Naturalness in Higgs inflation in a frame independent formalism

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Abstract. We make use of the frame and gauge independent formalism for scalar and tensor cosmological perturbations developed in Ref. [1] to show that the physical cutoff for 2-to-2 tree level scatterings in Higgs inflation is above the Planck scale $M_P = 1/\sqrt{8\pi G_N}$ throughout inflation. More precisely, we found that in the Jordan frame, the physical cutoff scale is $(\Lambda/a)_J \gtrsim \sqrt{M_P^2 + \xi \phi^2}$, while in the Einstein frame it is $(\Lambda/a)_E \gtrsim M_P$, where $\xi$ is the nonminimal coupling and $\phi$ denotes the Higgs vev during inflation. The dimensionless ratio of the physical cutoff to the relevant Planck scale is equal to one in both frames, thus demonstrating the physical equivalence of the two frames. Our analysis implies that Higgs inflation is unitary up to the Planck scale, and hence there is no naturalness problem in Higgs inflation. In this paper we only consider the graviton and scalar interactions.
1 Introduction

Higgs inflation \[2, 3\] \[4–10\] is perhaps the most economical approach to cosmological inflation: it identifies the only known scalar field in the Standard Model (SM) – the Higgs boson – with the inflaton, the scalar field that drives the inflationary expansion in the very early universe. The action for Higgs inflation features a non-minimal coupling of the Higgs field to the Ricci scalar. If the non-minimal coupling $\xi$ is large, it leads to a successful period of chaotic inflation, producing primordial power spectra that fit the observational bounds \[11\]. Hence, Higgs inflation can be considered as a "natural" scenario, since it does not seem to require the introduction of new physics to explain the inflationary expansion of the universe.

However, the naturalness of the Higgs inflation scenario has been under debate. Refs. \[12, 13\] used power-counting techniques to determine the energy scale $\Lambda$ at which perturbation theory breaks down, thus determining the range of validity of Higgs inflation. It was claimed that this cutoff scale $\Lambda$ lies dangerously close to the energy scale of inflation, thereby questioning the naturalness of Higgs inflation. Since then many works have appeared claiming that Higgs inflation is "natural" \[14, 15\] or "unnatural" \[16, 17\]. Perhaps the most complete treatment has been done in Ref. \[18\]. It was found that $\Lambda$ is generally field dependent and lies above the typical energy scales in different regions, such that the perturbative (semiclassical) expansion is valid in Higgs inflation.

Although there seems to be a consensus about the cutoff scale as computed in Ref. \[18\] (however, see the recent work \[19\] \footnote{The results presented in Ref. \[19\] use different techniques and arrive at results that are consistent with those presented in this paper and earlier in Ref. \[20\].}), we revisit the computation of the cutoff in this work. The reason is that there are some important aspects that have not been fully taken into account. The most important aspect is that General Relativity contains a large diffeomorphism symmetry, which when truncated resembles the symmetry of a gauge theory. This means that, like in QED, some of the degrees of freedom in the action are actually not physical. As a consequence, some of the interaction vertices obtained after a naive perturbative expansion of the action are gauge dependent, and any conclusion that one arrives at by using these vertices can be a gauge artifact. Moreover, it is possible that there are additional vertices that conspire to cancel dominant perturbative contributions. Accounting for these aspects can raise the cutoff scale. Even though it is possible to determine the physical cutoff scale within a gauge dependent formulation, by far the simplest and most reliable way to determine this scale
is to use the physical vertices, which can be obtained from the perturbative action in a manifestly gauge
invariant way (an alternative is to completely fix the gauge freedom and take account of contributions
from all vertices). This formulation of the action in terms of gauge invariant perturbations has been
found for a non-minimally coupled scalar field up to third order in perturbations in Refs. [1, 21, 22].
In this work we use these previously found results in order to demonstrate that the cutoff scale for
physical, that is, gauge invariant perturbations is always \( \geq M_P \). To be more precise, we find that
\[
\left( \frac{\Lambda}{a} \right)_J \gtrsim \sqrt{M_P^2 + \xi \phi^2}, \quad \left( \frac{\Lambda}{a} \right)_E \gtrsim M_P,
\]
(1.1)
where \( a_J \) and \( \phi \) are the background scale factor and scalar field in the Jordan frame, and \( a_E \) is the
Einstein frame scale factor. The extra scale factors have been overlooked in previous computations of
the cutoff scale. They appear since \( \Lambda^{-1} \) is a comoving scale, and therefore in an expanding universe
the corresponding physical length scale, \( a/\Lambda \), always includes a scale factor \( a \). The cutoffs in Jordan
and Einstein frame are different, simply because \( \Lambda \) is a dimensionful quantity which differs between
the frames, just like the effective Planck mass. If \( M_P \) is the energy scale where quantum gravity kicks
in in the Einstein frame, then \( M_{P,J} = \sqrt{M_P^2 + \xi \phi^2} \) can be identified with the scale of quantum gravity
in the Jordan frame. Thus in either frame the perturbative (semiclassical) treatment is valid all the
way up to the scale at which gravity becomes strong. This means that Higgs inflation is perfectly
natural and that no new physics is necessary to explain the inflationary expansion of the universe and
the anisotropies in the CMB.

In order to arrive at the physical cutoff scales (1.1), we first briefly discuss Higgs inflation and
physically equivalent frames in section 2. In section 3 we review the naturalness debate. In section 4
we discuss the concept of gauge and frame dependence in cosmology and quote from Ref. [1] the
action for gauge and frame invariant cosmological perturbations. Finally we compute the cutoff scale
for physical perturbations in section 5 and conclude in section 6.

2 Higgs inflation

Unfortunately, inflation in the pure SM has been ruled out by observations of the CMB, which have
put tight constraints on the inflationary potential. However, these constraints can be greatly relaxed
when the Higgs field \( H \) is quadratically coupled to the Ricci scalar \( R \) by a large non-minimal coupling
\( \xi \sim 10^4 \) [23–26]. The pure scalar and gravitational part of the action reads
\[
S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} M_P^2 R + \xi H^\dagger H R - g^{\mu\nu} (\partial_\mu H) \dagger (\partial_\nu H) - \lambda \left( H^\dagger H - \frac{v^2}{2} \right)^2 \right\}.
\]
(2.1)
Here the metric signature is \( \text{sign}[g_{\mu\nu}] = (-, +, +, +) \), the Higgs self-coupling is \( \lambda \) and the Higgs vacuum
expectation value (vev) is \( v = 246 \text{ GeV} \). Although the complex doublet \( H \) contains four degrees of
freedom, three of those can be absorbed by the gauge bosons (not shown in Eq. (2.1)). In unitary gauge
the remaining scalar degree of freedom with non-zero vev can be parametrized by \( H = (0, \Phi/\sqrt{2}) \),
such that the scalar action takes the Jordan frame form
\[
S = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ R F(\Phi) - g^{\mu\nu} (\partial_\mu \Phi)(\partial_\nu \Phi) - \frac{\lambda}{2} (\Phi^2 - v^2)^2 \right\},
\]
(2.2)
with \( F(\Phi) = M_P^2 + \xi \Phi^2 \). The claim is now that a successful period of (chaotic) inflation is possible when
the non-minimal coupling parameter is large, \( \xi \sim 10^4 \). Successful means that the model’s predictions
for the primordial power spectra of scalar and tensor perturbations agree with the observational
constraints on these spectra. The current $1\sigma$ constraints on the scalar spectral index and the tensor-to-scalar ratio are $n_s = 0.9603 \pm 0.0073$ and $r < 0.11$, respectively [11]. On the theory side, it is known that in a minimally coupled model ($\xi = 0$) both $n_s - 1$ and $r$ are proportional to slow-roll parameters. Moreover, since the Hubble parameter $H$ does not directly couple to the time dependent expectation value of the scalar field $\langle \Phi \rangle = \phi(t)$ in the Friedmann equations, it is possible to express the slow-roll parameters in terms of the slope of the potential. Hence, in a minimally coupled model the shape of the scalar potential provides an intuition for the successful realization of inflation. For instance, a quartic potential is excluded by CMB observations [11] because it is insufficiently flat in the inflationary regime, which supports the previously mentioned statement that inflation is not possible in the pure (i.e. minimally coupled) SM.

