A proof of Milnor conjecture in dimension 3

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Abstract. We present a proof of the Milnor conjecture in dimension 3 based on Cheeger–Colding theory on limit spaces of manifolds with Ricci curvature bounded below. It is different from [17] that relies on minimal surface theory.

1. Introduction

Milnor [18] in 1968 conjectured that any open $n$-manifold $M$ with non-negative Ricci curvature has a finitely generated fundamental group. This conjecture remains open. Anderson [1] and Li [16] independently proved that, for a manifold with Euclidean volume growth, its fundamental group is finite. Sormani proved that if the manifold has small linear diameter growth, or linear volume growth, then the Milnor conjecture holds [20].

For 3-manifolds, Schoen and Yau [19] developed minimal surface theory in dimension 3 and proved that any 3-manifold of positive Ricci curvature is diffeomorphic to the Euclidean space $\mathbb{R}^3$. Based on minimal surface theory, Liu [17] proved that for any 3-manifold with $\text{Ric} \geq 0$, either it is diffeomorphic to $\mathbb{R}^3$, or its universal cover splits off a line. In particular, this confirms the Milnor conjecture in dimension 3.

There are some interests to find a proof of the Milnor conjecture in dimension 3 not relying on minimal surface theory. Our main attempt is to accomplish this by using structure results for limit spaces of manifolds with Ricci curvature bounded below [2,3,7,8], equivariant Gromov–Hausdorff convergence [10] and the pole group theorem [20].

Theorem 1.1. Let $M$ be an open 3-manifold with $\text{Ric}_M \geq 0$, then $\pi_1(M)$ is finitely generated.

For any open 3-manifold $M$ of $\text{Ric}_M \geq 0$ and any sequence $r_i \to \infty$, by Gromov’s precompactness theorem [12], we can pass to some subsequences and consider tangent cones at infinity of $M$ and its Riemannian universal cover $\widetilde{M}$ coming from the sequence $r_i^{-1} \to 0$, via Gromov–Hausdorff convergence:

$$
\begin{array}{ccc}
(r_i^{-1}M, \bar{p}) & \xrightarrow{\text{GH}} & (C_\infty \widetilde{M}, \bar{o}) \\
\downarrow \pi & & \downarrow \pi \\
(r_i^{-1}M, p) & \xrightarrow{\text{GH}} & (C_\infty M, o).
\end{array}
$$
We roughly illustrate our approach to prove Theorem 1.1. If $\pi_1(M, p)$ is not finitely generated, then we derive a contradiction by choosing a special sequence $r_i \to \infty$ and eliminating all the possibilities regarding the dimension of $C_\infty \widehat{M}$ and $C_\infty M$ above in the Colding–Naber sense [8], which are integers 1, 2 or 3.

We also make use of reduction results by Wilking [21] and Evans–Moser [9]. The first reduces any non-finitely generated fundamental group to an abelian one in any dimension, while the latter further reduces any abelian non-finitely generated group to some subgroup of $\mathbb{Q}$, the additive group of rationals, in dimension 3. In particular, we can assume that $\pi_1(M)$ is torsion free if it is not finitely generated. One observation is that, if $\pi_1(M, p)$ is torsion free, then for any $r_i \to \infty$, the corresponding equivariant Gromov–Hausdorff convergent sequence [10]

$$(r_i^{-1} \widehat{M}, p, \pi_1(M, p)) \xrightarrow{GH} (C_\infty \widehat{M}, \partial, G)$$

satisfies that the orbit $G \cdot \partial$ is not discrete (see Corollary 2.4). This observation plays a key role in the proof.

2. Proof of Theorem 1.1

We start with the following reductions by Wilking and Evans–Moser.

**Theorem 2.1** ([21]). *Let $M$ be an open $n$-manifold with $\text{Ric}_M \geq 0$. If $\pi_1(M)$ is not finitely generated, then it contains a non-finitely generated abelian subgroup.*

**Theorem 2.2** ([9]). *Let $M$ be a 3-manifold. If $\pi_1(M)$ is abelian and not finitely generated, then $\pi_1(M)$ is torsion free.*

Evans–Moser [9] actually showed that $\pi_1(M)$ is a subgroup of the additive group of rationals. Being torsion free is sufficient for us to prove Theorem 1.1.

