Full-counting statistics of charge fluctuations in quantum Hall states

Clément Berthiere,1,2 Benoît Estienne,3 Jean-Marie Stéphan,4 and William Witczak-Krempa1,2,5

1 Université de Montréal, Département de Physique, Montréal, QC, Canada, H3C 3J7
2 Centre de Recherches Mathématiques, Université de Montréal, Montréal, QC, Canada, H3C 3J7
3 Sorbonne Université, CNRS, Laboratoire de Physique Théorique et Hautes Energies, LPTHE, F-75005 Paris, France
4 Univ Lyon, CNRS, Université Claude Bernard Lyon 1, Institut Camille Jordan, UMR5208, F-69622 Villeurbanne, France
5 Institut Courtois, Université de Montréal, Montréal, QC H2V 0B3, Canada

(Dated: November 11, 2022)

We study the cumulants of the charge distribution of a subregion for two-dimensional quantum Hall states of bosons and fermions at both integer and fractional fillings, focusing on subregions with corners. Even cumulants, which include the variance, satisfy an area law with subleading corrections sensitive to finer geometric details of the subregion such as corner contributions, while at the same time probing non-trivial sum rules for general correlation functions. We perform a systematic study of such corner terms, by a combination of analytic, numerically exact, and Monte Carlo computations. We also study odd charge cumulants, for which the area-law term vanishes and distinct corner contributions appear. The observed shape dependence of the third cumulant shows nearly universal behavior for integer and fractional Laughlin Hall states in the lowest Landau level. While these states serve as our main example, many of our finding are expected to hold in considerable generality. As an illustration, we discuss properties of gapless Dirac fermions, and more general conformal field theories.

A fundamental concept of quantum mechanics is the statistical nature of measurements of observables. Measuring the same observable in identically prepared systems generally leads to different outcomes governed by a probability distribution. This distribution is described by its cumulants. The higher the order of the cumulant one has access to, the more detailed information one obtains about correlations in the system that determine the distribution.

In certain experiments, one measures only a subregion of a physical system. In that case, one is interested in the cumulants of a given observable in a subregion of the full system. The cumulants $C_m(A)$ of an observable $O$ within a region $A$ are defined through the generating function $\chi_A(\lambda) = \langle e^{\lambda O_A}\rangle$ as

$$C_m(A) = \partial_A^m \log \chi_A(\lambda)|_{\lambda=0}.$$  \hfill (1)

The first cumulant is the mean $\langle O_A \rangle$. The second and third cumulants, $\langle (O_A - \langle O_A \rangle)^m \rangle$ with $m = 2, 3$, are the variance (or fluctuations) and the skewness, respectively. Both these cumulants can be written simply in terms of central moments, which is not true for higher-order cumulants that are more complicated polynomials in the moments. For instance, the fourth cumulant is given by

$$C_4(A) = \langle (O_A - \langle O_A \rangle)^4 \rangle - 3(\langle (O_A - \langle O_A \rangle)^2 \rangle)^2.$$  \hfill (2)

A non-zero skewness or higher-order cumulant is an indication of the non-Gaussianity of the probability distribution of an observable, since a Gaussian distribution has all cumulants of order three and above equal to zero.

As long as the total quantity $O = O_A + O_B$ does not fluctuate, one has $C_m(A) = (-1)^m C_m(B)$ for all cumulants except the mean $m = 1$. That is (odd) even cumulants are (anti) symmetric under exchange of the subregion $A$ and its complement $B$. For product states, all cumulants above the mean vanish. In the context of condensed matter physics, cumulants of conserved charges—in particular fluctuations—have recently received considerable attention [1–14]. Indeed, some of their properties mentioned above, especially for even cumulants, are reminiscent of those of entanglement entropy. As it turns out, for non-interacting fermions, the entanglement entropy as well as the full entanglement spectrum are entirely encoded in the even cumulants [2, 4–6]. Moreover, bipartite charge fluctuations can be shown to be proportional to the (Rényi) entanglement entropy for, e.g., conformal field theories in 1D or Fermi gases [2, 8]. Bipartite fluctuations (and few higher-order charge cumulants) have been measured in mesoscopic condensed-matter systems [15, 16] and in cold atomic gases [17, 18]. The success of entanglement entropy being widespread, relating it to measurable properties has long been the main motivation for studying bipartite fluctuations. More recently, fluctuations have been investigated for topological states and quantum critical systems [11, 13, 19–21], as well as in the context of random matrix theory [22, 23].

From general principles, cumulants of conserved observables in a pure state behave for large two-dimensional regions $A$ as

$$C_m(A) = c_m |\partial A| - a_m + \cdots,$$  \hfill (3)

for most physical systems, where the leading term scales with the area of the boundary of $A$, and $a_m$ is a subleading correction. As we noted above, a volume law cannot appear. Furthermore, an area law is expected for even cumulants from physical intuition and symmetry between subregion $A$ and its complement $B$: we consider a charge that is globally conserved in the system, such as the number of particles, but it is still allowed to fluctuate between subregions $A$ and $B$. Thus $A$ can trade particles with $B$ through their common boundary, hence
one would expect an area law. The area law for even cumulants can rigorously be shown to hold for integer quantum Hall (IQH) states [24]. It can also be shown to hold for general translation invariant interacting systems, under mild assumptions on the decay of connected correlation functions of the associated local charges, see Appendix I. From its definition (2), the fourth cumulant need not be positive. Moreover, each term in (2) will generally scale with the volume of the subregion, but it is this precise combination between the fourth central moment and the variance squared that cancels the volume dependence. Similar patterns occur for higher-order cumulants. Odd cumulants of conserved observables have been less studied but are also interesting. They are antisymmetric between the subregion and its complement $B$, i.e. $C_m(A) = -C_m(B)$ for $m > 1$ odd. Thus neither volume nor area-law terms can appear, that is $c_m = 0$ for $m$ odd in (3), and the leading contribution is $-a_m$.

The (subleading) piece $a_m$ in this expansion (3) is the most interesting, as it is sensitive to the presence of sharp corners in $A$. (If the boundary of $A$ is smooth, this term vanishes, and there are other contributions stemming from the curvature of the boundary [25], but those enter at an order which is proportional to the inverse size of $A$.) This corner contribution depends on the opening angle $\theta \in (0, 2\pi)$; if there are several corners, $a_m$ is obtained by summing over all corner contributions, and those are independent. In practice, isolating the contribution of a single corner can be challenging, but this was achieved in [13] for the variance (see also [26–30] for investigations of corner contributions in related quantities such as entanglement entropy).

In this paper, we study higher-order cumulants which are much more complicated, focusing on the example of quantum Hall states. We present results for the charge cumulants of the IQH groundstate at filling $\nu = 1$. The cumulants can be computed using standard free fermion methods [31, 32]. We also consider Laughlin states at filling fractions $\nu = 1/2$ (bosons), $\nu = 1/3$ (fermions) where fluctuations are accessible through Monte Carlo simulations.

**Even cumulants in quantum Hall states.** For large regions, the cumulants behave as (3). We denote by $a_m(\theta)$ the contribution of a single corner of opening angle $\theta$. All $c_m$ are known exactly for IQH [24], see also Appendix III. For example, the second cumulant has area-law coefficient [32] $c_2 = (2\pi)^{-3/2}$ for $\nu = 1$, where we work in units of the magnetic length $\ell_B = 1$, while the corner fluctuations function for general incompressible filling $\nu$ reads [13]

$$a_2(\theta) = \frac{\nu}{4\pi^2} \left( 1 + (\pi - \theta) \cot \theta \right).$$

Both coefficients $c_m, a_m$ can be numerically computed to high precision. They are obtained by following the method of [33], see Appendix III for more details. We report the values of $c_m$ for $m = 2, 4, \cdots, 28$ in Fig. S3 in Appendix III. We find that the area-law coefficients are not always positive, but rather alternate in sign as $(-1)^{m/2 - 1}$. The fourth cumulant is thus negative, the sixth positive, and so on. The coefficient $c_m$ takes its minimal value at $m = 8$, and then increases factorially with the order $m$. This factorial growth is expected from the definition of the cumulants, see (1) and (S46).

