Non-Adiabatic Transitions in Parabolic and Super-Parabolic $\mathcal{PT}$-Symmetric Non-Hermitian Systems in 1D Optical Waveguides

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Exceptional points, which are spectral degeneracy points in the complex parameter space, are fundamental to non-Hermitian quantum systems. The dynamics of non-Hermitian systems in the presence of exceptional points differ significantly from those of Hermitian ones. Here, non-adiabatic transitions in non-Hermitian $\mathcal{PT}$-symmetric systems are investigated, in which the exceptional points are driven through at finite speeds which are quadratic or cubic functions of time. Different transmission dynamics separated by exceptional points are identified, and analytical approximate formulas for the non-adiabatic transmission probabilities are derived. Possible experimental realizations with a $\mathcal{PT}$-symmetric non-Hermitian 1D tight-binding optical waveguide lattice are discussed through non-Hermitian Bloch oscillations between different bands.

1. Introduction

In recent years, the emerging field of non-Hermitian quantum systems,\textsuperscript{[1–14]} that is, open quantum systems which do not obey conservation laws because of the exchange of energy between the systems and the environment,\textsuperscript{[15–18]} has attracted great interest due to the potential for new quantum devices.\textsuperscript{[19–38]} This has opened up new challenges and opportunities for both theorists and experimentalists.\textsuperscript{[39–62]}

In conventional quantum mechanics, there is an axiom that the dynamics of a state of an isolated quantum system is governed by a Hermitian Hamiltonian ($\hat{H} = \hat{H}^\dagger$), which ensures real energy eigenvalues as well as a unitary time evolution for which the total probability of finding a particle anywhere in space is conserved.\textsuperscript{[8]} However, for an open system which exchanges energy with its environment, the total probability is in general not conserved, which yields a non-unitary time evolution described by a non-Hermitian Hamiltonian ($\hat{H} \neq \hat{H}^\dagger$).\textsuperscript{[18]} Remarkably, as shown by Bender and Boettcher,\textsuperscript{[1,2]} the reality of the eigenvalues is ensured by a wide class of non-Hermitian Hamiltonians which are symmetric under parity-time ($\mathcal{PT}$) transformations, that is, $[\mathcal{PT}, \hat{H}] = 0$. Here, the actions of the parity $\mathcal{P}$ and time $\mathcal{T}$ operators are defined as $\mathcal{P} \colon i \rightarrow -i, \hat{x} \rightarrow -\hat{x}, \hat{p} \rightarrow -\hat{p}$ and $\mathcal{T} \colon i \rightarrow i, e^{i\hat{H}_0\tau} \rightarrow e^{-i\hat{H}_0\tau}$, where $\hat{x}$ and $\hat{p}$ are position and momentum operators respectively.\textsuperscript{[1,2]}

Hence, the action of the parity-time operator is $\mathcal{PT} \colon i \rightarrow -i, \hat{x} \rightarrow -\hat{x}, \hat{p} \rightarrow \hat{p}$, where the operator $\mathcal{P}$ is linear, and the operator $\mathcal{T}$ is anti-linear, as it changes the sign of $i$. The operators $\mathcal{P}$ and $\mathcal{T}$ commute, that is, $[\mathcal{P}, \mathcal{T}] = 0$, and satisfy the relations $\mathcal{P}^2 = \mathcal{T}^2 = 1$, $\mathcal{P} = \mathcal{P}^3$, and $\mathcal{T} = \mathcal{T}^{-1}$.\textsuperscript{[5]} For a single particle in 1D space equipped with a Hamiltonian $\hat{H} \equiv \hat{p}^2/2m + \hat{V}(x)$, the condition of $\mathcal{PT}$ symmetry is equivalent to $\hat{V}(x) = \hat{V}^*(−x)$.\textsuperscript{[5]}

Similar to the connection between symmetries and degeneracies of energy levels in Hermitian systems, $\mathcal{PT}$ symmetries lead to a new type of spectral degeneracies in non-Hermitian systems, known as the exceptional points, at which not only at least two eigenvalues, but the associated eigenstates also coincide.\textsuperscript{[18]} In a striking contrast to the spectral degeneracies of a Hermitian Hamiltonian, at which the eigenstates can still be chosen to be orthogonal to one another, the spectral degeneracies induced by $\mathcal{PT}$ symmetries are peculiar because certain eigenstates become completely parallel and the Hamiltonian matrix becomes defective at the exceptional points.\textsuperscript{[17]}

The intriguing properties of $\mathcal{PT}$-symmetric non-Hermitian systems give rise to many counterintuitive features. A general $\mathcal{PT}$-symmetric Hamiltonian may undergo a parity-time symmetry breaking phase transition, in which complex eigenvalues appear.\textsuperscript{[18]} For a non-Hermitian two-state Hamiltonian $\hat{H}$ with eigenstates $\phi_1$ and $\phi_2$, and eigenvalues $\lambda_1 \neq \lambda_2$, the condition of $\mathcal{PT}$ symmetry leads to $\hat{H}(\mathcal{PT}\phi_1) = \lambda_1' (\mathcal{PT}\phi_1)$, and similarly for $\phi_2$. Clearly, the states $\mathcal{PT}\phi_1$ are also eigenstates of $\hat{H}$ with eigenvalues $\lambda_1''$. Hence, the simplest set of solutions is $\mathcal{PT}\phi_1 = \phi_1$, where $\lambda_1'' = \lambda_1'$, indicating the reality of the eigenvalues. However, there always exists another set of possible solutions, $\mathcal{PT}\phi_1 = \phi_2$, where $\lambda_1'' = \lambda_2'$, which shows that $\phi_1$ and $\phi_2$ are no longer the simultaneous eigenstates of the $\mathcal{PT}$ operator, and the associated eigenvalues form a complex conjugate pair.\textsuperscript{[8]} In this regard, even though the Hamiltonian still possesses $\mathcal{PT}$ symmetry, it is spontaneously broken in certain regions of the parameter space, accompanied by complex eigenvalue bifurcation. By changing the parameters, one may...
reveal the underlying eigenvalue topological structure of non-Hermitian systems, where the real and imaginary parts of the eigenvalues form a set of multi-sheet Riemann surfaces centered around the exceptional points in the parameter space.\(^1\) When encircling an exceptional point, there is an unconventional level-crossing behavior, accompanied by a phase change of one eigenstate but not of the other.\(^2\) A particularly intriguing level-crossing behavior, accompanied by a phase change of one encircling an exceptional point, there is an unconventional space. For non-hermiticity, there are two different regions in the parameter space, which have not been reported in previous studies. Unlike our previous studies on non-adiabatic transitions in Hermitian systems, which are quadratic or cubic functions of time. We consider the transition points separating different behaviors, such that the encircling direction of the exceptional point determines the final output state.\(^3\)

