Low-frequency anomalies in dynamic localization

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Abstract
Quantum mechanical spreading of a particle hopping on tight binding lattices can be suppressed by the application of an external ac force, leading to periodic wave packet reconstruction. Such a phenomenon, referred to as dynamic localization (DL), occurs for certain ‘magic’ values of the ratio $\Gamma = F_0/\omega$ between the amplitude $F_0$ and frequency $\omega$ of the ac force. It is generally believed that in the low-frequency limit ($\omega \to 0$) DL can be achieved for an infinitesimally small value of the force $F_0$, i.e. at finite values of $\Gamma$. Such normal behavior is found in homogeneous lattices as well as in inhomogeneous lattices of the Glauber–Fock (GF) type. Here we introduce a tight-binding lattice model with inhomogeneous hopping rates, referred to as a pseudo-GF lattice, which shows DL but fails to reproduce the normal low-frequency behavior of homogeneous and GF lattices. In pseudo-GF lattices, DL can be exactly realized; however, at the DL condition the force amplitude $F_0$ remains finite as $\omega \to 0$. Such an anomalous behavior is explained in terms of a $PT$ symmetry-breaking transition of an associated two-level non-Hermitian Hamiltonian that effectively describes the dynamics of the Hermitian lattice model.

Keywords: dynamic localization, quantum diffusion and coherent lattice transport, $PT$ symmetry breaking

(Some figures may appear in colour only in the online journal)
the so-called class of GF lattices [7], which show an inhomogeneous hopping rate \(^1\). An implication of equation (1) is that, at the low frequency limit \( \omega \rightarrow 0 \), DL can be achieved for an infinitesimally small value of the forcing amplitude, i.e.

\[
\frac{F_0}{\omega} = \Gamma
\]  

(2)

remains finite as \( \omega \rightarrow 0 \). Such a condition will be referred to as the 'normal' behavior of DL in the low-frequency limit.

In this work we present an ac-driven lattice model with inhomogeneous hopping rates, referred to as a pseudo-GF lattice, which shows exact DL but fails to follow the normal behavior of equation (2). In particular, at the DL point, \( F_0/\omega \) diverges like \( 1/\omega \) as \( \omega \rightarrow 0 \). Such anomalous behavior can be explained in terms of a \( PT \) symmetry-breaking transition of an associated two-level non-Hermitian Hamiltonian that effectively describes the dynamics of the pseudo-GF lattice model.

2. Lattice models

Let us consider the hopping motion of a quantum particle on a tight-binding lattice driven by an external ac force \( F(t) \) of frequency \( \omega \). Assuming a lattice period \( a = 1 \) and \( h = 1 \), in the nearest-neighbor approximation the amplitude probability \( c_n(t) \) to find the particle at lattice site \( n \) is described by the set of coupled equations [1]

\[
\frac{d c_n}{d t} = - \kappa_n c_n - \kappa_{n+1} c_{n+1} + n F(t) c_n
\]  

(3)

where \( \kappa_n \) is the hopping rate between sites \( n \) and \( (n-1) \). The regime of DL corresponds to periodic self-reconstruction of an arbitrary initial wave packet at integer multiplies of the forcing period \( T = 2\pi/\omega \), i.e.

\[
|c_n(T)|^2 = |c_n(0)|^2
\]  

(4)

for \( l = 1, 2, 3, \ldots \) and for an arbitrary initial state \( c_0(0) \). For a homogeneous lattice \( \kappa_n = \kappa \) it is well-known that DL is attained, provided that the following condition is satisfied [1]

\[
\int_0^T d t e^{i \int_0^t d t' F(t')} = 0.
\]  

(5)

For example, for a sinusoidal force \( F(t) = F_0 \cos(\omega t) \) the condition (5) yields

\[
J_0(\Gamma) = 0
\]  

(6)

where \( \Gamma = F_0/\omega \) and \( J_0 \) is the Bessel function of the first kind and of zero order. Taking the lowest root \( \Gamma \approx 2.405 \) of the Bessel function, it follows that DL is achieved for a force amplitude \( F_0 \) given by

\[
F_0 \approx 2.405 \omega.
\]  

(7)

