BINARY DIFFERENTIAL EQUATIONS WITH SYMMETRIES

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ABSTRACT. This paper introduces the study of occurrence of symmetries in binary differential equations (BDEs). These are implicit differential equations given by the zeros of a quadratic 1-form, $a(x, y)dy^2 + b(x, y)dxdy + c(x, y)dx^2 = 0$, for $a, b, c$ smooth real functions defined on an open set of $\mathbb{R}^2$. Generically, solutions of a BDE are given as leaves of a pair of foliations, and the action of a symmetry must depend not only whether it preserves or inverts the plane orientation, but also whether it preserves or interchanges the foliations. The first main result reveals this dependence, which is given algebraically by a formula relating three group homomorphisms defined on the symmetry group of the BDE. The second main result adapts methods from invariant theory of compact Lie groups to obtain an algorithm to compute general expressions of equivariant quadratic 1-forms under each compact subgroup of the orthogonal group $O(2)$.

1. Introduction. A binary differential equation on the plane, or a BDE, is an implicit quadratic differential equation

$$a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dx^2 = 0,$$

where the coefficients $a, b, c$ are real functions which we assume to be smooth on an open set $U \subseteq \mathbb{R}^2$. The function $\delta : U \to \mathbb{R}$, $\delta(x, y) = (b^2 - ac)(x, y)$, is the discriminant function and its zero set

$$\Delta = \{(x, y) \in U : (b^2 - ac)(x, y) = 0\}$$

is the discriminant set of the BDE. The investigation of occurrence of symmetries is converted in purely algebraic terms, so there is no loss of generality in assuming that $U$ is the whole plane, which we do from now on for simplicity.

At points where $\delta > 0$, (1) defines a pair of transversal directions, and by the configuration associated with this equation we mean the distribution of all solution curves tangent to these directions. The geometry of a BDE configuration is

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a subject of great interest, with important applications in differential geometry, as the equations of lines of curvatures, characteristic curves and asymptotic curves of smooth surfaces, and in control theory (see [20] and references therein). Conditions for local stability of positive binary differential equations \((\delta > 0)\) and their classification are given in [12] and [13], with a description of the topological patterns that bifurcate in one-parameter families of these equations. Singular points of a class of positive binary equations associated with a smooth surface are also studied in [12] and [19], the coefficients of BDE being given in terms of the coefficients of the first and second fundamental forms of the surface.

Determining models of configurations associated with BDEs has been also addressed in many works; for example, in [2, 4, 5, 6, 7, 14, 21, 22] the classification of BDEs is performed up to topological, formal, analytic or smooth equivalences. We refer to [20] for a survey on these topics.

This work is motivated by the recognition of symmetries in most normal forms that appear in the works mentioned above. In the linear case for example, namely when the coefficients of (1) are linear functions, presence of at least one nontrivial symmetry (namely minus identity) is a necessary condition. As a consequence, our study shows which symmetry groups are attained or never attained in the nonlinear cases. For a general approach of symmetries applied to differential equations, vector fields and differential forms we refer to [11] and [18] for example. The purpose of the present paper is to introduce the systematic study of symmetries in binary differential equations. [17] is a continuation of this study: we investigate symmetries of a BDE in connection with an associated pair of equivariant vector fields (see Remark 3); also, quadratic forms with homogeneous coefficients are studied through an analysis of the number of invariant lines that appear in the configuration space imposed by their group of symmetries; in addition, we analyse possible symmetry groups of BDEs with Morse type discriminant introduced in [4]. More recently we have driven attention to some questions relating symmetries in differential forms to pairs of foliations of special classes of surfaces; for example, for a given equivariant BDE, we ask whether this can be realized as an equation of lines of curvatures or of asymptotic lines of a surface immersed in \(\mathbb{R}^n\) for some \(n\).

To remark on evidences of occurrence of symmetries in configurations associated with BDEs, let us consider the configurations that appear in the pioneer work of Darboux [8] and also in [3] and [19], which are generic topological structures of

![Figure 1. Configurations of symmetric BDEs. In (a) and (b) the symmetry group is \(\mathbb{Z}_2 \times \mathbb{Z}_2\) and in (c) the symmetry group is \(D_6\).](image)
principal direction fields at umbilic points of surfaces on Euclidean spaces. The normal forms are given as triples \((a, b, c)\) for \(c = \pm a\) in (1) and their configurations are reproduced in Fig. 1, the so-called 1(a) lemon, 1(b) monstar and 1(c) star, or Darbouxians \(D_1, D_2\) and \(D_3\). The solution curves determine two foliations of the plane distinguished by blue dashed lines and magenta solid lines, and the black point is the discriminant set. The pictures clearly suggest an invariance of the three configurations under reflection with respect to the \(x\)-axis. There is another invariance with respect to the \(y\)-axis, which is given by this operation followed by a change of colour. As a consequence, the composition of these two elements (minus identity) must be a symmetry which interchanges colour. In fact, we should recognize \textit{a priori} minus identity in the set of symmetries of all these cases by their linearity, as mentioned above. The third picture has also six rotational symmetries, three of which are colour-preserving and the other three are colour-interchanging. In fact, the full symmetry group of pictures 1(a) and 1(b) is \(\mathbb{Z}_2 \times \mathbb{Z}_2\), generated by the reflections across the axes, and the full symmetry group of picture 1(c) is the dihedral group \(D_6\), generated by a reflection and a rotation of order six. As these examples illustrate, the group action must be defined taking colour changes into consideration at the region on the plane where (1) defines a bivalued direction field. As we shall see, the action of a symmetry group \(\Gamma\) of a binary differential equation must be defined on the tangent bundle through its representation on the plane combined with a group homomorphism \(\eta: \Gamma \to \mathbb{Z}_2 = \{\pm 1\}\) (Definition 2.1). This action is translated into an action of \(\Gamma\) on the space of symmetric matrices under conjugacy. This is then used to obtain the general forms of equivariant BDEs under all compact subgroups of \(\mathbb{O}(2)\) through invariant theory.

The paper is organized as follows: in Section 2 we introduce the notion of symmetries in a BDE, namely when the equation is invariant under the linear action of a subgroup \(\Gamma \leq \mathbb{O}(2)\). We formalize the concept using group representation theory on the tangent bundle on which the associated quadratic 1-form is defined. One of our two main results is Theorem 2.2, which establishes a formula that reveals the effect of a symmetry in the configuration geometry in simple algebraic terms. In Section 3 we generalize the results in [1] for \(\Gamma\)-equivariant mappings with distinct actions on the source and target. The results allow the computation of general forms of equivariant mappings using an algebraic algorithm (Algorithm 3.2). In Section 4 we use the previous section to deduce the general forms of equivariant BDEs under any compact subgroup of \(\mathbb{O}(2)\), which is our second main result, summarized in Table 1: in this section we include a number of examples.

2. The symmetry group of a binary differential equation. In this section we formalize the concept of a symmetric binary differential equation under the linear action of a compact Lie subgroup \(\Gamma\) of \(\mathbb{O}(2)\).

Let \(Q(\mathbb{R}^2)\) denote the set of real \(C^\infty\) quadratic differential forms on \(\mathbb{R}^2\), \(\omega: T\mathbb{R}^2 \to \mathbb{R}\),

\[
(2) \quad \omega(x, y, dx, dy) = a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dx^2,
\]

with \(a, b, c\) \(C^\infty\) functions on \(\mathbb{R}^2\).

Let \(\Gamma\) be a compact Lie group acting linearly on \(\mathbb{R}^2\). This induces an action of \(\Gamma\) on the tangent bundle \(T\mathbb{R}^2 = \{((x, y), (X, Y)) : (x, y), (X, Y) \in \mathbb{R}^2\}\),

\[
\Gamma \times T\mathbb{R}^2 \to T\mathbb{R}^2
\]

\(\gamma, (x, y), (X, Y) \mapsto \gamma \cdot ((x, y), (X, Y)) = (\gamma(x, y), (d\gamma)_{(x,y)}(X, Y))\),
where, \((d\gamma)(x,y)(X,Y)\) is in fact simply \(\gamma(X,Y)\) by the linearity of the action.