In this work, we consider the dynamics of a non-Hermitian \(PT\)-symmetric system which directly goes through an assembly of exceptional points. Despite its great importance, the non-Hermitian generalization of the two-level Landau-Zener paradigm has only recently been analyzed by Longstaff,\(^4\) and the associated non-Hermitian Landau–Zener–Stückelberg interferometry was analyzed by Shen.\(^5\) Here, we go one step further and study, both analytically and numerically, the non-Hermitian generalization of the parabolic and super-parabolic models, in which the exceptional points are driven through at finite speeds which are quadratic or cubic functions of time. We consider the case that the system is almost Hermitian when the parameters are far away from the exceptional points, such that the instantaneous eigenstates are nearly orthogonal. Specifically, in this case it is relevant to consider the transmission probabilities, which are the ratios of the transmission populations to the total population. We derive new analytical approximate formulas for the transmission probabilities as well as the transmission probabilities, which have not been reported in previous studies. Unlike our previous studies on non-adiabatic transitions in Hermitian systems,\(^6\) in which the transition points separating different dynamics regions can only be determined numerically, the benefit of our current approach is that the transmission dynamics can be approximated in terms of simple functions like hyperbolic or hypergeometric functions in the predetermined regions of broken and unbroken \(PT\) symmetry.

2. Non-Adiabatic Transitions in Non-Hermitian Two-Level Systems

To begin with, let us consider the following simple 2 × 2 non-Hermitian Hamiltonian matrix

\[
\hat{H} = \begin{pmatrix} -\nu & i\Gamma \\ i\Gamma & \nu \end{pmatrix}
\]  

(1)

which is \(PT\)-symmetric, that is, \([PT, \hat{H}] = 0\), where \(P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) and \(T\) performs complex conjugation.\(^6\) Here, the parameters \(\nu\) and \(\Gamma\) are both real, and \(\Gamma\) can be set to be positive without loss of generality. This Hamiltonian describes two states with an energy difference of \(2\nu\) and a nonreciprocal coupling \(\Gamma\). Because of the non-hermiticity, there are two different regions in the parameter space. For \(|\nu| > \Gamma\), the two eigenvalues \(\epsilon_+ \equiv \pm \sqrt{\nu^2 - \Gamma^2}\) are both real, which belong to the region of unbroken \(PT\) symmetry. On the other hand, for \(|\nu| < \Gamma\), the two eigenvalues \(\epsilon_\pm \equiv \pm i\sqrt{\Gamma^2 - \nu^2}\) are the complex conjugates of each other, which belong to the region of broken \(PT\) symmetry. The boundaries between the two regions, that is, \(|\nu| = \Gamma\), are an assembly of exceptional points at which the two eigenvalues coincide, and the associated eigenstates become completely parallel.

We may calculate the left and right instantaneous adiabatic eigenstates \(|\chi_\pm\rangle\) and \(|\phi_\pm\rangle\), which are defined by \(\hat{H}|\phi_\pm\rangle = \epsilon_\pm|\phi_\pm\rangle\) and \(\langle\chi_\pm|\hat{H} = \epsilon_\pm\langle\chi_\pm|\), and can be explicitly expressed as

\[
|\phi_\pm\rangle = \frac{1}{N_\pm} \begin{pmatrix} i \Gamma \\ \nu \mp \epsilon_\pm \end{pmatrix}, \quad \langle\chi_\pm\rangle = \frac{1}{N_\pm} \begin{pmatrix} \nu \mp \epsilon_\pm \\ i \Gamma \mp \epsilon_\pm \end{pmatrix}
\]  

(2)

where \(N_\pm \equiv \sqrt{2\epsilon_\pm(\nu \mp \epsilon_\pm)}\) are the normalization constants that give the inner products \(\langle\chi_\pm|\phi_\pm\rangle = 0\) and \(\langle\chi_\pm|\phi_\pm\rangle = 1\). As a result, we obtain the eigen-decomposition of the Hamiltonian, \(\hat{H} = \epsilon_\pm|\phi_\pm\rangle\langle\chi_\pm| + \epsilon_\pm|\phi_\pm\rangle\langle\chi_\pm|\), as long as there is no energy degeneracy, that is, \(\epsilon_+ \neq \epsilon_-\). The overlap of the instantaneous adiabatic eigenstates \(|\phi_\pm\rangle\) and \(|\phi_\pm\rangle\) has the form

\[
g \equiv \frac{|\langle\phi_\pm|\phi_\pm\rangle|}{\sqrt{|\langle\phi_\pm|\phi_\pm\rangle|}} = \begin{cases} \frac{\Gamma}{|\nu|}, & \text{for } |\nu| \geq \Gamma \\ \frac{|\nu|}{\Gamma}, & \text{for } |\nu| \leq \Gamma \end{cases}
\]  

(3)

which becomes unity at the exceptional points and vanishes when \(\nu\) drops to zero. As we can see from Figures 1 and 2, the overlaps of the adiabatic eigenstates exhibit non-analytic behaviors at the exceptional points, which are unique to non-Hermitian systems with \(PT\)-symmetry. Unlike Van Hove singularities in the density of states in band structures, which generally exist in periodic systems due to the presence of saddle points in the energy dispersion in momentum space, the non-analytic signatures in the overlaps of adiabatic eigenstates in non-Hermitian systems are due to \(PT\)-symmetric breaking, where real eigenvalues change into complex ones upon varying the external parameters of the Hamiltonian.

