PSL(2, Z) as a non distorted subgroup of Thompson’s group T

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Abstract
In this paper we characterize the elements of PSL(2, Z), as a subgroup of Thompson group T, in the language of reduced tree pair diagrams and in terms of piecewise linear maps as well. Actually, we construct the reduced tree pair diagram for every element of PSL(2, Z) in normal form. This allows us to estimate the length of the elements of PSL(2, Z) through the number of carets of their reduced tree pair diagrams and, as a consequence, to prove that PSL(2, Z) is a non distorted subgroup of T. In particular, we find non-distorted free non abelian subgroups of T.

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1 Introduction
Thompson’s group T was one of the first examples of finitely presented infinite simple groups. There are at least three different ways of representing the elements of Thompson’s group T (see [4] for a detailed introduction to Thompson’s groups). Probably the most common interpretation is as a subgroup of the group of homeomorphisms of the circle, thought as the unit interval with identified endpoints. Then, T is the group of orientation preserving piecewise linear homeomorphisms of the circle that are differentiable except at finitely many dyadic rational numbers, and such that, on intervals of differentiability, the derivatives are powers of 2 (see, for example, [7]).

Another interesting approach to Thompson’s group T describes it as equivalence classes of tree pair diagrams. A tree pair diagram is a pair of finite rooted binary trees with the same number of leaves, with a cyclic numbering system pairing the leaves in the two trees. We say that two different tree pair diagrams are equivalent if they have the same reduced representant. This combinatorial version of Thompson’s group T allowed Burillo, Cleary, Stein and Taback to discuss metric properties of T (see [2]). In particular, they found an estimation of the word length in terms of the number of carets in a reduced tree pair diagram, and showed that some subgroups of T are undistorted.

We need to mention a third viewpoint about Thompson’s group T. We can represent T as a subgroup of the piecewise projective orientation preserving homeomorphisms of the real projective line RP1. This result was claimed separately by Thurston (see [4]) and Kontsevich, and then proved by Imbert (see [11]) and Sergiescu (see [15]). It turns out that T is isomorphic to PPSL2(Z), which is the group of orientation preserving homeomorphisms of the real projective line which are piecewise PSL2(Z), and have a finite numbers of non differentiable points, all of them being rational numbers. This geometric approach is also related to the fact that one can actually see Thompson’s group T as the asymptotic mapping class group of an infinite surface of genus zero (see [6]).

The piecewise projective approach to Thompson’s group T invites us to study the projective special linear group PSL2(Z) as a subgroup of T using the combinatorial methods of tree pair diagrams. It turns out that all the reduced tree pair diagrams representing elements of PSL2(Z) as a subgroup of Thompson’s group T have exactly one leaf and one interior vertex for each level. In fact, we provide an explicit bijection between tree pair diagrams of elements in PSL2(Z) and the corresponding reduced words on the classical generators of PSL2(Z) of order 2 and 3 (see [10]). This characterization of PSL2(Z) in terms of tree pair diagrams together with the results of Burillo, Cleary, Stein and Taback in the word metric on T (see [2]) allow us to prove that PSL2(Z) is a non distorted subgroup of Thompson’s group T.
Although it was already known that $T$ has subgroups isomorphic to the free non abelian group of rank 2 (this is the usual way to proof the non amenability of $T$), it was not known if these subgroups are distorted. The piecewise projective approach together with our results give an easy example of a non distorted subgroup of $T$ isomorphic to $F_2$.

There are several normal subgroups of $\text{PSL}_2(\mathbb{Z})$ isomorphic to the free non abelian group of rank two. One particular example is the commutator group of $\text{PSL}_2(\mathbb{Z})$ (see [10]). In addition, this subgroup has finite index on $\text{PSL}_2(\mathbb{Z})$.

Now, using the fact that finite index subgroups are non distorted and our result that $\text{PSL}_2(\mathbb{Z})$ is non distorted as a subgroup of Thompson’s group $T$, we have constructed a non distorted subgroup of Thompson’s group $T$ isomorphic to the free non abelian group of rank 2.

1.1 Definitions and statements of the results

First we state the definitions related to the three different ways of representing Thompson’s group $T$ introduced above:

**Definition 1.** The piecewise linear Thompson’s group $T$ is the set of orientation preserving piecewise linear homeomorphisms of the circle $S^1 = [0,1]/0\sim 1$ that are differentiable except at finitely many dyadic rational numbers and such that, on intervals of differentiability, the derivatives are powers of 2.

For the combinatorial definition we need the following notion:

**Definition 2.** A tree pair diagram is a triple $(T_1, \sigma, T_2)$, where $T_1$ and $T_2$ are finite rooted binary trees with the same number $n$ of leaves, and $\sigma$ is a cyclic permutation of the set $\{1, \ldots, n\}$. The binary tree $T_1$ is called the source tree, and $T_2$ is called the target tree. A node of an ordered rooted binary tree together with its two downward directed edges is called a caret. A caret is called exposed if it contains two leaves of the tree. A tree pair diagram is reduced if, for all exposed caret of $T_1$, the images under $\sigma$ of their leaves do not form an exposed caret of $T_2$.

When elements of Thompson’s group $T$ are represented as equivalence classes of tree pair diagrams, the multiplication of $(T_1, \sigma_1, T_2)$ by $(T_3, \sigma_2, T_4)$ is described by the following procedure.

1. Let $T_{23}$ be an ordered rooted binary tree which is a common expansion of $T_2$ and $T_3$. This element always exists.
2. Let $(T_1', \sigma_1', T_{23})$ and $(T_{23}, \sigma_2', T_4')$ be the tree pair diagrams which are equivalent to $(T_1, \sigma_1, T_2)$ and $(T_3, \sigma_2, T_4)$, respectively.
3. Then, one sets $(T_1, \sigma_1, T_2) * (T_3, \sigma_2, T_4) = (T_1', \sigma_1' \sigma_2', T_4')$, and reduces the obtained tree pair diagram, if possible.

**Definition 3.** The combinatorial Thompson’s group $T$ is the set of equivalence classes of tree pair diagrams.

The third viewpoint of Thompson’s group $T$ connects it with the projective special linear group $\text{PSL}_2(\mathbb{Z})$ (see, for example, [13]).

