_2(x) = \cosh x - 1$ | $(\cosh_2)_+(y) = \int_0^y \sinh^{-1}(v) \, dv = y \sinh^{-1} y - \sqrt{1 + y^2}$ |
| $\text{gauss}_+(x) = \exp \left( \frac{1}{2} x^2 \right) - 1$ | no closed form |

that is, the Legendre transform coincides with the convex conjugate,

$$
\Psi(y) = xy - \Phi(\phi^{-1}(y)) = \inf_x (xy - \Phi(x)) , \quad (79)
$$

Table 2 collects the examples we are going to use in the following.

Given a probability space $(X, \mathcal{X}, \mu)$, we denote by $L^\mu(\mu)$ the space of $\mu$-classes of real random variables.

**Definition 12** Given a Young function $\Phi$ and a probability measure $\mu$, the Orlicz space $L_\Phi(\mu)$ is the vector subspace of $f \in L^\mu(\mu)$ such that $\int \Phi(\rho^{-1} f) \, d\mu$ is finite for some $\rho > 0$.

**Proposition 4** $L_\Phi(\mu)$ is a Banach space for the norm whose closed unit ball is $\{ f \in L^\mu(\mu) \mid \int \Phi(\rho^{-1} |f|) \, d\mu \leq 1 \}$.

The vector space property is easy to check by using the convexity of the Young function $\Phi$. The norm implicitly defined above is called Luxemburg norm. Explicitly,

$$
\|f\|_{L_\Phi(\mu)} \leq \rho \quad \text{if, and only if,} \quad \int \Phi(\rho^{-1} |f|) \, d\mu \leq 1 , \quad (80)
$$

that is,

$$
\|f\|_{L_\Phi(\mu)} = \inf \left\{ \rho \mid \int \Phi(\rho^{-1} |f|) \, d\mu \leq 1 \right\} . \quad (81)
$$

We refer to the standard monographs on Orlicz spaces for detailed proofs of the proposition above. The next remark list some special features of this class of Banach spaces.

**Remark 1**

1. If $c$ is constant function, then $\|c\|_{L_\Phi(\mu)} = c$ if, and only if $\Phi(1) = 1$, which is the case for power functions, but is not the case for the other examples in the table.

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22 Indeed, the previous theory is just a very special case of convex duality, cf. [24 Ch. I].

23 See [4 § 8.7-11] and [32 §I.1]. See a proof of completeness in [32 Th. 7.7].
2. In the case $\Phi(x) = |x|^\alpha$, $1 < \alpha < \infty$, the Luxemburg norm equals the Lebesgue norm. If $\Phi(x) = \alpha^{-1} |x|^\alpha$, then the Luxemburg norm equals $\alpha^{-1/\alpha} \times$ the Lebesgue norm.

3. We have assumed the reference measure $\mu$ to be a probability measure. This is not required in the general theory of Orlicz spaces. It is a specific feature of the application we are looking for.

4. The convergence of a sequence $(f_n)$ to zero in $L_\Phi(\mu)$, that is $\lim_{n\to\infty} \|f_n\|_{L_\Phi(\mu)} = 0$, is not equivalent $\lim_{n\to\infty} \int \Phi(f_n) \, d\mu = 0$. In fact, it is required that, for all $\epsilon > 0$, it holds $\|\epsilon^{-1} f_n\|_{L_\Phi(\mu)} \leq 1$ definitively. The condition of norm convergence in terms of integrals is

$$\int \Phi(\epsilon^{-1} f_n) \, d\mu \leq 1 \text{ definitively for all } \epsilon > 0.$$  \hspace{1cm} (82)

Now, for all $0 < \lambda < 1$ it holds $\Phi(\lambda x) \leq \lambda \Phi(x)$, so that

$$\int \Phi(\epsilon^{-1} f_n) \, d\mu \leq \lambda \int \Phi((\lambda \epsilon)^{-1} f_n) \, d\mu \leq \lambda \text{ definitively for all } \epsilon > 0.$$  \hspace{1cm} (83)

In conclusion,

$$\lim_{n\to\infty} \|f_n\|_{L_\Phi(\mu)} = 0 \iff \lim_{n\to\infty} \int \Phi(\epsilon^{-1} f_n) \, d\mu = 0, \quad \epsilon > 0.$$  \hspace{1cm} (84)

5. If a growth condition of the form $\Phi(ax) \leq C(a)\Phi(x)$, $a > 0$, holds, then the condition $\lim_{n\to\infty} \Phi(f_n) \, d\mu = 0$ clearly implies (84). This is the case of the power functions $|ax|^{\alpha} = a^{\alpha} |x|^\alpha$. This is not the case for $\exp_2(x)$ because $\exp_2(x)/\exp_1(x)$ is unbounded for $x \geq 0$. This issue is important for the duality between conjugate spaces, see below.

For each couple of conjugate Young function $\Phi$ and $\Psi = \Phi^*$ we have a couple of conjugate Orlicz spaces with a duality pairing. In fact, integration of the Young inequality (77) gives

$$\int |uv| \, d\mu \leq \int \Phi(|u|) \, d\mu + \int \Psi(|v|) \, d\mu.$$  \hspace{1cm} (85)

The duality pairing is

$$L_\Phi(\mu) \times L_\Psi(\mu) \ni (u, v) \mapsto \langle u, v \rangle_{\mu} = \int uv \, d\mu.$$  \hspace{1cm} (86)

If the norms of $u$ and $v$ in (85) are both 1, the LHSXSXS is bounded by 2, that is,

$$\langle u, v \rangle_{\mu} \leq 2 \|u\|_{L_\Phi(\mu)} \|v\|_{L_\Phi(\mu)}.$$  \hspace{1cm} (87)

Each element of an Orlicz space is associated via the duality pairing (86) to a linear continuous functional of the conjugate. However, an Orlicz space is the dual
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Banach space of of its conjugate in particular cases only, see below. However, an equivalent norm can be defined via the duality pairing, namely, the Orlicz norm,

$$\|f\|_{L^q(\mu)^*} = \sup \left\{ \langle f, g \rangle_{\mu} \mid \|f\|_{L^q(\mu)} \leq 1 \right\}. \quad (88)$$

By bounding the pairing with \((87)\), we see that \(\|f\|_{L^q(\mu)^*} \leq 2\|f\|_{L^q(\mu)}\). Because of this inequality, \((88)\) defines a continuous norm on \(L^q(\mu)\) and \(\langle f, g \rangle_{\mu} \leq \|f\|_{L^q(\mu)^*} \|g\|_{L^{q^*}(\mu)}\). More, the Luxembourg norm and the Orlicz norm are actually equivalent. Let us show that \(\|f\|_{L^q(\mu)^*} \leq 1\) whenever \(\|f\|_{L^q(\mu)} \leq 1\). The conjugation relation extends to integrals,

$$\int \Phi(f) \, d\mu = \sup \left\{ \langle f, g \rangle_{\mu} - \int \Psi(g) \, d\mu \mid g \in L^q(\mu) \right\}. \quad (89)$$

but we can compute the sup on a smaller set because

$$\sup \left\{ \langle f, g \rangle_{\mu} - \int \Psi(g) \, d\mu \mid \int \Psi(g) \, d\mu > 1 \right\} \leq \sup \left\{ \|g\|_{L^q(\mu)} - \int \Psi(g) \, d\mu \mid \int \Psi(g) \, d\mu > 1 \right\} \leq 0. \quad (90)$$

With that, \((89)\) becomes

$$\int \Phi(f) \, d\mu = \sup \left\{ \langle f, g \rangle_{\mu} - \int \Psi(g) \, d\mu \mid \|f\|_{L^q(\mu)} \leq 1 \right\} \leq 1 \quad (91)$$

and the bound is proved.

Other equivalent norms are of interest and will be discussed later in specific instances of the Young function or the base measure \(\mu\).

The domination relation between Young functions imply continuous injection properties for the corresponding Orlicz spaces. We will say that \(\Phi_2\) eventually dominates \(\Phi_1\), written \(\Phi_1 < \Phi_2\), if there are positive constants \(a, b\) and a non-negative \(x\) such that \(\Phi_1(x) \leq a\Phi_2(bx)\) for all \(x \geq \bar{x}\). As, in our case, \(\mu\) is a probability measure, the continuous embedding \(L_{\Phi_1}(\mu) \to L_{\Phi_2}(\mu)\) holds if, and only if, \(\Phi_1 < \Phi_2\). If \(\Phi_1 < \Phi_2\), then \((\Phi_2) \prec (\Phi_1)\). See [4, Th. 8.12] or [32, Th. 8.5].

When there exists a function \(C\) such that \(\Psi(ax) \leq C(a)\Psi(x)\) for all \(a \geq 0\). In such a case, the conjugate \(L_{\Psi^*}(\mu)\) is the dual Banach space of \(L^{\Psi}(\mu)\), and bounded functions are a dense set.\(^{24}\)

We now move to the discussion of the examples to be used in the one version of affine statistical manifold, that is, the cases from Tab.\textsuperscript{2}

The spaces corresponding to the power functions coincide with the ordinary Lebesgue spaces. The norm are related by

$$\|f\|_{L^a(\mu)} = a^{1/a} \|f\|_{L^a(\mu)} \quad (92)$$

\(^{24}\) We do not develop this classical topic here because it is not much relevant in our application to IG. See [4, §8.17-20].
The embedding conditions hold. The spaces are dual of each other.

The Young function \( \exp \) and \( \cosh \) are equivalent and the Orlicz spaces are isomorphic equal as vector spaces and isomorphic, \( L_{\exp}^2(\mu) \leftrightarrow L_{\cosh}^2(\mu) \). This example is of special interest for us as they provide the model spaces for a non-parametric version of Information Geometry, see Sec. 4 below. They both are eventually dominated by \( \text{gauss}_2 \) and eventually dominate all powers, that is,

\[
L^\infty(\mu) \leftrightarrow L_{\text{gauss}}^2(\mu) \leftrightarrow L_{\exp}^2(\mu) \approx L_{\cosh}^2(\mu) \\
L^\alpha(\mu) \leftrightarrow L^2(\mu) \leftrightarrow L^\beta(\mu) \\
L_{\exp}^\beta(\mu) \approx L_{\cosh}^\beta(\mu) \leftrightarrow L_{\text{gauss}}^\beta(\mu) \leftrightarrow L^1(\mu), \quad (93)
\]

where \( \beta > 2 \) and \( 1 < \beta < 2 \) are conjugate, \( 1/\alpha + 1/\beta = 1 \). Each space at the left of \( L^2(\mu) \) is the dual of one space at the right.

The Orlicz space \( L_{\exp}^2(\mu) = L_{\cosh}^2(\mu) \) is actually known with different many names in various chapters of Statistics. The proposition below provides equivalent definitions.

**Proposition 5**

1. A function belongs to the space \( L_{\cosh}^2(\mu) \) if, and only if, its moment generating function \( \mathbb{E}e^{-\mu|f|} \) is finite in a neighborhood of 0. In turn, this implies that the moment generating function is analytic at 0.

