GEOMETRY OF SINGULARITIES OF A PINCHUK’S MAP

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Abstract. In [VV], the authors associated to a nonvanishing jacobian polynomial map \(F : \mathbb{C}^2 \to \mathbb{C}^2\) singular varieties whose intersection homology describes the geometry of singularities of the map. We determine explicitly such a variety associated to the Pinchuk’s map given in [P] and calculate its intersection homology. That provides a counter-example for the “real version” of Anna and Guillaume Valette’s result in [VV].

1. INTRODUCTION

Let \(F : \mathbb{K}^n \to \mathbb{K}^n\), where \(\mathbb{K} = \mathbb{C}\) or \(\mathbb{K} = \mathbb{R}\), be a polynomial map and let us denote by \(JF(x)\) the Jacobian matrix of \(F\) at \(x \in \mathbb{K}^n\). The determinant \(\det(JF(x))\) is a polynomial map from \(\mathbb{K}^n\) to \(\mathbb{K}\). In 1939, O. H. Keller [K] stated a famous conjecture known nowadays as the Jacobian Conjecture, whose statement is the following:

“A polynomial map \(F : \mathbb{K}^n \to \mathbb{K}^n\) is nowhere vanishing jacobian, i.e. \(\det(JF(x)) \neq 0\), for every \(x \in \mathbb{K}^n\), if and only if it is a polynomial automorphism.”

The asymptotic variety of a given polynomial map \(F : \mathbb{K}^n \to \mathbb{K}^n\) is the smallest set \(S_F\) such that the map \(F : \mathbb{K}^n \setminus F^{-1}(S_F) \to \mathbb{K}^n \setminus S_F\) is proper. The Jacobian Conjecture reduces to show that the asymptotic variety of a polynomial map \(F : \mathbb{K}^n \to \mathbb{K}^n\) with nonzero constant Jacobian is empty.

In the complex case, the Jacobian Conjecture remains open today even for the dimension 2. However, in the 2-dimensional real case, the Real Jacobian Conjecture was solved negatively by Pinchuk [P] in the year 1994. In fact, Pinchuk provided counter-examples by giving a class of polynomial maps \(F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2\) satisfying the condition \(\det(JF(x, y)) > 0\) for every \((x, y) \in \mathbb{R}^2\) but \(F\) is not injective. Let us recall his construction: given \((x, y) \in \mathbb{R}^2\), let us denote

\[ t = xy - 1, \quad h = t(xt + 1), \quad f = (xt + 1)^2(t^2 + y), \quad P = f + h.\]

Notice that \(\deg h = 5\) and \(\deg f = 10\) then \(\deg P = 10\). The polynomial \(Q\) varies for different Pinchuk’s maps \((P, Q)\) but \(Q\) always has the form

\[ Q = -t^2 - 6th(h + 1) - u(f, h) \]

where \(u\) is an auxiliary polynomial in \(f\) and \(h\) is chosen so that

\[ J(P, Q) = t^2 + (t + (f(13 + 15h))^2 + f^2.\]

Then \(J(P, Q)(x, y) > 0\), for every \((x, y) \in \mathbb{R}^2\) since if \(t = 0\) then \(f = y = \frac{1}{2} \neq 0\). The original Pinchuk map in [P] is defined by choosing

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\[ Q = -r^2 - 6th(h + 1) - 170fh - 91h^2 - 195fh^2 - 69h^3 - 75fh^3 - \frac{75}{4}h^4. \]

The degree of this Pinchuk’s map is 25 and it is the minimum degree for any Pinchuk’s map.

The geometry of Pinchuk’s maps is a desired object to study, since the Jacobian condition “\( \det(JF(x)) \neq 0 \), for every \( x \in \mathbb{R}^n \)” in the complex case becomes “\( \det(JF(x)) \) is a non-zero constant polynomial” in the real case. In other words, the study of polynomial maps \( F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) with the condition \( \det(JF(x)) \equiv \text{const.} \neq 0 \) coincides with the study of the 2-dimensional Complex Jacobian Conjecture.

In the series of papers [C1, C2, C3, C4, C5], A. Campbell studied the behaviours of the Pinchuk’s map given in [P]. In this paper, we consider this Pinchuk’s map using the results of A. Campbell. The paper is inspired by the paper [VV]: in the year 2010, Anna Valette and Guillaume Valette gave a new approach to study the Complex Jacobian Conjecture. Given a polynomial map \( F : \mathbb{C}^n \rightarrow \mathbb{C}^n \), they constructed some real 2\( n \)-dimensional pseudomanifolds \( V_F \) contained in some euclidean space \( \mathbb{R}^n \), where \( n > 2n \) and such that the singular loci of these pseudomanifolds are contained in \((S_F \times K_0(F)) \times \{0_{\mathbb{R}^{2n}}\}\) where \( K_0(F) \) is the set of critical values of \( F \). In the case of dimension 2, they prove that for a nonvanishing jacobian polynomial map \( F : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) (i.e, \( K_0(F) = \emptyset \)), the condition “\( S_F = \emptyset \)” is equivalent to the condition “the intersection homology in dimension two and with any perversity of any constructed pseudomanifold \( V_F \) is trivial” (Theorem 3.2 in [VV]). This result is generalized in the case of higher dimension in [NVV]. In this paper, singular varieties \( V_F \) constructed by Anna and Guillaume Valette are called Valette varieties. We can also construct “Valette varieties” associated to real polynomial maps \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) (see Remark 2.7 of [VV] or Proposition 3.8 of [NVV]). Two natural questions then arise:

