POLAR VARIETIES, BERTINI’S THEOREMS AND NUMBER OF POINTS OF SINGULAR COMPLETE INTERSECTIONS OVER A FINITE FIELD

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Abstract. Let $V \subset \mathbb{P}^n(\mathbb{F}_q)$ be a complete intersection defined over a finite field $\mathbb{F}_q$ of dimension $r$ and singular locus of dimension at most $s$, and let $\pi : V \to \mathbb{P}^{s+1}(\mathbb{F}_q)$ be a “generic” linear mapping. We obtain an effective version of the Bertini smoothness theorem concerning $\pi$, namely an explicit upper bound of the degree of a proper Zariski closed subset of $\mathbb{P}^{s+1}(\mathbb{F}_q)$ which contains all the points defining singular fibers of $\pi$. For this purpose we make essential use of the concept of polar variety associated to the set of exceptional points of $\pi$. As a consequence of our effective Bertini theorem we obtain results of existence of smooth rational points of $V$, namely conditions on $q$ which imply that $V$ has a smooth $\mathbb{F}_q$–rational point. Finally, for $s = r - 2$ and $s = r - 3$ we obtain estimates on the number of $\mathbb{F}_q$–rational points and smooth $\mathbb{F}_q$–rational points of $V$, and we discuss how these estimates can be used in order to determine the average value set of “small” families of univariate polynomials with coefficients in $\mathbb{F}_q$.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of $q$ elements and let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. We denote by $\mathbb{P}^n(\mathbb{F}_q)$, $\mathbb{P}^n := \mathbb{P}^n(\overline{\mathbb{F}}_q)$, $\mathbb{A}^n(\mathbb{F}_q)$ and $\mathbb{A}^n := \mathbb{A}^n(\overline{\mathbb{F}}_q)$ the $n$–dimensional projective and affine spaces defined over $\mathbb{F}_q$ and $\overline{\mathbb{F}}_q$ respectively. For any affine or projective variety $V$ defined over $\mathbb{F}_q$, we denote by $V(\mathbb{F}_q)$ the set of $\mathbb{F}_q$–rational points of $V$, namely the set of points of $V$ with coordinates in $\mathbb{F}_q$, and by $|V(\mathbb{F}_q)|$ its cardinality. Observe that, for any $r \geq 0$, we have

$$p_r := |\mathbb{P}^r(\mathbb{F}_q)| = q^r + \cdots + q + 1.$$
Let $V \subset \mathbb{P}^n$ be an ideal–theoretic complete intersection defined over $\mathbb{F}_q$, of dimension $r$, multidegree $d := (d_1, \ldots, d_{n-r})$ and singular locus of dimension $s \geq 0$. The main results of this paper are estimates on $|V(\mathbb{F}_q)|$ and conditions on $q$ which imply that $V(\mathbb{F}_q)$ is not empty. All these estimates and conditions will be expressed in terms of $r$, $d$ and $s$.

In a fundamental work \cite{Deligne1980}, P. Deligne has shown that if $V$ is nonsingular, then

\begin{equation}
|V(\mathbb{F}_q)| - p_r \leq b'_r(n, d) q^{r/2},
\end{equation}

where $b'_r(n, d)$ is the $r$th primitive Betti number of any nonsingular complete intersection of $\mathbb{P}^n$ of dimension $r$ and multidegree $d$ (see, e.g., \cite{Ghorpade1997} Theorem 4.1) for an explicit expression of $b'_r(n, d)$ in terms of $n$, $r$ and $d$.

This result has been extended by C. Hooley and N. Katz to singular complete intersections. More precisely, in \cite{HooleyKatz1996} it is proved that if the singular locus of $V$ has dimension at most $s \geq 0$, then

\begin{equation}
|V(\mathbb{F}_q)| = p_r + O(q^{(r+s+1)/2}),
\end{equation}

where the constant implied by the $O$–notation depends only on $n$, $r$ and $d$, and it is not explicitly given.

In \cite{Ghorpade1996} (see also \cite{Ghorpade1997}), S. Ghorpade and G. Lachaud have obtained the following explicit version of (2):

\begin{equation}
|V(\mathbb{F}_q)| - p_r \leq b'_{r-s-1}(n - s - 1, d) q^{(r+s+1)/2} + C(n, r, d)q^{(r+s)/2},
\end{equation}

where $C(n, r, d) := 9 \cdot 2^{n-r}((n-r)d + 3)^{n+1}$ and $d := \max_{1 \leq i \leq n-r} d_i$.

From the point of view of possible applications of (3), the fact that $C(n, r, d)$ depends exponentially on $n$ may be an inconvenience. This is particularly the case if $V$ is a hypersurface, because $C(n, r, d)$ becomes exponential in the degree of $V$. For this reason, in \cite{HooleyKatz1996} it is shown that, if $V$ is normal, then one has

\begin{equation}
|V(\mathbb{F}_q)| - p_r \leq b'_{1}(n - r + 1, d) q^{-1/2} + 2((n-r)d\delta)^2 q^{-1},
\end{equation}

provided that $q > 2(n - r)d\delta + 1$ holds, where $\delta = d_1 \cdots d_{n-r}$ is the degree of $V$. This solves the exponential dependency on $n$ of the error term in (3) for $s = r - 2$ and $q$ large enough.

1.1. Our contributions. A fundamental tool for our work is an effective version of the Bertini smoothness theorem. The Bertini smoothness theorem asserts that a generic $(r-s-1)$–dimensional linear section of a complete intersection $V \subset \mathbb{P}^n$ of dimension $r$ and singular locus of dimension $s$ is nonsingular. With notations as above, an effective version of this result establishes a threshold $C(n, r, s, d)$ such that for $q > C(n, r, s, d)$ there exists a nonsingular unidimensional linear section defined over $\mathbb{F}_q$ of a complete intersection defined over $\mathbb{F}_q$. In this
paper we show the following theorem (see Theorem 6.3 and Corollary 6.5 below).

**Theorem 1.1.** Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension $r$, degree $\delta$, multidegree $d := (d_1, \ldots, d_{n-r})$ and singular locus of dimension at most $s \geq 0$. Let $D := \sum_{i=1}^{n-r} (d_i - 1)$. Then for $q > (n+1)^2 D^{r-s-1} \delta$ there exist nonsingular $(r-s-1)$-dimensional linear sections of $V$ defined over $\mathbb{F}_q$.

We remark that [1] and [9] provide effective versions of the Bertini smoothness theorem for hypersurfaces and normal complete intersections respectively. Theorem 1.1 significantly improves and generalizes both results.

The linear sections underlying Theorem 1.1 are obtained as (the Zariski closure of) fibers of a “generic” linear mapping $\pi : V \to \mathbb{P}^{s+1}$. For this purpose, it is necessary to analyze the set $S$ of critical points of $\pi$. Our treatment of the set $S$ relies on the notion of polar varieties. Polar varieties are a classical concept of projective geometry which, in its modern formulation, was introduced in the 1930’s by F. Severi and J. Todd. Around 1975 a renewal of the theory of polar varieties took place with essential contributions due to R. Piene [28], B. Teissier [33] and others (see [34] for a historical account and references). Our main result in connection with polar varieties is a genericity condition on $\pi$ which implies that the polar variety associated to the exceptional locus of $\pi$ has the expected dimension (Theorem 4.5).

More precisely, let $\lambda \in (\mathbb{P}^n)^{s+2}$ denote the matrix of coefficients of the linear forms defining $\pi$. We show that there exists a hypersurface of $(\mathbb{P}^n)^{s+2}$ which contains all the points $\lambda$ for which the exceptional locus of $\pi$ has not the expected dimension. In order to bound the degree of this hypersurface, we use tools from intersection theory for products of projective spaces, such as a multiprojective version of the Bézout theorem (see, e.g., [12, Theorem 1.11]). Combining this with the results on the number of $\mathbb{F}_q$-rational points of multiprojective hypersurfaces of Section 3 we obtain suitable bounds on the number of nonsingular linear sections of $V$ defined over $\mathbb{F}_q$.

Next we obtain conditions on $q$ which imply that the variety $V$ under consideration has a smooth $\mathbb{F}_q$-rational point. A classical problem is that of establishing conditions which imply that a given variety has a $\mathbb{F}_q$-rational point. Nevertheless, in several number-theoretical applications it is not just an $\mathbb{F}_q$-rational point what is required, but a smooth $\mathbb{F}_q$-rational point (see, e.g., [20], [37], [38]).

A standard approach to this question consists of combining a lower bound for the number of $\mathbb{F}_q$-rational points with an upper bound for the number of singular $\mathbb{F}_q$-rational points of $V$. Instead of doing this, we use our effective Bertini theorem, namely we obtain a condition on $q$ which implies that there exists a nonsingular $(r-s-1)$-dimensional
linear section $S$ of $V$ defined over $\mathbb{F}_q$, and apply Deligne’s estimate \(^{11}\) to this linear section. As the linear section $S$ is contained in the smooth locus $V_{\text{sm}} := V \setminus \text{Sing}(V)$, the existence of an $\mathbb{F}_q$-rational point of $S$ implies that of a smooth $\mathbb{F}_q$-rational point of $V$. More precisely, we obtain the following result (see Corollaries \(5,3\) and \(7,4\)).

**Theorem 1.2.** Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension $r$, degree $\delta$, multidegree $d$ and singular locus of dimension at most $s$. Let $D := \sum_{i=1}^{n-r}(d_i - 1)$. If either $s = r - 2$ and $q > 2(D + 2)^2\delta^2$, or $s = r - 3$ and $q > 3D(D + 2)^2\delta$ holds, then $V$ has a smooth $\mathbb{F}_q$-rational point.

Finally, we obtain estimates on the number of $\mathbb{F}_q$-rational points and smooth $\mathbb{F}_q$-rational points of a complete intersection for which the singular locus has dimension $s$ at most $r - 2$ or $r - 3$. For this purpose, assuming that there exists a linear mapping $\pi : V \rightarrow \mathbb{P}^{s+1}$ defined over $\mathbb{F}_q$ which is generic in the sense above, we express $V$ as the union of $p_{s+1} := |\mathbb{P}^{s+1}(\mathbb{F}_q)|$ linear sections of $V$ of dimension $r - s - 1$, namely the Zariski closure of the fibers of the points of $\mathbb{P}^{s+1}(\mathbb{F}_q)$ under $\pi$. “Most” fibers will be nonsingular and thus Deligne’s estimate can be applied to them, while the $\mathbb{F}_q$-rational points lying in the remaining fibers do not make a significant contribution to the estimate. Summarizing, we obtain the following result (see Corollaries \(8,3\) and \(8,4\) below).

**Theorem 1.3.** Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension $r$, degree $\delta$, multidegree $d$ and singular locus of dimension at most $s \in \{r - 3, r - 2\}$. Let $D := \sum_{i=1}^{n-r}(d_i - 1)$. Then, for $s \leq r - 2$, we have:

$$
\left| |V(\mathbb{F}_q)| - p_r \right| \leq (\delta(D - 2) + 2)q^{r-1/2} + 14D^2\delta^2q^{r-1}, \\
\left| |V_{\text{sm}}(\mathbb{F}_q)| - p_r \right| \leq (\delta(D - 2) + 2)q^{r-1/2} + 8(r + 1)D^2\delta^2q^{r-1}.
$$

On the other hand, for $s \leq r - 3$, we have:

$$
\left| |V(\mathbb{F}_q)| - p_r \right| \leq 14D^2\delta^2q^{r-1}, \\
\left| |V_{\text{sm}}(\mathbb{F}_q)| - p_r \right| \leq (34r - 20)D^2\delta^2q^{r-1}.
$$

Our estimates for the number of $\mathbb{F}_q$-rational points follow the pattern of \(8\) for $s = r - 2$ or $s = r - 3$, but differ from \(8\) in that the exponential dependency on $n$ is not present. In Section \(8\) we show that Theorem \(1.3\) yields a more accurate estimate than \(8\) in the case $s = r - 2$ and $s = r - 3$ for varieties of large dimension, say $r \geq (n+1)/2$, or small degree, say $\delta \leq (2(n + r))^{n-r}$. On the other hand, \(8\) may be preferable to Theorem \(1.3\) for varieties of small dimension and large degree. In this sense, we may say that Theorem \(1.3\) complements \(8\) for $s = r - 2$ and $s = r - 3$. Finally, Theorem \(1.3\) improves \(4\) for normal varieties in that it holds without any restriction on $q$, while the latter only holds for $q > 2(n - r)d\delta + 1$.

We end this paper discussing a problem that requires the estimates underlying Theorem \(1.3\) the average value set of “small” families of
polynomials. We sketch how we apply Theorem 1.3 in order to determine the average value set of families of univariate polynomials of \( \mathbb{F}_q[T] \) of fixed degree where certain coefficients are fixed. Since this problem is concerned with a complete intersection of “low” degree, our estimate yields a significant gain compared with what is obtained by means of (3).

2. Notions, notations and preliminary results

We use standard notions and notations of commutative algebra and algebraic geometry as can be found in, e.g., [18], [25] or [32].

Let \( K \) be any of the fields \( \mathbb{F}_q \) or \( \overline{\mathbb{F}_q} \). We say that \( V \) is a projective (affine) variety defined over \( K \) if it is the set of all common zeros in \( \mathbb{P}^n (\mathbb{A}^n) \) of a family of homogeneous polynomials \( F_1, \ldots, F_m \in K[X_0, \ldots, X_n] \) (of polynomials \( F_1, \ldots, F_m \in K[X_1, \ldots, X_n] \)). In the remaining part of this section, unless otherwise stated, all results referring to varieties in general should be understood as valid for both projective and affine varieties.

For a \( K \)–variety \( V \) in the \( n \)–dimensional (affine or projective) space, we denote by \( I(V) \) its defining ideal and by \( K[V] \) its coordinate ring. The \textit{dimension} \( \dim V \) of a \( K \)–variety \( V \) is the (Krull) dimension of its coordinate ring \( K[V] \). The \textit{degree} \( \deg V \) of an irreducible \( K \)–variety \( V \) is the maximum number of points lying in the intersection of \( V \) with a generic linear subspace \( L \) of codimension \( \dim V \) for which \( |V \cap L| < \infty \) holds. More generally, if \( V = V_1 \cup \cdots \cup V_s \) is the decomposition of \( V \) into irreducible \( K \)–components, we define the degree of \( V \) as \( \deg V := \sum_{i=1}^{s} \deg V_i \) (cf. [19]). We shall say that \( V \) has \textit{pure dimension} \( r \) if every irreducible \( K \)–component of \( V \) has dimension \( r \). A \( K \)–variety \( V \) is \textit{absolutely irreducible} if it is irreducible as \( \mathbb{F}_q \)–variety.

