HOMFLY-PT HOMOLOGY FOR GENERAL LINK DIAGRAMS AND BRAIDLKE
ISOTOPY

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ABSTRACT. Khovanov and Rozansky’s categorification of the HOMFLY-PT polynomial is invariant
under braided isotopies for any link diagram and Markov moves for braid closures. To define
HOMFLY-PT homology, they required a link to be presented as a braid closure, because they did not
prove invariance under the other oriented Reidemeister moves. In this text we prove that the Rei-
demeister IIb move fails in HOMFLY-PT homology by using virtual crossing filtrations of the author
and Rozansky. The decategorification of HOMFLY-PT homology for general link diagrams gives a
deformed version of the HOMFLY-PT polynomial, \( P^b(D) \), which can be used to detect nonbraided
isotopies. Finally, we will use \( P^b(D) \) to prove that HOMFLY-PT homology is not an invariant of
virtual links, even when virtual links are presented as virtual braid closures.

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1. INTRODUCTION

Khovanov and Rozansky in [12] introduced a triply-graded link homology theory categorifying
the HOMFLY-PT polynomial. The construction given in [12] of Khovanov-Rozansky HOMFLY-PT
homology, or briefly HOMFLY-PT homology, is an invariant of link diagrams up to braided iso-
topy (isotopies which locally resemble isotopies of a braid) and Markov moves for closed link
diagrams. However, Khovanov and Rozansky were not able to prove invariance under all ori-
ented Reidemeister moves. In particular, they could not prove the Reidemeister IIb move, and in
fact expected that it would fail in general. Because of this, they required that a link be presented
as a braid closure so that HOMFLY-PT homology would be an invariant of links. In this text we
will directly address this issue by proving the failure of the Reidemeister IIb move in HOMFLY-PT
homology, and explore the consequences of this failure.

The framework of HOMFLY-PT homology can be extended to include the use of “virtual cross-
ings”, degree 4 vertices which are not actually positive or negative crossings. The author and
Rozansky in [1] proved that a filtration can be placed on the chain complex whose homology is
HOMFLY-PT homology. The associated graded complex of this filtration is described using dia-
grams containing only virtual crossings. The filtration allows us to rewrite the chain complexes
in an illuminating manner, allowing us to see new isomorphisms which would be difficult to see
otherwise. Using the framework of virtual crossing filtrations we prove the following theorem.
Theorem 1.1 (See Theorem 4.4). Suppose \( D_1 \), \( D_2 \), and \( D_3 \) are oriented link diagrams which are identical except in the neighborhood of a single point. Suppose in the neighborhood of that single point, \( D_1 \) is \( \sim \), \( D_2 \) is \( \sim \), and \( D_3 \) is \( \sim \). \( \mathcal{H}(D_1) \simeq \mathcal{H}(D_3) \) up to a grading shift, while \( \mathcal{H}(D_1) \not\simeq \mathcal{H}(D_2) \) in general.

In Section 4 we prove the above theorem and give an explicit example of a diagram of the unknot that does not have the HOMFLY-PT homology of the unknot (Example 4.5). Recall \( \mathcal{H}(D) \) is a triply-graded vector space. Suppose \( d_{ijk} = \dim(\mathcal{H}(D)_{i,j,k}) \), then we can define the Poincaré series of \( \mathcal{H}(D) \) as

\[
P(D) = \sum_{i,j,k \in \mathbb{Z}} d_{ijk} q^i a^j t^k.
\]

Let \( P(D) \) denote the HOMFLY-PT polynomial of the link diagram \( D \). Murakami, Ohtsuki, and Yamada introduced a state sum formulation of the HOMFLY-PT polynomial commonly called the MOY construction [13]. Their approach resolves a link diagram into a \( \mathbb{Z}(q,a) \)-linear combination of oriented planar 4-regular graphs. They give relations which evaluate each such planar graph as an element of \( \mathbb{Z}(q,a) \). The resulting rational function from this process for any link diagram \( D \) is its HOMFLY-PT polynomial \( P(D) \).

We now define a deformed HOMFLY-PT polynomial \( P^b(D) = P(D)|_{t=-1} \). In the case that \( D \) is presented as a braid closure then \( P^b(D) = P(D) \). However, this is not true for general link diagrams. We collect properties of \( P^b(D) \) into the following theorem.

Theorem 1.2 (See Theorem 5.1). Let \( D \) be a link diagram. \( P^b(D) \) is an invariant of link diagrams up to braidlike isotopy satisfying the skein relation \( q P^b(\underset{\text{MOY 0}}{\sim}) - q^{-1} P^b(\underset{\text{MOY I}}{\sim}) = (q - q^{-1}) P^b(\underset{\text{MOY IIa}}{\sim}) \). Furthermore, \( P^b(D) \) satisfies the relations in Figure 1.

The relations in Figure 1 are not always enough to determine \( P^b(D) \), though in many examples they do suffice. In Section 5 we use \( P^b(D) \) to show that \( \mathcal{H}(D) \) is not an invariant of virtual links, even when presented as a virtual braid closure, by showing it violates the virtual exchange move.

Recent research involving annular link homology by Auroux-Grigsby-Wehrli in the \( sl_2 \) case [4] and Queffelec-Rose in the \( sl_n \) case [15] give some insight into why to expect this issue. Annular link homology theories, that is homology theories of closed braids in the thickened annulus \( D^2 \times S^1 \), are normally constructed via the use of Hochschild homology on chain complexes of...
bimodules associated to braids. Hochschild homology $\text{HH}(C)$ acts as a trace on the homotopy category of bimodules, but in general does not act like a Markov trace. In particular, if $\beta_1$ and $\beta_2$ are two braids which are Markov equivalent and $C(\beta_1)$ and $C(\beta_2)$ are their associated chain complexes of bimodules, then $\text{HH}(C(\beta_1))$ is not necessarily homotopy equivalent to $\text{HH}(C(\beta_2))$. This corresponds to the fact that even though the braid closures of $\beta_1$ and $\beta_2$ are isotopic as links in $S^3$, they may not be isotopic in $D^2 \times I$.

However, $\mathcal{H}(D)$ is not quite an annular invariant. Though $\mathcal{H}(D)$ can be constructed using Hochschild homology (see [10]), it is invariant under the second Markov move. This is why $\mathcal{H}(D)$ gives invariants of links in $S^3$ when $D$ is presented as the closure of a braid. Though it does show some behavior of an annular invariant as well. The Reidemeister IIb configuration $\begin{tikzpicture}[baseline] \draw (-0.5,0) -- (0.5,0); \draw (0,0) -- (0,0.5); \end{tikzpicture}$ can only appear in a braid closure when the braid axis is between the two strands. In the case of annular invariants the braid axis is an obstruction to isotopy, and disallows the isotopy $\begin{tikzpicture}[baseline] \draw (-0.5,0) -- (0.5,0); \draw (0,0) -- (0,0.5); \end{tikzpicture} \sim \begin{tikzpicture}[baseline] \draw (-0.5,0) -- (0.5,0); \draw (0,0) -- (0,-0.5); \end{tikzpicture}$. In an annular invariant the exchange move $\begin{tikzpicture}[baseline] \draw (-0.5,0) -- (0.5,0); \draw (0,0) -- (0,-0.5); \end{tikzpicture} \sim \begin{tikzpicture}[baseline] \draw (-0.5,0) -- (0.5,0); \draw (0,0) -- (0,0.5); \end{tikzpicture}$ is disallowed, however this move preserves the isomorphism type of $\mathcal{H}(D)$.

Outline of the paper. In Section 2 we review the definition of the HOMFLY-PT polynomial and the MOY construction of the HOMFLY-PT polynomial. We use nonstandard conventions in this text to illuminate the connections with HOMFLY-PT homology. In Section 3 we review the construction of HOMFLY-PT homology of links using closed braid diagrams. We also review some homological algebra, in particular properties of Koszul complexes. In Section 4 we explore the properties of HOMFLY-PT homology for general link diagrams. We introduce the role of virtual crossings in this framework and use virtual crossings as a tool to prove that HOMFLY-PT homology is not invariant under the Reidemeister IIb move. Finally, in Section 5 we explore the decategorification (Poincaré series) of HOMFLY-PT homology and use it to prove that HOMFLY-PT homology cannot be extended to an invariant of virtual links. We also include in Appendix A a description of the framework of virtual crossing filtrations and the necessary homological algebra.

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2. THE MOY CONSTRUCTION OF THE HOMFLY-PT POLYNOMIAL

We begin by recalling two constructions of the HOMFLY-PT polynomial for oriented links. Most of this material is well-known, but we introduce it with the purpose of setting our conventions for the sequel. The first construction is given by a skein relation and first appeared in [6]. The second construction, first introduced by Murakami, Ohtsuki, and Yamada in [13], constructs the HOMFLY-PT polynomial in terms of a state sum formula. It is this second construction which is categorified in the construction of Khovanov and Rozansky’s HOMFLY-PT homology.

2.1. The HOMFLY-PT polynomial of an oriented link diagram. Let $L$ denote a link in $\mathbb{R}^3$. In this text we will assume all links are oriented. Let $D$ denote a link diagram of $L$, that is a regular projection of $L$ onto a copy of $\mathbb{R}^2$. The HOMFLY-PT polynomial is an invariant of links which takes (oriented) link diagrams to elements of $\mathbb{Z}(q,a)$.