Unfortunately, we do not have the luxury of such an intuition in the case of (large) non-minimal coupling $\xi \gg 1$. The mixing of the gravitational and matter part of the action prevents us from expressing the slow-roll parameters in terms of the potential. Worse, it is not clear what are the slow-roll parameters in a non-minimally coupled model, that is, what are the parameters that remain small during the inflationary period, and neither is it clear how the primordial power spectra depend on these small parameters. The most straightforward way to see whether or not Higgs inflation works is therefore to compute the primordial power spectra for non-minimal coupling from scratch, which involves a derivation of the quadratic action for scalar and tensor perturbations in the Jordan frame [21]. Fortunately we can avoid this exercise by making use of a clever trick. If we redefine the metric, scalar field and scalar potential in the action as follows [21],

$$g_{\mu\nu,E} = \frac{F}{M_P^2} g_{\mu\nu},$$

$$\left( \frac{d\Phi_E}{d\Phi} \right)^2 = \frac{M_P^2}{F^2} \left( 1 + \frac{3}{2} F' r^2 \right),$$

$$V_E(\Phi_E) = \frac{M_P^4}{F^2} V(\Phi), \quad \text{(}2.3\text{)}$$

Since the metric has been rescaled by a factor it is commonly said that we are in another conformal frame; the specific frame for which the scalar field is minimally coupled to the Ricci scalar is called the Einstein frame, and quantities in this frame are indicated by a subscript $E$. Thus the action has been rewritten to the more familiar minimally coupled form. The crucial point is that the Jordan and Einstein frame are related by field redefinitions (2.3), and hence the two formulations are physically equivalent. Nature is indifferent to whether we use one set of variables to describe her phenomena, or a different, but related, set. Physical observables should therefore be invariant with respect to the frame in which you compute them [27]. Specifically, the primordial power spectra for scalar and tensor perturbations (or the predicted values of $n_s$ and $r$) are the same whether you compute them in the Jordan frame using variables $g_{\mu\nu}$ and $\Phi$, or in the Einstein frame using $g_{\mu\nu,E}$ and $\Phi_E$ related to the first set by Eqs. (2.3). This was first demonstrated in Refs. [28, 29], see also Ref. [21]. Hence, the more familiar and more intuitive Einstein frame formulation can be used to check the successfulness of Higgs inflation [2, 3]. In the Einstein frame it is intuitively clear that Higgs inflation works: the potential becomes exponentially flat in the large field limit. Predictions for the spectral index and scalar-to-tensor ratio in Higgs inflation are $n_s \approx 0.97$ and $r \approx 0.0032$, and are therefore well within the observational bounds.

\[ \text{– 3 –} \]
3 The naturalness debate

At first sight Higgs inflation is a very attractive scenario, because it does not seem to require the introduction of new physics to explain the exponential expansion of the early universe. However, this has been questioned by Refs. [12, 13], who looked at the ultraviolet cutoff scale \( \Lambda \) in Higgs inflation. \( \Lambda \) indicates the energy scale above which scattering amplitudes violate the unitarity bound, and therefore determines when perturbation theory breaks down. Such a cutoff can be found by power-counting techniques for which one can use the following recipe:

1. Perform a perturbative expansion of the fields in the action.
2. Rescale the fields such that their kinetic terms are canonically normalized.
3. Read off the cutoff scale from the interaction terms with \( D > 4 \) which are suppressed by \( \Lambda^{4-D} \).

In Refs. [12, 13] the cutoff scale was obtained by expanding the scalar field around its vev, \( \Phi = v + \varphi \), and the metric around Minkowski space, \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/M_P \), where the normalization \( M^{-1}_P \) is chosen such that the gravitational terms are canonically normalized. In the Jordan frame, the cutoff follows from the expansion of the \( \xi H^2 H R \) term, which gives the 5-dimensional interaction \( (\xi/M_P)\varphi^2 \delta H \). Reading off the cutoff scale gives \( \Lambda = M_P/\xi \). In the Einstein frame the cutoff follows from a small-field expansion of the potential, which gives \( V_E = \frac{1}{2}\lambda \varphi_E^2 - \lambda (\xi/M_P)^2 \varphi_E^0 + \ldots \). The dimension 6 terms are due to the small-field relation between Jordan and Einstein frame fields \( \Phi \approx \Phi_E[1 - (\xi/M_P)^2] \). Again, the cutoff scale from the dimension 6 term is \( \Lambda = M_P/\xi \). The breakdown of perturbation theory may signal the appearance of new physics, for example higher dimensional operators suppressed by \( \Lambda \), which solve the unitarity problems at energy scales above \( \Lambda \). Now, the point is that the cutoff scale \( \Lambda \) lies dangerously close to the energy scale of inflation, characterized by the Einstein frame Hubble parameter \( H_E \approx \sqrt{\lambda} M_P/\xi \). These higher dimensional operators may therefore enter the inflationary potential and affect inflationary predictions, or worse, spoil Higgs inflation. Thus we need knowledge of the ultraviolet completion of Higgs inflation in order to find out if inflation is successful, which obviously clashes with the original attractiveness of the scenario.

Since the appearance of the works [12, 13] there has been a debate in literature about the presence or absence of unitarity problems in Higgs inflation. Refs. [14, 15, 30] considered the Einstein frame analysis and objected, correctly, that the dimension 6 terms in the small field expansion of the potential only give rise to unitarity problems in the large field regime (\( \langle \Phi \rangle \gg M_P/\xi \) or \( \langle \Phi_E \rangle \geq M_P \)), where the small field expansion is no longer valid. Instead, one should perform a perturbative expansion around the field expectation value, which results in a field dependent cutoff scale.

Then, by making use of the equivalence between the Jordan and Einstein frame, it was argued that there are also no unitarity problems in the Jordan frame for single field inflation. Refs. [16, 17] showed subsequently that the unitarity bound \( M_P/\xi \) again appears when the Goldstone bosons are taken into account, both in the Jordan and Einstein frame. Ref. [16] added that even in unitary gauge, where the Goldstone bosons are eaten by the gauge bosons, the cutoff scale appears in Higgs-gauge interactions. Ref. [15, 30] argued that, instead of expanding around a small Higgs vev, the perturbative expansion should be performed around a large expectation value \( \phi \gg M_P/\xi \), which is the background relevant for inflation. Here \( \phi = \langle \Phi \rangle \), with \( \Phi \) the Higgs field in unitary gauge, \( H = (0, \Phi) \). It was shown that the cutoff scale following from the Einstein frame potential is \( M_P \) in the inflationary regime, but undergoes a transition to \( M_P/\xi \) for small field values \( \phi \ll M_P/\xi \). The authors then argued that this post-inflation unitary bound does not affect our ability to describe physical processes during inflation. Arguably the most complete treatment of the cutoff so far has been performed in Ref. [18]. There the metric was expanded around an expanding background \( g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \) and likewise the scalar field
was expanded around a time dependent background $\Phi = \phi + \varphi$. Next, the Jordan frame action was put into canonical form by a redefinition of the metric and scalar field perturbations. In the Jordan frame the cutoff originates from the $\xi \Phi^2 R$ term and was found to be $M_P/\xi$ for $\phi \ll M_P/\xi$, $\xi \phi^2 / M_P$ for $M_P/\xi < \phi < M_P/\sqrt{\xi}$ and $\sqrt{\xi} \phi$ for $\phi > M_P/\sqrt{\xi}$. In the Einstein frame the same results were obtained from the potential $^2$.

