A Full-Matrix Approach for Solving General Plasma Dispersion Relation

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A hitherto difficult and unsolved issue in plasma physics is how to give a general numerical solver for complicated plasma dispersion relation, although we have long known the general analytical forms. We transform the task to a full-matrix eigenvalue problem, which allows to numerically calculate all the dispersion relation solutions exactly free from convergence problem and give polarizations naturally for arbitrarily complicated multi-scale fluid plasma with arbitrary number of components. Attempt to kinetic plasma via N-point Padé approximation of plasma dispersion function also shows good results.

Since only few simple dispersion relations are analytically tractable in plasma physics, it is a historical issue to develop general numerical solvers for practical applications, especially that it is difficult to give a kinetic solver with general distribution function and general other effects. The first problem comes from the Landau contour integral (e.g., Landau damping) of the kinetic distribution function. Second problem is the infinity orders of Bessel function summation for magnetized plasma which is related to the cyclotron resonance (e.g., Bernstein modes). These two obstacles make it difficult to develop a root finding solver with good convergence.

WHAMP (Waves in Homogeneous Anisotropic Multi-component Magnetized Plasma) code by Ronmark[1-2] is an important step to that goal, which can be used to calculate general non-relativistic kinetic wave dispersion relation in plasmas with parallel beams. To calculate fast, Padé approximation is used for plasma dispersion function. However, the numerical convergence is still not as expectations, especially that it is difficult to give proper initial guesses for high frequency (e.g., \( \omega \geq 10 \Omega_{ci} \)) modes.

Bret[3] discussed how to derive and solve the 3-by-3 parallel beam-plasma dielectric tensor in fluid approximation with relativistic effect with the help of computer (Mathematica). However, it is still not easy to use this method if there are many components.

In our treatment, we do not need to derive the final 3-by-3 dispersion relation tensor matrix for \( \mathbf{E} = (E_x, E_y, E_z) \) as the conventional treatment, such as by Stix[4] and used by Ronmark[1-2] and Bret et al.[3,5], which is helpful to provide analytical insight but not a must for numerical solver.

In this manuscript (brief communication), as the first topic, we present a trivial multi-fluid non-relativistic magnetized arbitrary orbit warm beam plasma problem, which is very difficult via usual treatments, to show how our full-matrix approach works.

The original equations are

\[
\begin{align*}
\partial_t n_s & = -\nabla \cdot (n_s \mathbf{v}_s), \\
\partial_t \mathbf{v}_s & = -\mathbf{v}_s \cdot \nabla \mathbf{v}_s + \frac{\omega_p^2}{m_s} (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) - \frac{\Sigma v_s}{\rho_s}, \\
\partial_t \mathbf{E} & = c^2 \nabla \times \mathbf{B} - J/\epsilon_0, \\
\partial_t \mathbf{B} & = -\nabla \times \mathbf{E},
\end{align*}
\]

with

\[
\begin{align*}
\mathbf{J} & = \sum s q_s n_s \mathbf{v}_s, \\
d_t (p_s \rho_s^{-\gamma_s}) & = 0,
\end{align*}
\]

where \( \rho_s \equiv m_s n_s, p_s \equiv k_B T_s / m_s \) and \( c^2 = 1/\mu_0 \epsilon_0 \).

In cold plasma limit, for parallel beam, the usual 3-by-3 dispersion relation tensor \( \mathbf{D}(\mathbf{k}, \omega) \) can be derived as

\[
\begin{pmatrix}
K_{xx} - n^2 \cos^2 \theta & K_{yx} & K_{xz} - n^2 \sin \theta \cos \theta \\
K_{yx} & K_{yy} & K_{yz} \\
K_{xz} - n^2 \sin \theta \cos \theta & K_{yz} & K_{zz} - n^2 \sin^2 \theta
\end{pmatrix},
\]

with

\[
\begin{align*}
K_{xx} & = K_{yy} = 1 - \sum s \left( \frac{\omega_p^2}{\omega^2_s} - \frac{\omega_p^2}{\omega^2_{cs}} \right) \left( \frac{\omega^2_s}{\omega} \right)^2, \\
K_{zz} & = 1 - \sum s \omega_p^2 \left[ \frac{1}{\omega^2_s} - \frac{1}{\omega^2_s - \omega^2_{cs}} \right] \left( \frac{k v_{ds}}{\omega} \right)^2 \sin^2 \theta, \\
K_{yx} & = -K_{xy} = i \sum s \frac{\omega_{cs} \omega_p^2}{\omega^2_s - \omega^2_{cs}} \frac{\omega'}{\omega}, \\
K_{xx} & = K_{xz} = -i \sum s \frac{\omega_p^2}{\omega^2_s - \omega^2_{cs}} \frac{\omega'}{\omega} k v_{ds} \sin \theta, \\
K_{zy} & = -K_{yz} = -i \sum s \frac{\omega_p^2}{\omega^2_s - \omega^2_{cs}} \frac{\omega'}{\omega} v_{ds} \sin \theta,
\end{align*}
\]

and \( n \equiv ck/\omega, \omega' \equiv \omega - k \cdot \mathbf{v}_{s0}, \mathbf{B}_0 = (0, 0, B_0), \mathbf{v}_{s0} = (0, 0, v_{s0}), \mathbf{k} = (k_x, 0, k_z) = (k \sin \theta, 0, k \cos \theta), \omega_{cs} = q_s B_0 / m_s \) (note: \( q_c = -e \)) and \( \omega^2_p = n_{s0} q_e^2 / \epsilon_0 m_s \).

