Fully Algebraic Description of the Static Level Sets for the System of Two Particles under a Van der Waals Potential

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Abstract

We study the equipotential surfaces around of a two particles system in 3-d under a pairwise good potential as the one of Van der Waals. The level sets are completely determined by the solutions of polynomials of at most fourth degree that can be solved by standard algebraic methods. The distribution of real positive roots determines the level sets and provides a complete description of the map for the equipotential zones. Our methods can be generalized to a family of polynomials with degree multiple of 2, 3, or 4. Numerical simulations of 2-d and 3-d pictures depicting the true orbits and equipotential zones are provided.

Keywords: Pairwise good potential; solubility by radicals; Levels set; Equipotential Surfaces.

1 Introduction

The study of clusters of massless non-interactive particles under a potential fields has been approached, roughly speaking, under two viewpoints: the one that considers the dynamics given by equations of Newton associated to the potential, and the one that considers static particles. The latter is related among others, to theoretical and experimental research in crystal formation, and plasticity, whereas the former is connected with research in control theory. Another line of research involving pairwise good potentials and control of chaos concerns to the dynamical analysis and control of micro cantilevers as single mode approximation for tackling interactions under Van der Waals potential, see for instance [\textsuperscript{1}].
In the study of the N-body problem, several pairwise good potentials have been utilized for describing relative equilibria and central configurations, see for instance [3], the methods there include classical dynamical systems as well as non-linear geometric control techniques [2].

We are interested in clusters of non-interactive particles under pairwise good potentials, our research points to the optimal control problems of optimal path planning, collision free navigation and crystal formation on equipotential zones of clusters.

In this paper, we restrict ourselves to the study of the equipotential zones for a system of two static non-interactive particles.

In the literature, numerical methods of mathematical software is commonly used to generate graphics of levels set or orbits. However, there are not algebraic methods and techniques for determining the shape of the surfaces or curves or dots of the equipotential energy even for small clusters. The novelty of our approach is to solve the implicit function problem of the determination of the level sets using the algebraic methods for solving polynomials by radicals. With our methods, the exact solution of an orbit corresponds to roots of a polynomial with degree \( n \leq 4 \) as formulae involving its coefficients, arithmetic operations, and radicals.

The paper is organized as follows, in section 2 we depict the notation of the problem, the general description of our approach, and our main result. Section 3 contains the description of the algebraic procedures. In section 4 we present a comparison of numerical and algebraic results. At the end in section 5 we derive some conclusions.

## 2 Notation and Problems

Let be \( R^+ = [0, \infty) \). For a given \( d > 0 \), a smooth function \( P : R^+ \to R \) is well pairwise good potential if satisfies the following conditions:

1. Infinite rejection for avoiding, destroying or collapsing particles, \( \lim_{d \to 0} P(d) = \infty \).
2. A negative basin around a minimum distance \( d^* = \arg\min_{d \in (0, \infty)} P(d) \).
3. Asymptotic attraction, \( \lim_{d \to \infty} P(d) = 0^- \).

An elementary model for the Van der Waals that complies the previous properties is the following: \( B(d) = \frac{1}{d^4} - \frac{2}{d^2} \). This function has its minimum at \( d^* = 1 \), \( B(d^*) = -1 \). For a group of three particles \( p_i = (x_i, y_i, z_i) \in R^3 \), \( i = 1, 2, 3 \), with the Euclidean metric \( d_{ij} = d(p_i, p_j) \). We consider the problem of determining the orbits of a system of two given particles and one free under potential, hereafter shall be called \( B_2 \), that is, \( B_2(p_1, p_2, p_3) = \sum_{1 \leq i < j \leq 3} B(\sqrt{d_{ij}}) = \sum_{1 \leq i < j \leq 3} \left( \frac{1}{(d_{ij})^4} - \frac{2}{d_{ij}} \right) \).
Without lost of generality, we can assume \( p_1 = (-\frac{1}{2}, 0, 0), p_2 = (\frac{1}{2}, 0, 0) \) where \( l > 0 \) is the distance for a given system of two fixed particles. Let be \( p = (x, y, z) \in \mathbb{R}^3 \) a free particle, then the complete potential of this system is