1) How are the behaviours of Valette varieties associated to the Pinchuk’s map given in [P]?

2) Is there a “real version” of Anna and Guillaume Valette’s result, i.e if \( F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is a nowhere vanishing jacobian polynomial map then the condition \( S_F = \emptyset \) is equivalent to the condition \( IH^1_0(V_F) = 0 \)? (Notice that in this case, the dimension of \( V_F \) is 2, then there is only one perversity: the zero perversity).

Let us denote the Pinchuk’s map given in [P] by \( P \). Since \( P \) satisfies the condition \( \det(\mathcal{P}(x, y)) > 0 \), for every \( (x, y) \in \mathbb{R}^2 \), but \( P \) is not bijective then the asymptotic variety \( S_P \) is non-empty. We describe in this paper a Valette variety \( V_P \) associated to the Pinchuk’s map \( P \) and we calculate its intersection homology. The main result is the Theorem 5.3: the intersection homology of \( V_P \) in dimension one and with the zero perversity is trivial. The main tool to prove this result is the description of the behaviours of the asymptotic variety and the “asymptotic flower” (the inverse image of the asymptotic variety) of the Pinchuk’s map \( P \) described in the series of A. Campbell’s papers. The result describes the geometry of singularities of the Pinchuk’s map. It provides also a counter-example for the “real version” of Anna and Guillaume Valette’s result in [VV].

2. Intersection homology

Given a variety \( V \) in \( \mathbb{R}^2 \), we denote by \( \text{Reg}(V) \) and \( \text{Sing}(V) \) the regular and singular loci of the variety \( V \), respectively. Moreover, \( \overline{V} \) will stand for the topological closure of \( V \). The boundary of \( V \) will be denoted by \( \partial V \).
We briefly recall the definition of intersection homology. For details, the readers can see [GM1, GM2] or [B].

**Definition 2.1.** Let $V$ be an $m$-dimensional variety in $\mathbb{R}^k$. A locally topologically trivial stratification of $V$ is the data of a finite filtration

$$V = V_m \supset V_{m-1} \supset \cdots \supset V_0 \supset V_{-1} = \emptyset$$

such that

1) for every $i$, the set $S_i = V_i \setminus V_{i-1}$ is either empty or a topological manifold of dimension $i$,

2) for every $x \in V_i \setminus V_{i-1}$, for all $i \geq 0$, there is an open neighborhood $U_x$ of $x$ in $V$, a stratified set $L_i$ and a homeomorphism

$$h : U_x \to (0, 1)^i \times cL_i,$$

such that $h$ maps the strata of $U_x$ (induced stratification) onto the strata of $(0, 1)^i \times cL_i$ (product stratification).

A connected component of $S_i$ is called a stratum of $V$.

**Definition 2.3.** A perversity is an $(m+1)$-uple of integers $\bar{p} = (p_0, p_1, p_2, \ldots, p_m)$ such that $p_0 = p_1 = p_2 = 0$ and $p_{r+1} \in \{p_r, p_r + 1\}$ for $2 \leq r \leq m - 1$.

The perversity $\bar{0} = (0, \ldots, 0)$ is called the zero perversity.

In this paper, we consider the groups of $i$-dimensional PL chains $C_i(V)$ and we denote also by $c$ the support of a chain $c$. A chain $c$ is $(\bar{p}, i)$-allowable if $\dim(c \cap V_{m-r}) \leq i - r + p_r$, for all $r \geq 0$. It is easy to see that this condition holds always when $r = 0$. Define $IC^\bar{p}_i(V)$ to be the $\mathbb{R}$-vector subspace of $C_i(V)$ consisting of the chains $c$ such that $c$ is $(\bar{p}, i)$-allowable and its boundary $\partial c$ is $(\bar{p}, i - 1)$-allowable, that means

$$IC^\bar{p}_i(V) = \left\{ c \in C_i(V) : \begin{array}{l}
\dim(c \cap V_{m-r}) \leq i - r + p_r \\
\dim(\partial c) \cap V_{m-r} \leq (i - 1) - r + p_r
\end{array}, \forall r \geq 1 \right\}.$$

**Definition 2.5.** The $i^{th}$ intersection homology group with perversity $\bar{p}$, denoted by $IH^\bar{p}_i(V)$, is the $i^{th}$ homology group of the chain complex $IC^\bar{p}_i(V)$.

