Approximation numbers of composition operators on the Hardy space of the infinite polydisk

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Abstract. We study the composition operators of the Hardy space on \( D^\infty \cap \ell_1 \), the \( \ell_1 \) part of the infinite polydisk, and the behavior of their approximation numbers.

1 Introduction

Recently, in [2], we investigated approximation numbers \( a_n(C_\varphi), n \geq 1 \), of composition operators \( C_\varphi \), \( C_\varphi(f) = f \circ \varphi \), on the Hardy or Bergman spaces \( H^2(\Omega), B^2(\Omega) \) over a bounded symmetric domain \( \Omega \subseteq \mathbb{C}^d \). Assuming that \( \varphi(\Omega) \) has non-empty interior, one of the main results of this study was the following theorem.

Theorem 1.1 ([2]). Let \( C_\varphi : H^2(\Omega) \to H^2(\Omega) \) be compact. Then:

1) we always have \( a_n(C_\varphi) \geq ce^{-C n^{1/d}} \) where \( c, C \) are positive constants;
2) if \( \Omega \) is a product of balls and if \( \varphi(\Omega) \subseteq r \Omega \) for some \( r < 1 \), then:

\[
a_n(C_\varphi) \leq Ce^{-c n^{1/d}}.
\]

As a result, the minimal decay of approximation numbers is slower and slower as the dimension \( d \) increases, which might lead one to think that, in infinite-dimension, no compact composition operators can exist, since their approximation numbers will not tend to 0. After all, this is the case for the Hardy space of a half-plane, which supports no compact composition operator ([12], Theorem 3.1; in [9], it is moreover proved that \( \|C_\varphi\|_e = \|C_\varphi\| \) as soon as \( C_\varphi \) is bounded; see also [15] for a necessary and sufficient condition for \( H^2(\Omega) \) has compact composition operators, where \( \Omega \) is a domain of \( \mathbb{C} \)). We will see that this is not quite the case here, even though the decay will be severely limited. In particular, we will never have a decay of the form \( Ce^{-c n^{\delta}} \) for some \( c, C, \delta > 0 \).
2 Framework and reminders

2.1 Hardy spaces on $\mathbb{D}^\infty$

Let $\mathbb{T} = \partial \mathbb{D}$ be the unit circle of the set of complex numbers. We consider $\mathbb{T}^\infty$ and equip it with its Haar measure $m$. This is a compact Abelian group with dual $\mathbb{Z}^{(\infty)}$, the set of eventually zero sequences $\alpha = (\alpha_j)_{j \geq 1}$ of integers. We denote $L^2_{\mathbb{N}^{(\infty)}}(\mathbb{T}^\infty)$ the Hilbert subspace of $L^2(\mathbb{T}^\infty)$ formed by the functions $f$ whose Fourier spectrum is contained in $\mathbb{N}^{(\infty)}$:

$$\hat{f}(\alpha) := \int_{\mathbb{T}^\infty} f(z) z^\alpha \, dm(z) = 0 \quad \text{if} \ \alpha \notin \mathbb{N}^{(\infty)}.$$ 

The set $E := \mathbb{N}^{(\infty)}$ is called the narrow cone of Helson, and we also denote $L^2_{\mathbb{N}^{(\infty)}}(\mathbb{T}^\infty) = L^2_{E}(\mathbb{T}^\infty)$. Any element of that subspace can be formally written as:

$$f = \sum_{\alpha \geq 0} c_\alpha e_\alpha \quad \text{with} \quad c_\alpha = \hat{f}(\alpha) \quad \text{and} \quad \sum_{\alpha \geq 0} |c_\alpha|^2 < \infty.$$

Here, $(e_\alpha)_{\alpha \in \mathbb{N}^{(\infty)}}$ is the canonical basis of $L^2(\mathbb{T}^\infty)$ formed by characters, and accordingly $(e_\alpha)_{\alpha \in \mathbb{N}^{(\infty)}}$ is the canonical basis of $L^2_{E}(\mathbb{T}^\infty)$.

Now we consider $\Omega_2 = \mathbb{D}^\infty \cap \ell_2$.

