LARGE DEVIATIONS, SHARRON-MCMILLAN-BREIMAN THEOREM FOR SUPER-CRITICAL TELECOMMUNICATION NETWORKS

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Abstract. In this article we obtain large deviation asymptotics for supercritical communication networks modelled as signal-interference-noise ratio networks. To do this, we define the empirical power measure and the empirical connectivity measure, and prove joint large deviation principles (LDPs) for the two empirical measures on two different scales i.e. λ and λ²aλ, where λ is the intensity measure of the poisson point process (PPP) which defines the SINR random network. Using this joint LDPs we prove an asymptotic equipartition property for the stochastic telecommunication Networks modelled as the SINR networks. Further, we prove a Local large deviation principle (LLDP) for the SINR Network. From the LLDP we prove the a large deviation principle, and a classical McMillian Theorem for the stochastic SNIR network processes. Note, for typical empirical connectivity measure, qπ ⊗ π, we can deduce from the LLDP a bound on the cardinality of the space of SINR networks to be approximately equal to e^{λ²a₁∥qπ⊗π∥H(qπ⊗π∥qπ⊗π∥)}, where the connectivity probability of the network, Q_{zλ}, satisfies a⁻¹λ Q_{zλ} → q. Observe, the LDP for the empirical measures of the stochastic SINR network were obtained on spaces of measures equipped with the τ− topology, and the LLDPs were obtained in the space of SINR network process without any topological restrictions.

Keywords: Super-critical sinr networks, Poisson Point Process, Empirical power measure, Empirical connectivity measure, Large deviations, Relative entropy, Entropy

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1. Introduction and Background

1.1 Introduction. Large deviations may be regarded as a group of mathematical techniques (stochastic methods) often use to estimate asymptotic properties of increasingly rare events such as their empirical measures and most likely manner of occurrence. See, for example, [13]. There are many applications of large deviation techniques to SINR networks as a model for Telecommunication networks. Some of this applications include but not limited to the analysis of bi-stability in networks, example notorious bi-stability in multiple access protocols such as the Aloha, and the stochastic
We shall consider behaviour of ATM such as the admission control, sizing of internal buffers, and the simulation of ATM models. See [13].

The Shanno-MacMillian-Breiman (SMB) Theorem or the asymptotic equipartition property may be regarded as the strong law of large numbers in information theory. It says output source of a stochastic data source may be partition into two sets, namely the set of typical events and the set of atypical events. The SMB is the foundation of all approximate pattern matching and coding algorithms.

Researchers over the last two decades have given some large deviation analysis for telecommunication networks modelled as a sequence of i.i.d random variables and or markov chains in discrete and continuous times. See, [13] and reference therein. [11] and [12] defined empirical measures on the SINR network and proved some joint LDP results including the SMB and the classical MacMillian theorem for the dense or critical telecommunication networks modelled as the SINR network.

In this article we prove joint large deviation principles on the scales \( \lambda \) and \( \lambda^2 a_\lambda \), where \( \lambda \) is the intensity measure of the underlining PPP of the SINR network. See, [2] or [3] for similar results for the colored random graph models. From these LDPs we prove an asymptotic equipartition property, see example [8], for the SINR networks.

Further, we prove a local LDP for the SINR networks. See example, [10] or [9] and reference therein. From the local LDP we deduce asymptotic bounds on the cardinality of the set of SINR networks for a given typical empirical power measure. We also prove from the local LDP an LDP for the SINR network processes.

The remaining part of the article is organized in this manner: Section 2 contains the main results; Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4, Corollary 2.5 and Corollary 2.6. In Section 3 the main results of the article, Theorem 2.1, Section 3.3 contain proof of the SBM, see Theorem 2.3 and Section 3.4. Proof of Theorem 2.4, Corollary 2.3 and Corollary 2.6. Finally, we give the conclusion of the article in Section 6.

1.2 Background

We fix dimension \( d \in \mathbb{N} \) and some measurable set \( D \subset \mathbb{R}^d \) with respect to the Borel-Sigma algebra \( B(\mathbb{R}^d) \). For an intensity function, \( \lambda \pi : D \to [0,1] \), a transition kernel from \( D \) to \( (0,\infty) \), \( K \) and a path loss function, \( \beta(\ell) = \ell^{-r} \), where \( r \in (0,\infty) \), and some technical constants; \( \tau(\lambda), \gamma(\lambda) : (0,\infty) \to (0,\infty) \), we define the SINR network model as follows:

- We pick \( Z = (Z_i)_{i \in I} \) a Poisson Point process (PPP) with rate measure \( \lambda \pi : D \to [0,1] \).
- Given \( Z \), we assign each \( Z_i \) a power \( \eta(Z_i) = \eta_i \) independently according to the transition function \( K(\cdot, Z_i) \).
- For any two powered points \( ((Z_i, \eta_i), (Z_j, \eta_j)) \) we connect a link iff \( \text{SINR}(Z_i, Z_j, Z) \geq \tau(\lambda)(\eta_j) \) and \( \text{SINR}(Z_j, Z_i, Z) \geq \tau(\lambda)(\eta_i) \),

where

\[
\text{SINR}(Z_j, Z_i, Z) = \frac{\eta_i \beta(\|Z_i - Z_j\|)}{N_0 + \gamma(\lambda)(\eta_j) \sum_{i \in I \setminus \{j\}} \eta_i \beta(\|Z_i - Z_j\|)}
\]