Recall that a pseudomanifold $V$ is a variety such that its singular locus is of codimension at least 2 in $V$ and its regular locus is dense in $V$. Goresky and MacPherson [GM1, GM2] proved that the intersection homology of a pseudomanifold does not depend on a choice of a locally topologically trivial stratification (see also [B]).

In this article, we consider the intersection homology with real coefficients, i.e. the intersection homology groups $IH^\bar{p}_i(V, \mathbb{R})$. Moreover, we consider the groups of PL chains with both compact supports and closed supports. Given a triangulation of $V$, recall that a chain with compact support is a chain of the form $\sum c_{\sigma}$ for which all coefficients $c_{\sigma} \in \mathbb{Z}$ are zero but a finite number, where $\sigma$ are $i$-simplices. A chain with closed support is a locally finite linear combination $\sum c_{\sigma}$ with integer coefficients $c_{\sigma} \in \mathbb{Z}$. Notice that if $c$ is a chain with compact support, then $c$ is also a chain with closed support.

The homology groups with closed supports are called Borel Moore Homology, or homology groups of deuxième espèce in [Car]. The intersection homology groups with closed supports are
denoted by $IH^b_{cl}(V)$. The corresponding intersection homology groups with compact supports are denoted by $IH^b_{c}(V)$.

3. **The asymptotic variety**

Let $F: \mathbb{K}^n \to \mathbb{K}^n$ be a polynomial map, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. We denote by $S_F$ the set of points at which the map $F$ is not proper, i.e.

$$S_F = \{ a \in \mathbb{K}^n \text{ such that } \exists \{\xi_k\} \subset \mathbb{K}^n, |\xi_k| \to \infty, F(\xi_k) \to a \},$$

and call it the *asymptotic variety*. Notice that, by $|\xi_k|$ we mean the usual Euclidean norm of $\xi_k$ in $\mathbb{K}^n$. In the complex case, one has:

**Theorem 3.1 ([J1]).** If $F: \mathbb{C}^n \to \mathbb{C}^n$ is a generically finite polynomial map, then $S_F$ is either an $(n - 1)$ pure dimensional algebraic variety or the empty set.

Recall that one says that $F$ is generically finite if there exists a subset $U \subset \mathbb{C}^n$ dense in the target space $\mathbb{C}^n$ such that for any $a \in U$, the fiber $F^{-1}(a)$ is finite.

In the real case, if the asymptotic variety is non-empty, then its dimension can be any integer between 1 and $(n - 1)$:

**Theorem 3.2 ([J2]).** Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a non-constant polynomial map. Then the set $S_F$ is a closed, semi-algebraic set and for every non-empty connected component $S \subset S_F$, we have $1 \leq \dim S \leq n - 1$.

**Example 3.3.** Let $F: \mathbb{C}^3 \to \mathbb{C}^3$ such that

$$F(x, y, z) = (x, y, (x^2 + y^2)z).$$

It is easy to see that $F$ is generically finite since the subset $U = \{ (\alpha, \beta, \gamma) \in \mathbb{C}^3 : \alpha^2 + \beta^2 \neq 0 \}$ is dense in the target space and $F|_{F^{-1}(U)} : F^{-1}(U) \to U$ is bijective.

We determine now the asymptotic variety of $F$. Assume that $\{\xi_k\} = \{ (x_k, y_k, z_k) \}$ is a sequence tending to infinity in the source space such that its image does not tend to infinity. Hence the coordinates $x_k$ and $y_k$ cannot tend to infinity. Therefore, $z_k$ must tend to infinity. Since $(x_k^2 + y_k^2)z_k^2$ cannot tend to infinity, then $(x_k^2 + y_k^2)$ must tend to zero. Consequently, the asymptotic variety of $F$ is the algebraic variety of equation $\alpha^2 + \beta^2 = 0$.

As an illustration, let us take the sequence $\{(a + 1/k, b + 1/k, c)\}$ tending to infinity in the source space such that $a^2 + b^2 = 0$. Then its image tends to $(a, b, 2(a + b)c)$. This point belongs to the hypersurface $\alpha^2 + \beta^2 = 0$.

Notice that, if we replace $\mathbb{C}$ by $\mathbb{R}$, we get the same equation for the asymptotic variety. However, the equation $\alpha^2 + \beta^2 = 0$ reduce to a line in $\mathbb{R}^3$, which is not anymore a hypersurface.

