1. Introduction

One of the most important integrable equations of mathematical physics is the sine-Gordon equation (SG)

\[ u_{tt} - u_{xx} + \sin u = 0. \] (1.1)

It appeared in the XIX century in the theory of constant negative curvature surfaces in \( \mathbb{R}^3 \) in the light-cone representation

\[ u_{\xi\eta} = 4 \sin u. \]

Already in the XIX century it was shown, that the SG equation has some remarkable properties, in particular, sufficiently many exact solutions were constructed using the so-called Bäcklund transformations. This equation naturally arose in many areas of mathematics and physics.

The modern approach to the sine-Gordon equation is based on representation in the form of compatibility conditions for a pair of auxiliary linear problems. It was first found in \([1]\) by Ablowitz, Kaup, Newell and Segur. The \( \theta \)-functional formulas for finite-gap SG solutions were obtained by Its, Kozel and Kotlyarov in \([9], [10]\), the reality constraints on the spectral curves were also found in these papers. The reality constraints on the divisor turned out to be rather nontrivial, and they were found by Cherednik \([2]\). In \([2]\) it was also shown, that all real finite-gap SG solutions are automatically non-singular. Divisors, generating real solutions are called admissible. For a fixed spectral curve the number of connected components for the variety of admissible divisors may contain more that one connected components, the number of these components was also calculated in \([2]\). A characterization of these components in terms of the Jacobi variety of the spectral curve was obtained by Dubrovin and Natazon \([3]\) and Ercolani and Forest \([5]\).

It is natural to call SG solutions periodic in \( x \) with period \( T \) if the quantity \( e^{iu} \) is periodic with period \( T \) in the real variable \( x \in \mathbb{R} \). This definition does not assume that the function

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$u(x,t)$ is periodic in $x$, but

$$u(x+T,t) = u(x,t) + 2\pi n, \quad n \in \mathbb{Z}.$$  

The quantity $n$ is called the **topological charge** of $u$.

For generic real finite-gap SG solutions the function $e^{in(x,t)}$ is quasiperiodic in $x, t$, but $u(x,t)$ is not quasiperiodic. It is easy to show, that for real finite-gap SG solutions constructed by non-degenerate spectral curves the **density of topological charge**

$$\bar{n} = \lim_{T \to \infty} \frac{u(x+T,t) - u(x,t)}{2\pi T}$$

is well-defined. This density is the most basic and useful characteristic of real solutions. In particular it is a conservation law for the SG hierarchy “surviving” generic perturbations of the type

$$u_{tt} - u_{xx} + \sin u = \varepsilon F(\sin u, \cos u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots)$$

The problem of calculating the topological charge in terms of the finite-gap spectral data was first raised and partially solved in 1982 by Dubrovin and Novikov [4]. In [11] Novikov pointed out, that the formulas of [4] are meaningful only if the spectral curve is assumed to be sufficiently close to a degenerate one. A complete solution was obtained by Grinevich and Novikov only in 2001 [7], [8].

The calculation of topological charge density in [4], [11], [7], [8] was based on the so-called algebro-topological approach, and explicit $\theta$-functional formulas for $u(x,t)$ were not essentially used. Therefore a paradoxical situation arose: we have explicit $\theta$-functional formulae for the solution, but they do not help to answer the most basic questions about the solution. As pointed out by S.P. Novikov, if the $\theta$-functional form of solutions is to be considered as an effective one, then it should be possible to calculate $\bar{n}$ directly from the $\theta$-functional formulae. However, no way to achieve this goal was found until now.

In the present article, we suggest a new approach for studying the topological characteristics of real finite-gap SG solutions. It is based on the so-called **multiscale** or **elliptic limit**. In this limit the spectral curve $\Gamma$ is deformed to a singular nodal curve having elliptic curves as components (and possibly an additional hyperelliptic component). In particular, we demonstrate, that this approach allows us to extract the density of topological charge directly from the $\theta$-functional formulae.

The authors would like to express their gratitude to Professor Novikov for attracting their attention to this problem and stimulating discussions.

2. **Complex Finite-gap Sine-Gordon solutions**

We recall the construction of finite-gap SG solutions ([3], [7], [10]). The spectral data is a pair $(\Gamma, D)$, where $\Gamma$ is a nonsingular hyperelliptic curve with branching points $\{0, \infty\}$ and $\{E_1, E_2, \ldots, E_{2g}\}$,

$$(2.1) \quad \Gamma : \mu^2 = \lambda \prod_{i=1}^{2g} (\lambda - E_i) \quad \text{with } E_1, \ldots, E_{2g} \text{ distinct nonzero complex numbers}$$

