SOME RESULTS ON MAXIMUM LIKELIHOOD OF INCOMPLETE DATA: FINITE SAMPLE PROPERTIES, CONSISTENT SANDWICH ESTIMATOR OF COVARIANCE MATRIX AND RECURSIVE ALGORITHMS

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Abstract. This paper presents some new results on maximum likelihood of incomplete data. Finite sample properties of conditional observed information matrices (the conditional expectation of minus second derivatives of the log-likelihood given incomplete data) are established. In particular, they possess the same Loewner partial ordering properties as the expected information matrices do. In its new form, the observed Fisher information (OFI) simplifies conditional expectation of outer product of the complete-data score function appearing in the Louis (1982) general matrix formula. It verifies positive definiteness and consistency to the expected Fisher information as the sample size increases. Furthermore, it shows a resulting information loss presented in the incomplete data. For this reason, the OFI may not be the right (consistent and efficient) estimator to derive the standard error (SE) of maximum likelihood estimates (MLE) for incomplete data. A sandwich estimator of covariance matrix is developed to provide consistent and efficient estimates of SE. The proposed sandwich estimator coincides with the Huber sandwich estimator for model misspecification under complete data (Huber, 1967; Freedman, 2006; Little and Rubin, 2020). However, in contrast to the latter, the new estimator does not involve OFI which notably gives an appealing feature for application. Recursive algorithms for the MLE, the observed information and the sandwich estimator are presented. Application to parameter estimation of a regime switching conditional Markov jump process is considered to verify the results. The recursive equations for the inverse OFI generalize the algorithm of Hero and Fessler (1994). The simulation study confirms that the MLEs are accurate and consistent having asymptotic normality. The sandwich estimator produces standard error of the MLE close to their analytic values compared to those overestimated by the OFI.

1. Introduction

Large sample properties of maximum likelihood estimates (MLE) of statistical model parameters have been well documented in literature. See for e.g. Cramér (1946), Newey and McFadden (1994), and Van der Vaart (2000). The properties state that as the sample size increases, the MLE has asymptotic (multivariate) normal distribution with mean equal to the true parameter value whereas the covariance matrix is given by the inverse expected Fisher information of observed data. These fine properties of MLE were first shown by Fisher (1925) and later established rigorously, among others, by Cramér (1946). For unbiased estimators, the information matrix corresponds to the Cramér-Rao lower bound. See for e.g. p.489 of Cramér (1946) or p.2162 of Newey and McFadden (1994). For finite-sample application, the observed Fisher information has been widely used to find the standard error of the MLE with a good accuracy (Efron and Hinkley, 1978). Although the large sample properties have been well developed for complete data, some further studies are required to understand the finite/large sample properties of the MLE when applied to incomplete data.

The EM algorithm developed by Dempster et al.(1977) for maximum likelihood estimation under incomplete data has been widely used on various fields in literature. It is a robust and powerful tool for statistical analysis with missing data (Little and Rubin, 2020). The algorithm provides an iterative approach to obtain the maximum likelihood of a statistical model parameters in a way that avoids necessary regularity conditions on the log-likelihood function in terms of the existence of its
second derivative and invertibility of the corresponding Hessian matrix. Otherwise, if the regularity conditions are satisfied for each observation, one may employ the Newton-Raphson approach, i.e., the Fisher scoring method to find the MLE also iteratively, see for e.g. Osborne (1992), Hastie et al. (2009), and Takai (2020). There are two steps in the EM estimation. The first step, the \textit{E-step}, involves valuation under current parameter estimate of conditional expectation of the log-likelihood given the observed data, whereas the second step, the \textit{M-step}, deals with optimizing the conditional expectation. In each iteration, the algorithm increases the value of log-likelihood. The appealing monotone convergence property of the observed-data log-likelihood gives a higher degree of stability for the convergence of the EM algorithm (Wu, 1983). We refer to McLachlan and Krishnan (2008) for more details on recent developments, extensions and applications of the EM algorithm.

However, the EM algorithm only provides point estimates of parameters. Unlike the Fisher scoring method, it does not automatically produce the covariance matrix of the MLE. Additional steps are therefore required to find the covariance matrix. For finite and independent data, the expected information is replaced by the observed information matrix specified by the second derivative of the observed data log-likelihood function (Efron and Hinkley, 1978). In general, the derivative can be very difficult to evaluate directly. One of major contribution on the evaluation of the observed information was given by Louis (1982) in which a general matrix formula was proposed. Notice that the Louis’ formula involves conditional expectation of outer product of the complete-data score function which in general may be complicated to simplify. Meng and Rubin (1991) derived the covariance matrix using the fundamental identity given in Dempster et al. (1977) which relates the log-likelihood of observed data, the EM-update criterion function and the conditional expectation of conditional log-likelihood of complete data given its incomplete observation. In working out the information matrix, they applied in the M-step of the EM-algorithm a first-order Taylor approximation around current parameter estimate to the EM-update function. This approximation was first noted in Meilijison (1989) in an attempt to provide a fast improvement to the EM algorithm. The derived covariance matrix found in Meng and Rubin (1991) is given by the difference between the complete information matrix and an incomplete one, although in a slightly more complex form than Louis’ matrix formula. Based on the Taylor approximation discussed in Meilijison (1989), Jamshidian and Jennrich (2000) proposed a numerical differentiation method to evaluate the covariance matrix for the incomplete information. A rather direct calculation of observed information matrix was discussed in Oakes (1999) based on taking derivatives of the fundamental identity.

All above methods provide convergent and consistent MLE of the true parameter whose consistent limiting normal distribution has the mean equal to the true value, whereas the asymptotic covariance matrix is specified by the inverse observed Fisher information of the observed data. As a result, the covariance matrix estimator is determined by the inverse observed Fisher information. However, as will be shown in this paper, this turns out not necessarily to be the case for maximum likelihood estimation from incomplete data. In this case, the observed Fisher information may no longer be the right (consistent and efficient) estimator to derive the standard error of the MLE.

1.1. Organization of the paper. This paper is organized as follows. Section 2 discusses the problem of maximum likelihood estimation of incomplete data. Section 3 presents some preliminary results required to derive the main results and contributions of this paper presented in Section 4. An example on conditional Markov jump processes is presented in Section 5. A series of simulation studies based on Section 5 example are performed in Section 6 to verify the results of Sections 3-4. Section 7 concludes this paper. Some details of derivations are deferred to the Appendix.

2. Maximum likelihood estimation of incomplete data

Let $X$ and $Y$ be two random vectors defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $(\mathcal{X}, \mathcal{S})$ and $(\mathcal{Y}, \mathcal{T})$ the corresponding measurable state spaces of $X$ and $Y$ and by $T : \mathcal{X} \to \mathcal{Y}$ a many-to-one mapping from $\mathcal{X}$ to $\mathcal{Y}$. Suppose that the complete-data vector $x \in \mathcal{X}$ is only partially
observed through incomplete-data vector \( y = T(x) \) in \( \mathcal{Y} \). Assume that there exist probability density functions \( f_c(x|\theta) \) and \( f_o(y|\theta) \) corresponding to the complete data \( x \in \mathcal{X} \) and its incomplete observation \( y \in \mathcal{Y} \). Here \( \theta \) represents a vector of parameters on parameter space \( \Theta_d \), with \( d = |\theta| \), characterizing the distribution of \( X \). Define \( \mathcal{X}(y) = \{x \in \mathcal{X}: T(x) = y\} \in \mathcal{S} \). It follows that

\[
(1) \quad f_o(y|\theta) = \int_{\mathcal{X}(y)} f_c(x|\theta) \lambda(dx),
\]

where \( \lambda \) is a finite measure on \( \mathcal{S} \), absolutely continuous with respect to the probability distribution \( \mathbb{P} \circ T^{-1} \) with the density function \( f_c(x|\theta) \) (the Radon-Nikodym derivative). See e.g. Halmos and Savage (1949) for details. Following the identity \( (1) \), the conditional probability density function \( f(x|y,\theta) \) of the complete-data \( X \) given its incomplete observation \( Y \) is therefore given by

\[
(2) \quad f(x|y,\theta) = \frac{f_c(x|\theta)}{f_o(y|\theta)}.
\]

Suppose that a complete data \( X_1, \cdots, X_n \) were generated independently from the density \( f_c(x|\theta) \) under a pre-specified and unknown parameter value \( \theta^0 \in \Theta^d \). For the purpose of formality we assume that the parameter space \( \Theta^d \) is compact, and that each outcome \( X^k, k = 1, \cdots, n \) is only partially available given by an incomplete-data vector \( Y^k = T(X^k) \). We assume throughout that each respective observation \( Y^k \) of independent complete data \( X^k \) is also independent. Namely, if \( X^k \) is independent of \( X^\ell \), for \( k \neq \ell \), then \( Y^k \) is independent of \( Y^\ell \). The log-likelihood of \( Y = \bigcup_{k=1}^n Y^k \)

\[
\log f_o(Y|\theta) = \sum_{k=1}^n \log \int_{\mathcal{X}(y^k)} f_c(x^k|\theta) \lambda(dx^k),
\]

is generally difficult to evaluate to derive a maximum likelihood estimator \( \hat{\theta}^0 \) of the unknown true value \( \theta^0 \) and to obtain the corresponding standard error of \( \hat{\theta}^0 \). We are interested in estimating the true value \( \theta^0 \) based on the incomplete observation \( \{Y^k, k = 1, \cdots, n\} \) using the EM-criterion

\[
(3) \quad M_n(\theta) = \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ \log f_c(X^k|\theta)|Y^k, \theta^0 \right],
\]

where \( \mathbb{E}[\bullet|\theta^0] \) refers to the expectation operator associated with the underlying probability measure \( \mathbb{P}(\bullet|\theta^0) \) from which the complete-data \( \{X^k: 1 \leq k \leq n\} \) were generated under \( \theta^0 \). Note that the criterion function \( M_n(\theta) \) is defined slightly different from the common EM criterion function \( Q(\theta|\theta^0) \) used in the M-step of the EM algorithm to update a current estimate \( \theta^t \). Unlike the latter, \( M_n(\theta) \) emphasizes \( \theta \) as the parameter, whilst \( \theta^0 \) as a specified value. Furthermore, it is scaled by \( n^{-1} \).

The main results presented in this paper are derived based on the following assumption.

**Assumption 1.** Without loss of generality, we assume throughout the remaining of this paper that the log-likelihood function \( \log f_c(x|\theta) \) is twice continuously differentiable w.r.t \( \theta \) and for all \( \theta \in \Theta^d \),

\[
(4) \quad \mathbb{E} \left[ \frac{\partial^m \log f_c(x|\theta)}{\partial \theta^m} \bigg| Y = y, \theta^0 \right] = \int_{\mathcal{X}(y)} \frac{\partial^m \log f_c(x|\theta)}{\partial \theta^m} f(x|y,\theta) \lambda(dx) < \infty, \quad \text{for } m = 0, 1, 2.
\]

**Definition 1 (EM-estimator).** An estimator \( \hat{\theta}^0 \) which maximizes the EM-criterion \( M_n(\theta) \) over \( \theta \in \Theta^d \) is hereby referred to as the EM-estimator. It is equivalently defined as the solution of

\[
(5) \quad 0 = S_n(\theta) := \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ \frac{\partial \log f_c(X^k|\theta)}{\partial \theta} \bigg| Y^k, \theta^0 \right].
\]

Let \( S(\theta) := \mathbb{E} \left[ \frac{\partial \log f_c(X|\theta)}{\partial \theta} \bigg| \theta^0 \right] \), with \( S(\theta^0) = 0 \). The EM-estimator \( \hat{\theta}^0 \) can simply be written as

\[
(5) \quad \hat{\theta}^0 = S_n^{-1}(S(\theta^0)).
\]
In view of the EM algorithm, the E-step is concerned with the valuation of conditional expectation \( E \left[ \log f_c(X^k | \theta) | \theta^0 \right] \) given initial parameter value \( \theta^0 \). In the M-step, the update \( \hat{\theta}^0 \) is found by maximizing the criterion function \( M_n(\theta) \), i.e., \( \hat{\theta}^0 \) is set to be the solution of (4), given by (5).

**Remark 1.** At its convergence as the sample size \( n \) increases, \( \hat{\theta}^0 \) coincides with MLE \( \hat{\theta} \) satisfying

\[
0 = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ \frac{\partial \log f_c(X^k | \theta^0)}{\partial \theta} \right] | Y^k, \theta^0].
\]

Furthermore, on account that \( S(\theta^0) = 0 \), it follows from (6) that the MLE \( \hat{\theta} \) satisfies equation (7)

\[
\theta^0 = S_n^{-1}(S(\theta^0)).
\]

Thus, unlike the EM-estimator \( S_n^{-1}(S(\theta^0)) \), the MLE \( S_n^{-1}(S(\theta)) \) replaces \( \theta^0 \) by its estimate \( \hat{\theta} \).

