Hybrid Quantum Cosmology: Combining Loop and Fock Quantizations

G. A. Mena Marugán and M. Martín-Benito

1 Instituto de Estructura de la Materia, CSIC, Serrano 121, 28006 Madrid, Spain

As a necessary step towards the extraction of realistic results from Loop Quantum Cosmology, we analyze the physical consequences of including inhomogeneities. We consider in detail the quantization of a gravitational model in vacuo which possesses local degrees of freedom, namely, the linearly polarized Gowdy cosmologies with the spatial topology of a three-torus. We carry out a hybrid quantization which combines loop and Fock techniques. We discuss the main aspects and results of this hybrid quantization, which include the resolution of the cosmological singularity, the polymeric quantization of the internal time, a rigorous definition of the quantum constraints and the construction of their solutions, the Hilbert structure of the physical states, and the recovery of a conventional Fock quantization for the inhomogeneities.

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1. INTRODUCTION

In spite of the impressive progress that Cosmology has experienced in recent years, we are still missing a consistent explanation of the origin of the Universe and the formation of structures which is deduced entirely from a fundamental theory. General Relativity (GR) breaks down in the very initial instants of the history of the Universe, leading to a cosmological singularity of the big bang type [1]. In this regime GR cannot be trusted, and the very own predictability of the laws of physics is lost. One expects instead that the physics of the Primitive Universe belongs to the realm of Quantum Gravity, namely, a theory of the gravitational field which incorporates the quantum behavior of nature. One of the most promising candidates for such a theory is Loop Quantum Gravity [2].

At present, important efforts are being made in order to adapt the techniques of Loop Quantum Gravity to much simpler settings than those of the complete theory, which on the other hand remains to be concluded. This is the case of a series of cosmological models obtained from GR by symmetry reduction. The resulting field of research is known under the general name of Loop Quantum Cosmology (LQC) [3].

The first cosmological system whose quantization was performed to completion in LQC was the homogeneous, isotropic, and spatially flat model provided with a minimally coupled, homogeneous, and massless scalar field as matter content [4, 5, 6]. The geometry was polymerically quantized (i.e., using typical LQC methods), while the matter field was described using standard quantization methods. A thorough analysis of the resulting quantum dynamics [5, 6] showed that the initial singularity is successfully resolved. The classical big bang is replaced with a quantum big bounce which deterministically connects a semiclassical expanding universe with a previous semiclassical contracting one. After the pioneer study of this simple model, LQC has been further developed to describe other homogeneous isotropic systems [7], or homogenous and anisotropic cosmologies like, e.g., the Bianchi I model [8, 9, 10]. In all these works, the cosmological singularity is eluded in the quantum dynamics, a result which strengthens the validity of the singularity resolution mechanism and the relevance of LQC.

One may wonder whether the quantum resolution of the singularities of GR is just an artifact of the great symmetry of the models analyzed so far in the literature, and in particular of the homogeneity, or whether, on the contrary, singularities are removed as
well when inhomogeneities are present. In order to ask this question, it seems unavoidable to extend the quantization performed in LQC to inhomogeneous scenarios. Moreover, since inhomogeneities may have played a crucial role at the first instants of the Universe, one should take them into account, indeed, in the development of a realistic theory of quantum cosmology.

In order to progress in this direction, we will consider here the quantization of one of the simplest inhomogeneous cosmological systems, namely the linearly polarized Gowdy $T^3$ model [11]. This model is a natural test bed to incorporate inhomogeneities in LQC. On the one hand, its quantization by means of standard techniques has been discussed in detail [12], and a successful Fock (and Schrödinger) quantization has already been achieved [13, 14]. Even though this system has no timelike isometry and possesses an infinite number of degrees of freedom, this quantization has been shown to be unique (up to unitary equivalence) under certain reasonable requirements, namely, it is the unique Fock quantization with a unitary implementation of the dynamics and invariance under the action of diffeomorphisms (strictly speaking, of the only diffeomorphisms which remain in the system after a natural gauge fixing: the $S^1$ translations). On the other hand, generically, the classical solutions of the Gowdy model represent spacetimes with an initial curvature singularity [15, 16]. Besides, the subset formed by the homogeneous solutions represents Bianchi I spacetimes with a three-torus topology, a model which has been polymerically quantized and where the singularity has again been shown to be resolved as a result of the quantization [9, 10]. Therefore, it is natural to ask how the inclusion of the inhomogeneities affects the singularity resolution in this particular model. In view of the knowledge available about the system and the characteristics of the different gravitational degrees of freedom entering the model, the simplest possibility to investigate that question is to perform a hybrid quantization, which combines the polymeric quantization of the homogeneous degrees of freedom with the Fock quantization of the inhomogeneities. This idea constitutes the basis of our work, whose main aspects have already been presented in Ref. [17]. Actually, this hybrid quantization will allow us to investigate not only whether the singularity is resolved owing to quantum geometry effects, but, furthermore, to explore whether the loop quantization of just the degrees of freedom that parametrize the homogeneous solutions (zero modes in our description) may suffice to cure the big bang singularity. As we will see, the answer turns out to be in the affirmative for our
model. The simplicity of this result reinforces the interest of our hybrid approach, since the same strategy can be applied to the quantization of more general cosmologies.

In addition, the hybrid quantization of the Gowdy model provides a specially suitable arena for the analysis of other important issues in quantum gravity and cosmology, such as:

- The role of the internal time. We will select as emergent time a gravitational variable which behaves like a clock in the classical theory. However, in contrast with the situation studied in the recent literature [5, 6], we will quantize this emergent time using a polymeric representation, because it describes one of the degrees of freedom of the homogeneous sector of solutions. As a consequence of this polymeric quantization, we will see that, in principle, the evolution with respect to that time is not unitary.

- The recovery of the standard quantum field theory. One can regard the inhomogeneities as (conveniently scaled) gravitational waves propagating in a homogeneous curved background, namely a Bianchi I spacetime. In a standard quantization, the inhomogeneities can be described by means of a Fock space. One may then ask what is the status of this Fock description from the viewpoint of LQC, where one adopts a polymeric quantization inequivalent to the standard one. We will see that, with our hybrid approach, one indeed recovers the standard Fock description of the inhomogeneities, even though the background is quantized employing loop techniques.

The rest of the paper is organized as follows. In Sec. 2 we will introduce the classical system. We will partially fix the gauge and parametrize the degrees of freedom of the homogeneous solutions (what we will call the homogeneous sector) in Ashtekar variables, whereas the inhomogeneities will be described by nonzero Fourier modes of a certain scalar field (this will be called the inhomogenous sector). A global diffeomorphism constraint remains in the reduced model, affecting only the inhomogeneous sector. This sector will be quantized in Sec. 3 adopting a Fock representation. In addition, we will impose the quantum analog of the mentioned diffeomorphism constraint. On the other hand, a global Hamiltonian constraint, coupling the homogeneous and inhomogeneous sectors, is also present. In Sec. 4 we will construct a polymeric representation of the homogeneous sector,
which coincides with the phase space of a vacuum Bianchi I model. The Hamiltonian
constraint of the complete system will be imposed in Sec. 5, where we will also construct
the physical Hilbert space of the hybrid Gowdy model. Finally, in Sec. 6 we will discuss
the main results of our quantization.

