SYMMETRIES OF PDEs SYSTEMS IN SOLAR PHYSICS AND CONTACT GEOMETRY

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Abstract

One considers a class of PDEs systems (1) equivalent to (8) and one determines the PDEs system (10) which defines the associated symmetry group. Particularly, for the Blair system (2)+(3) equivalent to (14), one finds the symmetry Lie group $G$ (Theorem 5). One proves that the class of PDEs systems in the form $(8)+(14')$ which are invariant with respect to the Lie group $G$, is reduced to the Blair system (Theorem 6). One finds new solutions $B_2^{(2)}$ and $B_2^{(3)}$ of the Blair system, and thus new "force-free" model of solar physics.

Key-words: Blair system, infinitesimal symmetries, symmetry group, group-invariant solutions.

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Physics Subject Classification: 96.60.H

1 Introduction

The classical method for finding the symmetries of PDEs reduces to the Lie group method of infinitesimal transformations. The classical symmetries of PDEs are related to vector fields which are functions of the independent and dependent variables. A symmetry group is a Lie group which transforms the solutions of the system into itself, thus it provides a mean of classifying different symmetry classes of solutions, where two solutions are considered to be equivalent if one can be transformed into the other by the same of the group element. The infinitesimal generator is a vector field on the underlying manifold which determines a flow (1-parameter group of transformations). One can regard the entire group of symmetries as being generated in this manner by the composition of the basic flows of its infinitesimal generators. In [6] G.Bluman and J.D.Cole introduced the algorithm of finding the solutions of the system by using the transformations of symmetry. A modern presentation of this theory can be found for example in [12], [15], [19]. There are many applications...
of this theory in the study of the PDEs systems which arises of mathematics, mechanics and physics, for example [1]-[4], [6]-[19].

The solutions of the vector equations

(1) \( \text{curl } B = f \cdot B \)

and

(2) \( \text{div } B = 0, \)

where \( f \) is an arbitrary function, and

\[
B = u(x, y, z) \frac{\partial}{\partial x} + v(x, y, z) \frac{\partial}{\partial y} + w(x, y, z) \frac{\partial}{\partial z},
\]

is a differentiable vector field on a simply connected domain \( D \subset \mathbb{R}^3 \), define "force-free" model of solar physics. On the other hand, the vector equation

(3) \( \text{curl } B = |B| \cdot B, \)

was introduced by Blair in [5]: a solution of it gives a conformally flat contact metric structure on \( \mathbb{R}^3 \). The vector fields which satisfy the vector equation (3) are

\[
B_1 = \sin z \frac{\partial}{\partial x} + \cos z \frac{\partial}{\partial y},
\]

and

\[
B_2 = \frac{8(xz - y)}{(1 + x^2 + y^2 + z^2)^2} \frac{\partial}{\partial x} + \frac{8(x + yz)}{(1 + x^2 + y^2 + z^2)^2} \frac{\partial}{\partial y} + \frac{4(1 + z^2 - x^2 - y^2)}{(1 + x^2 + y^2 + z^2)^2} \frac{\partial}{\partial z},
\]

and \( B_1 \) is a solenoidal vector field. In the cylindrical coordinates, by using the method of sucsecutive aproximations, Blair gives a solution of the equation (3) for which (2) is also satisfied, thus a new solution for the "force-free" model equations. The finding of the PDEs system (2)+(3) solutions is still an open problem.

In this paper we shall determine infinitesimal symmetries associated to the PDEs system (1) for \( f = f(u, v, w) \) (Theorem 3) and in the particular case for \( f = |B| \) (Theorem 4). We call Blair system the PDEs system (2)+(3). Also we shall find the symmetry group \( G \) associated to it (Theorem 5). We shall prove that the only PDEs system (1)+(2) which is invariant with respect to the group \( G \) is the Blair system (Theorem 6). We shall prove that the known solutions are group-invariant solutions and by using the symmetry group \( G \) one finds new solutions of it: \( B^{(2)}_1 \) and \( B^{(3)}_1 \).

We make the remarks that there are many computational symbolic programs for finding the defining system of infinitesimal symmetries and ours calculus was verified by using the Head’s program LIE [10]. This paper gives a new point of
view for the PDEs systems which appear in the solar physics and as well in the contact geometry.

We adopt the notations of the Olver’s book [12]. We start to make a short presentation of the symmetry group theory in the general case of the PDEs system.

2 Symmetry group for PDEs system

Let us consider the PDEs system

\[ \Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \ldots, l, \]

with \( x = (x^1, \ldots, x^p) \) the independent variables, \( u = (u^1, \ldots, u^q) \) the dependent variables and

\[ \Delta(x, u^{(n)}) = (\Delta_1(x, u^{(n)}), \ldots, \Delta_l(x, u^{(n)})), \]

where we denote \( u^{(n)} \) all the partial derivatives of the function \( u \) to 0 up \( n \). Any function \( u = h(x) \),

\[ h : D \subset \mathbb{R}^p \to U \subset \mathbb{R}^q, \quad h = (h^1, \ldots, h^q), \]

induces the function

\[ pr^{(n)}h : D \to U^{(n)}, \]

called the \( n \)-th prolongation of \( h \), which for each \( x \in D \), \( pr^{(n)}h(x) \) is a vector whose \( q(p+n) \) entries represent the values of \( h \) and all its derivatives up to order \( n \) at the point \( x \).

The total space \( D \times U^{(n)} \), whose coordinates represent the independent variables, the dependent variables and the derivatives of the dependent variables up to order \( n \), is called the \( n \)-th order jet space of the underlying space \( D \times U \).

Thus \( \Delta \) is a map from the jet space \( D \times U^{(n)} \) to \( \mathbb{R}^l \). The PDEs system (4) determines a subvariety

\[ S = \{(x, u^{(n)}) | \Delta(x, u^{(n)}) = 0\} \]

of the total jet space \( D \times U^{(n)} \). One identifies the PDEs system (4) with its corresponding subvariety \( S \).

