Abelian Chern-Simons field theory
and anyon equation on a cylinder

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Abstract

We present the anyon equation on a cylinder and in an infinite potential wall from the abelian Chern-Simons theory coupled to non-relativistic matter field by obtaining the effective hamiltonian through the canonical transformation method used for the theory on a plane and on a torus. We also give the periodic property of the theory on the cylinder.
I. INTRODUCTION

It is well known that the Chern-Simons field theory\cite{1,2} provides an effective description of the statistical transmutation\cite{3,4} whose effect appears as the Aharonov-Bohm potential in the many-body equation and give rise to the anyonic property\cite{5}. For the non-relativistic case, this is explicitly demonstrated by obtaining the Schrödinger equation on an infinite plane\cite{8,9} and on compact spaces\cite{10,12} from the Chern-Simons gauge theory coupled to non-relativistic matter-field.

The derivation of the anyon equation on the compact space from the Chern-Simons field theory is not straight-forward due to the zero-modes and the boundary condition of the wavefunction is not well-understood. We have re-analyzed\cite{12} the Chern-Simons theory on a torus extending the canonical transformation method used in the plane\cite{6}. The canonical transformation decouples the gauge fields from the matter sector and leaves the effective hamiltonian in terms of matter fields only. This hamiltonian system is equivalent to the one where Gauss constraint is solved. In addition, the system has the manifest translation invariance along the non-contractible loops. From this we could obtain the many-body Schrödinger equation with simple periodic (not quasi-periodic) boundary condition. It does not exclude the possibility of the quasi-periodic boundary condition. Instead, depending on the definition of the wavefunction, one can have the quasi-periodic condition due to the vacuum degeneracy of the gauge sector\cite{13} as presented previously\cite{10}.

The canonical transformation avoids the ambiguity appearing in solving the Gauss constraint classically and quantizing the zero-modes and the matter fields afterwards. It is worthwhile to apply this method to other spaces with different topology and obtain its effective theory and many-body quantum mechanics. In this paper, we will consider the Chern-Simons theory on a cylinder and inside an infinite potential wall. The cylinder is compact in the one direction and infinite in the other. Therefore, it will share some of the properties on torus and on plane. In addition, it will be interesting how the results change if one introduces the infinite potential wall.
This paper is organized as follows. In section II, we consider the quantization of abelian pure Chern-Simons gauge field on a strip (with the cylindrical boundary condition) and obtain an effective hamiltonian of matter field in the fundamental domain of the cylinder by defining new fields through a canonical transformation. In section III, we construct the many-body Schrödinger equation on the strip. The equation possesses the Aharonov-Bohm potential which is adapted to the cylindrical boundary condition and is responsible for the statistical transmutation. In section IV, we consider the periodic property of the fields and the boundary condition of the wavefunction. In section V, we comment on the theory inside a (rectangular) hard wall. Section V is the summary of the result and discussion.

II. HAMILTONIAN ON A STRIP

Let us consider the system in the fundamental domain of the cylinder, \( D_0 = (0 \leq x^1 < L_1 \) and \( -\infty < x^2 < \infty \). The lagrangian density of abelian Chern-Simons gauge theory coupled to non-relativistic matter field is given as

\[
\mathcal{L} = \frac{\mu}{2} \epsilon^{\mu
u\rho} a_\mu \partial_\nu a_\rho + \psi^+ i \hbar D_0 \psi - \frac{\hbar^2}{2m} |\vec{D}\psi|^2, \tag{2.1}
\]

where \( i \hbar D_\mu \psi = (i \hbar \partial_\mu - \frac{e}{c} a_\mu) \psi \). We will choose the matter field \( \psi \) as a fermion-field for definiteness. The analysis goes same with the bosonic case.

The gauge field does not propagate and the hamiltonian system becomes a constraint one. The phase space variables of the gauge fields are \( a_1 \) and \( \pi_1 \), where

\[
\pi_1 = \frac{\partial \mathcal{L}}{\partial \dot{a}_1} = \frac{\mu}{c} a_2, \tag{2.2}
\]

and the gauge-field itself constitutes the phase space. The hamiltonian density of the system is written as

\[
\mathcal{H} = a_0(\mu b + \frac{e}{c} J_0) + \frac{1}{2m} |i \hbar D_\mu \psi|^2, \tag{2.3}
\]

where \( J_0 = \psi^+ \psi \) and \( b = -(\partial_1 a_2 - \partial_2 a_1) \). The operators satisfy the equal-time (anti-) commutation relations;

\[
\{ \psi(x), \psi^+(y) \} = \delta(\vec{x} - \vec{y})
\]
\[ [J_0(x), \psi^+(y)] = \psi^+(x)\delta(\vec{x} - \vec{y}) \]
\[ [a_1(x), a_2(y)] = i\frac{\hbar}{\mu} \delta(\vec{x} - \vec{y}) . \]  

(2.4)

Here \( x \) denotes the three vector \((ct, \vec{x})\) and \( \vec{x} = (x^1, x^2) = (-x_1, -x_2) \).

\( a_0(x) \) is treated as a Lagrange multiplier and commutes with \( a_i \)'s. The constraint is given as the Gauss law, \( \Gamma \cong 0 \) where
\[ \Gamma \equiv b + \frac{e}{\mu c} J_0 \]  

(2.5)

is the gauge generator. A physical state should be annihilated by \( \Gamma \), \( \Gamma|_{phys} > 0 \).