and $D = \gamma_1 + \gamma_2 + \cdots + \gamma_g$ is a divisor of points: with $\gamma_1, \cdots, \gamma_g \in \Gamma \setminus \{0, \infty\}$. For sufficiently small $(x,t)$ (made precise below), the finite-gap solution $u(x,t)$ obtained from this spectral data is non-singular i.e., takes values in the finite complex plane. In order to present the
formula for \(u(x, t)\) we need to introduce some notation. We choose a symplectic basis of cycles \(\{a_1, b_1, \ldots, a_g, b_g\}\) in \(H_1(\Gamma, \mathbb{Z})\) and also a basis for the \(g\)-dimensional space of holomorphic differentials on \(\Gamma\): \(\omega = (\omega_1, \ldots, \omega_g)\) normalized such that \(\int_{a_j} \omega_i = \delta_{ij}\). The Riemann matrix of \(\Gamma\) is the matrix defined by \(B_{ij} = \int_{b_i} \omega_j\). The matrix \(B\) is symmetric and its imaginary part is positive definite. The Riemann theta function associated with \(B\) is the entire function of \(g\) complex variables \(z = (z_1, \ldots, z_g)\) defined by

\[
(2.2) \quad \theta(z|B) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t B n) \exp(2\pi i n^t z)
\]

It satisfies the transformation rule

\[
(2.3) \quad \theta(z + N + BM) = \theta(z) \exp\left(-\pi i \left[2M^t z + M^t BM\right]\right) \quad \text{where} \quad N, M \in \mathbb{Z}^g
\]

The complex torus \(J(\Gamma) = \mathbb{C}^g/\{\mathbb{Z}^g + B\mathbb{Z}^g\}\) is the Jacobian variety of \(\Gamma\), and the Abel map \(A : \Gamma \rightarrow J(\Gamma)\) is defined by \(P \mapsto \int_{\infty}^P \omega\). Let \(K \in \mathbb{C}^g\) be the associated vector of Riemann constants. Let \(\omega_\infty\) and \(\omega_0\) be abelian differentials of the second kind having double poles at \(P = \infty\) and \(P = 0\) respectively with the principal parts being \(-\lambda d(1/\sqrt{\lambda})\) and \(-1/\lambda d\sqrt{\lambda}\) respectively and normalized to have zero \(a\)-periods. (where by \(\sqrt{\lambda}\) and \(1/\sqrt{\lambda}\) we mean local coordinates \(k_0\) and \(k_\infty\) at \(P = 0\) and \(P = \infty\) respectively with \(k_0^2 = \lambda\) and \(k_\infty^2 = 1/\lambda\). We define two vectors \(U, V \in \mathbb{C}^g\) by

\[
(2.4) \quad U_j = \frac{1}{2\pi} \int_{b_j} \omega_\infty, \quad V_j = \frac{1}{2\pi} \int_{b_j} \omega_0
\]

The \(\theta\)-functional formula for \(u(x, t)\) is given by (\([3], [10]\)):

\[
(2.5) \quad e^{iu(x,t)} = C_1^2 \frac{\theta(A(0) + z(x,t)) \theta(-A(0) + z(x,t))}{\theta^2(z(x,t))}
\]

where \(C_1^2 = \exp(\pi i e^t Be/2)\). We remark that \(A(0)\) is a half-period (because \(A(P) - A(Q)\) is always a half-period whenever \(P, Q\) are branch points of a 2-sheeted cover \(\Gamma \rightarrow \mathbb{P}^1\)). Let \(\Theta = \{z \in J(\Gamma) \mid \theta(z) = 0\}\) (which is well-defined by virtue of (2.3)) denote the \(\theta\)-divisor. Let \(\mathcal{U} \subset \mathbb{R}^2\) be a neighborhood of \((0, 0)\) in the \((x, t)\) plane for which the image in \(J(\Gamma)\) of \(z(x, t)\) is disjoint from the divisor \(\Theta \cup (\Theta + A(0))\). If \(\mathcal{U}\) is simply-connected, then (2.5) defines the function \(u(x, t)\) uniquely once \(u(0, 0)\) is specified. In the real case \(\mathcal{U}\) may be chosen to coincide with the whole \(\mathbb{R}^2\) (see below).

### 3. Real Tori and Real Solutions

In order to obtain real solutions \(u(x, t)\), we must impose some conditions on the spectral data. The reality condition on \(\Gamma\) (found in [9], [10]) is:

\[
(3.1) \quad \{E_1, \ldots, E_{2g}\} = \{E_1, \ldots, E_{2g}\} \quad \text{and} \quad E_i \in \mathbb{R} \Rightarrow E_i < 0
\]

Let \(2m\) denote the number of real \(E_i\). We order the real branch points as \(0 > E_1 > E_2 > \cdots > E_{2m}\) and also assume \(E_{2i} = E_{2i-1}\) for \(m + 1 \leq i \leq g\). The reality condition on the divisor found by Cherednik [2] is

\[
(3.2) \quad D + \tau D - 0 - \infty \sim K
\]
where $\tau$ is the anti-holomorphic involution of $\Gamma$ given by $\tau : (\lambda, \mu) \mapsto (\bar{\lambda}, \bar{\mu})$ and $\mathcal{K}$ is the canonical class, and $\sim$ denotes the relation of linear equivalence of divisors. Such a divisor will be called admissible. The Abel map $A : \Gamma \to J(\Gamma)$ extends to divisors by $A(\sum_i P_i - \sum_j Q_j) = \sum_i A(P_i) - \sum_j A(Q_j)$. It was shown in the works [2], [3], [5] that the image in $J(\Gamma)$ of the admissible divisors under the Abel map, consists of $2^m$ components each of which is a real $g$-dimensional torus. We use here a concrete description of these tori, similar to the one in [3].

