MEAN CURVATURE MOTION OF GRAPHS WITH
CONSTANT CONTACT ANGLE AND MOVING
BOUNDARIES

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Abstract

We consider the motion by mean curvature of an n-dimensional graph over a time-dependent domain in \( \mathbb{R}^n \), intersecting \( \mathbb{R}^n \) at a constant angle. In the general case, we prove local existence for the corresponding quasilinear parabolic equation with a free boundary, and derive a continuation criterion based on the second fundamental form. If the initial graph is concave, we show this is preserved, and that the solution exists only for finite time. This corresponds to a symmetric version of mean curvature motion of a network of hypersurfaces with triple junctions, with constant contact angle at the junctions.

1. Time-dependent graphs with a contact angle condition.

We consider a moving hypersurface \( \Sigma_t \) in \( \mathbb{R}^{n+1} \), with normal velocity equal to its mean curvature, assumed to be a graph over a time-dependent open set \( D(t) \subset \mathbb{R}^n \) (not necessarily bounded, or connected.) The (properly embedded) \((n-1)\)-submanifold of intersection:

\[ \Gamma(t) = \Sigma_t \cap \mathbb{R}^n = \partial D(t) \]

is a ‘moving boundary’. Along \( \Gamma(t) \) we impose a constant-angle condition:

\[ \langle N, e_{n+1} \rangle|_{\Gamma(t)} = \beta, \]

where \( 0 < \beta < 1 \) is a constant and \( N \) is the upward unit normal of \( \Sigma_t \).

‘Mean curvature motion’ is defined by the law:

\[ V_N = H, \]

where \( V_N = \langle V, N \rangle \), with \( V = \partial_t F \) the velocity vector in a given parametrization \( F(t) \) of \( \Sigma_t \) (\( V \) depends on the parametrization, while \( V_N \) does not). A particular parametrization yields ‘mean curvature flow’:

\[ \partial_t F = HN. \]

For graphs, it is natural to consider ‘graph mean curvature motion’: if \( \Sigma_t = \text{graph } w(t) \) for a function \( w(t) : D(t) \to \mathbb{R} \), imposing \( \langle \partial_t F, N \rangle = H \) with \( F(y, t) = [y, w(y, t)] \) for \( y \in D(t) \), we find:

\[ w_t = \sqrt{1 + |Dw|^2} H \]
(and the velocity is vertical, \( \partial_t F = w_t e_{n+1} \)). With the contact angle condition, we obtain a free boundary problem for a quasilinear PDE:
\begin{align*}
\begin{cases}
  w_t = g^{ij}(Dw)w_{ij} & \text{in } D(t), \\
  w = 0, & \beta \sqrt{1 + |Dw|^2} = 1 \text{ on } \partial D(t),
\end{cases}
\end{align*}

where \( g^{ij}(Dw) = \delta^{ij} - w_i w_j / (1 + |Dw|^2) \) is the inverse metric matrix.

Remark 1.1. It is easy to see that the constant-angle boundary condition is **incompatible** with mean curvature flow parametrized over a fixed domain \( D_0 \): on \( \partial D_0 \) we would have \( \langle F, e_{n+1} \rangle = 0 \), leading to \( \langle \partial_t F, e_{n+1} \rangle = 0 \), incompatible with \( \partial_t F = HN \) and \( \langle N, e_{n+1} \rangle = \beta \). If we parametrize over time-dependent domains, mean curvature flow and graph m.c.m. lead to identical normal velocities for the moving boundary (see section 2.)

To establish short-time existence (in parabolic Hölder spaces) we will work with a third realization of the motion, defined over a fixed domain:

\[ F(t) : D_0 \to \mathbb{R}^{n+1}, \quad F(x,t) = [\varphi(x,t), u(x,t)] \in \mathbb{R}^n \times \mathbb{R}, \]

where \( \varphi(t) : D_0 \to D(t) \) is a diffeomorphism and \( F \) is a solution of the parabolic system:

\[ F_t = g^{ij}(DF)F_{ij}, \]

where \( g_{ij} = \delta_{ij} + \langle F_i, F_j \rangle \) is the induced metric on \( \Sigma_t \) and \( g^{ij} \) is the inverse metric matrix.

In the first part of the paper (sections 3 to 8) we prove the following short-time existence theorem (on \( Q := D_0 \times [0,T] \)):

**Theorem 1.1.** Let \( \Sigma_0 \subset \mathbb{R}^{n+1} \) be a \( C^{3+\alpha} \) graph over \( D_0 \subset \mathbb{R}^n \) satisfying the contact and angle conditions at \( \partial D_0 \). There exists a parametrization \( F_0 = [\varphi_0, u_0] \in C^{2+\alpha}(D_0) \) of \( \Sigma_0 \), \( T > 0 \) depending only on \( F_0 \) and a unique solution \( F \in C^{2+\alpha,1+\alpha/2}(Q^T; \mathbb{R}^{n+1}) \) of the system:

\[ \begin{cases}
  \partial_t F - g^{ij}(DF)\partial_i \partial_j F = 0, \\
  u|_{\partial D_0} = 0, \quad N^{n+1}(D\varphi, Du)|_{\partial D_0} = \beta,
\end{cases} \]

with initial data \( F_0 \) and satisfying, in addition, the ‘orthogonality conditions’ at \( \partial D_0 \) (described in section 3.)

The system and boundary conditions are discussed in more detail in section 3. Sections 4, 5, and 6 deal with compatibility at \( t = 0 \), linearization and the verification that the boundary conditions satisfy ‘complementarity’. In particular, adjusting the initial diffeomorphism \( \varphi_0 \) to ensure compatibility (section 4) leads to the ‘loss of differentiability’ seen in theorem 1.1. The required estimates in Hölder spaces for the linearized system are described
in section 7, and the proof concluded (by a fixed-point argument) in section 8. While the general scheme is standard, due to the particular boundary conditions adopted many details had to be worked out from first principles. Free boundary-type problems for mean curvature motion of graphs have apparently not previously been considered.

We describe the evolution equations in the rotationally symmetric case in section 9 (including a stationary example for the exterior problem) and the extension to the case of a graph motion $\Sigma_t$ intersecting fixed support hypersurfaces orthogonally in section 10.

The original motivation for this work was to establish (by classical parabolic PDE methods) existence-uniqueness for mean curvature motion of networks of surfaces meeting along triple junctions with constant-angle conditions. One can use a motion $\Sigma_t$ of graphs with constant contact angle to produce examples of ‘triple junction motion’: three hypersurfaces moving by mean curvature meeting along an $(n - 1)$-dimensional submanifold $\Sigma(t)$ so that the three normals make constant angles (say, 120 degrees) along $\Gamma(t)$. The simplest way to do this is by reflection on $\mathbb{R}^n$, so the hypersurfaces are $\Sigma_t$, $\bar{\Sigma}_t$, and $\mathbb{R}^n - \bar{D}(t)$. If $\Sigma_t = \text{graph } w(t)$ with $w > 0$, the system is embedded in $\mathbb{R}^{n+1}$. This is mean curvature motion of a ‘symmetric triple junction of graphs’.

Short-time existence holds for general triple junctions of graphs moving by mean curvature with constant 120 degree angles at the junction, provided a compatibility condition holds along the junction (see section 15). The idea of proof is similar to the one given here; since the details are easier to understand in the symmetric case, we decided to do this first. In addition, in the present case it is possible to go a lot further towards a geometric global existence result. Motivated by recent work on ‘lens-type’ curve networks [5], in the second part of the paper (sections 11-14) we consider continuation criteria and preservation of concavity. Since we chose to develop these results for graph motion with a free boundary, although the general lines of proof (via maximum principles) have precedents, many details had to be developed anew. For example, section 13 contains an extension of the maximum principle for symmetric tensors with Neumann-type boundary conditions given in [9], which in our setting allows one to show preservation of weak concavity in general. The results obtained in sections 11-14 are summarized in the following theorem.
Theorem 1.2. Let $T_{\text{max}}$ be the maximal existence time for the evolution. Assuming $T_{\text{max}} < \infty$, the second fundamental form $h$ is unbounded at the junction $\Gamma_t$, as $t \to T_{\text{max}}$:

$$\limsup_{t \to T_{\text{max}}} (\sup_{\Gamma_t} |h|_g) = \infty.$$ 

If the mean curvature of the initial hypersurface is strictly negative ($\sup_{\Sigma_0} H = H_0 < 0$), then $T_{\text{max}}$ is finite. If $\Sigma_0$ is weakly concave ($h \leq 0$ at $t = 0$), this is preserved by the evolution.

The expected global existence result is that, assuming weak concavity, $\text{diam}(\Sigma_t) \to 0$ as $t \to T_{\text{max}}$.

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2. Normal velocity of the moving boundary. The evolution is naturally supplied with initial data $\Sigma_0$, a graph meeting $\mathbb{R}^{n+1}$ at the prescribed angle. Since we are interested in classical solutions in the parabolic Hölder space $C^{2+\alpha, 1+\alpha/2}$, we expect an additional compatibility condition at $t = 0$. We discuss this first for graph m.c.m. $w(y,t)$.

Denote by $\Gamma(t)$ a global parametrization of $\partial D(t)$ (with domain in a fixed manifold, and ‘space variables’ left implicit). Differentiating in $t$ the ‘contact condition’ $w(\Gamma(t), t) = 0$, we find:

$$w_t + \langle Dw, \dot{\Gamma}(t) \rangle = 0.$$ 

Denote by $n_t$ the unit normal vector field to $\Gamma(t)$, chosen so that $\langle n_t, Dw \rangle > 0$. The contact condition also implies the gradient of $w$ is purely normal:

$$Dw|_{\partial D(t)} = (D_{n_t} w)n_t.$$ 

Combining this with the angle condition, and bearing in mind that $D_{n_t} w|_{\Gamma(t)} > 0$, we find:

$$D_{n_t} w = \frac{\beta_0}{\beta} \text{ on } \partial D(t), \quad \beta_0 := \sqrt{1 - \beta^2}.$$
Thus, on $\partial D(t)$:

$$\frac{1}{\beta} H = \sqrt{1 + (D_n w)^2} = w_t = -\langle \hat{\Gamma}(t), n_t \rangle D_{nt} w = -\hat{\Gamma}_n(t) \frac{\beta_0}{\beta},$$

and we find the normal velocity of the moving boundary (independent of the parametrization of $\Gamma_t$):

$$\hat{\Gamma}_n = -\frac{1}{\beta_0} H|_{\Gamma(t)},$$

which in particular must hold at $t = 0$. Note that we don’t get a ‘compatibility condition’ in the usual sense (of a constraint on the 2-jet of the initial data), but instead an equation of motion for the moving boundary. (Later, in the fixed-domain formulation, we will have to deal with a real compatibility condition).

Now consider mean curvature flow parametrized over a time-dependent domain $D(t)$, with the boundary conditions:

$$\langle F(x,t), e_{n+1} \rangle = 0, \quad \langle N, e_{n+1} \rangle = \beta, \quad x \in \partial D(t).$$

Let $\nu = \nu_t$ be the inner unit normal to $D(t)$. Suppose $S(\theta,t), \theta \in S^{n-1}$, parametrizes $\partial D(t)$; thus the ‘junction’ $\partial \Sigma_t$ is parametrized by $\Gamma(\theta,t) = F(t, S(\theta,t))$, and (denoting partial $t$ derivatives with a dot):

$$\hat{\Gamma}(\theta,t) = \partial_t F(t, S(\theta,t)) + dF[S(\theta,t)] = HN + (\hat{S} \cdot \nu) \partial_n F$$

(where we used the fact that $\partial_\tau F = 0$ for any $\tau \in T \partial D(t)$.) Thus, using $\langle N, n \rangle = -\beta_0$:

$$\hat{\Gamma}_n := \hat{\Gamma}(\theta,t) \cdot n = -\beta_0 H + (\hat{S} \cdot \nu) \langle \partial_n F, n \rangle.$$

On the other hand, from $\langle F(t, S(\theta,t)), e_{n+1} \rangle \equiv 0$, we find by differentiation:

$$H \beta + (\hat{S} \cdot \nu) \langle \partial_n F, e_{n+1} \rangle = 0,$$

or $\hat{S} \cdot \nu = -H \beta / \langle \partial_n F, e_{n+1} \rangle$. Letting $T := \frac{\partial_n F}{|\partial_n F|}$ (tangent to $\Sigma_t$ at the interface), we have:

$$\hat{\Gamma}_n = -H(\beta_0 + \beta \frac{\langle T, n \rangle}{\langle T, e_{n+1} \rangle}).$$
Denoting by $N' = N - \beta e_{n+1}$ the $\mathbb{R}^n$ component of $N$, we clearly have $n = -(1/\beta_0)N'$, so $(T, n) = -(1/\beta_0)(T, N') = (\beta/\beta_0)(T, e_{n+1})$, and we conclude:

$$\dot{\Gamma}_n = -\frac{1}{\beta_0} H,$$

as before.

**Remark 2.1.** This is not unexpected, if we accept there is a reparametrization connecting the two motions, respecting the boundary conditions. That is, the ODE argument in [1] should also work in the presence of boundary conditions and moving boundaries.

**Remark 2.2.** We remark that for more general (non-symmetric, non-flat) triple junctions with 120 degree angles, the condition:

$$H^1 + H^2 = H^3 \text{ on } \Gamma(t)$$

must hold at the junction (for graphs, oriented by the upward normal), which in particular gives a geometric constraint on the initial data, for classical evolution in $C^{2+\alpha,1+\alpha/2}$. This is automatic in the symmetric case ($w^2 = -w^1$), since $H^3 = 0$ and $H^I = tr_{g^I} d^2 w^I$ for $I = 1, 2$.

**3. Choice of ‘gauge’**. It is traditional in moving boundary problems to parametrize the time-dependent domain $D(t)$ of the unknown $w(y, t)$ by a time-dependent diffeomorphism:

$$y = \varphi(x, t), \quad \varphi(t) : D_0 \to D(t),$$

and then derive the equation satisfied by the coordinate-changed function from the equation for $w$ (see e.g. [6] or [10]). Motivated by the work on curve networks ([7]) we will, instead, consider a general parametrization:

$$F : D_0 \times [0, T] \to \mathbb{R}^{n+1}, \quad F(x, t) = [\varphi(x, t), u(x, t)] \in \mathbb{R}^n \times \mathbb{R}$$

and derive an equation for $F$ directly from the definition of mean curvature motion:

$$\langle \partial_t F, N \rangle = H.$$

(We’ll still assume $\varphi(t) : D_0 \to D(t)$ is a diffeomorphism.) The first and second fundamental forms are given by:

$$g_{ij} = \langle F_i, F_j \rangle, \quad A(F_i, F_j) = \langle F_{ij}, N \rangle.$$
(Notation: $DF = F_i e_i, D^2 F(e_i, e_j) = F_{ij}, (e_i)$ is the standard basis of $\mathbb{R}^{n+1}$.)

The mean curvature is the trace of $A$ in the induced metric:

$$H = \langle g^{ij}(DF)F_{ij}, N \rangle.$$

The equation for $F$ is:

$$\langle \partial_t F - g^{ij}(DF)F_{ij}, N \rangle = 0.$$

There is a natural ‘gauge choice’ yielding a quasilinear parabolic system:

$$\partial_t F - g^{ij}(DF)F_{ij} = 0.$$

We will sometimes refer to this as the ‘split gauge’, since in terms of the components $F = [\varphi, u]$ we have the essentially decoupled system:

$$\begin{cases}
\partial_t u - g^{ij}(D\varphi, Du)u_{ij} = 0, \\
\partial_t \varphi - g^{ij}(D\varphi, Du)\varphi_{ij} = 0.
\end{cases}$$

The splitting is useful to state the boundary conditions:

$$u|_{\partial D_0} = 0 \text{ (‘contact’)},$$

$$N^{n+1}(D\varphi, Du)|_{\partial D_0} = \beta \text{ (‘angle’)}.$$

We immediately see there is a problem, since we have 2 scalar boundary conditions for $n + 1$ unknowns (and no moving boundary to help!) Our solution to this is to introduce $n - 1$ additional ‘orthogonality conditions’ at the boundary for the parametrization $\varphi(t)$. We impose:

$$\langle D_\tau \varphi, D_n \varphi \rangle|_{\partial D_0} = 0,$$

for any $\tau \in T\partial D_0$, where $n$ is the inward unit normal to $D_0$.

Geometrically, the ‘orthogonality’ boundary condition has precedent in a method often adopted when dealing with the evolution of hypersurfaces in $\mathbb{R}^{n+1}$ intersecting a fixed $n$-dimensional ‘support surface’ orthogonally (see e.g. [11]): one replaces vanishing inner product of the unit normals (a single scalar condition) by a stronger Neumann-type condition for the parametrization, corresponding to $n - 1$ scalar conditions. (More details are given in Section 10.)

The system must also be supplied with initial data. We assume given an initial hypersurface $\Sigma_0$, the graph of a $C^{3+\alpha}$ function $\tilde{u}_0(x)$ defined in the $C^{3+\alpha}$ domain $D_0 \subset \mathbb{R}^n$. (The reason for this choice of differentiability class
will be seen later.) It would seem natural to set \( \varphi_0 = \text{Id}_{D_0} \), but this causes problems (related to compatibility; see Section 4 below). We do require the 1-jet of \( \varphi_0 \) at the boundary to be that of the identity:

\[
\varphi_0|_{\partial D_0} = \text{Id}, \quad D\varphi_0|_{\partial D} = I.
\]

(In particular, the orthogonality condition holds at \( t = 0 \).)

We need a more explicit expression for the unit normal, and for that we use the ‘vector product’:

\[
\tilde{N}(D\varphi, Du) := (-1)^n \det \begin{bmatrix} e_1 & \cdots & e_{n+1} \\ DF^1 & \cdots & DF^{n+1} \end{bmatrix} = (-1)^n \det \begin{bmatrix} e_1 & \cdots & e_n & e_{n+1} \\ D\varphi^1 & \cdots & D\varphi^n & Du \end{bmatrix}
\]

\[
:= [J(D\varphi, Du), J_\varphi] \in \mathbb{R}^n \times \mathbb{R},
\]

where \( DF^i \in \mathbb{R}^n \) for \( i = 1, \ldots, n+1 \), \( J_\varphi > 0 \) is the jacobian of \( \varphi \) and \((-1)^n\) is introduced to make sure the last component is positive. \( J(D\varphi, Du) \) is an \( \mathbb{R}^n \)-valued multilinear form, linear in the components \( u_i \) of \( Du \) and of weight \( n-1 \) in the components of \( D\varphi \). It is easy to check that \( J(\text{Id}, Du) = -Du \).

