EXTENSIONS OF THE VIRASORO ALGEBRA
AND GAUGED WZW MODELS

Alexander Sevrin and Walter Troost

1. Department of Physics
   University of California at Berkeley
   and
   Theoretical Physics Group
   Lawrence Berkeley Laboratory
   Berkeley, CA 94720, U.S.A.

2. Instituut voor Theoretische Fysica
   Universiteit Leuven
   Celestijnenlaan 200D, B-3001 Leuven, Belgium

Abstract

To any non-trivial embedding of $sl(2)$ in a (super) Lie algebra, one can associate an extension of the Virasoro algebra. We realize the extended Virasoro algebra in terms of a WZW model in which a chiral, solvable group is gauged, the gauge group being determined by the $sl(2)$ embedding. The resulting BRST cohomology is computed and the field content of the extended Virasoro algebra is determined. The closure of the extended Virasoro algebra is shown. Applications such as the quantum Miura transformation and the effective action of the associated extended gravity theory are discussed.
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Extensions of the Virasoro algebra, for a review see [1], such as the $W_n$ algebras, the super-symmetric Virasoro algebra, etc., play a crucial role in the study of conformal field theories, 2D gravity and integrable systems. There is a close relation between extended Virasoro algebras and embeddings of $sl(2)$ in a (super) Lie algebra $\tilde{g}$. In [2], it was shown that given an affine Lie algebra $\hat{g}$ and an embedding of $sl(2)$ in $\hat{g}$, one recovers the Ward identities of an extended Virasoro algebra from the Ward identities of a WZW model by constraining the affine currents as $J_z \equiv \frac{1}{2} e_+ + T$ where $e_\pm$ are the $sl(2)$ generators and $T \in \ker \text{ad} e_\pm$. Almost all known extended Virasoro algebras can be obtained in this way. Some algebras which do not fall into this class can be obtained from an algebra which falls in this class by orbifolding. Another complication which arises is that for certain extended Virasoro algebras, currents of dimension $1/2$ appear. Dimension $1/2$ currents cannot be obtained by the method outlined above. However because of the fact that dimension $1/2$ currents can always be decoupled from the extended Virasoro algebra, [3], see also [4], these cases are covered as well. Many aspects of these reductions were studied in [6, 7]. Until recently, most considerations were classical. In [8, 9] important steps towards the understanding of the quantum theory were made.

Parallel to this issue was the development of induced and effective gravity theories (in the light-cone gauge) associated to extended Virasoro algebras [10] - [16].

In this letter we complete the program started in [16] which relates both issues discussed above. For an arbitrary embedding of $sl(2)$ in a Lie algebra $\tilde{g}$, we realize the corresponding extended Virasoro algebra in terms of a WZW model for which a chiral, solvable group has been gauged, the gauge group being determined by the $sl(2)$ embedding. Gauge invariance requires in certain cases the introduction of extra free fields. The generators of the extended Virasoro algebra are obtained by solving the BRST cohomology. Due to the presence of the extra, free fields, the BRST charge cannot be decomposed into two mutually anticommuting, nilpotent charges. So, the techniques of [8, 9] cannot be applied directly. However, because of the existence of a filtration of the BRST complex, spectral sequence techniques can still be used to solve the cohomology. The quantum Miura transformation immediately follows from this. Finally, choosing a different gauge yields effortlessly the effective action in the light-cone gauge of the corresponding extended gravity theory.

Given a (super) affine Lie algebra $\hat{g}$ of level $\kappa$, we call the finite dimensional subalgebra $\tilde{g}$. Consider a nontrivial embedding of $sl(2)$ in a (super) Lie algebra $\tilde{g}$. A thorough study of $sl(2)$ embeddings can be found in [17]. The adjoint representation of $\hat{g}$ branches into irreducible representations of $sl(2)$. For a given embedding, we denote the generators of $\tilde{g}$ by $t(jm,\alpha_j)$ where $j \in \frac{1}{2} \mathbb{N}$ labels the irreducible representation of $sl(2)$, $m$ runs from $-j$ to $j$ and $\alpha_j$ counts the multiplicity of the irreducible representation $j$ in the branching. The $sl(2)$ generators $e_\pm$ and $e_0$ are denoted by $e_\pm \equiv t(1\pm 1,0)/\sqrt{2}$ and $e_0 \equiv t(10,0)$. The $sl(2)$ algebra is given by $[e_0,e_\pm] = \pm 2e_\pm$.
and \([e_+, e_-] = e_0\]. The action of the \(sl(2)\) algebra on the other generators is given by
\[
[e_0, t(jm, \alpha_j)] = 2m t(jm, \alpha_j)
\]
\[
[e_\pm, t(jm, \alpha_j)] = (-)^{j+m-\frac{1}{2}+\frac{1}{2}} \sqrt{(j \mp m)(j \pm m + 1)} t(jm \pm 1, \alpha_j)
\]
(1)

