QCD IN THE REGGE LIMIT: FROM GLUON REGGEIZATION TO PHYSICAL AMPLITUDES

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This paper is a brief survey of the Balitskii-Fadin-Kuraev-Lipatov (BFKL) approach for the description of hard or semi-hard processes in the so-called Regge limit of perturbative QCD. The starting point is a fundamental property of perturbative QCD, the gluon Reggeization. This property, combined with $s$-channel unitarity, allows to predict the growth in energy of the amplitude of hard or semi-hard processes with exchange of vacuum quantum numbers in the $t$-channel. When also the so-called impact factors of the colliding particles are known, then not only the behavior with the energy, but the complete amplitude can be determined. This was recently done, for the first time with next-to-leading order accuracy and for a process with colorless external particles, in the case of the electroproduction of two light vector mesons.

1 Introduction

The BFKL equation became very popular in the last years due to the experimental results on deep inelastic scattering of electrons on protons obtained at the HERA collider. These results show a power-growth of the gluon density in the proton when the fraction of the proton momentum carried by the gluon (i.e. the $x$ Bjorken variable) decreases.

Together with the DGLAP evolution equation, the BFKL equation can be used for the description of structure functions for the deep inelastic electron-proton scattering at small values of the $x$ variable. It applies in general to all processes where a “hard” scale exists (the “hardness” can be supplied either by a large virtuality or by the mass of heavy quarks) which allows the use of perturbation theory. It is an iterative integral equation for the determination of the Green’s function for the elastic diffusion of two Reggeized gluons (Section 2). The knowledge of this Green’s function is enough for the determination of the growth in energy of the amplitude of hard or semi-hard processes and, for the case of deep-inelastic electron-proton scattering, for the growth in $x$ of the gluon density in the proton for decreasing $x$. The kernel of this equation was first derived in the leading logarithmic approximation (LLA), which means resummation of all terms of the type $(\alpha_s \ln s)^n$, where $\alpha_s$ is the QCD coupling constant and $s$ is the square of the c.m.s. energy. In this approximation total cross sections are predicted to grow at large $s$ with a power of the center-of-mass energy larger than for “soft” hadronic processes (Section 3).

Unfortunately, in this approximation neither the scale of $s$ nor the argument of the running coupling constant $\alpha_s$ can be fixed. So, in order to do accurate theoretical predictions, it was necessary to calculate the radiative corrections to the LLA (Section 4). These corrections turned out to be large and with negative sign with respect to the leading order and this has raised an ongoing debate on the reliability of perturbation theory in this context. Many attempts of improvement have been suggested, but a definite conclusion is still lacking, mainly because of the difficulty to translate any recipe designed to work for the BFKL Green’s function, which is an unphysical object, into a definite prediction, checkable by experiments.
The information which is needed together with the BFKL Green’s function in order to build a physical amplitude is given by the so-called “impact factors” of the colliding particles. At the next-to-leading order impact factors were calculated first for the case of colliding partons. Among the impact factors for transitions between colorless objects, the most important one from the phenomenological point of view is certainly the impact factor for the virtual photon to virtual photon transition, i.e. the $\gamma^* \rightarrow \gamma^*$ impact factor. Its determination would open the way to predictions of the $\gamma^* \gamma^*$ total cross section and would represent a necessary ingredient in the study of the $\gamma^* p$ total cross section, relevant for deep inelastic scattering. Its calculation is rather complicated and only after year-long efforts it is approaching completion.

A considerable simplification can be gained if one considers instead the impact factor for the transition from a virtual photon $\gamma^*$ to a light neutral vector meson $V = \rho^0, \omega, \phi$. In this case, indeed, a close analytical expression can be achieved in the NLA, up to contributions suppressed as inverse powers of the photon virtuality. The knowledge of the $\gamma^* \rightarrow V$ impact factor in the NLA allows to determine completely within perturbative QCD and with NLA accuracy the amplitude of a physical process, the $\gamma^* \gamma^* \rightarrow VV$ reaction (Section 5). This possibility is interesting first of all for theoretical reasons, since it could be used as a test-ground for comparisons with approaches different from BFKL, such as DGLAP, and with possible next-to-leading order extensions of phenomenological models. Moreover, it is useful for testing the improvement methods suggested at the level of the NLA Green’s function for solving the problem of the instability of the perturbative series. But it could be interesting also for possible applications to phenomenology. Indeed, the calculation of the $\gamma^* \rightarrow V$ impact factor is the first step towards the application of the BFKL approach to the description of processes such as the vector meson electro-production $\gamma^* p \rightarrow V p$, being carried out at the HERA collider, and the production of two mesons in the photon collision, $\gamma^* \gamma^* \rightarrow VV$ or $\gamma^* \gamma \rightarrow V J/\Psi$, which can be studied at high-energy $e^+e^-$ and $e^+e^-$ colliders.

