Macroscopic Quantum Fluctuations in the Josephson Dynamics of Two Weakly Linked Bose-Einstein Condensates

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Abstract

We study the quantum corrections to the Gross-Pitaevskii equation for two weakly linked Bose-Einstein condensates. The goals are: 1) to investigate dynamical regimes at the borderline between the classical and quantum behaviour of the bosonic field; 2) to search for new macroscopic quantum coherence phenomena not observable with other superfluid/superconducting systems. Quantum fluctuations renormalize the classical Josephson oscillation frequencies. Large amplitude phase oscillations are modulated, exhibiting collapses and revivals. We describe a new inter-well oscillation mode, with a vanishing (ensemble averaged) mean value of the observables, but with oscillating mean square fluctuations. Increasing the number of condensate atoms, we recover the classical Gross-Pitaevskii (Josephson) dynamics, without invoking the symmetry-breaking of the Gauge invariance.

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The experimental observation of the Bose-Einstein condensation of a trapped, dilute gas of alkali atoms [1], and the high accuracy of the engineering [2,3], are opening a new avenue to investigate the interplay between macroscopics and quantum coherence. Foundational problems of quantum theory [4,5] and condensed matter [4,6] can be addressed through real (and not just “gedanken”) experiments; dynamical regimes not accessible with other superconducting/superfluid systems might be testable.

The main goal of this work is to study the quantum corrections to the classical Gross-Pitaevskii dynamics [7] of two weakly linked Bose-Einstein condensates (BEC’s) forming a Josephson junction.

The Gross-Pitaevskii equation (GPE) has been shown to describe quite accurately the dynamical regimes experimentally investigated so far [8]. On the other hand, BEC’s can be experimentally created with a number of atoms ranging from few thousand to several millions, and in a wide variety of confining geometries [1–3]. This is opening the important possibility of studying dynamical regimes at the borderline between the quantum and classical nature of the bosonic field, and, more generally, to search for new macroscopic quantum phenomena not observables with other superfluid/superconducting systems.

Different Josephson dynamical regimes are characterized by the ratio of the “Josephson coupling energy” $E_J$ and the “on-site energy” $E_s$ [9–12]. In the limit $E_J \gg E_s$ (often referred in the literature as “classical” [10,11]), both the phase difference and the relative number of condensate atoms are well defined. Nevertheless, we will see that quantum corrections can significantly modify the classical dynamics even for $E_J/E_s \sim 10^2$, a regime accessible with current BEC technology. Quantum fluctuations renormalize the classical Josephson oscillation frequencies. Large amplitude phase oscillations are modulated by partial collapses and revivals. It is well known that the relative phase of two weakly linked systems diffuses [4,13–15] and subsequently revives [13] after the suppression of the Josephson coupling. In effect, even in two coupled condensates, the relative phase diffuses in the self-trapped, running-phase [16,17] regime. Then, partial or complete revivals could occur, due to the finite number of condensate atoms. An asymmetric potential can also induce phase diffusion
In the limit of large condensates, the dynamical equations for the mean values of the physical observables decouple from the equations governing the respective quantum fluctuations, with a smooth crossover from the quantum to the classical GPE regime.

The classical boson Josephson junction (BJJ) equations, derived by the GPE in the “two-mode” approximation \cite{16-20}, can be cast in terms of two canonically conjugate variables: the relative population $N$ and phase $\phi$ between the two traps. Quantizing BJJ, the c-numbers $N$ and $\phi$ are replaced by the corresponding operators, satisfying the commutation relation $[\hat{\phi}, \hat{N}] = i$ \cite{12,21}. Then the Hamiltonian of two weakly coupled condensates reads \cite{22}:

$$\hat{H} = \frac{E_s}{2} \hat{N}^2 - E_J \cos \hat{\phi} + \Delta E_0 \hat{N}$$

(1)

where $E_J (\sim N_T^{\alpha}; \alpha \sim 1)$ is the “Josephson coupling energy”; $E_s (\sim N_T^{-\beta}$, with $\beta = 3/5$ in 3$d$ traps) is the “on-site energy”, the analog of the charging energy in a superconducting Josephson Junction (SJJ); $N_T$ is the total number of condensate atoms. $\Delta E_0$ is the zero-point energy difference in two asymmetric traps \cite{10}, or an applied chemical potential difference (induced, for example, by the gravitational potential in vertical traps \cite{3}). The coefficients $E_J, E_s$ are determined by the BJJ geometry and the total number of atoms. They can be (consistently) calculated as overlap integrals of two orthogonal one-body Gross-Pitaevskii wave functions \cite{10}.

