On some conformally invariant fully nonlinear equations

YanYan Li

Department of Mathematics
Rutgers University
110 Frelinghuysen Rd.
Piscataway, NJ 08854

In this talk, we present some recent joint work with Aobing Li [15] on some conformally invariant fully nonlinear equations.

For \( n \geq 3 \), consider

\[
-\Delta u = \frac{n-2}{2} u^{\frac{n+2}{n-2}}, \quad \text{on } \mathbb{R}^n.
\]  

(1)

The celebrated Liouville type theorem of Caffarelli, Gidas and Spruck ([3]) asserts that positive \( C^2 \) solutions of (1) are of the form

\[
u(x) = (2n)^{\frac{n-2}{4}} \left( \frac{a}{1 + a^2|x-\bar{x}|^2} \right)^{\frac{n-2}{2}},
\]

where \( a > 0 \) and \( \bar{x} \in \mathbb{R}^n \). Under an additional decay hypothesis \( u(x) = O(|x|^{-n}) \), the result was proved by Obata ([20]) and Gidas, Ni and Nirenberg ([8]).

Let \( \psi \) be a M"obius transformation,

\[
\left( u^{\frac{n+2}{n-2}} \Delta u_{\psi} \right) = \left( u^{\frac{n+2}{n-2}} \Delta u \right) \circ \psi, \quad \text{on } \mathbb{R}^n,
\]

where \( u_{\psi} := |J_{\psi}|^\frac{n-2}{n} (u \circ \psi) \) and \( J_{\psi} \) denotes the Jacobian of \( \psi \). In particular, if \( u \) is a positive solution of (1), so is \( u_{\psi} \).

We call a fully nonlinear operator \( H(x, u, \nabla u, \nabla^2 u) \) conformally invariant on \( \mathbb{R}^n \) if for any Möbius transformation \( \psi \) and any positive function \( u \in C^2(\mathbb{R}^n) \)

\[
H(\cdot, u_{\psi}, \nabla u_{\psi}, \nabla^2 u_{\psi}) \equiv H(\cdot, u, \nabla u, \nabla^2 u) \circ \psi.
\]

(2)

We showed in [15] that \( H(x, u, \nabla u, \nabla^2 u) \) is conformally invariant if and only if

\[
H(x, u, \nabla u, \nabla^2 u) \equiv F(A^u),
\]
where

\[ A^u := -\frac{2}{n-2}u^{\frac{n+2}{n-2}}\nabla^2 u + \frac{2n}{(n-2)^2}u^{\frac{2n}{n-2}}\nabla u \otimes \nabla u - \frac{2}{(n-2)^2}u^{\frac{2n}{n-2}}|\nabla u|^2 I, \quad (3) \]

and \( F \) is invariant under orthogonal conjugations.

Let \( U \) be an open subset of \( n \times n \) symmetric matrices which is invariant under orthogonal conjugations (i.e. \( O^{-1}UO = U \) for all orthogonal matrices \( O \)) and has the property that \( U \cap \{ M + tN \mid 0 < t < \infty \} \) is convex for any \( n \times n \) symmetric matrix \( M \) and any \( n \times n \) positive definite symmetric matrix \( N \).

Let \( F \in C^1(U) \) be invariant under orthogonal conjugation and be elliptic, i.e.

\[ (F_{ij}(M)) > 0, \quad \forall M \in U, \]

where \( F_{ij}(M) := \frac{\partial F}{\partial M_{ij}}(M) \).

The following theorem extends the result of Obata and Gidas, Ni and Nirenberg to all conformally invariant operators of elliptic type.

**Theorem 1** ([15]) For \( n \geq 3 \), let \( U \) and \( F \) be as above, and let \( u \in C^2(R^n) \) be a positive solution of

\[ F(A^u) = 1, \quad \text{on} \ R^n. \]

Assume that \( u \) is regular at infinity, i.e., \( |x|^{2-n}u(x)/|x|^2 \) can be extended to a positive \( C^2 \) function near the origin. Then for some \( \bar{x} \in R^n \) and for some positive constants \( a \) and \( b \),

\[ u(x) \equiv \left( \frac{a}{1 + b^2|x - \bar{x}|^2} \right)^{\frac{n-2}{2}}, \quad \forall x \in R^n. \]

**Remark 1** In fact, as established in [15], the conclusion of the above theorem still holds when replacing the assumption \( u \in C^2(R^n) \) by a weaker assumption that \( u \in C^2(R^n \setminus \{0\}) \), \( u \) can be extended to a positive continuous function near the origin, and \( \lim_{x \to 0}(|x||\nabla u(x)|) = 0 \).