The corner term $a_m(\theta)$ behaves similarly. For fixed $\theta$, as the order $m$ increases, $a_m(\theta)$ first decreases to its minimal value and then increases factorially. However, $a_m(\theta)$ changes sign differently than $c_m$. Indeed, both $a_2$ and $a_4$ are positive functions, while $a_6$ is negative, $a_8$ positive, and the signs continue to alternate. As a consequence, in all the cases that we studied, $m \leq 28$, only the second cumulant (fluctuations) presents an area-law contribution of opposite sign compared to its subleading corner term. Starting with the fourth cumulant, area-law contribution and subleading corner contribution have the same sign. The heuristic argument that the subleading corner contribution should diverge as $1/\theta$ with the correct sign as $\theta \to 0$ in order to compensate the area law thus breaks down for higher cumulants. However, the $1/\theta$ divergence of the corner term should hold in considerable generality. We note that for $m > 2$, the coefficients $a_m$ and $c_m$ probe somewhat complicated sum rules for the connected $m$–point density function, as is explained in Appendix I. The sign of those sum rules cannot be easily understood from underlying general physical principles.

Even corner cumulant functions $a_m(\theta)$ share certain features, as can be seen in Fig. 1. First, they are monotonic functions for $0 < \theta < \pi$. They also share the complementarity property $a_m(\theta) = a_m(\pi - \theta)$, which is a consequence of the invariance of even cumulants under the exchange of region $A$ with its complement $B$. Coupled to the fact that $a_m(\pi) = 0$, this implies that they vanish quadratically in the smooth limit $\theta \sim \pi$.

$$a_m(\theta) \sim \sigma_m \cdot (\pi - \theta)^2, \quad (\theta \to \pi).$$

In the cusp limit $\theta \to 0$, we observe the divergence

$$a_m(\theta) \sim \kappa_m / \theta, \quad (\theta \to 0).$$

![FIG. 1. Normalized corner cumulant function $a_m(\theta)/\sigma_m$ for $m = 2, 4, \cdots, 10$. The variance stands out while the higher cumulants cluster.](image-url)
Both properties (5) and (6) can be explicitly verified for the variance, see (4), and also hold for the entanglement Rényi entropies at both integer [34] and fractional fillings [13], as well as in 2D CFTs. The values of the smooth $\sigma_m$ and cusp $\kappa_m$ coefficients for the cumulants $m = 2, 4, \cdots, 10$ may be found in Table I. Remarkably, we were able to find analytical expressions for $\kappa_m$, in terms of $m$–fold integrals (see Appendix II). Those integrals simplify nicely for $m = 2, 3, 4$, as reported in Table I. The smooth coefficient has been used to normalize the corner cumulant functions in Fig. 1, where we notice that the second cumulant stands out from the higher-order ones. It would be interesting to determine whether this dichotomy, and the clustering of higher-cumulants, hold in other quantum states.

**Fractional Quantum Hall.**— We now study fractional quantum Hall (FQH) states, focusing on Laughlin states in a disc geometry at fillings $\nu = 1/2$ (bosons) and $1/3$ (fermions), using large-scale Monte Carlo (MC) simulations. Computing cumulants is done by counting the number of particles in $A$ for each Monte Carlo sample. Extracting the corner contribution requires more work, and was done using the same method as in [13, 35], which focused on the variance and second Rényi entropy.

For fractions $\nu = 1/2, 1/3$, we found that the fourth cumulant is negative, as for the IQH state at $\nu = 1$, since the area-law coefficients have the same sign, $c_4 < 0$, Table SII. We show in Fig. 2 the corresponding corner function $a_4(\theta)$, which is positive for both FQH states, exactly as we found for $\nu = 1$. The angular dependence for fillings $\nu = 1, 1/2, 1/3$, although similar, is not superuniversal, in contrast with fluctuations where $\langle \rho^2(\theta) \rangle / \langle \rho(\theta)^2 \rangle = \nu_1 / \nu_2$ (the ratios are plotted in the inset of Fig. 2). This breakdown of superuniversality is expected. Intuitively, the constant term comes from the neighborhood of the corner, which looks like an infinite angular sector $A = \{ z, 0 \leq \arg z \leq \theta \}$ in 2D complex coordinates. The second cumulant involves the two-point translationally invariant density function $\langle \rho(\theta)\rho(z) \rangle$, and superuniversality essentially follows from scale invariance of the region $A$, see [13]. For higher cumulants, scale invariance is no longer sufficient to constrain the angular dependence of the corner term, because higher translationally invariant correlation functions depend on more than one variable. This situation is similar to what happens in massless field theories, where scale invariance is sufficient to fix the two-point function, but not higher-point functions. An analytical counterexample to superuniversality for $C_3$ is given in Appendix II, which relies on the two special points $\theta \to 0$ and $\theta = \pi/2$.

We observe that $|C_4|$ increases with the filling fraction, which is in agreement with the intuition that having a greater density of particles leads to greater particle fluctuations in $A$. In fact, the increase holds for both the area-law coefficient and $a_4(\theta)$. The same holds for the variance $C_2$, where the corner term scales exactly linearly with $\nu$. Interestingly, for filling fractions $\nu = 1/3, 1/2, 1$, we found that the area-law coefficient $c_2$ also depends linearly (within 0.2% relative error) on $\nu$: $c_2(\nu) \simeq 0.00653 + 0.05699 \nu$, see Table SII. No simple explanation for this observed linearity is known since $c_2$ depends on the entire static structure factor, not only its long-distance part [13]. For the fourth cumulant, $c_4$ is very close to such a linear behavior as well. We note that at integer fillings $\nu > 1$, the area-law coefficient $c_2$ increases with $\nu$, but sublinearly (see Fig. S3 in Appendix III). There is thus a striking difference in the behavior of $c_2$ between fillings $\nu \leq 1$ and integer ones $\nu > 1$.

**Odd cumulants in quantum Hall states.** As already mentioned, odd cumulants do not scale with the area of the boundary of $A$ at leading order, hence the first term in the expansion (3) vanishes. For geometries with corners, $a_m(\theta)$ thus enters at leading order in the large-region expansion. The odd cumulants being antisymmetric under

| $\nu = 1$ | $\nu = 1/2$ | $\nu = 1/3$ |
|-----------|------------|------------|
| $c_2$     | $(2\pi)^{-3/2} \simeq 0.0635$ | $0.0351$ | $0.0255$ |
| $c_4$     | $-0.00336$ | $-0.00219$ | $-0.00163$ |

**TABLE I.** Smooth and cusp coefficients for $\nu = 1$ cumulants.

| $\sigma_m$ | $\kappa_m$ |
|------------|------------|
| $C_2$      | $1/(12\pi^2)$ | $1/(4\pi) \simeq 0.07957$ |
| $C_4$      | $0.000364$ | $\frac{18+\pi-12\pi\theta}{4\pi^2} \simeq 0.00004$ |
| $C_6$      | $-0.000191$ | $-0.00459$ |
| $C_8$      | $0.000221$ | $0.00478$ |
| $C_{10}$   | $-0.000435$ | $-0.008$ |
| $C_2$      | $0.01462$ | $\frac{3\sqrt{2}}{4\pi^2} \simeq -0.05204$ |
| $C_4$      | $0.00254$ | $0.01168$ |
| $C_7$      | $-0.00171$ | $-0.00938$ |
| $C_9$      | $0.0023$ | $0.0015$ |

