Abstract
Orthogonality relations for Foulkes characters of full monomial groups are presented, along with three solutions to the problem of decomposing products of these characters, and new applications, including a product reformulation of a Markov chain for adding random numbers studied by Diaconis and Fulman, and a new proof of a theorem of Zagier which generalizes one of Harer and Zagier on the enumeration of Riemann surfaces of a given genus.

1 Introduction
Let \( \ell(\pi) \) denote the number of cycles of a permutation \( \pi \in S_n \). Let \( \phi_0, \phi_1, \ldots, \phi_{n-1} \) be the Foulkes characters of \( S_n \), so \( \phi_i \) is afforded by the sum of Specht modules \( V_\beta \) with \( \beta \) of border shape with \( n \) boxes and \( i + 1 \) rows. For history and properties, see Chapter 8 of Kerber’s book [9]. Our starting point is the classical fact that the \( \phi_i \)'s depend only on length in the sense that
\[
\phi_i(\sigma) = \phi_i(\tau) \quad \text{whenever} \quad \ell(\sigma) = \ell(\tau),
\]
and in fact the \( \phi_i \)'s form a basis for the space \( \text{CF}_\ell(S_n) \) of all class functions \( \vartheta \) that depend only on \( \ell \), with each \( \vartheta \in \text{CF}_\ell(S_n) \) decomposing uniquely as
\[
\vartheta = \sum_{i=0}^{n-1} \frac{\langle \vartheta, \epsilon_i \rangle}{\epsilon_i(1)} \phi_i,
\]
where \( \epsilon_i \) is the irreducible character \( \chi_\lambda \) for the hook shape \( \lambda = (n - i, 1^i) \), so \( \epsilon_i(1) = \binom{n-1}{i-1} \). Other important facts about the \( \phi_i \)'s include: They decompose the character \( \rho \) of the regular representation:
\( \phi_0 + \phi_1 + \ldots + \phi_{n-1} = \rho. \)  
(1.3)

Their degrees are Eulerian numbers:

\[ \phi_i(1) = |\{ \pi \in S_n \mid \text{des}(\pi) = i \}|, \quad \text{des}(\pi) = |\{ i \mid \pi(i) > \pi(i+1) \}|. \]  
(1.4)

They branch according to

\[ \phi_i|_{S_{n-1}} = (n-i)\phi_{i-1} + (i+1)\phi_i. \]  
(1.5)

And they even admit a closed-form expression:

\[ \phi_i(\pi) = \sum_{j=0}^{n-1} (-1)^{i-j} \binom{n+1}{i-j} (j+1)^{\ell(\pi)}. \]  
(1.6)

But two questions remain unanswered.

**Question 1** How does a product \( \phi_i \phi_j \) decompose into a sum of \( \phi_k \)'s?

**Question 2** What is the inner product \([ \cdot, \cdot \] \) with respect to which the \( \phi_i \)'s form an orthonormal basis?

We answer both of these questions in the next section.

For decomposing products, we present 3 solutions. The first is a combinatorial solution which follows from a recent result that interprets the values \( \phi_i(\pi) \) as coefficients of Loday’s Eulerian idempotents from cyclic homology [10] in certain sums in the group algebra \( \mathbb{C}[S_n] \). The second solution is an explicit closed-form solution using (1.6). The third solution is perhaps the most surprising, being a recursive solution given by Delsarte in 1976 in a context void of characters and groups, and given 4 years before the \( \phi_i \)'s were introduced by Foulkes in 1980. Delsarte’s work, which had been overlooked up to now, adds yet another surprising place where Foulkes characters arise.

A few years ago, Diaconis and Fulman connected the \( \phi_i \)'s with adding random numbers [4]. Denote by \( \Phi \) the character table

\[ \Phi = (\phi_i(C_{n-j}))_{0 \leq i, j \leq n-1}, \]

where

\[ C_i = \{ \pi \in S_n \mid \ell(\pi) = i \} \]

and for any \( \vartheta \in \text{CF}_\ell(S_n) \) we denote by \( \vartheta(C_i) \) the value \( \vartheta(\pi) \) for any \( \pi \in C_i \). Holte [8] studied the carries that occur when adding \( n \) random numbers in base \( b \), particularly the Markov chain with transition matrix

\[ M = (M(i, j))_{0 \leq i, j \leq n-1} \]
given by
\[ M(i, j) = \text{chance\{next carry is } j \mid \text{last carry is } i \}. \quad (1.7) \]

Diaconis and Fulman found that the transposed columns of \( \Phi \) are left eigenvectors, in particular
\[ \Phi^t M = D \Phi^t, \]
where \( D = \text{diag}(b^0, b^{-1}, \ldots, b^{-n+1}) \).

We consider not adding random numbers and keeping track of carries, but multiplying random \( n \)-cycles in \( S_n \) and counting factorizations. Let \( \sigma \) and \( \tau \) be \( n \)-cycles \((i_1 i_2 \ldots i_n)\) chosen uniformly at random from \( C_1 \), and consider the expected number of ways that the product \( \sigma \tau \) can be written as a product \( \alpha \beta \) with \( \alpha \in C_i \) and \( \beta \in C_j \), i.e.
\[ E|\sigma C_i \cap \tau C_j|. \]
Dividing by \( n! \) gives a probability distribution on pairs \((C_i, C_j)\), and our answer to Question 2 is that the \( \phi_i \)'s form an orthonormal basis with respect to the inner product on \( \text{CF}_\ell(S_n) \) defined by
\[ [\vartheta, \psi] = \frac{1}{|S_n|} \sum_{i,j=0}^{n-1} \vartheta(C_i) \overline{\psi}(C_j) E|\sigma C_i \cap \tau C_j|. \]

As a remarkable consequence, we find that the \( \phi_i \)'s arise in a natural way from multiplying random \( n \)-cycles: they result from the inner product \([\cdot, \cdot]\) by applying the Gram–Schmidt process to the natural basis of characters \( 1^\ell, 2^\ell, \ldots, n^\ell \) in \( \text{CF}_\ell(S_n) \). This is analogous to how the irreducible characters of \( S_n \) can be obtained by taking the usual inner product on class functions of \( S_n \), namely
\[ \langle \vartheta, \psi \rangle = \frac{1}{|S_n|} \sum_{K \in \text{Cl}(S_n)} \vartheta(K) \overline{\psi}(K) |K|, \]
using a natural choice of permutation characters indexed by partitions, namely \((1_{S_n})^S\) with a certain natural order, and applying the Gram–Schmidt process.

As a new application of Foulkes characters, we give a short proof of a celebrated result of Zagier which generalizes one of Harer and Zagier on the enumeration of Riemann surfaces of a given genus. We also rewrite the Markov chain for carries in terms of our inner product \([\cdot, \cdot]\) and products of characters in \( \text{CF}_\ell(S_n) \):
\[ M(i, j) = [\phi_i, b^{\ell-n} \phi_j], \]
which is not generally equal to \( M(j, i) \).
In the second part of the paper, Sect. 3, we answer Questions 1 and 2 for the full monomial groups \( G(r, 1, n) \) with \( r > 1 \). The author introduced analogues of Foulkes characters for these groups, as well as many other reflection groups, in [12], where they were constructed from certain reduced homology groups for subcomplexes of the Milnor fiber complex, which is a certain wedge of spheres that is an equivariant strong deformation retract of a Milnor fiber from the invariant theory of the group, and then used various machinery to prove, among other things [12–15], analogues of (1.1)–(1.6). The role of \( \ell \) is played by \( n - l \), where \( l \) is the most natural choice of “length”.

\[
[(x) = \min\{k \geq 0 \mid x = y_1 y_2 \ldots y_k \text{ for some reflections } y_i \in G(r, 1, n)\}.
\]

In addition to enjoying properties analogous to (1.1)–(1.6), the Foulkes characters of \( G(r, 1, n) \) were shown in [14] to play the role of irreducibles among the characters of \( G(r, 1, n) \) that depend only on \( l \) in the sense that the characters of \( G(r, 1, n) \) that depend only on \( l \) are precisely the unique non-negative integer linear combinations of the Foulkes characters. So our answers to Questions 1 and 2 round out a truly remarkable story for the groups \( G(r, 1, n) \) with \( r > 1 \), particularly the hyperoctahedral groups \( G(2, 1, n) \).

