Quantum complementarity and entanglement are fundamental features of Quantum Mechanics. The purity of a bipartite quantum state $|\Psi\rangle_{AB}$ was shown to imply in complete complementarity relations for the quantum $A$ or $B$. These triality equalities involve quantum coherence $C(\rho_A)$, the wave aspect of a quont, and quantum predictability $P(\rho_A)$ and quantum entanglement $E(|\Psi\rangle_{AB})$, the particle aspect of a quont. Recently, we reported a resource theory for quantum predictability, which quantifies the “path” information from the knowledge of the qont’s density matrix. In this article, we give an operational interpretation to predictability in entanglement swapping from initially partially entangled states. Entanglement swapping is an essential primitive for quantum communication where one starts with two pairs of entangled qubits and makes a Bell-basis measurement (BBM) on one qubit from each pair, ending up maximally entangling the other two qubits, that possibly have never interacted directly. Here we start by noticing that there is a plethora of partially entangled states as well as maximally entangled states that can be obtained after a BBM. Then, we show that the predictability of the pre-measurement one-qubit density matrix is directly related to the probability of the partially entangled component of the post-measurement state. Going even further, we analyze the entanglement swapping for partially entangled states in the light of complementarity relations and show that, in the cases where the entanglement increases after the BBM, the predictability is consumed when compared to the initially prepared state. Moreover, the triality relations will always remain valid before and after the BBM, which means the we have a interplay between predictability and entanglement before and after the BBM.

Keywords: Complementarity relations; Predictability; Entanglement; Entanglement swapping

I. INTRODUCTION

It is undeniable that the development of the field of Quantum Information has led to an ever increasing interest in the foundations of Quantum Mechanics (QM), or Quantum Foundations (QF). Since the beginning of QM, one of the pillars of QF is the famous Bohr’s complementarity principle [1], in which the wave-particle duality emerged as the main example. Ever since, countless paths have been traced for quantifying the wave-particle properties of a quantum system (quonts) in terms of an elegant complementarity relation between predictability and visibility [2–11]. The generalization of such complementarity relation for $d$-dimensional quonts (qudits) came with the realization that the quantum coherence [12] is the natural generalization for the visibility of an interference pattern [6, 7, 13]. As well, inspired by the mathematical development of QM and its consequent axiomatization, the authors in Ref. [10], by exploring the properties of the density matrix of a quont $A$, derived several complementarity relations between different measures of quantum coherence and predictability, with both satisfying the criteria established in Refs. [14, 15] for bone-fide measures of visibility and predictability. However, to fully characterize a quont, it is not enough to consider its wave-particle aspect, one also has to account its correlations with other systems. For this, triality relations, also known as complete complementarity relations (CCRs), involving predictability, visibility and entanglement, were first suggested by Jakob and Bergou [16] provided that a bipartite (or multipartite) quantum system $AB$ is in a pure state. In contrast, by taking the purity of the bipartite quantum system as the main hypothesis, the authors in Ref. [17] derived CCRs of the type

$$C_{re}(\rho_A) + P_{vn}(\rho_A) + S_{vn}(\rho_A) = \log_2 d_A,$$

(1)

where $C_{re}(\rho_A) := S_{vn}(\rho_{Adiag}) - S_{vn}(\rho_A)$ is the relative entropy of quantum coherence, with $\rho_{Adiag}$ being the diagonal part of $\rho_A$, $P_{vn}(\rho_A) = \log_2 d_A - S_{vn}(\rho_{Adiag})$ is the corresponding bone-fide predictability measure and $S_{vn}(\rho_A) = - \text{Tr} \rho_A \log_2 \rho_A$ is the von Neumann entropy, which is an entanglement monotone. Besides, the first order approximation of Eq. (1) was also obtained in Ref. [17], and it is the CCR based on the linear entropy: $C_{hs}(\rho_A) + P_1(\rho_A) + S_1(\rho_A) = (d_A - 1)/d_A$, where $C_{hs}(\rho_A)$ is Hilbert-Schmidt quantum coherence [12], $P_1(\rho_A) := (d_A - 1)/d_A - S_1(\rho_{Adiag})$ and $S_1(\rho_A) = 1 - \text{Tr} \rho_A^2$. Besides, the predictability measure together with an entanglement monotone can be considered as a measure of path distinguishability in an interferometer, as recently discussed in

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As well, the CCR given by Eq. (1) has several interesting properties that were explored recently in the literature: it is intrinsically connected to the notion of realism defined by Bilobran and Angelo [20], as shown in Ref. [21]; together with the CCR based on the linear entropy, it is invariant by global unitary operations, which implies that it is preserved under unitary evolution and it is invariant under Lorentz transformations [22], which by its turn made possible the extension of the validity of the CCRs to curved spacetimes [23].

