Hausdorff operators on holomorphic Hardy spaces and applications

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The aim of this paper is to characterize the non-negative functions $\varphi$ defined on $(0, \infty)$ for which the Hausdorff operator

$$H_\varphi f(z) = \int_0^\infty f\left(\frac{z}{t}\right) \frac{\varphi(t)}{t} \, dt,$$  

is bounded on the Hardy spaces of the upper half-plane $\mathcal{H}_p^0(\mathbb{C}_+), p \in [1, \infty]$. The corresponding operator norms and their applications are also given.

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1. Introduction and the main result

Let $\varphi$ be a locally integrable function on $(0, \infty)$. The Hausdorff operator $H_\varphi$ is then defined for suitable functions $f$ on $\mathbb{R}$ by

$$H_\varphi f(x) = \int_0^\infty f\left(\frac{x}{t}\right) \frac{\varphi(t)}{t} \, dt, \quad x \in \mathbb{R}. $$  

(1.1)

The Hausdorff operator is an interesting operator in harmonic analysis. There are many classical operators in analysis which are special cases of the Hausdorff operator if one chooses suitable kernel functions $\varphi$, such as the classical Hardy operator, its adjoint operator, the Cesàro type operators, the Riemann–Liouville fractional integral operator, etc., see the survey article [15] and the references therein. In the recent years, there is an increasing interest in the study of boundedness of the Hausdorff operator and its commuting with the Hilbert transform on the real Hardy spaces and on the Lebesgue spaces, see for example [1–4, 10, 13, 15–17, 20].
Let $C_+$ be the upper half-plane in the complex plane. For $0 < p \leq \infty$, the Hardy space $H^p_a(C_+)$ is defined as the set of all holomorphic functions $f$ on $C_+$ such that

$$
\|f\|_{H^p_a(C_+)} := \sup_{y > 0} \left( \int_{-\infty}^{\infty} |f(x + iy)|^p \, dx \right)^{1/p} < \infty
$$

if $0 < p < \infty$, and if $p = \infty$, then

$$
\|f\|_{H^\infty_a(C_+)} := \sup_{z \in C_+} |f(z)| < \infty.
$$

It is classical (see [6, 9]) that if $f \in H^p_a(C_+)$, then $f$ has a boundary value function $f^* \in L^p(\mathbb{R})$ defined by

$$
f^*(x) = \lim_{y \to 0} f(x + iy), \quad \text{a.e. } x \in \mathbb{R}.
$$

Let $p \in [1, \infty]$ and let $\varphi$ be a non-negative function in $L^1_{\text{loc}}(0, \infty)$ for which

$$
\int_0^\infty t^{1/p - 1} \varphi(t) \, dt < \infty.
$$

Then it is well-known (see [1]) that $H_\varphi$ is bounded on $L^p(\mathbb{R})$, and thus $H_\varphi(f^*) \in L^p(\mathbb{R})$ for any boundary value function $f^*$ of a function $f$ in $H^p_a(C_+)$. A natural question arises is whether the transformed function $H_\varphi(f^*)$ is also the boundary value function of a function in $H^p_a(C_+)$? In some special cases of $\varphi$ and $1 < p < \infty$, using the spectral mapping theorem and the Hille–Yosida–Phillips theorem, Arvanitidis and Siskakis [2] and Ballamoole et al. [3] studied and gave affirmative answers to this question.

In the present paper, we give an affirmative answer to the above question by studying a complex version of $H_\varphi$ defined by

$$
H_\varphi f(z) = \int_0^\infty f \left( \frac{z}{t} \right) \varphi(t) \, dt, \quad z \in C_+.
$$

Our main result reads as follows.

**Theorem 1.1.** Let $p \in [1, \infty]$ and let $\varphi$ be a non-negative function in $L^1_{\text{loc}}(0, \infty)$. Then $H_\varphi$ is bounded on $H^p_a(C_+)$ if and only if (1.2) holds. Moreover, in that case, we obtain

$$
\|H_\varphi\|_{H^p_a(C_+) \to H^p_a(C_+)} = \int_0^\infty t^{1/p - 1} \varphi(t) \, dt
$$

and, for any $f \in H^p_a(C_+)$,

$$
(H_\varphi f)^* = H_\varphi(f^*).
$$

It should be pointed out that some main results in [2, 3] (see [2, theorems 3.1, 3.3 and 4.1] and [3, theorem 3.4]) can be viewed as special cases of theorem 1.1 by choosing suitable kernel functions $\varphi$. In the setting of Hardy spaces $H^p(\mathbb{D})$ on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, Galanopoulos and Papadimitrakis ([8, theorems 2.3

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and 2.4) studied and obtained some similar results to theorem 1.1 for $1 < p < \infty$ while it is slightly different at the endpoints $p = 1$ and $p = \infty$ (see also the survey article [15]).

Furthermore, if we denote by $\mathcal{H}^1(\mathbb{R})$ the real Hardy space in the sense of Fefferman-Stein (see the last section), then by using theorem 1.1, we obtain the following result.

**Corollary 1.2** (see theorem 3.8). Let $\varphi$ be a non-negative function in $L^1_{\text{loc}}(0, \infty)$ such that $H_\varphi$ is bounded on $\mathcal{H}^1(\mathbb{R})$. Then,

$$\int_0^\infty \varphi(t) dt \leq \|H_\varphi\|_{\mathcal{H}^1(\mathbb{R})} < \infty.$$  

The above corollary not only gives an answer to a question posted by Liflyand [13, Problem 4], but also gives a lower bound for the norm of $H_\varphi$ on $\mathcal{H}^1(\mathbb{R})$. Another corollary of theorem 1.1 is:

**Corollary 1.3** (see theorem 3.1). Let $p \in (1, \infty)$ and let $\varphi$ be as in theorem 1.1. Then $H_\varphi$ is bounded on $L^p(\mathbb{R})$ if and only if (1.2) holds. Moreover, in that case,

$$\|H_\varphi\|_{L^p(\mathbb{R})} = \frac{1}{\|\varphi\|_{L^1(\mathbb{R})}} \int_0^\infty t^{1/p} \varphi(t) dt$$

and $H_\varphi$ commutes with the Hilbert transform $H$ on $L^p(\mathbb{R})$.

Throughout the whole article, we use the symbol $A \lesssim B$ (or $B \gtrsim A$) means that $A \leq CB$ where $C$ is a positive constant which is independent of the main parameters, but it may vary from line to line. If $A \lesssim B$ and $B \gtrsim A$, then we write $A \sim B$. For any $E \subset \mathbb{R}$, we denote by $\chi_E$ its characteristic function.

**2. Proof of Theorem 1.1**

In the sequel, we always assume that $\varphi$ is a non-negative function in $L^1_{\text{loc}}(0, \infty)$. Also we remark that, for any $f \in \mathcal{H}^p_a(\mathbb{C}_+)$, the function $\mathcal{H}_\varphi f$ is well-defined and holomorphic on $\mathbb{C}_+$ provided (1.2) holds, since

$$|f(x + iy)| \leq \left(\frac{2}{\pi y}\right)^{1/p} \|f\|_{\mathcal{H}^p_a(\mathbb{C}_+)} \tag{2.1}$$

and

$$|f'(x + iy)| \leq \frac{2}{y} \left(\frac{4}{\pi y}\right)^{1/p} \|f\|_{\mathcal{H}^p_a(\mathbb{C}_+)} \tag{2.2}$$

for all $z = x + iy \in \mathbb{C}_+$. See Garnett’s book [9, p. 57].

