BOIJ-SÖDERBERG DECOMPOSITIONS OF LEX-SEGMENT IDEALS

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Abstract. Boij-Söderberg theory describes the scalar multiples of Betti diagrams of graded modules over a polynomial ring as a linear combination of pure diagrams with positive coefficients. There are a few results that describe Boij-Söderberg decompositions explicitly. In this paper, we focus on the Betti diagrams of lex-segment ideals and describe the Boij-Söderberg decomposition of a lex-segment ideal in terms of Boij-Söderberg decompositions of some other related lex-segment ideals.

1. Introduction

Boij-Söderberg is very recent theory which addresses to the characterization of Betti diagrams of graded modules in polynomial rings. Its origins are in a pair of conjectures by Boij and Söderberg [2], whose proof is given by Eisenbud and Schreyer in [3], see also [4]. The result is a characterization of Betti tables of graded modules up to scalar multiples. For more information about Boij-Söderberg theory, we refer to [4]. There is not much known about the behavior of the Boij-Söderberg decomposition of an ideal in polynomial rings. Any characterization of Boij-Söderberg decompositions that one obtains will also assist to understand and interpret the more structural consequences of this decomposition of the Betti diagrams.

In this paper, we focus on behavior of the Boij-Söderberg decompositions of lex-segment ideals. Lex-segment ideals have very particular Betti diagrams. The Bigatti-Hulett-Pardue [1] theorem shows that lex-segment ideals have the largest Betti numbers among the ideals with the same Hilbert function. This pivotal property of lex-segment ideals makes their Boij-Söderberg decompositions worthy to study. The main goal is to obtain a pattern for the Boij-Söderbeg decomposition of a lex ideal by using the decompositions of some other related lex-segment ideals. We mainly restrict our attention to the pure Betti diagrams that occur as summands in the decomposition.

In what follows, let $R = k[x, y, z]$ be a polynomial ring of 3 variables, with the lexicographic order, $x >_{lex} y >_{lex} z$ and $L$ be a lex-segment ideal in $R$. The ideal $L$ can be decomposed as $L = xa + J$ where $a$ is also a lex-segment ideal in $R$ and $J$ is a lex-segment ideal in $k[y, z]$. We study some relations of the Betti numbers of the ideals $L$, $a$, and $J$ in Section 2. We describe the entire Betti diagram of the lex ideal $L$ in terms of the Betti numbers of the colon ideal $a = L : (x)$ and the stable ideal $J$. In Section 3, we describe the “beginning of the Boij-Söderberg decomposition” of $L$ in terms of the decomposition of $a$. The algorithm of Boij-Söderberg decomposition itself provides a chain of degree sequences. The first degree sequence in the chain is the top degree sequence of the Betti diagram of $L$. By the algorithm, the second degree sequences is the top degree sequence of the remaining diagram after the subtraction of the first pure diagram with a suitable coefficient from the Betti diagram. It continues until the Betti diagram is decomposed completely. Thus, by saying that “the beginning of the Boij-Söderberg decomposition”, we mean the beginning in the order of the chain of degree sequences in of $L$. Section 3 shows that if there are $t$ degree sequences of the length 3 in the Boij-Söderberg decomposition of $a = L : (x)$, we know the first $t$ degree sequences of length 3 in the decomposition of $L$. We also believe that the results shown in Section 3 could be generalized to the polynomial rings with $n$ variables for finite $n$.

Section 4 is devoted to the pure diagrams of the Boij-Söderberg decomposition of the Betti diagrams of $L$ and $(L, x)$ in the polynomial ring $R = k[x, y, z]$. Like in Section 3, we notice the similarity of the Boij-Söderberg decompositions of lex ideal $L$ and $(L, x)$. We reveal that the entire part of the Boij-Söderberg decomposition of $(L, x)$ containing all pure diagrams of length less than 3 shows up precisely as the last part of the Boij-Söderberg decomposition of $L$, that is, all pure diagrams of length less than 3.

One naturally hopes to obtain the description of entire Boij-Söderberg decomposition of lex-segment ideal $L$. Thus, Section 5 includes further observations for a possible way to describe the entire chain of top degree sequences in the Boij-Söderberg decomposition of $L$. In Sections 3 and 4, we study the cases of $R = k[x, y, z]$, we partly provide a description of the Boij-Söderberg decomposition of lex ideal $L$ in terms
of the lex ideals $a = L : (x)$ and $(L, x)$. However, most of the time, this description does not cover all pure diagrams in the decomposition of $L$ since there might be some pure diagrams of length 3 which are not described. The lexicographic order $x >_{\text{lex}} y >_{\text{lex}} z$ makes us to think about the colon ideals $b = L : (y)$ and $c = L : (z)$. Like for the case $a = L : (x)$ in section 3, one may expect similar results for the lex ideals $b$ and $c$. Indeed, we see a relation between the Boij-Söderberg decompositions of the lex ideal $L$ and the colon ideals $b$ and $c$. This allows us to almost give a full description of the pure diagrams appearing in the decomposition of $L$.

2. Background and Preliminaries

Throughout this section we assume that $R$ is a graded polynomial ring with $n < \infty$ variables over a field $k$ with each variable has degree one. In the case of $n = 3$, we will see the description of the Betti diagram $L = xa + J$ in terms of the Betti numbers of $a$ and $J$.

Let $M$ be a graded $R$-module. The minimal graded free resolution of $M$ is written as

$$
\begin{array}{cccccccc}
F_3 & \rightarrow & \cdots & \rightarrow & F_1 & \rightarrow & \cdots & \rightarrow \ F_0 & \rightarrow & M & \rightarrow & 0
\end{array}
$$

where $F_i = \bigoplus_{j \geq 0} R(-j)^{\beta_{i,j}}$. The numbers $\beta_{i,j}$ are the Betti numbers of $M$ and are considered in the Betti diagram $\beta(M)$ of $M$ whose entry in row $i$ and column $j$ is $\beta_{i,i+j}$.

Let $d = (d_0, d_1, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ be a sequence of non-negative integers of length $n + 1$ with $d_0 < \ldots < d_n$. The graded free resolution of $M$ is called a pure resolution of type $d = (d_0, \ldots, d_i, \ldots, d_n)$ if, for all $i = 0, 1, \ldots, n$, the $i$-th syzygy module of $M$ is generated only by elements of degree $d_i$, in other words, all Betti numbers are zero except $\beta_{i,d_i}(M)$. Then the Betti diagram of this module is called a pure diagram of type $d$. The formula for the pure diagram associated by $d$ is based on the Herzog and Kühl equations introduced in [7].

$$
\beta_{i,j} = \begin{cases} 
\lambda \prod_{i=0, i \neq j}^{n} \frac{1}{|d_i - d_j|} & \text{if } j = d_i \\
0 & \text{otherwise}
\end{cases}
$$

where $\lambda \in \mathbb{Q}$.

We define a partial order on the degree sequences so that $d^s < d^t$ if $d_i^s \leq d_i^t$ for all $i = 0, 1, \ldots, n$. The order on the degree sequences induces an order of the pure diagrams $\pi_{d^s} < \pi_{d^t}$ if $d^s < d^t$. Thus the Boij-Söderberg decomposition of a graded $R$-module $M$ gives an ordered decomposition of the Betti diagram,

$$
\beta(M) = \sum_s a_s \pi_{d^s} \quad \text{where } \pi_{d^s} < \pi_{d^t} \text{ if } s < t.
$$

For instance, let $I = (x^2, xy, xz, y^2)$ be an ideal in $k[x,y,z]$, the Boij-Söderberg decomposition of $R/I$ is given as following

$$
\beta(R/I) = (8)\pi_{d^0} + (4)\pi_{d^1}
$$

where

$$
\pi_{d^0} = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & \frac{1}{2} & - & - & - \\
1 & \frac{1}{4} & \frac{1}{3} & \frac{1}{5}
\end{pmatrix}
$$

and $\pi_{d^1} = \begin{pmatrix}
0 & 1 & 2 \\
\frac{1}{6} & - & - \\
1 & - & \frac{1}{2}
\end{pmatrix}$ as $d^0 = (0, 2, 3, 4) < d^1 = (0, 2, 3)$.

Consider now a monomial ideal $I$ in $R$. We will denote the set of minimal monomial generators of $I$ with $G(I)$ and then $G(I)_i$ will denote the subset of $G(I)$ containing the minimal generators of degree $i$. The notation $a(I)$ will be used for the initial degree of the monomials in $I$ and $e^+(I)$ will stand for the maximum degree of the monomials in $G(I)$ throughout the paper.

**Definition 2.1.** Let $m = x_1^{s_1} \ldots x_n^{s_n}$ and $n = x_1^{t_1} \ldots x_n^{t_n}$ be two monomials in $R = k[x_1, \ldots, x_n]$. If either $\deg m > \deg n$ or $\deg m = \deg n$ and $s_i - t_i > 0$ for the first nonzero index $i$, then it is said that $m >_{\text{lex}} n$ in lexicographic order.

**Definition 2.2.** Let $R$ be a polynomial ring and $L$ be a monomial ideal in $R$ generated by the monomials $m_1, \ldots, m_l$. The ideal $L$ is called a lex-segment ideal (lexicographic ideal, or lex ideal) in $R$ if for each monomial $m \in R$ the existence of some $m_i \in G(L)$ with $m >_{\text{lex}} m_i$ and $\deg(m) = \deg(m_i)$ implies $m \in L$.

For simplicity, we will use “$>$” for the lex order “$>_{\text{lex}}$” unless the order is different than lexicographic order.

In this section, we make some observations about the Betti diagrams of lex-segment ideals. We aim to get some relations between their Betti numbers.
Lemma 2.6. Let $L$ be a lex-segment ideal in $R = \mathbb{k}[x_1, \ldots, x_n]$. Consider the colon ideals $a_i = L : (x_i)$, for $i = 1, \ldots, n$. Then each $a_i$ is also lex-segment ideals in $R$.