4 Gauge invariance

In this work we revisit the computation of the cutoff scale in Higgs inflation. The reason is that all of the papers mentioned in the previous section have overlooked a crucial aspect of the computation of quantum corrections in general relativity. This crucial aspect is the fact that general relativity contains a large diffeomorphism symmetry. Since the diffeomorphism symmetry resembles many aspects of gauge symmetries, general relativity is often called a gauge theory and its degrees of freedom (dofs) are generally gauge dependent. Due to gauge dependence not all dofs are physical. Moreover, general relativity is a constrained theory, such that some of the dofs do not participate in the dynamics, but instead impose constraints on the system. Therefore, computing a cutoff by means of expanding $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ and $\Phi = \phi + \varphi$ and using the corresponding vertices, will generally give non-physical and incorrect answers. Instead one should first determine what are the truly physical degrees of freedom, and subsequently find their interaction vertices and the ultraviolet cutoff.

As a small sidestep, let us illustrate the above by looking at an analogy: the simpler and more familiar case of QED. QED is described by a vector field $A_\mu$, which naively contains 4 dofs. However, one of the 4 dofs (the longitudinal component of the spatial vector field) is not physical due to the gauge symmetry $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$. Moreover one of the components of the vector field is not dynamical: there are no time derivatives acting on $A_0$ in the Maxwell action. This becomes particularly clear when the theory is written in Hamiltonian form. The non-dynamical component (the Coulomb potential) can in fact be decoupled from the dynamical part of the action. Thus, out of 4 dofs in QED, there are only 2 remaining dynamical dofs, which correspond to the transverse components of the vector field, i.e. the two polarizations of the photon.

Let us now turn to Higgs inflation. The action for Higgs inflation, which basically constitutes of the Einstein-Hilbert action for general relativity, a scalar field in a potential and a coupling of the scalar field to the EH-action, contains naively 10 + 1 dynamical degrees of freedom (dofs) in the metric and scalar field. However, due to the diffeomorphism symmetry (invariance under coordinate reparametrizations $x^\mu \rightarrow x'^\mu + \xi^\mu$), 4 of these dofs are not physical. The analogy with QED becomes more apparent when we look at linearized perturbations. The action for these first order perturbations is invariant under the transformations $\delta g_{\mu\nu} \rightarrow \delta g_{\mu\nu} + 2 \nabla_\nu (\xi_\mu)$ and $\varphi \rightarrow \varphi + \dot{\varphi} \xi_0$. This closely resembles the gauge symmetry of electrodynamics, only in this case there are 4 gauge parameters, and thus 4 non-physical components of the metric.

Also, analogous to QED, 4 degrees of freedom in the metric are not dynamical. These are the 00 and 0i components of the ADM metric,

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt).$$

[^2]: The cutoffs in the Jordan and Einstein frames are related by the conformal factor $\sqrt{1 + \xi \phi^2 / M_P^2}$. This can be understood by realizing that the cutoff is some energy scale, or length scale, which is changed by the conformal transformation. In Ref. [18] the cutoffs are only found to be equivalent (note: up to factors of the self-coupling $\lambda$) once this factor is taken into account, see also [4, 7].
Here $g_{ij}$ is the spatial metric and $N$ and $N^i$ are the lapse and shift functions, respectively. In terms of the ADM metric the action (2.2) can be written as

$$S = \frac{1}{2} \int d^3x dt \sqrt{\mathcal{g}} \left\{ NR F(\Phi) + \frac{1}{N} (E^{ij} E_{ij} - E^2) F(\Phi) - \frac{2}{N} \mathcal{E} F'(\Phi) \left( \partial_t \Phi - N^i \partial_i \Phi \right) 
+ 2g^{ij} \nabla_i N \nabla_j F(\Phi) + \frac{1}{N} \left( \partial_t \Phi - N^i \partial_i \Phi \right)^2 - Ng^{ij} \partial_i \Phi \partial_j \Phi - 2NV(\Phi) \right\}, \quad (4.2)$$

where $F'(\Phi) = dF(\Phi)/d\Phi$ and the measure $\sqrt{\mathcal{g}}$, the Ricci scalar $R$ and covariant derivatives $\nabla_i$ are composed of the spatial part of the metric $g_{ij}$ alone. The quantities $E_{ij}$ and $E$ are related to the extrinsic curvature $K_{ij}$ as $E_{ij} = - NK_{ij}$, with

$$E_{ij} = \frac{1}{2} \left( \partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i \right) \quad (4.3)$$

$$E = g^{ij} E_{ij}. \quad (4.4)$$

From Eq. (4.2) it can clearly be seen that the lapse and shift functions are not dynamical (there are no kinetic terms for them). This becomes even clearer in a Hamiltonian formulation [21], in which the lapse and shift functions multiply the energy and momentum constraints. Because the lapse and shift functions are related to the constraints in general relativity and are non-dynamical, they are also called constraint, or auxiliary fields. Thus, out of the $10 + 1$ degrees of freedom (dofs) in the action (2.2), 4 are non-dynamical and related to the constraints, and 4 others are gauge dofs. We can therefore already argue that there are only 3 dynamical dofs in the perturbed action, and we should take this into account when computing the physical cutoff.

As a first example of how incorrect results are obtained by not taking into account the gauge freedom and constraints in general relativity, let us consider the case of quantum corrections in Higgs inflation by expanding around a Minkowsky background, as done in Refs. [12, 13]. The dominant contribution from the term $\xi \Phi^2 R$ is found to be $(\xi/M_P)\varphi^2 \Box h$. However, it is well known that the only dynamical dofs in Minkowsky space are the transverse traceless part of the metric, the graviton $\gamma_{ij}$, and the scalar field fluctuation $\varphi$ \footnote{Commonly, in Minkowski space the 00 and 0\(i\) components of the metric fluctuation $h_{\mu\nu}$ are zero (the solution for the auxiliary fields is zero), and – in the absence of matter fields – the gauge freedom is used to set the scalar and vectorial components of the spatial metric fluctuation $h_{ij}$ to zero. The only remaining degree of freedom is the traceless and traceless $\gamma_{ij}$.}. In that case, a term such as $h = \text{Tr}[h_{\mu\nu}]$ disappears from the action. The next most dominant term is then $(\xi/M_P^2)\varphi^2 \partial \gamma_{ij} \partial \gamma_{ij}$, which is a dimension 6 term with a cutoff scale $M_P/\sqrt{\xi}$, which is higher than the naive cutoff scale $M_P/\xi$. Although this is still an incorrect derivation of the cutoff – one should perform an expansion around an expanding universe and a time dependent background field – it already shows that taking into account the gauge symmetry can raise the cutoff scale.

