On the horizontal throughflow across a vertical porous wall

Antonio Barletta
Department of Industrial Engineering, Alma Mater Studiorum Università di Bologna,
Viale Risorgimento 2, 40136 Bologna, Italy
E-mail: antonio.barletta@unibo.it

Abstract. The stationary two-dimensional mixed convection flow in a vertical porous wall subject both to a temperature and to a pressure gap across its boundaries is studied. The instability of this flow is investigated by adopting a linear modal analysis. The two-dimensional transverse rolls are shown to be the preferred pattern at onset of instability. Transverse rolls turn out to be travelling modes. They are driven by the net vertical flow predicted by the basic solution. A numerical procedure based on the joint use of the Runge-Kutta method and on the shooting method is employed to solve the perturbation eigenvalue problem. This numerical solution yields both the neutral stability curves, defining the threshold of linear instability, and the critical Darcy-Rayleigh number yielding the onset condition of instability.

1. Introduction
When an infinite vertical porous layer saturated by a fluid is subject to a boundary temperature difference, a parallel vertical flow establishes under steady-state conditions. This flow is one of purely free convection, it is characterised by a zero net flow rate and by a linear velocity profile. Such a simple flow can be considered as a single convection cell with an infinite height developing along the porous layer. A pioneering study published by Gill [1] proved that this single-cell flow is linearly stable. This means that random perturbations of this flow having a very small amplitude are unable to change it significantly and permanently. Extensions of Gill’s work to a domain of perturbations with larger amplitude, namely to the nonlinear domain, were carried out later on. A significant finding on this issue was achieved by Straughan [2]. In fact, this author was able to prove that a nonlinear stability analysis, based on the energy method, leads to the same response declared by Gill [1]: the stationary single-cell flow in a vertical porous slab is stable.

The practical interest of Gill’s stability proof relies on the peculiar heat transfer characteristics of the single-cell flow in a vertical porous layer. In fact, as far as the vertical porous layer has an infinite height, the free convection flow does not modify the temperature profile and, hence, the heat transfer rate with respect to a case of pure heat conduction. The stability of this flow demonstrates the impossibility to enhance heat transfer beyond heat conduction. As pointed out by Gill [1], this information is definitely significant with respect to the design of thermal insulation in buildings.

A step further on this subject was taken in a recent paper [3] where the role of the momentum boundary conditions is pointed out. In particular, the author proved that the stability of the single-cell flow in a vertical porous layer strongly depends on the velocity boundary conditions. If the boundaries are assumed to be perfectly impermeable, then Gill’s proof applies and the single-cell flow is stable. If the conditions of impermeability are relaxed and replaced by conditions of open boundaries, then Gill’s proof does not hold and instability is possible. The instability
means transition from the single-cell flow regime to a multiple-cells flow pattern, occurring when the Darcy-Rayleigh number exceeds the threshold of 197.081 \[3\].

The aim of this contribution is to investigate a setup where the basic single-cell flow studied by Gill \[1\] is superposed to a prescribed horizontal throughflow across the vertical porous layer, so that the resulting stationary basic flow is two-dimensional instead of one-dimensional and parallel. This generalisation is a sensible one, having in mind applications to the building insulation industry such as the breathing walls \[4\].

2. Problem formulation

A porous vertical layer is considered. Its width is \(L\), so that the parallel vertical planes bounding the layer are at \(x = -L/2\) and \(x = L/2\) (see the sketch in Fig. 1). These boundaries are kept at different temperatures \(T_1\) and \(T_2\), respectively. On denoting by \(p\) the excess pressure over the hydrostatic pressure distribution, namely the hydrodynamic pressure, we assume a pressure gap acting across the layer and causing a forced horizontal throughflow. Hence, the hydrodynamic pressure on the boundary planes at \(x = -L/2\) and \(x = L/2\) is considered as uniform with asymmetric values \(p_1\) and \(p_2\), respectively. Hereafter, for the sake of brevity, \(p\) will be called pressure field.

The Cartesian coordinates \(x, y\) and \(z\) define the horizontal direction perpendicular to the boundaries, the vertical direction, and the horizontal direction parallel to the boundaries, respectively. The seepage velocity field \(u\) has components \((u, v, w)\) along the \((x, y, z)\)–axes, time is denoted by \(t\) and the temperature field by \(T\).