We shall consider \( Z^\lambda := Z^\lambda(\eta, K, \beta) = \{(Z_i, \eta_i), j \in I\}, E \) under the joint law of the powered Poisson Point Process and the Network. We will interpret \( Z^\lambda \) as an SINR Network and \((Z_i, \eta_i) := Z^\lambda_i\) as the
power type of device \(i\). We recall from [11] that the link/connectivity probability of the SINR network, \(Q^z\), is given by \(Q^z((x, \eta_x), (y, \eta_y)) = e^{-\lambda q^z(x, \eta_x, y, \eta_y)}\), where

\[
q^z((x, \eta_x), (y, \eta_y)) = \int_D \left[ \frac{e^{-(\lambda q^z(x, \eta_x) + ||x-y||)\eta}}{\lambda q^z(x, \eta_x) + ||x-y||} + \frac{e^{-(\lambda q^z(y, \eta_y) + ||y-x||)\eta}}{\lambda q^z(y, \eta_y) + ||y-x||} \right] \eta(dz).
\]

We have assumed there exists a sequence of real numbers \(a_\lambda\) and a function \(q : D \times \mathbb{R}_+ \to (0, \infty)\) such that \(a_\lambda \to \infty\) and \(\lim a_\lambda^{-1}Q^z((x, \eta_x), (y, \eta_y)) = q((x, \eta_x), (y, \eta_y))\).

Sakyi-Yeboah et. al [12] studied the critical SINR Networks (i.e. \(\lambda a_\lambda \to 1\)). In this paper we shall look at Sup-critical SINR Networks (i.e. \(\lambda a_\lambda \to \infty\)).

We define the set \(S(D)\) by

\[
S(D) = \bigcup_{x \in D} \{ x : |x \cap A| < \infty, \text{ for any bounded } A \subset D \}.
\]

Write \(W = S(D \times \mathbb{R}_+^+)\) and \(M(W)\), denote the space of positive measures on the space \(W\) equipped with \(\tau-\) topology. Note, \(W\) a locally finite subset of the set \(W\). See, example, [12] or [1]. For any SINR Network \(Z^{\lambda}\) we define a probability measure, the empirical power measure, \(U^\lambda_1 \in M(W)\), by

\[
U^\lambda_1((x, \eta_x)) := \frac{1}{\lambda} \sum_{i \in I} \delta_{Z^{\lambda}_i((x, \eta_x))}
\]

and a finite measure, the empirical connectivity measure \(U^\lambda_2 \in M(W \times W)\), by

\[
U^\lambda_2((x, \eta_x), (y, \eta_y)) := \frac{1}{\lambda^2 a_\lambda} \sum_{(i, j) \in E} \left[ \delta_{(Z^{\lambda}_i, Z^{\lambda}_j)} + \delta_{(Z^{\lambda}_{i}, Z^{\lambda}_{j})} \right]((x, \eta_x), (y, \eta_y)).
\]

Note that the total mass \(\|U^\lambda_1\|\) of the empirical power measure is 1 and total mass of the empirical link measure is 2\(|E|/\lambda^2 a_\lambda\).

### 2. Main Results

Theorem 2.1 is a Joint Large deviation principle for the empirical measures of the SINR network models. We recall from Subsection 1.2 the definition of \(q^z\) as

\[
q^z((x, \eta_x), (y, \eta_y)) = \int_D \left[ \frac{e^{-(\lambda q^z(x, \eta_x) + ||x-y||)\eta}}{\lambda q^z(x, \eta_x) + ||x-y||} + \frac{e^{-(\lambda q^z(y, \eta_y) + ||y-x||)\eta}}{\lambda q^z(y, \eta_y) + ||y-x||} \right] \eta(dz)
\]

and write

\[
q^z \otimes \eta((x, \eta_x), (y, \eta_y)) := q((x, \eta_x), (y, \eta_y)) \mu((x, \eta_x)) \mu((y, \eta_y)).
\]

**Theorem 2.1.** Let \(Z^{\lambda}\) is a super critical powered SINR network with rate measure \(\lambda \eta : D \to [0, 1]\) and a power probability function \(K(\cdot, \eta) = e^{-c_\eta \cdot \eta}, \eta \geq 0\) and path loss function \(\beta(r) = r^{-\ell}, \ell > 0\). Thus, the connectivity probability \(Q^{z^{\lambda}}\) of \(Z^{\lambda}\) satisfies \(a_\lambda^{-1}Q^{z^{\lambda}} \to q\) and \(\lambda a_\lambda \to \infty\). Then, as \(\lambda \to \infty\), the pair of measures \((U^\lambda_1, U^\lambda_2)\) satisfies a large deviation principle in the space \(M(W) \times M(W \times W)\)

(i) with speed \(\lambda\) and good rate function

\[
I_{Sc}(\pi, \nu) = \begin{cases} H \left( \pi \left| \eta \otimes q \right. \right) & \text{if } \nu = q^z \otimes \eta \\ \infty & \text{otherwise.} \end{cases}
\]

(2.1)
Theorem 2.4. Let the connectivity probability $Q^\lambda$ of $Z^\lambda$ satisfies $a_\lambda^{-1}Q^\lambda \to q$ and $\lambda a_\lambda \to \infty$. Let $Z^\lambda$ be a super critical powered SIR network with rate measure $\lambda \eta : D \to [0,1]$ and a power probability function $K(\cdot, \eta) = ce^{-c\eta}$, $\eta > 0$ and path loss function $\beta(r) = r^{-\ell}$, for $\ell > 0$. Thus, the connectivity probability $Q^\lambda_z$ of $Z^\lambda$ satisfies $a_\lambda^{-1}Q^\lambda_z \to q$ and $\lambda a_\lambda \to \infty$. Suppose the sequence $a_\lambda$ of $Z^\lambda$ is such that $\lambda a_\lambda \log \lambda \to \infty$ and $a_\lambda \log \lambda \to -1$. Then, we have

$$\lim_{\lambda \to \infty} \mathbb{P} \left\{ -\frac{1}{\alpha_\lambda^{2}\log \lambda} \log P(Z^\lambda) - \int_{W \times W} q((x, \eta_x), (y, \eta_y)) q(d\eta_x) q(d\eta_y) dx dy \geq \varepsilon \right\} = 0.$$