4. **VALETTE VARIETIES**

4.1. **Construction.** Valette varieties $V_F$ are constructed originally in [VV]. In this section, we recall briefly this construction: Let $F: \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map, the construction of Valette varieties associated to $F$ consists of the following steps:

1) We consider $F$ as a real map $F: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Determine the set of critical points $\text{Sing} F$ of $F$. 


2) Choose a covering \( \{U_1, \ldots, U_p\} \) of \( M_F = \mathbb{R}^{2n} \setminus \text{Sing}(F) \) by semi-algebraic open subsets (in \( \mathbb{R}^{2n} \)) such that on every element of this covering, the map \( F \) induces a diffeomorphism onto its image. Choose semi-algebraic closed subsets \( V_i \subset U_i \) (in \( M_F \)) which cover \( M_F \) as well.

3) For each \( i = 1, \ldots, p \), choose a Nash function \( \psi_i : M_F \to \mathbb{R} \), such that:
   (a) \( \psi_i \) is positive on \( V_i \) and negative on \( M_F \setminus U_i \).
   (b) \( \psi_i(\xi) \) tends to zero whenever \( \{\xi\} \subset M_F \) tends to infinity or to a point in \( \text{Sing}F \).

Recall that a Nash function \( \psi : W \to \mathbb{R} \) defined on \( W \subset \mathbb{R}^{2n} \), is an analytic function such that there exists a non trivial polynomial \( P : W \times \mathbb{R} \to \mathbb{R} \) such that \( P(x, \psi(x)) = 0 \).

4) Determine the closure of the image of \( M_F \) by \( (F, \psi_1, \ldots, \psi_p) \), we obtain a Valette variety associated to \( F \) and the chosen covering and Nash functions, that means

\[
V_F := (F, \psi_1, \ldots, \psi_p)(M_F).
\]

Notice that we may have a lot of ways to choose the covering \( \{U_1, \ldots, U_p\} \). Moreover, with each covering \( \{U_1, \ldots, U_p\} \), we may have a lot of choices of Nash functions \( \psi_1, \ldots, \psi_p \). Then we may have more than one Valette variety associated to a given polynomial map. However, the singular locus of any Valette variety is always contained in \( (K_0(F) \cup S_F) \times \{0_{\mathbb{R}^v}\} \), which depends only on the given map \( F \).

In the real case, i.e. if \( F : \mathbb{R}^n \to \mathbb{R}^n \), the real map in the first step is replaced by the map \( F \) itself and the construction follows the same process. However, in this case, Valette varieties are no longer pseudomanifolds in general. They are simply real semi-algebraic varieties (see Remark 2.7 of [VV] or Proposition 3.8 of [NVV]).

The idea of the construction of Valette varieties can be understood in the following way: that is to "separate" the images \( F(U_i) \) of the covering \( \{U_1, \ldots, U_p\} \) by adding to each \( F(U_i) \) one dimension, for \( i = 1, \ldots, p \), then "glue" the images \( F(U_i) \) together along \( (K_0(F) \cup S_F) \times \{0_{\mathbb{R}^v}\} \), this is done by the fact that we take the closure of \( (F, \psi_1, \ldots, \psi_p)(M_F) \).

Let us illustrate this construction by an example.

4.2. An example.

**Example 4.1.** Let \( F : \mathbb{R}_x \to \mathbb{R}_s \) be a polynomial map defined by \( F(x) = x^2 \). Notice that, by \( \mathbb{R}_x \) and \( \mathbb{R}_s \), we denote the source space and the target space, respectively. By an easy calculation, we have

\[
S_F = \emptyset, \quad \text{Sing}(F) = \{0\}, \quad K_0(F) = \{0\}.
\]

The set \( \text{Sing}(F) \) divide \( \mathbb{R}_x \) into two subsets:

\[
U_1 = \{x \in \mathbb{R}_x : x > 0\}, \quad U_2 = \{x \in \mathbb{R}_x : x < 0\}
\]

and \( F(U_1) = F(U_2) = \{a \in \mathbb{R}_s : a > 0\} \).

These subsets \( U_1 \) and \( U_2 \) are semi-algebraic open subsets in \( \mathbb{R}_x \) and on each \( U_\nu \) for \( i = 1, 2 \), the map \( F \) induces a diffeomorphism onto its image. In this case, we can choose \( \{U_1, U_2\} \) as a covering of \( M_F = \mathbb{R} \setminus \text{Sing}F = \mathbb{R} \setminus \{0\} \).
We choose \( V_1 = U_1 \) and \( V_2 = U_2 \). In this case, \( V_1 \) and \( V_2 \) are closed subsets of \( M_F \) and they cover \( M_F \) as well. Let us choose the Nash functions:

\[
\psi_1 : M_F \to \mathbb{R}, \quad \psi_1(x) := \frac{x}{1 + x^2} \quad \text{and} \quad \psi_2 : M_F \to \mathbb{R}, \quad \psi_2(x) := -\frac{x}{1 + x^2}.
\]

