HESSIAN OF THE BUSEMANN FUNCTION ON DAMEK-RICCI SPACES

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Abstract. In this note, we calculate the Hessian $H_\theta = \nabla db_\theta$ of the Busemann function $b_\theta$ on a Damek-Ricci space. We investigate eigenvalues of $H_\theta$ and show its positive definiteness.

1. Introduction

A Damek-Ricci space is a Riemannian manifold formed by a one-dimensional extension of a nilpotent Lie group known as the generalized Heisenberg group. This class of manifolds includes the complex hyperbolic space, quaternionic hyperbolic space, and the octonionic hyperbolic plane. As a special case, the real hyperbolic space can also be regarded as a Damek-Ricci space.

A Riemannian manifold $M$ is called a harmonic manifold if the volume density function of any geodesic sphere in $M$ depends only on the radius. A conjecture about harmonic manifolds states that harmonic manifolds are either rank one symmetric spaces or flat Euclidean spaces. This conjecture, known as the Lichnerowicz conjecture, has been proven true in the 4-dimensional case and the compact case. However, the Damek-Ricci spaces provide counterexamples to this conjecture in the non-compact case. For details on the Lichnerowicz conjecture, see [9].

The Busemann function $b_\gamma$, determined by a geodesic $\gamma$, acts like a distance function from infinity, plays a crucial role in the analysis of non-compact spaces. In spaces such as a Hadamard manifold $M$, where an ideal boundary $\partial M$ can be defined, the Busemann function can be regarded as a function on $M$ determined by a boundary point $\theta \in \partial M$, provided that a base point is fixed. The Busemann function is significant for analyzing the asymptotic behavior of geodesics and for understanding the structure of the ideal boundary.

A level hypersurface of a Busemann function is called a horosphere. The shape operator of a horosphere is determined by the Hessian of the Busemann function. Note that while the Hessian of a function is a $(0, 2)$-tensor field, it can be naturally regarded as a $(1, 1)$-tensor field via the metric $g$, as is customary. In rank one symmetric spaces of non-compact type, the Hessian $H_\theta$ of the Busemann function can be expressed as

$$H_\theta = g - db_\theta \otimes db_\theta + \sum_{i=1}^{d} db_\theta \circ J_i \otimes db_\theta \circ J_i,$$

where $\{J_i\}_{i=1,\ldots,d}$ is the associated complex ($d = 1$) or quaternionic ($d = 3$), octonionic ($d = 7$) structure. In the case of real hyperbolic space, the summation term vanishes. This indicates that horospheres are hypersurfaces with constant principal curvatures, and their spectra are common for all horospheres. Does this property characterize rank one symmetric spaces of non-compact type as the only...
non-compact Riemannian manifolds with such features? In \cite{Sato}, we investigated this question and obtained partial results. For example, suppose all horospheres in $M$ are umbilic at every point, and the values of the principal curvatures are the same for all horospheres. In that case, $M$ is isometric to real hyperbolic space \cite{Sato} Theorem 1.1.

In this paper, we compute the Hessian of the Busemann function on Damek-Ricci spaces and analyze the properties of its spectrum.

**Theorem 1.** Let $S$ be a Damek-Ricci space, which is a one-dimensional extension of the generalized Heisenberg group $N = \exp(v \oplus z)$ and $\partial S \cong N \cup \{\infty\}$ be the ideal boundary of $S$. Let $b_\theta$ be the Busemann function on $S$ determined by a boundary point $\theta \in \partial S$ and $H_\theta$ be its Hessian.

1. When $\theta = \infty$, $H_\theta$ has eigenvalues $0, \frac{1}{2}$ and 1 at any point $p \in S$, with multiplicities $1, k = \dim v$ and $m = \dim z$ respectively.

2. When $\theta = (v, z)$ ($\neq \infty$) and $p = (V, Z, a) \in S \cong v \times z \times \mathbb{R}_+$,
   (2-i) if $V = v - V = 0$ or $Z = z - Z - \frac{1}{2}[V, v] = 0$, then $H_\theta$ has eigenvalues $0, \frac{1}{2}$ and 1 at $p$, with multiplicities $1, k$ and $m$, respectively.

   (2-ii) if $V \neq 0$ and $Z \neq 0$, then
   (2-ii-a) The restriction $H_\theta|_{s_4}$ has eigenvalues $0, \frac{1}{2}$ and 1 at $p$, with multiplicities $1, 2$ and 1, respectively. Here $s_4$ is a subspace of $T_p S$ spaned by $\{V, JZV, Z, A\}$.

   (2-ii-b) The eigenvalues $\lambda$ of $H_\theta|_{s_4^1}$ are solutions of the following cubic equation
   $$\left(\lambda - \frac{1}{2}\right)^2 (\lambda - 1) = -(\mu + 1) \frac{\sigma^2|V|^4|Z|^2}{8F^3}$$
   (1)

   and these are all positive. Here $\mu$ is an eigenvalue of the symmetric operator $K^2$ (see Section 2.2 for the definition), and $F$ is a positive function on $S \times N$ (see Section 3.1.2 for the definition).

This theorem shows that if all horospheres in a Damek-Ricci space $S$ have constant principal curvatures, then $S$ is a rank one symmetric space of non-compact type.

**Remark 2.** We note that the foundational aspects of the Hessian computation in \cite{Sato} are based on earlier results from a preliminary version of this work. In \cite{Sato}, the focus is on the spectral analysis of the Hessian in connection with the rank of geodesics and the visibility axiom, utilizing a slightly different framework and notation. In contrast, this paper revisits the Hessian’s spectral properties from a broader perspective, further clarifying the eigenvalue structure and its implications in the geometry of Damek-Ricci spaces. These differences in approach provide complementary insights into the problem.

