Synchronization in the BCS Pairing Dynamics as a Critical Phenomenon

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Fermi gas with time-dependent pairing interaction hosts several different dynamical states. Coupling between the collective BCS pairing mode and individual Cooper pair states can make the latter either synchronize or dephase. We describe transition from phase-locked undamped oscillations to Landau-damped dephased oscillations in the collisionless, dissipationless regime as a function of coupling strength. In the dephased regime, we find a second transition at which the long-time asymptotic pairing amplitude vanishes. Using a combination of numerical and analytical methods we establish a continuous (type II) character of both transitions.

Recent discovery of BCS pairing in fermionic vapors \([1, 2]\), made possible by control of interactions in trapped cold gases \([3]\), has renewed interest in quantum collective phenomena \([4]\). Advanced detection techniques and long coherence times in vapors enable time-resolved studies of new collective modes, such as spin waves \([5]\) and the BCS pairing mode \([6]\).

Interaction between a collective mode and constituting particles is key for our understanding of dynamics in various systems, from plasma to quantum gases. One of the most surprising of these phenomena is Landau damping which occurs in a collisionless regime via direct dissipationless energy transfer from the collective mode to single particles. Its nondissipative and thus reversible character \([7]\) leads to a variety of regimes, notably to quenching of the damping, first explored in plasma physics \([8]\). Remarkably, a linearly damped mode can regrow and transform to a stationary oscillatory Bernstein-Greene-Kruskal mode. This fascinating prediction was confirmed experimentally only recently \([9]\).

Naturally, the richness of these nonlinear phenomena makes it tempting to look for their analogs in cold gases. Collisionless damping in cold gases was considered, in the linear regime, for optical excitations \([10]\), spin waves \([11, 12]\), and excitations in optical lattices \([13]\). Motivated by the work on fermion superfluidity \([1, 2, 6]\), here we focus on the pairing dynamics of fermions \([14, 15, 16, 17]\) induced by a sudden change of interaction. The collisionless regime becomes practical in this case due to long elastic collision times \(\tau_\text{el} \gg \hbar/\Delta\) \([14]\), where \(\Delta\) is the BCS gap. The pairing mode of a small amplitude oscillates at a frequency \(2\Delta/\hbar\) and exhibits collisionless dephasing \([15]\). These conclusions were extended recently to the nonlinear regime \([19]\).

This behavior changes drastically as the perturbation increases. The main result of this work, as summarized in Fig. 1, is prediction of a dynamical transition resulting from competition between synchronization and collisionless dephasing, taking place as a function of the initial pairing gap, \(\Delta_s\). We found three qualitatively different regimes (\(A, B,\) and \(C\)) with the critical points at \(\Delta_{AB} = e^{-\pi/2}\Delta_0\) and \(\Delta_{BC} = e^{\pi/2}\Delta_0\), where \(\Delta_0\) is the equilibrium pairing amplitude in the final BCS state. Below the \(A-B\) transition, \(\Delta_s < \Delta_{AB}\), individual Cooper pair states synchronize and the pairing amplitude oscillates between \(\Delta_s\) and \(\Delta_c\). In the dephased regime (\(B, C\)), the pairing amplitude saturates to a constant value, \(\Delta_d\), when \(\Delta_{AB} \leq \Delta_s < \Delta_{BC}\), and decreases to zero at \(\Delta_s \geq \Delta_{BC}\). Dashed line: The stationary gap value \(\Delta(T_s)\) reached in a closed system after equilibration.

![Synchronization](attachment:image.png)

**FIG. 1:** Three regimes of the pairing dynamics vs. the initial gap value \(\Delta_s\). In synchronized phase \((A)\), \(\Delta_s < \Delta_{AB}\), the pairing amplitude oscillates between \(\Delta_s\) and \(\Delta_c\). In the dephased regime \((B, C)\), the pairing amplitude saturates to a constant value, \(\Delta_d\), when \(\Delta_{AB} \leq \Delta_s < \Delta_{BC}\), and decreases to zero at \(\Delta_s \geq \Delta_{BC}\). Dashed line: The stationary gap value \(\Delta(T_s)\) reached in a closed system after equilibration.
Symmetry of the model. The interaction constants phase of $\Delta$ is a constant of motion due to the particle-hole regime similar to the evolution from the normal state [14].