In the phase representation, the operators in Eq. (1) are expressed as $\hat{N} = -i \frac{\partial}{\partial \phi}; \hat{\phi} = \phi$. Then the dynamical equation for the amplitude $\Psi(\phi, t)$ is ($\hbar = 1$):

$$i \frac{\partial \Psi(\phi, t)}{\partial t} = - \frac{\partial^2 \Psi(\phi, t)}{\partial \phi^2} - \Gamma \cos \phi \Psi(\phi, t) - i E_0 \frac{\partial \Psi(\phi, t)}{\partial \phi},$$

(2)

where $\Gamma = \frac{2 E_J}{E_s}$, $E_0 = \frac{2 \Delta E_0}{E_s}$ and the time has been rescaled as $\frac{E_s}{2} t \to t$. Since we are considering an isolated, energy conserving system, the “potential” is periodic and defined on a 2\pi-ring, with the wave-function boundary conditions $\Psi(\phi) = \Psi(\phi + 2\pi)$.

In the context of SJJ’s, (where only stationary regimes are experimentally accessible), Eq. (2) is the drosophila of low-capacitance systems, \cite{14,11}. The effect of dissipation on
their quantum statistical properties (like phase transitions from normal to superconducting phases) has been extensively studied [23]. Other typical effects include the renormalization of the critical Josephson current [24], and the macroscopic quantum tunneling among metastable minima of the "washboard" potential [9, 11].

In an BJJ, on the other hand, dynamical density oscillations can be studied by shifting the position of the laser barrier or tailoring the traps [2, 3]. (A similar argument holds when considering Raman transitions between two condensate in different hyperfine levels of a single traps). The small frequency, < kHz, oscillations of the population imbalance, and the corresponding mean square deviations, might be directly monitored by destructive or non-destructive techniques. It is worth stressing, then, that the set of experimentally accessible observables in BJJ is rather different from the SJJ one. New collective quantum phenomena, not accessible with other superconducting/superfluid systems, might be observed in BEC’s.

In this Letter we study analytically the quantum corrections to the classical Gross-Pitaevskii-Josephson equations, providing a simple framework to study quantitatively a mesoscopic BJJ [25].

We consider a time-dependent variational approach. The time evolution of the variational parameters is characterized by the minimization of an action with the effective Lagrangian:

\[
L(q_i, \dot{q}_i) = i\langle \dot{\Psi} \dot{\Psi} \rangle - \langle \dot{\Psi} \hat{H} \dot{\Psi} \rangle
\]

with \( \hat{H} = -\frac{\partial^2}{\partial \phi^2} - \Gamma \cos \phi - iE_0 \frac{\partial}{\partial \phi} \) and \( \Psi(\phi, q_i(t)) \), \( q_i \) being the time-dependent parameters. This provides the familiar Lagrange equations:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}
\]

We choose the time-dependent variational wave function as:

\[
\Psi(\phi, t) = R[\phi - \phi_c(t), \lambda(t)] e^{iS[\phi - \phi_c(t), \lambda(t), p(t), \delta(t)]}
\]

with the square root of the probability density \( R \) and the dynamical phase \( S \) being real, and:
\[ S = p(t) [\phi - \phi_c(t)] + \frac{\delta(t)}{2} [\phi - \phi_c(t)]^2. \] (6)

