The inhomogeneous Cauchy-Riemann equation for weighted smooth vector-valued functions on strips with holes

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Abstract
This paper is dedicated to the question of surjectivity of the Cauchy-Riemann operator \( \overrightarrow{\partial} \) on spaces \( \mathcal{E}(\Omega, E) \) of \( C^\infty \)-smooth vector-valued functions whose growth on strips along the real axis with holes \( K \) is induced by a family of continuous weights \( \mathcal{V} \). Vector-valued means that these functions have values in a locally convex Hausdorff space \( E \) over \( \mathbb{C} \). We derive a counterpart of the Grothendieck-Köthe-Silva duality \( \mathcal{O}(\mathbb{C} \setminus K)/\mathcal{O}(\mathbb{C}) \cong \mathcal{A}(K) \) with non-empty compact \( K \subset \mathbb{R} \) for weighted holomorphic functions. We use this duality and splitting theory to prove the surjectivity of \( \overrightarrow{\partial} : \mathcal{E}(\Omega, E) \to \mathcal{E}(\Omega, E) \) for certain \( E \).

This solves the smooth (holomorphic, distributional) parameter dependence problem for the Cauchy-Riemann operator on \( \mathcal{E}(\Omega, \mathbb{C}) \).

Keywords Cauchy-Riemann · Parameter dependence · Weight · Smooth · Solvability · Vector-valued

Mathematics subject classification 35A01 · 35B30 · 32W05 · 46A63 · 46A32 · 46E40

1 Introduction

The smooth (holomorphic, distributional) parameter dependence problem for the Cauchy-Riemann operator \( \overrightarrow{\partial} := (1/2)(\partial_1 + i\partial_2) \) on the space \( C^\infty(\Omega) \) of smooth complex-valued functions on an open set \( \Omega \subset \mathbb{R}^2 \) is whether for every family \( (f_\lambda)_{\lambda \in U} \) in \( C^\infty(\Omega) \) depending smoothly (holomorphically, distributionally) on a parameter \( \mathcal{E}(\Omega, E) \) there is a family \( (u_\lambda)_{\lambda \in U} \) in \( C^\infty(\Omega) \) with the same kind of parameter dependence such that

\[
\overrightarrow{\partial} u_\lambda = f_\lambda, \quad \lambda \in U.
\]

Here, smooth (holomorphic, distributional) parameter dependence of \( (f_\lambda)_{\lambda \in U} \) means that the map \( \lambda \mapsto f_\lambda(x) \) is an element of \( C^\infty(U) \) (of the space of holomorphic functions \( \mathcal{O}(U) \) on \( U \subset \mathbb{C} \) open, the space of distributions \( \mathcal{D}(V)' \) for open \( V \subset \mathbb{R}^d \) where \( U = \mathcal{D}(V) \)) for each \( x \in \Omega \).

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The parameter dependence problem for a variety of partial differential operators on several spaces of (generalised) differentiable functions has been extensively studied, see e.g. [4, 6, 7, 16, 31, 32] and the references and background in [3, 22]. The answer to this problem for the Cauchy-Riemann operator is affirmative since the Cauchy-Riemann operator

\[ \overline{\partial}^E : C^\infty(\Omega, E) \to C^\infty(\Omega, E) \]  

(1)
on the space \( C^\infty(\Omega, E) \) of \( E \)-valued smooth functions is surjective if \( E = C^\infty(U) \) (\( \mathcal{O}(U) \), \( \mathcal{D}(V') \)) by [8, Corollary 3.9, p. 1112] which is a consequence of the splitting theory of Bonet and Domaniński for PLS-spaces [3, 4], the topological isomorphy of \( C^\infty(\Omega, E) \) to Schwartz' \( \varepsilon \)-product \( C^\infty(\Omega) \circ E \) and the fact that \( \overline{\partial} : C^\infty(\Omega) \to C^\infty(\Omega) \) is surjective on the nuclear Fréchet space \( C^\infty(\Omega) \) (with its usual topology). More generally, the Cauchy-Riemann operator (1) is surjective if \( E \) is a Fréchet space by Grothendieck’s classical theory of tensor products [12] or if \( E = F_b' \) where \( F \) is a Fréchet space satisfying the condition \( (DN) \) by [31, Theorem 2.6, p. 174] or if \( E \) is an ultrabornological PLS-space having the property \( (PA) \) by [8, Corollary 3.9, p. 1112] since the space \( \mathcal{O}(\Omega) \) of \( C \)-valued holomorphic functions on \( \Omega \), i.e. the kernel of \( \overline{\partial} \), has the property \( (\Omega) \) by [31, Proposition 2.5 (b), p. 173]. The first and the last result cover the case that \( E = C^\infty(U) \) or \( \mathcal{O}(U) \) whereas the last covers the case \( E = \mathcal{D}(V)' \) as well. More examples of the second or third kind of such spaces \( E \) are arbitrary Fréchet–Schwartz spaces, the space \( S(\mathbb{R}^d)' \) of tempered distributions, the space \( \mathcal{D}(V)' \) of distributions, the space \( \mathcal{D}_{(n)}(V)' \) of ultradistributions of Beurling type and some more (see [4, 8, Corollary 4.8, p. 1116] and [22, Example 3, p. 7]).

In the present paper we consider the Cauchy-Riemann operator on weighted spaces \( \mathcal{E}\mathcal{V}(\Omega, E) \) of smooth \( E \)-valued functions where \( E \) is a locally convex Hausdorff space over \( \mathbb{C} \) with a system of seminorms \( (p_\gamma)_{\gamma \in \mathfrak{N}} \) generating its topology. These spaces consist of functions \( f \in C^\infty(\Omega, E) \) fulfilling additional growth conditions induced by a family \( \mathcal{V} := (v_n)_{n \in \mathbb{N}} \) of continuous functions \( v_n : \Omega \to (0, \infty) \) on a sequence of open sets \( (\Omega_n)_{n \in \mathbb{N}} \) with \( \Omega = \bigcup_{n \in \mathbb{N}} \Omega_n \) given by the constraint

\[ |f|_{n,m,\alpha} := \sup_{\gamma \in \mathfrak{N}_0} \left| \partial_\gamma^\gamma f(x) \right| v_n(x) < \infty \]

for every \( n \in \mathbb{N}, m \in \mathbb{N}_0 \) and \( \alpha \in \mathfrak{N} \), where \( (\partial_\gamma^\gamma)^\gamma f \) denotes the partial derivative of \( f \) w.r.t. the multi-index \( \gamma \). Our main goal is to derive sufficient conditions on \( \mathcal{V} \) and \( (\Omega_n)_{n \in \mathbb{N}} \) such that

\[ \overline{\partial}^E : \mathcal{E}\mathcal{V}(\Omega, E) \to \mathcal{E}\mathcal{V}(\Omega, E) \]

is surjective. We recall the main result of [22], which sets the course of the present paper.

**Theorem 1** [22, Theorem 5, p. 7-8] Let \( \mathcal{E}\mathcal{V}(\Omega) \) be a Schwartz space and \( \mathcal{E}\mathcal{V}_b(\Omega) \) a nuclear subspace satisfying property \( (\Omega) \). Assume that the \( C \)-valued operator \( \overline{\partial} : \mathcal{E}\mathcal{V}(\Omega) \to \mathcal{E}\mathcal{V}(\Omega) \) is surjective. Moreover, if

(a) \( E := F_b' \) where \( F \) is a Fréchet space over \( \mathbb{C} \) satisfying \( (DN) \), or
(b) \( E \) is an ultrabornological PLS-space over \( \mathbb{C} \) satisfying \( (PA) \),
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then

\[ \overline{\partial}^\pm : \mathcal{EV}(\Omega, E) \rightarrow \mathcal{EV}(\Omega, E) \]

is surjective.

Here \( \mathcal{EV}(\Omega) := \mathcal{EV}(\Omega, \mathbb{C}) \) and \( \mathcal{EV}_{\partial}(\Omega) \) is the kernel of \( \overline{\partial} \) in \( \mathcal{EV}(\Omega) \), i.e. its topological subspace

\[ \mathcal{EV}_{\partial}(\Omega) := \mathcal{O}(\Omega) \cap \mathcal{EV}(\Omega) \]

consisting of holomorphic functions.

We restrict to the case where the sequence \( (\Omega_n)_{n \in \mathbb{N}} \) is given by strips \( \Omega_n := S_n(K) \) along the real axis with holes around a compact set \( K \subset [-\infty, \infty] =: \mathbb{R} \), i.e. for \( t \in \mathbb{R}, t \geq 1 \), we define

\[ S_t(K) := \left( \mathbb{C} \setminus \overline{U_t(K)} \right) \cap \{ z \in \mathbb{C} \mid |\text{Im}(z)| < t \}, \quad t > 1, \quad \text{and} \quad S_1(K) := S_{3/2}(K), \]

where the closure is taken in \( \mathbb{C} \) and the open sets \( U_t(K) \) are given.

Fig. 1 \( U_t(K) \) for \( \pm \infty \in K \) (c.f. [19, Figure 3.1, p. 11])

Fig. 2 \( S_t(K) \) for \( \pm \infty \in K \) (c.f. [19, Figure 3.2, p. 12])
\[ U_t(K) := \{ z \in \mathbb{C} \mid d([z], K \cap \mathbb{C}) < 1/t \} \]

\[ \bigcup \begin{cases} \emptyset, & K \subset \mathbb{R}, \\ (t, \infty) + i(-1/t, 1/t), & \infty \in K, -\infty \notin K, \\ (-\infty, -t) + i(-1/t, 1/t), & \infty \notin K, -\infty \in K, \\ ((-\infty, -t) \cup (t, \infty)) + i(-1/t, 1/t), & \pm \infty \in K, \end{cases} \]

where \( d([z], K \cap \mathbb{C}) \) denotes the Euclidean distance of the sets \( \{z\} \) and \( K \cap \mathbb{C} \) (see Figs. 1, 2). We note that \( \bigcup_{n \in \mathbb{N}} S_n(K) = \mathbb{C} \setminus K \) and the definition of \( S_1(K) \) is motivated by \( (\mathbb{C} \setminus U_1(\overline{\mathbb{R}})) \cap \{ z \in \mathbb{C} \mid |\text{Im}(z)| < 1 \} = \emptyset \). As a further simplification we only consider weights of the form \( \nu(x, z) := \exp(a_n |\text{Re}(z)|^\beta), z \in \mathbb{C}, \) for all \( n \in \mathbb{N} \) for some \( 0 < \beta \leq 1 \) and some strictly increasing real sequence \( (a_n)_{n \in \mathbb{N}} \), and in combination with \( \Omega_n := S_n(K), n \in \mathbb{N} \), we fix the notation

\[ \mathcal{E}^\theta_{(a_n), \partial}(\overline{\mathbb{C}} \setminus K, E) := \mathcal{E} \mathcal{V}(\mathbb{C} \setminus K, E) \quad \text{and} \quad \mathcal{E}^\theta_{(a_n), \partial}(\overline{\mathbb{C}} \setminus K) := \mathcal{E} \mathcal{V}^\partial(\mathbb{C} \setminus K) \]

with \( \overline{\mathbb{C}} := \mathbb{R} + i\mathbb{R} \). In the case \( \beta = 1 \) these spaces are of interest because they are the basic spaces for the theory of vector-valued Fourier hyperfunctions, see e.g. [13–15, 17, 19, 24].

Looking at Theorem 1, the main obstacle is to prove that \( \mathcal{E}^\theta_{(a_n), \partial}(\overline{\mathbb{C}} \setminus K) \) satisfies property (\( \Omega \)). In [22, Corollary 14, p. 18] this is accomplished for \( K = \emptyset \) and sequences \( (a_n)_{n \in \mathbb{N}} \) such that \( a_n \not\to 0 \) or \( a_n \not\to \infty \). We will use this result and extend it to the case that \( K \subset \mathbb{R} \) is a non-empty compact set.

Let us summarise the content of our paper. In Sect. 2 we recall necessary definitions and preliminaries which are needed in the subsequent sections. Sect. 3 is dedicated to a counterpart for weighted holomorphic functions of the Silva-Köthe-Grothendieck duality

\[ \mathcal{O}(\mathbb{C} \setminus K)/\mathcal{O}(\mathbb{C}) \cong \mathcal{A}(K)_b' \]

where \( K \subset \mathbb{R} \) is a non-empty compact set and \( \mathcal{A}(K) \) is the space of germs of real analytic functions on \( K \) (see Theorem 11, Corollary 13, Corollary 15). In Sect. 4 we use this duality to show that \( \mathcal{E}^\theta_{(a_n), \partial}(\overline{\mathbb{C}} \setminus K) \) satisfies property (\( \Omega \)) under some restrictions on \( K \), or on \( (a_n)_{n \in \mathbb{N}} \) and \( \beta \) (see Corollary 19). The preceding conditions on \( K \), or on \( (a_n)_{n \in \mathbb{N}} \) and \( \beta \) are used in Theorem 20 to show that \( \overline{\partial}^E : \mathcal{E}^\theta_{(a_n), \partial}(\overline{\mathbb{C}} \setminus K, E) \to \mathcal{E}^\theta_{(a_n), \partial}(\overline{\mathbb{C}} \setminus K, E) \) is surjective if \( E := F_b' \)

where \( F \) is a Fréchet space over \( \mathbb{C} \) satisfying (\( DN \)), or if \( E \) is an ultrabornological PLS-space over \( \mathbb{C} \) satisfying (\( PA \)).

### 2 Notation and preliminaries

The notation and preliminaries are essentially the same as in [22, 23, Sect. 2]. We define the distance of two subsets \( M_0, M_1 \subset \mathbb{R}^2 \) w.r.t. the Euclidean norm \(| \cdot | \) on \( \mathbb{R}^2 \) via

\[ d(M_0, M_1) := \begin{cases} \inf_{x \in M_0, y \in M_1} |x - y|, & M_0, M_1 \neq \emptyset, \\ \infty, & M_0 = \emptyset \text{ or } M_1 = \emptyset. \end{cases} \]

Moreover, we denote by \( B_r(x) := \{ w \in \mathbb{R}^2 \mid |w - x| < r \} \) the Euclidean ball around \( x \in \mathbb{R}^2 \) with radius \( r > 0 \) and identify \( \mathbb{R}^2 \) and \( \mathbb{C} \) as (normed) vector spaces. We denote the closure of a subset \( M \subset \mathbb{R}^2 \) by \( \overline{M} \), the boundary of \( M \) by \( \partial M \) and for a function \( f : M \to \mathbb{C} \)
and $K \subset M$ we denote by $f|_K$ the restriction of $f$ to $K$. We write $C(\Omega)$ for the space of continuous $\mathbb{C}$-valued functions on a set $\Omega \subset \mathbb{R}^2$ and $L^1(\Omega)$ for the space of (equivalence classes of) $\mathbb{C}$-valued Lebesgue integrable functions on a measurable set $\Omega \subset \mathbb{R}^2$.

By $E$ we always denote a non-trivial locally convex Hausdorff space over the field $\mathbb{C}$ equipped with a directed fundamental system of seminorms $(p_a)_{a \in \mathcal{A}}$. If $E = \mathbb{C}$, then we set $(p_a)_{a \in \mathcal{A}} := \{| \cdot |\}$. Further, we denote by $L(F, E)$ the space of continuous linear maps from a locally convex Hausdorff space $F$ to $E$ and sometimes use the notation $(T, f) := T(f)$, $f \in F$, for $T \in L(F, E)$. If $E = \mathbb{C}$, we write $F' := L(F, \mathbb{C})$ for the dual space of $F$. If $F$ and $E$ are (linearly topologically) isomorphic, we write $F \cong E$. We denote by $L_0(F, E)$ the space $L(F, E)$ equipped with the locally convex topology of uniform convergence on the bounded subsets of $F$.

