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BURKHOLDER’S SUBMARTINGALES FROM A STOCHASTIC CALCULUS PERSPECTIVE

Giovanni PECCATI∗ and Marc YOR†

May 24, 2007

Abstract

We provide a simple proof, as well as several generalizations, of a recent result by Davis and Suh, characterizing a class of continuous submartingales and supermartingales that can be expressed in terms of a squared Brownian motion and of some appropriate powers of its maximum. Our techniques involve elementary stochastic calculus, as well as the Doob-Meyer decomposition of continuous submartingales. These results can be used to obtain an explicit expression of the constants appearing in the Burkholder-Davis-Gundy inequalities. A connection with some balayage formulae is also established.

Key Words: Balayage; Burkholder-Davis-Gundy inequalities; Continuous Submartingales; Doob-Meyer decomposition

AMS 2000 classification: 60G15, 60G44

1 Introduction

Let \( W = \{W_t : t \geq 0\} \) be a standard Brownian motion initialized at zero, set \( W^*_t = \max_{s \leq t} |W_s| \) and write \( \mathcal{F}^W_t = \sigma \{W_u : u \leq t\}, t \geq 0 \). In [3], Davis and Suh proved the following result.

Theorem 1 ([3, Th. 1.1]) For every \( p > 0 \) and every \( c \in \mathbb{R} \), set

\[
Y_t = Y_t(c, p) = (W^*_t)^{p-2} [W^2_t - t] + c (W^*_t)^p, \quad t > 0, \tag{1}
\]

1. For every \( p \in (0, 2] \), the process \( Y_t \) is a \( \mathcal{F}^W_t \)-submartingale if, and only if, \( c \geq \frac{2-p}{p} \).
2. For every \( p \in [2, +\infty) \), the process \( Y_t \) is a \( \mathcal{F}^W_t \)-supermartingale if, and only if, \( c \leq \frac{2-p}{p} \).

As pointed out in [3, p. 314] and in Section 4 below, part 1 of Theorem 1 can be used to derive explicit expressions of the constants appearing in the Burkholder-Davis-Gundy (BDG) inequalities (see [5], or [5, Ch. IV, §4]). The proof of Theorem 1 given in [3] uses several delicate estimates related to a class of Brownian hitting times: such an approach can be seen as a ramification of the discrete-time techniques developed in [2]. In particular, in [3] it is observed that the submartingale (or supermartingale) characterization of \( Y_t(c, p) \) basically relies on the properties of the random subset of \( [0, +\infty) \) composed of the instants \( t \) where \( |W_t| = W^*_t \). The aim of this note is to bring this last connection into further light, by providing an elementary proof of Theorem 1, based on a direct application of Itô formula and on an appropriate version of the Doob-Meyer decomposition of submartingales. We will see that our techniques lead naturally to some substantial generalizations (see Theorem 4 below).

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2 A general result

Throughout this section, $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$ stands for a filtration satisfying the usual conditions. We will write $X = \{X_t : t \geq 0\}$ to indicate a continuous $\mathcal{F}_t$-submartingale issued from zero and such that $\mathbb{P} \{X_t \geq 0, \forall t\} = 1$. We will suppose that the Doob-Meyer decomposition of $X$ (see for instance [4, Th. 1.4.14]) is of the type $X_t = M_t + A_t$, $t \geq 0$, where $M$ is a square-integrable continuous $\mathcal{F}_t$-martingale issued from zero, and $A$ is an increasing (integrable) natural process. We assume that $A_0 = M_0 = 0$; the symbol $\langle M \rangle = \{\langle M \rangle_t : t \geq 0\}$ stands for the quadratic variation of $M$. We note $X^*_t = \max_{s \leq t} X_s$, and we also suppose that $\mathbb{P} \{X^*_t > 0\} = 1$ for every $t > 0$. The following result is an extension of Theorem 1.

**Theorem 2** Fix $\varepsilon > 0$.

1. Suppose that the function $\phi : (0, +\infty) \mapsto \mathbb{R}$ is of class $C^1$, non-increasing, and such that

$$
\mathbb{E} \left[ \int_\varepsilon^T \phi(X^*_s)^2 \, d\langle M \rangle_s \right] < +\infty,
$$

for every $T > \varepsilon$. For every $x \geq z > 0$, we set

$$
\Phi(x, z) = -\int_z^x y \phi'(y) \, dy;
$$

then, for every $\alpha \geq 1$ the process

$$
Z_{\varepsilon}(\phi, \alpha; t) = \phi(X^*_t) (X_t - A_t) + \alpha \Phi(X^*_t, X^*_s), \quad t \geq \varepsilon,
$$

is a $\mathcal{F}_t$-submartingale on $[\varepsilon, +\infty)$.