By introducing the scaling,

\[
\frac{1}{L} (x, y, z) \rightarrow (x, y, z), \quad \frac{\alpha}{\sigma L^2} t \rightarrow t, \quad \frac{K}{\mu\alpha} (p - p_0) \rightarrow p, \quad \frac{L}{\alpha} u = \frac{L}{\alpha} (u, v, w) \rightarrow (u, v, w) = u, \quad \frac{T - T_0}{T_2 - T_1} \rightarrow T,
\]

one defines the dimensionless coordinates, time, and fields, so that the local mass, momentum

\[
\text{Figure 1. A sketch of the vertical porous layer and of the boundary conditions.}
\]
and energy balance equations can be written in a dimensionless form as

\[ \nabla \cdot \mathbf{u} = 0, \quad (2a) \]

\[ \mathbf{u} = -\nabla p + R T \hat{e}_y, \quad (2b) \]

\[ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nabla^2 T. \quad (2c) \]

In Eq. (1), \( \alpha \) is the average thermal diffusivity of the saturated porous medium, \( p_0 = (p_1 + p_2)/2 \) is the reference pressure, \( T_0 = (T_1 + T_2)/2 \) is the reference temperature, \( \mu \) is the dynamic viscosity, \( K \) is the permeability, and \( \sigma \) is the heat capacity ratio of the saturated porous medium. The latter quantity is defined as the ratio between the average volumetric heat capacity of the saturated medium and the volumetric heat capacity of the fluid.

The Darcy-Rayleigh number, \( R \), and the Péclet number, \( Q \), associated with the horizontal throughflow are defined as

\[ R = \frac{g \beta (T_2 - T_1) KL}{\nu \alpha}, \quad Q = \frac{(p_1 - p_2) K}{\mu \alpha}, \quad (3) \]

where \( \nu \) is the kinematic viscosity of the fluid. Physically, it is not restrictive to assume \( R > 0 \), provided that \( Q \) can be either positive or negative. Thus, all the forthcoming results will be discussed by tacitly assuming a positive value of \( R \).

The dimensionless boundary conditions are given by

\[ x = \pm \frac{1}{2} : \quad \mathbf{p} = \mp \frac{Q}{2}, \quad T = \pm \frac{1}{2}. \quad (4) \]

### 3. Basic solution

There exists a stationary solution of Eqs. (2) and (4) expressed as

\[ u_b = Q, \quad v_b = R \frac{e^{Qx} - \cosh(Q/2)}{2 \sinh(Q/2)}, \quad w_b = 0, \]

\[ T_b = \frac{v_b}{R} = \frac{e^{Qx} - \cosh(Q/2)}{2 \sinh(Q/2)}, \quad p_b = -Qx. \quad (5) \]

One may easily check that the basic solution given by Eq. (5) coincides with that assumed in the analysis by Gill [1] when \( Q \to 0 \). In fact, Gill’s basic flow is a parallel vertical flow with a vanishing mass flow rate, such that \( u_b = w_b = 0 \) and \( v_b = Rx \). On the other hand, Eq. (5) implies a two-dimensional velocity field with a non-vanishing vertical flow rate when \( Q \neq 0 \). Indeed, the average vertical flow rate is

\[ Q_v = \int_{-1/2}^{1/2} v_b \, dx = -\frac{R}{Q} \left[ \frac{Q}{2} \coth\left(\frac{Q}{2}\right) - 1 \right]. \quad (6) \]

The right hand side of Eq. (6) is positive, meaning a bulk upward flow, when \( Q \) is negative. On the other hand, the average vertical flow rate is negative, viz. the bulk flow is downward, if \( Q \) is positive.

Figures 2 and 3 illustrate the qualitative patterns of the streamlines for \( R = 10 \) and either positive (Fig. 2) or negative (Fig. 3) Péclet numbers, \( Q \). These figures reveal that an increasing value of \( |Q| \) gradually changes the streamlines from an almost straight vertical shape to an almost straight horizontal shape. The former shape corresponds to Gill’s parallel flow, while the latter suggests a forced convection regime where buoyancy is almost negligible and the horizontal throughflow prevails. In fact, Fig. 4 reveals that things are not exactly so since, with \( Q = \pm 50 \), a very thin buoyant boundary layer develops either close to \( x = -1/2 \) (for \( Q = -50 \)) or close to \( x = 1/2 \) (for \( Q = 50 \)).
Figure 2. Streamlines of the basic flow, with $R = 10$ and positive values of $Q$

Figure 3. Streamlines of the basic flow, with $R = 10$ and negative values of $Q$

Figure 4. Plots of $v_b/R$ with different values of $Q$
There is a symmetry underlying the basic solution given by Eq. (5). In fact, the solution (5) is left invariant by the transformation,

\[ u \rightarrow -u, \quad T \rightarrow -T, \quad x \rightarrow -x, \quad Q \rightarrow -Q. \tag{7} \]

Figures 2 and 3 show that the streamlines of the basic flow conform to this symmetry.