We recall that a function $f : \Omega \to E$ on an open set $\Omega \subset \mathbb{C}$ to $E$ is called holomorphic if the limit

$$\left( \frac{\partial}{\partial z} \right)^E f(z_0) := \lim_{\substack{h \to 0 \\ h \in \mathbb{C}, h \neq 0}} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists in $E$ for every $z_0 \in \Omega$. The linear space of all functions $f : \Omega \to E$ which are holomorphic is denoted by $\mathcal{O}(\Omega, E)$. For a compact set $K \subset \mathbb{R}$, $0 < \beta \leq 1$ and a strictly increasing real sequence $(a_n)_{n \in \mathbb{N}}$ we set

$$\mathcal{O}^\beta(a_n)(\overline{\mathbb{C} \setminus K}, E) := \{ f \in \mathcal{O}(\mathbb{C} \setminus K, E) \mid \forall n \in \mathbb{N}, \alpha \in \mathcal{A} : |f|_{n, \alpha} < \infty \}$$

where

$$|f|_{n, \alpha} := \sup_{z \in S_n(K)} p_a(f(z)) e^{a_n |\Re(z)|^\beta}.$$ 

The subscript $\alpha$ in the notation of the seminorms is omitted in the $\mathbb{C}$-valued case and we write $\mathcal{O}^\beta(a_n)(\overline{\mathbb{C} \setminus K}) := \mathcal{O}^\beta(a_n)(\overline{\mathbb{C} \setminus K}, \mathbb{C})$.

**Remark 2** We have $\mathcal{O}^\beta(a_n)(\overline{\mathbb{C} \setminus K}) = \mathcal{E}^\beta(a_n)(\overline{\mathbb{C} \setminus K})$ as Fréchet spaces by [22, Proposition 7 (b), p. 11] and [22, Example 6, p. 11].

Throughout the rest of the paper we make the following standing assumptions.

**Assumption 3**

(i) $E$ is sequentially complete,

(ii) $K \subset \mathbb{R}$ is a non-empty compact set,

(iii) $0 < \beta \leq 1$,

(iv) $(a_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence with $a_n < 0$ for all $n \in \mathbb{N}$ or $a_n \geq 0$ for all $n \in \mathbb{N}$, and $\lim_{n \to \infty} a_n = 0$ or $\lim_{n \to \infty} a_n = \infty$.

### 3 Duality

We recall the well-known topological Silva-Köthe-Grothendieck isomorphy
where $E$ is a quasi-complete locally convex Hausdorff space, $\emptyset \neq K \subset \mathbb{R}$ is compact, $\mathcal{O}(\mathbb{C} \setminus K, E)$ is equipped with the topology of uniform convergence on compact subsets of $\mathbb{C} \setminus K$, the quotient space with the induced quotient topology and $\mathcal{A}(K)$ is the space of germs of real analytic functions on $K$ with its inductive limit topology (see e.g. [29, p. 6], [11, Proposition 1, p. 46], [18, §27.4, p. 375-378], [27, Theorem 2.1.3, p. 25]). The aim of this section is to prove a counterpart of this isomorphy for weighted vector-valued holomorphic functions and non-empty compact $K \subset \overline{\mathbb{R}}$.

The spaces $\mathcal{O}^\beta_{\alpha_n}(\overline{\mathbb{C}} \setminus K, E)$ play the counterpart of $\mathcal{O}(\mathbb{C} \setminus K, E)$ for our version of the isomorphy (2). Next, we introduce the counterparts of $\mathcal{A}(K)$. Let $\Omega \subset \mathbb{C}$ be open and $f \in \mathcal{O}(\Omega)$. For $z \in \Omega$ and $n \in \mathbb{N}_0$ we denote the point evaluation of the $n$th complex derivative at $z$ by $\delta^{(n)}_z f := f^{(n)}(z)$.

**Proposition 4** For $n \in \mathbb{N}$ let

$$\mathcal{O}^\beta_{\alpha_n} \left( \overline{U_n(K)} \right) := \{ f \in \mathcal{O}(U_n(K)) \cap C\left( \overline{U_n(K)} \right) \mid \| f \|_{K,n} := \| f \|_n < \infty \}$$

where

$$\| f \|_{K,n} := \| f \|_n := \sup_{z \in U_n(K)} | f(z) | e^{-\alpha_n | \Re(z) |^\theta}$$

and the spectral maps for $n, k \in \mathbb{N}, n \leq k$, are given by the restrictions

$$\pi_{n,k} : \mathcal{O}^\beta_{\alpha_n} \left( \overline{U_n(K)} \right) \to \mathcal{O}^\beta_{\alpha_k} \left( \overline{U_k(K)} \right), \quad \pi_{n,k}(f) := f|_{U_k(K)}.$$

Then the following assertions hold.

(a) The inductive limit

$$\mathcal{O}^\beta_{\alpha_n}(K) := \lim_{n \to \mathbb{N}} \mathcal{O}^\beta_{\alpha_n} \left( \overline{U_n(K)} \right)$$

exists and is a DFS-space.

(b) The span of the set of point evaluations of complex derivatives $\{ \delta^{(n)}_{x_0} \mid x_0 \in K \cap \mathbb{R}, n \in \mathbb{N}_0 \}$ is dense in $\mathcal{O}^\beta_{\alpha_n}(K)$ if $K \subset \mathbb{H}$ or $K \cap \{ \pm \infty \}$ contains no isolated points in $K$.

**Proof** (a) It is easy to see that $\mathcal{O}^\beta_{\alpha_n}(U_n(K))$ is a Banach space for every $n \in \mathbb{N}$. Further, the maps $\pi_{n,m} : \mathcal{O}^\gamma_{\alpha_n} \left( \overline{U_n(K)} \right) \to \mathcal{O}^\beta_{\alpha_m} \left( \overline{U_m(K)} \right), n \leq m$, are injective by virtue of the identity theorem and the definition of sets $U_n(K)$. Thus the considered spectrum is an embedding spectrum. For all $M \subset U_n(K)$ compact and $f \in B_n := \{ g \in \mathcal{O}^\beta_{\alpha_n} \left( \overline{U_n(K)} \right) \mid \| g \|_n \leq 1 \}$ we have

$$\| f \|_M := \sup_{z \in M} | f(z) | = \sup_{z \in M} | f(z) | e^{-\alpha_n | \Re(z) |^\theta} e^{\alpha_n | \Re(z) |^\theta} \leq \sup_{z \in M} e^{\alpha_n | \Re(z) |^\theta} \| f \|_n \leq \sup_{z \in M} e^{\alpha_n | \Re(z) |^\theta}.$$
Thus $B_n$ is bounded in $\mathcal{O}(U_n(K))$ w.r.t. the system of seminorms generated by $\| \cdot \|_{\mathcal{A}'}$ for compact $M \subset U_n(K)$. As this space is a Fréchet-Montel space, $B_n$ is relatively compact and hence relatively sequentially compact in $\mathcal{O}(U_n(K))$.

What remains to be shown is that for all $n \in \mathbb{N}$ there exists $m > n$ such that $\pi_{n,m}$ is a compact map. Because the considered spaces are Banach spaces, it suffices to prove the existence of $m > n$ such that $(\pi_{n,m}(f_k))_{k \in \mathbb{N}}$ has a convergent subsequence in $\mathcal{O}_{a_{\mathcal{A}}}^{-\beta}(U_m(K))$ for every sequence $(f_k)_{k \in \mathbb{N}}$ in $B_n$. We choose $m := 2n$. For $\varepsilon > 0$ we set

$$Q := U_{2n}(K) \cap \{ z \in \mathbb{C} \mid \| \Re(z) \| \leq \max(0, \ln(\varepsilon)/(a_n - a_{2n})) + n \},$$

and get

$$\sup_{z \in U_{2n}(K) \setminus Q} e^{-a_{2n}|\Re(z)|^\beta} = \sup_{z \in U_{2n}(K) \setminus Q} e^{(a_n - a_{2n})|\Re(z)|^\beta} \leq \varepsilon. \quad (3)$$

Thus condition (RU) in [2, p. 67] is fulfilled and it follows analogously to the proof of [2, Theorem (b), p. 67-68] that every sequence $(f_k)_{k \in \mathbb{N}}$ in $B_n$ has a subsequence $(f_{k_n})_{n \in \mathbb{N}}$ such that $(\pi_{n,2n}(f_{k_n}))_{n \in \mathbb{N}}$ converges in $\mathcal{O}_{a_{\mathcal{A}}}^{-\beta}(U_{2n}(K))$, proving the compactness of $\pi_{n,2n}$. Hence the inductive limit $\mathcal{O}_{(a_{\mathcal{A}})}^{-\beta}(K)$ exists and is a DFS-space by [25, Proposition 25.20, p. 304].

(b) We set $F := \text{span}\{ \delta_{x_0}^{(n)} \mid x_0 \in K \cap \mathbb{R}, n \in \mathbb{N}_0 \}$. Let $x_0 \in K \cap \mathbb{R}$ and $n \in \mathbb{N}_0$. It follows from Cauchy's inequality that $\delta_{x_0}^{(n)}$ is continuous on $\mathcal{O}_{a_{\mathcal{A}}}^{-\beta}(U_{(a_{\mathcal{A}})}(K))$ for all $k \in \mathbb{N}$, implying $F \subset \mathcal{O}_{(a_{\mathcal{A}})}^{-\beta}(K)'$. As $\mathcal{O}_{(a_{\mathcal{A}})}^{-\beta}(K)$ is a DFS-space by part (a), it is reflexive by [25, Proposition 25.19, p. 303], which means that the canonical embedding $J : \mathcal{O}_{(a_{\mathcal{A}})}^{-\beta}(K) \to (\mathcal{O}_{(a_{\mathcal{A}})}^{-\beta}(K)'_{b})'_{b}$ is a topological isomorphism. We consider the polar set of $F$, i.e.

$$F^o := \{ y \in (\mathcal{O}_{(a_{\mathcal{A}})}^{-\beta}(K)')_{b} \mid \forall T \in F : y(T) = 0 \}.$$ 

Let $y \in F^o$. Then there is $f \in (\mathcal{O}_{(a_{\mathcal{A}})}^{-\beta}(K)')_{b}$ such that $y = J(f)$. For $T := \delta_{x_0}^{(n)} \in F$

$$0 = y(T) = J(f)(T) = T(f) = f^{(n)}(x_0)$$

is valid for any $n \in \mathbb{N}_0$. Thus $f$ is identical to zero on a neighbourhood of $x_0$ (by Taylor series expansion) since $f$ is holomorphic near $x_0 \in U_n(K)$. Due to the assumptions every component of $U_n(K)$ contains a point $x_0 \in K \cap \mathbb{R}$ so $f$ is identical to zero on $U_n(K)$ by the identity theorem and continuity, yielding to $y = 0$. Therefore $F^o = \{ 0 \}$ and thus $F$ is dense in $\mathcal{O}_{(a_{\mathcal{A}})}^{-\beta}(K)'_{b}$ by the bipolar theorem.

In the case $\beta := 1$ and $a_{\mathcal{A}} := -1/n$ for all $n \in \mathbb{N}$ the spaces $\mathcal{O}_{(a_{\mathcal{A}})}^{-1}(K)$ play an essential role in the theory of Fourier hyperfunctions and it is already mentioned in [17, p. 469] resp. proved in [15, 1.11 Satz, p. 11] and [19, 3.5 Theorem, p. 17] that they are DFS-spaces.

**Remark 5** If $K \subset \mathbb{R}$, then $\mathcal{O}_{(a_{\mathcal{A}})}^{-\beta}(K) \cong \mathcal{S}(K)$.

Now, we take a closer look at the sets $U_n(K)$ (c.f. [19, 3.3 Remark, p. 13]).

**Remark 6** Let $t \in \mathbb{R}, t \geq 1$. 

\[ \square \]
(a) The set $U_i(K)$ has finitely many components.
(b) Let $Z$ be a component of $U_i(K)$. We define $a := \min(Z \cap K)$ and $b := \max(Z \cap K)$ if existent (in $\mathbb{R}$).

(i) If $Z$ is bounded, there exists $0 < R \leq 1/t$ such that for all $0 < r \leq R$:
$$\{z \in \mathbb{C} \mid d([z], [a, b]) < r\} \subset Z$$
(ii) If $Z \cap \mathbb{R}$ is bounded from below and unbounded from above and $a$ exists, there exists
$$0 < R \leq 1/t$$
such that for all $0 < r \leq R$:
$$\{z \in \mathbb{C} \mid d([z], [a, \infty)) < r\} \subset Z$$
(iii) If $Z \cap \mathbb{R}$ is bounded from above and unbounded from below and $b$ exists, there exists
$$0 < R \leq 1/t$$
such that for all $0 < r \leq R$:
$$\{z \in \mathbb{C} \mid d([z], (-\infty, b)) < r\} \subset Z$$
(iv) If $Z \cap \mathbb{R}$ is unbounded from below and above, there exists $0 < R \leq 1/t$ such that for all $0 < r \leq R$:
$$\{z \in \mathbb{C} \mid d([z], \mathbb{R}) < r\} \subset Z$$
(v) If $Z \cap \mathbb{R}$ is bounded from below and unbounded from above and $a$ does not exist, then
$$Z = (t, \infty) + i(-1/t, 1/t).$$
If $Z \cap \mathbb{R}$ is bounded from above and unbounded from below and $b$ does not exist, then $Z = (-\infty, -t) + i(-1/t, 1/t)$.

**Proof** (a) We only consider the case $\infty \in K, -\infty \notin K$. Let $(Z_j)_{j \in J}$ denote the (pairwise disjoint) components of $U_i(K)$. Then $U_i(K) = \bigcup_{j \in J} Z_j$ and by definition of a component there is $k \in J$ such that $Z_k$ is the only component including $(t, \infty) + i(-1/t, 1/t)$. Furthermore there exists $m \in \mathbb{R}$ with $\bigcup_{j \in J \setminus \{k\}} (Z_j \cap \mathbb{R}) \subset \{m, t\}$ by assumption. For $j \neq k$ the length $\lambda(Z_j \cap \mathbb{R})$ of the interval $Z_j \cap \mathbb{R}$, where $\lambda$ denotes the Lebesgue measure, is estimated from below by $\lambda(Z_j \cap \mathbb{R}) \geq 2/t$ by definition of $U_i(K)$. Since all $Z_j$ are pairwise disjoint, this implies that $J$ has to be finite. The others cases follow analogously.

(b)(i) Since $Z \cap K$ is closed in $\mathbb{R}$ and therefore compact, $a$ and $b$ exist. Hence $[a, b] \subset Z$ by the definition of $U_i(K)$ and as $Z$ is connected, $[a, b]$ being a compact subset of the open set $Z$ implies that there is $0 < R < 1/t$ such that $([a, b] + i(-R, R)) \subset Z$ by the tube lemma, which completes the proof.

(ii) If $Z \cap K \cap (-\infty, t] \neq \emptyset$, then $a$ exists and analogously to (i) there exists $0 < R < 1/t$ such that for all $0 < r \leq R$
$$\{z \in \mathbb{C} \mid d([z], [a, t]) < r\} \subset Z.$$ By definition of $U_i(K)$ this brings forth $\{z \in \mathbb{C} \mid d([z], [a, \infty)) < r\} \subset Z$. If $Z \cap K \cap (-\infty, t] = \emptyset$ and $a$ exists, the desired $0 < R < 1/t$ exists by the definition of $U_i(K)$ since $t \notin Z \cap K$ and $Z \cap K$ is closed in $\mathbb{R}$, which implies $d([t], Z \cap K) > 0$. 

(iii) Analogously to (ii).