Given an holomorphic function $f$ on $\mathbb{C}_+$, we define the non-tangential maximal function of $f$ by

$$\mathcal{M}(f)(x) = \sup_{|t-x|<y} |f(t + iy)|, \quad x \in \mathbb{R}.$$  

The following lemma is classical and can be found in [6,9].
Lemma 2.1. Let $0 < p < \infty$. Then:

(i) For any $f \in \mathcal{H}_a^p(\mathbb{C}_+)$, we have

$$\|f^*\|_{L^p(\mathbb{R})} = \|f\|_{\mathcal{H}_a^p(\mathbb{C}_+)} \quad \text{and} \quad \lim_{y \to 0} \|f(\cdot + iy) - f^*(\cdot)\|_{L^p(\mathbb{R})} = 0.$$

(ii) $f \in \mathcal{H}_a^p(\mathbb{C}_+)$ if and only if $\mathcal{M}(f) \in L^p(\mathbb{R})$. Moreover,

$$\|f\|_{\mathcal{H}_a^p(\mathbb{C}_+)} \sim \|\mathcal{M}(f)\|_{L^p(\mathbb{R})}.$$

Lemma 2.2. Theorem 1.1 is true for $p = \infty$.

Proof. Suppose that $\int_0^\infty t^{-1} \varphi(t)dt$ is finite. Then, for any $f \in \mathcal{H}_a^\infty(\mathbb{C}_+),$

$$\|\mathcal{H}_\varphi f\|_{\mathcal{H}_a^\infty(\mathbb{C}_+)} = \sup_{z \in \mathbb{C}_+} \left| \int_0^\infty f \left( \frac{z}{t} \right) \frac{\varphi(t)}{t} dt \right| \leq \int_0^\infty t^{-1} \varphi(t)dt \|f\|_{\mathcal{H}_a^\infty(\mathbb{C}_+)}.$$

Therefore, $\mathcal{H}_\varphi$ is bounded on $\mathcal{H}_a^\infty(\mathbb{C}_+)$, moreover,

$$\|\mathcal{H}_\varphi\|_{\mathcal{H}_a^\infty(\mathbb{C}_+) \to \mathcal{H}_a^\infty(\mathbb{C}_+)} \leq \int_0^\infty t^{-1} \varphi(t)dt. \quad (2.3)$$

On the other hand, we have

$$\|\mathcal{H}_\varphi\|_{\mathcal{H}_a^\infty(\mathbb{C}_+) \to \mathcal{H}_a^\infty(\mathbb{C}_+)} \geq \frac{\|\mathcal{H}_\varphi(1)\|_{\mathcal{H}_a^\infty(\mathbb{C}_+)}}{\|1\|_{\mathcal{H}_a^\infty(\mathbb{C}_+)}} = \int_0^\infty t^{-1} \varphi(t)dt.$$

This, together with (2.3), implies that

$$\|\mathcal{H}_\varphi\|_{\mathcal{H}_a^\infty(\mathbb{C}_+) \to \mathcal{H}_a^\infty(\mathbb{C}_+)} = \int_0^\infty t^{-1} \varphi(t)dt.$$

Moreover, by the dominated convergence theorem, for any $x \neq 0$,

$$(\mathcal{H}_\varphi f)^*(x) = \lim_{y \to 0} \int_0^\infty f \left( \frac{x}{t} + \frac{y}{t} \right) \frac{\varphi(t)}{t} dt = \int_0^1 f^* \left( \frac{x}{t} \right) \frac{\varphi(t)}{t} dt = H_\varphi(f^*)(x).$$

Conversely, suppose that $\mathcal{H}_\varphi$ is bounded on $\mathcal{H}_a^\infty(\mathbb{C}_+)$. As the function $f(z) \equiv 1$ is in $\mathcal{H}_a^\infty(\mathbb{C}_+)$, we obtain that $\mathcal{H}_\varphi f = \int_0^1 t^{-1} \varphi(t)dt < \infty$. \hfill \Box

Lemma 2.3. Let $p \in [1, \infty)$ and let $\varphi$ be such that (1.2) holds. Then

(i) $\mathcal{H}_\varphi$ is bounded on $\mathcal{H}_a^p(\mathbb{C}_+)$, moreover,

$$\|\mathcal{H}_\varphi\|_{\mathcal{H}_a^p(\mathbb{C}_+) \to \mathcal{H}_a^p(\mathbb{C}_+)} \leq \int_0^\infty t^{1/p-1} \varphi(t)dt.$$

(ii) If $\text{supp } \varphi \subset [0, 1]$, then

$$\|\mathcal{H}_\varphi\|_{\mathcal{H}_a^p(\mathbb{C}_+) \to \mathcal{H}_a^p(\mathbb{C}_+)} = \int_0^1 t^{1/p-1} \varphi(t)dt.$$
(iii) For any $f \in \mathcal{H}_a^p(\mathbb{C}_+)$, we have

$$(\mathcal{H}_\varphi f)^* = H_\varphi(f^*) .$$

Proof. (i) For any $f \in \mathcal{H}_a^p(\mathbb{C}_+)$, we have

$$\mathcal{M}(\mathcal{H}_\varphi f)(x) = \sup_{|u-x|<y} \left| \int_0^\infty f \left( \frac{u+iy}{t} \right) \frac{\varphi(t)}{t} dt \right| \leq \int_0^\infty \sup_{|u/t-x/t|<y/t} \left| f \left( \frac{u}{t} + \frac{y}{t} i \right) \right| \frac{\varphi(t)}{t} dt = H_\varphi(\mathcal{M}f)(x)$$

for all $x \in \mathbb{R}$. Therefore, by the Minkowski inequality and lemma 2.1(ii),

$$\| \mathcal{H}_\varphi f \|_{\mathcal{H}_a^p(\mathbb{C}_+)} \lesssim \| \mathcal{M}(\mathcal{H}_\varphi f) \|_{L^p(\mathbb{R})} \leq \| H_\varphi(\mathcal{M}f) \|_{L^p(\mathbb{R})} \leq \| \mathcal{M}f \|_{L^p(\mathbb{R})} \leq \int_0^\infty t^{1/p-1}\varphi(t)dt \| f \|_{\mathcal{H}_a^p(\mathbb{C}_+)} .$$

This proves that $\mathcal{H}_\varphi$ is bounded on $\mathcal{H}_a^p(\mathbb{C}_+)$, moreover,

$$\| \mathcal{H}_\varphi \|_{\mathcal{H}_a^p(\mathbb{C}_+) \rightarrow \mathcal{H}_a^p(\mathbb{C}_+)} \lesssim \int_0^\infty t^{1/p-1}\varphi(t)dt . \quad (2.4)$$