Such a result indicates that, in the low frequency limit \( \omega \rightarrow 0 \), DL can be achieved for an infinitesimally small value of the force amplitude \( F_0 \). The possibility of realizing DL with an infinitesimally small amplitude of the forcing in the low modulation frequency limit will be referred to as the 'normal' behavior of DL at low frequencies. Of course the normal behavior of DL holds for other modulation profiles of the force, for example for a square-wave modulation. Remarkably, in a recent work [7] it was shown that DL can also be exactly realized for certain inhomogeneous semi-infinite lattices, the class of so-called GF lattices [16, 17]. In such lattices, the hopping rate is inhomogeneous and increases with the site number \( n \) according to the relation

\[
\kappa_n = \sigma \sqrt{n},
\]  

(8)

\( n = 0, 1, 2, 3, \ldots \). Noticeably, lattices with a square-root dependence on the site index were earlier introduced in [18] for studying the equivalence between continuous and discrete Schrödinger equations for a quantum particle in a dc field by preserving the Heisenberg equations of motion. As shown in [7], in such lattices DL can be realized under the same condition equation (5) of a homogeneous lattice. Such a result indicates that the normal regime of DL in the low frequency limit found in homogeneous lattices holds for the GF lattices as well.

Let us now consider another class of semi-infinite lattices with inhomogeneous hopping rates given by

\[
\kappa_n = \sigma n.
\]  

(9)

Such a class of lattices, that will be referred to as pseudo-GF lattices, differs from the GF lattice (equation (8)) because of a linear (rather than square root) increase of the hopping rate with the site index \( n \). Such a class of inhomogeneous lattices was previously introduced in [19] and shown to admit of exact Bloch oscillations in the presence of a dc force. Here we will investigate the case of an ac force. As shown in the next sections, DL can be realized for the pseudo-GF lattices as well; however, as opposed to the homogeneous or GF lattices, they show an anomalous behavior at the low frequency limit.

3. Dynamic localization condition

An important property of both GF and pseudo-GF lattices is that the single-particle hopping dynamics on the lattices, as described by equation (3), can be mapped onto the dynamics in Fock space of certain bosonic field models. For the GF lattices, let us consider a bosonic field described by the time-periodic Hamiltonian [7, 16, 17]

\[
\hat{H}(t) = F(t) \hat{a} \hat{a} - \sigma (\hat{a} + \hat{a}^\dagger)
\]  

(10)

where \( \hat{a}^\dagger \) and \( \hat{a} \) are the creation and destruction operators of the bosonic field, satisfying the usual commutation relations \([\hat{a}, \hat{a}^\dagger] = 1\) and \([\hat{a}, \hat{a}] = 0\). If we expand the state vector \( |\psi(t)\rangle \) of the system in Fock space as

\[
|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n(t) |n\rangle
\]  

(11)

where \( |n\rangle = 1/\sqrt{n!} \hat{a}^n\hat{a}^\dagger|0\rangle \), it can be readily shown that the evolution of the amplitude probabilities \( c_n(t) \) is governed

\( \text{\footnote{For a lattice with arbitrary inhomogeneous hopping rates, DL can be only approximately obtained in the high-frequency regime under the condition (1), whereas in the low-frequency regime DL cannot be realized.}} \)
by equation (3) with $\kappa^2$ given by equation (8) (GF lattices). Similarly, let us consider two bosonic fields described by the Hamiltonian

$$\tilde{H}(t) = \frac{F(t)}{2} \left( \tilde{a}^\dagger \tilde{a} + \tilde{b}^\dagger \tilde{b} \right) - \sigma \left( \tilde{a} \tilde{b} + \tilde{b}^\dagger \tilde{a}^\dagger \right)$$

(12)

where $\tilde{a}^\dagger$, $\tilde{a}$ and $\tilde{b}^\dagger$, $\tilde{b}$ are the creation and destruction operators of the two bosonic fields. If we only consider the dynamics of the bosonic fields in the subspace of Fock space corresponding to the states $|n\rangle = (1/n!)^{1/2} \tilde{a}^{\dagger n} |0\rangle$ with the same boson number $n$ in the two fields, i.e. if we expand the state vector $|\psi(t)\rangle$ of the fields as in equation (11), it readily follows that the evolution of the amplitude probabilities $c_n(t)$ is governed by equation (3) with $\kappa^2$ given by equation (9) (pseudo-GF lattices). Hence the dynamical properties of the GF and pseudo-GF lattices can be derived from the analysis of the Hamiltonians (10) and (12), which are quadratic in the bosonic field operators. Interestingly, the conditions of DL for the two lattice models can be obtained without the need to derive the explicit forms of the propagators for the two lattice systems. To this aim, let us note that the site occupation probabilities $|c_n|^2$ can be derived from the relation

$$|c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} dq S(q, t) \exp(-iqn)$$