Theorem 2.5. Let $Z^\lambda$ be a super critical powered SIR network with rate measure $\lambda \nu : D \to [0,1]$ and a power probability function $K(\eta) = ce^{-c\eta}$, $\eta > 0$ and path loss function $\beta(r) = r^{-\ell}$, for $\ell > 0$. Thus, the connectivity probability $Q^\lambda_z$ of $Z^\lambda$ satisfies $a_\lambda^{-1}Q^\lambda_z \to q$ and $\lambda a_\lambda \to \infty$. Then, for any functional $\nu \in \mathcal{M}_\nu$ and a number $\varepsilon > 0$, there exists a weak neighbourhood $B_\nu$ such that

$$\mathbb{P}_\nu \left\{ \mathcal{L}_\lambda^\lambda \in \mathcal{G}_\nu \left| L_\nu^\lambda \in B_\nu \right. \right\} \leq e^{-\frac{1}{2}\lambda^2 a_\lambda H(\nu||q\pi \otimes \pi) - \lambda a_\lambda \varepsilon}.$$

Theorem 2.6. Let $Z^\lambda$ be a super critical powered SIR network with rate measure $\lambda \nu : D \to [0,1]$ and a power probability function $K(\eta) = ce^{-c\eta}$, $\eta > 0$ and path loss function $\beta(r) = r^{-\ell}$, for $\ell > 0$. Thus, the connectivity probability $Q^\lambda_z$ of $Z^\lambda$ satisfies $a_\lambda^{-1}Q^\lambda_z \to q$ and $\lambda a_\lambda \to \infty$. Then, for any functional $\nu \in \mathcal{M}_\nu$, a number $\varepsilon > 0$ and a fine neighbourhood $B_\nu$, we have the estimate:

$$\mathbb{P}_\nu \left\{ \mathcal{L}_\lambda^\lambda \in \mathcal{G}_\nu \left| L_\nu^\lambda \in B_\nu \right. \right\} \geq e^{-\frac{1}{2}\lambda^2 a_\lambda H(\nu||q\pi \otimes \pi) + \lambda a_\lambda \varepsilon}.$$

We define for telecommunication networks an entropy $h : \mathcal{M}(W \times W) \to [0, \infty]$ by

$$h(\nu) := \left( ||\nu|| - ||\lambda \pi \otimes \pi|| - \left< \nu, \log \frac{\nu}{||q\pi \otimes \pi||} \right> \right)/2.$$
Corollary 2.5 (McMillian Theorem). Let $G_p$ be a super critical powered Sinr network with rate measure $\lambda \mu : D \to [0, 1]$ and a power probability function $K(\eta) = ce^{-\eta}, \eta > 0$ and path loss function $\beta(r) = r^{-\ell},$ for $\ell > 0.$ Thus, the connectivity probability $Q^{z^\lambda}$ of every $z^\lambda \in G_p$ satisfies $a_\lambda^{-1}Q^{z^\lambda} \to q$ and $\lambda a_\lambda \to \infty.$

(i) For any empirical connectivity measure $\nu$ on $W \times W$ and $\varepsilon > 0,$ there exists a neighborhood $B_\nu,$ such that
\[
\text{Card}\left(\{z^\lambda \in G_p \mid L^\lambda_2 \in B_\nu\}\right) \geq e^{\lambda^2 a_\lambda (h(\nu) - \varepsilon)}.
\]

(ii) for any neighborhood $B_\rho$ and $\varepsilon > 0,$ we have
\[
\text{Card}\left(\{z^\lambda \in G_p \mid U^\lambda_2 \in B_\nu\}\right) \leq e^{\lambda^2 a_\lambda (h(\nu) + \varepsilon)},
\]
where Card$(A)$ means the cardinality of $A.$

Remark 1 For $\nu = q\pi \otimes \pi,$ we have $\text{Card}\left(\{y \in G_p\}\right) \approx e^{\lambda^2 a_\lambda ||q\pi \otimes \pi|| h(q\pi \otimes \pi/||q\pi \otimes \pi||)}.$

Corollary 2.6. Let $Z^\lambda$ is a super critical powered Sinr network with rate measure $\lambda \eta : D \to [0, 1]$ and a power probability function $K(\eta) = ce^{-\eta}, \eta > 0$ and path loss function $\beta(r) = r^{-\ell},$ for $\ell > 0.$ Thus, the connectivity probability $Q^{z^\lambda}$ of $Z^\lambda$ satisfies $a_\lambda^{-1}Q^{z^\lambda} \to q$ and $\lambda a_\lambda \to \infty.$

- Let $F$ be closed subset $M_\pi.$ Then we have
\[
\limsup_{\lambda \to \infty} \frac{1}{\lambda^2 a_\lambda} \log \mathbb{P}_\pi\left\{Z^\lambda \in G_p \mid U^\lambda_2 \in F\right\} \leq -\frac{1}{4} \inf_{\pi \in \mathcal{F}} \left\{\mathcal{H}(\nu||q\pi \otimes \pi)\right\}.
\]

- Let $O$ be open subset $M_\mu.$ Then we have
\[
\liminf_{\lambda \to \infty} \frac{1}{\lambda^2 a_\lambda} \log \mathbb{P}_\pi\left\{Z^\lambda \in G_p \mid U^\lambda_2 \in O\right\} \geq -\frac{1}{4} \inf_{\nu \in \mathcal{O}} \left\{\mathcal{H}(\nu||q\pi \otimes \pi)\right\}.
\]