(iv) By the assumptions $Z \cap K \cap [-t, t] \neq \emptyset$. Analogously to (i) there exists $0 < R < 1/t$ such that for all $0 < r \leq R$
$$\{z \in \mathbb{C} \mid d([z], [-t, t]) < r\} \subset Z.$$ Like in (ii) and (iii) this brings forth $\{z \in \mathbb{C} \mid d([z], \mathbb{R}) < r\} \subset Z$.

(v) This follows directly from the definition of $U_i(K)$ and as $Z$ is a component of $U_i(K)$.

**Definition 7** Let $n \in \mathbb{N}$ and $(Z_j)_{j \in J}$ denote the components of $U_n(K)$. A component $Z_j$ of $U_n(K)$ fulfils one of the cases of Remark 6 (b) and so for $a = a_j, b = b_j$ (in the cases (i)-(iii)), for $0 < r_j < R_j = R$ (in the cases (i)-(iv)) resp. $0 < r_j < 1/n =: R_j$ (in the case (v)) we define
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\[ V_{r_j}(Z_j) := \begin{cases} 
\{ z \in \mathbb{C} | d(\{z\}, \{a_j, b_j\}) < r_j \} & Z_j \text{ fulfils (i)}, \\
\{ z \in \mathbb{C} | d(\{z\}, \{a_j, \infty\}) < r_j \} & Z_j \text{ fulfils (ii)}, \\
\{ z \in \mathbb{C} | d(\{z\}, (-\infty, b_j]) < r_j \} & Z_j \text{ fulfils (iii)}, \\
\{ z \in \mathbb{C} | d(\{z\}, \mathbb{R}) < r_j \} & Z_j \text{ fulfils (iv)}, \\
(1/r_j, \infty) + i(-r_j, r_j) & Z_j = (n, \infty) + i(-1/n, 1/n), \\
(-\infty, -1/r_j) + i(-r_j, r_j) & Z_j = (-\infty, -n) + i(-1/n, 1/n), 
\end{cases} \]

where \( Z_j \) fulfils (v) in the last two cases. By Remark 6 (a) there is w.l.o.g. \( k \in \mathbb{N} \) with \( U_n(K) = \bigcup_{j=1}^k Z_j \). We set \( r := (r_j)_{1 \leq j \leq k} \) and the path

\[ \gamma_{K,n,r} := \sum_{j=1}^k \gamma_j \]

where \( \gamma_j \) is the path along the boundary of \( V_{r_j}(Z_j) \) in \( \mathbb{C} \) in the positive sense (counterclockwise) (see Fig. 3).

**Proposition 8** Let \( n \in \mathbb{N} \) and \( \gamma_{K,n,r} \) be the path from Definition 7. Then the following assertions hold.

(a) \( F \cdot \varphi \) is Pettis-integrable along \( \gamma_{K,n,r} \) for all \( F \in \mathcal{O}_1^\beta(\overline{\mathbb{C} \setminus K}, E) \) and \( \varphi \in \mathcal{O}_a^{-\beta}\left(U_n(K)\right) \).

(b) There are \( m \in \mathbb{N} \) and \( C > 0 \) such that for all \( a \in \mathfrak{A} \), \( F \in \mathcal{O}_1^\beta(\overline{\mathbb{C} \setminus K}, E) \) and \( \varphi \in \mathcal{O}_a^{-\beta}\left(U_n(K)\right) \)

\[ p_a\left( \int_{\gamma_{K,n,r}} F(\xi)\varphi(\xi)d\xi \right) \leq C|F|_{m,a}||\varphi||_n. \]

(c) For all \( F \in \mathcal{O}_1^\beta(\overline{\mathbb{C} \setminus K}, E) \), \( \varphi \in \mathcal{O}_a^{-\beta}\left(U_n(K)\right) \) and \( \bar{T} := (\bar{r}_j)_{1 \leq j \leq k} \) with \( 0 < \bar{r}_j < R_j \) for all \( 1 \leq j \leq k \)

\[ \int_{\gamma_{K,n,r}} F(\xi)\varphi(\xi)d\xi = \int_{\gamma_{K,n,r}} F(\xi)\varphi(\xi)d\xi. \]

(d) For all \( F \in \mathcal{O}_1^\beta(\overline{\mathbb{C}, E}) \) and \( \varphi \in \mathcal{O}_a^{-\beta}\left(U_n(K)\right) \)

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**Fig. 3** Path \( \gamma_{K,n,r} \) for \( \pm \infty \in K \) (c.f. [19, Figure 4.1, p. 40])
\[ \int_{y_{K,n,r}} F(\zeta)\varphi(\zeta) d\zeta = 0. \]

**Proof** (a) + (b) We have to show that there is \( e_{K,n,r} \in E \) such that

\[ \langle e', e_{K,n,r} \rangle = \int_{y_{K,n,r}} \langle e', F(\zeta)\varphi(\zeta) \rangle d\zeta, \quad e' \in E', \]

which gives \( \int_{y_{K,n,r}} F(\zeta)\varphi(\zeta) d\zeta = e_{K,n,r} \).

First, let \( V_{r_j}(Z_j) \) be bounded for some \( 1 \leq j \leq k \). There is a parametrisation \( \gamma_j : [0, 1] \to \mathbb{C} \) which has a continuously differentiable extension \( \widetilde{\gamma}_j \) on \((-1, 2)\). As the map \( (e' \circ (F \cdot \varphi) \circ \gamma_j) \cdot \gamma'_j \) is continuous on \([0, 1]\) for every \( e' \in E' \), it is an element of \( L^1([0, 1]) \) for every \( e' \in E' \). Thus the map

\[ \mathcal{F}_j : E' \to \mathbb{C}, \quad \mathcal{F}_j(e') := \int_{\gamma_j} \langle e', F(\zeta)\varphi(\zeta) \rangle d\zeta = \int_0^1 \langle e', (F \cdot \varphi)(\gamma_j(t)) \rangle \gamma'_j(t) dt, \]

is well-defined and linear. We estimate

\[ |\mathcal{F}_j(e')| \leq \int_0^1 |\gamma'_j(t)| dt \sup_{z \in (F \cdot \varphi)(\gamma_j([0, 1]))} |e'(z)|, \quad e' \in E'. \]

Let us denote by \( \overline{\text{ac} \mathcal{X}}((F \cdot \varphi)(\gamma_j([0, 1]))) \) the closure of the absolutely convex hull of the set \((F \cdot \varphi)(\gamma_j([0, 1]))\). Since \( e' \circ (F \cdot \varphi) \circ \gamma_j \in C^1((-1, 2)) \) for every \( e' \in E' \), the absolutely convex set \( \overline{\text{ac} \mathcal{X}}((F \cdot \varphi)(\gamma_j([0, 1]))) \) is compact in the sequentially complete space \( E \) by [5, Proposition 2, p. 354], yielding \( \mathcal{F}_j \in (E'_K)' \cong E \) by the theorem of Mackey-Arens, i.e. there is \( e_j \in E \) such that

\[ \langle e', e_j \rangle = \mathcal{F}_j(e') = \int_{\gamma_j} \langle e', F(\zeta)\varphi(\zeta) \rangle d\zeta, \quad e' \in E'. \]

Therefore \( F \cdot \varphi \) is Pettis-integrable along \( \gamma_j \). Furthermore, we choose \( m_j \in \mathbb{N} \) such that \( (1/m_j) < r_j \) and for \( \alpha \in \mathfrak{A} \) we set \( B_{m_j} := \{ x \in E \mid p_{m_j}(x) < 1 \} \). We note that

\[ p_{m_j} \left( \int_{\gamma_j} F(\zeta)\varphi(\zeta) d\zeta \right) \]

\[ = \sup_{e' \in B_{m_j}'} \left| \langle e', \int_{\gamma_j} F(\zeta)\varphi(\zeta) d\zeta \rangle \right| \leq \ell(\gamma_j) \sup_{e' \in B_{m_j}''} \sup_{z \in (F \cdot \varphi)(\gamma_j([0, 1]))} |e'(z)| \varphi(z)| \]

\[ \leq \ell(\gamma_j) \sup_{w \in (F \cdot \varphi)(\gamma_j([0, 1]))} \left( e^{\alpha_{m_j} - \alpha_{n_j}} |\text{Re}(w)|^\theta \right) \sup_{e' \in B_{m_j}''} \sup_{z \in E_{m_j}''(K)} |e'(F(z)e^{\alpha_{m_j} |\text{Re}(z)|^\theta})||\varphi||n \]

\[ = \ell(\gamma_j) \sup_{w \in (F \cdot \varphi)(\gamma_j([0, 1]))} \left( e^{\alpha_{m_j} - \alpha_{n_j}} |\text{Re}(w)|^\theta \right) \|F|_{m_j,a} \| \varphi \|n \]

where we used [25, Proposition 22.14, p. 256] in the first and the last equation to get from \( p_{m_j} \) to \( \sup_{e' \in B_{m_j}''} \) and back. If \( K \subset \mathbb{R} \), then all \( V_{r_j}(Z_j), 1 \leq j \leq k \), are bounded and with the
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The choice $e_{k,n,r} := \sum_{j=1}^{k} e_j$, $m := \max_{1 \leq j \leq k} m_j$ and $C := k \max_{1 \leq j \leq k} \ell(\gamma_j) \sup_{u \in \gamma_j(0,1)} e^{(a_n-a_m) \Re (u)^\beta}$ we deduce our statement.

Second, let us consider the case $\infty \in K$, $-\infty \not\in K$. Let $Z_k$ be the unique unbounded component of $U_n(K)$. For $q \in \mathbb{N}$, $q > 1/r_k > n$, we denote by $\gamma_{k,q}$ the part of $\gamma_k$ in $\{z \in \mathbb{C} \mid \Re (z) \leq q\}$. Like in the first part the Pettis-integral

$$e_{k,q} := \int_{\gamma_{k,q}} F(\zeta) \varphi(\zeta) d\zeta$$

exists (in $E$) and for $\alpha \in \mathfrak{F}$ and $m_k \in \mathbb{N}$, $(1/m_k) < r_k$, we have

$$p_a \left( \int_{\gamma_{k,q}} F(\zeta) \varphi(\zeta) d\zeta \right) \leq \ell(\gamma_{k,q}) \sup_{u \in \gamma_{k,q}(0,1)} e^{(a_n-a_m) \Re (u)^\beta} |F|_{m_k,\alpha} \|\varphi\|_n.$$ 

Next, we prove that $(e_{k,q})_{q > 1/r_k}$ is a Cauchy sequence in $E$. We choose $M := \max(m_k, 2n)$. For $q, p \in \mathbb{N}$, $q > p > 1/r_k > n$, we obtain

$$p_a(e_{k,q} - e_{k,p})$$

and observe that $(\int_{0}^{q} e^{(a_n-a_m) \Re (u)^\beta} dt)_q$ is a Cauchy sequence in $\mathbb{C}$ because

$$\int_{0}^{\infty} e^{(a_n-a_m) \Re (u)^\beta} dt = \frac{\Gamma(1/\beta)}{\beta |a_n - a_m|^{1/\beta}}$$

where $\Gamma$ is the gamma function. Therefore $(e_{k,q})_{q > 1/r_k}$ is a Cauchy sequence in $E$, has a limit $e_k$ in the sequentially complete space $E$ and

$$e_k = \int_{\gamma_k} F(\zeta) \varphi(\zeta) d\zeta.$$ 

We fix $p \in \mathbb{N}$, $p > 1/r_k > n$, and conclude that

$$p_a \left( \int_{\gamma_k} F(\zeta) \varphi(\zeta) d\zeta \right) \leq p_a(e_k - e_{k,p}) + p_a(e_{k,p})$$

and consequently, our statement holds also in the case $\infty \in K$, $-\infty \not\in K$ and in the remaining cases it follows analogously.
(c) We note that
\[
\langle e', \int_{\gamma_{K,n,r}} F(\zeta)\varphi(\zeta) d\zeta - \int_{\tilde{\gamma}_{K,n}} F(\zeta)\varphi(\zeta) d\zeta \rangle = \int_{\gamma_{K,n,r} - \tilde{\gamma}_{K,n}} \langle e', F(\zeta)\varphi(\zeta) \rangle d\zeta
\]
for all $e' \in E'$. Thus statement (c) follows from Cauchy’s integral theorem and the Hahn-Banach theorem if $K \subseteq \mathbb{R}$. Now, let us consider the case $\infty \in K$, $-\infty \notin K$. We denote by $\gamma_k$ resp. $\tilde{\gamma}_k$ the part of $\gamma_{K,n,r}$ resp. $\tilde{\gamma}_{K,n}\tilde{\gamma}$ in the unbounded component of $U_n(K)$. It suffices to show that
\[
\int_{\gamma_k - \tilde{\gamma}_k} \langle e', F(\zeta)\varphi(\zeta) \rangle d\zeta = 0, \quad e' \in E'.
\]
Let $\varepsilon > 0$ and w.l.o.g. $r_k < \tilde{\gamma}_k$. We choose the compact set $Q \subseteq U_{2n}(K)$ as in the proof of Proposition 4 (b). Further, we take $q \in \mathbb{R}$ such that $q > 1/r_k$ and $q \in U_{2n}(K) \setminus Q$ and define the path $\gamma_0^+ : [r_k, \tilde{\gamma}_k] \rightarrow \mathbb{C}$, $\gamma_0^+(t) := q + it$. We deduce that for $m_k \in \mathbb{N}$, $(1/m_k) < \min(r_k, 1/(2n))$, and every $e' \in E'$
\[
\left| \int_{\gamma_{0,q}} \langle e', F(\zeta)\varphi(\zeta) \rangle d\zeta \right| \leq \int_{\tilde{\gamma}_k} e^{(a_n - a_m) \varepsilon} \Re(q + it)^\beta \, d|\varphi|_n |e'\circ F|_{m_k} \\
= (\tilde{\gamma}_k - r_k) e^{(a_n - a_m) \varepsilon} \|\varphi\|_n |e'\circ F|_{m_k} \leq (\tilde{\gamma}_k - r_k) \|\varphi\|_n |e'\circ F|_{m_k} \varepsilon.
\]
In the same way we obtain with $\gamma_{0,q}^- : [-\tilde{\gamma}_k, -r_k] \rightarrow \mathbb{C}$, $\gamma_{0,q}^-(t) := q + it$, that
\[
\left| \int_{\gamma_{0,q}^-} \langle e', F(\zeta)\varphi(\zeta) \rangle d\zeta \right| \leq (\tilde{\gamma}_k - r_k) \|\varphi\|_n |e'\circ F|_{m_k} \varepsilon.
\]
Hence we get (4) by Cauchy’s integral theorem and the Hahn-Banach theorem as well. The remaining cases follow similarly.