So let us continue by considering fluctuations on top of an expanding universe. In order to deal with the non-dynamical parts of the metric and the gauge freedom, it is convenient to separate the metric into different components and expand each component separately. We thus insert into the action (2.2) (or Eq. (4.2))

$$g_{ij} = a^2(t) e^{2\xi}(e^\alpha)_{ij}$$

$$\Phi = \phi + \varphi$$

$$N = \tilde{N} (1 + n)$$

$$N^i = a^{-1} \tilde{N}(t) (a^{-1} \partial_i s + n^T_i) \quad (4.5)$$

$$\nabla_i N^j (a^{-1} \partial_i s + n^T_i)$$
where \(a(t)\) is the scale factor and

\[
\alpha_{ij} = a^{-2} \partial_i \partial_j \tilde{h} + a^{-1} \partial_i (\tilde{h}^T)_{ji} + \gamma_{ij},
\]

with \(\gamma_{ii} = 0, \partial_j \gamma_{ji} = 0, \partial_j h^T_{ij} = 0\) and \(\partial_i n^T_i = 0\). Now, all fluctuations of the metric and scalar field are by themselves not invariant under the gauge transformations (coordinate reparametrizations) of general relativity. Thus interaction vertices for the fluctuations in Eq. (4.5) are generally not gauge invariant, and thus not physical. In order to find the physical interaction vertices, the best way to go is to derive the action for gauge invariant perturbations. These are variables that are by themselves invariant under coordinate reparametrizations, and hence their interaction vertices are physical. In Refs. [1, 21, 22] we have precisely done this: we have derived the quadratic and cubic actions for gauge invariant cosmological perturbations. We have found that, when written in terms of physical (gauge invariant) fields, the scalar-tensor theory (2.2) can be separated into dynamical and constraint parts, such that the constraint and dynamical fields decouple. The dynamical part contains one real scalar degree of freedom and one (traceless, transverse) tensor field with two degrees of freedom (polarizations). The non-dynamical part contains gauge invariant versions of the constraint fields in the action. There is one gauge invariant scalar lapse function and one gauge invariant shift vector field (with three components), which is usually split into a transverse vector and a (longitudinal) scalar. On-shell all four gauge invariant constraint fields are zero (zero gauge invariant constraint fields solve the corresponding equations of motion). This can also be understood by viewing the lapse and shift fields as auxiliary fields that can be solved for, order by order in perturbation theory, and their solution inserted in the action [1] (see also Ref. [33] for a gauge fixed approach). The gauge invariant lapse and shift functions being zero on-shell corresponds to solving the equation of motion for the lapse and shift function. Solving for the auxiliary fields is hence analogous to decoupling the fields in the action. Because of this decoupling, the considerations of the gauge invariant scalar-tensor theories significantly simplify, in the sense that it suffices to consider the action for the dynamical degrees of freedom only (truncated to a certain order). In Refs. [1, 21, 22], such an action was constructed up to cubic order in the fields for a class of scalar-tensor theories with general \(F(\Phi)\) term in Eq. (2.2). In order to do that, we had to make a choice for gauge invariant variables. The potential complication is the fact that, at second order in gauge transformations there are infinitely many such variables. However, one can show that the form of the cubic actions, when expressed in terms of different gauge invariant variables, differs only by surface terms [1, 22], which are irrelevant for the (bulk) evolution. Nevertheless, these surface terms can be of physical relevance for (the non-Gaussian part of) the initial state. One set of gauge invariant variables is special, in that the variables are also frame independent: they are invariant under a transformation from Jordan to Einstein frame. When expressed in terms of these variables, the cubic actions in the Einstein and Jordan frame are manifestly equal (when one takes account of the trivial frame transformation of the time dependent background quantities that appear as prefactors in front of the cubic vertices).

Therefore, in order to simplify the unitarity cutoff discussion it is natural to work with these unique frame (and gauge) independent variables. They are the gauge invariant scalar perturbation \(W_\zeta\) and the gauge invariant tensor (transverse, traceless graviton) perturbation \(\tilde{\gamma}_{ij}\). These are the second order generalizations of the gauge invariant Sasaki-Mukhanov field, \(w_\zeta = \zeta - (H/\dot{\phi})\varphi\), and the graviton perturbation \(\gamma_{ij}\), where \(H(t) = \dot{a}/a = \bar{N}(t)^{-1}d\ln(a)/dt\) is the Hubble parameter, and \(\phi(t)\) the background (inflaton) field. For the precise definitions of \(W_\zeta\) and \(\tilde{\gamma}_{ij}\) – which are given by \(w_\zeta\) and \(\gamma_{ij}\) plus corrections that are quadratic in perturbations – we refer to Ref. [1]. We have also given an explicit proof that these variables are frame independent, thus \(W_{\zeta,E} = W_\zeta\) and \(\tilde{\gamma}_{ij,E} = \tilde{\gamma}_{ij}\).

The complete action for these gauge and frame independent dynamical fields up to the cubic order for these variable can be found in Refs. [1, 21, 22], which we shall repeat here. The quadratic
The cubic action can be conveniently split into the scalar-scalar-scalar, scalar-scalar-tensor, scalar-tensor-tensor and tensor-tensor-tensor vertices,

\begin{align}
S^{(3)} = S^{(3)}_{W_\zeta W_\zeta W_\zeta} + S^{(3)}_{W_\zeta W_\zeta \gamma} + S^{(3)}_{W_\zeta \gamma \gamma} + S^{(3)}_{\gamma \gamma \gamma}.
\end{align}

Up to boundary terms, the scalar-scalar-scalar part of the action is (see Ref. [22] and Eq. (6.70) of Ref. [20]),

\begin{align}
S^{(3)}_{W_\zeta W_\zeta W_\zeta} &= \int d^3x dt \tilde{N} a^3 F \left\{ \frac{1}{4} \left[ \frac{z^2}{F^2} \left( W_\zeta \left( \tilde{W}_\zeta^2 + \left( \frac{\partial W_\zeta}{a} \right)^2 \right) - 2 \tilde{W}_\zeta \left( \frac{\partial W_\zeta}{\sqrt{2}} \right) \partial_\zeta W_\zeta \right) - \frac{1}{16} \frac{z^6}{F^3} W_\zeta \left( \tilde{W}_\zeta^2 - \left( \frac{\partial \partial_j W_\zeta}{\sqrt{2}} \right) \left( \frac{\partial \partial_j W_\zeta}{\sqrt{2}} \right) \right) \right] + \frac{z^2}{2 F} \left[ \frac{\tilde{z} - \frac{1}{2} \frac{\dot{F}}{F}}{H + \frac{1}{2} \frac{\dot{F}}{F}} \right] \tilde{W}_\zeta W_\zeta^2 \right\},
\end{align}

the scalar-scalar-tensor part of the action is (see Ref. [1] and Eqs. (7.12) and (7.47) of [20]),

\begin{align}
S^{(3)}_{W_\zeta W_\zeta \gamma} &= \int d^3x dt \tilde{N} a^3 \frac{1}{2} \left[ \frac{z^2}{8 F^2} \left( \gamma_{\zeta,ij} \left( \frac{\partial}{a} W_\zeta \right) \left( \frac{\partial}{a} W_\zeta \right) \right) - \frac{z^4}{8 F^2} W_\zeta \gamma_{\zeta,ij} \frac{\partial \partial_j}{\sqrt{2}} \partial_\zeta W_\zeta \right] + \frac{z^4}{8 F} \left( \frac{\partial \partial_j}{\sqrt{2}} \tilde{W}_\zeta \right) \left( \frac{\partial \partial_j}{\sqrt{2}} \tilde{W}_\zeta \right),
\end{align}

the scalar-tensor-tensor part of the action is (see Ref. [1] and Eq. (7.49) of [20]),

\begin{align}
S^{(3)}_{W_\zeta \gamma \gamma} &= \int d^3x dt \tilde{N} a^3 \left\{ \frac{z^2}{16} W_\zeta \left( \tilde{\gamma}_{\zeta,ij} \tilde{\gamma}_{\zeta,ij} + \left( \frac{\partial}{a} \tilde{\gamma}_{\zeta,ij} \right) \left( \frac{\partial}{a} \tilde{\gamma}_{\zeta,ij} \right) \right) - \frac{z^2}{8} \gamma_{\zeta,ij} \left( \frac{\partial \tilde{\gamma}_{\zeta,ij}}{a} \right) \left( \frac{\partial \tilde{\gamma}_{\zeta,ij}}{a} \right) \right\},
\end{align}

