AN ARCTIC CIRCLE THEOREM FOR GROVES

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ABSTRACT. In earlier work, Jockusch, Propp, and Shor proved a theorem describing the limiting shape of the boundary between the uniformly tiled corners of a random tiling of an Aztec diamond and the more unpredictable ‘temperate zone’ in the interior of the region. The so-called arctic circle theorem made precise a phenomenon observed in random tilings of large Aztec diamonds.

Here we examine a related combinatorial model called groves. Created by Carroll and Speyer as combinatorial interpretations for Laurent polynomials given by the cube recurrence, groves have observable frozen regions which we describe precisely via asymptotic analysis of a generating function. Our approach also provides another way to prove the arctic circle theorem for Aztec diamonds.

1. Introduction

Groves came into existence as combinatorial interpretations of the rational functions generated by the cube recurrence:

\[ f_{i,j,k} f_{i-1,j-1,k-1} = f_{i-1,j,k} f_{i,j-1,k-1} + f_{i,j-1,k} f_{i-1,j,k-1} + f_{i,j,k-1} f_{i-1,j-1,k}, \]

where some initial functions are specified. Typically, \( f_{i,j,k} := x_{i,j,k} \) for some choice of \((i, j, k) \in \mathbb{Z}^3\) called the initial conditions. Fomin and Zelevinsky [4] were able to show that for initial conditions satisfying some basic requirements, the rational functions generated by the cube recurrence are in fact Laurent polynomials in the \( x_{i,j,k} \). The introduction of groves by Carroll and Speyer [1] gave a combinatorial proof of the surprising fact that each term of these polynomials has coefficient +1. The main results in this paper only apply to the family of groves on standard initial conditions as described in Section 1.1.

Before getting into the details of groves, let us first describe the motivation for this paper: random domino tilings of large Aztec diamonds. An Aztec diamond of order \( n \) consists of the union of all unit squares with integer vertices contained in the planar region \( \{(x, y) : |x| + |y| \leq n + 1\} \). A domino tiling of an Aztec diamond is an arrangement of \( 2 \times 1 \) rectangles, or dominoes, that cover the diamond without any overlapping. A random domino tiling of a large Aztec diamond consists of two qualitatively different regions.\(^2\) As seen in the random tiling in Figure 1, the dominoes in the corners of the diamond are frozen in a brickwork pattern, whereas the dominoes in the interior have a more random, temperate behavior. It was shown in [5] and [2] that asymptotically, the boundary between the frozen and temperate regions in a random tiling is given by the circle inscribed in the Aztec diamond.

\(^1\)Herein we will invoke some of the basic properties of groves without proof. For such arguments, as well as a general treatment of groves and the cube recurrence, the reader is referred to [1].

\(^2\)By random we mean selected from the uniform distribution on all tilings of an Aztec diamond of order \( n \), though other probability distributions may be considered as well. See [2].
Since everything outside the circle is expected to be frozen, it is referred to as the arctic circle.

Figure 1. A random domino tiling of an Aztec diamond of order 64.

In this paper we shall see that groves on standard initial conditions exhibit a very similar behavior. A grove, however, is not a type of tiling. As the name may suggest, a grove is in fact a collection of trees. From our point of view, groves are spanning forests on a finite triangular lattice satisfying certain connectivity conditions on the boundary. We will show that outside of the circle inscribed in the triangle, the trees of a large random grove line up uniformly.

Despite their superficial differences, groves and random domino tilings of Aztec diamonds are linked by more than their asymptotic behavior. In fact it seems that their asymptotic behavior is similar because they share a deeper link. The paper of Carroll and Speyer [1] establishes that groves are encoded in the terms of a Laurent polynomial given by the cube recurrence. There is a more general form of the cube recurrence:

\[
{f_{i,j,k}}_{i-1,j-1,k-1} = \alpha f_{i-1,j,k} f_{i,j-1,k-1} + \beta f_{i,j-1,k} f_{i-1,j,k-1} + \gamma f_{i,j,k-1} f_{i-1,j-1,k}
\]

where \(\alpha, \beta, \gamma\) are constants. If \(\alpha = \beta = \gamma = 1\) we have the original form of the cube recurrence from whence come groves. If \(\alpha = \beta = 1\) and \(\gamma = 0\), we have (after re-indexing), the octahedron recurrence:

\[
g_{i,j,n+1} + g_{i,j,n-1} = g_{i-1,j,n} g_{i+1,j,n} + g_{i,j-1,n} g_{i,j+1,n},
\]
with which we may encode tilings of Aztec diamonds. In Section 3 we will show how the polynomial $g_{0,0,n}$ yields all tilings of an Aztec diamond of order $n$ and we will describe the role that this recurrence plays in the large scale behavior of such tilings.

While the octahedron recurrence is important to us, it has not been extant in the study of tilings of Aztec diamonds in the past. Rather, a local move called domino shuffling has been used. Domino shuffling was introduced in [3] and is generalized in [10]. It provides a method for generating tilings of successively larger Aztec diamonds uniformly at random, and has been at least implicit in all probabilistic analysis done to date. Section 1.3 will introduce an analogous local move for groves that we call grove shuffling. Like domino shuffling, it will be key to our analysis.

For each of the two models discussed we have a global perspective and a local perspective. Laurent polynomials tell the global story: all groves are encapsulated in $f_{0,0,0}$ (from the cube recurrence), all tilings in $g_{0,0,n}$ (from the octahedron recurrence). A specified shuffling algorithm tells the local story. In this paper we combine these two points of view to build generating functions (for tilings of Aztec diamonds as well as for groves), with which we can study asymptotic behavior.

![Figure 2. A portion of $G(5)$.](image-url)
1.1. Groves on standard initial conditions. The standard initial conditions of order \( n \) specify a vertex set \( \mathcal{I}(n) = \mathcal{C}(n) \cup \mathcal{B}(n) \) where \( \mathcal{C}(n) = \{(i, j, k) \in \mathbb{Z}^3 \mid -n-1 \leq i+j+k \leq -n+1, i, j, k \leq 0\} \) and \( \mathcal{B}(n) = \{(i, j, k) \in \mathbb{Z}^3 \mid i+j+k < -n-1; i, j, k \leq 0; \) and \( i, j, \) or \( k = 0\}\). We draw its projection onto the plane \( \mathbb{R}^3/(1,1,1) \) as shown in Figure 1 for the case \( n = 5 \). One way to generate all groves of order \( n \) is to set \( f_{i,j,k} := x_{i,j,k} \) for all \( (i, j, k) \in \mathcal{I}(n) \), and compute \( f_{0,0,0} \). Each term in the resulting Laurent polynomial defines a grove as follows. Let \( \mathcal{G}(n) \) be the graph on the vertex set \( \mathcal{I}(n) \) where vertex \( (i, j, k) \) has as its neighbors the vertices \( \mathcal{I}(n) \cap \{(i \pm 1, j \pm 1, k), (i \pm 1, j, k \pm 1), (i, j \pm 1, k \pm 1)\} \). Pictorially, edges of \( \mathcal{G}(n) \) connect vertices that lie diagonally across a rhombus. In Figure 2 the graph \( \mathcal{G}(5) \) is made up of the lighter edges and the dark vertices.

As established in [1], the terms in \( f_{0,0,0} \) are Laurent monomials of the form

\[
m(g) = \prod_{(i,j,k) \in \mathcal{I}(n)} x^{\text{deg}(i,j,k)-2},
\]

where \( \text{deg}(i,j,k) \in \{1, 2, \ldots, 6\} \) is the number of edges connected to vertex \( (i,j,k) \).

We have the following

**Definition 1.** The grove \( g \) defined by \( m(g) \) is the unique subgraph of \( \mathcal{G}(n) \) containing no crossing edges such that vertex \( (i, j, k) \in \mathcal{I}(n) \) has exactly \( \text{deg}(i,j,k) \) incident edges.

