ON THE CAUCHY PROBLEM FOR STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS IN THE SCALE SPACES OF GENERALIZED SMOOTHNESS

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Abstract. Parabolic integro-differential nondegenerate Cauchy problem is considered in the scale of $L_p$ spaces of functions whose regularity is defined by a Levy measure with O-regularly varying radial profile. Existence and uniqueness of a solution is proved by deriving apriori estimates. Some probability density function estimates of the associated Levy process are used as well.

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1. Introduction

Let $\sigma \in (0, 2)$ and $\mathfrak{N}^\sigma$ be the class of all nonnegative measures $\nu$ on $R^d_0 = R^d \setminus \{0\}$ such that $\int |y|^2 \wedge 1 d\nu < \infty$ and

$$\sigma = \inf \left\{ \alpha < 2 : \int_{|y| \leq 1} |y|^\alpha d\nu < \infty \right\}.$$
In addition, we assume that for $\nu \in \mathfrak{A}^\sigma$,

$$
\int_{|y| > 1} |y| \, d\nu < \infty \text{ if } \sigma \in (1, 2),
$$

$$
\int_{R < |y| \leq R'} y \, d\nu = 0 \text{ if } \sigma = 1 \text{ for all } 0 < R < R' < \infty.
$$

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration of $\sigma-$algebras on $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ satisfying the usual conditions. Let $\mathcal{R}(\mathbb{F})$ be the progressive $\sigma-$algebra on $[0, \infty) \times \Omega$. Let $(U, \mathcal{U}, \Pi)$ be a measurable space with $\sigma-$finite measure $\Pi, \mathcal{R}_0^d = \mathbb{R}^d \setminus \{0\}$. Let $p(dt, dz)$ be $\mathbb{F}$-adapted point measures on $([0, \infty) \times U, \mathcal{B}([0, \infty)) \otimes \mathcal{U})$ with compensator $\Pi(]dz) \, dt$. We denote the martingale measure $q(dt, dz) = p(dt, dz) - \Pi(]dz) \, dt$.

In this paper we consider the stochastic parabolic Cauchy problem

$$
(1.1) \quad du(t, x) = [-Lu(t, x) - \lambda u(t, x) + f(t, x)] \, dt + \int_{U} \Phi(t, x, z) q(dt, dz),
$$

$$
u(0, x) = g(x), \quad t \geq 0, x \in \mathbb{R}^d,
$$

with $\lambda \geq 0$ and integro-differential operator

$$
L \varphi(x) = L^\nu \varphi(x) = \int [\varphi(x + y) - \varphi(x) - \chi_{\sigma}(y) \cdot \nabla \varphi(x)] \, d\nu(y), \varphi \in C_0^\infty(\mathbb{R}^d),
$$

where $\nu \in \mathfrak{A}^\sigma$, $\chi_{\sigma}(y) = 0$ if $\sigma \in [0, 1)$, $\chi_{\sigma}(y) = 1_{\{|y| \leq 1\}}(y)$ if $\sigma = 1$ and $\chi_{\sigma}(y) = 1$ if $\sigma \in (1, 2)$. The symbol of $L$ is

$$
\psi(\xi) = \psi^\nu(\xi) = \int \left[e^{i2\pi \xi \cdot y} - 1 - 2i\pi \chi_{\sigma}(y) \xi \cdot y\right] \, d\nu(y), \xi \in \mathbb{R}^d.
$$

Note that $\nu(\mathbb{F}^\sigma) = dy/|y|^{d+\sigma} \in \mathfrak{A}^\sigma$ and, in this case, $L = L^\nu = c(\sigma, d)(-\Delta)^{\sigma/2}$, where $(-\Delta)^{\sigma/2}$ is a fractional Laplacian. The equation (1.1) is forward Kolmogorov equation for the Levy process associated to $\psi^\nu$. We assume that $g, f$ and $\Phi$ are resp. $\mathcal{F}_0 \otimes \mathcal{B}(\mathbb{R}^d)$-$\mathcal{F}_0 \otimes \mathcal{B}(\mathbb{R}^d)$-$\mathcal{F}$ measurable. $\Phi$ is $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$-measurable.

We define for $\nu \in \mathfrak{A}^\sigma$ its radial distribution function

$$
\delta(r) = \delta_\nu(r) = \nu \left( x \in \mathbb{R}^d : |x| > r \right), r > 0,
$$

and

$$
w(r) = w_\nu(r) = \delta_\nu^{-1}(r), r > 0.
$$

Let $\delta$ be continuous, and $\lim_{r \to 0} \delta(r) = \infty$. Our main assumption is that $w_\nu(r)$ is an $O$-$RV$ function at both infinity and at zero. That is

$$
r_1(x) = \lim_{\epsilon \to 0} \sup \frac{w_\nu(ex)}{w_\nu(\epsilon)} < \infty, \quad r_2(x) = \lim_{\epsilon \to \infty} \sup \frac{w_\nu(ex)}{w_\nu(\epsilon)} < \infty, x > 0.
$$

The purpose of this paper is twofold. First, it can be viewed as a generalization of the results in [7, 8] and [9]. Second, the results are complement
of those in [6] where the solutions of generalized Hölder class were discussed. Quasilinear and variable coefficient models will be discussed in a forthcoming paper.

2. Notations, Function Spaces, Main Results and Examples

2.1. Notation. The following notation will be used in the paper.

Let $\mathbb{N} = \{1, 2, \ldots\}, \mathbb{N}_0 = \{0, 1, \ldots\}, \mathbb{R}_0^d = \mathbb{R}^d \backslash \{0\}$. If $x, y \in \mathbb{R}^d$, we write

$$x \cdot y = \sum_{i=1}^{d} x_i y_i, \quad |x| = \sqrt{x \cdot x}.$$ 

We denote by $C_0^\infty(\mathbb{R}^d)$ the set of all infinitely differentiable functions on $\mathbb{R}^d$ with compact support.

We denote the partial derivatives in $x$ of a function $u(t, x)$ on $\mathbb{R}^{d+1}$ by $\partial_i u = \partial u / \partial x_i$, $\partial^2_{ij} u = \partial^2 u / \partial x_i \partial x_j$, etc.; $Du = \nabla u = (\partial^1 u, \ldots, \partial^d u)$ denotes the gradient of $u$ with respect to $x$; for a multiindex $\gamma \in \mathbb{N}_0^d$ we denote

$$D^\gamma_x u(t, x) = \frac{\partial |\gamma| u(t, x)}{\partial x_1^{\gamma_1} \ldots \partial x_d^{\gamma_d}}.$$ 

For $\alpha \in (0, 2]$ and a function $u(t, x)$ on $\mathbb{R}^{d+1}$, we write

$$\partial^\alpha u(t, x) = -F^{-1}[[|\xi|^\alpha F u(t, \xi)](x),$$

where

$$F h(t, \xi) = \hat{h}(\xi) = \int_{\mathbb{R}^d} e^{-i2\pi \xi \cdot x} h(t, x) dx, F^{-1} h(t, \xi) = \int_{\mathbb{R}^d} e^{i2\pi \xi \cdot x} h(t, \xi) d\xi.$$ 

For $\nu \in \mathcal{A}'$, we denote $Z^\nu_t, t \geq 0$, the Levy process associated to $L^\nu$, i.e., $Z^\nu_t$ is cadlag with independent increments and its characteristic function

$$\mathbb{E} e^{i2\pi \xi \cdot Z^\nu_t} = \exp \{ \psi^\nu(\xi) t \}, \xi \in \mathbb{R}^d, t \geq 0.$$ 

The letters $C = C(\cdot, \ldots, \cdot)$ and $c = c(\cdot, \ldots, \cdot)$ denote constants depending only on quantities appearing in parentheses. In a given context the same letter will (generally) be used to denote different constants depending on the same set of arguments.

2.2. Function Spaces. Let $S(\mathbb{R}^d)$ be the Schwartz space of real-valued rapidly decreasing functions. Let $V$ be a Banach space with norm $|\cdot|_V$. The space of $V$-valued tempered distribution we denote by $S'(\mathbb{R}^d, V) (f \in S'(\mathbb{R}^d, V)$ is a continuous $V$-valued linear functional on $S(\mathbb{R}^d))$. If $V = \mathbb{R}$, we write $S'(\mathbb{R}^d, V) = S'(\mathbb{R}^d)$ and denote by $\langle \cdot, \cdot \rangle$ the duality between $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$.

For a $V$-valued measurable function $h$ on $\mathbb{R}^d$ and $p \geq 1$ we denote

$$|h|_{V,p}^p = \int_{\mathbb{R}^d} |h(x)|_V^p dx.$$
We fix $\nu \in \mathbb{R}^\sigma$. Obviously, $\text{Re} \psi^p = \psi^{p,\text{sym}}$, where

$$\nu^{\text{sym}} (dy) = \frac{1}{2} [\nu (dy) + \nu (-dy)].$$

Let

$$Jf = J_\nu f = (I - L^{\nu^{\text{sym}}}) f = f - L^{\nu^{\text{sym}}} f, f \in \mathcal{S} \left( \mathbb{R}^d, V \right).$$

For $s \in \mathbb{R}$ set

$$J^s f = (I - L^{\nu^{\text{sym}}})^s f = \mathcal{F}^{-1} [(1 - \psi^{p,\text{sym}})^s \hat{f}], f \in \mathcal{S} \left( \mathbb{R}^d, V \right).$$

We fix $\nu \in \mathbb{R}$, $\nu \in \mathcal{D}$, following [4], the Bessel potential space $H^s_p (\mathbb{R}^d, V) = H^{p,s}_p (\mathbb{R}^d, V)$ as the closure of $\mathcal{S} (\mathbb{R}^d, V)$ in the norm

$$|f|_{H^s_p (\mathbb{R}^d, V)} = |J^s f|_{L^p (\mathbb{R}^d, V)} = \left| \mathcal{F}^{-1} [(1 - \psi^{p,\text{sym}})^s \hat{f}] \right|_{L_p (\mathbb{R}^d, V)} = \left| (I - L^{\nu^{\text{sym}}})^s f \right|_{L_p (\mathbb{R}^d, V)}, f \in \mathcal{S} (\mathbb{R}^d, V).$$

According to Theorem 2.3.1 in [4], $H^t_p (\mathbb{R}^d) \subseteq H^s_p (\mathbb{R}^d)$ is continuously embedded if $p \in (1, \infty), s, t \in \mathbb{R}, H^0_p (\mathbb{R}^d) = L_p (\mathbb{R}^d)$. For $s \geq 0, p \in [1, \infty)$, the norm $|f|_{H^s_p}$ is equivalent to (see Theorem 2.2.7 in [3])

$$|f|_{H^s_p} = |f|_{L_p} + \left| \mathcal{F}^{-1} [(-\psi^{p,\text{sym}})^s \mathcal{F} f] \right|_{L_p}.$$

Further, for a characterization of our function spaces we will use the following construction (see [2]).

**Remark 1.** For an integer $N > 1$ there exists a function $\phi = \phi^N \in C^\infty_0 (\mathbb{R}^d)$ (see Lemma 6.1.7 in [2]), such that $\text{supp} \ \phi = \{ \xi : \frac{1}{N} \leq |\xi| \leq N \}, \ \phi (\xi) > 0$ if $N^{-1} < |\xi| < N$ and

$$\sum_{j=-\infty}^{\infty} \phi (N^{-j} \xi) = 1 \quad \text{if} \ \xi \neq 0.$$

Let

$$\tilde{\phi} (\xi) = \phi (N \xi) + \phi (\xi) + \phi (N^{-1} \xi), \xi \in \mathbb{R}^d.$$  

Note that $\text{supp} \ \tilde{\phi} \subseteq \{ N^{-2} \leq |\xi| \leq N^2 \}$ and $\tilde{\phi} \phi = \phi$. Let $\varphi_k = \varphi^N_k = \mathcal{F}^{-1} \phi (N^{-k}), k \geq 1$, and $\varphi_0 = \varphi^N_0 \in \mathcal{S} (\mathbb{R}^d)$ is defined as

$$\varphi_0 = \mathcal{F}^{-1} \left[ 1 - \sum_{k=1}^{\infty} \phi (N^{-k}) \right].$$
Let \( \phi_0(\xi) = F\varphi_0(\xi) \), \( \hat{\phi}_0(\xi) = F\varphi_0(\xi) + F\varphi_1(\xi) \), \( \xi \in \mathbb{R}^d \), \( \varphi = F^{-1}\hat{\phi} \), \( \varphi \in F^{-1}\phi \), and

\[
\bar{\varphi}_k = \sum_{l=-1}^{1} \varphi_{k+l}, k \geq 1, \bar{\varphi}_0 = \varphi_0 + \varphi_1
\]

that is

\[
F\bar{\varphi}_k = \phi\left(N^{-k+1}\xi\right) + \phi\left(N^{-k}\xi\right) + \phi\left(N^{-k-1}\xi\right)
\]

\[
= \bar{\phi}\left(N^{-k}\xi\right), \xi \in \mathbb{R}^d, k \geq 1.
\]

Note that \( \bar{\varphi}_k = \varphi_k \ast \varphi_k, k \geq 0 \). Obviously, \( f = \sum_{k=0}^{\infty} f \ast \varphi_k \) in \( \mathcal{S}' \left( \mathbb{R}^d \right) \) for \( f \in \mathcal{S} \left( \mathbb{R}^d \right) \).

Let \( s \in \mathbb{R} \) and \( p, q \geq 1 \). For \( \nu \in \mathfrak{A}_\sigma \), we introduce the Besov space \( B_{pq}^s = B_{pq}^{s;\mathfrak{A}_\sigma}(\mathbb{R}^d, V) \) as the closure of \( \mathcal{S} \left( \mathbb{R}^d, V \right) \) in the norm

\[
|f|_{B_{pq}^s(\mathbb{R}^d, V)} = |f|_{B_{pq}^{s;\mathfrak{A}_\sigma}(\mathbb{R}^d, V)} = \left( \sum_{j=0}^{\infty} |J^s \varphi_j \ast f|^q_{L_p(\mathbb{R}^d, V)} \right)^{1/q},
\]

where \( J = J_\nu = I - L^{\nu_{\text{sym}}} \).

We introduce the corresponding spaces of generalized functions on \( E = [0, T] \times \mathbb{R}^d \). The spaces \( B_{pq}^{\mu;\mathfrak{A}_\sigma}(E, V) \) (resp. \( H_{\mu;\mathfrak{A}_\sigma}(E, V) \)) consist of all measurable \( B_{pq}^{\mu;\mathfrak{A}_\sigma}(\mathbb{R}^d, V) \) (resp. \( H_{\mu;\mathfrak{A}_\sigma}(\mathbb{R}^d, V) \) -valued functions \( f \) on \([0, T]\) with finite corresponding norms:

\[
|f|_{B_{pq}^s(E, V)} = |f|_{B_{pq}^{s;\mathfrak{A}_\sigma}(E, V)} = \left( \int_0^T |f(t, \cdot)|^q_{B_{pq}^{s;\mathfrak{A}_\sigma}(\mathbb{R}^d, V)} dt \right)^{1/q},
\]

\[
(2.2) \quad |f|_{H_{\mu}^s(E, V)} = |f|_{H_{\mu;\mathfrak{A}_\sigma}^s(E, V)} = \left( \int_0^T |f(t, \cdot)|^p_{H_{\mu;\mathfrak{A}_\sigma}^s(\mathbb{R}^d, V)} dt \right)^{1/p}.
\]

Similarly we introduce the corresponding spaces of random generalized functions.

Let \((\Omega, \mathcal{F}, \mathbf{P})\) be a complete probability spaces with a filtration of \( \sigma \)-algebras \( \mathcal{F} = (\mathcal{F}_t) \) satisfying the usual conditions. Let \( \mathcal{R}(\mathcal{F}) \) be the progressive \( \sigma \)-algebra on \([0, \infty) \times \Omega \).

The spaces \( B_{pp}^s(\mathbb{R}^d, V) \) and \( \mathbb{H}^s_p(\mathbb{R}^d, V) \) consists of all \( \mathcal{F} \)-measurable random functions \( f \) with values in \( B_{pp}^s(\mathbb{R}^d, V) \) and \( H_{\mu}^s(\mathbb{R}^d, V) \) with finite norms

\[
|f|_{B_{pp}^s(\mathbb{R}^d, V)} = \left\{ \mathbf{E} |f|^p_{B_{pp}^s(\mathbb{R}^d, V)} \right\}^{1/p}
\]

and

\[
|f|_{\mathbb{H}_p^s(\mathbb{R}^d, V)} = \left\{ \mathbf{E} |f|^p_{H_{\mu}^s(\mathbb{R}^d, V)} \right\}^{1/p}.
\]
The spaces $B^s_{pp}(E,V)$ and $H^s_p(E,V)$ consist of all $\mathcal{R}(\mathbb{F})$-measurable random functions with values in $B^s_{pp}(E,V)$ and $H^s_p(E,V)$ with finite norms

$$|f|_{B^s_{pp}(E,V)} = \left\{ E |f|^p_{B^s_{pp}(E,V)} \right\}^{1/p}$$

and

$$|f|_{H^s_p(E,V)} = \left\{ E |f|^p_{H^s_p(E,V)} \right\}^{1/p}.$$

**Remark 2.** (see [8]) For every $\varepsilon > 0$, $B^\nu_{pp,N:s+\varepsilon}(\mathbb{R}^d)$ is continuously embedded into $H^{\nu,0}_{pp}(\mathbb{R}^d)$; $p > 1$; for $p \geq 2$, $H^{\nu,0}_{pp}(\mathbb{R}^d)$ is continuously embedded into $B^\nu_{pp,N:s}(\mathbb{R}^d)$.