**Definition 4.** The piecewise projective Thompson’s group $T$ ($\text{PPSL}_2(\mathbb{Z})$) is the group of orientation preserving homeomorphisms of the real projective line $\mathbb{RP}^1$ which are piecewise $\text{PSL}_2(\mathbb{Z})$ and have a finite number of non differentiable points, all of them being rational numbers.

Now we need a few properties of the projective special linear group. The group $\text{PSL}_2(\mathbb{Z})$ is isomorphic to the free product $\mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/3\mathbb{Z}$, which means that it admits a generating set $\{a, b\}$, where $a^2 = 1$ and $b^3 = 1$ (see [10], page 20 example 1.5.3). Then, every element of $\text{PSL}_2(\mathbb{Z})$ has a normal form in the generators $a$ and $b$ given by a word of the form $a^{\epsilon_1} b^{\epsilon_2} a b^{\delta_1} a b^{\delta_2} a^{\epsilon_2} a^{\epsilon_3}$, where $k$ is a non negative integer, $\epsilon_1, \epsilon_2 \in \{0,1\}$ and $\delta_1, \ldots, \delta_k \in \{-1,1\}$. We will denote by $a$ and $b$ both the generators of $\text{PSL}_2(\mathbb{Z})$ as a group of matrices and their images in $T$ under the isomorphism between $\text{PPSL}_2(\mathbb{Z})$ and Thompson’s group $T$.

In this paper we give a characterization of the elements of $\text{PSL}_2(\mathbb{Z}) \subset T$ in terms of reduced tree pair diagrams. Before stating the theorem we need the following definition:

**Definition 5.** A finite rooted binary tree is called thin if all its caret have one leaf and one internal vertex except for the last caret, which is exposed.

**Remark 1.** It is easy to see that one can characterize thin trees with $n$ leaves by associating a weight $r_i \in \{-1,1\}$ to each one of its $n - 2$ internal vertices. Let $v_0, \ldots, v_{n-3}$ be the internal vertices of a thin tree. Denote by $v_{-1}$ its root. Then, $r_i = 1$ if $v_i$ is the left descendant of $v_{i-1}$ and $r_i = -1$ if $v_i$ is the right descendant of $v_{i-1}$.

The main result of this note is the following:
Theorem 1. Let \((T_1, \sigma, T_2)\) be the reduced tree pair diagram of an element \(f\) of Thompson’s group \(T\). Then, \(f\) belongs to the subgroup \(\text{PSL}_2(\mathbb{Z})\) if and only if \((T_1, \sigma, T_2)\) satisfies one of the following conditions:

1. the number of leaves of \(T_1\) and \(T_2\) is less than 4; or
2. the trees \(T_1\) and \(T_2\) are thin with associated weights \(r_0, \ldots, r_{k-1}\) and \(s_0, \ldots, s_{k-1}\) respectively, which verify the equations:
\[
\sum_{i=2}^{k-1} r_is_{k+1-i} = 2 - k,
\]
and
\[
l + \sigma(1) + \epsilon(s_0) \equiv \frac{3 - s_1}{2} \pmod{k + 2},
\]
where \(l - 1\) is the cardinal of the set \(\{i : r_i = -1, 0 \leq i \leq k - 1\}\), and \(\epsilon(x) = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{if } x = -1. \end{cases}\)

This answers a question posed by Vlad Sergiescu.

In fact, given an element of \(\text{PSL}_2(\mathbb{Z})\) as a subgroup of Thompson’s group \(T\) in its normal form on the generating set \(\{a, b\}\), we describe an easy way to construct its reduced tree pair diagram.

Proposition 1. Let \(w(a, b) = a^{\epsilon_1}b^{\delta_1}ab^{\delta_2}a \ldots ab^{\delta_k}a^{\epsilon_2}\) be a reduced word in the standard generators \(\{a, b\}\) of \(\text{PSL}_2(\mathbb{Z})\), viewed as a subgroup of Thompson’s group \(T\), i.e. \(\epsilon_1, \epsilon_2 \in \{0, 1\}\) and \(\delta_1, \ldots, \delta_k \in \{-1, 1\}\). Assume that \(k \geq 2\). Let \(T_1\) be the thin tree given by the weights \(r_0 = \epsilon^{-1}(\epsilon_1)\) and \(r_i = \delta_i\) for \(1 \leq i \leq k - 1\), and let \(T_2\) be the thin tree given by the weights \(s_0 = \epsilon^{-1}(\epsilon_2)\) and \(s_i = -\delta_{i+1}\) for \(1 \leq i \leq k - 1\). Let \(\sigma\) be the cyclic permutation defined by
\[
\sigma(1) \equiv \frac{3 - s_1}{2} - \epsilon(s_0) - l \pmod{k + 2},
\]
where \(l\) and \(\epsilon\) are defined as in theorem 1. Then, \((T_1, \sigma, T_2)\) is the reduced tree pair diagram for \(w\).

Furthermore, the weights \(r_0, \ldots, r_{k-1}\) and \(s_0, \ldots, s_{k-1}\) satisfy the equations of theorem 1.

This proposition proves the ‘if’ part of the main theorem. For the ‘only if’ part it suffices to show that the number of solutions to the equations \(1\) and \(2\) coincides with the number of words \(w(a, b) = a^{\epsilon_1}b^{\delta_1}ab^{\delta_2}a \ldots ab^{\delta_k}a^{\epsilon_2}\), with \(\epsilon_1, \epsilon_2 \in \{0, 1\}\) and \(\delta_1, \ldots, \delta_k \in \{-1, 1\}\).

Now, we can ask whether or not \(\text{PSL}_2(\mathbb{Z})\) is distorted as a subgroup of Thompson’s group \(T\). Recall that a finitely generated subgroup \(H\) of a finitely generated group \(G\) is distorted if there exists an infinite family \(\{h_n\}_{n \in \mathbb{N}}\) of elements of \(H\) such that

1. \(|h_n|_Z < |h_{n+1}|_Z\),
2. \(\lim_{n \to \infty} |h_n|_Z = \infty\), and
3. the limit \(\lim_{n \to \infty} \frac{|h_n|_Z}{|h_n|_Y}\) exists and is equal to zero,

where \(Z\) is a finite generating set for the group \(G\) which contains a finite generating set \(Y\) for the subgroup \(H\), and \(|.|_Z\) and \(|.|_Y\) denote the word metrics on the generating sets \(Z\) and \(Y\), respectively. Observe that \(|.|_Z \leq |.|_Y\). See \(\text{[8]}\) for another definition of distortion.