In order to show

$$\| \mathcal{H}_\varphi \|_{\mathcal{H}_a^p(\mathbb{C}_+) \rightarrow \mathcal{H}_a^p(\mathbb{C}_+)} \leq \int_0^\infty t^{1/p-1}\varphi(t)dt , \quad (2.5)$$

let us first assume that (iii) is proved. Then, by lemma 2.1(i) and the Minkowski inequality, we get

$$\| \mathcal{H}_\varphi f \|_{\mathcal{H}_a^p(\mathbb{C}_+)} = \| (\mathcal{H}_\varphi f)^* \|_{L^p(\mathbb{R})} = \| H_\varphi(f^*) \|_{L^p(\mathbb{R})}$$

$$\leq \int_0^\infty \left( \int_{\mathbb{R}} |f^*(\frac{x}{t})|^p dx \right)^{1/p} \frac{\varphi(t)}{t} dt$$

$$= \| f^* \|_{L^p(\mathbb{R})} \int_0^\infty t^{1/p-1}\varphi(t)dt$$

$$= \| f \|_{\mathcal{H}_a^p(\mathbb{C}_+)} \int_0^\infty t^{1/p-1}\varphi(t)dt .$$

This proves that (2.5) holds.
(ii) Let $\delta \in (0, 1)$ be arbitrary and let $\varphi_\delta(t) = \varphi(t)\chi_{[\delta, \infty)}(t)$ for all $t \in (0, \infty)$. Since (2.5) holds, we see that

$$\|\mathcal{H}_{\varphi_\delta}\|_{\mathcal{H}^p_0(\mathbb{C}_+)} \leq \int_0^\infty t^{1/p - 1}\varphi_\delta(t)dt = \int_\delta^1 t^{1/p - 1}\varphi(t)dt < \infty$$

and

$$\|\mathcal{H}_\varphi - \mathcal{H}_{\varphi_\delta}\|_{\mathcal{H}^p_0(\mathbb{C}_+)} \leq \int_0^\infty t^{1/p - 1}[\varphi(t) - \varphi_\delta(t)]dt = \int_\delta^1 t^{1/p - 1}\varphi(t)dt. \quad (2.6)$$

For any $\varepsilon > 0$, we define the function $f_\varepsilon : \mathbb{C}_+ \to \mathbb{C}$ by

$$f_\varepsilon(z) = \frac{1}{z + i}^{1/p+\varepsilon},$$

where, and in what follows, $\zeta^{1/p+\varepsilon} = |\zeta|^{1/p+\varepsilon}e^{(1/p+\varepsilon)\arg \zeta}$ for all $\zeta \in \mathbb{C}$. Then

$$\|f_\varepsilon\|_{\mathcal{H}^p_0(\mathbb{C}_+)} = \left(\int_{-\infty}^\infty \frac{1}{\sqrt{x^2 + 1}} dx\right)^{1/p} < \infty. \quad (2.7)$$

For all $z = x + iy \in \mathbb{C}_+$, we have

$$\mathcal{H}_{\varphi_\delta}(f_\varepsilon)(z) - f_\varepsilon(z) \int_0^\infty t^{1/p - 1}\varphi_\delta(t)dt = \int_\delta^1 [\phi_{\varepsilon, z}(t) - \phi_{\varepsilon, z}(1)]t^{1/p - 1}\varphi(t)dt,$$

where $\phi_{\varepsilon, z}(t) := ((t^z)/(z + it))^{1/p+\varepsilon}$. For any $t \in [\delta, 1]$, a simple calculus gives

$$|\phi_{\varepsilon, z}(t) - \phi_{\varepsilon, z}(1)| \leq |t - 1| \sup_{s \in [\delta, 1]} |\phi'_{\varepsilon, z}(s)| \leq \frac{\varepsilon\delta^{-1-1/p}}{\sqrt{x^2 + 1}^{1/p+\varepsilon}} + \frac{(1/p + \varepsilon)\delta^{-1-1/p}}{\sqrt{x^2 + 1}^{1+1/p+\varepsilon}}.$$

This, together with (2.7), yields

$$\frac{\|\mathcal{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon\|_{\mathcal{H}^p_0(\mathbb{C}_+)}}{\|f_\varepsilon\|_{\mathcal{H}^p_0(\mathbb{C}_+)}} \leq \int_\delta^1 t^{1/p - 1}\varphi(t)dt \left[\varepsilon\delta^{-1-1/p} + \frac{(1/p + \varepsilon)\delta^{-1-1/p}}{\left(\int_{-\infty}^\infty ((1/\sqrt{x^2 + 1}^{p+1})) dx\right)^{1/p}}\right] \to 0$$

as $\varepsilon \to 0$. As a consequence,

$$\int_\delta^1 t^{1/p - 1}\varphi(t)dt = \int_0^\infty t^{1/p - 1}\varphi(t)dt \leq \|\mathcal{H}_{\varphi_\delta}\|_{\mathcal{H}^p_0(\mathbb{C}_+) \to \mathcal{H}^p_0(\mathbb{C}_+)}. \quad (2.8)$$
This, combined with (2.6), allows us to conclude that
\[
\|\mathcal{H}_\varphi\|_{\mathcal{H}^p_0(C_+)\to\mathcal{H}^p_0(C_+)} \\
\geq \int_0^1 t^{1/p-1}\varphi(t)dt - 2 \int_0^\delta t^{1/p-1}\varphi(t)dt - \int_0^1 t^{1/p-1}\varphi(t)dt
\]
as \delta \to 0 since \(\int_0^1 t^{1/p-1}\varphi(t)dt < \infty\). Hence, by (2.5),
\[
\|\mathcal{H}_\varphi\|_{\mathcal{H}^p_0(C_+)\to\mathcal{H}^p_0(C_+)} = \int_0^1 t^{1/p-1}\varphi(t)dt.
\]

(iii) For any \(\sigma > 0\), it follows from (2.1) that the function
\[
f_\sigma(z) := f(z + i\sigma)
\]
is in \(\mathcal{H}^p_0(C_+) \cap \mathcal{H}^\infty_0(C_+)\). Let \(\delta\) and \(\varphi_\delta\) be as in (ii). Noting that
\[
\int_0^\infty t^{-1}\varphi_\delta(t)dt \leq \delta^{-1/p} \int_\delta^\infty t^{1/p-1}\varphi(t)dt < \infty,
\]
lemma 2.2(ii) gives \((\mathcal{H}_\varphi f_\sigma)^* = H_{\varphi_\delta}(f_\sigma^*)\). Therefore, by lemma 2.1(i), \[1,\, theorem 1\] and (2.4), we obtain that
\[
\|\mathcal{H}_\varphi f - H_\varphi(f^*)\|_{L^p(\mathbb{R})} \leq \|\mathcal{H}_\varphi - \mathcal{H}_{\varphi_\delta}\|_{\mathcal{H}^p_0(C_+) + \mathcal{H}^\infty_0(C_+)} + \|H_{\varphi_\delta}(f^* - f_\sigma^*)\|_{L^p(\mathbb{R})}
\]
\[
\leq \|f\|_{\mathcal{H}^p_0(C_+)} \int_0^\infty t^{1/p-1}[\varphi(t) - \varphi_\delta(t)]dt + \|f^* - f_\sigma^*\|_{L^p(\mathbb{R})} \int_0^\infty t^{1/p-1}\varphi_\delta(t)dt
\]
\[
\leq \|f\|_{\mathcal{H}^p_0(C_+)} \int_0^\delta t^{1/p-1}\varphi(t)dt + \|f^* - f_\sigma^*\|_{L^p(\mathbb{R})} \int_0^{\infty} t^{1/p-1}\varphi(t)dt \to 0
\]
as \(\sigma \to 0\) and \(\delta \to 0\). This completes the proof of Lemma 2.3.

