CONTROL OF FIXED POINTS AND EXISTENCE AND UNIQUENESS OF CENTRIC LINKING SYSTEMS

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ABSTRACT. A. Chermak has recently proved that to each saturated fusion system over a finite $p$-group, there is a unique associated centric linking system. B. Oliver extended Chermak’s proof by showing that all the higher cohomological obstruction groups relevant to unique existence of centric linking systems vanish. Both proofs indirectly assume the classification of finite simple groups. We show how to remove this assumption, thereby giving a classification-free proof of the Martino-Priddy conjecture concerning the $p$-completed classifying spaces of finite groups. Our main tool is a 1971 result of the first author on control of fixed points by $p$-local subgroups. This result is directly applicable for odd primes, and we show how a slight variation of it allows applications for $p = 2$ in the presence of offenders.

1. INTRODUCTION

Given a saturated fusion system $\mathcal{F}$ over a finite $p$-group $S$, A. Chermak showed how to construct a centric linking system for $\mathcal{F}$ that is unique up to isomorphism [Che13]. His construction is made possible by a delicate filtration of the collection of $\mathcal{F}$-centric subgroups, which makes use of the Thompson subgroup in a critical way, together with an iterative procedure for extending a linking system on a given collection of subgroups of $S$ to a linking system on a larger collection. Within a given step of the procedure, one is working locally in the normalizer of a $p$-subgroup, and the problem of carrying out the inductive step is reduced to the problem of extending an automorphism of a linking system of a constrained group to an automorphism of the group. It is at this place where an appeal to the classification of finite simple groups is needed, in the form of the General FF-module Theorem of Meierfrankenfeld and Stellmacher [MS12].

In a companion paper [Oli13], B. Oliver showed how to interpret Chermak’s proof in terms of the established Broto-Levi-Oliver obstruction theory for the existence and uniqueness of centric linking systems [BLO03b §3]. The obstruction groups appearing there are the higher derived limits over the orbit category of the fusion system, at the level of $\mathcal{F}$-centric subgroups, of the (contravariant) center functor $Z_\mathcal{F} : \mathcal{O}(\mathcal{F}^c) \to \text{Ab}$ which sends an $\mathcal{F}$-centric subgroup to its center. From the point of view of this obstruction theory, Chermak’s

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filtration gives a way of filtering the center functor $Z_F$ so that the higher limits of each subquotient functor in the filtration can be shown to vanish. Within Oliver’s proof, the problem is again reduced to the case where $F$ is the fusion system of a constrained group, and to showing that $\lim^1$ (when $p$ is odd) and $\lim^2$ (when $p = 2$) of certain explicit subquotient functors of the center functor on the orbit category of this group vanish. For this, an appeal to the General FF-module Theorem gives a list of the possible groups, and then the proof is finished by examining these cases.

We study the constrained case in Oliver’s proof of Chermak’s Theorem and give proofs of Proposition 3.3 of [Oli13] that do not depend on the classification of finite simple groups. When taken together with the reduction via Chermak’s filtration to this situation in [Oli13], we obtain a classification-free proof of existence and uniqueness of centric linking systems at all primes.

**Theorem 1.1** (Oliver [Oli13]). Let $F$ be a saturated fusion system over a finite $p$-group. Then $\lim_{\Omega(F^c)}^k Z_F = 0$ for all $k \geq 1$ if $p$ is odd, and for all $k \geq 2$ if $p = 2$.

*Proof.* When $p$ is odd, this follows from the proof of [Oli13, Theorem 3.4] and Proposition 3.11 below. When $p = 2$, it follows from the proof of [Oli13, Theorem 3.4] and Proposition 6.9 below. $\Box$

It was known very early on that $\lim_{\Omega(F^c)}^1 Z_F$ can be nonvanishing when $p = 2$. An example of this is given by the 2-fusion system $F$ of the alternating group $A_6$, where $\lim_{\Omega(F^c)}^1 Z_F$ is of order 2 [Oli06, Proposition 1.6, Ch.10].

**Theorem 1.2** (Chermak [Che13]). Each saturated fusion system has an associated centric linking system that is unique up to isomorphism.

*Proof.* This follows from Theorem 1.1 together with [BLO03b, Proposition 3.1]. $\Box$

For a finite group $G$, the canonical centric linking system for $G$ at the prime $p$ is so-named because it provides a link between the fusion system of $G$ and the $p$-completion, in the sense of Bousfield and Kan, of its classifying space. More precisely, Broto, Levi, and Oliver showed that two finite groups have homotopy equivalent $p$-completed classifying spaces if and only if there is an equivalence of categories between their centric linking systems [BLO03a]. The question of whether or not the fusion system alone is enough to recover the centric linking system, and thus the $p$-completed classifying space of the group, is known as the Martino-Priddy conjecture. A special case of Chermak’s Theorem, the Martino-Priddy conjecture was first proven by Oliver [Oli04, Oli06] by showing that Theorem 1.1 holds for the fusion system of a finite group using the classification of finite simple groups. Thus, one consequence of the results in this paper is a classification-free proof of this conjecture.

In addition to relying heavily on the reductions of Chermak and Oliver, our arguments use variations on techniques of the first author for studying when, for a finite group $G$
acting on an abelian $p$-group $D$, some subgroup $H$ controls fixed points in $G$ on $D$ – that is, when $C_D(H) = C_D(G)$. In particular, very general conditions were given in [Gla71 Theorem A1.4] under which this holds for a suitable $p$-local subgroup $H$ of $G$. This general result is the basis for the statement, also found in [Gla71], that the normalizer of the Thompson subgroup “controls weak closure of elements” when $p$ is odd. We refer to §14 of [Gla71] for more details on this relationship.

In order to explain how control of fixed points is helpful in computing limits of constrained groups, we fix a finite group $\Gamma$ having Sylow $p$-subgroup $S$ and a normal $p$-subgroup $Y$ containing its centralizer in $\Gamma$. The filtrations of the center functor that feature in Chermak’s proof correspond to objectwise filtrations of the collection of $\mathcal{F}$-centric subgroups. A collection $Q \subseteq F_c$ that is invariant under $\mathcal{F}$-conjugacy and closed under passing to overgroups corresponds to a quotient $Z_Q^F$ of the center functor. For such a collection, one can form the 

|**Theorem 1.3.** Let $G$ be a finite group, $S$ a Sylow 2-subgroup of $G$, and $D$ an abelian 2-group on which $G$ acts faithfully. Assume that $G$ has a minimal nontrivial offender on |
that is not solitary. Then there is a subgroup \( J \leq S \), generated by offenders and weakly closed in \( S \) with respect to \( G \), such that \( C_D(N_G(J)) = C_D(G) \).

**Proof.** This follows from Theorem 4.1 via Lemmas 4.6 and 4.15. □

This could be viewed as the main result of §4, although the more detailed information contained in Lemmas 4.6 and 4.15 is needed for the remainder. When interpreted, Theorem 1.3 gives the vanishing of \( \varprojlim \mathbb{Z} \mathcal{F} \mathcal{R} \) for \( \mathcal{F} \) the fusion system of any constrained group \( \Gamma \) as above with \( D = Z(Y) \) and \( G = \Gamma/C_{\Gamma}(D) \) having the prescribed action on \( D \), and with \( \mathcal{R} \) those subgroups containing \( Y \) whose images in \( G \) do not contain offenders on \( D \).

The results of §4 are then applied to upgrade the vanishing of \( \varprojlim \mathcal{F} \) to that of \( \varprojlim \mathcal{G} \) away from the canonical obstruction, and this is carried out in Theorem 5.4. For this and for the remaining arguments we work with the bar resolution for these limits. As a result, the arguments involve questions about realizing an automorphism of a locality as an inner automorphism of a group, and thus begin to resemble those appearing in Chermak’s paper [Che13]. We hope that the preliminary lemmas of §5 will make more clear this connection for those who are more familiar with Chermak’s group theoretic approach.

With a little more work it is seen that in a minimal counterexample, \( G \) is generated by its solitary offenders on \( D \), and in particular by transvections on \( \Omega_1(D) \). In this paper, by a **natural module** for a symmetric group \( S_m (m \geq 3) \), we mean the lone nontrivial irreducible composition factor of the standard permutation module for \( S_m \) over the field with two elements. This is of dimension \( m - 1 \) when \( m \) is odd, and of dimension \( m - 2 \) when \( m \) is even.

The following is Theorem 6.2 below.

**Theorem 1.4.** Let \( G \) be a finite group and \( D \) an abelian 2-group on which \( G \) acts faithfully. Assume that \( G \) has no nontrivial normal 2-subgroups and that \( G \) is generated by its solitary offenders on \( D \). Then \( G \) is a direct product of symmetric groups of odd degree at least three, and \( D/C_D(G) \) is a direct sum of natural modules.

Thus, ultimately, the General FF-module Theorem is replaced by an appeal (in the proof of Theorem 1.4) to McLaughlin’s classification of irreducible subgroups of \( SL_n(2) \) generated by transvections [McL69].

**Notation.** Conjugation maps and morphisms in fusion systems will be written on the right and in the exponent, while cocycles and cohomology classes for functors will be written on the left. Groups of cochains are written multiplicatively. However, on some occasions we express that a group, or a cocycle or a cohomology class, is trivial by saying that it is equal to 0.

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2. Functors on orbit categories

In this section, we recall the terminology and constructions in homological algebra that are needed. Our choice of notation follows [Oli13], and we also recall here some of the preliminary lemmas on cohomology that we use from that paper. Since we will have very little explicit need for the theory of fusion or linking systems, we refer to [AKO11] for the definitions and basic properties.

Fix a saturated fusion system $\mathcal{F}$ over a finite $p$-group $S$. We say that subgroups $P$ and $Q$ of $S$ are $\mathcal{F}$-conjugate if they are isomorphic in $\mathcal{F}$. A subgroup $P$ of $S$ is fully $\mathcal{F}$-normalized if $|N_S(P)| \geq |N_S(Q)|$ for every subgroup $Q$ of $S$ that is $\mathcal{F}$-conjugate to $P$. The subgroup $P$ is $\mathcal{F}$-centric if $C_S(Q) \leq Q$ for every subgroup $Q$ of $S$ that is $\mathcal{F}$-conjugate to $P$. We write $\mathcal{F}^f$ and $\mathcal{F}^c$ for the collection of fully $\mathcal{F}$-normalized and the collection of $\mathcal{F}$-centric subgroups of $S$, respectively. We also use $\mathcal{F}^c$ to denote the full subcategory of $\mathcal{F}$ with objects the $\mathcal{F}$-centric subgroups. Since morphisms are written on the right, $\text{Hom}_{\mathcal{F}}(P,Q)$ has a right action by $\text{Inn}(Q)$ for each pair of subgroups $P,Q \leq S$. The orbit category $\mathcal{O}(\mathcal{F}^c)$ of $\mathcal{F}$-centrics has as objects the set $\mathcal{F}^c$ and as morphisms the orbits under this action:

$$\text{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P,Q) = \text{Hom}_{\mathcal{F}}(P,Q)/\text{Inn}(Q).$$

The class of a morphism $\varphi \in \text{Hom}_{\mathcal{F}}(P,Q)$ is denoted by $[\varphi]$.

The center functor $\mathcal{Z}_F: \mathcal{O}(\mathcal{F}^c) \to \text{Ab}$ defined by $\mathcal{Z}_F(P) = Z(P)$ on objects. For a morphism $\varphi: P \to Q$ in $\mathcal{F}$, $\mathcal{Z}_F([\varphi])$ is the map from $Z(Q) = C_S(Q) \leq C_S(P^\varphi) = Z(P^\varphi)$ to $Z(P)$ induced by $\varphi^{-1}$.

Useful filtrations by subquotient functors of this functor often correspond to filtrations of the subgroups of $S$. Denote by $\mathcal{S}(S)$ the set of subgroups of $S$, and by $\mathcal{S}(S)_{\geq Y}$ the subset of those that contain a fixed subgroup $Y \leq S$.

**Definition 2.1.** A collection $\mathcal{R} \subseteq \mathcal{S}(S)$ is an interval if $P \in \mathcal{R}$ whenever $P_1, P_2 \in \mathcal{R}$ and $P_1 \leq P \leq P_2$. An interval $\mathcal{R}$ is $\mathcal{F}$-invariant if $P^\varphi \in \mathcal{R}$ whenever $P \in \mathcal{R}$ and $\varphi \in \text{Hom}_{\mathcal{F}}(P,S)$.

Given an $\mathcal{F}$-invariant interval $\mathcal{R}$, define the functor $\mathcal{Z}_F^\mathcal{R}: \mathcal{O}(\mathcal{F}^c) \to \text{Ab}$ by $\mathcal{Z}_F^\mathcal{R}(P) = Z(P)$ whenever $P \in \mathcal{R}$, and by 1 otherwise. Then $\mathcal{Z}_F^\mathcal{R}$ is a subfunctor of $\mathcal{Z}_F$ when $\mathcal{R}$ is closed under passing to (centric) subgroups, and a quotient functor of $\mathcal{Z}_F$ when $\mathcal{R}$ is closed under passing to overgroups in $S$—that is, when $S \in \mathcal{R}$.

Following [Oli13], we write

$$L^k(\mathcal{F}; \mathcal{R}) := \lim_{\leftarrow}^{\mathcal{R}} \mathcal{Z}_F^\mathcal{R}$$

for the higher derived limits of these functors, and we think of them as cohomology groups of the category $\mathcal{O}(\mathcal{F}^c)$ with coefficients in the functor $\mathcal{Z}_F^\mathcal{R}$. They are cohomology groups of
a certain cochain complex $C^*(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}^\mathcal{F})$, in which $k$-cochains are maps from sequences of $k$ composable morphisms in the category. A 0-cochain $u$ is a map sending $P \in \mathcal{F}^c$ to an element $u(P) \in \mathcal{Z}_{\mathcal{F}}^\mathcal{F}(P)$, and a 1-cochain $t$ is a map sending a morphism $P \xrightarrow{[\varphi]} Q$ to an element $t([\varphi]) \in \mathcal{Z}_{\mathcal{F}}^\mathcal{F}(P)$. We will be working in §5 with cochains for $\mathcal{Z}_Q^\mathcal{F}$ in the case where $Q$ is closed under passing to overgroups. With our notational conventions, the coboundary maps on such 0- and 1-cohains in this special case are as follows:

\begin{equation}
(2.2)
\begin{align*}
du([\varphi]) &= \begin{cases} 
  u(Q)^{-1}u(P)^{-1} & \text{if } P \in Q, \\
  1 & \text{otherwise; and}
\end{cases}
\end{align*}
\end{equation}

\begin{equation}
(2.3)
\begin{align*}
dt([\varphi][\psi]) &= \begin{cases} 
  t([\psi])^{-1}t([\varphi\psi])^{-1}t([\varphi]) & \text{if } P \in Q, \\
  1 & \text{otherwise,}
\end{cases}
\end{align*}
\end{equation}

for any sequence $P \xrightarrow{\varphi} Q \xrightarrow{\psi} R$ of composable morphisms in $\mathcal{F}^c$. We refer to [AKO11, §III.5.1] for more details on the bar resolution for these functors.

**Definition 2.4.** A general setup for the prime $p$ is a triple $(\Gamma, S, Y)$ where $\Gamma$ is a finite group, $S$ is a Sylow $p$-subgroup of $\Gamma$, and $Y$ is a normal $p$-subgroup of $\Gamma$ such that $C_{\Gamma}(Y) \leq Y$. A reduced setup is a general setup such that $Y = C_S(Z(Y))$ and $O_p(\Gamma/C_{\Gamma}(Z(Y))) = 1$.

