On the detectability of post-Newtonian effects in gravitational-wave emission of a coalescing binary

ANDRZEJ KRÓLAK\textsuperscript{a} KOSTAS D. KOKKOTAS\textsuperscript{b} GERHARD SCHÄFER\textsuperscript{c}

\textsuperscript{a}Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8, 00-950 Warsaw, Poland
\textsuperscript{b}Department of Physics, Section Astrophysics, Astronomy and Mechanics,
Aristotle University of Thessaloniki
540 06 Thessaloniki, Macedonia, Greece
\textsuperscript{c}Max-Planck-Research-Group Gravitational Theory
at the Friedrich-Schiller-University
07743 Jena, Germany

INTRODUCTION

It is currently believed that the gravitational waves that come from the final stages of the evolution of compact binaries just before their coalescence are very likely signals to be detected by long-arm laser interferometers \cite{1}. The two projects to build such detectors - LIGO and VIRGO - are already approved and they are rapidly progressing, and the third one - GEO600 - is likely to be funded soon. A standard optimal method to detect the signal from a coalescing binary in a noisy data set and to estimate its parameters is to correlate the data with the filter matched to the signal and vary the parameters of the filter until the correlation is maximal. It has recently been realized \cite{2} that the correlation is very sensitive even to very small variations of the phase of the filter because of the large number of cycles in the signal. Consequently, the addition of small corrections to the phase of the signal due to the post-Newtonian effects decreases the correlation considerably. Thus the post-Newtonian effects in the coalescing binary waveform can be detected and estimated to a much higher accuracy than it was thought before \cite{3}. We present an analysis of the estimation of parameters of the post-Newtonian signal. We also examine the detectability of the post-Newtonian signal and estimation of its parameters using the Newtonian waveform as a filter. This filter can be used as the simplest search template.

GRAVITATIONAL-WAVE SIGNAL FROM A BINARY

Let us first give the formula for the gravitational waveform of a binary with the currently known post-Newtonian corrections. In this paper we work within the so called “restricted” post-Newtonian approximation, i.e., we only include the

\footnotesize
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post-Newtonian corrections to the phase of the signal keeping the amplitude in its Newtonian form; this is because the effect of the phase on the correlation is dominant. The inclusion of post-Newtonian effects in amplitudes will not qualitatively change our results. Due to radiation reaction the orbit of the binary is rapidly circularized, nevertheless, we include the first order correction due to eccentricity for completeness\(^2\). The tidal effects are known to be very small and we neglect them. We also include the contributions due to dipole radiation predicted by the Jordan-Fierz-Brans-Dicke (JFBD) theory (assuming that they are small) to investigate the possibility of testing GR against alternative theories of gravity.

The analysis of the signal is best performed in the Fourier domain. The expression for the Fourier transform of our signal in the stationary phase approximation is given by

\[
\tilde{s} = \frac{1}{(30)^{1/2}} \frac{1}{\pi^{2/3}} \mu^{1/2} m^{1/3} R \tilde{h},
\]

where

\[
\tilde{h} = f^{-7/6} \exp[i(2\pi ft_c - \phi_c - \pi/4 -
\]

\[
\frac{7065}{187136} \frac{k_e}{(\pi f)^{3/9}} + \frac{3}{128} \frac{k}{(\pi f)^{5/3}} + \frac{5}{96} \frac{k_1}{(\pi f)^{1/3}} - \frac{3}{32} \frac{k_{3/2}}{(\pi f)^{2/3}} + \frac{3}{128} \frac{k_2}{(\pi f)^{1/3}}
\]

\[
- \frac{5}{14336} \frac{k_D}{(\pi f)^{7/3}},
\]

(for \(f > 0\), and by the complex conjugate of the above expression, for \(f < 0\)) where

\[
k = \frac{1}{\mu m^{2/3}}(1 - \frac{F(C_1, C_2, m_1, m_2)}{2 + \omega}),
\]

\[
k_e = \frac{1}{\mu m^{2/3}} e_o^2 (\pi f_o)^{1/9},
\]

\[
k_1 = \frac{1}{\mu} \left( \frac{743}{336} + \frac{11}{4} \frac{\mu}{m} \right),
\]

\[
k_{3/2} = \frac{m^{1/3}}{\mu} (4\pi - s_o),
\]

\[
k_2 = \frac{m^{4/3}}{\mu} \left( 3.0 + \frac{c_1}{m} + \frac{c_2}{m^2} + \frac{s_s}{\mu} \right),
\]

\[
k_D = \frac{1}{\mu m^{4/3}} \left( \frac{C_1 - C_2}{2 + \omega} \right),
\]

where \(e_o\) is the eccentricity of the binary at gravitational frequency \(f_o\) (\(k_e\) is constant to order \(e_o^2\)), \(s_o\) and \(s_s\) are spin-orbit and spin-spin parameters, respectively\(^4\). The constants \(c_1\) and \(c_2\) in the 2nd post-Newtonian contribution are only recently being calculated\(^5\) and calculation of further corrections is in progress. \(C_i, i=(1,2),\) is the sensitivity of the body \(i\) to changes of the scalar field, \(F(C_1, C_2, m_1, m_2)\) is a complicated function of sensitivities and masses of the bodies and \(\omega\) is the parameter of the JFBD theory. Current observational tests restrict \(\omega > 600\) and observations of the Hulse-Taylor binary pulsar restrict \(\omega > 200\).