\[
B_2(x, y, z) = \frac{1}{(x + \frac{1}{2})^2 + y^2 + z^2} - \frac{1}{(x - \frac{1}{2})^2 + y^2 + z^2},
\]

where \( K_l = \frac{1}{l} - \frac{3}{l^2} \) is the corresponding potential between \( p_1 \) and \( p_2 \), which is constant.

An orbit of \( B_2 \) of value \( G \) is the set \( \{(x, y, z) \in \mathbb{R}^3 | B_2(x, y, z) = G\} \), \( G \in [m_l, \infty) \) where \( l > 0 \), and \( m_l = \min_{(x, y, z) \in \mathbb{R}^3} B_2(x, y, z) \).

The symmetry of \( B_2(x, y, z) \) with respect to second and third axis allows the reduction of the problem to \( \mathbb{R}^2 \). Moreover, \( B_2(x, y) = B_2(x, -y) = B_2(-x, -y) = B_2(-x, y) \).

Therefore, it is only necessary to consider the orbits of \( B_2 \) of value \( G \) as the sets \( O(l, G) = \{(x, y) \in \mathbb{R}^2 | B_2(x, y) = G\} \), \( l > 0 \), \( G \in [m_l, \infty) \). The corresponding 3-d model can be constructed by appropriate rotations.

Our methodology is based on: (1) The algebraic methods of Cardano and Ferrari; (2) For a given \( G = G_x \), the equation \( B_2(x, y) = G \) yields a third or fourth degree polynomial with coefficients in the ring \( \mathbb{R}[x] \), that we solve to obtain an exact root \( r(x, G, l) \); (3) If \( r(x, G, l) \geq 0 \) is a root, then \( O(l, G) = \{(x, y) | x > 0, y = \sqrt{r(x, G, l)}\} \). We state now the main proposition

**Proposition 1.** Given a system of two particles, \( p_1 = (-\frac{1}{2}, 0, 0), p_2 = (\frac{1}{2}, 0, 0) \). The orbits \( O(l, G) \) of \( B_2 \), for \( G \in [m_l, \infty) \), correspond to the positive roots of the third and fourth degree polynomial obtained from the equation:

\[
B_2(x, y) - G = 0. \tag{2.1}
\]

**Proof.** The equation \( 2.1 \) gives:

\[
0 = \left( (x + \frac{1}{2})^2 + y^2 \right)^2 - 2 \left( (x + \frac{1}{2})^2 + y^2 \right) \left( (x - \frac{1}{2})^2 + y^2 \right)^2 - \left( (x - \frac{1}{2})^2 + y^2 \right) \left( (x + \frac{1}{2})^2 + y^2 \right)^2 - \left( (x - \frac{1}{2})^2 + y^2 \right)^2 - (K_l + G) \left( (x + \frac{1}{2})^2 + y^2 \right)^2 \left( (x - \frac{1}{2})^2 + y^2 \right)^2.
\]

It has power on \( x \) or \( y \) as 8, 6, 4, 2 when \( (K_l + G) \neq 0 \) and it has power on \( x \) or \( y \) as 6, 4, 2 when \( (K_l + G) = 0 \).