2. Suppose that the function $\phi : (0, +\infty) \mapsto \mathbb{R}$ is of class $C^1$, non-decreasing and such that (3) holds for every $T > \varepsilon$. Define $\Phi(\cdot, \cdot)$ according to (3), and $Z_{\varepsilon}(\phi, \alpha; t)$ according to (4). Then, for every $\alpha \geq 1$ the process $Z_{\varepsilon}(\phi, \alpha; t)$ is a $\mathcal{F}_t$-supermartingale on $[\varepsilon, +\infty)$.

**Remarks.** (i) Note that the function $\phi(y)$ (and $\phi'(y)$) need not be defined at $y = 0$.

(ii) In Section 3, where we will focus on the Brownian setting, we will exhibit specific examples where the condition $\alpha \geq 1$ is necessary and sufficient to have that the process $Z_{\varepsilon}(\alpha, \phi; t)$ is a submartingale (when $\phi$ is non-decreasing) or a supermartingale (when $\phi$ is non-increasing).

**Proof of Theorem 2.** (Proof of Point 1.) Observe first that, since $M_t = X_t - A_t$ is a continuous martingale, $X^*$ is non-decreasing and $\phi$ is differentiable, then a standard application of Itô formula gives that

$$
\phi(X^*_t) (X_t - A_t) = \phi(X^*_t) M_t - \phi(X^*_s) M_s
$$

$$
= \int_\varepsilon^t \phi(X^*_s) dM_s + \int_\varepsilon^t (X_s - A_s) \phi'(X^*_s) \, dX^*_s.
$$
The assumptions in the statement imply that the application \( \tilde{M}_{\varepsilon, t} := \int_{\varepsilon}^{t} \phi(X_{s}^{*})dM_{s} \) is a continuous square integrable \( F_{t} \)-martingale on \([\varepsilon, +\infty)\). Moreover, the continuity of \( \tilde{X} \) implies that the support of the random measure \( dX_{t}^{*} \) (on \([0, +\infty)\)) is contained in the (random) set \( \{ t \geq 0 : X_{t} = X_{t}^{*} \} \), thus yielding that

\[
\int_{\varepsilon}^{t} (X_{s} - A_{s}) \phi' (X_{s}^{*}) dX_{s}^{*} = \int_{\varepsilon}^{t} (X_{s}^{*} - A_{s}) \phi' (X_{s}^{*}) dX_{s}^{*} = - \int_{\varepsilon}^{t} A_{s} \phi' (X_{s}^{*}) dX_{s}^{*} - \Phi (X_{t}^{*}, X_{t}^{*}) ,
\]

where \( \Phi \) is defined in (3). As a consequence,

\[
Z_{\varepsilon} = \Phi (X_{t}^{*}, X_{t}^{*}) = \tilde{M}_{\varepsilon, t} + \int_{\varepsilon}^{t} (-A_{s} \phi' (X_{s}^{*}))dX_{s}^{*} + (\alpha - 1) \Phi (X_{t}^{*}, X_{t}^{*}) .
\]

Now observe that the application \( t \mapsto \Phi (X_{t}^{*}, X_{t}^{*}) \) is non-decreasing (a.s.-\( \mathbb{P} \)), and also that, by assumption, \(-A_{s} \phi' (X_{s}^{*}) \geq 0 \) for every \( s > 0 \). This entails immediately that \( Z_{\varepsilon} (\phi, \alpha; t) \) is a \( F_{t} \)-submartingale for every \( \alpha \geq 1 \).

(Proof of Point 2.) By using exactly the same line of reasoning as in the proof of Point 1., we obtain that

\[
Z_{\varepsilon} (\phi, \alpha; t) = \int_{\varepsilon}^{t} \phi (X_{s}^{*})dM_{s} + \int_{\varepsilon}^{t} (-A_{s} \phi' (X_{s}^{*}))dX_{s}^{*} + (\alpha - 1) \Phi (X_{t}^{*}, X_{t}^{*}) .
\]

Since (3) is in order, we deduce that \( t \mapsto \int_{0}^{t} \phi(X_{s}^{*})dM_{s} \) is a continuous (square-integrable) \( F_{t} \)-martingale on \([\varepsilon, +\infty)\). Moreover, \(-A_{s} \phi' (X_{s}^{*}) \leq 0 \) for every \( s > 0 \), and we also have that \( t \mapsto \Phi (X_{t}^{*}, X_{t}^{*}) \) is a.s. decreasing. This implies that \( Z_{\varepsilon} (\phi, \alpha; t) \) is a \( F_{t} \)-supermartingale for every \( \alpha \geq 1 \).

