Bulk viscous quintessential inflation

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The incorporation of bulk viscosity process to General Relativity leads to the appearance of nonsingular backgrounds that, at early and late times, depict an accelerated universe. These backgrounds could be analytically calculated and mimicked, in the context of General Relativity, by a single scalar field whose potential could also be obtained analytically. We will show that, we can build viable backgrounds that, at early times, depict an inflationary universe leading to a power spectrum of cosmological perturbations which match with current observational data, and after leaving the inflationary phase, the universe suffers a phase transition needed to explain the reheating of the universe via gravitational particle production, and finally, at late times, it enters into the de Sitter phase that can explain the current cosmic acceleration.

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Introduction.-- Recently, bulk viscosity phenomena (see, for instance, \cite{1} for a detailed description) applied to cosmology have attracted considerable interest since this dissipative mechanism in the Friedmann-Lemaître-Robertson-Walker space-time is able to explain the inflation \cite{2, 3} and the current cosmic acceleration \cite{4, 5}.

It is well-known that, in the context of bulk viscous cosmology, our universe attains two accelerated phases at early and late times (see \cite{6} for a review), where essentially the universe starts in an unstable de Sitter solution (early accelerated phase) and ends in a stable one (current accelerating phase). Unfortunately, the early time acceleration proposed in current models do not correspond to an inflationary phase \cite{8}.

For this reason, our main goal in the present letter, is to pledge simple models containing an isotropic viscous fluid filling the universe with a linear Equation of State (EoS) whose pressure was positive, depicting nonsingular backgrounds (without the big bang singularity) that have, at early times, an inflationary phase that ends in a sudden phase transition in order to produce enough particles to reheat the universe, and, at late times, renders an accelerated universe. With these nonsingular backgrounds, using the reconstruction method, we find inflationary quintessential potentials, that predict, at early times, a spectral index of scalar cosmological perturbations with running and its corresponding ratio of tensor to scalar perturbations that fit well with recent observational data. Moreover, these potentials have an absolute minimum at the corresponding de Sitter solution, meaning that the de Sitter solution is a late time attractor, and thus, the background will depict the current cosmic acceleration.

The units used throughout the letter are $\hbar = c = 8\pi G = 1$.

Bulk viscous cosmology.-- In cosmology, the simplest effective way to incorporate the bulk viscosity is to use Eckart theory \cite{9} (see also \cite{1}), where basically the pressure $p$ is replaced by $p = 3H\xi(H)$, where $\xi(H)$ is the so-called coefficient of bulk viscosity \cite{10}, which could depend on the Hubble parameter $H$, or, equivalently, of the energy density of the universe. Hence, in the flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, the classical Friedmann and Raychaudhuri’s equations are modified as

\begin{equation}
\rho = 3H^2, \quad \dot{H} = -\frac{1}{2}(p + \rho) + \frac{3}{2}H\xi(H),
\end{equation}

where $\rho$ is the energy density.

Now, assuming that the universe is filled with a barotropic fluid with Equation of State (EoS): $p = (\gamma - 1)\rho$, where the dimensionless parameter $\gamma$ has been chosen greater or equal to 1, in order to have a non-negative pressure. Hence, Raychaudhuri’s equation can be written as

\begin{equation}
\dot{H} = -\frac{3}{2}(1 + w_{\text{eff}}(H))H^2,
\end{equation}

where we have introduced the effective EoS parameter

\begin{equation}
w_{\text{eff}}(H) \equiv -1 - \frac{2\dot{H}}{3H^2} = -1 + \gamma \left(1 - \frac{\xi(H)}{\gamma H}\right). \tag{3}
\end{equation}

The idea is to find models, i.e., to find the coefficient of bulk viscosity, such that, the equation $1 + w_{\text{eff}}(H) = 0$ has two positive roots, namely $H_+ > H_-$, that will correspond to different de Sitter solutions, one of them will be an attractor and the other one must be a repeller. When $H_+$, which eventually could be $+\infty$, is the repeller and $H_-$ is an attractor, we will have a nonsingular background with $w_{\text{eff}}(H) > -1$, that starts at $H_+$ and ends at $H_-$, depicting an accelerated universe at early and late times. This is a candidate to depict our universe. However, to check its viability, one has to show that at early times, this accelerated phase is an inflationary one. That is what we will do in the next sections.