The uniqueness of the grove determined by each monomial is a consequence of Theorem 3 in [1]. For example, \( f_{0,0,0} \) on \( \mathcal{I}(2) \) is

\[
\frac{x_{-1,-1,0} x_{0,0,-1}}{x_{-1,-1,-1}} + \frac{x_{-1,0,-1} x_{0,-1,0}}{x_{-1,-1,-1}} + \frac{x_{0,-1,-1} x_{-1,0,0}}{x_{-1,-1,-1}},
\]

and the corresponding groves are shown in Figure 3.

![Figure 3. The three groves of order 2.](image)

For a more interesting example, one term of \( f_{0,0,0} \) on \( \mathcal{I}(5) \) is

\[
\frac{x_{-3,0,-2} x_{-2,-1,-1} x_{-1,-3,0} x_{0,-2,-2}}{x_{-3,-1,-2} x_{-2,-3,-1} x_{-1,-2,-2}}.
\]

Its corresponding grove, \( g \), is shown in Figure 4. We can observe some connectivity properties of this grove that in fact hold for all groves. Every vertex on the boundary of \( \mathcal{C}(n) \) (where cubes have been pushed down) is connected to another vertex on the boundary of \( \mathcal{C}(n) \) if and only if those vertices are equidistant to the nearest corner (i.e. where two coordinates are zero) of the grove. Groves are acyclic—every connected component of a grove is a tree. Lastly, each grove spans \( \mathcal{I}(n) \). These connectivity properties are in fact what distinguish groves from arbitrary subgraphs of \( \mathcal{G}(n) \), and so give us a combinatorial definition of groves.
Within a grove notice that there are two types of edges: long edges and short edges, depending on whether the long or short diagonal of a rhombus is used. For a vertex \( v = (i, j, k) \) in \( C(n) \), we say that \( v \) is:

- **up**: if \( i + j + k = -n + 1 \),
- **down**: if \( i + j + k = -n - 1 \),
- **flat**: if \( i + j + k = -n \),
- **even**: if \( i + j + k \) is even,
- **odd**: if \( i + j + k \) is odd.

Long edges connect flat vertices to flat vertices, and short edges connect up vertices to down vertices. Even vertices are only connected to even vertices and odd vertices are only connected to odd vertices. It is shown in [1] that every vertex in \( B(n) \) has degree 2 and only uses its short edges. As a result, there are only finitely many long edges, and these determine the grove. This observation leads to a more convenient way of looking at groves.

### 1.2. Simplified groves

We begin by constructing a modified form of the cube recurrence. Let \( a_{i,j}, b_{k,j}, c_{i,k} \) be long edge variables where \( -n = i + j + k \) is fixed. The variable \( a_{i,j} \) is the label for the edge between vertices \((i, j - 1, k + 1)\) and \((i - 1, j, k + 1)\), \( b_{k,j} \) is the label for the edge between \((i - 1, j, k + 1)\) and \((i, j, k)\), and \( c_{i,k} \) is the label for the edge between \((i, j, k)\) and \((i, j - 1, k + 1)\). We write a modified form of the cube recurrence as follows:

\[
 f_{i,j,k} f_{i-1,j-1,k-1} = b_{i,k} c_{i,j} f_{i-1,j-1,k} f_{i,j-1,k-1} + c_{i,j} a_{j,k} f_{i,j-1,k} f_{i-1,j,k-1} + a_{j,k} b_{i,k} f_{i,j,k-1} f_{i-1,j-1,k} 
\]

As we said, the long edges determine the grove, so rather than setting \( f_{i,j,k} := x_{i,j,k} \) for \((i, j, k) \in I(n)\), we set \( f_{i,j,k} := 1 \) for \((i, j, k) \in I(n)\). Then \( f_{0,0,0} \) is simply a polynomial in the edge variables \( a_{i,j}, b_{i,j}, c_{i,j} \), where the variables appear with
exponent +1 or 0, depending on whether the corresponding long edge is present or not. Each term describes a unique grove, and we still produce every grove. This form of the cube recurrence is called the edge variables version.

Taking inspiration from the edge variables version of the cube recurrence, we can draw a simpler picture of our groves by ignoring all short edges and all of the vertices incident with them. In other words, specify a subset of the standard initial conditions of order \( n \), called the simplified initial conditions:

\[
I'(n) = \{ (i, j, k) \in \mathbb{Z}^3 \mid i+j+k = -n, i, j, k \leq 0 \} \subset I(n).
\]

The simplified initial conditions are just all of the flat vertices. We now represent our groves as graphs on this vertex set—a triangular lattice shown in Figure 5. Also in Figure 5, we see the same grove as in Figure 4 but with only the long edges included. In terms of edge variables, this grove is given by

\[
a_{0,0}a_{0,1}a_{0,2}a_{1,0}a_{1,1}a_{2,0}b_{0,0}b_{0,1}c_{0,0}c_{0,1}c_{1,0}c_{2,0}.
\]

**Figure 5.** On the left: \( I'(5) \) compared to \( I(5) \). On the right: a standard grove and its corresponding simplified grove.

Another modification of the cube recurrence that we shall like to use is the edge-and-face variables version. In the original version of the cube recurrence, the variables \( x_{i,j,k} \) such that \( i+j+k = -n + 1 \) were vertex variables. In the simplified picture, we call them the face variables of order \( n \), for reasons that will become clear. Rather than setting \( f_{i,j,k} := 1 \) for all \( (i, j, k) \) in \( I(n) \), we give the face variables their formal weights. That is, we set \( f_{i,j,k} := 1 \) for \( (i, j, k) \in \{ (i, j, k) \in \mathbb{Z}^3 \mid -n - 1 \leq i+j+k \leq n, i, j, k \leq 0 \} \) and \( f_{i,j,k} := x_{i,j,k} \) for \( (i, j, k) \in \{ (i, j, k) \in \mathbb{Z}^3 \mid i+j+k = -n + 1, i, j, k \leq 0 \} \). Generating \( f_{0,0,0} \) using these initial conditions, we get a Laurent polynomial in the edge and face variables.
The vertices of the simplified initial conditions can be seen as forming \( n(n+1)/2 \) downward-pointing equilateral triangles, each with top-left vertex \((i, j - 1, k + 1)\), top-right vertex \((i - 1, j, k + 1)\), and bottom vertex \((i, j, k)\). The face variables then correspond to each of these downward-pointing triangles. The triangle with \((i, j, k)\) as its bottom vertex has face variable \(x_{i,j,k+1}\). The exponent of the face variable is \(-1, 0,\) or \(1\), corresponding to whether the downward-pointing triangle has, respectively, two, one, or zero edges present. There can’t be three edges, since that would introduce a cycle and we would no longer have a forest. Although the face variables don’t tell us anything new about a particular grove, they will be useful later in deriving probabilities of edges being present in random groves.

1.3. Grove shuffling. We have given one definition for what groves are, and how they may be generated. The methods and notation introduced in the previous section will be very helpful for later proofs. However, there is another tool we will like to use; an algorithm called grove shuffling (or cube-popping as in [1]). Grove shuffling not only gives a purely combinatorial definition of groves, but also a method for generating groves of order \(n\) uniformly at random. Its inspiration comes from domino shuffling, due to Elkies, Kuperberg, Larsen, and Propp [3]. The use to which we put grove shuffling is directly motivated by James Propp and his paper [16]. For proof that grove shuffling does indeed give rise to the same objects as the terms of the Laurent polynomials given by the cube recurrence, see Carroll and Speyer [1]. Here we will only include a description of the algorithm.

