Shooting with Degree Theory: Analysis of Some Weighted Polyharmonic Systems

John Villavert

Department of Applied Mathematics, University of Colorado at Boulder
Boulder, Colorado 80309, USA

Abstract

In this paper, the author establishes the existence of positive entire solutions to a general class of semilinear poly-harmonic systems under certain non-degeneracy conditions, which includes equations and systems of the weighted Hardy–Littlewood–Sobolev type as motivating examples. The novel method used implements the classical shooting method enhanced by topological degree theory. The key steps of the method are to first construct a target map which aims the shooting method and the non-degeneracy conditions guarantee the continuity of this map. With the continuity of the target map, we apply a topological argument from degree theory to show the existence of zeros of the target map. The existence of zeros of the map along with a non-existence theorem for the corresponding Navier boundary value problem imply the existence of positive solutions for the class of poly-harmonic systems.

Keywords: Degree theory; poly-harmonic equations; the shooting method; stationary Schrödinger system; weighted Hardy–Littlewood–Sobolev inequality.

Mathematics Subject Classification: 35B09; 35B33; 35J30; 35J48; 47H11; 47H11.

1 Introduction

A well-known difficulty in the study of nonlinear elliptic systems, or any nonlinear system of partial differential equations (PDEs) for that matter, is on the development of tools useful in their analysis. Such tools are usually limited to specific problems; that is, a technique for examining one class of problems may prove ineffective in examining other classes of problems. Our main objective of this article is to further develop and refine a novel approach—first introduced in [22]—for proving the existence of positive solutions and related properties to higher-order, nonlinear system of elliptic equations in the whole space. As we shall see below, the class of problems we examine will include the well-known Lane–Emden and Hardy–Littlewood–Sobolev type systems along with their weighted counterparts as motivating examples. Remarkably, the mathematical tools utilized within this framework are more or less elementary by themselves, but we combine them together to obtain some new and interesting results. Our first main result proves the existence of positive entire
solutions, under reasonable assumptions, to the general system
\[ (-\Delta)^k u_i = f_i(|x|, u_1, u_2, \ldots, u_L) \quad \text{in } x \in \mathbb{R}^n \setminus \{0\}, \quad \text{for } i = 1, 2, \ldots, L. \quad (1.1) \]

As we demonstrate below, proving the existence of positive solutions to this system of polyharmonic PDEs involves reformulating the problem in radial coordinates then applying the classical shooting method combined with a non-existence theorem for the corresponding Navier boundary value problem. Specifically, a natural ingredient of the proof entails constructing a continuous target map which aims the shooting method. Then a topological argument via degree theory is invoked to guarantee the existence of zeros of this target map, which enables us to identify the correct initial shooting positions for the shooting method. By combining this with a non-existence result for the corresponding boundary value problem, we obtain the existence of positive (radial) solutions to system (1.1).

In establishing our general results, the primary examples we consider are the weighted Hardy–Littlewood–Sobolev (HLS) equation
\[
\begin{align*}
(-\Delta)^{\gamma/2} u &= \frac{u^p}{|x|^{\sigma}} \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\
u > 0 &\quad \text{in } \mathbb{R}^n,
\end{align*}
\]
and the weighted HLS system,
\[
\begin{align*}
(-\Delta)^{\gamma/2} u &= \frac{u^p}{|x|^{\sigma_1}} \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\
(-\Delta)^{\gamma/2} v &= \frac{v^q}{|x|^{\sigma_2}} \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\
u, v > 0 &\quad \text{in } \mathbb{R}^n.
\end{align*}
\]
Here, \( n \geq 3, \gamma \in (0, n), \sigma, \sigma_1, \sigma_2 \in \mathbb{R} \) and \( p \) and \( q \) are positive exponents. Notice that when \( \gamma = 2 \) and \( \sigma_1 = \sigma_2 = 0 \), the weighted system reduces to the well-known Lane–Emden system
\[
\begin{align*}
-\Delta u &= v^q, \quad u > 0, \quad \text{in } \mathbb{R}^n, \\
-\Delta v &= u^p, \quad v > 0, \quad \text{in } \mathbb{R}^n,
\end{align*}
\]
or more generally to the HLS system when \( \gamma > 2 \) and \( \sigma_1 = \sigma_2 = 0 \):
\[
\begin{align*}
(-\Delta)^{\gamma/2} u &= v^q, \quad u > 0, \quad \text{in } \mathbb{R}^n, \\
(-\Delta)^{\gamma/2} v &= u^p, \quad v > 0, \quad \text{in } \mathbb{R}^n.
\end{align*}
\]
The Lane–Emden and HLS systems have received much attention in the past few decades. For instance, the scalar case was studied in \([2, 4, 17]\), and similar problems have been approached geometrically including the prescribing Gaussian and scalar curvature problems (cf. \([5, 7, 9]\)). Related systems including its generalized version, the HLS type systems, have been studied as well (cf. \([6–12], [14–15], [19–21], [23–28, 31, 35]\)). When \( \gamma \) is an even integer, \((1.5)\) is equivalent to the integral system
\[
\begin{align*}
u(x) &= \int_{\mathbb{R}^n} \frac{v(y)^q}{|x - y|^{n-\gamma}} dy, \quad u > 0 \quad \text{in } \mathbb{R}^n, \\
v(x) &= \int_{\mathbb{R}^n} \frac{u(y)^p}{|x - y|^{n-\gamma}} dy, \quad v > 0 \quad \text{in } \mathbb{R}^n,
\end{align*}
\]
in the sense that a solution of one system, multiplied by a suitable constant if necessary, is also a solution of the other when \( p, q > 1 \), and vice versa. Hence, the PDE system \((1.5)\) and
the integral system (1.6) are both referred to as the HLS system. Now, when studying the HLS system, the exponents \( p, q \) and \( \gamma \) play an essential role in determining the criteria for the existence and non-existence of solutions. More precisely, there are three important cases to consider: The HLS system is said to be in the **subcritical** case if \( \frac{1}{r} + \frac{1}{s} < \frac{n-2}{n} \), in the **critical** case if \( \frac{1}{r} + \frac{1}{s} = \frac{n}{n-2} \) and in the **supercritical** case if \( \frac{1}{r} + \frac{1}{s} > \frac{n}{n-2} \). In the special case of (1.4), the famous Lane–Emden conjecture—an analogue to the celebrated result of Gidas and Spruck in [17] for the scalar case—states that this elliptic system in the critical case has no classical solution. This has been completely settled for radial solutions (cf. [27, 33]), for dimensions \( n \leq 4 \) (cf. [30, 34, 37]), and for \( n \geq 5 \) but under certain subregions of subcritical exponents (cf. [3, 16, 27, 32, 37, 38]). With the help of the method of moving planes in integral form, the work in [13]—when combined with the non-existence results in [26]—provides a partial resolution of this conjecture as well. On the other hand, it is interesting to note that the results in this paper also include the existence of solutions to the Lane–Emden system in the non-subcritical case.