Now we are ready to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By lemmas 2.2 and 2.3, it suffices to prove that
\[
\int_0^\infty t^{1/p-1}\varphi(t)dt \leq \|\mathcal{H}_\varphi\|_{\mathcal{H}^p_0(C_+)\to\mathcal{H}^p_0(C_+)}
\]
whenever \(\mathcal{H}_\varphi\) is bounded on \(\mathcal{H}^p_0(C_+)\) for \(1 \leq p < \infty\).

Indeed, we first claim that
\[
\int_0^\infty t^{1/p-1}\varphi(t)dt < \infty.
\]
Assume (2.10) holds for a moment.
For any $m > 0$, we define $\phi_m(t) = \phi(mt)\chi((0,1])(t)$ for all $t \in (0,\infty)$. Then, by lemma 2.3(i), we see that

$$
\|H^2 - H^{\phi_m}(\pi)\|_{H^p_a(C_+)} = \|H^2 - H^{\phi_m}(\pi)\|_{H^p_a(C_+)} \\
= \int_0^\infty t^{1/p-1} \left[ \phi(t) - \phi_m \left( \frac{t}{m} \right) \right] dt \\
= \int_m^\infty t^{1/p-1} \phi(t)dt < \infty.
$$

(2.11)

Noting that

$$
\|f \left( \frac{t}{m} \right) \|_{H^p_a(C_+)} = m^{1/p} \|f(\cdot)\|_{H^p_a(C_+)} \quad \text{and} \quad H^{\phi_m}(\cdot/m) f = H^\phi f \left( \frac{\cdot}{m} \right)
$$

for all $f \in H^p_a(C_+)$, lemma 2.3(ii) gives

$$
\|H^\phi\|_{H^p_a(C_+)} \geq \int_0^\infty t^{1/p-1} \phi(t)dt - 2\int_m^\infty t^{1/p-1} \phi(t)dt. 
$$

as $m \to \infty$ since $\int_0^\infty t^{1/p-1} \phi(t)dt < \infty$. This proves (2.9).

Now we return to prove (2.10). Indeed, we consider the following two cases.

**Case 1: $p = 1$.** Take $f(z) = ((1)/((z + i)))^2$ for all $z \in C_+$. Then

$$
\|f\|_{H^1_a(C_+)} = \int_{-\infty}^\infty \frac{1}{x^2 + 1} dx < \infty.
$$

Therefore, by the Fatou lemma, we get

$$
\infty > \|H^\phi f\|_{H^1_a(C_+)} = \sup_{y > 0} \int_{-\infty}^\infty \left| \int_0^\infty \frac{1}{[x/t + i(y/t + 1)]^2} \frac{\phi(t)}{t} dt \right| dx \\
\geq 2 \sup_{y > 0} \int_{0}^\infty dx \int_0^\infty \frac{x/t (y/t + 1)}{[x/t]^2 + (y/t + 1)^2} \frac{\phi(t)}{t} dt \\
\geq 2 \int_{0}^\infty dx \int_0^\infty \frac{x/t}{[x/t]^2 + 1} \frac{\phi(t)}{t} dt \\
= 2 \int_{0}^\infty \frac{u}{[u^2 + 1]^2} du \int_0^\infty \phi(t) dt.
$$

This proves (2.10).
**Case 2:** $1 < p < \infty$. For any $0 < \varepsilon < 1 - 1/p$, take
\[
f_{\varepsilon}(z) = \left(\frac{1}{z + i\varepsilon}\right)^{1/p + \varepsilon}
\]
for all $z \in \mathbb{C}_+$. Then
\[
\|f_{\varepsilon}\|_{H^p_{\mathbb{C}_+}} = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2 + \varepsilon^2}^{1+p\varepsilon}} \, dx\right)^{1/p} < \infty
\]  
(2.12)
and
\[
\infty \geq \|\mathcal{H}_\varphi(f_{\varepsilon})\|_{H^p_{\mathbb{C}_+}}^p = \sup_{y > 0} \left| \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{1}{\sqrt{(x/t + iy/t + \varepsilon)^2 + \varepsilon^2}} \frac{\varphi(t)}{t} \, dt \right|^p \, dx
\]
\[
\geq \left| \int_{0}^{\infty} \int_{0}^{\infty} \frac{x/t}{\sqrt{(x/t)^2 + (y/t + \varepsilon)^2}} \frac{1}{\sqrt{(x/t)^2 + (y/t + \varepsilon)^2}^{1/p + \varepsilon}} \frac{\varphi(t)}{t} \, dt \right|^p \, dx,
\]
where we used the Fatou lemma and the fact that
\[
\Re \left(\frac{1}{x/t + i(y/t + \varepsilon)}\right)^{1/p + \varepsilon} \geq \frac{x/t}{\sqrt{(x/t)^2 + (y/t + \varepsilon)^2}} \frac{1}{\sqrt{(x/t)^2 + (y/t + \varepsilon)^2}^{1/p + \varepsilon}}
\]
for all $x, y, t > 0$ since $0 < 1/p + \varepsilon < 1$. This, together with (2.12), gives
\[
\|\mathcal{H}_\varphi\|_{H^p_{\mathbb{C}_+} \to H^p_{\mathbb{C}_+}}^p \geq \frac{\|\mathcal{H}_\varphi(f_{\varepsilon})\|_{H^p_{\mathbb{C}_+}}^p}{\|f_{\varepsilon}\|_{H^p_{\mathbb{C}_+}}^p}
\]
\[
\int_{1}^{\infty} \left(\int_{0}^{\infty} \frac{1}{\sqrt{(x/t)^2 + \varepsilon^2}} ((1)/\sqrt{(x/t)^2 + \varepsilon^2})^{1/p + \varepsilon} \right)^{1/p} \, dx
\]
\[
\geq \frac{2 \int_{0}^{\infty} (1/(\sqrt{x^2 + \varepsilon^2}))^{1+p\varepsilon} \, dx}{2^{(3+p(1+\varepsilon))/2}} \left(\int_{0}^{1/\varepsilon} t^{1/p - 1 + \varepsilon} \varphi(t) \, dt\right)^{p}
\]
\[
\times \frac{\int_{1}^{\infty} ((1)/(x^{1+p\varepsilon})) \, dx}{\varepsilon^{-p} \int_{0}^{\infty} ((1)/(\sqrt{x^2 + 1} + \varepsilon^{1+p\varepsilon})) \, dx}.
\]
Hence,
\[
\int_{0}^{1/\varepsilon} t^{1/p - 1 + \varepsilon} \varphi(t) \, dt \leq 2^{(3+p(1+\varepsilon))/2} \varepsilon^{-p} \left(\frac{\int_{0}^{\infty} (1/(\sqrt{x^2 + 1} + \varepsilon^{1+p\varepsilon})) \, dx}{\int_{1}^{\infty} ((1)/(x^{1+p\varepsilon})) \, dx}\right)^{1/p}
\]
\[
\times \|\mathcal{H}_\varphi\|_{H^p_{\mathbb{C}_+} \to H^p_{\mathbb{C}_+}}.
\]
Letting $\varepsilon \to 0$, we obtain
\[
\int_{0}^{\infty} t^{1/p-1} \varphi(t) dt \leq 2^{((3+p)/(2p))} \|\mathcal{H}_\varphi\|_{\mathcal{H}_p^a(C_+) \to \mathcal{H}_p^a(C_+)} < \infty.
\]
This proves (2.10), and thus ends the proof of Theorem 1.1. \(\Box\)