If $V_r = L_r(U,\mathcal{U},\Pi), r \geq 1$, the space of $r$-integrable measurable functions on $U$, and $V_0 = \mathbb{R}$, we write

$$B^s_{r,pp}(A) = B^s_{pp}(A,V), \quad B^s_{r,pp}(A) = B^s_{pp}(A,V),$$

$$H^s_{r,p}(A) = H^s_p(A,V), \quad H^s_{r,p}(A) = H^s_p(A,V),$$

and

$$L_{r,p}(A) = H^0_{r,p}(A), \quad L_{r,p}(A) = H^0_{r,p}(A),$$

where $A = \mathbb{R}^d$ or $E$. For scalar functions we drop $V$ in the notation of function spaces.

Let $U_n \in \mathcal{U}, U_n \subseteq U_{n+1}, n \geq 1, \bigcup_n U_n = U$ and $\Pi(U_n) < \infty, n \geq 1$. We denote by $C^\infty_{r,p}(E), 1 \leq p < \infty$, the space of all $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable $V_r$-valued random functions $\Phi$ on $E$ such that for every multiindex $\gamma \in \mathbb{N}_0^d$,

$$E \int_0^T \sup_{x \in \mathbb{R}^d} |D^\gamma \Phi(t,x)|_{V_r}^p dt + E \left[ |D^\gamma \Phi|_{L_p(E;V_r)}^p \right] < \infty,$$

and $\Phi = \Phi_{\chi_{U_n}}$ for some $n$ if $r = 2, p$. Similarly we define the space $C^\infty_{r,p}(\mathbb{R}^d)$ by replacing $\mathcal{R}(\mathbb{F})$ and $E$ by $\mathcal{F}$ and $\mathbb{R}^d$ respectively in the definition of $C^\infty_{r,p}(E)$.

### 2.3. Main Results

We set for $\nu \in \mathbb{R}^\nu$

$$\delta(r) = \delta_\nu(r) = \nu \left( x \in \mathbb{R}^d : |x| > r \right), \quad r > 0$$

$$w(r) = w_\nu(r) = \delta_\nu(r)^{-1}, \quad r > 0.$$

Our main assumption is that $w = w_\nu(r)$ is an O-RV function at both infinity and at zero. That is

$$r_1(x) = \limsup_{\epsilon \to 0} \frac{w(\epsilon x)}{w(\epsilon)} < \infty, \quad r_2(x) = \limsup_{\epsilon \to \infty} \frac{w(\epsilon x)}{w(\epsilon)} < \infty, x > 0.$$

By Theorem 2 in [1], the following limit exist:

$$(2.3) \quad p_1 = p_1^{w_\nu} = \lim_{\epsilon \to 0} \frac{\log r_1(\epsilon)}{\log \epsilon} \leq q_1 = q_1^{w_\nu} = \lim_{\epsilon \to \infty} \frac{\log r_1(\epsilon)}{\log \epsilon}.$$
and
\[(2.4) \quad p_2 = p_2^w = \lim_{{\epsilon \to 0}} \frac{\log r_2 (\epsilon)}{\log \epsilon} \leq q_2 = q_2^w = \lim_{{\epsilon \to \infty}} \frac{\log r_2 (\epsilon)}{\log (\epsilon)}.
\]

Note that \(p_1 \leq \sigma \leq q_1\) (see \[6\]). We will assume throughout this paper that \(p_1, p_2, q_1, q_2 > 0\). The numbers \(p_1, p_2\) are called lower indices and \(q_1, q_2\) are called upper indices of O-RV function.

When the context is clear, for a function \(f\) which is both O-RV at zero and infinity we always denote its lower index at zero by \(p_1\), upper index at zero by \(q_1\), lower index at infinity by \(p_2\) and upper index at infinity by \(q_2\).

If we wish to be precise, we will write \(p_1^f, p_2^f, q_1^f, q_2^f\).

The main result for (1.1) is the following statement.

**Theorem 1.** Let \(p \in (1, \infty), s \in \mathbb{R}, N > 1\). Let \(\nu \in \mathfrak{N}\), and \(w = w_\nu\) be continuous O-RV function at zero and infinity with \(p_i, q_i, i = 1, 2\), defined in (2.2), (2.4). Assume

\textbf{A.} for \(i = 1, 2\)
\[
0 < p_i \leq q_i < 1 \text{ if } \sigma \in (0, 1), 0 < p_i \leq 1 \leq q_i < 2 \text{ if } \sigma = 1,
\]
\[1 < p_i \leq q_i < 2 \text{ if } \sigma \in (1, 2).
\]

\textbf{B.}
\[
\inf_{R \in (0, \infty), |\hat{x}| = 1} \int_{|y| \leq 1} \left| \hat{x} \cdot y \right|^2 \tilde{\nu}_R (dy) > 0,
\]

where \(\tilde{\nu}_R (dy) = w (R) \nu (R dy)\). Then for each \(f \in L_{p,R}^{\nu,s+\frac{1}{p}} (E), g \in L_{pp,R}^{\nu,s+\frac{1}{p}} (\mathbb{R}^d), \)
\(\Phi \in L_{p,pp,R}^{\nu,s+1/p} (E) \cap L_{2}^{\nu,s+\frac{1}{2}} (E)\) if \(p \in [2, \infty)\) and \(\Phi \in L_{pp,R}^{\nu,s+\frac{1}{p}} (E)\)
if \(p \in (1, 2)\), there is a unique \(u \in L_{p,R}^{\nu,s+1} (E)\) solving (1.7). Moreover, there is \(C = C (d, p, \nu)\) such that for \(p \in [2, \infty), \)
\[
|L^\nu u|_{L_{p,R}^{\nu,s+\frac{1}{p}} (E)} \leq C \left[ |f|_{L_{p,R}^{\nu,s+\frac{1}{p}} (E)} + |g|_{L_{pp,R}^{\nu,s+1/p}} (\mathbb{R}^d) + |\Phi|_{L_{pp,R}^{\nu,s+\frac{1}{p}} (E)} \right],
\]
\[
|u|_{L_{p,R}^{\nu,s}} (E) \leq C [p, |f|_{L_{p,R}^{\nu,s+\frac{1}{p}} (E)} + |g|_{L_{p,R}^{\nu,s+\frac{1}{p}} (\mathbb{R}^d)} + |\Phi|_{L_{pp,R}^{\nu,s+\frac{1}{p}} (\mathbb{R}^d)}],
\]
and for \(p \in (1, 2), \)
\[
|L^\nu u|_{L_{p,R}^{\nu,s+\frac{1}{p}} (E)} \leq C \left[ |f|_{L_{p,R}^{\nu,s+\frac{1}{p}} (E)} + |g|_{L_{pp,R}^{\nu,s+1/p}} (\mathbb{R}^d) + |\Phi|_{L_{pp,R}^{\nu,s+\frac{1}{p}} (E)} \right],
\]
\[
|u|_{L_{p,R}^{\nu,s}} (E) \leq C [p, |f|_{L_{p,R}^{\nu,s+\frac{1}{p}} (E)} + |g|_{L_{p,R}^{\nu,s+\frac{1}{p}} (\mathbb{R}^d)} + |\Phi|_{L_{pp,R}^{\nu,s+\frac{1}{p}} (\mathbb{R}^d)}],
\]
where \(p, = 1/\lambda \wedge T.\)

Note that the assumption \textbf{A} depends on \(\nu\) only through \(w_\nu\).
Furthermore, we have the following estimate for time regularity for deterministic equation
\[
 du(t,x) = \left[ L^\nu u(t,x) - \lambda u(t,x) + f(t,x) \right] dt
\]
(2.5)
\[
u(0,x) = g(x), t \in [0,T], x \in \mathbb{R}^d,
\]
with \( \lambda \geq 0 \).

**Proposition 1.** Let \( \nu \in \mathfrak{A}^\sigma \) satisfying Assumption B, \( p \in (1, \infty), s \in \mathbb{R} \). Assume that \( w = w_\nu \) is continuous, O-RV and satisfies assumption A.

For each \( f \in H^{\nu,s}_p(E), g \in B^{\nu,N,s+1-1/p}_{pp} (\mathbb{R}^d) \), we denote \( u_f, u_g \in H^{\nu,s}_p(E) \) the unique solution of (2.5) with \( g = 0 \) and \( f = 0 \) respectively. Then the following estimate holds:

for \( \kappa \in \left( \frac{1}{p}, 1 \right], t, t' \in [0, T] \) there is \( C > 0 \) independent of \( \lambda \) and \( T \) such that
\[
|L^{\nu,1-\kappa} (u_f(t) - u_f(t'))|_{H^{\nu,s}_p(\mathbb{R}^d)} \leq C \left( t - t' \right)^{\left( \kappa - \frac{1}{p} \right)} |f|_{H^{\nu,s}_p(E)}
\]
Moreover for \( \mu_1 \in [0, \kappa - \frac{1}{p}) \) and \( \mu_2 \in [0, 1] \), there exists \( C > 0 \) independent of \( T \) and \( \lambda \) such that for any \( t', t \in [0, T] \), the following estimate holds
\[
|L^{\nu,1-\kappa} (u_g(t) - u_g(t'))|_{H^{\nu,s}_p(\mathbb{R}^d)} \leq C \left[ (t - t')^{\mu_1} + (t - t')^{\mu_2} \lambda^{\mu_2} \right] |g|_{B^{\mu_1+1,s}_p(\mathbb{R}^d)}.
\]

The following examples are taken from [6].

**Example 1.** According to [3] (pp. 70-74), any Levy measure \( \nu \in \mathfrak{A}^\sigma \) can be disintegrated as
\[
\nu(\Gamma) = -\int_0^\infty \int_{S_{d-1}} \chi_\Gamma (rw) \Pi(r,dw) d\delta_\nu(r), \Gamma \in \mathcal{B} \left( \mathbb{R}^d_0 \right),
\]
where \( \delta = \delta_\nu \), and \( \Pi(r,dw), r > 0 \) is a measurable family of measures on the unit sphere \( S_{d-1} \) with \( \Pi(r,S_{d-1}) = 1, r > 0 \). If \( w_\nu = \delta^{-1} \) is an continuous, O-RV and satisfies assumption A. Assume that \( |\{s \in [0,1] : r_i(s) < 1\}| > 0, i = 1, 2, \) and
\[
\inf \left| \xi \right| = 1 \int_{S_{d-1}} \left| \hat{\xi} \cdot \omega \right|^2 \Pi(r,d\omega) \geq c_0 > 0, \quad r > 0,
\]
hold, then all assumptions of Theorem 1 holds. (cf. Corollary 2 in the Appendix)

**Example 2.** Consider Levy measure in radial and angular coordinate in the form
\[
\nu(B) = \int_0^\infty \int_{|\omega|=1} 1_B (r\omega) a(r,\omega) j(r) r^{d-1} S(d\omega) dr, B \in \mathcal{B} \left( \mathbb{R}^d_0 \right),
\]
where \( S(d\omega) \) is a finite measure on the unit sphere.
Assume
(i) There is $C > 1, c > 0, 0 < \delta_1 \leq \delta_2 < 1$ such that
$$C^{-1} \phi (r^{-2}) \leq j (r) \leq C \phi (r^{-2})$$
and for all $0 < r \leq R$,
$$c^{-1} \left( \frac{R}{r} \right)^{\delta_1} \leq \frac{\phi (R)}{\phi (r)} \leq c \left( \frac{R}{r} \right)^{\delta_2}.$$  

(ii) There is a function $\rho_0 (\omega)$ defined on the unit sphere such that $\rho_0 (\omega) \leq a (r, \omega) \leq 1, \forall r > 0$, and for all $|\hat{\xi}| = 1$,
$$\int_{S^{d-1}} |\hat{\xi} \cdot \omega|^2 \rho_0 (\omega) S (d\omega) \geq c > 0.$$  

Under these assumptions it can be shown that $B$ holds, and $\delta_\nu$ is an O-RV function with $2\delta_1 \leq \delta_\nu \leq \delta_2, i = 1, 2$.

3. Auxiliary results

We start with

3.1. Some estimates of O-RV functions. We start with a simple but useful corollaries about functions that are O-RV at both zero and infinity. For $\nu \in A^\sigma$, $R > 0$, we denote $\tilde{\nu}_R (dy) = w_\nu (R \nu (Rdy)$.

Next we note scaling property of functions $f : (0, \infty) \to (0, \infty)$ that are non-decreasing, O-RV a both zero and infinity with strictly positive lower indices.

**Lemma 1.** Let $\nu \in A^\sigma$, and $w = w_\nu$ be continuous O-RV function at zero and infinity with $p_i, q_i, i = 1, 2$, defined in (2.3), (2.4), and assumption $A$ holds. Then for any $\alpha_1 > q_1 \lor q_2$, $0 < \alpha_2 < p_1 \land p_2$, there exist $c_1 = c_1 (\alpha_1), c_2 = c_2 (\alpha_2) > 0$ such that
$$c_1 \left( \frac{y}{x} \right)^{\alpha_2} \leq \frac{w (y)}{w (x)} \leq c_2 \left( \frac{y}{x} \right)^{\alpha_1}, 0 < x < y < \infty.$$  

**Proof.** Due to similarity, we only show the right hand side of the inequality. By Karamata characterization (see (1.7) of [1]) of O-RV functions, there are $0 < \eta_1 < \eta_2$ such that the RHS inequality holds if either $x \lor y \leq \eta_1$ or $x \land y \geq \eta_2$. If $y \geq \eta_2$ and $x \leq \eta_1$,
$$\frac{w (y)}{w (x)} = \frac{w (y)}{w (\eta_2)} \frac{w (\eta_2)}{w (\eta_1)} \frac{w (\eta_1)}{w (x)} \leq c \left( \frac{y}{x} \right)^{\alpha_1} \frac{w (\eta_2)}{w (\eta_1)},$$  

and similarly we consider other cases. $\square$

The following statement holds

**Lemma 2.** (Lemma 8 in [6]) Assume $w (r), r > 0$, is a non-negative non-decreasing O-RV function at zero with lower and upper indices $p_1, q_1$, that is,
$$r_1 (x) = \lim_{\varepsilon \to 0} \frac{w (\varepsilon x)}{w (\varepsilon)} < \infty, x > 0,$$  

and

\[ p_1 = \lim_{\epsilon \to 0} \frac{\log r_1(\epsilon)}{\log \epsilon} \leq q_1 = \lim_{\epsilon \to \infty} \frac{\log r_1(\epsilon)}{\log \epsilon}. \]

a) Let \( \beta > 0 \) and \( -\tau > -\beta p_1 \). There is \( C > 0 \) so that

\[ \int_0^x t^\tau w(t)^{\beta} \frac{dt}{t} \leq C x^\tau w(x)^{\beta}, x \in (0, 1], \]

and \( \lim_{x \to 0} x^\tau w(x)^{\beta} = 0. \)

b) Let \( \beta > 0 \) and \( -\tau < -\beta q_1 \). There is \( C > 0 \) so that

\[ \int_x^1 t^\tau w(t)^{\beta} \frac{dt}{t} \leq C x^\tau w(x)^{\beta}, x \in (0, 1], \]

and \( \lim_{x \to 0} x^\tau w(x)^{\beta} = \infty. \)

c) Let \( \beta < 0 \) and \( -\tau > -\beta q_1 \). There is \( C > 0 \) so that

\[ \int_x^\infty t^\tau w(t)^{\beta} \frac{dt}{t} \leq C x^\tau w(x)^{\beta}, x \in (0, 1], \]

and \( \lim_{x \to 0} x^\tau w(x)^{\beta} = 0. \)

d) Let \( \beta < 0 \) and \( -\tau < -\beta p_1 \). There is \( C > 0 \) so that

\[ \int_x^1 t^\tau w(t)^{\beta} \frac{dt}{t} = \int_1^{x^{-1}} t^{-\tau} w \left( \frac{1}{t} \right)^{\beta} \frac{dt}{t} \leq C x^\tau w(x)^{\beta}, x \in (0, 1], \]

and \( \lim_{x \to 0} x^\tau w(x)^{\beta} = \infty. \)

Similar statement holds for O-RV functions at infinity.

