ON A CLASS OF UNEQUIVOLUME PARTITIONS WITH SMALL EXPECTED $L_2$-DISCREPANCY

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Abstract. We study the expected $L_2$-discrepancy under a class of unequivol-
ume partitions, explicit formula is given. This result attains smaller expected
$L_2$-discrepancy than the use of jittered sampling and partition given in [9].

1. Introduction

It is well known that classical jittered sampling (JS) patterns perform better than
traditional Monte Carlo (MC) patterns in terms of convergence order, see [10, 12,
13]. This means stratified sampling is the refinement of the traditional Monte Carlo
method, which involves the measure of the irregularity of point distribution, volume
partition is adopted to study it, which also needs us to introduce the definition of
$L_2$-discrepancy.

$L_2$-discrepancy. $L_2$-discrepancy of a sampling set $P_{N,d} = \{t_1, t_2, \ldots, t_N\}$ is
defined by

$$L_2(D_N, P_{N,d}) = \left( \int_{[0,1]^d} |\lambda([0,z]) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(t_i)|^2 dz \right)^{1/2},$$

where $\lambda$ denotes the Lebesgue measure, $1_A$ denotes the characteristic function on set
$A$. For the applications of $L_2$-discrepancy, see [3, 5, 7].

In the definition of $L_2$-discrepancy, if we introduce the counting measure $\#$, (1.1)
can also be expressed as

$$L_2(D_N, P_{N,d}) = \left( \int_{[0,1]^d} |\lambda([0,z]) - \frac{1}{N} \#(P_{N,d} \cap [0,z])|^2 dz \right)^{1/2},$$

where $\#(P_{N,d} \cap [0,z])$ denotes the number of points falling into the set $[0, z]$.

To simplify the expression of $L_2$-discrepancy, we employ the discrepancy function
$\Delta(P_{N,d}, z)$ via:

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Discrepancy function (1.3) essentially reveals the difference between the ratio of the tested area to the total area and the ratio of random sample points falling within the area to the total sample points. Its $L^2$-norm ($L^2$-discrepancy) actually measures the uniformity of random sampling point set.

Accordingly, the $L_2$--discrepancy can be extended to a fixed compact convex set $K \subset \mathbb{R}^d$ with $\lambda(K) > 0$, see [9]. Discrepancy function in (1.3) of a finite set of points $P = \{x_1, x_2, \ldots, x_n\} \subset K$ is now given by

\begin{equation}
\Delta(P, x) = \frac{\lambda((-\infty, x] \cap K)}{\lambda(K)} - \frac{1}{N} \#(P \cap (-\infty, x]).
\end{equation}

The $L_2$-discrepancy is well established and has found applications in areas of number theory, see [3–5, 7]. The special deterministic point set constructions are called low discrepancy point sets, and the best known asymptotic upper bounds of $L_2$-discrepancy for these point sets are of the form $O((\ln N)^{d-1} 2^{-2d} N).$

In the present paper, we introduce random factors to study $L_2$-discrepancy. Former have conducted extensive research on $L_2$-discrepancy under random sampling. In [12], F. Pausinger and S. Steinerberger proved that the expected $L_2$-discrepancy of a point set generated from any equivolume partition is no larger than that from simple random sampling. The conclusion of 'strictly smaller' was given in [10]. Explicit $L_2$--discrepancy under simple random sampling $P_N$ was presented,

\begin{equation}
\mathbb{E}L^2_2(D_N, P_N) = \frac{1}{N^2} \left[ \frac{1}{2} \sum_{i=1}^{d} (m - 1) - \frac{1}{3} \right],
\end{equation}

for the corresponding sampling manner, see Figure 1.

Jittered sampling has been found that it was not the best equivolume partition manner if we use the $L_2$-discrepancy measure. This was proved in [9]. Recently, an explicit expected $L_2$-discrepancy formula has been given in [11], which is,

\begin{equation}
\mathbb{E}L^2_2(D_N, P_\Omega) = \frac{1}{m^2} \left[ \left(\frac{m - 1}{2}\right) + \frac{1}{2} \right]^d - \left(\frac{m - 1}{2} + \frac{1}{3}\right)^d,
\end{equation}

where $N = m^d$ and $P_\Omega$ is jittered sampling set, for the corresponding partition, see Figure 2.