Proof. Let $m' \in a_i$ be a monomial, for any $i = 1, \ldots, n$. Let $m$ be a monomial in $R$ and $\deg m = \deg m'$ and $m >_{\text{lex}} m'$. Then $x_i m' \in L$ as $a_i = L : (x_i)$, and $x_i m >_{\text{lex}} x_i m'$. This implies $x_i m \in L$ and hence $m \in L : (x_i) = a_i$. □

Let $u$ be a monomial in $R = \mathbb{k}[x_1, \ldots, x_n]$, we define $m(u)$ to be the largest index $i$ such that $x_i$ divides $u$. Recall that a monomial ideal $I$ is said to be stable if, for every monomial $u \in G(I)$ and all $i < m(u)$, $x_i u / x_{m(u)}$ is also in $G(I)$.

Proposition 2.4. (Eliahou-Kervaire formula, [5]) Let $I \subset R$ be a stable ideal. Then

(a) $\beta_{i,i+j}(I) = \sum_{u \in G(I)} \binom{m(u)}{i}$;
(b) $\text{proj dim } R/I = \max \{ m(u) : u \in G(I) \}$;
(c) $\text{reg } (I) = \max \{ \deg (u) : u \in G(I) \}$.

From now on, we assume $n = 3$, that is, $R = \mathbb{k}[x, y, z]$.

Lemma 2.5. If $L$ is lex-segment ideal in $R$, then there are unique monomial ideals $a \subset R$ and $J \subset \mathbb{k}[y, z]$ such that

$L = xa + J$.

Moreover, the ideal $a$ is also a lex-segment ideal since $a = L : (x)$ and $J$ is stable in $R$, and $G(L) = xG(a) \oplus G(J)$.

Lemma 2.6. Let $0 \to F_2 \to F_1 \to J$ and $0 \to G_3 \to G_2 \to G_1 \to a$ be graded free resolutions for the ideals $J$ and $a$. If $L = a(x) + J$, then there is a short exact sequence

$$0 \to J(-1) \to a(-1) \oplus J \to L \to 0$$

Moreover,

$$0 \to G_3(-1) \oplus F_2(-1) \to G_2(-1) \oplus F_2 \oplus F_1(-1) \to G_1(-1) \oplus F_1 \to L.$$  

is the graded minimal free resolution of $L$.

Proof. The form of the lex-segment ideal $L$ implies the short exact sequence [1]. The mapping cone for the short exact sequence provides a free resolution for $L$. If $m \in G(a) \cap G(J)$ then $mx \in G(L)$ and also $m \in G(L)$ but clearly if $m$ is a minimal generator of $L$ then $mx$ cannot be a minimal generator. Therefore the ideals $J$ and $a$ do not have common minimal generators. This tells us that there is no cancellation in the mapping cone structure. So the resulting graded free resolution for $L$ is minimal. □

First we analyze the Betti numbers of the ideals $L$, $a = L : (x)$ and $J$. We know that the lex-segment ideals $L$ and $a$ are stable and in addition to this, $J$ is a lex ideal in $\mathbb{k}[y, z]$. Thus, Eliahou-Kervaire formula gives rise to the following decomposition,

$$\beta_{i,i+j}(L) = \sum_{u \in G(L)_{j}} \binom{m(u) - 1}{i} + \sum_{u \in G(L)_{j} \text{ and } x_{|u}} \binom{m(u) - 1}{i}$$

$$= \beta_{i,i+j-1}(a) + \sum_{u \in G(L)_{j} \text{ and } x_{|u}} \binom{m(u) - 1}{i}$$

Let’s denote the initial degree of $J$, $a(J) := k$ and the Betti numbers of $\beta(a)$ and $\beta(J)$ as

$$a_{i,i+j} := \beta_{i,i+j}(a), \quad c_{i,i+j} := \beta_{i,i+j}(J).$$

The following remark gives some relations and identities about the Betti numbers of $L$, $a$ and $J$ that will help us to describe the entire Betti diagram of $L$ with respect to the Betti numbers of $a$ and $J$.

Remark 2.7. Recall that $L = xa + J$ in $R = \mathbb{k}[x, y, z]$.

(i) If $a(L) = 1$, then $a = 1$. If $a(L) \geq 2$, then $a(L) = a(a) + 1$ by stability of ideal $L$ and $a = L : (x) \neq 1$.  

(ii) $a(L) = a(a) + 1$.

(iii) $a(L) = a(a) + 1$.

(iv) $a(L) = a(a) + 1$.

(v) $a(L) = a(a) + 1$.

(vi) $a(L) = a(a) + 1$.

(vii) $a(L) = a(a) + 1$.

(viii) $a(L) = a(a) + 1$.

(ix) $a(L) = a(a) + 1$.

(x) $a(L) = a(a) + 1$.

(xi) $a(L) = a(a) + 1$.

(xii) $a(L) = a(a) + 1$.

(xiii) $a(L) = a(a) + 1$.

(xiv) $a(L) = a(a) + 1$.

(xv) $a(L) = a(a) + 1$.

(xvi) $a(L) = a(a) + 1$.

(xvii) $a(L) = a(a) + 1$.

(xviii) $a(L) = a(a) + 1$.

(xix) $a(L) = a(a) + 1$.

(xx) $a(L) = a(a) + 1$.

(xxi) $a(L) = a(a) + 1$.

(xxii) $a(L) = a(a) + 1$.

(xxiii) $a(L) = a(a) + 1$.

(xxiv) $a(L) = a(a) + 1$.

(xxv) $a(L) = a(a) + 1$.

(xxvi) $a(L) = a(a) + 1$.

(xxvii) $a(L) = a(a) + 1$.

(xxviii) $a(L) = a(a) + 1$.
We know that $\beta_{i,j}(L) = \beta_{i,i+1}(a) + D_{i,i}$. Thus, we observe that
if $j \leq k-1$, $D_{i,i} = 0$
if $j \geq k$, $D_{i,i} = \beta_{i,i+1}(J,x)$, it implies that $D_{0,j} = c_{0,j}, D_{1,j+1} = c_{0,j} + c_{1,j+1}$, and $D_{2,j+2} = c_{1,j+1}$.

(iii) The Elishau-Kervaire formula for $a$ gives

\begin{align*}
a_{o,j} = \begin{cases} a_{1,j+1} - a_{2,j+2} + 1 & \text{if } j = a(L) - 1 \\
a_{1,j+1} - a_{2,j+2} & \text{if } j > a(L) - 1 \end{cases}
\end{align*}

(iv) We have the following identities for the Betti numbers of the $J$

- $c_{0,k} = c_{1,k+1} + 1$
- $c_{0,j} = c_{1,j+1}$ for all $j \geq k + 1$
- $c_{0,k} = k + 1$ then $c_{1,k+1} = k$ and $c_{i,i+j} = 0$ for all $i = 0, 1$ and $j \geq k + 1$.

Remark 2.8. \(\min\{s|a_{1,s+1} \neq 0\} \leq \min\{s|a_{2,s+1} \neq 0\}\).

Proof. It follows from the fact that $a$ is stable. \(\square\)

Lemma 2.9. If $a_{o,j-1} = 0$ then $\beta_{0,j}(L) = 0$.

Proof. Let $a_{o,j-1} = 0$. Suppose that $c_{0,j} \neq 0$ so $c_{1,j+1} = c_{0,j} - 1 \geq 0$ and by Remark 2, $\beta_{0,j}(L) \neq 0$. Since $a_{o,j-1} = 0$ and $c_{0,j} \neq 0$, no minimal generator of degree $j$ is divisible by $x$. Thus any minimal generator of degree $j$ is of the form $y^m z^n$ where $m \geq 0, n \geq 0$ and $m + n = j$.

On the other hand, as $e^+(a) > j - 1$ there is a minimal generator $v \in G(L)_{e^+(a)+1}$ such that $x|v$.

Let $v = x^s y^t z^p$ where $s \geq 1$ and $s + t + p = e^+(a) + 1 > j$.

Now we can find a monomial such that $x^s y^t z^{j-s-r} \in L$ where $0 \leq r \leq t$ since $L$ is a lex-segment ideal and so $x^s y^t z^{j-s-r}|v$. Hence $v$ cannot be a minimal generator, that is, $a_{0,e^+(a)} = 0$. This contradicts our assumption. Thus, $c_{0,j} = 0$, then $\beta_{0,j+1} = 0$. \(\square\)

Lemma 2.10. \(\min\{s|a_{2,s+1} \neq 0\} < \min\{s|c_{1,s+1} \neq 0\}\)

Proof. Say $N := \min\{s|a_{2,s+1} \neq 0\}$ and $M := \min\{s|c_{1,s+1} \neq 0\}$. First, recall that $a(L) \geq 2$. Also, recall that $k = a(J)$. Then the Betti diagram for $J$ is

\[
\begin{array}{c|cc}
\beta(J) & 0 & 1 \\
\hline 
\text{k} & c_{0,k} & - \\
\text{k + 1} & - & - \\
\vdots & \vdots & \vdots \\
\text{M - 1} & - & - \\
\text{M} & c_{0,M} & c_{1,M+1} \neq 0 \\
\text{M + 1} & c_{0,M+1} & c_{1,M+2} \\
\vdots & \vdots & \vdots \\
\end{array}
\]

It shows that there exists at least one minimal generator of the form $y^m z^n \in L$ where $m + n = M$ and $n \geq 1$.

As $x^{2m+n-1} > y^m z^n$, $x^{2m+n-1} \in L$.

If $x^{2m+n-1}$ is a minimal generator in $L$, then $a_{2,M+1} = \sum_{u \in G(a)_{M-1}} (m(u)-1) \geq (3-1) \neq 0$.

Therefore, $\min\{s|a_{2,s+1} \neq 0\} \leq M - 1 < M = \min\{s|c_{1,s+1} \neq 0\}$.