**Lemma 3.** Assume \( w(r), r > 0, \) is a non-negative non-decreasing O-RV function at infinity with lower and upper indices \( p_2, q_2, \) that is,

\[ r_2(x) = \lim_{\epsilon \to \infty} \frac{w(\epsilon x)}{w(\epsilon)} < \infty, \ x > 0, \]

and

\[ p_2 = \lim_{\epsilon \to 0} \frac{\log r_2(\epsilon)}{\log \epsilon} \leq q_2 = \lim_{\epsilon \to \infty} \frac{\log r_2(\epsilon)}{\log \epsilon}. \]

a) Let \( \beta > 0 \) and \( -\tau > -\beta q_2 \). There is \( C > 0 \) so that

\[ \int_x^\infty t^\tau w(t)^{\beta} \frac{dt}{t} \leq C x^\tau w(x)^{\beta}, x \in [1, \infty), \]

and \( \lim_{x \to \infty} x^\tau w(x)^{\beta} = 0. \)

b) Let \( \beta > 0 \) and \( -\tau < -\beta p_2 \). There is \( C > 0 \) so that

\[ \int_1^x t^\tau w(t)^{\beta} \frac{dt}{t} \leq C x^\tau w(x)^{\beta}, x \in [1, \infty), \]

and \( \lim_{x \to \infty} x^\tau w(x)^{\beta} = \infty. \)

c) Let \( \beta < 0 \) and \( -\tau < -\beta p_2 \). There is \( C > 0 \) so that

\[ \int_1^\infty t^\tau w(t)^{\beta} \frac{dt}{t} \leq C x^\tau w(x)^{\beta}, x \in [1, \infty), \]
and \( \lim_{x \to \infty} x^\tau w(x)^\beta = 0 \).

d) Let \( \beta < 0 \) and \( \tau > -\beta q_2 \). There is \( C > 0 \) so that
\[
\int_1^x t^\tau w(t)^\beta \frac{dt}{t} \leq C x^\tau w(x)^\beta, \quad x \in [1, \infty),
\]
and \( \lim_{x \to \infty} x^\tau w(x)^\beta = \infty \).

**Proof.** The claims follow easily by Theorems 3, 4 in [1]. Because of the similarities, we will prove d) only. Let \( \beta < 0 \) and \( \tau > -\beta q_2 \). Then
\[
\lim_{t \to \infty} w(\varepsilon t)^\beta t^\tau w(t)^{\beta-\beta} = r_2 (\varepsilon^{-1})^{-\beta} < \infty, \varepsilon > 0.
\]
Hence \( w(t)^\beta, t \geq 1 \), is an O-RV function at infinity with
\[
p = \lim_{\varepsilon \to 0} \frac{\log r_2 (\varepsilon^{-1})^{-\beta}}{\log \varepsilon} = \beta q_2 \leq \beta p_2 = - \lim_{\varepsilon \to 0} \frac{\log r_2 (\varepsilon)^{-\beta}}{\log \varepsilon} = q.
\]

Then by Theorems 3 and 4 in [1], for \( \tau > -\beta q_2 \),
\[
\int_1^x t^\tau w(t)^\beta \frac{dt}{t} \leq C x^\tau w(x)^\beta, \quad x \geq 1,
\]
and \( \lim_{x \to \infty} x^\tau w(x)^\beta = \infty \).\( \square \)

**Lemma 4.** Let \( \nu \in \mathcal{A}, w = w_\nu \) be an O-RV at zero and infinity with indices \( p_1, q_1, p_2, q_2 \) defined in [2.3], [2.4]. Then for any \( \alpha_1 > q_1 \lor q_2 \) and \( 0 < \alpha_2 < p_1 \land p_2 \), there is \( C = C(w, \alpha_1, \alpha_2) > 0 \) such that
\[
\int_{|y| \leq 1} |y|^\alpha_1 \tilde{\nu}_R(dy) + \int_{|y| > 1} |y|^\alpha_2 \tilde{\nu}_R(dy) \leq C, R > 0.
\]

**Proof.** First,
\[
\int_{|y| \leq 1} |y|^\alpha_1 \tilde{\nu}_r(dy) = \delta (r)^{-1} r^{-\alpha_1} \int_{|y| \leq r} |y|^\alpha_1 \nu(dy)
\]
\[
= \delta (r)^{-1} r^{-\alpha_1} \alpha_1 \int_0^r s^{\alpha_1} [\delta(s) - \delta(r)] \frac{ds}{s}
\]
\[
= \delta (r)^{-1} r^{-\alpha_1} \alpha_1 \int_0^r \delta(s) s^{\alpha_1-1} ds - 1,
\]
and similarly,
\[
\int_{|y| > 1} |y|^\alpha_2 \tilde{\nu}_r(dy) = w(r) r^{-\alpha_2} \alpha_2 \int_0^\infty \delta(s \lor r) s^{\alpha_2} \frac{ds}{s}
\]
\[
= 1 + w(r) r^{-\alpha_2} \alpha_2 \int_r^\infty \delta(s) s^{\alpha_2} \frac{ds}{s}.
\]
Thus
\[
\int_{|y| \leq 1} |y|^{\alpha_1} \, \pi_r(d y) + \int_{|y| > 1} |y|^{\alpha_2} \, \pi_r(d y)
\]
\[= w(r) r^{-\alpha_1} \int_0^r w(s)^{-1} s^{\alpha_1} \frac{d s}{s} + w(r) r^{-\alpha_2} \int_r^\infty w(s)^{-1} s^{\alpha_2} \frac{d s}{s}
\]
\[= I_1 + I_2
\]
By Lemma 3, there is \(C\) so that
\[I_2 = w(r) r^{-\alpha_2} \int_r^\infty w(s)^{-1} s^{\alpha_2} \frac{d s}{s} \leq C, r \geq 1.
\]
By Lemma 2, there is \(C\) so that
\[w(r) r^{-\alpha_2} \int_r^1 w(s)^{-1} s^{\alpha_2} \frac{d s}{s} \leq C, r \in (0, 1).
\]
Hence there is \(C\) so that \(I_2 \leq C\) for all \(r > 0\). Similarly, using Lemmas 2-3, we estimate \(I_1\).

Remark 3. In particular, if \(w = w_\nu\) satisfies assumption \(A\), then we may choose in Lemma 4 \(\alpha_1, \alpha_2\) so that
\[(i) \quad \alpha_1, \alpha_2 \in (0, 1) \text{ if } \sigma \in (0, 1); \quad (ii) \quad \alpha_1, \alpha_2 \in (1, 2) \text{ if } \sigma \in (1, 2); \quad (iii) \quad \alpha_1 \in (1, 2] \text{ and } \alpha_2 \in [0, 1) \text{ if } \sigma = 1.
\]

Lemma 5. Let \(\nu \in A, w = w_\nu(r), r > 0\), be a continuous O-RV at zero and infinity with indices \(p_1, q_1, p_2, q_2\) defined in (2.3), (2.4), and \(p_1, p_2 > 0\). Let
\[a(r) = \inf \{ t > 0 : w(t) \geq r \}, r > 0.
\]
Then
\[(i) \quad w(a(t)) = t, t > 0, \text{ and }
\]
\[w(a(t) -) \leq t \leq w(a(t) +), t > 0,
\]
\[a(w(t) -) \leq t \leq a(w(t) +), t > 0.
\]
\[(ii) \quad a \text{ is O-RV at zero and infinity with lower indices } \bar{p}, \bar{q} \text{ and upper indices } p, q \text{ respectively so that}
\]
\[
\frac{1}{q_1} \leq p \leq q \leq \frac{1}{p_1},
\]
\[
\frac{1}{q_2} \leq \bar{p} \leq \bar{q} \leq \frac{1}{p_2}.
\]

Proof. (i) follow easily from the definitions,
(ii) By Theorem 3 in [1], for any \(\alpha \in (0, p_1)\) there is \(C > 0\) so that
\[\frac{w(x)}{x^\alpha} \leq C \frac{w(y)}{y^\alpha}, 0 < x \leq y \leq 1.
\]
Hence
\[\frac{a(y)}{y^\frac{1}{\alpha}} \leq C a(x) \frac{y^\frac{1}{\alpha}}{x^\frac{1}{\alpha}}, 0 < x \leq y \leq 1,
\]
and, by Theorem 3 in [1], $a$ is O-RV at zero with upper index,

$$q \leq \frac{1}{p_1}.$$ 

Similarly we find that the lower index at zero $p \geq \frac{1}{q_1}$, and determine that $a$ is O-RV at infinity with indices $\bar{p} \leq \bar{q}$ so that

$$\frac{1}{q_2} \leq \bar{p} \leq \frac{1}{p_2}.$$ 

□

The following claim is an obvious consequence of Lemmas [2,5].

**Corollary 1.** Let $\nu \in \mathfrak{U}, w = w_\nu (r), r > 0$, be a continuous O-RV at zero and infinity with indices $p_1, q_1, p_2, q_2$ defined in (2.3), (2.4), and $p_1, p_2 > 0$. Let

$$a (r) = \inf \{ t > 0 : w (t) \geq r \}, r > 0.$$ 

(i) For any $\beta > 0$ and $\tau < \frac{\beta}{q_1} \wedge \frac{\beta}{q_2}$ there is $C > 0$ such that

$$\int_0^r t^{-\tau} a (t)^\beta \frac{dt}{t} \leq Cr^{-\tau} a (r)^\beta, r > 0,$$

$$\lim_{r \to 0} r^{-\tau} a (r)^\beta = 0, \quad \lim_{r \to \infty} r^{-\tau} a (r)^\beta = \infty,$$

and for any $\beta < 0, \tau > \frac{-\beta}{p_1} \vee \frac{-\beta}{p_2}$ there is $C > 0$ such that

$$\int_0^r t^\tau a (t)^\beta \frac{dt}{t} \leq Cr^\tau a (r)^\beta, r > 0,$$

$$\lim_{r \to 0} r^\tau a (r)^\beta = 0, \quad \lim_{r \to \infty} r^\tau a (r)^\beta = \infty.$$

(ii) For any $\gamma > 0$ and $\tau > \frac{\gamma}{p_1} \vee \frac{\gamma}{p_2}$ there is $C > 0$ such that

$$\int_r^\infty t^{-\tau} a (t)^\gamma \frac{dt}{t} \leq Cr^{-\tau} a (r)^\gamma, r > 0,$$

$$\lim_{r \to 0} r^{-\tau} a (r)^\gamma = \infty, \lim_{r \to \infty} r^{-\tau} a (r)^\gamma = 0,$$

and for any $\gamma < 0$ and $\tau < \left( \frac{-\gamma}{q_1} \right) \wedge \left( \frac{-\gamma}{q_2} \right)$ there is $C > 0$ such that

$$\int_r^\infty t^\tau a (t)^\gamma \frac{dt}{t} \leq Cr^\tau a (r)^\gamma, r > 0,$$

$$\lim_{r \to 0} r^\tau a (r)^\gamma = \infty, \lim_{r \to \infty} r^\tau a (r)^\gamma = 0.$$
3.2. Estimates for probability density. In this section we derive some estimates of probability density of $Z^R_t$ (see below for detailed description), these preliminary estimates will be used in verifying Hörmander condition and stochastic Hörmander condition in \cite{7 9}. Thus, allowing us to derive apriori estimates.

Let $\nu \in \mathfrak{F}$ and $p(dt,dy)$ be a Poisson point measure on $[0, \infty) \times \mathbb{R}_0^d$ such that $\mathbb{E}p(dt,dy) = \nu(dy)dt$. Let $q(dt,dy) = p(dt,dy) - \nu(dy)dt$. We associate to $L^\nu$ the stochastic process with independent increments

$$Z_t = Z^\nu_t = \int_0^t \int \chi_\sigma(y)q(ds,dy) + \int_0^t \int (1 - \chi_\sigma(y))p(ds,dy), t \geq 0.$$  

By Ito formula,

$$\mathbb{E}e^{i2\pi \xi \cdot Z^\nu_t} = \exp \left\{ \psi^\nu(\xi) t \right\}, t \geq 0, \xi \in \mathbb{R}^d,$$

where

$$\psi^\nu(\xi) := \int \left[ e^{i2\pi \xi y} - 1 - i2\pi y \cdot \xi_\sigma(y) \right] \nu(dy).$$

Let $Z = Z^R = Z^\nu_R$ be the stochastic process with independent increments associated with $\tilde{\nu}_R = w_\nu(R)\nu_R$, i.e.,

$$\mathbb{E}e^{i2\pi \xi \cdot Z^R_t} = \exp \left\{ \psi^R(\xi) t \right\}$$

where

$$\psi^R(\xi) = \int \left[ e^{i2\pi \xi y} - 1 - i2\pi \chi_\sigma(y) y \cdot \xi \right] d\tilde{\nu}_R, \xi \in \mathbb{R}^d.$$

Note $Z^R = Z^\nu_R$ and $R^{-1}Z^\nu_{R^t}$, $t > 0$, have the same distribution. For $R > 0$, consider Levy measures $\nu^{R,0}(dy) = \chi_{\left\{ |y| \leq 1 \right\}} \nu_R(dy)$, i.e.,

$$\int \chi_\Gamma(y) \nu^{R,0}(dy) = \int_{|y| \leq 1} \chi_\Gamma(y) \tilde{\nu}_R(dy) = w(R) \int_{|y| \leq R} \chi_\Gamma(y/R) \nu(dy), \Gamma \in \mathfrak{B}_0 \left( \mathbb{R}^d \right).$$

Let $\eta = \eta^R_0$ be a random variable with characteristic function $\exp \left\{ \psi^{R,0}(\xi) \right\}$, where (with $\xi = \xi/|\xi|, \xi \in \mathbb{R}_0^d$)

$$\psi^{R,0}(\xi) = \int_{|y| \leq 1} \left[ e^{i2\pi \xi y} - 1 - i\chi_\sigma(y) \xi \cdot y \right] \tilde{\nu}_R(dy)$$

$$= \int_{|y| \leq 1} \left[ e^{i2\pi |\xi| y} - 1 - i\chi_\sigma(y) \xi \cdot y \right] \tilde{\nu}_R(dy)$$

$$= \frac{w(R)}{w(R/|\xi|^{-1})} \int_{|y| \leq |\xi|} \left[ e^{i2\pi \xi y} - 1 - i\chi_\sigma(y) \xi \cdot y \right] \tilde{\nu}_{R|\xi|^{-1}}(dy).$$

Lemma 6. Let $\nu \in \mathfrak{F}$, $w = w_\nu$ is a continuous $O$-RV function and $A, B$ hold. Then

(i)

$$\Re \psi^{R,0}(\xi) \leq -c |\xi|^\kappa, |\xi| \geq 1,$$

with some $c, \kappa > 0$ independent of $R$. 

(3.4)
(ii) \( \eta = \eta_0^R \) has a pdf \( p_{R,0}(x), x \in \mathbb{R}^d \), such that for any multi-index \( \beta \in \mathbb{N}^d_0 \) and a positive integer \( n \geq 0 \),

\[
\sup_x \left| \partial^\beta p_{R,0}(x) \right| + \int (1 + |x|^2)^n \left| \partial^\beta p_{R,0}(x) \right| dx \leq C, R > 0.
\]

**Proof.** (i) Let \( \alpha_2 \in (0, p_1 \land p_2) \). By Lemma 1, there is \( C \) so that

\[
\frac{w(R)}{w(R \mid |\xi|^{-1})} \geq C |\xi|^{\alpha_2}, R > 0, |\xi| \geq 1.
\]

Hence, according to (3.3), for \( |\xi| \geq 1, \hat{\xi} = \xi/|\xi| \),

\[
\Re \psi^{R,0}(\xi) \leq c \left| \xi \right|^{\alpha_2} \int_{|y| \leq 1/8} \left[ \cos \left( 2\pi \hat{\xi} \cdot y \right) - 1 \right] \tilde{\nu}_R(0, 1/8) dy,
\]

\[
\leq -c |\xi|^{\alpha_2} \inf_{R \in (0, \infty), |\hat{\xi}| = 1} \int_{|y| \leq 1} \left| \hat{\xi} \cdot y \right|^2 \tilde{\nu}_R(dy) = -c_0 |\xi|^{\alpha_2},
\]

and

\[
(3.5) \quad \int \exp \left\{ \Re \psi^{R,0}(\xi) \right\} d\xi < \infty.
\]

(ii) Since (3.5) holds, by Proposition I.2.5 in [11], \( \eta = \eta_0^R \) has a bounded density

\[
(3.6) \quad p_{R,0}(x) = \int e^{-i2\pi x \cdot \xi} \exp \left\{ \psi^{R,0}(\xi) \right\} d\xi, x \in \mathbb{R}^d.
\]

Moreover, by (i), for any multi-index \( \beta \in \mathbb{N}^d_0 \),

\[
\partial^\beta p_{R,0}(x) = \int e^{-i2\pi x \cdot \xi} (-i2\pi \xi)^\beta \exp \left\{ \psi^{R,0}(\xi) \right\} d\xi, x \in \mathbb{R}^d,
\]

is a bounded continuous function. The function \( (1 + |x|^2)^n \partial^\beta p_{R,0} \) is integrable if

\[
(3.7) \quad \left( -i2\pi x \right)^l (-i2\pi x_{\beta_k})^{2n} \partial^\beta p_{R,0}(x)
\]

\[
= \int \partial^l_{\xi_j} \partial^{2n}_{\xi_k} \psi^{R,0}(\xi) \exp \{ \psi^{R,0}(\xi) \} d\xi
\]

\[
= (-1)^{l+2n} \int e^{-i2\pi x \cdot \xi} \partial^l_{\xi_j} \partial^{2n}_{\xi_k} \psi^{R,0}(\xi) \exp \{ \psi^{R,0}(\xi) \} d\xi
\]

is bounded for all \( j, k \), and for \( l \leq d + 1 \). Since \( \partial^\mu \psi^{R,0}(\xi) \) is bounded for \( |\mu| \geq 2 \) and

\[
\int_{|y| \leq 1} \chi_\mu(y) |y| (|\xi| |y| \land 1) \tilde{\nu}_R(dy),
\]

with \( \zeta(\xi, \mu) = \int_{|y| \leq 1} \chi_\mu(y) |y| (|\xi| |y| \land 1) \tilde{\nu}_R(dy) \). The boundedness follows from (i) and Lemma 3 (see Remark 3). \( \square \)
Lemma 7. Let \( \nu \in \mathfrak{V}^\sigma, w = w_\nu \) is a continuous O-RV function and \( A, B \) hold.