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**Figure 6. Grove shuffling.**

Grove shuffling can be thought of as a local move on the downward-pointing triangles of a simplified grove according to whether a triangle has zero, one, or two edges present. See Figure 6. Let \(x\) be a generic downward-pointing triangle with possible edges \(a, b, c\) as shown, and let \(x'\) be an upward-pointing triangle, concentric with \(x\), with possible edges \(a', b', c'\) as shown. There are three configurations of
Grove shuffling takes each of these triangles and replaces them with an upward-pointing triangle \( x' \) having none of its possible edges present. There are three configurations of \( x \) with exactly one edge: \( ab, ac, bc \). Each of these is replaced by the upward-pointing triangle \( x' \) with only the parallel edge: \( a', b', c' \), respectively present. Lastly, there is one configuration of \( x \) with none of its possible edges present. This triangle is replaced with the upward-pointing triangle \( x' \) containing any two of its three possible edges: \( a'b', a'c', b'c' \), chosen randomly with probability \( 1/3 \). This last step is the only random part of the algorithm. After we have turned every downward-pointing triangle into an upward-pointing triangle, we add three new vertices to the corners of the grove so that we may shuffle again. For an example of grove shuffling, see Figure 7.

![Figure 7. A grove of order 4 shuffled into a grove of order 5.](http://ups.physics.wisc.edu/~hal/SSL/groveshuffler/)

There is a unique grove of order 1. It has one downward-pointing triangle with zero edges. We now give a purely combinatorial description of simplified groves on standard initial conditions of order \( n \): they are all the possible results of \( n - 1 \) iterations of grove shuffling, beginning with the grove of order 1. From looking at the cube recurrence, it is not hard to show that there are \( 3^{\lceil n^2/4 \rceil} \) groves of order \( n \).

We can now make the following claim about grove shuffling.

**Theorem 1.** Beginning with the unique grove of order one, any grove of order \( n \) will be generated after \( n - 1 \) iterations of grove shuffling with probability \( 1/3^{\lceil n^2/4 \rceil} \). In other words, grove shuffling can be used to generate groves uniformly at random.

**Proof.** Clearly the statement holds for \( n=2 \). Suppose that the claim holds for some \( k \geq 1 \). We would like to know the probability of an arbitrary grove of order \( k + 1 \) being generated. Fix such a grove and call it \( G(k + 1) \). Only a certain subset of the groves of order \( k \) can be shuffled to become \( G(k + 1) \). Call this set the shuffling pre-image of \( G(k + 1) \), denoted \( S^{-1}(G(k + 1)) \). Let \( G(k) \in S^{-1}(G(k + 1)) \). Let \( a \) be the number of downward-pointing triangles in \( G(k) \) with zero edges, let \( b \) be the number with exactly one edge, and \( c \) be the number of downward-pointing triangles with two edges.

From the rules of grove shuffling, we see that the order of \( S^{-1}(G(k + 1)) \) is \( 3^c \). Each pre-image is obtained by making different choices of the two edges appearing in each of the \( c \) downward-pointing triangles of \( G(k) \). So since we have

\[3^{\lceil n^2/4 \rceil}\]

To see grove shuffling in action, visit [http://ups.physics.wisc.edu/~hal/SSL/groveshuffler/](http://ups.physics.wisc.edu/~hal/SSL/groveshuffler/).
supposed the probability of generating a particular grove of order \( k \) to be uniform, the probability is
\[
\frac{3^c}{3^\left\lfloor \frac{k^2}{4} \right\rfloor}
\]
that after \( k \) shuffles we produce a grove in \( S^{-1}(G(k + 1)) \).

Let \( S(G(k)) = S(S^{-1}(G(k + 1))) \) be the set of groves of order \( k + 1 \) that can be obtained by shuffling a grove in \( S^{-1}(G(k + 1)) \). The order of \( S(G(k)) \) is \( 3^a \). This is because in each of the pre-images there are \( a \) downward-pointing triangles with no edges present, and every such triangle can be shuffled to any of three upward-pointing triangles. Furthermore, the only edges where the groves of \( S^{-1}(G(k + 1)) \) differ will be annihilated upon shuffling. So there is a \( 1/3^a \) chance that one of the pre-images of \( G(k + 1) \) will actually shuffle into \( G(k + 1) \). Therefore the probability that \( k + 1 \) iterations of grove shuffling yields \( G(k + 1) \) is
\[
\frac{1}{3^\left\lfloor \frac{k^2}{4} \right\rfloor} \cdot \frac{1}{3^{a-c}}
\]
Now we claim that \( a - c = \left\lfloor \frac{k+1}{2} \right\rfloor \). If so, then the probability computed above is equal to
\[
\frac{1}{3^\left\lfloor \frac{(k+1)^2}{4} \right\rfloor}
\]
as desired.

Let us make some basic observations from [1] or by easy induction. First, \( a + b + c = k(k + 1)/2 \); the total number of downward-pointing triangles in any grove of order \( k \). Secondly, \( b + 2c = \left\lfloor \frac{k^2}{2} \right\rfloor \); the total number of edges in any grove of order \( k \). Then \( a - c = k(k + 1)/2 - \left\lfloor \frac{k^2}{2} \right\rfloor = \left\lfloor \frac{k+1}{2} \right\rfloor \), and the theorem is proved.

1.4. Frozen regions. We now describe the phenomenon that we analyze in Section 2. First we observe that edges are indexed relative to the corners perpendicular to them, so in fact the edges \( a \) and \( a' \) in the description of grove shuffling have the same label: \( a = a' = a_{i,j} \). Horizontal edges are indexed relative to the bottom corner, and the diagonal edges are indexed relative to the top-right and top-left corners. In this way we can think of grove-shuffling as more akin to domino shuffling [16]. Rather than replacing edges with parallel edges, we “slide” edges toward the corners along perpendicular lines. When a downward-pointing triangle has two edges, we remove both of those edges because they “annihilate” each other. When a downward-pointing triangle has no edges, we create two new ones randomly.

With this viewpoint, we define an edge to be frozen if it cannot be annihilated under any further iterations of grove shuffling. Clearly the bottom corner edge, \( a_{0,0} \), is frozen when present. Then the edge \( a_{i,j} \) is frozen exactly when the edges \( a_{i',j'}, \) are frozen, \( i \leq i' \leq 0, j \leq j' \leq 0 \). Diagonal edges behave similarly. In Figure 8 all the highlighted edges are frozen.

We conclude this section by examining a picture of a large random grove generated by grove shuffling. In Figure 9 we see that outside of a certain region, all of the edges are parallel. Moreover, the boundary between the less uniform interior and the frozen regions in the corners seems to approximate a circle. Proving that this boundary approaches a circle in the limit is the main goal of this paper.
2. The arctic circle theorem

For any $n$, we can scale the initial conditions so that they resemble an equilateral triangle with sides of length $\sqrt{2}$, by mapping each vertex $(i, j, k)$ to $(i/n, j/n, k/n)$. The corner vertices $(-n, 0, 0), (0, -n, 0), (0, 0, -n)$ are scaled to $(-1, 0, 0), (0, -1, 0)$, and $(0, 0, -1)$. We will show that outside of the circle inscribed in this triangle, there is homogeneity of the edges in an appropriately scaled random grove of order $n$, with probability approaching 1 as $n \to \infty$. Specifically, we will examine the limiting probability of finding a particular type of edge in a given location outside of the inscribed circle.

2.1. Edge probabilities. Let $p_n(i, j) = p(i, j, k), k = -n-i-j$, be the probability that $a_{i,j}(n)$, the horizontal edge on triangle $x_{i,j,k+1}$, is present in a random grove of order $n$. Similarly define probabilities $q_n(k, i), r_n(k, j)$ for the diagonal edges.
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Define $E_n(i,j) = E(i,j,k+1) = 1 - p_n(i,j) - q_n(k,i) - r_n(k,j)$. The numbers $E_n(i,j)$ are analogous to the creation rates discussed in [5], [2], and [16]. We will also refer to them as creation rates. As proven below, we can also realize the number $E_n(i,j)$ as the expected value of the exponent of the face variable $x_{i,j,k+1}$. We prove the following formula for finding the edge probability $p_n(i,j)$ in terms of creation rates.

**Theorem 2.** The horizontal edge probabilities are given recursively by

$$p_n(i,j) = p_{n-1}(i,j) + \frac{2}{3}E_{n-1}(i,j).$$

Thus, $p_n(i,j) = \frac{2}{3} \sum_{l=1}^{n-1} E_l(i,j)$.