By independence of incomplete data \( \{Y^k\} \), the law of large number shows that \( S_n(\theta) \xrightarrow{p} S(\theta) \) for any \( \theta \in \Theta^d \). As the result of this, we have asymptotically that \( \theta \approx S_n^{-1}(S(\theta)) \). Following the latter, we consider throughout the remaining of this paper the \( \sqrt{n} \)-consistent limiting normal distribution and the asymptotic covariance matrix of the sequence of the random variables

\[
\sqrt{n} (S_n^{-1}(S(\theta)) - \theta^0),
\]

where \( \theta \) is either the EM-estimator \( \hat{\theta}^0 \) or the MLE \( \hat{\theta} \). Note that by the imposed Assumption 1, \( S_n(\theta) \) and \( S(\theta) \) are continuously differentiable in \( \theta \). By continuity of the inverse of continuous function, \( S_n^{-1}(S(\theta)) \) is continuous in \( \theta \). Hence, if \( \theta \xrightarrow{p} \theta^0 \), by continuous mapping theorem and central limit theorem, \( \sqrt{n} (S_n^{-1}(S(\theta)) - \theta^0) \) converges to the same distribution as \( \sqrt{n} (S_n^{-1}(S(\theta^0)) - \theta^0) \) does.

### 3. Preliminary Results

By regularity condition (A1) and independence of incomplete data \( \{Y^k\} \), the law of large numbers yields the convergence in probability of \( M_n(\theta) \xrightarrow{p} M(\theta) = \mathbb{E} \left[ \log f_c(X | \theta) | \theta^0 \right] \) for every \( \theta \in \Theta^d \). The result below shows that the true value \( \theta^0 \) is the global maximum of \( M(\theta) \) with \( M'(\theta^0) = 0 \).

**Lemma 1.** For a given \( \theta^0 \in \Theta^d \), \( M(\theta) \leq M(\theta^0) \) for all \( \theta \in \Theta^d \) with \( M'(\theta^0) = 0 \).

**Proof.** The proof follows from concavity of log-function and application of Jensen’s inequality, i.e.,

\[
M(\theta) - M(\theta^0) = \mathbb{E} \left[ \log \left( \frac{f_c(X | \theta)}{f_c(X | \theta^0)} \right) \right] | \theta^0 \leq \mathbb{E} \left[ \frac{f_c(X | \theta)}{f_c(X | \theta^0)} \right] | \theta^0 = 0,
\]

where the last equality is due to the fact that the likelihood ratio \( \frac{f_c(X | \theta)}{f_c(X | \theta^0)} \) corresponds to the Radon-Nikodym derivative of changing the underlying probability measure from \( \mathbb{P}\{\cdot | \theta^0\} \) to \( \mathbb{P}\{\cdot | \theta\} \) which results in \( \mathbb{E} \left[ \frac{f_c(X | \theta)}{f_c(X | \theta^0)} \right] | \theta^0 = 1 \). \( M'(\theta^0) = 0 \) follows from (A1) and that \( \mathbb{E} \left[ \frac{\partial \log f_c(X | \theta)}{\partial \theta} | \theta \right] = 0 \).

The result of Lemma 1 verifies consistency condition for the \( M \)-estimator, see Theorem 5.7 in Van der Vaart (2000). By this theorem, it follows from the law of large numbers that \( \hat{\theta}^0 \xrightarrow{p} \theta^0 \).

The following result plays an important role in verifying consistency of the EM-estimator \( \hat{\theta}^0 \), deriving the \( \sqrt{n} \)-consistent limiting normal distribution of \( \sqrt{n} (\theta^0 - \theta^0) \), the asymptotic covariance matrix \( \Sigma(\theta^0) \) and the corresponding sandwich estimator \( \Sigma_n(\hat{\theta}^0) \), and the recursive equations for \( \hat{\theta}^0 \).

**Proposition 1 (EM-Gradient scoring method).** Under Assumption 7 and by smoothness of the complete-data log-likelihood function \( \theta \rightarrow \log f_c(x | \theta) \), there exists \( \theta^0 \leq \theta \leq \hat{\theta}^0 \) in \( \Theta^d \) such that

\[
\hat{\theta}^0 = \theta^0 + \left( \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ - \frac{\partial^2 \log f_c(X^k | \theta)}{\partial \theta^2} | Y^k, \theta^0 \right] \right)^{-1} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ \frac{\partial \log f_c(X^k | \theta^0)}{\partial \theta} | Y^k, \theta^0 \right].
\]
Theorem 2. Conditional probability density (2) and the above identities, we derive the following result.\[ \frac{\partial}{\partial \theta} \left[ S_n(\theta) \right] = S_n(\theta) + S'_n(\overrightarrow{\theta})(\overrightarrow{\theta} - \theta), \]

from which the EM-estimator \( \theta^0 \) follows under (A1) after solving the equation for \( \theta^0 \).

Remark 2. In view of (8), the function \( S_n^{-1}(\theta) \) can be set without loss of generality as
\[ S_n^{-1}(\theta) = \theta + J_n^{-1}(\theta)S_n(\theta), \]
where \( J_n(\theta) \) is the observed information in \( (\theta) \), with \( \theta = \theta^0 \), and \( S_n(\theta) \) is defined in (4). Note that (9) leads to \( S_n^{-1}(\theta) = \theta \) if \( \theta^0 \) is replaced either by the MLE \( \theta \) or by the EM-estimator \( \theta^0 \).

The result below will be used to derive the observed Fisher information of incomplete data \( \{Y_k\} \).

**Lemma 2.** For any \( \theta \in \Theta_d \) and a given incomplete data \( y \in \mathcal{Y} \), the following equality holds
\[ \mathbb{E} \left[ \frac{\partial \log f_c(X|\theta)}{\partial \theta} \left| Y = y, \theta \right. \right] = \frac{\partial \log f_o(y|\theta)}{\partial \theta}. \]

Furthermore, since \( \theta^0 \rightarrow \theta^0 \), at its convergence \( \theta^0 \) coincides with the solution of equation:
\[ \frac{\partial \log f_o(y|\theta)}{\partial \theta} = \sum_{k=1}^{n} \frac{\partial \log f_o(y_k|\theta)}{\partial \theta} = 0 \quad \text{for any } y \in \mathcal{Y}. \]

Proof. See Appendix A for details of the proof and derivation.

To show that the sandwich estimator is less than the inverse of the incomplete-data observed Fisher information in the sense of Loewner partial matrix ordering, the result below is required.

**Lemma 3.** For any \( \theta \in \Theta_d \) and a given incomplete data \( y \in \mathcal{Y} \), the following equality holds
\[ \mathbb{E} \left[ \frac{1}{f_c(X|\theta)} \frac{\partial^2 f_c(X|\theta)}{\partial \theta^2} \left| Y = y, \theta \right. \right] = \mathbb{E} \left[ \frac{1}{f_o(y|\theta)} \frac{\partial^2 f_o(y|\theta)}{\partial \theta^2} \left| Y = y, \theta \right. \right]. \]

Proof. See Appendix B for details of the proof and derivation.

The following results are required to establish the fact that the sandwich estimator of covariance matrix and the observed Fisher information of incomplete data are both positive definite.

**Theorem 1.** For any \( \theta \in \Theta_d \) and a given incomplete data \( y \in \mathcal{Y} \), it holds true that
\[ \mathbb{E} \left[ \frac{\partial \log f(X|\theta)}{\partial \theta} \left| Y = y, \theta \right. \right] = 0. \]

Furthermore, the incomplete-data conditional information is positive definite satisfying
\[ \mathbb{E} \left[ - \frac{\partial^2 \log f(X|\theta)}{\partial \theta^2} \left| Y = y, \theta \right. \right] = \mathbb{E} \left[ \left( \frac{\partial \log f(X|\theta)}{\partial \theta} \right) \left( \frac{\partial \log f(X|\theta)}{\partial \theta} \right)^\top \left| Y = y, \theta \right. \right]. \]

Proof. See Appendix C for details of the proof and derivation.

Taking expectation w.r.t to probability measure \( \mathbb{P}(\bullet|\theta) \) on both sides of the identity leads to the fact that \( - \frac{\partial^2 \log f(X|\theta)}{\partial \theta^2} \) is the observed Fisher information, see Schervish (1995). By applying the conditional probability density \( \mathcal{E} \) and the above identities, we derive the following result.

**Theorem 2 (Loss of information in incomplete data).** For any \( \theta \in \Theta_d \) and \( y \in \mathcal{Y} \),
\[ J_c(\theta) := \mathbb{E} \left[ - \frac{\partial^2 \log f_c(X|\theta)}{\partial \theta^2} \left| Y = y, \theta \right. \right] \geq J_p(\theta) := - \frac{\partial^2 \log f_o(y|\theta)}{\partial \theta^2}. \]

Proof. See Appendix D for details of the proof and derivation.

As far as the existing literatures are concerned, see for e.g. Schervish (1995) and McLachlan and Krishnan (2008), the above conditional information matrices have not much been utilized.
From the result of Theorem 2 we deduce the following inequality corresponding to the resulting information loss presented in the incomplete-data $Y$. The inequality was discussed in Orchard and Woodbury (1972). See also Blahut (1987), Schervish (1995), and McLachlan and Krishnan (2008).

\[ I_c(\theta) := \mathbb{E} \left[ - \frac{\partial^2 \log f_c(X|\theta)}{\partial \theta^2} \right] \geq I_p(\theta) := \mathbb{E} \left[ - \frac{\partial^2 \log f_o(Y|\theta)}{\partial \theta^2} \right]. \]

**Theorem 3.** If the observed Fisher information $J_p(\theta)$ is positive definite, then for $y \in \mathcal{Y}, \theta \in \Theta^d$,

\[ \mathbb{E} \left[ - \frac{\partial^2 \log f_c(X|\theta)}{\partial \theta^2} \right] Y = y, \theta \geq \text{Var} \left[ \frac{\partial \log f_c(X|\theta)}{\partial \theta} \right] Y = y, \theta. \]

In particular, for an estimator $\hat{\theta}^0$ solving (11) at the convergence, it follows for any $y \in \mathcal{Y}$ that

\[ \mathbb{E} \left[ - \frac{\partial^2 \log f_c(X|\hat{\theta}^0)}{\partial \theta^2} \right] Y = y, \hat{\theta}^0 > \mathbb{E} \left[ \left( \frac{\partial \log f_c(X|\hat{\theta}^0)}{\partial \theta} \right) \left( \frac{\partial \log f_c(X|\hat{\theta}^0)}{\partial \theta} \right)^\top \right] Y = y, \hat{\theta}^0. \]

**Proof:** See Appendix E for details of the proof and derivation.

The above theorem complements existing result on the equivalence between unconditional variance of complete-data score function and the expected Fisher information, see Schervish (1995).

The following theorem plays an important role in deriving the results presented in this paper.

**Theorem 4 (Theorem 2.7 (iv) of Van der Vaart (2000)).** Denote by $\|x - y\|$ a distance between two vectors $x, y \in \mathbb{R}^d$, with $d \geq 1$. Let $X_n, X$ and $Y_n$ be $R^d$—valued random variables.

If $X_n \overset{d}{\longrightarrow} X$ and $\|X_n - Y_n\| \overset{p}{\longrightarrow} 0$, then $Y_n \overset{d}{\longrightarrow} X$.

4. THE MAIN RESULTS

The main results of this paper are given in terms of the observed information matrices below.

**Theorem 5.** 

(i) The information matrices $I_p(\theta)$ and $I_c(\theta)$ (14) are consistently estimated by

\[ J_p(\theta_i, \theta_j) = - \frac{1}{n} \sum_{k=1}^{n} \frac{\partial^2 \log f_o(Y_k|\theta)}{\partial \theta_i \partial \theta_j} - \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ \frac{\partial \log f_c(X_k|\theta)}{\partial \theta_i} \right] \frac{\partial \log f_c(X_k|\theta)}{\partial \theta_j} \]

\[ + \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ \frac{\partial \log f_c(X_k|\theta)}{\partial \theta_i} \right] Y_k, \theta \mathbb{E} \left[ \frac{\partial \log f_c(X_k|\theta)}{\partial \theta_j} \right] Y_k, \theta, \]

and

\[ J_c(\theta_i, \theta_j) := \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ - \frac{\partial^2 \log f_c(X_k|\theta)}{\partial \theta_i \partial \theta_j} \right] Y_k, \theta. \]

(ii) Furthermore, the two information matrices $J_p(\hat{\theta}^0)$ and $J_c(\hat{\theta}^0)$ are positive definite.