In our analysis of the BCS problem we employ the well known pseudospin formulation [21] in which spin 1/2 operators $s^\pm_p = s^+_p \pm is^0_p$ describe Cooper pairs $(p, -p)$. The BCS Hamiltonian takes the form

$$\mathcal{H} = -\sum_p 2\epsilon_p s^+_p - \lambda(t) \sum_{p,q} s^-_p s^+_q,$$

where $\epsilon_p = p^2/2m - \mu$ is the free particle spectrum with $\mu$ the Fermi energy. Here we consider the time evolution induced by an instantaneous change of interaction from $\lambda_s$ at $t < 0$ to $\lambda$ at $t > 0$. In the spin formulation, Eq. (1), the dynamics is of a Bloch form

$$\frac{dr_p}{dt} = 2b_p \times r_p, \quad b_p = -(\Delta_x, \Delta_y, \epsilon_p),$$

where $r_p = 2(s^+_p)$ are classical vectors, and the effective magnetic field $b_p$ depends on the pairing amplitude $\Delta$. The latter is defined self-consistently:

$$\Delta = \Delta_x + i\Delta_y = \frac{\lambda(t)}{2} \sum_p r^+_p, \quad r^+_p = r^x_p + ir^y_p.$$

We first present numerical results for the dynamics [2, 3]. The Runge-Kutta method of the 4th order was used with $N = 10^4, 10^5$ equally spaced discrete energy states within a band $W = 50\Delta_0$ with a constant density of states $\nu(E_F)$. As an initial state we take

$$r^+_p(0) = \frac{\Delta_0}{\sqrt{\Delta_0^2 + \epsilon_p^2}}, \quad r^-_p(0) = \frac{\epsilon_p}{\sqrt{\Delta_0^2 + \epsilon_p^2}}.$$ (4)

which describes the $T = 0$ paired ground state [21]. Without loss of generality we set $\Delta(t) = \Delta_s$, since the phase of $\Delta$ is a constant of motion due to the particle-hole symmetry of the model. The interaction constants $\lambda_s$ and $\lambda$ define, via the self-consistency relation [3], the initial and final equilibrium BCS gap values, $\Delta_s = W e^{-1/g_\nu}, \quad g_\nu = \lambda_s \nu \ll 1, \quad \Delta_0 = W e^{-1/g}, \quad g = \lambda \nu$, which we use to parameterize the system.

We observe three qualitatively different dynamical regimes. The initial states with a relatively small gap give rise to undamped oscillations (Fig. 3a). In this case $\Delta(t)$ oscillates non-harmonically between $\Delta_-$ and $\Delta_+$ (the regime $A$ in Fig. 1). Synchronization of different Cooper pair states results from their interaction with the mode singled out by BCS instability of the initial state, similar to the evolution from the normal state [14].

Desynchronization takes place at $\Delta_s \geq \Delta_{AB} = 0.21\Delta_0$ giving rise to two different regimes exhibiting dephasing, underdamped and overdamped ($B$ and $C$, Fig. 1). The former, illustrated in Fig. 3c, is simplest to understand for a small initial deviation, $\Delta_s \approx \Delta_0$ [18], by linearizing Bloch equations about the equilibrium state.

The analysis predicts damped oscillations at long times:

$$\Delta(t) = \Delta_s + A(t) \sin(2\Delta_s t + \alpha), \quad A(t) \propto t^{-1/2}.$$ (5)

The power-law decay of $A(t)$ was explained in Ref. [18] by interaction of the collective mode with the continuous spectrum of excitations with energies above $2\Delta_s$ and linked to the linear Landau damping. In the spin formulation, the dephasing results from the Larmor frequency of spin precession $b_p$ being a continuous function of $\epsilon_p$. An extension of this argument to the nonlinear regime was proposed recently in Ref. [10] which, however, did not clarify the range of its validity. The dephased time evolution similar to [18] was also reported in Refs. [16, 17, 22].

In the overdamped regime, $\Delta_s \geq \Delta_{BC} = 4.81\Delta_0$ (Fig. 3b), $\Delta(t)$ decays to zero without oscillations. This behavior can be understood in the limit $\Delta_s/\Delta_0 \gg 1$, i.e., when the coupling in the initial paired state in $\Delta_s$ is suddenly completely turned off. For different spins precessing freely and independently one obtains

$$r^+_p(t) = e^{-i2\epsilon_p t} r^+_p(0), \quad \Delta(t) \approx \Delta_s^{-1} \propto (\Delta_s t)^{-1/2} e^{-2\Delta_s t}.$$ (6)

The fast dephasing can also be understood by noting that the energy distribution in Fig. 1 corresponds to an effective temperature $T \sim \Delta_s$, which exceeds $T_c$ for $\Delta_0$ (see below).

