Application of a light-front coupled-cluster method to quantum electrodynamics

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Abstract

A field-theoretic formulation of the exponential-operator technique is applied to a Hamiltonian eigenvalue problem in electrodynamics, quantized in light-front coordinates. Specifically, we consider the dressed-electron state, without positron contributions but with an unlimited number of photons, and compute its anomalous magnetic moment. A simple perturbative solution immediately yields the Schwinger result of $\alpha/2\pi$. The nonperturbative solution, which requires numerical techniques, sums a subset of corrections to all orders in $\alpha$ and incorporates additional physics.

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I. INTRODUCTION

Although the nonperturbative light-front coupled-cluster (LFCC) method \[1\] is intended for strongly coupled theories, where perturbation theory is of limited use, we explore its utility in the context of a gauge theory by considering the dressed-electron state in quantum electrodynamics (QED) \[2\]. The method requires the light-front coordinates of Dirac \[3,4\], where the Hamiltonian evolves a state along the time direction $x^+ = t + z$. The spatial coordinates are $\mathbf{x} = (x^- \equiv t - z, \mathbf{x}_\perp \equiv (x, y))$. The light-front energy conjugate to the chosen time is $p^- = E - p_z$, and the corresponding light-front momentum is $\mathbf{p} = (p^+ \equiv E + p_z, \mathbf{p}_\perp \equiv (p_x, p_y))$. In these coordinates, the fundamental Hamiltonian eigenvalue problem is $H T^+ |\psi\rangle = \pm \sqrt{Z}e^T |\phi\rangle$ from a valence state $|\phi\rangle$ and an operator $T$ that increases particle number while conserving any quantum numbers of the valence state. The constant $Z$ is a normalization factor.

The valence state is then an eigenstate of an effective Hamiltonian $\mathcal{H} = e^{-T}H e^T$, which can be computed using a Baker–Hausdorff expansion $\mathcal{H} = \mathcal{H}^+ + [\mathcal{H}^+, T] + \frac{1}{2}[[\mathcal{H}^+, T], T] + \cdots$. We define $\Delta_i = e^{-T}[\hat{\mathcal{O}}_i, T] + \frac{1}{2}[[\hat{\mathcal{O}}_i, T], T] + \cdots$, and, to avoid the infinite sum in the denominator, $Z_i |\phi\rangle e^{T_\perp} e^{T_i} = \sqrt{Z_i} |\psi_i\rangle e^{T_i}$, where $P_i$ is the projection onto the valence sector.

A matrix element such as $\langle \psi_2 | \hat{\mathcal{O}} | \psi_1 \rangle$ can be calculated, with $|\psi_1\rangle = \sqrt{Z_1} e^{T_1} |\phi_1\rangle$ and $Z_1 = 1/\langle \phi_1 | e^{T_1} e^{T_\perp} | \phi_1 \rangle$. We define $\Delta_i = e^{-T}[\hat{\mathcal{O}}_i, T] = \hat{\mathcal{O}}_i + [\hat{\mathcal{O}}_i, T] + \frac{1}{2}[[\hat{\mathcal{O}}_i, T], T] + \cdots$ and, to avoid the infinite sum in the denominator,

$$
\langle \psi_1 | \hat{\mathcal{O}} | \psi_1 \rangle = \sqrt{Z_1/Z_2} \langle \psi_2 | \hat{\mathcal{O}}_2 e^{-T_2} e^{T_1} | \phi_1 \rangle = \sqrt{Z_1/Z_2} \langle \psi_1 | \hat{\mathcal{O}}_1 e^{-T_1} e^{T_2} | \phi_2 \rangle^*.
$$

We then have

$$
\langle \psi_2 | \hat{\mathcal{O}} | \psi_1 \rangle = \sqrt{Z_1/Z_2} \langle \psi_2 | \hat{\mathcal{O}}_2 e^{-T_2} e^{T_1} | \phi_1 \rangle = \sqrt{Z_1/Z_2} \langle \psi_1 | \hat{\mathcal{O}}_1 e^{-T_1} e^{T_2} | \phi_2 \rangle^*.
$$

In the diagonal case, this reduces to

$$
\langle \psi | \hat{\mathcal{O}} | \psi \rangle = \langle \tilde{\psi} | \hat{\mathcal{O}} | \phi \rangle.
$$

The $|\psi_i\rangle$ can be shown to be left eigenstates of the effective Hamiltonian. We apply this to QED in an arbitrary covariant gauge, for which the Pauli–Villars-regulated Lagrangian is $\mathcal{L}$.