2. The same property is equivalent to a large deviation inequality. Precisely, a function \( f \) belongs to \( L_{\cosh}^2(\mu) \) if, and only if, it is sub-exponential, that is, there exist constants \( C_1, C_2 > 0 \) such that

\[
\mu(|f| \geq t) \leq C_1 \exp(-C_2t), \quad t \geq 0. \quad (94)
\]

**Proof** The first statement is immediate. If \( \|f\|_{L_{\cosh}^2(\mu)} = \rho \), then \( \int e^{\rho-1|f|} \ d\mu \leq 4 \). It follows that

\[
\mu(|f| > t) = \mu\left(e^{\rho-1|f|} > e^{\rho-1t}\right) \leq \left(\int e^{\rho-1|f|} \ d\mu\right) e^{-\rho-1t} \leq 4e^{-\rho-1t}. \quad (95)
\]

The sub-exponential inequality holds with \( C_1 = 4 \) and \( C_2 = \|f\|_{L_{\cosh}^2(\mu)}^{-1} \). Conversely, for all \( \lambda > 0 \),

\[
\int e^{\lambda f} \ d\mu \leq \int_1^\infty \mu\left(e^{\lambda t} > t\right) \ dt \leq C_1 \int_0^\infty e^{-t(C_2t^{-1}-1)} \ ds. \quad (96)
\]

The right-hand side is finite if \( \lambda < C_2 \) and the same bound holds for \(-f\). \( \square \)

**Remark 2**

25 See [12] Ch. 2]

26 See [49] Ch. 2].
1. A sub-exponential random variable is of particular interest in applications because they admit an explicit exponential bound in the Law of Large Numbers. Another class of interest consists of the sub-Gaussian random variables, that is, those random variables whose square is sub-exponential.

2. The theory of sub-exponential random variables provides an equivalent norm for the space $L_{\text{cosh}_2}(\mu)$\footnote{See \cite{49}.}. The norm is

$$f \mapsto \sup_k \left(2^k t^{-1} \int f^{2k} \, d\mu\right)^{1/2k} = \| f \|_{\text{cosh}_2}.$$ \hspace{1cm} (97)

Let us prove the equivalence. If $\| f \|_{\text{cosh}_2}(\mu) \leq 1$, then

$$1 \geq \int \text{cosh}_2 f \, d\mu \geq \frac{1}{(2k)!} \int f^{2k} \, d\mu \text{ for all } k = 1, 2, \ldots,$$ \hspace{1cm} (98)

so that $1 \geq \| f \|_{\text{cosh}_2}$. Conversely, if the latter inequality holds, then

$$\int \text{cosh}_2 (f / \sqrt{2}) \, d\mu = \sum_{k=1}^{\infty} \frac{1}{(2k)!} \int f^{2k} \, d\mu \left(\frac{1}{2}\right)^k \leq 1,$$ \hspace{1cm} (99)

so that $\| f \|_{\text{cosh}_2}(\mu) \leq \sqrt{2}$.

3. It is convenient to introduce a further notation. For each Young function $\Phi$, the function $\Phi(x^2)$ is again a Young function such that $\| f \|_{\text{cosh}_2}(\mu) \leq \lambda$ if, and only if, $\| f \|_{\text{cosh}_2}(\mu) \leq \lambda^2$. We will denote the resulting space by $L_{\Phi}(\mu)$. For example, $\text{gauss}_2$ and $\text{cosh}_2$ are $\approx$-equivalent, hence the isomorphism $L_{\text{gauss}_2}(\mu) \leftrightarrow L_{\text{cosh}_2}(\mu)$. As an application of this notation, consider that for each increasing convex $\Phi$ it holds $\Phi(fg) \leq \Phi(f^2 + g^2)/2 \leq (\Phi(f^2) + \Phi(g^2))/2$. It follows that when the $L_{\Phi}(\mu)$-norm of $f$ and of $g$ is bounded by one, the $L_{\Phi}(\mu)$-norm of $f$, $g$, and $fg$ are all bounded by one. The space $L_{\text{cosh}_2}(\mu)$ has a continuous injection in the Fréchet space $L^{0-0}(\mu) = \cap_{\alpha>1} L^\alpha(\mu)$, which is an algebra. When we need the product, we can either assume the factors are both sub-Gaussian or move up the functional framework to the Lebesgue spaces’ intersection.

**Example 15 (Gaussian exponential Orlicz space)** Let us now discuss other special issues of Orlicz spaces by focusing on a case of specific interest in IG, that is, the Gaussian exponential Orlicz space $L_{\text{cosh}_2}(\gamma)$, with $\gamma$ the standard $n$-variate Gaussian density. We note that that dominated convergence does not hold in this space. In fact, the squared-norm function $f(x) = |x|^2$ belongs to the Gaussian exponential Orlicz space $L_{\text{cosh}_2}(\gamma)$ because

$$\int \text{cosh}_2(\lambda f(x)) \, \gamma(x) \, dx < \infty \text{ for all } \lambda < 1/2.$$ \hspace{1cm} (100)

$^{27}$ See \cite{49}.

$^{28}$ See \cite{13} or \cite{48}.
The sequence \( f_N(x) = f(x)(|x| \leq N) \) converges to \( f \) point-wise and in all \( L^\alpha(y) \), \( 1 \leq \alpha < \infty \). However, the convergence does not hold in the Gaussian exponential Orlicz space. We see that, for all \( \lambda \geq 1/2 \),
\[
\int \cosh_2(\lambda(f(x) - f_N(x))) \gamma(x)dx = \int_{|x| > N} \cosh_2(\lambda f(x)) \gamma(x)dx = \infty,
\]
while convergence would imply
\[
\limsup_{N \to \infty} \int \cosh_2(\lambda(f(x) - f_N(x))) \gamma(x)dx \leq 1 \quad \text{for all } \lambda > 0. \tag{101}
\]

In the same spirit, one must observe that the closure in \( L_{\cosh_2}(\gamma) \) of the vector space of bounded functions is called Orlicz class \( M_{\cosh_2}(\gamma) \) and is strictly smaller than the full Orlicz space. Precisely, one can prove that \( f \in M_{\cosh_2}(\gamma) \) if, and only if, the moment generating function \( \lambda \mapsto \int e^{\lambda f(x)} \gamma(x)dx \) is finite for all \( \lambda \). An example is \( f(x) = x \). Bounded convergence holds in the Orlicz class. Assume \( f \in M_{\cosh_2}(\gamma) \) and consider the sequence \( f_N(x) = (|x| \leq N)f(x) \). Now,
\[
\int \cosh_2(\lambda(f(x) - f_N(x))) \gamma(x)dx = \int_{|x| \geq N} \cosh_2(\lambda f(x)) \gamma(x)dx \leq 1 \quad \text{for all } \lambda > 0. \tag{102}
\]

converges to 0 as \( N \to \infty \).

### 4 Exponential statistical bundles

We now show how to apply the construction of the exponential and mixture affine bundles in the case where the fibers are Orlicz spaces as defined in the previous sections. Precisely, we consider two equivalent conjugate couples of Young function from Table 2.

\[
\exp_2(x) = e^x - 1 - x = \int_0^x (x - s) \exp s \, ds.
\tag{103}
\]

\[
\exp_2^*(y) = (1 + y) \log(1 + y) - y = \int_0^y \frac{y - s}{1 + s} \, ds.
\tag{104}
\]

\[
\cosh_2(x) = \cosh x - 1 = \int_0^x (x - s) \cosh s \, ds.
\tag{105}
\]

\[
\cosh_2^*(y) = y \sinh^{-1} y - \sqrt{1 + y^2} = \int_0^y \frac{y - s}{\sqrt{1 + s^2}} \, ds.
\tag{106}
\]

The integral form is convenient in proving useful inequalities. The identity of the Orlicz spaces from the Young functions (103) and (105) follows from the inequalities

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See [37].
cosh_2(x) \leq \exp_2(x) \leq 2 \cosh_2(x)$, for $x \geq 0$, which, in turn, follow from $\cosh s \leq \exp s \leq 2 \cosh s$. For all $a, y > 0$,

$$\exp^*_2(ay) = \int^y_0 \frac{ay - s}{1 + s} \, ds = a \int^y_0 \frac{a(y - t)}{1 + at} \, dt \leq \max(a, a^2) \int^y_0 \frac{y - t}{1 + t} \, dt ,$$

and similarly for $\cosh^*_2$. The growth bounds

$$\exp^*_2(ay) \leq a \max(1, a) \exp^*_2(y) \quad \text{and} \quad \cosh^*_2(ay) \leq a \max(1, a) \cosh^*_2(y)$$

imply a bound on the Luxemburg norm. If $\int \exp^*_2(f) \, dm < \infty$, then

$$\int \exp^*_2(\rho^{-1}f) \, dm \leq \rho^{-1} \max(1, \rho^{-1}) \int \exp^*_2(f) \, dm < \infty, \quad \rho > 0 .$$

It follows that

$$\|f\|_{\exp^*_2(m)}^{-1} \max\left(1, \|f\|_{\exp^*_2(m)}^{-1}\right) \int \exp^*_2(f) \, dm \geq \int \exp^*_2\left(\|f\|_{\exp^*_2(m)}^{-1} f\right) \, dm = 1 ,$$

and

$$\|f\|_{\exp^*_2(m)} \min\left(1, \|f\|_{\exp^*_2(m)}\right) \leq \int \exp^*_2(f) \, dm .$$

The bounds (108) imply that the conjugate space equals the dual Banach space. From the bound in Eq. (109), we see that the norm is defined with equality, that is, $\|f\|_{\exp^*_2(m)} = 1$ if, and only if, $\int \exp^*_2(f) \, dm = 1$. Now, Eq. (104) shows that $\exp^*_2(y)$ is smaller than $y^2/2 = \int^y_0 (y - s) \, ds$. Because of that, we have the injection $L^2(m) \hookrightarrow L_{\exp_2^*(m)}$, and hence the dual injection $\left(L_{\exp^*_2(m)}\right)^* \hookrightarrow (L^2(m))^* = L^2(m)$, so that each element of the dual is identified with a random variable. That is, a linear functional on $L_{\exp^*_2(m)}$ of norm $k$ is of the form $f \mapsto \langle f, g \rangle_m$ with $|\langle f, g \rangle_m| \leq k \|f\|_{\exp^*_2(m)}$. The dual is identified with $L_{\exp^*_2(m)}$ with the Orlicz norm. In conclusion

$$L_{\exp^*_2(\mu)} = (L_{\exp^*_2(\mu)})^* \quad \text{and} \quad L_{\cosh^*_2(\mu)} = (L_{\cosh^*_2(\mu)})^* .$$

The reverse duality, that is, reflexivity, does not hold unless the sample space is finite. That is, the dual of $L_{\exp^*_2(\mu)}$ contains functionals which are not representable as functions. This is similar to the well known case $(L^1(\mu))^* = L^\infty(\mu)$ and $(L^\infty(\mu))^* \subseteq L^1(\mu)$.