We choose a special basis of cycles $\{a_1, b_1, \ldots, a_g, b_g\}$ suggested in [4], and depicted in Fig. 2 (where the parameter $k$ must be taken to be 1). The picture shows the $\lambda$-plane with the thick-dashed lines representing the system of cuts, and the transition from solid to dashed line in the cycles $\{b_{m+1}, \ldots, b_g\}$ indicating change of sheet across a cut. The action of $\tau$ on $H_1(\Gamma, \mathbb{Z})$ is given by

$$
\begin{align*}
\tau a_i &= -a_i & 1 \leq i \leq g \\
\tau b_i &= b_i & 1 \leq i \leq m \\
\tau b_i &= b_i + a_i & m + 1 \leq i \leq g
\end{align*}
$$

This immediately implies that the effect of $\tau$ on the holomorphic differentials is given by $-\omega_j = \tau^* \omega_j$. Therefore

$$
(3.3) \quad A(\tau P) = -\overline{A(P)}
$$

$$
(3.4) \quad -\overline{B} = B + \begin{pmatrix} 0 & I_{g-m} \\ 0 & -I_{g-m} \end{pmatrix} \quad \text{or} \quad \text{Re}(B) = -1/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

where $I_{g-m}$ is the identity matrix of size $g - m$. (From here on, all vectors $v$ written as $(v_1, v_2)^t$ are understood to be split into blocks of length $m$ and $g - m$).

In order to describe the real tori, we need to calculate $A(\mathcal{K})$ and $A(0)$. The divisor $(d\lambda/\mu) = (2g-2) \cdot \infty$ is canonical hence we obtain $A(\mathcal{K}) = 0$. Let $2\pi i \alpha$ and $2\pi i \beta$ denote the vectors of $a$ and $b$-periods of the differential $\omega = \frac{1}{2} d\log \lambda$. From the bilinear relations of Riemann applied to the pair of differentials $\omega, \omega_i$ for $1 \leq i \leq g$, we immediately obtain $(\beta - B\alpha)/2 = A(0)$. The numbers $\alpha_j$ and $\beta_j$ are the winding numbers of the projection $\lambda(a_j)$ and $\lambda(b_j)$ around $\lambda = 0$, and are read off from Fig. 2 to be $\alpha = (1,0)^t$ and $\beta = (0,1)^t$. Thus we obtain:

$$
(3.5) \quad A(\mathcal{K}) = 0
$$

$$
(3.6) \quad A(0) = \frac{1}{2} \epsilon' + \frac{1}{2} B\epsilon, \quad \epsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \epsilon' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

Every $z \in J(\Gamma)$ can be written as $z = x + By$ for unique vectors $x, y \in \mathbb{R}^g/\mathbb{Z}^g$. We will denote the subset of elements of the form $\{x + B0\}$ as $T^g$. Clearly $T^g \subset J(\Gamma)$ is isomorphic to $\mathbb{R}^g/\mathbb{Z}^g$. Applying the Abel map to the Cherednik reality condition $D + \tau D - 0 - \infty \sim \mathcal{K}$ and using (3.3), (3.5) and (3.6) we obtain:

$$
(3.7) \quad A(D) = x + B \begin{pmatrix} s/4 \\ 1/2 \end{pmatrix}, \quad s = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{pmatrix}
$$

for some $x \in T^g$ and $s_1, \ldots, s_m = \pm 1$.

For each of the $2^m$ collection of symbols $s_1, \ldots, s_m = \pm 1$, let $T_s$ denote the subset of $J(\Gamma)$ consisting of elements of the form $-A(D) - \mathcal{K}$ where $A(D)$ is as in (3.7) and $\mathcal{K}$ is the vector.
of Riemann constants.
\[ T_s = -T^g - B \left( \frac{s/4}{1/2} \right) - K \]

The symbol \( s \) will be called the **topological type** of the admissible divisor \( D \). In the sequel, the terminology real tori will refer to the \( T_s \). We collect the essential facts (from [2]) about the real tori in the following lemma.