**Proof.** We wish to derive a relation between $p_n(i,j)$ and $p_{n-1}(i,j)$. In order to simplify notation, we let:

- $p = p_{n-1}(i,j)$
- $q = q_{n-1}(k,i)$
- $r = r_{n-1}(k,j)$
- $a = a_{i,j}(n-1)$
- $b = b_{k,i}(n-1)$
- $c = c_{k,j}(n-1)$
- $P = p_n(i,j)$
- $Q = q_n(k,i)$
- $R = r_n(k,j)$
- $A = a_{i,j}(n)$
- $B = b_{k,i}(n)$
- $C = c_{k,j}(n)$

See Figure 10.

![Figure 10. Labels for downward- and upward-pointing triangles.](image)

Let $pr(\cdot)$, where $\cdot$ is a subset of $\{a, b, c\}$, be the probability that a random grove contains that set of edges and not its compliment. Define $Pr(\cdot)$ similarly. Some observations that come directly from grove shuffling:

- $pr(ab) = pr(ac) = pr(bc)$
- $pr(abc) = 0$
- $pr(\emptyset) + pr(a) + pr(b) + pr(c) + pr(ab) + pr(ac) + pr(bc) = 1$
- $p = pr(a) + pr(ab) + pr(ac)$
- $q = pr(b) + pr(ab) + pr(bc)$
- $r = pr(c) + pr(ac) + pr(bc)$
- $Pr(A) = pr(a)$
- $Pr(B) = pr(b)$
- $Pr(C) = pr(c)$
- $Pr(AB) = Pr(AC) = Pr(BC) = 1/3pr(\emptyset)$

4Notice the similarity between this statement and equation 1.5 of [2].
We will now deduce \( P = p_n(i,j) \).

\[
P = \Pr(A) + \Pr(AB) + \Pr(AC) = \frac{p}{3} + 2/3\Pr(\emptyset) + \frac{2}{3}(1 - p - q - r) + \frac{2}{3}(p(ab) + \Pr(ac) + \Pr(bc))
\]

\[
= \frac{p}{3} + 2/3(1 - p + q - r + \Pr(ab) + \Pr(ac) + \Pr(bc)) + \frac{2}{3}(1 - p - q - r) + \frac{2}{3}(p(ab) + \Pr(ac) + \Pr(bc)) + \frac{2}{3}(1 - p - q - r)
\]

\[
= p + 2/3(1 - p - q - r)
\]

Let \( x = x_{i,j,k+2} \) be the face variable of the downward-pointing triangle in question. Notice that

\[
E(x) = \text{Expected value of exponent on } x = 1 \cdot \Pr(\emptyset) + 0 \cdot (\Pr(a) + \Pr(b) + \Pr(c)) - 1 \cdot (\Pr(ab) + \Pr(ac) + \Pr(bc))
\]

\[
= 1 - \Pr(a) - \Pr(b) - \Pr(c) - 2\Pr(ab) - 2\Pr(ac) - 2\Pr(bc)
\]

\[
= 1 - p - q - r
\]

\[
= E_{n-1}(i,j).
\]

Therefore, \( P = p + 2/3E(x) \). In the coordinate system, we have

\[
p_n(i,j) = n_p(i,j) + 2/3E_{n-1}(i,j) = \frac{2}{3} \sum_{i=1}^{n-1} E_i(i,j)
\]

and the theorem is proved. \( \square \)

2.2. A generating function. We now know that to compute the probability of a particular edge being present in a random grove, it will be enough to compute the creation rates \( E_i(i,j) \). In this section we derive a generating function for computing these numbers as well as the related generating function for the horizontal edge probabilities.

Let \( F(x,y,z) = \sum_{i,j,k \geq 0} E(-i,-j,-k)x^iy^jz^k \) be the generating function for the creation rates. First consider the uniformly weighted version of the cube recurrence:

\[
f_{i,j,k} f_{i-1,j-1,k-1} = \frac{1}{3} (f_{i-1,j,k} f_{i,j-1,k-1} + f_{i-1,j,k} f_{i-1,j,k-1} + f_{i,j,k-1} f_{i-1,j-1,k}).
\]

We will return to the convention of setting \( f_{i,j,k} = x_{i,j,k} \) for all \( (i,j,k) \in I(n) \). Using this recurrence to calculate \( f_{0,0,0} \) we will get each monomial weighted uniformly, so that if we set all the variables equal to 1, \( f_{0,0,0} = 1 \). If we want the expectation of the exponent of the face variable \( x = x_{i_0,j_0,k_0} \), we need only calculate the derivative of \( f_{0,0,0} \) with respect to this variable, then set all variables equal to one. In other words,

\[
E(i_0,j_0,k_0) = \frac{\partial}{\partial x} (f_{0,0,0}) \bigg|_{x_{i,j,k}=1}
\]

Furthermore, we can calculate the intermediate creation rates the same way.

Lemma 1. Fix \( x = x_{i_0,j_0,k_0} \) for \( i_0 + j_0 + k_0 = -n + 1 \). Then for any \( (i',j',k') \) such that \( i' + j' + k' = -n' + 1 \) with \( n' < n \), we have

\[
\frac{\partial}{\partial x} (f_{i',j',k'}) \bigg|_{x_{i,j,k}=1} = E(i_0 - i', j_0 - j', k_0 - k').
\]
Proof. First, we re-center our initial conditions to make the situation clear. Introduce the variables \( \tilde{x} \) by \( x_{i,j,k} \rightarrow \tilde{x}_{i-\ell,j-\ell',k-\ell'} \). In particular, \( x_{i_0,j_0,k_0} \rightarrow \tilde{x}_{i_0-\ell,j_0-\ell',k_0-\ell'} \). Then we have \( f_{\ell',\ell',k'} \rightarrow f_{0,0,0} \), a polynomial generated from a set of standard initial conditions of order \( n - n' - 1 \). Differentiating \( f_{\ell',\ell',k'} \) with respect to \( x_{i_0,j_0,k_0} \) and setting all the variables to one is equivalent to differentiating \( \tilde{f}_{0,0,0} \) with respect to \( \tilde{x}_{i_0-\ell,j_0-\ell',k_0-\ell'} \) and setting all variables equal to one. The latter action clearly gives \( E(i_0 - \ell', j_0 - \ell', k_0 - \ell') \), and the claim is proved. \( \square \)

With this in mind, let us differentiate the weighted cube recurrence with respect to \( x_{i_0,j_0,k_0} \):

\[
f'_{i,j,k} f_{i-1,j-1,k-1} + f_{i,j,k} f'_{i-1,j-1,k-1} = \frac{1}{3} \left( f'_{i-1,j,k} f_{i,j-1,k-1} + f_{i-1,j,k} f'_{i,j-1,k-1} + f'_{i,j,k-1} f_{i-1,j-1,k} + f_{i,j,k-1} f'_{i-1,j-1,k} \right).
\]

Now by setting \( x_{i,j,k} = 1 \) for all \((i,j,k)\), we get a linear recurrence for the expectations in question (where \( r = i_0 - i, s = j_0 - j, t = k_0 - k \)):

\[
E(r, s, t) + E(r + 1, s + 1, t + 1) = \frac{1}{3} \left( E(r + 1, s, t) + E(r, s + 1, t + 1) + E(r, s + 1, t) + E(r + 1, s, t + 1) + E(r, s, t + 1) + E(r + 1, t + 1) \right).
\]

The recurrence holds for any \( r, s, t < 0 \). Let us also observe some recurrences near the boundary.