2. THE CLASSICAL GOWDY MODEL

The Gowdy cosmologies are vacuum spacetimes with two spacelike commuting Killing
vector fields and spatial sections of compact topology\textsuperscript{11}. We consider the simplest of
these Gowdy models, namely, the case with the spatial topology of a three-torus, \(T^3\),
and with a linear polarization. In this case both Killing vectors are axial (because of the
topology) and hypersurface orthogonal (owing to the considered polarization).

Let us denote by \(\partial_\sigma\) and \(\partial_\delta\) the two Killing vector fields. To describe the system, we
choose global coordinates \(\{t, \theta, \sigma, \delta\}\) adapted to the symmetries, with \(\theta, \sigma, \delta \in S^1\). The
metric components depend on \(t\) and \(\theta\), being periodic in the latter coordinate. The space-
time is globally hyperbolic and, in a 3+1 decomposition, the metric can be described by
the densitized lapse function \(N\), the shift vector \(N^i\), and the three-dimensional metric
\(q_{ij}\) induced on the spatial slices which foliate the manifold, where \(i, j \in \{\theta, \sigma, \delta\}\). As a
consequence of the hypersurface orthogonality, the induced metric is such that \(q_{\sigma\delta} = 0\).

Besides, the conditions that \(q_{\theta\sigma}\) and \(q_{\theta\delta}\) vanish fix the gauge freedom associated with the
diffeomorphism constraints in \(\sigma\) and \(\delta\), the directions defined by the Killing vector fields
\textsuperscript{18}. In addition, the dynamical stability of these conditions imply that \(N^\sigma\) and \(N^\delta\) must
be equal to zero. At this stage, the induced metric is diagonal and therefore can be char-
acterized by three fields that describe the norm of one of the Killing vectors, the area of
the isometry group orbits, and the scale factor of the metric induced on the set of group
orbits. We further demand that both the generator of the conformal transformations of
this latter metric and the area of the isometry group orbits be homogeneous functions.
These conditions turn out to fix the gauge freedom associated with the nonzero (inhomo-
geneous) Fourier modes (with respect to the \(\theta\)-dependence) of both the diffeomorphism
constraint in the \(\theta\)-direction and the densitized Hamiltonian constraint. Besides, they
imply that \(N^\theta\) and \(N\) must be homogeneous functions \textsuperscript{13, 19}. The difference between
this gauge fixing and reduction procedure and the one considered in Ref. \textsuperscript{13} is that,
in that reference, the system was totally deparametrized, since the area of the isometry group orbits (which is a global time function) was entirely fixed, whereas now we leave unfixed its homogeneous part (the zero Fourier mode).

As a result of the almost complete gauge fixing that we have performed, all the gauge degrees of freedom are fixed except two. The resulting reduced phase space splits into two sectors, the homogeneous sector and the inhomogeneous one. The former of these sectors is formed by the degrees of freedom that describe homogeneous metric functions in our gauge fixing, together with their conjugate variables. This includes the zero mode of the only metric field which has not been fixed yet, and which provides the norm of one of the Killing vector fields (e.g., $\partial_\delta$). The inhomogeneous sector, on the other hand, contains the information about all the nonzero modes of this metric field and their conjugate momenta.

We next describe the homogeneous sector in terms of Ashtekar variables: an $SU(2)$ gravitational connection $A^a_i$ and a densitized triad $E^i_a$, both of them constant on spatial sections. It is worth commenting that this homogeneous sector can be interpreted as the phase space of a vacuum Bianchi I model whose compact sections are homeomorphic to a three-torus. These are precisely the spacetimes described by the considered degrees of freedom when the inhomogeneities vanish. Hence, the system is provided with a natural coordinate cell, namely the $T^3$-cell, with sides of coordinate length equal to $2\pi$. In a diagonal $[SU(2)]$ gauge, and choosing from now on the Euclidean metric as fiducial metric, the nontrivial components of the corresponding Ashtekar variables are

$$A_i^a = \frac{c^i}{2\pi} \delta_i^a, \quad E^i_a = \frac{p_i}{4\pi^2} \delta^i_a.$$ (1)

The nonvanishing Poisson brackets between these variables are $\{c^i, p_j\} = 8\pi G \gamma \delta^i_j$. Here, $a$ is an internal $SU(2)$ index, the symbol $G$ stands for the Newton constant, and $\gamma$ is the Immirzi parameter.

The rest of degrees of freedoms of our reduced system correspond to a metric field $\xi(\theta)$ and its conjugate momentum $P_\xi(\theta)$, both of them devoid of any zero mode contribution since these have already been included in the homogenous sector. Appropriate variables for the description of what we call the inhomogeneous sector are then all the nonzero Fourier modes $\{(\xi_m, P^m_\xi), m \in \mathbb{Z} - \{0\}\}$. Now, we introduce the creation and annihilation variables $\{(a_m, a^*_m)\}$ which would be naturally associated with $\xi(\theta)$ if this
were a free massless scalar field. These are defined by

$$a_m = \sqrt{\frac{\pi}{8G|m|}} \left( |m| \xi_m + i \frac{4G}{\pi} P_m^m \right),$$

(2)

and the complex conjugate relation, and are such that \(\{a_m, a_m^*\} = -i \delta_{m\tilde{m}}\).

The explicit relation between the chosen variables and the metric is [13, 19]

$$ds^2 = \frac{|p_\theta p_\sigma p_\delta|}{4\pi^2} \left[ e^{\tilde{\gamma}} \left( -\frac{N^2}{(2\pi)^4} dt^2 + \frac{d\theta^2}{p_\theta^2} \right) + e^{-2\pi \tilde{\xi}/\sqrt{|m|}} \frac{d\sigma^2}{p_\sigma^2} + e^{2\pi \tilde{\xi}/\sqrt{|m|}} d\delta^2 \right],$$

(3)

where

$$\tilde{\xi}(\theta) = \sum_{m \neq 0} \frac{\sqrt{G}}{\pi \sqrt{|m|}} (a_m + a_m^*) e^{im\theta},$$

(4)

and

$$\tilde{\gamma}(\theta) = \left( \frac{2c_\delta p_\delta}{c_\sigma p_\sigma + c_\delta p_\delta} - 1 \right) \frac{2\pi}{\sqrt{|p_\theta|}} \tilde{\xi}(\theta) - \frac{\pi^2}{|p_\theta|} \tilde{\xi}(\theta)^2 - \frac{8\pi G \gamma}{c_\sigma p_\sigma + c_\delta p_\delta} \zeta(\theta),$$

(5)

with

$$\zeta(\theta) = i \sum_{\tilde{m}, m \neq 0} \text{sign}(m + \tilde{m}) \frac{\sqrt{|m + \tilde{m}| \tilde{m}|m|}}{m} (a_{m+\tilde{m}}^* - a_{m}\tilde{m}) \left( a_{m+\tilde{m}} + a_{m-\tilde{m}}^* \right) e^{im\theta}. $$

(6)

If the inhomogeneities vanish, we recover the Bianchi I spacetime. Let us comment that \(\tilde{\gamma}\) contains a zero mode contribution arising from quadratic terms in the inhomogeneities. We note also that \(p_\theta\) is proportional to the time variable chosen in Ref. [13] to de-parametrize the system.