Let us further motivate the importance of the HLS system and its related systems in connection with the study of the classical HLS inequality. Recall the HLS inequality states that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x - y|^\lambda} \, dx \, dy \leq C_{s, \lambda, n} \|f\|_r \|g\|_s \quad \text{(1.7)}
\]

where \( 0 < \lambda < n \), \( 1 < s, r < \infty \), \( \frac{1}{r} + \frac{1}{s} = 2 \), \( f \in L^r(\mathbb{R}^n) \), and \( g \in L^s(\mathbb{R}^n) \) (cf. [18, 25, 36]). To find the best constant in the HLS inequality, one maximizes the associated HLS functional

\[
J(f, g) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x - y|^\lambda} \, dx \, dy \quad \text{(1.8)}
\]

under the constraint \( \|f\|_r = \|g\|_s = 1 \). Let \( p = \frac{1}{r} \), \( q = \frac{1}{s} \) and with a suitable scaling such as \( u = c_1 f^{r-1} \) and \( v = c_2 g^{s-1} \), the Euler–Lagrange equations are precisely the system of integral equations in (1.6). Here, \( u \in L^{p+1} \) and \( v \in L^{q+1} \) where the positive exponents \( p \) and \( q \) are in the critical case. In [25], Lieb proved the existence of positive solutions to (1.6) which maximize the corresponding functionals \( J(f, g) \) in the class of \( u \in L^{p+1} \) and \( v \in L^{q+1} \). In other words, there exist extremal functions of (1.8), thereby proving the existence of ground state solutions to the HLS system in the critical case. In addition, Hardy and Littlewood also introduced the following double weighted inequality which was later generalized by Stein and Weiss in [39]:

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x - y|^\lambda |y|^\beta} \, dx \, dy \leq C_{\alpha, \beta, s, \lambda, n} \|f\|_r \|g\|_s \quad \text{(1.9)}
\]

where \( \alpha + \beta \geq 0 \),

\[
1 - \frac{1}{r} - \frac{\lambda}{n} < \frac{\alpha}{n} < 1 - \frac{1}{s}, \quad \text{and} \quad \frac{1}{r} + \frac{1}{s} + \frac{\lambda + \alpha + \beta}{n} = 2.
\]

As before, to obtain the sharp constant in the double weighted HLS inequality, one maximizes the associated functional

\[
J(f, g) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x - y|^\lambda |y|^\beta} \, dx \, dy.
\]

The corresponding Euler–Lagrange equations for this functional is the system of integral
where $0 < p,q < \infty$, $0 < \lambda < n$, $\frac{a}{n} < \frac{1}{p+1} < \frac{\lambda+\alpha}{n}$, and $\frac{1}{1+p} + \frac{1}{1+q} = \frac{\lambda+\alpha+\beta}{n}$. In [11], Chen and Li examined this weighted HLS inequality and its corresponding Euler–Lagrange equations. As a result, the authors proved the uniqueness of solutions to the singular nonlinear system

$$\begin{cases}
-\Delta (|x|^\alpha u) = \frac{u^q}{|x|^\beta} 	ext{ in } \mathbb{R}^n \setminus \{0\}, \\
-\Delta (|x|^\beta v) = \frac{v^p}{|x|^\alpha} 	ext{ in } \mathbb{R}^n \setminus \{0\},
\end{cases}$$

(1.11)

and classified all the solutions for the case $\alpha = \beta$ and $p = q$, thereby obtaining the best constant in the corresponding weighted HLS inequality. Observe that if $f(x) = |x|^\gamma u(x)$ and $g(x) = |x|^\gamma v(x)$, then (1.11) becomes

$$\begin{cases}
-\Delta f(x) = \frac{g(x)^\theta}{|x|^\beta(\theta+1)} 	ext{ in } \mathbb{R}^n \setminus \{0\}, \\
-\Delta g(x) = \frac{f(x)^\theta}{|x|^\alpha(\theta+1)} 	ext{ in } \mathbb{R}^n \setminus \{0\},
\end{cases}$$

which is just a particular case of system (1.3). Let us remark on the case of supercritical exponents for the HLS system in relation to the work in this article. As a simple illustration, let $u = v$ and $p = q$ in (1.6) to obtain the following scalar integral equation,

$$u(x) = \int_{\mathbb{R}^n} \frac{u(y)^p}{|x - y|^{n-\gamma}} dy, \ u > 0 \text{ in } \mathbb{R}^n,$$

(1.12)

with the corresponding partial differential equation

$$(-\Delta)^k u(x) = u(x)^p, \ 2k < n, \ u > 0 \text{ in } \mathbb{R}^n.$$  

(1.13)

As before, both the integral and differential equations are called supercritical if $p > \frac{n+\alpha}{n-\gamma}$, critical if $p = \frac{n+\alpha}{n-\gamma}$, and subcritical if $p < \frac{n+\alpha}{n-\gamma}$. In the supercritical case with $k = 1$, the shooting method can be successfully applied to (1.13), however, much difficulty arises even in the scalar case for $k \geq 2$. In the results of this paper, we circumvent these difficulties by further developing our degree theoretic framework for the shooting method to handle even more general systems such as the weighted poly-harmonic systems, especially since such existence results are not so well developed for these problems. Hence, we shall determine the conditions on such systems which allow us to prove existence of solutions using our technique. In doing so, we demonstrate how to handle even the case of (1.2) and (1.3), which are not included in the results of [22] and [24]. In addition to the non-existence results, the difficulty in implementing our technique lies in determining the sufficient conditions which guarantee the continuity of the target map. This difficulty motivates our consideration of non-degeneracy conditions below. Specifically, we introduce non-degeneracy conditions each geared to handle HLS type systems with varying exponents and weights.

The rest of this manuscript is structured as follows. In section 2 we introduce some preliminary definitions and provide the precise statements of our main results. Section 3

4
gives the proof of our general existence theorem concerning the system (1.1). In section 4 we prove the existence theorems concerning equation (1.2) and system (1.3). In view of Theorem 1 to prove the existence of solutions for these weighted systems, some non-existence results for the corresponding boundary value problems are needed and whose proofs are also provided in the section.

## 2 Preliminaries and Main Results

Throughout this paper, we take $n \geq 3$, $x \in \mathbb{R}^n$, and set $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_+^L$ to be the usual Cartesian product of $\mathbb{R}_+$. For $v \in \mathbb{R}_+^L$, we say $v > 0$ if $v_j > 0$ for all $j = 1, 2, \ldots, L$. For $v \in \partial \mathbb{R}_+^L$, let $I_v^0$ and $I_v^+$ denote the set of indices $j \in \{1, 2, \ldots, L\}$ for which $v_j = 0$ and $v_j > 0$, respectively.

Now consider the system

$$\begin{cases}
(-\Delta)^k u_i = f_i(|x|, u_1, u_2, \ldots, u_L) & \text{in } \mathbb{R}^n \setminus \{0\}, \\
u_i > 0 & \text{in } \mathbb{R}^n, \text{ for } i = 1, 2, \ldots, L,
\end{cases} \tag{2.1}$$

with the following assumptions. From this point on, we always assume $k_i \geq 1$ and $F(|x|, u) = (f_1(|x|, u), f_2(|x|, u), \ldots, f_L(|x|, u))$ satisfies the following conditions:

(a) $F : (0, \infty) \times \mathbb{R}_+^L \rightarrow \mathbb{R}_+^L$ is a continuous vector-valued map,

(b) $F(|x|, u) > 0$ in the interior of $\mathbb{R}_+ \times \mathbb{R}_+^L$,

(c) $F$ is locally Lipschitz continuous in the second argument uniformly in the interior of $\mathbb{R}_+ \times \mathbb{R}_+^L$.

**Non-degeneracy condition 1.** Let $F = F(|x|, u)$ satisfy the following:

(i) For each non-zero $v \in \partial \mathbb{R}_+^L$ there are constants $\lambda = \lambda(v) > 0$, $\sigma = \sigma(v) > -2$ and a $\delta = \delta(v) > 0$ such that if $|v - w| < \delta$, then

$$\lambda(v)|x|^\sigma \leq \sum_{j \in I_v^0} f_j(|x|, w) \text{ for } x \in \mathbb{R}_+^L \setminus \{0\}.$$

(ii) If $\lim_{|x| \rightarrow \infty} F(|x|, v) = 0$, then $v \in \partial \mathbb{R}_+^L$.

**Non-degeneracy condition 2.** Let $F = F(v)$ be autonomous and satisfy the following:

(i) $v \in \partial \mathbb{R}_+^L \iff F(v) = 0$.