3. **Proof of Theorem 2.2(i)**

By Gartner-Ellis theorem and Method of Mixtures

3.1 Proof of Theorem 2.2(i)

Suppose $A_1, ..., A_n$ is a decomposition of the space $D \times \mathbb{R}^+.$ Observe that, for every $(x, y) \in A_i \times A_j, i, j = 1, 2, 3, ..., n,$ $\lambda U^\lambda_1(x, y)$ given $\lambda U^\lambda_1(x) = \lambda \mu(x)$ is binomial with parameters $\lambda^2 \mu(x)\mu(y)/2$ and $Q^{z^\lambda}(x, y).$ Let $q$ be the exponential distribution with parameter $c.$ We recall the function $q_\pi^D$ from the previous sections and note that Lemma 2.4 is key component in the application of the Gartner-Ellis Theorem. See [1].

Lemma 3.1. Let $Z^\lambda$ is a super critical powered Sinr network with rate measure $\lambda \mu : D \to [0, 1]$ and a power probability function $K(\eta) = ce^{-\eta}, \eta > 0$ and path loss function $\beta(r) = r^{-\ell},$ for $\ell > 0.$ Thus, the connectivity probability $Q^{z^\lambda}$ of $Z^\lambda$ satisfies $a_\lambda^{-1}Q^{z^\lambda} \to q$ and $\lambda a_\lambda \to \infty.$ Let $Z^\lambda$ be a supercritical SINR network, conditional on the event $U^\lambda_1 = \pi.$ Let $g : W \times W \to \mathbb{R}$ be bounded function. Then,
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{E}\left\{e^{\lambda(g, U^\lambda_1)|U^\lambda_1 = \pi}\right\} = \frac{1}{2} \lim_{n \to \infty} \sum_{j=1}^{n} \sum_{i=1}^{n} \left\langle g, q\pi \otimes \pi \right\rangle_{A_i \times A_j} = \frac{1}{2} \left\langle g, q\pi \otimes \pi \right\rangle_{W \times W}.
\]
Proof. Now we observe that
\[
\mathbb{E}\left\{ e^{\int \lambda g(x,y) U_1^2(\,dx,dy)/2} \, \bigg| \, U_1^\lambda = \pi \right\} = \mathbb{E}\left\{ \prod_{x \in W} \prod_{y \in W} e^{\lambda g(x,y) U_2^2(\,dx,dy)/2} \right\}
\]
\[
\mathbb{E}\left\{ \prod_{x \in W} \prod_{y \in W} e^{\lambda g(x,y) U_2^2(\,dx,dy)/2} \right\} = \prod_{i=1}^n \prod_{j=1}^n \prod_{\lambda \in A_i} \prod_{\lambda \in A_j} \mathbb{E}\left\{ e^{\lambda g(x,y) U_2^2(\,dx,dy)/2} \right\}
\]
\[
\log \left\{ e^{\lambda(X_2)/2} \left| U_1^\lambda = \pi \right\} = \sum_{j=1}^n \sum_{i=1}^n \int_{B_j} \int_{B_i} \log \left[ 1 - Q^\lambda(x,y) + e^{\lambda g(x,y)} Q^\lambda(x,y) \right] + o(n)
\]
By the dominated convergence theorem
\[
\frac{1}{\lambda} \log \mathbb{E}\left\{ e^{\lambda(X_2)/2} \left| U_1^\lambda = \pi \right\} = \frac{1}{\lambda} \lim_{\lambda \to \infty} \sum_{j=1}^n \sum_{i=1}^n \int_{A_i} \int_{A_j} \log \left[ 1 + g(x,y) q(x,y)/\lambda + o(\lambda)/\lambda \right] + o(n)/\lambda
\]
\[
\frac{1}{\lambda} \log \mathbb{E}\left\{ e^{\lambda(X_2)/2} \left| U_1^\lambda = \pi \right\} = \frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n \left\langle g, q \otimes \pi \right\rangle_{A_i \times A_j}
\]
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{E}\left\{ e^{\lambda(X_2)/2} \left| U_1^\lambda = \pi \right\} = \frac{1}{2} \lim_{n \to \infty} \sum_{j=1}^n \sum_{i=1}^n \left\langle g, q \otimes \pi \right\rangle_{A_i \times A_j}
\]
\[
= \frac{1}{2} \left\langle g, q \otimes \pi \right\rangle_{W \times W}.
\]
Hence, by Gartner-Ellis theorem, conditional on the event \( \{ U_1^\lambda = \mu \} \), \( U_2^\lambda \) obey a large deviation principle with speed \( \lambda \) and variational formulation of the rate function
\[
I_\mu(\pi) = \frac{1}{2} \sup_g \left\{ \left\langle g, \pi \right\rangle_{W \times W} - \left\langle g, q \otimes \pi \right\rangle_{W \times W} \right\}
\]
which when solved, see example [2], would clearly reduces to the good rate function given by
\[
I_\pi^1(\nu) = 0.
\]

3.2 Proof of Theorem [2,2](ii)
Similarly we take \( A_1, ..., A_n \) a decomposition of the space \( D \times \mathbb{R}_+ \). We recall the function \( h^D_\lambda \) from the previous sections and state the following Lemma. Lemma [3,2] is key component in the application of the Gartner-Ellis Theorem. See, [1].