The gauge fixing that we have performed is partial. This explains the appearance of a homogenous densitized lapse \(N\) in the metric, and the fact that is still allowed to redefine \(\theta\) by introducing a homogenous shift \(N^\theta\). As a consequence, two global constraints remain in the model. One of them is the zero Fourier mode of the diffeomorphism constraint in \(\theta\), \(C_\theta\), which generates translations in the circle. The other is the zero mode of the densitized Hamiltonian constraint, \(\tilde{C}_H\). The volume of the Bianchi I counterpart of the Gowdy spacetime (for vanishing inhomogeneities), which is given by \(V = \sqrt{|p_\theta p_\sigma p_\delta|}\) and that we will call the “homogeneous-volume”, allows us to define a global (nondensitized) Hamiltonian constraint, \(C_G\), instead of \(\tilde{C}_G\). It is obtained by choosing the (nondensitized) homogeneous lapse function \(N = V\tilde{N}\) rather than \(\tilde{N}\). The two remaining constraints can then be expressed as [17]

$$C_\theta = \sum_{m=1}^{\infty} m(a_m^* a_m - a_m^* a_{-m}) = 0,$$

(7)
\[ C_G = \frac{\dot{C}_G}{V} = C_{\text{BI}} + C_\xi = 0 , \]  
\[ C_{\text{BI}} = -\frac{2}{\gamma^2 V} [c_0 p_0 c_0 p_0 + c_0 p_0 c_3 p_3 + c_3 p_3 c_3 p_3] , \]  
\[ C_\xi = \frac{G}{V} \left[ \frac{(c_\sigma p_\sigma + c_3 p_3)^2}{\gamma^2 |p_\theta|} H_{\text{int}}^\xi + 32\pi^2 |p_\theta| H_0^\xi \right] . \]

\( C_{\text{BI}} \) is the classical Hamiltonian constraint of the Bianchi I model, whereas \( C_\xi \) is the inhomogeneous term, which couples in a nontrivial manner the homogeneous sector of the Gowdy model with the inhomogeneities. These are encoded in the contributions

\[ H_0^\xi = \sum_{m \neq 0} |m| a_m^* a_m , \quad H_{\text{int}}^\xi = \sum_{m \neq 0} \frac{1}{2|m|} \left[ 2a_m^* a_m + a_m a_{-m} + a_m^* a_{-m}^* \right] . \]

These terms represent, respectively, the Hamiltonian corresponding to a free massless scalar field and an interaction term quadratic in the field.

The choice of variables that we have made is the most suitable one for the subsequent hybrid quantization that we will perform. On the one hand, the homogeneous sector has been described in the Ashtekar formulation in order to prepare it for the polymeric quantization characteristic of LQC. On the other hand, for the inhomogeneous sector we have chosen the parametrization introduced in Ref. [13], since it is essentially the only one which admits a satisfactory Fock quantization, including a unitary implementation of the dynamics after completing the choice of time gauge (i.e. after a complete deparametrization). The coupling between the two sectors shows the interest (and nontriviality) of our approach because, owing to it, it is not straightforward that a well-defined hybrid quantization be viable.

### 3. FOCK QUANTIZATION

We will start the construction of our quantum model with the Fock quantization of the inhomogeneous sector and the imposition of the constraint defined by the generator of \( S^1 \)-translations, which depends only on the inhomogeneities. In doing so, we first choose as complex structure the one which is naturally associated with the identification of \( \{(a_m, a_m^*)\} \) as creation and annihilation operators, for all modes with \( m \neq 0 \) [14]. Using this complex structure, we construct the “one-particle” Hilbert space, the corresponding symmetric Fock space \( \mathcal{F} \), and a quantization of the variables \( \{(a_m, a_m^*)\} \) by standard
methods. The field dynamics is unitarily implemented in this quantization \cite{13}. Besides, the quantization provides also a natural unitary implementation of the gauge group of $S^1$-translations, since the resulting vacuum is invariant under that group. Imposing these two unitarity requirements, the Fock quantization turns out to be unique (up to equivalence) \cite{14}.

We will denote by $|\{n_m\}:=|...,n_{-m},...,n_m,...\rangle$ the corresponding $n$-particle states, which provide an orthonormal basis for the Fock space. Here $n_m < \infty$ is the occupation number of the $m$-th mode. In these states, only a finite set of these occupation numbers differ from zero. The dense set spanned by them will be denoted by $\mathcal{S}$.

The generator of $S^1$-translations, given in Eq. (7), can be promoted to the quantum operator:

$$\hat{C}_\theta = \sum_{m=1}^{\infty} m(\hat{a}_m^\dagger \hat{a}_m - \hat{a}_{-m}^\dagger \hat{a}_{-m}).$$

(12)

The proper Fock subspace annihilated by this constraint will be called $\mathcal{F}_p$. A basis for it is provided by the $n$-particle states which verify the condition

$$\sum_{m=1}^{\infty} m(n_m - n_{-m}) = 0.$$

(13)

Clearly, the vacuum is one of these states.

The subspace $\mathcal{F}_p$ is the physical Hilbert space of the deparametrized system quantized in Ref. \cite{13}. There, $\hat{C}_\theta$ was the only quantum constraint. In order to get the physical Hilbert space now, however, we need to impose also the quantum counterpart of the Hamiltonian constraint (8).

4. QUANTIZATION OF THE BIANCHI I MODEL

When the inhomogeneities vanish, the Gowdy model reduces to the vacuum Bianchi I model with $T^3$ topology. As a preliminary step before discussing the quantization of the complete Gowdy system, including both the homogeneous and inhomogenous sectors, we will focus our attention on the Bianchi I model, quantizing it polymerically. Here, we will summarize and revisit the analysis of Ref. \cite{9}, where we accomplished the loop quantization of this particular (homogeneous) subfamily of Gowdy spacetimes.
4.1. Quantum representation

In LQC the basic configuration variables are holonomies, whereas the basic momentum variables are fluxes. The holonomy along an edge of oriented coordinate length $2\pi \mu_i$ in the direction $i$ is $h_i^{\mu_i}(c^i) = e^{\mu_i c^i \tau_i}$, where $\tau_i$ are the $SU(2)$ generators proportional to the Pauli matrices, such that $[\tau_i, \tau_j] = \epsilon_{ijk} \tau_k$. On the other hand, the flux across a square $S^i$ of fiducial area $(2\pi \mu_i)^2$ and normal to the direction $i$ is $E(S^i) = p_i (\mu_i)^2$.

The configuration algebra for each fiducial direction $i$ is the algebra of almost periodic functions of $c^i$, which is generated by the matrix elements of the holonomies $N_{\mu_i}(c^i) = e^{i \mu_i c^i}$. We will call $\text{Cyl}_i$ the corresponding vector space. The kinematical Hilbert space $H_{\text{kin}}$ is the tensor product of three copies $H_{\text{kin}}^i$ (one for each fiducial direction) of the space $L^2(\mathbb{R}_B, d\mu_{\text{Bohr}})$, where $\mathbb{R}_B$ is the Bohr compactification of the real line, and $d\mu_{\text{Bohr}}$ is the normalized Haar measure on it [23]. In momentum representation, and employing the Dirac ket notation $|\mu_i\rangle$ to denote the states $N_{\mu_i}(c^i)$, the Hilbert space $H_{\text{kin}}^i$ can be seen as the completion of the algebra $\text{Cyl}_i$ with respect to the discrete inner product $\langle \mu_i | \mu_i' \rangle = \delta_{\mu_i \mu_i'}$.