To continue, let us denote the state of the system by \(|\psi(t)\rangle \equiv (\psi_1(t), \psi_2(t))\rangle\), where \(\psi_1(t)\) and \(\psi_2(t)\) are the wave amplitudes in the diabatic basis. When the conditions \(|\psi_1(-\infty)\rangle = 0\) and \(|\psi_2(-\infty)\rangle = 1\) are initially fulfilled, the problem of non-adiabatic transition is to determine the transmission probabilities at \(t \to +\infty\) given by

\[
P_1(t) = \frac{|\psi_1(t)|^2}{|\psi_1(t)|^2 + |\psi_2(t)|^2}, \quad P_2(t) = 1 - P_1(t)
\]  

(4)

From the form of the Hamiltonian in Equation (1), one obtains the Schrödinger equations for the two wave amplitudes \(\psi_1\) and \(\psi_2\)

\[
i\gamma \psi_1 = i\Gamma \psi_2 - \nu(t)\psi_1
\]

(5a)

\[
i\gamma \psi_2 = i\Gamma \psi_1 + \nu(t)\psi_2
\]  

(5b)

from which one immediately obtains the second-order differential equations that \(\psi_1\) and \(\psi_2\) obey

\[
\psi_1 + [\nu^2(t) - \Gamma^2 - i\nu(t)]\psi_1 = 0
\]  

(6a)

\[
\psi_2 + [\nu^2(t) - \Gamma^2 + i\nu(t)]\psi_2 = 0
\]  

(6b)

For unitary evolutions, \(|\psi_1(t)|^2\) can be calculated from \(|\psi_1(t)|^2\) due to the conservation of total population, that is, \(|\psi_1(t)|^2 + |\psi_2(t)|^2 = \text{const.}\) However, for non-unitary evolutions, the total
population is in general not conserved, and hence $|\psi_e(t)|^2$ has to be calculated separately even when $|\psi_e(t)|^2$ is known. Fortunately, for our non-Hermitian two level model, we have

$$\frac{d}{dt}|\psi_e|^2 = \frac{d}{dt}|\psi_s|^2 = \Gamma (\psi_e^* \psi_s + \psi_s^* \psi_e)$$

(7)

which implies that the difference between the level populations, $|\psi_s|^2 - |\psi_e|^2$, is still a constant. For the special case that the separation of diabatic energies varies linearly with time, that is, $v(t) = at$ and $\Gamma = \text{const}$, Equations (6a) and (6b) describe the non-Hermitian generalization of the Landau–Zener model, which has the following analytic solutions for the final level populations: $|\psi_s(\infty)|^2 = e^{i\beta t/\eta} - 1$ and $|\psi_e(\infty)|^2 = e^{i\beta t/\eta}$, provided that the system is initially subjected to the constraints $\psi_e(-\infty) = 0$ and $|\psi_e(-\infty)| = 1$. In general, when the linear separation of the diabatic energies is modified by an additional term $c_n t^n$, that is, $v(t) \equiv at + c_n t^n$ and $\Gamma \equiv \text{const}$, $\psi_1$ and $\psi_2$ are governed by

$$\begin{align*}
\dot{\psi}_1 + (c_n^2 t^{2n} + 2ac_n e^{i\beta t} - iac_n e^{-i\beta t} - a^2 t^2 - i\beta)\psi_1 &= 0 \\
\dot{\psi}_2 + (c_n^2 t^{2n} + 2ac_n e^{i\beta t} + iac_n e^{-i\beta t} + a^2 t^2 - i\beta)\psi_2 &= 0
\end{align*}$$

(8)

Clearly, unlike conventional Hermitian models associated with $n$ level-crossing points, there are in general $2n$ exceptional points for non-Hermitian systems.

For the parabolic case with $c_1 \equiv \beta \neq 0$, we obtain

$$\begin{align*}
\dot{\psi}_1 + (\beta^2 t^4 + 2a\beta t^3 + a^2 t^2 - 2i\beta t - \Gamma^2 - i\alpha)\psi_1 &= 0 \\
\dot{\psi}_2 + (\beta^2 t^4 + 2a\beta t^3 + a^2 t^2 + 2i\beta t - \Gamma^2 + i\alpha)\psi_2 &= 0
\end{align*}$$

After a change of variable $\tau \equiv t + \frac{\alpha}{2\beta}$, the equation that governs $\psi_1$ becomes

$$\frac{d^2\psi_1}{d\tau^2} + \left(\frac{\beta^2}{4\beta^2} - \Gamma^2 - 2i\beta \tau\right)\psi_1 = 0$$

(9)

where $\psi_2$ satisfies a similar equation with $-2i\beta \tau$ replaced by $2i\beta \tau$. For this system, there are at most four exceptional points located at $\tau = \pm \frac{1}{2|\beta|} \sqrt{a^2 + 41\beta^2}$. For $a^2 > 16\Gamma^2\beta^2$, there are four exceptional points; for $a^2 < 16\Gamma^2\beta^2$, by contrast, there are only two exceptional points; whereas for $a^2 = 16\Gamma^2\beta^2$, there are three exceptional points. Hence, $a^2 = 16\Gamma^2\beta^2$ defines a critical surface in the parameter space (see Figure 3a). After the transformation $z \equiv 2i\beta \tau/3$, Equation (9) becomes the second canonical form of the tri-confluent Heun equation

$$\frac{d^2\psi_1}{dz^2} + \left(\frac{\beta^2}{4\beta^2} - \Gamma^2 + \mu z - 3\zeta t^2 - 9\xi t^4\right)\psi_1 = 0$$