**Lemma 1.**

i) The real tori \( T_s \) do not intersect the divisor \( \Theta \cup (\Theta + A(0)) \)

ii) Let \( D \) be an admissible divisor of topological type \( s \), and let \( z(x, t) = -A(D) + x(V - U)/4 - t(U + V)/4 - K \). Then \( z(x, t) \in T_s \) for all \( x, t \).

iii) The real solutions \( u(x, t) \) are non-singular for all \( x, t \)

**Proof.** i) We use the fact ([6] pp 310) that \( \Theta \) consists of elements of the form \( A(\gamma_1 + \cdots + \gamma_{g-1}) + K \). From this fact (together with \( \Theta = -\Theta \)) it follows that for a divisor \( D \), the condition \(-A(D) - K \in \Theta \cup (\Theta + A(0))\) is equivalent to \( D \sim \gamma_1 + \cdots + \gamma_g \) for some \( \gamma_i \) with \( \gamma_g \in \{0, \infty\} \). Suppose such a divisor \( D \) is also admissible, i.e. \( D + \tau D = 0 - \infty \sim \mathcal{K} \).

Since the Cherednik condition depends only on the divisor class of \( D \), we may assume \( D = \gamma_1 + \cdots + \gamma_{g-1} + \gamma_g \) with \( \gamma_g \in \{0, \infty\} \). Let \( D_1 = \gamma_1 + \gamma_2 + \cdots + \gamma_{g-1} \). The admissibility condition can be rewritten as
\[ D_1 + \tau D_1 \pm (0 - \infty) = (\alpha) \]
for some abelian differential \( \alpha \)

If the differential \( \alpha \) is holomorphic then ([3]) implies that it has an odd number of zeros at \( \infty \). However every holomorphic differential on \( \Gamma \) has an even number of zeroes at \( \infty \).

Therefore \( \alpha \) is not holomorphic and ([3]) now implies that \( \alpha \) has a single pole, contradicting the residue theorem. Thus \( T_s \) does not intersect \( \Theta \cup (\Theta + A(0)) \).

ii) the second assertion follows from the first as soon as we show that the vectors \( U, V \) defined by (2.4) are purely real. It is easy to see from the definition of \( \omega_0, \omega_\infty \) that they are fixed by the involution \( \omega \mapsto \tau^* \omega \). Equation (2.4) together with the fact that \( \tau b_i = b_i \pmod{a_i} \), now implies that \( U \) and \( V \) are purely real.

iii) The formula (2.3) for \( u(x, t) \) shows that it is nonsingular as long as \( z(x, t) \notin \Theta \cup (\Theta + A(0)) \). Thus the first two assertions imply the third one.

To conclude this section, we calculate the vector of Riemann constants \( K \) for the chosen base point of Abel map \( (P = \infty) \), and the choice of basic cycles. A formula for \( K \) is well-known ([6], pp 325) when the base point of Abel map is a branch point of a 2-sheeted cover \( \lambda : \Gamma \to \mathbb{P}^1 \). It equals \( \sum_i A(P_i) \) where the sum is over those branch points \( P_i \) for which \( A(P_i) \) is an odd half-period (i.e. of the form \((n + Bm)/2\) with the scalar product \((n, m)\) being an odd integer). In the present case, we obtain:
\[ K = A(E_1) + A(E_3) + \cdots + A(E_{2g-1}) \]

Let \( 2\pi i \alpha' \) and \( 2\pi i \beta' \) be the vectors of \( a \) and \( b \)-periods of the differential \( \tilde{\omega} = \frac{1}{2} \partial \log \prod_{i=1}^g \left( \lambda - E_{2i-1} \right) \), which is a differential of the third kind having residues +1 at \( E_1, E_3, \cdots, E_{2g-1} \) and residue \(-g\) at \( \infty \). From the bilinear relations of Riemann applied to the pair of differentials \( \omega_i, \tilde{\omega} \) for \( 1 \leq i \leq g \), we obtain
\[ (\beta' - B\alpha')/2 = A(E_1) + A(E_3) + \cdots + A(E_{2g-1}) = K \]
The numbers $\alpha'_j$ and $\beta'_i$ being $\frac{1}{2\pi i} \int_{a_i} \omega$ and $\frac{1}{2\pi i} \int_{b_i} \omega$ respectively are equal to the sum of the winding numbers about $E_1, E_3, \ldots, E_{2g-1}$ of the projected curves $\lambda(a_i)$ and $\lambda(b_i)$ respectively. The winding numbers can be read off from Fig. 2. We obtain:

\begin{equation}
K = \frac{1}{2} \left( \frac{1}{\nu_2} \right) + \frac{1}{2} B \left( \frac{\nu_1}{\nu_2} \right) \quad \text{where} \quad \left( \frac{\nu_1}{\nu_2} \right) = \left( \frac{1}{2} z \right)
\end{equation}