**FIG. 2.** Fourth corner cumulant function for the Laughlin state at fillings $\nu = 1, 1/2, 1/3$, extracted from Monte Carlo data on $N = 64$ particles. The inset shows the ratio $a_2(\theta)/a_2^{-1}(\theta)$ for $\nu = 1/2, 1/3$.
exchange between $A$ and its complement, we have
\[ a_m(\theta) = -a_m(2\pi - \theta). \tag{7} \]

Due to this antisymmetry about $\pi$, the function $a_m(\theta)$ is expected to vanish linearly in the smooth regime, in contrast to the quadratic behavior obtained for even $m$,
\[ a_m(\theta \sim \pi) \simeq \sigma_m \cdot (\pi - \theta), \tag{8} \]

and this is indeed what we observe in the data, see Fig. 3. However, it still diverges as $1/\theta$ in the limit $\theta \to 0$, as in (6). We expect those two features to hold in considerable generality.

As for even cumulants, we observe a pattern of alternating sign of $a_m(\theta)$ as $(-1)^{(m-1)/2}$ in the range $0 < \theta \leq \pi$ for the IQH data (see Appendices). The odd corner cumulant functions appear to be all monotonic, an observation which is not unreasonable to expect.

**Fractional Quantum Hall.** — We performed Monte Carlo simulations on the third cumulant for the Laughlin states at filling fractions $\nu = 1/2, 1/3$. In Fig. 3, we show our results for $C_3(\theta)$, which is found to be positive for both fractional fillings just as was found for the IQH state at unit filling. We observe that $C_3$ increases with the filling fraction, as was the case for $C_2$ and $|C_4|$. Quite remarkably, the data suggest that the angular dependence of $C_3$ is universal for filling fractions $\nu = 1/3, 1/2, 1$, as can be seen from the inset of Fig. 3 where we have plotted the ratio $a_3^\nu(\theta)/a_3^{\nu=1}(\theta)$, which is constant over a wide range of angles. We find $a_3^\nu(\theta)/a_3^{\nu=1}(\theta) \simeq 0.67$ and 0.53 for $\nu = 1/2$ and $\nu = 1/3$, respectively. Furthermore, the dependence on $\nu$ is nearly linear:
\[ a_3^\nu(\theta)/a_3^{\nu=1}(\theta) \simeq \alpha \nu + \beta, \tag{9} \]

where $\alpha = 7/10 = 1 - \beta$ (within 3% error relative to the fit in Fig. 3). It is striking that $C_3$ displays such universality for the three states with $\nu \leq 1$ given their very different properties.

**Discussion.** We have studied the cumulants of the charge distribution in two-dimensional quantum Hall states at both integer and fractional fillings, focusing on subregions with corners. Even cumulants satisfy an area law whose coefficient increases factorially at large order, while for odd cumulants it vanishes. The (subleading) constant term is sensitive to finer geometric details of the subregion such as corner contributions, which we systematically investigated. We expect the observed behavior of the corner cumulants in the smooth and cusp regimes to hold in considerable generality, as well as their monotonicity for $0 < \theta \leq \pi$.

The variance (second cumulant) is known to be super-universal [13, 25], i.e. it takes the same form for a large class of unrelated systems. We have shown that super-universality breaks down for cumulants higher than the variance. However, we have discovered that the angle dependence of the third cumulant appears to be universal within error bars over a wide range of angles for quantum Hall states at fillings $\nu = 1, 1/2, 1/3$. These numerical results give access to information on sum rules for higher correlation functions, but in a more convoluted way than for the variance. Studying such subregion cumulants provides a new and useful method to understand such non-trivial sum rules, in particular the third cumulant which displays striking universality for quantum Hall states at fillings $\nu \leq 1$.

An interesting direction would be to study charge cumulants in other systems such as conformal field theories (CFTs) in $d \geq 2$ spatial dimensions, beyond the variance [13, 19, 20]. When considering a conserved charge $O$, the cumulant-generating function $\chi(i\lambda)$ is the expectation value of the so-called disorder operator $\exp(iO_A)$, which performs a symmetry transformation in subregion $A$. The expectation value of the disorder operator, which can be used to probe higher-form symmetries, was studied as a function of both $A$ and the corner angle near quantum critical points [19, 20, 36]. However, little is understood for higher cumulants $C_{m>2}$. For conserved currents of CFT, the three-point function vanishes at equal times [37]. The corresponding third cumulant thus vanishes as well, in contrast to what we found for quantum Hall states. For CFTs with charge conjugation symmetry $C$, all odd cumulants vanish as well since the charge density $J_0$ is odd under $C$. In fact, this vanishing of odd cumulants is general to $C$-symmetric systems, independent of details, since the charge density is always odd. This is for example the case for tight-binding models of hopping electrons that are particle-hole symmetric, see [14] for an example with 2D Dirac cones. In contrast, not much can be said for even cumulants beyond the variance, even about the sign. For instance, we considered a tight-binding model on the square lattice with Dirac cones, and found that the sign of the leading area-law coefficient $c_m$ for even charge cumulants is $+, +, -, +, +, -$, $\cdots$ for $m = 2, 4, 6, 8, 10, 12, \cdots$, displaying a pattern different than the simple one observed above for quantum Hall states $(+, -, +, -, +, - \cdots)$; the first two signs hold for
the FQH states as well). We have shown this by numerically calculating the cumulants, see Appendix IV. It would be interesting to extend this analysis to higher corner terms in Dirac semimetals and general CFTs, since they encode new universal information about the quantum critical degrees of freedom.

Acknowledgments. W.W.-K. thanks P.-G. Rozon for earlier collaboration on related topics. B.E. thanks N. Regnault for discussions on the effect of charge conjugation on the spectrum of the correlation matrix. C.B. was supported by a CRM-Simons Postdoctoral Fellowship at the Université de Montréal. B.E. was supported by Grant No. ANR-17-CE30-0013-01. J.-M.S. was supported by IDEX Lyon project ToRe (Contract No. ANR-16-IDEX-0005). W.W.-K. was funded by a Discovery Grant from NSERC, a Canada Research Chair, and a grant from the Fondation Courtois.