Remark 2. \(H\) is a non distorted subgroup of \(G\) if and only if there are constants \(K > 0\) and \(L\) such that
\[
\frac{1}{K}|h|_Y - L \leq |h|_Z \leq K|h|_Y + L
\]
holds for every \(h \in H\).

Using theorem 1 and proposition 1 we can estimate the length of the elements in this subgroup. Furthermore, Burillo, Cleary, Stein and Taback gave a general estimation of the length of elements in \(T\) in the number of carets in \(\text{[2]}\). As a consequence of both results, we obtain:

Proposition 2. The group \(\text{PSL}_2(\mathbb{Z})\) is a non distorted subgroup of Thompson’s group \(T\).
In particular, we derive:

**Corollary 1.** Let \( \{a, b\} \) be the standard generating set of \( \text{PSL}_2(\mathbb{Z}) \) as a subgroup of Thompson’s group \( T \). Then, the subgroup \( H = \langle abab, abab \rangle \) is a free non abelian group of rank 2 and it is non distorted in \( T \).

Finally, we consider the elements of \( \text{PSL}_2(\mathbb{Z}) \) as piecewise linear maps and we give a characterization in terms of the coordinates of their non differentiable points \((x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k)\). As the result is technical and needs both notation and definitions, the reader is referred to section 4 for the details.

**Structure of the paper.** This paper is structured in four sections. The group \( T \) is presented as the group of piecewise \( \text{PSL}_2(\mathbb{Z}) \) homeomorphisms of the real projective line in section two. The third section deals with the characterization of the elements of \( \text{PSL}_2(\mathbb{Z}) \) as a subgroup of \( T \), the proof that \( \text{PSL}_2(\mathbb{Z}) \) is a non distorted subgroup and the construction of a non distorted free group of rank two in \( T \). Finally, in section four we give a piecewise linear characterization of the elements of \( \text{PSL}_2(\mathbb{Z}) \).

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2 \( \text{PSL}(2, \mathbb{Z}) \) as a subgroup of Thompson’s group \( T \)

The projective special linear group \( \text{PSL}_2(\mathbb{Z}) \) is isomorphic to the free product of the cyclic groups of order 2 and 3 (see [16], page 20 example 1.5.3), i.e. it can be given by the following presentation

\[
\langle a, b | a^2 = b^3 = 1 \rangle,
\]

where \( a = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \) and \( b = \left( \begin{array}{cc} 0 & -1 \\ 1 & 1 \end{array} \right) \).

In order to connect Thompson’s group \( T \) with the modular group we need the Minkowsky question mark function (see [14] or [17]). Let \( \bar{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\} \) and let \( \mathbb{Q}_2 \) denote the dyadic rational numbers of the unit interval. Then, the Minkowski question mark function, \( ? : \bar{\mathbb{Q}} \rightarrow \mathbb{Q}_2 \), is defined recursively. The basic cases are

\[
?(-\infty) = \frac{1}{2}, \quad ?\left(\frac{0}{1}\right) = 0, \text{ and } \quad ?(\infty) = \frac{1}{2},
\]

where \( \infty \) will be represented by the fraction \( \frac{1}{0} \) and \( -\infty \) by \( \frac{-1}{0} \). Then, for each pair of reduced fractions, \( \frac{p}{q} \) and \( \frac{r}{s} \), satisfying \( |ps - qr| = 1 \), one defines the Minkowski question mark of their Farey mediant \( \frac{p}{q} \oplus \frac{r}{s} := \frac{p+r}{q+s} \) as

\[
? \left( \frac{p}{q} \oplus \frac{r}{s} \right) = \frac{1}{2} \left( ? \left( \frac{p}{q} \right) + ? \left( \frac{r}{s} \right) \right).
\]

The Minkowski question mark function is clearly a bijection between \( \bar{\mathbb{Q}} \) and \( \mathbb{Q}_2 \). Thus, using the density of \( \bar{\mathbb{Q}} \) in the real projective line \( \mathbb{R}P^1 = \mathbb{R} \cup \{\infty\} \) and \( \mathbb{Q}_2 \) in \([0, 1] \), respectively, the Minkowski question mark function can be extended to \( ? : \mathbb{R}P^1 \rightarrow [0, 1] \).

The pairs of reduced fractions, \( \frac{p}{q} \) and \( \frac{r}{s} \), satisfying \( |ps - qr| = 1 \) are called consecutive Farey numbers, and if \( \frac{p}{q} < \frac{r}{s} \), the interval \( \left[ \frac{p}{q}, \frac{r}{s} \right] \) is called a Farey interval. Every rational number appears as the Farey mediant of a Farey interval. See, for example, [1] section 5.4 or [9] chapter 3 for more details on Farey fractions.

**Lemma 1.** Let \( h \) be an element of \( \text{PPSL}_2(\mathbb{Z}) \) which is \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{PSL}_2(\mathbb{Z}) \) on the interval \([x_0, x_k] \) of the real projective line \( \mathbb{R}P^1 \). Let \( x_0 < x_1 < \ldots < x_k \) be a partition of \([x_0, x_k] \) such that every interval of the partition and every interval of the image partition \( h(x_0) < \ldots < h(x_k) \) is a Farey interval. Then,

\[
h(x_i \oplus x_{i+1}) = h(x_i) \oplus h(x_{i+1}),
\]

for all \( 0 \leq i < k \).
Proof. Let \( x_i = \frac{p}{q} \) and \( x_{i+1} = \frac{r}{s} \) be real rational numbers (i.e. \( q, s \neq 0 \)). Then 
\[
 h(x_i) = \frac{ap + bq}{cp + dq} \quad \text{and} \quad h(x_{i+1}) = \frac{ar + bs}{cr + ds}.
\]
Hence,
\[
h \left( \frac{p}{q} \oplus \frac{r}{s} \right) = \frac{a(p + r) + b(q + s)}{c(p + r) + d(q + s)} = h \left( \frac{p}{q} \right) \oplus h \left( \frac{r}{s} \right).
\]
If \( x_{i+1} = \frac{1}{0} \), then \( h(x_{i+1}) = \frac{a}{c} \) and
\[
h \left( \frac{p}{q} \oplus \frac{1}{0} \right) = \frac{a(p + 1) + bq}{c(p + 1) + dq} = h \left( \frac{p}{q} \right) \oplus h \left( \frac{1}{0} \right).
\]
Analogously if \( x_i = \frac{1}{0} \).