(ii) Let $m > 0$ be an arbitrary positive number. Suppose that for any $v = (v_1, v_2, \ldots, v_L) \in \mathbb{R}_+^L$ and any permutation $\{i_1, i_2, \ldots, i_L\}$ of the set $\{1, 2, \ldots, L\}$ such that

$$v_{i_k} \leq m \text{ for } k = 1, 2, \ldots, j \text{ and } v_{i_k} > m \text{ for } k = j + 1, \ldots, L \text{ (} 1 \leq j < L),$$

there exists a constant $C_m > 0$, depending only on $m$, such that

$$\sum_{k=j+1}^{L} f_{i_k}(v) \leq \frac{C_m}{L} \sum_{k=1}^{j} f_{i_k}(v). \tag{2.2}$$
Systems of the form (1.1) satisfying conditions (a)–(c) and the non-degeneracy ‘condition 1’ (or ‘condition 2’) are said to be non-degenerate ‘type I’ (or ‘type II’). The weighted HLS system is an example of a non-degenerate type I system and the stationary Schrödinger system,

\[
\begin{cases}
-\Delta u = u^s v^q, & u > 0, \quad \text{in } \mathbb{R}^n, \\
-\Delta v = v^t u^p, & v > 0, \quad \text{in } \mathbb{R}^n,
\end{cases}
\]

is an example of a non-degenerate type II system provided \( p \geq t \geq 0, \) and \( q \geq s \geq 0, \) but notice it is not type I.

The first theorem presented in this paper illustrates how the existence of solutions for non-degenerate systems (1.1) follows from the non-existence of solutions to the corresponding Navier boundary value problem

\[
\begin{cases}
(-\Delta)^k u_i = f_i(|x|, u_1, u_2, \ldots, u_L), & \text{in } B_R(0) \setminus \{0\}, \\
u_i > 0 & \text{in } B_R(0), \\
u_i = -\Delta u_i = \ldots = (-\Delta)^{k-1} u_i = 0 & \text{on } \partial B_R(0), \; i = 1, 2, \ldots, L,
\end{cases}
\]

for all \( R > 0. \) Here, \( B_R(0) \subset \mathbb{R}^n \) denotes the open ball of radius \( R \) centered at the origin with boundary \( \partial B_R(0). \) We shall only be concerned with type I systems in this paper, but Li and Villavert established, among other interesting results, analogous existence theorems for unweighted, non-degenerate type II systems in [24]. Namely, as a motivating example, one of the results in that work proved the following:

**Theorem.** The non-degenerate type II system (2.3) has a solution of class \( C^2(\mathbb{R}^n) \) provided that either

\[
\frac{1}{1 + q} + \frac{1}{1 + p} \leq \frac{n - 2}{n}, \quad \text{or} \quad \min\{s + q, t + p\} \geq \frac{n + 2}{n - 2},
\]

where \( q \geq t \geq 1 \) and \( p \geq s \geq 1. \)

Now we are ready to state our main results.

**Theorem 1.** The non-degenerate type I system (2.1) admits a radially symmetric solution of class \( C^{2k}(\mathbb{R}^n \setminus \{0\}) \) provided that (2.4) admits no radially symmetric solution of class \( C^{2k-1}(B_R(0) \setminus \{0\}) \cap C^{2k-1}(\overline{B_R(0)}) \) for all \( R > 0. \) Furthermore, the solution satisfies the following asymptotic property:

\[
u_i \longrightarrow 0 \quad \text{uniformly as } |x| \longrightarrow 0 \quad \text{for } i = 1, 2, \ldots, L.
\]

The following non-existence theorems serve as an important ingredient in proving the existence results for (1.2) and (1.3). It may be interesting to note that these non-existence results easily apply to any smooth, bounded star-shaped domain.

**Theorem 2.** Let \( k \in [1, n/2) \) be an integer, \( p > 0, \) and \( \sigma \in (-\infty, n). \) Then the 2k-th order equation

\[
\begin{cases}
(-\Delta)^k u = \frac{u^p}{|x|^{\sigma}}, & \text{in } B_R(0) \setminus \{0\}, \\
u > 0 & \text{in } B_R(0), \\
v = -\Delta v = \ldots = (-\Delta)^{k-1} v = 0 & \text{on } \partial B_R(0),
\end{cases}
\]
admits no radially symmetric solution of class $C^{2k}(B_R(0)\setminus \{0\}) \cap C^{2k-1}(\overline{B_R(0)})$ for any $R > 0$ provided that
\[ p \geq \frac{n+2k-2\sigma}{n-2k}. \] (2.7)

**Theorem 3.** Let $k \in [1, n/2)$ be an integer, $s,t,p,q \geq 0$, and $\sigma_1, \sigma_2 \in (-\infty, n)$. Then the $2k$-th order system
\[
\begin{cases}
(-\Delta)^k u = \frac{u^pv^q}{|x|^\sigma_1} & \text{in } B_R(0)\setminus \{0\}, \\
(-\Delta)^k v = \frac{v^pu^q}{|x|^\sigma_2} & \text{in } B_R(0)\setminus \{0\}, \\
u, v > 0 & \text{in } B_R(0), \\
u = -\Delta u = \cdots = (-\Delta)^{k-1}u = 0 & \text{on } \partial B_R(0), \\
v = -\Delta v = \cdots = (-\Delta)^{k-1}v = 0 & \text{on } \partial B_R(0),
\end{cases}
\] (2.8)
admits no radially symmetric solution of class $C^{2k}(B_R(0)\setminus \{0\}) \cap C^{2k-1}(\overline{B_R(0)})$ for any $R > 0$ provided that
\[ \frac{n-\sigma_1}{1+q} + \frac{n-\sigma_2}{1+p} \leq n - 2k. \] (2.9)

Theorems 1–3 have the following consequences.

**Corollary 1.** Let $k \in [1, n/2)$ be an integer, $p > 0$, and $\sigma \in (-\infty, 2)$. Then the $2k$-th order equation
\[
\begin{cases}
(-\Delta)^k u = \frac{u^p}{|x|^\sigma} & \text{in } \mathbb{R}^n\setminus \{0\}, \\
u > 0 & \text{in } \mathbb{R}^n, \\
u \to 0 \text{ uniformly as } |x| \to 0,
\end{cases}
\] (2.10)
admits a solution of class $C^{2k}(\mathbb{R}^n\setminus \{0\})$ provided that
\[ p \geq \frac{n+2k-2\sigma}{n-2k}. \]

**Corollary 2.** Let $k \in [1, n/2)$ be an integer, $p,q > 0$, and $\sigma_1, \sigma_2 \in (-\infty, 2)$. Then the $2k$-th order system
\[
\begin{cases}
(-\Delta)^k u = \frac{u^qv^q}{|x|^\sigma_1} & \text{in } \mathbb{R}^n\setminus \{0\}, \\
(-\Delta)^k v = \frac{v^pu^p}{|x|^\sigma_2} & \text{in } \mathbb{R}^n\setminus \{0\}, \\
u, v > 0 & \text{in } \mathbb{R}^n, \\
u, v \to 0 \text{ uniformly as } |x| \to 0,
\end{cases}
\] (2.11)
admits a solution of class $C^{2k}(\mathbb{R}^n\setminus \{0\})$ provided that
\[ \frac{n-\sigma_1}{1+q} + \frac{n-\sigma_2}{1+p} \leq n - 2k. \]
3 Proof of Theorem 1

In order to prove the first existence theorem, we must introduce several key ideas and lemmas. As mentioned earlier, the proof centers on a construction of a map which aims the shooting method. This section defines the target map and applies our method to prove Theorem 1 but the proof on the continuity of the target map is provided later in Section 4.

Before we can apply our method, we need to reduce the poly-harmonic system into a second-order system. For \( i = 1, 2, \ldots, L \) set \( w_{i,j} = (-\Delta)^{j-1}u_i, 1 \leq j \leq k_i \) and consider the system

\[
\begin{aligned}
-\Delta w_{i,1} &= w_{i,2}, -\Delta w_{i,2} = w_{i,3}, \ldots, -\Delta w_{i,k_i-1} = w_{i,k_i}, \\
-\Delta w_{i,k_i} &= f_i(x, w_{1,1}, w_{1,2}, \ldots, w_{L,1}) & \text{in } \mathbb{R}^n \setminus \{0\}, \\
w_{i,1}, w_{i,2}, \ldots, w_{i,k_i} &> 0 & \text{in } \mathbb{R}^n,
\end{aligned}
\]

where \( i = 1, 2, \ldots, L \).