Throughout the paper we will use projection operators \(\Pi\), e.g.\(\Pi_+ \bar{g} = \{t(jm, \alpha_j) | m > 0; \forall j, \alpha_j\}\), \(\Pi_{\geq m} \bar{g} = \{t(jm, \alpha_j) | n \geq m; \forall j, \alpha_j\}\), \(\Pi_m \bar{g} = \{t(jm, \alpha_j) | \forall j, \alpha_j\}\). Our conventions are as in [16].

The affine Lie algebra \(\hat{g}\) is realized by a WZW model with action \(\kappa S^-[g]\). The action \(S_1\)
\[
S_1 = \kappa S^-[g] + \frac{1}{\pi x} \int \text{str} A_z \left( J_z - \kappa e_- + \kappa \tau [\tau, e_-] \right) + \frac{\kappa}{4\pi x} \int \text{str} \tau, e_- \bar{\partial} \tau,
\]
(2)
with the affine currents \(J_z = \frac{\kappa}{2} \partial g g^{-1}\), the gauge fields \(A_z \in \Pi_+ \bar{g}\) and the “auxiliary” fields \(\tau \in \Pi_{1/2} \bar{g}\), is invariant under
\[
g \rightarrow h g \quad A_z \rightarrow \bar{\partial} h h^{-1} + h A_z h^{-1} \quad \tau \rightarrow \tau + \Pi_{1/2} \eta,
\]
(3)
where \(h = \exp \eta, \eta \in \Pi_+ \bar{g}\). Note that as bosonic irreducible representations of half integer spin always occur in pairs, the introduction of auxiliary fields can be avoided in the purely bosonic case [18] by further restricting the gauge group. In the supersymmetric case however, this is not true anymore. The simplest example, the standard \(N = 1\) Neveu-Schwarz algebra, which is based on the embedding of \(sl(2)\) in \(osp(2|1)\), already requires the introduction of one extra free fermion. For uniformity and simplicity, we always introduce these extra fields whenever representations of half-integer spin occur.

The gauge fields \(A_z\) are Lagrange multipliers which impose the constraint \(\Pi_- J_z = \frac{\kappa}{2} e_- + \frac{\kappa}{2} \tau [\tau, e_-]\). Call the constrained current \(J^c_z\). Performing the gauge transformation which brings \(J^c_z\) in the form \(T + \frac{\kappa}{2} e_-\) where \(T \in \text{ker} ade_+\), we get, by construction, the fields \(T\) which are gauge invariant modulo the constraints, i.e. modulo the equations of motion of the gauge fields \(A_z\). They are of the form \(T \propto \Pi_{\text{ker} ade_+} J_z + \cdots\)

We couple these currents to sources and modify the action to
\[
S_2 = S_1 + \frac{1}{4\pi xy} \int \text{str} \mu T,
\]
with the sources \(\mu \in \text{ker} ade_-\). We will show that \(T\) forms an extension of the Virasoro algebra.

As the polynomials are only gauge invariant modulo terms proportional to the equations of motion of the gauge fields, we cancel the resulting non-invariance terms in \(\delta S_2\) by modifying* 

*Except: in [16], auxiliary fields \(\tau, r\) and \(\bar{r}\) were introduced. The relation between the auxiliary fields here and those in [16] is \([\tau_{\text{here}}, e_-] = (\tau + r + \bar{r})_{\text{there}}\).
by: use the Batalin-Vilkovisky formalism, [19, 20]. We introduce ghostfields $c \in \Pi_+\bar{g}$ and anti-fields $J^*_z \in \bar{g}$, $A^*_z \in \Pi_-\bar{g}$, $\tau^* \in \Pi_{1/2}\bar{g}$ and $c^* \in \Pi_-\bar{g}$. The solution to the BV master equation is given by:

$$S_{BV} = S_2 - \frac{1}{2\pi x} \int \text{str} c^* c + \frac{1}{2\pi x} \int \text{str} J_z^* \left( \frac{\kappa}{2} \partial c + [c, J_z] \right) + \frac{1}{2\pi x} \int \text{str} \tau^* c + \frac{1}{2\pi x} \int \text{str} A_z^* \left( \partial \bar{c} + [c, A_z] + \mu \text{-dependent terms} \right). \quad (5)$$