Then for \( R > 0 \), \( Z_1^R \) has a bounded continuous probability density of the form
\[
p^R (1, x) = \int p_{R,0} (x - y) P_R (dy), x \in \mathbb{R}^d,
\]
where \( P_R \) is a probability distribution. Moreover, for any \( 0 < \alpha_2 < p_1^w \wedge p_2^w \) and multindex \( k \in \mathbb{N}_0^d \), there is \( C = C (k, \alpha_2, \nu) > 0 \) such that
\[
\int (1 + |x|^{\alpha_2}) \left| D^k p^R (1, x) \right| dx \leq C.
\]

Proof. We have
\[
\tilde{\nu}_R = \tilde{\nu}_{R,0} + \tilde{\nu}_{R,2},
\]
where \( \tilde{\nu}_{R,0} (dy) = \chi_{|y| \leq 1} \tilde{\nu}_R (dy) \) and hence,
\[
\psi^{\tilde{\nu}_R} (\xi) = \psi^{\tilde{\nu}_{R,0}} (\xi) + \psi^{\tilde{\nu}_{R,2}} (\xi), \xi \in \mathbb{R}^d.
\]
Denote \( \eta_0^R \) and \( \eta_2^R \) to be independent random variables with characteristic exponent \( \exp \{ \psi^{\tilde{\nu}_{R,0}} (\xi) \} \) and \( \exp \{ \psi^{\tilde{\nu}_{R,2}} (\xi) \} \) respectively. Obviously the distribution of \( Z_1^R \) coincides with the distribution of the sum \( \eta_0^R + \eta_2^R \). Therefore,
\[
p^R (1, x) = \int p_{R,0} (x - y) P_R (dy)
\]
where \( P_R (dy) \) is the probability distribution of \( \eta_2^R \). It is clear from 3.10 that for any multiindex \( \beta \in \mathbb{N}_0^d \),
\[
\partial^\beta p^R (1, x) = \int \partial^\beta p_{R,0} (x - y) P_R (dy)
\]
and from Lemma 6,
\[
\int \left| \partial^\beta p^R (1, x) \right| dx \leq C,
\]

and
\[
\sup_x \left| \partial^\beta p^R (1, x) \right| \leq C
\]

Moreover, using Lemma 4 and following the argument of Lemma 10 of 8, one derives easily the estimate 3.8.

We write \( \pi \in \mathfrak{A}_{\text{sign}} = \mathfrak{V}^\sigma - \mathfrak{V}^\sigma \) if \( \pi = \mu - \eta \) with \( \mu, \eta \in \mathfrak{V}^\sigma \), and \( L^\pi = L^\mu + L^\eta \). Given \( \pi \in \mathfrak{A}_{\text{sign}} \), we denote \( |\pi| \) its variation measure. Obviously, \( |\pi| \in \mathfrak{V}^\sigma \). Based on Lemma 7, we derive the following estimates similar to Lemma 12, and Lemma 13 of 8. Since the proof carry over to our setting in obvious ways we omit the proof. We denote \( p^{\nu} (t, \cdot) \) the probability density of the process \( Z_t^\nu, t > 0 \).
The fractional operator is defined in the following way. Let \( \mu \in A_\sigma = \{ \eta \in A_\sigma : \eta \text{ is symmetric}, \eta = \eta_{sym} \} \). Then for \( \delta \in (0,1) \) and \( f \in S(\mathbb{R}^d) \), we have
\[
-(-\psi^\mu(\xi)\delta \hat{f}(\xi)) = c_\delta \int_0^\infty t^{-\delta} [\exp(\psi^\mu(\xi) t) - 1] \frac{dt}{t} \hat{f}(\xi), \xi \in \mathbb{R}^d,
\]
and define
\[
(3.11) \quad L^{\mu;\delta} f(x) := \mathcal{F}^{-1} \left[ -(-\psi^\mu) \delta \hat{f} \right](x) = c_\delta \mathbb{E} \int_0^\infty t^{-\delta} [f(x + Z^\mu_t) - f(x)] \frac{dt}{t}, x \in \mathbb{R}^d.
\]
For \( \mu \in A_\sigma, \delta \in (0,1) \) we let \( \mu_{sym} = \frac{\mu(dy) + \mu(-dy)}{2} \) and define \( L^{\mu;\delta} = L^{\mu_{sym};\delta} \).

Following the arguments in Lemmas 11-13 \[8\], it is easy to show that the following statement holds.

**Lemma 8.** Let \( \nu \in A_\sigma, w = w_\nu \) is a continuous O-RV function and \( A, B \) hold. Let
\[
\alpha_1 > q_1^w \vee q_2^w, 0 < \alpha_2 < p_1^w \wedge p_2^w,
\]
and
\[
\alpha_2 \quad \text{if } \sigma \in (1,2),
\]
\[
\alpha_1 \leq 1 \text{ if } \sigma \in (0,1), \alpha_1 \leq 2 \text{ if } \sigma \in [1,2).
\]
Let \( \pi \in A_{sign} \) and assume that
\[
\int_{|y| \leq 1} |y|^{\alpha_1} d|\tilde{\pi}|_R + \int_{|y| > 1} |y|^{\alpha_2} d|\tilde{\pi}|_R \leq M, R > 0
\]
(i) There is \( C = C(k,w_\nu) \) so that for any multiindex \( k \), and \( \beta \in [0,\alpha_2] \),
\[
\int_{|z| > c} \left| L^\pi D^k p^\nu(t,z) \right| dz \leq CM t^{-1} a(t)^{\beta - |k|} c^{-\beta},
\]
\[
\int \left| L^\pi D^k p^\nu(t,z) \right| dz \leq CM t^{-1} a(t)^{-|k|},
\]
(ii) There exists \( C = C(w_\nu) > 0 \) such that
\[
\int_{\mathbb{R}^d} |L^\pi p^\nu(t,x-y) - L^\pi p^\nu(t,x)| dx \leq CM \frac{|y|}{ta(t)}, t > 0, y \in \mathbb{R}^d,
\]
where \( a(t) = \inf \{ r : w_\nu(r) \geq t \}, t > 0 \).
(iii) There is a constant \( C = C(w_\nu) \) such that
\[
(3.12) \quad \int_{2b}^\infty \int |L^\pi p^\nu(t-s,x) - L^\pi p^\nu(t,x)| dx dt \leq CM, |s| \leq b < \infty.
\]

Following the arguments in Lemmas 3-6 in \[9\], it is easy to show that the following statement holds.
Lemma 9. Let $\mu \in \mathfrak{M}_{\text{sym}}$, $\delta \in (0, 1)$. Let $\nu \in \mathfrak{M}$, $w = w_\nu$ is a continuous $O$-RV function and $A$, $B$ hold. Assume

$$\alpha_1 > q^w_1 \vee q^w_2, 0 < \alpha_2 < p^w_1 \wedge p^w_2,$$

$$\alpha_2 > 1 \text{ if } \sigma \in (1, 2),$$
$$\alpha_1 \leq 1 \text{ if } \sigma \in (0, 1), \alpha_1 \leq 2 \text{ if } \sigma \in [1, 2),$$

and

$$\int_{|y| \leq 1} |y|^{\alpha_1} d\mu_R + \int_{|y| > 1} |y|^{\alpha_2} d\mu_R \leq M, R > 0.$$ 

Let $p^\nu(t, x), x \in \mathbb{R}^d$, be the pdf of $Z^\nu_t, t > 0$.

(i) For any $p \geq 1$, and $\varepsilon > 0$ there is $C_\varepsilon > 0$ so that

$$\left| L^{\mu, \delta} f \right|_{L^p(\mathbb{R}^d)} \leq \varepsilon \left| L^f \right|_{L^p(\mathbb{R}^d)} + C \left| f \right|_{L^p(\mathbb{R}^d)}, f \in \mathcal{S}(\mathbb{R}^d).$$

(ii) There exists $C = C(w_\nu) > 0$ such that

$$\int_{\mathbb{R}^d} \left| L^{\mu, \delta} p^\nu(t, x - y) - L^{\mu, \delta} p^\nu(t, x) \right| dx \leq C M \left| y \right|_{p^\nu(a(t))}, t > 0, y \in \mathbb{R}^d,$$

where $a(t) = \inf \{ r : w(r) \geq t \}, t > 0$.

(iii) There is $C = C(w_\nu) > 0$ such that

$$\int_{2b}^{\infty} \left( \int_{|x| > c} \left| L^{\mu, \delta} p^\nu(t - s, x) - L^{\mu, \delta} p^\nu(t, x) \right| dx \right)^2 dt \leq C M, |s| \leq b < \infty.$$

(iv) For each $\beta \in [0, \delta \alpha_2)$ and for any multiindex $k$, there is $C = C(k, w_\nu)$ so that

$$\int_{|x| > c} \left| L^{\mu, \delta} D^k p^\nu(t, x) \right| dx \leq C M t^{-\delta} a(t)^{\beta - |k|} c^{-\beta},$$
$$\int \left| L^{\mu, \delta} D^k p^\nu(t, x) \right| dx \leq C M t^{-\delta} a(t)^{-|k|}.$$

4. Equivalent norms of function spaces

In this section, we discuss a characterization of Bessel potential spaces and Besov spaces.

Lemma 10. Let $w$ be a non-decreasing $O$-RV function at zero and infinity, with $p^w_1, p^w_2 > 0$. Let $\pi \in \mathfrak{M}$ and define $\tilde{\pi}_R(dy) = w(R) \pi(R dy)$.

(a) Assume there is $N_2 > 0$ so that

$$\int (|y| + 1) \tilde{\pi}_R(dy) \leq N_2 \text{ if } \sigma \in (0, 1),$$
$$\int (|y|^2 + 1) \tilde{\pi}_R(dy) \leq N_2 \text{ if } \sigma = 1,$$
$$\int_{|y| \leq 1} |y|^2 \tilde{\pi}_R(dy) + \int_{|y| > 1} |y| \tilde{\pi}_R(dy) \leq N_2 \text{ if } \sigma \in (1, 2)$$
for any $R > 0$. Then there is a constant $C_1$ so that for all $\xi \in \mathbb{R}^d$,

$$
\int [1 - \cos (2\pi x \xi)] \pi (dy) \leq C_1 N_2 w \left( |\xi|^{-1} \right)^{-1},
$$

$$
\int |\sin (2\pi \cdot y) - 2\pi \chi_{A} (y) \xi \cdot y | \pi (dy) \leq C_1 N_2 w \left( |\xi|^{-1} \right)^{-1},
$$

assuming $w \left( |\xi|^{-1} \right)^{-1} = 0$ if $\xi = 0$.

b) Let

$$
(4.2) \inf_{R \in (0, \infty), |\xi| = 1} \int_{|y| \leq 1} \hat{\xi} \cdot y \pi_R (dy) = c_0 > 0.
$$

Then there is a constant $c_2 = c_2 (w, c_0) > 0$ such that

$$
\int [1 - \cos (2\pi x \xi)] \pi (dy) \geq c_2 w \left( |\xi|^{-1} \right)^{-1}
$$

for all $\xi \in \mathbb{R}^d$, assuming $w \left( |\xi|^{-1} \right)^{-1} = 0$ if $\xi = 0$.

Proof. The following simple estimates hold:

$$
(4.3) \quad |\sin x - x| \leq \frac{|x|^3}{6}, 1 - \cos x \leq \frac{1}{2} x^2, x \in \mathbb{R},
$$

$$
1 - \cos x \geq \frac{x^2}{\pi} \quad \text{if} \quad |x| \leq \pi/2.
$$

Part a) was proved in Lemma 7 of [7].

b) By (4.3), for all $\xi \in \mathbb{R}^d$, by Lemma 11, we have

$$
\int [1 - \cos (2\pi x \xi)] \pi (dy)
$$

$$
= \int [1 - \cos (2\pi \hat{\xi} \cdot y)] \pi_{|\xi|^{-1}} (dy) \geq \int_{|y| \leq \hat{\xi}} 4\pi |\hat{\xi} \cdot y|^2 \pi_{|\xi|^{-1}} (dy)
$$

$$
= 4^{-1} \int_{|y| \leq 1} \pi |\hat{\xi} \cdot 4y|^2 \pi_{|\xi|^{-1}} (dy)
$$

$$
\geq cw \left( |4\xi|^{-1} \right)^{-1} \geq cw \left( |\xi|^{-1} \right)^{-1}.
$$

□

**Lemma 11.** Let $\nu \in \mathfrak{A}^\sigma$, $w = w_\nu$ is a continuous $O$-RV function and $A$, $B$ hold. Let $\zeta, \zeta_0 \in C^\infty_0 (\mathbb{R}^d)$ be such that $0 \notin \operatorname{supp} (\zeta)$. Let $\hat{\zeta} = \mathcal{F}^{-1} \zeta, \hat{\zeta}_0 = \mathcal{F}^{-1} \zeta_0$, and

$$
H^R (t, x) = E\hat{\zeta} (x + Z^R_t), t \geq 0, x \in \mathbb{R}^d,
$$

$$
H^R_0 (t, x) = E\hat{\zeta}_0 (x + Z^R_t), t \geq 0, x \in \mathbb{R}^d.
$$
(i) For any \(0 < \alpha < p_1 \wedge p_2\) there are constants \(C_0, C_1, C_2 > 0\) independent of \(R\) such that

\[
\int (1 + |x|^{\alpha_2}) |H_R(t, x)| \, dx \leq C_1 e^{-C_2 t}, \quad t \geq 0,
\]
\[
\int |x|^{\alpha_2} |H_R(t, x)| \, dx \leq C_0 (1 + t), \quad t \geq 0,
\]
\[
\int |H_R(t, x)| \, dx \leq C_0, \quad t \geq 0.
\]

(ii) There are constants \(C_1, C_2 > 0\) independent of \(R\) so that for \(y \in \mathbb{R}^d\),

\[
\int |H_R(t, x + y) - H_R(t, x)| \, dx \leq C_1 |y| e^{-C_2 t},
\]
\[
\int |H_0^R(t, x + y) - H_0^R(t, x)| \, dx \leq |y| \int |\nabla \tilde{\zeta}_0(x)| \, dx.
\]

Proof. (i) Note that

\[
F H_R(t, \xi) = \exp \left\{ \psi_{\tilde{\nu}_R}^{\tilde{\nu}_R}(\xi) t \right\} \tilde{\zeta}(\xi), \xi \in \mathbb{R}^d.
\]