The unit normal is:

\[
N(D\varphi, Du) = \tilde{N}(D\varphi, Du)/(|J(D\varphi, Du)|^2 + (J_\varphi)^2)^{1/2}.
\]

Thus the angle condition may be stated in the form:

\[
\beta[|J(Du, D\varphi)|^2 + (J_\varphi)^2]_{|\partial D_0} = J_\varphi|_{\partial D_0},
\]

and we lose nothing by squaring it:

\[
B(D\varphi, Du) := \beta^2|J(Du, D\varphi)|^2 - \beta_0^2(J_\varphi)^2_{|\partial D_0} = 0.
\]

4. Compatibility and the choice of \( \varphi_0 \). Assume \( D\varphi_0|_{\partial D_0} = \text{Id} \).

Differentiating in \( t \) the contact condition \( u|_{\partial D_0} = 0 \) and evaluating at \( t = 0 \), we find:

\[
0 = g^{ij}(\text{Id}, Du_0)u_{0ij} \equiv g^{ij}_0 u_{0ij} \text{ on } \partial D_0.
\]

To interpret this condition, consider the mean curvature at \( t = 0 \), on \( \partial D_0 \):

\[
H_0 = \frac{1}{v_0} [(J(\text{Id}, Du_0), g^{ij}_0 \varphi_{0ij}) + J_{\varphi_0}g^{ij}_0 u_{0ij}],
\]

where:

\[
v_0 = |(J(\text{Id}, Du_0)|^2 + J_{\varphi_0}^2|_{\partial D_0})^{1/2} = (|Du_0|^2 + 1)^{1/2} \frac{1}{\beta}.
\]
using (recall $\beta_0 := \sqrt{1 - \beta^2}$):

$$J(I, Du_0) = -Du_0 = -(D_nu_0)n = \frac{\beta_0}{\beta}n$$
on $\partial D_0$. Thus the compatibility condition is equivalent to:

$$H_{0|\partial D_0} = -\beta_0 g^{ij}_0 \langle \varphi_{0ij}, n \rangle_{\partial D_0}.$$

This implies we can’t choose $\varphi_0 \equiv Id$ (on all of $D_0$), unless $H_{0|\partial D_0} \equiv 0$, a constraint not present in the geometric problem (as seen above). Instead, regarding $H_0$ as given (by $\Sigma_0$), and using:

$$g^{ij}_0 = \delta_{ij} - \frac{u_0iu_0j}{v_0^2} = \delta_{ij} - \beta^2_0 n^in^j,$$

we find the compatibility constraint:

$$\langle (\delta_{ij} - \beta^2_0 n^in^j) \varphi_{0ij}, n \rangle = -\frac{1}{\beta_0} H_0 \text{ on } \partial D_0.$$

Given the zero and first order constraints on $\varphi_0$, this can also be written as:

$$n^in^j \langle \varphi_{0ij}, n \rangle = -\frac{1}{\beta^2_0} H_0 \text{ on } \partial D_0.$$

The next lemma shows this can be solved.

**Lemma 4.1.** Let $D_0 \subset \mathbb{R}^n$ be a uniformly $C^{3+\alpha}$ domain (possibly unbounded), $h \in C^{\alpha}(\partial D_0)$ ($0 < \alpha < 1$).

(i) One may find a diffeomorphism $\varphi \inDiff^{2+\alpha}(D_0)$ satisfying on $\partial D_0$:

$$\varphi = Id, \quad d\varphi = I, \quad n \cdot d^2 \varphi(n,n) = h.$$

(ii) More generally, given a non-vanishing vector field $e \in C^{1+\alpha}(\partial D_0; \mathbb{R}^n)$, one may find $\varphi \inDiff^{2+\alpha}(D_0)$ satisfying on $\partial D_0$:

$$\varphi = Id, \quad d_n\varphi = e, \quad n \cdot d^2 \varphi(n,n) = h.$$

If $\partial D_0$ has two components, we may even require $\varphi$ to satisfy the conditions in parts (i) and (ii) at $\partial_1 D_0, \partial_2 D_0$ (resp.), with different functions $h$. (This will be needed in section 11).

As usual, a domain is ‘uniformly $C^{3+\alpha}$’ if at each boundary point there are local charts to the upper half-space (of class $C^{3+\alpha}$), defined on balls of
uniform radius, and with uniform bounds on the $C^{3+\alpha}$ norms of the charts and their inverses.

Remarks: 4.1. The proof is given in Appendix 1.
4.2. Note that, in particular, $\varphi$ satisfies the orthogonality conditions at $\partial D_0$.
4.3. It is at this step in the proof that we have a drop in regularity: for $C^{2+\alpha}$ local solutions, we require $C^{3+\alpha}$ initial data. While this is not unexpected in free-boundary problems (see e.g. [6]), I don’t know a counterexample to the lemma if $D_0$ is assumed to be a $C^{2+\alpha}$ domain.
4.4. In our application of the lemma, we in fact have $h \in C^{1+\alpha}(\partial D_0)$, but this does not imply higher regularity for $\varphi$.

5. Linearization. The evolution equation and boundary conditions in ‘split gauge’ are:

\[
\begin{align*}
F_t - g^{ij}(DF)F_{ij} & = 0, \\
u_{|\partial D_0} & = 0, \\
B(D\varphi, Du)_{|\partial D_0} & = 0, \\
\mathcal{O}(D\varphi)_{|\partial D_0} & = 0,
\end{align*}
\]

where:

\[
\mathcal{O}(D\varphi) := \langle D^{T}\varphi, D_n\varphi \rangle.
\]

Here $D^{T}\varphi = D\varphi - (D_n\varphi)\langle \cdot, n \rangle$ is an $\mathbb{R}^n$-valued 1-form on $\partial D_0$. We’ll prove short-time existence for this system (with initial data $u_0, \varphi_0$) in $C^{2+\alpha,1+\alpha/2}$ by the usual fixed-point argument based on linear parabolic theory. Given $\bar{F} = [\bar{\varphi}, \bar{u}]$ in a suitable ball in this Hölder space with center $F_0 = [\varphi_0, u_0]$, it suffices to consider the ‘pseudolinearization’ of the system:

\[
F_t - g^{ij}(DF_0)u_{ij} = [g^{ij}(DF) - g^{ij}(DF_0)]F_{ij} := \mathcal{F}(\bar{F}, D\bar{F});
\]

a fixed point of the map $\bar{F} \mapsto F$ corresponds to a solution of the quasilinear equation.

For the nonlinear boundary conditions, we need the honest linearization at $F_0$. For the angle condition, a computation using the boundary constraints on $u_0$ and $\varphi_0$ yields:

\[
\frac{1}{2}\mathcal{L}_0 B[D\varphi, Du] = -\beta\beta_0 D_n u - \beta_0^2 \langle D_n \varphi, n \rangle.
\]

The corresponding linear boundary condition will be:

\[
\beta\beta_0 D_n u + (1 - \beta^2) \langle D_n \varphi, n \rangle = B(D\bar{F}, DF_0),
\]
where:

\[ 2\mathcal{B}(DF, DF_0) := B(D\varphi, Du) - B(Du_0, D\varphi_0) - \mathcal{L}_0 B[D(\varphi - \varphi_0), D(u - u_0)], \]

and we used:

\[ -\frac{1}{2} \mathcal{L}_0[D\varphi_0, Du_0]|_{\partial D_0} = \beta_0 D_n u_0 + (1 - \beta^2) \langle D_n \varphi_0, n \rangle|_{\partial D_0} = 0. \]

Also, \( B(D\varphi_0, Du_0)|_{\partial D_0} = 0 \), so at a fixed point \( B(D\varphi, Du)|_{\partial D_0} = 0 \).

Linearizing the orthogonality boundary condition, we find that \( \mathcal{L}_0 \mathcal{O}[D\varphi] \) is the 1-form on \( \partial D_0 \):

\[ \mathcal{L}_0 \mathcal{O}[D\varphi](v) = (\partial_j \varphi^i + \partial_i \varphi^j) n^j (\delta_{ik} - n_k n_i) v^k \]

(with sum over repeated indices.) The corresponding linear boundary condition is:

\[ \langle D_n \varphi, \text{proj}^T \rangle + \langle D^T \varphi, n \rangle = -\Omega(D\varphi_0, D\varphi_0), \]

where:

\[ \Omega(D\varphi, D\varphi_0) := \mathcal{O}(D\varphi) - \mathcal{O}(D\varphi_0) - \mathcal{L}_0 \mathcal{O}[D\varphi - D\varphi_0], \]

and we used:

\[ \mathcal{L}_0 \mathcal{O}[D\varphi_0]|_{\partial D_0} = \langle (D_n \varphi_0)^T, \cdot \rangle + \langle D^T \varphi_0, n \rangle|_{\partial D_0} = 0. \]

6. Complementarity. We wish to apply linear existence theory to the system:

\[ F_t - g^{ij}(DF_0) F_{ij} = \bar{F}, \]

with boundary conditions at \( \partial D_0 \):

\[ u = 0, \]

\[ \beta_0 D_n u + \beta^2 \langle D_n \varphi, n \rangle = \bar{B}, \]

\[ \langle D_n \varphi, \text{proj}^T \rangle + \langle D^T \varphi, n \rangle = -\bar{\Omega} \]

and initial conditions:

\[ u_{t=0} = u_0, \quad \varphi_{t=0} = \varphi_0. \]

It is easy to see that the initial data satisfy the linearized boundary conditions, and above we constructed \( \varphi_0 \) so as to guarantee \( g^{ij}(Du_0, D\varphi_0)u_{0ij}|_{\partial D_0} = 0. \)
0. (There is no first-order compatibility condition for $\varphi_0$.) Thus the linear system satisfies the required compatibility at $t = 0$.

Since the linearized boundary conditions are slightly non-standard, we must verify they satisfy the ‘complementarity’ (Lopatinski-Shapiro) conditions. We fix $x_0 \in \partial D_0$ and introduce adapted coordinates $(\rho, \sigma)$ in a neighborhood $N_0 \subset N$ of $x_0$ in $D_0$:

$$x = \Gamma_0 + \rho n(\sigma), \quad \sigma = (\sigma_a) \in U,$$

where $\Gamma_0 : U \to \mathbb{R}^n$ is a local chart for $\partial D_0$ at $x_0$ ($U \subset \mathbb{R}^{n-1}$ open). This defines a basis of tangential vector fields in $\Gamma_0(U)$, and we may assume that, at $x_0$: $\langle \tau_a, \tau_b \rangle = \delta_{ab}$ and $\nabla_{\tau_a} \tau_0(x_0) = 0$. Let $U$ and $\psi$ be defined in $(-\rho_1, 0) \times U \times [0, T]$ by:

$$U(\rho, \sigma, t) = u(\Gamma_0(\sigma) + \rho n(\sigma), t), \quad \psi(\rho, \sigma, t) = \varphi(\Gamma_0(\sigma) + \rho n(\sigma), t).$$

In these coordinates, the induced metric is written (in ‘block form’):

$$[g] = \begin{bmatrix} |\psi_\rho|^2 + (U_\rho)^2 & \langle \psi_\rho, \psi_a \rangle + U_\rho U_a \\ \langle \psi_\rho, \psi_a \rangle + U_\rho U_a & \langle \psi_a, \psi_b \rangle + U_a U_b \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta^2} & 0 \\ 0 & \mathbb{I}_{n-1} \end{bmatrix}$$

at $t = 0$ and $x_0$.

We have:

$$U_{\rho\rho} = D^2 u(n, n) \quad \text{(since } \nabla_n n = 0),$$

$$U_{ab} = D^2 u(\tau_a, \tau_b) + D u \cdot \nabla_{\tau_a} \tau_b = D^2 u(\tau_a, \tau_b) \quad \text{at } x_0,$$

and we don’t need $U_{\rho a}$, since $g_{\rho a} = 0$ at $x_0$.

Thus:

$$tr_{g_0} D^2 u(x_0) = \beta^2 D^2 u(n, n) + \sum_a D^2 u(\tau_a, \tau_a) = \beta^2 U_{\rho\rho} + \sum_a U_{aa} := \beta^2 U_{\rho\rho} + \Delta_\sigma U,$$

and, likewise:

$$tr_{g_0} D^2 \varphi(x_0) = \beta^2 \psi_{\rho\rho} + \Delta_\sigma \psi.$$

For the linearized orthogonality operator, note that, at $x_0$:

$$L_0 \mathcal{O}[D \psi] = \left( \langle \psi_\rho, \tau_a \rangle + \psi_a, n \right) \tau_a.$$ 

Putting everything together, the linear system to consider at $x_0$ is:

$$U_t - \beta^2 U_{\rho\rho} - \Delta_\sigma U = 0,$$

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\[ \psi_t - \beta^2 \psi_{\rho\rho} - \Delta_\sigma \psi = 0, \]

with boundary conditions:
\[ U|_{\rho=0} = 0, \]
\[ \beta_0 \langle \psi_\rho, \mu \rangle + \beta U_\rho|_{\rho=0} = b(\sigma, t), \]
\[ \langle \psi_\rho, \tau_a \rangle + \langle \psi_{\rho \tau_a} \rangle|_{\rho=0} = \omega_\rho(\sigma, t), \quad a = 1, \ldots n - 1. \]

Now take Fourier transform in \( \sigma \in \mathbb{R}^{n-1} \), Laplace transform in \( t \) to obtain:
\[ \hat{U}(\rho, \xi, p) \in \mathbb{C}, \hat{\psi}(\rho, \xi, p) \in \mathbb{C}^n; \quad \xi \in \mathbb{R}^{n-1}, p \in \mathbb{C}, \rho < 0. \]

In transformed variables, we obtain the system of linear ODE (in \( \rho < 0 \), for fixed \((\xi, p)\)):
\[ \beta^2 \hat{U}_{\rho\rho} - (p + |\xi|^2) \hat{U} = 0, \]
\[ \beta^2 \hat{\psi}_{\rho\rho} - (p + |\xi|^2) \hat{\psi} = 0. \]

Writing the solution in the form:
\[ \begin{bmatrix} U(\rho) \\ \hat{U}(0) \\ \psi(\rho) \end{bmatrix} = e^{i\rho \gamma} \begin{bmatrix} \hat{U}(0) \\ \psi(0) \end{bmatrix}, \]
we find the characteristic equation \( \beta^2 \gamma^2 + p + |\xi|^2 = 0 \), and choose the root \( \gamma \) so that \( i\gamma = (1/\beta) \sqrt{\Delta} \) (where \( \Delta = p + |\xi|^2 \) and we take the branch of \( \sqrt{\Delta} \) : \( \text{Re}(\sqrt{\Delta}) > 0 \)). Here \((p, \xi) \in A\), where:
\[ A = \{(p, \xi) \in \mathbb{C} \times \mathbb{R}^{n-1}; |p| + |\xi| > 0, \text{Re}(p) > -|\xi|^2\}. \]

Thus the solutions decay as \( \rho \to -\infty \). Let \( \mathcal{W}^+ \) be the space of such decaying solutions, \( \dim_{\mathbb{C}} \mathcal{W}^+ = n - 1 \). The relevant boundary operator on \( \mathcal{W}^+ \) is:
\[ B \left[ \begin{array}{c} \hat{U} \\ \hat{\psi} \end{array} \right] = \left[ \begin{array}{c} \beta_0 \langle \hat{\psi}_\rho, \mu \rangle + \beta \hat{U}_\rho \\ \langle \hat{\psi}_\rho, \tau_a \rangle + i\xi_a \langle \hat{\psi}, \mu \rangle \end{array} \right]|_{\rho=0} = \left[ \begin{array}{c} \hat{U}(0) \\ \beta_0 (i\gamma) \langle \hat{\psi}(0), \mu \rangle + i\beta \gamma \hat{U}(0) \\ (i\gamma) \langle \hat{\psi}(0), \tau_a \rangle + i\xi_a \langle \hat{\psi}(0), \mu \rangle \end{array} \right]. \]

(a vector in \( \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1} \)).

The ‘complementarity condition’ (see e.g. [3]) is the statement that \( B \) is a linear isomorphism from \( \mathcal{W}^+ \) to \( \mathbb{C}^{n+1} \). With respect to the basis \( \{\hat{U}(0), \langle \hat{\psi}(0), \mu \rangle, \langle \hat{\psi}(0), \tau_a \rangle\} \) of \( \mathcal{W}^+ \), the matrix of \( B \) is (in ‘block form’):
\[ [B] = \begin{bmatrix} 1/\sqrt{\Delta} & 0 & [0]_{1 \times (n-1)} \\ \beta_0 \sqrt{\Delta} & \beta_0 (i\gamma) \langle \hat{\psi}(0), \mu \rangle + i\beta \gamma \hat{U}(0) & [0]_{1 \times (n-1)} \\ [0]_{(n-1) \times 1} & [i\xi_a]_{(n-1) \times 1} & \sqrt{\Delta} / \beta_{n-1} \end{bmatrix}. \]
This is triangular with non-zero diagonal entries for every \((p,\xi) \in A\). Hence \(B\) is an isomorphism.

7. Estimates in Hölder spaces.

For the fixed-point argument based on the linear system, we need estimates for \(\|\mathcal{F}\|_\alpha, \|\mathcal{B}\|_{1+\alpha}, \|\Omega\|_{1+\alpha}\), of two types: ‘mapping’ and ‘contraction’ estimates.

A bit more precisely, for \(T > 0, R > 0\) and \(Q^T = D_0 \times [0,T]\) consider the open ball:

\[
B^T_R = \{ F \in C^{2+\alpha,1+\alpha/2}(Q^T,\mathbb{R}^{n+1}); \|F - F_0\|_{2+\alpha} < R, F|_{t=0} = F_0 \}.
\]

\((F_0 = [\varphi_0, u_0] \text{ is defined from the initial surface } \Sigma_0, \text{ via Lemma 4.1.})\) Solving the linear system with ‘right-hand side’ defined by \(F \in B^T_R\) defines a map \(F : \bar{F} \mapsto F\), and we need to verify that, for suitable choices of \(T\) and \(R\), \(F\) maps into \(B^T_R\) and is a contraction.

Remark: The argument that follows is standard, and the experienced reader may want to skip to the statement of local existence at the end of the next section. On the other hand the result is not covered by any general theorem proved in detail in a reference known to the author, and some readers may find it useful to have all the details included. Another reason is that, although the ‘right hand sides’ are clearly quadratic, without explicit expressions one might run into trouble with compositions (which behave poorly in Hölder spaces), or when appealing to ‘Taylor remainder arguments’ if the domain is not convex.

For ‘mapping’, we need estimates of the form:

\[
\|\mathcal{F}(\bar{F}, F_0)\|_\alpha + \|\mathcal{B}(DF, DF_0)\|_{1+\alpha} + \|\Omega(D\varphi, D\varphi_0)\|_{1+\alpha} \text{ decays as } T \to 0_+,
\]

and for ‘contraction’:

\[
\|\mathcal{F}(F^1, F^0) - \mathcal{F}(F^2, F^0)\|_\alpha + \|\mathcal{B}(DF^1, DF^2)\|_{1+\alpha} + \|\Omega(D\varphi^1, D\varphi^2)\|_{1+\alpha} \leq \mu(T)\|F^1 - F^2\|_{2+\alpha},
\]

where \(\mu(T) \to 0\) as \(T \to 0_+\).