We see that \( \psi_i \) is positive on \( V_i = U_i \) and negative on \( M_F \setminus U_i \), for \( i = 1, 2 \) and \( j = \{1, 2\} \setminus \{i\} \). Moreover, if \( \{x_k\} \) is a sequence tending to infinity or zero, then \( \psi_i(x_k) \) tends to zero. Now, in order to determine the Valette variety associated to the given \( F \) and the chosen Nash functions, we need to calculate the closure of \( (F, \psi_1, \psi_2)(M_F) \), that means, we have to calculate

\[
(F, \psi_1, \psi_2)(M_F) = \left\{ \left( x^2, \frac{x}{1 + x^2}, -\frac{x}{1 + x^2} \right) : x \in M_F \right\},
\]

or

\[
V_F = \left\{ \left( x^2, \frac{x}{1 + x^2}, -\frac{x}{1 + x^2} \right) : x \in \mathbb{R} \setminus \{0\} \right\} \cup \{ (\alpha, 0, 0) : \exists \{x_k\} \subset \mathbb{R} \text{ such that } x_k \to 0 \text{ and } F(x_k) \to \alpha \}.
\]

It is easy to see that if a sequence \( \{x_k\} \) tends to zero, then \( (F, \psi_1, \psi_2)(x_k) \) tends to the origin of \( \mathbb{R}^3 \), which coincides with \( (F, \psi_1, \psi_2)(0) \). Hence, the closure of \( (F, \psi_1, \psi_2)(M_F) \) is, in fact, the set \( (F, \psi_1, \psi_2)(\mathbb{R}^2) \), which is smooth. Then a Valette variety \( V_F \) associated to \( F = x^2 \) is smooth and it is easily graphed \(^1\) as in the Figure 1.

\[\text{Figure 1. A Valette variety of the polynomial map } F(x) = x^2\]

Roughly speaking, in this case \( F(U_1) \) and \( F(U_2) \) coincide, then we separate \( F(U_1) \) and \( F(U_2) \) by adding to each \( F(U_i) \) one more dimension (thanks to the Nash functions \( \psi_1 \) and \( \psi_2 \)). That means, we embed \( F(U_1) \) em \( \mathbb{R} \oplus \mathbb{R} \), and \( F(U_2) \) em \( \mathbb{R} \oplus \mathbb{R} \). We get the variety \( F(U_1) \cup F(U_2) \) in \( \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \). Then, we glue \( F(U_1) \) and \( F(U_2) \) along \( (K_0(F) \cup S_f) \times \{0_{\mathbb{R}^2}\} \), i.e., glue \( F(U_1) \) and \( F(U_2) \) at the origin of \( \mathbb{R}^3 \). The Valette variety considered is a space parabol in \( \mathbb{R}^3 \).

\(^1\)The figure is graphed by the Software online: http://www.math.uri.edu/bkaskosz/flashmo/parcur/
This example shows that in general the singular locus of a Valette variety does not always coincide with $K_0(F) \cup S_F$.

4.3. **Intersection homology of Valette singular varieties.** The intersection homology of Valette semi-algebraic pseudomanifolds $V_F$ associated to a given nonvanishing jacobian polynomial map $F : \mathbb{C}^2 \to \mathbb{C}^2$ describes the geometry of singularities at infinity of the map (Theorem 3.2 of [VV]). This result is generalized in [NVV] in the case of higher dimensions. We simply recall these results. Notice that the following results hold for any Valette variety associated to a given polynomial map. Notice also that the following theorems hold for the intersection homology groups with both compact supports and closed supports.

**Theorem 4.2 ([VV]).** Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map with nowhere vanishing Jacobian. The following conditions are equivalent:

1. $F$ is non proper.
2. $IH^p_F(V_F, \mathbb{R}) \neq 0$ for any (or some) perversity $p$.

**Theorem 4.3.** [NVV] Let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with nowhere vanishing Jacobian. If $\text{rank}_\mathbb{C}(D\hat{F}_i)|_{α} > n - 2$, where $\hat{F}_i$ is the leading form of $F_i$ and $D\hat{F}_i$ is the (first) derivative of $\hat{F}_i$, then the following conditions are equivalent:

1. $F$ is non proper.
2. $IH^p_F(V_F, \mathbb{R}) \neq 0$ for any (or some) perversity $p$.
3. $IH^2_{2n-2}(V_F, \mathbb{R}) \neq 0$, for any (or some) perversity $p$.