The symmetry group of a binary differential equation is a subgroup of \(O(2)\) that leaves invariant the configuration of its integral curves. Now, there is an action of \(\Gamma\) on \(Q(\mathbb{R}^2)\) given by \(\gamma \omega(X) = \omega(\gamma^{-1}X)\), \(X \in \mathbb{R}T^{2}\). For each \(\gamma \in \Gamma\), we have that \(\gamma \omega = \pm \omega\), since the only nontrivial one-dimensional representation of a compact Lie group is the sign representation. This representation on \(\mathbb{R}\) is then defined as \(\eta: \Gamma \rightarrow \mathbb{Z}_2 = \{\pm 1\}\) by requiring that \(\eta(\gamma)\omega = \gamma \omega\).

In a geometrical point of view, the tangent space at \((x,y)\) is divided into cones \(C_{\pm}\) such that \(\omega(X) > 0\) (respectively \(< 0\)) on the interior of \(C_{\pm}\) (respectively \(C_{-}\)). We change sign when tangent cones are mapped to opposite sign cones by \(\gamma\). Of course, the foliations (determined by cone boundaries) do not depend on sign, which is why we can use a bigger group of symmetries. We then define:

**Definition 2.1.** Let \(\Gamma\) be a compact Lie group acting linearly on \(\mathbb{R}^2\) and \(\eta: \Gamma \rightarrow \mathbb{Z}_2 = \{\pm 1\}\) a one-dimensional representation of \(\Gamma\). An element \(\omega \in Q(\mathbb{R}^2)\) is \(\Gamma\)-equivariant if, for all \(\gamma \in \Gamma\),

\[
\omega(\gamma \cdot (x,y,dx,dy)) = \eta(\gamma)\omega(x,y,dx,dy).
\]

If \(\omega\) is \(\Gamma\)-equivariant, then the equation (1) is \(\Gamma\)-invariant or, as we shall also say, \(\Gamma\) is the symmetry group of (1).

The group of symmetries of a BDE generally admits, by its nature, an order-2 normal subgroup, which is precisely the case when the group homomorphism \(\eta\) in Definition 2.1 is nontrivial. If \(\eta\) is trivial, then \(\omega \in Q(\mathbb{R}^2)\) is \(\Gamma\)-invariant.

**Remark 1.** Let \(\Sigma(\Delta) \leq O(2)\) denote the group of symmetries of the discriminant set \(\Delta\), namely, the subgroup of elements of \(O(2)\) that leave \(\Delta\) setwise invariant. Then

\[
\Gamma \leq \Sigma(\Delta).
\]

In other words, symmetries of a BDE are at most the symmetries of the discriminant set. This can be of practical use when detecting the symmetry group of the equation if we know the shape of \(\Delta\).

Solutions of (1) are nonoriented curves, associated with direction fields. At the region on the plane where the discriminant \(\delta\) is positive, these form a pair of transverse foliations \(F_1\) and \(F_2\). The set of symmetries of each foliation, that is, the set of elements that leave them setwise invariant is a subgroup of \(\Gamma\). The structure of this subgroup is discussed below. At the points where the function \(a\) does not vanish, this pair is associated with two oriented foliations given as integral curves of the vector fields

\[
F_i(x,y) = (a(x,y), -b(x,y) + (-1)^i \sqrt{\delta(x,y)}), \quad i = 1, 2.
\]

Consider for example the equation

\[
\omega(x,y,dx,dy) = ydy^2 + 2xdxdy - ydx^2 = 0,
\]

for which the associated vector fields are

\[
F_i(x,y) = (y, -x + (-1)^i \sqrt{x^2 + y^2}), \quad i = 1, 2.
\]

The configuration of this equation is given in Fig. 1(a). Both \(F_1\) and \(F_2\) are reversible-equivariant vector fields under the action of the group \(\mathbb{Z}_2\) generated by the reflection \(\kappa_x\) with respect to the \(x\)-axis,

\[
F_i(\kappa_x(x,y)) = -\kappa_x F_i(x,y), \quad i = 1, 2.
\]
As the picture suggests, this reflection is in fact a symmetry of the BDE. Now, the combination of the two foliations adds symmetries to the whole picture, leading to a configuration which is also symmetric with respect to the reflection on the y-axis. By the nature of this additional symmetry, this element should invert foliations. In fact, we prove that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the symmetry group of the BDE. However, it is more subtle to realize how each element should act on the form $\omega$, in the sense that it is not obvious whether $\eta(\gamma) = 1$ or $-1$ for each $\gamma$ in the group. As we shall see in Theorem 2.2, this depends not only whether $\eta$ preserves or interchanges foliations, but also whether it preserves or inverts orientation on the plane.

Before we state the result, we introduce another homomorphism: Consider the open region in $\mathbb{R}^2$

$$\Omega = \{(x, y) \in \mathbb{R}^2 : \delta(x, y) > 0\},$$

and consider the restriction of the action of $\Gamma$ on $\Omega$. This is well-defined since the discriminant set $\Delta$ is $\Gamma$-invariant and splits the plane into two invariant regions, where $\delta$ is positive or negative. For BDEs (1) for which $\Omega$ is non-empty, we introduce the homomorphism $\lambda : \Gamma \rightarrow \mathbb{Z}_2 = \{1, -1\}$,

$$\lambda(\gamma) = \begin{cases} 1, & \gamma(F_1) = F_i \\ -1, & \gamma(F_1) = F_j, \ j \neq i, \end{cases}$$

(6) $i, j \in \{1, 2\}$. It follows directly from this definition that the subgroup of symmetries of each foliation $F_i$, $i = 1, 2$, is $\Sigma(F_i) = \ker \lambda$.

**Theorem 2.2.** Let $\eta, \lambda : \Gamma \rightarrow \mathbb{Z}_2$ be the two group homomorphisms of Definition 2.1 and of (6). Then, for all $\gamma \in \Gamma$,

$$\lambda(\gamma) = \det(\gamma) \eta(\gamma).$$

**Proof of Theorem 2.2.** Let $(dx_1, dy_1)$ and $(dx_2, dy_2)$ denote two pairs of vectors in the tangent directions of the pair of foliations $F_1, F_2$ at $(x, y)$, respectively. For $\gamma \in \Gamma$, let $(\tilde{d}x_1, \tilde{d}y_1)$ and $(\tilde{d}x_2, \tilde{d}y_2)$ denote their image under $\gamma$ in the tangent directions of $F_1, F_2$ at $\gamma(x, y)$, respectively.

If $\lambda(\gamma) = 1$, then

$$\begin{vmatrix} \tilde{d}x_1 & \tilde{d}x_2 \\ \tilde{d}y_1 & \tilde{d}y_2 \end{vmatrix} = \gamma \begin{vmatrix} dx_1 & dx_2 \\ dy_1 & dy_2 \end{vmatrix} \det(\gamma) \begin{vmatrix} dx_1 & dx_2 \\ dy_1 & dy_2 \end{vmatrix}.$$

If $\lambda(\gamma) = -1$ then the two columns of the determinant in the middle above interchange. Hence,

$$\begin{vmatrix} \tilde{d}x_1 & \tilde{d}x_2 \\ \tilde{d}y_1 & \tilde{d}y_2 \end{vmatrix} = \lambda(\gamma) \det(\gamma) \begin{vmatrix} dx_1 & dx_2 \\ dy_1 & dy_2 \end{vmatrix}.$$