Theorem 1 indicates that behavior of solutions to conformally invariant equations is very rigid. Thus, we expect some good theories for conformally invariant uniformly elliptic fully nonlinear equations. Let \( F \) be \( C^\infty \) functions defined on \( n \times n \) real symmetric matrices, and let \( F \) be invariant under orthogonal conjugations. We assume that for some constants \( 0 < \lambda \leq \Lambda < \infty \),

\[ \lambda I \leq (F_{ij}(M)) \leq \Lambda I, \quad \text{for all} \ n \times n \text{ symmetric matrices}. \]

We raise the following
**Question 1** Let $F$ be as above, and let $B_1$ be a unit ball in $\mathbb{R}^n$ and $a > 0$ be some constant. Are there some positive constants $\alpha$ and $C$, depending only on $F$, $a$ and $n$ such that for any positive $C^\infty$ solution $u$ of

$$F(A^u) = 0, \quad \text{in } B_1$$

satisfying

$$\min_{B_1} u \geq a, \quad \|u\|_{C^2(B_1)} \leq \frac{1}{a},$$

we have

$$\|u\|_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C?$$

Other interesting questions include to understand behavior near an isolated singularities of a solution in a punctured disc of this subclass of uniformly elliptic equations and to establish some removable singularity results.

Let $(M, g)$ be an $n$–dimensional smooth Riemannian manifold without boundary, consider the Schouten tensor

$$A_g = \frac{1}{n-2} \left( Ric_g - \frac{R_g}{2(n-1)} g \right),$$

where $Ric_g$ and $R_g$ denote respectively the Ricci tensor and the scalar curvature associated with $g$.

For $1 \leq k \leq n$, let

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n,$$

denote the $k$–th symmetric function, and let $\Gamma_k$ denote the connected component of $\{ \lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0 \}$ containing the positive cone $\{ \lambda \in \mathbb{R}^n \mid \lambda_1, \cdots, \lambda_n > 0 \}$.

It is known (see, e.g., [2]) that $\Gamma_k$ is a convex cone with its vertex at the origin,

$$\Gamma_n \subset \cdots \subset \Gamma_2 \subset \Gamma_1,$$

and

$$\frac{\partial \sigma_k}{\partial \lambda_i} > 0 \quad \text{in } \Gamma_k, \quad 1 \leq i \leq n,$$

and

$$\sigma_k^\dagger$$

is concave in $\Gamma_k$.

Fully nonlinear elliptic equations involving $\sigma_k(D^2 u)$ have been investigated in the classical and pioneering paper of Caffarelli, Gidas and Nirenberg [2]. For extensive studies and outstanding results on such equations, see, e.g., Guan and Spruck [10], Trudinger
On Riemannian manifolds of nonnegative curvature, Li studied in [17] equations

\[ \sigma_k \left( \lambda(\nabla^2 g u) \right) = \psi(x, u), \]  

(4)

where \( \lambda(\nabla^2 g u + g) \) denotes eigenvalues of \( \nabla^2 g u + g \) with respect to \( g \). On general Riemannian manifolds, Viaclovsky introduced and systematically studied in [28] and [27] equations

\[ \sigma_k \left( \lambda(A g) \right) = \psi(x, u), \]  

(5)

where \( \lambda(A g) \) denotes the eigenvalues of \( A g \) with respect to \( g \). On 4-dimensional general Riemannian manifolds, remarkable results on (5) for \( k = 2 \) were obtained by Chang, Gursky and Yang in [4] and [5], which include Liouville type theorems, existence and compactness of solutions, as well as applications to topology. On the other hand, works on the Yamabe equation by Caffarelli, Gidas and Spruck ([3]), Schoen ([22] and [23]), Li and Zhu ([19]), and Li and Zhang ([18]), have played an important role in our approach to the study of (5) as developed in [15].