\[
E(r, 0, 0) = \frac{1}{3} E(r + 1, 0, 0), \\
E(0, s, 0) = \frac{1}{3} E(0, s + 1, 0), \\
E(0, 0, t) = \frac{1}{3} E(0, 0, t + 1), \\
E(r, s, 0) = \frac{1}{3} \left( E(r + 1, s, 0) + E(r, s + 1, 0) + E(r + 1, s + 1, 0) \right), \\
E(r, 0, t) = \frac{1}{3} \left( E(r + 1, 0, t) + E(r, 0, t + 1) + E(r + 1, 0, t + 1) \right), \\
E(0, s, t) = \frac{1}{3} \left( E(0, s + 1, t) + E(0, s, t + 1) + E(0, s + 1, t + 1) \right).
\]

After computing \( E(0,0,0) = 1 \), we can form the rational generating function in the variables \( x, y, z \):

\[
F(x, y, z) = \sum_{i,j,k \geq 0} E(-i, -j, -k) x^i y^j z^k = \frac{1}{1 + xyz - \frac{1}{3}(x + y + z + xy + xz + yz)}.
\]
Now using the fact that $p(i, j, k) = p(i, j, k + 1) + (2/3)E(i, j, k + 2)$, we can derive the formula for the generating function we want:

$$G(x, y, z) = \sum_{i, j, k \geq 0} p(-i, -j, -k)x^iy^jz^k$$

$$= \frac{2z^2F(x, y, z)}{3(1 - z)}.$$

2.3. **Asymptotic analysis.** With our generating function in hand, we can prove our main theorem. First let us embed a triangle in three-space by $T := \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \leq 0, x + y + z = -1\}$. This is the triangle that we will scale $\mathcal{I}(n)$ to fit. A point $(x, y, z) \in T$ is outside of the inscribed circle (what will show is the arctic circle) if and only if the angle between the vector $(x, y, z)$ and vector $(-1, -1, -1)$ is greater than $\cos^{-1}(\sqrt{2/3})$.

Notice that for any point $(x, y, z)$ outside of the inscribed circle, we have either $x, y > -1/2$, $x, z > -1/2$, or $y, z > -1/2$. We call any coordinate with a value strictly greater than $-1/2$ a small coordinate. We now state

**Theorem 3** (Weak Arctic Circle). Let $(x_0, y_0, z_0)$ be a point in $T$ outside of the inscribed circle for which $z_0$ is a small coordinate. Let $(i_n, j_n, k_n)$, $i_n + j_n + k_n = -n$, be a sequence of nonpositive integer triples such that

$$\lim_{n \to \infty} \frac{1}{n}(i_n, j_n, k_n) = (x_0, y_0, z_0)$$

Then $\lim_{n \to \infty} p(i_n, j_n, k_n) = 0$.

In other words, the theorem states that in the upper two regions of $T$ outside of the arctic circle, the probability of finding a horizontal edge goes to zero as the order of a (scaled) random grove goes to infinity. By symmetry, there can be no diagonal edges in the lower region, and in order to satisfy the connectivity properties of groves, all the edges in the lower region must be horizontal. The following lemma is the heart of the proof.

**Lemma 2.** Fix a point $(x_0, y_0, z_0)$ in $T$ outside of the inscribed circle for which $z_0$ is a small coordinate. Then there are real constants $A, B, C$ such that

$$p(-i, -j, -k) = O(e^{-(A+ijk+Bj+Ck)})$$

for all $i, j, k \geq 0$ and $Ax_0 + By_0 + Cz_0 < 0$.

Proving Lemma 2 consumes most of the rest of this section. Let us suppose the lemma is true and present the proof of the theorem.

**Proof of Theorem 3**. By Lemma 2 $p(i_n, j_n, k_n) = O(e^{Ain+Bjn+Ckn})$, so we will have that $p(i_n, j_n, k_n) \to 0$ if $Ai_n + Bj_n + Ck_n \to -\infty$.

Say $Ax_0 + By_0 + Cz_0 = d < 0$ and let

$$a_n = \frac{Ain}{n} + \frac{Bjn}{n} + \frac{Ckn}{n}.$$ 

Then for any $\epsilon > 0$ there is some $N > 0$ such that for all $n \geq N$, $a_n \in B(d, \epsilon) = \{x : |d - x| < \epsilon\}$. So if we take $n$ sufficiently large, $a_n < 0$, and $|d + \epsilon|n < |a_n|n$. Since $|d + \epsilon|n \to \infty$, by comparison we have

$$na_n = Ai_n + Bj_n + Ck_n \to -\infty,$$

and the theorem is proved. □
To facilitate the proof of Lemma 2, we will use the following claims.

Claim 1. Let $f(x, y, z)$ be an analytic function. Let $r, s, t$ be positive real numbers such that $f(x, y, z) \neq 0$ for $|x| \leq r$, $|y| \leq s$, $|z| \leq t$. If

$$G(x, y, z) = \frac{1}{f(x, y, z)} = \sum_{i,j,k \geq 0} a_{i,j,k} x^i y^j z^k,$$

then $a_{i,j,k} = O(r^{-i}s^{-j}t^{-k})$.

Proof of Claim 1 Define the loops $\gamma = \{|z| = t\}$, $\gamma' = \{|y| = s\}$, and $\gamma'' = \{|x| = r\}$. Then we have

$$a_{i,j,k} = \frac{1}{i!j!k!} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} \frac{\partial}{\partial z^k} G(x, y, z) \bigg|_{(0,0,0)} = \frac{1}{(2\pi)^3} \int \int \int \frac{G(x, y, z)}{x^i y^j z^k} \, dx \, dy \, dz \leq \frac{M}{(2\pi)^3} \int \int \int \frac{1}{x^i y^j z^k} \, dx \, dy \, dz \leq \frac{M}{r^i s^j t^k}.

$$

(Since $G(x, y, z)$ is bounded on the compact set $\gamma \times \gamma' \times \gamma''$.)

In other words, $a_{i,j,k} = O(r^{-i}s^{-j}t^{-k})$ and the claim is proved.

Claim 2. Let $(x_0, y_0, z_0)$ be as in Lemma 2. Without loss of generality, say $y_0$ is the second small coordinate. Then there exists a vector $(A_0, B_0, C_0) \in \mathbb{R}^3$ such that

- $A_0 x_0 + B_0 y_0 + C_0 z_0 < 0$
- $A_0 B_0 + A_0 C_0 + B_0 C_0 > 0$
- $B_0, C_0 < 0$.

Proof of Claim 2 Let $\phi$ be the angle between $(x_0, y_0, z_0)$ and $(-1, -1, -1)$. The assumption that $(x_0, y_0, z_0)$ lies outside the Arctic circle means $\phi > \sin^{-1}(\sqrt{1/3})$. Take $(A_0, B_0, C_0)$ in the plane spanned by $(-1, -1, -1)$ and $(x_0, y_0, z_0)$ and lying on the other side of $(-1, -1, -1)$ from $(x_0, y_0, z_0)$. Let $\theta$ be the angle between $(A_0, B_0, C_0)$ and $(-1, -1, -1)$, so $\theta + \phi$ is the angle between $(A_0, B_0, C_0)$ and $(x_0, y_0, z_0)$. If $\phi = \sin^{-1}(\sqrt{1/3}) + \epsilon > 0$, we can choose $\theta = \cos^{-1}(\sqrt{1/3} - \epsilon/2)$ so that $\theta + \phi = \cos^{-1}(\sqrt{1/3} + \epsilon/2) > \cos^{-1}(\sqrt{1/3} + \sin^{-1}(\sqrt{1/3} = \pi/2).

From $\theta < \cos^{-1}(\sqrt{1/3}$, we deduce

$$\frac{(A_0 + B_0 + C_0)^2}{3(A_0^2 + B_0^2 + C_0^2)} = \frac{\langle(A_0, B_0, C_0), (-1, -1, -1) \rangle^2}{\|(A_0, B_0, C_0)\|^2 \|(-1, -1, -1)\|^2} = \cos^2 \theta > \frac{1}{3},$$

so $(A_0 + B_0 + C_0)^2 > A_0^2 + B_0^2 + C_0^2$ which is equivalent to $A_0 B_0 + A_0 C_0 + B_0 C_0 > 0$.