On the other hand, we write

$$a(x, y) \omega(x, y, dx, dy) = (ady + (b - \sqrt{\delta})dx)(ady + (b + \sqrt{\delta})dx),$$

with $a, b, c$ on the right computed at $(x, y)$. Here we assume w.l.g. that $a$ does not vanish at some point on $\Omega$; otherwise, the factorization of $\omega$ has just another expression and we proceed in the same way. We then have

$$ady_1 + (b - \sqrt{\delta})dx_1 = ady_2 + (b + \sqrt{\delta})dx_2 = 0.$$

Now, $(\tilde{d}x_1, \tilde{d}y_1)$ and $(\tilde{d}x_2, \tilde{d}y_2)$ are the two solution directions of $\omega = 0$ at $\gamma(x, y)$, and, from (3),

$$a(x, y) \omega(\gamma(x, y), d\tilde{x}, d\tilde{y}) = \eta(\gamma)(ady + (b - \sqrt{\delta})dx)(ady + (b + \sqrt{\delta})dx).$$
with the functions on the right-hand side computed at \((x, y)\). Since \(\eta^2(\gamma) = 1\), we can rewrite
\[
\eta(\gamma)a(x, y) \omega(\gamma(x, y), d\tilde{x}, d\tilde{y}) = (\eta(\gamma)ady + (\eta(\gamma)b - \sqrt{3})dx)(\eta(\gamma)ady + (\eta(\gamma)b + \sqrt{3})dx),
\]
which implies that the (ordered) pair \((dx_1, dy_1), (dx_2, dy_2)\) is mapped into \((d\tilde{x}_1, d\tilde{y}_1), (d\tilde{x}_2, d\tilde{y}_2)\) if \(\eta(\gamma) = 1\) and into \((dx_1, dy_1), (dx_1, dy_1)\) if \(\eta(\gamma) = -1\), and so
\[
\begin{vmatrix}
   d\tilde{x}_1 & d\tilde{x}_2 \\
   d\tilde{y}_1 & d\tilde{y}_2 
\end{vmatrix} = \eta(\gamma) \begin{vmatrix}
   dx_1 & dx_2 \\
   dy_1 & dy_2 
\end{vmatrix}.
\]

\[\Box\]

**Remark 2.** In practice, Theorem 2.2 adds information to the inclusion (4) when detecting the symmetry group \(\Gamma\) of a BDE. In fact, it provides the construction of the homomorphism \(\eta\) by the geometrical investigation of whether each element \(\gamma \in \Gamma\) preserves the foliations \((\lambda(\gamma) = 1)\) or interchanges the foliations \((\lambda(\gamma) = -1)\). To illustrate, consider the pictures in Fig. 1. In 1(a) and 1(b), foliations are interchanged by \(\kappa_y\), whereas they are preserved by \(\kappa_x\), and now we use \(\text{det}(\kappa_x) = \text{det}(\kappa_y) = -1\) to conclude by Theorem 2.2 that \(\eta(\kappa_x) = -\eta(\kappa_y) = -1\). These are the generators of the symmetry group \(\mathbb{Z}_2 \times \mathbb{Z}_2\), and so the homomorphism \(\eta\) is well-established for these examples. In 1(c) foliations are interchanged by \(\kappa_y\) and rotation of \(\pi/3\); since these are orientation preserving and orientation preserving respectively, it follows that \(\eta\) assumes 1 and \(-1\), respectively. These are the generators of the symmetry group \(D_6\), and so the homomorphism \(\eta\) is well-established.

The following property is direct from Theorem 2.2:

**Corollary 1.** If \(\ker \lambda \cap \ker \eta\) is finite, then it is a cyclic subgroup of \(\Gamma\).

**Remark 3.** Although Definition 2.1 is given here for quadratic 1-forms, the equivariance condition (3) can be given for general 1-forms. For the linear case, it follows that
\[
\alpha(x, y, dx, dy) = A(x, y)dy + B(x, y)dx
\]
is \(\Gamma\)-equivariant if, and only if, the associated planar mapping \(F = (A, B)\) is \(\Gamma\)-equivariant in the sense of Definition 3.1 below. For an investigation of pairs of symmetric linear 1-forms and their connection with symmetries of an associated BDE we refer to [17].

A quadratic differential form (2) is associated to the matrix-valued mapping \(B : \mathbb{R}^2 \rightarrow \text{Sym}_2\),
\[
B(x, y) = \begin{pmatrix}
   c(x, y) & b(x, y) \\
   b(x, y) & a(x, y)
\end{pmatrix},
\]
where \(\text{Sym}_2\) denotes the space of symmetric matrices of order 2. Then (2) can be written as
\[
\omega = \left( \begin{array}{c}
   dx \\
   dy 
\end{array} \right)^t B(x, y) \left( \begin{array}{c}
   dx \\
   dy 
\end{array} \right),
\]
where superscript \(t\) denotes transposition. From (3), it follows that symmetries of (1) are given by the following equivariance condition of \(B\), with the action on the target given by the homomorphism \(\eta\) and conjugacy:
\[
B(\gamma(x, y)) = \eta(\gamma)\gamma B(x, y)\gamma^t, \ \forall \ \gamma \in \Gamma.
\]
From now on we shall use this matricial notation to investigate symmetries in BDEs.
3. **Invariant theory.** In this section we register the generalization of the results of [1] for \( \Gamma \)-equivariant mappings \( V \to W \) (possibly distinct representations in the source and target). Proofs here adapt straightforwardly from [1]. The results are then applied to the systematic study of matrix-valued mappings (7) satisfying (8), for which \( V = \mathbb{R}^2 \), \( W = \text{Sym}^2 \) with the action of \( \Gamma \) on \( W \) defined from the representation of \( \Gamma \) on \( V \) by conjugacy. These are the basics of the deduction of general forms of equivariant differential forms developed in Section 4.

We start with a brief summary about algebraic invariant theory of compact Lie groups. We consider a compact Lie group \( \Gamma \) acting linearly on real vector spaces of finite dimension. For a given action of \( \Gamma \) on a vector space \( V \) of dimension \( n \), \( \Gamma \times V \to V \), \( (\gamma, x) \mapsto \gamma x \), there is a \( \Gamma \)-invariant inner product on \( V \) under which the associated representation \( \rho : \Gamma \to \text{GL}(n) \), \( \rho(\gamma)x = \gamma x \), is orthogonal, i.e. for \( \gamma \in \Gamma \), \( \rho(\gamma) \in O(n) \), the group of orthogonal matrices of order \( n \) ([11, XII, Proposition 1.3]). Hence, Lie groups in this paper are the closed subgroups of \( O(n) \). In what follows we sometimes use the pair \((V, \rho)\) to stress that the vector space \( V \) is under the action of \( \Gamma \) associated with \( \rho \).

A real function \( f : V \to \mathbb{R} \) is \( \Gamma \)-invariant if
\[
f(\rho(\gamma)x) = f(x), \quad \forall \gamma \in \Gamma, \; \forall x \in V.
\]

The set \( \mathcal{P}(\Gamma) \) of \( \Gamma \)-invariant polynomials is a ring over \( \mathbb{R} \). A finite set \( \{u_1, \ldots, u_s\} \) of \( \Gamma \)-invariants generating this ring is called a Hilbert basis. The existence of a Hilbert basis was proved by Weyl in 1946, and Schwarz proved in 1975 that the same set generates the ring of \( C^\infty \) \( \Gamma \)-invariant germs (see [11, XII, Proposition 4.2]).

For \( (\rho, V) \) and \( (\nu, W) \) representations of \( \Gamma \), a mapping \( g : V \to W \) is \( \Gamma \)-equivariant if
\[
g(\rho(\gamma)x) = \nu(\gamma)g(x), \quad \forall \gamma \in \Gamma, \; \forall x \in V.
\]

The set \( \overrightarrow{\mathcal{P}}(\Gamma) \) of \( \Gamma \)-equivariant mappings \( (V, \rho) \to (W, \nu) \) with polynomial entries is a module over \( \mathcal{P}(\Gamma) \). Poénaru in 1976 [11, XII, Proposition 6.8] proved that \( \overrightarrow{\mathcal{P}}(\Gamma) \) is finitely generated over the ring \( \mathcal{P}(\Gamma) \) and that the same set generates the module of equivariant \( C^\infty \) germs over the ring of invariant germs.