On the strip with (period) length \( L_1 \) (the fundamental domain of the cylinder), the gauge operators are decomposed as
\[ a_i(x) = \frac{1}{\sqrt{2\pi L_1}} \sum_{n_1} \int dp \sqrt{2} \left[ \frac{p_1}{\mu} g(\vec{p}) + i\frac{\epsilon_{ij} p_j}{2} h(\vec{p}) \right] e^{-ip^\cdot x} + c.c \right|_{p^\rho = |\vec{p}|} , \]  

(2.6)

where \( n_1 = \frac{L_1 p_1}{2\pi} \) is an integer. \( g \) and \( h \) satisfy the commutation relation,
\[ [g(\vec{p}), h^+(\vec{q})] = \hbar \delta_{p_1 q_1} \delta(p_2 - q_2) \]
\[ [g(\vec{p}), g^+(\vec{q})] = [h(\vec{p}), h^+(\vec{q})] = 0 . \]

(2.7)

\( a_0(x) \) commutes with other fields and is expressed as
\[ a_0(x) = \lambda + \frac{1}{\sqrt{2\pi L_1}} \sum_{n_1} \int dp \sqrt{2} \left[ \frac{g(\vec{p})}{\mu} e^{-ip^\cdot x} + c.c \right] |_{p^\rho = |\vec{p}|} , \]  

(2.8)

where \( \lambda \) is a constant (time-independent) Lagrange multiplier. The statistical magnetic field \( b \) is given as
\[ b(x) = \frac{1}{\sqrt{2\pi L_1}} \sum_{n_1} \int dp \sqrt{2} \left[ \frac{p_1^0}{2} h(\vec{p}) e^{-ip^\cdot x} + c.c \right] |_{p^\rho = |\vec{p}|} . \]  

(2.9)

The above mode expansions are constructed such that \([a_0, a_i] = 0\), and the Lorentz gauge fixing condition is satisfied identically, \( \partial_\mu a^\mu = 0 \). There is still left a residual gauge degree of freedom with the Lorentz gauge fixing condition maintained: \( a_\mu \rightarrow a_\mu + \partial_\mu \Lambda \) where \( \partial^2 \Lambda = 0 \). This merely redefines \( g(\vec{p}) \) in Eqs. (2.6, 2.8) satisfying the same commutation relations in
Eq. (2.7). In our analysis, we do not need the specific form of \( g(\vec{p}) \) and therefore, we can proceed in the residual gauge independent manner.

The mode decomposition of the gauge field \( a_i(x) \) in Eq. (2.6) contains the zero-modes \( (n_1 = 0) \), which give rise to the mean field contribution to the \( b \)-field in Eq. (2.9). This is in contrast with the case on a torus, where we have to introduce the mean field mode in the canonical transformation formalism. On the other hand, the natural occurrence of the mean-field contribution is similar to the case on the infinite plane. This feature leads us to follow the canonical formalism used in the infinite plane case. We will consider two procedures adopted for the infinite plane and on the torus separately and compare the distinct features.

Let us try first the canonical transformation following the method \([6]\) used in the field theory on a plane to get the effective hamiltonian and decouple the gauge field from the physical Hilbert space. The canonical transformation operator is given as

\[
V_1(t) = \exp i \frac{e}{\hbar c} \int \int d\vec{x}d\vec{y} J_0^{(1)}(\vec{x}, t) G_{cyl}(\vec{x}, \vec{y}) \partial_k a_k^{(1)}(\vec{y}, t) .
\] (2.10)

\( G_{cyl} \) is the periodic (single-valued on the cylinder) Green’s function satisfying

\[
\nabla^2 G_{cyl}(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y})
\] (2.11)

\[
G_{cyl}(\vec{x}, \vec{y}) = G_{cyl}(\vec{x}, \vec{y} + L_1 \hat{e}_1) ; \quad G_{cyl}(\vec{x}, \vec{y}) = G_{cyl}(\vec{y}, \vec{x})
\] (2.12)

whose explicit form is given as

\[
G_{cyl}(\vec{x}, \vec{y}) = \frac{1}{4\pi} \ln |\sin \pi z|^2
\] (2.13)

where \( z = [(x^1 - y^1) + i(x^2 - y^2)]/L_1 \). We regularize the Green’s function as

\[
\epsilon_{ij} \partial_x^j G_{cyl}(\vec{x}, \vec{y}) \big|_{\vec{x}=\vec{y}} = 0 .
\] (2.14)

The original fields are given in terms of the new fields as

\[
\psi(x) = V_1(t) \psi^{(1)}(x) V_1^+(t)
\]

\[
= \psi^{(1)}(x) \exp -\frac{ie}{\hbar c} \int d\vec{y} G_{cyl}(\vec{x}, \vec{y}) \partial\psi a_i^{(1)}(y) ,
\]
\[ a_i(x) = V_i(t)a_i^{(1)}(x)V_i^+(t) = a_i^{(1)}(x) - \frac{e}{\mu c}\epsilon_{ij}\partial_{x^j} \int d\vec{y}G_{cyl}(\vec{x}, \vec{y})J_0^{(1)}(y), \] (2.15)

where \( J_0(x) = J_0^{(1)}(x) \). The new fields satisfy the same form of the commutation relations given in Eq. (2.4).