We next state the three preliminary lemmas from [Oli13] that are needed later.

**Lemma 2.5.** Let $\mathcal{F}$ be a saturated fusion system over a $p$-group $S$, and let $Q \subseteq \mathcal{F}^c$ be an $\mathcal{F}$-invariant interval such that $S \in Q$. Let $\mathcal{F}_Q$ be the full subcategory of $\mathcal{F}$ with object set $Q$.

(a) The inclusion $\mathcal{F}_Q \to \mathcal{F}^c$ induces an isomorphism of cochain complexes $C^*(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_Q) \xrightarrow{\cong} C^*(\mathcal{O}(\mathcal{F}_Q); \mathcal{Z}_Q^\mathcal{F}|_Q)$, and in particular an isomorphism $L^*(\mathcal{F}; Q) \xrightarrow{\cong} L^*(\mathcal{F}_Q; \mathcal{Z}_Q^\mathcal{F}|_Q)$.

(b) If $(\Gamma, S, Y)$ is a general setup, $\mathcal{F} = \mathcal{F}_S(\Gamma)$, and $Q = \mathcal{S}(S)_{\geq Y}$, then

$$L^k(\mathcal{F}; Q) = \begin{cases} 
  C_{Z(Y)}(\Gamma) & \text{if } k = 0; \\
  0 & \text{otherwise.}
\end{cases}$$

**Proof.** This is Lemma 1.6 of [Oli13], with the additional information in part (a) shown in its proof. \qed

Part (b) of the following lemma gave us the first concrete indication that questions regarding control of fixed points by $p$-local subgroups would be relevant to Theorem 1.1. It is the starting point for nearly all the arguments to follow.

**Lemma 2.6.** Let $\mathcal{F}$ be a saturated fusion system over a $p$-group $S$. Let $Q, R \subseteq \mathcal{F}^c$ be $\mathcal{F}$-invariant intervals such that

(i) $Q \cap R = \emptyset$,

(ii) $Q \cup R$ is an interval,

(iii) $Q \in Q$, $R \in R$ implies $Q \nsubseteq R$. 


Then $Z^R_F$ is a subfunctor of $Z^Q_{F\cup R}$, $Z^Q_{F\cup R}/Z^R_F \cong Z^Q_F$, and there is a long exact sequence
\[
0 \rightarrow L^0(F; R) \rightarrow L^0(F; Q \cup R) \rightarrow L^0(F; Q) \rightarrow \cdots
\]
\[
\rightarrow L^{k-1}(F; Q) \rightarrow L^k(F; R) \rightarrow L^k(F; Q \cup R) \rightarrow L^k(F; Q) \rightarrow \cdots.
\]
In particular, if $(\Gamma, S, Y)$ is a general setup, $D = Z(Y)$, $F = F_S(\Gamma)$ and $Q \cup R = \mathcal{I}(S)_{\geq Y}$, then
\begin{itemize}
  \item[(a)] $L^{k-1}(F; Q) \cong L^k(F; R)$ for each $k \geq 2$, and
  \item[(b)] there is a short exact sequence
\[
1 \rightarrow C_D(\Gamma) \rightarrow C_D(\Gamma^*) \rightarrow L^1(F; R) \rightarrow 1,
\]
where $\Gamma^*$ is the set of $g \in \Gamma$ such that there exists $Q \in Q$ with $Q^g \in q$.
\end{itemize}

Proof. This is nearly Lemma 1.7 of [Oli13]. Our part (a) is stated in the situation of a general setup, and it follows from the long exact sequence and Lemma 2.5(b). \hfill \Box

**Lemma 2.7.** Let $(\Gamma, S, Y)$ be a general setup, $\Gamma_0$ a normal subgroup of $\Gamma$ containing $Y$, and $S_0 = S \cap \Gamma_0$. Set $F = F_S(\Gamma)$ and $F_0 = F_{S_0}(\Gamma_0)$. Let $Q \subseteq \mathcal{I}(S)_Y$ be an $F$-invariant interval such that $S \subseteq Q$, and such that $\Gamma_0 \cap Q \in Q$ whenever $Q \in Q$. Set $Q_0 = \{Q \in Q \mid Q \leq \Gamma_0\}$. Then restriction induces an injection
\[
L^1(F; Q) \rightarrow L^1(F_0; Q_0).
\]

Proof. This is Lemma 1.13 of [Oli13]. \hfill \Box

## 3. The norm argument and the odd case

For a finite group $G$ with action on an abelian group $V$ (written multiplicatively), and a subgroup $H$ of $G$, the norm map $\mathfrak{N}_H^G : C_V(H) \rightarrow C_V(G)$ is defined by
\[
\mathfrak{N}_H^G(v) = \prod_{g \in [G/H]} v^g
\]
for each $v \in C_V(H)$, where $v^g$ denotes the image of $v$ under $g$, and where $[G/H]$ is a set of right coset representatives for $H$ in $G$. We say that $\mathfrak{N}_H^G = 1$ on $V$ if this map is constant, equal to the identity of $V$. Since $\mathfrak{N}_H^K = \mathfrak{N}_K^G \mathfrak{N}_H^K$ whenever $H \leq K \leq G$, one sees that $\mathfrak{N}_H^G = 1$ on $V$ whenever either of $\mathfrak{N}_K^K$ or $\mathfrak{N}_H^K$ is 1 on $V$.

In this section, we give some sufficient conditions for determining that the norm map $\mathfrak{N}_H^G$ is constant for suitable $p$-local subgroups $H$ of $G$, and then apply these results in Proposition 3.11 to obtain a proof of Theorem 1.1 for odd primes.

A subgroup $A$ of $G$ acts *quadratically* on $V$ if $[V, A, A] = 1$ but $[V, A] \neq 1$. In particular, when $V$ is elementary abelian, each element of such a subgroup has quadratic or linear minimum polynomial in its action on $V$.

**Lemma 3.1.** Suppose that $A$ is a $p$-group acting on an elementary abelian $p$-group $V$. 
(a) if $p$ is odd and $A$ acts quadratically on $V$, then $\mathfrak{N}_{A_0}^A = 1$ on $V$ for every proper subgroup $A_0$ of $A$.

(b) if $p = 2$, then $\mathfrak{N}_{A_0}^A = 1$ on $V$ for every subgroup $A_0$ of $A$ satisfying one of the following conditions:

(i) $|A : A_0| \geq 2$ and $C_V(A_0) = C_V(A)$, or

(ii) $|A : A_0| \geq 4$ and $A$ acts quadratically on $V$.

Proof. We view elements of $A$ as endomorphisms of $V$. Suppose first that $p$ is odd. Let $A_0 \leq A_1 \leq A$ with $A_1$ of index $p$ in $A$, and $a \in A - A_1$. Then $(1 - a)^2 = 0$ in $\text{End}(V)$ by assumption and $pa = 0$ in $\text{End}(V)$ since $V$ is elementary abelian. Hence

$$\mathfrak{N}_{A_1}^A(v) = v^{1+a+\cdots+a^{p-1}} = v^{(1-a)^p-1} = 1$$

for all $v \in C_V(A_1)$, since $p - 1 \geq 2$.

Let $p = 2$. Under assumption (i), choose coset representatives $\{1, a\}$ for a maximal subgroup $A_1$ of $A$ containing $A_0$, and then $\mathfrak{N}_{A_1}^A(v) = vr^a = v^2 = 1$ for all $v \in C_V(A_1) = C_V(A)$, since $V$ is elementary abelian. Under assumption (ii), fix a subgroup $A_1$ containing $A_0$ with index 4 in $A$ and choose coset representatives for $A_1$ in $A$ as $\{1, a, b, ab\}$ for suitable $a, b \in A - A_1$. Then $(1 + a + b + ab = (1 - a)(1 - b) = 0$ in $\text{End}(V)$ since the action is quadratic, and so $\mathfrak{N}_{A_1}^A(v) = v^{1+a+b+ab} = 1$ for all $v \in C_V(A_1)$ as required. \hfill \Box

**Theorem 3.2.** Suppose $G$ is a finite group, $S \in \text{Syl}_p(G)$, and $D$ is a $p$-group on which $G$ acts. Let $\mathfrak{A}$ be a nonempty set of subgroups of $S$, and set $J = \langle \mathfrak{A} \rangle$. Assume that $J$ is weakly closed in $S$ with respect to $G$ and that

$$(3.3) \quad \text{whenever } A \in \mathfrak{A}, g \in G, A \not\leq S^g, \text{ and } V \text{ is a composition factor of } D \text{ under } G, \text{ then } \mathfrak{N}_{A\cap S^g}^A = 1 \text{ on } V.$$

Then $C_D(N_G(J)) = C_D(G)$.

Proof. This is Theorem A1.4 of [Gla71]. \hfill \Box

We refer to Theorem 3.2 and also to Theorem 4.1 below, as the norm argument for short. It is usually applied with $p$ odd and in the presence of quadratic elements in $G$ on $D$ (cf. Lemma 3.1(a)).

**Definition 3.4.** For a general setup $(\Gamma, S, Y)$, set $D = Z(Y)$ and $G = \Gamma/C_T(D)$. Let $\mathfrak{A}$ be a set of abelian subgroups of $G$ that is invariant under $G$-conjugation. For any subgroup $H$ of $G$, set $\mathfrak{A} \cap H = \{A \in \mathfrak{A} \mid A \leq H\}$ and $J_\mathfrak{A}(H) = \langle \mathfrak{A} \cap H \rangle$. For a subgroup $H$ of $\Gamma$, we let $J_\mathfrak{A}(H, D)$ be the preimage in $H$ of $J_\mathfrak{A}(HC_T(D)/C_T(D))$. Often $J_\mathfrak{A}(H, D)$ will be abbreviated to $J_\mathfrak{A}(H)$ when $D$ is understood.

The collection $\mathfrak{A}$ of Definition 3.4 will generally be some subset of the collection of nontrivial best offenders in $G$ on $D$, as defined below.
**Definition 3.5.** Let $G$ be a finite group and let $D$ be an abelian $p$-group on which $G$ acts faithfully. An abelian $p$-subgroup $A \leq G$ is an offender on $D$ if
\[ |A||C_D(A)| \geq |D| \]
and a best offender if
\[ |A||C_D(A)| \geq |B||C_D(B)| \]
for every subgroup $B \leq A$. An offender $A$ is nontrivial if $A \neq 1$. We write $A_D(G)$ for the collection of nontrivial best offenders in $G$ on $D$.

A best offender is, in particular, an offender, as can be seen by taking $B = 1$ in the above definition. Conversely, each nontrivial offender $A$ contains a nontrivial best offender, which can be obtained as a subgroup $B$ such that the quantity $|B||C_D(B)|$ is maximal among all nontrivial subgroups of $A$. In turn, by the Timmesfeld Replacement Theorem [Tim82], each nontrivial best offender contains a nontrivial quadratic best offender, namely a best offender that acts quadratically on $D$. We include a short proof of this, using the Thompson Replacement Theorem, in the form which is needed here.

**Lemma 3.6.** Suppose $A$ is a nontrivial offender on $D$. Let $B$ be a nontrivial subgroup of $A$ that is minimal under inclusion subject to $|B||C_D(B)| \geq |A||C_D(A)|$. Then $B$ is a quadratic best offender on $D$.

**Proof.** It follows from the choice of $B$ that $B$ is a best offender, so we need only show that it acts quadratically on $D$. We work in the semidirect product $DB$, where we set $C = BC_D(B)$.

We first show that $C$ is an abelian subgroup of $DB$ of maximum possible order. Suppose that $C_1$ is an abelian subgroup of $DB$ such that $|C_1| \geq |C|$, and let $B_1$ be the image of $C_1$ under the projection of $DB$ onto $B$. Then $C_1 \cap D \leq C_D(C_1) = C_D(B_1)$, and so
\[ |C| \leq |C_1| = |C_1/(C_1 \cap D)||C_1 \cap D| = |B_1||C_1 \cap D| \leq |B_1||C_D(B_1)| \leq |B||C_D(B)| = |C| \]
with the last inequality since $B$ is a best offender on $D$. Therefore equality holds everywhere, which yields
\[ |B_1||C_D(B_1)| = |B||C_D(B)| \]
(3.7)

This shows that $C$ is an abelian subgroup of maximal order in $DB$.

Note that if $D$ normalizes $C$, then $[D,C] \leq C$, and so $[D,B,B] = [D,C,C] \leq [C,C] = 1$ since $C$ is abelian. Hence $B$ acts quadratically on $D$ in this case.

Suppose that $D$ does not normalize $C$. Then by Thompson’s Replacement Theorem [Gor80, Theorem 8.2.5], there exists an abelian subgroup $C_1$ of $DB$ such that $|C_1| = |C|$, $C_1 \cap D > C \cap D$, and $C_1$ normalizes $C$. Take $B_1$ as above. Then by (3.7),
\[ |B_1||C_D(B_1)| = |B||C_D(B)| \]
and
\[ |B_1| = |B||C_D(B)|/|C_D(B_1)| = |B||C \cap D||C_1 \cap D| < |B|. \]
By the minimal choice of $B$, we have that $B_1 = 1$ and therefore that $C_1 = C_D(B_1) = D$. This shows that $D$ normalizes $C$, a contradiction which completes the proof of the lemma. \hfill \Box

**Lemma 3.8.** Let $(\Gamma, S, Y)$ be a general setup for the prime $p$, set $D = Z(Y)$, and use bar notation for images modulo $C_T(D)$. If $Q$ is a subgroup of $S$ containing $C_S(D)$, then $N_{\Gamma}(Q) = N_{\Gamma}(Q)$.

**Proof.** Let $N$ be the preimage of $N_{\Gamma}(Q)$. Then $C_T(D)N_{\Gamma}(Q) \leq N$ and we must show that $N \leq C_T(D)N_{\Gamma}(Q)$. Now $QC_T(D)$ is normal in $N$, and $QC_T(D) \cap S = QC_S(D) = Q$ is a Sylow $p$-subgroup of $N$. By the Frattini argument,

$$N = (QC_T(D))N_{SN}(Q) \leq QC_T(D)N_{\Gamma}(Q) = C_T(D)N_{\Gamma}(Q)$$

as desired. \hfill \Box

The following proposition is a generalization of [Oli13, Proposition 3.2] for odd primes.

**Proposition 3.9.** Let $(\Gamma, S, Y)$ be a general setup for the prime $p$. Set $D = Z(Y)$, $G = \Gamma/C_T(D)$, and $\mathcal{F} = \mathcal{F}_S(\Gamma)$. Let $\mathcal{A}$ be a $G$-invariant collection of $p$-subgroups of $G$, each of which acts nontrivially and quadratically on $D$. Let $\mathcal{R} \subseteq \mathcal{F}(S)_{\geq Y}$ be an $\mathcal{F}$-invariant interval such that $Y \in \mathcal{R}$ and $J_A(S) \notin \mathcal{R}$. If $p$ is odd, then $L^1(\mathcal{F}; \mathcal{R}) = 0$.

**Proof.** Set $Q = \mathcal{F}(S)_{\geq Y} - \mathcal{R}$. Let $\Gamma^*$ be the subset of $\Gamma$ consisting of those $g \in \Gamma$ for which there is $Q \in Q$ with $Q^g \in Q$. Then $Q$ and $\mathcal{R}$ are $\mathcal{F}$-invariant intervals that satisfy the hypotheses of Lemma 2.6 so

\begin{equation}
(3.10) \quad \text{it suffices to show that } C_D(\Gamma) = C_D(\Gamma^*)
\end{equation}

by part (b) of that lemma.