4. Linear stability analysis

We perturb the basic state (5) with small-amplitude disturbances given by

\[ u = u_b + \varepsilon \tilde{u}, \quad p = p_b + \varepsilon \tilde{p}, \quad T = T_b + \varepsilon \tilde{T}. \tag{8} \]

Here, the perturbation parameter \( \varepsilon \) is such that \( \varepsilon > 0 \) and \( \varepsilon \ll 1 \), while \( \tilde{u} = (\tilde{u}, \tilde{v}, \tilde{w}), \tilde{p} \) and \( \tilde{T} \) are the perturbation fields. Substitution of Eq. (8) into Eqs. (2) and (4), by taking into account Eq. (5) yields

\[ \nabla \cdot \tilde{u} = 0, \quad \tilde{u} = -\nabla \tilde{p} + R \tilde{T} \hat{e}_y, \tag{9a} \]

\[ \frac{\partial \tilde{T}}{\partial t} + Q \frac{\partial \tilde{T}}{\partial x} + v_b(x) \frac{\partial \tilde{T}}{\partial y} + T_b'(x) \tilde{u} = \nabla^2 \tilde{T}, \tag{9b} \]

\[ x = \pm \frac{1}{2}: \quad \tilde{p} = 0, \quad \tilde{T} = 0. \tag{9c} \]

where prime means derivative with respect to \( x \). We may adopt a pressure-temperature formulation, by evaluating the divergence of Eq. (9b) and by taking into account Eq. (9a). Thus, we obtain

\[ \nabla^2 \tilde{p} = R \frac{\partial \tilde{T}}{\partial y}, \tag{10a} \]

\[ \frac{\partial \tilde{T}}{\partial t} + Q \frac{\partial \tilde{T}}{\partial x} + R T_b(x) \frac{\partial \tilde{T}}{\partial y} - T_b'(x) \frac{\partial \tilde{p}}{\partial x} = \nabla^2 \tilde{T}, \tag{10b} \]

\[ x = \pm \frac{1}{2}: \quad \tilde{p} = 0, \quad \tilde{T} = 0. \tag{10c} \]

By employing Eq. (5), one may show that Eqs. (10) are invariant under the transformation,

\[ x \rightarrow -x, \quad y \rightarrow -y, \quad Q \rightarrow -Q, \quad \tilde{p} \rightarrow -\tilde{p}. \tag{11} \]

A stability analysis versus normal modes is carried out by writing

\[ \tilde{p} = f(x) e^{i(k_y y + k_z z)} e^{\lambda t}, \quad \tilde{T} = h(x) e^{i(k_y y + k_z z)} e^{\lambda t}, \tag{12} \]

where \( (k_y, k_z) \) is the wave vector, and \( k = (k_y^2 + k_z^2)^{1/2} \) is the wave number. While \( k_y \) and \( k_z \) are real, \( \lambda \) is a complex parameter whose real part, \( \Re(\lambda) \), expresses the time-growth rate of the disturbance. On the other hand, the angular frequency is given by \( \omega = -\Im(\lambda) \), namely by the negative imaginary part of \( \lambda \). On account of Eq. (12), Eqs. (10) give rise to an ordinary differential eigenvalue problem

\[ f'' - k^2 f - ik_y R h = 0, \tag{13a} \]

\[ h'' - Q h' - [k^2 + \lambda + ik_y R T_b(x)] h + T_b'(x) f' = 0, \tag{13b} \]

\[ x = \pm \frac{1}{2}: \quad f = 0, \quad h = 0. \tag{13c} \]
We note that the eigenvalue problem (13) is endowed with a symmetry. In fact, on account of Eq. (5), Eqs. (13) are left invariant by the transformation,

\[ x \rightarrow -x, \quad Q \rightarrow -Q, \quad k_y \rightarrow -k_y, \quad f \rightarrow -f. \quad (14) \]

