On the Design of Channel Estimators for given Signal Estimators and Detectors
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Abstract—The fundamental task of a digital receiver is to decide the transmitted symbols in the best possible way, i.e., with respect to an appropriately defined performance metric. Examples of such performance metrics are the probability of error and the Mean Square Error (MSE) of a symbol estimator. In a coherent receiver, the symbol decisions are made based on the use of a channel estimate. This paper focuses on examining the optimality of such estimators such as the minimum variance unbiased (MVU) and the minimum mean square error (MMSE) estimators for these metrics and on proposing better estimators whenever it is necessary. For illustration purposes, this study is performed on a toy channel model, namely a single input single output (SISO) flat fading channel with additive white Gaussian noise (AWGN). In this way, this paper highlights the design dependencies of channel estimators on target performance metrics.

Index Terms—Minimum mean square error (MMSE), minimum variance unbiased (MVU), probability of error, single input single output (SISO).

I. INTRODUCTION

SIGNAL estimation and detection are two main concerns in the course of designing a communication system [5], [13], [22]. The main goal is to design optimal demodulators at the receiver side providing the detector with the necessary sufficient statistics for its decision on the transmitted symbol at a specific observation interval. Furthermore, the optimization of the decision device is also a target, i.e., its design based on such statistical tests which rely on sufficient statistics and minimize the probability of error. A different setup of optimal designs related to radar and sonar systems is to detect the presence of either a deterministic or random signal in noise with least probability of error or false alarm [17]. Although the two aforementioned setups have conceptual differences, they are usually treated in the same fashion. First, an optimal demodulator is necessary to deliver the sufficient statistics to the decision device. Then, the decision device, that optimally uses these sufficient statistics, has to be derived. The optimal design of the decision device is formulated in any case as a hypotheses testing problem. Moreover, the optimization of the transmitter is another related problem. In this case, the problem turns to be the design of optimal transmission sets, such that the end performance metric, i.e., the probability of error is minimized.

Depending on the degree of knowledge about the transmission channel at the receiver side, the detector can be coherent, semi-coherent or noncoherent [22]. The more information about the transmission channel is available, the better the receiver’s performance will be. This justifies the fact that the receivers usually have a built-in channel estimator. In the communication and signal processing literature, the usual channel estimators are the minimum variance unbiased (MVU) and the minimum mean square error (MMSE) estimators [16]. The combination of these channel estimators with the optimal decision devices is usually considered to address the problem of determining the optimal receiver.

Current physical layer (PHY) standards that have attracted a lot of attention both from the mobile industry and the research community are the Wireless Interoperability for Microwave Access (WiMAX), the Long Term Evolution (LTE) and the Digital Video Broadcasting (DVB) either in its terrestrial (DVB-T) or its Handheld (DVB-H) versions [2], [7], [8], [9], [21], [24], [26]. These standards are orthogonal frequency division multiple access (OFDMA) based and they can satisfy the need for shorter communication links to provide truly broadband connectivity services. In these systems, either MVU/least squares (LS) or MMSE channel estimators are used, usually employing some sort of estimate interpolation through the frame if the goal is to track a time-varying channel [1], [4], [11], [18], [19], [20], [23], [25], [27].

In this paper, we re-examine the validity of the common belief that the MVU and MMSE channel estimators are the best choices to be combined with the optimal detectors, delivering an overall optimal receiver, when finite-sample training is used to estimate the channel. To this end, ideas originating from the system identification field are employed. Recent results in optimal experiment design indicate that it is better to design the optimal training for the estimation of a certain set of unknown parameters with respect to optimizing the end performance metric rather than the mean square error of the parameter estimator itself [3], [10], [14], [15]. We will slightly modify this idea and we will examine if the aforementioned channel estimators are the best choices, when the selection of the channel estimator is made with respect to an appropriately defined end performance metric. For illustration purposes, this study is performed on a toy channel model, namely a single input single output (SISO) flat fading channel with additive white Gaussian noise (AWGN). The initial focus is on two different MSE criteria. These MSE criteria serve to demonstrate the dependence of the optimal channel estimators

1In this sense, the asymptotic efficiency of the maximum likelihood (ML) estimator together with its invariance property are irrelevant.

2In this toy model, the MVU estimator coincides with the LS and the ML channel estimators.
on the end performance metrics. Their choice is based on the simplicity of the analysis that they allow. Then, using the obtained results, we will examine the case of the error probability as the performance metric of interest. We show that for several performance metrics examined in this paper, the MVU and MMSE channel estimators are suboptimal, while we propose ways to obtain better channel estimators. Finally, we numerically compare the performances of the derived channel estimators with those of the MVU and MMSE channel estimators for all performance metrics in this paper. These comparisons verify that the optimality of the usual channel estimators with respect to common end performance metrics is questionable.

This paper is organized as follows: Section II defines the problem of designing the channel estimator with respect to the end performance metric. Section III presents some results and comments that will be useful in the rest of the paper, while it introduces approximations of the performance metrics that the rest of the analysis will be based on. The optimality of the MVU and MMSE channel estimators with respect to the minimization of the symbol estimate MSE is examined in Section IV and subsections therein, while uniformly better channel estimators are also proposed. The same analysis as in Section IV is pursued in Section V for a differently defined symbol estimate MSE and in Section VI for a rough approximation (variation) of the error probability performance metric. Section VII illustrates the validity of the derived results. Finally, Section VIII concludes the paper.

II. PROBLEM STATEMENT

The received signal model for a SISO system, when the channel is considered to be narrowband block fading, is given as follows:

\[ y(n) = hx(n) + w(n), \]  

where \( y(n) \) is the observed signal at the receiver side at time instant \( n \), \( h \) is the complex channel impulse response coefficient, \( x(n) \) is the transmitted symbol at the same time instant taken from an M-ary constellation \( \mathcal{X} = \{x_1, x_2, \ldots, x_M\} \) and \( w(n) \) is complex, circularly symmetric, Gaussian noise with zero mean and variance \( \sigma_w^2 \). Given an equiprobable distribution on the constellation symbols, we further assume that \( E[x(n)] = 0 \) and \( E[|x(n)|^2] = \sigma_x^2 \), while our modulation method is memoryless. In addition, \( w(n) \) and \( x(n) \) are independent random sequences, while \( w(n) \) is a white random sequence.