The next result allows to characterize the nature of the process \( Z \) appearing in (3), on the whole positive axis. Its proof can be immediately deduced from formulae (8) (for Part 1) and (9) (for Part 2).

**Proposition 3** Let the assumptions and notation of this section prevail.

1. Consider a decreasing function \( \phi : (0, +\infty) \rightarrow \mathbb{R} \) verifying the assumptions of Part 1 of Theorem 2 and such that

\[
\Phi (x, 0) := - \int_{0}^{x} y \phi' (y) dy \text{ is finite } \forall x > 0.
\]

Assume moreover that

\[
\mathbb{E} \left[ \int_{0}^{T} \phi (X_{s}^{*})^{2} d \langle M \rangle_{s} \right] < +\infty,
\]

and also

\[
\phi (X_{s}^{*}) M_{t} = \phi (X_{s}^{*}) (X_{t} - A_{t}) \text{ converges to zero in } L^{1} (\mathbb{P}) , \text{ as } \varepsilon \downarrow 0 ,
\]

\[
\Phi (X_{t}^{*}, 0) \in L^{1} (\mathbb{P}) .
\]

Then, for every \( \alpha \geq 1 \) the process

\[
Z (\phi, \alpha; t) = \begin{cases} 
0 & \text{for } t = 0 \\
\phi (X_{t}^{*}) (X_{t} - A_{t}) + \alpha \Phi (X_{t}^{*}, 0) & \text{for } t > 0
\end{cases}
\]

is a \( F_{t} \)-submartingale.

2. Consider an increasing function \( \phi : (0, +\infty) \rightarrow \mathbb{R} \) as in Part 2 of Theorem 2 and such that assumptions (3)–(7) are satisfied. Then, for every \( \alpha \geq 1 \) the process \( Z (\phi, \alpha; t) \) appearing in (7) is a \( F_{t} \)-supermartingale.
Remarks. (i) A direct application of the Cauchy-Schwarz inequality shows that a sufficient condition to have $(10)$ is the following:

$$\lim_{\varepsilon \to 0} \mathbb{E} \left[ \phi \left( X^*_\varepsilon \right)^2 \right] \times \mathbb{E} \left[ M^2 \right] = \lim_{\varepsilon \to 0} \mathbb{E} \left[ \phi \left( X^*_\varepsilon \right)^2 \right] \times \mathbb{E} \left[ \langle M \rangle \varepsilon \right] = 0 \quad (13)$$

(observe that $\lim_{\varepsilon \to 0} \mathbb{E} \left[ M^2 \right] = 0$, since $M_0 = 0$ by assumption). In other words, when $(13)$ is verified the quantity $\mathbb{E} \left[ M^2 \right]$ ‘takes care’ of the possible explosion of $\varepsilon \to \mathbb{E} \left[ \phi \left( X^*_\varepsilon \right)^2 \right]$ near zero.

(ii) Let $\phi$ be non-increasing or non-decreasing on $(0, +\infty)$, and suppose that $\phi$ satisfies the assumptions of Theorem 3 and Proposition 3. Then, the process $t \mapsto \int_0^t \phi(X^*_s)dM_s$ is a continuous square-integrable $\mathcal{F}^W_t$-martingale. Moreover, for any choice of $\alpha \in \mathbb{R}$, the process $Z (\phi, \alpha; t)$, $t \geq 0$, defined in $(12)$ is a semimartingale, with canonical decomposition given by

$$Z (\phi, \alpha; t) = \int_0^t \phi(X^*_s)dM_s + \int_0^t ((\alpha - 1)X^*_s - A_s) \phi' \left( X^*_s \right) dX^*_s.$$