On the basis states $|\mu_i\rangle$, the action of the basic operators $\hat{p}_i$ (associated with fluxes) and $\hat{N}_{\mu_i}$ (associated with holonomies) reads

$$\hat{p}_i |\mu_i\rangle = 4\pi \gamma l_{\text{Pl}}^2 p_i |\mu_i\rangle, \quad \hat{N}_{\mu'_i} |\mu_i\rangle = |\mu_i + \mu'_i\rangle. \quad (14)$$

Here, $l_{\text{Pl}} = \sqrt{G\hbar}$ is the Planck length.

4.2. Hamiltonian constraint

The classical Hamiltonian constraint (9) of the Bianchi I model leads to the following form when expressed in terms of the basic variables of LQC:

$$C_{BI} = -\frac{2}{\gamma^2} \left[ \Lambda_\theta \Lambda_\sigma \left( \frac{1}{\sqrt{|p_\theta|}} \right) + \Lambda_\phi \Lambda_\delta \left( \frac{1}{\sqrt{|p_\phi|}} \right) + \Lambda_\sigma \Lambda_\delta \left( \frac{1}{\sqrt{|p_\theta|}} \right) + \Lambda_\phi \Lambda_\sigma \left( \frac{1}{\sqrt{|p_\phi|}} \right) \right], \quad (15)$$

where

$$\Lambda_i = -\lim_{\mu'_i \to 0} \frac{i}{2\mu'_i} \text{sign}(p_i)(N_{2\mu'_i} - N_{-2\mu'_i}). \quad (16)$$

The existence in Loop Quantum Gravity of a minimum nonzero eigenvalue for the area, $\Delta = 2\sqrt{3}\pi \gamma l_{\text{Pl}}^2$, has been argued to imply that the limit $\mu'_i \to 0$ is not feasible and
that, for each fiducial direction, there exists in fact a minimum nonzero edge length for the holonomy, $\bar{\mu}_i$. Therefore, in order to obtain the quantum counterpart of $\Lambda_i$ in LQC, the above classical limit is replaced by evaluation at $\bar{\mu}_i$. On the other hand, to determine this minimum fiducial length we adopt the proposal presented in Ref. [8], which leads to the condition $\bar{\mu}_i^2|p_i| = \Delta$. Thus, the value of $\bar{\mu}_i$ depends on that of $|p_i|$, which is one of the degrees of freedom of the physical metric. Quantum mechanically, this translates into the operator relation

$$\frac{\hat{1}}{\bar{\mu}_i} = \frac{\sqrt{|p_i|}}{\sqrt{\Delta}}.$$  \hspace{1cm} (17)

Taking then a suitable symmetric factor ordering in the quantum counterpart of (16), we finally arrive at the operator

$$\hat{\Lambda}_i = -\frac{i}{4\sqrt{\Delta}} \sqrt{|p_i|} \left[ (\hat{\mathcal{N}}_{2\bar{\mu}_i} - \hat{\mathcal{N}}_{-2\bar{\mu}_i}) \text{sign}(p_i) + \text{sign}(p_i) (\hat{\mathcal{N}}_{2\bar{\mu}_i} - \hat{\mathcal{N}}_{-2\bar{\mu}_i}) \right] \sqrt{|p_i|}.$$  \hspace{1cm} (18)

In principle, given relation (17), the operator $\hat{\mathcal{N}}_{\bar{\mu}_i}$ should produce a state-dependent shift $\bar{\mu}_i(\mu_i)$ on the basis states $|\mu_i\rangle$. It is then most convenient to relabel this basis introducing an affine parameter $v_i$ such that the action of $\hat{\mathcal{N}}_{\bar{\mu}_i}$ is defined to cause just a constant shift in the new label. Such an affine reparametrization is possible (see e.g. Ref. [9]), resulting in the following action of the basic operators on the states $|v_i\rangle$ of the relabeled basis:

$$\hat{\rho}_i |v_i\rangle = 3^{1/3} \Delta \text{sign}(v_i) |v_i|^{2/3} |v_i\rangle, \quad \hat{\mathcal{N}}_{\bar{\mu}_i} |v_i\rangle = |v_i + 1\rangle.$$ \hspace{1cm} (19)

Starting with $\hat{\rho}_i$, we can define the operators $\sqrt{|p_i|}$ and $\text{sign}(p_i)$ via the spectral theorem [24]. It is then straightforward to compute the action on the above basis of the operator $\hat{\Lambda}_i$, given in Eq. (18). To promote the Hamiltonian constraint (15) to an operator, we also need to represent the inverse of $\sqrt{|p_i|}$. In this case, the spectral theorem is not applicable since zero is in the discrete spectrum of $\hat{\rho}_i$. To overcome this problem, one introduces the quantum analog of the classical identity [21]

$$\left( \frac{1}{\sqrt{|p_i|}} \right) = \frac{\text{sign}(p_i)}{2\pi G \bar{\mu}_i} \text{tr} \left( \tau_i h_i^{\bar{\mu}_i} (c^l) \{ h_i^{-\bar{\mu}_i} (c^r), \sqrt{|p_i|} \} \right).$$ \hspace{1cm} (20)

The result is a regularized operator whose action on the basis states turns out to be diagonal:

$$\left[ \sqrt{\frac{1}{|p_i|}} \right] |v_i\rangle = b(v_i) |v_i\rangle, \quad b(v_i) = \frac{3^{7/12} \sqrt{8\pi} |v_i|^{1/3} |v_i + 1|^{1/3} - |v_i - 1|^{1/3} |}{\sqrt{74p_i}}.$$  \hspace{1cm} (21)
In this way, the Hamiltonian constraint of the Bianchi I model is represented by the symmetric operator

\[
\hat{C}_{BI} = -\frac{2}{\gamma^2} \left\{ \hat{\Lambda}_\theta \hat{\Lambda}_\sigma \left[ \frac{1}{\sqrt{|p_\delta|}} \right] + \hat{\Lambda}_\theta \hat{\Lambda}_\delta \left[ \frac{1}{\sqrt{|p_\sigma|}} \right] + \hat{\Lambda}_\sigma \hat{\Lambda}_\delta \left[ \frac{1}{\sqrt{|p_\theta|}} \right] \right\},
\]

which is densely defined on the domain \( Cyl_S = \text{span}\{ |v_\theta, v_\sigma, v_\delta\rangle \} = |v_\theta\rangle \otimes |v_\sigma\rangle \otimes |v_\delta\rangle \} \).

### 4.3. Singularity resolution

Owing to the factor ordering chosen in our definitions, our symmetric Hamiltonian constraint annihilates all the states in the basis \( \{ |v_\theta, v_\sigma, v_\delta\rangle \} \) with any of the \( v_i \)'s equal to zero. We will call them “zero volume states”, since they are eigenstates of the (homogeneous-)volume operator \( \hat{V} = \otimes_i \sqrt{|p_i|} \) with vanishing eigenvalue. Furthermore, the complement of the subspace spanned by these zero volume states is invariant under the action of the constraint operator (22). Therefore, the restriction of the constraint to this complement is well defined. We can say that, when the constraint is imposed, this complement decouples from the space of zero volume states. Then, we limit our considerations to it in the following in order to find nontrivial solutions to the constraint. We will call \( \tilde{C}_{yl_S} \) the linear span of tensor products of states \( |v_i\rangle \) such that none of the \( v_i \)'s vanishes, whereas \( \tilde{H}_{\text{Kin}} \) will denote the corresponding Hilbert space of nonzero volume states.

