GROWTH OF HOMOLOGY TORSION IN FINITE COVERINGS AND HYPERBOLIC VOLUME

THANG T. Q. LÊ

ABSTRACT. We give an upper bound for the growth of homology torsions of finite coverings of irreducible 3-manifolds with tori boundary in terms of hyperbolic volume.

1. Introduction

1.1. Growth of homology torsions in finite coverings. Suppose $X$ is a finite CW-complex with fundamental group $\Pi = \pi_1(X)$. For a subgroup $\Gamma < \Pi$ of finite index let $X_{\Gamma}$ be the covering of $X$ corresponding to $\Gamma$, and $t_j(\Gamma)$ be the size of the $\mathbb{Z}$-torsion part of $H_j(X_{\Gamma}, \mathbb{Z})$. We want to study the growth of $t_j(\Gamma)$.

A sequence of subgroups $(\Gamma_k)_{k=1}^\infty$ of $\Pi$ is nested if $\Gamma_{k+1} < \Gamma_k$ and it is exhausted if $\bigcap_k \Gamma_k = \{e\}$, where $e$ is the unit of $\Pi$. It is known that the fundamental group of any compact 3-manifold is residually finite [He], i.e. it has an exhausted nested sequence of normal subgroups of finite index.

We will prove the following result and its generalizations.

Theorem 1. Suppose $X$ is an orientable irreducible compact 3-manifold whose boundary $\partial X$ is either empty or a collection of tori. Let $(\Gamma_k)_{k=1}^\infty$ be an exhausted nested sequence of normal subgroups of $\Pi$ of finite index. Then

$$\limsup_{n \to \infty} \frac{\ln t_1(\Gamma_k)}{[\Pi : \Gamma_k]} \leq \frac{\text{vol}(X)}{6\pi}.$$ 

Here vol$(X)$ is defined as follows. If the fundamental group of $X$ is finite, let vol$(X) = 0$. Suppose the fundamental group of $X$ is infinite. By geometrization results of Thurston and Perelman (see e.g. [Boi, BBB+]), every piece in the Jaco-Shalen-Johansson decomposition of $X$ (satisfying the assumption of Theorem 1) is either Seifert fibered or hyperbolic. Define vol$(X)$ as the sum of the volumes of all hyperbolic pieces.

1.2. More general limit: trace limit. For any group $G$ with unit $e$ define a trace function $\text{tr}_G : G \to \mathbb{Z}$ by $\text{tr}_G(g) = 1$ if $g = e$ and $\text{tr}_G(g) = 0$ if $g \neq e$. This extends to a $\mathbb{C}$-linear map $\text{tr}_G : \mathbb{C}[G] \to \mathbb{C}$. Here $\mathbb{C}[G]$ is the group ring of $\Pi$ with complex coefficients.

Assume $X$ is a 3-manifold satisfying the assumption of Theorem 1 with $\Pi = \pi_1(X)$. Let $\mathcal{G}$ be the set of all finite index subgroups of $\Pi$. For any $\Gamma \in \mathcal{G}$, $\Pi$ acts on the right on the finite set $\Gamma \backslash \Pi$ of right cosets of $\Gamma$, and the action gives rise to an action of $G$ on $\mathbb{C}[\Gamma \backslash \Pi]$, the $\mathbb{C}$-vector space with base $\Gamma \backslash \Pi$. For $g \in \Pi$, define

$$\text{tr}_{\Gamma \backslash \Pi}(g) = \frac{\text{tr}(g, \mathbb{C}[\Gamma \backslash \Pi])}{[\Pi : \Gamma]}.$$ 

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where \( \text{tr}(g, C[\Gamma \setminus \Pi]) \) is the trace of the operator \( g \) acting on the vector space \( C[\Gamma \setminus \Pi] \). Note that when \( \Gamma \) is a normal subgroup of \( \Pi \), then \( G := \Gamma \setminus \Pi \) is a group, and \( \text{tr}_{\Gamma \setminus \Pi}(g) \) coincides with the above definition of \( \text{tr}_C(p(g)) \), where \( p(g) \) is the image of \( g \) in the quotient group \( G \).

Suppose \( (\Gamma_k)_{k=1}^\infty \) is a sequence of subgroups of \( \Pi \) of finite index, i.e. \( \Gamma_k \in \mathcal{G} \). We say \( \Gamma_k \overset{\text{tr}}{\longrightarrow} 1 \) if for any \( g \in \Pi \),

\[
\lim_{k \to \infty} \text{tr}_{\Gamma_k \setminus \Pi}(g) = \text{tr}_{\Pi}(g).
\]

For example, if \( (\Gamma_k)_{k=1}^\infty \) is a nested exhausted sequence of subgroups of \( \Pi \) of finite index, then \( \Gamma_k \overset{\text{tr}}{\longrightarrow} 1 \).

We have the following stronger version of Theorem 1.

**Theorem 2.** Suppose \( X \) is an orientable irreducible compact 3-manifold whose boundary \( \partial X \) is either empty or a collection of tori.

Then

\[
\limsup_{\mathcal{G} \ni \Gamma \to 1} \frac{\ln t_1(\Gamma)}{[\Pi : \Gamma]} \leq \frac{\text{vol}(X)}{6\pi}.
\]

Here \( \limsup_{\mathcal{G} \ni \Gamma \to 1} f(\Gamma) \), for a function \( f : \mathcal{G} \to \mathbb{R} \), is defined to be the infimum of the set of all values \( L \) such that for any sequence \( \Gamma_k \in \mathcal{G} \) with \( \Gamma_k \overset{\text{tr}}{\longrightarrow} 1 \), one has \( \limsup_{k \to \infty} f(\Gamma_k) \leq L \).

**Conjecture 1.** Suppose \( X \) is a 3-manifold satisfying the assumption of Theorem 2.

(a) One has

\[
\limsup_{\mathcal{G} \ni \Gamma \to 1} \frac{\ln t_1(\Gamma)}{[\Pi : \Gamma]} = \frac{\text{vol}(X)}{6\pi}.
\]

(b) (Stronger version) Let \( \mathcal{G}^N \) be the set of all normal subgroups of \( \Pi \) of finite index. One has

\[
\limsup_{\mathcal{G}^N \ni \Gamma \to 1} \frac{\ln t_1(\Gamma)}{[\Pi : \Gamma]} = \frac{\text{vol}(X)}{6\pi}.
\]

Similar conjectures were formulated independently by Bergeron and Venkatesh [BV] and Lück. It is clear that the left hand side of \( (1) \) is non-negative. Hence we have the following result.

**Corollary 1.1.** The strong version of Conjecture 1 holds true for 3-manifold \( X \) satisfying the assumption of Theorem 2 with \( \text{vol}(X) = 0 \).

For example, if \( X \) is the complement in \( S^3 \) of a tubular neighborhood of a torus link, then the strong conjecture holds for \( X \). A 3-manifold \( X \) satisfying the assumption of Theorem 2 has \( \text{vol}(X) = 0 \) if and only if \( X \) is spherical or a graph manifold.

1.3. Branched covering. A version of Theorem 2 is the following. Suppose \( K \) is a knot in \( S^3 \) and \( X = S^3 \setminus N(K) \), where \( N(K) \) is an open tubular neighborhood of \( K \). For a subgroup \( \Gamma \leq \Pi = \pi_1(X) \) let \( \hat{X}_\Gamma \) be the branched \( \Gamma \)-covering of \( S^3 \), branched along \( K \). This means, \( \hat{X}_\Gamma \) is obtained from \( X_\Gamma \) by attaching solid tori to \( \partial X_\Gamma \) in such a way that any lift of a meridian of \( K \) bounds a disk in \( \hat{X}_\Gamma \).

**Theorem 3.** Suppose \( K \) is a knot in \( S^3 \). For a subgroup \( \Gamma \leq \Pi = \pi_1(S^3 \setminus K) \) let \( \hat{X}_\Gamma \) be the branched \( \Gamma \)-covering of \( S^3 \), branched over \( K \). Then

\[
\limsup_{\mathcal{G} \ni \Gamma \to 1} \frac{\ln |\text{Tor}_2 H_1(\hat{X}_\Gamma, \mathbb{Z})|}{[\Pi : \Gamma]} \leq \frac{\text{vol}(X)}{6\pi}.
\]

Here \( |\text{Tor}_2 H_1(\hat{X}_\Gamma, \mathbb{Z})| \) is the size of the \( \mathbb{Z} \)-torsion part of \( H_1(\hat{X}_\Gamma, \mathbb{Z}) \).
1.4. **On trace convergence.** In the case when \((\Gamma_k)_{k=1}^{\infty}\) is a nested sequence of subgroups of \(\Pi\) of finite index, the definition of \(\Gamma_k \xrightarrow{\text{tr}} 1\) was introduced in [Fa]. Even when \((\Gamma_k)_{k=1}^{\infty}\) is not nested, \(\Gamma_k \xrightarrow{\text{tr}} 1\) if and only if the representations \(\rho_k : \Pi \to \mathbb{C}[\Gamma/\Pi]\) with \(k = 1, 2, \ldots\), form an **arithmetic approximation of** \(\Pi\) in the sense of [Fa, Definition 9.1].

For a discrete group \(\Pi\), there exists a sequence \((\Gamma_k)_{k=1}^{\infty}\) of subgroups of finite index such that \(\Gamma_k \xrightarrow{\text{tr}} 1\) if and only if \(\Pi\) is residually finite.

It turns out that the limit \(\Gamma_k \xrightarrow{\text{tr}} 1\), for not necessarily nested sequences, is closely related to known limits in the literature.

**The case of normal subgroups.** When each of \(\Gamma_k\) is a normal subgroup of \(\Pi\) of finite index, i.e. \(\Gamma_k \in \mathcal{G}^N\), the definition of \(\Gamma_k \xrightarrow{\text{tr}} 1\) simplifies. Suppose \((\Gamma_k)_{k=1}^{\infty}\) is a nested sequence of normal subgroups of \(\Pi\) of finite index. Then \(\Gamma_k \xrightarrow{\text{tr}} 1\) if and only if \(\cap_{m=1}^{\infty} \cup_{k=m}^{\infty} \Gamma_k = \{e\}\). This means, \(\Gamma_k \xrightarrow{\text{tr}} 1\) if and only if for every non-trivial \(g \in \Pi\), \(g\) eventually does not belong to \(\Gamma_k\).

**Relation to sofic approximation.** Suppose \(\Gamma_k\) is a sequence of subgroups of finite index of \(\Pi\). Then \(\Gamma_k \xrightarrow{\text{tr}} 1\) if and only if \(g \in \Pi\), \(g\) eventually does not belong to \(\Gamma_k\).