Definition 2.1. Let $D$ be a link diagram and let $O$ be a simple closed curve in the plane of the link diagram. We define the HOMFLY-PT polynomial, $P(D) \in \mathbb{Z}(q,a)$, via the following relations:

(1) $P(\emptyset) = 1$, $P(O) = \frac{1 + aq^2}{1 - q^2}$
(2) $P(D \sqcup O) = P(D)P(O)$
(3) $qP(D_+) - q^{-1}P(D_-) = (q - q^{-1})P(D_0)$, where $D_+, D_-$, and $D_0$ are link diagrams which are the same except in the neighborhood of a single point where $D_+ = \bigcirc$, $D_- = \bigcirc$, and $D_0 = \downarrow$. We will call a crossing which locally looks like $\bigcirc$ a positive crossing, and a crossing that locally looks like $\bigcirc$ a negative crossing. Let $f, g \in \mathbb{Z}(q, a)$ be nonzero. We will write $f \equiv g$ if $f = (-1)^ia^jq^kg$ for some $i, j, k \in \mathbb{Z}$. In other words, we write $f \equiv g$ if $f/g$ is a unit in $\mathbb{Z}[q^{\pm 1}, a^{\pm 1}]$.

**Theorem 2.2** (HOMFLY [6], PT [14]). Let $D$ and $D'$ be two link diagrams of a link $L$. Then $P(D) = \pm P(D')$. Furthermore, $P(D_2) = -q^{-2}P(D_1)$ and $P(D_3) = qa^2P(D_1)$, where $D_1, D_2,$ and $D_3$ are link diagrams which are the same except in the neighborhood of a single point where they are as in Figure 2.

We will often denote the HOMFLY-PT polynomial of a link by $P(L)$, suppressing the choice of link diagram. In this case $P(L)$ is well-defined up to a unit in $\mathbb{Z}[q^{\pm 1}, a^{\pm 1}]$.

![Figure 2. The diagrams $D_1, D_2,$ and $D_3$ from Theorem 2.2](image)

**Remark 2.3.** We are using non-standard conventions for the HOMFLY-PT polynomial in this text. The HOMFLY-PT polynomial as defined here is not a polynomial, but rather is a rational function. One may choose a different normalization where both $P(D) \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$ is honestly a (Laurent) polynomial and $P(D_1) = P(D_2) = P(D_3)$ where $D_1, D_2,$ and $D_3$ are as in Figure 2. The choice of normalization here coincides with our conventions for HOMFLY-PT homology in the sequel.

2.2. **The MOY construction of the HOMFLY-PT polynomial.** Murakami, Ohtsuki, and Yamada in [13] give a construction of the $sl_n$ polynomial, $P_n(L) \in \mathbb{Z}[q, q^{-1}]$, of a link $L$ using evaluations of oriented colored trivalent plane graphs. These trivalent plane graphs correspond to the intertwiners between tensor powers of fundamental representations of $U_q(sl_n)$. The $sl_n$ polynomial is actually a specialization of the HOMFLY-PT polynomial, that is $P_n(L)(q) = P(L)(q, a = q^{2-2n})$ (in our conventions). We may adjust the MOY construction of the $sl_n$ polynomial to compute the HOMFLY-PT polynomial. We will replace the “wide edge” graph of $[13]$ with a single degree 4 vertex (see Figure 3) which we will call a MOY vertex.

![Figure 3. The MOY wide edge graph and our MOY vertex](image)

The MOY state model of the HOMFLY-PT polynomial writes a link diagram as a formal $\mathbb{Z}(q, a)$-linear combination of planar, oriented, 4-regular graphs. The orientation locally at each vertex is the same as the orientation of the MOY vertex in Figure 3. We call such planar, oriented, 4-regular graphs MOY graphs.

We now define the MOY construction of the HOMFLY-PT polynomial. Let $D$ be a link diagram. We can resolve any crossing $c$ into either an oriented smoothing $\downarrow$ or a MOY vertex $\bigcirc$ (with consistent orientation). To each resolution of $c$ we associate a weight. If we smooth the crossing then the resolution has weight 0. If we replace the crossing with a MOY vertex, then the weight is $-2$ if the crossing was positive and 0 if the crossing was negative. A resolution chart is given in
Figure 4 for reference. We define a state $\sigma$ of $D$ as a choice of resolution for every crossing $c$ in $D$. If $D$ has $n$ crossings, then it has $2^n$ possible states. We define the weight of a state $\mu(\sigma)$ to be the sum of the weights of the chosen resolutions of $\sigma$. Finally we will set $\nu(\sigma)$ to be the number of MOY vertices in $\sigma$.

**Definition 2.4.** The MOY polynomial, $\bar{P}(D)$, is given by

$$\bar{P}(D) = \sum_{\sigma} (-1)^{\nu(\sigma)} q^{\mu(\sigma)} \bar{P}(D_{\sigma})$$

Where $\bar{P}(D_{\sigma})$ is the evaluation of the MOY graph given by the state $\sigma$ using the relations in Figure 4.

**Theorem 2.5** (Murakami-Ohtsuki-Yamada [13]). The relations given in Figure 4 are sufficient to compute $\bar{P}(D_{\sigma})$ as an element of $\mathbb{Z}(q,a)$ for any link diagram $D$ and any state $\sigma$. Furthermore, $\bar{P}(D) = P(D)$ for any link diagram.

**Example 2.6.** As an example we will work out $\bar{P}(D)$ for the diagram of the left-handed trefoil knot given in Figure 5. We resolve the diagram into its eight resolutions. Many of the resolutions give the same graph due to symmetries in the diagram. Therefore,

$$\bar{P}(D) = \bar{P}(\Gamma_0) - 3\bar{P}(\Gamma_1) + 3\bar{P}(\Gamma_2) - \bar{P}(\Gamma_3),$$

where the graphs $\Gamma_i$ are shown in Figure 5.
Using (MOY II), we see that $\bar{P}(\Gamma_2) = (1 + q^2) \bar{P}(\Gamma_1)$ and $\bar{P}(\Gamma_3) = (1 + q^2) \bar{P}(\Gamma_2) = (1 + q^2)^2 \bar{P}(\Gamma_1)$. Using (MOY I) and (MOY 0) we see that

$$\bar{P}(\Gamma_1) = 1 + aq^2 \frac{1}{1 - q^2} \bar{P}(O) = (1 + aq^2)(1 + aq^4) \frac{1}{(1 - q^2)^2}, \quad \bar{P}(\Gamma_0) = \left(1 + aq^2\right)^2.$$

Therefore, combining everything, we compute

$$\bar{P}(D) = \frac{1 + aq^2}{1 - q^2} \left(1 + aq^2 - \frac{3(1 + aq^4)}{1 - q^2} + \frac{3(1 + q^2)(1 + aq^4)}{1 - q^2} - \frac{(1 + q^2)^2(1 + aq^4)}{1 - q^2}\right)$$

which simplifies to

$$\bar{P}(D) = \left(1 + aq^2\right) \left(q^2 + aq^2 + aq^6\right). \tag{2.2}$$

3. HOMFLY-PT HOMOLOGY FOR CLOSED BRAID DIAGRAMS

In this section we introduce the construction of Khovanov and Rozansky’s HOMFLY-PT homology. The approach of this construction is to associate a chain complex of modules to every MOY graph and a bicomplex of modules to every link diagram. Our approach in this section is most similar to the approach of Rasmussen in [16] where we ignore his “$sl_n$” differential, as it is not needed in the construction of HOMFLY-PT homology.

3.1. Koszul complexes. Before introducing HOMFLY-PT homology, we recall some terminology and notation involving Koszul complexes. Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a $\mathbb{Z}$-graded commutative $\mathbb{Q}$-algebra and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a $\mathbb{Z}$-graded $R$-module. It will be instructive to keep the example of $R = \mathbb{Q}[x, y]$ in mind, where $x$ and $y$ are finite lists of variables (not necessarily of the same length). We define the grading shift functor $\bullet(k)$ by $M(k)_j = M_{j-k}$ for all $j \in \mathbb{Z}$. We will commonly use a nonstandard notation for grading shifts. In particular, we will set $q^k M := M(k)$ and say $\deg_q(x) = q^j$ if $x \in M_j$.

**Definition 3.1.** Let $p \in R$ be an element of degree $k$. The Koszul complex of $p$ is defined as the chain complex

$$[p]_R = q^k R_1 \xrightarrow{p} R_0,$$

where $p$ is used to denote the algebra endomorphism of $R$ given by multiplication by $p$. Here $R_0 = R_1 = R$ and the subscript is simply used to denote the homological degree of the module. We will often write $[p] = [p]_R$ when there can be no confusion. Now let $p = p_1, \ldots, p_k$ be a sequence of elements in $R$. Then we define the Koszul complex of $p$ as the complex

$$\left[\begin{array}{c}
[p_1] \\
\vdots \\
[p_k]
\end{array}\right] = [p_1] \otimes_R \cdots \otimes_R [p_k]$$

**Figure 5.** Diagram of the left-handed trefoil knot and its resolutions
where \( \otimes_R \) denotes the ordinary tensor product of chain complexes.

As a convention, we will call the homological grading in Koszul complexes the **Hochschild grading** and denote it by \( \deg_a \). We write \( \deg_a(x) = a^k \) to say that \( x \) is in Hochschild degree \( k \) and similarly write \( a^k M \) to denote that \( M \) is being shifted \( k \) in Hochschild degree.

We say a sequence of elements \( p = p_1, \ldots, p_k \) in \( R \) is a regular sequence if \( p_m \) is not a zero divisor in \( R/(p_1, \ldots, p_{m-1}) \) for all \( m = 1, \ldots, k \). The following proposition is a standard fact in homological algebra and is proven in many introductory texts such as [17].

**Proposition 3.2.** Let \( p = p_1, \ldots, p_n \) be a regular sequence in \( R \). Then the Koszul complex of \( p \) is a graded free \( R \)-module resolution of \( R/(p_1, \ldots, p_n) \).

The notation we use for Koszul complexes is reminiscent of the notation for a row vector in \( R^\oplus n \). Note that we will always use square brackets for Koszul complexes and round brackets for row vectors in \( R^\oplus n \) to eliminate any confusion. Along these lines, we can look at “row operations” on Koszul complexes.