Following Lemma 2 of [8] (see also Corollary 2 of [6]), we choose \(\epsilon > 0\) so that \(\text{supp} (\tilde{\zeta}) \subset \{ \xi : |\xi| \leq \epsilon^{-1} \}\). Let \(\tilde{\nu}_{R,\epsilon}(dy) = \chi_{\{|y| \leq \epsilon\}} \tilde{\nu}_R(dy), R \in [0, 1]\). Then for \(\xi \in \text{supp} (\tilde{\zeta}), |y| \leq \epsilon\) and \(R \in (0, 1]\),

\[
1 - \cos (\xi \cdot y) \geq \frac{1}{\pi} |\xi \cdot y|^2 = \frac{|\xi|^2}{\pi} \left| \frac{\xi}{|\xi|} \cdot y \right|^2, \quad \hat{\xi} = \xi / |\xi|,
\]

and for some \(c_0 = c_0(\epsilon)\),

\[
-\text{Re} \psi_{\tilde{\nu}^{R,\epsilon}}^{\tilde{\nu}_R}(\xi) = \int_{|y| \leq \epsilon} [1 - \cos (\xi \cdot y)] \tilde{\nu}_{R,\epsilon}(dy)
\]

\[
\geq \frac{|\xi|^2}{\pi} \int_{|y| \leq \epsilon} \left| \frac{\xi}{|\xi|} \cdot y \right|^2 \tilde{\nu}_R(dy)
\]

\[
= \frac{|\xi|^2}{\pi} \int_{|y| \leq 1} \epsilon^2 \left| \frac{\xi}{|\xi|} \cdot y \right|^2 \frac{w(R)}{w(Re)} \tilde{\nu}_{\epsilon R}(dy) \geq c_0 |\xi|^2
\]

for all \(\xi \in \mathbb{R}^d\). Hence,

\[
H^R(t, \cdot) = F(t, \cdot) * P_t,
\]

where

\[
F(t, x) = \mathcal{F}^{-1} \left\{ \exp \left\{ \psi_{\tilde{\nu}^{R,\epsilon}}^{\tilde{\nu}_R}(t) \right\} \tilde{\zeta} \right\}(x) = E \tilde{\zeta} \left( x + Z_t^{\tilde{\nu}_{R,\epsilon}} \right), \quad t \geq 0, \quad x \in \mathbb{R}^d,
\]
and $P_t(dy)$ is the distribution of $Z_t^{\tilde{R}-\tilde{R}_R}$. From by Plancherel for any multiindex $\gamma \in \mathbb{N}^d_0$ and (4.4),

$$\int |x^\gamma F(t, x)|^2 \, dx \leq C \int |D^\gamma \left[ \zeta(\xi) \exp \{ \psi^{\tilde{R}_R, c}(\xi) t \} \right]|^2 \, d\xi$$

$$\leq C e^{-C_2 t}, \quad t \geq 0.$$  

By Cauchy-Schwarz inequality, with $d_0 = \left[ \frac{d}{2} \right] + 1$,

$$\int \left( 1 + |x|^2 \right) |F(t, x)| \, dx$$

$$= \int \left( 1 + |x|^2 \right) (1 + |x|)^{-d_0} |F(t, x)| (1 + |x|)^{d_0} \, dx$$

$$\leq \left( \int (1 + |x|)^{-2d_0} \, dx \right)^{1/2} \left( \int (1 + |x|)^4 |F(t, x)|^2 (1 + |x|)^{2d_0} \, dx \right)^{1/2}$$

$$\leq C \left( \int F(t, x)^2 (1 + |x|^2)^{d_0+2} \, dx \right)^{1/2} \leq C \exp \{-C_2 t\}, \quad t \geq 0.$$  

Let $0 < \alpha_2 < p_1 \land p_2$. By Lemma 4 and Lemma 17 of [8], there is $C > 0$ so that

$$\mathbb{E} \left[ \left| Z_t^{\tilde{R}-\tilde{R}_R} \right|^{\alpha_2} \right] = \int |y|^{\alpha_2} P_t(dy) \leq C (1 + t), \quad t \geq 0.$$  

Hence there are constants $C_1, C_2$ so that

$$\int (1 + |x|^{\alpha_2}) |H^R(t, x)| \, dx = \int (1 + |x|^{\alpha_2}) \left| \int F(t, x-y) P_t(dy) \right| \, dx$$

$$\leq \int \int (1 + |x-y|^{\alpha_2}) |F(t, x-y)| P_t(dy) \, dx$$

$$+ \int \int |y|^{\alpha_2} |F(t, x-y)| P_t(dy) \, dx$$

$$\leq C_1 e^{-C_2 t}, \quad t \geq 0,$$  

and

$$\int |x|^{\alpha_2} |H^R_0(t, x)| \, dx = \int |x|^{\alpha_2} \mathbb{E}_{\tilde{\zeta}_0} \left( x + Z^R_t \right) \, dx$$

$$\leq \mathbb{E} \int |x + Z^R_t|^{\alpha_2} \tilde{\zeta}_0(x + Z^R_t) \, dx + \mathbb{E} \left[ |Z^R_t|^{\alpha_2} \right] \int |\tilde{\zeta}_0(x)| \, dx$$

$$\leq C (1 + t).$$  

The last inequality is trivial.
(ii) Similarly as in part (i), for \( y \in \mathbb{R}^d \),
\[
\int |H^R (t, x + y) - H^R (t, x)| \, dx \\
= \int \left| \int \int_0^1 \nabla F (t, x + sy - z) \cdot ydsP_k (dz) \right| \, dx \\
\leq |y| \int |DF (t, x)| \, dx \leq C_1 |y|e^{-\zeta t}, t > 0,
\]
and directly
\[
\int |H_0^R (t, x + y) - H_0^R (t, x)| \, dx \leq |y| \int \left| \nabla \tilde{\zeta}_0 (x) \right| \, dx.
\]
\[\square\]

Lemmas 10, 11, and 12 are key ingredients in proving the following characterization of our Bessel potential spaces and Besov spaces. The proof follows verbatim as given in [8]. Therefore we omit it, and simply state the result.

**Proposition 2.** Let \( \nu \in \mathfrak{A}^s \), \( w = w_\nu \) is a continuous O-RV function and \( A, B \) hold. Let \( s \in \mathbb{R}, p, q \in (1, \infty) \). Define for \( N > 1 \), \( \tilde{H}_p^{\nu,N,s} (\mathbb{R}^d), \tilde{H}_p^{\nu,N,s} (\mathbb{R}^d; l_2) \), \( \tilde{B}_p^{\nu,N,s} (\mathbb{R}^d) \) as the spaces of all generalized function with finite norm
\[
|f|_{\tilde{H}_p^{\nu,N,s} (\mathbb{R}^d)} = \left( \sum_{j=0}^{\infty} \left| w_\nu (N^{-j})^{-s} f * \varphi_j \right|^2 \right)^{\frac{1}{2}}_{L_p(\mathbb{R}^d)}
\]
\[
|f|_{\tilde{H}_p^{\nu,N,s} (\mathbb{R}^d; l_2)} = \left( \sum_{k,j=0}^{\infty} \left| w_\nu (N^{-j})^{-s} f_k * \varphi_j \right|^2 \right)^{\frac{1}{2}}_{L_p(\mathbb{R}^d)}
\]
\[
|f|_{\tilde{B}_p^{\nu,N,s} (\mathbb{R}^d)} = \left( \sum_{j=0}^{\infty} \left| w_\nu (N^{-j})^{-sq} f * \varphi_j \right|^q \right)^{\frac{1}{q}}_{L_p(\mathbb{R}^d)}
\]
respectively. Then we have the following equivalence of norms
(i) \( \tilde{B}_p^{\nu,N,s} (\mathbb{R}^d) = B_p^{\nu,N,s} (\mathbb{R}^d) \) and the norms are equivalent
(ii) \( \tilde{H}_p^{\nu,N,s} (\mathbb{R}^d) = H_p^{\nu,s} (\mathbb{R}^d) \) and the norms are equivalent
(iii) \( \tilde{H}_p^{\nu,N,s} (\mathbb{R}^d; l_2) = H_p^{\nu,s} (\mathbb{R}^d; l_2) \) and the norms are equivalent.

Moreover, for all \( s, s' \in \mathbb{R} \), \( J^s : A^{s'} \rightarrow A^{s'-s} \) is an isomorphism where \( A^s = \tilde{B}_p^{\nu,N,s} (\mathbb{R}^d), \tilde{H}_p^{\nu,N,s} (\mathbb{R}^d) \) or \( \tilde{H}_p^{\nu,N,s} (\mathbb{R}^d; l_2) \).
5. Proof of the main results

Let $\nu \in \mathfrak{A}$, and $Z_t = Z_t^\nu$, $t \geq 0$, be the Levy process associated to it. Let $P_t(dy)$ be the distribution of $Z_t^\nu$, $t > 0$, and for a measurable $f \geq 0$,

$$T_t f(x) = \int f(x + y) P_t(dy), (t, x) \in E.$$ 

We represent the solution to (1.1) with smooth input functions using the following operators:

$$T^\lambda g(x) = e^{-\lambda t} \int g(x + y) P_t(dy), (t, x) \in E, \quad g \in \tilde{C}^\infty_{0,p}(\mathbb{R}^d), p > 1,$$

$$R_\lambda f(t, x) = \int_0^t e^{-\lambda(t-s)} \int f(s, x + z) P_{t-s}(dz) ds, (t, x) \in E, \quad f \in \tilde{C}^\infty_{0,p}(E), p > 1,$$

and

$$\tilde{R}_\lambda \phi(t, x) = \int_0^t e^{-\lambda(t-s)} \int U \phi(s, x + z) P_{t-s}(dz) ds, (t, x) \in E,$$

$$\phi \in \tilde{C}^\infty_{p,p}(E) \cap \tilde{C}^\infty_{2,p}(E) \text{ if } p \geq 2, \phi \in \tilde{C}^\infty_{p,p}(E) \text{ if } p \in (1, 2).$$

Lemma 12. (see Lemmas 7, 8 in [9]) Let $f \in \tilde{C}^\infty_{0,p}(E), g \in \tilde{C}^\infty_{0,p}(\mathbb{R}^d), \Phi \in \tilde{C}^\infty_{2,p}(E) \cap \tilde{C}^\infty_{p,p}(E)$ for $p \in [2, \infty)$ and $\Phi \in \tilde{C}^\infty_{p,p}(E)$ for $p \in (1, 2)$, then there is unique $u \in \tilde{C}^\infty_{0,p}(E)$ solving (1.1). Moreover,

$$u(t, x) = T^\lambda g(x) + R_\lambda f(t, x) + \tilde{R}_\lambda \Phi(t, x), (t, x) \in E,$$

and $u_1(t, x) = T^\lambda g(x), (t, x) \in E$, solves (1.1) with $f = 0, \Phi = 0, u_2 = R_\lambda f$ solves (1.1) with $g = 0, \Phi = 0$, and $u_3 = \tilde{R}_\lambda \Phi$ solves (1.1) with $g = 0, f = 0$. Moreover, the following estimates hold for any multiindex $\gamma$:

(i) $\mathbb{P}$-a.s.

$$|D^\gamma T^\lambda g|_{L^p(E)} \leq \rho^{\frac{1}{p}} |D^\gamma g|_{L^p(\mathbb{R}^d)}, \quad g \in \tilde{C}^\infty_{0,p}(\mathbb{R}^d), p \geq 1,$$

$$|D^\gamma R_\lambda f|_{L^p(E)} \leq \rho |D^\gamma f|_{L^p(E)}, \quad f \in \tilde{C}^\infty_{0,p}(E), p \geq 1,$$

and

$$|D^\gamma R_\lambda f(t, \cdot)|_{L^p(\mathbb{R}^d)} \leq \int_0^t |D^\gamma f(s, \cdot)|_{L^p(\mathbb{R}^d)} ds, t \geq 0,$$

$$|T^\lambda g|_{L^p(\mathbb{R}^d)} \leq e^{-\lambda t} |g|_{L^p(\mathbb{R}^d)}, t \geq 0, p \geq 1;$$
(ii) For each \( p \geq 2 \),
\[
\left| D^n \tilde{R}_\lambda \Phi \right|^{p}_{L_p(E)} \leq C \left[ \rho_\lambda \mathbb{E} \int_0^T |D^n \Phi (s, \cdot)|^p_{L_2, p}(\mathbb{R}^d) \, ds + \rho_\lambda |D^n \Phi|^{p}_{L_{p,p}(E)} \right],
\]
where \( \Phi \in \tilde{C}_{2,p}^\infty (E) \cap \tilde{C}_{p,p}^\infty (E) \), and for each \( p \in (1, 2) \),
\[
\left| D^n \tilde{R}_\lambda \Phi \right|^{p}_{L_p(E)} \leq C \rho_\lambda |D^n \Phi|^{p}_{L_{p,p}(E)}, \Phi \in \tilde{C}_{p,p}^\infty (E),
\]
where \( \rho_\lambda = T \wedge \frac{1}{\lambda} \). Moreover,
\[
\left| D^n \tilde{R}_\lambda \Phi (t, \cdot) \right|^{p}_{L_p(\mathbb{R}^d)} \leq C \left[ \mathbb{E} \left[ \left( \int_0^t |D^n \Phi (s, \cdot)|^2_{L_2, p}(\mathbb{R}^d) \, ds \right)^{p/2} \right] + \mathbb{E} \int_0^t |D^n \Phi (s, \cdot)|^{p}_{L_{p,p}(\mathbb{R}^d)} \, ds \right],
\]
if \( p \geq 2 \), and
\[
\left| D^n \tilde{R}_\lambda \Phi (t, \cdot) \right|^{p}_{L_p(\mathbb{R}^d)} \leq C \mathbb{E} \int_0^t |D^n \Phi (s, \cdot)|^{p}_{L_{p,p}(\mathbb{R}^d)} \, ds, t > 0,
\]
if \( p \in (1, 2) \).

5.1. Initial value part estimate. The solution associated to the initial value function is given explicitly by
\[
T_\lambda g = G^\lambda_t * g, \quad g \in \tilde{C}_{0,p}^\infty (\mathbb{R}^d),
\]
where \( G^\lambda_t (x) := \exp (-\lambda t) p^{\nu^*} (t, x), \nu^* (dy) = \nu (-dy) \).

We prove that for each \( \omega \in \Omega \), there is \( C = C (\nu, d, p) \) so that
\[
|L^n T_\lambda g|_{H_{p,s}^{\nu,s}(E)} \leq C |g|_{B_{p,p}^{\nu,N:s+1-1/p}(\mathbb{R}^d)}.
\]

Since by Proposition 2, \( J_{\nu}^t : H_{p,s}^{\nu,s}(\mathbb{R}^d) \to H_{p,s-t}^{\nu,s-t}(\mathbb{R}^d) \) and \( J_{\nu}^t : B_{p,p}^{\nu,N:s}(\mathbb{R}^d) \to B_{p,p}^{\nu,N:s-t}(\mathbb{R}^d) \) are isomorphisms for any \( s, t \in \mathbb{R} \), it is enough to derive the estimate for \( s = 0 \).

Lemma 13. Let \( \nu \in \mathfrak{A}^\nu, w = w_\nu \) is a continuous O-RV function and \( A, B \) hold. Then there is \( \bar{C} = C (\nu, d, p) \) so that
\[
|L^n T_\lambda g|_{L_p(E)} \leq \bar{C} |g|_{B_{p,p}^{\nu,N:1-1/p}(\mathbb{R}^d)}, g \in \tilde{C}_{0,p}^\infty (\mathbb{R}^d).
\]

Proof. We will use an equivalent norm. Let \( N > 1 \) be an integer. There exists a function \( \phi \in C_0^\infty (\mathbb{R}^d) \) such that \( \text{supp} \phi = \{ \xi : 1 \leq |\xi| < N \} \), \( \phi (\xi) > 0 \) if \( N^{-1} \leq |\xi| < N \) and
\[
\sum_{j=-\infty}^{\infty} \phi (N^{-j} \xi) = 1 \quad \text{if } \xi \neq 0.
\]
Let
\[ \tilde{\phi}(\xi) = \phi(N\xi) + \phi(\xi) + \phi(N^{-1}\xi), \xi \in \mathbb{R}^d. \]

Note that \( \text{supp} \, \tilde{\phi} \subseteq \{N^{-2} \leq |\xi| \leq N^2\} \) and \( \tilde{\phi} \phi = \phi \). Let \( \varphi_k = \mathcal{F}^{-1} \phi(N^{-k} \cdot), k \geq 1, \) and \( \varphi_0 \in \mathcal{S}(\mathbb{R}^d) \) is defined as

\[ \varphi_0 = \mathcal{F}^{-1} \left[ 1 - \sum_{k=1}^{\infty} \phi(N^{-k} \cdot) \right]. \]

Let \( \phi_0(\xi) = \mathcal{F} \varphi_0(\xi), \tilde{\phi}_0(\xi) = \mathcal{F} \varphi_0(\xi) + \mathcal{F} \varphi_1(\xi), \xi \in \mathbb{R}^d, \tilde{\varphi} = \mathcal{F}^{-1} \tilde{\varphi}, \varphi = \mathcal{F}^{-1} \varphi \). Let

\[ \tilde{\varphi}_k = \sum_{l=-1}^{1} \varphi_{k+l}, k \geq 1, \tilde{\varphi}_0 = \varphi_0 + \varphi_1. \]