5. **Intersection Homology of a Valette variety associated to the Pinchuk’s map $\mathcal{P}$**

5.1. **A lemma.** In order to prove the main result of this paper (Theorem 5.3), we need the following lemma:

**Lemma 5.1.** Let $F : \mathbb{K}^n \to \mathbb{K}^n$ be a polynomial map, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Let $V_F \subset \mathbb{R}^{2n+p}$ if $\mathbb{K} = \mathbb{C}$, or $V_F \subset \mathbb{R}^{n+p}$ if $\mathbb{K} = \mathbb{R}$, be a Valette variety associated to $F$. Then $V_F$ is non-compact and

$$V_F = (F, \psi_1, \ldots, \psi_p)(M_F) \cup ((K_0(F) \cup S_F) \times \{0_{R^p}\}).$$

**Proof.** Assume that the image of any sequence (in the source space) tending to infinity converges to a point in $S_F$. Then $F$ is a rational polynomial map, which is a contradiction. Thus, there exists at least a sequence tending to infinity such that its image tends to infinity. By the definition of Valette varieties, $V_F$ must be non-compact.

We prove at first that the subset $S_F \times \{0_{R^p}\}$ is contained in $V_F$. Let us take a point $α \in S_F$. Then there exists a sequence $\{ξ_k\} \subset \mathbb{K}^n$ (in the source space) tending to infinity such that $F(ξ_k)$ tends to $α$. Let $\{η_k\}$ be a subsequence of $\{ξ_k\}$ consisting of all points which do not belong to $\text{Sing}F$, then $\{η_k\} \subset M_F$. Notice that the sequence $\{η_k\}$ also tends to infinity. Moreover, $F(η_k)$ tends to $α$. By the construction of Valette varieties (see section 4.1), the Nash function $ψ_i(η_k)$ tends to zero, for $i = 1, \ldots, p$. Then $(F, ψ_1, \ldots, ψ_p)(η_k)$ tends to $(α, 0_{R^p}) \in S_F \times \{0_{R^p}\}$. Hence $(α, 0_{R^p})$ is an accumulation point of $(F, ψ_1, \ldots, ψ_p)(M_F)$. By the definition of $V_F$, the point $(α, 0_{R^p})$ belongs to $V_F$. Consequently, the subset $S_F \times \{0_{R^p}\}$ is contained in $V_F$. 


We prove now that $K_0(F) \times \{0_{\mathbb{R}^p}\}$ is contained in $V_F$. For this, let us take a point $\alpha \in K_0(F)$. Then there exists a point $x \in \operatorname{Sing}F$ such that $\alpha = F(x)$. Take a sequence $\{\xi_k\} \subset M_F$ such that $\xi_k$ tends to $x$. Again, by the construction of Valette varieties, the image by $\xi_k$ of the chosen Nash function $\psi_i$ must tend to zero, for $i = 1, \ldots, p$. Then $(F, \psi_1, \ldots, \psi_p)(\xi_k)$ tends to $(F(x), 0_{\mathbb{R}^p}) = (\alpha, 0_{\mathbb{R}^p})$. That means $(\alpha, 0_{\mathbb{R}^p})$, which is a point in $K_0(F) \times \{0_{\mathbb{R}^p}\}$, is an accumulation point of $(F, \psi_1, \ldots, \psi_p)(M_F)$. Consequently, $(\alpha, 0_{\mathbb{R}^p})$ belongs to $V_F$.

It is obvious that $(F, \psi_1, \ldots, \psi_p)(M_F) \subset V_F$, then $(F, \psi_1, \ldots, \psi_p)(M_F) \cup ((K_0(F) \cup S_F) \times \{0_{\mathbb{R}^p}\}) \subset V_F$. We prove now that $V_F \subset (F, \psi_1, \ldots, \psi_p)(M_F) \cup ((K_0(F) \cup S_F) \times \{0_{\mathbb{R}^p}\})$. We need to prove that if $\beta \in \mathbb{R}^{2n+p}$ is a point of $V_F$ but $\beta$ does not belong to $(F, \psi_1, \ldots, \psi_p)(M_F)$ then $\beta$ belongs to $(K_0(F) \cup S_F) \times \{0_{\mathbb{R}^p}\}$. There exists a sequence $\{\xi_k\} \subset M_F$ such that $(F, \psi_1, \ldots, \psi_p)(\xi_k)$ tends to $\beta$. Assume that $\xi_k$ tends to $x$. If $x$ is neither infinity nor a point in $\operatorname{Sing}F$, then $x$ belongs to $M_F$. Since $F$ is a polynomial map and $\psi_1, \ldots, \psi_p$ are Nash functions, then $(F, \psi_1, \ldots, \psi_p)(\xi_k)$ tends to $(F, \psi_1, \ldots, \psi_p)(x)$. Hence $\beta = (F, \psi_1, \ldots, \psi_p)(x)$ and $\beta$ belongs to $(F, \psi_1, \ldots, \psi_p)(M_F)$, that provides a contradiction. Then $\xi_k$ must tend to infinity or to a point of $\operatorname{Sing}F$. If $\xi_k$ tends to infinity, then since $(F, \psi_1, \ldots, \psi_p)(\xi_k)$ does not tend to infinity, $F(\xi_k)$ must tend to a point in $S_F$. If $\xi_k$ tends to a point $x \in \operatorname{Sing}F$, then $F(\xi_k)$ tends to a point in $K_0(F)$. In both of these cases, $\psi_i(\xi_k)$ tends to zero, for $i = 1, \ldots, p$ and $\beta$ is a point of the variety $(K_0(F) \cup S_F) \times \{0_{\mathbb{R}^p}\}$.