Let \( M \subset D \times U \) be an open set. We suppose \( X \) is a vector field on \( M \), with corresponding local 1-parameter group \( \exp(\varepsilon X) \). The \( n \)-th prolongation of \( X \), denoted by \( pr^{(n)}X \), is a vector field on the \( n \)-jet space \( M^{(n)} \), and is defined to be the infinitesimal generator of the corresponding prolonged 1-parameter group \( pr^{(n)}[\exp(\varepsilon X)] \):

\[ pr^{(n)}X_{(x, u^{(n)})} = \frac{d}{d\varepsilon} pr^{(n)}[\exp(\varepsilon X)](x, u^{(n)})|_{\varepsilon=0} \]

for any \( (x, u^{(n)}) \in M^{(n)} \)
The PDEs system (4) is called to be of maximal rank if the Jacobian matrix
\[ J_\Delta(x, u^{(n)}) = \left( \frac{\partial \Delta_\nu}{\partial x^i}, \frac{\partial \Delta_\nu}{\partial u^\alpha_j} \right) \]
of \( \Delta \), with respect to all the variables \((x, u^{(n)})\), is of rank \( l \) whenever
\[ \Delta(x, u^{(n)}) = 0. \]

**Theorem 1.** Let
\[ X = \sum_{i=1}^{p} \zeta^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \Phi^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \]
be a vector field on open set \( M \subset D \times U \). The \( n \)-th prolongation of \( X \) is the vector field
\[ \text{pr}^{(n)} X = X + \sum_{\alpha=1}^{q} \sum_{J} \Phi_{J}^\alpha(x, u^{(n)}) \frac{\partial}{\partial u^\alpha_{J,i}}, \]
defined on the corresponding jet space \( M^{(n)} \subset D \times U^{(n)} \), the second summation being over all multi-indices \( J = (j_1, ..., j_k) \) with \( 1 \leq j_k \leq p \), \( 1 \leq k \leq n \). The coefficient functions \( \Phi_{J}^\alpha \) of \( \text{pr}^{(n)} X \) are given by the following formula
\[ \Phi_{J}^\alpha(x, u^{(n)}) = D_J \left( \phi_\alpha - \sum_{i=1}^{p} \zeta^i u^\alpha_i \right) + \sum_{i=1}^{p} \zeta^i u^\alpha_{J,i}, \]
where \( u^\alpha_i = \frac{\partial u^\alpha}{\partial x^i} \), \( u^\alpha_{J,i} = \frac{\partial u^\alpha_{J,i}}{\partial x^i} \).

**Theorem 2.** (Infinitesimal criterion of invariance): Let us consider the PDEs system (4) of maximal rank defined over \( M \subset D \times U \). If \( G \) is a local group of transformations acting on \( M \), and
\[ \text{pr}^{(n)} X[\Delta_\nu(x, u^{(n)})] = 0, \quad \nu = 1, ..., l, \]
whenever \( \Delta_\nu(x, u^{(n)}) = 0 \), for every infinitesimal generator \( X \) of \( G \), then \( G \) is a symmetry group of the system.

**Proposition 1.** If the PDEs system (4) defined on \( M \subset D \times U \) is of maximal rank, then the set of infinitesimal symmetries of the PDEs system forms a Lie algebra on \( M \). Moreover, if this algebra is finite-dimensional, then the symmetry group of PDEs system is a Lie group of local transformations on \( M \).

**Algorithm for determination of the symmetry group \( G \) associated to the PDEs system (4):**
- one considers the field \( X \) on \( M \) and one writes the infinitesimal invariance condition (6):
-one eliminates any dependence between partial derivatives of the functions $u^\alpha$, determined by the PDEs system (4);
- one writes the condition (6) like polynomials in the partial derivatives of $u^\alpha$;
- one equates with zero the coefficients of partial derivatives of $u^\alpha$ in (6), written as polynomials in the derivatives of the functions $u^\alpha$; it follows a PDEs system with respect to the unknown functions $\zeta^i$, $\phi^\alpha$ and this system defines the Lie symmetry group $G$ of the given PDEs system.

In general, for each $s$-parameter subgroup $H$ of the full symmetry group $G$ associated to the PDEs system (in $p > s$ independent variables), there will correspond a family of group-invariant solutions. Thus, a classification of these solutions is obtained by using an optimal system of group-invariant solutions from which any other solution can be derived.

**Proposition 2.** If $u = h(x)$ is an $H$-invariant solution of the PDEs system and $g \in G$, then the composed function $v = \tilde{h}(x) = g \cdot h(x)$ is an $\tilde{H}$-invariant solution, where $\tilde{H} = gHg^{-1}$ is the conjugate subgroup under $g$.

The problem of classifying group-invariant solutions reduces to the problem of classifying subgroups of the full symmetry group $G$ under conjugation and this is equivalent with the classifying subalgebras of the Lie algebra $g$ of the group $G$ under the adjoint representation.

An optimal system of $s$-parameter subgroups is a list of conjugacy inequivalent $s$-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of $s$-parameter subalgebras forms an optimal system of $s$-parameter subalgebras if every $s$-parameter subalgebra of $g$ is equivalent to a unique member of the list under some element of the adjoint representation.