The change \( k_y \rightarrow -k_y \) is equivalent to \( \omega \rightarrow -\omega \) as its effect is just reversing the direction of the wave vector. Therefore, we can reformulate the symmetry of the eigenvalue problem (13) as an invariance under the transformation,

\[ x \rightarrow -x, \quad Q \rightarrow -Q, \quad \omega \rightarrow -\omega, \quad f \rightarrow -f. \quad (15) \]

Longitudinal rolls are normal modes with \( k_y = 0 \), oblique rolls are such that \( 0 < k_y < k \), while transverse rolls are defined by \( k_y = k \). It is immediately verified that oblique or longitudinal rolls may be mapped onto equivalent transverse rolls by a suitable rescaling of the Rayleigh number,

\[ S = \frac{k_y}{k} R, \quad (16) \]

so that Eq. (13) can be rewritten as

\[ f'' - k^2 f - i k S h = 0, \quad (17a) \]

\[ h'' - Q h' - \left[ k^2 + \lambda + ik ST_b(x) \right] h + T_b'(x)f' = 0, \quad (17b) \]

\[ x = \pm \frac{1}{2} : \quad f = 0, \quad h = 0. \quad (17c) \]

The Squire-like transformation given by Eq. (16) was previously adopted by Rees [5]. An immediate consequence of this transformation is that longitudinal rolls \((k_y = 0)\) are equivalent to transverse rolls with a zero Darcy-Rayleigh number. This means that no convective instability can be induced by longitudinal rolls, since no buoyancy force exists when the Darcy-Rayleigh number is zero. Moreover, oblique rolls are equivalent to transverse rolls having a smaller Darcy-Rayleigh number. This implies that transverse rolls are selected at onset of convective instability, being the modes which magnify the action of the buoyancy force.

5. Numerical solution

Equations (17) are homogeneous. Hence, they obviously admit the trivial solution \( f = h = 0 \). Besides this solution, nontrivial solutions are allowed for special choices of the parameters \((k, \lambda, S, Q)\). In other words, Eqs. (17) define an eigenvalue problem. Its solutions can be found numerically by the combined use of an initial value solver for ordinary differential equations, based on an explicit Runge-Kutta method, and the shooting method. The steps to be made are well-known in the literature (see, for instance, § 19.1 of [6]). A way to formulate them is the following:

(i) Complete Eqs. (17) so that the boundary value problem is transformed into an initial value problem. In other words, we replace Eq. (17c) with

\[ f(-1/2) = 0, \quad f'(-1/2) = \eta_1 + i\eta_2, \quad h(-1/2) = 0, \quad h'(-1/2) = 1, \quad (18) \]

where \( h'(-1/2) = 1 \) is a scale-fixing condition intended to break the scale-invariance of any solution of Eqs. (17), and \((\eta_1, \eta_2)\) are unknown real constants to be determined by the shooting method.

(ii) Solve the initial value problem given by Eqs. (17a), (17b) and (18) for prescribed values of \((k, S, Q)\), and guessed values of \((\lambda, \eta_1, \eta_2)\).

(iii) Determine the actual values of the four real parameters \( \Re(\lambda), \Im(\lambda) = -\omega, \eta_1 \) and \( \eta_2 \) by employing the shooting method to satisfy the target conditions, \( f(1/2) = 0 \) and \( h(1/2) = 0 \), implied by Eq. (17c). As \( f \) and \( h \) are complex-valued functions, these target conditions are, in fact, four real equations to be satisfied by the four unknowns \( \Re(\lambda), \Im(\lambda) = -\omega, \eta_1 \) and \( \eta_2 \).
This method can be efficiently implemented within the computational software program Mathematica 10 (© Wolfram Research), by using the built-in functions NDSolve and FindRoot. The main aim is the determination of the neutral stability curves in the parametric plane \((k, S)\), for different prescribed values of \(Q\). This task is accomplished by setting \(\Re(\lambda) = 0\), viz. by determining the locus of vanishing growth rate, and thus the curve dividing stable \((\Re(\lambda) < 0)\) from unstable \((\Re(\lambda) > 0)\) conditions.

6. Discussion of the results

The neutral stability curves are plotted, for different positive values of \(Q\) in Fig. 5. The essential meaning is that unstable conditions, for a given \(Q\), lie in the region above the corresponding neutral stability curve. The region below the curve is that defining the parametric conditions of linear stability. As a consequence, the absolute minimum of the neutral stability curve yields the so-called critical values of \(k\), \(S\) and \(\omega\). These values are denoted as \(k_c\), \(S_c\) and \(\omega_c\).