2 Gluon Reggeization in perturbative QCD

The key role in the derivation of the BFKL equation is played by the gluon Reggeization. “Reggeization” of a given elementary particle usually means that the amplitude of a scattering process with exchange of the quantum numbers of that particle in the $t$-channel goes like $s^{j(t)}$ in the Regge limit $s \gg |t|$. The function $j(t)$ is called “Regge trajectory” of the given particle and takes the value of the spin of that particle when $t$ is equal to its squared mass. In perturbative QCD, the notion of gluon Reggeization is used in a stronger sense. It means not only that a Reggeon exists with the quantum numbers of the gluon and with a trajectory $j(t) = 1 + \omega(t)$ passing through 1 at $t = 0$, but also that this Reggeon gives the leading contribution in each order of perturbation theory to the amplitude of processes with large $s$ and fixed (i.e. not growing with $s$) squared momentum transfer $t$.

To be definite, let us consider the elastic process $A + B \rightarrow A' + B'$ with exchange of gluon quantum numbers in the $t$-channel, i.e. for octet color representation in the $t$-channel and negative signature (see Fig. 1). Gluon Reggeization means that, in the

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*aThe negative (positive) signature part of an amplitude is the part of the amplitude which is odd (even) under the exchange of the Mandelstam variables $s$ and $u$.**
Regge kinematical region $s \simeq -u \to \infty$, $t$ fixed (i.e. not growing with $s$), the amplitude of this process takes the form

$$\left(A^{-}_{8}\right)_{AB}^{A'B'} = \Gamma^{c}_{A'A} \left[ \left(-\frac{s}{-t}\right)^{j(t)} - \left(\frac{s}{-t}\right)^{j(t)} \right] \Gamma^{c}_{B'B}.$$  \hspace{1cm} (1)

Here $c$ is a color index and $\Gamma^{c}_{P'P}$ are the particle-particle-Reggeon (PPR) vertices, not depending on $s$. This form of the amplitude has been proved rigorously to all orders of perturbation theory in the LLA. In this approximation the Reggeized gluon trajectory, $j(t) \equiv 1 + \omega(t)$, enters with 1-loop accuracy and we have

$$\omega^{(1)}(t) = \frac{g^2 t}{2} \int \frac{d^{D-2} k_{\perp}}{2^{D-1} q_{\perp}^{2}} = -\frac{g^2 N (1 - \epsilon) \Gamma^2(\epsilon)}{(4\pi)^{2/2}} \Gamma^2(2\epsilon) (-q_{\perp}^{2})^{\epsilon}.$$  \hspace{1cm} (4)

Here $D = 4 + 2\epsilon$ has been introduced in order to regularize the infrared divergences and the integration is performed in the space transverse to the momenta of the initial colliding particles. In the NLA the form (1) has been checked in the first three orders of perturbation theory and is only assumed to be valid to all orders.

### 3 BFKL in the LLA and the “bootstrap”

Amplitudes with quantum numbers in the $t$-channel different from the gluon ones are obtained in the BFKL approach by means of unitarity relations, thus calling for inelastic amplitudes. In the LLA, the main contributions to the unitarity relations from inelastic amplitudes come from the multi-Regge kinematics, i.e. when rapidities of the produced particles are strongly ordered and their transverse momenta do not grow with $s$. In the

\[ p_{\perp} = -\vec{p}_{\perp} \] 

will be also used.
multi-Regge kinematics, the real part of the production amplitudes takes a simple factorized form, due to gluon Reggeization,