Any $f \sim \sum_{\alpha \geq 0} c_\alpha e_\alpha \in L^2_{E}(\mathbb{T}^\infty)$ defines an analytic function on the infinite-dimensional Reinhardt domain $\Omega_2$ by the formula:

$$(2.1) \quad f(z) = \sum_{\alpha \geq 0} c_\alpha z^\alpha$$

where the series is absolutely convergent for each $z = (z_j)_{j \geq 1} \in \Omega_2$, as the pointwise product of two square-summable sequences. Indeed, using an Euler type formula, we get for $z \in \Omega_2$:

$$\sum_{\alpha \geq 0} |z^\alpha|^2 = \prod_{j=1}^{\infty} (1 - |z_j|^2)^{-1} < \infty ,$$

and hence, by the Cauchy-Schwarz inequality:

$$\sum_{\alpha \geq 0} |c_\alpha z^\alpha| \leq \left( \sum_{\alpha \geq 0} |c_\alpha|^2 \right)^{1/2} \left( \sum_{\alpha \geq 0} |z^\alpha|^2 \right)^{1/2} < \infty .$$

If $\alpha \in E$ and $z \in \Omega_2$, we have set, as usual, $z^\alpha = \prod_{j \geq 1} z_j^{\alpha_j}$.

This shows that $L^2_{E}(\mathbb{T}^\infty)$ can be identified with $H^2(\Omega_2)$, the Hardy-Hilbert space of analytic functions $f(z) = \sum_{\alpha \geq 0} c_\alpha z^\alpha$ on $\Omega_2$ with

$$\|f\|^2 := \sum_{\alpha \geq 0} |c_\alpha|^2 < \infty.$$
This setting is customary in connection with Dirichlet series (see [7]).

In this paper, for a technical reason appearing below in the proof of Proposition 2.5, we will consider, instead of \( \Omega_2 = D_\infty \cap \ell_2 \), the sub-domain:

\[
\Omega = D_\infty \cap \ell_1,
\]
i.e. the open subset of \( \ell_1 \) formed by the sequences:

\[
z = (z_n)_{n \geq 1} \quad \text{such that} \quad |z_n| < 1, \forall n \geq 1,
\]
and the restrictions to \( \Omega \) of the functions \( f \in H^2(\Omega_2) \). We denote \( H^2(\Omega) \) the space of such restrictions.

Hence \( f \in H^2(\Omega) \) if and only if:

\[
f(z) = \sum_{\alpha \geq 0} c_\alpha z^\alpha \quad \text{with} \ z \in \Omega,
\]
and \( \|f\|^2 := \sum_{\alpha \geq 0} |c_\alpha|^2 < \infty \).

We now identify the space \( L^2_\mathbb{E}(T^\infty) \) with the space \( H^2(\Omega) \).

We more generally define Hardy spaces \( H^p(\Omega) \), for \( 1 \leq p < \infty \), in the usual way:

\[
H^p = H^p(\Omega) = \{ f : \Omega \to \mathbb{C} ; \|f\|_p < \infty \},
\]
where \( f \) is analytic in \( \Omega \) and \( \|f\|_p = \sup_{0 < r < 1} M_p(r,f) = \lim_{r \to 1} M_p(r,f) \) with:

\[
M_p(r,f) = \left( \int_{T^\infty} |f(rz)|^p \, dm(z) \right)^{1/p}, \quad 0 < r < 1.
\]

We have \( \|f\| = \|f\|_2 \). Moreover, \( H^q \) contractively embeds into \( H^p \) for \( p < q \).

2.2 Singular numbers

We begin with a reminder of operator-theoretic facts. We recall that the approximation numbers \( a_n(T) = a_n \) of an operator \( T : H \to H \) (with \( H \) a Hilbert space) are defined by:

\[
a_n = \inf_{\text{rank } R \leq n} \|T - R\|.
\]

According to a 1957’s result of Allahverdiev (see [3], page 155), we have \( a_n = s_n \), the \( n \)-th singular number of \( T \). We also recall a basic result due to H. Weyl and one obvious consequence:

**Theorem 2.1.** Let \( T : H \to H \) be a compact operator with eigenvalues \( (\lambda_n) \) rearranged in decreasing order and singular numbers \( (a_n) \). Then:

\[
\prod_{j=1}^n |\lambda_j| \leq \prod_{j=1}^n a_j \quad \text{for all } n \geq 1.
\]

As a consequence:

\[
|\lambda_{2n}|^2 \leq a_1 a_{2n}.
\]
2.3 Spectra of projective tensor products

The following operator-theoretic result will play a basic role in the sequel. Let \( E_1, \ldots, E_n \) be Banach spaces and let \( E = \bigotimes_{i=1}^n E_i \) their projective tensor product (the only tensor product we shall use). If \( T_i \in \mathcal{L}(E_i) \), we define as usual their projective tensor product \( T = \bigotimes_{i=1}^n T_i \in \mathcal{L}(E) \) by its action on the atoms of \( E \), namely:

\[
T(\bigotimes_{i=1}^n x_i) = \bigotimes_{i=1}^n T_i(x_i).
\]

Denote in general \( \sigma(x) \) the spectrum of \( x \in A \) where \( A \) is a unital Banach algebra. We recall ([13], chap.11, Theorem 11.23) the following result.