**Relation to Benjamini-Schramm convergence.** Suppose \(X\) is a hyperbolic 3-manifold. Raimbault observed that \(\Gamma_k \xrightarrow{\text{tr}} 1\) if and only if \(X_{\Gamma_k}\) Benjamini-Schramm (BS) converge to the hyperbolic space \(\mathbb{H}^3\) in the sense of [ABB+]. The notion of BS convergence was introduced in [ABB+] for more general sequences of manifolds, not necessarily coverings of a fixed manifold.

1.5. **Growth of Betti numbers.** Suppose \(X\) is a connected finite CW-complex with fundamental group \(\Pi\). For a subgroup \(\Gamma \leq \Pi\) of finite index let \(b_j(\Gamma)\) be the \(j\)-th Betti number of \(X_{\Gamma}\).

If \((\Gamma_k)_{k=1}^{\infty}\) is an exhausted nested sequence of normal subgroups of \(\Pi\), then Kazhdan [Ka, Gro] showed that

\[
\limsup_{k \to \infty} \frac{b_j(X_{\Gamma_k})}{[\Pi : \Gamma_k]} \leq b_j^{(2)}(\tilde{X}),
\]

where \(b_j^{(2)}(\tilde{X})\) is the \(L^2\)-Betti number of the universal covering \(\tilde{X}\). For the definition of \(L^2\)-Betti number and \(L^2\)-invariants in general, see [Li3]. Lück [Li1] then showed that one actually has a much stronger result

\[
\lim_{k \to \infty} \frac{b_j(X_{\Gamma_k})}{[\Pi : \Gamma_k]} = b_j^{(2)}(\tilde{X}).
\]

Farber extended Lück’s result [3] to the trace convergence, \(\Gamma_k \xrightarrow{\text{tr}} 1\) (no normal subgroups). The following is a special case of [Fa, Theorem 9.2]: If \((\Gamma_k)_{k=1}^{\infty}\) is a sequence of subgroups of \(\Pi\) of finite index such that \(\Gamma_k \xrightarrow{\text{tr}} 1\), then

\[
\lim_{k \to \infty} \frac{b_j(X_{\Gamma_k})}{[\Pi : \Gamma_k]} = b_j^{(2)}(\tilde{X}).
\]
Theorem 2 is an analog of Kazhdan’s inequality for the growth of the torsion part of the homology, and Conjecture I asks for analogs of Lück’s and Farber’s equalities. In general, questions about the growth of the torsion part of the homology are much more difficult than similar questions for the free part of the homology.

1.6. Abelian covering. Suppose $\tilde{X} \to X$ is a regular covering with the group of deck transformations equal to $\mathbb{Z}^n$. Here $X$ is a finite CW-complex. We choose generators $z_1, \ldots, z_n$ of $\mathbb{Z}^n$ and identify $\mathbb{Z}[\mathbb{Z}^n] \cong \mathbb{Z}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$.

Let $G$ in this case be the set of all subgroups $\Gamma \leq \mathbb{Z}^n$ of finite index. Again $X_\Gamma$ is the corresponding finite covering, and $b_j(\Gamma)$ and $t_j(\Gamma)$, for $\Gamma \in G$, are respectively the rank and the size of the $\mathbb{Z}$-torsion part of $H_j(X_\Gamma, \mathbb{Z})$. In this abelian case, $\Gamma_k \xrightarrow{tr} 1$ is equivalent to $\lim_{k \to \infty} (\Gamma_k) = \infty$, where $(\Gamma)$, for $\Gamma \leq \mathbb{Z}^n$ is the smallest among norms of non-zero elements in $\Gamma$.

Since $\mathbb{Z}^n$ acts on $\tilde{X}$, the homology group $H_j(\tilde{X}, \mathbb{Z})$ is a finitely-generated $\mathbb{Z}[\mathbb{Z}^n]$-module. Let $R = \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. For any finitely-generated $R$-module $M$ its rank $\text{rk}_R(M)$ is defined as the $F(R)$-dimension of $M \otimes_R F(R)$, where $F(R)$ is the fractional field of $R$. There are defined the Alexander polynomials $\Delta_k(M)$, $k = 0, 1, 2, \ldots$, see e.g. [Le2, Tu]. Each $\Delta_k(M)$ belongs to $\mathbb{Z}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ and is defined up to a unit of $\mathbb{Z}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. One has that $\Delta_{k+1}(M)$ divides $\Delta_k(M)$, and the smallest number $r$ for which $\Delta_r(M) \neq 0$ is equal to the rank of $M$. Let $\Delta(M) = \Delta_r(M)$, the first non-zero Alexander polynomial of $M$.

In this abelian case, the $L^2$-Betti number $b_j^{(2)}(\tilde{X} \to X)$ is equal to the rank of $H_j(\tilde{X}, \mathbb{Z})$, as a module over $R$, see [Li, Li3]. Lück’s result shows that

$$\lim_{\mathbb{Z}^n \ni \Gamma \to 1} \frac{b_j(X_\Gamma)}{[\Pi : \Gamma]} = \text{rk}_R \left( H_j(\tilde{X}, \mathbb{Z}) \right).$$

(5)

In this case, we have a positive answer to an analog of Conjecture II as follows. For any $j \geq 0$,

$$\lim_{\mathbb{Z}^n \ni \Gamma \to 1} \frac{\ln t_j(X_\Gamma)}{[\mathbb{Z}^n : \Gamma]} = \mu \left( \Delta(H_j(\tilde{X}, \mathbb{Z})) \right).$$

(6)

Here $\mu(f)$, for a non-zero $f \in R = \mathbb{Z}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, is the Mahler measure of $f$ defined by

$$\mu(f) = \int_{\mathbb{T}^n} \ln |f| d\sigma,$$

where $\mathbb{T}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_j| = 1 \ \forall j = 1, \ldots, n\}$ is the unit $n$-dimensional torus, and $d\sigma$ is the invariant measure on $\mathbb{T}^n$ with total mass 1. For a proof of (6) and discussions of related results, see [Le2]. The proof in [Le2] uses tools from algebraic geometry, commutative algebra, and also results from algebraic dynamics and cannot be generalized to non-commutative cases.

For an arbitrary finite CW-complex with fundamental group $\Pi$ and a fixed index $j$, we don’t know what would be the upper limit of the left hand side of (6), with $\mathbb{Z}^n$ replaced by $\Pi$. Only when $X$ has some geometric structure, like 3-dimensional manifolds, do we have results like Theorem 2.

1.7. Related results. The growth of torsion parts of homology of coverings or more generally bundles over a fixed manifold has attracted a lot of attention lately. For related results see e.g. [ABB+, BD, BV, BSV, MG, MiP, McP]. Probably C. Gordon [Go] was the first to ask about the asymptotic growth of torsion of homology in finite, albeit abelian, coverings.
1.8. Organization. In Section 2 we recall the definition of geometric determinant, introduce the notion of metric abelian groups, and use them to get an estimate for the torsion parts in exact sequences. In Section 3 we recall the definition of the Fuglede-Kadison determinant and prove an upper bound for growth of geometric determinants by Fuglede-Kadison determinant. Finally in Section 4 we give proofs of the main results.

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2. Geometric determinant, lattices, and volume in inner product spaces

In this section we recall the definition of the geometric determinant, introduce the notion of metric abelian groups, and establish certain results on upper bounds for torsion parts in exact sequences.

For a finitely-generated abelian group $A$ let $\text{Tor}_\mathbb{Z}A$ be the torsion subgroup of $A$ and $\text{Fr}(A) := A/\text{Tor}_\mathbb{Z}A$, which is a free abelian group. Denote by $\text{rk}(A)$ and $t(A)$ respectively the rank of $A$ and the size of the torsion part of $A$.

2.1. Geometric determinant. For a linear map $f : V_1 \to V_2$, where each $V_i$ is a finite-dimensional inner product space the geometric determinant $\det'(f)$ is the product of all non-zero singular values of $f$. Recall that $x \in \mathbb{R}$ is singular value of $f$ if $x \geq 0$ and $x^2$ is an eigenvalue of $f^*f$. By convention $\det'(f) = 1$ if $f$ is the 0 map. Thus we always have $\det'(f) > 0$.

Since the maximal singular value of $f$ is the norm $||f||$, we have

$$\det'(f) \leq ||f||^{\text{rk}(f)} \quad \text{with convention } 0^0 = 1.$$  

Remark 2.1. The geometric meaning of $\det'f$ is the following. The map $f$ restricts to a linear isomorphism $f'$ from $\text{Im}(f^*)$ to $\text{Im}(f)$, each is a inner product space. Then $\det'(f) = |\det(f')|$, where the ordinary determinant $\det(f')$ is calculated using orthonomal bases of the inner product spaces.

Suppose for each $i = 1, 2$ one has a linear map $f_i : V_i \to W_i$ between inner product spaces. The direct sums $V_1 \oplus V_2$ and $W_2 \oplus W_2$ have natural inner product structure, and we have

$$\det'(f_1 \oplus f_2) = \det'(f_1) \det'(f_2).$$

Lemma 2.2. Suppose $g$ is a generator of a cyclic group $A$ of order $l$. Define an inner product structure on $\mathbb{C}[A]$ such that $A$ is an orthonormal basis. Let $A$ acts on $\mathbb{C}[A]$ be left multiplication. Then $\det'(1 - g) = l$.

Proof. Let $f = 1 - g$. Then $f^* = 1 - g^{-1}$, and $f^*f = 2 - g - g^{-1}$. Let $\zeta = \exp(2\pi i/l)$. Since $\zeta^k$, with $k = 0, 1, \ldots, l - 1$, are all eigenvalues of $g$, the eigenvalues of $f^*f$ are $2 - \zeta^k - \zeta^{-k} = |1 - \zeta^k|^2$, with $k = 0, 1, \ldots, l - 1$. Excluding the 0 value, we have

$$\det'(f) = \prod_{j=1}^{l-1} |1 - \zeta^j| = l,$$
where the last identity follows since for any complex number \( z \) one has
\[
\prod_{j=1}^{l-1} (z - \zeta^k) = \sum_{k=0}^{l-1} z^k = \frac{z^l - 1}{z - 1}.
\]
(The above holds since \( \zeta^k \), with \( k = 1, \ldots, l - 1 \), are roots of \( \frac{z^l - 1}{z - 1} \).) \( \square \)

2.2. Volume of lattices. We recall here some well-known facts about volumes of lattices in inner product spaces.

Suppose \( V \) is a finite-dimensional inner space. For a \( \mathbb{Z} \)-submodule (also called a lattice) \( \Lambda \subset V \) with \( \mathbb{Z} \)-basis \( v_1, \ldots, v_l \) define
\[
\text{vol}(\Lambda) = |\det \left( (v_i, v_j)_{i,j=1}^l \right)|^{1/2} > 0.
\]
That is, \( \text{vol}(\Lambda) \) is the volume of the parallelepiped spanned by a set of basis vectors. By convention, the volume of the 0 space is 1.

