Operator-Schmidt decompositions
and the Fourier transform,
with applications to the operator-Schmidt
numbers of unitaries

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Abstract
The operator-Schmidt decomposition is useful in quantum information
theory for quantifying the nonlocality of bipartite unitary operations.
We construct a family of unitary operators on $\mathbb{C}^n \otimes \mathbb{C}^n$ whose
operator-Schmidt decompositions are computed using the discrete Fourier
transform. As a corollary, we produce unitaries on $\mathbb{C}^3 \otimes \mathbb{C}^3$ with
operator-Schmidt number $S$ for every $S \in \{1, ..., 9\}$. This corollary was
unexpected, since it contradicted reasonable conjectures of Nielsen et al [Phys.
Rev. A 67 (2003) 052301] based on intuition from a striking result in
the two-qubit case. By the results of Dür, Vidal, and Cirac [Phys. Rev.
Lett. 89 (2002) 057901], who also considered the two-qubit case, our
result implies that there are nine equivalence classes of unitaries on $\mathbb{C}^3 \otimes \mathbb{C}^3$
which are probabilistically interconvertible by (stochastic) local operations
and classical communication. As another corollary, a prescription is
produced for constructing maximally-entangled unitaries from biunimodular
functions. Reversing tact, we state a generalized operator-Schmidt
decomposition of the quantum Fourier transform considered as an operator
$\mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2} \to \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$, with $M_1 M_2 = N_1 N_2$. This decomposition
shows (by Nielsen’s bound) that the communication cost of the QFT re-
mains maximal when a net transfer of qudits is permitted. In an appendix,
a canonical procedure is given for removing basis-dependence for results
and proofs depending on the “magic basis” introduced in [S. Hill and W.
Wootters, “Entanglement of a pair of quantum bits,” Phys Rev. Lett 78
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1 Introduction

This paper addresses some open problems (questions 1-3 below) concerning the operator-Schmidt decomposition [1] (see definition 1 below), which is useful in quantum information theory [2] [3] for quantifying nonlocality of bipartite unitary operations. Our main results are obtained by constructing a family of unitaries on \( \mathbb{C}^N \otimes \mathbb{C}^N \) with computable operator-Schmidt decompositions, a result which should facilitate further study of this decomposition.

Dür, Vidal, and Cirac [4] used operator-Schmidt numbers to determine when there exists a probabilistic \(^1\) simulation of a unitary \( \tilde{U} \) on \( \mathbb{C}^d \otimes \mathbb{C}^d \) with a single application of a given unitary \( U \) on \( \mathbb{C}^d \otimes \mathbb{C}^d \) aided by (stochastic) local operations, classical communication, and ancilla. In particular, they show that this simulation can occur iff \( \text{Sch}(U) \geq \text{Sch}(\tilde{U}) \), where \( \text{Sch}(U) \) is the number of nonzero operator-Schmidt coefficients of \( U \) (see definition 1 below).\(^2\)

Intriguingly, Dür, Vidal, and Cirac observed that a unitary acting on two qubits may have operator-Schmidt number 1, 2, or 4, but not 3.\(^3\) Thus there exist three equivalence classes of two qubit unitary operations under probabilistic local interconversion \([using \,(S)LOCC]\), with the successive classes represented by the identity, CNOT, and SWAP operation, respectively. Their observation followed immediately from the canonical decomposition of two-qubit unitaries\(^4\): \(^4\) any two-qubit unitary operation \( U_{AB} \in \text{SU}(4) \) can be written in the following standard form

\[
U_{AB} = (V_A \otimes W_B) \exp \left( i \sum_{k=1}^{3} \mu_k \sigma_k^A \otimes \sigma_k^B \right) \left( \tilde{V}_A \otimes \tilde{W}_A \right),
\]

(1)

where \( V_A, W_B, \tilde{V}_A, \) and \( \tilde{W}_B \) are local unitaries and where the \( \sigma_k \) are the Pauli operators with \( \sigma_0 = 1 \), and

\[
\pi/4 \geq \mu_1 \geq \mu_2 \geq |\mu_3| \geq 0.
\]

(Since the Schmidt coefficients of \( U_{AB} \) are unaffected by the local \( V_A, ..., \tilde{W}_B \), their claim reduced to a simple calculation of the operator-Schmidt coefficients of the exponential.)

An interesting problem posed by Nielsen et al [5] is to find the allowed operator-Schmidt numbers of unitaries on \( \mathbb{C}^n \otimes \mathbb{C}^m \). Since there is no known generalization of the canonical decomposition (1) to unitaries on \( \mathbb{C}^n \otimes \mathbb{C}^m \) for \( \max(n,m) > 2 \),\(^5\) at present a different method is required to solve this problem. (The \( n = m = 3 \) case is solved below.)

\(^1\)i.e. succeeding with a nonzero probability

\(^2\)Dür et al. note that, for example, entanglement purification is a probabilistic process, so it is natural to consider probabilistic simulation of its component gates.

\(^3\)This fact was rediscovered by Nielsen et al.\(^5\).

\(^4\)Kraus and Cirac have a constructive “magic basis” proof [7]. The invariants of this decomposition were first discovered by Makhlin [8].

\(^5\)An interesting restriction of this open problem is to illuminate the nonlocal structure of maximally-entangled bipartite unitaries. (See definition 2.)
The operator-Schmidt decomposition was introduced by Nielsen [1] in consideration of the following problem of coherent communication complexity:

Suppose Alice has $n_a$ qubits and Bob has $n_b$ qubits, and they wish to perform some general unitary operation $U$ on their $n_a + n_b$ qubits. How many qubits of quantum communication are required to achieve this goal?

Nielsen proved that the minimum number $Q_0(U)$ of such qubits satisfies the following bound [9]:

$$K_{\text{har}}(U) \leq Q_0(U) \leq 2 \min(n_a, n_b),$$  

where the Hartley strength $K_{\text{har}}$ satisfies

$$K_{\text{har}}(U) = \log_2(\text{Sch}(U)),$$

where $\text{Sch}(U)$, defined in definition 1 below, is the number of nonzero operator-Schmidt coefficients of $U$. It was assumed that Alice and Bob have the use of ancilla, but they must separately retain their (modified) data qubits at the end of the computation. The upper bound of (2) is trivial, for Alice could simply send all her bits to Bob and let him send them back, or vice-versa. We emphasize that the communication complexity $Q_0(U)$ is the communication cost of exact computation of one application of $U$. An interesting open problem is to consider the communication cost of approximate computation of $U^\otimes M$, where the error goes to zero in some appropriate sense for large $M$.