It is worth noticing that, since zero volume states have been removed from our kinematical Hilbert space, so that the kernel of all the operators \( \hat{p}_i \) is empty, there is no longer any quantum analog of the classical cosmological singularity, where some of the triads \( p_i \) vanish classically. In this sense, the singularity is resolved in our Bianchi I model. We will discuss this issue in more detail in Sect. 6, where we will also analyze the fate of the singularity in the Gowdy model.

### 4.4. Densitized Hamiltonian constraint

In principle, solutions to the Hamiltonian constraint do not need to be normalizable in the kinematical Hilbert space, which is mainly a mathematical tool introduced to construct a representation of the holonomy-flux algebra and the constraints of the model, but which does not take into account the dynamics. We will look for solutions in the much
bigger space $(\tilde{\text{Cyl}}_S)^*$, the algebraic dual of the dense domain of definition of the Hamiltonian constraint, whose elements will be denoted by $(\psi|$. Instead of looking directly for solutions to the constraint (22), which are difficult to determine, we find more convenient to densitize this constraint by means of the following bijection in the dual space $(\tilde{\text{Cyl}}_S)^*$:

$$(\psi| \rightarrow (\psi| \sqrt[\gamma]{\frac{1}{V}}),$$

where the operator that represents the inverse of the (homogeneous-)volume is

$$\sqrt[\gamma]{\frac{1}{V}} = \otimes_i \sqrt[\gamma]{\frac{1}{|p_i|}}.$$

The transformed physical states are now annihilated by the (adjoint of the) symmetric densitized Hamiltonian constraint, defined as

$$\hat{c}_{\text{BI}} = \left(\sqrt[\gamma]{\frac{1}{V}}\right) - \frac{1}{2} \hat{c}_\text{BI} \left(\sqrt[\gamma]{\frac{1}{V}}\right) - \frac{1}{2} = -\frac{2}{\gamma^2} \left[\hat{\Theta}_\theta \hat{\Theta}_\sigma + \hat{\Theta}_\theta \hat{\Theta}_\delta + \hat{\Theta}_\sigma \hat{\Theta}_\delta\right],$$

where $\hat{\Theta}_i$ is the symmetric operator

$$\hat{\Theta}_i = \left[\sqrt[\gamma]{\frac{1}{\sqrt{|p_i|}}} \hat{\Lambda}_i \sqrt[\gamma]{\frac{1}{\sqrt{|p_i|}}}\right]^{-\frac{1}{2}}.$$

Note that the operator $[1/\sqrt{|p_i|}]^{-1/2}$ is well defined once we have restricted the study to (a dense domain in) the space of nonzero volume states. It is also worth noticing that the above kind of densitization procedure can be similarly applied if $\tilde{\text{Cyl}}_S$ is replaced with another dense domain for the Hamiltonian constraint such that the inverse (homogeneous)-volume is still a bijection in the corresponding dual.

From Eq. (25), we see that all the operators $\hat{\Theta}_i$ are Dirac observables in the Bianchi I model, because they commute with the constraint. This fact simplifies enormously the resolution of the constraint, which can be seen as an algebraic equation in the eigenstates that are allowed simultaneously for the three operators $\hat{\Theta}_i$, one for each direction. Since the three operators are formally identical, the problem is reduced to a one-dimensional problem. With this motivation, we will now analyze the properties of $\hat{\Theta}_i$. 


4.4.1. Superselection and no-boundary description

As one can check by direct calculation, the operator \( \hat{\Theta}_i \) is a difference operator acting on the basis states \( |v_i\rangle \):

\[
\hat{\Theta}_i |v_i\rangle = -i \frac{\Delta}{2\sqrt{3}} \left[ f_+(v_i)|v_i + 2\rangle - f_-(v_i)|v_i - 2\rangle \right],
\]

(27)

where

\[
f_{\pm}(v_i) = g(v_i \pm 2)s_{\pm}(v_i)g(v_i), \quad s_{\pm}(v_i) = \text{sign}(v_i \pm 2) + \text{sign}(v_i),
\]

(28)

and

\[
g(v_i) = \begin{cases} 
\left| \frac{1}{1 + \frac{1}{v_i}} \right|^\frac{1}{2} - \left| \frac{1}{1 - \frac{1}{v_i}} \right|^\frac{1}{2} & \text{if } v_i \neq 0, \\
0 & \text{if } v_i = 0.
\end{cases}
\]

(29)

The presence of signs in Eq. (28) has important consequences. Namely, it turns out that the function \( f_+(v_i) \) (\( f_-(v_i) \)) vanishes in the whole interval \([-2, 0] \cup [0, 2]\)). As a result, in terms of the label \( v_i \), the action of the difference operator \( \hat{\Theta}_i \) does not mix any of the following semilattices:

\[
L_{\pm}^{\pm} = \{ \pm(\varepsilon_i + 2k), k = 0, 1, 2... \}, \quad \varepsilon_i \in (0, 2].
\]

(30)

From now on, we will call \( \mathcal{H}_{\pm}^{\pm} \) the corresponding subspaces of states with support in these semilattices. Then, each of them provides a superselection sector.

If we compare the classical densitized constraint [see Eqs. (8) and (9)] with its quantum counterpart, given in Eq. (25), we conclude that the operator \( \hat{\Theta}_i \) is the quantum analog of the classical quantity \( c'p_i \) [21]. Hence, the Wheeler-DeWitt analog of \( \hat{\Theta}_i \) would be the first order differential operator \( \hat{\Theta}_i = i8\pi\gamma l_p^2 p_i \partial p_i \). One would expect that \( \hat{\Theta}_i \) could be recovered from \( \hat{\Theta}_i \) in a suitable semiclassical limit. However, its action (27) indicates that \( \hat{\Theta}_i \) is instead a second-order difference operator. This apparent conflict, nonetheless, does not break physical consistency. Because of the property \( f_{\pm}(\varepsilon_i) = 0 \), when one hits the origin in \( \mathcal{H}_{\pm}^{\pm} \), one gets a relation which constrains the data at \( v_i = \pm\varepsilon_i \) and \( v_i = \pm(\varepsilon_i + 2) \).

Hence, the eigenstates of \( \hat{\Theta}_i \) in \( \mathcal{H}_{\pm}^{\pm} \) depend in fact on a single piece of initial data (as it would correspond to a first-order operator), given just by the projection on the \( v_i = \pm\varepsilon_i \) “slice” of the semilattice.
In this sense, besides, the Hamiltonian constraint provides a no-boundary description: physical states will not only have no contribution from the slice $v_i = 0$ (for any $i=1, 2$ or 3), which would correspond to the classical singularity, but they also get no contribution with the opposite sign of the label $v_i$—namely the opposite triad orientation—without the need to impose any boundary condition at the initial slice.