Solutions of (3.1) are clearly solutions of (2.1), so it will suffice to show the existence of solutions to (3.1) instead. We can express the above system into the more general form:

\[
\begin{aligned}
-\Delta w_1 &= f_1(r, w), -\Delta w_2 = f_2(r, w), \\
-\Delta w_3 &= f_4(r, w), \ldots, -\Delta w_{L-1} = f_{L-1}(r, w), \\
-\Delta w_L &= f_L(r, w) & \text{in } \mathbb{R}^n \setminus \{0\}, \\
w_1, w_2, \ldots, w_L &> 0 & \text{in } \mathbb{R}^n,
\end{aligned}
\]

where we still use \( L \) to represent a generic positive integer. We shall work with (3.2) instead when proving Theorem 1 and the continuity of the target map below, but note that the non-degeneracy condition 1 still holds true for this reduced second-order system. Now let us define the aforementioned target map. For any strictly positive initial value \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_L) \), consider the IVP

\[
\begin{aligned}
w_i''(r) + \frac{n-1}{r}w_i'(r) &= -f_i(r, w(r)), \\
w_i'(0) = 0, \quad w_i(0) = \alpha_i & \text{for } i = 1, 2, \ldots, L.
\end{aligned}
\]

Clearly (3.3) is equivalent to (3.2) in radial coordinates.

**Definition.** Define the target map \( \psi : \mathbb{R}^L_+ \to \mathbb{R}^L_+ \) as follows. For \( \alpha \in \text{int}(\mathbb{R}^L_+) \), the interior of \( \mathbb{R}^L_+ \),

\( a \) \( \psi(\alpha) = w(r_0) \) where \( r_0 \) is the smallest such \( r \) for which \( w_{i_0}(r) = 0 \) for some \( 1 \leq i_0 \leq L \),

\( b \) otherwise, if no such \( r_0 \) exists, then \( \psi(\alpha) = \lim_{r \to \infty} w(r) \).

\( c \) \( \psi \equiv \text{Identity on the boundary } \partial \mathbb{R}^L_+ \).

**Remark 1.** We may think of (a) as the case when the solution hits the wall for the first time and (b) is the case where it never hits the wall. Observe also that \( \psi \) is equivalent to the identity map on the wall. This property is crucial when we apply the tools from topological degree theory.

The next lemma is a standard result from Brouwer topological degree theory and can be found in various literature (cf. [1] and [29] for instance).
Lemma 1 (Dependence on boundary values). Let \( U \subset \mathbb{R}^n \) be a bounded open set and \( f, g : \overline{U} \to \mathbb{R}^n \) are continuous maps. Suppose that \( f \equiv g \) on \( \partial U \) and \( a \notin f(\partial U) = g(\partial U) \), then \( \text{degree}(f, U, a) = \text{degree}(g, U, a) \).

One may recall the important property that if \( \text{degree}(f, U, a) \neq 0 \), then there exists a point \( x \in U \) such that \( f(x) = a \).

Lemma 2. The target map \( \psi : \mathbb{R}_+^L \to \partial \mathbb{R}_+^L \) is continuous.

We give the proof of this later in the final section.

Lemma 3. For every \( a > 0 \), there exists an \( \alpha_a \in A_a \) where

\[
A_a := \left\{ \alpha \in \mathbb{R}_+^L \mid \sum_{i=1}^{L} \alpha_i = a \right\}
\]

such that \( \psi(\alpha_a) = 0 \).

Proof of Lemma 3 Define the set \( B_a \) as follows

\[
B_a := \left\{ \alpha \in \partial \mathbb{R}_+^L \mid \sum_{i=1}^{L} \alpha_i \leq a \right\}.
\]

It follows that \( \psi \) maps \( A_a \) into \( B_a \) due to the non-increasing property of solutions. Now define the continuous map \( \phi : B_a \to A_a \) by

\[
\phi(\alpha) = \alpha + \frac{1}{L} \left( a - \sum_{i=1}^{L} \alpha_i \right) (1,1,\ldots,1)
\]

with continuous inverse \( \phi^{-1} : A_a \to B_a \) defined

\[
\phi^{-1}(\alpha) = \alpha - \left( \min_{i=1,\ldots,L} \alpha_i \right) (1,1,\ldots,1).
\]

Set \( \eta = \phi \circ \psi : A_a \to A_a \). Then \( \eta \) is continuous on \( A_a \) and is equivalent to the identity map on the boundary of \( A_a \). By Lemma 1, the index of the map satisfies \( \text{degree}(\eta, A_a, \alpha) = \text{degree}(\text{Identity}, A_a, \alpha) = 1 \neq 0 \) for any interior point \( \alpha \in \text{int}(A_a) \). So \( \eta \) is onto, and thus \( \psi \) is onto. Then there exists an \( \alpha_a \in A_a \) such that \( \psi(\alpha_a) = 0 \).

Proof of Theorem 1 For fixed \( a > 0 \), let \( w = w(r) \) be the solution of (3.3) with initial condition \( w(0) = \alpha_a \) as given by Lemma 3. We claim this solution must never hit the wall. Otherwise, if this was the case, then there would be a smallest finite value \( r = r_0 \) such that \( w(r_0) = \psi(\alpha_a) = 0 \). But this would imply that \( w = w(|x|) \) is a radially symmetric solution to (2.4) with \( R = r_0 \), which contradicts the non-existence assumption on all ball domains. Hence, the solution must never hit the wall, which implies that \( w = w(|x|) \) is a radially symmetric solution of (2.4). Furthermore, the definition of the target map implies that \( w \to 0 \) uniformly as \( |x| \to \infty \).
Remark 2. In the proof of Theorem 1, we are using the fact that the Navier boundary value problem \( (2.4) \) is equivalent, under classical solutions, to the problem

\[
\begin{cases}
- \Delta w_{1,1} = w_{1,2}, \\
- \Delta w_{1,2} = w_{1,3}, \\

\vdots \\
- \Delta w_{i,k_i-1} = w_{i,k_i}, \\
- \Delta w_{i,k_i} = f_i(|x|, w_{1,1}, w_{2,1}, \ldots, w_{L,1}) \quad \text{in} \quad B_R(0) \setminus \{0\}, \\
w_{i,1}, w_{i,2}, \ldots, w_{i,k_i} > 0 \quad \text{in} \quad B_R(0), \\
w_{i,1} = w_{i,2}, \ldots, w_{i,k_i} = 0 \quad \text{on} \quad \partial B_R(0),
\end{cases}
\]

where \( w_{i,j} = (-\Delta)^{-1} u_i \) for \( j = 1, 2, \ldots, k_i \) and \( i = 1, 2, \ldots, L \). To see this, first observe that if \( u_1 = w_{1,1}, u_2 = w_{2,1}, \ldots, u_L = w_{L,1} \) where the \( w_{i,j} \)'s satisfy \( (2.4) \), then \( u_1, u_2, \ldots, u_L \) must satisfy \( (2.4) \). Conversely, suppose \( u_1, u_2, \ldots, u_L \) satisfy \( (2.4) \) and let

\[
w_{i,j} = (-\Delta)^{-1} u_i \quad \text{for} \quad j = 1, 2, \ldots, k_i \quad \text{and} \quad i = 1, 2, \ldots, L.
\]

Notice that it suffices to show the following super polyharmonic property:

\[
w_{i,j} > 0 \quad \text{in} \quad B_R(0) \quad \text{for} \quad j = 1, 2, \ldots, k_i, \quad \text{and} \quad i = 1, 2, \ldots, L,
\]

since this would imply \( w = (w_{i,j}) \) under \( (3.5) \) solves \( (3.4) \). Let us sketch the proof of this super poly-harmonic property. Since \( w_{1,1}, w_{2,1} \) and \( w_{L,1} \) are positive in \( B_R(0) \), we have that \( -\Delta w_{i,k_i} > 0 \) in \( B_R(0) \) for \( i = 1, 2, \ldots, L \). The boundary conditions along with the strong maximum principle imply that \( w_{i,k_i} > 0 \) in \( B_R(0) \), which in turn, implies that \( -\Delta w_{i,k_i-1} > 0 \) in \( B_R(0) \). Again, the boundary conditions and the strong maximum principle imply that \( w_{i,k_i-1} > 0 \) in \( B_R(0) \) for \( i = 1, 2, \ldots, L \). Obviously, we can repeat this argument inductively to show the remaining components of \( w = (w_{i,j}) \) are positive in \( B_R(0) \).

