Production of non-Abelian Tensor Gauge Bosons

Tree Amplitudes and BCFW Recursion Relation

George Georgiou and George Savvidy

Demokritos National Research Center
Institute of Nuclear Physics
Ag. Paraskevi, GR-15310 Athens, Greece
E-mail: georgiou@inp.demokritos.gr, savvidy@inp.demokritos.gr

Abstract

The BCFW recursion relation is used to calculate tree-level scattering amplitudes in generalized Yang-Mills theory and, in particular, four-particle amplitudes for the production rate of non-Abelian tensor gauge bosons of arbitrary high spin in the fusion of two gluons. The consistency of the calculations in different kinematical channels is fulfilled when all dimensionless cubic coupling constants between vector bosons and high spin non-Abelian tensor gauge bosons are equal to the Yang-Mills coupling constant. We derive a generalization of the Parke-Taylor formula in the case of production of two tensor gauge bosons of spin-s and N gluons (jets). The expression is holomorphic in the spinor variables of the scattered particles, exactly as the MHV gluon amplitude is, and reduces to the gluonic MHV amplitude when s=1.
1 Introduction

The Lagrangian of non-Abelian tensor gauge fields describes the interaction of the Yang-Mills quanta with massless tensor gauge bosons of increasing spins \([1, 2, 3, 4]\). The characteristic property of generalized Yang-Mills theory is that all interaction vertices between Yang-Mills and high-spin fields have dimensionless coupling constants in four-dimensional space-time. That is, the cubic interaction vertices have only first order derivatives and the quartic vertices have no derivatives at all.

One of the first calculations of tree level scattering amplitudes in generalized Yang-Mills theory was made in a series of articles \([41, 42]\) where the authors considered the creation of tensor gauge bosons in the annihilation processes of quarks and gluons. The main problem in this calculation was the necessity to sum over infinitely many diagrams even in the lowest order of the perturbation theory because the kinetic term of the Lagrangian contains non-diagonal transitions. Our intension in this article is to use a different technique to resolve these difficulties.

Recently, a very powerful technique was developed for the calculation of high order tree level diagrams in Yang-Mills and other supersymmetric theories \([5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]\). It uses spinor representation of the scattering amplitudes and dramatically simplifies the calculations \([9, 13, 14, 15]\). The advantage of this approach is that it allows the computation of high order scattering amplitudes in terms of lower ones, expressing any tree amplitude as a sum over terms constructed from products of two amplitudes of fewer particles multiplied by a Feynman propagator. The two amplitudes in each term are physical, in the sense that all particles are on-shell and momentum conservation is preserved \([13, 14, 15]\). Here the use of the complex momenta allows to write non-vanishing three-particle on-shell vertices as well as to deform two of the momenta in an arbitrary scattering amplitude along a complex direction defined by the deformation parameter \(z\). Any tree-level amplitude becomes a rational function of the complex parameter \(z\), with at most simple poles and if the amplitude vanishes at large \(z\), then it can be computed by knowing the position of the poles and the value of the residues. These are on-shell data that are completely specified by the three-point on-shell vertices.

The application of the BCFW recursion relation to calculate four-particle amplitudes allows to derive the production rate of non-Abelian tensor gauge bosons of arbitrary high spin in the fusion of two gluons \(G + G \rightarrow T + T\). The consistency of the calculations in different kinematical channels is fulfilled when all cubic coupling constants between vector bosons (gluons) and high spin tensor bosons are of the generalized Yang-Mills type \([1, 2, 3, 4]\) and are equal to the Yang-Mills coupling constant

\[
g_{1ss} = g_{1ss-2} = g_{YM}, \quad s = 2, 3, \ldots
\]  

We have checked that the amplitude vanishes quickly enough as deformation parameter \(z\) tends to infinity, so that there is no contribution from the contour at infinity. The result can be expressed in a compact form

\[
d\sigma_{+s} = \left(\frac{1 - \cos \theta}{1 + \cos \theta}\right)^{2s-2} d\sigma_{+1}, \quad s = 1, 2, 3, \ldots
\]

where \(d\sigma_{+1}\) is polarized cross section of two gluons into two gluons \(G + G \rightarrow G + G\) and \(\theta\) is the scattering angle. The formula demonstrates the complete dependence of the cross section on the spin of the tensor gauge bosons and allows to sum contributions from all spins.
We derive the four-particle cross section for the scattering of high spin tensor gauge bosons of the helicities \( h_1, h_2, h_3, h_4 \) \( (h_1 + h_2 + h_3 + h_4 = 0, \ h_1 > 0, \ h_2 < 0) \), which has the following form:

\[
dσ_{h_1,h_2,h_3,h_4} = \frac{(1 - \cosθ)^{2h_1+2h_2+2h_3-2}}{2^{2h_1-2h_2-4}(1 + \cosθ)^{2h_3-2}}dσ_1
\]  

and spectacularly falls exponentially as the spins of scattering particles increases.

We also derive a generalization of the Parke-Taylor formula in the case of production of two tensor gauge bosons of spin-\( s \) and \( (n - 4) \) gluons (jets) in the amplitude \( G + G \to T + T + (n - 4) \ G \). The result reads:

\[
M^{(n)}(1^+, \ldots i^-, \ldots, k^+, \ldots, j^-^s, \ldots, n^+) = g^{n-2} < ij >^4 \prod_{l=1}^{n} < ll+1 > \left( < ik > \right)^{2s-2}, \quad (4)
\]

where \( n \) is the total number of particles, and the dots stand for the positive helicity gluons. Furthermore, \( i \) is the position of the negative helicity gluon, while \( k \) and \( j \) are the positions of the particles with helicities \( +s \) and \( -s \) respectively. This expression is holomorphic in the spinors of the particles, exactly as the MHV gluon amplitude and for \( s = 1 \) the second fraction in (4) is absent and (4) reduces to the well-known result for the MHV amplitude [10].

In the next section, we shall review the spinor representation of scattering amplitudes. In the third section, we discuss the three-point on-shell vertices for complex momenta and in the forth section we describe a class of the three-point on-shell vertices which have dimensionless coupling constants corresponding to cubic vertices of the generalized Yang-Mills theory. In the fifth section, we apply the Benincasa-Cachazo recursion relation to compute four-particle scattering amplitude of gluons and tensor gauge bosons using three-point on-shell vertices of the generalized Yang-Mills theory. The production cross sections (2) of the non-Abelian tensor gauge bosons of arbitrary spin-\( s \) and the general formula (3) are derived in the sixth section and the generalization of the Parke-Taylor formula (4) in the last, seventh section.

## 2 Scattering Amplitudes in Spinor Representation

Let us consider a scattering amplitude for massless particles of momenta \( p_i \) and polarization tensors \( ε_i \) \( (i=1,...,n) \), which are described by irreducible massless representations of Poincaré group and are classified by their helicities \( h = ±s \), where \( s \) is an integer,

\[
M_n = M_n(p_1, ε_1; \ p_2, ε_2; \ldots; \ p_n, ε_n).
\]  

We are interested in representing the momenta \( p_i \) and polarization tensors \( ε_i \) in terms of spinors and the above scattering amplitude in terms of rational functions of spinor products [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20].