[1] I. Klich, G. Refael, and A. Silva, “Measuring entanglement entropies in many-body systems,” Phys. Rev. A 74 no. 3, (2006) 032306, arXiv:cond-mat/0603004.
[2] I. Klich and L. Levitov, “Quantum Noise as an Entanglement Meter,” Phys. Rev. Lett. 102 (2009) 100502, arXiv:0804.1377.
[3] B. Hsu, E. Grosfeld, and E. Fradkin, “Quantum noise and entanglement generated by a local quantum quench,” Phys. Rev. B 80 (Dec, 2009) 235412, arXiv:0908.2622.
[4] H. F. Song, S. Rachel, and K. Le Hur, “General relation between entanglement and fluctuations in one dimension,” Phys. Rev. B 82 no. 1, (2010) 012405, arXiv:1002.0825.
[5] H. F. Song, C. Flindt, S. Rachel, I. Klich, and K. Le Hur, “Entanglement entropy from charge statistics: Exact relations for noninteracting many-body systems,” Phys. Rev. B 83 (2011) 161408, arXiv:1008.5191.
[6] H. F. Song, S. Rachel, C. Flindt, I. Klich, N. Laflorencie, and K. Le Hur, “Bipartite Fluctuations as a Probe of Many-Body Entanglement,” Phys. Rev. B 85 (2012) 035409, arXiv:1109.1001.
[7] B. Swingle and T. Senthil, “Universal crossovers between entanglement entropy and thermal entropy,” Phys. Rev. B 87 no. 4, (2013) 045123, arXiv:1112.1069.
[8] P. Calabrese, M. Mintchev, and E. Vicari, “Exact relations between particle fluctuations and entanglement in Fermi gases,” EPL 98 no. 2, (2012) 20003, arXiv:1111.4836.
[9] S. Rachel, N. Laflorencie, H. F. Song, and K. Le Hur, “Detecting Quantum Critical Points Using Bipartite Fluctuations,” Phys. Rev. Lett. 108 no. 11, (2012) 116401, arXiv:1110.0743.
[10] A. Petrescu, H. F. Song, S. Rachel, Z. Ristivojevic, C. Flindt, N. Laflorencie, I. Klich, N. Regnault, and K. Le Hur, “Fluctuations and entanglement spectrum in quantum Hall states,” J. Stat. Mech. 2014 no. 10, (2014) 10005, arXiv:1405.7816.
[11] L. Herviou, K. Le Hur, and C. Mora, “Bipartite fluctuations and topology of Dirac and Weyl systems,” Phys. Rev. B 99 no. 7, (2019) 075133, arXiv:1809.08252.
[12] M. T. Tan and S. Ryu, “Particle number fluctuations, Rényi entropy, and symmetry-resolved entanglement entropy in a two-dimensional fermi gas from multidimensional bosonization,” Phys. Rev. B 101 (2020) 235169, arXiv:1911.01451.
[13] B. Estienne, J.-M. Stéphan, and W. Witzczak-Krempa, “Cornering the universal shape of fluctuations,” Nature Commun. 13 no. 1, (2022) 287, arXiv:2102.06223.
[14] V. Crépel, A. Hackenbroich, N. Regnault, and B. Estienne, “Universal signatures of Dirac fermions in entanglement and charge fluctuations,” Phys. Rev. B 103 no. 23, (2021) 235108, arXiv:2102.09571.
[15] M. Esposito, U. Harbola, and S. Mukamel, “Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems,” Rev. Mod. Phys. 81 no. 4, (2009) 1665–1702, arXiv:0811.3717.
[16] D. Kambly, C. Flindt, and M. Büttiker, “Factorial cumulants reveal interactions in counting statistics,” Phys. Rev. B 83 no. 7, (2011) 075432, arXiv:1012.0756.
[17] V. Gritsev, E. Altman, E. Demler, and A. Polkovnikov, “Full quantum distribution of contrast in interference experiments between interacting one-dimensional Bose liquids,” Nature Physics 2 no. 10, (2006) 705–709, arXiv:cond-mat/0602475.
[18] I. Bloch, J. Dalibard, and W. Zwerger, “Many-body physics with ultracold gases,” Rev. Mod. Phys. 80 no. 3, (2008) 885–964, arXiv:0704.3011.
[19] Y.-C. Wang, M. Cheng, and Z. Y. Meng, “Scaling of the variance,” Phys. Rev. A 83 no. 2, (2011) 023824, arXiv:1006.3738.
[20] X.-C. Wu, C.-M. Jian, and C. Xu, “Universal features of higher-form symmetries at phase transitions,” SciPost Phys. 11 no. 2, (2021) 033, arXiv:2011.10342.
[21] B. Oblak, N. Regnault, and B. Estienne, “Equipartition of entanglement in quantum Hall states,” Phys. Rev. B 105 no. 11, (2022) 115131, arXiv:2112.13854 [cond-mat.str-el].
[22] B. Lacroix-A-Chez-Toine, S. N. Majumdar, and G. Schehr, “Rotating trapped fermions in two dimensions and the complex Ginibre ensemble: Exact results for the entanglement entropy and number variance,” Phys. Rev. A 99 no. 2, (Feb., 2019) 021602, arXiv:1809.05835 [cond-mat.stat-mech].
[23] B. Lacroix-A-Chez-Toine, J. A. M. Garzó, C. S. H. Calva, I. P. Castillo, A. Kundu, S. N. Majumdar, and G. Schehr, “Intermediate deviation regime for the full eigenvalue statistics in the complex Ginibre ensemble,” Phys. Rev. E 100 no. 1, (July, 2019) 012137, arXiv:1904.01813 [cond-mat.stat-mech].
[24] L. Charles and B. Estienne, “Entanglement Entropy and Berezin-Toeplitz Operators,” Commun. Math. Phys. 376 no. 1, (2019) 521–554, arXiv:1803.03149.
[25] C. Berthiere, B. Estienne, J.-M. Stéphan, and W. Witzczak-Krempa, “Super-universal fluctuations,” in preparation (2022).
[26] H. Casini and M. Huerta, “Universal terms for the entanglement entropy in 2+1 dimensions,” Nucl. Phys. B764 (2007) 183–201, arXiv:hep-th/0606256.

[27] A. B. Kallin, E. M. Stoudenmire, P. Fendley, R. R. P. Singh, and R. G. Melko, “Corner contribution to the entanglement entropy of an O(3) quantum critical point in 2 + 1 dimensions,” J. Stat. Mech. 6 (2014) 06009, arXiv:1401.3504.

[28] P. Bueno, R. C. Myers, and W. Witczak-Krempa, “Universality of corner entanglement in conformal field theories,” Phys. Rev. Lett. 115 (2015) 021602, arXiv:1505.04804.

[29] C. Berthiere, “Boundary-corner entanglement for free bosons,” Phys. Rev. B99 (2019) 165113, arXiv:1811.12875.

[30] C. Berthiere and W. Witczak-Krempa, “Entanglement of Skeletal Regions,” Phys. Rev. Lett. 128 no. 24, (2022) 240502, arXiv:2112.13931.

[31] I. Peschel, “Calculation of reduced density matrices from correlation functions,” J. Phys. A 36 no. 14, (2003) L205–L208, arXiv:cond-mat/0212631.

[32] B. Sirois, L. M. Fournier, J. Leduc, and W. Witczak-Krempa, “Geometric entanglement in integer quantum Hall states,” Phys. Rev. B 103 no. 11, (2021) 115115, arXiv:2009.02337.

[33] I. D. Rodríguez and G. Sierra, “Entanglement entropy of integer quantum Hall states in polygonal domains,” J. Stat. Mech. 2010 no. 12, (2010) 12033, arXiv:1007.5356.

[34] I. Peschel, “Lower entropy bounds and particle number fluctuations in a Fermi sea,” J. Phys. A 39 (2006) L85–L92, arXiv:quant-ph/0406068.

[35] I. D. Rodríguez and G. Sierra, “Entanglement entropy of integer quantum Hall states in polygonal domains,” J. Stat. Mech. 2010 no. 12, (2010) 12033, arXiv:1007.5356.