4 The Remaining Proofs

This section first proves the non-existence theorems for the Navier boundary value problems. Then, the proof of Lemma \( \text{[2]} \) concerning the continuity of the target map is given followed by the proofs of Corollaries \( \text{[1]} \) and \( \text{[2]} \).

4.1 Non-existence of Solutions on Bounded Domains

As demonstrated in Remark \( \text{[2]} \) at the end of Section 3, the proof of Theorem \( \text{[2]} \) reduces to showing the equivalent system,

\[
\begin{cases}
- \Delta w_1 = w_2, \quad -\Delta w_2 = w_3, \ldots, -\Delta w_{k-1} = w_k, \\
- \Delta w_k = \frac{w_k^p}{|x|^q} \quad \text{in} \quad B_R(0) \setminus \{0\} \\
w_1, w_2, \ldots, w_k > 0 \quad \text{in} \quad B_R(0) \\
w_1 = w_2 = \cdots = w_k = 0 \quad \text{on} \quad \partial B_R(0),
\end{cases}
\]

\( (4.1) \)
admits no solution of class $C^2(B_R(0)\setminus \{0\}) \cap C^1(\overline{B_R(0)})$ for any $R > 0$. The key ingredients for this non-existence result center on establishing a Pohozaev type identity and the following identity.

**Lemma 4.** Let $w_j \ (j = 1, 2, \ldots, k)$ solve (4.1). Then

$$\int_{B_R(0)} \frac{u_j^{p+1}}{|x|^\sigma} \, dx = \int_{B_R(0)} \nabla w_j \cdot \nabla w_{k+1-j} \, dx$$

$$= \int_{B_R(0)} w_{j+1} w_{k+1-j} \, dx =: E_1,$$  

(Here, it should be understood that $w_{k+1} := \frac{u_p}{|x|^\sigma} = -\Delta w_k$).

**Proof.** To prove this lemma, multiply the $k$-th equation in (4.1) by $w_1$ then integrate over $B_R(0)$. The repeated application of integration by parts with the boundary conditions imply

$$\int_{B_R(0)} \frac{w_j^{p+1}}{|x|^\sigma} \, dx = \int_{B_R(0)} \nabla w_1 \cdot \nabla w_k \, dx$$

$$= - \int_{B_R(0)} w_k \Delta w_1 \, dx = \int_{B_R(0)} w_k w_2$$

$$= - \int_{B_R(0)} w_2 \Delta w_{k-1} \, dx = \int_{B_R(0)} \nabla w_2 \cdot \nabla w_{k-1} \, dx$$

$$= - \int_{B_R(0)} w_{k-1} \Delta w_2 \, dx = \int_{B_R(0)} w_{k-1} w_3 \, dx$$

$$\vdots$$

$$= \int_{B_R(0)} \nabla w_j \cdot \nabla w_{k+1-j} \, dx = \int_{B_R(0)} w_{j+1} w_{k+1-j} \, dx.$$

**Remark 3.** Let us be more precise in the calculations found in our proof of Lemma 4 since we employ similar calculations below. For instance, when we multiply, say, the $k$-th equation $-\Delta w_k = |x|^{-\sigma} u_k^p$ by $w_1$ then integrate over the ball $B_R(0)$, this should be understood implicitly in the following way: We integrate over $B_R(0) \setminus B_\epsilon(0)$ for $0 < \epsilon < R$ and use an integration by parts to obtain

$$\int_{B_R(0) \setminus B_\epsilon(0)} \frac{w_j^{p+1}}{|x|^\sigma} \, dx = - \int_{B_R(0) \setminus B_\epsilon(0)} w_1 \Delta w_k \, dx$$

$$= - \int_{\partial B_\epsilon(0)} w_1 \frac{\partial w_k}{\partial \nu} \, ds + \int_{B_R(0) \setminus B_\epsilon(0)} \nabla w_1 \cdot \nabla w_k \, dx,$$

where $\nu$ is the inward unit normal vector along $\partial B_\epsilon(0)$. By taking the limit as $\epsilon$ tends to zero, the surface integral vanishes since the $w_i$’s are of the class $C^1(\overline{B_R(0)})$. Then we obtain

$$\int_{B_R(0)} \frac{w_j^{p+1}}{|x|^\sigma} \, dx = \int_{B_R(0)} \nabla w_1 \cdot \nabla w_k \, dx.$$

All such calculations including those found in the proof of Theorems 2 and 3 below should be understood in this way.
Proof of Theorem 2.} The proof is by contradiction. Assume \( w \) is a solution of (4.1). For \( j = 1, 2, 3, \ldots, k \), multiply the \( j \)-th equation in (4.1) by \( x \cdot \nabla w_{k+1-j} \), integrate over \( B_R(0) \), then integrate by parts to obtain
\[
- \int_{\partial B_R(0)} \frac{\partial w_j}{\partial n} \frac{\partial w_{k+1-j}}{\partial n} (x \cdot n) \, ds + \int_{B_R(0)} \nabla w_j \cdot \nabla w_{k+1-j} \, dx + \int_{B_R(0)} x \cdot \nabla (w_{k+1-j})_x, u_{j,x} \, dx
\]
\[
= \int_{B_R(0)} w_{j+1}(x \cdot \nabla w_{k+1-j}) + w_{k+2-j}(x \cdot \nabla w_j) \, dx, \tag{4.3}
\]
where \( n \) is the outward pointing unit normal vector. Similarly, multiply the \( (k + 1 - j) \)-th equation in (4.1) by \( x \cdot \nabla w_{j} \), integrate over \( B_R(0) \), then integrate by parts to obtain
\[
- \int_{\partial B_R(0)} \frac{\partial w_j}{\partial n} \frac{\partial w_{k+1-j}}{\partial n} (x \cdot n) \, ds + \int_{B_R(0)} \nabla w_j \cdot \nabla w_{k+1-j} \, dx + \int_{B_R(0)} x \cdot \nabla (w_{j})_x, w_{k+1-j,x} \, dx
\]
\[
= \int_{B_R(0)} w_{j+1}(x \cdot \nabla w_{k+1-j}) + w_{k+2-j}(x \cdot \nabla w_j) \, dx. \tag{4.4}
\]
By adding (4.3) and (4.4) together and using integration by parts, we obtain
\[
- \int_{\partial B_R(0)} \frac{\partial w_j}{\partial n} \frac{\partial w_{k+1-j}}{\partial n} (x \cdot n) \, ds + (2 - n) \int_{B_R(0)} \nabla w_j \cdot \nabla w_{k+1-j} \, dx
\]
\[
= \int_{B_R(0)} w_{j+1}(x \cdot \nabla w_{k+1-j}) + w_{k+2-j}(x \cdot \nabla w_j) \, dx. \tag{4.5}
\]
In addition, observe that integration by parts and the boundary conditions yield the identity
\[
\int_{B_R(0)} x \cdot (w_j \nabla w_{k+2-j} + w_{k+2-j} \nabla w_j) \, dx = \int_{B_R(0)} x \cdot \nabla (w_j w_{k+2-j}) \, dx
\]
\[
= -n \int_{B_R(0)} w_j w_{k+2-j} \, dx.
\]
With this identity, summing (4.5) over \( j = 1, 2, 3, \ldots, k \) gives us the Pohozaev type identity
\[
(2 - n) \sum_{j=1}^k \int_{B_R(0)} \nabla w_j \cdot \nabla w_{k+1-j} \, dx + \sum_{j=1}^{k-1} n \int_{B_R(0)} w_{j+1} w_{k+1-j} \, dx + \frac{2(n-\sigma)}{1+p} \int_{B_R(0)} \frac{w_j^{p+1}}{|x|^\sigma} \, dx
\]
\[
= \sum_{j=1}^k \int_{\partial B_R(0)} \frac{\partial w_j}{\partial n} \frac{\partial w_{k+1-j}}{\partial n} (x \cdot n) \, ds.
\]
Observe that the right hand side of this inequality must be strictly positive by the non-increasing property of the positive radial solutions. Hence, Lemma 4 implies that
\[
\left\{ k(2 - n) + n(k - 1) + \frac{2(n-\sigma)}{1+p} \right\} \cdot E_1 > 0,
\]
and we arrive at a contradiction. \( \square \)