Thus we compute the adjoint representation $Ad G$ of the underlying Lie group $G$, by using the Lie series:

\begin{equation}
Ad(exp(\varepsilon X)Y) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(adX)^n(Y) = Y - \varepsilon[X,Y] + \frac{\varepsilon^2}{2}[X,[X,Y]] - ...
\end{equation}

### 3 Symmetries of PDEs systems in solar physics and contact geometry

Let us consider

\[ B = u(x, y, z) \frac{\partial}{\partial x} + v(x, y, z) \frac{\partial}{\partial y} + w(x, y, z) \frac{\partial}{\partial z}, \]

a differentiable vector field on a simply connected domain $D \subset \mathbb{R}^3$. Let

\begin{align*}
\begin{cases}
    w_y - v_z &= uf \\
    u_z - w_x &= vf \\
    v_x - u_y &= wf,
\end{cases}
\end{align*}

(8)
be the PDEs system associated with the vector equation (1), where \( f \) is an arbitrary differentiable function of \( u, v, w \). We denote
\[
\Delta_1 = w_y - v_z - u f, \quad \Delta_2 = u_z - w_x - v f, \quad \Delta_3 = v_x - u_y - w f,
\]
and compute the partial derivatives
\[
\frac{\partial \Delta_1}{\partial u} = -f - u f_u, \quad \frac{\partial \Delta_1}{\partial v} = -u f_v, \quad \frac{\partial \Delta_1}{\partial w} = -u f_w, \quad \frac{\partial \Delta_1}{\partial v_z} = -1, \quad \frac{\partial \Delta_1}{\partial w_y} = 1,
\]
\[
\frac{\partial \Delta_2}{\partial u} = -v f_u, \quad \frac{\partial \Delta_2}{\partial v} = -v f_v, \quad \frac{\partial \Delta_2}{\partial w} = -v f_w, \quad \frac{\partial \Delta_2}{\partial u_z} = 1, \quad \frac{\partial \Delta_2}{\partial w_x} = -1,
\]
\[
\frac{\partial \Delta_3}{\partial u} = -w f_u, \quad \frac{\partial \Delta_3}{\partial v} = -w f_v, \quad \frac{\partial \Delta_3}{\partial w} = -f - w f_w, \quad \frac{\partial \Delta_3}{\partial v_x} = -1, \quad \frac{\partial \Delta_3}{\partial w_x} = 1.
\]
Let \( J_\Delta \) be the Jacobi matrix of the function \( \Delta \). It results that \( \text{rank} J_\Delta = 3 \) and thus the PDEs system (8) is of maximal rank. Let us consider
\[
X = \zeta \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \theta \frac{\partial}{\partial z} + \phi \frac{\partial}{\partial u} + \lambda \frac{\partial}{\partial v} + \psi \frac{\partial}{\partial w},
\]
a vector field on an open set \( M \subset D \times U \) of the space of the independent and dependent variables of the system, where \( \zeta, \eta, \theta, \phi, \lambda, \psi \) are functions of \( x, y, z, u, v, w \).

If we consider \( X \) to be the infinitesimal generator of the symmetry group of PDEs system (8), then the first prolongation of it is the next vector field
\[
pr^{(1)} X = X + \Phi^x \frac{\partial}{\partial u_x} + \Phi^y \frac{\partial}{\partial u_y} + \Phi^z \frac{\partial}{\partial u_z} + \Lambda^x \frac{\partial}{\partial u_x} + \Lambda^y \frac{\partial}{\partial u_y} + \Lambda^z \frac{\partial}{\partial u_z} +
\]
\[
+ \Psi^x \frac{\partial}{\partial w_x} + \Psi^y \frac{\partial}{\partial w_y} + \Psi^z \frac{\partial}{\partial w_z},
\]
where
\[
\Phi^x = \phi_x + u_x (\phi_u - \zeta_x) - u_y \eta_x - u_z \theta_x + \phi_v v_x + \phi_w w_x - u^2 \phi_{w_x} -
\]
\[
- u_x u_y \eta_u - u_x u_z \theta_u - u_x v_x \zeta_v - u_y v_x \eta_v - v_x u_x \theta_x - u_x w_x \zeta_w -
\]
\[
- u_y w_x \eta_w - u_z w_x \theta_w,
\]
\[
\Phi^y = \phi_y - u_x \zeta_y + u_y (\phi_u - \eta_y) - u_z \theta_y + v_y \phi_v + w_y \phi_w - u_x u_y \zeta_u - u^2 \eta_u -
\]
\[
- u_y u_z \theta_u - u_x v_y \zeta_v - u_y v_y \eta_v - u_z v_y \theta_v - u_x w_y \zeta_w - u_y w_y \eta_w - u_z w_y \theta_w,
\]
\[
\Phi^z = \phi_z + u_x \zeta_z - u_y \eta_z + u_z (\phi_u - \eta_z) + v_z \phi_v + w_z \phi_w - u_x u_z \zeta_u - u_y u_z \zeta_u -
\]
\[
- u^2 \theta_u - u_x v_z \zeta_v - u_y v_z \eta_v - u_z v_z \theta_v - u_x w_z \zeta_w - u_y w_z \eta_w - u_z w_z \theta_w,
\]
\[
\Lambda^x = \lambda_x + u_x \lambda_u + v_x (\lambda_v - \zeta_x) - v_y \eta_x - v_z \theta_x + w_x \lambda_w - u_x v_x \zeta_u - u_x v_y \eta_u -
\]
\[
- u_x v_z \theta_u - u^2 \zeta_v - v_x v_y \eta_v - v_z v_x \theta_v - v_x w_x \zeta_w - u_y w_x \eta_w - v_z w_x \phi_w,
\]
\[ 
\Lambda^y = \lambda_y + u_y \lambda_u - v_x \zeta_y + v_y (\lambda_v - \eta_y) - v_z \theta_y + \lambda_w y_w - u_y v_x \zeta_u - \\
- u_y v_y \eta_u - u_y v_z \theta_u - v_x v_y \zeta_v - v_y^2 \eta_v - v_y v_z \theta_u - v_x w_y \zeta_w - \\
- v_y w_y \eta_w - v_z w_y \theta_w,
\]
\[ 
\Lambda^z = \lambda_z + u_z \lambda_u - v_x \zeta_z - v_y \eta_z + v_z (\lambda_v - \theta_z) + w_z \lambda_w - v_x u_z \zeta_u - v_y u_z \eta_u - \\
- v_z u_z \theta_u - v_x w_z \zeta_w - v_y w_z \eta_w - v_z w_z \theta_w - v_y v_z \zeta_v - v_y v_z \theta_v - v_x w_z \zeta_w - \\
- \psi_1 w_1 \zeta_1 - w_x w_y \eta_1 - w_x w_z \theta_1,
\]
\[ 
\Psi^x = \psi_x + u_x \psi_u + u_x \psi_v + w_x (\psi_w - \zeta_x) - u_x \psi_1 \eta_1 - u_x \psi_1 \theta_1 - u_x w_z \zeta_u - \\
- u_x w_y \eta_u - u_x w_x \zeta_x - u_x w_y \zeta_v - w_x w_y \eta_w - v_x w_x \zeta_x - \\
- w_2 \zeta_w - w_x w_y \eta_1 - w_x w_z \theta_1,
\]
\[ 
\Psi^y = \psi_y + u_y \psi_u + v_y \psi_v + w_y (\psi_w - \zeta_y) - w_z \theta_y - w_x u_y \zeta_u - \\
- w_y u_y \eta_u - w_x u_y \theta_u - w_x v_y \zeta_v - w_y v_y \eta_w - u_x w_y \zeta_u - w_x w_y \zeta_w - \\
- w_2 \zeta_w - w_y w_z \theta_1,
\]
\[ 
\Psi^z = \psi_z + u_z \psi_u + v_z \psi_v + w_z (\psi_w - \theta_z) - w_z \zeta_z - \eta_z w_y - u_z w_z \zeta_u - u_z w_y \eta_u - \\
- \theta_u w_z - v_z w_x \zeta_x - v_z w_y \eta_x - v_z w_z \theta_x - w_x w_z \zeta_w - w_z w_y \eta_w - w_2 \zeta_w.
\]