Expressing the Hamiltonian density \( H \) in terms of the new fields, we have

\[ H = \mu a_0 b^{(1)} + \frac{1}{2m} \left| (i\hbar\partial_i - \frac{e}{c}a_i^{(\text{eff})})\psi^{(1)} \right|^2. \] (2.16)

Implicit ordering of \( \psi^{(1)+} \) to the left of \( a_i^{(\text{eff})} \) is assumed. Here \( b^{(1)} = -(\partial_1a_2^{(1)} - \partial_2a_1^{(1)}) \) is the gauge generator since \( b^{(1)} = b + \frac{e}{\mu c}J_0 = \Gamma \) from Eq. (2.15). \( a_i^{(\text{eff})} \) is an effective gauge-like term,

\[ a_i^{(\text{eff})}(x) = \epsilon_{ij}\partial_{x^j} \int d\vec{y}G_{cyl}(\vec{x}, \vec{y})(b^{(1)}(y) - \frac{e}{\mu c}J_0(y)). \] (2.17)

This is obtained by noting that (see Eq. (2.10))

\[ a_i^{(1)}(x) = \epsilon_{ij}\partial_{x^j} \int d\vec{y}G_{cyl}(\vec{x}, \vec{y})b^{(1)}(x) + \partial_{x^i} \int d\vec{y}G_{cyl}(\vec{x}, \vec{y})\partial_y a_j^{(1)}(y). \] (2.18)

We emphasize that the readers should not confuse the gauge field \( a_i^{(1)}(x) \) with \( a_i^{(\text{eff})}(x) \). \( a_i^{(1)}(x) \)'s are satisfying the same commutation relation as that given in Eq. (2.4). However, the commutation relation between \( a_i^{(\text{eff})}(x) \)'s is given as

\[ [a_1^{(\text{eff})}(x), a_2^{(\text{eff})}(y)] = 0. \] (2.19)

The gauge transformation operator of this system is given as

\[ B \equiv i\frac{\mu}{\hbar} \int d\vec{x}b^{(1)}(\vec{x})\Lambda(x), \] (2.20)

where \( \Lambda(x) \) is an arbitrary real periodic function. \( a_i^{(1)}(x), a_i(x) \) and \( \psi(x) \) are gauge-dependent since

\[ [B, a_i^{(1)}(x)] = \partial_i\Lambda(x), \quad [B, a_i(x)] = \partial_i\Lambda(x), \]
\[ [B, \psi(x)] = \frac{-ie}{\hbar c} \Lambda(x) \psi(x), \tag{2.21} \]

whereas \( \psi^{(1)}(x) \) is gauge invariant

\[ [B, \psi^{(1)}(x)] = 0. \tag{2.22} \]

Since the gauge generator \( b^{(1)}(x) \) commutes with any other operators in \( \mathcal{H} \) and since we consider the gauge invariant operators and physical states only, we may drop \( b^{(1)}(x) \) completely in \( \mathcal{H} \). Therefore, the gauge fields disappear from the hamiltonian density,

\[ \mathcal{H}(x) = \frac{1}{2m} \left| (i\hbar \partial_i - \frac{e}{c} A_i^{[1]}(x)) \psi^{(1)}(x) \right|^2, \tag{2.23} \]

where

\[ A_i^{[1]}(x) = -\frac{e}{\mu c} \epsilon_{ij} \partial_j \int d\vec{y} G_{cyl}(\vec{x}, \vec{y}) J_0(y). \tag{2.24} \]

The effective hamiltonian density is the desired form, which amounts to solving the Gauss constraint and the newly defined fields enable us to construct the physical state by simply applying operators consisting of \( \psi^{(1)} + \) on the vacuum \( |0> \) which satisfies \( b^{(1)}|0> = J_0|0> = 0. \)

The canonical transformation is successful in obtaining the effective interaction. One may wonder if there is any subtleties observed in the canonical formalism on a compact space such as on a torus since a cylinder has a finite dimension in one direction and an infinite on the other. We note that one of the subtleties arises on the boundary condition of the Green’s function on the compact space. To see this more explicitly, we may re-investigate the new gauge field \( a_i^{(1)}(x) \) given in Eq. (2.15). To obtain this, we discarded the boundary term considering that it vanishes. It really vanishes at the edge of the strip since the Green’s function is given in Eq. (2.13). On the other hand, at infinity of the strip \( G_{cyl} \) increases linearly with the coordinate, but one still regards the contributions at \( \pm \infty \) cancel each other.

One may avoid the linear growth of the Green’s function by removing the asymptotically increasing term and define another periodic Green’s function.
\begin{equation}
G_p(\vec{x}, \vec{y}) = \frac{1}{4\pi} \ln |\sin \pi z|^2 - \frac{1}{2}|x_2 - y_2|, \quad (2.25)
\end{equation}

satisfying
\begin{equation}
\nabla^2 G_p(\vec{x}, \vec{y}) = (\delta(x^1 - y^1) - \frac{1}{L_1})\delta(x^2 - y^2). \quad (2.26)
\end{equation}

We can consider another canonical operator \( V_2(t) \) in analogy as in the torus [12] using the Green’s function \( G_p(\vec{x}, \vec{y}) \),
\begin{equation}
V_2(t) = \exp \left( \frac{e}{\hbar c} \int d\vec{x} d\vec{y} J_0^{(2)}(\vec{x}, t) \left[ g_2(\vec{x}, \vec{y})\theta_2^{(2)}(\vec{y}, t) + G_p(\vec{x}, \vec{y})\partial_y a_k^{(2)}(\vec{y}, t) \right] \right). \quad (2.27)
\end{equation}