Note that \( \varphi_k = \tilde{\varphi}_k \ast \varphi_k, k \geq 0. \) Obviously, \( g = \sum_{k=0}^{\infty} g \ast \varphi_k \) in \( \mathcal{S}'(\mathbb{R}^d) \) for \( g \in \mathcal{S}(\mathbb{R}^d) \). For \( j \geq 1, \)

\[
\mathcal{F}[L^\nu T_\lambda g(t, \cdot) \ast \varphi_j] = w(N^{-j})^{-1} \tilde{\varphi}_N^{-j}(N^{-j}\xi) \exp \left\{ w(N^{-j})^{-1} \tilde{\varphi}_N^{-j}(N^{-j}\xi) t - \lambda t \right\} 
\]

\[
\times \tilde{\phi}(N^{-j}\xi) \tilde{g}_j(\xi),
\]

and

\[
\mathcal{F}[L^\nu T_\lambda g(t, \cdot) \ast \varphi_0] = \psi(\xi) \exp \{ \psi(\xi) t - \lambda t \} \tilde{\phi}_0(\xi) \tilde{g}_0(\xi),
\]

where \( g_j = g \ast \varphi_j, j \geq 0. \)

Let \( Z^j = Z^\nu N^{-j}, j \geq 1. \) Let \( \tilde{\phi} \in C_0^\infty(\mathbb{R}^d), 0 \notin \text{supp} \, (\tilde{\phi}) \) and \( \tilde{\phi} \tilde{\phi} = \tilde{\phi}, \tilde{\eta} = \mathcal{F}^{-1} \tilde{\nu}_0, \) we have

\[ L^\nu T_\lambda g(t, \cdot) \ast \varphi_j = w(N^{-j})^{-1} \tilde{H}_t^\lambda, j \geq 1,
\]

\[ L^\nu T_\lambda g(t, \cdot) \ast \varphi_0 = \tilde{H}_t^\lambda, 0 \ast g_0, t > 0,
\]

where for \( j \geq 1, \)

\[
\tilde{H}_t^\lambda(x) = N^{jd} H_{w(N^{-j})^{-1}t}(N^j x), (t, x) \in E,
\]

\[
H_t^\lambda = e^{-\lambda \nu(N^{-j})t} L^\nu_{N^{-j}} \tilde{\eta} \tilde{\nu}(\cdot + Z_t^j), t > 0,
\]

and

\[
\tilde{H}_t^\lambda(x) = e^{-\lambda t} L^\nu \mathbb{E} \tilde{\eta}(\cdot + Z_t^\nu), (t, x) \in E.
\]

By Lemma 5 in [6],

\[
\sup_j \int |L^\nu_{N^{-j}} \tilde{\eta}| \, dx < \infty.
\]

Hence by Lemma [11]

\[
\int |H_t^\lambda| \, dx \leq \int |L^\nu_{N^{-j}} \tilde{\eta}| \, dx \int |\tilde{\varphi}(\cdot + Z_t^\nu)| \, dx \leq C e^{-ct}, t > 0, j \geq 1,
\]
and hence,
\[(5.4) \quad \int |\tilde{H}_t^{\lambda,j}| \, dx \leq C \exp \left\{ -cw (N^{-j})^{-1} t \right\}, \quad t > 0, j \geq 1.\]
and by Lemma \[5\]
\[(5.5) \quad \int |\tilde{H}_t^{\lambda,0}| \, dx \leq C \left( \frac{1}{t} \wedge 1 \right), \quad t > 0.\]

It follows by Proposition \[2\] and \[(5.3)\] that
\[|L^\mu T_\lambda g(t)|_{L^p(\mathbb{R}^d)}^p \leq C \left( \sum_{j=1}^{\infty} \left| w (N^{-j})^{-1} \tilde{H}_t^{\lambda,j} * g_j \right| \right)^{1/2p} \left. \right|_{L^p(\mathbb{R}^d)}^{1/2p} \]
\[+ C \int |\tilde{H}_t^{\lambda,0} * g_0|^p \, dx.\]

Hence
\[|L^\mu T_\lambda g(t)|_{L^p(\mathbb{R}^d)}^p \leq C \sum_{j=0}^{\infty} \left| w (N^{-j})^{-1} \tilde{H}_t^{\lambda,j} * g_j \right|^p \left. \right|_{L^p(\mathbb{R}^d)} \text{ if } p \in (1, 2],\]
and, by Minkowski inequality,
\[|L^\mu T_\lambda g(t)|_{L^p(\mathbb{R}^d)}^p \leq C \left( \sum_{j=1}^{\infty} \left( \int \left| w (N^{-j})^{-1} \tilde{H}_t^{\lambda,j} * g_j \right|^p \, dx \right)^{2/p} \right)^{p/2} \]
\[+ C \int |\tilde{H}_t^{\lambda,0} * g_0|^p \, dx\]
if \(p > 2\). Now, by \[(5.4)\],
\[(5.6) \quad \int \left| w (N^{-j})^{-1} \tilde{H}_t^{\lambda,j} * g_j \right|^p \, dx \leq \left( \int |\tilde{H}_t^{\lambda,j}| \, dx \right)^p \int \left| w (N^{-j})^{-1} g_j \right|^p \, dx \leq C w (N^{-j})^{-p} \exp \left\{ -cw (N^{-j})^{-1} t \right\} \left| g_j \right|_{L^p}^p \text{ if } j \geq 1,\]
and, by \[(5.5)\],
\[(5.7) \quad \int |\tilde{H}_t^{\lambda,0} * g_0|^p \, dx \leq C \left( \frac{1}{t} \wedge 1 \right)^p \int |g_0|^p \, dx.\]
Therefore for \(p \in (1, 2]\),
\[\int_0^\infty |L^\mu T_\lambda g(t)|_{L^p(\mathbb{R}^d)}^p \, dt \leq C \sum_{j=0}^{\infty} \left| w (N^{-j})^{-1-1/p} \left| g_j \right|_{L^p(\mathbb{R}^d)} \right|^p,\]
and \[(5.2)\] follows by Proposition \[2\].
Let $p > 2$. In this case,

$$\int_0^\infty |L^\nu T_\lambda g(t)|_{L_p(\mathbb{R}^d)}^p dt \leq C[G + |g_0|_{L_p(\mathbb{R}^d)}^p],$$

where

$$G = \int_0^\infty \left( \sum_{j=1}^\infty \exp \left\{ -cw (N^{-j})^{-1} t \right\} k_j^2 \right)^{p/2} dt$$

with $c > 0$ and

$$k_j = w (N^{-j})^{-1} |g_j|_{L_p(\mathbb{R}^d)}, j \geq 1.$$

Now, let $B = \{ j : a(t) \leq N^{-j} \}$ where $a(t) = \inf \{ r : w(r) \geq t \}, t > 0$. Then

$$\sum_{j=1}^\infty e^{-cw(N^{-j})^{-1} t} k_j^2 = \sum_{j \in B} \ldots + \sum_{j \notin B} \ldots = D(t) + E(t), t > 0.$$

Let $0 < \frac{\beta_0}{2} < p_1 \land p_2$, by Hölder inequality,

$$D(t) \leq C \sum_{j=1}^\infty \chi_{\{ j : a(t) \leq N^{-j} \}} \left( \frac{a(t)}{N^{-j}} \right)^\beta \left( \frac{a(t)}{N^{-j}} \right)^{-\beta} k_j^2$$

$$\leq C \left( \sum_{j=1}^\infty \chi_{\{ j : a(t) \leq N^{-j} \}} \left( \frac{a(t)}{N^{-j}} \right)^{\beta \frac{p-2}{p-\beta}} \right)^{1-\frac{2}{p}} \left( \sum_{j=1}^\infty \chi_{\{ j : j \leq N^{-j} \}} \left( \frac{a(t)}{N^{-j}} \right)^{-\beta \frac{p}{2}} k_j^p \right)^{\frac{2}{p}}$$

$$= CD_1^p D_2^p.$$

Denoting $\beta' = \beta p/(p-2)$, we have for $t > 0$,

$$D_1(t) = \sum_{j=1}^\infty \chi_{\{ j : \frac{a(t)}{N^{-j}} \leq 1 \}} \left( \frac{a(t)}{N^{-j}} \right)^{\beta'}$$

$$\leq C \int_0^\infty \chi_{\{ j : \frac{a(t)}{N^{-j}} \leq 1 \}} \left( \frac{a(t)}{N^{-j}} \right) \frac{\beta'}{N^{-x}} \frac{dx}{y} \leq C \int_0^1 y^{\beta'} dy < \infty.$$

Applying Corollary with $1 > \beta \cdot \frac{p}{2} \cdot \left( \frac{1}{p_1} \lor \frac{1}{p_2} \right)$, and arbitrary $\rho > 1$,

$$\int_0^\infty D_2^{p/2} dt \leq C \sum_{j=1}^\infty \int_0^\infty \chi_{\{ j : \frac{a(t)}{N^{-j}} \leq 1 \}} \left( \frac{a(t)}{N^{-j}} \right)^{-\beta \frac{p}{2}} k_j^p dt$$

$$\leq C \sum_{j=1}^\infty \int_0^{\kappa(N^{-j})^\rho} \left( \frac{a(t)}{N^{-j}} \right)^{-\beta \frac{p}{2}} dt k_j^p \leq C \sum_{j=1}^\infty \kappa(N^{-j}) \left( \frac{a(t)}{N^{-j}} \right)^{-\beta \frac{p}{2}} k_j^p.$$
Hence
\[ \int_0^\infty D_2^{p/2} dt \leq C \sum_{j=1}^\infty \kappa \left( N^{-j} \right) k_j^p. \]

Now, we estimate the second term \( E(t), t > 0 \). By Hölder inequality, for \( t > 0 \),
\[
E(t) = \sum_{a(t) > N^{-j}} e^{-cw(N^{-j})^{-1}t} w(N^{-j})^{-2} |g_j|^2_{L_p}
\leq \left( \sum_{a(t) > N^{-j}} e^{-cw(N^{-j})^{-1}t} \right)^{p-2} \left( \sum_{a(t) > N^{-j}} e^{-cw(N^{-j})^{-1}t} k_j^p \right)^{\frac{2}{p}}.
\]

Changing the variable of integration \( y = \frac{1}{a(t)N^j} \), we have for some \( l > 0 \) (see Lemma 1),
\[
\sum_{a(t) > N^{-j}} e^{-cw(N^{-j})^{-1}t} \leq \sum_{a(t) \geq N^{-j}} \exp \left\{ -c \frac{w(a(t))}{w(N^{-j})} \right\}
\leq \sum_{a(t) \geq N^{-j}} \exp \left\{ -c \left( \frac{a(t)}{N^{-j}} \right)^l \right\} = \sum_{N^j a(t) \geq 1} \exp \left\{ -c (N^j a(t))^l \right\}
\leq C \int_{N^j a(t) \geq 1} \exp \left\{ -c (N^j a(t))^l \right\} dx = C \int_1^\infty \exp \left\{ -cy^l \right\} dy
\]
Hence
\[ \int_0^\infty E(t)^{p/2} dt \leq C \int_0^\infty \sum_{\kappa(N^{-j})^{-1}t \geq 1} e^{-cw(N^{-j})^{-1}t} k_j^p dt \leq C \sum_j w(N^{-j}) k_j^p. \]

The estimate (5.2) is proved. \( \Box \)

5.2. Estimates of \( R_\lambda f, \tilde{R}_\lambda \Phi \), verification of Hörmander conditions. First we show that for each \( p > 1 \) there is \( C > 0 \) so that
\[ |L^{\nu} R_\lambda f|_{\tilde{H}^0_p(E)} \leq C |f|_{\tilde{H}^0_p(E)} \quad f \in \tilde{C}_0^\infty (E). \]

According to Lemma 1
\[ \frac{w(\epsilon r)}{w(r)} \leq C \epsilon^{\alpha_2} \vee \epsilon^{\alpha_1}, r, \epsilon > 0. \]
Thus \( w \) is a scaling function with scaling factor \( \epsilon^{\alpha_2} \vee \epsilon^{\alpha_1} \) as defined in [7] and [8]. Therefore, we may apply Calderon-Zygmund theorem (see Theorem 5 in [7]) to derive (5.8). For \( p = 2 \), the estimate (5.8) follows by Plancherel identity (see [8]). We remind some simple facts that for a non-decreasing continuous function \( w : (0, \infty) \rightarrow (0, \infty) \) such that \( \lim_{t \rightarrow 0} w(\epsilon) = 0 \) and...
Lemma 14. Let \( \nu \in \mathbb{A}^\sigma \), \( w = w_\nu \) is a continuous O-RV function and \( A, B \) hold. Let \( \alpha_1 > q_1 \lor q_2 \), \( 0 < \alpha_2 < p_1 \land p_2 \), and
\[
\alpha_2 > 1 \text{ if } \sigma \in (1, 2),
\alpha_1 \leq 1 \text{ if } \sigma \in (0, 1), \alpha_1 \leq 2 \text{ if } \sigma \in [1, 2).
\]
Let \( \pi \in \mathbb{A}^\sigma_{|\pi|} \), and assume that
\[
\int_{|y| \leq 1} |y|^{\alpha_1} d|\pi|_R + \int_{|y| > 1} |y|^{\alpha_2} d|\pi|_R \leq M, R > 0.
\]
Let
\[
K^\sigma_{\nu} (t, x) = e^{-\lambda t} L^\pi p^{\nu^*} (t, x) \chi_{[t, \infty]} (t), t > 0, x \in \mathbb{R}^d,
\]
where \( \nu^* (dy) = \nu (-dy) \). There exist \( C_0 > 1 \) and \( C \) so that
\[
(5.9) \quad \mathcal{I} = \int \chi_{Q_{C_0 \eta} (0)} \left( t, x \right) |K^\sigma_{\nu} (t-s, x-y) - K^\sigma_{\nu} (t, x)| \, dx \, dt \leq CM
\]
for all \( |s| \leq w (\eta) \), \( |y| \leq \eta, \eta > 0 \), where \( Q_{C_0 \eta} (0) = (-w (C_0 \eta), w (C_0 \eta)) \times \{ x : |x| \leq C_0 \eta \} \).

Proof. Fix \( \rho > 1 \) throughout the proof. By Lemma 1 we choose \( C_0 > 3 \) such that \( w (C_0 \eta) > 3w (\eta), \eta > 0 \). We split
\[
\mathcal{I} = \int_{-\infty}^{2|s|} \int \cdots \int_{-\infty}^{2|s|} \int \cdots = \mathcal{I}_1 + \mathcal{I}_2
\]
Since \( w (C_0 \eta) > 3w (\eta), \eta > 0 \), it follows by Lemma 5 Corollary 1 and Lemma 10 for \( A \), denoting \( k_0 = C_0 - 1 \),
\[
\mathcal{I}_1 \leq C \int_{0}^{3|s|} \int_{|x| > k_0 a(|s|)} |L^\pi p^{\nu^*} (t, x)| \, dx \, dt
\]
\[
\leq CM \frac{1}{a(|s|)^{\alpha_2}} \int_{0}^{3|s|} t^{-1} a (t) \, dt \leq CM \frac{a (3 |s|)^{\alpha_2}}{a (|s|)^{\alpha_2}} = CM.
\]
Now,  
\[
\mathcal{I}_2 \leq \int_{2|s|}^{\infty} \int \chi_{Q_{\epsilon_0}(0)} \left| L^\pi p^{\nu^*} (t-s, x-y) - L^\pi p^{\nu^*} (t-s, x) \right| \, dx \, dt \\
+ \int_{2|s|}^{\infty} \int \chi_{Q_{\epsilon_0}(0)} \chi_{[\epsilon, \infty)} (t-s) L^\pi p^{\nu^*} (t-s, x) \\
- \chi_{[\epsilon, \infty)} (t) L^\pi p^{\nu^*} (t, x) \, dx \, dt \\
= I_{2,1} + I_{2,2}.
\]

We split the estimate of $I_{2,1}$ into two cases.

Case 1. Assume $|y| \leq a (2 |s| + )$. Then, by Lemma III, Corollary [1], and Lemma [1]

\[
I_{2,1} \leq C M |y| \int_{2|s|}^{\infty} (t-s)^{-1} a (|t-s|)^{-1} \, dt \\
= C M |y| a (2 |s|) \leq C M |y| a (|s|)^{-1} \\
\leq C M \frac{|y|}{a (2 |s|)} \leq C M.
\]

Case 2. Assume $|y| > a (2 |s| + )$, i.e. $\eta \geq |y| > a (2 |s| + )$ and $a^{-1} (\eta) \geq a^{-1} (|y|) \geq 2 |s|$. We split

\[
I_{2,1} = \int_{2|s|}^{2|s| + a^{-1}(|y|)} + \int_{2|s| + a^{-1}(|y|)}^{\infty} \int ... = I_{2,1,1} + I_{2,1,2}.
\]

If $2 |s| \leq t \leq 2 |s| + a^{-1} (|y|)$, then $0 \leq t \leq 3 a^{-1} (\eta) \leq 3 w (\eta) \leq w (C_0 \eta)$.