The pairs of time-dependent parameters \( \phi_c(t), p(t) \) and \( \lambda(t), \delta(t) \) are canonically conjugate:

\[
\begin{align*}
\dot{\phi}_c &= \frac{\partial H}{\partial p} = 2p + E_0 \quad \text{(7a)} \\
\dot{p} &= -\frac{\partial H}{\partial \phi_c} = -\frac{\partial}{\partial \phi_c} \langle V(\phi) \rangle \quad \text{(7b)} \\
\dot{\lambda} &= \frac{\partial H}{\partial \delta} = 4\lambda \delta \quad \text{(7c)} \\
\dot{\delta} &= -\frac{\partial H}{\partial \lambda} = -2\delta^2 + \frac{\partial}{\partial \lambda} \left[ \int_{\phi_c}^{\phi_c+\pi} RR'' d\phi - \langle V(\phi) \rangle \right] \quad \text{(7d)}
\end{align*}
\]

with the effective Hamiltonian:

\[ H = \langle T \rangle + \langle V \rangle = p^2 + 2\lambda \delta^2 - \int_{\phi_c}^{\phi_c+\pi} RR'' d\phi + \langle V(\phi) \rangle + E_0 p. \] (8)

Thus \( p(t) \) is the momentum associated with the center of mass motion \( \phi_c(t) = \int_{\phi_c}^{\phi_c+\pi} R^2(\phi - \phi_c) \phi \ d\phi \), and \( \delta(t) \) is the conjugate momentum of the width of the wave-function, which is proportional to \( \lambda(t) = \frac{1}{2} \int_{\phi_c}^{\phi_c+\pi} R^2 \phi^2 \ d\phi \). The \( \langle \ldots \rangle \) means \( \int_{\phi_c}^{\phi_c+\pi} |\Psi(\phi - \phi_c)|^2 \ldots d\phi \), and the prime \( ' \) stands for \( \frac{\partial}{\partial \phi} \). The mean value of the population imbalance between the two traps is \( N(t) = \langle \Psi(\phi, t) \hat{N} \Psi(\phi, t) \rangle = p(t) \), the relative phase is \( \phi(t) = \phi_c(t) \), and the corresponding mean square deviations are \( \sigma_N^2(t) = \langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2 = \langle T(t) \rangle \) and \( \sigma_{\phi}^2(t) = 2\lambda(t) \).

For \( \Gamma \gg 1 \), \( R[\phi - \phi_c(t), \lambda(t)] \) can be well approximated by a Gaussian:

\[ R(\phi, t) = (4\pi \lambda)^{-1/4} e^{-\frac{\sigma^2_{\phi}(t)^2}{2}} \] (9)

with the caveat that during the dynamics its width \( 2\sigma_{\phi}(t) = 2\sqrt{2\lambda(t)} \ll 2\pi \).

The equations of motion become:

\[
\begin{align*}
\dot{N} &= -\Gamma \sin \phi \ e^{-\frac{\sigma^2_{\phi}}{2}} \quad \text{(10a)} \\
\dot{\phi} &= 2N + E_0 \quad \text{(10b)} \\
\dot{\sigma}_{\phi} &= 2\sigma_{\phi} \delta \quad \text{(10c)} \\
\dot{\delta} &= -2\delta^2 + \frac{1}{2\sigma^4_{\phi}} - \Gamma \cos \phi \ e^{-\frac{\sigma^2_{\phi}}{2}} \quad \text{(10d)}
\end{align*}
\]
with the total (conserved) energy and the relative population dispersion:

\[ H = N^2 + \sigma_N^2 - \Gamma \cos \phi \ e^{-\sigma_N^2/2} + E_0 \ N \]  
(11)

\[ \sigma_N = \frac{1}{2 \sigma_\phi} \sqrt{1 + 4 \sigma_\phi^4 \delta^2} \]  
(12)

The canonically conjugate dynamical variables are \( N, \phi \), as in the classical Josephson Hamiltonian, and the pair \( \sigma_\phi^2, \delta = \frac{1}{\sigma_\phi^2} \sqrt{\sigma_N^2 - \frac{1}{4 \sigma_\phi^2}} \), characterizing the respective quantum fluctuations. As expected, \( \sigma_N \sigma_\phi \geq \frac{1}{2} \) during the dynamics. The classical Josephson equations are recovered from Eqs.(10 a,b) in the limit \( \sigma_\phi \rightarrow 0 \). We will discuss more about the transition from the quantum to the classical regimes in the following.