Question 1 for \( G(r, 1, n) \) has answers that are similar to our answers for \( S_n \). Question 2 for \( G(r, 1, n) \) is more complicated than for \( S_n \), but our answer is of a similar flavor and simplifies in the case of the hyperoctahedral group. As in the case of type A, benefits include a probability distribution on the analogues of the pairs \((C_i, C_j)\), and a new construction of the Foulkes characters of \( G(r, 1, n) \) in terms of multiplying random elements and applying the Gram–Schmidt process to a natural basis. Another application is a remarkable rewriting of a Markov chain studied by Diaconis and Fulman for adding random numbers in balanced ternary, a number system that both reduces carries and, in the words of Donald Knuth, is “perhaps the prettiest number system of all.”

\section*{2 Type A}

\subsection*{2.1 The inner product}

We start with our answer to Question 2 for \( S_n \). Given a subset \( A \) of a group \( G \), we denote by \( \sum_{a \in A} a \) in \( \mathbb{C}[G] \).

\textbf{Definition 2.1} For \( \vartheta, \psi \in \text{CF}_{\ell}(S_n) \), and for \( n \)-cycles \( \sigma \) and \( \tau \) chosen uniformly at random from \( C_1 \), we define

\[
[\vartheta, \psi] = \frac{1}{|S_n|} \sum_{i,j=1}^n \vartheta(C_i) \psi(C_j) \mathbb{E}[\sigma C_i \cap \tau C_j].
\]
Proposition 2.2 For \( \vartheta, \psi \in \text{CF}_\ell(S_n) \),

\[
[\vartheta, \psi] = \sum_{\chi} \left< \vartheta, \frac{\chi}{\chi(1)} \right| \left< \psi, \frac{\chi}{\chi(1)} \right>,
\]

(2.1)

where the sum is over the irreducible characters \( \epsilon_i = \chi_{(n-i,1^i)}, 0 \leq i \leq n-1 \).

**Proof** Denoting the regular representation of \( S_n \) by \( \text{Reg} \), we have

\[
[\vartheta, \psi] = \frac{1}{|S_n|^2} \text{Tr} \circ \text{Reg} \left( \sum_{i,j=1}^{n} \vartheta(C_i) \psi(C_j) \frac{C_i^2 C_j}{|C_1|^2} \right)
\]

\[
= \frac{1}{|S_n|^2} \sum_{\chi \in \text{Irr}(S_n)} \sum_{i,j=1}^{n} \frac{\vartheta(C_i) \psi(C_j)}{|C_1|^2} \chi(1) \frac{\chi(C_i)^2 |C_1|^2}{\chi(1)^2} \sum_{x \in C_i} \chi(x) \sum_{y \in C_j} \chi(y)
\]

\[
= \frac{1}{|S_n|^2} \sum_{\chi \in \text{Irr}(S_n)} \chi(C_1)^2 \sum_{x \in C_i} \frac{\vartheta(C_i) \chi(x)}{\chi(1)} \sum_{y \in C_j} \frac{\psi(C_j) \chi(y)}{\chi(1)}
\]

\[
= \sum_{\chi \in \text{Irr}(S_n)} \left< \vartheta, \frac{\chi}{\chi(1)} \right| \left< \psi, \frac{\chi}{\chi(1)} \right>
\]

where the last sum is over all \( \chi_{(n-i,1^i)} \) with \( 0 \leq i \leq n-1 \).

**Theorem 2.3** The characters \( \phi_0, \phi_1, \ldots, \phi_{n-1} \) form an orthonormal basis for the Hilbert space \( \text{CF}_\ell(S_n) \) with inner product \( \left< -, - \right> \).

**Proof** By property (1.2) and Proposition 2.2.

A natural choice of basis for \( \text{CF}_\ell(S_n) \) that is composed of characters is \( 1^\ell, 2^\ell, \ldots, n^\ell \), the character \( k^\ell : \pi \mapsto k^\ell(\pi) \) being afforded by \( (\mathbb{C}^k)^\otimes n \) with

\[
\pi.(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = v_{\pi^{-1}(1)} \otimes v_{\pi^{-1}(2)} \otimes \ldots \otimes v_{\pi^{-1}(n)}.
\]

**Theorem 2.4** The characters \( \phi_0, \phi_1, \ldots, \phi_{n-1} \) result from the inner product \( \left< -, - \right> \) by applying the Gram–Schmidt process to the characters \( 1^\ell, 2^\ell, \ldots, n^\ell \).

**Proof** By (1.6), we have \( \Phi = LV \) with

\[
L = \left( (-1)^{i-j} \binom{n+1}{i-j} \right)_{0 \leq i, j \leq n-1}, \quad V = \left( (i+1)^\ell(n-j) \right)_{0 \leq i, j \leq n-1},
\]

so \( L \) is lower unitriangular and \( V \) is the character table of the \( k^\ell \). But this means that the rows of \( \Phi \) are obtained by applying the Gram–Schmidt process to the rows of \( V \) using the inner product with respect to which the rows of \( \Phi \) are orthonormal.
2.2

We remark on a formula for Foulkes characters that is similar to some well-known formulas for various systems of orthogonal polynomials, including Legendre polynomials, Hermite polynomials \((2X - \frac{d}{dX})^n, 1\), and Laguerre polynomials \(\frac{1}{n!}(\frac{d}{dX} - 1)^n X^n\).

It appears in the work of Diaconis and Fulman [4] in a slightly different form.

Let
\[
A_n = \sum_{\pi \in S_n} X^{\text{des}(\pi)},
\]
so
\[
A_0 = 1, \quad A_1 = 1, \quad A_2 = 1 + X, \quad A_3 = 1 + 4X + X^2, \quad \ldots,
\]
and
\[
\left(1 + X \frac{d}{dX}\right)^n \frac{1}{1 - X} = \frac{A_n}{(1 - X)^{n+1}}. \tag{2.2}
\]

**Theorem 2.5** (Diaconis–Fulman) For \(1 \leq j \leq n\),
\[
\sum_{i=0}^{n-1} \phi_i(C_j)X^i = (1 - X)^n \left(1 + X \frac{d}{dX}\right)^j \frac{1}{1 - X}. \tag{2.3}
\]

**Proof** Denoting by \(\phi_i^{(n)}\) and \(C_j^{(n)}\) the \(\phi_i\) and \(C_j\) for \(S_n\), we have [6,9,12]
\[
\phi_i^{(n)}(C_j^{(n)}) = \phi_i^{(n-1)}(C_j^{(n-1)}) - \phi_{i-1}^{(n-1)}(C_j^{(n-1)}) \tag{2.4}
\]
for \(0 \leq i \leq n - 1\) and \(1 \leq j \leq n - 1\), where we take \(\phi_{-1}^{(n-1)} = \phi_{n-1}^{(n-1)} = 0\). So the \(S_{n-1}\) cases of (2.3) imply the first \(n - 1\) cases of (2.3) for \(S_n\), while equality holds for \(j = n\) by (1.4) and (2.2). \(\square\)

2.3 Decomposing products of Foulkes characters

We now present three solutions to computing \([\phi_i \phi_j, \phi_k]\) for \(0 \leq i, j, k \leq n - 1\).