The model and its dynamical analysis.-- Due to the equivalence between bulk viscous and open cosmology, where isentropic particle production is allowed \cite{7}, our first attempt is to choose a coefficient of bulk viscosity \cite{11}

\begin{equation}
\xi(H) = -\xi_0 + mH + \frac{n}{H}, \tag{4}
\end{equation}

where $m$ and $n$ are two arbitrary positive constants, $\xi_0$ is a positive constant that could be chosen positive or negative, and

\begin{equation}
\xi(H) = -\frac{1}{2}\left(1 - \frac{\xi_0}{H}\right) + \frac{n}{H}.
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\begin{equation}
\xi(H) = -\frac{1}{2}\left(1 - \frac{\xi_0}{H}\right) + \frac{n}{H}.
\end{equation}
where $\xi_0$, $m$ and $n$ are some positive parameters.

As a consequence of the entropy growth in irreversible processes of energy dissipation, $\xi(H)$ must be positive (see section 49 of [12]), then taking into account its minimum achieved at $\sqrt{\frac{m}{n}}$, its positivity condition is satisfied when $\xi_0^2 \leq \frac{m}{n}$.

After a dynamical analysis, one can show that, in order to have a nonsingular dynamics without the big bang singularity where the universe depicts two accelerated phases, one at early times, and one at late-times (current accelerating phase) the parameters $\xi_0$, $m$ and $n$ must belong to the following domain of $\mathbb{R}^3$

$$W = \{ \xi_0 > 0, m \geq \gamma, n > 0, \text{ and }, \frac{4(\gamma - m)n}{\xi_0^2} > -1 \}. \quad (5)$$

In that domain, the critical points (de Sitter solutions) of the system, i.e., the values of $H$ that are the solutions of the equation $1 + w_{eff}(H) = 0$, are

$$H_{\pm} = \frac{\xi_0}{2(m - \gamma)} \left(1 \pm \sqrt{1 + \frac{4(\gamma - m)n}{\xi_0^2}}\right). \quad (6)$$

where $H_+$ is a repeller and $H_-$ is an attractor. Then, the dynamics going from $H_+$ to $H_-$ becomes nonsingular. Moreover, since at the critical points $w_{eff} = -1$, one will have a universe starting and ending in an accelerated phase.

Assuming that, $n \ll \xi_0^2 \min \left(m - \gamma, \frac{m}{n}\right)$, one obtains

$$H_+ \approx \frac{\xi_0}{m - \gamma}, \quad H_- \approx \frac{n}{\xi_0}, \quad \text{with } 0 < H_- < H_+, \quad \text{and consequently, the universe could start at very high energies and ends at low ones. In particular, for } m = \gamma, \quad \text{one has } H_+ = \infty,$$

but there is no big bang singularity, because in that case, $H = -\frac{3}{2}(\xi_0 H - n)$, which for large values of $H$ gives

$$H \approx -\frac{3}{2}\xi_0 H \iff H(t) \approx H_0 e^{-\frac{2}{\xi_0}(t-t_0)}, \quad (7)$$

meaning that $H$ diverges only when $t = -\infty$.

In fact, for our model, the Raychaudhuri equation (2) can be analytically solved for $m > \gamma$ as

$$H(t) = \frac{H_+ e^{-\frac{2}{\xi_0}(H_+ - H_-)(m - \gamma)t} + H_- e^{\frac{2}{\xi_0}(H_+ - H_-)(m - \gamma)t}}{2 \cosh \left(\frac{2}{\xi_0}(H_+ - H_-)(m - \gamma)t\right)} \quad (8)$$

and,

$$H(t) = \xi_0 e^{-\frac{2}{\xi_0}H_0t} + \frac{n}{\xi_0} \quad (9)$$

for $m = \gamma$.