The dispersion relations is

\[
D(\mathbf{k}, \omega) \equiv \det [\mathbf{D}(\mathbf{k}, \omega)] = 0.
\]

The summation \( \sum_s \) in \( K_{ij} \) (\( i, j = x, y, z \)) with \( \omega \) in the denominator, which is singularity at \( \omega = 0 \) and \( \omega^2 = \)

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\[ \omega^2_m, \text{ makes a general good convergence numerical solver very difficult. And, we need also give good initial guesses when using special root finding solver such as Newton’s iterative method or give a guess domain in complex plane using such as Davies’ method}. \]

We can get an explicit polynomial form dispersion relation equation from (5) for \( k(\omega, \theta) \) easily. While, it is very cumbersome to calculate an explicit form for \( \omega(k, \theta) \), even though with the aids of computer (e.g., using Mathematica). Without beam (\( \nu_{ds} = 0 \)) and for only one ion species (\( s = e, i \)), a fifth order explicit form polynomial for \( \omega^2(k, \theta) \) is given in Swanson’s textbook[9], which is simplified from hundreds of terms.

The above investigations imply that it is not a satisfactory choice to solve the dispersion relation using (5) directly as usual treatment.

An alterant method is using the original full dispersion relation matrix and then treating the task as a matrix eigenvalue problem, which need neither derive the final dispersion relation equation nor worry about how to solve it.

The linearized version of (11) with \( f = f_0 + f_1 e^{ik \cdot r - i \omega t} \) is equivalent to a matrix eigenvalue problem (similar treatment can be found in [8] for MHD equations)

\[ \lambda X = M \cdot X, \quad (6) \]

with \( \lambda = -i \omega \) the eigenvalue and corresponding \( X \) eigen vector, which represents the polarization information of each normal/eigen mode solution.

For (11), \( X \) is

\[ \{n_{s1}, v_{s1x}, v_{s1y}, v_{s1z}\}, E_{1x}, E_{1y}, E_{1z}, B_{1x}, B_{1y}, B_{1z}\}^T, \]

and \( M \) is

\[ \begin{bmatrix}
-k \cdot v_{s0} & -ik_x n_{s0} & 0 & -ik_z n_{s0} \\
-ik_x c_s^2/\rho s & -ik \cdot v_{s0} & \omega_c & 0 \\
0 & -\omega_c & -ik \cdot v_{s0} & 0 \\
-ik_z c_s^2/\rho s & 0 & 0 & -ik \cdot v_{s0}
\end{bmatrix}
\]

\[ \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
q_{x0} & 0 & 0 & q_{x0} & 0 & 0 \\
0 & q_{z0} & 0 & 0 & q_{z0} & 0 \\
-q_{x0} q_{y0} & q_{x0} & 0 & 0 & 0 & 0 \\
0 & 0 & -q_{z0} & 0 & q_{z0} & 0 \\
0 & 0 & 0 & 0 & q_{z0} c^2 & 0
\end{bmatrix}, \quad (7)
\]

\[ \text{TABLE I: Comparing the cold plasma solutions using matrix method and Swanson’s polynomial. Only positive solutions are shown, with } \omega^2 = \omega^2_m. \]

| \( \omega^M \) | \( \omega^S \) | \( \omega^M \) | \( \omega^S \) |
|----------------|----------------|----------------|----------------|
| 10.5152        | 10.5152        | 1.1330E-4      | 1.1330E-4+11E-16 |
| 10.0031        | 10.0031        | -              | 1E-32+1E-18     |
| 9.5158         | 9.5158         | 0              | 2.4020E-4-2.4020E-4+i3E-17 |

where we have used \( p_{s0} = \gamma_s p_{00} \rho_{s0}/\rho_{s0} \), which is from [25], and \( c_s^2 \equiv \gamma_s p_{s0}/\rho_{s0} \). The effects from non-zero equilibrium quantities (\( J_b \) and \( v_{s0} \times B_0 \)) are omitted here (Note: This is another annoying unsolved problem in literatures. Principally, we need treat it as inhomogeneous problem.).

For \( s \) kinds of species, the dimensions of \( M \) are \((4s + 6) \times (4s + 6)\). Here, we can get all the solutions of the above system exactly (without convergence problem) via standard matrix eigenvalue solver, e.g. function \emph{eig()} in MATLAB.

\[ \text{Cold limit (} p_{s0} = 0, \text{ without beam (} v_{s0} = 0), s = e, i, \text{ numerical solutions of } (6) (\omega^M) \text{ and Swanson’s polynomial } (\omega^S) \text{ are given in Table I with } kc = 0.1, \theta = \pi/3, m_i/m_e = 1836 \text{ and } \omega_{pe} = 10 \omega_{ci}. \text{ The results are consistent with each other exactly, except some small (<10^{-15}) numerical errors.} \]

A 3-component result with perpendicular electron beams is shown in Fig[1].

