Improved Bohr inequalities for certain class of harmonic univalent functions

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ABSTRACT
Let $\mathcal{H}$ be the class of complex-valued harmonic mappings $f = h + \bar{g}$ defined in the unit disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$, where $h$ and $g$ are analytic functions in $D$ with the normalization $h(0) = 0 = h'(0) - 1$ and $g(0) = 0$. Let $\mathcal{H}_0 = \{ f = h + \bar{g} \in \mathcal{H} : g'(0) = 0 \}$. Let $P_{\mathcal{H}}^0(M) := \{ f = h + \bar{g} \in \mathcal{H}_0 : \Re(zh''(z)) > -M + |zg''(z)|, z \in D \text{ and } M > 0 \}$.

be the class of harmonic univalent mappings in the unit disk $D$ [Ghosh N, Allu V. On some subclasses of harmonic mappings. Bull Aust Math Soc. 2020;101:130–140.]. In this paper, we obtain the sharp Bohr–Rogosinski inequality, improved Bohr inequality, refined Bohr inequality and Bohr-type inequality for the class $P_{\mathcal{H}}^0(M)$.

1. Introduction
The classical Bohr inequality (see [1]) states that if $f$ is an analytic function with the power series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $D$ where $D := \{ z \in \mathbb{C} : |z| < 1 \}$ such that $|f(z)| \leq 1$ for all $z \in D$, then

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1, \quad \text{for all } |z| = r \leq \frac{1}{3}, \quad (1)$$

and the constant $1/3$ cannot be improved. The constant $r_0 = 1/3$ is known as Bohr’s radius, while the inequality $\sum_{n=0}^{\infty} |a_n| r^n \leq 1$ is known as Bohr inequality. Bohr actually obtained the inequality (1) for $r \leq 1/6$ and later Weiner, Riesz and Schur have independently proved it for $1/3$.

The Bohr inequality can be written in terms of distance as

$$d\left(\sum_{n=0}^{\infty} |a_n z^n|, |a_0|\right) = \sum_{n=1}^{\infty} |a_n z^n| \leq 1 - |f(0)| = d(f(0), \partial f(D)),$$
where $d$ is the Euclidean distance and $\partial f(\mathbb{D})$ is the boundary of $f(\mathbb{D})$. The notion of Bohr inequality can be generalized to any domain $\Omega$ to find the largest radius $r_{\Omega} > 0$ such that

$$
d \left( \sum_{n=0}^{\infty} |a_n z^n|, |a_0| \right) = \sum_{n=1}^{\infty} |a_n z^n| \leq d(f(0), \partial f(\Omega)) \tag{2}
$$

holds for all $|z| = r \leq r_{\Omega}$, and for all functions analytic in $\mathbb{D}$ and such that $f(\mathbb{D}) \subseteq \Omega$. Interestingly enough, it was exhibited in [2] that if $\Omega$ is convex then the inequality (2) holds for all $|z| \leq 1/3$, and this radius is the best possible. Thus if $\Omega$ is convex, then $r_{\Omega}$ coincides with Bohr’s radius and most notably, the radius does not depend on $\Omega$. In 2010, Abu-Muhana [3] established a result showing that when $\Omega$ is a simply connected domain with $f(\mathbb{D}) \subset \Omega$, then the inequality (2) holds for $|z| \leq 3 - 2\sqrt{2} = 0.1715 \cdots$ and this radius is sharp for the Koebe function $k(z) = z/(1 - z)^2$.

Operator algebraists began to get interested in the inequality after Dixon [4] exposed a connection between the inequality and the characterization of Banach algebras that satisfy von Neumann inequality. The generalization of Bohr’s theorem for different classes of analytic functions becomes now a days an active research area [5–7]. For example, for the holomorphic functions, in 2001, Aizenberg et al. [8], in 2013, Aytuna and Djakov [9] studied Bohr phenomenon, and for the class of starlike logarithmic mappings, in 2016, Ali et al. [10] found Bohr radius. In 2018, Ali and Ng [11] extended the classical Bohr inequality in Poincaré disk model of the hyperbolic plane. In 2018, Kayumov and Ponnusamy [5] introduced the notion of $p$-Bohr radius for harmonic functions, and established result obtaining $p$-Bohr radius for the class of odd analytic functions. Powered Bohr radius for the class of all self-analytic maps on $\mathbb{D}$ has been studied in [12] while several different improved versions of the classical Bohr inequality were proved in [6]. In this connection, Kayumov et al. [13] ascertained Bohr radius for the class of analytic Bloch functions and also for certain harmonic Bloch functions.

In addition to the Bohr radius, the concept of the Rogosinski radius is also used in [14], which is defined as follows: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ and its corresponding partial sum of $f$ is defined by $S_N(z) := \sum_{n=0}^{N-1} a_n z^n$. Then, for every $N \geq 1$, we have $|\sum_{n=0}^{N-1} a_n z^n| < 1$ in the disk $|z| < 1/2$ and the radius $1/2$ is sharp. Motivated by Rogosinski radius for bounded analytic functions in $\mathbb{D}$, Kayumov and Ponnusamy [15] have introduced Bohr–Rogosinski radius and considered the Bohr–Rogosinski sum $R^f_N(z)$ which is defined by

$$
R^f_N(z) := |f(z)| + \sum_{n=N}^{\infty} |a_n||z|^n. \tag{3}
$$

It is important to note that $|S_N(z)| = |f(z) - \sum_{n=0}^{\infty} a_n z^n| \leq |R^f_N(z)|$. Thus it is easy to see that the validity of Bohr-type radius for $R^f_N(z)$, which is related to the classical Bohr sum (Majorant series) in which $f(0)$ is replaced by $f(z)$, gives Rogosinski radius in the case of bounded analytic functions in $\mathbb{D}$. We have the following interesting result by Kayumov and Ponnusamy [15].
Theorem 1.1 ([15]): Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be analytic in \( \mathbb{D} \) and \(|f(z)| \leq 1 \). Then
\[
|f(z)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq 1 \tag{4}
\]
for \(|z| = r \leq R_N\), where \( R_N \) is the positive root of the equation \( \psi_N(r) = 0, \psi_N(r) = 2(1 + r) r^N - (1 - r)^2. \) The radius \( R_N \) is the best possible. Moreover,
\[
|f(z)|^2 + \sum_{n=N}^{\infty} |a_n||z|^n \leq 1 \tag{5}
\]
for \( R_N' \), where \( R_N' \) is the positive root of the equation \((1 + r) r^N - (1 - r)^2 = 0. \) The radius \( R_N' \) is the best possible.

The main aim of this paper is to establish several improved versions of Bohr inequality, refined Bohr inequality and Bohr–Rogosinski inequality, finding the corresponding sharp radius for the class \( P_0^M(M) \) which has been studied by Ghosh and Vasudevarao in [16]
\[
P_0^M(M) = \{ f = h + g \in H_0 : \text{Re} \left( zh'''(z) \right) > -M + |zg''(z)|, z \in \mathbb{D} \text{ and } M > 0 \},
\]
where
\[
f(z) = h(z) + g(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n. \tag{6}
\]
To study Bohr inequality and Bohr radius for functions in \( P_0^M(M) \), we require the coefficient bounds and growth estimate of functions in \( P_0^M(M) \). We have the following result on the coefficient bounds and growth estimate for functions in \( P_0^M(M) \).