Similarly, the key ingredients in the proof of Theorem 3 is a Pohozaev type identity and the following identity.

\[ 12 \]
Lemma 5. Let \( w_j (j = 1, 2, \ldots, 2k) \) solve

\[
\begin{cases}
- \Delta w_1 = w_2, \ldots, -\Delta w_{k-1} = w_k, \\
- \Delta w_k = \frac{w_k^q}{|x|^\sigma}, \\
- \Delta w_{k+1} = w_{k+2}, \ldots, -\Delta w_{2k-1} = w_{2k}, \\
- \Delta w_{2k} = \frac{w_{k+1}^p}{|x|^\sigma} & \text{in } B_R(0) \setminus \{0\}, \\
w_1, w_2, \ldots, w_{2k} > 0 & \text{in } B_R(0), \\
w_1 = w_2 = \cdots = w_{2k} = 0 & \text{on } \partial B_R(0).
\end{cases}
\]

(4.6)

Then

\[
\int_{B_R(0)} \frac{u^s v^{q+1}}{|x|^\sigma_1} \, dx = \int_{B_R(0)} \frac{v^t u^{p+1}}{|x|^\sigma_2} \, dx = \int_{B_R(0)} \nabla w_j \cdot \nabla w_{2k+1-j} \, dx
\]

\[
= \int_{B_R(0)} w_{j+1} w_{2k+1-j} \, dx =: E_2.
\]

(4.7)

(Here, it should be understood that \( w_{2k+1} := v^t u^p/|x|^\sigma_2 = -\Delta w_{2k} \).

Proof. To prove this lemma, multiply the 2\(k\)-th equation in (4.6) by \( w_1 \) then integrate over \( B_R(0) \). The repeated application of integration by parts along with the boundary conditions yield

\[
\int_{B_R(0)} \frac{w_{k+1}^t w_1^{p+1}}{|x|^\sigma_2} \, dx = \int_{B_R(0)} \nabla w_1 \cdot \nabla w_k \, dx
\]

\[
= - \int_{B_R(0)} w_{2k} \Delta w_1 \, dx = \int_{B_R(0)} w_{2k} w_2 \nabla w_2 \cdot \nabla w_{2k-1} \, dx
\]

\[
= - \int_{B_R(0)} w_{2k-1} \Delta w_2 \, dx = \int_{B_R(0)} w_{2k-1} w_3 \nabla w_3 \cdot \nabla w_{2k-2} \, dx
\]

\[
\vdots
\]

\[
= \int_{B_R(0)} \nabla w_k \cdot \nabla w_{k+1} \, dx = - \int_{B_R(0)} w_{k+1} \Delta w_k \, dx
\]

\[
= \int_{B_R(0)} \frac{w_k^t w_{k+1}^{p+1}}{|x|^\sigma_1} \, dx.
\]

Proof of Theorem 3. Assume that \( w = (w_j) \) is a solution of (4.6) with non-negative exponents satisfying (2.9). For \( j = 2, 3, \ldots, k - 1 \), multiply the \( j \)-th equation in (4.6) by
Let us calculate $I_1$ and $I_2$. Using integration by parts,

$$I_1 = -\int_{\partial B_R(0)} \frac{\partial w_1}{\partial n} \frac{\partial w_2}{\partial n} (x \cdot n) \, ds + \int_{B_R(0)} \nabla w_1 \cdot \nabla w_2 \, dx + \int_{B_R(0)} x_i \frac{\partial w_2}{\partial x_j} \left( \frac{\partial^2 w_1}{\partial x_j \partial x_i} \right) \, dx,$$

and

$$I_2 = \frac{1}{1 + p} \int_{B_R} \frac{w_{k+1}^{t} w_{1}^{p+1}}{|x|^\sigma_2} \, dx \cdot x_i.
\frac{w_{k+1}^{t} w_{1}^{p+1}}{|x|^\sigma_2} \, dx - \frac{t}{1 + p} \int_{B_R} \frac{w_{k+1}^{t} w_{1}^{p+1}}{|x|^\sigma_2} \cdot \nabla w_{k+1} \, dx.$$

Now multiply the first equation by $x \cdot \nabla w_2$ and integrate over $B_R(0)$ to obtain

$$\int_{B_R(0)} (x \cdot \nabla w_2) \Delta w_1 \, dx = \int_{B_R(0)} (x \cdot \nabla w_2) w_2 \, dx.$$
We use integration by parts to rewrite $I_1$ as follows.

$$I_1 = -\int_{\partial B_R(0)} \frac{\partial w_{2k}}{\partial n} \frac{\partial w_1}{\partial n} (x \cdot n) \, ds + \int_{B_R(0)} \nabla w_1 \cdot \nabla w_{2k} \, dx + \int_{B_R(0)} x_j \frac{\partial w_1}{\partial x_j} \left( \frac{\partial^2 w_{2k}}{\partial x_j \partial x_i} \right) \, dx.$$  

By summing together the two equations $I_1 = I_2$ and $II = II_1$ and using the fact that

$$\int_{B_R(0)} x \cdot \nabla (\nabla w_1 \cdot \nabla w_{2k}) \, dx = \int_{\partial B_R(0)} \frac{\partial w_{2k}}{\partial n} \frac{\partial w_1}{\partial n} (x \cdot n) \, ds - n \int_{B_R(0)} \nabla w_1 \cdot \nabla w_{2k} \, dx,$$

we obtain the identity

$$(2 - n) \int_{B_R} \nabla w_{2k} \cdot \nabla w_1 \, dx + \frac{n - \sigma_2}{1 + p} \int_{B_R} \frac{w_{k+1}^{q+1}}{|x|^\sigma_2} \, dx = \int_{\partial B_R} \frac{\partial w_{2k}}{\partial n} \frac{\partial w_1}{\partial n} (x \cdot n) \, ds \tag{4.10}$$

$$+ \int_{B_R} w_2(x \cdot \nabla w_{2k}) \, dx - \frac{t}{1 + p} \int_{B_R} \frac{w_{k+1}^{q+1} w_{1}^{s-1}}{|x|^\sigma_1} (x \cdot \nabla w_{k+1}) \, dx.$$

Multiply the $k$-th and $(k+1)$-th equations in (4.10) by $x \cdot \nabla w_{k+1}$ and $x \cdot \nabla w_k$, respectively, and integrate over $B_R(0)$. Using similar calculations to those used in deriving (4.10), we obtain

$$(2 - n) \int_{B_R} \nabla w_k \cdot \nabla w_{k+1} \, dx + \frac{n - \sigma_1}{1 + q} \int_{B_R} \frac{w_{k+1}^{q+1}}{|x|^\sigma_1} \, dx = \int_{\partial B_R} \frac{\partial w_k}{\partial n} \frac{\partial w_{k+1}}{\partial n} (x \cdot n) \, ds \tag{4.11}$$

$$+ \int_{B_R} w_{k+2}(x \cdot \nabla w_k) \, dx - \frac{s}{1 + q} \int_{B_R} \frac{w_{k+1}^{q+1} w_{1}^{s-1}}{|x|^\sigma_1} (x \cdot \nabla w_1) \, dx.$$