3 A generalization of Theorem 1

The forthcoming Theorem 4 is a generalization of Theorem 1. Recall the notation: $W$ is a standard Brownian motion issued from zero, $W^*_t = \max_{s \leq t} |W_s|$ and $\mathcal{F}^W_t = \sigma \{ W_u : u \leq t \}$. We also set for every $m \geq 1$, every $p > 0$ and every $c \in \mathbb{R}$:

$$J_t \equiv J_t (m, c, p) = (W^*_t)^{p-m} \left[ |W_t|^m - A_{m,t} \right] + c (W^*_t)^p, \quad t > 0, \quad J_0 (m, c, p) = J_0 = 0, \quad (14)$$

where $t \mapsto A_{m,t}$ is the increasing natural process in the Doob-Meyer decomposition of the $\mathcal{F}^W_t$-submartingale $t \mapsto |W_t|^m$. Of course, $J_t (2, c, p) = Y_t (c, p)$, as defined in $(1)$.

Theorem 4 Under the above notation:

1. For every $p \in (0, m]$, the process $J_t$ is a $\mathcal{F}^W_t$-submartingale if, and only if, $c \geq \frac{m-p}{p}$.

2. For every $p \in [m, +\infty)$, the process $J_t$ is a $\mathcal{F}^W_t$-supermartingale if, and only if, $c \leq \frac{m-p}{p}$.

Proof. Recall first the following two facts: (i) $W^*_t \overset{law}{=} \sqrt{t}W^*_1$ (by scaling), and (ii) there exists $\eta > 0$ such that $\mathbb{E} \left[ \exp (\eta (W^*_1)^{-2}) \right] < +\infty$ (this can be deduced e.g. from [2, Ch. II, Exercice 3.10]), so that the random variable $(W^*_1)^{-1}$ has finite moments of all orders. Note also that the conclusions of both Point 1 and Point 2 are trivial in the case where $p = m$. In the rest of the proof we will therefore assume that $p \neq m$.

To prove Point 1, we shall apply Theorem 3 and Proposition 1 in the following framework: $X_t = |W_t|^m$ and $\phi (x) = x^{-\frac{m}{p}} = x^{-\frac{m}{m-p}}$. In this case, the martingale $M_t = |W_t|^m - A_{m,t}$ is such that $\langle M \rangle_t = m^2 \int_0^t W^{2m-2}_s ds$, $t \geq 0$, and $\Phi (x, z) = - \int_x^z y \phi' (y) dy = - \left( \frac{z}{m} - 1 \right) \int_x^z y^{-\frac{m}{m-p}} dy = \frac{m-p}{m} \left( z^{\frac{m}{m-p}} - z^{-\frac{m}{m-p}} \right)$. Also, for every $T > \varepsilon > 0$

$$\mathbb{E} \left[ \int_\varepsilon^T \phi (X_s^*)^2 d\langle M \rangle_s \right] = m^2 \mathbb{E} \left[ \int_\varepsilon^T (W^*_s)^{2p-2m} W_s^{2m-2} ds \right] \leq m^2 \mathbb{E} \left[ \int_\varepsilon^T (W^*_s)^{2p-2} ds \right] = m^2 \mathbb{E} \left[ (W^*_1)^{2p-2} \right] \int_\varepsilon^T s^{-\frac{m}{m-p}} ds, \quad (15)$$