Consider

\[ \sigma_k \left( \lambda(A g) \right) = 1, \]  

(6)

together with

\[ \lambda(A g) \in \Gamma_k. \]  

(7)

Let \( g_1 = u^{\frac{4}{n-2}} g_0 \) be a conformal change of metrics, then (see, e.g., [28]),

\[ A_{g_1} = -\frac{2}{n-2} u^{-1} \nabla^2 g_0 u + \frac{2n}{(n-2)^2} u^{-2} \nabla_{g_0} u \otimes \nabla_{g_0} u - \frac{2}{(n-2)^2} u^{-2} |\nabla_{g_0} u|^2 u_0 + A_{g_0}. \]

Let \( g = u^{\frac{4}{n-2}} g_{flat} \), where \( g_{flat} \) denotes the Euclidean metric on \( R^n \). Then by the above transformation formula,

\[ A_g = u^{\frac{4}{n-2}} A^u_{ij} dx^i dx^j, \]

where \( A^u \) is given by (3).

Equations (6) and (7) take the form

\[ \sigma_k(\lambda(A^u)) = 1, \quad \text{on} \quad R^n, \]  

(8)

and

\[ \lambda(A^u) \in \Gamma_k, \quad \text{on} \quad R^n. \]  

(9)

Our next result extends the Liouville type theorem of Caffarelli, Gidas and Spruck to all \( \sigma_k, 1 \leq k \leq n \). For \( k = 1 \), equation (8) is (1).
Theorem 2 ([15]) For $n \geq 3$ and $1 \leq k \leq n$, let $u \in C^2(\mathbb{R}^n)$ be a positive solution of (8) satisfying (9). Then for some $a > 0$ and $\bar{x} \in \mathbb{R}^n$,

$$u(x) = c(n, k) \left( \frac{a}{1 + a^2|x - \bar{x}|^2} \right)^{\frac{n-2}{2}}, \quad \forall \ x \in \mathbb{R}^n,$$

where $c(n, k) = 2^{(n-2)/4} \binom{n}{k}^{(n-2)/4k}$.

The case $k = 2$ and $n = 4$ was obtained by Chang, Gursky and Yang ([5]). More recently, they ([6]) have independently established the result for $k = 2$ and $n = 5$, and they also established the result for $k = 2$ and $n \geq 6$ under an additional hypothesis $\int_{\mathbb{R}^n} u \frac{2}{|x|^2} \, dx < \infty$. Under an additional hypothesis that $\frac{1}{|x|^2} u \left( \frac{x}{|x|^2} \right)$ can be extended to a $C^2$ positive function near $x = 0$, the case $2 \leq k \leq n$ was obtained by Viaclovsky ([28], [29]).

As mentioned above, the case $k = 1$ was obtained by Caffarelli, Gidas and Spruck, while under an additional hypothesis that $\frac{1}{|x|^2} u \left( \frac{x}{|x|^2} \right)$ is bounded near $x = 0$, the case $k = 1$ was obtained by Obata, and by Gidas, Ni and Nirenberg.

The methods of Chang, Gursky and Yang in [5] and [6] include an ingenious way of using the Obata technique which, as they pointed out, allows the possibility to be generalized to establish the uniqueness of solutions on general Einstein manifolds. Our proof of Theorem 2 is very different from that of [5] and [6]. A crucial ingredient in our proof is the following Harnack type inequality.

Theorem 3 ([15]) For $n \geq 3$, $1 \leq k \leq n$, and $R > 0$, let $B_{3R} \subset \mathbb{R}^n$ be a ball of radius $3R$ and let $u \in C^2(B_{3R})$ be a positive solution of

$$\sigma_k(A^u) = 1, \quad \text{in} \ B_{3R},$$

satisfying

$$\lambda(A^u) \in \Gamma_k, \quad \text{in} \ B_{3R}.$$ 

Then

$$\left( \max_{B_R} u \right) \left( \min_{B_{2R}} u \right) \leq C(n) R^{2-n}.$$

The above Harnack type inequality for $k = 1$ was obtained by Schoen ([23]) based on the Liouville type theorem of Caffarelli, Gidas and Spruck. An important step toward our proof of Theorem 3 was taken in an earlier work of Li and Zhang ([18]), where they gave a different proof of Schoen’s Harnack type inequality without using the Liouville type theorem.