We shall also consider a one-dimensional representation of \( \Gamma \),
\[
\eta : \Gamma \to \mathbb{Z}_2 = \{\pm 1\}, \tag{9}
\]

which is a group homomorphism with \( \Gamma_+ = \ker \eta \) a normal subgroup of \( \Gamma \) of index 2 if \( \eta \) is nontrivial. The \( \eta \)-dual representation of \( (\nu, W) \), denoted by \( \nu_\eta \), is defined by the product
\[
\gamma \mapsto \nu_\eta(\gamma) = \eta(\gamma)\nu(\gamma).
\]

**Definition 3.1.** Let \( \eta : \Gamma \to \mathbb{Z}_2 \) be a group homomorphism as in (9) and \( (\rho, V) \) and \( (\nu, W) \) representations of \( \Gamma \). We define the \( \mathcal{P}(\Gamma) \)-modules of polynomials
\[
\mathcal{P}^\eta(\Gamma) := \{ f : V \to \mathbb{R} : f(\rho(\gamma)x) = \eta(\gamma)f(x), \quad \forall \gamma \in \Gamma, \; \forall x \in V \}
\]
and
\[
\overrightarrow{\mathcal{P}}^\eta(\Gamma) := \{ g : V \to W : g(\rho(\gamma)x) = \eta(\gamma)\nu(\gamma)g(x), \quad \forall \gamma \in \Gamma, \; \forall x \in V \}.
\]

A polynomial function \( f \in \mathcal{P}^\eta(\Gamma) \) is \( \Gamma \)-equivariant from \( (\rho, V) \) to \( (\eta, \mathbb{R}) \), and a polynomial mapping \( g \in \overrightarrow{\mathcal{P}}^\eta(\Gamma) \) is \( \Gamma \)-equivariant from \( (\rho, V) \) to \( (\nu_\eta, W) \), so the finitude of generators for each as \( \mathcal{P}(\Gamma) \)-modules follows by Poénaru’s theorem mentioned above. It also follows that generators of equivariant polynomials are also generators
Let $\Gamma$ be a closed subgroup of $\text{Algorithm 3.2.}$

This is done through an algebraic algorithm to compute generators of $\overline{P}(\Gamma)$ from the knowledge of generators of $\overline{P}(\Gamma_+)$, when source and target spaces are the same.

In Proposition 1 and Algorithm 3.2 we generalize this, with a similar algorithm to compute generators of $\Gamma$-equivariants with possibly distinct source and target.

We follow the notation used in [1] to introduce the Reynolds operators $R : P(\Gamma+) \to P(\Gamma+)$ and $\overline{R} : \overline{P}(\Gamma+) \to \overline{P}(\Gamma+)$,

$$R(f)(x) = 1/2(f(x) + f(\rho(\delta)x)) \quad \overline{R}(g)(x) = 1/2(g(x) + \nu(\delta)^{-1}g(\rho(\delta)x)),$$

and, the $\eta$-Reynolds operators on $P(\Gamma_+)$ and on $\overline{P}(\Gamma_+)$, $S : P(\Gamma_+) \to P(\Gamma_+)$ and $\overline{S} : \overline{P}(\Gamma_+) \to \overline{P}(\Gamma_+)$,

$$S(f)(x) = 1/2(f(x) - f(\rho(\delta)x)) \quad \overline{S}(g)(x) = 1/2(g(x) - \nu(\delta)^{-1}g(\rho(\delta)x)),$$

for an arbitrary fixed $\delta \in \Gamma \setminus \Gamma_+$.

Let us denote by $I_{P(\Gamma_+)}$ and $I_{\overline{P}(\Gamma_+)}$ the identity maps on $P(\Gamma_+)$ and on $\overline{P}(\Gamma_+)$, respectively.

**Proposition 1.** The operators above satisfy the following:

(a): They are homomorphisms of $P(\Gamma)$-modules and

$$R + S = I_{P(\Gamma_+)} \quad \text{and} \quad \overline{R} + \overline{S} = I_{\overline{P}(\Gamma_+)}.$$

(b): They are idempotent projections and the following direct sum decompositions of $P(\Gamma)$-modules hold:

$$P(\Gamma_+) = P(\Gamma) \oplus P^\eta(\Gamma) \quad \text{and} \quad \overline{P}(\Gamma_+) = \overline{P}(\Gamma) \oplus \overline{P}^\eta(\Gamma). \quad (10)$$

**Proof of Proposition 1.** Analogous to Propositions 2.3 and 2.4 in [1].

The algorithm is based on the decompositions (10) and on the projection operators $S$ and $\overline{S}$ applied to a given Hilbert basis of $P(\Gamma_+)$ and a set of generators of $\overline{P}(\Gamma_+)$. The procedure is:

**Algorithm 3.2.** Let $\Gamma$ be a closed subgroup of $O(n)$ and $\eta : \Gamma \to \mathbb{Z}_2$ a homomorphism with $\ker \eta = \Gamma_+$, $\{u_1, \ldots, u_s\}$ a Hilbert basis of $P(\Gamma_+)$ and $\{H_0, \ldots, H_r\}$ a generator set of $\overline{P}(\Gamma_+)$ as a $P(\Gamma_+)$-module;

1: Fix $\delta \in \Gamma \setminus \Gamma_+$ arbitrary;
2: For $i \in \{1, \ldots, s\}$, do $\tilde{u}_i = S(u_i), \tilde{u}_0 := 1$;
3: For $i \in \{1, \ldots, s\}$ and $j \in \{0, \ldots, r\}$, do $H_{ij} = \tilde{u}_i H_j$;
4: For $i \in \{1, \ldots, s\}$ and $j \in \{0, \ldots, r\}$, do $\tilde{H}_{ij} = \tilde{S}(H_{ij})$.

**Result:** $\{\tilde{H}_{ij} : 0 \leq i \leq s, 0 \leq j \leq r\}$ is a generator set of $\overline{P}^\eta(\Gamma)$ as a $P(\Gamma)$-module.

As proved in [1], step 2 above provides a generator set of the $P(\Gamma)$-module $P^\eta(\Gamma)$ (these are the anti-invariants in that paper). What we also remark at this point is that replacing $\overline{S}$ by the projection operator $\overline{R}$ in step 4 we obtain, as expected, a direct way to compute a set of generators for the equivariants under the whole group
Γ from the knowledge of equivariants under the subgroup Γ+. This is formalized below:

**Proposition 2.** Let Γ be a compact Lie group acting on V and on W and \( \{ H_{ij} = \hat{u}_i H_j, 0 \leq i \leq s, 0 \leq j \leq r \} \) a generator set of \( \overrightarrow{P}(\Gamma) \) as a \( \mathcal{P}(\Gamma) \)-module given by step 3 in Algorithm 3.2. Then

\[
\{ \overrightarrow{R}(H_{i,j}), 0 \leq i \leq s, 0 \leq j \leq r \}
\]
generates \( \overrightarrow{P}(\Gamma) \) as a \( \mathcal{P}(\Gamma) \)-module.

**Proof of Proposition 2.** Let \( g \in \overrightarrow{P}(\Gamma) \subset \overrightarrow{P}(\Gamma) \). Then

\[
g = \sum_{i,j} p_{ij} H_{ij}, \quad p_{ij} \in \mathcal{P}(\Gamma), \quad 0 \leq i \leq s, 0 \leq j \leq r.
\]

Since \( \overrightarrow{R} \) is a \( \mathcal{P}(\Gamma) \)-homomorphism and \( \overrightarrow{R}(g) = g \), then

\[
g = \overrightarrow{R}(g) = \overrightarrow{R} \left( \sum_{i,j} p_{ij} H_{ij} \right) = \sum_{i,j} p_{ij} \overrightarrow{R}(H_{ij}).
\]

\[\square\]

4. **General forms of symmetric BDEs.** The aim of this section is to present the algebraic forms of BDEs symmetric under the compact subgroups of \( O(2) \) with its standard action on the plane. These are derived from generator sets of the modules of equivariant quadratic forms on the plane

\[
B(x, y) = \begin{pmatrix} c(x, y) & b(x, y) \\ b(x, y) & a(x, y) \end{pmatrix}.
\]

These modules are \( \overrightarrow{P}(\Gamma) \) or \( \overrightarrow{P}(\Gamma) \) of Subsection 3. If the group homomorphism \( \eta : \Gamma \to \mathbb{Z}_2 \) is nontrivial we apply Algorithm 3.2.