Nielsen applied his abstract bound to show that the communication complexity of the quantum Fourier transform is maximal, first in the case of $n_a = n_b$ [9][1], and then (with collaborators) in the case $n_a \leq n_b$ [5], where Alice holds the most-significant qubits. This was extended to $n_a > n_b$ and to arbitrary qudits in [10]. In section 5 we extend this result to the case that a net transfer of data qudits is permitted.

1.1 Results

The main result of this paper is the construction of a family of unitaries on $\mathbb{C}^N \otimes \mathbb{C}^N$ whose Schmidt decompositions are computable using Fourier analysis. Specifically, Theorem 7 gives a set $\{\Phi_{\alpha\beta}\}$ of vectors in a tensor product of two Hilbert spaces of dimension $N$ such that the Schmidt-coefficients of the diagonal operator

$$D = \sum_{\alpha,\beta=0}^{N-1} \lambda(\alpha, \beta) \left| \Phi_{\alpha\beta} \right\rangle \langle \Phi_{\alpha\beta} \right|$$




6It would be very interesting to know if this asymptotic cost for approximate computation depends only on the operator-Schmidt coefficients of $U$. The reader is warned that the entanglement $K_{\text{Sch}}(U)$ of $U : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B}$ considered as an element of the vector-space $\mathcal{B}(\mathcal{A}) \otimes \mathcal{B}(\mathcal{B})$ (see definition 2) was shown by Nielsen et al. [5] not to satisfy the chaining property. In particular, there exist $U, V$ such that $K_{\text{Sch}}(UV) > K_{\text{Sch}}(U) + K_{\text{Sch}}(V)$.

7See section 5 for a precise statement of this problem.
are the nonzero values of $|\hat{\lambda}(\alpha, \beta)|$, where $\hat{\lambda}$ is the discrete Fourier transform.

Furthermore, this paper addresses the following questions concerning the operator-Schmidt decomposition:

1. What operator-Schmidt numbers $S$ occur in unitary operators on $\mathbb{C}^3 \otimes \mathbb{C}^3$?
2. How can one construct maximally-entangled unitaries on $\mathbb{C}^N \otimes \mathbb{C}^N$?
3. Can one generalize the results of [9], [1], [5], and [10] to show that the communication cost of quantum Fourier transform for data shared between two parties remains maximal if a net transfer of data qudits is allowed to occur?

The cases $S \notin \{2, 4\}$ of question 1 are resolved using [3] by considering the cardinalities of the support of Fourier transforms of phase-valued functions $\lambda(\alpha, \beta) = \exp(i\theta_{\alpha, \beta})$. Resolving the remaining cases $S \in \{2, 4\}$ by inspection, Theorem 10 shows that there exists unitaries on $\mathbb{C}^3 \otimes \mathbb{C}^3$ with arbitrary Schmidt number $S \in \{1, \ldots, 9\}$. By the work of Dür, Vidal, and Cirac [4], this result implies that there are nine equivalence classes of unitaries on $\mathbb{C}^3 \otimes \mathbb{C}^3$ which are probabilistically interconvertible by (stochastic) local operations and classical communication.

Using the diagonal operator [3], question 2 is partially answered by existing mathematical studies of biunimodular functions, that is phase-valued functions whose discrete Fourier transforms are also phase-valued. Using a construction of Björck and Saffari [11], uncountably-many maximally entangled unitaries on $\mathbb{C}^N \otimes \mathbb{C}^N$ may be constructed for $N$ divisible by a square. However, it remains to check whether any two of the constructed $A$, $B$ are inequivalent in the sense that

$$A = (U \otimes W) B (X \otimes Y),$$

for local unitaries $U$, $W$, $X$, and $Y$. The problem of how to verify such equivalences completely has not (to our knowledge) been worked out for general $A$ and $B$, and is left as an open problem. (This is related to [12], however.)

Question 3 is answered by computing the generalized operator-Schmidt decomposition of the quantum Fourier transform as a map from $\mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2}$ to $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ and applying a slight modification of Nielsen’s bound.

In the appendix we remark on the “magic basis” of Hill and Wootters [13].

1.2 Definitions and Notation

Definition 1 Let $A$, $A'$, $B$, and $B'$ be finite-dimensional Hilbert spaces, and let $F : A \otimes B \rightarrow A' \otimes B'$ be a nonzero linear transformation. The Hilbert-Schmidt space $B(A \rightarrow A')$ is the Hilbert space of linear transformations from $A$ to $A'$ under the Hilbert-Schmidt inner product

$$\langle C, D \rangle_{B(A \rightarrow A')} = \text{Tr}_{A} C^\dagger D,$$
where
\[ \langle C^\dagger \psi, \phi \rangle_A = \langle \psi, C\phi \rangle_{A'} \]
for all \( \phi \in A \) and \( \psi \in A' \). For simplicity, we define \( B(\mathcal{H}) = B(\mathcal{H} \to \mathcal{H}) \). A **generalized operator-Schmidt decomposition** of \( F \) is a decomposition of the form
\[
F = \sum_{k=1}^{\text{Sch}(F)} \lambda_k A_k \otimes B_k, \quad \lambda_k > 0, \tag{5}
\]
where the \( \{A_k\}_{k=1}^{\text{Sch}(F)} \) and \( \{B_k\}_{k=1}^{\text{Sch}(F)} \) are orthonormal subsets of \( B(\mathcal{A} \to \mathcal{A'}) \) and \( B(\mathcal{B} \to \mathcal{B'}) \), respectively.\(^8\) The quantity \( \text{Sch}(F) \) is the **Schmidt number**, and the \( \lambda_k \) are the Schmidt coefficients. Equation (5) is an **operator-Schmidt decomposition** when restricted to the special case \( \mathcal{A} = \mathcal{A'} \) and \( \mathcal{B} = \mathcal{B'} \).