and finally the tensor-tensor-tensor part of the action is (see Ref. [1] and Eq. (7.39) of [20]),

\begin{align}
S^{(3)}_{\gamma \gamma \gamma} &= \int d^3x dt \tilde{N} a^3 \frac{F}{8} \left\{ \tilde{\gamma}_{\zeta,ij} \frac{\partial \tilde{\gamma}_{\zeta,kl}}{a} \frac{\partial \tilde{\gamma}_{\zeta,kl}}{a} + \gamma_{\zeta,kl} \left( \frac{\partial \tilde{\gamma}_{\zeta,kl}}{a} \right) \left( \frac{\partial \tilde{\gamma}_{\zeta,kl}}{a} \right) \right\}.
\end{align}
When written in terms of the frame independent variables \( W_\zeta \) and \( \tilde{\gamma}_{ij} \), we proved in [1] that the action (4.7–4.15) is frame independent. Since the frame independent perturbations themselves coincide in Jordan and Einstein frame, \( W_\zeta = W_\zeta,E \) and \( \tilde{\gamma}_{ij} = \tilde{\gamma}_{ij,E} \), the Einstein frame and Jordan frame actions coincide as well. The frame independence becomes manifest when one expresses the background quantities in the Jordan frame \( a, H, z \) in terms of their Einstein frame counterparts in Eqs. (4.7–4.15) by using the relations (2.3) at background level. This gives the following identities

\[
F^{1/2} \tilde{N} = M_P \tilde{N}_E, \quad F^{1/2} a = M_P a_E, \quad F^{-1/2} \tilde{W}_\zeta = M_P^{-1} \tilde{W}_{\zeta,E}, \quad \frac{H + \frac{\dot{E}}{2F}}{M_P} = H_E
\]

\[
\frac{\dot{\phi}^2 + \frac{3}{2} \frac{\dot{F}^2}{F}}{F^2} = \frac{\dot{\phi}_E^2}{M_P^2}, \quad \frac{z}{\sqrt{F}} = \frac{z_E}{M_P}, \quad \frac{\frac{\ddot{z}}{\dot{F}} - \frac{1}{2} \frac{\dot{F}^2}{F}}{H + \frac{\dot{F}}{2F}} = \frac{1}{M_P} \frac{\dot{z}}{z_E}
\]

(4.16)

where all subscripts \( E \) denote quantities in the Einstein frame, \( z_E \equiv \dot{\phi}_E/H_E \) and a dotted derivative denotes \( \dot{X} = dX/(\dot{N}dt) \) and \( \dot{X}_E = dX_E/(\dot{N}_E dt) \) for Jordan and Einstein frame quantities respectively. Hence there is no need to quote the action in the Einstein frame (in which \( z^2/F = z_E^2/M_P^2 = \dot{\phi}_E^2/(M_P^2 H_E^2) \)).

In order to compute the cutoff in a frame independent formulation, it is convenient to transform to canonically normalized variables

\[
V_\zeta = a z W_\zeta = a \sqrt{2\epsilon_E} F W_\zeta, \quad \Gamma_{\zeta,ij} = \frac{1}{2} a \sqrt{F} \tilde{\gamma}_{\zeta,ij},
\]

(4.17)

where we have used

\[
\epsilon_E = -\frac{\dot{H}_E}{H_E^2} = \frac{z_E^2}{2M_P^2} = \frac{z^2}{2F},
\]

(4.18)

which relates \( z \) (defined in Eq. (4.10)) to the slow-roll parameter \( \epsilon_E \) in the Einstein frame. In terms of the canonical fields the second order action (4.7–4.9) in conformal time \( \tau (\tilde{N}(t) \rightarrow a(\tau)) \) become

\[
S^{(2)}_{V_\zeta} = \frac{1}{2} \int d^3xd\tau \left[ \frac{V_{\zeta}^2}{2} - (\partial_i V_\zeta)^2 + \frac{(az)^\nu}{az} V_\zeta \right] - \frac{1}{2} \int d^3x \left[ \frac{(az)^\nu}{az} V_\zeta \right]^2_{\tau_{\text{fin}}},
\]

(4.19)

\[
S^{(2)}_{\Gamma_{\zeta,ij}} = \frac{1}{2} \int d^3xd\tau \left[ \frac{(\partial_i \Gamma_{\zeta,ij})^2}{2} + \frac{(az)^\nu}{a\sqrt{F}} \Gamma_{\zeta,ij} \right] - \frac{1}{2} \int d^3x \left[ \frac{(az)^\nu}{a\sqrt{F}} \Gamma_{\zeta,ij} \right]^2_{\tau_{\text{fin}}},
\]

(4.20)

where a prime denotes a derivative with respect to conformal time \( \tau \). The boundary terms in (4.19–4.20) do not contribute to the propagator equation of motion and thus they can be discarded. Furthermore, the transformation to the canonically normalized fields has led to generation of time dependent (negative) mass terms in (4.19–4.20). However, these terms can be neglected on energy and momentum scales far above the Hubble scale. Namely, \((az)^\nu/(az) = a H + (1/2) \phi^\nu \left[ d\ln (F)/d\phi \right] + (1/2) \epsilon_E^\nu/\epsilon_E \). Now, since the latter two terms are suppressed by slow roll parameters, the first term, \( a H = \mathcal{H} \) (here and subsequently \( \mathcal{H} \) denotes a conformal Hubble rate) is the dominant term. Likewise, one can show that the dominant term in \((az)^\nu/(az) = 2a^2 H^2 \) plus slow roll suppressed terms, such that when the (conformal) energy scale, \( E_c \gg \mathcal{H} \), this term can be neglected. Because \((a \sqrt{F})^\nu/(a \sqrt{F}) = 2a^2 H^2 \) plus slow roll suppressed terms, the same consideration applies to canonically normalized gravitons. This means that the canonically normalized scalar and graviton propagator behave in the ultraviolet (on scales \( E_c = |k^0| \gg \mathcal{H} \) and \( |\vec{k}| \gg \mathcal{H} \)) as, \( \sim 1/(\eta_{\mu\nu} k^\mu k^\nu) \) plus corrections of the order \( \mathcal{H}^2 \), which we shall neglect in the following considerations.
When the cubic action \((4.12-4.15)\) is expressed in terms of the canonical variables \((4.17)\), one gets for the pure scalar cubic action

\[
S_{V_{(3)}}^{(3)} \simeq \int d^3xd\tau \sqrt{\epsilon_E \frac{8a^2F}{\epsilon_a}} \left\{ V_{\xi} \left[ (V_{\xi}')^2 - 2 \frac{(az)'}{az} V_{\xi}' + \frac{[(az)']^2}{(az)^2} V_{\xi}^2 + (\partial_l V_{\xi})^2 \right] \right. \\
- \frac{2}{\sqrt{\epsilon_a}} \left( V_{\xi}' - \frac{(az)'}{az} V_{\xi} \right) \left[ \frac{\partial_l}{\sqrt{\epsilon_a}} \left( V_{\xi}' - \frac{(az)'}{az} V_{\xi} \right) \right] \left( \partial_l V_{\xi} \right) \\
- \frac{\epsilon E}{2} V_{\xi} \left[ \left( V_{\xi}' - \frac{(az)'}{az} V_{\xi} \right)^2 - \frac{\partial_l \partial_j}{\sqrt{\epsilon_a}} \left( V_{\xi}' - \frac{(az)'}{az} V_{\xi} \right) - \frac{\partial_l \partial_j}{\sqrt{\epsilon_a}} \left( V_{\xi}' - \frac{(az)'}{az} V_{\xi} \right) \right] \\
+ \frac{1}{\epsilon E} \left( \frac{\epsilon E}{2H} \right) \left( V_{\xi}' - \frac{(az)'}{az} V_{\xi} \right) \left( V_{\xi}' - \frac{(az)'}{az} V_{\xi} \right) \right\}. \tag{4.21}
\]