Gromov [11] introduced the notion of short generators of $\pi_1(M, p)$. By path lifting, $\pi_1(M, p)$ acts on $\widehat{M}$ isometrically. We say that $\{\gamma_1, \ldots, \gamma_i, \ldots\}$ is a set of short generators of $\pi_1(M, p)$, if

$$d(\gamma_1 \tilde{p}, \tilde{p}) \leq d(\gamma_i \tilde{p}, \tilde{p})$$

for all $\gamma \in \pi_1(M, p)$, and for each $i$, $\gamma_i \in \pi_1(M, p) - \{\gamma_1, \ldots, \gamma_{i-1}\}$ satisfies

$$d(\gamma_i \tilde{p}, \tilde{p}) \leq d(\gamma_{i-1} \tilde{p}, \tilde{p})$$

for all $\gamma \in \pi_1(M, p) - \{\gamma_1, \ldots, \gamma_{i-1}\}$.

where $\{\gamma_1, \ldots, \gamma_{i-1}\}$ is the subgroup generated by $\gamma_1, \ldots, \gamma_{i-1}$.

Let $M$ be an open 3-manifold with $\text{Ric}_M \geq 0$. We always denote $\pi_1(M, p)$ by $\Gamma$. Suppose that $\Gamma$ is not finitely generated, then by Theorems 2.1 and 2.2, we can assume that $\Gamma$ is torsion free. Let $\{\gamma_1, \ldots, \gamma_{i-1}\}$ be an infinite set of short generators at $p$. Since $\Gamma$ is a discrete group acting freely on $\widehat{M}$, we have $r_i = d(\tilde{p}, \gamma_i \tilde{p}) \to \infty$. When considering a tangent cone at infinity of $\widehat{M}$ coming from the sequence $r_i^{-1} \to 0$, we also take $\Gamma$-action into account. Passing to some subsequences if necessary, we assume that the following sequence converges
in equivariant Gromov–Hausdorff topology \([10]\):
\[
(r_i^{-1}M, \tilde{p}, \Gamma) \xrightarrow{\text{GH}} (\tilde{Y}, \tilde{y}, G)
\]
\[
(2.1)
\]
\[
(r_i^{-1}M, p) \xrightarrow{\text{GH}} (Y = \tilde{Y} / G, y).
\]

Based on Cheeger–Colding’s work \([4]\), Colding–Naber \([8]\) showed that the isometry group of any Ricci limit space is a Lie group. In particular, \(G\) above, as a closed subgroup of \(\text{Isom}(\tilde{Y})\), is a Lie group.

We recall the dimension of Ricci limit spaces in the Colding–Naber sense \([8]\). A point \(x\) in some Ricci limit space \(X\) is \(k\)-regular, if any tangent cone at \(x\) is isometric to \(\mathbb{R}^k\). Colding–Naber showed that there is a unique \(k\) such that \(\mathbb{R}^k\), the set of \(k\)-regular points, has full measure in \(X\) with respect to any limit renormalized measure (see \([3, 8]\)). We regard such \(k\) as the dimension of \(X\) and denote it by \(\text{dim}(X)\). It is unknown whether in general the Hausdorff dimension of \(X\) equals \(\text{dim}(X)\). For Ricci limit spaces coming from 3-manifolds, the dimension in the Colding–Naber sense equals the Hausdorff dimension, which follows from \([3, \text{Theorem 3.1}]\) and \([13]\).

As indicated in the introduction, we prove Theorem 1.1 by eliminating all the possibilities regarding the dimension of \(Y\) and \(\tilde{Y}\) in \((2.1)\). There are three possibilities:

- **Case 1**: \(\text{dim}(\tilde{Y}) = 3\) (Lemma 2.7);
- **Case 2**: \(\text{dim}(Y) = \text{dim}(\tilde{Y}) = 2\) (Lemma 2.8);
- **Case 3**: \(\text{dim}(Y) = 1\) (Lemma 2.10).

We rule out each of them, which finishes the proof of Theorem 1.1.

**Lemma 2.3.** Let \((M_i, p_i)\) be a sequence of complete \(n\)-manifolds and let \((\tilde{M}_i, \tilde{p}_i)\) be their universal covers. Suppose that the following sequence converges:
\[
(\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{\text{GH}} (\tilde{X}, \tilde{p}, G),
\]
where \(\Gamma_i = \pi_1(M_i, p_i)\) is torsion free for each \(i\). If the orbit \(G \cdot \tilde{p}\) is discrete in \(\tilde{X}\), then there is an integer \(N\) such that
\[
\#\Gamma_i(1) \leq N
\]
for all \(i\), where \(\#\Gamma_i(1)\) is the number of elements in
\[
\Gamma_i(1) = \{y \in \Gamma_i \mid d(y \tilde{p}_i, \tilde{p}_i) \leq 1\}.
\]