**Proof:** See Appendix E for details of the proof and derivation.

Remark 3. The observed Fisher information $J_p(\theta^0)$ (16) takes a slightly different form than the Louis (1982) general matrix formula. The main differences with the latter is that it simplifies the calculation of conditional expectation of the outer product of the complete-data score function appeared in the Louis’ matrix formula. And most notably, it directly verifies the asymptotic consistency of $J_p(\theta^0)$ to the incomplete-data Fisher information $I_p(\theta^0)$ as the sample size $n$ increases.
Definition 2 (Finite-sample sandwich estimator). Let $\widehat{\theta}^0$ be the EM-estimator \cite{5} and define
\begin{equation}
\Sigma_n(\widehat{\theta}^0) = J_c^{-1}(\widehat{\theta}^0)\Lambda_n(\widehat{\theta}^0)J_c^{-1}(\widehat{\theta}^0),
\end{equation}
a finite-sample consistent sandwich estimator of covariance matrix of $\widehat{\theta}^0$ with $\Lambda_n(\theta)$ specified by
\begin{equation}
\Lambda_n(\theta) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ \left( \frac{\partial \log f_c(X_k|\theta)}{\partial \theta} \right) \left( \frac{\partial \log f_c(X_k|\theta)}{\partial \theta} \right)^\top \right] Y^k, \theta).
\end{equation}

By the regularity condition of Assumption (A1), we assume that $J_c(\theta)$ and $\Lambda_n(\theta)$ are continuous.

Notice that the proposed sandwich estimator \cite{18, 19} coincides with the Huber sandwich estimator $V_n(\widehat{\theta}) = J_p^{-1}(\widehat{\theta})K_n(\widehat{\theta})J_p^{-1}(\widehat{\theta})$, with $K_n(\theta) = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{\partial \log f_c(Y_k|\theta)}{\partial \theta} \right) \left( \frac{\partial \log f_c(Y_k|\theta)}{\partial \theta} \right)^\top$ for model misspecification when the complete data $X$ is fully observed, namely when $T$ is an one-to-one mapping, i.e., $Y = T(X) = X$. See e.g. Huber (1967), Freedman (2006), Little and Rubin (2020). In contrast to the Huber sandwich estimator, \cite{18} does not involve the observed Fisher information $J_p(\theta^0)$ which may be degenerate and therefore more difficult to evaluate and to invert than $J_c(\theta^0)$.

**Theorem 6 ($\sqrt{n}$-consistent limiting normal distribution of $\widehat{\theta}^0$).** Under Assumption \cite{4}

\begin{enumerate}[(i)]
\item The EM estimator $\widehat{\theta}^0$ \cite{5} specifies a consistent estimate of $\theta^0$, i.e., $\widehat{\theta}^0 \xrightarrow{P} \theta^0$.
\item Furthermore, $\widehat{\theta}^0$ has the $\sqrt{n}$-consistent limiting normal distribution:
\begin{equation*}
\Sigma_n^{-1/2}(\widehat{\theta}^0) \sqrt{n}(\widehat{\theta}^0 - \theta^0) \xrightarrow{d} N(0, I).
\end{equation*}
\end{enumerate}

**Proof:** [(i)] Consistency of the MLE $\widehat{\theta}^0$ can be verified in two different ways. First, by \cite{A1} and bounded convergence theorem, it follows that $S_n(\theta) \xrightarrow{P} S(\theta)$ for every $\theta \in \Theta^d$. On account that $\widehat{\theta}^0$ solves the equation $S_n(\widehat{\theta}^0) = 0$ and that $\theta^0$ is the global maximum of $M(\theta)$ satisfying $S(\theta^0) = 0$, by writing $\widehat{\theta}^0 = S_n^{-1}(S(\theta^0))$ it follows that $\widehat{\theta}^0 \xrightarrow{P} \theta^0$. Secondly, the convergence $\widehat{\theta}^0 \xrightarrow{P} \theta^0$ follows from \cite{8}, \cite{A1} and bounded convergence theorem. To be more precise, by the regularity condition (A1) and the convexity of absolute function \cite{9}, it follows from Jensen’s inequality that $\frac{1}{n} \sum_{k=1}^{n} \left| \frac{\partial \log f_c(Y_k|\theta^0)}{\partial \theta} \right| < \infty$, and hence by identity \cite{10} and independence of $\{Y_k\}_{k \geq 1}$, $\frac{1}{n} \sum_{k=1}^{n} \left| \frac{\partial \log f_c(Y_k|\theta^0)}{\partial \theta} \right| \xrightarrow{P} \mathbb{E} \left[ \left| \frac{\partial \log f_c(Y_k|\theta^0)}{\partial \theta} \right| \right] = 0$ which results following \cite{8} in consistency of $\widehat{\theta}^0$.

[(ii)] Asymptotic normality of the EM-estimator $\widehat{\theta}^0$ is established using identity \cite{5}, consisten- cy of $\widehat{\theta}^0$, \cite{15} and Theorem 4. Namely, since $\left\| \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ \left( \frac{\partial \log f_c(X_k|\theta)}{\partial \theta} \right) \right] Y^k, \theta \right\| \xrightarrow{P} 0$ and that by the central limit theorem $\xrightarrow{d} N(0, \Lambda(\theta^0))$, with $\Lambda_n(\theta) \xrightarrow{P} \Lambda(\theta)$, Theorem 4 yields $\sqrt{n}(\widehat{\theta}^0 - \theta^0) = J_c^{-1}(\theta^0)\sqrt{n}(S_n(\theta^0) - S(\theta^0)) \xrightarrow{d} N(0, \Sigma(\theta^0))$, with $\Sigma(\theta^0) = I_c^{-1}(\theta^0)\Lambda(\theta^0)I_c^{-1}(\theta^0)$, where the convergence is due to \cite{18}, \cite{19} and by independence of $\{Y_k\}$.

The result of Theorem 3 shows that the sandwich estimator $\Sigma_n(\widehat{\theta}^0)$ has smaller standard error of the EM-estimator $\widehat{\theta}^0$ than those given by the inverse observed Fisher information $J_p^{-1}(\widehat{\theta}^0)$.

**Corollary 1.** The sandwich estimator $\Sigma_n(\widehat{\theta}^0)$ satisfies the Loewner partial ordering
\begin{equation*}
0 < \Sigma_n(\widehat{\theta}^0) \leq J_p^{-1}(\widehat{\theta}^0),
\end{equation*}
i.e., the covariance matrix $\Sigma(\theta^0)$ is smaller than the Cramér-Rao lower bound $I_p^{-1}(\theta^0)$.

**Proof:** The proof follows from applying \cite{15} and \cite{13} to \cite{18} on account of Theorem 7.2.1 on p. 438 of Horn and Johnson (2013) that every positive definite matrix is invertible and the inverse itself is positive definite. The last statement is due to the consistency of $\widehat{\theta}^0$, $\Sigma_n(\theta)$ and $J_p(\theta)$. ■
4.1. Recursive algorithm for obtaining the EM estimator \( \hat{\theta}_0 \). The algorithm is derived from [5] whereby \( \theta_0 \) represents a current estimate \( \hat{\theta}_0 \), whereas \( \theta_0 \) gives the update \( \hat{\theta}_{t+1} \). Therefore,

\[
(20) \quad \hat{\theta}_{t+1} = S_n^{-1}(S(\hat{\theta}_t)).
\]

Equivalently in terms of [8] the recursive estimator \( \hat{\theta}_t \) can be conveniently written as follows by replacing \( \theta \) by \( \theta_0 \). The replacement is justified by the fact that \( \theta_0 \leq \theta \leq \hat{\theta}_0 \) and that \( \theta_0 \xrightarrow{P} \theta_0 \).

\[
(21) \quad \hat{\theta}_{t+1} = \hat{\theta}_t + J_c^{-1}(\hat{\theta}_t)S_n(\hat{\theta}_t),
\]

The recursive equation corresponds to the EM-Gradient algorithm proposed by Lange (1995). The recursion provides the fastest Newton-Raphson algorithm for solving the M-step iteratively which has quadratic convergence compared to the linear convergence in the EM algorithm. See Wu (1983) for the convergence properties of the EM algorithm. The EM-Gradient algorithm [21] yields an estimate for \( \hat{\theta}_0 \) which serves as the lower bound to that of given by the incomplete-data Fisher scoring method, see for e.g. Osborne (1992), McLachlan and Krishnan (2008) and Takai (2020), whose asymptotic covariance matrix is given by the inverse expected Fisher information \( I_p^{-1}(\theta_0) \).

It follows from (21) that at its convergence, i.e. when \( ||\hat{\theta}_{t+1} - \hat{\theta}_t|| < \epsilon \), \( \hat{\theta}_t \) converges to the MLE.

4.2. Recursive algorithms for \( J_p^{-1}(\theta_0) \) and \( \Sigma_n(\theta_0) \). In their recent work, Hero and Fessler (1994) proposed a recursive equation for the valuation of the inverse of the expected incomplete-data Fisher information \( I_p(\theta_0) \). The method uses the information matrix \( I_p(\theta_0) \) and the inverse of the expected complete-data Fisher information \( I_c(\theta_0) \). Thus, the method avoids taking the inverse of \( I_p(\theta_0) \) which may be more difficult to invert than \( I_c(\theta_0) \). However, the result is not immediately applicable for incomplete data in general since the expected information matrices \( I_p(\theta_0) \) and \( I_c(\theta_0) \) may not be available in closed form. To overcome this problem, we generalize their result for the inverse observed Fisher information \( J_p^{-1}(\theta) \) based on the observed information matrices \( J_p(\theta) \) [16] and \( J_c(\theta_0) \) [17], which were not discussed in Hero and Fessler (1994). In addition, recursive equation for the sandwich estimator \( \Sigma_n(\theta_0) \) [18] of the covariance matrix \( \Sigma(\theta_0) \) is also presented.

**Theorem 7.** Let \( \{\Psi_t\}_{t \geq 0} \) be a sequence of \((d \times d)\)-matrices, with \( d = |\theta_0| \), and \( \Psi_0 = 0 \) satisfying

\[
(22) \quad \Psi_{t+1} = A(\theta_0)\Psi_t + B(\theta_0),
\]
for a given \( \Lambda(\bar{\theta}) \), \( \Sigma(\bar{\theta}) \) \( \in \mathbb{R}^{d \times d} \). Then, \( \{\Psi_\ell\}_{\ell \geq 1} \) converges with root of convergence \( \rho(\Lambda(\bar{\theta})) \) to

(i) \( \Psi = J_p^{-1}(\bar{\theta}) \) for specified \( \Lambda(\bar{\theta}) = [I - J_c^{-1}(\bar{\theta})]J_p(\bar{\theta}) \) and \( \Sigma(\bar{\theta}) = J_c^{-1}(\bar{\theta}) \).

(ii) \( \Psi = \Sigma^{-1}(\bar{\theta}) \) for specified \( \Lambda(\bar{\theta}) = [I - \Lambda_n(\bar{\theta})]I_c^{-1}(\bar{\theta}) \) and \( \Sigma(\bar{\theta}) = J_c(\bar{\theta}) \).