We let now \( u = y^2 \) the following third and fourth degree equations are
obtained:

\[
0 = (1024Gx^2 + 256G + 1280 + 1024x^2)u^3 + \\
(352 + 256Gx^2 + 3328x^2 + 1536Gx^4 + 1536x^4 + 96G)u^2 + \\
(-256Gx^4 + 1024Gx^6 - 576x^2 - 48 - 64Gx^2 + 2816x^4 + 16G + 1024x^6)u - 15 + \\
G - 16Gx^2 + 96Gx^4 - 256Gx^6 + 256Gx^8 - 848x^2 - 672x^4 + 768x^6 + 256x^8, \tag{2.2}
\]

\[
0 = (256G + 256Kl)u^4 + \\
(1024Gx^2 + 256G + 1280 + 1024x^2)u^3 + \\
(-256Gx^4 + 1024Gx^6 - 576x^2 - 48 - 64Gx^2 + 2816x^4 + 16G + 1024x^6)u - 15 + \\
G - 16Gx^2 + 96Gx^4 - 256Gx^6 + 256Gx^8 - 848x^2 - 672x^4 + 768x^6 + 256x^8. \tag{2.3}
\]

The roots of 2.2 and 2.3 are constructed by the methods of Cardano and Ferrari in the complex plane \( \mathbb{C} \). Therefore, \( O(l, G) = \{ (x, y) | x > 0, y = \sqrt{r(x, G, l)}, r(x, G, l) \geq 0 \} \).

**Remark 2.** The construction of the roots is not done by mathematical software, but it is done by finding and replacing the parameters of the formulas of the methods of Cardano and Ferrari with the coefficients of the polynomials of the equations 2.2 and 2.3. See Appendix 1.

**Proposition 3.** For the system of the previous proposition, when \( l = 1 \), we have

1. \( m_1 = -3 \).

2. \( O(1, -3) = \{ (0, \sqrt[3]{3}) \} \). It is a point the root of fourth degree equation for \( G_1 = -3 \).

3. There are two orbits from the roots of fourth degree equation with \( G_1 \in [m, -1) \).

4. There is one orbit from the roots of fourth degree equation with \( G_1 \in (-1, \infty) \).

**Proof.** It follows by direct substitution of \( l = 1 \) in the resulting roots of the previous proposition.

**Remark 4.** Also for \( l = 1 \) the figure \( \square \) depicts some examples of the orbits. The free point and the other two particles form an equilateral triangle of size 1, and minimum is \( B_2(0, \frac{\sqrt{3}}{2}) = -3 = m_1 \). There is one orbit from the roots of third degree equation with \( G_1 = -1 \), the details are given in section 4. It is easy to prove that polynomials \( Ax^{2k} + Bx^k + C, Ax^{3k} + Bx^{2k} + Cx^k + D, Ax^{4k} + Bx^{3k} + Cx^{2k} + Dx^k + E \ k > 1 \) are solvable by radicals.
3 Solving Polynomial by radicals

The Galois theory is the algebraic framework for the study of roots of polynomials. It focuses on the construction of formulae using radicals instead of in numerical estimations. For numerical estimation there is for instance the well know method of Newton–Raphson. It is known that there is no a general method or formula for finding the roots of a polynomial with degree \( n \geq 5 \). Here, we are interested in applying the known methods of Cardano and Ferrari for polynomials of degree 3 and 4 which are solvable by radicals. Appendix 1 depicts the formulas of Cardano and Ferrari.

The cubic polynomial \( f(x) = x^3 + ax^2 + bx + c \), was solved at the 16th century by more complicated formula found simultaneously by Ferro and Tartaglia. For the fourth degree polynomial \( f(x) = x^4 + ax^3 + bx^2 + cx + d \), Ferrari provided a procedure. These methods were published by Cardano in the Ars Magna at 1545. In 18th century, Lagrange unified these methods for a polynomial with degree \( n \leq 4 \) using what now is know as the resolvent of Lagrange. This method involves an auxiliary equation with a polynomial with degree less than one, i.e., it uses, for example, a third degree polynomial to solve the fourth degree polynomial. In fact, this is the procedure for solving the quadratic polynomial. However, this method fails for the polynomials of degree 5, because the auxiliary polynomial is of degree 6. Ruffini at 1799, and Abel at 1824 proved that there is not a general formula using radicals for finding the roots of quintic polynomial. Galois by 1832 showed how to associate to each polynomial \( f(x) \) a subgroup \( \text{Gal}(f) \) of the symmetric group, call the Galois group of \( f(x) \), and established the following result, for details see for instance [4]

**Theorem 5.** (Galois) A polynomial \( f \) is soluble by radicals if and only if its group \( \text{Gal}(f) \) is soluble.

