In Hermitian quantum mechanics the differential geometric notion of Chern number arises in the context of adiabatic transport of eigenstates in Hilbert space [1]. The change in the overall phase of a state after it is transported around a closed loop contains a purely geometric contribution associated with curvature of the Hilbert space. Quantized integer Chern numbers are associated only with an extended state, and serve as a nontrivial diagnostic of localization. This procedure has been used to study localization in the integer quantum Hall effect [2]. Physically, the Chern number reflects the ability of the state to transport charge across the system, and is proportional to its Hall conductance [1].

Recently there has been much interest in localization properties of certain non-Hermitian operators, for example, operators governing effective two-dimensional quantum descriptions of pinning of tilted flux lines by extended defects in three-dimensional type-II superconductors [3], and passive scalar transport in random vorticity fields [4]. Both these examples have close ties to the problem of electron localization in random magnetic fields [1]. In this paper we will consider the generalization of Chern numbers to non-Hermitian operators. We review the non-Hermitian Berry phase [3] and how it gives rise to Chern numbers through eigenvalue degeneracies. Application is then made to a class of operators that interpolate, as a function of a parameter $0 \leq \theta < \pi/2$, between a special case of the Hermitian random flux model and the non-Hermitian flux line and passive scalar problems.

Consider a family $L(X)$ of (possibly non-Hermitian) operators, where $X = (X_1, X_2, \ldots)$ is a set of fixed real parameters spanning a manifold. Let $|m; X\rangle$ and $\langle m; X|$ be the right and left eigenvectors of $L(X)$ satisfying

$$L(X)|m; X\rangle = E_m(X)|m; X\rangle \quad \text{and} \quad \langle m; X|L(X) = E_m(X)\langle m; X|.$$  

The eigenstates are determined up to an arbitrary complex amplitude, and we make a fixed arbitrary choice $\langle m; X| = \langle m|$. For any given operator $L(t)$, we define the dynamics of states $|\phi(t)\rangle$ and $\langle \phi(t)|$ via

$$i\partial_t|\phi(t)\rangle = L(t)|\phi(t)\rangle, \quad -i\partial_t\langle \phi(t)| = \langle \phi(t)|L(t).$$

This dynamics preserves all inner products $\langle \phi_1(t)|\phi_2(t)\rangle$. Let $X(\alpha), 0 \leq \alpha \leq 1$, $X(0) = X(1)$, parameterize a closed path, $C$, in parameter space. We define parallel transported eigenstates $|m; \alpha\rangle = b_m(\alpha)|m; X(\alpha)\rangle$ along the path $X(\alpha)$ by the rule

$$\partial_\alpha b_m(\alpha) = -\langle m; X(\alpha)|\partial_\alpha|m; X(\alpha)\rangle b_m(\alpha),$$  

with $b_m(0) = 1$. The solution is

$$b_m(\alpha) = \exp \left[i \int_{C: X(0)} A_m(X) \cdot dX \right].$$

$$A_m(X) \equiv \langle m; X(\alpha)|i\partial_\alpha|m; X(\alpha)\rangle.$$  

Direct substitution shows that this parallel transport may be generated dynamically using the operator $L(t) = i(\partial_\alpha P_m(X(t)), P_m(X(t))|$ (with $\alpha = t$) [5]. For a closed loop one obtains