For the computations below we shall use the action of (subgroups of) \( O(2) \) on \( \mathbb{R}^2 \approx \mathbb{C} \) with the usual semi-direct product of \( SO(2) \) and \( \mathbb{Z}_2(\kappa) \), using complex coordinates,

\[
\theta \cdot z = e^{i\theta} z, \quad \kappa z = \bar{z}, \quad \theta \in [0, 2\pi], \quad z \in \mathbb{C}.
\]

In Subsections 4.1 and 4.2 we derive the general forms of symmetric BDEs under the cyclic group \( \mathbb{Z}_n, n \geq 3 \), and in Subsection 4.3 the general forms under \( \mathbb{Z}_2 \), for all possible homomorphisms \( \eta \). For the other compact subgroups of \( O(2) \), and their related homomorphisms \( \eta \) the computations are similar and shall be omitted. In Subsection 4.4 all general forms are given in Table 1.

For a given choice of a group \( \Gamma \) and a homomorphism \( \eta \) we use the notation \( \Gamma[\ker \eta] \) when \( \eta \) is nontrivial. This notation is motivated by the fact that the definition of \( \eta \) is determined by the subgroup \( \ker \eta \), which also motivates the use of the notation \( \mathcal{P}[\Gamma, \ker \eta] \) for \( \mathcal{P}(\Gamma) \) and \( \overrightarrow{P}[\Gamma, \ker \eta] \) for \( \overrightarrow{P}(\Gamma) \).

4.1. **\( \mathbb{Z}_n \)-equivariant quadratic forms.** Here we consider the cyclic group \( \mathbb{Z}_n \), \( n \geq 3 \), with \( \eta \) trivial. We compute generators of \( \overrightarrow{P}(\mathbb{Z}_n) \) by computing generators of \( \mathcal{M}(\mathbb{Z}_n) \), the module of \( \mathbb{Z}_n \)-equivariant matrix-valued mappings \( \mathbb{R}^2 \to M_2(\mathbb{R}^2) \), and projecting onto the space of mappings \( \mathbb{R}^2 \to Sym_2 \). In complex coordinates we write any element of \( \mathcal{M}(\mathbb{Z}_n) \) as

\[
z \mapsto \alpha(z) w + \beta(z) \bar{w}, \forall w \in \mathbb{R}^2,
\]

(12)
for functions $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$, with $\alpha_j, \beta_j, j = 1, 2$, real functions. Associating it with the real matrix
\[
M = \begin{pmatrix}
\alpha_1 + \beta_1 & \beta_2 - \alpha_2 \\
\alpha_2 + \alpha_1 & \alpha_1 - \beta_1
\end{pmatrix},
\]
the desired quadratic forms are obtained by the projection
\[
M \mapsto B = 1/2(M + M^t),
\]
after imposing the $\mathbb{Z}_n$-symmetry condition. Write (12) as
\[
M(z)w = \sum \alpha_{jk}z^j\bar{z}^kw + \sum \beta_{jk}z^j\bar{z}^k\bar{w}, \quad \alpha_{jk}, \beta_{jk} \in \mathbb{C}.
\]
The equivariance with respect to $\theta \in \mathbb{Z}_n$ gives
\[
M(z)w = \sum \alpha_{jk}e^{i\theta(j-k)}z^j\bar{z}^kw + \sum \beta_{jk}e^{i\theta(j-k-2)}z^j\bar{z}^k\bar{w}.
\]
So $\alpha_{jk} = \alpha_{jk}e^{i\theta(j-k)}$ and $\beta_{jk} = \beta_{jk}e^{i\theta(j-k-2)}$ for $\theta = 2k\pi/n$, $k = 1, \ldots, n$, and so
\[
\alpha_{jk} = 0 \text{ ou } j \equiv k \pmod{n} \quad \text{and} \quad \beta_{jk} = 0 \text{ ou } j \equiv k + 2 \pmod{n}.
\]
A Hilbert basis for $\mathcal{P}(\mathbb{Z}_n)$ is given in [9],
\[
\{u_1 = z\bar{z}, u_2 = z^n + \bar{z}^n, u_3 = i(z^n - \bar{z}^n)\}.
\]

Factor out $z\bar{z}$ in (14) and use (15) to get
\[
M(z)w = \sum \alpha_{jk}(z\bar{z})^{k-j}z^j\bar{z}^kw + \sum \beta_{jk}(z\bar{z})^{k-j-1}z^j\bar{z}^k\bar{w} + \sum \beta_{jk}(z\bar{z})^{j-1}z^j\bar{z}^kw + \sum \beta_{jk}(z\bar{z})^{j-2}z^j\bar{z}^k\bar{w} + \sum \beta_{jk}(z\bar{z})^{j-3}z^j\bar{z}^k\bar{w} + \sum \beta_{jk}(z\bar{z})^{j-4}z^j\bar{z}^k\bar{w}.
\]
where $c' \in \mathbb{C}$, $l, l_2 \in \mathbb{N}$, $t = 1, \ldots, 4$, $l_1, l_2, l_3 \geq 0$ and $l_4 \geq 1$. We now use the identities
\[
\begin{align*}
z^{ln} &= (z^n + \bar{z}^n)z^{(l-1)n} - (z\bar{z})^n z^{(l-2)n} \\
\bar{z}^{ln} &= (z^n + \bar{z}^n)\bar{z}^{(l-1)n} - (z\bar{z})^n \bar{z}^{(l-2)n} \\
z^{ln+2} &= (z^n + \bar{z}^n)z^{(l-1)n+2} - (z\bar{z})^n z^{(l-2)n+2} \\
\bar{z}^{ln-2} &= (z^n + \bar{z}^n)\bar{z}^{(l-1)n-2} - (z\bar{z})^n \bar{z}^{(l-2)n-2} \\
z^n &= (z^n + \bar{z}^n) - z^n \\
z^{n+2} &= (z^n + \bar{z}^n)z^2 - (z\bar{z})^2 z^n \\
\bar{z}^{l+1-n} &= (z^n + \bar{z}^n)\bar{z}^{(l-1)n} - (z\bar{z})^n \bar{z}^{(l-2)n+2} \\
\bar{z}^{l+1-n-2} &= (z^n + \bar{z}^n)\bar{z}^{(l-1)n-2} - (z\bar{z})^n \bar{z}^{(l-2)n+2}
\end{align*}
\]
to conclude that a set of generators of $\mathcal{M}_2(\mathbb{Z}_n)$ over $\mathcal{P}(\mathbb{Z}_n)$ is given by the elements
\[
M_1(z)w = w, \quad M_2(z)w = iw, \quad M_3(z)w = z^2\bar{w}, \quad M_4(z)w = i\bar{z}^2\bar{w}, \quad M_5(z)w = \bar{z}^{-2}\bar{w}, \quad M_6(z)w = \bar{z}^{-2}\bar{w}, \quad M_7(z)w = \bar{z}^n w, \quad M_8(z)w = iz^n w.
\]
We now apply the projection (13) to the elements above to find generators of $\overline{\mathcal{P}}(\mathbb{Z}_n)$,
\[
B_1(z)w = w, \quad B_3(z)w = z^2\bar{w}, \quad B_4(z)w = iz^2\bar{w}, \quad B_5(z)w = \bar{z}^{-2}\bar{w}, \quad B_6(z)w = iz^{-2}\bar{w}, \quad B_7(z)w = \bar{z}^n w, \quad B_8(z)w = iz^n w.
\]
4.2. $\mathbb{Z}_n[\mathbb{Z}_n/2]$-equivariant quadratic forms, for $n \geq 4$ even. In this case, ker $\eta = \mathbb{Z}_n/2$. From the preceding subsection we extract

$$H_0(z)w = w, H_1(z)w = z^2 \tilde{w}, H_2(z)w = iz^2 \tilde{w}, H_3(z)w = z^{n/2-2} \tilde{w}, H_4(z)w = iz^{n/2-2} \tilde{w}$$

as generators of $\mathcal{P}(\mathbb{Z}_n/2)$ over the ring $\mathcal{P}(\mathbb{Z}_n/2)$ whose Hilbert basis is

$$\{u_1(z) = z \tilde{z}, u_2(z) = z^{n/2} + \tilde{z}^{n/2}, u_3(z) = i(z^{n/2} - \tilde{z}^{n/2})\},$$

We now apply Algorithm 3.2:

