Generalization of Faddeev–Popov Rules in Yang–Mills Theories: N=3,4

BRST Symmetries

Alexander Reshetnyak

Institute of Strength Physics and Materials Science of Siberian Branch of RAS,
634055, Tomsk, Russia

Abstract

The Faddeev–Popov rules for a local and Poincare-covariant procedure of Lagrangian quantization for a gauge theory with gauge group are generalized to the case of an invariance of the respective quantum actions, $S_{(N)}$, with respect to $N$-parametric Abelian SUSY transformations with odd-valued parameters $\lambda_p$, $p = 1, \ldots, N$ and anticommuting generators $s_p$: $s_p s_q + s_q s_p = 0$, for $N = 3,4$, implying the substitution of an $N$-plet of ghost fields, $C^p$, instead of the parameter, $\xi$, of the infinitesimal gauge transformations: $\xi = C^p \lambda_p$. The total configuration spaces of field variables for a quantum theory of the same classical model coincide in the $N = 3$ and $N = 4$ symmetric cases. For the $N = 3$-parametric SUSY transformations the superspace of the irreducible representation includes, in addition to Yang–Mills fields $A^\mu$, also 3 ghost odd-valued fields $C^p$, as well as 3 new even-valued $B^{pq} = -B^{qp}$ and 1 odd-valued $\bar{B}$ fields for $p, q = 1, 2, 3$. It is shown, that in order to construct the quantum action, $S_{(3)}$, a gauge-fixing procedure achieved by adding to the classical action of an $N = 3$-exact gauge-fixing term (without introduction of non-degenerate odd supermatrix) additionally requires a 1 antighost field, 3 even-valued $B^p$ and 3 odd-valued $\bar{B}^{pq}$ fields, as well as the Nakanishi–Lautrup field $B$. The action of $N = 3$ transformations in the space of additional fields, $\Phi_{(3)} = (\bar{C}, B^\mu, \bar{B}^{pq}, B)$, not being entangled with the fields $\Phi_{(3)}$ of $N = 3$-irreducible representation space is realized as well. These transformations are the $N = 3$ BRST symmetry transformations for the vacuum functional, $Z_4(0) = \int d\Phi_{(3)} d\tilde{\Phi}_{(3)} \exp\left\{\left(\i/\hbar\right) S_{(3)}\right\}$. It is shown that the total configuration space of the fields $(\Phi_{(3)}, \tilde{\Phi}_{(3)})$, as the space of reducible $N = 3$ BRST symmetry transformations, proves to be the space of an irreducible representation of the fields $\Phi_{(4)}$, for $N = 4$-parametric SUSY transformations, which contains, in addition to $A^\mu$ the $(4 + 6 + 4 + 1)$ ghost-antighost, $C^r = (C^r, \bar{C})$, new even-valued, $B^{rs} = -B^{sr} = (B^{pq}, B^{pq} = B^{pq})$, odd-valued $\bar{B}^r = (\bar{B}, \bar{B}^{pq})$ fields and $B$ for $r, s = 1, 2, 3, 4, r = (p, 4)$. The quantum action $S_{(4)}$ is constructed by adding to the classical action an $N = 4$-exact gauge-fixing term with a gauge boson, $F_{(4)}$ as the $s_r$-potential as compared to a gauge fermion $\Psi_{(3)}$ for $N = 3$ case. It is proved that the $N = 4$-parametric SUSY transformations are by $N = 4$ BRST transformations for the vacuum functional, $Z_4(0) = \int d\Phi_{(4)} \exp\left\{\left(\i/\hbar\right) S_{(4)}\right\}$. The procedures are valid for any admissible gauge. The equivalence with $N = 1$ and $N = 2$ BRST-invariant quantization methods are explicitly established. The finite $N = 3, 4$ BRST transformations are derived from the algebraic SUSY transformations. The Jacobians for a change of variables related to finite $N = 3, 4$ SUSY transformations with field-dependent parameters in the respective path integral are calculated. The Jacobians imply the presence of a corresponding modified Ward identity which reduces to a new form of the standard Ward identities in the case of constant parameters and describe the problem of a gauge-dependence. The gauge-independent Gribov-Zwanziger models with local $N = 3, 4$ BRST symmetries are proposed. An introduction into diagrammatic Feynman techniques for $N = 3, 4$ BRST invariant quantum actions for Yang–Mills theory is suggested. A generalization to the case of $N = 2K - 1$ and $N = 2K$, $K > 2$ BRST transformations is discussed.\footnote{The paper is dedicated to the memory of the outstanding Soviet and Russian theoretical physicist and mathematician, Academician Ludwig Dmitrievich Faddeev (1934-2017)}

*e-mail address: reshet@ispms.tsc.ru
1 Introduction

The problem of Lorentz-covariant quantization for gauge theories with a non-Abelian gauge group is a long-standing one, starting with the lecture of R. Feynman [2], showing that the naive one-loop calculation within perturbative techniques with a propagator constructed, according to quantum electrodynamic, for the photon field $A_\mu$ in the form

$$G_{\mu\nu}(k) = \frac{1}{k^2 + i0} \left( \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \beta(k) \frac{k_\mu k_\nu}{k^2}, \quad (1.1)$$

turns out to be incorrect. A modification of calculations for reconstructing the one-loop contribution from the tree diagrams, using unitarity and analyticity [2], makes it possible to interpret the additional contributions as an input from a scalar particle, which should be, however, considered as a fermion due to the "-" sign before this summand. The solution of this problem was found by L. Faddeev and V. Popov in their celebrated work [3] by means of a trick known as the insertion of unity, providing the existence of a path integral for Yang–Mills fields, $A_\mu(x) = A_\mu^m(x) t^m$, given in Minkowski space-time $\mathbb{R}^{1,3}$ and taking values in a compact Lie group $G$, with generators $t^m$ for its Lie algebra $\mathcal{G}$, in the form

$$Z^L_0 = \int dA\delta(\partial^\mu A_\mu) \det M(A) \exp \left\{ \frac{i}{\hbar} S_0(A) \right\}, \quad (1.2)$$

$$Z^F_0 = \int dA dB \det M(A) \exp \left\{ \frac{i}{\hbar} S_0(A) + \int d^4x (\partial^\mu A_\mu + g^2 B) B \right\}, \quad (1.3)$$

respectively, for the Landau gauge, $\chi(A) = 0$, $\chi(A) = \partial^\mu A_\mu$, and then with the use of the proposal of 't Hooft [5] for the Feynman gauge $\chi(A,B)$, $\chi(A,B) = \partial^\mu A_\mu + g^2 B$, with an arbitrary field $B = B^m t^m$ known as Nakanishi-Lautrup field [6, 7]. This representation, with a gauge-invariant classical action $S_0$, in comparison with the case of an Abelian $U(1)$ gauge group, essentially includes a determinant of an non-degenerated operator $M(A)$:

$$M(A) = \partial^\mu D_\mu = \partial^\mu (\partial_\mu - [A_\mu],) \quad (1.4)$$

known as the Faddeev–Popov operator (having multiple zero-mode eigenfunctions as compared to the Abelian case, known as Gribov copies [8]). In [9] (see the review [10]), it was shown, with the use of F. Berezin [11] generalization of the Gaussian integral over Grassmann variables, that the representations may be equivalently presented in a local form by using fictitious scalar Grassman-odd fields $(C(x), \overline{C}(x)) = (C^m(x), \overline{C}^m(x)) t^m$

$$Z^L_0 = \int dA dC d\overline{C} dB \exp \left\{ \frac{i}{\hbar} S^L_{FP}(A, C, \overline{C}, B) \right\} \text{ with } S^L_{FP} = S_0 + \int d^4x \left\{ \overline{C} M(A) C + \chi(A) B \right\}, \quad (1.5)$$

and similarly for $Z^F_0$, where, instead of the quantum action $S^L_{FP} = S^L_{FP}(A, C, \overline{C}, B)$, one should use the action $S^L_{FP} = S^L_{FP}\big|_{\chi(A)\to\overline{\chi}(A,B)}$ given in the Feynman gauge. Independently, the diagrammatic technique without using Grassman-odd fictitious fields was suggested by B. DeWitt [4]. The representation (1.4) allows one to replace gauge transformations for Yang–Mills fields with arbitrary scalar functions $\xi(x) = \xi^m(x) t^m$ by global transformations in the total configuration space $\mathcal{M}_{tot}$ of fields $\Phi^A = (A, C, \overline{C}, B)$, with a constant Grassman-odd parameter $\mu$, $\mu^2 = 0$ by the rule $\xi(x) = C(x) \mu$, being an invariance transformation for the quantum action and for the integral measure in (1.5), which is known as a BRST symmetry transformation [12, 13]. The BRST symmetry allows

\[2\text{Let us point out that the elements of the scattering matrix, among the physical states, do not depend on the value of } \beta(k) \]

\[3\text{Because of the integration in (1.3) in powers of } B \text{ is gaussian, the only way to get after integration the gauge-fixed term: } -\frac{1}{2 g^2} (\partial^\mu A_\mu)^2 \text{ to restore coupling constant in the Feynman gauge as it was done above through field } B. \]

\[4\text{Here the notation for } M(A) \text{ introduced in [3] was used. In what follows we will use the definition of the covariant derivative } D_\mu \text{ with opposite sign: } D_\mu = \partial_\mu + [A_\mu,]. \]
one to prove the gauge-invariant renormalizability of a quantum Yang–Mills theory \cite{14, 15}, as well as the path integral independence from a choice of the gauge condition for small variations. This also makes it possible to obtain the Ward identities for generating functionals of Green’s functions \cite{16}. In \cite{17, 18} it was shown that the Faddeev–Popov representation \cite{17, 18} admits the form \cite{17, 18} for an anti-BRST symmetry transformation with another Grassmann parameter, $\mu$: $\xi(x) = C(x)^{\mu}$, which may be considered within the $N = 2$ BRST (BRSTantiBRST) symmetry \cite{19} for Yang–Mills theories, describing ghost and antighost fields as an $Sp(2)$-doublet $C^{\mu\nu}(x)$ of fields: $(C^{\mu1}, C^{\mu2}) = (C^{\mu}, C^\nu)$, as well as the parameters $(\mu, \bar{\mu}) = (\mu_1, \mu_2)$, which follows from the substitution $\xi^n(x) = C^{\mu\nu}(x)\mu_\alpha$ (with summation over repeated indices). The lifting of $N = 1, 2$ BRST symmetry transformations, given originally in an infinitesimal algebraic form, to a finite group-like form, with finite field-dependent parameters $\mu(\Phi)$, $\mu_\alpha(\Phi)$ has been introduced for $N = 1$ case in \cite{23, 24} (for gauge theories with a closed algebra and general gauge theories, see \cite{25}), for $N = 2$ case in \cite{26} (as well as for constrained dynamical systems and general gauge theories in \cite{28, 29, 30, 31} with references therein), which allows one to establish that the path integral in different gauges, and field-dependent parameters and respective group-like elements being functionals of field variables. We have studied some physical consequences of these transformations at the path integral level. As a consequence, we are interested in the following question.

Is it possible to find a general solution for the non-local Faddeev–Popov path integral representations \cite{12, 13} in a local form which admits an extended $N = k$ global SUSY transformation with $k \geq 3$ Grassmann-odd parameters, such as those realized by $N = 1, 2$ BRST symmetries? In the case of a positive solution, which depends on a possibility to realize on an appropriate $N = k$ SUSY irreducible representation space $N = k$-invariant gauge-fixing procedure to construct $N = k$-invariant quantum action, $S_N$, we are interested in investigating such physical consequences as gauge-dependence, unitarity, renormalizability and Ward identities for the Feynman diagrams in the corresponding path integral with local $N = 3$ and $N = 4$-BRST invariant quantum actions.

The paper is devoted to the solution of the problem in question and is organized as follows. In Section 2 we expound a generalization of the non-local Faddeev–Popov path integral realization in Subsection 2.2 starting from the review of $N = 1, 2$ cases in Subsection 2.1. We derive a local Faddeev–Popov path integral, $Z_3$, over fields composing total configuration space, which is the reducible representation superspace of $N = 3$ SUSY transformations being explicitly constructed both for the fields of $N = 3$ irreducible representation superspace and for auxiliary fields from non-minimal sector in Subsection 2.3 so as to formulate an $N = 3$ BRST invariant gauge-fixing procedure without a special odd supermatrix. In Section 3 we consider the fields of $N = 3$ irreducible and additional representation superspaces on equal footing within explicitly constructed $N = 4$ SUSY transformations, and formulate $N = 4$ SUSY invariant gauge-fixing procedure for local path integral, $Z_4$, in Section 4 for which these transformations are $N = 4$ BRST symmetry transformations. In Section 5 we determine infinitesimal and finite group-like $N = k$ BRST symmetry transformations, for $k = 3, 4$, with constant and field-dependent parameters and compute respective Jacobians for changes of variables in the path integrals. In Section 6 we apply the results concerning the Jacobians so as to relate the respective path integral in different gauges, and to obtain new Ward identities, accompanied by the study of gauge dependence and gauge-invariant Gribov–Zwanziger formulation both within $N = 3$ and $N = 4$ BRST local quantum actions for Yang–Mills theories. The introduction into Feynman diagrammatic technique in $N = 3, 4$ BRST quantum perturbative formulations for Yang–Mills theory is the basic point of Section 7. The results are summarized in Conclusions. The proof of an impossibility to realize $N = 3$ BRST invariant gauge-fixing on the configuration space consisting of only the fields of $N = 3$ irreducible representation superspace without an odd nondegenerate supermatrix (based on an explicit construction of quantum action and $N = 3$ BRST transformations) is given in Appendix A. The details of derivation

\footnote{For superfield and geometrical interpretation of anti-BRST symmetry see e.g. \cite{20, 21, 22} and references therein}
of \( N = 4 \) BRST invariant quantum gauge-fixed action in \( R^\mu \)-like gauges is considered in Appendix B.

We use the DeWitt condensed notation [32]. We denote by \( \epsilon(F) \) the value of Grassmann parity of a quantity \( F \) and also use \( \eta_{\mu\nu} = \text{diag}(-,+,\ldots,+) \) for the metric tensor of a \( d \)-dimensional Minkowski space-time (generalizing the case of \( d = 4 \)), with the Lorentz indices \( \mu, \nu = 0, 1, \ldots, d - 1 \). A local orthonormal basis \( m^n \) in the semi-simple Lie algebra \( G \) of \( G \) is normalized by the Killing metric \( \epsilon_{m^n m^n} = - \frac{1}{2} \delta^{mn} \). Derivatives with respect to the field variables \( \Phi^A \) and sources \( J_A \) are denoted by \( \tilde{\partial}^A (\tilde{\partial}^A) \) for right (left) derivatives and \( \bar{\partial}^A (\bar{\partial}^A) \) for left ones. The symmetrized and antisymmetrized in \( p \) and \( q \) products of the tensor quantities, \( F^p \) and \( G^q \) are denoted as: \( F^{[pG^q]} = F^pG^q + F^qG^p; F^{[pG^q]} = F^pG^q - F^qG^p. \) The raising and lowering of \( \text{Sp}(2) \) indices, \( \left( \hat{s}^a, \hat{s}_a \right) = (\epsilon^{ab} s_b, \epsilon_{ab} s^b) \), is carried out by a constant antisymmetric tensor \( \epsilon^{ab} \), \( \epsilon^{ac}\epsilon_{cb} = \delta^a_b, \epsilon^{12} = 1. \)

### 2 Generalization of the Faddeev–Popov method

Let us consider a configuration space of fields \( A^i = A^n(x) = A^{\mu n}(x)t^n \) in \( \mathbb{R}^{1:d-1} \), taking their values \( A^{\mu n}(x) \) in a Lie algebra \( \mathfrak{g} = \text{su}(\mathcal{N}) \) of a gauge group \( G = SU(\mathcal{N}) \) for \( n = 1, \ldots, \mathcal{N}^2 - 1 \), with an action \( S_0(A) \) invariant under gauge transformations, in the condensed notations in finite and infinitesimal form \( \delta A^i = R^i_{\gamma}(A)\xi^\gamma \), with the generators \( R_{\mu}(A) \) of the gauge transformations:

\[
S_0(A) = \frac{1}{2g^2} \int d^dx \text{tr} G_{\mu\nu}(x)G^{\mu\nu}(x), \quad G_{\mu\nu} = \partial_{[\mu}A_{\nu]}(x) + [A_\mu(x), A_\nu(x)], \quad (2.1)
\]

\[
A_\mu(x) \rightarrow A_\mu^\Omega(x) = \Omega(x)A_\mu(x)\Omega^{-1}(x) + \partial_\mu\Omega(x)\Omega^{-1}(x) \Rightarrow G_{\mu\nu} \rightarrow G_{\mu\nu}^\Omega = G_{\mu\nu}\Omega^\Omega, \quad \Omega \in SU(\mathcal{N}). \quad (2.2)
\]

\[
S_0(A) = -\frac{1}{4g^2} \int d^dx G_{\mu\nu}(x)G^{\mu\nu}(x), \quad G_{\mu\nu} = G_{\mu\nu}^m(x)t^m, \quad G_{\mu\nu}^m = \partial_{[\mu}A_{\nu]}^m + f_{mn}^a A_{\mu}^a A_{\nu}^m, \quad (2.3)
\]

\[
\delta A_\mu^m(x) = D_\mu^m(x)\xi^m(x) = \int d^dy R_{\mu}^m(x;y)\xi^m(y), \quad \text{where } i = (\mu, m, x), \alpha = (n, y). \quad (2.4)
\]

Here \( G_{\mu\nu}(x), \Omega(x) \), \( g \) and \( D_{\mu}^m(x) = \delta^{mn}\partial_n + f_{mn}^a A_{\mu}^a(x) \) are by the field strength, arbitrary gauge function taking theirs values in \( SU(\mathcal{N}) \), (dimensionless for \( d = 4 \)) coupling constant, covariant derivative with completely antisymmetric structural constants \( f_{mn}^{\alpha} \): \( g^{[m} t^n] = g f^{mn\alpha}_{\mu} \) of \( \text{su}(\mathcal{N}) \) and local generators of gauge transformations, \( R_{\mu}^m(x;y) = D_\mu^m(x)\delta(x-y) \), whereas for the infinitesimal gauge transformations \( \delta A_\mu^m(x) = \delta A^m_{\mu}(x) \) the representation, \( \Omega(x) = 1 + \zeta^m(x)t^m \) holds.

#### 2.1 Review of \( N = 1, 2 \) BRST symmetry

In the case of usual BRST symmetry, the path integral, be it in Landau [12], Feynman [13], or arbitrary admissible gauges, may be uniquely presented using a local quantum action, \( S_\Phi = S_\Phi(\Phi) \) in the space \( \mathcal{M}^{(N=1)}_\text{tot} \equiv \mathcal{M}_\text{tot} \) of fields \( \Phi^A \):

\[
Z_\Phi = \int d\Phi \exp \left\{ \frac{i}{\hbar} S_\Phi(\Phi) \right\}, \quad \text{with } S_\Phi = S_0 + \Psi(\Phi)\hat{\Psi} = S_0 + \left\{ \mathcal{C}M(A)C + \chi(A, B)B \right\}, \quad (2.5)
\]

for \( M(A) = \int dy (\partial_{A}^\mu(y)\chi(A, B))D^\mu, \) with the help of a gauge fermion \( \Psi(\Phi) \), encoding the gauge by a gauge function \( \chi(A, B) \) linear in the fields \( A_\mu, B \):

\[
\Psi(\Phi) = \mathcal{C}\chi(A, B) + \hat{\Psi}(\Phi), \quad \epsilon(\Psi) = 1, \quad \text{for } \text{deg}_\phi \hat{\Psi} > 2, \text{ deg}_\phi \chi(A, B) = 1 \quad (2.6)
\]

with the use, first, the condensed notations in \( \mathcal{C} \) and \( \mathcal{C} \), implying the integration over some region in \( \mathbb{R}^{1:d-1} \) and trace over \( \text{su}(\mathcal{N}) \) indices, second, of a nilpotent Grassmann-odd “right-hand” (left-hand) Slavnov generator \( \frac{\hat{\psi}}{s} \) (s), \( \frac{s}{\hat{s}} = 0 \), \([13]\) of \( N = 1 \) BRST transformations acting on the local coordinates
The quantum action $S_\Psi$ and the integration measure $d\Phi$ are invariant under BRST transformations $\Phi^A \to \Phi'{}^A$ with a constant parameter $\mu$,

$$\Phi'^A = \Phi^A(1 + \frac{\lambda}{s} \mu) : \delta_\mu \Phi^A = \Phi^A \frac{\lambda}{s} \mu \implies \delta_\mu S_\Psi = 0, \quad \text{sdet}(\delta\Phi'/\delta\Phi) = 1, \quad (2.8)$$

providing the invariance of the integrand in $Z_\Psi$ with respect to these transformations. In turn, for the generating functionals of Green’s functions, as well as of correlated and one-particle irreducible Green’s functions (known as well as, the effective action $\Gamma(\Psi^A)$), depending, respectively, on the external sources $J_A$, $\epsilon(J_A) = \epsilon_A$ and mean fields, $\langle \Phi^A \rangle$, we have

$$Z(J) = \int d\Phi \exp \left\{ \frac{i}{\hbar} S_\Psi(\Phi) + J_A \Phi^A \right\} = \exp \left\{ \frac{i}{\hbar} W(J) \right\}, \quad \Gamma(\langle \Phi^A \rangle) = W(J) - J_A \langle \Phi^A \rangle \quad (2.9)$$

by means of a Legendre transformation of $W(J)$ with respect to $J_A$, for $\langle \Phi^A \rangle = \frac{\partial}{\partial J_A} W$, with $J_A = -(\delta \Gamma / \delta \langle \Phi^A \rangle)$. $N = 1$ BRST transformations lead to the presence of respective Ward identities:

$$J_A \langle \Psi^A \rangle_{\Psi, J} = 0, \quad J_A \langle \langle \Phi^A \rangle_{\Psi, J} \rangle_{\Psi, J} = 0, \quad \frac{\delta \Gamma}{\delta \langle \Phi^A \rangle} \langle \langle \Phi^A \rangle_{\Psi, J} \rangle_{\Psi, J} = 0, \quad (2.10)$$

with respective normalized average expectation values $\langle \langle L \rangle_{\Psi, J} \rangle, \langle \langle L \rangle_{\Psi, J} \rangle, \langle \langle L \rangle_{\Psi, J} \rangle, \langle \langle \Phi^A \rangle_{\Psi, J} \rangle$ for a functional $L = L(\Phi)$ calculated using $Z(J)$, $W(J)$, $\Gamma$ for a given gauge fermion $\Psi$, with the external sources $J_A$ and $\langle \Phi^A \rangle$.