---

50. This is a general result, see [32, §13] or [4, §8.17–20]. Here we sketch the argument in our special case.

51. See the general references already cited above.
Remark 3 (Analytic bi-lateral Laplace transform) The random variable $u$ belongs to $L_{\exp_2}(m)$ if, and only if, the Laplace transform of the image probability measure $u_* (m)$ is finite on an open interval containing 0. In such a case, the Laplace transform itself is analytic at 0\footnote{See, for example, \cite{31}.}

Remark 4 (Densities with finite entropy) Let us discuss the relation between the following properties: 1) the probability density $p$ belongs to the conjugate space $L_{\exp_2}(m)$; 2) the density $p$ has integrable logarithm $-\infty < \int \log p \, dm \leq 0$; 3) the density $p$ has finite entropy $0 \leq \int p \log p \, dm < +\infty$\footnote{A classical reference is \cite{18}.}. The function $y \mapsto y \log y$, $y > 0$, is convex, hence the increment is bounded by the derivatives at the extreme points, log $y + 1 \leq (1 + y) \log (1 + y) - y \log y \leq \log (1 + y) + 1 \leq y + 1$. From the upper bound, integration gives $\int \exp_2^2(p) \, dm \leq \int p \log p \, dm + 1$, that is, 3) implies 1). From the lower bound, if 2) holds, then $\int p \log p \, dm \leq \int \exp_2^2(p) \, dm - \int \log p \, dm$, that is, 1) and 2) imply 3).

In the next section, we define a class of densities such that the affine structure based on these Orlicz spaces applies.

### 4.1 Maximal exponential model

In this section, we apply the general methods of Sec. 2 to the specific case of a Banach manifold modeled on the Orlicz space $L_{\cosh_2}(\mu)$\footnote{The idea of maximal exponential model has been introduced in \cite{41}, \cite{16}, \cite{40}; the extension to the statistical bundle was introduced in \cite{26}, \cite{30} and \cite{37}.}.

First, we define the moment functional and the cumulant functional. These are non-parametric versions of the moment generating function and the cumulant generating function, respectively. Given a probability measure $\mu$ on the measurable space $(\Omega, \mathcal{B})$, we define $B_\mu = \{ u \in L_{\cosh_2}(\mu) \mid \int u \, d\mu = 0 \}$. It is a Banach space when the Luxemburg norm of $L_{\cosh_2}(\mu)$ is restricted to the sub-space. The moment functional is the convex mapping

$$M_\mu: B_\mu \ni u \mapsto \int e^u \, d\mu \in [0, \infty].$$

(113)

The proper domain of $M_\mu$, $\{ u \in B_\mu \mid M_\mu(u) < \infty \}$, is a convex subset of $B_\mu$ that contains the unit ball of $L_{\cosh_2}(\mu)$. In fact, $\int \cosh_2 u \, d\mu \leq 1$ implies $\int e^u \, d\mu = M_\mu(u) \leq 4$. It follows that the interior of the proper domain of $M_\mu$ is an open convex set, $S(\mu) = \{ u \in B_\mu \mid k_\mu(u) < \infty \}$.

The cumulant functional is defined by for $u \in \int_\mu$ by $K_\mu(u) = \log M_\mu(u)$. For all $u \in \int_\mu$ and $h \in B_\mu$, the mapping $t \mapsto M_\mu(u + th) = \int e^{t \hat{h}} e^u \, d\mu$ is the Laplace transform of the random variable $h$ with respect to the finite measure $e^u \cdot d\mu$ and it is
Affine statistical bundles defined on a neighborhood of 0. It follows from standard results that the mapping is infinitely differentiable at $t = 0$ with $k$'th derivative

$$\left. \frac{\partial M_\mu(u + \sum_{j=1}^k t_j h_j)}{\partial t_1 \cdots \partial t_k} \right|_{t_1, \ldots, t_k=0} = \int h_1 \cdots h_k e^u \, d\mu . \quad (114)$$

The multi-linear mapping $B_\mu \times \cdots \times B_\mu \ni (h_1, \ldots, h_k) \mapsto h_1 \cdots h_k e^u$ is bounded into $L^1(\mu)$. In fact,

$$\int |h_1 \cdots h_k| e^u \, d\mu \leq \left( \int |h_1 \cdots h_k|^n \, d\mu \right)^{1/n} \left( \int e^{nu/(n-1)} \, d\mu \right)^{(n-1)/n} \quad (115)$$

and we can chose $n$ such that $nu/(n-1) \in s_\mu$. For such an $n$, the first factor is bounded by

$$\int |h_1 \cdots h_k|^n \, d\mu \leq n^k \int e^{h_1+\cdots+h_k} \, d\mu , \quad (116)$$

where the RHS integral is bounded if $h_1+\cdots+h_k \in S(\mu)$. This proves the boundedness.

In other words, the mapping $f_\mu \ni u \mapsto M_\mu(u)$ is infinitely Gateaux-differentiable and the derivatives are continuous linear operators. We will denote the derivative by

$$D^k M_\mu(u) \cdot (h_1 \cdots h_k) = \int h_1 \cdots h_k e^u \, d\mu . \quad (117)$$

The relevance for us of the moment functional and of the cumulant functional lies in the fact they are the normalising constants (or partition functions) in probability densities of exponential form. The support $M$ of the affine manifold of interests here is the *maximal exponential model* $E(\mu)$ consisting of all probability densities of the exponential form

$$q = M_\mu(u)^{-1} e^u = \exp \left( u - K_\mu(u) \right) , \quad u \in f_\mu . \quad (118)$$

The following "portmanteau" proposition is crucial in proving the consistency of the affine structure of our Orlicz space setup. It shows the existence of a statistical bundle with base $E(\mu)$, fibers which are closed subspaces of $L_{\cosh^2}(\mu)$, and a proper co-cycle of parallel transports.

**Proposition 6** For all densities $p, q \in E(\mu)$ the following propositions are equivalent.

1. $q = e^{u-K_\mu(u)} \cdot p$, \quad (119)

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35 See, for example, [12].
36 We do not discuss here stronger differentiability conditions, see [15]. In fact, the moment functional is Frechét-differentiable on $S(\mu)$ and analytic on the open unit ball.
37 The idea to study the geometry of statistical models by embedding them in a larger exponential family is due to [21] and [22]. The possibility of a non-parametric set-up for the statistical bundle was suggested first in [19] and [20]. See the review paper [28].
where \( u \in L_{(\cosh^{-1})} (\mu) \), \( \int u \, p \, d\mu = 0 \), and \( u \) belongs to the interior of the proper domain of the convex function \( K_p = \int e^{u} \, p \, d\mu \).

2. The densities \( p \) and \( q \) are connected by an open exponential arc, that is, there exists a one-dimensional exponential family \( r_\theta \propto e^{\theta t} \) with \( \theta \in I \), \( r_0 = p \), \( r_1 = q \), and \([0, 1] \subset I \).

3. \( L_{\cosh^2} (p) = L_{\cosh^2} (q) \) and the norms are equivalent;

4. \( p/q \in \cup_{\alpha > 1} L^\alpha (q) \) and \( q/p \in \cup_{\alpha > 1} L^\alpha (p) \).

5. The mapping \( v \mapsto \frac{d}{p} v \) is an isomorphism of \( L_{\cosh^2} (p) \) onto \( L_{\cosh^2} (q) \).

**Proof (Main argument only)** We give only part of the proof. Precisely, we prove a generalization of the implication \[3\Rightarrow4\] Let \( F \) be logarithmically convex on \( R \) and such that \( \Phi = F - 1 \) is a Young function. For example, the assumption holds for both \( F(x) = \cosh x \) and \( F(x) = e^{x^2/2} \). For all real \( A \) and \( B \), the function

\[
\mathbb{R}^2 \ni (\lambda, t) \mapsto F(\lambda A) e^{tB} = \exp (\log F(\lambda A) + tB) \tag{120}
\]

is convex and so is the integral

\[
C(\lambda, t) = \int F(\lambda f(x)) \, e^{t u(x)} \, p(x) \, \mu(dx) , \tag{121}
\]

where \( f \in L_{\Phi}(\gamma) \) with and \( u \in L_{\cosh^2} (p) \) with \( \int u(x) \, p(x) \, dx = 0 \). Without restriction of generality, assume \( \|f\|_{L_{\Phi}(p)} = 1 \). Let us derive two marginal inequalities. First, for \( t = 0 \), the definition of Luxemburg norm gives

\[
C(\lambda, 0) = \int F(\lambda f) \, p(x) \, \mu(dx) \leq 2 , \quad -1 \leq \lambda \leq 1 . \tag{122}
\]

Second, for \( \lambda = 0 \), consider \( K_p(tu) = \log \int e^{tu} \, p(x) \mu(dx) , \) where \( t \) belongs to an open interval \( I \) containing \([0, 1] \) and such that \( K_p(tu) < +\infty \). It follows that

\[
C(0, t) = \int e^{tu} \, p(x) \mu(dx) = e^{K_p(tu)} < +\infty . \tag{123}
\]

Choose a \( t > 1 \) in \( I \) and consider the convex combination

\[
\left( \frac{t - 1}{t}, 1 \right) = \frac{t - 1}{t} (1, 0) + \frac{1}{t} (0, t) \tag{124}
\]

and the inequality

\[
C \left( \frac{t - 1}{t}, 1 \right) \leq \frac{t - 1}{t} C(1, 0) + \frac{1}{t} C(0, t) \leq 2 \frac{t - 1}{t} + \frac{1}{t} e^{K_1(tu)} . \tag{125}
\]

Now,\[38\] See [16][44][45][48] for a detailed proof and some further developments.
[\int \Phi \left( \frac{t-1}{t} f(x) \right) e^{u(x)-K_1(u)} p(x) \mu(dx) = \right. \\
\left. \int F \left( \frac{t-1}{t} f(x) \right) e^{\alpha(x)-K_\rho(u)} p(x) \mu(dx) - 1 = \right. \\
\left. e^{-\alpha(u)} C \left( \frac{t-1}{t}, 1 \right) - 1 \leq e^{-\alpha(u)} \left( 2 - \frac{t-1}{t} + \frac{1}{t} e^{K_\rho(tu)} \right) - 1. \right]

As the RHS is finite, we have proved that \( f \in L_\Phi(q) \) for \( q = e^{u-\alpha(u)} \). Conversely, a similar argument shows the other implication. We have proved that all Orlicz spaces \( L_\Phi(p) \), \( p \in E(\gamma) \) are equal. In turn, equality of spaces implies the equivalence norms. It is possible to derive explicit bounds by choosing a \( t \) such that the RHS is smaller or equal to 1.

\[ \square \]

If \( p, q \) are densities in the maximal exponential model, then there is an \( \epsilon > 0 \) such that the combination \( r = (1 - \lambda) p + \lambda q, \) \( -\epsilon < \lambda < 1 + \epsilon \) is there too. That is, the maximal exponential model is a convex set and is open on lines. This is proved by Prop. \( \text{[64]} \).