In this case, the infinitesimal invariance condition (6) implies

\[
\begin{align*}
-(f + u f_u) \phi - u f_v \lambda - u f_w \psi + \Psi^y - \Lambda^z &= 0 \\
-v f_u \phi - (f + v f_v) \lambda - v f_w \psi + \Phi^z - \Psi^x &= 0 \\
-w f_u \phi - w f_v \lambda - (f + w f_w) \psi + \Phi^x - \Psi^y &= 0.
\end{align*}
\]

Substituting the functions \( \Phi^y, \Phi^z, \Lambda^x, \Lambda^x, \Psi^x, \Psi^y \) and after eliminating any dependencies among the derivatives of the \( u, v, w \) caused by the PDEs system (8) itself, we find the following PDEs system which defines the symmetry group of the studied PDEs system

\[
\begin{align*}
\psi_y - \lambda_z - \phi(u f_u + f) - u \lambda f_v - w \psi f_w - v f \lambda_u + w f \zeta_x + u f (\psi_w - \eta_y) + v w f^2 \zeta_u - u^2 f^2 \eta_u + \\
+ u_y (\psi_u + \zeta_z + v f \zeta_u - u f \eta_u) + v_y (\psi_v + \eta_z + v f \eta_u - u f \eta_v) + v_z (\theta_x - \lambda_v + \psi_w - \eta_y + \\
+ v f \theta_u + w f \zeta_w - 2 u f \eta_u) - w_x (\lambda_u + \zeta_u - w f \zeta_u + u f \zeta_w) - w_z (\lambda_w + \theta_y - w f \zeta_w + u f \theta_u) + \\
+ u_y w_z (\zeta_w - \theta_u) + v_y w_z (\eta_u - \theta_v) + u_y v_z (\zeta_u - \eta_u) + u_x v_z (\zeta_v - \eta_u) + u_x w_z (\theta_u - \zeta_v) + v_y w_z (\eta_u - \zeta_v) = 0.
\end{align*}
\]

\[
\begin{align*}
\phi_z - \psi_x - v f_u \phi - (v f_u + f) \lambda - v f \psi v + v f (\phi_u - \zeta_x) - w f \psi_v + u f \eta_u - v^2 f^2 \theta_u + u w f^2 \eta_u - \\
- u x (\zeta_x + \psi_u + v f \zeta_u - u f \eta_u) - u_y (\eta_z + \psi_v + v f \eta_u - u f \eta_v) + v_z (\phi_v + \eta_x - v f \theta_v + w f \eta_v) + \\
+ w_z (\zeta_x - \psi_w + \phi_u - \theta_x - 2 v f \theta_u + w f \zeta_w + u f \eta_w) + w_x (\phi_w + \theta_u - v f \theta_u + w f \theta_v) + \\
+ u_x v_z (\eta_u - \zeta_v) + u_x w_z (\theta_u - \zeta_v) + u_y v_z (\theta_u - \eta_u) + w_2^2 (\zeta_w - \eta_u) + w_x u_y (\zeta_v - \\
- \eta_u) + w_x v_z (\eta_u - \theta_v) = 0.
\end{align*}
\]

\[
\begin{align*}
\lambda_x - \phi_y - w \phi f_u - w \lambda f_v - (w f_u + f) \psi + v f \theta_y + w f (\lambda_v - \zeta_x) - u f \phi_u - w^2 f^2 \zeta_v + w v^2 \theta_w + \end{align*}
\]
\[ u_x(\lambda_u + \zeta_y + u f \zeta_w - w f \zeta_v) + u_y(\eta_y - \phi_u + \lambda_v - \zeta_x + v f \theta_u + u f \eta_w - 2 w f \zeta_u) - \\
- v_y(\phi_v + \eta_x - v f \theta_v + w f \eta_v) - v_z(\theta_x + \phi_w + w f \theta_v - v f \theta_u) + w_x(\lambda_w + \theta_v - \\
- w f \zeta_w + u f \theta_w) + u_y^2(\eta_u - \zeta_v) + u_x v_y(\zeta_v - \eta_u) + u_x v_z(\zeta_w - \theta_u) + w_x v_y(\theta_v - \eta_u) + \\
+ w_x u_y(\theta_u - \zeta_w) + u_y v_z(\eta_w - \theta_v) = 0. \]