If $x^{2m+n-1}$ is not a minimal generator, then $L$ contains a minimal generator that divides $x^{2m+n-1}$ and since $x \notin L$.

There is a minimal generator of the form $x^t$ where $t < m + n - 1 = M - 1$. Then it follows that $2t \neq 0$ and so $\min\{s|a_{2,s+1} \neq 0\} < t < M = \min\{s|c_{1,s+1} \neq 0\}$.

\(\square\)

Lemma 2.11. If $a_{1,j} = 0$ then $\beta_{1,j+1}(L) = 0$

Proof. First of all, if $a_{1,j} = 0$ then $a_{2,j+1} = 0$.

If $a_{0,j-1} = 0$ then by Lemma 2.9, $\beta_{0,j}(L) = 0$, so $\beta_{1,j+1}(L) = 0$.

If $a_{0,j-1} \neq 0$ it is easy to see that the only minimal generator of $a$ of degree $j - 1$ is $x^{j-1}$ since $a_{1,j} =
Then, $a(L) = j$. If $c_{0,j} = 0$ then $c_{1,j+1} = 0$ and therefore $\beta_{1,j+1}(L) = a_{1,j} + c_{0,j} + c_{1,j+1} = 0$. Suppose $c_{0,j} \neq 0$, and as $a(L) = j$, $y^j \in G(L)_j$ but also $x^j \in G(L)_j$. Then by lex-order $xy^{j-1} \in G(L)_j$. This contradicts $a_{1,j} = 0$. □

**Lemma 2.12.** $a(J) \geq e^+(a) + 1$ where $J \neq 0$.

**Proof.** Say $e^+(a) = t$.

Suppose $k = a(J) < t$, then $y^k \in G(L)_k$. So, by lex-order, all monomials $u$ of degree $k$ divisible by $x$ are in $L$. Thus, $u$ is in the form $x^i y^j z^s$ where $s \geq 1$, $i + j + s = k$. As $e^+(a) = t > k$, there is a minimal generator $v \in L$ of degree $t+1$ such that $x|v$. Therefore, $v$ can be written as a product of two monomials $w_1$ and $w_2$ such that $w_2$ is divisible by $x$ and the degree of $w_1$ is $k$, and $w_2$ has degree $t - k$. Since all degree $k$ monomials divisible by $x$ are in $L$, $v$ cannot be a minimal generator.

Thus $k \geq t$.

Now, we need to show that the equality is not possible. Suppose $k = t$.

So $y^k$ is a minimal generator in $L$ and since $t = k$ we can find at least one minimal generator $u$ of $a$ with degree $k$ then $xu$ becomes a minimal generator in $L$ of degree $k + 1$. However all monomials $v$ of degree $k$ divisible by $x$ are in $L$. Then there is a monomial $w$ such that $v = xw$ and $w|u$, but this contradicts that $u$ is a minimal generator of $a$.

Hence $k \neq t$. i.e. $k \geq t + 1$ □

Lemma 2.12 tells us if the Betti diagrams of the ideals $a$ and $J$ overlap then they do only at the $k^{th}$ row of the $\beta(L)$. So if we have the following diagrams for $a$ and $J$; respectively.

| $\beta(a)$ | 0     | 1     | 2     |
|-------------|-------|-------|-------|
| 1           | $a_{0,1}$ | $a_{1,2}$ | $a_{2,3}$ |
| 2           | $a_{0,2}$ | $a_{1,2}$ | $a_{2,4}$ |
| $\vdots$   | $\vdots$ | $\vdots$ | $\vdots$ |
| $k-1$       | $a_{0,k-1}$ | $a_{1,k}$ | $a_{2,k+1}$ |
| $k$         | $a_{0,k}$ | $a_{1,k+1}$ | $a_{2,k+2}$ |

| $\beta(J)$ | 0     | 1     | 2     |
|-------------|-------|-------|-------|
| $k$         | $c_{0,k}$ | $c_{0,k}-1$ | $\vdots$ |
| $k+1$       | $c_{0,k+1}$ | $c_{0,k+1}$ | $\vdots$ |
| $\vdots$   | $\vdots$ | $\vdots$ | $\vdots$ |
| $e^+(J)$    | $c_{0,e^+(L)}$ | $c_{0,e^+(L)}$ | $\vdots$ |

**Table 1.** Betti diagrams of $a$ and $J$.

Then, the Betti diagram for $L$ has the following form:

| $\beta(L)$ | 0     | 1     | 2     |
|-------------|-------|-------|-------|
| 2           | $a_{0,1}$ | $a_{1,2}$ | $a_{2,3}$ |
| 3           | $a_{0,2}$ | $a_{1,2}$ | $a_{2,4}$ |
| $\vdots$   | $\vdots$ | $\vdots$ | $\vdots$ |
| $k-1$       | $a_{0,k-2}$ | $a_{1,k-1}$ | $a_{2,k}$ |
| $k$         | $a_{0,k-1} + c_{0,k}$ | $a_{1,k} + 2c_{0,k} - 1$ | $a_{2,k+1} + c_{0,k} - 1$ |
| $k+1$       | $c_{0,k+1}$ | $2c_{0,k+1}$ | $c_{0,k+1}$ |
| $\vdots$   | $\vdots$ | $\vdots$ | $\vdots$ |
| $e^+(L) = e^+(J)$ | $c_{0,e^+(L)}$ | $2c_{0,e^+(L)}$ | $c_{0,e^+(L)}$ |

**Table 2.** Betti diagram of $L$.

3. **The Boij-Söderberg Decompositions of $L$ and $L : (x)$**

In this section we identify the beginning of the Boij-Söderberg decomposition of a lex-segment ideal. More precisely, the next theorem shows that if $d^0 < d^1 < \ldots < d^t < \ldots < d^i$ is the chain of all length 3 top degree sequences in the Boij-Söderberg decomposition of the Betti diagram of $a = L : (x)$ and the chain of the first $t + 1$ top degree sequences of the Boij-Söderberg decomposition of the Betti diagram of
L is $\overline{d}^0 < \overline{d}^1 < ... < \overline{d}^i < ... < \overline{d}^t$ then $\overline{d}^i = d^i + 1 = (d^i_0 + 1, d^i_1 + 1, d^i_2 + 1)$ for all $i = 0, 1, ..., t$ with exactly the same coefficients, except possibly the coefficient of $\pi_{d^i}$.

**Theorem 3.1.** Let $R = k[x, y, z]$ and $L$ be a lex-segment ideal of codimension 3 in $R$. Suppose $1 \neq a = L : (x)$.

Write the Boij-Söderberg decomposition of $a$ as

$$\beta(a) = \sum_{i=0}^{t} \alpha_i \pi_{d^i} + R_a,$$

where $d^0 < d^1 < ... < d^t$ are all top degree sequences of length 3, that is, $d^i = (d^i_0, d^i_1, d^i_2)$ for $i = 0, 1, ..., t$, and $R_a$ is the linear combination of the pure diagrams greater than $\pi_{d^i}$. Then the Boij-Söderberg decomposition of $L$ has the form

$$\beta(L) = \sum_{i=0}^{t} \tilde{\alpha}_i \pi_{\overline{d}^i} + R_L$$

where $\overline{d}^i = d^i + 1 = (d^i_0 + 1, d^i_1 + 1, d^i_2 + 1)$, and $\tilde{\alpha}_i = \alpha_i$ for $i = 0, 1, ..., t$ and $\tilde{\alpha}_t \geq \alpha_t$, and $R_L$ is a linear combination of pure diagrams greater than $\pi_{d^t}$.

**Proof.** Recall that, for a given top degree sequence $d = (d_0, d_1, d_2)$, the “normalized” pure diagram $\pi_{d}$ can be obtained as following:

$$\tilde{\beta}_{i, i+j}(\pi_{d}) = \begin{cases} 0 & \text{if } i + j \neq d_i \\ \prod_{r=0, r \neq i}^2 \frac{\lambda}{d_i - d_r} & \text{if } i + j = d_i, \text{ where } \lambda = \text{lcm} \left( \prod_{r=0, r \neq i}^2 |d_i - d_r|, i = 0, 1, 2 \right). \end{cases}$$

Thus, this formula provides pure diagrams with integer entries. From now on, we always consider “normalized” pure diagrams, that is, pure diagrams with integer entries.

Let $d^0 = (d^0_0, d^0_1, d^0_2)$ be the top degree sequence for the Betti diagram of $a$. If $d^0_2 < k + 1$, that is, $d^0_0 < d^0_1 < d^0_2 < k + 1$, so $d^0_0 < k - 1$. Then we see that $\tilde{\beta}_{i, i+j}(a) = \beta_{i, i+j+1}(L)$ for all $j = 0, 1, ..., k - 2$ since the Betti diagrams of $a$ and $J$ may only overlap on the $k$-th row in the Betti diagram of $L$. As $L = xa + J$ and degree shift due to multiplication by $x$ the top degree sequence of $\beta(L)$ will be $d^0$. Thus $\beta(L) - \alpha_0 \pi_{d^0+1}$ becomes the first step of the Boij-Söderberg-decomposition of $\beta(L)$. Actually we generalize this for all degree sequence $d^a$ such that $d^a_2 < k + 1$.