Lemma 1.2 ([16]): Let \( f = h + g \in P_0^M(M) \) for some \( M > 0 \) and be of the form (6). Then for \( n \geq 2, \)
\begin{itemize}
  \item[(i)] \(|a_n| + |b_n| \leq \frac{2M}{n(n-1)};\)
  \item[(ii)] \(||a_n| - |b_n|| \leq \frac{2M}{n(n-1)};\)
  \item[(iii)] \(|a_n| \leq \frac{2M}{n(n-1)}\).
\end{itemize}
The inequalities are sharp with extremal function \( f \) given by \( f'(z) = 1 - 2M \ln (1 - z). \)

Lemma 1.3 ([16]): Let \( f \in P_0^M(M) \). Then
\[
|z| + 2M \sum_{n=2}^{\infty} \frac{(-1)^{n-1}|z|^n}{n(n-1)} \leq |f(z)| \leq |z| + 2M \sum_{n=2}^{\infty} \frac{|z|^n}{n(n-1)}. \tag{7}
\]
Both inequalities are sharp for the function \( f_M \) given by \( f_M(z) = z + 2M \sum_{n=2}^{\infty} \frac{z^n}{n(n-1)}. \)

The organization of this paper is as follows: In Section 2, we prove the sharp Bohr–Rogosinski radius for the class \( P_0^M(M) \). In Section 3, we prove the sharp results on
2. Bohr–Rogosinski radius for the class $\mathcal{P}_{\mathcal{H}}^0(M)$

We prove the following sharp Bohr–Rogosinski inequality for function in the class $\mathcal{P}_{\mathcal{H}}^0(M)$.

**Theorem 2.1**: Let $f \in \mathcal{P}_{\mathcal{H}}^0(M)$ be given (6) with $0 < M < 1/(2(\ln 4 - 1))$. Then for $N \geq 3$, we have

$$ |f(z)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)|z|^n \leq d(f(0), \partial f(\mathbb{D})) $$

(8)

for $|z| = r \leq r_N(M)$, where $r_N(M)$ is the smallest root of the equation

$$ r - 1 + 2M \left(2r - 1 + 2(1 - r) \ln ((1 - r)) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} + \ln 4 \right) = 0. $$

(9)

The radius $r_N(M)$ is the best possible.

**Remark 2.1**: In the study of the roots $r_N(M)$ of Equation (9), the following interesting facts can be observed. For the values of

$$ M \geq \frac{1}{2(2\ln 2 - 1)} $$

the equation

$$ H_{N,M}(r) := r - 1 + 2M \left(2r - 1 + 2(1 - r) \ln ((1 - r)) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} + \ln 4 \right) = 0 $$

does not have any solutions in $(0, 1)$. For all odd $N$, $H_{N,M}(r) \neq 0$ for $r \in (0, 1)$ although the function $H_{N,M}(r)$ has some finite global maximum at $r = 1$ and for all even $N$, equation $H_{N,M}(r) = 0$ has no roots in $\mathbb{R}$ when $M \geq 1/(2(2\ln 2 - 1))$ (Figures 1 and 2) and Table 1).

By considering power of $|f(z)|$ for the functions in the class $\mathcal{P}_{\mathcal{H}}^0(M)$, we obtain the following sharp result showing that the radius is different to compare with Theorem 2.1 (Tables 2 and 3).
Theorem 2.2: Let $f \in \mathcal{P}_{F_k}(M)$ be given by (6) with $0 < M < 1/(2 \ln 4 - 1)$. Then for $N \geq 3$,

$$|f(z)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|)|z|^n \leq d(f(0), \partial f(\mathbb{D}))$$

(10)

for $r \leq r_N(M)$, where $r_N(M)$ is the smallest root of the equation

$$(r + 2M(r + (1 - r) \ln(1 - r)))^2 - 1$$

$$+ 2M \left( r - 1 + (1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1) + \ln 4} \right) = 0.$$  

(11)

Table 2. The roots $r_N(M)$ of (11) for $N = 4, 6, 8$ when $M < 1/(2 \ln 2 - 1)$.

| $M$  | 0.2  | 0.4  | 0.6  | 0.8  | 1.0  | 1.29 |
|------|------|------|------|------|------|------|
| $r_4(M)$ | 0.697 | 0.537 | 0.412 | 0.300 | 0.189 | 0.003 |
| $r_6(M)$ | 0.704 | 0.542 | 0.415 | 0.301 | 0.189 | 0.003 |
| $r_8(M)$ | 0.706 | 0.542 | 0.414 | 0.300 | 0.189 | 0.003 |

Figure 1. The graph of $H_{N,M}(r)$ of (9) when $M < 1/(2(2 \ln 2 - 1))$.

Figure 2. The root $r_{6}(M)$ of (9) when $M < 1/(2(2 \ln 2 - 1))$. 
Table 3. The roots $r_N(M)$ of (11) for $N = 3, 5, 6, 9, 10$ when $M < 1/(2(2 \ln 2 - 1))$.

| $M$   | 0.2 | 0.3 | 0.5 | 0.6 | 0.8 | 0.9 | 1.0 | 1.1 | 1.29 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $r_3(M)$ | 0.739 | 0.668 | 0.559 | 0.512 | 0.425 | 0.381 | 0.334 | 0.279 | 0.053 |
| $r_5(M)$ | 0.751 | 0.681 | 0.517 | 0.524 | 0.435 | 0.390 | 0.342 | 0.286 | 0.054 |
| $r_6(M)$ | 0.753 | 0.682 | 0.572 | 0.525 | 0.436 | 0.391 | 0.342 | 0.286 | 0.054 |
| $r_9(M)$ | 0.754 | 0.684 | 0.573 | 0.525 | 0.436 | 0.391 | 0.342 | 0.286 | 0.054 |
| $r_{10}(M)$ | 0.755 | 0.684 | 0.573 | 0.525 | 0.436 | 0.391 | 0.342 | 0.286 | 0.054 |

Figure 3. The roots $r_M(M)$ of (11) for $M < 1/(2(2 \ln 2 - 1))$.

The radius $r_N(M)$ is the best possible.

We prove the following sharp Bohr–Rogosinski inequality by considering power of $z$ in $|f(z)|$ for the functions in the class $P_0^H(M)$ (Figure 3).

Theorem 2.3: Let $f \in P_0^H(M)$ be given by (6) with $0 < M < 1/(2(\ln 4 - 1))$. Then for $N \geq 3$,

$$|f(z^n)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|) |z|^n \leq d(f(0), \partial f(D))$$

for $r \leq r_{m,N}(M)$, where $r_{m,N}(M) \in (0,1)$ is the smallest root of the equation

$$r^m - 1 + 2M \left( r^m + r - 1 + \psi_m(r) + \psi_1(r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} + \ln 4 \right) = 0,$$

and $\psi_m(r) := (1 - r^m) \ln (1 - r^m)$. The radius $r_{m,N}(M)$ is the best possible.

3. Improved Bohr radius for the class $P_0^H(M)$

In 2017, Kayumov and Ponnusamy [15] proved the following improved version of Bohr’s inequality.
Theorem 3.1 ([15]): Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be analytic in \( \mathbb{D} \), \( |f(z)| \leq 1 \) and \( S_r \) denotes the area of the image of the subdisk \( |z| < r \) under mapping \( f \). Then

\[
B_1(r) := \sum_{n=0}^{\infty} |a_n| r^n + \frac{16}{9} \left( \frac{S_r}{\pi} \right) \leq 1 \quad \text{for } r \leq \frac{1}{3}
\]

and the numbers 1/3 and 16/9 cannot be improved. Moreover,

\[
B_2(r) := |a_0|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \frac{9}{8} \left( \frac{S_r}{\pi} \right) \leq 1 \quad \text{for } r \leq \frac{1}{2}
\]

and the numbers 1/2 and 9/8 cannot be improved.