**Inflationary quintessential potential.** In this section, we will see under which conditions a scalar field $\phi$ with potential $V(\phi)$ could mimic the dynamics of a perfect fluid with bulk viscosity in order to provide viable backgrounds that could depict our universe correctly.

It is well-known that the energy density, namely $\rho_\phi$, and pressure, namely $p_\phi$, of the scalar field minimally coupled with gravity are given by

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi), \quad (10)$$

To show the equivalence with the bulk viscous system (1), we perform the replacement

$$\rho \rightarrow \rho_\phi, \quad p \rightarrow p_\phi - 3H\xi(H),$$

(11)

to recover the standard Friedmann and Raychaudhuri equations for a universe filled by an scalar field

$$3H^2 = \rho_\phi, \quad 2\dot{H} = -\dot{\phi}^2. \quad (12)$$

Note that, since the equations in (12) are the usual equations for a single scalar field, this means that the dynamics driven by a fluid with bulk viscosity with an effective EoS parameter greater than -1, could be mimicked by a single scalar field in the context of General Relativity.

Combining the Friedmann and Raychaudhuri equations (1), (3) and (12), one easily obtains

$$\dot{\phi} = \sqrt{-2H} \sqrt{3\gamma H^2 \left[1 - \frac{\xi(H)}{\gamma H}\right]} \quad (13)$$

and

$$V(\phi) = \frac{3H^2}{2} \left[(2 - \gamma) + \frac{\xi(H)}{H}\right] \quad (14)$$

Then, to reconstruct the potential, the first step is to integrate (13). Therefore, performing the change of variable $dt = \frac{dH}{H}$, we obtain

$$\varphi = -\int \sqrt{-\frac{2}{H}} dH = -\frac{2}{\sqrt{3}} \int \frac{dH}{\sqrt{\gamma H^2 - \xi(H)H}} \quad (15)$$

For our model, where $\xi(H) = -\xi_0 + mH + n/H$, one has

$$\varphi = -\frac{2}{\sqrt{3}} \int \frac{dH}{\sqrt{(\gamma - m)H^2 + \xi_0 H - n}} \quad (16)$$

and, when the parameters belong to the domain $W$, for $m < \gamma$ one has (see formula 2.261 of [13])

$$\varphi = \frac{2}{\sqrt{3(m - \gamma)}} \arcsin \left(\frac{2(\gamma - m)H + \xi_0}{\sqrt{\xi_0^2 + 4(\gamma - m)n}}\right) \quad (17)$$

defined in $\left(\frac{-\pi}{\sqrt{3(m - \gamma)}}, \frac{\pi}{\sqrt{3(m - \gamma)}}\right)$, and when $m = \gamma$

$$\varphi = -\frac{4}{\sqrt{3\xi_0}} \sqrt{\xi_0 H - n} $$

(18)

which is defined in the domain $(-\infty, 0)$.

Finally, isolating $H$ and inserting it in (14), one obtains the corresponding potentials. Once the potential has been reconstructed, one has the corresponding conservation equation

$$\dot{\varphi} + \sqrt{3}\sqrt{\frac{\xi_0}{2} + V(\varphi)}\dot{\varphi} + V_\varphi(\varphi) = 0\quad (19)$$

whose solutions depict different backgrounds, and where one of them is the solution of (3).
For example, in the simplest case $m = \gamma$, one easily obtains

$$V(\varphi) = \frac{27}{256} \xi_0^2 \varphi^4 + \frac{9}{8} \left(n - \frac{\xi_0^2}{4}\right) \varphi^2 + \frac{3n^2}{80} \varphi^3 \xi_0^2,$$