Suppose $d^a_2 < k + 1$ for all $s = 0, 1, ..., l - 1$, then we have a chain $d^a_2 < d^a_2 < ... < d^a_{l-1} < k + 1$. Therefore, after $l$ steps of the algorithm, we would get the remaining diagram

$$\beta(a) - \sum_{s=0}^{l-1} \alpha_s \pi_{d^s} =: \tilde{\beta}(a) \quad \text{and} \quad \beta(L) - \sum_{s=0}^{l-1} \alpha_s \pi_{d^s+1} =: \tilde{\beta}(L).$$

Let $d^1 = (d^1_0, d^1_1, d^1_2)$ be the next top degree sequence of the Betti diagram for $a$ and $d^1_2 = k + 1$ so above paragraph shows that $d^1 + 1$ becomes the next top degree sequence of Betti diagram for $L$. Therefore the remaining diagrams after the first $l$ steps of the Boij-Söderberg decompositions for both $a$ and $L$ look like as following,

| $\beta(a)$ | 0   | 1   | 2   |
|------------|-----|-----|-----|
| $\beta_{0, d^0}(a)$ | -   | -   | -   |
| $\beta_{1, d^1}(a)$ | -   | -   | -   |
| $\beta_{2, d^2}(a)$ | -   | -   | -   |

| $\beta(a)$ | $d^1_0$ | $d^1_1$ | $d^1_2 - 1$ | $a_{0,d^1_0}$ | $a_{1,d^1_1}$ | $a_{2,d^1_2}$ |
|------------|---------|---------|-------------|-------------|-------------|-------------|
| $\beta(a)$ | $\beta_{0, d^1}(a)$ | $\beta_{1, d^1}(a)$ | $\beta_{2, d^1}(a)$ | -           | -           | -           |
| $\beta(a)$ | $d^1_0 - 1$ | $a_{0,d^1_0-1}$ | $a_{1,d^1_0-1}$ | $a_{2,d^1_0-1}$ |

**Table 3.** Remaining diagram after $l$ steps for $\beta(a)$

and similarly,
already shows that
Moreover we assume that it will vanish in the (l + 1)-th row of the Boij-Söderberg decomposition of L, \( \beta(L) \).

In other words, we only need to think about the top degree sequences \( d_0 + 1 < k \) and \( d_1 < k \)

The decomposition algorithm exposes the coefficient of the pure diagram \( \pi_{d^l} \) to be

\[
\alpha_l = \min \left\{ \frac{\hat{\beta}_0, d_{l+1}^l (\pi_{d^l})}{\hat{\beta}_0, d_{l}^l (\pi_{d^l})}, \frac{\hat{\beta}_1, d_{l+1}^l (\pi_{d^l})}{\hat{\beta}_1, d_{l}^l (\pi_{d^l})}, \frac{a_{2, d^l_1} + c_{1, d^l_1}}{a_{2, d^l_1}} \right\}
\]

and similarly for the Boij-Söderberg-decomposition of \( \beta(L) \) there is a rational number \( \hat{\alpha}_l \) as the coefficient of the pure diagram \( \pi_{d^l+1} \) such that

\[
\hat{\alpha}_l = \min \left\{ \frac{\hat{\beta}_0, d_{l+1}^l (\pi_{d^l})}{\hat{\beta}_0, d_{l}^l (\pi_{d^l})}, \frac{\hat{\beta}_1, d_{l+1}^l (\pi_{d^l})}{\hat{\beta}_1, d_{l}^l (\pi_{d^l})}, \frac{a_{2, d^l_1} + c_{1, d^l_1}}{a_{2, d^l_1}} \right\}
\]

Hence we just need to look at the k-th row of the Betti diagram of L if \( \beta(a) \) and \( \beta(L) \) overlap. Thus, we only need to think about the top degree sequences \( d^l \) of length 3 of \( \beta(a) \) such that \( d^l_2 = k + 1 \).

**CASE I:** Let \( a_{2, k+1} \) be eliminated in the \( (l + 1) \)-th step of the decomposition algorithm of \( \beta(a) \).

In other words, \( d^l = (d^l_0, d^l_1, d^l_2) \) is of length 3, but \( d^{l+1} = (d^{l+1}_0, d^{l+1}_1, d^{l+1}_2) \) has length 2. It shows that \( d^0 < d^1 < ... < d^l < ... < d^{l+1} \) are all length 3 degree sequences in the decomposition of \( \beta(a) \). Hence, Boij-Söderberg-decomposition of \( \beta(a) \) is as

\[
\beta(a) = \sum_{s=0}^{l} \alpha_s \pi_{d^s} + [\text{all pure diagrams of length less than 3}].
\]

So we do not need to pay attention to the \( (l + 2) \)-th step in the decomposition. Besides the diagram already shows that \( d^l + 1 \) is top degree sequence of the remaining diagram of \( L, \beta(L) \). Therefore the first \( (l + 1) \)-th top degree sequences of Boij-Söderberg decomposition of \( \beta(L) \) is

\[
d^0 + 1 < d^1 + 1 < ... < d^l + 1
\]

where the coefficients \( \hat{\alpha}_s = \alpha_s \) for \( i = 0, 1, ..., l - 1 \).

**CASE II:** Suppose that \( a_{2, k+1} \) is not eliminated in the \( (l + 1) \)-th step of the decomposition of \( \beta(a) \).

Moreover we assume that it will vanish in the \( (t + 1) \)-th step for some \( t > 1 \). That is, the chain of the degree sequences in the Boij-Söderberg decomposition of \( \beta(a) \) is

\[
d^0 < d^1 < ... < d^t < ... < d^{l+1}
\]

where,

- for \( s = 0, 1, ..., l - 1 \), \( d^s = (d^s_0, d^s_1, d^s_2) \) has length 3 such that \( d^s_2 < k + 1 \),
- for \( s = l, ..., t \), \( d^s = (d^s_0, d^s_1, d^s_2) \) has length 3 such that \( d^s_2 = k + 1 \) and
- for \( s = t + 1, ..., n \), \( d^s = (d^s_0, d^s_1) \) has length 2.

As the entries only above the \((k-1)\)-th row are eliminated until the \((t+1)\)-th step of the decomposition, it is not difficult to guess the remaining diagram of \( L \).

In the previous section we have seen that the entries above the \( k \)-th in \( \beta(a) \) are the same entries in \( \beta(L) \). Let the remaining diagram of \( \beta(a) \) after subtracting the first \( t \) pure diagrams be

| \( \beta(L) \) | 0 | 1 | 2 |
|---|---|---|---|
| \( d^l_0 + 1 \) | \( \hat{\beta}_{0, d^l_0+1}(L) \) | - | - |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( d^l_1 \) | \( a_{0, d^l_1-1} \) | \( \hat{\beta}_{1, d^l_1+1}(L) \) | - |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( d^l_2 \) | \( a_{0, d^l_2} + c_{0, d^l_2} \) | \( a_{1, d^l_2} + c_{1, d^l_2} \) | \( \hat{\beta}_{2, d^l_2+1}(L) \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |

Table 4. Remaining diagram after \( l \) step for \( \beta(L) \)
\[ \beta(a) - \sum_{s=0}^{t-1} \alpha_s \pi_{d^s} = \begin{array}{c|cc}
\beta_1(a) & 0 & 1 & 2 \\
\hline
\beta''_0(d'_0) & - & - \\
\vdots & \vdots & \vdots & \vdots \\
\beta_{1,d'_1}(a) & - \\
\beta''_2(d''_2) & \beta_{2,d''_2}(a) \\
\end{array} \]

where \( \beta_{i,d'_i}(a) = \beta_{1,d'_1}(a) - \sum_{s=0}^{t-1} \alpha_s \beta_{i,d'_i}(\pi_{d^s}) \), for \( i = 0, 1, 2 \).

Furthermore, as in (2) and (3), we can observe similar relations between the coefficients in both Boij-Söderberg decomposition of \( \beta(a) \) and \( \beta(L) \) during their first \( t \) steps. The coefficients of the pure diagrams \( \pi_{d^s} \) in the decomposition of \( \beta(a) \) for \( s = l, \ldots, t - 1 \) is

\[ \alpha_s = \min \{ \frac{\beta_{i,d'_i}(a)}{\beta_{i,d'_i}(\pi_{d^s})} \} \quad \text{for} \quad i = 0, 1, 2 \} \]

Similarly, the corresponding coefficient \( \tilde{\alpha}_s \) of the pure diagram \( \pi_{d^s+1} \) in the decomposition of \( \beta(L) \) becomes

\[ \tilde{\alpha}_s = \min \left\{ \frac{\beta_{i,d'_i+1}(L)}{\beta_{i,d'_i+1}(\pi_{d^s+1})} \right\} \quad \text{for} \quad i = 0, 1, 2 \}

\[ = \min \left\{ \frac{\beta_{0,d'_0}(a)}{\beta_{0,d'_0}(\pi_{d^s})} \beta_{1,d'_1}(a) \beta_{2,d'_2}(a) + c_{1,d'_2} \right\} \]

We assume that any of the entries corresponding to \( d''_s \) for \( i = 0, 1 \) would be eliminated where \( s = l, \ldots, t - 1 \). Thus

\[ \alpha_s < \frac{\beta_{2,d''_2}(a)}{\beta_{2,k+1}(\pi_{d^s})} \quad \text{where} \quad d''_s = k + 1. \]

So it follows that

\[ \frac{\beta_{2,d'_2}(a)}{\beta_{2,k+1}(\pi_{d^s})} < \frac{\beta_{2,d'_2}(a) + c_{1,k+1}}{\beta_{2,k+1}(\pi_{d^s})} \]

Hence \( \tilde{\alpha}_s = \alpha_s \) for \( s = l, \ldots, t - 1 \). However, this situation may change for the coefficients \( \alpha_i \) and \( \tilde{\alpha}_i \) since \( \beta_{2,d'_2}(a) \) will be eliminated in the next step. So \( \alpha_i \leq \tilde{\alpha}_i \).