2.3.1 First solution

Our first solution is a combinatorial solution in terms of descents, and it is a corollary of an earlier theorem involving Loday’s Eulerian idempotents [10]. Writing
\[
\mathcal{D}_i = \sum_{\pi \in S_n, \text{des(}\pi\text{)} = i} \pi,
\]
the Eulerian idempotents $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{n−1} \in \mathbb{Q}[S_n]$ are defined by

$$\sum_{i=0}^{n-1} \binom{n + n - 1 - i}{n} \mathcal{D}_i = \sum_{i=0}^{n-1} \mathcal{E}_{n-1-i} X^{n-i}, \quad (2.5)$$

and the following is a special case of Theorem 9 in [13].

**Theorem 2.6** $\Phi^t$ is the transition matrix from

$$\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_{n-1}$$

to

$$\mathcal{E}_{n-1}, \mathcal{E}_{n-2}, \ldots, \mathcal{E}_0,$$

so

$$\mathcal{D}_i = \sum_{j=0}^{n-1} \phi_i(C_{n-j}) \mathcal{E}_{n-1-j} \quad (2.6)$$

and

$$\Phi^{-t} = \left( \text{Coeff. of } X^{n-j} \text{ in } \binom{n + n - 1 - i}{n} \right)_{0 \leq i, j \leq n-1}. \quad (2.7)$$

As a consequence of Theorem 2.6, we have the following.

**Theorem 2.7** For any fixed $z \in S_n$ with $\text{des}(z) = k$,

$$[\phi_i \phi_j, \phi_k] = | \{(x, y) \in S^2_n \mid \text{des}(x) = i, \text{des}(y) = j, xy = z\}|. \quad (2.8)$$

**Proof** The $\mathcal{D}_i$’s form a basis for a subalgebra of $\mathbb{C}[S_n]$, and the $\mathcal{E}_i$’s are idempotents, so by (2.6), $[\phi_i \phi_j, \phi_k]$ is the coefficient of $\mathcal{D}_k$ in $\mathcal{D}_i \mathcal{D}_j$. Hence (2.8). \hfill $\Box$

### 2.3.2 Second solution

Our second solution is a closed-form solution which uses the decomposition in (1.2), the explicit expression for $\phi_i(\pi)$ in (1.6), and the fact that, for any $\chi_\lambda \in \text{Irr}(S_n)$,

$$\langle X^\ell, \chi_\lambda \rangle = \prod_{b \in \lambda} \frac{X + c(b)}{h(b)}, \quad (2.9)$$

where for a box $b \in \lambda$ located in the $i$-th row and $j$-th column,

$$c(b) = j - i, \quad h(b) = \lambda_i - j + 1 + |\{k > i \mid \lambda_k \geq j\}|.$$
Theorem 2.8

\[
[\phi_i \phi_j, \phi_k] = \sum_{0 \leq u \leq i, v \leq j} (-1)^{i-u} (-1)^{j-v} \binom{n+1}{i-u} \binom{n+1}{j-v} \binom{uv + u + v + n - k}{n}.
\]

(2.10)

Proof  By (1.2) and (1.6),

\[
[\phi_i \phi_j, \phi_k] = \left( \phi_i \phi_j, \frac{\epsilon_k}{\epsilon_k(1)} \right) = \sum_{u,v=0}^{n-1} (-1)^{i-u} (-1)^{j-v} \binom{n+1}{i-u} \binom{n+1}{j-v} ((u+1)(v+1))^{1/2} \binom{u}{\epsilon_k} \binom{v}{\epsilon_k(1)},
\]

and by (2.9),

\[
\left( (u+1)(v+1))^{1/2} \binom{u}{\epsilon_k} \binom{v}{\epsilon_k(1)} \right) = \binom{(u+1)(v+1) + n - 1 - k}{n}.
\]

\[\square\]

2.3.3 Third solution

The third solution is a recursive solution due to P. Delsarte [2]. For Foulkes characters $\phi_i, \phi_j, \phi_k$ of $S_n$, let us write

\[
\phi_i \phi_j = \sum_{k=0}^{n-1} c_{ijk}^{(n)} \phi_k.
\]

Delsarte defines recursively certain values $F(i, k, n)$, $0 \leq i, k \leq n$, that depend on a parameter $q$ and initial conditions $F(0, k, m)$ with $0 \leq k \leq m$, and he considers the matrix $P_{n-1} = (F(i, k, n - 1))_{0 \leq i \leq k \leq n-1}$. Although Delsarte did not specialize in this way, taking $q = 1$ and $F(0, k, m)$ to be the Eulerian number $|\{ \pi \in S_{m+1} : \text{des}(\pi) = k \}|$, and then comparing Delsarte’s definition with (1.4) and (2.4), we find that the transpose of Delsarte’s matrix $P_{n-1}$ becomes the Foulkes character table $\Phi$ of $S_n$, so that

\[
\phi_i (C_{n-j}) = F(j, i, n - 1), \quad 0 \leq i, j \leq n - 1.
\]

(2.11)

In addition to finding very general expressions for the $F(i, k, n)$ and the determinant of $P_{n-1}$ in Theorems 2 and 3 of [2], Delsarte also found a recursive solution for calculating the $c_{ijk}^{(n)}$'s, since $c_{ij0}^{(n)} = [\phi_i, \phi_j] = \delta_{ij}$. 

\[\square\] Springer
Theorem 2.9 (Delsarte)

\[ e_{i,j,k}^{(n+1)} - e_{i+1,j+1,k+1}^{(n+1)} = -e_{i,j,k}^{(n)} - e_{i+1,j+1,k}^{(n)} + e_{i,j+1,k}^{(n)} - e_{i+1,j+1,k}^{(n)}. \]

2.4

Before moving on, we give another useful consequence of Theorem 2.6. Let

\[ \phi = \phi_0 + X\phi_1 + X^2\phi_2 + \ldots + X^{n-1}\phi_{n-1}. \]  \hspace{1cm} \text{(2.12)}

Theorem 2.10 For any two sequences \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \),

\[ \sum_{i=1}^{n} a_i X_i = \sum_{k=1}^{n} b_k \binom{X+n-k}{n} \]

if and only if

\[ \sum_{i=1}^{n} a_i \phi(C_i) = \sum_{k=1}^{n} b_k X^{k-1}. \]

Proof Let

\[ a = \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_1 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad x = \begin{pmatrix} X_n \\ X^{n-1} \\ \vdots \\ X^1 \end{pmatrix}, \quad y = \begin{pmatrix} (X+n-1) \\ X^{n-2} \\ \vdots \\ X_n \end{pmatrix}. \]

Then (2.7) can be rewritten as

\[ \Phi^{-t} x = y. \]

So

\[ a^t x = b^t y \Leftrightarrow a^t x = b^t \Phi^{-t} x \Leftrightarrow a^t \Phi^t x = b^t x \Leftrightarrow a^t \Phi^t z = b^t z, \]

where \( z = (1, X, \ldots, X^{n-1})^t \) is obtained from \( x \) by replacing \( X \) by \( X^{-1} \) and then multiplying by \( X^n \).