We say that a \( K \)–variety \( V \) of dimension \( r \) in the \( n \)–dimensional space is an (ideal–theoretic) \textit{complete intersection} if its ideal \( I(V) \) over \( K \) can be generated by \( n-r \) polynomials. If \( V \subset \mathbb{P}^n \) is a complete intersection defined over \( K \), of dimension \( r \) and degree \( \delta \), and \( F_1, \ldots, F_{n-r} \) is a system of generators of \( I(V) \), the degrees \( d_1, \ldots, d_{n-r} \) depend only on \( V \) and not on the system of generators. Arranging the \( d_i \) in such a way that \( d_1 \geq d_2 \geq \cdots \geq d_{n-r} \), we call \( d := (d_1, \ldots, d_{n-r}) \) the \textit{multidegree} of \( V \). In particular, it follows that \( \delta = \prod_{i=1}^{n-r} d_i \) holds.

An important tool for our estimates is the following \textit{Bézout inequality} (see [19], [15], [36]): if \( V \) and \( W \) are \( K \)–varieties, then the following inequality holds:

\[
\deg(V \cap W) \leq \deg V \cdot \deg W.
\]

Let \( \phi : V \rightarrow W \) be a regular linear map of \( K \)–varieties. Then we have (see, e.g., [3], Lemma 2.1)):

\[
\deg \overline{\phi(V)} \leq \deg V.
\]
For a given variety $V$, we denote by $V(\mathbb{F}_q)$ the set of $\mathbb{F}_q$-rational points of $V$, namely, $V(\mathbb{F}_q) := V \cap \mathbb{F}^n_q$ in the projective case and $V(\mathbb{F}_q^0) := V \cap \mathbb{A}^n(\mathbb{F}_q)$ in the affine case. For a projective variety $V$ of dimension $r$ and degree $\delta$ we have the upper bound (see [17 Proposition 12.1] or [9 Proposition 3.1])

\[
|V(\mathbb{F}_q)| \leq \delta r.
\]

2.1. Multiprojective space. Let $\mathbb{N} := \mathbb{Z}_{\geq 0}$ be the set of nonnegative integers. For $\mathbf{n} := (n_1, \ldots, n_m) \in \mathbb{N}^m$, we define $|\mathbf{n}| := n_1 + \cdots + n_m$ and $\mathbf{n}! := n_1! \cdots n_m!$. Given $\alpha, \beta \in \mathbb{N}^m$, we write $\alpha \geq \beta$ whenever $\alpha_i \geq \beta_i$ holds for $1 \leq i \leq m$. For $\mathbf{d} := (d_1, \ldots, d_m) \in \mathbb{N}^m$, the set $\mathbb{N}^d_{\mathbf{n}+1} := \mathbb{N}^n_{d_1+1} \times \cdots \times \mathbb{N}^n_{d_m+1}$ consists of the elements $\mathbf{a} := (a_1, \ldots, a_m) \in \mathbb{N}^n_{d_1+1} \times \cdots \times \mathbb{N}^n_{d_m+1}$ with $|a_i| = d_i$ for $1 \leq i \leq m$.

For $\mathbb{K} := \mathbb{F}_q$ or $\mathbb{K} := \mathbb{F}_q$, we denote by $\mathbb{P}^n(\mathbb{K})$ the multiprojective space $\mathbb{P}^n(\mathbb{K}) := \mathbb{P}(\mathbb{K}) \times \cdots \times \mathbb{P}(\mathbb{K})$ defined over $\mathbb{K}$. We shall use the notation $\mathbb{P}^n := \mathbb{P}^n(\mathbb{F}_q)$. For $1 \leq i \leq m$, let $X_i := \{X_{i,0}, \ldots, X_{i,n_i}\}$ be group of $n_i + 1$ variables and let $X := \{X_1, \ldots, X_m\}$. A multihomogeneous polynomial $F \in \mathbb{K}[X]$ of multidegree $\mathbf{d} := (d_1, \ldots, d_m)$ is a polynomial which is homogeneous of degree $d_i$ in $X_i$ for $1 \leq i \leq m$. An ideal $I \subset \mathbb{K}[X]$ is multihomogeneous if it is generated by a family of multihomogeneous polynomials. For any such ideal, we denote by $V(I) \subset \mathbb{P}^n$ the variety defined over $\mathbb{K}$ as its set of common zeros. In particular, an hypersurface in $\mathbb{P}^n$ defined over $\mathbb{K}$ is the set of zeros of a multihomogeneous polynomial of $\mathbb{K}[X]$. The notions of irreducible variety and dimension of a subvariety of $\mathbb{P}^n$ are defined as in the projective space.

Let $V \subset \mathbb{P}^n$ be an irreducible variety of dimension $r$ and let $I(V) \subset \mathbb{F}_q[X]$ be its multihomogeneous ideal. The quotient ring $\mathbb{F}_q[X]/I(V)$ is multigraded and its part of multidegree $\mathbf{b} \in \mathbb{N}^m$ is denoted by $(\mathbb{F}_q[X]/I(V))_\mathbf{b}$. The Hilbert–Samuel function of $V$ is the function $H_V : \mathbb{N}^m \to \mathbb{N}$ defined as $H_V(\mathbf{b}) := \dim(\mathbb{F}_q[X]/I(V))_\mathbf{b}$. It turns out that there exist $\mathbf{d}_0 \in \mathbb{N}^m$ and a unique polynomial $P_V \in \mathbb{Q}[z_1, \ldots, z_m]$ of degree $\mathbf{r}$ such that $P_V(\mathbf{d}) = H_V(\mathbf{d})$ for every $\mathbf{d} \in \mathbb{N}^m$ with $\mathbf{d} \geq \mathbf{d}_0$ (see, e.g., [12 Proposition 1.8]). For $\mathbf{b} \in \mathbb{N}^m$, we define the mixed degree of $V$ of index $\mathbf{b}$ as the nonnegative integer

$$\text{deg}_\mathbf{b}(V) := \mathbf{b}! \cdot \text{coeff}(P_V).$$

This notion can be extended to equidimensional varieties and, more generally, to equidimensional cycles (formal linear combination with integer coefficients of subvarieties of equal dimension) by linearity.

The Chow ring of $\mathbb{P}^n$ is the graded ring

$$A^*(\mathbb{P}^n) := \mathbb{Z}[\theta_1, \ldots, \theta_m]/(\theta_1^{n_1+1}, \ldots, \theta_m^{n_m+1}),$$

where each $\theta_i$ denotes the class of the inverse image of a hyperplane of $\mathbb{P}^n$, under the projection $\mathbb{P}^n \to \mathbb{P}^m$. Given a variety $V \subset \mathbb{P}^n$ of pure
dimension \( r \), its class in the Chow ring is

\[
[V] := \sum_{b} \deg_b(V)\theta_{r_1}^{n_1 - b_1} \cdots \theta_{r_m}^{n_m - b_1} \in A^*(\mathbb{P}^n),
\]

where the sum is over all \( b \in \mathbb{N}^m \) with \( b \leq n \). This is an homogeneous element of degree \(|n| - r\). In particular, if \( \mathcal{H} \subset \mathbb{P}^n \) is an hypersurface and \( F \in \mathbb{F}[X] \) is a polynomial of minimal degree defining \( \mathcal{H} \), then

\[
[\mathcal{H}] := \sum_{i=1}^m \deg X_i(F) \theta_i
\]

(see, e.g., [12, Proposition 1.10]).

3. Number of zeros of multihomogeneous hypersurfaces

Let \( n := (n_1, \ldots, n_m) \in \mathbb{N}^m \) and let \( \mathbb{P}^n(\mathbb{F}_q) \) be the multiprojective space defined over \( \mathbb{F}_q \). For \( 1 \leq i \leq m \), let \( X_i := \{X_{i,0}, \ldots, X_{i,n_i}\} \) be a group of \( n_i + 1 \) variables and let \( X := \{X_1, \ldots, X_m\} \). Let \( F \in \mathbb{F}[X] \) be a multihomogeneous polynomial of multidegree \( d := (d_1, \ldots, d_m) \). In this section we collect basic facts concerning the number of \( \mathbb{F}_q \)-rational zeros of \( F \).

For \( \alpha \in \mathbb{N}^m \), we use the notations \( d^{\alpha} := d_1^{\alpha_1} \cdots d_m^{\alpha_m} \) and \( p_{n-\alpha} := p_{n_1-\alpha_1} \cdots p_{n_m-\alpha_m} \) for \( n \geq \alpha \). We have the following result.

**Proposition 3.1.** Let \( F \in \mathbb{F}_q[X] \) be a multihomogeneous polynomial of multidegree \( d \) with \( \max_{1 \leq i \leq m} d_i < q \) and let \( N \) be the number of zeros of \( F \) in \( \mathbb{P}^n(\mathbb{F}_q) \). Then

\[
N \leq \eta_m(d, n) := \sum_{\varepsilon \in \{0,1\}^m \setminus \{0\}} (-1)^{|\varepsilon|+1} d^\varepsilon p_{n-\varepsilon}.
\]

**Proof.** We argue by induction on \( m \). The case \( m = 1 \) of the statement is [7].

Suppose that the statement holds for \( m - 1 \) and let \( F \in \mathbb{F}_q[X] \) be an \( m \)-homogeneous polynomial of multidegree \( d := (d_1, \ldots, d_m) \). Let \( N \) be the number of zeros of \( F \) in \( \mathbb{P}^n(\mathbb{F}_q) \), and let \( Z_m \) be the subset of \( \mathbb{P}^{nm}(\mathbb{F}_q) \) which consists of the elements \( x_m \) such that the substitution \( F(X_1, \ldots, X_{m-1}, x_m) \) of \( x_m \) for \( X_m \) in \( F \) yields the zero polynomial of \( \mathbb{F}_q[X_1, \ldots, X_{m-1}] \). According to [7] we have \( |Z_m| \leq d_m p_{n_m-1} \).

Fix \( x_m \in \mathbb{P}^{nm}(\mathbb{F}_q) \setminus Z_m \) and denote by \( N_{m-1} \) the number of zeros of \( F(X_1, \ldots, X_{m-1}, x_m) \) in \( \mathbb{P}^{n_1}(\mathbb{F}_q) \times \cdots \times \mathbb{P}^{n_{m-1}}(\mathbb{F}_q) \). Then the inductive hypothesis implies

\[
N_{m-1} \leq \eta_{m-1}(d', n'),
\]

where \( d' := (d_1, \ldots, d_{m-1}) \) and \( n' := (n_1, \ldots, n_{m-1}) \). By the induction hypothesis, it follows that

\[
N \leq \sum_{x_m} \sum_{x_m} \eta_{m-1}(d', n') \leq \eta_m(d, n),
\]

as desired.
where \( \mathbf{d}^* := (d_1, \ldots, d_{m-1}) \) and \( \mathbf{n}^* := (n_1, \ldots, n_{m-1}) \). As a consequence, we obtain

\[
N \leq |Z_m| \cdot p_{n_1} \cdot \cdots \cdot p_{n_{m-1}} + (p_{n_m} - |Z_m|) \cdot \eta_{m-1}(\mathbf{d}^*, \mathbf{n}^*) \leq \eta_m(\mathbf{d}, \mathbf{n}).
\]

This completes the proof of the proposition. □

Since the proof of Proposition 3.1 is concerned with hypersurfaces, in order to bound from above the number of \( \mathbb{F}_q \)-rational points of a projective hypersurface \( \mathcal{H} \subset \mathbb{P}^n \) of degree \( \delta \) one may use the Serre bound (see [31]):

\[
|\mathcal{H}(\mathbb{F}_q)| \leq \delta q^{n-1} + p_{n-2}.
\]

Although (8) is stated for hypersurfaces defined over \( \mathbb{F}_q \) in [31], it is easy to see that it also holds for hypersurfaces defined over \( \mathbb{F}_q \). Using (8) the upper bound of Proposition 3.1 can be slightly improved. In particular, it may be worthwhile to remark that, if \( \mathbf{d}, \mathbf{n} \in \mathbb{N}^m \) are of the form \( \mathbf{d} = (d, \ldots, d) \) and \( \mathbf{n} := (n, \ldots, n) \), by using (8) we obtain

\[
N \leq p_m^m - (q^n - (d-1)q^{n-1})^m.
\]

A similar argument as in the proof of Proposition 3.1 yields the following upper bound for the number \( N_a \) of zeros of \( F \) in \( \mathbb{F}_q^{n+1} := \mathbb{F}_q^{n_1+1} \times \cdots \times \mathbb{F}_q^{n_m+1} \):

\[
N_a \leq \eta^{\mathbb{F}_q} \cdot (\mathbf{d}, \mathbf{n}) := \sum_{\epsilon \in \{0, 1\}^m \setminus \{0\}} (-1)^{|\epsilon|+1} d^\epsilon q^{n+1-\epsilon},
\]

where \( \mathbf{q}, \mathbf{1} \in \mathbb{N}^m \) are defined by \( \mathbf{q} := (q, \ldots, q) \) and \( \mathbf{1} := (1, \ldots, 1) \).

We end this section with the following consequence of Proposition 3.1.

**Corollary 3.2.** Let \( F \in \mathbb{F}_q[\mathbf{X}] \) be a multihomogeneous polynomial of multidegree \( \mathbf{d} \) and let \( d := \max_{1 \leq i \leq m} d_i \). If \( q > d \), then there exists \( \mathbf{x} \in \mathbb{P}^n(\mathbb{F}_q) \) such that \( F(\mathbf{x}) \neq 0 \) holds.

**Proof.** It suffices to show that there exists \( \mathbf{x} \in \mathbb{F}_q^{n+1} \) for which \( F(\mathbf{x}) \neq 0 \) holds. For this purpose, according to (10) we have that the number of elements of \( \mathbb{F}_q^{n+1} \) not annihilating \( F \) is bounded from above by the following quantity:

\[
q^{n+1} \cdot \eta^\mathbb{F}_q \cdot (\mathbf{d}, \mathbf{n}) = \sum_{\epsilon \in \{0, 1\}^m} (-1)^{|\epsilon|} d^\epsilon q^{n+1-\epsilon} = q^n \prod_{j=1}^{m} (q - d_j).
\]

Our hypothesis implies that the right–hand side of the previous identities is strictly positive, which immediately yields the corollary. □
4. Polar varieties

Let $V \subset \mathbb{P}^n$ be a variety of pure dimension $r$ and degree $\delta$. Let $\Sigma \subset V$ denote the singular locus of $V$ and let $V_{\rm sm} := V \setminus \Sigma$. For each integer $s$ with $0 \leq s \leq r - 2$ and for $x \in V_{\rm sm}$, a linear variety $L \subset \mathbb{P}^n$ of dimension $n - s - 2$ meets $T_x V \subset \mathbb{P}^n$ in dimension at least $r - s - 2$. The set of points $x \in V_{\rm sm}$ such that the dimension of the intersection is greater than or equal to $r - s - 1$ is called the $s$th polar variety of $V$ with respect to $L$ and is denoted by $M(L)$. In symbols,

$$M(L) := \{ x \in V_{\rm sm} : \dim(T_x V \cap L) \geq r - s - 1 \}.$$  

This is a classical notion of projective geometry. The polar variety $M(L)$ is empty or of pure dimension at least $s$. In fact, following [24], we have that, for a generic $L$, the polar variety $M(L)$ has dimension $s$.