Our aim is to prove a harmonic analogue of Theorem 3.1 for functions \( f \in P^0_H(M) \). It will be interesting to investigate Theorem 3.1 in powers of \( S_r/\pi \). Therefore, in order to generalize Theorem 3.1, we consider an \( N \)th degree polynomial in \( S_r/\pi \) as follows:

\[
P \left( \frac{S_r}{\pi} \right) = \left( \frac{S_r}{\pi} \right)^N + \left( \frac{S_r}{\pi} \right)^{N-1} + \cdots + \frac{S_r}{\pi}.
\]

In this regard, we recall the polylogarithm function which is defined by a power series in \( z \), a Dirichlet series in \( s \).

\[
\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots,
\]

and this definition is valid for arbitrary complex order \( s \) and for all complex arguments \( z \) with \( |z| < 1 \). Therefore, the dilogarithm, denoted as \( \text{Li}_2(z) \), is a particular case of the polylogarithm.

Theorem 3.2: Let \( f \in P^0_H(M) \) be given by (6) with \( 0 < M < 1/(2(\ln 4 - 1)) \). Then for \( N \geq 3 \)

\[
r + \sum_{n=2}^{\infty} \left( |a_n| + |b_n| \right) r^n + P \left( \frac{S_r}{\pi} \right) \leq d \left( f(0), \partial f(\mathbb{D}) \right)
\]

for \( |z| = r \leq r_N(M) \), where \( P(w) = w^N + w^{N-1} + \cdots + w \), a polynomial in \( w \) of degree \( N \), and \( r_N(M) \in (0, 1) \) is the smallest root of the equation

\[
r - 1 + 2M \left( r - 1 + (1 - r) \ln(1 - r) + 2 \ln 2 + P \left( r^2 + 4M^2 G(r) \right) \right) = 0,
\]

where \( G(r) \) is defined by

\[
G(r) := r^2 (\text{Li}_2(r^2) - 1) + (1 - r^2) \ln (1 - r^2).
\]

The radius \( r_N(M) \) is the best possible.

By considering powers in the coefficients, we prove the following sharp result for functions in the class \( P^0_H(M) \).
Theorem 3.3: Let \( f \in \mathcal{P}_{\mathcal{H}}^0 (M) \) be given by (6) with \( 0 < M < 1/(2(\ln 4 - 1)) \). Then for \( N \geq 3 \)

\[
r + \sum_{n=2}^{\infty} (|a_n| + |b_n| + n(n-1)(|a_n| + |b_n|)^2) r^n \leq d \left( f(0), \partial f(\mathbb{D}) \right), \quad \text{for} \ r \leq r_M, \tag{18}
\]

where \( r_M \in (0, 1) \) is the smallest root of the equation

\[
r - 1 + 2M(2M + 1) (r + (1 - r) \ln (1 - r)) - 2M(1 - \ln 4) = 0. \tag{19}
\]

The radius \( r_M \) is the best possible.

4. Refined Bohr radius and Bohr-type inequalities for the class \( \mathcal{P}_{\mathcal{H}}^0 (M) \)

In this section, we prove sharp results for refined Bohr radius and certain Bohr-type inequalities for the class \( \mathcal{P}_{\mathcal{H}}^0 (M) \).

Theorem 4.1: Let \( f \in \mathcal{P}_{\mathcal{H}}^0 (M) \) be given by (6) with \( 0 < M < 1/(2(\ln 4 - 1)) \). Then for integer \( p \geq 1, \ N \geq 3 \) and \( t = [(N - 1)/2] \), we have

\[
|f(z)|^p + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n + \sum_{n=2}^{t} (|a_n| + |b_n|)^2 \frac{r^n}{1 - r} \\
+ \frac{1}{1 - r} \sum_{n=t+1}^{\infty} n(n - 1)(|a_n| + |b_n|)^2 r^{2n} \\
\leq d \left( f(0), \partial f(\mathbb{D}) \right), \quad \text{for} \ r \leq r_{p,t,N} M, \tag{20}
\]

where \( r_{p,t,N} M \in (0, 1) \) is the smallest root of the equation

\[
(r + 2M(r + (1 - r) \ln(1 - r)))^p + 2M \left( r + (1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} \right) \\
+ \sum_{n=2}^{t} \frac{2M}{n(n-1)} \frac{r^n}{1 - r} + 4M^2 \left( r^2 + (1 - r^2) \ln(1 - r^2) - \sum_{n=1}^{t} \frac{r^{2n}}{n(n-1)} \right) \\
- 1 - 2M(1 - \ln 4) = 0. \tag{21}
\]

The radius \( r_{p,t,N} (M) \) is the best possible.

In 2018, Liu et al. [17] proved the following result computing Bohr-type radius for the analytic functions \( f(z) \) for which \( |a_0| \) and \( |a_1| \) are replaced by \( |f(z)| \) and \( |f'(z)| \) respectively.

Theorem 4.2 ([17]): Suppose that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is analytic in \( \mathbb{D} \) and \( |f(z)| < 1 \) in \( \mathbb{D} \). Then

\[
|f(z)| + |f'(z)||z| + \sum_{n=2}^{\infty} |a_n||z|^n \leq 1 \quad \text{for} \ |z| = r \leq \frac{\sqrt{17} - 3}{4}.
\]

The radius \( (\sqrt{17} - 3)/4 \) is the best possible.
We prove the following sharp Bohr-type inequality by considering Jacobian in place of derivative for the functions of the class $\mathcal{P}_{\mathcal{H}}^0(M)$ which is a harmonic analogue of Theorem 4.2.

**Theorem 4.3:** Let $f \in \mathcal{P}_{\mathcal{H}}^0(M)$ is given by (6) with $0 < M < 1/(2(\ln 4 - 1))$. Then for $N \geq 3$

$$|f(z)|^p + \sqrt{|Jf(z)|}r + \sum_{n=2}^\infty (|a_n| + |b_n|)r^n + \frac{1}{1 - r^N} \sum_{n=2}^\infty n(n - 1)(|a_n| + |b_n|)^2r^{2n}$$

$$\leq d(f(0), \partial f(\mathbb{D})) \quad (22)$$

for $|z| = r \leq r_{p,N}(M)$, where $r_{p,N}(M) \in (0, 1)$ is the smallest root of the equation

$$(r + 2M(r + (1 - r) \ln(1 - r)))^p + (1 - 2M \ln(1 - r))r + 2M(r + (1 - r) \ln(1 - r))$$

$$+ \frac{4M^2}{1 - r^N} (r^2 + (1 - r^2) \ln(1 - r^2)) - 1 - 2M(1 - \ln 4) = 0. \quad (23)$$

The radius $r_{p,N}(M)$ is the best possible.