The following result, due to Imbert, Kontsevich and Sergiescu, identifies the two groups encountered above (see [11], theorem 1.1).

**Theorem 2.** ([11], theorem 1.1) The group \( \text{PPSL}_2(\mathbb{Z}) \) is isomorphic to Thompson’s group \( T \).

Proof. We claim that the homomorphism \( \text{Inn} : \text{PPSL}_2(\mathbb{Z}) \to T \), given by \( \text{Inn}(g) = \circ g \circ \circ^{-1} \) is an isomorphism. First, we consider the generating set \( \{a, b\} \) of \( \text{PSL}_2(\mathbb{Z}) \) and calculate \( \text{Inn}(a) \) and \( \text{Inn}(b) \). By lemma [1] it suffices to find Farey partitions of \( \mathbb{Q} \) whose images are also Farey partitions. Then, applying the Minkowski question mark function we will obtain partitions of the unit interval characterizing finite binary trees with the same number of leaves, thus elements in \( T \). The following table summarizes this procedure.

| Generator | Farey partitions | Piecewise linear map | Reduced tree pair diagram |
|-----------|-----------------|----------------------|--------------------------|
| \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\] | \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] | \[
\begin{align}
& x + \frac{1}{2}, \quad 0 \leq x \leq \frac{1}{2} \\
& x - \frac{1}{2}, \quad \frac{1}{2} \leq x \leq 1
\end{align}
\] | ![Reduced tree pair diagram 1](image_1.png) |
| \[
\begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}
\] | \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & -1 \\
1 & -1
\end{bmatrix}
\] | \[
\begin{align}
& x + \frac{3}{4}, \quad 0 \leq x \leq \frac{1}{2} \\
& 2x - 1, \quad \frac{1}{2} \leq x \leq \frac{3}{4} \\
& x - \frac{1}{4}, \quad \frac{3}{4} \leq x \leq 1
\end{align}
\] | ![Reduced tree pair diagram 2](image_2.png) |

Thanks to this and lemma [1] the map \( \text{Inn}_T \) is well defined. Thus, \( \text{Inn}_T \) is a homomorphism. The injectivity is also a consequence of lemma [1]. Let \( A, B \) and \( C \) be the three classical generators of Thompson’s group \( T \), i.e. the elements with the following reduced tree pair diagrams (see [4] for details).

![](image_3.png)

Note that \( \text{Inn}_T(b) = C \) and \( \text{Inn}_T(a) = CA \). In order to prove that \( \text{Inn}_T \) is surjective, we have to find an element \( d \)
in $\text{PPSL}_2(\mathbb{Z})$ such that $\text{Inn}_\gamma(d) = B$. This element is

$$
d(x) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } 0 \leq x \leq \frac{1}{6}, \\
\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, & \text{if } \frac{1}{6} \leq x \leq \frac{1}{4}, \\
\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}, & \text{if } \frac{1}{4} \leq x \leq \frac{3}{2}, \\
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{if } \frac{3}{2} \leq x \leq 0.
\end{cases}
$$

As a direct consequence of Theorem 2, $\text{PSL}_2(\mathbb{Z})$ can be considered as a subgroup of Thompson’s group $T$. By abuse of notation, from now on we will denote the generators in $\text{PSL}_2(\mathbb{Z})$ and its images in $T$ with the same symbol, $a$ and $b$, and the subgroup $\text{Inn}_\gamma(\text{PSL}_2(\mathbb{Z}))$ will be simply $\text{PSL}_2(\mathbb{Z})$.

**Remark 3.** Let $A$, $B$ and $C$ be the three classical generators of Thompson’s group $T$ (see [4] for details). It is easy to see that $a = CA$ and $b = C$. Hence, $\text{PSL}_2(\mathbb{Z}) \subset T$ is the subgroup generated by $A$ and $C$.

## 3 Tree diagram characterization of $\text{PSL}_2(\mathbb{Z})$

Recall that an ordered rooted binary tree is thin if all its carets have one leave and one internal vertex, except for the last caret. Furthermore, there exists a bijection between the set of thin trees with $n$ leaves and $\{-1, 1\}^{n-2}$, by associating a weight $r_i \in \{-1, 1\}$ to every internal vertex $v_i$, for $0 \leq i \leq n-3$, in the following way: $r_i = 1$ if $v_i$ is the left descendant of $v_{i-1}$ and $r_i = 1$ if $v_i$ is the right descendant of $v_{i-1}$, where $v_{n-1}$ denotes the root.

**Remark 4.** A thin tree with $n$ leaves has exactly $n - 2$ associated weights and $n - 1$ carets. Furthermore, if $l$ is the number of the left leaf of the exposed caret, then $l - 1$ is the cardinal of the set \{i : r_i = 1, 0 \leq i \leq k - 1\}.

Recall that we want to characterize the elements of $\text{PSL}_2(\mathbb{Z})$ as a subgroup of Thompson’s group $T$ in terms of reduced tree pair diagrams.

**Theorem 1.** Let $(T_1, \sigma, T_2)$ be the reduced tree pair diagram of an element $f$ of Thompson’s group $T$. Then, $f$ belongs to the subgroup $\text{PSL}_2(\mathbb{Z})$ if and only if $(T_1, \sigma, T_2)$ satisfies one of the following conditions:

1. the number of leaves of $T_1$ and $T_2$ is less than 4; or
2. the trees $T_1$ and $T_2$ are thin with associated weights $r_0, \ldots, r_{k-1}$ and $s_0, \ldots, s_{k-1}$ respectively, which verify the equations:

$$
\sum_{i=2}^{k-1} r_i s_{k+1-i} = 2 - k,
$$

and

$$
l + \sigma(1) + \epsilon(s_0) \equiv \frac{3 - s_1}{2} \pmod{k + 2},
$$

where $l - 1$ is the cardinal of the set \{i : r_i = 1, 0 \leq i \leq k - 1\}, and $\epsilon(x) = \begin{cases} 1 & \text{if } x = 1 \\
0 & \text{if } x = -1.\end{cases}$

In particular, there is a relation between reduced words on the generators $a$ and $b$ of $\text{PSL}_2(\mathbb{Z})$ and reduced tree pair diagrams representing them as elements of Thompson’s group $T$. We will denote the inverse of the generator $b$ by $b^{-1}$ or $b$.