The paper is organized as follows. In Section 2, we provide the necessary preliminaries, including a review of generalized Heisenberg groups, the operator $K$, and the definition of Damek-Ricci spaces. Section 3 focuses on the Hessian $H_\theta$ of the Busemann function on Damek-Ricci spaces, where we analyze its components and eigenvalues in detail for both $\theta = \infty$ and $\theta \neq \infty$. The proof of the main theorem, Theorem 1, is presented here, including the spectral properties of $H_\theta$. 
2. Preliminaries

2.1. Generalized Heisenberg groups. Let $(\mathfrak{n},[\cdot,\cdot]_n)$ denote a 2-step nilpotent Lie algebra with a positive definite inner product $\langle\cdot,\cdot\rangle_n$. Let $\mathfrak{z}$ denote the center of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement. For $Z \in \mathfrak{z}$, we define a skew-symmetric linear map $J_Z : \mathfrak{v} \to \mathfrak{v}$ by $\langle J_Z V_1, V_2 \rangle_n = \langle Z, [V_1, V_2]_n \rangle_n$ for $V_1, V_2 \in \mathfrak{v}$. If for every $Z \in \mathfrak{z}$, 

$$(J_Z)^2 = -|Z|^2 \text{id}_\mathfrak{v} \quad (2)$$

holds, then we say that $\mathfrak{n} = (\mathfrak{n},[\cdot,\cdot]_n,\langle\cdot,\cdot\rangle_n)$ is a generalized Heisenberg algebra. Here $\text{id}_\mathfrak{v}$ is the identity map on $\mathfrak{v}$.

**Lemma 3** ([3, p.24–25]). Let $(\mathfrak{n},[\cdot,\cdot]_n,\langle\cdot,\cdot\rangle_n)$ be a generalized Heisenberg algebra. For $V, V_i \in \mathfrak{v}$ and $Z, Z_i \in \mathfrak{z}$ ($i = 1, 2$), the following equations hold:

(i) $J_{Z_1} \circ J_{Z_2} + J_{Z_2} \circ J_{Z_1} = -2(Z_1, Z_2)\text{id}_\mathfrak{v}$

(ii) $\langle J_{Z_1} V_1, J_{Z_2} V_2 \rangle + \langle J_{Z_2} V_1, J_{Z_1} V_2 \rangle = 2\langle V_1, V_2 \rangle (Z_1, Z_2)$

(iii) $\langle J_{Z_1} V_1, J_{Z_2} V_2 \rangle = |Z|^2 \langle V_1, V_2 \rangle$

(iv) $\langle J_{Z_1} V, J_{Z_2} V \rangle = |V|^2 (Z_1, Z_2)$

(v) $\langle J_{Z_1} V_1, V_2 \rangle = |V|^2 (Z_1, Z_2)$

(vi) $\langle J_{Z_1} V_1, J_{Z_2} V_2 \rangle = |V|^2 (Z_1, Z_2)$

(vii) $\langle J_{Z_1} V_1, J_{Z_2} V_2 \rangle = |V|^2 (Z_1, Z_2)$

(viii) $\langle V, J_{Z_1} V \rangle = |V|^2 (Z_1, Z_2)$

**Proof.** Using the polarization identity, (i) can be derived from (2). (ii) is obtained from the skew-symmetry of $J_Z$ and (i). In (ii), replacing $Z_1$ and $Z_2$ with $Z$ yields (iii). Similarly, in (ii), replacing $V_1$ and $V_2$ with $V$ yields (iv).

Equations (v) and (i) are equivalent. In (v), replacing $Z$ with $Z_1$, $V_1$ with $V$, and $V_2$ with $J_{Z_2} V$ yields (vi). In (v), replacing $V$ with $J_{Z_1} V$ also yields (vi). Additionally, in (v), replacing both $V_1$ and $V_2$ with $V$ yields (vi).

We denote the formal adjoint map of $\text{ad}(V) = [V, \cdot] : \mathfrak{v} \to \mathfrak{z}$ by $\text{ad}(V)^*$. Then, (v) and (viii) in lemma 3 are written as

$$\text{ad}(V_1) \circ \text{ad}(V_2)^* + \text{ad}(V_2) \circ \text{ad}(V_1)^* = 2\langle V_1, V_2 \rangle \text{id}_\mathfrak{z}, \quad (3)$$

$$\text{ad}(V) \circ \text{ad}(V)^* = |V|^2 \text{id}_\mathfrak{z}. \quad (4)$$

Fix a non-zero vector $V \in \mathfrak{v}$, we obtain an orthogonal decomposition of $\mathfrak{v}$ as follows:

$$\mathfrak{v} = \text{Ker}(\text{ad}(V)) \oplus J_{Z_1} V, \quad J_{Z_1} V := \{J_{Z_1} V | Y \in \mathfrak{z}\}. \quad (5)$$

Since $V \in \text{Ker}(\text{ad}(V))$, $\text{Ker}(\text{ad}(V))$ can be decomposed as the orthogonal direct sum $\mathbb{R}V$ and its orthogonal complement in $\text{Ker}(\text{ad}(V))$, denoted by $\text{Ker}(\text{ad}(V))_0$:

$$\mathfrak{v} = \mathbb{R}V \oplus \text{Ker}(\text{ad}(V))_0 \oplus J_{Z_1} V. \quad (6)$$

Moreover, fix a non-zero vector $Z \in \mathfrak{z}$, we obtain an orthogonal decomposition

$$J_{Z_1} V = \mathbb{R}J_Z V \oplus J_{Z_1} \perp V, \quad \perp = \{Y \in \mathfrak{z} | \langle Z, Y \rangle_n = 0\}. \quad (7)$$

Hence we have

$$\mathfrak{v} = \mathbb{R}V \oplus \mathbb{R}J_Z V \oplus \text{Ker}(\text{ad}(J_Z V))_0 \oplus J_{Z_1} \perp (J_Z V). \quad (8)$$

By using $J_Z V$ instead of $V$, we obtain

$$\mathfrak{v} = \mathbb{R}V \oplus \mathbb{R}J_Z V \oplus \text{Ker}(\text{ad}(J_Z V))_0 \oplus J_{Z_1} (J_Z V). \quad (9)$$

From (8) and (9), we have

$$\mathfrak{v} = \mathbb{R}V \oplus \mathbb{R}J_Z V \oplus \text{span}\{J_{Z_1} V, J_{Z_1} (J_Z V)\} \oplus (\text{Ker}(\text{ad}(V))_0 \cap \text{Ker}(\text{ad}(J_Z V))_0). \quad (10)$$

If $\mathfrak{n}$ satisfies the $J^2$-condition, i.e., for any orthogonal vectors $Z_1, Z_2 \in \mathfrak{z}$, there exists $Z_3 \in \mathfrak{z}$ such that $J_{Z_1} \circ J_{Z_2} = J_{Z_3}$, then we can find that

$$J_{Z_1} V = J_{Z_1} (J_Z V), \quad \text{Ker}(\text{ad}(V))_0 = \text{Ker}(\text{ad}(J_Z V))_0.$$
2.2. The operator $K$ and the map $\overline{K}$. In this section, we choose arbitrary unit vectors $V \in \mathfrak{v}, Z \in \mathfrak{z}$ and fix them.