It is clear that if \( \Lambda_1 \subset \Lambda_2 \) are lattices in \( V \) of the same rank, then
\[
|\Lambda_2/\Lambda_1| = \frac{\text{vol}(\Lambda_1)}{\text{vol}(\Lambda_2)}.
\]

Let \( V \) and \( W \) be finite-dimensional inner product spaces. Suppose \( \Lambda \) is a lattice in \( V \) of rank equal the dimension of \( V \) and \( f : \Lambda \to W \) is an abelian group homomorphism. The kernel \( \ker(f) \) and the image \( \text{Im}(f) \) are lattices in respectively \( V \) and \( W \).

Note that \( f \) extends uniquely to a linear map \( \tilde{f} : V \to W \), and we put \( \det'(f) = \det'(\tilde{f}) \). We have
\[
\text{vol}(\ker(f)) \text{vol}(\text{Im}(f)) = \det'(f) \text{vol}(\Lambda).
\]

We record here one more property of volumes which we will need. Suppose \( \Lambda \) is a lattice in an inner product space \( V \) such that there is a \( \mathbb{Z} \)-basis of \( \Lambda \) which is an orthonormal basis of \( V \). Assume \( W \subset V \) is a subspace which has a basis whose elements are in \( \Lambda \). Let \( W^\perp \) be the orthogonal complement of \( W \) in \( V \) with respect to the inner product. Then (see [Ber])
\[
\text{vol}(W \cap \Lambda) = \text{vol}(W^\perp \cap \Lambda).
\]

2.3. Metric abelian groups. For a finitely-generated abelian group \( A \) recall that \( \text{Fr}(A) := A/\text{Tor}_\mathbb{Z}(A) \).

Definition 1. A metric abelian group is a finitely generated abelian group \( A \) equipped with an inner product on \( \text{Fr}(A) \otimes_\mathbb{Z} \mathbb{R} = A \otimes_\mathbb{Z} \mathbb{R} \). A large metric abelian group is a metric abelian group \( A \) such that \( \text{vol}(\Lambda) \geq 1 \) for any lattice \( \Lambda \subset \text{Fr}(A) \).

Example 2.3. Suppose \( A \) is a finitely generated free abelian group with a fixed set of generators. Define an inner product on \( A \otimes_\mathbb{Z} \mathbb{R} \) such that the set of generators is an orthonormal basis. This inner product gives \( A \) a metric. The metric is large, since if \( \Lambda \subset A \) is a sub lattice, then \( \text{vol}(\Lambda) \) is a positive integer, and must be \( \geq 1 \).

Suppose \( B \leq A \), where \( A \) is a metric abelian group. Then \( \text{Fr}(B) \otimes_\mathbb{Z} \mathbb{R} = B \otimes_\mathbb{Z} \mathbb{R} \) naturally embeds in \( A \otimes_\mathbb{Z} \mathbb{R} = \text{Fr}(A) \otimes_\mathbb{Z} \mathbb{R} \). The restriction of the inner product of \( A \otimes_\mathbb{Z} \mathbb{R} \) on \( \text{Fr}(B) \otimes_\mathbb{Z} \mathbb{R} = B \otimes_\mathbb{Z} \mathbb{R} \) gives a metric structure on \( B \), called the induced metric from \( A \). If the metric on \( A \) is large, then the induced metric on \( B \) is also large.
Furthermore, \((A/B) \otimes \mathbb{Z} \mathbb{R}\) is naturally isomorphic to \((B \otimes \mathbb{Z} \mathbb{R})^\perp\), the orthogonal complement of \(B \otimes \mathbb{Z} \mathbb{R}\) in \(A \otimes \mathbb{Z} \mathbb{R}\). Using this natural isomorphism on can define a metric on the quotient group \(A/B\) by restricting the inner product on the space \((B \otimes \mathbb{Z} \mathbb{R})^\perp \cong (A/B) \otimes \mathbb{Z} \mathbb{R}\).

Suppose \(X\) is a finite CW-complex. By definition, the cellular \(\mathbb{Z}\)-complex \(C(X)\) of \(X\) consists of free abelian groups \(C_j(X)\), which are free abelian group with basis the set of \(j\)-cells of \(X\). We equip \(C_j(X)\) with a metric such that the mentioned basis consisting of \(j\)-cells is an orthonormal basis. There are induced metrics on the homology groups \(H_j(X, \mathbb{Z})\) for every \(j = 0, 1, 2, \ldots\). As observed, all these metrics are large.

If \(X’\) is a finite covering of \(X\), which is a finite CW-complex, then \(X’\) has a CW-complex structure inherited from that of \(X\). Hence we can define metrics on \(H_j(X’, \mathbb{Z})\).

Suppose \(\alpha : A \rightarrow B\) is a group homomorphism between metric abelian groups. Then \(\alpha \otimes \text{id} : A \otimes \mathbb{Z} \mathbb{R} \rightarrow B \otimes \mathbb{Z} \mathbb{R}\) is a linear map between finite-dimensional inner product space. We define

\[
\det'(\alpha) := \det'(\alpha \otimes \text{id}).
\]

2.4. Torsion estimate. Recall that for a finitely-generated abelian group \(A\), \(t(A)\) is the size of its \(\mathbb{Z}\)-torsion part.

Lemma 2.4. (a) Suppose \(A\) and \(B\) are free, large metric abelian groups and \(\alpha : A \rightarrow B\) is a group homomorphism. Then

\[ t(\text{coker}(\alpha)) \leq \text{vol}(\text{Im}(\alpha)) \leq \det'(\alpha)\text{vol}(A). \tag{11} \]

(b) Suppose \(A \xrightarrow{\alpha} B \rightarrow C \rightarrow 0\) is an exact sequence of large metric abelian groups. Then

\[ t(C) \leq t(B)\det(\alpha)\text{vol}(\text{Fr}(A)). \tag{12} \]

(c) Suppose \(F_1 \xrightarrow{\beta} F_2 \rightarrow A \rightarrow B\) is an exact sequence of metric abelian groups, where \(F_1\) and \(F_2\) are free. Then

\[ t(A) \leq t(B)\det'(\beta)\text{vol}(F_1). \]

Proof. (a) Let \(\overline{\text{Im}}\alpha = (\text{Im}\alpha \otimes \mathbb{Z} \mathbb{Q}) \cap B\), where both \((\text{Im}\alpha) \otimes \mathbb{Z} \mathbb{Q}\) and \(B\) are considered as subsets of \(B \otimes \mathbb{Z} \mathbb{Q}\). Then \(\text{Tor}_Z(\text{coker}\alpha) = \overline{\text{Im}}\alpha/\text{Im}\alpha\). Hence, \(\text{vol}(\text{Im}(\alpha)) = t(\text{coker}(\alpha))\text{vol}(\overline{\text{Im}}\alpha)\). Since \(\text{vol}(\overline{\text{Im}}\alpha) \geq 1\) due to largeness of the metric, we have

\[ t(\text{coker}(\alpha)) \leq \text{vol}(\text{Im}(\alpha)). \]

By (\ref{eq:lemma2.4a}), \(\text{vol}(\text{Im}(\alpha))\text{vol}(\text{ker}(\alpha)) = \det'(\alpha)\text{vol}(A)\), from which we get the second inequality of (\ref{eq:lemma2.4a}) since \(\text{vol}(\text{ker}(\alpha)) \geq 1\).

(b) There is an isomorphism \(B \cong B_{fr} \oplus \text{Tor}_Z(B)\), which we use to identify \(B\) with \(B_{fr} \oplus \text{Tor}_Z(B)\). Then there is \(\alpha_{fr} : A \rightarrow \text{Tor}_Z(B)\) such that \(\alpha = \alpha_{fr} \oplus \alpha_{t}\). Then

\[ C \cong B_{fr}/\text{Im}(\alpha_{fr}) \oplus \text{Tor}_Z(B)/\text{Im}(\alpha_{t}). \]

Hence,

\[ \text{Tor}_Z(C) \cong \text{Tor}_Z(\text{coker}(\alpha_{fr})) \oplus \text{Tor}_Z(B)/\text{Im}(\alpha_{t}). \]

It follows that

\[ |\text{Tor}_Z(C)| = t(\alpha_{fr})|\text{Tor}_Z(B)/\text{Im}(\alpha_{t})| \leq t(\alpha_{fr})|\text{Tor}_Z(B)| \leq |\text{Tor}_Z(B)|\det(\alpha_{fr})\text{vol}(A), \]

where the last inequality follows from (\ref{eq:lemma2.4a}).

(c) First we observe that if

\[ 0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0 \]

Then there is an isomorphism \(A_2 \cong A_1 \oplus A_3\), which we use to identify \(A_2\) with \(A_1 \oplus A_3\). Then

\[ C \cong A_1/\text{Im}(\alpha_{fr}) \oplus \text{Tor}_Z(A_3)/\text{Im}(\alpha_{t}). \]

Hence,

\[ \text{Tor}_Z(C) \cong \text{Tor}_Z(\text{coker}(\alpha_{fr})) \oplus \text{Tor}_Z(A_3)/\text{Im}(\alpha_{t}). \]

It follows that

\[ |\text{Tor}_Z(C)| = t(\alpha_{fr})|\text{Tor}_Z(A_3)/\text{Im}(\alpha_{t})| \leq t(\alpha_{fr})|\text{Tor}_Z(A_3)| \leq |\text{Tor}_Z(A_3)|\det(\alpha_{fr})\text{vol}(A), \]

where the last inequality follows from (\ref{eq:lemma2.4a}).
is an exact sequence of finitely generated abelian groups, then
\[ t(A_2) \leq t(A_1)t(A_3). \]

From the assumption we have the following exact sequence
\[ 0 \to F_2/\text{Im}(\beta) \to A \to B' \to 0, \]
where \( B' \leq B \). By the above observation,
\[ t(A) \leq t(B')t(F_2/\text{Im}(\beta)) = t(B)t(\text{coker}\beta) \leq t(B) \det'(\beta) \text{vol}(F_1), \]
where the last inequality follows from part (a). This completes the proof of the lemma. \( \square \)

3. Fuglede-Kadison determinant

We recall here the definition and establish some properties of the Fuglede-Kadison determinant. We will introduce the Fuglede-Kadison determinant only for a class of operators which we will need in this paper. For a detailed treatment of the Fuglede-Kadison determinant, the reader should consult the book [L"u3]. We prove that the Fuglede-Kadison determinant of a matrix with entries in \( \mathbb{Z}[\Pi] \) serves as an upper bound for the growth of the geometric determinants of the action of the same matrix on finite quotients. This extends a result of L"uck.

