Erratum: Thermal decoupling of WIMPs from first principles

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Here we address three issues that we recently identified in our original article [1]. The first concerns the derivation of the Liouville operator and is purely conceptual; no result or equation derived in our paper is affected. The second concerns an overall factor of 2 in the definition of the collision term; including it slightly increases the kinetic decoupling temperature, in typical cases by a factor of $2^{1/4} \approx 1.19$. The third point in the list is not an actual correction but a brief update about more recent work that generalized our main results by lifting some of the specific assumptions that we made in the interest of a clear and pedagogic derivation.

1. Contrary to the impression given in appendix A, we are throughout using physical rather than co-moving momenta. The text after eq. (A.1), and until the end of that paragraph (“... drop the bars over the $p$”), should thus for consistency be replaced by the following:

Here, the Liouville operator $\hat{L}$ is the covariant generalization of the convective derivative familiar from hydrodynamics, or — in more technical terms — the variation with respect to an affine parameter $\lambda$ along a geodesic:

$$\hat{L}[f] = \frac{df}{d\lambda} = \frac{dx^i}{d\lambda} \frac{\partial f}{\partial x^i} + \frac{dp^i}{d\lambda} \frac{\partial f}{\partial p^i} = p^i \frac{\partial f}{\partial x^i} - \Gamma^i_{\rho\sigma} p^\rho p^\sigma \frac{\partial f}{\partial p^i},$$ (1)
where in the last step we have chosen $\lambda = \tau$ (i.e. the eigentime). Note that the sum is only over spatial momenta $p^i$ since we consider on-shell particles and hence $p^0$ is not an independent variable of $f$; likewise, we only sum over spatial coordinates $x^i$ because we consider a Hamiltonian system where $f$ cannot have an explicit time-dependence.

In a (flat) Friedmann-Robertson-Walker (FRW) spacetime,

$$ ds^2 = dt^2 - a^2(t)dx^2, \quad (2) $$

the homogeneity of space implies $f = f(|p|, t)$ and eq. (1) becomes

$$ \hat{L}[f] = -2p^0Hp \cdot \nabla p f(p)|_{p=p(t)} \quad (3) $$

$$ = p^0(\partial_t - H p \cdot \nabla p) f(t, p)|_{p=p(t)}, \quad (4) $$

where we have introduced $H \equiv \dot{a}/a$. In the second step, we have adopted the standard convention of treating $f$ explicitly as a function of two $(t, p)$ rather than only one $(p)$ variable, using that to leading order the time-dependence of $p$ derives exclusively from the scalefactor $a$, i.e. $p(t) = \bar{p}(a(t))$. Writing $f$ as a function of the co-moving momenta $\bar{p}$ instead, $f = f(t, \bar{p})$, the Liouville operator would simply become $\tilde{L} = p^0 \partial_t$.

2. For the choice of $\lambda = \tau$ adopted for the Liouville operator, the correctly normalized collision term reads [2, 3]

$$ C = \frac{1}{2g_\chi} \int \frac{d^3k}{(2\pi)^32\omega} \int \frac{d^3\bar{k}}{(2\pi)^32\omega} \int \frac{d^3\bar{p}}{(2\pi)^32E} (2\pi)^4 \delta^{(4)}(\bar{p} + \bar{k} - p - k)|M|^2 J, \quad (5) $$

where $M$ is the scattering amplitude, summed over final and initial spin states, and

$$ J \equiv [(1 \mp g^\pm)(\omega) g^\pm(\bar{\omega}) f(\bar{p}) - (1 \mp g^\omega)(\bar{\omega}) g^\pm(\omega) f(p)]. \quad (6) $$

This is a factor of 2 smaller than the collision term stated in appendix B of [1] (note that we are now adopting a slightly different convention for the spin sum). The need for this additional factor is most easily seen by integrating the Boltzmann equation with the homogeneity of space implies $f = f(|p|, t)$ and eq. (1) becomes

$$ \hat{L}[f] = -2p^0Hp \cdot \nabla p f(p)|_{p=p(t)} \quad (3) $$

$$ = p^0(\partial_t - H p \cdot \nabla p) f(t, p)|_{p=p(t)}, \quad (4) $$

where we have introduced $H \equiv \dot{a}/a$. In the second step, we have adopted the standard convention of treating $f$ explicitly as a function of two $(t, p)$ rather than only one $(p)$ variable, using that to leading order the time-dependence of $p$ derives exclusively from the scalefactor $a$, i.e. $p(t) = \bar{p}(a(t))$. Writing $f$ as a function of the co-moving momenta $\bar{p}$ instead, $f = f(t, \bar{p})$, the Liouville operator would simply become $\tilde{L} = p^0 \partial_t$.

$$ C(T) = \frac{1}{96\pi^3TM_\chi^2} \int d\omega k^4g^\pm(\omega) \left(1 \mp g^\pm\right) |M|^2_{s=2} \sum_{s=M_\chi^2+2M_\omega+M_3M} \frac{J(\omega)}{J(\omega) + J(-\omega)} f(p) \quad (7) $$

$$ \equiv 2M_\chi\gamma(T) \left[M_\chi T \Delta_p + p \cdot \nabla p + 3\right] f(p). \quad (8) $$