Another remarkable aspect of Eq. (1) is that both quantum coherence and quantum entanglement are properties of quantons that can be used as resources for tasks in Quantum Information and Quantum Computation. This motivated us [24] to put forward a quantum resource theory of predictability by relating the predictability of a given state with reference to an observable X with quantum coherence with reference to observables mutually unbiased (MU) to X of the state obtained from a non-revealing von Neumann measurement of X as well as identifying its free states, free operations and resource states together with resource measures, in which $P_{\text{en}}(\rho_A)$ and $P(\rho_A)$ are examples. In this manuscript, we give another step in this direction and report an operational interpretation of predictability using a well known protocol of quantum information and relating the predictability of the pre-measurement one-qubit state with the probability of obtaining the maximally entangled component of the post-measurement state. To do this we use the protocol of entanglement swapping from initially partially entangled pure states.

Entanglement swapping is an essential primitive for quantum communication where one starts with two pairs of entangled qubits and makes a Bell-basis measurement (BBM) on one qubit from each pair, ending up maximally entangling the other two qubits, that possibly have never interacted directly [25]. Entanglement swapping from initially partially entangled states was first studied in Ref. [26], for a particular class of initial states. In this article we make a thorough analysis by studying a more general class of initially partially entangled states and by showing that there is a plethora of partially entangled states as well as maximally entangled states that can be obtained after a BBM. Besides, for the partially entangled states obtained after a BBM, it is possible, by local operations, to purify this entanglement into a smaller number of maximally entangled pairs of quantons [27]. This, as we will show, lead to a naturally operational interpretation for predictability. Going even further, we analyze the entanglement swapping for partially entangled states in the light of complementarity relations and show that, in the cases where the entanglement increases after the BBM, the predictability is consumed when compared to the initially prepared state. Moreover, the triality relations will always remain valid before and after the BBM, which means we have an interplay between predictability and entanglement before and after the BBM process.

### II. ENTANGLEMENT SWAPPING FROM PARTIALLY ENTANGLED PURE STATES

Let us consider three laboratories handled by Alice, Bob, and Charlie. In addition, let us consider that Darwin, in another laboratory, prepares two pairs of qubits in a partially entangled pure state:

$$|\xi\rangle_{AC} = \sqrt{p}|00\rangle_{AC} + \sqrt{1-p}|11\rangle_{AC},$$  \hspace{1cm} (2)

$$|\eta\rangle_{C'B} = \sqrt{q}|00\rangle_{C'B} + \sqrt{1-q}|11\rangle_{C'B}.$$  \hspace{1cm} (3)

with $p, q \in [0,1]$. The quantons $C$ and $C'$ are sent to Charlie, who makes a selective Bell basis measurement (BBM) on them. The quanton $A$ is sent to Alice and the quanton $B$ is sent to Bob. It is worth mentioning here that, in Ref. [26], the authors studied entanglement swapping of partially entangled pure states only for $p = q$. Therefore, the analysis of entanglement swapping of partially entangled pure states made in this manuscript is more general. The composed state of the four qubits can be written as

$$\sqrt{2}|\xi\rangle_{AC}|\eta\rangle_{C'B} =$$ \hspace{1cm} (4)

$$|\Phi_+\rangle_{CC'}(\sqrt{pq}|00\rangle_{AB} + \sqrt{(1-p)(1-q)}|11\rangle_{AB}) +$$

$$|\Phi_-\rangle_{CC'}(\sqrt{pq}|00\rangle_{AB} - \sqrt{(1-p)(1-q)}|11\rangle_{AB}) +$$

$$|\Psi_+\rangle_{CC'}(\sqrt{p(1-q)}|01\rangle_{AB} + \sqrt{(1-p)q}|10\rangle_{AB}) +$$

$$|\Psi_-\rangle_{CC'}(\sqrt{p(1-q)}|01\rangle_{AB} - \sqrt{(1-p)q}|10\rangle_{AB}).$$

After Charlie makes a BBM, the possible post-measurement states are given by

$$|\phi_\pm\rangle_{AB} = \frac{1}{N_\phi}(\sqrt{pq}|00\rangle_{AB} \pm \sqrt{(1-p)(1-q)}|11\rangle_{AB}),$$ \hspace{1cm} (5)

$$|\psi_\pm\rangle_{AB} = \frac{1}{N_\psi}(\sqrt{p(1-q)}|01\rangle_{AB} \pm \sqrt{(1-p)q}|10\rangle_{AB}),$$ \hspace{1cm} (6)

with $N_\phi = \sqrt{pq + (1-p)(1-q)}$ and $N_\psi = \sqrt{p(1-q) + (1-p)q}$. The states $|\phi_\pm\rangle_{AB}$ and $|\psi_\pm\rangle_{AB}$ are obtained with probabilities:

$$Pr(\phi_\pm) = \frac{1}{2}(pq + (1-p)(1-q)), \hspace{1cm} (7)$$

$$Pr(\psi_\pm) = \frac{1}{2}(p(1-q) + (1-p)q), \hspace{1cm} (8)$$

which are the same as the probabilities for Charlie to obtain the Bell states $|\Phi_\pm\rangle_{CC'}$ and $|\Psi_\pm\rangle_{CC'}$ respectively. Besides, it is easy to see that if we set $p = q = 1/2$, the probability of any of the BBM outcomes is 1/4 and the quantons $A$ and $B$ end up in a Bell (maximally entangled) state. As for the general case, we have that the von Neumann entropy of the local reduced state, $\rho_A^\phi = Tr_B(\rho_\phi_{AB} \langle \phi_\pm \rangle)$ and $\rho_A^\psi = Tr_B(\rho_\psi_{AB} \langle \psi_\pm \rangle)$, is an entanglement monotone given, respectively, by $S_{\text{en}}(\rho_A^\phi)$ =
are entangled in general, except when $\rho_{AB}$ will end up in a partially entangled state. For instance, if $p = 0.1$ and $q = 0.75$, the initial states $|\xi\rangle_{AC}$ and $|\eta\rangle_{CD}$ have $S_{vn}(\rho_B^\phi) = S_{vn}(\rho_C^\phi) \approx 0.4689$ and $S_{vn}(\rho_B^\psi) = S_{vn}(\rho_C^\psi) \approx 0.8112$, respectively. Meanwhile, Alice and Bob will obtain the post-measurement states $|\phi^\pm\rangle_{AB}$ and $|\psi^\pm\rangle_{AB}$ with $S_{vn}(\rho_B^\phi) = S_{vn}(\rho_B^\psi) \approx 0.8112$ with probability $Pr(\phi^\pm) = 0.15$ and $S_{vn}(\rho_B^\phi) = S_{vn}(\rho_B^\psi) \approx 0.2222$ with $Pr(\psi^\pm) = 0.35$. Thus, one can see that Alice and Bob obtain the post-measurement states with the lower entanglement more frequently. However, if there is a large enough ensemble of partially entangled quantons in the beginning of the protocol, it is possible, by local operations, to purify this entanglement into a smaller number of maximally entangled pairs of quantons.

Now, for completeness, let us consider the case specified by $p = 1 - q$, for which

\[
|\xi\rangle_{AC}|\eta\rangle_{CD} = \\
\sqrt{(1-q)q}|\Phi^+_\psi\rangle_{AB} + \sqrt{(1-q)q}|\Phi^-\psi\rangle_{AB} + |\Psi^+_\psi\rangle_{CC} - \frac{1}{\sqrt{2}}(\sqrt{(1-q)^2}|01\rangle_{AB} + \sqrt{(1-q)^2}|01\rangle_{AB}) + |\Psi^-\psi\rangle_{CC} - \frac{1}{\sqrt{2}}(\sqrt{(1-q)^2}|01\rangle_{AB} - \sqrt{(1-q)^2}|01\rangle_{AB}).
\]

If Charlie makes a BBM, then he, as well as Alice and Bob, will end up with the state $|\Phi^\pm\rangle$ with the probability $Pr(\Phi^\pm) = q(1 - q)$. In the other cases, Charlie will end up with states $|\Psi^\pm\rangle$ with probabilities $Pr(\Phi^\pm) = \frac{1}{2}(1 - q^2 + q^2)$ and Alice and Bob will share a partially entangled state given by $|\psi^\pm\rangle$. For $q = 0$, $Pr(\Phi^\pm) = 0.4901$, $Pr(\Phi^\pm) = 0.0099$ while for $q = 0.75$, $Pr(\Phi^\pm) = 0.3125$, $Pr(\Phi^\pm) = 0.1875$. Actually, one can see in Fig. 2(a), $Pr(\Phi^\pm) > Pr(\Phi^\pm)$ for all $q \in [0, 1]$. However, there is a non-null probability that Alice and Bob will end up with a maximally entangled state even when the initial states are partially entangled. As already noticed in Ref. [26], she actually increases the magnitude of entanglement she shares with Bob. However, the authors did not explain from where this extra entanglement came from. And one of main goals of this article
is to explain the origin of this extra entanglement. In the other cases, Alice and Bob will end up with lower entanglement states in comparison with the initial ones. However, as was noticed in Ref. [26], if there is a large enough ensemble of partially entangled quantons in the beginning of the protocol, it is possible, by local operations, to change the states of a certain fraction of the shared pairs of Alice and Bob to Bell states at the cost of decreasing the entanglement of the other shared pairs even further.