Notation: The \((\alpha,\alpha/2)\) norms are taken on \(Q^T\), the \((1 + \alpha, (1 + \alpha)/2)\) norms on \(\partial D_0 \times [0,T]\). Double bars without an index refer to the \((2+\alpha, 1+\alpha/2)\) norm, single bars to supremum norms over \(Q^T\), and parabolic norms are indexed by their spatial regularity (\(\alpha\) for \((\alpha,\alpha/2)\), etc.) In general, we use brackets for Hölder-type difference quotients.
We deal with the estimates for the ‘forcing term’ \( \mathcal{F} \) first. Consider the map
\[
\mathcal{G} : \text{Imm}(\mathbb{R}^n, \mathbb{R}^{n+1}) \to GL_n
\]
which associates to the linear immersion \( A \) the inverse matrix of \( (\langle A_i, A_j \rangle)_{i=1}^n \), inner products of the rows of \( A \). \( \mathcal{G} \) is smooth, in particular locally Lipschitz in the space \( \mathcal{W} \) of linear immersions. Hence if \( F^1, F^2 \) are maps \( Q^T \to \mathbb{R}^{n+1} \), such that \( DF^i \in C^{\alpha,\alpha/2}(Q^T) \) and \( DF^i(z) \in K \) for all \( z \in Q^T \), where \( K \subset \mathcal{W} \) is a fixed compact set, we have the bound:
\[
||\mathcal{G}(DF^1) - \mathcal{G}(DF^2)||_{\alpha} \leq c_K ||D(F^1 - F^2)||_{\alpha}.
\]
In fact our maps \( F^i \) are in \( C^{2+\alpha,1+\alpha/2} \), so \( DF^i \in C^{1+\alpha,1+\alpha/2} \). From this higher regularity we obtain the decay as \( T \to 0^+ \). Assuming \( F^1|_{t=0} = F^2|_{t=0} \), we have:
\[
|D(F^1 - F^2)| \leq [D(F^1 - F^2)](1+\alpha/2)T^{1+\alpha/2}.
\]

To continue, we recall an elementary fact for Hölder spaces:

Let \( D \subset \mathbb{R}^n \) be a uniformly \( C^1 \) domain (not necessarily convex or bounded). Then if \( f \in C^1(D) \) and \( \alpha \in (0,1) \), we have:
\[
[f]^{(\alpha)} \leq C_D ||f||_{C^1}.
\]

Here ‘uniformly \( C^1 \)’ means \( D \) can be covered by countably many balls of a fixed radius, which are domains of \( C^1 \) manifold-with-boundary local charts for \( D \), with uniform \( C^1 \) bounds for the charts and their inverses. The constant \( C_D \) depends on those bounds. Applying the lemma to \( DF \), where \( F = F^1 - F^2 \) vanishes identically at \( t = 0 \), and assuming \( T < 1 \):
\[
[DF]_{\alpha}^{(\alpha)} \leq c(||DF| + |D^2F|) \leq c([DF]_{1/2}^{(1+\alpha/2)}T^{1+\alpha/2} + [D^2F]_{T}^{(\alpha/2)}T^{\alpha/2}) \leq c||F||T^{\alpha/2},
\]
(where \( c \) depends on \( D_0 \)) and similarly for the oscillation in \( t \):
\[
[DF]_{t}^{(\alpha)} \leq [DF]_{T}^{(1+\alpha/2)}T^{1/2} \leq ||F||T^{1/2},
\]
so we have:
\[
||D(F^1 - F^2)||_{\alpha} \leq c||F^1 - F^2||T^{\alpha/2}.
\]
We conclude, under the assumption \( F^1 = F^2 \) at \( t = 0 \):
\[
||\mathcal{G}(DF^1) - \mathcal{G}(DF^2)||_{\alpha} \leq c_K ||F^1 - F^2||T^{\alpha/2}.
\]
In particular, applying this to $\bar{F}$ and $F_0$, we find:

$$|| (\mathcal{G}(D\bar{F}) - \mathcal{G}(DF_0)) D^2 F ||_\alpha \leq c_K ||\bar{F} - F_0|| T^{\alpha/2} ||\bar{F}||,$$

and for $F^1$ and $F^2$ coinciding at $t = 0$:

$$|| (\mathcal{G}(DF^1) - \mathcal{G}(DF^2)) D^2 F^1 ||_\alpha \leq c_K ||F^1 - F^2|| T^{\alpha/2} ||F^1||,$$

as well as:

$$|| (\mathcal{G}(DF^2) - \mathcal{G}(DF_0)) (D^2 F^1 - D^2 F^2) ||_\alpha \leq c_K ||F^2 - F_0|| T^{\alpha/2} ||F^1 - F^2||,$$

so we have the mapping and contraction estimates for $\mathcal{F}(\bar{F}, F_0)$ and $\mathcal{F}(F^1, F_0) - \mathcal{F}(F^2, F_0)$.

**Lemma 7.1.** Assume $\bar{F}, F_0, F^1, F^2$ are in $C^{2+\alpha, 1+\alpha/2}(Q^T, \mathbb{R}^{n+1})$ and have the same initial values, and that $D\bar{F}, DF_0, DF^1, DF^2$ all take values in the compact subset $K$ of $Imm(\mathbb{R}^n, \mathbb{R}^{n+1})$. Then:

$$||\mathcal{F}(\bar{F}, F_0)||_\alpha \leq c_K ||\bar{F} - F_0|| ||\bar{F}|| T^{\alpha/2},$$

$$||\mathcal{F}(F^1, F_0) - \mathcal{F}(F^2, F_0)||_\alpha \leq c_K (||F^1|| + ||F^2 - F_0||) T^{\alpha/2} ||F^1 - F^2||.$$

In particular, if $\bar{F} \in B_R^T$:

$$||\mathcal{F}(\bar{F}, F_0)||_\alpha \leq c_0 R T^{\alpha/2}.$$

If $\bar{F}^1, \bar{F}^2 \in B_R^T$, we have:

$$||\mathcal{F}(\bar{F}^1, F_0) - \mathcal{F}(\bar{F}^2, F_0)||_\alpha \leq c_0 T^{\alpha/2} ||\bar{F}^1 - \bar{F}^2||.$$  

(The constant $c_0$ depends only on the data at $t = 0$, and we assume $T < 1$, $R < 1$).

Turning to the orthogonality boundary condition, first observe that:

$$\Omega(D\varphi^1, D\varphi^2) = \langle DT\varphi^1, D_n\varphi^1 \rangle - \langle DT\varphi^2, D_n\varphi^2 \rangle - \mathcal{L}_0 \mathcal{O} [D\varphi^1 - D\varphi^2]$$

$$= \langle DT(\varphi^1 - \varphi^2), D_n\varphi^1 \rangle + \langle DT\varphi^2, D_n(\varphi^1 - \varphi^2) \rangle - \langle D_n(\varphi^1 - \varphi^2), DT\varphi_0 \rangle - \langle DT(\varphi^1 - \varphi^2), D_n\varphi_0 \rangle$$

$$= \langle DT\varphi^1 - DT\varphi^2, D_n\varphi^1 - D_n\varphi_0 \rangle + \langle D_n\varphi^1 - D_n\varphi^2, DT\varphi^2 - DT\varphi_0 \rangle,$$

which has quadratic structure. Using a local frame $(\tau_a)_{a=1}^{n-1}$ for $T\partial D_0$, we find the components $\Omega_a$:

$$\Omega_a(D\varphi^1, D\varphi^2) = [D_i(\varphi^1 - \varphi^2)D_j(\varphi^1 - \varphi_0) + D_j(\varphi^1 - \varphi^2)D_i(\varphi^2 - \varphi_0)] n^i \tau_a.$$
(summation convention, $i, j = 1, \ldots, n$), so $\Omega_a$ is a sum of terms of the form:

$$b(x)D(\varphi^1 - \varphi^2)D(\varphi^3 - \varphi^4),$$

where $b(x) = n^j\tau^i_a$ and the $\varphi^I$ coincide at $t = 0$. It is then not hard to show that:

$$||b(x)D(\varphi^1 - \varphi^2)D(\varphi^3 - \varphi^4)||1+\alpha \leq c||b||1+\alpha||\varphi^1 - \varphi^2||||\varphi^3 - \varphi^4||T^\alpha,$$

with $c$ depending on the $C^1$ norms of local charts for $D_0$. To bound the norm $||n \otimes \tau_a||1+\alpha$, note $|n||\tau_a| \leq 1$, $|D(n \otimes \tau_a)| \leq |Dn| + |D\tau_a|$ and $|D(n \otimes \tau_a)|_x(\alpha) \leq [Dn]_x(\alpha) + |D\tau_a|_x^\alpha$. Since $n = -(\beta/\beta_0)Du_0$ on $\partial D_0$ (and $\partial D_0$ is a level set of $u_0$), we clearly have:

$$||Dn||_\alpha + ||D\tau_a||_\alpha \leq c||D^2u_0||_\alpha \leq c||u_0||.$$

We summarize the conclusion in the following lemma.

**Lemma 7.2.** Assume $\bar{\varphi}, \varphi_0 \in C^{2+\alpha,1+\alpha/2}(Q^T; \mathbb{R}^n)$ have the same initial values. Then:

$$||\Omega(D\bar{\varphi}, D\varphi_0)||1+\alpha \leq c_0||u_0||||\bar{\varphi} - \varphi_0||^2T^\alpha$$

and

$$||\Omega(D\varphi^1, D\varphi^2)||1+\alpha \leq c_0||u_0||(||\varphi^1 - \varphi_0|| + ||\varphi^2 - \varphi_0||)T^\alpha||\varphi^1 - \varphi^2||,$$

with $c_0$ depending only on the data at $t = 0$. In particular, if $\bar{F} = [\bar{\varphi}, \bar{u}] \in B^T_R$, we have:

$$||\Omega(D\bar{\varphi}, D\varphi_0)||1+\alpha \leq c_0R^2T^\alpha,$$

and for $\bar{F}^I = [\bar{\varphi}^I, \bar{u}^I] \in B^T_R$, $I = 1, 2$:

$$||\Omega(D\bar{\varphi}^1, D\varphi^2)||1+\alpha \leq c_0RT^\alpha||\bar{\varphi}^1 - \varphi^2||.$$

To explain the estimates for the angle condition, we write the normal vector as a multilinear form on $DF^i$:

$$\hat{N}(DF) = J_n(DF) := (-1)^n \sum_{i=1}^{n+1} (-1)^{i-1}(DF^1 \wedge \ldots \hat{DF}^i \wedge \ldots DF^{n+1})e_i \in \mathbb{R}^{n+1}$$

($DF^i$ omitted in the $i^{th}$ term of the sum), where $DF^i \in \mathbb{R}^n$ for $i = 1, \ldots, n+1$ and we identify the $n$-vector in $\mathbb{R}^n$ with a scalar, using the standard volume form. The angle condition has the form:

$$\beta^2|\hat{N}|^2 - \langle \hat{N}, e_{n+1} \rangle^2 = 0 \text{ on } \partial D_0.$$
and we set:

\[ B(DF) := \beta^2 |J_n(DF)|^2 - \langle J_n(DF), e_{n+1} \rangle^2, \]

with linearization at \( DF_0 = [I_n |Du_0] \):

\[ \mathcal{L}_0 B[DF] = 2 \beta^2 \langle J_n(DF_0), DJ_n(DF_0)[DF] \rangle - 2 \langle J_n(DF_0), e_{n+1} \rangle \langle DJ_n(DF_0)[DF], e_{n+1} \rangle. \]

Under the assumption \( F^1 = F^2 \) at \( t = 0 \), we need an estimate in \( C^{1+\alpha, \frac{1+\alpha}{2}} \) for:

\[ B(DF^1, DF^2) := B(DF^1) - B(DF^2) - \mathcal{L}_0 B[DF^1 - DF^2] \]

\[ = \beta^2 (|J_n(DF^1)|^2 - |J_n(DF^2)|^2 - 2 \langle J_n(DF_0), DJ_n(DF_0)[DF^1 - DF^2] \rangle) \]

\[ - (\langle J_n(DF^1), e_{n+1} \rangle^2 - \langle J_n(DF^2), e_{n+1} \rangle^2 - 2 \langle J_n(DF_0), e_{n+1} \rangle \langle DJ_n(DF_0)[DF^1 - DF^2], e_{n+1} \rangle). \]

It will suffice to estimate the expression in the first parenthesis; the second is analogous.

We need the following algebraic observation: if \( T_0 = [I_n |Du_0] \) and \( T \) are \( n \times (n + 1) \) matrices, the expression:

\[ |J_n(T_0 + T)|^2 - |J_n(T_0)|^2 - 2 \langle J_n(T_0), DJ_n(T_0)[T] \rangle \]

is a linear combination (with constant coefficients) of terms of the form:

\[ u_{0i} p_{(2)}(T), \quad u_{0i} u_{0j} p_{(2)}(T), \quad p_{(2)}(T), \]

where the \( p_{(2)}(T) \) are polynomials in the entries of \( T \) (with constant coefficients), with terms of degree: \( 2 \leq \text{deg} \leq 2n \).

Thus \( B(DF^1, DF^2) \) is a linear combination (with constant coefficients) of terms of:

\[ u_{0i} p_{(2)}(DF^1 - DF^2), \quad u_{0i} u_{0j} p_{(2)}(DF^1 - DF^2), \quad p_{(2)}(DF^1 - DF^2), \]

with the \( p_{(2)} \) as described; and hence is a linear combination of terms of the form:

\[ u_{0i} (F^{1j}_k - F^{2j}_k)^d, \quad u_{0i} u_{0j} (F^{1j}_k - F^{2j}_k)^d, \quad (F^{1j}_k - F^{2j}_k)^d \]

(where \( 2 \leq d \leq 2n, 1 \leq j \leq n + 1, 1 \leq i, l, k \leq n \)), which we write symbolically as:

\[ B(DF^1, DF^2) \sim \sum_{2 \leq d \leq 2n} b(x)(DF^1 - DF^2)^d, \]
where $b(x)$ is constant or $u_{0_i}(x)$ or $u_{0_j}(x)u_{0_j}(x)$. For the degree $d$ terms $G^{(d)} \sim b(x)(DF^1 - DF^2)^d$, it is not hard to show the bound:

$$||G^{(d)}||_{1+\alpha} \leq c||b||_{1+\alpha}||F^1 - F^2||^d, \quad 2 \leq d \leq 2n.$$ 

We conclude:

**Lemma 7.3.** Assume $\bar{F}, F_0, F^1, F^2$ are in $C^{2+\alpha,1+\alpha/2}(Q^T ; \mathbb{R}^{n+1})$ and have the same initial values. Then:

$$||B(D\bar{F}, DF_0)||_{1+\alpha} \leq c(1 + ||u_0||^2)(1 + ||\bar{F} - F_0||^{2n-2}\alpha)\bar{F} - F_0||^2.$$ 

$$||B(DF^1, DF^2)||_{1+\alpha} \leq c(1 + ||u_0||^2)(1 + ||F^1 - F^2||^{2n-2}\alpha)\bar{F} - F_0||^2,$$

with $c$ depending only on $F_0$. In particular, if $\bar{F} \in B_R^T$:

$$||B(D\bar{F}, DF_0)||_{1+\alpha} \leq c_0 R^2 T^2,$$

and if $\bar{F}^1, \bar{F}^2 \in B_R^T$:

$$||B(D\bar{F}^1, D\bar{F}^2)||_{1+\alpha} \leq c_0 T^2 ||\bar{F}^1 - \bar{F}^2||,$$

with $c_0$ depending only on $F_0$.

8. **Local existence.**

Let $D_0 \subset \mathbb{R}^n$ be a uniformly $C^{2+\alpha}$ domain, not necessarily bounded or connected (note: we define our norms as the sum of the norms on each connected component).

Given a $C^{3+\alpha}$ graph $\Sigma_0$ over $D_0$ satisfying the contact and angle conditions, let $\varphi_0 \in Diff^{2+\alpha}$ be a diffeomorphism given by lemma 4.1 (with the 1-jet of the identity at $\partial D_0$ and 2-jet determined by the mean curvature of $\Sigma_0$ at $\partial D_0$). Then find $u_0 \in C^{2+\alpha}(D_0)$ so that $F_0 = [\varphi_0, u_0]$ parametrizes $\Sigma_0$ over $D_0$.

(Precisely, if $[z, \bar{u}_0(z)]$ parametrizes $\Sigma_0$ as a graph, and $\varphi_0$ is given by lemma 4.1, let $u_0 = \bar{u}_0 \circ \varphi_0$; so $u_0 \in C^{2+\alpha}$.)

We obtained in section 7 all the estimates needed for a fixed-point argument in the set:

$$B_R^T = \{ F \in C^{2+\alpha,1+\alpha/2}(Q^T, \mathbb{R}^{n+1}); ||F - F_0|| < R, F|_{t=0} = F_0 \}.$$

Choose $R < 1$ and $T_0 < 1$ small enough (depending only on $F_0$) so that, for $F \in B_R^T$, $F(t) = [\varphi(t), u(t)]$ defines an embedding of $D_0$, with $\varphi(t)$ a diffeomorphism onto its image $D(t)$. Let $K \subset Imm(\mathbb{R}^n, \mathbb{R}^{n+1})$ be a compact set containing $DF(z)$ for all $F \in B_R, z \in Q^{T_0}$. Now consider $T < T_0$. 

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Given $\bar{F} \in B^T_R$, solve the linear system (with initial data $F_0$) to obtain $F \in C^{2+\alpha,1+\alpha/2}(Q^T)$. (This is possible since the complementarity and compatibility conditions hold for the linear system.) This defines a map $F: \bar{F} \mapsto F$.

From linear parabolic theory (e.g. [3], thm VI.21):

$$\|F - F_0\| \leq M(\|F(\bar{F},F_0)\|_\alpha + \|\mathcal{B}(DF,DF_0)\|_{1+\alpha} + \|\Omega(D\bar{\varphi},D\varphi_0)\|_{1+\alpha}),$$

where $M > 0$ depends on the $C^{\alpha,\alpha/2}$ norm of the coefficients of the linear system, that is, ultimately on $\|F_0\|$.

From lemmas 7.2-7.4 in section 7, it follows that:

$$\|F - F_0\| \leq M_0(RT^{\alpha/2} + R^2T^\alpha) < R$$

provided $T$ is chosen small enough (depending only on $F_0$). Thus $F$ maps $B^T_R$ to itself.

Similarly, if $F(\bar{F}^i) = F^i$ for $i = 1,2$, standard estimates for the linear system solved by $F^1 - F^2$ give:

$$\|F^1 - F^2\| \leq M(\|F(F^1,F^2)\|_\alpha + \|\mathcal{B}(DF^1,DF^2)\|_{1+\alpha} + \|\Omega(D\bar{\varphi}^1,D\varphi^2)\|_{1+\alpha})$$

Again the estimates in lemmas 7.2-7.4 imply:

$$\|F^1 - F^2\| \leq M_0(T^{\alpha/2} + T^\alpha)|\bar{F}^1 - \bar{F}^2| < \frac{1}{2}||\bar{F}^1 - \bar{F}^2||,$$

assuming $T$ is small enough (depending only on $F_0$). This concludes the argument for local existence.

**Theorem 8.1.** Let $\Sigma_0 \subset \mathbb{R}^{n+1}$ be a $C^{3+\alpha}$ graph over $D_0 \subset \mathbb{R}^n$ satisfying the contact and angle conditions at $\partial D_0$ ($\Sigma_0$ may be unbounded or not connected). There exists a parametrization $F_0 = [\varphi_0, u_0] \in C^{2+\alpha}(D_0)$ of $\Sigma_0$, $T > 0$ depending only on $F_0$ and a unique solution $F \in C^{2+\alpha,1+\alpha/2}(Q^T; \mathbb{R}^{n+1})$ of the system:

$$\begin{cases}
\partial_t F - g^{ij}(DF)\partial_i\partial_j F = 0, \\
u|_{\partial D_0} = 0, \quad N^{n+1}(D\varphi,Du)|_{\partial D_0} = \beta,
\end{cases}$$

with initial data $F_0$. For each $t \in [0,T)$, $F(t)$ is a $C^{2+\alpha}$ embedding parametrizing a surface $\Sigma_t$ which satisfies the contact and angle conditions and moves by mean curvature. In addition, $F(t)$ satisfies the orthogonality condition at $\partial D_0$. 