Using this method for other arbitrarily complicated fluid system is just straightforward, due to that it is just a directly rewriting of original linearized equations without any approximations.

As a second topic, we try this matrix method for kinetic plasma.

A possible general method is discretizing[10] [12] the distribution function in \( v \)-space or using basis function expansion[11] to transform the equations to matrix form. However, it is shown that for both Vlasov-Possion[10] and Vlasov-Ampere[12] systems, the (Landau) damping (\( 3(\omega) < 0 \)) normal modes are not eigenmodes but just related to spectral density accumulating of eigenmodes, though those methods can give some correct solutions for growth modes. Another problem is that the matrix
dimension should be very large to calculate accurately since discrete of $v$ is introduced.

Another attempt is the initial value version\cite{12,13} of this matrix method, which can give correct kinetic Landau damping, but gives only few largest imaginary part modes and can not overcome multi-scale problem.

Here, as our first successful attempt to kinetic plasma, we limit to simple multi-component 1D electrostatic (ES1D) problem with drift Maxwellian distribution, with dispersion relation

$$D = 1 - \sum_s \frac{1}{(k\lambda_{Ds})^2} \frac{Z'(\zeta_s)}{2} = 0,$$

(8)

where $\lambda_{Ds}^2 = \frac{t_a k_o T_e}{n_e q_e^2}$, $\nu_t = \sqrt{\frac{2k_o T_e}{m_e}}$ and $\zeta_s = \frac{\omega - kv_s}{kv_s}$, and use $N$-point Padé approximation of plasma dispersion function as by Ronnmark \cite{11,12}

$$Z'(s) = \sum_j \frac{b_j}{(s - c_j)^2},$$

(9)

where $N = 8$ is used, which shows to be very accurate for most domain in upper plane (except $y < \sqrt{x^2 - x^2}$, $x \gg 1$, with $s = x + iy$). Bad performances are just for strong damping domain, for which we have little interest.

Combining (8) and (9), gives a very similar dispersion relation equation

$$1 - \sum_s \sum_j \frac{b_{sj}}{(\omega - c_{sj})^2} = 0,$$

(10)

as the one of the fluid ES1D beam plasma, with $b_{sj} = \frac{b_1 v_j v_e^2}{2\lambda_{Ds}^2}$ and $c_{sj} = k(v_{s0} + v_{ts}c_j)$, which can help us transform the problem to an equivalent linear system

$$\omega A_{sj} = b_{sj} B_{sj} + c_{sj} A_{sj},$$

(11a)

$$\omega B_{sj} = c_{sj} B_{sj} + C,$$

(11b)

$$C = \sum_{sj} A_{sj},$$

(11c)

which is an eigenvalue prolem of a $2sN \times 2sN$ dimensions eigen matrix $M$, with $sN = s \times N$. The sigularity of dominator, which meet in conventional treatment, is canceled after this transformation. And the matrix method can support multi-component very easily and naturally.

For Langmuir wave Landau damping, calculating the largest imaginary part solution using matrix method ($\omega^M$) and original $Z(\zeta)$ function ($\omega^Z$) are shown in Table II

| $k\lambda_{De}$ | $\omega^M$ | $\omega^Z$ | $\omega^M$ | $\omega^Z$ |
|----------------|-----------|-----------|-----------|-----------|
| 0.1            | 1.0152    | 0.0000    | 1.0152    | -4.8E-15  |
| 0.5            | 1.4157    | -0.1533   | 1.4157    | -0.1534   |
| 1.0            | 2.0458    | -0.8513   | 2.0458    | -0.8513   |
| 2.0            | 3.1897    | -2.8278   | 3.1891    | -2.8272   |

For two-frequency-scale ion acoustic mode, besides the Langmuir mode $\omega = 2.0458 - 0.8513i$, the largest imaginary part solution in matrix method is also consistent with $Z(\zeta)$ function solution, e.g., $T_i = T_e$, $m_i = 1836m_e$, $k\lambda_{De} = 1$, gives $\omega = 0.0420 - 0.0269i$.

First four largest imaginary part solutions of electron bump-on-tail mode, with ion effects, are shown in Fig. 3 for $T_h = T_e = T_i$, $v_b = 5v_{te}$ and $n_b = 0.1$. For this test run, matrix method calculates very fast and automatically, and can give all the solutions we want. While, using $Z(\zeta)$ function, we need test different initial guess one by one to select different modes (not shown here).

Detail features and benchmarks of highly nontriv-
FIG. 3: ES1D electron bump-on-tail modes with ion effect.

In summary, a very efficacious method is presented to solve both fluid and (simple) kinetic plasma dispersion relation, which have overcome many (almost all) troubles in conventional treatments.

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Codes for solving (6) and (11) are provided as supplementary materials for one who hopes to quickly access to this matrix approach.

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