**Definition 6.** We define an endomorphism $K = K_{V,Z} : Z^\perp \to Z^\perp$ by

$$K(X) := \text{ad}(V) \circ \text{ad}(JZV)^\ast(X) = [V, J_X J_Z V].$$

We can immediately see that $K_{V,Z}$ is skew-symmetric with respect to $\langle \cdot, \cdot \rangle_n$.

The operator $K$ can be interpreted as follows. From the decomposition (8) of $\mathfrak{v}$, we find that for $X \in Z^\perp$ there exist $U \in \text{Ker}(\text{ad}(V))_0$ and $Z' \in Z^\perp$ such that $J_X J_Z V = U + J_Z V$. Then, by using lemma (3) (viii), we have

$$K(X) = [V, J_X J_Z V] = [V, J_Z V + U] = [V, J_Z V] = |V|^2 Z' = Z',$$

from which we have

$$J_X J_Z V = J_{K_{V,Z}(X)} V + U.$$ (11)

Therefore, $J_{K_{V,Z}(X)} V$ is the $J_S V$-part of $J_X J_Z V$.

**Remark 5.** Since $K^2$ is a symmetric operator, the eigenvalues of $K^2$ are all real numbers and it is also known that they lie within $[-1,0]$. Moreover, the eigenspace $L_\mu$, corresponding to a non-zero eigenvalue $\mu$ has even dimension, because if $K^2(X) = \mu X$, then $K^2(K(X)) = \mu K(X)$. See [3] pp. 33–34 for details.

**Definition 6.** We define a linear map $\overline{K}_{V,Z} : \text{Ker}(\text{ad}(V))_0 \to \text{Ker}(\text{ad}(V))_0$.

**Lemma 7.** The operator $K = K_{V,Z}$ and the map $\overline{K} = \overline{K}_{V,Z}$ satisfy

$$\overline{K} \circ \overline{K}^\ast = \text{id}_{Z^\perp} + K^2,$$ (12)

where the asterisk $\ast$ means the formal adjoint map with respect to $\langle \cdot, \cdot \rangle_n$.

**Proof.** At first, we remark that $\text{ad}(JZV)^\ast(Z^\perp)$ is not a subset of $\text{Ker}(\text{ad}(V))_0$. Hence, we have $\overline{K} = \pi_0 \circ \text{ad}(JZV)^\ast$, where $\pi_0$ is the projection from $\mathfrak{v}$ onto $\text{Ker}(\text{ad}(V))_0$. From (11), we have for any $X \in Z^\perp$

$$\overline{K} \circ \overline{K}^\ast(X) = \text{ad}(JZV) \circ \pi_0 \circ \text{ad}(JZV)^\ast(X)$$

$$= \text{ad}(JZV) \circ \pi_0(J_X J_Z V)$$

$$= \text{ad}(JZV)(J_X J_Z V - J_{K_{V,Z}(X)} V)$$

$$= [J_Z V, J_X J_Z V] - [J_Z V, J_{K_{V,Z}(X)} V]$$

$$= |J_Z V|^2 X - [J_{K_{V,Z}(X)} J_Z V, V]$$

$$= X + [V, J_{K_{V,Z}(X)} J_Z V] = X + K^2_{V,Z}(X),$$

from which (12) is obtained. \qed

**Remark 8.**

(i) The subspace $\text{Ker}(\text{ad}(V))_0 \cap \text{Ker}(\text{ad}(JZV))_0$ in (10) is the kernel of $\overline{K}_{V,Z}$.

(ii) If $\mathfrak{n}$ satisfies the $J^2$-condition, then $\overline{K}^2_{V,Z} = 0$. Hence, in this case, $K^2_{V,Z} = -\text{id}_{Z^\perp}$.

2.3. Damek-Ricci spaces. Let $(\mathfrak{n}, [\cdot, \cdot]_n, \langle \cdot, \cdot \rangle_n)$ be a generalized Heisenberg algebra. Let $\mathfrak{s} := \mathfrak{n} \oplus \mathfrak{a}$, where $\mathfrak{a} = RA$ is a one dimensional real vector space with generator $A$, and define a bracket product $[\cdot, \cdot]_a$ and an inner product $\langle \cdot, \cdot \rangle_a$ by

$$[U_1 + Y_1 + s_1 A, U_2 + Y_2 + s_2 A]_a = s_1 U_2 - \frac{s_2}{2} U_1 + s_1 Y_2 - s_2 Y_1 + [U_1, U_2]_a, \quad (13)$$

$$\langle U_1 + Y_1 + s_1 A, U_2 + Y_2 + s_2 A \rangle_a = \langle U_1, U_2 \rangle_n + \langle Y_1, Y_2 \rangle_n + s_1 s_2,$$ (14)

where $U_i \in \mathfrak{v}, Z_i \in \mathfrak{z}, s_i \in \mathbb{R}$ ($i = 1, 2$), respectively. We find immediately that the derived subalgebra $[\mathfrak{s}, \mathfrak{s}]_a$ of $\mathfrak{s}$ is equal to $\mathfrak{n}$. This shows that $\mathfrak{s}$ is a solvable Lie algebra.
Definition 9 ([4]). Let $S$ be a simply connected Lie group whose Lie algebra is $(s, [\cdot, \cdot]_s)$ equipped with the left invariant metric $g$ induced by $\langle \cdot, \cdot \rangle_s$. We call $(S, g)$ a Damek-Ricci space.

If we identify $S$ with $\mathfrak{v} \times \mathfrak{z} \times \mathbb{R}_+$ via the exponential map, the group structure on $S$ is given by

$$
(V_1, Z_1, a_1) \cdot (V_2, Z_2, a_2) = \left( V + \sqrt{a_1} V_2, Z + a_1 Z_2 + \frac{\sqrt{a_1}}{2} [V_1, V_2], a_1 a_2 \right). 
$$

(15)

We refer to [3, Chap. 4] for details.