The infinitesimal field-dependent (FD) $N = 1$ BRST transformations with a functional parameter $\mu(\Phi) = (1/\hbar) \delta \Psi$ allow one to establish gauge-independence for the path integral $Z_\Psi$ under an infinitesimal variation of the gauge condition, $\Psi \to \Psi + \delta \Psi$, due to an input from the superdeterminant of the change of variables (2.8), $\text{sdet}||\Phi^A \frac{\partial}{\partial \Phi}|| = (1 - \mu(\Phi)) \frac{\lambda}{s}$, in the integrand of $Z_{\Psi + \delta \Psi}$:

$$Z_{\Psi + \delta \Psi} = \int d\Phi \text{sdet}||\Phi^A \frac{\partial}{\partial \Phi}|| \exp \left\{ \frac{i}{\hbar} S_{\Psi + \delta \Psi}(\Phi) \right\} = Z_\Psi. \quad (2.11)$$

In turn, finite FD $N = 1$ BRST transformations, whose set enlarges the Abelian supergroup, $G(1) = \{ g(\mu), g(\mu) = 1 + \frac{\lambda}{s} \mu \}$, acting in $\mathcal{M}_{tot}$ and providing an non-Abelian supergroup, $G(1) = \{ \tilde{g}(\mu), \tilde{g}(\mu) = 1 + \frac{\lambda}{s} \mu(\Phi) \}$ with $\tilde{g}(\mu_1) \tilde{g}(\mu_2) = \tilde{g}(\mu_1 \tilde{g}(\mu_2)) \tilde{g}(\mu_2 \tilde{g}(\mu_1)) = \tilde{g}(\mu_2) \tilde{g}(\mu_1)$, introduced for the first time in [23], allow one to obtain a new form of the Ward identities, depending on an FD parameter, and to establish gauge-independence for the path integral $Z_\Psi$ under a finite change of the gauge, $\Psi \to \Psi + \Psi': \quad Z_\Psi = Z_{\Psi + \Psi'}$. In this case, the superdeterminant of a change of variables (2.8), $\text{sdet}||\Phi^A \frac{\partial}{\partial \Phi}|| = (1 + \mu(\Phi)) \frac{\lambda}{s}^{-1}$, calculated in [24] – see also [25] for general gauge theories – implies a modified Ward identity:

$$\langle \exp \left\{ \frac{i}{\hbar} J_A \Phi^A \tilde{\Psi} \mu(\Psi') \right\} (1 + \mu(\Psi')) \frac{\lambda}{s}^{-1} \rangle_{\Psi, J} = 1, \quad (2.12)$$

for $\mu(\Psi') = \frac{i}{\hbar} g(y) \Psi'$, $\left[ y, g(y) \right] = \left[ i/\hbar \right] \Psi' \frac{\lambda}{s}, \quad (1 - \exp\{y\})/y$, (2.13)

and leads to a solution of the gauge dependence problem for the generating functional $Z_\Psi(J)$:

$$Z_{\Psi + \Psi'}(J) - Z_\Psi(J) = \frac{i}{\hbar} J_A \left\{ \Phi^A \tilde{\Psi} \mu(\Phi - \Psi') \right\}_{\Psi, J} = \left( Z_{\Psi + \Psi'}(J) - Z_\Psi(J) \right)_{J=0} = 0. \quad (2.14)$$
For an \( N = 2 \) BRST symmetry realization for the quantum local action we, once again, follow the Faddeev–Popov proposal \( \text{(1.2)} \), where, instead of the gauge function \( \chi(\mathcal{A}) \), a Grassmann-even gauge functional \( Y(\mathcal{A}) \), \( \epsilon(\mathcal{Y}) = 0 \), is utilized:

\[
Z_L = \int d\mathcal{A} \left( \frac{1}{\hbar} \mathcal{Y}(\mathcal{A}) \right) \det \mathcal{M}(\mathcal{A}) \exp \left\{ -\frac{i}{\hbar} S_0(\mathcal{A}) \right\} ; \quad \text{for } \mathcal{Y}(\mathcal{A}) = \frac{\delta \mathcal{Y}}{\delta A^\mu} D^\mu = \partial_\mu A^\mu \Leftrightarrow \chi_a^Y = Y_i \mathcal{F}_i^a \quad (2.15)
\]

(for \( Y_i \equiv \delta \mathcal{Y}/\delta A^i \), \( A^i = A^\mu(x) \)) which leads to a local representation for the path integral in the same configuration space \( \mathcal{M}_{(N=2)}(N=1) \) of fields \( \Phi^A \), arranged into \( \text{Sp}(2) \)-doublet as \( \Phi^A = (A_\mu, C^a, B, C^{\text{conf}}, B^m) \)

\[
Z_Y = \int d\Phi \exp \left\{ \frac{1}{\hbar} S_Y(\Phi) \right\} , \quad \text{with } S_Y = S_0 - \frac{1}{4} Y_s \frac{\mathcal{F}_a}{\mathcal{F}_s} a \text{ and } -\frac{1}{4} Y_s \frac{\mathcal{F}_a}{\mathcal{F}_s} a = \Psi(\Phi)^{\mathcal{F}_s}. \quad (2.16)
\]

The functional \( (2.16) \), in the Feynman gauge condition, providing a particular representative (for \( \xi = 1 \)) from the class of \( R_\xi \)-gauges, \( \partial_\mu A^\mu + \xi g^2 B \) (Landau gauge for \( \xi = 0 \)), takes the form

\[
Z_{Y_\xi} = \int d\Phi \exp \left\{ \frac{1}{\hbar} S_{Y_\xi}(\Phi) \right\} ; \quad \text{for } Y_\xi(\Phi) = \frac{1}{2} \int d^dx \left( -\partial_\mu A^\mu + \xi g^2 \varepsilon_{ab} C^a C^b \right),
\]

\[
S_{Y_\xi}(\Phi) = S_0 - \frac{1}{4} Y_{\xi} \frac{\mathcal{F}_a}{\mathcal{F}_s} a = S_0 + S_{\text{gf}} + S_{\text{gh}} + S_{\text{add}}, \quad (2.17)
\]

where the gauge-fixing term \( S_{\text{gf}} \) and the ghost term \( S_{\text{gh}} \) coincide with \( N = 1 \) BRST exact term \( \Psi(\Phi) \frac{\mathcal{F}_s}{\mathcal{F}_s} \) in the \( N = 1 \) BRST invariant quantum action \( S_{Y_\xi} \), for \( \xi = 1 \), whereas the interaction term \( S_{\text{add}} \), quartic in ghosts \( C^{sa} \), specific for the \( N = 2 \) BRST symmetry, is given by

\[
S_{\text{gf}} + S_{\text{gh}} = \int d^dx \left[ \partial_\mu A^\mu + \xi g^2 B^m \right] B^m + \frac{1}{2} \int d^dx \left( \partial_\mu C^{ma} \right) D_{\mu mn} C^{nb} \varepsilon_{ab}, \quad (2.19)
\]

\[
S_{\text{add}} = -\frac{\xi g^2}{24} \int d^dx \ f_{\text{ermal}} f_{\text{lers}} C^{sa} C^{rb} C^{md} \varepsilon_{a b c d}. \quad (2.20)
\]

The quantum action and integration measure are invariant with respect \( N = 2 \) BRST symmetry transformations at the algebraic level, with right-hand Grassmann-odd generators \( \frac{\mathcal{F}_a}{\mathcal{F}_s} b + \frac{\mathcal{F}_b}{\mathcal{F}_s} a = 0, a, b = 1, 2 \)

for \( \Phi A \rightarrow \Phi(\mathcal{F}_s) \mu_a \)

\[
(A_\mu, C^a, B) \frac{\mathcal{F}_a}{\mathcal{F}_s} = \left( D_\mu C^a, \varepsilon_{ba} B + \frac{1}{2} [C^b, C^a], \frac{1}{2} ([B, C^a] + \frac{1}{2} [C^b, C^a]) \varepsilon_{ab} \right). \quad (2.21)
\]

As in the \( N = 1 \) BRST case, this invariance, for the corresponding generating functionals of Green’s functions, \( Z_Y(J) = \exp \{ (\alpha g^2) W_Y(J) \}, \Gamma_Y(\Phi) \) constructed by the rules \( (2.9) \) with a given gauge condition \( Y(\Phi) \), leads to the presence of an \( \text{Sp}(2) \)-doublet of Ward identities:

\[
J_A(\langle \Phi A \frac{\mathcal{F}_a}{\mathcal{F}_s} \rangle)_{Y,J} = 0, \quad J_A(\langle \Phi(\mathcal{F}_s) \mu_a \rangle)_{Y,J} = 0, \quad \frac{\delta \Gamma_Y}{\delta(\Phi A)}(\langle \Phi(\mathcal{F}_s) \mu_a \rangle)_{Y,(\Phi)} = 0, \quad (2.22)
\]

with respective normalized average expectation values \( \langle L \rangle_{Y,J}, \langle (L) \rangle_{Y,J}, \langle \langle L \rangle \rangle_{Y,(\Phi)} \) for a functional \( L = L(\Phi) \) calculated using \( Z_Y(J), W_Y(J), \Gamma_Y \) for a given gauge boson \( Y \) in the presence of external sources \( J_A \) and mean fields \( \langle \Phi A \rangle \). The gauge independence of the path integral \( Z_Y(0) \) under an infinitesimal variation of the gauge condition, \( Y \rightarrow Y + \delta Y \), is established using the infinitesimal field-dependent (FD) \( N = 2 \) BRST transformations \( \text{(3.3), (3.4)} \) with the functional parameters \( \mu_a(\Phi) = (\alpha g^2) \delta \mathcal{Y} \frac{\mathcal{F}_a}{\mathcal{F}_s} a \) which induce the superdeterminant for the change of variables \( \text{(2.8)} \) made in the integrand of \( Z_{Y + \delta Y}, \text{sdet} \| \Phi A \frac{\mathcal{F}_s}{\mathcal{F}_s} B \| = 1 - \mu_a(\Phi) \frac{\mathcal{F}_s}{\mathcal{F}_s} a \), as follows:

\[
Z_{Y + \delta Y} = \int d\Phi \text{sdet} \| \Phi A \frac{\mathcal{F}_s}{\mathcal{F}_s} B \| \exp \left\{ -\frac{i}{\hbar} S_{Y + \delta Y}(\Phi) \right\} = Z_Y. \quad (2.23)
\]

\[\text{For } g = 1, \text{ the expressions for } S_{\text{gf}} \text{ (2.19) and } S_{\text{add}} \text{ (2.20) coincide with ones in (26) after rescaling } \xi \rightarrow \frac{1}{4} \xi.\]
The finite $N = 2$ BRST transformations acting in $\mathcal{M}_{\text{tot}}$, whose set forms an Abelian supergroup,

$$G(2) = \left\{ g(\mu_a) : g(\mu_a) = 1 + \frac{1}{2} \phi^a \mu_a + \frac{1}{4} \phi^a \phi^b \mu_a \mu_b = \exp \left( \phi^{a} \phi^{a}_a \right) \right\}, \quad (2.24)$$

are restored from the algebraic $N = 2$ BRST transformations according to [26]:

$$\{K(g(\mu_a)\Phi) = K(\Phi) \text{ and } K\phi^a = 0\} \Rightarrow g(\mu_a) = \exp \left( \phi^{a} \phi^{a}_a \right), \quad (2.25)$$

where $K = K(\Gamma)$ is an arbitrary regular functional, and $\phi^a, \phi^a = \phi^a \phi^a_a$ are the generators of BRST-antiBRST and mixed BRST-antiBRST transformations in the space of $\Phi^A$. These finite transformations, in a manifest form [26], for $\Delta \Phi^A = \Phi^A - \Phi^A$, read as follows:

$$\Delta A_a = D_a C^o \mu_a - \frac{1}{2} \left[ D_a B + \frac{1}{2} [C^o, D_a C^b] \epsilon_{ab} \right] \mu^2,$$

$$\Delta B = \frac{1}{2} \left[ [B, C^o] + \frac{1}{6} [C^o, C^b] \epsilon_{cb} \right] \mu_a ,$$

$$\Delta C^b = \epsilon^{ba} B + \frac{1}{2} [C^b, C^a] \mu_a + \frac{1}{2} \left( [B, C^b] + \frac{1}{6} [C^c, C^b] \epsilon_{cb} \right) \mu^2 ,$$

and cannot be presented as group elements (in terms of an exp-like relation) for an Sp(2)-doublet $\mu_a(\Phi)$ which is not closed under BRST-antiBRST transformations: $\mu_a(\Phi) \phi^a b \neq 0$. Once again, the finite FD $N = 2$ BRST transformations with functionally-dependent parameters $\mu_a = \Lambda^a \phi^a_a$ allow one to derive a new form of the Ward identities, depending on FD parameters, and to study gauge-independence for the generating functionals, e.g., $Z_Y(J)$ and $Z_{Y'}$, under a finite change of the gauge, $Y \rightarrow Y + Y'$, $Z_Y \rightarrow Z_{Y']}. \quad (2.27)

Now, the superdeterminant for a change of variables: $\Phi^A \rightarrow \Phi'^A = \Phi^A g(\mu_a(\Phi))$, $\mathrm{sdet} ||g(\mu_a(\Phi))|| = (1 - \frac{1}{2} \Lambda(\Phi) \phi^a \phi^a_a)^{-2}$, calculated in [26] – see also [31] for general gauge theory and general form of FD parameters $\mu_a$ – leads to a modified Ward identity depending on the parameters $\mu_a(Y') = \mu_a(Y')$, for $y \equiv (i/4\hbar)Y' \phi^a_a$, \quad $\Lambda(\Phi|Y') = \frac{i}{4\hbar} g(y) Y' \phi^a_a$, \quad [30], [31]

$$\langle \left\{ 1 + \frac{1}{2} J_A \Phi^A \left[ \phi^a \mu_a(\Lambda) + \frac{1}{2} \phi^a \phi^2 \mu^2(\Lambda) \right] - \frac{1}{2} \left( \frac{1}{2} \right)^2 J_A \Phi^A \phi^a \mu_a \left( \Phi \Phi^B \phi^a \phi^B \phi^a \right) \mu^2(\Lambda) \right\} \rangle (2.29)$$

$$\times (1 - \frac{1}{2} \Lambda(\phi^a \phi^a_a)^{-2})_{Y,J} = 1,$$

$$Z_{Y + Y'}(J) - Z_Y(J) = \langle \frac{1}{2} J_A \Phi^A \left[ \phi^a \mu_a (\Phi - Y') + \frac{1}{2} \phi^a \phi^2 \mu^2 (\Phi - Y') \right] \rangle \quad \text{vanishing on the mass shell determining by the hypersurface } J_A = 0.$$

Now, we have all the things prepared to generalize the Faddeev–Popov procedure in order to realize a more general case of $N = 3$ BRST symmetry for an appropriate local quantum action depending on the entire set of fields, on which the latter symmetry transformations are defined.

### 2.2 Proposal for non-local Faddeev–Popov path integral with $N = 3$ BRST symmetry

There are many ways to present the functionals [12], [13] without using a determinant and a functional $\delta$-function within perturbation techniques. In the case of Landau and Feynman gauges, we generalize the path integral [12], [13] by the rule

$$Z_0^F = \int dA \delta(\chi(A)) \det M(A) \det^k M(A) \det^{-k} M(A) \exp \left\{ \frac{i}{\hbar} S_0(A) \right\}, \quad k \geq 0, \quad (2.31)$$

$$Z_0^F = \int dAB \det M(A) \det^k M(A) \det^{-k} M(A) \exp \left\{ \frac{i}{\hbar} S_0(A) + \int d^4x \left( \partial^\mu A_\mu + g^2 B \right) \right\}. \quad (2.32)$$
The path integral formulations with local quantum action exist for any $k \in \mathbb{N}_0$ as follows, e.g. for (2.31):

$$Z_0^L = \int dAdB \prod_{l=0}^{k} dC^l d\overline{C}^l \prod_{l=1}^{k} dB^l d\overline{B}^l \exp \left\{ \frac{i}{\hbar} \tilde{S}_L^{(k)}(A, C^0, \overline{C}^0, C^{|k|}, \overline{C}^{|k|}, B^{|k|}, \overline{B}^{|k|}, B) \right\}$$  \hspace{1cm} (2.33)

with $\tilde{S}_L^{(k)} = S_0 + \int d^d x \left\{ \sum_{l=1}^{k} \left( \overline{C}^0 M(A) C^l + \overline{B}^l M(A) B^l \right) + \overline{C}^0 M(A) C^0 + \chi(A) B \right\}$, \hspace{1cm} (2.34)

for $D^{(|k|)} = (D^1, ..., D^k)$, $D \in \{C, \overline{C}, B, \overline{B}\}$, $(C^0, \overline{C}^0) \equiv (C, \overline{C})$,

where odd-valued fields $C^{|k|}, \overline{C}^{|k|}$ and even-valued fields $B^{|k|}, \overline{B}^{|k|}$ taking values in Lie group $G$, whose numbers coincide.

However, it is not for any $k$ there exists a local representation for the path integral (2.33) such that the total set of fields, $\tilde{\Phi}^{(k)} = (A, C^0, \overline{C}^0, C^{|k|}, \overline{C}^{|k|}, B^{|k|}, \overline{B}^{|k|}, B)$ forms the representation space of Abelian group of SUSY transformations, like $N = 1, 2$ BRST symmetry, for $k = 0$, but with larger numbers of $N \geq 3$, so that the Grassmann-odd: with $C^l, \overline{C}^l$; Grassmann-even: with $B^l, \overline{B}^l$ ghost actions with Faddeev-Popov operator and gauge-fixed term with $\chi(A) B$ would be generated as the exact terms with respect to the action of being searched $N$-parametric generators of BRST symmetry transformations.

More exactly, the fact is that

**Statement 1:** In order the action functional $\tilde{S}_L^{(k)}$, (2.33) to be given on the configuration space of fields $\tilde{\Phi}^{(k)} = (A, C^0, \overline{C}^0, C^{|k|}, \overline{C}^{|k|}, B^{|k|}, \overline{B}^{|k|}, B)$ permitting the local presentation for the path integral, $Z^L(0)$, (2.31) in the form (2.33) will be invariant with respect to $N = N(k)$-parametric Abelian SUSY transformations with Grassmann-odd generators $\overline{s}^k q_k$, $s^k \overline{q}_k \overline{s}^k q_k + \overline{s}^k q_k \overline{s}^k p_k = 0$, $q_k, p_k = 1, ..., N$, and will be presented in the form:

$$S_{(N(k))}^L(\Phi_{(N(k))}) = S_0(A) - \frac{(-1)^N}{N!} F_{(N(k))}^{(k)}(\Phi_{(N(k))}) \prod_{\epsilon=1}^{N} \overline{s}^k p_k \epsilon_{\overline{s}^k p_k}^{p_k} = 1, ..., N, \epsilon(F_{(N(k))}) = N, \hspace{1cm} (2.36)$$

with completely antisymmetric $N(k)$-rank (Levi-Civita) tensors $\epsilon_{\overline{s}^k p_k}^{p_k} \overline{s}^k p_k = N!$ for $k > 2$,

$$\epsilon_{\overline{s}^k p_k}^{p_k} \overline{s}^k p_k = N! \text{ for } k > 2, \hspace{1cm} (2.37)$$

with some gauge-fixing functional, $F_{(N(k))}$ corresponding to the Landau gauge, so that the fields $\Phi_{(N(k))}$ should parameterize the irreducible representation superspace of the Abelian superalgebra $G(N(k))$ of $N(k)$-parametric SUSY transformations, the spectra of integer $k = k(N)$ should be found as:

$$1) \hspace{1cm} k(1) = 0, \hspace{0.5cm} k(N) = 2^{N-2} - 1, \hspace{0.5cm} \text{for } N \geq 2. \hspace{1cm} (2.38)$$

If in addition, the gauge-fixing functional $F_{(N(k))}$ should be determined without introducing auxiliary Grassmann-odd scalar or supermatrix the spectra of integer $k = k_u(N)$ is determined by the relation:

$$2) \hspace{1cm} k_u(1) = 0, \hspace{0.5cm} k_u(N) = 2^{|\frac{N-1}{2}|} - 1, \hspace{0.5cm} \text{for } N \geq 2, \hspace{1cm} (2.39)$$

for integer part, $[x]$, of real $x$.

Note, the local path integral $Z_{0}^{L(\Phi_{(N(k))})} = \int d\Phi_{(N(k))} \exp \left\{ \frac{i}{\hbar} \tilde{S}_L^{(k)}(\Phi_{(N(k))}) \right\} \neq Z_0^L$ for $N(k) > 2$ due to the presence of possible additional vertexes in fictitious fields in $S_{(N(k))}^L$. In addition, in the second case

$^7$When the exponential index $k$ in the representations (2.31), (2.32) is related to $N = N(k)$-parametric SUSY transformations we denote the fields parameterizing configuration space, the quantum action and gauge-fixing functional as, $\Phi_{(N(k))}, S_{(N(k))}^L, F_{(N(k))}$ in opposite case we add “tilde” over it: $\tilde{\Phi}^{(k)}$, $\tilde{S}_L^{(k)}$, $\tilde{F}^{(k)}$ so that for $N(k) = k$ in general:

$\Phi_{(N(k))} \neq \Phi^{(k)}, S_{(N(k))}^L \neq S_L^{(k)}$. 

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the requirement of the irreducibility of the \( G(N(k)) \) (finite-dimensional) representations for each \( N(k) \) is weakened. The irreducibility will be hold only for even \( N: N = 2K, K \in \mathbb{N} \).

Indeed, this leads, for \( N = 1, k(1) = 0 \), to the standard Faddeev–Popov representation (1.5) with the BRST symmetry, whereas, for \( k(2) = 0 \), this leads to the \( N = 2 \) BRST symmetry with a local path integral (2.16).

For \( N = 3 \), \( k(3) = 1 \), there arises a first non-trivial case for the case 1 (2.38) and \( k_u(3) = 3 \) for the case 2 (2.39) of the Statement 1. For \( N = 4 \) for both cases we have from (2.38), (2.39): \( k(4) = k_u(4) = 3 \).

The validity of the first part (2.38) follows from the simple fact that any field finite-dimensional irreducible tensor representation superspace of the Abelian superalgebra \( G(N) \) with Grassmann-odd generators \( \bar{u}^p; \prod_{p=1}^{N+1} \bar{u}^p = 0 \), contains in addition to the gauge fields \( A^\mu \), whose infinitesimal gauge transformations are changed on the global transformations with constant Grassmann-odd parameters, \( \lambda_p; \epsilon(\lambda_p) = 1 \):

\[
\delta A^\mu(x) = D^\mu \xi(x) = D^\mu C^\mu(x) \lambda_p = \delta_A A^\mu(x) = A^\mu(x) \bar{u}^p \lambda_p;
\]

(2.40)

(where the summation with respect to repeating indices, \( p \), is implied) the \( N \)-plet of Grassmann-odd fields, \( C^\mu, \frac{1}{2} N(N-1) \) new Grassmann-even fields, \( B_{P^1P^2} \), and so on up to \( N \)-plet of new fields, \( B_{P^1P^2...P^{N-1}} \), \( (\epsilon(B_{P^1P^2...P^{N-1}}) = N - 1 \) and new single field, \( B_{(N)} \), \( (\epsilon(B_{(N)}) = N) \). All the new fields take theirs values in \( su(\tilde{N}) \) and appear from the chain:

\[
A^\mu \bar{u}^p = D^\mu C^\mu, \quad C^\mu \bar{u}^p = B_{P^1P^2} + O(CB), \ldots, \quad (2.41)
\]

\[
B_{P^1P^2...P^{N-1}} \bar{u}^p = B_{P^1P^2...P^{N-1}} + O(CB), \quad B_{P^1P^2...P^{N-1}} \bar{u}^p = B_{P^1P^2...P^{N-1}} + O(CB), \quad (2.42)
\]

\[
B_{(N)} \bar{u}^p = O(CB), \quad (2.43)
\]

generated by a nilpotent of the order \( (N + 1) \) differential-like element, \( \bar{d} : \bar{d} = \sum_p \bar{u}^p \), the such that \( (\bar{d})^{N+1} = 0 \). The length of the chain, \( l \), is equal to, \( l = (N + 1) \), whereas its non-vanishing linear part in fields \( C^\mu, B_{P^1...P_l}, B \), for \( l = 2, ..., N - 1 \), due to the last equation in (2.43) has the length, \( l_{\text{fin}} = N \). The Grassmann-odd and Grassmann-even numbers of new degrees of freedom for additional to \( A^\mu \) fields in the multplet \( \Phi(N(k)) \) of the irreducible representation of the superalgebra \( G(N) \) without decomposition in \( su(\tilde{N}) \) generators \( \bar{u}^m \) are equal to, \( (N-1, 2N-1 - 1) \). Indeed, for \( N = 1 \), the only ghost field \( C(x) \) contains in \( N = 1 \) irreducible multiplet. For \( N = 2 \), two ghost-antighost \( C^a, A^a, a = 1, 2 \) and Wakanishi-Lautrup, \( B_2 = B \) fields. Then, first, extracting the degrees of freedom relating to the usual ghost and antighost \( C, \bar{G} \) and \( B \) fields, second, dividing any subset on pairs of Grassmann-odd (and Grassmann-even) fields as it is given in (2.34), we get to the value of \( k = k(N) \) for the respective exponent of the determinants in (2.34):

\[
\begin{eqnarray}
(2N-1, 2N-1 - 1) \to (2N-1 - 2, 2N-1 - 2) \to \frac{1}{2} (2N-1 - 2, 2N-1 - 2) = 2.38 \quad (2N-2 - 1, 2N-2 - 1). \quad (2.44)
\end{eqnarray}

However, we meet the problem when going to construct the action functional (quantum action), \( S_{L(N(k))} \), by the rule (2.34) for odd \( N = 2K - 1 \), in particular, for \( N = 3 \) SUSY transformations on the \( G(N) \)-irreducible representation superspace. Indeed, the respective gauge-fixing functional, \( F_{(3)}(\Phi_{(3)}) \) due to the linear part of the \( N = 3 \) SUSY transformations (2.41), (2.43) should be, at least, quadratic in the fields \( A^\mu \). The fact that, the Grassmann parity of \( F_{(3)} \) determines it as the fermion, \( \epsilon(F_{(3)}) = 1 \), means the necessity to introduce some additional Grassmann odd non-degenerate supermatrix in order to realize the prescription (2.20) for the quantum action. The details of using of such kind of odd supermatrix, which should both to determine the required Grassmann parity of \( F_{(3)} \) and to change the basis of additional fields \( (C, B) \) in the configuration space parameterized by \( \Phi_{(3)} \) to construct \( N = 3 \) SUSY invariant action \( S_{L(3)}(\Phi_{(3)}) \) for \( k(3) = 1 \) are considered in the Appendix A.

For \( k(N)|_{N=5,6,7...} = 7; 15, \ldots, \) etc, the situation is more involved, and we leave its detailed consideration out of the paper scope (see as well comments in the Conclusions).
The validity of the second part (2.39) of the Statement we consider here only for \( N = 3 \) case, whereas for even, \( N = 2K \), case its both parts (2.38) and (2.39) coincide.

To do so we should determine the total configuration space, \( \mathcal{M}^{(N=3)}_{\text{tot}} \equiv \mathcal{M}^{(3)}_{\text{tot}} \), of fields parameterizing it, \( (\Phi(3), \bar{\Phi}(3)) = \Phi(3) \) being sufficient to construct the (bare) quantum action, \( S^L_{(3)} \), which must form a finite-dimensional field completely reducible representation of Abelian \( G(3) \) superalgebra. That means, that on the fields \( \Phi(3) \) it will be realized the another irreducible representation of \( N = 3 \)-parametric \( G(3) \) superalgebra not being entangled with the irreducible \( G(3) \)-representation acting on the fields \( \Phi(3) \).