**5. Proof of the main results**

Before starting the proof of the main results, we recall here growth formula and distance bound for the class $\mathcal{P}_{\mathcal{H}}^0(M)$. For $f \in \mathcal{P}_{\mathcal{H}}^0(M)$, we have

$$|f(z)| \geq |z| + 2M \sum_{n=2}^\infty \frac{(-1)^n}{n(n - 1)} |z|^n \quad \text{for } |z| < 1. \quad (24)$$

Then the Euclidean distance between $f(0)$ and the boundary of $f(\mathbb{D})$ is given by

$$d(f(0), \partial f(\mathbb{D})) = \liminf_{|z| \to 1} |f(z) - f(0)|. \quad (25)$$

Since $f(0) = 0$, from (7) and (25) we obtain

$$d(f(0), \partial f(\mathbb{D})) \geq 1 + 2M \sum_{n=2}^\infty \frac{(-1)^{n-1}}{n(n - 1)} = 1 + 2M(1 - \ln 4). \quad (26)$$

**Proof of Theorem 2.1:** Let $f \in \mathcal{P}_{\mathcal{H}}^0(M)$ be given by (6). Using Lemmas 1.2 and 1.3, for $|z| = r$, we obtain

$$|f(z)| + \sum_{n=N}^\infty (|a_n| + |b_n|)r^n$$

$$\leq r + \sum_{n=2}^\infty \frac{2Mr^n}{n(n - 1)} + \sum_{n=N}^\infty \frac{2Mr^n}{n(n - 1)}$$
\[= r + 2M(r + (1 - r) \ln(1 - r)) + 2M\left(r + (1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)}\right)\]

\[= r + 2M\left(2r + 2(1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)}\right).\]  (27)

A simple computation shows that

\[r + 2M\left(2r + 2(1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)}\right) \leq 1 + 2M(1 - 2 \ln 2)\]  (28)

for \(r \leq r_N(M)\), where \(r_N(M)\) is the smallest root of \(F_1(r) = 0\) in \((0, 1)\), here \(F_1 : [0, 1) \to \mathbb{R}\) is defined by

\[F_1(r) := r - 1 + 2M\left(2r - 1 + 2(1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} + 2 \ln 2\right).\]

The existence of the root \(r_N(M)\) is guaranteed by the following fact that \(F_1\) is a continuous function satisfies \(F_1(0) = -1 - 2M(1 - 2 \ln 2) < 0\) for \(M < 1/(2(\ln 4 - 1))\) and \(\lim_{r \to 1^-} F_1(r) > 0\) for \(N \geq 3\). Indeed, we note that

\[\lim_{r \to 1^-} (1 - r) \ln(1 - r) = 0\quad \text{and} \quad \sum_{n=2}^{N-1} \frac{1}{n(n-1)} = \frac{N - 2}{N - 1} \quad \text{for} \quad N \geq 3,\]  (29)

which leads to

\[\lim_{r \to 1^-} F_1(r) = 2M\left(1 - \frac{N - 2}{N - 1} + 2 \ln 2\right) = 2M\left(\frac{1}{N - 1} + 2 \ln 2\right) > 0 \quad \text{for} \quad N \geq 3.\]

Let \(r_N(M)\) to be the smallest root of \(F_1(r) = 0\) in \((0, 1)\). Then we have \(F_1(r_N(M)) = 0\). That is

\[r_N(M) + 2M\left(2r_N(M) + (1 - r_N(M)) \ln(1 - r_N(M)) - \sum_{n=2}^{N-1} \frac{r^n(M)}{n(n-1)}\right)\]

\[= 1 + 2M(1 - 2 \ln 2)\]  (30)

It follows from (26), (27) and (28) for \(|z| = r \leq r_N(M)\), we obtain

\[|f(z)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)r^n \leq d(f(0), \partial f(\mathbb{D})).\]

In order to show that \(r_N(M)\) is the best possible radius, we consider the following function \(f = f_M\) be defined by

\[f_M(z) = z + \sum_{n=2}^{\infty} \frac{2Mz^n}{n(n-1)}.\]  (31)

At \(z = -r\), it is easy to see that

\[|f_M(-r)| = | - r + 2M \sum_{n=2}^{\infty} \frac{(-r)^n}{n(n-1)} | = r + 2M \sum_{n=2}^{\infty} \frac{(-1)^{n-1}r^n}{n(n-1)}\]
Therefore, the distance is defined by
\[
d(f_M(0), \partial f_M(D)) = \liminfr\to1 |f(-r)| = 1 + 2M \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n(n-1)} = 1 + 2M(1 - 2\ln 2).
\]

It is easy to show that \( f_M \in H^0_\mathcal{N}(M) \) and for \( f = f_M \), we have
\[
d(f_M(0), \partial f_M(\mathbb{D})) = 1 + 2M(1 - 2\ln 2). \tag{32}
\]

For the function \( f = f_M \) and \( z = r \), a simple computation shows that
\[
|f_M(r)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)r^n = r + \sum_{n=2}^{\infty} \frac{2Mr^n}{n(n-1)} + \sum_{n=N}^{\infty} \frac{2Mr^n}{n(n-1)}. \tag{33}
\]

Now for \( r > r_N(M) \), we obtain
\[
r + \sum_{n=2}^{\infty} \frac{2Mr^n}{n(n-1)} + \sum_{n=N}^{\infty} \frac{2Mr^n}{n(n-1)} > r_N(M) + \sum_{n=2}^{\infty} \frac{2Mr^n_N(M)}{n(n-1)} + \sum_{n=N}^{\infty} \frac{2Mr^n_N(M)}{n(n-1)}
\]
\[
= r_N(M) + 2M(r_N(M) + (1 - r_N(M)) \ln(1 - r_N(M)))
\]
\[
+ 2M \left( r_N(M) + (1 - r_N(M)) \ln(1 - r_N(M)) - \sum_{n=2}^{N-1} \frac{r^n_N(M)}{n(n-1)} \right)
\]
\[
= r_N(M) + 2M \left( 2r_N(M) + 2(1 - r_N(M)) \ln(1 - r_N(M)) - \sum_{n=2}^{N-1} \frac{r^n_N(M)}{n(n-1)} \right). \tag{34}
\]

Using (30), (32) and (34) in (33) for \( r > r_N(M) \), we obtain
\[
|f_M(r)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)r^n
\]
\[
> r_N(M) + 2M \left( 2r_N(M) + 2(1 - r_N(M)) \ln(1 - r_N(M)) - \sum_{n=2}^{N-1} \frac{r^n_N(M)}{n(n-1)} \right)
\]
\[
= 1 + 2M(1 - 2\ln 2)
\]
\[
= d(f_M(0), \partial f_M(\mathbb{D})).
\]

Therefore, the radius \( r_N(M) \) is the best possible. This completes the proof. \( \blacksquare \)
**Proof of Theorem 2.2:** Let $f \in P_{\mathcal{H}}^0(M)$ be given by (6). Using Lemmas 1.2 and 1.3 for $|z| = r$, we obtain

$$|f(z)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|)|z|^n$$

$$\leq \left( r + \sum_{n=2}^{\infty} \frac{2M2^n}{n(n-1)} \right)^2 + \sum_{n=N}^{\infty} \frac{2M2^n}{n(n-1)}$$

$$= (r + 2M(r + (1 - r) \ln(1 - r)))^2 + 2M \left( r + (1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} \right).$$

(35)

An elementary calculation shows that

$$(r + 2M(r + (1 - r) \ln(1 - r)))^2 + 2M \left( r + (1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} \right)$$

$$\leq 1 + 2M(1 - 2 \ln 2)$$

(36)

for $r \leq r_N(M)$, where $r_N(M)$ is the smallest root of $F_2(r) = 0$ in $(0, 1)$, here

$$F_2(r) := (r + 2M(r + (1 - r) \ln(1 - r)))^2 - 1$$

$$+ 2M \left( r - 1 + (1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} + 2 \ln 2 \right).$$

The existence of the root $r_N(M)$ is guaranteed by the following fact that $F_2$ is a continuous function with $F_2(0) = -1 - 2M(1 - 2 \ln 2) < 0$ for $M < 1/(2(\ln 4 - 1))$ and $\lim_{r \to 1^-} F_2(r) > 0$ for $N \geq 3$. Indeed, (29) gives

$$\lim_{r \to 1^-} F_2(r) = 4M^2 + 2M \ln 4 + \frac{2MN}{N-1} > 0 \quad \text{for } N \geq 3.$$ 