Hence the remaining diagram of \( \beta(L) \) is

\[ \tilde{\beta}(L) := \beta(L) - \sum_{s=0}^{t-1} \alpha_0 \pi_{d^s+1} = \begin{array}{c|cc}
\tilde{\beta}_1(a) & 0 & 1 & 2 \\
\hline
\tilde{\beta}_{0,d'_0+1}(L) & - & - \\
\vdots & \vdots & \vdots & \vdots \\
\tilde{\beta}_{1,d'_1+1}(L) & - \\
\tilde{\beta}_{2,d'_2+1}(L) & \beta_{2,d'_2}(a) + c_{1,d'_2} \\
\end{array} \quad \text{where} \quad d''_s = k < \beta_{0,d'_0+1}(L) = \beta_{0,d'_0}(a) \quad \text{and} \quad \tilde{\beta}_{1,d'_1+1}(L) = \tilde{\beta}_{1,d'_1}(a) \quad \text{as} \quad d'_0 + 1 < k \quad \text{and} \quad d'_1 < k \]

This will bring us back to Case I, \( d^t = (d'_0, d'_1, d''_2) \) is the last top degree sequence of length 3 in the Boij-Söderberg decomposition of \( \beta(a) \). Above remaining diagram clearly shows us that \( d^t + 1 = (d'_0 + 1, d'_1 + 1, d''_2 + 1) \) is a degree sequence in the Boij-Söderberg decomposition of \( \beta(L) \).

As a summary, if \( d^0 < d^1 < \ldots < d^t \) is the chain of the all top degree sequences of length 3 in the Boij-Söderberg-decomposition of \( \beta(a) \) with coefficients \( \alpha_s \) for \( s = 0, 1, \ldots, t \). Then \( d^0 + 1 < d^1 + 1 < \ldots < d^t + 1 \) becomes the initial \( t \) top degree sequences of length 3 in the Boij-Söderberg-decomposition of \( \beta(L) \) with \( \tilde{\alpha}_s = \alpha_s \) if \( s < t \) and \( \tilde{\alpha}_t \geq \alpha_t \).
Remark 3.2. We believe that this result can be generalized to the lex-ideals in \( k[x_1, \ldots, x_n] \).

Let \( L = (x_1) + J \) in \( R = k[x_1, x_2, \ldots, x_n] \) be a lex-segment ideal, then \( a \) is also lex-segment ideal in \( R \) and \( J \) turns out to be a stable ideal of codim \( n-1 \) in \( k[x_2, \ldots, x_n] \). Suppose

\[
F_{n-1} \to \cdots \to F_1 \to J \to 0
\]

\[
G_n \to \cdots \to G_i \to \cdots \to G_1 \to a \to 0
\]

are the minimal free resolutions of \( J \) and \( a \), respectively. We get the same short exact sequence like in Lemma \( \text{[26]} \) then by mapping cone we have the following minimal free resolution for \( L \)

\[
0 \to G_n(-1) \oplus F_{n-1} \to \cdots \to G_2(-1) \oplus F_2 \oplus F_1(-1) \to G_1(-1) \oplus F_1 \to L.
\]

So it yields \( \beta_{i,i+j}(L) = \left\{
\begin{array}{ll}
\beta_{i,i+1}(a) & \text{where } i = 0, 1, \ldots, n-1 \text{ and } i+j < a(J), \\
\beta_{i,i+1}(a) + \sum_{t=i-1}^{i} \beta_{t,i+1}(J) & \text{where } i = 0, 1, \ldots, n-1 \text{ and } i+j \geq a(J).
\end{array}
\right.
\]

By using lex-order properties of \( L \) and \( a \), as we did in case \( n = 3 \), we conclude that the Betti diagrams of \( a \) and \( J \) either overlap only on the \( a(J) \)-th row of the Betti diagram of \( L \) or do not overlap at all. Identify \( k := a(J) \). Therefore, the Betti diagram of \( L \) in \( k[x_1, \ldots, x_n] \) is

| \( \beta(L) \) | 0 | 1 | 2 | \ldots | n-1 |
|---|---|---|---|---|---|
| 2 | \( a_{0,1} \) | \( a_{1,2} \) | \( a_{2,3} \) | \ldots | \( a_{n-1,n} \) |
| 3 | \( a_{0,2} \) | \( a_{1,3} \) | \( a_{2,4} \) | \ldots | \( a_{n-1,n+1} \) |
| \vdots | \vdots | \vdots | \vdots | \ldots | \vdots |
| \( k-1 \) | \( a_{0,k-2} \) | \( a_{1,k-1} \) | \( a_{2,k} \) | \ldots | \( a_{n-1,k-1-n} \) |
| \( k \) | \( a_{0,k-1} + c_{0,k} \) | \( a_{1,k} + c_{0,k} + c_{1,k} \) | \( a_{2,k+1} + c_{1,k+1} + c_{2,k} \) | \ldots | \( a_{n-1,k+n-2} + c_{n-1,k+n-1} \) |
| \( k+1 \) | \( c_{0,k+1} \) | \( c_{0,k+1} + c_{1,k+2} \) | \( c_{1,k+2} + c_{2,k+3} \) | \ldots | \( c_{n-1,k+n-1} \) |
| \( \beta(L) \) | \( c_{0,e^+(L)} \) | \( c_{0,e^+(L)} + c_{1,e^+(L)} \) | \( c_{1,e^+(L)} + c_{2,e^+(L)} \) | \ldots | \( c_{n-1,e^+(L)+n-1} \) |

Table 5. Betti diagram of \( L \) in \( k[x_1, \ldots, x_n] \)

We believe that the proof of Theorem \( \text{[31]} \) can be modified for the polynomial ring of \( n \) variables. Hence one concludes that if \( \pi_{d^0} < \pi_{d^1} < \ldots < \pi_{d^t} \) are all pure diagrams of length \( n \) in the Boij-Söderberg decomposition of \( a \), where \( d^i = (d^i_0, d^i_1, \ldots, d^i_{n-1}) \) for \( i = 0, 1, \ldots, t \). Then the chain of pure diagrams

\[
\pi_{d^0} < \pi_{d^1} < \ldots < \pi_{d^t}
\]

appears in the beginning of the Boij-Söderberg decomposition of \( L \) such that \( d^i = d^{i-1} + 1 \). However, the details require further work.

4. The Boij-Söderberg Decomposition for \((L, x)\)

In Section 3, we described the beginning of the chain of the degree sequences in the Boij-Söderberg decomposition of \( \beta(L) = xa + J \) in terms of the decomposition of \( \beta(a) \). Now we consider the end of the Boij-Söderberg decomposition of \( L \) in \( R = k[x, y, z] \).

We conjecture that all degree sequences of length less than 3 in the decomposition of \( \beta(L, x) = \beta(J, x) \) occur precisely as all degree sequences of length less than 3 in the decomposition for \( L \).

We give the proof of this statement for all Artinian lex-segment ideals \( L = a(x) + J \) except the ones of the form \( L = x(x, y, z^t) + J \) where \( J \) is different that \( (y, z)^{a(J)} \) and \( 1 < t < k-1 \). Actually we believe that the statement is also true of this particular case of \( L \), however proof of this particular case requires a case analyzing which becomes infeasible.

Theorem 4.1. Let \( L \subset R = k[x, y, z] \) be an Artinian lex-segment ideal of codimension 3. Suppose that \( L \) cannot be decomposed as \( L = x(x, y, z^t) + J \) where \( J \) is different that \( (y, z)^{a(J)} \) and \( 1 < t < k-1 \).

Let \( a = L : (x) \) be a lex-segment ideal of \( R \). Then \( L = xa + J \) where \( J \in k[y, z] \) is also a stable ideal of codim 2. The ideal \( (J, x) = (L, x) \) is also a codim 3 Artinian, lex-segment ideal in \( R \).

\[
\beta(L, x) = R_{(L, x)} + \sum_{i=1}^{n} a_i \pi_{d^i}
\]
where \(d^{t+1} < d^{t+2} < \ldots < d^n\) are all top degree sequences of length less than 3, with the coefficients \(\alpha_i\), \(i = t + 1, \ldots, n\). \(R_{(L,x)}\) is the linear combination of the pure diagrams associated with the degree sequences of length 3.

Then the Boij-Söderberg decomposition of \(L\) is

\[
\beta(L) = R_L + \sum_{i=t+1}^{n} \alpha_i \pi_{d^i}
\]

the chain \(d^{t+1} < d^{t+2} < \ldots < d^n\) of degree sequences of length 2 and 1 exactly with the same coefficients \(\alpha_i\) and \(R_L\) is the linear combination of the pure diagrams associated with the degree sequences of length 3.

Proof. First let’s observe the decomposition of the Betti diagram of \((L, x)\). Say \(e^+(L, x) = e^+(L) = n\). Suppose \(k = a(J) > 2\) and \(n \geq k + 1\). So the diagram has the following form:

| \(\beta(L, x)\) | 0 | 1 | 2 |
|-----------------|---|---|---|
| 1               | 1 | – | – |
| 2               | – | – | – |
| \(k - 1\)       | – | – | – |
| \(k\)           | \(c_{0,k}\) | \(2c_{0,k} - 1\) | \(c_{0,k} - 1\) |
| \(k + 1\)       | \(2c_{0,k+1}\) | \(c_{0,k+1}\) | \(\vdots\) |
| \(n\)           | \(c_{0,n}\) | \(2c_{0,n}\) | \(c_{0,n}\) |

We aim to show that before the entry \(\beta_{2,k+3}(L, x) = c_{0,k} - 1\) gets eliminated, \(\beta_{0,1}(L, x) = 1\) is eliminated.