$$b_m(1) = e^{i\gamma_m(C)}, \quad \gamma_m(C) = \int_C dX \cdot A_m(X).$$
In the Hermitian case $A_m$, and hence $\gamma(C)$, is real. In
the non-Hermitian case they are both generally complex.
It is easy to see that $\gamma(C)$ is independent of the arbitrar-
iness of normalization of the states $|m;X\rangle$. Any change
of normalization, $e^{\lambda(X)}$, appears as a gauge transforma-
tion $A' = A - \partial_X \lambda$, under which the integral in (3) is
invariant. Provided no singularities are encountered we may
write $\gamma(C)$ as the integral of the gauge invariant flux
using Stokes theorem,
\[
F_m(X) = \nabla \times A_m(X) \nonumber \\
= i[\partial_X |m_0;X\rangle \times |\partial_X |m_0;X\rangle],
\]
over any surface $S$ bounded by $C$ [3]:
\[
\gamma_m(C) = \int_S d\Sigma \cdot F_m(X).
\]
By construction one has $\nabla \cdot F_m = 0$, as is required for
$\gamma(C)$ to be independent of the choice of $S$.
By inserting a complete set of states, the flux may be writ-
ten in the form
\[
F_{m_0}(X) = i \sum_n \left< m;X | \partial_X L | m;X \right> \times \left| m;X | \partial_X L | m;X \right> \nonumber \\
\left( E_n - E_m \right)^2
\]
where the prime indicates that the term $n = m$ is omit-
ted. Equation (8) demonstrates explicitly that so long as
$L(X)$ is smooth, singularities in $F_m(X)$ can only occur when
two eigenvalues cross: $E_n - E_m \to 0$ for some $n \neq m$.
In the Hermitian case eigenvalue crossings, and there-
fore singularities in $F_m(X)$, generically occur at isolated
points in the three dimensional space of $X$. One may
choose the surface $S$ in [3] to avoid such points by pass-
ing either over or under them. On the one hand, the dif-
ference of the integrals over two such surfaces $S_1$ and
$S_2$ is
\[
\gamma_m(S) = \gamma_m(S_1) - \gamma_m(S_2) = \int_S d\Sigma \cdot F_m,
\]
where $S = S_1 \cup S_2$ is a closed surface enclosing the singu-
larities. On the other hand, $e^{i\gamma_m(S_1)} = e^{i\gamma_m(S_2)}$ since both
surfaces are bounded by the same curve $C$. We conclude
that for any closed surface $S$, $\gamma_m(S) = 2\pi q$ for some in-
teger Chern number $q$ [3]. Eigenvalue degeneracies act as a
quantized point sources of flux for $F_m$.
This argument does not rely directly on Hermiticity, but
only on the result that energy level degeneracies occur
generically at points in three dimensional space, so that
these singularities can be enclosed by singularity-
free closed surfaces. We now address the issue of how to
generalize this result to the non-Hermitian case. In order
to study a degeneracy in more detail, we project $L$ onto
the (complex) two-dimensional subspace spanned by the
eigenvectors $|m;X\rangle$ and $|n;X\rangle$. If $P_{mn} = |m\rangle \langle m| + |n\rangle \langle n|$ is
the corresponding orthogonal projection, we consider the matrix
\[
L_{mn}(X) = P_{mn}(X)L(X)P_{mn}(X) \\
= \alpha_0(X)I + \vec{\alpha}(X) \cdot \vec{\sigma}
\]
where $\vec{\sigma}$ are the Pauli matrices, and $\alpha_0$ and $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are four (at this stage arbitrary) complex numbers. In the Hermitian case these numbers are all real. The eigenvalues of $L_{mn}$ are $E_{\pm} = \alpha_0 \pm iE$, where $\Delta E = \sqrt{\Delta^2 E}$ the left and right eigenvectors are easily computed as well, and the the fluxes are
\[
F_{\pm}(X) = \frac{\alpha_1 a_2 \times a_3 + \alpha_2 a_3 \times a_1 + \alpha_3 a_1 \times a_2}{2\Delta E^3}.
\]
where $a_i(X) = \partial_X \alpha_i(X)$.
Equation (11) can be used to explore the nature of the
singularities in $F_{\pm}$ in the neighborhood of degeneracies,
$\Delta E \to 0$. The condition for $\Delta E = 0$ is simply $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0$. In the Hermitian case where the $\alpha_i$ are all real, this condition requires that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, but in general this equality places only two conditions on the six parameters (the real and imaginary parts of each $\alpha_i$, $i = 1, 2, 3$). In the parameter space $X$ degeneracies will then generically occur on a submanifold of codimension
2, i.e., on lines in the three dimensional space.
Since the matrix $L_{mn}$ is no longer Hermitian, there is
no guarantee that there will be two independent eigenvectors
associated with the two degenerate eigenvalues. Generically
$L_{mn}$ will be upper-triangular (with $\alpha_1 + i\alpha_2 = 0$ at the same time that $\alpha_3 = 0$, but $\alpha_1 - i\alpha_2 \neq 0$), and only one eigenvector will exist, with
the orthogonal direction spanned by a generalized eigen-
vector [10]. We shall call this situation a generalized de-
generacy. What we call a true degeneracy exists only when
both $\alpha_1 \pm i\alpha_2 = 0$, so that $L_{mn} = E I$ is propor-
tional to the identity matrix (this differs from the Her-
mitian case only in that $\alpha_0 = E$ may be complex), and
places six conditions on the occurrence of a true degener-
acy, which will generically not occur for a three dimen-
sional parameter space $X$, unless further restrictions are
placed on the class of matrices (such as self-adjointness).
In what follows we do not impose any such further restric-
tions.
If $X_q$ be a point of generalized degeneracy, set $x = X - X_q$. From (11) we obtain for small $x$:
\[
F_{\pm} = \mp \frac{v}{2^{5/2}|w \cdot x|^3/2},
\]
with complex vectors
\[
v = \alpha_1 a_2 \times a_3 + \alpha_2 a_3 \times a_1 + \alpha_3 a_1 \times a_2 \\
w = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3,
\]
with $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0$. The easily proven result $v \cdot w = 0$ guarantees $V \cdot F_{\pm} = 0$.
The condition $w \cdot x = 0$ is satisfied locally if $x$ is or-
thogonal to both $w R \equiv R w$ and $w L \equiv L w$, show-
ing that the line of generalized degeneracies is locally tan-
gent to $u = w R \times w J$. Note that unless it is real, $v$ will
not be parallel to \( u \). This line can pierce the integration surface \( S \) at singular points where \( F_\pm \) will have the asymptotic behavior \([12]\). Since the surface integral is two dimensional, the singularity will be integrable.