**Lemma 2.2.** Let \( A \) be a unital Banach algebra, and \( x_1, \ldots, x_n \) be pairwise commuting elements of \( A \). Then:

\[
\sigma(x_1 \cdots x_n) \subseteq \prod_{i=1}^n \sigma(x_i).
\]

Here, \( \prod_{i=1}^n \sigma(x_i) \) is the product in the Minkowski sense, namely:

\[
\prod_{i=1}^n \sigma(x_i) = \left\{ \prod_{i=1}^n \lambda_i : \lambda_i \in \sigma(x_i) \right\}.
\]

As a consequence, we then have the following lemma due to Schechter, which we prove under a weakened form, sufficient here, and which is indeed already in [1] (we just add a few details because this is a central point in our estimates).

**Lemma 2.3.** Let \( F \) be a Banach space, \( T_1, \ldots, T_n \in \mathcal{L}(F) \) and \( T = \bigotimes_{i=1}^n T_i \). Then \( \sigma(T) \subseteq \prod_{i=1}^n \sigma(T_i) \).

**Proof.** To save notation, we assume \( n = 2 \). Let \( x_1 = T_1 \otimes I_2 \) and \( x_2 = I_1 \otimes T_2 \) where \( I_i \) is the identity of \( E_i \). Clearly,

\[
x_1 x_2 = x_2 x_1 = T_1 \otimes T_2 = T \quad \text{and} \quad \sigma(x_i) = \sigma(T_i)
\]

where the spectrum of \( x_i \) is in the Banach algebra \( \mathcal{L}(E) \) and that of \( T_i \) in \( \mathcal{L}(E_i) \). Lemma 2.2 now gives:

\[
\sigma(T) = \sigma(x_1 x_2) \subseteq \sigma(x_1) \sigma(x_2) = \sigma(T_1) \sigma(T_2),
\]

hence the result. \( \square \)

2.4 Schur maps and composition operators

We now pass to some general facts on composition operators \( C_\varphi \), defined by \( C_\varphi(f) = f \circ \varphi \), associated with a Schur map, namely a non-constant analytic self-map \( \varphi \) of \( \Omega \). We say that \( \varphi \) is a symbol for \( H^2(\Omega) \) if \( C_\varphi \) is a bounded linear operator from \( H^2(\Omega) \) into itself.

The differential \( \varphi'(a) \) of \( \varphi \) at some point \( a \in \Omega \) is a bounded linear map \( \varphi'(a) : \ell^1 \to \ell^1 \).
Definition 2.4. The symbol $\varphi$ is said to be truly infinite-dimensional if the differential $\varphi'(a)$ is an injective linear map from $\ell^1$ into itself for at least one point $a \in \Omega$.

In finite dimension, this amounts to saying that $\varphi(\Omega)$ has non-void interior.

We have the following general result.

Proposition 2.5. Let $(\varphi_j)_{j \geq 1}$ be a sequence of analytic self-maps of $\mathbb{D}$ such that $\sum_{j \geq 1} |\varphi_j(0)| < \infty$. Then, the mapping $\varphi: \Omega \to \mathbb{C}^\infty$ defined by the formula $\varphi(z) = (\varphi_j(z_j))_{j \geq 1}$ maps $\Omega$ to itself and is a symbol for $H^2(\Omega)$.

Proof. First, the Schwarz inequality:

$$|\varphi_j(z_j) - \varphi_j(0)| \leq 2|z_j|$$

shows that $\varphi(z) \in \Omega$ when $z \in \Omega$. To see that $\varphi$ is moreover a symbol for $H^2(\Omega)$, we use the fact ([8]) that:

(2.2) $\|C_{\varphi_j}\| \leq \sqrt{\frac{1+|\varphi_j(0)|}{1-|\varphi_j(0)|}}$.

Now, by the separation of variables and Fubini’s theorem, we easily get:

(2.3) $\|C_{\varphi}\| \leq \prod_{j \geq 1} \|C_{\varphi_j}\| < \infty$.

As $\sum_{j \geq 1} |\varphi_j(0)| < \infty$, by hypothesis, the infinite product

$$\prod_{j \geq 1} \sqrt{\frac{1+|\varphi_j(0)|}{1-|\varphi_j(0)|}}$$

converges and, in view of (2.2) and (2.3), $C_{\varphi}$ is bounded.

We also have the following useful fact.

Lemma 2.6. The automorphisms of $\Omega$ act transitively on $\Omega$ and define bounded composition operators on $H^2(\Omega)$.