We include a proof of this result for the sake of completeness (see also [28, Transversality Lemma 1.3]).

Proposition 4.1. For a generic linear variety $L \subset \mathbb{P}^n$ of dimension $n - s - 2$, the polar variety $M(L)$ has dimension $s$.

Proof. Let $\mathbb{G}(r, n)$ denote the Grassmannian of $r$-planes in $\mathbb{P}^n$. We consider the Gauss map $G : V_{\rm sm} \rightarrow \mathbb{G}(r, n)$, which maps a point $x$ into the tangent space $T_x V$. Let $S \subset \mathbb{G}(r, n)$ be the following Schubert variety:

$$S = \{ \Lambda \in \mathbb{G}(r, n) : \dim(\Lambda \cap L) \geq r - s - 1 \}.$$  

First we observe that $S$ has dimension $\dim(\mathbb{G}(r, n)) - (r - s)$ (see, e.g., [18, Example 11.42]). Furthermore, it is clear that $M(L) = G^{-1}(S \cap \mathbb{G}(V_{\rm sm}))$. Let $i : S \rightarrow \mathbb{G}(r, n)$ denote the standard inclusion mapping. We claim that the polar variety $M(L)$ coincides with the fiber product $V_{\rm sm} \times_{\mathbb{G}(r, n)} S$. Indeed,

$$V_{\rm sm} \times_{\mathbb{G}(r, n)} S = \{(x, \Lambda) \in V_{\rm sm} \times S : T_x V = \Lambda\} = \{ x \in V_{\rm sm} : \dim(T_x V \cap L) \geq r - s - 1 \} = M(L).$$  

The general linear group acts transitively on $\mathbb{G}(r, n)$, and with respect to this action $S$ is in general position because $L$ is so by hypothesis. Therefore, [23, Theorem 2] shows that $M(L)$ is of pure dimension

$$\dim M(L) = \dim V_{\rm sm} + \dim S - \dim \mathbb{G}(r, n) = s.$$  

This finishes the proof of the proposition. \qed

Set $X := (X_0, \ldots, X_n)$. For $\mu := (\mu_0 : \cdots : \mu_n) \in \mathbb{P}^n$, we shall use the notation $\mu \cdot X := \mu_0 X_0 + \cdots + \mu_n X_n$. Let $\lambda_0, \ldots, \lambda_{s+1}$ be linearly independent elements of $\mathbb{P}^n$ and let $L \subset \mathbb{P}^n$ be the linear space of dimension $n - s - 2$ defined by

$$L := \{ x \in \mathbb{P}^n : \lambda_0 \cdot x = \cdots = \lambda_{s+1} \cdot x = 0 \}. \quad (11)$$

For $x \in V_{\rm sm}$, let $\varphi_x : T_x V \rightarrow \mathbb{P}^{s+1}$ be the linear mapping defined by $Y_i := \lambda_i \cdot X$ ($0 \leq i \leq s + 1$). Observe that $\varphi_x$ may be seen
For any \( x \), \( \lambda \) is dependent and the tangent space defined by homogeneous polynomials \( \in x \) of their Jacobian in the context of efficient real elimination.

Remark 4.2. The polar variety \( M(L) \) coincides with the set of points \( x \in V_{sm} \) such that the dimension of \( E_x \) is at least \( r - s - 1 \).

A critical point for our approach is that the polar variety \( M(L) \) can be defined in terms of the vanishing of certain minors involving the partial derivatives of the polynomials defining \( V \) and \( L \). This has the advantage of providing an explicit system of equations defining the polar variety \( M(L) \). In the series of papers [4], [5], [6], [2], [3] polar varieties are locally described by regular sequences consisting of the polynomials defining \( V \) and certain well-determined maximal minors of their Jacobian in the context of efficient real elimination.

Assume that \( V \) is an ideal–theoretic complete intersection in \( \mathbb{P}^n \) defined by homogeneous polynomials \( F_1, \ldots, F_{n-r} \in \mathbb{E}[X_0, \ldots, X_n] \) of degrees \( d_1 \geq \cdots \geq d_{n-r} \geq 2 \) respectively and let \( D := \sum_{i=1}^{n-r} (d_i - 1) \). For any \( x \in V_{sm} \), the gradients \( \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \) are linearly independent and the tangent space \( T_x V \) is the \( r \)-dimensional linear variety

\[
T_x V = \{ v \in \mathbb{P}^n : \nabla F_1(x) \cdot v = \cdots = \nabla F_{n-r}(x) \cdot v = 0 \}.
\]

Let \( 0 \leq s \leq r - 2 \) and consider the \((n - s - 2)\)-dimensional linear variety \( L \) of \( \mathbb{P}^n \). Write \( \lambda_i := (\lambda_{i,0}, \ldots, \lambda_{i,n}) \) for \( 0 \leq i \leq s + 1 \), \( \lambda := (\lambda_0, \ldots, \lambda_{s+1}) \) and consider the matrix

\[
M(X, \lambda) := \begin{pmatrix}
\frac{\partial F_1}{\partial X_0} & \cdots & \frac{\partial F_1}{\partial X_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_{n-r}}{\partial X_0} & \cdots & \frac{\partial F_{n-r}}{\partial X_n} \\
\lambda_{0,0} & \cdots & \lambda_{0,n} \\
\vdots & \ddots & \vdots \\
\lambda_{s+1,0} & \cdots & \lambda_{s+1,n}
\end{pmatrix}.
\]

The dimension of \( T_x V \cap L \) is equal to \( r - s - 2 \) if and only if \( M(x, \lambda) \) has maximal rank. Equivalently, \( M(x, \lambda) \) has no maximal rank if and only if the dimension of \( T_x V \cap L \) is at least \( r - s - 1 \). As a consequence, if we denote by \( \Delta_1(x, \lambda), \ldots, \Delta_N(x, \lambda) \) the maximal minors of \( M(x, \lambda) \), then the polar variety \( M(L) \) is given by

\[
M(L) = \{ x \in V_{sm} : \Delta_1(x, \lambda) = \cdots = \Delta_N(x, \lambda) = 0 \}.
\]

Proposition 4.3. Suppose that \( M(L) \) has dimension \( s \) and denote by \( \Sigma \) the singular locus of \( V \). Then there exists a subvariety \( Z(L) \subset V \) of pure dimension \( s \) and degree at most \( D^{r-s} \delta \) such that \( M(L) \cup \Sigma \subset Z(L) \) holds.
Proof. Since $M(L) \cup \Sigma = \{ x \in V : \Delta_1(x, \lambda) = \cdots = \Delta_N(x, \lambda) = 0 \}$ has dimension at most $s \leq r - 2$, there exists $x \in V \setminus (M(L) \cup \Sigma)$. For each such $x$, there exists at least a maximal minor $\Delta_j$ of the matrix $M(X, \lambda)$ of (12) with $\Delta_j(x, \lambda) \neq 0$, and thus an $\mathbb{F}_q$-linear combination $G_1 := \sum_{j=1}^N \gamma_j \Delta_j$ with $G_1(x, \lambda) \neq 0$. This implies that $G_1 \in \mathbb{F}_q[X_0, \ldots, X_n]$ is a nonzero polynomial of degree $D$ vanishing on $M(L) \cup \Sigma$ and not vanishing identically on $V$. The absolute irreducibility of $V$ implies that $V^{(1)} := V \cap \{ G^{(1)} = 0 \}$ is a projective variety of pure dimension $r - 1$ for which $M(L) \cup \Sigma \subset V^{(1)}$ holds. By the Bézout inequality (5) we deduce that $\deg V^{(1)} \leq D\delta$.

Let $V^{(1)} = \bigcup_{i=1}^t C_i$ be the decomposition of $V^{(1)}$ into absolutely irreducible components. Since $\dim(M(L) \cup \Sigma) \leq s < r - 1$ holds, we may choose regular points $x_i \in C_i \setminus M(L)$ for $1 \leq i \leq t$. Arguing as above, we conclude that there exist $\gamma_1^{(2)}, \gamma_2^{(2)} \in \mathbb{F}_q$ such that the nonzero polynomial $G^{(2)} := \sum_{j=1}^N \gamma_j^{(2)} \Delta_j(X, \lambda)$ does not vanish on $x_i$ for $1 \leq i \leq t$. Therefore $V^{(2)} := V^{(1)} \cap \{ G^{(2)} = 0 \}$ is a projective variety of pure dimension $r - 2$ and degree at most $D^2\delta$ with $M(L) \cup \Sigma \subset V^{(2)}$.

Applying successively this argument we finally obtain a projective variety $V^{(r-s)}$ of pure dimension $s$ and degree at most $D^{r-s}\delta$ with $M(L) \cup \Sigma \subset V^{(r-s)}$. The proof of the proposition finishes setting $Z(L) := V^{(r-s)}$. \qed

In the sequel we shall encounter several times a similar situation as in Proposition 4.3, namely a projective or multiprojective subvariety $W_1$ of a pure dimensional variety $W$, which is defined as the zero locus in $W$ of homogeneous or multihomogeneous polynomials $H_1, \ldots, H_M$. If $m$ denotes the codimension of $W_1$ in $W$, arguing as in the proof of Proposition 4.3 we shall conclude that there exist $m$ generic linear combinations $H^1, \ldots, H^m$ of $H_1, \ldots, H_M$ such that the zero locus $W_2$ of $H^1, \ldots, H^m$ in $W$ has pure codimension $m$ in $W$ and contains $W_1$.

As established in Proposition 4.4 for a generic choice of $L$ the dimension of $M(L)$ is equal to $s$. Our next goal is to obtain conditions on $\lambda_0, \ldots, \lambda_{s+1} \in \mathbb{P}^n$ which imply that the polar variety $M(L)$ has dimension $s$. For $0 \leq i \leq s + 1$ we shall denote by $\Lambda_i := (\Lambda_{i,0}, \ldots, \Lambda_{i,n})$ a group of $n + 1$ variables and set $\Lambda := (\Lambda_0, \ldots, \Lambda_{s+1})$. We consider the so-called generic polar variety, namely

$$W := (V_{nm} \times \mathcal{U}) \cap \{ \Delta_1(X, \Lambda) = \cdots = \Delta_N(X, \Lambda) = 0 \},$$

where $\mathcal{U} \subset (\mathbb{P}^n)^{s+2}$ is the Zariski open subset consisting of all the $(s + 2) \times (n + 1)$-matrices of maximal rank and $\Delta_1, \ldots, \Delta_N$ are the maximal minors of the generic version $M(X, \Lambda)$ of the matrix $M(X, \lambda)$ of (12).

Proposition 4.4. Let $t := n(s + 2)$. Then $W$ is an irreducible variety of $\mathbb{P}^n \times \mathcal{U}$ of dimension $s + t$. 

Proof. Let $\pi_1 : W \to V_{sm}$ be the linear projection $\pi_1(x, \lambda) := x$. Fix $x \in V_{sm}$ and consider the fiber $\pi_1^{-1}(x)$. We have that $\pi_1^{-1}(x) = \{x\} \times \mathcal{L}$, where $\mathcal{L} \subset \mathcal{U}$ denotes the set of matrices $\lambda := (\lambda_0, \ldots, \lambda_{s+1})$ for which the matrix $M(x, \lambda)$ is not of full rank. This is the same as saying that

$$\langle \lambda_0, \ldots, \lambda_{s+1} \rangle \cap \langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \rangle \neq \emptyset,$$

where $\langle v_0, \ldots, v_m \rangle \subset \mathbb{A}^{n+1}$ is the linear variety spanned by $v_0, \ldots, v_m$ in $\mathbb{A}^{n+1}$. Equivalently, the vectors $\lambda_0, \ldots, \lambda_{s+1}$ are not linearly independent in the quotient $\mathbb{F}_q$-vector space

$$\mathcal{V} := \mathbb{A}^{n+1}/\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \rangle.$$

This shows that $\mathcal{L}$ is, modulo $\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \rangle$, isomorphic to the Zariski open set $L_{s+1}(\mathbb{A}^{s+2}, \mathcal{V}) \cap \mathcal{U}$, where

$$L_{s+1}(\mathbb{A}^{s+2}, \mathcal{V}) := \{ f \in \text{Hom}_{\mathbb{F}_q}(\mathbb{A}^{s+2}, \mathcal{V}) : \text{rank}(f) \leq s + 1 \}.$$

According to [8, Proposition 1.1], $L_{s+1}(\mathbb{A}^{s+2}, \mathcal{V})$ is an irreducible variety of dimension $(s + 1)(r + 2)$. Taking into account that we are considering subspaces of $\mathbb{A}^{n+1}$ of dimension $s + 2$ modulo a subspace $\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \rangle$ of dimension $n - r$, we see that $\mathcal{L}$ is an irreducible variety of $\mathbb{P}^n \times \mathbb{A}^{(n+1)(s+2)}$ of dimension $(s + 1)(r + 2) + (n - r)(s + 2) = (n + 1)(s + 2) - r - s$. We may also rephrase this conclusion saying that $\pi_1^{-1}(x) = \{x\} \times \mathcal{L}$ is an irreducible variety of $(\mathbb{P}^n)^{s+3}$ of dimension $t + s - r$.

Our previous arguments shows that $\pi_1 : W \to V_{sm}$ is surjective. Then the proof of [32, §I.6.3, Theorem 8] shows that $W$ is an irreducible variety of $V_{sm} \times \mathcal{U}$ of dimension $s + t$. \hfill $\square$

In the sequel, we shall associate each point $\lambda := (\lambda_0, \ldots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2}$ with the linear space $L := \{ x \in \mathbb{P}^n : \lambda_0 : x = \cdots = \lambda_{s+1} : x = 0 \}$.

Theorem 4.5. There exists an hypersurface $\mathcal{H}_1 \subset (\mathbb{P}^n)^{s+2}$, defined by a multihomogeneous polynomial of degree at most $(n-s)(r-s)D^{s-1}\delta+1$ in each group of variables $\Lambda_i$, such that for any $\lambda \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}_1$ the polar variety $M(L)$ has dimension at most $s$.