$$A_{AB}^{A'B'} = 2s \Gamma_{AA}^{c_{-1}} \prod_{i=1}^{n} \frac{P_i}{\gamma_{c_i c_{i+1}}(q_i, q_{i+1})} \left( \frac{s_i}{s_R} \right) \frac{1}{t_i} \left( \frac{s_{n+1}}{s_R} \right) \frac{\omega_{n+1}}{\Gamma_{BB}^{c_{n+1}}} ,$$

where $s_R$ is an energy scale, irrelevant in the LLA, $\gamma_{c_i c_{i+1}}(q_i, q_{i+1})$ is the (non-local) effective vertex for the production of the particles $P_i$ with momenta $k_i = q_i - q_{i+1}$ in the collisions of Reggeons with momenta $q_i$ and $-q_{i+1}$ and color indices $c_i$ and $c_{i+1}$, $q_0 \equiv p_A$, $q_{n+1} \equiv -p_B$, $s_i = (k_i - 1 + k_i)^2$, $k_0 \equiv p_A$, $k_{n+1} \equiv p_B$ and $\omega_i$ stands for $\omega(t_i)$, with $t_i = q_i^2$. In the LLA, $P_i$ can be only the state of a single gluon (see Fig. 2). By using $s$-channel unitarity and the previous expression for the production amplitudes, the amplitude of the elastic scattering process $A + B \rightarrow A' + B'$ at high energies can be written as

$$A_{AB}^{A'B'} = \frac{is}{(2\pi)^{D-1}} \int \frac{d^{D-2}q_1}{q_1^2} \int \frac{d^{D-2}q_2}{q_2^2} \int \frac{d^{D-2}q_3}{q_3^2} \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{\sin(\pi\omega)} \sum_{R, \nu} \Phi^{(R, \nu)}_{A'A} (q_1; q_2; s_0) \times \left[ \left( \frac{-s}{s_0} \right) \tau \left( \frac{s}{s_0} \right) \right] G^{(R)}_{\nu} (q_1, q_2, q) \Phi^{(R, \nu)}_{B'B} (-q_2; -q; s_0) .$$

Here and below $q_i^2 \equiv q_i - q$, $q \sim q_{i\perp}$ is the momentum transfer in the process, the sum is over the irreducible representations $R$ of the color group obtained in the product of two adjoint representations and over the states $\nu$ of these representations, $\tau$ is the signature equal to $+1 (-1)$ for symmetric (antisymmetric) representations and $s_0$ is an artificial energy scale, which disappears in the full expression of the amplitude at each fixed order.

The imaginary part gives a next-to-next-to-leading contribution in the unitarity relations.
of approximation. \( \Phi_{A' A}^{(R, \nu)} \) and \( \Phi_{B' B}^{(R, \nu)} \) are the so-called impact factors in the \( t \)-channel color state \( (R, \nu) \). The first of them is related to the convolution of the PPR effective vertices \( \Gamma_{AA} \) and \( \Gamma_{A' \tilde{A}} \), the second to the convolution of \( \Gamma_{\tilde{B} B} \) and \( \Gamma_{B' \tilde{B}} \). \( G^{(R)} \) is the Mellin transform of the Green’s functions for Reggeon-Reggeon scattering (see Fig. 3).

The dependence from \( s \) is determined by \( G^{(R)} \), which obeys the equation (see Fig. 4)

\[
\omega G^{(R)}(\vec{q}_1, \vec{q}_2; \vec{q}) = \frac{\omega^2}{2} \delta^{(D-2)}(\vec{q}_1 - \vec{q}_2) + \int \frac{d^{D-2}q_f}{q_f^2(\vec{q}_f - \vec{q})^2} K^{(R)}(\vec{q}_1, \vec{q}_2; \vec{q}) G^{(R)}(\vec{q}_f, \vec{q}_2; \vec{q}) ,
\]

whose integral kernel,

\[
K^{(R)}(\vec{q}_1, \vec{q}_2; \vec{q}) = [\omega (-\vec{q}_1^2) + \omega (-\vec{q}_1^2)] \delta^{(D-2)}(\vec{q}_1 - \vec{q}_2) + K^{(R)}(\vec{q}_1, \vec{q}_2; \vec{q}) ,
\]

is composed by a “virtual” part, related to the gluon trajectory, and by a “real” part, \( K^{(R)} \), related to particle production in Reggeon-Reggeon collisions. In the LLA, the “virtual” part of the kernel takes contribution from the gluon Regge trajectory with 1-loop accuracy, \( \omega^{(1)} \), while the “real” part takes contribution from the production of one gluon in the Reggeon-Reggeon collision at Born level, \( K^{(B)} \). The BFKL equation is given by Eq. (4) when \( t = 0 \) and for singlet color representation in the \( t \)-channel, otherwise Eq. (4) is called the “generalized” BFKL equation.