Combining this explicit formula with the partition manner $\Omega^*_\times$ in [9], it can be proved
\[
\mathbb{E} L_2^2(D_N, P_{\nu'}) = \frac{1}{m^{2d}} \left[ \left( \frac{m - 1}{2} + \frac{1}{2} \right)^d - \left( \frac{m - 1}{2} + \frac{1}{3} \right)^d \right] - \frac{2}{5} \cdot \frac{1}{3^d} \cdot \frac{1}{m^{3d}},
\]

where \( N = m^d \), for the corresponding partition, see Figure 3.

In present paper, we study a class of unequivolume partitions, see Figure 4, which acquires better \( L_2 \)-discrepancy than the use of jittered sampling and stratified sampling sets formed by stratified manner in [9], explicit formula is given.

The rest of this paper is organized as follows. Section 2 presents preliminaries on random sampling. Section 3 presents our main result, which provides explicit expected \( L_2 \)-discrepancy for a certain class of partitions. Section 4 includes the proofs of the main results. Finally, in section 5 we conclude the paper with a short summary.

2. Preliminaries on random sampling

Before introducing the main result, we list the preliminaries used in this paper.

2.1. Simple random sampling. In a sense, simple random sampling is Monte Carlo sampling. Uniform distributed point set is selected in \([0, 1]^d\), see Figure 1. In terms of the discrepancy under simple random sampling, see [1,8].

2.2. Jittered sampling. Jittered sampling is a type of grid-based equivolume partition. \([0, 1]^d\) is divided into \( m^d \) axis parallel boxes \( Q_i, 1 \leq i \leq N \), each with sides \( \frac{1}{m} \), see illustration of Figure 2. Research on the jittered sampling are extensive, see [2,6,9,10,12].
2.3. **Partition model in [9].** For a grid-based equivolume partition in two dimension, the two squares in the upper right corner are merged to form a rectangle

\[ I = [a_1, a_1 + 2b] \times [a_2, a_2 + b], \]

where \(a_1, a_2, b\) are three positive constants. The diagonal of \(I\) is the partition line, which constitutes a special partition mode. We set it

\[ \Omega_\lambda = (\Omega_{1\lambda}, \Omega_{2\lambda}, Q_3, \ldots, Q_N), \]

where \(\Omega_{2\lambda} = I \setminus \Omega_{1\lambda}\).

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**Figure 2.** Jittered sampling formed by isometric grid partition.

**Figure 3.** The designed partition model in [9].
2.4. Class of partition model. For the rectangle

\[ I = [a_1, a_1 + 2b] \times [a_2, a_2 + b], \]

where \( a_1, a_2, b \) are three positive constants. A straight line partition is used to divide the rectangle into two parts if we set the straight line parallel to the diagonal of \( I \), and the distance from the intersection point \( Q \) to the endpoint at the upper right corner of \( I \) is \( b \in (0, \frac{2}{m}) \), and we set the original of \( I \) is \( O' \), see Figure 4.

For convenience of notation, we set this partition model

\[ \Omega_{b,\sim} = (\Omega_{1, b,\sim}, \Omega_{2, b,\sim}, Q_3, \ldots, Q_N), \]

where \( \Omega_{2, b,\sim} = I \setminus \Omega_{1, b,\sim} \).

![Figure 4. Unequivolume partition](image)

Now, consider \( d \)-dimensional cuboid

\[ I_d = I \times \prod_{i=3}^{d} [a_i, a_i + b] \]

and its two partitions \( \Omega'_{\sim} = (\Omega'_{1, \sim}, \Omega'_{2, \sim}) \) and \( \Omega'_{b,\sim} = (\Omega'_{1, b,\sim}, \Omega'_{2, b,\sim}) \) into two closed, superconvex bodies with

\[ \Omega'_{1,\sim} = \Omega_{1,\sim} \times \prod_{i=3}^{d} [a_i, a_i + b], \]

and
\[ \Omega_{1,b,\sim} = \Omega_{1,b,\sim} \times \prod_{i=3}^{d} [a_i, a_i + b]. \]

We choose \( a_1 = \frac{m-2}{m}, a_2 = \frac{m-1}{m}, b = \frac{1}{m} \) in \( \Omega_{1,\sim} \) and \( \Omega_{1,b,\sim} \), denoted by \( \Omega_{1,\sim}^* \) and \( \Omega_{1,b,\sim}^* \), then we obtain

\[ (2.3) \quad \Omega_{\sim}^* = (\Omega_{1,\sim}^*, \Omega_{2,\sim}^*, Q_3 \ldots, Q_N), \]

and

\[ (2.4) \quad \Omega_{b,\sim}^* = (\Omega_{1,b,\sim}^*, \Omega_{2,b,\sim}^*, Q_3 \ldots, Q_N). \]

3. Explicit Expected \( L_2 \)-discrepancy for stratified random sampling formed by a class of partitions

In this section, explicit \( L_2 \)-discrepancy formula is given for a class of partition model 2.4.