[36] B. Sirois, L. M. Fournier, J. Leduc, and W. Witczak-Krempa, “Geometric entanglement in integer quantum Hall states,” Phys. Rev. B 103 no. 11, (2021) 115115, arXiv:2009.02337.
Supplemental Material: Full-counting statistics of charge fluctuations in quantum Hall states

Clément Berthiere\textsuperscript{1,2}, Benoit Estienne\textsuperscript{3}, Jean-Marie Stéphan\textsuperscript{4} and William Witczak-Krempa\textsuperscript{1,2,5}

\textsuperscript{1}Université de Montréal, Département de Physique, Montréal, QC, Canada, H3C 3J7
\textsuperscript{2}Centre de Recherches Mathématiques, Université de Montréal, Montréal, QC, Canada, H3C 3J7
\textsuperscript{3}Sorbonne Université, CNRS, Laboratoire de Physique Théorique et Hautes Energies, LPTHE, F-75005 Paris, France
\textsuperscript{4}Univ Lyon, CNRS, Université Claude Bernard Lyon 1, Institut Camille Jordan, UMR5208, F-69622 Villeurbanne, France
\textsuperscript{5}Institut Courtois, Université de Montréal, Montréal, QC H2V 0B3, Canada

(Dated: November 11, 2022)

Contents

I. Scaling of cumulants from general principles 7
1. Disentangling geometry and correlation functions 7
2. Sum rules and asymptotic expansion of the cumulants 8
3. Examples of exact geometric formulas 8

II. Some exact results for corner terms in the integer quantum Hall effect 9
1. Corner terms in a parallelogram 10
2. $1/\theta$ divergence for a single corner 10
3. $\theta = \pi/2$ for a single corner 11
4. Breakdown of superuniversality for cumulants higher than variance 11

III. Cumulants for IQH states on a cylinder from the overlap matrix 12

IV. Cumulants of massless Dirac fermions 14

I. Scaling of cumulants from general principles

In this appendix, we gather several general results regarding the scaling of cumulants in general interacting theories.

1. Disentangling geometry and correlation functions

Consider a general interacting system in the continuum, and $A$ a region of $\mathbb{R}^d$. The $m$–th cumulant for fluctuations may be written as

$$C_m(A) = \int_A d\mathbf{r}_1 \ldots d\mathbf{r}_m \langle \rho(\mathbf{r}_1) \ldots \rho(\mathbf{r}_m) \rangle_c,$$  \hspace{1cm} (S1)

where $\rho(\mathbf{r})$ is the local density associated to the conserved quantity, and $\langle \rho(\mathbf{r}_1) \ldots \rho(\mathbf{r}_m) \rangle_c$, its connected $m$–point function. In the following, we assume that the theory under consideration is invariant with respect to translations

$$\langle \rho(\mathbf{r}_1) \ldots \rho(\mathbf{r}_m) \rangle_c = f(\mathbf{r}_2 - \mathbf{r}_1, \ldots, \mathbf{r}_m - \mathbf{r}_1),$$  \hspace{1cm} (S2)

where notice $f$ has $m - 1$ variables. We further assume that $f(s_2, \ldots, s_m)$ decays faster than any power law when any of its argument has large modulus. All bulk quantum Hall states considered in this paper satisfy those two requirements.

Using the first assumption, and following the approach put forward in [38, 39] to study a similar problem, the $m$–th cumulant can be rewritten after change of variables as

$$C_m(A) = \int_{\mathbb{R}^{(m-1)d}} ds_2 \ldots ds_m \mathcal{G}_A(s_2, \ldots, s_m) f(s_2, \ldots, s_m),$$  \hspace{1cm} (S3)

where $\mathcal{G}_A$ is defined as

$$\mathcal{G}_A(s_2, \ldots, s_m) = \int_{\mathbb{R}^d} ds \mathbf{1}_{\{\mathbf{s} \in A\}} \mathbf{1}_{\{\mathbf{s}_2 + \mathbf{s} \in A\}} \ldots \mathbf{1}_{\{\mathbf{s}_m + \mathbf{s} \in A\}}.$$  \hspace{1cm} (S4)
Here $1_{\{c\}}$ evaluates to one if condition $c$ is satisfied, zero otherwise. $\mathcal{G}_A$ is a purely geometric quantity, in fact it is nothing but the volume of the region $A \cap (A - s_2) \cap \ldots \cap (A - s_m)$.

With this at hand and using our second assumption, finding a full asymptotic expansion of the cumulants $C_m(\lambda A)$ as $\lambda \to \infty$ boils down to finding an asymptotic expansion for $\mathcal{G}_{\lambda A}$ as $\lambda \to \infty$. Since

$$\mathcal{G}_{\lambda A}(s_2, \ldots, s_m) = \lambda^d \mathcal{G}_A(s_2/\lambda, \ldots, s_m/\lambda),$$

(S5)

this only requires understanding the expansion of $\mathcal{G}_A$ for small arguments. The first term is simply proportional to volume, since by definition $\mathcal{G}_A(0, \ldots, 0) = \text{vol} A$. For regions with a smooth boundary, or for polygons, the next order term is [39]

$$\mathcal{G}_{\lambda A}(s_2, \ldots, s_m) = \lambda^d \text{vol} A - \lambda^{d-1} \int_{\partial A} d\sigma \max(0, s_2 \cdot \mathbf{n}_\sigma, \ldots, s_m \cdot \mathbf{n}_\sigma) + o(\lambda^{d-1}),$$

(S6)

where the integral is on the boundary of $A$, $\partial A$, and $\mathbf{n}_\sigma$ is the unit normal at a given point of $\partial A$ parameterized by $\sigma$. Higher order corrections have also been studied for smooth boundaries [40, 41]. The result takes the form of a full asymptotic series, but explicit expressions for each term become very quickly cumbersome as order increases. For polytopes, the higher order expansion takes a different form, in particular it terminates at order $\lambda^0$, due to the fact that the intersection of translated polytopes is still a polytope. As already alluded to, these expansions for $\mathcal{G}_{\lambda A}$ may then be plugged in (S3) to get an asymptotic expansion for the cumulants.

2. Sum rules and asymptotic expansion of the cumulants

While the general structure of the asymptotic expansion was described above, certain terms might vanish due to the symmetries of the physical model under consideration. For example, particle number conservation imposes the sum rule

$$\int_{\mathbb{R}^d} d\mathbf{r}_2 f(\mathbf{r}_2, \ldots, \mathbf{r}_m) = 0,$$

(S7)

which means the volume term vanishes for all cumulants in case particle number is conserved. This is the famous area-law scaling for even cumulants. For odd cumulants, one can show by similar symmetry considerations that the area-law term also vanishes.

3. Examples of exact geometric formulas

Besides the general asymptotic result (S6), there are several geometries for which $\mathcal{G}_A(\mathbf{r})$, or even $\mathcal{G}_A(\mathbf{r}_2, \ldots, \mathbf{r}_m)$ can be computed in closed form. We discuss two of them below, the circle and the square. Before doing so, let us mention the following “linear transformation formula”:

$$\mathcal{G}_{u(A)}(\mathbf{r}_2, \ldots, \mathbf{r}_m) = (\det u) \mathcal{G}_A(u^{-1}(\mathbf{r}_2), \ldots, u^{-1}(\mathbf{r}_m)),$$

(S8)

where $u$ is any linear map with strictly positive determinant. The proof of this formula follows from either linearity and change of variables, or the interpretation as volume.

a. The circle

As suggested by Fig. S1, establishing a simple formula for all cumulants seems very complicated, but for the second cumulant one can establish this

$$\mathcal{G}_A(\mathbf{r}) = 2R^2 \arccos \frac{r}{2R} - Rr \sqrt{1 - \left(\frac{r}{2R}\right)^2},$$

(S9)

for a disc of radius $R$. Here $r = |\mathbf{r}|$, and $0 \leq r \leq R$. As explained in the previous subsection, a full asymptotic expansion of the variance is obtained by large $R$ (or small $r$) expansion of the above formula, which reads

$$\mathcal{G}_A(\mathbf{r}) = \pi R^2 - R^2 \sum_{n \geq 0} \alpha_n \left(\frac{r}{R}\right)^2^{n+1},$$