To fully exhibit phase locking in the synchronized regime which abruptly disappears in the dephased regime, we now explore the phase dynamics. It is convenient to measure precession angles relative to time-independent $b_p = (\Delta_a, 0, \epsilon_p)$, where $\Delta_a$ is the asymptote $\Delta(t \rightarrow \infty)$ in the regimes $B$ and $C$, and the average
value of oscillating $\Delta(t)$ in A (dash-dotted line in Fig.4). The angle and frequency of precession are defined by
\begin{equation}
n_p^+ = n_p^x + in_p^y \propto e^{-i\phi_p(t)}, \quad \omega_p(t) = d\phi_p/dt,
\end{equation}
where the vectors $n_p$ are obtained from $r_p$ by a rotation about the $y$ axis which maps $\hat{z}$ onto $\hat{r}_p$: $n_p^y = r_p^y$, $n_p^x + in_p^y = e^{i\phi_p(r_p^x + ir_p^y)}$, with the rotation angle defined by $\tan \theta_p = \Delta_0/|\epsilon_p|$. The phase evolution, which becomes linear at long times $\tau \gg \Delta_0^{-1}$ (Figs.3d), can be characterized by average frequency (phase slope) $\omega_\phi = (d\phi_p/dt) = (\phi_p(\tau) - \phi_p(0))/\tau$. While in the regime A different $\mathbf{p}$ states phase lock (Fig.3c), in the regimes B, C the frequencies $\omega_\phi$ have dispersion (Fig.3d). The latter reproduces quasiparticle spectrum, $\omega_p = 2(\epsilon_p^2 + \Delta_0^2)^{1/2}$ with the long-time asymptote $\Delta_0$ which vanishes in the overdamped regime C.

We observe a qualitative change in behavior, with $\omega_p$ dispersing in the regions B, C and phase locking in the region A. However, the oscillation amplitude $\Delta(\Delta_+ - \Delta_-)$ in A and the asymptotic value $\Delta_0$ in B vanish continuously at the critical points $\Delta = \Delta_{AB}, \Delta_{BC}$ (Fig.4), indicating a type II transition.

Until now, we considered the dissipationless dynamics at times shorter than the quasiparticle relaxation time, $t \lesssim \tau_\phi$. Using the energy balance argument, one can determine the system state at long times, $t \gg \tau_\phi$. To account for system equilibration, one needs to consider the full many-body Hamiltonian which enables elastic scattering of individual quasiparticles, omitted in Eq.1. For a closed system, such as an atomic trap, the final temperature $T_f$ and the gap $\Delta_\phi$ can be determined from the energy conservation condition $E_{\geq \tau_\phi}(T_f) = E_0$, where $E_0$ is the energy immediately after interaction switching,
\begin{equation}
E_0 = \sum_p (\epsilon_p - \bar{\epsilon}_p) + \frac{\Delta_0^2}{\lambda_s} \left( 2 - \frac{\lambda}{\lambda_s} \right),
\end{equation}
with the spectrum $\bar{\epsilon}_p = (\epsilon_p^2 + \Delta_0^2)^{1/2}$, and $E_{\geq \tau_\phi}(T_f)$ is the energy of the final state:
\begin{equation}
E_{\geq \tau_\phi}(T_f) = \sum_p [\epsilon_p - (1 - 2n_p)\bar{\epsilon}_p(\Delta_\phi)] + \frac{\Delta_0^2}{\lambda}.
\end{equation}
Here $n_p = 1/(1 + e^{\epsilon_p(\Delta_-)/T_f})$ describes equilibrium with $T = T_f$, $\bar{\epsilon}_p(\Delta_\phi) = (\epsilon_p^2 + \Delta_\phi^2)^{1/2}$. After integrating over $\epsilon_p$, we arrive at the equation for $T_f$:
\begin{equation}
F \left( \frac{\Delta_\phi}{2T_f} \right) = 1 - \left( \frac{\Delta_\phi}{\Delta} \right)^2 + \alpha \left( \frac{\Delta_\phi}{\Delta} \right)^2 \ln \left( \frac{\Delta_\phi}{\Delta_0} \right)^2,
\end{equation}
where $F(u) = 2\int_0^u dx \cosh 2x [1 - \tanh (u \cosh x)]$, and $\alpha = 1 - g \ln(D_\phi/D_0)$. From Eq.4 we obtain $T_f$ and the equilibrium gap $\Delta_\phi = \Delta(T_f)$. They are approximately constant, with $T_f \approx 0.72T_c$, $\Delta(T_f) \approx 0.81\Delta_0$ in the regime A, are described by a non-monotonic function in B, and vanish in C. The system turns normal in the final state for $\Delta_\phi \geq f(g)\Delta_0$, $f(g) = 2.2 + 0.86g + O(g^2)$. Numerical solution of Eq.4 at $g = 0.26$ is displayed in Fig.4 (dashed line).

