HOLOMORPHIC PROJECTION AND DUALITY
FOR DOMAINS IN COMPLEX PROJECTIVE SPACE

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ABSTRACT. We show that the efficiency of a natural pairing between certain projectively
invariant Hardy spaces on dual strongly C-linearly convex real hypersurfaces in complex
projective space is measured by the norm of the corresponding Leray transform.

1. INTRODUCTION

Let $S$ be a smooth compact real hypersurface in complex projective space $\mathbb{CP}^n$. $S$ is said
to be strongly $C$-linearly convex if all complex tangent hyperplanes to $S$ lie to one side of $S$
with minimal order of contact (see §5 below). When $S$ is strongly $C$-linearly convex, the
set of all complex tangents hyperplanes to $S$ form a smooth strongly $C$-linearly convex
real hypersurface $S^*$ in the dual projective space $\mathbb{CP}^n^*$. There is a natural
$C$-bilinear pairing between the space of square-integrable sections over $S$ of the $n$th
power of the tautological line bundle and the corresponding space of sections over $S^*$. The natural generalization to this setting of the one-dimensional Cauchy
transform is the Leray transform defining a projection operator from the $L^2$ section spaces
just described onto the corresponding Hardy spaces. The Leray transform is in a suitable
sense self-adjoint with respect to the pairing mentioned above.

In this paper we show that the norm of the Leray transform measures the effectiveness
of the induced pairing on Hardy spaces. The sharp form of this result requires the use of
specific $L^2$ norms. The norms we use are based on Fefferman’s measure weighted with a
certain geometric invariant of $S$ and $S^*$. The norms along with every other construct
mentioned so far are Möbius-invariant (that is, invariant under automorphisms of $\mathbb{CP}^n$), and
in fact these particular norms are the only Möbius- and duality-invariant norms based on
second-order geometric data with the desired properties.

The paper is organized as follows. In §2 we adapt a standard treatment of line bundles on $\mathbb{CP}^n$ to fit the needs of the current paper. §3 contains a brief account of the one-
dimensional versions (where available) of the constructions from later in paper. In §4 we
explain how Fefferman’s measure can be used to set up invariantly defined $L^2$ spaces and
the norm of the corresponding Hardy spaces (of line bundle sections) on strongly pseudoconvex hyper-
surfaces. This is followed in §5 by a discussion of the Möbius-invariant geometry of real
hypersurfaces and in §6 by an account of duality between real hypersurfaces in $\mathbb{CP}^n$. In
§7 we construct our $C$-bilinear pairing between $L^2$ spaces on dual hypersurfaces. (This

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1
pairing is connected with the Fantappi\`e transform [Fan] and in particular with the variant given in §3.2 of [APS]. The pairing is not a duality pairing between the $L^2$ spaces constructed in [4]; we remedy this by introducing appropriate (invariantly) weighted $L^2$ spaces and corresponding Hardy spaces in [S]. In [8] we bring the Leray transform $L_S$ into the picture, proving in particular the connection mentioned above between the norm of $L_S$ and the pairing of the weighted Hardy spaces.

The theory we construct is in some respects simpler and in some respects richer in dimension two than in higher dimensions. The paper is written to make it reasonably easy for the reader to focus primarily on the two dimensional case, and some readers will want to exercise this option on a first pass.

2. THE BUNDLES $\mathcal{O}(j,k)$

(Compare [GrHa, §1.3] or [APS, §3.2].)

We define a family $\mathcal{O}(j,k)$ of $\mathbb{C}$-bundles over complex projective space $\mathbb{C}P^n$. (The bundles will be holomorphic only when $k = 0$.)

A section of $\mathcal{O}(j,k)$ over a subset $E$ of $\mathbb{C}P^n$ is given by a complex-valued function $F$ on the corresponding dilation-invariant subset of $\mathbb{C}^{n+1} \setminus \{0\}$ satisfying the homogeneity condition

$$F(\lambda \zeta) = \lambda^j \overline{\lambda}^k F(\zeta).$$

(In this paper, $j$ and $k$ will generally be integers but in fact it suffices to have $j, k \in \mathbb{R}$, $j - k \in \mathbb{Z}$.)

We denote by $\Gamma(E; j,k)$ the space of continuous sections of $\mathcal{O}(j,k)$ over $E$.

If $F \in \Gamma(E; j,k)$ then $\overline{F} \in \Gamma(E; k,j)$. If $F_1 \in \Gamma(E; j_1,k_1)$ and $F_2 \in \Gamma(E; j_2,k_2)$ then $F_1 F_2 \in \Gamma(E; j_1+j_2, k_1+k_2)$.

It makes sense to declare that $F \in \mathcal{O}(j,j)$ is positive when $F$ takes values in $\mathbb{R}_+$. If $F \in \mathcal{O}(j,j)$ is positive then $F^{k/j} \in \mathcal{O}(k,k)$ is well-defined and positive.

Similar remarks apply to sections which are $\geq 0$.

If $F \in \Gamma(E; j,k)$ then we may define

$$|F| = (\overline{F} F)^{1/2} \in \Gamma \left( E; j+k, j+k \right).$$

The bundle $\mathcal{O}(-n-1,0)$ may be identified with the canonical bundle of $(n,0)$-forms by identifying a form written as

$$f(z_1, \ldots, z_n) \, dz_1 \wedge \ldots \wedge dz_n$$

in standard affine coordinates with $F \in \Gamma(E; -n-1,0)$ via the formulae

$$F(\zeta_0, \zeta_1, \ldots, \zeta_n) = \zeta_0^{-n-1} f(\zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0)$$

$$f(z_1, \ldots, z_n) = F(1, z_1, \ldots, z_n).$$

Similarly, a volume form

$$f(z_1, \ldots, z_n) \, dz_1 \wedge \ldots \wedge dz_n \wedge \overline{dz_1 \wedge \ldots \wedge dz_n}$$
may be identified with a section \( F \in \Gamma(E; -n - 1, -n - 1) \). For \( n \) even our notion of positivity of \( F \) coincides with the usual notion of positivity of the corresponding volume form; thus \( F \geq 0 \) implies \( \int_E f(z_1, \ldots, z_n) \, dz_1 \wedge \ldots \wedge dz_n \geq 0 \). For \( n \) odd we have instead that \( F \geq 0 \) implies \( \int_E i f(z_1, \ldots, z_n) \, dz_1 \wedge \ldots \wedge dz_n \geq 0 \).

For \( F \in \Gamma(E; j, k) \) the above remarks allow us to interpret \( \langle FF^{\frac{n+1}{j+k}} \rangle = |F|^{-\frac{2n+1}{j+k}} \) as a volume form; it is guaranteed that \( \langle FF^{\frac{n+1}{j+k}} \rangle \geq 0 \).

Automorphisms of \( \mathbb{C}P^n \) are induced by matrices \( M \in SL(n + 1, \mathbb{C}) \). The pullback operation \( M^* : F \mapsto F \circ M \) induces lifted automorphisms of the line bundles \( \mathcal{O}(j, k) \) respecting the conjugation and multiplication operations. The lifted automorphisms are in general not unique: each automorphism of \( \mathbb{C}P^n \) is represented by \( (n + 1) \) distinct choices of \( M \) differing by roots of unity, and these give rise to \( \frac{n+1}{\gcd(j-k,n+1)} \) distinct lifted automorphisms of \( \mathcal{O}(j, k) \). Note that the lifts are in fact unique for the the canonical bundle \( \mathcal{O}(n + 1, 0) \) as well as for bundles of the form \( \mathcal{O}(j, j) \) or \( \mathcal{O}(j(n + 1), k(n + 1)) \).

**Remark 1.** All the bundles \( \mathcal{O}(j, k) \) are in fact trivial over \( \mathbb{C}^n \) (identified with \( \{ (\zeta_0 : \zeta_1 : \cdots : \zeta_n) \in \mathbb{C}P^n : \zeta_0 \neq 0 \} \)). In particular, a section of \( \mathcal{O}(j, k) \) over \( E \subset \mathbb{C}^n \) may be identified with a scalar function on \( E \) via the formulae

\[
F(\zeta_0, \zeta_1, \ldots, \zeta_n) = \frac{\zeta_j^k}{\zeta_0} f(\zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0) \\
f(z_1, \ldots, z_n) = F(1, z_1, \ldots, z_n).
\]

In this notation we have

\[
M^* f = (M_{0,0} + M_{0,1} z_1 + \ldots M_{0,n} z_n)^j \cdot (M_{0,0} + M_{0,1} z_1 + \ldots M_{0,n} z_n)^k \cdot (f \circ \Psi_M)
\]

where \( M = (M_{j,k})_{j,k=0}^n \) and

\[
\Psi_M(z) = \\
\begin{pmatrix}
M_{1,0} + M_{1,1} z_1 + \ldots M_{1,n} z_n & M_{n,0} + M_{n,1} z_1 + \ldots M_{n,n} z_n \\
M_{0,0} + M_{0,1} z_1 + \ldots M_{0,n} z_n & M_{0,0} + M_{0,1} z_1 + \ldots M_{0,n} z_n
\end{pmatrix}.
\]

In view of the identification of \( \mathcal{O}(-n - 1, 0) \) with the canonical bundle, it is convenient to use the notation

\[
f(z_1, \ldots, z_n) (dz_1 \wedge \ldots \wedge dz_n)^{\frac{j}{n+1}} (dz_1 \wedge \ldots \wedge dz_n)^{\frac{k}{n+1}}
\]

for sections of \( \mathcal{O}(j, k) \).