### 4.4.2. Spectrum and eigenfunctions

Let us call $\text{Cyl}_{v_i}^\pm$ the linear span of the $v_i$-states in the semilattice $L_{v_i}^\pm$. Then, the operator $\hat{\Theta}_{v_i}$, with domain $\text{Cyl}_{v_i}^\pm$, is essentially self-adjoint on $\mathcal{H}_{v_i}^\pm$. In addition, its spectrum has been completely characterized: it is absolutely continuous, nondegenerate, and coincides with the real line. Furthermore, the generalized eigenfunction with generalized eigenvalue $\lambda \gamma l_p^2$, denoted by $e^\pm_{\lambda v_i}(v_i)$, is completely determined by the data on the initial slice, $e^\pm_{\lambda v_i}(\pm v_i)$, as we pointed out before. One can show that the explicit expression e.g. of $e^v_{\lambda v_i}(v_i)$ is\footnote{\textbf{Equation 31}}

$$e^v_{\lambda v_i}(v_i) = \sum_{O(M)} \left[ \prod_{\{r_k\}} \frac{f_-(v_i + 2r_k + 2)}{f_+(v_i + 2r_k + 2)} \right] \left[ \prod_{\{s_l\}} \frac{-i 2\sqrt{3} \lambda \gamma l_p^2}{\Delta f_+(v_i + 2s_l)} \right] e^v_{\lambda v_i}(v_i).$$

Here, $O(M)$ denotes the set of all possible ways to move from 0 to $M$ by jumps of one or two steps. For each element in $O(M)$, $\{r_k\}$ is the subset of integers followed by a jump of two steps, whereas $\{s_l\}$ is the subset of integers followed by a jump of only one step. Note that, up to a constant phase, these complex coefficients oscillate from real to imaginary when $v_i$ varies along the considered semilattice.

After a suitable (delta-)normalization of the generalized eigenstates $|e^\pm_{\lambda v_i}\rangle$ (in ket notation), the spectral resolution of the identity in the kinematical Hilbert space $\mathcal{H}_{v_i}^\pm$ associated with $\hat{\Theta}_{v_i}$ is given then by

$$\mathbb{I}_{v_i}^\pm = \int_\mathbb{R} d\lambda |e^\pm_{\lambda v_i}\rangle \langle e^\pm_{\lambda v_i}|.$$

### 4.5. Physical states for Bianchi I

Let us restrict our study to any specific superselection sector determined by the three numbers $(\varepsilon_\theta, \varepsilon_\sigma, \varepsilon_\delta)$ and by a sign in each direction for the orientation of the triad. In the following, for simplicity, we will choose positive orientations. Since the $\hat{\Theta}_{v_i}$'s are Dirac
observables and we know their associated resolution of the identity, it is really easy to find the physical Hilbert space. We can follow two strategies. On the one hand, since \( \hat{\Theta}_i \) is essentially self-adjoint in \( \text{Cyl}^+_e \), the constraint operator \( \hat{C}_{\text{BI}} \), given in Eq. (25), is essentially self-adjoint in the tensor product of these spaces, and we can apply the group averaging procedure \([25, 26]\) to determine the solutions to the Hamiltonian constraint and their Hilbert structure. On the other hand, we can also solve directly the constraint and determine the Hilbert structure of the solutions by choosing a complete set of real observables and imposing that they be represented as self-adjoint operators.

In both cases, one concludes that the solutions to the Hamiltonian constraint have the form

\[
\psi(v_\theta, v_\sigma, v_\delta) = \int_{\mathbb{R}^2} d\lambda_\sigma d\lambda_\delta e^{\epsilon_\sigma}_{\lambda_\sigma} (v_\sigma) e^{\epsilon_\delta}_{\lambda_\delta} (v_\delta) \tilde{\psi}(\lambda_\sigma, \lambda_\delta),
\]

with

\[
\lambda_\theta[\lambda] = -\frac{\lambda_\delta \lambda_\sigma}{\lambda_\delta + \lambda_\sigma}.
\]

Here, the physical states \( \tilde{\psi}(\lambda_\sigma, \lambda_\delta) \) belong to the Hilbert space

\[
\mathcal{H}^\text{BI} = L^2 \left( \mathbb{R}^2, |\lambda_\sigma + \lambda_\delta| d\lambda_\sigma d\lambda_\delta \right).
\]

### 4.5.1. Evolution and observables

In the above expression for the solutions, we have eliminated the dependence on \( \lambda_\theta \), determined in terms of \( \lambda_\sigma \) and \( \lambda_\delta \). This is just a matter of convention: we could choose to fix any of these three variables in terms of the other two. In view of our choice, we can interpret \( v_\theta \) as an internal time and regard physical states as evolving with respect to it \([27]\). At a particular time \( v_\theta^o \), the solution can be written as

\[
\psi(v_\sigma, v_\delta)|_{v_\theta^o} = \int_{\mathbb{R}^2} d\lambda_\sigma d\lambda_\delta e^{\epsilon_\sigma}_{\lambda_\sigma} (v_\sigma) e^{\epsilon_\delta}_{\lambda_\delta} (v_\delta) \tilde{\psi}(\lambda_\sigma, \lambda_\delta)|_{v_\theta^o},
\]

where

\[
\tilde{\psi}(\lambda_\sigma, \lambda_\delta)|_{v_\theta^o} = e^{\epsilon_\sigma}_{\lambda_\sigma}|_{v_\sigma^o} e^{\epsilon_\delta}_{\lambda_\delta}|_{v_\delta^o} \tilde{\psi}(\lambda_\sigma, \lambda_\delta)
\]

belongs to the “\( v_\theta^o \)-slice” Hilbert space

\[
\mathcal{H}_{v_\theta^o} = L^2 \left( \mathbb{R}^2, \left|\frac{\lambda_\sigma + \lambda_\delta}{e^{\epsilon_\theta}_{\lambda_\theta}|_{v_\theta^o}^2} \right|^2 d\lambda_\sigma d\lambda_\delta \right).
\]
A complete set of observables is given by the constants of motion \( \hat{\Theta}_\delta \) and \( \hat{\Theta}_\sigma \), which act on the physical states just by multiplication by \( \lambda_\sigma \gamma_2 l^2_{Pl} \) and \( \lambda_\delta \gamma_2 l^2_{Pl} \) respectively, and by the observables at fixed time \( \hat{v}_\sigma |v_\sigma^o\rangle \) and \( \hat{v}_\delta |v_\delta^o\rangle \), whose action on solutions is given by

\[
\hat{v}_\alpha |v_\sigma^o\rangle \psi(v_\sigma, v_\delta) |v_\sigma^o\rangle = v_\alpha \psi(v_\sigma, v_\delta) |v_\sigma^o\rangle, \quad \alpha = \sigma, \delta. \tag{39}
\]

On the \( v_\sigma^o \)-slice Hilbert space, on the other hand, the corresponding action is

\[
\hat{v}_\sigma |v_\sigma^o\rangle \tilde{\psi}(\lambda_\sigma, \lambda_\delta) |v_\sigma^o\rangle = \int_{\mathbb{R}} d\tilde{\lambda}_\sigma \langle e^{e_{\lambda_\sigma}} |v_\sigma| e^{e_{\lambda_\delta}} \rangle_{\text{kin}} \tilde{\psi}(\tilde{\lambda}_\sigma, \lambda_\delta) |v_\sigma^o\rangle, \tag{40}
\]

and similarly for \( \hat{v}_\delta |v_\delta^o\rangle \).