The variational ground state energy of Eq. (2) is given by:

\[ E_{gs} = \frac{1}{4 \sigma_{\phi,s}^4} - \Gamma e^{-\sigma_{\phi,s}^2/2} + \frac{E_0^2}{2} = \sigma_{N,s}^2 - \Gamma e^{-1/8 \sigma_{N,s}^2} - \frac{E_0^2}{4} \]  
(13)

where \( \sigma_{\phi,s}, \sigma_{N,s} \) are the solution of:

\[ 2 \sigma_{\phi,s}^4 e^{-\sigma_{\phi,s}^2/2} = e^{-\frac{1}{8 \sigma_{N,s}^2}} = \frac{1}{\Gamma} \]  
(14)

The stationary results were first discussed in [11], where Eqs. (13,14) were obtained minimizing the ground state energy with a Gaussian trial wave function in the case \( E_0 = 0 \). Linearizing Eq. (10) for small amplitude \( \phi \) oscillations, we have: \( \ddot{\phi} = -2 \Gamma e^{-\sigma_{\phi,s}^2/2} \phi \). The condensate atoms oscillate coherently with a frequency (unscaled):

\[ \omega_q = \sqrt{E_s E_J} \ e^{-\sigma_{\phi,s}^2/4} \]  
(15)

where the classical Josephson relation gives \( \omega_c = \sqrt{E_s E_J} \). The quantum fluctuations renormalize the Josephson plasma frequency, with \( \frac{\omega_q}{\omega_c} = \exp(-\sigma_{\phi,s}^2/4) = (\sigma_{\phi,s}^2 \sqrt{2 \Gamma})^{-1} \). Notice that in the linear regime, the current-phase Eqs.(10 a,b) are effectively decoupled from the dynamics of the respective fluctuations Eqs.(10 c,d). On the contrary, for large amplitude \( \phi \) oscillations, Eqs. (14) cannot be decoupled. In this case the exponential factor modulates the amplitude and the frequencies of the oscillations, inducing partial collapses and revivals. This can be seen in Fig.(1 a-d), where we show the population imbalance, the relative phase and the respective mean square deviations as a function of time.
Above the critical point \((N = 0, \phi = \pi/2)\), the phase \(\phi(t)\) starts running, Fig.(2b), and the system is set into macroscopic quantum self-trapping (MQST) mode \([16,17]\). The width of the wave function grows and the amplitude of oscillations ‘collapses’, Fig.(2a). In the deep MQST regime, when \(N(t) \simeq N(t = 0)\), the phase diffuses as \(\sigma_{\phi}(t) \simeq \sigma_{\phi,s}^{2} + \frac{E_{s}^{2}}{4\sigma_{\phi,s}^{4}}t^{2}\), Fig(2d), regardless of the initial value of \(N(t = 0)\) \([27]\). The relative population oscillations collapse with a life time \(\tau \simeq 2E_{J}^{-\frac{1}{2}}E_{s}^{-\frac{3}{4}}\), while the \(\sigma_{N}^{2}(t)\) tends to a constant value, Fig(2c). However, since the total number of condensate atoms is finite, the phase can eventually revive partially or completely. This can be seen rewriting the wave function in the \(N\) representation: \(\Psi(\phi,t) = \sum_{N} a_{N} \Phi_{N}(\phi)e^{-\frac{i}{\hbar}E_{N}t}\), where \(E_{N}\) are the eigenenergies of Eq. (2). For instance, in the limit \(E_{s}N_{T} \ll E_{J}\), the eigenvalue spectrum is approximately linear \(E_{N} \sim \frac{E_{J}}{N_{T}}N\), and the revival time is \(\tau_{R} \sim \hbar N_{T}/E_{J}\). More generally, the occurrence of a complete or partial revival, or the complete destruction of it, depends on the detailed eigenspectrum of the Hamiltonian. We note that Eq. (8) cannot describe the revival after a complete collapse since the Gaussian ansatz Eq. (3) (and, consequently, the semiclassical approximation underlying it) breaks down when \(\sigma_{\phi} \simeq \pi\).