**Proposition 3.3.** Let \( p = p_1, \ldots, p_k \) be a sequence of elements in \( R \), and let \( \lambda \in \mathbb{Q} \), then

\[
\begin{pmatrix}
\vdots \\
p_i \\
\vdots
\end{pmatrix} \otimes_R \begin{pmatrix}
\vdots \\
p_i + \lambda p_j \\
\vdots
\end{pmatrix} \cong \begin{pmatrix}
\vdots \\
\vdots \\
p_j
\end{pmatrix}
\]

A homotopy equivalence of this form will be called a change of basis.

**Proof.** We will omit grading shifts in the proof for clarity. We consider the map \( \Phi : [p_i] \otimes_R [p_j] \to [p_i + \lambda p_j] \otimes_R [p_j] \) given by

\[
\Phi \begin{pmatrix}
\vdots \\
[\lambda p_i] \\
\vdots
\end{pmatrix} = R \begin{pmatrix}
p_i & p_j
\end{pmatrix} \begin{pmatrix}
1 & \lambda \\
0 & 1
\end{pmatrix} \begin{pmatrix}
-p_j & p_i
\end{pmatrix}
\]

This map is clearly invertible. \( \square \)

### 3.2. Marked MOY graphs.

A **marked MOY graph** is a MOY graph \( \Gamma \) (possibly with boundary) with markings such that the marks partition the graph into some combination of **elementary MOY graphs** as shown in Figure 6. We label the marks and the endpoints of the graph (if any) with variables. Typically, though not necessarily, we will label outgoing edges by variables \( y_i \), incoming edges by variables \( x_i \), and internal marks by variables \( t_i \). An example of this process is given in Figure 6.

To a marked MOY graph \( \Gamma \), we will associate a collection of rings. Let \( x, y, t \) denote the lists of incoming, outgoing, and internal variables respectively. We first define the **total ring** of \( \Gamma \), \( E^t(\Gamma) \) as the polynomial ring \( \mathbb{Q}[x, y, t] \) containing all variables. We make this ring into a graded ring by setting \( \deg_q(x_i) = \deg_q(y_i) = \deg_q(t_i) = q^2 \). We call this grading the **internal** or **quantum grading**. We also suppose that all elements in \( E^t(\Gamma) \) have Hochschild degree \( a^0 \). The other rings we will
define will be subrings of $E^t(\Gamma)$. The edge ring, $E(\Gamma)$, is the polynomial ring of incoming and outgoing ("edge") variables $\mathbb{Q}[x, y]$. $E^t(\Gamma)$ has a natural (possibly infinite rank) free $E(\Gamma)$-module structure. We also define the incoming ring (resp. outgoing ring) by $E^i(\Gamma) = \mathbb{Q}[x]$ (resp. $E^o(\Gamma) = \mathbb{Q}[y]$). Since $E(\Gamma) \cong E^i(\Gamma) \otimes_{\mathbb{Q}} E^o(\Gamma)$ as $\mathbb{Q}$-algebras, then any $E(\Gamma)$-module can be considered as an $E^i(\Gamma)$-$E^o(\Gamma)$-bimodule. Note that if $\Gamma$ does not have any boundary (e.g. if it is a resolution of a link diagram), then $E(\Gamma) \cong E^i(\Gamma) \cong E^o(\Gamma) \cong \mathbb{Q}$. We list the four rings associated to $\Gamma$ in Figure 7 for easy reference.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Notation & Name & Ring & Variables Included \\
\hline
$E^t(\Gamma)$ & Total ring & $\mathbb{Q}[x, y, t]$ & incoming, outgoing, internal \\
$E(\Gamma)$ & Edge ring & $\mathbb{Q}[x, y]$ & incoming, outgoing \\
$E^i(\Gamma)$ & Incoming ring & $\mathbb{Q}[x]$ & incoming \\
$E^o(\Gamma)$ & Outgoing ring & $\mathbb{Q}[y]$ & outgoing \\
\hline
\end{tabular}
\caption{The four rings associated to a marked MOY graph $\Gamma$}
\end{table}

We will now define complexes $C(\Gamma)$ of free $E(\Gamma)$-modules associated to a marked MOY graph $\Gamma$. The chain modules of $C(\Gamma)$ will be direct sums of shifted copies of $E^t(\Gamma)$. We do this by first defining Koszul complexes associated to the elementary MOY graphs and then give rules for how gluing the graphs together affects the complexes associated to them. We will use the symbols $\hat{}$ and $\hat{x}$ to denote the elementary arc and vertex MOY graphs. To the arc, we associate the Koszul complex of $E^t(\hat{x}) = E(\hat{x}) = \mathbb{Q}[x, y]$-modules,

\begin{equation}
C(\hat{x}) = [y - x]_{E^t(\hat{x})} = q^2 a E(\hat{x}) \xrightarrow{y - x} E(\hat{x})
\end{equation}

and to the vertex graph, we associate the Koszul complex of $E^t(\hat{x}) = E(\hat{x}) = \mathbb{Q}[x_1, x_2, y_1, y_2]$-modules,

\begin{equation}
C(\hat{x}) = \left[ \frac{y_1 + y_2 - x_1 - x_2}{(y_1 - x_1)(y_1 - x_2)} \right]_{E^t(\hat{x})} = q^6 a^2 E(\hat{x}) \xrightarrow{A} q^4 a E(\hat{x}) \oplus q^2 a E(\hat{x}) \xrightarrow{B} E(\hat{x}),
\end{equation}

where

$$A = \left( \frac{y_1 + y_2 - x_1 - x_2}{(y_1 - x_1)(y_1 - x_2)} \right), \quad B = (y_1 - x_1)(y_1 - x_2) \quad y_1 + y_2 - x_1 - x_2$$

Now suppose $\Gamma$ is a marked MOY graph with edge ring $E$ and total ring $E^t$. Also let $\Gamma'$ be another marked MOY graph with edge ring $E'$ and total ring $E'^t$. The disjoint union of these graphs $\Gamma \sqcup \Gamma'$ has edge ring $E'' \cong E \otimes_{\mathbb{Q}} E'$ and total ring $E''^t \cong E^t \otimes_{\mathbb{Q}} E'^t$. To the marked MOY graph $\Gamma \sqcup \Gamma'$ we will associated the complex of $E''$-modules $C(\Gamma \sqcup \Gamma') := C(\Gamma) \otimes_{\mathbb{Q}} C(\Gamma')$. A picture of the corresponding diagram is shown in Figure 8.
Finally, we define a complex for when we glue two marked MOY graphs together. Let $\Gamma$ and $\Gamma'$ be two marked MOY graphs. We can glue outgoing edges of $\Gamma$ to incoming edges of $\Gamma'$ (or vice versa) to get a new marked MOY graph. First suppose only one endpoint from each graph are being glued together. Suppose that endpoint in both $\Gamma$ and $\Gamma'$ is labeled by the variable $z$ (that is, $z \in E^i(\Gamma) \cap E^i(\Gamma')$ or $z \in E^i(\Gamma') \cap E^i(\Gamma)$). Then we define the new graph $\Gamma \cup_z \Gamma'$ by identifying the endpoints labeled by $z$ and associate to $\Gamma \cup_z \Gamma'$ the complex

\[
C(\Gamma \cup_z \Gamma') := C(\Gamma) \otimes_{Q[z]} C(\Gamma')
\]

The edge ring of $\Gamma \cup_z \Gamma'$ is $E(\Gamma \cup_z \Gamma') = (E(\Gamma) \otimes_{Q[z]} E(\Gamma'))/(z)$ and the total ring is $E^t(\Gamma \cup_z \Gamma') = E^t(\Gamma) \otimes_{Q[z]} E^t(\Gamma')$. Note that after gluing, $z$ is no longer in the edge ring as it is an internal variable. We may glue multiple edges as once in a similar manner. If $z = z_1, \ldots, z_n$ are the variables at the marked endpoints being identified, then we define $C(\Gamma \cup_z \Gamma') := C(\Gamma) \otimes_{Q[z]} C(\Gamma')$. Similar to the case where we only identified one pair of edges, The edge ring of $\Gamma \cup_z \Gamma'$ is $E(\Gamma \cup_z \Gamma') = (E(\Gamma) \otimes_{Q[z]} E(\Gamma'))/(z_1, \ldots, z_n)$ and the total ring is $E^t(\Gamma \cup_z \Gamma') = E^t(\Gamma) \otimes_{Q[z]} E^t(\Gamma')$.

We can also describe disjoint union and gluing of marked MOY graphs in terms of Koszul complexes. Suppose $C(\Gamma)$ and $C(\Gamma')$ are given by the Koszul complexes

\[
C(\Gamma) = \begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix}_{E^t(\Gamma)} \quad \text{and} \quad C(\Gamma') = \begin{bmatrix} p'_1 \\ \vdots \\ p'_n \end{bmatrix}_{E^t(\Gamma')}
\]

We can present $C(\Gamma \sqcup \Gamma')$ and $C(\Gamma \cup_z \Gamma')$ as the following Koszul complexes:

\[
C(\Gamma \sqcup \Gamma') = \begin{bmatrix} p_1 \\ \vdots \\ p_m \\ p_{1'} \\ \vdots \\ p_{n'} \end{bmatrix}_{E^t(\Gamma \sqcup \Gamma')} \quad \text{and} \quad C(\Gamma \cup_z \Gamma') = \begin{bmatrix} p_1 \\ \vdots \\ p_m \\ p_{1'} \\ \vdots \\ p_{n'} \end{bmatrix}_{E^t(\Gamma \cup_z \Gamma')}
\]

Here the distinction comes from the difference in total and edge rings. $C(\Gamma \sqcup \Gamma')$ is a chain complex of free $E(\Gamma \sqcup \Gamma')$-modules with the chain modules as direct sums of shifted copies of $E^t(\Gamma \sqcup \Gamma')$. However $C(\Gamma \cup_z \Gamma')$ is a chain complex of free $E(\Gamma \cup_z \Gamma')$-modules with the chain modules as direct sums of shifted copies of $E^t(\Gamma \cup_z \Gamma')$. We now give another useful technique for simplifying the complexes associated to marked MOY graphs, called mark removal.