\( \theta_i \) is the analogue of the zero-mode on a torus,
\begin{align*}
\theta_1(x) &= \frac{1}{2 \sqrt{2\pi} L_1} \int dp_2[ie^{-i(p \cdot x)} + c.c]|_{p_0=|p_2|, p_1=0} \\
\theta_2(x) &= \frac{1}{\mu \sqrt{2\pi} L_1} \int dp_2[ie^{-i(p \cdot x)} + c.c]|_{p_0=|p_2|, p_1=0} \quad (2.28)
\end{align*}

which satisfy the equal-time commutation relation
\begin{equation}
[\theta_1(x), \theta_2(y)] = \frac{i\hbar}{\mu L_1} \delta(x_2 - y_2). \quad (2.29)
\end{equation}

\( g_2(\vec{x}, \vec{y}) \) is a function satisfying \( \partial_2 g_2(\vec{x}, \vec{y}) = \delta(x_2 - y_2) \), whose explicit form we may choose as
\begin{equation}
g_2(\vec{x}, \vec{y}) = \frac{1}{2} \epsilon(x^2 - y^2). \quad (2.30)
\end{equation}

The canonical operator \( V_2(t) \) reduces to \( V_1(t) \) if we neglect the boundary term: \( V_2(t) \) is an improved operator.

The relation of the new fields with the original ones is given as
\begin{align*}
\psi(x) &= V_2(t)\psi_2^{(2)}(x)V_2^+(t) \\
&= \psi_2^{(2)}(x) \exp \left( \frac{ie}{\hbar c} \int d\vec{y} \left( g_2(\vec{x}, \vec{y})\theta_2(y) + G_p(\vec{x}, \vec{y})\partial_y a_i^{(2)}(y) \right) \right), \\
a_i(x) &= V(t)a_i^{(2)}(x)V^+(t) \\
&= a_i^{(2)}(x) - \frac{e}{\mu c} \epsilon_{ij} \partial_x^j \int d\vec{y} G_{cy1}(\vec{x}, \vec{y}) J_0^{(2)}(y),
\end{align*}
\begin{equation}
(2.31)
\end{equation}
where \( J_0(x) = J_0^{(2)}(x) \) and \( b^{(2)} = -\left( \partial_1 a_2^{(2)} - \partial_2 a_1^{(2)} \right) \). Expressing the hamiltonian density \( \mathcal{H} \) in terms of the new fields, we have

\[
\mathcal{H} = \mu a_0 b^{(2)} + \frac{1}{2m} \left| (i\hbar \partial_t - \frac{e}{c} a_i^{(\text{eff})}) \psi^{(2)} \right|^2. \tag{2.32}
\]

Implicit ordering of \( \psi^{(2)+} \) to the left of \( a_i^{(\text{eff})} \) is assumed. Here \( b^{(2)} = \Gamma = b^{(1)} \) is the gauge generator since \( b^{(2)} = b + \frac{e}{\mu c} J_0 \) from Eq. (2.31). \( a_i^{(\text{eff})} \) is the same effective gauge-like term given in Eq. (2.17). The effective hamiltonian does reduce to the same form as obtained from the canonical transformation using \( V_1 \). Restricting the system to the physical Hilbert space we decouple the gauge field from the system and regain the hamiltonian given in Eq. (2.23).

### III. MANY-BODY QUANTUM MECHANICS

The many-body quantum mechanics can be obtained from the field theory using the Heisenberg equation of motion. Let us define the \( N \)-particle wavefunction as

\[
\Phi(1, \ldots, N) \equiv \langle 0 | \psi^{(2)}(x^{(1)}) \cdots \psi^{(2)}(x^{(N)}) | N \rangle. \tag{3.1}
\]

We assume that all the coordinates of the particles lie in a fundamental domain. \(|0>\) is the vacuum which satisfies \( J_0 |0> = 0 \) and \(|N>\) is the \( N \)-body Heisenberg state vector. The wavefunction is gauge invariant since \( \psi^{(2)}(x) \) is gauge invariant. Then the Schrödinger equation is given as

\[
i\hbar \frac{\partial \Phi}{\partial t}(1, \ldots, N) = \sum_{p=1}^{N} <0|\psi^{(2)}(x^{(1)}) \cdots \hbar a_i \psi^{(2)}(x^{(p)}) \cdots \psi^{(2)}(x^{(N)}) | N >. \tag{3.2}
\]

The time evolution of the matter-field operator is given as the Heisenberg equation of motion,

\[
i\hbar \frac{\partial \psi^{(2)}(x)}{\partial t} = [\psi^{(2)}(x), H]
\]

\[
= \frac{1}{2m} \left| (i\hbar \partial_x - \frac{e}{e} A_i^{[1]}(x))^2 + \frac{(e)}{c})^2 \int d\vec{y} \psi^{(2)+}(y) K_i(y, x) K_i(y, x) \psi^{(2)}(y) \right.
\]

\[
- \frac{(e)}{c}) \int d\vec{y} \psi^{(2)+}(y)(i\hbar \partial_y - \frac{(e)}{c})A_i^{[1]}(y)) K_i(y, x) \psi^{(2)}(y)
\]
\[- \left( \frac{e}{c} \int d\vec{y} \psi^{(2)+}(y) \right) K_i(y, x) (i\hbar \partial_{y^i} - \left( \frac{e}{c} A_i^{[1]}(y) \right) \psi^{(2)}(y)) \psi^{(2)}(x) \right],
\]

where

\[ K_i(x, y) = - \frac{e}{\mu c} \epsilon_{ij} \partial_{x^j} G_{cyl}(\vec{x}, \vec{y}). \]