$$
\mathcal{L} = \sum_{i=0}^2 (-1)^i \left[ -\frac{1}{4} F_{i\mu\nu} F_{i\mu\nu} + \frac{1}{2} \mu_i^2 A_i^\mu A_i^\mu - \frac{1}{2} \zeta (\partial^\mu A_i^\mu)^2 \right] + \sum_{i=0}^2 (-1)^i \bar{\psi}_i (i\gamma^\mu \partial_\mu - m_i) \psi_i - e \bar{\psi} \gamma^\mu \psi A_\mu.
$$
Here the fundamental physical \((i = 0)\) and Pauli–Villars \((i = 1)\) fields appear in null combinations

\[
\psi = \sum_{i=0}^{2} \sqrt{\beta_i} \psi_i, \quad A_\mu = \sum_{i=0}^{2} \sqrt{\xi_i} A_{i\mu}, \quad F_{i\mu\nu} = \partial_\mu A_{i\nu} - \partial_\nu A_{i\mu}. \tag{1.7}
\]

The coupling coefficients \(\xi_i\) and \(\beta_i\) are constrained by

\[
\xi_0 = 1, \quad \sum_{i=0}^{2} (-1)^i \xi_i = 0, \quad \beta_0 = 1, \quad \sum_{i=0}^{2} (-1)^i \beta_i = 0. \tag{1.8}
\]

To fix \(\xi_2\) and \(\beta_2\), we require chiral symmetry restoration in the zero-mass limit \([6]\) and a zero photon mass \([7]\). The light-front Hamiltonian, without antifermion terms, is then found to be \([3]\).

\[
P^- = \sum_{i} \int dp \frac{m_i^2 + p^2}{p^+} (-1)^i b^\dagger_{is}(p) b_{is}(p) + \sum_{i\lambda} \int dk \frac{p_{i\lambda}^2 + k_\perp^2}{k^+} (-1)^i \epsilon^\lambda a^\dagger_{i\lambda}(k) a_{i\lambda}(k) \tag{1.9}
\]

\[
+ \sum_{ijls\lambda} \int dydk_\perp \frac{dp}{\sqrt{16\pi^3 p^+}} \left\{ h_{ijl}^{\sigma\lambda}(y, k_\perp) a^\dagger_{i\sigma}(y, k_\perp; p) b^\dagger_{j\lambda}(1 - y, -k_\perp; p) b_{j\sigma}(p)
\right.
\]

\[
+ h_{ijl}^{\sigma\lambda}(y, k_\perp) b^\dagger_{j\lambda}(p) b_{j\sigma}(1 - y, -k_\perp; p) a_{i\sigma}(y, k_\perp; p) \right\},
\]

with \(\epsilon^\lambda = (-1, 1, 1, 1)\) and the \(h_{ijl}^{\sigma\lambda}\) known vertex functions.

II. THE DRESSED-ELECTRON STATE

The right and left-hand valence states \((P^-\) is not Hermitian!) are \(|\phi^\pm_a\rangle = \sum_i z_{ai} b^\dagger_{i\pm}(P) |0\rangle\) and \(\langle \tilde{\phi}^\pm_a | = \langle 0 | \sum_i \tilde{z}_{ai} b_{i\pm}(P)\). We approximate the \(T\) operator with the simplest form

\[
T = \sum_{ijl\sigma\lambda} \int dydk_\perp \int \frac{dp}{\sqrt{16\pi^3 p^+}} \sqrt{p^+} t_{ijl}^{\sigma\lambda}(y, k_\perp) a^\dagger_{i\sigma}(yp^+, y\bar{p}_\perp + k_\perp)
\]

\[
\times b^\dagger_{j\lambda}(1 - y)p^+, (1 - y)\bar{p}_\perp - k_\perp) b_{j\sigma}(p). \tag{2.1}
\]

The effective Hamiltonian \(P^-\) can then be constructed \([2]\). From this effective Hamiltonian, the right and left-hand valence-sector equations become, for \(a = 0, 1\),

\[
m_i^2 z_{ai}^\pm + \sum_j I_{ji} z_{aj}^\pm = M_a^2 z_{ai}^\pm \quad \text{and} \quad m_i^2 z_{ai}^\pm + \sum_j (-1)^{i+j} I_{ji} z_{aj}^\pm = M_a^2 z_{ai}^\pm, \tag{2.2}
\]

with \(M_a\) the \(a\)th eigenmass and the self-energy given by

\[
I_{ji} = (-1)^i \sum_{\ell ls\lambda} h_{ijl}^{\sigma\lambda}(y, k_\perp) t_{ijl}^{\sigma\lambda}(y, k_\perp). \tag{2.3}
\]