Suppose now that \( x \) circles a singular line in the plane formed by \( w_R \) and \( w_I \). For example, let \( w \cdot x = \eta e^{i\theta} \) with \( \eta \) fixed and \( -\pi < \theta \leq \pi \) determined by \( w_R \cdot x = \eta \cos(\theta) \) and \( w_I \cdot x = \eta \sin(\theta) \). Then \( F_\pm \propto e^{-i\theta/2} \), so that single valuedness can be maintained only if a surface \( \text{of branch cuts} \) extends out of the singular line. It is straightforward to show that this branch sheet joins \( F_+ \) and \( F_- \): tracing the closed path causes \( E_m \) and \( E_n \) to exchange positions by circling each other in the complex plane. \( F_\pm (X) \) forms a single complex function on a two-sheeted Riemann surface (see Fig. 2 below).

In the Hermitian case one may deform the integration surface to small spheres enclosing the point singularities. In the generic case the deformed surface wraps around those portions of the lines and their branch surfaces that are internal to the original surface.

In order to understand the result of the integration \([12]\) we consider a model problem with the exact form
\[
\alpha = (\alpha_1 + a_1 \cdot x, \alpha_2 + a_2 \cdot x, \alpha_3 + a_3 \cdot x),
\]
so that a true degeneracy exists at \( x = 0 \) when \( \alpha^0 = (\alpha_1, \alpha_2, \alpha_3) = 0 \). Near \( 0 \) the flux is
\[
F_\pm (x) = \mp \frac{D_0 x}{2(Sx \cdot Sx)^{3/2}}
\]
(14)
where \( S \) is the matrix whose rows are the \( a_i \), and \( D_0 = \det S = a_1 \cdot a_2 \times a_3 \).