so that $\phi$ verifies (2) and (5). Relations (8) and (11) are trivially satisfied. To see that (10) holds, use the relations
\[
\mathbb{E} \left\{ |\phi (X^*_c) (X^* - A^*_c)| \right\} = \mathbb{E} \left\{ |(W^*_c)^{p-m} |W^*_m - A^*_m,| \right\} \\
= \mathbb{E} \left\{ |(W^*_c)^{p-m} M_c| \right\} \leq \mathbb{E} \left\{ (W^*_c)^{2p-2m} \right\}^{1/2} \mathbb{E} \left\{ \langle M \rangle_c \right\}^{1/2} \\
= m \mathbb{E} \left\{ W^{2m-2}_c \right\}^{1/2} \mathbb{E} \left\{ (W^*_c)^{2p-2m} \right\}^{1/2} \epsilon^m \frac{1}{p} \left( \int_0^\varepsilon s^{m-1} ds \right)^{1/2} \\
\rightarrow 0, \text{ as } \varepsilon \downarrow 0.
\]
From Point 1 of Proposition 3, we therefore deduce that the process $Z(t)$ defined as $Z(0) = 0$ and, for $t > 0$,
\[
Z(t) = \phi ((W^*_c)^m |W^*_m - A^*_m,| + \alpha \Phi ((W^*_c)^m, 0) \\
= (W^*_c)^{p-m} |W^*_m - A^*_m,| + \alpha \frac{m-p}{p} (W^*_c)^p,
\]
is a $\mathcal{F}_t^W$-submartingale for every $\alpha \geq 1$. By writing $c = \alpha \frac{m-p}{p}$ in the previous expression, and by using the fact that $\frac{m-p}{p} \geq 0$ by assumption, we deduce immediately that $J_t (m, c, p)$ is a submartingale for every $c \geq \frac{m-p}{p}$. Now suppose $c < \frac{m-p}{p}$. One can use formulae (3), (6) and (7) to prove that
\[
J_t (m, c, p) = \int_0^t \phi (X^*_c) dM_s + \int_0^t [-A_{m,s} \phi' ((W^*_c)^m)] d(W^*_c)^m + (\alpha - 1) \Phi ((W^*_c)^m, 0) \\
= \int_0^t (W^*_c)^{p-m} dM_s \\
+ \left( \frac{p}{m} - 1 \right) \int_0^t [(1 - \alpha) (W^*_c)^m - A_{m,s}] (W^*_c)^{p-2m} d(W^*_c)^m,
\]
where $1 - \alpha = 1 - pc/(m-p) > 0$. Note that $\int_0^t (W^*_c)^{p-m} dM_s$ is a square-integrable martingale, due to (13). To conclude that, in this case, $J_t (m, c, p)$ cannot be a submartingale (nor a supermartingale), it is sufficient to observe that (for every $m \geq 1$ and every $\alpha < 1$) the paths of the finite variation process
\[
t \mapsto \int_0^t [(1 - \alpha) (W^*_c)^m - A_{m,s}] (W^*_c)^{p-2m} d(W^*_c)^m
\]
are neither non-decreasing nor non-increasing, with $\mathbb{P}$-probability one.

To prove Point 2, one can argue in exactly the same way, and use Point 2 of Proposition 3 to obtain that the process $Z(t)$ defined as $Z(0) = 0$ and, for $t > 0$,
\[
Z(t) = (W^*_c)^{p-m} |W^*_m - A^*_m,| + \alpha \frac{m-p}{p} (W^*_c)^p
\]
is a $\mathcal{F}_t^W$-supermartingale for every $\alpha \geq 1$. By writing once again $c = \alpha \frac{m-p}{p}$ in the previous expression, and since $\frac{m-p}{p} \leq 0$, we immediately deduce that $J_t (m, c, p)$ is a supermartingale for every $c \leq \frac{m-p}{p}$. One can show that $J_t (m, c, p)$ cannot be a supermartingale, whenever $c > \frac{m-p}{p}$, by using arguments analogous to those displayed in the last part of the proof of Point 1.

The following result is obtained by specializing Theorem 4 to the case $m = 1$ (via Tanaka’s formula).

**Corollary 5** Denote by $\{ \ell_t : t \geq 0 \}$ the local time at zero of the Brownian motion $W$. Then, the process
\[
J_t (p) = (W^*_c)^{p-1} |W^*_m - \ell_t| + c (W^*_c)^p, \quad t > 0, \\
J_0 = 0,
\]
is such that: (i) for $p \in (0,1)$, $J_t (p)$ is a $\mathcal{F}_t^W$-submartingale if, and only if, $c \geq 1/p - 1$, and (ii) for $p \in [1, +\infty)$, $J_t (p)$ is a $\mathcal{F}_t^W$-supermartingale if, and only if, $c \leq 1/p - 1$. 

5
4 Burkholder-Davis-Gundy (BDG) inequalities

We reproduce an argument taken from [3, p. 314], showing that the first part of Theorem 3 can be used to obtain a strong version of the BDG inequalities (see e.g. [3, Ch. IV, §4]).