First of all, let us find exactly the action of the generators \( \mathcal{S}^p \) of \( \mathcal{G}(N) \)-representation for \( N = 3 \) on the fields \( \Phi(3) \), \( \Phi(3) = (A^\mu, C^{p1}, B^{p1p2}, B^{(3)} = \hat{B}) \) parameterizing irreducible representation superspace from (2.41) – (2.43).

**Lemma 1:** The action of the generators \( \mathcal{S}^p \) of the Abelian superalgebra \( G(3) \) on the fields \( \Phi(3) \) is given by the relations:

\[
\begin{align*}
\downarrow &\leftarrow \mathcal{S}^p \\
A^\mu &\mapsto D^\mu(A)C^p \\
C^{p1} &\mapsto B^{p1p2} + B^{P1p2} + \frac{1}{2}[C^p, C^p] + \frac{1}{6}[C^p, [C^p, C^p]] \\
B^{p1p2} &\mapsto \frac{1}{2}[\hat{B}, C^p] - \left\{ \frac{1}{2}[B^{p1p2}, C^p], C^p \right\} + \frac{1}{6}[B^{p1p2}, C^p, C^p] \epsilon_{p1p2p3} \\
\end{align*}
\] (2.45)

The respective \( N = 3 \) SUSY transformations with triplet of anticommuting parameters, \( \lambda_p \), on the fields \( \Phi(3) \) are determined as: \( \delta_\lambda \Phi(3) = \Phi(3) \mathcal{S}^p \lambda_p \).

To prove the representation (2.45) we start from the boundary condition for such transformation inherited from the gauge transformations for \( A^\mu \) (2.40) and present the realization for the sought-for generators as series:

\[
\mathcal{S}^p \equiv \sum_{\epsilon \geq 0} \mathcal{S}^p : A^\mu \mapsto A^\mu \mathcal{S}^p = A^\mu \mathcal{S}^p = D^\mu(A)C^p \text{ and } C^{p1} \mathcal{S}^p = 0. \] (2.46)

Since, first,

\[
A_\mu \left( \mathcal{S}^p \mathcal{S}^r \mathcal{S}^s + \mathcal{S}^r \mathcal{S}^s \mathcal{S}^p \right) \neq 0,
\] (2.47)

we must add to \( \mathcal{S}^p \) the nontrivial action of new \( \mathcal{S}^p \) on \( C^{p1} \) (vanishing when acting on \( A_\mu \): \( A_\mu \mathcal{S}^p = 0 \)), starting from the Grassmann-even triplet of the fields \( B^{p1p2} = B^{p1p2}_{p1p2} \) (BRST-like variation of \( C^{p1} \))

\[
C^{p1} \mathcal{S}^{p2} = B^{p1p2} + (\kappa_{C^1})^{p1p2}_{r1r2} [C^{r1}, C^{r2}], \quad \text{for } B^{p1p2} = -B^{p2p1} = (B^{12}, B^{13}, B^{23}), \quad \epsilon(B^{p1p2}) = 0
\] (2.48)

(where the summation with respect to repeated indices is assumed) with unknown real numbers: \( \kappa_{C^1}^{p1p2} = \kappa_{C^1}^{p1p2}_{r1r2} \), to be determined from the consistency of \( 3 \times 3 \) equations:

\[
A_\mu \left( \mathcal{S}^{p1} \mathcal{S}^{p2} \mathcal{S}^{p3} \right) = 0, \quad \text{where } \mathcal{S}^{p1} \mathcal{S}^{p2} \mathcal{S}^{p3} = \sum_{n \geq 0} \mathcal{S}^{s1}_{n}, \quad \text{and } C^{p1} \mathcal{S}^{p2} = 0
\] (2.49)

from which, in fact, follows the property of antisymmetry for \( B^{p1p2} \) in the indices \( p1, p2 \). The solution for (2.49) determines:

\[
\kappa_{C^1}^{p1p2}_{r1r2} = -1/4 \delta^{p1}_{r1} \delta^{p2}_{r2},
\] (2.50)

providing the validity of the 2-nd row in the table (2.45). Having in mind, that any completely antisymmetric tensor, \( \sigma^{p1...p_n} \) of the \( n \)-th rank, is vanishing for \( n > 3 \), there are only the third-rank independent
completely antisymmetric constant tensor with upper, \( \varepsilon^{p_1p_2p_3} = -\varepsilon^{p_1p_3p_2} = -\varepsilon^{p_2p_1p_3} \), and lower, \( \varepsilon_{p_1p_2p_3} \), indices, which are normalized by the conditions (according with (2.37))

\[
\varepsilon^{123} = 1, \quad \varepsilon^{p_1p_2p_3} \varepsilon_{r_1r_2p_3} = (\delta^{p_1}_{r_1} \delta^{p_2}_{r_2} - \delta^{p_1}_{r_2} \delta^{p_2}_{r_1}), \quad \varepsilon^{p_1p_2p_3} \varepsilon_{r_1r_2p_3} = 2 \delta^{p_1}_{r_1}.
\]  

(2.51)

Second, because of

\[
C^{p} \left( \tilde{s}^{p}_{[1]} \varepsilon^{p}_{[1]} + \varepsilon^{p}_{[1]} \tilde{s}^{p}_{[1]} \right) \neq 0,
\]

(2.52)

we should determine, for a nontrivial action of \( \tilde{s}^{p}_{3} \) on \( B^{p_1p_2} \) (vanishing when acting on \( A_{\mu}, C^{p} : (A_{\mu}, C^{p_1}) \tilde{s}^{p}_{2} = 0 \), in the form of a general anzatz, starting from the new Grassmann-odd field variables \( \tilde{B} = B^{p_1p_2} \) (BRST-like variation of \( B^{p_1p_2} \) \((2.43)\)) up to the third power in \( C^{p} \) with a preservation of Grassmann homogeneity in each summand, as in the \((2.48)\),

\[
B^{p_1p_2} \tilde{s}^{p}_{2} = \varepsilon^{p_1p_2p_3} \tilde{B} + (k_{B_1})^{p_1p_2p_3}_{r_1r_2r_3} \left[ B^{r_1r_2}, C^{r_3} \right] + (k_{B_2})^{p_1p_2p_3}_{r_1r_2r_3} \left[ C^{r_1}, [C^{r_2}, C^{r_3}] \right], \quad \epsilon(\tilde{B}) = 1.(2.53)
\]

with unknown real numbers: \( (k_{B_1})^{p_1p_2p_3}_{r_1r_2r_3} = -(k_{B_3})^{p_1p_2p_3}_{r_1r_2r_3}, \quad j = 1, 2; \quad (k_{B_1})^{p_1p_2p_3}_{r_1r_2r_3} = -(k_{B_1})^{p_1p_2p_3}_{r_2r_1r_3}; \quad (k_{B_2})^{p_1p_2p_3}_{r_1r_2r_3} = (k_{B_2})^{p_1p_2p_3}_{r_1r_2r_3} \), to be determined from the fulfillment of the \( 3 \times 3 \times 3 \) equations

\[
C^{p} \left( \tilde{s}^{p}_{2} \tilde{s}^{p}_{[2]} + \tilde{s}^{p}_{3} \tilde{s}^{p}_{[3]} \right) = 0, \quad \text{where } B^{p_1p_2} \tilde{s}^{p}_{l} = 0, \quad l = 0, 1.
\]

(2.54)

Its general solution has the form:

\[
(k_{B_1})^{p_1p_2p_3}_{r_1r_2r_3} = \frac{1}{4} \delta^{p_1}_{r_1} \delta^{p_2}_{r_2} \delta^{p_3}_{r_3}; \quad (k_{B_1})^{p_1p_2p_3}_{r_1r_2r_3} = \frac{1}{2} \left[ B^{p_1p_2}, C^{p_3} \right],
\]

(2.55)

\[
(k_{B_2})^{p_1p_2p_3}_{r_1r_2r_3} = -\frac{1}{12} \delta^{p_1}_{r_1} \delta^{p_2}_{r_2} \delta^{p_3}_{r_3}; \quad (k_{B_2})^{p_1p_2p_3}_{r_1r_2r_3} = -\frac{1}{12} \left[ C^{p_1}, [C^{p_2}, C^{p_3}] \right].
\]

(2.56)

providing the validity of the 3-rd row in the \((2.45)\).

Third, due to

\[
B^{p_1p_2} \left( \tilde{s}^{p}_{3} \tilde{s}^{p}_{[3]} + \tilde{s}^{p}_{4} \tilde{s}^{p}_{[4]} \right) \neq 0,
\]

(2.57)

we should determine for a nontrivial action of \( \tilde{s}^{p}_{3} \) on \( \tilde{B} \), (vanishing when acting on \( A_{\mu}, C^{p}, B^{p_1p_2}, (A_{\mu}, C^{p}, B^{p_1p_2}) \tilde{s}^{p}_{2} = 0 \) a general ansatz without using the new field variables (due to the 4-th order nilpotency of \( \tilde{s}^{p}_{3} \); \( \prod_{l=1}^{2} \tilde{s}^{p}_{l} = 0 \)) up to the fourth order in \( C^{p} \) with a preservation of Grassmann homogeneity in each summand, as in the case of \((2.48)\) and \((2.53)\),

\[
\tilde{B} \tilde{s}^{p}_{3} = (\sigma_{B_1})^{p}_{r_1r_2r_3r_4} \left[ B^{r_1r_2}, C^{r_3r_4} \right] + \left( \sigma_{B_2}^{p} \right)^{p}_{r_1r_2r_3r_4} \left[ C^{r_1}, [C^{r_2}, C^{r_3r_4}] \right] + \left( \sigma_{B_3}^{p} \right)^{p}_{r_1r_2r_3r_4} \left[ B^{r_1r_2}, B^{r_3r_4} \right],
\]

(2.58)

Here unknown real numbers \( \sigma_{B_1}, \sigma_{B_2}, \sigma_{B_3} \), \( \left( \sigma_{B_1}^{p} \right)^{p}_{r_1r_2r_3r_4} = -(\sigma_{B_2}^{p} )^{p}_{r_1r_2r_3r_4}, \sigma_{B_3}^{p} = -(\sigma_{B_3}^{p} )^{p}_{r_1r_2r_3r_4} = -(\sigma_{B_3}^{p} )^{p}_{r_1r_2r_3r_4} = -(\sigma_{B_3}^{p} )^{p}_{r_1r_2r_3r_4} = -(\sigma_{B_3}^{p} )^{p}_{r_1r_2r_3r_4}, \sigma_{B_4}^{p} = -(\sigma_{B_4}^{p} )^{p}_{r_1r_2r_3r_4} = (\sigma_{B_4}^{p} )^{p}_{r_1r_2r_3r_4} \), should be determined from the \( 3 \times 3 \times 3 \) equations:

\[
B^{p_1p_2} \left( \tilde{s}^{p}_{3} \tilde{s}^{p}_{[3]} + \tilde{s}^{p}_{4} \tilde{s}^{p}_{[4]} \right) = 0, \quad \text{where } \tilde{B} \tilde{s}^{p}_{l} = 0, \quad l = 0, 1, 2
\]

(2.59)

Its general solution looks as

\[
(\sigma_{B_1})^{p}_{r_1r_2r_3r_4} = \frac{1}{2}, \quad (\sigma_{B_2}^{p} )^{p}_{r_1r_2r_3r_4} = -\frac{1}{8} \varepsilon_{r_1r_2r_3r_4} \delta^{p}_{r_3r_4} - \frac{1}{12} \tilde{s}^{p}_{r_1r_2r_3r_4} \delta^{p}_{r_3r_4} = \sigma_{B_3}^{p} = \sigma_{B_4}^{p} = 0,
\]

(2.60)

providing the validity of the last row in the table \((2.45)\). In deriving \((2.60)\) the use has been made of the symmetry for the commutator \([C^{p}, C^{r}] = [C^{r}, C^{p}]\), and the following relations

\[
[[B^{p_1p_2}, C^{p_3}], C^{p_4}] + [[B^{p_1p_2}, C^{p_3}], C^{p_4}] = \varepsilon^{p_1p_2p_3} C^{p_4},
\]

(2.61)

\[
[[B^{p_1p_2}, C^{p_3}], C^{p_4}] - [[B^{p_1p_2}, C^{p_3}], C^{p_4}] + [[C^{p_3}, B^{p_1p_2}], C^{p_4}] = \varepsilon^{p_1p_2} Q^{p_4},
\]

(2.62)

for \( P^{p_3} = \frac{1}{2} [[B^{p_1p_2}, C^{p_3}], C^{p_4}] \varepsilon_{pp_1p_2} \), and \( Q^{p_3} = [B^{p_1p_2}, C^{p_3}], C^{p_4} \varepsilon_{pp_1p_2} \).

(2.63)
as well as the Jacobi identities, which establish the absence of the 4-th power in the fields $C^p$ in the transformation for $\hat{\overline{B}}$ (2.55):

\[
B^{p_1p_2} \left( \frac{\hat{s}}{s} \hat{p}_3 \frac{s}{s} + \frac{s}{s} \hat{p}_3 \frac{\hat{s}}{s} \right) \big|_{(\hat{\overline{B}}=-\hat{B}=0)} = \frac{-1}{12} \left[ C^{[p_1}, [C^{p_3}, \{C^{p_2}, C^{p_1]} \right] + \{C^{p_3}, [C^{p_2}, C^{p_1]} \right] \\
+ \{C^{p_2}, [C^{p_3}, C^{p_1}} \right] \big|_{(\hat{\overline{B}}=-\hat{B}=0)} = \varepsilon^{p_1p_2}(\hat{p}_3 \hat{\overline{B}} \hat{p}_3 \hat{p}_1) \big|_{(\hat{\overline{B}}=-\hat{B}=0)} = 0,
\]

(2.64)

meaning that we may put $p_{\hat{B}4} = 0$. One can easily see that the $3 \times 3$ equations (2.55) considered for $\hat{B}$ are fulfilled as well:

\[
\hat{B} \left( \frac{s}{s} \hat{p}_3 \frac{\hat{s}}{s} + \frac{\hat{s}}{s} \hat{p}_3 \frac{s}{s} \right) = 0 \iff \varepsilon^{p_1p_2}(\hat{p}_3 \hat{p}_1) = 0.
\]

(2.65)

Therefore, $\hat{s}^p = \hat{s}^p_{[3]}$ are the generators of the irreducible representation of $G(3)$ superalgebra of $N = 3$-parametric transformations in the field superspace, $M^{(3)}_{\text{min}}$, parameterized by the fields, $\Phi^{(3)}_{\text{A}}$. That fact completes the proof of the Lemma 1.

Thus, in order to have the superspace of irreducible representation being closed with respect to the action of abelian Lie superalgebra $G(3)$ with Grassmann odd scalar generators $\hat{s}^p$ this superspace should parameterized by the set of fields:

\[
\{ \Phi^{(3)}_{\text{A}} \} = \{ \Phi^{(3)}_{\text{A}}, C^p, B^{p_1p_2}, \hat{B} \} = \{ \Phi^{(3)}_{\text{A}}, C^p, B^{p_1p_2}, \hat{B}^{n} \} \quad (2.66)
\]

used as local coordinates in the configuration space $M^{(3)}_{\text{min}}$ with dimension: $\dim M^{(3)}_{\text{min}} = (\hat{N}^2 - 1)(d + 3, 3 + 1)$, for an irreducible gauge theory of the fields $\Phi^{(3)}_{\text{A}}$ with a non-Abelian gauge group $SU(\hat{N})$. It is obvious that $M^{(3)}_{\text{min}} \supset M^{(i)}_{\text{tot}}$ for $i = 1, 2$. We will call $M^{(3)}_{\text{min}}$, as the minimal configuration space.

Now, due to insufficiency of the $M^{(3)}_{\text{min}}$ to provide gauge-fixing procedure without using of additional odd supermatrix or Grassmann-odd parameter let us extend the $M^{(3)}_{\text{min}}$ by the fields $\Phi^{(3)}_{\text{A}}$ of so-called non-minimal sector, starting from a new antighost field, $\overline{C}(x) = \overline{C}^m(x) \tau^m$, to provide a determination of the gauge fermion $F(3) \equiv \Psi^{(3)}_{\text{A}}$ as the quadratic functional for the Landau gauge, $\chi(\mathcal{A}) = 0$:

\[
\Psi^{(3)}_{\text{L}}(\overline{C}, \mathcal{A}) = \int d^d x \ tr \overline{C} \chi(\mathcal{A}).
\]

(2.67)

Properly the fields $\Phi^{(3)}_{\text{A}}$ contain the Nakaniishi-Lautrup fields, $B$, and have the contents

\[
\Phi^{(3)}_{\text{A}} = (\overline{C}, B^p, \hat{B}^{p_1p_2}, \hat{B}), \quad \epsilon(\overline{C}, \hat{B}^{p_1p_2}) + (1, 1) = \epsilon(B^p, \hat{B}) = (0, 0)
\]

(2.68)

with even and odd degrees of freedom, $(3 + 1, 1 + 3)$ (modulo general factor with $\dim SU(\hat{N})$) and determine the action of generators $\hat{s}^p_{(n)}$ of the representation of the Abelian superalgebra $G(3)$ in the superspace, $M^{(3)}_{\text{min}}$, with the local coordinates $\Phi^{(3)}_{\text{A}}$.

**Lemma 2:** The action of the generators $\hat{s}^p_{(n)}$ of the Abelian superalgebra $G(3)$ on the fields $\Phi^{(3)}_{\text{A}}$ is determined by the relations:

\[
\overline{C}^p_{(n)} = B^p, \quad B^{p_1} \overline{s}^p_{(n)} = \hat{B}^{p_1p_2} \overline{s}^p_{(n)} = \varepsilon^{p_1p_2p} B, \quad B^p_{(n)} = 0.
\]

(2.69)

The respective $N = 3$ SUSY transformations with triplet of anticommuting parameters, $\lambda_p$, on the fields $\Phi^{(3)}_{\text{A}}$ are given by the rule: $\delta \chi^{(3)} = \delta \lambda^{(3)} \hat{s}^p_{(n)} \lambda_p$.

Indeed, the relations (2.69) repeat by its form linearized chain (2.41) - (2.43) without non-linear terms. It easy to check, that the generators $\hat{s}^p_{(n)}$ satisfy to the defining relations:

\[
\hat{s}^{p_1}_{(n)} \hat{s}^{p_2}_{(n)} + \hat{s}^{p_2}_{(n)} \hat{s}^{p_1}_{(n)} = 0, \quad \prod_{l=1}^{4} \hat{s}^{p_l}_{(n)} = 0.
\]

(2.70)
In particular, we have the exact sequence
\[
(\mathcal{C}, B^p, \bar{B}^{p+2}, B) \xrightarrow{\bar{\Upsilon}^p_{(3)}} (B^{p+2}, \bar{B}^{p+3}, \varepsilon^{p+2} B, 0) \xrightarrow{\bar{\Upsilon}^{p+3}_{(3)}} (\bar{B}^{p+6}, \varepsilon^{p+6} B, 0, 0, 0) \xrightarrow{\bar{\Upsilon}^{p+6}_{(3)}} 0
\]
of the length, equal to 4.

We will call the representation (2.69) as the $N = 3$ trivial representation of the superalgebra $\mathcal{G}(3)$.

Finally, we construct the reducible representation of the superalgebra $\mathcal{G}(3)$ in the total configuration space, $\mathcal{M}^{(3)}_{\text{tot}}$, parameterized by the fields, $(\Phi^{(3)}_3, \bar{\Phi}^{(3)}_3) = (\Phi^{(3)})$, with dimension in each space-time point $x \in \mathbb{R}^{1,d-1}$,
\[
\dim \mathcal{M}^{(3)}_{\text{tot}} = (N^2 - 1)(d + 3^3 - 1, 2^3).
\]
The generators of this representation we will denote as, $\bar{\Upsilon}^p_{\text{tot}} = \bar{\Upsilon}^p + \bar{\Upsilon}^{p+3}$, and then we will omit index "tot" in it as it done for the generally-adopted notations in $N = 1, 2$ BRST symmetry cases. The action of $\bar{\Upsilon}^p_{\text{tot}}$ is completely determined by (2.45) and (2.69).

Now, let us turn to the gauge-fixing procedure, construction of the quantum action and path integral, whose integrand will be invariant with respect to derived $N = 3$ SUSY transformations.

### 2.3 $N = 3$ BRST-invariant path integral and quantum action

Let us determine the local path integral, $Z_3$, and generating functionals of Green functions in any admissible gauge, turning to the non-degenerate Faddeev-Popov matrix, for Yang-Mills theory underlying above constructed explicit $N = 3$ SUSY invariance (2.45), (2.69) in the total configuration space $\mathcal{M}^{(3)}_{\text{tot}}$, with triplet of anticommuting parameters $\lambda_p$ and the local quantum action $S_\Psi^{(3)}(\Phi^{(3)}_3, \bar{\Phi}^{(3)}_3)$ given by the prescription (2.36) as follows:
\[
Z_3(\Psi(0) = \int d\Phi^{(3)}_3 d\bar{\Phi}^{(3)}_3 \exp \left\{ \frac{i}{\hbar} S_\Psi^{(3)}(\Phi^{(3)}_3, \bar{\Phi}^{(3)}_3) \right\} \text{ with } S_\Psi^{(3)} = S_0(A) + \frac{1}{3!} \Psi^{(3)} \bar{\Upsilon}^p \bar{\Upsilon}^q \bar{\Upsilon}^r \varepsilon_{pqr}, \tag{2.73}
\]
\[
Z_3(\Psi(\bar{J})) = \int d\Phi^{(3)}_3 d\bar{\Phi}^{(3)}_3 \exp \left\{ \frac{i}{\hbar} S_\Psi^{(3)}(\Phi^{(3)}_3, \bar{\Phi}^{(3)}_3) + J_3(\Phi^{(3)}_3, \bar{\Phi}^{(3)}_3) \right\} = \exp \left\{ \frac{i}{\hbar} W_{\Psi}(\bar{J}) \right\}, \tag{2.74}
\]
with gauge fermion functional, $\Psi^{(3)} = \Psi^{(3)}(\Phi^{(3)}_3)$, depending on the fields $\Phi^{(3)}_3$ as follows (confer with (2.16)):
\[
\Psi^{(3)}(\Phi^{(3)}_3) = \mathcal{C}_3(\lambda, A, B) + \bar{\Psi}^{(3)}(\bar{\Phi}^{(3)}_3), \text{ for } \text{deg}_{\Phi^{(3)}_3}(A, B) > 2, \text{ deg}_{\lambda}(A, B) = 1, \tag{2.75}
\]
and external sources $\bar{J}_{A_3} = (\bar{J}_{A_3}, \bar{J}_{A_3}^\uparrow)$ to the respective Green functions related to the fields $\Phi^{(3)}_3, \bar{\Phi}^{(3)}_3$ with the same Grassmann parities: $\epsilon(J_{A_3}) = \epsilon(\Phi^{(3)}_3), \epsilon(J_{A_3}^\uparrow) = \epsilon(\bar{\Phi}^{(3)}_3)$.