Observe also that integrating by parts and using the boundary conditions, we obtain

$$\int_{B_R(0)} x \cdot (w_{j+1} \nabla w_{k+1-j} + w_{2k+1-j} \nabla w_{j+1}) \, dx = \int_{B_R(0)} x \cdot \nabla (w_{j+1} w_{2k+1-j}) \, dx$$

$$= - n \int_{B_R(0)} w_{j+1} w_{2k+1-j} \, dx.$$

Using this identity and summing (4.9) over $j = 2, 3, \ldots, k - 1$ along with (4.10) and (4.11), we arrive at the following Pohozaev type identity:

$$(2 - n) \sum_{j=1}^{k} \int_{B_R} \nabla w_j \cdot \nabla w_{k+1-j} \, dx + \frac{n - \sigma_1}{1 + q} \int_{B_R} \frac{w_{k+1}^{q+1}}{|x|^\sigma_1} \, dx$$

$$+ \frac{n - \sigma_2}{1 + p} \int_{B_R} \frac{w_{k+1}^{q+1}}{|x|^\sigma_2} \, dx + \sum_{j=1}^{k-1} n \int_{B_R} w_{j+1} w_{2k+1-j} \, dx$$

$$= \sum_{j=1}^{k} \int_{\partial B_R} \frac{\partial w_j}{\partial n} \frac{\partial w_{k+1-j}}{\partial n} (x \cdot n) \, ds - \left\{ \left( \frac{s}{1 + q} \right) \int_{B_R} \frac{w_{k+1}^{q+1} w_1^{s-1}}{|x|^\sigma_1} (x \cdot \nabla w_1) \, dx \right. $$

$$+ \frac{t}{1 + p} \int_{B_R} \frac{w_{k+1}^{q+1} w_1^{s-1}}{|x|^\sigma_2} (x \cdot \nabla w_{k+1}) \, dx \}.$$

Observe that the right hand side of this Pohozaev type identity must be strictly positive by the non-increasing property of the positive radial solutions. Hence, Lemma [3] implies that

$$\left\{ k(2 - n) + \frac{n - \sigma_1}{1 + q} + \frac{n - \sigma_2}{1 + p} + (k - 1)n \right\} \cdot E_2 > 0.$$
In other words, we have

\[ \frac{n - \sigma_1}{1 + q} + \frac{n - \sigma_2}{1 + p} > n - 2k, \]

but this contradicts with (2.9). \qed

4.2 Continuity of the Target Map

Proof of Lemma 2. Fix an \( \epsilon > 0 \) and choose any \( \overline{\alpha} \in \mathbb{R}_+^L \). To show the continuity of the target map at \( \overline{\alpha} \), there are three cases to consider:

Case (1): \( \overline{\alpha} \in \partial \mathbb{R}_+^L \),

Case (2): \( \overline{\alpha} \in \text{int}(\mathbb{R}_+^L) \) and the solution of (3.3) with initial position \( \overline{\alpha} \) touches the wall,

Case (3): \( \overline{\alpha} \in \text{int}(\mathbb{R}_+^L) \) and the solution of (3.3) with initial position \( \overline{\alpha} \) never touches the wall.

Case (1): By definition, \( \psi(\overline{\alpha}) = \overline{\alpha} \) for this case. If \( \overline{\alpha} = 0 \), the continuity of \( \psi \) at \( \overline{\alpha} \) follows easily from the non-increasing property of solutions since we can choose \( \delta = \epsilon \) so that \( |\psi(\alpha) - \psi(\overline{\alpha})| = |\psi(\alpha)| \leq \alpha < \epsilon \) whenever \( |\alpha - \overline{\alpha}| < \delta \).

So, we can assume \( \overline{\alpha} \in \partial \mathbb{R}_+^L \) is a non-zero boundary point. We can find a \( \delta_1 > 0 \) with \( \delta_1 < \epsilon \) such that for \( |\overline{\alpha} - \alpha| < \delta_1 = \delta_1(\overline{\alpha}) \), the non-degeneracy condition 1 implies that

\[ \sum_{j \in I_0} f_j(r, \alpha) \geq \lambda(\overline{\alpha})r^\sigma \text{ for } r \ll 1. \]

We can find \( \delta_2 > 0 \) such that \( |\overline{\alpha} - w(r, \alpha)| < \delta_1 \) for \( r < \delta_2 \) and \( |\alpha - \overline{\alpha}| < \delta_2 \) before the solution hits the wall. Let

\[ W_0(r, \alpha) := \sum_{j \in I_0} w_j(r, \alpha), \]

then the non-degeneracy condition 1 and (3.3) imply that

\[ -\frac{d}{dr} \left( r^{n-1} \frac{d}{dr} W_0(r, \alpha) \right) \geq \lambda(\overline{\alpha})r^{n-1+\sigma}. \]

Integrating this twice with respect to \( r \) yields

\[ W_0(r, \alpha) \leq \left( \sum_{j \in I_0} \alpha_j \right) - \frac{\lambda(\overline{\alpha})}{(2 + \sigma)(n + \sigma)} r^{2+\sigma}, \]

thus

\[ w_j(r, \alpha) \leq W_0(r, \alpha) \leq \left( \sum_{j \in I_0} \alpha_j \right) - \frac{\lambda(\overline{\alpha})}{(2 + \sigma)(n + \sigma)} r^{2+\sigma} \text{ for } j \in I_0. \]

We can then choose \( \delta > 0 \) sufficiently small so that \( |\alpha - \overline{\alpha}| < \delta \), then there is a smallest value \( r_\alpha < \delta_2 \) for which \( w_{j_0}(r_\alpha, \alpha) = 0 \) for some \( j_0 \in I_0 \). Hence,

\[ |\psi(\alpha) - \psi(\overline{\alpha})| = |\psi(\alpha) - \overline{\alpha}| \leq |w(r_\alpha, \alpha) - \overline{\alpha}| < \epsilon \text{ whenever } |\alpha - \overline{\alpha}| < \delta. \]
Case (2): Since the source terms \( f_i \) are non-negative, \( u'_i(r_0, \overline{\alpha}) < 0 \) by a direct computation or simply by Hopf’s Lemma. This transversality condition along with the ODE stability imply that for \( \alpha \) sufficiently close to \( \overline{\alpha} \), the solution to this perturbed IVP must hit the wall and \( \psi(\alpha) \) must be close to \( \psi(\overline{\alpha}) \).

Case (3): First, observe that elementary ODE or elliptic theory implies that \( F(\psi(\overline{\alpha})) = 0 \) for this case, which further implies that \( \psi(\overline{\alpha}) \in \partial \mathbb{R}_{L}^{+} \) from the non-degeneracy conditions. In fact, we claim that \( \psi(\overline{\alpha}) = 0 \). To see this, assume the contrary i.e. \( \psi(\overline{\alpha}) \) is a non-zero boundary point. We can assume from the non-degeneracy conditions that there is a \( j_0 \in I^{+}_{\psi(\overline{\alpha})} \) such that

\[
-d \left( r^{n-1} \frac{d}{dr} w_{j_0}(r, \overline{\alpha}) \right) = r^{n-1} f_{j_0}(r, w(r, \overline{\alpha})) \geq \lambda r^{n-1+\sigma} \quad \text{for} \quad r \gg 1,
\]

where \( \lambda \) and \( \sigma \) are suitable constants depending on \( \psi(\overline{\alpha}) \). From this, we have that

\[
w_{j_0}(r, \overline{\alpha}) \leq C - \frac{\lambda}{(n+\sigma)(2+\sigma)} r^{2+\sigma}
\]

for some constant \( C > 0 \). However, this implies that the solution must eventually hit the wall, which contradicts that \( w(r, \overline{\alpha}) \) is a positive entire solution of (3.3). Thus, \( \psi(\overline{\alpha}) = 0 \).