Let us recall a few notation from Sec. \( \text{[2]} \)

\[ B_p = \{ u \in L_{\cosh^2}(\mu) \mid \int u \ p \ d\mu = 0 \} \]  
\( (126) \)

\[ e_p(u) = \exp(u - K_p(u)) \cdot p \]  
\( (127) \)

\[ e^\mu_p q u = u - \int u \ p \ d\mu \]  
\( (128) \)

\[ e^\mu_p q u = \frac{q}{p} u \]  
\( (129) \)

The spaces \( B_p, p \in E(\mu) \), of Eq. \( (126) \) will be the fibers of the statistical bundle. Given \( p, q \in E(\mu) \), Prop. \( \text{[64]} \) shows that the Banach spaces \( B_p \) and \( B_q \) are vector sub-spaces of co-dimension 1 of the two isomorphic space \( L_{\cosh^2}(p) \approx L_{\cosh^2}(q) \). The mapping \( e^\mu_p : B_p \rightarrow B_q \) of Eq. \( (128) \) is indeed such an automorphism.

According to Prop. \( \text{[61]} \), for each given \( p \in E(\mu) \), every other \( q \in E(\mu) \) is of the form \( e_p(u) \) for some \( u \in B_p \). Precisely, \( \log \frac{p}{q} = u - K_p(u) \) with \( \int \log \frac{p}{q} \ p \ d\mu = -K_p(u) \) and \( u = \log \frac{p}{q} - \int \log \frac{p}{q} \ p \ d\mu \). It follows that the mapping

\[ E(\mu)^2 \ni (p, q) \mapsto \mathcal{S}(p, q) = \log \frac{q}{p} - \int \log \frac{q}{p} \ p \ d\mu \in B_p \]  
\( (130) \)

is well defined and, moreover,

\[ s_p : E(\mu)^2 \ni q \mapsto \mathcal{S}(p, q) \in B_p \]  
\( (131) \)

is 1-to-1 with image the open set \( \{ u \in B_p \mid K_p(u) < \infty \}^\circ \).

\( ^{\text{[59]}} \) See the detailed proof in \( \text{[44]} \)
Eq. (128) defines a linear continuous invertible operator from $B_p$ onto $B_q$. The co-cycle properties hold: $e_\mathcal{U}_p^q$ is the identity and $e_\mathcal{U}_q^q e_\mathcal{U}_p^q = e_\mathcal{U}_p^q$. In turn, we see that $(e_\mathcal{U}_p^q)^{-1} = e_\mathcal{U}_q^p$.

We have thus proved that the statistical bundle
\[ S \mathcal{E} (\mu) = \{(p,u) \mid p \in \mathcal{E} (\mu), u \in B_p\} \] (132)
admits the family of parallel transports $e_\mathcal{U}_p^q: B_p \to B_q$. With respect to this family, the map $S$ of Eq. (130) is easily seen to be an affine displacement. Moreover, the image of the chart an open set. All requirements for an affine statistical manifold are met.

Let us discuss the duality. Define
\[ *B_p = \left\{ v \in L_{\exp_2^p} (\mu) \left| \int v p \, d\mu = 0 \right. \right\} . \] (133)
It is a Banach space for the restriction of the Luxemburg norm. We use the pre-script notation to remember that $B_p$ is the dual of $^*B_p$, that is, $(^*B_p)^* = B_p$. In the pairing $B_p$, $^*B_p$, it holds
\[ \langle e_\mathcal{U}_p^q u, v \rangle_q = \langle u, m_{\mathcal{U}_q^p} v \rangle_p , \] (134)
for all $p, q \in \mathcal{E} (\mu), u \in B_p$ and $v \in ^*B_p$.

We can define the conjugate affine system with
\[ *s_p : q \mapsto \frac{q}{p} - 1 . \] (135)
Let us check that $*s_p(q) \in ^*B_p$. Clearly, $\int \left( \frac{q}{p} - 1 \right) p \, d\mu = 0$. From $\exp_2^p(y) = (1 + y) \log(1 + y) − y$, we find
\[ \int \exp_2^p(s_p(q)) p \, d\mu = \int \frac{q}{p} \log \frac{q}{p} p \, d\mu - \int \frac{q}{p} \, d\mu + 1 = - \int \frac{p}{q} q \, d\mu . \] (136)
The last term of the equality is finite because it is the opposite of the normalising constant of the exponential representation of $p$ with respect to $q$. The existence of a family of transports is shown in Prop. (6).

We give below a few results about the first three derivatives of the cumulant functional. As the values of the Gateaux derivatives are directional derivatives, all equalities below reduce to well known properties of the usual cumulant generating functions.
\[ DK_p(u)[h] = \int h e^{u-K_p(u)} \, d\mu = \int h e_p(u) \, d\mu . \] (137)
Example 16 (First and second variation of the KL divergence) presented in our formalism.

\[
D^2 K_p(u)[h_1, h_2] = \int (e^{\mathcal{U}_p(u)} h_1)(e^{\mathcal{U}_p(u)} h_2) e_p(u) \, d\mu = \left\langle e^{\mathcal{U}_p(u)} h_1, e^{\mathcal{U}_p(u)} h_2 \right\rangle_{e_p(u)}. \tag{138}
\]

\[
D^3 K_p(u)[h_1, h_2, h_3] = \int (e^{\mathcal{U}_p(u)} h_1)(e^{\mathcal{U}_p(u)} h_2)(e^{\mathcal{U}_p(u)} h_3) e_p(u) \, d\mu. \tag{139}
\]

Remark 5 The equations above show that the expected value \( \mathbb{E}_q[h] \), the covariance \( \text{Cov}_q(h_1, h_2) \), and the triple covariance \( \text{Cov}_q(h_1, h_2, h_3) \) all actually depend on a convex functions, namely, the cumulant functional. In fact, we have here is a special case of Hessian geometry.

Remark 6 (Entropy: cf. Ex. 6 and Rem. 4) The entropy is finite on all of \( \mathcal{E}(\mu) \) because \(-\mathcal{H}(q) = D(q|1)\) and its expression in terms of \( K_1 \) and \( DK_1 \) is

\[
\int q \log q \, d\mu = \int (u - K_1(u)) e_1(u) \, d\mu = DK_1(u)[u] - K_1(u). \tag{140}
\]

The random variable \( \log q = u - K_1(u) \) is integrable, with \( \int \log q \, d\mu = -K_p(u) \). It follows from the argument in Remark 4 that each density \( p \in \mathcal{E}(\mu) \) is a an element of \( L^{\exp_1}(\mu) \).

We have the mapping between convex sets

\[
B_1 \supset f_\mu \ni u \mapsto e_1(u) \in \mathcal{E}(\mu) \subset ^*B_1 + 1. \tag{141}
\]

It should be noted that \( p \mapsto ^*s(p) = p - 1 \in ^*B_1 \) is not a chart of \( \mathcal{E}(\mu) \) because the image is not an open set in general. It is in fact the restriction of an affine chart defined on a larger base set, the affine subspace of \( L^{\exp_{\mu}}_2(\mu) \) generated the maximal exponential model, that is, \( ^*B_1 + 1 \).

We conclude this section with examples showing how a classical topics of IG are presented in our formalism.

Example 16 (First and second variation of the KL divergence) Given \( p, q \in \mathcal{E}(m) \), we can write \( q = e^{u - K_p(u)} \cdot p = e_p(u) \), that is, \( u = s_p(q) \). It follows that the value at \( u \) of the expression of the Kullback-Leibler divergence \( q \mapsto D(q|p) \) in the chart \( s_p \) is

\[
D(q|p) = \int \log \frac{q}{p} \, q \, dm = \int (u - K_p(u)) e_p(u) \, p \, dm = \int u \, e_p(u) \, p \, dm - K_p(u) = DK_p(u)[u] - K_p(u). \tag{142}
\]

The derivative in the direction \( h \) is

\[\text{See [47].}\]
D(DK_p(u)[u] - K_p(u))[h] = D^2 K_p(u)[u, h] + DK_p(u)[h] - DK_p(u)[h] = D^2 K_p(u)[u, h] = \langle \mathcal{g}_p u, \mathcal{g}_p h \rangle_q = \langle \mathcal{g}_p u, \mathcal{g}_p h \rangle_p. \quad (143)

The second derivative in the directions $h$ and $k$ is

$$D^2 (DK_p(u)[u] - K_p(u))[h, k] = D(DK_p(u)[u, h])[k] = D^3 K_p(u)[u, h, k] + D^2 K_p(u)[k, h] =$$

$$\int (\mathcal{g}_p^2 u)(\mathcal{g}_p^2 h)(\mathcal{g}_p^2 k) q \, d\mu + \int (\mathcal{g}_p^3 h)(\mathcal{g}_p^2 k) q \, d\mu \quad (144)$$

Both $\mathcal{g}_p^2 h$ and $\mathcal{g}_p^2 k$ are in the fiber $B_q \subset L_{\text{cosh}_2}(q)$ but, in general, their product $(\mathcal{g}_p^2 h)(\mathcal{g}_p^2 k)$ is an element of $\cap_{a > 1} L^a(q)$ only. If actually $h, k \in L_{\text{gauss}}(q)$, a simple algebraic expansion presents the symmetric part of the bilinear operator as a function of the product $hk - \int hk \, p \, dm \in B_p$.

**Example 17 (Sub-models with constant expectation)** Let be given a constant $b \in \mathbb{R}$, a reference density $p \in \mathcal{E}(m)$, and a random variable $f \in L_{\text{cosh}_2}(m)$ such that $\int f \, p \, dm = b$. Consider the subset $E(f, p)$ of the maximal exponential model $\mathcal{E}(m)$ consisting of all densities $q$ such that $\int f \, q \, dm = \int f \, p \, dm$. It is a relatively open convex set. Note that $f - b \in B_p$ and the condition can be equivalently rewritten in terms of the coordinate $u = s_p(q)$. Namely, $e_p(u) = e^{u-K_p(u)} \cdot p = q \in E(B, p)$, if, and only if,

$$0 = \int (f - b) \, e_p(u) \, dm = DK_p(u)[f - b]. \quad (145)$$

Any tangent vector $h$ satisfies $D^2 K_p(u)[h, f - b] = \text{Cov}_q(h, f - b) = 0$ for a smooth $F: \mathcal{E}(m) \rightarrow \mathbb{R}$, the expression at $p$ is $F_p(u) = F \circ e_p(u)$. Any extremal point of $F$ restricted to $E(f, p)$ satisfies $DF_p(u)[h] = 0$.

Consider the entropy $\mathcal{H}(q) = \int \log q \, q \, dm$. As seen in Rem. 4 we have $\mathcal{H}_p(u) = DK_1(u)[u] - K_1(u)$ and $D \mathcal{H}_1(u)[h] = \text{Cov}_{e_1(u)}(h, u)$, so that the stationarity condition is $\text{Cov}_{e_1(u)}(h, f) = 0$ whenever $\text{Cov}_{e_1(u)}(h, f - b)$. In conclusion $u \propto (f - b)$ and the stationary point has the form $q = e^{u(f - b) - K_p(u(f - b))}$.

A similar argument holds in the case of a finite number of random variables with given expected values. In the case of an infinite dimensional subspace of random variables, the subspace must be splitting.

**Example 18 (Pythagorean theorem for the KL divergence)** The Kullback-Leibler divergence is $D(p|q) = \int \log \frac{p}{q} \, p \, dm$. Consider positive densities $q = e^{u-K_p(u)} \cdot p$, $q \in \mathcal{E}(p)$, and $r = (1 + v) \cdot p$, $v \in \ast B_p$. A simple computation shows that

$$D(r|q) + \langle u, v \rangle_p = D(r|p) + D(p|q).$$

The case where $\langle u, v \rangle_p = 0$ is sometimes called Pythagorean Theorem for divergences. The result actually implies a conjugation statement. In fact, $\langle u, v \rangle_p \leq D(r|p) + D(p|q)$ and $r = q$ gives $\langle u, v \rangle_p = D(q|p) + D(p|q)$. 