If we equate the coefficients of the remaining unconstrained partial derivatives of \( u, v, w \) to zero, we obtain the following PDEs system which defines the symmetry group of the studied PDEs system:

\[ \begin{align*}
\psi_y - \lambda_z - \phi(u f_u + f) - u \lambda f_v - v \lambda f_w - v f \lambda u + w f \zeta_v + \\
&+ u f(\psi_w - \eta_y) + w v f^2 \zeta_u - u^2 f^2 \eta_w = 0 \\
\phi_z - \psi_x - v f u \phi - (v f_v + f) \lambda - v f w \psi + v f(\phi_u - \theta_z) - \\
&- w f \psi_v + u f \eta_x - v^2 f^2 \theta_u + u w f^2 \eta_v = 0 \\
\lambda_x - \phi_y - w f f_u - w \lambda f_v - (w f_w + f) \psi + v f \theta_y + w f(\lambda_v - \zeta_x) - \\
&- u f \phi_w - w^2 f^2 \zeta_v + u w f^2 \theta_w = 0 \\
\phi_v = - \eta_x + f(\psi_w - \eta_w) \\
\end{align*} \]

(10)

\[ \begin{align*}
\phi_w = - \theta_x + f(v \theta_w - w \theta_v) \\
\lambda_u = - \zeta_y + f(w \zeta_u - u \zeta_w) \\
\lambda_w = - \theta_y + f(w \theta_u - u \theta_w) \\
\psi_u = - \zeta_z + f(u \zeta_v - v \zeta_u) \\
\psi_v = - \eta_z + f(w \eta_u - v \eta_w) \\
\phi_u - \psi_w = - \zeta_x + \theta_z + f(2 v \theta_u - w \zeta_v - u \eta_w) \\
\psi_w - \lambda_v = - \theta_z + \eta_y + f(2 u \eta_u - v \theta_u - w \eta_v) \\
\eta_u = \zeta_v \\
\theta_v = \eta_w \\
\zeta_w = \theta_u. \\
\end{align*} \]

It results the next theorem

**Theorem 3.** The general vector field which describes the algebra of infinitesimal symmetries associated to PDEs system (8) is

\[ X = \zeta \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \theta \frac{\partial}{\partial z} + \phi \frac{\partial}{\partial u} + \lambda \frac{\partial}{\partial v} + \psi \frac{\partial}{\partial w}, \]

where the functions \( \zeta, \eta, \theta, \phi, \lambda \) and \( \psi \) satisfy the PDEs system (10).
Now let us consider the PDEs system

\[
\begin{align*}
  w_y - v_z &= u\sqrt{u^2 + v^2 + w^2} \\
  u_z - w_x &= v\sqrt{u^2 + v^2 + w^2} \\
  v_x - u_y &= w\sqrt{u^2 + v^2 + w^2},
\end{align*}
\]

(11)

associated to the vector equation (3).

We obtain

**Theorem 4.** A Lie algebra of infinitesimal symmetries associated to the PDEs system (11) is described by the next vector fields

\[
X_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}, \quad X_2 = -z\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - w\frac{\partial}{\partial u} + v\frac{\partial}{\partial w}, \\
X_3 = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z} - w\frac{\partial}{\partial u} + u\frac{\partial}{\partial w}, \quad X_4 = \frac{\partial}{\partial x}, \quad X_5 = \frac{\partial}{\partial y}, \quad X_6 = \frac{\partial}{\partial z}.
\]

(12)

\[
X_7 = \frac{x}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w}, \\
X_8 = 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (z^2 - x^2 - y^2)\frac{\partial}{\partial z} + 2(xw - zw)\frac{\partial}{\partial u} + 2(yw - zv)\frac{\partial}{\partial v} - 2(xu + yv + zw)\frac{\partial}{\partial w}, \\
X_9 = xy\frac{\partial}{\partial x} + \frac{1}{2}(y^2 - x^2 - z^2)\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z} + (xw - zw)\frac{\partial}{\partial v} - (xu + yv + zw)\frac{\partial}{\partial w}, \\
X_{10} = \frac{1}{2}(x^2 - y^2 - z^2)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z} - (xu + yv + zw)\frac{\partial}{\partial u} + (yu - xv)\frac{\partial}{\partial v} + (zu - xw)\frac{\partial}{\partial w}.
\]

**Proof.** We substitute \( f = \sqrt{u^2 + v^2 + w^2} \) in the PDEs system (10). If we consider \( \zeta, \eta, \theta, \phi, \lambda, \) and \( \psi \) polynomial functions of \( x, y, z, u, v, w, \) we get the following solution:

\[
\begin{align*}
  \zeta &= C_7x - C_1y - C_3z + 2C_8xz + C_9xy + \frac{1}{2}C_{10}(x^2 - y^2 - z^2) + C_4 \\
  \eta &= C_1x + C_7y - C_2z + 2C_8yz + \frac{1}{2}C_9(y^2 - x^2 - z^2) + C_{10}xy + C_5 \\
  \theta &= C_3x + C_2y + C_7z + C_8(x^2 - x^2 - y^2) + C_9yz + C_{10}xz + C_6 \\
  \phi &= -C_7u - C_1v - C_3w + 2C_8(xw - zw) + C_9(xv - yu) - C_{10}(xu + yv + zw) \\
  \lambda &= C_1u + C_7v = C_3w + 2C_8(yw - zw) - C_9(xu + yv + zw) + C_{10}(yu - xv) \\
  \psi &= C_3u + C_2v - C_7w - 2C_8(xw + yv + zw) + C_9(zv - yu) + C_{10}(zu - xw),
\end{align*}
\]

(13)
with $C_i \in \mathbb{R}$, $i = 1, \ldots, 10$. The infinitesimal generator of the associated symmetry subgroup is $X = \sum_{i=1}^{10} C_i X_i$, where the vector fields $X_i$ are given by the relation (12).

