Small x Physics and Why It’s Interesting

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Abstract: I discuss small x physics and its implications at very high energies. At very high energy the density of partons becomes so large that much of the physics can be described using weak coupling methods in QCD. This may allow for a solution to problems such as the asymptotic nature of deep inelastic scattering at fixed $Q^2$, unitarization, and multiparticle production.

1 Some Old Problems We Would Like to Solve

Small x physics is the study of the wee parton structure of hadronic wavefunctions. If we could understand this aspect of hadrons, there are a variety of problems which we might understand:

1.1 Total Cross Sections

How do cross sections behave as the center of mass energy $E \to \infty$? There are a variety of arguments based on unitarity which suggest that the total $pp$ cross section at high energies behaves as $\sigma \sim ln^2(E)$. Is the coefficient for this $ln^2(E)$ behaviour universal and independent of what type of hadrons scatter? In deep inelastic scattering at fixed $Q^2$ how does the cross section behave at high energy? Is this related in any simple way to hadronic cross sections?

1.2 Cross Sections and Structure Functions at Small x

Before addressing this issue, we should define $x$. Let us work in a reference frame where a hadron has a very high longitudinal momentum $P_h$. Let the longitudinal momentum of a constituent be $p$. In this frame

$$x = p / P_h$$

(1)
It is of course possible to generalize this variable to a Lorentz invariant variable, but working in the large momentum frame is conceptually useful. We will do this generalization later.

Small $x$ is of course the region where $x \ll 1$. HERA measures the small $x$ part of parton structure functions.

We can look at this relationship for a component of the hadron wavefunction which has small longitudinal momentum, almost at rest. This hadron sees the valence part of the hadron wavefunction as Lorentz contracted to a scale size of order $x$.

In deep inelastic scattering, one measures the cross section for a photon of virtuality $Q^2$ to scatter from a hadron as a function of energy. Increasing the energy corresponds to making $x$ smaller. This cross section is proportional to the structure functions.

It is useful to introduce a rapidity variable

$$y = y_{\text{hadron}} - \ln(1/x)$$

The variable $y_{\text{hadron}}$ can be chosen to be the smallest value of $x$ which is measured in a given experiment. It is arbitrary in this context. If you like, it is the energy of the hadron in the large momentum frame.

The small $x$ problem is the following: If we plot a parton distribution function

$$\frac{dN}{dy} = x \frac{dN}{dx}$$

as a function of $y$, then at small $x$, $dN/dy$ grows, perhaps as fast as

$$dN/dy \sim \exp^{\kappa(y_{\text{hadron}} - y)}$$

The problem with this growth is it would seem that adding more constituents to a hadron would increase the cross section at fixed $Q^2$. This follows since the cross section is proportional to the structure functions. The question we must ask is: How can the rapid small $x$ growth seen at Hera be consistent with the slow growth of hadronic cross sections expected from general arguments based on unitarity?

### 1.3 Particle Production

How are parton distribution functions related to particles which are produced in hadron hadron interactions? We know the answer to this question for high transverse momentum jets which are produced in hadron-hadron collisions. What about the great majority of particles which are not produced at huge values of transverse momenta? Can we predict $dN/dy$ for produced hadrons as $y_{\text{proj}} \to \infty$?

What about fluctuations and correlations? For example, does measuring the two particle rapidity correlation function $dN/dy_1 dy_2$ tell us anything simple about the small $x$ hadron wavefunction? Is there a correlation between transverse momenta and the rapidity density which one can measure with $dN/dy dp_T$?
1.4 Asymptotic Nature of the Quark Sea

What is the intrinsic nature of the quark sea? At small $x$, what is the ratio of light quarks to glue? What is the ratio of heavy quarks to light? What is the intrinsic transverse momentum of constituents of the sea?

1.5 Initial Conditions for Heavy Ion Collisions

Typically in heavy ion collisions, we describe the matter after some formation time $t_{\text{formation}}$. This is supposedly the time at which particles are produced. At this time, the initial conditions for cascade and hydrodynamic simulations are formulated by some recipe. Can one determine these initial conditions from first principles?