Proof. Let $a = (a_j)_{j \in \mathbb{N}} \in \Omega$ and let $\Psi_a: \Omega \to \mathbb{C}^\infty$ be defined by:

$$\Psi_a(z) = (\Phi_{a_j}(z_j))_{j \geq 1}$$

where in general $\Phi_a: \mathbb{D} \to \mathbb{D}$ is defined by $\Phi_a(z) = (z - a)/(1 - az)$. The Schwarz lemma gives $|\Phi_{a_j}(z_j) + a_j| \leq 2|z_j|$, and shows that $\Psi_a$ maps $\Omega$ to itself. Clearly, $\Psi_a$ is an automorphism of $\Omega$ with inverse $\Psi_{-a}$ and $\Psi_a(a) = 0$. The fact that the composition operator $C_{\Psi_a}$ is bounded on $H^2(\Omega)$ is a consequence of Proposition 2.5.
3 Spectrum of compact composition operators

We begin with the following definition, following [10].

**Definition 3.1.** Let $\varphi : \Omega \to \Omega$ be a truly infinite-dimensional symbol. We say that $\varphi$ is compact if $\varphi(\Omega)$ is a compact subset of $\Omega$.

We then have the following result.

**Lemma 3.2.** If $\varphi : \Omega \to \Omega$ is a compact mapping, then:

1) $C_\varphi : H^2(\Omega) \to H^2(\Omega)$ is bounded and moreover compact.

2) If $a \in \Omega$ a fixed point of $\varphi$, $\varphi'(a) \in \mathcal{L}(\ell^1)$ is a compact operator.

**Proof.** 1) follows from a H. Schwarz type criterion via an Ascoli-Montel type theorem: every sequence $(f_n)$ of $H^2(\Omega)$ bounded in norm contains a subsequence which converges uniformly on compact subsets of $\Omega$. Indeed, we have the following ([4], chap. 17, p. 274): if $A$ is a locally bounded set of holomorphic functions on $\Omega$, then $A$ is locally equi-Lipschitz, namely every point $a \in \Omega$ has a neighbourhood $U \subset \Omega$ such that:

$$z, w \in U \quad \text{and} \quad f \in A \implies |f(z) - f(w)| \leq C_{A,U} \|z - w\|. $$

The Ascoli-Montel theorem easily follows from this. Then, if $f_n \in H^2(\Omega)$ converges weakly to 0, it converges uniformly to 0 on compact subsets of $\Omega$; in particular on $\varphi(\Omega)$. This means that $\|C_\varphi(f_n)\|_\infty = \|f_n \circ \varphi\|_\infty \to 0$, implying $\|f_n \circ \varphi\|_2 \to 0$ and the compactness of $C_\varphi$.

Actually, $C_\varphi$ is compact on every Hardy space $H^p(\Omega)$, $1 \leq p \leq \infty$. This observation will be useful later on.

For 2), we may indeed dispense ourselves with the invariance of $a$, and force $a = 0$ to be a fixed point of $\varphi$. Indeed, we can replace $\varphi$ by $\psi = \Psi_b \circ \varphi \circ \Psi_a$ where $b = \varphi(a)$ is arbitrary, and use Lemma 2.6 as well as the ideal property of compact linear operators. We set $A = \varphi'(0)$. Expanding each coordinate $\varphi_j$ of $\varphi$ in a series of homogeneous polynomials, we may write (since $\varphi(0) = 0$):

$$\varphi(z) = \sum_{|\alpha| = 1} c_\alpha z^\alpha + \sum_{s=2}^{\infty} \left( \sum_{|\alpha| = s} c_\alpha z^\alpha \right) = A(z) + \sum_{s=2}^{\infty} \left( \sum_{|\alpha| = s} c_\alpha z^\alpha \right),$$

where $c_\alpha = (c_{\alpha,j})_{j \geq 1} \in \mathbb{C}^\infty$. We clearly have (looking at the Fourier series of $\varphi(z e^{i\theta})$):

$$\|z\|_1 < 1 \implies z \in \Omega \implies A(z) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(z e^{i\theta}) e^{-i\theta} d\theta.$$

Since $\varphi$ is compact, this clearly implies, with $B$ the open unit ball of $\ell^1$, that $A(B)$ is totally bounded, proving the compactness of $A$. \square

The following extension of results of [11], then [1] and [6], which themselves extend a theorem of G. Königs ([14], p. 93) will play an essential role for lower bounds of approximation numbers.
Theorem 3.3. Let \( \varphi : \Omega \to \Omega \) be a compact symbol. Assume there is an \( a \in \Omega \) such that \( \varphi(a) = a \) and that \( \varphi'(a) \in \mathcal{L}(\ell^1) \) is injective. Then, the spectrum of \( C_{\varphi} : H^2(\Omega) \to H^2(\Omega) \) is exactly formed by the numbers \( \lambda^\alpha, \alpha \in \mathbb{N}^{(\infty)}, \) and 0, 1, where \( (\lambda_j)_{j \geq 1} \) denote the eigenvalues of \( A := \varphi'(a) \) and:

\[
\lambda^\alpha = \prod_{j \geq 1} \lambda_j^{\alpha_j} \quad \text{if} \quad \alpha = (\alpha_j)_{j \geq 1} \in \mathbb{N}^{(\infty)}.
\]