Recall that \( \Pi \) is the fundamental group of a finite CW-complex. For a ring \( R \) let \( \text{Mat}(n \times m, R) \) denote \( R \)-algebra of all \( n \times m \) matrices with entries in \( R \). We will mostly consider the cases \( R = \mathbb{Z}, \mathbb{C}, \mathbb{Z}[\Pi], \mathbb{C}[\Pi]. \)

3.1. Fuglede-Kadison determinant of a density function. We modify the following definition from [L"u1].

**Definition 2.** (a) A right continuous function
\[ F : [0, \infty) \to [0, \infty) \]
is called a density function if
(i) \( F \) is increasing (i.e. \( F(\lambda) \leq F(\lambda') \) if \( \lambda \leq \lambda' \)), and
(ii) There is a constant \( C \) such that \( F(\lambda) = F(C) \) for all \( \lambda > C \).

(b) A density function \( F \) is said to be in the determinantal class if the integral \( \int_0^\infty \ln(\lambda)dF \) exists as a real number. If \( F \) is in the determinantal class, define its determinant by
\[ \det(F) := \exp\left( \int_0^\infty \ln(\lambda)dF \right). \]
If \( F \) is not in the determinantal class, define \( \det(F) = 0. \)

Let \( C \) be the number in Condition (ii) of Definition 2. If \( F \) is a density function in the determinantal class, then one has (see [L"u3, Lemma 3.15])
\[ \ln \det(F) = (F(C) - F(0)) \ln(C) - \int_{0^+}^C \frac{F(\lambda) - F(0)}{\lambda} d\lambda. \]
Actually, \( F \) is in the determinantal class if and only if the integral on the right hand side of (13) exists as a real number.

For an increasing function \( F \) we define
\[ F^+(\lambda) = \lim_{\varepsilon \to 0^+} F(\lambda + \varepsilon). \]
If \( F : [0, \infty) \to [0, \infty) \) is a not necessarily right continuous function satisfying conditions (i)–(ii), then \( F^+ \), which is \( F \) made right continuous, is a density function.
Proposition 3.1. Let \((F_k)_{k=1}^{\infty}\) be a sequence of density functions in the determinantal class and 
\[ F(\lambda) = \lim_{k \to \infty} F_k(\lambda). \]
Assume that \(F^+\) is in the determinantal class, and
(a) there is a constant \(C\) such that \(F_k(\lambda) = F_l(\lambda)\) for any \(k, l \geq 1\) and \(\lambda > C\), and
(b) \(\lim_{k \to \infty} F_k(0) = F^+(0)\).
Then
\[ \limsup_{k \to \infty} \det(F_k) \leq \det(F^+). \]

Proof. The proof is almost contained in [Lü1, Sections 2 & 3], and the proof here is a modification of the one in [Lü1].
From Levi’s Theorem of monotonous convergence, one can easily show that for any sequence of increasing functions \(f_k\), one has
\[ \int_0^C \liminf_{k \to \infty} f_k(\lambda) d\lambda \leq \liminf_{k \to \infty} \int_0^C f_k(\lambda) d\lambda. \]

Since all the functions \(F_k\) are the same when \(\lambda > C\), condition (ii) in the definition of a density function shows that we can change \(C\) if necessary such that \(F_k(\lambda) = F_l(\lambda) = F_k(C)\) for all \(k, l \geq 1\) and \(\lambda \geq C\). It follows that \(F(\lambda) = F(C)\) for all \(\lambda \geq C\), and \(F(C) = \lim_{k \to \infty} F_k(C)\).

By definition \(F(0) = \liminf_{k \to \infty} F_k(0)\). Condition (b) implies that \(F(0) = F^+(0) = \lim_{k \to \infty} F_k(0)\).
By (13)
\[ \det(F^+) = (F(C) - F(0)) \ln(C) - \int_0^C \frac{F^+(\lambda) - F^+(0)}{\lambda} d\lambda \]
(15)
\[ = \lim_{k \to \infty} (F_k(C) - F_k(0)) \ln C - \int_0^C \frac{F^+(\lambda) - F^+(0)}{\lambda} d\lambda \]

By [Lü1, Lemma 3.2], we have the first equality in the following.
\[ \int_0^C \frac{F^+(\lambda) - F^+(0)}{\lambda} d\lambda = \int_0^C \frac{F(\lambda) - F^+(0)}{\lambda} d\lambda \]
\[ = \int_0^C \frac{F(\lambda) - F(0)}{\lambda} d\lambda \]
\[ = \int_0^C \liminf_{k \to \infty} \frac{F_k(\lambda) - F_k(0)}{\lambda} d\lambda \]
\[ \leq \liminf_{k \to \infty} \int_0^C \frac{F_k(\lambda) - F_k(0)}{\lambda} d\lambda \]
(16)
where the last inequality follows from (13).
Using Inequality (16) in (15), we have
\[ \det(F^+) \geq \lim_{k \to \infty} (F_k(C) - F_k(0)) \ln C - \limsup_{k \to \infty} \int_0^C \frac{F_k(\lambda) - F_k(0)}{\lambda} d\lambda \]
\[ = \limsup_{k \to \infty} \left\{ (F_k(C) - F_k(0)) \ln C - \int_0^C \frac{F_k(\lambda) - F_k(0)}{\lambda} d\lambda \right\} = \limsup_{k \to \infty} \det(F_k), \]
where in the last equality we again use (13). This completes the proof of the proposition. \(\square\)
3.2. Relation between Fuglede-Kadison determinant and geometric determinant. Suppose $D : V_1 \to V_2$ is a $\mathbb{C}$-linear operator between finite-dimensional Hermitian spaces. For $\lambda \geq 0$ let $F_D(\lambda)$ be the number of eigenvalues of $D^*D$ which are less than or equal to $\lambda$, counted with multiplicity. Since all eigenvalues of $D^*D$ have absolute value less than or equal to the norm $\|D^*D\|$, if $\lambda > \|D^*D\|$ then $F_D(\lambda) = \dim(V_1)$, a constant.

Then $F_D : [0, \infty) \to [0, \infty)$ is a bounded, right continuous, step function. It is easy to see that $F(\lambda)$ is a density function in the determinantal class and

$$\sqrt{\det(F_D)} = \det'(D). \quad (17)$$

3.3. Von Neumann algebra of a group and trace function. Let $\ell^2(\Pi)$ be the Hilbert space with orthonormal basis $\Pi$. In other words, $\ell^2(\Pi)$ is the set of all formal sums $\sum_{g \in \Pi} c_g g$, with $c_g \in \mathbb{C}$ and $\sum_{g \in \Pi} |c_g|^2 < \infty$, with inner product $\langle g, g' \rangle = \delta_{g,g'}$. For every positive integer $n$, $(\ell^2(\Pi))^n$ inherits a Hilbert structure, where

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_{j=1}^{n} \langle x_j, y_j \rangle.$$ 

We will consider $(\ell^2(\Pi))^n$ as a left $\Pi$-module by the left multiplication.

By definition, the von Neumann algebra $\mathcal{N}(\Pi)$ of $\Pi$ is the $\mathbb{C}$-algebra of bounded $\Pi$-equivariant operators from $\ell^2(\Pi)$ to $\ell^2(\Pi)$.

For $f \in \mathcal{N}(\Pi)$ its trace is defined by

$$\text{tr}_\Pi(f) = \langle e, f(e) \rangle,$$

where $e$ is the unit of $\Pi$. More generally, suppose $f : (\ell^2(\Pi))^n \to (\ell^2(\Pi))^n$ is a bounded $\Pi$-equivariant operator, define its trace by

$$\text{tr}(f) = \sum_{j=1}^{n} \langle e_j, fe_j \rangle,$$

where $e_j = 0^{j-1} \times e \times 0^{n-j} \in (\ell^2(\Pi))^n$.

3.4. Fuglede-Kadison determinants of matrices in $\text{Mat}(n \times m, \mathbb{C}[\Pi])$. Suppose $B$ is an $n \times m$ matrix with entries in $\mathbb{C}[\Pi]$. Let $T : (\ell^2(\Pi))^n \to (\ell^2(\Pi))^m$ be the bounded $\mathbb{Z}[\Pi]$-linear operator defined by $x \mapsto xB$.

Let $\{P(\lambda), \lambda \in [0, \infty)\}$ be the right continuous spectral family of the positive operator $T^*T$. The spectral density function of $B$ is defined by

$$F(\lambda) = F_B(\lambda) := \text{tr}_\Pi(P(\lambda)).$$

Then $F : [0, \infty) \to [0, \infty)$ is a density function (Definition 2). We say that $B$ is in the determinantal class if $F$ is in the determinantal class. Define

$$\det_\Pi(B) = \sqrt{\det(F)}.$$ 

When $B$ is in the determinantal class $\det_\Pi(B)$ is a positive real number.

**Example 3.2.** Suppose $\Pi$ is the trivial group. Then $\mathbb{C}[\Pi] = \mathbb{C}$. Assume $B \in \text{Mat}(n \times m, \mathbb{C}[\Pi])$, i.e. $B$ is an $n \times m$ matrix with complex entries. In this case, $B$ is in the determinantal class, and (see [Lit3 Example 3.12])

$$\det_\Pi(B) = (\det'(B))^{1/n}. \quad (18)$$
3.5. **Universal bound for norm.** One can easily find an universal upper for the norm of all operators induced from one acting on $\mathbb{Z}[\Pi]_n$.

**Lemma 3.3.** Suppose $B \in \text{Mat}(n \times n, \mathbb{C}[\Pi])$. There is a constant $C$ (depending on $B$) such that
\[
\|B_\Gamma\| < C,
\]
for any $\Gamma \in \mathcal{G}$. Here $B_\Gamma : \mathbb{C}[\Gamma \setminus \Pi]^n \to \mathbb{C}[\Gamma \setminus \Pi]^m$ is the induced $\mathbb{C}$-homomorphism defined by $x \to xB$.

(Recall that $\mathbb{C}[\Gamma \setminus \Pi]$ is equipped with the Hermitian structure in which $\Gamma \setminus \Pi$ is an orthonormal basis.)