As each element of $\mathcal{A}$ acts quadratically on $D$, Lemma 3.1(a) shows that (3.3) is satisfied. Hence, $C_D(\Gamma) = C_D(N_{\Gamma}(J_A(S)))$ by Theorem 3.2 and Lemma 3.8 (where the latter applies by Definition 3.4). Since $J_A(S) \in \mathcal{Q}$ by assumption, we have $N_{\Gamma}(J_A(S)) \leq \Gamma^*$. Hence

$$C_D(\Gamma) = C_D(N_{\Gamma}(J_A(S))) \geq C_D(\Gamma^*) \geq C_D(\Gamma)$$

and (3.10) complete the proof. \hfill \Box

When compared with [Oli13, Proposition 3.2], the increased generality of Proposition 3.9 allows some simplifications in obtaining [Oli13, Proposition 3.3] when $p$ is odd. We point out those simplifications now.

**Proposition 3.11.** Let $(\Gamma, S, Y)$ be a general setup for the prime $p$. Set $\mathcal{F} = \mathcal{F}_S(\Gamma)$, $D = Z(Y)$, and $G = \Gamma/C_T(D)$. Let $\mathcal{R} \subseteq \mathcal{F}(S)_{\geq Y}$ be an $\mathcal{F}$-invariant interval such that for all $Q \in \mathcal{F}(S)_{\geq Y}$, $Q \in \mathcal{R}$ if and only if $J_{A_{\mathcal{D}}(G)}(Q) \in \mathcal{R}$. If $p$ is odd, then $L^k(\mathcal{F}; \mathcal{R}) = 0$ for all $k \geq 1$. 
Proof. Let \((\Gamma, S, Y, \mathcal{R}, k)\) be a counterexample for which the four-tuple \((k, |\Gamma|, |\Gamma/Y|, |\mathcal{R}|)\) is minimal in the lexicographic ordering. Steps 1 and 2 in the proof of \cite[Proposition 3.3]{Oli13} show that \(\mathcal{R} = \{P \leq S \mid J_{\mathcal{R}}(G)(P) = Y\}\) and \(k = 1\) (since \(p\) is odd).

Let \(\mathcal{A}\) be the set of nontrivial best offenders in \(G\) on \(D\) that are minimal under inclusion. Each best offender contains a member of \(\mathcal{A}\) as a subgroup, and so \(J_{\mathcal{A}}(P) \in \mathcal{R}\) if and only if \(J_{\mathcal{A}}(S) \notin \mathcal{R}\). By Lemma \ref{lem:quad}, each member of \(\mathcal{A}\) acts quadratically on \(D\).

If \(S \in \mathcal{R}\), then \(\mathcal{R} = \mathcal{R}(S)\) since \(\mathcal{R}\) is an interval, and \(L^k(\mathcal{F}; \mathcal{R}) = 0\) for all \(k \geq 1\) by Lemma \ref{lem:quad}(b). Hence \(S \notin \mathcal{R}\) and so \(J_{\mathcal{A}}(S) \notin \mathcal{R}\). Now Proposition \ref{prop:3.9} shows that \((\Gamma, S, Y, \mathcal{R}, 1)\) is not a counterexample. \(\square\)

4. Norm arguments for \(p = 2\)

In this section, we define the notion of a solitary offender and prove the lemmas necessary to obtain Theorem \ref{thm:1.3}. These results are used in \cite{Fou13} to give a proof of \cite[Proposition 3.2]{Oli13} except in the case where every minimal offender under inclusion is solitary.

The following version of Theorem \ref{thm:3.2} is easier to apply in applications for \(p = 2\). The proof is very similar to that of Theorem \ref{thm:3.2} and is given in Appendix \ref{sec:app1}.

**Theorem 4.1.** Suppose \(G\) is a finite group, \(S\) is a Sylow 2-subgroup of \(G\), and \(D\) is an abelian \(2\)-group on which \(G\) acts. Let \(\mathcal{A}\) be a nonempty set of subgroups of \(S\), and set \(J = \langle \mathcal{A} \rangle\). Let \(H\) be a subgroup of \(G\) containing \(N_G(J)\). Set \(V = \Omega_1(D)\). Assume that \(J\) is weakly closed in \(S\) with respect to \(G\), and whenever \(A \in \mathcal{A}\), \(g \in G\), and \(A \not\trianglelefteq H^g\), then \(\mathfrak{N}_{A \cap H^g}^1 = 1\) on \(V\),

\begin{equation}
\text{or more generally,}
\end{equation}

\begin{equation}
\text{whenever } g \in G \text{ and } J \not\trianglelefteq H^g, \text{ then } \mathfrak{N}_{J \cap H^g}^1 = 1 \text{ on } V.
\end{equation}

Then \(C_D(H) = C_D(G)\).

Throughout the remainder of this section, we fix a finite group \(G\), a Sylow 2-subgroup \(S\) of \(G\), an abelian 2-group \(D\) on which \(G\) acts faithfully, and we set \(V = \Omega_1(D)\).

**Definition 4.4.** Set

\[ \hat{\mathcal{A}}_D(G) = \{ A \in \mathcal{A}_D(G) \mid |A||C_D(A)| > |D| \}, \]

the members of which are sometimes called over-offenders.

Denote by \(\mathcal{A}_D(G)\) the set of those members of \(\mathcal{A}_D(G)\) that are minimal under inclusion, and denote by \(\hat{\mathcal{A}}_D(G)\) those members of \(\hat{\mathcal{A}}_D(G)\) that are minimal under inclusion.

For a positive integer \(k\), write

\[ \mathcal{A}_D(G)_2 = \{ A \in \mathcal{A}_D(G) \mid |A| = 2 \}; \]

\[ \mathcal{A}_D(G)_4 = \{ A \in \mathcal{A}_D(G) \mid |A| \geq 4 \}. \]
It may help to reiterate that a member of $\hat{A}_D(G)^\circ$, while minimal under inclusion in the collection $\hat{A}_D(G)$, may not be minimal under inclusion in $A_D(G)$. By Lemma 3.6 each member of $A_D(G)^\circ$ acts quadratically on $D$.

**Remark 4.5.** Assume $G$ is faithful on $D$ and $p = 2$. If $A \in A_D(G)_2$, then $|D/C_D(A)| = 2$. In particular, every member of $\hat{A}_D(G)$ is of order at least 4.

**Lemma 4.6.** Let $A = A_D(G)_{\geq 4} \cup \hat{A}_D(G)^\circ$. Assume that $A$ is not empty and $H$ is a subgroup of $G$ containing $N_G(J_A(S))$. Then $A \cap S$ satisfies (1.2).

**Proof.** Fix $A \in A \cap S$ and $g \in G$ with $A \not\leq H^g$, and let $A_0$ be a subgroup of $A$ of index 2 that contains $A \cap H^g$. Suppose $\mathfrak{A}^A_{A \cap H^g}$ is not 1 on $V$. Since $\mathfrak{A}^A_{A \cap H^g} = \mathfrak{A}^A_{A_0} \mathfrak{A}^A_{A \cap H^g}$, we have $C_V(A) < C_V(A_0)$ by Lemma 3.1. It follows that (4.7)

$$|A_0||C_D(A_0)| \geq \frac{1}{2}|A| \cdot 2|C_D(A)| = |A||C_D(A)|.$$ 

If $A \in A_D(G)^\circ_{\geq 4}$, then $A$ is a best offender minimal under inclusion subject to $A \neq 1$, so we have $A_0 = 1$ and $|A| = 2$. This contradicts $|A| \geq 4$. Hence $A \in A_D(G)^\circ$ and in particular $|A| \geq 4$. But then $A_0 \in \hat{A}_D(G)$ by (4.7), contradicting the minimality of $A$. \hfill \Box

**Lemma 4.8.** Let $A = A_D(G)_2$. Assume that $\hat{A}_D(G) = \emptyset$ and that $A, B \in A$. Then

(a) if $[A, B] = 1$ and $A \neq B$, then $C_D(A) \neq C_D(B)$ and $AB$ is quadratic on $D$;

(b) if $[A, B]$ is a 2-group, then $[A, B] = 1$;

(c) $J_A(S)$ is elementary abelian; and

(d) if $[A, B] \neq 1$, then $L := [A, B] \cong S_3$, $[D, L]$ is elementary abelian of order 4, and $D = [D, L] \times C_D(L)$.

**Proof.** Since $A$ and $B$ lie in $A_D(G)_2$, (4.9)

$$[D, A, A] = 1 = [D, B, B] \text{ and } |D/C_D(A)| = 2.$$

Suppose that $[A, B] = 1$ and $A \neq B$, but that $C_D(A) = C_D(B)$. Then $AB$ is of order 4 and so $C_D(C) = C_D(A)$ for every nontrivial subgroup $C$ of $AB$. Since

$$|AB||C_D(AB)| = |A||B||C_D(A)| = 2|D| > |D|,$$

we have $AB \in \hat{A}_D(G)$ contrary to assumption. Thus, the first statement of (a) holds. Since $A \in A$, $[D, A]$ is $B$-invariant and of order 2. Thus $[D, A, B] = 1$. By symmetry $[D, B, A] = 1$. By a commutator identity and (1.9), $[D, AB, AB] = 1$, which establishes the second conclusion of (a).

Suppose that $X := [A, B]$ is a 2-group and $A \neq B$. Then $|D/C_D(X)| = 4$, and $X$ acts on $D/C_D(X)$. Since $X$ is a 2-group, there is an $X$-invariant $D_1$ with $C_D(X) < D_1 < D$.

Set $C = [A, B]$ and suppose that $C \neq 1$. Provided it is shown that $[D, A, B] = 1 = [D, B, A]$, then the Three Subgroups Lemma gives $[C, D] = [A, B, D] = 1$ and consequently the contradiction that $C = 1$ by faithfulness of the action. Thus, we may assume that (4.10)

$$[D, A, B][D, B, A] \neq 1.$$
Since $D_1$ is $X$-invariant and $D/D_1$ is of order 2, we have $[D, A] \leq D_1$ and $[D, B] \leq D_1$. Also, $C_D(X) \leq C_D(A)$. If $C_D(A) \neq D_1$, then $[D, A] \leq C_D(A) \cap D_1 \leq C_D(X)$, and hence $[D, A, B] = 1$. Similarly, if $C_D(B) \neq D_1$, then $[D, B, A] = 1$. Without loss of generality we may assume $C_D(A) = D_1$ by (4.10). Then $[C_D(A), A, B] = 1$, and $[C_D(A), B, A] = 1$ as $C_D(A)/C_D(X)$ is of order 2 and $X$-invariant. By the Three Subgroups Lemma, $[C_D(A), C] = 1$ and so $C_D(A) \leq C_D(C)$. But then

(4.11) \[ C_D(A) = C_D(C) \]

because $C \neq 1$ and the action is faithful. Hence $C \in \mathcal{A}$ since $\hat{A}_D(G) = \emptyset$. In particular $|C| = 2$ and $[A, C] = 1$. Now (4.11) contradicts part (a) since $A \neq C$, so (b) is established. Part (c) then follows immediately from (b).

To prove (d), let $L = \langle A, B \rangle$. Then $L$ is a dihedral group, and $L$ is not a 2-group by (b). Let $K$ be the largest odd order subgroup of the cyclic maximal subgroup of $L$. Then

$$|D/C_D(K)| \leq |D/C_D(L)| \leq |D/C_D(A)||D/C_D(B)| = 4.$$

Since $K$ has odd order and $D$ is a 2-group, $K$ acts faithfully on $D/C_D(K)$ and $D = [D, K] \times C_D(K)$. So $D/C_D(K)$ is elementary abelian of order 4, $|K| = 3$, and $C_D(K) = C_D(L)$. Hence $L$ acts faithfully on $[D, K]$. As $\text{Aut}([D, K]) \cong S_3$, we see that $L \cong S_3$.

**Definition 4.12.** Let $\mathcal{A} = \mathcal{A}_D(G)_2$, $J = J_\mathcal{A}(S)$, and $T \in \mathcal{A} \cap S$. We say that $T$ is solitary in $G$ relative to $S$ if there is a subgroup $L$ of $G$ containing $T$ such that

(S1) $L \cong S_3$;
(S2) $J = T \times C_J(L)$ and $C_J(L) = \langle (\mathcal{A} \cap S) - \{T\} \rangle$; and
(S3) $D = [D, L] \times C_D(L)$ and $[D, L, C_J(L)] = 1$.

For a subgroup $S_0$ of $S$, we say that $T \leq S_0$ is semisolitary relative to $S_0$, if there are subgroups $W$ and $X$ of $D$ that are normalized by $\langle \mathcal{A} \cap S_0 \rangle$, such that

(SS2) $\langle \mathcal{A} \cap S_0 \rangle = T \times \langle (\mathcal{A} \cap S_0) - \{T\} \rangle$; and
(SS3) $W$ is elementary abelian of order 4, $D = W \times X$, $T$ centralizes $X$, and $\langle \mathcal{A} \cap S_0 - \{T\} \rangle$ centralizes $W$.

Denote by $\mathcal{T}_D(G)$ the collection of subgroups of $G$ that are solitary relative to some Sylow 2-subgroup of $G$. Likewise, a member of $\mathcal{A}_D(G)_2$ is said to be semisolitary if it is semisolitary relative to some Sylow 2-subgroup of $G$.

**Remark 4.13.** If $\hat{A}_D(G) = \emptyset$, then given a subgroup $T$ which is solitary in $G$ relative to $S$, and given $L$ as in Definition 4.12 we may take $W = [D, L]$ and $X = C_D(L)$ and see from Lemma 1.8(d) that $T$ is semisolitary relative to $S$.

A solitary offender is of order 2 (by definition), and thus is generated by a transvection when $D$ is elementary abelian. If $G = S_3$ and $D = C_2 \times C_2$, then one may take $L = G$ to see that each subgroup of order 2 in $G$ is solitary. More generally, if $G$ a symmetric group of odd degree and $D$ is a natural module for $G$, then each subgroup generated by a transposition
is solitary. Indeed, given a Sylow 2-subgroup $S$ of $G$ containing the transposition, $L$ may be taken in this case to move only three points, namely the two points moved by the transposition and the unique point fixed by $S$. On the other hand, (S2) implies that $SL_n(2) \ (n \geq 3)$ and even degree symmetric groups, for example, have no solitary offenders on their respective natural modules, despite being generated by transvections. We refer Lemma 6.1 for more details.

There is no “(SS1)” in Definition 4.12 because we view (SS2) and (SS3) as weakenings of (S2) and (S3). The reader should view the introduction of semisolitary offenders as auxiliary. They are used in relative situations when the connection between solitary offenders in $G$ and those in a subgroup $H$ is difficult to ascertain. The following elementary lemma addresses a similar uncertainty.

**Lemma 4.14.** Assume that $\hat{A}_D(G) = \emptyset$. Then $T_D(G) \cap S$ is the set of subgroups that are solitary in $G$ relative to $S$.

**Proof.** One containment follows from the definition. For the other containment, set $A = A_D(G)_2$ and assume $T \in T_D(G) \cap S$. Suppose $T$ is solitary relative to the Sylow 2-subgroup $S_1$ of $G$, and let $g \in G$ with $S_1^g = S$. Set $J_1 = \langle A \cap S_1 \rangle$ and $J = \langle A \cap S \rangle$. Then $J_1$ is elementary abelian by Lemma 4.8 and $T^g \leq J_1 = J$. Fix a subgroup $L_1 \cong S_3$ containing $T$ so that (S1)-(S3) holds with $S_1$ and $J_1$ in the roles of $S$ and $J$, respectively. Since $J$ is abelian and weakly closed in $S$ with respect to $G$, there is $h \in N_G(J)$ with $T^gh = T$ (by Lemma B.1). Setting $L = L_1^gh$, one establishes the validity of (S1)-(S3) in Definition 4.12 for $L$, $S$, and $J$ from their validity for $L_1$, $S_1$, and $J_1$. □

The following lemma will be applied later with $J^*$ equal to the subgroup generated by members of $A_D(G)_2 \cap S$ that are not semisolitary relative to $S$.