\( G_{cyl}(\vec{x}, \vec{y}) \) is given in Eq. (2.13). We put \( \psi^{(2)+} \) and \( A_i^{[1]} \) to the far left in each term such that the operators vanish when they act on the vacuum \( <0| \) and used the identity

\[ [\psi^{(2)}(x), A_i^{[1]}(y)] = K_i(y, x) \psi^{(2)}(x)]. \]

The one-particle wavefunction satisfies the Schrödinger equation

\[ i\hbar \frac{\partial \Phi}{\partial t}(x) = <0| i\hbar \frac{\partial \psi^{(2)}(x)}{\partial t}|1> = - \frac{\hbar^2}{2m} \nabla^2 \Phi(x). \]

The equation does reduce to the free one. For the two-body case, we have

\[ i\hbar \frac{\partial \Phi}{\partial t}(1, 2) = <0| \psi^{(2)}(x^{(1)}) i\hbar \frac{\partial \psi^{(2)}(x^{(2)})}{\partial t} + i\hbar \frac{\partial \psi^{(2)}(x^{(1)})}{\partial t} \psi^{(2)}(x^{(2)})|2> \]

\[ = \frac{1}{2m} \{ (i\hbar \partial_{x^{(1)}} - \frac{e}{c} A_i(1, 2))^2 + (i\hbar \partial_{x^{(2)}} - \frac{e}{c} A_i(2, 1))^2 \} \Phi(1, 2), \]

where

\[ A_i(p, r) = - \frac{e}{\mu c} \epsilon_{ik} \partial_k^{(p)} G_{cyl}(p, r), \]

is the Aharonov-Bohm potential. In general for the \( N \)-particle case, we have

\[ i\hbar \frac{\partial \Phi}{\partial t}(1, \cdots, N) = \frac{1}{2m} \sum_{p=1}^{N} \left\{ i\hbar \partial_{x^{(p)}} - \frac{e}{c} \sum_{r=1(\neq p)}^{N} A_i(p, r) \right\}^2 \Phi(1, \cdots, N). \]

The Aharonov-Bohm gauge potential can be transformed away through the singular gauge transformation as in the infinite plane case. Explicitly, the gauge potential becomes

\[ A_i(p, r) = - \partial_{x^{(p)}}^{(p)} \left[ \frac{i e}{4 \pi \mu c} \ln \frac{\sin(\pi z_{pr})}{\sin(\pi z_{pr})} \right], \]

when we neglect the singular parts at the coincident points. As the result, the Schrödinger equation becomes the free one, and the transformed wave-function (anyonic wavefunction) is multivalued

\[ \Phi^{(anyon)} (1, \cdots, N) = \prod_{p > r} \left( \frac{\sin(\pi z_{pr})}{\sin(\pi z_{pr})} \right) \frac{\nu}{\sin(\pi z_{pr})} \Phi (1, \cdots, N), \]

where \( \nu = e^2/(2\pi \hbar \mu c^2) \) and \( \frac{e}{\mu c} = \nu \phi_0 \) with the unit flux quantum \( \phi_0 = \hbar c/e. \)
IV. PERIODIC PROPERTY

Until now we have considered the theory on the fundamental domain of the cylinder. To find the boundary condition of the anyon equation, we have to consider the theory on the covering space, which consists of the repeated domains of the fundamental one with the boundary identified. In this scheme, the non-contractible loop on the cylinder is identified with a line on the fundamental domain from one edge to the other. To describe the hamiltonian density outside the fundamental domain, let us denote $D_m$ for the domain with $(x'^1 = x^1 + mL_1, x'^2 = x^2)$ where $x^i$ lies on the fundamental domain $D_0$ and $m$ is an integer. (In the following, we reserve the unprimed coordinates for the ones in the fundamental domain $D_0$ and primed for the domain $D_m$).

Obviously, the effective hamiltonian density on $D_m$ can be written as

$$
H(x') = \frac{1}{2m} \left| i\hbar \partial_i - \frac{e}{c} A_i^{[1]}(x') \right| \psi^{(2)}(x') \right|^2 ,
$$

where

$$
A_i^{[1]}(x') = -\frac{e}{\mu c} \epsilon_{ij} \partial_{x^j} \int d\bar{y} G_{cyl}(\bar{x}', \bar{y}') J_0(y') .
$$

Noting the relation, $G_{cyl}(\bar{x}, \bar{y}) = G_{cyl}(\bar{x}', \bar{y}')$, we have

$$
A_i^{[1]}(x') = -\frac{e}{\mu c} \epsilon_{ij} \partial_{x^j} \int d\bar{y} G_{cyl}(\bar{x}, \bar{y}) J_0(y') .
$$