**Theorem 3.1.** For partition \( \Omega_{b,\sim}^* \) of \([0,1]^d\) and \( m \geq 2, b \in \left[ \frac{3}{2m}, \frac{2}{m} \right) \), then

\[ (3.1) \quad \mathbb{E} L_2^2(D_N, P_{\Omega_{b,\sim}^*}) = \frac{1}{m^{2d}} \left[ \left( \frac{m-1}{2} + \frac{1}{2} \right)^d - \left( \frac{m-1}{2} + \frac{1}{3} \right)^d \right] - \frac{P_0(b)}{2^d \cdot m^{3d}} - \frac{P_1(b)}{3^d \cdot m^{3d}}, \]

where

\[ P_0(b) = \frac{8 - m^2 b^2}{3} - \frac{16}{24 - 3m^2 b^2}, \]

\[ P_1(b) = \frac{m^4 b^4}{40} + \frac{114m^2 b^2}{40} + \frac{19}{5} - \frac{6m^3 b^3 - 3m^5 b^5 + 352}{40 - 5m^2 b^2}. \]

**Remark 3.2.** Actually, from the model of 2.4, formula (3.1) presents explicit result for division line above diagonal, the same argument can be applied for the divide line below the diagonal, however, could not get better results than partition model 2.3. Noticing that the parameter \( b \in \left[ \frac{3}{2m}, \frac{2}{m} \right) \), within this range, for any \( d \), we obtain a class of better \( L_2 \)-discrepancy result than the use of partition model 2.3.

**Remark 3.3.** For \( b \in (0, \frac{3}{2m}) \), \( P_0(b) \) is a decreasing function, thus \( P_0(b) > 0 \), using (3.1) minus (1.7), we obtain

\[ -\frac{P_0(b)}{2^d \cdot m^{3d}} - \frac{P_1(b)}{3^d \cdot m^{3d}} + \frac{2}{5 \cdot 3^d \cdot m^{3d}}, \]

thus \( -\frac{P_0(b)}{2^d \cdot m^{3d}} < 0 \), we only consider \( \frac{2}{5} - P_1(b) \), for \( b \in \left[ \frac{3}{2m}, \frac{2}{m} \right) \), we can assure this quantity less than 0, which proves a smaller expected \( L_2 \)-discrepancy, while \( b \in (0, \frac{3}{2m}) \)
is an undetermined situation, which may need consider \(3.1\) as a whole, this depends on the dimensions \(d\).

**Corollary 3.4.** Let \(m, d \in \mathbb{N}\) with \(m \geq 2, b \in \left[\frac{3}{2m}, \frac{2}{m}\right]\). Let \(N = m^d\) and jitted sampling set \(P_{\Omega} = \{x_1, x_2, \ldots, x_N\}\). Stratified random \(d\)-dimension point sets \(P_{\Omega_{\sim}^b} = \{s_1, s_2, \ldots, s_N\}\) and \(P_{\Omega_{\sim}^*} = \{y_1, y_2, \ldots, y_N\}\) are uniformly distributed in the stratified subsets of \(\Omega_{\sim}^b\) and \(\Omega_{\sim}^*\) respectively, then

\[
(3.2) \quad \mathbb{E}L_2^{b^2}(D_N, P_{\Omega_{\sim}^b}) < \mathbb{E}L_2^{b^2}(D_N, P_{\Omega_{\sim}^*}) < \mathbb{E}L_2^{b^2}(D_N, P_{\Omega}).
\]

**Remark 3.5.** In \(3.1\), Let \(b \in \left[\frac{3}{2m}, \frac{2}{m}\right]\), for any fixed \(d\), using \(3.1\) minus \(1.7\), easy to show that the desired result.

## 4. Proofs

We prove Theorem 3.1 by the definition of \(L_2\)-discrepancy, we first give some lemmas.