5.2. The asymptotic variety and the “asymptotic flower” of $P$. As we refer in the introduction, in this paper we considered the Pinchuk’s map given in [P], denoted by $P$. In the series of papers [C1, C2, C3, C4, C5], A’ Campbell describes the behaviours of the asymptotic variety and the “asymptotic flower” (the inverse image of the asymptotic variety) of the Pinchuk’s map $P$.

We recall briefly some important properties of these varieties that we will use to prove the main result in the section 5.3.

**Proposition 5.2.** [C1, C2]

1. (see [C1], page 1) The asymptotic variety $S_P$ of the Pinchuk’s map $P$ is the curve whose the bijective polynomial parametrization by $s \in \mathbb{R}$ is:

$$p(s) = s^2 - 1, \quad q(s) = -75s^5 + \frac{345}{4}s^4 - 29s^3 + \frac{117}{2}s^2 - \frac{163}{4}.$$  

Moreover, this curve intersects the vertical axis at $(0, 0)$ and $(0, 208)$ and its leftmost point is $(-1, -163/4)$. This leftmost point is also the only singular point of the curve $S_P$.

2. (see [C2], page 11) Exactly two points, the leftmost point $(-1, -163/4)$ and the origin $(0, 0)$, have no inverse images. Moreover, every other point of the asymptotic variety $S_P$ has exactly one inverse image.

3. (see [C2], page 12) The asymptotic flower is the union of three curves $D_1, D_2$ and $D_3$. Each of these curves divides the source space into simply connected parts, which may be described as the regions left and right of the curve, using the induced orientations to define left and right. Removing the curves $D_1, D_2$ and $D_3$, it thus leaves exactly four simply connected open components: two regions $A$ and two regions $B$ (see Figure 2).
Moreover, the restriction of \( \mathcal{P} \) to each of these regions is homeomorphism. Furthermore, the three curves \( D_1, D_2, D_3 \) are the inverse images of \( C_1, C_2 \) and \( C_3 \), respectively in the following way: two points \((-1, -163/4)\) and \((0, 0)\) that have no inverse images from the asymptotic variety break up the asymptotic flower into three connected curves \( D_1, D_2 \) and \( D_3 \) such that each point of each of the three curves \( C_1, C_2 \) and \( C_3 \) has exactly one inverse image.

5.3. Main result. We calculate in this section the intersection homology groups with both compact supports and closed supports of a Valette variety associated to the Pinchuk’s map \( \mathcal{P} \).

\textbf{Theorem 5.3.} Let \( \mathcal{P} \) be the Pinchuk’s map given in [P]. Then there exists a Valette variety \( V_{\mathcal{P}} \) associated to \( \mathcal{P} \) such that \( IH_{1}^{0, c}(V_{\mathcal{P}}) = IH_{1}^{0, d}(V_{\mathcal{P}}) = 0. \)

\textbf{Proof.} Since \( K_0(\mathcal{P}) = \emptyset \), then the singular locus \( \text{Sing} \mathcal{P} \) of \( \mathcal{P} \) is empty, hence \( M_{\mathcal{P}} = \mathbb{R}^2 \). We need to determine the covering \( \{U_1, \ldots, U_p\} \) of \( M_{\mathcal{P}} \) by semi-algebraic open subsets (in \( \mathbb{R}^{2n} \)) as in the construction of Valette varieties (see section 4.1). By the item (3) of the Proposition 5.2, each of three curves \( D_1, D_2 \) and \( D_3 \) of the asymptotic flower divides the source space into simply connected parts (see Figure 2). Removing the curves \( D_1, D_2 \) and \( D_3 \), it thus leaves exactly four simply connected open components: two regions \( A \) and two regions \( B \). Let us call call \( A_1 \) (resp., \( B_1 \)) is the region \( A \) (resp., \( B \)) “on the left” and \( A_2 \) (resp., \( B_2 \)) is the region \( A \) (resp., \( B \)) “on the right”, using the induced orientations to define left and right (see figure 2). We use the Figure 2 and Figure 3 in [C2] of the “asymptotic flower” and the asymptotic variety, respectively, but we labeled \( A_1, A_2, B_1 \) and \( B_2 \) as the four regions.