Before this time there is of course the wavefunction for the hadron. If one understands this wavefunction, then surely one can determine the initial conditions.

At early times the problem is complicated by two types of coherence. The first is quantum mechanical. This is what forbids us to use cascade simulations up to $t = 0$ since in a cascade one simultaneously specifies the momentum and coordinates of a participant in the cascade. The second type of coherence is charge coherence. During the early stage of the collision, the density of the hadron constituents of the nuclear wavefunction is very large, $\sim \gamma$ where $\gamma$ is the nuclear Lorentz gamma factor. However, due to classical charge coherence, this high density cannot produce much effect. Because there is both positive and negative color charge, disturbances at large wavelength cannot be generated. This would not be the case in a cascade where charge coherence is ignored (all scattering is proportional to matrix elements squared).

1.6 Universality

Is there universal behaviour of cross sections at high energy? Are all measurable functions only of the local density of partons? If we let $R$ be the radius of the hadron under consideration, are all observables only functions of the local rapidity density per unit area

$$\Lambda^2 = \frac{1}{\pi R^2} \frac{dN}{dy}$$

1.7 UNIVERSALITY

Is the theory which describes the small x distribution functions a theory at a critical point? Are all observables determined by the universal behaviour of this theory? In the limit of $x \to 0$ are all observables therefore determined by the symmetries of the theory which describes these observables and UNIVERSALITY?
1.8 Claim

MOST (MAYBE ALL) OF THE ABOVE MAY BE POSSIBLE TO UN-
DERSTAND AS A RESULT OF THE HERA MEASUREMENTS OF
SMALL X DISTRIBUTION FUNCTIONS

We will argue below that the results of the HERA experiments imply that the
partons become very dense at small x. This requires that the theory of small x
distribution functions has a small strong coupling constant, \( \alpha_S \ll 1 \). If this is the
case, then we should be able to compute properties of hadrons at small x from first
principles in QCD.

2 A Theoretical Sidebar

Before proceeding further, we must develop more refined variables for describing the
physics at small x. These will be light cone coordinates and momenta. If we let the
energy be \( P^0 \) and longitudinal momentum be \( P^z \) then we define

\[
P^\pm = \frac{1}{\sqrt{2}}(P^0 \pm P^z)
\]

\[
X^\pm = \frac{1}{\sqrt{2}}(t \pm z)
\]

The dot product is

\[
P \cdot X = p_T \cdot x_T - P^+ X^- - P^- X^+
\]

We take the variable \( P^- \) to be an energy variable. The variable \( X^+ \) will therefore
be the time variable conjugate to \( P^- \). States of the theory will be eigenstates of \( P^- \).

The longitudinal momenta will be taken to be \( P^+ \). The longitudinal spatial
variable conjugate to \( P^+ \) will be \( X^- \).

The uncertainty principle requires that

\[
P^+ X^- \geq 1
\]

The x variable may now be defined in a Lorentz invariant way using light cone
coordinates. We take ratio of \( p^+ \) of a constituent to that of the hadron \( P^+_{hadron} \)

\[
x = \frac{p^+}{P^+_{hadron}}
\]

It is easy to check that this is Lorentz boost invariant.

Two useful variables are the momentum space rapidity \( y_{mom} \) and the space-
time rapidity \( y_{st} \). We will construct these variables from longitudinal momenta and
coordinates as

\[
y_{mom} = y_{hadron} - \ln(1/x)
\]
and

\[ y_{\text{st}} = y_{\text{hadron}} - \ln(x^- p^+_\text{hadron}) \]  

(11)

On account of the uncertainty principle, for the quantum mechanical state which describes small \( x \) physics, we expect that \( x^- \sim 1/p^+ \). Therefore at small \( x \) where \( y_{\text{mom}} \gg 1 \), we can take

\[ y_{\text{mom}} \sim y_{\text{st}} \]  

(12)

that is the momentum space and space-time rapidity must be equal up to a unit or so of rapidity. We will therefore use momentum space and space-time rapidity interchangeably and define

\[ y = y_{\text{st}} \]  

(13)

3 The Only Scale in the Problem

In hadron-hadron collisions, rapidity correlations are measured. These correlations are measured by

\[ \frac{dN}{dy_1dy_2} \]  

(14)

(The length of the correlation may grow slightly with energy.) It would be a miracle if there were correlations in the wavefunction which did not appear in the final state distribution of particles. We will therefore assume that the wavefunction is more or less locally defined in rapidity.