**Lemma 4.1.** Let \(\Omega_{\sim}^b\) of \([0, 1]^d\) and \(m \geq 2, b \in \left[\frac{3}{2m}, \frac{2}{m}\right]\), stratified random point sets \(P_{\Omega_{\sim}^b} = \{s_1, s_2, \ldots, s_N\}\), assume \(I_{\sim} = \Omega_{\sim}^b \cup \Omega_{\sim}^b\), if we divide \(I_{\sim}\) into three area \(I, II, III\) as Figure 5, then for \(z \in I \cup II\) we have

\[
(4.1) \quad \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z)}(s_i) = \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z) \setminus [O',z)}(s_i) dz
\]

where \(\text{Var}\) denotes the variance.

**Lemma 4.2.** Let \(\Omega_{\sim}^b\) of \([0, 1]^d\) and \(m \geq 2, b \in \left[\frac{3}{2m}, \frac{2}{m}\right]\), stratified random point sets \(P_{\Omega_{\sim}^b} = \{s_1, s_2, \ldots, s_N\}\), assume \(I_{\sim} = \Omega_{\sim}^b \cup \Omega_{\sim}^b\), if we divide \(I_{\sim}\) into three area \(I, II, III\) as Figure 5, then for \(z \in III\) we have

\[
\int_{P_{\Omega_{\sim}^b}} \int_{I+II} |\lambda([0,z)) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z)}(s_i)|^2 dz dw - \int_{I+II} \text{Var}(\frac{1}{N} \sum_{i=1}^{N} 1_{[0,z) \setminus [O',z)}(s_i)) dz
\]

\[
= \left[\left(-\frac{b^4 m^4}{24} + \frac{2b^3 m^3}{3} - 2b^2 m^2 + 4\right) \cdot \frac{4}{8 - b^2 m^2}\right] \cdot \left(\frac{1}{2m^2}\right)^d
\]

\[
\left[\left(-\frac{b^4 m^4}{160} + \frac{3b^5 m^5}{20} - \frac{3b^4 m^4}{2} + 6b^3 m^3 - 6b^2 m^2 + 8\right) \cdot \left(\frac{1}{3m^3}\right)^d\right],
\]

where \(\text{Var}\) denotes the variance.
\begin{align*}
\int_{P_{\Omega_z}} \int_{\Omega_z^*} |\lambda([0, z]) - \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{[0, z]}(s_i)|^2 dz d\omega &- \int_{III} \text{Var}\left(\frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{[0, z]}(O', z)(s_i)\right) dz \\
= \left[\frac{24 b^2 m^2 - 8 b^3 m^3}{24 - 3 b^2 m^2}\right] \cdot \frac{4}{8 - b^2 m^2} + \frac{b^2 m^2}{6} \cdot \left(\frac{1}{2 m^3}\right)^d \\
+ \left(1 - \frac{8}{b^2 m^2}\right) \cdot \frac{9 b^6 m^6}{960} - \frac{1}{8 - b^2 m^2} \cdot \left(\frac{b^8 m^8}{80} - \frac{3 b^7 m^7}{40} + \frac{9 b^6 m^6}{8} - 6 b^5 m^5 + 9 b^4 m^4\right) \\
+ 2\left(1 - \frac{4}{b^2 m^2}\right) (1 - \frac{4}{8 - b^2 m^2}) \cdot \left(-\frac{9 b^6 m^6}{1152} - \frac{9 b^5 m^5}{240} + \frac{9 b^4 m^4}{48}\right) \cdot \left(\frac{1}{3 m^3}\right)^d,
\end{align*}

where \text{Var} denotes the variance.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{division.pdf}
\caption{Division of the integral region I}
\end{figure}

**Proof of Lemma 4.1** Let \(\Omega_z = [0, z) \setminus [O', z)\), then we have
\[
\int_{P_{\Omega_z}} \int_{I+II} |\lambda([0, z]) - \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{[0, z]}(s_i)|^2 dz d\omega
\]
\[
= \int_{P_{\Omega_z}} \int_{I+II} \lambda(\Omega_z) + \lambda([O', z]) - \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\Omega_z}(s_i) - \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{[O', z]}(s_i)|^2 dz d\omega.
\]
Therefore, we compute
\begin{align*}
\int_{P_{\Omega_z}} \int_{I+II} |\lambda(\Omega_z) + \lambda([O', z]) - \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\Omega_z}(s_i) - \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{[O', z]}(s_i)|^2 dz d\omega \\
&= \int_{I+II} \int_{P_{\Omega_z}} |\lambda(\Omega_z) + \lambda([O', z]) - \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\Omega_z}(s_i) - \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{[O', z]}(s_i)|^2 d\omega dz.
\end{align*}
Besides,

\begin{equation}
\mathbb{E}\left(\frac{1}{N} \sum_{i=1}^{N} 1_{\Omega_z}(s_i)\right) = \lambda(\Omega_z),
\end{equation}

and

\begin{equation}
\mathbb{E}\left(\frac{1}{N} \sum_{i=1}^{N} 1_{[O',z]}(s_i)\right) = \frac{4m^{d-2}}{8m^{d-2} - Nb^2} \cdot \lambda([O', z]).
\end{equation}