Assume that a maximum energy \( \mathcal{E} \) and a training length of \( B \) time slots are available at the transmitter for training. We can collect the received samples corresponding to training in one vector:

\[ y_{tr} = hx_{tr} + w_{tr}, \]  

where \( y_{tr} = [y(l-B+1), y(l-B+2), \ldots, y(l)]^T \) is the vector of \( B \) received samples corresponding to training, \( x_{tr} = [x(l-B+1), x(l-B+2), \ldots, x(l)]^T \) is the vector of \( B \) training symbols and \( w_{tr} = [w(l-B+1), w(l-B+2), \ldots, w(l)]^T \) is the vector of \( B \) noise samples. Considering the class of linear channel estimators, the channel is estimated as follows:

\[ \hat{h} = f^H y_{tr} = h f^H x_{tr} + f^H w_{tr}, \]

where \( f \) is a \( B \times 1 \) channel estimating filter.

With the assumptions in [1], if the constellation symbols are equiprobable and the channel is perfectly known, the ML detector is optimal [5], [22]. This is with respect to minimizing the probability that a different symbol from the one transmitted is decided given the transmitted symbol. The ML decision rule is given by the following expression:

\[ \text{dec} \{x(n)\}(h) = \arg \min_{\hat{x}(n) \in \mathcal{X}} |y(n) - h \hat{x}(n)|^2. \]

Here, \( \text{dec} \{x(n)\}(h) \) denotes the decision of the detector, when the transmitted symbol is \( x(n) \). In essence, the ML detector minimizes the probability of error, when the transmitted symbols are equiprobable. When the receiver has a channel estimate \( \hat{h} \), \( h \) is replaced by \( \hat{h} \) in the last expression.

A different kind of performance metric is the MSE of a linear symbol estimator. In this paper, we will call the symbol estimator an equalizer. The equalizer uses the channel knowledge and delivers a soft decision of the transmitted symbol, i.e., a symbol estimate. We will call clairvoyant the equalizer that has perfect channel knowledge. Denoting this equalizer by \( \tilde{c}(h) \), we can find its mathematical expression as follows:

\[ \tilde{c}(h) = \arg \min_{c(h)} E \left[ (c(h)y(n) - x(n))^2 \right], \]

where the expectation is taken over the statistics of \( x(n) \) and \( w(n) \). If we set the derivative of the last expression with respect to \( c(h) \) to zero and we solve for \( c(h) \), then the optimal clairvoyant equalizer is given by the expression

\[ \tilde{c}(h) = \frac{\sigma_x^2 h^*}{|h|^2 \sigma_x^2 + \sigma_w^2}. \]

We will call this the MMSE clairvoyant equalizer. We observe that as the SNR increases, i.e., \( \sigma_w^2 \rightarrow 0, \tilde{c}(h) \rightarrow 1/h \). We will call \( \tilde{c}(h) = 1/h \) the Zero Forcing (ZF) clairvoyant equalizer. Using the above definitions and assuming that the receiver has only an estimate of the channel, the system performance metric is the symbol estimate MSE:

\[ \text{MSE}_x = E \left[ |c(\hat{h})y(n) - x(n)|^2 \right]. \]

The MSE given by (7) can be defined in two different ways:

If we assume that the channel is an unknown but otherwise deterministic quantity, then the expectation in (7) does not consider \( h \). This leads to an MSE expression dependent on the unknown channel \( h \). In this case, only the channel estimators that treat the channel as an unknown deterministic variable or as a random variable are meaningful. If we assume that the unknown channel is a random variable, then we can average the MSE expression over \( h \). In this case, both the estimators that treat the channel as an unknown deterministic variable or as a random variable are meaningful. The former represents the case where the system designer chooses to ignore the knowledge of the channel
statistics in the selection of the channel estimator for some reason.

In the following, we focus on the ZF equalizer, which becomes optimal as the SNR increases. This choice is made to preserve the simplicity of this paper and to highlight the derived results.

The previous MSE definition implies the definition of yet another MSE that is meaningful in the context of communication systems. Given an equalizer, we can define the excess of the symbol estimate based on an equalizer that only knows a channel estimate over the equalizer with perfect channel knowledge, thus leading to

\[ \text{MSE}_{x_e} = E \left[ \left( c(\hat{h})y(n) - c(h)y(n) \right)^2 \right]. \quad (8) \]

In the sequel, this metric will be called excess MSE.

Our goal will be to determine the optimal channel estimators for fixed training sequences so that each performance metric based on a given equalizer is minimized. To this end, the following section presents some useful ideas.

III. PRELIMINARY RESULTS

Consider the MVU estimator. Since it is an unbiased estimator, it satisfies \( f^H \mathbf{x}_{tr} = 1 \). This condition implies that \( E[\hat{h}] = h \). For our problem assumptions, the MVU estimator can be found by solving the following optimization problem:

\[ \min_f \sigma_w^2 ||f||^2 \]

\[ \text{s.t.} \quad f^H \mathbf{x}_{tr} = 1. \quad (9) \]

Forming the Lagrangian for this problem and zeroing its gradient with respect to \( f \), we get:

\[ f_{MVU} = \frac{\mathbf{x}_{tr}}{||\mathbf{x}_{tr}||^2}. \quad (10) \]

For the sake of completeness, this estimator coincides with the ML and LS channel estimators under our assumptions.

If we assume that the prior distribution of \( h \) is known, then instead of the MVU one could use the MMSE channel estimator. With our assumptions and the extra assumption that \( E[h] = 0 \), one can obtain \( \sigma_w^2 \)

\[ f_{MMSE} = \frac{E[|h|^2] \mathbf{x}_{tr}}{E[|h|^2] ||\mathbf{x}_{tr}||^2 + \sigma_w^2}. \quad (11) \]

The \( \text{MSE}_x \) of the ZF equalizer using a deterministic channel (“dc”) assumption is

\[ \text{MSE}_{x}^{dc} (ZF) = E \left[ \frac{\hat{h} - h}{h} \right]^2 \sigma_w^2 + \sigma_w^2 E \left[ \frac{1}{|h|^2} \right], \quad (12) \]

the corresponding for random channel (“rc”) is:

\[ \text{MSE}_{x}^{rc} (ZF) = \text{E}_h \left[ \frac{\hat{h} - h}{h} \right]^2 \sigma_w^2 + \sigma_w^2 \text{E}_h \left[ \frac{1}{|h|^2} \right], \quad (13) \]

while for the \( \text{MSE}_{x_e} \) we accordingly have:

\[ \text{MSE}_{x_e}^{rc} (ZF) = E \left[ \frac{\hat{h} - h}{h} \right]^2 \left( \sigma_x^2 + \frac{\sigma_w^2}{|h|^2} \right), \quad (14) \]

(c.f. \( \text{MSE}_{x}^{rc} \)). The \( \text{MSE}_{x_e}^{rc} \) is obtained by averaging the last expression over \( h \).