4.2 Covariant derivatives, tensor bundle, acceleration

In discussing higher-order geometry, one needs to define bundles whose fibers are the product of multiple copies of the mixture and exponential fibers.

As a first example, the full bundle is

\[ 1S^1E(\mu) = \{(q, \eta, w) \mid q \in E(\mu), \eta \in ^*S_qE(\mu), w \in S_qE(\mu)\}. \quad (146) \]

There is a duality pairing \(^*S_qE(\mu) \times S_qE(\mu) \ni (\eta, w) \mapsto \langle \eta, w \rangle_q\) and the dual of \(^*S_qE(\mu)\) is \(S_qE(\mu)\). The full bundle is our setup to discuss second-order geometry.

More generally, \(h^k\) will denote the case with \(h\) mixture factors and \(k\) exponential factors. We use both notations for expected values in the following sections, \(\int F \ pd\mu = \mathbb{E}_p[F]\).

Let us compute the expression of the velocity at time \(t\) of a smooth curve in the exponential bundle:

\[ t \mapsto \gamma(t) = (q(t), w(t)) \in SE(\mu) = 0S^1E(\mu), \quad (147) \]

where \(q(t) \in E(\mu)\) and \(w(t) \in S_qE(\mu)\) is a \(q(t)\)-centered random variable in the Orlicz space \(L_{\text{cosh}}(\mu)\).

In the chart centered at \(p\), the expression of the curve is

\[ \gamma_p(t) = s_p(\gamma(t)) = \left(s_p(q(t)), e^{\int_q^p w(t)}\right), \quad (148) \]

and, consequently, the time derivative has two components,

\[ \frac{d}{dt}s_p(q(t)) = \frac{d}{dt} \left( \log \frac{q(t)}{p} - \mathbb{E}_p \left[ \log \frac{q(t)}{p} \right] \right) = \frac{\dot{q}(t)}{q(t)} - \mathbb{E}_p \left[ \frac{\dot{q}(t)}{q(t)} \right] = e^{\int_q^p \frac{\dot{q}(t)}{q(t)}} = e^{\int_q^p \frac{\dot{q}(t)}{q(t)}} \frac{d}{dt} \log q(t), \quad (149) \]

and

\[ \frac{d}{dt} e^{\int_q^p w(t)} = \frac{d}{dt} \left( w(t) - \mathbb{E}_p \left[ w(t) \right] \right) = \dot{w}(t) - \mathbb{E}_p \left[ \dot{w}(t) \right]. \quad (150) \]

By expressing the tangent at each time \(t\) in the chart centered at the current position \(q(t)\), from the first component we obtain the velocity,

\[ \dot{q}(t) = e_q^{\int_q^p} \frac{d}{dt} s_p(q(t)) = \hat{u}(t) - \mathbb{E}_q[\hat{u}(t)] = \frac{d}{dt} \log q(t) = \frac{\dot{q}(t)}{q(t)}. \quad (151) \]
Notice that \( t \mapsto (q(t), \dot{q}(t)) \) is a curve in the exponential statistical bundle whose expression in the chart centered at \( p \) is \( t \mapsto (u(t), \dot{u}(t)) \). The mapping \( q \mapsto (q, \dot{q}) \) is the lift of the curve to the statistical bundle.

Let us turn to the interpretation of the second component in (150). In terms of the exponential parallel transport in (62), we define an exponential covariant derivative by setting

\[
\frac{D}{dt} w(t) = \left( e^{\mathcal{U}_p^q(t)} \frac{d}{dt} e^{\mathcal{U}_p^q(t)} w(t) = e^{\mathcal{U}_p^q(t)} (\dot{w}(t) - \mathbb{E}_p [\dot{w}(t)]) = \dot{w}(t) - \mathbb{E}_q(t) [\dot{w}(t)]. \right. \tag{152}
\]

The notation \( \frac{D}{dt} \) will generically denote the covariant time derivative in a given transport or connection, whose choice will depend on the context. When necessary, we use \( D_v, D_m \), or similar notations.

Let us do now the computation in the mixture bundle. The curve is

\[
t \mapsto \zeta(t) = (q(t), \eta(t)) \in SE(\mu) = \mathcal{S}^0 E(\mu) \tag{153}
\]

The computation in the first component is the same as above. The expression of the second component in chart is \( m^{\mathcal{U}_p^q(t)} \eta(t) = \frac{q(t)}{p} \eta(t) \). This gives the derivation

\[
\frac{d}{dt} m^{\mathcal{U}_p^q(t)} \eta(t) = \frac{d}{dt} \frac{q(t)}{p} \eta(t) = \frac{1}{p} (q(t) \dot{\eta}(t) + q(t) \dot{\eta}(t)) \tag{154}
\]

which, in turn, defines the mixture covariant derivative as

\[
\frac{D}{dt} \eta(t) = m^{\mathcal{U}_p^q(t)} \frac{d}{dt} m^{\mathcal{U}_p^q(t)} \eta(t) = \frac{p}{q(t)} \frac{1}{p} (q(t) \eta(t) + q(t) \dot{\eta}(t)) = \frac{p}{q(t)} (q(t) \eta(t) + \dot{\eta}(t)). \tag{155}
\]

A basic computation in the full statistical bundle is the the variation of the duality pairing. The couple of covariant derivatives in (152), (155) are compatible with the duality pairing.

\textbf{Proposition 7 (Duality of the covariant derivatives)} For each smooth curve in the full statistical bundle,

\[
t \mapsto (q(t), \eta(t), w(t)) \in \mathcal{S}_E(\mu), \tag{156}
\]

it holds

\[
\frac{d}{dt} \langle \eta(t), w(t) \rangle_{q(t)} = \left< \frac{D_m}{dt} \eta(t), w(t) \right>_{q(t)} + \left< \eta(t), \frac{D_v}{dt} w(t) \right>_{q(t)}. \tag{157}
\]

\textbf{Proof} In an arbitrary reference density \( p \),
Affine statistical bundles

\[ \frac{d}{dt} \langle \eta(t), w(t) \rangle_{q(t)} = \frac{d}{dt} \left( m_{u_1D45E q(t)} \eta(t), e_{u_1D45E q(t)} w(t) \right)_p = \left( \frac{d}{dt} m_{u_1D45E q(t)} \eta(t), e_{u_1D45E q(t)} w(t) \right)_p + \left( m_{u_1D45E q(t)} \eta(t), \frac{d}{dt} e_{u_1D45E q(t)} w(t) \right)_p = \left( m_{u_1D45E q(t)} \eta(t), d_{u_1D45E q(t)} w(t) \right)_p , \]  

which is Eq. (157).

\[ \text{Remark 7 (Riemannian derivative)} \] In [7] another covariant derivative is defined for \( t \mapsto (q(t), w(t)) \in SE(\mu) \). Because of the embedding \( SE(\mu) \subset *SE(\mu) \), we can define

\[ \frac{D_0}{dt} w(t) = \frac{1}{2} \left( \frac{D_m}{dt} w(t) + \frac{D_e}{dt} w(t) \right) \]  

The remarkable property of this derivative is the compatibility with the inner product. If \( t \mapsto (q(t), v(t), w(t)) \in 0S^2 E(\mu) \), a straightforward computation shows that

\[ \frac{d}{dt} \langle v(t), w(t) \rangle_{q(t)} = \left( \frac{D_0}{dt} v(t), w(t) \right)_{q(t)} + \left( v(t), \frac{D_0}{dt} w(t) \right)_{q(t)} \]  

Let us show that both the covariant derivatives we have defined in Eqs (152) and (155) actually deserve their name. All curves and fields are assumed to be smooth.

\[ \text{Proposition 8} \] Both exponential and mixture covariant derivatives satisfy the following equations.

\[ \frac{D}{dt} (X(t) + Y(t)) = \frac{D}{dt} X(t) + \frac{D}{dt} Y(t) \]  

\[ \frac{D}{dt} f(t)v(t) = \dot{f}(t)X(t) + f(t)\frac{D}{dt} X(t) \]  

\[ \text{Proof} \] Both equations follow immediately from the definitions, that is \( \frac{D}{dt} X(t) = \bigcup_{u_1D45E q(t)} \frac{d}{dt} m_{u_1D45E q(t)} X(t) \).

\[ \text{Remark 8} \] Let us justify our unusual presentation of a classical topic such as covariant derivatives in short. As the manifold structure we discuss is actually an affine space with global charts, a number of notions are presented in a direct way, namely, the geometry of the tangent bundle of the manifold is described in terms of explicitly defined parallel transports on its expression as a statistical bundle. Because of that, and because of the non-parametric set-up, and because of the statistical application of interest, the covariant derivatives are given as operations on smooth curves.

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41 We adapt the presentation of [14] § 2.2.
42 This was done first in [26].
The usual covariant derivatives of vector fields could be defined, whether it is needed, as follows. If $F$ is a smooth section of the statistical bundle, $F(q)$ in the fiber at $q$, for each smooth curve $t \mapsto q(t)$ one could compute $\frac{d}{dt}F(q(t))$ and looks at its representation as a linear operator $DF(q(t))$ applied to $\dot{q}(t)$. Here $DF$ would be the covariant derivative as defined on smooth sections. As we have a specific representation of the linear operators of fibers represented precisely by the full statistical bundle $\mathcal{S}^1 \mathcal{E}(\mu)$, we prefer to talk about covariant gradients instead of covariant derivatives. This is done in the following section.

We now define the second statistical bundle to be

$$0 \mathcal{S}^3 \mathcal{E}(\mu) = \left\{ (q, w_1, w_2, w_3) \left| (q \in \mathcal{E}(\mu), w_1, w_2, w_3 \in \mathcal{S}_q \mathcal{E}(\mu) \right. \right\}, \quad (163)$$

with charts centered at $p \in \mathcal{E}(\mu)$ defined by

$$s_p(q, w_1, w_2, w_3) = \left( s_p(q), \mathcal{E}(\mu)_q w_1, \mathcal{E}(\mu)_q w_2, \mathcal{E}(\mu)_q w_3 \right). \quad (164)$$

The second bundle is an expression of the statistical bundle of the exponential statistical bundle, $\mathcal{S}\mathcal{S}\mathcal{E}(\mu)$. That is, for each curve $t \mapsto \gamma(t) = (q(t), w(t)) \in \mathcal{S}\mathcal{E}(\mu)$, we define its velocity at $t$ to be

$$\dot{\gamma}(t) = \left( q(t), w(t), \dot{q}(t), \frac{d}{dt}w(t) \right), \quad (165)$$

because $t \mapsto \dot{\gamma}(t)$ is a curve in the second statistical bundle and that its expression in the chart at $p$ has the last two components equal to the values given in (149) and (150), respectively. The corresponding notion of gradient will be discussed in the next section.