Fix \( p \in (0, 2) \) and define \( c = (2 - p)/p = 2/p - 1 \). Since, according to the first part of Theorem 3, \( Y_t = Y_t(c, p) \) is a \( \mathcal{F}_t^W \)-submartingale starting from zero, we deduce that, for every bounded and strictly positive \( \mathcal{F}_t^W \)-stopping time \( \tau \), one has \( \mathbb{E}(Y_\tau) \geq 0 \). In particular, this yields

\[
\mathbb{E} \left( \frac{\tau}{(W_\tau^*)^{2-p}} \right) \leq \frac{2}{p} \mathbb{E}((W_\tau^*)^p).
\]  

(18)

Formula [18], combined with an appropriate use of Hölder’s inequality, entails finally that, for \( 0 < p < 2 \),

\[
\mathbb{E} \left( \tau^{\frac{p}{2}} \right) \leq \left[ \frac{2}{p} \mathbb{E}((W_\tau^*)^p) \right]^{\frac{p}{2}} \mathbb{E}((W_\tau^*)^p) = \left[ \frac{2}{p} \right]^{\frac{p}{2}} \mathbb{E}((W_\tau^*)^p).
\]  

(19)

Of course, relation (19) extends to general stopping times \( \tau \) (not necessarily bounded) by monotone convergence (via the increasing sequence \( \{ \tau \wedge n : n \geq 1 \} \)).

Remark. Let \( \{ \mathcal{A}_n : n \geq 0 \} \) be a discrete filtration of the reference \( \sigma \)-field \( \mathcal{A} \), and consider a \( \mathcal{A}_n \)-adapted sequence of measurable random elements \( \{ f_n : n \geq 0 \} \) with values in a Banach space \( \mathbf{B} \). We assume that \( f_n \) is a martingale, i.e. that, for every \( n \), \( \mathbb{E}[f_n - f_{n-1} | \mathcal{A}_{n-1}] = \mathbb{E}[d_n | \mathcal{A}_{n-1}] = 0 \), where \( d_n := f_n - f_{n-1} \). We note

\[
S_n(f) = \sqrt{\sum_{k=0}^{n} |d_k|^2} \quad \text{and} \quad f_n^* = \sup_{0 \leq m \leq n} |f_m|,
\]

and write \( S(f) \) and \( f^* \), respectively, to indicate the pointwise limits of \( S_n(f) \) and \( f_n^* \), as \( n \to +\infty \). In [2], D.L. Burkholder proved that

\[
\mathbb{E}(S(f)) \leq \sqrt{3} \mathbb{E}(f^*),
\]  

(20)

where \( \sqrt{3} \) is the best possible constant, in the sense that for every \( \eta \in (0, \sqrt{3}) \) there exists a Banach space-valued martingale \( f(\eta) \) such that \( \mathbb{E}(S(f(\eta))) > \eta \mathbb{E}(f(\eta)) \). As observed in [3], Burkholder’s inequality (20) should be compared with (19) for \( p = 1 \), which yields the relation \( \mathbb{E}(\tau^{1/2}) \leq \sqrt{2} \mathbb{E}(W_\tau^*) \) for every stopping time \( \tau \). This shows that in such a framework, involving uniquely continuous martingales, the constant \( \sqrt{3} \) is no longer optimal.

5 Balayage

We reproduce an argument taken from [3, p. 314], showing that the first part of Theorem 3 can be used to obtain a strong version of the BDG inequalities (see e.g. [3, Ch. IV, §4]).

Fix \( p \in (0, 2) \) and define \( c = (2 - p)/p = 2/p - 1 \). Since, according to the first part of Theorem 3, \( Y_t = Y_t(c, p) \) is a \( \mathcal{F}_t^W \)-submartingale starting from zero, we deduce that, for every bounded and strictly positive \( \mathcal{F}_t^W \)-stopping time \( \tau \), one has \( \mathbb{E}(Y_\tau) \geq 0 \). In particular, this yields

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\]  

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\]  

(19)

Of course, relation (19) extends to general stopping times \( \tau \) (not necessarily bounded) by monotone convergence (via the increasing sequence \( \{ \tau \wedge n : n \geq 1 \} \)).

Remark. Let \( \{ \mathcal{A}_n : n \geq 0 \} \) be a discrete filtration of the reference \( \sigma \)-field \( \mathcal{A} \), and consider a \( \mathcal{A}_n \)-adapted sequence of measurable random elements \( \{ f_n : n \geq 0 \} \) with values in a Banach space \( \mathbf{B} \). We assume that \( f_n \) is a martingale, i.e. that, for every \( n \), \( \mathbb{E}[f_n - f_{n-1} | \mathcal{A}_{n-1}] = \mathbb{E}[d_n | \mathcal{A}_{n-1}] = 0 \), where \( d_n := f_n - f_{n-1} \). We note

\[
S_n(f) = \sqrt{\sum_{k=0}^{n} |d_k|^2} \quad \text{and} \quad f_n^* = \sup_{0 \leq m \leq n} |f_m|,
\]

and write \( S(f) \) and \( f^* \), respectively, to indicate the pointwise limits of \( S_n(f) \) and \( f_n^* \), as \( n \to +\infty \). In [2], D.L. Burkholder proved that

\[
\mathbb{E}(S(f)) \leq \sqrt{3} \mathbb{E}(f^*),
\]  