We remark that the generalized operator-Schmidt decomposition is just a special case of the well-known Schmidt decomposition \(^9\)
\[
\psi = \sum_{k=1}^{\text{Sch}(\psi)} \lambda_k e_k \otimes f_k, \quad \lambda_k > 0 \tag{6}
\]
of a vector \( \psi \in \mathcal{H} \otimes \mathcal{K} \), where the \( \{e_k\} \) and \( \{f_k\} \) are orthonormal. In particular, one sets \( \mathcal{H} = B(\mathcal{A} \to \mathcal{A'}) \), \( \mathcal{K} = B(\mathcal{B} \to \mathcal{B'}) \), and \( \psi = F \in B(\mathcal{A} \otimes \mathcal{B} \to \mathcal{A'} \otimes \mathcal{B'}) \). The decomposition (6) is then obtained by identifying \( B(\mathcal{A} \to \mathcal{A'}) \otimes B(\mathcal{B} \to \mathcal{B'}) \) with \( B(\mathcal{A} \otimes \mathcal{B} \to \mathcal{A'} \otimes \mathcal{B'}) \) under the natural isomorphism.\(^9\) Note that \( \text{Sch}(\psi) \) and the set \( \{\lambda_k\}_{k=1,...,\text{Sch}(\psi)} \) are independent of the choice of decomposition, since they are just the rank and set of singular values of the map \( |f\rangle_{\mathcal{K}} \mapsto \langle \psi| |f\rangle_{\mathcal{K}} : \mathcal{K} \to \mathcal{H}^* \), respectively.\(^{10}\) Furthermore,

**Definition 2** The **Schmidt strength** \( K_{\text{Sch}}(F) \) of \( F : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A'} \otimes \mathcal{B'} \)\(^2\) is the entanglement of \( F \) considered as an element of \( B(\mathcal{A} \to \mathcal{A'}) \otimes B(\mathcal{B} \to \mathcal{B'}) \).\(^{11}\) \( F \) is said to be **maximally entangled** if \( K_{\text{Sch}}(F) \) is maximized or, equivalently, if \( \text{Sch}(F) = \min (\text{dim} (\mathcal{A}) \text{dim} (\mathcal{A'}), \text{dim} (\mathcal{B}) \text{dim} (\mathcal{B'})) \) and all the operator-Schmidt coefficients are equal.

We note that the Schmidt-number condition on maximally-entangled operators implies that they have maximal communication cost by Nielsen’s bound  \(^{2}\) (see also the slight modification, \(^{21}\) below).

\(^8\)But not necessarily bases.

\(^9\)In particular, there exists a unique unitary \( \Xi : B(\mathcal{A} \to \mathcal{A'}) \otimes B(\mathcal{B} \to \mathcal{B'}) \to B(\mathcal{A} \otimes B \to \mathcal{A'} \otimes \mathcal{B'}) \) such that \( \Xi(A \otimes B) = (Af) \otimes (Bg) \) for all \( f \in A \) and \( g \in B \).

\(^{10}\)See definition \(^3\) for the Hilbert-space structure of the dual space \( \mathcal{H}^* \).

\(^{11}\)Hence \( K_{\text{Sch}}(F) = S(\text{Tr}_B(A \to A') \langle F | F \rangle) \), where \( S \) is the von-Neuman entropy \( S(\rho) = -\text{Tr} \rho \log \rho \).
2 Schmidt decompositions given by the Fourier transform

The goal of this section is to construct the family of diagonal operators \( \mathcal{D} \), whose operator-Schmidt coefficients are computed using the discrete Fourier transform. There are two ingredients in this construction:

1. The well-known isomorphism between \( \mathcal{H} \otimes \mathcal{H}^* \) (defined below) and \( B(\mathcal{H}) \), which allows application of the tools of operator theory to the study of bipartite Hilbert spaces.

2. The characterization (up to a phase) of the discrete Fourier transform by its action by conjugation on the Heisenberg-Weyl algebra.

Definition 3 Let \( \mathcal{H} \) be a Hilbert space of dimension \( N \) with inner product\(^{12} \) \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \), and let \( \mathcal{H}^* \) be its dual.\(^ {13} \) Define the natural antilinear map \( f \mapsto \bar{f} : \mathcal{H} \rightarrow \mathcal{H}^* \) by

\[
\bar{\psi} = \langle \psi | \quad (7)
\]

and endow \( \mathcal{H}^* \) with the inner product \( \langle \bar{f}, \bar{g} \rangle_{\mathcal{H}^*} = \langle g, f \rangle_{\mathcal{H}} \). For a linear operator \( A : \mathcal{H} \rightarrow \mathcal{H} \), define the adjoint \( A^\dagger : \mathcal{H} \rightarrow \mathcal{H} \) by

\[
\langle f, Ag \rangle_{\mathcal{H}} = \langle A^\dagger f, g \rangle_{\mathcal{H}}
\]

and the conjugate \( \bar{A} : \mathcal{H}^* \rightarrow \mathcal{H}^* \) by\(^ {14} \)

\[
\bar{A}f = \overline{Af} \quad (8)
\]

The natural isomorphism \( A \mapsto |A\rangle\rangle_{\mathcal{H} \otimes \mathcal{H}^*} : B(\mathcal{H}) \rightarrow \mathcal{H} \otimes \mathcal{H}^* \) is the unitary map satisfying

\[
A = |f\rangle \langle g| \implies |A\rangle\rangle_{\mathcal{H} \otimes \mathcal{H}^*} = f \otimes \bar{g},
\]

for all \( f, g \in \mathcal{H} \), where \( \otimes \) on the right-hand-side is the defining formal Hilbert-space tensor product of \( \mathcal{H} \otimes \mathcal{H}^* \).\(^ {15} \) Let \( \mathbb{Z}_N = \{0, \ldots, N-1\} \) and \( \mathbb{Z}_N^2 = \mathbb{Z}_N \times \mathbb{Z}_N \). The computational basis is denoted by \( \{|j\rangle\rangle_{j \in \mathbb{Z}_N} \subseteq \mathcal{H} \).\(^ {16} \)

The following lemma is a basis-free version of equations 6 and 10 of \cite{15}, with a similar proof:

Lemma 4 Let \( A, B, C \in B(\mathcal{H}) \). Then \( (A \otimes \bar{B}) |C\rangle\rangle_{\mathcal{H} \otimes \mathcal{H}^*} = |ACB^\dagger\rangle\rangle_{\mathcal{H} \otimes \mathcal{H}^*} \). Furthermore, \( |C\rangle\rangle_{\mathcal{H} \otimes \mathcal{H}^*} \) is maximally entangled iff \( C \) is a nonzero scalar multiple of a unitary.