Finally, we must show that we can choose $B_0, C_0 < 0$. Let $\alpha$ be the distance between $(A_0, B_0, C_0)$ and $1/3(-1, -1, -1)$. Let $\beta > \sqrt{3}/6$ be the distance between $(x_0, y_0, z_0)$ and $1/3(-1, -1, -1)$. We have, for $(A_0, B_0, C_0)$ satisfying the first two
conditions and lying in the plane $x + y + z = -1$, that $\alpha \beta > 1/6$. Now let $\vec{v} = (x_0, y_0, z_0) + 1/3(1, 1, 1)$ so that $|\vec{v}| = \beta$. Then choosing

$$(A_0, B_0, C_0) = -\alpha \frac{\vec{v}}{|\vec{v}|} - \frac{1}{3}(1, 1, 1) = -\frac{\alpha}{\beta} \vec{v} - \frac{1}{3}(1, 1, 1)$$

has all the desired properties. The first two are obvious by construction, and using the facts that $-\alpha < -1/(6\beta)$, $\beta > \sqrt{3}/6$ and, because $z_0$ is a small coordinate, $-z_0 < 1/2$, we can verify that

$$C_0 = -\frac{\alpha}{\beta}(z_0 + 1/3) - 1/3$$

$$= \frac{1}{36\beta^2} - 1/3$$

$$< 1/3 - 1/3 = 0$$

If $y_0$ is the other small coordinate then an identical computation shows $B_0 < 0$. □

Claim 3. Let $(A, B, C) \in \mathbb{R}^3$ with $C < 0$. Suppose that

$$g(x, y, z) = 1 + xyz - (1/3)(x + y + z + xy + xz + yz)$$

is not zero for $(x, y, z) \in [0, e^A] \times [0, e^B] \times [0, e^C]$. Then $g(x, y, z)$ is not zero for any $(x, y, z) \in \mathbb{C}^3$ with $(x, y, z) \in B(0, e^A) \times B(0, e^B) \times B(0, e^C) = \{ (x, y, z) : |x| \leq e^A, |y| \leq e^B, |z| \leq e^C \}$. 

---

**Figure 11.** The relative positions of $(x_0, y_0, z_0)$ and $(A_0, B_0, C_0)$. 

---
Proof of Claim 3. Suppose for contradiction that there is a complex zero \((x, y, z)\) of \(g\) with \(((|x|, |y|, |z|) \in [0, e^A] \times [0, e^B] \times [0, e^C], \) but no real zeros in the same region. As the zero locus of \(g\) is closed, we may assume that there is no complex zero \((x', y', z')\) with \(|x'| < |x|\), \(|y'| < |y|\) and \(|z'| < |z|\). So the power series of

\[
G(x, y, z) = \frac{1}{4(1-z)g(x, y, z)}
\]

converges on \(B(0, |x|) \times B(0, |y|) \times B(0, |z|)\) (we used \(C < 0\) to conclude that the \((1-z)\) term doesn’t vanish) and blows up to \(\infty\) as we approach \((x, y, z)\).

But the coefficients of \(G(x, y, z)\) are all positive as they are probabilities. So the series must also blow up as we approach \(((|x|, |y|, |z|) \) and thus \(g(|x|, |y|, |z|) = 0\) contradicting our assumption that there are no zeroes in \([0, e^A] \times [0, e^B] \times [0, e^C]\). □

Proof of Lemma. We now apply the claims to the edge probability generating function:

\[
G(x, y, z) = \sum_{i,j,k \geq 0} p(-i, -j, -k)x^iy^jz^k
\]

\[
= \frac{z^2}{4(1-z)(1+xyz - \frac{1}{3}(x+y+z+xy+xz+yz))}
\]

\[
= \frac{z^2}{4(1-z)g(x, y, z)}.
\]

By Claim 3 we only need to show that we can choose real numbers \(A, B, C\) so that

- \(Ax_0 + By_0 + Cz_0 < 0\)
- Both \(1-z\) and \(g(x, y, z)\) are not equal to zero for any \((x, y, z) \in \{ (x, y, z) \in \mathbb{C}^3 : |x| \leq e^A, |y| \leq e^B, |z| \leq e^C \}\).

We will now show that, for \((A_0, B_0, C_0)\) as in Claim 2 and \(t\) positive and sufficiently small, \((A, B, C) = t(A_0, B_0, C_0)\) has the desired properties. We have \(Ax_0 + By_0 + Cz_0 < 0\) and \(C < 0\) because the analogous properties hold for \((A_0, B_0, C_0)\). All that remains is to show that \(g(x, y, z)\) does not vanish for \((x, y, z) \in B(0, e^A) \times B(0, e^B) \times B(0, e^C)\). By Claim 3 it is enough to show that \(g\) has no zeroes on \([0, e^A] \times [0, e^B] \times [0, e^C]\). The identity

\[
g(x, y, z) = \frac{(1-x)(1-yz) + (1-y)(1-xz) + (1-z)(1-xy)}{3}
\]

shows that \(g(x, y, z) \neq 0\) on \([0, 1]^3\). Writing \(x = e^\alpha, y = e^\beta, z = e^\gamma\), we have \(g(x, y, z) = \alpha\beta + \alpha\gamma + \beta\gamma + O(\alpha + \beta + \gamma)^3\). So, near \((1, 1, 1)\), the zero locus of \(g\) looks like the cone \(\alpha\beta + \alpha\gamma + \beta\gamma = 0\).

Let \(L = \{ (x, y, z) : g(x, y, z) = 0 \}\) be the zero locus. We want to show that there is a number \(t > 0\) such that the point \((e^{tA_0}, e^{tB_0}, e^{tC_0}) = (e^{tA_0}, e^{tB_0}, e^{tC_0})\) is inside of \(L\). We can write

\[
g(e^{tA_0}, e^{tB_0}, e^{tC_0}) = t(A_0B_0 + A_0C_0 + B_0C_0) + t^3O(A_0 + B_0 + C_0)^3
\]

\[
= t(A_0B_0 + A_0C_0 + B_0C_0) + t^3O(-1)
\]

\[
= t(A_0B_0 + A_0C_0 + B_0C_0 + t^2O(-1)).
\]

For fixed \((A_0, B_0, C_0)\), we can certainly choose a \(t > 0\) small enough to guarantee that \(A_0B_0 + A_0C_0 + B_0C_0 + t^2O(-1) > 0\). Therefore we have \((e^A, e^B, e^C) \notin L\) and moreover it is on the ‘inside’ of \(L\) in that it is closer to the origin than the locus.
Now we’d like to say $g$ will not vanish on $[0, e^A] \times [0, e^B] \times [0, e^C]$. Since it is nonzero on $[0, 1]^3$, we just need to check that $g$ is not zero on $[1, e^A] \times [0, e^B] \times [0, e^C]$, where we can take $B, C < 0$. Let $x = x(y, z)$ be a parameterization of the zero locus. Then we have

$$x(y, z) = \frac{1/3(y + z + yz) - 1}{yz - 1/3(1 + y + z)}.$$ 

It will be enough to show that for any $(y, z) \in [0, e^B] \times [0, e^C],

e^A < x(e^B, e^C) \leq x(y, z).$

Let $(y, z)$ be any pair such that $0 \leq y \leq e^B$, $0 \leq z \leq e^C$ and $e^A < x(y, z)$. The pair $(e^B, e^C)$ is such an example. Then we will show that $x(y, z - \epsilon) > x(y, z)$ for any $0 < \epsilon \leq z$, and similarly, $x(y - \epsilon, z) > x(y, z)$ for any $0 < \epsilon \leq y$. We have

$$x(y, z - \epsilon) = \frac{1/3(y + z - \epsilon + y(z - \epsilon)) - 1}{y(z - \epsilon) - 1/3(1 + y + z - \epsilon)} = \frac{1/3(y + z + yz) - 1 - 1/3\epsilon(y + 1)}{yz - 1/3(1 + y + z) - \epsilon(y - 1/3)}.$$

Because both $y$ and $z$ are less than one, the numerator and denominator of $x(y, z)$ are both negative. It now suffices to check that $\epsilon(y - 1/3) < 1/3\epsilon(y + 1)$. This amounts to saying $y < 1$, which is true by supposition. Hence, $x(y, z - \epsilon) > x(y, z)$. By symmetry, $x(y - \epsilon, z) > x(y, z)$ as well, and the lemma is proved.