4. Basic charges and Topological Charge in two special cases

In order to calculate the topological charge density $\bar{n}$ defined in [1.2] for the finite-gap solution $u(x, t)$ [23], following [2] we introduce some integer quantities $(n_1, n_2, \ldots, n_g)$ associated with each real torus $T_s$. We call the quantities $n_j$ as basic charges. Let $e_j$ denote the $j$-th standard basis vector of $\mathbb{R}^g$ and let $\{z_j(T) = -A(D) - Te_j - K \mid 0 \leq T \leq 1\}$ denote the $j$-th basic cycle of the real torus $T_s$. We define the basic charges $n_j$ by:

\begin{equation}
n_j = \frac{1}{2\pi i} \int_{T=0}^{T=1} d \log e^{i u_j(T)}, \quad \text{where} \quad \hat{e}^{i u_j(T)} = C_1 \frac{\theta(A(0) + z_j(T)) \theta(-A(0) + z_j(T))}{\theta^2(z_j(T))}
\end{equation}

The density of topological charge $\bar{n}$ for the solution real finite-gap solution $u(x, t)$ depends only on its topological type, and is related to the basic charges by (see [3]):

\begin{equation}
\bar{n} = \sum_{j=1}^{g} (U_j - V_j) n_j / 4
\end{equation}

It will be convenient to replace the quantity $\epsilon = (1, 0)^T$ in the formula (3.6) for $A(0)$ by the quantity $\hat{\epsilon}$ defined by $\hat{\epsilon}_j = (-1)^j s_j$ for $j \leq m$ and $\hat{\epsilon}_j = 0$ for $j > m$. Using this new expression for $A(0)$ in equation (4.1) and using the $\theta$-transformation rule (2.3) we obtain the following formula for $n_j$:

\begin{equation}
n_j = -\hat{\epsilon}_j + \frac{2}{2\pi i} \int_{T=0}^{T=1} d \log \left( \frac{\theta(B \hat{\epsilon}/2 - Te_j - K - B (s/4))}{\theta(-Te_j - K - B (s/4))} \right)
\end{equation}

We calculate the basic charges $n_j$ for two special cases of spectral curves $\Gamma$. In the first case $m = 0$, i.e. none of the $E_i$ are real. In this case all components of the quantity $\hat{\epsilon}$ defined above are zero. Thus both terms in (4.3) vanish and we obtain $n_j = 0$. In the second case we consider $g = m = 1$, i.e. elliptic curves with all branching points real. We denote the Riemann matrix $B$ by $\tau$. For this case, (4.3) can be rewritten as:

\begin{equation}
n_1 = s_1 + \frac{2}{2\pi i} \int_{T=0}^{T=1} d \log \left( \frac{\theta(-T - (1 + \tau)/2 - s_1 \tau/4 - s_1 \tau/2)}{\theta(-T - (1 + \tau)/2 - s_1 \tau/4)} \right)
\end{equation}

The integral term is equal to $2s_1$ times the number of zeroes of $\theta(z|\tau)$ in the region $R = \{z \in \mathbb{C} \mid -\tau/4 \leq \text{Im}(z) \leq \tau/4, -1 \leq \text{Re}(z) \leq 0\}$. Since the elliptic theta function $\theta(z|\tau)$ vanishes if and only if $z \sim (1 + \tau)/2$ and the region $R$ does not contain any such $z$, it follows that the integral term in (4.4) is zero and we obtain $n_1 = s_1$. For use later in Section 6, we will calculate a related quantity $n_1$ given by:

\begin{equation}
\tilde{n}_1 = -s_1 + \frac{2}{2\pi i} \int_{T=0}^{T=1} d \log \left( \frac{\theta(-T - 1/2 - s_1 \tau/4 + s_1 \tau/2)}{\theta(-T - 1/2 - s_1 \tau/4)} \right)
\end{equation}
In this case the integral term is equal to $-2s_1$ times the number of zeroes $\theta(z|\tau)$ in the region $R$. Hence the integral term drops out and we obtain $\check{n}_1 = -s_1$.

5. Multiscale Limit

In this section we construct a deformation of the spectral curve $\Gamma$ (2.1) satisfying the reality condition (3.1). For each $k \in [1, \infty)$, let $\Gamma(k)$ be the real hyperelliptic curve:

$$
\Gamma(k) : \mu^2 = \lambda \prod_{i=1}^{m} (\lambda - k^{i-1}E_{2i-1})(\lambda - k^{i-1}E_{2i}) \prod_{i=2m+1}^{2g} (\lambda - k^{m}E_{i})
$$