3. Some applications

Let $1 \leq p < \infty$, we define (see [19]) the Hilbert transform of $f \in L^p(\mathbb{R})$ by
\[
\mathcal{H}(f)(x) := \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy, \quad x \in \mathbb{R}.
\]

**Theorem 3.1.** Let $p \in (1, \infty)$ and let $\varphi$ be as in theorem 1.1. Then $\mathcal{H}_\varphi$ is bounded on $L^p(\mathbb{R})$ if and only if (1.2) holds. Moreover, in that case,
\[
\|\mathcal{H}_\varphi\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} = \int_{0}^{\infty} t^{1/p-1} \varphi(t) dt
\]
and $\mathcal{H}_\varphi$ commutes with the Hilbert transform $\mathcal{H}$ on $L^p(\mathbb{R})$.

In order to prove theorem 3.1, we need the following lemmas.

**Lemma 3.2** (see [6, 9]). Let $1 < p < \infty$. Then:

(i) If $g \in L^p(\mathbb{R})$, then $f^* := g + i\mathcal{H}(g)$ is the boundary value function of some function $f \in \mathcal{H}_p^a(C_+)$.

(ii) Conversely, if $f^*$ is a boundary value function of $f \in \mathcal{H}_p^a(C_+)$, then there exists a real-valued function $g \in L^p(\mathbb{R})$ such that $f^* = g + i\mathcal{H}(g)$.

Moreover, in those cases,
\[
\|g\|_{L^p(\mathbb{R})} \sim \|g + i\mathcal{H}(g)\|_{L^p(\mathbb{R})} = \|f^*\|_{L^p(\mathbb{R})} = \|f\|_{\mathcal{H}_p^a(C_+)}.\]

**Lemma 3.3** (see [1, 20]). Let $p \in (1, \infty)$ and let $\varphi$ be such that (1.2) holds. Then:

(i) $\mathcal{H}_\varphi$ is bounded on $L^p(\mathbb{R})$, moreover,
\[
\|\mathcal{H}_\varphi\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq \int_{0}^{\infty} t^{1/p-1} \varphi(t) dt.
\]

(ii) If $\text{supp} \ \varphi \subset [0, 1]$, then
\[
\|\mathcal{H}_\varphi\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} = \int_{0}^{1} t^{1/p-1} \varphi(t) dt.
\]
Proof of Theorem 3.1. Suppose that (1.2) holds. By lemma 3.3(i),
\[ \|H_\varphi\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq \int_0^\infty t^{1/p-1} \varphi(t) dt. \tag{3.1} \]

Conversely, suppose that \( H_\varphi \) is bounded on \( L^p(\mathbb{R}) \). We first claim that
\[ \int_0^\infty t^{1/p-1} \varphi(t) dt < \infty. \tag{3.2} \]
Assume (3.2) holds for a moment.

For any \( m > 0 \), take \( \varphi_m \) as in the proof of Theorem 1.1. Then, by a similar argument to the proof of Theorem 1.1, we get
\[
\|H_\varphi\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \geq \|H_{\varphi_m(\cdot/m)}\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} - \|H_\varphi - H_{\varphi_m(\cdot/m)}\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \\
\geq \int_0^\infty t^{1/p-1} \varphi(t) dt - 2 \int_m^\infty t^{1/p-1} \varphi(t) dt \to \int_0^\infty t^{1/p-1} \varphi(t) dt
\]
as \( m \to \infty \). This, together with (3.1), yields
\[
\|H_\varphi\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} = \int_0^\infty t^{1/p-1} \varphi(t) dt.
\]

Now let us return to prove (3.2). Indeed, for any \( \epsilon \in (0,1) \), take
\[
f_\epsilon(x) = |x|^{-1/p-\epsilon} \chi_{\{y \in \mathbb{R} : |y| > 1\}}(x)
\]
and
\[
g_\epsilon(x) = |x|^{-1/p+\epsilon} \chi_{\{y \in \mathbb{R} : |y| < 1\}}(x)
\]
for all \( x \in \mathbb{R} \). Then some simple computations give
\[
\|H_\varphi\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \geq \frac{\|H_\varphi(f_\epsilon)\|_{L^p(\mathbb{R})}}{\|f_\epsilon\|_{L^p(\mathbb{R})}} \geq \epsilon^{1/p} \int_1^\infty t^{1/p-1} \varphi(t) dt
\]
and
\[
\|H_\varphi\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \geq \frac{\|H_\varphi(g_\epsilon)\|_{L^p(\mathbb{R})}}{\|g_\epsilon\|_{L^p(\mathbb{R})}} \geq \epsilon^{1/p} \int_\epsilon^1 t^{1/p-1} \varphi(t) dt.
\]
Letting \( \epsilon \to 0 \), we get
\[
\int_1^\infty t^{1/p-1} \varphi(t) dt \leq \|H_\varphi\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} < \infty
\]
and
\[
\int_0^1 t^{1/p-1} \varphi(t) dt \leq \|H_\varphi\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} < \infty.
\]
This proves (3.2).
Finally, we need to show that $H_\varphi$ commutes with the Hilbert transform $H$ on $L^p(\mathbb{R})$. To this end, it suffices to show
\[ H_\varphi(H(f)) = H(H_\varphi(f)) \tag{3.3} \]
for all real-valued functions $f$ in $L^p(\mathbb{R})$. Indeed, by theorem 1.1 and lemma 3.2, there exists a real-valued function $g$ in $L^p(\mathbb{R})$ such that
\[ g + iH(g) = H_\varphi(f + iH(f)). \]
This proves (3.3), and thus completes the proof of Theorem 3.1.

Let $1 < p < \infty$, we denote by $H^p_+(\mathbb{R})$ and $H^p_-(\mathbb{R})$ the subspaces of $L^p(\mathbb{R})$ consisting of those functions whose Poisson extensions to the upper half-plane $\mathbb{C}_+$ are holomorphic and anti-holomorphic, respectively.

It is well-known (see [6, 9, 19]) that
\[ H^p_+(\mathbb{R}) = \{ f + iH(f) : f \in L^p(\mathbb{R}) \} \tag{3.4} \]
and
\[ H^p_-(\mathbb{R}) = \{ f - iH(f) : f \in L^p(\mathbb{R}) \}. \tag{3.5} \]
Moreover, $L^p(\mathbb{R}) = H^p_+(\mathbb{R}) \oplus H^p_-(\mathbb{R})$.

**Theorem 3.4.** Let $p \in (1, \infty)$ and let $\varphi$ be such that (1.2) holds. Then $H_\varphi$ is bounded on the space $H^p_+(\mathbb{R})$, moreover,
\[ \|H_\varphi\|_{H^p_+(\mathbb{R}) \to H^p_+(\mathbb{R})} = \int_0^\infty t^{1/p-1} \varphi(t)dt \]
and $H_\varphi$ commutes with the Hilbert transform $H$ on $H^p_+(\mathbb{R})$.