**Proof.** When $\Gamma$ is a normal subgroup, the statement was proved in [Lü1] with $C = nm \max_{i,j} |O_{ij}|_1$), where for an element $x = \sum c_g g \in \mathbb{Z}[\Pi]$ one sets $|x|_1 = \sum |c_g|$. The easy proof in [Lü1] also works for our more general case. $\Box$

3.6. **Upper limit of growth of determinants.** Suppose $B \in \text{Mat}(m \times n, \mathbb{Z}[\Pi])$ and $\Gamma \leq \Pi$ is a subgroup of finite index, i.e. $\Gamma \in \mathcal{G}$. The map $\mathbb{C}[\Pi]^n \to \mathbb{C}[\Pi]^m$, given by $x \to xB$, descends to a $\mathbb{C}$-linear map $B_\Gamma : \mathbb{C}[\Gamma \setminus \Pi]^n \to \mathbb{C}[\Gamma \setminus \Pi]^m$. The inner product on $\mathbb{C}[\Gamma \setminus \Pi]$, defined so that $\Gamma \setminus \Pi$ is an orthonormal basis, induces an inner product on $\mathbb{C}[\Gamma \setminus \Pi]^n$. One can now define $\det'(B_\Gamma)$.

**Theorem 3.4.** Let $B \in \text{Mat}(n \times m, \mathbb{Z}[\Pi])$ be in the determinantal class. One has
\[
\limsup_{\mathcal{G} \ni \Gamma \to 1} \frac{\ln \det'(B_\Gamma)}{[\Pi : \Gamma]} \leq \ln \det_{\Pi}(B).
\]

**Proof.** Suppose $(\Gamma_k)_{k=1}^\infty$ is a sequence in $\mathcal{G}$ such that $\Gamma_k \xrightarrow{\text{tr}} 1$. We have to show that
\[
(19) \quad \limsup_{k \to \infty} \frac{\ln \det'(B_{\Gamma_k})}{[\Pi : \Gamma_k]} \leq \ln \det_{\Pi}(B).
\]

Let
\[
F_k(\lambda) = F_{B_{\Gamma_k}}(\lambda)/[\Pi : \Gamma_k],
\]
where $F_{B_{\Gamma_k}}$ is defined as in Section 3.2. Recall that each $F_k$ is a density function in the determinantal class. By Lemma 3.3, there is a constant $C$ such that $\|B_{\Gamma_k}^* B_{\Gamma_k}\| < C$ for all $k$. It follows that if $\lambda > C$, then $F_k(\lambda) = n$ for all $k$.

Define
\[
F(\lambda) = \liminf_{k \to \infty} F_k(\lambda).
\]

**Lemma 3.5.** (a) The function $F^+$ is the spectral density function of $B$.

(b) Besides,
\[
\lim_{k \to \infty} F_k(0) = F(0) = F^+(0).
\]

The proof of the lemma is given below. We assume the lemma for now.

Now we can apply Proposition 3.1 to the sequence $F_k$ and get
\[
\limsup_{k \to \infty} \ln \det(F_k) \leq \ln \det(F^+).
\]

Since $\ln \det(F_k) = 2[\Pi : \Gamma_k] \det'(B_{\Gamma_k})$ (by (17)) and $\ln \det(F) = \ln \det_{\Pi}(B)$, we get (19). $\Box$
Proof of Lemma 3.5. Lemma 3.5 can be obtained as a special case of results of Farber [Fa] as follows. First of all the inferior limit \( \lim \inf \) is a special case of \( \lim_{\omega} \) of [Fa]. Then [Fa, Theorem 8.3] implies Lemma 3.5(a). Actually, [Fa, Theorem 8.3] is formulated only for matrix \( B \) which is the boundary operator of the universal complex of a finite CW-complex, but the proof there does not use the fact that \( B \) comes from a CW-complex and works for all matrix \( B \in \text{Mat}(n \times m, \mathbb{Z}[\Pi]) \).

As mentioned in Introduction, \( \Gamma_k \xrightarrow{t_{\ast}} 1 \) implies that the representations \( \Pi \to \mathbb{C}[\Gamma_k \backslash \Pi] \) form an arithmetic approximation of \( \Pi \) in the sense of [Fa]. Then [Fa, Theorem 9.2(i)] implies Lemma 3.5(b). Thus, we can get Lemma 3.5 as special cases of Farber’s results. □

Remark 3.6. The lemma, for the case when \( (\Gamma_k) \) is an exhausted nested sequence of normal subgroups, was part of the main technical result of [Lü1] (see Theorem 2.3 there). What Farber did in [Fa] is to generalize Lück’s result to more general settings. Since our case is a special case of Farber, we don’t need the full proof of Farber’s. Actually, a small modification of Lück proof is enough for our case, as follows.

The only place where normality of subgroups were used in [Lü1] is Lemma 2.6 there, which is the following claim.

Claim 1. Suppose \( p(\lambda) \) is a polynomial. Then there exists a number \( m \), depending on \( (\Gamma_k) \), such that if \( k > m \) then
\[
\text{tr}_\Pi(p(T^*T)) = \frac{1}{[\Pi : \Gamma_k]} \text{tr}(p(B_{\Gamma_k}^r B_{\Gamma_k})).
\]

Here on the right hand side, \( \text{tr} \) stands for the usual trace of an endomorphism of a finite-dimensional vector space.

In our setting, Claim 1 does not hold in general. However, the following weaker statement holds.

Claim 2. Suppose \( p(\lambda) \) is a polynomial. Then
\[
\text{tr}_\Pi(p(T^*T)) = \lim_{k \to \infty} \frac{1}{[\Pi : \Gamma_k]} \text{tr}(p(B_{\Gamma_k}^r B_{\Gamma_k})).
\]

Proof of Claim 2. Let \( B^* \) be the matrix obtained from \( B \) by transpose following by conjugation, and \( A = p(B^* B) \). Then \( p(T^*T) : (\ell^2(\Pi))^n \to (\ell^2(\Pi))^n \) is given by \( x \to xA \), and \( p(B_{\Gamma_k}^r B_{\Gamma_k}) = A_{\Gamma_k} \).

We have then
\[
\text{tr}_\Pi(p(T^*T)) = \sum_{j=1}^n \text{tr}_\Pi(A_{jj}) = \lim_{k \to \infty} \sum_{j=1}^n \text{tr}_{\Gamma_k \backslash \Pi}(A_{jj}) = \lim_{k \to \infty} \frac{1}{[\Pi : \Gamma_k]} \text{tr}(A_{\Gamma_k}),
\]

where the second equality follows from \( \Gamma_k \xrightarrow{\text{tr}} 1 \), and the third equalities follows from the definition of \( \text{tr}_{\Gamma_k \backslash \Pi} \). This proves Claim 2.

With Claim 2 replacing Lemma 2.6 of [Lü1], the proof of [Lü1, Theorem 2.3] goes through verbatim, and we get Lemma 3.5.

3.7. Hyperbolic volume and Fuglede-Kadison determinant. Suppose \( X \) is an irreducible orientable compact 3-manifold with infinite fundamental group and with boundary \( \partial X \) either empty or a collection of tori. We define now good presentations of \( \Pi = \pi_1(X) \) and their reduced Jacobians.

First assume that \( \partial X \neq \emptyset \). Then \( X \) is homotopic to a 2-dimensional finite CW-complex \( Y \) which has one zero-cell. Suppose there are \( n \) two-cells. Since the Euler characteristic is 0, there must be \( n + 1 \) one-cells, denoted by \( a_1, \ldots, a_{n+1} \). The CW-structure of \( Y \) gives rise to a presentation of \( \Pi = \Pi_1(X) = \Pi_1(Y) \) of the form
\[
\Pi = \langle a_1, \ldots, a_n, a_{n+1} \mid r_1, \ldots, r_n \rangle,
\]
where \( r_j \) is the boundary of the \( j \)-th two-cell. We call such a presentation of \( \Pi = \pi_1(X) \) a good presentation for \( \Pi \). For a good presentation (20), define its reduced Jacobian to be the square matrix

\[
J = \left( \frac{\partial r_i}{\partial a_j} \right)_{i,j=1}^n \in \text{Mat}(n \times n, \mathbb{Z}[\Pi]),
\]

where \( \frac{\partial r_i}{\partial a_j} \) is the Fox derivative.

Now assume that \( \partial X = \emptyset \). A presentation of \( \Pi = \pi_1(X) \),

\[
(21) \quad \Pi = \langle a_1, \ldots, a_n, a_{n+1} | r_1, \ldots, r_n, r_{n+1} \rangle,
\]

is called a good presentation if it comes from a Heegaard splitting of \( X \). For a good presentation (21), define its reduced Jacobian to be the square matrix

\[
J = \left( \frac{\partial r_i}{\partial a_j} \right)_{i,j=1}^n \text{Mat}(n \times n, \mathbb{Z}[\Pi]).
\]

We quote here an important result which follows from a result of Lück and Schick (appeared as Theorem 4.3 in [Lü3] and Lück [Lü3, Theorem 4.9].

**Theorem 3.7.** Suppose \( X \) is an irreducible orientable compact 3-manifold with infinite fundamental group and with boundary either empty or a collection of tori. Let \( J \) be the reduced Jacobian of a good presentation of \( \Pi = \pi_1(X) \). Then

\[
\ln \det(\Pi)(J) = \frac{\text{vol}(X)}{6\pi}.
\]

**Proof.** In the proof of Theorem 4.9 of [Lü3] appeared in [Lü2] it was explicitly shown that \( \ln \det(\Pi)(J) = -\rho^{(2)}(\tilde{X}) \), where \( \rho^{(2)}(\tilde{X}) \) is the (additive) \( L^2 \)-torsion of the universal covering of \( X \).

By [Lü3, Theorem 4.3], \( \rho^{(2)}(\tilde{X}) = -\text{vol}(X)/6\pi \). Hence \( \ln \det(\Pi)(J) = \frac{\text{vol}(X)}{6\pi}. \)

**Remark 3.8.** In the formulation of [Lü3, Theorem 4.3] there is an assumption that \( X \) satisfies the conclusion of Thurston Geometrization Conjecture, which is redundant now due the Perelman’s celebrated result. Besides, there is a requirement that the boundary of \( X \) to be incompressible. But if a torus component of \( X \) is compressible, then \( X \) must be a solid torus, for which all the results are trivial.

4. **Proofs of Theorem 2 and Theorem 3**

4.1. **Growth of functions.** Recall that \( \mathcal{G} \) is the set of all subgroups of \( \Pi \) of finite index. Suppose \( f, g : \mathcal{G} \to \mathbb{R}_{>0} \) are functions on \( \mathcal{G} \) with positive values. We say that \( f \) has negligible growth if

\[
\limsup_{\Gamma \to 1} (f(\Gamma))^{\frac{1}{[\Pi:\Gamma]}} \leq 1.
\]

We will write \( f \preceq g \) if \( f/g \) has negligible growth.