This hamiltonian density on $D_m$ is canonically related with the one on $D_0$

$$
H(x') = T_m H(x) T_m^+ ,
$$

where

$$
J_0(x') = T_m J_0(x) T_m^+ ,
$$

$$
b^{(2)}(x') = T_m b^{(2)}(x) T_m^+ ,
$$

$$
\psi^{(2)}(x') = T_m \psi^{(2)}(x) T_m^+ .
$$

The explicit translation operator along the non-contractible loop is given as $T_m = T_1^m$, where
\[ T_1 \equiv \exp -L_1 \int d\vec{y} \{ \psi^{(2)+}(y) \partial_y^{\dagger} \psi^{(2)}(y) + i \frac{\mu}{\hbar} \partial_1^{(2)}(y) \partial_y^{\dagger} a_1^{(2)}(y) \}. \] (4.5)

We also note that the translation operator connects the gauge field as

\[ a_i^{(2)}(x') = T_m a_i^{(2)}(x) T^+_m. \] (4.6)

On the cylinder one can leave the hamiltonian density \( \mathcal{H} \) invariant under the translation along the non-contractible loops as given in Eq. (4.4) if one defines the matter field as \( \psi^{(2)}(x') = \psi^{(2)}(x) \exp i C_m \), where \( C_m \) is a constant on \( D_m \). We can choose the constant \( C_m = 0 \) without losing any generality.

\[ \psi^{(2)}(x') = \psi^{(2)}(x). \] (4.7)

This definition leads to the periodic condition on \( J_0 \) and \( b^{(2)} \) as

\[ J_0(x') = J_0(x); \quad b^{(2)}(x') = b^{(2)}(x). \] (4.8)

This makes the effective gauge field \( A_i^{[1]}(x') \) manifestly translation-invariant,

\[ A_i^{[1]}(x') = A_i^{[1]}(x), \] (4.9)

and the translation-invariance of the hamiltonian density follows:

\[ [T_m, \mathcal{H}(x)] = 0. \] (4.10)

We note that the vacuum defined on the fundamental domain remains the same on the covering space due to Eq. (4.8). Therefore, the physical state is the same on all the covering space and the Gauss constraint is identically satisfied. The equal-time commutation relation of \( J_0(x) \) with \( \psi^{(2)}(y') \) is given as

\[ [J_0(x), \psi^{(2)}(y')] = [J_0(x), \psi^{(2)}(y)] = -\psi^{(2)}(y) \delta(\vec{x} - \vec{y}). \] (4.11)

One can also define the periodic property of the gauge field \( a_i^{(2)} \) given in Eq. (4.6) up to a gauge transformation, which leaves the field strength \( b^{(2)} \) invariant as given in Eq. (4.8),

\[ a_i^{(2)}(x') = T_m a_i^{(2)}(x) T^+_m. \] (4.6)
\[ a_i^{(2)}(x') = a_i^{(2)}(x) + \partial_i \Omega(x), \]  

(4.12)

where \( \Omega(x) \) is an arbitrary periodic function.

The original fields can be obtained using the canonical operator,

\[
V_2^{(m)}(t) = \exp \frac{i}{\hbar c} \int d\vec{x}d\vec{y} J_0(x') \left[ g_2(\vec{x}, \vec{y}) \theta_2(y') + G_p(\vec{x}, \vec{y}) \partial_y a^{(2)}_i(y') \right],
\]

(4.13)

where the integration \( \int d\vec{x} \) denotes for \( \int_{0 \leq x_1 < L_1, -\infty < x_2 < \infty} d\vec{x} \). The operator is the same as in Eq. (2.27) except that the field operators are replaced by the ones on the domain \( D_m \). The hamiltonian density becomes

\[
\mathcal{H}(x') = \frac{1}{2m} \left| \left( i\hbar \partial_i - \frac{e}{c} a_i(x') \right) \psi(x') \right|^2,
\]

(4.14)

where

\[
\psi(x') = \exp -i \frac{e}{\hbar c} \int d\vec{y} \left[ g_2(\vec{x}, \vec{y}) \theta_2(y') + G_p(\vec{x}, \vec{y}) \partial_y a^{(2)}_i(y') \right] \psi^{(2)}(x'),
\]

\[
a_i(x') = a_i^{(2)}(x') - \frac{e}{\mu c} \epsilon_{ij} \partial_{x^j} \int d\vec{y} G_{cyl}(\vec{x}, \vec{y}) J_0(y).
\]

(4.15)

The periodic property of the fields are easily deduced using Eqs. (4.7, 4.12),

\[
\psi(x') = \psi(x) \exp -i \frac{e \Omega(x)}{\hbar c},
\]

\[
a_i(x') = a_i(x) + \partial_i \Omega(x),
\]

(4.16)

which demonstrates that the periodic property for \( \psi(x) \) and \( a_i(x) \) is given as a local gauge transformation. The commutation relations of the fields are given as

\[
\{ \psi(x'), \psi(y'') \} = 0
\]

\[
\{ \psi(x'), \psi^+(y'') \} = \delta(\vec{x} - \vec{y})
\]

\[
[a_1(x'), a_2(y'')] = i \frac{\hbar}{\mu} \delta(\vec{x} - \vec{y}),
\]

where \( x' \) and \( y'' \) may lie in different domains.