The fact that the dependence of \( |e^{e_{\lambda_\sigma}}|_{\lambda_\sigma}(v_\sigma) \) on \( \lambda_\sigma \) and \( \lambda_\delta \) varies with the value of \( v_\theta \) precludes one establishing a unitary relation between the different \( v_\theta \)-slice Hilbert spaces, obtained for other choices of \( v_\theta \) instead of \( v_\sigma^o \), so that one does not get a unitary evolution in this emergent time. This result is a mere consequence of the nature of the polymeric quantization performed on the variable that plays the role of internal time \[27\].

5. HYBRID QUANTIZATION OF THE GOWDY MODEL

Once we have represented the homogeneous sector of the Gowdy model following the prescriptions of LQC, and the inhomogeneous sector employing the Fock quantization, we are ready to construct the quantum counterpart of the Hamiltonian constraint of the Gowdy model, \( C_G \), within our hybrid approach. The kinematical Hilbert space is just the tensor product of the kinematical Hilbert spaces for each sector, that is \( \mathcal{H}_{\text{kin}} \otimes \mathcal{F} \).

Taking into account Eqs. (8)-(11), it is straightforward to represent the constraint \( C_G \) as an operator. In fact, in the previous section we have already represented the quantum analog \( \hat{C}_{\text{BI}} \) of the homogeneous term \( C_{\text{BI}} \). In addition, we have constructed the inverse homogeneous-volume operator, given in Eq. (24), and obtained the quantum counterpart \( \hat{\Theta}_i \) of the quantity \( c^i p_i \). Therefore, we can also promote the inhomogeneous term \( C_\xi \) to an operator. Adopting the same kind of factor ordering used for the homogeneous term, we get the following symmetric operator:

\[
\hat{C}_\xi = \left[ \frac{1}{V} \right]^{\frac{1}{2}} \hat{\Theta}_i \left[ \frac{1}{V} \right]^{\frac{1}{2}}, \quad \hat{C}_\xi = l^2_{Pl} \left[ \frac{\hat{\Theta}_\sigma + \hat{\Theta}_\delta}{\gamma^2} \right]^2 \left( \frac{1}{\sqrt{|p_\theta|}} \right)^2 \hat{H}_{\text{int}} + 32\pi^2 |\widehat{p_\theta}| \hat{H}_0 \tag{41}
\]
where the free Hamiltonian $\hat{H}_0^\xi$ and the interaction term $\hat{H}_{\text{int}}^\xi$ are normal ordered:
\[
\hat{H}_0^\xi = \sum_{m \neq 0} |m| \hat{a}_m^\dagger \hat{a}_m, \quad \hat{H}_{\text{int}}^\xi = \sum_{m \neq 0} \frac{1}{2|m|}(2\hat{a}_m^\dagger \hat{a}_m + \hat{a}_m \hat{a}_{-m} + \hat{a}_m^\dagger \hat{a}_{-m}).
\] (42)

Similarly to what happens with the Hamiltonian constraint of Bianchi I, $\hat{C}_{\text{BI}}$, the inhomogeneous term $\hat{C}_\xi$ annihilates the subspace spanned by the zero homogeneous-volume states and leaves invariant its complement. Therefore, this subspace decouples from its complement also in the complete Gowdy model when the constraint is imposed. Thus, we can again restrict our study to the kinematical Hilbert space $\tilde{H}_{\text{kin}}$ in the homogeneous sector. Moreover, the Hilbert spaces $\otimes_i \mathcal{H}_{\xi_i}^\pm \otimes \mathcal{F}$ are also superselected by the Hamiltonian constraint of the Gowdy model, so that for practical purposes we can further restrict our discussion to any of these kinematical Hilbert spaces. We continue choosing positive orientation of the triads for simplicity and, in principle, define our operators in the corresponding dense set
\[
\otimes_i \text{Cyl}_{\xi_i}^+ \otimes \mathcal{S} = \text{span}\{|v_\theta, v_\sigma, v_\delta\} \otimes |\{n_m\}; \quad v_i \in \mathcal{L}_{\xi_i}^+\}. \tag{43}
\]

As in the Bianchi I case, it is preferable to densitize the Hamiltonian constraint, in particular because it is then straightforward to recognize some Dirac observables, what facilitates the resolution of the constraint. We introduce again the bijective map (23) [in principle in the dual $(\otimes_i \text{Cyl}_{\xi_i}^+ \otimes \mathcal{S})^*$, but the dual of any other dense set where the inverse homogenous-volume operator provides a bijection would be acceptable as well]. This leads to “transformed” physical states which are annihilated just by the densitized Hamiltonian constraint $\hat{C}_G = \hat{C}_{\text{BI}} + \hat{C}_\xi$, with $\hat{C}_{\text{BI}}$ given in Eq. (22) and $\hat{C}_\xi$ in Eq. (41). It is worth noting that, although the constraint operator $\hat{C}_G$ couples both the homogenous and inhomogeneous sectors of the Gowdy model in a highly nontrivial way, it is indeed a well-defined symmetric operator with domain $\otimes_i \text{Cyl}_{\xi_i}^+ \otimes \mathcal{S}$.

In the Gowdy model, the operators $\hat{\Theta}_\sigma$ and $\hat{\Theta}_\delta$ are still Dirac observables. Nevertheless, the densitized Hamiltonian constraint depends now on $|\hat{p}_\theta|$ and $1/\sqrt{|\hat{p}_\theta|}$, so that $\hat{\Theta}_\theta$ no longer commutes with this constraint and fails to be a Dirac observable. It is then convenient to use the basis of states $|v_\theta\rangle \otimes |e_{\lambda_\sigma}^\xi\rangle \otimes |e_{\lambda_\delta}^\xi\rangle$ for the homogeneous sector. In doing so, the densitized constraint becomes a difference equation in $v_\theta$ and can be regarded as an evolution equation in this parameter, what means that $p_\theta$ plays the role of internal time. As we have already commented, this was essentially the time choice.
adopted to deparametrize the system in the Fock quantization of Ref. [13]. Apart from a global factor and some suitable rescalings, the inhomogeneous part of the constraint, \( \hat{C}_\xi \), coincides in each “generalized eigenspace” of \( \hat{\Theta}_\sigma \) and \( \hat{\Theta}_\delta \) with the inhomogeneous Hamiltonian of the deparametrized (gauge-fixed) model [13]. The time dependence of that Hamiltonian becomes a dependence on \( p_\theta \) in our case.

5.1. Solutions to the eigenvalue equation for the Hamiltonian constraint

Let us consider the (complex) eigenvalue equation for \( \hat{C}_G \),

\[
(\psi | \hat{C}_G = \rho l^1_{\psi}(\psi). \tag{44}
\]