### 3. Dimension One

Let \( \gamma \) be a smooth oriented simple closed curve in the Riemann sphere \( \mathbb{C}P^1 \), and let \( \Omega_+ \) and \( \Omega_- \) denote the components of \( \mathbb{C}P^1 \setminus \gamma \) enclosed positively and negatively, respectively, by \( \gamma \).

Using the conventions of Remark\[\Pi\], a section \( f \) of \( \mathcal{O}(-1, 0) \) will be notated as \( f(z) \sqrt{dz} \), where \( z \) is the standard affine coordinate on \( \mathbb{C} \subset \mathbb{C}P^1 \).

We have the hermitian pairing

\[
\langle f(z) \sqrt{dz}, g(z) \sqrt{dz} \rangle = \int_\gamma f(z) \overline{g(z)} \, |dz|
\]
on \( \Gamma(\gamma; -1, 0) \). We denote the resulting Hilbert space by \( L^2(\gamma; -1, 0) \).

We denote by \( \mathcal{H}_+(\gamma) \) and \( \mathcal{H}_-(\gamma) \) the Hardy spaces consisting of boundary values in \( L^2(\gamma; -1, 0) \) of holomorphic sections of \( \mathcal{O}(-1, 0) \) on \( \Omega_+ \) and \( \Omega_- \), respectively.

We also have a \( C \)-bilinear pairing

\[
\langle \langle f(z) \sqrt{dz}, g(z) \sqrt{dz} \rangle \rangle = \int_{\gamma} f(z) g(z) \, dz
\]

on \( L^2(\gamma; -1, 0) \).

We define the inner and outer Cauchy transforms \( C_\pm \) by

\[
C_\pm \left( f(w) \sqrt{dw} \right) = \frac{1}{2} f(z) \sqrt{dz} \pm \frac{1}{2\pi i} \text{P. V.} \left \langle \left \langle f(w) \sqrt{dw}, \frac{\sqrt{dw} \sqrt{dz}}{w - z} \right \rangle \right \rangle ;
\]

here the pairing is taken with respect to the \( w \) variable and P. V. denotes the principal value of the singular integral. The Cauchy transforms define bounded projection operators

\[
C_\pm : L^2(\gamma; -1, 0) \to \mathcal{H}_\pm(\gamma).
\]

We have

\[
\langle \langle C_+ f, g \rangle \rangle = \langle \langle f, C_- g \rangle \rangle = \langle \langle C_+ f, C_- g \rangle \rangle.
\]

The norm \( \| C_+ \| = |C_-| \) measures the effectiveness of the pairing \( \langle \langle \cdot, \cdot \rangle \rangle \) between \( \mathcal{H}_+(\gamma) \) and \( \mathcal{H}_-(\gamma) \):

\[
\frac{1}{\| C_\pm \|} = \inf_{f \in \mathcal{H}_+(\gamma), \|f\| = 1} \sup_{h \in \mathcal{H}_-(\gamma), \|h\| \leq 1} |\langle \langle f, h \rangle \rangle|.
\]

(The proof follows that of Corollary 32 below.) The norm \( \| C_\pm \| \) will equal 1 if and only if \( \gamma \) is a circle (or extended line). (This follows from [KeSt2, §7].)

The constructions above are all invariant under the Möbius group of automorphisms of \( \mathbb{C}P^1 \). As explained in §2 there is a \( \pm \) ambiguity in the lifting of an automorphism of \( \mathbb{C}P^1 \) to the bundle \( \mathcal{O}(-1, 0) \), but the \( \pm \) signs all cancel in formulae such as (3.1), (3.2), (3.4) and (3.5).

There are a number of basic one-dimensional results that do not admit higher-dimensional versions in the theory developed below. They include

- the formula \( \langle \langle f, h \rangle \rangle = 0 \) for \( f, h \in \mathcal{H}_+(\gamma) \) or \( f, h \in \mathcal{H}_-(\gamma) \) (following from Cauchy’s theorem)

and

- the identity \( f = C_+ f + C_- f \) exhibiting \( L^2(\gamma; -1, 0) \) as the algebraic direct sum of \( \mathcal{H}_+(\gamma) \) and \( \mathcal{H}_-(\gamma) \).

4. AN INVARIANT HARDY SPACE

In this section we set up a natural hermitian inner product on \( \Gamma(S; -n, 0) \) over a smooth compact pseudoconvex real hypersurface \( S \) in \( \mathbb{C}P^n \).
Let $J : T\mathbb{CP}^n \to T\mathbb{CP}^n$ be the complex structure tensor and let $HS = TS \cap JTS$ denote the maximal complex subspace of $TS$. Let $L : HS \times HS \to TS/HS$ be the Levi-form of $S$. $L$ can be defined in terms of the Lie bracket by the formula

$$L(X_1, X_2) \equiv [X_1, JX_2] \mod HS$$

for $HS$-valued vector fields $X_1$ and $X_2$. Alternately, $L$ can be constructed by taking the hermitian part of the restriction to $HS$ of the geometric second fundamental form of $S$ and then applying $J$.

### 4.1. Dimension two.

Given a $(2, 2)$-form $\omega \geq 0$ defined on a set containing $S$ we define a 3-form $\sigma_\omega$ on $S$ itself by the formula

$$\sigma_\omega \left( X, JX, L(X, X) \right) = \left( \omega \left( X, JX, L(X, X), JL(X, X) \right) \right)^{2/3}$$

for $X \in HS$. The formula above determines $\sigma_\omega$ uniquely and consistently on the Levi-nondegenerate portion of $S$ (see [Bar], §2) and $\sigma_\omega$ vanishes where the Levi-form vanishes.

We define the $L^2$-norm of a section $f$ of $O(-2, 0)$ on $S$ by the formula

$$\|f\|^2_S \overset{\text{def}}{=} 2^{-5/3} \int_S \sigma(f\overline{f})^{3/2}.$$ 

The corresponding inner product is given by

$$\langle f, g \rangle_S = \int_{S \setminus \{g = 0\}} \left( \frac{f}{g} \right) \sigma(\overline{g})^{3/2}$$

$$= \int_{S \setminus \{f = 0\}} \left( \frac{g}{f} \right) \sigma(\overline{f})^{3/2}.$$ 

If $S$ is strongly pseudoconvex then this norm will be comparable with (but not identical to) a norm based on surface area. In particular, if $S \subset \mathbb{C}^2 \subseteq \mathbb{CP}^2$ then a section of $F \in \Gamma(E; -2, 0)$ corresponding to the form $f(z_1, z_2) \, dz_1 \wedge dz_2 = F(1, z_1, z_2) \, dz_1 \wedge dz_2$ has square norm equal to

$$2^{-1/3} \int_S |f|^2 |L|^{1/3} \, dS$$

where $|L|$ is the euclidean norm of the Levi-form and $dS$ is euclidean surface area; the inner product is given similarly by

$$2^{-1/3} \int_S \left( \frac{g}{f} \right) |L|^{1/3} \, dS.$$ 

The norm (4.2) (and the higher-dimensional version given below) first appeared (in different notation) in work of Fefferman [Fef, p. 259].

We denote by $L^2(S; -2, 0)$ the Hilbert space arising from the norm just constructed.

**Remark 2.** The same set-up constructs spaces $L^2(S; j, k)$ for $j + k = -2$. We note for future reference that conjugation defines an isometry between $L^2(S; 0, -2)$ and $L^2(S; -2, 0)$. 
From [2] we see that each automorphism of \( \mathbb{C}P^2 \) gives rise to three lifted automorphisms of \( \mathcal{O}(-2,0) \) yielding three isometries of \( L^2(S;-2,0) \). Note that a change in the choice of lift has the effect of multiplying the pull-back of \( f \) by a root of unity which disappears when a norm or inner product is computed. A similar remark applies to the inner product.

4.2. Higher dimension. In higher dimension we convert an \((n,n)\)-form \( \omega \geq 0 \) on a set containing \( S \) to a \((2n-1)\)-form \( \sigma_\omega \) on \( S \) itself by setting

\[
\sigma_\omega \left( X_1, JX_1, \ldots, X_{n-1}, JX_{n-1}, \det \frac{1}{n+1} (L_{j,k}) \right)
= \left( i^{n^2} \omega \left( X_1, JX_1, \ldots, X_{n-1}, JX_{n-1}, \det \frac{1}{n+1} (L_{j,k}), J \det \frac{1}{n+1} (L_{j,k}) \right) \right)^{\frac{n}{n+1}};
\]

here we have chosen a complex basis \( X_1, \ldots X_{n-1} \) of \( HS \) and set

\[
L_{j,k} = \mathcal{L}(X_j, X_k) - i \mathcal{L}(JX_j, X_k)
\]

so that \( (L_{j,k}) \) is the (complex-)hermitian matrix representing \( \mathcal{L} \) with respect to the given basis [Bar, §2].

We then define the \( L^2 \)-norm of a section \( f \) of \( \mathcal{O}(-n,0) \) on \( S \) by

\[
\|f\|_S^2 \overset{\text{def}}{=} 2 \frac{1-n-1}{n+1} \int_S \left| f \right|^2 \sigma(f) \frac{\partial}{\partial \nu} \left( \frac{n-1}{n+1} \right) dS
\]

with corresponding formulae for the inner product. In euclidean terms this norm coincides with

\[
2 \frac{1-n}{n+1} \int_S |f|^2 |\det \mathcal{L}| \frac{\partial}{\partial \nu} dS = 2 \int_S \frac{|f|^2}{d\mathcal{r} \cdot N} \det \left( \begin{array}{cc} 0 & r_j \\ r_k & r_{jk} \end{array} \right) \frac{1}{n+1} dS
\]

where \( r \) is a defining function for \( S \) and \( N \) is the outward unit normal.