Lemma 3.2. Let \( Z^\lambda \) is a super critical powered Sinr network with rate measure \( \lambda \mu : D \to [0,1] \) and a power probability function \( K(\eta) = ce^{-\eta}, \eta > 0 \) and path loss function \( \beta(r) = r^{-\ell}, \ell > 0 \). Thus, the connectivity probability \( Q^\lambda \) of \( Z^\lambda \) satisfies \( a_{\lambda \lambda}^1 Q^\lambda \to q \) and \( \lambda a_{\lambda \lambda} \to \infty \). Let \( Z^\lambda \) be a supercritical SINR network, conditional on the event \( U_1^\lambda = \pi \). Let \( g : W \times W \to \mathbb{R} \) be bounded function. Then,
Hence, by Gartner-Ellis theorem, conditional on the event 

\[ \lim_{\lambda \to \infty} \frac{1}{\lambda^2 a_\lambda} \log E\{e^{\lambda^2 a_\lambda (g, U_2^\lambda)} \mid U_1^\lambda = \pi\} = -\frac{1}{2} \lim_{n \to \infty} \sum_{j=1}^{n} \sum_{i=1}^{n} \left\langle 1 - e^g, q_\pi \otimes \pi \right\rangle_{A_i \times A_j} \]

\[ = -\frac{1}{2} \left\langle 1 - e^g, q_\pi \otimes \pi \right\rangle_{W \times W}. \]

Proof. Now we observe that

\[ \mathbb{E}\left\{ e^{\int \lambda^2 a_\lambda g(x, y) U_2^\lambda(dx, dy)/2} \mid U_1^\lambda = \pi \right\} = \mathbb{E}\left\{ \prod_{x \in W} \prod_{y \in W} e^{\lambda^2 a_\lambda g(x, y) U_2^\lambda(dx, dy)/2} \right\} \]

\[ \mathbb{E}\left\{ \prod_{x \in W} \prod_{y \in W} e^{g(x, y) U_2^\lambda(dx, dy)/2} \right\} = \prod_{i=1}^{n} \prod_{j=1}^{n} \mathbb{E}\left\{ e^{\lambda^2 a_\lambda g(x, y) U_2^\lambda(dx, dy)/2} \right\} \times e^{o(n)} \]

\[ \log \left\{ e^{\lambda^2 a_\lambda (g, U_2^\lambda)/2} \right\} = \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{A_i} \int_{A_j} \log \left[ 1 - Q^\lambda(x, y) + Q^\lambda(x, y)e^{g(x, y)} \right]^{\lambda^2 \pi \otimes \pi(dx, dy)/2} + o(n) \]

By the dominated convergence theorem

\[ \frac{1}{\lambda^2 a_\lambda} \log E\{e^{\lambda^2 a_\lambda (g, U_2^\lambda)/2} \mid U_1^\lambda = \pi\} = \frac{1}{\lambda^2 a_\lambda} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{A_i} \int_{A_j} \log \left[ 1 - (1 - e^{g(x, y)}) Q^\lambda(x, y) \right]^{\lambda^2 \pi \otimes \pi(dx, dy)/2} + o(n)/\lambda^2 a_\lambda \]

\[ \frac{1}{\lambda^2 a_\lambda} \log \mathbb{E}\{e^{\lambda^2 a_\lambda (g, U_2^\lambda)/2} \mid U_1^\lambda = \pi\} = \lim_{\lambda \to \infty} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{A_i} \int_{A_j} \log \left[ 1 - (1 - e^{g(x, y)}) Q^\lambda(x, y) \right]^{\lambda^2 \pi \otimes \pi(dx, dy)/2} + o(n)/\lambda^2 a_\lambda \]

\[ \lim_{\lambda \to \infty} \frac{1}{\lambda^2 a_\lambda} \log \mathbb{E}\{e^{\lambda^2 a_\lambda (g, U_2^\lambda)/2} \mid U_1^\lambda = \pi\} = -\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{A_i} \int_{A_j} \left[ (1 - e^{g(x, y)}) q(x, y) \pi \otimes \pi(dx, dy) \right] \]

\[ \lim_{\lambda \to \infty} \frac{1}{\lambda^2 a_\lambda} \log \mathbb{E}\{e^{\lambda^2 a_\lambda (g, U_2^\lambda)/2} \mid U_1^\lambda = \pi\} = -\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \left\langle 1 - e^g, q_\pi \otimes \pi \right\rangle_{A_i \times A_j} \]

\[ = -\frac{1}{2} \left\langle 1 - e^g, q_\pi \otimes \pi \right\rangle_{W \times W} \]

Hence, by Gartner-Ellis theorem, conditional on the event \( \{M_1^\lambda = \mu\} \), \( U_2^\lambda \) obey a large deviation principle with speed \( \lambda \) and variational formulation of the rate function

\[ I_0(\pi) = \frac{1}{2} \sup_{g} \left\{ \left\langle g, \pi \right\rangle_{W \times W} + \left\langle 1 - e^g, q_\pi \otimes \pi \right\rangle_{W \times W} \right\} \]

which when solved, see example [2], would clearly reduces to the good rate function given by

\[ I_0^2(\nu) = \frac{1}{2} \mathcal{H}(\nu \| q_\pi \otimes \pi). \]
3.3 Proof of Theorem 2.1 by Method of Mixtures. For any \( \lambda \in (0, \infty) \) we define

\[
\mathcal{M}_\lambda(W) := \left\{ \mu \in \mathcal{M}(W) : \lambda \mu(x) \in \mathbb{N} \text{ for all } x \in W \right\},
\]

\[
\mathcal{M}_\lambda(W \times W) := \left\{ \pi \in \mathcal{M}(W \times W) : \lambda \pi(x, y) \in \mathbb{N}, \text{ for all } x, y \in W \right\}.
\]