If we set

\[ P(s_i \in V_i) = \frac{\lambda(V_i)}{\lambda(\Omega_{i,b,\sim})} \]

for \( i = 1, 2 \), and

\[ P(s_i \in U_i) = \frac{\lambda(U_i)}{\lambda(Q_i)} \]

for \( i = 3, 4, \ldots, N \).

Then, (4.3) can be converted to

\begin{equation}
\int_{I+II} \int_{\Omega_{i,b,\sim}} \left[ \mathbb{E}\left(\frac{1}{N} \sum_{i=1}^{N} 1_{\Omega_z}(s_i)\right) + \mathbb{E}\left(\frac{1}{N} \sum_{i=1}^{N} 1_{[O',z]}(s_i)\right) + \left(1 - \frac{4m^{d-2}}{8m^{d-2} - Nb^2}\right) \cdot \lambda([O', z]) \right]
\end{equation}

\[ - \frac{1}{N} \sum_{i=1}^{N} 1_{\Omega_z}(s_i) - \frac{1}{N} \sum_{i=1}^{N} 1_{[O',z]}(s_i) \right|^2 d \omega dz
\]

\[ = \int_{I+II} \text{Var}\left(\frac{1}{N} \sum_{i=1}^{N} 1_{[0, z) \setminus [O', z]}(s_i)\right) dz + \int_{I+II} \text{Var}\left(\frac{1}{N} \sum_{i=1}^{N} 1_{[O',z]}(s_i)\right) dz
\]

\[ + |(1 - \frac{4m^{d-2}}{8m^{d-2} - Nb^2}) \cdot \lambda([O', z])|^2 dz.
\]

\[ \text{Var}\left(\frac{1}{N} \sum_{i=1}^{N} 1_{[O',z]}(s_i)\right) = \frac{1}{N^2} \cdot \frac{\lambda([O', z])}{\lambda(I + II)} \cdot (1 - \frac{\lambda([O', z])}{\lambda(I + II)}).
\]

Thus,

\[ \lambda(I + II) = \frac{2}{N} \cdot \frac{b^2}{4} \cdot \frac{1}{m^{d-2}}. \]
\[ \text{Var}(\frac{1}{N} \sum_{i=1}^{N} 1_{(r,z)}(s_i)) + |(1 - \frac{4m^{d-2}}{8m^{d-2} - Nb^2}) \cdot \lambda([O', z])|^2 \]
\[ = \frac{1}{N} \cdot \frac{4}{8 - m^2b^2} \cdot \lambda([O', z]) + (1 - \frac{8}{8 - m^2b^2}) \cdot \lambda^2([O', z]). \]

The equation of the dividing line is the following

\[ Z_2 = -\frac{1}{2}Z_1 + \frac{3}{2} - \frac{b}{2}. \]

Thus,

\[ \int_I \lambda([O', z]) = \int_{1/2}^{1-b} \int_{1/3}^{1} (Z_1 - 1 + \frac{2}{m})(Z_2 - 1 + \frac{1}{m})dZ_1dZ_2. \]

\[ = \frac{b^2m^2 - 4bm + 4}{2m^2} \cdot (\frac{1}{2m^2})^d. \]

\[ \int_I \lambda^2([O', z]) = \int_{1/2}^{1-b} \int_{1/3}^{1} (Z_1 - 1 + \frac{2}{m})^2(Z_2 - 1 + \frac{1}{m})^2dZ_1dZ_2. \]

\[ = (-bm^3 + 6b^2m^2 - 12bm + 8) \cdot (\frac{1}{3m^3})^d. \]
\[ \int_{II} \lambda^2([O', z]) = \int_{1-b}^1 \int_{1-\frac{1}{m}}^{\frac{Z_1}{2} + \frac{3}{2} - \frac{b}{2}} (Z_1 - 1 + \frac{2}{m})^2 (Z_2 - 1 + \frac{1}{m})^2 dZ_1 dZ_2. \]