The spinor representation of momenta \( pμ \) and polarization tensors \( ε_i \) can be constructed as follows. The spinors \( \{λ_a, \bar{λ}_a\} \) transform in the representation \( (1/2, 0) \) and \( (0, 1/2) \) of the universal cover of the Lorentz group, \( SL(2, C) \), respectively. Invariant tensors are \( e^{\dot{a}b} \), \( \tilde{e}^{\dot{a}b} \) and \( (σ^μ)_{\dot{a}b} \), where \( σ^μ = (1, σ) \). The basic Lorentz invariant spinor products can be constructed as follows:

\[
λ_aλ_b^\prime ε^{\dot{a}b} ≜ < λ, λ^\prime >, \quad \bar{λ}_a\bar{λ}_b^\prime ε^{\dot{a}b} ≜ [ λ, λ^\prime ].
\]
The scalar product of two vectors \( p^\mu \) and \( q^\nu \) is given by the product
\[
2(p \cdot q) = \langle \lambda^\mu, \lambda^\nu \rangle [\bar{\lambda}^\mu, \bar{\lambda}^\nu].
\]

Using the third invariant tensor one can define \( p^\mu = \lambda^a (\sigma^\mu)_{a\dot{a}} \bar{\lambda}^{\dot{a}} \) and find out the corresponding spinor representation of massless particle momentum in the form
\[
p_{a\dot{a}} = \lambda_a \bar{\lambda}_{\dot{a}}.
\]

The corresponding polarization vectors of spin-1 particles are given by
\[
\varepsilon^-_{a\dot{a}}(p) = \frac{\lambda_{a\dot{a}}}{[\lambda, \bar{\mu}]} \quad \varepsilon^+_{a\dot{a}}(p) = \frac{\mu_a \bar{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}
\]

with \( \mu_a \) and \( \bar{\mu}_{\dot{a}} \) as arbitrary reference spinors. One can check that in this representation the following basic properties of polarization tensors are fulfilled:
\[
p \cdot \varepsilon^-(p) = p \cdot \varepsilon^+(p) = 0
\]
\[
\varepsilon^-(p) \cdot \varepsilon^-(p) = \varepsilon^+(p) \cdot \varepsilon^+(p) = 0
\]
\[
\varepsilon^+(p) \cdot \varepsilon^-(p) = -1
\]

The polarization tensors of massless particles of integer spins can be expressed in terms of spin-1 particle as follows:
\[
\varepsilon^-_{a_1\dot{a}_1, \ldots, a_s\dot{a}_s}(p) = \prod_{i=1}^s \varepsilon^-_{a_i\dot{a}_i}, \quad \varepsilon^+_{a_1\dot{a}_1, \ldots, a_s\dot{a}_s}(p) = \prod_{i=1}^s \varepsilon^+_{a_i\dot{a}_i}.
\]

The presence in (7) of arbitrary reference spinors \( \mu_a \) and \( \bar{\mu}_{\dot{a}} \) means that polarization tensors are not uniquely fixed once \( \{\lambda, \bar{\lambda}, h\} \) is given. Let us consider the transformation
\[
\mu_{\dot{a}} \to \bar{\mu}_{\dot{a}} + \eta \mu_{\dot{a}} + \eta' \bar{\lambda}_{\dot{a}}.
\]

The action of this transformation on the polarization vector \( \varepsilon^-_{a\dot{a}}(p) \) can be computed:
\[
\varepsilon^-_{a\dot{a}} \to \frac{\lambda_a (\mu_{\dot{a}} + \eta \mu_{\dot{a}} + \eta' \bar{\lambda}_{\dot{a}})}{[\lambda, \mu + \eta \mu + \eta' \bar{\lambda}]} = \varepsilon^-_{a\dot{a}} + \frac{\eta'}{(1 + \eta)[\lambda, \bar{\mu}]} \lambda_a \bar{\lambda}_{\dot{a}} = \varepsilon^-_{a\dot{a}} + w p_{a\dot{a}},
\]

where
\[
w = \frac{\eta'}{(1 + \eta)[\lambda, \bar{\mu}]}, \quad \lambda_a \bar{\lambda}_{\dot{a}} = p_{a\dot{a}}
\]

and therefore represents a gauge transformation.

Thus the scattering amplitude of massless bosons \( M_n (5) \) can be considered as a function of spinors \( \lambda_i, \bar{\lambda}_i \) and helicities \( h_i \):
\[
M_n = M_n (\lambda_{1}, \bar{\lambda}_{1}, h_{1}; \ldots; \lambda_{n}, \bar{\lambda}_{n}, h_{n}).
\]

The Poincaré invariance requires that \( M_n \) gives the same answer independently of the choice of reference spinors \( \mu_i \) [21, 22, 23].

\(^{1}\)

In labeling helicities, we consider all particles to be outgoing.
Let us consider scaling transformation of spinors which leaves the momenta in (6) intact:

\[ \lambda \to u\lambda, \quad \tilde{\lambda} \to u^{-1}\tilde{\lambda}, \]

then

\[ \varepsilon^{-}_{a\bar{a}} \to u^{2}\varepsilon^{-}_{a\bar{a}}, \quad \varepsilon^{+}_{a\bar{a}} \to u^{-2}\varepsilon^{+}_{a\bar{a}} \]

and the scattering amplitude \( M_{n} \) should be a homogeneous function of these spinors of order \( 2h_{i} \) [12]:

\[
(\lambda^{a}_{i} \frac{\partial}{\lambda^{a}_{i}} - \tilde{\lambda}^{a}_{i} \frac{\partial}{\tilde{\lambda}^{a}_{i}})M_{n}(\ldots \{\lambda_{i}, \tilde{\lambda}_{i}, h_{i}\} \ldots) = -2h_{i} M_{n}(\ldots \{\lambda_{i}, \tilde{\lambda}_{i}, h_{i}\} \ldots). \tag{10}
\]

From this equation one can derive a general structure of the three-particle amplitudes \( M_{3} \) in spinor representation [15]. This will provide us with unique information about the structure of three-point on-shell vertices \( V_{3}(p_{1}, p_{2}, p_{3}) \) in high spin quantum field theory.

3 Three-Point On-Shell Vertices

In the generalized Yang-Mills theory [1, 2, 3, 4], the three-point on-shell vertices can be computed as a product of a three-particle vertex \( V_{3} \) times three polarization tensors \( \varepsilon [43, 44]: \)

\[
M_{3}(p_{1}, \varepsilon_{1}; p_{2}, \varepsilon_{2}; p_{3}, \varepsilon_{3}) = V_{3}(p_{1}, p_{2}, p_{3}) \varepsilon_{1}(p_{1}) \varepsilon_{2}(p_{2}) \varepsilon_{3}(p_{3}), \tag{11}
\]

where the mass-shell conditions and momentum conservation are \( p_{1}^{2} = p_{2}^{2} = p_{3}^{2} = 0 \) and \( p_{1} + p_{2} + p_{3} = 0 \). The three point scattering amplitude for on-shell massless particles is equal to zero for the real momenta \( p_{i} \), but if one allows complex momenta or a different space-time signature [12, 13, 14, 15, 43, 44] then these matrix elements will have nontrivial behavior. Indeed, the three-point on-shell vertices are naturally non-zero if one chooses to work with complex momenta \( p^{\mu} \) [12, 15]. In that case \( (\lambda_{a})^{*} \neq \pm \tilde{\lambda}_{a} \), and the spinors of each particle, \( \lambda_{i} \) and \( \tilde{\lambda}_{i} \), are independent vectors in \( C^{2} \).