The representation \( \Phi^{(R)} \) of the elastic amplitude, \( A + B \rightarrow A' + B' \), derived from \( s \)-channel unitarity, for the part with gluon quantum numbers in the \( t \)-channel \( (R = 8, \tau = -1) \), must reproduce the representation \( \Phi^{(R)} \) with one Reggeized gluon exchange in the \( t \)-channel, with LLA accuracy. This consistency is called “bootstrap” and was checked in the LLA already in Ref. \[^1\]. Subsequently, a rigorous proof of the gluon Reggeization in the LLA was constructed.\[^2\]

The part of the representation \( \Phi^{(R)} \) with vacuum quantum numbers in the \( t \)-channel \( (R = 0, \tau = +1) \) for the case of zero momentum transfer is relevant for the total cross section of the scattering of particles \( A \) and \( B \), via the optical theorem:

\[
\sigma_{AB}(s) = \frac{I m_s A_{AB}^{(R, \nu)}}{s} = \int \frac{d^2q_1}{2\pi} \Phi_A(\vec{q}_1, s_0) \int \frac{d^2q_2}{2\pi} \Phi_B(\vec{q}_2, s_0) \left( \frac{s}{s_0} \right)^\omega G_\omega(\vec{q}_1, \vec{q}_2) ,
\]

where \( D \) has been put equal to 4, assuming that the state \( A \) and \( B \) are colorless and have therefore good infrared behavior.\[^3\]

\[^3\]For the precise definition of impact factors, as well as of the BFKL kernel \( K^{(R)} \) which appears below, see Ref. \[^2\].
This Green’s function satisfies the equation
\[ \omega G(\vec{q}_1, \vec{q}_2) = \delta^2(\vec{q}_1 - \vec{q}_2) + \int d^2 \vec{q} K(\vec{q}_1, \vec{q}) G(\vec{q}, \vec{q}_2), \] (7)
where
\[ K(\vec{q}_1, \vec{q}_2) = \frac{K^{(0)}(\vec{q}_1, \vec{q}_2)}{q_1^2 q_2^2}. \] (8)

The eigenfunctions and the corresponding eigenvalues of the LLA singlet kernel are known (see, for instance, Ref.14):
\[ \int d^2 \vec{q}_2 K(\vec{q}_1, \vec{q}_2)(q_2^2)_{\gamma - 1} = \frac{N_c \alpha_s}{\pi} \chi(1/2 + i \nu), \] (9)
with
\[ \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma), \quad \psi(\gamma) = \Gamma'(\gamma) / \Gamma(\gamma). \]

The set of functions \((q_2^2)^{\gamma - 1}\) with \(\gamma = 1/2 + i \nu, -\infty < \nu < \infty\) forms a complete set, so that
\[ \sigma_{AB}(s) = \int_{\delta^+ i \infty}^{\delta^- i \infty} \frac{d\omega}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi^2 \omega [\omega - \frac{N_c \alpha_s}{\pi} \chi(1/2 + i \nu)]} \left[ \Phi_A(\vec{q}_1, s_0) \int \frac{d^2 \vec{q}_2}{2\pi} \Phi_B(-\vec{q}_2, s_0) \right] \left( \frac{s}{s_0} \right)^{\omega} (q_1^2)^{-i\nu-3/2} (q_2^2)^{+i\nu-3/2}. \] (9)

The maximal value of \(\chi(\gamma)\) on the integration contour is \(\chi(1/2) = 4 \ln 2\), which corresponds to the maximal eigenvalue of the kernel \(\omega_\beta^B = (\alpha_s N_c / \pi)4 \ln 2\), where the subscript “\(P\)” stands for “Pomeron”. It is then easy to see that Eq. (9) leads, by saddle point evaluation of the \(\nu\)-integration, to the following result:
\[ \sigma^{LLA} \sim s^{\omega_\beta^B} / \sqrt{\ln s}. \] (10)

For \(\alpha_s = 0.15\) one gets \(\omega_\beta^B \simeq 0.40\), much larger than the corresponding value for the cross section of soft hadronic processes \((\simeq 0.08)\), but in rough agreement with the power-growth in \(x\) observed for the gluon density in the proton at small \(x\) and large virtuality \(Q^2\).