Depending on the probability distributions of \( |\hat{h}| \) and \( |h| \), the above MSE expressions may fail to exist. The MSEs will be finite if the probability distribution function (pdf) of \( |h| \) is of order \( O(|h|^2) \) as \( h \to 0 \). A similar condition should hold for the pdf of \( |\hat{h}| \) in the case of \( \text{MSE}_{x_e}^{rc} \). In the opposite case, we end up with an infinite moment problem. In order to obtain well-behaved channel estimators that will be used in conjunction with the actual performance metrics, some sort of regularization is needed. Some ideas for appropriate regularization techniques to use may be obtained by modifying robust estimators (against heavy-tailed distributions), e.g., by trimming a standard estimator, if it gives a value very close to zero \([12]\).

An example of such a trimmed estimator is given as follows:

\[ \hat{h} = \begin{cases} f^H \mathbf{y}_{tr}, & \text{if} \ |f^H \mathbf{y}_{tr}| > \lambda \\ \lambda f^H \mathbf{y}_{tr}, & \text{if} \ |f^H \mathbf{y}_{tr}| \leq \lambda \\ \text{o.w.} \end{cases}, \quad (15) \]

where \( f \) can be any estimator and \( \lambda \) a regularization parameter.

Remark: Clearly, the reader may observe that the definition of the trimmed \( \hat{h} \) preserves the continuity at \( f^H \mathbf{y}_{tr} = \lambda \). Additionally, the event \( \{f^H \mathbf{y}_{tr} = 0\} \) has zero probability since the distribution of \( f^H \mathbf{y}_{tr} \) is continuous. Therefore, in this case \( \hat{h} \) can be arbitrarily defined, e.g., \( \hat{h} = \lambda \).

We focus now on the \( \text{MSE}_{x_e}^{rc} (ZF) \). Assume a fixed \( \lambda \). In the appendix, we show that, for a sufficiently small \( \lambda \) and a sufficiently high SNR during training, minimizing \( \text{MSE}_{x_e}^{rc} (ZF) \) is equivalent to minimizing the following approximation

\[ \text{E}_h \left[ |\hat{h} - h|^2 \right] \sigma_w^2 + \sigma_w^2 \frac{1}{E[|h|^2]}. \quad (16) \]

Following similar steps and using some minor additional technicalities, we can work with

\[ \text{E}_h \left[ |\hat{h} - h|^2 \right] \sigma_w^2 + \sigma_w^2 \frac{1}{E[|h|^2]}, \quad (17) \]

instead of \( \text{MSE}_{x_e}^{rc} (ZF) \). Moreover, \( \text{MSE}_{x_e}^{dc} (ZF) \) and \( \text{MSE}_{x_e}^{rc} (ZF) \) can be defined accordingly. We will call the last approximations zeroth order symbol estimate MSEs and excess MSEs, respectively. The following analysis and results will be based on the zeroth order metrics and they will reveal the dependency of the channel estimator’s selection on the considered (any) end performance metric.

Remarks:

1) A useful, alternative way to consider the zeroth order MSEs is to view them as affine versions of normalized channel MSEs, where the actual true channel is \( \hat{h} \) and the estimator is \( h \).

2) In the definition of \( (16) \), one can observe that after approximating the mean value of the ratio by the ratio\footnote{This parameter can be tuned via cross-validation or any other technique, although in the simulation section we empirically select it for simplicity purposes.}.
of the mean values the infinite moment problem is eliminated. In the following, all zeroth order metrics will be defined based on the non-trimmed $\hat{h}$ to ease the derivations. This treatment is approximately valid when $\lambda$ is sufficiently small as it is actually shown in eq. (22) of the appendix.

IV. MINIMIZING THE ZEROTH ORDER SYMBOL ESTIMATE MSE

We now examine the zeroth order symbol estimate MSE in the case of the ZF equalizer. The optimality of the MVU and MMSE channel estimators will be investigated. Additionally, the training sequence is assumed fixed.

A. ZF Equalization

The channel is considered either deterministic or random, depending on the available knowledge of a priori channel statistics and the will of the system designer to ignore or to exploit this knowledge.

1) Deterministic Channel: The expectation operators in Eq. (16) are with respect to $u_{tr}, x(n)$ and $w(n)$. We have:

$$\text{MSE}_{\text{ZF}}(ZF) = \sigma_w^2 \frac{\|h\|^2 \|f^H x_{tr} - 1\|^2 + \sigma_w^2 \|f\|^2}{\|h\|^2 \|f^H x_{tr}\|^2 + \sigma_w^2 \|f\|^2}.$$  (18)

The numerator of the gradient of the above expression with respect to $f$ discarding the outer $\sigma_w^2$ is given by the following expression:

$$\left[|h|^2 \varphi^2 + \sigma_w^2 \|f\|^2 \right] \left[|h|^2 (\varphi - 1)^* x_{tr} + \sigma_w^2 f \right] - \left[|h|^2 \varphi^* x_{tr} + \sigma_w^2 f \right] \left[\frac{\sigma_w^2}{\sigma_w^2} + |h|^2 |\varphi - 1|^2 + \sigma_w^2 \|f\|^2 \right],$$  (19)

where $\varphi = f^H x_{tr}$. Setting $f = f_{\text{MVU}}$, we obtain:

$$- \frac{\sigma_w^2}{\sigma_w^2} \left[|h|^2 x_{tr} + \sigma_w^2 \frac{x_{tr}}{\|x_{tr}\|^2} \right] \neq 0.$$  (20)

Note that no choice of $x_{tr}$ will zero this expression for any $|h|^2, \sigma_w^2$. Therefore, the MVU is not an optimal channel estimator in this case. We can state this result more formally:

**Proposition 1:** The MVU estimator is not an optimal channel estimator for the task of minimizing $\text{MSE}_{\text{ZF}}(ZF)$, when the channel is considered deterministic but otherwise an unknown quantity.