2.5 Zagier's result

As an application of Foulkes characters, particularly the formula of Diaconis and Fulman in Theorem 2.5 and our inversion result in Theorem 2.10, we give a new short proof of a well-known result of Zagier [16].
Theorem 2.11 (Main theorem of Zagier) For any conjugacy class $K \in \text{Cl}(S_n)$ and any $n$-cycle $\sigma \in S_n$, let

$$p_i(K) = \frac{|\{\tau \in K \mid \tau \sigma \text{ has } i \text{ cycles}\}|}{|K|}.$$ 

The numbers $p_i(K)$ are determined by

$$\sum_{i=1}^{n} p_i(K) p_i(X) = \frac{\varphi(K, X)}{(1 - X)^{n+2}},$$

where $\varphi(K, X) = \det(1 - \tau X, \mathbb{C}^n)$ is the characteristic polynomial of an element $\tau \in K$ under the permutation representation $\tau \mapsto (\delta_{i \tau(j)})_{i,j}$ and

$$P_1(X) = \frac{1}{(1 - X)^2}, \quad P_2(X) = \frac{1 + X}{(1 - X)^3}, \quad P_3(X) = \frac{1 + 4X + X^2}{(1 - X)^4}, \ldots$$

are the polynomials in $\frac{1}{1 - X}$ defined by $P_i(X) = \sum_{k=1}^{\infty} k^i X^{k-1} \in \mathbb{Z}[[X]]$.

Proof Writing $L(X) = \sum_{\pi \in S_n} \pi X^\ell(\pi)$, and denoting the regular representation by $\text{Reg}$, we have

$$\sum_{i=1}^{n} p_i(K) X^i = \frac{1}{|S_n|} \frac{1}{|K|} \text{Tr} \circ \text{Reg}(K \sigma L(X))$$

$$= \frac{1}{|S_n|} \frac{1}{|K|} \sum_{\chi \in \text{Irr}(S_n)} \chi(1) \frac{\chi(K)|K|}{\chi(1)} \chi(\sigma) \sum_{\pi \in S_n} \frac{\chi(\pi) X^{\ell(\pi)}}{\chi(1)}$$

$$= \sum_{k=0}^{n-1} (-1)^k \epsilon_k(K) \left( \frac{\epsilon_k}{\epsilon_k(1)}, X^\ell \right)$$

$$= \sum_{k=0}^{n-1} (-1)^k \epsilon_k(K) \left( X + n - k - 1 \right).$$

Equivalently, by Theorem 2.10,

$$\sum_{i=1}^{n} p_i(K) \phi(C_i) = \sum_{k=0}^{n-1} (-1)^k \epsilon_k(K) X^k = \frac{\varphi(K, X)}{1 - X}.$$ \hspace{1cm} (2.14)

By Theorem 2.5, (2.14) is equivalent to

$$\sum_{i=1}^{n} p_i(K) p_i(X) = \frac{\varphi(K, X)}{(1 - X)^{n+2}}.$$ \hspace{1cm} (2.15)
2.6 Carries in terms of products

As another application of our framework for Foulkes characters, we give an interesting reformulation of the Markov chain studied by Holte [8] and Diaconis and Fulman [4] in terms of our inner product $[\cdot, \cdot]$ and products of characters in $\text{CF}_\ell(S_n)$. Let $M$ be the transition matrix given in (1.7).

**Theorem 2.12** $M(i, j) = [\phi_i, b^{\ell-n} \phi_j]$.

**Proof** Diaconis and Fulman showed that, for $0 \leq j \leq n-1$, the row vector $(\phi_0(C_{n-j}), \phi_1(C_{n-j}), \ldots, \phi_{n-1}(C_{n-j}))$ is a left eigenvector of the transition matrix $M$ with eigenvalue $b^{-j}$, so

$$\Phi^t M = D \Phi^t,$$

where $D$ is the diagonal matrix $\text{diag}(1, b^{-1}, b^{-2}, \ldots, b^{-n+1})$.

For $0 \leq i, j \leq n-1$, let

$$\alpha_{ij} = \frac{1}{|S_n|} \sum_{\pi \in C_{n-j}} \frac{\epsilon_i(\pi)}{\epsilon_i(1)}, \quad \epsilon_i = \chi_{(n-i,1)}(\pi),$$

and let $\Lambda$ be the matrix

$$\Lambda = (\alpha_{ij})_{0 \leq i, j \leq n-1}.$$

Then, using (1.2),

$$\Lambda \Phi^t = \left(\left(\frac{\epsilon_i}{\epsilon_i(1)}, \phi_j\right)_{0 \leq i, j \leq n-1}\right) = I.$$ (2.17)

Hence, by (2.16) and (2.17),

$$M = \Lambda D \Phi^t = \left(\left(\frac{\epsilon_i}{\epsilon_i(1)} b^{\ell-n}, \phi_j\right)_{0 \leq i, j \leq n-1}\right) = \left(\phi_i, b^{\ell-n} \phi_j\right)_{0 \leq i, j \leq n-1}.$$

\[\square\]

3 Type B and the other full monomial groups

We begin by fixing an integer $r > 1$, a primitive $r$-th root of unity $\zeta$, the cyclic group $Z = \langle \zeta \rangle$, and a full monomial group

$$G_n = G(r, 1, n),$$
so the elements of $G_n$ are the $n$-by-$n$ matrices $x$ with exactly one nonzero entry in each row and each column, and with $r$-th roots of unity for the nonzero entries. Equivalently, the elements $x \in G_n$ are the products

$$x = D.A_\pi$$

where $D$ is a diagonal matrix $\text{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ with $\xi_i \in \mathbb{Z}$, and $A_\pi = (\delta_{i\pi(j)})_{1 \leq i, j \leq n}$ is the usual matrix of a permutation $\pi \in S_n$. By the type of $x$ we shall mean the partition-valued function

$$\lambda : \text{Cl}(\mathbb{Z}) \to \mathcal{P}$$

which takes $\{\zeta^j\}$, $0 \leq j \leq r - 1$, to the partition $\lambda^j$ whose parts are the periods of the cycles $(i_1 i_2 \ldots i_k)$ of $\pi$ such that $x_{i_1i_2i_3 \ldots i_k} = \zeta^j$, so two elements of $G_n$ belong to the same conjugacy class if and only if they have the same type. We shall denote by $K_\lambda$ the class of elements of type $\lambda$. Identifying $\lambda$ with the $r$-tuple of partitions $\lambda^j$, we shall write

$$\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{r-1}) \in \mathcal{P}^r$$

and

$$\|\lambda\| = \sum_{i=0}^{r-1} |\lambda^i| = n.$$

In general, for any partition-valued function $f$ on a finite set $\mathcal{P}$, we write

$$\|f\| = \sum_{s \in \mathcal{P}} |f(s)|.$$

There is also the natural bijection [11] between irreducible characters $\chi_\lambda$ of $G_n$ and partition-valued functions

$$\lambda : \text{Irr}(\mathbb{Z}) \to \mathcal{P}$$

with $\|\lambda\| = n$. Denoting by $\varphi_k$ the irreducible character of $\mathbb{Z}$ given by

$$\varphi_k(\zeta^s) = \zeta^{ks},$$

and identifying $\lambda$ with the $r$-tuple of values $\lambda(\varphi_i)$, we shall write

$$\lambda = (\lambda(\varphi_0), \lambda(\varphi_1), \ldots, \lambda(\varphi_{r-1})).$$

With $G_n$ being a reflection group, there is the natural length function

$$l(x) = \min\{k \geq 0 \mid x = y_1y_2 \ldots y_k \text{ for some reflections } y_i \in G_n\}.$$
For our purposes, we will instead work with another length function $\ell$. We define, for $x \in G_n$ of type $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{r-1}) \in \mathcal{P}^r$,

$$\ell(x) = \text{number of parts of } \lambda^0.$$  

By Proposition 2 of [13],

$$\ell(x) = n - l(x) = \dim \ker(x - 1), \quad (3.1)$$

so studying $\ell$ is equivalent to studying $l$. In particular, a function $f$ depends only on $\ell$, in the sense that $f(x) = f(y)$ whenever $\ell(x) = \ell(y)$, if and only if $f$ depends only on $l$.