**Lemma 3.4.** Suppose that $z$ is an internal variable of a marked MOY graph $\Gamma$ and $C(\Gamma)$ is presented as a Koszul complex of the sequence $p = p_1, \ldots, z - p_1, \ldots, p_k$ where $p_1, \ldots, p_1, \ldots, p_k \in E(\Gamma)$. Let $\psi : E^t(\Gamma) \rightarrow$
be the quotient map identifying \( z \) with \( p_i \). Then, as complexes of \( E(\Gamma) \)-modules,

\[
C(\Gamma) \simeq \psi \left( \left[ \begin{array}{c} P_1 \\ \vdots \\ z - p_i \\ \vdots \\ P_k \end{array} \right] \right) \simeq \left[ \begin{array}{c} P_1 \\ \vdots \\ z - p_i \\ \vdots \\ P_k \end{array} \right] E^i(\Gamma)/(z - p_i)
\]

omitting the term \( z - p_i \) from the sequence.

Various forms of this lemma are proven in other texts on HOMFLY-PT homology, such as [12, 16]. We refer the reader to [16] for this exact form. Lemma 3.4 allows us to freely add in or remove marks without changing the homotopy type of the complex (as a complex of \( E(\Gamma) \)-modules). This implies the following very helpful statement.

**Corollary 3.5.** Let \( \Gamma \) and \( \Gamma' \) be two marked MOY graphs whose underlying (unmarked) MOY graphs are the same (isomorphic as oriented graphs). Then \( C(\Gamma) \simeq C(\Gamma') \) as complexes of \( E(\Gamma) = E(\Gamma') \)-modules.

**Example 3.6.** Consider the marked MOY graph from Figure 9. The marks partition the MOY graphs into six elementary MOY graphs (three MOY vertices and three arcs) which are drawn below in Figure 9.

![Figure 9](image_url)

**Figure 9.** The marked MOY graph in Example 3.6 and its elementary MOY graphs

We can write \( \Gamma \) as \( (\Gamma_1 \cup \Gamma_4) \cup_{t_1,t_2,t_3} (\Gamma_2 \cup \Gamma_5) \cup_{t_4,t_5,t_6} (\Gamma_3 \cup \Gamma_6) \), and therefore

\[
C(\Gamma) = C(\Gamma_1 \cup \Gamma_4) \otimes_{\mathbb{Q}[t_1,t_2,t_3]} C(\Gamma_2 \cup \Gamma_5) \otimes_{\mathbb{Q}[t_4,t_5,t_6]} C(\Gamma_3 \cup \Gamma_6)
\]

We can write \( C(\Gamma) \), after some applications of mark removal, as

\[
C(\Gamma) \simeq \left[ \begin{array}{c} y_1 + y_2 - t_1 - t_4 \\ y_1 y_2 - t_1 t_3 \\ t_4 + y_3 - t_2 - x_3 \\ t_4 y_3 - t_2 x_3 \\ t_1 + t_2 - x_1 - x_2 \\ t_1 t_2 - x_1 x_2 \end{array} \right]_{\mathbb{Q}[x_1,x_2,x_3,y_1,y_2,y_3,t_1,t_2,t_4]}
\]

We invite the reader to finish the process of removing the internal variables \( t_1, t_2 \) and \( t_4 \) to get a finite rank complex of \( \mathbb{Q}[x_1,x_2,x_3,y_1,y_2,y_3] \)-modules.
3.3. MOY braid graphs. A MOY braid graph is a graph formed by taking a braid and replacing every crossing with a MOY vertex, whose incoming and outgoing edges are consistent with the orientation of the braid. The complexes associated to MOY braid graphs and their “braid closures” satisfy the following local relations (as proven in [12, 16]):

**Proposition 3.7.** Let $\Gamma_0, \Gamma_{1a}, \Gamma_{1b}, \Gamma_{2a}, \Gamma_{2b}, \Gamma_{3a}, \Gamma_{3b}, \Gamma_{3c}, \Gamma_{3d}$ be MOY graphs as in Figure 10 below. Then

\[
\begin{align*}
C(\Gamma_0) & \simeq \bigoplus_{i=0}^{\infty} q^{2i}(Q \oplus aq^2Q) \\
C(\Gamma_{1a}) & \simeq \bigoplus_{i=0}^{\infty} q^{2i}(C(\Gamma_{1b}) \oplus aq^4C(\Gamma_{1b})) \\
C(\Gamma_{2a}) & \simeq C(\Gamma_{2b}) \oplus q^2C(\Gamma_{2b}) \\
C(\Gamma_{3a}) \oplus q^2C(\Gamma_{3b}) & \simeq q^2C(\Gamma_{3c}) \oplus C(\Gamma_{3d})
\end{align*}
\]

![Figure 10. MOY graphs for Proposition 3.7](image)

To compare the isomorphisms in Proposition 3.7 to the relations in Figure 4, we introduce the notation of a “Laurent series shift functor”. Suppose $F(q, a) \in \mathbb{N}[q^{\pm 1}, a^{\pm 1}]$, that is that $F(q, a) = \sum_{i,j \in \mathbb{Z}} c_{ij}q^ia^j$. Also suppose $M$ is a $\mathbb{Z} \times \mathbb{Z}$-graded $R$-module with grading shifts denoted by $q^ia^j$. Then

\[F(q, a)M := \bigoplus_{i,j \in \mathbb{Z}} q^ia^jM^{\oplus c_{ij}}.\]

With this notation in mind, we can rewrite the isomorphisms (3.5) and (3.6). First looking at (3.5):

\[C(\Gamma_0) \simeq \bigoplus_{i=0}^{\infty} q^{2i}(Q \oplus aq^2Q) = \bigoplus_{i=0}^{\infty} q^{2i}(1 + aq^2)Q = (1 + aq^2) \sum_{i=0}^{\infty} q^{2i}Q = \frac{1 + aq^2}{1 - q^2}Q.\]

Next we consider the case of (3.6):

\[C(\Gamma_{1a}) \simeq \bigoplus_{i=0}^{\infty} q^{2i}(C(\Gamma_{1b}) \oplus aq^4C(\Gamma_{1b})) = (1 + aq^4) \sum_{i=0}^{\infty} q^{2i}C(\Gamma_{1b}) = \frac{1 + aq^4}{1 - q^2}C(\Gamma_{1b}).\]

We invite the reader to compare the rewritten relations (3.10) and (3.11) to (MOY 0) and (MOY 1) from Figure 4. The same comparison can be made for (3.7) and (3.8) to (MOY 2a) and (MOY 3).
3.4. Khovanov-Rozansky HOMFLY-PT homology. We now have all of the necessary tools to define Khovanov-Rozansky HOMFLY-PT homology, or briefly HOMFLY-PT homology. We first define two $q$-degree 0 maps $\chi_i : \{ \} \rightarrow q^{-2} Q$ and $\chi_o : Q \rightarrow \{ \}$. Set $E = \mathbb{Q}[x_1, x_2, y_1, y_2]$ to be the edge ring of both $\{ \}$ and $Q$. Then we define $\chi_i$ and $\chi_o$ by

$$
\chi_i : \{ \} \rightarrow a^2q^4E \xrightarrow{(y_1-x_1 \ y_2-x_2)} aq^2E \oplus aq^2E \xrightarrow{(x_2-y_2 \ y_1-x_1)} E
$$

(3.12)

$$
\chi_o : Q \rightarrow a^2q^6E \xrightarrow{(y_1-x_1 \ y_1+y_2-x_1-x_2 \ y_1+y_2-x_1-x_2)} aE \oplus aq^2E \xrightarrow{(x_1+x_2-y_2-y_1 \ y_1-x_1 \ y_1-x_1) \ y_1-x_1} E
$$

(3.13)

We now define two bicomplexes of free $E$-modules for the positive crossing $q^{-2} \times$ and the negative crossing $\times$:

$$
C(q^{-2} \times) := C(\{ \}) \xrightarrow{\chi_i} tq^{-2}C(Q)
$$

(3.14)

$$
C(\times) := t^{-1}C(Q) \xrightarrow{\chi_o} C(\{ \})
$$

(3.15)

Note that in the above diagram we use the notation $t^kC(\Gamma)$ to mean that the complex for $\Gamma$ sits in homological degree $k$. This is a different homological degree than our Hochschild degree we introduced earlier. We will simply denote this degree by $\text{deg}_q$ and call it the homological degree. We will say that $x$ has (total) degree $\text{deg}(x) = q^i a^j b^k$ if it has quantum degree $i$, Hochschild degree $k$, and homological degree $j$. We will denote the differential in the complexes for the MOY graphs as $d_q$ and the differentials in the complexes (built from $\chi_i$ and $\chi_o$) associated to crossings as $d_c$. Both $C(q^{-2} \times)$ and $C(\times)$ are bicomplexes with commuting differentials $d_q$ and $d_c$. We define rings associated to each marked tangle diagram $\tau$ in a similar manner to our constructions for marked MOY graphs. We do not discuss this further but reference the reader to the table in Figure 11.

| Notation | Name            | Ring           | Variables Included          |
|----------|-----------------|----------------|-----------------------------|
| $E^I(\tau)$ | Total ring     | $\mathbb{Q}[x, y, t]$ | incoming, outgoing, internal |
| $E(\tau)$    | Edge ring       | $\mathbb{Q}[x, y]$  | incoming, outgoing          |
| $E^I(\tau)$ | Incoming ring   | $\mathbb{Q}[x]$    | incoming                     |
| $E^o(\tau)$  | Outgoing ring   | $\mathbb{Q}[y]$    | outgoing                     |