As an application of these algebraic techniques of Cardano and Ferrari, we have the following result.

**Proposition 6.** Given a system of two particles, \( p_1 = (-\frac{l}{2}, 0, 0), p_2 = (\frac{l}{2}, 0, 0) \), the critical points of \( B_2(x, y) \) are the following

1. \((0, 0) \forall l \geq 0.\)
2. \((\pm x^*, 0), x^* \) is a positive root of the polynomial obtained from substituting \( y = 0 \) in \( \frac{\partial}{\partial x} B_2(x, y) \) with \( l > 2 \). The points \((\pm x^*, 0)\) are collinear with \( p_1, p_2 \).
3. \((0, \pm y^*) \) where \( y^* \) is a positive root of the polynomial obtained from substituting \( x = 0 \) in \( \frac{\partial}{\partial y} B_2(x, y) \). The points \((0, \pm y^*)\) correspond to the opposite vertex of an isosceles triangle of the side with vertices \( p_1, p_2 \).

**Proof.** The polynomial to solve comes from the first optimality condition: \( \frac{\partial}{\partial x} B_2(x, y) = 0 \) and \( \frac{\partial}{\partial y} B_2(x, y) = 0 \), where
\[
\frac{\partial}{\partial x} B_2(x, y) = -4 \frac{(x \pm \frac{l}{2})}{((x \pm \frac{l}{2})^2 + y^2)^2} + \frac{4(x \pm \frac{l}{2})}{((x \pm \frac{l}{2})^2 + y^2)^3},
\]
\[
\frac{\partial}{\partial y} B_2(x, y) = -4 \frac{y}{((x \pm \frac{l}{2})^2 + y^2)^3} + \frac{4y}{((x \pm \frac{l}{2})^2 + y^2)^2}.
\]

Note that from the previous equations, we have that \( \nabla B_2(0, 0) = 0 \), for all \( l > 0 \).

Using that \( y = 0 \) gives \( \frac{\partial}{\partial y} B_2(x, 0) = 0 \), the equation for \( \frac{\partial}{\partial x} B_2(u^2, 0) = 0 (x = u^2 \) is changed) is
\[
-64u^3 + (64 - 16l^2) u^2 + (20l^4 + 160l^2) u + (20l^4 - 3l^6) = 0. \tag{3.1}
\]

The positive roots of the previous third degree equation give the optimal points. In a similar way, using that \( x = 0 \) gives \( \frac{\partial}{\partial x} B_2(0, u^2) = 0 \), the equation for \( \frac{\partial}{\partial x} B_2(0, u^2) = 0 \) gives
\[
4y^2 + l^2 - 4 = 0 \tag{3.2}
\]

The real roots of the previous second degree equation give the optimal points for \( l \in [0, 2) \). The interval of \( l \) come from the restriction of the discriminant of the quadratic equation.

\[ \square \]

4 Levels set of the potential \( B_2 \)

The proposition 6 gives similar results for \( B_2 \) instead of LJ 12-6 to the triangular and collinear choreographies of the 3-body system discussed in [3]. We remark that our approach is done by basic analysis and algebraic techniques for the static case.

**Proposition 7.** Given a system of two particles of the proposition 6 and \( l \in (2, 0) \). If a free particle is at \( (0, y) \), where \( y = \pm \frac{1}{2} \sqrt{2^2 - l^2} \). Then the free particle is on a local point. Moreover, from the free particle’s point of view, the two particles act as one virtual particle of double \( B \) potential when \( l \to 0 \).