The question that arises in this case is how to find the optimal channel estimator in this setup or generally how to determine a uniformly better channel estimator for minimizing $\text{MSE}_{\text{ZF}}(ZF)$. Equating (19) to 0 and taking the inner product of both sides with $f$, we obtain the following necessary condition that every optimal channel estimating filter $f$ must satisfy given the training sequence $x_{tr}$:

$$f^H x_{tr} = \left(1 + \frac{\sigma_w^2}{\sigma_w^2 |h|^2} \right).$$  (21)

A possible $f$ that satisfies this condition is

$$f_{\text{opt}}^{\text{ZF}} = \left(1 + \frac{\sigma_w^2}{\sigma_w^2 |h|^2} \right) \frac{x_{tr}}{\|x_{tr}\|^2} = \left(1 + \frac{\sigma_w^2}{\sigma_w^2 |h|^2} \right) f_{\text{MVU}},$$  (22)

which becomes:

$$f_{\text{opt}}^{\text{ZF}} = \left(1 + \frac{\sigma_w^2}{\sigma_w^2 |h|^2} \right) \frac{1}{x_{tr}}.$$  (23)

for $B = 1$. Clearly, (22) is sufficient for (19) to become zero. However, (22) has another problem, namely that the optimal solution depends on the unknown channel $h$.

In order to deal with the dependence of the optimal estimator on the unknown channel, we will resort to a stochastic approach. We will assume a noninformative prior distribution for the unknown channel. If the real and imaginary parts of the channel are considered bounded in the intervals $I_R \subset \Re$ and $I_I \subset \Re$, the receiver can treat them as independent random variables uniformly distributed on $I_R$ and $I_I$, respectively. The $\text{MSE}_{\text{ZF}}(ZF)$ is now replaced by $E[h] \left\{ \text{MSE}_{\text{ZF}}(ZF) \right\}_0$, where $E[h]\{ \cdot \}$ denotes the expectation with respect to the joint (uniform) distribution of the real and imaginary parts of $h$. Applying again the zeroth order approximation and following the above analysis we can easily show that the eqs. (21), (22) and (23) give again the necessary condition and optimal estimators in this case with the substitution of $|h|^2$ by $E_{h}^d(|h|^2)$.

2) Random channel: In this case, the actual prior statistics of the channel are known. The zeroth order symbol estimate MSE is given by

$$\text{MSE}_x(\text{ZF})_0 = \frac{\sigma_w^2 \|E[|h|^2] f^H x_{tr} - 1\|^2 + \sigma_w^2 \|f\|^2}{\|E[|h|^2] f^H x_{tr}\|^2 + \sigma_w^2 \|f\|^2}.$$  (24)

Differentiating this expression with respect to $f$, we get the numerator of the gradient which is given by (19) but with $|h|^2$ replaced by $E[|h|^2]$. It can be easily shown that this numerator is different from zero if $f = f_{\text{MVU}}$ or $f = f_{\text{MMSE}}$. We therefore have a formal statement of this result:

**Proposition 2:** The MVU and MMSE estimators are not optimal channel estimators for the task of minimizing $\text{MSE}_x(\text{ZF})_0$, when the prior channel distribution is known. The optimal channel estimator $f_{\text{opt}}^{\text{ZF}}$ satisfies (21), (22) and (23), but with $|h|^2$ replaced by $E_{h}(|h|^2)$.

Remarks:

1) Considering (22) and the corresponding expression for the random channel case, we observe that the design of the estimator with respect to the end performance metric introduces a bias to the MVU estimator in the form of scaling, leading to a smaller value of the end performance metric than the one that we would obtain by using the MVU estimator. This bias introduction mechanism has similarities with the introduction of bias in estimators to reduce their MSE (the MSE here is the

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5 This assumption is usually reasonable in practice.

6 Part of this analysis is presented in Subsection V.A.

7 ignoring all the positive scaling terms
average square distance of the parameter estimator from the true value of the parameter) [6]. Nevertheless, the reader may observe the conceptual differences in the motivation and goals behind the end performance metric estimator designs presented in this paper and the ideas in [6].

2) The claimed optimality of the derived estimators in this section but also in this paper is with respect to the zeroth order performance metrics. These estimators turn out to be uniformly better than the MVU and MMSE estimators also when comparing against the true and end performance metrics as we demonstrate in the simulation section.

3) An alternative way to express eq. (22) is

$$f_{\text{opt}}^{\text{ZF}} = (1 + \alpha) f_{\text{MVU}},$$  

where $\alpha = \sigma_w^2 / (\sigma_x^2 |h|^2)$ is the inverse SNR at the receiver side. Depending on how we implement the last estimator in practice, $\alpha$ turns to a tuning parameter controlling the introduction of bias in the MVU estimator. We numerically demonstrate this very interesting aspect of the derived estimators in Figs. 7 and 8.

### B. Discussion on the Optimal Training

Since the channel estimator is selected in order to optimize the final performance metric of the communication system, one may consider the problem of selecting optimally the training vector $x_{tr}$ under a training energy constraint $\|x_{tr}\|^2 \leq \mathcal{E}$ to serve the same purpose. To optimize the training vector, one should first fix the channel estimator. This is a “complementary” problem with respect to the approach that we have followed so far. Suppose that we use either the MVU or the MMSE channel estimators. One can observe that for $B = 1$ the problem of selecting optimally the training vector is meaningless. Therefore, we will end up using an inferior channel estimator (i.e., the MVU or the MMSE) than the one given by (23) and its random channel counterpart. In the case that $B > 1$, fixing for example $f = f_{\text{MVU}}$ one can observe that again the problem of selecting optimally the training vector is meaningless. Consider for example the case of $\text{MSE}_{xe}^{\text{ZF}} (\hat{h})$.  

$$\text{MSE}_{xe}^{\text{ZF}} (\hat{h}) = \frac{\sigma_x^2 \|x_{tr}\|^2 + \sigma_w^2}{E[|h|^2]} + \frac{\sigma_a^2}{E[|h|^2]},$$  

which only depends on $\|x_{tr}\|^2$. Furthermore, setting $\theta = \|x_{tr}\|^2$, it follows that $\partial \text{MSE}_{xe}^{\text{ZF}} (\theta) / \partial \theta < 0$ at sufficiently high SNR, i.e., $\text{MSE}_{xe}^{\text{ZF}} (\theta)$ is minimized when $\|x_{tr}\|^2 = \mathcal{E}$, which is intuitively appealing. Therefore, any $x_{tr}$ with energy equal to $\mathcal{E}$ is an equally good training vector for the MVU estimator. Thus, for the same $x_{tr}$, the estimator $f = f_{\text{opt}}^{\text{ZF}}$ will be better than the MVU. Similar conclusions can be reached for the MMSE estimator, as well.