Since \( \psi(\overline{\alpha}) = 0 \), we can choose a sufficiently large \( R > 0 \) so that \( |u(R, \overline{\alpha})| < \epsilon/2 \). Then, by ODE stability, we can choose a \( \delta > 0 \) for which

\[
u(r, \alpha) > 0 \quad \text{and} \quad |u(r, \alpha) - u(r, \overline{\alpha})| < \epsilon/2 \quad \text{on} \quad [0, R] \quad \text{whenever} \quad |\overline{\alpha} - \alpha| < \delta.
\]

Hence,

\[
|\psi(\overline{\alpha}) - \psi(\alpha)| \leq |u(R, \alpha)| \leq |u(R, \overline{\alpha}) - u(R, \alpha)| + |u(R, \overline{\alpha})| < \epsilon.
\]

This completes the proof that \( \psi \) is continuous at \( \overline{\alpha} \in \mathbb{R}_{L}^{+} \).

4.3 Proof of Corollaries 1 and 2

It is straightforward to check that equation (2.10) and system (2.11) are non-degenerate type I for \( k > 1 \); however, condition 1-(ii) does not necessarily hold if \( k = 1 \). Nevertheless, the continuity of the target map still remains true for this case. More precisely, by adopting similar arguments used in the proof of Lemma 2 for Case (iii), we can still show that the entire positive solutions must decay to zero at infinity. Thus, we arrive at the same conclusion that \( \psi(\overline{\alpha}) = 0 \) and the continuity of the target map holds for this case. The continuity of \( \psi \) at \( \overline{\alpha} \) under Cases (i) and (ii) follows the same exact arguments used in the proof of Lemma 2. Subsequently, Corollary 1 is a direct consequence of Theorems 1 and 2 and Corollary 2 is a consequence of Theorems 1 and 3.

References

[1] A. Ambrosetti and A. Malchiodi. Nonlinear Analysis and Semilinear Elliptic Problems, volume 104. Cambridge University Press, 2007.
[2] H. Brezis and L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(4):437–477, 1983.

[3] J. Busca and R. Manasevich. A Liouville-type theorem for Lane–Emden systems. *Indiana Univ. Math. J.*, 51:37–51, 2002.

[4] L. Caffarelli, B. Gidas, and J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.*, 42:615–622, 1989.

[5] A. Chang and P. C. Yang. On uniqueness of solutions of n-th order differential equations in conformal geometry. *Math. Res Lett.*, 4:91–102, 1997.

[6] W. Chen and C. Li. Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.*, 63:615–622, 1991.

[7] W. Chen and C. Li. A note on the Kazdan–Warner type conditions. *J. Differential Geom.*, 41:259–268, 1995.

[8] W. Chen and C. Li. What kinds of singular surfaces can admit constant curvature? *Duke Math. J.*, 78:437–451, 1995.

[9] W. Chen and C. Li. *A priori* estimates for prescribing scalar curvature equations. *Ann. of Math.*, 145:547–564, 1997.

[10] W. Chen and C. Li. Prescribing scalar curvature on $S^n$. *Pacific J. Math.*, 199:61–78, 2001.

[11] W. Chen and C. Li. The best constant in a weighted Hardy–Littlewood–Sobolev inequality. *Proc. AMS*, 136:955–962, 2008.

[12] W. Chen and C. Li. Classification of positive solutions for nonlinear differential and integral systems with critical exponents. *Acta Math. Sci.*, 4(29B):949–960, 2009.

[13] W. Chen and C. Li. An integral system and the Lane–Emden conjecture. *Disc. & Cont. Dynamics Sys.*, 4(24):1167–1184, 2009.

[14] W. Chen, C. Li, and B. Ou. Qualitative properties of solutions for an integral equation. *Disc. & Cont. Dynamics Sys.*, 12:347–354, 2005.

[15] W. Chen, C. Li, and B. Ou. Classification of solutions for an integral equation. *Comm. Pure Appl. Math.*, 59:330–343, 2006.

[16] D. G. De Figueiredo and P. L. Felmer. A Liouville-type theorem for elliptic systems. *Ann. Sc. Norm. Sup. Pisa*, 21:387–397, 1994.

[17] B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 34(4):525–598, 1981.

[18] G. H. Hardy, J. E. Littlewood, and G. Pólya. Some properties of fractional integral (1). *Math. Zeitschr.*, 27:565–606, 1928.

[19] Y. Lei and C. Li. Sharp criteria of Liouville type for some nonlinear systems, preprint. 2013. http://arxiv.org/abs/1301.6235.

[20] Y. Lei, C. Li, and C. Ma. Decay estimation for positive solution of a $\gamma$-Laplace equation. *Disc. & Cont. Dynamics Systems*, 30:547–558, 2011.
[21] Y. Lei, C. Li, and C. Ma. Regularity of solutions for an integral system of Wolff type. 
Adv. in Math., 226:2676–2699, 2011.

[22] C. Li. A degree theory approach for the shooting method, preprint. 2013. 
http://arxiv.org/abs/1301.6232.

[23] C. Li and J. Villavert. An extension of the Hardy–Littlewood–Pólya inequality. Acta 
Math. Sci., 31(6):2285–2288, 2011.

[24] C. Li and J. Villavert. A degree theory framework for semilinear elliptic systems. 
preprint, 2013.

[25] E. Lieb. Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities. 
Ann. of Math., 118:349–374, 1983.

[26] E. Mitidieri. A Rellich type identity and applications. Comm. Partial Differential 
Equations, 18(1–2):125–151, 1993.

[27] E. Mitidieri. Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$. 
Differ. Integral Equations, 9:465–480, 1996.

[28] W.M. Ni and J. Serrin. Nonexistence theorems for singular solutions of quasilinear 
partial differential equations. Comm. Pure Appl. Math., 39(3):379–399, 1986.

[29] L. Nirenberg. Topics in Nonlinear Functional Analysis. Notes by R. A. Artino, volume 
6 of Courant Lecture Notes in Mathematics. The American Mathematical Society, 2001.

[30] P. Poláčik, P. Quittner, and P. Souplet. Singularity and decay estimates in superlinear 
problems via Liouville-type theorems, I: Elliptic equations and systems. Duke Math. 
J., 139(3):555–579, 2007.

[31] P. Pucci and J. Serrin. A general variational identity. Indiana Univ. J. Math., 35:681– 703, 1986.

[32] W. Reichel and H. Zou. Non-existence results for semilinear cooperative elliptic systems 
via moving spheres. J. Differential Equations, 161(1):219–243, 2000.

[33] J. Serrin and H. Zou. Non-existence of positive solutions of semilinear elliptic systems. 
Discourses in Mathematics and its Applications, 3:55–68, 1994.

[34] J. Serrin and H. Zou. Non-existence of positive solutions of Lane–Emden systems. 
Differ. Integral Equations, 9:635–654, 1996.

[35] J. Serrin and H. Zou. Cauchy–Liouville and universal boundedness theorems for quasi-
linear elliptic equations and inequalities. Acta Math., 189(1):79–142, 2002.

[36] S. L. Sobolev. On a theorem of functional analysis. Transl. Amer. Math. Soc. (2), 
34:39–68, 1963. Translated from Math. Sb. (N.S.), 4(46)(1938), pp. 471–497.

[37] P. Souplet. The proof of the Lane–Emden conjecture in four space dimensions. Adv. 
in Math., 221(5):1409–1427, 2009.

[38] M. A. S. Souto. A priori estimates and existence of positive solutions of nonlinear 
cooperative elliptic systems. Differ. Integral Equations, 8:1245–1245, 1995.

[39] E. B. Stein and G. Weiss. Fractional integrals in $n$-dimensional Euclidean space. J. 
Math. Mech., 7:681–703, 1958.