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The hypersurfaces $\Sigma_t$ are graphs. For each $t \in [0, T)$, $\varphi(t) : D_0 \to D(t)$ is a diffeomorphism and $\Sigma_t = \text{graph}(w(t))$, for $w(t) : D(t) \to \mathbb{R}$ given by $w(t) = u(t) \circ \varphi^{-1}(t)$. (We have $w(t) \in C^{2+\alpha}(D(t))$, ‘less regular’ than $u(t)$ or $\varphi(t)$.) $D(t)$ is a uniformly $C^{2+\alpha}$ domain.

Remark 8.1. This theorem does not address geometric uniqueness of the motion, given $\Sigma_0$. It only asserts uniqueness for solutions of the parametrized flow (including the orthogonality boundary condition) in the given regularity class.

9. Rotational symmetry. In this section we record the equations for two rotationally symmetric instances of the problem: (i) $D_0$ and $D(t)$ are disks, and $u > 0$ (‘lens’ case); (ii) $D_0$ and $D(t)$ are complements of disks in $\mathbb{R}^n$ (‘exterior’ case). For simplicity we restrict to $n = 2$.

Let $F(r) = [\varphi(r), u(r)]$ parametrize a hypersurface $\Sigma$, where $\varphi(r) = \phi(r)e_r$ is a diffeomorphism onto its image. Here $e_r, e_\theta$ are orthonormal vectors, outward normal (resp. counterclockwise tangent) to the circles $r=\text{const}$. The unit upward normal vector and mean curvature are:

$$N = \frac{-u_r e_r, \phi_r}{\sqrt{u_r^2 + \phi_r^2}},$$

$$H = \frac{1}{(\phi_r^2 + u_r^2)^{3/2}} \langle \phi_r \mathcal{M}(\phi_r, u_r)[D^2u] - \langle u_r e_r, \mathcal{M}(\phi_r, u_r[D^2\varphi]) \rangle,$$

where:

$$\mathcal{M}(\phi_r, u_r)[D^2u] = u_{rr} + (\phi_r^2 + u_r^2) \frac{u_r \phi_r}{\phi^2},$$

$$\mathcal{M}(\phi_r, u_r)[D^2\varphi] = [\phi_{rr} + (\phi_r^2 + u_r^2)(\frac{r \phi_r}{\phi^2} - \frac{1}{\phi})] e_r.$$

Simplifying:

$$H = \frac{1}{(\phi_r^2 + u_r^2)^{3/2}} \frac{u_r u_{rr} - u_r \phi_{rr} + (\phi_r^2 + u_r^2) \frac{u_r}{\phi}}{\phi}.$$

Now consider the time-dependent case $F(r, t) = [\phi(r, t)e_r, u(r, t)]$. From the above expressions, one finds easily that the equation $\langle \partial_t F, N \rangle = H$ takes the form:

$$\phi_r(u_t - \frac{1}{\phi_r^2 + u_r^2} \mathcal{M}(\phi_r, u_r)[D^2u]) = u_r \langle e_r, \varphi_t \rangle - \frac{1}{\phi_r^2 + u_r^2} \mathcal{M}(\phi_r, u_r)[D^2\varphi].$$
In ‘split gauge’, we consider the system:

\[
\begin{align*}
    u_t - \frac{1}{\phi_r^2 + u_r^2} \mathcal{M}(\phi_r, u_r)[D^2 u] &= 0, \\
    \varphi_t - \frac{1}{\phi_r^2 + u_r^2} \mathcal{M}(\phi_r, u_r)[D^2 \varphi] &= 0.
\end{align*}
\]

Note that \(\phi(r, t) = r\) solves the \(\phi\) equation, and that in this case the \(u\) equation becomes:

\[
w_t - \frac{w_{rr}}{1 + w_r^2} - \frac{w_r}{r} = 0.
\]

This can be compared with the equation for curve networks:

\[
w_t - \frac{w_{xx}}{1 + w_x^2} = 0.
\]

The boundary conditions are easily stated (we assume \(D_0\) is the unit disk or its complement).

The ‘contact condition’ at \(r = 1\) is \(u = 0\). For the ‘angle condition’ at \(r = 1\), we find:

\[
u_r^2 = \frac{\beta_0^2}{\beta^2} \phi_r^2, \quad \beta_0 := \sqrt{1 - \beta^2}.
\]

Assuming \(\phi_r > 0\) at \(r = 1\), this resolves as:

\[
\begin{align*}
    \beta u_r + \beta_0 \phi_r &= 0 \text{ at } r = 1 \text{ (lens case)}; \\
    \beta u_r - \beta_0 \phi_r &= 0 \text{ at } r = 1 \text{ (exterior case)}.
\end{align*}
\]

(For lenses, one also has at \(r = 0\): \(u_r = 0\) and \(\phi_r = 1\).) Thus in both cases one can work with linear Dirichlet/Neumann-type boundary conditions.

One reason to consider the exterior case is that (unlike the lens case) it admits stationary solutions. Geometrically, one just has to consider one-half of a catenoid, truncated at an appropriate height. For example, for 120 degree junctions the equation for stationary solutions:

\[
\begin{align*}
    \frac{u_{rr}}{1 + u_r^2} + \frac{u_r}{r} &= 0 \text{ in } \{r > 1\}, \\
    u_r |_{r=1} &= \sqrt{3}, \quad u |_{r=1} = 0
\end{align*}
\]

admits the explicit solution:

\[
u(r) = \frac{\sqrt{3}}{2} (\ln(2r + \sqrt{4r^2 - 3}) - \ln 3), \quad r > \sqrt{3}/2.
\]
Problem. It would be interesting to consider the nonlinear dynamical stability of this solution (even linear stability is yet to be considered.) One may even work with bounded domains, by introducing a fixed boundary at some $R > 1$, intersecting the surface orthogonally (see Section 10.)

10. Fixed supporting hypersurfaces. Extending the local existence theorem to the case of hypersurfaces intersecting a fixed hypersurface $S$ orthogonally presents no essential difficulty. The case of vertical support surface leads directly to graph evolution with a standard Neumann condition on a fixed boundary; we consider the complementary case where $S$ is a graph. Let $S \subset \mathbb{R}^{n+1}$ be a $C^4$ embedded hypersurface (not necessarily connected), the graph over $D \subset \mathbb{R}^n$ of $B \in C^4(D)$, oriented by the upward unit normal:

$$\nu(y) := \frac{1}{v_B} \tilde{\nu}(y), \quad \tilde{\nu}(y) := [-DB(y), 1] \in \mathbb{R}^n \times \mathbb{R}, \quad v_B := \sqrt{1 + |DB(y)|^2}.$$  

$\nu$ is assumed to be nowhere vertical in $D$ ($DB \neq 0$). To state the problem in the graph parametrization, we consider a time-dependent domain $D(t) \subset \mathbb{R}^n$ with boundary consisting of two components $\partial_1 D(t)$ and $\partial_2 D(t)$, both moving. The hypersurface $\Sigma_t$ is the graph of $w(\cdot, t)$ over $D(t)$, solving the parabolic equation:

$$w_t - g^{ij} (Dw) w_{ij} = 0 \quad \text{in} \quad E := \bigcup_{t \in [0,T]} D(t) \times \{t\} \in \mathbb{R}^{n+1} \times [0,T],$$

with boundary conditions:

$$w(\cdot, t)|_{\partial_1 D(t)} = 0, \quad \sqrt{1 + |Dw|^2}|_{\partial_1 D(t)} = 1/\beta$$

(as before), and on $\partial_2 D(t)$:

$$w = B, \quad \nabla w \cdot \nabla B = -1.$$  

(The first-order condition on $\partial_2 D(t)$ is equivalent to $\langle \nu, N \rangle = 0$).

Differentiating in $t$ the boundary condition $w = B$ leads easily to an equation for the normal velocity of the interface $\Gamma(t) = \partial_2 D(t)$:

$$\Gamma'_n = \frac{vH}{B_n - w_n}.$$  

Note that $w_n$ at $\partial_2 D(t)$ can be computed from $B_n$, since:
\[-1 = \nabla w \cdot \nabla B = w_n B_n + |\nabla^T B|^2;\]

in particular neither \(B_n\) nor \(w_n\) can vanish (so both have constant sign on connected components of \(\partial_2 D\)), and one easily computes: \(w_n - B_n = -v_B^2/B_n\).

Let \(\Lambda = \Sigma \cap S\) be the \((n-1)\)-manifold of intersection, the graph of \(w\) (or \(B\)) over \(\partial_2 D\). Given the graph parametrizations of \(\Sigma\) and \(S\):

\[
G(y) = [y, w(y)], \quad \mathbb{B}(y) = [y, B(y)], \quad y \in \partial_2 D,
\]

and \(\tau \in T\partial_2 D\), we have the tangent vectors:

\[
G_n := [n, w_n] \in T\Sigma, \quad G_B := [\nabla B, -1] = -v_B \nu \in T\Sigma, \quad G_\tau := [\tau, \nabla w \cdot \tau] \in T\Lambda,
\]

and the second fundamental forms of \(\Sigma\) and \(S\) (for \(e \in \mathbb{R}^n\) arbitrary):

\[
A(dG_n, dG_n) = \frac{1}{v} d^2 w(e, e), \quad A(d\mathbb{B}, d\mathbb{B}) = \frac{1}{v_B} d^2 B(e, e).
\]

From \(\langle \nu, N \rangle = 0\) at \(\partial_2 D\), it follows easily that (cp. \([9]\)):

\[
A(G_\tau, \nu) = -A(G_\tau, N), \quad \tau \in T\partial D.
\]

For the remainder of this section, we concentrate on the boundary conditions at \(\partial_2 D_0\), and denote this boundary component simply by \(\partial D_0\). To establish short-time existence, we consider as before the parametrized flow:

\[
F_t - tr_g d^2 F = 0, \quad g = g(dF), \quad F = [\varphi, u].
\]

The contact and angle boundary conditions are:

\[
u|_{\partial D_0} = B \circ \varphi|_{\partial D_0}, \quad \langle N, \nu \circ \varphi \rangle|_{\partial D_0} = 0.
\]

Again we have two scalar boundary conditions for \(n+1\) components. Here the solution is easier than at the junction. With the notation \(F_n = dF_n = [\varphi_n, u_n]\), we replace the angle condition by the 'vector Neumann condition':

\[
F_n \perp TS, \text{ or } F_n = -\alpha v_B \nu \text{ on } \partial D_0,
\]

where \(\alpha : \partial D_0 \to \mathbb{R}\), or equivalently (since this leads to \(\alpha = -u_n\)):

\[
\varphi_n = -u_n(\nabla B \circ \varphi) \text{ on } \partial_2 D_0.
\]
Clearly the Neumann condition implies the angle condition \( \langle N, \nu \circ \varphi \rangle = 0 \), but not conversely. This linear Neumann-type condition can easily be incorporated into the fixed-point existence scheme described earlier.

There is one issue to consider: the 0 and 1st-order compatibility conditions must hold at \( \partial D_0 \), at \( t = 0 \). The initial hypersurface \( \Sigma_0 \) uniquely determines \( w_0 \) and \( D_0 \subset \mathbb{R}^n \) (satisfying \( w_0 = B \) and \( \nabla w_0 \cdot \nabla B = -1 \) on \( \partial D_0 \)), and then once \( \varphi_0 \in Diff(D_0) \) is fixed, \( u_0 = w_0 \circ \varphi_0 \) is also determined. We may assume:

\[
\varphi_0 = id, \quad \varphi_n = \nabla B \text{ on } \partial D_0,
\]

so:

\[
u_0 = \nabla w_0 \cdot \varphi_0 = \nabla w_0 \cdot \nabla B = -1 \text{ on } \partial D_0,
\]

and then the Neumann condition \( F_{0|\partial_2 D_0} = -v_B \nu \) holds at \( t = 0 \), on \( \partial D_0 \).

The first-order compatibility condition is:

\[
tr_g d^2 u_0 = u_t = \nabla B \cdot \varphi_t = \nabla B \cdot tr_g d^2 \varphi_0 \text{ on } \partial D_0,
\]

or equivalently:

\[
tr_g \langle \nu, d^2 F_0 \rangle = 0 \text{ on } \partial D_0.
\]

(This is not a mean curvature condition; the mean curvature of \( \Sigma_0 \) is \( H = tr_g \langle N, d^2 F_0 \rangle \).)

From now on we omit the subscript 0, but continue to discuss compatibility at \( t = 0 \). First observe that the Neumann condition leads to a splitting of the induced metric. Given \( \tau \in T\partial D_0 \), let \( F_\tau = dF \tau \in T\Lambda \). Then (recalling \( u_n = -1 \) on \( \partial D_0 \)):

\[
\langle F_\tau, F_n \rangle = \langle \tau, dB \tau \rangle = \langle \varphi_n, u_n \rangle = \nabla B \cdot \tau - \nabla B \cdot \tau = 0.
\]

Thus we have:

\[
tr_g \langle \nu, d^2 F \rangle = g^{ab} \langle \nu, d^2 F(\tau_a, \tau_b) \rangle + g^{nn} \langle \nu, d^2 F(F_n, F_n) \rangle,
\]

for a local basis \( \{ T_a = dF \tau_a \}_{a=1}^{n-1} \) of \( T\Lambda \), with \( g_{ab} = \langle T_a, T_b \rangle \) and \( g_{nn} = |F_n|^2 = v_B^2 \).

Differentiating in \( n \) the condition \( u_n = \nabla w \cdot \varphi_n \) (assuming, as usual, \( n \) extended to a tubular neighborhood \( N \) of \( \partial D_0 \) as a self-parallel vector field), we find:

\[
u_{nn} = d^2 w(n, \nabla B) + \nabla w \cdot d^2 \varphi(n, n).
\]
This is used to compute:

\[
\langle \nu, d^2 F(n, n) \rangle = \frac{1}{v_B} \left[ u_{nn} - \nabla B \cdot d^2 \varphi(n, n) \right]
\]

\[
= \frac{1}{v_B} \left[ d^2 w(n, \nabla B) + (\nabla w - \nabla B) \cdot d^2 \varphi(n, n) \right]
\]

\[
= -vA(G_n, \nu) + \frac{1}{v_B} (w_n - B_n) n \cdot d^2 \varphi(n, n).
\]

Bearing in mind the expression for \( w_n - B_n \) found earlier, the compatibility condition may be stated in the form:

\[
\frac{v_B}{B_n} n \cdot d^2 \varphi(n, n) = -vA(G_n, \nu) + g^{ab} \langle d^2 F(\tau_a, \tau_b), \nu \rangle.
\]

We are now in the same situation as in section 4: given the 1-jet of \( \varphi_0 \) on \( \partial D_0 \), we extend \( \varphi_0 \) to a tubular neighborhood \( \mathcal{N} \) of \( \partial D_0 \) (and then to all of \( D_0 \)), so that \( n \cdot d^2 \varphi(n, n) \) has on \( \partial D_0 \) the value dictated by the compatibility condition (using Lemma 4.1(ii)). We just need to verify that the right-hand side of the above expression depends only on \( \Sigma^0 \), \( S \), and the 1-jet of \( \varphi_0 \) over \( \partial D_0 \). Clearly only the term \( g^{ab} \langle d^2 F(\tau_a, \tau_b), \nu \rangle \) is potentially an issue.

Fix \( p \in \partial D_0 \), and let \( \{\tau_a\} \) be an orthonormal frame for \( T\partial D_0 \) near \( p \), parallel at \( p \) for the connection induced on \( \partial D_0 \) from \( \mathbb{R}^n \). If \( \mathcal{K} \) denotes the second fundamental form of \( \partial D_0 \) in \( \mathbb{R}^n \), we have:

\[
\tau_a(\tau_b) = \mathcal{K}(\tau_a, \tau_b) n \quad \text{ (at } p)\]

(on the left-hand-side, \( \tau_b \) is regarded as a vector-valued function in \( \mathbb{R}^n \)). Still computing at \( p \), this implies:

\[
d^2 F(\tau_a, \tau_b) = \tau_a(dF\tau_b) - dF(\tau_a(\tau_b))
\]

\[
= \tau_a(d\mathbb{B}\tau_b) - \mathcal{K}(\tau_a, \tau_b) F_n
\]

\[
= d^2 \mathbb{B}(\tau_a, \tau_b) + \mathcal{K}(\tau_a, \tau_b) \mathbb{B}_n - \mathcal{K}(\tau_a, \tau_n) F_n,
\]

where \( F_n = -v\nu \) and \( \mathbb{B}_n = d\mathbb{B}n \in TS \). Hence:

\[
\langle \nu, d^2 F(\tau_a, \tau_b) \rangle = \langle \nu, d^2 \mathbb{B}(\tau_a, \tau_b) \rangle + v\mathcal{K}(\tau_a, \tau_b) = A(T_a, T_b) + v\mathcal{K}(\tau_a, \tau_b).
\]

This clearly depends only on \( S \) and on \( \Sigma^0 \). We summarize the discussion in a lemma.
Lemma 10.1 Let $\Sigma_0 = \text{graph}(w_0)$ be a $C^3$ graph over $D_0 \subset \mathbb{R}^n$ (a uniformly $C^3$ domain), intersecting a fixed hypersurface $S = \text{graph}(B)$ over $\partial D_0$. Consider the parametrized mean curvature motion with Neumann boundary condition:

$$F \in C^{2,1}(D_0 \times [0,T]) \to \mathbb{R}^{n+1}, \quad F = [\varphi, u]$$

$$F_t - tr_g d^2 F = 0, \quad g = g(dF), \quad u \circ \varphi = B \text{ and } F_n \perp TS \text{ on } \partial D_0.$$ 

Then $\varphi_0 \in Diff(D_0)$ can be chosen so that (with $u_0 = w_0 \circ \varphi_0$) the initial data $F_0 = [\varphi_0, u_0]$ satisfies the order zero and the first-order compatibility conditions at $t = 0$ and $\partial D_0$:

$$\varphi_{0n} = -u_{0n}(\nabla B \circ \varphi_0), \quad \langle \nu \circ \varphi_0, tr_{g_0} d^2 F_0 \rangle = 0.$$ 

Remark 10.1. Differentiating $dw_{\tau_a} = dB_{\tau_a}$ along $\tau_b$, we find:

$$d^2 w(\tau_a, \tau_b) - d^2 B(\tau_a, \tau_b) = (w_n - B_n)K(\tau_a, \tau_b)$$

(reminding us that, although $w \equiv B$ on $\partial D_0$, the tangential components of their Hessians do not coincide.) From this follows the expression for $K$ in terms of $A$ and $\Lambda$:

$$K(\tau_a, \tau_b) = \frac{1}{w_n - B_n}[vA(T_a, T_b) - v_B A(T_a, T_b)].$$

It is also easy to express the corresponding traces in terms of the mean curvatures $H^\Lambda$ and $\mathcal{H}^\Lambda$ of $\Lambda$ in $\Sigma$ and $S$:

$$H^\Lambda = \frac{v}{v_B}g^{ab}A(T_a, T_b), \quad \mathcal{H}^\Lambda = \frac{v_B}{v}g^{ab}A(T_a, T_b).$$

11. A continuation criterion. Once local existence has been established, it is easier to obtain geometric estimates (in particular using the maximum principle) for the solution in the graph parametrization. (From this point on, we focus on the ‘lens’ case, without fixed support hypersurfaces.)