Proof. According to Proposition 4.4 for a generic matrix $\lambda \in \mathcal{U}$ the polar variety $M(L)$ is of pure dimension $s \geq 0$. Then the projection mapping $\pi_2 : W \to (\mathbb{P}^n)^{s+2}$ defined by $\pi_2(x, \lambda) := \lambda$ is dominant. This implies that the field extension $\mathbb{F}_q(\Lambda) \hookrightarrow \mathbb{F}_q(W)$ has transcendence degree $s + 1$, and therefore, there exist indices $i_0, \ldots, i_s$ such that the coordinate functions of $\mathbb{F}_q(W)$ defined by $X_{i_0}, \ldots, X_{i_s}$ form a transcendence basis of this field extension.

Fix $i \in \Gamma := \{0, \ldots, n\} \setminus \{i_0, \ldots, i_s\}$ and consider the linear mapping $\pi^i : W \to \mathbb{P}^{s+1} \times (\mathbb{P}^n)^{s+2}$ defined by $X_{i_0}, \ldots, X_{i_s}, X_i$ and $\Lambda$. Then the Zariski closure $W_i \subset \mathbb{P}^{s+1} \times (\mathbb{P}^n)^{s+2}$ of $\pi^i(W)$ is an hypersurface. Since $F_1, \ldots, F_{n-r}$ define a subvariety of $(\mathbb{P}^n)^{s+3}$ of pure dimension $r + n(s + 2)$, from Proposition 4.3 we conclude that there exist $r - s$
generic linear combinations, say \( \Delta_1, \ldots, \Delta^{r-s} \), of \( \Delta_1, \ldots, \Delta_N \), such that \( F_1, \ldots, F_{n-r}, \Delta_1, \ldots, \Delta^{r-s} \) define a subvariety \( W' \) of \((\mathbb{P}^n)^{s+3}\) of pure dimension \( s + n(s + 2) \) containing \( W \). In particular, the Zariski closure \( W' \) of \( \pi'(W') \) is an hypersurface of \( \mathbb{P}^{s+3} \times (\mathbb{P}^n)^{s+2} \) which contains \( W_1 \).

Next we estimate the multidegree of \( W' \), and hence of \( W_i \). For this purpose, we consider the class \([W']\) of \( W' \) in the Chow ring \( \mathcal{A}^*((\mathbb{P}^n)^{s+3}) \) of \((\mathbb{P}^n)^{s+3}\). Denote by \( \theta_{j-2} \) the class of the inverse image of a hyperplane of \( \mathbb{P}^n \) under the \( j \)th canonical projection \( (\mathbb{P}^n)^{s+3} \to \mathbb{P}^n \) for \( 1 \leq j \leq s+3 \). According to the multihomogeneous Bézout theorem (see, e.g., [12, Theorem 1.11]), we have

\[
[W'] = \prod_{i=1}^{n-r}(d_i \theta_{-1}) \prod_{k=1}^{r-s}(D \theta_{-1} + \theta_0 + \cdots + \theta_{s+1})
\]

\[
= \delta D^{r-s-1} (D(\theta_{-1})^{n-s} + (r-s)(\theta_{-1})^{n-s-1}(\theta_0 + \cdots + \theta_{s+1}))
\]

\[
+ \mathcal{O}((\theta_{-1})^{n-s-2}),
\]

where \( \mathcal{O}((\theta_{-1})^{n-s-2}) \) represents a sum of terms of degree at most \( n - s - 2 \) in \( \theta_{-1} \). On the other hand, by definition \([W'] = \deg_X m'_i \theta_{-1} + \deg_{\lambda_0} m_i \theta_0 + \cdots + \deg_{\lambda_{s+1}} m_i \theta_{s+1} \), where \( m'_i \in \mathbb{F}_q[X_{i_0}, \ldots, X_{i_s}, X, \Lambda] \) is a polynomial of minimal degree defining \( W'_i \). Let \( \mathcal{A}^*(\mathbb{P}^{s+1}) \to \mathcal{A}^*((\mathbb{P}^n)^{s+3}) \) be the injective \( \mathbb{Z} \)-map \( P \mapsto (\theta_{-1})^{n-s-1}P \) induced by \( \pi_i \). Then [12, Proposition 1.16] shows that \( g([W'_i]) \leq [W'] \), where the inequality is understood in a coefficient–wise sense. This implies \( \deg_{\Lambda_i} m'_i \leq (r-s)D^{r-s-1}\delta \) for \( 0 \leq j \leq s + 1 \).

Let \( m_i \in \mathbb{F}_q[X_{i_0}, \ldots, X_{i_s}, X, \Lambda] \) be the polynomial defining \( W_i \). Observe that \( D_i := \deg_X m_i > 0 \) holds. Let \( A_i \in \mathbb{F}_q[X_{i_0}, \ldots, X_{i_s}, \Lambda] \) be the (nonzero) polynomial arising as the coefficient of \( X_i^{D_i} \) in \( m_i \), considered as an element of the polynomial ring \( \mathbb{F}_q[X_{i_0}, \ldots, X_{i_s}, \Lambda][X_i] \). Further, let \( A_i^* \in \mathbb{F}_q[\Lambda] \) be a nonzero coefficient of \( A_i \), considering \( A_i^* \) as an element of the polynomial ring \( \mathbb{F}_q[\Lambda][X_{i_0}, \ldots, X_{i_s}] \). Finally, let \( A_0 \in \mathbb{F}_q[\Lambda] \) denote an arbitrary maximal minor of the generic matrix \( (\Lambda_{ij})_{0 \leq i \leq s+1, 0 \leq j \leq n} \) and set \( A := A_0 \cdot \prod_{i \in \Gamma} A_i^* \in \mathbb{F}_q[\Lambda] \). We claim that the hypersurface \( \mathcal{H}_1 \subset (\mathbb{P}^n)^{s+2} \) defined by the zero locus of \( A \) satisfies the requirements of the theorem.

In order to show this claim, let \( \lambda := (\lambda_0, \ldots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2}\setminus \mathcal{H}_1 \) and denote \( m_i(\lambda) := m_i(X_{i_0}, \ldots, X_{i_s}, X_i, \lambda) \). Since \( A_0(\lambda) \neq 0 \) holds, we have that \( \lambda \in \mathcal{U} \). Then \( m_i(\lambda) \) is a nonzero polynomial of \( \mathbb{F}_q[X_{i_0}, \ldots, X_{i_s}, X_i, \Lambda] \) with \( \deg_X m_i(\lambda) > 0 \) vanishing on \( \mathcal{M}(L) \) for any \( i \in \Gamma \), where \( L \) is the linear variety associated with \( \lambda \). This implies that the coordinate function of \( \mathcal{M}(L) \) defined by \( X_i \) satisfies a nontrivial algebraic equation over \( \mathbb{F}_q(X_{i_0}, \ldots, X_{i_s}) \) for any \( i \in \Gamma \). As a consequence, it follows that \( \mathcal{M}(L) \) has dimension at most \( s \).

Since \( A_i^* \) is a multihomogeneous polynomial of \( \mathbb{F}_q[\Lambda] \) with \( \deg_{\Lambda_i} A_i^* \leq (r-s)D^{r-s-1}\delta \) and \( |\Gamma| = n - s \) holds, we obtain the upper bound
 deg_\Lambda_i A \leq (n - s)(r - s)D^{r - s - 1}\delta + 1. This finishes the proof of the theorem. \hfill \Box

5. ON THE EXISTENCE OF NONSINGULAR LINEAR SECTIONS

In this section we shall establish a Bertini–type theorem, namely we shall show the existence of nonsingular linear sections of a given variety. Combining the main result of this section and Theorem 4.5 we shall be able to obtain an effective Bertini smoothness theorem suitable for our purposes.

A version of the Bertini theorem asserts that a generic hyperplane section of a nonsingular variety \( V \) is nonsingular. A more precise variant of this result asserts that, if \( V \subset \mathbb{P}^n \) is a projective variety with a singular locus of dimension at most \( s \), then a section of \( V \) defined by a generic linear space of \( \mathbb{P}^n \) of codimension at least \( s + 1 \) is nonsingular (see, e.g., [16, Proposition 1.3]). In this section, we shall consider the existence of nonsingular sections of codimension \( s + 2 \). Identifying each section of this type with a point in the multiprojective space \( (\mathbb{P}^n)^{s+2} \), we shall show the existence of an hypersurface of \( (\mathbb{P}^n)^{s+2} \) containing all the linear subvarieties of codimension \( s + 2 \) of \( (\mathbb{P}^n)^{s+2} \) which yield singular sections of \( V \). We shall further provide an estimate of the multidegree of this hypersurface.

We remark that an effective version of a weak form of a Bertini theorem is obtained in [1]. Nevertheless, the bound given in [1] is exponentially higher than ours and therefore is not suitable for our purposes.

Let \( V \subset \mathbb{P}^n \) be an ideal–theoretic complete–intersection defined over \( \mathbb{F}_q \), of dimension \( r \) and degree \( \delta \). Let \( F_1, \ldots, F_{n-r} \in \mathbb{F}_q[X_0, \ldots, X_n] \) be homogeneous polynomials of degrees \( d_1 \geq \ldots \geq d_{n-r} \geq 2 \) respectively, which generate the ideal \( I(V) \) of \( V \). Let \( \Sigma \subset V \) be the singular locus of \( V \) and suppose that it has dimension at most \( s \leq r - 2 \). As asserted above, our goal is to obtain a condition on \( \lambda_0, \ldots, \lambda_{s+1} \in \mathbb{P}^n \) which implies that \( V \cap L \) is a nonsingular variety of dimension \( r - s - 2 \), where \( L := \{ \lambda_i \cdot X = 0 : 0 \leq i \leq s + 1 \} \subset \mathbb{P}^n \).

We recall the notations \( D := \sum_{i=1}^{n-r}(d_i - 1) \) and \( t := n(s + 2) \). We start with two technical results.

Lemma 5.1. There exists an hypersurface \( \mathcal{H}_2' \subset (\mathbb{P}^n)^{s+2} \), defined by a multihomogeneous polynomial of \( \mathbb{F}_q[\Lambda] \) of multidegree at most \( \delta \) in each group of variables \( \Lambda_i \), with the following property: let \( \lambda \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}_2' \), let \( (Y_0, \ldots, Y_{s+1}) := \lambda \cdot X \), and let \( \pi : V \to \mathbb{P}^{s+1} \) be the linear mapping defined by \( Y_0, \ldots, Y_{s+1} \). Then the Zariski closure \( V_y \) of any fiber \( \pi^{-1}(y) \) is of pure dimension \( r - s - 1 \) and the set of exceptional points of \( \pi \) is of pure dimension \( r - s - 2 \).

Proof. Let \( U_0, \ldots, U_r \) be \( r + 1 \) groups of \( n + 1 \) indeterminates over \( \mathbb{F}_q[X_0, \ldots, X_n] \), where \( U_i := (U_{i,0}, \ldots, U_{i,n}) \), and let \( U := (U_0, \ldots, U_r) \).
Denote by \( F_V \in \mathbb{E}_q[U] \) the Chow form of \( V \) (see, e.g., [21], [29]). This is an irreducible polynomial of \( \mathbb{E}_q[U] \) which characterizes the set of overdetermined linear systems over \( V \). Furthermore, \( F_V \) is homogeneous in each group of variables \( U_i \) and satisfies the identities \( \deg_{U_i} F_V = \deg_{U_i} F_V = \delta \) for \( 0 \leq i \leq r \).

Consider \( F_V \) as a polynomial of \( \mathbb{E}_q[U_0, \ldots, U_{s+1}][U_{s+2}, \ldots, U_r] \) and fix \( u_{s+2}, \ldots, u_r \in \mathbb{P}^n \) such that \( B := F_V(U_0, \ldots, U_{s+1}, u_{s+2}, \ldots, u_r) \) does not vanish. We claim that any \( \lambda := (\lambda_0, \ldots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2} \) with \( B(\lambda) \neq 0 \) satisfies the requirements of the lemma.

Indeed, by the definition of \( \lambda \) and \( u := (u_{s+2}, \ldots, u_r) \) we have that the mapping \( \pi_r : V \to \mathbb{P}^r \) defined by the linear forms \( \lambda_0 \cdot X, \ldots, \lambda_{s+1} \cdot X, u_{s+2} \cdot X, \ldots, u_r \cdot X \) is a finite morphism. Let \( \pi : V \to \mathbb{P}^{s+1} \) be the mapping defined by \( \lambda_0 \cdot X, \ldots, \lambda_{s+1} \cdot X \). Then the Zariski closure \( V_0 \) of any fiber \( \pi^{-1}(y) \) agrees with the inverse image by \( \pi_r \) of a linear variety of \( \mathbb{P}^r \) of dimension \( r - s - 1 \), and hence is of pure dimension \( r - s - 1 \). On the other hand, the fact that \( V \cap \{ \lambda_0 \cdot X = \cdots = \lambda_{s+1} \cdot X = u_{s+2} \cdot X = \cdots = u_r \cdot X = 0 \} \) is empty immediately implies that the set of exceptional points \( V \cap \{ \lambda_0 \cdot X = \cdots = \lambda_{s+1} \cdot X = 0 \} \) of \( \pi \) is of pure dimension \( r - s - 2 \).

As a consequence, defining \( \mathcal{H}_2' \subset (\mathbb{P}^n)^{s+2} \) as the zero locus of the polynomial \( B \in \mathbb{E}_q[U_0, \ldots, U_{s+1}] \) finishes the proof of the lemma. \( \square \)

**Lemma 5.2.** There exists an hypersurface \( \mathcal{H}_2'' \subset (\mathbb{P}^n)^{s+2} \), defined by a multihomogeneous polynomial of \( \mathbb{E}_q[\Lambda] \) of multidegree at most \( D^{r-s-1} \delta \) in each group of variables \( \Lambda_i \), with the following property: if \( \lambda \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}_2'' \), then \( \Sigma \cap \Lambda_i \) is empty.

**Proof.** Arguing as in the proof of Proposition 5.3 we see that there exists a projective variety \( Z \subset \mathbb{P}^n \) of pure dimension \( s + 1 \) and degree at most \( D^{r-s-1} \delta \) with \( \Sigma \subset Z \). Let \( \mathcal{F}_Z \in \mathbb{E}_q[\Lambda] \) be the Chow form of \( Z \). We have that \( \mathcal{F}_Z \) is homogeneous in each group of variables \( \Lambda_i \) and satisfies the upper bound \( \deg_{\Lambda_i} \mathcal{F}_Z \leq D^{r-s-1} \delta \) for \( 0 \leq i \leq s + 1 \).