**Proof.** It follows from theorem 3.1 that $H_\varphi(f)$ belongs to $H^p_+(\mathbb{R})$ for all $f \in H^p_+(\mathbb{R})$, and thus
\[ \|H_\varphi\|_{H^p_+(\mathbb{R}) \to H^p_+(\mathbb{R})} \leq \|H_\varphi\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} = \int_0^\infty t^{1/p-1} \varphi(t)dt. \tag{3.6} \]

For any $\varepsilon > 0$, by theorem 1.1, there exists $f_\varepsilon \in \mathcal{H}_0^p(\mathbb{C}_+)$ for which
\[ \frac{\|H_\varphi(f_\varepsilon)\|_{L^p(\mathbb{R})}}{\|f_\varepsilon\|_{L^p(\mathbb{R})}} = \frac{\|H_\varphi(f_\varepsilon)^s\|_{L^p(\mathbb{R})}}{\|f_\varepsilon^s\|_{L^p(\mathbb{R})}} = \frac{\|H_\varphi(f_\varepsilon)\|_{\mathcal{H}_0^p(\mathbb{C}_+)}}{\|f_\varepsilon\|_{\mathcal{H}_0^p(\mathbb{C}_+)}} \geq \int_0^\infty t^{1/p-1} \varphi(t)dt - \varepsilon. \]
This, together with (3.6), allows us to conclude that
\[ \|H_\varphi\|_{H^p_+(\mathbb{R}) \to H^p_+(\mathbb{R})} = \int_0^\infty t^{1/p-1} \varphi(t)dt. \]

Finally, $H_\varphi$ commutes with the Hilbert transform $H$ on $H^p_+(\mathbb{R})$ is followed from theorem 3.1 and (3.4).
Theorem 3.5. Let $p \in (1, \infty)$ and let $\varphi$ be such that \eqref{eq:1.2} holds. Then $H_\varphi$ is bounded on the space $H^p_+(\mathbb{R})$, moreover,

$$\|H_\varphi\|_{H^p_+(\mathbb{R}) \to H^p_+(\mathbb{R})} = \int_0^\infty t^{1/p-1} \varphi(t) \, dt$$

and $H_\varphi$ commutes with the Hilbert transform $H$ on $H^p_+(\mathbb{R})$.

Proof. It follows from theorem 3.4 and the fact that $f \in H^p_+(\mathbb{R})$ if and only if $\overline{f} \in H^p_-(\mathbb{R})$. \hfill \Box

Let $\Phi$ be in the Schwartz space $S(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \Phi(x) \, dx \neq 0$. For any $t > 0$, set $\Phi_t(x) := t^{-1} \Phi(x/t)$. Following Fefferman and Stein [7, 19], we define the real Hardy space $H^1_+(\mathbb{R})$ as the set of all functions $f \in L^1(\mathbb{R})$ such that

$$\|f\|_{H^1_+(\mathbb{R})} := \|M_\Phi(f)\|_{L^1(\mathbb{R})} < \infty,$$

where $M_\Phi(f)$ is the smooth maximal function of $f$ defined by

$$M_\Phi(f)(x) = \sup_{t > 0} |f \ast \Phi_t(x)|, \quad x \in \mathbb{R}.$$

Remark that the norm $\| \cdot \|_{H^1_+(\mathbb{R})}$ depends on the choice of $\Phi$, but the space $H^1_+(\mathbb{R})$ does not depend on this choice (see proposition 3.7 below).

The following lemma is well-known.

Lemma 3.6 (see [6, 9, 18]).

(i) If $g \in H^1_+(\mathbb{R})$, then $f^* := g + iH(g)$ is the boundary value function of some function $f \in \mathcal{H}^1_+(\mathbb{C}_+)$. 

(ii) Conversely, if $f^*$ is a boundary value function of $f \in \mathcal{H}^1_+(\mathbb{C}_+)$, then there exists a real-valued function $g \in H^1(\mathbb{R})$ such that $f^* = g + iH(g)$.

Moreover, in those cases,

$$\|g\|_{H^1_+(\mathbb{R})} \sim \|g + iH(g)\|_{H^1_+(\mathbb{R})} = \|f^*\|_{H^1_+(\mathbb{R})} \sim \|f^*\|_{L^1(\mathbb{R})} = \|f\|_{\mathcal{H}^1_+(\mathbb{C}_+)}.$$ 

Let $P_t(x) = 1/\pi((t)/(x^2 + t^2))$ be the Poisson kernel on $\mathbb{R}$. For any $f \in L^1(\mathbb{R})$, we denote $u(y,t) = f \ast P_t(y)$. Then, set

$$M_{P_t}(f)(x) = \sup_{|y-x|<t} |u(y,t)| \quad \text{and} \quad S(f)(x)$$

$$= \left[ \iint_{|y-x|<t} (|u_t(y,t)|^2 + |u_y(y,t)|^2) \, dy \, dt \right]^{1/2}.$$ 

A function $a$ is called an $\mathcal{H}^1$-atom related to the interval $B$ if

- $\text{supp } a \subset B$;
- $\|a\|_{L^\infty(\mathbb{R})} \leq |B|^{-1}$. 

We define the Hardy space $H^1_{at}(\mathbb{R})$ as the space of functions $f \in L^1(\mathbb{R})$ which can be written as $f = \sum_{j=1}^{\infty} \lambda_j a_j$ with $a_j$’s are $H^1$-atoms and $\lambda_j$’s are complex numbers satisfying $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. The norm on $H^1_{at}(\mathbb{R})$ is then defined by

$$
\|f\|_{H^1_{at}(\mathbb{R})} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.
$$

The following proposition is classical and can be found in Stein’s book [19].

**Proposition 3.7.** Let $f \in L^1(\mathbb{R})$. Then the following conditions are equivalent:

(i) $f \in H^1(\mathbb{R})$.

(ii) $M_\Phi(f)(\cdot) = \sup_{|\cdot| < t} |f * \Phi_t(\cdot)| \in L^1(\mathbb{R})$.

(iii) $M_P(f) \in L^1(\mathbb{R})$.

(iv) $S(f) \in L^1(\mathbb{R})$.

(v) $f \in H^1_{at}(\mathbb{R})$.

(vi) $H(f) \in L^1(\mathbb{R})$.

Moreover, in those cases,

$$
\|f\|_{H^1(\mathbb{R})} \sim \|M_\Phi(f)\|_{L^1(\mathbb{R})} \sim \|M_P(f)\|_{L^1(\mathbb{R})} \sim \|S(f)\|_{L^1(\mathbb{R})} \sim \|f\|_{H^1_{at}(\mathbb{R})} \sim \|f\|_{L^1(\mathbb{R})} + \|H(f)\|_{L^1(\mathbb{R})}.
$$

Of course, the above constants are depending on $\Phi$.

The following gives a lower bound for the norm of $H_\varphi$ on $H^1(\mathbb{R})$.