The only variable therefore which is local which can describe this part of the hadron wavefunction is \( \Pi \)

\[ \Lambda^2 = \frac{1}{\pi R^2} \frac{dN}{dy} \]  

(15)

Here \( R \) is the hadron radius. Since the gluon density grows at small \( x \), we have

\[ \lim_{x \to 0} \Lambda^2 \gg \Lambda_{\text{QCD}}^2 \]  

(16)

Therefore at small \( x \), the strong coupling constant becomes small \( \alpha_S(\Lambda) \ll 1 \).

We arrive at one consequence of this trivially. There should be universality of physical quantities in terms of the parton density per unit area.

Because QCD is almost a scale invariant theory, the intrinsic parton transverse momenta must grow at small \( x \)

\[ p_T^2 \sim \Lambda^2 \]  

(17)

Since the intrinsic scale of \( p_T \) grows, the sea must become flavor symmetric. Whenever \( \Lambda \gg m_{\text{quark}} \) the quark is essentially massless and should contribute to the wavefunction as any light quark would contribute.
4 Hadrons vs Nuclei

What is the fundamental difference between hadrons and nuclei at small x? Since everything depends only upon the parton density per unit area, there must be no fundamental difference.

There is a huge practical difference: The rapidity distribution is proportional to

\[ dN/dy \sim Ax^{-\delta} \]  

where \( \delta \sim 0.2 - 0.5 \). Therefore

\[ \Lambda^2 \sim A^{1/3}x^{-\delta} \]  

The gluon density at fixed x is much greater in a nucleus than in a hadron. We see that reducing x by 2 - 5 orders of magnitude corresponds to using lead instead of a proton!

If the universality of parton distribution functions at small x is established, then using a nucleus to study small x is more efficient than a proton. This is because to get the same physics, one must go to much smaller x in a proton, and therefore the energy of an accelerator must be much higher.

5 Transverse Momentum Broadening and Unitarity

We can now understand how the constraint of unitarity is satisfied in deep inelastic scattering.\[2\] The deep inelastic scattering cross section at fixed \( Q^2 \) is proportional to the parton density. This is the integral over all partons with intrinsic transverse momenta less than that \( Q \),

\[ xG(x, Q^2) = \int_0^{Q^2} d^2p_T \frac{dN}{dyd^2p_T} \]  

The quantity which is computed from QCD is the density of partons per unit area per unit \( d^2p_T \),

\[ \frac{1}{\pi R^2} \frac{dN}{dyd^2p_T} \]  

At large \( p_T \), the distribution of gluons is given by bremsstrahlung and should go as \( 1/p_T^2 \). There must be a coefficient of \( \Lambda^2 \) in order to have the correct dimensions. We can compute this coefficient in the McLerran-Venugopalan model.\[1\] In this model, the sea quark density follows the gluon density.

At smaller values of \( p_T \), the distribution softens, to something which is at most logarithmically varying in \( p_T \). On dimensional grounds, the \( \Lambda \) dependence must also be weak. The value of \( p_T \) where \( \Lambda^2/p_T^2 \) softens is \( p_T \sim \Lambda \).
We can now understand the x dependence of $xG(x, Q^2)$. Suppose that $Q^2 \gg \Lambda^2$. Then the integral above for $xG(x, Q^2)$ gets most of its contribution for $p_T \gg \Lambda$. $xG$ will be proportional to $\Lambda^2$ and this will increase rapidly as $x$ decreases. This happens until $Q^2 \leq \Lambda^2$. At this point the integral is dominated by the small $p_T$ region (but still much larger than $\Lambda_{QCD}$). It has a very slowly varying dependence on $\Lambda$. The fixed $Q^2$ cross section must therefore be slowly rising.