Momentum conservation and mass-shell conditions imply that \( p_{i} \cdot p_{j} = 0 \), that is, in spinor representation we have

\[
< 1, 2 > < 1, 2 ] = 0, \quad < 2, 3 > [ 2, 3 ] = 0, \quad < 3, 1 > [ 3, 1 ] = 0 \tag{12}
\]

and either spinors \( \lambda_{i} (i = 1, 2, 3) \) or spinors \( \tilde{\lambda}_{i} (i = 1, 2, 3) \) are collinear. This means that the non-trivial solution of (12) is either

\[
< 1, 2 > = < 2, 3 > = < 3, 1 > = 0
\]

or

\[
[ 1, 2 ] = [ 2, 3 ] = [ 3, 1 ] = 0.
\]

Therefore the three-point on-shell vertex, \( M_{3}(\ldots \lambda_{i}, \tilde{\lambda}_{i}, h_{i} \ldots) \), which is restricted to be a function of \( < i, j > \) and \( [i, j] \), splits into two independent parts [15]

\[
M_{3} = A(< 1, 2 >, < 2, 3 >, < 3, 1 >) + B( [1, 2 ] , [2, 3 ] , [3, 1 ] ).
\]

As a result, equation (10) also splits into two equations for amplitudes \( A \) and \( B \)

\[
\lambda^{a}_{i} \frac{\partial}{\lambda^{a}_{i}} A(< 1, 2 >, < 2, 3 >, < 3, 1 >) = -2h_{i} A(< 1, 2 >, < 2, 3 >, < 3, 1 >)
\]
\[
\tilde{\lambda}^a_i \frac{\partial}{\partial \lambda_i^a} B( [1, 2], [2, 3], [3, 1] ) = 2h_i B( [1, 2], [2, 3], [3, 1] ).
\]

Looking for a polynomial solution of the first equation in the form \(<1, 2>^d_3 <2, 3>^{d_1}_1 <3, 1>^{d_2}_1 >d_2\) and of the second one in the form \([1, 2]^{-d_3}[2, 3]^{-d_1}[3, 1]^{-d_2}\) one can see that if

\[
d_1 = h_1 - h_2 - h_3, \quad d_2 = h_2 - h_3 - h_1, \quad d_3 = h_3 - h_1 - h_2,
\]

then these polynomials represent true solutions. For the three-point on-shell vertices therefore we shall have [15]

\[
M_3 = f <1, 2>^d_3 <2, 3>^{d_1}_1 <3, 1>^{d_2}_1 + k [1, 2]^{-d_3}[2, 3]^{-d_1}[3, 1]^{-d_2},
\]

where \(f\) and \(k\) are momentum independent constants.

The amplitude should vanish in the limit when all momenta \(p^\mu\) become real, in that case both spinor products \(<i, j>\) and \([i, j]\) tend to zero. This means that if

\[
D = d_1 + d_2 + d_3 = -h_1 - h_2 - h_3 > 0,
\]

then one should take \(k = 0\), while if

\[
-D = d_1 + d_2 + d_3 = -h_1 - h_2 - h_3 < 0
\]

one should take \(f = 0\) in order to avoid singularities. Because the dimensionality of spinors \(\lambda\) and \(\tilde{\lambda}\) in formula (6) is \([mass]^{1/2}\), the dimensionality of the three-point on-shell vertex \(M_3\) is \([mass]^D\).

The earlier investigation of the three-point vertices in the light-front formulation of relativistic dynamics and derivation of restrictions imposed on the helicities of scattered particles by the Poincaré group were made in [24, 25, 26]. In covariant formulation the interaction vertices were studied in [27, 28, 29], see also [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. The advantage of the spinor formulation is that it gives non-perturbative expressions.

In the generalized Yang-Mills theory [1, 2, 3, 4] all interaction vertices between high-spin fields have dimensionless coupling constants in four-dimensional space-time. That is, the cubic interaction vertices have only first order derivatives and that the quartic vertices have no derivatives at all. Our intension therefore will be to find out constraints on helicities of the scattered particles, when the amplitudes \(M_3\) in (14) have dimensionality of \(mass\), that is, \(D = 1\) and therefore the corresponding vertices \(V_3\) have a dimensionless coupling constants in four-dimensional space-time as it is the case in the generalized Yang-Mills theory.

## 4 Cubic Vertices with Dimensionless Coupling Constants

The three-point on-shell vertex \(M_3\) in (14) will have dimensionality of \(mass\) if \(D = 1\). In four dimensional space-time this will correspond to the three-particle interaction vertex \(V_3\), which has dimensionless coupling constant, the property of the main importance [1, 2, 3, 4]. This condition can be expressed as

\[
D = d_1 + d_2 + d_3 = -h_1 - h_2 - h_3 = 1,
\]

(15)
when \( k = 0 \) and as
\[
D = -(d_1 + d_2 + d_3) = h_1 + h_2 + h_3 = 1, \tag{16}
\]
when \( f = 0 \). We can now express the helicity of the third particle in terms of the first two independent helicities. In the case \( D = d_1 + d_2 + d_3 = -h_1 - h_2 - h_3 = 1 \) we have
\[
d_1 = h_1 - h_2 - h_3 = h_1 - h_2 + (h_1 + h_2 + 1) = 2h_1 + 1 \\
d_2 = h_2 - h_1 - h_3 = h_2 - h_1 + (h_1 + h_2 + 1) = 2h_2 + 1 \\
d_3 = h_3 - h_1 - h_2 = -(h_1 + h_2 + 1) - h_1 - h_2 = -2h_1 - 2h_2 - 1
\]
and for the amplitude we shall get:
\[
M_3 = f < 1, 2 > ^{-2h_1 - 2h_2 - 1 < 2, 3 > ^{2h_1 + 1 < 3, 1 > ^{2h_2 + 1}}, \quad h_3 = -1 - h_1 - h_2. \tag{17}
\]
For the second solution, when \( D = -(d_1 + d_2 + d_3) = h_1 + h_2 + h_3 = 1 \), we have
\[
d_1 = h_1 - h_2 - h_3 = h_1 - h_2 - (-h_1 - h_2 + 1) = 2h_1 - 1 \\
d_2 = h_2 - h_1 - h_3 = h_2 - h_1 - (-h_1 - h_2 + 1) = 2h_2 - 1 \\
d_3 = h_3 - h_1 - h_2 = (-h_1 - h_2 + 1) - h_1 - h_2 = -2h_1 - 2h_2 + 1
\]
and for the amplitude we shall get:
\[
M_3 = k[1, 2] ^{2h_1 + 2h_2 - 1 [2, 3] ^{-2h_1 + 1 [3, 1] ^{-2h_2 + 1}}, \quad h_3 = 1 - h_1 - h_2. \tag{18}
\]
The formulas (17) and (18) give a general expression for the amplitude \( M_3 \) in terms of two independent helicities \( h_1 \) and \( h_2 \). It allows to choose any \( h_1 \) and \( h_2 \) and then find out \( h_3 \) for which the three-particle interaction vertex in four-dimensional space-time will have dimensionless coupling constants \( f \) or \( k \). We conclude that the general three-point on-shell vertices with dimensionless coupling constants are the vertices (17) and (18).