Proof. This is proved in [1] for the unit ball \( B_E \) of an arbitrary Banach space \( E \) and for the space \( H^{\infty}(B_E) \), in four steps which are the following:

1. If \( \varphi(B_E) \) lies strictly inside \( B_E \) (namely if \( \varphi(B_E) \subseteq r B_E \) for some \( r < 1 \)), in particular when \( \varphi \) is compact, \( \varphi \) has a unique fixed point \( a \in B_E \), according to a theorem of Earle and Hamilton.

2. The spectrum of \( C_{\varphi} \) contains the numbers \( \lambda \) where \( \lambda \) is an eigenvalue of \( \varphi'(a) \) or \( \lambda = 0, 1 \).

3. It is then proved that the spectrum of \( C_{\varphi} \) contains the numbers \( \lambda^\alpha \) and 0, 1.

4. It is finally proved that spectrum of \( C_{\varphi} \) is contained in the numbers \( \lambda^\alpha \) and 0, 1.

Here, handling with the domain \( \Omega \), we see that:

1. True or not for \( \Omega \), the Earle-Hamilton theorem is not needed since we will force, by a change of the compact symbol \( \varphi \) in another compact symbol \( \psi = \Psi_b \circ \varphi \circ \Psi_a \), the point 0 to be a fixed point. Moreover \( A = \psi'(0) \) is injective if \( \varphi'(a) \) is, since \( \Psi_b \) and \( \Psi'_a \) are invertible.

2. The second step (non-surjectivity) is valid for any domain and for \( H^2(\Omega) \), or \( H^p(\Omega) \), in exactly the same way.

3. The third step consists of proving \( \{ \lambda^\alpha \} \subseteq \sigma(C_{\varphi}) \).

For that purpose, assume that \( \lambda^\alpha = \prod_{i=1}^m \lambda_i \neq 0 \) with \( \lambda_i \) an eigenvalue of \( \varphi'(0) \) and with repetitions allowed. As we already mentioned, under the assumption of compactness of \( \varphi \), \( C_{\varphi} \) is compact on \( H^p(\Omega) \) as well, for any \( p \geq 1 \). We take here \( p = 2m \). Step 2 provides us with non-zero functions \( f_i \in H^p(\Omega) \) such that \( f_i \circ \varphi = \lambda_i f_i, \) \( 1 \leq i \leq m \), since for the compact operator \( C_{\varphi} : H^p \to H^p, \) non-surjectivity implies non-injectivity. Let \( f = \prod_{1 \leq i \leq m} f_i \). Then, using the integral representation of the norm and the Hölder inequality, we see that \( f \in H^2(\Omega) \), \( f \neq 0 \) and \( f \circ \varphi = \lambda_f f \), proving our claim.

4. The fourth step is valid as well, with a slight simplification: we have to show that, if \( \mu \neq 1 \) is not of the form \( \lambda^\alpha \), then \( C_{\varphi} - \mu I \) is injective. Let \( f \in H^2(\Omega) \) satisfying \( f \circ \varphi = \mu f \) and let:

\[
f(z) = \sum_{m=0}^{\infty} \frac{d^m f(0)}{m!} (z^m)
\]

be the Taylor expansion of \( f \) about \( z = 0 \) (observe that \( \Omega \) is a Reinhardt domain). As usual, \( d^m f(0) := L_m \) is an \( m \)-linear symmetric form on \( F = \ell^1 \) and the notation \( L_m(z, z, \ldots, z) \) means \( L_m(z, z, \ldots, z) \).
Observe that $L_m$ can be isometrically identified with an element (denoted $L_m$) of $\mathcal{L}(F^\otimes n)$ defined by the formula:

$$L_m(x_1 \otimes \cdots \otimes x_n) = L_m(x_1, \ldots, x_m).$$

We will prove by induction that $L_n = 0$ for each $n$. For this, we can avoid the appeal to transposes of $[1]$ as follows: if the result holds for $L_m$ with $m < n$, one gets (comparing the terms in $z^n$ in both members of $f \circ \varphi = \mu f$):

$$\mu A = A \circ B \quad \text{where} \quad A = L_m \quad \text{and} \quad B = \varphi'(0)\otimes^n.$$

That is $A(B - \mu I) = 0$ where $I$ is the identity map of $F^\otimes n$. Now, $B - \mu I$ is invertible in $\mathcal{L}(F)$ by Lemma 3.3, so that $A = A(B - \mu I)(B - \mu I)^{-1} = 0$.