2.4. The Busemann function. Let $(M, g)$ be a complete, simply connected, non-compact Riemannian manifold with no focal points. Here no focal point means that every geodesic has no focal point as a one dimensional submanifold (see [5]). Then, for a geodesic $\gamma : \mathbb{R} \to M$ with unit speed, we can define a function $b_\gamma$ on $M$, called the Busemann function, by

$$
b_\gamma(x) = \lim_{t \to \infty} (d(\gamma(t), x) - t).
$$

(16)

The Busemann function is a $C^2$-convex function [2]. Then we can define the Hessian $\nabla db_\gamma$ of the Busemann function and find that $\nabla db_\gamma$ is positive semi-definite.

Remark 10. The gradient vector field $\nabla b_\gamma$ of the Busemann function $b_\gamma$ is a unit vector field. Hence, we find that the Hessian of $b_\gamma$ has 0-eigenvalue associated with $\nabla b_\gamma$.

Every Riemannian manifold $M$, such as those described above, carries its ideal boundary, denoted by $\partial M$, which is a quotient space of all geodesic rays on $M$ divided by an equivalence relation $\sim$ defined as follows: $\gamma_1 \sim \gamma_2$ if and only if $d(\gamma_1(t), \gamma_2(t)) < \infty$ on $t \in [0, \infty)$. If $\gamma_1 \sim \gamma_2$, then $b_{\gamma_1} - b_{\gamma_2}$ is a constant function on $M$. Hence, we find that if $\gamma_1 \sim \gamma_2$, then the Hessian of $b_{\gamma_1}$ coincides with that of $b_{\gamma_2}$. Namely, the Hessian of the Busemann function, denoted by $H_{b_\gamma}$, is determined for a boundary point $\theta \in \partial M$.

We refer to [5] for details about the Busemann function.

A Damek-Ricci space $S$ is a Hadamard manifold, i.e., a complete, simply connected, non-compact Riemannian manifold whose sectional curvature is non-positive. It is known that such a space has no focal points (see [5]). The ideal boundary of $S$ is identified with $N \cup \{\infty\}$ (see [1, 7]). Let $b_\theta$ be the Busemann function of a Damek-Ricci space $S$. We refer to a group-theoretic representation of the Busemann function to [7] as follows.

Proposition 11 ([7, Theorem 4]). For $p = (V, Z, a) \in S \simeq \mathfrak{v} \times \mathfrak{z} \times \mathbb{R}$, the Busemann function is represented by

$$
b_\theta(p) = \begin{cases} 
\log \left( \frac{(a + \frac{1}{2} |V|^2 + |Z - \frac{1}{2} [V, V]|^2)}{a \left(1 + \frac{1}{2} |V|^2 + |Z|^2\right)} \right), & \theta = (v, z) \\
-\log a, & \theta = \infty 
\end{cases}
$$

(17)

3. Hessian of the Busemann function on $S$

In this section, we compute the Hessian $H_{b_\theta} = \nabla db_\theta$ of the Busemann function $b_\theta$ on a Damek-Ricci space $S$. We consider both cases: $\theta = \infty$ and $\theta \neq \infty$, and analyze the eigenvalues of $H_{b_\theta}$ in detail. The map $K^2$ and its spectral properties play a key role in our computations.
3.1. Components of $H_\theta$. To proceed, we introduce a global coordinate system for $S \simeq \mathbb{v} \times \mathbb{3} \times \mathbb{R}_+$. Let $\{e_i\}_{i=1, \ldots, k}$ and $\{e_{k+r}\}_{r=1, \ldots, m}$ be orthonormal bases of $\mathbb{v}$ and $\mathbb{3}$, respectively, with $V^i, i = 1, \ldots, k; \ Z^r, r = 1, \ldots, m$ as the corresponding coordinates on $N \simeq \mathbb{v} \times \mathbb{3}$. Here $k = \dim \mathbb{v}$ and $m = \dim \mathbb{3}$. Moreover, we set $a : RA \ni sA \mapsto e^i \in \mathbb{R}_+$. Then, a left-invariant orthonormal frame field of $S$ at $p = (V, Z, a)$ is written as

$$
E_0 = a \frac{\partial}{\partial a}, \quad E_i = \sqrt{a} \frac{\partial}{\partial V^i} - \frac{\sqrt{a}}{2} \sum_{j=1}^k \sum_{r=1}^m A_{ij}^r V^j \frac{\partial}{\partial Z^r}, \quad (i = 1, 2, \ldots, k), \quad (18)
$$

$$
E_{k+r} = a \frac{\partial}{\partial Z^r}, \quad (r = 1, 2, \ldots, m),
$$

where $A_{ij}^r := \langle [e_i, e_j]_n, e_{k+r} \rangle$ (see [3 4.1.5]). Then, we have

$$
\nabla E_i E_j = \frac{1}{2} \sum_r A_{ij}^r E_{k+r} + \frac{1}{2} \delta_{ij} E_0, \quad \nabla E_i E_{k+r} = \nabla E_{k+r} E_i = - \frac{1}{2} \sum_j A_{ij}^r E_j ,
$$

$$
\nabla E_0 E_i = - \frac{1}{2} E_i, \quad \nabla E_{k+r} E_{k+l} = \delta_{rl} E_0, \quad \nabla E_{k+r} E_0 = - E_{k+r}, \quad \nabla E_a E_0 = 0,
$$

where $1 \leq i, j \leq k, 1 \leq r, l \leq m$ and $1 \leq \alpha \leq k + m$ (see [3 4.1.6]).

Now we compute components of $H_\theta = (b_{\alpha, \beta})$ with respect to $\{E_\alpha\}_{\alpha=0,1,\ldots,k+m}$, i.e.,

$$
b_{\alpha, \beta} := \langle \nabla E_\alpha, db_\beta \rangle (E_\beta) = E_\alpha (E_\beta b_\beta) - \langle \nabla E_\alpha, E_\beta \rangle b_\beta.
$$

3.1.1. The case of $\theta = \infty$. From Proposition[11] we immediately obtain the following.

**Theorem 12.** If $\theta = \infty$, then we have

$$
b_{ij} = \frac{1}{2} \delta_{ij}, \quad b_{k+r, k+l} = \delta_{rl}, \quad (\text{otherwise}) = 0.
$$

From this theorem, we obtain Theorem[1](1). This result aligns with the known structure of horospheres in symmetric spaces.