\(^{12} \)We take linear products to be linear in the second argument.

\(^{13} \)The dual space \( \mathcal{H}^* \) is the set of linear functionals \( \ell : \mathcal{H} \rightarrow \mathbb{C} \). In Dirac notation, \( \mathcal{H}^* \) is the space of bras.

\(^{14} \)The suggestive use of bar-notation in \( \bar{f} \) is motivated by the following formulas:

\[
\psi = \sum_k a_k |k\rangle \Rightarrow \bar{\psi} = \sum_k \bar{a}_k |k\rangle \\
A |j\rangle = \sum_k a_{jk} |k\rangle \Rightarrow \bar{A} |j\rangle = \sum_k \bar{a}_{jk} |k\rangle.
\]

\(^{15} \)The double-ket notation goes back to \cite{14}. Equivalently, \( \langle f \otimes \bar{g}| |A\rangle\rangle_{\mathcal{H} \otimes \mathcal{H}^*} = \langle f, Ag \rangle_{\mathcal{H}} \) for all \( f, g \in \mathcal{H} \), where \( \cdot \) is the inner product on \( \mathcal{H} \otimes \mathcal{H}^* \).
The second ingredient in our construction is the following

**Theorem 5 (H. Weyl)** Let $\mathcal{H}$ and $N$ be as in definition 3. Then any irreducible unitary representation of the group generated by the discrete Weyl relations

\[ R^n = I \text{ iff } n \in N\mathbb{Z} \quad (9) \]
\[ T^n = I \text{ iff } n \in N\mathbb{Z} \quad (10) \]
\[ RT = \exp\left(-\frac{2\pi i}{N}\right) TR. \quad (11) \]

is unitarily equivalent to one in which $R$ and $T$ are represented on $\mathcal{H}$ as the right-shift operator and twist operator, respectively:

\[ R \ket{j} = \ket{j + 1 \text{ mod } N}, \ j \in \mathbb{Z}_N \quad (12) \]
\[ T \ket{j} = \exp\left(\frac{2\pi i j}{N}\right) \ket{j}. \quad (13) \]

Furthermore, if $F$ satisfies the associated Fourier relations

\[ FRF^{-1} = T \]
\[ FT F^{-1} = R^{-1} \]

then $F$ will be simultaneously represented (up to a scalar factor $\lambda$) as the discrete Fourier transform:

\[ \langle j | F | k \rangle = \frac{\lambda}{\sqrt{N}} \exp\left(\frac{2\pi i N jk}{N}\right). \]

The first part of the theorem is given in [16]. The second part follows trivially from Schur’s lemma. Weyl considered the representations of the discrete Weyl relations because they are a finite-dimensional analogue of the canonical commutation relation $[P, Q] = -i$ for self-adjoint $P$ and $Q$.

**Definition 6** The discrete Fourier transform of functions on $\mathbb{Z}_N^2$ is given by

\[ \hat{\lambda}(a, b) = \frac{1}{N} \sum_{\alpha, \beta = 0}^{N-1} \exp\left(\frac{2\pi i}{N} (\alpha a + \beta b)\right) \lambda(\alpha, \beta). \]

\[ \text{See [18] for the representations of the infinite-dimensional Weyl relations (due to von Neumann). See [17] for their relationship to the CCR, and for an example (essentially due to Ed Nelson) of an irreducible representation of the CCR on } L^2(\mathbb{R}) \text{ that is not unitarily equivalent to } Q = x, P = -i \frac{d}{dx}. \]
Theorem 7 Take \( R, T \in B(\mathcal{H}) \) to be given by \( \| \mathbb{I}_2 - i \mathbb{I}_3 \| \). Let
\[
\Phi_{\alpha\beta} = N^{-1/2} | T^\alpha R^{-\beta} \rangle \rangle_{\mathcal{H} \otimes \mathcal{H}^*},
\]
for \( \alpha, \beta \in \mathbb{Z}_N \). Then the \( \Phi_{\alpha\beta} \) form a maximally entangled orthonormal basis of \( \mathcal{H} \otimes \mathcal{H}^* \). Furthermore, for an arbitrary function \( \lambda : \mathbb{Z}_N^2 \to \mathbb{C} \), the diagonal operator \( D \)
\[
D = \sum_{\alpha, \beta = 0}^{N-1} \lambda(\alpha, \beta) | \Phi_{\alpha\beta} \rangle \langle \Phi_{\alpha\beta} | : \mathcal{H} \otimes \mathcal{H}^* \to \mathcal{H} \otimes \mathcal{H}^*,
\]
satisfies the relation
\[
D = \frac{1}{N} \sum_{a,b=0}^{N-1} \hat{\lambda}(a,b) \times (R^a T^b) \otimes (\overline{R^a T^b}).
\]
In particular, a Schmidt decomposition of \( D \) is given by
\[
D = \sum_{a,b} \hat{\lambda}(a,b) \times \left( \frac{\hat{\lambda}(a,b)}{\hat{\lambda}(a,b)} \frac{1}{\sqrt{N}} R^a T^b \right) \otimes \left( \frac{1}{\sqrt{N}} R^a T^b \right),
\]
where the summation is over the \( a, b \in \mathbb{Z}_N \) such that \( \hat{\lambda}(a,b) \neq 0 \).

Proof. It was observed by Schwinger [19] that the set \( \{ N^{-1/2} T^\alpha R^{-\beta} \}_{\alpha, \beta \in \mathbb{Z}_N} \) is an orthonormal basis of \( B(\mathcal{H}) \). That the \( \Phi_{\alpha\beta} \) form an orthonormal basis of \( \mathcal{H} \otimes \mathcal{H}^* \) follows by the natural isomorphism. Maximal entanglement follows from the second part of lemma 4.