The proof is complete.

The following theorem summarizes and exploits the preceding theorem. Possibly, some restrictions can be removed, and we could only assume the compactness of $C_\varphi$, not of $\varphi$ itself. After all, in dimension one, there are symbols $\varphi$ with $\|\varphi\|_\infty = 1$ for which $C_\varphi : H^2 \to H^2$ is compact.

**Theorem 3.4.** Let $\varphi : \Omega \to \Omega$ be a truly infinite-dimensional compact mapping of $\Omega$. Then:

1) $C_\varphi : H^2(\Omega) \to H^2(\Omega)$ is bounded and even compact.

2) $A = \varphi'(0)$ is compact.

3) No $\delta > 0$ can exist such that $a_n(C_\varphi) \leq Ce^{-c n^\delta}$ for all $n \geq 1$. More precisely, the numbers $a_n$ satisfy:

$$\sum_{n \geq 1} \frac{1}{\log^p(1/a_n)} = \infty \quad \text{for all} \quad p < \infty.$$

**Proof.** The proof is based on the previous Theorem 3.3. Without loss of generality, we can assume that $\varphi(0) = 0$ and $\varphi'(0)$ is injective, by using a point $a$ at which $\varphi'(a)$ is injective, and then the fact that automorphisms of $\Omega$ act transitively on $\Omega$, act boundedly on $H^2(\Omega)$, and the ideal property of approximation numbers. More precisely, we pass to $\Psi = \Psi_b \circ \varphi \circ \Psi_a$ with $b = \varphi(a)$ and get:

$$\Psi(0) = 0 \quad \text{and} \quad \Psi'(b) = \Psi'_b(b) \varphi'(a) \Psi'_a(0)$$

injective, since $\Psi'_b(b)$ and $\Psi'_a(0)$ are, and $\Psi_a$ and $\Psi_b$ are automorphisms of $\Omega$.

We now have the following simple but crucial lemma.

**Lemma 3.5.** Whatever the choice of the numbers $\lambda_j$ with $0 < |\lambda_j| < 1$, denoting by $(\delta_n)_{n \geq 1}$ the non-increasing rearrangement of the numbers $\lambda^n$, one has:

$$\sum_{n \geq 1} \frac{1}{\log^p(1/\delta_n)} = \infty \quad \text{for all} \quad p < \infty.$$
Proof of the Lemma. For any positive integer \( p \), we set:

\[ q = 2p, \quad \log 1/|\lambda_j| = A_j, \]

and we use that:

\[ \sum_{1 \leq j \leq q} \alpha_j A_j \leq \left( \sum_{1 \leq j \leq q} \alpha_j^2 \right)^{\frac{1}{2}} \left( \sum_{1 \leq j \leq q} A_j^2 \right)^{\frac{1}{2}} =: C_q \left( \sum_{1 \leq j \leq q} \alpha_j^2 \right) = C_q \| \alpha \|^2, \]

where \( \| . \| \) stands for the euclidean norm in \( \mathbb{R}^q \). We then get:

\[ \sum_{n \geq 1} \frac{1}{\log^p (1/\delta_n)} = \sum_{n \geq 1} \frac{1}{\log^p (1/|\lambda^n|)} \]

\[ \geq \sum_{\alpha_j \geq 1, 1 \leq j \leq q} \frac{1}{\log^p (1/|\lambda_1^{\alpha_1} \cdots |\lambda_q^{\alpha_q}|)} \]

\[ = \sum_{\alpha_j \geq 1, 1 \leq j \leq q} \frac{1}{(\alpha_1 A_1 + \cdots + \alpha_q A_q)^p} \]

\[ \geq C_q^{-p} \sum_{\alpha_j \geq 1, 1 \leq j \leq q} \frac{1}{(\alpha_1^2 + \cdots + \alpha_q^2)^p} \]

\[ = C_q^{-p} \sum_{\alpha_j \geq 1, 1 \leq j \leq q} \frac{1}{\| \alpha \|^q} = \infty, \]

because:

\[ \int_{x \in \mathbb{R}^q, \| x \| \geq 1} \frac{1}{\| x \|^q} \, dx = c_q \int_1^{\infty} r^{q-1} \frac{1}{r^q} \, dr = \infty. \]

This proves the lemma. \( \square \)

This can be transferred to the approximation numbers \( a_n = a_n(C_\varphi) \) to end the proof of Theorem 3.4. Indeed, we know from Lemma 3.5 that the non-increasing rearrangement \( (\delta_n) \) of the eigenvalues \( \lambda^n \) of \( C_\varphi \) satisfies

\[ \sum_{n \geq 1} \frac{1}{\log^p (1/\delta_n)} = \infty. \]