The structure constants of the Lie algebra generated by the vector fields (12) are given by the next table

| $\ldots$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $X_1$    | 0     | $X_3$ | $-X_2$| $-X_5$| $X_4$ | 0     | 0     | 0     | $X_{10}$| $-X_9$ |
| $X_2$    | $-X_3$| 0     | $X_1$ | 0     | $-X_6$| $X_5$ | 0     | 2$X_9$| $-\frac{1}{2}X_8$| 0     |
| $X_3$    | $X_2$ | $-X_1$| 0     | $-X_6$| 0     | $X_4$ | 0     | 2$X_{10}$| 0     | $-\frac{1}{2}X_8$|
| $X_4$    | $X_5$ | 0     | $X_6$ | 0     | 0     | 0     | $X_4$ | $-2X_3$| $-X_1$| $X_7$ |
| $X_5$    | $-X_4$| $X_6$ | 0     | 0     | 0     | 0     | $X_5$ | $-2X_2$| $X_9$| $X_1$ |
| $X_6$    | 0     | $-X_5$| $-X_4$| 0     | 0     | 0     | 0     | $X_6$ | $2X_2$| $X_2$| $X_3$ |
| $X_7$    | 0     | 0     | 0     | $-X_4$| $-X_5$| $-X_6$| 0     | $X_8$| $X_9$| $X_{10}$|
| $X_8$    | 0     | $-2X_9$| $-2X_{10}$| $2X_3$| $2X_2$| $-2X_7$| $-X_8$| 0    | 0    | 0     |
| $X_9$    | $-X_{10}$| 0     | $0$   | $X_1$ | $-X_9$| $-X_2$| $-X_9$| 0    | 0    | 0     |
| $X_{10}$ | $X_9$ | 0     | $\frac{1}{2}X_8$| $-X_7$| $-X_1$| $-X_3$| $-X_{10}$| 0    | 0    | 0     |

For the Blair system

$$
\begin{align*}
\begin{cases}
  w_y - v_z &= uf \\
  u_z - w_x &= vf \\
  v_x - u_y &= w_f \\
  u_x + v_y + w_z &= 0,
\end{cases}
\end{align*}
$$

(14)

where

$$
(14')
\begin{align*}
u_x + v_y + w_z &= 0,
\end{align*}
$$

is equivalent to (2), it results

**Theorem 5.** The following vector fields

$$
\begin{align*}
X_1 &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v}, \\
X_2 &= -z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} - w \frac{\partial}{\partial u} + v \frac{\partial}{\partial w}, \\
X_3 &= -z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z} - w \frac{\partial}{\partial u} + u \frac{\partial}{\partial w}, \\
X_4 &= \frac{\partial}{\partial x}, \ X_5 = \frac{\partial}{\partial y}, \ X_6 = \frac{\partial}{\partial z}.
\end{align*}
$$

(15)
\[ X_7 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} - w \frac{\partial}{\partial w}. \]

describe the Lie algebra \( g \) of the infinitesimal symmetries associated to the Blair system (14).

**Proof.** The PDEs system (14) is of maximal rank. By using the above algorithm, we find that

\[
\begin{align*}
\zeta &= C_7 x - C_1 y - C_3 z + C_4 \\
\eta &= C_1 x + C_7 y - C_2 z + C_5 \\
\theta &= C_3 x + C_2 y + C_7 z + C_6 \\
\phi &= -C_7 u - C_1 v - C_3 w \\
\lambda &= C_1 u - C_7 v - C_2 w \\
\psi &= C_3 u + C_2 v - C_7 w,
\end{align*}
\]

where \( C_i \in \mathbb{R}, \ i = 1, \ldots, 7, \) is the solution of the PDEs system which defines the symmetry group of Blair system. It results that \( X = \sum_{i=1}^7 C_i X_i \) is the infinitesimal generator of it, where the vector fields \( X_i \) are given by (15).

**Remark.** The Lie algebra \( g \) associated to the symmetry group of the Blair system is a subalgebra of the algebra of infinitesimal symmetries associated to PDEs system (11). Using the Lie series, one determines the adjoint representation \( AdG \) of the Lie symmetry group \( G \):

| \( Ad \) | \( X_1 \) | \( X_2 \) | \( X_3 \) |
| --- | --- | --- | --- |
| \( X_1 \) | \( X_1 \) | \( X_2 \cos \varepsilon - X_3 \sin \varepsilon \) | \( X_3 \cos \varepsilon + X_2 \sin \varepsilon \) |
| \( X_2 \) | \( X_1 \cos \varepsilon + X_3 \sin \varepsilon \) | \( X_2 \) | \( X_3 \cos \varepsilon - X_1 \sin \varepsilon \) |
| \( X_3 \) | \( X_1 \cos \varepsilon - X_2 \sin \varepsilon \) | \( X_2 \cos \varepsilon + X_1 \sin \varepsilon \) | \( X_3 \) |
| \( X_4 \) | \( X_1 - \varepsilon X_5 \) | \( X_2 \) | \( X_3 - \varepsilon X_6 \) |
| \( X_5 \) | \( X_1 + \varepsilon X_4 \) | \( X_2 - \varepsilon X_6 \) | \( X_3 \) |
| \( X_6 \) | \( X_4 \) | \( X_2 + \varepsilon X_5 \) | \( X_3 + \varepsilon X_4 \) |
| \( X_7 \) | \( X_1 \) | \( X_2 \) | \( X_3 \) |

| \( Ad \) | \( X_4 \) | \( X_5 \) | \( X_6 \) | \( X_7 \) |
| --- | --- | --- | --- | --- |
| \( X_1 \) | \( X_4 \cos \varepsilon + X_5 \sin \varepsilon \) | \( X_5 \cos \varepsilon - X_4 \sin \varepsilon \) | \( X_6 \) | \( X_7 \) |
| \( X_2 \) | \( X_4 \) | \( X_5 \cos \varepsilon + X_6 \sin \varepsilon \) | \( X_6 \cos \varepsilon - X_5 \sin \varepsilon \) | \( X_7 \) |
| \( X_3 \) | \( X_4 \cos \varepsilon + X_5 \sin \varepsilon \) | \( X_5 \) | \( X_6 \cos \varepsilon - X_4 \sin \varepsilon \) | \( X_7 \) |
| \( X_4 \) | \( X_2 \) | \( X_5 \) | \( X_6 \) | \( X_7 - \varepsilon X_4 \) |
| \( X_5 \) | \( X_4 \) | \( X_5 \) | \( X_6 \) | \( X_7 - \varepsilon X_5 \) |
| \( X_6 \) | \( X_4 \) | \( X_5 \) | \( X_6 \) | \( X_7 - \varepsilon X_6 \) |
| \( X_7 \) | \( \varepsilon^2 X_4 \) | \( \varepsilon X_5 \) | \( \varepsilon X_6 \) | \( \varepsilon X_7 \) |
Remark. If \((u = f(x, y, z), v = g(x, y, z), w = h(x, y, z))\) is a solution of the Blair system, then the following triples