Next, the scalar-scalar-tensor cubic action \((4.13)\) can be written as,

\[
S_{V_{(3)}V_{(3)}}^{(3)} = \int d^3xd\tau \left\{ \frac{1}{a\sqrt{F}} \Gamma_{\xi,i} \left( \partial_j V_{\xi} \right) \left( \partial_j V_{\xi} \right) \right. \\
- \frac{z^2}{4aF^{3/2}} V_{\xi} \left( \Gamma_{\xi,i} - \frac{(a\sqrt{F})'}{a\sqrt{F}} \Gamma_{\xi,i} \right) \left[ \frac{\partial_l \partial_j}{\sqrt{\epsilon_a}} \left( V_{\xi}' - \frac{(az)'}{az} V_{\xi} \right) \right] \\
+ \frac{z^2}{4aF^{3/2}} \left[ \frac{\partial_l \partial_j}{\sqrt{\epsilon_a}} \left( V_{\xi}' - \frac{(az)'}{az} V_{\xi} \right) \right] \left[ \frac{\partial_l \partial_j}{\sqrt{\epsilon_a}} \left( V_{\xi}' - \frac{(az)'}{az} V_{\xi} \right) \right] \left( \partial_l \Gamma_{\xi,i,j} \right) \right\}, \tag{4.22}
\]

the scalar-tensor-tensor part of the action \((4.14)\) becomes,

\[
S_{V_{(3)}\Gamma_{(3)}}^{(3)} = \int d^3xd\tau \left\{ \frac{z}{4aF} V_{\xi} \left[ \left( \Gamma_{\xi,i} - \frac{(a\sqrt{F})'}{a\sqrt{F}} \Gamma_{\xi,i} \right) \left( \Gamma_{\xi,i} - \frac{(a\sqrt{F})'}{a\sqrt{F}} \Gamma_{\xi,i} \right) + \left( \partial_l \Gamma_{\xi,i,j} \right) \left( \partial_l \Gamma_{\xi,i,j} \right) \right] \right. \\
- \frac{z}{2aF} \left( \Gamma_{\xi,i} - \frac{(a\sqrt{F})'}{a\sqrt{F}} \Gamma_{\xi,i} \right) \left( \partial_l \Gamma_{\xi,i,j} \right) \left[ \frac{\partial_l \partial_j}{\sqrt{\epsilon_a}} \left( V_{\xi}' - \frac{(az)'}{az} V_{\xi} \right) \right] \right\}, \tag{4.23}
\]

and finally the pure tensor part of the action \((4.15)\) becomes

\[
S_{\Gamma_{(3)}\Gamma_{(3)}}^{(3)} = \int d^3xd\tau \left\{ \frac{1}{a\sqrt{F}} \left[ \left( \partial_l \Gamma_{\xi,k,l} \right) \left( \partial_l \Gamma_{\xi,i,l} \right) + \left( \partial_l \Gamma_{\xi,k,l} \right) \left( \partial_l \Gamma_{\xi,i,k} \right) \right] \left( \partial_l \Gamma_{\xi,i,j} \right) \left( \partial_l \Gamma_{\xi,i,k} \right) \right\}. \tag{4.24}
\]

We shall use the cubic action \((4.21-4.24)\) in section 5 to analyse the unitarity cutoff implied by the \(2 \rightarrow 2\) tree-level scattering processes. Of course, to complete the analysis, one would also need the gauge invariant quartic vertices, which are currently not available. We do not expect however, that quartic vertices will change in any way the qualitative discussion we present in the next section.

5 The cutoff scale revisited

A usual requirement for renormalizable theories is that the unitarity bound is not violated in perturbation theory, i.e. scattering amplitudes should not become bigger than unity. In particular there is the requirement of tree unitarity \([31]\), which states that \(N\) particle tree amplitudes \(A_N\) should not grow more rapidly than \(E^{4-N}\), where \(E\) is the center of mass energy. If the amplitude grows faster, perturbation theory fails at some cutoff scale \(\Lambda\). This usually means that some new physics should
enter at this energy scale. When the relevant energy scales of the theory under consideration are well below the cutoff scale, the theory can be considered 'natural', in the sense that perturbation theory is valid and there is no need for new physics. Conversely, there is a naturalness problem if typical energy scales are higher than the cutoff scale.

The cutoff is most easily computed when the perturbative action is written in canonical form, such that the propagator goes as $1/k^2$, where $k^2 = \eta_{\mu\nu}k^\mu k^\nu$ and $k^\mu$ is a conformal 4-momentum. Next, the cutoff can be read off from the vertices of dimension higher than 4, which should be suppressed as $\Lambda^{-D}$. This has been done for Higgs inflation in Ref. [1] in both the Einstein and Jordan frame, and we outline the derivation here. In the Jordan frame the starting point is the action (2.2) with $F(\Phi) = M_P^2 + \xi \Phi^2$, which is the action for Higgs inflation in unitary gauge and gauge interactions are neglected. Next, following Ref. [32], and analogously as it was done in (4.17), generic (non-invariant) perturbations $\delta g_{\mu\nu} = \bar{g}_{\mu\nu} - \hat{g}_{\mu\nu}$ and $\varphi = \Phi(x) - \phi(t)$ can be rescaled to canonical variables $\hat{g}_{\mu\nu}$ and $\hat{\varphi}$ for which the corresponding quadratic actions are canonically normalized. Here $\bar{g}_{\mu\nu}$ denotes a background metric (which in the UV can be approximated by Minkowski metric) and $\phi(t)$ is a background field. The dominant term in the action with dimension higher than 4 and in the Jordan frame is of the order

$$\xi \varphi^2 \Box \delta g,$$

(5.1)

where $\delta g = \bar{g}^{\mu\nu} \delta g_{\mu\nu}$. When reexpressed in terms of the canonically normalized fields $\hat{g}_{\mu\nu}$ and $\hat{\varphi}$, this term becomes

$$\frac{\xi \sqrt{M_P^2 + \xi \phi^2}}{M_P^2 + \xi \phi^2 + 6\xi^2 \phi^2} \hat{\varphi}^2 \Box \delta \hat{g}.$$

(5.2)

At high energies this vertex scales as $E^2$, where $E = |k^0|$ denotes the energy scale. If we consider a $2 \rightarrow 2$ scattering process of $\hat{\varphi}$ via exchange of a gravitational scalar $\delta \hat{g}$, we see that the total amplitude scales as $E^2/\Lambda^2$ at high energies. The cutoff scale is precisely the inverse of the operator above. The cutoff in the Jordan frame is thus

$$\Lambda \sim \frac{M_P^2 + \xi \phi^2 + 6\xi^2 \phi^2}{\xi \sqrt{M_P^2 + \xi \phi^2}}.$$

(5.3)

In Ref. [18] the cutoff was also computed via the Einstein frame. In that frame the non-minimal coupling term is absent and the gravitational and field kinetic terms are canonical. Still, the cutoff scale reappears in the non-polynomial potential and shows a similar behavior as above (though not exactly equal). From (5.3) one sees that $\Lambda \propto \sqrt{\xi} \hat{\varphi}$ in the regime where $\hat{\varphi} \gg M_P/\sqrt{\xi}$, $\Lambda \sim \xi \hat{\varphi}^2/M_P$ for $M_P/\sqrt{\xi} \gg \phi \gg M_P/\xi$ and $\Lambda \sim M_P/\xi$ when $\phi \ll M_P/\xi$. The authors of Ref. [32] then argue that all relevant energy scales in these regimes are lower than the cutoff scale, such that the perturbative expansion is valid and Higgs inflation is natural.