The $\mu$-dependent terms proportional to $A_z^*$ absorb all complications arising from the non-invariance of $T$. Determining them requires an explicit knowledge of the extra terms which were added to the transformation rule of $A_z$. We will not need this here.

We now perform a canonical transformation which changes $A_z^*$ into a field, the antighost $b \in \Pi_-\bar{g}$, and $A_z$ into an antifield $b^*$. This amounts to the gauge choice $A_z = 0$. The gauge-fixed action reads:

$$S_{gf} = \kappa S^-[g] + \frac{\kappa}{4\pi x} \int \text{str} [\tau, e_-] \partial \bar{\tau} + \frac{1}{2\pi x} \int \text{str} b \partial \bar{c} + \frac{1}{4\pi x} \int \text{str} \mu \hat{T}, \quad (6)$$

and the BRST charge is:

$$Q = \frac{1}{4\pi i x} \int \text{str} \left\{ c \left( J_z - \frac{\kappa}{2} e_- - \frac{\kappa}{2} [\tau, e_-] + \frac{1}{2} J_z^g \right) \right\}, \quad (7)$$

where $J_z^g = \frac{1}{2} \{b, c\}$. It is nilpotent. The total current $\hat{J}_z = J_z + J_z^g$ satisfies the same operator algebra as $J_z$, except for the central extension. If we write $\hat{J}_z = t^A \hat{J}_{zA}$, $\Pi_-\hat{J}_z = t^a \hat{J}_{za}$ and $(1 - \Pi_-)\hat{J}_z = t^a \hat{J}_{za}$, we find

$$\hat{J}_{za}(x)\hat{J}_{zb}(y) = (x - y)^{-1} f_{ab}^c \hat{J}_{zc}(y)$$

$$\hat{J}_{za}(x)\hat{J}_{zb}(y) = \left( -\frac{\kappa}{2} g_{ab} + (-)^{(c)} f_{ca}^d f_{db}^e \right) (x - y)^{-2} + (x - y)^{-1} f_{ab}^c \hat{J}_{zc}(y)$$

$$\hat{J}_{za}(x)\hat{J}_{zb}(y) = \left( -\frac{\kappa}{2} g_{ab} + (-)^{(c)} f_{ca}^d f_{db}^e \right) (x - y)^{-2} + (x - y)^{-1} f_{ab}^e \hat{J}_{ze}(y). \quad (8)$$

The only unknown in the action is the current $\hat{T}$. This again reflects the fact that we did not specify the explicit form of the $\mu$ dependent terms in eq. (3). For $\mu = 0$ the action is BRST invariant. In order to guarantee BRST invariance for $\mu \neq 0$, the currents $\hat{T}$ themselves have to be BRST invariant. This determines them up to BRST exact pieces.
We now study the BRST cohomology in detail. Our methods are inspired by [8, 21, 4, 22]. However the analysis in [8, 21, 4] is based on the presence of a double complex. As we will see, the \( \tau \) fields obstruct the existence of a double complex. Nevertheless, spectral sequence techniques [22] are still applicable.

Consider the algebra \( \mathcal{A} \) generated by the basic fields \( \{b, \bar{J}_z, \tau, c\} \), which consists of all regularized products of the basic fields and their derivatives modulo the usual relations [23, 24] between different orderings, derivatives, etc. To every field \( \Phi \), we assign a double grading \( [\Phi] = (k, l), k, l \in \frac{1}{2} \mathbb{Z} \), with \( k + l \in \mathbb{Z} \) the ghostnumber: \( [J_z] = (m, -m) \) for \( J_z \in \Pi_m \bar{g}, m \in \frac{1}{2} \mathbb{Z} \), \( [b] = (-m, m - 1) \) for \( b \in \Pi_{-m} \bar{g}, m > 0 \), \( [c] = (m, -m + 1) \) for \( c \in \Pi_m \bar{g}, m > 0 \) and \( [\tau] = (0, 0) \). Through this \( \mathcal{A} \) acquires a double grading:

\[
\mathcal{A} = \bigoplus_{m,n \in \frac{1}{2} \mathbb{Z}} \mathcal{A}_{(m,n)}.
\]

The operator product expansions (OPE) are compatible with the grading.