(20)

Recall that, the positivity of $\xi(H)$ implies $\frac{\xi_0^2}{4} \leq n$, and note that, when $n \geq \frac{\xi_0^2}{4}$, the unique minimum of the potential (20) is achieved at $\varphi = 0$. On the contrary, when $n < \frac{\xi_0^2}{4}$, the potential has a maximum at $\varphi = 0$. Then, since we want that $H_\ast$ must be an attractor, we will have to choose $n \geq \frac{\xi_0^2}{4}$. Moreover, as the fluid has positive pressure ($m = \gamma \geq 1$), this condition is compatible with the positivity of $\xi(H)$ because $\frac{\xi_0^2}{4} \leq n$.

Unfortunately, the models we have chosen have two important defaults:

1.- As we well see, the main slow-roll parameter is of the order $\frac{\xi_0^2}{4}$, this means that, in order to match with current observational data, the observable modes must leave the Hubble radius at scales of the order $H \sim 10^3 \xi_0 \leq 10^3 H_\ast \ldots$ However, since $H_{\ast -}$ has to be close to the current value of the Hubble parameter, the condition $H \lesssim 10^3 H_\ast$ is compatible with the fact that the observational modes must leave the Hubble radius at high energy densities (few orders below Planck’s one).

2.- There is no mechanism to reheat the universe, because neither oscillations nor abrupt phase transitions at high scales, that breakdown the adiabaticity to produce enough amount of particles that thermalize the universe after inflation, occur in these models.

A simple way to overpass both problems consists of introducing a phase transition at early times when the universe ceases to accelerate. More precisely, we will choose the phase transition when the universe is radiation dominated.

Taking $\gamma = \frac{1}{3} \iff p = \frac{1}{3}, \ n = \frac{3\xi_0^2}{4} \iff \xi(H) \geq 0$, our continuous coefficient of bulk viscosity is improved as follows

$$\xi(H) = \begin{cases} -\xi_0 + \frac{1}{3} H + \frac{3\xi_0^2}{10H}, & \text{for } H \geq H_E, \\ \frac{\xi_0^2}{\xi_1} - \frac{1}{4}, & \text{for } H \leq H_E, \end{cases}$$

(21)

where $0 < \xi_1 \ll \xi_0$, and,

$$H_E = \frac{3}{8} (\xi_1 + \xi_0) \left(1 + \sqrt{1 - \frac{\xi_0^2}{(\xi_1 + \xi_0)^2}}\right) \approx \frac{3\xi_0}{8}.$$  

The corresponding potential has the form

$$V(\varphi) = \begin{cases} \frac{27\xi_0^2}{256} \varphi^4 - \frac{9}{8} \varphi^2 + 1, & \text{for } \varphi \leq \varphi_E \\ H(\varphi) \left(H(\varphi) + \frac{3\xi_0}{2}\right), & \text{for } \varphi \geq \varphi_E, \end{cases}$$

(22)

where $\varphi_E = -\sqrt{\frac{16H_E}{3\xi_0}} - 1 \approx -1$ and

$$H(\varphi) = \frac{A^2 e^{-\frac{2\varphi}{H}\varphi - \varphi}}{16} + \frac{3\xi_0}{8} \varphi^2 + \frac{3\xi_1}{8},$$

(23)

with

$$A \equiv \frac{4\sqrt{4H_E^2 - 3\xi_1 H_E + 8H_E - 3\xi_1}}{6\xi_0}. \quad (24)$$

The corresponding conservation equation (19) provides backgrounds that could depict our universe, and one of them is the solution of (24)

$$H(t) = \begin{cases} \left(\frac{H_E - \frac{3\xi_0}{16}}{e^{-\frac{9\xi_1}{16}(t-t_E) + \frac{3\xi_0}{16}}}, & \text{for } t \leq t_E, \\ \frac{3\xi_0}{H_E - (H_{\ast -} - 3\xi_1) e^{-\frac{9\xi_1}{16}(t-t_E)}}, & \text{for } t \geq t_E, \end{cases} \quad (25)$$

where $t_E$ is the phase transition time, and whose viability will be checked in the next section.