**Proof.** We claim that if a sequence \(\gamma_i \in \Gamma_i\) converges to \(g \in G\) with \(g\) fixing \(\tilde{p}\), then \(g = e\), the identity element, and \(\gamma_i = e\) for all \(i\) sufficiently large. In fact, suppose that \(\gamma_i \neq e\) for some subsequence. Since \(\gamma_i\) is torsion free, we always have \(\text{diam}(\gamma_i) = \infty\). Together with \(d(\gamma_i \tilde{p}_i, \tilde{p}_i) \rightarrow 0\), we see that \(G \cdot \tilde{p}\) cannot be discrete, a contradiction to the assumption.

Therefore, there exists \(i_0\) large such that for all \(g \in G(2)\) and any two sequences with
\[
\gamma_i \xrightarrow{\text{GH}} g \quad \text{and} \quad \gamma'_i \xrightarrow{\text{GH}} g,
\]
\[
\gamma_i = \gamma'_i \text{ holds for all } i \geq i_0. \text{ In particular, we conclude that }
\# \Gamma_i(1) \leq \# G(2) < \infty
\]
for all \(i \geq i_0\).

**Corollary 2.4.** Let \((M, p)\) be an open \(n\)-manifold with \(\text{Ric}_M \geq 0\) and let \((\tilde{M}, \tilde{p})\) be its universal cover. Suppose that \(\Gamma = \pi_1(M, p)\) is torsion free. Then for any \(s_i \to \infty\) and any convergent sequence
\[
(s_i^{-1}\tilde{M}, p, \Gamma) \xrightarrow{GH} (C_{\infty}\tilde{M}, \tilde{o}, G),
\]
the orbit \(G \cdot \tilde{o}\) is not discrete.

**Proof.** The proof follows directly from Lemma 2.3. If \(G \cdot \tilde{o}\) is discrete, then there is \(N\) such that \(\# \Gamma(s_i) \leq N\) for all \(i\). On the other hand, \(\# \Gamma(s_i) \to \infty\) because \(\Gamma\) is torsion free, a contradiction.

**Lemma 2.5.** Let \((M, p)\) be an open \(n\)-manifold with \(\text{Ric}_M \geq 0\) and let \((\tilde{M}, \tilde{p})\) be its universal cover. Suppose that \(\Gamma = \pi_1(M, p)\) has infinitely many short generators \(\{\gamma_1, \ldots, \gamma_i, \ldots\}\). Then in the following tangent cone at infinity of \(\tilde{M}\),
\[
(r_i^{-1}\tilde{M}, p, \Gamma) \xrightarrow{GH} (\tilde{Y}, \tilde{y}, G),
\]
the orbit \(G \cdot \tilde{y}\) is not connected, where \(r_i = d(\gamma_i \tilde{p}, \tilde{p}) \to \infty\).

**Proof.** On \(r_i^{-1}\tilde{M}, \gamma_i\) has displacement 1 at \(\tilde{p}\). By basic properties of short generators, \(\gamma_i \cdot \tilde{p}\) has distance 1 from the orbit \(H_i \cdot \tilde{p}\), where \(H_i = \langle \gamma_1, \ldots, \gamma_{i-1} \rangle\). From the equivariant convergence
\[
(r_i^{-1}\tilde{M}, \tilde{p}, H_i, \gamma_i) \xrightarrow{GH} (\tilde{Y}, \tilde{y}, H_\infty, \gamma_\infty),
\]
we conclude \(d(\gamma_\infty \cdot \tilde{y}, H_\infty \cdot \tilde{y}) = 1\). Moreover, for any element \(g \in G - H_\infty\), we can find a sequence \(g_i \in \Gamma - H_i\) such that
\[
(r_i^{-1}\tilde{M}, \tilde{p}, g_i) \xrightarrow{GH} (\tilde{Y}, \tilde{y}, g).
\]
Again by the basic properties of short generators, we see that \(d(g \cdot \tilde{y}, H_\infty \cdot \tilde{y}) \geq 1\). We divide the orbit \(G \cdot \tilde{y}\) into two non-empty subsets: \(H_\infty \cdot \tilde{y}\) and \((G - H_\infty) \cdot \tilde{y}\). Since these two subsets have distance 1 between them, we conclude that the orbit \(G \cdot \tilde{y}\) must be non-connected.\(^1\)

We recall the cone splitting principle, which follows from the splitting theorem for Ricci limit spaces [2].