We denote by \( \Theta_\lambda := \mathcal{M}_\lambda(W) \) and \( \Theta := \mathcal{M}(W) \). We write

\[
P^{(\lambda)}(\mu) := \mathbb{P}\left\{ U_1^\lambda = \pi \mid U_2^\lambda = \eta \right\},
\]

\[
P^{(\lambda)}(\mu) := \mathbb{P}\left\{ U_1^\lambda = \pi \right\}.
\]

The joint distribution of \( U_1^\lambda \) and \( U_2^\lambda \) is the mixture of \( P^{(\lambda)}(\mu) \) with \( P^{(\lambda)}(\mu) \), as follows:

\[
d \tilde{P}^{\lambda}(\mu, \eta, \pi) := dP^{(\lambda)}(\mu) dP^{(\lambda)}(\mu).
\]  

(Biggins, Theorem 5(b), 2004) gives criteria for the validity of large deviation principles for the mixtures and for the goodness of the rate function if individual large deviation principles are known. The following three lemmas ensure validity of these conditions.

Observe that the family of measures \( (P^{(\lambda)} : \lambda \in (0, \infty)) \) is exponentially tight on \( \Theta \).

**Lemma 3.3.**

(i) The family of measures \( (\tilde{P}^{\lambda} : \lambda \in (0, \infty)) \) is exponentially tight on \( \Theta \times \mathcal{M}_\lambda(W \times W) \).

(ii) The family measures \( (Q^{\lambda} : \lambda \in (0, \infty)) \) is exponentially tight on \( \Theta \times \mathcal{M}_\lambda(W \times W) \).

We refer to [11, Lemma 4.3] for similar proof for Large deviation Principle on the scale \( \lambda^2 \).

Define the function \( I_1^{Sc} : \Theta \times \mathcal{M}_\lambda(W \times W) \rightarrow [0, \infty] \), by

\[
I_1^{Sc}(\pi, \nu) = \begin{cases} H(\pi \mid \eta \otimes q) & \text{if } \nu = q\pi \otimes \pi \\ \infty & \text{otherwise}. \end{cases}
\]  

\[
I_2^{Sc}(\pi, \nu) = \frac{1}{2} H(\nu \| q\pi \otimes \pi).
\]

**Lemma 3.4.**

(i) \( I_1^{Sc} \) is lower semi-continuous.

(ii) \( I_2^{Sc} \) is lower semi-continuous.

By (Biggins, Theorem 5(b), 2004) the two previous lemmas, the LDP for the empirical power measure, see, [11, Theorem 2.1] and the large deviation principles we have established Theorem 2.2 ensure that under \( (\tilde{P}^{\lambda}) \) and \( Q^{\lambda} \) the random variables \( (\pi_\lambda, \eta_\lambda) \) satisfy a large deviation principle on \( \mathcal{M}(W) \times \mathcal{M}_\lambda(W \times W) \) and \( \Theta \times \mathcal{M}_\lambda(W \times W) \) on the speeds \( \lambda \) and \( \lambda^2 a_\lambda \) with good rate functions \( I_1^{Sc} \) and \( I_2^{Sc} \) respectively, which ends the proof of Theorem 2.1.

4. Proof of Theorem 2.3 by Large deviations

In order to establish the asymptotic equipartition property, we first prove a weak law of large numbers for the empirical power measure and the empirical connectivity measure of the SINR network.
Lemma 4.1. Let $Z^\lambda$ is a super critical powered Sinr network with rate measure $\lambda\mu : D \to [0, 1]$ and a power probability function $K(\eta) = ce^{-c\eta}, \eta > 0$ and path loss function $\beta(r) = r^{-\ell}$, for $\ell > 0$. Thus, the connectivity probability $Q^{\lambda^*}$ of $Z^\lambda$ satisfies $a^{\lambda_1}Q^{\lambda^*} \to q$ and $\lambda\alpha \to \infty$. Then,

$$\lim_{\lambda \to \infty} \mathbb{P}\left\{ \sup_{(x, \eta_x) \in \mathcal{W}} \left| L^\lambda_1(x, \eta_x) - \mu \otimes K(x, \eta_x) \right| > \varepsilon \right\} = 0$$

and

$$\lim_{\lambda \to \infty} \mathbb{P}\left\{ \sup_{(x, \eta_x) \in \mathcal{W}} \left| L^\lambda_2([x, \eta_x], [y, \eta_y]) - q\mu \otimes K \times \mu \otimes K([x, \mu_x], [y, \mu_y]) \right| > \varepsilon \right\} = 0$$

Proof. Let

$$F_{1,\mathcal{W}} = \left\{ \pi : \sup_{(x, \eta_x) \in \mathcal{W}} |\pi(x, \eta_x) - m \otimes K(x, \eta_x)| > \varepsilon \right\},$$

$$F_{2,\mathcal{W}} = \left\{ \nu : \sup_{(x, \eta_x) \in \mathcal{W}} |\nu([x, \eta_x], [y, \eta_y]) - q\mu \otimes K \times \mu \otimes K([x, \eta_x], [y, \eta_y])| > \varepsilon \right\}$$

and $F_{3,\mathcal{W}} = F_{1,\mathcal{W}} \cup F_{2,\mathcal{W}}$. Now, observe from Theorem 2.1 that

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{P}\left\{ (L^\lambda_1, L^\lambda_2) \in F_{3,\mathcal{W}}^c \right\} \leq - \inf_{(\pi, \varpi) \in F_{3,\mathcal{W}}^c} I(\pi, \varpi).$$

It suffices for us to show that $I$ is strictly positive. Suppose there is a sequence $(\pi_n, \varpi_n) \to (\pi, \varpi)$ such that $I(\pi_n, \varpi_n) \downarrow I(\pi, \varpi) = 0$. This implies $\pi = \mu \otimes K$ and $\varpi = q\mu \otimes K \times \mu \otimes K$ which contradicts $(\pi, \varpi) \in F_{3,\mathcal{W}}^c$. This ends the proof of the Lemma.