**Figure 11.** Rings associated to a marked tangle diagram $\tau$

We can now build a bicomplex for any tangle diagram (and link diagram) in a similar manner to what we did in Section 3.3 for MOY graphs. To a disjoint union of (marked) tangles $\tau = \tau_1 \sqcup \tau_2$ we
associate the bicomplex of $E(\tau)$-modules,

$$C(\tau) := C(\tau_1) \otimes_{\mathbb{Q}} C(\tau_2)$$

Where the tensor product of bicomplexes is defined by considering a bicomplex as a complex of complexes and then applying the tensor product of chain complexes. Similarly if we are gluing two tangles $\tau_1$ and $\tau_2$ at the marked points $z = z_1, \ldots, z_k$ in such a way that the orientations are consistent, then we define a bicomplex of $E(\tau_1 \cup_z \tau_2)$-modules

$$C(\tau_1 \cup_z \tau_2) = C(\tau_1) \otimes_{\mathbb{Q}[z]} C(\tau_2)$$

We omit the rest of the details in this case, and leave it to the reader to compare with the analogous conventions for marked MOY graphs. Now let $\beta \in \text{Br}_n$ be a braid with $n$ strands. We can mark $\beta$ in such a way that we partition it into arcs and crossings of the form $\backslash /$ or $/ \backslash$ and we label the endpoints and markings in a similar manner to our conventions for marked MOY graphs. Therefore we can use the rules of disjoint unions and gluing of tangles to write a bicomplex $C(\beta)$ of $E(\beta)$-modules. In this case, $E(\beta) = \mathbb{Q}[x, y]$ where $|x| = |y| = n$.

We now describe the construction of the HOMFLY-PT homology of a link $L$. Suppose that $\beta \in \text{Br}_n$ is a braid representative of $L$, that is $L$ is the circular closure of $\beta$ in $\mathbb{R}^3$. Then we can describe the bicomplex $C(L_\beta) = C(\beta) \otimes_{\mathbb{Q}[x, y]} C(1_n)$, where $1_n$ denotes the identity braid (oriented downwards) with the top endpoints labeled by $x$ and the bottom endpoints labeled by $y$. Here we use the notation $L_\beta$ to remember the choice of braid representative. We refer the reader to Figure 12 for an example of this decomposition of a braid closure.

**Figure 12.** A link presented as a braid closure and the constituent tangles.

**Definition 3.8.** Suppose $L$ is a link with braid representative $\beta \in \text{Br}_n$. The HOMFLY-PT homology of $L_\beta$ is $\mathcal{H}(L_\beta) = H_{d,e}(H_{d_\alpha}(C(L_\beta)))$.

**Remark 3.9.** $\mathcal{H}(L_\beta)$, as defined above, arises as the $E^2$-page of a spectral sequence. Let $L$ be an $n$-component link. It is easily shown that the $E^\infty$-page of that spectral sequence is the homology of the $n$-component unlink (up to a grading shift). In particular, $H_{d,e}(C(L))$ is isomorphic to the $E^\infty$-page.

**Theorem 3.10 ([12]).** Suppose $\beta \in \text{Br}_n$ and $\beta' \in \text{Br}_{n'}$ are two braid representatives of a link $L$. Then $\mathcal{H}(L_\beta) \cong \mathcal{H}(L_{\beta'})$ up to a grading shift. Furthermore, suppose the Poincaré series (see (1.1)) of $\mathcal{H}(L_\beta)$ is given by

$$P(L_\beta) = \sum_{i,j,k \in \mathbb{Z}} d_{i,j,k} q^i a^j t^k,$$

where $d_{i,j,k} = \dim_{\mathbb{Q}}(\mathcal{H}(L_\beta))_{i,j,k}$

then

$$P(L_\beta)|_{t=-1} = \sum_{i,j,k \in \mathbb{Z}} d_{i,j,k} q^i a^j (-1)^k = P(L_\beta)$$
4. HOMFLY-PT HOMOLOGY FOR GENERAL LINK DIAGRAMS

In this section we study what happens when we consider general link diagrams in the construction of HOMFLY-PT homology. We will see that not all Reidemeister moves are respected, and that in general HOMFLY-PT homology is only an invariant up to braidlike isotopy.

4.1. Virtual crossings and marked MOY graphs. We start by introducing virtual crossings into the framework of (marked) MOY graphs. We will not fully discuss virtual knot theory here, but rather refer the reader to Kauffman [9]. Virtual crossings were first considered as a tool in HOMFLY-PT and $sl(n)$ homologies by Khovanov and Rozansky in [11], and studied further by the author and Rozansky in [1].

A virtual MOY graph is a MOY graph where we allow the underlying graph to be nonplanar. Such a graph can always be drawn where the intersections forced by the projection onto the plane are transverse double points. An example of this is given in Figure 13. To the marked virtual crossing graph we associate the following complex of free $E(\mathfrak{g}) = \mathbb{Q}[x_1, x_2, y_1, y_2]$-modules:

\[
C(\mathfrak{g}) = \left[\begin{array}{c}
 y_1 - x_2 \\
 y_2 - x_1
\end{array}\right] E(\mathfrak{g}) = q^4 E(\mathfrak{g}) \xrightarrow{A} q^2 E(\mathfrak{g}) \oplus q^2 E(\mathfrak{g}) \xrightarrow{B} E(\mathfrak{g})
\]

where

\[
A = \left(\begin{array}{c}
 y_1 - x_2 \\
 y_2 - x_1
\end{array}\right), \quad B = (x_1 - y_2 \quad y_1 - x_2).
\]

Note that $C(\mathfrak{g})$ resembles $C(\mathfrak{g}')$ except for a transposition of $x_1$ and $x_2$ in the definition of the complexes. In this sense, we can think of a virtual crossing as being a permutation of strands with no additional crossing data or vertex at the intersection.

**Figure 13.** A (marked) virtual crossing and an example of a virtual MOY graph

**Proposition 4.1.** The moves in Figure 14 preserve the homotopy equivalence type of $C(\Gamma)$.

**Proof.** We shall not prove all of the isomorphisms due to the similarity of their proofs. We shall prove (VMOY2a) and leave the rest of the proofs to the reader. The left hand side of (VMOY2a) is presented as the Koszul complex of the sequence $y_1 - t_1, y_1 - t_2, t_1 + t_2 - x_1 - x_2, (t_1 - x_1)(t_1 - x_2)$. Let $w = y_1 + y_2 - x_1 - x_2$, then

\[
\left[\begin{array}{c}
 y_2 - t_1 \\
 y_1 - t_2 \\
 t_1 + t_2 - x_1 - x_2 \\
 (t_1 - x_1)(t_1 - x_2)
\end{array}\right]_{\mathbb{Q}[x,y,t]} \cong \left[\begin{array}{c}
 y_2 - t_1 \\
 y_1 - t_2 \\
 t_1 - x_1 \\
 (t_1 - x_1)(t_1 - x_2)
\end{array}\right]_{\mathbb{Q}[x,y,t]} \cong \left[\begin{array}{c}
 y_1 - t_2 \\
 w \\
 (y_1 - x_1)(y_1 - x_2)
\end{array}\right]_{\mathbb{Q}[x,y,t]} \cong \left[\begin{array}{c}
 w \\
 (y_1 - x_1)(y_1 - x_2)
\end{array}\right]_{\mathbb{Q}[x,y]}.
\]
The first isomorphism is a change of basis and the other three isomorphisms are mark removals. The last term is the Koszul complex for $C(\text{VMOY2a})$. The second isomorphism in (VMOY2a) is proven by a similar argument. □

For any virtual link diagram $D$, that is a link diagram with virtual crossings, we can repeat the procedure from Section 3.4 to build a bicomplex of $E(D)$-modules. We now record the additional “virtual” Reidemeister moves.

**Proposition 4.2.** The moves in Figure 15 preserve the homotopy equivalence type of $C(D)$. The isomorphisms (VR1),(VR2a),(VR2b), (VR3) and (SVR) are called virtual Reidemeister moves, and the isomorphisms (Z1±) and (Z2±) are called Z-moves.

**Figure 15.** Virtual Reidemeister moves and Z-moves
The proofs of (VR1), (VR2a), (VR2b), and (VR3) are similar to the proof presented for (VMOY2a) in Proposition 4.1. The proofs for (SVR), and the Z-moves are included in [1]. They are easy to see after resolving the crossings in terms of MOY graphs and using the virtual MOY moves.

4.2. Failure of Reidemeister IIb for $\mathcal{H}(D)$. Now with the introduction of virtual crossings, we can study the exact outcome of allowing general link diagrams in the computation of $\mathcal{H}(D)$. We first look at the analogue of (MOYIIb) from Figure 4.

**Lemma 4.3.** Let $\tilde{C}(\downarrow) = \frac{q^2 + aq}{1 - q^2} C(\downarrow)$, then the following diagram is commutative:

$$
\begin{array}{c}
\tilde{C}(\downarrow) \otimes 1 \\
\downarrow \cong \\
C(\downarrow) \oplus \tilde{C}(\downarrow) \\
\downarrow \cong \\
C(\downarrow)
\end{array}
$$

where $\iota$ includes $\tilde{C}(\downarrow)$ as a subcomplex of $C(\downarrow)$, $\pi$ projects $C(\downarrow)$ onto its quotient complex $C(\downarrow)$ and $*$ represents some map from $\tilde{C}(\downarrow)$ to $C(\downarrow)$.

We prove Lemma 4.3 in Section A.3 using the framework of virtual crossing filtrations from [1].

We now state the main result of this section.

**Theorem 4.4.** The Reidemeister IIb isomorphism fails: $C(\downarrow) \not\cong C(\downarrow)$. In particular, $C(\downarrow) \simeq tq^{-2}C(\downarrow)$.