By lemma 4 and the Weyl relations (11), each \( \Phi_{\alpha\beta} \) is an eigenvector of each \( (R^a T^b) \otimes (R^a T^b) \):
\[
(R^a T^b) \otimes (R^a T^b) \Phi_{\alpha\beta} = N^{-1/2} \left| R^a T^b T^\alpha R^{-\beta} (R^a T^b)^\dagger \rightangle \\
= \exp \left( -\frac{2\pi i}{N} (a\alpha + b\beta) \right) \Phi_{\alpha\beta}.
\]
Since the \( \Phi_{\alpha\beta} \) form an orthonormal basis, (17) becomes
\[
(R^a T^b) \otimes (R^a T^b) = \sum_{\alpha, \beta = 0}^{N-1} \exp \left( -\frac{2\pi i}{N} (a\alpha + b\beta) \right) | \Phi_{\alpha\beta} \rangle \langle \Phi_{\alpha\beta} |.
\]
By the Fourier inversion theorem,
\[
| \Phi_{\alpha\beta} \rangle \langle \Phi_{\alpha\beta} | = \frac{1}{N^2} \sum_{a,b=0}^{N-1} \exp \left( \frac{2\pi i}{N} (a\alpha + b\beta) \right) \ (R^a T^b) \otimes (R^a T^b).
\]
Hence

\[ D = \sum_{\alpha, \beta = 0}^{N-1} \lambda(\alpha, \beta) \left| \Phi_{\alpha \beta} \right\rangle \left\langle \Phi_{\alpha \beta} \right| \]

\[ = \sum_{\alpha, \beta = 0}^{N-1} \lambda(\alpha, \beta) \frac{1}{N^2} \sum_{a, b = 0}^{N-1} \exp\left(\frac{2\pi i}{N} (a\alpha + b\beta)\right) (R^aT^b) \otimes (R^aT^b) \]

\[ = \frac{1}{N} \sum_{a, b = 0}^{N-1} \hat{\lambda}(a, b) (R^aT^b) \otimes (R^aT^b). \]

By the orthonormality of the \( N^{-1/2} R^aT^b \), (16) is a Schmidt decomposition. ■

**Remark 8** By the lemmas used in [7] to prove the canonical decomposition (11) one has the following fact: Up to local unitaries in the sense of (4), for \( N = 2 \) every unitary on \( \mathcal{H} \otimes \mathcal{H}^* \) is of the form (14), even if the \( \Phi_{\alpha \beta} \) are replaced by an arbitrary maximally-entangled basis.

### 3 Application to Schmidt numbers of unitaries.

It this section we produce the allowed Schmidt numbers of unitaries on \( \mathbb{C}^3 \otimes \mathbb{C}^3 \), solving a special case of the problem of Nielsen et al [5] which prompted our investigations here. By Theorem 7 one may produce a unitary of Schmidt number \( S \) from a “unimodular” function \( \lambda : \mathbb{Z}_N^2 \to \{|z| = 1\} \) whose Fourier transform \( \hat{\lambda} \) has support of cardinality \( S \).

**Lemma 9** There exists a function \( \lambda : \mathbb{Z}_3^2 \to \{|z| = 1\} \) such that the support of \( \hat{\lambda} \) has cardinality \( S \) iff \( S \in \{1, 3, 5, 6, 7, 8, 9\} \).

**Proof.** Define \( g_1 \) and \( g_3 : \mathbb{Z}_3 \to \{|z| = 1\} \) by declaring \( g_1 \) identical and choosing and \( g_3 \) such that \( \text{supp} g_3 = \mathbb{Z}_3 \). Then the support of the Fourier transform of \( (g_a \otimes g_b) (j, k) = g_a (j) g_b (k) \) has cardinality \( S = ab \in \{1, 3, 9\} \). For a function \( \lambda : \mathbb{Z}_3^2 \to \mathbb{C} \), let \( \Gamma \lambda \) be the \( 3 \times 3 \) matrix whose \( j, k \) entry is \( \lambda (j, k) \), \( j, k \in \mathbb{Z}_3 \). Setting

\[ \omega = \exp\left(\frac{2\pi i}{3}\right), \]
one has the following table of unimodular $\lambda_S$ such that the support of $\hat{\lambda}_S$ has cardinity $S$:

| $S$ | $\Gamma \lambda_S$ | $\Gamma \hat{\lambda}_S$ |
|-----|---------------------|--------------------------|
| 5   | $\begin{bmatrix} 1 & -1 & 1 \\ \omega & -\omega & \omega^2 \\ \omega^2 & -\omega^2 & \omega \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 1 & \omega^2 & \omega \\ 0 & 1-\omega & 1-\omega^2 \end{bmatrix}$ |
| 6   | $\begin{bmatrix} 1 & 1 & 1 \\ \omega & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 1 & \omega^2 & \omega \\ 2 & 1+\omega & 1+\omega^2 \end{bmatrix}$ |
| 7   | $\begin{bmatrix} 1 & \omega & \omega^2 \\ 1 & -\omega^2 & \omega \\ -1 & \omega^2 & -\omega \end{bmatrix}$ | $\frac{1}{3} \begin{bmatrix} 0 & 0 & 3 \\ -2+2\omega & 1+2\omega & 7+2\omega \\ 2-2\omega & -1-2\omega & -1-2\omega \end{bmatrix}$ |
| 8   | $\begin{bmatrix} \omega & \omega & \omega^2 \\ 1 & -\omega^2 & \omega^2 \\ -1 & 1 & -\omega \end{bmatrix}$ | $\frac{1}{3} \begin{bmatrix} 0 & -3+3\omega & 3 \\ -2+2\omega & 1+2\omega & 4+5\omega \\ -1+\omega & -1-2\omega & -1-2\omega \end{bmatrix}$ |

Now let $P \subseteq \mathbb{Z}_3^2$ have cardinality $S = 2$ or $4$. We claim that there exists a nonzero $v \in \mathbb{Z}_3^2$ such that there exits a unique $x \in P$ such that $x+v \mod 3\mathbb{Z}_2^2 \in P$. For $S = 2$ this fact is trivial. For $S = 4$, by a modular translation and a rotation, one can assume without loss of generality that the points $(0,0)$ and $(0,1)$ are in $P$. But then either $(0,2) \in P$ or there is another adjacent pair $(s,t)$ and $(s,t+1) \in P$. In either case a contradiction follows by inspection.