The Foulkes characters of $G_n$ were introduced in [12], where they were constructed from certain reduced homology groups coming from the associated Milnor fiber complex, which is a certain wedge of spheres that is a strong deformation retract of a Milnor fiber $f_1^{-1}(1)$ coming from the invariant theory of $G_n$. They are denoted

$$\phi_0, \phi_1, \ldots, \phi_n,$$

and they were shown in [12] to have some remarkable properties that are analogous to the type A properties stated in (1.1)–(1.6).

The $\phi_i$’s form a basis for the space $\mathcal{CF}_{\ell}(G_n)$ of all class functions $\vartheta$ that depend only on $\ell$, with each $\vartheta \in \mathcal{CF}_{\ell}(G_n)$ decomposing uniquely as

$$\vartheta = \sum_{i=0}^{n} \frac{\langle \vartheta, \epsilon_i \rangle}{\epsilon_i(1)} \phi_i, \quad (3.2)$$

where $\epsilon_i$ is the irreducible character $\chi_{((n-i),1^i),\emptyset,\emptyset,\ldots,\emptyset}$, so $\epsilon_i(1) = \binom{n}{i}$. They decompose the character $\rho$ of the regular representation:

$$\phi_0 + \phi_1 + \ldots + \phi_n = \rho. \quad (3.3)$$

Their degrees are the natural analogues of Eulerian numbers given by Steingrímsson’s notion of descent:

$$\phi_i(1) = |\{x \in G_n \mid \text{des}(x) = i\}|. \quad (3.4)$$

They branch according to

$$\phi_i|_{G_{n-1}} = ((n + 1)r - (ri + 1))\phi_{i-1} + (ri + 1)\phi_i. \quad (3.5)$$

And they admit closed-form expressions:

$$\phi_i(x) = \sum_{j=0}^{n} (-1)^{i-j} \binom{n + 1}{i - j} (r j + 1)^{\ell(x)}. \quad (3.6)$$
We shall denote by $\Phi$ the character table

$$\Phi = (\phi_i(C_{n-j}))_{0 \leq i, j \leq n},$$

where, for $0 \leq i \leq n$,

$$C_i = \{ x \in G_n \mid \ell(x) = i \},$$

and for any $\vartheta \in \text{CF}_\ell(G_n)$ we denote by $\vartheta(C_i)$ the value $\vartheta(x)$ for any $x \in C_i$.

### 3.1 Fourier transform of $X^\ell$

The Fourier transform of the class function $X^\ell : x \mapsto X^\ell(x)$ will play an important role in what follows. Given $\lambda \in \mathcal{P}^r$, by $b \in \lambda$ we shall mean a box $b$ contained in the Young diagram of some $\lambda^j$, and by $c(b)$ and $h(b)$ we shall mean the usual content and hook-length associated to the box $b$ in $\lambda^j$. Given $\lambda \in \mathcal{P}^r$ and $b \in \lambda$, we define

$$\delta_0(b) = \begin{cases} 1 & \text{if } b \in \lambda^0, \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 3.1** For any $\lambda \in \mathcal{P}^r$ with $||\lambda|| = n$,

$$\langle X^\ell, \chi_\lambda \rangle = \prod_{b \in \lambda} \frac{X - 1}{r( X - 1 ) + c(b) + \delta_0(b) h(b)},$$

(3.7)

$$\left( X^\ell, \frac{\chi_\lambda}{\chi_\lambda(1)} \right) = \frac{1}{n!} \prod_{b \in \lambda} \left( \frac{X - 1}{r( X - 1 ) + c(b) + \delta_0(b)} \right),$$

(3.8)

and, for $\mathcal{X}_\lambda : G_n \rightarrow \text{GL}_d(\mathbb{C})$ affording $\chi_\lambda$,

$$\sum_{x \in G_n} X^{\ell(x)} \mathcal{X}_\lambda(x) = \prod_{b \in \lambda} (X + r c(b) + r \delta_0(b) - 1) \mathcal{X}_\lambda(1).$$

(3.9)

**Proof** For any non-negative integer $k$, define

$$\chi_{n,k}(x) = (kr + 1)^{\ell(x)}, \quad x \in G_n.$$

By Proposition 6 and Proposition 7 of [13], in the standard notation, see [13], we have

$$\sum_{n \geq 0} \text{ch}(\chi_{n,k}) X^n = H(\varphi_0)^{k+1} \prod_{j=1}^{r-1} H(\varphi_j)^k.$$  

(3.10)
On Foulkes characters

For any $\varphi \in \text{Irr}(Z)$, by [11, p. 66], we have

$$H(\varphi)^k = \sum_{\mu \in \mathcal{P}} a_\mu s_\mu(\varphi) X^{\mu},$$

(3.11)

where

$$a_\mu = \prod_{b \in \mu} \frac{k + c(b)}{h(b)}.$$  

(3.12)

Hence

$$\sum_{n \geq 0} \text{ch}(\chi_{n,k}) X^n = \sum_{v \in \mathcal{P}^r} a_v S_v X^{[v]},$$

(3.13)

where

$$a_v = \prod_{b \in v} \frac{k + c(b) + \delta_0(b)}{h(b)}.$$  

(3.14)

Equivalently, for any $v \in \mathcal{P}^r$ with $|v| = n$,

$$\langle (kr + 1)^\ell, \chi_v \rangle = \prod_{b \in v} \frac{k + c(b) + \delta_0(b)}{h(b)}.$$  

(3.15)

This holds for all non-negative integers $k$, so it holds as an equality of polynomials in $\mathbb{C}[k]$, and upon replacing $k$ by $\frac{Xr - 1}{r}$, we get (3.7).

The equality in (3.8) follows from (3.7), since

$$\chi_\lambda(1) = \left(\frac{0}{|\lambda^0|}, \frac{1}{|\lambda^1|}, \ldots, \frac{r-1}{|\lambda^{r-1}|}\right) \prod_{i=0}^{r-1} \chi_{\lambda^i}(1) = n! \prod_{i=0}^{r-1} \frac{\chi_{\lambda^i}(1)}{|\lambda^i|} = \frac{n!}{\prod_{b \in \lambda} h(b)}.$$

(3.16)

For (3.9), let $L(X) = \sum_{x \in G_n} x X^{\ell(x)}$. $L(X)$ is central, so

$$\chi_\lambda(L(X)) = \alpha \chi_\lambda(1)$$  

(3.17)

for some polynomial $\alpha$. Taking the trace on both sides of (3.17) and dividing by $\chi_\lambda(1)$ gives

$$\alpha = n! r^n \left( X^\ell, \frac{\chi_\lambda}{\chi_\lambda(1)} \right).$$  

(3.18)
By (3.8) and (3.18),

$$\alpha = \prod_{b \in \lambda} (X - 1 + rc(b) + r\delta_0(b)).$$

$$\square$$

### 3.2

There are four important consequences of Theorem 3.1.

#### 3.2.1

For $0 \leq k \leq n - 1$, we shall write

$$\eta_{s,k} = \chi_\emptyset \cdots \chi_\emptyset \chi_{(n-k,1^k)} \chi_\emptyset \cdots \chi_\emptyset,$$

where the hook-shaped partition $(n - k, 1^k)$ is in position $0 \leq s \leq r - 1$.