For a time interval $I = (t_0, t_1) \subset [0,T]$ set:

$$E = \{z = (y, t) \in \mathbb{R}^n \times I; y \in D(t)\}, \quad S = \{(y, t); t \in I, y \in \partial D(t)\}.$$ 

Let $w$ be a solution in $E$ of:

$$w_t - g^{ij}(Dw)D^2w_{i,j} = 0$$
with boundary conditions on $S$:

$$w = 0, \quad D_nw = \beta_0 v, \quad \beta_0 := \sqrt{1 - \beta^2}.$$ 

For the remainder of the paper we assume $D(t)$ is bounded, for each $t \in I$. $n$ denotes the $(t$-dependent) inner unit normal at $\partial D(t)$, extended to a $C^{2,1}$ unit vector field in a tubular neighborhood of $D(t)$ so that $D_n n = 0$.

Denote by $L$ the operator $L = \partial_t - g^{ij}(Dw)\partial_i\partial_j$, so $Lw = 0$ in $E$. The following height bound is immediate.

**Lemma 11.1.** Assume $0 < w_0 < M$ in $D(t_0)$. (If there is a support surface $S$, we assume $B(y) > 0$ in $D$ and $M < \sup_D B$.) Then $0 < w < M$ in $\bar{E}$.

**Proof.** Follows from the maximum principle applied to $L$, since $0 \leq w \leq M$ holds on the parabolic boundary $\partial_p E$.

It is well-known that the function $v = \sqrt{1 + |Dw|^2}$ solves the evolution equation (assuming $Dw \in C^{2,1}(\bar{E})$, see e.g. [4]):

$$Lv + \frac{2}{v}g^{ij}v_i w_j = -v|A|^2_g.$$ 

From the maximum principle, we have the following global bound on $v$ (equivalently, on $|Dw|$).

**Lemma 11.2** Assume $w$ is a solution with $Dw \in C^{2,1}(\bar{E})$. Then we have on $\bar{E}$:

$$v(z) \leq \max\{\sup_{D(t_0)} v(x, t_0), \frac{1}{\beta}\}.$$ 

**Proof.** By the maximum principle, $\max_{\bar{E}} v = \max_{\partial_p E} v$. Note $v|_S \equiv \frac{1}{\beta}$.

It follows from this lemma that $g_{ij}(t)$ is uniformly equivalent to the euclidean metric in $D(t)$: if $v \leq \bar{v}$ in $E$, and $X$ is a vector field in $D(t)$:

$$|X|^2_e \leq |X|^2_g = g_{ij}X^i X^j = |X|^2_e + (X \cdot Dw)^2 \leq |X|^2_e (1 + |Dw|^2) \leq \bar{v}^2 |X|^2_e.$$ 

Also, if $\omega := v^{-1} Dw$:

$$|\omega|^2_e = \frac{|Dw|^2_e}{v^2} = 1 - \frac{1}{v^2} \leq 1 - \frac{1}{\bar{v}^2}.$$ 

This equivalence of norms clearly extends to tensors, in particular to $h$:

$$\frac{1}{c_n} |h|^2_e \leq |h|^2_g \leq c_n |h|^2_e,$$

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where (throughout this section) $c_n$ denotes a constant depending only on $n$ and $\bar{v}$. More generally, defining:

$$|\partial h|^2_e := \sum_{i,j,k} (\partial_k h_{ij})^2, \quad |\partial h|^2_g := \sum_{i,j,k,l} g^{kl}(\partial_k h_{ij})(\partial_l h_{ij}),$$

we have, for each pair $i,j$:

$$\sum_k (\partial_k h_{ij})^2 = \sum_{k,l} \delta_{kl} \partial_k h_{ij} \partial_l h_{ij} = \sum_{k,l} (g^{kl} + \omega^k \omega^l) \partial_k h_{ij} \partial_l h_{ij}$$

$$= (d_\omega h_{ij})^2 + \sum_{k,l} g^{kl} \partial_k h_{ij} \partial_l h_{ij} \leq \sum_{k,l} g^{kl} \partial_k h_{ij} \partial_l h_{ij} + (1 - \frac{1}{\bar{v}^2}) \sum_k (\partial_k h_{ij})^2,$$

and hence, adding over $i,j$ we have:

$$|\partial h|^2_e \leq |\partial h|^2_g \leq \bar{v}^2 |\partial h|^2_g.$$

The same argument works for second derivatives. The norms defined by:

$$|\partial^2 h|^2_e := \sum_{i,j,k,m} (\partial_m \partial_k h_{ij})^2, \quad |\partial^2 h|^2_g := \sum_{i,j,k,m,n} g^{mn}(\partial_m \partial_k h_{ij})(\partial_n \partial_k h_{ij})$$

are uniformly equivalent in $E$:

$$|\partial^2 h|^2_e \leq |\partial^2 h|^2_g \leq \bar{v}^2 |\partial^2 h|^2_g.$$

The point is that these euclidean norms satisfy easily computed evolution equations. Using the results in Appendix 2, it is straightforward to see that:

$$L[|h|^2_e] = -2|\partial h|^2_g + 2 \sum_{i,j} C_{ij} h_{ij}, \quad C_{ij} := L[h_{ij}],$$

$$L[|\partial h|^2_e] = -2|\partial^2 h|^2_g + 2 \sum_{i,j,k} (\partial_k C_{ij})(\partial_k h_{ij}) + 2 \sum_{i,j,k,m,n} (\partial_k g^{mn})(\partial_m \partial_n h_{ij})(\partial_k h_{ij}).$$

In symbolic notation, we have:

$$C_{ij} \sim h \ast h \ast h, \quad \partial_k C_{ij} \sim (\partial h) \ast h \ast h,$$

which combined with the previous remarks implies:

$$\sum_{ij} (C_{ij})^2 \leq c_n |h|^6_g, \quad \sum_{i,j,k} (\partial_k C_{ij})^2 \leq c_n |\partial h|^2_g |h|^4_g,$$
for a constant $c_n$ as above. In addition, from (see Appendix 2):
\[ \partial_k g^{mn} = h^m_k \omega^n + h^n_k \omega^m \text{ and } |g^{ij}| \leq 2, |\omega^n| < 1, \]
we have $|\partial_k g^{mn}| \leq c_n |h|_g$ for each $m,n,k$. We conclude:
\[
L[|h|^2] \leq -2|\partial h|^2_g + c_n |h|^4_g,
\]
\[
L[|\partial h|^2] \leq -2|\partial^2 h|^2_g + c_n |\partial h|^2_g |h|^2_g + c_n |h|_g |\partial h|_g |\partial^2 h|_g.
\]
These differential inequalities imply the continuation criterion given in Proposition 11.3.

Recall that for mean curvature flow (or mean curvature motion) of graphs, interior estimates for $v$ imply interior estimates for $A$ and its covariant derivatives $\nabla^m A$ of any order (see [1] or [2]). In the following ‘continuation criterion’, global bounds are needed.

**Proposition 11.3.** Assuming $T_{\text{max}}$ is finite, let $w : E^{T_{\text{max}}} \to \mathbb{R}$ be a maximal solution, defined for $t \in [0, T_{\text{max}})$. Then:
\[
\limsup_{t \to T_{\text{max}}} (\sup_{(y,t) \in E} |h|_g + \sup_{y \in \partial D(t)} |\nabla h|_g(y,t)) = \infty.
\]

**Proof.** By contradiction, assume we have bounds in $[t_0, t_1]$ (for $t_1$ arbitrarily close to $T_{\text{max}}$):
\[
\sup_{z \in E} |h|_g \leq a_0, \quad \sup_{z \in S \cup \partial D(0)} |\nabla h|_g \leq b_0.
\]
For $\alpha > 0$ to be chosen (small), define the function on $E$:
\[
f(x,t) = \alpha |\partial h|^2_c + |h|^2_c.
\]
Then, for any $\eta > 0$:
\[
L[f] \leq -2\alpha |\partial^2 h|^2_g - 2|\partial h|^2_g + c_n (a_0^4 + \alpha a_0^2 |\partial h|^2_g + \alpha a_0 |\partial^2 h|_g |\partial h|_g)
\]
\[
\leq -2\alpha |\partial^2 h|^2_g - 2|\partial h|^2_g + c_n a_0 \alpha |\partial^2 h|^2_g + (c_n \alpha a_0^2 + \frac{c_n a_0 \alpha}{\eta}) |\partial h|^2_g + c_n a_0^4.
\]
Choosing $\eta$ so that $c_n a_0 \eta \leq 1$, then $\alpha$ so that $c_n \alpha a_0^2 + \frac{c_n a_0 \alpha}{\eta} < 1$, we ensure that:
\[
L[f - c_n a_0^4 t] \leq 0
\]
in $E$. By the maximum principle:

$$\alpha \sup_E |\partial h| \leq \sup_E f \leq \sup_{\partial E} f \leq c_n a_0^4 T \leq c_n (a_0^2 + \alpha b_0^2 + a_0^4 T).$$

This implies a uniform $C^3(\bar{D}(t))$ bound for $w$ in $\bar{E}$, and hence (by linear parabolic theory, given the uniform bound on $|Dw|$ from lemma 10.2) a $C^{3+\alpha}$ bound for some $0 < \alpha < 1$. So we can apply the local existence theorem with initial data $\Sigma_{t_1}$, to continue the solution for a time depending only on bounds at $t_0$, contradicting the maximality of $T_{\text{max}}$.

Lemma 12.2 (in the next section) implies the conclusion can be strengthened: only a uniform bound on tangential covariant derivatives of the second fundamental form $K$ of the moving boundary (in $\mathbb{R}^n$) is needed:

**Proposition 11.4.** Assuming $T_{\text{max}}$ is finite:

$$\limsup_{t \to T_{\text{max}}} \left[ \sup_{(y,t) \in \bar{E}} |h| + \sup_{y \in \partial D(t)} |\nabla_y K| \right] = \infty.$$  

It is possible to strengthen this further and show that:

$$\limsup_{t \to T_{\text{max}}} \left[ \sup_{y \in \partial D(t)} |h| + \sup_{y \in \partial D(t)} |\nabla_y K| \right] = \infty.$$  

That is, the supremum of $|h|$ on the moving boundary controls its value in the interior. The reason is that we already have a bound on $\sup_E v$; as remarked earlier, it is a well known fact for mean curvature flow of graphs that this implies interior bounds for the second fundamental form and its covariant derivatives ([1], [2]). In the next lemma we describe a global argument for mean curvature motion of graphs with moving boundaries.

**Proposition 11.5.** Let $w : E \to \mathbb{R}$ be a solution of graph m.c.m in a spacetime domain $E \subset \mathbb{R}^n \times [0, T]$, where $T < \infty$. Assume the first derivative bound $v(x,t) \leq \bar{v}$ holds globally in $E$. Then if the bound $|h| \leq h_0$ holds on the parabolic boundary $\partial_p E$, we also have the global bound:

$$|h| \leq a_0 \quad \text{in } \bar{E},$$  

for a constant $a_0$ depending only on $n, \bar{v}, h_0, T$ and the initial data of $w$.

**Proof.** The idea is to consider a function in $E$ of the form:

$$\varphi = Av^p + |h|^2 v^p + B|h|^2,$$
where $A$, $B$ and $p$ are positive constants. We claim it is possible to choose these constants, and also $C > 0$ (all depending only on $n$ and $\bar{v}$) so that:

$$L[\varphi] \leq C \text{ in } E.$$ 

Thus $L[\varphi - Ct] \leq 0$ in $E$, and hence by the maximum principle:

$$\sup_{E} B|h|_{g}^{2} \leq \sup_{E} \varphi \leq \sup_{\partial E, E} \varphi + CT,$$

which clearly implies the bound claimed in the proposition.

The proof that $\varphi$ as above exists is (of course) based on the evolution equations for $|h|_{g}$ and $v$ (see Appendix 2), which imply (for constants $c_{n}, d_{n}$ depending only on $n$):

$$L[|h|_{g}^{2}] \leq -2|\nabla h|_{g}^{2} + c_{n}|h|_{g}^{4},$$

$$L[v^{p}] = -pv^{p}|h|_{g}^{2} - (p(p - 1)v^{p-2}|\partial v|_{g}^{2} - 2pv^{p-2}g^{ij}v_{i}w_{j}.$$

Here $|\partial v|_{g}^{2} := g^{kl}v_{k}v_{l}$, and we have the bounds:

$$L[v^{p}] \leq -pv^{p}|h|_{g}^{2} - p(p - 1)v^{p-2}2pv^{p-2}|\partial v|_{g}^{2} + d_{n}pv^{p-1}|\partial v|_{g},$$

$$L[v^{p}] \leq -pv^{p}|h|_{g}^{2} - [p(p - 1) - \frac{1}{4}]|\partial v|_{g}^{2} + d_{n}pv^{p-2}v^{2(p-1)}.$$

The main term in $L[\varphi]$ is:

$$L[|h|_{g}^{2}v^{p}] = L[|h|_{g}^{2}v^{p} + L[v^{p}]|h|_{g}^{2} - 4g^{kl}pv^{p-1}|\partial h v(h, \nabla h)|_{g} := (I) + (II) + (III),$$

where:

$$(I) \leq -2v^{p}|\nabla h|_{g}^{2} + c_{n}v^{p}|h|_{g}^{4};$$

$$(II) \leq -pv^{p}|h|_{g}^{4} - p(p - 1)v^{p-2}|\partial v|_{g}^{2}|h|_{g}^{2} + d_{n}pv^{p-1}|\partial v|_{g}|h|_{g}^{2};$$

$$(III) \leq 4pv^{p-1}|\partial v|_{g}|h|_{g}^{2}\nabla h|_{g} \leq \frac{4}{\gamma} \frac{p}{p - 1}pv^{p}v|\nabla h|_{g}^{2} + \gamma p(p - 1)v^{p-2}|\partial v|_{g}^{2}|h|_{g}^{2},$$

for an arbitrary constant $\gamma \in (0, 1)$. With $\eta > 0$ to be chosen sufficiently small later, we estimate the last term in (II):

$$d_{n}pv^{p-1}|\partial v|_{g}|h|_{g}^{2} \leq \eta d_{n}pv^{p-2}|\partial v|_{g}^{2}|h|_{g}^{2} + \frac{2}{\eta}d_{n}pv^{p}|h|_{g}^{2}.$$

Adding to these estimates for (I)+(II)+(III) the term $L[B|h|_{g}^{2}]$, we have:

$$L[|h|_{g}^{2}v^{p} + B|h|_{g}^{2}] \leq \frac{4p}{\gamma(p - 1)} - 2 - \frac{2}{v^{p}}B|v^{p}|\nabla h|_{g}^{2} + [c_{n} - p + \frac{c_{n}}{v^{p}}B]|v^{p}|h|_{g}^{4}.$$
Given $\gamma \in (0, 1)$ arbitrary, we choose $p > 0$ so large that $(2/\gamma)c_n < p - 1,$
then $B > 0$ so that:

$$\frac{2p}{\gamma(p - 1)} < 1 + \frac{B}{v^p} < 1 + \frac{B}{v^p} < 1 + B < \frac{p}{c_n}.$$ 

In this way we ensure that, in the expression above, the coefficients in the
first two square brackets are negative. Choosing $\eta > 0$ sufficiently small
(depending on $\gamma$ and $p$), the same holds for the third square bracket. Finally,
in view of the second estimate given above for $L[v^p]$, if we add $L[Av^p]$ with
$A > (2/\eta)d_n$ we also take care of the last square bracket (we also assume
$p(p - 1) > 1/4$), and then:

$$L[Av^p + |h|^2_g v^p + B|h|^2_g] \leq C := Ad_n^2 p^2 (\bar{v})^{2(p-1)},$$

concluding the proof.

12. Boundary conditions for the second fundamental form.

Proving global existence for the mean curvature motion of graphs over
time-dependent domains requires estimates for the second fundamental form.

The simplest form of the evolution equations for $h_{ij}$ and $H = g^{ij} h_{ij}$ is
given in terms of the differential operator on functions: $L[f] = \partial_t f - \text{tr}_d d^2 f$.

The evolution equations for $h_{ij}$ and $H$ are given in Appendix 2. In
this section we derive boundary conditions for $h$ and $H$; the development is
similar to work of A. Stahl [9] for MCF of hypersurfaces intersecting a fixed
boundary orthogonally.

It is easy to see that $h$ splits on $\partial D(t)$: if $\tau \in T\partial D(t)$ is a tangential
vector field, and $n = n_t$ is the inner unit normal:

$$h(n, \tau) = \frac{1}{v} d^2 w(n, \tau) = \frac{1}{v}(\tau(w_n) - Dw \cdot \nabla_\tau n) = 0 \text{ on } \partial D(t),$$

since $w_n \equiv \beta_0/\beta$ on the boundary and $\nabla_\tau n \in T\partial D(t)$ ($\nabla$ is the euclidean
connection.) In particular, it follows that $h(Dw, \tau) = 0$ on $\partial D(t)$.

**Boundary condition for $H$.** We derived in section 2 the equation for the
normal velocity of the moving boundary $\Gamma_t = \partial D(t)$. Letting $\Gamma(\theta, t), \theta \in S^{n-1},$ be any parametrization of $\Gamma_t,$ we find for $\hat{\Gamma}_n := \partial_t \Gamma \cdot n$:

$$\hat{\Gamma}_n = -\frac{v}{w_n} H = -\frac{1}{\beta_0} H \text{ at } \partial D(t).$$
Since \( \langle N, e_{n+1} \rangle(t, \Gamma(t)) \equiv \beta \) on \( \partial D(t) \) we have:

\[
\langle \partial_t N, e_{n+1} \rangle = -\langle \partial_t N, e_{n+1} \rangle \dot{\Gamma}^k,
\]

where \( \partial_t N = -g^{ij} h_{ik} G_j \), with \( e_{n+1} \) component:

\[
\langle \partial_t N, e_{n+1} \rangle = -g^{ij} w_j h_{ik} = -\frac{1}{v^2} h(Dw, \partial_k) = -\frac{1}{v^2} w_n h(n, \partial_k).
\]

Hence we find, on \( \partial D(t) \):

\[
\langle \partial_t N, e_{n+1} \rangle = w_n \frac{1}{v^2} h(n, \dot{\Gamma}) = w_n \frac{1}{v^2} h(n, n) = -\beta H h_{nn}.
\]

On the other hand, using \( \partial_t N = -\nabla^\Sigma H - Hv^{-1}\nabla^\Sigma v \), combined with the expressions (valid on \( \partial D(t) \)):

\[
\langle \nabla^\Sigma H, e_{n+1} \rangle = g^{ij} H_i (G_j, e_{n+1}) = g^{ij} w_j H_i = \frac{1}{v^2} w_i H_i = \frac{w_n}{v^2} H_n = \beta_0 H_n,
\]

\[
\langle \nabla^\Sigma v, e_{n+1} \rangle = \frac{v_n w_n}{v^2} = \frac{w_n^2}{v^2} h_{nn} = \beta_0^2 h_{nn},
\]

we find on \( \partial D(t) \):

\[
\langle \partial_t N, e_{n+1} \rangle = -\beta \beta_0 (H_n + \beta_0 H h_{nn}).
\]

Comparing these two expressions for \( \langle \partial_t N, e_{n+1} \rangle \) yields:

\[
H_n = \frac{\beta_0^2}{\beta_0} H h_{nn},
\]

a Neumann-type condition for \( H \) on \( \partial D(t) \).