Now suppose $\lambda$ is unimodular and $\hat{\lambda}$ has cardinality 2 or 4. Let $P$ be the support of $\lambda$ and take $v$ and $x$ to be as in the previous paragraph. Then

$$0 = \delta_{v,0} = (\overline{\lambda}\lambda)(-v \mod 3\mathbb{Z}_2^2)$$

$$= \frac{1}{N} \sum_{w \in \mathbb{Z}_3^2} \overline{\lambda}(v+w \mod 3\mathbb{Z}_2^2) \hat{\lambda}(w)$$

$$= \frac{1}{N} \overline{\lambda}(v+x \mod N\mathbb{Z}_2^2) \hat{\lambda}(x) \neq 0,$$

yielding a contradiction. ■
**Theorem 10** There exist unitary operators on $\mathbb{C}^3 \otimes \mathbb{C}^3$ with Schmidt number $S$, for every $S \in \{1, \ldots, 9\}$.

**Proof.** By Theorem 7 and lemma 9, all that remains is to check that there exist unitaries on $\mathbb{C}^3 \otimes \mathbb{C}^3$ with the Schmidt numbers 2 and 4. Setting

\[ P_1 = \text{diag}(1,0,0), \quad P_2 = \text{diag}(0,1,1), \]

both of the following unitary operators have Schmidt number 2:

\[ U = P_1 \otimes R + P_2 \otimes I \]
\[ V = R \otimes P_1 + I \otimes P_2, \]

where $R$ is given by (12). Furthermore, their product

\[ UV = P_1 R \otimes RP_1 + P_1 \otimes RP_2 + P_2 R \otimes P_1 + P_2 \otimes P_2 \]

has Schmidt number 4, since this is already a Schmidt decomposition, except for normalizations. ■

### 4 A connection between maximally entangled unitaries and biunimodular functions

Theorem 7 gives some insight into the problem of constructing maximally-entangled unitaries on $\mathbb{C}^N \otimes \mathbb{C}^N$. The best-known example of such a unitary is the SWAP operator $f \otimes g \mapsto g \otimes f$ on $\mathbb{C}^N \otimes \mathbb{C}^N$, with Schmidt decomposition

\[ \text{SWAP} = \sum_{j=1}^{N^2} A_j \otimes A_j^*, \quad (18) \]

where $\{A_j\}_{j=1}^{N^2}$ is any orthonormal basis of $B(\mathbb{C}^N)$.

Furthermore, corollary 15 below, shows that the quantum Fourier transform $\mathcal{F}_{M_1 \times M_2 \rightarrow N_1 \times N_2} : \mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2} \rightarrow \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ is maximally-entangled in many cases, including the case where only one species of qudit is present.

Theorem 7 shows that the diagonal operator $D$ on $\mathbb{C}^N \otimes \mathbb{C}^N$ is maximally entangled iff $\lambda : \mathbb{Z}_N^2 \rightarrow \mathbb{C}$ is biunimodular, i.e. both $\lambda$ and $\lambda$ have ranges lying in the circle $\{|z|=1\}$. To characterize the biunimodular functions on $\mathbb{Z}_N^2$ is a generalization of a studied problem of considerable difficulty: to characterize the biunimodular functions on $\mathbb{Z}_N$.

Known examples of biunimodular functions on $\mathbb{Z}_N^2$ come as tensor products $f(x)g(y)$ of biunimodular functions $f$ and $g$ on $\mathbb{Z}_N$. The first examples of biunimodular functions on $\mathbb{Z}_N$ were known to Gauss: for odd $N$ there are the

\[ \langle A_j \otimes A_k^*, \text{SWAP} \rangle_{B(\mathbb{C}^n \otimes \mathbb{C}^n)} = \langle A_j, A_k \rangle_{B(\mathbb{C}^n \otimes \mathbb{C}^n)} = \delta_{jk}, \]

equation (15) is just the coordinate expansion of SWAP in the orthonormal basis $\{A_j \otimes A_k^*\}$ of $B(\mathbb{C}^n \otimes \mathbb{C}^n)$.

Special cases of the general result were given in [11] [10].
biunimodular Gaussians \( g_{N,a,b} : \mathbb{Z}_N \to \mathbb{C} \), for \( a, b \in \mathbb{Z}_N \) with \( a \) coprime to \( N \), given by
\[
g_{N,a,b}(k) = \exp \left( \frac{2\pi i}{N} (ak^2 + bk) \right),
\]
and for even \( N \) one has
\[
g_N(k) = \exp \left( \frac{2\pi i}{N} k^2 \right).
\]
For \( N \) divisible by a square, these Gaussian examples are special cases of the following theorem:

**Theorem 11 (Björck and Saffari [11])** Let \( n^2 \) be the largest square dividing \( N \). If \( n > 1 \) then there exist infinitely many biunimodular functions on \( \mathbb{Z}_N \). In particular, setting \( m = N/n \),

**Case 1:** Either \( n \) is even or \( m \) is odd. An infinite set of biunimodular functions \( f_{\tau,C,\rho} : \mathbb{Z}_N \to \mathbb{C} \) is given by
\[
f_{\tau,C,\rho}(k) = c_h \rho^{\tau(h) + nr(r-1)/2}, \tag{19}
\]
where \( k \) has “mixed-decimal” expansion \( k = nr + h \) (with \( 0 \leq h < n \), \( 0 \leq r < m \)), where \( C = (c_0, ..., c_{n-1}) \) is an arbitrary unimodular sequence of length \( n \), \( \tau \) is any permutation of \( \{0, 1, ..., n-1\} \), and \( \rho \) is any primitive \( m \)th root of unity.

**Case 2:** \( n \) is odd and \( m \) is even. Each function \( g_{\tau,C,\rho} : \mathbb{Z}_N \to \mathbb{C} \) of the following form is biunimodular:
\[
g_{\tau,C,\rho}(k) = z_{k \mod 2} \times f_{\tau,C,\rho}(k \mod (N/2)),
\]
where \( z \) is the sequence \( z = (1, i) \) and where \( f_{\tau,C,\rho} : \mathbb{Z}_{N/2} \to \mathbb{C} \) is a biunimodular function generated using case 1.

For further results on biunimodular functions, see [20], [21], [22], and [23].

5 Generalized Schmidt decomposition of the quantum Fourier transform

In this section, we consider the communication complexity of the bipartite quantum Fourier transform when a net transfer of data is allowed to occur between the two parties, generalizing the decompositions of [11] [15] [10].