Now, the distribution of the marked PPP $P(x) = \mathbb{P}\left\{ X^\lambda = x \right\}$ is given by

$$P_\lambda(x) = \prod_{i=1}^I \left| \mu \otimes K(x_i, \eta_i) \right| \prod_{(i,j) \in E} \frac{Q^{\lambda^*}([x_i, \mu_i], [y_j, \mu_j])}{1 - Q^{\lambda^*}([x_i, \mu_i], [y_j, \mu_j])} \prod_{(i,j) \in E} (1 - Q^{\lambda^*}([x_i, \mu_i], [y_j, \mu_j])) \prod_{i=1}^I (1 - Q^{\lambda^*}([x_i, \mu_i], [y_j, \mu_j]))$$

$$- \frac{1}{a_\lambda \lambda^2 \log \lambda} \log P_\lambda(x) = \frac{1}{a_\lambda \lambda \log \lambda} \left\langle - \log \mu \otimes Q(L^\lambda_1), L^\lambda_2 \right\rangle + \frac{1}{a_\lambda \lambda \log \lambda} \left\langle - \log \left( \frac{Q^{\lambda^*}}{1 - Q^{\lambda^*}} \right), L^\lambda_2 \right\rangle$$

$$+ \frac{1}{a_\lambda \lambda \log \lambda} \left\langle - \log(1 - Q^{\lambda^*}), L^\lambda_1 \otimes L^\lambda_1 \right\rangle + \frac{1}{a_\lambda \lambda \log \lambda} \left\langle - \log(1 - Q^{\lambda^*}), L^\lambda_1 \right\rangle$$

Notice,

$$\lim_{\lambda \to \infty} \frac{1}{a_\lambda \lambda \log \lambda} \left\langle - \log \mu \otimes K, L^\lambda_1 \right\rangle = \lim_{\lambda \to \infty} \frac{1}{\lambda} \left\langle - \log(1 - Q^{\lambda^*}), L^\lambda_1 \right\rangle = \lim_{\lambda \to \infty} \frac{1}{a_\lambda \lambda \log \lambda} \left\langle - \log(1 - Q^{\lambda^*}), L^\lambda_1 \otimes L^\lambda_1 \right\rangle = 0.$$

Using, Lemma 4.1 we have

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \left\langle - \log \left( \frac{Q^{\lambda^*}}{1 - Q^{\lambda^*}} \right), L^\lambda_2 \right\rangle = \left\langle 1, q\mu \otimes K \times \mu \otimes K \right\rangle$$

which concludes the proof of Theorem 2.1.
5. Proof of Theorem 2.4, Corollary 2.5, Corollary 2.6

For \( \pi \in \mathcal{M}(W) \) we define the spectral potential of the marked SINR graph \((Z^\lambda)\) conditional on the event \( \{L_1^\lambda = \pi\} \), \( \rho_q(g, \pi) \) as

\[
\phi_q(g, \pi) = \left( - (1 - e^g) , q \pi \otimes \pi \right). \tag{5.1}
\]

Note that remarkable properties of a spectral potential, see or \[11\] holds for \( \phi_q \).

For \( \pi \in \mathcal{M}(W \times W) \), we observe that \( I_\pi(\pi) \) is the Kullback action of the marked SINR graph \( Z^\lambda \).

**Lemma 5.1.** The following hold for the Kullback action or divergence function \( I_\pi(\pi) \):

- \( I_{Sc}(\pi) = \sup_{g \in C} \{ \langle g, \pi \rangle - \phi_q(g, \pi) \} \)
- The function \( I_{Sc}(\pi) \) is convex and lower semi-continuous on the space \( \mathcal{M}(W \times W) \).
- For any real \( \alpha \), the set \( \{ \pi \in \mathcal{M}(W \times W) : I_{Sc}(\pi) \leq \alpha \} \) is weakly compact.

The proof of Lemma 5.1 is omitted from the article. Interested readers may refer to \[10\] for similar proof for empirical measures of the Typed Random Graph Processes or See, for example \[7\] for the multitype Galton-Watson processes and/or the reference therein, \[5\], for proof of the lemma for empirical measures on measurable spaces.

Note from Lemma 5.1 that, for any \( \varepsilon > 0 \), there exists some function \( g \in W \times W \) such that

\[
I_{Sc}(\pi) - \frac{\varepsilon}{2} < \langle g, \pi \rangle - \phi_q(g, \pi).
\]

We define the probability distribution of the powered \( Z \) by \( P_\pi \) by

\[
P_\pi(z) = \prod_{(i,j) \in E} e^{g(x_i,x_j)} \prod_{(i,j) \in E} e^{h_\lambda(x_i,x_j)},
\]

where

\[
h_\lambda(x, y) = \frac{1}{a_\lambda} \log \left[ 1 - Q_{\mathcal{L}^\lambda}(x, y) + Q_{\mathcal{L}^\lambda}(x, y) e^{g(x,y)} \right]
\]

Then, observe that

\[
\frac{dP_\pi}{dP_\pi}(z) = \prod_{(i,j) \in E} e^{-g(x_i,x_j)} \prod_{(i,j) \in E} e^{-h_\lambda(x_i,x_j) a_\lambda}
\]

\[
= e^{-\lambda^2 a_\lambda \left( \frac{1}{2} g(L_2^\lambda) - \lambda^2 a_\lambda \left( \frac{1}{2} h_\lambda(L_1^\lambda) \right) \right)}
\]