Since a divergent series of non-negative and non-increasing numbers \( u_n \) satisfies \( \sum u_{2n} = \infty \), we further see that:

\[ \sum_{n \geq 1} \frac{1}{\log^p (1/\delta_{2n})} = \infty \quad \text{for all} \quad p < \infty. \]

Moreover, by Theorem 2.1 we have:

\[ (3.4) \quad \left( 2 \log 1/\delta_{2n} \right)^p \leq \left( \log 1/(a_1 a_n) \right)^p. \]
Since $1/(\log 1/a_1a_n) \sim 1/(\log 1/a_n)$, Lemma 3.5 then gives the result. This clearly prevents an inequality of the form $a_n \leq Ce^{-c n^\delta}$ for some positive numbers $c, C, \delta$ and all $n \geq 1$. Indeed, this would imply:

$$\sum_{n \geq 1} \frac{1}{\log^p(1/a_n)} < \infty \quad \text{for} \quad p > 1/\delta,$$

contradicting (3.3).

Remarks. Let us briefly comment on the assumptions in Theorem 3.4.

1) We do not need the Earle-Hamilton theorem under our assumptions. The Schauder-Tychonoff theorem gives the existence (if not the uniqueness) of a fixed point for $\varphi$ in $\Omega$ (bounded and convex).

2) The Earle-Hamilton theorem is in some sense more general (for analytic maps) since it remains valid when $\varphi(\Omega)$ is only assumed to lie strictly inside $\Omega$, i.e. when $\varphi(\Omega) \subseteq r\Omega$ for some $r < 1$. But this assumption does not ensure the compactness of $C_\varphi$ as indicated by the simple example $\varphi(z) = rz$, $0 < r < 1$. The coordinate functions $z \mapsto z_n$ converge weakly to 0, while $\|C_\varphi(z_n)\|_{H^2(\Omega)} = r$.

3) The mere assumption that $\text{cl}(\varphi(\Omega))$ is compact is not sufficient either. Just take:

$$\varphi(z) = \left(\frac{1 + z_1}{2}, 0, \ldots, 0, \ldots\right).$$

Since the composition operator $C_{\varphi_1}$ associated with $\varphi_1(z) = \frac{1 + z}{2}$ is notoriously non-compact on $H^2(\mathbb{D})$, neither is $C_\varphi$ on $H^2(\Omega)$. Yet, $\varphi(\Omega)$ is obviously compact in $l^1$.

4 Possible upper bounds

Recall that $\Omega = \mathbb{D}^\infty \cap l^1$.

4.1 A general example

Theorem 4.1. Let $\varphi((z_j)_j) = (\lambda_j z_j)_j$ with $|\lambda_j| < 1$ for all $j$, so that $\varphi(\Omega) \subseteq \Omega$ and $\varphi'(0)$ is the diagonal operator with eigenvalues $\lambda_j$, $j \geq 1$, on the canonical basis of $l^1$. Let $p > 0$. Then:

$$(\lambda_j)_j \in l^p \implies C_\varphi \in S_p.$$

In particular, there exist truly infinite-dimensional symbols on $\Omega$ such that the composition operator $C_\varphi : H^2(\Omega) \to H^2(\Omega)$ is in all Schatten classes $S_p$, $p > 0$. 

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Proof. Since $C_\varphi$ is diagonal on the orthonormal basis $(z_\alpha)_\alpha$ of the Hilbert space $H^2(\Omega)$, with $C_\varphi(z_\alpha) = \varphi^\alpha$, its approximation numbers are the non-increasing rearrangement of the moduli of eigenvalues $\lambda^\alpha$, so that an Euler product-type computation gives:

$$\sum_{n=1}^{\infty} a_n^p = \sum_{\alpha \in E} |\lambda^\alpha|^p = \sum_{\alpha_j \in \mathbb{N}} |\lambda_j|^{|\alpha_j|} = \prod_{j=1}^{\infty} (1 - |\lambda_j|^p)^{-1} < \infty .$$

To obtain $C_\varphi \in \bigcap_{p>0} S_p$, just take $\lambda_n = e^{-n}$. This ends the proof. \hfill \Box

4.2 A sharper upper bound

By making a more quantitative study, we can prove the following result.