**Proof.** We first write $C(\downarrow)$ using (3.14) and (3.15),

$$C(\downarrow) = t^{-1}C(\downarrow) \xrightarrow{(1 \otimes \chi_0)} C(\downarrow) \oplus q^{-2}C(\downarrow) \xrightarrow{(\chi_i \otimes 1 - 1 \otimes \chi_0)} t^{-2}C(\downarrow)$$

We use the isomorphisms from Proposition 3.7 and Lemma 4.3 to rewrite the complex as

$$t^{-1}q^{-2}\tilde{C}(\downarrow) \xrightarrow{(1 \otimes \chi_0)} q^{-2}\tilde{C}(\downarrow) \oplus C(\downarrow) \oplus q^{-2}C(\downarrow) \xrightarrow{(\star 1 - 1 \otimes \chi_0)} tC(\downarrow)$$

where the maps $\star$ are irrelevant to our discussion. After performing Gaussian elimination (see Section A.1) on the complex above, we are left with

$$C(\downarrow) \simeq q^{-2}C(\downarrow) \xrightarrow{1 \otimes \chi_0} t^{-2}C(\downarrow) \simeq t^{-2}C(\downarrow) \otimes (t^{-1}C(\downarrow) \xrightarrow{\chi_0} C(\downarrow)).$$

Where the tensor product above is over the appropriate ring of internal variables. □

**Example 4.5.** We now give an example where the local failure of Reidemeister IIIb gives a failure of isotopy invariance for a certain link diagram. Let $D$ be the diagram for the unknot given in Figure 16 and let $O$ denote the standard diagram of the unknot as a circle bounding a disc in $\mathbb{R}^2$. Then we have the following chain of isomorphims

$$\mathcal{H}(D) \simeq t^2q^{-4}\mathcal{H}(D') \simeq t^2q^{-4}\mathcal{H}(D'') \simeq t^2q^{-4}\mathcal{H}(D''').$$
The first isomorphism is given by applying Theorem 4.4 twice. The second isomorphism following from applying \((\mathbb{Z}_2 -)\) from Proposition 4.1. The last isomorphism follows from applying \((\text{VR2b})\) from Proposition 4.1. \(D''\) is the diagram of the left-handed trefoil knot, and we computed \(P(D'')\) in Example 2.6. This is enough to show that \(\mathcal{H}(D) \not\simeq \mathcal{H}(O)\), however it is an easy exercise to show that
\[
\mathcal{H}(D'') = (aq^2 + t^2q^2 + t^2aq^4)\frac{1 + aq^2}{1 - q^2} \not\simeq \frac{1 + aq^2}{1 - q^2} \simeq \mathcal{H}(O).
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{failure_of_reidemeister_IIb.png}
\caption{The failure of Reidemeister IIb for an unknot diagram. The above diagrams all have the same HOMFLY-PT homology up to a grading shift.}
\end{figure}

4.3. **Braidlike isotopy and \(\mathcal{H}(L)\).** As we saw in Example 4.5, HOMFLY-PT homology for general link diagrams is *not* an isotopy invariant. However, it was proven in [12] that it was a *braidlike isotopy* invariant. We now carefully recall the definition of braidlike isotopy

**Definition 4.6.** Two link diagrams \(D\) and \(D'\) are said to represent *braidlike isotopic* links if they differ by a sequence of planar isotopies and the following moves in Figure 17. Such a sequence of moves will be called a *braidlike isotopy*.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{braidlike_reidemeister_moves.png}
\caption{Braidlike Reidemeister moves}
\end{figure}

**Theorem 4.7** (Khovanov-Rozansky[12]). Let \(D\) and \(D'\) be two braidlike isotopic link diagrams, then \(\mathcal{H}(D) \simeq \mathcal{H}(D')\).
Braidlike isotopy is an important notion in studying links in the solid torus. It is a well-known fact that two braid closures in the solid torus give isotopic links if and only if the braids are equivalent, or rather if they are braidlike isotopic. Audoux and Fiedler in [3] give a deformation of Khovanov homology which detects braidlike isotopy of links in $\mathbb{R}^3$. Their invariant decategorifies to a deformation of the Jones polynomial which can be computed using a Kauffman bracket-like relation. In the case of closed braid diagrams, their invariant corresponds with the homology theory studied in [2] and the decategorification corresponds with the polynomial invariant studied in [7].

We can now interpret Example 4.5 in the following manner: The unknot diagram $D$ shown in Figure 16 is isotopic, but not braidlike isotopic to the standard unknot diagram $O$. In particular, there is no sequence of Reidemeister moves transforming $D$ to $O$ which does not contain the Reidemeister IIb move. In this sense, we see that $H(D)$ can detect nonbraidlike isotopy. Note that $H(D)$ is isomorphic to the homology of the left-handed trefoil knot (after two negative Reidemeister I moves), though clearly $D$ is not a diagram for the left-handed trefoil knot. Therefore, this viewpoint of $H(D)$ may not be useful in determining isotopy type of general diagrams, but can be very useful when we know the two diagrams are of the same isotopy type and we wish to determine if they are of the same braidlike isotopy type.

5. Decategorification of $H(D)$ for general link diagrams

In this section we study the decategorification of $H(D)$, which we will denote as $P^b(D)$. As we saw in Example 4.5, $P^b(D) \neq P(D)$ in general. In particular, when this occurs, this implies that $D$ is not braidlike isotopic to a closed braid presentation of a link. We will end this section with a note on virtual links and give an explicit example of where $P^b(D)$ is not invariant under the virtual exchange move, which implies that $H(D)$ cannot be extended to a virtual link invariant.

5.1. A deformation of the HOMFLY-PT polynomial. Let $D$ be a link diagram, then we will define our deformed HOMFLY-PT polynomial as

$$P^b(D) = P(D)|_{t=-1} \in \mathbb{Z}(q,a).$$

**Theorem 5.1.** Let $D$ and $D'$ be two link diagrams which are braidlike isotopic. Then $P^b(D) = P^b(D')$. $P^b$ satisfies the following skein relation:

$$q P^b(\raisebox{-1pt}{\includegraphics{figure1.png}}) - q^{-1} P^b(\raisebox{-1pt}{\includegraphics{figure2.png}}) = (q-q^{-1}) P^b(\raisebox{-1pt}{\includegraphics{figure3.png}}).$$

**Proof.** The first statement is an immediate corollary of Theorem 4.7. For the second part of the statement, note that we have a map $\psi: tq^{-1}C(\raisebox{-1pt}{\includegraphics{figure4.png}}) \to qC(\raisebox{-1pt}{\includegraphics{figure5.png}})$ of homological degree 1 given by

$$\psi: \begin{array}{c}
tq^{-1}C(\raisebox{-1pt}{\includegraphics{figure4.png}}) \xrightarrow{\chi_0} \xrightarrow{\chi_i} \to qC(\raisebox{-1pt}{\includegraphics{figure5.png}}),
\end{array}$$

The mapping cone of $\psi$, after Gaussian elimination, is homotopy equivalent to

$$qC(\raisebox{-1pt}{\includegraphics{figure6.png}}) \xrightarrow{\chi_0 \chi_i} tq^{-1}C(\raisebox{-1pt}{\includegraphics{figure7.png}}).$$
and therefore

\[(5.3) \quad \text{Cone}(tq^{-1}C(\bigotimes) \xrightarrow{\psi} C(\bigotimes)) \simeq (qC(\bigotimes) \xrightarrow{\chi} tq^{-1}C(\bigotimes))\]

Properties of Poincaré series and (5.3) gives us the relation (5.2) as desired. □

**Corollary 5.2.** If \(D\) is a link diagram presented as a braid closure, then \(P^b(D) = P(D)\). Equivalently if \(D\) is a link diagram for some link \(L\) and \(P^b(D) \neq P(D)\), then \(D\) is not braidlike isotopic to a braid presentation of \(L\).

We can also give a MOY-style construction for \(P^b(D)\). In particular, we address the relation (MOY IIb) from Figure 4. As we saw in Lemma 4.3, the categorified MOY IIb relation does not hold as we would expect. However, we can decategorify the results in Proposition 3.7, Proposition 4.1, Theorem 4.4, and Lemma 4.3 in a natural way.

**Proposition 5.3.** Let \(D\) be a link diagram. The relations in Figure 18 hold for \(P^b(D)\).

\[\begin{align*}
\bigotimes = 1 & \quad \bigotimes = 1 - \bigotimes \\
\bigcirc = \frac{1 + aq^2}{1 - q^2} & \quad \bigotimes = \frac{1 + aq^4}{1 - q^2} & \quad \bigotimes = (1 + q^2) \\
\bigotimes = q^2 + \frac{aq^6}{1 - q^2} & \quad q^2 + \frac{aq^7}{1 - q^2} + \bigotimes = \bigotimes
\end{align*}\]

(Figure 18. MOY relations for \(P^b(D)\) and the deformed Reidemeister IIb relation. The notation \(P^b(\cdot)\) is omitted for readability.

**Example 5.4.** Let \(D\) be the diagram of the \((2, 2k + 1)\)-torus knot given in Figure 19. First note that we can transform \(D\) to \(D'\) using (RIIb') so that \(P^b(D) = q^{-4}P^b(D')\). \(D'\) is isotopic to the \((2, 2k - 1)\)-torus knot via a planar isotopy and a (braidlike) Reidemeister IIa move. Therefore via a straightforward calculation similar to Example 2.6,

\[P^b(D) = \frac{1 + aq^2}{1 - q^2} \left( (a + 1) \sum_{i=1}^{k-1} q^{-i} + q^{-4k} \right) .\]

However, if \(\tilde{D}\) is a braid presentation of the \((2, 2k + 1)\)-torus knot, then

\[P^b(\tilde{D}) = \frac{1 + aq^2}{1 - q^2} \left( (a + 1) \sum_{i=1}^{k} q^{-i-2} + q^{-4k-2} \right) .\]

Therefore \(D\) is not braidlike isotopic to a braid presentation of the \((2, 2k + 1)\)-torus knot for all \(k \geq 1\).
5.2. Obstruction to extending HOMFLY-PT homology to virtual links. Finally we wish to present an argument showing that the current definition of $H(D)$ cannot be extended to virtual links, even when they are presented as closures of virtual braids. We will ultimately use $P^b(D)$ to justify this statement.