Furthermore, the convergence is monotone in the sense that

\[ \Psi_\ell < \Psi_{\ell+1} \leq \Psi \quad \text{for } \ell = 0, 1, \ldots. \]

**Proof:** [(i)] Since by Theorem 5.6.16 of Horn and Johnson (2013), all eigenvalues of \( I - J_c^{-1}(\bar{\theta})J_p(\bar{\theta}) \) are positive and strictly less than one. See Corollary 1.3.4 in Horn and Johnson (2013). By Corollary 5.6.16 of Horn and Johnson (2013), it leads to

\[ \Psi = [I - J_c^{-1}(\bar{\theta})(J_c(\bar{\theta}) - J_p(\bar{\theta}))]^{-1}J_c^{-1}(\bar{\theta}) = \left( \sum_{\ell=0}^{\infty} [I - J_c^{-1}(\bar{\theta})J_p(\bar{\theta})]\right)^{-1}J_c^{-1}(\bar{\theta}). \]

Since all eigenvalues of \( I - J_c^{-1}(\bar{\theta})J_p(\bar{\theta}) \) are positive and strictly less than one, we then obtain

\[ \Psi_{\ell+1} - \Psi = [I - J_c^{-1}(\bar{\theta})J_p(\bar{\theta})] \Psi_\ell + J_c^{-1}(\bar{\theta}) - \Psi = \left[D - J_c^{-1}(\bar{\theta})J_p(\bar{\theta})]\right] \Psi_\ell \to 0 \quad \text{as } \ell \to \infty, \]

with root of convergence factor given by the maximum absolute eigenvalues of \( I - J_c^{-1}(\bar{\theta})J_p(\bar{\theta}) \). Thus, \( \{\Psi_\ell\}_{\ell \geq 1} \) converges to \( \Psi \) as \( \ell \to \infty \). To show the convergence is monotone, recall that

\[ \Psi_{\ell+1} - \Psi_\ell = [I - J_c^{-1}(\bar{\theta})J_p(\bar{\theta})] \Psi_\ell - \Psi_{\ell-1} = [I - J_c^{-1}(\bar{\theta})J_p(\bar{\theta})] (\Psi_1 - \Psi_0) \]

\[ = [I - J_c^{-1}(\bar{\theta})J_p(\bar{\theta})] (J_c^{-1}(\bar{\theta}) > 0, \]

which is positive definite by Corollary 7.7.4(a) of Horn and Johnson (2013) and [15].

The assertion of the second claim [(ii)] follows from applying similar lines of arguments. \( \blacksquare \)

5. **Example: Regime-switching conditional Markov jump process**

To exemplify the results of Section 4, we consider maximum likelihood estimation of the distribution parameters of a regime-switching conditional Markov jump process (RSCMJP) \( X = \{X_t : t \geq 0\} \) introduced recently in Surya (2021). RSCMJP is a complex stochastic model which can be used to describe a sequence of events where the occurrence of an event depends not only on the current state, but also on the current time and observation of past events. It may be considered as a natural
generalization of the Markov jump process (for e.g. Norris, 2009) and has distributional equivalent stochastic representation with the general mixture of Markov jump processes proposed in Frydman and Surya (2021). It allows the process to switch the transition rates from a finite number of transition matrices \( Q_m = \{q_{xy,m} : x, y \in S\}, m = 1, \cdots, M \) when it moves from any phase \( x \) of the state space \( S = \{1, \cdots, p\} \), with \( 1 < p \in \mathbb{N} \), to another state \( y \in S \) with switching probability depending on the current state, time and its past information. The latter summarizes observable quantities of \( X \) concerning the number of transitions \( N_{xy} \) between states \( (x, y) \in S \), occupation time \( T_x \) in each state \( x \in S \), and initial state indicator \( B_x \) having value one if \( X_0 = x \), or zero otherwise. Figure 1 depicts the transition diagram of a RSCMJ. Beside the transition matrices \( \{Q_m : m = 1, \cdots, M\} \), the distribution of \( X \) is characterized by an initial probability \( \alpha \) with \( \alpha_x = \mathbb{P}\{X_0 = x\} \) satisfying \( \sum_{x=1}^{p} \alpha_x = 1 \), and regime-switching probability \( \phi_{x,m} = \mathbb{P}\{X_0 = X_0(m) | X_0 = x\} \) which is the probability of making an initial transition w.r.t. a Markov process \( \Phi = \{X_t(m) : t \geq 0\} \), with transition matrix \( Q_m \), starting from a state \( x \in S \). As the underlying Markov processes \( \{X(m), Q_m\}, m = 1, \cdots, M \) are defined on the same state space \( S \), there is a hidden information \( \Phi \) regarding which underlying Markov process that drives the movement of \( X \) when it makes a jump from one state to another. The random variable \( \Phi \) has a multinomial distribution with \( \mathbb{P}\{\Phi = m\} = p_m \) and \( \sum_{m=1}^{M} p_m = 1 \). The pair \( \{X^k, \Phi^k\} \) accounts for complete observation of \( k \)th paths. Define \( \Phi_{k,m} = 1_{\{\Phi^k = m\}} \). Consider \( n \) independent paths \( \{X^k : k = 1, \cdots, n\} \) of \( X \) (generated data or real dataset). See Surya (2021) for more details on algorithm of generating sample paths of RSCMJ. Following Frydman and Surya (2021), the log-likelihood of all realized paths \( \mathcal{X} = \bigcup_{k=1}^{n} X^k \) of \( X \) is

\[
\log f_o(\mathcal{X}|\theta) = \sum_{k=1}^{n} \sum_{x=1}^{p} B_x^k \log \alpha_x + \sum_{k=1}^{n} \log \sum_{m=1}^{M} f_c(X^k, \Phi^k = m|\theta)
\]

where \( f_o(X^k, \Phi^k|\theta) \) is the log-likelihood of a complete information \( \{(X^k, \Phi^k) : k = 1, \cdots, K\} \),

\[
\log f_c(X^k, \Phi^k|\theta) = \sum_{m=1}^{M} \Phi_{k,m} \left( \sum_{x=1}^{p} B_x^k \log \phi_{x,m} + \sum_{x=1}^{p} \sum_{y \neq x,y=1}^{p} \left[ N_{xy}^k \log q_{xy,m} - q_{xy,m}T_x^k \right] \right)
\]

where \( \{\phi_{x,m} : x \in S, 1 \leq m \leq M\} \) are subject to the constraint \( \sum_{m=1}^{M} \phi_{x,m} = 1 \) for each \( x \in S \). Note that \( N_{xy}^k, T_x^k \) and \( B_x^k \) are the respective observable quantities for the \( k \)th sample path \( X^k \). It follows from the log-likelihood \( \log f_c(X^k|\theta) \) of \( \Phi \) that \( \alpha_x \) can be estimated separately by \( \hat{\alpha}_x = \frac{1}{n} \sum_{k=1}^{n} B_x^k \).

As it does not involve the hidden information \( \Phi \), it is therefore excluded from estimation of the other parameters \( \theta^0 = (\phi^0_{x,m}, q^0_{xy,m} : x, y \in S, 1 \leq m \leq M) \). We are interested in applying the EM-Gradient algorithm to derive an explicit form of the MLE \( \hat{\theta}^0 \) and to evaluate it iteratively.

5.1. Finding the EM-estimator \( \hat{\theta}^0 \) and MLE \( \hat{\theta} \). From the complete-data log-likelihood \( \log f_c(X^k, \Phi^k|\theta) \) and on account of the constraint \( \sum_{m=1}^{M} \phi_{x,m} = 1 \) for each \( x \in S \), we rewrite

\[
\sum_{m=1}^{M} \sum_{x=1}^{p} \Phi_{k,m} B_x^k \log \phi_{x,m} = \sum_{m=1}^{M-1} \sum_{x=1}^{p} \Phi_{k,m} B_x^k \log \phi_{x,m} + \sum_{x=1}^{p} \Phi_{k,M} B_x^k \log \phi_{x,M}.
\]

The corresponding score functions for \( \phi_{x,m} \) and \( q_{xy,m} \) and their derivatives w.r.t each variable are

\[
\frac{\partial \log f_c(X^k, \Phi^k|\theta)}{\partial \phi_{x,m}} = \frac{\Phi_{k,m} B_x^k}{\phi_{x,m}}, \quad \frac{\partial^2 \log f_c(X^k, \Phi^k|\theta)}{\partial \phi_{x,m} \partial \phi_{y,\ell}} = \frac{\Phi_{k,\ell} \delta_{m}(\ell)}{\phi_{y,\ell}^2} + \frac{\Phi_{k,M} \delta_{y,\ell}(\ell)}{\phi_{y,\ell}^2} B_y^k \delta_x(y),
\]

\[
\frac{\partial \log f_c(X^k, \Phi^k|\theta)}{\partial q_{xy,m}} = \frac{\Phi_{k,m} q_{xy,m}^{k}}{q_{xy,m}}, \quad \frac{\partial^2 \log f_c(X^k, \Phi^k|\theta)}{\partial q_{xy,m} \partial q_{rv,\ell}} = \frac{\Phi_{k,\ell} N_{ry}^k}{q_{xy,m} q_{xy,m}} \delta_x(y) \delta(y) \delta_{m}(\ell),
\]
where \(-\frac{\partial^2 \log f_c(X^k, \Phi^k|\theta)}{\partial \phi_{x,m} \partial q_{rv,\ell}} = 0\), \(\Psi_{x,m|M} = \Phi_{k,m} - \frac{\phi_{x,m}}{\phi_{x,M}} \Phi_{k,M}\), \(A_{xy,m}^k = N_{xy,m}T_x^k\), and \(\delta_x(y)\) defines

\[
\delta_x(y) = \begin{cases} 
1, & \text{if } y = x \\
0, & \text{if } y \neq x.
\end{cases}
\]

Using Proposition C1 in Frydman and Surya (2021), one can verify that the score functions of \(\phi_{x,m}\) and \(q_{xy,m}\) have zero mean, and that the mean of minus second derivatives equal to the covariance of the respective score functions. For the EM-estimator \(\hat{\theta}\), define \(\hat{\Phi}_{k,m}(\theta) = \mathbb{E}(\Phi_{k,m}|X, \theta)\), i.e.,

\[
\hat{\Phi}_{k,m}(\theta) = \frac{f_c(X^k, \Phi^k = m|\theta)}{\sum_{m=1}^M f_c(X^k, \Phi^k = m|\theta)}.
\]

One can verify from the log-likelihood \(\log f_c(X^k, \Phi^k = m)\) that the function \(\theta \rightarrow \hat{\Phi}_{k,m}(\theta)\) is continuous and nonzero. Define the quantities \(\bar{B}_x = \sum_{k=1}^n B_x^k\) and \(\bar{B}_{x,m}(\theta) = \sum_{k=1}^n \hat{\Phi}_{k,m}(\theta)B_x^k\). Similarly defined, \(\hat{N}_{xy,m}(\theta) = \sum_{k=1}^n \hat{\Phi}_{k,m}(\theta)N_{xy,m}^k\) and \(\hat{T}_{x,m}(\theta) = \sum_{k=1}^n \hat{\Phi}_{k,m}(\theta)T_x^k\). Then, from (4) we obtain

\[
\frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ \frac{\partial \log f_c(X^k, \Phi^k|\theta)}{\partial \phi_{x,m}} \right] X^k, \theta^0 = \frac{\hat{B}_{x,m}(\theta^0)}{n \phi_{x,m}} - \frac{\hat{B}_{x,m}(\theta)}{n \phi_{x,M}}, \quad 1 \leq m \leq M - 1,
\]

\[
\frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ \frac{\partial \log f_c(X^k, \Phi^k|\theta)}{\partial q_{xy,m}} \right] X^k, \theta^0 = \frac{\hat{N}_{xy,m}(\theta^0)}{n q_{xy,m}} - \frac{\hat{T}_{x,m}(\theta^0)}{n}, \quad 1 \leq m \leq M,
\]

for \(x, y \in S\). On account that \(\hat{\phi}_{x,M} = 1 - \sum_{m=1}^{M-1} \hat{\phi}_{x,m}\) and \(\bar{B}_{x,M}(\theta^0) = \bar{B}_x - \sum_{m=1}^{M-1} \bar{B}_{x,m}(\theta^0)\), the EM-estimator \(\hat{\theta}\) is found following (4) as the root of the conditional score functions, i.e.,

\[
\hat{\theta}^0 = \frac{\hat{B}_{x,m}(\theta^0)}{\bar{B}_x} \quad \text{and} \quad \hat{q}_{xy,m}^0 = \frac{\hat{N}_{xy,m}(\theta^0)}{\hat{T}_{x,m}(\theta^0)}.
\]

These are the components of the EM-estimator \(\hat{\theta}^0 = S_n^{-1}(S(\theta^0))\) which are continuous in \(\theta^0\). Subsequently, by (7) the MLE \(\hat{\theta}\) is given by replacing the true value \(\theta^0\) by the estimator \(\hat{\theta}\), s.t.

\[
\hat{\phi}_{x,m} = \frac{\hat{B}_{x,m}(\theta)}{\bar{B}_x} \quad \text{and} \quad \hat{q}_{xy,m} = \frac{\hat{N}_{xy,m}(\theta)}{\hat{T}_{x,m}(\theta)}.
\]

The MLEs \(\hat{\theta} = \{\hat{\phi}_{x,m} : x \in S, 1 \leq m \leq M - 1\} \cup \{\hat{q}_{xy,m} : x, y \in S, y \neq x, 1 \leq m \leq M\}\) coincide with those given in Frydman and Surya (2021) in which the consistency of \(\hat{\theta}^0\) was proved.