\[ \int_{1-\frac{1}{m}}^1 \int_{1-\frac{1}{m}}^1 \ldots \int_{1-\frac{1}{m}}^1 \prod_{i=3}^d (Z_i - 1 + \frac{1}{m}) \]

\[ = (-\frac{b^6 m^6}{160} + \frac{3b^5 m^5}{20} - \frac{3b^4 m^4}{2} + 7b^3 m^3 - 15b^2 m^2 + 12 bm) \cdot \left(\frac{1}{3m^3}\right)^d. \]

Combining with (4.6) and (4.7), we obtain the desired result.

**Proof of Lemma 4.2** For area \( z \in \text{III} \), we have,

\[ \lambda(\Omega_{2,1}) = \frac{1}{4} (Z_1 + 2Z_2 + b - 3)^2 \cdot \prod_{i=3}^d (Z_i - 1 + \frac{1}{m}), \]

and

\[ \lambda(\Omega_{2,2}) = (Z_1 - 1 + \frac{2}{m}) (Z_2 - 1 + \frac{1}{m}) \cdot \prod_{i=3}^d (Z_i - 1 + \frac{1}{m}) - \lambda(\Omega_{2,1}). \]

**Figure 6.** Division of the integral region II
\[
\int_{P_{b_{10}}} \int_{III} |\lambda(\Omega_z) + \lambda([O', z]) - \frac{1}{N} \sum_{i=1}^{N} 1\Omega_z(s_i) - \frac{1}{N} \sum_{i=1}^{N} 1_{[O', z]}(s_i)|^2 dz d\omega = \int_{III} \int_{P_{b_{10}}} |\lambda(\Omega_z) + \lambda([O', z]) - \frac{1}{N} \sum_{i=1}^{N} 1\Omega_z(s_i) - \frac{1}{N} \sum_{i=1}^{N} 1_{[O', z]}(s_i)|^2 d\omega dz \\
(4.8)
\int_{III} \text{Var}(\frac{1}{N} \sum_{i=1}^{N} 1_{[O', z]}(s_i)) dz + \int_{III} \text{Var}(\frac{1}{N} \sum_{i=1}^{N} 1_{[O', z]}(s_i)) \\
+ |(1 - \frac{4m^{d-2}}{Nb^2}) \cdot \lambda(\Omega_{2,1}) + (1 - \frac{4m^{d-2}}{8m^{d-2} - Nb^2}) \cdot \lambda(\Omega_{2,2})|^2 dz.
\]

Besides, \( (4.9) \)
\[
\text{Var}(\frac{1}{N} \sum_{i=1}^{N} 1_{[O', z]}(s_i)) + |(1 - \frac{4m^{d-2}}{Nb^2}) \cdot \lambda(\Omega_{2,1}) + (1 - \frac{4m^{d-2}}{8m^{d-2} - Nb^2}) \cdot \lambda(\Omega_{2,2})|^2 \\
= \text{Var}(\frac{1}{N} \sum_{i=1}^{N} 1_{[O', z]}(s_i)) + |(1 - \frac{4m^{d-2}}{Nb^2}) \cdot \lambda(\Omega_{2,1})|^2 + |(1 - \frac{4m^{d-2}}{8m^{d-2} - Nb^2}) \cdot \lambda(\Omega_{2,2})|^2 \\
+ 2((1 - \frac{4m^{d-2}}{Nb^2}) \cdot \lambda(\Omega_{2,1}))(1 - \frac{4m^{d-2}}{8m^{d-2} - Nb^2}) \cdot \lambda(\Omega_{2,2})) = \frac{4}{Nm^2b^2}\lambda(\Omega_{2,1}) + \frac{4}{8N - Nm^2b^2}\lambda(\Omega_{2,2}) + (1 - \frac{8}{m^2b^2})\lambda^2(\Omega_{2,1}) + (1 - \frac{8}{8 - m^2b^2})\lambda^2(\Omega_{2,2}) \\
+ 2((1 - \frac{4m^{d-2}}{Nb^2}) \cdot \lambda(\Omega_{2,1}))(1 - \frac{4m^{d-2}}{8m^{d-2} - Nb^2}) \cdot \lambda(\Omega_{2,2})).
\]