The important conclusion which one can draw from the above result is that if all three massless particles have the same spin \( s_1 = s_2 = s_3 = s \) then the only vertex with dimensionless coupling constant will be the one with \( s = 1 \). This is the case of Yang-Mills theory where indeed the trilinear interaction has dimensionless coupling constant. But already for spin 2 the three-particle vertex has dimensional coupling constant as it is the case in the general relativity.

Because (17) and (18) allow many other solutions with nonequal spins \( s_1 \neq s_2 \neq s_3 \), we can conclude that there exist infinitely many trilinear interaction vertices between high spin fields with dimensionless coupling constants as it is the case in the generalized Yang-Mills theory [1, 2, 3, 4]. Indeed, let us first consider the Yang-Mills theory where all spins are equal to one \( s_1 = s_2 = s_3 = 1 \). For \( h_1 = \pm 1, h_2 = -1 \) we get \( h_3 = \mp 1 \) and the amplitude (17) yields
\[
M_3^{a_1 a_2 a_3} (+1, -1, -1) = g_{111} \ f^{a_1 a_2 a_3} {< 2, 3 > ^{4 < 1, 2 >> < 2, 3 >> < 3, 1 >> }, \tag{19}
\]
\[
M_3^{a_1 a_2 a_3} (-1, -1, +1) = g_{111} \ f^{a_1 a_2 a_3} {< 1, 2 > ^{4 < 1, 2 >> < 2, 3 >> < 3, 1 >> }.
\]
where \( f^{abc} \) are the structure constants of the internal gauge group \( G \). For \( h_1 = \pm 1, h_2 = 1 \) we get \( h_3 = \mp 1 \) and the amplitude (18) gives

\[
M_3^{a_1a_2a_3}(+1,+1,-1) = g_{111} f^{a_1a_2a_3} \frac{[1,2]^4}{[1,2][2,3][3,1]},
\]

\[
M_3^{a_1a_2a_3}(-1,+1,+1) = g_{111} f^{a_1a_2a_3} \frac{[2,3]^4}{[1,2][2,3][3,1]}.
\]

(20)

We are interested in studying all possible dimensionless interactions between spin 1 and spin 2 particles. There are no solutions with \( s_1 = 1, s_2 = 1, s_3 = 2 \), but in the generalized Yang-Mills theory \([1, 2, 3, 4, 43, 44]\) there are solutions with one vector and two tensors \( s_1 = 2, s_2 = 1, s_3 = 2 \), they are:

\[
M_3^{a_1a_2a_3}(+2,-1,-2) = g_{122} f^{a_1a_2a_3} \frac{<2,3>^6}{<1,2><3,1>},
\]

\[
M_3^{a_1a_2a_3}(+2,+1,-2) = g_{122} f^{a_1a_2a_3} \frac{[1,2]^6}{[1,2][2,3]^3[3,1]}.
\]

(21)

Thus there is a nontrivial interaction between spin 1 and two spin 2 massless bosons with dimensionless coupling constant exactly as in the generalized Yang-Mills theory \([1, 2, 3, 4]\). There exists a non-trivial cubic vertex of spin-1 and spin-s bosons with second spin-s boson

\[
M_3^{a_1a_2a_3}(-s,-1,+s) = g_{1ss} f^{a_1a_2a_3} \frac{<1,2>^{2s+2}}{<1,2><3,1>},
\]

\[
M_3^{a_1a_2a_3}(+s,-1,-s) = g_{1ss} f^{a_1a_2a_3} \frac{<2,3>^{2s+2}}{<1,2><3,1>},
\]

\[
M_3^{a_1a_2a_3}(+s,+1,-s) = g_{1ss} f^{a_1a_2a_3} \frac{[1,2]^{2s+2}}{[1,2][2,3]^{2s-1}[3,1]},
\]

\[
M_3^{a_1a_2a_3}(-s,+1,+s) = g_{1ss} f^{a_1a_2a_3} \frac{[2,3]^{2s+2}}{[1,2]^{2s-1}[2,3][3,1]}.
\]

(22)

as well as with spin-(s-2) and spin-(s+2) bosons

\[
M_3^{a_1a_2a_3}(-s,+1,+s-2) = g_{1ss-2} f^{a_1a_2a_3} \frac{<1,2>^{2s-3}}{<2,3>^{2s}},
\]

\[
M_3^{a_1a_2a_3}(+s,-1,-s+2) = g_{1ss-2} f^{a_1a_2a_3} \frac{[1,2]^{2s-3[2,3][3,1]^3}}{[2,3]^{2s}},
\]

\[
M_3^{a_1a_2a_3}(+s,+1,-s-2) = g_{1ss+2} f^{a_1a_2a_3} \frac{<2,3>^{2s+2}}{<1,2>^{2s-3}<2,3><3,1>},
\]

\[
M_3^{a_1a_2a_3}(-s,-1,+s+2) = g_{1ss+2} f^{a_1a_2a_3} \frac{[2,3]^{2s+2}}{[1,2]^{2s-3[2,3][3,1]^3}}.
\]

(23)

With these vertices in hand we can compute the gluon fusion amplitudes into two high spin-s gauge bosons \( s = 2, 3, ... \). There are two on-mass-shell diagrams in Fig.1 which contribute to this process.
In this section, we intend to calculate the polarized cross sections for the reaction $V + V \to T + T$, to the lowest order in cubic interaction coupling constants. There are two lowest-order diagrams contributing to the annihilation process of a pair of vector bosons (gluons) into a pair of tensor gauge bosons as shown in Fig.1. Vector gauge bosons $V$ carry helicities $h_{1,2} = \pm 1$ and tensor gauge bosons $T$ carry helicities $h_{3,4} = \pm s$.

We shall choose to deform the momenta of the initial gluons $p_1(z) = (\lambda_1 + z\lambda_2)\tilde{\lambda}_1$, $p_2(z) = \lambda_2(\lambda_2 - z\tilde{\lambda}_1)$ of helicities $h_1 = +1$, $h_2 = -1$ and shall leave the final tensor bosons momenta $p_3, p_4$ of helicities $h_3 = +s$, $h_4 = -s$ undeformed. The contribution of two diagrams takes the form

$$M_{4}^{abcd}(+1, -1, +s, -s) = 2i \ g_{1ss}^2 \ f^{ade} f^{bce} \ \begin{array}{c} (-1)^{s+1} \ [1, 3]^{s+1} < 2, 4 >^{s+1} \\ p^2_{12} p^2_{14} [1, 4]^{s-1} < 2, 3 >^{s-1} \end{array} +$$

$$+ 2i \ g_{1ss}^2 f^{ace} f^{bde} \ \begin{array}{c} (-1)^{s+1} \ [1, 3]^{s+1} < 2, 4 >^{s+1} \\ p^2_{12} p^2_{13} [1, 4]^{s-1} < 2, 3 >^{s-1} \end{array},$$

which can be written in a following factorized form:

$$M_{4}^{abcd}(+1, -1, +s, -s) = 2i \ g_{1ss}^2 (-1)^{s+1} [1, 3]^{s+1} < 2, 4 >^{s+1} \ \begin{array}{c} f^{ade} f^{bce} \\ p^2_{12} p^2_{14} + f^{ace} f^{bde} \\ p^2_{12} p^2_{13} \end{array}$$

where in the brackets we have a pure gluonic amplitude. The alternative helicity amplitude $M_{4}^{abcd}(+1, -1, -s, +s)$ can be found in a similar way.