**Proposition 2.6.** Let \((X, p)\) be the limit of a sequence of complete \(n\)-manifolds \((M_i, p_i)\) of \(\text{Ric}_{M_i} \geq 0\). Suppose that \(X = \mathbb{R}^k \times C(Z)\) is a metric cone with vertex \(p = (0, z)\). If there is an isometry \(g \in \text{Isom}(X)\) with \(g(0, z) \notin \mathbb{R}^k \times \{z\}\), then \(X\) splits isometrically as \(\mathbb{R}^{k+1} \times C(Z')\).

\(^1\) In an early version of this paper posted on arXiv:1703.08143v1, the proof of Lemma 2.5 uses the fact that \(G\) is a Lie group. Xiaochun Rong pointed out that whether \(G\) is a Lie group or not is not relevant in Lemma 2.5.
**Lemma 2.7.** Case 1 can not happen.

*Proof.* When \( \dim(\widetilde{Y}) = 3 \), \( \widetilde{Y} \) is a non-collapsing limit space \([3]\), that is, there is \( v > 0 \) such that

\[
\text{vol}(B_1(\tilde{p}, r_i^{-1}\tilde{M})) \geq v
\]

for all \( i \). By relative volume comparison, this implies that \( \widetilde{M} \) has Euclidean volume growth

\[
\lim_{r \to \infty} \frac{\text{vol}(B_r(\tilde{p}))}{r^n} \geq v.
\]

By \([3]\), \( \widetilde{Y} \) is a metric cone \( \mathbb{R}^k \times C(Z) \) with vertex \( \tilde{y} = (0, z) \), where \( C(Z) \) has vertex \( z \) and \( \text{diam}(Z) < \pi \). We rule out all the possibilities of \( k \in \{0, 1, 2, 3\} \).

If \( k = 3 \), then \( \widetilde{Y} = \mathbb{R}^3 \). Thus \( \tilde{M} \) is isometric to \( \mathbb{R}^3 \) (see \([7]\)).

If \( k = 2 \), then according to co-dimension 2 of singular sets \([3]\), actually \( \widetilde{Y} = \mathbb{R}^3 \) holds.

If \( k = 1 \), then \( Y = \mathbb{R} \times C(Z) \). By Proposition 2.6, the orbit \( G \cdot \tilde{y} \) is contained in \( \mathbb{R} \times \{z\} \). Applying Lemma 2.5, we see that \( G \cdot \tilde{y} \) is not connected. Note that a non-connected orbit in \( \mathbb{R} \) is either a \( \mathbb{Z}_2 \)-translation orbit, or a \( \mathbb{Z}_2 \)-reflection orbit. In particular, the orbit \( G \cdot \tilde{y} \) must be discrete. This contradicts with Corollary 2.4.

If \( k = 0 \), then \( Y = C(Z) \) with no lines. Again by Proposition 2.6, the orbit \( G \cdot \tilde{y} \) must be a single point \( \tilde{y} \), a contradiction to Lemma 2.5.

\( \Box \)

**Lemma 2.8.** Let \((M, p)\) be an open n-manifold with \( \text{Ric}_M \geq 0 \) and let \((\tilde{M}, \tilde{p})\) be its universal cover. Assume that \( \Gamma = \pi_1(M, p) \) is torsion free. Then for any \( s_i \to \infty \) and any convergent sequence

\[
\begin{array}{ccc}
(s_i^{-1}\tilde{M}, \tilde{p}, \Gamma) & \xrightarrow{GH} & (C_\infty \tilde{M}, \tilde{o}, G) \\
\downarrow \pi & & \downarrow \pi \\
(s_i^{-1}M, p) & \xrightarrow{GH} & (C_\infty M, o),
\end{array}
\]

\( \dim(C_\infty \tilde{M}) = \dim(C_\infty M) \) can not happen. In particular, Case 2 can not happen.

*Proof.* We claim that \( G \) is a discrete group when \( \dim(C_\infty \tilde{M}) = \dim(C_\infty M) = k \). If this claim holds, then the desired contradiction follows from Corollary 2.4.