**ESTIMATION OF THE POST-NEWTONIAN PARAMETERS**

A basic method proposed to detect the gravitational-wave signal of a coalescing binary and to estimate its parameters is matched-filtering and maximum likelihood

\(^2\)This correction was derived by N. Wex
(ML) estimation. The performance of the method is determined by signal-to-noise ratio \( d \) and the Fisher information matrix \( \Gamma \). Signal-to-noise ratio determines the probability of detection of the signal and the diagonal elements of the inverse of the Fisher matrix are approximately the variances of the maximum likelihood estimators of the parameters for large \( d \).

We have considered three representative binary systems with parameters summarized in Table I.

### Table I. Parameters of Neutron Star (NS) and Black-Hole (BH) Binary Systems

| Binary | \( m_1 \) \( M_\odot \) | \( m_2 \) \( M_\odot \) | \( M_\odot \) | \( s \) | \( kM_{10}^{-5/3} \) | \( k_1M_{10}^{-1} \) | \( k_{3/2}M_{10}^{-2/3} \) |
|--------|----------------|----------------|---------|------|----------------|----------------|----------------|
| NS-NS  | 1.4            | 1.4            | 1.2     | 2.4 \times 10^{-2} | 0.72            | 4.1             | 25              |
| NS-BH  | 1.4            | 10             | 3.0     | 4.0  | 0.16           | 2.0             | 16              |
| BH-BH  | 10             | 10             | 8.7     | 3.9  | 2.7 \times 10^{-2} | 0.58            | 4.7             |

Assuming that the noise is Gaussian, neglecting the effects of eccentricity, 2nd post-Newtonian effects, dipole radiation, and assuming that binaries are located at a distance of 200Mpc we have calculated the signal-to-noise ratios and approximate values of the variances of the ML estimators. The results are summarized in Table II (cf. Table I).

### Table II. Signal-to-Noise Ratio and RMS Errors for Parameters of the 3 Binaries

| Binary | \( S/N \) | \( \sqrt{n} \) | \( \Delta t_\text{ms} \) | \( \Delta \mathcal{M}/\mathcal{M} \) | \( \Delta \mu/\mu \) | \( \Delta m/m \) | \( \Delta s/s \) |
|--------|-----------|----------------|----------------|----------------|----------------|----------------|----------------|
| NS-NS  | 15        | 32             | 0.75           | 0.023%        | 6.4%           | 9.6%           | 33 \times 10^{2/3} |
| NS-BH  | 32        | 15             | 0.65           | 0.061%        | 4.8%           | 7.0%           | 8.2%           |
| BH-BH  | 77        | 6              | 0.46           | 0.19%         | 16%            | 23%            | 38%            |

The number \( \sqrt{n} \) gives the improvement of the signal-to-noise ratio due to matched filtering.

We have also examined the effects of eccentricity and dipole radiation. We have added their contributions to the phase of the 3/2 post-Newtonian signal separately and we have calculated the Fisher matrix and its inverse. For a NS-NS binary the relative rms error in eccentricity is given by \( \Delta e_e = \frac{0.63}{a_e} \times 10^{-6} \). For the Hulse - Taylor binary pulsar we have \( a_e = c_0 f_0^{19/9} = 1.8 \times 10^{-13} \). Thus the effects of eccentricity would be practically undetectable for such a binary. For the NS-BH binary the relative rms error in the dipole radiation coefficient \( k_D \) in JFBD theory is given by \( \Delta k_D = 0.75 \left( \frac{a_e}{100} \right) \). Thus, the effects of the dipole radiation could be determined to about the same accuracy as from timing of the binary pulsar in our Galaxy.

It is well known that the rms errors of the estimators of the parameters increase with the number of parameters. We have investigated this effect with the increasing number of post-Newtonian parameters. We have considered a reference binary of \( \mathcal{M} = 1M_\odot \) located at the distance of 100Mpc.

### Table III RMS Errors for Parameters at Various post-Newtonian Orders

| \( \Delta t_e \) msec | \( \Delta \phi_e \) | \( \Delta kM_{10}^{-5/3} \) | \( \Delta k_1M_{10}^{-1} \) | \( \Delta k_{3/2}M_{10}^{-2/3} \) | \( \Delta k_{2M}^{-17/3} \) | \( \Delta k_{eM}^{-19/9} \) |
|----------------------|------------------|----------------|----------------|----------------|----------------|----------------|
| 0.17                 | 0.10             | 8.3 \times 10^{-6} | -              | -              | -              | -              |
| 0.27                 | 0.33             | 4.0 \times 10^{-5} | 5.8 \times 10^{-3} | -              | -              | -              |
| 0.54                 | 1.9              | 1.7 \times 10^{-4} | 0.70 \times 10^{-1} | 0.52           | -              | -              |
| 1.6                  | 24               | 6.6 \times 10^{-4} | 0.50           | 7.2            | 28             | -              |
| 2.3                  | 45               | 2.3 \times 10^{-3} | 1.3            | 17             | 59             | 1.2 \times 10^{-6} |
We were able to include the 2nd post-Newtonian correction because the Fourier transform of the signal is linear in the mass parameters \(k_i\) and consequently, the Fisher matrix for these parameters does not depend on their numerical values. In Table IV we show the degradation of accuracy of estimation of the chirp mass, the reduced mass, and the total mass with the increasing number of parameters in the template for the NS-NS binary at a distance of 200Mpc.