**Proposition 3.2**

$$\langle X^\ell, \frac{\eta_{s,k}}{\eta_{s,k}(1)} \rangle = \begin{cases} \frac{x-1+n-k}{r} & \text{if } s = 0, \\
\frac{x-1+n-k-1}{r+n} & \text{if } s \neq 0. \end{cases}$$

(3.19)

**Proof** By (3.8) of Theorem 3.1. $$\square$$

**Proposition 3.3**

$$\langle \phi_i, \frac{\eta_{s,k}}{\eta_{s,k}(1)} \rangle = \begin{cases} \delta_{i,k} & \text{if } s = 0, \\
\delta_{i,k+1} & \text{if } s \neq 0. \end{cases}$$

(3.20)

**Proof** For $0 \leq u, v \leq n$, we have [13, Eq. 18]

$$\sum_{j=0}^{n} (-1)^{u-j} \binom{n+1}{u-j} \binom{n+j-v}{n} = \delta_{uv}.$$ (3.21)

See also [10, Eq. 1.6.1] and [12, Eqs. 9 and 11].
By (3.6), (3.19), and (3.21),

\[
\left\langle \phi_i, \frac{\eta_{s,k}}{\eta_{s,k}(1)} \right\rangle = \sum_{j=0}^{n} (-1)^{i-j} \binom{n+1}{i-j} (rj+1)^\ell, \frac{\eta_{s,k}}{\eta_{s,k}(1)} \\
= \sum_{j=0}^{n} (-1)^{i-j} \binom{n+1}{i-j} (n+j-k-1+\delta_{0s})
\]

\[
= \left\{ \begin{array}{ll}
\delta_{ik} & \text{if } s = 0, \\
\delta_{i,k+1} & \text{if } s \neq 0.
\end{array} \right.
\]

\[
\left\langle \phi_i, \frac{\eta_{s,k}}{\eta_{s,k}(1)} \right\rangle = \sum_{j=0}^{n} (-1)^{i-j} \binom{n+1}{i-j} (n+j-k-1+\delta_{0s})
\]

3.2.2

For \(0 \leq k \leq n\), let

\[
\epsilon_k = \chi((n-k),(1^k),\emptyset,\emptyset,...,\emptyset).
\]  

(3.22)

**Proposition 3.4**

\[
\left\langle X^\ell, \frac{\epsilon_k}{\epsilon_k(1)} \right\rangle = (\frac{X-1}{r} n + n - k).
\]  

(3.23)

**Proof** By (3.8) of Theorem 3.1.

\[
\left\langle \phi_i, \frac{\epsilon_k}{\epsilon_k(1)} \right\rangle = \delta_{ik}.
\]  

(3.24)

**Proposition 3.5**

**Proof** By (3.6), (3.23), and (3.21),

\[
\left\langle \phi_i, \frac{\epsilon_k}{\epsilon_k(1)} \right\rangle = \sum_{j=0}^{n} (-1)^{i-j} \binom{n+1}{i-j} (rj+1)^\ell, \frac{\epsilon_k}{\epsilon_k(1)} \\
= \sum_{j=0}^{n} (-1)^{i-j} \binom{n+1}{i-j} (n+j-k) \\
= \delta_{ik}.
\]  

\[
\left\langle \phi_i, \frac{\epsilon_k}{\epsilon_k(1)} \right\rangle = \sum_{j=0}^{n} (-1)^{i-j} \binom{n+1}{i-j} (rj+1)^\ell, \frac{\epsilon_k}{\epsilon_k(1)} \\
= \sum_{j=0}^{n} (-1)^{i-j} \binom{n+1}{i-j} (n+j-k) \\
= \delta_{ik}.
\]  

□
3.3 Orthogonality relations

Definition 3.6 Let \( a \) be a sequence of classes

\[ K_1, K_2, \ldots, K_m \in \text{Cl}(G_n). \]

Let \( k_i \) be chosen uniformly at random from \( K_i \), for \( 1 \leq i \leq m \), and consider the expected number of ways that the random product \( k_1 k_2 \ldots k_m \) can be written as \( ab \) with \( a \in C_i \) and \( b \in C_j \), i.e.

\[ \mathbb{E}|k_1 k_2 \ldots k_mC_i \cap C_j|. \quad (3.25) \]

For \( \vartheta, \psi \in \text{CF}_\ell(G_n) \), let

\[ [\vartheta, \psi]_a = \frac{1}{|G_n|} \sum_{i,j=0}^n \vartheta(C_i)\psi(C_j) \mathbb{E}|k_1 k_2 \ldots k_mC_i \cap C_j|. \quad (3.26) \]

Our inner product will be a certain convex combination of \([-,-]\)'s. We shall denote the expectation in (3.25) by

\[ \mu_a(C_i, C_j) = \mathbb{E}|k_1 k_2 \ldots k_mC_i \cap C_j|. \]

Proposition 3.7

\[ [\vartheta, \psi]_a = \sum_{\chi \in \text{Irr}(G_n)} \frac{\chi(K_1)\chi(K_2)\ldots\chi(K_m)}{\chi(1)^{m-2}} \left( \vartheta, \frac{\chi}{\chi(1)} \right) \left( \psi, \frac{\chi}{\chi(1)} \right). \quad (3.27) \]

Proof Writing

\[ \mathcal{K} = \prod_{i=1}^m \frac{\mathcal{K}_i}{|K_i|}, \]

the right-hand side of (3.26) equals

\[ \frac{1}{|G_n|^2} \text{Tr} \circ \text{Reg} \left( \sum_{i,j=0}^n \vartheta(C_i)\psi(C_j)\mathcal{K}^* C_i C_j \right), \quad (3.28) \]

which in turn equals

\[ \frac{1}{|G_n|^2} \sum_{i,j=1}^n \sum_{\chi \in \text{Irr}(G_n)} \vartheta(C_i)\psi(C_j) \chi(1)^2 \omega_\chi(\mathcal{K}^* C_i C_j), \quad (3.29) \]
where $\omega_{\chi}(W)$ denotes the scalar by which a central element $W$ of $\mathbb{C}[G_n]$ acts on a module affording $\chi$, so

$$\omega_{\chi}(\mathcal{C}_i \mathcal{C}_j) = \left( \prod_{u=1}^{m} \frac{\chi(K_u)}{\chi(1)} \right) \sum_{x \in \mathcal{C}_i} \chi(1) \sum_{y \in \mathcal{C}_j} \chi(1).$$

(3.30)

Hence

$$[\vartheta, \psi]_a = \sum_{\chi \in \text{Irr}(G_n)} \frac{\chi(K_1) \chi(K_2) \ldots \chi(K_m)}{\chi(1)^{m-2}} \left( \vartheta, \frac{\chi}{\chi(1)} \right) \left( \psi, \frac{\chi}{\chi(1)} \right).$$

(3.31)

Proposition 3.8 Let $a = (K_1, K_2, \ldots, K_m)$ be a sequence of classes of $G_n$ such that $K_1 = K_{\lambda}$ for some $\lambda$ with $\lambda^s = (n)$ for some $s$. Then

$$[\phi_i, \phi_j]_a = \delta_{ij} \xi_a(i),$$

(3.32)

where

$$\xi_a(i) = \sum_{\chi \in H_i} \frac{\chi(K_1) \chi(K_2) \ldots \chi(K_m)}{\chi(1)^{m-2}},$$

and

$$H_i = \begin{cases} \{\eta_{0,0}\} & \text{if } i = 0, \\ \{\eta_{0,i}\} \cup \{\eta_{1,i-1}, \eta_{2,i-1}, \ldots, \eta_{r-1,i-1}\} & \text{if } 0 < i < n, \\ \{\eta_{1,n-1}, \eta_{2,n-1}, \ldots, \eta_{r-1,n-1}\} & \text{if } i = n. \end{cases}$$

Proof By Proposition 3.7, the analogue of Murnaghan–Nakayama for $G_n$ given by Ariki and Koike [1], and Proposition 3.3. $\square$