The idea of spectral sequences is now to compute the cohomology in steps. One starts by working to 'leading order' only in the first component of the grading, and improves on this successively.

The grading splits the BRST charge into three parts \( Q = Q_0 + Q_1 + Q_2 \), with \( [Q_0] = (1, 0) \), \( [Q_1] = (\frac{1}{2}, \frac{1}{2}) \) and \( [Q_2] = (0, 1) \). The operators \( Q_0, Q_1 \) and \( Q_2 \), map \( \mathcal{A}_{(m,n)} \) to \( \mathcal{A}_{(m+1,n)}, \mathcal{A}_{(m+\frac{1}{2},n+\frac{1}{2})} \) and \( \mathcal{A}_{(m,n+1)} \) respectively:

\[
Q_0 = -\frac{\kappa}{8\pi ix} \int \text{str} ce_-
\]
\[
Q_1 = -\frac{\kappa}{8\pi ix} \int \text{str} \tau, e_-
\]

 Nilpotency of \( Q \) implies that \( Q_0^2 = Q_2^2 = \{Q_0, Q_1\} = \{Q_1, Q_2\} = Q_1^2 + \{Q_0, Q_2\} = 0 \), but

\[
Q_1^2 = -\{Q_0, Q_2\} = \frac{\kappa}{32\pi ix} \int \text{str} \left\{ c \left[ \Pi_{1/2} c, e_- \right] \right\},
\]

This does not vanish - this is the obstruction to the existence of a double complex. The action of \( Q_0, Q_1 \) and \( Q_2 \) on the basic fields is given by

\[
Q_0 : \begin{array}{c|c|c|c}
\text{Field} & b & c & \bar{J}_z \\
\hline
\text{Operator} & -\frac{\kappa}{2} e_- & 0 & -\frac{\kappa}{2} [\tau, e_-] \\
\hline
\end{array}
\]
\[
Q_1 : \begin{array}{c|c|c|c}
\text{Field} & b & c & \bar{J}_z \\
\hline
\text{Operator} & -\frac{\kappa}{2} e_- & 0 & -\frac{\kappa}{2} [\tau, e_-] \\
\hline
\end{array}
\]
\[
Q_2 : \begin{array}{c|c|c|c}
\text{Field} & b & c & \bar{J}_z \\
\hline
\text{Operator} & -\frac{\kappa}{4} [\tau, e_-] & 0 & -\frac{\kappa}{2} [\Pi_{1/2} c, \bar{J}_z] + \frac{\kappa}{2} \partial c \\
\hline
\end{array}
\]
\[
\tau \rightarrow 0, \quad \tau \rightarrow \frac{1}{2} \Pi_{1/2} c, \quad \tau \rightarrow 0,
\]

\[\uparrow\text{We use the standard point-splitting regularization: (AB)(z) = \frac{1}{2\pi i} \int dz' (z' - z)^{-1} A(z') B(z).}\]
where $\Pi_- \equiv 1 - \Pi_-$ and $[A, B]$ stands for
\[
[X, Y] = (-)^{(AB)} (X^A Y^B) f_{AB} C t_C, \tag{13}
\]
where $(X^A Y^B)$ is a regularized product. The BRST charge $Q$ acts as a derivation on a regularized product of fields.