**Remark 5.** For a given true value \(\theta^0\), the EM-estimator \(\hat{\theta}^0\) and \(\hat{q}_{xy,m}^0\) are calculated using (23) and their standard error are computed based on the sandwich estimator (18) and (19), or iteratively using (22). If \(\theta^0\) is unknown, \(\hat{\phi}_{x,m}^0\) and \(\hat{q}_{xy,m}^0\) are estimated using (21), or the EM-algorithm

\[
\hat{\phi}_{x,m}^{(\ell+1)} = \frac{\hat{B}_{x,m}(\theta^\ell)}{\bar{B}_x} \quad \text{and} \quad \hat{q}_{xy,m}^{(\ell+1)} = \frac{\hat{N}_{xy,m}(\theta^\ell)}{\hat{T}_{x,m}(\theta^\ell)},
\]

with \(\theta^0\) is chosen based on the data. See Frydman and Surya (2021) for details of the EM algorithm.

### 5.2. Asymptotic covariance matrix \(\Sigma(\theta^0)\) of \(\sqrt{n}(\hat{\theta}^0 - \theta^0)\).

It was shown in Frydman and Surya (2021) using consistency of \(\hat{\theta}^0\) and application of Theorem 4 and multivariate central limit theorem,
where $\Sigma_{ij}(\theta^0)$ denotes the $(i,j)$-element of $\Sigma(\theta^0)$ and $S_m^0 = \text{diag}(\phi_{1,m}^0, \cdots, \phi_{p,m}^0)$. The result generalizes that of Albert (1962) for a general heterogeneous mixture of Markov jump processes.

The goal of the simulation study is to test the goodness-of-fit of the estimator $S_n^{-1}(S(\theta))$ under

$$\sqrt{n}(S_n^{-1}(S(\theta)) - \theta^0) \xrightarrow{H_0} N(0, \Sigma(\theta^0)),$$

where $\theta$ is the estimator of $\theta^0$ derived either from [23] under the true value $\theta^0$ or from applying the EM algorithm [25] for which the true value is unknown and thus estimated from the observation.

5.3. Finite-sample sandwich estimator of covariance matrix. This section discusses the components of the sandwich estimator of covariance matrix $\Sigma_n(\theta^0) = J_c^{-1}(\theta^0)\Lambda_n(\theta^0)J_c^{-1}(\theta^0)$.

Lemma 4. The $(i,j)$-elements of the observed information matrices $\Lambda_n(\theta)$ and $J_c(\theta)$ are

$$\Lambda_n(\phi_{x,m}, \phi_{y,\ell}) = \delta_x(y) \left( \frac{\delta_m(\ell) \hat{\beta}_{y,\ell}(\theta)}{n\phi_{x,m}\phi_{y,\ell}} + \frac{\hat{\beta}_{y,M}(\theta)}{n\phi_{x,M}\phi_{y,M}} \right);$$

$$J_c(\phi_{x,m}, \phi_{y,\ell}) = \Lambda_n(\phi_{x,m}, \phi_{y,\ell});$$

$$\Lambda_n(q_{xy,m}, q_{rv,\ell}) = \frac{\delta_m(\ell)}{n q_{xy,m} q_{rv,\ell}} \sum_{k=1}^n \tilde{\Phi}_{k,\ell}(\theta) A_{xy,m}^k A_{rv,\ell}^k;$$

$$J_c(q_{xy,m}, q_{rv,\ell}) = \frac{\delta_x(r)\delta_y(v)\delta_m(\ell)}{n q_{xy,m} q_{rv,\ell}} \hat{N}_{rv,\ell}(\theta);$$

$$\Lambda_n(\phi_{x,m}, q_{rv,\ell}) = \frac{\delta_m(\ell)}{n \phi_{x,m} q_{rv,\ell}} \sum_{k=1}^n \tilde{\Phi}_{k,\ell}(\theta) A_{rv,\ell}^k B_{x}^k;$$

$$J_c(\phi_{x,m}, q_{rv,\ell}) = 0.$$

Proof. See Appendix [G] for details of the proof and derivation. □

Proposition 2. The sandwich estimator $\Sigma_n(\hat{\theta}(\theta)) = J_c^{-1}(\hat{\theta}(\theta))\Lambda_n(\hat{\theta}(\theta))J_c^{-1}(\hat{\theta}(\theta)) \xrightarrow{p} \Sigma(\theta)$ [26].

Proof. See Appendix [H] for details of the proof and derivation. □

6. Simulation study

This section verifies the results of Sections 3-4 for the conditional Markov jump process discussed in Section 5. For this purpose, the model parameters are set to have the following values.

| State (x) | $\alpha_x^0$ | $\phi_{x,1}^0$ | $\phi_{x,2}^0$ | $\phi_{x,3}^0$ |
|-----------|---------------|---------------|---------------|---------------|
| 1         | 1/3           | 0.5           | 0.3           | 0.2           |
| 2         | 1/3           | 0.25          | 0.55          | 0.2           |
| 3         | 1/3           | 0.6           | 0.1           | 0.3           |

Table 1. Parameter values for $\alpha_x^0$ and $\phi_{x,m}^0$, $m = 1, 2, 3$.

6.1. Specification of initial value of distribution parameters $\theta^0$. For simulation study, let $S = \{1, 2, 3\}$ and $M = 3$. The value of initial probabilities $\alpha_x^0$ and $\phi_{x,m}^0$ are presented in Table 1.

The intensity matrices $Q_1^0$, $Q_2^0$, and $Q_3^0$ for the regime membership $X^{(1)}$, $X^{(2)}$, and $X^{(3)}$ are

$$Q_1^0 = \begin{pmatrix} -2.0 & 1.2 & 0.8 \\ 0.2 & -0.4 & 0.2 \\ 1.2 & 1.8 & -3.0 \end{pmatrix}, \quad Q_2^0 = \begin{pmatrix} -3.0 & 2.4 & 0.6 \\ 0.2 & -0.4 & 0.2 \\ 0.4 & 1.6 & -2.0 \end{pmatrix},$$

and

$$Q_3^0 = \begin{pmatrix} -4.0 & 1.6 & 2.4 \\ 0.2 & -0.4 & 0.2 \\ 3.0 & 2.0 & -5.0 \end{pmatrix},$$
respectively. We see following intensity matrices $Q^0_m$, $m = 1, 2, 3$, that each regime $X^{(m)}$ has different expected state occupation time and the probability of making a jump from one state to another, except for the transition from state 2. In the latter case, it is difficult to identify which underlying Markov jump process that drives the dynamics of $X$ when it moves out of state 2.

6.2. Simulation and estimation results. Based on the above parameters, a specified $n$ independent sample paths $\{X^k : k = 1, \ldots, n\}$ of $X$ are generated. See Surya (2021) for the algorithm of generating sample paths of RSCMJP. Using these $n$ independent observations of incomplete data $Y = \bigcup_{k=1}^n Y^k$, where $Y^k = \{N^k_{xy}, T^k_x, B^k_x : (x,y) \in S\}$ is the realizations of the $k$–th paths, the EM and EM-Gradient algorithms (25) and (21) are applied to obtain maximum likelihood estimate $\hat{\theta}^0$ of the initial value $\theta^0$. In carrying out the statistical computation, the R language (2013) was used.

Figure 2 compares in each iteration the value of incomplete-data log-likelihood $\log f_o(Y|\hat{\theta}^\ell)$ as a function of the current estimate $\hat{\theta}^\ell$ obtained using the EM algorithm (25) against that of derived using the EM-Gradient algorithm (21). We observe that the latter reaches its convergence faster than the EM algorithm as it requires less iterations to converge. Figure 3 shows estimation error $\|\hat{\theta}^{\ell+1} - \hat{\theta}^\ell\|$ of the two two algorithms. As we can see from the figure that the EM-Gradient algorithm is able to correct estimation error faster than the EM algorithm to reach its convergence.

To verify the asymptotic properties of the estimator $S^{-1}_n(S(\hat{\theta}^0))$, a set of $K = 200$ independent sample paths of size $n = 4000$ each are generated. We compare the results when the estimate $\hat{\theta}^0$
is derived from (23) given the true value $\theta^0$ against that of when $\hat{\theta}^0$ is estimated using the EM-algorithm (25). To each set $K = 200$ observations of sample paths, the algorithm (23), subsequently (25), is applied to get $K$ independent sets of estimates $\{\hat{\theta}_k^0 : k = 1, \ldots, K\}$. The estimate $\hat{\theta}^0 = K^{-1} \sum_{k=1}^{K} \hat{\theta}_k^0$ and the mean squared errors $\text{MSE}(\hat{\theta}^0) = \frac{1}{K} \sum_{k=1}^{K} (S_n^{-1}(S(\hat{\theta}_k^0)) - \theta^0)^2$ are calculated. Subsequently, the information matrices $J_p(\hat{\theta}_k^0)$ (16) and $J_c(\hat{\theta}_k^0)$ (17) are evaluated for each $\hat{\theta}_k^0$. Then each set of information matrices $\{J_p(\hat{\theta}_k^0), k = 1, \ldots, K\}$ and $\{J_c(\hat{\theta}_k^0), k = 1, \ldots, K\}$ are averaged. The inverse of estimated Fisher information $\hat{I}_p^{-1}(\hat{\theta}^0) := \frac{1}{K} \sum_{k=1}^{K} J_p(\hat{\theta}_k^0)$ is used to get estimated standard error. The standard error derived from $\hat{I}_p^{-1}(\hat{\theta}^0)$ is compared against that of derived from the sandwich estimator $\Sigma_n(\hat{\theta}^0) = \hat{I}_c^{-1}(\hat{\theta}^0) \Lambda_n(\hat{\theta}^0) \hat{I}_c^{-1}(\hat{\theta}^0)$, where $\hat{I}_c(\hat{\theta}) := \frac{1}{K} \sum_{k=1}^{K} J_c(\hat{\theta}_k^0)$, and against $\hat{V}(\hat{\theta}^0)$ derived from the recursive equation (22). As it involves exponential matrix, the analytic covariance matrix $\Sigma(\theta^0)$ (26) was computed based on the algorithm in Van Loan (1978).

Table 2 presents the corresponding results for $S_n^{-1}(S(\hat{\theta}^0))$ where the estimator $\hat{\theta}^0$ was derived from the EM algorithm (25). See Frydman and Surya (2021) and Surya (2021) for further details of the algorithm. The table shows that the parameter estimate $\hat{\theta}^0$ shows its convergence to the true value $\theta^0$. Also, the iterative estimator $\sqrt{V_\ell}$ of the standard error (22) converges after $\ell = 20$ iterations to that of given by the inverse observed Fisher information $\hat{I}_p^{-1}(\hat{\theta}^0)$. Furthermore, the table shows that the RMSE are much closer to the standard errors produced by $\hat{I}_p^{-1}(\hat{\theta}^0)$ and

**Figure 3.** Estimation error $||\hat{\theta}^{\ell+1} - \hat{\theta}^\ell||$ of the MLE $\hat{\theta}^0$ using the EM algorithm (solid line) against that of derived by the EM-Gradient algorithm (dashed line).
larger than those provided by the sandwich estimator $\Sigma_n(\hat{\theta}^0)$ [18], despite its convergence to the asymptotic covariance matrix $\Sigma(\theta^0)$. This is in line with what was discussed in Remark 4. Namely, the replacement of the true value $\theta^0$ in $S_n^{-1}(S(\theta^0))$ by the MLE $\hat{\theta}$ results in changing the limiting distribution of the estimator $\hat{\theta}$ whose standard errors are larger than the analytic ones $\Sigma(\theta^0)$. The difference is attributed by the resulting information loss presented in the incomplete data [13].