The physics of this is easy to understand. The small $x$ enhancement corresponds to adding more higher $p_T$ components to the hadron wavefunction. These components have smaller cross sections $\sim 1/p_T^2$. We are adding more constituents, but they are smaller and smaller so that there is no conflict with unitarity.

6 How Does a Parton See a Nucleus?

The gluon field which is responsible for generating the gluon density is produced by fluctuations in the color charge. These arise from sources with rapidity much greater than that at which we are measuring the field. Since the variation in the gluon field is generated by interactions, the typical length scale for variation in rapidity of the gluon density must be of order $1/\alpha_S$. This must also be the typical length scale over which charges affect the field. Since rapidity intervals of order $1/\alpha_S$ translate into momentum ratios of order $e^{1/\alpha_S}$, the typical sources which generate the gluon field must be Lorentz contracted down to infinitesimal width. Moreover since the sources move very close to the speed of light, they should be effectively recoileless.

The typical transverse momentum is $p_T \sim \Lambda$. This is $p_T \gg 1/R$. therefore the concept of a hadron size is well defined. Variations of the transverse shape of the distribution will be very small compared to the typical parton wavelength. We can effectively treat the nucleus as of infinite transverse extent.

The source which generates the gluon field is therefore sitting on an infinitesmally thin sheet of infinite transverse extent propagating at the speed of light.

Let us probe the distribution of color charge on the surface. On transverse resolution scales larger than a fermi, the sheet is neutral. On scales $\Delta x \ll 1Fm$, one resolves sources coming from individual nucleons. These come from sources which are far separated in rapidity, and to a first approximation at least should be uncorrelated. That is, the source is random and the fluctuations are controlled by a Gaussian weight.

On transverse scales $\Delta x \gg 1/\sqrt{\rho}$, where $\rho$ is the average color charge squared from all sources at rapidity greater than the rapidity at which we compute the field,

\[(\Delta x)^2 \rho = Q^2 \gg 1 \quad (22)\]

This means that the color charge is in a high dimensional representation of the color group and can be treated classically.

The theory which describes these fields is given by the path integral [1]

\[\int [dA][d\rho] e^{iS[A] + iA^- \rho - \frac{1}{2} \int dy d^2 x_T \rho^2(y,x_T)/\mu^2(y)} \quad (23)\]
where here \( \rho \) is the charge per unit area per unit rapidity. The parameter \( \mu^2(y) \) is the average charge squared per unit rapidity. The total charge squared at rapidities larger than that of interest is

\[
\chi = \int_y^{y_{\text{proj}}} dy' \mu^2(y') \tag{24}
\]

This theory enables one to compute the intrinsic \( p_T \) distribution. It has the features described above in the section about unitarity in deep inelastic scattering. The parameter \( \Lambda^2 \sim \chi \)

The question to ask is what determines \( \mu^2(y) \)? To understand this, we must recognize that the above theory is an effective theory valid only for rapidities close to those for which we measure the field. If we compute quantum corrections to this theory, we get big corrections of order \( \alpha_S Y \) where \( Y \) is a cutoff on the maximal allowed rapidity.

To compute the effective theory, we must integrate out the high rapidity modes. This can be done in weak coupling if we sequentially integrate out higher momentum modes to finally arrive at a low energy Lagrangean. This procedure is the Wilson-Kadanoff renormalization group. The Lagrangean above and \( \mu^2(y) \) are determined by this procedure. In fact higher order terms in \( \rho \) might be generated as well (although these are not important at \( p_T \sim \Lambda \)). All of these coefficients are determined by the renormalization group. Perhaps the entire dependence of the structure functions on rapidity may be determined by universality and the symmetries of this theory.