**Proposition 1.** Let $w(a, b) = a^{\epsilon_1} b^{\delta_1} a^{\epsilon_2} b^{\delta_2} \cdots a^{\epsilon_k} b^{\delta_k} a^{\epsilon_2}$ be a reduced word on the generators \{a, b\} of $\text{PSL}_4(\mathbb{Z})$, where $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and $\delta_1, \ldots, \delta_k \in \{-1, 1\}$. Assume that $k \geq 2$. Let $T_1$ be the thin tree given by the weights $r_0 = \epsilon^{-1}(\epsilon_1)$
and \( r_i = \delta_i \) for \( 1 \leq i \leq k - 1 \), and let \( T_2 \) be the thin tree given by the weights \( s_0 = \epsilon^{-1}(\epsilon_2) \) and \( s_i = -\delta_{k+1-i} \) for \( 1 \leq i \leq k - 1 \). Let \( \sigma \) be the cyclic permutation defined by

\[
\sigma(1) \equiv \frac{3 - s_1}{2} - \epsilon(s_0) - l \pmod{k + 2},
\]

where \( l \) and \( \epsilon \) are defined as in theorem 1. Then, \((T_1, \sigma, T_2)\) is the reduced tree pair diagram for \( w \).

Furthermore, the weights \( r_0, \ldots, r_{k-1} \) and \( s_0, \ldots, s_{k-1} \) satisfy equations (1) and (2) of theorem 1.

**Proof.** We will proceed by induction. First, we consider the eight cases with \( k = 2 \) and \( \epsilon_2 = 0 \), so that the reduced word is \( w = a^{\epsilon_1}b^h_1ab^{h_2} \).

The weights of their reduced tree pair diagrams verify the equations of the proposition

\[
\begin{align*}
r_0 &= \epsilon^{-1}(\epsilon_1) \\
r_1 &= \delta_1 \\
s_0 &= -1 \\
s_1 &= -\delta_2,
\end{align*}
\]

and

\[
\sigma(1) \equiv \frac{3 - s_1}{2} - l \pmod{k + 4}.
\]

Now, let \( w(a, b) = a^{\epsilon_1}b^h_1ab^{h_2}a \ldots ab^{h_k}a^{\epsilon_2} \) be a reduced word verifying the induction hypothesis. We will see that all reduced words of type \( w' = wy \), where \( y \in \{a, b, \bar{b}\} \), also satisfy the equations of the proposition. There are several cases to be considered.

a.- Suppose that \( \epsilon_2 = 0 \). Then, \( w' = wa \). By the induction hypothesis, \( wa \) has the reduced tree pair diagram of figure 1.

Figure 1: Tree pair diagrams multiplication for \( wa \).
The tree pair diagram for $wa$ in figure 1 is reduced because the tree pair diagram for $w$ in the same figure was reduced. As a consequence, $r'_i = r_i$ for $0 \leq i \leq k - 1$, $s'_0 = 1$, $s'_i = s_i$ for $1 \leq i \leq k - 1$, $l' = l$ and $\sigma'(1) \equiv \sigma(1) - 1 \pmod{k + 2}$. Since $w' = wa$, we have $c'_1 = c_1$, $c'_2 = 1$ and $\delta'_i = \delta_i$ for $1 \leq i \leq k$. Thus, equation (1) for $w'$ follows from equation (1) for $w$. Further, $l' + \sigma'(1) + \epsilon(\epsilon'_2) = l + \sigma(1) - 1 + 1$, which is congruent to $\frac{3 - s'_1}{2}$ by the induction hypothesis, thereby proving that equation (2) is also verified.

b.- Suppose that $\epsilon_2 = 1$. Then, by the induction hypothesis $w$ has the reduced tree pair diagram $(T_1, \sigma, T_2)$ in figure 2.

Figure 2: Reduced tree pair diagram for $w$.

Note that in order to apply $b$ or $\bar{b}$ to $T_2$ we need to split the $(k + 2)$-th vertex of $T_2$. That makes us split the $p$-th vertex of $T_1$ as well, where $p$ satisfies $\sigma(p) = k + 2$. Since $\sigma(p) = \sigma(1) + p - 1$, then $\sigma(1) + p - 1 = k + 2$, i.e. $p \equiv k + 3 - \sigma(1) \equiv 1 - \sigma(1) \pmod{k + 2}$. We want to see that $p \in \{l, l + 1\}$. By the induction hypothesis we have $\sigma(1) \equiv 1 - \delta_1 = 1$ and $\sigma(1) \equiv -l$ if $\delta_k = -1$. Thus, $p \equiv l$ if $\delta_k = 1$ and $p \equiv l + 1$ if $\delta_k = -1$.

Now, we must consider four subcases:

| b1. $\delta_k = 1$, $w' = wb$ ($\delta_{k+1} = 1$) | b2. $\delta_k = 1$, $w' = \bar{w}b$ ($\delta_{k+1} = -1$) |
| b3. $\delta_k = -1$, $w' = wb$ ($\delta_{k+1} = 1$) | b4. $\delta_k = -1$, $w' = \bar{w}b$ ($\delta_{k+1} = -1$) |
It is clear that the tree pair diagrams obtained for \( w' \) are reduced because the tree pair diagram for \( w \) was reduced. Note also that all four subcases satisfy the following relations:

\[
\begin{align*}
(r1) & \quad k' = k + 1, \\
(r2) & \quad r_i' = r_i, \text{ for } 0 \leq i \leq k - 1, \\
(r3) & \quad r_k' = \delta_k, \\
(r4) & \quad s_0' = -1, \\
(r5) & \quad s_i' = -\delta_{k+1}, \text{ and} \\
(r6) & \quad s_i' = s_{i-1}, \text{ for } 2 \leq i \leq k.
\end{align*}
\]

As a consequence of (r1), (r2) and (r6) we obtain

\[
\sum_{i=2}^{k'-1} r_i' s_{k'+1-i} = \sum_{i=2}^{k-1} r_i s_{k+1-i} + r_k' s_2'.
\]

Thus, applying the induction hypothesis we get

\[
\sum_{i=2}^{k'-1} r_i' s_{k'+1-i} = 2 - r_k' \delta_k.
\]

Therefore, equation (1) holds if and only if \( r_k' \delta_k = 1 \), and this follows from (r3).