Remark 3. The dimensional constants in (4.3) and (4.4) were unspecified in [Fef] and differ from those used in a related construction in [Bar]. They have been chosen to simplify formulas below (in particular those in Remarks 21 and 22).

Remark 4. It is not so crucial here that the ambient space be precisely \( \mathbb{C}P^n \); the bundle \( \mathcal{O}(-n,0) \) could be replaced by a holomorphic \( \mathbb{C} \)-bundle \( B \) equipped with an isomorphism between \( B^{n+1} \) and the \( n^{th} \) power of the canonical bundle (possibly tensored with a flat \( S^1 \) bundle).

4.3. The Hardy space. For \( S \) arising as the boundary of strongly pseudoconvex domain \( \Omega \) we define the Hardy space \( \mathcal{H}(S) \) to be \( L^2(S; -n,0) \)-closure of

\[
\{ f \in \Gamma(\Omega \cup S; -n,0) : f \text{ holomorphic on } \Omega \}.
\]

(In this setting, this construction will agree with other standard definitions of the Hardy space.)

The corresponding Szegő kernel \( K_S(z,w) \) is a section of \( \mathcal{O}(-n,0) \times \mathcal{O}(0,-n) \) on \( \Omega \times \Omega \) which is holomorphic in the first factor and anti-holomorphic in the second factor. \( K_S(z,w) \) is characterized by the conditions \( K_S(z,w) \in \mathcal{H}(S) \) for fixed \( w \in \Omega \); \( K_S(w,z) = \overline{K_S(z,w)} \); \( (f(w), K_S(w,z))_S = f(z) \) for \( f \in \mathcal{H}(\Omega), z \in \Omega \).
For $M \in SL(n + 1, \mathbb{C})$ inducing an automorphism of $\mathbb{CP}^n$ taking $S_1$ to $S_2$ we have an induced isometry $M^* : L^2(S_2; -n, 0) \to L^2(S_1; -n, 0)$ taking $\mathcal{H}(S_2)$ to $\mathcal{H}(S_1)$. Again, $M^*$ is not uniquely determined by the underlying automorphism of $\mathbb{CP}^n$. To be precise, if $M$ is replaced by $\omega M$ with $\omega^{n+1} = 1$ then the induced map on $L^2(S_2; -n, 0)$ picks up a factor of $\omega^{-n} = \omega$. But note that the induced map on $L^2(S_2; 0, -n)$ similarly picks up a factor of $\overline{\omega} = \omega^{-1}$, and this guarantees that the pullback operation in the Szegö transformation law

$$(4.6) \quad K_{S_1} = M^* K_{S_2}$$

is unambiguously determined by the underlying automorphism of $\mathbb{CP}^n$.

**Remark 5.** In the notation of Remark 4 if $\Psi_M$ maps a compact strongly pseudoconvex $S_1 \subset \mathbb{C}^n$ to $S^2 \subset \mathbb{C}^n$ then (4.6) reads

$$k_{S_1}(z, w) = k_{S_2}(\Psi_M(z), \Psi_M(z)) (|\det \Psi_M'(z)|)^{n/(n+1)} (|\det \Psi_M'(w)|)^{n/(n+1)}$$

in affine coordinates.

The transformation law (4.6) will apply to more general biholomorphic maps in the setting of Remark 4 if we assume that the map admits a suitable lift to a bundle map from $B_1$ to $B_2$. In the euclidean case, the condition is just that $\det \Psi'$ admit an $(n + 1)^{st}$ root.

**Question 6.** Which smoothly bounded weakly pseudoconvex domains have the property that functions in the space $\mathcal{H}$ are uniformly bounded on compact subsets of $\Omega$ by their $L^2(S; -n, 0)$-norms, allowing us to construct $\mathcal{H}(S)$ and $K_S$ as above?

5. Invariant geometry of hypersurfaces

For smooth real hypersurfaces $S \subset \mathbb{CP}^n$ it turns out that the restriction of the geometric second fundamental form to $HS$ is Möbius-invariant [Bol2, §2]. Applying the complex structure tensor $J$ we obtain a Möbius-invariant tensor mapping $HS \times HS$ to $TS/HS$. The unique decomposition of this tensor into its hermitian and anti-hermitian parts is also Möbius-invariant. As indicated in §4 the hermitian part is just the Levi-form $\mathcal{L}$. We denote the anti-hermitian part of this decomposition by $\mathcal{Q}$; thus we have

$$(5.1a) \quad \mathcal{L}(JX_1, JX_2) = \mathcal{L}(X_1, X_2)$$

$$(5.1b) \quad \mathcal{Q}(JX_1, JX_2) = -\mathcal{Q}(X_1, X_2).$$

5.1. Dimension two. Let $p$ be a point of a smooth real hypersurface $S \subset \mathbb{CP}^2$. After applying an automorphism of $\mathbb{CP}^2$ we may assume that $p = 0 \in \mathbb{C}^2 \subsetneq \mathbb{CP}^2$ with $T_0S = \mathbb{C} \times \mathbb{R}$, $H_0S = \mathbb{C} \times \{0\}$ and moreover that $S$ is locally the zero set of a defining function $\rho$ of the form

$$(5.2) \quad \rho(z_1, u + iv) = v - \alpha|z_1|^2 - \text{Re}(\beta z_1^2) + O(|z_1|^3 + |z_1|u + u^2);$$

here $\alpha \in \mathbb{R}$ corresponds to the Levi-form $\mathcal{L}$ and $\beta \in \mathbb{C}$ corresponds to the anti-hermitian form $\mathcal{Q}$. 
When $|\beta/\alpha| \neq 1$ then the normalizing coordinates in (5.2) are unique up to transformations of the form

\begin{align*}
  z_1 &= \frac{A\tilde{z}_1 + C\tilde{z}_2}{1 + D\tilde{z}_1 + E\tilde{z}_2} \\
  z_2 &= \frac{B\tilde{z}_2}{1 + D\tilde{z}_1 + E\tilde{z}_2} \quad (B > 0)
\end{align*}

(5.3)

with new coefficients

\begin{align*}
  \tilde{\alpha} &= \frac{|A|^2}{B} \alpha \\
  \tilde{\beta} &= \frac{A^2}{B} \beta.
\end{align*}

(5.4)

The epigraph of $S$ will be strongly pseudoconvex if and only if $\alpha > 0$. When this holds, $Q/L$ defines an $\mathbb{R}$-valued function on each $H_z S \setminus \{0\}$ which is $\mathbb{R}_+$-homogeneous of degree 0 and has $|\beta/\alpha|$ as its maximum absolute value. Thus the scalar function $|\beta/\alpha|$ defines a Möbius invariant of $S$.

A strongly pseudoconvex hypersurface $S$ is said to be strongly $C$-linearly convex when $|\beta/\alpha| < 1$. (See [APS] and [Hör] for more on this notion and its history, including variations in terminology. In particular, this notion is also known as strong (or strict) “lineal convexity.”) $S$ is strongly $C$-linearly convex in a neighborhood of $p \in S$ if and only if there is a Möbius transformation making $S$ strongly convex near $p$. Working globally, a compact hypersurface $S$ will be strongly $C$-linearly convex if and only if the complex hyperplanes in $\mathbb{CP}^n$ tangent to $S$ do not enter the pseudoconvex side of $S$ and have minimal order of contact with $S$ in all directions at the point of tangency. (See [APS, Remark 2.5.11].)

When $S$ is strongly $C$-linearly convex, the level sets of the tensor $L + Q$ are ellipses in each $H_z S$ with major-to-minor axis ratio equal to $\frac{|\alpha| + |\beta|}{|\alpha| - |\beta|} = \frac{1 + |\beta/\alpha|}{1 - |\beta/\alpha|}$.

5.2. A Beltrami-like tensor. Let $S$ be a smooth real hypersurface in $\mathbb{CP}^2$ and let $\rho$ be a smooth defining function for $S$, viewed as a real-valued section of $\mathcal{O}(0,0)$. Multiplying $\rho$ by $\|z\|^4$ we may insist instead that $\rho$ be a real-valued section of $\mathcal{O}(2,2)$.

**Proposition 7.** If $S$ is strongly pseudoconvex then the quotient

\begin{equation}
  -4 \det (\rho_{jk}) / \det \left( \rho_{jk} \right)
\end{equation}

defines a section $\mathcal{B}_S \in \Gamma(S; -3, 3)$. (Here the subscripts denote differentiation and range over $0, 1, 2$.)

The section $\mathcal{B}_S$ is independent of the choice of defining function $\rho$.

For $M \in \text{SL}(3, \mathbb{C})$ we have the transformation law

\begin{equation}
  \mathcal{B}_S = M^* \mathcal{B}_{M(S)}
\end{equation}

(5.6)

**Proof.** The $\rho_{j,k}$ are in $\Gamma(S; 0, 2)$, so the numerator is in $\Gamma(S; 0, 6)$. The $\rho_{j,k}$ are in $\Gamma(S; 1, 1)$, so the denominator is in $\Gamma(S; 3, 3)$. Thus the quotient is in $\Gamma(S; -3, 3)$ (where defined).
Applying an automorphism induced by $M \in SL(3, \mathbb{C})$ we find that the matrix in the numerator of (5.3) is multiplied on the right by $M$ and on the left by $M^*$; similarly, the matrix in the denominator is multiplied on the right by $M$ and on the left by $M^*$. The transformation law (5.6) follows immediately. (Note that the lift $M^*$ is uniquely determined in this situation.)