The results for $S_n^{-1}(S(\theta^0))$ with the estimate $\hat{\theta}^0$ derived from (23) under the true value $\theta^0$ is presented in Table 3. The table shows the accuracy of the estimate $\hat{\theta}^0$ as it is very close to the actual value $\theta^0$. The recursive covariance matrix estimator $\hat{\Sigma}(\theta^0) := \Psi^{-1}_\ell$ [22] converges to the sandwich estimator $\Sigma_n(\theta^0)$ after $\ell = 20$ iterations. From the table we observe that the RMSE is much closer to those provided by $\Sigma_n(\theta^0)$ and $\hat{\Sigma}(\theta^0)$ [22] than that of given by the inverse observed Fisher information $\hat{I}_n^{-1}(\theta^0)$. It is well noticed that the standard errors given by $\Sigma_n(\theta^0)$ and $\hat{\Sigma}(\theta^0)$ are reasonably close to their asymptotic values $\Sigma(\theta^0)$. The dominance of $\hat{I}_n^{-1}(\theta^0)$ over $\Sigma_n(\theta^0)$ is quite apparent which indeed confirms the theoretical result. The last two columns of the table correspond to the p-values of the Kolmogorov-Smirnov statistics and the Z-test for the

| $\theta^0$ | True Value | Estimate $\hat{\theta}$ | RMSE | $\sqrt{\hat{I}_n^{-1}(\hat{\theta})}$ | $\sqrt{\Psi_\ell}$ | $\sqrt{\Sigma_n(\hat{\theta})}$ | True SE $\sqrt{\Sigma(\theta)}$ |
|------------|-------------|-------------------------|------|-------------------------------------|-----------------|---------------------------------|-----------------------------|
| 1          | 0.50000     | 0.49982                 | 1.94518 | 1.92844                               | 1.92446        | 1.36699                         | 1.36931                     |
| 2          | 0.30000     | 0.30048                 | 1.76564 | 1.76445                               | 1.76024        | 1.25309                         | 1.25499                     |
| 3          | 0.25000     | 0.25100                 | 1.79320 | 1.81141                               | 1.80659        | 1.18532                         | 1.18585                     |
| 4          | 0.55000     | 0.54903                 | 1.83888 | 1.85454                               | 1.85020        | 1.36098                         | 1.36244                     |
| 5          | 0.60000     | 0.60021                 | 1.75665 | 1.76453                               | 1.76252        | 1.33936                         | 1.34164                     |
| 6          | 0.10000     | 0.10067                 | 1.26765 | 1.24622                               | 1.24339        | 0.81833                         | 0.82158                     |
| 7          | 1.20000     | 1.19923                 | 1.57419 | 1.59330                               | 1.59620        | 1.27628                         | 1.27706                     |
| 8          | 0.80000     | 0.79964                 | 1.21567 | 1.22437                               | 1.22426        | 1.03948                         | 1.04271                     |
| 9          | 0.20000     | 0.19955                 | 0.25729 | 0.25131                               | 0.25129        | 0.21854                         | 0.21904                     |
| 10         | 0.20000     | 0.20009                 | 0.24928 | 0.25105                               | 0.25103        | 0.21898                         | 0.21904                     |
| 11         | 1.20000     | 1.20181                 | 2.26457 | 2.05913                               | 2.05539        | 1.55727                         | 1.55610                     |
| 12         | 1.80000     | 1.80045                 | 2.25882 | 2.15657                               | 2.15057        | 1.90778                         | 1.90583                     |
| 13         | 2.40000     | 2.39657                 | 4.03942 | 4.06048                               | 4.05040        | 2.96510                         | 2.97067                     |
| 14         | 0.60000     | 0.59883                 | 1.77408 | 1.78900                               | 1.78848        | 1.47943                         | 1.48534                     |
| 15         | 0.20000     | 0.20036                 | 0.28331 | 0.28859                               | 0.28858        | 0.25326                         | 0.25266                     |
| 16         | 0.20000     | 0.19999                 | 0.27589 | 0.28837                               | 0.28835        | 0.25269                         | 0.25266                     |
| 17         | 0.40000     | 0.39834                 | 1.42977 | 1.34891                               | 1.34576        | 1.00406                         | 1.00630                     |
| 18         | 1.60000     | 1.60167                 | 2.30828 | 2.28304                               | 2.28240        | 2.01524                         | 2.01260                     |
| 19         | 1.60000     | 1.60077                 | 2.33304 | 2.47066                               | 2.47065        | 2.32054                         | 2.32033                     |
| 20         | 2.40000     | 2.39907                 | 3.28214 | 3.30419                               | 3.30418        | 2.84465                         | 2.84181                     |
| 21         | 0.20000     | 0.20003                 | 0.29895 | 0.32010                               | 0.32010        | 0.29799                         | 0.29727                     |
| 22         | 0.20000     | 0.20050                 | 0.33870 | 0.32037                               | 0.32037        | 0.29828                         | 0.29727                     |
| 23         | 3.00000     | 2.99797                 | 4.13302 | 4.02493                               | 4.02490        | 3.53318                         | 3.53811                     |
| 24         | 2.00000     | 1.99994                 | 3.03314 | 3.06672                               | 3.06672        | 2.89133                         | 2.88886                     |
Table 3. Comparison between the true value $\theta^0$, the EM estimator $S_{n}^{-1}(S(\hat{\theta}^0))$ where $\hat{\theta}^0$ is estimated by (25) given $\theta^0$, the standard error based on the RMSE, inverse observed Fisher information $\hat{I}^{-1}_p(S(\hat{\theta}^0))$, the recursive estimator $\hat{\Psi}^{-1}_\ell$, the sandwich estimator $\Sigma_n(\hat{\theta}^0)$, and the asymptotic covariance $\Sigma(\theta^0)$ for $K = 200$ generated sample paths of size $n = 4000$ each. The two columns provide the p-values of Kolmogorov-Smirnov statistics and the Z-test.

standardized biases $z_k = (\hat{\theta}^0_k - \theta^0_k)/\sigma^0_k$, for each $k = 1, \ldots, d$ where $\sigma^0_k$ is the true standard deviation of the element $\hat{\theta}^0_k$ derived from the asymptotic covariance matrix $\Sigma(\theta^0)$ (26). All p-values are larger than 5%. Hence, the asymptotic normality of the biases and the null hypothesis $H_0 : \theta_k = \theta^0_k$ are both statistically significant at the acceptance level $\alpha = 5\%$. This conclusion confirms the $\sqrt{n}$-consistent limiting normal distribution of the EM-estimator $\hat{\theta}^0$ with covariance matrix $\Sigma(\theta^0)$.

7. Concluding Remarks

This paper developed some new results on maximum likelihood of incomplete data which extend the existing works found in literature. The novelty of the approach is based on the Halmos and Savage (1949) work on the theory of sufficient statistics. Conditional observed information matrices are introduced and utilized to their greater extent to which their finite-sample properties are derived. They possess the same Loewner partial matrix ordering properties as the expected Fisher information matrices do. In its new form, the observed Fisher information of incomplete data simplifies the conditional expectation of the outer product of the complete-data score function in
the Louis (1982) general matrix formula and directly verifies its asymptotic consistency. Using the conditional observed information matrices, a consistent sandwich estimator of covariance matrix is proposed. It extends the Huber sandwich estimator (Huber, 1967; Freedman, 2006; Little and Rubin, 2020) to model misspecification under incomplete data. The main appealing feature of the proposed sandwich estimator is that unlike its counterpart, it does not involve the inverse observed Fisher information. The standard errors provided by the latter prove to be larger than those given by the proposed sandwich estimator. The difference is attributed to the resulting information loss presented in the incomplete data. Recursive algorithms are developed for iterative estimation of the true parameter, the inverse observed Fisher information and the sandwich estimator. The simulation study on a complex stochastic model of regime-switching conditional Markov jump processes confirms the results presented in this paper. Most notably, the sandwich estimator gives an accurate standard errors of the model parameters estimates under their true values being close to their asymptotic covariance. We believe that the results should offer potential for variety of applications for maximum likelihood estimation of statistical model parameters for incomplete data.

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Appendix A. Proof of Lemma 2

By assumption [A1], the conditional expectation of complete-data score function is

$$\mathbb{E}\left[ \frac{\partial \log f_c(X|\theta)}{\partial \theta_i} \bigg| Y = y, \theta \right] = \int_{X(y)} \frac{\partial \log f_c(x|\theta)}{\partial \theta_i} \frac{f(x,y,\theta)}{\theta_i} \lambda(dx)$$

$$= \int_{X(y)} \frac{\partial f_c(x|\theta)}{\partial \theta_i} \frac{f(x,y,\theta)}{f_c(x|\theta)} \lambda(dx)$$

$$= \frac{1}{\theta_o(y|\theta)} \int_{X(y)} \frac{\partial f_c(x|\theta)}{\partial \theta_i} \lambda(dx)$$

$$= \frac{1}{\theta_o(y|\theta)} \frac{\partial}{\partial \theta_i} \int_{X(y)} f_c(x|\theta) \lambda(dx),$$

from which assertion of the claim (10) is accomplished on account of (1) and (2). Following the identities (10) and (8), the EM-estimator $\hat{\theta}_0$ coincides with the MLE $\hat{\theta}$. The MLE is found by replacing the unknown parameter $\theta^0$ by the estimator $\hat{\theta}_0$ in (4) or in the identity (8). ■

Appendix B. Proof of Lemma 3

By assumption [A1] and the identity (2), it follows that

$$\mathbb{E}\left[ \frac{1}{f_c(X|\theta)} \frac{\partial^2 f_c(X|\theta)}{\partial \theta^2} \bigg| Y = y, \theta \right] = \int_{X(y)} \frac{1}{f_c(x|\theta)} \frac{\partial^2 f_c(x|\theta)}{\partial \theta^2} \frac{f(x,y,\theta)}{\theta^2} \lambda(dx)$$

$$= \frac{1}{\theta_o(y|\theta)} \int_{X(y)} \frac{\partial^2 f_c(x|\theta)}{\partial \theta^2} \lambda(dx)$$

$$= \frac{1}{\theta_o(y|\theta)} \frac{\partial^2}{\partial \theta^2} \int_{X(y)} f_c(x|\theta) \lambda(dx),$$

where the last equality is due to [A1], which by (1) completes the proof. ■
APPENDIX C. PROOF OF THEOREM 1

[(i)] The assertion of the first claim is rather straightforward, namely that
\[
E\left[ \frac{\partial \log f(X|Y, \theta)}{\partial \theta} \mid Y = y, \theta \right] = \int_{X(y)} \frac{\partial \log f(x|y, \theta)}{\partial \theta} f(x|y, \theta) \lambda(dx)
\]
\[
= \int_{X(y)} \frac{\partial f(x|y, \theta)}{\partial \theta} \lambda(dx),
\]
from which the claim follows on account of \([1]\) and \([2]\), i.e., \(f_{X(y)} f(x|y, \theta) \lambda(dx) = 1\). Alternatively to the above, the proof is established on account of identity \([10]\) and taking the logarithm of \([2]\):
\[
\frac{\partial \log f_o(x|\theta)}{\partial \theta} = \frac{\partial \log f_o(y|\theta)}{\partial \theta} + \frac{\partial \log f(x|y, \theta)}{\partial \theta}.
\]

[(ii)] To prove the second claim, recall that the following identity holds for any \(\theta \in \Theta^d\) and \(y \in \mathcal{Y}\),
\[
\left( \frac{\partial^2 \log f(x|y, \theta)}{\partial \theta^2} \right) f(x|y, \theta) = \frac{\partial}{\partial \theta} \left[ \left( \frac{\partial \log f(x|y, \theta)}{\partial \theta} \right) f(x|y, \theta) \right]
\]
\[
- \left( \frac{\partial \log f(x|y, \theta)}{\partial \theta} \right) \left( \frac{\partial \log f(x|y, \theta)}{\partial \theta} \right)^\top f(x|y, \theta).
\]
Therefore, using the above the l.h.s of the second claim can be worked out using Fatou’s lemma as
\[
E\left[ \frac{\partial^2 \log f(X|Y, \theta)}{\partial \theta^2} \mid Y = y, \theta \right] = \int_{X(y)} \left( \frac{\partial^2 \log f(x|y, \theta)}{\partial \theta^2} \right) f(x|y, \theta) \lambda(dx)
\]
\[
= \int_{X(y)} \frac{\partial}{\partial \theta} \left[ \left( \frac{\partial \log f(x|y, \theta)}{\partial \theta} \right) f(x|y, \theta) \right] \lambda(dx)
\]
\[
- \int_{X(y)} \left( \frac{\partial \log f(x|y, \theta)}{\partial \theta} \right) \left( \frac{\partial \log f(x|y, \theta)}{\partial \theta} \right)^\top f(x|y, \theta) \lambda(dx)
\]
\[
= \frac{\partial}{\partial \theta} E\left[ \frac{\partial \log f(X|Y, \theta)}{\partial \theta} \mid Y = y, \theta \right]
\]
\[\quad - E\left[ \left( \frac{\partial \log f(X|Y, \theta)}{\partial \theta} \right) \left( \frac{\partial \log f(X|Y, \theta)}{\partial \theta} \right)^\top \mid Y = y, \theta \right],
\]
from which we arrive at the the second identity taking into account of the first one. ■