The relation (10) shows that unitarity is violated, since the cross section overcomes the Froissart-Martin bound. This is obvious since, in the LLA, only a definite set of intermediate states, as we have seen, contributes to the \(s\)-channel unitarity relation. This means that the BFKL approach cannot be applied at asymptotically high energies. In order to identify the applicability region of the BFKL approach, it is necessary to know the scale of \(s\) and the argument of the running coupling constant, which are not fixed in the LLA.
4 BFKL in the NLA

In the NLA, the Regge form of the elastic amplitude (1) and of the production amplitudes (2), implied by gluon Reggeization, has been checked only in the first three orders of perturbation theory. In order to derive the BFKL equation in the NLA, gluon Reggeization is assumed to be valid to all orders of perturbation theory. It becomes important, therefore, to check the validity of this assumption.

In the NLA it is necessary to include into the unitarity relations contributions which differ from those in the LLA by having one additional power of $\alpha_s$ or one power less in $\ln s$. The first set of corrections is realized by performing, only in one place, one of the following replacements in the production amplitudes (2) entering the $s$-channel unitarity relation:

$$\omega^{(1)} \rightarrow \omega^{(2)}, \quad \Gamma_{cP}^{c\text{(Born)}} \rightarrow \Gamma_{cP}^{c\text{(1-loop)}}, \quad \gamma_{c_{i+1}}^{G_{i_{1}}\text{(Born)}} \rightarrow \gamma_{c_{i+1}}^{G_{i_{1}}\text{(1-loop)}}.$$

The second set of corrections consists in allowing the production in the $s$-channel intermediate state of one pair of particles with rapidities of the same order of magnitude, both in the central or in the fragmentation region (quasi-multi-Regge kinematics). This implies one replacement among the following ones in the production amplitudes (2) entering the $s$-channel unitarity relation:

$$\Gamma_{cP}^{c\text{(Born)}} \rightarrow \Gamma_{cP}^{c\{f\}P}, \quad \gamma_{c_{i+1}}^{G_{i_{1}}\text{(Born)}} \rightarrow \gamma_{c_{i+1}}^{Q_{i_{1}}Q\text{(Born)}}, \quad \gamma_{c_{i+1}}^{G_{i_{1}}\text{(Born)}} \rightarrow \gamma_{c_{i+1}}^{GG\text{(Born)}}.$$

Here $\Gamma_{cP}^{\{f\}P}$ stands for the production of a state containing an extra particle in the fragmentation region of the particle $P$ in the scattering off the Reggeon, $\gamma_{c_{i+1}}^{Q_{i_{1}}Q\text{(Born)}}$ and $\gamma_{c_{i+1}}^{GG\text{(Born)}}$ are the effective vertices for the production of a quark anti-quark pair and of a two-gluon pair, respectively, in the collision of two Reggeons.

The detailed program of next-to-leading corrections to the BFKL equation was formulated in Ref.15 and was carried out over a period of several years (for an exhaustive review, see Ref.14). It turns out that also in the NLA the amplitude for the high energy elastic process $A + B \rightarrow A' + B'$ can be represented as in Eq. (3) and in Fig. 3. The Green’s function obeys an equation with the same form as Eq. (4), with a kernel having the same structure as in Eq. (5). Here the “virtual” part of the kernel takes also the contribution from the gluon trajectory at 2-loop accuracy, $\omega^{(2)}$11, while the “real” part of the kernel takes the additional contribution from one-gluon production in the Reggeon-Reggeon collisions at 1-loop order, $K_{RRG}^{(1)}$16,17,18,19, from quark anti-quark pair production at Born level, $K_{RRQQ}^{(B)}$20, for the forward case, Ref.21 for the non-forward case) and from two-gluon pair production at Born level, $K_{RRGG}^{(B)}$ (Ref.22 for the forward case, Ref.23 for the non-forward, octet case, Ref.24 for the non-forward, singlet case).