(d) The proof is similar to (c). Let $F \in \mathcal{O}_{(a_n)}(\overline{C}, E)$. Again, it suffices to prove that
\[
\int_{\gamma_{K,n,r}} \langle e', F(\zeta)\varphi(\zeta) \rangle d\zeta = 0, \quad e' \in E'.
\]
This follows from Cauchy’s integral theorem and the Hahn-Banach theorem if $K \subseteq \mathbb{R}$. Again, we only consider the case $\infty \in K$, $-\infty \notin K$ and only need to show that
\[
\int_{\gamma_k} \langle e', F(\zeta)\varphi(\zeta) \rangle d\zeta = 0, \quad e' \in E',
\]
where $\gamma_k$ is the part of $\gamma_{K,n,r}$ in the unbounded component of $U_n(K)$. Let $\varepsilon > 0$ and choose $q$ as in (c). Then we have with $\gamma_{0,q}^+ : [-r_k, r_k] \rightarrow \mathbb{C}$, $\gamma_{0,q}^+(t) := q + it$, that
\[
\left| \int_{\gamma_{0,q}^+} \langle e', F(\zeta)\varphi(\zeta) \rangle d\zeta \right| \leq 2r_k \|\varphi\|_n |e'\circ F|_{2n} \varepsilon
\]
for every $e' \in E'$. Cauchy’s integral theorem and the Hahn-Banach theorem imply our statement.
An essential role in the proof of $\mathcal{O}(\mathbb{C} \setminus K, E)/\mathcal{O}(\mathbb{C}, E) \cong L_b(\mathcal{A}(K), E)$ for non-empty compact $K \subset \mathbb{R}$ and quasi-complete $E$ (see (2)) plays the fundamental solution $z \mapsto 1/(z)\partial$ of the Cauchy-Riemann operator. By the identity theorem we can consider $\mathcal{O}_a^{\beta}(\mathbb{C}, E)$ as a subspace of $\mathcal{O}_a^{\beta}(\mathbb{C} \setminus K, E)$ and we equip the quotient space $\mathcal{O}_a^{\beta}(\mathbb{C} \setminus K, E)/\mathcal{O}_a^{\beta}(\mathbb{C}, E)$ with the induced locally convex quotient topology (which may not be Hausdorff, see Remark 14). We want to prove the isomorphy

$$\mathcal{O}_a^{\beta}(\mathbb{C} \setminus K, E)/\mathcal{O}_a^{\beta}(\mathbb{C}, E) \cong L_b(\mathcal{O}_a^{\beta}^{-}(K), E)$$

for non-empty compact $K \subset \mathbb{R}$ under some assumptions on $K, \beta$ and $(a_n)_{n \in \mathbb{N}}$. Since we have to deal with functions having some growth given by our exponential weights, we have to use the adapted fundamental solution $z \mapsto e^{-\frac{1}{\beta}}/(\pi z)$ of the Cauchy-Riemann operator.

**Proposition 9** Let $\gamma_{K,n,r}$ be the path from Definition 7. The map

$$H_K : \mathcal{O}_a^{\beta}(\mathbb{C} \setminus K, E)/\mathcal{O}_a^{\beta}(\mathbb{C}, E) \to L_b(\mathcal{O}_a^{\beta}^{-}(K), E)$$

given by

$$H_K(f)(\varphi) := \int_{\gamma_{K,n,r}} F(\zeta) \varphi(\zeta) d\zeta$$

for $f = \lfloor F \rfloor \in \mathcal{O}_a^{\beta}(\mathbb{C} \setminus K, E)/\mathcal{O}_a^{\beta}(\mathbb{C}, E)$ and $\varphi \in \mathcal{O}_a^{\beta}^{-}(U_n(K)), n \in \mathbb{N}$, is well-defined, linear and continuous. For all non-empty compact sets $K_1 \subset K$ it holds that

$$H_K|_{\mathcal{O}_a^{\beta}(\mathbb{C} \setminus K, E)/\mathcal{O}_a^{\beta}(\mathbb{C}, E)} = H_{K_1}$$

on $\mathcal{O}_a^{\beta}^{-}(K)$.

**Proof** In the following we omit the index $K$ of $H_K$ if no confusion seems to be likely. Let $f = \lfloor F \rfloor \in \mathcal{O}_a^{\beta}(\mathbb{C} \setminus K, E)/\mathcal{O}_a^{\beta}(\mathbb{C}, E)$ and $\varphi \in \mathcal{O}_a^{\beta}^{-}(U_n(K))$. Due to Proposition 8 (a) and (d) $H(f)(\varphi) \in E$ and $H(f)$ is independent of the representative $F$ of $f$. From Proposition 8 (c) it follows that $H(f)$ is well-defined on $\mathcal{O}_a^{\beta}(K)$, i.e. for all $k \in \mathbb{N}, k \geq n$, and $\varphi \in \mathcal{O}_a^{\beta}^{-}(U_n(K))$ it holds that

$$H(f)(\varphi) = H(f)(\varphi|_{U_n(K)}) = H(f)(\pi_{n,k}(\varphi)).$$

For all $n \in \mathbb{N}$ there are $m \in \mathbb{N}$ and $C > 0$ such that

$$p_{\alpha}(H(f)(\varphi)) \leq C|f|_{m,n} \|\varphi\|_n$$

for all $f = \lfloor F \rfloor \in \mathcal{O}_a^{\beta}(\mathbb{C} \setminus K, E)/\mathcal{O}_a^{\beta}(\mathbb{C}, E), \varphi \in \mathcal{O}_a^{\beta}^{-}(U_n(K))$ and $\alpha \in \mathcal{A}$ by Proposition 8 (b), which implies that $H(f) \in L(\mathcal{O}_a^{\beta}^{-}(U_n(K)), E)$ for every $n \in \mathbb{N}$. We deduce that $H(f) \in L(\mathcal{O}_a^{\beta}^{-}(K), E)$ by [9, 3.6 Satz, p. 117]. Let

$$q : \mathcal{O}_a^{\beta}(\mathbb{C} \setminus K, E) \to \mathcal{O}_a^{\beta}(\mathbb{C} \setminus K, E)/\mathcal{O}_a^{\beta}(\mathbb{C}, E), q(F) := \lfloor F \rfloor,$$

denote the quotient map. We equip the quotient space with its usual quotient topology generated by
Now, let $M \subset \mathcal{O}^{-\beta}(K)$ be a bounded set. Since the sequence $(B_n)_{n\in\mathbb{N}}$ of closed unit balls $B_n$ of $\mathcal{O}^{-\beta}(U_n(K))$ is a fundamental system of bounded sets in $\mathcal{O}^{-\beta}(K)$ by [25, Proposition 25.19, p. 303], there exist $n \in \mathbb{N}$ and $\lambda > 0$ with $M \subset \lambda B_n$. We derive from (7) that

$$\sup_{\varphi \in \mathcal{M}} p_\alpha(H(f)(\varphi)) \leq |\lambda| C |f|_{l,a}^\wedge,$$

proving the continuity of $H$.

Moreover, let $K_1 \subset \overline{\mathbb{R}}$ be compact and $K_1 \subset K$. We observe that for every $F \in \mathcal{O}^{-\beta}(\overline{\mathbb{C}} \setminus K_1, E)$ and $\varphi \in \mathcal{O}^{-\beta}(U_n(K))$, $n \in \mathbb{N}$, it holds that

$$H_{K_1}([F])(\varphi) = \int_{T_{K_1,n}} F(\zeta)\varphi(\zeta)d\zeta = \int_{T_{K_1,n}} F(\zeta)\varphi(\zeta)d\zeta = H_{K_1}([F])(\varphi)$$

by Cauchy’s integral theorem and the Hahn-Banach theorem like in Proposition 8 (c) and (d). This yields to

$$H_{K_1}|_{\mathcal{O}^{-\beta}((\overline{\mathbb{C}} \setminus K_1), E) / \mathcal{O}^{-\beta}(\overline{\mathbb{C}}, E)} = H_{K_1}$$

on $\mathcal{O}^{-\beta}(K)$. $\square$

Now, we take a closer look at the potential inverse of $H_{K_1}$.

**Proposition 10** The map

$$\Theta_K : L_b(\mathcal{O}^{-\beta}(K), E) \to \mathcal{O}^{-\beta}(\overline{\mathbb{C}} \setminus K, E) / \mathcal{O}^{-\beta}(\overline{\mathbb{C}}, E)$$

given by

$$\Theta_K(T) := \left[ \mathbb{C} \setminus K \ni z \mapsto \frac{1}{2\pi i} \left\{ T, \frac{e^{-(z-j)^2}}{z-j} \right\} \right], \quad T \in L_b(\mathcal{O}^{-\beta}(K), E),$$

is well-defined, linear and continuous.

**Proof** We start with the proof that the map $\Theta_K$ is well-defined and take a closer look at its components. For $z, \zeta \in \mathbb{C}$ we set $G(z, \zeta) := e^{-(z-j)^2}$ and note that $\frac{\partial}{\partial z} G(z, \zeta) = -2(z-j)G(z, \zeta)$. We remark that for all $z = z_1 + iz_2 \in \mathbb{C}$ and all $n \in \mathbb{N}$
\[ \|G(z, \cdot)\|_n = \sup_{\zeta \in \overline{U}_n(K)} e^{-\text{Re}((z-\zeta)^2)} e^{-\alpha_n |\text{Re}(\zeta)|^p} \]
\[ \leq \sup_{\zeta_1 + i\zeta_2 \in \overline{U}_n(K)} e^{-\zeta_1^2 + (z_2 - \zeta_2)^2} e^{\alpha_n (1+|\zeta_1|)} \]
\[ \leq e^{(\|z_2|+1)^2 - \zeta_1^2 + \alpha_n} \sup_{\zeta_1 + i\zeta_2 \in \overline{U}_n(K)} e^{-\zeta_1^2 + (2|z_2|+|\alpha_n|)|\zeta_1|} \]
\[ \leq e^{(\|z_2|+1)^2 - \zeta_1^2 + \alpha_n} e^{-(|z_2|+|\alpha_n|)^2 + (2|z_2|+|\alpha_n|)(|z_2|+|\alpha_n|)/2} \]

and we deduce that \( G(z, \cdot) \in \mathcal{O}^{-\rho}_{\alpha_n}(\overline{U}_n(K)) \), in particular \( G(\mathbb{B}_{1/2}(z), \cdot) \subset \mathcal{O}^{-\rho}_{\alpha_n}(\overline{U}_n(K)) \). For \( \zeta = \zeta_1 + i\zeta_2 \in \overline{U}_n(K) \) and \( h \in \mathbb{C} \), \( 0 < |h| \leq 1 \), we observe that

\[ \frac{G(z+h, \zeta) - G(z, \zeta)}{h} = \left( e^{-2(z-\zeta)h} \right) \frac{G(z, \zeta)}{G(z, \zeta)} e^{-\alpha_n |\text{Re}(\zeta)|^p} \]
\[ \leq |h| \left( 1 + \sum_{k=2}^{\infty} \frac{1}{k!} (2z - \zeta) + 1 \right) |G(z, \zeta)| e^{-\alpha_n |\text{Re}(\zeta)|^p} \]
\[ \leq |h| e^{2|z-\zeta| + 1 - e^{-2(z-\zeta)^2}} e^{-\alpha_n |\text{Re}(\zeta)|^p} \]
\[ \leq |h| e^{2|z_2|+|\alpha_n|} e^{-(z_2 - \zeta_2)^2 + (z_2 - \zeta_2)^2} e^{-\alpha_n |\zeta_1|} \]
\[ \leq |h| e^{2|z_2|+|\alpha_n|} e^{-(z_2 - \zeta_2)^2 + (z_2 - \zeta_2)^2} e^{-\alpha_n |\zeta_1|} \]
\[ \leq |h| e^{2|z_2|+|\alpha_n|} e^{-(z_2 - \zeta_2)^2 + (z_2 - \zeta_2)^2} e^{-\alpha_n |\zeta_1|} \]

yielding to

\[ \left\| \frac{G(z+h, \cdot) - G(z, \cdot)}{h} - (2z-\cdot)G(z, \cdot) \right\|_n \leq |h| C_0 \quad h \to 0. \]

We conclude that \( \frac{d}{d\zeta} G(z, \cdot) = -2(z-\cdot)G(z, \cdot) \in \mathcal{O}^{-\rho}_{\alpha_n}(\overline{U}_n(K)) \) (inequality above and triangle inequality holds).

For \( z \in \mathbb{C} \setminus K \) and \( \zeta \in \mathbb{C} \setminus \{z\} \) we define
\[ g(z, \zeta) := \frac{G(z, \zeta)}{z-\zeta} = \frac{e^{-|z-\zeta|^2}}{z-\zeta} \]
and note that \( g(z, \cdot) \in \mathcal{O}(\mathbb{C} \setminus \{z\}) \). For \( z \in \mathbb{C} \setminus K \) there is \( k = k(z) \in \mathbb{N} \) such that
\[ d_k := d(\mathbb{B}_{1/k}(z), \overline{U}_k(K)) > 0 \]
and we obtain
\[ \|g(w, \cdot)\|_k = \sup_{\zeta \in \overline{U}_k(K)} \frac{1}{|w - \zeta|} |G(w, \zeta)| e^{-\alpha_n |\text{Re}(\zeta)|^p} \leq \frac{1}{d(\{w\}, \overline{U}_k(K))} \|G(w, \cdot)\|_k < \infty \]
for all \( w \in B_{1/k}(z) \). We deduce that 
\[
g(w, \cdot) \in \mathcal{O}^{-\beta}_{\alpha_k}\left( \overline{U}_k(K) \right)
\]
for all \( w \in B_{1/k}(z) \). Moreover, we observe that
\[
\frac{\partial}{\partial z} g(z, \zeta) = \frac{\partial}{\partial z} G(z, \zeta) - G(z, \zeta) \frac{1}{(z - \zeta)^2} = -\left(2 + \frac{1}{(z - \zeta)^2}\right) G(z, \zeta)
\]
for all \( \zeta \in \overline{U}_k(K) \). Let \( h \in \mathbb{C} \) with \( 0 < |h| < 1/k \). Then
\[
\frac{1}{|z + h - \zeta|} - \frac{1}{|z|} = \frac{-h}{(z + h - \zeta)(z - \zeta)} \leq \frac{|h|}{d_k^2}
\]
and
\[
\frac{1}{h} \left( \frac{1}{|z + h - \zeta|} - \frac{1}{|z - \zeta|} \right) + \frac{1}{(z - \zeta)^2} = \frac{h}{(z + h - \zeta)(z - \zeta)^2} \leq \frac{|h|}{d_k^3}
\]
for all \( \zeta \in \overline{U}_k(K) \), which implies
\[
\frac{1}{|h|} \left| \frac{G(z + h, \zeta) - G(z, \zeta)}{h} - \frac{\partial}{\partial z} g(z, \zeta) \right| \leq \frac{1}{|z + h - \zeta|} \left| \frac{G(z + h, \zeta) - G(z, \zeta)}{h} - \frac{\partial}{\partial z} G(z, \zeta) \right| + \frac{\partial}{\partial z} G(z, \zeta) \left| \frac{1}{|z + h - \zeta|} - \frac{1}{|z - \zeta|} \right|
\]
for all \( \zeta \in \overline{U}_k(K) \), which implies
\[
\frac{1}{|h|} \left| \frac{G(z + h, \cdot) - G(z, \cdot)}{h} - \frac{\partial}{\partial z} g(z, \cdot) \right| \leq \frac{1}{d_k} \left| \frac{G(z + h) - G(z)}{h} - \frac{\partial}{\partial z} G(z, \cdot) \right| + \frac{\partial}{\partial z} G(z, \cdot) \left| \frac{1}{|k|} \frac{|h|}{d_k} + \left| G(z) \right| \frac{|h|}{d_k^3} \right|
\]
We conclude that \( \frac{\partial}{\partial z} g(z, \cdot) \in \mathcal{O}^{-\beta}_{\alpha_k}\left( \overline{U}_k(K) \right) \) and \( \frac{g(z + h, \cdot) - g(z, \cdot)}{h} \) converges to \( \frac{\partial}{\partial z} g(z, \cdot) \) in the space \( \mathcal{O}^{-\beta}_{\alpha_k}\left( \overline{U}_k(K) \right) \) as \( h \to 0 \) by (9). Hence for all \( T \in L(\mathcal{O}^{-\beta}_{\alpha_k}(K), E) \) the limit
\[
\lim_{h \to 0, h \in \mathbb{C}, h \neq 0} \frac{\langle T, g(z + h, \cdot) \rangle}{h} = \left\langle \lim_{h \to 0, h \in \mathbb{C}, h \neq 0} \frac{g(z + h, \cdot) - g(z, \cdot)}{h} \right\rangle = \left\langle T, \frac{\partial}{\partial z} g(z, \cdot) \right\rangle
\]
exists in \( E \), meaning that \( (z \mapsto \frac{1}{2\pi i} \langle T, \frac{g(z, \cdot) - g(\zeta, \cdot)}{z - \zeta} \rangle) \in \mathcal{O}(\mathbb{C} \setminus K, E) \).