(13)

where the spectrum $S(q, t)$ is given by

$$S(q, t) = \sum_{n=\pm} \exp(iqn) \cdot |c_n|^2 \exp(-iqn).$$

(14)

From equations (13) and (14) it follows that the condition for DL (equation (4)) can be written as

$$S(q, t) = S(q, 0)$$

(15)

($l = 1, 2, 3, \ldots$). On the other hand, the spectrum $S(q, t)$ can be determined as the expectation value of the operator $\exp(iq \tilde{a} \tilde{a}^\dagger)$, i.e.

$$S(q, t) = \langle \psi(t) | \exp(iq \tilde{a} \tilde{a}^\dagger) | \psi(t) \rangle$$

(16)

where $|\psi(t)\rangle$ is the state vector of the bosonic fields, with $|n\rangle = (1/n!)^{1/2} \tilde{a}^{\dagger n} |0\rangle$ for the GF lattice and $|n\rangle = (1/n!)^{1/2} \tilde{a}^{\dagger n} \tilde{b} |0\rangle$ for the pseudo-GF lattice. Let us indicate by $\tilde{U}(t)$ the propagator associated with the time-periodic Hamiltonian $\tilde{H}(t)$, i.e. $i(d\tilde{U}/dt) = \tilde{H}(t) \tilde{U}(t)$ and $|\psi(t)\rangle = \tilde{U}(t) |\psi(0)\rangle$, and let us indicate by $\tilde{A}_c(t) = \tilde{U}^\dagger(t) \tilde{A} \tilde{U}(t)$ the Heisenberg operator associated with any given operator $\tilde{A}$ acting on the bosonic fields with $\tilde{A}(0) = \tilde{A}$. For a time-independent operator $\tilde{A}$, the operator $\tilde{A}_c(t)$ then satisfies the Heisenberg equation of motion

$$i \frac{d}{dt} \tilde{A}_c = \left[ \tilde{A}_c, \tilde{H}_c \right].$$

(17)

In the Heisenberg representation, the spectrum $S(q, t)$ as given by equation (16) can be computed according to the relation

$$S(q, t) = \langle \psi(0) | \exp \left\{ i q \tilde{A}_c(t) \right\} | \psi(0) \rangle$$

(18)

and hence the condition (15) for DL is realized provided that

$$\tilde{A}_c(T) = \tilde{A}_c(0) \exp(i\varphi) = \tilde{A} \exp(i\varphi)$$

(19)

where $\varphi$ is an arbitrary phase constant.

Let us first consider the DL problem for the GF lattice, which was previously studied in [7]. In this case, from equation (17) with $\tilde{A} = \tilde{a}$ one obtains

$$i \frac{d\tilde{a}(t)}{dt} = F(t) \tilde{a} - \sigma$$

(20)

which is readily solved with the initial condition $\tilde{a}(0) = \tilde{a}$, yielding

$$\tilde{a}(t) = \exp \left\{ -i \int_0^t dt' F(t') \right\} \tilde{a} + i \alpha \int_0^t dt' \exp \left\{ i \int_0^{t'} d\xi F(\xi) \right\}.$$  

(21)

Note that if condition (5) is satisfied, taking into account that

$$\int_0^T dt F(t) = 0,$$

one has $\tilde{a}(T) = \tilde{a}$. This shows that, as previously proven in [7], DL occurs in GF lattices for the same driving conditions as in the homogeneous lattices.