Let $r_N(M)$ to be the smallest root of $F_2(r) = 0$ in $(0, 1)$. Therefore, we have $F_2(r_N(M)) = 0$. That is

$$(r_N(M) + 2M(r_N(M) + (1 - r_N(M)) \ln(1 - r_N(M))))^2 - 1$$

$$+ 2M \left( r_N(M) - 1 + (1 - r_N(M)) \ln(1 - r_N(M)) - \sum_{n=2}^{N-1} \frac{r_N^n(M)}{n(n-1)} + 2 \ln 2 \right) = 0$$

(37)

which is equivalent to

$$(r_N(M) + 2M(r_N(M) + (1 - r_N(M)) \ln(1 - r_N(M))))^2$$

$$+ 2M \left( r_N(M) + (1 - r_N(M)) \ln(1 - r_N(M)) - \sum_{n=2}^{N-1} \frac{r_N^n(M)}{n(n-1)} \right)$$
From (26), (35) and (36) for \(|z| = r \leq r_N(M)\), we obtain

\[
|f(z)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|)|z|^n \leq d(f(0), \partial f(D)).
\]

To show that the radius \(r_N(M)\) is the best possible, we consider the function \(f = f_M\) given by (31). For the function \(f = f_M\) and \(z = r\), a simple computation using shows that

\[
|f_M(r)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|)r^n = \left( r + \sum_{n=2}^{\infty} \frac{2Mr^n}{n(n-1)} \right)^2 + \sum_{n=N}^{\infty} \frac{2Mr^n}{n(n-1)}. \quad (39)
\]

For \(r > r_N(M)\), it is easy to see that

\[
\left( r + \sum_{n=2}^{\infty} \frac{2Mr^n}{n(n-1)} \right)^2 + \sum_{n=N}^{\infty} \frac{2Mr^n}{n(n-1)}
\]

\[
> \left( r_N(M) + \sum_{n=2}^{\infty} \frac{2Mr^n(M)}{n(n-1)} \right)^2 + \sum_{n=N}^{\infty} \frac{2Mr^n(M)}{n(n-1)}
\]

\[
= (r_N(M) + 2M(r_N(M) + (1 - r_N(M)) \ln(1 - r_N(M))))^2
\]

\[
+ 2M \left( r_N(M) + (1 - r_N(M)) \ln(1 - r_N(M)) - \sum_{n=2}^{N-1} \frac{r_n(M)}{n(n-1)} \right).
\]

Using (32), (38) and (40) in (39) for \(r > r_N(M)\), we obtain

\[
|f_M(r)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|)r^n
\]

\[
> (r_N(M) + 2M(r_N(M) + (1 - r_N(M)) \ln(1 - r_N(M))))^2
\]

\[
+ 2M \left( r_N(M) + (1 - r_N(M)) \ln(1 - r_N(M)) - \sum_{n=2}^{N-1} \frac{r_n(M)}{n(n-1)} \right)
\]

\[
= 1 + 2M(1 - 2 \ln 2)
\]

\[
= d(f_M(0), \partial f_M(D)).
\]

Hence, the radius \(r_N(M)\) is the best possible. This completes the proof. ■

**Proof of Theorem 2.3:** Let \(f \in P^0_{\mathcal{P}}(M)\) be given by (6). Using Lemmas 1.2 and 1.3 for \(|z| = r\), we obtain

\[
|f(z^m)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|) |z|^n
\]
\[\begin{align*}
&\leq r^m + \sum_{n=2}^{\infty} \frac{2M(r^m)^n}{n(n-1)} + \sum_{n=N}^{\infty} \frac{2Mr^n}{n(n-1)} \\
&= r^m + 2M(r^m + (1 - r^m) \ln(1 - r^m)) + 2M \left( r + (1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} \right) .
\end{align*}\] (41)

A simple computation shows that
\[ r^m + 2M(r^m + (1 - r^m) \ln(1 - r^m)) + 2M \left( r + (1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} \right) \leq 1 + 2M(1 - 2 \ln 2) \] (42)

for \( r \leq r_{m,N}(M) \), where \( r_{m,N}(M) \) is the smallest root of \( F_3(r) = 0 \) in \( (0, 1) \), here
\[
F_3(r) := r^m - 1 + 2M \left( r^m + r - 1 + (1 - r^m) \ln(1 - r^m) + (1 - r) \ln(1 - r) \right) \\
- 2M \left( \sum_{n=2}^{N-1} \frac{r^n}{n(n-1)} - 2 \ln 2 \right) .
\]

The existence of the root \( r_{m,N}(M) \) is guaranteed by the following fact that \( F_3 \) is a continuous function with \( F_3(0) = -1 - 2M(1 - 2 \ln 2) < 0 \) for \( M < 1/(2(\ln 4 - 1)) \) and \( \lim_{r \to 1^-} F_3(r) > 0 \) for \( N \geq 3 \). Indeed, (29) together with \( \lim_{r \to 1^-} (1 - r^m) \ln(1 - r^m) = 0 \) gives
\[
\lim_{r \to 1^-} F_3(r) = 2M \ln 4 + \frac{2M}{N-1} > 0 \quad \text{for} \quad N \geq 3.
\]

Let \( r_{m,N}(M) \) to be the smallest root of \( F_3(r) = 0 \) in \( (0, 1) \). Therefore, we have \( F_3(r_{m,N}(M)) = 0 \). That is
\[
2M(r_{m,N}^m(M) + r_{m,N}(M) - 1 + (1 - r_{m,N}^m(M)) \ln(1 - r_{m,N}^m(M)) \\
+ (1 - r_{m,N}(M)) \ln(1 - r_{m,N}(M))) \\
+ r_{m,N}^m(M) - 1 - 2M \left( \sum_{n=2}^{N-1} \frac{r_{m,N}^n(M)}{n(n-1)} - 2 \ln 2 \right) = 0
\] (43)

which is equivalent to
\[
\begin{align*}
&\quad r_{m,N}^m(M) + 2M(r_{m,N}^m(M) + (1 - r_{m,N}^m(M)) \ln(1 - r_{m,N}^m(M))) \\
&\quad + 2M \left( r_{m,N}(M) + (1 - r_{m,N}(M)) \ln(1 - r_{m,N}(M)) - \sum_{n=2}^{N-1} \frac{r_{m,N}^n(M)}{n(n-1)} \right) \\
&= 1 + 2M(1 - 2 \ln 2) .
\end{align*}\] (44)
From (26), (41) and (42) for $|z| = r \leq r_{m,N}(M)$, we obtain

$$|f(z^n)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(D)).$$

To show that $r_{m,N}(M)$ is the best possible radius, we consider the function $f = f_M$ defined by (31). For $f = f_M$ and $z = r$, a simple computation shows that

$$|f_M(r^n)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|) |z|^n = r^n + \sum_{n=2}^{\infty} \frac{2M(r^n)^n}{n(n-1)} + \sum_{n=N}^{\infty} \frac{2M r^n}{n(n-1)}. \quad (45)$$