1. Fix $\delta = e^{2\pi i/n} \in \mathbb{Z}_n \setminus \mathbb{Z}_n/2$.

2. Generators of $\mathcal{P}[\mathbb{Z}_n, \mathbb{Z}_n/2]$ over $\mathcal{P}(\mathbb{Z}_n)$:

$$\tilde{u}_1(z) = S(u_1)(z) = \frac{1}{2}(z \tilde{z} - (e^{2\pi i/n}z)(e^{-2\pi i/n} \tilde{z})) = 0.$$

$$\tilde{u}_2(z) = S(u_2)(z) = \frac{1}{2}(z^{n/2} + \tilde{z}^{n/2} - (e^{\pi i}z^{n/2} + e^{-\pi i} \tilde{z}^{n/2}) = z^{n/2} + \tilde{z}^{n/2}.$$

$$\tilde{u}_3(z) = S(u_3)(z) = \frac{1}{2}(i(z^{n/2} - \tilde{z}^{n/2}) - i((e^{\pi i}z^{n/2} - e^{-2\pi i} \tilde{z}^{n/2})) = i(z^{n/2} - \tilde{z}^{n/2}).$$

3. Generators of $\mathcal{P}[\mathbb{Z}_n, \mathbb{Z}_n/2]$ over $\mathcal{P}(\mathbb{Z}_n/2)$: set $\tilde{u}_0(z) = 1$,

$$H_{0j}(z)w = \tilde{u}_0(z)H_j(z)w = H_j(z)w, \ j = 0, ..., 4;$$

$$H_{1j}(z)w = \tilde{u}_1(z)H_j(z)w = 0, \ j = 0, ..., 4;$$

$$H_{20}(z)w = \tilde{u}_2(z)H_0(z)w = (z^{n/2} + \tilde{z}^{n/2})w;$$

$$H_{21}(z)w = \tilde{u}_2(z)H_1(z)w = (z^{n/2+2} + (z \tilde{z})^2z^{n/2-2}) \tilde{w};$$

$$H_{22}(z)w = \tilde{u}_2(z)H_2(z)w = i(z^{n/2+2} + (z \tilde{z})^2z^{n/2-2}) \tilde{w};$$

$$H_{23}(z)w = \tilde{u}_2(z)H_3(z)w = (z^{n/2-2} + (z \tilde{z})^{n/2-2}z^2) \tilde{w};$$

$$H_{24}(z)w = \tilde{u}_2(z)H_4(z)w = i(z^{n/2-2} + (z \tilde{z})^{n/2-2}z^2) \tilde{w};$$

$$H_{30}(z)w = \tilde{u}_3(z)H_0(z)w = i(z^{n/2} - \tilde{z}^{n/2})w;$$

$$H_{31}(z)w = \tilde{u}_3(z)H_1(z)w = i(z^{n/2+2} - (z \tilde{z})^2z^{n/2-2}) \tilde{w};$$

$$H_{32}(z)w = \tilde{u}_3(z)H_2(z)w = -(z^{n/2+2} + (z \tilde{z})^2z^{n/2-2}) \tilde{w};$$

$$H_{33}(z)w = \tilde{u}_3(z)H_3(z)w = i(-z^{n-2} + (z \tilde{z})^{n/2-2}z^2) \tilde{w};$$

$$H_{34}(z)w = \tilde{u}_3(z)H_4(z)w = (z^{n-2} - (z \tilde{z})^{n/2-2}z^2) \tilde{w},$$

which, as an intermediate step, we simplify to the reduced list

$$H_{00}(z)w = w, H_{01}(z)w = z^2 \tilde{w}, H_{02}(z)w = iz^2 \tilde{w}, H_{03}(z)w = z^{n/2-2} \tilde{w},$$

$$H_{04}(z)w = iz^{n/2-2} \tilde{w}, H_{20}(z)w = (z^{n/2} - \tilde{z}^{n/2})w,$$

$$H_{21}(z)w = z^{n/2+2} \tilde{w}, H_{22}(z)w = iz^{n/2+2} \tilde{w},$$

$$H_{23}(z)w = z^{n-2} \tilde{w}, H_{24}(z)w = iz^{n-2} \tilde{w}, H_{30}(z)w = i(z^{n/2} - \tilde{z}^{n/2})w.$$
4. Generators of $\mathcal{P}[\mathbb{Z}_n, \mathbb{Z}_{n/2}]$ over $\mathcal{P}(\mathbb{Z}_n)$:

$H_{00}(z)w = \tilde{H}_{01}(z)w = \tilde{H}_{02}(z)w = 0$

$\tilde{H}_{03}(z)w = \tilde{z}^{n/2} - z^{n/2}w$

$\tilde{H}_{04}(z)w = i\tilde{z}^{n/2} - z^{n/2}w$

$\tilde{H}_{20}(z)w = (\tilde{z}^{n/2} + \tilde{z}^{n/2})w$

$\tilde{H}_{21}(z)w = z^{n/2} + \tilde{z}^{n/2}w$

$\tilde{H}_{22}(z)w = iz^{n/2} + \tilde{z}^{n/2}w$

$\tilde{H}_{23}(z)w = \tilde{H}_{24}(z)w = 0$

$\tilde{H}_{20}(z)w = i(\tilde{z}^{n/2} - z^{n/2})w$

Therefore, $\mathcal{P}[\mathbb{Z}_n, \mathbb{Z}_{n/2}]$ is the $\mathcal{P}(\mathbb{Z}_n)$-module generated by

$\tilde{B}_1(z)w = \tilde{z}^{n/2} - z^{n/2}w$, $\tilde{B}_2(z)w = iz^{n/2} - \tilde{z}^{n/2}w$, $\tilde{B}_3(z)w = (\tilde{z}^{n/2} + \tilde{z}^{n/2})w$

$\tilde{B}_4(z)w = z^{n/2} + \tilde{z}^{n/2}w$, $\tilde{B}_5(z)w = iz^{n/2} + \tilde{z}^{n/2}w$, $\tilde{B}_6(z)w = i(\tilde{z}^{n/2} - z^{n/2})w$.

4.3. $\mathbb{Z}_2$-equivariant quadratic forms. Let $\mathbb{Z}_2$ be the group generated by the reflection $\kappa_x$ on the $x$-axis. First we consider $\eta : \mathbb{Z}_2(\kappa_x) \to \mathbb{Z}_2$ trivial. Imposing the $\mathbb{Z}_2$-equivariance to (11) gives

$$
\begin{pmatrix}
  c(x, -y) & b(x, -y) \\
  b(x, -y) & a(x, -y)
\end{pmatrix} =
\begin{pmatrix}
  c(x, y) & -b(x, y) \\
  -b(x, y) & a(x, y)
\end{pmatrix}.
$$

This is to say that $a$ and $c$ are $\mathbb{Z}_2[1]$-equivariant and $b$ is $\mathbb{Z}_2$-anti-invariant. Therefore, the generators of $\mathcal{P}(\mathbb{Z}_2)$ under $\mathcal{P}(\mathbb{Z}_2)$ are

$$(x, y) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, (x, y) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, (x, y) \mapsto \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}.$$ 

Assume now $\eta$ nontrivial, so $\ker \eta = 1$. Imposing the $\mathbb{Z}_2[1]$-equivariance to (11) gives

$$
\begin{pmatrix}
  c(x, -y) & b(x, -y) \\
  b(x, -y) & a(x, -y)
\end{pmatrix} =
\begin{pmatrix}
  -c(x, y) & b(x, y) \\
  b(x, y) & -a(x, y)
\end{pmatrix}.
$$

Hence $b$ is $\mathbb{Z}_2$-invariant and the functions $a$ and $c$ are $\mathbb{Z}_2[1]$-equivariant. Therefore, the generators for $\mathcal{P}(\mathbb{Z}_2, 1)$ under $\mathcal{P}(\mathbb{Z}_2)$ are

$$(x, y) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (x, y) \mapsto \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, (x, y) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

4.4. Summarizing table and illustrations. In this subsection we present the general forms of symmetric quadratic differential 1-forms $\omega = a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dy^2$ under compact subgroups of $\mathcal{O}(2)$. Table 1 shows each group $\Gamma$ with all possible values of $\ker \eta$, denoted by $\Gamma[\ker \eta]$. Following the previous notation, when $\eta$ is trivial the group is denoted simply by $\Gamma$. Also, $\mathcal{D}_n(\kappa_x)$ and $\mathcal{D}_n(\kappa_y)$ shall denote the dihedral groups generated by the rotation of angle $2\pi/n$ and by the reflections with respect to the $x$-axis or $y$-axis, respectively.