The asymptotics of multivariate generating functions is described in some generality in the sequence of papers [13, 14, 15], by Robin Pemantle and Mark Wilson. Unfortunately, their methods do not apply to our generating function. Refer to the complex 2-manifold given by $1 + xyz - 1/3(x + y + z + xy + xz + yz) = 0$ as the singular variety; what we refer to above as the zero locus. Our analysis requires studying the behavior of this surface near the point $(1, 1, 1)$. This point is a singularity of the variety that, as we have seen, resembles a cone point locally. The papers of Pemantle and Wilson give asymptotics near smooth points of the singular variety and near multiple points of the singular variety, but not near singular points of the singular variety as in our case. We hope that an extension of their techniques can be used to prove a stronger version of Theorem 8.5 In particular, we hope for a theorem that describes the asymptotic probabilities in any location (rather than just outside of the circle), similar to Theorem 1 of [2].

3. Domino tilings of Aztec diamonds

We now draw parallels between the examination of the behavior of large groves on standard initial conditions, and the behavior of tilings of large Aztec diamonds. This approach yields no new results for Aztec diamonds, but presents an alternative approach to their study. In this section we derive a generating function for the probabilities $p_n(i, j)$ that position $(i, j)$ in a tiling of an Aztec diamond of order $n$ is covered by a particular type of horizontal domino. Weak asymptotics for the function we will derive are discussed as an example in [13], and ongoing work of Henry Cohn and Pemantle.

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5The introduction to [13] refers to the case of a singular point of the singular variety as a tractable problem to be handled in future work. Personal communication from Pemantle reveals that work on the case where the singular point locally resembles a quadratic surface, e.g. as in our case of the cone, may be finished very soon. See section 3 and mention of current work of Cohn and Pemantle.
Cohn and Robin Pemantle seeks to give a full asymptotic expansion. The first derivation of the function is due to James Propp and Alexandru Ionescu, though their (different) derivation has never been published. Some recursive formulas for $p_n(i,j)$ are given in [16], and are the inspiration for our derivation of the edge probabilities for groves.

3.1. Tilings of Aztec diamonds and the octahedron recurrence. We begin by describing precisely how tilings of Aztec diamonds are encoded in the terms of polynomials generated by the octahedron recurrence. First of all, rather than considering tilings of an Aztec diamond, we prefer to consider perfect matchings of the dual graph of this region. We call such a graph an Aztec diamond graph. The Aztec diamond graph of order $n$ has as its vertices the set \{(i ± 1/2, j ± 1/2) | i, j ∈ \mathbb{Z}, |i| + |j| ≤ n − 1\}. In other words, it is the set of centers of the unit squares that compose the Aztec diamond of order $n$. Each vertex is connected with an edge to its nearest horizontal and vertical neighbors. The faces of the Aztec diamond graph are the points $(i, j)$ such that $|i| + |j| ≤ n$. For example, see Figure 12.

Recall that the octahedron recurrence is:

$$g_{i,j,n}g_{i,j,n-2} = g_{i-1,j,n-1}g_{i+1,j,n-1} + g_{i,j-1,n-1}g_{i,j+1,n-1}.$$ 

We initialize the octahedron recurrence by setting $g_{i,j,n} = x_{i,j,n}$ for $n = 0, -1$. Then

$$g_{0,0,n} = \sum_{\text{tilings } T \text{ of order } n} m(T),$$

Figure 12. Aztec diamond graphs of order $n=1$ and $n=2$. 

---

\[6\]Thorough analysis of the generating function presented in this section requires analysis of the singular variety near a point that is locally a cone.

\[7\]See Speyer’s paper, [17], for more on encoding graphs with this recurrence.
where $m(T)$ is a Laurent monomial in the variables $x_{i,j,\delta}$ where $\delta = 0$ if $i + j + n$ even, $\delta = -1$ if $i + j + n$ odd. Each monomial is of the form

$$m(T) = \prod_{|i|+|j|\leq n} x_{i,j,\delta}^{1-\deg(i,j)}$$

where $\deg(i,j) \in \{0, 1, 2\}$ is the number of edges surrounding face $(i,j)$ in the dual matching to the tiling $T$. For example, in Figure 13 we see the tiling associated with the monomial

$$\frac{x_{-1,-1,0}x_{1,-1,0}x_{1,1,0}}{x_{0,-1,x_{0,1,-1,x_{0,0,0}}}}.$$

**Figure 13.** Tiling of an Aztec diamond of order 2 and its dual matching.

The assertion that Aztec diamond tilings can be so encoded is just a special case of Theorem 5.5 of Eric Kuo’s paper on graphical condensation [12]. Here we take the weight of a tiling $T$ of order $n$ to be $m(T)$ as above.

**3.2. Domino shuffling.** Domino shuffling is the name given to the local move that can be used to generate random tilings of Aztec diamonds. In a tiling of an Aztec diamond there are four types of dominoes, which may be characterized as follows. Make a checkerboard coloring of the Aztec diamond of order $n$ by making the leftmost square in each row of the top half of the diamond be white. A horizontal domino is *north-going* if its leftmost square is white, *south-going* if its leftmost square is black. Similarly, a vertical domino is *east-going* if its topmost square is black, *west-going* if its topmost square is white. Domino shuffling is described in detail in [16], but basically each domino slides in the direction indicated by its name. North-going dominoes take one step north, south-going dominoes head south, and so on. However, two dominoes may not pass through one another. If a north- and south- or east- and west-going pair of dominoes collide, they annihilate one another, and the resulting hole and any other holes opened by sliding dominoes is filled in with a pair of dominoes: two horizontals with probability 1/2, or two verticals with probability 1/2. Using this description we have,

**Theorem 4.** The horizontal edge probabilities are given recursively by $p_n(i,j) = p_{n-1}(i,j) + \frac{1}{2}E_{n-1}(i,j)$. Thus, $p_n(i,j) = \frac{1}{2} \sum_{l=1}^{n-1} E_l(i,j)$. 

The proof of the theorem can be found in [10]. It follows more or less directly from the definition of domino shuffling, where $E_n(i, j)$ is the net creation rate (see [3, 16]).

3.3. Another generating function. By differentiating the uniformly weighted version of the octahedron recurrence

$$g_{i,j,n+1}g_{i,j,n-1} = \frac{1}{2}(g_{i-1,j,n}g_{i+1,j,n} + g_{i,j-1,n}g_{i,j+1,n}),$$

and because

$$E_n(i_0, j_0) = \frac{\partial}{\partial x}(g_{0,0,n})\bigg|_{x_{i,j}=1}$$

we obtain

$$E_{n+1}(i, j) + E_{n-1}(i, j) = \frac{1}{2} (E_n(i - 1, j) + E_n(i + 1, j)) + \frac{1}{2} (E_n(i, j - 1) + E_n(i, j + 1)).$$

From this recurrence and Theorem 4 we get the generating function:

$$G(x, y, z) = \sum_{n\geq0} \sum_{|i|+|j|\leq n} p_n(i, j)x^iy^jz^n$$

$$= \frac{z/2}{(1-yz)(1 + z^2 - \frac{1}{2}(x + x^{-1} + y + y^{-1})).}$$

This is the form of the generating function used as an example in [13]. A weak arctic circle theorem like our Theorem 3 follows directly from that example. Probabilities throughout the diamond could be extracted from this function in principle, and current work of Cohn and Pemantle seeks to carry out this more difficult analysis.