It is easy to check that both the functional measure, $d\Phi^{(3)}_3 d\bar{\Phi}^{(3)}_3 = d\bar{\Phi}^{(3)}_3$, as well as the quantum action, $S_\Psi^{(3)}$, are invariant with respect to the change of variables, $\Phi^{(3)}_3 \to \bar{\Phi}^{(3)}_3$ generated by $N = 3$ SUSY transformations (2.45), (2.69), with accuracy up to the first order in constant $\lambda_p$ (equally with infinitesimal $\lambda_p$):
\[
\bar{\Phi}^{(3)}_3 = \bar{\Phi}^{(3)}_3(1 + \bar{\Upsilon}^p \lambda_p) : \delta \lambda \bar{\Phi}^{(3)}_3 = \bar{\Phi}^{(3)}_3 \bar{\Upsilon}^p \lambda_p \Longrightarrow \delta \lambda S_\Psi^{(3)} = o(\lambda), \text{ sdet } \left\| \delta \bar{\Phi}^{(3)}_3/\delta \Phi^{(3)}_3 \right\| = 1 + o(\lambda), \tag{2.76}
\]
We will call, therefore, the transformations:
\[
\delta \lambda \bar{\Phi}^{(3)}_3 = \bar{\Phi}^{(3)}_3 - \bar{\Phi}^{(3)}_3 = \bar{\Phi}^{(3)}_3 \bar{\Upsilon}^p \lambda_p, \tag{2.77}
\]
with the explicit action of the generators $\bar{\Upsilon}^p$, (2.45), (2.69) on the component fields as $N = 3$-parametric BRST transformations.
The particular representations for the path integrals \(2.73, 2.74\) in the Landau and Feynman gauges are easily obtained within the same \(R\) family of the gauges as for the \(N = 1\) BRST invariant case \(2.5\) due to obvious coinciding choice of the gauge functions, \(\chi_{(3)}(A, B)\), for \(\hat{\Psi}_{(3)} = 0\), in \(2.75\) with one, \(\chi(A, B) = (\partial^\mu A_\mu + \xi g^2 B) = 0\), in \(2.6\). The quantum action, \(S_{\Psi_{(3)}(\hat{\Psi}_{(3)})}\), has the representation:

\[
S_{\Psi_{(3)}(\hat{\Psi}_{(3)})} = S_0 + \frac{1}{3!} \psi_{(3)} \xi^\mu \xi^p \xi^q \xi^r \epsilon_{pqr} = S_0 + S_{gf(3)} + S_{gb(3)} + S_{add(3)},
\]

\[
S_{gf(3)} = \int d^4x \text{tr} \left[ \partial^\mu A_\mu + \xi g^2 B \right],
\]

\[
S_{gb(3)} = \int d^4x \text{tr} \left[ B M(A) \hat{B} + \frac{1}{2} \left\{ B^p M(A) B^p + \hat{B}^p M(A) C^r \right\} \epsilon_{pqr} \right],
\]

\[
S_{add(3)} = \frac{1}{6} \int d^4x \epsilon \left\{ -3 (\partial_\mu B^p) \left[ D^\mu (A) C^q, C^r \right] - (\partial_\mu C^r) \left\{ 2 \left[ D^\mu (A) C^q, B^p \right] + \left[ D^\mu (A) B^p, C^r \right] + \left[ \left[ D^\mu (A) C^p, C^q \right], C^r \right] \right\} \epsilon_{pqr},
\]

where we have used the identities,

\[
M(A) \xi s^p = [M(A), C^p] \Rightarrow M^{mn} (A; x, y) \xi s^p = f^{mn} M^s (A; x, y) C^p (y),
\]

\[
(M(A) C^q) \xi s^p = -\partial_\mu \left[ D^\mu (A) C^q, C^p \right] + M(A) (C^q \xi s^p)
\]

\[
= -[M(A) C^p, C^q] - D^\mu (A) C^p, \partial_\mu C^q + M(A) \left\{ B^p + \frac{1}{2} \left[ C^q, C^p \right] \right\},
\]

\[
(AB) \xi s^p = (A \xi s^p) B (-1)^{\epsilon(B)} + A (B \xi s^p),
\]

\[
(AB) \xi s^p \xi s^q \epsilon_{pqr} = \left[ A \xi s^p \xi s^q B + 2 A \xi s^p (B \xi s^q) (-1)^{\epsilon(B)} + A (B \xi s^p \xi s^q) \right] \epsilon_{pqr},
\]

\[
(AB) \xi s^p \xi s^q \xi s^r \epsilon_{pqr} = \left[ A \xi s^p \xi s^q \xi s^r B (-1)^{\epsilon(B)} + 3 A \xi s^p (B \xi s^q \xi s^r) (-1)^{\epsilon(B)}
\]

\[
+ 3 A \xi s^p \xi s^q (B \xi s^r) + A (B \xi s^p \xi s^q \xi s^r) \right] \epsilon_{pqr},
\]

where the latter relations \(2.84-2.86\) appear by readily established Leibnitz-like properties of the generators of \(N = 3\) BRST transformations, \(\xi s^p\) acting on the product of any functions \(A, B\) with definite Grassmann parities depending on the fields \(\Phi_{(3)}\). Indeed, e.g. the validity of \(2.84\) follows from the calculation of variations:

\[
\delta_\lambda A = A \partial_\xi A \left( \Phi_{(3)} \xi s^p \right) \lambda_p \Rightarrow A \partial_\xi A \left( \Phi_{(3)} \xi s^p \right) \equiv A \xi s^p,
\]

\[
\delta_\lambda (AB) = (\delta_\lambda A) B + A (\delta_\lambda B) = (A \xi s^p \lambda_p) B + A (B \xi s^p \lambda_p) = \left\{ A \xi s^p B (-1)^{\epsilon(B)} + A (B \xi s^p) \lambda_p \right\} \epsilon_{pqr},
\]

and the same for the second: \(\delta_\lambda \delta_\lambda (AB)\), and third: \(\delta_\lambda \epsilon_\lambda \lambda (AB)\) variations.

For instance, the ghost-dependent functional, \(S_{add(3)}\), with cubic and quartic in fictitious fields terms is derived from the expression:

\[
S_{add(3)} = \frac{1}{6} \int d^4x \text{tr} \left\{ 3 B^p \left\{ [M(A) C^q, C^r] + \left[ D^\mu (A) C^q, \partial_\mu C^r \right] \right\}
\]

\[
+ \left[ C^q, C^r \right] \epsilon_{pqr} \equiv 0, \text{ and have used of the antisymmetry in } p, q, r \text{ as well as the integration by parts. The representation } 2.81 \text{ immediately follows from } 2.83. \text{ Note, the each term in } S_{add(3)}, \text{ which determine the interaction vertexes from the sector of fictitious fields, contains}
the space-time differential operator for any gauge from \( R_\xi \)-gauge, that looks as more nontrivial analog of \( S_{\text{add}} \) for \( N = 2 \) BRST symmetry.

Let us study some consequences of the suggested \( N = 3 \) BRST transformations. As in the \( N = 1, 2 \) BRST case, the \( N = 3 \) invariance, for the corresponding generating functionals of Green's functions, \( Z_3|\Psi(J) \), \( W_3|\Psi(J) \) and effective action, \( \Gamma_3|\Psi(\tilde{\Phi}(3)) \):

\[
\Gamma_3|\Psi(\tilde{\Phi}(3)) = W_3|\Psi(J) - \tilde{J}_A^I \tilde{\Phi}(A^I) - \tilde{J}_A^I = -\delta\Gamma_3|\Psi/\delta(\tilde{\Phi}(A^I)), \quad \tilde{\Phi}(A^I) = \tilde{\partial}^I/J W_3|\Psi(J),
\]

(2.90)

with a given gauge condition \( \Psi(3) \tilde{\Phi}(3) \), leads to the presence of an \( G(3) \)-triplet of Ward identities:

\[
\tilde{J}_A^I (\tilde{\Phi}(3)^I \tilde{s}^p) \Psi(3)\tilde{J} = 0, \quad \tilde{J}_A^I (\langle \tilde{\Phi}(3)^I \tilde{s}^p \rangle) \Psi(3)\tilde{J} = 0, \quad \frac{\delta\Gamma_3|\Psi}{\delta(\tilde{\Phi}(3)^I)} (\langle \tilde{\Phi}(3)^I \tilde{s}^p \rangle) \Psi(3)\tilde{J} = 0,
\]

(2.91)

with respective normalized average expectation values \( \langle M \rangle_{\Psi(3)\tilde{J}} \), \( \langle \langle M \rangle \rangle_{\Psi(3)\tilde{J}} \), \( \langle \{ M \} \rangle_{\Psi(3)\tilde{J}} \), so that\( \langle \{ 1 \} \rangle_{\Psi(3)\tilde{J}} = 1 \), for a functional \( M = M(\tilde{\Phi}(3)) \) calculated using \( Z_3|\Psi(J) \), \( W_3|\Psi(J) \), \( \Gamma_3|\Psi \) for a given gauge fermion \( \Psi(3) \) in the presence of external sources \( \tilde{J}_A^I \) and mean fields \( \langle \tilde{\Phi}(3)^I \rangle \). The gauge independence of the path integral \( Z_3|\Psi(0) = Z_3|\Psi(3) \) under an infinitesimal variation of the gauge condition, \( \Psi(3) \rightarrow \Psi(3) + \delta\Psi(3) \):

\[
Z_3|\Psi(\xi)(0) = Z_3|\Psi(3)(0).
\]

(2.92)

is established using the infinitesimal FD \( N = 3 \) BRST transformations with the functional parameters,

\[
\lambda_p(\bar{\Phi}) = \frac{1}{3!}\left(\gamma/\hbar\right)\delta\Psi(3)\tilde{\Phi}(\bar{\Phi})\bar{s}^q\bar{s}^r\bar{s}^t \delta_{pqr},
\]

(2.93)

which we consider in details in the Section [4].

The equivalence of \( N = 3 \) and \( N = 1 \) BRST invariant path integrals \( Z_3|\Psi(0) \) \( Z_3 \Psi(3) \), e.g in \( R_\xi \)-like gauges immediately follows from the structure of the quantum action \( S_{\Psi(3)\xi} \) \( S_{\Psi(3)\xi} \). Indeed, integrating by the fields \( \tilde{B}^{pq} \), second, with respect to \( C^p \), then trivially with respect to \( B^p \) and \( B^{pq} \) we get:

\[
Z_3|\Psi(\xi)(0) = \int d\tilde{\Phi}(3)\tilde{s} d\tilde{B}^{pq} d\tilde{B}^{pq} \det^3 M(A)\delta(C^p) \exp\left\{ \frac{i}{\hbar} S_{\Psi(3)\xi}(\bar{\Phi}) - \frac{1}{2} \int d^d x \text{ tr } \tilde{B}^{pq} M(A) C^r \delta_{pqr} \right\}
\]

\[
\int d\tilde{B}^{pq} d\tilde{B}^{pq} \det^3 M(A) \det^{-3} M(A) \delta(B^{pq}) \exp\left\{ \frac{i}{\hbar} S_{\Psi(\xi)}(\bar{\Phi}) \right\} = Z_3 \Psi,
\]

(2.94)

where, e.g. \( \delta(C^p) = \prod_k \prod_{k=1}^3 \delta(C^k(x)) \) appears by the functional \( \delta \)-function and \( S_{\Psi(\xi)}(\bar{\Phi}) \) is the \( N = 1 \) BRST invariant quantum action \( S_{\Psi(\xi)} \) given in the \( R_\xi \)-gauge. The functional \( Z_3 \Psi \) coincides with one given in \( S_{\Psi(\xi)} \) after identification for the field \( \tilde{B} \) as \( \tilde{B} = C \) which plays now the role of the ghost field.

The crucial point of the found \( N = 3 \) BRST symmetry transformations in \( M_{\text{tot}} \) that the whole fields \( \tilde{\Phi}(3) \) due to the relations \( S_{\Psi(3)\xi} \) \( S_{\Psi(3)\xi} \) of the Statement leading to: \( k_u(3) = k(4) = k_u(4) \), maybe organized in the respective multiplet of \( N = 4 \) field irreducible SUSY transformations with constant 4 Grassmann-odd parameters, \( \lambda_r, \tau = 1, 2, 3, 4 \). The construction of the respective \( N = 4 \) SUSY transformations will be the main aim of the next Section.

### 3 \( N = 4 \) global SUSY transformations

Before introducing the \( N = 4 \) SUSY transformations we consider additional \( N = 1 \)-parametric SUSY transformations in \( M_{\text{tot}} \) with new Grassmann-odd nilpotent generator, \( \bar{C} \), parameter, \( \bar{\lambda} : (\bar{\lambda}^2, \bar{\lambda}^2) = 0 \), anticommuting with triplet of \( \lambda^p : \lambda^p + \lambda^p \lambda = 0 \), where as for \( N = 1 \) antiBRST transformations \[17, 18\] the role of the antighost field \( \bar{C} \), as well as the rest multiplet \( \bar{\Phi}(3) \) from the non-minimal sector should be considered in opposite way ac compared to the multiplet \( \Phi(3) \) = \( (A^p, C^p_1, B^{p_1 p_2}, \bar{B}) \) from the \( G(3) \) irreducible (minimal) representation.
3.1 Additional $N = 1$ BRST transformations on the fields of $N = 3$ representation

It is valid the following

**Lemma 3**: The action of the generator $\vec{s}$ of the Abelian superalgebra $\mathcal{G}(1)$ on the fields $(\Phi_{(3)}, \Phi_{(3)})$ parameterizing $\mathcal{M}^{(3)}_{\text{tot}}$ is determined by the relations:

\[
\begin{array}{c|c}
A^µ & D^µ(A) \bar{C} \\
\hline
\bar{C} & \frac{1}{2} [\bar{C}, \bar{C}] \\
B^{p_1} & [B^{p_1}, \bar{C}] \\
\bar{B}^{p_1} & [\bar{B}^{p_1}, \bar{C}] \\
C^{p_1} & B^{p_1} + [C^{p_1}, \bar{C}] \\
\bar{B}^{p_1} & [\bar{B}^{p_1}, C] \\
\bar{B}^{p_1} & B \\
B & 0 \\
\hline
\end{array}
\] (3.1)

The respective $N = 1$ SUSY transformations with anticommuting parameter, $\lambda$, on the fields $\vec{\Phi}_{(3)}$ are given by the rule: $\delta \vec{\Phi}_{(3)} = \vec{\Phi}_{(3)} \vec{s} \lambda$.

Note, the transformations (3.1) reflects the fact that only the field $\bar{C}(x)$ appears by the active (as compared to $C^p$) connection.

To prove the correctness of (3.1) it is sufficient to check, the nilpotency of $\vec{s}$ on each field from the multiplet, because of the homogeneity in Grassmann grading is obvious. The nilpotency calculated on the gauge field $A^µ$: $A^µ \vec{s}^2 = 0$, means that the set of local generators of the gauge transformations, $R^µ_\alpha(A) = R^µ_n(x, y)$ (2.4), for $\epsilon_\alpha = 0$, forms the local algebra Lie (as well as for the case of Lemma 1 (2.49) but for $\vec{s}^p$):

\[
R^µ_\alpha(A) \overleftrightarrow{\partial_j} R^µ_\beta(A) - (-1)^{e_\alpha e_\beta} R^µ_\beta(A) \overleftrightarrow{\partial_j} R^µ_\alpha(A) = - F^µ_\alpha F^µ_\beta, \text{ for } F^µ_\alpha = f^m n \delta(x - z) \delta(x - y). \quad (3.2)
\]

The nilpotency on any other fields follows, first, from the Leibnitz rule of acting of $\vec{s}$ on the commutator of any functions $A, B$ with definite Grassmann parities:

\[
[A, B] \vec{s} = \left[ A \vec{s}, B \right] (-1)^{e(B)} + \left[ A, B \vec{s} \right], \quad (3.3)
\]

second, from the Jacobi identity:

\[
[[A, C], C] (-1)^{e(A)} - [[C, A], C] + [[C, C], A] (-1)^{e(A)} = 0, \quad (3.4)
\]

for any $A \in \{ C, B^{p_1}, C^{p_1}, \bar{B}^{p_1}, \bar{B}, B \}$. E.g. for Grassmann-even $A = B^{p_1 p_2}$ we have,

\[
B^{p_1 p_2} \vec{s}^2 = \left( B^{p_1 p_2} + [B^{p_1 p_2}, C] \right) \vec{s} \\
= [B^{p_1 p_2}, C] - [B^{p_1 p_2} + [B^{p_1 p_2}, C], C] + \frac{1}{2} [B^{p_1 p_2}, [C, C]] \\
= - \frac{1}{2} \left( [B^{p_1 p_2}, C] - [C, B^{p_1 p_2}, C] + [C, B^{p_1 p_2}], C] \right) = 0, \quad (3.5)
\]

where we have used the relations (3.1), linearity and Leibnitz rule (3.3) for $\vec{s}$, Jacobi identity (3.4) and generalized antisymmetry for the (super)commutator.

The transformations,

\[
\vec{\Phi}_{(3)} \rightarrow \vec{\Phi}'_{(3)} = \vec{\Phi}_{(3)} (1 + \vec{s} \lambda) = \vec{\Phi}_{(3)} + \delta \vec{\Phi}_{(3)}, \quad (3.6)
\]
appear by the invariance transformations of following path integral and quantum action:

\[ Z_{1|\Psi}(0) = \int d\Phi(3) d\bar{\Phi}(3) \exp \left( \frac{i}{\hbar} S_{\Psi(1)}(\Phi(3), \bar{\Phi}(3)) \right), \text{ with } S_{\Psi(1)} = S_0(A) + \Psi(1) \bar{\Psi}, \quad (3.7) \]

with a new gauge fermion functional, \( \Psi(1) = \Psi_{(1)}(\Phi(3)) \), which should determine a non-degenerate quantum action \( S_{\Psi(1)} \) on the \( \mathcal{M}_{tot}^{(3)} \), i.e. with non-degenerate supermatrix of the second derivatives in \( \Phi(3), \bar{\Phi}(3) \) of \( S_{\Psi(1)} \) evaluated on a some vicinity of the solutions, \( \Phi_{(0)} = (A^\mu_0, 0, ..., 0) \) of the respective equations of motions: \( S_0 \frac{\partial}{\partial j} = 0 \):

\[ \Psi_{(1)} = \tilde{B}_X(1) (A, B) + C^p B^{\alpha r} \varepsilon_{pqr} + \tilde{\Psi}(1)(\Phi(3)), \text{ for } \deg_\Phi \Psi_{(1)} > 2, \quad \deg_\Phi \chi_{(1)}(A, B) = 1. \quad (3.8) \]

Indeed, from the invariance of the integration measure, \( d\Phi(3) \), and quantum action, \( S_{\Psi(1)} \), due to the same reason as for the standard \( N = 1 \) BRST realization in \( \mathcal{M}_{tot} (2.8) \):

\[ \delta_{\chi} S_{\Psi(1)} = 0, \quad d\Phi'(3) = d\Phi'(3) \det \left\| \frac{\delta \Phi'(3)}{\delta \Phi(3)} \right\| = d\Phi(3), \quad (3.9) \]

it follows the invariance of the integrand in \( Z_{1|\Psi}(0) \) with respect to these transformations. It justifies a definition of the transformations (3.6) as \( N = 1 \) antiBRST symmetry transformations in \( \mathcal{M}_{tot}^{(3)} \).

Choosing, \( \tilde{\Psi}(1)(\Phi(3)) = 0 \) in (3.8) for the quadratic gauge functional, \( \Psi_{(1)} \), (in particular, for \( R_\xi \)-gauges: \( \chi_{(1)}(A, B) = \chi(A, B) \)) we find for the quantum action, \( S_{\Psi(1)} \), the representation:

\[ S_{\Psi(1)} = S_0(A) + \int d^d x \left( \partial^\mu A^\mu + \xi g^2 B + \tilde{B} M(A) \bar{C} + \{ B^p B^{\alpha r} + C^p \tilde{B}^{\alpha r} \} \varepsilon_{pqr} \right) + S_{\text{add}(1)} \quad (3.10) \]

\[ S_{\text{add}(1)} = \int d^d x \left( C^p [B^{\alpha r}, \bar{C}] + [C^p, \bar{C}] B^{\alpha r} \right) \varepsilon_{pqr}. \quad (3.11) \]

Integrating out of \( B^p, \tilde{B}^{\alpha r} \) fields we get for the path integral:

\[ Z_{1|\Psi}(0) = \int dA d\tilde{B} dC dB dC^p dB^{\alpha r} \delta(B^{\alpha r+1}) \delta(C^p) \exp \left\{ \frac{i}{\hbar} \left( S_\Psi(\Phi(1)) + S_{\text{add}(1)} \right) \right\} \quad (3.12) \]

\[ = \int dA d\tilde{B} d\bar{C} dB \exp \left( \frac{i}{\hbar} S_\Psi(\Phi(1)) \right) \text{ with } S_\Psi = S_0(A) + \Psi(\Phi(1)) \bar{\Psi}, \Psi(\Phi(1)) = \tilde{B}_X(1)(A, B). \quad (3.13) \]

where the resulting (after integration) fields \( \Phi^A(1) \), in fact, coincide with the fields given by the local formulation for the path integral (2.5) within Faddeev-Popov rules with \( N = 1 \) BRST symmetry, in particular, for the Landau gauge (1.25) under identification:

\[ \Phi^A(1) = (A, \tilde{B}, \bar{C}, B) \rightarrow \Phi^A = (A, C, \bar{C}, B). \quad (3.14) \]

The only difference consists in the realization \( N = 1 \) antiBRST symmetry for \( Z_{1|\Psi}(0) \) given in \( \mathcal{M}_{tot}^{(3)} \) and of \( N = 1 \) BRST symmetry for \( Z_\Psi \) (2.5) determine over \( \mathcal{M}_{tot} \). After replacing \( (\tilde{B}, \bar{C}) \rightarrow (\bar{C}, C) \) the above path integral will coincide exactly.

Thus, we reached the validity of the

**Statement 2:** The path integral, \( Z_{1|\Psi}(0) \), (3.7) with the quantum action, \( S_{\Psi(1)} \), (3.10) at least, for the special quadratic gauge fermion, \( \Psi_{(1)} \), (3.8) with \( \tilde{\Psi}(1) = 0 \) determined in \( N = 3 \) reducible representation space, \( \mathcal{M}_{tot}^{(3)} \), of \( G(3) \) superalgebra, but with realization of the additional \( N = 1 \) antiBRST symmetry (3.1), (3.9) coincide with respective path integral (3.12), with the quantum action, \( S_\Psi \), (3.13) obtained with use of \( N = 1 \) antiBRST symmetry transformations acting in the standard configuration space, \( \mathcal{M}_{tot} \).

Now, we may reveal the physical contents of the fields spectrum for the \( Z_{1|\Psi}(0) \) (2.77), \( S_{\Psi(1)}(\Phi(3)) \) (2.78) (2.81) being invariant with respect to \( N = 3 \) BRST symmetry transformations (2.45), (2.69).
Namely, the fields $\hat{B}, \overline{C}$ from $\mathcal{M}^{(3)}_{\text{tot}}$ space correspond respectively to the pair of ghost field $C$ inheriting the gauge symmetry and antighost field, $\overline{C}$, introducing the gauge into the quantum action. The triplet of the ghost fields $C^p$ and triplet of dual to $\overline{B}^{p_1+p_2}$ fields: $\overline{B}_{p_3} = \frac{1}{2} \xi_{p_1+p_2} \overline{B}^{p_1+p_2} = (B^{23}, B^{31}, B^{12})$ are organized into the pairs of $N = 3$ triplet of Grassmann-odd ghost-antighost pairs: $(C^p, \overline{B}_{p})$. The triplet of the Grassmann-even fictitious fields $B^p$ and triplet of dual to $B^{p_1+p_2}$ fields: $\overline{B}_{p_3} = \frac{1}{2} \xi_{p_1+p_2} B^{p_1+p_2} = (B^{23}, B^{31}, B^{12})$ forms the pairs of $N = 3$ triplet of Grassmann-even ghost-antighost pairs: $(B^p, \overline{B}_{p})$. The role of the Nakahshi-Lautrup field $B$ remains the same as in case of standard $N = 1$ BRST symmetry formulation, i.e. as the Lagrangian multiplier (at least for Landau gauge) introducing the gauge into the quantum action.

Because of, the term in the ghost part, $S_{\text{gh}(3)}$, (2.80) with Grassmann-even triplet of ghost-antighost pairs maybe presented as follows,

$$\frac{1}{2} \int d^4 x \ tr B^p M(A) B^r \varepsilon_{pqr} \equiv \int d^4 x \ tr B^p M(A) \overline{B}_p \tag{3.15}$$

we can immediately identify the fields, $(C^0, \overline{C}; C^3, \overline{C}^3; B^{[3]}, \overline{B}^{[3]})$ in the quantum action (2.33) for the local representation (2.33) of the generalized path integral (2.31), for $k_u(3) = 3$, with singlet and Grassmann-odd and Grassmann-even triplets of ghost pairs as follows:

$$(C^0, \overline{C}; C^3, \overline{C}^3; B^{[3]}, \overline{B}^{[3]}) = (\hat{B}, \overline{C}; C^p, \overline{B}_p; \overline{B}_p, B^p). \tag{3.16}$$

Note, first, that for $N = 1$ antiBRST symmetry realization in the configuration space $\mathcal{M}^{(3)}_{\text{tot}}$ it is possible in addition to the path integral formulation (3.14) introduce all necessary for diagrammatic Feynman technique generating functionals of Green functions as it was done for $N = 1$ and $N = 3$ BRST symmetry case in the Subsections 2.3.1 and 2.3.2 and study theirs respective properties (Ward identities, gauge-independence problem). Second, as for the above developed $N = 1$ antiBRST symmetry concept in $\mathcal{M}^{(3)}_{\text{tot}}$ it is possible to construct a so-called $N = 3$ antiBRST symmetry transformations as the $N = 3$ SUSY transformations of $\mathcal{G}(3)$ superalgebra with the triplets of the anticommuting generators $\epsilon \overline{s}_p$ with lower indices $p = 1, 2, 3$ and Grassmann-odd parameters, $\lambda^p$. Doing so we should, to change all the Grassmann-odd and Grassmann-even ghosts on their antighosts in the $N = 3$ SUSY transformations described by Lemmas 1, 2, starting from the change for the gauge parameters $\xi$: $\xi = \hat{B}_p \lambda^p$ and the first relations in a chain of these transformation

$$A^\mu \overline{s}_p = D^\mu(A) \overline{B}_p, \overline{B}_q \overline{s}_p = \varepsilon_{pqr} B^r + \frac{1}{2}[\overline{B}_q, \overline{B}_p], \ldots, \tag{3.17}$$

and finishing with the construction of the respective path integral, whose action and functional measure should be invariant with respect to these transformations. We leave the details of this interesting concept out of the paper scope.