For $r > r_{m,N}(M)$, we obtain

$$r^n + \sum_{n=2}^{\infty} \frac{2M(r^n)^n}{n(n-1)} + \sum_{n=N}^{\infty} \frac{2M r^n}{n(n-1)} > r_{m,N}(M) + \sum_{n=2}^{\infty} \frac{2M(r_{m,N}(M))^n}{n(n-1)} + \sum_{n=N}^{\infty} \frac{2M r_{m,N}(M)}{n(n-1)}$$

$$= r_{m,N}(M) + 2M(r_{m,N}(M) + (1 - r_{m,N}(M)) \ln(1 - r_{m,N}(M))$$

$$+ 2M \left( r_{m,N}(M) + (1 - r_{m,N}(M)) \ln(1 - r_{m,N}(M)) - \sum_{n=2}^{N-1} \frac{r_{m,N}(M)}{n(n-1)} \right). \quad (46)$$

Using (32), (44) and (46) in (45) for $r > r_{m,N}(M)$, we obtain

$$|f_M(r^n)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|) |z|^n$$

$$> r_{m,N}(M) + 2M(r_{m,N}(M) + (1 - r_{m,N}(M)) \ln(1 - r_{m,N}(M))$$

$$+ 2M \left( r_{m,N}(M) + (1 - r_{m,N}(M)) \ln(1 - r_{m,N}(M)) - \sum_{n=2}^{N-1} \frac{r_{m,N}(M)}{n(n-1)} \right)$$

$$= 1 + 2M(1 - 2 \ln 2)$$

$$= d(f_M(0), \partial f_M(D)).$$

Hence, the radius $r_{m,N}(M)$ is the best possible. This completes the proof. \[ \blacksquare \]

**Proof of Theorem 3.2:** Let $f \in P_0^0(M)$ be given by (6). It is well known that

$$S_r = \iint_{D_r} \left( |h'(z)|^2 - |g'(z)|^2 \right) \, dx \, dy, \quad (47)$$

$$\frac{1}{\pi} \iint_{D_r} |h'(z)|^2 \, dx \, dy = \sum_{n=1}^{\infty} n |a_n|^2 r^{2n} \quad (48)$$

$$\frac{1}{\pi} \iint_{D_r} |g'(z)|^2 \, dx \, dy = \sum_{n=2}^{\infty} n |b_n|^2 r^{2n}. \quad (49)$$
Then, we in view of Lemma 1.2 and using (47), (48) and (49), we obtain

\[
\frac{S_r}{\pi} = \frac{1}{\pi} \int \int_{D_r} \left( |h'(z)|^2 - |g'(z)|^2 \right) \, dx \, dy
\]

\[
= r^2 + \sum_{n=2}^{\infty} n|a_n|^2 r^{2n} - \sum_{n=2}^{\infty} n|b_n|^2 r^{2n}
\]

\[
= r^2 + \sum_{n=2}^{\infty} n (|a_n| + |b_n|) (|a_n| - |b_n|) r^{2n}
\]

\[
\leq r^2 + \sum_{n=2}^{\infty} \frac{4M^2 r^{2n}}{n(n-1)^2}
\]

\[
= r^2 + 4M^2 \left( r^2 (\text{Li}_2(r^2) - 1) + (r^2 - 1)(1 - r^2) \right).
\]

Using Lemma 1.2 for \(|z| = r\), we obtain

\[
r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n + P \left( \frac{S_r}{\pi} \right) \leq r + \sum_{n=2}^{\infty} \frac{2Mr^n}{n(n-1)} + P(r^2 + 4M^2G(r))
\]

\[
= r + 2M(r + (1 - r) \ln(1 - r)) + P(r^2 + 4M^2G(r)).
\] (50)

where \(G(r)\) is defined by

\[G(r) := r^2 (\text{Li}_2(r^2) - 1) + (r^2 - 1) \ln(1 - r^2).\]

A simple computation shows that

\[r + 2M(r + (1 - r) \ln(1 - r)) + P(r^2 + 4M^2G(r)) \leq 1 + 2M(1 - 2 \ln 2)\] (51)

for \(r \leq r_N(M)\), where \(r_N(M)\) is the smallest root of \(F_4(r) = 0\) in \((0, 1)\), here

\[F_4(r) := r - 1 + 2M(r - 1 + (1 - r) \ln(1 - r) + P(r^2 + 4M^2G(r)) + 2 \ln 2)\.
\]

We note that \(\text{Li}_2(0) = 0\) and \(\text{Li}_2(1) = \frac{\pi^2}{6} - 1\), which implies that

\[G(0) = 0 \quad \text{and} \quad \lim_{r \to 1^-} G(r) = \frac{\pi^2}{6} - 1.\] (52)

Therefore, we have

\[F_4(0) = -1 + 2M \left( -1 + P(4M^2G(0)) + \ln 4 \right) = -1 + 2M \left( -1 + P(0) + \ln 4 \right).\]
Since $P(w) = w^N + w^{N-1} + \cdots + w$ and $P(0) = 0$, we have $F_4(0) = -1 - 2M(1 - 2\ln 2) < 0$ for $M < 1/(2(\ln 4 - 1))$. On the other hand, using (52), we obtain

$$\lim_{r \to 1^-} F_4(r) = 2M \left( P \left( 1 + 4M^2 \lim_{r \to 1^-} G(r) \right) + \ln 4 \right)$$

$$= 2M \left( P \left( 1 + 4M^2 \left( \frac{\pi^2}{6} - 1 \right) \right) + \ln 4 \right).$$

It is easy to see that $1 + 4M^2(\pi^2/6 - 1) + \ln 4 > 0$ and hence, $P(1 + 4M^2(\pi^2/6 - 1) + \ln 4) > 0$, which shows that $\lim_{r \to 1^-} F_4(r) > 0$. The existence of the root $r_N(M)$ is guaranteed by the following fact that $F_4$ is a continuous function with the properties $F_4(0) = -1 - 2M(1 - 2\ln 2) < 0$ for $M < 1/(2(\ln 4 - 1))$ and $\lim_{r \to 1^-} F_4(r) > 0$ for $N \geq 3$. Let $r_N(M)$ to be the smallest root of $F_4(r) = 0$ in $(0, 1)$. Therefore, we have $F_4(r_N(M)) = 0$. Then we have

$$r_N(M) - 1 + 2M \left( r_N(M) - 1 + (1 - r(M)) \ln(1 - r_N(M)) \right)$$

$$+ 2M \left( P(r_N^2(M) + 4M^2G(r_N(M))) + 2\ln 2 \right) = 0 \quad (53)$$

which is equivalent to

$$r_N(M) + 2M(r_N(M) + (1 - r_N(M)) \ln(1 - r_N(M))) + P \left( r_N^2(M) + 4M^2G(r_N(M)) \right)$$

$$= 1 + 2M(1 - 2\ln 2). \quad (54)$$

Using (26), (51) and (53) for $|z| = r \leq r_N(M)$, we obtain

$$r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n + P \left( \frac{S_r}{\pi} \right) \leq d(f(0), \partial f(D)).$$

To show that $r_N(M)$ is the best possible radius, we consider the function $f = f_M$ defined by (31). Then for $|z| = r$, a simple computation shows that

$$r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n + P \left( \frac{S_r}{\pi} \right) = r + \sum_{n=2}^{\infty} \frac{2Mr^n}{n(n-1)} + P(r^2 + 4M^2G(r)) \quad (55)$$