For each smooth curve $t \mapsto q(t)$, the velocity of its lift $t \mapsto \chi(t) = (q(t), \dot{q}(t))$ is

$$\dot{\chi}(t) = \left( q(t), \dot{q}(t), \ddot{q}(t), \dddot{q}(t) \right), \quad (166)$$

where $\dddot{q}(t)$ defines the exponential acceleration at $t$,

$$\dddot{q}(t) = \frac{d}{dt} \dot{q}(t) = \frac{d}{dt} \frac{d}{dt} q(t) - \mathcal{E}_{q(t)} \left[ 0 \right] = \frac{d}{dt} \frac{d}{dt} q(t) - \mathcal{E}_{q(t)} \left[ \frac{d}{dt} \frac{d}{dt} q(t) \right] = \frac{d}{dt} \frac{d}{dt} q(t) - \mathcal{E}_{q(t)} \left[ \left( \frac{d}{dt} \frac{d}{dt} q(t) \right)^2 - \frac{d}{dt} \frac{d}{dt} q(t) \right]. \quad (167)$$

As the two middle components of the RHS of Eq. (166) are equal, the acceleration is actually defined in $0 \mathcal{S}^2 \mathcal{E}(\mu)$.

Remark 9 (e-Geodesic) The acceleration defined above has the one-dimensional exponential families as (differential) geodesic. Every exponential (Gibbs) curve $t \mapsto q(t) = e_p(tu)$ has velocity $\dot{q}(t) = u - dK(tu)[u]$, so that the acceleration is $\dddot{q}(t) = 0$. Conversely, if one writes $v(t) = \log q(t)$, then

As described, for example, in [30 Ch. VIII]
so that \(v(x; t) = tv(x) + c\).

We have defined the exponential acceleration \(\frac{d}{dt} \dot{q}(t) = \ddot{q}(t)\) via exponential transport in (167). Similarly, we define the mixture acceleration, via mixture transport, as

\[
\frac{d_m}{dt} \dot{q}(t) = \nabla^p \left( \frac{d}{dt} \nabla^p \dot{q}(t) = \ddot{q}(t)/q(t) .
\]

(169)

**Example 19 (The SIR model as a second order equation)** The Kermack and McKendrick SIR model is a differential equation on the positive probability densities on \(\{S, I, R\}\). As an equation in the statistical bundle is

\[
\begin{align*}
\dot{\mathbf{p}}(S; t) &= -\beta p(I; t) \\
\dot{\mathbf{p}}(I; t) &= \beta p(S; t) - \gamma \\
\dot{\mathbf{p}}(R; t) &= \gamma p(I; t)/p(R; t)
\end{align*}
\]

(170)

The mixture acceleration \(m \frac{d^2}{dt^2} p = \ddot{p}/p\) as a function of \((p(t), \dot{p}(t))\) is given by an equation which is linear in each fiber:

\[
m \frac{d^2}{dt^2} p(t) = \begin{bmatrix}
-\beta p(I; t) & -\beta p(I; t) & 0 \\
\beta p(S; t) & (\beta p(S; t) - \gamma) & 0 \\
0 & \gamma & 0
\end{bmatrix} \dot{p}(t)
\]

(171)

### 4.3 Gradients on the Tensor Bundles

Given a scalar field \(F: \mathcal{E}(\mu) \rightarrow \mathbb{R}\) the gradient of \(F\) is the section \(q \mapsto \nabla F(q)\) of the mixture bundle \(\mathcal{S} \mathcal{E}(\mu)\) such that for all smooth curve \(t \mapsto q(t) \in \mathcal{E}(\mu)\) it holds

\[
\frac{d}{dt} F(q(t)) = \left\langle \nabla F(q(t)), \dot{q}(t) \right\rangle_{q(t)} .
\]

(172)

**Example 20 (Gradient of the entropy)** The natural gradient can be computed in some cases without recourse to the computation in charts. For example, if conditions for existence and smoothness are satisfied, the derivative of the entropy function \(\mathcal{H}(q(t)) = \mathbb{E}_1 [q(t) \log q(t)]\) along the curve \(t \mapsto q(t)\) is

\[
\frac{d}{dt} \mathcal{H}(q(t)) = -\mathbb{E}_1 [q(t) \log q(t)] = -\mathbb{E}_1 [\dot{q}(t)(\log q(t) + 1)] = -\mathbb{E}_{q(t)} \left[ \log q(t) \dot{q}(t) \right] = \left\langle -\log q(t) - \mathcal{H}(q(t)), \dot{q}(t) \right\rangle_{q(t)} ,
\]

(173)

hence \(\nabla \mathcal{H}(q) = -\log q - \mathcal{H}(q)\).
More precisely, in coordinates, we have

\[
\mathcal{H}(e_p(u(t))) = - \int (u(t) - K_p(u(t)) e_p(u(t)) \, d\mu = -DK_p(u(t))[u(t)] + K_p(u(t)). \tag{174}
\]

With cancellations and centering, the derivative is

\[
D^2K_p(u(t))[u(t), \dot{u}(t)] = \left(-\log q(t) - \mathbb{E}_{q(t)} [-\log q(t)] \right. \left. \frac{\partial}{\partial \theta} \right)_{q(t)} \tag{175}
\]

In chart, the gradient is expressed as a function of the ordinary gradient \(\nabla F_p\) of \(F\). In the generic chart at \(p\), with \(q \equiv e_p(u)\) and \(F(q) = F_p(u)\), it holds

\[
\left\langle \text{grad} F(q(t)), \frac{\partial}{\partial \theta} \right\rangle_{q(t)} = \frac{d}{dt} F(q(t)) = \frac{d}{dt} F_p(u(t)) = DF_p(u(t))[\dot{u}(t)] = \left\langle \frac{\partial}{\partial \theta} \text{grad} F_p(u(t)), \frac{\partial}{\partial \theta} \right\rangle_{q(t)} = \left\langle \frac{\partial}{\partial \theta} \text{grad} F_p(u(t)), \frac{\partial}{\partial \theta} \right\rangle_{q(t)} \tag{176}
\]

where, in the pairing, mixture and exponential transports consistently acts on the fibers.

**Remark 10 (Natural gradient)** The definition of gradient above is a non parametric version of the natural gradient introduced by S-I Amari.\(^{44}\) Consider a \(d\)-dimensional statistical model \(\Theta \ni \theta \mapsto q(\theta) \in \mathcal{E}(\mu), \Theta \subset \mathbb{R}^d\) open. The variation along the curve \(\theta_t \mapsto q(\theta)\) is

\[
\frac{\partial}{\partial \theta_t} F(q(\theta)) = \left\langle \text{grad} F(\theta), \frac{\partial}{\partial \theta_t} \log q(\theta) \right\rangle_{q(\theta(\theta_t))}. \tag{177}
\]

Assume there is a \(q(\theta)\)-orthogonal projection of \(\text{grad} F(q(\theta))\) onto the space generated by \(\frac{\partial}{\partial \theta_1} \log q(\theta), \ldots, \frac{\partial}{\partial \theta_d} \log q(\theta)\).

\[
\frac{\partial}{\partial \theta_j} F(q(\theta)) = \sum_{j=1}^d \tilde{F}_j(\theta) \left\langle \frac{\partial}{\partial \theta_j} \log q(\theta), \frac{\partial}{\partial \theta_i} \log q(\theta) \right\rangle_{q(\theta(\theta_t))} \quad \text{for } j = 1, \ldots, d,
\]

where the matrix \(I(\theta)\) is the Fisher matrix of the given model. The last equation presents the natural gradient \(\tilde{F}_j(\theta)\) as the inverse Fisher matrix applied to the ordinary gradient with respect to the parameters. Notice that the non-parametric setup clarifies an explicit set of assumptions to justify the computation.

\(^{44}\) See [8].
Example 21 (Gradient Flow of the entropy: cf. Ex. 2) The integral curves of the gradient flow equation
\[ \dot{q}(t) = \text{grad} \mathcal{H}(q(t)) \] (179)
are exponential families of the form \( q(t) \propto q(0)e^{\langle t \rangle} \). In fact, if we write the equation in \( \mathbb{R}^N \), we get the quasi-linear ODE
\[ \frac{d}{dt} \log q(t) = -\log q(t) - \mathcal{H}(q(t)) \] (180)
and, in turn,
\[ \log q(t) = e^{-t}\log q(0) - e^{-t} \int_0^t e^{s}\mathcal{H}(q(s)) \, ds . \] (181)
The behaviour as \( t \to \pm \infty \) and other properties follow easily.45
Given a section \( q \mapsto F(q) \in S_q \mathcal{E}(\mu) \), the variation of the entropy along the integral curves, \( \dot{q}(t) = F(q(t)) \), is
\[ \frac{d}{dt} \mathcal{H}(q(t)) = \langle \text{grad} \mathcal{H}(q(t)) , F(q(t)) \rangle_{q(t)} \]
\[ = -\langle \log q(t) + \text{grad} \mathcal{H}(q(t)) , F(q(t)) \rangle_{q(t)} . \] (182)
For example, the condition for entropy production becomes
\[ \langle \log q + \mathcal{H}(q(t)) , F(q) \rangle_q = \mathbb{E}_q [\log qF(q)] < 0 . \] (183)

The definition of the natural gradient can be generalized to functions defined on the full statistical bundle \( \mathbb{S}^1 \mathcal{E}(\mu) \). Precisely, let be given a real function \( F : \mathbb{S}^1 \mathcal{E}(\mu) \times \mathcal{D} \to \mathbb{R} \), where \( \mathcal{D} \) a domain of \( \mathbb{R}^k \). For a generic smooth curve
\[ t \mapsto (q(t), \eta(t), w(t), c(t)) \in \mathbb{S}^1 \mathcal{E}(\mu) \times \mathcal{D} , \] (184)
we want to write
\[ \frac{d}{dt} F(q(t), \eta(t), w(t), c(t)) = \]
\[ \left\langle \text{grad} F(q(t), \eta(t), w(t), c(t)), \dot{q}(t) \right\rangle_{q(t)} + \left\langle \frac{D}{dt} \eta(t), \text{grad}_\eta F(q(t), \eta(t), w(t), c(t)) \right\rangle_{q(t)} + \]
\[ \left\langle \text{grad}_c F(q(t), \eta(t), w(t), c(t)), \frac{D}{dt} w(t) \right\rangle_{q(t)} + \nabla F(q(t), \eta(t), w(t), c(t)) \cdot \dot{c}(t) , \] (185)
where the four components of the gradient are

45 See [35].
Proposition 9 The total derivative \(185\) holds true, where

1. \(\text{grad} F(q, \eta, w, c)\) is the natural gradient of

\[
q \mapsto F(q, \cup^q \xi, \cup^q p, v, c),
\]

that is, with the representation in \(p\)-chart

\[
F_p(u, \xi, w, c) = F(e_p(u), \cup^{e_p(u)} \xi, \cup^{e_p(u)} p, v, c),
\]

it is defined by

\[
\left(\text{grad} F(q, \xi, w, c), q^*\right)_q = d_t F_p(u, \xi, w, c) \left[\cup^{e_p(u)} q\right], \quad (q, q^*) \in SE(\mu)
\]

2. \(\text{grad}_w F(q, \eta, w, c)\) and \(\text{grad}_c F(q, \eta, w, c)\) are the fiber gradients;

3. \(\nabla F(q, \eta, w, c)\) is the Euclidean gradient w.r.t. the last variable.