(S10)

for coefficients $\alpha_n$ which can easily be computed. Notice the absence of constant term in the series. Generalization to an ellipse can be done using (S8), see [25] for a further discussion.
choices of vectors $r_j$ where $M$ and $K$ are small compared to the size of $G$ for intersections of translated parallelograms still give a parallelogram, and one can exploit this to get a simple explicit formula for $G_A(r_2, \ldots, r_n)$ for any $m \geq 2$, see (S13). The asymptotic regime we are interested in corresponds to the case where all $r_j$ are small compared to the size of $A$.

b. The parallelogram

For the interval $[0, 1]$, one can show that

$$G_{[0,1]}(x_1, \ldots, x_n) = 1 - M[x_1, \ldots, x_n],$$

where $M$ is defined as

$$M[x_1, \ldots, x_n] = \max(0, x_1, \ldots, x_n) - \min(0, x_1, \ldots, x_n),$$

see, e.g., [38]. Equation (S11) holds provided $M[x_1, \ldots, x_n] \leq 1$—which is always true in any relevant asymptotic regime—otherwise $G_{[0,1]} = 0$. Using this result one can deduce the analog formula for a square, and then the parallelogram by using the linear transform formula (S8). For example, if $A = u(S)$ is the image of the square $S = [0, L]^2$ through the linear map $u((x, y)) = (x + ay, y)$, then $u(S)$ is a parallelogram with angles $\pi \pm \theta$, where $\cot \theta = a$. In this case we obtain

$$G_{u(S)} = (L - M[x_1 - ay_1, \ldots, x_n - ay_n])(L - M[y_1, \ldots, y_n]),$$

provided the right-hand side is positive (otherwise the result is zero). This result will play a key role in Appendix II. Notice another difference with the disc: for a parallelogram, the smallest order in the expansion is $L^0$, which means the asymptotic expansion terminates at order $L^0$. The constant term can be identified as a corner term, which is absent in the disc expansion.

II. Some exact results for corner terms in the integer quantum Hall effect

In this appendix, we apply the general results of Appendix I to the integer quantum Hall effect, which is a bulk 2D free fermions system with two–point function (or kernel) given in symmetric gauge by

$$K(z, w) = \frac{1}{\pi} e^{-\frac{1}{2}(|z-w|^2 - z^* w + w^* z)}.$$

From now on, we use complex numbers $z = x + iy$ instead of the vector notation $r = (x, y)$. One way to use our general results is to reconstruct the connected $m$–point density function using Wick’s theorem. While this is in principle doable, we find it more convenient to use the well known result [2]

$$C_m = \left. \text{Tr} \partial^m \lambda \log [1 + (e^\lambda - 1)K] \right|_{\lambda=0}$$

$$= \sum_{p=1}^{m} \beta_{pm} \text{Tr} K^p,$$

for known coefficients $\beta_{pm} = \frac{1}{p} \sum_{j=0}^{p} (-1)^{j+1} C_p^{j} m^j$, which implies that $C_m$ is the trace of a polynomial of degree $m$ in $K^1$, and one can compute each trace $\text{Tr} K^p$ separately. Our main expansion result can then be applied because $K(z_1, z_2) \ldots K(z_{m-1}, z_m)K(z_m, z_1)$ is indeed translational invariant, so we may use our main formula (S3) with

$$f(z_2, \ldots, z_m) = K(0, z_2)K(z_2, z_3) \ldots K(z_{m-1}, z_m)K(z_m, 0).$$

The first few are $C_2 = \text{Tr}(K - K^2)$, $C_3 = \text{Tr}(K - 3K^2 + 2K^3)$, $C_4 = \text{Tr}(K - 7K^2 + 12K^3 - 6K^4)$, etc.
1. Corner terms in a parallelogram

In this subsection, $A$ is the image of the square $S = [0, L]^2$ through the map $u((x, y)) = (x + ay, y)$, that is, $A = u(S)$ is a parallelogram with four angles $\theta, \pi - \theta, \theta, \pi - \theta$, where $\cot \theta = a$, see Fig. S2. $a \to \infty$ corresponds to $\theta \to 0$, while $a = 0$ corresponds to $\theta = \pi/2$. Using equations (S3) and (S13), $\text{Tr} K^m$ has full asymptotic expansion

$$\text{Tr} K^m = \int dx_2 \cdots dx_m dy_2 \cdots dy_m (L - M[x_2 - ay_2, \ldots, x_m - ay_m]) (L - M[y_2, \ldots, y_m]) f(x_2 + iy_2, \ldots, x_m + iy_m) + O(L^{-\infty}),$$

(S18)

where $f$ is given by (S14) and (S17), and all integrals are over $\mathbb{R}$. The constant term in this asymptotic expansion is

$$k_m(\theta) = \int dx_2 \cdots dx_m dy_2 \cdots dy_m M[x_2 - ay_2, \ldots, x_m - ay_m] M[y_2, \ldots, y_m] f(x_2 + iy_2, \ldots, x_m + iy_m)$$

(S19)

for $m \geq 2$, and $k_1(\theta) = 0$. Therefore, the constant contribution to the $m$–th cumulant on the parallelogram $u(S)$ may be reconstructed as

$$d_m(\theta) = \sum_{p=1}^{m} \beta_{pm} k_m(\theta).$$

(S20)

Recall that we are after the contribution of a single corner $a_m(\theta)$. In terms of the single corner function, $d_m$ reads

$$d_m(\theta) = -2a_m(\theta) - 2a_m(\pi - \theta),$$

(S21)

where the minus signs come from the definition of $a_m$, see (3). Therefore, the above analytical result for $d_m(\theta)$ is not sufficient to fully reconstruct $a_m(\theta)$ in general. In Appendix III, we demonstrate how this may be circumvented by studying IQH on a cylinder geometry. Using this method, one can obtain numerical estimates for all $a_m$, essentially to arbitrary precision. However, fully analytical calculations seem to be difficult with this approach.

There are nevertheless two interesting cases where the limitations of the parallelogram results can be avoided. The first is the regime $\theta \to 0$ (i.e. $a \to \infty$): since $a_m(\pi) = 0$ (no corner), one can infer the behavior of $a_m(\theta)$ as $\theta \to 0$ from $d_m(\theta)$ alone (Appendix II 2). The other one is $\theta = \pi/2$, in which case $d_m(\pi/2) = -4a_m(\pi/2)$ (Appendix II 3).

2. $1/\theta$ divergence for a single corner

Let us denote by $\tilde{k}_m$ the coefficient of the divergence of $k_m(\theta)$ as $\theta \to 0$, $k_m(\theta) \sim \tilde{k}_m/\theta$. Since $1/\theta \sim a$, we can use the estimate

$$M[x_2 - ay_2, \ldots, x_m - ay_m] \sim aM[y_2, \ldots, y_m]$$

(S22)

as $a \to \infty$ to obtain

$$\tilde{k}_m = \int_{\mathbb{R}^{m-2}} dx_2 \cdots dx_m dy_2 \cdots dy_m M[y_2, \ldots, y_m]^2 f(x_2 + iy_2, \ldots, x_m + iy_m)$$

(S23)

$$= \frac{1}{\sqrt{2\pi m}} \left( \frac{2}{\pi} \right)^{\frac{m}{2}} \int_{\mathbb{R}^{m-1}} dy_2 \cdots dy_m M[y_2, \ldots, y_m]^2 \exp \left[ -2 \sum_{j=2}^{m} y_j^2 + \frac{2}{m} \left( \sum_{j=2}^{m} y_j \right)^2 \right]$$