To verify that $B_S$ is independent of the choice of defining function, we may use the transformation law (5.6) and the set-up for (5.2) above to reduce to checking at $p = (\zeta_0 : 0 : 0)$ with $p$ given by

$$\mu \cdot \left( -\frac{i}{2} \zeta_0^2 \zeta_1 \zeta_2 + \frac{i}{2} \zeta_0 \zeta_1 \zeta_2 - \alpha \zeta_0 \zeta_1 \zeta_2 - \frac{\beta}{2} \zeta_0^2 \zeta_2 - \frac{\beta}{2} \zeta_0 \zeta_1 \zeta_2 + \ldots \right)$$

where $\mu$ is a smooth positive section of $\mathcal{O}(0,0)$ near $p$. Using $\mu_0 = \mu_{\zeta_0} = 0$ at $p$ we find that the expression in (5.5) is

$$\frac{\mu \beta \zeta_0^3}{\mu \alpha \zeta_0^3} = \frac{\beta \zeta_0^3}{\alpha \zeta_0^3}$$

at $p$. Since this is independent of $\mu$, the quotient is indeed independent of the choice of defining function. This computation also shows that the denominator cannot vanish. □

**Addendum to Proposition 7.** The scalar function $|B_S| \in \Gamma(S; 0, 0)$ coincides with the scalar invariant $|\beta/\alpha|$ discussed in §5.1 above.

On $\mathbb{C}^2$ we may use the convention of Remark 1 to write $B_S$ in the form

$$b_S(z_1, z_2) \frac{dz_1 \wedge dz_2}{dz_1 \wedge dz_2}$$

which describes a scalar-valued function on $\mathbb{C}$-linearly independent pairs $X, Y \in T_z \mathbb{C}^2$. Here $b_S(z_1, z_2) = B_S(1, z_2, z_2)$ and $B_S(\zeta_0, \zeta_1, \zeta_2) = \zeta_0^{-3} \zeta_0^2 b_S(\zeta_1 / \zeta_0, \zeta_2 / \zeta_0)$. We have

$$b_S(z_1, z_2) \frac{dz_1 \wedge dz_2}{dz_1 \wedge dz_2}(\lambda X, Y) = \lambda \lambda^{-1} b_S(z_1, z_2) \frac{dz_1 \wedge dz_2}{dz_1 \wedge dz_2}(X, Y)$$

$$= b_S(z_1, z_2) \frac{dz_1 \wedge dz_2}{dz_1 \wedge dz_2}(X, \lambda Y)$$

for $\lambda \in \mathbb{C} \setminus \{0\}$.

Writing $\rho(\zeta_0, \zeta_1, \zeta_2) = \frac{\zeta_0^2 \zeta_1}{\zeta_0^3} r(\zeta_1 / \zeta_0, \zeta_2 / \zeta_0)$ we find that

$$b_S(z_1, z_2) = -\det \left( \begin{array}{ccc} 0 & r_1 & r_2 \\ r_1 & r_{11} & r_{12} \\ r_2 & r_{12} & r_{22} \end{array} \right) \cdot \det \left( \begin{array}{ccc} 0 & r_1 & r_2 \\ r_1 & r_{11} & r_{12} \\ r_2 & r_{12} & r_{22} \end{array} \right).$$

**Remark 8.** This type of differential is reminiscent (up to conjugation or inversion) of the Beltrami differentials $\overline{\partial} f / \partial f$ prominent in the study of quasiconformal mappings in one complex variable. (Compare [KoRe].)

In the case where $S$ is strongly $\mathbb{C}$-linearly convex (i.e., when $|B_S| < 1$) we have already seen at the end of §5.1 that $|B_S| = |\beta/\alpha|$ may be identified with the eccentricity data for a families of ellipses in each $H_2 S$. The “argument” of $B_S$ is similarly determined by the following condition:
when \( X \in H_2S \setminus \{0\}, Y \in T_2S \setminus H_2S \) then

\[
B_S(z_1, z_2) \frac{dz_1 \wedge dz_2}{dz_1 \wedge dz_2}(X, Y)
\]

will be positive precisely when \( X \) points in the direction of the minor axis of the family of ellipses in \( H_2S \).

(Of course, this condition becomes undefined when the ellipses are circles, i.e., when \( \mathcal{B}_S \) vanishes.)

5.3. Examples.

(1) The unit spheres

\[
\Sigma_{\rho}^{(1)} \overset{\text{def}}{=} \{|z_1|^p + |z_2|^p = 1\}
\]

of the two-dimensional \( L^p \) spaces are strongly \( \mathbb{C} \)-linearly convex for \( p > 1 \) and \( z_1z_2 \neq 0 \), with

\[
\mathcal{B}_S = \frac{2 - p}{p} \frac{dz_1 \wedge dz_2}{z_1z_2} \frac{\overline{z_1z_2}}{\overline{dz_1} \wedge dz_2}.
\]

(2) The hypersurfaces

\[
\Sigma_{\gamma}^{(2)} \overset{\text{def}}{=} \{\text{Im } z_2 = |z_1|^{\gamma}\}
\]

are strongly \( \mathbb{C} \)-linearly convex for \( \gamma > 1 \) and \( z_1 \neq 0 \), with

\[
\mathcal{B}_S = \gamma - \frac{2}{\gamma} \frac{dz_1 \wedge dz_2}{z_1} \frac{\overline{z_1}}{\overline{dz_1} \wedge dz_2}.
\]

(3) The hypersurfaces

\[
\Sigma_{\alpha, \beta}^{(3)} \overset{\text{def}}{=} \{\text{Im } z_2 = \alpha |z_1|^2 + \text{Re } \beta z_1^2\}
\]

are strongly \( \mathbb{C} \)-linearly convex for \( |\beta| < \alpha \), with

\[
\mathcal{B}_S = \frac{\beta}{\alpha} \frac{dz_1 \wedge dz_2}{dz_1 \wedge dz_2}.
\]

(4) For tube hypersurfaces \( S \subset \mathbb{C}^2 \) invariant under all real translations we have

\[
\mathcal{B}_S = -\frac{dz_1 \wedge dz_2}{dz_1 \wedge dz_2};
\]

such hypersurfaces are never strongly \( \mathbb{C} \)-linearly convex.

Remark 9. Any \( S \) with \( \mathcal{B}_S \equiv 0 \) is a Möbius image of a sphere ([DeTr], [Bol2]). Any \( S \) with \( \mathcal{B}_S = K \frac{dz_1 \wedge dz_2}{dz_1 \wedge dz_2}, K \) constant, \( |K| \neq 0, 1 \) is an affine image of a hypersurface of the form \( \Sigma_{\alpha, \beta}^{(3)} \) above [Bol3].

The examples listed above have the property that \( \mathcal{B}_S \) extends to a constant times a meromorphic \( (2, 0) \)-form divided by its conjugate (reminiscent of Teichmüller differentials in one complex variable). This does not hold in general.

The following result gives an indication of the restrictions that \( \mathcal{B}_S \) must satisfy.
Theorem 10. A section \( \lambda(z_1) \frac{dz_1 \wedge dz_2}{dz_1 \wedge dz_2} \) of \( O(-3,3) \) will arise as \( B_S \) for a strongly \( C \)-linearly convex hypersurface

\[
(5.8) \quad \text{Im } z_2 = f(z_1)
\]

invariant under real \( z_2 \)-translations if and only if the coefficient \( \lambda(z_1) \) satisfies

\[
(5.9a) \quad \text{Im } \left( \lambda z_1 \bar{z}_1 - \bar{\lambda} \lambda z_1 z_1 + \frac{\bar{\lambda} \lambda^2 z_1 + \lambda \bar{\lambda} z_1}{1 - \lambda \bar{\lambda}} \right) = 0,
\]

\[
(5.9b) \quad |\lambda| < 1.
\]

More precisely, if \( U \) is open in \( C \) and \( S \subset U \times C \) is a strongly \( C \)-linearly convex hypersurface given by \( (5.8) \) then the coefficient \( \lambda(z_1) \) of \( B_S \) must satisfy \( (5.9) \) on \( U \). Conversely, when \( U \) is simply-connected then any solution of \( (5.9) \) gives rise to a corresponding \( S \subset U \times C \).

Note that \( (5.9a) \) may be viewed as an underdetermined hyperbolic system in the two \( \mathbb{R} \)-valued unknowns \( \text{Re } \lambda, \text{Im } \lambda \).

Remark 11. Hypersurfaces of the form \( (5.8) \) are often known as rigid [BRT].

Proof of Theorem 10. The inequality \( (5.9b) \) is already accounted for in the definition of a strongly \( C \)-linearly convex hypersurface.

Substituting \( r = \text{Im } z_2 - f(z_1) \) into \( (5.7) \) we find that the question of solving \( B_S = \lambda(z_1) \frac{dz_1 \wedge dz_2}{dz_1 \wedge dz_2} \) with \( S \) as in \( (5.8) \) reduces to that of the solvability of

\[
(5.10a) \quad f_{z_1z_1} = \lambda(z_1) f_{z_1z_1}
\]

for \( \mathbb{R} \)-valued \( f \). For typographical simplicity we drop subscripts to rewrite this as

\[
(5.10a) \quad f_{zz} = \lambda(z) f_{zz}
\]

(5.10b) \quad \overline{f} = f.