**Proof.** Without loss of generality, let \( y_1 = \frac{1}{2} \sqrt{2^2 - l^2} \). The free particle is perpendicular to middle point of the side with vertices \( (p_1, p_2) \). The result follows immediately from proposition 6 and the equation 3.2 which corresponds to the condition \( \nabla B_2(0, y_1) = 0 \). Also, the free particle position goes to \( (0, 2) \) when \( l \to 0 \), which is the double of the optimal distance \( (d^* = 1) \) of the function \( B \).

**Proposition 8.** Given a system of two particles of the proposition 6 with \( l = 1 \). Then the minimal points of a free particle is at \( (0, y) \), where \( y = \pm \sqrt{2} \). The particles are the vertices of an equilateral triangle.
Figure 1: The true orbits for $O(1, -3)$, $O(1, -2.75)$, $O(1, -2.43755)$, $O(1, -2)$, $O(1, -1)$, $O(1, 0)$, and $O(1, 100)$.

Figure 2: 3-d model of the level sets of $B_2(x, y, z) = G$, where $G \in [3, \infty)$.

**Proof.** Without loss of generality, let $y_1 = \frac{1}{2} \sqrt{3}$, then the distance between the three particles is 1. The Hessian of the system at $(0, y_1)$ is

$$|\nabla^2 B_2(0, \frac{\sqrt{3}}{2})| = -64 \times (-3).$$

The optimality follows from proposition 6 and $|\nabla^2 B_2(0, y_1)| > 0$. \hfill \QED

**Proposition 9.** Given a system of two particles of the proposition 6, and $l \in (2, 0)$. Then the orbit of $B_2$ of value $G_y = B_2(0, y)$ is the pointed set $O(l, G_y) = \{(0, y) \in \mathbb{R}^+ \}$ where $y = \frac{1}{2} \sqrt{2^2 - l^2}$.

**Proof.** By the symmetry the space is restricted to $\mathbb{R}^+$. The Hessian of the system at $(0, y)$ is

$$|\nabla^2 B_2(0, \frac{1}{2} \sqrt{2^2 - l^2})| = -64l^2 \times (l^2 - 4).$$

The polynomial $-64l^2 \times (l^2 - 4)$ is strictly positive for $l \in (0, 2)$. This means that the positive root of the equation 3.2 corresponds to the optimality conditions $\nabla B_2(0, y) = 0$ and $|\nabla^2 B_2(0, y)| > 0$. Therefore $O(l, G_y)$ is the minimal point $(0, y)$.

For $G = -3$ the four roots of fourth degree equation 2.3 (with $l = 1$) are

- $r_1(x) = \frac{1}{4} - x^2 - \frac{1}{2} \sqrt{4x^2 - 4ix + 1}$,
- $r_2(x) = \frac{1}{4} \sqrt{4x^2 - 4ix + 1} - x^2 + \frac{1}{4}$,
- $r_3(x) = \frac{1}{4} - x^2 - \frac{1}{2} \sqrt{4x^2 + 4ix + 1}$,
- $r_4(x) = \frac{1}{2} \sqrt{4x^2 + 4ix + 1} - x^2 + \frac{1}{4}$.

Only for $x = 0$, $r_2(0) = \frac{3}{4} > 0$, therefore the orbit of $O(1, -3)$ is the point set $\{(0, \frac{\sqrt{3}}{2})\}$. Which is the optimal point of the equilateral triangle of size $= 1$.\hfill \QED
Here, our method gives the result without appealing to the optimal conditions as in proposition \textsuperscript{8} and in proposition \textsuperscript{9}.

The detailed analysis of the third and fourth degree equations of proposition \textsuperscript{11} for \( l = 1 \) was done for verifying the following:

1. \( m_1 = -3 \).
2. \( O(1, -3) = \{(0, \frac{\sqrt[3]{3}}{3})\} \), i.e., it is a point the root of fourth degree equation for \( G_1 = -3 \).
3. There are two orbits from the roots of fourth degree equation with \( G_1 \in [m, -1) \)
4. There is one orbit from the roots of third degree equation with \( G_1 = -1 \).
5. There is one orbit from the roots of fourth degree equation with \( G_1 \in (-1, \infty) \).