(20)

where \( \sqrt{3} \) is the best possible constant, in the sense that for every \( \eta \in (0, \sqrt{3}) \) there exists a Banach space-valued martingale \( f(\eta) \) such that \( \mathbb{E}(S(f(\eta))) > \eta \mathbb{E}(f(\eta)) \). As observed in [3], Burkholder’s inequality (20) should be compared with (19) for \( p = 1 \), which yields the relation \( \mathbb{E}(\tau^{1/2}) \leq \sqrt{2} \mathbb{E}(W_\tau^*) \) for every stopping time \( \tau \). This shows that in such a framework, involving uniquely continuous martingales, the constant \( \sqrt{3} \) is no longer optimal.

5 Balayage

Keep the assumptions and notation of Section 3 and Theorem 3, fix \( \varepsilon > 0 \) and consider a finite variation function \( \psi : (0, +\infty) \to \mathbb{R} \). In this section we focus on the formula

\[
\psi(X_\tau^*) (X_t - A_t) - \psi(X_\tau^*) (X_\varepsilon - A_\varepsilon) = \int_{\varepsilon}^{\tau} \psi(X_s^*) d(X_s - A_s) + \int_{\varepsilon}^{\tau} (X_s^* - A_s) d\psi(X_s^*),
\]  

(21)

where \( \varepsilon > 0 \). Note that by choosing \( \psi = \phi \) in (21), where \( \phi \in C^1 \) is monotone, one recovers formula (3), which was crucial in the proof Theorem 2. We shall now show that (21) can be obtained by means of the balayage formulae proved in [3].

To see this, let \( U = \{ U_t : t \geq 0 \} \) be a continuous \( \mathcal{F}_t \)-semimartingale issued from zero. For every \( t > 0 \) we define the random time

\[
\sigma(t) = \sup \{ s < t : U_s = 0 \}.
\]  

(22)

The following result is a particular case of [3, Th. 1].
Proposition 6 (Balayage Formula) Consider a stochastic process \( \{K_t : t \geq 0\} \) such that the restriction \( \{K_t : t \geq \varepsilon\} \) is locally bounded and \( \mathcal{F}_t \)-predictable on \([\varepsilon, +\infty)\) for every \( \varepsilon > 0 \). Then, for every fixed \( \varepsilon > 0 \), the process \( K_{\sigma(t)} : t \geq \varepsilon \), is locally bounded and \( \mathcal{F}_t \)-predictable, and moreover

\[
U_t K_{\sigma(t)} = U_{\varepsilon} K_{\sigma(\varepsilon)} + \int_{\varepsilon}^{t} K_{\sigma(s)} dU_s.
\] (23)

To see how (21) can be recovered from (23), set \( U_t = X_t - X^*_t \) and \( K_t = \psi(X^*_t) \). Then, \( K_t = K_{\sigma(t)} = \psi(X^*_\sigma(t)) \) by construction, where \( \sigma(t) \) is defined as in (22). As a consequence, (23) gives

\[
\psi(X^*_t)(X_t - X^*_t) = \psi(X^*_\varepsilon)(X_\varepsilon - X^*_\varepsilon) + \int_{\varepsilon}^{t} \psi(X^*_s)d(X_s - X^*_s).
\]

Finally, a standard integration by parts applied to \( \psi(X^*_t)(X_t - A_t) \) yields

\[
\psi(X^*_t)(X_t - A_t) = \psi(X^*_t)(X_t - X^*_t) + \psi(X^*_t)(X^*_t - A_t)
\]

\[
= \psi(X^*_\varepsilon)(X_\varepsilon - X^*_\varepsilon) + \int_{\varepsilon}^{t} \psi(X^*_s)d(X_s - X^*_s)
\]

\[
+ \psi(X^*_\varepsilon)(X^*_\varepsilon - A_\varepsilon) + \int_{\varepsilon}^{t} \psi(X^*_s)d(X^*_s - A_s)
\]

\[
+ \int_{\varepsilon}^{t} (X^*_s - A_s)d\psi(X^*_s),
\]

which is equivalent to (21).

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