Let us turn to the continuity of \( \Theta_K \). Let \( n \in \mathbb{N} \). We note that for \( \zeta_1, z_1 \in \mathbb{R} \)
\[
-a_{2n} |\zeta_1|^\beta + a_{2n} |z_1|^\beta \leq |a_{2n}| |z_1 - \zeta_1|^\beta \leq |a_{2n}| (1 + |z_1 - \zeta_1|).
\]
It follows that
The inhomogeneous Cauchy-Riemann equation for weighted smooth

\[ \sup_{z \in S_p(K)} \| G(z, \cdot) \|_{K, 2n} e^{a_n |\text{Re} (z)|^p} = \sup_{z \in S_p(K)} \sup_{\xi \in U_p(K)} e^{-\text{Re} ((\xi - z)^\beta)} e^{-a_n |\text{Re} (\xi)|^p} e^{a_n |\text{Re} (z)|^p} \]
\[ \leq e^{(n+1/2)(2n)^2} \sup_{z \in S_p(K)} e^{-e^{-((\xi - z_1)^2 + a_n |\xi - z_1|^2})} \]
\[ \leq e^{(n+1/2)(2n)^2 + a_n |\xi |^2} \sup_{z \in S_p(K)} e^{-e^{-((\xi - z_1)^2 + a_n |\xi - z_1|)^p}} \]
\[ = e^{(n+1/2)(2n)^2 + a_n |\xi |^2} + a_n |\xi |^2 \]

which yields, in particular, that \( G(S_p(K), \cdot) \subset \mathcal{O}^\beta_{a_n} \left( U_{2n}(K) \right) \). Moreover, there is \( k \in \mathbb{N} \) such that

\[ D_{n,k} := d(S_p(K), \overline{U}_{2n}(K)) > 0. \]

Again, it follows that \( g(S_p(K), \cdot) \subset \mathcal{O}^\beta_{a_n} \left( U_{2n}(K) \right) \) with \( m := \max(2n, k) \). Furthermore, we observe that \( M := \{ g(z, \cdot) e^{a_n |\text{Re} (z)|^p} | z \in S_p(K) \} \subset \mathcal{O}^\beta_{a_n} \left( U_{2n}(K) \right) \) and

\[ \sup_{\varphi \in M} \| \varphi \|_m = \sup_{z \in S_p(K)} \| g(z, \cdot) \|_{K, 2n} e^{a_n |\text{Re} (z)|^p} \leq \frac{1}{D_{n,k}} \sup_{z \in S_p(K)} \| G(z, \cdot) \|_{K, 2n} e^{a_n |\text{Re} (z)|^p} \leq \infty, \]

showing that \( M \) is bounded in \( \mathcal{O}^\beta_{a_n} (K) \) by \([25, \text{Proposition 25.19, p. 303}]\). For every \( \alpha \in \mathfrak{A} \) and \( T \in L(\mathcal{O}^\beta_{a_n}(K), E) \) we have

\[ |\Theta_K(T)|_{n, \alpha} \leq \left| z \mapsto \frac{1}{2\pi i} \langle T, g(z, \cdot) \rangle \right|_{n, \alpha} = \frac{1}{2\pi} \sup_{\varphi \in \mathfrak{A}} p_{\alpha}(T(\varphi)) \]

and therefore the map \( \Theta_K : L_b(\mathcal{O}^\beta_{a_n}(K), E) \to \mathcal{O}^\beta_{a_n}(\overline{\mathbb{C}} \setminus K, E) \cap \mathcal{O}^\beta_{a_n}(\overline{\mathbb{C}}, E) \) is well-defined, clearly linear and continuous. \( \square \)

The map \( \Theta_K \) is sometimes called (weighted) Cauchy transformation for obvious reasons (see \([26, \text{p. 84}]\)).

**Theorem 11** If \( K \subset \mathbb{R} \) or \( K \cap \{ \pm \infty \} \) has no isolated points in \( K \), then the map

\[ H_K : \mathcal{O}^\beta_{a_n}(\overline{\mathbb{C}} \setminus K, E) / \mathcal{O}^\beta_{a_n}(\overline{\mathbb{C}}, E) \to L_b(\mathcal{O}^\beta_{a_n}(K), E) \]

is a topological isomorphism with inverse \( \Theta_K \).

**Proof** As before we omit the index \( K \) of \( H_K \) and \( \Theta_K \) if it is not necessary. As a consequence of Proposition 9 and Proposition 10 the maps \( H \) and \( \Theta \) are linear and continuous. First, we prove that \( \Theta \circ H = \text{id} \) on \( \mathcal{O}^\beta_{a_n}(\overline{\mathbb{C}} \setminus K, E) / \mathcal{O}^\beta_{a_n}(\overline{\mathbb{C}}, E) \), which implies the injectivity of \( H \).

Let \( p \in \mathbb{N} \), \( p \geq 2 \). We choose \( n \in \mathbb{N} \) such that \( d(S_p(K), \overline{U}_{2n}(K)) > 0 \). We define the path \( I_p := I^- - I^+ \) with \( I_{\pm} : \mathbb{R} \to \mathbb{C}, I_{\pm}(t) := t \pm ip, \)

\[ \Gamma : \mathbb{R} \to \mathbb{C}, \Gamma_{\pm}(t) := t \pm ip, \]

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Further, we choose \( m \in \mathbb{N} \) such that \( 1/m < \min_{1 \leq j \leq k} r_j < 1/n \) and \( m > p \) where \( r = (r_j)_{1 \leq j \leq k} \) is from the path \( \gamma_{k,n,r} \) in the definition of \( H \). Due to this choice \( \Gamma_\pm \) and \( \gamma_{k,n,r} \) are within \( S_m(K) \).

Let \( f = [F] \in \mathcal{O}^\beta_{(a_u)}(\mathbb{C} \setminus K, E)/\mathcal{O}^\beta_{(a_u)}(\mathbb{C}, E) \) and \( z = x + iy \in S_p(K) \). Let \( u \in \mathbb{H}, \ u \neq x, \) and \([t_0, t_1] \subset [-p, p]\) such that the path \( \gamma_u : [t_0, t_1] \to \mathbb{C}, \ \gamma_u(t) := u + it, \) is within \( S_m(K) \). The map \( \zeta \mapsto F(\zeta) e^{-(z-\zeta)^2} \) is holomorphic on \( \mathbb{C} \setminus \{z\} \) with values in \( E \) and like in Proposition 8 (a) and (b) we deduce that it is Pettis-integrable along \( \gamma_u \) and \( I^{\pm}([s_0, s_1]) \subset \mathbb{H} \) using [5, Proposition 2, p. 354] and the Mackey-Arens theorem. Then we have

\[
\left| \left\langle e', \int_{\gamma_u} F(\zeta) \frac{e^{-(z-\zeta)^2}}{\zeta - z} \, d\zeta \right\rangle \right| \\
\leq |e' F|_m \int_{t_0}^{t_1} \left| e^{-(z-(u+it))^2} |e^{-u|a_n| |\text{Re}(u+it)|^\beta} \right| \frac{1}{|z - u - it|} \, dt \\
\leq |e' F|_m (t_1 - t_0) e^{-(x-u)^2+(y-t)^2} \frac{1}{|x - u|} \\
\leq \frac{1}{|x - u|} |e' F|_m (t_1 - t_0) e^{(y-t)^2-x^2+a_n|a_n|} \sup_{w \in \mathbb{R}} e^{-t^2+(2|x|+|a_n|)w} \\
\quad = \frac{1}{|x - u|} |e' F|_m (t_1 - t_0) e^{(y-t)^2-x^2+a_n|a_n|} e^{-(|x|+|a_n|/2)^2+(2|x|+|a_n|)|x|+|a_n|/2} \to 0
\]

for all \( e' \in E' \). Hence we derive from Cauchy’s integral formula that

\[
\left\langle e', F(z) \right\rangle = \frac{1}{2\pi i} \int_{\Gamma_p - \gamma_{k,n,r}} \left\langle e', F(\zeta) \frac{e^{-(z-\zeta)^2}}{\zeta - z} \right\rangle \, d\zeta = -\frac{1}{2\pi i} \int_{\Gamma_p - \gamma_{k,n,r}} \left\langle e', \frac{F(\zeta) e^{-(z-\zeta)^2}}{z - \zeta} \right\rangle \, d\zeta
\]

for all \( e' \in E' \) and \( z \in S_p(K) \). Thus we have

\[
F(z) = -\frac{1}{2\pi i} \int_{\Gamma_p - \gamma_{k,n,r}} F(\zeta) \frac{e^{-(z-\zeta)^2}}{z - \zeta} \, d\zeta
\]

for all \( z \in S_p(K) \). By (the proof of) Proposition 10 the function \( g(z, \cdot) = \frac{e^{-(z-\cdot)^2}}{z-\cdot} \in \mathcal{O}^\beta_{(a_u)}(K) \) for all \( z \in \mathbb{C} \setminus K \) and

\[
W : \mathbb{C} \setminus K \to E, \ W(z) := \frac{1}{2\pi i} H([F]) \left( \frac{e^{-(z-\cdot)^2}}{z-\cdot} \right) - F(z),
\]

is an element of \( \mathcal{O}^\beta_{(a_u)}(\mathbb{C} \setminus K, E) \) since \( T := H([F]) \in L(\mathcal{O}^\beta_{(a_u)}(K), E) \) by Proposition 9. It follows that

\[
W(z) = \frac{1}{2\pi i} \int_{\gamma_{k,n,r}} F(\zeta) \frac{e^{-(z-\zeta)^2}}{z - \zeta} \, d\zeta + \frac{1}{2\pi i} \int_{\Gamma_p - \gamma_{k,n,r}} F(\zeta) \frac{e^{-(z-\zeta)^2}}{z - \zeta} \, d\zeta
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma_p} F(\zeta) \frac{e^{-(z-\zeta)^2}}{z - \zeta} \, d\zeta =: W_p(z)
\]

(11)

for all \( z \in S_p(K) \). But the right-hand side \( W_p \) of (11), as a function in \( z \), is weakly holomorphic on \( S_p(\mathbb{C}) = \{z \in \mathbb{C} \mid |\text{Im}(z)| < p\} \), which follows from
\[ \langle e', W_p(z) \rangle = \left\langle e', \frac{1}{2\pi i} \int_{\Gamma_p} F(\zeta) \frac{e^{-(-z-\zeta)^2}}{z-\zeta} d\zeta \right\rangle \]
\[ = \frac{1}{2\pi i} \int_{\Gamma_p} e'(F(\zeta)) \frac{e^{-(-z-\zeta)^2}}{z-\zeta} d\zeta, \quad e' \in E'. \]
and differentiation under the integral sign. The weak holomorphy and the sequential completeness of \( E \) imply that \( W_p \) is holomorphic on \( S_p(\mathcal{O}) \) by [10, Corollary 2, p. 404]. Thus \( W \) is extended by \( W_p \) to a function in \( \mathcal{O}(\mathbb{C}, E) \) and the extensions for each \( p \in \mathbb{N} \) coincide because of the identity theorem. We denote this extension by \( W \) as well.

Then we have for \( z = x + iy \in S_{1/n}^1 := \{ w \in \mathbb{C} \mid |\operatorname{Im}(w)| \leq (1/n) \} \subset S_{2n}(\mathcal{O}) \)
\[ 2\pi |\langle e', W(z) \rangle| = 2\pi |\langle e', W_{2n}(z) \rangle| = \left| \int_{\Gamma_n} e'(F(\zeta)) \frac{e^{-(-z-\zeta)^2}}{z-\zeta} d\zeta \right| \]
\[ \leq \int_{-\infty}^{\infty} |e'(F(t + 2ni))| \frac{|e^{-(-z-(t-2ni))^2}|}{|z-(t-2ni)|} dt + \int_{-\infty}^{\infty} |e'(F(t + 2ni))| \frac{|e^{-(-z-(t+2ni))^2}|}{|z-(t+2ni)|} dt \]
\[ \leq \left( \frac{1}{|y+2n|} + \frac{1}{|y-2n|} \right) \max_{-\infty}^{\infty} |e^{-(-z-(t\pm2ni))^2}| |e^{-a_n|\operatorname{Re}(z-(t\pm2ni))|}| dt |e'F|_{K\mathcal{A}n} \]
\[ \leq 2\frac{1}{2n-\frac{1}{n}} \max_{-\infty}^{\infty} |e^{-(-z-(t\pm2ni))^2}| |e^{-a_n|\operatorname{Re}(z-(t\pm2ni))|}| dt |e'F|_{K\mathcal{A}n}. \]
Moreover, in combination with the estimate
\[ \int_{-\infty}^{\infty} e^{-(-z-(t\pm2ni))^2} |e^{-a_n|\operatorname{Re}(z-(t\pm2ni))|} e^{|a_n| |\operatorname{Re}(z)|} | dt \]
\[ = \int_{-\infty}^{\infty} e^{-Re((z-(t\pm2ni)^2)} e^{a_n|x|} e^{-a_n|\operatorname{Re}(z)|} \leq e^{(y\mp2n)^2} \int_{-\infty}^{\infty} e^{-|x-t|^2} e^{-|a_n| \cdot |x-t|^2} dt \]
\[ \leq e^{(1/n+2n)^2} e^{-a_n^2} \int_{-\infty}^{\infty} e^{-|x-t|^2} e^{-|a_n| \cdot |x-t|^2} dt = 2e^{(1/n+2n)^2} e^{-a_n^2} \int_{-\infty}^{\infty} e^{-t^2} dt \]
\[ \leq 2\sqrt{\pi} e^{(1/n+2n)^2} e^{-a_n^2} \leq C \]
we get for all \( \alpha \in \mathcal{X} \)
\[ \sup_{z \in S_{1/n}^1} p_\alpha(W(z)) e^{a_n|\operatorname{Re}(z)|} = \sup_{e' \in B_{\mathcal{A}n}^\circ} \sup_{0 \leq |y| \leq \frac{1}{n}} |\langle e', W(x+iy) \rangle| e^{a_n|\operatorname{Re}(x+iy)|} \]
\[ \leq \frac{C|F|_{K\mathcal{A}n,\alpha}}{\pi \left( 2n - \frac{1}{n} \right)} \]
yielding to
\[ |W|_{\mathcal{A}n,\alpha} = \sup_{z \in S_{1/n}^1} p_\alpha(W(z)) e^{a_n|\operatorname{Re}(z)|} \leq \max \left( |W|_{\mathcal{A}n,\alpha}, \sup_{z \in S_{1/n}^1} \sup_{\alpha \in \mathcal{X}} p_\alpha(W(z)) e^{a_n|\operatorname{Re}(z)|} \right) < \infty. \]
Hence \( W \in \mathcal{O}_{\mathcal{A}n}(\mathbb{C}, E) \) and thus
\((\Theta \circ H)(f) = \left[ z \mapsto \frac{1}{2\pi i} H([F]) \left( e^{(z-\cdot)^2} \right) - F(z) \right] + f = [W] + f = f \),

i.e. \(H\) is injective.

Second, we prove that \(H \circ \Theta = \text{id} \) on \(L(O^-_{\beta}(K), E)\), which implies the surjectivity of \(H\). Due to the Hahn-Banach theorem this is equivalent to the condition that

\[ e'(H(\Theta)(T)(\varphi)) = e'(T(\varphi)) \]

holds for all \(T \in L(O^-_{\beta}(K), E)\), \(\varphi \in O^-_{\beta}(K)\) and \(e' \in E'\). As

\[ e'(H(\Theta)(T)(\varphi)) = (H(\Theta)(e' \circ T)(\varphi) \]

for all \(T \in L(O^-_{\beta}(K), E)\), \(\varphi \in O^-_{\beta}(K)\) and \(e' \in E'\), it suffices to show the result for \(E = \mathbb{C}\).