The valence eigenvectors are orthonormal and complete in the following sense:

\[
\sum_i (-1)^i z_{ai}^\pm z_{bi}^\pm = (-1)^a \delta_{ab} \quad \text{and} \quad \sum_a (-1)^a z_{ai}^\pm z_{aj}^\pm = (-1)^i \delta_{ij}. \tag{2.4}
\]
The \( t \) functions satisfy the projection of the effective eigenvalue problem onto one-electron/one-photon states, orthogonal to \( |\phi\rangle \), which gives \[2\]

\[
\sum_i (-1)^i z_{ai}^\pm \left\{ t_i^{\pm s \lambda}(y, \vec{k}_\perp) + \frac{1}{2} V_i^{\pm s \lambda}(y, \vec{k}_\perp) + \left[ \frac{m_i^2 + k_i^2}{1 - y} + \frac{\mu_i^2 + k_i^2}{y} - m_i^2 \right] t_i^{\pm s \lambda}(y, \vec{k}_\perp) \right\} 
+ \frac{1}{2} \sum_{i'} I_{ijl} \frac{1}{1 - y} t_{i'l}^{\pm s \lambda}(y, \vec{k}_\perp) - \sum_{j'} (-1)^{i + j'} t_{ijl}^{\pm s \lambda}(y, \vec{k}_\perp) I_{j'l} = 0,
\]

with the vertex correction

\[
V_{ijkl}^{\sigma s \lambda}(y, \vec{k}_\perp) = \sum_{i', j', l', \sigma, s, \lambda} (-1)^{i' + j' + l'} e^{x'} \int \frac{dy'dk_i'}{16\pi^3} \frac{\theta(1 - y - y')}{(1 - y')^{3/2}(1 - y)^{3/2}} \times h_{ijl}^{\sigma s \lambda}(y, \vec{k}_\perp) + \frac{y'}{1 - y} \sum_{l'} t_{ijl}^{\sigma s \lambda}(y, \vec{k}_\perp) + \frac{y}{1 - y} \sum_{l'} t_{ijl}^{\sigma s \lambda}(y, \vec{k}_\perp).
\]

To partially diagonalize in flavor, we define \( C_{abl}^{\pm s \lambda}(y, \vec{k}_\perp) = \sum_{ij} (-1)^{i + j} z_{ai}^\pm z_{bj}^\pm t_{ijl}^{\pm s \lambda}(y, \vec{k}_\perp) \). With analogous definitions for \( H, I, \) and \( V \), we have

\[
\left[ M_a^2 - \frac{M_i^2 + k_i^2}{1 - y} - \frac{\mu_i^2 + k_i^2}{y} \right] C_{abl}^{\pm s \lambda}(y, \vec{k}_\perp) = \sum_{b'} I_{b'b} \frac{1}{1 - y} C_{abl}^{\pm s \lambda}(y, \vec{k}_\perp)
\]

(2.7)

to be solved simultaneously with the valence sector equations, which depend on \( C/t \) through the self-energy matrix \( I \). Notice that the physical mass \( M_b \) has replaced the bare mass in the kinetic energy term, without any need for sector-dependent renormalization \[8\].

In order to compute matrix elements, such as appear in the computation of form factors, we need the left-hand eigenstate. The dual to \( \langle \tilde{\psi} | = \sqrt{Z} \langle \psi | e^T \) is a right eigenstate of \( \mathcal{P}^{-1} \)

\[
|\tilde{\psi}^\sigma_a(\mathcal{P})\rangle = |\tilde{\phi}^\sigma_a(\mathcal{P})\rangle + \sum_{j\lambda s} \int dyd\vec{k}_\perp \sqrt{\frac{P^+}{16\pi^3}} \alpha^\sigma_{j\lambda}(y, \vec{k}_\perp) a_{j\lambda}^\dagger(y, \vec{k}_\perp; \mathcal{P}) b_{j\sigma}^\dagger(1 - y, -\vec{k}_\perp; \mathcal{P}) |0\rangle,
\]

(2.8)

The flavor-diagonal left-hand wave functions are \( D_{abl}^{\pm s \lambda}(y, \vec{k}_\perp) \equiv \sum_j (-1)^j z_{bj}^s t_{ijl}^{\pm s \lambda}(y, \vec{k}_\perp) \). They satisfy the coupled equations \[2\]

\[
\left[ M_a^2 - \frac{M_i^2 + k_i^2}{1 - y} - \frac{\mu_i^2 + k_i^2}{y} \right] D_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp) = \sum_{b'} J_{b'a} \tilde{H}_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp) + W_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp),
\]