It remains to verify the claim. Suppose that \( G_0 \) is non-trivial, then we pick \( g \neq e \) in \( G_0 \). We first show that there is a \( k \)-regular point \( \tilde{q} \in C_\infty \tilde{M} \) such that \( d(g\tilde{q}, \tilde{q}) > 0 \) and \( \tilde{q} \) projects to a \( k \)-regular point \( q \in C_\infty M \). In fact, let \( \mathcal{R}_k(C_\infty M) \) be the set of \( k \)-regular points in \( C_\infty M \). Since \( \mathcal{R}_k(C_\infty M) \) is dense in \( C_\infty M \), its pre-image \( \pi^{-1}(\mathcal{R}_k(C_\infty M)) \) is also dense in \( C_\infty \tilde{M} \). Let \( \tilde{q} \) be a point in the pre-image such that \( d(g\tilde{q}, \tilde{q}) > 0 \). Note that any tangent cone at \( \tilde{q} \) splits an \( \mathbb{R}^k \)-factor isometrically. By \([14, \text{Proposition } 3.78]\) (see also \([15, \text{Corollary } 1.10]\)), it follows that any tangent cone at \( \tilde{q} \) is isometric to \( \mathbb{R}^k \). In other words, \( \tilde{q} \) is \( k \)-regular.

Along a one-parameter subgroup of \( G_0 \) containing \( g \), we can choose a sequence of elements \( g_j \in G_0 \) with \( d(g_j\tilde{q}, \tilde{q}) = 1/j \to 0 \). We consider a tangent cone at \( \tilde{q} \) and \( q \), respectively, coming from the sequence \( j \to \infty \). Passing to some subsequences if necessary, we
obtain

\[(jC_\infty \tilde{M}, \tilde{q}, G, g_j) \xrightarrow{\text{GH}} (C_\tilde{q}C_\infty \tilde{M}, \tilde{o}', H, h)\]

\[\downarrow \pi \hspace{1cm} \downarrow \pi\]

\[ (jC_\infty M, q) \xrightarrow{\text{GH}} (C_qC_\infty M, o')\]

with \(C_qC_\infty \tilde{M}/H = C_qC_\infty M\) and \(d(ho', \tilde{o}') = 1\). On the other hand, since both \(q\) and \(\tilde{q}\) are \(k\)-regular, we have

\[C_qC_\infty \tilde{M} = C_qC_\infty M = \mathbb{R}^k.\]

This is a contradiction to \(H \neq \{e\}\). Hence the claim holds.

To rule out the last case \(\dim(Y) = 1\), we recall Sormani’s pole group theorem [20]. We say that a length space \(Y\) has a pole at \(y \in Y\), if for any \(x \neq y\), there is a ray starting from \(y\) and going through \(x\).

**Theorem 2.9** ([20]). Let \((M, p)\) be an open \(n\)-manifold with \(\text{Ric}_M \geq 0\) and let \((\tilde{M}, \tilde{p})\) be its universal cover. Suppose that \(\Gamma = \pi_1(M, p)\) has infinitely many short generators \(\{\gamma_1, \ldots, \gamma_i, \ldots\}\). Then in the following tangent cone at infinity of \(M\),

\[(r_i^{-1} M, p) \xrightarrow{\text{GH}} (Y, y),\]

\(Y\) can not have a pole at \(y\), where \(r_i = d(\gamma_i, \tilde{p}, \tilde{p}) \to \infty\).

**Lemma 2.10.** Case 3 can not happen.

**Proof.** By [13] (also see [6]), \(Y\) is a topological manifold of dimension 1. Since \(Y\) is non-compact, \(Y\) is either a line \((-\infty, \infty)\) or a half line \([0, \infty)\). By Theorem 2.9, \(Y\) can not have a pole at \(y\). Thus there is only one possibility left: \(Y = [0, \infty)\) but \(y\) is not the endpoint \(0 \in [0, \infty)\). Put \(d = d_Y(0, y) > 0\). We rule out this case by a rescaling argument and Lemmas 2.7 and 2.8. (In general, it is possible for an open manifold to have a tangent cone at infinity as \([0, \infty)\), with base point not being 0; see Example 2.11.)