4.2. **Chain complexes of covering.** Suppose \( X \) is a connected finite CW-complex with fundamental group \( \Pi \), and \( \tilde{X} \) is its universal covering. Then \( \tilde{X} \) inherits a CW-complex structure from \( X \), where the cells of \( \tilde{X} \) are lifts of cells of \( X \). The action of \( \Pi \) on \( \tilde{X} \) preserves the CW-structure and commutes with the boundary operators. Let \( C(\tilde{X}) \) be the chain \( \mathbb{Z} \)-complex of the CW-structure of \( \tilde{X} \). Then \( C_j(\tilde{X}) \) is the free \( \mathbb{Z} \)-module with basis the set of all \( j \)-cells of \( \tilde{X} \). We will identify \( C_j(\tilde{X}) \) with \( \mathbb{Z}[\Pi]^n \), a free \( \mathbb{Z}[\Pi] \)-module, as follows.

We assume that

(i) \( X \) has only one 0-cell \( e^0 \), and
(ii) for any \( j \)-cell of \( X \) with \( j \geq 1 \), one has \( e^0 = \chi_e((1,0,\ldots,0)) \), where \( \chi_e : D^j \to X \) is the characteristic map of \( e \). Here \( D^j = \{ x \in \mathbb{R}^j, ||x|| \leq 1 \} \) is the standard unit \( j \)-disk.

For example, if \( X \) is a compact connected manifold and the CW-structure of \( X \) is obtained from a Morse function having only one minimum using the stable spheres of a gradient-like flow in general position \([X] \), then we have (i) and (ii). The 0-cell is in the boundary from a Morse function having only one minimum using the stable spheres of a gradient-like flow in \( C \), where \( \partial \) is the boundary operator.

In general, any finite connected CW-complex is homotopic to one satisfying the above conditions (i) and (ii). Choose a lift \( \tilde{e}^0 \) of \( e^0 \). Condition (ii) shows that for every \( j \)-cell \( e \) there is a unique lift \( \tilde{e} \) defined by using the lift of the characteristic map which sends \((1,0,\ldots,0) \) to \( \tilde{e}^0 \).

Let \( e_1^j, \ldots, e_{n_j}^j \) be an ordered set of all \( j \)-cells of \( X \), then

\[
C_j(\tilde{X}) = \bigoplus_{i=1}^{n_j} \mathbb{Z}[\Pi] \cdot \tilde{e}_i^j,
\]

and we use the above equality to identify \( C_j(\tilde{X}) \) with \( \mathbb{Z}[\Pi]^{n_j} \). Thus, the identification \( C_j(\tilde{X}) \equiv \mathbb{Z}[\Pi]^{n_j} \), for a CW-complex \( X \) satisfying (i) and (ii) depends on the choice of a lift of \( e^0 \) and an ordering on the set of \( j \)-cells, for every \( j \geq 1 \).

We will write an element \( x \in \mathbb{Z}[\Pi]^n \) as a row vector \( x = (x_1, \ldots, x_n) \) where each \( x_j \in \mathbb{Z}[\Pi] \). Note that \( \mathbb{Z}[\Pi]^n \) can be considered as a left \( \mathbb{Z}[\Pi] \)-module or a right \( \mathbb{Z}[\Pi] \)-module. We will consider \( \mathbb{Z}[\Pi]^n \) as a left \( \mathbb{Z}[\Pi] \)-module unless otherwise stated. If \( B \in \text{Mat}(n \times m, \mathbb{Z}[\Pi]) \) is a \( n \times m \) matrix with entries in \( \mathbb{Z}[\Pi] \), then the right multiplication by \( B \) defines a \( \mathbb{Z}[\Pi] \)-linear map from \( \mathbb{Z}[\Pi]^n \) to \( \mathbb{Z}[\Pi]^m \), and every \( \mathbb{Z}[\Pi] \)-linear map \( \mathbb{Z}[\Pi]^k \to \mathbb{Z}[\Pi]^l \) arises in this way.

The boundary operator \( \partial_j : C_j(\tilde{X}) \to C_{j-1}(\tilde{X}) \) is given by a \( n_j \times n_{j-1} \) matrix with entries in \( \mathbb{Z}[\Pi] \); by abusing notation we also use \( \partial_j \) to denote this matrix. The chain complex \( C(\tilde{X}) \) has the form

\[
C(\tilde{X}) = \left( \cdots \to \mathbb{Z}[\Pi]^{n_{j+1}} \xrightarrow{\partial_{j+1}} \mathbb{Z}[\Pi]^{n_j} \xrightarrow{\partial_j} \mathbb{Z}[\Pi]^{n_{j-1}} \to \cdots \mathbb{Z}[\Pi]^{n_1} \xrightarrow{\partial_1} \mathbb{Z}[\Pi]^{n_0} \xrightarrow{\partial_0} 0 \right).
\]

Suppose \( \Gamma \leq \Pi \) is a subgroup and \( X_\Gamma \) the corresponding covering. Then \( X_\Gamma \) inherits a CW-structure from \( X \), and its chain \( \mathbb{Z} \)-complex is exactly \( \mathbb{Z}[\Gamma \backslash \Pi] \otimes_{\mathbb{Z}[\Pi]} C(\tilde{X}) \). Here we consider \( \mathbb{Z}[\Gamma \backslash \Pi] \) as a right \( \mathbb{Z}[\Pi] \)-module.

In general, if \( C \) is a chain complex over \( \mathbb{Z}[\Pi] \) of left \( \mathbb{Z}[\Pi] \)-modules and \( \Gamma \leq \Pi \), then we denote by \( C_\Gamma \) the chain \( \mathbb{Z} \)-complex \( \mathbb{Z}[\Gamma \backslash \Pi] \otimes_{\mathbb{Z}[\Pi]} C \).

4.3. Circle complex. Fix a non-trivial element \( g \) of a residually finite group \( \Pi \). Let \( S \) be the following chain \( \mathbb{Z}[\Pi] \)-complex

\[
0 \to \mathbb{Z}[\Pi] \xrightarrow{\partial_1} \mathbb{Z}[\Pi] \xrightarrow{\partial_0} 0,
\]

where \( \partial_1 = 1 - g \). For every \( \Gamma \in \mathcal{G} \), one has the \( \mathbb{Z} \)-complex \( S_\Gamma = \mathbb{Z}[\Gamma \backslash \Pi] \otimes_{\mathbb{Z}[\Pi]} S \). Since \( S_\Gamma \) is a finite CW-complex, there is defined a metric on each \( H_j(S_\Gamma) \), see Section 2.3. Recall that \( b_j(S_\Gamma) \) is the rank of \( H_j(S_\Gamma) \).

Lemma 4.1. (a) One has

\[
\limsup_{g \in \Gamma \to 1} \frac{b_0(S_\Gamma)}{[\Pi : \Gamma]} = \limsup_{g \in \Gamma \to 1} \frac{b_1(S_\Gamma)}{[\Pi : \Gamma]} = 0.
\]

(b) For each \( i = 0, 1 \), the the function \( \Gamma \to \text{vol}(H_i(S_\Gamma)) \) is negligible.
Proof. (a) The chain complex $S_\Gamma$ has the form
\[
0 \to \mathbb{Z} / [\Gamma \setminus \Pi] \xrightarrow{1-g_\Gamma} \mathbb{Z} / [\Gamma \setminus \Pi] \xrightarrow{\partial_0} 0,
\]
where $g_\Gamma$ denotes the action of $g$ on $\mathbb{Z} / [\Gamma \setminus \Pi]$. Then $H_1(C_\Gamma) = \ker(1 - g_\Gamma)$ and $H_0(S_\Gamma) = \coker(1 - g_\Gamma)$. From the Euler characteristic consideration, one has $b_1(S_\Gamma) = b_0(S_\Gamma)$.

The element $g_\Gamma$ acts on $C[\Gamma \setminus \Pi]$ by permuting the basis $\Gamma \setminus \Pi$. Suppose as a permutation, $g_\Gamma$ has $d$ cycles with length $l_1, \ldots, l_d$. Then $d = b_1(S_\Gamma) = \text{tr}_{\Gamma \setminus \Pi}(g)$. Hence, by definition of trace convergence,
\[
b_1(S_\Gamma) \left[ \Pi : \Gamma \right] = b_0(S_\Gamma) \left[ \Pi : \Gamma \right] = d(\Gamma) \left[ \Pi : \Gamma \right] = \text{tr}_{\Gamma \setminus \Pi}(g) \to \text{tr}_{\Pi}(g) = 0 \quad \text{as} \quad \Gamma \to 1.
\]
This proves (a).

(b) Let $N = N(\Gamma) := [\Pi : \Gamma]$.

Claim 1. The function $(N/d)^d$ on $G$ is negligible.

Proof of Claim 1. We already saw in part (a) that $d/N \to 0$ as $\Gamma \to 1$. It follows that $(N/d)^d \to 1$ as $\Gamma \to 1$. This proves Claim 1.

Claim 2. The function $\det'(1 - g_\Gamma)$ on $G$ is negligible.

Proof of Claim 2. Since $g_\Gamma$ has $d$ cycles of length $l_1, \ldots, l_d$, Lemma 2.2 shows that
\[
\det'(1 - g_\Gamma) = \prod_{j=1}^{d} l_j.
\]

Since $\sum_{j=1}^{d} l_j = N$, the arithmetic-geometric mean inequality implies
\[
\det'(1 - g_\Gamma) = \prod_{j=1}^{d} l_j \leq (N/d)^d,
\]
which, in light of Claim 1, proves Claim 2.