These results are interesting, but the question arises whether they can be trusted. In the following we point at the principal potential problems with the analysis outlined above, which put into question the reliability of the cutoff scale in Eq. (5.3)

(a) The computation of the cutoff is gauge dependent.

Both the metric fluctuations $\delta g_{\mu\nu}$ and scalar field perturbation $\varphi$ are gauge dependent, hence the vertex (5.1) is also gauge dependent, and has on its own no physical meaning. Indeed, it is well known that, when working with gauge dependent quantities, one can reach realizable conclusions only when one takes account of all terms up to this order. Namely, different gauge dependent terms can cancel each other, which was not accounted for in Ref. [32], see e.g. Ref. [19]. Furthermore, when written in terms of gauge invariant variables, gauge dependent vertices (such
as \( \varphi^2 \triangleq \delta g \) can be absorbed into the lower order action, such that they simply ‘disappear’ from a gauge invariant action. Next, some metric perturbations are non-dynamical in the sense that they act as auxiliary (constraint) fields and should be solved for, which generate additional vertices that may cancel the problematic vertex. In fact, this already happens at the level of the quadratic action. Naïvely, the field perturbation has an effective mass term 
\[
m^2_{\text{eff}} \varphi^2 = (-\xi \bar{R} + V'') \varphi^2 + \ldots\]
where \( \bar{R} = 6(2H^2 + \dot{H}) \) is the background Ricci scalar. Such a mass term is huge, in the sense that \( |\xi \bar{R}| \gg H^2 \). However, only a light inflaton field, with \( m^2_{\text{eff}} \ll H^2 \) can generate a nearly scale invariant power spectrum. Thus, such a mass term is disastrous for the model and naïvely rules out Higgs inflation. But, when contributions from the auxiliary fields are taken into account, the problematic \( \xi \bar{R} \) contribution gets canceled, leaving only a light effective mass for the inflaton field. Similar cancellations occur at higher orders in a gauge invariant formulation.

b) The canonical redefinition mixes up frames.

A crucial step in the computation of the cutoff was the definition of new perturbations \( \hat{\varphi} \) and \( \delta \hat{g} \) which canonically normalize the kinetic terms. However, this redefinition is in fact nothing more than a transformation from the Jordan to the Einstein frame at the level of perturbations.\(^4\) On the other hand, the cutoff is computed from the term \( \xi \Phi^2 R \) in the Jordan frame action. So somehow one computes a cutoff from a Jordan frame vertex using Einstein frame perturbations, which is very bizarre. Moreover, we would like to emphasize that, although the Einstein frame is often referred to as the frame in which both the gravitational action and scalar field action are written in canonical form, \( \textit{the Einstein frame is not canonical}. \) Canonical formulation means that the kinetic sectors for different fields are decoupled (and canonically normalized), such that one can straightforwardly extract the canonical momentum and quantize the theory. However, in the "canonical" Einstein frame the gravitational field still couples to the scalar sector as \( \sqrt{-g_E} g^E_{\mu \nu} \partial_\mu \Phi_E \partial_\nu \Phi_E \). Conversely, the scalar field still couples to the kinetic term for the metric perturbations in the \( \sqrt{-g_E} R_E \) term via the auxiliary fields in the metric (which contain \( \varphi_E \) in its first order solution). Hence, the Einstein frame is formulated in a non-canonical way, just like the Jordan frame. The true canonical formulation is only reached once one inserts perturbations and decouples physical and non-physical degrees of freedom, which was done in Ref. [1], and the results of which are summarized in section 4. When using the frame independent variables \( W_\zeta \) and \( \tilde{\gamma}_{ij,E} \) the Jordan frame quadratic and cubic actions becomes manifestly equal to the Einstein frame actions, see Eq. (4.16). Thus in a frame independent formulation the notion of frames becomes meaningless.

c) Inequivalence of Jordan and transformed Einstein cutoff.

As we have mentioned, the cutoffs computed directly in the Jordan frame, or via the Einstein frame, are similar, but not exactly the same. However, they should coincide, because no physical content is lost in the frame transformation. Of course the origin of this problem is related to the already mentioned problems, namely a non-invariant formulation and a mixing-up of different frames.

\(^4\)This can be explicitly checked by comparing Eqs. (2.9) and (2.10) in Ref. [18] to the transformations

\[
\varphi_E = \frac{d \varphi_E}{d \varphi} \varphi + O(\varphi^2), \quad \zeta_E = \zeta + \frac{1}{2F} \frac{d F}{d \varphi} \varphi + O(\varphi^2), \quad \gamma_{ij,E} = \gamma_{ij}
\]

and the expansion of \( g_{\mu \nu,E} = \Omega^2 g_{\mu \nu} \) to first order in perturbations.
5.1 Frame independent computation of cutoff

This subsection we show that the above problems become all obsolete once the theory is written in a manifestly gauge invariant and frame independent way. Firstly, when one makes use of gauge invariant variables, all vertices are physical vertices. Moreover, the gauge invariant perturbations decouple in the quadratic action, which makes it very easy to write the theory in canonically normalized form. And obviously, when frame independent perturbations are used, results in the Jordan and Einstein frames become manifestly equivalent.

As mentioned above, a simple way of estimating the unitarity cutoff scale for tree tree-level scattering amplitudes is to work with the canonically normalized fields (4.17), for which the propagator in the ultraviolet acquires a simple form:
\[ \sim (\eta_{\mu\nu} k^\mu k^\nu)^{-1} \]
plus corrections that are suppressed by
\[ H^2 = (aH)^2, \]
where \( k^\mu = (E_c/c, \vec{k}) \) denotes a conformal energy and momentum (the corresponding physical energy and momentum are given by \( E = E_c/a \) and \( \vec{k}/a \)). Up to boundary terms, the cubic actions for various combinations of scalar and tensor vertices are given in Eqs. (4.21–4.24).

Let us first consider the scalar cubic action (4.21). The cubic vertex \( V \) from the first two lines is of the order,
\[ \text{Scalar cubic vertex : } V \sim \epsilon E \max[E_c^2, \|\vec{k}\|^2] / a^{3/2}, \]
where we made use of \( \partial \rightarrow E_c, \partial_i \rightarrow k_i \) and we took ultraviolet limit in which \( \epsilon \leq E_c, \|\vec{k}\| \). The terms in the third line in (4.21) lead to a vertex that is in addition suppressed by \( \epsilon \), which is during inflation less than unity, and hence can be neglected. Finally, the vertex contribution from the fourth line is of the order \( \epsilon^{-1/2} \epsilon E / (\epsilon E) (E_c) \), that means it is proportional to a third order slow roll parameter, \( \epsilon^{(3)} = [\epsilon_E / (\epsilon_E H)] / \epsilon_E H \), and which can be assumed to be small during inflation. More precisely, this vertex contribution is suppressed with respect to (5.4) if
\[ \max[E_c^2, \|\vec{k}\|^2] / E_c \gg [\epsilon_E / (\epsilon_E H)] / \epsilon_E \]
which one can safely assume to be the case throughout inflation. On the other hand, we know that in four space-time dimensions a 2-to-2 scattering amplitude scales as,
\[ \max[E_c^2, \|\vec{k}\|^2] \sim \max[E_c^2, \|\vec{k}\|^2]. \]
Upon combining this with (5.4) we finally get for the physical cutoff in the Jordan frame,
\[ \text{Scalar cubic vertex : } \left(\frac{\Lambda}{a}\right)_J \sim \sqrt{\frac{M_P^2 + \xi \phi^2}{\epsilon_E}} \geq \sqrt{M_P^2 + \xi \phi^2}, \]
where we took account of \( F = M_P^2 + \xi \phi^2 \) and \( \epsilon_E \ll 1 \) during inflation. In conclusion, we have found that during entire Higgs inflation for scatterings mediated by the scalar cubic interactions the unitarity bound is above the Planck scale. In fact, the cutoff in Higgs inflation is higher than the unitarity cutoff in the minimally coupled inflation, \( \sim M_P \), which is obtained by simply setting \( \xi \rightarrow 0 \) in (5.6).