This renormalization group procedure at intermediate and large \( p_T \geq \Lambda \) is a linear equation. It reduces in various limits to the BFKL and DGLAP equations for the evolution of structure functions. At small \( p_T \), the evolution equations become non-linear and presumably saturate, that is, the distribution functions cease evolving.

### 7 What Does the Gluon Field Look Like?

The equations of motion for the gluon field are the non-abelian Yang-Mills equations in the presence of a source localized in \( x^- \). The solution is easy to construct. Suppose the field is a pure two dimensional gauge transform of vacuum field for \( x^- > 0 \) and another gauge transform for \( x^- < 0 \)

\[
A^i(x^-, x_T) = \theta(x^-)\alpha^i_1(x_T) + \theta(-x^-)\alpha^i_2(x_T) \tag{25}
\]

where

\[
\alpha^i_j(x_T) = iU_j(x_T)\nabla^i U^\dagger_j(x_T) \tag{26}
\]

We take \( A^\pm = 0 \).
At $x^- \neq 0$, the solution has zero field strength. The field strength is concentrated at $x^- = 0$. The discontinuity conditions and a boundary condition at say $x^- \to -\infty$ determine the $U_j$.

The only big field strength is $F^{\pm \pm}$. This field strength gives $E \perp B$ with both perpendicular to the longitudinal direction. These are the precise non-abelian analogs of the Lienard-Wiechert potentials and Weizsacker-Williams fields of electrodynamics for a fast moving source of electromagnetic charge.

8 Hadron-Hadron Scattering

From our knowledge of the non-abelian Lienard-Wiechert potentials, we can now construct solutions for the two hadron scattering problem. Initially we have two infinitesimal sheets of charge approaching one another at the speed of light. On either side of the hadrons and in the region between them we have three separate fields which are two dimensional gauge transforms of vacuum. At $t = 0$ there is a singularity of the equations of motion. This form of the solution no longer solves the equations of motions after the collision.

The solution which solves the equations of motion after $t = 0$, and the boundary conditions at $t = 0$ is easy to construct. If we ignore a weak dependence on rapidity, the solution is

$$A^\pm = x^\pm \beta^\pm (\tau, x_T)$$
$$A^i = \beta^i (\tau, x_T)$$

Here $\tau = \sqrt{t^2 - x^2}$. This is the solution in the forward light cone.

This solution has the property that distributions of particles are Lorentz invariant. It inevitably leads to hydrodynamics equations of the form discussed by Bjorken.

At early times, the equations are non-linear. The initial fields are processed by these non-linearities. At large $\tau \gg 1/\Lambda$, the equations linearize. This is because the form of these fields describes an expanding matter distribution. When the equations become linear, one has plane waves of gluons. These form the initial conditions for a cascade calculation. There is a high density of weakly interacting gluons.

The initial energy density at the time $\tau \sim 1/\Lambda$ is

$$\epsilon \sim \Lambda^4 \sim A^{2/3}$$

At LHC energies, we will see that this corresponds to about 100 $GeV/Fm^3$.

9 Conclusions

M. Gyulassy and I have recently estimated reasonable numbers for the parameter $\chi$, the charge squared per unit rapidity at rapidities greater than the center of mass rapidity for hadron-hadron scattering. This parameter sets the scale of momentum.
We find that
\[ \chi = \left( \frac{A}{200} \right)^{1/3} \left( \frac{10^{-2}}{x} \right)^{\delta} (1\text{GeV})^2 \]  
where \( \delta \sim 0.2 \). This follows from the Gluck-Reya-Vogt parameterization of structure functions.\[7\]

This corresponds to 300-400 MeV at RHIC energies and about 1 GeV at LHC energies. Neither of these numbers is large enough so that a a weak coupling treatment will be absolutely reliable. It should be semi-quantitative at LHC, and perhaps qualitative at RHIC.

If the ideas above can be tested semi-quantitatively and qualitatively, then one will be confident one understands the physics. Then at least at asymptotically high energies, one will have a fundamental understanding of hadronic interactions.

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