Proof Let us fix a reference density \(p\) and express both the given function and the generic curve in the chart at \(p\). We can write the total derivative as

\[
\frac{d}{dt} F(q(t), \eta(t), w(t), c(t)) = \frac{d}{dt} F_p(u(t), \xi(t), v(t), c(t)) = \\
D_1 F_p(u(t), \xi(t), v(t), c(t)) \left[\dot{u}(t)\right] + D_2 F_p(u(t), \xi(t), v(t), c(t)) \left[\dot{\xi}(t)\right] + \\
D_3 F_p(u(t), \xi(t), v(t), c(t)) \left[\dot{v}(t)\right] + D_4 F_p(u(t), \xi(t), v(t), c(t)) \left[\dot{c}(t)\right].
\]

In the equation above, \(D_j\), with \(j = 1, \ldots, 4\), denotes the partial derivative with respect to the \(j\)-th variable of \(F_p\), which is intended to provide a linear operator represented by the appropriate dual vector, that is, the value of the proper gradient.

The last term of the total derivative does not require any comment and we can write it as the ordinary Euclidean gradient:

\[
D_4 F_p(u(t), \xi(t), v(t), c(t)) \left[\dot{c}(t)\right] = \nabla F_p(u(t), \xi(t), v(t), c(t)) \cdot \dot{c}(t).
\]

Let us consider together the second and the third term. This is a computation of the fiber derivative and does not involve the representation in chart. Given \(\alpha \in \ast SE_p(\mu)\) and \(\beta \in SE_p(\mu)\), that is, \((\alpha, \beta) \in \mathcal{S}_p(\mu)\), we have

\[
\frac{d}{dt} (\alpha, \beta) = D_1 (\alpha, \beta) \left[\dot{u}(t)\right] + D_2 (\alpha, \beta) \left[\dot{\xi}(t)\right] + \\
D_3 (\alpha, \beta) \left[\dot{v}(t)\right] + D_4 (\alpha, \beta) \left[\dot{c}(t)\right].
\]
followed by proper way to compute the first gradient is to consider the function on the relevant gradients. Consider that the inner product always has where function given by the pairing $\langle \mathbb{F}_p^q \alpha, \mathbb{F}_p^q \beta \rangle_q$, which is expressed, in turn, with the relevant gradients. Consider that the inner product always has $S_q \mathbb{E}(\mu)$ first, followed by $S_q \mathbb{E}(\mu)$ and that the subscript to the grad symbol displays which component of the full bundle is considered.

We have that

$$\frac{d}{dt} F(q(t), \eta(t), w(t), c(t)) = D_1 F_p(u(t), \xi(t), v(t), c(t)) \left[ \mathbb{F}_p^{p(u(t))} \frac{\partial}{\partial \epsilon} \mathbb{F}_p^{p(u(t))} \right]_{\epsilon=0} + \left\langle \nabla F(q(t), \eta(t), w(t), c(t)), \nabla F(q(t), \eta(t), w(t), c(t)) \right\rangle_{q(t)}$$

Putting together all the contributions, we have proved that

$$\frac{d}{dt} F(q(t), \eta(t), w(t), c(t)) =$$

$$D_2 F_p(u, \xi, v, c)[\alpha] + D_3 F_p(u, \xi, v, c)[\beta] = \frac{d}{dt} F_p(u, \xi + t\alpha, w + t\beta, c)$$

where $\mathbb{F}$ denotes the fiber derivative in $S^1 q \mathbb{E}(\mu)$, which is expressed, in turn, with the relevant gradients. Consider that the inner product always has $S_q \mathbb{E}(\mu)$ first, followed by $S_q \mathbb{E}(\mu)$ and that the subscript to the grad symbol displays which component of the full bundle is considered.

We have that

$$\frac{D}{dt} w(t) = \mathbb{F}_p^q (t) \dot{v}(t), \quad \frac{D}{dt} \eta(t) = \mathbb{F}_p^q (t) \dot{\xi}(t).$$

Putting together all the contributions, we have proved that

$$\frac{d}{dt} F(q(t), \eta(t), w(t), c(t)) =$$

$$D_1 F_p(u(t), \xi(t), v(t), c(t)) \left[ \mathbb{F}_p^{p(u(t))} \frac{\partial}{\partial \epsilon} \mathbb{F}_p^{p(u(t))} \right]_{\epsilon=0} + \left\langle \nabla F(q(t), \eta(t), w(t), c(t)), \nabla F(q(t), \eta(t), w(t), c(t)) \right\rangle_{q(t)}$$

To identify the first term in the total derivative above, consider the “geodesic” case,

$$q(t) = e_p(u(t)), \quad \eta(t) = \mathbb{F}_p^{p(u(t))} \xi, \quad w(t) = \mathbb{F}_p^{p(u(t))} v, \quad c(t) = c,$$  \hspace{1cm} (192)

so that the first term reduces to $D_1 F_p(u(t), \xi, v, c) \left[ \mathbb{F}_p^{p(u(t))} \right]_{\epsilon=0}$. It follows that the proper way to compute the first gradient is to consider the function on $\mathbb{E}(\mu)$ defined by

$$q \mapsto F_{\xi, v, c}(q) = F(q, \mathbb{F}_p^q \xi, \mathbb{F}_p^q v, c)$$  \hspace{1cm} (193)

which has a natural gradient whose chart representation is precisely that first term. □

We have so concluded the computation of the total derivative of a parametric function on the full bundle. Notice that the computation of the natural gradient for grad$_m F(q, \eta, w, c)$ and grad$_c F(q, \eta, w, c)$ is done by fixing the variables in the fibers to be translations of fixed ones.

We are going to discuss the following examples of gradient flow on the full statistical bundle: the scalar function $L(q, w) = \frac{1}{2} \langle w, w \rangle_q$; the cumulant function $L(q, w) = K_q(w)$; the conjugate cumulant function $H(q, \eta) = \mathbb{E}_q [(1 + \eta) \log(1 + \eta)]$.

**Example 22 (Scalar function) $\frac{1}{2} \langle w, w \rangle_q$** On the statistical bundle, consider the scalar function given by the pairing
The derivative with respect to

\[ L(q, w) = \frac{1}{2} \langle w, w \rangle_q. \]  

(194)

In chart, we have \( L_p(u, v) = L \left( e_{p(u), e^{\mathcal{K}_p(u)}_p(v)} \right) = D^2 K_p(u)[v, v] \) from Eq. (139), where \( K_p(u) \) is the expression in the chart at \( p \) of Kullback–Leibler divergence of \( q \mapsto D(p|q) \).

From Eq. (139), we write the derivative with respect to \( u \) in the direction \( h \) as

\[
\frac{1}{2} D^3 K_p(u)[v, v, h] = \frac{1}{2} \mathbb{E}_{p(u)} \left[ \left( v - \mathbb{E}_{p(u)} [v] \right)^2 e^{\mathcal{K}_p(u)}_p h \right] = \frac{1}{2} \langle w^2 - \mathbb{E}_q [w]^2, e\mathcal{U}_p h \rangle_q \]  

(195)

which, in turn, identifies the gradient as \( \text{grad} \frac{1}{2} \langle w, w \rangle_q = \frac{1}{2} (w^2 - \mathbb{E}_q [w^2]) \in \mathcal{E} \mu \). The exponential gradient is \( \text{grad} \frac{1}{2} \langle w, w \rangle_q = w \).

Example 23 (Cumulant functional) If \( L(q, w) = K_q(w) \), then

\[
L_p(u, v) = K_p(u) \left( e^{\mathcal{K}_p(u)}_p(v) \right) = \log \mathbb{E}_{p(u)} \left[ e^{v - \mathbb{E}_{p(u)} [v]} \right] x = \log \mathbb{E}_p \left[ e^{u - K_p(u) + v - \mathbb{E}_{p(u)} [v]} \right] = \log \mathbb{E}_p \left[ e^{u + v - K_p(u) - dK_p(u)[v]} \right] = K_p(u + v) - K_p(u) - dK_p(u)[v]. \]  

(196)

The derivative with respect to \( u \) in the direction \( h \) is

\[
D K_p(u + v)[h] - D K_p(u)[h] - D^2 K_p(u)[v, h] = \mathbb{E}_{p(u+v)} [h] - \mathbb{E}_{p(u)} [h] - \mathbb{E}_{p(u)} \left[ \left( \mathcal{U}^\mathcal{K}_p(u) v \right) \left( \mathcal{U}^\mathcal{K}_p(u) h \right) \right] = \mathbb{E}_{p(u)} \left[ \frac{e_p(u + v) h}{e_p(u) h} \right] - \mathbb{E}_{p(u)} [h] - \mathbb{E}_{p(u)} \left[ \mathcal{U}^\mathcal{K}_p(u) h \right] = \mathbb{E}_{p(u)} \left[ \frac{e_p(u + v)}{e_p(u)} \left( \mathcal{U}^\mathcal{K}_p(u) h \right) \right] - \left[ \mathcal{U}^\mathcal{K}_p(u) h \right]_q. \]  

(197)

The expected value of the factor \( \frac{e_p(u + v)}{e_p(u)} \) in the first term of the RHS equals

\[
\mathbb{E}_{p(u)} \left[ e^{v - (K_p(u) + v) - K_p(u)} \left( \mathcal{U}^\mathcal{K}_p(u) h \right) \right] = \mathbb{E}_{p(u)} \left[ e^{v - (K_p(u) + v) + dK_p(u)[v]} \left( \mathcal{U}^\mathcal{K}_p(u) h \right) \right] = \mathbb{E}_{p(u)} \left[ e^{v - K_p(u) - (\mathcal{U}^\mathcal{K}_p(u) h)} - \left( \mathcal{U}^\mathcal{K}_p(u) h \right)_q \right]. \]  

(198)
In conclusion, \( \text{grad} K_q(w) = \left( \frac{e_p(w)}{q} - 1 \right) - w \). The expectation gradient is easily seen to be

\[
\text{grad}_q K_q(w) = \frac{e_p(w)}{q} - 1 .
\]  

(199)

**Example 24 (Conjugate cumulant functional)** The conjugate cumulant functional

\[
^* \mathcal{E}(\mu) : (q, \eta) \mapsto H(q, \eta) = \mathbb{E}_q \left[ (1 + \eta) \log(1 + \eta) \right] , \quad \eta > -1 ,
\]  

(200)
is the Legendre transform of the cumulant function \( K_q \),

\[
H(q, \eta) = \langle \eta, (\text{grad} K_q)^{-1}(\eta) \rangle_q - K_q \left( (\text{grad} K_q)^{-1}(\eta) \right) .
\]  

(201)

In particular, the fiber gradient of \( H_q \) is

\[
\text{grad}_m H(q, \eta) = \log(1 + \eta) - \mathbb{E}_q \left[ \log(1 + \eta) \right]
\]  

which is the inverse of the fiber gradient of \( K_q \). Notice that \( r = (1 + \eta)q \) is a density, and \( D (r|q) = H(q, \eta) \).