### V. MINIMIZING THE ZEROTH ORDER EXCESS MSE

We now examine the zeroth order excess MSE in the case of the ZF equalizer.

#### A. ZF Equalizer with a deterministic channel

In this case, we have:

$$\text{MSE}_{xe}^{\text{ZF}} (\hat{h}) = \frac{|h|^2 |f^H \hat{x}_{tr} - 1|^2}{|h|^2 |f^H x_{tr}|^2 + \sigma_w^2 \|f\|^2} \left( \frac{\sigma_x^2}{|h|^2} + \frac{\sigma_w^2}{\|f\|^2} \right),$$  

(26)

The numerator of the gradient of the above expression with respect to $f$ is given by the following expression:

$$\frac{\partial}{\partial f} \text{MSE}_{xe}^{\text{ZF}} (\hat{h}) = \frac{|h|^2 |\hat{x}_{tr}|^2 + \sigma_w^2}{|h|^2 |\hat{x}_{tr}|^2 + \sigma_w^2 \|f\|^2} \left( 2 \frac{\sigma_x^2}{|h|^2} \right) - \frac{2 \sigma_a^2 \|f\|^2}{|h|^2 |\hat{x}_{tr}|^2 + \sigma_w^2 \|f\|^2},$$  

(27)

Setting $f = f_{MVU}$, one can easily check that the above expression becomes zero. Therefore:

**Proposition 3:** The MVU is an optimal channel estimator for the task of minimizing $\text{MSE}_{xe}^{\text{ZF}} (\hat{h})$, when the channel is considered a deterministic but otherwise unknown quantity.

**Remark:** Note that even if $\text{MSE}_{xe}^{\text{ZF}} (\hat{h})$ depends on the unknown channel $h$, the optimal channel estimator does not in this case.

#### B. ZF Equalizer with a random channel

In this case, the prior statistics of the channel are known. The zeroth order excess MSE is given by:

$$\text{MSE}_{xe}^{\text{ZF}} (\hat{h}) = \frac{|\hat{x}_{tr} - 1|^2}{|h|^2 |f^H \hat{x}_{tr} - 1|^2 + \sigma_w^2 \|f\|^2} \left( \frac{\sigma_x^2}{|h|^2} + \frac{\sigma_w^2}{\|f\|^2} \right) + \frac{\sigma_a^2 \|f\|^2}{|h|^2 |\hat{x}_{tr}|^2 + \sigma_w^2 \|f\|^2},$$  

(28)

Differentiating this expression w.r.t. $f$ and setting $f = f_{MVU}$ we get the following gradient.

**Proposition 4:** The MVU is an optimal channel estimator for the task of minimizing $\text{MSE}_{xe}^{\text{ZF}} (\hat{h})$, when the channel is considered random.

**Remark:** This result is counterintuitive: it says that when one has knowledge of the channel statistics but uses a ZF equalizer, one should ignore these statistics in choosing a channel estimator for minimizing the zeroth order excess MSE.

### VI. MINIMIZING THE ZEROTH ORDER PROBABILITY OF ERROR FOR THE ML DETECTOR

It is straightforwar to see that the decision rule given by (24) is equivalent to:

$$\text{dec} \left[ x(n) \right] (h) = \arg \min_{\hat{x}(n) \in \mathbb{X}} \left| \frac{y(n)}{h} - \hat{x}(n) \right|^2 \tag{29}$$

With a given channel estimate, $h$ is replaced by $\hat{h}$ in the last expression.\(^\dagger\)

\(^\dagger\)discarding the positive scalars and considering again the corresponding (hermitian) transpositions.

\(^\dagger\)Notice that this does not generalize to ISI and/or MIMO channels.
In the case of a perfectly known channel, the division \( y(n) / h = x(n) + w(n) / h \) results in an AWGN channel with information bearing signal power \( \sigma_w^2 \) and noise variance \( \sigma_e^2 \). If only an estimate of the channel, \( h = h + \epsilon \), is available, then the division results in \( y(n) / h = x(n) + (w(n) - cx(n)) / h \). Here, \( \epsilon \) is the channel estimation error, which is Gaussian distributed according to our assumptions with \( E[\epsilon] = h \left( f^H x_\epsilon - 1 \right) \) and variance \( \sigma_\epsilon^2 = \sigma_e^2 \| f \|^2 \).

Also, \( E[|\epsilon|^2] = \| h \|^2 \left( f^H x_\epsilon - 1 \right)^2 + \sigma_e^2 \| f \|^2 \).

For the case of most common constellations and an AWGN channel, the error probability is given by [22]:

\[
P_e \approx aQ \left( b \sqrt{\text{SNR}} \right) \tag{30}
\]

where \( Q(x) = (1/\sqrt{2\pi}) \int_x^{\infty} e^{-t^2/2} dt \) and \( a, b \) are positive constants depending on the geometry of the constellation. With a channel estimation error, the useful signal power is again \( \sigma_w^2 \). The noise variable is now \( w(n)' = (w(n) - cx(n)) / h \) and therefore \( E[w(n)'] = 0 \). For the power of the noise component, we have:

\[
E \left[ |w(n)|^2 \right] = E \left[ \left( w(n) / h \right)^2 \right] + E \left[ \left( \epsilon / h \right)^2 \right] \sigma_w^2 \tag{31}
\]