(10)

where $\mu \equiv \Gamma^2$, $\nu \equiv \sqrt{6i\beta}$, and $\xi \equiv -i\alpha^2/(2\beta)$. For the super-parabolic case with $c_1 \equiv \gamma \neq 0$, we obtain

$$\begin{align*}
\dot{\psi}_1 + (\gamma^2 t^6 + 2\gamma \theta t^4 + (\alpha^2 - 3\beta^2) t^2 - \Gamma^2 - i\alpha)\psi_1 &= 0 \\
\dot{\psi}_2 + (\gamma^2 t^6 + 2\gamma \theta t^4 + (\alpha^2 - 3\beta^2) t^2 + \Gamma^2 + i\alpha)\psi_2 &= 0
\end{align*}$$

where $\gamma^2 = c_n^2 t^{2n}$, $v \equiv -\sqrt{6i\beta}$, and $\xi \equiv -i\alpha^2/(2\beta)$. For the super-parabolic case with $c_1 \equiv \gamma \neq 0$, we obtain

$$\begin{align*}
\dot{\psi}_1 + (\gamma^2 t^6 + 2\gamma \theta t^4 + (\alpha^2 - 3\beta^2) t^2 - \Gamma^2 - i\alpha)\psi_1 &= 0 \\
\dot{\psi}_2 + (\gamma^2 t^6 + 2\gamma \theta t^4 + (\alpha^2 - 3\beta^2) t^2 + \Gamma^2 + i\alpha)\psi_2 &= 0
\end{align*}$$

where $\gamma^2 = c_n^2 t^{2n}$, $v \equiv -\sqrt{6i\beta}$, and $\xi \equiv -i\alpha^2/(2\beta)$. For the super-parabolic case with $c_1 \equiv \gamma \neq 0$, we obtain

$$\begin{align*}
\dot{\psi}_1 + (\gamma^2 t^6 + 2\gamma \theta t^4 + (\alpha^2 - 3\beta^2) t^2 - \Gamma^2 - i\alpha)\psi_1 &= 0 \\
\dot{\psi}_2 + (\gamma^2 t^6 + 2\gamma \theta t^4 + (\alpha^2 - 3\beta^2) t^2 + \Gamma^2 + i\alpha)\psi_2 &= 0
\end{align*}$$

where $\gamma^2 = c_n^2 t^{2n}$, $v \equiv -\sqrt{6i\beta}$, and $\xi \equiv -i\alpha^2/(2\beta)$.
second canonical form of the bi-confluent Heun equation \( \Gamma = 1 \).

After another change of variable \( \xi \equiv \sqrt{-i\eta/2\tau} \), it becomes the second canonical form of the bi-confluent Heun equation

\[
\frac{d^2 U_1}{d\xi^2} + \left( \frac{3}{16\tau^2} - \frac{\Gamma^2 + i\eta}{4\tau} + \frac{\alpha^2 - 3i\eta}{4} + \frac{\alpha \tau}{2} + \frac{\tau^2 - \eta^2}{4} \right) U_1 = 0
\]

After another change of variable \( \xi \equiv \sqrt{-i\eta/2\tau} \), it becomes the second canonical form of the bi-confluent Heun equation

\[
\frac{d^2 U_1}{d\xi^2} + \left( \frac{1 - \mu^2}{4\xi^2} - \frac{\eta \xi}{2\xi} + \lambda - \frac{\nu^2}{4} - \nu \xi - \xi^2 \right) U_1 = 0
\]

where \( \mu = -\frac{1}{2}, \nu = \alpha \sqrt{\frac{2\tau}{\gamma}}, \lambda = \frac{1}{2} \) and \( \eta = -\frac{\tau}{2}(1 + \frac{\tau^2}{4}) \).

3. Analytical Approximations to the Transmission Probabilities

To visualize and analyze the dynamics of such a non-Hermitian system, one may introduce four real variables, that is, \( S_0 \equiv |\psi_1|^2 + |\psi_2|^2 \), \( S_1 \equiv \psi_1^* \psi_1 + \psi_2^* \psi_2 \), \( S_2 \equiv -i(\psi_2^* \psi_1 - \psi_1^* \psi_2) \), and \( S_3 \equiv |\psi_2|^2 - |\psi_1|^2 \), which obey \( S_0^2 - S_1^2 - S_2^2 = S_3^2 = \text{const} \).

One may write \( S_1 \equiv (S_0^2 - S_3^2)^{1/2} \cos \Theta \) and \( S_2 \equiv (S_0^2 - S_3^2)^{1/2} \sin \Theta \), where \( \Theta \equiv \arg \psi_1 - \arg \psi_2 \) is the relative phase between the two wave amplitudes \( \psi_1 \) and \( \psi_2 \), and \( S_1 \) and \( S_2 \) are related to the total level population \( S_0 \) and the relative phase \( \Theta \). For the case when the system is initially in one of the instantaneous eigenstates, we obtain \( S_1 = \pm 1 \). Hence, both \( |\psi_2|^2 \equiv \frac{1}{2}(S_0 \mp 1) \) and \( |\psi_1|^2 \equiv \frac{1}{2}(S_0 \mp 1) \) can be determined from the total level population \( S_0 \).