**Theorem 3.8.** Let $\| \cdot \|_*$ be one of the six norms in proposition 3.7. Assume that $H_\varphi$ is bounded on $(H^1(\mathbb{R}), \| \cdot \|_*)$. Then,

$$
\int_0^{\infty} \varphi(t) dt \leq \| H_\varphi \|_{(H^1(\mathbb{R}), \| \cdot \|_*) \rightarrow (H^1(\mathbb{R}), \| \cdot \|_*)} < \infty
$$

and $H_\varphi$ commutes with the Hilbert transform $H$ on $H^1(\mathbb{R})$.

It should be pointed out that, when $\text{supp} \varphi \subset [1, \infty)$ and $\| \cdot \|_* = \| \cdot \|_{H^1_{at}(\mathbb{R})}$, the above theorem is due to Xiao [20, p. 666] (see also [12, 14]).

In order to prove theorem 3.8, we need the following lemma.

**Lemma 3.9.** Let $\varphi$ be such that $\int_0^{\infty} \varphi(t) dt < \infty$ and $\text{supp} \varphi \subset [0, 1]$. Then,

$$
\int_0^{1} \varphi(t) dt \leq \| H_\varphi \|_{(H^1(\mathbb{R}), \| \cdot \|_*) \rightarrow (H^1(\mathbb{R}), \| \cdot \|_*)} < \infty.
$$
Proof. It is well-known (see \cite{1, 10, 16}) that if \( \int_0^\infty \varphi(t)dt < \infty \), then \( H\varphi \) is bounded on \( \mathcal{H}^1(\mathbb{R}) \), moreover,

\[
\|H\varphi\|_{(\mathcal{H}^1(\mathbb{R}),\|\cdot\|_*) \to (\mathcal{H}^1(\mathbb{R}),\|\cdot\|_*)} \lesssim \int_0^\infty \varphi(t)dt = \int_0^1 \varphi(t)dt.
\]

We now show that

\[
\int_0^1 \varphi(t)dt \leq \|H\varphi\|_{(\mathcal{H}^1(\mathbb{R}),\|\cdot\|_*) \to (\mathcal{H}^1(\mathbb{R}),\|\cdot\|_*)}.
\]

Indeed, let \( \delta \in (0, 1) \) and \( \varphi_\delta \) be as in the proof of Lemma 2.3(ii). For any \( \varepsilon > 0 \), define the function \( f_\varepsilon : \mathbb{C}_+ \to \mathbb{C} \) by

\[
f_\varepsilon(z) = \frac{1}{(z + i)^{1+\varepsilon}}.
\]

Then, by lemma 2.3(iii), lemma 3.6, proposition 3.7 and (2.8),

\[
\left\| H\varphi_\delta (f_\varepsilon^*) \right\|_{\mathcal{H}^1(\mathbb{C}_+)} \rightarrow 0
\]

as \( \varepsilon \to 0 \). This implies that

\[
\int_0^1 \varphi(t)dt = \int_0^\infty \varphi_\delta(t)dt \leq \|H\varphi_\delta\|_{(\mathcal{H}^1(\mathbb{R}),\|\cdot\|_*) \to (\mathcal{H}^1(\mathbb{R}),\|\cdot\|_*)},
\]

and thus

\[
\int_0^1 \varphi(t)dt \leq \|H\varphi\|_{(\mathcal{H}^1(\mathbb{R}),\|\cdot\|_*) \to (\mathcal{H}^1(\mathbb{R}),\|\cdot\|_*)}
\]

since

\[
\|H\varphi - H\varphi_\delta\|_{(\mathcal{H}^1(\mathbb{R}),\|\cdot\|_*) \to (\mathcal{H}^1(\mathbb{R}),\|\cdot\|_*)} \lesssim \int_0^\infty (\varphi(t) - \varphi_\delta(t))dt = \int_0^\delta \varphi(t)dt \rightarrow 0
\]

as \( \delta \to 0 \). This ends the proof of Lemma 3.9. \( \square \)

Proof of Theorem 3.8. It follows from \cite[theorem 3.3]{10} that

\[
\int_0^\infty \varphi(t)dt < \infty.
\]

For any \( m > 0 \), let \( \varphi_m \) be as in the proof of Theorem 1.1. Then, by (3.7),

\[
\left\| H\varphi - H\varphi_m(\cdot/m)\right\|_{(\mathcal{H}^1(\mathbb{R}),\|\cdot\|_*) \to (\mathcal{H}^1(\mathbb{R}),\|\cdot\|_*)} \leq \int_0^\infty \left[ \varphi(t) - \varphi_m \left( \frac{t}{m} \right) \right] dt = \int_m^\infty \varphi(t)dt,
\]

where the constant is independent of \( m \).
Noting that
\[
\|f\left(\frac{\cdot}{m}\right)\|_* = m\|f(\cdot)\|_* \quad \text{and} \quad H_{\varphi_m}(\frac{\cdot}{m})f = H_{\varphi_m}f\left(\frac{\cdot}{m}\right)
\]
for all \(f \in \mathcal{H}^1(\mathbb{R})\), lemma 3.9 gives
\[
\left\|H_{\varphi_m}(\frac{\cdot}{m})\right\|_{(\mathcal{H}^1(\mathbb{R}),\|\cdot\|_\ast) \to (\mathcal{H}^1(\mathbb{R}),\|\cdot\|_\ast)} = m\left\|H_{\varphi_m}\right\|_{(\mathcal{H}^1(\mathbb{R}),\|\cdot\|_\ast) \to (\mathcal{H}^1(\mathbb{R}),\|\cdot\|_\ast)} \geq m \int_0^1 \varphi_m(t)dt = \int_0^m \varphi(t)dt.
\]
This, together with (3.8) and \(\lim_{m \to \infty} \int_0^\infty \varphi(t)dt = 0\), allows us to conclude that
\[
\int_0^\infty \varphi(t)dt \leq \left\|H_{\varphi}\right\|_{(\mathcal{H}^1(\mathbb{R}),\|\cdot\|_\ast) \to (\mathcal{H}^1(\mathbb{R}),\|\cdot\|_\ast)}.
\]

Using the Fourier transform, Liflyand and Móricz proved in [17] that \(H_{\varphi}\) commutes with the Hilbert transform \(H\) on \(\mathcal{H}^1(\mathbb{R})\). However, we also would like to give a new proof of this fact here. It suffices to prove
\[
H_{\varphi}(H(f)) = H(H_{\varphi}(f)) \tag{3.9}
\]
for all real-valued functions \(f\) in \(\mathcal{H}^1(\mathbb{R})\). Indeed, by theorem 1.1 and lemma 3.6, there exists a real-valued function \(g\) in \(\mathcal{H}^1(\mathbb{R})\) such that
\[
g + iH(g) = H_{\varphi}(f + iH(f)).
\]
This proves (3.9), and thus completes the proof of Theorem 3.8. \(\square\)

Let \(a : (0, \infty) \to [0, \infty)\) be a measurable function. Following Carro and Ortiz-Caraballo [5], we define
\[
\mathcal{S}_a F(z) = \int_0^\infty F(tz)a(t)dt, \quad z \in \mathbb{C}_+,
\]
for all holomorphic functions \(F\) on \(\mathbb{C}_+\); and define
\[
S_a f(x) = \int_0^\infty f(tx)a(t)dt, \quad x \in \mathbb{R},
\]
for all measurable functions \(f\) on \(\mathbb{R}\).