Since the span of the set of point evaluations of complex derivatives \(\{\delta_{x_0}^{(n)} \mid x_0 \in K \cap \mathbb{R}, n \in \mathbb{N}_0\}\) is dense in \(O^-_{\beta}(K)\) by virtue of Proposition 4 (b), we only need to show that \((H(\Theta)(\delta_{x_0}^{(n)})(\varphi) = \langle \delta_{x_0}^{(n)}, \varphi \rangle\) for all \(x_0 \in K \cap \mathbb{R}, n \in \mathbb{N}_0\) and \(\varphi \in O^-_{\beta}(K)\).

Let \(x_0 \in K \cap \mathbb{R}\) and \(n \in \mathbb{N}_0\). Now, we have

\[ (H(\Theta)(\delta_{x_0}^{(n)})(\varphi) = \frac{1}{2\pi i} \int_{\mathcal{C}_k} \left( \delta_{x_0}^{(n)}, \frac{e^{-(z-\cdot)^2}}{z-\cdot} \right) \varphi(z)dz \]  (12)

for all \(\varphi \in O^-_{\beta}(K), k \in \mathbb{N}\). Let us take a closer look at the integral on the right-hand side of (12). Let \(m \in \mathbb{N}, m \geq 2\). Then \(e^{(\cdot-\cdot)^2}_{\mathbb{C}} \in O(\mathbb{C}_{1/m}(x_0))\) for every \(z \in S_m(\{x_0\})\). We fix the notation \(g_{\zeta}(\zeta) := \frac{e^{(\cdot-\cdot)^2}_{\mathbb{C}}}{\zeta-\cdot}\) for \(z \in S_m(\{x_0\})\) and \(\zeta \in \mathbb{B}_{1/m}(x_0)\). Then we get by Cauchy’s inequality

\[ |g_{\zeta}^{(n)}(x_0)| \leq n!(2m)^n \max_{\zeta \in \mathbb{B}_{1/m}(x_0)} \left| \frac{e^{-(\cdot-\cdot)^2}_{\mathbb{C}}}{\zeta-\cdot} \right| \leq n!(2m)^{n-1} \max_{\zeta \in \mathbb{B}_{1/m}(x_0)} \left| \frac{e^{-(\cdot-\cdot)^2}_{\mathbb{C}}}{\zeta-\cdot} \right| \]

for every \(z \in S_m(\{x_0\})\). We deduce that

\[ \sup_{z \in S_m(\{x_0\})} |g_{\zeta}^{(n)}(x_0)| e^{\alpha_{\text{Re}(\zeta)}} \]

\[ \leq n!(2m)^{n-1} \sup_{z \in S_m(\{x_0\})} \max_{\zeta \in \mathbb{B}_{1/m}(x_0)} |e^{-(\cdot-\cdot)^2}_{\mathbb{C}}| e^{\alpha_{\text{Re}(\zeta)}} \]

\[ \leq n!(2m)^{n-1} \sup_{z \in S_m(\{x_0\})} \max_{\zeta \in \mathbb{B}_{1/m}(x_0)} |e^{-(\cdot-\cdot)^2}_{\mathbb{C}}| e^{\alpha_{2m_{\text{Re}(\zeta)}}} e^{\alpha_{\text{Re}(\zeta)}} \]

\[ \leq n!(2m)^{n-1} \sup_{\zeta \in \mathbb{B}_{1/m}(x_0)} e^{\alpha_{2m_{\text{Re}(\zeta)}}} \sup_{z \in S_m(\{x_0\})} \left\| e^{-(\cdot-\cdot)^2}_{\mathbb{C}} \right\|_{\{x_0\}, 2m} e^{\alpha_{\text{Re}(\zeta)}} < \infty, \]

implying \((z \mapsto (\delta_{x_0}^{(n)}(\frac{e^{-(\cdot-\cdot)^2}_{\mathbb{C}}}{\zeta-\cdot})) \in O^-_{\beta}(\mathbb{C}_{\cdot} \setminus \{x_0\})\). This means that the path of the integral on the right-hand side of (12) can be deformed using Cauchy’s integral theorem (like in Proposition 8 (a) and (b)) and we get with \(s := \min_j r_j > 0\) for \(r = (r_j)\).
$$\frac{1}{2\pi i} \int_{\gamma_{K,r}} \left( \delta_{x_0} - z \cdot \right) e^{-(z-z^2)^2} \varphi(z) dz = \frac{1}{2\pi i} \int_{\partial B_z(x_0)} \left( \delta_{x_0} - z \cdot \right) e^{-(z-z^2)^2} \varphi(z) dz$$

$$= \frac{1}{2\pi i} \int_{\partial B_z(x_0)} g^{(n)}_z(x_0) \varphi(z) dz$$

for all $\varphi \in \mathcal{O}^{-\beta}_{\alpha_i} \left( \overline{U_1(K)} \right)$, $k \in \mathbb{N}$. The Laurent series of $e^{-(z-z^2)^2}$ in $\zeta \neq z$ is

$$\frac{e^{-(z-z^2)^2}}{z - \zeta} = \frac{1}{z - \zeta} + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} (z - \zeta)^{2j-1}$$

and so we have for the $n$th complex derivative of $g_z$ at $x_0$

$$g^{(n)}_z(x_0) = \frac{n!}{(z-x_0)^{n+1}} + h(z, x_0)$$

with an entire function $h(\cdot, x_0)$. By Cauchy’s integral theorem and Cauchy’s integral formula for derivatives we have

$$(H \circ \Theta)(\delta_{x_0}^{(n)})(\varphi) = \frac{1}{2\pi i} \int_{\partial B_z(x_0)} g^{(n)}_z(x_0) \varphi(z) dz$$

$$= \frac{1}{2\pi i} \int_{\partial B_z(x_0)} \left( \frac{n!}{(z-x_0)^{n+1}} + h(z, x_0) \right) \varphi(z) dz$$

$$= \frac{n!}{2\pi i} \int_{\partial B_z(x_0)} \frac{\varphi(z)}{(z-x_0)^{n+1}} dz = \varphi^{(n)}(x_0) = \langle \delta_{x_0}^{(n)}, \varphi \rangle$$

for all $\varphi \in \mathcal{O}^{-\beta}_{\alpha_i} \left( \overline{U_1(K)} \right)$, $k \in \mathbb{N}$.

**Remark 12** If $K \subset \mathbb{R}$, then Theorem 11 is also valid for locally complete $E$ because Proposition 8 still holds due to [5, Proposition 2, p. 354].

If $K \cap \{\pm \infty\}$ has isolated points in $K$, e.g. $K = \{+\infty\}$, then we cannot apply the preceding theorem directly since a counterpart for Proposition 4 (b) is missing. However, we can make use of the relation (5) if $\mathcal{O}^{-\beta}_{(a_j)}(\mathbb{R})$ is dense in $\mathcal{O}^{-\beta}_{(a_j)}(K)$.

**Corollary 13** If $\mathcal{O}^{-\beta}_{(a_j)}(\mathbb{R})$ dense in $\mathcal{O}^{-\beta}_{(a_j)}(K)$, then the map

$$H_K : \mathcal{O}^{\beta}_{(a_j)}(\mathbb{C} \setminus K, E) / \mathcal{O}^{\beta}_{(a_j)}(\mathbb{C}, E) \to L_b(\mathcal{O}^{-\beta}_{(a_j)}(K), E)$$

is a topological isomorphism with inverse $\Theta_K$ and

$$\Theta_K(T) = \Theta_{\mathbb{R}}(T), \quad T \in L(\mathcal{O}^{-\beta}_{(a_j)}(K), E). \quad (13)$$

**Proof** $H_K$ and $\Theta_K$ are well-defined, linear and continuous maps by Proposition 9 and Proposition 10. $H_K$ is a topological isomorphism with inverse $\Theta_{\mathbb{R}}$ by Theorem 11. The embedding of $\mathcal{O}^{-\beta}_{(a_j)}(\mathbb{R})$ into $\mathcal{O}^{-\beta}_{(a_j)}(K)$ is continuous and dense, hence defines the embedding of $L(\mathcal{O}^{-\beta}_{(a_j)}(K), E)$ into $L(\mathcal{O}^{-\beta}_{(a_j)}(\mathbb{R}), E)$ (the density of the first embedding implies the injectivity of the latter one) and we have
\[ \Theta_K(T) = \Theta_{\mathbb{R}}(T), \quad T \in L(O^{-\beta}_{(a_n)}(K), E), \]

by the definition of \( \Theta_K \) and \( \Theta_{\mathbb{R}} \). Furthermore, it follows from (5) that

\[ H_{\mathbb{R}[O^\beta(\mathbb{C} \setminus K, E)/O^\beta_{(a_n)}(\mathbb{C}, E)]} = H_K \]

on \( O^{-\beta}_{(a_n)}(\mathbb{R}) \). We conclude for every \( f \in O^\beta_{(a_n)}(\mathbb{C} \setminus K, E)/O^\beta_{(a_n)}(\mathbb{C}, E) \)

\[ (\Theta_K \circ H_K)(f) = \Theta_{\mathbb{R}}(H_K(f)) = \Theta_{\mathbb{R}}(H_{\mathbb{R}}(f)) = f \]

and for every \( T \in L(O^{-\beta}_{(a_n)}(K), E) \)

\[ (H_K \circ \Theta_K)(T) = H_{\mathbb{R}}(\Theta_K(T)) = H_{\mathbb{R}}(\Theta_{\mathbb{R}}(T)) = T \]

by Theorem 11. Thus \( H_K \) is bijective and \( \Theta_K \) its inverse. \( \square \)

**Remark 14** Under the conditions of Theorem 11 resp. Corollary 13 it follows that \( O^\beta_{(a_n)}(\mathbb{C} \setminus K, E)/O^\beta_{(a_n)}(\mathbb{C}, E) \) is Hausdorff since \( E \) and thus \( L_b(O^{-\beta}_{(a_n)}(K), E) \) is Hausdorff. In particular, \( O^\beta_{(a_n)}(\mathbb{C}, E) \) is closed in \( O^\beta_{(a_n)}(\mathbb{C} \setminus K, E) \) by [25, Lemma 22.9, p. 254].

**Corollary 15** If \( (a_n)_{n \in \mathbb{N}} \) is strictly increasing, \( a_n < 0 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} a_n = 0 \), then the map

\[ H_K : O^1_{(a_n)}(\mathbb{C} \setminus K, E)/O^1_{(a_n)}(\mathbb{C}, E) \to L_b(O^{-1}_{(a_n)}(K), E) \]

is a topological isomorphism with inverse \( \Theta_K \).

**Proof** We only need to prove that the condition of Corollary 13 is fulfilled. Due to [17, Theorem 2.2.1, p. 474] (and its correction in [28, Remark, p. 247-248]) the space \( O^{-1}_{(a_n)}(\mathbb{R}) \) is dense in \( O^{-1}_{(a_n)}(K) \) (where \( O^{-1}_{(a_n)}(\mathbb{R}) \) is called \( \mathcal{P}_a \)). \( \square \)

The isomorphy \( O^1_{(a_n)}(\mathbb{C} \setminus K, E)/O^1_{(a_n)}(\mathbb{C}, E) \cong L_b(O^{-1}_{(a_n)}(K), E) \) in Corollary 15 is already known for special cases like \( E = \mathbb{C} \) [17, Theorem 3.2.1, p. 480] and Fréchet spaces \( E \) [15, 3.9 Satz, p. 41] but the proof is of homological nature. In the special case \( K = [a, \infty), a \in \mathbb{R}, \) and \( E = \mathbb{C} \) the duality was proved in [26, Theorem 3.3, p. 85-86] and served as an initial point to prove Corollary 15 for complete \( E \) in [19, 4.1 Theorem, p. 41].

4 \( (\Omega) \) for \( O^\beta_{(a_n)} \)-spaces on strips with holes

In this section we derive sufficient conditions on \( K, (a_n)_{n \in \mathbb{N}} \) and \( \beta \) such that \( O^\beta_{(a_n)}(\mathbb{C} \setminus K) \) satisfies (\( \Omega \)). The basic idea is to prove that the strong dual \( O^{-\beta}_{(a_n)}(K)' \) satisfies (\( \Omega \)). Then we use the duality \( O^\beta_{(a_n)}(\mathbb{C} \setminus K)/O^\beta_{(a_n)}(\mathbb{C}) \cong O^{-\beta}_{(a_n)}(K)' \) from the preceding section to obtain (\( \Omega \)) for \( O^\beta_{(a_n)}(\mathbb{C} \setminus K) \). Let us recall that a Fréchet space \( F \) with an increasing fundamental system of seminorms \( (\| \cdot \|_k)_{k \in \mathbb{N}} \) satisfies (\( \Omega \)) by [25, Chap. 29, Definition, p. 367] if

\[ \forall p \in \mathbb{N} \exists q \in \mathbb{N} \forall k \in \mathbb{N} \exists n \in \mathbb{N}, C > 0 \forall r > 0 : U_q \subset C r^p U_k + \frac{1}{r} U_p \]

where \( U_k := \{ x \in F \mid \| x \|_k \leq 1 \} \).
We start with a helpful observation concerning the inductive limit $O^{-\beta}_{(a_n)}(K)$, namely, that the choice of the sequence $(1/n)_{n\in\mathbb{N}}$ for the neighbourhoods $U_n(K) = U_{1/(1/n)}(K)$ is irrelevant.

**Remark 16** Let $(c_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence in $\mathbb{R}$ with $c_n \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} c_n = 0$. For $n \in \mathbb{N}$ let

$$O^{-\beta}_{a_n} \left( U_{1/c_n}(K) \right) := \{ f \in O(U_{1/c_n}(K)) \cap C \left( U_{1/c_n}(K) \right) \mid \| f \|_{1/c_n} < \infty \}$$

where

$$\| f \|_{1/c_n} := \sup_{z \in U_{1/c_n}(K)} | f(z) | e^{-a_n | \text{Re}(z) |^\beta}$$

and the spectral maps for $n, k \in \mathbb{N}$, $n \leq k$, be given by the restrictions

$$\tilde{\pi}_{n,k} : O^{-\beta}_{a_n} \left( U_{1/c_n}(K) \right) \to O^{-\beta}_{a_k} \left( U_{1/c_k}(K) \right), \quad \tilde{\pi}_{n,k}(f) := f|_{U_{1/c_k}(K)}.$$

Then

$$O^{-\beta}_{(a_n)}(K) \cong \lim_{n \to \infty} O^{-\beta}_{a_n} \left( U_{1/c_n}(K) \right).$$

**Proof** It follows directly from Proposition 4 (a) and [9, 4.2 Satz, p. 122].

We recall an equivalent description of the property $(\Omega)$. By [25, Lemma 29.13, p. 369] a Fréchet space $F$ with an increasing fundamental system of seminorms $(\| \cdot \|_k)_{k \in \mathbb{N}}$ satisfies $(\Omega)$ if and only if

$$\forall \ p \in \mathbb{N} \ \exists \ q \in \mathbb{N} \ \forall \ k \in \mathbb{N} \ \exists \ 0 < \theta < 1, \ C > 0 \ \forall \ y \in F^\prime : \| y \|_q^* \leq C \| y \|_p^{1-\theta} \| y \|_k^\theta$$

holds where

$$\| y \|_k^* := \sup \{ |y(x)| \mid \| x \|_k \leq 1 \} \in \mathbb{R} \cup \{ \infty \}$$

is the dual norm.