The consistency between the representation (3) of the elastic amplitude, $A + B \rightarrow A' + B'$, derived from $s$-channel unitarity, for the part with gluon quantum numbers in the $t$-channel ($R = 8$, $\tau = -1$), and the representation (11) with one Reggeized gluon exchange in the $t$-channel (“bootstrap”) is of crucial importance in the NLA. In this approximation, indeed, gluon Reggeization was only assumed in order to derive the BFKL
equation. Moreover, the check of the bootstrap is also a (partial) check of the correctness of calculations which were performed mostly by one research group. In the NLA, the bootstrap leads to two conditions to be fulfilled\textsuperscript{12}, one on the NLA octet kernel, the other on the NLA octet impact factors, which have both been verified\textsuperscript{21,25,28}.

Another set of bootstrap conditions, proposed in Ref.\textsuperscript{26}, was introduced. They were called “strong” bootstrap conditions, since their fulfillment implies that of the bootstrap conditions considered so far. They constrain the form of the octet impact factors and of the octet BFKL kernel. Their fulfillment has been verified in Refs.\textsuperscript{27,28}. Recently it has been understood that these strong bootstrap conditions are a subset of those which arise from the request of consistency with $s$-channel unitarity of the inelastic amplitudes in the Regge form which enter the BFKL approach\textsuperscript{29}. Their fulfillment amounts to prove the gluon Reggeization in the NLA\textsuperscript{29}.

As for the exchange of vacuum quantum numbers in the $t$-channel, the NLA corrections to the BFKL kernel lead to a large correction to the BFKL Pomeron intercept\textsuperscript{30,31}. Indeed, one gets now\textsuperscript{14}

$$\int d^2\vec{q}_2 K(\vec{q}_1, \vec{q}_2)(\vec{q}_2^2)^{\gamma-1} = \frac{N_c \alpha_s(\vec{q}_2^2)}{\pi} \left( \chi(\gamma) + \frac{\alpha_s(\vec{q}_1^2) N_c}{\pi} \chi^{(1)}(\gamma) \right) (\vec{q}_1^2)^{\gamma-1},$$

where $\chi^{(1)}(\gamma)$ is a known function\textsuperscript{14}. We observe that now the argument of $\alpha_s$ is not undefined as in the LLA case. The relative correction $\mathcal{R}(\gamma)$ defined as $\chi^{(1)}(\gamma) = -\mathcal{R}(\gamma) \chi(\gamma)$ in the point $\gamma = 1/2 + i\nu$ turns out to be very large\textsuperscript{30,31},

$$\mathcal{R}(1/2) \approx 6.46 + 0.05 \frac{n_f}{N_c} + 0.96 \frac{n_f}{N_c^3}$$

and leads to a NLA Pomeron intercept

$$\omega_P = \omega_P^B (1 - 2.4 \omega_P^B), \quad \omega_P^B = 4 \ln 2 N_c \alpha_s(\vec{q}_1^2) / \pi .$$

A lot of papers have been devoted to the problem of this large correction (see, for instance\textsuperscript{32}). As anticipated in the Introduction, the understanding of the way to treat these large corrections can be helped by considering the full NLA amplitude of hard QCD physical processes, instead of limiting the attention to the unphysical BFKL Green’s function. The construction of a physical NLA amplitude in the BFKL approach, however, calls for the determination of the NLA impact factors of the colorless particles involved in the process. This determination can be achieved by means of perturbation theory only in case a hard enough scale exists, such as a large photon virtuality or the mass of a heavy quark.

5 The inclusion of impact factors: the amplitude for the electroproduction of two light vector mesons

Recently, the impact factor for the transition from a virtual photon with longitudinal polarization to a light vector meson with longitudinal polarization was calculated with
It can be used, together with the NLA BFKL Green’s function to build with NLA accuracy the amplitude for the production of two light vector mesons ($V = \rho^0, \omega, \phi$) in the collision of two virtual photons. In the kinematics $s \gg Q_{1,2}^2 \gg \Lambda_{QCD}^2$, where $Q_{1,2}^2$ are the photon virtualities, other helicity amplitudes are indeed power suppressed, with a suppression factor $\sim m_V/Q_{1,2}$ and the light vector meson mass can be put equal to zero.