### 3.2 $N = 4 = 3 + 1$ SUSY transformations

Now, we can consider the triplte of the Grassmann-odd ghost fields $C^p$ and singlet $\overline{C}$, triplets of the Grassmann-even ghost fields $B^p$, triplet of new Grassmann-odd ghost fields $\overline{B}^{pq}$ and singlet $\hat{B}$ on the equal footing within corresponding Grassmann-odd quartet, $C^r$, Grassmann-even sextet, $B^{r_1 r_2}$, and Grassmann-odd quartet, $B^{r_1 r_2 r_3}$ for $r, r_1, r_2, r_3 = 1, 2, 3, 4$ as the elements (with the fields $A^\mu, B$) of the irreducible tensor representation of the Abelian $\mathcal{G}(4)$ superalgebra. In fact, the $N = 3$ and $N = 1$ representations of $\mathcal{G}(3)$ and $\mathcal{G}(1)$ superalgebra in the same $\mathcal{G}(3)$-representation space of the fields $\Phi^{(3)}$ are nontrivially entangled in unique $N = 4$ irreducible representation in the same representation space $\mathcal{M}^{(3)}_{\text{tot}} = \mathcal{M}^{(4)}_{\text{tot}}$ whose local coordinates (fields) are organized into $\mathcal{G}(4)$-irreducible antisymmetric tensors,
as well as the parameters and generators have the structures:

$$
\begin{align*}
\mathbf{g}(4) &= \left( \mathbf{C}', \mathbf{B}^{r_1 r_2}, \mathbf{B}^{r_1 r_2 r_3}, B \right) = \left( \mathbf{C}' \mathbf{C}, \left( \mathbf{B}^{p_1 p_2}, \mathbf{B}^{p_1} \right), \left( \mathbf{B}, \mathbf{B}^{p_1 p_2} \right), B \right), \\
\lambda_r &= (\lambda_p, \tilde{\lambda}); \quad \bar{\sigma}^s_r = \left( \bar{\sigma}^s, \bar{\sigma}^s \right); \quad r = (p, 4) = (1, 2, 3, 4).
\end{align*}
$$

**Lemma 4:** The action of the generators $\bar{\sigma}^s_r$ of $N = 4$-parametric Abelian superalgebra $\mathbf{g}(4)$ on the fields $\Phi_{(4)} = (A^\mu, C^r, B^{r_1 r_2}, B^{r_1 r_2 r_3}, B)$ is given by the relations:

$$
\begin{array}{c|c}
\hline
\downarrow & \bar{\sigma}^s_r \\
\hline
A^\mu & D^\mu(A)C^r \\
C^{r_1} & B^{r_1 r_2} + \frac{1}{2} \left( [B^{r_1 r_2}, C^r] - \frac{1}{3} [C^{r_1}, [C^{r_2}, C^r]] \right) \\
B^{r_1 r_2 r_3} & \frac{1}{2} \left( B^{r_1 r_2}, C^r \right) - \frac{1}{3} \left( [B^{r_1 r_2 r_3}, C^r], C^r \right) \tilde{e}_{r_1 r_2 r_3 r_4} \\
\hline
\end{array}
$$

with

$$
\sum_{r} (-1)^{P(r_1, r_2, r_3)} X^{r_1 r_2 r_3 r} = X^{r_1 r_2 r_3} - X^{r_2 r_1 r_3} + X^{r_2 r_1 r_3} + \ldots,
$$

where the sign, $\sum_{r} (-1)^{P(r_1, r_2, r_3)} X^{r_1 r_2 r_3 r}$ means the summation over all (odd with sign ” − “) and even with ” += “) 3! permutations of the indices $(r_1, r_2, r_3)$. The respective $N = 4$ SUSY transformations with quartet of anticommuting parameters, $\lambda_r$, on the fields $\Phi_{(4)}$ are determined as: $\delta_{\lambda} \Phi_{(4)} = \Phi_{(4)} \tilde{\sigma}^s \lambda_r$.

The form of the transformations (3.20) follows from the chain (2.31), (2.33) for $N = 4$. To prove the Lemma we will follow the algorithm elaborated when the Lemma 1 was proved. We start from the boundary condition for the transformations (3.20) inherited from the gauge transformations for $A^\mu$ (2.40) and present the realization for the sought-for generators as series:

$$
\bar{\sigma}^s_r = \sum_{s \geq 0} \bar{\sigma}^s_r s^0 : \quad A^\mu \bar{\sigma}^s_r = A^\mu \bar{\sigma}^s_r s^0 = D^\mu(A)C^r \text{ and } C^{r_1} \bar{\sigma}^s_r s^0 = 0.
$$

(3.21)

Then, because of,

$$
A_{\mu} \left( \bar{\sigma}^s_{r_1} \bar{\sigma}^s_{r_2} + \bar{\sigma}^s_{r_2} \bar{\sigma}^s_{r_1} \right) \neq 0,
$$

(3.22)

we should add to $\bar{\sigma}^s_r s^0$ the nontrivial action of new part $\bar{\sigma}^s_1$ on $C^{r_1}$ (vanishing when acting on $A_{\mu}$; $A_{\mu} \bar{\sigma}^s_1 \equiv 0$), starting from the Grassmann-even sextet of the fields $B^{r_1 r_2} = B^{r_1 r_2 m l} m$ (BRST-like variation of $C^{r_1}$) (2.31)

$$
C^{r_1} \bar{\sigma}^s_1 = B^{r_1 r_2} + (\eta C_1)^{r_1 r_2} [C^{s_1}, C^{s_2}], \quad \text{for } B^{r_1 r_2} = - C^{r_2 r_1}, \quad \epsilon(B^{r_1 r_2}) = 0
$$

(3.23)

with unknown real numbers: $(\eta C_1)^{r_1 r_2} = (\eta C_1)^{r_1 r_2 s_1 s_2}$, to be determined from the consistency of 4 $\times$ 4 equations:

$$
A_{\mu} \left( \bar{\sigma}^s_{r_1} \bar{\sigma}^s_{r_2} + \bar{\sigma}^s_{r_2} \bar{\sigma}^s_{r_1} \right) = 0, \quad \text{where } \bar{\sigma}^s_{r_1} \equiv \sum_{n \geq 0} \bar{\sigma}^s_{r_1 n}, \quad \text{and } C^{r_1} \bar{\sigma}^s_{r_2 s_0} = 0,
$$

(3.24)

from which follows the antisymmetry for $B^{r_1 r_2}$ in the indices $r_1, r_2$. The solution for (3.24) looks as:

$$
(\eta C_1)^{r_1 r_2} = \frac{1}{4} \delta^r_1 s_2, \quad \text{for } \delta^r_1 s_2 = \delta^r_1 s_2 = \delta^r_1 s_2 + \delta^r_1 s_2 s_1,
$$

(3.25)

that proves the validity of the 2-nd row in the table (3.20).
Second, in view of
\[ C^r \left( \frac{\delta_{i[1]} r_{[1]} r_{[1]} + \delta_{s[1]} r_{[1]}}{s_{[1]}} \right) \neq 0, \]  
(3.26)
we should determine, for a nontrivial action of \( \frac{s_{[1]}}{s_{[1]}} \) on \( B^{r_1 r_2} \) (vanishing when acting on \( A_\mu, C^r: (A_\mu, C^{r_{[1]}}) \frac{s_{[1]}}{s_{[1]}} \equiv 0 \)), in the form of a general anzatz, starting from the Grassmann-odd field variables \( B^{r_1 r_2 r_{[1]}} = B^{r_1 r_2 r_{[1]} m} \) (BRST-like variation of \( B^{r_1 r_2} \)) (2.43) up to the third power in \( C^r \) with a preservation of Grassmann homogeneity in each summand, as in the (3.23).

\[ B^{r_1 r_2} \frac{s_{[1]}}{s_{[1]}} = B^{r_1 r_2 r_{[1]} + (\eta B_1)^{r_1 r_2 r_{[1]} [B^{s_{[1]}}]} + (\eta B_2)^{r_1 r_2 r_{[1]} [C^{r_{[1]}}, [C^{r_{[2]}}, C^{r_{[3]}]}]}, \]  
with unknown real numbers: \( (\eta B_1)^{r_1 r_2 r_{[1]}}, (\eta B_2)^{r_1 r_2 r_{[1]}}, j = 1, 2 \); satisfying the same antisymmetry properties as for \( (\kappa B_1)^{r_1 r_2 r_{[1]}} \) in (2.33) and to be determined from the solution of the 4 \( \times \) 4 \( \times 4 \) equations

\[ C^{r_{[1]}} \left( \frac{\delta_{i[2]} r_{[2]} r_{[2]} + \delta_{s[2]} r_{[2]}}{s_{[2]}} \right) = 0, \]  
(3.28)
Its general solution has the form:

\[ (\eta B_1)^{r_1 r_2 r_{[1]} [B^{s_{[1]}}]} = \frac{1}{4} \delta_{i[1]} r_{[1]} [s_{[1]} r_{[1]} r_{[1]} : (\eta B_1)^{r_1 r_2 r_{[1]} [B^{s_{[1]}}]} = \frac{1}{2} \]  
(3.29)
\[ (\eta B_2)^{r_1 r_2 r_{[1]} [C^{r_{[1]}}, [C^{r_{[2]}}, C^{r_{[3]}]}]} = \frac{1}{12} [C^{r_{[1]}}, [C^{r_{[2]}}, C^{r_{[3]}]}]. \]  
providing the validity of the 3\( \times \)rd row in the table (3.20).

Third, there are only the fourth-rank independent completely antisymmetric constant tensors with upper, \( \varepsilon^{r_1 r_2 r_{[1]} r_{[2]}}, \) and lower, \( \varepsilon_{r_1 r_2 r_{[1]} r_{[2]}}, \) indices, which are normalized by the conditions (according with (2.37))

\[ \varepsilon^{i[2] j[2] k[2] l[2]} = 1, \varepsilon^{r_1 r_2 r_{[1]} r_{[2]} \varepsilon^{s_{[1]} s_{[2]} s_{[3]} s_{[4]} = \delta \left\{ \delta_{i[1]} r_{[1]} r_{[1]} \right\}, i, j, k, l, = 1, 2, 3; \]
\[ \varepsilon^{r_1 r_2 r_{[1]} r_{[2]} \varepsilon^{s_{[1]} s_{[2]} s_{[3]} r_{[4]} = 2 \delta_{i[1]} r_{[1]} r_{[1]} - \delta_{i[1]} r_{[1]} r_{[1]} ; \varepsilon^{r_1 r_2 r_{[1]} r_{[2]} \varepsilon^{s_{[1]} s_{[2]} s_{[3]} r_{[4]} = 6 \delta_{i[1]} r_{[1]} r_{[1]} \right\}, \right\}, \]  
(3.31)
Due to

\[ B^{r_1 r_2} \left( \frac{s_{[2]} r_{[2]} + s_{[2]} r_{[2]} r_{[2]} \right) \neq 0, \]  
(3.32)
we should determine for a nontrivial action of \( \frac{s_{[1]}}{s_{[1]}} \) on \( B^{r_1 r_2 r_{[1]}}, \) (vanishing when acting on \( A_\mu, C^r, B^{r_1 r_2}: (A_\mu, C^r, B^{r_1 r_2}) \frac{s_{[1]}}{s_{[1]}} \equiv 0 \) a general ansatz with use of the new Grassmann-even field variable, \( B, \)

\[ B^{r_1 r_2 r_{[1]} = \varepsilon^{r_1 r_2 r_{[1]} B + (\sigma B_1)^{r_1 r_2 r_{[1]} [B^{s_{[1]}} s_{[1]} s_{[2]} s_{[3]}}, C^{r_{[1]}}, (\sigma B_2)^{r_1 r_2 r_{[1]} [B^{s_{[1]}} s_{[1]} s_{[2]} s_{[3]}}, C^{r_{[1]}}, [C^{r_{[2]}}, C^{r_{[3]}}, C^r])]. \]  
(3.33)
Here, the unknown real numbers \( (\sigma B_1)^{r_1 r_2 r_{[1]} s_{[1]} s_{[2]} s_{[3]}}, i = 1, 2, 3, 4 \), obey the analogous properties of (anti)symmetry as for the coefficients \( (\sigma B_2)^{r_1 r_2 r_{[1]} s_{[1]} s_{[2]} s_{[3]}}, i = 1, 2, 3, 4 \) in the respective lower and upper indices that is now dictated by antisymmetry for \( B^{r_1 r_2 r_{[1]}}, B^{s_{[1]}} s_{[1]} s_{[2]} s_{[3]} \) and symmetry for \( [C^{r_{[1]}}, C^r] \) in G(4)-indices. They should be determined from the 6 \( \times \) 4 \( \times 4 \) equations:

\[ B^{r_1 r_2}(\frac{s_{[3]} r_{[3]} + s_{[3]} r_{[3]} r_{[3]}(s_{[3]} r_{[3]} r_{[3]})) = 0, \]  
(3.34)
Its general solution looks as

\[ (\sigma B_1)^{r_1 r_2 r_{[1]} [B^{s_{[1]} s_{[2]} s_{[3]}}, C^r] = \frac{1}{2} [B^{r_1 r_2 r_{[1]}}, C^r], \]  
(3.35)
\[ (\sigma B_2)^{r_1 r_2 r_{[1]} [B^{s_{[1]} s_{[2]} s_{[3]}}, C^r]] = - \sum \left\{ (-1)^{(r_1, r_2, r_{[1]}) (1/2) \left( [B^{r_1 r_2}, C^{r_{[1]}}, C^r] + \frac{1}{2} \left( [B^{r_1 r_2}, C^{r_{[2]}}, C^{r_{[3]}}, C^r] \right) \right) \right\}, \]  
(3.36)
\[ (\sigma B_3)^{r_1 r_2 r_{[1]} = (\sigma B_4)^{r_1 r_2 r_{[1]} = 0. \]  
(3.37)
providing the validity of the 4-th row in the table (3.20). In deriving (3.35)–(3.37), the use has been made of the symmetry for the commutator \([C^p, C^r] = [C^r, C^p]\), Jacobi identities both for \((B^{r_1}, B^{r_2}, C^{r_3})\) and for \((C^{p_1}, A^{p_2}, C^{p_3})\), which establish the absence of the 4-th power in the fields \(C^r\) in the transformation for \(B^{r_1r_2r_3}\) (3.38) completely repeating the equations (2.64) for \(N = 3\) case, but with replacement: \((B^{r_1p_2}, B^{r_2}, \tilde{s}^{[3]}_{[3]} p)\) on \((B^{r_1r_2}, B^{r_1r_2r_3}, C^r, \tilde{s}^{[3]}_{[3]} r)\).

Fourth, because of,

\[
B^{r_1r_2r_3} \left( \tilde{s}^{[3]}_{[3]} r \tilde{s}^{[3]}_{[3]} r + \tilde{s}^{[3]}_{[3]} r \tilde{s}^{[3]}_{[3]} r \right) \neq 0,
\]

we should determine for a nontrivial action of \(\tilde{s}^{[3]}\) on \(B\), (vanishing when acting on \(A^r, C^r, B^{r_1}, B^{r_1r_2}, B^{r_1r_2r_3}\); (\(A^r, C^r, B^{r_1r_2}, B^{r_1r_2r_3}\))^\rightarrow 0\) a general ansatz without new Grassmann-odd field variable due to 5-th order nilpotency for \(\tilde{s}^{[3]} r\) \((\prod_{r=1}^3 \tilde{s}^{[3]} r \equiv 0)\) up to the fifth order in \(C^r\) with a preservation of Grassmann homogeneity in each summand, as in the case of (3.35)–(3.37).

\[
B^{[3]}_{[3]} = (\sigma_{B1})^{[3]}_{[3]} [B, C^r] + (\sigma_{B2})^{[3]}_{[3]} \{B^{r_1} s^{[3]} r_2 s^{[3]} r_3, C^s, C^s\} + (\sigma_{B3})^{[3]}_{[3]} \{B^{r_1} s^{[3]} r_2 s^{[3]} r_3, C^s, C^s\},
\]

(3.39)

The above unknown real numbers, \((\sigma_{B1})^{[3]}_{[3]}, (\sigma_{B2})^{[3]}_{[3]}, i = 2, 3, 4\), obey the obvious properties of (anti)symmetry, e.g. as for the coefficients \((\sigma_{B2})^{[3]}_{[3]} = 0\), \((\sigma_{B3})^{[3]}_{[3]} = 0\). They should be determined from the 4 \(\times 4\) \(\times 4\) equations:

\[
B^{[3]}_{[3]} (\tilde{s}^{[3]} r_{[4]} C^s r_{[4]} + \tilde{s}^{[3]} r_{[4]} C^s r_{[4]} ) = 0, \quad B^{[3]}_{[3]} = 0, \quad i = 0, 1, 2, 3,
\]

(3.40)

whose general solution has the form

\[
\begin{align*}
(\sigma_{B1})^{[3]}_{[3]} &= \frac{1}{2} \delta_{[3]}^{[3]}, \quad (\sigma_{B2})^{[3]}_{[3]} \equiv 0, \quad (\sigma_{B3})^{[3]}_{[3]} = 0,
\end{align*}
\]

(3.41)

providing the validity of the last row in the table (3.20). In deriving (3.41), we have used the above mentioned properties found when establishing (3.35)–(3.37), as well as the Grassmann identity for the fields \((B^{r_1}, B^{r_1r_2}, C^{r_3})\) with the following representations for "4-cocycles", i.e. for 5-th rank tensors being antisymmetric in 4 indices:

\[
\frac{1}{3!} \sum_p (-1)^{P(r_1r_2r_3)} \left[[B^{r_1r_2}, C^{r_3}], C^{r_4}\right] = \varepsilon^{r_1r_2r_3r_4} P^r_4 \quad \text{for} \quad P^r_4 = \frac{1}{3!} \left[[B^{r_1r_2}, C^{r_3}], C^{r_4}\right] \varepsilon^{r_1r_2r_3r_4},
\]

(3.42)

\[
\frac{1}{2!} \sum_p (-1)^{P(r_1r_2r_3)} \left[[B^{r_1}, C^{r_2}], C^{r_4}\right] = \varepsilon^{r_1r_2r_3r_4} Q^r_4 \quad \text{for} \quad Q^r_4 = \frac{1}{2} \left[[B^{r_1}, C^{r_2}], C^{r_3}\right] \varepsilon^{r_1r_2r_3r_4},
\]

(3.43)

so that the latter quantities, \(Q^r_4\), (3.43) do not presented in the transformations for \(B\) in (3.20) as compared for the \(N = 3\) quantities, \(Q^p_4 \equiv Q^p_{[4]}\) (2.62), (2.63), which are non-vanishing when enter into the transformations for \(B\) (2.42). One can immediately check that the equations (3.30) considered for \(B\), instead of \(B^{r_1r_2r_3}\), are fulfilled as well:

\[
B \left( \tilde{s}^{[4]} \tilde{s}^{[4]} + \tilde{s}^{[4]} \tilde{s}^{[4]} \right) = 0 \Leftrightarrow \tilde{s}^{[4]}_{[4]} \tilde{s}^{[4]}_{[4]} = 0.
\]

(3.44)

Therefore, \(\tilde{s}^{[4]}\) are the generators of the irreducible representation of \(\mathfrak{G}(4)\) superalgebra of \(N = 4\)-parametric SUSY transformations in the field superspace, \(\mathcal{M}^{(4)}\), parameterized by the fields, \(\Phi^{A_4}_{(4)}\). That fact completes the proof of the Lemma 4.

Note, first, that the transformations on the fields \(B^{r_1r_2}, B^{r_1r_2r_3}, B\) do not contain the terms more than cubic in the fictitious fields, whereas they depend linearly on the fields \(B^s\) in the cubic terms. Second, the quantities, \(Q^r_4\) do not enter into the transformations for Grassmann-even field \(B\) as compared to its \(N = 3\) analogs, \(Q^p_4\), which are essentially presented in the transformations for Grassmann-odd \(\tilde{B}\).

Now, we have all necessary to construct \(\mathfrak{G}(4)\)-invariant quantum action for the Yang–Mills theory.

21
4 N=4 BRST invariant gauge-fixing procedure and local path integral

Let us determine according to the prescription (2.33), (2.34) the local path integral, $Z_4$, generating functionals of Green functions in any admissible gauge, turning to the non-degenerate Faddeev-Popov matrix, for Yang-Mills theory underlying above constructed explicit $N = 4$ SUSY invariance (3.20) in the total configuration space $\mathcal{M}_\text{tot}^{(4)}$, $\mathcal{M}_\text{tot}^{(3)}$, with quartet of anticommuting parameters $\lambda_r$ and the local quantum action $S_{Y(4)}(\Phi(4))$ as follows:

$$Z_4[Y(0)] = \int d\Phi(4) \exp \left\{ \frac{1}{\hbar} S_{Y(4)}(\Phi(4)) \right\}, \text{ with } S_{Y(4)}(\Phi(4)) = S_0(\mathcal{A}) - \frac{1}{4!} Y(4) \bar{\omega}^s_{\kappa_1} \bar{\omega}^s_{\kappa_2} \bar{\omega}^s_{\kappa_3} \bar{\omega}^s_{\kappa_4} \varepsilon_{[\kappa_1]},$$  \hspace{1cm} (4.1)

$$Z_4[Y(J(4))] = \int d\Phi(4) \exp \left\{ \frac{1}{\hbar} S_{Y(4)}(\Phi(4)) + J(4)\Phi(4) \right\} = \exp \left\{ \frac{1}{\hbar} W_4 Y(J(4)) \right\}.$$  \hspace{1cm} (4.2)

with use of the compact notation for, $\varepsilon_{\kappa_1\kappa_2\kappa_3\kappa_4} \equiv \varepsilon_{[\kappa_1]}$. Here, $W_4 Y(J(4))$ is the generating functional of connected correlated Green functions and gauge boson functional, $F(4) = Y(4) = Y(4)(\Phi(4))$, depends on the fields $\Phi(4)$ as follows (confer with $Y_\xi$ (2.17) for $N = 2$ BRST symmetry):

$$Y(4)(\Phi(4)) = Y^0(4)(\Phi(4)) + \hat{Y}(4)(\Phi(4)), \text{ for } deg_2 \hat{Y}(4) > 2, \text{ } deg_2 Y^0(4)(\Phi(4)) = 2,$$  \hspace{1cm} (4.3)

and $J_{A(4)}$ are the external sources (coinciding with ones for $N = 3$ case, $J_{A(4)}$) to the Green functions related to $\Phi(4)^A$ with the same Grassmann parities: $\epsilon(J_{A(4)}) = \epsilon(\Phi(4)^A)$.

It is not difficult to check that both the functional measure, $d\Phi(4)$, as well as the quantum action, $S_{Y(4)}$, are invariant with respect to the change of variables, $\Phi(4)^A \rightarrow \Phi(4)^A + \delta \lambda S_{Y(4)}$ generated by $N = 4$ SUSY transformations (3.20) with accuracy up to the first order in constant $\lambda_\rho$ (equally with infinitesimal $\lambda_\rho$):

$$\Phi(4)^A = \Phi(4)^A(1 + \bar{\omega}^s \lambda_r) : \delta \lambda \Phi(4)^A = \Phi(4)^A \bar{\omega}^s \lambda_r \implies \delta \lambda S_{Y(4)} = o(\lambda), \text{ sdet} \left\| \delta \Phi(4)/\delta \Phi(4) \right\| = 1 + o(\lambda).$$  \hspace{1cm} (4.4)

These properties justify the definition of the transformations:

$$\delta \lambda \Phi(4)^A = \Phi(4)^A - \Phi(4)^A = \Phi(4)^A \bar{\omega}^s \lambda_r,$$  \hspace{1cm} (4.5)

with the explicit action of the generators $\bar{\omega}^s$ (3.20) on the component fields as $N = 4$-parametric BRST transformations for the functionals $Z_4[Y(0)], Z_4[Y(J(4))]$.