(2.9)

where \( W_{abl}^{\sigma s \lambda} \) is a vertex-correction analog of \( V_{abl}^{\sigma s \lambda} \), though linear in \( D \), and \( J_{b'a}^\sigma \) is a self-energy analog of \( I_{ba} \). Solutions for \( M_a, z_{ai}^\sigma, z_{ai}^\overline{\sigma}, \) and \( C_{abl}^{\sigma s \lambda} \) are used as input.
III. ANOMALOUS MAGNETIC MOMENT

We compute the anomalous moment $a_e$ from the spin-flip matrix element $[9]$ of the current $J^+ = \bar{\psi} \gamma^+ \psi$ coupled to a photon of momentum $q$ in the Drell–Yan ($q^+ = 0$) frame $[10]

$$16\pi^3 \langle \psi_\sigma^e (P + q) | J^+ (0) | \psi_\sigma^\pm (P) \rangle = 2\delta_{\sigma\pm} F_1 (q^2) \pm \frac{q^1 \pm iq^2}{M_a} \delta_{\sigma\mp} F_2 (q^2).$$

(3.1)

In the limit of infinite Pauli–Villars masses, and with $M_0 = m_e$, the electron mass, we find $[2]

$$F_1 (q^2) = \frac{1}{N} \left[ 1 + \sum_s \int \frac{dyd\vec{k}_1}{16\pi^3} \left\{ \sum_{\lambda=\pm} b_{0,00}^{\pm s\lambda*} (y, \vec{k}_1 - y\vec{q}_1) t_{0,00}^{\pm s\lambda} (y, \vec{k}_1) \right. \right.$$

$$\left. \left. - \sum_{\lambda=0}^3 e^{\lambda} b_{0,00}^{\pm s\lambda*} (y, \vec{k}_1) t_{0,00}^{\pm s\lambda} (y, \vec{k}_1) \right\} \right].$$

(3.2)

and

$$F_2 (q^2) = \pm \frac{2m_e}{q^1 \pm iq^2} \frac{1}{N} \sum_s \sum_{\lambda=\pm} \int \frac{dyd\vec{k}_1}{16\pi^3} b_{0,00}^{\pm s\lambda*} (y, \vec{k}_1 - y\vec{q}_1) t_{0,00}^{\pm s\lambda} (y, \vec{k}_1),$$

(3.3)

with

$$N = 1 - \sum_s \sum_{\lambda=0,3} e^{\lambda} \int \frac{dyd\vec{k}_1}{16\pi^3} b_{0,00}^{\pm s\lambda*} (y, \vec{k}_1) t_{0,00}^{\pm s\lambda} (y, \vec{k}_1).$$

(3.4)

A second term is absent in $F_2$ because $l$ and $t$ are orthogonal for opposite spins. The $q^2 \to 0$ limit can be taken, to find $F_1 (0) = 1$ and

$$a_e = F_2 (0) = \pm m_e \sum_{s\lambda} e^{\lambda} \int \frac{dyd\vec{k}_1}{16\pi^3} b_{0,00}^{\pm s\lambda*} (y, \vec{k}_1) \left( \frac{\partial}{\partial k_1^\pm} \mp i \frac{\partial}{\partial k_2} \right) t_{0,00}^{\pm s\lambda} (y, \vec{k}_1).$$

(3.5)

As a check, we can consider a perturbative solution

$$t_{0,00}^{\sigma\lambda} = l_{0,00}^{\sigma\lambda} = h_{0,00}^{\sigma\lambda} / \left[ m_e^2 - \frac{m_e^2 + k_1^2}{1 - y} - \frac{\mu_\lambda^2 + k_2^2}{y} \right].$$

(3.6)

Substitution into the expression for $a_e$ gives immediately the Schwinger result $[11] \alpha / 2\pi$, in the limit of zero photon mass, for any covariant gauge.

IV. SUMMARY

The LFCC method provides a nonperturbative approach to bound-state problems in quantum field theories without truncation of the Fock space and without the uncanceled divergences and spectator dependence that such truncation can cause. The approximation is instead a truncation of the operator $T$ that generates contributions from higher Fock states. It is systematically improvable through the addition of more terms to $T$, with increasing numbers of particles created and annihilated.

To complete the application to the dressed-electron state, we need to solve numerically the coupled systems that determine the $t$ and $l$ functions and to use these solutions to compute...
the anomalous moment. Within the arbitrary-gauge formulation, we can test directly for
gauge dependence [5]. A more complete investigation of QED would include consideration of
the dressed-photon state, contributions from electron-positron pairs to the dressed-electron
state, and true bound states such as muonium and positronium. These will provide some
guidance for applications to quantum chromodynamics, particularly in extensions of the
holographic model for mesons [12].

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