Let us consider the many anyon wavefunctions whose coordinates are located in different domains. Similarly in Eq. (3.1), we define the wavefunction as
\[ \Phi(\{1\}, \ldots, \{N\}) \equiv 0|\psi^{(2)}(\{1\}) \ldots \psi^{(2)}(\{N\})|N >, \tag{4.17} \]

where the coordinates \( \{i\} \equiv (x_1^{(i)} + m^{(i)}L_1, x_2^{(i)}) \) with \( m^{(i)} \) integer may lie in any mixed domain. The Schrödinger equation is the same as the one given in Eq. (3.8) since the hamiltonian density \( H(x') \) is translation-invariant:

\[ i\hbar \frac{\partial \Phi}{\partial t}(\{1\}, \ldots, \{N\}) = \frac{1}{2m} \sum_{p=1}^{N} i\hbar \partial_x^{(p)} - \frac{e}{c} \sum_{r=1(\neq p)}^{N} A_r(p, r) \Phi(\{1\}, \ldots, \{N\}). \tag{4.18} \]

The hamiltonian has the periodic gauge-like potential since it contains the coordinate function of \( x^{(i)} \) rather than \( x^{(i)'} \). One should note that the periodic property of the matter field in Eq. (4.7) maintains the trivially periodic wavefunction

\[ \Phi(\{1\}, \ldots, \{N\}) = \Phi(1, \ldots, N), \tag{4.19} \]

and the fermionic exchange property

\[ \Phi(\ldots, \{i\}, \ldots, \{j\}, \ldots) = -\Phi(\ldots, \{j\}, \ldots, \{i\}, \ldots). \tag{4.20} \]

**V. HARD WALL BOUNDARY CONDITION**

The analysis for the torus and cylinder can be easily extended to the box surrounded by the hard wall. Let us consider the system in a box \((0 \leq x^1 < L_1 \text{ and } 0 \leq x^2 < L_2)\) with infinite potential barrier at the edge. The gauge mode is expanded as

\[ a_i(x) = \frac{2}{\sqrt{L_1 L_2}} \sum_{n_1, n_2} \frac{1}{\langle \vec{p} \rangle} \left[ \frac{p_i}{\mu} g(\vec{p}) + i \frac{\epsilon_{ij} \rho^j}{2} h(\vec{p}) \right] \sin p_1 x_1 \sin p_2 x_2 e^{-ip_0 x_0} + c.c \right]_{\rho^0 = |\vec{p}|}, \tag{5.1} \]

where \( n_i = \frac{L_i \rho^i}{\pi} \) is a positive integer, \( g \) and \( h \) satisfy the discrete version of the commutation relation given in Eq. (2.7). \( a_0(x) \) is expressed as

\[ a_0(x) = \lambda + \frac{2}{\sqrt{L_1 L_2}} \sum_{n_1, n_2} \frac{g(\vec{p})}{\mu} \sin p_1 x_1 \sin p_2 x_2 e^{-ip_0 x_0} + c.c \right]_{\rho^0 = |\vec{p}|}. \tag{5.2} \]

For the hamiltonian density \( \mathcal{H} \) given in Eq. (2.3), we consider a canonical transformation operator
\[ W(t) = \exp \frac{i}{\hbar c} \int d\vec{x} d\vec{y} J_0^{(1)}(\vec{x}, t) G_{\text{box}}(\vec{x}, \vec{y}) \partial_k a_k^{(1)}(\vec{y}, t). \] (5.3)

\( G_{\text{box}} \) is the Green's function, which vanishes at the edge of the box, whose form is written as

\[ G_{\text{box}}(\vec{x}, \vec{y}) = \frac{1}{4\pi} \ln \left| f(z, \bar{z}; \zeta, \bar{\zeta}) \right|^2, \] (5.4)

where \( z = (x^1 + ix^2)/L_1, \zeta = (y^1 + iy^2)/L_1. \) Explicitly,

\[ f(z, \bar{z}; \zeta, \bar{\zeta}) = \frac{\theta_1(\frac{z-\zeta}{2}\tau)\theta_1(\frac{z+\zeta}{2}\tau)}{\theta_1(\frac{\bar{z}-\bar{\zeta}}{2}\tau)\theta_1(\frac{\bar{z}+\bar{\zeta}}{2}\tau)} \] (5.5)

where \( \theta_1 \) is the odd Jacobi theta function and \( \tau = iL_2/L_1. \) One may also express this Green’s function in terms of the Weierstrass’ associated \( \sigma \) function,

\[ f(z, \bar{z}; \zeta, \bar{\zeta}) = \frac{\sigma(z - \zeta)\sigma(z + \zeta)}{\sigma(z - \zeta)\sigma(z + \zeta)}. \] (5.6)

The original fields are given in terms of the new fields as

\[ \psi(x) = W(t)\psi^{(1)}(x)W^+(t) \]
\[ = \psi^{(1)}(x) \exp \left\{ -i \left\{ \frac{e}{\hbar c} \int d\vec{y} G_{\text{box}}(\vec{x}, \vec{y}) \partial_y a_y^{(1)}(y) \right\} \right\}, \]

\[ a_i(x) = W(t)a_i^{(1)}(x)W^+(t) \]
\[ = a_i^{(1)}(x) - \frac{e}{\mu c} \epsilon_{ij} \partial_x \int d\vec{y} G_{\text{box}}(\vec{x}, \vec{y}) J_0^{(1)}(y), \] (5.7)

where \( J_0(x) = J_0^{(1)}(x). \)