First degree sequence is \(\bar{d}^0 = (1, k + 1, k + 2)\), then we have \(\beta(L, x) - \gamma_0 \pi_{\bar{d}^0}\) where \(\pi_{\bar{d}^0} = \begin{pmatrix}
1 & 1 & 2 \\
0 & 2 & 1 \\
1 & \frac{k}{3} & \frac{k+2}{3}
\end{pmatrix}\) and \(\gamma_0 = \min\{1, \frac{2c_{0,k}-1}{k+1}, \frac{c_{0,k}-1}{k}\} = \frac{c_{0,k}-1}{k}\). The next degree sequence becomes \(\bar{d}^1 = (1, k + 1, k + 3)\) and then the pure diagram is \(\pi_{\bar{d}^1} = \begin{pmatrix}
1 & 1 & 2 \\
0 & 2 & 1 \\
1 & \frac{k}{3} & \frac{k+2}{3}
\end{pmatrix}\). Then the coefficient can be obtained as

\[
\gamma_1 = \min\left\{\frac{1}{2} - \frac{c_{0,k}-1}{2k}, \frac{c_{0,k}}{k}, \frac{c_{0,k+1}}{k+1}, \frac{c_{0,k+2}}{k+2}\right\} = \frac{1}{2} - \frac{c_{0,k}-1}{2k}.
\]

Hence we have eliminated the entry \(\beta_{0,1}(L, x)\), then the remaining diagram is

\[
\beta(L, x) - \gamma_0 \pi_{\bar{d}^0} - \gamma_1 \pi_{\bar{d}^1} = \begin{pmatrix}
0 & 1 & 2 \\
k & c_{0,k} & \frac{(k+2)(c_{0,k+1} - k)}{k} - \frac{1}{2} - \frac{c_{0,k+1} - 1}{2k} \\
k+1 & c_{0,k+1} & 2c_{0,k+1} \\
k & \vdots & \vdots \\
k+1 & \vdots & \vdots \\
k+2 & \vdots & \vdots \\
k+3 & \vdots & \vdots \\
n & \frac{c_{0,n}}{k} & \frac{2c_{0,n}}{k} \\
n & \vdots & \vdots \\
n & \frac{c_{0,n}}{k} & \frac{2c_{0,n}}{k}
\end{pmatrix}
\]

Next, \(\bar{d}^2 = (k, k + 1, k + 3)\) and \(\pi_{\bar{d}^2} = \begin{pmatrix}
0 & 1 & 2 \\
k & 2 & 3 \\
k+1 & 2 & 1 \\
2k & \frac{c_{0,k+1}}{k+1} & \frac{c_{0,k+1}}{k+1}
\end{pmatrix}\). The corresponding coefficient is \(\gamma_2 = \min\{\frac{3(c_{0,k+1}-k)}{6}, \frac{k+1-c_{0,k}}{2}\}\). This brings us to two separate cases:

(i) If \(c_{0,k+1} < \frac{k}{3}\), then \(\gamma_2 = c_{0,k+1} - \frac{k+1-c_{0,k}}{2}\). Then,

\[
\beta(L, x) - \sum_{i=0}^{2} \gamma_i \pi_{\bar{d}^i} = \begin{pmatrix}
0 & 1 & 2 \\
k & k+1 - 2c_{0,k} & k - 3c_{0,k+1} \\
k+1 & c_{0,k+1} & 2c_{0,k+1} \\
k & \vdots & \vdots \\
k+1 & \vdots & \vdots \\
k+2 & \vdots & \vdots \\
n & c_{0,n} & 2c_{0,n} \\
n & \vdots & \vdots \\
n & c_{0,n} & c_{0,n}
\end{pmatrix}
\]

(ii) If \(\frac{k}{3} < c_{0,k+1}\), then \(\gamma_2 = \frac{3(c_{0,k}-1) - k}{6}\). So, we obtain
\[ \beta(L, x) - \sum_{i=0}^{2} \gamma_i \pi_{d^i!} = \begin{array}{ccc} k & 0 & 1 \\
 \frac{k+1}{2} + c_{0,k} & c_{0,k+1} & - \\
 & 2c_{0,k+1} & c_{0,k+1} \\
 & : & : \\
 n & c_{0,n} & 2c_{0,n} \\
 \end{array} \]

Now we examine the Boij-Söderberg decomposition of the lex ideal \( L \). First of all, as a trivial case, we must notice that if \( a(L) = 1 \), then the statement is vacuously true since \( L = (L, x) \).

We induct on the the difference of the initial degrees \( a(J) - a(\mathbf{a}) \geq 1 \).

**Base Step:** In this step, we show that the statement is true for the lex ideals \( L = xa + J \) when \( a(J) - a(\mathbf{a}) = 1 \). That is, if \( a(J) = k \geq 2 \) then \( a(\mathbf{a}) = k - 1 \). So \( \mathbf{a} = (x, y, z)^{k-1} \) since \( L \) is a lex ideal.

Thus we modify the Betti diagram of \( L \) in the Table (2) to this case,

\[ \beta(L) = \begin{array}{ccc} k & 0 & 1 \\
 \frac{k(k+1)}{2} + c_{0,k} & (k-1)(k+1) + 2c_{0,k} - 1 & \frac{k(k-1)}{2} + c_{0,k} - 1 \\
 & c_{0,k+1} & c_{0,k+1} \\
 & : & : \\
 n & c_{0,n} & 2c_{0,n} \\
 \end{array} \]

Then obviously \( d^0 = (k, k+1, k+2) \) and \( c_0 = \frac{k(k-1)}{2} + c_{0,k} - 1 \). Then \( d^1 = (k, k+1, k+3) \) becomes the next degree sequence with the coefficient \( c_1 = \min\{ \frac{k}{2}, c_{0,k+1} \} \).

(i) If \( \alpha_1 = c_{0,k+1} < \frac{k}{2} \), then

\[ \beta(L) - \sum_{i=0}^{1} \alpha_i \pi_{d^i!} = \begin{array}{ccc} k & 0 & 1 \\
 k + 1 - 2c_{0,k+1} & k - 3c_{0,k+1} & - \\
 & c_{0,k+1} & - \\
 & : & : \\
 n & c_{0,n} & 2c_{0,n} \\
 \end{array} \]

(ii) If \( \alpha_1 = \frac{k}{2} < c_{0,k+1} \) then the remaining diagram of of \( \beta(L) \) after three steps becomes

\[ \beta(L) - \sum_{i=0}^{2} \alpha_i \pi_{d^i!} = \begin{array}{ccc} k & 0 & 1 \\
 \frac{k+1}{2} + c_{0,k+1} & 2c_{0,k+1} & c_{0,k+1} - \frac{k}{2} \\
 & : & : \\
 n & c_{0,n} & 2c_{0,n} \\
 \end{array} \]

Thus, \( \beta(L) \) and \( \beta(L, x) \) have exactly the same remaining diagrams in the decomposition. Hence, the statement holds for the case of \( a(J) - a(\mathbf{a}) = 1 \).

**Induction Hypothesis:** Let the statement be true for all lex ideals \( L = xa + J \) satisfying \( a(J) - a(\mathbf{a}) = N \geq 1 \). We need to show that it is also true for the lex ideals satisfying \( a(J) - a(\mathbf{a}) = N + 1 \). We identify the initial degrees of \( J \) and \( \mathbf{a} \) by \( a(J) = k \) and \( a(\mathbf{a}) = m \).

Suppose that \( L = xa + J \) is a lex ideal such that \( k - m = N + 1 \). So \( k - m = N + 1 \geq 2 \). We prove this into two cases.

**CASE I:** If \( y^m \not\in \mathbf{a} \). Since \( \mathbf{a} \) is a lex ideal, we write \( \mathbf{a} = xb + I \). Then we notice that \( a(I) \neq k \) otherwise it contradicts to \( y^k \in G(J) \). Thus \( k \geq a(I) \geq m \) as \( y^m \not\in \mathbf{a} \).

Define \( \tilde{a} \subset \mathbf{a} \) as the ideal containing all monomials of \( \mathbf{a} \) of degree greater or equal to \( m + 1 \). One can easily check that \( \tilde{a} \) is also a lex ideal with \( a(\tilde{a}) = m + 1 \). Define \( \tilde{L} = x\tilde{a} + J \) and it is a lex ideal with \( a(J) - a(\tilde{a}) = k - (m + 1) = k - m - 1 = N + 1 - 1 = N \). Therefore by the induction hypothesis, \( \beta(\tilde{L}) \) and \( \beta(L, x) \) have the same ends in their Boij-Söderberg decompositions, i.e. same pure diagrams of length less than 3 with same coefficients,

\[ \beta(\tilde{L}) - \sum_{d^i!} \alpha_{d^i!} = \beta(L, x) - \sum_{d^i!} \gamma_{d^i!}. \]

On the other hand, \( \tilde{a} \) can be decomposed as \( \tilde{a} = x\tilde{b} + \tilde{I} \). It is easy to see that \( \tilde{I} = I \) as \( y^m \not\in \mathbf{a} \) and \( a(\tilde{b}) = m \). Clearly, \( a(I) - a(\tilde{b}) \leq (k - 1) - m = N \). Thus, again by the hypothesis Boij-Söderberg decompositions of \( \beta(\tilde{a}) \) and \( \beta(I, x) \) have the same ends.
Recall that \( a = xb + I \), so we get \( a(I) - a(b) \leq (k - 1) - (m - 1) = k - m = N + 1 \).

Suppose that \( a(I) - a(b) < N + 1 \), then the hypothesis provide the results, that is, Boij-Söderberg decompositions of \( a \) and \( (I, x) \) have the same ends, so do \( a \) and \( \tilde{a} \). That is,

\[
D := \beta(a) - \sum_{\text{all length 3 pure diagrams}} \beta(\tilde{a}) - \sum_{\text{all length 3 pure diagrams}}.
\]

Also using the Theorem 3.1 Boij-Söderberg decompositions for the ideals \( L \) and \( \tilde{L} \) can be observed as following:

\[
\beta(L) = \sum_{\text{all length 3 pure diagrams}} \alpha_i \pi_{\text{diag}}, \quad \beta(\tilde{L}) = \sum_{\text{all length 3 pure diagrams}} \tilde{\alpha}_i \pi_{\text{diag}}
\]

This shows that \( \beta(L) \) and \( \beta(\tilde{L}) \) have same ends but we also know that \( \beta(L) \) and \( \beta(L, x) \) have same ends. Hence the statement is true.