It is important that a different choice of the momenta deformations will give the same result. Let us deform the momenta of the vector and tensor bosons $p_1(z) = (\lambda_1 + z\lambda_4)\tilde{\lambda}_1$, $p_2(z) = \lambda_4(\lambda_4 - z\tilde{\lambda}_1)$ of helicities $h_1 = +1$, $h_4 = -s$ and leave the momenta $p_2, p_3$ of helicities $h_2 = -1$, $h_3 = +s$ undeformed. The contribution of the first diagram gives

$$2i \ g_{1111} g_{1ss} f^{abe} f^{dce} [1, 3]^2 < 2, 4 >^{2s} \ \begin{array}{c} p^2_{12} [3, 2] \end{array} < 3, 2 >^{2s-1},$$

multiplying it by $([1, 3] < 1, 4 > ([1, 3] < 1, 4 >)^{s-1}$ and combining different terms one can get

$$2i \ g_{1111} g_{1ss} f^{abe} f^{dce} \ \begin{array}{c} (-1)^{s+1} \ [1, 3] < 2, 4 >^{s+1} \end{array} \ \begin{array}{c} ([1, 4] < 2, 4 >)^{s-1} \\ p^2_{12} p^2_{23} [14] < 2, 3 >^{s-1} \end{array}.$$

From the momentum conservation it follows that the ratio in the last brackets is equal to one. Using the identity $f^{abe} f^{dce} = + f^{ade} f^{bce} - f^{ace} f^{bde}$ one can see that the term with $f^{ade} f^{bce}$...
of this amplitude coincides with the first term in (24) if the coupling constant \( g_{1ss} \) coincides with the Yang-Mills coupling constant \( g_{YM} \)

\[
g_{1ss} = g_{111} \equiv g_{YM}, \quad s = 1, 2, 3, \ldots
\]  

(27)

The remaining piece of the amplitude from the first diagram is

\[
-2i \ g_{111}^2 g_{1ss} f^{ace} f^{bde} \frac{(-1)^{s+1} ([1, 3] < 2, 4 >)^{s+1}}{p_{12}^2 p_{23}^2 ([14] < 2, 3 >)^{s-1}}.
\]  

(28)

The second diagram gives the following contribution:

\[
-2i \ g_{1ss}^2 f^{ace} f^{bde} \frac{(-1)^{s+1} ([1, 3] < 2, 4 >)^{s+1}}{p_{13}^2 p_{14}^2 ([14] < 2, 3 >)^{s-1}} \frac{([3, 4] < 2, 3 >)^{s-1}}{([1, 4] < 1, 2 >)^{s-1}}
\]  

(29)

and from momentum conservation it follows that the ratio in the last brackets is equal to one. The sum of this amplitude with the remaining piece (28) from the first diagram gives

\[
2i \ g_{1ss}^2 f^{ace} f^{bde} \frac{([1, 3] < 2, 4 >)^{s+1}}{([14] < 2, 3 >)^{s-1}} \frac{([1, 4] < 1, 2 >)^{s-1}}{([23] < 2, 4 >)^{s-1}}
\]  

and coincides with the second term in (24).

The third possibility is to deform the momenta of the tensor and vector bosons \( p_3(z) = (\lambda_3 + z\lambda_4)\tilde{\lambda}_3, \quad p_2(z) = \lambda_2(\tilde{\lambda}_2 - z\tilde{\lambda}_3) \) of helicities \( h_3 = +s, \ h_2 = -1 \) and leave the momenta \( p_1, p_4 \) of helicities \( h_1 = +1, \ h_2 = -s \) undeformed. The contribution of the first diagram gives

\[
-2i \ g_{111}^2 g_{1ss} f^{cde} f^{bae} \frac{< 2, 4 >^{2s-2}}{< 2, 3 >^{2s-2}} \frac{< 2, 4 >^4}{< 1, 2 > < 2, 3 > < 3, 4 > < 4, 1 >}.
\]  

(30)

This is exactly the contribution of the first diagram of the previous deformation (26), because using momentum conservation one can derive the following identities

\[
\frac{[1, 3]}{[1, 2]} = < 4, 2 > < 4, 3 >^s, \quad \frac{[1, 3]}{[2, 3]} = < 2, 4 > < 1, 4 >
\]

and then plugging them into the equation (26) we shall get (30). Similarly, the second diagram yields

\[
-2i \ g_{1ss}^2 f^{cae} f^{bde} \frac{< 2, 4 >^{2s-2}}{< 2, 3 >^{2s-2}} \frac{< 2, 4 >^4}{< 1, 1 > < 2, 3 > < 2, 4 >}.
\]  

(31)

Using the identity \([1, 3]/[1, 4] = < 2, 4 > / < 2, 3 >\) it is straightforward to see that this result coincides with the contribution of the second diagram (29) of the previous deformation. We conclude that the third deformation gives the same result for the scattering amplitude as previous two deformations.

The important observation which follows from the above consideration is that when we deform the momenta of gluons, the exchanged particles have helicities \( h = +s \) and \( h = -s \), while when we deform the momenta of gluon and tensor bosons, the exchanged particles have helicities \( h = 1 \) and \( h = -s \), therefore, as we have seen, a full consistency between different kinematical channels of the scattered particles exists only if the high spin coupling
constants $g_{1\alpha}$ fulfill the relations (27), that is, they all coincide with the Yang-Mills coupling constant $g_{YM}$.

A second comment concerns the deformations performed above. In all three cases, we have checked that the amplitude vanishes quickly enough as $z \to \infty$, so that there is no contribution from the contour at infinity and that there is no additional poles in the complex plane. The third comment concerns a possible deformation of the tensor particles $p_3(z) = (\lambda_3 + z\lambda_4)\tilde{\lambda}_3$, $p_4(z) = \lambda_4(\tilde{\lambda}_4 - z\tilde{\lambda}_3)$ of helicities $h_3 = +s, h_4 = -s$ leaving the gluons momenta $p_1, p_2$ of helicities $h_3 = +1, h_4 = -1$ undeformed. Let us inspect the amplitude (25) performing the above deformation $\lambda_3 \to \lambda_3 + z\lambda_4$, $\lambda_4 \to \tilde{\lambda}_4 - z\tilde{\lambda}_3$. As one can see the amplitude develops the higher order poles in the complex plane of the deformation parameter $z$ and the standard recurrence prescription should be modified.