The non-Hermitian two-level dynamics may be visualized on a hyperboloid of two sheets with \( S_0 \) in the horizontal direction (see Figure 4a,c), which is described by the following set of differential equations

\[
\dot{S}_0 = 2\Gamma S_1
\]

\[
\dot{S}_1 = 2\Gamma S_0 - 2\nu(t) S_2
\]

\[
\dot{S}_2 = 2\nu(t) S_1
\]
After the first exceptional point is reached, the total population \( S_0 \) for \( t \leq t \) can be approximated by

\[
S_0(t) \approx S_0(t_1) \cos h(2\Gamma(t-t_1)) + S_1(t_1) \sin h(2\Gamma(t-t_1))
\]

and

\[
S_1(t) \approx S_0(t_1) \sin h(2\Gamma(t-t_1)) + S_1(t_1) \cos h(2\Gamma(t-t_1))
\]

where \( S_1(t_1) = S_0(t_1) \). After leaving the last exceptional point at \( t = t_2 \), the total population approaches its stationary value in the region of unbroken \( \mathcal{PT} \) symmetry (see Figure 4b), which may be understood by neglecting the term \( 2\Gamma S_0 \) in Equation (14b).

Defining \( \Phi(t) \equiv 2 \int_{t_1}^t \psi(t)dt \), one obtains

\[
S_1(t) \approx S_1(t_2) \cos \Phi(t) - S_2(t_2) \sin \Phi(t)
\]

\[
S_2(t) \approx S_1(t_2) \sin \Phi(t) + S_2(t_2) \cos \Phi(t)
\]

\[
S_0(t) \approx S_0(t_2) + 2\Gamma S_1(t_2) \int_{t_1}^t \cos \Phi(s)ds
\]

\[
- 2\Gamma S_1(t_1) \int_{t_1}^t \sin \Phi(s)ds
\]

where the functions \( S_i(t_1) \) are evaluated using Equation (15). For a large negative initial time \( t \to -\infty \), if the system is initially in an instantaneous eigenstate with \( |\psi_1(-\infty) = 0 \) and \( |\psi_2(-\infty) = 1 \), we have \( S_1, S_2, S_0 = (0, 0, 1) \). Hence, we obtain a simple analytical formula for the total population for \( t \to -\infty \):

\[
S_0(\infty) \approx \cosh(2\Gamma\Delta t) + 2\Gamma \sinh(2\Gamma\Delta t) \int_{t_1}^\infty \cos \Phi(t)dt
\]

\[
- 4\Gamma \int_{t_1}^\infty \psi(t) \sinh(2\Gamma(t-t_1))dt \int_{t_1}^\infty \sin \Phi(t)dt
\]

where \( \Delta t \equiv t_2 - t_1 \) is the size of the region of broken \( \mathcal{PT} \) symmetry. In particular, from Equation (17), we recover the result \( S_0(\infty) = 1 \) for \( \Gamma = 0 \). In general, when there are more than two exceptional points, analytical approximations to the final transmission probabilities can also be obtained by neglecting either of the terms \( 2\Gamma S_0 \) or \( -2\Gamma S_1 \) in Equation (14b) in the regions of unbroken or broken \( \mathcal{PT} \) symmetry, and gluing the solutions at the boundaries between different regions. In Figure 7, we depict the transmission probabilities \( P_1 \) and \( P_2 \), based on the analytical formulas Equations (15)-(17), and compare the results to numerical simulations. Figure 7a,b show that the final transmission probabilities are well-approximated by Equation (17) for the parabolic case for both initial instantaneous eigenstates and randomly selected initial states. However, as one may see from Figure 7c,d,
there is an overestimation of \( P_\gamma \) and an underestimation of \( P_\lambda \) for the super-parabolic case, which are possibly caused by neglecting the contribution from the coupling between the two states just before reaching the first exceptional point.

In order to reduce the accumulated errors in the final transmission probabilities, one may add a transition region in front of the first exceptional point. As one may see from Equations (6a) – (6b), the term \( v^3 - \Gamma^3 \) is exactly zero at the exceptional point and gradually increases until it balances the terms \( \pm i\). Hence, the boundaries of the transition regions, which are referred to as the transition points, may be determined by the condition \( |v(t)| = |v^3(t) - \Gamma^3| \). In particular, for \( v(t) = at + \gamma t^3 \), the transition points are the real roots of the sextic equation \( r^6 + 2\gamma r^4 + (a^2 + 3\gamma) r^2 - \Gamma^3 = 0 \). Let us denote the transition point before the first exceptional point \( t_0 \) as \( t_0 \). For the transition region between \( t_0 \) and \( t_1 \), we may assume that \( v \approx -\Gamma \), so that Equations (14a)–(14c) are replaced by \( \delta = 2\Gamma S_0, S_1 = 2\Gamma(S_0 + S_2) \) and \( S_2 = -2\Gamma S_1 \). As a result, \( S_0 + S_2 \) becomes a constant, and so does \( S_1 \). Hence, the total population and the other two variables \( S_1 \) and \( S_2 \) can be approximated by

\[
S_1(t) \approx 2\Gamma(S_0(t_0) + S_2(t_0))t + S_1(t_0) \quad \text{(17a)}
\]

\[
S_2(t) \approx -2\Gamma^2(S_0(t_0) + S_2(t_0))t^2 - 2\Gamma S_1(t_0)t + S_1(t_0) \quad \text{(17b)}
\]

\[
S_0(t) \approx 2\Gamma^2(S_0(t_0) + S_2(t_0))t^2 + 2\Gamma S_1(t_0)t + S_0(t_0) \quad \text{(17c)}
\]

Here, \( S_0(t) = S_0(t_0) \) when the system is initially in an instantaneous eigenstate. For other cases, \( S_1(t_0) \) are determined by Equation (16) with \( S_0(t) \) replaced by \( S_0(t) \). In particular, for the special case that \( |\psi_1(infty) = 0 \) and \( |\psi_2(infty) = 1 \), we have \( S_0(infty) = 1 \) and \( S_1(infty) = S_2(infty) = 0 \). Hence, the modified analytical formula for the total population for \( t \to \infty \) is

\[
S_0(\infty) \approx (1 + 2\Gamma^2 t_0^2) \cos h(2\Gamma \Delta t) - 2\Gamma t_1 \sin h(2\Gamma \Delta t)
\]

\[
+ 2\Gamma(1 + 2\Gamma t_1^2) \sin h(2\Gamma \Delta t) - 2\Gamma t_1 \cos h(2\Gamma \Delta t) \int_{t_0}^{t_1} \cos \Phi(t) dt
\]

\[
- 4\Gamma(1 + 2\Gamma^2 t_1^2) \int_{t_0}^{t_1} \sin(h(2\Gamma(t - t_1))) dt - \Gamma^2 t_1^2
\]

\[
+ 2\Gamma t_1 \int_{t_0}^{t_1} \sin(h(2\Gamma(t - t_1))) dt \int_{t_0}^{t_1} \sin \Phi(t) dt \quad \text{(18)}
\]