Figure \textsuperscript{11} depicts some examples of the orbits that comply the previous statements. Figure \textsuperscript{22} depicts a 3-d model of the level sets for \( l = 1 \) and \( G \in [-3, \infty) \).

5 Conclusions

To our knowledge this is the first totally complete description of the level sets of a system of two particles under a Van der Waals Potential.

We have presented a novel approach for the study of orbits of systems of non interactive particles. We tackle the problem by reducing the analysis of the equipotential zones to the one of finding the roots of certain polynomials. A constructive method provided by the methods of Cardano and Ferrari, yields a complete factorization of the polynomials and consequently an analytical description of equipotential zones.

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Appendix 1. Formulae of Cardano and Ferrari for \textsuperscript{21}

The third degree polynomial is obtained from the equation \textsuperscript{2223} with \( u = y^2 \). Its polynomial corresponds to \( Ax^3 + Bx^2 +Cx + D = 0 \). And its root come from the following formula of Cardano:

\[
x_1 = \left(\left(\frac{1}{3}((B)(A)^{-1})^3-\frac{1}{3}((B)(A)^{-1})((C)(A)^{-1})+(D)(A)^{-1})\right)+\left(\left(\frac{A}{2}\right)((B)(A)^{-1})^3-\frac{1}{3}((B)(A)^{-1})((C)(A)^{-1})+(D)(A)^{-1})\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}
\]

The fourth degree polynomial is obtained from the equation \textsuperscript{23}. For the equation \textsuperscript{23} with polynomial type \( Ax^4 + Bx^3 + Cx^2 + Dx + E = 0 \) the four corresponding roots come from the following four formulae of Ferrari.
The first root is

\[
x_1 = -\frac{1}{3}B(A)^{-1} + \frac{1}{4}\left(\left(-\frac{3}{B}(B)^2(A)^{-2} - (C)(A)^{-1}\right)^2 + 2\left(-\frac{1}{3}\left(-\frac{3}{B}(B)^2(A)^{-2} + (C)(A)^{-1}\right)\right)^2 + \left(-\frac{3}{B}(B)^2(A)^{-2} + (C)(A)^{-1}\right)^2\right)^{1/2}
\]
\((A)^{-4} + \frac{1}{A}(C)(B)^2(A)^{-3} - \frac{1}{A}(B)(D)(A)^{-2} + (E)(A)^{-1})\))^{\frac{1}{2}} - 1)^{\frac{1}{2}} - 1)\).

The second root is

\[
x_2 = -\frac{1}{A}(B)(A)^{-1} + \frac{1}{A}(B)(D)(A)^{-2} + (E)(A)^{-1})\))^{\frac{1}{2}} - 1)^{\frac{1}{2}} - 1)\).
\]
\[ (C)(A)^{-2} + (D)(A^{-1})^2)^2 + \frac{1}{4}(-\frac{1}{2}(\frac{1}{3}B)(A^{-2} + (C)(A^{-1})^2 - (-\frac{2}{9}(B)^4 (A^{-4} + \frac{1}{2}(C)(B)^2(A^{-3} - \frac{1}{2}(B)(D)(A^{-2} + (E)(A^{-1})^2)^2))^{-1})^{-1})^{-1})^2 \].