3.1.2. The case of $\theta \neq \infty$. To compute $H_\theta$ in the case $\theta \neq \infty$, it is convenient to define certain maps on $S \times N$ by

$$
\mathbb{V} : S \times N \to \mathbb{v}; \quad \mathbb{V}(p, \theta) = v - V,
$$

$$
\mathbb{Z} : S \times N \to \mathbb{3}; \quad \mathbb{Z}(p, \theta) = z - Z - \frac{1}{2}[V, v]_n,
$$

and positive functions $f, F : S \times N \to \mathbb{R}_+$

$$
f(p, \theta) = a + \frac{1}{4} |\mathbb{V}(p, \theta)|^2, \quad F(p, \theta) = f(p, \theta)^2 + |\mathbb{Z}(p, \theta)|^2,
$$

where $x = (V, Z, a) \in S$ and $\theta = (v, z) \in N \simeq \mathbb{v} \times \mathbb{3}$. Then, we have

$$
b_\theta(p) = \log F(p, \theta) - \log a + C(\theta).
$$

Here $N$ means $\partial S \setminus \{\infty\}$ and $C(\theta)$ is a certain function on $N$. Below, we will omit the variables $(p, \theta)$ and denote simply as $\mathbb{V}, \mathbb{Z}, f$ and $F$.

Easy computation shows that

$$
E_0 f = a, \quad E_i f = - \sqrt{a} (\mathbb{V}, e_i), \quad E_{k+r} f = 0,
$$

$$
E_0 F = 2a f, \quad E_i F = - \sqrt{a} (f \mathbb{V} - J_2 \mathbb{V}, e_i), \quad E_{k+r} F = - 2a (\mathbb{Z}, e_{k+r}),
$$

where $J_2$ represents the orthogonal projector on $\mathbb{Z}$. Finally, we have

$$
H_\theta = (\frac{1}{2} f) f_\theta - C(\theta),
$$

and

$$
\frac{1}{2} f = b_\theta.
$$
from which, by using lemma 3 we have
\[ b_{0,0} = \frac{2a}{F^2} (FF + aF - 2a^2), \quad (22) \]
\[ b_{0,i} = -\frac{\sqrt{a}}{2F^2} \left( (FF + 2aF - 4af^2) \langle V, e_i \rangle + (4af - F) \langle JZ, e_i \rangle \right), \quad (23) \]
\[ b_{0,k+r} = \frac{2a}{F^2} (2af - F) \langle Z, e_{k+r} \rangle, \quad (24) \]
\[ b_{i,j} = \frac{a}{2F} (\langle V, e_i \rangle \langle V, e_j \rangle + \langle [e_i, V]_n, [e_j, V]_n \rangle) \]
\[ - \frac{a}{F^2} (fV - JZV, e_i) (fV - JZV, e_j) + \frac{1}{2} \delta_{ij}, \quad (25) \]
\[ b_{i,k+r} = -\frac{2a\sqrt{a}}{F^2} (fV - JZV, e_i) \langle Z, e_{k+r} \rangle \]
\[ -\frac{\sqrt{a}}{2F} \langle [e_i, f - 2a]V - JZV]_n, e_{k+r} \rangle, \quad (26) \]
\[ b_{k+r,k+l} = -\frac{4a^2}{F^2} \langle Z, e_{k+r} \rangle \langle Z, e_{k+l} \rangle + \frac{1}{F} (F - 2af + 2a^2) \delta_{rl}. \quad (27) \]

**Proposition 13.** Assume that \( \theta \neq \infty \).
(I) If \( V = 0 \) and \( Z = 0 \), then we have
\[ b_{i,j} = \frac{1}{2} \delta_{ij}, \quad b_{k+r,k+l} = \delta_{rl}, \quad (\text{otherwise}) = 0. \]

(II) If \( V = 0 \) and \( Z \neq 0 \), then we have
\[ b_{0,0} = \frac{4a}{F^2} (F - a^2), \quad b_{i,j} = \frac{1}{2} \delta_{ij}, \quad b_{k+r,k+l} = -\frac{4a^2}{F^2} \langle Z, e_{k+r} \rangle \langle Z, e_{k+l} \rangle + \delta_{rl}, \quad (28) \]
\[ b_{0,k+r} = \frac{2a}{F^2} (2a^2 - F) \langle Z, e_{k+r} \rangle, \quad (\text{otherwise}) = 0. \]

(III) If \( V \neq 0 \) and \( Z = 0 \), then we have
\[ b_{0,0} = \frac{2a(f - a)}{F^2}, \quad b_{i,j} = \frac{a}{2F^2} (-\langle V, e_i \rangle \langle V, e_j \rangle + \langle [e_i, V]_n, [e_j, V]_n \rangle) + \frac{1}{2} \delta_{ij}, \]
\[ b_{0,i} = -\frac{\sqrt{a}(f - 2a)}{2F^2} \langle V, e_i \rangle, \quad b_{i,k+r} = \frac{\sqrt{a}(f - 2a)}{2F^2} \langle e_i, J_1 e_{k+r} \rangle \]
\[ b_{k+r,k+l} = \frac{1}{F} (F - 2af + 2a^2) \delta_{rl}, \quad (\text{otherwise}) = 0. \]

**Proof.** From (20), if \( V = 0 \) and \( Z = 0 \), then we have \( f = a \) and \( F = f^2 \), respectively. Substituting these equations into (22) – (27), we obtain our lemma. \( \square \)

From the above proposition, we obtain Theorem 1 (2-i)

**Proof of Theorem 1 (2-i).** (I) Case of \( V = 0 \) and \( Z = 0 \) : Our assertion is trivial from lemma 3 (i).

(II) Case of \( V = 0 \) and \( Z \neq 0 \) : We choose an orthonormal basis \( \{e_{k+r}\}_{r=1,\ldots,m} \) of \( Z \) satisfying
\[ e_{k+1} = \frac{1}{|Z|} Z, \quad \text{and} \quad e_{k+r} \in Z^\perp, r = 2, \ldots, m. \quad (28) \]

Then, we can write the matrix \( (b_{\alpha,\beta}) \), of which some rows and columns are interchanged, as
\[
\begin{pmatrix}
B_1 & O & O \\
O & \frac{1}{2} id_0 & O \\
O & O & id_{Z^\perp}
\end{pmatrix}
\]
where
\[
B_1 = \begin{pmatrix}
    b_{0,0} & b_{0,k+1} \\
    b_{k+1,0} & b_{k+1,k+1}
\end{pmatrix}
= \frac{1}{F^2} \begin{pmatrix}
    2a \alpha^2 (F - a^2) & 2a \beta (2a^2 - F) \sqrt{F - a^2} \\
    2a \beta (2a^2 - F) \sqrt{F - a^2} & (F - 2a^2)^2
\end{pmatrix}.
\]

Easy computation shows that \(\det(B_1) = 0\) and \(\det(B_1 - I_2) = 0\) from which we find that eigenvalues of the \(2 \times 2\)-matrix \(B_1\) are 0 and 1. Hence, in this case the matrix \((b_{0,0})\) has eigenvalues \(0, \frac{1}{2} \) and 1 whose multiplicities are 1, \(k\) and \(1 + (m - 1) = m\), respectively.