Thanks again to the item (3) of the Proposition 5.2, the restriction of \( \mathcal{P} \) to each of these regions \( A_i \) and \( B_i \) is homeomorphism, for \( i = 1, 2 \). Moreover, the three curves \( D_1, D_2 \) and \( D_3 \) are the inverse images of \( C_1, C_2 \) and \( C_3 \), respectively, such that each point of each of the three curves \( C_1, C_2 \) and \( C_3 \) has exactly one inverse image, then we can choose an open covering \( \{U_i\}_{i=1, \ldots, 4} \) of \( \mathbb{R}^2 \) such that \( U_i \supset A_i, U_{i+2} \supset B_i \) for \( i = 1, 2 \), and the restriction \( \mathcal{P}|_{U_i} \) is homeomorphism, for \( i = 1, \ldots, 4 \). In this case, the covering by semi-algebraic closed subset \( \{V_i\}_{i=1, \ldots, 4} \) in the construction 4.1 may be chosen in the following way: \( V_i = U_i \cup D_j \), where \( j \in \{1, 2, 3\} \) such that \( V_i \) is closed.

By the construction of Valette varieties and the Lemma 5.1, the Valette variety associated the covering \( \{U_i\}_{i=1, \ldots, 4} \) of the Pinchuk’s map \( \mathcal{P} \) has four 2-dimensional smooth, non-compact, connected components glued along the asymptotic variety \( S_{\mathcal{P}} \) in \( \mathbb{R}^6 \). This gluing can be described as the following: \( F(A_2) \) and \( F(B_1) \) are glued along the curve \( C_1 \), \( F(A_2) \) and \( F(B_2) \) are glued along the curve \( C_2 \), \( F(A_1) \) and \( F(B_1) \) are glued along the curve \( C_3 \).

Let us denote by \( L = (-1, -163/4, 0_{R^4}) \) and \( O = (0, 0, 0_{R^4}) \). The singular locus of \( V_{\mathcal{P}} \) is two points \( L \) and \( O \). Notice that \( V_{\mathcal{P}} \subset \mathbb{R}^6 \) and can be represented by the figure 3.

The Valette variety \( V_{\mathcal{P}} \) is a pseudomanifold and the stratification

\[ V_{\mathcal{P}} \supset V_0 = \{L, O\} \supset \emptyset \]

is a locally topologically trivial stratification.

We emphasize again that since \( \dim_{R} V_{\mathcal{P}} = 2 \), then we have only one perversity: the zero perversity \( \bar{0} \) and \( p_0 = p_1 = p_2 = 0 \). We look for the 1-dimensional allowable chains. For this, we
have to verify the condition (2.4), i.e. for a 1-dimensional chain $c$ be $(0,1)$-allowable, at first, $c$ must be satisfied the condition

$$\dim(c \cap V_{-2}) \leq 1 - 2 + 0.$$ 

Hence, if $c$ is $(0, 1)$-allowable then $\dim(c \cap V_0) \leq -1$. Consequently, the chain $c$ cannot contain neither $L$ nor the origin $O$. In this case, the boundary $\partial c$ of $c$, which consists of points, does not meet the set $\{L, O\}$. Thus $\partial c$ is also $(0, 1)$-allowable, since $\dim(\partial c \cap V_0) = \dim \emptyset = -\infty$. Then the 1-dimensional allowable chains of $V_{\mathcal{P}}$ are the chains of types $c_1$ and $c_2$ (see figure 3) in such a way that: assume that $\tau_i$ is a 2-dimensional chain such that the boundary of $\tau_i$ is $c_i$, i.e. $\partial \tau_i = c_i$, for $i = 1, 2$, then $\tau_1$ contains either the point $L$ or the point $O$ or both of them and $\tau_2$ does not contain the two points $L$ and $O$. Notice that $c_1$ is a chain with closed support and $c_2$ is a chain with compact support.

We see that $\tau_1$ meets the stratum $V_0$ at $L$ or $O$ or both of them and the chain $\tau_2$ does not meet the stratum $V_0$. The condition (2.4) holds for all these two chains, i.e. the chain $\tau_i$ is $(0, 2)$-allowable, for $i = 1, 2$. Then we have $IH_{1, 0}^{\partial \mathcal{P}}(V_{\mathcal{P}}) = IH_{1, 1}^{\partial \mathcal{P}}(V_{\mathcal{P}}) = 0$. 

\[\square\]
Remark 5.4. Since the stratification used in the proof is a locally topologically trivial stratification then the result of the Theorem 5.3 is an invariant for any (another) locally topologically trivial stratification.

Corollary 5.5. The Valette variety associated to the Pinchuk map \( P \) constructed in the proof of the Theorem 5.3 is a counter example for the “real version” of the Valette’s Theorem 4.2.

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Figure 3. The Valette variety associated to the Pinchuk’s map with the chosen covering and some types of allowable chains for the intersection homology \( IH^0(V_P) \). The chains \( c_1 \) and \( l \) are homologous for the intersection homology.
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