Our next result concerns existence and compactness of solutions.
Theorem 4 ([15]) For \( n \geq 3 \) and \( 1 \leq k \leq n \), let \((M,g)\) be an \( n \)-dimensional smooth compact locally conformally flat Riemannian manifold without boundary satisfying
\[
\lambda(A_g) \in \Gamma_k, \quad \text{on} \; M.
\]
Then there exists some smooth positive function \( u \) on \( M \) such that \( \hat{g} = u^{\frac{4}{n-2}}g \) satisfies
\[\lambda(A_{\hat{g}}) \in \Gamma_k, \quad \sigma_k(\lambda(A_{\hat{g}})) = 1, \quad \text{on} \; M.\] (14)
Moreover, if \((M,g)\) is not conformally diffeomorphic to the standard \( n \)-sphere, all solutions of the above satisfy, for all \( m \geq 0 \), that
\[\|u\|_{C^m(M,g)} + \|u^{-1}\|_{C^m(M,g)} \leq C,\]
where \( C \) depends only on \((M,g)\) and \( m \).

For \( k = 1 \), it is the Yamabe problem for locally conformally flat manifolds with positive Yamabe invariants, and the result is due to Schoen ([21]-[22]). The Yamabe problem was solved through the work of Yamabe [30], Trudinger [24], Aubin [1], and Schoen [21]. For \( k = 2 \) and \( n = 4 \), the result was proved without the locally conformally flatness hypothesis by Chang, Gursky and Yang [5]. For \( k = n \), the existence result was established by Viaclovsky [27] for a class of manifolds which are not necessarily locally conformally flat. For \( k \neq \frac{n}{2} \), the result is independently obtained by Guan and Wang in [12] using a heat flow method. More recently, Guan, Viaclovsky and Wang [9] have proved that \( \lambda(A_g) \in \Gamma_k \) for \( k \geq \frac{n}{2} \) implies the positivity of the Ricci tensor, and therefore, by classical results, \((M,g)\) is conformally covered by \( S^n \) and the existence and compactness results in this case follow easily.

Our proof of Theorem 1, different from the ones in [20], [8], [3], [28] and [29], is in the spirit of the new proof of the Liouville type theorem of Caffarelli, Gidas and Spruck given by Li and Zhu in [19]. We also make use of the substantial simplifications of Li and Zhang in [18] to the proof in [19]. The proof is along the line of the pioneering work of Gidas, Ni and Nirenberg [8], which in particular does not need the kind of divergence structure needed for the method of Obata [20] and therefore can be applied in much more generality.

In our proofs blow up arguments are used, which require local derivative estimates of solutions. For \( \sigma_1 \) (the Yamabe equation), such estimates follow from standard elliptic theories. Guan and Wang [11] established local gradient and second derivative estimates for \( \sigma_k, k \geq 2 \). Global gradient and second derivative estimates for \( \sigma_k \) were obtained by Viaclovsky [27]. For the related equation (4) on manifolds of nonnegative curvature, global gradient and second derivative estimates were obtained by Li in [17]. By the concavity of \( \sigma_k^\frac{1}{2}, C^{2,\alpha} \) estimates hold due to the classical work of Evans [7] and Krylov [14]. For the proof of the existence part of Theorem 4, we introduce a homotopy \( \sigma_k(t\lambda + (1-t)\sigma_1(\lambda)) \), defined on \((\Gamma_k)_t = \{ \lambda \in \mathbb{R}^n \mid t\lambda + (1-t)\sigma_1(\lambda) \in \Gamma_k \}\), which establishes a natural link
between (14) and the Yamabe problem. We extend the local estimates in [11] for \( \sigma_k \) to \( \sigma_k (t\lambda + (1 - t)\sigma_1 (\lambda)) \), with estimates uniform in \( 0 \leq t \leq 1 \). The compactness results as stated in Theorem 4 were established in [15] along the homotopy. The compactness results for the Yamabe problem was established by Schoen [22]. We gave a different proof which does not rely on the Liouville type theorem, which allows us to establish existence results for more general \( f \) than \( \sigma_k \) for which Liouville type theorems are not available. The existence results follow from the compactness results with the help of the degree theory for second order fully nonlinear elliptic operators ([16]) as well as the degree counting formula for the Yamabe problem ([22]).