### III. THE OPERATIONAL INTERPRETATION OF PREDICTABILITY

It is intriguing the fact that, for $p = q$ or $p = 1 - q$, there is a non-null probability of obtaining a maximally entangled state when we start with initial states that partially (even almost not) entangled. However, this fact turns out to be very reasonable provided that we are partially (even almost not) entangled. However, this can be done for instance, if, initially, we have an interplay between predictability and entanglement before and after the BBM, which means the we have an interplay between predictability and entanglement before and after the BBM.

Beyond that, if $p = 1 - q$ the predictability of the pre-measurement one-qubit density matrix is directly related to the probability of Charlie obtaining the states $|\Psi_{\pm}\rangle$ and Alice and Bob sharing the state the states $|\psi_{\pm}\rangle$, i.e., by making a first order expansion of Eq.(11), it is possible to see that

$$P_{vn}(\rho_j^i) = \frac{1}{\ln 2} (q^2 + (1 - q)^2) + \left(1 - \frac{1}{\ln 2}\right) + O(q^3)$$

$$= \frac{1}{\ln 2} (Pr(\Psi_+) + Pr(\Psi_-)) + \left(1 - \frac{1}{\ln 2}\right) + O(q^3),$$

for $j = A, B$, where $Pr(\Psi_{\pm}) = \frac{1}{2} (q^2 + (1 - q)^2)$ is the probability of Charlie ending up with the one of the states $|\Psi_{\pm}\rangle$ and we used the fact that $\ln x \simeq x - 1$. As pointed out in the introduction, the linear predictability $P_l$ can be seen as the first order expansion of $P_{vn}$, which implies that we have the following exact result:

$$Pr(\Psi_{\pm}) = \frac{1}{2} \left(\frac{1}{2} + P_l(\rho_j^i)\right).$$

For the maximally entangled components of the state in Eq. (9), we have

$$Pr(\Phi_{\pm}) = \frac{1}{2} \left(\frac{1}{2} - P_l(\rho_j^i)\right).$$

For instance, if, initially, we have $P_l(\rho_j^i) = 0$, it means that $p = q = 1/2$ and the initial states are maximally entangled. Therefore, the probability of any of the BBM outcomes is 1/4 and the quantons $A$ and $B$ end up in a Bell (maximally entangled) state. In the other cases, we can see that the probability of obtaining partially entangled states after BBM is directly proportional to the predictability of the initial one-qubit states, as one can see in Fig. 2(a). This means that we have the following operational interpretation of predictability in terms of entanglement: the pre-measurement predictability is directly related to the probability of Alice and Bob obtaining partially entangled states after Charlie makes a BBM, which, by its turn, with a large enough copies, can be used for entanglement distillation. Otherwise, if the initial predictability is zero, Alice and Bob will end up with a maximally entangled state and they will not need to purify their sample. In other words, if, initially, Alice, Bob and Charlie have a large enough ensemble of quantum system prepared in the state given by Eq. (9) and, for instance, if such states have a high local predictability (which means that the entanglement is initially small), after Charlie makes a BBM on these states, Alice and Bob will end up with a large fraction of partially entangled states that, in turn, can be used for entanglement distillation, and therefore, a certain fraction of the shared pairs of Alice and Bob will be maximally entangled. The same interpretation holds for $p = q$. 

$$P_{vn}(\rho_A^i) = P_{vn}(\rho_B^i),$$

$$P_{vn}(\rho_B^i) + S_{vn}(\rho_B^i) = 1.$$
IV. CONCLUSIONS

In this work, we gave an operational interpretation of the predictability using a well-known protocol of quantum information, i.e., we used the protocol of entanglement swapping from initially partially entangled states. To do this, we studied such protocol by considering a more general class of initially partially entangled states than was regarded in Ref. [26]. We have found that there is a plethora of partially entangled states as well as maximally entangled states that can be obtained after a Bell-basis measurement (BBM). Besides, for the partially entangled states obtained after the BBM, it is possible, by local operations, to purify this entanglement into a smaller number of maximally entangled pairs of quantons. This, as we showed, led to a natural operational interpretation for predictability of the prepared states in terms of the probability of Alice and Bob obtaining partially entangled states after Charlie makes a BBM, which can be used for entanglement distillation. In contrast, if the initial predictability is zero, Alice and Bob will end up with a maximally entangled state and they will not need to purify their qubits. In addition, we analyzed the entanglement swapping for partially entangled states in the light of complete complementarity relations and showed that, in the cases where the entanglement increases after the BBM, the predictability is consumed when compared to the initially prepared state. Moreover, $P_{vn}(\rho_j) + S_{vn}(\rho_j) = 1$ will always remain valid before and after the BBM, which means the we have a interplay between predictability and entanglement before and after the BBM.

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