Differentiating \( (5.10a) \) with respect to \( \overline{z} \) we get

\[
f_{z\overline{z}} = \lambda f_{z\overline{z}} + \lambda z f_{z\overline{z}}.
\]

Conjugating and applying \( (5.10b) \) we have

\[
f_{z\overline{z}} = \overline{\lambda} f_{z\overline{z}} + \overline{\lambda}_z f_{z\overline{z}}.
\]

Since we are assuming in particular that \( S \) is strongly pseudoconvex we have \( f_{z\overline{z}} > 0 \). Setting \( h = \log f_{z\overline{z}} \) we have

\[
(5.11a) \quad h_z = \lambda h_{\overline{z}} + \lambda_{\overline{z}}
\]

(5.11b) \quad h_{\overline{z}} = \overline{\lambda} h_z + \overline{\lambda}_z

(5.11c) \quad \overline{h} = h.
Using linear algebra to isolate $h_z$ and $h_{\bar{z}}$ this may be rewritten as

\[(5.12a) \quad h_z = \frac{\lambda_z + \lambda_{\bar{z}}}{1 - \lambda \lambda} \]
\[(5.12b) \quad h_{\bar{z}} = \frac{\lambda \lambda_z + \bar{\lambda}_z}{1 - \lambda \lambda} \]
\[(5.12c) \quad \bar{h} = h. \]

Differentiating (5.12a) with respect to $z$ and matching this with the result of differentiating (5.12b) with respect to $z$ we obtain (5.9a).

For the converse, note that from the previous paragraph we see that (5.9a) is precisely the condition guaranteeing that the form

\[(5.13) \quad \lambda z + \lambda_{\bar{z}} - \lambda \lambda_z \bar{z} + \lambda \lambda_{\bar{z}} \bar{z} \]

is $d$-closed. Since (5.13) is self-conjugate we see that (5.9a) is precisely the condition required to solve (5.12) on simply-connected $U$.

It remains to show that solutions of (5.12) (equivalently, of (5.11)) give rise to solutions of (5.10). We begin by solving $g_{zz} = e^h, \bar{g} = g$ on $U$. We have

\[(g_{zz} - \lambda g_{z\bar{z}})z = e^h (h_z - \lambda_z - \lambda h_{\bar{z}}) = 0, \]

so we may write

\[g_{zz} = \lambda g_{z\bar{z}} - H_{zz} \]

with $H$ holomorphic. Writing

\[f = g + H + \bar{H} \]

we have $f = f$ and $f_{zz} = g_{zz} + H_{zz} = \lambda g_{z\bar{z}} = \lambda f_{z\bar{z}}$ as required. \qed

**Question 12.** What conditions must $\mathcal{B}_S$ satisfy for general (strongly $\mathcal{C}$-linearly convex) $S$?

**5.4. Higher dimension.** In higher dimension we say that $S$ is strongly $\mathcal{C}$-linearly convex when $L + Q$ is positive definite on each $H_zS$; in view of (5.1) this is equivalent to the condition

\[Q(X, X) < L(X, X) \text{ for } X \in H_zS \setminus \{0\}. \]

Instead of directly generalizing the Beltrami-type tensor $\mathcal{B}_S$ we introduce a scalar invariant

\[(5.14) \quad \varphi_S = \frac{16}{9} \left| \det^{-2} \left( \rho_{jk} \right) \cdot \det \left( \frac{\rho_{jk}}{\rho_{jk}} \right) \right|. \]

Writing $\rho(\zeta_0, \zeta_1, \ldots, \zeta_n) = \zeta_0^{r_{j_0}} r(\zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0)$ we find (after a few row operations) that

\[(5.15) \quad \varphi_S = \left| \det^{-2} \left( \begin{array}{c} r_j \\ r_k \end{array} \right) \cdot \det \left( \begin{array}{ccc} 0 & r_k & 0 \\ 0 & 0 & r_k \\ r_j & 0 & r_{jk} \\ 0 & r_k & r_{jk} \end{array} \right) \right|. \]
It is easy to check from the latter formula that \( \varphi_S \) is independent of the choice of defining function.

If \( S \) is defined by

\[
\operatorname{Im} z_n = \sum_{j,k=1}^{n-1} \alpha_{j,k} z_j \overline{z}_k + \operatorname{Re} \left( \sum_{j,k=1}^{n-1} \beta_{j,k} z_j \overline{z}_k \right)
\]

in affine coordinates near 0 we may diagonalize \( Q \) with respect to an \( L \)-orthogonal basis to reduce to the case where \( S \) has the form

\[
(5.16) \quad \operatorname{Im} z_n = \sum_{j=1}^{n-1} \alpha_j |z_j|^2 + \operatorname{Re} \left( \sum_{j=1}^{n-1} \beta_j z_j^2 \right) + O \left( ||z_1, \ldots, z_{n-1}||^3 + ||z_1, \ldots, z_{n-1}|| u + u^2 \right).
\]

Then

\[
(5.17) \quad \varphi_S = \prod_{j=1}^{n-1} \left( 1 - \frac{\beta_j}{\alpha_j} \right)
\]

at 0.

When \( S \) is strongly \( C \)-linearly convex we have \( \varphi_S > 0 \). In this situation \( \sigma \) may be viewed as the ratio of the volume of a sublevel set of \( L \) to the volume of the corresponding sublevel set of \( L + Q \).

In the case \( n = 2 \) we have \( \varphi_S = 1 - |B_S|^2 \).

**Remark 13.** The first part of Remark 9 generalizes to say that any strongly \( C \)-linearly convex \( S \) with \( \varphi_S \equiv 1 \) is a Möbius image of a sphere ([DeTr], [Bol2]).

**Remark 14.** If \( S \) is a compact strongly \( C \)-linearly convex real hypersurface in \( \mathbb{CP}^n \) then a suitable perturbation of a complex tangent hyperplane \( \mathcal{H}_z S \) will be disjoint from \( S \); sending the perturbed hyperplane to infinity we find that \( S \) is Möbius-equivalent to a compact hypersurface in \( \mathbb{C}^n \subseteq \mathbb{CP}^n \).

6. Dual Hypersurfaces

Let \( \mathbb{CP}^{n*} \) denote the projective space dual to \( \mathbb{CP}^n \). Each point \( \zeta^* = (\zeta_0^* : \ldots : \zeta_n^*) \) in \( \mathbb{CP}^{n*} \) determines a hyperplane

\[
\mathfrak{h}_{\zeta^*} \overset{\text{def}}{=} \{ \zeta \in \mathbb{CP}^n : \zeta_0 \zeta_0^* + \cdots + \zeta_n \zeta_n^* = 0 \}
\]

in \( \mathbb{CP}^n \); conversely, for \( \zeta \in \mathbb{CP}^n \) the set \( \{ \zeta^* \in \mathbb{CP}^{n*} : \zeta \in \mathfrak{h}_{\zeta^*} \} \) defines a hyperplane \( \mathfrak{h}_{\zeta^*} \) in \( \mathbb{CP}^{n*} \). We identify \( \mathbb{CP}^{n**} \) with \( \mathbb{CP}^n \).

For \( M \in \text{SL}(n + 1, \mathbb{C}) \) the automorphisms \( \zeta \mapsto M \zeta \) of \( \mathbb{CP}^n \) and \( \zeta \mapsto {}^t M^{-1} \zeta \) of \( \mathbb{CP}^{n*} \) satisfy

\[
M \zeta \in \mathfrak{h}_{M^{-1} \zeta} \text{ if and only if } \zeta \in \mathfrak{h}_{\zeta^*}.
\]
A smooth real hypersurface \( S \subset \mathbb{CP}^n \) induces a map \( D_S : S \to \mathbb{CP}^{n*} \) defined by the rule that \( h_{D_S(\zeta)} \) is the unique complex hyperplane in \( \mathbb{CP}^n \) that is tangent to \( S \) at \( \zeta \). (For \( S \subset \mathbb{C}^n \subset \mathbb{CP}^n \) we may write this in affine coordinates as \( h_{D_S(z)} = z + H_z(S) \).) This construction transforms properly under the \( SL(n+1, \mathbb{C}) \) action on \( \mathbb{CP}^n \times \mathbb{CP}^{n*} \) described in the previous paragraph.

**Theorem 15.** When \( S \) is compact and strongly \( \mathbb{C} \)-linearly convex then \( D_S \) maps \( S \) diffeomorphically onto a smooth strongly \( \mathbb{C} \)-linearly convex hypersurface \( S^* \subset \mathbb{CP}^{n*} \). This map satisfies the contact condition

\[
(6.1) \quad D'_S(\zeta) \left( H_{\zeta} S \right) = H_{D_S(\zeta)} S^*
\]

but this map is never \( \mathbb{C} \)-linear and thus \( D_S \) is never a CR map.

Moreover, \( S^{**} = S \) and \( D_{S^*} \circ D_S = I_S \).

We further have

\[
(6.2) \quad |B_{S^*}| \circ D_S = |B_S|
\]

for \( n = 2 \) and

\[
(6.3) \quad \varphi_{S^*} \circ D_S = \varphi_S
\]

in general.

Little or none of this is new, but for convenience we provide below a presentation of the purely local parts of this result. For the global aspects see [APS, S 2.5] (as well as [MT]).

**6.1. Examples.** Returning to the examples of \[5.3\] it is not hard to verify that the dual of \( \Sigma_p^{(1)} \) is Möbius-equivalent to the standard dual \( \Sigma_p^{(1)/(p-1)} \); that the dual of \( \Sigma^2_\gamma \) is Möbius-equivalent to \( \Sigma^2_{\gamma/(\gamma-1)} \) and that \( \Sigma^2_{\alpha,\beta} \) is self-dual (up to Möbius equivalence).