**Boundary conditions for \( h_{ij} \).** Fix \( p \in \partial D(t) \) and let \( \tau_a \) be an orthonormal frame for \( T_p \partial D(t) \) (in the induced metric), satisfying \( \nabla^\Gamma_{\tau_a} \tau_b(p) = 0 \) (\( \nabla^\Gamma \) is the connection induced on \( \Gamma_t \) by \( \nabla \), or, equivalently, by \( \nabla \), the Levi-Civita connection of the metric \( g \) in \( D(t) \)); we extend the \( \tau_a \) to a tubular neighborhood so that \( \nabla_n \tau_a = 0 \). Differentiating \( h(n, \tau_b) = 0 \) along \( \tau_a \), we find:

\[
(\nabla_{\tau_a} h)(n, \tau_b) = -h(\nabla_{\tau_a} n, \tau_b) - h(n, \nabla_{\tau_a} \tau_b).
\]

The second fundamental form \( K(\tau, \tau') \) of \( \Gamma \) in \( (D(t), eucl) \) (equivalently, in \( (D(t), g) \)) is defined by:

\[
\bar{\nabla}_{\tau_a} \tau_b = \nabla^\Gamma_{\tau_a} \tau_b + K(\tau_a, \tau_b)n \quad \text{on} \ \partial D(t).
\]
To relate $K$ to $h|_{\partial D(t)}$, note that since $w = 0$ on $\partial D(t)$:

$$h(\tau_a, \tau_b) = \langle [\bar{\nabla}_{\tau_a} \tau_b, 0], N \rangle = -\bar{\nabla}_{\tau_a} \tau_b \cdot \frac{D}{Du} = -\beta h(\tau_a, \tau_b).$$

(So we see that $\Gamma_t$ convex with respect to $n$ corresponds to $\Sigma_t$ concave over $D(t)$, as expected). In the appendix we observe that $\nabla_\partial \tau_j = (h_{ij}/v) Dw$. Then:

$$\nabla_{\tau_a} \tau_b = \tau^i_a ((\tau^j_b) \partial_j + \tau^j_b \nabla_\partial \tau_j) = \nabla_{\tau_a} \tau_b + \frac{1}{v} \tau^i_a \tau^j_b h_{ij} Dw$$

$$= \nabla^\Gamma_{\tau_a} \tau_b + K(\tau_a, \tau_b)n + \frac{w_{ij}}{v} h(\tau_a, \tau_b)n = (-\frac{1}{\beta} + \beta_0) h(\tau_a, \tau_b)n = -\frac{\beta^2}{\beta_0} h(\tau_a, \tau_b)n$$

at $p$, given our assumption $\nabla^\Gamma_{\tau_a} \tau_b(p) = 0$. We use this immediately to compute, at $p$:

$$\nabla_{\tau_a} n = \langle \nabla_{\tau_a} n, \tau_b \rangle g_{\tau_b} = -\langle n, \nabla_{\tau_a} \tau_b \rangle g_{\tau_b} = \beta^2 |n|^2 h(\tau_a, \tau_b) \tau_b = \frac{1}{\beta_0} h(\tau_a, \tau_b) \tau_b,$$

since $|n|^2_g = g_{ij} n^i n^j = 1 + w^2_n = \beta^{-2}$ at $p$. We conclude, using the Codazzi equations:

$$(\nabla_n h)(\tau_a, \tau_b) = (\nabla_{\tau_a} h)(n, \tau_b) = -\frac{1}{\beta_0} \sum c h(\tau_a, \tau_c) h(\tau_c, \tau_b) + \frac{\beta^2}{\beta_0} h(\tau_a, \tau_b) n_{mn}.$$  

This can also be written in the form:

$$\beta_0 (\nabla_n h)(\tau, \tau') = -(h^{tan})^2(\tau, \tau') + \beta^2 h_{nn} h(\tau, \tau').$$

It turns out the expression for covariant derivative of $h$ with respect to the euclidean connection $\bar{\nabla}$ is exactly the same (at $\partial D(t)$):

$$\beta_0 (\nabla_n h)(\tau, \tau') = -(h^{tan})^2(\tau, \tau') + \beta^2 h_{nn} h(\tau, \tau').$$

The reason is that $\nabla_n \tau_a = 0$ at the boundary, also for the $g$-connection:

$$\nabla_n \tau_a = \bar{\nabla}_n \tau_a + n^j \tau^i_a \nabla_\partial \tau_j = 0 + \frac{1}{v} h(n, \tau_a) Dw = 0,$$

so in fact:

$$(\nabla_n h)(\tau_a, \tau_b) = n(h(\tau_a, \tau_b)) = (\bar{\nabla}_n h)(\tau_a, \tau_b).$$

As done in [9], we combine this with the result for $H_n$ to compute $(\nabla_n h)(n, n)$. From:

$$H_n = \nabla_n (tr_g h) = tr_g (\nabla_n h) = \beta^2 (\nabla_n h)(n, n) + \sum_a (\nabla_n h)(\tau_a, \tau_a),$$
we find:
\[ \beta^2 (\nabla h)(n,n) = \frac{\beta^2}{\beta_0} H h_{nn} + \frac{1}{\beta_0} |h_{\text{tan}}|^2 - \frac{\beta^2}{\beta_0} (H - \beta^2 h_{nn}) h_{nn} \]
\[ = \frac{1}{\beta_0} (|h_{\text{tan}}|^2 + \beta^2 h_{nn}^2) = \frac{1}{\beta_0} |h|^2_g, \]

since \( g_{nn} = \beta^2 \) at \( \partial D \). Equivalently:
\[ \beta_0 (\nabla h)(n,n) = \frac{1}{\beta^2} |h|^2_g \quad \text{on} \ \partial D(t). \]

It is easy to obtain the corresponding expression for the euclidean connection. Noting that at \( \partial D(t) \):
\[ \nabla_n n = \nabla_n n + n^i n^j v_i h_{ij} D w = \beta_0 h_{nn} n, \]
we find:
\[ (\nabla h)(n,n) = n(h_{nn}) = (\nabla h)(n,n) + 2h(\nabla n n, n) = (\nabla h)(n,n) + 2\beta_0 h_{nn}^2, \]
so that:
\[ \beta_0 (\nabla h)(n,n) = \frac{1}{\beta^2} |h|^2_g + 2\beta_0 h_{nn}^2 \quad \text{on} \ \partial D(t). \]

It turns out that the expressions just derived, combined with the maximum principle proved in [9], are not enough to establish that concavity is preserved. We derive a suitable maximum principle in section 13. The result of the next lemma yields a continuation criterion stated earlier (Prop. 11.4).

**Lemma 12.2.** Let \( w(y,t) \) be a solution of graph MCM, with constant-angle boundary conditions, in \( E \subset \mathbb{R}^n \times [0,T) \). Denote by \( K \) the second fundamental form of \( \Gamma_t = \partial D(t) \) in \( \mathbb{R}^n \). Suppose that, for some \( a_0 > 0 \):
\[ \sup\{ |A|(y,t) + |\nabla K|(y,t); y \in \partial D(t), t \in [0,T), \tau \in T_y \partial D(t), |\tau| = 1 \} \leq a_0. \]
Then also:
\[ \sup\{ |
abla A|; y \in \partial D(t), t \in [0,T) \} < \infty. \]

**Proof.** From the boundary conditions computed above for \( \nabla h \), we have at boundary points:
\[ |(\nabla h)(\tau, \tau)| + |(\nabla h)(n,n)| + |(\nabla h)(n, \tau)| \leq c_0, \]
where \( c_0 \) depends only on \( \beta \) and \( a_0 \). The remaining components of \( \nabla h \) are:

\[
(\nabla_{\tau} h)(n, n) = (\nabla_{\tau} h)(\tau, n) \quad \text{and} \quad (\nabla_{\tau} h)(\tau, \tau),
\]

and since \( h^{tan} = -\beta_0 K \) at boundary points, the last one is assumed bounded in \([0, T]\). In addition, on \( \partial D(t) \):

\[
\tau(H) = \beta^2(\nabla_{\tau} h)(n, n) + 2\beta^2 h(\nabla_{\tau} n, n) + \sum_a [(\nabla_{\tau} h)(\tau_a, \tau_a) + 2h(\nabla_{\tau} \tau_a, \tau_a)],
\]

with all terms on the right bounded, except for the first one. Thus a bound on \( (\nabla_{\tau} h)(n, n) \) would follow from a bound on \( \tau(H) \). But this follows from the uniform gradient estimates (up to the boundary) of linear parabolic theory, since \( H \) is a solution of (see Appendix 2; \( \omega := Dw/v \)):

\[
\partial_t H - \text{tr}_g d^2 H = |h|^2 H + H h^2(\omega, \omega) - H^2 h(\omega, \omega), \quad H_n|_{\partial D(t)} = \frac{\beta^2}{\beta_0} H h_{nn},
\]

in which all the coefficients are uniformly bounded in \([0, T]\). The bound depends only on \( a_0 \) and the initial data. (The hypotheses of the proposition imply that the necessary regularity conditions on \( \partial E \) are satisfied.)

**Finite existence time.**

In the next section we show that weak concavity at \( t = 0 \) is preserved by the evolution. Assuming this, it is not difficult to derive that the flow is defined only for finite time.

**Lemma 12.4.** Let \( w(y, t), (y, t) \in E \subset \mathbb{R}^n \times [0, T) \) define a graph MCM \( \Sigma_t \) with constant-angle boundary conditions on a moving boundary. Assume \( \Sigma_0 \) (and hence \( \Sigma_t \), for all \( t \)) is weakly concave. Then:

Assume \( H_{|t=0} \leq H_0 < 0 \) (where \( H_0 \) is a negative constant). Then \( T \leq t_* = \frac{1}{2 \beta^2 c_n} \) (we are assuming \( T = \sup \{ t \in [0, T); D(t) \neq \emptyset \} \)). Here \( c_n > 0 \) depends only on \( n \) and an upper bound for \( v \) in \( E \).

The proof is based on the evolution equation and boundary condition for \( H \) (see Appendix 2; \( \omega = Dw/v \)):

\[
L[H] = |h|^2 H + H h^2(\omega, \omega) - H^2 h(\omega, \omega), \quad H_n = (\beta^2 / \beta_0) H h_{nn}.
\]

Since \( h^2(\omega, \omega) \geq 0 \), \( |h|^2 \geq (1/n) H^2 \) and (given that \( h \leq 0 \)) \( h(\omega, \omega) \geq |Dw|^2 H \), we have:

\[
L[H] \leq \frac{1}{n} H^3 + |Dw|^2 H^3 \leq c_n H^3,
\]

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where $c_n$ depends on $n$ and on $\sup_E |v|$ (already known to be finite). Let $\phi(t)$ solve the o.d.e. $\dot{\phi} = c_n \phi^3, \phi(0) = H_0$:

$$\phi(t) = H_0[1 - 2c_n H_0^2 t]^{-1/2}, \quad 0 \leq t < t_* := \frac{1}{2H_0^2 c_n}.$$ 

Then with $\psi := (1/n)(H^2 + H\phi + \phi^2) > 0$, setting $\chi = H - \phi$:

$$L[\chi] \leq \psi \chi \quad \text{in } E;$$

$$\chi_n = \frac{\beta^2}{\beta_0} (\chi + \phi) h_{nn} \geq \frac{\beta^2}{\beta_0} \chi \quad \text{on } \partial_l E$$

(since $\phi < 0$ and $h_{nn} \leq 0$). Given that $\chi \leq 0$ at $t = 0$, it follows from the maximum principle that $\chi \leq 0$, or $H \leq \phi$ in $[0, \min\{T, t_*\})$. This shows $t_* < T$ is impossible, since $\phi \to -\infty$ as $t \to t_*$. 

**Remark 12.1.** It would be natural to try to show that a negative upper bound $H_0$ on the mean curvature (at $t = 0$) is preserved, at least under the assumption of concavity. Unfortunately, the evolution equation for $H$ (under graph m.c.m.) does not lend itself to a maximum principle argument. Letting $u := H - H_0$, we have

$$L[u] = |h|^2 u + uh^2(\omega, \omega) - u(H + H_0)h(\omega, \omega) + H_0 Q \quad \text{in } E,$$

$$Q := |h|^2 + h^2(\omega, \omega) - H_0 h(\omega, \omega).$$

At a point where $u = 0$, we would need to show $L[u] \leq 0$. But it is not true that $Q \geq 0$ at such a point, even when $n = 2$. ($u_n \geq 0$ does hold at boundary points.)

**13. A maximum principle for symmetric 2-tensors.**

In this section we prove a weak maximum principle for the parabolic evolution of symmetric two-tensors on bounded euclidean domains, with moving boundaries and Neumann-type boundary conditions. The hypotheses are as follows.

Let $E \subset \mathbb{R}^n \times [0, T]$ be connected, open and bounded (with $C^2$ boundary), with $D(t) = E \cap (\mathbb{R}^n \times \{t\})$ bounded, open, connected for each $t \in [0, T]$. Fix $R > 0$ so that $E \subset B_R(0) \times [0, T]$.

On the ‘lateral boundary’ of $E$:

$$\partial_l E := \{z = (x, t); t \in [0, T], x \in \partial D(t)\},$$
we define the inner unit normal \( n = n_t \in \mathbb{R}^n \). Extend \( n_t \) to a vector field in all of \( \bar{D}(t) \) (so that it is in \( C^{2,1}(\bar{E}, \mathbb{R}^n) \), arbitrarily except for the requirements that \( |n| \leq 1 \) pointwise and \( \nabla_n n = 0 \) in a tubular neighborhood of \( \partial D(t) \). (Here \( \nabla \) denotes the euclidean connection, so this requirement can be written \( n^i \partial_i n^j = 0 \) for each \( j \).)

The assumptions on the coefficients are given next.

\( g = g_t \) is a \( t \)-dependent Riemannian metric in \( \bar{D}(t) \), uniformly equivalent to the euclidean metric for \( t \in [0,T] \);

\( X = X_t \) is a bounded \( t \)-dependent vector field in \( \bar{D}(t) \), satisfying \( \nabla X \cdot n \geq 0 \) for \( z \in \partial \bar{E} \);

\( q = q(z,m) \) assigns to each \( z \in \bar{E} \) and each \( m \) in \( S \) (the space of quadratic forms in \( \mathbb{R}^n \)) a quadratic form \( q \in S \). \( q \) is assumed to be \( C^{2,1} \) in \( z \), locally Lipschitz in \( m \) (uniformly in \( z \in \bar{E} \));

\( b = b(z,m) \in S \) is defined for \( z \in \partial \bar{E} \), with the same regularity assumptions.

**Theorem 13.1.** Assume \( m \in C^{2,1}(\bar{E}; \mathbb{S}) \) satisfies in \( E \) the differential inequality:

\[
\partial_t m_{ij} - \text{tr}_g d^2 m_{ij} \leq X \cdot dm_{ij} + q_{ij}(\cdot, m(\cdot)),
\]

and on \( \partial E \) the boundary condition:

\[
n \cdot dm_{ij}(z) \geq b_{ij}(z, m(z)).
\]

Suppose the functions \( q \) and \( b \) satisfy the following ‘null eigenvector conditions’:

if, for some \( \hat{m} \in S \), \( V \in \mathbb{R}^n \) is a null eigenvector of \( \hat{m} \) (\( \hat{m}_{ij} V^j = 0 \forall i \)), then, for any \( z \in \bar{E} \) (resp. any \( z \in \partial E \)):

\[
q_{ij}(z, \hat{m}) V^i V^j \leq 0 \quad (\text{resp. } b_{ij}(z, \hat{m}) V^i V^j \geq 0).
\]

Then weak concavity of \( m \) at \( t = 0 \) is preserved:

\[
m \leq 0 \text{ in } D(0) \Rightarrow m \leq 0 \text{ in } \bar{E}.
\]

**Proof.** The assumptions imply there is \( K > 0 \) (depending only on \( E \) and on the functions \( X, g, n, q \) and \( b \)) satisfying:

\[
|n|_{C^{2,1}(\bar{E})} \leq K, \quad |X(z)|_{\text{eucl}} \leq K, \quad |g| + |g^{-1}(z)| \leq K, \quad z \in \bar{E};
\]
and if \( m, \hat{m} \in C^2(\bar{E}, \mathbb{S}) \) satisfy (for some \( \mu : \bar{E} \to \mathbb{R}_+ \)):

\[
-\mu(z) \mathbb{I} \leq m(z) - \hat{m}(z) \leq \mu(z) \mathbb{I}
\]

(where \( \mathbb{I} = (\delta_{ij}) \) and the inequality of quadratic forms has the usual meaning), then also:

\[
q(z, m(z)) \leq q(z, \hat{m}(z)) + K\mu(z) \mathbb{I}, \quad z \in \bar{E},
\]

\[
b(z, m(z)) \geq b(z, \hat{m}(z)) - K\mu(z) \mathbb{I}, \quad z \in \partial E.
\]

Now define, for \( z \in \bar{E} \):

\[
\varphi(z) := -2Kn(z) \cdot x := 2Ks(z),
\]

where we use the euclidean inner product and, on \( \partial E \), \( s \) is the ‘support function’ of \( \partial D(t) \) (positive if \( D(t) \) is convex and contains the origin). It is clear we may find \( M = M(R, K) > 0 \) depending only on \( K, R \) and \( |n|_{C^2} \) so that:

\[
|\varphi|_{C^2} \leq M, \quad |d\varphi|^2_g + |\text{tr}g d^2\varphi| \leq M, \quad |X \cdot d\varphi| \leq M.
\]

We assume also \( M \geq K \). Now, given \( m \) as in the statement of the theorem and given constants \( \epsilon > 0, \gamma > 0 \) and \( \delta > 0 \), define for \( z \in E^\delta := E \cap \{ t < \delta \} \):

\[
\hat{m}(z) := m(z) - (\epsilon t + \gamma e^{\varphi(z)}) \mathbb{I}, \quad z \in E^\delta.
\]

Clearly \( \hat{m} \in C^2(\bar{E}^\delta; \mathbb{S}) \). We now derive the constraints on \( \delta, \epsilon \) and \( \gamma \). It will turn out that \( \delta \) must be taken small enough (depending only on \( K, R \)), \( \epsilon > 0 \) is arbitrary and \( \gamma \) is \( \epsilon \) times a constant depending only on \( K, R \).