In [4] the authors consider BDEs whose discriminant function $\delta = b^2 - ac$ is of Morse type. In this case, the discriminant set is a pair of transversal straight lines by the origin or the origin itself. They prove that these BDEs are topologically equivalent to their linear part. We remark that all the normal forms that they obtain must be equivariant under a finite symmetry group. In fact, it follows from Table 1 that there are no linear BDEs with infinite group of symmetries. As it appears
in [4], the Morse condition is given in terms of the coefficients of the linear part of the smooth functions \(a, b\) and \(c\). More precisely, if we write
\[
a = a_1 x + a_2 y + \omega(2),
\]
\[
b = b_1 x + b_2 y + \omega(2)
\]
and
\[
c = c_1 x + c_2 y + \omega(2),
\]
then the condition is
\[
(c_2 a_1 - c_1 a_2)^2 - 4(b_2 a_1 - b_1 a_2)(c_2 b_1 - c_1 b_2) \neq 0.
\]
From Table 1, the possible symmetry groups of BDEs whose linear parts satisfy (16) are
\[
Z_3, Z_6[Z_3], D_3, D_3[Z_3], D_6[D_3]
\]
(17) or
\[
Z_2, Z_2[1], Z_2 \times Z_2[Z_2(\kappa_x)]
\]
(18)

Recall from Remark 1 that the set of all symmetries of a BDE is at most the symmetry group \(\Sigma(\Delta)\) of the discriminant set. Hence, for the Morse cases it follows that if \(\Delta\) is the origin, then the possible nontrivial symmetry groups are the ones in (17), whereas the groups listed in (18) are the possible groups when the discriminant set is a pair of transversal straight lines. We also point out that the finiteness of the symmetry group also holds for equations with constant coefficients. A classification of these two types of BDEs is done in [17], including an analysis of the corresponding group of symmetries of the equation with possible number of invariant lines in the associated configuration.

**Remark 4.** The symmetry group of the configuration shown in Figure 1(c) is \(D_6[D_3(\kappa_y)]\), whose quadratic form \((y, x, -y)\) appears in Table 1 by interchanging the variables \(x\) and \(y\) and taking \(p_1 \equiv 1\) and \(p_2 \equiv p_3 \equiv 0\) in the general form for the group \(D_6[D_3(\kappa_x)]\). Similarly, the symmetry group of the configurations in Figure 1(a) and 1(b) is \(Z_2 \times Z_2[Z_2(\kappa_x)]\), whose quadratic forms appear from the data for \(Z_2 \times Z_2[Z_2(\kappa_x)]\) in Table 1 by interchanging \(x\) and \(y\) and taking \(p_1 \equiv p_2 \equiv 1, p_3 \equiv -1\), and \(p_1 \equiv 1, p_2 \equiv \frac{1}{2}, p_3 \equiv -1\), respectively.

We finish this paper with an example of each symmetry type given in Table 1. Let us point out that some of these configurations can be realized for example as lines of curvatures or as asymptotic lines of surfaces immersed in \(\mathbb{R}^3\) or \(\mathbb{R}^4\), whereas some others cannot. This is an interesting issue in differential geometry that we have started to investigate in presence of symmetries. For the context without symmetry we cite [16, 10, 15].

![Figure 2](image-url)

**Figure 2.** Configurations with symmetry (a) \(\text{SO}(2)\), (b) \(\text{O}(2)\) and (c) \(\text{O}(2)[\text{SO}(2)]\).

For \(\text{SO}(2)\), we choose \(p_1 \equiv p_2 \equiv p_3 \equiv 1\) in Table 1, so that the differential form is
\[
1 + y^2 - x^2 + 2xy, x^2 - y^2 + 2xy, 1 + x^2 - y^2 - 2xy.
\]
| $\Gamma[\ker \eta]$ | $\ker \lambda$ | General form |
|----------------------|-----------------|--------------|
| SO(2)                | SO(2)           | $a = p_1 + (y^2 - x^2)p_2 + 2xyp_3$; $b = 2xyp_2 + (x^2 - y^2)p_3$; $c = p_1 + (x^2 - y^2)p_2 - 2xyp_3$, $p_i \in \mathcal{P}(\text{SO}(2))$, $i = 1, 2, 3$. |
| O(2)                 | SO(2)           | $a = p_1 + (y^2 - x^2)p_2$; $b = 2xyp_2$; $c = p_1 + (x^2 - y^2)p_2$, $p_i \in \mathcal{P}(\text{O}(2))$, $i = 1, 2, 3$. |
| O(2)|SO(2)| $a = 2xyp$; $b = (x^2 - y^2)p$; $c = -2xyp$, $p \in \mathcal{P}(\text{O}(2))$. |
| $\mathbb{Z}_n$, $n \geq 3$ | $\mathbb{Z}_n$ | $a = p_1 + (y^2 - x^2)p_2 + 2xyp_3 - A_1p_4 - A_2p_5$; $b = 2xyp_2 + (x^2 - y^2)p_3 + A_1p_5 - A_2p_4$; $c = p_1 + (x^2 - y^2)p_2 - 2xyp_3 + A_1p_4 + A_2p_5$, $p_i \in \mathcal{P}(\mathbb{Z}_n)$, $i = 1, ..., 5$. |
| $\mathbb{Z}_n[\mathbb{Z}_n/2]$, $n \geq 4$ even | $\mathbb{Z}_{n/2}$ | $a = -A_3p_1 - A_4p_2 + A_5p_3 - A_6p_4 + A_7p_5 + A_8p_6 + A_9p_7$; $b = A_4p_1 + A_5p_2 + A_6p_3 + A_7p_4 + A_8p_5 + A_9p_6$; $c = A_3p_1 + A_4p_2 + A_5p_3 + A_6p_4 - A_7p_5 - A_8p_6 - A_9p_7$, $p_i \in \mathcal{P}(\mathbb{Z}_n)$, $i = 1, ..., 6$. |
| $\mathbb{D}_n$, $n \geq 3$ | $\mathbb{D}_n$ | $a = p_1 + (y^2 - x^2)p_2 - A_1p_3$; $b = 2xyp_2 - A_3p_3$; $c = p_1 + (x^2 - y^2)p_2 + A_1p_3$, $p_i \in \mathcal{P}(\mathbb{D}_n)$, $i = 1, 2, 3$. |
| $\mathbb{D}_n[\mathbb{Z}_n]$, $n \geq 3$ | $\mathbb{D}_n$ | $a = 2xyp_1 - A_3p_2 + A_5p_3$; $b = (x^2 - y^2)p_1 + A_1p_3$; $c = -2xyp_1 + A_2p_2 + A_3p_3$, $p_i \in \mathcal{P}(\mathbb{D}_n)$, $i = 1, 2, 3$. |
| $\mathbb{D}_n[\mathbb{D}_n/2(\kappa_4)]$, $n \geq 4$ even | $\mathbb{D}_{n/2}(\kappa_4)$ | $a = -A_3p_1 + A_4p_2 + A_5p_3$; $b = A_4p_1 + A_5p_2 + A_6p_3$; $c = -A_3p_1 + A_4p_2 + A_5p_3 + A_6p_4 - A_7p_5 - A_8p_6 - A_9p_7$, $p_i \in \mathcal{P}(\mathbb{D}_n)$, $i = 1, 2, 3$. |
| $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $a = py_1$; $b = px_1$; $c = y_1$, $p_i \in \mathcal{P}(\mathbb{Z}_2)$, $i = 1, 2, 3$. |
| $\mathbb{Z}_2[1]$ | $\mathbb{Z}_2$ | $a = py_1$; $b = px_1$; $c = y_1$, $p_i \in \mathcal{P}(\mathbb{Z}_2)$, $i = 1, 2, 3$. |
| $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2(-1)$ | $a = p_1 + b = xyp_2$; $c = p_3$, $p_i \in \mathcal{P}(\mathbb{Z}_2)$, $i = 1, 2, 3$. |
| $\mathbb{Z}_2 \times \mathbb{Z}_2[\mathbb{Z}_2(-1)]$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $a = xyp_1$; $b = px_1$; $c = yx_1$, $p_i \in \mathcal{P}(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $i = 1, 2, 3$. |
| $\mathbb{Z}_2 \times \mathbb{Z}_2[\mathbb{Z}_2(\kappa_4)]$ | $\mathbb{Z}_2(\kappa_4)$ | $a = px_1$; $b = yx_1$; $c = px_3$, $p_i \in \mathcal{P}(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $i = 1, 2, 3$. |

Table 1. General forms of equivariant quadratic differential forms on the plane under closed subgroups of $\text{O}(2)$.