Definition 3.9 Let $a_1, a_2, \ldots, a_5$ be the sequences

$$a_1 = \left( K_{(n), \emptyset, \emptyset, \ldots, \emptyset}, K_{(n), \emptyset, \emptyset, \emptyset, \ldots} \right),$$

$$a_2 = \left( K_{(n), \emptyset, \emptyset, \ldots, \emptyset}, K_{(n), \emptyset, \emptyset, \ldots} \right),$$

$$a_3 = \left( K_{(n), \emptyset, \emptyset, \ldots, \emptyset}, K_{(n-1), (1), \emptyset, \ldots, \emptyset} \right),$$

$$a_4 = \left( K_{(n), \emptyset, \emptyset, \emptyset, \ldots}, K_{(n), \emptyset, \emptyset, \ldots} \right),$$

$$a_5 = \left( K_{(n), \emptyset, \emptyset, \emptyset, \ldots}, K_{(n-1), (1), \emptyset, \ldots, \emptyset} \right),$$

and let

$$\mu_i = \mu_{a_i}.$$
If \( n = 1 \), so \( G_n \) is cyclic, let
\[
\mu = \frac{2}{r} \mu_1 + \frac{r - 2}{r} \mu_2.
\]

If \( n \geq 2 \), let
\[
\mu = \frac{1}{r} \mu_1 + \frac{r - 2}{2r} \mu_2 + \frac{1}{4} \mu_3 + \frac{1}{2r} \mu_4 + \frac{r - 2}{4r} \mu_5.
\] (3.33)

Define, for \( \vartheta, \psi \in CF_\ell(G_n) \),
\[
[\vartheta, \psi] = \frac{1}{|G_n|} \sum_{i, j=0}^{n} \vartheta(C_i) \overline{\psi(C_j)} \mu(C_i, C_j). \] (3.34)

It should be noted that the expression for \( \mu \) given in (3.33) simplifies in the case of the hyperoctahedral group. If \( r = 2 \) and \( n \geq 2 \), then
\[
\mu = \frac{1}{2} \mu_1 + \frac{1}{4} \mu_3 + \frac{1}{4} \mu_4. \] (3.35)

As in the case of \( S_n \), we have the following properties.

**Proposition 3.10** For all \( 0 \leq i, j \leq n \),
\[
\mu(C_i, C_j) = \mu(C_j, C_i) \geq 0, \] (3.36)
\[
\sum_{i=0}^{n} \mu(C_i, C_j) = |C_j|, \] (3.37)
and
\[
\sum_{i, j=0}^{n} \frac{\mu(C_i, C_j) |G_n|}{|G_n|} = 1. \] (3.38)

**Proof** These follow from the definition of \( \mu \). \( \square \)

**Theorem 3.11** The characters \( \phi_0, \phi_1, \ldots, \phi_n \) form an orthonormal basis for the Hilbert space \( CF_\ell(G_n) \) with inner product \([- , -]\).

**Proof** The \( n = 1 \) case is a simple calculation, so assume \( n \geq 2 \). By the expression for \( \xi_{a(i)} \) in (3.32) and the analogue of Murnaghan–Nakayama for \( G_n \) given by Ariki and Koike [1], we have the values in Tables 1 and 2 for \( \xi_{a(i)} \).

By the definition of \([- , -]\) in (3.34), the orthogonality relation in (3.31), and the values in Tables 1 and 2, we conclude that the \( \phi_i \)'s are an orthonormal basis for the Hilbert space \( CF_\ell(G_n) \) with inner product \([- , -]\). \( \square \)
A natural choice of basis for $\text{CF}_\ell(G_n)$ that is composed of characters is

$$1^\ell, (r+1)^\ell, (2r+1)^\ell, \ldots, (nr+1)^\ell.$$ 

For the fact that these are characters, see Proposition 6 in [13].

**Theorem 3.12** The characters $\phi_0, \phi_1, \ldots, \phi_n$ result from the inner product $[-, -]$ by applying the Gram–Schmidt process to the characters

$$1^\ell, (r+1)^\ell, \ldots, (rn+1)^\ell.$$ 

**Proof** By (3.6), we have $\Phi = LV$ with

$$L = \left( (-1)^{i-j} \binom{n+1}{i-j} \right)_{0 \leq i,j \leq n}, \quad V = \left( (ri+1)^\ell C_{n-j} \right)_{0 \leq i,j \leq n},$$

so the rows of $\Phi$ are obtained by applying the Gram–Schmidt process to the rows of $V$ using the inner product with respect to which the rows of $\Phi$ are orthonormal. \(\square\)

We include the following orthogonality relation of independent interest.

**Proposition 3.13** Let $\sigma$ be an element of some class $K_\lambda$ of $G_n$ with $\lambda^i = (n)$ for some $i > 0$. Then

$$\frac{1}{|G_n|} \sum_{u,v=0}^n \phi_i(C_u)\phi_j(C_v)|\sigma C_u \cap C_v| = (-1)^{i \binom{n}{i}} \delta_{ij}. \quad (3.39)$$

**Proof** By Proposition 3.8 with $m = 1$ and $K_1 = K_\lambda$, and using the analogue of the Murnaghan–Nakayama rule for $G_n$. \(\square\)
3.4 Decomposing products of Foulkes characters

We give two solutions to computing $[\phi_i\phi_j, \phi_k]$. The first is a closed-form solution, and the second is a combinatorial solution.

3.4.1 First solution

**Theorem 3.14**

$$[\phi_i\phi_j, \phi_k] = \sum_{0 \leq u \leq i}^{0 \leq v \leq j} (-1)^i - u (-1)^j - v \binom{n + 1}{i - u} \binom{n + 1}{j - v} \binom{ruv + u + v + n - k}{n}.$$

(3.40)

**Proof** By (3.6) and Proposition 3.5,

$$[\phi_i\phi_j, \phi_k] = \left\langle \phi_i\phi_j, \epsilon_k \epsilon_k(1) \right\rangle,$$

$$= \sum_{u, v = 0}^{n} (-1)^{i - u} (-1)^{j - v} \binom{n + 1}{i - u} \binom{n + 1}{j - v} \left( ((ru + 1)(rv + 1))^{\ell} \frac{\epsilon_k}{\epsilon_k(1)} \right),$$

and by Proposition 3.4,

$$\left( ((ru + 1)(rv + 1))^{\ell} \frac{\epsilon_k}{\epsilon_k(1)} \right) = \binom{ruv + u + v + n - k}{n}.$$

(3.41)

$\square$

3.4.2 Second solution

Just as for $S_n$, our combinatorial solution for computing $[\phi_i\phi_j, \phi_k]$ is in terms of descents and certain idempotents. Using Steingrímsson’s notion of descent for $G_n$ and writing

$$D_i = \sum_{x \in G_n, \text{des}(x) = i} x,$$

the Eulerian idempotents $e_0, e_1, \ldots, e_n \in \mathbb{Q}[G_n]$ for $G_n$ are defined by

$$\sum_{i=0}^{n} \binom{n + \frac{X - 1}{r} - i}{n} D_i = \sum_{j=0}^{n} e_{n-j} X^{n-j},$$

(3.42)

see [13] and references therein. The following is a special case of [13, Thm. 9].
Theorem 3.15 \( \Phi^t \) is the transition matrix from
\[
D_0, D_1, \ldots, D_n
\]
to
\[
E_n, E_{n-1}, \ldots, E_0,
\]
so
\[
D_i = \sum_{j=0}^{n} \phi_i(C_{n-j})E_{n-j}
\]
and
\[
\Phi^{-t} = \left( \text{Coeff. of } X^{n-j} \text{ in } \left( n + \frac{x-1}{r} - i \right) \right)_{0 \leq i, j \leq n}.
\]

Our combinatorial solution is an immediate corollary of Theorem 3.15.