The following inequalities are easily derived:

\[
q(z, m(z)) \leq q(z, \hat{m}(z)) + K(\epsilon t + \gamma e^{\varphi(z)}) \mathbb{I};
\]

\[
X \cdot dm = X \cdot d\hat{m} + \gamma(e^\varphi X \cdot d\varphi) \mathbb{I} \leq X \cdot d\hat{m} + (\gamma e^\varphi M) \mathbb{I};
\]

\[
\partial_t \hat{m} = \partial_t m - \epsilon \mathbb{I} - (\gamma e^\varphi \partial_t \varphi) \mathbb{I} \leq \partial_t m + (\gamma e^\varphi M) \mathbb{I} - \epsilon \mathbb{I};
\]

\[
\text{tr}_g d^2 \hat{m} = \text{tr}_g d^2 \hat{m} - \gamma e^\varphi (|d\varphi|^2_g + \text{tr}_g d^2 \varphi) \mathbb{I} \geq \text{tr}_g d^2 m - (\gamma e^\varphi M) \mathbb{I}.
\]

We use this to compute:

\[
\partial_t \hat{m} - \text{tr}_g d^2 \hat{m} \leq \partial_t m - \text{tr}_g d^2 m + (2\gamma e^\varphi M) \mathbb{I} - \epsilon \mathbb{I}
\]
\[ \leq q(z, m(z)) + X \cdot dm + (2\gamma e^\varphi M)I - \epsilon I \]
\[ \leq q(z, \hat{m}(z)) + X \cdot d\hat{m} + K(\epsilon t + \gamma e^\varphi)I + (3M\gamma e^\varphi)I \]
\[ \leq q(z, \hat{m}(z)) + X \cdot d\hat{m} + M\epsilon tI + 4M\gamma e^\varphi I - \epsilon I. \]

We conclude the inequality:
\[ \partial_t \hat{m} - \text{tr}_g d^2 \hat{m} \leq q(z, \hat{m}(z)) + X \cdot d\hat{m} - (\epsilon/2)I \]
will hold in \( E^\delta \), provided the constants are selected so that, for \( z \in E^\delta \):
\[ 4M\gamma e^\varphi(z) + M\epsilon t \leq \epsilon/2. \quad (A) \]

Turning to boundary points \( z = (x, t) \in \partial_t E \), note that \( d_n \varphi = -2K \), so that:
\[ d_n \hat{m}(z) = d_n m(z) - (\gamma e^\varphi(z) d_n \varphi(z))I \geq b(z, m(z)) - (\gamma e^\varphi(z) d_n \varphi(z))I \]
\[ \geq b(z, \hat{m}(z)) - K(\epsilon t + \gamma e^\varphi(z))I - (\gamma e^\varphi(z) d_n \varphi(z))I \]
\[ \geq b(z, \hat{m}(z)) + K(\gamma e^\varphi(z) - \epsilon t)I, \]
so that the inequality:
\[ d_n \hat{m}(z) \geq b(z, \hat{m}), \quad z \in \partial_t E^\delta, \]
will hold provided the constants are chosen so that, on \( \partial_t E^\delta \):
\[ \epsilon t \leq \gamma e^\varphi(z). \quad (B) \]

Bearing in mind that, on \( E \): \( e^{-2KR} \leq e^\varphi(z) \leq e^{2KR} \), it is not hard to arrange for (A) and (B) to hold, or equivalently, for:
\[ \epsilon t \leq \gamma e^\varphi(z), \quad 10M\gamma e^\varphi(z) \leq \epsilon. \]

Given \( \epsilon > 0 \), define \( \gamma \) so that \( 10M\gamma e^{2KR} = \epsilon \). Then the second inequality holds, and so will the first, provided:
\[ \epsilon t \leq \gamma e^{-2KR} = (\epsilon/10M)e^{-4KR}, \]
which is true for any \( \epsilon > 0 \), if \( \delta \) is defined via \( \delta := e^{-4KR}/10M \) (recall \( t \in [0, \delta] \)).

Note that, since \( m \geq 0 \) at \( t = 0 \), it follows that \( \hat{m} \) is negative-definite at \( t = 0 \), and hence also for small time, and we claim that this persists throughout \( \bar{E}^\delta \), so that (letting \( \epsilon \to 0 \)) \( m \leq 0 \) in \( \bar{E}^\delta \). Restarting the argument at \( t = \delta \), we see this is enough to prove the theorem.
To prove this claim, suppose (by contradiction) \( \hat{m} \) acquires a null eigenvector \( 0 \neq V \in \mathbb{R}^n \) at a point \( z_1 = (x_1, t_1) \in \tilde{E}^\delta \), with \( t_1 \in (0, \delta] \) the first time this happens.

Let \( \hat{f}(z) := \hat{m}_{ij}V^iV^j \), \( z \in E^\delta \) (that is, we ‘extend’ \( V \) to \( E^\delta \) as a constant vector.) It follows from the preceding that \( \hat{f} \) satisfies in \( E^\delta \):

\[
\partial_t \hat{f} \leq tr_g d^2 \hat{m}_{ij}V^iV^j + X \cdot d\hat{m}_{ij}V^iV^j + q_{ij}(\cdot, \hat{m})V^iV^j - \frac{\epsilon}{2} |V|_{eul}^2.
\]

Noting that \( tr_g d^2 \hat{m}_{ij}V^iV^j = tr_g d^2 \hat{f} \) and \( X \cdot d\hat{m}_{ij}V^iV^j = d\hat{f} \cdot X \), and using the null eigenvector condition for \( q \), we find that \( \hat{f} \) satisfies in \( E^\delta \) the strict inequality:

\[
\partial_t \hat{f} < tr_g d^2 \hat{f} + d\hat{f} \cdot X.
\]

This shows \( x_1 \) cannot be an interior point of \( D(t_1) \), for then (as a first-time interior maximum point for \( \hat{f} \) we would have \( tr_g d^2 \hat{f}(z_1) \leq 0 \) and \( d\hat{f}(z_1) = 0 \), contradicting \( \partial_t \hat{f}(z_1) \geq 0 \). Thus \( x_1 \in \partial D(t_1) \). Since \( \hat{f} \) satisfies the differential inequality just stated and \( z_1 = (x_1, t_1) \) is a first-time boundary maximum in \( \tilde{E}^\delta \), the parabolic Hopf lemma implies \( d_n \hat{f}(z_1) < 0 \). On the other hand, as seen above:

\[
d_n \hat{f} = d_n \hat{m}_{ij}V^iV^j \geq b_{ij}(z_1, \hat{m}(z_1))V^iV^j \geq 0,
\]

from the boundary null-eigenvector condition. This contradiction concludes the proof.

**Corollary 13.2.** Suppose \( m \in C^{2,1}(\tilde{E}, \mathbb{S}) \) satisfies the same differential inequality, with the same hypotheses on the coefficients as in the theorem (including the null eigenvector condition for \( q \)), and the boundary conditions:

\[
m(z)(n, \tau) = 0, \quad \forall z = (x, t) \in \partial E, \tau \in T_x \partial D(t);
\]

\[
n^i n^j d_n m_{ij}(z) = (\nabla_n m)(n, n) \geq b_{nn}(z, m(z));
\]

\[
\tau^i \tau^j d_n m_{ij} = (\nabla_n m)(\tau, \tau) \geq b^{tan}(z, m(z))(\tau, \tau), \quad \tau \in T_x \partial D(t),
\]

for functions \( b_{nn}(z, \hat{m}) \) from \( E \times \mathbb{S} \) to \( \mathbb{R} \) and \( b^{tan} \) assigning to \( (z, \hat{m}), z = (x, t) \), a quadratic form in \( T_x \partial D(t) \). Suppose \( b_{nn} \geq 0 \) in \( E \times \mathbb{S} \) and \( b^{tan} \) satisfies:

\[
\hat{m}(\tau, \tau) = 0 \text{ for some } \tau \in T_x \partial D(t) \Rightarrow b^{tan}(z, \hat{m})(\tau, \tau) \geq 0.
\]

Then, as in the theorem, concavity is preserved:

\[
m \leq 0 \text{ at } t = 0 \Rightarrow m \leq 0 \text{ in } \tilde{E}.
\]
Proof. This is proved as the theorem, with the following change in the last part of the proof: if $0 \neq V \in \mathbb{R}^n$ is a null eigenvector of $\hat{m}$ (defined as in the proof of the theorem) at a boundary point $z_1 = (a_1, t_1) \in \partial E$, write:

$$V = V^* n + V^T, \quad V^T \in T_{x_1} \partial D(t_1).$$

Assume first $V^* \neq 0$. Then (noting that $\hat{m}$ splits at the boundary if $m$ does), we see that $n$ is a null eigenvector of $\hat{m}$ at $z_1$, so we define $\hat{f}(z) = m_{ij}(\hat{z}) n^i(z_1)n^j(z_1)$ and repeat the argument. At $z_1$, $(\nabla_n \hat{m})(n, n) = b_{nn}(z_1, \hat{m}(z_1)) \geq 0$ leads to a contradiction with the parabolic Hopf lemma, as before.

If $V^* = 0$, then $V^T \in T_{x_1} \partial D(t_1)$ must be a null eigenvector of $\hat{m}$ at the boundary point $z_1$, and then we run the argument with $\hat{f}(z) = \hat{m}(z)(V^T, V^T)$, leading to a contradiction, as before.

**Corollary 13.3.** For MCM of graphs with constant-angle boundary conditions, weak concavity is preserved:

$$h \leq 0 \text{ at } t = 0 \Rightarrow h \leq 0 \text{ in } \bar{E}.$$

**Proof.** The conditions of the theorem hold, and the expressions obtained for $\nabla_n h$ in the preceding section easily imply that the boundary conditions in Corollary 13.2 are satisfied; hence the claim follows from Corollary 13.2.

**Remark 13.1.** It seems plausible that a slightly different version of the result in this section could be used to strengthen the conclusions of [9]. This is currently being considered.

14. An improved continuation criterion.

In this section we improve the continuation criterion: if $\sup_E |h|_g = a_0$ is finite, the solution can be continued past $T$.

The argument given below works in all dimensions, but for simplicity of notation we deal here only with the two-dimensional case: $\Sigma_t$ is a surface, the moving boundary $\Gamma_t$ is a curve in $\mathbb{R}^2$. Assuming such a bound on $|h|_g$, given the results in section 12 all we have to do is bound $(\nabla_\tau h)(\tau, \tau)$ and $(\nabla_\tau h)(n, n)$, where $\tau = \tau_t$ is a unit vector field tangent to $\Gamma_t$. At the moving boundary: $H = \beta^2 h(n, n) + h(\tau, \tau)$, and we already showed $\tau(H)$ is bounded, so it suffices to bound one of these quantities.

We adopt the notation: $f \sim g$ if $f - g$ is bounded in $E$ by constants depending only on the initial data and $a_0$. 

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Consider the vector fields in $D(t) \subset \mathbb{R}^2$:

$$\omega = \frac{1}{v}[w_1, w_2], \quad \tilde{\omega} = v\omega^\perp = [-w_2, w_1].$$

It is easy to verify the following:

$$\langle \omega, \tilde{\omega} \rangle_g = 0, \quad |\omega|_g^2 = |\tilde{\omega}|_g^2 = |D\omega|_e^2 := w_1^2 + w_2^2.$$

Thus we may think of $\{\omega, \tilde{\omega}\}$ as a 'conformal pseudo-frame' ($\omega$ and $\tilde{\omega}$ vanish when $D\omega = 0$), defined on all of $D(t)$. Moreover, at the boundary $\partial D(t)$:

$$\omega = \beta_0 n, \quad \tilde{\omega} = \frac{\beta_0}{\beta} n^\perp := \frac{\beta_0}{\beta} \tau,$$

where $\{\tau, n\}$ is an euclidean-orthonormal frame along $\Gamma_t$. Thus $\omega$ and $\tilde{\omega}$ supply 'canonical' extensions of $n, \tau$ to the interior of $D(t)$, as uniformly bounded vector fields.

Recall the boundary conditions for $h$:

$$h(\omega, \tilde{\omega}) = 0, \quad (\nabla_n h)(\omega, \omega) \sim 0, \quad (\nabla_n h)(\tilde{\omega}, \tilde{\omega}) \sim 0 \quad \text{on} \ \partial D(t).$$

These give the boundary conditions for the components $h_{11}, h_{12}, h_{22}$ of $h$ in the standard basis of $\mathbb{R}^2$. As shown in appendix 2, these three functions are solutions of a linear parabolic system in $E \subset \mathbb{R}^2 \times [0, T]$ (a non-cylindrical domain), with bounded coefficients:

$$\partial_t (h_{ij}) - g^{kl}(h_{ij})_{kl} + 2h^k_i d_\omega(h_{kj}) + 2h^k_j d_\omega(h_{ik}) = C_{ij}.$$

The boundary conditions also have bounded coefficients:

$$n^1 n^2 (h_{22} - h_{11}) + [(n^1)^2 - (n^2)^2] h_{12} = 0,$$

$$\begin{align*}
(n^1)^2 d_n(h_{11}) + 2n^1 n^2 d_n(h_{12}) + (n^2)^2 d_n(h_{22}) &= b_1, \\
(n^2)^2 d_n(h_{11}) - 2n^1 n^2 d_n(h_{12}) + (n^1)^2 d_n(h_{22}) &= b_2.
\end{align*}$$

The only thing left to do is to argue that this set of linear equations and boundary conditions define a parabolic system. Then it follows from the 'global gradient bounds' of linear theory that also the tangential derivatives $d_\tau h_{ij}$ are bounded on $\partial_t E$, which leads quickly to the desired conclusion.

We need to verify the 'complementarity conditions' hold for this system, so we proceed as in Section 6 (up to a point.) Fix a point $z_0 = (y_0, t_0) \in \partial E$ and 'freeze coefficients' there. Consider a manifold-with-boundary chart
\((y, t) \mapsto (\rho, \sigma, s)\) mapping a neighborhood of \(z_0\) in \(E\) to \(\{\rho > 0\} \times \mathbb{R} \times \mathbb{R}_+\). Here \(\rho\) is the coordinate normal to \(\partial D(t)\), \(\sigma\) parametrizes \(\partial D(t)\) and slices \(\{s = \text{const.}\}\) correspond to \(\{t = \text{const.}\}\).

Let \(\hat{h}_{ij}(\rho, \sigma, s) = h_{ij}(y, t)\) be the unknown functions in the new coordinates. The corresponding system is:

\[
\partial_s \hat{h}_{ij} - (\beta^2(\hat{h}_{ij})_{\rho\rho} + (\hat{h}_{ij})_{\sigma\sigma}) + c_i^k(\hat{h}_{jk})_\rho + c_j^k(\hat{h}_{ik})_\rho = \varphi_{ij}.
\]

Here the \(c_i^k\) are constants. The boundary conditions can also be easily written down (freeze the \(n^i\) to their value \(n_0^i\) at \(z_0\) and replace \(d_n(h_{ij})\) by \(\hat{h}_\rho\).) It is natural to consider the linear transformation of the unknown functions:

\[
(\hat{h}_{11}, \hat{h}_{12}, \hat{h}_{22}) \mapsto (f_{11}, f_{12}, f_{22}), \quad f_{ij} = f_{ij}(\rho, \sigma, s)
\]

defined by:

\[
f_{11} = (n_0^1)^2 \hat{h}_{11} + 2n_0^1n_0^2 \hat{h}_{12} + (n_0^2)^2 \hat{h}_{22}
\]

\[
f_{22} = (n_0^2)^2 \hat{h}_{11} - 2n_0^1n_0^2 \hat{h}_{12} + (n_0^1)^2 \hat{h}_{22}
\]

\[
f_{12} = (n_0^1n_0^2)(\hat{h}_{22} - \hat{h}_{11}) + [(n_0^1)^2 - (n_0^2)^2] \hat{h}_{22}.
\]

Since the principal part of the linear system for the \(\hat{h}_{ij}\) is diagonal, the principal part of the system for \(f_{ij}\) is exactly the same (while the lower-order terms have different values):

\[
\partial_s f_{ij} - (\beta^2(f_{ij})_{\rho\rho} + (f_{ij})_{\sigma\sigma}) + c_i^k(f_{jk})_{\rho} + c_j^k(f_{ik})_{\rho} = \varphi_{ij}.
\]

This linear transformation is invertible, with inverse given by:

\[
\hat{h}_{11} = (n_0^1)^2 f_{22} + 2n_0^1n_0^2 f_{12} + (n_0^2)^2 f_{11}
\]

\[
\hat{h}_{22} = (n_0^2)^2 f_{22} - 2n_0^1n_0^2 f_{12} + (n_0^1)^2 f_{11}
\]

\[
\hat{h}_{12} = n_0^1n_0^2(f_{11} - f_{22}) + [(n_0^1)^2 - (n_0^2)^2] f_{12}
\]

Thus the original system frozen at \(z_0\) (for the \(\hat{h}_{ij}\)), with its boundary conditions, satisfies the complementarity condition if an only if the same holds for the \(f_{ij}\) system, with the transformed boundary conditions. But these take a very simple form:

\[
f_{12}(\rho = 0) = 0, \quad d_{\rho}(f_{11})(\rho = 0) = b_1, \quad d_{\rho}(f_{22})(\rho = 0) = b_2.
\]

These are standard Dirichlet (resp. Neumann) boundary conditions for a standard \(3 \times 3\) parabolic system (decoupled to highest order). Hence the
original system (for the $h_{ij}$) with boundary conditions satisfies ‘complemen-
tarity’ at each point of $\partial_l E$.

In particular, the global gradient estimates hold for the linear system (with uniformly bounded coefficients and boundary conditions) in the unknowns $h_{ij}$, and we have:

$$|d_\tau h_{ij}| \leq M \quad \text{in } \partial_l E,$$

for any tangential unit vector field $\tau$, for some $M$ depending only on $a_0$ and the initial data. This clearly implies bounds on $(\nabla_\tau h)(\tau, \tau)$ and $(\nabla_\tau h)(n,n)$. Combining with the results in section 12, we have the following conclusion:

**Proposition 14.1.** Assume the maximal existence time $T_{\text{max}}$ is finite. Then:

$$\limsup_{t \to T_{\text{max}}} \sup_{\partial D(t)} |h|_g = \infty.$$

**Remark.** It should be clear that the argument works in all dimensions; this will be included in the final version of the paper.

15. Final comments.

1. The main step missing for the global existence result

$$\lim_{t \to T_{\text{max}}} \text{diam}(\Sigma_t) = 0$$

(in the concave case) is showing that a lower bound on diameter gives an upper bound for $|h|_g$. This may follow from properties of the support function (based on a point in $\mathbb{R}^n$ common to all $\Sigma_t$), but remains to be addressed (work in progress). If confirmed, this would correspond to Theorem 1 in [5] for lens-type curve networks. An issue apparently completely unexplored in dimensions above 1 is existence-uniqueness of ‘homothetic solutions’ for this problem.

2. We state here the local existence theorem for configurations of graphs over domains with moving boundaries. In this setting, a **triple junction configuration** consists of three embedded hypersurfaces $\Sigma^1, \Sigma^2, \Sigma^3$ in $\mathbb{R}^{n+1}$, graphs of functions $w^i$ defined over time-dependent domains $D^1(t), D^2(t) \subset \mathbb{R}^n$ ($D^1$ covered by one graph, $D^2$ by two graphs), satisfying the following conditions: (1) The $\Sigma^I$ intersect along an $(n-1)$-dimensional graph $\Lambda(t)$ (the ‘junction’), along which the upward unit normals satisfy the relation:
\[ N_1 + N_2 = N_3. \]

(2) If a fixed support hypersurface \( S \subset \mathbb{R}^{n+1} \) is given (also a graph, not necessarily connected), the \( \Sigma^I \) intersect \( S \) orthogonally.

Topologically, in the case of bounded domains one has the following examples: (i) (`lens’ type) 2 disks (or two annuli) covering \( D^2(t) \) and one annulus covering \( D^1(t) \); (ii) (`exterior’ type) two annuli covering \( D^2(t) \) and one disk covering \( D^1(t) \). The boundary component of the annuli disjoint from the junction intersects the support hypersurface \( S \) orthogonally for each \( t \).