However, we must still explain the case when \( a(I) - a(b) = (k - 1) - (m - 1) = k - m = N + 1 \), which means \( a(I) = k - 1 \). It follows \( I = (y, z)^{k-1} \) since \( a(J) = k \). Then \( a = xb + I \) and \( b = x\overline{b} + \overline{T} \) where \( a(\overline{T}) - a(\overline{b}) \geq (k - 2) - (m - 2) \geq N + 1 \). If it is a strict inequality then the same process as we have done for \( L \) can be applied to \( a \) to prove the statement. If there is an equality, we end up with the same situation. \( L = xa + J \) where \( a(J) = k \), \( a(a) = m \) and \( k - m = N + 1 \), and \( a = xb + I \) where \( I = (y, z)^{k-1} \), \( a(b) = m - 1 \), and \( b = x\overline{b} + \overline{T} \) where \( T = (y, z)^{k-2} \), \( a(\overline{T}) = m - 2 \). We repeat this until we get

\[ c = x(y, z)^{t-1} + K \]

where \( K = (y, z)^s \), \( s = k - m + 1, 1 \leq t \leq k - m \).

For this form of the lex ideal, one can check the Boij-Söderberg decomposition of the ideal \( c \).

\[
\beta(c) = \begin{bmatrix}
0 & 1 & 2 \\
2 & 1 & 1 \\
\vdots & \vdots & \vdots \\
t & 1 & 2 \\
\vdots & \vdots & \vdots \\
s & s + 1 & 2s + 1 \\
\end{bmatrix}
\]

\[
= \frac{1}{t} \begin{bmatrix} 2 : t - 1 & t & - \\
\vdots & \vdots & \vdots \\
t & - & t - 1 \\
\end{bmatrix} + \frac{1}{t} \begin{bmatrix} 2 : 1 & - & - \\
\vdots & \vdots & \vdots \\
t & - & t - 1 \\
\end{bmatrix}
\]

\[
+ \frac{1}{s} \begin{bmatrix} 2 : s - t + 1 & - & - \\
\vdots & \vdots & \vdots \\
s & - & s - t - 1 \\
\end{bmatrix} + \frac{t-1}{s} \begin{bmatrix} 2 : 1 & - & - \\
\vdots & \vdots & \vdots \\
s & - & s - s - 1 \\
\end{bmatrix}
\]

\[
+ \frac{1}{s} \begin{bmatrix} t & 1 & - \\
\vdots & \vdots & \vdots \\
s & - & s - t + 1 \\
\end{bmatrix} + \frac{s}{s} \begin{bmatrix} s : 1 & 1 & 1 \\
\vdots & \vdots & \vdots \\
[s : 1] & 1 \\
\end{bmatrix}
\]

Therefore the statement is true for the ideal \( c \). So we may assume that, without loss of generality, \( a \) can be assumed as in the form of \( c \), that is, \( a(I) - a(a) < N + 1 \). This observation completes the proof for Case I.

**CASE II:** Let \( y^n \in a \).
We construct similar short exact sequences like (1) for the ideals $b$ segment ideals such that $\beta(I) - a(\tilde{a}) = m - (m - 1) = 1$. By the base case, the decompositions of $\beta(\tilde{a})$ and $\beta(I, x)$ have the same ends. Hence,

$$\beta(\tilde{a}) - \sum \text{all length 3 pure diagrams} = \beta(I, x) - \sum \text{all length 3 pure diagrams} = \beta(\tilde{a}) - \sum \text{all length 3 pure diagrams}.$$ 

Similar to the Case I, consider the lex ideal $\tilde{L} = \tilde{x}a + J$ and $y^{a(\tilde{a})} = y \notin \tilde{a}$. Thus by the result of the Case I, the statement is true for $\tilde{L}$. We do exactly the same trick as in Case I to show that $\beta(L)$ and $\beta(\tilde{L})$ have the same ends and it follows that the statement holds for $L$.

(ii) If $m = 1$; that is, $a = (x, y, z^t)$ where $1 \leq t \leq k - 1$. In the Case I we have already shown that the Boij-Söderberg decomposition of the $\beta(L)$ satisfy the statement if $L = x(x, y, z^t) + J$ where $J = (y, z)^k$. Nevertheless, for more general stable ideal $J \subset k[y, z]$ we had already assumed that $L$ cannot be in that form in the statement.

\[\square\]

**Conjecture 4.2.** The statement of Theorem 4.1 holds for all Artinian lex-ideals in $k[x, y, z]$. 

Theorem 4.1 shows that the ends of the Boij-Söderberg decompositions of $L$ and $(L, x) = (J, x)$ are exactly the same for all Artinian lex ideals $L$ in $R$ except the ones in the form of $L = x(x, y, z^t) + J$ where $J$ is different from $(y, z)^m(J)$ and $1 < t < k - 1$. However, based on the computations we have done using the Boij-Söderberg packages of the computer algebra software Macaulay2, we believe that this result is also true for the lex ideals in that particular form.

5. **Further Observations and Examples**

For an Artinian lex ideal $L \subset k[x, y, z]$ of codimension 3, we have shown that the summands of length $3$ pure diagrams of the Boij-Söderberg decomposition of $a$ where $a = L : (x)$, and the summands of pure diagrams of length less than $3$ in the Boij-Söderberg decomposition of $(L, x)$ appear in the decomposition of the ideal $L = a(x) + J$ in the beginning and the end, respectively.

$$\beta(L) = \left[ \begin{array}{c} \text{length 3 degree sequences coming from } a \\ \text{extra length 3 degree sequences} \\ \text{all length < 3 degree sequences coming from } (L, x) \end{array} \right]$$

There might be also some other pure diagrams of length $3$ other than the ones coming from the Boij-Söderberg decomposition of $a$. However, how this middle part containing pure diagrams of length $3$ comes out is not quite clear. One might ask whether or not the ideals $b = L : (y)$ and $c = L : (z)$ help to describe the middle part. In fact, examples show that there is a quite strong relation between them. Nevertheless, there are some cases, the diagrams obtained from the decompositions of $\beta(b)$ and $\beta(c)$ do not cover the entire middle part of the decomposition of $\beta(L)$ or the Boij-Söderberg decomposition of $b$ and $c$ may have pure diagrams which do not appear in the decomposition of $\beta(L)$.

Now in this section we illustrate the possible relation between the Boij-Söderberg decompositions of the ideals $b$, $c$ and $L$ via examples.

**Example 5.1.** $L = (x^2, xy^2, xy, xz^2, y^6, y^3z, y^6z, y^5z^2, y^4z^3, y^3z^5, y^2z^6, y^2z^7, y^7, z^8)$ is a lex segment ideal in $R$. Then $a = L : z = (x^2, y^2, z^2)$ is lex segment ideal such that $L = xa + J$ where $J = (y, z)^8$ is stable in $R$ and lex segment in $k[y, z]$.

Similarly the ideals $b = L : y = (x^2, xy, xz, y^7, y^6z, y^5z^2, y^4z^3, y^3z^4, y^2z^5, y^2z^6, z^7)$ and $c = L : z = c$ are lex segment ideals such that $L = yb + I = zc + K$ where $I = (x^2, xz^2, z^8)$ and $K = (x^2, xy^2, y^8)$. We construct similar short exact sequences like (1) for the ideals $b$ and $c$. Unlike the case for $a$, we might have cancellations in the mapping cone of the short exact sequences for ideals. It means we can have cancellations in the Betti diagram since the mapping cone structure may not yield the minimal free resolution This situation causes different degree sequences that do not appear in Boij-Söderberg decomposition of $L$.

Now, first we notice that $b = c$ and find the Boij-Söderberg decomposition of $\beta(a)$

$$\beta(a) = (1)\pi(1, 3, 4) + \text{[pure diags. of length < 3]},$$

Then we consider the short exact sequence for the ideal $b$
Observe that one cancellation occurs in the mapping cone process of each ideal example. This example will show that some different situations might occur other than the previous example.

Therefore, 

\[ \beta(b) = \beta(c) = (1)\pi_{(2,3,4)} + \left(\frac{1}{4}\right)\pi_{(2,3,9)} + \left(\frac{8}{7}\right)\pi_{(2,8,9)} + \text{[pure diags. length < 3]} \]

The impressive point of this example is that one might expect to deduce a structural meaning from the description of the chain of degree sequences in Boij-Söderberg decomposition of \( \beta(b) \). Indeed, the Boij-Söderberg decomposition of \( L \) is,

\[ \beta(L) = (1)\pi_{(2,4,5)} + \left(\frac{2}{7}\right)\pi_{(3,4,10)} + \left(\frac{9}{7}\right)\pi_{(3,9,10)} + (8)\pi_{(8,9)} + (1)\pi_{(8)} \]

The mapping cone of the short exact sequence for ideal \( b \) (so the same for \( c \)) ends up with "one" cancellation in the first degree. So we interpret this as ignoring one pure diagram at the beginning, which is the one corresponding to the degree sequence \( (2,3,4) \) at the beginning of the decomposition of \( \beta(b) \). Therefore,

\[ \beta(L) = [\text{length 3 pure diags.}] + (8)\pi_{(8,9)} + (1)\pi_{(8)} \]

The pure diagrams of length less than 3 are coming from the ideal \( b \). Hence we claim that the summands (with coefficients) in the Boij-Söderberg decomposition of \( \beta(L) \) are 

\[ \beta(L) \approx (1)\pi_{(2,4,5)} + (\alpha_2)\pi_{(3,4,10)} + (\alpha_3)\pi_{(3,9,10)} + (8)\pi_{(8,9)} + (1)\pi_{(8)} \]

for some coefficients \( \alpha_2, \alpha_3 \) in \( \mathbb{Q} \).