To exploit the double grading, we need one more preparation. For a fixed ghost number $k$, the sequence of gradings $(m, k - m)$ is unbounded because the first component is negative for some currents and for the $b$-field. The application of spectral sequence techniques requires this set to be finite. This is remedied by noticing that the subcomplex $\mathcal{A}^{(1)}$, generated by $\{b, \Pi_- \hat{J}_2 - \frac{1}{2} [\tau, e_-] \}$, has a trivial cohomology $H^*(\mathcal{A}^{(1)}; Q) = C$. From this and eq. (12) we find that the double grading on $\mathcal{A}^{(1)}$, generated by $\{\Pi_- \hat{J}_2, \tau, c \}$, as $H^*(\mathcal{A}) = H^*(\mathcal{A}_0)$. The OPEs close on the reduced complex. The double grading on $\mathcal{A}$ induces a double grading on $\mathcal{A}$: $\hat{\mathcal{A}}^m \equiv \bigoplus_{k \in \frac{1}{2} \mathbb{Z}} \mathcal{A}_{(k,l)}$. Now leads to a spectral sequence $(E_r, d_r)$, $r \geq 1$, converging to $H^*(\mathcal{A}; Q)$. Each term in the sequence is the cohomology of the previous term with a derivation that represents the effective action of the BRST operator at that level: $E_r = H^*(E_{r-1}; d_{r-1})$. To start with, this means that one neglects terms that are one half unit lower in the first grading component. The first term in the sequence is then $E_0 = \hat{\mathcal{A}}$, $d_0 = Q_0$.

The next term is $E_1 = H^*(\mathcal{A}; Q_0)$. The derivation operator at this level will act like $Q$ up to terms that have the first grading component at least one unit lower. Thus on $E_1$, the $Q_0$ cohomology classes, it acts like $Q_1$, i.e. if $\Phi \in \hat{\mathcal{A}}, [\Phi] \in E_1, d_1[\Phi] = [Q_1] \Phi$. Note that although $Q_1^2 \neq 0$, nevertheless $d_1^2 = 0$, since $Q_1^2 = -Q_0 Q_2 - Q_2 Q_0$ implies that on the $Q_0$ cohomology classes it always results in the trivial class.

The next term is then $E_2 = H^*(E_1; d_1)$. The $d_2$ derivation can be computed as follows. Let $\Phi$ represent some class $[[\Phi]] \in E_2$. Then $Q_0(\Phi) = 0$, and $Q_1(\Phi) = Q_0(\Psi)$ for some $\Psi \in \hat{\mathcal{A}}$. The action of $d_2$ is $d_2[[\Phi]] = [[\Phi_2 \equiv Q_2(\Phi) - Q_2(\Psi)]]$. One may check that this is a proper operation in $E_2$, since $Q_0(\Phi_2) = 0$ and $Q_1(\Phi_2) = Q_0(\Phi_2)$. It is equally trivial to verify that $d_2^2 = 0$. The $d_2$-cohomology gives the next term in the sequence, and so on, until the sequence stabilizes (i.e. the derivation operator vanishes).

Applying this to $H^*(\mathcal{A}; Q)$, using eq. (12), one finds that $E_1$ is represented by
\[
E_1 \simeq \hat{\mathcal{A}} \left[ \Pi_{\text{ker ade}} J_2 \right] \otimes \hat{\mathcal{A}}[\tau] \otimes \hat{\mathcal{A}} \left[ \Pi_{\frac{1}{2} c} \right], \tag{15}
\]

\footnote{From now on $J_z$ stands for $\Pi_- J_z$.}
where we denoted the subalgebra of $\hat{A}$ generated by $\Phi$ by $\hat{A}[\Phi]$. Using eq. (12) again, one finds that $E_2$ is represented by

$$E_2 \simeq \hat{A} \left[ \Pi_{\ker ad_e} \left( \hat{J}_z + \frac{K}{4} [\tau, [e_-, \tau]] \right) \right],$$

and one has explicitly

$$Q_1 \Pi_{\ker ad_e} \left( \hat{J}_z + \frac{K}{4} [\tau, [e_-, \tau]] \right) = Q_0 \Pi_{\ker ad_e} \left[ \tau, \hat{J}_z \right].$$

(17)

It turns out that $E_2$ has only ghost number zero elements. Therefore, $d_2$ is actually the zero map and the sequence has already stabilized at the previous level. This gives the main result:

$$H^*(\mathcal{A}; Q) \simeq E_2 = H^*(\hat{A}, Q_0, d_1).$$

(18)