We suppose that the \( a_i \) are all real, so that \( \alpha^0 \) represents a (generally non-Hermitian) perturbation of an Hermitian problem. On transformation to real coordinates \( y = Sx \), the flux takes the classic Coulomb form
\[
F_\pm (y) = \mp \sigma(D_0) y / 2 |y|^3,
\]
where \( \sigma(D_0) \) is the sign of \( D_0 \). It follows immediately that the integrated flux through any surface \( S \) enclosing the origin is \( \gamma(S) = \mp 2\pi \sigma(D_0) \).

On including the constant shift \( \alpha^0 \), the flux becomes
\[
F_\pm (y) = \mp \sigma(D_0) \frac{\alpha^0 + y}{2(\alpha_0 + y) \cdot (\alpha_0 + y)^{3/2}}.
\]
(15)

For very large \( |y| \) this form reduces to \([12]\). The flux through a surface at infinity remains \( \gamma(S) \). Since \( \nabla_y \cdot F_\pm (y) = 0 \), this result holds for any surface that encloses all singularities in \( F_\pm \). These singularities occur at the zeros of \( (\alpha^0 + y)^2 \), i.e., for \( |y + \text{Re} \alpha^0|^2 = |\text{Im} \alpha^0|^2 \) and \( \text{Im} \alpha^0 \cdot (y + \text{Re} \alpha^0) = 0 \), which constitute a circular ring of radius \( \alpha = |\text{Im} \alpha^0| \) formed by the intersection of the sphere of radius \( \alpha \) centered at \( -\text{Re} \alpha^0 \) with the plane normal to \( \text{Im} \alpha^0 \) passing through this center. In \( x \)-space the ring becomes an ellipse. A ring must be spanned by a branch surface in order to maintain single-valuedness of \( F_\pm \).

We have therefore established that under non-Hermitian perturbations the usual quantized point charges expand into closed loops, with net flux quantized by the original Hermitian Chern number. If a loop grows to pierce the surface of integration, only a fraction of the total flux will be enclosed, and integer quantization is lost. This fraction depends on the geometry of the branch cut, which is itself arbitrary, except for its end points. Only the total flux, integrated over all sheets of the Riemann surface, remains quantized \([11]\). The latter will correspond to the net Chern number of the colliding eigenvalues, which are now topologically “entangled.” The entanglement becomes more pronounced as the non-Hermiticity increases, and may involve more than two eigenvalues that collide with each other sequentially as \( X \) is varied, generating further Riemann sheets.

We illustrate this formal discussion by a specific example, a family of operators in two dimensions:
\[
L = -e^{i\theta} D \nabla^2 + iA(r) \cdot \nabla,
\]
in which \( 0 \leq \theta < 2\pi \) controls the degree of non-Hermiticity \( L \) is Hermitian only for \( \theta = 0, \pi \). \( D \) is a constant coefficient, and \( A \) is an incompressible vector field: \( \nabla \cdot A = 0 \). For \( \theta = 0 \), this operator corresponds to the random flux problem in the Coulomb gauge \([3]\), with magnetic field \( B = \nabla \times A \), and a particular choice \( V = -A^2 / 4D \) for the scalar potential. For \( \theta = \pi / 2 \) this operator represents a model of flux lines in a three-dimensional superconductor with extended defects, (where \( A \) corresponds to the horizontal components of the magnetic field \([3]\), or a two-dimensional passive-scalar transport model, (where \( A \) is the velocity field, and \( D \) is the diffusion constant). We set \( D = 1 \) and implement \([10]\) numerically by discretizing it \([4]\) on a small \((4 \times 4)\) lattice. The components of the vector field \( A \) are initially chosen to be independent random variables chosen uniformly on an interval \([-W, W] \), with \( W = 40 \). Incompressibility is imposed by subtracting from \( A \) the function \( \nabla a \), where \( a(r) \) satisfies \( \nabla^2 a = \nabla \cdot A \) with periodic boundary conditions.