We note that the original spectral curve $\Gamma = \Gamma(1)$. The basic cycles and the system of cuts on $\Gamma(k)$ are shown in Fig. 2. If $\Pi_j(k)$ for $m + 1 \leq j \leq g$ denotes the cut on the $\lambda$-plane joining the complex conjugate branch points $k^mE_{2j-1}$ and $k^mE_{2j}$, then we require $\Pi_j(k) = k^m\Pi_j(1)$. The remaining cuts lie on the negative real line of the $\lambda$-plane and are obvious from Fig. 2. We will use the notation $\Gamma(k)^+$ to denote the sheet for which $\mu > 0$ when $\lambda > 0$. As indicated in Fig. 2, the cycles $a_j(k), b_j(k)$ are completely determined by their projections $\lambda(a_j(k)), \lambda(b_j(k))$ to the $\lambda$-plane. We specify the latter by:

$$
\lambda_j(a_j(k)) = \lambda(a_j(1)), \quad \lambda_j(b_j(k)) = \lambda(b_j(1))
$$

where $\lambda_j = \begin{cases} \lambda/k^{j-1} & \text{if } j \leq m \\ \lambda/k^m & \text{if } j > m \end{cases}$ and $\lambda : \Gamma(k) \to \mathbb{P}^1$ are projections

where we have introduced rescaled versions $\lambda_j$ of the coordinate function $\lambda$. We also associate with the deformation $\Gamma(k)$, $m$ elliptic curves $C_1, \ldots, C_m$ and a hyperelliptic curve $C_{m+1}$.

$$
C_j : y_j^2 = P_j(x_j) = x_j(x_j - E_{2j-1})(x_j - E_{2j}), \quad 1 \leq j \leq m
$$

$$
C_{m+1} : y_{m+1}^2 = P_{m+1}(x_{m+1}) = x_{m+1} \prod_{i=2m+1}^{2g} (x_{m+1} - E_{i}).
$$

The system of cuts on these curves is shown in Fig. 1. The cuts decompose each $C_j$ into two sheets. By the sheet $C_j^+$ we will mean the sheet for which $y_j > 0$ when $x_j > 0$.

![Figure 1](image_url)

**Figure 1.** Basic cycles and cuts on $C_j$, $1 \leq j \leq m$ and $C_{m+1}$. 7
We will be concerned with the limit of the pair \((\Gamma(k), B(k))\) as \(k \to \infty\). To this end, we decompose \(\Gamma(k)\) into \(m + 1\) open sets \(R_j\), and we also introduce certain rescaled versions of the coordinate function \(\mu\) on \(\Gamma(k)\):

\[
R_1 = \{(\lambda, \mu) \in \Gamma(k) | 0 \leq |\lambda_1| < k^{2/3}\}
\]

\[
R_j = \{(\lambda, \mu) \in \Gamma(k) | k^{-1/2} < |\lambda_j| < k^{2/3}\}, \quad \text{for } 2 \leq j \leq m
\]

\[
R_{m+1} = \{(\lambda, \mu) \in \Gamma(k) | k^{-1/2} < |\lambda_{m+1}| \leq \infty\}
\]

\[
(5.5) \quad \mu_j = \begin{cases} 
\frac{\mu}{\sqrt[k]{k^{3(j-1)} k^{(j+\cdots+m-1)} k^m (\prod_{l=j+1}^{2g} E_l)^{0.5}}} & \text{if } 1 \leq j \leq m \\
\frac{\mu}{\sqrt[k]{k^m}} & \text{if } j = m + 1
\end{cases}
\]

where all square-roots occurring in \((5.5)\) are positive real numbers. We define maps \(\phi(k) = (\phi_1(k), \phi_2(k), \ldots, \phi_{m+1}(k))\) from \(\Gamma(k)\) to \((\mathbb{P}^2)^{m+1}\) given by:

\[
(5.6) \quad \phi_j(k)(\lambda, \mu) = (\lambda_j : \mu_j : 1)
\]

We consider the restriction of the maps \(\phi_j(k)\) to the regions \(R_i\). Using \((5.4)\) and \((5.5)\) in \((5.6)\) we obtain:

\[
(5.7) \quad \phi_j(k)|_{R_i}(\lambda, \mu) = \begin{cases} 
(\alpha(1/k) : \mu(1/k) : 1) & \text{if } i < j \\
(\alpha(1/k) : \mu(1/k)) & \text{if } i = j + 1
\end{cases}
\]

\[
(\lambda_j : \sqrt[k]{P_j(\lambda_j)} (1 + \alpha(1/k)) : 1) & \text{if } i = j
\]

as \(k \to \infty\)

Figure 2. Basic cycles and cuts on \(\Gamma(k)\) for \(g = 4, m = 2\)