4. Biased groves, or, groves with a drift

![Figure 14. Groves with bias $(\alpha, \beta, \gamma)$.]
Another variation on the model presented in this paper would be to study “groves with a drift,” i.e., rather than having the random choice in grove shuffling be uniform, we make a biased choice. For example, say we choose the two diagonal edges with probability $\alpha$, the horizontal and one of the diagonal edges with probability $\beta$, and the horizontal and the other diagonal edge with probability $\gamma = 1 - \alpha - \beta$. This bias is reflected in the cube recurrence by setting

$$f_{i,j,k}f_{i-1,j-1,k-1} = \alpha f_{i-1,j,k}f_{i,j-1,k} + \beta f_{i,j-1,k}f_{i-1,j,k-1} + \gamma f_{i,j,k-1}f_{i-1,j-1,k}.$$  

The generating function for biased creation rates is then

$$F_{\alpha,\beta,\gamma}(x, y, z) = \sum_{i,j,k \geq 0} E_{\alpha,\beta,\gamma}(-i, -j, -k)x^i y^j z^k = \frac{1}{1 + xyz - \alpha(x + yz) - \beta(y + xz) - \gamma(z + xy)}.$$  

We can also derive the generating function for biased horizontal edge probabilities:

$$G_{\alpha,\beta,\gamma}(x, y, z) = \sum_{i,j,k \geq 0} p_{\alpha,\beta,\gamma}(-i, -j, -k)x^i y^j z^k = \frac{(\beta + \gamma)z^2 F_{\alpha,\beta,\gamma}(x, y, z)}{1 - z}.$$  

As seen in Figure 14, the arctic circle is just a special case of what one might call the “arctic ellipse,” dependent on the free parameters $\alpha, \beta$. Using the same approach as in the unbiased case where $\alpha = \beta = 1/3$, we can prove the following theorem without too much difficulty.

**Theorem 5.** The boundary of the frozen region for (rescaled) groves with a drift is given by the intersection of the plane $x + y + z = -1$ with the surface

$$rs + rt + st = \frac{r^2 + s^2 + t^2}{2},$$

where $r = (\beta + \gamma)x$, $s = (\alpha + \gamma)y$, and $t = (\alpha + \beta)z$.

Aside from simply describing the shape of the frozen region in the biased situation, it may also be useful to observe how the area of the temperate zone varies in $\alpha$ and $\beta$ (with respect to the area of the entire grove). Specifically, we can compute the ratio of the area of the temperate zone, $A_\circ$, to the area of the entire grove, $A_\nabla$, as a function of $\alpha$ and $\beta$. Given that $\gamma = 1 - \alpha - \beta$,

$$\rho(\alpha, \beta) = \frac{A_\circ}{A_\nabla} = \frac{\pi(\alpha + \beta)(\alpha + \gamma)(\beta + \gamma)}{((\alpha + \beta)(\alpha + \gamma) + (\alpha + \beta)(\beta + \gamma) + (\alpha + \gamma)(\beta + \gamma))^{3/2}}.$$ 

5. Speculation on statistics of groves

As mentioned, we hope to apply the methods of Pemantle and Wilson to determine asymptotic probabilities throughout a random grove. Based on computer experiments and the similarity of groves and Aztec diamond tilings seen so far, we believe a formula for such probabilities exists. Another future aim is to apply the methods of growth models and statistical mechanics to groves, in the style of Johansson [6], [7], or more recently Kenyon, Okounkov, and Sheffield [8], [9], [10], [11].

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8See [2] for analysis of the same variation in Aztec diamond tilings.
5.1. Randomly growing Young diagrams. Perhaps something can be proved about the variance of the boundary circle by interpreting groves in a more familiar setting. One clever way for determining the boundary of the frozen region for Aztec diamond tilings is to look at a frozen corner as a randomly growing Young diagram. See [5], [6], [7] for this interpretation. A nearly identical projection of the frozen region of a grove yields some sort of randomly growing Young diagram, but it seems to follow more intricate rules of growth than those of Aztec diamond tilings.

Figure 15. Projecting a frozen corner to a Young diagram.

If we project the grove onto the plane $\mathbb{R}^3/(0,0,1)$ we see a triangular array of boxes with two types of diagonal edges. Let us put the corner box at the upper-left and index it as $(0,0)$. Then we index the rows by $0 \leq i \leq n-1$, the columns by $0 \leq j \leq n-1$, so that each box has index $(i,j)$ with $i+j \leq n-1$. A diagonal edge from the top left corner to the bottom right corner of a box corresponds to a short edge in the grove. Edges from the top right to bottom left correspond to long edges in the grove. The box $(i,j)$ is frozen if it contains a long edge and all the boxes $(i',j')$ contain long edges, $0 \leq i' \leq i$, $0 \leq j' \leq j$. Clearly the collection of all frozen boxes is a Young diagram. We would like to be able to describe how this Young diagram grows under grove shuffling.

In a randomly growing Young diagram, we call box $(i,j)$ a growth position if boxes $(i-1,j)$ and $(i,j-1)$ are both frozen and $(i,j)$ is not frozen (we use the convention that boxes of the form $(i,-1), (-1,j)$ are always frozen). In the case of Aztec diamond tilings, the growth of a frozen corner under domino shuffling corresponds exactly to adding a new box at each growth position independently with probability 1/2. With groves this is not the case, though it may not be clear from Figure 16. In fact, two groves may have the same Young diagram projection, yet grow very differently upon shuffling. It can happen that, at a particular growth position, one grove will not permit the addition of a new box under any circumstances, whereas the other will add a box with probability 2/3.

Another difference seems to suggest the need for a new definition of growth position. It is possible that a grove can project to a Young diagram with growth position $(i,j)$, but after just one iteration of grove shuffling the projection adds a box not only to position $(i,j)$, but also position $(i+1,j)$. In fact for any $k$ there is a grove of some (perhaps large) order $n$ with growth position $(i,j)$, so that with positive
Figure 16. On the left, the Young diagram corresponding to a random grove of order 100. On the right, the Young diagram corresponding to a random tiling of an Aztec diamond of order 100.

probability its projection to a Young diagram adds boxes \((i, j), (i+1, j), \ldots, (i+k, j)\) upon one iteration of grove shuffling. Perhaps such situations are outliers, but there is still much work to do in this direction.

5.2. The nexus. Much of the statistical study of groves is motivated by analogy with statistics for domino tilings of Aztec diamonds. But there is at least one interesting feature of groves that seems to have no analogy in the realm of Aztec diamonds. We conclude the paper with some observations about a unique vertex that is present in every grove. We call this special vertex the nexus. Loosely speaking, the nexus is the vertex at the “middle” of the unique tree connected to all three sides of the initial conditions. If the nexus and its incident edges are removed from the grove, then all three sides become disconnected from one another. In Figure 17 we see the nexus highlighted.

Figure 17. The nexus of a grove.

**Definition 2.** Let \(g\) be a grove of order \(2n\) or \(2n+1\), \(n = 1, 2, \ldots\). The nexus of \(g\) is the (even) vertex \(v\) which is connected to each of the three midpoints: \((-n, -n, 0)\), \((-n, 0, -n)\), and \((0, -n, -n)\), and for which each midpoint lies on a distinct branch of the tree rooted at \(v\).

We would like to understand how the nexus moves during grove shuffling. The nexus takes some kind of random walk in the initial conditions, but it is not a simple
random walk. In some sense the nexus is a “stuttering” random walker. In some situations the nexus takes a step in one of three directions with equal probability, in others it does not move at all, and in still others it moves deterministically. Specifically, if the nexus is at an “up” vertex (see section 1.1), then after one iteration of grove shuffling it will take a step in one of three directions, each with equal probability. If the nexus is at a “down” vertex, then after one iteration of grove shuffling the nexus will not move (and so becomes a “flat” vertex in the new grove). Strangely, if the nexus is at a flat vertex then depending on which edges lead from the nexus to the midpoints, the nexus will either move deterministically one step or it will not move. Ultimately we want to say something about the distance of the nexus from the center of the grove as the size of the grove gets very large. How far from home does the nexus roam?

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