Example 5.2. This example will show that some different situations might occur other than the previous example. Let \( L = (x^2, xy, yz, xz^2, y^4, y^2z, yz^2, z^9) \) be lex-segment ideal in \( R \). Then \( a = L : x = (x, y^2, yz, z^2), b = L : y = (x^2, xy, xz, y^3, y^2z, yz^2, z^9) \) and \( c = L : z = (x^2, xy, xz, y^3, y^2z, yz^2, z^8) \). We observe that one cancellation occurs in the mapping cone process of each ideal \( b \) and \( c \). Boij-Söderberg decompositions of \( a, b, c \) and \( (L,x) \) are

- \( \beta(a) = 1\pi_{(1,3,4)} + [\text{pure diags. length < 3}] \)
- \( \beta(b) = 1\pi_{(2,3,4)} + \frac{1}{3}\pi_{(2,3,5)} + \frac{5}{6}\pi_{(2,4,5)} + \frac{1}{4}\pi_{(2,4,8)} + \frac{7}{20}\pi_{(3,4,8)} + \frac{1}{10}\pi_{(3,7,8)} + [\text{pure diags. length < 3}] \)
- \( \beta(c) = 1\pi_{(2,3,4)} + \frac{1}{3}\pi_{(2,3,5)} + \frac{1}{3}\pi_{(2,4,5)} + \frac{1}{2}\pi_{(2,4,8)} + \frac{1}{10}\pi_{(3,4,8)} + \frac{1}{10}\pi_{(3,7,8)} + \frac{3}{14}\pi_{(3,7,10)} + \frac{1}{42}\pi_{(3,9,10)} + [\text{pure diags. length < 3}] \)

So, the Boij-Söderberg decomposition for the ideal \( L \) is likely to be.
\[ \beta(L) \approx 1\pi(2,4,5) + \alpha_2 \pi(3,4,6) + \alpha_3 \pi(3,5,6) + \alpha_4 \pi(3,5,9) + \alpha_5 \pi(4,5,9) + \alpha_6 \pi(4,8,9) + \alpha_7 \pi(4,8,11) + \alpha_8 \pi(4,10,11) + 1\pi(4,10) + 1\pi(7,10) + 1\pi(9), \]

where \( \alpha_i \in \mathbb{Q}, \quad i = 2, \ldots, 8. \)

Thus it seems that we almost obtain the actual Boij-Söderberg decomposition for \( L \) which is

\[
\beta(L) = \frac{1}{3} \pi(2,4,5) + \frac{2}{3} \pi(3,4,6) + \frac{2}{3} \pi(3,5,6) + \frac{1}{2} \pi(3,5,9) + \frac{3}{10} \pi(4,5,9) + \frac{1}{20} \pi(4,8,9) + \frac{1}{4} \pi(4,8,11) + \frac{1}{10} \pi(4,10) + \frac{1}{10} \pi(7,10) + \pi(9),
\]

from \( a(-1) \) and \( c(-1) \) from \( b(-1) \) and \( c(-1) \) from \( b(-1) \) and \( c(-1) \) from \( (L,x) \).

Apparently, the Boij-Söderberg decomposition of \( c \) provides an additional pure diagram, \( \pi(4,10,11) \), which does not appear in the Boij-Söderberg decomposition of \( L \). Nevertheless it still supports the idea of the connection of the middle part of the decomposition of \( \beta(L) \) and the decompositions of \( \beta(b) \) and \( \beta(c) \).

**Example 5.3.** In the previous example we saw that one can obtain a longer chain of the degree sequences for \( L \) that the actual chain of the degree sequences via the Boij-Söderberg decomposition of the ideals \( a, b, c \) and \( (L,x) \). This example shows

Consider the lex-segment ideal \( L = (x^2, xy, xz^2, y^6, y^5 z, y^4 z^3, y^3 z^4, y^2 z^5, y z^6, z^9) \) in \( R \). Then the colon ideals are \( a = L : x = (x, y, z^2) \), \( b : y = (x, y^5, y^4 z, y^3 z^3, y^2 z^4, y z^5, z^9) \), and \( c = L : z = (x^2, xy, xz, y^6 z, y^5 z^2, y^4 z^3, y^3 z^4, y^2 z^5, y z^6) \). The mapping cone for the ideal \( c \) requires two cancellations, so we ignore the first two degree sequences. Then,

\[
\beta(a) = \frac{1}{3} \pi(1,2,4) + \frac{1}{3} \pi(1,3,4) + \text{[pure diags. of length < 3]},
\]

\[
\beta(b) = \frac{1}{3} \pi(1,6,7) + \frac{9}{35} \pi(1,6,8) + \frac{2}{7} \pi(1,7,8) + \frac{1}{3} \pi(1,7,8) + \text{[pure diags. of length < 3]},
\]

\[
\beta(c) = \frac{1}{3} \pi(2,3,4) + \frac{1}{3} \pi(2,3,5) + \frac{1}{3} \pi(2,6,8) + \frac{9}{35} \pi(2,7,8) + \frac{1}{3} \pi(2,7,10) + \frac{1}{3} \pi(5,7,10) + \text{[pure diags. of length < 3]},
\]

and \( \beta(L,x) = \text{[pure diags. length 3]} + \frac{1}{2} \pi(6,8) + 2 \pi(7,8) + 2 \pi(7,10) + \pi(9) \).

Then, we get the following chain of degree sequences in order to set up the approximate Boij-Söderberg decomposition for \( L \)

\[
\beta(L) \approx \begin{cases} 
(2,3,5) \quad \text{from } a(-1) \\
(2,7,8) \quad \text{from } b(-1) \\
(2,7,9) \quad \text{from } c(-1) \\
(2,8,9) \quad \text{from } (L,x) \\
(6,8,9) \quad \text{from } (L,x) \\
\end{cases}
\]

However, the degree sequences in the decomposition must be a partial ordered chain, so we have to eliminate the ones that violate the partial order. From the decomposition of \( \beta(c) \), we get \((3,7,9)\) as the first degree sequence, but \((2,8,9)\) and \((6,8,9)\) cannot be before \((3,7,9)\). So we have to ignore the sequences \((2,8,9)\) and \((6,8,9)\). Then we get an approximate decomposition such as

\[
\beta(L) \approx \frac{1}{3} \pi(2,3,5) + \frac{1}{3} \pi(2,4,5) + \frac{1}{3} \pi(2,7,8) + \frac{1}{3} \pi(2,7,9) + \frac{1}{3} \pi(3,7,9) + \frac{1}{3} \pi(3,8,9) + \frac{1}{3} \pi(3,8,11) + \frac{1}{3} \pi(5,7,9) + 2 \pi(6,9) + 2 \pi(8,11) + 1 \pi(10).
\]

The Boij-Söderberg decomposition of \( \beta(L) \) is

\[
\beta(L) = \frac{1}{3} \pi(2,3,5) + \frac{1}{3} \pi(2,4,5) + \frac{1}{3} \pi(2,4,5) + \frac{1}{3} \pi(2,7,8) + \frac{1}{3} \pi(2,7,9) + \frac{1}{3} \pi(3,7,9) + \frac{1}{3} \pi(3,8,9) + \frac{1}{3} \pi(3,8,11) + \frac{1}{3} \pi(5,7,9) + 2 \pi(6,9) + 2 \pi(8,11) + 1 \pi(10).
\]

The degree sequence \((3,8,11)\) associated with \((2,7,10)\), which is coming from the decomposition of \( \beta(c) \), does not show up in the decomposition of \( \beta(L) \), similar to the situation in Example 5.2. Moreover, for this lex-segment ideal \( L \), we realize another different situation. The degree sequence \((2,4,8)\) shows up in the chain of the Boij-Söderberg decomposition of \( \beta(L) \), but \((2,4,8)\) does not appear in any of the decompositions of \( \beta(a), \beta(b) \) and \( \beta(c) \).

An explanation for that extra degree sequence \((2,4,8)\) might be possible for this example. We see that \((2,4,5)\) is the last degree sequence coming from \( a(-1) \) and the next degree sequence \((2,7,8)\) is from \( b(-1) \). If we assume that there is no other degree sequence between \((2,4,5)\) and \((2,7,8)\), it implies that simultaneous elimination of the entries in the positions of \( \beta_{1,4} \) and \( \beta_{2,5} \) in the Betti diagram of \( L \) by the
algorithm of Boij-Söderberg decomposition. However, this is not possible because otherwise there would not be a pure diagram of length 2 in the Boij-Söderberg decomposition of $a$. Hence again by the partial order, it must be $(2, 4, 5) < (2, 4, 8) < (2, 7, 8)$.

The examples 5.2 and 5.3 show that the Boij-Söderberg decompositions of $a$, $b$, $c$ and $(L, x)$ may not be enough to provide the entire chain of degree sequences in the Boij-Söderberg decomposition of $L$. There might be some gaps and redundant degree sequences. With the explanations, such as the cancellations in mapping cone, the necessity of the order of the chain of the degree sequences, we provide the entire chain of degree sequences in the Boij-Söderberg decomposition of $L$.

**Remark 5.4.** The relation between Boij-Söderberg decompositions of a (Artinian) lex-segment ideal $L$ and the lex ideals $a = L : x$ and $(L, x)$ is pointed out in Theorems 3.1 and 4.1. Furthermore, the examples that we have observed show that if we know the Boij-Söderberg decompositions of the colon ideals $b = L : y$ and $c = L : z$, almost the entire chain of the degree sequences of the decomposition for the lex-segment ideal $L$ is revealed. In other words, we formalize the full chain of degree sequences of the Boij-Söderberg decomposition of the ideal $L$ by using the chains of the Boij-Söderberg decompositions of the colon ideals $a, b, c$ and the lex ideal $(L, x)$. Proving what the examples indicate is a further direction for our research on Boij-Söderberg decomposition of lex-segment ideals.

A natural question which asks to describe the whole Boij-Söderberg decomposition of a lex ideal $L$ in terms of the decompositions of its colon ideals and $(L, x)$ arises at this point. So the coefficients of the pure diagrams in the decompositions come into play. As noticed throughout the paper, we narrow our attention on the degree sequences, that is, pure diagrams. Although the results involve the coefficients as well, we do not have a precise relation between the coefficients in the decompositions of $b$, $c$ and $L$.

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