(S24)

$$= \frac{(m-1)!}{2\sqrt{\pi m} \pi \frac{m}{2}} \int_{y_2 < \cdots < y_m} dy_2 \cdots dy_m \left( \max(0, y_m) - \min(0, y_2) \right)^2 \exp \left[ -\sum_{j=2}^{m} y_j^2 + \frac{1}{m} \left( \sum_{j=2}^{m} y_j \right)^2 \right].$$

(S25)
The above can be simplified to
\[
\bar{k}_m = \frac{m!}{2\sqrt{\pi m\pi^2}} \int_{0<y_2<\ldots<y_m} dy_2 \ldots dy_m y_m^2 \exp \left[ -\sum_{j=2}^{m} y_j^2 + \frac{1}{m} \left( \sum_{j=2}^{m} y_j \right)^2 \right] \tag{S26}
\]
after further manipulations. We managed to compute the integrals analytically for \( m \in \{2, 3, 4\} \) and numerically otherwise. The result can be used to determine the coefficient of the corner divergence \( \kappa_m \) for the \( m \)-th cumulant, once again by expressing the cumulant in terms of traces of powers of \( K \). We obtain in particular
\[
\kappa_2 = \frac{\bar{k}_2}{2} = \frac{1}{4\pi} \simeq 0.07957747155 , \tag{S27}
\]
\[
\kappa_3 = -\frac{-3\bar{k}_2 + 2 \bar{k}_3}{2} = -\frac{3\sqrt{3} - \pi}{4\pi^2} \simeq -0.05204260692 , \tag{S28}
\]
\[
\kappa_4 = -\frac{-7\bar{k}_2 + 12 \bar{k}_3 - 6 \bar{k}_4}{2} = -\frac{12\sqrt{3} + \pi + 18}{4\pi^2} \simeq 0.00904284082 , \tag{S29}
\]
\[
\kappa_5 = -\frac{-15\bar{k}_2 + 50 \bar{k}_3 - 60 \bar{k}_4 + 24 \bar{k}_5}{2} \simeq 0.01168589257 , \tag{S30}
\]
and so on. Those are in perfect agreement with the numerically exact results of the main text, obtained using the method described in Appendix III. Notice the global factor \( 1/2 \) for all cumulants, which accounts for the fact that the parallelogram has two corners with small angle \( \theta \) as \( \theta \to 0 \).

3. \( \theta = \pi/2 \) for a single corner

This case simply corresponds to the square \( a = 0 \). We obtain
\[
k_m(\pi/2) = \int_{\mathbb{R}^{2m-2}} dx_2 \ldots dx_m dy_2 \ldots dy_m M[x_2, \ldots, x_m]M[y_2, \ldots, y_m] f(x_2 + iy_2, \ldots, x_m + iy_m). \tag{S31}
\]
One gets
\[
k_2(\pi/2) = \frac{1}{\pi^2} , \quad k_3(\pi/2) = \frac{3(1 + \sqrt{5}) + \log(\sqrt{5} - 2)}{4\pi^2} , \tag{S32}
\]
after a very long calculation for \( m = 3 \). In terms of pure corner terms for the cumulants, we obtain
\[
a_2(\pi/2) = \frac{h_2}{4} = \frac{1}{4\pi^2} \simeq 0.02533029591 , \tag{S33}
\]
\[
a_3(\pi/2) = -\frac{-3h_2 + 2h_3}{4} = -\frac{3(\sqrt{5} - 1) + \log(\sqrt{5} - 2)}{8\pi^2} \simeq -0.0286810945 , \tag{S34}
\]
where the factor \( 1/4 \) accounts for the fact that there are four corners with angle \( \pi/2 \) in the square.

4. Breakdown of superuniversality for cumulants higher than variance

Consider the ratios \( \kappa_m/a_m(\pi/2) \) which compare the behavior at \( \theta = \pi/2 \) and \( \theta \to 0 \). For IQH states, our previous results imply
\[
\frac{\kappa_2}{a_2(\pi/2)} = \pi , \tag{S35}
\]
\[
\frac{\kappa_4}{a_3(\pi/2)} = \frac{6\sqrt{3} - 2\pi}{3\sqrt{5} - 3 + \log(\sqrt{5} - 2)} \simeq 1.814526527 . \tag{S36}
\]
The ratio for the second cumulant is always \( \pi \) for any theory provided \( f \) does not decay too slowly, due to superuniversality [13]. There is no reason why this would hold for higher cumulants. A counter-example is provided by a free fermion theory with pure gaussian kernel
\[
V(z, w) = \frac{1}{\pi} e^{-|z-w|^2/2} , \tag{S37}
\]
which is similar to the IQH kernel (S14), but simpler for cumulants higher than the variance. Using similar techniques as those described above we get

$$\frac{\kappa_2}{a_2(\pi/2)} = \pi,$$

(S38)

$$\frac{\kappa_3}{a_3(\pi/2)} = \frac{48\sqrt{3} + 5\pi}{45} \approx 2.196586712.$$

(S39)

Hence the ratio for the third cumulant differs from that of IQH, demonstrating a breakdown of superuniversality.

From a technical standpoint, the pure gaussian kernel is significantly simpler than its IQH counterpart at angle $\pi/2$, because it is translationally invariant. In this case it is possible to exploit the results of [38] even further, and get the formula

$$k_m(\pi/2) = \left[ \frac{2m/2-1}{\pi} \sum_{p=1}^{m-1} \frac{1}{\sqrt{p(m-p)}} \right]^2,$$

(S40)

from which one can reconstruct all $a_m(\pi/2)$. Note that for $m$ very large, $k_m(\pi/2) \approx 2m^{-2}$.

### III. Cumulants for IQH states on a cylinder from the overlap matrix

The single-electron Hamiltonian in the Landau gauge for IQH states is given by

$$H = \frac{p_x^2}{2m_e} + \frac{(p_y + eBx)^2}{2m_e},$$

(S41)

where $m_e$ is the effective mass of the electron. The orientation is chosen so that $eB > 0$, and we rescale $x$ and $y$ to set the magnetic length $\ell_B = \sqrt{\hbar/eB}$ to unity. On a two-dimensional cylinder of circumference $l_y$, the eigenstates of $H$ are organized into the Landau level (LL) labelled by $n \in \mathbb{N}$, with wavefunctions

$$\phi_{n,k}(x,y) = \frac{e^{iky}}{\sqrt{2^n n! \ell_y \sqrt{\pi}}} H_n(x + k)e^{-(x+k)^2/2}, \quad k \in 2\pi \mathbb{Z}/\ell_y,$$

(S42)

where $H_n(x)$ are the Hermite polynomials. The many-body IQH state at filling fraction $\nu \in \mathbb{N}^*$ is obtained by entirely filling all LLs with $n < \nu$.

**FIG. S3.** Left: Area-law coefficient $c_2$ for the variance as a function of integer filling $\nu$. Right: Area-law coefficient $c_m$ (orange) and corner cumulant function $|a_m(\pi/2)|$ (blue) for $m = 2, 4, \ldots, 28$. The data ranges over 8 orders of magnitude (log-linear plot). The two solid lines show an ansatz of the form $d_1(m-1)!/d_2^m$, where $d_1$, $d_2$ are fitting parameters.

---

2 If $V(z, w) = V(z - w)$, then $V(z_1 + w, z_2 + w) \cdots V(z_m + w, z_1 + w) = V(z_1, z_2) \cdots V(z_m, z_1)$, but the converse is not true, a counterexample being provided precisely by IQH. Why translation kernel are special is nicely explained in [38].
where aometry of odd cumulants for conserved charges, which translates for corners as from the square. We may play this game for different shapes.