Study of random operators like \([14]\) is motivated by questions of localization \([2, 3]\). Extended states are distinguished from localized states by their sensitivity to boundary conditions. Let \( E_n(\phi_x, \phi_y) \), \( \psi_n^{\phi_x, \phi_y}(x, y) \) be the eigenvalues and eigenstates of \( L \) on an \( L_x \times L_y \) system with boundary conditions \( \psi_n^{\phi_x, \phi_y}(x + L_x, y) = e^{i\phi_x} \psi_n^{\phi_x, \phi_y}(x, y) \), \( \psi_n^{\phi_x, \phi_y}(x, y + L_y) = e^{i\phi_y} \psi_n^{\phi_x, \phi_y}(x, y) \). The two-dimensional space \( \{\phi_x, \phi_y\} \) has the geometry of a torus that is embedded in a three-dimensional parameter space \( X \). The Berry flux through the torus is the integral of \([3]\): \( F_n(\phi_x, \phi_y) = \langle \partial_{\phi_x} \psi_n | \partial_{\phi_y} \psi_n \rangle - \langle \partial_{\phi_y} \psi_n | \partial_{\phi_x} \psi_n \rangle \), and if the integral of the flux over the surface (the Chern number) is non-zero, then the eigenstate is extended.

Fig. 6 displays the variation of eigenvalues with \( \theta \) for \( \phi_x = \phi_y = 0 \). In Fig. 2 we plot the imaginary parts of \( F_x(\phi_x, \phi_y) \) for the labeled eigenvalues in Fig. 2 and for two values of \( \theta \); revealing the emergence of a branch cut and showing \( F_x \) are different sheets of the same analytic function. The log–log plot exhibits the 3/2-law \([12]\) as an endpoint of a branch cut is approached.
The action corresponding to (16) at $\theta = \pi/2$ is the Wick rotation of the action at $\theta = 0$. A natural question is whether this analytic continuation to imaginary time is accompanied by a smooth variation in properties of eigenstates, or whether a phase transition, e.g., metal to insulator, occurs as a function of $\theta$ (see ref. [4]). For finite $\theta$, we have shown that the geometry of eigenvalue degeneracies can change dramatically, but the notion of Chern number is still relevant. Whether the entanglement of eigenstates becomes vanishingly small. In the non-Hermitean case this fails in general because $E_m$ is no longer real, and an initially infinitesimal amount of tunneling may grow exponentially instead of simply oscillating. For our purposes the connection with the adiabatic limit is not necessary. It matters only that parallel transport defined by (1) along an arbitrary curve $C$ be implemented dynamically by some choice of $L(t)$.

[8] We assume henceforth that $X$ is a 3-dimensional vector.

[9] Since $\gamma_m(S_1)$ and $\gamma_m(S_2)$ differ, $A_m$ must have a line singularity that emerges from the singularity in $F_m$ and passes through $S_1$ or $S_2$. This is not a true singularity, but a coordinate singularity, as occurs in polar coordinates at the poles of a sphere, and its position varies with gauge choice. These singularities do not appear in $F_m$ since it is gauge invariant.

[10] Let $\{-\}$ be the eigenvector obeying $L_m|\{-\} = E|\{-\}$. At a generic degeneracy one has $|\{+\} = |\{-\}$, and there exists an independent generalized eigenvector (also called a cyclic vector) $|g\}$ that obeys $L_m|g\} = |g\}$ and may be chosen so that $g|\{\pm\} = 0$. For details see an advanced text on linear algebra.

[11] To prove this, observe that the Berry flux through a branch surface is identical for the two eigenvalues involved, but the surface normals are opposite. Branch surfaces make no contribution and the total integral is conserved and equal to the sum of the Chern numbers before the Chern loop pierced the integration surfaces.