**Lemma 4.15.** Set $A = A_D(G)_2$, $T = T_D(G)$, and $B = A - T$. Assume

(a) $\hat{A}_D(G) = \emptyset$; and
(b) $B \neq \emptyset$.

Then for each subgroup $J^* \leq J_B(S)$ that is weakly closed in $S$ with respect to $G$, $A \cap S$ satisfies (4.3) with $H = N_G(J^*)$.

**Proof.** Set $J = J_A(S)$ for short, and let $J^*$ be a subgroup of $J_B(S)$ that is weakly closed in $S$ with respect to $G$. Then $J^* \leq J_B(S) \leq J$ since $B \subseteq A$, and all of these are weakly closed in $S$ with respect to $G$, since $A$ and $B$ are $G$-invariant. From (a) and Lemma 4.8(c)

\begin{equation}
J \text{ is elementary abelian.}
\end{equation}

Since $J^*$ is weakly closed,

\begin{equation}
S \leq N_G(J) \leq N_G(J^*) = H.
\end{equation}

Assume (4.3) is false. That is, let $g \in G$ with $J \not\subseteq H^g$, set $I = J \cap H^g$, and suppose $\mathfrak{N}_I \neq 1$ on $V$. 

Suppose first that $|J : I| \geq 4$. Let $J_0$ be a subgroup of $J$ with $I \leq J_0$ and $|J : J_0| = 4$. By (4.16), and since $J$ is generated by members of $A_D(G)_2$, we may write $J = J_0 \times A \times B$ with $A, B \in A$. Then $\mathfrak{H}_I^J = \mathfrak{H}_h^J \mathfrak{H}_h^J$ and so $\mathfrak{H}_J^J \neq 1$ on $V$. However, $AB$ acts quadratically on $C_\nu(J_0)/C_\nu(J)$ by Lemma 4.3(a). Thus, $\mathfrak{H}_h^J = 1$ on $V$ by Lemma 4.3(b)(ii), a contradiction. We conclude that

$$|J : I| = 2.$$  

Fix $T \in A$ with $J = IT = I \times T$. Let $A \in A \cap S$ and suppose that $A \neq T$. Let $B = AT \cap I$. Then $B$ is of order 2 and $|D/C_D(B)| \leq |D/C_D(A)||D/C_D(T)| = 4$. If $|D/C_D(B)| = 4$, then

$$C_D(I) \leq C_D(B) = C_D(A) \cap C_D(T) \leq C_D(T).$$

Consequently, $J = IT$ centralizes $C_D(I)$, and $C_D(I) = C_D(J)$. Hence, also $C_\nu(I) = C_\nu(J)$. So $\mathfrak{H}_h^J = 1$ on $V$ in this case by Lemma 3.1(b)(i), a contradiction. Thus, $|D/C_D(B)| = 2$. That is, $B \in A$. This shows

$$\text{if } A \in A \cap S, \text{ then } A = T \text{ or there exists } B \in A \cap I \text{ such that } AT = BT.$$  

In particular, $J = \langle A \cap S \rangle = \langle A \cap I \rangle T$, and so

$$I = \langle A \cap I \rangle.$$

Recall that $I = J \cap H^g$, so that $I^{g^{-1}} \subseteq H$. Let $h \in H$ such that $I^{g^{-1}h} \subseteq S$. Then we have

$$I^{g^{-1}h} = \langle A \cap I \rangle^{g^{-1}h} \leq \langle A \cap S \rangle = J$$

by (4.20). As $H^{h^{-1}g} = H^g$, we may replace $g$ by $h^{-1}g$ for convenience so that

$$I = J \cap H^g \leq J \cap J^g.$$  

We now show that $T$ is solitary in $G$ relative to $S$. Since $J^g \neq J$, we may choose $U \in (A \cap S)^g$ such that $U \neq J$. Then $U \leq J^g$. Let $L = \langle T, U \rangle$. Since $T \leq J$ and $J$ is abelian,

$$[I, L] = 1$$

by (4.21). We first show that $T$ and $U$ do not commute. Suppose on the contrary that $[T, U] = 1$. Then $IL$ is a 2-group generated by members of $A$, and hence is conjugate to a subgroup of $J$. Then since $J = IT \leq IL$, we see that $J = IL$. Thus $U \leq L \leq J$, contrary to the choice of $U$.

Thus,

$$L \cong S_3, \quad |D, L| = 4, \quad \text{and } D = [D, L] \times C_D(L)$$

by (a) and Lemma 4.8(d). Further, $C_J(L) = I$ by the structure of $L$, and so $J = T \times C_J(L)$ by (4.22).
Since $[D,T]$ is of order 2 and $I$-invariant, $[D,T,I] = 1$. Since $I^x = I$ and $[D,T^x] = [D,T]^x \leq C_D(I^x) = C_D(I)$ for every $x \in L$,

\[(4.24) \quad [D,L] \leq C_D(I) = C_D(C_J(L)).\]

We have shown that (S1), (S3), and half of (S2) hold in Definition 4.12; it remains to prove (4.24)

\[(4.20) \quad [T] \leq C_D(I) = C_D(C_J(L)).\]

Thus, $T$ completes the proof of (S2). Consequently $J$ is weakly closed in $G$ and $J$ is of order 2 and $(4.20)$ completes the proof of (S2). Thus, $T \in T$.

Recall that $J^* \leq J_B(S)$ where $B$ consists of the members of $A$ that are not solitary in $G$ relative to $S$. We have just shown that $J_B(S) \leq I \leq J \cap J^g$, so that $(J^*)^{g^{-1}} \leq J_B(S)^{g^{-1}} \leq J$.

Since $J^*$ is weakly closed in $S$ with respect to $G$, it follows that $g \in N_G(J^*) = H$ and consequently that $J \leq H = H^g$. This contradicts the choice of $g$ and completes the proof of the lemma.

\[\Box\]

\textbf{Lemma 4.25.} Let $T$ be a subset of $A_D(G)_2 \cap S$ and $\mathcal{Y} = \{[D,T] \mid T \in T\}$. Fix $A \in A_D(G)$ and set $B = C_A(\langle \mathcal{Y} \rangle)$. Assume that every member of $T$ is semisolitary relative to $S$. Then

(a) $\langle \mathcal{Y} \rangle$ is the direct product of $[D,T]$ for $T \in T$; and

(b) if $A$ acts transitively on $T$ by conjugation, then either $A = B$, or $|\mathcal{Y}| = 2$, $B$ has index 2 in $A$, $C_D(A)$ has index 2 in $C_D(B)$, and each element of $A - B$ induces a transposition on $T$.

\textbf{Proof.} Whenever $T \in T$, set $Z_T = [D,T]$ for short. To prove (a),

\[(4.26) \quad Z_T \cap \prod_{R \neq T} Z_R = 1 \text{ for each } T \in T.\]

For each $R$ in $T$, let $W_R$ and $X_R$ be as in Definition 4.12 in the roles of $W$ and $X$. For each $R \in T$,

$Z_R = [D,R] = [W_RX_R,R] = [W_R,R] \leq W_R$,

so for each $T \in T$ different from $R$,

\[(4.27) \quad Z_T = [D,R] = [W_TX_T,R] = [X_T,R] \leq X_T,\]

since $R$ centralizes $W_T$ and normalizes $X_T$ by Definition 4.12. Therefore,

$Z_T \cap \prod_{R \neq T} Z_R \leq W_T \cap X_T = 1,$

by (SS3) and part (a) now follows from (4.26).
Assume $A$ acts transitively on $T$ by conjugation. Then $A$ acts on $\mathcal{Y}$ in the same way. Now $\langle \mathcal{Y} \rangle$ is a transitive permutation module for $A$ by part (a), so $|C_{\mathcal{Y}}(A)| = 2$. Suppose $A > B$ and set $m = |A/B|$. Then $|\mathcal{Y}| = m > 1$ and $|A||C_D(A)| \geq |B||C_D(B)|$, so that

\begin{equation}
(4.28) \quad m = |A/B| \geq |C_D(B)/C_D(A)| \geq |C_{\mathcal{Y}}(B)/C_{\mathcal{Y}}(A)| = 2^m/2.
\end{equation}

Hence $m = 2$ and equality holds in (4.28), so $|C_D(B)/C_D(A)| = 2$. Since $A \in A_D(G)$, an element of $A - B$ must act as a transposition on $\mathcal{Y}$ and on $T$. \hfill $\Box$

**Lemma 4.29.** Let $P \subseteq S$ and let $\mathcal{T} \subseteq A_D(G)_2 \cap P$ be the collection of semisolitary subgroups relative to $P$. If $A \in A_D(G)$ normalizes $P$, then $A$ normalizes every element of $\mathcal{T}$.

**Proof.** Let $\mathcal{Y} = \{[D, T] \mid T \in \mathcal{T}\}$, let $B = C_A(\langle \mathcal{Y} \rangle)$, and set $Z_T = [D, T]$ whenever $T \in \mathcal{T}$, for short. By assumption $A$ acts on $\mathcal{T}$. By Lemma 4.25(b), each orbit of $A$ on $\mathcal{T}$ has size at most 2. We assume that $\{T, R\}$ is a nontrivial orbit and aim for a contradiction. Since $T$ is semisolitary relative to $P$, we may choose subgroups $W$ and $X$ of $D$ such that $D = W \times X$, $|W| = 4$, $T$ centralizes $X$, $[W, T] = Z_T = C_W(T)$, and $R$ centralizes $W$.

We claim that $C_W(A) = 1$; assume otherwise. Since $A$ transposes $Z_T$ and $Z_R$ and $T$ does not centralize $W$, it follows that that $T$ does not centralize $C_W(A)$. Therefore, $[C_W(A), T] = Z_T$ and so for $a \in A - N_A(T)$,

$$Z_T = ([Z_T]a = [C_W(A), T^a] = [C_W(A), R] \leq [W, R] = 1,$$

a contradiction. Thus $C_W(A) = 1$. Since $|W| = 4$, we have

$$|C_D(B)/C_D(A)| \geq 4$$

counter to Lemma 4.25. \hfill $\Box$

5. Reduction to the transvection case

For this section, we fix a finite group $\Gamma$ with Sylow $p$-subgroup $S$, we set $\mathcal{F} = \mathcal{F}_S(\Gamma)$, and we let $Q \subseteq F^c$ be an $\mathcal{F}$-invariant interval such that $S \in Q$. In this situation, define $\Gamma^*$ to be the set of elements of $\Gamma$ that conjugate some member of $Q$ into $Q$.

The objective of this section is to give a proof of [Oli13 Proposition 3.2] in the case where some minimal offender under inclusion is not solitary. This result is obtained in Theorem 5.4.

We say that a 1-cocycle for the functor $Z^Q_\mathcal{F}$ is inclusion-normalized if it sends the class $[\iota^Q_P] \in \text{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P, Q)$ of any inclusion $\iota^Q_P$ to the identity element of $Z(P)$ for each $P, Q \in Q$.

In what follows, we only specify 0- and 1-cochains for the functor $Z^Q_\mathcal{F}$ on subgroups in $Q$, and it is to be understood that they are the identity on $\mathcal{F}$-centric subgroups outside $Q$. Alternatively, apply the isomorphism of cochain complexes in Lemma 2.5(a) to view these cochains as restrictions to the full subcategory of $\mathcal{O}(\mathcal{F}^c)$ with objects in $Q$.

The reader may wish to recall the coboundary maps for 0- and 1-cochains in our right-handed notation from (2.2) and (2.3).
Lemma 5.1. Each 1-cocycle for $\mathcal{Z}_F^Q$ is cohomologous to an inclusion-normalized 1-cocycle. If $t$ is an inclusion-normalized 1-cocycle, then

(a) $t([\varphi_1]) = t([\varphi_2])$ for each commutative diagram

$$
\begin{array}{ccc}
P_2 & \xrightarrow{\varphi_2} & Q_2 \\
\downarrow{\iota_{P_2}} & & \downarrow{\iota_{Q_2}} \\
P_1 & \xleftarrow{\varphi_1} & Q_1
\end{array}
$$

in $\mathcal{F}$ among subgroups of $Q$;

(b) the function $\tau: \Gamma^* \to \Gamma^*$ defined by the rule

$$
g^\tau = t([c_g])g,
$$

is a bijection that restricts to the identity map on $S$, and

$$(g_1 g_2 \cdots g_n)^\tau = g_1^\tau g_2^\tau \cdots g_n^\tau$$

for each collection of elements $g_i \in \Gamma^*$ with the property that there is $Q \in Q$ such that $Q^{g_1 \cdots g_n} \in Q$ for all $1 \leq i \leq n$; and

(c) $t = 0$ if and only if $\tau$ is the identity on $\Gamma^*$.

Proof. Given a 1-cocycle for $\mathcal{Z}_F^Q$, define a 0-cochain $u$ by $u(P) = t([\iota_P])$ for each $P \in Q$. Then for any inclusion $\iota_P^Q$ in $\mathcal{F}$ with $P, Q \in Q$, we see that

$$
du(P \xrightarrow{\iota_P^Q} Q) = u(Q)u(P)^{-1} = t([\iota_Q^S])t([\iota_P^S])^{-1},
$$

so that

$$(t du)(P \xrightarrow{\iota_P^Q} Q) = t([\iota_Q^S])t([\iota_P^S])^{-1} = t([\iota_P^S])t([\iota_P^S])^{-1} = 1$$

by the 1-cocycle identity. Hence, $t du$ is inclusion-normalized.

Assume now that $t$ is inclusion-normalized, and let $P_i, Q_i$, and $\varphi_i$ be as in (a). Since $t$ sends inclusions to the identity, the 1-cocycle identity yields

$$
t([\varphi_1 \iota_{Q_1}]) = t([\iota_{Q_1}])^{\varphi_1^{-1}}t([\varphi_1]) = t([\varphi_1]),$$

and

$$
t([\iota_{P_1}^P \varphi_2]) = t([\varphi_2])^{(\iota_{P_1}^P)^{-1}}t([\iota_{P_1}^P]) = t([\varphi_2])$$

and so (a) follows by commutativity of the diagram.

Let $\tau$ be given by (b). Since $g \in \Gamma^*$, the conjugation map $c_g: \#S \cap S \to S \cap S^g$ is a map between subgroups in $Q$. Part (a) shows that $t([c_g])$ agrees with the value of $t$ on the class of each restriction of $c_g$ provided that the source and target of such a restriction lie in $Q$. This shows that $\tau$ is well defined. Then $\tau$ is a bijection since its inverse is induced by $t^{-1}$ (which is inclusion-normalized) in the same way. Further, for $s \in S$, $[c_s] = [\text{id}_S]$ is the identity in the orbit category, and so $\tau$ is the identity map on $S$, since $t$ is normalized.

Let $g_1, g_2 \in \Gamma^*$ and $Q \in Q$ with $Q^{g_1} \leq S$ and $Q^{g_1 g_2} \leq S$. Then by the 1-cocycle identity,

$$
g_1^\tau g_2^\tau = t([c_{g_1}])g_1 t([c_{g_2}])g_2 = t([c_{g_1}])t([c_{g_2}])g_1 g_2 = t([c_{g_1} g_2])g_1 g_2 = (g_1 g_2)^\tau.
$$
Lemma 5.2. Let \( t \) be an inclusion-normalized 1-cocycle for the functor \( Z^Q_F \) and let \( \tau \) be the rigid map associated with \( t \).