Using the basis of states introduced above for the homogeneous sector and substituting the formal expansion

\[
(\psi | = \sum_{v_\theta \in L^+ \sigma} \int_{\mathbb{R}^2} d\lambda_\sigma d\lambda_\delta \langle v_\theta \mid \otimes e^{\xi_\sigma} \otimes \langle e^{\xi_\delta} \otimes (\psi_{\lambda_\sigma, \lambda_\delta} (v_\theta) |, \tag{45}
\]

we get the solution [17, 19]

\[
(\psi_{\lambda_\sigma, \lambda_\delta}(\varepsilon_\theta + 2M) | n_m \rangle = (\psi_{\lambda_\sigma, \lambda_\delta}(\varepsilon_\theta) \sum_{O(M)} \prod_{\{r_k\}} f_-(\varepsilon_\theta + 2r_k + 2) \prod_{\{s_l\}} f_+(\varepsilon_\theta + 2r_k + 2) \right) \times \mathcal{P} \left[ \prod_{\{s_l\}} \tilde{H}_\rho^\xi (v_\theta + 2s_l, \lambda_\sigma, \lambda_\delta) \right] | n_m \rangle, \tag{46}
\]

where \( O(M) \), \( \{r_k\} \) and \( \{s_l\} \) have the same meaning as in Sec. 4.4.4.4.2. Besides, the symbol \( \mathcal{P} \) denotes path ordering, and \( \tilde{H}_\rho^\xi (v_\theta, \lambda_\sigma, \lambda_\delta) \) has the form:

\[
\tilde{H}_\rho^\xi (v_\theta, \lambda_\sigma, \lambda_\delta) = \frac{i}{2\pi(\lambda_\sigma + \lambda_\delta f_+(v_\theta)} \left[ \rho + 2\lambda_\sigma \lambda_\delta - \frac{(\lambda_\sigma + \lambda_\delta)^2}{\gamma} b_2^2(v_\theta) \tilde{H}_{\int}^\xi - 32\pi^2 3^{1/3} \gamma \Delta |v_\theta|^{2/3} \tilde{H}_0^\xi \right]. \tag{47}
\]

5.2. Physical states and observables

The operator \( \hat{C}_G \) is essentially self-adjoint if and only if there is no normalizable solution of the form (46) when \( \rho = \pm i \). In principle, one can make use in that case of the group averaging method [25, 26] to construct the physical Hilbert space. In practice,
nevertheless, to apply this method we should first determine and adopt an invariant
domain for the constraint operator $\hat{C}_G$, a task which seems difficult to accomplish given
the complexity of the action of the constraint and the infinite number of degrees of freedom
of the system. Therefore, instead of following that approach, we will directly employ our
knowledge of the formal solutions to the densitized Hamiltonian constraint, deduced in
the previous subsection, and determine a physical Hilbert structure for them by requiring
the self-adjointness of a complete set of real observables.

Solutions to the constraint are formally given by expression (46) with $\rho = 0$. As we
can see, the initial data $(\psi_{\lambda, \lambda}(\varepsilon))$ completely determines the formal solution, so that
we can identify the latter with the data on the initial section $\psi_\theta = \varepsilon_\theta$. To construct the
physical Hilbert space, we then simply provide the vector space of initial data with a
Hilbert structure. The desired inner product can be fixed by choosing a complete set
of real classical observables and requiring that their quantum analogs, which act on the
initial data, be self-adjoint operators. Such a complete set is provided by the observables
introduced for Bianchi I in Sec. (44.5.4.5.1) (with a trivial action on the inhomogeneous
sector) and by a complete set acting on the inhomogeneous modes, which can be, for
instance, the set of operators that represent the Fourier sine and cosine coefficients of the
nonzero modes. Up to irrelevant constant real factors, these operators are
\[ \left\{ (\hat{a}_m + \hat{a}_m^\dagger) \pm (\hat{a}_{-m} + \hat{a}_{-m}^\dagger), \quad i[(\hat{a}_m - \hat{a}_m^\dagger) \pm (\hat{a}_{-m} - \hat{a}_{-m}^\dagger)]; \quad m \in \mathbb{N} \right\}. \] (48)

Actually, they are self-adjoint operators in the standard Fock space $\mathcal{F}$. Therefore, we
conclude that the Hilbert space of initial data picked up by our conditions is, up to
equivalence, $L^2(\mathbb{R}^2, |\lambda_\sigma + \lambda_\delta|d\lambda_\sigma d\lambda_\delta) \otimes \mathcal{F}$.

Finally, to obtain the true physical Hilbert space of the model, we still have to impose
the $S^1$-symmetry generated by the constraint $\hat{C}_\theta$, which commutes with the Hamiltonian
constraint. This symmetry is encoded in the condition (13). Taking this into account, we
arrive in the end to the physical Hilbert space
\[ \mathcal{H}^G = L^2(\mathbb{R}^2, |\lambda_\sigma + \lambda_\delta|d\lambda_\sigma d\lambda_\delta) \otimes \mathcal{F}_\rho. \] (49)
6. CONCLUDING REMARKS

We have carried out a thorough quantization of the Gowdy spacetimes with the spatial section of a three-torus and linear polarization by combining the loop quantization of the homogeneous Bianchi I cosmology (with compact sections) with the Fock quantization of the inhomogeneities in a totally deparametrized Gowdy system. In this way, we have constructed a hybrid quantum model for this family of cosmological spacetimes in vacuo which incorporates the presumably most relevant effects of the quantization of the geometry (at least in scenarios that can be considered close to homogeneity), while allowing the treatment of an infinite number of degrees of freedom.

Even though the Hamiltonian constraint couples in a nontrivial manner the polymeric quantum homogeneous sector with the inhomogeneous sector, quantized with standard Fock methods, we have been able to find the formal solutions to this constraint. The constraint can be regarded as an evolution equation in an emergent time, and the explicit expression of the solutions shows that they are in fact completely determined by their data at an initial value of this time.

The emergent time corresponds to one of the variables that have been quantized with loop techniques. Because of this fact, a naive and straightforward definition of the evolution with respect to it is not implemented as a unitary transformation in the quantum theory. We note, in this sense, that the results discussed at the end of Sec. for Bianchi I may be extrapolated (at least in principle) to the case of the hybrid Gowdy model, because this model admits observables which are just the tensor product of Bianchi I observables and the identity on the inhomogeneous sector.

Owing to the polymeric quantization carried out in the homogeneous sector, we have been able to decouple the space of states in the kernel of any of the triad operators and rigorously remove it from our quantization. Therefore, there is no longer a quantum analog of the (generic) classical cosmological singularity in the system. This result, originally obtained for the Bianchi I model, extends also to the Gowdy model. Hence, we conclude that the loop quantization of the considered zero modes suffice to avoid the singularity in this inhomogeneous model. This robust result is different and simpler than other suggested possibilities for avoiding cosmological singularities in inhomogeneous scenarios, like e.g. by appealing to the BKL (Belinsky, Khalatnikov, and Lifshitz) conjecture. In particular,
the persistence of the resolution mechanism found in homogenous LQC is a global result, in
the sense that one does not need to analyze the approach to the singularity independently
at each point of the corresponding spatial section. Furthermore, no particular boundary
condition has to be imposed in the construction of the quantum states in order to avoid
the singularity and prevent the emergence of contributions with different triad orientation.
From this perspective, one can say that the developed formalism provides a no boundary
description.

In addition to all the above, we have been able to complete the hybrid quantization by
determining the Hilbert space of physical states and providing a complete set of observ-
ables acting on it. Remarkably, the physical Hilbert space which results of this hybrid
quantization is (equivalent to) the tensor product of the physical Hilbert space for Bianchi
I in LQC and the Fock space which describes the quantization of the inhomogeneities in
the totally deparametrized system [13]. Therefore, we can say that one recovers the stan-
dard quantum field theory for the inhomogeneities on a polymerically quantized Bianchi
I background. Finally, let us mention that there are other interesting issues for future
investigation for which our approach may be specially appropriate, like e.g. the analysis
of the semiclassical behavior of physical states, or the implementation of perturbative
approaches which deal with the inhomogeneities as perturbations around the Bianchi I
background.

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