Let us compute the gradient. The expression of the conjugate cumulant functional in the chart at \( p \) is

\[
H_p(u, \zeta) = \mathbb{E}_{e_p(u)} \left[ \left( 1 + \frac{p}{e_p(u)} \zeta \right) \log \left( 1 + \frac{p}{e_p(u)} \zeta \right) \right] = \mathbb{E}_p \left[ \left( \frac{e_p(u)}{\zeta} + 1 \right) \log \left( 1 + \frac{p}{e_p(u)} \zeta \right) \right] .
\]  

(202)

As, for \( h \in S_p \mathcal{E}(\mu) \),

\[
D \left( \frac{e_p(u)}{\zeta} + \zeta \right) [h] = \frac{e_p(u)}{\zeta} m_{\mathbb{E}_{e_p(u)} h} ,
\]  

(203)

\[
D \left( 1 + \frac{p}{e_p(u)} \zeta \right) [h] = -\frac{p}{e_p(u)} m_{\mathbb{E}_{e_p(u)} h} ,
\]  

(204)

the derivative of \( H_p \) with respect to \( u \) in the direction \( h \) is given by

\[
DH_p(u, \zeta)[h] = \mathbb{E}_p \left[ \left( \frac{e_p(u)}{\zeta} m_{\mathbb{E}_{e_p(u)} h} \right) \log \left( 1 + \frac{p}{e_p(u)} \zeta \right) \right] - \mathbb{E}_p \left[ \left( \frac{e_p(u)}{\zeta} + \zeta \right) \left( 1 + \frac{p}{e_p(u)} \zeta \right)^{-1} \frac{p}{e_p(u)} m_{\mathbb{E}_{e_p(u)} h} \right] = \mathbb{E}_q \left[ \log(1 + \eta) m_{\mathbb{E}_{e_p(u)} h} \right] - \mathbb{E}_q \left[ \zeta m_{\mathbb{E}_{e_p(u)} h} \right] ,
\]  

(205)

hence \( \text{grad} H(q, \eta) = \log(1 + \eta) - \mathbb{E}_q \left[ \log(1 + \eta) \right] - \eta \).
4.4 Lagrangian and Hamiltonian Formalisms on the full Statistical Bundle

The dually affine geometry of the statistical bundle is naturally well suited for describing the dynamics of probability densities in a Lagrangian and Hamiltonian formalism. This is apparent from the previous examples.

The Lagrangian formulation of mechanics derives the fundamental laws of force balance from variational principles. In our context, the exponential model $E(\mu)$ corresponds to the configuration space, while the statistical bundle is associated to the velocity phase space. For a given smooth curve $q: [0, 1] \ni t \mapsto q(t)$ in $E(\mu)$ and its lift $t \mapsto (q(t), \dot{q}(t)) \in SE(\mu)$, we introduce a generic Lagrangian function

$$L(q(t), \dot{q}(t)): SE(\mu) \times [0, 1] \rightarrow \mathbb{R}$$

and define an action as the integral of the Lagrangian along the curve over the fixed time interval $[0, 1]$,

$$q \mapsto A(q) = \int_0^1 L(q(t), \dot{q}(t), t) \, dt \ .$$

Hamilton's principle states that this function has a critical point at a solution within the space of curves on $E(\mu)$. We have

Proposition 10 (Euler-Lagrange equation) If $q$ is an extremal of the action integral, then

$$\frac{D}{dt} \text{grad}_q L(q(t), \dot{q}(t), t) = \text{grad} L(q(t), \dot{q}(t), t) \ .$$

Proof Let us express the action integral in the exponential chart $s_p$ centered at $p$. If $q(t) = e^{u(t)} - K_{p}(u(t)) \cdot p$, with $t \mapsto u(t) \in S_p E(\mu)$, we have

$$L(q(t), \dot{q}(t), t) = L\left( e_p(u(t)), e^{e_{p}(u(t))} \dot{u}(t), t \right)$$

so that the expression of the Euler-Lagrange equation in chart is given by

$$D_1L_p(u(t), \dot{u}(t), t)[h] = \frac{d}{dt}D_2L_p(u(t), \dot{u}(t), t)[h] \ . \ t \in [0, 1] \ , \ h \in S_p E(\mu) \ .$$

(210)

Consider first the RHS of (210). From Proposition (9) we have

$$D_1L_p(u(t), \dot{u}(t), t)[h] = \left( \text{grad} L(q(t), \dot{q}(t), t), e^{e_{p}(u(t))} \dot{h} \right)_{q(t)} \ .$$

(211)

On the left hand side, we have

$$D_2L_p(u(t), \dot{u}(t))[h] = \left( \text{grad}_q L(q(t), \dot{q}(t), t), e^{e_{p}(u(t))} \dot{h} \right)_{q(t)} \ .$$

(212)

See [38][17]
The derivation formula of (157) gives
\[
\frac{d}{dt} D_2 L_p(u(t), \dot{u}(t), t)[h] = \frac{d}{dt} \left( \text{grad}_q L(q(t), \dot{q}(t), t), e^{U_p^{q(\cdot)}} h \right)_{q(t)} = 0.5 \left( \frac{d}{dt} \text{grad}_q L(q(t), \dot{q}(t), t), e^{U_p^{q(\cdot)}} h \right)_{q(t)} = 0.
\]

because \( \frac{d}{dt} e^{U_p^{q(\cdot)}} h = 0 \). As \( h \) is arbitrary, the conclusion follows. \( \square \)

4.4.1 Hamiltonian mechanics

At each fixed density \( q \in E(\mu) \), and each time \( t \), the partial mapping \( S_q E(\mu) \ni w \mapsto L_q(t)(w) = L(q, w, t) \) is defined on the vector space \( S_q E(\mu) \), and its gradient mapping in the duality of \( *S_q E(\mu) \times S_q E(\mu) \) is \( w \mapsto \text{grad}_q L(q, w, t) \). The standard Legendre transform argument provides the intrinsic form of the Hamilton equations under the following assumption.

**Assumption** We restrict our attention to Lagrangians such that the fiber gradient mapping at \( q, w \mapsto \eta = \text{grad}_q L_q(w) \) is a 1-to-1 mapping from \( S_q E(\mu) \) to \( *S_q E(\mu) \). In particular, this true when the partial mappings \( w \mapsto L_q(w) \) are strictly convex for each \( q \).

In the finite dimensional context, this assumption is equivalent to the assumption that the fiber gradient is a diffeomorphism of the statistical bundles \( \text{grad}_q L : S_q E(\mu) \rightarrow *S_q E(\mu) \). This is related to the properties of regularity and hyper-regularity, cf. [11] § 3.6]. The bilinear form \( *S_q E(\mu) \times S_q E(\mu) \ni (\eta, w) \mapsto \langle \eta, w \rangle_q = E_q[\eta w] \) will always be written in this order. The Legendre transform of \( L_q(t) \) is defined for each \( \eta \in *S_q E(\mu) \) of the image of \( \text{grad}_q L(q, \cdot, t) \), so that the Hamiltonian is
\[
H(q, \eta, t) = \langle \eta, (\text{grad}_q L_q(t))^{-1}(\eta) \rangle_q - L(q, (\text{grad}_q L_q(t))^{-1}(\eta)).
\] (213)

If \( t \mapsto q(t) \) a solution of Euler-Lagrange (208), the curve \( t \mapsto \zeta(t) = (q(t), \eta(t)) \) in \( *S_q E(\mu) \), where \( \eta(t) = \text{grad}_q L(q(t), \dot{q}(t), t) \) is the momentum. The mixture bundle \( *S_q E(\mu) \) then plays the role of the cotangent bundle in mechanics.

**Proposition 11 (Hamilton equations)** When (1) holds, the momentum curve satisfies the Hamilton equations,
\[
\begin{cases}
\frac{D}{dt} \eta(t) = - \text{grad} H(q(t), \eta(t), t) \\
\dot{q}(t) = \text{grad}_m H(q(t), \eta(t), t).
\end{cases}
\] (214)
Moreover,
\[ \frac{d}{dt} H(q(t), \eta(t), t) = \frac{\partial}{\partial t} H(q(t), \eta(t), t). \] (215)

The special intrinsic form of the Hamilton equations is obtained by the use of the covariant derivatives and the gradients of the statistical bundles.

**Example 25 (Mechanics of \( \frac{1}{2} \langle w, w \rangle_q \))** The scalar function \( \frac{1}{2} \langle w, w \rangle_q \) of Example 22 corresponds to the kinetic energy Lagrangian in mechanics. In this case, as first shown in [Ref.], the Euler–Lagrange equations are equivalent to the equations of geodesic motion, whose solution coincides with the one-dimensional exponential families.

Now, if \( L(q, w) = \frac{1}{2} \langle w, w \rangle_q \) is our lagrangian, then via Legendre transform we obtain the Hamiltonian \( H(q, \eta) = \frac{1}{2} \langle \eta, \eta \rangle_q \). The gradients are
\[
\text{grad} H(q, \eta) = -\frac{1}{2} \left( \eta^2 - \mathbb{E}_q [\eta^2] \right)
\]
\[
\text{grad}_m H(q, \eta) = \eta
\]
\[
\text{grad}_L L(q, w) = \frac{1}{2} (w^2 - \mathbb{E}_q [w^2])
\]
\[
\text{grad}_e L(q, w) = w
\]

For \( \dot{\eta} = w \in \mathcal{SE} (\mu) \), the Euler-Lagrange equation is
\[
\frac{D}{dt} \dot{\eta} = \frac{1}{2} \left( \ddot{\eta}^2 - \mathbb{E}_{\eta(t)} [\ddot{\eta}(t)^2] \right),
\]
where the covariant derivative is computed in \( \mathcal{SE} (\mu) \), that is, \( \frac{D}{dt} \dot{\eta} = \ddot{\eta}/\dot{\eta} \). In terms of the exponential acceleration \( \dddot{\eta}(t) = \ddot{\eta}(t)/\dot{\eta}(t) - \left( \ddot{\eta}(t)^2 - \mathbb{E}_{\eta(t)} [\ddot{\eta}(t)^2] \right) \), the Euler-Lagrange equation reads
\[
\dddot{\eta}(t) = -\frac{1}{2} \left( (\dddot{\eta}(t))^2 - \mathbb{E}_{\eta(t)} [(\dddot{\eta}(t))^2] \right),
\]
consistently with the result in Example 22.

The Hamilton equations are
\[
\begin{align*}
\frac{D}{dt} \eta(t) &= \frac{1}{2} \left( \eta^2 - \mathbb{E}_{\eta(t)} [\eta^2] \right), \\
\dot{\eta}(t) &= \eta(t)
\end{align*}
\]
with the covariant derivative again computed in \( \mathcal{SE} (\mu) \).

The conserved energy is
\[
H(q(t), \eta(t)) = \frac{1}{2} \left( \dot{\eta}(t), \dot{\eta}(t) \right)_{q(t)} = \frac{1}{2} \mathbb{E}_{\eta} \left[ \frac{\ddot{\eta}(t)^2}{\dot{\eta}(t)} \right].
\]
which reflects in the conservation of the *Fisher information*.
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