It is instructive to compare our results to the spectral analysis based on the integrability of the BCS Hamiltonian [20, 23, 24]. There is an infinite number of commuting integrals of motion, $H_p = H_{p'p}^p$, parameterized by $p$, where following Ref.21 we employ the Lax vector,
\begin{equation}
L_p = \hat{z} + \lambda \sum_{p' \neq p} \frac{S_{pp'}}{\epsilon_p - \epsilon_{p'}}.
\end{equation}
We will need \( L_y^2 = 4(L_p s_p)^2 \) which is also conserved (due to the Pauli matrices algebra combined with Eq. (11)). The mean-field expressions are obtained by substituting the averages \( \langle s_p \rangle \) instead of \( s_p \).

The spectral polynomial defining the evolution of individual states is proportional to the square of the Lax vector \( L \), \( Q(\epsilon) = L_y^2(\epsilon) \prod_p (\epsilon - \epsilon_p)^2 \), where \( L(\epsilon_p) = L_p \). The pairs of complex roots of the spectral equation \( Q(\epsilon) = 0 \) uniquely determine the long-time dynamics of the system \( [20] \). Evaluation of \( L_y^2(y) \) for the initial state \( [11] \) is straightforward:

\[
L_y^2(y)/g^2 = \left( \frac{\Delta_y}{\Delta_0} + y G(y) \right)^2 + G^2(y),
\]

where \( y = \epsilon/\Delta_s \) and

\[
G(y) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{y - \sinh x} = \frac{1}{2} \frac{\ln y + \sqrt{1 + y^2}}{y - \sqrt{1 + y^2}},
\]

where \( y \) is complex. One can show that the roots of \( L_y^2(y) \) lie on the imaginary axis. Upon variation of \( \Delta_s/\Delta_0 \) they disappear at \( y = 0 \). Expanding about \( y = 0 \), we obtain

\[
L_y^2(y)/g^2 = \left( \frac{\Delta_y}{\Delta_0} - \frac{\pi}{2} \right) \left( \frac{\Delta_y}{\Delta_0} + \frac{\pi}{2} \right) + O(y),
\]

Thus \( L_y^2(y) \) has no complex roots at \( \Delta_s/\Delta_0 \geq e^{\pi/2} \), one pair of roots \( y = \pm iu \) when \( e^{-\pi/2} \leq \Delta_s/\Delta_0 \leq e^{\pi/2} \), with another pair appearing at \( \Delta_s/\Delta_0 \leq e^{-\pi/2} \).

There is a direct correspondence between this behavior of the roots and the dynamical regimes \( A, B, \) and \( C \) observed numerically. The pairing amplitude is subject to fast dephasing and tends to zero when \( L_y^2(y) \) does not have complex roots. A pair of complex roots \( y = \pm iu \Delta_s/\Delta_0 \) defines the long-time asymptote \( \Delta(t) \approx \Delta_y \). Two pairs of roots, \( y = \pm iu_1 \) and \( y = \pm iu_2 \), correspond to the parameters \( \Delta_{\pm} = (u_1 \pm u_2) \Delta_s \) of the elliptic function which defines the asymptotic behavior:

\[
\Delta(t) = \Delta_{\pm} \text{dn} \left[ \Delta_{\pm} (t - \tau_0), k \right], \quad k = 1 - \Delta_{\pm}^2/\Delta_{\pm}^2,
\]

where the time lag \( \tau_0 \) is a half of the period. As illustrated in Fig. \[11\], Eq. (12) agrees well with \( \Delta(t) \) found numerically. Thus the spectral analysis is in accord with the simulation of Bloch dynamics. It confirms the existence of the three regimes and also provides the exact values \( \Delta_{AB} = e^{-\pi/2} \Delta_0 \) and \( \Delta_{BC} = e^{\pi/2} \Delta_0 \).

Finally, we estimate the change of scattering length required to cross the \( A-B \) and \( B-C \) transitions. Using the BCS gap in a weakly interacting Fermi gas \([22]\), \( \Delta = 0.49 E_F e^{-1/g}, \quad g = \frac{2}{\pi} k_F |a| \), we see that the conditions \( \Delta_s/\Delta_0 = e^{\pm \pi/2} \), written as \( 1/g - 1/g_s = \pm \pi/2 \), translate into \( 1/a - 1/a_s = \pm k_F \). At weak coupling this corresponds to a small change of scattering length, \( \delta a/a \approx \pm k_F a \), easily achievable for magnetically tunable Feshbach resonance.

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