Finally, note that, at late time, the system has a critical point at $H_{\ast -} = \frac{3\xi_1}{12} \iff \varphi = \varphi_E + \frac{2}{\sqrt{3}} \ln \left(\frac{A}{\sqrt{3}}\right)$. To show that it is an attractor, we have to calculate, applying the chain rule, $V_{\varphi \varphi}$ at this critical point, leading to

$$V_{\varphi \varphi} = \frac{2}{\varphi^2} - 3\xi_1 \frac{\varphi H_H}{\varphi_H^3} = \frac{9}{8} \left(\frac{8H}{3} - \xi_1\right) = \frac{9\xi_1}{8} > 0.$$

(26)

meaning that the potential has a minimum at the critical point, and consequently, it is an attractor.

A final remark is in order: When one considers the case $m \geq 1 = \frac{\gamma}{3}$, and assumes a phase transition as in the model (21), the bulk viscous Raychaudhuri equation (4) leads to a nonsingular solution that starts at $H_{\ast +} = \frac{\xi_0}{m - \frac{\gamma}{3}}$ and ends at $H_{\ast -} = \frac{3\xi_1}{12}$. Then, the corresponding quintessential inflationary potential will have a maximum at $H_{\ast +}$ (unstable) and a minimum at $H_{\ast -}$ (attractor), that is, some backgrounds models which, at early time, are close to our nonsingular background (25) given by (19), leave the de Sitter phase $H_\ast$ at early-times, and suffer a sudden phase transition when the universe starts to decelerate, and finally, enters into the stable de Sitter phase $H_{\ast -}$. In fact, the shape of the potential can easily be imagined: From equation (17), one can deduce that, before the phase transition ($\varphi < \varphi_E$), the potential has a sinusoidal form with period $\frac{4m}{\Delta m}$, and, after the phase transition, it has the same shape as (22).

Slow roll phase.– We know that, at early time, our background (25) satisfy $w_{\text{eff}}(H) \equiv -1$, this means that the universe is quasi de Sitter, and we aim to check whether this background could lead to a power spectrum of cosmological perturbations that fit well with current observational data (14).

Hence, we consider the slow roll parameters (15)

$$\epsilon = -\frac{\dot{H}}{H^2}, \quad \eta = 2\epsilon - \frac{i}{2H}, \quad (27)$$

that allows us to calculate the spectral index, its running and the ratio of tensor to scalar perturbations

$$n_s - 1 = -6\epsilon + 2\eta, \quad \alpha_s = \frac{H\dot{n_s}}{H^2 + H}, \quad r = 16\epsilon. \quad (28)$$
At early times, i.e., when $H > H_E$, introducing the notation $x \equiv \frac{3c_0}{2H}$, one has
\[ \epsilon = x \left( 1 - \frac{x}{8} \right), \quad \eta = \epsilon + \frac{x}{2}, \quad (29) \]
and, as a consequence, $n_s - 1 = -3x + \frac{x^2}{2}$. Conversely,
\[ x = 3 \left( 1 - \sqrt{1 - \frac{2(1 - n_s)}{9}} \right). \quad (30) \]

Then, given the observational values of the spectral index, one can obtain the range of $x$. Recent PLANCK+WP 2013 data (see table 5 of [14]) predicts the spectral index at 1σ Confidence Level (C.L.) to be $n_s = 0.9583 \pm 0.0081$, which means that, at 2σ C.L., one has $0.9885 \leq x \leq 0.0193$, and thus, $0.1344 < r = 16\epsilon < 0.3072$.