For $r > r_N(M)$, it is easy to see that

$$r + \sum_{n=2}^{\infty} \frac{2Mr^n}{n(n-1)} + P(r^2 + 4M^2G(r))$$

$$> r_N(M) + \sum_{n=2}^{\infty} \frac{2Mr_n^2(M)}{n(n-1)} + P(r_N^2(M) + 4M^2G(r_N(M)))$$

$$= r_N(M) + 2M(r_N(M) + (1 - r_N(M)) \ln(1 - r_N(M))) + P \left( r_N^2(M) + 4M^2G(r_N(M)) \right). \quad (56)$$
Using (32), (54) and (56) in (55) for \( r > r_N(M) \), we obtain
\[
\begin{align*}
& r + \sum_{n=2}^{\infty} ((|a_n| + |b_n|)r^n + P\left(\frac{S_r}{\pi}\right)) \\
& > r_N(M) + 2M(r_N(M) + (1 - r_N(M))\ln(1 - r_N(M))) + P\left(r_N^2(M) + 4M^2G(r_N(M))\right) \\
& = 1 + 2M(1 - 2\ln 2) \\
& = d(f_M(0), \partial f_M(\mathbb{D})).
\end{align*}
\]
Hence the radius \( r_N(M) \) is the best possible. This completes the proof. \[\Box\]

**Proof of Theorem 3.3:** Let \( f \in \mathcal{P}_{t^*}^0(M) \) be given by (6). Using Lemma 1.2 for \( |z| = r \), we obtain
\[
\begin{align*}
& r + \sum_{n=2}^{\infty} ((|a_n| + |b_n| + n(n - 1)(|a_n| + |b_n|)^2) r^n) \\
& \leq r + \sum_{n=2}^{\infty} \frac{2Mr^n}{n(n - 1)} + \sum_{n=2}^{\infty} \frac{4M^2r^n}{n(n - 1)} \\
& = r + 2M(1 + 2M)(r + (1 - r)\ln(1 - r)).
\end{align*}
\]
(57)

A simple computation shows that
\[
r + 2M(1 + 2M)(r + (1 - r)\ln(1 - r)) \leq 1 + 2M(1 - 2\ln 2)
\]
for \( r \leq r_M \), where \( r_M \) is the smallest root of \( F_5(r) = 0 \) in \((0, 1)\), here
\[
F_5(r) := r - 1 + 2M(2M + 1)(r + (1 - r)\ln(1 - r)) - 2M(1 - 2\ln 2).
\]
The existence of the root \( r_M \) is guaranteed by the following fact that \( F_5 \) is a continuous function with satisfies \( F_5(0) = -1 - 2M(1 - 2\ln 2) < 0 \) for \( 0 < M < 1/(2(\ln 4 - 1)) \) and \( \lim_{r \to 1^-} F_5(r) > 0 \). Indeed, (29) gives
\[
\lim_{r \to 1^-} F_5(r) = 2M(2M + \ln 4) > 0 \quad \text{for} \quad M > 0.
\]
Let \( r_M \) to be the smallest root of \( F_5(r) = 0 \) in \((0, 1)\). Then we have \( F_5(r_M) = 0 \). That is,
\[
r_M - 1 + 2M(2M + 1)(r_M + (1 - r_M)\ln(1 - r_M)) - 2M(1 - 2\ln 2) = 0
\]
(59)
which is equivalent to
\[
r_M + 2M(1 + 2M)(r_M + (1 - r_M)\ln(1 - r_M)) = 1 + 2M(1 - 2\ln 2).
\]
(60)
From (26), (57) and (58) for \( |z| = r \leq r_M \), we obtain
\[
r + \sum_{n=2}^{\infty} ((|a_n| + |b_n| + n(n - 1)(|a_n| + |b_n|)^2) r^n \leq d(f(0), \partial f(\mathbb{D})).
\]
In order to show that \( r_M \) is the best possible radius, we consider the function \( f = f_M \) defined by (31). For the function \( f = f_M \) and \( |z| = r \), a simple computation shows that

\[
 r + \sum_{n=2}^{\infty} \left( |a_n| + |b_n| + n(n - 1)(|a_n| + |b_n|)^2 \right) r^n
 = \left( r + \sum_{n=2}^{\infty} \frac{2M r^n}{n(n - 1)} + \sum_{n=2}^{\infty} \frac{4M^2 r^n}{n(n - 1)} \right) r_M.
\]

(61)

For \( r > r_M \), we obtain

\[
 r + \sum_{n=2}^{\infty} \frac{2M r^n}{n(n - 1)} + \sum_{n=2}^{\infty} \frac{4M^2 r^n}{n(n - 1)} > r_M + \sum_{n=2}^{\infty} \frac{2M r_M^n}{n(n - 1)} + \sum_{n=2}^{\infty} \frac{4M^2 r_M^n}{n(n - 1)}
 = r_M + 2M(1 + 2M)(r_M + (1 - r_M) \ln(1 - r_M)).
\]

(62)

Using (32), (60) and (62), from (61), for \( r > r_M \), we obtain

\[
 r + \sum_{n=2}^{\infty} \left( |a_n| + |b_n| + n(n - 1)(|a_n| + |b_n|)^2 \right) r^n
 > r_M + 2M(1 + 2M)(r_M + (1 - r_M) \ln(1 - r_M))
 = 1 + 2M(1 - 2\ln 2)
 = d(f_M(0), \partial f_M(\mathbb{D})).
\]

This shows that the radius \( r_M \) is the best possible. This completes the proof. \[\Box\]

**Proof of Theorem 4.1:** Let \( f \in \mathcal{P}_H^0(M) \) be given by (6). Using Lemmas 1.2 and 1.3 for \( |z| = r \), we obtain

\[
 |f(z)|^p + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n + \frac{t}{r} \sum_{n=2}^{\infty} (|a_n| + |b_n|)^2 \frac{r^n}{1 - r}
 + \frac{1}{1 - r} \sum_{n=t+1}^{\infty} n(n - 1)(|a_n| + |b_n|)^2 r^{2n}
 \leq |f(z)|^p + \sum_{n=N}^{\infty} \frac{2M}{n(n - 1)} r^n + \sum_{n=2}^{\infty} \frac{4M^2}{n^2(n - 1)^2} \frac{r^n}{1 - r}
 + \frac{1}{1 - r} \sum_{n=t+1}^{\infty} \frac{4M^2}{n(n - 1)} r^{2n}
 = (r + 2M(r + (1 - r) \ln(1 - r)))^p + 2M \left( r + (1 - r) \ln(1 - r) - \sum_{n=2}^{N-1} \frac{r^n}{n(n - 1)} \right)
\]
\[ + \sum_{n=2}^{t} \frac{4M^2}{n^2(n-1)^2} \frac{r^n}{1-r} + \frac{4M^2}{1-r} \left( r^2 + (1-r^2) \ln(1-r^2) - \sum_{n=2}^{t} \frac{r^{2n}}{n(n-1)} \right) \]

\[ := C_{p,t,N,M}(r). \] (63)

It is easy to see that

\[ C_{p,t,N,M}(r) \leq 1 + 2M(1 - 2 \ln 2) \] (64)

for \( r \leq r_{p,t,N}(M) \), where \( r_{p,t,N}(M) \) is the smallest root of \( F_6(r) = 0 \) in \((0, 1)\), here \( F_6 : [0, 1) \to \mathbb{R} \) is defined by

\[ F_6(r) := C_{p,t,N,M}(r) - 1 - 2M(1 - 2 \ln 2). \]

The existence of the root \( r_{p,t,N}(M) \) can be shown by the similar argument as in the proof of Theorem 2.1. Clearly, we have \( F_6(r_{p,t,N}(M)) = 0 \). That is

\[ C_{p,t,N,M}(r_{p,t,N}(M)) = 1 + 2M(1 - 2 \ln 2). \] (65)