The value of \(S_c\) is specially important. In fact, convective instability takes place whenever \(S > S_c\). Thus, Fig. 5 discloses an important feature: the basic flow is stabilised by increasing the value of \(Q\). In other words, the basic horizontal throughflow has a stabilising effect. A similar feature was pointed out, in the case of a horizontal porous layer with a vertical throughflow, by Homsy and Sherwood [7], and by Jones and Persichetti [8].

![Figure 5. Plots of the neutral stability curves in the \((k, S)\)-plane with \(Q = 0\) (dashed line) and \(Q > 0\) (solid lines).](image)

Numerical values of \(k_c\), \(S_c\) and \(\omega_c\) versus \(Q\) are reported in Table 1. The data reported in this table corroborate the behaviour revealed by Fig. 5, namely that \(S_c\) monotonically increases with \(Q\). There is an additional information concerning the values of \(\omega_c\). The neutral stability condition with \(Q = 0\) is marked by \(\omega = 0\), meaning that instability is driven by non-travelling modes [3]. This feature is altered by the presence of a horizontal throughflow. In fact, with \(Q > 0\), Table 1 reports negative values of \(\omega_c\). Having in mind transverse rolls, this means a negative vertical phase velocity, \(\omega_c/k_c < 0\), so that unstable transverse rolls travel in the downward direction. Physically, this behaviour is reasonable since Eq. (6) predicts a net basic vertical flow velocity \(Q_v\) whenever \(Q > 0\). Thus, it is the net basic downflow which determines the direction of the travelling perturbation modes. With reference to transverse rolls, a comparison between the plots of \([Q_v]\), evaluated with \(R = R_c\), and \([\omega_c/k_c]\) versus \(Q\) is displayed in Fig. 6. This figure reveals that the values of the phase velocity are slightly larger than those of the basic vertical flow velocity, so that unstable transverse rolls travel faster than the basic net flow rate in the vertical direction. One must be careful on converting this information from dimensionless velocities to dimensional velocities. In fact, in Eq. (1), the scale adopted for velocity is not consistent with the ratio between the scales of length and time.
Table 1. Critical values of $k_c$, $S_c$ and $\omega_c$ for different values of $Q$.

| $Q$ | $k_c$    | $S_c$    | $\omega_c$ |
|-----|----------|----------|------------|
| 0   | 1.059498 | 197.0812 | 0.000000   |
| 0.1 | 1.059540 | 197.1448 | -2.241053  |
| 0.5 | 1.060546 | 198.6713 | -11.24815  |
| 1   | 1.063750 | 203.4443 | -22.76430  |
| 2   | 1.077505 | 222.5464 | -47.67209  |
| 3   | 1.103842 | 254.1847 | -76.86275  |
| 4   | 1.147907 | 297.3569 | -112.4635  |
| 5   | 1.216155 | 349.7842 | -156.5670  |
| 6   | 1.314163 | 408.1566 | -211.1184  |
| 7   | 1.442516 | 469.2641 | -277.4873  |
| 8   | 1.593426 | 531.2067 | -355.7898  |
| 9   | 1.755564 | 593.5152 | -445.1874  |
| 10  | 1.922379 | 656.2993 | -545.1334  |
| 11  | 2.092200 | 719.6239 | -655.6562  |
| 12  | 2.264884 | 783.4235 | -776.9483  |
| 13  | 2.440254 | 847.5836 | -909.1300  |
| 14  | 2.617907 | 911.9994 | -1052.227  |
| 15  | 2.797377 | 976.5929 | -1206.219  |

Figure 6. Transverse rolls: plots of $|Q_v|$ with $R = R_c$ (dashed line) and of the modulus of the phase velocity $|\omega_c/k_c|$ (solid line).

Thus, our conclusion that the phase velocity is larger than the basic vertical flow velocity holds in dimensional terms if $\sigma \leq 1$, while it may hold or not if $\sigma > 1$.