Hence $|x| > C_0 \eta \geq a (2 |s|) + |y|$ and

\[
|x - y| \geq (C_0 - 1) \eta = k_0 \eta \geq \frac{k_0}{2} [a (2 |s|) + |y|] \\
\geq a (2 |s|) + |y| \text{ if } (t, x) \notin Q_{C_0 \eta} (0).
\]

Also,

\[(5.10) \quad 2 \geq \frac{2 |s| + a^{-1} (|y|)}{2 |s| + a^{-1} (|y|) - s} \geq \frac{2}{3},\]

and, by Lemma IV

\[(5.11) \quad \frac{a (3 |s| + a^{-1} (|y|))}{a (2 |s|) + |y|} \leq \frac{a (\frac{5}{2} a^{-1} (|y|))}{a (a^{-1} (|y|))} \leq C.
\]
By Lemma 8 Corollary 1 and (5.11),

\[ I_{2,1,1} \leq C \int_{2|s|}^{2|s| + a^{-1}(|y|)} \int_{|x| > a(2|s|) + |y|} |L^\pi p^{\nu^*} (t - s, x)| \, dx \, dt \]

\[ \leq \frac{CM}{a(2|s|) + |y|^{\alpha_2}} \int_{0}^{2|s| + a^{-1}(|y|)} (t - s)^{-1} a(|t - s|)^{\alpha_2} \, dt \]

\[ \leq CM \frac{a(3|s| + a^{-1}(|y|))^{\alpha_2}}{a(2|s|) + |y|^{\alpha_2}} \leq CM \]

Then, by Lemma 8 Corollary 1 and (5.10)

\[ I_{2,1,2} \leq \int_{2|s| + a^{-1}(|y|)}^{\infty} \left[ \int_{\mathbb{R}^d} |L^\pi p^{\nu^*} (t - s, x - y) - L^\pi p^{\nu^*} (t - s, x)| \, dx \right] \, dt \]

\[ = CM |y| \int_{2|s| + a^{-1}(|y|)}^{\infty} (t - s)^{-1} a(|t - s|)^{-1} \, dr \]

\[ \leq CM |y| a(2|s| + a^{-1}(|y|) - s)^{-1} \leq CM |y| a(a^{-1}(|y|) + |s|)^{-1} \]

\[ \leq CM |y| |y|^{-1} \leq CM \]

Hence, \( I_{2,1} \leq C. \)

Finally, by Lemma 8

\[ I_{2,2} \leq \int_{2|s|}^{\infty} \int_{\mathbb{R}^d} \chi_{Q_y(0)} \left| L^\pi p^{\nu^*} (t - s, x) - L^\pi p^{\nu^*} (t, x) \right| \, dx \, dt \]

\[ + \int_{\varepsilon<s} \int \left| L^\pi p^{\nu^*} (t, x) \right| \, dx \, dt \]

\[ \leq CM. \]

The proof is complete.

\[ \square \]

**Remark 4.** Although we write Lemma 12 in this paper we only need the result for \( \pi = \nu. \)

Now we will show that for \( p \in [2, \infty), \) there is \( C \) so that for all \( \Phi \in \tilde{C}^{\infty}_{2,p}(E) \cap \tilde{C}^{\infty}_{p,p}(E), \)

(5.12) \[ \left| L^\nu \tilde{R}^\pi \Phi \right|_{E_{p;}^{\nu_a}(E)} \leq C \left[ \left| \Phi \right|_{E_{p;}^{\nu_a+1/2}(E)} + \left| \Phi \right|_{E_{2,p;}^{\nu_a+1/2}(E)} \right], \]

and for \( p \in (1, 2), \) there is \( C \) so that for all \( \Phi \in \tilde{C}^{\infty}_{p,p}(E), \)

(5.13) \[ \left| L^\nu \tilde{R}^\pi \Phi \right|_{E_{p;}^{\nu_a}(E)} \leq C \left| \Phi \right|_{E_{p;}^{\nu_a+1/2}(E)} \]

\[ \leq C \rho_\lambda^{1/p} \left| \Phi \right|_{E_{2,p;}^{\nu_a}(R^d)}. \]
It is enough to consider the case \( s = 0 \). Let \( \varepsilon > 0 \),

\[ G_{s,t}^{\lambda,\varepsilon} (x) = \exp \left( -\lambda (t-s) \right) p^{\nu^*} (t-s, x) \chi_{[\varepsilon,\infty]} (t-s), 0 < s < t, x \in \mathbb{R}^d, \]

where \( \nu^* (dy) = \nu (-dy) \). Denote

\[ Q (t,x) = \int_0^t \int_U \Phi_{\varepsilon} (s,x,\nu) q (ds,d\nu), (t,x) \in E, \]

\[ \Phi_{\varepsilon} (s,x,\nu) = \int \left( L^{\nu} G_{s,t}^{\lambda,\varepsilon} \right) (x-y) \Phi (s,y,\nu) dy, (s,x) \in E. \]

Obviously, with

\[ K_{\lambda}^{\varepsilon} (t,x) = e^{-\lambda \frac{1}{2} \nu^* (t,x)} \chi_{[\varepsilon,\infty]} (t), t > 0, x \in \mathbb{R}^d, \]

we have

\[
(5.14) \quad \mathbb{E} \left[ \int_0^t \int_U \left( L^{\nu} G_{s,t}^{\lambda,\varepsilon} \right) (x-y) \Phi (s,y,\nu) dy q (ds,d\nu) \right]^{p}_{L_p (E)} \\
= \mathbb{E} \left[ \int_0^t \int_U L^{\nu^{1/2}} G_{s,t}^{\lambda,\varepsilon} (x-y) L^{\nu^{1/2}} \Phi (s,y,\nu) dy q (ds,d\nu) \right]^{p}_{L_p (E)} \\
= \mathbb{E} \left[ \int_0^t \int_U K_{\lambda}^{\varepsilon} (t-s,x-y) L^{\nu^{1/2}} \Phi (s,y,\nu) dy q (ds,d\nu) \right]^{p}_{L_p (E)}.
\]

If \( 2 \leq p < \infty \), then

\[
\mathbb{E} \int_0^T |Q (t,\cdot)|^p_{L_p (\mathbb{R}^d)} dt \\
\leq C \mathbb{E} \left\{ \int_0^T \left[ \int_0^t \int_U \Phi_{\varepsilon} (s,\cdot,\nu)^2 \Pi (d\nu) ds \right]^{1/2}_{L_p (\mathbb{R}^d)} dt \right\} \\
+ C \mathbb{E} \left\{ \int_0^T \int_0^t \int_U \Phi_{\varepsilon} (s,\cdot,\nu)^p_{L_p (\mathbb{R}^d)} \Pi (d\nu) ds dt \right\} = C (\mathbb{E} I_1 + \mathbb{E} I_2).
\]

If \( 1 < p < 2 \), then

\[
\mathbb{E} \int_0^T |Q (t,\cdot)|^p_{L_p (\mathbb{R}^d)} dt \leq C \mathbb{E} I_2.
\]

**Estimate of \( \mathbb{E} I_2.** Let \( B_{t}^{\lambda} g (x) = e^{-\lambda} \mathbb{E} g (x + Z_t^{\nu}), (t,x) \in E, g \in \tilde{C}_{0,p} ^{\infty} (\mathbb{R}^d). \)

Then
\[ I_2 = \int_0^T \int_0^t \int_U \left[ \Phi_x (s, \cdot, \nu) \right]_{L_p (\mathbb{R}^d)}^p \Pi (d\nu) \, ds \, dt \]
\[ = \int_0^T \int_0^t \int_U \left[ L^\nu G_{s,t} \Phi (s, \cdot, \nu) \right]_{L_p (\mathbb{R}^d)}^p \Pi (d\nu) \, ds \, dt \]
\[ \leq \int_0^T \int_0^t \int_U \left[ L^\nu B_{t-s} \Phi (s, \cdot, \nu) \right]_{L_p (\mathbb{R}^d)}^p \Pi (d\nu) \, ds \, dt \]
\[ = \int_U \int_0^T \int_S \left[ L^\nu B_{t-s} \Phi (s, \cdot, \nu) \right]_{L_p (\mathbb{R}^d)}^p dt \, ds \, \Pi (d\nu) \]

It follows from Proposition 2 and Lemma 13 that for \( p > 1 \),

\[ E I_2 \leq E \int_U \int_0^T \int_S \left[ L^\nu B_{t-s} \Phi (s, \cdot, \nu) \right]_{L_p (\mathbb{R}^d)}^p dt \, ds \, \Pi (d\nu) \]
\[ \leq C E \int_U \int_0^T \int_0^\infty \left[ \kappa (N^{-j})^{-1/(1-p)} \right] \left[ \Phi (s, \cdot, \nu) \right]_{L_p (\mathbb{R}^d)}^p \, ds \, \Pi (d\nu) \]
\[ = C \left[ \Phi \left[ \mathbb{P}^{\nu,N:1-1/p} (E) \right] \right] \cdot \]

Hence (5.13) holds for \( p \in (1, 2) \). We prove that for \( p \geq 2 \),

(5.15) \[ E I_1 \leq C \left[ \Phi \left[ \mathbb{P}^{\nu,1/2} (E) \right] \right], \Phi \in \mathbb{E} \infty (E), \]

by verifying Hörmander condition (5.16) (see 6.2.2, [9]) in Lemma 15 below.

In the following statement we show that a version of the stochastic Hörmander condition holds which implies (5.15) in Lemma 14. Due to similarity to Lemma 14 (following the same splitting), we skip some details.

**Lemma 15.** Let \( \nu \in \mathcal{A}^\sigma, w = w_\nu \) is a continuous O-RV function and \( A, B \) hold. Let

\[ K^\nu_\lambda (t, x) = e^{-\lambda t} L^\nu_2 p^{\nu^*} (t, x) \chi_{[\kappa, \infty)} (t), t > 0, x \in \mathbb{R}^d, \]

where \( \nu^* (dy) = \nu (-dy) \). There exists \( C_0 > 0 \) and \( N > 0 \) such that for all \( |s| \leq w (\eta), |y| \leq \eta, \eta > 0 \), we have

(5.16) \[ \mathcal{I} = \int_0^T \left( \int \chi_{Q_\eta (t)} \right) \left| K^\nu_\lambda (t - s, x - y) - K^\nu_\lambda (t, x) \right| \, dx \, dt \leq N, \]

where \( Q_\eta (0) = (-w (C_0 \eta), w (C_0 \eta)) \times \{ x : |x| < C_0 \eta \} \).

**Proof.** By Lemma 11 we choose \( C_0 > 3 \) such that \( w (C_0 \eta) > 3w (\eta), \eta > 0 \). We split

\[ \mathcal{I} = \int_{-\infty}^{2|s|} \left( \int \ldots \right) \, dt + \int_{2|s|}^{\infty} \left( \int \ldots \right) \, dt = \mathcal{I}_1 + \mathcal{I}_2. \]
Since $\kappa (C_0 \eta) > 3 \kappa (\eta), \eta > 0$, it follows by Lemmas 9, 1 and Corollary 1 with $k_0 = C_0 - 1$ and $\beta \in (0, \frac{\mu_2}{2})$,

$$|I_1| \leq C \int_0^3 |s| \left[ \int_{|x| > k_0 a(|s|)} |L^{\nu_2 \frac{1}{2} p \nu}(t, x)| \, dx \right]^2 \, dt$$

$$\leq C \int_0^3 |s| \left( t^{\frac{1}{2} a (t) \beta (k_0 a(|s|))^{-\beta}} \right)^2 \, dt \leq C a (|s|)^{-2 \beta} \int_0^3 a(t)^{2 \beta} \frac{dt}{t}$$

$$\leq C a (s)^{-2 \beta} a (3 |s|)^{2 \beta} \leq C$$

Now,

$$I_2 = 2 \int_{2|s|}^{\infty} \left[ \int \chi_{Q_{C_0 \eta}^0 (0)} \left| L^{\nu_2 \frac{1}{2} p \nu}(t - s, x - y) - L^{\nu_2 \frac{1}{2} p \nu}(t - s, x) \right| \, dx \right]^2 \, dt$$

$$+ 2 \int_{2|s|}^{\infty} \left\{ \int \chi_{Q_{C_0 \eta}^0 (0)} \chi_0 (t - s) L^{\nu_2 \frac{1}{2} p \nu}(t - s, x) - \chi_0 (t) L^{\nu_2 \frac{1}{2} p \nu}(t, x) \, dx \right\}^2 \, dt$$

$$= I_{2,1} + I_{2,2}.$$

We split the estimate of $I_{2,1}$ into two cases.

**Case 1.** Assume $|y| \leq a (2 |s| +)$. Then by Lemmas 9, 1 and Corollary 1

$$I_{2,1} \leq C \int_{2|s|}^{\infty} \frac{|y|^2}{(t - s) a (t - s)} \, dt$$

$$\leq C |y|^2 \int_{2|s|}^{\infty} \frac{1}{a (t - s)^2} \, dt$$

$$\leq C |y|^2 \int_{2|s|}^{\infty} a (t) -2 dt \leq C |y|^2 a (|s|)^{-2} \leq C$$

**Case 2.** Assume $|y| > a (2 |s| +)$ i.e. $\eta \geq |y| > a (2 |s| +)$ and $a^{-1} (\eta) \geq a^{-1} (|y|) \geq 2 |s|$. We split

$$I_{2,1} = \int_{2|s|}^{2|s| + a^{-1}(|y|)} \left[ \int \ldots \right]^2 + \int_{2|s| + a^{-1}(|y|)}^{\infty} \left[ \int \ldots \right]^2 = I_{2,1,1} + I_{2,1,2}.$$

Hence, with $\beta \in (0, \frac{\mu_2}{2})$, by Lemmas 9, 1 and Corollary 1

$$I_{2,1,1} = C \int_{2|s|}^{2|s| + a^{-1}(|y|)} \left[ \int_{|x| > a (2 |s| +) + |y|} \left| L^{\nu_2 \frac{1}{2} p \nu}(t - s, x) \right| \, dx \right]^2 \, dt$$

$$\leq \frac{C}{(a (2 |s| + |y|)^{2 \beta} \int_{2|s|} \frac{a (t)^{2 \beta} dt}{(t - s)}}$$

$$\leq \frac{C}{(a (2 |s| + |y|)^{2 \beta} \int_{2|s|} a (3 |s| + a^{-1}(|y|))^{2 \beta}} \leq C$$
Then by Lemma 9 and Corollary 1

$$I_{2,1,2} \leq C \int_{2|s|+a^{-1}(|y|)}^{\infty} \left[ \int_{|y|^{-2}}^{2|s|+a^{-1}(|y|)-2} \frac{dy}{a(t-s)^2} \right]^2 \ dt$$

$$\leq C \int_{2|s|+a^{-1}(|y|)}^{\infty} \frac{|y|^2}{a(t-s)^2} \ dt \leq C |y|^2 a (2|s| + a^{-1}(|y|) - s)^{-2}$$

Hence, $I_{2,1} \leq C$. Since

$$I_{2,2} \leq \int_{2|s|}^{\infty} \left[ \int \chi_{Q_{c_{(0)}}} L^{\nu} d \nu \right] \left( t-s, x \right) - L^{\nu} \left( t, x \right) \ dx \right]^2 \ dt$$

$$\leq C |y|^2 a (|s| + a^{-1}(|y|))^{-2} \leq C$$

Clearly, $I_{2,2,2} \leq C$ as well. 

Now, Lemmas 12, 15 allow us to derive all the estimates of $|u|_{L^p(E)}$ and $|L^\nu u|_{L^p(E)} \ (s = 0)$ claimed in Theorem 1 for the solution $u$ of (1.1) with smooth input functions. Uniqueness is based on Ito formula by repeating the arguments of Lemma 8 of [9]. The result for a general input functions follows by passing to the limit in a standard way (see [8], [9]). The result for an arbitrary $s \in \mathbb{R}$ follows by Proposition 2 (see proof of Theorem 1 pp. 27–28 of [9] for details).

5.3. Proof of Proposition 1, time regularity. Let $\nu \in \mathbb{R}^d, w = w_\nu$ is a continuous O-RV function and $A, B$ hold. We discuss time-regularity of the deterministic equations,

$$du(t, x) = L^\nu u(t, x) - \lambda u(t, x) + f(t, x)$$

(5.17)

$$u_0 = g$$

with $f \in L^\nu_p(E)$ and $g \in B^{\nu_{1-\frac{1}{p}}} (\mathbb{R}^d)$. We write the solution of (5.17) as

$$u = u_f(t) + u_g(t), t \geq 0,$$

with (skipping spatial variable $x$)

$$u_f(t) = \int_0^t G_{t-s} f(s) \ ds, u_g(t) = G_t^\lambda * g,$$

$$G_t^\lambda (x) = e^{-\lambda t} \nu^* (t, x) = e^{-\lambda t} G_t (x), t > 0, x \in \mathbb{R}^d,$$
where \( p^\nu \) is the probability density associated to \( Z_t^\nu \). Here we prove Lemma 11. Clearly, by Proposition 2, it suffices to consider the case \( s = 0 \).