From (26), (63) and (64) for \( |z| = r \leq r_{p,t,N}(M) \), we obtain

\[
[f(z)]^p + \sum_{n=N}^{\infty} (|a_n| + |b_n|)r^n + \sum_{n=2}^{t} (|a_n| + |b_n|)^2 \frac{r^n}{1-r} \\
+ \frac{1}{1-r} \sum_{n=t+1}^{\infty} n(n-1)(|a_n| + |b_n|)^2 r^{2n} \\
\leq d(f(0), \partial f(\mathbb{D})).
\] (66)

To show that \( r_{p,t,N}(M) \) is the best possible radius, we consider the function \( f = f_M \) defined by (31). For \( f = f_M \) and \( z = r \), it is easy to see that

\[
[f_M(r)]^p + \sum_{n=N}^{\infty} (|a_n| + |b_n|)r^n + \sum_{n=2}^{t} (|a_n| + |b_n|)^2 \frac{r^n}{1-r} \\
+ \frac{1}{1-r} \sum_{n=t+1}^{\infty} n(n-1)(|a_n| + |b_n|)^2 r^{2n} \\
= [f_M(r)]^p + \sum_{n=N}^{\infty} \frac{2M}{n(n-1)} r^n + \sum_{n=2}^{t} \frac{4M^2}{n^2(n-1)^2} \frac{r^n}{1-r} \\
+ \frac{1}{1-r} \sum_{n=t+1}^{\infty} \frac{4M^2}{n(n-1)} r^{2n}. \] (66)
Therefore, for \( r > r_{p,t,N}(M) \), it is easy to see that

\[
|f_M(r)|^p + \sum_{n=N}^{\infty} \frac{2M}{n(n-1)} r^n + \sum_{n=2}^{t} \frac{4M^2}{n^2(n-1)^2} \frac{r^n}{1-r} \\
+ \frac{1}{1-r} \sum_{n=t+1}^{\infty} \frac{4M^2}{n(n-1)} r^{2n} \geq |f_M(r_p,t,N)(M)|^p + \sum_{n=N}^{\infty} \frac{2M}{n(n-1)} r_{p,t,N}^n(M) + \sum_{n=2}^{t} \frac{4M^2}{n^2(n-1)^2} \frac{r_{p,t,N}^n(M)}{1-r_{p,t,N}(M)} \\
+ \frac{1}{1-r_{p,t,N}(M)} \sum_{n=t+1}^{\infty} \frac{4M^2}{n(n-1)} r_{p,t,N}^{2n}(M) = C_{p,t,N,M}(r_{p,t,N}(M)). \tag{67}
\]

Using (32), (65) and (67) in (66) for \( r > r_{p,t,N}(M) \), we obtain

\[
|f_M(r)|^p + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n + \sum_{n=2}^{t} (|a_n| + |b_n|)^2 \frac{r^n}{1-r} \\
+ \frac{1}{1-r} \sum_{n=t+1}^{\infty} n(n-1)(|a_n| + |b_n|)^2 r^{2n} \geq C_{p,t,N,M}(r_{p,t,N}(M)) \\
= 1 + 2M(1 - 2 \ln 2) \\
= d(f_M(0), \partial f_M(D)).
\]

Hence the radius \( r_{p,t,N}(M) \) is the best possible. This completes the proof. \( \blacksquare \)

**Proof of Theorem 4.3:** Let \( f \in \mathcal{P}^0_{H^r}(M) \) be given by (6). The Jacobian of complex-valued harmonic function \( f = h + \overline{g} \) has the following property:

\[
|J_f(z)| \leq |h'(z)|^2 - |g'(z)|^2 \leq |h'(z)|^2 \leq \left( 1 + 2M \sum_{n=1}^{\infty} \frac{r^n}{n} \right)^2. \tag{68}
\]

Using Lemmas 1.2 and 1.3 and (68) for \( |z| = r \), we obtain

\[
|f(z)|^p + r \sqrt{|J_f(z)|} + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n + \frac{1}{1-r} \sum_{n=2}^{\infty} n(n-1)(|a_n| + |b_n|)^2 r^{2n} \leq \left( r + \sum_{n=2}^{\infty} \frac{2Mr^n}{n(n-1)} \right)^p + r \left( 1 + 2M \sum_{n=1}^{\infty} \frac{r^n}{n} \right) + \sum_{n=2}^{\infty} \frac{2Mr^n}{n(n-1)} \\
+ \frac{1}{1-r} \sum_{n=2}^{\infty} \frac{4M^2(r^2)^n}{n(n-1)}.
\]
A simple computation shows that

\[ B_{M,N,p}(r) \leq 1 + 2M(1 - 2 \ln 2) \]  

for \( r \leq r_{p,N}(M) \), where \( r_{p,N}(M) \) is the smallest root of \( F_7(r) = 0 \) in \((0, 1)\), here

\[ F_7(r) := B_{M,N,p}(r) - 1 - 2M(1 - 2 \ln 2). \]

The existence of the root \( r_{p,N}(M) \) can be shown by the same argument used in the proof of Theorem 2.1. Therefore, we have \( F_7(r_{p,N}(M)) = 0 \). That is

\[ B_{M,N,p}(r_{p,N}(M)) = 1 + 2M(1 - 2 \ln 2). \]

From (26), (69) and (70) for \(|z| = r \leq r_{p,N}(M)\), we obtain

\[
|f(z)|^p + r \sqrt{|J_f(z)|} + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n + \frac{1}{1 - r^N} \sum_{n=2}^{\infty} n(n-1)(|a_n| + |b_n|)^2 r^{2n} \leq d(f(0), \partial f(\mathbb{D})).
\]

To show that \( r_{p,N}(M) \) is the best possible radius, we consider the function \( f = f_M \) defined by (31). For \( f = f_M \) and \( z = r \), a simple calculation shows that

\[
|f_M(r)|^p + r \sqrt{|J_f(z)|} + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n + \frac{1}{1 - r^N} \sum_{n=2}^{\infty} n(n-1)(|a_n| + |b_n|)^2 r^{2n} = \left( r + 2Mr^2 \right)^p + r \left( 1 + 2M \sum_{n=1}^{\infty} \frac{r^n}{n} \right) + \frac{2Mr^2}{n(n-1)} + \frac{4M^2(r^2)^n}{n(n-1)} \]

Now for \( r > r_{p,N}(M) \), it is easy to see that

\[
\left( r + 2Mr^n \right)^p + r \left( 1 + 2M \sum_{n=1}^{\infty} \frac{r^n}{n} \right) + \frac{2Mr^n}{n(n-1)} + \frac{4M^2(r^2)^n}{n(n-1)} > \left( r_{p,N}(M) + 2Mr_{p,N}(M) \right)^p + r_{p,N}(M) \left( 1 + 2M \sum_{n=1}^{\infty} \frac{r_{p,N}(M)}{n} \right) + \frac{2Mr_{p,N}(M)}{n(n-1)}
\]
\[ + \frac{1}{1 - r_{p,N}(M)} \sum_{n=2}^{\infty} \frac{4M^2(r^2(M))^n}{n(n - 1)} \]
\[ = B_{M,N,p}(r_{p,N}(M)). \quad (73) \]

Using (32), (71) and (73) in (72) for \( r > r_{p,N}(M) \), we obtain

\[
|f_M(r)|^p + r\sqrt{|f(z)|} + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n + \frac{1}{1 - r_{p,N}} \sum_{n=2}^{\infty} n(n - 1)(|a_n| + |b_n|)^2 r^{2n}
\]
\[
> B_{M,N,p}(r_{p,N}(M))
\]
\[
= 1 + 2M(1 - 2 \ln 2)
\]
\[
= d(f_M(0), \partial f_M(\mathbb{D})).
\]

Therefore, the radius \( r_{p,N}(M) \) is the best possible. This completes the proof. ■

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