(III) Case \(\mathcal{V} \neq 0\) and \(\mathcal{Z} = 0\) : We choose an orthonormal basis of \(\mathfrak{v}\) satisfying
\[
\begin{align*}
    e_1 &= \mathcal{V} = \frac{1}{|\mathcal{V}|} \mathcal{V}, \\
    e_i &\in \ker(\text{ad}(\mathcal{V}))_0, \\
    e_{(k-m)+r} &= J_{e_k+} \mathcal{V} = \text{ad}(\mathcal{V})^r(e_{k+r}) \in J_3 \mathcal{V}, \quad r = 2, \ldots, m - k,
\end{align*}
\]

We remark that in this case \(e_{k+r} = \text{ad}(\mathcal{V})(e_{(k-m)+r})\). Then, we can write the matrix \(H_\theta = (b_{0,0})\) of which some rows and columns are interchanged, as
\[
\begin{pmatrix}
    B_2 & O & O \\
    O & \frac{1}{2} \text{id}_{\ker(\text{ad}(\mathcal{V}))_0} & O \\
    O & O & B_3
\end{pmatrix},
\]
where
\[
B_2 = \begin{pmatrix}
    b_{0,0} & b_{0,1} \\
    b_{1,0} & b_{1,1}
\end{pmatrix}
= \frac{1}{F^2} \begin{pmatrix}
    2a(f - a) & -\sqrt{a(f - a)(f - 2a)} \\
    -\sqrt{a(f - a)(f - 2a)} & \frac{1}{2} (f - 2a)^2
\end{pmatrix},
\]
and
\[
B_3 = \begin{pmatrix}
    (b_{k-m+r,k-m+r})_{1,1} & (b_{k-m+r,k+m+r})_{1,1} \\
    (b_{k-r,k+m+r})_{1,1} & (b_{k+r,k+m+r})_{1,1}
\end{pmatrix}
= \frac{1}{F^2} \begin{pmatrix}
    2a(f - a) + \frac{1}{2} f^2 & \sqrt{a(f - a)(f - 2a)} \text{id}_{2\mathfrak{z}} \\
    -\sqrt{a(f - a)(f - 2a)} & f^2 - 2a f - 2a^2 \text{id}_{2\mathfrak{z}}
\end{pmatrix}.
\]

Easy computation shows that \(\det(B_2) = 0\) and \(\det(B_2 - \frac{1}{2} I_2) = 0\) from which we find that eigenvalues of the \(2 \times 2\)-matrix \(B_2\) are 0 and \(\frac{1}{2}\). On the other hand, by appropriately permuting the rows and columns of \(B_3\), it becomes a block diagonal matrix with the following \(2 \times 2\)-matrix
\[
B_3' = \frac{1}{F^2} \begin{pmatrix}
    2a(f - a) + \frac{1}{2} f^2 & -\sqrt{a(f - a)(f - 2a)} \\
    -\sqrt{a(f - a)(f - 2a)} & f^2 - 2a f - 2a^2
\end{pmatrix}.
\]

It is easy to check that satisfies \(B_3'\) satisfies \(\det(B_3' - I_2) = \det(B_3' - \frac{1}{2} I_2) = 0\), which means that \(B_3\) has eigenvalues \(\frac{1}{2}\) and 1 whose multiplicities are \(m\). Hence, in this case the matrix \((b_{0,0})\) has eigenvalues \(0, \frac{1}{2} \) and 1 whose multiplicities are 1, \(1 + (k - m - 1) + m = k\) and \(m\), respectively.

From these computations, we find that when \(\mathcal{V} = 0\) or \(\mathcal{Z} = 0\), the eigenvalues remain \(0, \frac{1}{2}\), and 1. For the general case, the eigenvalues are solutions of a cubic equation, indicating a richer structure.
3.1.3. The case of \( \theta \neq \infty, V \neq 0 \) and \( Z \neq 0 \). We choose orthonormal basis \( \{ e_i, e_{k+r} \}_{1 \leq i, l \leq k, 1 \leq r \leq m} \) of \( n = v \oplus z \) satisfying (28) and (29).

We set a 4 \( \times \) 4 matrix \( B_0 \) defined by

\[
B_0 = \begin{pmatrix}
  b_{0,0} & b_{0,1} & b_{0,k-m+1} & b_{0,k+1} \\
  b_{1,0} & b_{1,1} & b_{1,k-m+1} & b_{1,k+1} \\
  b_{k-m+1,0} & b_{k-m+1,1} & b_{k-m+1,k-m+1} & b_{k-m+1,k+1} \\
  b_{k+1,0} & b_{k+1,1} & b_{k+1,k-m+1} & b_{k+1,k+1}
\end{pmatrix},
\]

where

\[
\begin{align*}
  b_{0,0} &= \frac{2a}{F^2} (fF + aF - 2af^2), \\
  b_{1,0} &= -b_{k+1,k-m+1} = -\frac{1}{F^2} (fF + 2af - 4af^2) \sqrt{a(f-a)}, \\
  b_{k-m+1,0} &= b_{k+1,1} = -\frac{1}{F^2} (4af - F) \sqrt{a(f-a)}(F-f^2), \\
  b_{k+1,0} &= \frac{2a}{F^2} (2af - F) \sqrt{F-f^2}, \\
  b_{1,1} &= -\frac{2a}{F^2} (f-a)(2f^2 - F) + \frac{1}{2}, \\
  b_{k-m+1,1} &= \frac{4af}{F^2} (f-a) \sqrt{F-f^2}, \\
  b_{k-m+1,k-m+1} &= \frac{1}{2} + \frac{2a}{F^2} (f-a)(2f^2 - F), \\
  b_{k+1,k+1} &= 1 - \frac{2a}{F^2} (fF + aF - 2af^2).
\end{align*}
\]

Theorem 14. The eigenvalues of \( B_0 \) are 0, \( \frac{1}{2} \) and 1 whose multiplicities are 1, 2 and 1, respectively.