Furthermore, we have,
\[
\int_{III} \lambda(\Omega_{2,2})dZ_1dZ_2 \ldots Z_d = \frac{-2b^3m^3 + 6b^2m^2}{3}(\frac{1}{2m^2})^d, \\
\int_{III} \lambda^2(\Omega_{2,2})dZ_1dZ_2 \ldots Z_d = \frac{(b^6m^6 - 9b^5m^5 + 9b^4m^4 - 18b^3m^3 + 9b^2m^2)(\frac{1}{3m^3})^d}{80} \\
- \frac{9b^5m^5}{120} + \frac{9b^4m^4}{8} - \frac{18b^3m^3}{3} + 9b^2m^2)(\frac{1}{3m^3})^d, \\
\int_{III} \lambda(\Omega_{2,1})dZ_1dZ_2 \ldots Z_d = \frac{b^4m^4}{24}(\frac{1}{2m^2})^d, \\
\int_{III} \lambda^2(\Omega_{2,1})dZ_1dZ_2 \ldots Z_d = \frac{9b^6m^6}{960}(\frac{1}{3m^3})^d.
\]
\[
\int_{III} \lambda(\Omega_2,1) \lambda(\Omega_2,2) dZ_1 dZ_2 \ldots dZ_d \\
= \left( -\frac{9b^6 m^6}{1152} - \frac{9b^5 m^5}{240} + \frac{9b^4 m^4}{48} \right) (\frac{1}{3m^3})^d.
\]

Combining with (4.8) and (4.9), we obtain the desired result.

**Proof of Theorem 3.1** From the definition of \( L_2 \)-discrepancy.

For point set \( P_{\Omega^*_{b,\sim}} = \{s_1, s_2, \ldots, s_N\} \), we have

\[
(4.10) \quad E L^2_2(D_N, P_{\Omega^*_{b,\sim}}) = \int_{P_{\Omega^*_{b,\sim}}} \int_{[0,1]^d} |\lambda([0, z)) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z)}(s_i)|^2 dz d\omega.
\]

Therefore, from (4.10), we get

\[
(4.11) \quad E L^2_2(D_N, P_{\Omega^*_{b,\sim}}) = \int_{P_{\Omega^*_{b,\sim}}} \int_{[0,1]^d} |\lambda([0, z)) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z)}(s_i)|^2 dz d\omega \\
\quad + \int_{P_{\Omega^*_{b,\sim}}} \int_{I^\sim} |\lambda([0, z)) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z)}(s_i)|^2 dz d\omega.
\]

Now, we first only care about \( I^\sim \), then we have

\[
(4.12) \quad \int_{P_{\Omega^*_{b,\sim}}} \int_{I^\sim} |\lambda([0, z)) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z)}(s_i)|^2 dz d\omega \\
\quad = \int_{P_{\Omega^*_{b,\sim}}} \int_{I^\sim} |\lambda([0, z) \setminus [O', z)) + \lambda([O', z)) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z) \setminus [O', z)}(s_i) - \frac{1}{N} \sum_{i=1}^{N} 1_{[O', z)}(s_i)|^2 dz d\omega.
\]

From Lemma 4.1 and 4.2, we obtain
\[(4.13)\]
\[
E L^2_2(D_N, P_{\Omega_{b,\sim}}) = \int_{P_{\Omega_{b,\sim}}} \int_{[0,1]^d} |\lambda([0, z]) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(s_i)|^2 dz d\omega
\]
\[
= \int_{P_{\Omega_{b,\sim}}} \int_{[0,1]^d \setminus I_{\sim}} |\lambda([0, z]) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(s_i)|^2 dz d\omega
\]
\[
+ \int_{P_{\Omega_{b,\sim}}} \int_{I_{\sim}} |\lambda([0, z]) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(s_i)|^2 dz d\omega
\]
\[
= \int_{P_{\Omega_{b,\sim}}} \int_{[0,1]^d \setminus I_{\sim}} |\lambda([0, z]) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(s_i)|^2 dz d\omega
\]
\[
+ \int_{I_{\sim}} \text{Var}(\frac{1}{N} \sum_{i=1}^{N} 1_{[0,z] \setminus [O',z]}(s_i)) dz + \frac{P_2(b)}{3} \cdot \left(\frac{1}{3m^3}\right)^d
\]
\[
= \int_{P_{\Omega_{b,\sim}}} \int_{[0,1]^d \setminus I_{\sim}} |\lambda([0, z]) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(s_i)|^2 dz d\omega
\]
\[
+ \int_{I_{\sim}} \text{Var}(\frac{1}{N} \sum_{i=1}^{N} 1_{[0,z] \setminus [O',z]}(s_i)) dz + \frac{P_2(b)}{3} \cdot \left(\frac{1}{3m^3}\right)^d
\]