Let \(B(k)\) denote the Riemann matrix of \(\Gamma(k)\) with respect to this choice of basic cycles. We will be concerned with the limit of the pair \((\Gamma(k), B(k))\) as \(k \to \infty\). To this end, we decompose \(\Gamma(k)\) into \(m + 1\) open sets \(R_j\), and we also introduce certain rescaled versions of the coordinate function \(\mu\) on \(\Gamma(k)\):

\[
R_1 = \{(\lambda, \mu) \in \Gamma(k) | 0 \leq |\lambda_1| < k^{2/3}\}
\]

\[
R_j = \{(\lambda, \mu) \in \Gamma(k) | k^{-1/2} < |\lambda_j| < k^{2/3}\}, \quad \text{for } 2 \leq j \leq m
\]

\[
R_{m+1} = \{(\lambda, \mu) \in \Gamma(k) | k^{-1/2} < |\lambda_{m+1}| \leq \infty\}
\]

\[
(5.5) \quad \mu_j = \begin{cases} 
\frac{\mu}{\sqrt[k]{k^{3(j-1)} k^{(j+\cdots+m-1)} k^m (\prod_{l=j+1}^{2g} E_l)^{0.5}}} & \text{if } 1 \leq j \leq m \\
\frac{\mu}{\sqrt[k]{k^m}} & \text{if } j = m + 1
\end{cases}
\]

where all square-roots occurring in \((5.5)\) are positive real numbers. We define maps \(\phi(k) = (\phi_1(k), \phi_2(k), \ldots, \phi_{m+1}(k))\) from \(\Gamma(k)\) to \((\mathbb{P}^2)^{m+1}\) given by:

\[
(5.6) \quad \phi_j(k)(\lambda, \mu) = (\lambda_j : \mu_j : 1)
\]

We consider the restriction of the maps \(\phi_j(k)\) to the regions \(R_i\). Using \((5.4)\) and \((5.5)\) in \((5.6)\) we obtain:

\[
(5.7) \quad \phi_j(k)|_{R_i}(\lambda, \mu) = \begin{cases} 
(\alpha(1/k) : \mu(1/k) : 1) & \text{if } i < j \\
(\alpha(1/k) : \mu(1/k)) & \text{if } i = j
\end{cases}
\]

\[
(\lambda_j : \sqrt[k]{P_j(\lambda_j)} (1 + \alpha(1/k)) : 1) & \text{if } i = j
\]

as \(k \to \infty\)

where, for \((\lambda, \mu) \in \Gamma(k)^+\) the expression \((\lambda_j, \sqrt[k]{P_j(\lambda_j)}) \in \mathcal{C}_j^+\). This can be seen by noting that, for \(j \leq m\), \(\text{sgn}(-\mu) = \text{sgn}(\lambda^{-1}) = (-1)^{j-1}\) on \(\Gamma(k)^+\) over \((E_{2j}, E_{2j-1})\), and therefore \((5.5)\) implies that \(\text{sgn}(\mu_j) = -1\) on \(\Gamma(k)^+\) over \((E_{2j}, E_{2j-1})\). Also \(\text{sgn}(\sqrt[k]{P_j(x_j)}) = -1\) on \(\mathcal{C}_j^+\) over \(x_j \in (E_{2j}, E_{2j-1})\). Similarly if \(j = m + 1\), then both quantities \(\mu, \lambda^m\) and therefore \(\mu_{m+1}\)
are positive on $\Gamma(k)^+$ over the positive real axis, and $\sqrt{P_{m+1}(x_{m+1})}$ is also positive on $C_{m+1}$ over the positive real axis.

We next consider embeddings $\phi_i : C_i \to (\mathbb{P}^2)^{m+1}$ given by:

$$\phi_i : (x_i, y_i) \mapsto \{(0 : 1 : 0)\}^{i-1} \times (x_j : y_j : 1) \times \{(0 : 0 : 1)\}^{m+1-i}$$

($\phi_{m+1}$ is singular at $\infty \in C_{m+1}$ if $g - m > 1$, but this is inessential for us). Let $C \subset (\mathbb{P}^2)^{m+1}$ be defined as $\cup_{i=1}^{m+1} \phi_i(C_i)$. Clearly $C$ is a nodal hyperelliptic curve of genus $g$ having $m$ double points $\{(0 : 1 : 0)\}^i \times \{(0 : 0 : 1)\}^{m+1-i}$ for $1 \leq i \leq m$. It is also clear from (5.7) that:

$$\lim_{k \to \infty} \phi(k)(\Gamma(k)) = C \subset (\mathbb{P}^2)^{m+1}$$

with

$$\lim_{k \to \infty} \phi(k)(R_i) = \phi_i(C_i) \subset C$$

and

$$\lim_{k \to \infty} \phi(k)(R_i \cap R_{i+1}) = \{(0 : 1 : 0)\}^i \times \{(0 : 0 : 1)\}^{m+1-i} \quad \text{(the m double points)}$$