**6.2. Dimension two.** To study the above construction locally near a point \( \zeta \in S \subset \mathbb{CP}^2 \) we may first apply an automorphism of \( \mathbb{CP}^2 \) to reduce to the case where \( \zeta = (1 : 0 : 0) \) and \( S \) is given locally by \[5.2\] with respect to affine coordinates \( z_1 = \zeta_1/\zeta_0, u + iv = z_2 = \zeta_2/\zeta_0 \). Setting \( \zeta^* = D_S(\zeta) \) and using affine coordinates \( \eta_1 = -\zeta_1*/\zeta_2*, \eta_2 = \zeta_0*/\zeta_2^* \) we have from \( \zeta \in h_{\zeta^*} \) that

\[
(6.4a) \quad z_2 + \eta_2 = z_1\eta_1.
\]

Since this line is parallel in \( \mathbb{C}^2 \) to \( H_zS \) we have

\[
(6.4b) \quad dz_2 = \eta_1 \, dz_1 \text{ on } H_zS.
\]

For \( z \in S \) near zero, \(6.4b\) determines \( \eta_1 \) and then \(6.4a\) determines \( \eta_2 \). From this and \[5.2\] we deduce that the derivative of \( D_S \) at 0 takes the form

\[
D'_S(0) : T_0S = \mathbb{C} \times \mathbb{R} \to T_{0S^*} = \mathbb{C} \times \mathbb{R}
\]

\[
(6.5) \quad \begin{pmatrix} z_1 \\ u \end{pmatrix} \mapsto \begin{pmatrix} 2i\beta z_1 + 2i\alpha \zeta_1 \\ -u \end{pmatrix}.
\]
In particular we see that $\mathcal{D}_S'(0)$ maps $H_0 S = C \times \{0\}$ to $H_0 S^* = C \times \{0\}$ but the assumption that $\alpha \neq 0$ guarantees that this last map is not $C$-linear; it follows that $\mathcal{D}_S$ is contact but not $CR$. Differentiating (6.4a) along $S$ and applying (6.4b) and the contact condition (6.1) we find that

$$d\eta_2 = z_1 d\eta_1 \text{ on } H_\eta S^*. \tag{6.4c}$$

The equations (6.4c) and (6.4a) allow us to determine $z \in S$ from $\eta \in S^*$. The symmetry of the equations (6.4) shows that $\mathcal{D}_S^* \circ \mathcal{D}_S = I_S$ and hence that $S^{**} = S$.

Remark 16. The equations (6.4) are reminiscent of the equations describing the Legendre transform of a strongly convex real planar curve (see for example [Hör, p. 18]).

Remark 17. The projective or even affine invariance properties of the dualization are not so transparent from this formulation, but for future reference we note that the transformation (5.3) dualizes to

$$\eta_1 = \frac{B \tilde{\eta}_1 + BD \tilde{\eta}_2}{1 + C \tilde{\eta}_1 + \left( \frac{CD}{\alpha} - E \right) \tilde{\eta}_2} \tag{6.6}$$

$$\eta_2 = \frac{B \tilde{\eta}_2}{1 + C \tilde{\eta}_1 + \left( \frac{CD}{\alpha} - E \right) \tilde{\eta}_2}.$$ 

The description of $\mathcal{D}_S'(0)$ above allows us to deduce that $S^*$ has the form

$$\text{Im} \eta_2 = -\frac{\alpha}{4(\alpha^2 - |\beta|^2)} |\eta_1|^2 - \text{Re} \frac{\bar{\beta}}{4(\alpha^2 - |\beta|^2)} \eta_1^2 + O \left( |\eta_1|^3 + |\eta_1 \text{ Re } \eta_2| + (\text{Re } \eta_1)^2 \right). \tag{6.7}$$

The transformation law (6.2) follows immediately.

6.3. Higher dimension. Using affine coordinates $z_j = \zeta_j / \zeta_0$ for $1 \leq j \leq n$, $\eta_j = -\zeta_j^* / \zeta_n^*$ for $1 \leq j \leq n - 1$ and $\eta_n = \zeta_0^* / \zeta_n^*$ we find that (6.4) is replaced by

$$z_n + \eta_n = z_1 \eta_1 + \cdots + z_{n-1} \eta_{n-1} \tag{6.8a}$$

$$dz_n = \eta_1 dz_1 + \cdots + \eta_{n-1} dz_{n-1} \text{ on } H_z S \tag{6.8b}$$

$$d\eta_n = z_1 d\eta_1 + \cdots + z_{n-1} d\eta_{n-1} \text{ on } H_\eta S^*. \tag{6.8c}$$

To study the duality near a general point of $S$ we may normalize as before to reduce to the study of (5.16) near 0. It will be convenient to apply suitable coordinate dilations so as to further assume that

$$\alpha_j^2 - |\beta_j|^2 = 1/4. \tag{6.9}$$
Then (6.5) is replaced by
\[
D'_S(0): T_0S = \mathbb{C}^{n-1} \times \mathbb{R} \rightarrow T_0S^* = \mathbb{C}^{n-1} \times \mathbb{R}
\]
\[
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_{n-1} \\
  u
\end{pmatrix} 
\mapsto 
\begin{pmatrix}
  2i\beta_1 z_1 + 2i\alpha_1 \bar{z}_1 \\
  \vdots \\
  2i\beta_{n-1} z_{n-1} + 2i\alpha_{n-1} \bar{z}_{n-1} \\
  -u
\end{pmatrix}
\]
(6.10)
and (6.7) is replaced by
\[
\text{Im} \eta_n = -\sum_{j=1}^{n-1} a_j |\eta_j|^2 - \text{Re} \left( \sum_{j=1}^{n-1} \bar{\beta}_j \eta_j^2 \right)
\]
\[
+ O \left( \|\eta_1, \ldots, \eta_{n-1}\| \|\eta_1, \ldots, \eta_{n-1}\| |\eta_n| + (|\eta_n|)^2 \right).
\]
(6.11)

The transformation law (6.3) now follows from (5.17).

Remark 18. From (6.10) we see that the maps (6.1) will be anti-$\mathbb{C}$-linear for all $\zeta \in S$ if and only if the $\beta_j$ always vanish, that is, if and only if $\varphi_S \equiv 1$. From Remark 13 it follows that $D_S$ is anti-CR if and only if $S$ is a Möbius image of a sphere.

7. Transfer and pairing between dual hypersurfaces

7.1. Dimension two. Let $S \subset \mathbb{CP}^2$ be a compact strongly $\mathbb{C}$-linearly convex real hypersurface.

Theorem 19. The map $D_S : S \rightarrow S^*$ lifts to a bundle isomorphism between $\mathcal{V}(0,-2)$ over $S$ and $\mathcal{V}(-2,0)$ over $S^*$ such that the corresponding map $\mathcal{I}_S : \Gamma(S^*; -2,0) \rightarrow \Gamma(S; 0,-2)$ has the following properties:

1. $\| (1 - |B_S|^2)^{1/3} \mathcal{I}_S f \|_S = \| f \|_{S^*}$;
2. The construction of $\mathcal{I}_S$ respects the $\text{SL}(3, \mathbb{C})$-action on $\mathbb{CP}^2 \times \mathbb{CP}^{2*}$ described in 6;
3. $\mathcal{I}_S^* \left( \mathcal{I}_S f \right) = (1 - |B_S|^2)^{-2/3} f$.
4. If $\mathcal{B}_S f^3 \in \Gamma(S^*; -3, -3)$ is $\geq 0$ then $B_S (\mathcal{I}_S f)^3 \in \Gamma(S; -3, -3)$ is $\geq 0$.

Proof. We describe the map at a pair of corresponding points $\zeta \in S, \zeta^* = D_S(\zeta) \in S^*$. After applying an automorphism of $\mathbb{CP}^2$ we may assume that $\zeta = (1 : 0 : 0), \zeta^* = (0 : 0 : 1)$ with $S$ and $S^*$ given locally in terms of the corresponding affine coordinates by (5.2) and (6.7). We define the map by setting

\[
\mathcal{I}_S: \left( f (d\eta_1 \wedge d\eta_2)^2 \right)^{2/3} \mapsto (2\alpha)^{2/3} f \frac{dz_1 \wedge dz_2}{2/3}
\]
at 0.
To check that this is a valid definition we must see that it transforms properly under the coordinate changes given in (5.3), (6.6) which linearize to $z_1 = A\tilde{z}_1 + C\tilde{z}_2$, $z_2 = B\tilde{z}_2$, $\eta_1 = \frac{B}{A}\tilde{\eta}_1 + \frac{BD}{A}\tilde{\eta}_2$, $\eta_2 = B\tilde{\eta}_2$. Combining these with (5.4) we see that the definition of $T_S$ in the new coordinates is consistent with that in the old coordinates. Also note that the “lift ambiguity” for the bundles in question may introduce a cube-root-of-unity factor $\omega$ into the left side of (7.1) which cancels with the corresponding factor of $\omega^{-1} = \omega$ introduced into the right side.

The invariance property (2) is clear from the construction.

Condition (1) above follows from a computation based on (6.7) together with (4.1) or (4.2), condition (3) follows from a similar computation also based on (6.7), and condition (4) follows in the same way from (5.7), (5.2) and (6.7).