The hamiltonian density \( \mathcal{H} \) becomes of the form in Eq. (2.16) and the effective gauge field is given as

\[ a_i^{(\text{eff})}(x) = \epsilon_{ij} \partial_x \int d\vec{y} G_{\text{box}}(\vec{x}, \vec{y})(b^{(1)}(y) - \frac{e}{\mu c} J_0(y)). \] (5.8)

As done in section 2, we may drop \( b^{(1)}(x) \) completely since it generates the gauge transformation and commutes with other modes. Therefore, the hamiltonian density becomes,

\[ \mathcal{H}(x) = \frac{1}{2m} \left| (i\hbar \partial_i - \frac{e}{c A_1^{(1)}(x)}) \psi^{(1)}(x) \right|^2, \] (5.9)
where

\[ A^{[1]}(x) = -\frac{e}{\mu c} \epsilon_{ij} \partial_x \int d\vec{y} G_{\text{box}}(\vec{x}, \vec{y}) J_0(y). \] (5.10)

Now the many-anyon equation can be obtained easily. We have only to replace the Green’s function in Eq. (3.4) by the one given in Eq. (5.4). The Aharonov-Bohm potential appears adapted to the boundary condition, which can be transformed away leaving the multi-valued wavefunction as

\[ \Phi^{(\text{anyon})}(1, \cdots, N) = \prod_{p>r} \left( \frac{f(z_p, \bar{z}_p; z_r, \bar{z}_r)}{f^*(z_p, \bar{z}_p; z_r, \bar{z}_r)} \right)^{\frac{\mu}{2}} \Phi(1, \cdots, N). \]

Finally, one may also apply the theory inside strip with the hard-wall at edge. In this case the Green’s function becomes of the form

\[ G_{\text{strip}}(\vec{x}, \vec{y}) = \frac{1}{4\pi} \ln |F(z, \zeta, \bar{\zeta})|^2; \] (5.11)

where

\[ F(z, \zeta, \bar{\zeta}) = \prod_{n=-\infty}^{\infty} \frac{z - \zeta - 2n}{z + \zeta - 2n}. \]

From this one can obtain easily the anyon equation following the above analysis.

VI. SUMMARY AND DISCUSSION

We have analyzed the Chern-Simons theory coupled to non-relativistic matter field on a cylinder in analogy with the case for the infinite plane and on the torus. Quantizing the field first and performing canonical transformation we obtain an effective theory in terms of matter field such that it manifests the gauge invariance and the translation invariance along the non-contractible loop. The periodic property of the original field along the non-contractible loop is determined up to the gauge transformation, which is demonstrated from the simple periodic property of the new fields through the canonical transformation. The canonical transformation operator can be constructed either like in the infinite plane case \( V_1(t) \) in Eq. (2.10) or in the torus case \( V_2(t) \) in Eq. (2.27), which result in the same effective hamiltonian and anyon equation.
Unlike on a torus the gauge field on the cylinder in Eq. (2.6) does not need the extra mean magnetic field contribution since both the mean field contribution and the analogue of the zero-mode $\theta_i(x)$ as given in Eq. (2.28) are contained as necessary ingredients to satisfy the commutation relation given in Eq. (2.4). This mean field contribution has the important role to decouple the gauge field in the canonical transformation. The typical physical state constructed as

$$|N > \sim \int d\vec{x}(1) \cdots d\vec{x}(N) \psi^{(2)+}(1) \cdots \psi^{(2)+}(N)|0 >,$$

$(\psi^{(2)+}$ is the gauge invariant field defined in Eq. (2.31)) satisfies the Gauss constraint automatically and flux quantization is not needed since $\int dy b(y)|N > = -e/c N|N >$ due to the mean field contribution in the gauge field and $Q|N > = N|N >$ where $Q = \int d\vec{x} J_0(x)$.

One may take the Wilson loop operator into consideration, which is defined as $W(t) = \exp i \int_{-\infty}^{\infty} dy \theta_i^{(2)}(x, y, t) \kappa(y)$, where $\kappa(y)$ is an arbitrary distribution. One can easily check that $W$ is gauge invariant and is a constant of motion,

$$[W, \Gamma] = 0; \quad [W, H] = 0.$$

Therefore, one may define the wavefunction as

$$\Phi^{(w)}(1, \cdots, N) \equiv <0|W \psi^{(2)}(x^{(1)}) \cdots \psi^{(2)}(x^{(N)})|N >.$$

instead of the definition given in Eq. (3.1). This newly-defined wavefunction is also gauge invariant and satisfies the same Schrödinger equation in Eq. (3.8). It is obvious that this wavefunction has the simple periodic condition as given in Eq. (4.19).

We also considered the system inside a box with the hard wall boundary condition, where the zero-modes do not appear. There are mean magnetic field modes in the gauge field, which play the similar role of $\theta_i$ on the cylinder and decouple the gauge fields from the physical states after the canonical transformation. The canonical transformation operator is similar to the one given in the plane case. The resulting many-body Schrödinger equation contains the Aharonov-Bohm potential adapted to the hard-wall boundary condition.
Finally, we note that even though the many-body equation looks very simple (it becomes free equation when the potential is singular gauged away), its solution is not easy to get at. In this situation, to understand the clear connection between the degenerate ground states of the anyons on the infinite plane with magnetic field and the eigenstates of the Calogero model [14] will be helpful for obtaining the idea about the spectrum and wavefunctions of the many-body states in general.

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