The first step in our proof of the Liouville type Theorem 2 is to establish the Harnack type inequality (Theorem 3), from which we obtain sharp asymptotic behavior at infinity of an entire solution. Then we establish Theorem 2 by distinguishing into two cases. In the case \( k > \frac{n}{2} \), Theorem 2 is proved by using the sharp asymptotic behavior of an entire solution and Theorem 1-Remark 1, together with a result of Trudinger and Wang ([26]). In the case \( 1 \leq k \leq \frac{n}{2} \), Theorem 2 is proved by the sharp asymptotic behavior of an entire solution together with the Obata type integral formula of Viaclovsky ([28]). For the second case, divergence structure of the equation is used.

Theorem 2, Theorem 3 and Theorem 4 are established for more general nonlinear \( f \) than \( \sigma_k \) in [15], including those for which no divergence structure is available.

References

[1] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. 55 (1976), 269-296.

[2] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations, III: Functions of the eigenvalues of the Hessian. Acta Math. 155 (1985), 261-301.

[3] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (1989), 271-297.

[4] S.Y. A. Chang, M. Gursky and P. Yang, An equation of Monge-Ampere type in conformal geometry, and four-manifolds of positive Ricci curvature, Ann. of Math., to appear.

[5] S.Y. A. Chang, M. Gursky and P. Yang, An a priori estimate for a fully nonlinear equation on four-manifolds, preprint.

[6] S.Y. A. Chang, M. Gursky and P. Yang, Entire solutions of a fully nonlinear equation, preprint.
[7] L.C. Evans, Classical solutions of fully nonlinear, convex, second-order elliptic equations, Comm. Pure Appl. Math. 35 (1982), 333-363.

[8] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979), 209-243.

[9] P. Guan, J. Viaclovsky and G. Wang, Some properties of the Schouten tensor and applications to conformal geometry, preprint.

[10] B. Guan and J. Spruck, Boundary-value problems on $S^n$ for surfaces of constant Gauss curvature, Ann. of Math. 138 (1993), 601-624.

[11] P. Guan and G. Wang, Local estimates for a class of fully nonlinear equations arising from conformal geometry, preprint.

[12] P. Guan and G. Wang, A fully nonlinear conformal flow on locally conformally flat manifolds, preprint.

[13] M. J. Gursky and J. Viaclovsky, Fully nonlinear equations on Riemannian manifolds with negative curvature, preprint.

[14] N.V. Krylov, Boundedly inhomogeneous elliptic and parabolic equation in a domain, Izv. Akad. Nauk SSSR 47 (1983), 75-108.

[15] A. Li and Y.Y. Li, On some conformally invariant fully nonlinear equations, preprint.

[16] Y.Y. Li, Degree theory for second order nonlinear elliptic operators and its applications, Comm. in Partial Differential Equations 14 (1989), 1541-1578.

[17] Y.Y. Li, Some existence results of fully nonlinear elliptic equations of Monge-Ampere type, Comm. Pure Appl. Math. 43 (1990), 233-271.

[18] Y.Y. Li and L. Zhang, Liouville type theorems and Harnack type inequalities for semilinear elliptic equations, Journal d’Analyse Mathematique, to appear.

[19] Y.Y. Li and M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math. J. 80 (1995), 383-417.

[20] M. Obata, The conjecture on conformal transformations of Riemannian manifolds, J. Diff. Geom. 6 (1971), 247-258.

[21] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Diff. Geom. 20 (1984), 479-495.

[22] R. Schoen, On the number of constant scalar curvature metrics in a conformal class, Differential Geometry: A symposium in honor of Manfredo Do Carmo (H.B. Lawson and K. Tenenblat, eds), Wiley, 1991, 311-320.
[23] R. Schoen, Courses at Stanford University, 1988, and New York University, 1989.

[24] N. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Cl. Sci. (3) 22 (1968), pp. 265-274.

[25] N.S. Trudinger, On the Dirichlet problem for Hessian equations, Acta Math. 175 (1995), 151-164.

[26] N.S. Trudinger and X. Wang, Hessian measures II, Ann. of Math. 150 (1999), 579-604.

[27] J. Viaclovsky, Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds, Comm. Anal. Geom., to appear.

[28] J. Viaclovsky, Conformal geometry, contact geometry, and the calculus of variations, Duke Math. J. 101 (2000), 283-316.

[29] J. Viaclovsky, Conformally invariant Monge-Ampere equations: global solutions, Trans. Amer. Math. Soc. 352 (2000), 4371-4379.

[30] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. 12 (1960), 21-37.