As a bit of after-the-fact motivation for the definition of $T_S$ we note that with the same normalizations as above, the pullback via $D_S$ at 0 of a $(2, 0)$-form $f \, d\eta_1 \wedge d\eta_2$ (restricted to $S^*$) may be written in the form $-2i\beta f \, dz_1 \wedge dz_2 - 2iaf \frac{dz_1}{dz_2} \wedge dz_2$ (restricted to $S$); it is easy to check that this decomposition of the pullback into the sum of a $(2, 0)$-form and a $(0, 2)$-form is unique. This construction defines projected pull-back operators $T_{1,S} : \Gamma(S^*; -3, 0) \to \Gamma(S^*; -3, 0)$ and $T_{2,S} : \Gamma(S^*; -3, 0) \to \Gamma(S^*; 0, -3)$.

We have

$$ (T_S F)^3 = - \left( T_{2,S} f^{3/2} \right)^2. $$

The map $T_S$ is determined up to a cube root of unity by (7.2); alternately, conditions (1), (2) and (4) of Theorem 19 determine $T_S$ up to a cube root of unity on the portion of $S$ where $B_S \neq 0$. In any event, in the sequel we use the version of $T_S$ given by (7.1) above.

We now define a $C$-bilinear pairing between $f \in \Gamma(S; -2, 0)$ and $g \in \Gamma(S^*; -2, 0)$ by setting

$$ \langle\langle f, g \rangle\rangle = \langle f, T_S g \rangle. $$

where $\langle\cdot, \cdot \rangle_S$ is the hermitian pairing from (4)

Using the polarized version

$$ \langle(1 - |B_S|^2)^{1/3}T_S f, (1 - |B_S|^2)^{1/3}T_S g \rangle_S = \langle f, g \rangle_S, $$

of (1) from Theorem 19 together with its neighbor (3) we have

$$ \langle g, T_S f \rangle_S = \langle(1 - |B_S|^2)^{1/3}T_S f, (1 - |B_S|^2)^{1/3}T_S g \rangle_S = \langle(1 - |B_S|^2)^{-1/3} f, (1 - |B_S|^2)^{1/3} g \rangle_S $$

$$ = \langle f, g \rangle_S $$

$$ = \langle\langle f, g \rangle\rangle; $$

thus $S$ and $S^*$ play symmetric roles in the construction of $\langle\langle \cdot, \cdot \rangle\rangle$. 
7.2. Higher dimension. The higher-dimensional version of Theorem 19 is the following.

**Theorem 20.** If \( S \subset \mathbb{C}P^n \) is a compact strongly \( C \)-linearly convex real hypersurface then the map \( D_S : S \to S^* \) lifts to a bundle isomorphism between \( \mathcal{O}(0,-n) \) over \( S \) and \( \mathcal{O}(-n,0) \) over \( S^* \) such that the corresponding map \( \mathcal{I}_S : \Gamma(S^*;-n,0) \to \Gamma(S;0,-n) \) has the following properties:

1. \( \|\varphi_S^{2(n+1)} \mathcal{I}_S f\|_S = \|f\|_S \);
2. the construction of \( \mathcal{I}_S \) respects the \( SL(n+1,\mathbb{C}) \)-action on \( \mathbb{C}P^n \times \mathbb{C}P^{n*} \) described in [6];
3. \( \mathcal{I}_S^* \left( \mathcal{I}_S f \right) = \varphi_S^{-\frac{n}{n+1}} f \).

To construct \( \mathcal{I}_S \) we use the normalized coordinates used in [6.3] to set

\[
(7.5) \quad \mathcal{I}_S : \left( f \left( d\eta_1 \wedge \ldots \wedge d\eta_n \right)^{\frac{n}{n+1}} \right) \mapsto 2^{-\frac{n(n-1)}{n+1}} \left( \prod_{j=1}^{n-1} \alpha_j \right)^{\frac{n}{n+1}} \left( f \left( dz_1 \wedge \ldots \wedge dz_n \right)^{\frac{n}{n+1}} \right)
\]

at 0.

We now set up our \( C \)-bilinear pairing between \( f \in \Gamma(S;-n,0) \) and \( g \in \Gamma(S^*;-n,0) \) as we did in (7.3).

**Remark 21.** Using the normalized coordinates, the element of integration for \( \langle \langle f, g \rangle \rangle \) at 0 is 
\[
f(0)g(0)2^{n-1} \prod_{j=1}^{n-1} \alpha_j dS = f(0)g(0)\varphi_S^{-1/2} dS.
\]
(Compare (3.2.6) in [APS].)

8. A weighted norm

In general we will refer to a \( C \)-bilinear pairing \( \langle \langle \cdot, \cdot \rangle \rangle \) between Hilbert spaces \( H_1 \) and \( H_2 \) as a **duality pairing** if

\[
\|x\| = \sup_{y \in H_2, \|y\| \leq 1} |\langle \langle x, y \rangle \rangle|
\]

and

\[
\|y\| = \sup_{x \in H_1, \|x\| \leq 1} |\langle \langle x, y \rangle \rangle|
\]

hold for \( x \in H_1, y \in H_2 \).

It turns out that the pairing \( \langle \langle \cdot, \cdot \rangle \rangle \) constructed in the previous section is not a duality pairing between \( L^2(S;-n,0) \) and \( L^2(S^*;-n,0) \) (except in the spherical case \( \sigma_S \equiv 1 \)). To remedy this we introduce the weighted inner product

\[
(8.1) \quad \langle f, g \rangle_S^\sharp = \langle (1 - |B_S|^2)^{-1/3} f, g \rangle_S = \langle f, (1 - |B_S|^2)^{-1/3} g \rangle_S
\]

for \( n = 2 \) and

\[
(8.2) \quad \langle f, g \rangle_S^\sharp = \langle \varphi_S^{-\frac{n}{2(n+1)}} f, g \rangle_S = \langle f, \varphi_S^{-\frac{n}{2(n+1)}} g \rangle_S
\]

in general with corresponding weighted norm

\[
(8.3) \quad \left( \|f\|_S^\sharp \right)^2 = \langle f, f \rangle_S^\sharp.
\]
Let $L^2_q(S; -n, 0)$ denote the modified Hilbert space.

**Remark 22.** Using normalized coordinates as before, the element of integration for $\langle f, g \rangle_S$ at 0 is

$$f(0)g(0)2^{n-1}\prod_{j=1}^{n-1} \alpha_j dS = f(0)g(0)\varphi_S^{-1/2} dS.$$

Focusing for the moment on $n = 2$ and noting that (1) from Theorem 19 also yields

$$\| (1 - |B_S|^2)^{1/3} f \|_S^0 = \| f \|_S^0,$$

we have

$$\langle \langle f, g \rangle \rangle = \langle f, \overline{T_S g} \rangle_S$$

$$= \langle f, (1 - |B_S|^2)^{1/3} \overline{T_S g} \rangle_S$$

$$\leq \| f \|_S^0 \| (1 - |B_S|^2)^{1/3} \overline{T_S g} \|_S$$

$$= \| f \|_S^0 \| g \|_S^0;$$

(8.4)

that is, $\langle \langle \cdot, \cdot \rangle \rangle$ satisfies the Cauchy-Schwarz inequality with respect to the norms $\| \cdot \|_S^0$ and $\| \cdot \|_S^0$.

For $f \in L^2_q(S; -2, 0) \setminus \{0\}$ let $g = \frac{(1 - |B_S|^2)^{1/3} f}{\| f \|_S^0}$. Then $\| g \|_S^0 = 1$ and using (7.4) we have

$$\langle \langle f, g \rangle \rangle = \langle g, (1 - |B_S|^2)^{1/3} \overline{T_S f} \rangle_S$$

$$= \left( \frac{\| (1 - |B_S|^2)^{1/3} \overline{T_S f} \|_S^0}{\| f \|_S^0} \right)^2$$

$$= \| f \|_S^0.$$

(8.5)

Combining (8.4) with (8.5) we have the following.

**Proposition 23.** For $f \in L^2_q(S; -2, 0)$ we have

$$\| f \|_S^0 = \max_{g \in L^2_q(S^*; -2, 0), \| g \|_S^0 \leq 1} |\langle \langle f, g \rangle \rangle|.$$

Reversing this argument and combining results we have the following.

**Proposition 24.** The pairing $\langle \langle \cdot, \cdot \rangle \rangle$ constructed in §7 is a duality pairing between $L^2_q(S; -2, 0)$ and $L^2_q(S^*; -2, 0)$.

The argument easily generalizes to show that $\langle \langle \cdot, \cdot \rangle \rangle$ is a duality pairing between $L^2_q(S; -n, 0)$ and $L^2_q(S^*; -n, 0)$ in dimension $n$.

We define the modified Hardy space $\mathcal{H}_q(S)$ as a subspace of $L^2_q(S; -n, 0)$ as in §4.
The construction of \( L^2_t(S; -n, 0) \) and \( \mathcal{H}_t(S) \) is invariant under automorphisms of \( \mathbb{CP}^n \) but does not enjoy the broader biholomorphic invariance properties mentioned in Remark 4.

Question 6 can be repeated for \( L^2_t(S; -n, 0) \) and \( \mathcal{H}_t(S) \).

**Remark 25.** In work on two-dimensional Reinhardt domains in [BaLa] the norm \( \| \cdot \|_S^\sharp \) is obtained by integrating with respect to the measure \( \mu_0 \) discussed in §8 of that work.