Having established the cohomology of $Q$, we now turn to the explicit construction of its generators. The cohomology is generated by $\hat{T} \equiv \sum_{j, \alpha_j} \hat{T}^{(j, \alpha_j)} t_{(jj; \alpha_j)} \in \ker ad_e$ and $\hat{T}^{(j, \alpha_j)}$ has the form

$$\hat{T}^{(j, \alpha_j)} = 2^j \sum_{r=0}^{2j} \hat{T}_r^{(j, \alpha_j)},$$

(19)

where $\hat{T}_r^{(j, \alpha_j)}$ has grading $(j - \frac{r}{2}, -j + \frac{r}{2})$. From the previous discussion, we know that the leading term $\hat{T}_0^{(j, \alpha_j)}$ is of the form

$$\hat{T}_0^{(j, \alpha_j)} = C^j \left\{ j^j_{(jj; \alpha_j)} + \frac{K}{4} \sum_{\alpha_0} \delta_{j,0} \delta_{\alpha_j, \alpha_0} [\tau, [e_-, \tau]]^{(00; \alpha_0)} \right\}$$

(20)

where the normalization constant $C$ will be fixed later on. The remaining terms are recursively determined by a generalized tic-tac-toe construction. We have

$$Q_0 \hat{T}_r^{(j, \alpha_j)} = -Q_1 \hat{T}_{r-1}^{(j, \alpha_j)} - Q_2 \hat{T}_{r-2}^{(j, \alpha_j)},$$

(21)

where $\hat{T}_r^{(j, \alpha_j)} = 0$ for $r < 0$ or $r > 2j$. This determines $\hat{T}_r^{(j, \alpha_j)}$ modulo, the addition of an arbitrary functional of $\tau$. As a functional of $\tau$ only has grading $(0,0)$, it can only appear in $\hat{T}_{2j}^{(j, \alpha_j)}$. It gets determined by the final recursion relation: $Q_1 \hat{T}_{2j}^{(j, \alpha_j)} + Q_2 \hat{T}_{2j-1}^{(j, \alpha_j)} = 0$. As an example we compute $\hat{T}^{(0, \alpha_0)}$ and $\hat{T}^{(1/2, \alpha_{1/2})}$. From eqs. (12) and (17), one immediately gets

$$\hat{T}^{(0, \alpha_0)} = \hat{T}_0^{(0, \alpha_0)} = \hat{j}_z^{(00; \alpha_0)} + \frac{K}{4} [\tau, [e_-, \tau]]^{(00; \alpha_0)}.$$
For $j = 1/2$, one has $\hat{T}^{(1/2,\alpha_1/2)}_0 = \sqrt{C}j_z^{(1/2,1/2;\alpha_1/2)}$. From eq. (17), it follows that $\hat{T}^{(1/2,\alpha_1/2)}_1 = -\sqrt{C}[\tau,\hat{J}_z]^{(1/2,1/2;\alpha_1/2)} + f(\tau)$. The unknown function $f$ gets determined by the next recursion relation $Q_2\hat{T}^{(1/2,\alpha_1/2)}_0 + Q_1\hat{T}^{(1/2,\alpha_1/2)}_1 = 0$: $f(\tau) = \sqrt{C}\left(\frac{\kappa}{6}[\tau, [\tau, e_-]] - \frac{\kappa}{2}\partial\tau - [\Pi_-(t^A), [\Pi_+(t_A), \partial\tau]]\right)$. We have that $\hat{T}^{(1/2,\alpha_1/2)}(\tau,\hat{J}_z) = \hat{T}^{(1/2,\alpha_1/2)}_0 + \hat{T}^{(1/2,\alpha_1/2)}_1$.

The explicit form of all generators $\hat{T}^{(j,\alpha)}$ must obviously be computed on a case by case basis. A useful tool for this is the fact that one can always construct $\hat{J}_z' \in \Pi_{\text{gerade}} \hat{J}_z$. This implies that $c$ can be written as $c = -\frac{2}{3}Q_0(\hat{J}_z') + 2Q_1(\tau)$. Note also that $Q_2(\hat{J}_z)$ can be rewritten as $Q_2(\hat{J}_z) = \frac{1}{4}[c, \mathbb{P}_- \hat{J}_z] - \frac{1}{4}[[\mathbb{P}_- \hat{J}_z, c] + \kappa + \tilde{h}\partial c + \frac{1}{4}[[\Pi_0(t^A), [\Pi_0(t_A), \partial c]]$.