APPENDIX D. PROOF OF THEOREM 2

By taking derivative w.r.t \(\theta\) on \([27]\), we have after taking conditional expectation \(E[\bullet|Y, \theta]\),
\[
E\left[ - \frac{\partial^2 \log f_c(x|\theta)}{\partial \theta^2} \mid Y, \theta \right] = - \frac{\partial^2 \log f_o(Y|\theta)}{\partial \theta^2} + E\left[ - \frac{\partial^2 \log f(X|Y, \theta)}{\partial \theta^2} \mid Y, \theta \right]
\]
\[
= - \frac{\partial^2 \log f_o(Y|\theta)}{\partial \theta^2} + E\left[ \left( \frac{\partial \log f(X|Y, \theta)}{\partial \theta} \right) \left( \frac{\partial \log f(X|Y, \theta)}{\partial \theta} \right)^\top \mid Y, \theta \right]
\]
\[
\geq - \frac{\partial^2 \log f_o(Y|\theta)}{\partial \theta^2},
\]
where the last equality is due the fact that the matrix \(S_{x|y}S_{x|y}^\top\), with \(S_{x|y} = \frac{\partial \log f(X|Y, \theta)}{\partial \theta}\), is positive definite since for any \(0 \neq z \in \mathbb{R}^d\), \(d = |\theta|\), the Euclidean norm \(z^\top S_{x|y}S_{x|y}^\top z = [S_{x|y}^\top z]^\top S_{x|y}^\top z > 0\). ■
APPENDIX E. PROOF OF THEOREM E

By the chain rule of partial derivative, it is straightforward to show that
\[- \frac{\partial^2 \log f_c(X|\theta)}{\partial \theta^2} = \left( \frac{\partial \log f_c(X|\theta)}{\partial \theta} \right) \left( \frac{\partial \log f_c(X|\theta)}{\partial \theta} \right)^\top - \frac{1}{f_c(X|\theta)} \frac{\partial^2 f_c(X|\theta)}{\partial \theta^2}.\]

Notice that the first identity in (14) follows from the above on account that \(\mathbb{E}_{f_c(X|\theta)} \left[ \frac{1}{f_c(X|\theta)} \frac{\partial^2 f_c(X|\theta)}{\partial \theta^2} \right] = 0\). Applying the identity (12) it follows after taking conditional expectation \(\mathbb{E}_{\hat{Y} = y, \theta} \left[ \frac{\partial^2 \log f_c(X|\theta)}{\partial \theta^2} \right| Y = y, \theta\) that
\[\mathbb{E}_{\hat{Y} = y, \theta} \left[ \frac{\partial^2 \log f_c(X|\theta)}{\partial \theta^2} \right| Y = y, \theta = \mathbb{E} \left[ \left( \frac{\partial \log f_c(X|\theta)}{\partial \theta} \right) \left( \frac{\partial \log f_c(X|\theta)}{\partial \theta} \right)^\top \right| Y = y, \theta \]
\[- \frac{1}{f_o(y|\theta)} \frac{\partial^2 f_o(y|\theta)}{\partial \theta^2}.\]

After some calculations and rearrangement, we can rewrite the second term on r.h.s as
\[(28) \quad \frac{1}{f_o(y|\theta)} \frac{\partial^2 f_o(y|\theta)}{\partial \theta^2} = \frac{\partial^2 \log f_o(y|\theta)}{\partial \theta^2} + \left( \frac{\partial \log f_o(y|\theta)}{\partial \theta} \right) \left( \frac{\partial \log f_o(y|\theta)}{\partial \theta} \right)^\top.\]

Since \(\hat{\theta}^0\) solves equation (11), \(\hat{\theta}^0\) coincides with the MLE of \(\theta^0\), i.e., \(\hat{\theta}^0\) is the maximizer of the log-likelihood function \(\log f_o(y|\theta)\). The latter implies that \(\frac{\partial^2 \log f_o(y|\theta)}{\partial \theta^2} < 0\) by which (15) follows.

The fact that the \(\Lambda_n(\hat{\theta}^0)\) is positive definite follows from the fact that \(\left( \frac{\partial \log f_o(X|\theta)}{\partial \theta} \right) \left( \frac{\partial \log f_o(X|\theta)}{\partial \theta} \right)^\top\) is positive definite given that it is of the form \(S_xS_x^\top\), where \(S_x = \frac{\partial \log f_o(X|\theta)}{\partial \theta}\) is a \((d \times 1)\)-column vector, with \(d = |\theta|\). That is for any \(0 \neq z \in \mathbb{R}^d\), \(z^\top S_xS_x^\top z = \left( S_x^\top z \right) \left( S_x^\top z \right) > 0\). From Theorem 7.2.1 on p. 438 of Horn and Johnson (2013), every positive definite matrix is invertible and the inverse itself is again positive definite. Thus, efficiency of \(\hat{\theta}^0\) over the MLE \(\hat{\theta}\) is justified. ■

APPENDIX F. PROOF OF THEOREM F

[(i)] Following the likelihood function (2) and the identity (10), we have
\[\frac{\partial \log f_o(y|\theta)}{\partial \theta_j} = \mathbb{E}_{X|y} \left[ \frac{\partial \log f_c(X|\theta)}{\partial \theta_j} \right| Y = y, \theta] = \int_{X|y} \frac{\partial \log f_c(x|\theta)}{\partial \theta_j} f(x|y, \theta) \lambda(dx).\]

By linearity of the partial derivative and expectation operators, the proof is established for a single complete-data \(X\) and its respective incomplete-data \(Y\). Differentiating both sides w.r.t \(\theta_i\) gives
\[\frac{\partial^2 \log f_o(y|\theta)}{\partial \theta_i \partial \theta_j} = \int_{X|y} \left( \frac{\partial^2 \log f_c(x|\theta)}{\partial \theta_i \partial \theta_j} + \frac{\partial \log f(x|y, \theta)}{\partial \theta_i} \frac{\partial \log f_c(x|\theta)}{\partial \theta_j} \right) f(x|y, \theta) \lambda(dx)\]
\[= \int_{X|y} \left( \frac{\partial^2 \log f_c(x|\theta)}{\partial \theta_i \partial \theta_j} + \left[ \frac{\partial \log f_c(x|\theta)}{\partial \theta_i} - \frac{\partial \log f_o(y|\theta)}{\partial \theta_i} \right] \frac{\partial \log f_c(x|\theta)}{\partial \theta_j} \right) f(x|y, \theta) \lambda(dx)\]
\[= \mathbb{E} \left[ \frac{\partial^2 \log f_c(X|\theta)}{\partial \theta_i \partial \theta_j} \right| Y = y, \theta] + \mathbb{E} \left[ \left( \frac{\partial \log f_c(X|\theta)}{\partial \theta_i} \right) \left( \frac{\partial \log f_c(X|\theta)}{\partial \theta_j} \right) \right| Y = y, \theta]\]
\[= \mathbb{E} \left[ \frac{\partial \log f_c(X|\theta)}{\partial \theta_i} \right| Y = y, \theta] \mathbb{E} \left[ \frac{\partial \log f_c(X|\theta)}{\partial \theta_j} \right| Y = y, \theta],\]

where the third conditional expectation on the last equality is due to the identity (10). The proof is established on account of independence of the incomplete data \(\{Y^k\}_{k \geq 1}\), namely
\[\frac{1}{n} \sum_{k=1}^{n} \frac{\partial^2 \log f_o(Y^k|\theta)}{\partial \theta_i \partial \theta_j} \Rightarrow \mathbb{E} \left[ \frac{\partial^2 \log f_o(Y|\theta)}{\partial \theta_i \partial \theta_j} \right| \theta].\]
Similarly, by independence of \( \{Y^k\}_{k \geq 1} \) and applying iterated law of conditional expectation,

\[
\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\left[ \frac{\partial^2 \log f_c(X^k|\Theta)}{\partial \theta_i \partial \theta_j} \left| Y^k, \Theta \right. \right] \xrightarrow{P} \mathbb{E}\left[ \frac{\partial^2 \log f_c(X|\Theta)}{\partial \theta_i \partial \theta_j} \left| \Theta \right. \right],
\]

leading to the complete-data information after rearranging the sum. This ends the proof of [(i)].

[(ii)] The claim is established following the identities (15)-(16) and (13).

**APPENDIX G. PROOF OF LEMMA 4**

The proofs are presented for the matrix components \( \Lambda_n(\phi_{x,m}, \phi_{y,\ell}) \) and \( J_c(\phi_{x,m}, \phi_{y,\ell}) \), while the others can be derived by applying similar arguments. Following the log-likelihood function and the fact that for \( 1 \leq \ell \leq M - 1 \), \( \Phi_{k,m} \Phi_{k,\ell} = \delta_m(\ell) \Phi_{k,\ell} \) and that for \( x, y \in \mathbb{S} \), \( B^k_x B^k_y = \delta_x(y) B^k_y \),

\[
\left( \frac{\partial \log f_c(X^k|\Theta)}{\partial \phi_{x,m}} \right) \left( \frac{\partial \log f_c(X^k|\Theta)}{\partial \phi_{y,\ell}} \right) = \left( \frac{\Phi_{k,m} B^k_x}{\phi_{x,m}} - \frac{\Phi_{k,M} B^k_x}{\phi_{x,M}} \right) \left( \frac{\Phi_{k,\ell} B^k_y}{\phi_{y,\ell}} - \frac{\Phi_{k,M} B^k_y}{\phi_{y,M}} \right)
\]

\[
= \frac{\Phi_{k,\ell} B^k_y}{\phi_{x,M} \phi_{y,\ell}} \delta_x(y) \delta_m(\ell) + \frac{\Phi_{k,M} B^k_y}{\phi_{x,M} \phi_{y,M}} \delta_x(y).
\]

Since \( B^k_x \) is observable from the path, the element \( \Lambda_n(\phi_{x,m}, \phi_{y,\ell}) \) is obtained by taking conditional expectation \( \mathbb{E}[\bullet|X^k, \Theta] \). The same arguments applied to get the expression for \( J_c(\phi_{x,m}, \phi_{y,\ell}) \).

**APPENDIX H. PROOF OF PROPOSITION 2**

The assertion of the claims requires the consistency of the estimator \( \hat{\theta}^0 \), continuous mapping theorem and the following results which are due to Proposition C1 in Frydman and Surya (2021)

\[
\mathbb{E}\left[ \Phi_{k,\ell} N^k_{x,y,\ell}|\Theta^0 \right] = q^0_{xy,\ell} \alpha_0^0 \mathbb{S}^0_{\ell} \left( \int_0^T e^{Q^0_{\ell} u} du \right) e_r,
\]

\[
\mathbb{E}\left[ \Phi_{k,\ell} A^k_{x,y,m} A^k_{x,y,\ell}|\Theta^0 \right] = \delta_x(r) \delta_y(v) q^0_{xy,\ell} \alpha_0^0 \mathbb{S}^0_{\ell} \left( \int_0^T e^{Q^0_{\ell} u} du \right) e_r,
\]

\[
\mathbb{E}\left[ \Phi_{k,\ell} B^k_x B^k_y \right] = 0.
\]

By independence of \( \{Y^k\} \), the law of large numbers leads to the following convergence of

\[
\Lambda_n(q^0_{xy,m}, q^0_{x,y,\ell}) = \frac{\delta_m(\ell)}{q^0_{xy,m} q^0_{x,y,\ell}} \frac{1}{n} \sum_{k=1}^{n} \Phi_{k,\ell}(\hat{\theta}^0) A^k_{x,y,m} A^k_{x,y,\ell} \xrightarrow{P} \delta_m(\ell) \mathbb{E}\left[ \Phi_{k,\ell} A^k_{x,y,m} A^k_{x,y,\ell}|\Theta^0 \right] \mathbb{S}^0_{\ell} \left( \int_0^T e^{Q^0_{\ell} u} du \right) e_r,
\]

\[
J_c(q^0_{xy,m}, q^0_{x,y,\ell}) = \frac{\delta_x(r) \delta_y(v) \delta_m(\ell)}{q^0_{xy,m} q^0_{x,y,\ell}} \frac{1}{n} \sum_{k=1}^{n} \Phi_{k,\ell}(\hat{\theta}^0) N^k_{x,y,\ell} \xrightarrow{P} \delta_x(r) \delta_y(v) \delta_m(\ell) \mathbb{E}\left[ \Phi_{k,\ell} N^k_{x,y,\ell}|\Theta^0 \right] \mathbb{S}^0_{\ell} \left( \int_0^T e^{Q^0_{\ell} u} du \right) e_r,
\]

from which we arrive at

\[
\Sigma_n(q^0_{xy,m}, q^0_{x,y,\ell}) = J^{-1}_c(q^0_{xy,m}) \Lambda_n(q^0_{xy,m}, q^0_{x,y,\ell}) J^{-1}_c(q^0_{x,y,\ell}).
\]
Furthermore, since

$$
\Phi_k,\ell (\theta_0) A_{\nu,\ell} D_k \xrightarrow{P} \Phi_k,\ell A_{\nu,\ell} D_k [\theta_0] = 0,
$$

$$
\Lambda_n (\bar{\phi}_{x,m}, \bar{\rho}_{x,M}) = \frac{1}{n} \sum_{k=1}^{n} \Phi_k,\ell (\theta_0) A_{\nu,\ell} D_k \xrightarrow{P} \frac{1}{\phi_{x,m} D_{x \ell}} \mathbb{E} \left[ \Phi_k,\ell A_{\nu,\ell} D_k [\theta_0] \right] = 0,
$$

$$
\Sigma_n (\bar{\phi}_{x,m}, \bar{\rho}_{x,M}) = J^{-1} (\bar{\phi}_{x,m}) \Lambda_n (\bar{\phi}_{x,m}, \bar{\rho}_{x,M}) J^{-1} (\bar{\rho}_{x,M}) \xrightarrow{P} 0 = \Sigma (\phi_{x,m}^{0}, \phi_{y,\ell}^{0}).
$$