As we have seen before, the forward amplitude may be presented as follows

$$Im_s(A) = \frac{s}{(2\pi)^2} \int \frac{d^2 \vec{q}_1}{\vec{q}_1^2} \Phi_1(\vec{q}_1, s_0) \int \frac{d^2 \vec{q}_2}{\vec{q}_2^2} \Phi_2(-\vec{q}_2, s_0) \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \left( \frac{s}{s_0} \right)^\omega G_\omega(\vec{q}_1, \vec{q}_2).$$

This representation for the amplitude is valid with NLA accuracy. Here $\Phi_1(\vec{q}_1, s_0)$ and $\Phi_2(-\vec{q}_2, s_0)$ are the impact factors describing the transitions $\gamma^*(p) \to V(p_1)$ and $\gamma^*(p') \to V(p_2)$, respectively. The Green’s function in (11) obeys the BFKL equation (7). The scale $s_0$ is artificial and must disappear in the full expression for the amplitude at each fixed order of approximation\(^*\). Using the result for the meson NLA impact factor such cancellation was demonstrated explicitly in Ref.\(^6\) for the process in question.

The impact factors are known as an expansion in $\alpha_s$ (see Ref.\(^6\))

$$\Phi_{1,2}(\vec{q}) = \alpha_s D_{1,2} \left[ C_{1,2}^{(0)}(\vec{q}^2) + \tilde{\alpha}_s C_{1,2}^{(1)}(\vec{q}^2) \right], \quad D_{1,2} = -\frac{4\pi \alpha_s f_V}{N_c Q_{1,2}} \sqrt{N_c^2 - 1},$$

where $f_V$ is the meson dimensional coupling constant ($f_\rho \approx 200$ MeV) and $\alpha_q$ should be replaced by $e/\sqrt{2}$, $e/(3\sqrt{2})$ and $-e/3$ for the case of $\rho^0, \omega$ and $\phi$ meson production, respectively.

Using the known impact factors and the NLA Green’s function, spectrally decomposed on the basis of the LLA kernel eigenfunctions, one gets\(^8\)

$$\frac{Im_s(A)}{D_1 D_2} = \frac{s}{(2\pi)^2} \int_{-\infty}^{+\infty} d\nu \left( \frac{s}{s_0} \right) \tilde{\alpha}_s(\mu_R) \chi(\nu) \alpha_s^2(\mu_R) c_1(\nu)c_2(\nu)$$

$$\times \left[ 1 + \tilde{\alpha}_s(\mu_R) \left( \frac{c_1^{(1)}(\nu)}{c_1(\nu)} + \frac{c_2^{(1)}(\nu)}{c_2(\nu)} \right) \right]$$

$$+ \tilde{\alpha}_s^2(\mu_R) \ln \left( \frac{s}{s_0} \right) \left[ \chi(\nu) + \beta_0 \frac{3}{8N_c} \chi(\nu) \left[ -\chi(\nu) + \frac{10}{3} + i \frac{d\ln(c_2(\nu))}{d\nu} + 2 \ln(\mu_R^2) \right] \right],$$

where $c_{1,2}(\nu)$ ($c_{1,2}(\nu)$) are the coefficient of the expansion of the LLA (NLA) impact factors for the two photons in the basis formed by the eigenfunctions of the LLA kernel.

\(^*\)For understanding the reasons why this scale was introduced and enters also the definition of the impact factors, see Refs.\(^\text{[12-14]}\).
All the other functions of $\nu$ in (12) are known (see Ref.8). It is possible to write this amplitude in the form of a series, as

$$\frac{Q_1 Q_2 \mathcal{I} m_s A}{D_1 D_2 s} = \frac{1}{(2\pi)^2} \alpha_s(\mu_R)^2 \times \left[ b_0 + \sum_{n=1}^{\infty} \bar{\alpha}_s(\mu_R)^n b_n \left( \ln \left( \frac{s}{s_0} \right)^n + d_n(s_0, \mu_R) \ln \left( \frac{s}{s_0} \right)^{n-1} \right) \right],$$

(14)

where the $b_n$ and $d_n$ coefficients can be easily determined by comparison with (13). One should stress that both representations of the amplitude (14) and (13) are equivalent with NLA accuracy, since they differ only by next-to-NLA (NNLA) terms. It is easily seen from Eq. (14) that the amplitude is independent in the NLA from the choice of energy and strong coupling scales. Indeed, with the required accuracy,

$$\bar{\alpha}_s(\mu_R) = \bar{\alpha}_s(\mu_0) \left( 1 - \frac{\bar{\alpha}_s(\mu_0) \beta_0}{4N_c} \ln \left( \frac{\mu_R^\beta}{\mu_0^\beta} \right) \right),$$

(15)

and therefore terms $\bar{\alpha}_s^n \ln^{n-1} s \ln s_0$ and $\bar{\alpha}_s^n \ln^{n-1} s \ln \mu_R$ cancel in (14).