Finally, we must verify that our four subcases satisfy equation (2). Note that (r4) implies \( \epsilon(s_0') = 0 \), so the left-hand-side of the equation (2) for \( w' \) is equal to \( l' + \sigma'(1) \). The reduced tree pair diagrams show that

\[
l' + \sigma'(1) = \begin{cases} 
  l + \sigma(1) + 2, & \text{in case b1} \\
  l + \sigma(1) + 1, & \text{in case b2} \\
  l + \sigma(1) + 3, & \text{in case b3} \\
  l + \sigma(1) + 2, & \text{in case b4}.
\end{cases}
\]

On the other hand, since \( w \) verifies equation (2) we obtain

\[
l + \sigma(1) + 1 \equiv \begin{cases} 
  1, & \text{if } \delta_k = -1 \text{ (cases b1 and b2)} \\
  2, & \text{if } \delta_k = 1 \text{ (cases b3 and b4)}
\end{cases} \pmod{k + 2}.
\]

Furthermore, \( 1 \leq l \leq k + 1 \) and \( 1 \leq \sigma(1) \leq k + 2 \), so

\[
l + \sigma(1) + 1 = \begin{cases} 
  k + 3, & \text{if } \delta_k = -1 \text{ (cases b1 and b2)} \\
  k + 4, & \text{if } \delta_k = 1 \text{ (cases b3 and b4)}
\end{cases}
\]

Hence, equation (2) holds for \( w' \).

Proof. (of theorem 1). First, we list all reduced tree pair diagrams with less than 4 leaves and their reduced words in the generators \( a \) and \( b \) of \( PSL_2(\mathbb{Z}) \) (see the table below). Then, we focus on tree pair diagrams with thin trees and more than 3 leaves. The ‘if’ part coincides with proposition 1. The ‘only if’ part will be proved by counting the number of elements satisfying the two equations and showing this is exactly the number of elements that come from reduced words in \( a \) and \( b \).
Given a thin source tree, equation (1) implies that we can only have four different thin target trees, namely those with \( s_0 \in \{0,1\} \) and \( s_1 \in \{-1,1\} \). Moreover, given a pair of compatible thin trees, \( l \) is determined by the source tree and all other terms of equation (2) except \( \sigma(1) \) are determined by the target tree. Then, there exists a unique permutation \( \sigma \) satisfying equation (2). Therefore, given a thin tree \( T_1 \) there exist exactly four reduced tree pair diagrams having \( T_1 \) as the source tree and satisfying equations (1) and (2), which is exactly the same number of reduced word in \( a,b \) whose associated source tree is \( T_1 \).

\begin{proof}
\end{proof}

\textbf{Remark 5.} From theorem 1 and proposition 1 we can read the reduced word in generators \( a,b \) of any element of \( \text{PSL}_2(\mathbb{Z}) \) directly from its reduced tree pair diagram.

Next, we wonder whether \( \text{PSL}_2(\mathbb{Z}) \) is distorted as a subgroup of Thompson’s group \( T \).

\textbf{Proposition 2.} The group \( \text{PSL}_2(\mathbb{Z}) \) is a non distorted subgroup of Thompson’s group \( T \).

For the proof, we will need the following theorem due to Burillo, Cleary, Stein and Taback.

\textbf{Theorem 2.} ([10], theorem 5.1) Let \( N(w) \) be the number of carets of the reduced tree pair diagram representing \( w \in T \), and let \( |w|_{A,B,C} \) denote the word length of \( w \) in the classical generating set \( \{A,B,C\} \). Then, there exists a positive constant \( K \) such that

\[
\frac{N(w)}{K} \leq |w|_{A,B,C} \leq KN(w).
\]

\textbf{Proof.} (of proposition 2) Let \( w = a^{\delta_1} b^{\delta_2} a \ldots b^{\delta_k} a^{\epsilon_2} \) be a reduced word in the generators \( a \) and \( b \) of \( \text{PSL}_2(\mathbb{Z}) \), where \( \delta_i \in \{-1,1\} \) and \( \epsilon_j \in \{0,1\} \). We want to find constants \( K' > 0 \) and \( L \) such that \( \frac{|w|_{a,b}}{K'} - L \leq |w|_{A,B,C,a,b} \leq K'|w|_{a,b} + L \) (see remark 1). Let \( N(w) \) be the number of carets of the reduced tree pair diagram representing \( w \in T \). Recall that \( a = CA \) and \( b = C. \) Thus,

\[
\frac{|w|_{A,B,C}}{2} \leq |w|_{A,B,C,a,b} \leq |w|_{A,B,C}.
\]

Then, applying theorem 2 we obtain

\[
\frac{N(w)}{2K} \leq |w|_{A,B,C,a,b} \leq 2KN(w). \tag{4}
\]

On the other hand,

\[
|w|_{a,b} = |a^{\epsilon_1} b^{\epsilon_2} a \ldots b^{\delta_k} a^{\epsilon_2}|_{a,b} = 2k - 1 + \epsilon_1 + \epsilon_2.
\]

From proposition 1 and theorem 1 the reduced tree pair diagram of \( w \) has exactly \( k + 2 \) leaves and \( k + 1 \) carets. Thus,

\[
2N(w) - 3 \leq |w|_{a,b} \leq 2N(w) - 1,
\]

which implies that

\[
\frac{|w|_{a,b} + 1}{2} \leq N(w) \leq \frac{|w|_{a,b} + 3}{2}. \tag{5}
\]

Therefore, from the equations (4) and (5) we obtain the inequalities

\[
\frac{|w|_{a,b} + 1}{4K} \leq |w|_{A,B,C,a,b} \leq K(|w|_{a,b} + 3),
\]

which yield

\[
\frac{|w|_{a,b}}{K'} - L \leq |w|_{A,B,C,a,b} \leq K'|w|_{a,b} + L,
\]

where \( K' = 4K \) and \( L = 3K \).

\textbf{Corollary 1.} There exists a non distorted subgroup of Thompson’s group \( T \) isomorphic to the free non abelian group of rank 2.