Having established the neutral stability condition when $Q > 0$, one may question on the conditions for the onset of convective instability when $Q < 0$. The answer readily comes from the symmetry (15) governing the stability eigenvalue problem. When $Q < 0$, the neutral stability curves coincide with the corresponding curves obtained for $Q > 0$. This means that $S_c$ does not change if $Q \to -Q$. Besides that, Eq. (15) implies that the angular frequency $\omega$ change its sign at neutral stability, when $Q \to -Q$. Thus, the same values of $|\omega_c|$ reported in Table 1 are retrieved by changing the sign of $Q$. One may conclude that, when $Q < 0$, unstable modes travel in the upward direction, since $\omega > 0$.

Figure 7 is relative to transverse rolls ($k_y = k$). This figure displays the streamlines and the
isotherms of transverse rolls at onset of convection under critical conditions \((k = k_c, R = R_c)\), with either \(Q = 0, Q = 1\) or \(Q = 5\). Displaying explicitly the orientation of the streamlines is unnecessary as corotating and counterrotating cells are alternately piled up within the vertical layer, as one may devise from Eq. (12). The comparison between the three cases is definitely interesting as the perturbation patterns gradually change from perfectly symmetric when \(Q = 0\), to slightly asymmetric when \(Q = 1\) and, eventually, to markedly asymmetric when \(Q = 5\).

![Figure 7](image-url)

**Figure 7.** Transverse rolls: plots of the streamlines (left) and isotherms (right) at a fixed time and critical conditions \((k = k_c, R = R_c)\) for \(Q = 0\), \(Q = 1\) and \(Q = 5\).

Trying to figure out the corresponding perturbation patterns when \(Q = -1\) and \(Q = -5\) is relatively easy if one applies the transformation (11). This means that reflections of the horizontal \(x\)-axis and of the vertical \(y\)-axis are needed, so that a gradual compression of the isotherms to \(x = -1/2\), instead of \(x = 1/2\), occurs as \(|Q|\) increases.

The behaviour of the isotherms of the secondary flow, superposed to the basic single-cell flow, suggests that the Dirichlet boundary conditions for the temperature may become harder and harder to be sustained when \(|Q|\) increases. In fact, the strong temperature gradient at the left/right boundary of the porous layer might induce deviations from the isothermal boundary conditions assumed in our model. To address this issue, it seems reasonable a future improvement of the model employed for the temperature boundary conditions. One can assume Robin temperature conditions instead of the Dirichlet boundary conditions, and introduce suitable Biot numbers to modulate the heat transfer rate to the external fluid environments.

7. Conclusions

The effects of an imposed horizontal pressure gradient on the buoyant flow in a plane vertical porous layer have been studied. The plane boundaries of the layer have been modelled as isobaric and isothermal, with both a pressure gap and a temperature gap imposed across the layer. The basic stationary mixed convection flow is two-dimensional, consisting of a parallel vertical buoyant flow superposed to a uniform horizontal throughflow. The governing parameters are the Darcy-Rayleigh number, \(R\), and the Péclet number, \(Q\), associated with the horizontal throughflow. Without any loss of generality, the study has been carried out by keeping \(R > 0\), and by allowing \(Q\) to be either positive or negative. It has been shown that, when \(Q > 0\), a net vertical downflow occurs with non-dimensional average velocity \(Q_v < 0\). On the other hand, when \(Q < 0\), a net vertical upflow occurs with non-dimensional average velocity \(Q_v > 0\).

Suitable symmetries of the governing equations have been established allowing one to gather information on the regime \(Q < 0\) from the results obtained when \(Q > 0\).

A modal linear stability analysis has been performed to determine the neutral stability condition for the onset of convective instability. A Squire-like transformation has been defined
in order to map the analysis of any normal mode onto that of two-dimensional transverse rolls. According to this transformation, three-dimensional normal modes are always more stable than transverse rolls. Thus, convective instability is triggered by two-dimensional perturbation patterns superposed to the two-dimensional basic flow. Neutral stability curves for some values of $Q$ have been drawn, showing that the effect of an increasing value of $|Q|$ is stabilising. Transverse rolls turned out to be travelling in the same direction of the net vertical downflow/upflow, having modulus $|Q_v|$, associated with the basic solution. However, the dimensionless phase velocity of the transverse rolls has a modulus slightly larger than $|Q_v|$. The presented stability analysis is pivotal to a fully nonlinear analysis of the supercritical regime which is the natural future development of this research. In particular, expected results from the nonlinear study regard the stationary or non-stationary flow patterns spawned when $R > R_c$, as well as the evaluation of the heat transfer rate across the wall. The latter can be a crucial information for an efficient design of building elements such as the porous dynamic insulation media, well-known as breathing walls.

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