**Lemma 17** There is a strictly decreasing sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $c_n \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} c_n = 0$ such that

$$\forall \ p, q, k \in \mathbb{N}, \ p < q < k \ \exists \ C > 0 \ \forall \ \zeta \in \mathbb{R}, \ |\zeta| \geq 1 + c_k^{-1} :$$

$$\left( \sup_{z \in \mathbb{C}, |z-\zeta| \leq c_p} e^{a_p | \text{Re}(z) |^\beta} \right)^\theta \left( \sup_{z \in \mathbb{C}, |z-\zeta| \leq c_q} e^{a_q | \text{Re}(z) |^\beta} \right)^{1-\theta} \leq C \inf_{z \in \mathbb{C}, |z-\zeta| \leq c_p} e^{a_p | \text{Re}(z) |^\beta}$$

with $\theta := \frac{\ln(c_p/c_q)}{\ln(c_p/c_q)}$. 

\[\small\]
Proof Let \( c_n := \exp(1/a_n) \) for all \( n \in \mathbb{N} \) if \( a_n < 0 \) for all \( n \in \mathbb{N} \) and \( c_n := \exp(-a_n) \) for all \( n \in \mathbb{N} \) if \( a_n \geq 0 \) for all \( n \in \mathbb{N} \). Then \((c_n)_{n \in \mathbb{N}}\) is a strictly decreasing sequence, \( c_n \leq 1 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} c_n = 0 \). Let \( p, q, k \in \mathbb{N} \) such that \( p < q < k \) and \( \theta := \frac{\ln c_p / \ln c_q}{\ln c_q / c_k} \). Let \( \zeta \in \mathbb{C} \) with \( |\zeta| \geq 1 + c_k^{-1} \). For \( z \in \mathbb{C} \) with \( |z - \zeta| \leq c_n \), \( n \in \{p, q, k\} \), we deduce from the inequalities

\[
| |z| - |\text{Re}(z) - \zeta| | \beta \leq | \text{Re}(z) | \beta \leq (| \text{Re}(z) - \zeta | + |\zeta|)^\beta \leq (c_n + |\zeta|)^\beta
\]

and

\[
|\zeta| - |\text{Re}(z) - \zeta| \geq |\zeta| - c_n \geq 1 + c_k^{-1} - c_n \geq c_k^{-1} > 0
\]

that

\[
\inf_{z \in \mathbb{C},|z - \zeta| \leq c_q} e^{a_q |\text{Re}(z)|^\beta} \geq e^{a_q(c_q + |\zeta|)^\beta}
\]

and

\[
\sup_{z \in \mathbb{C},|z - \zeta| \leq c_q} e^{\beta a_q |\text{Re}(z)|^\beta} \sup_{z \in \mathbb{C},|z - \zeta| \leq c_p} e^{(1-\theta) a_p |\text{Re}(z)|^\beta} \leq e^{\beta a_q(|\zeta| - c_q)^\beta + (1-\theta)a_p(|\zeta| - c_p)^\beta},
\]

if \( a_n < 0 \), as well as

\[
\inf_{z \in \mathbb{C},|z - \zeta| \leq c_q} e^{a_q |\text{Re}(z)|^\beta} \geq e^{a_q(|\zeta| - c_q)^\beta}
\]

and

\[
\sup_{z \in \mathbb{C},|z - \zeta| \leq c_q} e^{\beta a_q |\text{Re}(z)|^\beta} \sup_{z \in \mathbb{C},|z - \zeta| \leq c_p} e^{(1-\theta) a_p |\text{Re}(z)|^\beta} \leq e^{\beta a_q(c_q + |\zeta|)^\beta + (1-\theta)a_p(c_q + |\zeta|)^\beta},
\]

if \( a_n \geq 0 \). Now, we only need to prove that there is \( C > 0 \) such that

\[
e^{\beta a_q(|\zeta| - c_q)^\beta + (1-\theta)a_p(|\zeta| - c_p)^\beta} \leq Ce^{a_q(c_q + |\zeta|)^\beta}, \quad a_n < 0,
\]

resp.

\[
e^{\beta a_q(c_q + |\zeta|)^\beta + (1-\theta)a_p(c_q + |\zeta|)^\beta} \leq Ce^{a_q(|\zeta| - c_q)^\beta}, \quad a_n \geq 0.
\]

If \( a_n < 0 \), we observe that

\[
\theta a_k(|\zeta| - c_k)^\beta + (1-\theta)a_p(|\zeta| - c_p)^\beta - a_q(c_q + |\zeta|)^\beta \\
\leq \theta a_k(|\zeta| - c_p)^\beta + (1-\theta)a_p(|\zeta| - c_p)^\beta - a_q(c_q + |\zeta| - c_p)^\beta - a_q(c_q + c_p)^\beta \\
= (\theta a_k + (1-\theta)a_p - a_q)(|\zeta| - c_p)^\beta - a_q(c_p + c_q)^\beta
\]

and, if \( a_n \geq 0 \), that

\[
\theta a_k(c_k + |\zeta|)^\beta + (1-\theta)a_p(c_p + |\zeta|)^\beta - a_q(|\zeta| - c_q)^\beta \\
\leq \theta a_k(c_p + |\zeta|)^\beta + (1-\theta)a_p(c_p + |\zeta|)^\beta - a_q(c_p + |\zeta| + c_p)^\beta - |c_p + c_q|^\beta \\
\leq (\theta a_k + (1-\theta)a_p - a_q)(c_p + |\zeta|)^\beta + a_q(c_p + c_q)^\beta.
\]

What remains to be shown is that

\[\text{Springer}\]
\[ 0 \geq \theta a_k + (1 - \theta) a_p - a_q \tag{15} \]
because then we are done with \( C := \exp(|a_q|(c_p + c_q)^{\theta}) \). If \( a_n < 0 \), then
\[ \theta = \frac{\ln(c_p/c_q)}{\ln(c_p/c_k)} = \frac{(1/a_p) - (1/a_q)}{(1/a_p) - (1/a_k)} = \frac{a_k(a_q - a_p)}{a_q(a_k - a_p)} \]
and (15) is equivalent to
\[ 0 \geq \frac{a_k^2(a_q - a_p)}{a_q(a_k - a_p)} + \left(1 - \frac{a_k(a_q - a_p)}{a_q(a_k - a_p)} \right) a_p - a_q, \]
which holds if and only if
\[
\begin{align*}
0 &\leq a_k^2(a_q - a_p) + (a_q(a_k - a_p) - a_k(a_q - a_p)) a_p - a_k^2(a_k - a_p) \\
&= a_k^2(a_q - a_p) + (a_k - a_q)a_p^2 - a_k^2(a_k - a_q + a_q - a_p) \\
&= (a_k^2 - a_q^2)(a_q - a_p) + (a_k - a_q)(a_p^2 - a_q^2) \\
&= (a_k - a_q)(a_k + a_q)(a_q - a_p) - (a_k - a_q)(a_q - a_p)(a_p + a_q)
\end{align*}
\]
as \( a_k(a_k - a_p) < 0 \). Since \( a_k - a_q > 0 \) and \( a_q - a_p > 0 \), this is equivalent to
\[ 0 \leq (a_k + a_q) - (a_p + a_q) = a_k - a_p, \]
which is true. If \( a_n \geq 0 \), then
\[ \theta = \frac{\ln(c_p/c_q)}{\ln(c_p/c_k)} = \frac{a_q - a_p}{a_k - a_p} \]
and (15) is equivalent to
\[ 0 \geq \frac{a_q - a_p}{a_k - a_p} a_k + \left(1 - \frac{a_q - a_p}{a_k - a_p} \right) a_p - a_q, \]
which holds, as \( a_k - a_p > 0 \), if and only if
\[
\begin{align*}
0 &\geq (a_q - a_p)a_k + (a_k - a_p - (a_q - a_p)) a_p - (a_k - a_p)a_q \\
&= a_qa_k - a_p a_k + a_k a_p - a_qa_p + a_k a_q + a_p a_q = 0.
\end{align*}
\]
\[\square\]

We note that \( \theta \) in the lemma above fulfils \( 0 < \theta < 1 \) and state the following improvement of \([19, 5.21 \text{ Lemma, p. } 88]\).

**Lemma 18** The following assertions hold.

(a) \( \forall p, q, k \in \mathbb{N}, p < q < k \\exists 0 < \theta < 1, C > 0 \forall f \in C^{\theta}_a(U, \mathbb{C}) : \)
\[ ||f||_{q, k, \varepsilon} \leq C||f||_{p, \varepsilon}^{1-\theta} ||f||_{k, \varepsilon}^{\theta} \]
with \( c_n \) from Lemma 17 if \( K \cap \{\pm \infty\} \neq \emptyset \) resp. \( c_n := 1/n, n \in \mathbb{N}, \text{if } K \subset \mathbb{R} \).
(b) $O_\beta(a_n)(K)^h$ satisfies $(\Omega)$.  

**Proof** (a) Let $p, q, k \in \mathbb{N}$, $p < q < k$, and $f \in O_{\alpha_p}(U_{1/e_p}(K))$. Considering the components of $U_{1/e_p}(K)$ we have to distinguish three different cases.

(i) Let $Z_p$ be a bounded component of $U_{1/e_p}(K)$. By Remark 6 (a) there are only finitely many components $Z_q$ of $U_{1/e_q}(K)$ with $Z_q \subset Z_p$. For every such component $Z_q$ we choose $\zeta \in Z_q \cap K$, which exists since $Z_q$ is bounded. Let $Z_k$ be the (unique) component of $U_{1/e_k}(K)$ which contains $\zeta$. $Z_p$ is a proper simply connected subset of $\mathbb{C}$. Thus there exists a biholomorphic map $\tilde{\psi} : Z_p \to \mathbb{B}_1(0)$ with $\tilde{\psi}(\zeta) = 0$ due to the Riemann mapping theorem (and Möbius transformation). In addition, $Z_p$ and $\mathbb{B}_1(0)$ are Jordan domains (for the definition see [1, 2.8.5 Lemma, p. 193, 1.8.5 Jordan Curve Theorem, p. 68]) and so there exists a homeomorphism $\psi : \overline{Z_p} \to \overline{\mathbb{B}_1(0)}$ such that $\psi|_{Z_p} = \tilde{\psi}$ by [1, 2.8.8 Theorem (Caratheodory), p. 195]. Since $\psi(Z_p) \subset \psi(Z_q) = \mathbb{B}_1(0)$ and $\psi(\overline{Z_q})$ is compact, as $\overline{Z_q}$ is compact and $\psi$ continuous, there is $0 < r < r_q$ such that $\psi(\overline{Z_q}) \subset \mathbb{B}_q(0)$. Moreover, there exists $0 < r_k < r_q$ such that $\mathbb{B}_{r_k}(0) \subset \psi(Z_k)$ since $0 \in \psi(Z_k)$, $\psi(Z_k)$ is open by the open mapping theorem (from complex analysis) and $\psi(Z_k) \subset \psi(Z_q)$. The function $u := f\circ(\psi^{-1})$ is holomorphic on $\mathbb{B}_1(0)$ and continuous on $\overline{\mathbb{B}_1(0)}$, in particular, $u$ is subharmonic on $\mathbb{B}_1(0)$ and continuous on $\overline{\mathbb{B}_1(0)}$. Setting

$$M(r) := \sup_{|z|=r} |u(z)|, \quad 0 < r \leq 1,$$

we obtain by virtue of [1, 4.4.32 Proposition (Hadamard’s Three Circles Theorem), p. 338]

$$\ln(M(r_q)) \leq \frac{\ln(1/r_q)}{\ln(1/r_k)} \ln(M(r_k)) + \frac{\ln(r_q/r_k)}{\ln(1/r_k)} \ln(M(1))$$

and hence

$$M(r_q) \leq M(r_k)^\theta M(1)^{1-\theta}$$

with $\theta := \frac{\ln(1/r_k)}{\ln(1/r_q)}$. We note that $0 < \theta < 1$ because $0 < r_k < r_q < 1$. By the maximum principle we have

$$M(r_q) = \sup_{|z|=r_q} |u(z)| \geq \inf_{|z| \leq r_q} e^{\alpha_q} |\Re(\psi^{-1}(z))|^\theta \sup_{|z| \leq r_q} |f(\psi^{-1}(z))| e^{-\alpha_q} |\Re(\psi^{-1}(z))|^\theta$$

$$\geq \inf_{\psi(Z_q) \subset \mathbb{B}_q(0)} \sup_{|z| \leq r_q} |f(z)| e^{-\alpha_q} |\Re(z)|^\theta$$

$$=: c_q > 0$$

as well as

\[ \text{Springer} \]
\[ M(r_k)^\theta M(1)^{-\theta} = \sup_{|z| \leq r_k} |u(z)|^\theta \sup_{|z| \leq 1} |u(z)|^{1-\theta} \leq \left( \sup_{|z| \leq r_k} e^{a_k |\text{Re}(\psi^{-1}(z))|^\beta} \right)^\theta \left( \sup_{|z| \leq 1} |f(\psi^{-1}(z))| e^{-a_k |\text{Re}(\psi^{-1}(z))|^\beta} \right)^{1-\theta} \cdot \left( \sup_{|z| \leq 1} e^{a_k |\text{Re}(\psi^{-1}(z))|^\beta} \right)^{1-\theta} \left( \sup_{|z| \leq 1} |f(\psi^{-1}(z))| e^{-a_k |\text{Re}(\psi^{-1}(z))|^\beta} \right) \leq \frac{C_1}{C_0} \left( \sup_{z \in \mathbb{Z}_q} |f(z)| e^{-a_k |\text{Re}(z)|^\beta} \right)^\theta \left( \sup_{z \in \mathbb{Z}_p} |f(z)| e^{-a_k |\text{Re}(z)|^\beta} \right)^{1-\theta}, \]

and therefore

\[ \sup_{z \in \mathbb{Z}_q} |f(z)| e^{-a_k |\text{Re}(z)|^\beta} \leq \frac{C_1}{C_0} \left( \sup_{z \in \mathbb{Z}_q} |f(z)| e^{-a_k |\text{Re}(z)|^\beta} \right)^\theta \left( \sup_{z \in \mathbb{Z}_p} |f(z)| e^{-a_k |\text{Re}(z)|^\beta} \right)^{1-\theta}. \tag{16} \]

\text{(ii) Let } K \cap \{ \pm \infty \} \neq \emptyset. \text{ Let } Z_p \text{ be an unbounded component of } U_{1/c_p}(K), \text{ w.l.o.g. the real part of } Z_p \text{ is bounded from below and unbounded from above. Let } \zeta \in \mathbb{R} \text{ such that } \zeta \geq 1 + c_k^{-1}. \text{ Then we have } \mathbb{B}_{c_j}(\zeta) \subset \{ (c_j^{-1}, \infty) + i(-c_j, c_j) \} \text{ for } j \in \{ p, q, k \} \text{ since } c_p^{-1} < c_q^{-1} < c_k^{-1} \text{ and } c_j \leq 1. \text{ Applying Hadamard’s Three Circles Theorem to } u := |f|, \text{ we get } M(c_q) \leq M(c_k)^{\theta} M(c_p)^{1-\theta} \text{ with } \theta := \frac{\ln(c_q/c_p)}{\ln(c_p/c_k)} \text{ fulfilling } 0 < \theta < 1. \text{ Like in (i) we obtain } \]

\[ M(c_q) \geq \inf_{|z-\zeta| \leq c_q} e^{a_k |\text{Re}(z)|^\beta} \sup_{|z-\zeta| \leq c_q} |f(z)| e^{-a_k |\text{Re}(z)|^\beta}, \]

and

\[ M(c_k)^\theta M(c_p)^{1-\theta} \leq \left( \sup_{|z-\zeta| \leq c_k} e^{a_k |\text{Re}(z)|^\beta} \right)^\theta \left( \sup_{|z-\zeta| \leq c_p} e^{a_k |\text{Re}(z)|^\beta} \right)^{1-\theta} \left( \sup_{|z-\zeta| \leq c_k} |f(z)| e^{-a_k |\text{Re}(z)|^\beta} \right)^\theta \left( \sup_{|z-\zeta| \leq c_p} |f(z)| e^{-a_k |\text{Re}(z)|^\beta} \right)^{1-\theta}. \]