As we shall see below (see formulas (38) and (39)) the amplitudes given above can be recast as purely holomorphic functions of the spinor variables $\lambda_i$ and will be used in the last section to generalized the MHV amplitudes (56). The necessary condition for the n-particle amplitudes to be holomorphic $\partial M^{(n)}/\partial \lambda_i = 0$ can be found using the equation (10) and the fact that the amplitudes are Lorentz $J_{ab}M^{(n)} = 0$ and scale $D\hat{M}^{(n)} = 0$ invariant functions, from which it follows that $\frac{1}{2}\sum_i \lambda_i^\alpha \partial M^{(n)} = -(n-4)M^{(n)}[12]$. So that the amplitudes are holomorphic if particle helicities fulfill the equation

$$h_1 + h_2 + \ldots + h_n = n - 4$$

and are anti-holomorphic if

$$h_1 + h_2 + \ldots + h_n = -n + 4.$$  \hspace{1cm} (33)

The exceptionally interesting non-MHV amplitude which involves two vector bosons and two tensor bosons of the different spins $s$ and $s - 2$ is:

$$M_4^{abcd}(+1, -s, +1, s - 2) = 2i g^2 f^{ace} f^{bde} <2, 4>^3_{<1, 2><2, 3><3, 1>} -$$

$$- f^{ade} f^{bce} <2, 4>^4_{<1, 2><2, 3><2, 4><4, 1>} \times \left( <2, 3>_{<3, 4>} \right)^{2s-1},$$  \hspace{1cm} (34)

where $s = 2, 3, \ldots$. This amplitude is holomorphic and is of special interest because it has only one particle of negative helicity. In comparison, the n-gluon tree amplitudes for $n \geq 4$ with all but one gluon of positive helicity vanish [10, 12]. Thus the n-particle tree amplitudes in generalized Yang-Mills theory have more rich structure. It is well known that the tree level n-gluon scattering amplitudes with all positive helicities are also vanish [10, 12]. In generalized Yang-Mill theory this statement remains true and can be prove by induction. The 3-particle amplitudes in generalized Yang-Mills theory with all positive helicities vanish (17) and (18). Suppose that this is true for n particle amplitude, then the tree amplitude for $n+1$ particles is a sum over terms constructed from products of two amplitudes of fewer particles multiplied by a Feynman propagator. One of these fewer amplitudes is always with all positive helicity particles and therefore vanish. Thus in generalized Yang-Mills theory the tree level n-particle scattering amplitudes with all positive helicities vanish. $A^{(n)}(+, \ldots, +) = 0$, but tree amplitudes with one negative helicity particle are already nonzero (34).

Finally one can compute scattering amplitudes of four arbitrary non-Abelian tensor gauge bosons of helicities $h_1, h_2, h_3, h_4$. In the first diagram of Fig.1 the intermediate helicity is
\[ h = 1 + h_2 + h_3 = 1 - h_1 - h_4 \] and in the second \[ h = 1 + h_2 + h_4 = 1 - h_1 - h_3, \] thus one should have always

\[ h_1 + h_2 + h_3 + h_4 = 0. \] (35)

We can always choose \( h_1 > 0, \ h_2 < 0, \) and the total amplitude takes the form:

\[ M_{abcd}^{h_1, h_2, h_3, h_4} = \]

\[ = 2i \ g^2 \ f_{ace} f_{bde} \frac{( -1 )^{h_2 + h_3} \ [1, 3]^{2h_1 + h_2 + h_3} < 2, 4 > h_3 - h_2 < 3, 4 > 2h_2 + 2}{p_{12}^2 p_{14}^2 \ [1, 4]^{h_2 + h_3} < 2, 3 > h_2 + h_3 [3, 4]^{2h_1 - 2}} \]

\[ + 2i \ g^2 \ f_{ace} f_{bde} \frac{( -1 )^{1-h_3} \ [1, 3]^{h_1 + h_3} < 2, 4 > h_1 + h_3 < 4, 3 > 2h_2 + 2}{p_{12}^2 p_{13}^2 \ [1, 4]^{h_3 - h_1} < 2, 3 > h_1 + 2h_2 + h_3 [4, 3]^{2h_1 - 2}}. \] (36)

The conservation of momenta gives

\[ \frac{[1, 3]}{[1, 4]} = \frac{< 2, 4 >}{< 2, 3 >}, \quad \frac{[1, 3]}{[3, 4]} = \frac{< 2, 4 >}{< 1, 2 >}, \quad \frac{[3, 4]}{[1, 4]} = \frac{< 1, 2 >}{< 2, 3 >}, \] (37)

and can be used to obtain purely holomorphic expression:

\[ M_{abcd}^{h_1, h_2, h_3, h_4} = \]

\[ = -2i \ g^2 \ f_{ace} f_{bde} \left( \frac{< 24 > 2h_1}{< 21 >} \right) \left( \frac{< 34 > 2h_2}{< 32 >} \right) \left( \frac{< 24 > 2h_3}{< 23 >} \right) \frac{< 12 > < 34 >}{< 14 > < 32 >} \]

\[ -2i \ g^2 \ f_{ace} f_{bde} \left( \frac{< 24 > 2h_1}{< 21 >} \right) \left( \frac{< 34 > 2h_2}{< 32 >} \right) \left( \frac{< 24 > 2h_3}{< 23 >} \right) \frac{< 12 > < 34 >}{< 13 > < 24 >}. \] (38)

Further it can be written in the following factorized form

\[ M_{abcd}^{h_1, h_2, h_3, h_4} = -2i \ g^2 \left( \frac{< 24 > 2h_1 + 2h_3 - 4}{< 23 > 2h_2 + 2h_3} \frac{< 34 > 2h_2 + 2}{< 12 > 2h_1 - 2} \times \right) \]

\[ \times \left( \frac{f_{ace} f_{bde} < 24 > 4}{\Pi < ii + 1 >} + f_{ace} f_{bde} < 24 > 4 \right) \frac{\Pi_{3+4} < ii + 1 >}{\Pi_{3+4} < ii + 1 >}, \] (39)

where in the brackets is the MHV amplitude for the spin-1 gauge bosons times a factor which is the contribution of the high spin gauge bosons. We have to notice that the derivation of this amplitude in different channels is consistent if the coupling constants fulfill the following relations:

\[ g_{s_1, s_4, |1-s_1+s_4|} = g_{s_1, s_4, |1+s_1-s_4|} = g_{s_2, s_3, |1-s_2+s_3|} = g_{s_2, s_3, |1+s_2-s_3|} = g_{YM}. \] (40)

In the next section we shall use these amplitudes to calculate the production cross sections of non-Abelian tensor gauge bosons.
Figure 2: The scattering $V + V \rightarrow T + T$, shown in the center-of-mass frame. The $p_1, p_2$ are momenta of the vector gauge bosons $V$ and $p_3, p_4$ are momenta of the tensor gauge bosons $T$.

6 Tensor Bosons Production Cross Sections

The scattering process is illustrated in Fig.2. Working in the center-of-mass frame, we make the following assignments:

$p_1 = E(1, 0, 0, 1)$, $p_2 = E(1, 0, 0, -1)$, $p_3 = E(1, \sin \theta, 0, \cos \theta)$, $p_4 = E(1, -\sin \theta, 0, -\cos \theta)$, where $p_{1,2}$ are momenta of the vector bosons $V + V$ and $p_{3,4}$ are momenta of the tensor gauge bosons $T + T$. All particles are massless: $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$.