**Definition 12** The quantum Fourier transform \( \mathcal{F}_{M_1,M_2 \to N_1,N_2} : \mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2} \to \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \), with \( N = N_1N_2 = M_1M_2 \), is the unitary map satisfying
\[
N_1 \langle j |_{N_2} \langle k |_{M_2} | \ell \rangle_{M_1} | m \rangle_{M_2} = \frac{1}{\sqrt{N}} \exp \left( \frac{2\pi i}{N} (jN_2 + k) (\ell M_2 + m) \right).
\]
The communication cost of a unitary operation $U : \mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2} \rightarrow \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ is given by

$$Q_0 (U) = \min \sum_{d=2}^\infty N_d \log_2 (d),$$

where the minimum is over all protocols to compute $U$ using ancilla, local operations, and the transmission of $N_d$ qudits of dimension $d$, for $d \geq 2$. The communication cost of $U$ is said to be maximal if $Q_0 (U) = \log_2 \min (M_1, N_1, M_2, N_2)$.

This is just the usual quantum Fourier transform, with the data shared by Alice and Bob using mixed-decimals $|\ell\rangle_{M_1} |m\rangle_{M_2} \leftrightarrow |\ell M_2 + m\rangle_{N_1}$ before (and $|j\rangle_{N_1} |k\rangle_{N_2} \leftrightarrow |j N_2 + k\rangle_{N_2}$ after) the computation. We note that the dimensions of Alice and Bob’s local Hilbert spaces change upon each communication of a qudit, although the product of the dimensions remains constant. Furthermore, the trivial bound

$$Q_0 (U) \leq \log_2 \min (M_1 N_1, M_2 N_2), \quad (20)$$

for any $U : \mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2} \rightarrow \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ follows from the fact that Alice could send all her qudits to the Bob, who would perform the computation and send back the required ones (or vice-versa).

We now state the generalized Schmidt decomposition of $F_{M_1, M_2 \rightarrow N_1, N_2}$. A proof and derivation are not included, as they scarcely differ from those in [10].

**Definition 13** Let $\mathbb{Z}_N = \{0, ..., N - 1\}$. The equivalence classes of $\mathbb{Z}_{N_2} \times \mathbb{Z}_{M_2}$ mod $(M_1, N_1)$ consist of all sets of the form

$$C = \{(a + M_1 k_1, b + M_2 k_2) | a \in \mathbb{Z}_{N_2}, b \in \mathbb{Z}_{M_2}, k_1, k_2 \in \mathbb{Z}\} \cap (\mathbb{Z}_{N_2} \times \mathbb{Z}_{M_2})$$

where addition is NOT modular.

Note that we do not consider equivalence classes of $\mathbb{Z}_{N_2} \times \mathbb{Z}_{M_2}$ mod $(N_1, M_1)$: the order of the $N$’s and $M$’s switches.

**Theorem 14** Let $N = N_1 N_2 = M_1 M_2$. Then $F_{M_1, M_2 \rightarrow N_1, N_2}$ has generalized Schmidt decomposition

$$F_{M_1, M_2 \rightarrow N_1, N_2} = \sum_C \lambda_C A_C \otimes B_C,$$

where the summation is over equivalence classes $C$ of $\mathbb{Z}_{N_2} \times \mathbb{Z}_{M_2}$ mod $(M_1, N_1)$, and where

$$\lambda_C = \sqrt{\frac{N_1 M_1 \text{Card} (C)}{N}},$$

$$(A_C)_{jk} = \frac{1}{\sqrt{N_1 M_1}} \exp \left[ \frac{2\pi i}{N} (N_2 M_2 j k + M_2 k \hat{s} + N_2 j \hat{t}) \right], \quad \text{for} \ (\hat{s}, \hat{t}) \in \mathbb{Z},$$

$$(B_C)_{jk} = \frac{1}{\sqrt{\text{Card} (C)}} \times \left\{ \begin{array}{ll} \exp \left( \frac{2\pi i}{N} j k \right) & \text{if} \ (j, k) \in C \\ 0 & \text{otherwise} \end{array} \right..$$
with Card(\(C\)) denoting the cardinality of \(C\). Note that the definition of \(A_C\) is independent of the choice of \((\hat{s}, \hat{t}) \in C\).

**Corollary 15** In all cases

\[
\text{Sch}(\mathcal{F}_{M_1M_2\to N_1N_2}) = \min (M_1N_1, M_2N_2).
\]  

(21)

In particular, the communication cost of \(Q_0(\mathcal{F}_{M_1M_2\to N_1N_2})\) is maximal in all cases. Furthermore, \(\mathcal{F}_{M_1M_2\to N_1N_2}\) is maximally-entangled iff

\[
(M_1 \text{ is a factor of } N_2 \text{ or } M_1 > N_2)
\]

and

\[
(N_1 \text{ is a factor of } M_2 \text{ or } N_1 > M_2).
\]  

(22)

Otherwise \(\mathcal{F}_{M_1M_2\to N_1N_2}\) has at most four distinct Schmidt coefficients (of various multiplicities), taking values of the form

\[
\sqrt{\frac{N_1M_1}{N}a_+b_+},
\]

where we ignore Schmidt coefficients stated as zero, and where

\[
a_+ = \lfloor N_2/M_1 \rfloor, \quad a_- = \lceil N_2/M_1 \rceil
\]

\[
b_+ = \lceil M_2/N_1 \rceil, \quad b_- = \lfloor M_2/N_1 \rfloor.
\]

**Proof.** Equation (21) follows by a simple counting argument. That the communication cost is maximal then follows from a slight modification of the work of Nielsen et al in [5], as follows.\(^{19}\) Replacing the Schmidt decomposition by the generalized Schmidt decomposition in the definition of the Hartley strength [5] and replacing the SWAP operator in section III.B.3 of [5] by communication operators\(^{20}\)

\[
C : (C^{d_1} \otimes C^{d_2}) \otimes C^{d_3} \to C^{d_1} \otimes (C^{d_2} \otimes C^{d_3})
\]

\[
(f \otimes g) \otimes h \mapsto f \otimes (g \otimes h),
\]  

(23)

one immediately obtains the following version of Nielsen’s bound [2]:

\[
K_{\text{hat}}(\mathcal{F}_{M_1M_2\to N_1N_2}) \leq Q_0(\mathcal{F}_{M_1M_2\to N_1N_2}).
\]  

(24)

Hence the left-hand side of (24) equals the right-hand-side of (20), proving the communication cost is maximal, as claimed. The rest of this corollary is trivial. 