\textbf{Proof.} Let \( H \) be the subgroup of \( \text{PSL}_2(\mathbb{Z}) \) generated by \( g = abab \) and \( h = abab \), which is isomorphic to the free non abelian group of rank 2 (see [5], page 26 or [12] proposition III.12.3). The subgroup \( H \) is the kernel of the morphism

\[
\begin{array}{c}
\text{PSL}_2(\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \\
a \mapsto (1,0) \\
b \mapsto (0,1),
\end{array}
\]

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so $H$ is of finite index in $\text{PSL}_2(\mathbb{Z})$. Thus, $H$ is non distorted in $\text{PSL}_2(\mathbb{Z})$ and using the proposition $\Box H$ is non distorted in $T$.

This is related to recent results of Calegari and Freedman $\Box$, who analyzed distorted cyclic subgroups of more general homeomorphisms groups.

## 4 Piecewise linear characterization of $\text{PSL}_2(\mathbb{Z})$

In this section we will characterize the elements of $\text{PSL}_2(\mathbb{Z})$ as piecewise linear maps of the unit interval with identified endpoints. Note that a piecewise linear map of $[0, 1]$ with identified endpoints $f$ can be given by the coordinates of its non differentiable points $(x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k)$. Furthermore, $x_0 = 0$, $x_k = 1$ and there exists $i \in \{0, \ldots, k\}$ such that $y_i = 0$. Let $\Delta_j(x)$ denote the difference $x_j - x_{j-1}$ for $1 \leq j \leq k$, and $\Delta_j(y)$ denote $y_j - y_{j-1}$. Thus, we can represent the map $f$ by the sequence

$$S(f) = \left\{ \Delta_1(y), \Delta_2(y), \ldots, \Delta_i(y), \Delta_{i+1}(y) \circ \Delta_{i+1}(y), \ldots, \Delta_k(y) \right\},$$

since, for $0 \leq j \leq k$ we have

$$x_j = \sum_{i=1}^{j} \Delta_i \quad \text{and} \quad y_j = \sum_{l=1+1}^{i+j} \Delta_{l(\text{mod} \ k)}.$$

Note that the mark $\circ$ on the sequence $S(f)$ denotes the cyclical permutation which prevents us to have $y_0 = 0$. We will denote by $\Delta_z(f)$ the sequence $\Delta_1(x), \Delta_2(x), \ldots, \Delta_k(x)$, by $\Delta_y(f)$ the sequence $\Delta_1(y), \Delta_2(y), \ldots, \Delta_k(y)$ and by $\Delta^0_{\sigma}(f)$ the sequence $\Delta_{i+1}(y), \Delta_{i+2}(y), \ldots, \Delta_k(y), \Delta_1(y), \ldots, \Delta_i(y)$. We will denote by $\Delta(f)$ either $\Delta_z(f), \Delta_y(f)$ or $\Delta^0_{\sigma}(f)$.

**Remark 6.** A piecewise linear map $f$ is an element of Thompson group $T$ if and only if for all $j \in \{1, \ldots, k\}$, $x_j, y_j$ are dyadic rational numbers and the fraction $\frac{\Delta_j(y)}{\Delta_j(z)}$ is a power of two.

**Definition 6.** Let $\sigma$ be a permutation of the set $\{1, \ldots, k\}$. Then, $\sigma$ is $k$--extremal if:

1. $\sigma(1) = 1$ or $\sigma(1) = k$;
2. for $2 \leq i \leq k - 2$, $\sigma(i)$ is the maximum or the minimum of the set $\{1, \ldots, k\} \setminus \bigcup_{j=1}^{i-1} \sigma(j)$;
3. $\sigma(k-1)$ is the minimum and $\sigma(k)$ is the maximum of the pair $\{1, \ldots, k\} \setminus \bigcup_{j=1}^{k-2} \sigma(j)$.

**Example:** The permutation $\sigma = (1, 2, 3, 4, 5) \mapsto (5, 1, 2, 3, 4)$ is 5--extremal.

**Definition 7.** The sequence $\Delta$ is $k$--thin if there exists $\sigma$ which is $k$--extremal such that:

(t1) $\Delta_{\sigma^{-1}(i)} = 2^{-i}$, for $1 \leq i \leq k - 1$, and

(t2) $\Delta_{\sigma^{-1}(k)} = 2^{-k+1}$.

**Lemma 2.** There exists a canonical bijection between $k$--thin sequences and thin trees with $k$ leaves.

**Proof.** Let $\Delta$ be a $k$--thin sequence. Note that $\Delta = \{\Delta^{(-1)}, 2^{-k+1}, 2^{-k+1}, \Delta^{(1)}\}$, where $\Delta^{(-1)}$ and $\Delta^{(1)}$ are subsequences of $\Delta$. For $0 \leq i \leq k - 3$ we define $r_i = j$ if $\Delta_{\sigma^{-1}(i+1)} \in \Delta^{(j)}$. Then, we associate the thin tree with weights $r_0, \ldots, r_{k-3}$ to $\Delta$. Finally, we remark that the number of thin trees with $k$ leaves and the number of $k$--thin permutations are both equal to $2^{k-2}$. $\Box$

**Example:** A 5--thin sequence and its associated thin tree are shown below.
\[ \Delta = \{2^{-2}, 2^{-4}, 2^{-4}, 2^{-3}, 2^{-1}\}. \]
\[ \sigma = (1, 2, 3, 4, 5) \mapsto (5, 1, 4, 2, 3). \]
\[ \Delta^{(-1)} = \{2^{-2}\} \Rightarrow r_1 = -1. \]
\[ \Delta^{(1)} = \{2^{-3}, 2^{-1}\} \Rightarrow r_2 = r_0 = 1. \]

**Definition 8.** The sequence \( S(f) \) representing a piecewise linear map \( f \) is \( k \)-good if

**(g1)** the sequence \( \Delta_k(f) \) is \( k \)-thin,

**(g2)** the sequence \( \Delta^k(f) \) is \( k \)-thin, and

**(g3)** for \( 3 \leq j \leq k-2 \), the fraction \( \frac{2^{-k-1+j}}{2^{-j}} \) appears in \( S(f) \).

**Example:** The sequence \( S(f) = \left\{ \frac{2^{-5}}{2^{-3}}, \frac{2^{-3}}{2^{-2}}, \frac{2^{-2}}{2^{-2}}, \frac{2^{-1}}{2^{-1}}, \frac{2^{-1}}{2^{-1}}, \frac{2^{-1}}{2^{-1}} \right\} \) is 7-good.