\[
\begin{align*}
(u^{(1)}) & = af(ax + by, -bx + ay, z) - bg(ax + by, -bx + ay, z) \\
v^{(1)} & = bf(ax + by, -bx + ay, z) + ag(ax + by, -bx + ay, z) \\
w^{(1)} & = h(ax + by, -bx + ay, z),
\end{align*}
\]

\[
\begin{align*}
(u^{(2)}) & = f(x, ay + bz, -by + az) \\
v^{(2)} & = ag(x, ay + bz, -by + az) - bh(x, ay + bz, -by + az) \\
w^{(2)} & = bg(x, ay + bz, -by + az) + ah(x, ay + bz, -by + az),
\end{align*}
\]

\[
\begin{align*}
(u^{(3)}) & = af(ax + bz, y, -bx + az) - bh(ax + bz, y, -bx + az) \\
v^{(3)} & = g(ax + bz, y, -bx + az) \\
w^{(3)} & = b(f(ax + bz, y, -bx + az) + ah(ax + bz, y, -bx + az),
\end{align*}
\]

\[
\begin{align*}
(u^{(4)}) & = f(x - \varepsilon, y, z) \\
v^{(4)} & = g(x - \varepsilon, y, z) \\
w^{(4)} & = h(x - \varepsilon, y, z),
\end{align*}
\]

\[
\begin{align*}
(u^{(5)}) & = f(x, y - \varepsilon, z) \\
v^{(5)} & = g(x, y - \varepsilon, z) \\
w^{(5)} & = h(x, y - \varepsilon, z),
\end{align*}
\]

\[
\begin{align*}
(u^{(6)}) & = f(x, y, z - \varepsilon) \\
v^{(6)} & = g(x, y, z - \varepsilon) \\
w^{(6)} & = h(x, y, z - \varepsilon),
\end{align*}
\]

\[
\begin{align*}
(u^{(7)}) & = e^{-\varepsilon}f(e^{-\varepsilon}x, e^{-\varepsilon}y, e^{-\varepsilon}z) \\
v^{(7)} & = e^{-\varepsilon}g(e^{-\varepsilon}x, e^{-\varepsilon}y, e^{-\varepsilon}z) \\
w^{(7)} & = e^{-\varepsilon}h(e^{-\varepsilon}x, e^{-\varepsilon}y, e^{-\varepsilon}z),
\end{align*}
\]

are also solutions of the Blair system, where \(\varepsilon \in \mathbb{R}, \cos \varepsilon = a\) and \(\sin \varepsilon = b\).

By using the adjoint representation \(AdG\) of the grup Lie \(G\) we can find an optimal system of \(s\)-parameter subalgebras associated to the Blair system, with \(s = 1, 2\). For example, it results six 2-dimensional optimal subalgebras, which are generated by the next vector fields:

\[
X_4, X_5; \quad X_5, X_6; \quad X_4, X_6,
\]

and respectively

\[
X_1, X_6; \quad X_2, X_5; \quad X_3, X_4.
\]

1. Let us consider the symmetry subgroup for which the infinitesimal generators are the vector fields \(X_4\) and \(X_5\) and \(F = G(z, u, v, w)\) the invariant function of it. By substituting

\[
u = g(z), \quad v = h(z), \quad w = k(z),
\]

in the Blair system, we get the following system of differential equations:

\[
\begin{align*}
k &= 0 \\
-h' &= g\sqrt{g^2 + h^2} \\
g' &= h\sqrt{g^2 + h^2}.
\end{align*}
\]

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By using a coordinate transformation, the solution of it is

\[ u(z) = \sin z, \quad v(z) = \cos z, \quad w = 0, \]

and it results the vector field \( B_1 \).

2. Now we determine the group-invariant solutions for the symmetry subgroup with the infinitesimal generators \( X_1 \) și \( X_6 \). The invariant function is

\[ F = G(\sqrt{x^2+y^2}, xu+yv, xv-yu, w). \]

If we consider

\[ xu + yv = g(\sqrt{x^2+y^2}), \quad xv - yu = h(\sqrt{x^2+y^2}), \quad w = \gamma(\sqrt{x^2+y^2}), \]

the Blair system turns in

\[
\begin{align*}
g &= 0 \\
-\gamma' &= h\sqrt{\left(\frac{\beta}{r}\right)^2 + \gamma^2} \\
\beta' &= \gamma\sqrt{\left(\frac{\beta}{r}\right)^2 + \gamma^2},
\end{align*}
\]

where \( r = \sqrt{x^2+y^2} \), or in an equivalent form

\[
\begin{align*}
-\gamma' &= \beta\sqrt{\beta^2 + \gamma^2} \\
\frac{1}{r}\beta + \beta' &= \gamma\sqrt{\beta^2 + \gamma^2},
\end{align*}
\]

by using the change of function \( \beta = \frac{h}{r} \). The solution of this system was determined by Blair using the method of successive approximations [5].

By circular permutation of \( x, y, z \), one determines other solutions of the Blair system.