The particular representations for the path integrals (1.1), (1.2) in the Landau and Feynman gauges may be obtained within the same $R_C$-family of the gauges as for the $N = 1, 2, 3$ BRST invariant cases (2.5), (2.17), (2.73). To do so we determine the quadratic gauge boson functional, $Y^0_{(4)}(\xi(\Phi(4)))$, which should generate $R_C$-like gauges as follows:

$$Y^0_{(4)}(\xi(\Phi(4))) = Y^0_{(4)}(\mathcal{A}) + Y^B_{(4)}(B^{rs}) = \int d^d x \left\{\frac{1}{2} A_\mu A^\mu - \frac{\xi g^2}{4!} B^{rs} B^{rs} \xi_{[\kappa_1]} \right\}.$$  \hspace{1cm} (4.6)

The quantum action, $S_{Y(4)\xi}$, has the representation:

$$S_{Y(4)\xi}(\Phi(4)) = S_0 - \frac{1}{4!} Y(4) \bar{\omega}^s_{\kappa_1} \bar{\omega}^s_{\kappa_2} \bar{\omega}^s_{\kappa_3} \bar{\omega}^s_{\kappa_4} \varepsilon_{[\kappa_1]} = S_0 + S_{gf(4)} + S_{gh(4)} + S_{add(4)},$$  \hspace{1cm} (4.7)

$$S_{gf(4)} = \int d^d x \left\{ 0^{\mu} A_\mu + \xi g^2 B \right\},$$  \hspace{1cm} (4.8)

$$S_{gh(4)} = \int d^d x \left[ \frac{1}{3!} B^{r_1 r_2 r_3} M(A) C^{r_4} + \frac{1}{8} B^{r_1 r_2} M(A) B^{r_3} \right] \varepsilon_{[\kappa_1]}.$$  \hspace{1cm} (4.9)

\footnote{Instead of the functional $\hat{Y}^B_{(4)}(B^{rs})$ which generates the $\xi$-dependent term it is possible to consider the functional $\hat{Y}^B_{(4)}(C, B^{rs}) = \frac{\xi g^2}{4!} \int d^d x tr C^{\kappa_1} B^{rs} B^{rs} \xi_{[\kappa_1]}$, still leading to the same quadratic term: $\xi g^2 B^2$ in $S_{gf(4)}$, but with another non-quadratic in the fictitious fields summands in $S_{add(4)}$.}
\[ S_{\text{add}}(4) = \int d^4x \, tr \frac{1}{4!} \left\{ (\partial^\mu A_\mu)(2[B^{r_1 r_2 r_3}, C^{r_4}] - [[B^{r_1 r_2}, C^{r_3}], C^{r_4}]) - B^{r_1 r_2} \left( [C^{r_3}, M(A)C^{r_4}] + 4[\partial_\mu C^{r_3}, D^\mu C^{r_4}] + [D^\mu C^{r_2}, B^{r_3 r_4}] + [[D^\mu C^{r_2}, C^{r_3}], C^{r_4}] \right) \right. \\
\left. + \frac{\delta g^2}{4} \left( \frac{1}{4} B^{r_1 r_2}, B^{r_3 r_4} \right) [B^{r_2 q_1}, B^{r_3 r_4}] + \frac{1}{(3!)^2} \left[ C^{q_1}, [[C^{q_2}, C^{r_3}], C^{r_4}] \right] \times \left[ C^{q_3}, [[C^{q_4}, C^{r_3}], C^{r_4}] \right] \right\} \epsilon_{[\rho]4} + \tilde{S}_\xi, \right. \\
(4.10) \]

with some Grassmann-even functional \( \tilde{S}_\xi \) vanishing in the Landau gauge (\( \xi = 0 \)). To derive (4.7)–(4.10) we have used the relations (2.82)–(2.86), (4.5) being adapted for \( N = 4 \) BRST symmetry, as well as the following from (2.86) Leibnitz-like property of the generators, \( s^r \) acting on the product of any functions \( A, B \) with definite Grassmann grading:

\[ (AB)^{s^1 r_1} s^2 s^3 s^4 r_4 \epsilon_{[\rho]4} = \left[A^{s^1 r_1} s^2 s^3 s^4 r_4 B + 4 A^{s^1 r_1} (B^{s^2 r_2} s^3 s^4 r_4) (-1)^{(B)} \right] \epsilon_{[\rho]4} + 6 A^{s^1 r_1} s^2 s^3 s^4 r_4 (B^{s^2 r_2} s^3 s^4) (-1)^{(B)} + A (B^{s^2 r_2} s^3 s^4 s^4 r_4) \epsilon_{[\rho]4}. \]

The detailed derivation for the quantum action, structure of the additional \( \xi \)-dependent term, \( \tilde{S}_\xi \), are considered in the Appendix. Note, the each term in \( S_{\text{add}}(4) \) contains space-time derivative and, in particular, the second-order differential operator (Faddeev-Popov operator) for any gauge from \( \delta \xi \)-gauge, as for the \( S_{\text{add}}(3) \) (2.89) for \( N = 3 \) BRST symmetry. For the Landau gauge, the summands in \( S_{\text{add}}(4) \) proportional to the Lorentz condition: \( (\partial^\mu A_\mu) = 0 \), may be omitted therein due to the presence of \( \delta((\partial^\mu A_\mu)) \) in the functional integral (4.1) after integrating over the fields \( B \).

The equivalence of \( N = 4 \) and \( N = 1 \) BRST invariant path integrals \( Z_{4Y}(0) \) (4.11), \( Z_\Psi \) (2.5), e.g in the Landau gauge determined by the gauge functional \( Y^0_4(A) \) (4.6) follows analogously to the derivation (2.91) for \( N = 3 \) case from the structure of the quantum action \( S_{Y(4)\xi} \) (4.7)–(4.10). Indeed, using the representation for \( S_{Y(4)\xi} \) (4.9) in terms of dual \( G(4) \)-tenors fields \( B_{r_1 r_2}, C_r \) (2.20), (2.21) let us divide the quartets of ghost Grassman-odd fields \( C_r, C^r \) as \( G(3) \)-triplets and singlets which permits to present the respective term in the ghost part of the action as:

\[ (C_r; C^r) = (\overline{C}; C_p); (C; C^p) \Rightarrow C_r M(A)C^r = \overline{C} M(A)C + C_p M(A)C^p, \]

(4.12)

for \( r = (1, p), p = 2, 3, 4 \) and \( \overline{C} \equiv -B^{234} \). Because of the remark above we may omit the terms with \( (\partial^\mu A_\mu) \) with exception for Nakanishi-Lautrup field \( B \) and therefore integrate by the fields \( C_p \), second, with respect to \( C^p \), and then trivially with respect to \( B^{r_1 r_2} \) and \( B_{r_1 r_2} \) for \( 1 \leq r_1 < r_2 \leq 3 \) as follows:

\[ Z_{4Y(0)}(0) = \int d\Phi dC dB^{r_1 r_2} dC_{r_1 r_2} \text{det}^3 M(A) \delta(C^p) \exp \left\{ \frac{1}{\hbar} \left\{ S_{Y(4)\xi}(\Phi(4)) - \int d^4x \, tr C_p M(A)C^p \right\} \right\} = \int d\Phi dB_{r_1 r_2} \text{det}^3 M(A) \delta(C^p) \delta(B^{r_1 r_2}) \exp \left\{ \frac{1}{\hbar} S_{\Psi(\Phi)}(\Phi)|_{\xi=0} \right\} = Z_{\Psi}. \]

(4.13)

The functional \( \Psi \) exactly coincides with one given in (2.5) in the Landau gauge.

Again, the \( N = 4 \) BRST invariance, for the corresponding generating functionals of Green’s functions, \( Z_{4Y}(J(4)) \), \( W_{4Y}(J(4)) \) and effective action, \( \Gamma_{4Y}(\Phi(4)) \) determined by the same rule as for its \( N = 3 \) analog (2.90) with a given gauge condition \( Y(4)(\Phi(4)) \), leads to the presence of a \( G(4) \)-quartet of Ward identities:

\[ J_{A_1^r}(\Phi(4); s^r) Y(4); J = 0, \quad J_{A_1^r}(\Phi(4); s^r) Y(4); J = 0, \quad \frac{\delta \Gamma_{4Y}}{\delta A_1^r} \left( \langle \Phi(4) s^r \rangle \right) Y(4)(\Phi) = 0, \]

(4.14)
with corresponding normalized average expectation values (as in (4.11)) in the presence of the external sources \(J_{A_i}^r\) and mean fields \(\Phi_{(4)}\). The gauge independence of the path integral \(Z_{4|Y(0)} = Z_{4|Y(0)}\) under an infinitesimal variation of the gauge condition, \(Y(4) \to Y(4) + \delta Y(4)\):

\[
Z_{4|Y(4) + \delta Y(4)}(0) = Z_{4|Y(4)}(0)
\]

is established using the infinitesimal FD \(N = 4\) BRST transformations with the functional parameters,

\[
\lambda_{r_1}(\Phi_{(4)}) = \frac{1}{4!} \left[ \left( \frac{i}{\hbar} \right) \delta Y_{(4)}(\Phi_{(4)}) \right] \prod_{k=2}^{4} \tilde{s}_k \varepsilon_{[r_1]} ,
\]

which will be carefully elaborated in the next Section 5 as well as some important consequences of the suggested \(N = 3\) and \(N = 4\) BRST transformations, respective quantum actions and gauge-fixing procedures.

5  \(N = k, k = 3, 4\) infinitesimal and finite BRST transformations and their Jacobians

Here, we consider the algorithm of construction of finite \(N = k\) BRST transformations starting from its algebraic (infinitesimal) proposals respectively for \(k = 3, 4\) cases and calculate theirs Jacobians together with some physical corollaries.

5.1  \(N = 3\) BRST transformations

The finite \(N = 3\) BRST transformations acting on the fields \(\Phi_{(3)}\), parameterizing configuration space \(\mathcal{M}_{\text{tot}}^{(3)}\), are restored from the algebraic (equivalently, infinitesimal for small \(\lambda_p\)) \(N = 3\) BRST transformations, generalizing the recipe [26] for \(N = 2\) BRST symmetry and following to [27], [35] in two equivalent ways. First, the derivation is based on the condition which follows for any \(\tilde{s}^p\)-closed regular functional \(K(\tilde{\Phi}_{(3)})\) to be invariant with respect to right-hand supergroup transformations and, second, from the Lie equations:

1) \(\{K(g(\lambda_p)\tilde{\Phi}_{(3)}) = K(\tilde{\Phi}_{(3)})\text{ and } K\tilde{s}^p = 0\} \Rightarrow g(\lambda_p) = \exp \left\{ \frac{\tilde{s}^p}{s} \lambda_p \right\} , \) \(5.1\)

2) \(\tilde{A}_{(3)}(\tilde{\Phi}_{(3)}|\lambda) \tilde{s}^p = \tilde{A}_{(3)}(\tilde{\Phi}_{(3)}|\lambda) \tilde{s}^p\) \(\left(\text{for } \tilde{s}^p \equiv \frac{\partial}{\partial \lambda_p}\right)\) \(5.2\)

whose set forms an Abelian 3-parametric supergroup,

\[
G(3) = \left\{ g(\lambda_p) : g(\lambda_p) = 1 + \sum_{c=1}^{3} \frac{1}{c!} \prod_{i=1}^{c} \tilde{s}^{p_{i}} \lambda_{p_{i}} = \exp \left\{ \frac{\tilde{s}^p}{s} \lambda_p \right\} \right\} ,
\]

\(5.3\)

where \(\tilde{s}^p, \tilde{s}^{p_1}, \tilde{s}^{p_2} \varepsilon_{[p_3]}\) and \(\tilde{s}^{p_1}, \tilde{s}^{p_2}, \tilde{s}^{p_3} \varepsilon_{[p_3]}\) are respectively the generators of \(N = 3\) BRST, quadratic mixed and cubic mixed \(N = 3\) BRST transformations in the space of fields \(\tilde{\Phi}_{(3)}\).

For the field-dependent \(G(3)\) triplet of odd-valued functionals \(\lambda_p(\tilde{\Phi}_{(3)})\), which is not closed under \(N = 3\) BRST transformations, \(\lambda_p \tilde{s}^p \neq 0\), but for, \(\partial / \partial x^\mu \lambda_p = 0\), the finite element \(g(\lambda_p(\tilde{\Phi}_{(3)}))\) cannot be

\(\tilde{\Phi}_{(3)}\) and \(\tilde{A}_{(3)}(\tilde{\Phi}_{(3)}|\lambda)\), the form of Lie equations: \(\tilde{A}_{(3)}(\tilde{\Phi}_{(3)}|\lambda) = \tilde{A}_{(3)}(\tilde{\Phi}_{(3)}|\lambda) \tilde{s}^p \lambda_p\), is equivalent to \((5.2)\) with a formal solution for constant \(\lambda_p^0\): \(\tilde{A}_{(3)}(\tilde{\Phi}_{(3)}|\lambda) = \tilde{A}_{(3)}(\tilde{\Phi}_{(3)}|\lambda) \tilde{s}^p \lambda_p\).
presented as group element (using an exp-like relation) in \((5.3)\). In this case, the set of algebraic elements \(\mathcal{G}(3) = \{ \hat{g}_{lin}(\lambda(\mathbf{F}(3))) := 1 + \mathcal{E}^p \lambda_p\} \) forms a non-linear superalgebra which corresponds to a set of formal group-like finite elements:

\[
\mathcal{G}(3) = \left\{ \hat{g}(\lambda_p(\mathbf{F}(3))) : \hat{g} = 1 + \mathcal{E}^p \lambda_p + \frac{1}{2} \mathcal{E}^p \mathcal{E}^q \lambda_q \lambda_p + \frac{1}{3!} \mathcal{E}^p \mathcal{E}^q \mathcal{E}^r \lambda_r \lambda_q \lambda_p \right\}, \tag{5.4}
\]

with loss of the commutativity property: \([\hat{g}(\lambda_p(\mathbf{F}(3))), \hat{g}(\lambda_q(\mathbf{F}(3)))] \neq 0\). The Jacobian of a change of variables: \(\mathbf{F}_A^{(3)} \to \mathbf{F}_B^{(3)} = \mathbf{F}_A^{(3)} \tilde{g}(\lambda_p(\mathbf{F}(3)))\), in \(\mathcal{M}_{tot}^{(3)}\), in the path integral \(Z_{tot}(0)\) \((2.73)\) generated by finite FD \(N = 3\) BRST transformations may be calculated explicitly, following a generalization of the recipe proposed in \([26]\) for an irreducible gauge theory with a closed algebra (including the Yang-Mills theory, see as well \([31]\) in the \(N = 2\) case, or following the recipe of \([27]\) for \(N = m\) finite FD SUSY transformations. The results are as follows:

\[
\text{sdet} \left| \frac{\delta}{\delta \mathbf{F}_B^{(3)}} \tilde{g}(\lambda_p(\mathbf{F}(3))) \right| = \exp \left\{ - \text{tr}_{G(3)} \ln \left( (e + m)^p \right) \right\}, \quad \text{for } \{ (e^p, m^p) = (s^p, \lambda_q \mathcal{E}^q) \}, \tag{5.5}
\]

where \(\text{tr}_{G(3)}\) denotes trace over matrix \(G(3)\)-indices. Representation \((5.3)\) is based on the explicit calculation which generalizes the algorithm for the Jacobian of the change of variables generated by \(N = 2\) BRST transformations for Yang-Mills theory \([31], [89]\) as follows:

\[
\text{sdet} \left| \frac{\delta}{\delta \mathbf{F}_B^{(3)}} \tilde{g}(\lambda_p(\mathbf{F}(3))) \right| = \exp \left\{ \text{Str} \ln \left( \delta_{A_3} + M_{A_3} \right) \right\}, \quad \text{for } M_{A_3} = P_{B_3}^{A_3} + \sum_{i=1}^3 (Q_i)_{A_3} B_3 \tag{5.6}
\]

\[
\begin{align}
& \left\{ \begin{array}{l}
P_{A_3}^{B_3} = \mathbf{F}_A^{(3)} \mathcal{E}^p (\lambda_q \mathcal{E}^q) \mathbf{F}_B^{(3)}; \\
(1)_{A_3}^{B_3} = \lambda_p \left\{ \mathbf{F}_A^{(3)} \mathcal{E}^p (\mathbf{F}_B^{(3)} \mathcal{E}^q) \mathbf{F}_B^{(3)} - \mathbf{F}_A^{(3)} \mathcal{E}^q \mathcal{E}^p \right\} (-1)^{\epsilon_{A_3} + 1} , \\
(2)_{A_3}^{B_3} = \frac{1}{2} \lambda_p \lambda_q \left\{ \mathbf{F}_A^{(3)} \mathcal{E}^p (\mathbf{F}_B^{(3)} \mathcal{E}^q) \mathbf{F}_B^{(3)} - \frac{1}{3!} \mathcal{E}^{pqr} \mathbf{F}_A^{(3)} \mathbf{F}_B^{(3)} (\lambda_r \mathcal{E}^q) \right\} , \\
(3)_{A_3}^{B_3} = \frac{1}{3!} (\lambda^3 (\mathbf{F}_A^{(3)} \mathcal{E}^q) \mathbf{F}_B^{(3)} (-1)^{\epsilon_{A_3}+1} ,
\end{array} \right.
\end{align}
\]

\[
\Rightarrow \text{Str}(P + \sum_{i=1}^3 Q_i)^n = \text{Str}(P + 2 \sum_{i=1}^2 Q_i)^n + n \text{Str}P^{n-1}Q_3, \tag{5.7}
\]

\[
\text{Str}(P + 2 \sum_{i=1}^3 Q_i)^n = \text{Str}P^n + \text{Str}F_n(P, Q_1, Q_2) \quad \text{with } F_n(P, Q_i)|_{Q_i=0} = 0, \tag{5.8}
\]

(where we imply: \(A_i^3 \equiv A_i; (\lambda)^3 = \lambda \lambda \lambda \epsilon^{[a]} \) and \((\mathcal{E})^3\) given by \((5.13)\)), so that the only supermatrix \(P\) gives the non-vanishing contribution into the Jacobian \((5.5)\):

\[
\text{sdet} \left| \frac{\delta}{\delta \mathbf{F}_B^{(3)}} \tilde{g}(\lambda_p(\mathbf{F}(3))) \right| = \exp \left\{ - \sum_{n=1}^\infty \frac{(-1)^n}{n} \text{Str}P_{B_3}^{A_3} \right\}, \tag{5.10}
\]

as compared with the nilpotent supermatrices \((Q_i)_{A_3}^{B_3}\) (entering in \(F_n\) \((5.9)\)) which do not contribute to the Jacobian \((5.5)\) due to: \(\prod_{k=1}^n \lambda_{p_k} = 0\) for \(m > 3\).

For functionally-independent FD \(\lambda_p(\mathbf{F}(3))\), the Jacobian \((5.5)\) is not \(\mathcal{E}^p\)-closed in general. For \(\mathcal{E}^p\)-potential (thereby, functionally-dependent) parameters

\[
\hat{\lambda}_p(\mathbf{F}(3)) = \frac{1}{2!} \Lambda(\mathbf{F}(3)) \epsilon^{[a]} \mathcal{E}^p \mathcal{E}^q \mathcal{E}^r, \tag{5.11}
\]
with an arbitrary potential being by Grassmann-odd-valued functional $\Lambda(\tilde{\Phi}(3))$ the Jacobian simplifies to $N = 3$ BRST exact functional determinant:

$$J_{(3)}(\tilde{\Phi}(3)) = \text{sdet} \left| \tilde{\Phi}_{(3)}^{i} \tilde{\partial}_{B_{3}} \right| \left\{ 1 + \frac{1}{3!} \Lambda(\tilde{\Phi}(3)) \varepsilon_{[p]} \prod_{k=1}^{3} \tilde{S}_{B_{p}^{k}} \right\}^{-3}, J_{(3)}(\tilde{\Phi}(3)) \tilde{S}^{p} = 0, \quad (5.12)$$

by virtue of the fact that the tensor quantity $(\tilde{S}_{B_{1}}^{p_{1}} \tilde{S}_{B_{2}}^{p_{2}} \tilde{S}_{B_{3}}^{p_{3}})$ is completely antisymmetric in $(p_{1}, p_{2}, p_{3})$ indices and can be presented as:

$$\tilde{S}_{B_{1}}^{p_{1}} \tilde{S}_{B_{2}}^{p_{2}} \tilde{S}_{B_{3}}^{p_{3}} = \frac{1}{3!} \varepsilon_{[p]} \tilde{S}^{3} \quad \text{for} \quad (\tilde{S})^{3} = \tilde{S}_{q_{1}} \tilde{S}_{q_{2}} \tilde{S}_{q_{3}} \varepsilon_{[q]}^{3} \quad (5.13)$$

which permits, because of: $\prod_{k=1}^{3} \tilde{S}_{B_{k}}^{q_{k}} = 0$, to have the representation

$$\delta_{q}^{p} + \lambda_{q}(\tilde{\Phi}(3)) \tilde{S}^{p} = \delta_{q}^{p} + \frac{1}{2!} \Lambda(\tilde{\Phi}(3)) \varepsilon_{q p q_{2}} \tilde{S}_{B_{2}}^{q_{2}} \tilde{S}_{B_{3}}^{p_{3}} \tilde{S}_{B_{1}}^{p_{1}} \quad (5.14)$$

$$\Rightarrow \text{tr}_{G(3)} \left( \delta_{q}^{p} + \lambda_{q}(\tilde{\Phi}(3)) \tilde{S}^{p} \right) = \text{tr}_{G(3)} \left( \delta_{q}^{p} \left[ 1 + \frac{1}{3!} \Lambda(\tilde{S})^{3} \right] \right) = \delta_{q}^{p} \left[ 1 + \frac{1}{3!} \Lambda(\tilde{S})^{3} \right] \quad (5.15)$$

that proves (5.12).

In the case of $\tilde{S}^{p}q_{1}$-closed parameters $\lambda_{p}$, $\lambda_{p} \tilde{S}^{q} = 0$, including constant $\lambda_{p}$, i.e., for $G(3)$ group elements, the Jacobian becomes trivial: $J_{(3)} = 1$. In turn, for the infinitesimal FD triplet $\tilde{\lambda}_{p}(\tilde{\Phi}(3))$ (5.11) the Jacobian (5.12) reduces to:

$$J_{(3)}(\tilde{\Phi}(3)) = 1 - \frac{1}{2} \Lambda(\tilde{\Phi}(3))(\tilde{S})^{3} + o(\Lambda) = \exp \left\{ - \frac{1}{2} \Lambda(\tilde{\Phi}(3))(\tilde{S})^{3} \right\} + o(\Lambda), \quad (5.16)$$

which permits to justify the gauge independence for the path integral $Z_{\Psi(3)}$ (and therefore for the conventional S-matrix) under small variation of the gauge condition: $\Psi(3) \rightarrow \tilde{\Psi}(3) + \delta \Psi(3)$, announced in (2.89) because of

$$Z_{3|\Psi(3)+\delta\Psi(3)}(0) = \int d\tilde{\Phi}(3) \text{sdet} ||\tilde{\Phi}_{(3)}^{i} \tilde{\partial}_{B_{3}}|| \exp \left\{ \frac{i}{\hbar} S_{\Psi(3)+\delta\Psi(3)}(\tilde{\Phi}) \right\} = Z_{3|\Psi(3)}(0). \quad (5.17)$$

in accordance with the choice (2.53) for $\delta \Psi(3)$ in terms of $\Lambda(\tilde{\Phi}(3))$ and therefore of $\tilde{\lambda}_{p} = \tilde{\lambda}_{p}(\Lambda)$

$$\Lambda(\tilde{\Phi}(3)|\delta\Psi(3)) = \frac{1}{3!} (\delta_{r}/\hbar) \delta\Psi(3)(\tilde{\Phi}(3)) \Rightarrow \tilde{\lambda}_{p}(\Lambda) = \frac{1}{3!} (\delta_{r}/\hbar) \delta\Psi(3)(\tilde{S}^{q} \tilde{S}^{r} \varepsilon_{pq r}). \quad (5.18)$$

Another properties for the generating functionals of Green functions related to the finite FD $N = 3$ BRST transformations we will consider in the Section [6]

5.2 $N = 4$ BRST transformations

The results of the above subsection are easily adapted for $N = 4$ BRST transformations with some specificity. Thus, the finite $N = 4$ BRST transformations acting on the fields $\Phi_{(4)}^{A}$, parameterizing configuration space $M_{tot}^{(4)}$ coinciding with $M_{tot}^{(3)}$ by dimension, are restored from the algebraic $N = 4$ BRST transformations by two equivalent ways: or from the condition which follows for any $\tilde{S}^{r}$-closed regular functional $K(\Phi_{(4)})$ to be invariant with respect to right-hand supergroup transformations $\{g(\lambda_{r})\}$, $r = 1, 2, 3, 4$, or from the Lie equations:

1) $\left\{ K \left( g(\lambda_{r})\Phi_{(4)} \right) = K \left( \Phi_{(4)} \right) \quad \text{and} \quad K \tilde{\partial}^{r} = 0 \right\} \Rightarrow g(\lambda_{r}) = \exp \left\{ \tilde{S}^{r} \lambda_{r} \right\}, \quad (5.19)$

2) $\Phi_{(4)}^{A}(\Phi_{(4)}|\lambda) \tilde{\partial}^{r} = \Phi_{(4)}^{A}(\Phi_{(4)}|\lambda) \tilde{S}^{r}$ \quad \text{(for} $\tilde{\partial}^{r} \equiv \frac{\partial}{\partial \lambda_{r}} \text{).} \quad (5.20)$
The set of such \( \{ g(\lambda_r) \} \) forms an Abelian 4-parametric supergroup,

\[
G(4) = \left\{ g(\lambda_r) : g(\lambda_r) = 1 + \sum_{e=1}^{4} \frac{1}{e!} \prod_{i=1}^{e} \frac{\lambda_{r_i}}{s^{r_i}} \lambda_{r_e} = \exp \left( \frac{1}{s} \lambda_r \right) \right\},
\]

(5.21)

For the \textit{field-dependent} \( G(4) \) quartet of odd-valued functionals \( \lambda_r(\Phi(4)) \), which is not closed under \( N = 4 \) BRST transformations, \( \lambda_r \frac{s}{s} \neq 0 \), the finite element \( g(\lambda_r(\Phi(4))) \) cannot be presented as group element in (5.21). The set of algebraic elements \( \hat{G}(4) = \{ \hat{g}_{lin}(\lambda(\Phi(4))) := 1 + \frac{s}{s} \lambda_r(\Phi(4)) \} \) forms a non-linear superalgebra which again corresponds to a set of formal group-like finite elements:

\[
\hat{G}(4) = \left\{ \hat{g}(\lambda_r(\Phi(4))) : \hat{g} = 1 + \sum_{e=1}^{4} \frac{1}{e!} \prod_{k=1}^{e} \frac{\lambda_{r_{k+1}}}{s^{r_k}} \prod_{k=1}^{e} \lambda_{r_e} \right\},
\]

(5.22)

The Jacobian of a change of variables: \( \Phi_i^{A^i} \rightarrow \Phi_i^{A^i} = \Phi_i^{A^i}(\lambda_r(\Phi(4))) \), in \( M_{4\text{tot}}(\Phi(4)) \), the path integral \( Z_{4|Y(0)}(4.1) \) and in \( Z_{4|Y}(J(4)) (4.2) \) generated by finite FD \( N = 4 \) BRST transformations may be calculated explicitly following to the same way as for the Jacobian (5.5) in \( N = 3 \) case:

\[
\text{sdt} \left\| \Phi_i^{A^i}(\lambda_r(\Phi(4))) \frac{\delta}{\delta \Phi_i^B(4)} \right\| = \exp \left\{ - \text{tr}_{G(4)} \ln \left( \left[ e + m \right]_{r_2} \right) \right\}, \quad \text{for} \quad (e_{r_1}, m_{r_2}) \equiv (\delta_{r_1}, \lambda_{r_2} \frac{s}{s} \lambda_{r_1}),
\]

(5.23)

where \( \text{tr}_{G(4)} \) denotes trace over matrix \( G(4) \)-indices. The justification of the representation (5.23) is based on the same points (5.6–5.10) as for its \( N = 3 \) analog (5.5), whose detailed calculation we leave out of the paper scope.