The energy-momentum tensor $\hat{T}^{(1,0)}_\text{EM}$ itself can be computed. It is given by

$$\hat{T}^{\text{EM}} = \frac{\kappa}{x(\kappa + \tilde{h})}\left[\text{str}\{\hat{J}_z e_+\} + \text{str}\{[\tau, e_-]\hat{J}_z\} + \frac{1}{\kappa}\text{str}\left\{\Pi_0(\hat{J}_z)\Pi_0(\hat{J}_z)\right\}\right] + \frac{\tilde{h}}{\kappa}\text{str}\{e_-\partial\hat{J}_z'\}
$$

$$+ \frac{1}{\kappa}\text{str}\left\{[\Pi_0(t^A), [\Pi_0(t_A), \partial\hat{J}_z']\} e_-\right\} - \frac{\kappa + \tilde{h}}{4}\text{str}\{[\tau, e_-]\partial\tau\}.$$  

(23)

The first term is $\hat{T}_0^{(1,0)}$, the second term $\hat{T}_1^{(1,0)}$ and the remainder forms $\hat{T}_2^{(1,0)}$.

It is not hard to verify that $\hat{T}^{\text{IMP}}$,

$$\hat{T}^{\text{IMP}} \equiv \frac{1}{x(\kappa + \tilde{h})}\text{str}\,J_z J_z - \frac{1}{8xy}\text{str}e_0 \partial J_z - \frac{\kappa}{4x} tr\left([\tau, e_-]\partial\tau\right)
$$

$$+ \frac{1}{4x}\text{str}b[e_0, \partial c] - \frac{1}{2x}\text{str}b\partial c + \frac{1}{4x}\text{str}b[e_0, c],$$  

(24)

is also BRST invariant. It differs from $\hat{T}^{\text{EM}}$ by a BRST exact term $Q\left(-\frac{2}{x(\kappa + \tilde{h})}\text{str}b\hat{J}_z + \cdots\right)$.

Eq. (24) is particularly useful to compute the central extension $[\hat{g}]$. One finds:

$$c = \frac{1}{2}c_{\text{crit}} - \frac{(d_B - d_F)\tilde{h}}{\kappa + \tilde{h}} - 6y(\kappa + \tilde{h}),$$  

(25)

where $d_B$, $d_F$ respectively, is the number of bosonic, fermionic respectively, generators of $\hat{g}$, $c_{\text{crit}}$ is the critical value of the central charge for the extension of the Virasoro algebra under consideration:

$$c_{\text{crit}} = \sum_{j, \alpha_j}(-)^{(\alpha_j)}(12j^2 + 12j + 2).$$  

(26)

and $y$ is the index of embedding, which in the case $\tilde{h} \neq 0$ is given by

$$y = \frac{1}{3\tilde{h}}\sum_{j\alpha_j}(-)^{(\alpha_j)}j(j + 1)(2j + 1)$$  

(27)
and \((-)^{(\alpha)} = +1 (-1)\) if \(t_{j(m,\alpha)}\) is bosonic (fermionic). Note that the requirement that \(\hat{T}^{EM}\) generates the Virasoro algebra in the standard normalization fixes the normalization constant \(C\) to be
\[
C = \frac{4y\kappa}{\sqrt{2(\kappa + \tilde{h})}}. \tag{28}
\]
Knowing the leading term of the currents, eq. (20), we find that the conformal dimension of \(\hat{T}_{j(m,\alpha)}\) is given by \(j + 1\).

We already observed that the OPEs close on \(\hat{A}\) and preserve the grading. From this we deduce that the OPEs of the generators \(\hat{T}\) close modulo BRST exact terms. However, we work on the reduced complex \(\hat{A}\), which has no states of negative ghostnumber. So there are no BRST exact terms at ghost number zero. We conclude that the OPEs of the generators \(\hat{T}\) close.

The quantum Miura transformation for the generators of the extended Virasoro algebra is also easily obtained. As the OPEs preserve the grading, the part of \(\hat{T}\) with grading \((0,0)\), \(\hat{T}_{(0,0)}\), closes among themselves. So the map \(\hat{T} \to \hat{T}_{(0,0)}\) is an algebra homomorphism. In order to prove that the map \(\hat{T} \to \hat{T}_{(0,0)}\) is an algebra isomorphism, we have to show that each generator of the extended Virasoro algebra has a non-vanishing component of grading \((0,0)\). This is shown following a reasoning similar to the one in [21]. Consider the mirror of the spectral sequence, \(i.e.\) the one associated to the filtration
\[
\hat{A}^m \equiv \bigoplus_{l \in \frac{1}{2} \mathbb{Z}} \bigoplus_{k \geq m} \hat{A}_{(k,l)}. \tag{29}
\]
We already know that \(E_\infty^r\) for this spectral sequence vanishes unless the ghostnumber is zero. Using eq. (12), one shows that \(E_1 = H^*(\hat{A}; Q_2)\) is only non-vanishing at grading \((\frac{m}{2}, \frac{m}{2})\), \(m \geq 0\). This implies that \(E_\infty\) is only non-vanishing at grading \((0,0)\). This proves that \(\hat{T} \to \hat{T}_{(0,0)}\) is indeed an algebra isomorphism.