9. **The Invariant Projection Operator**

We let

\[
\Phi(\zeta, \zeta^*) = \left( \sum_{j=0}^n \zeta_j \zeta_j^* \right)^{-n};
\]

\( \Phi \) may be viewed as a meromorphic section of \( \mathcal{O}(-n, 0) \times \mathcal{O}(-n, 0) \) on \( \mathbb{CP}^n \times \mathbb{CP}^n \) with pole along \( \{ (\zeta, \zeta^*) : \zeta^* \in \mathfrak{h}_\zeta \} \). We may use \( \Phi \) to define \( \Phi_{\zeta} : \zeta^* \mapsto \Phi(\zeta, \zeta^*) \).

**Theorem 26.** If \( S \) be a compact strongly \( \mathbb{C} \)-linearly convex real hypersurface in \( \mathbb{CP}^n \) then the formula

\[
(L_S F)(\zeta) = \frac{1}{2} F(\zeta) + \frac{(n-1)!}{2} \left( \frac{i}{\pi} \right)^n \cdot \text{P. V.} \langle \langle F, \Phi_{\zeta} \rangle \rangle
\]

defines a bounded projection operator

\[
L_S : L^2_t(S; -n, 0) \to \mathcal{H}_t(S; -n, 0).
\]

(Here \( \text{P. V.} \) denotes the principal value of the singular integral.)

For any lift (as in §2) \( M^* \) of an automorphism of \( \mathbb{CP}^n \) mapping \( S_1 \) to \( S_2 \) we have

\[
M^*(L_{S_2} F) = L_{S_1}(M^* F).
\]

**Remark 27.** For \( n = 1 \) the affinizations from §6.3 yield \( D_S : z \mapsto -z \) and \( S^* = -S \). It is easy to check that \( L_S = C_+ \) as defined in 3.3.

**Proof of Theorem 26.** The invariance is clear from the construction.

Using the invariance together with Remark 14 we may assume that \( S \subset \mathbb{C}^n \). We claim that after converting to affine coordinates as in §6.3 and using the standard trivializations of \( \mathcal{O}(-n, 0) \) over \( \mathbb{C}^n \), the operator \( L_S \) coincides now with the classic Leray transform \( \tilde{L}_S \) ([Ler], see also [Aîz1]) defined by

\[
(\tilde{L}_S f)(z) = \frac{1}{2} f(z) + (2\pi i)^{-n} \text{P. V.} \int_{w \in S} f(w) \frac{\partial \rho(w) \wedge (\bar{\partial} \rho(w))^{n-1}}{(\partial \rho(w)[w - z])^n}
\]

where \( \partial \rho(w)[w - z] = \frac{\partial \rho}{\partial w_j}(w)(w_j - z_j) \) and \( \text{P. V.} \) again denotes the principal value of the singular integral; then the remaining claims follow from well-known facts about \( \tilde{L}_S \) ([KeSt1] see also [Han]).
It suffices to show that both integral terms match for \( z, w \in S \). Since \( \tilde{L}_S \) is known to have the same invariance property noted above for \( L_S \) \[Bol2\] it will suffice to check this under the assumption that \( w = 0 \) and \( S \) is given near 0 by (5.16) with further normalization (6.9). Then routine computation reveals that in either formulation the contribution to the integral term at \( w \) is
\[
\frac{(n-1)!f(0)}{4} \left( \frac{2i}{\pi z_n} \right)^n \left( \prod_{j=1}^{n-1} \alpha_j \right) dS
\]
where \( dS \) is euclidean surface area. □

**Theorem 28.** For \( S \) as above we have
\[
\langle \langle L_S F, G \rangle \rangle = \langle \langle F, L_S^* G \rangle \rangle = \langle \langle L_S F, L_S^* G \rangle \rangle.
\]
(9.5)

See the proof of Theorem 26 in [Lin] for closely related facts.

**Proof.** The first equality follows from (9.1) and Fubini’s theorem. (To accommodate the use of singular integrals here, we do this for a sequence of modified versions of \( \Phi \) truncated near the singularity, then pass to the limit.) Then we also have \( \langle \langle L_S F, G \rangle \rangle = \langle \langle L^2_S F, G \rangle \rangle = \langle \langle L_S F, L_S^* G \rangle \rangle \). □

The conditions (9.2) and (9.5) characterize \( L_S \).

**Corollary 29.** In the above setting we have
\[
\| L_S \|_\# = \sup_{\substack{f \in L^2(S; -n,0), \|f\|_1 \leq 1 \\ g \in L^2(S^*; -n,0), \|g\|_1 \leq 1}} |\langle \langle L_S f, g \rangle \rangle |
\]
\[
= \sup_{\substack{f \in L^2(S; -n,0), \|f\|_1 \leq 1 \\ g \in L^2(S^*; -n,0), \|g\|_1 \leq 1}} |\langle \langle f, L_S^* g \rangle \rangle |
\]
\[
= \sup_{\substack{f \in L^2(S; -n,0), \|f\|_1 \leq 1 \\ g \in L^2(S^*; -n,0), \|g\|_1 \leq 1}} |\langle \langle L_S f, L_S^* g \rangle \rangle |
\]
\[
= \| L_{S^*} \|_\#.
\]
(9.6)

**Proof.** This follows from the (general dimension version of) Proposition 24 and Theorem 28. □

**Corollary 30.** For \( f \in \mathcal{H}_\sharp(S) \) we have
\[
\frac{\|f\|_\#}{\|L_S\|_\#^2} \leq \sup_{h \in \mathcal{H}_\sharp(S^*), \|h\|_1 \leq 1} |\langle \langle f, h \rangle \rangle | \leq \|f\|_\#.
\]

**Proof.** The right-hand inequality follows from (8.4).
For the left-hand inequality we cite Proposition 24, Theorem 26 and Theorem 28 to obtain

\[
\|f\|^{\sharp} = \sup_{g \in L^2(S; -2, 0), \|g\|^\sharp \leq 1} |\langle\langle f, g \rangle\rangle| = \sup_{g \in L^2(S; -2, 0), \|g\|^\sharp \leq 1} |\langle\langle L_S f, g \rangle\rangle| = \sup_{g \in L^2(S; -2, 0), \|g\|^\sharp \leq 1} |\langle\langle f, L_S^* g \rangle\rangle| \\
\leq \sup_{h \in \mathcal{H}_2(S^*), \|h\|^\sharp \leq \|L_S\|^\sharp} |\langle\langle f, h \rangle\rangle| = \|L_S\|^\sharp \sup_{h \in \mathcal{H}_2(S^*), \|h\|^\sharp \leq 1} |\langle\langle f, h \rangle\rangle|.
\]

\[\square\]

Remark 31. Corollary 30 may be viewed as a Hardy space version of the duality theorem of Martineau [Mar] and Aizenberg [Aiz1] – see also [APS, Chapter 3] and especially [Lin, Thm. 26].

Corollary 32. In the above setting we have

\[
(9.7) \quad \inf_{f \in \mathcal{H}_2(S), \|f\|^\sharp = 1} \sup_{h \in \mathcal{H}_2(S^*), \|h\|^\sharp \leq 1} |\langle\langle f, h \rangle\rangle| = \frac{1}{\|L_S\|^\sharp}.
\]

Proof. Corollary 30 shows that the left-hand side of (9.7) is \(\geq \frac{1}{\|L_S\|^\sharp}\).

For the other half we note that for small \(\varepsilon > 0\) we may pick \(\tilde{f} \in L^2(S; -n, 0)\) and \(f = L_S \tilde{f} \in \mathcal{H}_2(S)\) with \(\|f\|^\sharp = 1, \|\tilde{f}\|^\sharp \leq \frac{1}{\|L_S\|^\sharp - \varepsilon}\). Then

\[
\sup_{h \in \mathcal{H}_2(S^*), \|h\|^\sharp \leq 1} |\langle\langle f, h \rangle\rangle| = \sup_{h \in \mathcal{H}_2(S^*), \|h\|^\sharp \leq 1} |\langle\langle L_S \tilde{f}, h \rangle\rangle| = \sup_{h \in \mathcal{H}_2(S^*), \|h\|^\sharp \leq 1} |\langle\langle \tilde{f}, h \rangle\rangle| \\
\leq \|\tilde{f}\|^\sharp \leq \frac{1}{\|L_S\|^\sharp - \varepsilon}.
\]

Since \(\varepsilon > 0\) was arbitrary we have

\[
\inf_{f \in \mathcal{H}_2(S), \|f\|^\sharp = 1} \sup_{h \in \mathcal{H}_2(S^*), \|h\|^\sharp \leq 1} |\langle\langle f, h \rangle\rangle| \leq \frac{1}{\|L_S\|^\sharp}
\]

as required.  \[\square\]
Thus \( \|L_S\|^2 \) measures efficiency of the pairing between \( H_\ast (S) \) and \( H_\ast (S^\ast) \). If \( S \) is the Möbius image of a sphere then so is \( S^\ast \), and using Remark 18 we find that we are essentially pairing \( H(S) \) with its conjugate; thus the pairing is perfectly efficient and \( \|L_S\|^2 = 1 \).

On the other hand, it follows from work of Bolt ([Bol1], [Bol2]) that \( \|L_S\|^2 = 1 \) implies that \( S \) is the Möbius image of a sphere.

**Remark 33.** It follows from results in [BaLa] that

\[
\|L_S\|^2 \geq \max \left\{ \left( 1 - |B_S(z)|^2 \right)^{-1/2} : z \in S \right\}
\]

when \( S \) is the smooth boundary of a strongly convex Reinhardt domain in \( \mathbb{C}^2 \).

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