Here, we face again the infinite moment problem. Using again similar arguments in the appendix for approximating \( E[X/Y] \) by \( E[X] / E[Y] \) when \( Y = |h|^2 \), we define the corresponding zeroth order version of \( E \left[ |w(n)|^2 \right] \):

\[
\left\{ \begin{array}{l}
E \left[ \left( \frac{w(n)}{h} \right)^2 \right] \\
E \left[ \left( \frac{\epsilon}{h} \right)^2 \right]
\end{array} \right\} = \frac{\sigma_w^2}{E[|h|^2]} = \frac{\| h \|^2 \| f^H x_\epsilon \|^2 + \sigma_w^2 \| f \|^2}{\| h \|^2 \| x_\epsilon \|^2 + \sigma_e^2 \| f \|^2} \tag{32}
\]

A variation of the error probability performance metric for any of the commonly used linear modulation schemes, named zeroth order error probability, will be given by the following expression:

\[
\begin{align*}
\left[ P_e \right]_0 &= aQ \left( b \frac{\| h \|^2 \left( \| f^H x_\epsilon \|^2 + \sigma_w^2 \| f \|^2 \right)}{\sigma_w^2 (\sigma_w^2 + \| h \|^2 \| f^H x_\epsilon \|^2 + \sigma_w^2 \| f \|^2)} \right) \\
&= aQ \left( b \frac{\sigma_w^2}{\text{MSE}_{zc}(ZF)} \right) \tag{33}
\end{align*}
\]

where we have used the zeroth order SNR approximation given by:

\[
\text{SNR}_0 = \frac{\sigma_w^2}{\text{MSE}_{zc}(ZF)}.
\]

Clearly, \( \text{SNR}_0 \) is an artificial performance metric that appears in this paper for the sake of our arguments. It is used as a variation of the error probability to help us extract useful conclusions.

Since \( Q(x) \) is a strictly decreasing function, the zeroth order probability of error for a given channel \( h \) is minimized when \( \text{MSE}_{zc}(ZF) \) is minimized. Therefore, the results of Subsection IV-A apply:

**Proposition 5:** The MVU estimator is not an optimal channel estimator for the task of minimizing \( \left[ P_e \right]_0 \) given the true channel, when using any of the well-known digital modulations in a flat-fading AWGN channel.

Suppose now that we average \( \left[ P_e \right]_0 \) with respect to any given channel distribution. We can then make the following statement:

**Proposition 6:** The MVU estimator is not an optimal channel estimator for the task of minimizing the average \( \left[ P_e \right]_0 \), when using any of the well-known digital modulations in a flat-fading AWGN channel.

**Proof:** Assume that the pdf of the fading coefficient magnitude is \( p(|h|) \) and \( |h| \in [\alpha, \beta], \alpha, \beta \geq 0, \beta \) possibly equal to \( +\infty \). The average \( \left[ P_e \right]_0 \) is given by the expression:

\[
\left[ P_e \right]_0 = \int_0^{\beta} aQ \left( b \sqrt{\text{SNR}_0} \right) p(|h|) d|h| \tag{34}
\]

Assuming that the differentiation and integral operators can be interchanged, we can set the gradient of the above expression with respect to \( f^H \) to zero to get the equation:

\[
\nabla_{f^H} \int_0^{\beta} aQ \left( b \sqrt{\text{SNR}_0} \right) p(|h|) d|h| = 0^H
\]

\[
\int_0^{\beta} aQ \left( x = b \sqrt{\text{SNR}_0} \right) \frac{b \nabla_{f^H} \text{SNR}_0}{2b \sqrt{\text{SNR}_0}} p(|h|) d|h| = 0^H
\]

where in the second equation we have used the chain rule of differentiation. \( Q(x) \) is strictly decreasing in \( x \), thus

\[
\nabla_{x, Q} \left( x = b \sqrt{\text{SNR}_0} \right) < 0
\]

for any value of \( h \). Also \( p(|h|) \geq 0 \) for every value of \( h \) since it is a distribution function. Additionally, \( \text{SNR}_0 \geq 0 \) for every value of \( h \). Finally, the numerator of \( \nabla_{f^H} \text{SNR}_0 \) for the MVU estimator is given by [20] multiplied by \(-1\) and by a positive scalar [22]. Therefore, it is either positive or negative with respect to \( h \) in a componentwise fashion depending on the sign of the corresponding element in \( x, f \). These arguments verify that \( \nabla_{f^H} \left[ P_e \right]_0 \neq 0 \). This concludes the proof.

If we assume that the prior distribution of \( h \) is known, then instead of the MVU, one could use the MMSE channel estimator. Plugging \( f_{\text{MMSE}} \) into the negative of [19], one can obtain that \( \nabla_{f^H} \text{SNR}_0 \neq f_{\text{MMSE}} \). Since in the case of the MMSE estimator, the assumption is that we always know the prior channel fading distribution, we can make the following statement:

**Proposition 7:** The MMSE estimator is not optimal for the task of minimizing \( \left[ P_e \right]_0 \) when using any of the well-known digital modulations in a flat-fading AWGN channel.

**Proof:** The result follows along the same lines as in Proposition 6.

The problems of determining the optimal channel estimator for the task of minimizing \( \left[ P_e \right]_0 \) for a given channel \( h \) and \( \left[ P_e \right]_0 \) was already solved in Subsections IV-A.1 and IV-A.2 and

---

10 The denominator is always positive as a squared term.
11 Some of the entries of \( x_\epsilon \) may be zero but not all of them simultaneously.
respectively. In the case of $P_x$, we can only assess their optimality analytically, using the following argument: We use the upper bound\(^{12}\) $Q(x) < (1/x)(1/\sqrt{2\pi})e^{-x^2/2}, x > 0,$ which becomes tight as $x$ increases \([22]\). In our case, $x = b\sqrt{\text{SNR}}_0$, and we have already assumed high SNR, therefore high $|\text{SNR}|_0$, to justify the use of the ZF equalizer. Using this bound and the relationship between $|\text{SNR}|_0$ and $\text{MSE}^{\text{ZF}}_x(0)$, we get:

$$Q(b\sqrt{|\text{SNR}|_0}) < \sqrt{\frac{\text{MSE}^{\text{ZF}}_x(0)}{b\sigma_x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{b^2\sigma_x^2}{2|\text{SNR}|_0}}$$

(35)

where the last inequality holds for large SNR and therefore small $\text{MSE}^{\text{ZF}}_x(0)$. The right hand side function is concave with respect to $\text{MSE}^{\text{ZF}}_x(0) > 0$, therefore averaging over any channel distribution, we get:

$$E_h \left[ Q \left( b\sqrt{|\text{SNR}|_0} \right) \right] < \sqrt{E_h \left[ \frac{\text{MSE}^{\text{ZF}}_x(0)}{b\sigma_x} \right]}$$

We can use one more time the zeroth order approximation to approximate $E_h \left[ \text{MSE}^{\text{ZF}}_x(0) \right]$. The right hand side is minimized when this last zeroth order approximation is minimized. Thus, the estimators derived in Subsections IV-A.1 and IV-A.2 are optimal for the task of minimizing $P_x$, in the sense that they minimize an upper bound to $E_h \left[ Q \left( b\sqrt{|\text{SNR}|_0} \right) \right]$.