Proof. We can compute that

\[
\det(B_0) = \det(B_0 - I_4) = \det \left( B_0 - \frac{1}{2} I_4 \right) = 0 \quad \text{and} \quad \text{tr}(B_0) = 2,
\]

from which we obtain our result. Here \( I_n \) is the \( n \times n \) identity matrix. \( \square \)

The above theorem implies Theorem 11 (2-ii-a).

From (20), for \( 2 \leq i \leq k-m \) and \( 2 \leq r \leq m \), we have

\[
b_{i,k+r} = \frac{\sqrt{a}}{2F} \langle [e_i, J_Z V]_a, e_{k+r} \rangle = -\frac{\sqrt{a}}{2F} |Z|(K_{V,z}(e_i), e_{k+r}).
\]

On the other hand, from (20), for \( 2 \leq r, l \leq m \),

\[
b_{(k-m)+l,k+r} = -\frac{\sqrt{a}}{2F} \langle [e_{k+l}, J_Z V], (f-2a) V - J_Z V ]_a, e_{k+r} \rangle
\]

\[
= -\frac{\sqrt{a}}{2F} \left\{ (f-2a) [J_{e_{k+l}}, \nabla]_a V, e_{k+r} \right\} - \langle [J_{e_{k+l}}, \nabla], J_Z V \rangle_1 e_{k+r} - \langle [\nabla, J_{e_{k+l}}], J_Z V \rangle_1 e_{k+r} + \langle [\nabla, J_Z V], (e_{k+r}, e_{k+r}) \rangle.
\]

Proposition 15. We set

\[
\begin{align*}
  b_1 &= \frac{1}{2} + \frac{2a}{F^2} (f-a) = \frac{1}{2} + \frac{a}{2F} |V|^2, \\
  b_2 &= 1 - \frac{2a}{F^2} (f-a) = 1 - \frac{a}{2F} |V|^2.
\end{align*}
\]
Moreover, $B = (b_{\alpha, \beta})$, of which some rows and columns are interchanged can be described as follows:

$$H_{\theta} = \begin{pmatrix} B_0 & O \\ O & B \end{pmatrix},$$

where

$$B = \begin{pmatrix} \frac{1}{2} \text{id}_{\ker(\text{ad}(\overline{V}))} & O \\ O & -b_4 K_{\overline{V}, \overline{Z}} \\ -b_4 K_{\overline{V}, \overline{Z}} & (b_3 \text{id}_{\overline{Z}^\perp} + b_4 K_{\overline{V}, \overline{Z}}) \circ \text{ad}(\overline{V}) \end{pmatrix}.$$

Moreover, $b_i, i = 1, 2, 3, 4$ satisfy

$$b_1 b_2 - b_3^2 - b_4^2 = 1, \quad b_1 + b_2 = \frac{3}{2}.$$

Here, we examine the eigenvalues and eigenvectors of $B$. Let

$$x = U + J_{Z_1} \overline{V} + Z_2 = \begin{pmatrix} U \\ J_{Z_1} \overline{V} \\ Z_2 \end{pmatrix} \in T_pS$$

be an eigenvector of $B$ corresponding to the eigenvalue $\lambda$, where $U \in \ker(\text{ad}(\overline{V}))_0$, $Z_1, Z_2 \in \mathbb{Z}^\perp$. The equation $(B - \lambda I)x = 0$ is equivalent to the following three equations:

$$\left(\frac{1}{2} - \lambda\right)U - b_4 K_{\overline{V}, \overline{Z}} Z_2 = 0,$$

$$(b_1 - \lambda) J_{Z_1} \overline{V} + \text{ad}(\overline{V})^* \circ (b_3 \text{id}_{\overline{Z}^\perp} - b_4 K_{\overline{V}, \overline{Z}}) Z_2 = 0,$$

$$-b_4 K_{\overline{V}, \overline{Z}} U + (b_3 \text{id}_{\overline{Z}^\perp} + b_4 K_{\overline{V}, \overline{Z}}) Z_1 + (b_2 - \lambda) Z_2 = 0.$$

From these equations, we derive the conditions that $U, Z_1$, and $Z_2$ must satisfy.

By applying $\text{ad}(\overline{V})$ to both sides of (37), we have

$$(b_1 - \lambda) Z_1 + (b_3 \text{id}_{\overline{Z}^\perp} - b_4 K_{\overline{V}, \overline{Z}}) Z_2 = 0,$$

which means that $Z_1$ is expressed by using $Z_2$. By multiplying $(\frac{1}{2} - \lambda) (b_1 - \lambda)$ to (38) and eliminating $U$ and $Z_1$ using (36) and (39), we obtain

$$0 = (b_1 - \lambda) b_4^2 (\text{id}_{\overline{Z}^\perp} + K_{\overline{V}, \overline{Z}}^2) Z_2 - \left(\frac{1}{2} - \lambda\right) \left(b_3^2 \text{id}_{\overline{Z}^\perp} - b_4^2 K_{\overline{V}, \overline{Z}}^2\right) Z_2$$

$$+ \left(\frac{1}{2} - \lambda\right) \lambda^2 - \frac{3}{2} \lambda + b_1 b_2) Z_2$$

$$= (b_1 - \lambda) + \left(\frac{1}{2} - \lambda\right) b_4^2 \left(\text{id}_{\overline{Z}^\perp} + K_{\overline{V}, \overline{Z}}^2\right) Z_2$$

$$= \left\{- (b_1 - \lambda) + \left(\frac{1}{2} - \lambda\right) b_4^2 \left(\text{id}_{\overline{Z}^\perp} + K_{\overline{V}, \overline{Z}}^2\right) Z_2$$
properties of a determinant and spectral analysis of \( K \),

\[
K^2 V Z_2 = - \left( 1 - \frac{(\frac{1}{2} - \lambda)^2 (1 - \lambda)}{b^2_2 (b_1 - \frac{1}{2})} \right) Z_2
\]

i.e.,

\[
K^2 V Z_2 = - \left\{ 1 + \frac{8 F^3}{a^2 |V|^2 |Z|^2} \right\} \left( \lambda - \frac{1}{2} \right)^2 (\lambda - 1) Z_2,
\]

from which we obtain Theorem II (2-ii-b). From the above, the following can be concluded about \( Z_1, Z_2 \in Z^\perp; \)

(i) \( Z_2 \in L_\mu \), where

\[
\mu = - \left\{ 1 + \frac{8 F^3}{a^2 |V|^2 |Z|^2} \right\} \left( \lambda - \frac{1}{2} \right)^2 (\lambda - 1),
\]

(ii) from (39), \( Z_1 \) is determined by \( Z_2 \) and it can be seen that also \( Z_1 \in L_\mu \).