A virtual braid is a braid in which we allow virtual crossings alongside positive and negative crossings. Two virtual braids are said to be equivalent if they differ by the moves from Figure 17 and the moves (VR2a), (VR3), and (SVR) from Figure 15. Kauffman in [9] proves that every virtual link can be presented as the closure of a virtual braid. There is also an analogue of the Markov theorem for virtual links.

**Theorem 5.5 (Kamada [8]).** Let $\beta$ and $\beta'$ be two virtual braids. Their braid closures are equivalent virtual links if and only if they differ by a sequence of virtual braid equivalence moves (the braidlike Reidemeister moves, (VR2a), (VR3), and (SVR)), the Markov moves, and the virtual exchange move. The Markov moves and virtual exchange move are pictured in Figure 20.

Now we show by example that $P^b(D)$ is not invariant under the virtual exchange move. Therefore $H(D)$ is not an invariant of virtual links, even when the links are presented as virtual braid closures.

**Example 5.6.** Let $L$ be a connected sum of two virtual Hopf links as shown in Figure 21. $\beta_1$ and $\beta_2$ are two virtual braids whose closures are equivalent as virtual links to $L$. In particular, $\beta_1$ and
$\beta_2$ are related by Markov move I and the virtual exchange move shown in Figure 20. Using the relations from Figure 18 we can directly compute that

\[
P^b(D_1) = \frac{1 + aq^2}{1 - q^2} \left( 1 - q^{-2} \left( \frac{1 + aq^4}{1 - q^2} \right) \right)^2
\]

\[
P^b(D_2) = \frac{1 + aq^2}{1 - q^2} \left( aq^2 + 2 \left( \frac{1 + aq^4}{1 - q^2} \right) - q^{-2} \left( \frac{1 + aq^4}{1 - q^2} \right)^2 \right)
\]

It is easy to see that $P^b(D_2) \neq P^b(D_1)$. In particular,

\[
P^b(D_2) - P^b(D_1) = \frac{q^2(1 + a + aq^2 + a^2q^2)}{1 - q^2}
\]

Therefore, $P^b(D)$ is not an invariant of virtual links and thus neither is $H(D)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure21.png}
\caption{A connected sum of two virtual Hopf links, $L$, and two braid presentations of $L$.}
\end{figure}

Appendix A. Virtual filtrations on HOMFLY-PT homology

In this appendix we present the necessary tools to understand our application of virtual crossing filtrations from [11]. We begin by reviewing relevant tools from homological algebra and then introduce virtual crossing filtrations. We end the appendix by presenting a proof of Lemma 4.3.

A.1. Homological algebra of twisted complexes. For this section we assume that all chain complexes are chain complexes of objects in some additive category. The main examples to keep in mind are $R$-mod, the category of $R$-modules, and its homotopy category $K(R$-mod).

Definition A.1. A twisted complex $C_\bullet$ is a collection of chain complexes $C_i$ with maps $f_{ji} : C_i \to C_j$ for all $i, j \geq 0$ such that

1. Only finitely many of the $C_i$ are nonzero.
2. $f_{ji} \in \bigoplus_{k \in \mathbb{Z}} \text{Hom}(C_{i,k+(j-i)}, C_{j,k})$ if $j > i$ and $f_{ji} = 0$ otherwise.
3. $d_{C_j}f_{ji} + (-1)^{j-i+1}f_{ji}d_{C_i} = \sum_{j > k > i} f_{jk}f_{ki}$.

We will say the length of a map $f_{ji}$ in a twisted complex is the number $j - i$. From condition (3) we can see that maps of length one are chain maps, and that the composition of consecutive length one maps $f_{j+1,j}f_{j,j-1}$ is homotopic to the zero map with homotopy $f_{j+1,j-1}$. Similar statements can be deduced for any length map.
Definition A.2. Let $C_\bullet$ be a twisted complex such that $C_k = 0$ for all $k > n+1$, the convolution $\text{Con}(C_\bullet)$ is the chain complex which is isomorphic to $\bigoplus_{i \geq 0} C_i[i]$ as a module with differential

$$d_{\text{Con}(C_\bullet)} = \begin{pmatrix} dC_0 & 0 & \cdots & \cdots & 0 \\ f_{10} & -dC_1 & 0 & \cdots & 0 \\ f_{20} & f_{21} & dC_2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ f_{n0} & f_{n1} & \cdots & f_{nk} & (−1)^n dC_n \end{pmatrix}.$$  

Note that $d_{\text{Con}(C_\bullet)}^2 = 0$ by condition (3) of Definition A.1. We invite the reader to check that in the case that $n = 1$, $\text{Con}(C_\bullet) \simeq \text{Cone}(f_{10})$, and in the case that $n = 2$ that $\text{Con}(C_\bullet)$ is homotopy equivalent to the iterated mapping cone $\text{Cone}(\text{Cone}(f_{10}) \xrightarrow{f_{20}+f_{21}} C_2[1]) \simeq \text{Cone}(C_0 \xrightarrow{f_{20}+f_{10}} \text{Cone}(f_{21}))$.

We also recall a couple of facts which will be useful in simplifying twisted complexes. Recall a deformation retract $Ψ : C \to C'$ is a map $Ψ$ such that there exists a triple $(Ψ : C \to C', Ψ' : C' \to C, ψ : C \to C)$ where $Ψ'$ is a chain map such that

1. $Ψ'Ψ = dCψ + ψdC + IdC$
2. $ΨΨ' = IdC'$
3. $Ψψ, ψΨ'$ and $ψψ$ are zero maps.

We will often refer to the triple $(Ψ, Ψ', ψ)$ as SDR data for $Ψ : C \to C'$. We now recall a special type of deformation retract.

Proposition A.3. Consider the complex

$$A \xrightarrow{(α)} B \oplus C \xrightarrow{(φ \lambda \mu ν)} D \oplus E \xrightarrow{(ε)} F,$$

where $φ : B \to D$ is an isomorphism and all other maps are arbitrary up to the condition that $d^2 = 0$. Then there exists a deformation retract

$$A \xrightarrow{(α)} B \oplus C \xrightarrow{(φ^{-1} \lambda \mu ν)} D \oplus E \xrightarrow{(ε)} F$$

We call this deformation retract Gaussian elimination.

A proof of this proposition can be found in other texts on link homology such as [5]. We now consider the effect of applying deformation retracts to constituent complexes in a twisted complex. We write the following fact in the case that the original twisted complex has only maps of length 1, but the result can be extended to the general case.

Proposition A.4. Let $C_i$ be chain complexes for $i = 1, \ldots, n$ and suppose that $(Ψ_i, Ψ'_i, ψ_i)$ is SDR data for the deformation retract $Ψ_i : C_i \to C'_i$. Suppose that $C_\bullet = (C_i, f_{ji})$ is a twisted complex such that $f_{ji} = 0$.
for \( j - i \geq 2 \). Then there exist maps \( f_{ji} : C_i \to C_j \) such that \( C_i = (C_i, f_{ji}) \) is a twisted complex and there exists a deformation retract \( \nabla : \text{Con}(C_i) \to \text{Con}(C_j) \). Explicitly, the maps \( f_{ji} \) are given by the formula

\[
f_{ji} = \Psi_j f_{j,j-1} \psi_{j-1} f_{j-1,j-2} \psi_{j-2} \cdots f_{i+1,i} \psi_i.
\]

A.2. Virtual filtration. Recall the complexes \( C(\cdot, \cdot), C(\mathcal{X}), \) and \( C(\mathcal{X}') \) given by

\[
C(\cdot, \cdot) = \left[ \begin{array}{c} y_1 - x_1 \\ y_2 - x_2 \end{array} \right]_E, \quad C(\mathcal{X}) = \left[ \begin{array}{c} y_1 - x_2 \\ y_2 - x_1 \end{array} \right]_E, \quad C(\mathcal{X}') = \left[ \begin{array}{c} y_1 + y_2 - x_1 - x_2 \\ (y_1 - x_1)(y_1 - x_2) \end{array} \right]_E,
\]

where \( E = \mathbb{Q}[x_1, x_2, y_1, y_2] \).

Definition A.5. Let \( R \) be a commutative ring and let \( C \) and \( D \) be chain complexes of objects in an additive category. Let \( \bullet[i] \) denote the homological shift functor given by \( C_{j+i}[i] = C_j \). Define \( \text{Hom}^k(C, D) \) to be the \( \mathbb{Z} \)-module of chain maps \( f : C \to D[-k] \) quotiented by the submodule of chain maps homotopic to the zero map.

In [1] Khovanov and Rozansky make the following observation.

Proposition A.6. There exists a unique map \( F \in \text{Hom}^1(C(\cdot, \cdot), q^2C(\mathcal{X})) \), up to rescaling, such that \( \text{Cone}(F) \) is homotopy equivalent to \( C(\mathcal{X}) \). Likewise there exists a unique map, up to rescaling, \( G \in \text{Hom}^1(C(\mathcal{X}), q^2C(\cdot, \cdot)) \) such that \( \text{Cone}(G) \) is homotopy equivalent to \( C(\mathcal{X}') \).

We will call the maps \( F \) and \( G \) virtual saddle maps. In our presentation of \( C(\cdot, \cdot) \) and \( C(\mathcal{X}) \) we can write the virtual saddle maps explicitly. We write \( G \) below and leave it to the reader to write the analogous map \( F \).

\[
a^2q^4E \xrightarrow{a \rightarrow \psi} aq^2E \oplus aq^2E \xrightarrow{\phi} aq^2E \oplus aq^2E \xrightarrow{\psi} q^2E.
\]

Figure 22. An explicit presentation of the virtual saddle map \( G \).