According with the above remark, using the relations (17), one finds new solutions of the system. For example in the case of the vector field

\[ B_1 = \sin z \frac{\partial}{\partial x} + \cos z \frac{\partial}{\partial y}, \]

the following vector fields are also solutions

\[ B_1^{(1)} = \sin(z - \varepsilon) \frac{\partial}{\partial x} + \cos(z - \varepsilon) \frac{\partial}{\partial y} = B_1^{(6)}, \]

\[ B_1^{(2)} = \sin(az - by) \frac{\partial}{\partial x} + a \cos(az - by) \frac{\partial}{\partial y} + b \cos(az - by) \frac{\partial}{\partial z}, \]

\[ B_1^{(3)} = a \sin(az - bx) \frac{\partial}{\partial x} + \cos(az - bx) \frac{\partial}{\partial y} + b \cos(az - bx) \frac{\partial}{\partial z}, \]
\[ B_1^{(7)} = e^{-\varepsilon} \sin(e^{-\varepsilon} z) \frac{\partial}{\partial x} + e^{-\varepsilon} \cos(e^{-\varepsilon} z) \frac{\partial}{\partial y}. \]

Because \( B_1^{(4)} = B_1^{(5)} = B_1 \), we get the next result: the solution \( B_1 \) is invariant with respect to the 2-parameter subalgebra described by \( X_4 \) and \( X_5 \). Analogously, one finds new solutions of the Blair system by using the Blair solution.

We make the remark that the vector field

\[ B_2 = \frac{8(xz - y)}{(1 + x^2 + y^2 + z^2)^2} \frac{\partial}{\partial x} + \frac{8(x + yz)}{(1 + x^2 + y^2 + z^2)^2} \frac{\partial}{\partial y} + \frac{4(1 + z^2 - x^2 - y^2)}{(1 + x^2 + y^2 + z^2)^2} \frac{\partial}{\partial z}, \]

which is a solution of the vector equation (3), can be found by using the subgroup generated by \( X_1 \), because \( B_2^{(1)} = B_2 \).

Now we study the inverse of the Theorem 5.

**Theorem 6.** The only PDEs system \((8)+(14')\) which is invariant with respect to the symmetry group \( G \) associated to the Blair system is the Blair system.

**Proof.** One writes the infinitesimal conditions (6) in the case of the PDEs system \((8)+(14')\) and one substitutes the vector fields given by the relation (15). It results the next PDEs system

\[
\begin{align*}
uf_u + vf_v + wf_w & = f \\
u f_v - v f_u & = 0 \\
v f_w - w f_v & = 0 \\
w f_w - w f_u & = 0,
\end{align*}
\]

with the solution \( f(u, v, w) = \sqrt{u^2 + v^2 + w^2} \).

We consider that the study of the PDEs system \((1)+(2)\) from the point of view of the symmetry group theory is very interesting, because for this PDEs system are many known solutions, but we don’t have any classification of them.

**References**

[1] N.BILĂ - *Symmetry Lie groups of PDE of surfaces with constant Gaussian curvature*, Scientific Bulletin, University Politehnica of Bucharest, Series A, 61, 1-2(1999), in press.

[2] N.BILĂ - *Lie groups applications to minimal surfaces PDE*, Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, BSG Proceedings 3(1999), Geometry Balkan Press, Editor: Gr. Tsagas, 197-205.

[3] N.BILĂ - *Symmetry groups and Lagrangians associated to Țițeica surfaces*, to appear.
[4] N.BILă, C.UDRăTE - Infinitesimal symmetries of Camassa-Holm equation, to appear.

[5] D.E.BLAIR - On the existence of conformally flat contact metric manifolds, preprint.

[6] G.BLUMAN, J.D.COLE - Similarity Methods for Differential Equations, Applied Mathematical Sciences, 13, Springer Verlag, New York, 1974.

[7] G.CAVIGLIA - Symmetry transformations, isovectors and conservation laws, J. Math. Phys. 27, 4(1986), 972-978.

[8] P.A.CLARKSON, E.L.MANSFIELD, T.J.PRIESTLEY - Symmetries of a class of nonlinear third order partial differential equations, Mathematical and Computer Modelling, 25, 8-9(1997), 195-212.

[9] B.K.HARRISON, F.B.ESTABROOK - Geometric approach to invariance groups and solution of partial differential systems, J. Math. Phys. 12, 4(1971), 653-666.

[10] A.K.HEAD - Program LIE for analysis of differential equations on IBM type PCs, Melbourne, Australia, "http://archives.math.utk.edu/software/msdos/adv.diff.equations/lie/.html".

[11] P.METZGER - Quelques autres exemples de groupes d’invariance d’équations aux dérivées partielles, C.R. Acad. Sci., Paris, Sér.A, 279(1974), 193-196.

[12] P.J.OLVER - Applications of Lie Groups to Differential Equations, Graduate Texts in Math., 107, Springer Verlag, New York Inc., 1986.

[13] P.J.OLVER - Symmetry groups and group invariant solutions of partial differential equations, J. Diff. Geom., 14(1979), 497-542.

[14] P.J.OLVER, P.ROSENAU - Group invariant solutions of differential equations, SIAM J. Appl. Math., 47, 2(1987), 263-278.

[15] D.OPRİŞ, I.BUTULESCU - Metode geometrice în studiul sistemelor de ecuații diferențiale, Editura Mirton, Timișoara, 1997.

[16] S.STEINBERG - Symmetry methods in differential equations, Technical Report 367, University of New Mexico, 1979.

[17] C.UDRÎŞTE, N. BILĂ - Symmetry group of Tîţeica surfaces PDE, Balkan Journal of Geometry and Its Applications, 4, 1(1999), in press.

[18] C.UDRÎŞTE, N.BILĂ - Symmetry Lie groups of the Monge-Ampère equation, Balkan Journal of Geometry and Its Applications, 3, 2(1998), 121-133.
[19] A.M.VINOGRADOV - *Symmetries and conservation laws of partial differential equations: Basic notions and results*, Acta Appl. Math., 15(1989), 3-21.

[20] G.VRÂNCEANU - *Lecții de geometrie diferențială*, Editura Didactică și Pedagogică, București, 1976.

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