**Theorem 3.** An element \( f \) of \( T \) belongs to \( PSL_2(\mathbb{Z}) \) if and only if the sequence \( S(f) \) is \( k \)-good.

**Proof.** For the ‘if’ part, we first enumerate the 2-good and 3-good sequences \( S(f) \) with their corresponding elements of \( PSL_2(\mathbb{Z}) \) in the following table.

| \( \frac{2^{-k}}{2^{-1}} \) | \( \frac{2^{-k+1}}{2^{-1}} \) | \( \frac{2^{-k+i-2}}{2^{-i-1}} \) | \( \frac{2^{-k+i-1}}{2^{-i-1}} \) | \( \frac{2^{-i-1}}{2^{-1}} \) | \( \frac{2^{-i}}{2^{-1}} \) | \( \frac{2^{-i+1}}{2^{-1}} \) | \( \frac{2^{-i+2}}{2^{-1}} \) |
|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|
| \( \sigma \) | \( \sigma \) | \( \sigma \) | \( \sigma \) | \( \sigma \) | \( \sigma \) | \( \sigma \) | \( \sigma \) |
| \( \bar{a} \) | \( \bar{a} \) | \( \bar{a} \) | \( \bar{a} \) | \( \bar{a} \) | \( \bar{a} \) | \( \bar{a} \) | \( \bar{a} \) |
| \( \bar{b} \) | \( \bar{b} \) | \( \bar{b} \) | \( \bar{b} \) | \( \bar{b} \) | \( \bar{b} \) | \( \bar{b} \) | \( \bar{b} \) |

Now suppose that we are given a \( k \)-good sequence \( S(f) \) with \( k \geq 4 \). Let \( r_0, \ldots, r_{k-3} \) be the weights associated to \( \Delta(f) \) and \( s_0, \ldots, s_{k-3} \) be the weights associated to \( \Delta^k(f) \) using the lemma [2]. First, we will prove that \( r_0, \ldots, r_{k-3} \) and \( s_0, \ldots, s_{k-3} \) satisfy the equation [1] of Theorem 1 by showing that \( r_j = -s_{k-1-j} \) for \( 2 \leq j \leq k-3 \).

From (g3) we know that, for \( 3 \leq j \leq k-2 \), \( S(f) \) contains the term \( \frac{2^{-k+j-1}}{2^{-1}} \). Let \( 2 \leq i, j \leq k-3 \). Thus, we have one of the following situations:

1. \( S(f) = \left\{ \frac{2^{-k+1}}{2^{-1}}, \frac{2^{-k+1}}{2^{-2}}, S^{(1)}, \frac{2^{-k+i-2}}{2^{-i-1}}, S^{(2)}, \frac{2^{-i+1}}{2^{-i+1}}, S^{(3)}, \frac{2^{-i+2}}{2^{-i+2}}, S^{(4)}, \right\}, \)
or
2. \( S(f) = \left\{ S^{(1)}, \frac{2^{-k+i-2}}{2^{-i-1}}, S^{(2)}, \frac{2^{-k+i-1}}{2^{-i-1}}, S^{(3)}, \frac{2^{-k+i}}{2^{-i-1}}, S^{(4)}, \frac{2^{-k+i-1}}{2^{-i-1}}, S^{(5)}, \frac{2^{-k+i}}{2^{-i-1}}, S^{(6)}, \right\}, \)
or
3. \( S(f) = \left\{ \frac{2^{-k+1}}{2^{-1-\delta}}, S^{(1)}, \frac{2^{-k+i-2}}{2^{-i-1}}, S^{(2)}, \frac{2^{-k+i-1}}{2^{-i-1}}, S^{(3)}, \frac{2^{-k+i}}{2^{-i-1}}, S^{(4)}, \frac{2^{-k+i-1}}{2^{-i-1}}, S^{(5)}, \frac{2^{-k+i}}{2^{-i-1}}, S^{(6)}, \right\}, \)

where \( \delta \in \{0, 1\} \) and \( S^{(q)} \) are subsequences, for \( 1 \leq q \leq 4 \). Then, in each one of the above cases we have \( r_i = -1 = -s_{k-1-i} \) and \( r_j = 1 = -s_{k-1-j} \). Now we will find a cyclic permutation \( \sigma \) such that \( s_0, s_1 \) and \( \sigma(1) \) satisfy the equation [2] of Theorem 1. Let \( p \) be the position of the mark on \( \Delta_\sigma(f) \). Define \( \sigma(1) \) as the integer satisfying \( \sigma(1) + p - 1 \equiv 1 \pmod{k} \). Let \( l \) be the position of the first \( 2^{-k+1} \) on \( \Delta_\sigma(f) \). Since \( S(f) \) is \( k \)-good we have four possible cases:
1. \( S(f) = \left\{ S^{(1)}, \frac{2^{-k+1}}{2}, \frac{2^{-2}}{2}, S^{(2)} \right\}, \) so that \( s_0 = s_1 = -1 \) and \( l \equiv p \pmod{k} \).

2. \( S(f) = \left\{ S^{(1)}, \frac{2^{-k+1}}{2}, \frac{2^{-2}}{2}, S^{(2)} \right\}, \) so that \( s_0 = 1, s_1 = -1 \) and \( l \equiv p - 1 \pmod{k} \).

3. \( S(f) = \left\{ S^{(1)}, \frac{2^{-2}}{2}, \frac{2^{-1}}{2}, S^{(2)} \right\}, \) so that \( s_0 = -1, s_1 = 1 \) and \( l \equiv p - 1 \pmod{k} \).

4. \( S(f) = \left\{ S^{(1)}, \frac{2^{-2}}{2}, \frac{2^{-1}}{2}, \frac{2}{2}, S^{(2)} \right\}, \) so that \( s_0 = s_1 = 1 \) and \( l \equiv p - 2 \pmod{k} \).

which correspond exactly with equation (2).

Finally, for the ‘only if’ part, it suffices to remark that given a \( k \)-thin sequence \( \Delta_x(f) \), there are exactly four \( k \)-good sequences \( S(f) \). This coincides with the number of thin marked trees satisfying the equations of theorem 1 that could be a target tree for a given thin source tree.

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