Let \(\alpha(t)\) be a unit speed ray in \(M\) starting from \(p\), and converging to the unique ray from \(y\) in \(Y = [0, \infty)\) with respect to the sequence

\[(r_i^{-1} M, p) \xrightarrow{\text{GH}} (Y, y).\]

Let \(x_i \in r_i^{-1} M\) be a sequence of points converging to \(0 \in Y\), then \(r_i^{-1}d_M(p, x_i) \to d\). For each \(i\), let \(c_i(t)\) be a minimal geodesic from \(x_i\) to \(\alpha(dr_i)\), and let \(q_i\) be a closest point to \(p\) on \(c_i\). We re-parametrize \(c_i\) so that \(c_i(0) = q_i\). With respect to the sequence

\[(r_i^{-1} M, p) \xrightarrow{\text{GH}} (Y, y),\]

c_i subconverges to the unique segment between 0 and \(2d \in [0, \infty)\). Clearly,

\[r_i^{-1}d_M(x_i, \alpha(dr_i)) \to 2d, \quad r_i^{-1}d_i \to 0,\]

where \(d_i = d_M(p, c_i(0))\).
If \( d_i \to \infty \), then we rescale \( M \) and \( \tilde{M} \) by \( d_i^{-1} \to 0 \). Passing to some subsequences if necessary, we obtain

\[
\begin{align*}
(d_i^{-1}\tilde{M}, \tilde{\rho}, \Gamma) \xrightarrow{\text{GH}} (\tilde{Y}', \tilde{y}', G')
\end{align*}
\]

If \( \dim(Y') = 1 \), then we know that \( Y' = (-\infty, \infty) \) or \([0, \infty)\). On the other hand, since

\[
d_i^{-1}d_M(c_i(0), x_i) \to \infty, \quad d_i^{-1}d_M(c_i(0), \alpha(dr_i)) \to \infty, \quad d_i^{-1}d_M(c_i, p) = 1,
\]

\( c_i \) subconverges to a line \( c_{\infty} \) in \( Y' \) with \( d(c_{\infty}, y') = 1 \). Clearly this can not happen in \( Y' = (-\infty, \infty) \) nor \([0, \infty)\). If \( \dim(\tilde{Y}') = 3 \), then \( \tilde{M} \) has Euclidean volume growth. Thus with the sequence \( r_i^{-1} \), the corresponding limit spaces \( Y \) and \( \tilde{Y} \) must satisfy \( \dim(Y) = 1 \) and \( \dim(\tilde{Y}) = 3 \), which is already covered in Lemma 2.7. Therefore, the only possibility left is \( \dim(\tilde{Y}') = \dim(Y') = 2 \). By Lemma 2.8, this also leads to a contradiction. In conclusion, \( d_i \to \infty \) can not happen.

If there is some \( R > 0 \) such that \( d_i \leq R \) for all \( i \), then on \( M \), \( c_i \) subconverges to a line \( c \) with \( c(0) \in B_{2R}(p) \). Consequently, \( M \) splits off a line isometrically [5], a contradiction to \( Y = [0, \infty) \). This completes the proof. \( \square \)

**Example 2.11.** We construct a surface \((S, p)\) isometrically embedded in \( \mathbb{R}^3 \) such that \( S \) has a tangent cone at infinity as \([0, \infty)\), but \( p \) does not correspond to 0. We first construct a subset of the \( xy \)-plane by gluing intervals. Let \( r_i \to \infty \) be a positive sequence with \( r_i+1/r_i \to \infty \). Starting with an interval \( I_1 = [-r_1, r_2] \), we attach a second interval \( I_2 = [-r_3, r_4] \) perpendicularly to \( I_1 \) by identifying \( r_2 \in I_1 \) and \( 0 \in I_2 \). Repeating this process, suppose that \( I_k \) is attached, then we attach the next interval \( I_{k+1} = [-r_{2k+1}, r_{2k+2}] \) perpendicularly to \( I_k \) by identifying \( r_{2k} \in I_k \) and \( 0 \in I_{k+1} \). In this way, we construct a subset \( T \) in the \( xy \)-plane consisting of segments. We can smooth the \( \epsilon \)-neighborhood of \( T \) in \( \mathbb{R}^3 \) so that it has sectional curvature \( \geq -C \), where \( \epsilon, C > 0 \). We call this surface \( S \). Let \( p \in S \) be a point closest to \( 0 \in I_1 \) as the base point. If we rescale \((S, p)\) by \( r_{2k+1}^{-1} \to 0 \), then

\[
(r_{2k+1}^{-1} S, p) \xrightarrow{\text{GH}} ([-1, \infty), 0)
\]

because \( r_{i+1}/r_i \to \infty \). In other words, \( S \) has a tangent cone at infinity as the half line, but the base point does not correspond to the end point in this half line.

**Acknowledgement.** The author would like to thank Professor Xiaochun Rong and Professor Jeff Cheeger for suggestions during the preparation of this paper.

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Eingegangen 17. Februar 2017, in revidierter Fassung 19. Oktober 2017