For \( \frac{s}{s} \)-potential, therefore functionally-dependent parameters

\[
\lambda_{r_1} = \frac{1}{3!} \Lambda(\Phi(4)) \varepsilon_{[r_1]} \frac{s}{s} \frac{s}{s} \frac{s}{s} \lambda_{r_1},
\]

(5.24)

with an arbitrary potential being by Grassmann-even-valued functional \( \Lambda(4) = \Lambda(4)(\Phi(4)) \) the Jacobian (5.23) reduces to \( N = 4 \) BRST exact functional determinant:

\[
J(4)(\Phi(4)) = \text{sdt} \left\| \Phi_i^{A^i}(\lambda_r(\Phi(4))) \frac{\delta}{\delta \Phi_i^B(4)} \right\| = \left\{ 1 + \frac{1}{4!} \Lambda(4)(\Phi(4)) \left( \frac{s}{s} \right)^4 \right\}^{-4}, \quad J(4)(\Phi(4)) \frac{s}{s} = 0,
\]

(5.25)

where we have used the property for tensor quantity \( \prod_{k=1}^{4} \frac{s}{s} r_k \) to be completely antisymmetric in \( (r_1, r_2, r_3, r_4) \) indices that makes natural the definition:

\[
\prod_{k=1}^{4} \frac{s}{s} r_k = \frac{1}{4!} \varepsilon_{[r]} \left( \frac{s}{s} \right)^4 \quad \text{for} \quad \left( \frac{s}{s} \right)^4 = \prod_{k=1}^{4} \frac{s}{s} r_k \varepsilon_{[r]}.
\]

(5.26)

Again, for the case of \( \frac{s}{s} \)-closed parameters \( \lambda_r, \lambda_{r_2} \frac{s}{s} = 0 \), including constant \( \lambda_r \), i.e., for \( G(4) \) group elements, the Jacobian becomes trivial: \( J(4) = 1 \), whereas for the infinitesimal FD quartet \( \lambda_r(\Phi(4)) \) (5.24) the Jacobian (5.25) reduces to:

\[
J(4)(\Phi(4)) = 1 - \frac{1}{3!} \Lambda(4)(\Phi(4)) \left( \frac{s}{s} \right)^4 + o(\Lambda(4)) = \exp \left\{ - \frac{1}{3!} \Lambda(4)(\Phi(4)) \left( \frac{s}{s} \right)^4 \right\} + o(\Lambda(4)),
\]

(5.27)

which immediately leads to the gauge independence of the path integral \( Z_{4|Y(4)}(0) \) (and therefore for the conventional S-matrix) under small variation of the gauge condition: \( Y(4) \rightarrow Y(4) + \delta Y(4) \), announced in (4.15) because of

\[
Z_{4|Y(4) + \delta Y(4)}(0) = \int d\Phi(4) \text{sdt} \left\| \Phi_i^{A^i} \frac{s}{s} \frac{s}{s} \frac{B_i}{s} \right\| \exp \left\{ \frac{1}{n} S_{Y(4) + \delta Y(4)}(\Phi(4)) \right\} = Z_{4|Y(4)}(0).
\]

(5.28)

27
according to the choice \( \delta Y_4 \) in terms of \( \Lambda_4(\Phi(4)) \) and therefore of \( \hat{\lambda}_r = \hat{\lambda}_r(\Lambda(4)) \)

\[
\Lambda_4(\Phi(4) \delta Y_4) = -\frac{1}{4}(i/\hbar)\delta Y_4(\Phi(4)) \implies \hat{\lambda}_r(\Lambda(4)) = \frac{1}{4!}(i/\hbar)\delta Y_4(\Phi(4)) \prod_{k=2}^{4} S^{r_k \varepsilon_{[r_k]}}, \tag{5.29}
\]

6. Correspondence between the gauges, Ward identities, gauge dependence, gauge-invariant Gribov–Zwanziger model.

Here we consider the physical properties of the respective \( N = 3, N = 4 \) finite BRST transformations, including extended by sources (antifields) to the \( N = 3 \) or \( N = 4 \) BRST transformations effective actions in the Subsection 6.1 and its applications in the Subsection 6.2 to the Gribov–Zwanziger model [36] and therefore the corresponding triplet of field-dependent parameters have the form according to the choice (4.16) for fermionic functionals \( \Psi^{(3)}_0 \).

The coincidence of the vacuum functionals \( \Psi^{(3)}_0, z^{(3)} \) for \( \xi = 0 \). The other one \( \Psi^{(3)}_0 + \Psi^{(3)}_0 \) corresponds to any family from the gauges within the \( \Psi^{(3)}_0(\Phi(3)) \), including \( \tilde{\xi} \)-gauges for \( \Phi(3) = 0 \) in (2.75) and for \( \chi(A, B) = (\partial^\mu A_\mu + \xi g^2 B = 0) \) within the functional \( \Psi^{(3)}_0(\Phi(3)) \). To this end, we use a finite \( N = 3 \) BRST transformation with functionally-dependent parameters \( \hat{\lambda}_p(\Lambda(\Phi(3)) \) (6.11), the \( N = 3 \) BRST invariance of the quantum action, \( S^{(4)}(\Phi(3)) \) (5.29) for \( \xi = 0 \), and the form of the Jacobian, \( J^{(3)}(\Phi(3)) \), (5.12) of a corresponding change of variables, \( \Phi(3) \to \Phi(3)\hat{g}(\hat{\lambda}) \), given as follows

\[
\begin{align*}
Z^{(3)}_p|\Psi^{(3)}_0, \Phi^{(3)}_0 \to \Phi^{(3)}_0 \hat{g}(\hat{\lambda}) & = \int d\Phi^{(3)}_0 \exp \left\{ \frac{i}{\hbar} \left[ S^{(3)}_{\Psi^{(3)}_0 + \Psi^{(3)}_0} + 3\hbar \ln \left( 1 + \frac{1}{3!} \Lambda^{(\Phi(3))}(\mathcal{S})^3 \right) \right] \right\} \\
& = \int d\Phi^{(3)}_0 \exp \left\{ \frac{i}{\hbar} \left[ S^{(3)}_{\Psi^{(3)}_0 + \Psi^{(3)}} + 3\hbar \ln \left( 1 + \frac{1}{3!} \Lambda^{(\Phi(3))}(\mathcal{S})^3 \right) - \frac{1}{3!} \Psi^{(3)}_0(\mathcal{S})^3 \right] \right\}. \tag{6.1}
\end{align*}
\]

The coincidence of the vacuum functionals \( Z^{(3)}_p|\Psi^{(3)}_0, \Phi^{(3)}_0 \to \Phi^{(3)}_0 \hat{g}(\hat{\lambda}) \), evaluated with the respective fermionic functionals \( \Psi^{(3)}_0 \) and \( \Psi^{(3)}_0 + \Psi^{(3)}_0 \), takes place in case there holds a compensation equation for an unknown Fermionic functional \( \Lambda = \Lambda^{(\Phi(3))} \):

\[
3\hbar \ln \left( 1 + \frac{1}{3!} \Lambda^{(\mathcal{S})^3} \right) = \frac{1}{3!} \Psi^{(3)}_0(\mathcal{S})^3 \iff \frac{1}{3!} \Lambda^{(\mathcal{S})^3} = \exp \left( -\frac{i}{3! \cdot 3\hbar} \Psi^{(3)}_0(\mathcal{S})^3 \right) - 1. \tag{6.2}
\]

The solution of equation (6.2) for an unknown \( \Lambda^{(\Phi(3))} \), which determines \( \hat{\lambda}_p(\Phi(3)) \), according to (5.11), with accuracy up to \( N = 3 \) BRST exact terms, is given by

\[
\Lambda^{(\Phi(3))}|\Psi^{(3)}_0 = -\frac{i}{3\hbar} g(y)\Psi^{(3)}_0, \text{ for } g(y) = [\exp(y) - 1]/y \text{ and } y = -\frac{i}{3\hbar} \Psi^{(3)}_0(\mathcal{S})^3, \tag{6.3}
\]

and therefore the corresponding triplet of field-dependent parameters have the form

\[
\hat{\lambda}_p^{(\Phi(3))}|\Psi^{(3)}_0, \Psi^{(3)}_0 = -\frac{i}{3\hbar} g(y)\Psi^{(3)}_0 \mathcal{S}^{q} \mathcal{S}^{r} \varepsilon^{pqr}, \tag{6.4}
\]
whose approximation linear in $\Psi'_3$ is given by

$$\hat{\lambda}_p \left( \Phi(3) \Psi' \right) = -\frac{i}{3!\hbar} \left( \Psi(3) \bar{\Psi} \bar{S}^\nu \bar{S}^\tau \varepsilon_{pqr} \right) + o(\Psi'_3), \tag{6.5}$$

with opposite sign than in $\Psi'_3(6.4)$ because of we started here from the gauge determined by $\Psi'_3(3)0$ instead of $\Psi'_3(3)0 + \Psi'_3(3)$ in $\Psi'_3$. Therefore, for any change $\Psi'_3(3)$ of the gauge condition $\Psi'_3(3)0 \rightarrow \Psi'_3(3)0 + \Psi'_3(3)$, we can construct a unique FD $N = 3$ BRST transformation with functionally-dependent parameters $\Psi'_3(3)0$ that allows one to preserve the form of the path integral (6.1) for the same Yang–Mills theory. On the other hand, if we consider the inverse form of compensation equation (6.2) for an unknown gauge variation $\Psi'_3$ with a given $\Lambda(\Phi(3))$, we can present it in the form

$$3 \cdot 3!\hbar \ln \left( 1 + \frac{1}{3!} \Lambda(\bar{S})^3 \right) = \Psi'_3(3) \left( \frac{\bar{S}}{3} \right)^3 \leftrightarrow 3 \cdot 3!\hbar \left[ \sum_{n=1} (-1)^n \frac{(\Lambda(\bar{S})^3)^{n-1} \Lambda}{(3!)^{n-1}n} \right] \left( \bar{S} \right)^3 = \Psi'_3(3) \left( \frac{\bar{S}}{3} \right)^3, \tag{6.6}$$

whose solution, with accuracy up to an $\frac{1}{3!\hbar}$-exact term, is given by

$$\Psi'_3(\Phi(3)|\Lambda) = 3 \cdot 3!\hbar \left[ \sum_{n=1} (-1)^n \frac{(\Lambda(\bar{S})^3)^{n-1} \Lambda}{(3!)^{n-1}n} \right] = 3i\hbar \left[ \sum_{n=1} (-1)^n \frac{\Lambda(\bar{S})^3}{(3!)^{n-1}n} \right] = 1, \tag{6.7}$$

Thereby, for any change of variables in the path integral $Z_{\Psi_3(3)}$ given by finite FD $N = 3$ BRST transformations with the parameters $\hat{\lambda}_p = \frac{1}{3!}\Lambda(\bar{S})^3 \varepsilon_{pqr}$, we obtain the same path integral $Z_{\Psi_3(3)0 + \Psi'_3(3)}$, evaluated, however, in a gauge determined by the Fermionic functional $\Psi_3(3)|\Lambda + \hat{\lambda}_p(\bar{S})$, in complete agreement with (6.7).

This latter, in particular, implies that we are able to reach any gauge condition for the partition function within the $R_\xi$-like family of gauges, starting, e.g., from the Landau gauge and choosing: $\Psi'(3) = \xi g^2 \int d^4x tr' (\nabla B)$ (for $\xi = 1$ in the Feynman gauge).

Making in $Z_{\Psi_3(3)}(\tilde{J}_3(3))$ an FD $N = 3$ BRST transformation, $\tilde{\Phi}(3) \rightarrow \tilde{\Phi}(3)\tilde{g}(\tilde{\lambda})$ and using the relations (5.12) and (6.3), we obtain a modified Ward (Slavnov–Taylor) identity:

$$\left\langle \exp \left\{ \frac{i}{\hbar} \tilde{J}_3 \tilde{C}_3^i \tilde{g} \left[ \tilde{g} \left( \tilde{\Phi}(3)|\Lambda \right) \right] - 1 \right\} \right\rangle \left\langle \left( 1 + \frac{1}{3!} \Lambda(\bar{S})^3 \right)^{-3} \right\rangle_{\Psi'(3), \tilde{J}_3(3)} = 1, \tag{6.8}$$

where the source-dependent average expectation value corresponding to a gauge-fixing $\Psi_3(\Phi(3))$, as in (2.91), explicitly for regular functional $L = L(\tilde{\Phi}(3))$:

$$\langle L \rangle_{\Psi_3(3), \tilde{J}_3(3)} = Z_{3!\Psi_3(3)}^{-1} \left( \tilde{J}_3(3) \right) \int d\tilde{\Phi}(3) \left. L \exp \left\{ \frac{i}{\hbar} \left[ \tilde{S}_{\Phi(3)} + \tilde{J}_3 \tilde{C}_3^i \tilde{g} \left[ \tilde{g} \left( \tilde{\Phi}(3)|\Lambda \right) \right] \right] \right\} \right|_{\Psi_3(3), \tilde{J}_3(3)} = 1. \tag{6.9}$$

Due to the presence of $\Lambda(\tilde{\Phi}(3))$, which implies functionally dependent $\hat{\lambda}_p(\Lambda)$, the modified Ward identity depends on a choice of the gauge Fermion $\Psi_3(\tilde{\Phi}(3))$ for non-vanishing $\tilde{J}_3$, according to (6.3), (6.4), and therefore the corresponding Ward identities for Green’s functions, obtained by differentiating (6.8) with respect to the sources, contain the functionals $\hat{\lambda}_p(\Lambda)$ and their derivatives as weight functionals. Due to (6.8) for constant $\lambda_p$, the usual $G(3)$-triplet of the Ward identities (2.91) for $Z_{3!\Psi_3(3)}(\tilde{J}_3(3))$ follow from the first order in $\lambda_p$.

Then, taking account of (6.4), we find that (6.8) implies a relation which describes the gauge dependence of $Z_{3!\Psi_3(3)}(\tilde{J}_3(3))$ for a finite change of the gauge, $\Psi_3(3) \rightarrow \Psi_3(3) + \Psi'_3(3)$:

$$Z_{3!\Psi_3(3) + \Psi'_3(3)}(\tilde{J}_3(3)) = Z_{3!\Psi_3(3)}(\tilde{J}_3(3)) \left\langle \exp \left\{ \frac{i}{\hbar} \tilde{J}_3 \tilde{C}_3^i \tilde{g} \left[ \tilde{g} \left( \tilde{\Phi}(3)|\Lambda \right) - \Psi'_3(3) \right] \right\} - 1 \right\rangle_{\Psi'_3(3), \tilde{J}_3(3)} = 1. \tag{6.10}$$
so that on the mass-shell for $Z_{3|\Psi(3)}(\tilde{J}(3))$: $\tilde{J}(3) = 0$, the path integral (and therefore the conventional physical S-matrix) does not depend on the choice of $\Psi'_{(3)}(\tilde{\Phi}(3))$.

Let us introduce extended generating functionals of Green’s functions by means of sources $K_{C^3_{3|p}} = -K_{C^3_{3|p}}$, $K_{C^3_{3|pq}} = -K_{C^3_{3|pq}}$, $\epsilon(K_{C^3_{3|p}}) = \epsilon(K_{C^3_{3|pq}}) + 1 = \epsilon(\tilde{\Phi}^{C^3_{3|p}}) + 1$, introduced respectively to $N = 3$ BRST variations $\tilde{\Phi}^{C^3_{3|p}}_3$, $\tilde{\Phi}^{C^3_{3|pq}}_3$, and $\tilde{\Phi}^{C^3_{3|pq}}_3$.

\[
Z_{3|\Psi(3)}(\tilde{J}(3), K_p, K_{pq}, K) = \int d\tilde{\Phi}^{C^3_{3|p}}_3 \exp \left\{ \frac{i}{\hbar} \left[ S_{\Psi(3)}(\tilde{\Phi}^{C^3_{3|p}}_3) + K_{C^3_{3|p}}(\tilde{\Phi}^{C^3_{3|p}}_3) \right] + \frac{1}{3!} \epsilon^{pq} K_{C^3_{3|pq}}(\tilde{\Phi}^{C^3_{3|pq}}_3) \right\} \text{ for } Z_{3|\Psi(3)}(\tilde{J}(3), 0, 0, 0) = Z_{3|\Psi(3)}(\tilde{J}(3)). \tag{6.11}
\]

If we make in (6.11) a change of variables in the extended space of $(\tilde{\Phi}^{C^3_{3|p}}_3, K_{C^3_{3|p}}, K_{C^3_{3|pq}}, K)$:

\[
\tilde{\Phi}^{C^3_{3|p}}_3 \to \tilde{\Phi}^{C^3_{3|p}}_3(\lambda), \quad \tilde{\Phi}^{C^3_{3|pq}}_3 \to \tilde{\Phi}^{C^3_{3|pq}}_3(\lambda), \quad K_{C^3_{3|p}} \to K_{C^3_{3|p}}, \tag{6.12}
\]

for $\tilde{J}^{C^3_{3|p}}_3 = 0$, with finite constant parameters $\lambda_p$, we find that the integrand in (6.11) is unchanged, due to $(\tilde{\Phi}^{C^3_{3|p}}_3)$ is invariant with accuracy up to $\mathcal{O}(\lambda)$ justifying to call them as the algebraic extended $N = 3$ BRST transformations.

Making in (6.11) a change of variables, which corresponds only to $N = 3$ BRST transformations $\tilde{\Phi}^{C^3_{3|p}}_3 \to \tilde{\Phi}^{C^3_{3|p}}_3(\lambda)$ with an arbitrary functional $\lambda_p(\tilde{\Phi}(3))$ from (6.12), we obtain a modified Ward identity for $Z_{3|\Psi(3)}(\tilde{J}(3), K_p, K_{pq}, K)$:

\[
\langle \exp \left\{ \frac{i}{\hbar} \left[ J^{C^3_{3|p}}_3(\lambda) \tilde{\Phi}^{C^3_{3|p}}_3(\lambda) \right] - 1 \right\} + K_{C^3_{3|p}}(\tilde{\Phi}^{C^3_{3|p}}_3) \right\} \left\{ 1 + \frac{1}{3!} \lambda^{pq} \right\} = 1, \tag{6.13}
\]

where the symbol “$\langle L\rangle_{\Psi(3), \tilde{J}(3), K_p, K_{pq}, K}$” for any $L = L(\tilde{\Phi}(3), K_p, K_{pq}, K)$ stands for a source-dependent average expectation value for a gauge $\Psi(3)$ in the presence of sources (extended Zinn–Justin fields) $K_{C^3_{3|p}}, K_{C^3_{3|pq}}, K$.

For constant parameters $\lambda_p$, we deduce from (6.13), in the first order in $\lambda_p$,

\[
\langle \tilde{J}^{C^3_{3|p}}_3(\lambda) \tilde{\Phi}^{C^3_{3|p}}_3(\lambda) + K_{C^3_{3|p}}(\tilde{\Phi}^{C^3_{3|p}}_3(\lambda)) \rangle_{\Psi(3), \tilde{J}(3), K_p, K_{pq}, K} = 0. \tag{6.15}
\]

Identities (6.15) can be presented as

\[
\left[ \frac{\delta}{\delta K_{C^3_{3|p}}} - K_{C^3_{3|p}} \delta K_{C^3_{3|pq}}^{C^3_{3|pq}} + \frac{1}{3!} \epsilon^{pq} K_{C^3_{3|pq}}(\tilde{\Phi}^{C^3_{3|pq}}_3) \right] \ln Z_{3|\Psi(3)}(\tilde{J}(3), K_p, K_{pq}, K) = 0. \tag{6.16}
\]
Let us consider an extended generating functional of vertex Green's functions, \( \Gamma(\langle \Phi(3) \rangle, K_p, K_{pq}, \bar{K}) \), being a functional Legendre transform of \( \ln Z_{3|\Psi(3)}(\langle J(3) \rangle, K_p, K_{pq}, \bar{K}) \) with respect to the sources \( \langle J(3) \rangle \):

\[
\Gamma(\langle \Phi(3) \rangle, K_p, K_{pq}, \bar{K}) = \frac{\hbar}{i} \ln Z_{3|\Psi(3)}(\langle J(3) \rangle, K_p, K_{pq}, \bar{K}) - \frac{\hbar}{i} \langle \Phi(3) \rangle,
\]

(6.17)

where \( \tilde{J}_{C_3} = -\Gamma(\langle \Phi(3) \rangle, K_p, K_{pq}, \bar{K}) \frac{\delta}{\delta \langle \Phi(3) \rangle} \) and \( \langle \Phi(3) \rangle = \frac{\hbar}{i} \ln Z_{3|\Psi(3)}(\langle J(3) \rangle, K_p, K_{pq}, \bar{K}) \).

From (6.16)–(6.18), we deduce for \( \Gamma_3 = \Gamma(\langle \Phi(3) \rangle, K_p, K_{pq}, \bar{K}) \) an \( G(3) \)-triplet of independent Ward identities:

\[
\frac{\Gamma_3}{\langle \Phi(3) \rangle} \frac{\delta}{\delta \langle \Phi(3) \rangle} + \left\{ \frac{\delta}{\delta K_{C_3}^a} \frac{\delta}{\delta K_{C_3}^b} - \frac{1}{3!} \varepsilon_{pq} \frac{\delta}{\delta K_{C_3}^c} \frac{\delta}{\delta K_{C_3}^d} \right\} \Gamma_3 = \frac{1}{2} (\Gamma_3, \Gamma_3)_{(3)}^p + V_3^p \Gamma_3 = 0,
\]

(6.19)

for \( p = 1, 2, 3 \), in terms of \( G(3) \)-triplets of extended antibrackets, \( (\cdot, \cdot)_{(3)}^p \), and operators \( V_3^p \), extending the familiar \( Sp(2) \)-covariant Lagrangian quantization for gauge theories \( 33, 34 \) (see also \( 40, 41, 42 \) as well as \( 28, 29, 30 \)) in the \( N = 2 \) case, introduced for general gauge theories.

The Ward identities (6.19) are interesting as they remind of the behavior of the extended quantum action \( S_{3|\Psi(3)}(\tilde{F}, \tilde{G}, K_p, \bar{K}) \) – being the tree approximation for the extended effective action \( \Gamma_3 \) within the loop expansion – and serve as generating equations for a corresponding \( G(3) \)-covariant method of Lagrangian quantization, covering the case more general than a gauge group.