Theorem 3.16 For any fixed \( z \in G_n \) with exactly \( k \) descents,
\[
[\phi_i \phi_j \phi_k] = |\{(x, y) \in G_n \times G_n | \text{des}(x) = i, \text{des}(y) = j, xy = z\}|.
\]

Proof As in the proof of Theorem 2.7, the \( D_i \)'s span a subalgebra of \( \mathbb{C}[G_n] \), and the \( E_i \)'s are idempotents, so by (3.43), \([\phi_i \phi_j \phi_k]\) is the coefficient of \( D_k \) in \( D_i D_j \).

3.5

We end with \( G_n \) analogues of three earlier results for \( S_n \), namely, the formula of Diaconis and Fulman in Theorem 2.5, the useful inversion result in Theorem 2.10, and the reformulation of Holte’s Markov chain for adding random numbers in terms of products and Foulkes characters.

3.5.1

Writing
\[
A_{r,n} = \sum_{x \in G_n} X^{\text{des}(x)},
\]
the analogue of (2.2) is
\[
\left[ \left( 1 + Y \frac{d}{dY} \right)^n \frac{1}{1 - Y^r} \right]_{Y=X^{1/r}} = \frac{A_{r,n}}{(1 - X)^{n+1}},
\]
(3.46)
and the analogue of Theorem 2.5 is the following, with a version of the hyperoctahedral case already appearing in earlier work of Diaconis and Fulman [5].

**Theorem 3.17** For $0 \leq j \leq n$,

$$
\sum_{i=0}^{n} \phi_i(C_j)X^i = \left[ (1 - Y^r)^{n+1} \left( 1 + Y \frac{d}{dY} \right)^{j} \frac{1}{1 - Y^r} \right]_{Y=X^{1/r}}. 
$$

**(Proof)** The proof follows just as for $S_n$. Denoting by $\phi_i^{(n)}$ and $C_j^{(n)}$ the $\phi_i$ and $C_j$ for $G_n$, we have [12, Theorem 7]

$$
\phi_i^{(n)}(C_j^{(n)}) = \phi_i^{(n-1)}(C_j^{(n-1)}) - \phi_{i-1}^{(n-1)}(C_j^{(n-1)})
$$

for $0 \leq i \leq n$ and $0 \leq j \leq n - 1$, where we take $\phi_{-1}^{(n-1)} = \phi_{n-1}^{(n-1)} = 0$. So the $G_{n-1}$ cases of (3.47) imply the first $n$ cases of (3.47) for $G_n$, while equality holds for $j = n$ by (3.4), which is Corollary 8.1 in [12], and (3.46).

3.5.2

For the analogue of Theorem 2.10, let

$$
\phi = \phi_0 + X\phi_1 + X^2\phi_2 + \ldots + X^n\phi_n.
$$

**Theorem 3.18** For any two sequences $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$,

$$
\sum_{i=0}^{n} a_i X^i = \sum_{i=0}^{n} b_i \left( n + \frac{X-1}{r} - i \right)
$$

if and only if

$$
\sum_{i=0}^{n} a_i \phi(C_i) = \sum_{i=0}^{n} b_i X^i.
$$

**(Proof)** This follows from (3.44) just as Theorem 2.10 followed from (2.7).

3.5.3 Carries in terms of products

In [5], Diaconis and Fulman connected the hyperoctahedral Foulkes characters with adding an even number $N$ of random numbers using balanced digits and odd base $b$.

We shall denote the transition matrix of the Diaconis–Fulman Markov chain by $M_B$, so

$$
M_B = (M_B(i, j))_{0 \leq i, j \leq N}
$$
with

\[ M_B(i, j) = \text{chance}\left\{ \text{next carry is } j - \frac{N}{2} \mid \text{last carry is } i - \frac{N}{2} \right\}. \]

We rewrite this Markov chain in terms of our inner product and products involving Foulkes characters of the hyperoctahedral group \( B_N = G(2, 1, N) \).

**Theorem 3.19** Let \( \phi_0, \phi_1, \ldots, \phi_N \) be the Foulkes characters of \( B_N \). Then

\[ M_B(i, j) = [\phi_i, b^{\ell-N} \phi_j]. \]

**Proof** Denoting by \( \Phi_B \) the Foulkes character table \( (\phi_i(C_{N-j}))_{0 \leq i, j \leq N} \) for \( B_N \), Diaconis and Fulman showed that

\[ \Phi_B^t M_B = D \Phi_B^t, \quad D = \text{diag}(b^0, b^{-1}, \ldots, b^{-N}). \]

For \( 0 \leq i, j \leq N \), let

\[ \alpha_{ij} = \frac{1}{|B_N|} \sum_{x \in C_{N-j}} \frac{\epsilon_i(x)}{\epsilon_i(1)}, \quad \epsilon_i = \chi((N-i),(1^i),\emptyset,\emptyset,\ldots,\emptyset), \]

and let

\[ \Lambda = (\alpha_{ij})_{0 \leq i, j \leq N}. \]

Then

\[ \Lambda \Phi_B^t = \left( \left( \frac{\epsilon_i}{\epsilon_i(1)} , \phi_j \right) \right)_{0 \leq i, j \leq N} = I. \]

Hence

\[ M_B = \Lambda D \Phi_B^t = \left( \left( \frac{\epsilon_i}{\epsilon_i(1)} b^{\ell-N} , \phi_j \right) \right)_{0 \leq i, j \leq N} = \left[ [\phi_i, b^{\ell-N} \phi_j] \right]_{0 \leq i, j \leq N}. \]

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**References**

1. Ariki, S., Koike, K.: A Hecke algebra of \( (\mathbb{Z}/r\mathbb{Z}) \wr S_n \) and construction of its irreducible representations. Adv. Math. **106**, 216–243 (1994)
2. Delsarte, P.: Properties and applications of the recurrence \( F(i+1, k+1, n+1) = q^{k+1} F(i, k+1, n) - q^k F(i, k, n) \). SIAM J. Appl. Math. **31**, 262–270 (1976)
3. Diaconis, P.: Group Representations in Probability and Statistics. Lecture Notes-Monograph Series **11** (1988)
4. Diaconis, P., Fulman, J.: Foulkes characters, Eulerian idempotents, and an amazing matrix. J. Algebr. Comb. **36**, 425–440 (2012)
5. Diaconis, P., Fulman, J.: Combinatorics of balanced carries. Adv. Appl. Math. **59**, 8–25 (2014)
6. Foulkes, H.O.: Eulerian numbers, Newcomb’s problem and representations of symmetric groups. Discr. Math. **30**, 3–49 (1980)
7. Harer, J., Zagier, D.: The Euler characteristic of the moduli space of curves. Invent. Math. **85**, 457–485 (1986)
8. Holte, J.M.: Carries, combinatorics, and an amazing matrix. Am. Math. Monthly **104**, 138–149 (1997)
9. Kerber, A.: Applied Finite Group Actions, 2nd edn. Springer, New York (1999)
10. Loday, J.-L.: Opérations sur l’homologie cyclique des algèbres commutatives. Invent. Math. **96**, 205–230 (1989)
11. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford University Press, Oxford (1995)
12. Miller, A.R.: Foulkes characters for complex reflection groups. Proc. Am. Math. Soc. **143**, 3281–3293 (2015)
13. Miller, A.R.: Some characters that depend only on length. Math. Res. Lett. **24**, 879–891 (2017)
14. Miller, A.R.: Walls in Milnor fiber complexes. Doc. Math. **23**, 1247–1261 (2018)
15. Miller, A.R.: Milnor fiber complexes and some representations, In: Topology of Arrangements and Representation Stability, pp. 43–123, Oberwolfach reports **15**(1) (2018)
16. Zagier, D.: On the distribution of the number of cycles of elements in symmetric groups. Nieuw Arch. Wisk. **13**, 489–495 (1995)

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