In the center-of-mass frame the momenta satisfy the relations $\vec{p}_1 = -\vec{p}_2$, $\vec{p}_3 = -\vec{p}_4$. The invariant variables of the process are:

$s = 2(p_1 \cdot p_2)$, $t = -\frac{s}{2}(1 - \cos \theta)$, $u = -\frac{s}{2}(1 + \cos \theta)$,

where $s = (2E)^2$ and $\theta$ is the scattering angle. It is convenient to write the differential cross section in the center-of-mass frame with tensor boson produced into the solid angle $d\Omega$ as

$$d\sigma = \frac{1}{2s}|M|^2 \frac{1}{32\pi^2} d\Omega,$$

where the final-state density is $d\Phi = \frac{1}{32\pi} d\Omega$. The spinor representation of momenta is

$$\lambda_1 = (\sqrt{2E}, 0), \quad \lambda_2 = (0, \sqrt{2E}),$$

$$\lambda_3 = (\sqrt{E(1 + \cos \theta)}, \frac{E \sin \theta}{\sqrt{E(1 + \cos \theta)}}), \quad \lambda_4 = (\frac{-E \sin \theta}{\sqrt{E(1 + \cos \theta)}}, \sqrt{E(1 + \cos \theta)}),$$

and it allows to calculate all spinor invariant products in (24),(25):

$$M_{4}^{abcd}(+1, -1, +s, -s) = i g^2 \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)^{s-1} \times$$

$$\times \left[ f^{ade} f^{bce} \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)^2 + f^{ace} f^{bde} (1 - \cos \theta) \right],$$

as well as the alternative helicity amplitude

$$M_{4}^{abcd}(+1, -1, -s, +s) = i g^2 \left( \frac{1 + \cos \theta}{1 - \cos \theta} \right)^{s-1} \times$$

$$\times \left[ f^{ade} f^{bce} (1 + \cos \theta) + f^{ace} f^{bde} \left( \frac{1 + \cos \theta}{1 - \cos \theta} \right)^2 \right].$$
To compute the cross section, we must square matrix elements (42), (43) and then average over the symmetries of the initial bosons and sum over the symmetries of the final tensor gauge bosons. This gives

\[ |M_{++\rightarrow ss}|^2 = \frac{g^4}{d(G)} C_2^2(G) \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)^{2s-2} \left( \frac{3 + \cos^2 \theta}{\sin^4 \theta} \right) (1 - \cos \theta)^4. \]  

\[ |M_{+-\rightarrow ss}|^2 = \frac{g^4}{d(G)} C_2^2(G) \left( \frac{1 + \cos \theta}{1 - \cos \theta} \right)^{2s-2} \left( \frac{3 + \cos^2 \theta}{\sin^4 \theta} \right) (1 + \cos \theta)^4, \]

where the invariant operator \( C_2 \) is defined by the equation \( t^a t^a = C_2 \) and \( d(G) \) is the dimension of the internal group \( G \). Plugging squared matrix elements into the cross-section formula (41) yields:

\[ d\sigma_{++\rightarrow ss} = \frac{\alpha^2}{s} \frac{C_2^2(G)}{d(G)} \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)^{2s-2} \left( \frac{3 + \cos^2 \theta}{\sin^4 \theta} \right) (1 - \cos \theta)^4 \, d\Omega, \]

\[ d\sigma_{+-\rightarrow ss} = \frac{\alpha^2}{s} \frac{C_2^2(G)}{d(G)} \left( \frac{1 + \cos \theta}{1 - \cos \theta} \right)^{2s-2} \left( \frac{3 + \cos^2 \theta}{\sin^4 \theta} \right) (1 + \cos \theta)^4 \, d\Omega, \]

where \( \alpha = \frac{g^2 M_t}{4\pi} \). It is instructive to compare the above cross sections with the corresponding cross sections for the vector gauge bosons. Indeed, the last formulas can be written in the factorized form stressing the similarity with the gluon annihilation cross section

\[ d\sigma_{++} = \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)^{2s-2} d\sigma_{++}, \quad d\sigma_{+-} = \left( \frac{1 + \cos \theta}{1 - \cos \theta} \right)^{2s-2} d\sigma_{+-}, \quad s = 1, 2, 3, \ldots \]  

(48)

where \( \sigma_{++\rightarrow ss} \equiv \sigma_{++} \) and \( \sigma_{+-\rightarrow ss} \equiv \sigma_{+-} \). It also shows explicitly the spin dependence of the cross sections which has an amazingly simple form. These cross sections have standards infrared singularity in the forward and backward directions due to the massless character of the spectrum of non-Abelian tensor gauge bosons. One can sum the production rate of high spin tensor gauge bosons. This gives

\[ \sum_{s=1}^{N} \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)^{2s-2} = \frac{(1 + \cos \theta)^2}{4 \cos \theta (1 + \cos \theta)^{2N-2}} \]  

(49)

and taking the limit \( N \rightarrow \infty \) one can get

\[ d\sigma_{\text{tot}} = \sum_{s=1}^{\infty} \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)^{2s-2} = \frac{(1 + \cos \theta)^2}{4 \cos \theta} d\sigma_{++}, \]

(50)

which is valid in the region of the scattering angle

\[ \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right) < 1, \quad 0 \leq \theta < \pi/2. \]  

(51)

The characteristic feature of the total cross section is that it increases in the transverse direction and tends to infinity already in the transversal plane \( \theta \sim \pi/2 \).

In the general case of scattering of tensor gauge bosons of helicities \( h_1, h_2, h_3, h_4 \) \( (h_1 + h_2 + h_3 + h_4 = 0, \ h_1 > 0, \ h_2 < 0) \), one can also compute the cross section using the amplitude (38), (39). The cross section takes the form:

\[ d\sigma_{h_1, h_2, h_3, h_4} = \frac{(1 - \cos \theta)^{2h_1 + 2h_2 + 2h_3 - 2}}{2^{2h_1 - 2h_2 - 4}(1 + \cos \theta)^{2h_3 - 2}} d\sigma_{++}, \]

and spectacularly falls exponentially as the spins of scattering particles increase.
7  Production of Tensor Gauge Bosons and $(n - 4)$ Jets

In this section, we shall focus ourselves on color-ordered scattering amplitudes involving two tensor particles of helicity $+s$ and $-s$ respectively, one negative helicity gluon and any number of gluons with positive helicity. In the case where $s = 1$ one has the MHV amplitudes for the scattering of vector bosons (gluons). The expression for this amplitude is given by the famous Parke-Taylor formula [10]. We shall see that it is possible to write a generalization of the Parke-Taylor formula for the case of arbitrary spin $s$.