Since PLANCK+WP 2013 data provides the constrain $r \leq 0.25$, at 95.5% C.L., then when $0.0085 \leq x \leq 0.0156$, i.e., for the modes that leave the Hubble radius at scales $2 \times 10^{30} \xi_0 \leq \rho \leq 9 \times 10^{39} \xi_0$, the spectral index belongs to the 1-dimensional marginalized 95.5% C.L., and also $r \leq 0.25$, at 95.5% C.L.

For the running at 1σ C.L., PLANCK+WP 2013 data gives $\alpha_s = -0.021 \pm 0.012$, and our background [25] leads to the theoretical value $\alpha_s = \frac{x}{2(1 - n_s)}$. Consequently, at the scales we are dealing with, $10^{-5} \leq \alpha_s \leq 6 \times 10^{-5}$, and thus, the running also belongs to the 1-dimensional marginalized 95.5% C.L.

Note also that, we have the relation $w_{\text{eff}}(H) = -1 + \frac{1}{2} \epsilon$. Therefore, if we assume that the slow-roll ends when $\epsilon = 1$, and let $H_{\text{end}}$ be the value of the Hubble parameter when the slow roll ends, then the slow roll will end when $w_{\text{eff}}(H_{\text{end}}) = -\frac{1}{2}$, i.e., when the universe will start to decelerate.

On the other hand, the number of e-folds from observable scales exiting the Hubble radius to the end of inflation, namely $N(H)$, could be calculated using the formula $N(H) = -\int^{H}_{H_{\text{end}}} \frac{H}{H^2} dH$, leading to
\[ N(x) = \frac{1}{x - x_{\text{end}}} + \frac{1}{16} \ln \left( \frac{16 - x}{16 - x_{\text{end}}} \right), \quad (31) \]
where $x_{\text{end}} = 4(1 - \sqrt{1/2}) \approx 1.1715$, is the value of the parameter $x$ when inflation ends. For our values of $x$ that allow to fit well with the theoretical value of the spectral index, its running and the tensor/scalar ratio with their observable values, we will obtain $64 \leq N \leq 117$.

To determine the value of $\xi_0$, one has to take into account the theoretical [13] and the observational [10] value of the power spectrum
\[ P \cong \frac{H^2}{8\pi^2} = \frac{9\varepsilon_0^2}{32\pi^2 \epsilon_x} = \frac{18\varepsilon_0^2}{\rho_{\text{pl}} \epsilon x} \cong 2 \times 10^{-9}, \quad (32) \]
where we have explicitly introduced the Planck’s energy density, which in our units is $\rho_{\text{pl}} = 64\pi^2$. Using the values of $x$ in the range $[0.0085, 0.0156]$, we can conclude that
\[ 10^{-7} \sqrt{\rho_{\text{pl}}} \leq \xi_0 \leq 10^{-6} \sqrt{\rho_{\text{pl}}}. \quad (33) \]

Summing up, the observable modes in our model leave the Hubble radius at scales
\[ 2 \times 10^{-9} \rho_{\text{pl}} \leq \rho \leq 9 \times 10^{-7} \rho_{\text{pl}}. \quad (34) \]
and since the sudden transition occurs at $H_E \cong \frac{3c_0}{8} \implies \rho_{\text{pl}} \approx 10^{-1} \xi_0$, one can deduce that the universe pre-heats, due to the gravitational particle production, at scales (the same result was obtained in formula (15) of [17])
\[ 10^{-15} \rho_{\text{pl}} \leq \rho \leq 10^{-13} \rho_{\text{pl}}. \quad (35) \]

These particles will interact exchanging gauge bosons [18] to reach a thermal equilibrium temperature which in the radiation dominated era, is of the order $T_R \sim 10^3$ GeV (see for details [17]), matches with the hot Friedmann universe, and finally, at late times, due to the expansion of the universe, the energy density of the matter will become sub-dominant and the scalar field will come back to dominate the evolution of the universe, leading to the current cosmic acceleration.

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