Here some numerical results are presented for the amplitude given in Eq. (14) for the $Q_1 = Q_2 \equiv Q$ kinematics, i.e. in the “pure” BFKL regime. The other interesting regime, $Q_1 \gg Q_2$ or vice-versa, where collinear effects could come heavily into the game, will not be considered here. In the numerical analysis presented below the series in the R.H.S. of Eq. (14) has been truncated to $n = 20$, after having verified that this procedure gives a very good approximation of the infinite sum for $Y \leq 10$.

The $b_n$ and $d_n$ coefficients for $n_f = 5$ and $s_0 = Q^2 = \mu_R^2$ can be calculated numerically and turn to be the following ones:

$$
\begin{align*}
    b_0 &= 17.0664 \\
    b_1 &= 34.5920 \\
    b_2 &= 40.7609 \\
    b_3 &= 33.0618 \\
    b_4 &= 20.7467 \\
    b_5 &= 10.5698 \\
    b_6 &= 4.54792 \\
    b_7 &= 1.69128 \\
    b_8 &= 0.554475
\end{align*}
$$

(16)

These numbers make visible the effect of the NLA corrections: the $d_n$ coefficients are negative and increasingly large in absolute values as the perturbative order increases. In this situation the optimization of perturbative expansion, in our case the choice of the renormalization scale $\mu_R$ and of the energy scale $s_0$, becomes an important issue. Below the principle of minimal sensitivity (PMS)8 is adopted. Usually PMS is used to fix the value of the renormalization scale for the strong coupling. Here this principle is used in a broader sense, requiring the minimal sensitivity of the predictions to the change of both the renormalization and the energy scales, $\mu_R$ and $s_0$. More precisely, we replace in (14) $\ln(s/s_0)$ with $Y - Y_0$, where $Y = \ln(s/Q^2)$ and $Y_0 = \ln(s_0/Q^2)$, and study the dependence of the amplitude on $Y_0$.

It has been found that, for several fixed values of $Q^2$ and $n_f$, at any value of the energy $Y$ there are wide regions of values of the parameters $Y_0$ and $\mu_R$ where the amplitude is
practically flat, this allowing to assign a value for the amplitude for each energy $Y$. In Fig. 1 the behavior of the amplitude for the $Q^2=24$ GeV$^2$ and $n_f = 5$ is presented. This result shows that, although the NLA corrections are large and with opposite sign with respect to the LLA result, it is possible to give a definite meaning to a perturbative series, by the request of renorm-invariance and of stability with respect to the artificial parameter $s_0$.

It turns out, however, that the optimal values for $Y_0$ and $\mu_R$ are far from the “natural” choices $s_0 \simeq \mu_R^2 \simeq Q^2$. In the case of $Q^2=24$ GeV$^2$ and $n_f = 5$, we have, indeed, $Y_0 \simeq 2$ and $\mu_R \simeq 10Q$. These values look unnatural since there is no other scale for transverse momenta in the problem at question except $Q$. Moreover one can guess that at higher orders the typical transverse momenta are even smaller than $Q$ since they “are shared” in the many-loop integrals and the strong coupling grows in the infrared. The large values of $\mu_R$ found is not an indication of the appearance of a new scale, but is rather a manifestation of the nature of the BFKL series. The fact is that NLA corrections are large and then, necessarily, since the exact amplitude should be renorm- and energy scale invariant, the NNLA terms should be large and of the opposite sign with respect to the NLA. It can be guessed that if the NNLA corrections were known and if PMS were applied to the amplitude constructed as LLA + NLA-corrections + NNLA-corrections, more “natural” values of $\mu_R$ would be obtained.

It is a great pleasure for me to contribute to this volume in honor of Adriano Di Giacomo. His dedication to science is an example to me.

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