Let \( \lambda \in (\mathbb{P}^n)^{s+2} \) be such that \( \mathcal{F}_Z(\lambda) \neq 0 \) holds and let \( L := \{ \lambda_i \cdot X = 0 \ (0 \leq i \leq s+1) \} \). Then \( Z \cap L \) is empty and hence so is \( \Sigma \cap L \). Therefore, defining \( \mathcal{H}_2'' \subset (\mathbb{P}^n)^{s+2} \) as the zero locus of \( \mathcal{F}_Z \) finishes the proof of the lemma. \( \square \)

Similarly to Section 4 we consider the following incidence variety:

\[
W_s := (V_{sm} \times \mathcal{U}) \cap \{ \Lambda_0 \cdot X = 0, \ldots, \Lambda_{s+1} \cdot X = 0, \quad \Delta_1(\Lambda, X) = 0, \ldots, \Delta_N(\Lambda, X) = 0 \},
\]

where \( \mathcal{U} \subset (\mathbb{P}^n)^{s+2} \) is the Zariski open subset of \( (s + 2) \times (n + 1) \)-matrices of maximal rank and \( \Delta_1, \ldots, \Delta_N \) are the maximal minors of the generic version \( M(X, \Lambda) \) of the matrix of \( (12) \).

**Proposition 5.3.** \( W_s \) is an irreducible subvariety of \( \mathbb{P}^n \times \mathcal{U} \) of dimension \( t - 1 \).
of $V$ According to [8, Proposition 1.1], $L$ is an irreducible variety of dimension $(s + 1) + \lambda_1$. This shows that the Zariski open set $U \subset L$ such that $W(\lambda, 1) \cap V \neq \emptyset$, or equivalently, an irreducible variety of dimension $t = r - 1$.

We first prove that $W_s$ is an irreducible variety of dimension $t - 1$. According to [8, Proposition 1.1], $L_{s+1}(A^{s+2}, \mathbb{W})$ is an irreducible variety of dimension $(s + 1)(r + 1)$. Since we are considering subspaces of $\mathbb{W}$ of dimension $s + 2$ modulo $\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \rangle$, which has dimension $n - r$, it follows that $\pi_1^{-1}(x) = \{x\} \times L$ is an open dense subset of an irreducible variety of $\mathbb{P}^n \times A^{(s+2)}$ of dimension $(s + 1)(r + 1) + (n - r)(s + 2) = (n + 1)(s + 2) - r - 1$, or equivalently, an irreducible variety of $\mathbb{P}^n \times U$ of dimension $t = r - 1$.

Combining these arguments with the proof of [32, §1.6.3, Theorem 8] we conclude that $W_s$ is an irreducible variety of dimension $t - 1$. □

An immediate consequence of Proposition [5.3] is that the Zariski closure of the image of the projection $\pi_2 : W_s \to U$ is an irreducible variety of dimension at most $t - 1$. Our next result strengthens somewhat this conclusion and provides further quantitative information.

**Proposition 5.4.** Let $H_s \subset (\mathbb{P}^n)^{s+2}$ be the Zariski closure of the image of $\pi_2 : W_s \to U$. Then $H_s$ is an hypersurface of $(\mathbb{P}^n)^{s+2}$, defined by a multihomogeneous polynomial of $\mathbb{P}^n[A]$ of degree at most $\delta_D^r - s^2(D + r - s - 1)$ in each group of variables $\Lambda_i$.

**Proof.** We first prove that $H_s$ is an hypersurface. For this purpose, it suffices to show that there exists a zero dimensional fiber of $\pi_2$, because the theorem on the dimension of fibers (see, e.g., [32, §6.2, Theorem 7]) readily implies our assertion.

Fix generic linear forms $\lambda_0 \cdot X, \ldots, \lambda_s \cdot X$. By the Bertini theorem in the form of [16, Proposition 1.3] we have that $V \cap \{\lambda_0 \cdot X = \cdots = \lambda_s \cdot X = 0\}$ is nonsingular of pure dimension $r - s - 1$. Choose $\lambda_{s+1} \in \mathbb{P}^n$ such that $V \cap \{\lambda_0 \cdot X = \cdots = \lambda_{s+1} \cdot X = 0\}$ is singular. From [22, Appendix, Theorem 2] it follows that the singular locus of $V \cap \{\lambda_0 \cdot X = \cdots = \lambda_{s+1} \cdot X = 0\}$ has dimension zero. Since such a singular locus is
isomorphic to the fiber $\pi^{-1}_2(\lambda_0, \ldots, \lambda_{s+1})$, we deduce the existence of a zero–dimensional fiber of $\pi_2$, which completes the proof of the first assertion.

Next, fix $r - s - 1$ generic linear combinations, say $\Delta^1, \ldots, \Delta^{r-s-1}$, of $\Delta_1(\Lambda, X), \ldots, \Delta_N(\Lambda, X)$, such that the subvariety $W'_s \subset (\mathbb{P}^n)^{s+3}$ defined by the set of common zeros of the equations

$$F_1 = 0, \ldots, F_{n-r} = 0, \Lambda_0 \cdot X = 0, \ldots, \Lambda_{s+1} \cdot X = 0,$$

is of pure dimension $t - 1$ and let $\mathcal{H}'_s \subset (\mathbb{P}^n)^{s+2}$ be the union of the components of the Zariski closure of $\pi_2(W'_s)$ of dimension $t - 1$. Then $\mathcal{H}'_s$ is an hypersurface containing $\mathcal{H}_s$.

Finally, we estimate the multidegree of $\mathcal{H}'_s$. For this purpose, we consider the class $[W'_s]$ of $W'_s$ in the Chow ring $\mathcal{A}^t((\mathbb{P}^n)^{s+3})$ of $(\mathbb{P}^n)^{s+3}$. Denote by $\theta_{j-2}$ the class of the inverse image of a hyperplane of $\mathbb{P}^n$ under the $j$th canonical projection $(\mathbb{P}^n)^{s+3} \to \mathbb{P}^n$ for $1 \leq j \leq s + 2$. Then the multihomogeneous Bézout theorem (see, e.g., [12, Theorem 1.1]) asserts that

$$[W'_s] = \prod_{i=1}^{n-r} (d_i \theta_{i-1}) \prod_{j=0}^{s+1} (\theta_{j-1} + \theta_j) \prod_{k=1}^{r-s-1} (D\theta_{j-1} + \theta_0 + \cdots + \theta_{s+1})$$

$$= \delta D^r s^{-2}(D + r - s - 1)(\theta_{j-1})^n(\theta_0 + \cdots + \theta_{s+1})$$

+ terms of lower degree in $\theta_{j-1}$.

On the other hand $[\mathcal{H}'_s] = \deg \chi H'_s \theta_{j-1} + \deg \Lambda_0 H'_s \theta_0 + \cdots + \deg \Lambda_{s+1} H'_s \theta_{s+1}$, where $H'_s \in \mathbb{F}_q[\Lambda]$ is a polynomial of minimal degree defining $\mathcal{H}'_s$. Let $j : \mathcal{A}^t((\mathbb{P}^n)^{s+2}) \hookrightarrow \mathcal{A}^t((\mathbb{P}^n)^{s+3})$ be the injective $\mathbb{Z}$–map $P \mapsto (\theta_{j-1})^nP$ induced by $\pi_2$. Then [12, Proposition 1.16] shows that $j([\mathcal{H}'_s]) \leq [W'_s]$, where the inequality is understood in a coefficient–wise sense. This implies $\deg \Lambda_0 H'_s \leq \delta D^r s^{-2}(D + r - s - 1)$ for $0 \leq j \leq s + 1$, finishing thus the proof of the proposition.

Finally, combining Lemmas 5.1 and 5.2 and Proposition 5.4 we obtain the main result of this section.

**Corollary 5.5.** There exists an hypersurface $\mathcal{H}_2 \subset (\mathbb{P}^n)^{s+2}$, defined by a multihomogeneous polynomial of degree at most $\left(D^r s^{-2}(2D + r - s - 1) + 1\right)\delta$ in each group of variables $\Lambda_i$, with the following property: if $\lambda \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}_2$, then $V \cap L$ is nonsingular of pure dimension $r - s - 2$ and $\lambda$ satisfies the conditions in the statements of Lemmas 5.1 and 5.2.

6. **An effective Bertini theorem**

This section is devoted to obtain an effective version of the Bertini smoothness Theorem. The Bertini smoothness theorem (see, e.g., [32, II.6.2, Theorem 2]) asserts that, given a dominant morphism $f : V_1 \to V_2$ of irreducible varieties defined over a field of characteristic zero with
V_1 nonsingular, there exists a dense open set U of V_2 such that the fiber f^{-1}(y) is nonsingular for every y \in U. An effective version of this result provides an upper bound of the degree of the subvariety of V_2 consisting of the points defining singular fibers. The effective version we shall obtain holds without any restriction on the characteristic of the ground field and generalizes significantly [9, Theorem 5.3].

Let V \subset \mathbb{P}^n be an ideal–theoretic complete intersection defined over \mathbb{F}_q of dimension r and degree \delta. Let F_1, \ldots, F_{n-r} \in \mathbb{F}_q[X_0, \ldots, X_n] be homogeneous polynomials of degrees d_1 \geq \cdots \geq d_{n-r} \geq 2 respectively, which generate the ideal I(V) of V. We assume that the singular locus \Sigma of V has dimension at most s \leq r - 2.

Let \lambda := (\lambda_0, \ldots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}_2, where \mathcal{H}_2 \subset (\mathbb{P}^n)^{s+2} is the hypersurface of the statement of Corollary 5.5 and let Y_j := \lambda_j \cdot X for 0 \leq j \leq s + 1. Let \pi : V \rightarrow \mathbb{P}^{s+1} be the linear mapping defined by Y_0, \ldots, Y_{s+1}. The set of exceptional points of \pi is equal to V \cap L, where L := \{Y_0 = \cdots = Y_{s+1} = 0\}.

Remark 6.1. With assumptions and notations as above, \Sigma \cap L is empty, and thus, M(L) \cap L = \text{Sing}(V \cap L) is also empty.

Proof. The fact that \lambda \notin \mathcal{H}_2 implies that \lambda \notin \mathcal{H}_2^{s+2}, where \mathcal{H}_2^{s+2} \subset (\mathbb{P}^n)^{s+2} is the hypersurface of the statement of Lemma 5.2. Then \Sigma \cap L is empty.

Observe that \text{Sing}(V \cap L) is the set of points of V_{sm} \cap L = V \cap L where the intersection is not transversal. For a point x \in V \cap L, the intersection is not transversal if and only if \dim(T_x V \cap L) > r + (n - s - 2) - n = r - s - 2. This shows that x \in \text{Sing}(V \cap L) if and only if x \in M(L) \cap L, namely M(L) \cap L = \text{Sing}(V \cap L).

Finally, \lambda \notin \mathcal{H}_s, where \mathcal{H}_s \subset (\mathbb{P}^n)^{s+2} is the hypersurface of the statement Proposition 5.4 which implies that V \cap L is nonsingular. \square

We shall prove that, for a generic choice of Y_0, \ldots, Y_{s+1}, there exists a nonempty open subset U of \mathbb{P}^{s+1} such that the Zariski closure V_y of \pi^{-1}(y) is nonsingular for every y \in U. Furthermore, we shall provide an estimate on the degree of the generic condition underlying the choice of Y_0, \ldots, Y_{s+1} and the degree of the variety \mathbb{P}^{s+1} \setminus U yielding nonsingular fibers.

The first step is to obtain a sufficient condition for the nonsingularity of the linear section V_y of V defined by a point y \in \mathbb{P}^{s+1}. Fix y := (y_0 : \cdots : y_s) \in \mathbb{P}^{s+1} and assume without loss of generality that y_0 \neq 0 holds. Then

V_y = \{x \in V : y_j Y_0(x) - y_0 Y_j(x) = 0 \text{ for } 1 \leq j \leq s\}.

In particular, we have that V \cap L \subset V_y. Since \lambda \notin \mathcal{H}_2, where \mathcal{H}_2 \subset (\mathbb{P}^n)^{s+2} is the hypersurface of Corollary 5.5 it turns out that \Sigma \cap L is empty. Furthermore, by Remark 6.1 we have that V \cap L is nonsingular.
This in particular implies that any point of $V \cap L$ is a nonsingular point of $V_y$.

Now we can state and prove a sufficient condition for the nonsingularity of the linear section $V_y$ of $V$. For this purpose, we shall consider as before the linear mapping $\varphi : T_y V \rightarrow \mathbb{P}^{s+1}$ defined by $Y_0, \ldots, Y_{s+1}$.

**Lemma 6.2.** Let $y$ be a point of $\mathbb{P}^{s+1}$ such that for every $x \in \pi^{-1}(y)$ the following conditions hold:

(i) $x$ is a point of $V$,

(ii) the set of exceptional points of $\varphi_x$ has dimension at most $r - s - 2$.

Then $V_y$ is a nonsingular variety.

**Proof.** Since $V_y$ is of pure dimension $r - s - 1$, it suffices to prove that for every $x \in V_y$ the tangent space $T_y V_y$ has dimension at most $r - s - 1$. Fix $x \in \pi^{-1}(y)$. Condition (i) implies that the tangent space $T_x V$ has dimension $r$. Consider the linear mapping

$$\varphi_x|_{T_y V_y} : T_x V_y \rightarrow \mathbb{P}^{s+1}$$

$$v \mapsto (Y_0(v) : \cdots : Y_{s+1}(v)).$$

It is clear that the set $E_{x,y}$ of exceptional points of $\varphi_x|_{T_y V_y}$ is contained in the set $E_x$ of exceptional points of $\varphi_x$. From the fact that the restriction $\pi|_{V_y} : V_y \rightarrow \mathbb{P}^{s+1}$ maps $V_y$ to the point $y$ it follows that the dimension of $\varphi_x(T_x V_y)$ is equal to 0. By the Dimension theorem of linear algebra (see, e.g., [20, Chapter 8, Section 4]) we have that

$$\dim T_x V_y = \dim E_{x,y} + \dim \varphi_x(T_x V_y) + 1.$$

From this and condition (ii) we deduce that

$$\dim T_x V_y \leq \dim E_x + 1 \leq r - s - 1.$$

We conclude that $\dim T_x V_y \leq r - s - 1$ and therefore $x$ is a regular point of $V_y$.

Finally, let $x \in V_y \setminus \pi^{-1}(y)$. Then $x \in V \cap L$ and is a regular point of $V \cap L$. From the fact that $F_1, \ldots, F_{n-r}, Y_0, \ldots, Y_{s+1}$ define the ideal of $V \cap L$ we easily deduce that $x$ is also a regular point of $V_y$, which finishes the proof of the lemma. $\square$

The critical point of our effective Bertini theorem is the analysis of the set of regular points $x \in V$ for which the dimension of the set of exceptional points of $\varphi_x$ has dimension at least $r - s - 1$, where $Y_0, \ldots, Y_{s+1}$ are suitably chosen. Remark [1.2] asserts that this set is the polar variety $M(L)$, where $L := \{Y_0 = \cdots = Y_{s+1} = 0\}$.

Now we are ready to state our effective version of the Bertini smoothness theorem.