Now making use of (4.16), from which we see that
\[ a_E = a_F \frac{F^{1/2}}{M_P}, \]
the cutoff (5.6) can be written in the Einstein frame as
\[ \left(\frac{\Lambda}{a}\right)_E \sim \frac{M_P}{\epsilon_E} \geq M_P. \]
The difference between the frames can be attributed to the frame dependence of the physical cutoff scale $\Lambda/a$, and has no physical meaning. Therefore, to make a meaningful comparison of the Einstein and Jordan frame cutoffs, the rescaling (5.7) has to be taken account of. Hence, we conclude that there is a perfect agreement in the cutoff scale in two different frames, and in both frames the physical cutoff is above the Planck scale $M_P = 1/(8\pi G_N)^{1/2}$.

Let us now turn our attention to other cubic vertices. From Eqs. (4.22–4.24) we obtain the dominant contributions for the other types of vertices,

\[
\begin{align*}
\text{Scalar} - \text{scalar} - \text{tensor vertex} & : \quad V_{\xi\xi\gamma} \sim \frac{1}{aF^{1/2}} \times \max \left[ \epsilon_E E_c^2, \|\vec{k}\|^2 \right] \\
\text{Scalar} - \text{tensor} - \text{tensor vertex} & : \quad V_{\xi\gamma\gamma} \sim \frac{\epsilon_E^{1/2}}{aF^{1/2}} \times \max \left[ E_c^2, \|\vec{k}\|^2 \right] \\
\text{Pure tensor vertex} & : \quad V_{\gamma\gamma\gamma} \sim \frac{\|\vec{k}\|^2}{aF^{1/2}} \quad (5.9)
\end{align*}
\]

When these are inserted into (5.5) one obtains the following results for the cutoff scales from different types of vertices,

\[
\begin{align*}
\text{Scalar} - \text{scalar} - \text{tensor vertex} & : \quad \left( \frac{\Lambda}{a} \right)_J \sim \sqrt{M_P + \xi \phi^2} \times \min \left[ \epsilon_E, 1 \right] \gtrsim \sqrt{M_P + \xi \phi^2} \\
\text{Scalar} - \text{tensor} - \text{tensor vertex} & : \quad \left( \frac{\Lambda}{a} \right)_J \sim \sqrt{M_P + \xi \phi^2} \times \min \left[ \epsilon_E^{1/2}, 1 \right] \gtrsim \sqrt{M_P + \xi \phi^2} \quad (5.10) \\
\text{Pure tensor vertex} & : \quad \left( \frac{\Lambda}{a} \right)_J \sim \sqrt{M_P + \xi \phi^2} \times \min \left[ E_c^2/\|\vec{k}\|^2, 1 \right] \gtrsim \sqrt{M_P + \xi \phi^2},
\end{align*}
\]

where the first expression in square brackets gives a cutoff for the case when $E_c \gg \|\vec{k}\|$, and the latter for the case when $E_c \ll \|\vec{k}\|$. Analogous conclusions as in (5.10) are reached when one considers $2 \to 2$ scatterings composed by two classes of vertices, e.g. a combination of a scalar-scalar-graviton and a pure graviton vertex.

In summary, we have shown that in all cases, for all kinds of vertices and throughout inflation the physical cutoff in the Jordan frame is above the scale $\sqrt{M_P + \xi \phi^2}$, while in the Einstein frame it is above $M_P$ (the difference has no physical meaning and it is attributed to the non-invariant definition of the physical cutoff in the two frames) \(^5\).

The Jordan frame cutoff is shown in figure 1. The important point is that also here the cutoff is never smaller than the Planck scale. This means that the perturbative expansion is valid at least up to an energy scale of the order the Planck scale for a theory with some non-minimal coupling to gravity, such as Higgs inflation. Thus, the above analysis strongly suggests that, at least for the class of tree level $2 \to 2$ scattering processes considered here, there is no naturalness problem in Higgs inflation.

There are caveats to this statement however. Firstly, we have only considered the scalar-gravity sector of Higgs inflation, but neglected interactions with e.g. gauge fields. In Ref. [16] it was stated that the (low) cutoff scale of $M_P/\xi$ also appears in the Higgs-gauge interactions. However, these vertices are gauge dependent as well, so the problem may be absent in a gauge invariant formulation.

---

\(^5\)It comes as no surprise that the canonically normalized scalar-scalar-graviton and pure graviton vertices are not suppressed by a factor of $\sqrt{\epsilon_E}$. The reason is that in the de Sitter limit $\epsilon_E \to 0$ the curvature perturbation $\zeta$ becomes a pure gauge mode, and is completely absorbed by the gauge invariant lapse and shift perturbations. The only remaining dynamical perturbations are the scalar $\varphi$ and the graviton $\gamma_{ij}$. The term $g^{\mu\nu} \partial_\mu \varphi \partial_\nu \Phi$ in the original action then gives the interaction term $\gamma_{ij} \partial_i \varphi \partial_j \varphi$, which is not $\epsilon_E$ suppressed. Likewise, the pure graviton vertices are always present and are not suppressed by powers of $\epsilon_E$. 

---
that includes the gauge fields. Secondly, for the computation of the cutoff we have used the partially integrated actions (4.21–4.24). Had we not made use of partial integrations, we would have found disastrous terms in the action. For example, before any partial integration, the leading term in the pure cubic scalar action for \( V_\zeta = W_\zeta/(az) \) from Ref. [1] contributes (in the limit \( E_c \gg \|\vec{k}\|\)) to the cubic vertex as, \((\epsilon_E F/a) \times \max[E_c^2, \|\vec{k}\|^2]\), implying a physical cutoff (in the Jordan frame), \((\Lambda/a)_J \sim \sqrt{\epsilon_E(M_P^2 + \phi^2)} \times H/E_c\), which is much below the Planck scale, and hence disastrous. Similar problems occur when the scalar-graviton-graviton vertices before partial integrations are considered. Therefore, for a more complete understanding of naturalness it is of crucial importance to understand the role of the boundary terms (on equal time hypersurfaces).

6 Discussion

We have used the frame and gauge independent formalism for scalar and tensor cosmological perturbations of Ref. [1] to show that the physical cutoff for 2-to-2 tree level scatterings in Higgs inflation is above the Planck scale \( M_P = 1/\sqrt{8\pi G_N} \) throughout inflation. More precisely, we found that in the Jordan frame, the physical cutoff scale is \((\Lambda/a)_J \gtrsim \sqrt{M_P^2 + \phi^2}\), while in the Einstein frame it is \((\Lambda/a)_E \gtrsim M_P\), where \(\xi\) is the nonminimal coupling and \(\phi(t)\) denotes the Higgs vev. The physical cutoff in the Jordan frame is illustrated in figure 1. The difference between the two frames is immaterial in that it can be fully attributed to the frame dependence of the (physical) cutoff, see Eq. (5.7). Our results are incomplete, in that we have not discussed the relevance of:
• quartic vertices and loops,
• vertices containing gauge and fermionic fields,
• boundary terms on equal time hypersurfaces (that result from partial integrations),

for the question of naturalness in Higgs inflation. We do however believe that the principal conclusion reached in this paper will not change when these contributions are fully accounted for.

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