**Lemma 16.** For \( \kappa \in (\frac{1}{p}, 1] \), there exists \( C > 0 \) independent of \( \nu \) and \( \lambda \) such that for any \( t', t \in [0, T] \),

\[
|L^{\nu;1-\kappa} (u_f (t) - u_f (t'))|_{L_p(R^d)} \leq C (t-t')^{(\kappa - \frac{1}{p})} |f|_{L_p(E)}
\]

Moreover for \( \mu_1 \in [0, \kappa - \frac{1}{p}) \) and \( \mu_2 \in [0, 1] \), there exists \( C > 0 \) independent of \( \nu \) and \( \lambda \) such that for any \( t', t \in [0, T] \),

\[
|L^{\nu;1-\kappa} (u_g (t) - u_g (t'))|_{L_p(R^d)} \leq C \left[ (t-t') + (t-t')^{\mu_1} + (t-t')^{\mu_2} \lambda \mu_2 \right] |g|_{B_p^1}^{\frac{1}{1-p}} (R^d)
\]

**Proof.** Throughout this proof the constant \( C \) change from line to line but remains independent of \( \lambda \) and \( T \).

1. Estimate of \( u_f (t) \) following closely the proof of Proposition 2 of [10], for \( r, l \geq 0 \),

\[
u\nu\nu u_f (l+r) - u_f (l) = \int_0^{l+r} e^{-\lambda (l+r-s)} G_{l+r-s} \ast f (s) \, ds - \int_0^l e^{-\lambda (l-s)} G_{l-s} \ast f (s) \, ds = \int_l^{l+r} e^{-\lambda (l+r-s)} G_{l+r-s} \ast f (s) \, ds + \left( e^{-\lambda r} - 1 \right) \int_0^l e^{-\lambda (l-s)} G_{l+r-s} \ast f (s) \, ds + \int_0^l e^{-\lambda (l-s)} [G_{l+r-s} - G_{l-s}] \ast f (s) \, ds = A_1 + A_2 + A_3.
\]

Using the fact that \( G_{l+s} = G_t \ast G_s, t, s > 0 \),

\[
u\nu\nu A_3 = \int_0^l e^{-\lambda (l-s)} [G_{l+r-s} - G_{l-s}] \ast f (s) \, ds = G_r \ast u_f (l) - u_f (l)
\]

Examining the solution of the initial value function, for an appropriate function \( h \),

\[
u\nu\nu G_r \ast h - h = \int_0^r L^\nu G_s \ast h \, ds
\]

For \( \kappa \in (0, 1] \), we have, by Lemma [9]

\[
u\nu|G_r \ast h - h|_{L_p(R^d)} \leq \int_0^r |L^\nu G_s \ast h|_{L_p(R^d)} \, ds \leq \int_0^r |L^{\nu;1-\kappa} G_s \ast L^{\nu;\kappa} h|_{L_p(R^d)} \, ds \leq \int_0^r s^{\kappa-1} \, ds |L^{\nu;\kappa} h|_{L_p(R^d)} = r^{\kappa} |L^{\nu;\kappa} h|_{L_p(R^d)}
\]

(5.18)

Let \( \kappa \in (\frac{1}{p}, 1] \). Applying (5.18) to \( A_3 \),
We now estimate $A_1$. By Lemma 8 with $1/p + 1/q = 1$,

$$\left| L^{\nu;1-\kappa} A_1 (l,r) \right|_{L^p(\mathbb{R}^d)}^P \leq C r^{p\kappa} \left| L^{\nu} u_f (l) \right|_{L^p(\mathbb{R}^d)}^P$$

Following Proposition 2 of [10], and applying Theorem 11 we record the following estimate,

$$(t-t')^{\mu p-1} \int_0^{t-t'} \frac{dr}{r^{1+\mu p}} \int_0^{t-r} \left| L^{\nu;1-\kappa} A_1 (r,l) \right|_{L^p(\mathbb{R}^d)}^P + \left| L^{\nu;1-\kappa} A_3 (r,l) \right|_{L^p(\mathbb{R}^d)}^P \
$$

$$(5.19)$$

$$\leq C (t-t')^{\kappa p-1} \left| f \right|_{L^p(\mathbb{E})}^P$$

We deal with $A_2$, with $\epsilon \in [0,1]$, by Lemma 8 and Hölder inequality, for any $\eta \in [0,1]$ we have

$$\left| L^{\nu;1-\kappa} A_2 (l,r) \right|_{L^p(\mathbb{R}^d)} = \left| \left( e^{-\lambda r} - 1 \right) \int_0^l e^{-\lambda (l-s)} L^{\nu;1-\kappa} G_{l+r-s} * f (s) \ ds \right|_{L^p(\mathbb{R}^d)}$$

$$\leq \left| \left( e^{-\lambda r} - 1 \right) \int_0^l e^{-\lambda (l-s)} \left| L^{\nu;1-\kappa} G_{l+r-s} \right|_{L^p(\mathbb{R}^d)} \left| f (s) \right|_{L^p(\mathbb{R}^d)} \ ds \right|$$

$$\leq \left| e^{-\lambda r} - 1 \right| r^{\kappa-1} \left[ \int_0^l e^{-\lambda (l-s)} \ ds \right]^{\frac{1}{\eta}} \left[ \int_0^l \left| f (s) \right|_{L^p(\mathbb{R}^d)}^p \ ds \right]$$

Choosing $\eta = \frac{1}{q}$, then $\eta + \kappa - 1 = \frac{1}{q} + \kappa - 1 = \kappa - \frac{1}{p} > 0$.

We have

$$\left(5.20\right) \quad \left| L^{\nu;1-\kappa} A_2 (l,r) \right|_{L^p(\mathbb{R}^d)} \leq r^{\kappa-\frac{1}{p}} \left| f \right|_{L^p(\mathbb{E})}$$

Repeating argument in Proposition 2 of [10], precisely applying Lemma 11 therein, combining estimates for $A_1, A_2, A_3$, we summarize that for all
\( \kappa \in (\frac{1}{p}, 1] \) there is \( C \) independent of \( T \) and \( \lambda \),

\[
\left| L^\nu,1-\kappa \left( u_f (t) - u_f (t') \right) \right|_{L^p (\mathbb{R}^d)} \leq C (t - t')^{\left( \frac{\kappa - 1}{p} \right)} \left| f \right|_{L^p (E)}, \kappa \in \left( \frac{1}{p}, 1 \right].
\]

2. Estimate of \( u_g (t) = G_t^\lambda g, \quad 0 \leq t \leq T \). We have for \( t', t \in [0, T] \),

\[
\left| u_g (t) - u_g (t') \right|_{L^p (\mathbb{R}^d)} = \left| G_t^\lambda g - G_t^\lambda g \right|_{L^p (\mathbb{R}^d)}
\]

\[
\leq \left| e^{-\lambda t} \sum_{j=0}^{\infty} G_t g \star \varphi_j \varphi_j - e^{-\lambda t'} G_t g \star \varphi_j \varphi_j \right|_{L^p (\mathbb{R}^d)}
\]

\[
\leq \sum_{j=0}^{\infty} \left| e^{-\lambda t} G_t \varphi_j - e^{-\lambda t'} G_t \varphi_j \right|_{L^1 (\mathbb{R}^d)} \left| g \star \varphi_j \right|_{L^p (\mathbb{R}^d)}
\]

We note that

\[
e^{-\lambda t} G_t \varphi_j - e^{-\lambda t'} G_t \varphi_j = e^{-\lambda t} E \varphi \left( N^j x + Z_{w_j^t}^{P_{N-j}} \right) - e^{-\lambda t'} E \varphi \left( N^j x + Z_{w_j^t}^{P_{N-j}} \right), \quad j \geq 1,
\]

with \( w_j = w (N^{-j})^{-1} \), \( j \geq 1 \), and

\[
e^{-\lambda t} G_t \varphi_0 - e^{-\lambda t'} G_t \varphi_0 = e^{-\lambda t} E \varphi_0 \left( x + Z_{t'}^{P_{N-j}} \right) - e^{-\lambda s} E \varphi_0 \left( x + Z_{t'}^{P_{N-j}} \right)
\]

Hence, for \( j \geq 1 \)

\[
\left| e^{-\lambda t} G_t \varphi_j - e^{-\lambda t'} G_t \varphi_j \right|_{L^1 (\mathbb{R}^d)} = \left| e^{-\lambda t} E \varphi \left( \cdot + Z_{w_j^t}^{P_{N-j}} \right) - e^{-\lambda t'} E \varphi \left( \cdot + Z_{w_j^t}^{P_{N-j}} \right) \right|_{L^1 (\mathbb{R}^d)}
\]

\[
\leq e^{-\lambda t} \left| E \varphi \left( \cdot + Z_{w_j^t}^{P_{N-j}} \right) - E \varphi \left( \cdot + Z_{w_j^t}^{P_{N-j}} \right) \right|_{L^1 (\mathbb{R}^d)}
\]

\[
+ \left| e^{-\lambda t} - e^{-\lambda t'} \right|_L \left| E \varphi \left( \cdot + Z_{w_j^t}^{P_{N-j}} \right) \right|_{L^1 (\mathbb{R}^d)}
\]

\[
= I_1^j + I_2^j
\]

By Lemma [11]

\[
I_1^j \leq C e^{-\lambda t} \int_{w_j^t}^{w_j} e^{-\kappa r} dr = C e^{-\lambda t} e^{-\kappa w_j} \left[ 1 - e^{-\kappa w_j} \right]
\]

\[
\leq w_j^\mu (t - s)^\mu; \quad \mu \in [0, 1] ; \quad j \geq 1,
\]

and

\[
I_2^j \leq e^{-\kappa w_j} \left( e^{-\lambda t} - e^{-\lambda s} \right) \leq \lambda^\mu (t - s)^\mu; \quad \mu \in [0, 1] ; \quad j \geq 1
\]

and corresponding terms for \( j = 0 \)

\[
I_1^0 \leq \left| \int_{t'}^{t} L^\nu \varphi_0 \left( x + Z_{t'}^{P_{N-j}} \right) dr \right|_{L^1} \leq C (t - t')
\]

\[
I_2^0 \leq \lambda^\mu (t - t')^\mu; \quad \mu \in [0, 1].
\]
Hence

\[
|L^{\nu;1-\kappa}u_g(t) - L^{\nu;1-\kappa}u_g(t')|_{L^p(\mathbb{R}^d)} \leq C(t-t')^{\mu_1} \sum_{j \geq 1} w_j^{\mu_1} |L^{\nu;1-\kappa}g \ast \varphi_j|_{L^p(\mathbb{R}^d)} \\
\quad \quad \quad \quad + C(t-t')^{\mu_2} \lambda^{\mu_2} \sum_{j \geq 0} |L^{\nu;1-\kappa}g \ast \varphi_j|_{L^p(\mathbb{R}^d)} \\
\quad \quad \quad \quad + C(t-t') \|g \ast L^{\nu;1-\kappa}\varphi_0\|_{L^p(\mathbb{R}^d)} \\
= A + B + D
\]

For \( \kappa \in \left(\frac{1}{p}, 1\right], \mu_1, \mu_2 \in [0, 1], \mu_1 < \kappa - \frac{1}{p} \), by Hölder inequality and Proposition 2

\[
A \leq C(t-t')^{\mu_1} \sum_{j=1}^{\infty} w_j^{\mu_1+\frac{1}{p}-\kappa} |L^{\nu;1-\kappa}g \ast \varphi_j|_{L^p(\mathbb{R}^d)}^{1/p} \\
\leq C(t-t')^{\mu_1} \left( \sum_{j=1}^{\infty} w_j^{-\left(\mu_1+\frac{1}{p}-\kappa\right)} \right)^{1/q} \left( \sum_{j=1}^{\infty} \left( \frac{\nu}{\nu-j+1} \right)^p |L^{\nu;1-\kappa}g \ast \varphi_j|_{L^p(\mathbb{R}^d)}^p \right)^{1/p} \\
\leq C(t-t')^{\mu_1} \|g\|_{B_{pp}^{1-\frac{1}{p}}(\mathbb{R}^d)}.
\]

Now, obviously,

\[
D \leq C(t-t') \|g\|_{L^p(\mathbb{R}^d)}.
\]

Similarly for \( B \), by Hölder inequality,

\[
B \leq C(t-t')^{\mu_2} \lambda^{\mu_2} \sum_{j=0}^{\infty} |L^{\nu;1-\kappa}g \ast \varphi_j|_{L^p(\mathbb{R}^d)}^{1/p} \\
\leq C(t-t')^{\mu_2} \lambda^{\mu_2} \left( \sum_{j=0}^{\infty} w_j^{-\left(\kappa+\frac{1}{p}\right)} \right)^{1/q} \left( \sum_{j=0}^{\infty} \left( \frac{\nu}{\nu-j+1} \right)^p |L^{\nu;1-\kappa}g \ast \varphi_j|_{L^p(\mathbb{R}^d)}^p \right)^{1/p} \\
\leq C(t-t')^{\mu_2} \lambda^{\mu_2} \|g\|_{B_{pp}^{1-\frac{1}{p}}(\mathbb{R}^d)}.
\]

Summarizing for \( \frac{1}{p} < \kappa \leq 1, \mu_1 \in [0, \kappa - \frac{1}{p}] \) and \( \mu_2 \in [0, 1], \)

\[
|L^{\nu;1-\kappa}(u(t) - u(t'))|_{L_p(\mathbb{R}^d)} \leq C[(t-t')^{\mu_1} + (t-t')^{\mu_2} \lambda^{\mu_2}] \|g\|_{B_{pp}^{1-\frac{1}{p}}(\mathbb{R}^d)}.
\]

\( \square \)
Corollary 2. (Corollary 6 of [6]) Let $\nu \in \mathcal{R}^\sigma$,
\[ \nu(\Gamma) = - \int_0^\infty \int_{S_{d-1}} \chi_{\Gamma}(rw) \Pi(r, dw) \, d\delta(r), \Gamma \in \mathcal{B}(\mathbb{R}^d), \]
where $\delta = \delta_\nu, \Pi(r, dw), r > 0$ is a measurable family of measures on $S_{d-1}$ with $\Pi(r, S_{d-1}) = 1, r > 0$. Assume that $w = w_\nu = \delta_\nu^{-1}$ is an $O$-RV function at zero satisfying
\[ \inf_{|\xi| = 1} \int_{S_{d-1}} |\xi \cdot w|^2 \Pi(r, dw) \geq c_0 > 0, \quad r > 0 \]
and $|\{s \in [0,1] : r_i(s) < 1\}| > 0, i = 1, 2$. Then assumption B holds. That is
\[ \inf_{R \in (0,\infty), |\xi| = 1} \int_{|y| \leq 1} |\xi \cdot y|^2 \tilde{\nu}_R(dy) > 0 \]

Proof. The proof is taken from the proof of Corollary 6 of [1] with obvious modification.

Indeed, for $|\hat{\xi}| = 1, R > 0$, with $C > 0$,
\[ \int_{|y| \leq 1} |\xi \cdot y|^2 \nu_R(dy) \]
\[ = R^{-2} \int_{|y| \leq R} |\xi \cdot y|^2 \nu(dy) = - R^{-2} \int_0^R \int_{S_{d-1}} |\xi \cdot w|^2 \Pi(r, dw) r^2 d\delta(r) \]
\[ \geq - R^{-2} c_0 \int_0^R r^2 d\delta(r) = c_0 R^{-2} \int_{|y| \leq R} |y|^2 \nu(dy) = c_0 \int_{|y| \leq 1} |y|^2 \nu_R(dy) \]
\[ \int_{|y| \leq 1} |\xi \cdot y|^2 \tilde{\nu}_R(dy) \geq c_0 \int_{|y| \leq 1} |y|^2 \tilde{\nu}_R(dy) \]

Computing like in the proof of Lemma [4],
\[ \int_{|y| \leq 1} |y|^2 \tilde{\nu}_R(dy) = w(R) \int_{|y| \leq 1} |y|^2 \nu_R(dy) \]
\[ = 2 R^{-2} \int_0^R s^2 \left[ \frac{w(R)}{w(s)} - 1 \right] \frac{ds}{s} \]
\[ = 2 \int_0^1 s^2 \left[ \frac{w(R)}{w(Rs)} - 1 \right] \frac{ds}{s} \]

By Fatou’s Lemma,
\[ \liminf_{R \to 0} \int_{|y| \leq 1} |y|^2 \tilde{\nu}_R(dy) \geq 2 \int_0^1 s^2 \left[ \frac{1}{r_1(s)} - 1 \right] \frac{ds}{s} = c_1 > 0, \]
\[ \liminf_{R \to \infty} \int_{|y| \leq 1} |y|^2 \tilde{\nu}_R(dy) \geq 2 \int_0^1 s^2 \left[ \frac{1}{r_2(s)} - 1 \right] \frac{ds}{s} = c_2 > 0, \]
if $|\{s \in [0,1] : r_i(s) < 1\}| > 0, i = 1, 2$, completing the proof. □

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