In Figure 8a,b, the analytical approximations to the transmission probabilities are depicted for the super-parabolic case, after
Equations (18a)–(18c) for \( t_0 \leq t \leq t_1 \) are taken into account. The result shows that the final transmission probabilities are well-approximated by the modified analytical formulas for both initial instantaneous eigenstates and randomly selected initial states.

Finally, as a remark, one can readily show that the dynamics of the two wave amplitudes \( \psi_1 \) and \( \psi_2 \) show different characters when the system passes through the parameter space with different numbers of exceptional points. To show these, one needs to depict the wave amplitude \( \psi_1 \) or \( \psi_2 \) in the complex plane. As shown in Figure 9b, when the system possesses only a pair of exceptional points, the trajectory of \( \psi_2 \) directly connects the inner and outer limit cycles which represent the initial and final level populations. By contrast, as shown in Figure 9d, when the system possesses three pairs of exceptional points, the trajectory of \( \psi_2 \) follows an outward propagating path back and forth several times before reaching the outer limit cycle. Similar behaviors are found in the wave amplitude \( \psi_1 \), as shown in Figure 9a,c. To be more precise, one may depict the winding numbers of the paths associated with the wave amplitudes \( \psi_1 \) or \( \psi_2 \). After expressing the paths of the wave amplitudes \( \psi_i \) \((i = 1, 2)\) in the polar form as \((r_i(t), \theta_i(t))\), one may calculate the winding numbers of the paths \((r_i(t), \theta_i(t))\) via the formula \( \frac{1}{2\pi} \left( \theta_i(t_c) - \theta_i(-t_c) \right) \), where \( t_c \) represent the largest and smallest exceptional points respectively. As shown in Figure 10b, when the system possesses only a pair of exceptional points, the winding number of the path associated with \( \psi_1 \) in the region \( |t| \leq t_1 \) is a quadratic function of time. But, as shown in Figure 10d, when the system possesses three pairs of exceptional points, the winding number of the path associated with \( \psi_1 \) in the region \( |t| \leq t_1 \) is a quartic function of time. Similarly, as shown in Figure 10a,c, the winding numbers of the paths associated with \( \psi_2 \) in the region \( |t| \leq t_1 \) are linear and cubic functions of time when the system possesses one and three pairs of exceptional points respectively. This shows that the dynamics of the two wave amplitudes in \( PT \)-symmetric non-Hermitian two-level systems exhibit different characters when the system passes through the parameter space with different number of exceptional points.

4. Application to \( PT \)-Symmetric Tight-Binding Lattice

We now discuss how the parabolic and super-parabolic models studied in the last sections can be realized in a \( PT \)-symmetric non-Hermitian 1D tight-binding optical waveguide lattice with an index gradient, where the hopping dynamics of a single
Figure 6. Schematic of the transition probabilities and the level populations as functions of time. a,c) the transmission probabilities given by $P_1 \equiv |\psi_1|^2 / (|\psi_1|^2 + |\psi_2|^2)$ and $P_2 \equiv |\psi_2|^2 / (|\psi_1|^2 + |\psi_2|^2)$ are shown as blue and red solid lines respectively. In (a), there are two exceptional points located at $t = \pm 0.68233$, which are shown as dashed lines. In Figure 6c, there are six exceptional points located at $t = \pm 2.60944, \pm 2.08406$, and $\pm 0.52537$. b,d) the level populations $|\psi_1|^2$ and $|\psi_2|^2$ are shown as blue and red solid lines respectively.

A particle on the lattice is described by the Hamiltonian\[14,68–73\]

$$\hat{H} \equiv \sum_n \left\{-\kappa ([n] \langle n+1 | + | n+1 \rangle \langle n |) + [i\Gamma (-1)^n + F n] [n] \langle n | \right\}$$

(19)

where $\kappa$ is the hopping rate between adjacent sites, $\Gamma (-1)^n$ is an alternating gain or loss of the site energy, which may be implemented by metal-cladding on waveguides with odd values of $n$, and $F$ is an index gradient along the lattice, which may be experimentally achieved by bending the waveguides.\[72\] The 1D lattice described by the Hamiltonian (Equation (20)) can be used to achieve one-way robust light transport in the present of disorder.\[71,72\]

We now study the non-Hermitian system in the basis of the Bloch states $|k\rangle \equiv \frac{1}{\sqrt{2\pi}} \sum_n e^{i k n} |n\rangle$, where $k \in [-\pi, \pi]$ is the crystal momentum. In the absence of index gradient ($F \equiv 0$), the Bloch state $\psi(k)$ in the crystal momentum representation obeys $i d/dt \psi(k) = -2\kappa \cos k \psi(k) + i\Gamma \psi(k + \pi)$, and the state $\psi(k + \pi)$ obeys $i d/dt \psi(k + \pi) = 2\kappa \cos k \psi(k + \pi) + i\Gamma \psi(k)$. Hence, one may introduce a two-component state vector $|\Psi(k)\rangle \equiv (\psi(k), \psi(k + \pi))^T$, whose time evolution is governed by the Bloch Hamiltonian

$$\hat{h}(k) = \begin{pmatrix} -2\kappa \cos k & -Fq \\ i\Gamma & 2\kappa \cos k \end{pmatrix}$$