Due to Lemma 17 there is \( C_2 > 0 \), independent of \( \zeta \), such that

\[ \sup_{|z-\zeta| \leq c_q} |f(z)| e^{-a_k |\text{Re}(z)|^\beta} \leq C_2 \left( \sup_{|z-\zeta| \leq c_k} |f(z)| e^{-a_k |\text{Re}(z)|^\beta} \right)^\theta \left( \sup_{|z-\zeta| \leq c_p} |f(z)| e^{-a_k |\text{Re}(z)|^\beta} \right)^{1-\theta} \]

and thus
(iii) Let $K \cap \{\pm \infty\} \neq \emptyset$ and $Z_p$ be w.l.o.g. like in (ii). We define $\tilde{Z}_p := Z_p \cap ((-\infty, 1 + c_k^{-1}) + i\mathbb{R})$. By Remark 6 (a) there are only finitely many components $\tilde{Z}_q$ of $U_{1/c_k}(K) \cap ((-\infty, 1 + c_k^{-1}) + i\mathbb{R})$ with $\tilde{Z}_q \subset \tilde{Z}_p$. For every such component $\tilde{Z}_q$ we choose $\zeta \in \tilde{Z}_q \cap (K \cup \{x \in \mathbb{R} \mid x > c_k^{-1}\})$. Let $\tilde{Z}_k$ be the (unique) component of $U_{1/c_k}(K) \cap ((-\infty, 1 + c_k^{-1}) + i\mathbb{R})$ which contains $\zeta$. The rest is analogous to (i) and thus there are $\tilde{C}_0, \tilde{C}_1 > 0$ and $0 < \theta < 1$ such that

$$\sup_{z \in \tilde{Z}_q} |f(z)| e^{-a_z |\mathrm{Re}(z)|^\theta} \leq \frac{\tilde{C}_1}{\tilde{C}_0} \|f\|_{k,c_k} \|f\|_{p,c_p}^{1-\theta}. \quad (18)$$

(iv) First, let us remark the following. Let $B$ be a set, $B_0 \subset B$, $0 < \theta_0 < \theta_1 < 1$, $h : B_0 \to [0, \infty)$, $g : B \to [0, \infty)$ and $h \leq g$ on $B_0$. Then

$$\left(\sup_{z \in B_0} h(z)\right)^{\theta_1} \left(\sup_{z \in B} g(z)\right)^{1-\theta_1} \leq \left(\sup_{z \in B_0} h(z)\right)^{\theta_0} \left(\sup_{z \in B} g(z)\right)^{1-\theta_0}.$$

Now, we take the minimum of all the $\theta$s which appear in (i)-(iii). There are finitely many of them and denote their minimum again with $\theta$. Take the maximum of the constants $\frac{\tilde{C}_1}{\tilde{C}_0}, C_2$ and $\frac{\tilde{C}_1}{\tilde{C}_0}$ which appear in (i)-(iii). There are again finitely many of them and denote their maximum with $C$. We apply the remark above to $B_0 := U_{1/c_k}(K)$, $B := U_{1/c_p}(K)$, $h(z) := |f(z)| e^{-a_z |\mathrm{Re}(z)|^\theta}$ and $g(z) := |f(z)| e^{-a_z |\mathrm{Re}(z)|^\theta}$. Then we deduce from (16), (17) and (18) that

$$\|f\|_{q,c_q} \leq C \|f\|_{k,c_k} \|f\|_{p,c_p}^{1-\theta}.$$

(b) We recall Remark 16 and identify both inductive limits. Let $p \in \mathbb{N}$ and choose $q \in \mathbb{N}$, $q > p$. Let $k \in \mathbb{N}$. If $k \leq p$, then we get for any $0 < \theta < 1$ and all $y \in (\mathcal{O}_{(a_k)}(K)_b)'$ by definition of the dual norm

$$\|y\|_{k,c_k}^\ast \leq \|y\|_{p,c_p}^\ast \leq \|y\|_{k,c_k}^\ast 1-\theta \|y\|_{k,c_k}^\ast \theta \leq \|y\|_{p,c_p}^\ast 1-\theta \|y\|_{k,c_k}^\ast \theta.$$

Let $k > p$. If $k \leq q$, we have for any $0 < \theta < 1$ and all $y \in (\mathcal{O}_{(a_k)}(K)_b)'$ by definition of the dual norm

$$\|y\|_{k,c_k}^\ast \leq \|y\|_{k,c_k}^\ast 1-\theta \|y\|_{k,c_k}^\ast \theta \leq \|y\|_{p,c_p}^\ast 1-\theta \|y\|_{k,c_k}^\ast \theta.$$
Let \( k > q \) and \( y \in (\mathcal{O}^{-\beta}_{(a_n)}(K))' \). If \( \|y\|_{p,c_p}^* = \infty \), then (14) is obviously fulfilled. Let \( \|y\|_{p,c_p}^* < \infty \). As \( \mathcal{O}^{-\beta}_{(a_n)}(K) \) is a DFS-space by Proposition 4 (a), the sets
\[
B_n := \{ f \in \mathcal{O}^{-\beta}_{a_n}(\overline{U}_{1/c_n}(K)) \mid \|f\|_{n,c_n} \leq 1 \}, \quad n \in \mathbb{N},
\]
are a fundamental system of bounded sets of \( \mathcal{O}^{-\beta}_{(a_n)}(K) \) by [25, Proposition 25.19, p. 303] and hence the seminorms
\[
\|x\|_n := \sup_{f \in B_n} |x(f)|, \quad x \in \mathcal{O}^{-\beta}_{(a_n)}(K)',
\]
form a fundamental system of seminorms of \( \mathcal{O}^{-\beta}_{(a_n)}(K)' \). Furthermore, \( \mathcal{O}^{-\beta}_{(a_n)}(K) \) is reflexive and thus there is a unique \( f \in \mathcal{O}^{-\beta}_{(a_n)}(K) \) such that \( y(x) = x(f) \) for all \( x \in \mathcal{O}^{-\beta}_{(a_n)}(K) \). Then we obtain by [25, Proposition 22.14, p. 256] for all \( n \in \mathbb{N}, n \geq p, \)
\[
\infty > \|y\|_{p,c_p}^* \geq \|y\|_{n,c_n}^* = \sup\{ |y(x)| \mid \|x\|_n \leq 1 \} = \sup\{ |x(f)| \mid x \in B_n^a \}
\]
\[
= \inf\{ t > 0 \mid f \in tB_n \}.
\]
In particular, this means that \( \{ t > 0 \mid f \in tB_n \} \neq \emptyset \) and thus we have \( f \in \mathcal{O}^{-\beta}_{a_n}(\overline{U}_{1/c_n}(K)) \) as well as
\[
\|y\|_{n,c_n}^* = \inf\{ t > 0 \mid f \in tB_n \} = \|f\|_{n,c_n}
\]
for all \( n \geq p \). So by part (a), there are \( C > 0 \) and \( 0 < \theta < 1 \), only depending on \( p, q \) and \( k \), such that
\[
\|y\|_{q,c_q}^* = \|f\|_{q,c_q} \leq C\|f\|_{p,c_p}^{1-\theta}\|f\|_{k,c_k}^\theta = C\|y\|_{p,c_p}^{1-\theta}\|y\|_{k,c_k}^\theta.
\]
\( \square \)

The idea to use Hadamard’s Three Circles Theorem in the proof of Lemma 18 (a) is taken from the proof of [30, Lemma 5.2 (a)(3), p. 263-264]. If \( K \subset \mathbb{R} \) is non-empty and compact, Lemma 18 (b) is already known. Indeed, the space \( \mathcal{O}(\mathbb{C} \setminus K) \) satisfies \( (\Omega) \) by [31, Proposition 2.5 (b), p. 173] and thus the quotient space \( \mathcal{O}(\mathbb{C} \setminus K)/\mathcal{O}(\mathbb{C}) \) as well as \( (\Omega) \) is a linear-topological invariant by [25, Lemma 29.11 (2), p. 368]. Since \( (\Omega) \) is a linear-topological invariant by [25, Lemma 29.11 (1), p. 368], it follows from \( \mathcal{O}^{-\beta}_{(a_n)}(K)' \cong \mathcal{O}(\mathbb{C})' \cong \mathcal{O}(\mathbb{C})/\mathcal{O}(\mathbb{C}) \) by (2) that \( \mathcal{O}^{-\beta}_{(a_n)}(K)' \) also satisfies \( (\Omega) \). Combining our duality result with the preceding lemma, we get a generalisation of [19, 5.22 Theorem, p. 92].

**Corollary 19** If

(i) \( K \subset \mathbb{R}, \) or \( K \cap \{ \pm \infty \} \) has no isolated points in \( K, \) or

(ii) \( K \) is arbitrary, \( a_n < 0 \) for all \( n \in \mathbb{N}, \) \( \lim_{n \to \infty} a_n = 0 \) and \( \beta = 1, \)

then \( \mathcal{O}^{-\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K) \) satisfies \( (\Omega) \).

**Proof** The spaces \( \mathcal{O}^{-\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K) \) and \( \mathcal{O}^{-\beta}_{(a_n)}(\overline{\mathbb{C}}) \) are Fréchet spaces which is easily checked (similar to [20, 3.7 Proposition, p. 240]). By Theorem 11 in (i) resp. Corollary 15 in (ii) \( \mathcal{O}^{-\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K)/\mathcal{O}^{-\beta}_{(a_n)}(\overline{\mathbb{C}}) \) is topologically isomorphic to \( \mathcal{O}^{-\beta}_{(a_n)}(K)' \), in particular, the quotient is
a Fréchet space as $\mathcal{O}^\beta_{(a_n)}(K)$ is a DFS-space by Proposition 4 (a). Since $(\mathcal{O})$ is a linear-topological invariant by [25, Lemma 29.11 (1), p. 368], $\mathcal{O}^\beta_{(a_n)}(\overline{C} \setminus K)/\mathcal{O}^\beta_{(a_n)}(\overline{C})$ satisfies $(\mathcal{O})$ due to Lemma 18 (b). The sequence

$$0 \to \mathcal{O}^\beta_{(a_n)}(\overline{C}) \xrightarrow{i} \mathcal{O}^\beta_{(a_n)}(\overline{C} \setminus K) \xrightarrow{q} \mathcal{O}^\beta_{(a_n)}(\overline{C} \setminus K)/\mathcal{O}^\beta_{(a_n)}(\overline{C}) \to 0$$

is an exact sequence of Fréchet spaces where $i$ means the inclusion and $q$ the quotient map. $\mathcal{O}^\beta_{(a_n)}(\overline{C})$ satisfies $(\mathcal{O})$ by [22, Corollary 14, p. 18] combined with Assumption 3 (iii)+(iv) and $\mathcal{O}^\beta_{(a_n)}(\overline{C} \setminus K)/\mathcal{O}^\beta_{(a_n)}(\overline{C})$ as well, thus $\mathcal{O}^\beta_{(a_n)}(\overline{C} \setminus K)$ by [33, 1.7 Lemma, p. 230], too.

\[ \square \]

5 Surjectivity of the Cauchy-Riemann operator

In our last section we prove our main result on the surjectivity of the Cauchy-Riemann operator on $\mathcal{E}^\beta_{(a_n)}(\overline{C} \setminus K, E)$. This is done by using the results obtained so far and splitting theory. We recall that a Fréchet space $(F, (\|\cdot\|_k)_{k \in \mathbb{N}})$ satisfies $(DN)$ by [25, Chap. 29, Definition, p. 359] if

$$\exists \; p \in \mathbb{N} \forall \; k \in \mathbb{N} \exists \; n \in \mathbb{N}, \; C > 0 \; \forall \; x \in F : \; \|x\|_k^2 \leq C \|x\|_p \|x\|_n.$$ 

A **PLS-space** is a projective limit $X = \lim_{N \in \mathbb{N}} X_N$, where the $X_N = \lim_{N \in \mathbb{N}} (X_{N,n}, \|\cdot\|_{N,n})$ are DFS-spaces, and it satisfies (PA) if

$$\forall \; N \exists \; M \forall \; K \exists \; n \forall \; m \forall \; \eta > 0 \exists \; k, C, r_0 > 0 \forall \; r > r_0 \forall \; x' \in X'_N :$$

$$\|x' \circ i_M^{\eta}\|_{M,n} = C \left( r^\eta \|x' \circ i_K^{\eta}\|_{K,n} + \frac{1}{r} \|x'\|_{N,n} \right)$$

where $\|\cdot\|^*$ denotes the dual norm of $\|\cdot\|$ and $i_M^{\eta}, i_K^{\eta}$ the linking maps (see [4, Sect. 4, Eq. (24), p. 577]).

**Theorem 20** Let $(a_n)_{n \in \mathbb{N}}$ be strictly increasing, $a_n < 0$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} a_n = 0$. If

(i) $K \subset \mathbb{R}$, or $K \cap \{ \pm \infty \}$ has no isolated points in $K$, or

(ii) $K$ is arbitrary and $\beta = 1$, and

(a) $E := F'_1$ where $F$ is a Fréchet space over $\mathbb{C}$ satisfying (DN), or

(b) $E$ is an ultrabornological PLS-space over $\mathbb{C}$ satisfying (PA),

then

$$\mathcal{E}^\beta_{(a_n)}(\overline{C} \setminus K, E) \to \mathcal{E}^\beta_{(a_n)}(\overline{C} \setminus K, E)$$

is surjective.
The inhomogeneous Cauchy-Riemann equation for weighted smooth…

Proof We only need to check that the conditions of Theorem 1 are fulfilled. $\mathcal{E}^{\theta}_{(a_n)}(\overline{\mathbb{C}} \setminus K)$ is nuclear, in particular a Schwartz space, and thus its subspace $\mathcal{E}^{\theta}_{(a_n),\delta}(\overline{\mathbb{C}} \setminus K)$ as well by [21, Theorem 3.1, p. 188], [21, 2.8 Example (ii), p. 179], [21, Remark 2.7, p. 178-179] and [21, Remark 2.3 (b), p. 177]. Furthermore, $\mathcal{E}^{\theta}_{(a_n),\delta}(\overline{\mathbb{C}} \setminus K) = O^{\theta}_{(a_n)}(\overline{\mathbb{C}} \setminus K)$ by Remark 2. Due to Corollary 19 the space $O^{\theta}_{(a_n)}(\overline{\mathbb{C}} \setminus K)$ satisfies $(\Omega)$. The Cauchy-Riemann operator $\partial : \mathcal{E}^{\theta}_{(a_n)}(\overline{\mathbb{C}} \setminus K) \to \mathcal{E}^{\theta}_{(a_n)}(\overline{\mathbb{C}} \setminus K)$ in the $\mathbb{C}$-valued case is surjective by [23, Corollary 5.6, p. 27] which follows from [23, Example 5.7 (a), p. 27-28] in the case that $K \subset \mathbb{R}$ or $K \cap \{\pm \infty\}$ has no isolated points in $K$. If $K \cap \{\pm \infty\}$ has isolated points in $K$, then the proof that the conditions of [23, Corollary 5.6, p. 27] are fulfilled is verbatim as in [23, Example 5.7 (a), p. 27-28]. Hence all conditions of Theorem 1 are fulfilled. $\square$

Theorem 20, together with [22, Corollary 18, p. 21] (K = $\emptyset$), generalises [19, 5.24 Theorem, p. 95] which is case (ii) above.

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