\(^{19}\)This idea was mentioned vaguely in footnote 10 of [5] and in footnotes 1 and 6 of [10].

\(^{20}\)As stated in [10], the communication operator has generalized Schmidt-decomposition

\[
C = \sum_{k=1}^{d_2} \sqrt{d_3} A_k \otimes B_k, \text{ where } A_k = d_1^{1/2} \sum_{i=1}^{d_1} |i\rangle \langle ik| \colon C^{d_1} \otimes C^{d_2} \to C^{d_1} \text{ and } B_k = d_3^{1/2} \sum_{i=1}^{d_3} |ki\rangle \langle i| \colon C^{d_3} \to C^{d_2} \otimes C^{d_3}.
\]
6 Appendix: The magic basis, without the basis.

The natural isomorphism $A \mapsto |A\rangle\rangle : B(\mathcal{H}) \to \mathcal{H} \otimes \mathcal{H}^*$ has allowed us application of the tools of operator theory on $B(\mathcal{H})$ to the study of bipartite tensor product spaces in a natural manner. In this spirit we list below the properties of the gradient of the determinant, which we will relate to "conjugation in the magic basis" of Hill and Wootters [13].

**Theorem 16** The determinant is everywhere-differentiable on $B(\mathcal{H})$. In particular, the determinant has a gradient $\mathcal{G} : B(\mathcal{H}) \to B(\mathcal{H})$ such that

$$\frac{d}{dt} \det(A) = \left\langle \frac{\mathcal{G}(A)}{\mathcal{D}(A)} , \frac{dA}{dt} \right\rangle_{B(\mathcal{H})}$$

for differentiable functions $A : \mathbb{R} \to B(\mathcal{H})$. Define the corresponding map $\mathcal{D} : \mathcal{H} \otimes \mathcal{H}^* \to \mathcal{H} \otimes \mathcal{H}^*$ by

$$\mathcal{D} |A\rangle\rangle = |\mathcal{G}(A)\rangle\rangle.$$ 

Taking $\lambda \in \mathbb{C}$, $A, B \in B(\mathcal{H})$, $\psi \in \mathcal{H} \otimes \mathcal{H}^*$, and $N = \dim(\mathcal{H})$, the functions $\mathcal{G}$ and $\mathcal{D}$ have the following properties:

1. $\mathcal{G}$ is the continuous extension of the map $A \mapsto ((\det A)^{-1})^\dagger$ from invertible $A$ to all $A$.

2. $\mathcal{G}(AB) = \mathcal{G}(A)\mathcal{G}(B)$ and $\mathcal{G}(A^\dagger) = (\mathcal{G}(A))^\dagger$. In particular, $\mathcal{G}$ acts independently on the factors of the polar decomposition.

3. $\mathcal{D} ((A \otimes B) \psi) = \left( \mathcal{G}(A) \otimes \mathcal{G}(B) \right) \mathcal{D}(\psi)$. In particular, if $A$ and $B$ are unitary then $\mathcal{D} ((A \otimes B) \psi) = (\det A^\dagger B) (A \otimes B) \mathcal{D}(\psi)$.

4. $\mathcal{D}(\psi) = \alpha \psi$ for some $\alpha \in \mathbb{C}$ iff $\psi$ is maximally entangled or zero. Furthermore, for $N \geq 3$ the maximizers of $\|\mathcal{D}(\psi)\|_{\mathcal{H} \otimes \mathcal{H}^*} / \|\psi\|_{\mathcal{H} \otimes \mathcal{H}^*}$ are precisely the maximally entangled states.

5. Temporarily allowing Schmidt coefficients to vanish, the product of the Schmidt coefficients of $\psi$ is given by $N^{-1} |\langle \psi, \mathcal{D} \psi \rangle|_{\mathcal{H} \otimes \mathcal{H}^*}$.

6. Furthermore, if $N = 2$ then

(a) $\mathcal{G}$ and $\mathcal{D}$ are conjugations, i.e. antunitary maps squaring to the identity.

(b) $\psi$ is separable iff $\langle \psi, \mathcal{D} \psi \rangle = 0$.

(c) Denote $|ij\rangle \equiv |i\rangle \otimes |j\rangle \in \mathcal{H} \otimes \mathcal{H}^*$. Then each of the following vectors are invariant under $\mathcal{D}$:

$$\{|00\rangle + |11\rangle , i\langle 00| - i\langle 11| , i\langle 01| + i\langle 10| , |0\rangle - |1\rangle\}.$$ 

Furthermore, they form an orthonormal basis.
(d) If $A = e^{i\theta}UP$, where $U \in SU(2)$ and $P = \text{diag}(\lambda_1, \lambda_2)$ is positive, then $\mathcal{G}(A) = e^{-i\theta}U \text{diag}(\lambda_2, \lambda_1)$. In particular, $\mathcal{D}$ preserves Schmidt coefficients.

For $N = 2$, $\mathcal{D}$ is an analogue of conjugation of coordinates in the so-called “magic basis” of [13], which is recovered by simply removing the bars from (25). We note that Vollbrecht and Werner [24] make the following point:

The remarkable properties of the [magic] basis...are in some sense not so much a property of that basis, but of the antiunitary operation of complex conjugation in [that] basis.

In particular, one may canonically translate the results and the magic-basis or magic-conjugation proofs of [13] [26] [8] [7] on $\mathbb{C}^2 \otimes \mathbb{C}^2$ into basis-free results and proofs on $\mathcal{H} \otimes \mathcal{H}^*$. Hence it is apparent that choice of a basis (or of a basis-dependent conjugation) is necessary in the cited proofs because choice is necessary to select an isomorphism between $\mathcal{H} \otimes \mathcal{H}$ and $\mathcal{H} \otimes \mathcal{H}^*$.

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21 For $N = 2$ the properties [3] [5] [6a] and [6b] are just the $\mathcal{H} \otimes \mathcal{H}^*$ analogues of the useful properites of the magic basis.
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