Remark: Although, we have shown that the MVU and MMSE estimators are not optimal for the task of minimizing the zeroth order probability of error, we will see in the simulation section that their actual probability of error performance is almost identical with that of the optimal estimators for the zeroth order probability of error. This is due to two facts: first, the zeroth order probability of error is a variation of the actual probability of error and second, in practice the difference in the channel estimates must be large enough to give rise to a notable difference in the probability of error. Nevertheless, we conjecture that such a difference may be more clear in the case of multiple input multiple output (MIMO) systems if tight approximations of the error probability functions are used to derive the corresponding channel estimators.

VII. SIMULATIONS

In this section we present numerical results to verify our analysis. In all figures, $h \sim \mathcal{CN}(0, 1)$ and QPSK modulation is assumed. The SNR during training highlights how good the channel estimate is. The parameter $\lambda$ has been empirically selected to be 0.1. All schemes in Figs. 4\textsuperscript{13} and 5 use $15$ for the same $\lambda$. In Figs. $4$\textsuperscript{16} $E_h^{\text{vd}}[|h|^2]$ is chosen to be $3E[|h|^2] = 3$, i.e., the real and imaginary parts of $h$ are assumed i.i.d. following a uniform distribution in $[-3/\sqrt{2}, 3/\sqrt{2}]$. In Figs. 7\textsuperscript{17} and 8 $E_h^{\text{vd}}[|h|^2]$ equals $1/2$ and $1/6$, respectively.

\(^{12}\)The usual Chernoff bound can also be used.
Fig. 4. \( \text{MSE}^{e}_e(ZF) \) with SNR during training equal to 0 dB, \( B = 5 \) and \( \lambda = 0.1 \). Moreover, \( E_{h}^{ud} \|h\|^2 = 3 \).

Fig. 5. \( \text{MSE}^{e}_e(ZF) \) with SNR during training equal to 0 dB, \( B = 2 \) and \( \lambda = 0.1 \). Moreover, \( E_{h}^{ud} \|h\|^2 = 3 \).

Fig. 6. Average \( P_e \) with SNR during training equal to 10 dB, \( B = 5 \) and \( \lambda = 0.1 \). Moreover, \( E_{h}^{ud} \|h\|^2 = 3 \).

Fig. 7. \( \text{MSE}^{e}_e(ZF) \) with SNR during training equal to 0 dB, \( B = 5 \) and \( \lambda = 0.1 \). Moreover, \( E_{h}^{ud} \|h\|^2 = 1/2 \).

Fig. 8. \( \text{MSE}^{e}_e(ZF) \) with SNR during training equal to 0 dB, \( B = 2 \) and \( \lambda = 0.1 \). Moreover, \( E_{h}^{ud} \|h\|^2 = 1/6 \).

In Fig. [1] \( \text{MSE}^{e}_e(ZF) \) is presented for \( B = 5 \) and SNR during training equal to 0 dB. The derived optimal estimators in this paper are better than the MVU and MMSE estimators. Additionally, the MVU estimator appears to be better than the MMSE estimator for this performance metric. This is a new observation contradicting what one would expect and verifying the motivation of this paper.

Fig. [2] presents the corresponding results for \( \text{MSE}^{e}_e(ZF) \). The MVU is the best estimator as proved. This is another example contradicting what one would expect and verifying the motivation of this paper.

Furthermore, Fig. [3] shows the performance of all schemes in the case of an approximation to the error probability equal to \( Q \left( \sqrt{\frac{a}{b}} \text{MSE}^{e}_e(ZF) \right) \). Here, we have assumed that the constants \( a, b \) are equal to 1, since their specific values are irrelevant to the purpose of this simulation plot. The derived estimators in this paper are better than the MVU and MMSE estimators as proved in the previous section. The difference of the curves is present in the low SNR regime.

We now examine the performance of the derived estimators in this paper for the true performance metrics. All the estimators are implemented based on (13) to combat the infinite moment problems.
In Fig. 4, $\text{MSE}_{\text{cr}}(\text{ZF})$ is presented for $B = 5$ and SNR during training equal to 0 dB. The derived optimal estimators in this paper are better than the MVU and MMSE estimators. We can see that the zeroth order approximations in this case are satisfactory even for a low SNR during training, in the sense that the corresponding optimal estimators outperform the MVU and MMSE estimators for the true performance metric. Additionally, the MVU estimator appears to be better than the MMSE estimator for this performance metric. This is yet a new snapshot contradicting what one would naturally expect.

Fig. 5 presents the corresponding results for $\text{MSE}_{\text{cr}}(\text{ZF})$. The MVU is better than the MMSE estimator, coinciding with the analysis based on the zeroth order approximation. Note however that the other two estimators appear to be better than the MVU. To obtain a well-behaved $\text{MSE}_{\text{cr}}(\text{ZF})$ in this case, regularization of the same form as in (15) is applied to $h$ to avoid values around zero. In this sense, Fig. 5 serves more as a proof that the application-oriented estimator selection is valid and less as an actual scenario present in the real world.

Furthermore, Fig. 6 shows the performance of all schemes in the case of the error probability performance metric. Monte Carlo simulations have been used to compute the actual error probability. All schemes coincide because the differences in the channel estimators are not so large to appear in the error probability. Nevertheless, these differences may clearly appear in a MIMO scenario if tight approximations of the error probability function are used to derive the corresponding channel estimators.