(20)

When a static force is applied to the lattice by engineering the refractive index of the waveguides, an initial state that is close to an eigenstate of the Bloch Hamiltonian (Equation (21)) would experience non-adiabatic transitions between the energy bands, which are non-Hermitian generalizations of conventional Bloch oscillations, which correspond to a splitting of the beam in real space. In such a case, the Hamiltonian which governs the two-component state vector becomes

$$\hat{h}(k, q) = \begin{pmatrix} -2\kappa \cos k - F q \\ i\Gamma & 2\kappa \cos k - F q \end{pmatrix}$$

(21)

where $q \equiv id/dk$ is canonically conjugate to $k$, that is, $[q, k] = i$. In Hermitian band theory, the expectation value of the crystal momentum obeys the acceleration theorem, that is, $(k)_t = (k)_0 + Ft$. As shown by Longstaff and Graefe,\[14\] the acceleration theorem can be applied to non-Hermitian systems, as long as the initial
The uncertainty in the crystal momentum is negligible. Hence, the non-adiabatic transition dynamics can be effectively described by the Hamiltonian (Equation (21)), with the crystal momentum \( k \) replaced by its expectation value \( \langle k \rangle_t = \langle k \rangle_0 + Ft \). One may then Taylor expand the effective Hamiltonian around the band edge \( k = \pi/2 \), and obtain (after shifting the time origin)\[^{22}\]

\[
\hat{h}(t') \approx \begin{pmatrix} -at' - \gamma t'^3 + i\Gamma & i\Gamma \\ i\Gamma & at' + \gamma t'^3 \end{pmatrix}
\]

where \( \alpha \equiv 2\kappa F, \gamma \equiv -\kappa F^3/3, \ t' \equiv t - t_0, \) and \( t_0 \equiv \frac{\pi}{2}(\sigma - 2\langle k \rangle_0) \). The resulting Hamiltonian (23) is then equivalent to the non-Hermitian Hamiltonian (Equation (1)) for the super-parabolic case. Interestingly, the number of exceptional points of the Hamiltonian (Equation (23)) is irrelevant to the amplitude of the static force \( F \). For the case that \( \kappa F < 0 \) and \( |\kappa| < 3\Gamma/4\sqrt{2} \), there are two exceptional points; when \( |\kappa| > 3\Gamma/4\sqrt{2} \), there are six exceptional points; when \( |\kappa| = 3\Gamma/4\sqrt{2} \), there are four exceptional points. Similarly, one may Taylor expand the effective Hamiltonian around \( k = 0 \), and obtain (after subtracting a constant of \( -2\kappa + \kappa\langle k \rangle_0^2 \) from the Hamiltonian)\[^{23}\]

\[
\hat{h}(t) \approx \begin{pmatrix} -at - \beta t^2 & i\Gamma \\ i\Gamma & at + \beta t^2 \end{pmatrix}
\]
where $\alpha \equiv -2\kappa(k)\gamma F$ and $\beta \equiv -2\kappa F^2$. The resulting Hamiltonian (Equation (24)) is then equivalent to the non-Hermitian Hamiltonian (Equation (1)) for the parabolic case. Similar to the super-parabolic case, the number of exceptional points for Hamiltonian (Equation (24)) is irrelevant to the amplitude of the static force $F$. For $|\kappa| < 2\Gamma/(k)^2_0$, there are two exceptional points; for $|\kappa| > 2\Gamma/(k)^2_0$, there are four exceptional points; for $|\kappa| = 2\Gamma/(k)^2_0$, there are three exceptional points.

5. Conclusion

We discussed the non-Hermitian dynamics of a two-level quantum system driven through an assembly of exceptional points at finite speed which are quadratic or cubic functions of time. We derived analytical approximate formulas for the non-adiabatic transmission probabilities for both parabolic and super-parabolic cases. Moreover, we demonstrated possible experimental realizations in 1D graded index photonic crystal waveguides, which may be applied to achieve unidirectional light transport in modulated waveguides. The non-adiabatic transitions through an assembly of exceptional points may be realized by non-Hermitian Bloch oscillations between different bands. We found that for both parabolic and super-parabolic cases, the number of exceptional points increases as the hopping rate between the neighboring sites increases. In future studies, we may extend our current approximation procedure to the cases where both the amplitude of the alternating gain and loss, and the index gradient are time dependent.

While preparing the manuscript, we noticed some recent relevant publications on nonlinear waves,[74–83] which are, although not directly associated to $P\bar{T}$ symmetric non-Hermitian systems, of special interest. In our future studies, we shall explore related nonlinear wave dynamics in $P\bar{T}$ symmetric non-Hermitian systems.

Appendix: Evaluation of Equation (17) for the Parabolic Model

For $v(t) \equiv at + \beta t^2$ with $a^2 < 16\Gamma^2\beta^2$, the integral which involves the hyperbolic sine function in Equation (17) may be evaluated using the following indefinite integral

$$\int v(t) \sin h(2\Gamma(t-t_1)) dt$$

$$= \left( \frac{v}{2\Gamma} + \frac{\bar{v}}{8\Gamma^2} \right) \cos h(2\Gamma(t-t_1)) - \frac{\bar{v}}{4\Gamma^2} \sin h(2\Gamma(t-t_1)) \quad (A1)$$
which yields

\[ I_1 \equiv \int_{t_1}^{t_2} v(t) \sin h(2\Gamma(t - t_1)) dt \]

\[ = \left( \frac{v(t_2)}{2\Gamma} + \frac{\beta}{4\Gamma^3} \right) \left( \cos h(2\Gamma\Delta t) - 1 \right) - \frac{v(t_2)}{2\Gamma} \sin h(2\Gamma\Delta t) \]

\[ = \left( \pm \frac{1}{2} + \frac{\beta}{4\Gamma^3} \right) \left[ \cos h \left( \frac{2\Gamma}{\beta} \sqrt{\alpha^2 + 4\beta^2} \right) - 1 \right] \]

\[ - \sqrt{\alpha^2 + 4\beta^2} \sin h \left( \frac{2\Gamma}{\beta} \sqrt{\alpha^2 + 4\beta^2} \right) \]

(A2)

where we have used \( v(t_2) = v(t_1) = \pm \Gamma \), \( v(t_2) = \pm \sqrt{\alpha^2 + 4\beta^2} \) and \( \Delta t = \pm \frac{1}{\beta} \sqrt{\alpha^2 + 4\beta^2} \) for \( \beta = \pm |\beta| \). In particular, for \( \Gamma \to 0 \), we have \( I_1 \to \pm \Gamma^2 \alpha^2 / \beta^2 \).