Finally, since such an arrow-shaped region possesses two corners of opening angles $\theta$ and $2\pi - \theta$, we use the symmetry for even cumulants $a_m(\theta) = a_m(2\pi - \theta)$ and divide our result by two to get $a_m(\theta)$.

To obtain the corner contribution for odd cumulants, one must work a little more. Indeed, because of the antisymmetry of odd cumulants for conserved charges, which translates for corners as $a_m(\theta) = -a_m(2\pi - \theta)$, we cannot use arrow-shaped regions to extract $a_m(\theta)$. Instead, we use combinations of different geometries. For example, starting with a square, we obtain $a_m(\pi/2)$. Next, we consider an isosceles right triangle, for which the corners contribution reads $a_m(\pi/2) + 2a_m(\pi/4)$. We deduce $a_m(\pi/4)$ by subtracting the contribution of the angle $\pi/2$ previously obtained from the square. We may play this game for different shapes.

We present in Fig. S4 the normalized corner cumulants functions $a_m(\theta)/\sigma_m$ for $m = 2, 3, 4, \cdots, 10$. One clearly recognizes that the variance stands out. The data of $a_3(\theta)$ and $a_4(\theta)$ for IQH and FQH can be found in Table SI.
IV. Cumulants of massless Dirac fermions

Consider a two-dimensional square lattice, infinite in one direction, say $x$, and impose antiperiodic boundary conditions in the other, $y$. We want to compute the charge cumulants of a section $A$ of the infinite cylinder, i.e. $A$ is a finite cylinder of length $\ell_x$ and circumference $\ell$. We may then take advantage of the symmetry and perform a dimensional reduction along the transverse direction $y$. The lattice Hamiltonian of a 2D free massless Dirac fermion reads

$$H = \frac{i}{2} \sum_{i,j} \left[ \Psi_{i,j}^\dagger \gamma^0 \Psi_{i+1,j} \Psi_{i,j} - \Psi_{i,j} \gamma^0 \Psi_{i,j+1} - \Psi_{i,j} \right] - \text{h.c.},$$

where we set the lattice spacing to unity. The two-dimensional matrices $\gamma^0$ and $\gamma^j$ are proportional to Dirac matrices (e.g. $\gamma^0 = \sigma_3$ and $\gamma^1 = i\sigma_1$, $\gamma^2 = i\sigma_2$, with $\sigma_{1,2,3}$ the Pauli matrices). After dimensional reduction along $y$ (indexed by $j$), the resulting Hamiltonian consists in a sum of decoupled 1D massive free Dirac fermions, $H = \sum_k H_k$,

$$H_k = \sum_i \left[ -\frac{i}{2} \left( \Psi_{i}^\dagger \gamma^0 \Psi_{i+1} - \Psi_{i} \right) - \text{h.c.} \right] + m_k \Psi_{i}^\dagger \gamma^0 \Psi_{i},$$

where $m_k = \sin k_y$ and $k_y = 2\pi(y - 1/2)/\ell$, with $\ell$ the length of the subregion $A$ along $y$. The eigenvalues of the reduced density matrix can be related to those of the correlation matrix $\langle \Psi^\dagger_i \Psi_j \rangle_{A}$ restricted to a region $A$, see [31, 43]. The correlator for the 1D infinite chain is given by

$$\langle \Psi_i^\dagger \Psi_{i'} \rangle = \frac{1}{2} \delta_{ij} + \frac{1}{4\pi} \int_{-\pi}^{\pi} dx \frac{\gamma^0 \gamma^2 m_k + \gamma^0 \gamma^1 \sin x}{\sqrt{m_k^2 + \sin^2 x}} e^{ix(i-i')}.\quad (S49)$$

The expression for the 1D charge cumulants in terms of the eigenvalues $\nu_k$ of $\langle \Psi^\dagger_i \Psi_j \rangle_{A}$ reads

$$C^{(1d)}_m = \sum_i \partial^n \log [1 + (e^\lambda - 1)\nu_k],$$

$$\text{TABLE SI. Cumulant corner function } a_m(\theta), \text{ where } m = 3, 4, \text{ for IQH } \nu = 1 \text{ state (exact), and FQH at } \nu = 1/2, 1/3 \text{ (MC).}$$

| $\theta$ | $\nu = 1$ | $\nu = 1/2$ | $\nu = 1/3$ |
|----------|----------|-------------|-------------|
| $\pi/32$ | $-0.5310420927$ | $-0.1055$ | $-0.08836$ |
| $\pi/16$ | $-0.266641236$ | $-0.07484$ | $-0.05880$ |
| $2\pi/20$ | $-0.1673981982$ | $-0.05617$ | $-0.04479$ |
| $3\pi/20$ | $-0.1120365260$ | $-0.04666$ | $-0.03559$ |
| $4\pi/20$ | $-0.0839258568$ | $-0.03673$ | $-0.02933$ |
| $5\pi/20$ | $-0.066401863$ | $-0.03081$ | $-0.02463$ |
| $6\pi/20$ | $-0.0547456733$ | $-0.02618$ | $-0.02094$ |
| $7\pi/20$ | $-0.0459251086$ | $-0.02242$ | $-0.01794$ |
| $8\pi/20$ | $-0.0390255227$ | $-0.01923$ | $-0.01536$ |
| $9\pi/20$ | $-0.033406126$ | $-0.01652$ | $-0.01318$ |
| $10\pi/20$ | $-0.026810945$ | $-0.01408$ | $-0.01123$ |
| $11\pi/20$ | $-0.0246030765$ | $-0.01190$ | $-0.00953$ |
| $12\pi/20$ | $-0.0210052160$ | $-0.00992$ | $-0.00794$ |
| $13\pi/20$ | $-0.0177700173$ | $-0.00633$ | $-0.00509$ |
| $14\pi/20$ | $-0.0148115522$ | $-0.00469$ | $-0.00374$ |
| $15\pi/20$ | $-0.0120647447$ | $-0.00311$ | $-0.00249$ |
| $16\pi/20$ | $-0.0097996342$ | $-0.00154$ | $-0.00124$ |
| $17\pi/20$ | $-0.0070181942$ | $-0.0000$ | $-0.0000$ |
| $18\pi/20$ | $-0.0042960357$ | $-0.00101$ | $-0.00047$ |
| $19\pi/20$ | $-0.0023015727$ | $-0.00311$ | $-0.00311$ |
| $31\pi/32$ | $-0.00143681734$ | $-0.0000$ | $-0.0000$ |
Since we have performed a dimensional reduction, the charge cumulants are obtain by summing over the modes as \( C_m(A) = \sum_k C_m^{(1d)}(k) \), where \( C_m^{(1d)}(k) \) is the cumulant for the \( k \)th mode associated to \( H_k \). Note that due to the fermion doubling on the lattice, one has to divide the lattice results by 4 to get the charge cumulants corresponding to a Dirac field in the continuum limit.

Since the spectrum of the correlation matrix is symmetric around 1/2, the odd cumulants vanish exactly, no matter the subregion \( A \) one chooses, as expected from charge conjugation symmetry. In contrast, in the limit of large \( \ell, \ell_x \) we find that the even cumulants satisfy an area law, \( C_m(A) = c_m 2\ell + \cdots \), whose coefficients have the following signs: +, +, −, +, −, · · · for \( m = 2, 4, 6, 8, 10, 12 \).

| \( m \) | \( c_m \) |
|---|---|
| 2 | 0.020621681 |
| 4 | 0.0084800 |
| 6 | −0.009303 |
| 8 | 0.00471 |
| 10 | 0.014 |
| 12 | −0.07 |