Let us now consider the pseudo-GF lattice. In this case, coupled equations for the operators $\tilde{a}_c, \tilde{b}_c$ (and similarly for $\tilde{b}_c, \tilde{a}_c$) can be derived from equation (17) by taking $\tilde{A}_c = \tilde{a}_c$ and $\tilde{A}_c = \tilde{b}_c$. One then obtains

$$i \frac{d}{dt} \begin{pmatrix} \tilde{a}_c \\ \tilde{b}_c \end{pmatrix} = M(t) \begin{pmatrix} \tilde{a}_c \\ \tilde{b}_c \end{pmatrix}$$

(22)

where the $2 \times 2$ matrix $M(t)$ is given by

$$M(t) = \begin{pmatrix} F(t)/2 & -\sigma \\ \sigma & -F(t)/2 \end{pmatrix}.$$  

(23)

Note that the matrix $M(t)$ is non-Hermitian, in spite of the fact that the original Hamiltonian $\tilde{H}(t)$ of the pseudo-GF lattice (equation (12)) is Hermitian. Since $M(t)$ is periodic with period $T = 2\pi/\omega$, Floquet theory applies. By indicating the monodromy matrix of the linear periodic system (22) with $U$, i.e. the propagator from $t = 0$ to $t = T$

$$\begin{pmatrix} \tilde{a}_c(T) \\ \tilde{b}_c(T) \end{pmatrix} = U \begin{pmatrix} \tilde{a}_c \\ \tilde{b}_c \end{pmatrix}.$$  

(24)

Floquet theory ensures that $U$ can be written in the form

$$U = F \exp(-iJT) F^{-1}$$

(25)

where $F$ is a non-singular $2 \times 2$ matrix and $J$ is a Jordan normal form. The elements (eigenvalues) $\mu_1$ and $\mu_2$ on the diagonal of $J$ are the Floquet exponents (quasi-energies) of the periodic system and are defined to within integer multiples of $\omega$. Here we will assume for the sake of definiteness that the real parts of $\mu_1, 2$ are inside the interval $(-\omega/2, \omega/2)$. Hence the DL condition (equation (19)) for the pseudo-GF lattice requires

$$\mu_1 = \mu_2.$$  

(26)

The quasi-energies (Floquet exponents) $\mu_1$ and $\mu_2$, and thus the DL points corresponding to the quasi-energy crossing, can
be numerically computed by solving the system (22) over one oscillation cycle using standard methods. Owing to the form of the matrix $\mathcal{M}(t)$, it turns out that $\mu_2 = -\mu_1$. Note that, since $\mathcal{M}(t)$ is non-Hermitian, the quasi-energies are generally complex numbers.

4. Low-frequency anomalies of DL in the pseudo-GF lattices

An important property of pseudo-GF lattices is that DL can never be realized at the low-frequency limit $\omega \to 0$ by a forcing amplitude $F_0$ which vanishes like $\sim \omega$, i.e. pseudo-GF lattices fail to show the normal low-frequency behavior of DL of homogeneous and GF lattices. To prove this statement, let us notice that in the low-frequency limit $\omega \to 0$ the quasi-energies $\mu_{1,2}$ can be asymptotically determined by a Wentzel-Kramers-Brillouin (WKB) analysis of equation (22), yielding

$$\mu_{1,2} = \pm \frac{1}{T} \int_0^T dt \sqrt{\frac{F^2(t)}{4} - \sigma^2}. \quad (27)$$

This expression of the quasi-energies is accurate provided that no turning points arise over the oscillation cycle. If the forcing amplitude is small enough such that $|F(t)| < 2\sigma$ over the entire oscillation cycle, there are no turning points and the quasi-energies, as given by equation (27), are manifestly imaginary and complex conjugate in relation to each other, so an energy crossing $\mu_1 = \mu_2$, which is required for DL, cannot be realized. Hence DL in pseudo-GF lattices shows an anomalous behavior as $\omega \to 0$. As an example, in figure 1 we show the numerically computed behavior of the quasi-energies versus $\Gamma = F_0/\omega$ for a sinusoidal forcing $F(t) = F_0 \cos(\omega t)$ and for decreasing values of the modulation frequency $\omega_0$, normalized to $\sigma$. The quasi-energies turn out to be imaginary, so that in the figure the imaginary parts of $\mu_1$ and $\mu_2$ are shown. DL is obtained for the normalized forcing amplitudes $\Gamma$ where the quasi-energies cross. The lowest value $\Gamma = \Gamma_0$ of level crossing is indicated by an arrow in figure 1. Note that, as $\omega_0\sigma$ becomes small (from figures 1(a) to (d)), the amplitude $\Gamma_0$ at the lowest quasi-energy crossing becomes large, indicating the anomalous low-frequency behavior of DL for the pseudo-GL lattices. Conversely, in the large frequency modulation regime, DL is attained at values of $\Gamma$ that are (as expected) close to the roots $J_d(\Gamma) = 0$ (see figure 1(a)). In figure 2 we also show, as an example, the self-imaging property of the pseudo-GF lattice in the DL regime corresponding to single site excitation of the lattice at initial time.