Let \( \Sigma^I_0 \) \((I = 1, 2, 3)\) be graphs of \( C^{3+\alpha} \) functions over \( C^{3+\alpha} \) domains \( D^1_0, D^2_0 \subset \mathbb{R}^n \), defining a triple junction configuration and satisfying the compatibility condition for the mean curvatures on the common boundary \( \Gamma_0 \) of \( D^1_0 \) and \( D^2_0 \):

\[ H^1 + H^2 = H^3. \]

Then there exists \( T > 0 \) depending only on the initial data, and functions \( w^I \in C^{2+\alpha, 1+\alpha/2}(Q^I), Q^I \subset \mathbb{R}^n \times [0, T) \), so that the graphs of \( w^I(., t) : D^I(t) \to \mathbb{R} \) define a triple junction configuration for each \( t \in [0, T) \), moving by mean curvature.

The proof will be given elsewhere.

3. An interesting issue we have not addressed here is whether one has breakdown of uniqueness for initial data of lower regularity, or if the ‘orthogonality condition’ at the junction is removed. For curve networks, non-uniqueness has been considered in [S]; but neither a drop in regularity (from initial data to solution, in Hölder spaces) nor the orthogonality condition play a role in the case of curves.

**Appendix 1: Proof of lemma 4.1.**

Throughout the proof, \( n \) denotes the inner unit normal at \( \partial D \), extended to a tubular neighborhood \( \mathcal{N} \) so that \( D_n n = 0 \). Since \( D \) is uniformly \( C^{3+\alpha} \), it follows that \( n \in C^{2+\alpha}(\partial D) \), with uniform bounds. Denote by \( \rho \) the distance to the boundary (so \( D\rho = n \) in \( \mathcal{N} \)). Let \( \zeta \in C^3(\bar{D}) \) be a cutoff function, with \( \zeta \equiv 1 \) in \( \mathcal{N}_1 \subset \mathcal{N} \), \( \zeta \equiv 0 \) in \( D \setminus \mathcal{N} \).

We find \( \varphi \) of the form:

\[ \varphi(x) = x + \zeta(x)f(x)n(x) \]

with \( f \in C^{2+\alpha}(\mathcal{N}) \). The 1-jet conditions on \( \varphi \) at \( \partial D \) translate to the conditions on \( f \):

\[ f|_{\partial D} = 0, \quad Df|_{\partial D} = 0, \quad D^2f(n, n)|_{\partial D} = \Delta f|_{\partial D} = h. \]
Now use:

**Lemma A.1.** Let $D$ be a uniformly $C^{3+\alpha}$ domain with boundary distance function $\rho > 0$. Let $h \in C^\alpha(\partial D)$ be a bounded function. Then there exists an extension $g \in C^\infty(D) \cap C(\bar{D})$ so that $g|_{\partial D} = h$, $\sup_D |g| \leq \sup_{\partial D} |h|$ and $\rho^2 g \in C^{2+\alpha}(\bar{D})$.

Given this lemma, all we have to do is set $f = (1/2)\rho^2 g$, which clearly satisfies all the requirements (in particular, $\Delta f = h$ at $\partial D$.)

To verify that $\varphi$ is a diffeomorphism, it suffices to check that $|\zeta_{fn}|_{C^1}$ (in $\mathcal{N} \subset \{\rho < \rho_0]\}$) is small if $\rho_0$ is small. This is easily seen:

\[
|\zeta_{fn}|_{C^0} \leq (1/2)\rho_0^2 |g|_{C^0}; \\
|D\zeta| \leq c\rho_0^{-1} \Rightarrow |fD\zeta| \leq c\rho_0 |g|_{C^0}. \\
|Df| \leq (1/2)\rho_0^\alpha ||g||_{C^{2+\alpha}(\bar{D})}
\]
on $\mathcal{N}$, since $Df \in C^{1+\alpha}(\bar{D})$ and $Df|_{\partial D} = 0$. And finally, with $\mathcal{A}$ the second fundamental form of $\partial D$:

\[
|Dn| \leq |\mathcal{A}|_{C^0} \Rightarrow |fDn| \leq (1/2)\rho_0^2 |g|_{C^0} |\mathcal{A}|_{C^0}.
\]

A word about Lemma A.1. (This is probably in the literature, but I don’t know a reference.) If $D$ is the upper half-space, we solve $\Delta g = 0$ in $D$ with boundary values $h$. Then the estimate

\[
[D^2(\rho^2 P \ast h)]^{(\alpha)}(\bar{D}) \leq c|h|_{C^\alpha(\partial D)}
\]
follows by direct computation with the Poisson kernel $P$; for the rest of the norm, use interpolation. Then transfer the estimate to a general domain using ‘adapted local charts’, in which $\rho$ in $D$ corresponds to the vertical coordinate in the upper half-space. (It is easy to see that at each boundary point there is a $C^{2+\alpha}$ adapted chart, with uniform bounds.)

**Appendix 2: Evolution equations for the second fundamental form.**

We consider mean curvature motion of graphs:

\[
G(y,t) = [y, w(y,t)], \quad y \in D(t) \subset \mathbb{R}^n, \\
w_t = g^{ij} w_{ij} = vH, \quad v = \sqrt{1 + |Dw|^2}.
\]
In this appendix we include evolution equations for geometric quantities, in terms of the operators:

\[ \partial_t - \Delta_g, \quad L = \partial_t - tr_g d^2. \]

It is often convenient to use the vector field in \( D(t) \):

\[ \omega := \frac{1}{v} Dw. \]

Since \(-\omega\) is the \( \mathbb{R}^n \) component of the unit normal \( N \) and \( L[N] = |h|^2_N N \), we have:

\[ L[\omega^i] = |h|^2 g^i_j \omega^j, \quad |h|^2 := g^{ik} g^{jl} h_{ij} h_{kl}. \]

Here \( h = (h_{ij}) \) is the pullback to \( D(t) \) of the second fundamental form \( A \):

\[ h(\partial_i, \partial_j) = h_{ij} = A(G_i, G_j) = \frac{1}{v} w_{ij}. \]

First, denoting by \( \nabla \) the pullback to \( D(t) \) of the induced connection \( \nabla^\Sigma \) (that is, \( G^*_s(\nabla X Y) = \nabla^\Sigma_{G^*_s X} G^*_s Y \) for any vector fields \( X, Y \) in \( D(t) \)), and using the definition:

\[ \nabla^\Sigma G_i G_j = G_{ij} - (G_{ij}, N) N = [0, w_{ij}] - \frac{1}{v^2} w_{ij} [-Dw, 1] = \frac{w_{ij}}{v^2} [Dw, |Dw|^2] = \frac{w_{ij}}{v^2} G_s Dw, \]

we conclude:

\[ \nabla_{\partial_i} \partial_j = \frac{1}{v} h_{ij} Dw = h_{ij} \omega. \]

From this one derives easily a useful expression relating the Laplace-Beltrami operator and the operator \( tr_g d^2 \) acting on functions:

\[ \Delta_g f = tr_g d^2 f - \frac{H}{v} w_{ij} f = tr_g d^2 f - H d_\omega f. \]

We also have, for the covariant derivatives of \( h \) with respect to the euclidean connection and to \( \nabla = \nabla^g \):

\[ \partial_m (h_{ij}) = \nabla_m h_{ij} + [h_{jm} h_{ik} + h_{im} h_{jk}] \omega^k. \]

(Here \( \nabla h \) is the symmetric \((3,0)\)-tensor with components: \( \nabla_m h_{ij} = (\nabla \partial_m h)(\partial_i, \partial_j) \).)

Iterating this and taking \( g \)-traces yields (using the Codazzi identity and the easily verified relation \( \partial_i \omega^k = h_i^k := g^{jk} h_{ij} \)):

\[ tr_g d^2 (h_{ij}) = g^{mk} \partial_m (\partial_k (h_{ij})) = g^{mk} (\nabla^2_{\partial_m, \partial_k} h)(\partial_i, \partial_j) \]

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\[ + H \nabla_\omega h_{ij} + 2[h^k_{i} \nabla_k h_{jp} + h^k_{j} \nabla_k h_{ip}] \omega^p + [H_i h_{jp} + H_j h_{ip}] \omega^p \]
\[ + 2[h_{ip}(h^2)_{jq} + (h^2)_{ip} h_{jq} + H h_{ip} h_{jq}] \omega^p \omega^q + 2(h^3)_{ij} + 2(h^2)_{ij} h(\omega, \omega). \]

Here the powers \( h^2 \) and \( h^3 \) of \( h \) are the symmetric 2-tensors defined used the metric:

\[
(h^2)_{ij} := g^{kp} h_{ik} h_{jp}, \quad (h^3)_{ij} := g^{kp} g^{jq} h_{ik} h_{jpl} h_{aq}.
\]

Note also that:

\[
[h^k_{i} \nabla_k h_{jp} + h^k_{j} \nabla_k h_{ip}] \omega^p = \nabla_\omega (h^2)_{ij},
\]

using the Codazzi identity.

\textbf{Evolution equations for } \( h \).

Starting from \( G_t = vHc_{n+1} = H(N + \frac{1}{v}[Dw, |Dw|^2]) = HN + HG_\omega \)
and \( N_t = -\nabla^\Sigma H - Hv^{-1} \nabla^\Sigma v \) (where \( \nabla^\Sigma f = g^{ij} f_j G_i \) and \( \nabla f = g^{ij} f_j \partial_i \)) we have:

\[
\partial_t (h_{ij}) = \langle (HN)_{ij}, N \rangle - \langle G_{ij}, \nabla^\Sigma H \rangle - \frac{H}{v} \langle G_{ij}, \nabla^\Sigma v \rangle + \langle (HG_\omega)_{ij}, N \rangle.
\]

Using the easily derived facts:

\[
\langle N_{ij}, N \rangle = -h^2(\partial_i, \partial_j),
\]

\[
H_{ij} - \langle G_{ij}, \nabla^\Sigma H \rangle = (\nabla dH)(\partial_i, \partial_j),
\]

\[
\frac{1}{v} \langle G_{ij}, \nabla^\Sigma v \rangle = h(\omega, \omega) h_{ij},
\]

we obtain:

\[
\partial_t (h_{ij}) = (\nabla dH)(\partial_i, \partial_j) - H h^2(\partial_i, \partial_j) - H h(\omega, \omega) h_{ij} + \langle (HG_\omega)_{ij}, N \rangle,
\]

where:

\[
\langle (HG_\omega)_{ij}, N \rangle = H_i \langle (G_\omega)_{j}, N \rangle + H_j \langle (G_\omega)_{i}, N \rangle + H \langle (G_\omega)_{ij}, N \rangle.
\]

To identify the terms, computation shows that:

\[
\langle (G_\omega)_{i}, N \rangle = h(\omega, \partial_i),
\]

and hence, using also:

\[
\nabla^\Sigma_{G_i}(G_\omega) = G_\omega(\nabla \partial_i \omega), \quad \nabla \partial_i \omega = (h^p_i + \omega^q h_{iq} \omega^p) \partial_p = \sum_p h_{ip} \partial_p,
\]

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we obtain (using $\omega^k \partial_j (h_{ik}) = \nabla_\omega h_{ij} + 2h(\partial_i, \omega)h(\partial_j, \omega)$):

$$
\langle (G* \omega)_{ij}, N \rangle = \partial_j (\omega^k h_{ik}) - \langle \nabla_{Gi}(G* \omega), \partial_j N \rangle = h_j^k h_{ik} + \omega^k \partial_j (h_{ik}) + h(\partial_j, \nabla \partial_i \omega)
$$

$$
= (\nabla_\omega h)_{ij} + (h^2)_{ij} + 2h(\omega, \partial_i)h(\omega, \partial_j) + \sum_p h_{ip} h_{jp}
$$

$$
= (\nabla_\omega h)_{ij} + 2(h^2)_{ij} + 3h(\omega, \partial_i)h(\omega, \partial_j),
$$

since $\sum_p h_{ip} h_{jp} = (h^2)_{ij} + h(\omega, \partial_i)h(\omega, \partial_j)$. Combining all the terms yields the result:

$$
\partial_t (h_{ij}) = (\nabla dH)(\partial_i, \partial_j) + H \nabla_\omega h_{ij} + H_i h(\omega, \partial_j) + H_j h(\omega, \partial_i)
$$

$$
+ H(h^2)_{ij} + 3Hh(\omega, \partial_i)h(\omega, \partial_j) - Hh(\omega, \omega)h_{ij}.
$$

From this expression and Simons’ identity (in tensorial form):

$$
\nabla dH = \Delta_g h + |h|^2 g - H h^2,
$$

we obtain easily a tensorial ‘heat equation’ for $h$:

$$
[(\partial_t - \Delta_g) h]_{ij} = H \nabla_\omega h_{ij} + H_i h(\omega, \partial_j) + H_j h(\omega, \partial_i)
$$

$$
+ |h|^2 h_{ij} + 3Hh(\partial_i, \omega)h(\partial_j, \omega) - Hh(\omega, \omega)h_{ij}.
$$

Using the earlier computation relating $\Delta_g h$ (the tensorial Laplacian of $h$) and $tr_g d^2 h$, we obtain from this the evolution equation in terms of $L$:

$$
L[h_{ij}] = -2\nabla_\omega (h^2)_{ij} + C_{ij},
$$

$$
C_{ij} := -2[h(\partial_i, \omega)h^2(\partial_j, \omega) + h^2(\partial_i, \omega)h(\partial_j, \omega)] - 2(h^3)_{ij} - 2(h^2)_{ij} h(\omega, \omega)
$$

$$
+ |h|^2 h_{ij} + Hh(\partial_i, \omega)h(\partial_j, \omega) - Hh(\omega, \omega)h_{ij}.
$$

Time derivatives and evolution equations for $\omega$ and $g$.
It is sometimes convenient to use the ‘Weingarten operator’:

$$
S(X) := S(X^i \partial_i) = h_j^i X^j \partial_i.
$$
The time derivative of $\omega$ is simply minus the time derivative of the $R^n$ component of $N$. In addition, one computes easily that $\frac{\nabla v}{v} = S(\omega)$, so we have:

$$\partial_t \omega = \nabla H + \frac{H}{v} \nabla v = \nabla H + HS(\omega).$$

For the metric and ‘inverse metric’ tensors we have: from $\partial_t g_{ij} = (w_i w_j)_t$ and $w_{it} = (v H)_i$:

$$\partial_t g_{ij} = v^2 (H_i \omega^j + H_j \omega^i) + v^2 H (h(\omega, \partial_i) \omega^j + h(\omega, \partial_j) \omega^i),$$

and then, using $\partial_t g^{ij} = -g^{ik} \partial_t g_{kl} g^{lj}$:

$$\partial_t g^{ij} = -[(\nabla H)^i \omega^j + (\nabla H)^j \omega^i] - H [S(\omega)^i \omega^j + S(\omega)^j \omega^i].$$

Since we know the evolution equation of $\omega$, it is easy to obtain that of $g^{ij}$:

$$L[g^{ij}] = -L[\omega^i \omega^j] = -L[\omega^j] \omega^i + 2g^{kl} (\partial_k \omega^i)(\partial_l \omega^j) - \omega^i L[\omega^j].$$

Using $\partial_k \omega^i = h_k^i$, we find:

$$L[g^{ij}] = -2|h^2 g^{ij} \omega^j + 2(h^2)^{ij}. $$

It is also easy to see that $\partial_k g^{ij} = -(h_k^i \omega^j + h_k^j \omega^i)$.

**Evolution of mean curvature.**

To compute the evolution equation for $H = g^{ij} h_{ij}$, we just need to remember $g^{ij}$ is time-dependent:

$$(\partial_t - \Delta_g) H = (\partial_t g^{ij})(h_{ij}) + tr_g[(\partial_t - \Delta_g) h] = -2h(\nabla H, \omega) + 2H h^2(\omega, \omega) + tr_g[(\partial_t - \Delta_g) h].$$

The result is:

$$(\partial_t - \Delta_g) H = H d\omega H + |h|^2 H + H h^2(\omega, \omega) - H^2 h(\omega, \omega).$$

Since $L[f] = (\partial_t - \Delta_g)f - H d\omega f$ (for any $f$), we see that the equation in terms of $L$ has no first-order terms:

$$L[H] = |h|^2 H + H h^2(\omega, \omega) - H^2 h(\omega, \omega)$$

**Remark.** One can also find $L[H]$ starting from the expression:

$$L[g^{ij} h_{ij}] = L[g^{ij} h_{ij}] + g^{ij} L[h_{ij}] - 2g^{kl}(\partial_k g^{ij})(\partial_l h_{ij}).$$

This may be used to check the calculation.
Using the expressions given earlier, one easily finds:

\[
\Delta g h_j^k = g^{ik} \Delta g h_{ij}, \quad \text{or } \langle (\Delta g S)X,Y \rangle_g = (\Delta g h)(X,Y).
\]

The evolution equation is easily obtained:

\[
(\partial_t - \Delta g) h_j^k = \partial_t g^{ik} h_{ij} + g^{ik} (\partial_t - \Delta g) h_{ij}
\]

\[
= H \nabla_{\omega} h_j^k + h_j^k \omega^j - H h_j^l \omega^k + |h|_g^2 h_j^k + 2HS(\omega)^k h(\omega, \partial_j) - Hh(\omega,\partial_j)h_j^k - H h(S(\omega), \partial_j)\omega^k.
\]

**Remark:** Since the components of \( \nabla S \) are given by:

\[
(\nabla_{\omega} S)(\partial_j) = (\nabla_{\omega} h_j^k) \partial_k, \quad \nabla_{\omega} h_j^k = d_{\omega}(h_j^k) + h^2(\omega, \partial_j)\omega^k - h(\omega, \partial_j) S(\omega), \omega^k,
\]

we see that upon setting \( j = k \) and adding over \( k \) we recover the evolution equation for \( H \).

The evolution equation for \( h_j^k \) in terms of \( L \) follows from the calculation:

\[
L[h_j^k] = L[g^{ik}] h_{ij} + g^{ik} L[h_{ij}] - 2g^{mn} (\partial_m g^{ik})(\partial_n h_{ij})
\]

\[
= -2(\nabla_{\omega} h_j^k) h_{m}^j + (\partial_j |h|^2)^k + |h|_g^2 h_j^k - H h(\omega,\omega) h_j^k + HS(\omega)^k h(\partial_j,\omega) + 2h^3(\partial_j,\omega)\omega^k - 2(h^2)^k \omega^p h(\partial_j,\omega).
\]

Setting \( j = k \) and adding over \( k \), we recover the earlier expression for \( L[H] \).

**Evolution of \( |h|^2 \).**

The fact that \( g^{ij} \) is time-dependent introduces an additional term in the usual expression:

\[
(\partial_t - \Delta g)|h|^2_g = -2|\nabla h|^2_g + 2\langle h, (\partial_t - \Delta_g)h \rangle_g + 2(\partial_t g^{ij})(h^2)_{ij}.
\]

Using the expressions given earlier, one easily finds:

\[
(\partial_t - \Delta g)|h|^2_g = -2|\nabla h|^2_g + H d_{\omega}|h|^2_g + 2|h|^4_g - 4H h^3(\omega,\omega) - 2H|h|^2_g h(\omega,\omega),
\]

\[
L[|h|^2_g] = -2|\nabla h|^2_g + 2|h|^4_g - 4H h^3(\omega,\omega) - 2H|h|^2_g h(\omega,\omega).
\]
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