Eqs. (14) admit, as a dynamical solution, a quite peculiar oscillation mode, with zero relative phase and population imbalance, but oscillating fluctuations, according to:

\[
\begin{align*}
N(t) &= 0 \\
\phi(t) &= 0 \\
\sigma_{N}^{2}(t) &= \frac{1}{4\sigma_{\phi}^{2}} + \frac{\dot{\sigma}_{\phi}^{2}}{4} \\
\ddot{\sigma}_{\phi} &= \frac{1}{\sigma_{\phi}^{3}} - 2\Gamma\sigma_{\phi} e^{-\frac{\sigma_{\phi}^{2}}{4}}
\end{align*}
\]

with initial conditions \(N(t = 0) = 0, \phi(t = 0) = 0; E_{0} = 0\) and arbitrary \(\sigma_{\phi}(t = 0)\). This collective oscillation mode can be experimentally observed by lowering the height of the barrier of a BJJ ensemble in thermodynamic equilibrium. This corresponds to changing \(\Gamma\) in Eq. (14) from its initial value. The temporal evolution of the ensemble averaged observables and the respective mean square deviations can be calculated by tracing the
dynamical $N(t), \phi(t)$ trajectories of each junction.

*Classical limit.* Increasing the number of atoms, $\sigma_\phi \rightarrow 0$ as $\Gamma^{-\frac{1}{4}} \sim N_T^{-\frac{1}{4}(\alpha+\beta)}$. Moreover, for a given initial value $N(t=0)$, the amplitude of the “particle” oscillations in the $\phi$-potential of Eq. (2), decreases as $\phi_{\text{max}} \sim \Gamma^{-\frac{1}{2}} \sim N_T^{-\frac{1}{2}(\alpha+\beta)}$. Then Eqs. (10a,b) decouple from Eqs. (10c,d), and the time evolution of the mean values of current and phase become independent of the corresponding dispersions. In the MQST regime, the collapse time (and, consequently, the time over which the semiclassical predictions are reliable), increases as $\tau \sim N_T^{\frac{1}{2}(3\beta-\alpha)}$. In this framework the classical limit emerges naturally, without invoking any symmetry breaking argument, as discussed in [6].

*Numerical estimates.* Following the analytical estimations of the Josephson coupling energy and the on-site energy given for two weakly coupled condensates in [19], we have:

$$\Gamma \simeq 1.7 N_T \frac{a_0}{a_s} \exp(-S) \frac{1}{\tanh(S/2)} ,$$

with $a_0, a_s$ the trap length and the scattering length, respectively, and $S \sim \frac{1}{\hbar} \sqrt{2m \sigma_B^2 (V_0 - \mu)}$. $\sigma_B$ is the width of the barrier, $V_0$ its height, and $\mu$ the chemical potential. For typical traps and condensates, $a_0 \sim 10^4$ Å, $a_s \sim 50$ Å and $\sigma_B \sim 5$ μm. With a height of the barrier such that $(V_0 - \mu) \sim 10$ nK, we have $\Gamma \sim 10 - 100$ for $N_T \sim 1000$. Varying the width and/or the height of the barrier, and the total number of condensate atoms, the system might span from the $\Gamma \ll 1$ to the $\Gamma \gg 1$ limits. The temperature should be small compared to the Josephson coupling energy [19] to avoid destroying the quantum fluctuations. Damping effects are also reduced by decreasing the total number of atoms. Such regimes (vanishing small temperatures, and small population per site $\sim 1000$ in an optical array) are under current investigations [3]. Concluding, we remark that Eqs. (10) can be straightforwardly generalized to describe interwell tunneling in an array of trapped condensates. Work in this direction as well as on the quantitative analysis of the effects of temperature and damping on the quantum dynamics are in progress.

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FIGURES

FIG. 1. The population imbalance $N$, the relative phase $\phi$, and the corresponding fluctuations $\sigma_N$ and $\sigma_\phi$ as a function of time. The initial conditions are $N = 4$, $\phi = 0$, $\Gamma = 100$, $\delta = 0$, $\sigma_\phi = 0.26$ and $E_0 = 0$.

FIG. 2. $N, \phi, \sigma_N, \sigma_\phi$ as a function of time. The initial conditions are the same as in Fig.(1) except $N(t = 0) = 50$. 