It is easy to see that
\[
\mathcal{S}_a F = H_{\varphi} F \quad \text{and} \quad S_a f = H_{\varphi} f,
\]
where \(\varphi(t) = t^{-1}a(t^{-1})\) for all \(t \in (0, \infty)\). Hence, it follows from theorems 1.1, 3.1 and 3.8 that:
Theorem 3.10. Let $p \in [1, \infty]$ and let $a : (0, \infty) \to [0, \infty)$ be a measurable function. Then $S_a$ is bounded on $\mathcal{H}^p_a(\mathbb{C}_+)$ if and only if
\[
\int_0^\infty t^{-1/p}a(t)dt < \infty.
\]
Moreover, when (3.10) holds, we obtain
\[
\|S_a\|_{\mathcal{H}^p_a(\mathbb{C}_+) \to \mathcal{H}^p_a(\mathbb{C}_+)} = \int_0^\infty t^{-1/p}a(t)dt
\]
and, for any $f \in \mathcal{H}^p_a(\mathbb{C}_+)$,
\[
(S_af)^* = S_a(f^*).
\]

Theorem 3.11. Let $p \in (1, \infty)$ and let $a$ be as in theorem 3.10. Then $S_a$ is bounded on $L^p(\mathbb{R})$ if and only if (3.10) holds. Moreover, in that case,
\[
\|S_a\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} = \int_0^\infty t^{-1/p}a(t)dt\]
and $S_a$ commutes with the Hilbert transform $H$ on $L^p(\mathbb{R})$.

Theorem 3.12. Let $a$ be as in theorem 3.10. Then $S_a$ is bounded on $\mathcal{H}^1(\mathbb{R})$ if and only if $\int_0^\infty t^{-1}a(t)dt < \infty$. Moreover, in that case,
\[
\int_0^\infty t^{-1}a(t)dt \leq \|S_a\|_{\mathcal{H}^1(\mathbb{R}) \to \mathcal{H}^1(\mathbb{R})} < \infty
\]
and $S_a$ commutes with the Hilbert transform $H$ on $\mathcal{H}^1(\mathbb{R})$.

Also it is easy to see that if (1.2) holds for $1 < p < \infty$, then
\[
\int_{\mathbb{R}} H_\varphi f(x)g(x)dx = \int_{\mathbb{R}} f(x)S_\varphi g(x)dx
\]
whenever $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, $q = p/(p-1)$. Namely, $S_\varphi$ can be viewed as the Banach space adjoint of $H_\varphi$ and vice versa. Therefore, by theorems 3.8, 3.12 and [12, theorem 1], a duality argument gives:

Theorem 3.13.

(i) $S_\varphi$ is bounded on $BMO(\mathbb{R})$ if and only if $\int_0^\infty \varphi(t)dt < \infty$. Moreover, in that case,
\[
\|S_\varphi\|_{BMO(\mathbb{R}) \to BMO(\mathbb{R})} = \int_0^\infty \varphi(t)dt.
\]

(ii) $H_\varphi$ is bounded on $BMO(\mathbb{R})$ if and only if $\int_0^\infty t^{-1}\varphi(t)dt < \infty$. Moreover, in that case,
\[
\|H_\varphi\|_{BMO(\mathbb{R}) \to BMO(\mathbb{R})} = \int_0^\infty t^{-1}\varphi(t)dt.
\]
Here the space $BMO(\mathbb{R})$ (see [7, 11]) is the dual space of $\mathcal{H}^1(\mathbb{R})$ defined as the space of all functions $f \in L^1_{\text{loc}}(\mathbb{R})$ such that

$$\|f\|_{BMO(\mathbb{R})} := \sup_B \frac{1}{|B|} \int_B |f(x) - \frac{1}{|B|} \int_B f(y) dy| dx < \infty,$$

where the supremum is taken over all intervals $B \subset \mathbb{R}$.

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References
1. K. F. Andersen. Boundedness of Hausdorff operators on $L^p(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$, and $BMO(\mathbb{R}^n)$. Acta Sci. Math. (Szeged) 69 (2003), 409–418.
2. A. G. Arvanitidis and A. G. Siskakis. Cesàro operators on the Hardy spaces of the half-plane. Can. Math. Bull. 56 (2013), 229–240.
3. S. Ballamoole, J. O. Bonyo, T. L. Miller and V. G. Miller. Cesàro-like operators on the Hardy and Bergman spaces of the half plane. Complex Anal. Oper. Theory 10 (2016), 187–203.
4. A. Brown, P. Halmos and A. Shields. Cesàro operators. Acta Sci. Math. (Szeged) 26 (1965), 125–137.
5. M. J. Carro and C. Ortiz-Caraballo. Boundedness of integral operators on decreasing functions. Proc. R. Soc. Edinburgh Sect. A 145 (2015), 725–744.
6. P. L. Duren. Theory of $H^p$ spaces: pure and applied mathematics, vol. 38 (New York-London: Academic Press, 1970).
7. C. Fefferman and E. M. Stein. $H^p$ spaces of several variables. Acta Math. 129 (1972), 137–193.
8. P. Galanopoulos and M. Papadimitrakis. Hausdorff and quasi-Hausdorff matrices on spaces of analytic functions. Can. J. Math., 58 (2006), 548–579.
9. J. B. Garnett. Bounded analytic functions. Graduate texts in Mathematics, Revised 1st edn, vol. 236 (New York: Springer, 2007).
10. H. D. Hung, L. D. Ky and T. T. Quang. Norm of the Hausdorff operator on the real Hardy space $H^1(\mathbb{R})$. Complex Anal. Oper. Theory 12 (2018), 235–245.
11. F. John and L. Nirenberg. On functions of bounded mean oscillation. Comm. Pure Appl. Math. 14 (1961), 415–426.
12. A. K. Lerner and E. Liflyand. Multidimensional Hausdorff operators on the real Hardy space. J. Aust. Math. Soc. 83 (2007), 79–86.
13. E. Liflyand. Open problems on Hausdorff operators. Complex analysis and potential theory, pp. 280–285 (Hackensack, NJ: World Sci. Publ., 2007).
14. E. Liflyand. Boundedness of multidimensional Hausdorff operators on $H^1(\mathbb{R}^n)$. Acta Sci. Math. (Szeged). 74 (2008), 845–851.
15. E. Liflyand. Hausdorff operators on Hardy spaces. Eurasian Math. J. 4 (2013), 101–141.
16. E. Liflyand and F. Móricz. The Hausdorff operator is bounded on the real Hardy space $H^1(\mathbb{R})$. Proc. Am. Math. Soc. 128 (2000), 1391–1396.
17. E. Liflyand and F. Móricz. Commuting relations for Hausdorff operators and Hilbert transforms on real Hardy spaces. Acta Math. Hungar. 97 (2002), 133–143.
18. T. Qian, Y. Xu, D. Yan, L. Yan and B. Yu. Fourier spectrum characterization of Hardy spaces and applications. Proc. Am. Math. Soc. 137 (2009), 971–980.
19. E. M. Stein. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals (Princeton, NJ: Princeton University Press, 1993).
20. J. Xiao. $L^p$ and $BMO$ bounds of weighted Hardy-Littlewood averages. J. Math. Anal. Appl. 262 (2001), 660–666.