The mapping cone presentations give rise to filtrations. In particular, \( \text{Cone}(F) \) has \( q^2C(\mathcal{X}) \) as a submodule and \( C(\cdot, \cdot) \) as the quotient \( \text{Cone}(F)/q^2C(\mathcal{X}) \). We call this filtration the negative filtration and denote it as \( C_-(\mathcal{X}) \). Likewise \( \text{Cone}(G) \) has \( q^2C(\cdot, \cdot) \) as a submodule and \( C(\mathcal{X}) \) as the quotient \( \text{Cone}(F)/q^2C(\cdot, \cdot) \). We call this filtration the positive filtration and denote it as \( C_+(\mathcal{X}) \).

We will often identify \( C_-(\mathcal{X}) \) with \( \text{Cone}(F) \) and \( C_+(\mathcal{X}) \) with \( \text{Cone}(G) \) and consider the filtered complexes as mapping cones. This process simplifies the differential \( d_c \) so that it can be presented in the following manner (as proven in [1]).

Proposition A.7. \( C(\mathcal{X}) \) is homotopy equivalent to the bicomplex \( C(\cdot, \cdot) \xrightarrow{\phi} tq^{-2}C_+(\mathcal{X}), \) where \( \phi_i \) is the canonical inclusion of \( C(\cdot, \cdot) \) into \( \text{Cone}(G) \). Suppose \( C(\cdot, \cdot) \) has the trivial filtration, then \( \phi_i \) is a filtered map with respect to the filtration on \( C_+(\mathcal{X}) \) and thus \( C(\mathcal{X}) \) is a filtered bicomplex.

Likewise \( C(\mathcal{X}) \) is homotopy equivalent to the bicomplex \( t^{-1}C(\mathcal{X}) \xrightarrow{\phi} C(\cdot, \cdot) \), where \( \phi_i \) is the canonical projection of \( C(\cdot, \cdot) \) from \( \text{Cone}(F) \) and \( C(\mathcal{X}) \) is a filtered bicomplex.
We can extend this filtration to any tangle or link diagram via the tensor product filtration. We will also refer to the given filtration on the bicomplex associated to a tangle as a virtual crossing filtration. The following theorem was the main focus of [1].

**Theorem A.8.** Let $\beta$ be a braid on $n$ strands, and $L_\beta$ denote its circular closure. Then the virtual crossing filtration on $C(\beta)$ is invariant under Reidemeister IIA and is violated by at most two levels by Reidemeister III. Furthermore, the virtual crossing filtration on $H(L_\beta)$ is invariant under the Markov moves (up to a possible shift in filtration).

We now focus on describing the filtrations on MOY graphs, which will be useful in proving Lemma 4.3. A signed MOY graph is a MOY graph where each vertex is marked by a sign $+$ or $-$. Suppose $\Gamma$ is a signed MOY graph marked so that it is partitioned into graphs of the form $\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\node (v2) at (1,0) {$-$};
\end{tikzpicture}$, then we can define a filtration on $C(\Gamma)$. To each MOY vertex marked with a $+$ we associate $C_+(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture})$, and to each MOY vertex marked with a $-$ we associate $C_-(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture})$. We give the trivial filtration to $C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture})$. Then the filtration on $C(\Gamma)$ is given by the tensor product filtration.

If we choose different sign assignments, then we receive homotopy equivalent complexes for $C(\Gamma)$, but not necessarily filtered homotopy equivalent complexes (for example, $C_-(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture})$ and $C_+(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture})$ are not filtered homotopy equivalent). For this reason, if $\epsilon$ is a assignment of signs to each MOY vertex of $\Gamma$, then we will write $C_\epsilon(\Gamma)$ for the filtered complex we get from the above construction.

With this construction in mind, we may present every MOY graph as an iterated mapping cone or convolution of graphs with only virtual crossings and no MOY vertices. To do this we use twisted complexes.

**Example A.9.** We now consider the signed MOY graph $\Gamma = \begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\node (v2) at (1,0) {$-$};
\end{tikzpicture}$ whose left vertex is labeled by $+$ and right vertex is labeled by $-$. We can present $C_+(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture})$ as the convolution of the following twisted complex:

\[
\begin{array}{c}
q^4C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture}) \\
G \otimes 1 \\
\end{array} \rightarrow 
\begin{array}{c}
q^2C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\node (v2) at (1,0) {$+$};
\end{tikzpicture}) \oplus q^2C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\node (v2) at (1,0) {$-$};
\end{tikzpicture}) \\
- G \otimes 1 \oplus F \otimes 1 \\
\end{array} \rightarrow C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture})
\]

Note that this twisted complex does not have any nonzero maps of length $\geq 2$.

We end this section with a couple of simplifications which will be necessary in the next section. The following lemma is proven in [1].

**Lemma A.10.** Let $\hat{G}$ and $\hat{F}$ denote the partial braid closure of $\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture}$ and $\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture}$ respectively (see Figure 23). We write $\hat{G} : C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture}) \rightarrow q^2C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture})$ for the map induced by $G$ under braid closure, and similarly for $F$. Then

\begin{enumerate}
\item $\text{Cone}(\hat{G}) \simeq \text{Cone}(0) \simeq C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture}) \oplus q^2C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\node (v2) at (1,0) {$+$};
\end{tikzpicture})$
\item $\text{Cone}(\hat{F}) \simeq \text{Cone}\left(\left.\begin{array}{c}
q^2C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture}) \\
\end{array}\right| 1 + aq^2 C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture}) \rightarrow q^2C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture})\right) \right)$
\item $\text{Cone}(\hat{F}) \simeq \text{Cone}(\hat{G}) \simeq \frac{1 + aq^4}{1 - q^2} C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture})$
\end{enumerate}

Furthermore, the homotopy equivalences in (1) and (2) are filtered.

**A.3. A proof of Lemma 4.3.** We begin by restating Lemma 4.3.

**Lemma A.11 (See Lemma 4.3).** Let $\hat{C}(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture}) \rightarrow q^2C(\begin{tikzpicture}[baseline=-.5ex]
\node (v1) at (0,0) {$+$};
\end{tikzpicture})$, then the following diagram is commutative:
where \( \iota \) includes \( \tilde{C}(\{ \}) \) as a subcomplex of \( C(\opaX{0}) \). \( \pi \) projects \( C(\opaX{0}) \) onto its quotient complex \( C(\opaX{0}) \) and \( * \) represents some map from \( \tilde{C}(\{ \}) \) to \( C(\opaX{0}) \).

**Proof.** We prove the center isomorphism as keeping track of the explicit maps from application of Gaussian elimination constructs the maps on the bottom row of the commutative diagram above. Consider \( C(\opaX{0}) \) with the virtual filtration given by marking the left vertex by \( + \) and the right vertex by \( - \), then \( C(\opaX{0}) \) can be written as the convolution of the twisted complex given in (A.1). We can rewrite the twisted complex using the maps \( \hat{F} \) and \( \hat{G} \) from above, to get the twisted complex

\[
q^4 C(\hat{\opaX}) \otimes_{\mathbb{Q}} C(\{ \}) \xrightarrow{\begin{pmatrix} 1 \otimes F \\ \hat{G} \otimes 1 \end{pmatrix}} q^2 C(\opaX{0}) \oplus q^2 C(\opaX{1} \opaX{0}) \xrightarrow{\begin{pmatrix} -G \otimes 1 & 1 \otimes \hat{F} \\ \hat{G} \otimes 1 & 1 \otimes \hat{F} \end{pmatrix}} C(\{ \}) \otimes_{\mathbb{Q}} C(\hat{\opaX})
\]

where \( C(\opaX{0}) \) can be viewed as both \( C(\hat{\opaX}) \otimes_{\mathbb{Q}} C(\{ \}) \) and \( C(\{ \}) \otimes_{\mathbb{Q}} C(\hat{\opaX}) \). Lemma A.10 states that \( \hat{G} \) is homotopy equivalent to the zero map, so we can simplify the above complex to

\[
q^4 C(\hat{\opaX}) \otimes_{\mathbb{Q}} C(\{ \}) \xrightarrow{\begin{pmatrix} 1 \otimes F \\ 0 \end{pmatrix}} q^2 C(\opaX{0}) \oplus q^2 C(\opaX{1} \opaX{0}) \xrightarrow{\begin{pmatrix} -G \otimes 1 & 1 \otimes \hat{F} \\ \hat{G} \otimes 1 & 1 \otimes \hat{F} \end{pmatrix}} C(\{ \}) \otimes_{\mathbb{Q}} C(\hat{\opaX})
\]

Here the arrow \( \phi \) represents a map of length two (as described by Proposition A.4) which has yet to be determined explicitly. We then use (2) of Lemma A.10 to rewrite the above chain complex as

---

**Figure 23.** Partial braid closure of a virtual braidlike MOY graph.
Now we apply Proposition A.3 to the above twisted complex to get the following twisted complex

\[ q^4 C(\hat{\mathcal{X}}) \otimes \mathbb{Q} C(\downarrow) \xrightarrow{(1 \otimes F)} q^2 C(\hat{\mathcal{X}} \mathcal{Y}) \oplus C(\uparrow) \otimes \mathbb{Q} (aq^2 C(\downarrow) \oplus \frac{1+aq^4}{1-q^2} C(\downarrow)) \xrightarrow{(-G \otimes 1 \otimes F)} C(\uparrow \downarrow) \]

Note that we now omit \( \phi \) as it is clear it must be zero. The convolution of the above twisted complex is homotopy equivalent to \( C(\hat{\mathcal{X}} \mathcal{Y}) \oplus \frac{q^2 + aq^6}{1-q^2} C(\uparrow \downarrow) \) as desired.

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