Simplifying (4.13) and let
\[
P_2(b) = \frac{m^2b^2}{3} + \frac{16}{24 - 3m^2b^2} + \frac{4}{3},
\]
\[
P_3(b) = -\frac{m^4b^4}{40} - \frac{114m^2b^2}{40} - \frac{352}{40} + \frac{6m^6b^3 - 3m^5b^5 + 352}{40 - 5m^2b^2}.
\]

We obtain
\[(4.14)\]
\[
E L^2_2(D_N, P_{\Omega_{b,\sim}}) = \int_{P_{\Omega_{b,\sim}}} \int_{[0,1]^d \setminus I_{\sim}} |\lambda([0, z]) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(s_i)|^2 dz d\omega
\]
\[
+ \int_{I_{\sim}} \text{Var}(\frac{1}{N} \sum_{i=1}^{N} 1_{[0,z] \setminus [O',z]}(s_i)) dz + \frac{P_2(b)}{3} \cdot \left(\frac{1}{2m^3}\right)^d + \frac{P_3(b)}{3} \cdot \left(\frac{1}{3m^3}\right)^d.
\]

For jittered grid area $[0,1]^d \setminus I_{\sim}$ and $[0,z] \setminus [O',z)$, we obtain
\[ (4.15) \]
\[
\int_{P_{\Omega}^*} \int_{[0,1]^d \setminus I_{\sim}} |\lambda([0, z])) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(x_i) |^2 dz \, d\omega + \int_{I_{\sim}} \text{Var}\left( \frac{1}{N} \sum_{i=1}^{N} 1_{[0, z)\setminus(O', z)}(x_i) \right) dz
\]
\[= \int_{P_{\Omega}} \int_{[0,1]^d \setminus I_{\sim}} |\lambda([0, z])) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(x_i) |^2 dz \, d\eta + \int_{I_{\sim}} \text{Var}\left( \frac{1}{N} \sum_{i=1}^{N} 1_{[0, z)\setminus(O', z)}(x_i) \right) dz.\]

For jittered sampling point set \( P_{\Omega} = \{x_1, x_2, \ldots, x_N\} \),

\[ (4.16) \]
\[
E L^2_2(D_N, P_{\Omega}) = \int_{P_{\Omega}} \int_{[0,1]^d} |\lambda([0, z])) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(x_i) |^2 dz \, d\eta
\]
\[
= \int_{P_{\Omega}} \int_{[0,1]^d \setminus I_{\sim}} |\lambda([0, z])) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(x_i) |^2 dz \, d\eta
\]
\[+ \int_{P_{\Omega}} \int_{I_{\sim}} |\lambda([0, z])) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(x_i) |^2 dz \, d\eta.\]

Besides,

\[ (4.17) \]
\[
\int_{P_{\Omega}} \int_{I_{\sim}} |\lambda([0, z])) - \frac{1}{N} \sum_{i=1}^{N} 1_{[0,z]}(x_i) |^2 dz \, d\eta
\]
\[= \int_{I_{\sim}} \text{Var}\left( \frac{1}{N} \sum_{i=1}^{N} 1_{[0, z)\setminus(O', z)}(x_i) \right) dz + \int_{I_{\sim}} \text{Var}\left( \frac{1}{N} \sum_{i=1}^{N} 1_{[O', z)}(x_i) \right) dz.\]

Easy to obtain

\[ (4.18) \]
\[
\int_{I_{\sim}} \text{Var}\left( \frac{1}{N} \sum_{i=1}^{N} 1_{[O', z)}(x_i) \right) dz = 4 \cdot \frac{1}{2^d} \cdot \frac{1}{N^3} - 5 \cdot \frac{1}{3^d} \cdot \frac{1}{N^3}.\]

Therefore, from (4.14), (4.15), (4.16), (4.17) and (4.18), we have the desired result.

5. Conclusion

We study expected \( L_2 \)–discrepancy under a class of unequivoolume partitions, we give an explicit formula, this result improves expected \( L_2 \)–discrepancy under partition manner [9]. In future, we will study the optimal partition under this class and give corresponding explicit \( L_2 \)–discrepancy formula.
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