(a) For each \( Q \in \mathcal{Q} \cap \mathcal{F}^f \), there is \( z \in Z(N_S(Q)) \) such that \( \tau \) is conjugation by \( z \) on \( N_\Gamma(Q) \);

(b) if \( z \in Z(S) \) and \( u \) is the constant 0-cochain defined by \( u(Q) = z \) for each \( Q \in \mathcal{Q} \), then \( du \) is inclusion-normalized and the rigid map \( \upsilon \) associated with \( du \) is conjugation by \( z \) on \( \Gamma^* \); and

(c) if \( \mathcal{C} \subseteq \mathcal{F}^f \) is a conjugation family for \( \mathcal{F} \) and \( \tau \) is the identity on \( N_\Gamma(Q) \) for each \( Q \in \mathcal{C} \cap \mathcal{Q} \), then \( \tau \) is the identity on \( \Gamma^* \).

Proof. We give two proofs for part (a). The first one uses elementary group-theoretic arguments and the norm map, and is given by Lemma 5.1(b) in the appendix. The second is modeled on part of the proof of [AOV12, Lemma 4.2] and given now. From Lemma 5.1(b), \( \tau \) induces an automorphism of \( N_\Gamma(Q) \) that is the identity on \( N_S(Q) \). By [OV09, Lemma 1.2] and its proof, there is a commutative diagram

\[
\begin{array}{cccccccc}
1 & \longrightarrow & Z^1(\text{Out}_\Gamma(Q); Z(Q)) & \longrightarrow & \text{Aut}(N_\Gamma(Q), Q) & \longrightarrow & \text{Out}(Q) \\
 & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H^1(\text{Out}_\Gamma(Q); Z(Q)) & \longrightarrow & \text{Out}(N_\Gamma(Q), Q) & \longrightarrow & \text{Out}(Q)
\end{array}
\]

with exact rows, where \( \text{Aut}(N_\Gamma(Q), Q) \) is the subgroup of automorphisms of \( N_\Gamma(Q) \) that leave \( Q \) invariant (and similarly for \( \text{Out}(N_\Gamma(Q), Q) \)). Also, \( \tilde{\eta} \) maps the restriction of \( t \) (to \( \text{Out}_\Gamma(Q) \)) to the restriction of \( \tau \) to \( N_\Gamma(Q) \). The restriction map \( H^1(\text{Out}_\mathcal{F}(Q); Z(Q)) \rightarrow H^1(\text{Out}_S(P); Z(P)) \) is injective since \( \text{Out}_S(Q) \) is a Sylow \( p \)-subgroup of \( \text{Out}_\mathcal{F}(Q) \) by assumption on \( Q \). Hence \( t \) represents the zero class in \( H^1(\text{Out}_\mathcal{F}(P); Z(P)) \) since \( t \) is zero on \( \text{Out}_S(P) \). It then follows from (5.3) that \( \tau \) induces an inner automorphism of \( N_\Gamma(Q) \). Hence \( \tau \) is conjugation by an element in \( Z(N_S(Q)) \), since \( Q \in \mathcal{F}^c \) and \( \tau \) is the identity on \( N_S(Q) \).

With \( u \) as in (b), we see that \( du(P^{[g^Q]} \rightarrow Q) = u(Q)u(P)^{-1} = zz^{-1} = 1 \), for any inclusion among subgroups when \( P \in \mathcal{Q} \) (and when \( P \notin \mathcal{Q} \) by (2.2)). Also, for \( g \in \Gamma^* \),

\[
g^\upsilon = du([g])g = z^{-1}z^{-1}g = g^z
\]

so (b) holds.

Part (c) follows directly from Lemma 5.1(b); we give the details. Fix a conjugation family \( \mathcal{C} \subseteq \mathcal{F}^f \), and suppose that \( \tau \) is the identity on \( N_\Gamma(T) \) for each \( T \in \mathcal{C} \). Fix \( g \in \Gamma^* \), and choose \( Q \in \mathcal{Q} \) with \( Q^g \subseteq S \). Then there are a positive integer \( n \), subgroups \( T_1, \ldots, T_n \in \mathcal{C} \),
and elements \( g_i \in N_T(T_i) \) such that \( g = g_1 \cdots g_n, \ Q \leq T_1, \) and \( Q^{g_1 \cdots g_i} \leq T_{i+1} \) for each \( i = 1, \ldots, n - 1. \) As \( Q \) is \( F \)-invariant and closed under passing to overgroups, \( T_i \in Q \) for each \( i. \) Since \( \tau \) fixes \( g_i \) for each \( i \) by assumption, \( \tau \) fixes \( g \) by Lemma 5.1(b). \( \square \)

**Theorem 5.4.** Let \( (\Gamma, S, Y) \) be a reduced setup for the prime 2. Set \( D = Z(Y), \ F = F_S(\Gamma), \ G = \Gamma/C_T(D), \ A = A_D(G)^{\circ} \cup A_D(G)^{\circ}, \ T = T_D(G), \ B = A - T, \) and \( R = \{ P \in \mathcal{F}(S) \geq Y \mid J_A(P) = Y \}. \)

Assume \( B \) is not empty. Then \( L^2(F; R) = 0. \)

**Proof.** If \( R = \mathcal{F}(S) \geq Y \), then \( L^2(F; R) = 0 \) by Lemma 2.5(b) so we may assume \( Q := \mathcal{F}(S) \geq Y - R \) is not empty. That is, \( A \) is not empty. Since \( Q \) is closed under passing to overgroups, \( S \in Q \) and \( J_A(Q) \in Q \) for each \( Q \in Q. \)

We will show \( L^1(F; Q) = 0. \) Since \( Q \) and \( R \) are \( F \)-invariant intervals that together satisfy the hypotheses of Lemma 2.6, the result then follows from part (a) of that lemma. Fix a 1-cocycle \( t \) for the functor \( Z^2 Q. \) To show that \( t \) is cohomologous to 0, we may assume by Lemma 5.1 that \( t \) is inclusion-normalized. Let \( \tau: \Gamma^* \to \Gamma^* \) be the rigid map associated with \( t. \)

The proof splits into two cases. In Case 1, some member of \( A \) has order at least 4. In Case 2, every member of \( A \) has order 2. We now fix notation for each case. Use bars to denote images modulo \( C_T(D). \) If \( P \) is a Sylow \( 2 \)-subgroup of a subgroup \( H \leq \Gamma \) such that every member of \( A \cap \bar{P} \) has order 2, then define \( B(P, H) \) to be the collection of subgroups in \( A \cap \bar{P} \) that are not solitary in \( \bar{H} \) relative to \( \bar{P}. \) In any situation, let \( B(P) \) denote the set of subgroups in \( A \cap \bar{P} \) that are not semisolitary relative to \( \bar{P}, \) and set \( A_{\geq 4} = \{ A \in A \mid |A| \geq 4 \}. \)

Define

\[
J_1(P) = J_{A_{\geq 4}}(P); \quad \text{and} \quad J_2(P) = J_{A_{\geq 4} \cup B(P)}(P),
\]

so that \( Y \leq J_1(P) \leq J_2(P) \leq J_A(P) \) whenever \( P \geq Y. \) We define two subgroup mappings \( W_1 \) and \( W_2 \) on \( \mathcal{F}(S), \) to be employed in the respective cases. In all cases, set \( W_1(P) = P \) if \( P \) does not contain \( Y. \) For \( P \geq Y, \) set

\[
W_1(P) = \begin{cases} 
J_1(P) & \text{if } A_{\geq 4} \cap \bar{P} \neq \emptyset; \\
J_2(P) & \text{if } A_{\geq 4} \cap \bar{P} = \emptyset \text{ and } B(P) \neq \emptyset; \quad \text{and} \\
J_A(P) & \text{otherwise,
}
\end{cases}
\]

and

\[
W_2(P) = \begin{cases} 
J_{B(S,\Gamma)}(S) & \text{whenever } J_{B(S,\Gamma)}(S) \leq P; \quad \text{and} \\
J_A(P) & \text{otherwise.
}
\end{cases}
\]

In any case, \( W_1(P) \in Q \) and \( W_i(W_i(P)) = W_i(P) \) whenever \( P \in Q. \)
Set $W = W_i$ for $i = 1$ or 2. Then $W(S)$ is normal in $S$, so the restriction of $\tau$ to $N_\Gamma(W(S))$ is conjugation by an element $z \in Z(S)$ by Lemma 5.2. Upon replacing $t$ by $tdu$ where $u$ is the constant 0-cochain defined by $u(Q) = z^{-1}$ for each $Q \in Q$, and upon replacing $\tau$ by the rigid map associated with $tdu$, we may assume by Lemma 5.2(b) that

5.4.1. $\tau$ is the identity on $N_\Gamma(W(S))$.

Now $W_1$ is a $\Gamma$-conjugacy functor on $\mathcal{S}(S)$ in the sense of Appendix A since $W_1(P) \neq 1$ whenever $P \neq 1$ and since all subcollections used in defining $W_1$ are $G$-invariant. In case $W = W_2$ and $A = A_D(G)_2$, $W_2$ is a $\Gamma$-conjugacy functor on $\mathcal{S}(S)$ as then $\mathcal{B}(S, \Gamma)$ is weakly closed in $S$ with respect to $\Gamma$ by Lemma 1.14. By Theorem 5.6, the collection $\mathcal{C}$ of subgroups of $S$ that are well-placed with respect to $W$ forms a conjugation family for $\mathcal{S}$. If a subgroup $P$ is well-placed with respect to $W$, then so is $W(P)$ because $W(W(P)) = W(P)$.

Set

$$W = \{Q \in \mathcal{C} \cap Q \mid W(Q) = Q\}.$$ 

If $Q \in \mathcal{C} \cap Q$, then $W(Q) \in W$, and $N_\Gamma(Q) \leq N_\Gamma(W(Q))$. Therefore, it follows from Lemma 5.2(c) that

5.4.2. if $\tau$ is the identity on $N_\Gamma(Q)$ for each $Q \in W$, then $\tau$ is the identity on $\Gamma^*$.

For each $Q \in W$, the restriction of $\tau$ to $H = N_\Gamma(Q)$ is conjugation by an element $z_H \leq Z(N_S(Q)) \leq Z(Y) = D$ by Lemma 5.2(a). Hence, if $N_H(W(N_S(Q)))$ fixes $z_H$, and and this normalizer controls fixed points in $H$ on $D$, then $\tau$ is the identity on $H$. We record this important observation as follows.

5.4.3. Assume $Q \in W$ and $H = N_\Gamma(Q)$. If $\tau$ is the identity on $N_\Gamma(W(N_S(Q)))$ and $C_D(H) = C_D(N_H(W(N_S(Q))))$, then $\tau$ is the identity on $H$.

We now distinguish between the two cases. In each case, we prove that $\tau$ is the identity on $N_\Gamma(Q)$ for each $Q \in W$ by induction on the index of $N_S(Q)$ in $S$. Fix $Q \in Q$ and set $S^* = N_S(Q)$ and $H = N_\Gamma(Q)$ for short.

The norm arguments in 4.1 are statements about control of fixed points in $G$. Each member of $Q$ contains $Y = C_S(D)$ since $(\Gamma, S, Y)$ is a reduced setup, and so Lemma 3.8 provides the transition from control of fixed points by normalizers within $G$ and those within $\Gamma$. We apply Lemma 3.8 implicitly for this transition in the arguments that follow.

Case 1: Some member of $\mathcal{A}$ has order at least 4.

Put $W = W_1$. Assume first that $S^* = S$. Since Case 1 holds, the collection $\mathcal{A}_{\geq 4} = A_D(G)^\circ_{\geq 4} \cup \mathcal{A}_D(G)^\circ$ is not empty. Hence $W(S^*) = J_1(S)$ by definition of $W$. By Lemma 4.6, (4.12) is satisfied with $(\bar{H}, \bar{S}, D, A_{\geq 4} \cap \bar{H}, N_H(J_1(S)))$ in the role of the five-tuple $(G, S, D, A, H)$ of that lemma. Hence $C_D(H) = C_D(N_H(J_1(S)))$ by Theorem 4.1. Further, $\tau$ is the identity on $N_\Gamma(W(S)) = N_\Gamma(J_1(S))$ by 5.4.1, so that $\tau$ is the identity on $H$ by 5.4.3.

Assume now that $S^* < S$. If $A_{\geq 4} \cap S^* \neq \emptyset$ (that is, if $J_1(S^*) > Y$), then $W(S^*) = J_1(S^*)$, and $C_D(H) = C_D(N_H(W(S^*)))$ by Lemma 4.6 and Theorem 4.1 as before. Since
\[ S^* < N_S(W(S^*)) \text{ (by Lemma \[ B.3.5\)(c))}, \tau \text{ is the identity on } N_H(W(S^*)) \text{ by induction, so that } \tau \text{ is the identity on } H \text{ by } \[ 5.4.3\]. \]

Assume for the remainder of Case 1 that \( \mathcal{A}_{2^4} \cap S^* \) is empty. In particular, \( \hat{\mathcal{A}}_D(G) \cap \bar{S}^* \) is empty. However, \( \mathcal{A} \cap S^* \) is not empty since \( S^* \ni Q \in \mathcal{Q} \), and every member of \( \mathcal{A} \cap S^* \) is of order 2. Moreover,

\[ (5.5) \quad Q = W(Q) \leq J_A(Q) \leq J_A(S^*) \quad \text{and} \quad J_A(S^*) \text{ is elementary abelian} \]

from Lemma \[ 4.8\)(c). Assume next that \( \mathcal{B}(S^*) \) is empty. Then \( W(S^*) = J_2(S^*) > Y \). Since no member of \( \mathcal{B}(S^*) \) is solitary in \( \bar{H} \) relative to \( \bar{S}^* \) by Remark \[ 4.13\], we see that \( \mathcal{B}(S^*) \subseteq \mathcal{B}(S^*, H) \), and that

\[ W(S^*) = J_2(S^*) = J_{\mathcal{B}(S^*)}(S^*) \leq J_{\mathcal{B}(S^*, H)}(S^*). \]

Since \( J_A(S^*) \) is elementary abelian and semisolitary subgroups relative to \( S^* \) are invariant under conjugation in \( N_H(J_A(S^*)) \), Lemma \[ B.3.1\] shows that \( W(S^*) \) is weakly closed in \( S^* \) with respect to \( H \). Apply Lemma \[ 4.15\] with \((\bar{H}, \bar{S}^*, D, \mathcal{A} \cap \bar{S}^*, \bar{W}(S^*))\) in the role of the five-tuple \((G, S, D, \mathcal{A}, J^*)\) of that lemma to obtain the hypotheses of Theorem \[ 4.1\] which gives \( C_D(H) = C_D(N_H(W(S^*))) \) as before. However, \( S^* < N_S(W(S^*)) \) since \( S^* < S \) (by Lemma \[ B.5\)(c)). Since \( \tau \) is the identity on \( N_H(W(S^*)) \) by induction, we have that \( \tau \) is the identity on \( H \) by \[ 5.4.3\].