(iii) In addition, when \( \lambda = \frac{1}{2} \), it follows from (39) that \( Z_2 \) is not only in \( L_{-1} \) but also in \( \text{Ker}(K^2 V Z) \).

Remark 16. Since \( \mu \in [-1, 0] \) and \( \frac{F^3}{a^2 |V|^2 |Z|^2} > 1 \), we find that \( 1 \geq \lambda \geq \lambda_0 (0 > 0) \), where \( \lambda_0 \) is the solution of the cubic equation

\[
\left( \lambda - \frac{1}{2} \right)^2 (\lambda - 1) = -\frac{1}{8},
\]

(42)

The condition that \( U \in \text{Ker}(\text{ad}(V)) \) must satisfy depends on whether the value of \( \lambda \) is \( \frac{1}{2} \) or not. In the case \( \lambda = \frac{1}{2} \), substituting (39) into (38), and from (35) and \( K^2 V Z_2 = -Z_2 \), we have \( K^2 V U = 0 \), i.e., \( U \in \text{Ker}(\text{ad}(V)) \). In the case of \( \lambda \neq \frac{1}{2} \), from (39), we find that \( U \) can be expressed using \( Z_2 \) as

\[
U = -\frac{b_1}{\lambda - \frac{1}{2}} K^2 V Z_2.
\]

(43)

Remark 17. From the definition of \( b_1 \), we have \( 0 < b_1 < 1 \), so it is possible that \( \lambda = b_1 \). However, in this case, \( Z_2 = 0 \) follows from equation (39). \( U = 0 \) follows from (35), and \( Z_1 = 0 \) follows from (35). Therefore, \( b_1 \) cannot be an eigenvalue of \( B \).

Remark 18. Since \( H_\theta \) is positive semi-definite, it suffices to show that the determinant of \( B \) is positive to show that all eigenvalues of \( H_\theta \) are positive. From properties of a determinant and spectral analysis of \( K^2 \), we have

\[
\det(B) = \det \begin{pmatrix} \frac{1}{e} & \text{id} & 0 & -b_4 K^2 \\ O & b_1 \text{id} & (b_3 \text{id} + b_4 K) & \text{ad}(V)^* \circ (b_3 \text{id} - b_4 K) \\ -b_4 K^2 & (b_3 \text{id} + b_4 K) & \text{id} & \text{ad}(V)^* \circ (b_3 \text{id} - b_4 K) \\ \frac{1}{e} & O & b_1 \text{id} & -b_4 K^2 \\ O & (b_3 \text{id} + b_4 K) & \text{ad}(V)^* \circ (b_3 \text{id} - b_4 K) & b_2 \text{id} - 2b_4 K^2 K \\ \frac{1}{e} & O & b_1 \text{id} & \text{ad}(V)^* \circ (b_3 \text{id} - b_4 K) \\ O & b_2 \text{id} - 2b_4 K^2 K & \frac{1}{b_4} (b_3 \text{id} + b_4 K) & \text{ad}(V)^* \circ (b_3 \text{id} - b_4 K) \\ O & O & b_2 \text{id} - 2b_4 K^2 K & \frac{1}{b_4} (b_3 \text{id} + b_4 K) \circ (b_3 \text{id} - b_4 K) \end{pmatrix}
\]
\[
= \frac{1}{2^{k-m-1}} \det \left( b_1 b_2 \text{id} - 2 b_1 b_2^2 (\text{id} + K^2) - (b_3^2 \text{id} - b_4^2 K^2) \right)
\]
\[
= \frac{1}{2^{k-m-1}} \det \left( (b_1 b_2 - b_3^2 - b_4^2) \text{id} - 2 b_1 b_2^2 (\text{id} + K^2) + b_4^2 (\text{id} + K^2) \right)
\]
\[
= \frac{1}{2^{k-m-1}} \det \left( \frac{1}{2} \text{id} - (2b_1 - 1)b_2^2 (\text{id} + K^2) \right)
\]
\[
= \frac{1}{2^{k-2}} \det \left( \text{id} - \frac{a^2 |V|^4 |Z|^2}{2F^3} (\text{id} + K^2) \right)
\]

Since the eigenvalues $\mu$ of $K^2$ satisfies $-1 \leq \mu \leq 0$ and $\frac{a^2 |V|^4 |Z|^2}{2F^3} < \frac{1}{2}$ holds, we find that $\det(B) > 0$.

References

[1] J.-P. Anker, E. Damek and C. Yacoub, Spherical analysis on harmonic AN groups, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 23 (1996), pp.643–679.
[2] W. Ballmann, M. Gromov ans V. Schroeder, Manifolds of nonpositive curvature, Progr. Math. 61, Birkhäuser Boston, Inc., Boston, MA, 1985.
[3] J. Berndt, F. Tricerri and L. Vanhecke, Generalized Heisenberg groups and Damek-Ricci harmonic spaces, Lecture Notes in Math. 1598, Springer-Verlag, Berlin, 1995.
[4] E. Damek and F. Ricci, A class of nonsymmetric harmonic Riemannian spaces, Bull. Amer. Math. Soc. 27 (1992), pp.139–142.
[5] N. Innami, Convexity in Riemannian Manifolds without Focal Points, Adv. Stud. Pure Math. 3 (1984), pp.311–332.
[6] M. Itoh, S. Kim, J.H. Park and H. Satoh, Hessian of Busemann functions and rank of Hadamard manifolds, [arXiv:1702.03646] (2017).
[7] M. Itoh and H. Satoh, Information Geometry of Poisson Kernels on Damek-Ricci Spaces, Tokyo J. Math. 33 (2010), pp.129–144.
[8] M. Itoh and H. Satoh, Horospheres and Hyperbolic Spaces, Kyushu J. Math. 67 (2013), pp.309–326.
[9] G. Knieper, A survey on noncompact harmonic and asymptotically harmonic manifolds, In: Aravinda CS, Farrell FT, Lafont J-F, eds. Geometry, Topology, and Dynamics in Negative Curvature. London Mathematical Society Lecture Note Series. Cambridge University Press, 2016, pp.146–197.

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