3. Subsequently to the original publication [1], the limitation to relativistic scattering partners was overcome [8]. Including the above-mentioned factor of 1/2, the collision term stated in eq. (B.20) then becomes (initial and final spin states are again being summed over)

$$ C(T) = \frac{1}{96\pi^3TM_\chi^2} \int d\omega k^4g^\pm(\omega) \left(1 \mp g^\pm\right) |M|^2_{s=2} \sum_{s=M_\chi^2+2M_\omega+M_3M} \frac{J(\omega)}{J(\omega) + J(-\omega)} f(p) \quad (7) $$

$$ \equiv 2M_\chi\gamma(T) \left[M_\chi T \Delta_p + p \cdot \nabla p + 3\right] f(p). \quad (8) $$

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The above expression is only valid, to lowest order in the expansion parameter $T^3/M\chi$, if $|M|^2$ is Taylor expandable around $t = 0$ (in the sense that $|M|^2 - |M|_{t=0}^2 \lesssim T^3/M^2 |M|_{t=0}^2$). If this assumption is relaxed, the collision term still takes the above form, but with the replacement [10, 11]

$$|M|_{t=0}^2 \rightarrow \frac{1}{8k^4} \int_{-4k^2}^0 |M|^2 (-t) dt,$$

which describes an effective average over $t$.

Finally, a simplification in the original treatment [1] was to assume a constant effective number of heat bath degrees of freedom, which was later generalized [8, 9]. Assuming no non-standard entropy production, in particular, it is advantageous to consider the dimensionless parameters $x \equiv M\chi/T$ and $y \equiv M\chi T s^{-2/3}$, because the main process equation — eqs. (3) and (A.12) in [1] — then takes a particularly simple form [9]:

$$\frac{1}{y} \frac{dy}{dx} = - \left( 1 - \frac{x}{3 g_*} \frac{dg_*}{dx} \right) \frac{4M^2\gamma(T)}{Hx} \left( 1 - \frac{y_{eq}}{y} \right),$$

where $g_*$ is the effective number of entropy degrees of freedom of the heat bath. This equation, with the collision term stated above, is valid in full generality (as long as the co-moving DM number density does not change during or after kinetic decoupling, like for example in the presence of the Sommerfeld effect [9]). The kinetic decoupling temperature, in analogy with eq. (5), is then obtained as

$$x_{kd} = \left. \frac{M\chi}{T_{kd}} \right|_{x \rightarrow \infty} = \frac{s^{2/3}}{T^{2/3}} \bigg|_{T=T_{kd}}.$$ 

Note that $y(x)$ typically converges very quickly to a constant value for $x > x_{kd}$, such that this definition is both unique and rather independent of the assumed cosmological evolution (in contrast to what is indicated in [12]), cf. figure 1 in [8].

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References

[1] T. Bringmann and S. Hofmann, Thermal decoupling of WIMPs from first principles, JCAP 04 (2007) 016 [hep-ph/0612238] [inSPIRE].
[2] E.W. Kolb and M.S. Turner, The early universe, Addison Wesley, Boston U.S.A. (1990).
[3] P. Gondolo and G. Gelmini, Cosmic abundances of stable particles: Improved analysis, Nucl. Phys. B 360 (1991) 145 [inSPIRE].
[4] M.M. Müller, Comparison of Boltzmann Kinetics with Quantum Dynamics for Relativistic Quantum Fields, Ph.D. Thesis, Technical University of Munich (2006).
[5] J. Berges and M.M. Muller, Nonequilibrium quantum fields with large fluctuations, hep-ph/0209026 [inSPIRE].
[6] J. Berges and S. Borsányi, Range of validity of transport equations, *Phys. Rev. D* 74 (2006) 045022 [hep-ph/0512155] [inSPIRE].

[7] M. Garny, A. Hohenegger, A. Kartavtsev and M. Lindner, Systematic approach to leptogenesis in nonequilibrium QFT: Self-energy contribution to the CP-violating parameter, *Phys. Rev. D* 81 (2010) 085027 [arXiv:0911.4122] [inSPIRE].

[8] T. Bringmann, Particle Models and the Small-Scale Structure of Dark Matter, *New J. Phys.* 11 (2009) 105027 [arXiv:0903.0189] [inSPIRE].

[9] L.G. van den Aarssen, T. Bringmann and Y.C. Goedecke, Thermal decoupling and the smallest subhalo mass in dark matter models with Sommerfeld-enhanced annihilation rates, *Phys. Rev. D* 85 (2012) 123512 [arXiv:1202.5456] [inSPIRE].

[10] P. Gondolo, J. Hisano and K. Kadota, The Effect of quark interactions on dark matter kinetic decoupling and the mass of the smallest dark halos, *Phys. Rev. D* 86 (2012) 083523 [arXiv:1205.1914] [inSPIRE].

[11] J. Kasahara, Neutralino Dark Matter: The Mass of the smallest Halo and the golden Region, Ph.D. Thesis, University of Utah (2009).

[12] L. Visinelli and P. Gondolo, Kinetic decoupling of WIMPs: analytic expressions, *Phys. Rev. D* 91 (2015) 083526 [arXiv:1501.02233] [inSPIRE].