| pN order | \(\Delta M/M\) | \(\Delta \mu/\mu\) | \(\Delta m/m\) |
|----------|----------------|----------------|----------------|
| 1 pN     | 0.0054%        | 0.55%          | 0.81%          |
| 3/2 pN   | 0.023%         | 6.4%           | 9.6%           |
| 2 pN     | 0.080%         | 42%            | 63%            |

The rms errors in \(\mu\) and \(m\) do not depend on the value of 2nd post-Newtonian mass parameter \(k_2\).

THE NEWTONIAN FILTER

On the one hand the correlation of the signal with the template is very sensitive to small corrections in the phase of the signal on the other hand the accuracy of estimation of the parameters is significantly degraded with increasing number of corrections even though a correction may be small. Moreover we cannot entirely exclude unpredicted small effects in the gravitational-wave emission (e.g. corrections to general theory of gravity) that we present cannot model. Thus, there is a need for simple filters or search templates that will unable to scan the data effectively and isolate stretches of data where the signal is most likely to be present. The simplest such filter is just a Newtonian waveform \(h_N\) the Fourier transform of which, in stationary phase approximation, is given by

\[
\tilde{h}_N = f^{-7/6} \exp\left[2\pi ft_c - \phi_c - \pi/4 + k_3 \frac{3}{16}(\pi f)^{-5/3}\right].
\]

Using such a suboptimal filter decreases the correlation and, consequently, decreases the signal-to-noise ratio. Also the estimate of the \(k\) parameter is shifted by a definite amount depending on the noise of the detector and the parameters of the binary. In Table V we have given the drop in signal-to-noise given by

\[
FF = \frac{(h|h_N)/(h|h)}{(h|h_o)/(h|h)} = l \times d.
\]

In the case of the Newtonian filter probability of detection is determined by two parameters: optimal signal-to-noise ratio \(d = \sqrt{(h|h)}\) and \(d_o = \sqrt{(h|h_o)} = l \times d\). We have also calculated the shift in the estimate of the \(k\) parameter and the accuracy of the determination of \(k\). We have included the 1st and the 3/2 post-Newtonian corrections.

| Binary | \(l\) | FF  | Shift \(\delta kM_\odot^{3/2}\) | Accuracy \(\Delta kM_\odot^{3/2}\) |
|--------|------|-----|-------------------------------|-----------------------------|
| NS-NS  | 0.90 | 0.81| 0.01560                       | 0.13 \times 10^{-3}        |
| NS-BH  | 0.87 | 0.76| 0.004898                      | 0.026 \times 10^{-3}       |
| BH-BH  | 0.98 | 0.96| 0.001209                      | 0.011 \times 10^{-3}       |

When the amplitude and phase modulations are taken into account it was shown that in the worst case \(FF = 0.39 (l = 0.63)\). Using the Newtonian filter we would not like to loose any signals. We can achieve this by suitably lowering the detection threshold when filtering the data.
with the Newtonian filter. By this procedure we would isolate stretches of data with all the signals that would be detected with optimal filter and also an increased number of false alarms. The next step would be to analyse the reduced set of data with accurate templates and the initial threshold to make the final detection and estimate the parameters of the signal.

In Table VI we have given examples of the performance of the above procedure. We assume the detection threshold $T = 5$ and we assume that we have one signal for the optimal signal-to-noise ratio $d$. $N$ is the expected number of detected signals with the optimal filter, $N_F$ is number of false alarms, $N_N$ is the number of detected signals with the Newtonian filter, $T_N$ is the lowered threshold, $N_L$ is number of signals with the lowered threshold and $N_{FL}$ is number of false alarms with the lowered threshold.

### Table VI

| $d$ | FF | N   | $N_F$ | $N_N$ | $T_N$ | $N_L$ | $N_{FL}$ |
|-----|----|-----|-------|-------|-------|-------|----------|
| 15  | .81| 27  | 0.055 | 20    | 4.5   | 28    | 0.16     |
| 15  | .36| 27  | 0.055 | 20    | 5.6   | 28    | 2.1      |
| 30  | .81| 225 | 1.1   | 165   | 4.5   | 230   | 2.2      |
| 30  | .25| 225 | 1.1   | 31    | 2.875 | 229   | 32       |

A different search template consisting of the post-Newtonian waveform with spin effects stripped off has been analysed in Ref.[7].

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