Failure of the normal behavior of DL in the low frequency limit can be elegantly explained on the basis of the non-Hermitian nature of the Heisenberg equations (22–23) describing the evolution of the bosonic field operators $\hat{a}_d(t)$ and $\hat{b}_d(t)$. Following [1], for a sinusoidal force $F(t) = F_0 \cos(\omega t)$ in the limit $\omega \to 0$ and for a force amplitude $F_0$ that is finite or is vanishing, but of lower order than $\omega_0$, for times $t \ll T$ the particle dynamics on the lattice can be described by replacing the ac force with the dc force $F = F_0$. At this limit, for a homogeneous lattice, it is well known that the particle undergoes Bloch oscillations with periodicity $T_B = 2\pi F_0$, with no restrictions on the smallness of the force $F_0$. Such a result holds for the GF lattice as well, as can be readily proven from equation (21). Conversely, for the pseudo-GF lattice, taking a dc force $F(t) = F_0$ from equations (22) and (23), one obtains

$$\dot{\hat{a}}_d(t) = \left\{ \cos(\lambda t) - i \frac{F_0}{2\lambda} \sin(\lambda t) \right\} \hat{a} + i \frac{\sigma}{\lambda} \sin(\lambda t) \hat{b}^\dagger \quad (28)$$

where

$$\lambda = \sqrt{(F_0/2)^2 - \sigma^2} \quad (29)$$

and $\pm i\lambda$ are the two eigenvalues of the non-Hermitian matrix $\mathcal{M}$ for $F(t) = F_0$. From equation (28) it follows that if the eigenvalues $\pm i\lambda$ are complex, i.e. for $F_0 < 2\sigma$, the self-imaging condition $\hat{a}_d(t) = \hat{a} \exp(i\sigma t)$ is not satisfied at any time $t$, whereas

\[\sqrt{\langle F(t)/2 \rangle^2 - \sigma^2} \]

Figure 1. Numerically computed behavior of the quasi-energies $\mu_{1,2}$ (imaginary parts) versus $\Gamma = F_0/\omega$ for the non-Hermitian two-level system (22) with a sinusoidal driving force $F(t) = F_0 \cos(\omega t)$ for decreasing values of $\omega_0\sigma$: (a) $\omega_0\sigma = 5$, (b) $\omega_0\sigma = 1$, (c) $\omega_0\sigma = 0.4$ and (d) $\omega_0\sigma = 0.2$. The vertical arrows in the panels show the lowest value $\Gamma = \Gamma_0$ at which level crossing, corresponding to DL, is to be found. The thin dotted curves in (c) and (d), almost overlapped with the solid ones, show the behavior of quasi-energies as predicted by the WKB analysis (equation (27)). The quasi-energies at the WKB limit are plotted for forcing amplitudes $F_0$ smaller than $2\sigma$ to avoid turning points.

2 For the homogeneous and GF lattices $\Gamma_0$ does not depend on $\omega_0$ and is equal to 2.405.
where the eigenvalues $\pm \lambda$ are real, i.e. for $F_0 > 2\sigma$, self-imaging, corresponding to Bloch oscillations, are observed with the periodicity $T_B = \pi/\lambda$, i.e.