Finally, assume that \( \mathcal{B}(S^*) \) is empty. Then \( W(S^*) = J_A(S^*) \) by definition of \( W \), and every element of \( \mathcal{A} \cap \bar{S}^* \) is semisolitary relative to \( S^* \). We will show this leads to a contradiction – this is a critical step in the proof. Since Case 1 holds, \( J_A(S^*) < J_A(S) \). Now \( J_A(S^*) < J_A(N_S(J_A(S^*))) \) (by Lemma \[ B.3.5\)(b)), and so there exists \( A \subseteq S \) with \( A \in \mathcal{A} \cap \bar{S}^* \) such that

\[ (5.6) \quad A \subseteq J_A(N_S(J_A(S^*))), \text{ but } A \not\subseteq J_A(S^*). \]

It follows from the definitions that each subgroup of \( S^* \) that is semisolitary relative to \( S^* \) is also semisolitary relative to \( J_A(S^*) \). As \( A \) normalizes \( J_A(S^*) \), we see that \( \hat{A} \) permutes the elements of \( \mathcal{A} \cap \bar{S}^* = \mathcal{A} \cap \bar{J_A}(S^*) \) by conjugation. We are thus in the situation of Lemma \[ 4.29\] with \( \bar{J_A}(S) \) in the role of \( P \), and \( \mathcal{A} \cap \bar{S}^* \) in the role of \( \mathcal{T} \) there. By that lemma, \( \hat{A} \) normalizes every member of \( \mathcal{A} \cap \bar{S}^* \). Thus \( A \) normalizes each of their preimages in \( S \). However \( Q \) is generated by the preimages of a subset of \( \mathcal{A} \cap \bar{S}^* \) by \[ 5.5\], where \( W(Q) = J_A(Q) \) in the current situation. Hence \( A \) normalizes \( Q \). But then \( A \subseteq J_A(S^*) \), contrary to \( (5.6) \). This contradiction completes the proof of Case 1.

**Case 2:** Each member of \( \mathcal{A} \) is of order 2.

Put \( W = W_2 \) and assume first that \( S^* = S \). By assumption, \( \mathcal{A} = \mathcal{A}_D(G)_2 \), and so \( \hat{\mathcal{A}}_D(G) = \varnothing \) by Remark \[ 4.5\]. Set \( J = J_{\mathcal{B}(S,H)}(S) \) and note that \( W(S) = J_{\mathcal{B}(S,\Gamma)}(S) \) by definition of \( W_2 \). By Definition \[ 4.12\] every element of \( \mathcal{A} \) that is solitary in \( \bar{H} \) relative to \( \bar{S} \) is also solitary in \( G \) relative to \( S \). Thus \( \mathcal{B}(S, \Gamma) \subseteq \mathcal{B}(S, H) \), and

\[ W(S) \subseteq J \subseteq J_A(S). \]
We saw earlier (after 5.4.1) that \(W(S)\) is weakly closed in \(S\) with respect to \(\Gamma\) whenever \(A = A_0(G)_2\), which holds in the present case. So we may apply Lemma 4.15 with \((\bar{H}, S, D, A \cap \bar{H}, W(S))\) in the role of the five-tuple \((G, S, D, A, J^*)\) of that lemma to obtain the hypotheses of Theorem 4.11 which yields \(C_D(H) = C_D(N_H(W(S)))\). Further, \(\tau\) is the identity on \(N_H(W(S))\) by 5.4.1 so that \(\tau\) is the identity on \(H\) by 5.4.3.

Assume now that \(S^* < S\). As \(\overline{J_A(S)}\) is elementary abelian and \(Q \in W\),

\[Q = W(Q) \leq \overline{J_A(Q)} \leq \overline{J_A(S)},\]

and \(\overline{J_A(S)}\) centralizes \(\bar{Q}\). Hence \(\overline{J_A(S)} \leq S^*\) so that

\[\overline{J_A(S^*)} = \overline{J_A(S)}\]

Since \(J_{B(S,\Gamma)}(S) \leq \overline{J_A(S)}\), this shows that \(W(S^*) = J_{B(S,\Gamma)}(S) = W(S)\). As in the situation where \(Q\) was normal in \(S\), we have that \(B(S,\Gamma) \subseteq B(S^*,H)\) and that \(W(S^*)\) is weakly closed in \(S^*\) with respect to \(\bar{H}\). Apply Lemma 4.15 with \((\bar{H}, \bar{S}^*, D, A \cap \bar{H}, W(S^*))\) in the role of the five-tuple \((G, S, D, A, J^*)\) of that lemma to obtain the hypotheses of Theorem 4.11 which yields \(C_D(H) = C_D(N_H(W(S^*)))\) as before. Further, since \(W(S^*) = W(S)\), \(\tau\) is the identity on \(N_H(W(S^*))\) by 5.4.1 so that \(\tau\) is the identity on \(H\) by 5.4.3. This concludes the proof in Case 2.

The theorem now follows from 5.4.2 and Lemma 5.1(c). □

6. Transvections

The aim of this section is to give a proof of Proposition 3.3 of [Oli13] for \(p = 2\) in Theorem 6.9. This result and the proof of [Oli13] Theorem 3.4] give Theorem 1.1 when \(p = 2\).

Using McLaughlin’s classification of irreducible subgroups of \(SL_n(2)\) generated by transvections, we first classify in Theorem 6.2 those finite groups which have no nontrivial normal 2-subgroups and are generated by solitary offenders. Recall that by a natural \(S_m\)-module \((m \geq 3)\), we mean the nontrivial composition factor of the standard permutation for \(S_m\) over the field with two elements.

**Lemma 6.1.** Let \(G\) be a finite group acting irreducibly on an elementary abelian 2-group \(W\). Assume that \(G\) is generated by transvections. Then \(\mathcal{T}_W(G)\) is not empty if and only if \(G\) is isomorphic to a symmetric group of odd degree and \(W\) is a natural module for \(G\). Moreover, in this case, \(\mathcal{T}_W(G)\) is the set of transpositions.

**Proof.** Assume first that \(G\) is generated by transvections on the irreducible module \(W\). By a result of McLaughlin [McL69], \(G\) is isomorphic to \(SL(W)\), or the dimension \(n\) of \(W\) is even and at least 4 and \(G\) is isomorphic to \(Sp(W), O^-(W), O^+(W), S_{n+1}, \text{or } S_{n+2}\). For the classical groups, \(A_W(G)_2\) is the set of transvections; for the symmetric groups, \(A_W(G)_2\) is the set of transpositions. In all cases, \(A_W(G)_2\) is a single \(G\)-conjugacy class.

Fix a Sylow 2-subgroup \(S\) of \(G\), and assume first that \(G = SL(W)\) with \(n \geq 3\). Since \(A_W(G)_2\) is a single conjugacy class, either \(\mathcal{T}_W(G)\) is empty or \(\mathcal{T}_W(G) = A_W(G)_2\). Since
S is itself generated by transvections, (S2) forces S to be abelian in the latter case, a contradiction.

Assume that G is a symmetric group of degree \( n + 2 \geq 6 \). We may assume S stabilizes the partition \( \{1, 2, \ldots, n + 1, n + 2\} \), and then \( \langle A_W(G)_2 \cap S \rangle \) is the centralizer of this partition. Fix \( A = \langle (2j - 1, 2j) \rangle \in A_W(G)_2 \cap S \), and let \( L \cong S_3 \) be a subgroup of G containing A. Then all members of \( A_W(G)_2 \cap L \) are conjugate, and so the support of \( L \) is a three-element set, say \( \{2j - 1, 2j, k\} \). Hence \( L \) does not centralize the element of \( A_W(G)_2 \cap S \) moving \( k \), and thus \( A \) is not solitary in G relative to \( S \).

Assume that \( G = Sp(W) \) preserves the symplectic form b. Fix a maximal isotropic subspace \( W_0 \) stabilized by S, and let \( U \) be the unipotent radical of its stabilizer in G. Then all members of \( A_W(G)_2 \cap S \) are contained in \( U \). Let \( A \) be one of them, having center \( \langle e \rangle \subseteq W_0 \), and let \( L \cong S_3 \) be a subgroup of G containing A. Since two symplectic transvections commute if and only if their centers are orthogonal with respect to \( b \), \( [W, L] \) is a hyperbolic line. Since \( n \geq 4 \), we may find \( e' \in [W, L]^{\perp} \cap W_0 - \langle e \rangle \), and then \( L \) does not centralize the member of \( A_W(G) \cap S \) with center \( \langle e + e' \rangle \). So \( A \) is not solitary.

Assume that \( G \) is an orthogonal group preserving the quadratic form \( q \) with associated symplectic form \( b \). If \( n = 4 \), since then \( G \) is generated by transvections, \( G \cong S_5 \). Thus, we may assume that \( n \geq 6 \). Choose a maximal isotropic subspace \( W_0 \) (with respect to \( b \)) stabilized by \( S \) and such that \( W_0 \) contains a nonsingular vector. Let \( U \) be the unipotent radical of the stabilizer of the radical of \( q|_{W_0} \). Fix a nonsingular vector \( e \in W_0 \), let \( A \subseteq U \) be generated by the transvection with center \( \langle e \rangle \), and let \( L \cong S_3 \) be a subgroup of G such that \( L \) contains A. As before, the restriction of \( b \) to \( [W, L] \) is nondegenerate, and as \( L \) acts on \( [W, L] \), \( q|_{[W, L]} \) is of minus type. As \( \dim(W_0) \geq 3 \), there is a singular vector \( e' \in [W, L]^{\perp} \cap W_0 \), and then \( L \) does not centralize the orthogonal transvection with center \( \langle e + e' \rangle \). So \( A \) is not solitary.

Therefore, \( G \) is a symmetric group of odd degree and \( W \) is a natural module for \( G \).

For the converse, let \( G = S_{2n+1} \) \((n \geq 1)\), \( \Omega \) the standard \( G \)-set, and identify \( W \) with the set of even order subsets of \( \Omega \). To show that each transposition generates a solitary subgroup, we may restrict our attention to \( T = \langle (2n - 1, 2n) \rangle \). Consider the partition \( \{\{2i - 1, 2i\} : 1 \leq i \leq n\} \cup \Omega - \{2n + 1\} \), and let \( S \) be a Sylow 2-subgroup of G stabilizing this partition. Then \( A_W(G)_2 \cap S = \{(1, 2), \ldots, (2n - 1, 2n)\} \). Hence, taking \( L \) be the symmetric group induced on \( \{2n - 1, 2n, 2n + 1\} \), it is easily checked that (S1)-(S3) hold in Definition 12 so that \( T \) is solitary in \( G \) relative to \( S \). \( \square \)

**Theorem 6.2.** Let \( G \) be a finite group, let \( D \) be an abelian 2-group on which \( G \) acts faithfully, and set \( T = T_D(G) \). Assume that \( O_2(G) = 1 \) and that \( G = \langle T \rangle \) is generated by its solitary offenders. Then there exist a positive integer \( r \) and subgroups \( E_1, \ldots, E_r \) such that

(a) \( G = E_1 \times \cdots \times E_r \), \( T = (T \cap E_1) \cup \cdots \cup (T \cap E_r) \), and \( E_i \cong S_{m_i} \) with \( m_i \) odd for each \( i \); and
(b) $D = V_1 \times \cdots \times V_r \times C_D(G)$ with $V_i = [D, E_i]$ a natural $S_{m_i}$-module, and with $[V_i, E_j] = 1$ for $j \neq i$.

Proof. Set $W = [D, G]C_D(G)/C_D(G)$, and let $(G, D)$ be a counterexample with $|G| + |D|$ minimal. We show in Step 1 that $[D, G, G] = [D, G]$, in Step 2 that $C_{D/C_D(G)}(G) = 1$, in Step 3 that $G$ is faithful on $W$ with $W = [W, G]$ and $C_W(G) = 1$, in Step 4 that the theorem for $(G, W)$ implies the theorem for $(G, D)$, and in Step 5 that $W$ is irreducible. The theorem then follows from Lemma 6.1. We prefer to give essentially complete proofs from first principles.

When $D > D_1 > \cdots > D_k > 1$ is a chain of $G$-invariant subgroups of $D$ (or of any other abelian $p$-group with faithful action from $G$) we say for short that a subgroup $K$ of $G$ acts nilpotently on the chain if $K$ acts trivially on successive quotients. If a subgroup $K$ acts nilpotently on such a chain and $k \geq 1$, then $K \leq O_2(G)$ by [Gor80, Theorem 5.3.3].

Step 1: For each $T \in \mathcal{T}$, choose a subgroup $L$ of $G$ such that $L \supseteq T$ and $L \cong S_3$ as in Definition 4.12. Then $|D, T| \leq |D, L| \leq |D, G|$ and $|D, L, L| = |D, L|$ by (S3), so that $[L, T] \leq [D, L, L] \leq [D, G, G]$. Hence $T$ centralizes $G/[D, G, G]$. It follows that $G$ centralizes $G/[D, G, G]$ since $G = \langle T \rangle$ and the choice of $T$ was arbitrary. That is, $[D, G] \leq [D, G, G]$.

Step 2: Let $D_1$ be the preimage of $C_{D/C_D(G)}(G)$ in $D$. Suppose that $D_1 > C_D(G)$. Then $C_D(G) > 1$. As before fix $T \in \mathcal{T}$ and choose $L$ as in Definition 4.12 for $T$. Then $O^2(L)$ acts nilpotently on the chain $D_1 > C_D(G) > 1$. Since $O^2(L)$ is of odd order, it centralizes $D_1$. Hence, by (S3),

$$D_1 \leq C_D(O^2(L)) = C_D(L) \leq C_D(T).$$

That is, $T$ centralizes $D_1$. We conclude that $G$ centralizes $D_1$ since $G = \langle T \rangle$, and since the choice of $T$ was arbitrary. This contradicts $D_1 > C_D(G)$. Therefore, $D_1 = C_D(G)$.

Step 3: Set $W = [D, G]C_D(G)/C_D(G)$ as above.

Using faithfulness of $G$ on $D$ and the assumption $O_2(G) = 1$, we see that

$$(6.3) \quad W > 1$$

since otherwise $G$ acts nilpotently on $D > C_D(G) > 1$. Let $K$ be the kernel of the action of $G$ on $W$. Then $K$ acts nilpotently on the chain $D \supseteq [D, G]C_D(G) > C_D(G) \geq 1$, with the strict inclusion from (6.3), and hence $K \leq O_2(G) = 1$. We conclude that

$$(6.4) \quad G \text{ is faithful on } W.$$

By Steps 1 and 2,

$$(6.5) \quad W = [W, G] \quad \text{and} \quad C_W(G) = 1.$$
(b). Set \( V_1 = [D, E_1] \). Then under the projection from \( D \) onto \( D/C_D(G) \), \( V_1 \) maps onto \([W, E_1] = W_1\). On the other hand, we may choose \( m_1 - 1 \) elements \( T_1, \ldots, T_{m_1 - 1} \in \mathcal{T} \cap E_1 \) corresponding to adjacent transpositions that generate \( E_1 \), and see that
\[
|V_1| = |([D, T_1], \ldots, [D, T_{m_1 - 1}])| \leq \prod_{1 \leq i \leq m_1 - 1} |[D, T_i]| = 2^{m_1 - 1} = |W_1|
\]
since \( |[D, T_i]| = 2 \) for each \( 1 \leq i \leq m_1 - 1 \). Hence \( V_1 \cong W_1 \) is a natural \( E_1 \)-module. It follows that \( C_D(E_1) \cap V_1 = 1 \). Moreover,
\[
|D/C_D(E_1)| = |D/\langle \cap_i C_D(T_i) \rangle| \leq \prod_{1 \leq i \leq m_1 - 1} |D/C_D(T_i)| = 2^{m_1 - 1} = |V_1|,
\]
since \( |D/C_D(T_i)| = 2 \) for each \( i \). We conclude that \( D = V_1 \times C_D(E_1) \). In the case that \( r = 1 \), this shows that (b) holds for \( G \) and \( D \). Otherwise, apply induction (on \( r \)) to \( E_2 \cdots E_r \), \( \mathcal{T} \cap E_2 \cdots E_r \), and \( C_D(E_1) \) to obtain
\[
D = V_1 \times \cdots \times V_r \times C_D(E_1 \cdots E_r)
\]
yielding part (b) in general for \( G \) and \( D \).