Finally, Figs. 7 and 8 demonstrate the validity of the Remark 3 in the end of subsection V-A. These plots correspond to Figs. 4 and 5 but with $E_{\text{cr}}^{(0)}[|h|^2] = 1/2$ and 1/6, respectively. They verify that the zeroth order metrics used in this paper are good approximations in terms of indicating the structure of uniformly better estimators than the MVU and MMSE. Nevertheless, the zeroth order metrics cannot really determine the best possible bias with respect to the MVU estimator that the estimators in this paper must have in order to yield the best possible performance against the true performance metrics. The bias terms are only optimal with respect to the zeroth order metrics.

VIII. CONCLUSIONS

In this paper, application-oriented channel estimator selection has been compared with common channel estimators such as the MVU and MMSE estimators. We have shown that the application-oriented selection is the right way to choose estimators in practice. We have verified this observation based on three different performance metrics of interest, namely, the symbol estimate MSE, the excess symbol estimate MSE and the error probability.

APPENDIX

This section proposes a simplification of the $\text{MSE}_{\text{cr}}(\text{ZF})$ metric for the estimator given in (15) with a fixed $\lambda$. Due to the Gaussianity of $y_{\text{tr}}$, $\text{MSE}_{\text{cr}}(\text{ZF}) = \infty$ for any $f \neq 0$ (infinite moment problem). Using (15), the corresponding mean square error becomes:

$$E \left[ \sigma_n^2 \left| \frac{f^H y_{\text{tr}}}{f^H y_{\text{tr}}} \right| + \frac{\sigma_n^2}{\lambda^2} \left| f^H y_{\text{tr}} \right|^2 \right] \rightarrow \text{Pr} \left\{ \left| f^H y_{\text{tr}} \right| > \lambda \right\}.$$

$E \left[ \sigma_n^2 \frac{1 - \frac{h}{f^H y_{\text{tr}}}}{f^H y_{\text{tr}}} \right]^2 \frac{\sigma_n^2}{\lambda^2} \left| f^H y_{\text{tr}} \right|^2 \frac{\sigma_n^2}{\lambda^2} \left| f^H y_{\text{tr}} \right|^2 \right] + \text{Pr} \left\{ \left| f^H y_{\text{tr}} \right| \leq \lambda \right\}.$

$$\frac{1}{\lambda^2} \left( \frac{\sigma_n^2}{\lambda^2} \left| f^H y_{\text{tr}} \right|^2 \frac{\sigma_n^2}{\lambda^2} \left| f^H y_{\text{tr}} \right|^2 \right) \rightarrow \frac{1}{\lambda^2} \left( \sigma_n^2 \left| f^H y_{\text{tr}} \right|^2 \frac{\sigma_n^2}{\lambda^2} \left| f^H y_{\text{tr}} \right|^2 \right).$$

(37)

The same holds even if $f^H y_{\text{tr}}$ is a biased estimator of $h$ at high training SNR and $|f^H y_{\text{tr}}|$ tends to concentrate around a value $\alpha$ bounded away from $|h|$ (and of course from 0).

To show the last claim, we set $X = \left| f^H y_{\text{tr}} \right| - \alpha^2$ and $Y = \left| f^H y_{\text{tr}} \right|^2$. Since $Y > \lambda^2$, it also holds that $E[Y] > \lambda^2$. Furthermore, it can be seen that

$$E \left[ \frac{X}{Y} - \frac{E[X]}{E[Y]} \right] \leq \frac{1}{\lambda^2} E \left[ \left| X E[Y] - Y E[X] \right| \right].$$

(38)

At high training SNR, $X \rightarrow E[X]$ and $Y \rightarrow E[Y]$ in the mean square sense, $E[X^2] \rightarrow E^2[X]$, $E[Y^2] \rightarrow E^2[Y]$ and $E[X Y] \rightarrow E[X E[Y]]$. The Cauchy-Schwarz inequality yields

$$\frac{1}{\lambda^4} E \left[ \left| X E[Y] - Y E[X] \right| \right] \leq \frac{1}{\lambda^4} \left( E \left[ X E[Y] - Y E[X] \right]^2 \right)^{1/2}$$

$$= \frac{1}{\lambda^4} \left( E^2[Y] E[X^2] + E[Y^2] E^2[X] - 2 E[X Y] E[X E[Y]] \right)^{1/2}.$$
$E[X|Y]$. For the last case, notice that

$$\|E[XY] - E[X]E[Y]\| = E[|X - E[X]| |Y - E[Y]|]$$

$$\leq E[|X - E[X]| |Y - E[Y]|]$$

$$\leq \sqrt{E[|X - E[X]|^2] E[|Y - E[Y]|^2]}$$

(40)

where the last inequality follows again from the Cauchy-Schwarz inequality. By the mean square convergence of $X$ to $E[X]$ and $Y$ to $E[Y]$ the right hand side of (40) tends to 0. Therefore, the right hand side of (39) tends to 0.

Furthermore, under the high SNR assumption the conditional expectations can be approximated by their unconditional ones, since for a sufficiently small $\lambda$ their difference is due to an event of probability $O(\lambda^2)$. Therefore,

$$E \left[ \sigma_x^2 \left( 1 - \frac{h}{f^H y_{tr}} \right)^2 + \frac{\sigma_x^2}{f^H y_{tr}^2} \right] E[|h^H y_{tr}| > \lambda] \approx$$

$$\frac{\sigma_x^2 E[|h^H y_{tr} - h|^2] + \sigma_y^2}{E[|h^H y_{tr}|^2]} + O(\lambda^2).$$

(41)

Combining all the above results yields

$$\text{MSE}_{x}^{de}(ZF) \approx \left\{ \frac{\sigma_x^2 E[|f^H y_{tr} - h|^2] + \sigma_y^2}{E[|h^H y_{tr}|^2]} \right\} + O(1).$$

(42)

The $O(1)$ term is not negligible but for sufficiently small $\lambda$ its dependence on $f$ is insignificant. Hence, for a sufficiently small $\lambda$ and a sufficiently high SNR during training, minimizing $\text{MSE}_{x}^{de}(ZF)$ is equivalent to minimizing the following approximation

$$\left. \text{MSE}_{x}^{de}(ZF) \right|_0 = \frac{E[|h - h_0|^2]}{E[|h|^2]} \sigma_x^2 + \frac{1}{E[|h|^2]}.$$ 

(43)

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