$$T_B = \frac{\pi}{\sqrt{(F_0/2)^2 - \sigma^2}}$$  \hfill (30)

which agrees with the result previously reported in [19]. Hence in the pseudo-GF lattices self-imaging is found when the eigenvalues of the non-Hermitian matrix $M$ become real, which requires a minimum value $F_0 = 2\sigma$ of the forcing amplitude. Below such a value, i.e. for $F_0 < 2\sigma$, the eigenvalues of the matrix $M$ are complex conjugates and self-imaging cannot be realized. Hence the occurrence of self-imaging in the pseudo-GF lattice can be traced back to a $\mathcal{PT}$ symmetry breaking transition [20] at $F_0 = 2\sigma$ of the non-Hermitian system describing the evolution of the bosonic operators in the Heisenberg picture. For a dc field, the $\mathcal{PT}$ symmetry breaking transition at $F_0 = 2\sigma$ is also associated with a metal-insulator phase transition of the eigenstates of the pseudo-GF lattice Hamiltonian $\tilde{H}$, which can be calculated in a closed form as shown in [19]: while all the eigenstates of $\tilde{H}$ are extended for $F_0 < 2\sigma$ (continuous spectrum), they all become localized for $F_0 > 2\sigma$ (point spectrum).

5. Conclusions

Quantum diffusion of a particle hopping on a tight-binding lattice is known to be suppressed by application of an ac force. Such a phenomenon, referred to as dynamic localization, corresponds to periodic wave packet relocalization, induced by the ac force at certain ‘magic’ amplitudes, and is thus conceptually very different from other forms of localization, such as Anderson localization. Recent experimental observations of DL for matter and classical waves [12, 14] have renewed interest in such a rather old problem. One general belief about DL is that in the low-frequency limit suppression of quantum diffusion can be achieved for an infinitesimally small value of the force amplitude. This normal behavior occurs in homogeneous lattices [1, 2] and in other integrable inhomogeneous lattice models which are known to show exact DL, such as GF lattices [7]. In this work we have introduced a tight-binding lattice model with inhomogeneous hopping rates, referred to as a pseudo-GF lattice, in which DL is exact but shows an anomalous behavior at the low-frequency limit. In particular, at the DL point the force amplitude $F_0$ should remain finite, regardless of the smallness of the modulation frequency $\omega$. Such anomalous behavior has been explained in terms of a $\mathcal{PT}$ symmetry breaking transition [20] of an associated two-level non-Hermitian Hamiltonian which effectively describes the dynamics of the Hermitian lattice model. At the limit of a dc force, the $\mathcal{PT}$ symmetry breaking transition of the non-Hermitian two-level system is associated with a metal-insulator phase transition of the lattice eigenstates from extended to localized. Light transport in sinusoidally curved optical waveguide lattices with engineered hopping rates can provide a possible physical system for the experimental observation of low-frequency anomalies of DL [14, 15]. In such a system, the hopping rates can be tailored by a suitable control of the waveguide separation, whereas a sinusoidal ac field can be mimicked by sinusoidally curving the waveguide axis along the propagation direction; a possible scheme is the zigzag array geometry discussed in [19]. To observe the anomalous behavior of DL in the low frequency regime, one can design different sets of waveguide arrays corresponding to a decreasing value of $\omega/\sigma$; for each set, output distributions of light intensity can be measured, after one period of the oscillation cycle, for single-site input excitation at different values of the bending amplitude (see, for instance, figure 6 of the experiment discussed in [13]). The low-frequency anomaly of DL can be thus demonstrated by the increase of $I$ at the DL point when $\omega/\sigma$ is decreased. An experimentally challenging issue is the ability to precisely control the tunneling rates over several waveguides, thus realizing arrays with a sufficient number of waveguides that reproduce the square-root law of hopping rates and avoid truncation effects at the right edge. In the zigzag geometry discussed in [19], a maximum number of waveguides of $\sim 70$ is expected to be feasible, which is enough to show the increase of $I$ from $\sim 2.405$ in the high $\omega/\sigma$ regime (see figure 1(a)) to $\sim 3.353$ at $\omega/\sigma = 1$ (see figures 1(b) and 2).

Our results provide novel insights into the phenomenon of dynamic localization and show a noteworthy example where a symmetry-breaking transition in a non-Hermitian model associated with a Hermitian system can reveal a qualitative change of the energy spectrum and particle dynamics.  

3 Another interesting example, where the Heisenberg equations of a Hermitian quantum system are non-Hermitian and the associated exceptional points for a qualitative change of the behavior of the particle motion, has been recently published for a single particle in a harmonic trap with time-dependent frequency (see [21]).
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