Now define the neighbourhood of \( \nu \), \( B_\nu \) by

\[
B_\nu := \{ \omega \in \mathcal{M}(W \times W) : \langle g, \omega \rangle - \rho_q(g, \pi) > \langle g, \nu \rangle - \rho_q(g, \pi) - \varepsilon / 2 \}
\]

Note that under the condition \( L_2^\lambda \in B_\nu \) we have

\[
\frac{dP_\pi}{dP_\pi}(z) < e^{-\lambda^2 a_\lambda \left( \frac{1}{2} g(L_2^\lambda) - \lambda^2 a_\lambda \left( \frac{1}{2} h_\lambda(L_1^\lambda) \right) \right)} < e^{-\lambda^2 a_\lambda I_{Sc}(\nu) + \lambda^2 a_\lambda \varepsilon}
\]
Therefore, we obtain

\[ P_\pi \left\{ Z^\lambda \in \mathcal{G}_P \mid L_2^\lambda \in B_\nu \right\} \leq \int \mathbb{I}_{\{L_2^\lambda \in B_\nu\}} d\tilde{P}_\pi (z^\lambda) (z) \leq \int e^{-\lambda^2 a_\lambda I_{S_\nu}(\nu) - \lambda \varepsilon} d\tilde{P}_\pi (z^\lambda) \leq e^{-\lambda^2 a_\lambda I_{S_\nu}(\nu) - \lambda^2 a_\lambda \varepsilon}. \]

Note that \( I_{S_\nu}(\nu) = \infty \) implies Theorem 2.3 (ii), hence it sufficient for us to deduce that the result is true for a probability distribution of the form \( \nu = e^g \pi \otimes \pi \) and for \( I_{S_\nu}(\nu) = \frac{1}{2} \mathcal{H}(\nu \parallel q \pi \otimes \pi) \). Fix any number \( \varepsilon > 0 \) and any neigbourhood \( B_\nu \subset \mathcal{M}(W \times \mathcal{W}) \). Now define the sequence of sets

\[ \mathcal{G}_\nu^\lambda = \left\{ y \in \mathcal{G}_\nu : L_2^\lambda (y) \in B_\nu \left| \langle g, L_2^\lambda \rangle - \phi_q (g, \pi) \right| \leq \frac{\varepsilon}{2} \right\}. \]

Note that for all \( y \in \mathcal{G}_\nu^\lambda \) we have

\[ \frac{dP_\pi}{d\tilde{P}_\pi} > e^{-\lambda^2 a_\lambda \left( \frac{1}{2} g, \nu \right) + \lambda^2 a_\lambda \phi_q (g, \pi) + \lambda^2 a_\lambda \frac{\varepsilon}{2}}. \]

This yields

\[ P_\pi (\mathcal{G}_\nu^\lambda) = \int_{\mathcal{G}_\nu^\lambda} dP_\pi (y) \geq \int e^{-\lambda^2 a_\lambda \left( \frac{1}{2} g, \nu \right) + \lambda^2 a_\lambda \phi_q (g, \pi) + \lambda^2 a_\lambda \frac{\varepsilon}{2}} d\tilde{P}_\pi (y) \geq e^{-\lambda^2 a_\lambda \frac{1}{2} \mathcal{H}(\nu \parallel q \pi \otimes \pi) + \lambda^2 a_\lambda \varepsilon} \tilde{P}_\pi (\mathcal{G}_\nu^\lambda). \]

Applying the law of large numbers, we have that \( \lim_{\lambda \to \infty} \tilde{P}_\pi (\mathcal{G}_\nu^\lambda) = 1 \). This completes of the Theorem.

**Proof of Corollary 2.5**

The proof of Corollary 2.5 follows from the definition of the Kullback action and Theorem 2.4 if we set \( \pi = \rho \) and \( \lambda \pi \otimes \pi (a, b) = \| \lambda \pi \otimes \pi \| \), for all \( (a, b) \in \mathcal{Y} \times \mathcal{Y} \).

**Proof of Corollary 2.6**

We observe that, by Lemma 3.3 the law of empirical connectivity measure is exponentially tight. Henceforth, without loss of generality we can assume that the set \( F \) in Corollary 2.6(ii) above is relatively compact. If we choose any \( \varepsilon > 0 \); then for each functional \( \nu \in F \) we can find a weak neighbourhood such that the estimate of Theorem 2.4(i) above holds. From all these neighbourhood, we choose a finite cover of \( \mathcal{G}_F \) and sum up over the estimate in Corollary 2.6(i) above to obtain

\[ \limsup_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{P}_\pi \left\{ Z^\lambda \in \mathcal{G}_F \mid L_2^\lambda \in F \right\} \leq - \inf_{\nu \in F} I_\pi (\nu) + \varepsilon. \]

As \( \varepsilon \) was arbitrarily chosen and the lower bound in Theorem 2.6(ii) implies the lower bound in Theorem 2.2(i) we get the desired results which completes the proof.

6. Conclusion

In this article we have presented a joint large deviation principle for the empirical power measure and the empirical connectivity measure of telecommunication networks in the \( \tau - \) topology. From this large deviation principle we deduce an asymptotic equipartition property for the telecommunication network modelled as the SINR network model.

We have also presented a Local large deviation principle for the empirical connectivity measure given the empirical power measure and from this result we had deduce the classical MacMillan theorem and an asymptotic bound for the set all posible SINR network process. Finally, we the authors had also
presented a large deviation principle for the SINR networks. This article might be regarded as a first step in the proof of a Lossy asymptotic equipartition property for the SINR networks. See, [6] and [7] for similar results for the networked data structures modelled as colored random graph process and for the hierarchical data structure modelled as Galton-Watson tree process.

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