Finally, the method of working we described is particularly useful to compute the effective action in the light-cone gauge, \(W[\hat{T}]\), of the corresponding gravity theory, [16]. The effective action is defined by
\[
\exp -W[\hat{T}] = \int [\delta gg^{-1}] [d\tau] [dA_\tau] [d\mu] \left( \text{Vol} (\Pi_+ \tilde{g}) \right)^{-1} \exp - \left( S_2 - \frac{1}{4\pi xy} \int str \mu \tilde{g} \right). \tag{30}
\]
The effective action is most easily computed by making a different gauge choice. We make a canonical transformation in eq. (2) which interchanges fields and anti-fields for \(\{\tau, \tau^*\}\) and \(\{\Pi_+[e_+, J_2], \Pi_-[e_-, J_2^*]\}\). This corresponds to choosing the gauge \(\tau = \Pi_+[e_+, J_2] = 0\). We find
\[
W[\hat{T}] = \kappa e S_-[g] \tag{31}
\]
where $\kappa_c = \kappa + 2\hat{h}$ and we used $[\delta g g^{-1}] = [dJ_z] \exp \left(-2\hat{h} S^{-}[g]\right)$. From eq. (24) we get the level as a function of the central charge:

$$12y\kappa_c = 12y\hat{h} - \left(c - \frac{1}{2} c_{\text{crit}}\right) - \sqrt{\left(c - \frac{1}{2} c_{\text{crit}}\right)^2 - 24(d_B - d_F)\hat{h}y}$$  \hspace{1cm} (32)

Eq. (32) provides an all-order expression for the coupling constant renormalization. The WZW model in eq. (31) is constrained by

$$\partial gg^{-1} + \frac{1}{4xy} \text{str} \left\{ \Pi_{\text{NA}} \left(\partial gg^{-1}\right) \Pi_{\text{NA}} \left(\partial gg^{-1}\right) \right\} e_+ = e_- + \frac{1}{\kappa + \hat{h}} \sum_{j,\alpha} \frac{1}{2^{j-1} y_j} \tilde{T}^{(j\alpha)} t_{(jj,\alpha)},$$  \hspace{1cm} (33)

where $\Pi_{\text{NA}} g$ is the projection on the centralizer of $sl(2)$ in $g$. We also used $J_z = \frac{\alpha}{2}\partial gg^{-1}$ with $\alpha_{\kappa} = \kappa + \hat{h}$. For a detailed discussion on the value of $\alpha_{\kappa}$, see [3] and in particular [25].

The strategy developed in this paper has numerous applications. In particular, the representation theory of the extended Virasoro algebra should be closely related to the representation theory of the corresponding affine Lie algebra.

The approach followed here was closely related to 2D gravity in the light-cone gauge. In order to study e.g. non-critical strings based on some extended Virasoro algebra, one needs a covariant formulation. It is clear from eq. (32) that a minimal $(p, q)$ matter sector is realized in terms of a gauged WZW model with level $\kappa_M = p/q - \hat{h}$ and in order to cancel the conformal anomaly we need a gauge sector based on a WZW model with level $\kappa_M = -p/q - \hat{h}$. The gauge invariant coupling between the matter and the gauge sector is performed in a way somewhat similar to the path followed in [26]. The main problem is to construct the analogue of eq. (4) in which both the left and the right moving extended Virasoro are coupled to sources in a gauge invariant way, i.e. we need an action of the form $\mathcal{S}_2 = \mathcal{S}_1 + \frac{1}{4xy} \int \text{str}(\mu T + \bar{\mu} \bar{T})$. This can be achieved by following a path inspired by the methods developed in [27]. Details about this will be reported on elsewhere [28].

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