The third root is

\[ x_3 = -\frac{1}{4}(B)(A^{-1}) - \frac{1}{4}((-((\frac{1}{3}B)^2(A^{-2} + (C)(A^{-1})^{-2} + 2(-\frac{2}{9}(B)(A^{-2} + (C)(A^{-1})^{-3} + \frac{1}{2}(\frac{1}{3}B)(A^{-2} + (C)(A^{-1})^{-3} - \frac{1}{2}(B)(D)(A^{-2} + (E)(A^{-1})^{-1})^{-1})^{-1})^{-1}))^{-1} + \frac{1}{4}(-\frac{1}{8}(\frac{1}{3}B)^2(A^{-2} + (C)(A^{-1})^{-2} + (D)(A^{-1})^{-2} + \frac{1}{8}(\frac{1}{3}B)^2(A^{-2} + (C)(A^{-1})^{-2} - \frac{1}{2}B)(D)(A^{-2} + (E)(A^{-1})^{-1})^{-1} - \frac{1}{8}(\frac{1}{3}B)^2(A^{-2} + (C)(A^{-1})^{-2} + (D)(A^{-1})^{-2} + \frac{1}{8}(\frac{1}{3}B)^2(A^{-2} + (C)(A^{-1})^{-2} - \frac{1}{2}B)(D)(A^{-2} + (E)(A^{-1})^{-1})))^{-1})^{-1})^{-1} \]
\[
(A)^{-4} + \frac{1}{16}(C)(B)^2(A)^{-3} - \frac{3}{8}(B)(D)(A)^{-2} + (E)(A)^{-1} - \frac{3}{8}(B)^3(A)^{-3} - \frac{1}{8}(B)(C)(A)^{-2} + (D)(A)^{-1} + 2(\frac{3}{8}(B)^2(A)^{-2} + (C)(A)^{-1}) + ((-\frac{3}{8}(B)^2(A)^{-2} + (C)(A)^{-1}) + \frac{3}{8}(B)^3(A)^{-3} - \frac{3}{8}(B)(C)(A)^{-2} + (D)(A)^{-1})^2 + \frac{1}{8}((\frac{3}{8}(B)^2(A)^{-2} + (C)(A)^{-1}) + \frac{3}{8}(B)^3(A)^{-3} - \frac{3}{8}(B)(C)(A)^{-2} + (D)(A)^{-1}))^2).
\]

Finally, the fourth root is

\[
x_4 = -\frac{1}{4}(B)(A)^{-1} + \frac{1}{4}(-(\frac{3}{8}(B)^2(A)^{-2} + (C)(A)^{-1}) + 2(\frac{3}{8}(B)^2(A)^{-2} + (C)(A)^{-1}) + ((-\frac{3}{8}(B)^2(A)^{-2} + (C)(A)^{-1}) + \frac{3}{8}(B)^3(A)^{-3} - \frac{3}{8}(B)(C)(A)^{-2} + (D)(A)^{-1}))^2 + \frac{1}{8}((\frac{3}{8}(B)^2(A)^{-2} + (C)(A)^{-1}) + \frac{3}{8}(B)^3(A)^{-3} - \frac{3}{8}(B)(C)(A)^{-2} + (D)(A)^{-1}))^2).
\]
\[ (\frac{1}{4}(-\frac{1}{16}(\frac{-3}{8}(B)^2(A)^{-2}+(C)(A)^{-1})^3+\frac{1}{4}(\frac{-3}{8}(B)^2(A)^{-2}+(C)(A)^{-1})(\frac{1}{4}(B)^4(A)^{-4}+\frac{1}{16}(C)(B)^2(A)^{-3}-\frac{1}{4}(B)(D)(A)^{-2}+(E)(A)^{-1})-\frac{1}{8}(\frac{1}{8}(B)^2(A)^{-2}+(C)(A)^{-1})^2-\frac{1}{8}(C)(B)^2(A)^{-3}-(\frac{1}{4}(B)(D)(A)^{-2}+(E)(A)^{-1}))^3+\frac{1}{8}(E)(A)^{-1}))+\frac{1}{8}(E)(A)^{-1})). \]

Remark 10. The construction of the roots is done by a text editor, replacing the parameters \((A, B, C, D, \text{ and } E)\) of the previous root formulae with the coefficients of given polynomials. The parenthesis are necessary to avoid errors when complex term are replaced, by example, replacing \(-256Gx^4+1024Gx^6-576x^2-48-64Gx^2+2816x^4+16G+1024x^6\) by \(D\) for the formulae of Ferrari.

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