Let $a_j, b_j$ denote the basic cycles on $\phi_j(C_j) \subset C$ (or $\phi_{m+1}(C_{m+1}) \subset C$ if $j > m$). Let $\omega_j$ be the holomorphic differential on the elliptic curve $C_j$ (for $j \leq m$) satisfying $\int a_j \omega_j = 1$, and define $\tau_j = \int b_j \omega_j$. We define $B_1$ to be the diagonal matrix $\text{diag}(\tau_1, \cdots, \tau_m)$. Similarly, let $\omega_{m+j}$ for $1 \leq j \leq g - m$ be holomorphic differentials on the hyperelliptic curve $C_{m+1}$ satisfying $\int a_{m+j} \omega_{m+i} = \delta_{ij}$ and let $(B_2)_{ij} = \int b_{m+j} \omega_{m+i}$ for $1 \leq i, j \leq g - m$. Define $B_\infty$ to be the block-diagonal matrix $B_\infty = \text{diag}(B_1, B_2)$. It is the Riemann matrix of $C$ with respect to the basic cycles $\{a_1, b_1, \cdots, a_g, b_g\}$. Using (5.7) again, it follows that the component of $\lim_{k \to \infty} \phi(k)(a_j(k))$ in $\phi_i(C_i)$ is homologous to $a_i$ if $i = j$ and homologous to zero if $i \neq j$. Similarly the component of $\lim_{k \to \infty} \phi(k)(b_j(k))$ in $\phi_i(C_i)$ is homologous to $b_i$ if $i = j$ and homologous to zero if $i \neq j$.

$$\lim_{k \to \infty} \phi(k)_*(a_j(k)) = a_j, \quad \text{and} \quad \lim_{k \to \infty} \phi(k)_*(b_j(k)) = b_j$$

therefore

$$\lim_{k \to \infty} B(k) = B_\infty$$

6. Calculation of Topological Charge

We reduce the calculation of the basic charges $n_j$ in the general case to the two special cases computed in Section 4. As proved in Lemma 1, the real tori $T_s$ do not intersect the divisor $\Theta \cup (\Theta + A(0))$. Therefore the integral term in (1.3) stays nonsingular through the deformation $B(k)$. It is also constant because it is continuous in the deformation parameter $k$ and it is integer valued. In other words the basic charges $n_j$ may be calculated from (1.3) using $B_\infty$ in place of $B$. Also, it follows from the definition (2.2) that $\theta((z_1, z_2)^t \mid \text{diag}(B_1, B_2)) = \theta(z_1 \mid B_1) \theta(z_2 \mid B_2)$. Using this and the computation (3.11) for $K$ in the formula (1.3), we obtain

$$n_j = \begin{cases} s_j + \frac{2}{\pi i} \int_{T=0}^{T=1} d \log \left( \frac{\theta(-T-(1+\tau_j)/2-s_j \tau_j/4-s_j \tau_j/2)}{\theta(-(1+\tau_j)/2-s_j \tau_j/4)} \right) = s_j & \text{if } j \leq m \text{ is odd} \\ -s_j + \frac{2}{\pi i} \int_{T=0}^{T=1} d \log \left( \frac{\theta(-T-1/2-s_j \tau_j/4+s_j \tau_j/2)}{\theta(-T-1/2-s_j \tau_j/4)} \right) = -s_j & \text{if } j \leq m \text{ is even} \\ 0 & \text{if } j > m \end{cases}$$

(6.1)

Thus we have proved:
Theorem 1. The topological charge density \( \bar{n} \) for the real finite-gap solution \( u(x,t) \) for the spectral data \((\Gamma, D)\) with \( D \) corresponding to the real torus \( T_s \) is given by:

\[
\bar{n} = \sum_{j=1}^{g} \frac{(U_j - V_j) n_j}{4}
\]

where the basic charges \( n_j \) are:

\[
(6.2) \quad n_j = \begin{cases} 
(-1)^{j-1} s_j & \text{if } 1 \leq j \leq m \\
0 & \text{if } j > m
\end{cases}
\]

Remarks

(1) In [8], the admissible divisors were characterized by certain symbols \( \{s'_1, \ldots, s'_m\} \in \{\pm 1\}^m \) defined as follows. Given an admissible divisor \( D = \{(\lambda_i, \mu_i) \mid 1 \leq j \leq g\} \) let \( P(\lambda) \) be the unique polynomial of degree \( g - 1 \) interpolating the \( g \) points \( (\lambda_i, \mu_i/\lambda_i) \). Then \( P(\lambda) \) is real and \( s'_j \) is defined to be the sign of \( P(\lambda) \) over \([E_{2j}, E_{2j-1}]\). It was shown in [7] that the charges \( n_j \) are equal to \((-1)^{j-1} s'_j\) for \( j \leq m \) and \( n_j = 0 \) for \( j > m \). Comparing with formula \((6.2)\), it follows that the symbols \( s'_j \) and \( s_j \) coincide.

(2) The multiscale limit of the spectral curve constructed above was used only for a topological argument. The sine-Gordon solutions \( u(x,t,k) \) associated with the spectral curve \( \Gamma(k) \) (and admissible divisors \( D(k) \)) depend on the vectors \( U(k) \) and \( V(k) \) mentioned in the Section 3. As \( k \to \infty \), some component of \( U(k) \) will diverge to \( \infty \). Thus there is no limiting solution. However asymptotic expansion in the parameter \( k \) of \( u(x,t,k) \) involving elliptic (genus 1) solutions can be written. This will be investigated in a future work.

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