**Theorem 6.3.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the hypersurfaces of $(\mathbb{P}^n)^{s+2}$ of the statements of Theorem 4.5 and Corollary 5.2 respectively, let $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2$ and let $\Lambda := (\lambda_0, \ldots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}$. Let $Y_j := \lambda_j \cdot X$ for
0 ≤ j ≤ s + 1, let $L := \{Y_0 = \cdots = Y_{s+1} = 0\}$ and let $\pi : V \to \mathbb{P}^{s+1}$ be the linear mapping defined by $Y_0, \ldots, Y_{s+1}$. Then there exists a closed set $\tilde{W}(L) \subset \mathbb{P}^{s+1}$ of dimension at most $s$ and degree at most $D^{r-s}\delta$ such that for every $y \in \mathbb{P}^{s+1} \setminus \tilde{W}(L)$ the linear section $V_y$ of $V$ is nonsingular of pure dimension $r - s - 1$.

**Proof.** Since $\lambda \notin \mathcal{H}$, by Theorem 4.5 it follows that the polar variety $\mathcal{M}(L)$ has dimension $s$. Let $Z(L) \subset V$ be the subvariety of dimension $s$ and degree at most $D^{r-s}\delta$ whose existence was established in Proposition 4.3. Then

$$\mathcal{M}(L) \cup \Sigma \subset Z(L).$$

Defining $W(L) := \pi(Z(L))$ it turns out that $W(L) \subset \mathbb{P}^{s+1}$ has dimension at most $s$. Furthermore, by (6) we have that $W(L)$ has degree at most $D^{r-s}\delta$.

Let $y \in \mathbb{P}^{s+1} \setminus W(L)$. By Corollary 5.3 we have that $V_y$ is of pure dimension $r - s - 1$. Furthermore, the conditions of the statement of Lemma 6.2 are satisfied, and hence $V_y$ is nonsingular. This finishes the proof of the theorem. \qed

**Remark 6.4.** With notations and assumptions as in Theorem 6.3, for $\lambda \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}$ and $y \in \mathbb{P}^{s+1} \setminus W(L)$, the linear section $V_y$ is contained in $V_{\text{sm}}$. Indeed, by the choice of $y$ it turns out that any point $x \in \pi^{-1}(y)$ is a regular point of $V$. On the other hand, if $x \in V_y \setminus \pi^{-1}(y)$, then $x \in V \cap L$, and $V \cap L \subset V_{\text{sm}}$ by Remark 6.1.

Since the linear section $V_y$ is a nonsingular projective complete intersection for $y \notin W(L)$, the Hartshorne connectedness theorem (see, e.g., [25, VI, Theorem 4.2]) shows that $V_y$ is connected, which implies that it is absolutely irreducible.

In what follows, we shall frequently use the notation

$$B_{d,s} := D^{r-s-2}\delta \left((n-s)(r-s)+2D+r-s-1\right) + \delta + 1.$$

**Corollary 6.5.** For $q > \max\{B_{d,s}, D^{r-s}\delta\}$, there exists $y \in \mathbb{P}^{s+1}(\mathbb{F}_q)$ such that $V_y$ is a nonsingular $\mathbb{F}_q$-variety of pure dimension $r - s - 1$. In other words, $V$ has a nonsingular linear section of pure dimension $r - s - 1$ defined over $\mathbb{F}_q$.

**Proof.** According to Theorem 4.3 and Corollary 5.3 there exist hyper-surfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $(\mathbb{P}^n)^{s+2}$ such that for any $\lambda \in (\mathbb{P}^n)^{s+2}\setminus(\mathcal{H}_1 \cup \mathcal{H}_2)$, the associated linear space $L$ satisfy the requirements of the statements of Theorem 4.5 and Corollary 5.3. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined by multihomogeneous polynomials of $\mathbb{F}_q[\Lambda]$ of degree at most $(n-s)(r-s)D^{r-s-1}\delta + 1$ and $\delta(D^{r-s-2}(2D+r-s-1)+1)$ in each group of variables $\Lambda_i$ respectively, it follows that $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2$ is defined by a multihomogeneous polynomial of $\mathbb{F}_q[\Lambda]$ of degree at most $B_{d,s}$ in each group of variables $\Lambda_i$. By Corollary 3.2 we conclude that, if $q > B_{d,s}$, then there exists a point $\lambda \in (\mathbb{P}^n(\mathbb{F}_q))^{s+2} \setminus \mathcal{H}$. Let
\( \pi : V \to \mathbb{P}^{s+1} \) be the linear mapping defined by the corresponding linear forms \( Y_0, \ldots, Y_{s+2} \). Then Theorem 6.3 shows that there exists an hypersurface \( W(L) \subset \mathbb{P}^{s+1} \) of degree at most \( D^{s-\delta} \) such that, for \( y \in \mathbb{P}^{s+1} \setminus W(L) \), the linear section \( V_y \) is nonsingular. Since \( q > D^{s-\delta} \), we see that there exists \( y \in \mathbb{P}^{s+1}(\mathbb{F}_q) \setminus W(L) \), from which the corollary follows.

\[ \square \]

7. Results of existence of smooth \( \mathbb{F}_q \)-rational points

In this section we obtain results of existence of smooth \( \mathbb{F}_q \)-rational points of a complete intersection \( V \subset \mathbb{P}^n \) defined over \( \mathbb{F}_q \), of dimension \( r \), degree \( \delta \), multidegree \( d := (d_1, \ldots, d_{n-r}) \) with \( d_1 \geq \cdots \geq d_{n-r} \geq 2 \) and singular locus \( \Sigma \) of dimension at most \( s \). More precisely, we establish conditions on \( q \) which imply that \( V_{\text{sm}}(\mathbb{F}_q) \) is not empty.

The usual approach to this kind of results relies on a combination of the available estimates on the number of \( \mathbb{F}_q \)-rational points and upper bounds for the number of singular \( \mathbb{F}_q \)-rational points. Instead of doing this, we shall use the effective version of the Bertini smoothness theorem of Section 6 in order to establish the existence of a nonsingular linear section of \( V \) defined over \( \mathbb{F}_q \). Such a singular section will be contained in \( V_{\text{sm}} \). We shall combine this result with the following well-known estimate on the number of \( \mathbb{F}_q \)-rational points of a nonsingular complete intersection \( W \subset \mathbb{P}^n \) defined over \( \mathbb{F}_q \), of dimension \( r \), degree \( \delta \) and multidegree \( d \) due to P. Deligne [13]:

\[
||W(\mathbb{F}_q)| - p_r| \leq b_r'(n, d) q^{r/2},
\]

where \( b_r'(n, d) \) denotes the \( r \)th primitive Betti number of any nonsingular complete intersection of \( \mathbb{P}^n \) of dimension \( r \) and multidegree \( d \).

In the sequel we shall frequently use the following explicit expressions for \( b_r'(n, d) \) with \( r \in \{1, 2\} \) (see, e.g., [16, Theorem 4.1]):

\[
b_1'(n, d) = (d_1 \cdots d_{n-1})(d_1 + \cdots + d_{n-1} - n - 1) + 2,
\]

\[
b_2'(n, d) = (d_1 \cdots d_{n-2}) \left( \frac{n+1}{2} - (n+1) \sum_{1 \leq i \leq n-2} d_i + \sum_{1 \leq i \leq j \leq n-2} d_id_j \right) - 3.
\]

**Remark 7.1.** Let \( V \subset \mathbb{P}^n \) be a nonsingular complete intersection defined over \( \mathbb{F}_q \), of dimension 2 and multidegree \( d := (d_1, \ldots, d_{n-2}) \). Let \( D := \sum_{i=1}^{n-2} (d_i - 1) \). Observe that \( \deg V = d_1 \cdots d_{n-2} \) and the following upper bound holds:

\[
b_2'(n, d) \leq (n-1)D^2 \deg V.
\]

Indeed, we have

\[
-(n+1) \sum_{1 \leq i \leq n-2} d_i + \sum_{1 \leq i \leq j \leq n-2} d_id_j \leq \sum_{i=1}^{n-2} d_i \left( \sum_{i=1}^{n-2} d_i - n-1 \right) = \sum_{i=1}^{n-2} d_i(D-3).
\]
Using the inequality $\sum_{i=1}^{n-2} d_i \leq (n - 1)D$, we obtain

$$b'_q(n, d) \leq \deg V \left( \binom{n+1}{2} + (n - 1)D(D - 3) \right) \leq (n - 1)D^2 \deg V.$$  

This shows \([15]\).

Let $B_{d,s} := D^{r-s-2}\delta\left( ((n-s)(r-s) + 2) + r - s - 1 \right) + \delta + 1$. According to Corollary \([6,5]\) if $q > \max\{B_{d,s}, D^{r-s}\}$ then there exists a nonsingular linear section $S$ of $V$ defined over $\mathbb{F}_q$ of pure dimension $r - s - 1$ contained in $V_{sm}$. We are going to prove that the number of $\mathbb{F}_q$-rational points in $S$ is strictly positive, showing thus that $V$ has smooth $\mathbb{F}_q$-rational points. We have the following result.

**Theorem 7.2.** Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension $r \geq 2$, degree $\delta$, multidegree $d$ and singular locus $\Sigma$ of dimension at most $s \leq r - 2$. If $q > \max\{B_{d,s}, D^{r-s}\}, (b'_{r-s-1}(n - s - 1, d))^{2/(r-s-1)}\}$, then $V$ has a smooth $\mathbb{F}_q$-rational point.

**Proof.** Let $S$ be the nonsingular linear section of $V$ whose existence is assured by Corollary \([6,5]\). Since $S$ is a nonsingular complete intersection defined over $\mathbb{F}_q$ of dimension $r - s - 1$, from \([14]\) it follows that

$$|S(\mathbb{F}_q)| \geq q^{r-s+1} - b'q^{r-s+1} > q^{r-s+1}(q^{r-s+1} - b'),$$

where $b' := b'_{r-s-1}(n - s - 1, d)$. Our conditions immediately implies that the right-hand side in the previous expression is positive. Furthermore, according to Remark \([6,4]\) we have that $S \subset V_{sm}$, finishing the proof of the theorem.

Next we discuss two particular instances of this result.

**Corollary 7.3.** With notations and assumptions as in Theorem \([7,2]\) if

$$q > \begin{cases} 
\left( \delta(D - 2) + 2 \right)^2, & \text{for } D \geq 5 \text{ or } D = 4 \text{ and } n - r > 1, \\
(2(n - r + 3)D + 2)\delta + 1, & \text{otherwise}, 
\end{cases}$$

then $V$ has a smooth $\mathbb{F}_q$-rational point.

**Proof.** Observe that $b'_{r-s+1}(n - r + 1, d) = \delta(D - 2) + 2$. Therefore, applying Theorem \([7,2]\) with $s = r - 2$, we conclude that, if

$$q > \max\left\{ \left( (2(n - r + 3)D + 2)\delta + 1, D^2\delta, (\delta(D - 2) + 2)^2 \right\},$$

then $V$ has a smooth $\mathbb{F}_q$-rational point. For $D \leq 2$ we have $D^2\delta \leq (2(n - r + 3)D + 2)\delta + 1$, while $D^2\delta \leq (\delta(D - 2) + 2)^2$ for $D \geq 3$. As a consequence, we see that \((16)\) is equivalent to

$$q > \max\left\{ \left( (2(n - r + 3)D + 2)\delta + 1, (\delta(D - 2) + 2)^2 \right\}.$$
If $D \geq 6$, then we have
\[(\delta(D - 2) + 2)^2 \geq (2(D + 3)D + 2)\delta + 1 \geq (2(n - r + 3)D + 2)\delta + 1.\]
Combining this inequality with (17) and elementary calculations, we deduce the statement of the corollary.

\[
\square
\]

**Corollary 7.4.** Let notations and assumptions be as in Theorem 7.3. Suppose further that the singular locus of $V$ has dimension at most $r - 3 \geq 0$. If $q > 3D(D + 2)^2\delta$, then $V$ has a smooth $\mathbb{F}_q$–rational point.

**Proof.** We apply Theorem 7.2 with $s = r - 3$. According to Remark 7.1, we have $b_d(n - r + 2, d) \leq (n - r + 1)D^2\delta$. Therefore, Theorem 7.2 shows that a sufficient condition for the existence of a smooth $\mathbb{F}_q$–rational point of $V$ is
\[
q > \max\{D^3\delta, D\delta((3(n - r + 3) + 2)D + 2) + \delta + 1\}.
\]
Using the inequality $n - r \leq D$, we deduce that
\[
D\delta((3(n - r + 3) + 2)D + 2) + \delta + 1 \leq 3D(D + 2)^2\delta,
\]
which immediately implies the statement of the corollary. \qed

8. Estimates on the Number of $\mathbb{F}_q$–Rational Points

In this section we obtain estimates on $|V(\mathbb{F}_q)|$ for a complete intersection $V \subset \mathbb{P}^n$ of dimension $r$, degree $\delta$ and multidegree $d := (d_1, \ldots, d_{n-r})$ with $d_1 \geq \cdots \geq d_{n-r} \geq 2$ for which the singular locus has codimension at least 2 or 3.

Fix $s \in \{r - 2, r - 3\}$. Let $D := \sum_{i=1}^{n-r}(d_i - 1)$. Then Theorem 4.5 and Corollary 5.3 show that there exists an hypersurface $H := H_1 \cup H_2 \subset (\mathbb{P}^n)^{s+2}$, defined by a multihomogeneous polynomial of $\mathbb{F}_q[\Lambda]$ of degree at most
\[
B_{d,s} := D^{r-s-2}\delta (((n-s)(r-s) + 2)D + r - s - 1) + \delta + 1
\]
in each group of variables $\Lambda_i$, with the following property: for any $\lambda := (\lambda_0, \ldots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2} \setminus H$, let $Y_j := \lambda_j \cdot X$ for $0 \leq j \leq s + 1$, let $\pi : V \to \mathbb{P}^{s+1}$ be the linear mapping defined by $Y_0, \ldots, Y_{s+1}$ and let $L := \{Y_0 = \cdots = Y_{s+1} = 0\} \subset \mathbb{P}^n$. Then the following conditions hold:

(i) the polar variety $M(L)$ has dimension $s$,
(ii) any fiber $\pi^{-1}(y)$ is of pure dimension $r - s - 1$,
(iii) the set of exceptional points of $\pi$ is nonsingular of pure dimension $r - s - 2$.

For such a matrix $\lambda$, our effective version of the Bertini smoothness theorem (Theorem 6.3) asserts that there exists a variety $W_L := W(L) \subset \mathbb{P}^{s+1}$ of dimension at most $s$ and degree at most $D^{r-s}\delta$ such that for every $y \in \mathbb{P}^{s+1} \setminus W_L$, the fiber $\pi^{-1}(y)$ is a nonsingular complete intersection. Since $\pi^{-1}(y)$ is $\mathbb{F}_q$–definable for every $y \in \mathbb{P}^{s+1}(\mathbb{F}_q)$, we can estimate the number $N_y := |V_y(\mathbb{F}_q)|$ for $y \in \mathbb{P}^{s+1}(\mathbb{F}_q) \setminus W_L$ using
Deligne’s estimate \([14]\). On the other hand, fibers of points in \(W_L\) do
not make a significant contribution to the asymptotic behavior of the
number of rational points of \(V\). More precisely, we have the following
result.