We begin by giving the result which reads:

\[ M^{(n)}(1^+, ...i^-, ..., k^+, ...j^-, ..., n^+) = g^{n-2} \frac{\langle ij \rangle^4}{\prod_{l=1}^{n} \langle ll + 1 \rangle} \langle ik \rangle^{2s-2}, \]  

(53)

where $n$ is the total number of particles, and the dots stand for the positive helicity gluons. Finally, $i$ is the position of the negative helicity gluon, while $k$ and $j$ are the positions of the particles with helicities $+s$ and $-s$ respectively. A first comment is that this expression is holomorphic in the spinors of the particles, exactly as the MHV gluon amplitude. A second comment is that for $s = 1$ the second fraction in (53) is absent and (53) reduces to the well-known result for the MHV amplitude.

We now proceed to prove (53) by induction. For simplicity we shall look at the case in which the negative helicity gluon sits at position 2 while the particles with helicities $s$ and $-s$ sit at positions $n - 1$ and $n$, respectively. During the proof, it will become apparent that placing the particles at these special positions plays no role and is primarily a matter of convenience.

Let us start with the case of five particles. We shall deform the momentum of positive helicity gluon sitting at position 1 and the negative helicity gluon sitting at position 2, as in section 5. With this deformation there are two diagrams contributing. The first diagram has three external particles attached to the left vertex and two external particles to the right vertex while the diagram on the right vice versa. The right vertex of the first diagram gives a non-zero result only if the exchanged particle has helicity $h > h_2 + h_3 = -1 + 1 = 0$. Thus, the only possibility is $h = 1$. But for $h = 1$ the sum of the helicities of the left vertex is 2 and such a vertex does not exist (a vertex with 4 particles is non-zero only if the sum of the helicities is zero (35)). Consequently, the whole contribution to the five leg amplitude comes solely from the second diagram. As before, the left vertex is non-zero only if $h > -(h_1 + h_4) = s - 1$, which means that $h = s$.

The BCFW recursion relation allow us to write the amplitude as

\[ M^{(5)}(\hat{1}, 5, -\hat{P}_{51}) = \frac{1}{\hat{P}_{51}^2} M^{(4)}(\hat{P}_{51}, 4, 3, \hat{2}). \]  

(54)

The expressions for $M^{(3)}(\hat{1}, 5, -\hat{P}_{51})$ and $M^{(4)}(\hat{P}_{51}, 4, 3, \hat{2})$ are given by (22)

\[ M^{(3)}(\hat{1}, 5, -\hat{P}_{51}) = g^{\frac{[\hat{P}_{51}^{\hat{1}}]^{2s+1}}{[5\hat{P}_{51}][15]^{2s-1} \hat{P}_{51}^2}}, \]  

(55)

and (38), (39)

\[ M^{(4)}(\hat{P}_{51}, 4, 3, \hat{2}) = -g^2 \frac{\langle 2\hat{P}_{51} \rangle^4}{\prod_{i \in R} \langle ii + 1 \rangle} \langle 24 \rangle^{2s-2}, \]  

(56)
where the product $\prod_{i \in R} < ii + 1 >$ is over the spinors of the right vertex of the diagram. Plugging these expressions in (54) we obtain

$$M^{(5)} = -g^3 \frac{< 25 ^4 >}{\prod_{i=1}^5 < ii + 1 >} \times \frac{\hat{P}_{51} ^{2s-2}}{[15]^{2s-2}} \times \frac{< 2 \hat{P}_{51} > ^{2s-2}}{< 24 > ^{2s-2}}. \tag{57}$$

The first fraction in (57) is the MHV amplitude for gluons and can be obtained by isolating from (54) the terms for $s = 1$. What remains are the last two fractions of (57). The next step involves using

$$Y = \frac{\hat{P}_{51} ^{1} < 2 \hat{P}_{51} >} {[15] < 24 >} = \frac{< 2 |P_{15}| 1 / w < 2 |P_{15}| 1 / w} {[15] < 24 >} = \frac{< 2 |P_{5}| 1 < 2 |P_{5}| 1} {[15] < 24 >} \frac{1}{w \bar{w}}, \tag{58}$$

where

$$w \bar{w} = [\hat{P}_{51} 1] < 2 \hat{P}_{51} >= < 2 |P_{15}| 1]. \tag{58}$$

Plugging the last formula in $Y$ we get

$$Y = -\frac{< 25 >}{< 24 >}.$$

Altogether the five-particle amplitude becomes\(^2\)

$$M^{(5)}(1^+, 2^-, 3^+, 4^{++}, 5^{-}) = g^3 \frac{< 25 ^4 >}{\prod_{i=1}^5 < ii + 1 >} \times \frac{< 25 ^{2s-2} >}{< 24 ^{2s-2}>}. \tag{59}$$

We are, now, ready to argue that the only contribution to the $n$-particle amplitude comes from one diagram. The one which has a 3-particle vertex on the left part of the diagram and a $(n-1)$-particle vertex on the right part. All other diagrams with different distribution of particles between the left and right vertex give, in fact, zero. There are several ways to see this.

One way to get convinced is by considering the six-particle amplitude. There are three diagrams which can potentially contribute. The first one has a 3-particle vertex on the right. As above, this is non-zero only if $h > 0$ which implies that $h = 1$. But then the left vertex has particles with helicities $(1, -s, s, 1, 1)$ and such a vertex does not exist. The second diagram has two vertices with 4 particles each. The corresponding helicities are $(1, -s, s, h)$ for the left vertex and $(-1, -h, 1, 1)$ for the right one. Thus $h$ can be +1 or −1. But for $h = 1$ the left vertex does not exist since the sum of the helicities is non-zero (35). Similarly, if the exchanged particle has helicity $h = -1$ it is the right vertex that does not exist. Thus, the only contributing diagram is the last one. One can go to seven or more particles and by using similar arguments verify that there is only one contributing diagram.

Thus, there is only one diagram contributing to the $(1, -1, 1, ..., s, -s)$ amplitude. It is the diagram where the left vertex is $(1, -s, s)$ while the right one is $(-1, -s, s, 1, ...).$ It is now obvious that the proof of the validity of (53) can be completed by induction. The proof goes precisely as in the case for five particles. The only difference is that one allows for $n-1$ particles in the right vertex. Then the only thing needed is to substitute 5 with $n$ and 4 with $n-1$ in (54), (57) and (58).

\(^2\)The 4- and 5-particle amplitudes correspond to 2-jet and 3-jet production in hadronic collisions at very high energies, and so are of phenomenological importance.
As we already notice the amplitude $M_{4abcd}^{+1, -s, +1, s-2}$ (34) is a first example of non-MHV amplitude in generalized Yang-Mills theory. With the amplitudes (53) in hands one can try to calculate the n-particle amplitudes with more negative helicity particles generalizing the calculation of [16, 17, 18, 19].

In conclusion we would like to remark that one can perform a Fourier transformation of the holomorphic scattering amplitudes (53) from momentum space to twistor space, in the spirit of [12]. Since these amplitudes are holomorphic in the spinor variables this means that they will be supported on an algebraic curve in twistor space of degree one and genus zero, or in given case on a straight line. Therefore one can try to find out if there exists a tensionless string theory [45, 46], which will be the analog of the topological B-model for the case of $N = 4SY M$, that can reproduce the tree level amplitudes of the generalized Yang-Mills theory. It would be interesting to see if this possibility is realized.

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