**Step 5:** We next show that \( W \) is irreducible for \( G \). Assume on the contrary that \( W_1 \) is a nontrivial proper \( G \)-invariant subgroup of \( W \). Set

\[
\mathcal{T}_1 = \{ T \in \mathcal{T} \mid [W, T] \leq W_1 \},
\]

\[
\mathcal{T}_2 = \{ T \in \mathcal{T} \mid [W_1, T] = 1 \},
\]

\( G_1 = \langle \mathcal{T}_1 \rangle \), and \( G_2 = \langle \mathcal{T}_2 \rangle \). Then \( G_1 \) and \( G_2 \) are normal in \( G \).

Let \( T \in \mathcal{T} - \mathcal{T}_2 \). Then
\[
1 < [W_1, T] \leq [W, T] \leq [D, T]C_D(G)/C_D(G),
\]
so as \( |[D, T]| = 2 \), all these inclusions are equalities. In particular, \( [W, T] = [W_1, T] \leq W_1 \), which yields \( T \in \mathcal{T}_1 \). Hence

\[
(6.6) \quad \mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \quad \text{and} \quad G = G_1G_2.
\]

Set \( K = C_G(W/W_1) \cap C_G(W_1) \). Then \( K \) acts nilpotently on the chain \( W > W_1 > 1 \), and so \( K \leq O_2(G) = 1 \) by (6.6) and assumption on \( G \). Since \( [G_1, G_2] \leq G_1 \cap G_2 \leq K \), we see that

\[
(6.7) \quad \mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset \quad \text{and} \quad G = G_1 \times G_2
\]
from (6.6).

Now \( [W, G_1, G_2] = 1 \) by construction and \( [G_1, G_2, W] = 1 \) from (6.7), so \( [W, G_2, G_1] = 1 \) by the Three Subgroups Lemma. Hence \( [W, G_1G_2] = [W, G_1][W, G_2] \). Further, \( [W, G_1] \cap [W, G_2] \leq C_W(G_1G_2) \), which is the identity by (6.6), and so

\[
(6.8) \quad W = [W, G] = [W, G_1] \times [W, G_2]
\]
Theorem 5.4 shows that \( G \) module for \( \bar{\tau} \) again by (6.5) and (6.6). Similarly, \( \mathcal{T}_t \) is not empty. Hence \( 1 < |G_1| < |G| \) and \( 1 < |G_2| < |G| \). One then checks that \( T \in \mathcal{T}_k \) is solitary in \( G_k \) (on \( W_k \), \( k = 1, 2 \)), using (6.8) and the fact that an \( L \cong S_3 \) containing \( T \) in \( G \) is generated by \( G \)-conjugates of \( T \).

Induction applied to \( (G_1, W_1) \) and \( (G_2, W_2) \) now yields the theorem for \( (G, W) \). By Step 4, \( (G, D) \) is not a counterexample. We conclude that \( W \) is irreducible for \( G \) as desired. In particular, \( W \) is elementary abelian, and each element of \( \mathcal{T} \) induces a transvection on \( W \).

Now Lemma 6.11 shows that \( G \) is a symmetric group of odd degree and \( W \) is a natural module for \( G \), and \( \mathcal{T} \) is the collection of subgroups generated by a transposition. Again by Step 4, \( (G, D) \) is not a counterexample.

**Proposition 6.9.** Let \( (\Gamma, S, Y) \) be a general setup for the prime 2. Set \( \mathcal{F} = \mathcal{F}_S(\Gamma) \), \( D = Z(Y) \), and \( G = \Gamma/C_{\Gamma}(D) \). Let \( \mathcal{R} \subseteq \mathcal{S}(S)_\geq Y \) be an \( \mathcal{F} \)-invariant interval such that for each \( Q \in \mathcal{S}(S)_\geq Y \), \( Q \in \mathcal{R} \) if and only if \( J_{A_D(G)}(Q) \in \mathcal{R} \). Then \( L^k(\mathcal{F}; \mathcal{R}) = 0 \) for all \( k \geq 2 \).

**Proof.** Assume the hypotheses of the proposition, but assume that the conclusion is false. Let \( (\Gamma, S, Y, \mathcal{R}, k) \) be counterexample for which the four-tuple \( (k, |\Gamma|, |\Gamma/Y|, |\mathcal{R}|) \) is minimal in the lexicographic ordering. Steps 1–3 in the proof of [Oli13, Proposition 3.3] show that \( \mathcal{R} = \{ P \leq S \mid J_{A_D(G)}(P) = Y \} \), \( k = 2 \), and \( (\Gamma, S, Y) \) is a reduced setup.

Let \( Q = \mathcal{S}(S)_\geq Y - \mathcal{R} \) and \( A = A_D(G)^\circ \cup \hat{A}_D(G)^\circ \). Since \( (\Gamma, S, Y) \) is a counterexample, Theorem 5.4 shows that

6.9.1. \( \mathcal{A} = \mathcal{T}_D(G) \).

That is, \( \hat{A}_D(G) \) is empty and every best offender minimal under inclusion is solitary in \( G \) relative to \( \bar{S} \). Since \( L^2(\mathcal{F}; \mathcal{R}) \neq 0 \), we see that

6.9.2. \( L^1(\mathcal{F}, Q) \neq 0 \).

from Lemma 2.6 We prove next that

6.9.3. \( G = \langle \mathcal{A} \rangle \).

**Proof.** Let \( G_0 = \langle \mathcal{A} \rangle \). Let \( \Gamma_0 \) be the preimage of \( G_0 \) in \( \Gamma \), set \( S_0 = S \cap \Gamma_0 \), set \( \mathcal{F}_0 = \mathcal{F}_{S_0}(G_0) \), and set \( \mathcal{Q}_0 = \mathcal{S}(S_0)_{\geq Y} \cap \mathcal{Q} \). Then \( \Gamma_0 \leq \Gamma \) and by (6.9.1), we have \( Y \leq S_0 \). Further, \( (\Gamma_0, Y, S_0) \) is a reduced setup and \( \mathcal{Q}_0 \) is an \( \mathcal{F}_0 \)-invariant interval. Since each member of \( \mathcal{A} \) is contained in \( G_0 \), we have \( \Gamma_0 \cap Q \in \mathcal{Q}_0 \) for each \( Q \in \mathcal{Q} \). By Lemma 2.7 the restriction map induces an injection \( L^1(\mathcal{F}; Q) \to L^1(\mathcal{F}_0; \mathcal{Q}_0) \) and so \( L^1(\mathcal{F}_0; \mathcal{Q}_0) \neq 0 \) by 6.9.2. Hence \( \Gamma = \Gamma_0 \) by minimality of \( |\Gamma| \).

Therefore, \( G \) and its action on \( D \) are described by Proposition 6.2 We adopt the notation in that proposition for the remainder of the proof. In the decomposition of part (b) there, each \( V_i \) is \( G \)-invariant and so each is \( S \)-invariant. Thus, the centralizer of \( S \) in \( D \) factors as

6.9.4. \( C_D(S) = C_{V_1}(S) \times \cdots \times C_{V_r}(S) \times C_D(G) \).
Fix an inclusion-normalized 1-cocycle $t$ for $\mathcal{Q}^\mathcal{F}$ representing a nonzero class in $L^1(\mathcal{F}; \mathcal{Q})$ by $6.9.2$ and let $\tau: \Gamma^* \to \Gamma^*$ be the rigid map associated with $t$. We show next that

$6.9.5. \ r = 1.$

Proof. We assume $r > 1$ and aim for a contradiction. Let $G_1 = E_1$ and $G_2 = E_2 \cdots E_r$. For $i = 1, 2$, let $K_i$ be the preimage of $G_i$ in $\Gamma$, and set $\Gamma_i = K_iS$ and $\mathcal{F}_i = \mathcal{F}_S(\Gamma_i)$. We have that $\Gamma_i < \Gamma$ by assumption and that $(\Gamma_i, S, Y)$ is a general setup. Hence $L^1(\mathcal{F}_i; \mathcal{Q}) \cong L^2(\mathcal{F}_i; \mathcal{R}) = 0$ by minimality of $|\Gamma|$. As the restriction of $t$ to $\mathcal{O}(\mathcal{F}_i)$ represents the zero class, there are elements $z_i \in C_D(S)$ by $6.9.4$ such that $z_i \in [D, G_i] \leq C_D(G_{3-i})$ and such $\tau$ is conjugation by $z_i$ on $(\Gamma_i)^*$. Set $t' = t d u$ where $u$ is the constant 0-cochain defined by $u(P) = (z_1z_2)^{-1}$ for each $P \in \mathcal{Q}$. Then by Lemma $5.2(b)$, upon replacing $t$ by $t'$ and $\tau$ by the rigid map $\tau'$ associated with $t'$, we may assume that $\tau$ is the identity when restricted to $(\Gamma_i)^*$ for each $i = 1, 2$.

The objective is now to show that $\tau$ is the identity on $\Gamma^*$. Let $W$ be the $\Gamma$-conjugacy functor defined by $W(P) = J_4(P)$ for each $P \geq Y$, and by $W(P) = P$ otherwise, and let $\mathcal{C}$ be the collection of subgroups $P$ of $S$ such that $W(P) = P$ and $P$ is well-placed with respect to $W$. Since $N_\Gamma(P) \leq N_\Gamma(W(P))$ and $W(W(P)) = W(P)$ for each $P \leq S$, the collection $\mathcal{C}$ is a conjugation family for $\mathcal{F}$. It thus suffices to show that $\tau$ is the identity on $N_\Gamma(Q)$ for each $Q \in \mathcal{C} \cap \mathcal{Q}$ by Lemma $5.2(c)$.

Let $Q \in \mathcal{C} \cap \mathcal{Q}$, so that the image of $Q$ in $G$ is generated by members of $\mathcal{A}$. It then follows from Proposition $5.2(a)$ that $Q = Q_1Q_2$ with $Q_1 \cap Q_2 = Y$, where $\bar{Q}_1$ and $\bar{Q}_2$ are the projections in $G_1$ and $G_2$ of $Q$. If it happens that $Q_i \notin \mathcal{Q}$ for $i = 1$ or 2, this means that $Q_1 = Y$. Since $Y \notin \mathcal{Q}$ but $Q \in \mathcal{Q}$, we may assume without loss that $Q_2 \in \mathcal{Q}$. Now $Q$ is well-placed and $W(Q) = Q$, so that $Q$ is fully $\mathcal{F}$-normalized. A straightforward argument shows that $Q_2$ is also fully $\mathcal{F}$-normalized.

Let $g \in N_\Gamma(Q_2)$, and write $g = g_1g_2$ with $g_i \in K_i$. Since $N_\mathcal{S}(Q_2)$ is a Sylow 2-subgroup of $N_\Gamma(Q_2)$, we have by Lemma $3.8$ that $\overline{N_\Gamma(Q_2)} = N_\mathcal{G}(Q_2)$, and the latter is $G_1 \times N_\mathcal{G}_2(Q_2)$. Since $\bar{g}_1 \in G_1$ centralizes $\bar{Q}_2$, we may write $g_1 = h_1c_1$ where $h_1 \in C_{\Gamma_1}(Q_2/Y) \leq N_{\Gamma_1}(Q_2)$ and $c_1 \in C_\Gamma(D)$. So as $h_1 \in (\Gamma_1)^*$ and $c_1g_2 \in (\Gamma_2)^*$ (both send $Q_2$ to $Q_2$), we see that $\tau$ fixes $g$. We conclude that $\tau$ is the identity on $N_\Gamma(Q_2)$. However, then $\tau$ is the identity on $N_\Gamma(Q)$ since $N_\Gamma(Q) \leq N_\Gamma(Q_2)$. We conclude that $\tau$ is the identity on $\Gamma^*$, a contradiction. Thus, $r = 1$ as desired. \qed

By $6.9.5$ we may fix $m = 2n + 1$ such that $G = E_1 \cong S_m$ and write $\Omega$ for the set of even order subsets of $\{1, \ldots, m\}$. Identify $G$ with $S_m$ and $V := V_1$ with $\Omega$. We may assume that $S$ stabilizes the collection $\{\{2i-1, 2i\} \mid 1 \leq i \leq n\}$. For each $1 \leq i \leq n$, set $z_i = \{1, \ldots, 2i\}$, set $z_i' = \{2i + 1, \ldots, 2n\}$, and let $Q_i$ be the preimage in $S$ of $\langle \{1, 2\}, \ldots, (2i-1, 2i) \rangle$. Then $C_V(N_\Gamma(Q_n)) = \langle z_n \rangle$, and

$6.10 \ C_V(N_\Gamma(Q_i)) = \langle z_i, z_i' \rangle = C_V(N_\Gamma(Q_i) \cap N_\Gamma(Q_n)) \ 	ext{and} \ z_n = z_iz_i' \ 	ext{for all} \ 1 \leq i \leq n - 1.$
Set $\Gamma_n = C_\Gamma(z_n)$ (the setwise stabilizer of $z_n$). Then $S \subseteq N_\Gamma(Q_n) \subseteq \Gamma_n$, $C_V(S) = \langle z_n \rangle$, and the image of $\Gamma_n$ in $G$ is isomorphic with $S_{2n}$. Since $(\Gamma_n, S, Y)$ is a general setup and not a counterexample, we may adjust $t$ by a coboundary and assume that

(6.11) \[ \tau \text{ is the identity on } \Gamma_n. \]

Arguing as before, it suffices to show that $\tau$ is the identity on $N_\Gamma(Q)$ for each $Q \in Q \cap F^j$ with $Q = J_A(Q)$. Each $Q$ with these properties is $F$-conjugate to $Q_i$ for some $1 \leq i \leq n$, so it suffices to prove that $\tau$ is the identity on $N_\Gamma(Q_i)$ for each such $i$.

Now for $i = n$, we have that $N_\Gamma(Q_n) \subseteq \Gamma_n$, and so $\tau$ is the identity on this normalizer by (6.11). Hence $n \geq 2$ since $\Gamma$ is a counterexample. Since $z_{n-1}$ centralizes $N_\Gamma(Q_{n-1})$ and $z_n = z_{n-1}z_n^{-1}$ by (6.11), if necessary we may replace $t$ by $t' = tdu$ where $u(P) = z_n$ for each $P \in Q$, and obtain that $\tau$ is the identity on $N_\Gamma(Q_{n-1})$. Then $\tau$ remains the identity on $\Gamma_n$, since $z_n \in C_V(\Gamma_n)$, and since the rigid map associated with $du$ is conjugation by $z_n$ on $\Gamma_n$. Thus, $n \geq 3$ since $\Gamma$ is a counterexample.