We now evaluate the two definite integrals \( \int_{t_2}^{t_1} \cos \Phi(t) dt \) and \( \int_{t_2}^{t_1} \sin \Phi(t) dt \) in Equation (17), where \( \Phi(t) \equiv 2 \int_{t_2}^{t} v(s) ds \). To begin with, let us consider the integral

\[ I_2 \equiv \int_{t_2}^{t_1} e^{i\Phi(t)} dt = e^{-i(\alpha^2 t_1^2 + \frac{\beta}{2} t_2)} \int_{t_2}^{t_1} e^{i(\alpha^2 t^2 + \frac{\beta}{2} t^2)} dt \]

(A3)

After the change of variable \( \tau \equiv t + \frac{\alpha}{2\beta} \), Equation (A3) becomes

\[ I_2 = e^{-i\frac{3}{2}\int_{t_2}^{t_1} (\alpha^2 t^2 + \frac{\beta}{2} t^2)} \int_{t_2}^{t_1} e^{i\frac{3}{2} (\alpha^2 t^2 + \frac{\beta}{2} t^2)} d\tau \]

(A4)

where \( k \equiv \frac{\alpha}{2\beta} \). After the transformation \( x \equiv \frac{2\beta}{\alpha} t \) and \( \lambda \equiv 3k^2 (\frac{2\beta}{\alpha})^{2/3} \), Equation (A4) becomes

\[ I_2 = \left( \frac{3}{\sqrt{2\beta}} \right) \int_{\lambda x_2}^{\lambda x_1} e^{-i(\gamma^2 - \lambda x^3)} dx \]

(A5)

We now evaluate the generalized Fresnel integrals in terms of the confluent hypergeometric functions

\[ \int x^n e^{-\alpha x} dx = \frac{x^{n+1}}{m+1} F_1 \left( \frac{m+1}{m+1} | ix^n \right) \]

(A6)
Using the asymptotic expansion of the confluent hypergeometric function

$$\frac{e^{\frac{\lambda_{x}x}{3}}}{m!} F_1\left(\frac{m+1}{n} - \frac{1}{n}\right) \approx \frac{1}{n} \left(\frac{m+1}{n}\right) e^{\frac{\lambda_{x}x}{3}(m+1)/(2n)} \quad (A7)$$

Equation (A5) becomes

$$I_x = \left(\frac{3}{2\beta}\right)^{1/3} e^{\frac{\lambda_{x}x}{6}} e^{-\frac{i\lambda_{x}x}{3}} \sum_{m=0}^{\infty} \frac{(-i\lambda_{x}x)^{m}}{m!} \frac{x}{m+1} F_1\left(\frac{1}{3} - i\lambda_{x}x\right) \quad (A8)$$

Substituting Equations (A2) and (A8) into Equation (17), the total population at $t \rightarrow \infty$ becomes

$$S_0(\infty) \approx \cos k \left(\frac{2\Gamma}{\beta} \sqrt{x^2 + 4\Gamma^2}\right) \pm 2\Gamma \sin k \left(\frac{2\Gamma}{\beta} \sqrt{x^2 + 4\Gamma^2}\right)
$$

$$+ \left(\frac{3}{2\beta}\right)^{1/3} \operatorname{Re} \left(\frac{e^{\frac{\lambda_{x}x}{6}}}{3} e^{-\frac{i\lambda_{x}x}{3}} \sum_{m=0}^{\infty} \frac{(-i\lambda_{x}x)^{m}}{m!} \frac{x}{m+1} F_1\left(\frac{1}{3} - i\lambda_{x}x\right)\right)
$$

$$- \frac{\left(\pm 4\Gamma + i\beta \right)}{\Gamma} \cos k \left(\frac{2\Gamma}{\beta} \sqrt{x^2 + 4\Gamma^2}\right) - 1
$$

$$- \sqrt{x^2 + 4\Gamma^2} \sin k \left(\frac{2\Gamma}{\beta} \sqrt{x^2 + 4\Gamma^2}\right)
$$

$$\left(\frac{3}{2\beta}\right)^{1/3} \operatorname{Im} \left(\frac{e^{\frac{\lambda_{x}x}{6}}}{3} e^{-\frac{i\lambda_{x}x}{3}} \sum_{m=0}^{\infty} \frac{(-i\lambda_{x}x)^{m}}{m!} \frac{x}{m+1} F_1\left(\frac{1}{3} - i\lambda_{x}x\right)\right)
$$

$$- \frac{\left(\pm 4\Gamma + i\beta \right)}{\Gamma} \cos k \left(\frac{2\Gamma}{\beta} \sqrt{x^2 + 4\Gamma^2}\right) - 1
$$

$$- \sqrt{x^2 + 4\Gamma^2} \sin k \left(\frac{2\Gamma}{\beta} \sqrt{x^2 + 4\Gamma^2}\right)
$$

(A9)

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**Conflict of Interest**

The authors declare no conflict of interest.

**Keywords**

analytical approximations, exceptional points, non-adiabatic transitions, non-Hermitian PT-symmetric systems, optical waveguide lattices

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