**Theorem 8.1.** Let \(V \subset \mathbb{P}^n\) be a complete intersection defined over
\(\mathbb{F}_q\), of dimension \(r \geq 2\), degree \(\delta\), multidegree \(d\) and singular locus of
dimension at most \(s \in \{r - 2, r - 3\}\). Then the following estimate holds:

\[
|V(\mathbb{F}_q)| - p_r \leq b'_{r-s-1}q^{(r+s+1)/2} + A(n, s, d) q^{r-1},
\]

where \(A(n, s, d) := 2b'_{r-s-1} + 2(\delta D^{r-s} + 1)(\delta - 1)\) and \(b'_{r-s-1} :=
\(b'_{r-s-1}(n - s - 1, d)\) is the \((r - s - 1)\)th primitive Betti number of any
nonsingular complete intersection of \(\mathbb{P}^{n-s-1}\) of dimension \(r - s - 1\) and
multidegree \(d\).

**Proof.** First we observe that, if \(D = 1\), then \(V\) is a quadric, and the
statement of the theorem follows from known results on the number of
rational points of quadrics (see, e.g., [30, Theorem 2E] or [27, Section
6.2]).

Next we claim that we may assume without loss of generality that \(q >
B_{d,s}\) holds. Indeed, suppose that \(q \leq B_{d,s}\) holds. For \(D = 2\) we have
that \(V\) is either a cubic hypersurface or an intersection of two quadrics.
In both cases we have the upper bound \(|V(\mathbb{F}_q)| \leq \delta q^r + p_{r-1}\) (see [31]
and [14]), which implies \(|V(\mathbb{F}_q)| - p_r \leq (\delta - 1)q^r \leq B_{d,s}(\delta - 1)q^{r-1}.
Then the inequality \(B_{d,s} \leq 10 \cdot 2^{-s} \cdot \delta + 1\) completes the proof of
the theorem in this case.

On the other hand, for \(D \geq 3\), according to [27] we have \(|V(\mathbb{F}_q)| \leq \delta p_r,
and therefore \(|V(\mathbb{F}_q)| - p_r \leq (\delta - 1)p_r \leq 2B_{d,s}(\delta - 1)q^{r-1}\. As a
consequence, from the inequality \(B_{d,s} \leq 7D^{r-s}\delta + 1\) we easily deduce the
statement of the theorem in this case. This finishes the proof of
our claim.

For \(q > B_{d,s}\), combining Theorem 4.5 and Corollary 5.3 with Corollary 8.2 we
deduce that there exists \(\lambda \in (\mathbb{P}^n(\mathbb{F}_q))^{s+2}\) such that conditions (i)–(iii)
above are satisfied.

Let \(V_y\) be the linear section of \(V\) which is obtained as the Zariski
closure of the fiber \(\pi^{-1}(y)\) of an arbitrary point \(y \in \mathbb{P}^{s+1}\). We express
\(|V(\mathbb{F}_q)|\) in terms of the quantities \(N_y := |V_y(\mathbb{F}_q)|\) with \(y \in \mathbb{P}^{s+1}(\mathbb{F}_q)\):

\[
|V(\mathbb{F}_q)| = \sum_{y \in \mathbb{P}^{s+1}(\mathbb{F}_q)} (N_y - e) + e = \sum_{y \in \mathbb{P}^{s+1}(\mathbb{F}_q)} N_y - (p_{s+1} - 1)e,
\]

where \(e := |(V \cap L)(\mathbb{F}_q)|\). Since \(V \cap L\) has dimension \(r - s - 2\), we have
that \(e \leq \delta p_{r-s-2}\), and thus \(|e - p_{r-s-2}| \leq (\delta - 1)p_{r-s-2}\), holds.
Subtracting $p_r$ at both sides of (19) and taking into account the identity $p_r = p_{s+1}p_{r-s-1} - (p_{s+1} - 1)p_{r-s-2}$, we obtain:

$$||V(F_q)| - p_r|| \leq \sum_{y \in \mathbb{P}^{s+1}(F_q)} |N_y - p_{r-s-1}| + (p_{s+1} - 1)(\delta - 1)p_{r-s-2}$$

(20) $$\leq \sum_{y \in \mathbb{P}^{s+1}(F_q)} |N_y - p_{r-s-1}| + 2(\delta - 1)q^{r-1}. $$

Let $W_L \subset \mathbb{P}^{s+1}$ be the variety of the statement of Theorem 6.3. We can decompose the first term of the right-hand side of (20) as:

$$\sum_{y \in \mathbb{P}^{s+1}(F_q)} |N_y - p_{r-s-1}| = \sum_{y \notin W_L(F_q)} |N_y - p_{r-s-1}| + \sum_{y \in W_L(F_q)} |N_y - p_{r-s-1}|. $$

In order to estimate the first term in the right-hand side of the above expression, Theorem 6.3 asserts that, if $y \notin W_L(F_q)$, then $V_y$ is a nonsingular complete intersection of $\mathbb{P}^{n-s-1}$ defined over $F_q$, of pure dimension $r - s - 1$, degree $\delta$ and multidegree $d$. By (14) we have $|N_y - p_{r-s-1}| \leq b'_{r-s-1}q^{(r-s-1)/2}$. Therefore, we obtain

(21) $$\sum_{y \notin W_L(F_q)} |N_y - p_{r-s-1}| \leq b_{r-s-1}'q^{-\frac{r-s-1}{2}}p_{s+1} \leq b'_{r-s-1}q^{-\frac{r-s+1}{2}} + 2b'_{r-s-1}q^{r-1}. $$

On the other hand, for $y \in W_L(F_q)$ we have $N_y \leq \delta p_{r-s-1}$. Hence, taking into account that $\delta \geq 2$ holds, we obtain $|N_y - p_{r-s-1}| \leq (\delta - 1)p_{r-s-1}$. From (7) it follows that $|W_L(F_q)| \leq \deg W_L \cdot p_s$ holds and thus

(22) $$\sum_{y \in W_L(F_q)} |N_y - p_{r-s-1}| \leq (\delta - 1)p_{r-s-1} \cdot \deg W_L \cdot p_s \leq 4(\delta - 1) \deg W_L \cdot q^{r-1}. $$

Combining (20), (21), (22), we conclude that

$$||V(F_q)| - p_r|| \leq b'_{r-s-1}q^{-\frac{r-s+1}{2}} + 2 \left( b'_{r-s-1} + (2D^{r-s}\delta + 1)(\delta - 1) \right) q^{r-1}. $$

From this estimate we easily deduce the statement of the theorem. □

Our next result is concerned with the number of smooth $\mathbb{F}_q$-rational points.

**Theorem 8.2.** Let notations and assumptions be as in Theorem 8.1. Then the following estimate holds:

$$||V_{sm}(\mathbb{F}_q)| - p_r|| \leq b'_{r-s-1}q^{-\frac{r-s+1}{2}} + B(n, s, d)q^{r-1}, $$

where $B(n, s, d) := 2b'_{r-s-1} + 2(2D^{r-s}\delta + 1)(\delta - 1) + 2s + 2(\delta - 1)B_{d,s}$.

**Proof.** Let $\mathcal{H}_1 \subset (\mathbb{P}^n)^{s+2}$ and $\mathcal{H}_2 \subset (\mathbb{P}^n)^{s+2}$ be the hypersurfaces of Theorem 4.3 and Corollary 5.5 respectively, and let $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2 \subset \mathbb{P}^{n}$. Since $\mathcal{H}_1 \subset (\mathbb{P}^n)^{s+2}$ is contained in a hyperplane, we have $\dim \mathcal{H}_1 = n - s - 1$. Therefore, by Theorem 5.5 we have $|\mathcal{H}(F_q)| \leq b'_{n-s-1}q^{-\frac{n-s}{2}}$.

Combining (19), (20), (21), we obtain

$$||V(F_q)| - p_r|| \leq b'_{r-s-1}q^{-\frac{r-s+1}{2}} + b'_{n-s-1}q^{-\frac{n-s}{2}} + 2b'_{r-s-1}q^{r-1}.$$
(\mathbb{P}^n)^{s+2}. Recall that \( \mathcal{H} \) is defined by a multihomogeneous polynomial of \( \mathbb{F}_p[\Lambda] \) of degree at most \( B_{d,s} \) in each group of variables \( \Lambda_i \). We have

\[
||V_{\text{sm}}(\mathbb{F}_q)|| - p_r
\]

\[
= \frac{1}{p_n^{s+2}} \left( \sum_{\lambda \in ((\mathbb{P}^n)^{s+2}\setminus \mathcal{H})(\mathbb{F}_q)} ||V_{\text{sm}}(\mathbb{F}_q)|| - p_r \right)\]

\[
\leq \frac{1}{p_n^{s+2}} \left( \sum_{\lambda \in ((\mathbb{P}^n)^{s+2}\setminus \mathcal{H})(\mathbb{F}_q)} ||V_{\text{sm}}(\mathbb{F}_q)|| - p_r + |\mathcal{H}(\mathbb{F}_q)|(\delta - 1)p_r \right).
\]

By [1] it follows that \( |\mathcal{H}(\mathbb{F}_q)| \leq p_n^{s+2} - (q^n - \min\{q, B_{d,s}\}q^{n-1})^{s+2}. \)

Hence,

\[
|\mathcal{H}(\mathbb{F}_q)| \leq 2(s + 2)(\delta - 1)B_{d,s}q^{-1}.
\]

For each \( \lambda \in ((\mathbb{P}^n)^{s+2}\setminus \mathcal{H})(\mathbb{F}_q) \), Theorem [5.3] shows that there exists a variety \( W_L \subset \mathbb{P}^{s+1} \) of dimension at most \( s \) and degree at most \( D^{r-s}\delta \) such that for every \( y \in \mathbb{P}^{s+1}\setminus W_L \), the Zariski closure \( V_y \) of the fiber \( \pi^{-1}(y) \) is a nonsingular complete intersection contained in \( V_{\text{sm}} \). Then, arguing as in the proof of Theorem [8.1], we obtain

\[
\frac{1}{p_n^{s+2}} \sum_{\lambda \in \mathcal{H}(\mathbb{F}_q)} \left( ||V_{\text{sm}}(\mathbb{F}_q)|| - p_r \right) \leq \left( b_{r-s-1}q^{r+1+s-1} + 2(b_{r-s-1} + (2D^{r-s}\delta + 1)(\delta - 1))q^{r-1} \right).
\]

From this inequality we easily deduce the statement of the theorem. \( \square \)

8.1. An estimate for a normal complete intersection. In this section we consider the case \( s := r - 2 \) of Theorems [8.1] and [8.2]. We have the following result.

**Corollary 8.3.** Let \( V \subset \mathbb{P}^n \) be a normal complete intersection defined over \( \mathbb{F}_q \), of dimension \( r \geq 2 \), degree \( \delta \) and multidegree \( d \). Then we have

(23) \( ||V(\mathbb{F}_q)|| - p_r \) \( \leq \) \( (\delta(D - 2) + 2)q^{r-1/2} + 14D^2\delta^2q^{r-1}, \)

(24) \( ||V_{\text{sm}}(\mathbb{F}_q)|| - p_r \) \( \leq \) \( (\delta(D - 2) + 2)q^{r-1/2} + 11(r + 1)D^2\delta^2q^{r-1}. \)

**Proof.** Applying Theorems [8.1] and [8.2] with \( s = r - 2 \), we obtain

\[
||V(\mathbb{F}_q)|| - p_r \leq b'_1q^{r-1/2} + A(n, r - 2, d)q^{r-1},
\]

\[
||V_{\text{sm}}(\mathbb{F}_q)|| - p_r \leq b'_1q^{r-1/2} + B(n, r - 2, d)q^{r-1},
\]

where \( b'_1 := b'_1(n - r + 1, d), \)

\[
A(n, r - 2, d) := 2b'_1 + 2(7D^2\delta + 1)(\delta - 1),
\]

\[
B(n, r - 2, d) := 2b'_1 + 2(2D^2\delta + 1)(\delta - 1) + 2r(\delta - 1)B_{d,r-2}.
\]
Taking into account the identity $b_1' = \delta(D - 2) + 2$ we easily deduce (23). On the other hand, using the inequality $n - r \leq D$ we readily obtain (24).

For a normal complete intersection $V$ as in Corollary 8.3, we have the following estimate (see [16, Corollary 6.2]):

\begin{equation}
\sum_{i=1}^{n-r} d_i \leq (\frac{n-r}{n-r})^{n-r} \prod_{i=1}^{n-r} d_i^{r+1} \leq (2(n-r))^{n-r} D^2 \delta \left( \sum_{i=1}^{n-r} d_i^{r-1} \right),
\end{equation}

where the mid inequality is a consequence of the AM–GM inequality.

From the previous inequalities we draw several conclusions. First of all, for varieties of high dimension, say $r \geq (n+1)/2$, (23) and (26) are clearly preferable to (25). In particular, for hypersurfaces the second term in the right–hand side of both (23) and (26) is roughly quartic in $\delta$ while the one (25) contains an exponential term $\delta^{n+1}$. On the other hand, for varieties of low dimension the second term in the right–hand side of (25) might be preferable to (23) and (26). In particular, for curves the former is roughly linear in $\delta$ while the latter is quadratic in $\delta$. In this sense, we may say that (23)–(26) somewhat complement (25). Finally, we observe that the right–hand side of (26) is slightly lower than that of (23) but holds only for $q > 2(n-r)d\delta + 1$, while (26) holds without any restriction on $q$.

8.2. An estimate for a complete intersection regular in codimension 2. In this section we consider the case of a complete intersection which is regular in codimension 2, namely $s \leq r - 3$. We have the following result.

**Corollary 8.4.** Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension $r \geq 3$, degree $\delta$ and multidegree $d$ for which the singular locus has dimension at most $r - 3$. Then we have the following
estimates:

\begin{align}
  (27) & \quad ||V(\mathbb{F}_q)| - p_r| \leq 14D^3\delta^2 q^{r-1}, \\
  (28) & \quad ||V_{sm}(\mathbb{F}_q)| - p_r| \leq (34r - 20)D^3\delta^2 q^{r-1}.
\end{align}

**Proof.** By Theorems 8.1 and 8.2 it follows that

\begin{align*}
  ||V(\mathbb{F}_q)| - p_r| & \leq A(n, r - 3, d) q^{r-1}, \\
  ||V_{sm}(\mathbb{F}_q)| - p_r| & \leq B(n, r - 3, d) q^{r-1},
\end{align*}

where

\begin{align*}
  A(n, r - 3, d) & := 3b'_2 + 2(7D^3\delta + 1)(\delta - 1), \\
  B(n, r - 3, d) & := 3b'_2 + 2(2D^3\delta + 1)(\delta - 1) + 2(r - 1)(\delta - 1)B_{d, r-3},
\end{align*}

and \( b'_2 := b'_2(n - r + 2, d) \). According to Remark 7.1 we have \( b'_2 \leq (n - r + 1) D^2 \delta \leq (D + 1) D^2 \delta \). Therefore, a simple calculation shows the statement of the corollary.