It remains to show that $\Sigma_n (\bar{\phi}_{x,m}, \bar{\rho}_{x,M}) = J^{-1} (\bar{\phi}_{x,m}, \bar{\rho}_{y,\ell}) \xrightarrow{P} \Sigma (\phi_{x,m}^{0}, \phi_{y,\ell}^{0})$. Since the matrix $J_c (\bar{\phi}_{x,m}, \bar{\rho}_{y,\ell}) = J_n (\bar{\phi}_{x,m}, \bar{\rho}_{y,\ell})$ is of block diagonal type, we verify the result for the number of mixtures $M = 2, 3$ and leave the case for $M \geq 4$ to the numerical study. Recall that for $M = 2$, the parameter for regime probability is $\phi = (\phi_{1,1}, \phi_{2,1}, \phi_{3,1}, \phi_{3,2})$. In this case, $\phi_{x,m} + \phi_{x,M} = 1$. Moreover, by independence of $\{Y_k\}$,

$$
\hat{B}_{x,m} (\theta_0) = \frac{1}{n} \sum_{k=1}^{n} \Phi_k,\ell (\theta_0) B_k \xrightarrow{P} \mathbb{E} \left[ \Phi_k,\ell B_k [\theta_0] \right] = \alpha_{x}^{0} \phi_{x,m}^{0} \phi_{x,\ell}^{0}.
$$

Note that for $M = 3$, $\phi = (\phi_{1,1}, \phi_{1,2}, \phi_{2,1}, \phi_{2,2}, \phi_{3,1}, \phi_{3,2})$. To establish the claim, we apply $\hat{\alpha}_{x} = n^{-1} \hat{B}_x$ and $n^{-1} \hat{B}_{x,m} (\theta_0) = \alpha_{x}^{0} \phi_{x,m}^{0} \phi_{x,\ell}^{0}$ for $x \in \mathbb{S}$ and $1 \leq m \leq M$ with $\bar{\phi}_{x,m}^{0}$ satisfying $\phi_{x,m}^{0} = 1 - (\phi_{x,m}^{0} + \phi_{x,\ell}^{0})$ for $m, \ell = 1, 2, m \neq \ell$. Notice that $\Lambda_n (\bar{\phi})$ is a block diagonal matrix with

$$
\Lambda_n (\bar{\phi}_{x,m}, \bar{\phi}_{x,\ell}) = \left( \begin{array}{cc}
\delta_m (\ell) n^{-1} \hat{B}_{x,m} (\theta_0) & \delta_m (\ell) n^{-1} \hat{B}_{x,m} (\theta_0) \\
\phi_{x,m}^{0} \phi_{x,\ell}^{0} & \phi_{x,m}^{0} \phi_{x,\ell}^{0}
\end{array} \right) = \alpha_{x}^{0} \left( \begin{array}{cc}
\delta_m (\ell) \phi_{x,\ell}^{0} + \frac{1}{\phi_{x,M}^{0}} & \frac{1}{\phi_{x,M}^{0}} \\
\phi_{x,m}^{0} \phi_{x,\ell}^{0} \phi_{x,\ell}^{0} & \phi_{x,m}^{0} \phi_{x,\ell}^{0} \phi_{x,\ell}^{0}
\end{array} \right).
$$

Thus, the block diagonal submatrix $\Lambda_n (\bar{\phi}_{x,m})$ of $\Lambda_n$ is of $(2 \times 2)$-dimension defined by

$$
\Lambda_n (\bar{\phi}_{x,m}) = \left( \begin{array}{cc}
\Lambda_n (\bar{\phi}_{x,m}, \bar{\phi}_{x,m}) & \Lambda_n (\bar{\phi}_{x,m}, \bar{\phi}_{x,\ell}) \\
\Lambda_n (\bar{\phi}_{x,\ell}, \bar{\phi}_{x,m}) & \Lambda_n (\bar{\phi}_{x,\ell}, \bar{\phi}_{x,\ell})
\end{array} \right) = \alpha_{x}^{0} \left( \begin{array}{cc}
\frac{1}{\phi_{x,m}^{0}} + \frac{1}{\phi_{x,M}^{0}} & \frac{1}{\phi_{x,m}^{0}} \\
\frac{1}{\phi_{x,\ell}^{0}} & \frac{1}{\phi_{x,M}^{0}} + \frac{1}{\phi_{x,\ell}^{0}}
\end{array} \right).
$$

Since $\phi_{x,m}^{0} + \phi_{x,\ell}^{0} = 1 - \phi_{x,M}^{0}$, the determinant $D = |\Lambda_n (\bar{\phi}_{x,m})|$ is given after some calculations by

$$
D = \frac{(\alpha_{x}^{0})^{2}}{\phi_{x,m}^{0} \phi_{x,\ell}^{0} \phi_{x,M}^{0}}.
$$

Therefore, on account that $J_c (\bar{\phi}_{x,m}^{0}, \bar{\rho}_{x,\ell}^{0}) = \Lambda_n (\bar{\phi}_{x,m}, \bar{\rho}_{x,\ell})$, we obtain the inverse $J_c^{-1} (\bar{\phi}_{x,m}^{0})$ as

$$
\frac{1}{\phi_{x,m}^{0} \phi_{x,\ell}^{0} \phi_{x,M}^{0}} \left( \begin{array}{cc}
\frac{1}{\phi_{x,m}^{0}} & \frac{1}{\phi_{x,M}^{0}} \\
\phi_{x,\ell}^{0} & \phi_{x,\ell}^{0}
\end{array} \right) = \frac{\phi_{x,m}^{0} (1 - \phi_{x,m}^{0})}{\phi_{x,\ell}^{0}} \frac{\phi_{x,\ell}^{0} (1 - \phi_{x,\ell}^{0})}{\phi_{x,M}^{0}} \phi_{x,M}^{0},
$$

which in turn by consistency of the EM-estimator $\hat{\theta}_0$ leading to $J_c^{-1} (\bar{\phi}_{x,m}^{0}) \xrightarrow{P} \Sigma (\phi_{x,m}^{0})$.

REFERENCES

[1] Albert, A. (1962). Estimating the infinitesimal generator of a continuous time, finite state Markov process. *Annals of Mathematical Statistics* **38**, 727-753.

[2] Blahut, R. (1987). *Principles and Practice of Information Theory*. Addison-Wesley.

[3] Cramér, H. (1946). *Mathematical Methods of Statistics*. Princeton University Press.

[4] Dempster, A. P., Laird, N.M. and Rubin, D. B. (1977). Maximum likelihood from incomplete data via the EM algorithm (with discussion). *Journal of Royal Statistical Society: Series B (Statistical Methodology)*, **39**, p.1-38.
[5] Efron, B. and Hinkley, D. V. (1978). Assessing the accuracy of the maximum likelihood estimator: Observed versus expected Fisher information. Biometrika 65(3), 457-482.

[6] Fisher, R. A. (1925). Theory of statistical estimation. Proceedings of the Cambridge Philosophical Society 22, 700-725.

[7] Freedman, D. A. (2006). On the so-called "Huber Sandwich Estimator" and "Robust Standard Errors". The American Statistician, 60(4), 299-302.

[8] Frydman, H. and Surya, B. A. (2021). Maximum likelihood estimation for a general mixture of Markov jump processes. Preprint. https://arxiv.org/pdf/2103.02755.pdf

[9] Halmos, P. R., and Savage, L. J. (1949). Application of the Radon-Nikodym theorem to the theory of sufficient statistics. The Annals of Mathematical Statistics, 20(2), 225-241.

[10] Hastie, T., Tibshirani, R. and Friedman, J. (2009). The Elements of Statistical Learning: Data, Mining, Inference and Prediction 2nd Edition, Springer-Verlag.

[11] Hero, A. and Fessler, J. A. (1994). A recursive algorithm for computing Cramer-Rao-type bounds on estimator covariance. IEEE Transactions on Information Theory, 40(4), 1205-1210.

[12] Horn, R. A. and Johnson, C. R. (2013). Matrix Analysis. Cambridge University Press.

[13] Huber, P. J. (1967). The behavior of maximum likelihood estimates under nonstandard conditions. Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, 1, 221-233.

[14] Lange, K. (1995). A gradient algorithm locally equivalent to the EM algorithm. Journal of Royal Statistical Society: Series B (Statistical Methodology), 57(2), 425-437.

[15] Little, R. J. A. and Rubin, D. B. (2020). Statistical Analysis with Missing Data. John Wiley & Sons, Inc.

[16] Louis, T. A. (1982). Finding the observed information matrix when using the EM algorithm. Journal of Royal Statistical Society: Series B (Statistical Methodology), 44, 226-233.

[17] McLachlan, G. J. and Krishnan, T. (2008). The EM Algorithm and Extensions. John Wiley & Sons, Inc.

[18] Meiljison, I. (1989). A fast improvement to the EM algorithm on its own terms. Journal of Royal Statistical Society: Series B (Statistical Methodology), 51(1), 127-138.

[19] Meng, X. L. and Rubin, D. B. (1991). Using EM to obtain asymptotic variance-covariance matrices: the SEM algorithm. Journal of the American Statistical Association, 86, 899-909.

[20] Newey, W.K. and McFadden, D. (1994). "Chapter 36: Large sample estimation and hypothesis testing”. In Engle, Robert; McFadden, Dan (eds.). Handbook of Econometrics, Vol.4. Elsevier Science, 2111–2245.

[21] Norris, J. R. (2009). Markov Chains. Cambridge University Press, 15th printing.

[22] Oakes, D. (1999). Direct calculation of the information matrix via the EM algorithm. Journal of Royal Statistical Society: Series B (Statistical Methodology), 61, 479-482.

[23] Orchard, T. and Woodbury, M. A. (1972). A missing information principle: Theory and applications. Proceedings of the 6th Berkeley Symposium on Mathematical Statistics and Probability, Vol. 1, Berkeley, CA: University of California, 607-715.

[24] Osborne, R. (1992). Fisher’s method of scoring. International Statistical Review, 60(1), 99-117.

[25] R Core Team. (2013). R: A language and environment for statistical computing. R foundation for Statistical Computing, Vienna, Austria. https://www.R-project.org

[26] Schervish, M. J. (1995). Theory of Statistics. Springer.

[27] Surya, B. A. (2021). A new class of conditional Markov jump processes with regime switching and path dependence: Properties and maximum likelihood estimation. Preprint. https://arxiv.org/abs/2107.07026

[28] Takai, K. (2020). Incomplete-data Fisher scoring method with steplength adjustment. Statistics and Computing, 30, 871-886.

[29] Van der Vaart, A. W. (2000). Asymptotic Statistics. Cambridge University Press.

[30] Van der Vaart, A. W. (1997). Superefficiency, in Pollard, David, Torgersen, Erik, Yang, Grace L. (Eds.) Festschrift for Lucien Le Cam: Research Papers in Probability and Statistics, Springer.

[31] Van Loan, C. F. (1978). Computing integrals involving the matrix exponential. IEEE Transactions on Automatic Control 23 (3), 395-404.

[32] Wu, C. F. J. (1983). On the convergence properties of the EM algorithm. The Annals of Statistics 11 (1), 95-103.

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