The homomorphism $\lambda$ is trivial and the discriminant function is $\text{O}(2)$-invariant given by

$$\delta(x, y) = 2(x^2 + y^2)^2 - 1.$$ 

This is illustrated in Fig. 2(a).

For $\text{O}(2)$, we choose $p_1 \equiv 0$, $p_2 \equiv 1$, so the differential form is

$$(y^2 - x^2, 2xy, x^2 - y^2).$$
The homomorphism $\lambda$ is such that $\ker \lambda = \text{SO}(2)$ and the discriminant function is $\text{O}(2)$-invariant given by

$$\delta(x, y) = (x^2 + y^2)^2.$$  

This is illustrated in Fig. 2(b).

The configuration in Fig. 2(c) is $\text{O}(2)[\text{SO}(2)]$-symmetric, whose quadratic form has been chosen by taking $p \equiv 1$ in Table 1, that is,

$$(2xy, x^2 - y^2, -2xy).$$

The homomorphism $\lambda$ is trivial and the discriminant function is the $\text{O}(2)$-invariant given by

$$\delta(x, y) = (x^2 + y^2)^2.$$  

![Figure 3. Configurations with symmetry group given by (a) $\mathbb{Z}_5$ and (b) $\mathbb{Z}_4[\mathbb{Z}_2]$.](image)

We now consider $\mathbb{Z}_5$ taking $p_1 \equiv p_2 \equiv p_3 \equiv p_5 \equiv 1$ and $p_3 \equiv p_4 \equiv p_5 \equiv 0$ in Table 1, so that the differential form is

$$(1 + y^2 - x^2 - 3x^2y + y^3, 2xy + x^3 - 3xy^2, 1 + x^2 - y^2 + 3x^2y - y^3).$$

The homomorphism $\lambda$ is necessarily trivial. The discriminant function is the $\mathbb{Z}_5$-invariant given by

$$\delta(x, y) = (x^2 + y^2)^3 + 10x^4y - 20x^2y^3 + 2y^5 + (x^2 + y^2)^2 - 1.$$  

The picture for this case is shown in Fig. 3(a). The star shape of the discriminant set is in fact $\mathbb{Z}_5$-symmetric without reflectional symmetries, as it is easily checked by direct calculation.

Fig. 3(b) is a $\mathbb{Z}_4[\mathbb{Z}_2]$ case, considering $p_1 \equiv p_2 \equiv p_4 \equiv p_5 \equiv 1$ and $p_3 \equiv p_6 \equiv 0$ in Table 1, so that the differential form is

$$a(x, y) = -x^4 + 6x^2y^2 - y^4 + 4x^3y - 4xy^3, \quad b(x, y) = x^4 - 6x^2y^2 + y^4 + 4x^3y - 4xy^3,$$

and

$$c(x, y) = x^4 - 6x^2y^2 + y^4 - 4x^3y + 4xy^3.$$  

The homomorphism $\lambda$ must be such that $\ker \lambda = \mathbb{Z}_2$. The discriminant set is just the origin, given as the zero set of (the $\text{O}(2)$-invariant)

$$\delta(x, y) = 2(x^2 + y^2)^4.$$  

For $\text{D}_5$ in Table 1, we take $p_1 \equiv p_2 \equiv p_3 \equiv 1$, so that the differential form is

$$(1 + y^2 - x^2 - x^3 + 3xy^2, 2xy - 3x^2y + y^3, 1 - y^2 + x^2 + x^3 - 3xy^2).$$

In this case, $\ker \lambda = \mathbb{Z}_5$ and the discriminant function is the $\text{D}_5$-invariant given by

$$\delta(x, y) = (x^2 + y^2)^3 + 2x^5 - 20x^3y^2 + 10xy^4 + (x^2 + y^2)^2 - 1.$$
Figure 4. Configurations with symmetry groups $D_5$, $D_6[Z_6]$ and $D_6[D_3(\kappa_x)]$.

This is illustrated in Fig. 4(a).

We now consider $D_6[Z_6]$ choosing $p_1 \equiv p_2 \equiv 1$ and $p_3 \equiv 0$ in Table 1, so that the form is

$$(2xy - 4x^3y + 4xy^3, x^2 - y^2 + x^4 - 6x^2y^2 + y^4, -2xy + 4x^3y - 4xy^3).$$

In this case $\lambda$ is trivial and the discriminant function is $D_6$-invariant and given by

$$\delta(x, y) = (x^2 + y^2)^4 + 2x^6 - 30x^4y^2 + 30x^2y^4 + 2y^6 + (x^2 + y^2)^2.$$ 

The picture is given in Fig. 4(b).

We now turn to $D_6[D_3(\kappa_x)]$ taking $p_1 \equiv 1$ and $p_2 \equiv p_3 \equiv 0$ in Table 1, so that the form is

$$(-x, -y, x).$$

In this case, $\ker \lambda = D_3(\kappa_y)$ and the discriminant set is the origin, given by the zero set of

$$\delta(x, y) = x^2 + y^2.$$ 

The picture is given in Fig. 4(c).

Figure 5. Configurations with symmetry groups $Z_2$ and $Z_2[1]$.

Consider now $Z_2$ for $p_1 \equiv p_2 \equiv p_3 \equiv 1$ from Table 1, so that the form is

$$(1, y, 1).$$

We have $\ker \lambda = 1$ and the discriminant function is $Z_2$-invariant and given by

$$\delta(x, y) = y^2 - 1.$$ 

See the illustration of this case in Fig. 5(a).
Fig. 5(b) is a $\mathbb{Z}_2[1]$ case, for which we have chosen $p_1 \equiv p_2 \equiv 1$ and $p_3 \equiv -1$ in Table 1, so that the form is

$$(y, 1, -y).$$

The homomorphism $\lambda$ is trivial and the discriminant function is $\mathbb{Z}_2$-invariant and given by

$$\delta(x, y) = y^2 + 1.$$

Figure 6. Configurations with symmetry groups $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2[\mathbb{Z}_2(-I)]$ and $\mathbb{Z}_2 \times \mathbb{Z}_2[\mathbb{Z}_2(\kappa_x)]$.

For $\mathbb{Z}_2 \times \mathbb{Z}_2$ in Table 1, we take $p_1 \equiv p_2 \equiv 1$ and $p_3 \equiv -1$, so that the differential form is

$$(1, xy, -1).$$

In this case $\ker \lambda = \mathbb{Z}_2$ and the discriminant function is the $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant given by

$$\delta(x, y) = x^2 y^2 + 1.$$

This is illustrated in Fig. 6(a).

We now consider $\mathbb{Z}_2 \times \mathbb{Z}_2[\mathbb{Z}_2(-I)]$ choosing $p_1 \equiv p_2 \equiv 1$ and $p_3 \equiv -1$ in Table 1, so that the form is

$$(xy, 1, -xy).$$

In this case $\lambda$ is trivial and the discriminant function is $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant and given by

$$\delta(x, y) = x^2 y^2 + 1.$$

The picture is given in Fig. 6(b).

Finally, consider $\mathbb{Z}_2 \times \mathbb{Z}_2[\mathbb{Z}_2(\kappa_x)]$ taking $p_1 \equiv p_2 \equiv p_3 \equiv 1$ in Table 1, so that differential form is

$$(x, y, x).$$

In this case $\ker \lambda = \mathbb{Z}_2(\kappa_y)$ and the discriminant function is given by

$$\delta(x, y) = y^2 - x^2.$$

See the illustration of this case in Fig. 6(c).
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