**Theorem 4.2.** For any $0 < \delta < 1$, there exists a compact composition operator on $H^2(\Omega)$, with a truly infinite-dimensional symbol, such that, for some positive constants $c, C, b$, we have:

$$a_n(C_\varphi) \leq C \exp\left(-c e^{b \log n^{\delta}}\right).$$

Proof. Take the same operator $C_\varphi$ as in Theorem 4.1, with $\lambda_n = e^{-A_n}$ where the positive numbers $A_n$ have to be adjusted. Its approximation numbers $a_N$ are then the non-increasing rearrangement of the sequence of numbers $(\varepsilon_n)_n := (\lambda^\alpha)_\alpha$. This suggests using a generating function argument, namely considering $\sum \varepsilon_n x^n$, but the rearrangement perturbs the picture. Accordingly, we follow a slightly different route. Fix an integer $N \geq 1$ and a real number $r > 0$. Observe that, following the proof of Theorem 4.1:

$$N a_N^r \leq \sum_{n=1}^{N} a_n^r \leq \sum_{n=1}^{\infty} a_n^r = \prod_{n=1}^{\infty} (1 - e^{-r A_n})^{-1}.$$

First, consider the simple example $A_n = n$. We get:

$$N a_N^r \leq \eta\left(e^{-r}\right)$$

where $\eta$ is the Dedekind eta function (see [5]) given by:

$$\eta(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1} = \sum_{n=0}^{\infty} p(n) x^n , \quad |x| < 1 ,$$

where $p(n)$ is the number of partitions of the integer $n$. It is well-known ([5], Ch. 7, p. 169) that $\eta(e^{-r}) \leq e^{D/r}$ with $D = \pi^2/6$, so that:

$$a_n \leq \exp\left(\frac{D}{r^2} - \frac{\log N}{r}\right).$$

Optimizing with $r = 2D/\log N$, we get:

$$a_N \leq \exp(-c \log^2 N) ,$$

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with $c = 1/4D$. This is more precise than Theorem 4.1.

We now show that if $A_n$ increases faster, we can achieve the decay of Theorem 4.2. As before, we get in general:

\[
a_N \leq \inf_{x > 1} \left( \exp \left[ x (\log F(x^{-1}) - \log N) \right] \right),
\]

where

\[
F(r) = \prod_{n=1}^{\infty} (1 - e^{-rA_n})^{-1}.
\]

We have:

\[
\log F(r) = \sum_{n=1}^{\infty} \frac{\sum_{m=1}^{\infty} e^{-rmA_n}}{m} = \sum_{n=1}^{\infty} \frac{1}{m} \left( \sum_{n=1}^{\infty} e^{-rmA_n} \right).
\]

Now, take $A_n = e^{\alpha n}$ where $\alpha > 0$ is to be chosen. We have:

\[
\sum_{n=1}^{\infty} e^{-rm} e^{\alpha n} \leq \int_{0}^{\infty} e^{-rm e^{\alpha t}} dt =: \int_{m(r)}.
\]

Standard estimates now give, for $r < 1$:

\[
I_m(r) = \int_{1}^{\infty} e^{-rm} \frac{1}{\alpha (\log x)^{\alpha - 1}} \frac{dx}{x} = \int_{rm}^{\infty} e^{-y} \frac{1}{\alpha} \left( \log \frac{y}{rm} \right)^{\alpha - 1} \frac{dy}{y}
\]

\[
\leq \left( \log \frac{1}{r} \right)^{\alpha - 1} \int_{rm}^{\infty} e^{-y} \frac{dy}{y} \leq \int_{rm}^{\infty} e^{-y} \left( \log \frac{1}{r} \right)^{\frac{1}{\alpha}}
\]

so that:

\[
\log F(r) \leq \left( \log \frac{1}{r} \right)^{\frac{1}{\alpha}} \sum_{m=1}^{\infty} m^{-1} e^{-rm} \leq \left( \log \frac{1}{r} \right)^{\frac{1}{\alpha} + 1}.
\]

Going back to (4.1), we get, for some constant $C > 0$, and for $x = 1/r > 1$:

\[
a_N \leq C \exp \left[ C \left( (\log x)^{\frac{1}{\alpha} + 1} - \log N \right) \right].
\]

Adjusting $x = x_N > 1$ so as to have $(\log x)^{\frac{1}{\alpha} + 1} = \log N - 1$, that is:

\[
x_N = \exp \left[ (\log (N/e))^{\frac{\log \alpha}{\alpha + 1}} \right],
\]

we get $a_N \leq C e^{-c x_N}$, which is the claimed result with $\delta = \alpha/(\alpha + 1)$.

This $\delta$ can be taken arbitrarily in $(0, 1)$ by choosing $\alpha$ suitable, and we are done.

**Remark.** Of course, $\delta = 1$ is forbidden, because this would give $a_n \leq C e^{-c n^b}$, implying:

\[
\sum_{n=1}^{\infty} \frac{1}{(\log 1/a_n)^p} \leq \sum_{n=1}^{\infty} n^{-b p} < \infty,
\]

for large $p$, and contradicting Theorem 3.4.
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