Let us continue with the proof of the lemma. Let $W \subset \mathbb{C}[\Gamma \setminus \Pi]$ be the subspace span by $\text{Im}(1 - g_\Gamma)$. Then one can identify $\text{Fr}(H_0(S_\Gamma)) = W^\perp \cap \mathbb{Z} / [\Gamma \setminus \Pi]$, and the metric on $\text{Fr}(H_0(S_\Gamma))$ is exactly the metric of $\mathbb{C}[\Gamma \setminus \Pi]$ restricted on $W^\perp$. We have
\[
\text{vol}(H_0(S_\Gamma)) = \text{vol}(W^\perp \cap \mathbb{Z} / [\Gamma \setminus \Pi]) = \text{vol}(W \cap \mathbb{Z} / [\Gamma \setminus \Pi]),
\]
where the second identity follows from [11]. Since $\text{Im}(1 - g_\Gamma)$ is a sublattice of $W \cap \mathbb{Z} / [\Gamma \setminus \Pi]$, we have $\text{vol}(W \cap \mathbb{Z} / [\Gamma \setminus \Pi]) \leq \text{vol}(\text{Im}(1 - g_\Gamma))$. Hence,
\[
\text{(22) vol}(H_0(S_\Gamma)) \leq \text{vol}(\text{Im}(1 - g_\Gamma)).
\]

Recall that $H_1(S_\Gamma) = \ker(1 - g_\Gamma)$. Using (22), we have
\[
\text{vol}(H_1(S_\Gamma)) \text{vol}(H_0(S_\Gamma)) \leq \text{vol}(\ker(1 - g_\Gamma)) \text{vol}(\text{Im}(1 - g_\Gamma)) = \det'(1 - g_\Gamma) \quad \text{by (19)}.
\]
Since each of $\text{vol}(H_1(S_\Gamma))$ and $\text{vol}(H_0(S_\Gamma))$ is $\geq 1$, the above shows that
\[
1 \leq \text{vol}(H_1(S_\Gamma)) \leq (N/d)^d, \quad 1 \leq \text{vol}(H_0(S_\Gamma)) \leq (N/d)^d.
\]
From Claim 2 we can conclude that both $\text{vol}(H_0(S_\Gamma))$ and $\text{vol}(H_1(S_\Gamma))$ are negligible. \qed
4.4. Proof of Theorem 2, the case $\partial X$ is non-empty.

Proof. Suppose $X$ satisfies the assumption of Theorem 2 and besides, $\partial X$ is not empty. As in Section 3.7, choose a finite 2-dimensional CW-complex $Y$ which is homotopic to $X$. We assume $Y$ has one zero-cell denoted by $c^0$, $n + 1$ ordered one-cells $a_1, \ldots, a_{n+1}$, and $n$ ordered two-cells $y_1, \ldots, y_n$. Then we have the following good presentation (see Section 3.7) for $\Pi$:

$$\Pi = \langle a_1, \ldots, a_n, a_{n+1} | r_1, \ldots, r_n \rangle,$$

where $r_j$ comes from the boundary of $y_j$.

The universal covering $\tilde{Y}$ of $Y$ has a CW-structure induced from that of $Y$, on which $\Pi$ acts freely on the left. Let $\mathcal{C} = \mathcal{C}(\tilde{Y})$ be the chain $\mathbb{Z}$-complex of the CW-structure of $\tilde{Y}$. Each $\mathcal{C}_j(\tilde{Y})$ is a free $\mathbb{Z}[\Pi]$-module. Using the order on the set of cells of $Y$, we identify $\mathcal{C}_0(\tilde{Y}), \mathcal{C}_1(\tilde{Y}), \mathcal{C}_2(\tilde{Y})$ with respectively $\mathbb{Z}[\Pi], \mathbb{Z}[\Pi]^{n+1}, \mathbb{Z}[\Pi]^n$, see Section 4.2.

The complex $\mathcal{C}$ now becomes (see for example [11n, Claim 16.6])

$$\mathcal{C} = \left( 0 \to \mathbb{Z}[\Pi]^n \xrightarrow{\partial_2} \mathbb{Z}[\Pi]^{n+1} \xrightarrow{\partial_1} \mathbb{Z}[\Pi] \to 0 \right),$$

where $\partial_2 \in \text{Mat}(n \times (n + 1), \mathbb{Z}[\Pi])$ is the $n \times n + 1$ with entries $(\partial_2)_{ij} = \frac{\partial a}{\partial a_j}$ and $\partial_1 \in \text{Mat}(n + 1 \times 1, \mathbb{Z}[\Pi])$ is a column vector with entries $(\partial_1)_{ij} = 1 - a_j$. Recall that $\frac{\partial a}{\partial a_j}$ is the Fox derivative, with value $\frac{\partial a}{\partial a_j} \in \mathbb{Z}[\Pi]$.

Recall that the reduced Jacobian $J$, defined in Section 3.7, is the matrix obtained from $\partial_2$ be removing the last column, which will be denoted by $c$.

Then $\mathcal{C} = \mathcal{C}(\tilde{Y})$ is the middle row of the following commutative diagram

$$
\begin{array}{cccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}[\Pi] & \xrightarrow{1-a_{n+1}} & \mathbb{Z}[\Pi] & \xrightarrow{\partial_0} & 0 \\
\downarrow & & \downarrow & & \downarrow \iota & & \downarrow \text{id} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z}[\Pi]^n & \xrightarrow{\partial_2} & \mathbb{Z}[\Pi]^{n+1} & \xrightarrow{\partial_1} & \mathbb{Z}[\Pi] & \xrightarrow{\partial_0} & 0 \\
\downarrow & & \downarrow \text{id} & & \downarrow \tau & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{Z}[\Pi]^n & \xrightarrow{J} & \mathbb{Z}[\Pi]^n & \longrightarrow & 0 & \xrightarrow{\partial_0} & 0 \\
\end{array}
$$

where $\iota: \mathbb{Z}[\Pi] \to \mathbb{Z}[\Pi]^{n+1}$ is the embedding into the last component, and $\tau: \mathbb{Z}[\Pi]^{n+1} \to \mathbb{Z}[\Pi]^n$ is the projection onto the first $n$ components. Each row is a chain $\mathbb{Z}[\Pi]$-complex. Denote the chain complex of the first row and the third row by respectively $\mathcal{K}$ and $\mathcal{Q}$. The operators $\partial_0$ on the diagram indicate how to index the components of the complexes. For example, $\mathcal{Q}_2 = \mathcal{Q}_3 = \mathbb{Z}[\Pi]$.

The sequence

$$0 \to \mathcal{K} \to \mathcal{C} \to \mathcal{Q} \to 0$$

is split exact. Hence, for every $\Gamma \in \mathcal{G}$, one has the following exact sequence of $\mathbb{Z}$-complexes

$$0 \to \mathbb{K}_\Gamma \to \mathcal{C}_\Gamma \to \mathbb{Q}_\Gamma \to 0. \quad (23)$$

Note that $\mathcal{C}_\Gamma$ is the chain $\mathbb{Z}$-complex of $Y_\Gamma$, and its homology groups are the homology groups $H_\ast(Y_\Gamma, \mathbb{Z}) = H_\ast(X_\Gamma, \mathbb{Z})$. As each of $\mathbb{K}_\Gamma, \mathcal{C}_\Gamma, \mathbb{Q}_\Gamma$ is a free, based $\mathbb{Z}$-complex, each homology group of these complexes comes equipped with a metric, which is large (see Section 23).

The short exact sequence $\mathbb{K}_\Gamma \xrightarrow{\mathbb{B}_\Gamma} \mathbb{H}_1(\mathbb{K}_\Gamma) \to \mathbb{H}_1(\mathcal{C}_\Gamma) \to \mathbb{H}_1(\mathbb{Q}_\Gamma)$ generates a long exact sequence, part of it is

$$H_2(\mathbb{Q}_\Gamma) \xrightarrow{\partial_\Gamma} H_1(\mathbb{K}_\Gamma) \to H_1(\mathcal{C}_\Gamma) \to H_1(\mathbb{Q}_\Gamma).$$
Applying Lemma 2.4(c) to the above exact sequence, we get
\[ t(H_1(C_G)) \leq \det'(\beta_G) t(H_1(Q_G)) \text{vol}(H_2(Q_G)). \]
Since \( H_1(Q_G) = \text{coker}(J_G) \text{ and } H_2(Q_G) = \text{ker}(J_G) \), by Lemma 2.4(a), we have
\[ t(H_1(Q_G)) \text{vol}(H_2(Q_G)) = t(\text{coker}(J_G)) \text{vol}(\text{ker}(J_G)) \leq \det'(J_G). \]
Using the above inequality in (24), and \( t_1(\Gamma) = t(H_1(C_G)) \), we get
\[ t_1(\Gamma) \leq \det'(\beta_G) \det'(J_G). \]

Claim. \( \det'(\beta_G) \) is a negligible function on \( \mathcal{G} \).

Proof of Claim. We estimate \( \det'(\beta_G) \) by using upper bounds for the rank and the norm of \( \beta_G \). First, the rank of \( \beta_G \) is less than or equal to rank of its codomain, which is \( b_1(K_G) \).

The connecting homomorphism \( \beta \) is the restriction of \( \tilde{\beta} : \mathbb{Z}[\Pi^\prime]^n \rightarrow \mathbb{Z}[\Pi] \) given by \( \tilde{\beta}(x) = x \cdot c \), where \( c \) is the last column of \( \partial_2 \). It follows that \( \|\beta_G\| \leq \|\tilde{\beta}\| < C \) for some constant \( C \) not depending on \( \Gamma \) (see Lemma 3.3). Hence
\[ \det'(\beta_G) \leq C^{b_1(K_G)}. \]
By Lemma 4.1 \( \lim b_1(K_G)/[\Pi : \Gamma] = 0 \). It follows that \( \det'(\beta_G) \) is negligible. This completes the proof of the claim. \( \square \)

From (25) and the above claim, we have
\[ t_1(\Gamma) \leq \det'J_G. \]
Hence
\[ \limsup_{G \ni \Gamma \rightarrow 1} \ln \frac{t_1(\Gamma)}{[\Pi : \Gamma]} \leq \limsup_{G \ni \Gamma \rightarrow 1} \frac{\ln \det'(J_G)}{[\Pi : \Gamma]} \leq \det \Pi J = \frac{\text{vol}(X)}{6\pi}, \]
where we use Theorem 3.4 in the second inequality and Theorem 3.7 in the last equality. \( \square \)

4.5. Proof of Theorem 2, the case \( \partial X = \emptyset \text{ and } |\Pi| < \infty \). In this case, by Poincare’s conjecture (now a theorem due to Perelman), the universal covering \( \tilde{X} \) of \( X \) is \( S^3 \). Hence \( H_1(\tilde{X}, \mathbb{Z}) = 0 \).

Suppose \( (\Gamma_k)_{k=1}^\infty \) is a sequence of subgroups of \( \Pi \) such that \( \Gamma_k \rightarrow^\text{tr} 1 \). Since \( \Pi \) is finite, \( \Gamma_k \rightarrow^\text{tr} 1 \) implies that \( \Gamma_k = \{e\} \) for large enough \( k \). Thus, for large enough \( k \), \( t_1(\Gamma_k) = 1 \), and
\[ \lim_{k \rightarrow \infty} \ln \frac{t_1(\Gamma_k)}{[\Pi : \Gamma_k]} = 0 = \text{vol}(X). \]

4.6. Proof of Theorem 2, the case \( \partial X \) is empty and \( |\Pi| = \infty \). Suppose now \( \partial X = \emptyset \) and \( \Pi \) is infinite. Let \( X = H_1 \cup H_2 \) be a Heegaard decomposition of genus \( n+1 \) of \( X \). Assume \( a_1, \ldots, a_{n+1} \) are the cores of \( H_1 \) and \( b_1, \ldots, b_{n+1} \) are the cores of \( H_2 \). We assume that \( a_1, \ldots, a_{n+1} \) and \( b_1, \ldots, b_{n+1} \) are loops based at a point \(* \) which is on \( \partial H_1 = \partial H_2 \).