Finally, fix $i$ with $1 \leq i \leq n - 2$. Then $N_\Gamma(Q_i) = N_G(Q_i)$ by Lemma 3.3. Since $Q_i = \langle (1, 2), \ldots, (2i - 1, 2i) \rangle$, we see that $N_G(Q_i) = G_i \times G'_i$ where $G_i \cong C_2 \wr S_i$ moves the first $2i$ points, and where $G'_i \cong S_{m-2i}$ moves the remaining points in the natural action. Thus, we may write $N_\Gamma(Q_i) = \Gamma_i \Gamma'_i$ with $\Gamma_i \cap \Gamma'_i = C_\Gamma(D) \cap N_\Gamma(Q_i)$, $\Gamma_i = G_i$, and $\Gamma'_i = G'_i$. Fix $g \in N_\Gamma(Q_i)$ and write $g = g_i g'_i$ where $g_i \in \Gamma_i$ and $g'_i \in \Gamma'_i$. Then $g_i \in \Gamma_n$, and $\tau$ is the identity on $\Gamma_n$, so $\tau$ fixes $g_i$. But $\Gamma'_i$ is generated by $\Gamma_n \cap \Gamma'_i$ and $N_\Gamma(Q_{n-1}) \cap \Gamma'_i$ by Lemma [B.3], and so $\tau$ fixes $g'_i$ as well. Hence $\tau$ is the identity on $N_\Gamma(Q_i)$, and together with our reductions, this implies that $\tau$ is the identity on $\Gamma^*$. Now Lemma 5.1(c) shows that this is contrary to our choice of $t$. \hfill \Box

APPENDIX A. MODIFIED NORM ARGUMENT

Given a group $G$ and two nonempty subsets $X$ and $Y$ of $G$, define the product set

$$X \cdot Y = \{xy \mid x \in X, y \in Y\}.$$ 

**Definition A.1.** Let $G$ be a finite group and $V$ an abelian group on which $G$ acts. Let $X$ be a subset of $G$.

(a) A subset $Y$ of $G$ is a **transversal** to $X$ in $G$ if for each $g \in G$, there are unique $x \in X$ and $y \in Y$ such that $g = xy$.

(b) The **norm** from $X$ to $G$ relative to the transversal $Y$ is the group homomorphism

$$\mathfrak{N}^{G}_{X,Y} : C_V(X) \to C_V(G) \text{ given by } v \mapsto \prod_{y \in Y} v^y.$$ 

Given a subset $X$, a transversal $Y$ to $X$ in $G$, and an element $g \in G$, one sees that the map $y \mapsto yg$ is a bijection $Y \to Y$, where $yg \in Y$ is the unique element such that $yg = xyg$ for some $x \in X$. Hence the image of $\mathfrak{N}^{G}_{X,Y}$ does indeed lie in $C_V(G)$. 

Lemma A.2. Let $P$ be a finite $p$-group, let $V$ an abelian group on which $P$ acts, and let $Q$ and $R$ be subgroups of $P$. Then there exists a transversal to $Q \cdot R$ in $P$, and $\mathfrak{N}_R = \mathfrak{N}_R^{P} \mathfrak{N}_Q^{Q} \mathfrak{N}_R^{C_V(R)}$ for any such transversal $Y$.

Proof. This is a combination of Lemmas A1.1 and A1.2 in [Gla71], with the statement on norms following from Lemma A1.1(a) there and Definition A.1(b) here. □

Theorem A.3. Suppose $G$ is a finite group, $S$ is a Sylow $p$-subgroup of $G$, and $D$ is an abelian $p$-group on which $G$ acts. Let $A$ be a nonempty set of subgroups of $S$, and set $J = \langle A \rangle$. Let $H$ be a subgroup of $G$ containing $N_G(J)$, and set $V = \Omega^1(D)$. Assume that $J$ is weakly closed in $S$ with respect to $G$, and that

$$\text{(A.4)} \quad \text{whenever } A \in A, \ g \in G, \text{ and } A \not\leq H^g, \text{ then } \mathfrak{N}_{A \cap H^g} = 1 \text{ on } V,$$

or more generally,

$$\text{(A.5)} \quad \text{whenever } g \in G \text{ and } J \not\leq H^g, \text{ then } \mathfrak{N}_{J \cap H^g} = 1 \text{ on } V.$$

Then $C_D(H) = C_D(G)$.

Proof. We follow the argument from [Gla71] Theorem A1.4]. Let $H$ be a subgroup of $G$ containing $N_G(J)$. Then $S \leq H$ since $J$ is weakly closed in $S$ with respect to $G$.

In the situation of (A.5), there is $A \in A$ with $A \not\leq J \cap H^g$, since $J = \langle A \rangle$. Then $A \cap (J \cap H^g) = A \cap H^g$, and we see that (A.5) follows from (A.4) upon applying Lemma A.2 with $J$, $A$, and $J \cap H^g$ in the roles of $P$, $Q$, and $R$, respectively.

Thus, we assume (A.5) and prove $C_D(H) = C_D(G)$ by induction on the order of $D$. We may assume $D > 1$. The $p$-th power homomorphism on $D$ has kernel $V$ and image $\mathfrak{U}^1(D)$, and so $D/V \cong \mathfrak{U}^1(D)$. Since $\mathfrak{U}^1(D) < D$ and $\Omega^1(\mathfrak{U}^1(D)) \leq V$, the pair $(G, \mathfrak{U}^1(D))$ satisfies the hypotheses of the theorem in place of $(G, D)$. Thus

$$\text{(A.6)} \quad C_{D/V}(G) = C_{D/V}(H).$$

by induction.

Let $z \in C_D(H)$ and suppose first that $\langle V, z \rangle < D$. The coset $Vz$ is fixed by $H$, and so it is fixed by $G$ by (A.6). Thus, $\langle V, z \rangle$ is $G$-invariant. Apply induction with $\langle V, z \rangle$ in the role of $D$ to obtain that $z \in C_D(G)$ as required.

Next assume that $\langle V, z \rangle = D$ and $V < D$. Then $C_V(H) = C_V(G)$ by induction. Set $z' = \mathfrak{N}^G_{H^g}(z)$. Then $z' \in C_D(G) \leq C_D(H)$. Since $Vz$ is $G$-invariant, $z' \equiv z^{[G:H]}$ modulo $V$. Then as $|G : H|$ is prime to $p$, we see that $\langle V, z' \rangle = D$ and

$$z \in C_D(H) = C_V(H) \langle z' \rangle = C_V(G) \langle z' \rangle = C_D(G)$$

as required.

Finally assume that $V = D$. Given a set $[H/G/J]$ of $H-J$ double coset representatives in $G$ containing the identity, and a transversal $[J/J \cap H^g]$ to $J \cap H^g$ in $J$ for each $g \in [H/G/J]$,
then the disjoint union of $g[J/J \cap H^g]$ as $g$ ranges over $[H \setminus G/J]$ is a transversal to $H$ in $G$. Further, $\mathfrak{N}_{J/H}^I(z) = \mathfrak{N}_J^I(z) = z$. Thus, the norm map decomposes as

\[(A.7) \quad \mathfrak{N}^I_{J/H}(z) = \prod_{g \in [H \setminus G/J]} \mathfrak{N}^I_{J \cap H^g}(z^g) = z \prod_{g \in [H \setminus G/J] - \{1\}} \mathfrak{N}^I_{J \cap H^g}(z^g).\]

If $g \in [H \setminus G/J] - \{1\}$ and $J \leq H^g$, then we may choose $h \in H$ such that $J^{g^{-1}h} \leq S$. Then $g^{-1}h \in N_G(J) \leq H$, since $J$ is weakly closed in $S$ with respect to $G$, and so $HgJ = Hg^{-1}hJ = H$ yields $g = 1$ by our choice, a contradiction. Thus, $J \not\leq H^g$ for each $g \in [H \setminus G/J] - \{1\}$. We conclude that $\mathfrak{N}^I_{J \cap H^g}(z^g) = 1$ for each such $g$ from (A.5), and then $z = \mathfrak{N}^I_{J/H}(z) \in C_V(G)$ from (A.7). \[\square\]

### Appendix B. Conjugacy and conjugacy functors

We give here some elementary lemmas from finite group theory that are needed at various places in the paper. We also discuss the notion of a $\Gamma$-conjugacy functor $W$, and describe how it gives rise to the $\Gamma$-conjugation family of subgroups well-placed with respect to $W$, which is also a conjugation family for the fusion system of $\Gamma$.

**Lemma B.1** (Burnside). Let $G$ be a finite group and $S$ a Sylow $p$-subgroup of $G$. Assume that $J$ is an abelian subgroup of $S$ that is weakly closed in $S$ with respect to $G$ and that $X$ and $Y$ are subgroups of $J$. If $X$ and $Y$ are conjugate in $G$, then they are conjugate in $N_G(J)$.

**Proof.** Assume $X$ and $Y$ are conjugate in $G$, and fix $g \in G$ with $X^g = Y$. Since $\langle J, J^g \rangle \leq C_G(Y)$, we may choose $h \in C_G(Y)$ such that $\langle J^h, J^g \rangle$ is a $p$-group. Choose $g_1 \in G$ with $\langle J^{gh_1}, J^{gh_1} \rangle \leq S$. Then $J^{gh_1} = J = J^{gh_1}$ since $J$ is weakly closed in $S$ with respect to $G$. Thus, $gh^{-1} \in N_G(J)$, and $X^{gh^{-1}} = Y^{h^{-1}} = Y$ since $h^{-1}$ centralizes $Y$. \[\square\]

The following lemma gives an alternative, more elementary, argument for Lemma 5.2(a) using the norm map. One should apply it there by taking $(N_Γ(Q), N_S(Q), Q, τ)$ in the role of $(Γ, S, Y, τ)$ below.

**Lemma B.2.** Let $(Γ, S, Y)$ be a general setup for the prime $p$ and $τ$ an automorphism of $Γ$ centralizing $S$ and having order a power of $p$. Then $τ$ is conjugation by an element of $Z(S)$.

**Proof.** Set $D = Z(Y)$ for short. Denote by $Γ = Γ(τ)$ the semidirect product of $Γ$ by $⟨τ⟩$, and set $S = S(τ) = S × ⟨τ⟩$ and $D = D(τ) = D × ⟨τ⟩$. Then $Y$ and $D$ are normal in $Γ$, and $D = C_Γ(Y)$, so that $D$ is normal abelian in $Γ$.

Consider the norm $𝓡 := 𝓡_ cây : C_D(⟨S⟩) → C_δ(Γ)$, and set $n = |Γ : S|$. Since $Γ$ centralizes $D/D$ and since $n$ is prime to $p$, the restriction $𝓡|_{⟨τ⟩}$ is injective. Choose an integer $m$ such that $mn = 1 \pmod{|D|}$. Then $𝓡(τ) ≡ τ^n \pmod{D}$ and $𝓡(τ)^m ≡ τ \pmod{D}$. Let $σ = 𝓡(τ)^m$, and choose $d ∈ D$ with $σ = τd$. Then $τ ≡ d^{-1}$ modulo $Z(Γ)$ since $Γ$ centralizes $σ$. Further, since $⟨S⟩$ centralizes $τ$, we see that $d ∈ C_Γ(⟨S⟩) \cap D = Z(S)$, as desired. \[\square\]
Lemma B.3. Let \( n \geq 2 \) and let \( G \) be the symmetric group \( S_{2n+1} \). Set
\[
R_1 = \langle (1, 2), (3, 4), \ldots, (2n - 1, 2n) \rangle, \quad \text{and} \\
R_2 = \langle (1, 2), (3, 4), \ldots, (2n - 3, 2n - 2) \rangle.
\]
Then \( G \) is generated by \( N_G(R_1) \) and \( N_G(R_2) \).

Proof. Let \( H = \langle N_G(R_1), N_G(R_2) \rangle \) and \( \Omega = \{ 1, 2, \ldots, 2n + 1 \} \). Now \( N_G(R_1) \) is transitive on \( \Omega - \{ 2n + 1 \} \). Similarly, \( N_G(R_2) \) is transitive on \( \Omega - \{ 2n - 1, 2n, 2n + 1 \} \) and contains a subgroup inducing the symmetric group on \( \{ 2n - 1, 2n, 2n + 1 \} \). Therefore, \( H \) is transitive on \( \Omega \) and the stabilizer of \( 2n + 1 \) in \( H \) is transitive on \( \Omega - \{ 2n + 1 \} \). Since \( H \) contains the transposition \( (2n, 2n + 1) \) and is 2-transitive on \( \Omega \), it contains all transpositions. Hence, \( H = G \).

We next give the background on conjugacy functors and well-placed subgroups, which are used in §5 and §6.

Definition B.4. Let \( \Gamma \) be a finite group with Sylow \( p \)-subgroup \( S \). A \( \Gamma \)-conjugacy functor on \( \mathcal{S}(S) \) is a mapping \( W : \mathcal{S}(S) \to \mathcal{S}(S) \) such that for all \( P \leq S \),

(a) \( W(P) \leq P \);
(b) \( W(P) \neq 1 \) whenever \( P = 1 \); and
(c) \( W(P)^g = W(P^g) \) whenever \( g \in \Gamma \) with \( P^g \leq S \).

Lemma B.5. Let \( \Gamma \) be a finite group, \( S \) a Sylow \( p \)-subgroup of \( \Gamma \), and \( W \) a \( \Gamma \)-conjugacy functor on \( \mathcal{S}(S) \). Then for all \( P \leq S \),

(a) \( N_S(P) \leq N_S(W(P)) \);
(b) \( W(P) = W(N_S(W(P))) \) if and only if \( W(P) = W(S) \); and
(c) \( P = N_S(W(P)) \) if and only if \( P = S \).

Proof. Let \( P \leq S \) and \( T = N_S(W(P)) \). Part (a) holds by Definition B.4(c). If \( W(T) = W(P) \), then by (a), \( N_S(T) \leq N_S(W(T)) = N_S(W(P)) = T \), so that \( T = S \) and \( W(P) = W(T) = W(S) \). Now (b) holds since the converse is clear.

If \( P = T \), then again \( N_S(P) \leq T = P \), so that \( P = S \). Now (c) holds since the converse is clear.

A \( \Gamma \)-conjugacy functor \( W \) on \( \mathcal{S}(S) \) can be uniquely extended to a \( \Gamma \)-conjugacy functor \( \hat{W} \) in the sense of [Gla71, §5]: given a \( p \)-subgroup \( P \) of \( \Gamma \), choose \( g \in \Gamma \) with \( P^g \leq S \) and define \( \hat{W}(P) = W(P^g)^{g^{-1}} \). Then \( \hat{W} \) is a mapping on all \( p \)-subgroups of \( \Gamma \) which is uniquely determined by (b).

Each \( \Gamma \)-conjugacy functor \( W \) gives rise to a conjugation family via its well-placed subgroups. A conjugation family for the fusion system \( \mathcal{F} \) over \( S \) is a collection \( \mathcal{C} \) of subgroups of \( S \) such that every morphism in \( \mathcal{F} \) is a composition of restrictions of \( \mathcal{F} \)-automorphisms of the members of \( \mathcal{C} \). A conjugation family \( \mathcal{C} \) for \( S \) in \( \Gamma \) in the sense of [Gla71, §3] is itself a conjugation family for \( \mathcal{F}_S(\Gamma) \) in the above sense.
For a $\Gamma$-conjugacy functor $W$ on $\mathcal{S}(S)$ and a subgroup $P \leq S$, define $W_1(P) = P$ and, for all $i \geq 2$, define inductively $W_i(P) = W(N_S(W_{i-1}(P)))$. Then $P$ is said to be well-placed (with respect to $W$) if $W_i(P)$ is fully $\mathcal{F}_S(\Gamma)$-normalized for all $i \geq 1$.

**Theorem B.6.** Let $\Gamma$ be a finite group, $S$ a Sylow $p$-subgroup of $\Gamma$, and $W$ a $\Gamma$-conjugacy functor on $\mathcal{S}(S)$. Then every subgroup of $S$ is $\Gamma$-conjugate to a well-placed subgroup of $S$. The set of well-placed subgroups of $S$ forms a conjugation family for $\mathcal{F}_S(\Gamma)$.

**Proof.** This is a combination of Lemma 5.2 and Theorem 5.3 of [Gla71], given the above remarks. □

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