With hypothesis as in the statement of Corollary 8.4, the following estimate holds (16, Theorem 6.1):

\begin{equation}
  (29) \quad ||V(\mathbb{F}_q)| - p_r| \leq b'_2(n - r + 2, d) q^{r-1} + 9 \cdot 2^{n-r} \cdot ((n-r)d+3)^{n+1}q^{r-3/2}.
\end{equation}

In the comparison of (27) and (29) similar remarks can be made as in the case of normal complete intersections: for high-dimensional varieties (27) might be preferable, while for low-dimensional varieties (29) is likely to be better. Nevertheless, the presence of exponentials in the second term of the right-hand side of (29) may difficult the application of such an estimate even for low-dimensional varieties. As an illustration of this phenomenon, we sketch an application which requires an estimate like (27).

8.2.1. **The average value set of “small” families of polynomials.** Let \( T \) be an indeterminate over \( \mathbb{F}_q \) and let \( f \in \mathbb{F}_q[T] \). We define the value set \( N(f) \) of \( f \) as \( N(f) := \{|f(c) : c \in \mathbb{F}_q|\} \) (cf. [27]). Birch and Swinnerton–Dyer established the following significant result [7]: if \( f \in \mathbb{F}_q[T] \) with \( \deg(f) = d \geq 1 \) is a generic polynomial, then

\[ N(f) = \mu_d q + O(q^{1/2}), \]

where \( \mu_d := \sum_{r=1}^{d} (-1)^{r-1}/r! \) and the constant underlying the \( O \)-notation depends only on \( d \). Results on the average value \( N(d, 0) \) of \( N(f) \) when \( f \) ranges over all monic polynomials in \( \mathbb{F}_q[T] \) with \( f(0) = 0 \) of fixed degree were obtained by Uchiyama [35] and improved by Cohen [11]. More precisely, in [11 §2] it is shown that

\[ N(d, 0) = \sum_{r=1}^{d} (-1)^{r-1} \binom{q}{r} q^{1-r} = \mu_d q + O(1). \]

However, if some of the coefficients of \( f \) are fixed, the results on the average value of \( N(f) \) are less precise. In fact, Uchiyama [35] and
Cohen [10] obtain the result that we now state. Let be given \(s \leq d - 2\) and elements \(a_{d-1}, \ldots, a_{d-s} \in \mathbb{F}_q\). Let \(\alpha^r := (a_{d-s-1}, \ldots, a_1)\) and let \(f_{\alpha^r} := T^d + \sum_{i=1}^{d-1} a_i T^i\). Then for \(p := \text{char(}\mathbb{F}_q) > d\),

\[
N(d, s) = N(d, s; a_{d-1}, \ldots, a_{d-s}) := \frac{1}{q^{d-s-1}} \sum_{\alpha^r \in \mathbb{F}_q^{d-s-1}} N(f_{\alpha^r}) = \mu_d q + \mathcal{O}(q^{1/2}),
\]

the constant underlying the \(\mathcal{O}\)–notation depends only on \(d\) and \(s\). Our aim is to obtain an strengthened explicit version of (30) which holds without any restriction on \(p\).

For this purpose, we sketch our approach, which relies essentially on Corollary 8.4 (proofs will appear elsewhere). We start with the following identity:

\[
N(d, s) = \sum_{r=1}^{d-s} (-1)^{r-1} \binom{q}{r} q^{1-r} + \frac{1}{q^{d-s-1}} \sum_{r=d-s+1}^{d} (-1)^{r-1} \chi(d, s, r),
\]

where \(\chi(d, s, r)\) denotes the number of subsets \(\mathcal{X}_r\) of \(\mathbb{F}_q\) with exactly \(r\) elements such that \(f := T^d + \sum_{i=1}^{d-s} a_i T^i\) admits an interpolant \(g \in \mathbb{F}_q[T]\) of degree at most \(d - s - 1\), namely \(f|_{\mathcal{X}_r} = g|_{\mathcal{X}_r}\). A subset \(\mathcal{X}_r \subset \mathbb{F}_q\) with this property is called allowable for \(f\).

In this way, we see that the asymptotic behavior of \(N(d, s)\) is determined by that of \(\chi(d, s, r)\) for \(d-s+1 \leq r \leq d\). Concerning the latter, we have the following result.

**Theorem 8.5.** Fix \(r\) with \(d-s+1 \leq r \leq d\) and let \(X_1, \ldots, X_r\) be indeterminates over \(\mathbb{F}_q\). Suppose that \(1 \leq s \leq d/2\). Then there exist polynomials \(R_{d-s}, \ldots, R_{r-1} \in \mathbb{F}_q[X_1, \ldots, X_r]\) with the following properties:

- \(\mathcal{X}_r := \{x_1, \ldots, x_r\}\) is allowable for \(f\) if and only if the equality \(R_j(x_1, \ldots, x_r) = 0\) holds for \(d-s \leq j \leq r - 1\).
- Let \(V \subset \mathbb{P}^r\) the projective closure of the affine variety of \(\mathbb{F}_q\) defined by the polynomials \(R_{d-s}, \ldots, R_{r-1}\). Then \(V\) is an ideal–theoretic complete intersection defined over \(\mathbb{F}_q\) of dimension \(d-s\) and degree \(s!(d-r)\), whose singular locus has codimension at least 3.
- Let \(V^0 := V \cap \{X_0 = 0\} \subset \mathbb{P}^{r-1}\). Then \(V^0\) is an ideal–theoretic complete intersection defined over \(\mathbb{F}_q\) of dimension \(d-s-1\) and degree \(s!(d-r)\), whose singular locus has codimension at least 3.

Combining Corollary 8.4 and Theorem 8.5, we obtain precise information about the asymptotic behavior of \(\chi(d, s, r)\) for \(d-s+1 \leq r \leq d\), and thus of \(N(d, s)\). More precisely, we have the following result.

**Theorem 8.6.** Let assumptions and notations be as in Theorem 8.5 and set \(D(s, d, r) := \sum_{j=d-r+1}^{d} (j-1)\) and \(\delta(s, d, r) := \prod_{j=d-r+1}^{d} j\). We
have the following estimate:
\[
\left| \chi(d, s, r) - \frac{q^{d-s}}{r!} \right| \leq \frac{15}{r!} D(s, d, r)^3 \delta(s, d, r)^2 q^{d-s-1}.
\]

From Theorem 8.6 we obtain the following result concerning the behavior of \(N(d, s)\).

**Corollary 8.7.** With assumptions and notations as in Theorems 8.5 and 8.6, we have
\[
|N(d, s) - \mu_d q| \leq E(s, d) := \frac{e^{-1}}{2} + 16 \sum_{r=d-s+1}^{d} \frac{D(s, d, r)^3 \delta(s, d, r)^2}{r!} + \frac{2d}{q}.
\]

A rough upper bound for the sum in the right-hand side of the previous expression is \(10d^7 2^{-d} e^{2\sqrt{d}}\), which is easily seen to tend to 0 as \(d\) tends to infinity.

Corollary 8.7 strengthens (30) in several aspects. The first one is that our result holds without any restriction on the characteristic \(p\) of \(\mathbb{F}_q\), while (30) holds for \(p > d\). The second aspect is that Corollary 8.7 shows that \(N(d, s) = \mu_d q + \mathcal{O}(1)\), while (30) only asserts that \(N(d, s) = \mu_d q + \mathcal{O}(q^{1/2})\). Finally, we obtain an explicit expression for the constant underlying the \(\mathcal{O}\)–notation with a good behavior, while (30) does not provide an explicit expression for the corresponding constant. On the other hand, it must be said that our result holds for \(s \leq d/2\), while (30) holds without restrictions on \(s\).

**References**

[1] E. Ballico, *An effective Bertini theorem over finite fields*, Advances in Geometry 3 (2003), 361–363.

[2] B. Bank, M. Giusti, J. Heintz, M. Safey El Din, and E. Schost, *On the geometry of polar varieties*, Appl. Algebra Engrg. Comm. Comput. 21 (2010), no. 1, 33–83.

[3] B. Bank, M. Giusti, J. Heintz, L. Lehmann, and L.M. Pardo, *Algorithms of intrinsic complexity for point searching in compact real singular hypersurfaces*, Found. Comput. Math. 12 (2012), no. 1, 75–122.

[4] B. Bank, M. Giusti, J. Heintz, and G.M. Mbakop, *Polar varieties and efficient real equation solving: The hypersurface case*, J. Complexity 13 (1997), no. 1, 5–27.

[5] ---, *Polar varieties and efficient real elimination*, Math. Z. 238 (2001), no. 1, 115–144.

[6] B. Bank, M. Giusti, J. Heintz, and L.M. Pardo, *Generalized polar varieties: Geometry and algorithms*, J. Complexity 21 (2005), no. 4, 377–412.

[7] B. Birch and H. Swinnerton-Dyer, *Note on a problem of Chowla*, Acta Arith. 5 (1959), no. 4, 417–423.

[8] W. Bruns and U. Vetter, *Determinantantal rings*, Lecture Notes in Math., vol. 1327, Springer, Berlin Heidelberg New York, 1988.

[9] A. Cafure and G. Matera, *An effective Bertini theorem and the number of rational points of a normal complete intersection over a finite field*, Acta Arith. 130 (2007), no. 1, 19–35.
[10] S. Cohen, Uniform distribution of polynomials over finite fields, J. Lond. Math. Soc. (2) 6 (1972), no. 1, 93–102.
[11] ______, The values of a polynomial over a finite field, Glasg. Math. J. 14 (1973), no. 2, 205–208.
[12] C. D’Andrea, T. Krick, and M. Sombra, Heights of varieties in multiprojective spaces and arithmetic Nullstellensätze, Ann. Sci. Éc. Norm. Supér. (4) 46 (2013), no. 4, 571–649.
[13] P. Deligne, La conjecture de Weil. I, Inst. Hautes Études Sci. Publ. Math. 43 (1974), 273–307.
[14] F. Edoukou, S. Ling, and C. Xing, Intersection of two quadrics with no common hyperplane in $\mathbb{P}^n(k)$, Preprint [arxiv:0907.4556v1 [math.CO]], 2009.
[15] W. Fulton, Intersection theory, Springer, Berlin Heidelberg New York, 1984.
[16] S. Ghorpade and G. Lachaud, Étale cohomology, Lefschetz theorems and number of points of singular varieties over finite fields, Mosc. Math. J. 2 (2002), no. 3, 589–631.
[17] ______, Number of solutions of equations over finite fields and a conjecture of Lang and Weil, Number Theory and Discrete Mathematics (Chandigarh, 2000) (New Delhi) (A.K. Agarwal et al., ed.), Hindustan Book Agency, 2002, pp. 269–291.
[18] J. Harris, Algebraic geometry: a first course, Grad. Texts in Math., vol. 133, Springer, New York Berlin Heidelberg, 1992.
[19] J. Heintz, Definability and fast quantifier elimination in algebraically closed fields, Theoret. Comput. Sci. 24 (1983), no. 3, 239–277.
[20] W. Hodge and D. Pedoe, Methods of algebraic geometry. Vol. I, Cambridge Math. Lib., Cambridge Univ. Press, Cambridge, 1968.
[21] ______, Methods of algebraic geometry. Vol. II, Cambridge Math. Lib., Cambridge Univ. Press, Cambridge, 1968.
[22] C. Hooley, On the number of points on a complete intersection over a finite field, J. Number Theory 38 (1991), no. 3, 338–358.
[23] S. Kleiman, The transversality of a general translate, Compos. Math. 28 (1974), no. 2, 287–297.
[24] ______, The enumerative theory of singularities, Real and Complex Singularities, Oslo 1976: Proceedings of the 9th Nordic Summer School/NAVF Symposium in Mathematics, Oslo, Aug. 5–25, 1976 (P. Holm, ed.), Sijthoff & Noordhoff, 1976, pp. 297–396.
[25] E. Kunz, Introduction to commutative algebra and algebraic geometry, Birkhäuser, Boston, 1985.
[26] D. Lewis and S. Schuur, Varieties of small degree over finite fields, J. Reine Angew. Math. 262/263 (1973), 293–306.
[27] R. Lidl and H. Niederreiter, Finite fields, Addison–Wesley, Reading, Massachusetts, 1983.
[28] R. Piene, Polar classes of singular varieties, Ann. Scient. Éc. Norm. Sup. Sér. 4 11 (1978), no. 2, 247–276.
[29] P. Samuel, Méthodes d’algèbre abstraite en géométrie algébrique, Springer, Berlin Heidelberg New York, 1967.
[30] W. Schmidt, Équations over finite fields. An elementary approach, Lectures Notes in Math., no. 536, Springer, New York, 1976.
[31] J-P. Serre, Lettre à M. Tsfasman, Astérisque 198-200 (1991), 351–353.
[32] I.R. Shafarevich, Basic algebraic geometry: Varieties in projective space, Springer, Berlin Heidelberg New York, 1994.
[33] B. Teissier, Variétés polaires. II: Multiplicités polaires, sections planes et conditions de Whitney, Algebraic geometry, Proc. Int. Conf., La Rábida/Spain
1981 (Berlin Heidelberg New York) (J. Aroca, R. Buchweitz, M. Giusti, and M. Merle, eds.), Lect. Notes Math., vol. 961, Springer, 1982, pp. 314–491.

[34] ________, Quelques points de l’histoire des variétés polaires, de Poncelet à nos jours, Séminaire d’analyse: 1987–1988 (Clermont–Ferrand), vol. 4, Univ. Blaise–Pascal, 1988.

[35] S. Uchiyama, Note on the mean value of \( V(f) \). II, Proc. Japan Acad. 31 (1955), no. 6, 321–323.

[36] W. Vogel, Results on Bézout’s theorem, Tata Inst. Fundam. Res. Lect. Math., vol. 74, Tata Inst. Fund. Res., Bombay, 1984.

[37] T. Wooley, Artin’s Conjecture for septic and undecic forms, Acta Arith. 133 (2008), no. 1, 25–35.

[38] J. Zahid, Nonsingular points on hypersurfaces over \( \mathbb{F}_q \), J. Math. Sci. (N. Y.) 171 (2010), no. 6, 731–735.

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