As the Heegaard splitting can be obtained by a Morse function on \( X \) with one minimum point, we can assume that the CW-structure satisfies conditions (i) and (ii) of Section 4.2. The CW-complex \( X \) has one zero-cell \( e^0 \), \( n+1 \) one-cells \( a_1, \ldots, a_{n+1} \), \( n+1 \) two-cells \( r_1, \ldots, r_{n+1} \) which are the co-core of \( b_1, \ldots, b_{n+1} \), and one three-cell \( e^3 \). The universal covering \( \tilde{X} \) has an induced CW-structure, whose chain \( \mathbb{Z} \)-complex \( \mathcal{D} \) is a free chain \( \mathbb{Z}[\Pi] \)-complex. We identify \( \mathcal{D}_j \equiv \mathbb{Z}[\Pi]^\prime \) as in Section 4.2. Here \( n_0 = n_3 = 1 \) and \( n_1 = n_2 = n+1 \).

Then \( \mathcal{D} \) is the middle row of the following commutative diagram...
0 \longrightarrow 0 \longrightarrow \mathbb{Z}[[\Pi]]^n \xrightarrow{\partial_2'} \mathbb{Z}[[\Pi]]^{n+1} \xrightarrow{\partial_1} \mathbb{Z}[[\Pi]] \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow \mathbb{Z}[[\Pi]] \xrightarrow{\partial_3} \mathbb{Z}[[\Pi]]^{n+1} \xrightarrow{\partial_2} \mathbb{Z}[[\Pi]]^{n+1} \xrightarrow{\partial_1} \mathbb{Z}[[\Pi]] \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow \mathbb{Z}[[\Pi]] \xrightarrow{1-b_n+1} \mathbb{Z}[[\Pi]] \longrightarrow 0 \longrightarrow 0 \longrightarrow 0

where
\begin{align*}
\iota(x_1, \ldots, x_n) &= (x_1, \ldots, x_n, 0) \\
\tau(x_1, \ldots, x_{n+1}) &= x_{n+1} \\
\partial_3 &= (1-b_1, \ldots, 1-b_{n+1}) \in \text{Mat}(1 \times (n+1), \mathbb{Z}[[\Pi]]) \\
\partial_2 &= (\frac{\partial r_i}{\partial a_j})_{i,j=1}^{n+1} \in \text{Mat}((n+1) \times (n+1), \mathbb{Z}[[\Pi]]) \\
\partial_1 &= (1-a_1, \ldots, 1-a_{n+1})^T \in \text{Mat}((n+1) \times 1, \mathbb{Z}[[\Pi]]),
\end{align*}

(with $T$ denoting the transpose) and the matrix $\partial_3'$ is obtained from $\partial_2$ by removing the last row.

Denote the chain complex of the first row and the third row by respectively $\mathcal{C}$ and $\mathcal{E}$. The sequence
\[0 \rightarrow \mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E} \rightarrow 0\]
is split exact. Hence, for every $\Gamma \in \mathcal{G}$, one has the following exact sequence of $\mathbb{Z}$-complex
\begin{equation}
0 \rightarrow \mathcal{C}_\Gamma \rightarrow \mathcal{D}_\Gamma \rightarrow \mathcal{E}_\Gamma \rightarrow 0.
\end{equation}
Since $H_1(\mathcal{E}_\Gamma) = 0$, a portion of the long exact sequence derived from (26) is
\[H_2(\mathcal{E}_\Gamma) \xrightarrow{\alpha} H_1(\mathcal{C}_\Gamma) \rightarrow H_1(\mathcal{D}_\Gamma) \rightarrow H_1(\mathcal{E}_\Gamma) = 0.
\]
where $\alpha$ is the connecting homomorphism. From Lemma 2.4(b) one has
\begin{equation}
t_1(\mathcal{D}_\Gamma) \leq t_1(\mathcal{C}_\Gamma) \text{vol}(H_2(\mathcal{E}_\Gamma)) \text{det}'(\alpha).
\end{equation}

Claim. As a function on $\mathcal{G}$, $\text{det}'(\alpha)$ is negligible.

Proof of Claim. We will find an upper bound for $\text{det}'(\alpha)$ by looking at its norm and its rank. By definition, $H_2(\mathcal{E}_\Gamma) = \text{coker}(1-b_{n+1})) = \mathbb{Z}[[\Pi]]/\text{Im}(1-b_{n+1})$ is a quotient of $\mathbb{Z}[[\Pi]]$. Besides, $H_1(\mathcal{C}_\Gamma)$ is a quotient of $\text{ker}(\partial_1, \Gamma)$. The connecting homomorphism $\alpha : H_2(\mathcal{E}_\Gamma) \rightarrow H_1(\mathcal{C}_\Gamma)$ is induced from
\[\tilde{\alpha} : \mathbb{Z}[[\Pi]] \rightarrow \mathbb{Z}[[\Pi]]^{n+1}, \quad x \rightarrow \tilde{\alpha}(x) = xr_{n+1},
\]
where $r_{n+1}$ is the last row of $\partial_2$. Hence $\|\alpha\| \leq \|\tilde{\alpha}\| \leq C$ for some constant $C$ not depending on $\Gamma$ (see Lemma 3.3). It follows that $\text{det}'(\alpha) \leq C \text{rk}H_2(\mathcal{E}_\Gamma)$. Since $\lim_{\mathcal{G} \ni \Gamma \rightarrow 1} \text{rk}H_2(\mathcal{E}_\Gamma)/[\Pi : \Gamma] = 0$ by Lemma 4.1, we see that $\text{det}'(\alpha)$ is negligible.  

Since $\text{det}'(\alpha)$ and $\text{vol}(H_2(\mathcal{E}_\Gamma))$ are negligible (by Claim and Lemma 4.1), from (27) we have
\[t_1(\mathcal{D}_\Gamma) \leq t_1(\mathcal{C}_\Gamma)
\]
and hence
\[\limsup_{\mathcal{G} \ni \Gamma \rightarrow 1} \frac{\ln t_1(\mathcal{D}_\Gamma)}{[\Pi : \Gamma]} \leq \limsup_{\mathcal{G} \ni \Gamma \rightarrow 1} \frac{\ln t_1(\mathcal{C}_\Gamma)}{[\Pi : \Gamma]} \leq \text{det}_\Pi(J) = \frac{\text{vol}(X)}{6\pi},
\]
where the second inequality follows from the proof in Section 4.4 and the last equality is Theorem 5.7. This completes the proof of Theorem 2.

4.7. Proof of Theorem 3. Let $X = S^3 \setminus N(K)$, where $N$ is a small tubular open neighborhood of $K$. Then $\partial X$ is a torus. We will use the notations of Section 4.4. We can further assume that $a_{n+1}$ is a simple closed curve on $\partial X$.

Suppose $\Gamma \in \mathcal{G}$. Let $p : X_\Gamma \to X$ be the covering map. Then $p^{-1}(\partial X)$ consists of tori, and $p^{-1}(a_{n+1})$ is a collection of simple closed curves $C_1, \ldots, C_I$ on $p^{-1}(\partial X)$. From the definition, $H_1(\tilde{X}_\Gamma, \mathbb{Z}) = H_1(X_\Gamma, \mathbb{Z})/U$, where $U$ is the subgroups of $H_1(X_\Gamma, \mathbb{Z})$ generated by $C_1, \ldots, C_I$. The curves $C_1, \ldots, C_I$ are made up from all the lifts of $a_{n+1}$.

The exact sequence (28) gives rise to a long exact sequence

$$\ldots \to H_1(K_\Gamma) \xrightarrow{\gamma} H_1(C_\Gamma) \to H_1(Q_\Gamma) \to H_0(K_\Gamma) \to \ldots$$

Recall that we use the identification $C_1 \equiv \mathbb{Z}[\Pi]^{n+1}$ via $C_1 = \bigoplus_{j=1}^{n+1} \mathbb{Z}[\Pi] \cdot \tilde{a}_j$. Correspodingly, the identification $K_1 \equiv \mathbb{Z}[\Pi]$ is via $K_1 = \mathbb{Z}[\Pi] \cdot \tilde{a}_{-n+1}$. Under these identifications, one has $U \equiv \text{Im}(\gamma)$.

Hence, from the exact sequence (28), one has the following exact sequence

$$0 \to H_1(\tilde{X}_\Gamma, \mathbb{Z}) \to H_1(Q_\Gamma) \to H_0(K_\Gamma) \to \ldots$$

Since $H_0(K_\Gamma)$ is a free abelian group, the above exact sequence implies that

$$\text{Torsion}(H_1(\tilde{X}_\Gamma, \mathbb{Z})) = \text{Torsion}(H_1(Q_\Gamma)).$$

Applying (29) to the map $J_r : \mathbb{Z}[\Gamma/\Pi]^n \to \mathbb{Z}[\Gamma/\Pi]^n$, we get

$$|\text{Torsion}(H_1(\tilde{X}_\Gamma, \mathbb{Z}))| = |\text{Torsion}(H_1(Q_\Gamma))| = \frac{\det'(J_r)}{\text{vol}(\ker(J_r)) \text{vol}(\text{Im}(J_r))} \leq \det'(J_r).$$

Here $\text{Im}(J_r) := (\text{Im}(J_r) \otimes \mathbb{Z} \otimes \mathbb{Q}) \cap \mathbb{Z}[\Gamma/\Pi]^n$.

Theorem 3.4 and Theorem 5.7 show

$$\lim_{\gamma \in \Gamma \setminus \mathbb{Z} \setminus 1} \frac{\ln |\text{Torsion}(H_1(\tilde{X}_\Gamma, \mathbb{Z}))|}{|\Pi : \Gamma|} \leq \lim_{\gamma \in \Gamma \setminus \mathbb{Z} \setminus 1} \frac{\ln \det'(J_r)}{|\Pi : \Gamma|} \leq \ln \det(\Pi) = \frac{\text{vol}(X)}{6\pi}.$$

This completes the proof of Theorem 3.

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SCHOOL OF MATHEMATICS, 686 CHERRY STREET, GEORGIA TECH, ATLANTA, GA 30332, USA

E-mail address: letu@math.gatech.edu