In turn, the case of \( N = 4 \) finite BRST transformations permits to get the same results with some peculiarities. We restrict ourselves by only derivation of the respective modified Ward identity and description of the gauge dependence problem, being based on the solution of the compensation equation from the change of variables in \( Z_{4|Y(0)} \) (4.11) generated by FD \( N = 4 \) BRST transformations with quartet of the parameters \( \lambda_r(\Phi(4)) \) (5.24) with Jacobian \( J_{(4)}(\Phi(4)) \) (5.25).

\[
4i\hbar \ln \left( 1 + \frac{i}{4!} \Lambda(\Phi(4))(\bar{\Phi})^4 \right) = \frac{1}{4!} \Lambda(\Phi(4))(\bar{\Phi})^4 \iff \frac{i}{4!} \Lambda(\Phi(4))(\bar{\Phi})^4 = \exp \left( \frac{i}{4!} \cdot 4i\hbar Y_4^0(\bar{\Phi})^4 \right) - 1.
\]

(6.21)

to guarantee the coincidence of the path integrals, \( Z_{4|Y(0)} \), (4.11) and \( Z_{4|Y+Y'}(0) \) evaluated in different admissible gauges corresponding to the Bosonic gauge functionals \( Y_4(\Phi(4)) \) (e.g. for the Landau gauge \( Y_4^0(\Phi(4)) \) (4.13)) and, \( Y_4 + Y'_4 \), (e.g. for the Feynman gauge \( Y_4^0(\Phi(4)) \) (4.10) within \( R_\xi \)-like gauges for \( \xi = 1 \) for finite \( Y_4^0(\Phi(4)) \)). The solution of (6.21) for an unknown \( \Lambda(\Phi(4)) \) and hence of \( \lambda_r(\Phi(4)) \), with accuracy up to \( N = 4 \) BRST exact terms, is given in terms of the function \( g(z) \) (6.32):

\[
\Lambda(\Phi(4))Y_4^0 = \frac{i}{4!} g(z) Y_4^0(\bar{\Phi})^4, \quad z \equiv \frac{i}{4!} \cdot 4i\hbar Y_4^0(\bar{\Phi})^4,
\]

(6.22)

\[
\lambda_{r_1}(\Phi(4))Y_4^r = \frac{i}{4!} g(z) Y_4^r \prod_{k=2}^4 \bar{\Phi}^{r_k} \varepsilon_{[r]}^4.
\]

(6.23)

31
whose approximation linear in \(Y'_4\) coincide with \((4.10)\) with opposite sign because of we started here from the gauge determined by \(Y_4\) instead of \(Y_4 + \hat{Y}'_4\) in \((4.15)\). From the inverse form of compensation equation \((6.24)\) for an unknown gauge variation \(Y'_4\) with a given \(\Lambda(\Phi_4)\):

\[
4!4\hbar \ln \left( 1 + \frac{1}{4!} \Lambda(\hat{s})^4 \right) = -Y'_4(\hat{s})^4 \iff 4!4\hbar \left[ \sum_{n=1}^{\infty} \frac{(-1)^n}{(4!)^n} \left( \Lambda(\hat{s})^4 \right)^{n-1} \Lambda \right] \left( \hat{s} \right)^4 = -Y'_4(\hat{s})^4
\]

we find with accuracy up to an \(\hat{s}^{-r}\)-exact term, that

\[
Y'_4(\Phi_4|\Lambda) = 4 \cdot 4\hbar \left[ \sum_{n=1}^{\infty} \frac{(-1)^n}{(4!)^n} \left( \Lambda(\hat{s})^4 \right)^{n-1} \Lambda \right] = 4\hbar \left[ \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n-1} \left( \hat{\lambda}_r \hat{s}^{-r} \right)^{n-1} \Lambda(\Phi_4) \right]. \quad (6.25)
\]

Thus, for any change of variables in the path integral \(Z_{4|Y_4}(J_4)\) given by finite FD \(N = 4\) BRST transformations with the parameters \(\hat{\lambda}_r (5.24)\), we obtain the same path integral \(Z_{4|Y_4} + Y'_4\), evaluated, however, in a gauge determined by the Bosonic functional \(Y_4 + \hat{Y}'_4\).

Making in \(Z_{4|Y_4}(J_4)\) an FD \(N = 4\) BRST transformation, \(\Phi_4 \rightarrow \Phi_4(\tilde{\lambda})\) and using the relations \((5.25), (6.22)\) and \((6.23)\), we obtain a \(N = 4\) modified Ward (Slavnov–Taylor) identity:

\[
\left\langle \exp \left\{ \frac{i}{\hbar} \int_{\mathcal{L}_4} \Phi_4 \left[ \tilde{g}(\hat{\lambda}_r (\Phi_4|\Lambda)) - 1 \right] \right\} \left( 1 + \frac{1}{4!} \Lambda(\hat{s})^4 \right)^{-4} \right\rangle_{Y_4,J_4} = 1. \quad (6.26)
\]

where the source-dependent average expectation value corresponding to a gauge-fixing \(Y_4(\Phi_4)\) is determined as in \((6.9)\) for \(N = 3\) case. Due to \(\Lambda(\Phi_4)\), which implies functionally dependent \(\hat{\lambda}_r (\Lambda)\), the modified Ward identity depends on a choice of the gauge Boson \(Y_4(\Phi_4)\) for non-vanishing \(J_4\), according to \((6.22), (6.23)\) with the same as for \(N = 3\) case interpretation for the modified Ward identities for the Green functions. Due to \((6.26)\) for constant \(\lambda_r\), the usual \(G(4)\)-quartet of the Ward identities \((4.11)\) for \(Z_{4|Y_4}(J_4)\) follow from the first order in \(\lambda_r\).

Then, taking account of \((6.23)\), we find that \((6.26)\) implies a relation which describes the gauge dependence of \(Z_{4|Y_4}(J_4)\) for a finite change of the gauge, \(Y_4 \rightarrow Y_4 + \hat{Y}'_4\):

\[
Z_{4|Y_4} + Y'_4(J_4) = Z_{4|Y_4}(J_4) \left\langle \exp \left\{ \frac{i}{\hbar} \int_{\mathcal{L}_4} \Phi_4 \left[ \tilde{g}(\hat{\lambda}_r (\Phi_4|Y'_4)) - 1 \right] \right\} \right\rangle_{Y_4,J_4}, \quad (6.27)
\]

so that on the mass-shell for \(Z_{4|Y_4}(J_4)\): \(J_4 = 0\), the path integral (and therefore the conventional physical S-matrix) does not depend on the choice of \(Y'_4(\Phi_4)\).

### 6.2 Gauge-independent Gribov-Zwanziger model with local \(N = 3, 4\) BRST symmetries

Finally, we turn to the Gribov copies problem \([8]\) within the Gribov–Zwanziger model \([34]\) with a gauge-invariant horizon functional, \(H(A^h)\), recently proposed to be added to an \(N = 1\) BRST invariant Yang–Mills quantum action \([37]\) in Landau gauge with the use of the gauge-invariant (thereby, invariant with respect to a local \(N = 1, 2\) BRST invariance, as it was shown in \([39]\), Eq. (36)–(40)) transverse fields \(A^h_\mu = (A^h)_{\mu}^{\text{tr}} \) \([33]\):

\[
A_\mu = A^h_\mu + A^t_\mu : A^t_\mu = (\eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{2} (A^\nu - ig [A^\nu, A^\mu - \frac{1}{2} \partial^\nu \partial_\mu A^h])) + O(A^3) : A^h_\mu \delta^{sp} = 0, \quad (6.28)
\]

\[
H(A^h) = \gamma^2 \int d^3x (d^3y f^{mnk}(A^h)^n_{\mu}(x)(M^{-1})^{ml}(A^h)^{l}_{\nu}(x) f^{ijk}(A^h)^{j}_{\mu}(y) + d(N^2 - 1)). \quad (6.29)
\]
Note, that the systematic study for the original Gribov–Zwanziger model \[33\] with not BRST-invariant horizon, $H(A)$, within Lagrangian BRST quantization of gauge theories \[44, 45\] from the viewpoint of so-called soft BRST symmetry breaking was initiated in \[40\]. Then, as in the case of $N = 1, 2$ BRST symmetry, the gauge and $N = 1, 2$ BRST invariant extension of the respective quantum Yang–Mills action within the $R_c$-family of gauges with a gauge fermion $\Psi_\xi$ and a boson $Y_\xi$ prescribed by the Gribov–Zwanziger actions are given by

$$
\hat{S}_{GZ}(\Phi) = S_0 + \Psi_\xi \bar{\psi} + H(A^h), \quad \text{for} \quad \hat{S}_{GZ}(\Phi(1 + \bar{\psi} \mu)) = \hat{S}_{GZ}(\Phi),
$$

(6.30)

$$
\hat{S}_{GZ}(\Phi(2)) = S_0 - \frac{1}{2} Y_\xi \bar{a} \bar{a} + H(A^h), \quad \text{for} \quad \hat{S}_{GZ}(\Phi(2) g(\mu_a)) = \hat{S}_{GZ}(\Phi(2)),
$$

(6.31)

with allowance made for \[1.5, 2.0\] and \[2.10\], \[2.17\] the same may be done in $N = 3$ and $N = 4$ BRST invariant formulations of the respective quantum actions $S_{\Psi_\xi}(\tilde{\Phi}(3))$ \[2.78\] and $S_{Y_\xi}(\tilde{\Phi}(4))$ \[1.7\]. Therefore, the $N = 3$ and $N = 4$ BRST invariant and gauge independent Gribov–Zwanziger actions within $\Psi_\xi$ and respectively within $Y_\xi$-family of gauges related to $R_c$-gauges are given by

$$
\hat{S}_{GZ}(\tilde{\Phi}(3)) = S_0 + \frac{1}{2} \Psi_\xi(3) + H(A^h), \quad \text{for} \quad \hat{S}_{GZ}(\tilde{\Phi}(3) g(\lambda_\rho)) = \hat{S}_{GZ}(\tilde{\Phi}(3)),
$$

(6.32)

$$
\hat{S}_{GZ}(\tilde{\Phi}(4)) = S_0 - \frac{1}{2} Y_\xi(4) + H(A^h), \quad \text{for} \quad \hat{S}_{GZ}(\tilde{\Phi}(4) g(\lambda_\gamma)) = \hat{S}_{GZ}(\tilde{\Phi}(4)).
$$

(6.33)

As in the case of the $N = 1, 2$ BRST symmetry, one may expect the unitarity of the theory within the suggested $N = 3$, $N = 4$ BRST symmetry generalizations of the Faddeev–Popov quantization rules \[3\]. These problems are under study.

The same results concerning the problems of unitarity and gauge-independence may be achieved within the local formulations of Gribov–Zwanziger theory \[33\] when the horizon functional is localized (in the path integral) by means of a quartet of auxiliary fields $\phi_{aux} = (\varphi^{mn}_{\mu}, \varphi^{mn}_{\bar{\mu}}, \omega^{mn}_{\mu}, \bar{\omega}^{mn}_{\bar{\mu}})$, having opposite Grassmann parities, $\epsilon(\varphi, \bar{\varphi}) = \epsilon(\omega, \bar{\omega}) + 1 = 0$, and being antisymmetric in $su(N)$ indices $m, n$. We suggest here the only $N = 1$ BRST invariant formulation,

$$
\hat{S}_{GZ}(\Phi(1), \phi_{aux}) = S_0(A) + \Psi_\xi(\Phi(1)) \bar{\psi} + S_\gamma(A^h, \phi_{aux}).
$$

(6.34)

$$
S_\gamma = \int d^4x \left\{ \varphi^{mn}_{\mu} m^{ml}(A^h) \varphi^{ml}_{\mu - \omega}^{mn} M^{ml}(A^h) \omega_{\mu}^{mn} + \gamma f^{ml}(A^h) \omega_{\mu}^{mn} \omega_{\mu}^{mn} + \gamma^2 d(\hat{N}^2 - 1) \right\},
$$

(6.35)

with additional non-local $N = 1$ BRST transformations for the fields $\phi_{aux}$ with untouched ones for $\Phi(1)$ \[2.7\]

$$
\phi_{aux}(\bar{\psi}^{mn}_{\mu}, \varphi^{mn}_{\bar{\mu}}, \omega^{mn}_{\mu}, \bar{\omega}^{mn}_{\bar{\mu}}) = (0, \varphi^{mn}_{\mu} \varphi^{mn}_{\bar{\mu}}, \omega^{mn}_{\mu} \omega^{mn}_{\bar{\mu}}, \bar{\omega}^{mn}_{\mu} \bar{\omega}^{mn}_{\bar{\mu}}).
$$

(6.36)

The part $S_\gamma$ in case of $N = 3$ and $N = 4$ BRST formulation for the quantum actions (as well as for the $N = 2$ case) should be modified due to another spectra for the auxiliary fields $\phi_{aux}$.

Finally, the non-local gauge-invariant transverse fields, $A^h_{\mu}$, \[6.23\] can also be localized by using complex $SU(\hat{N})$-valued auxiliary field, $h(x)$, with non-trivial own gauge and $N = 1$ BRST transformations \[1.7\] in order to reach really localized Gribov–Zwanziger model still $N = 1, 2, 3, 4$ BRST invariant without Gribov ambiguity, whose properties are now under study.

### 7 On Feynman diagrammatic technique in $N = 3$, $N = 4$ BRST quantization

Here, we introduce some new definitions to develop a Feynman diagrammatic technique for the Yang–Mills theory within suggested $N = 3$ and $N = 4$ BRST invariant formulations for the non-renormalized...
quantum actions $S_{\psi, (3)}$ given by (2.73)–(2.81), and $S_{Y, (4)}$ determined by (1.7)–(1.10). To be complete, we compare the graphs which contain additional lines related to new fictitious fields to ones with known, i.e. ghost, $C(x)$, antighost, $\overline{C}(x)$, fields in $N = 1$ BRST setup and with duplet of ghost-antighost fields, $C^a(x)$, $a = 1, 2$ in $N = 2$ BRST setup, having in mind that usually the Nakanishi-Lautrup field $B(x)$ is integrated out from the quantum actions.

We present the generating functions of Green functions in $R_{\xi}$-gauges $Z(J)$ (2.49), $Z_Y(J)$ determined with the quantum action $S_{\psi, (3)}$ (2.18), $Z_{3\psi, (4)}(J)$ (2.24), $Z_{4\psi, (4)}(J)$ (1.2) respectively for $N = 1, 2, 3, 4$ BRST symmetry within the perturbation theory according to [38] but for $d$-dimensional space-time

\[
Z(J) = \exp \left\{ V \left( \frac{h}{i} \frac{\delta}{\delta J_{\mu}} \right) \right\} \exp \left\{ \frac{i}{2\hbar} \int d^d x d^d y \text{tr} \left[ J_{\mu}(x) D^{\mu\nu}(x-y) J_{\nu}(y) + 2J(z(x)) D_{CB}(x-y)\right] \right\},
\]

\[
Z_{Y, (3)}(J) = \exp \left\{ V_{Y, (3)} \left( \frac{h}{i} \frac{\delta}{\delta J_{\mu}} \right) \right\} \exp \left\{ \frac{i}{2\hbar} \int d^d x d^d y \text{tr} \left[ J_{\mu}(x) D^{\mu\nu}(x-y) J_{\nu}(y) + J_{\mu}(x) D^{\mu\nu}(x-y) J_{\nu}(y) \right] \right\},
\]

\[
Z_{3\psi, (4)}(J) = \exp \left\{ V_{3\psi, (4)} \left( \frac{h}{i} \frac{\delta}{\delta J_{\mu}} \right) \right\} \exp \left\{ \frac{i}{2\hbar} \int d^d x d^d y \text{tr} \left[ J_{\mu}(x) \times D^{\mu\nu}(x-y) J_{\nu}(y) + \frac{1}{3} J_{(r_1)\nu}^{(B)} D^{(r_1)\nu}_{CB}(x-y) J_{(r_1)\nu}(y) + \frac{1}{4} J_{(r_1)\nu}^{(B)} D^{(r_1)\nu}_{BB}(x-y) J_{(r_1)\nu}(y) \right] \right\},
\]

for $Sp(2)$-duplet of sources $J_a = (J_1, J_2) \equiv (J, \overline{J})$ to ghost-antighost fields $C^a$, for $G(3)$-triplets of Grassmann-odd $J_{(p)}^{(B)}$, Grassmann-even $J_{(p)}^{(B)}$ and Grassmann-odd singlets $J_{(r_1)}, J_{(r_1)}^{(B)}$ of sources for the respective fields $B_{(p)}^{(B)}, B_{(r_1)}^{(B)}, \overline{B}$ mentioned in the round brackets in the indices and for $G(4)$-quartets of Grassmann-odd $J_{(r_1)}, J_{(r_1)}^{(B)}$ and sextet of Grassmann-even $J_{(r_1)}^{(B)}$ sources for the fields $C^{r_1}, B^{(r_1)}, B^{(r_1)\nu}$. The causal Green functions for the vector field $A_{\mu}$: $D^{\mu\nu}(x)$ [35] and for the respective fictitious pair of fields $D(x)$, $D^{\mu\nu}(x)$ for the fictitious Grassmann-odd fields in $N = 2$; for $D_{CB}(x)$, $D_{CB}^{(r_1)}(x)$ for Grassmann-odd, $D_{BB}(x)$ for Grassmann-even fields in $N = 3$; $D_{CB}^{(r_1)}(x)$, $D_{BB}^{(r_1)}(x)$ respectively for Grassmann-odd and Grassmann-even fields in $N = 4$ cases are determined in terms of the Feynman propagators in momentum representation:

\[
D^{\mu\nu}(x) = \frac{1}{(2\pi)^d} \int d^d p e^{-ipx} D^{\mu\nu}(p), \text{ for } D^{\mu\nu}(p) = -\left( \eta_{\mu\nu} - \frac{p_{\mu}p_{\nu}(1 - \xi)}{p^2 + i0} \right) \frac{1_{su(N)}}{p^2 + i0},
\]

\[
D(x) = \frac{1}{(2\pi)^d} \int d^d p e^{-ipx} D(p), \text{ for } D(p) = \frac{1_{su(N)}}{p^2 + i0}, \quad 1_{su(N)} \equiv \|\delta^{mn}\|,
\]

\[
(D^{ab}; D_{CB}^{(r_1)}, D_{CB}^{(r_1)}, D_{CB}^{(r_1)}, D_{BB}^{(r_1)})(x) = \left( \varepsilon^{ab}; 1, \varepsilon^{[r_1]}, \varepsilon_{[r_1]}, \varepsilon^{[r_{1}]} \right) D(x).
\]
And the respective vertexes look as

\[ V(A_\mu; C, \overline{C}) = \frac{1}{4} \int d^d x \, tr \left\{ 2 \partial^{[\mu} A^{\nu]} [A_\mu, A_\nu] + [A_\mu, A_\nu]^2 + 4C \partial^\mu [A_\mu, C] \right\}, \tag{7.8} \]

\[ V_3(A_\mu; C^n) = V(A_\mu; C, \overline{C}) \big|_{C, \overline{C} \to C^n} - \frac{\xi}{24} \int d^d x \, tr \left[ C^n, C^e \right] [C^b, C^d] \varepsilon_{abecd}, \tag{7.9} \]

\[ V_3(A_\mu; C, \overline{C}) = V(A_\mu; C, \overline{C}) \big|_{C, \overline{C} \to (\overline{B}, \overline{C})} + \frac{1}{2} \int d^d x \, tr \left\{ B^{[\mu} \partial^\nu [A_\mu, B^{\nu\rho\delta}] + \hat{B}^{[\rho\delta]} \partial^\mu [A_\mu, C^{\rho\delta}] \right\} \varepsilon_{[\rho\delta]} + S_{\text{add}(3)}, \tag{7.10} \]

\[ V_4(A_\mu; C, \overline{C}, B^{[\rho\delta]}; B^{[\mu\nu]} = V(A_\mu; 0, 0) + \int d^d x \, tr \left\{ \frac{1}{8} B^{[\mu\nu]} \partial^\rho [A_\mu, B^{\rho\delta\epsilon}] \right\} \varepsilon_{[\rho\delta\epsilon]} + S_{\text{add}(4)}, \tag{7.11} \]

where each \( su(\hat{N}) \)-commutator implicitly contains interaction coupling \( g \) as multiplier, all the integrations above satisfy to the Feynman boundary conditions and the respective expressions \( (2.81), (4.10) \), for \( S_{\text{add}(3)}, S_{\text{add}(4)} \) were used.

The expansion of the functionals \( (7.1) \), \( (7.4) \) generates the respective diagrammatic techniques, known for \( N = 1 \) BRST symmetric formulation \( (7.1) \), e.g. from \[48\]. The basic elements for each \( N = m, m = 1, 2, 3, 4 \) we list in the momentum representation, first, for \( N = 1 \):

\[ D^{\mu\nu}(p) \equiv \]

\[ D(p) \equiv \]

**Figure 1:** Propagators for the vector field \( A_\mu \) and for the ghost fields \( \overline{C}, C \).

Second, for \( N = 2 \) case for only different propagator for \( Sp(2) \)-duplet of ghost-antighost field \( C^a \) and quartic in \( C^a, C^b, C^c, C^d \) \( (a, b, c, d = 1, 2) \) interaction vertex \( V(y_2)C^aC^bC^cC^d(p) \) obtained from

\[ trV(y_2)C^aC^bC^cC^d(x) \left[ C^aC^c \right] [C^b, C^d] = \frac{1}{(2\pi)^d} \int d^d p \, e^{-ipx} V(y_2)C^aC^bC^cC^d(p) \left\{ C^d \right\} (p) C^c, C^d(p) C^b, C^c(p) \]

\[ \tag{7.12} \]

\[ D^{\alpha\beta}(p) = \langle C^\alpha C^\beta \rangle_0 \equiv \]

\[ V^{\alpha\beta\gamma\delta}_{\langle y_2 \rangle} C^\alpha C^\beta C^\gamma C^\delta = \frac{\xi}{24} f^{\alpha\gamma\delta} f^{\mu\alpha\beta} \varepsilon_{abecd} \equiv \]

**Figure 2:** Propagators for the fields \( C^a \) and self-interaction vertex quartic in the ghost fields \( C^a \).

Third, for \( N = 3 \) propagators for the fictitious fields and with account for antisymmetry \( \hat{B}, B \)^{\gamma\rho} = -(\hat{B}, B)^{\gamma\rho}
Figure 3: Propagators for the fictitious Grassman-odd $G(3)$ singlets, $\mathcal{C}, \mathcal{B}$, 3 pairs of triplets $C^p, \mathcal{B}^{qr}$ and 3 pairs of Grassmann-even triplets $B^p, B^{qr}$.

Fourth, for $N = 4$ propagators of the fictitious fields with account for antisymmetry of $(B^{r_1 r_2 r_3}, B^{r_1 r_2}) = -(B^{r_2 r_3 r_1}, B^{r_2 r_1})$

\[
D_{\mathcal{C}B}(p) = \langle \mathcal{C} \mathcal{B} \rangle_0 \equiv \nabla \nabla \nabla \nabla \nabla \nabla \nabla
\]

\[
D_{C \mathcal{B}}^{qr}(p) = \langle C^p \mathcal{B}^{qr} \rangle_0 \equiv \nabla \nabla \nabla \nabla \nabla
\]

\[
D_{B \mathcal{B}}^{qr}(p) = \langle B^p B^{qr} \rangle_0 \equiv \nabla \nabla \nabla \nabla
\]

Figure 4: Propagators for the fictitious four pairs of Grassmann-odd $G(4)$-quartets $C^r, B^{[r]s} \equiv 3$ pairs of Grassmann-even sextet $B^{r_1 r_2}, B^{r_3 r_4}$.

And, for some $N = 1, 3, 4$ vertexes of the gauge vector fields $A_\mu$ with respective fictitious fields (Grassmann-odd for $N = 1, 4$ and Grassmann-even for $N = 3$) BRST symmetric formulations from the quadratic in the fictitious fields terms with Faddeev-Popov operator $M(A)$ in the momentum representation found as in (7.12) in the Figures 5, 6.

Note, starting from the $N = 2$ case we have introduced the additional notation of the respective (being valid for free (quadratic) theory) averaging fields $\langle \_ \_ \_ \rangle_0$ written under the respective propagator’s line to distinguish different fictitious fields corresponding, in fact, to the same function, $D(p)$. From the $N = 3$ case the propagator’s line for the Grassmann-even (Bose) fictitious particle is given by ”dash with dot” as compared to the standard ”dash” notations for the Grassmann-odd (Fermi) fictitious particle. There are 3 independent propagators for Grassmann-odd fields among $(C^p \mathcal{B}^{qr})_0$ and 3 ones for Grassmann-even from $\langle B^p B^{qr} \rangle_0$ in the Figure 5 which are $(C^1 \mathcal{B}^{23})_0, (C^2 \mathcal{B}^{31})_0, (C^3 \mathcal{B}^{12})_0$ and $\langle B^1 B^{23} \rangle_0, \langle B^2 B^{31} \rangle_0, \langle B^3 B^{12} \rangle_0$, i.e. for $\{p, q, r\} = \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. For $N = 4$ BRST symmetric case the Figure 6 contains 4 independent propagators for Grassmann-odd fields $\langle C^p B^{[r]s} \rangle_0$ and 3 ones for Grassmann-even from $\langle B^{r_1 r_2} B^{r_3 r_4} \rangle_0$: $(C^1 B^{234})_0, (C^2 B^{314})_0, (C^3 B^{124})_0, (C^4 B^{132})_0$ and $\langle B^{12} B^{34} \rangle_0, \langle B^{13} B^{42} \rangle_0, \langle B^{14} B^{23} \rangle_0$.

There exist more additional vertexes from (7.9), (7.10), (7.11) which can be analogously represented as in the Figures 5-6.

8 Conclusion

In the present work a generalization of the Faddeev–Popov proposal presenting the Lagrangian path integral for the Yang–Mills theory in Landau and Feynman gauges [3], [9] is proposed for non-local form by inserting the special unity, $\det^k M(A) \det^{-k} M(A)$, depending on non-negative integer $k$ in (2.1).
\[ V_{AC\xi}(p) = f^{mn} p_\mu \equiv \]

\[ V_{(\xi,AB\rho B\rho \rho)}(p) = \frac{1}{2} f^{mn} p_\mu \varepsilon_{[\rho]} = \]

\[ V_{4[\xi,AC\rho B\rho \rho]3}(p) = -\frac{1}{3!} f^{mn} p_\mu \varepsilon_{[\rho]} = \]

Figure 5: Interaction vertexes of \( A_\mu \): with \( C, C \) fields in \( N = 1 \), with any pair from \( G(3) \)-triplets of even (Bose) fictitious fields \( B^p, B^p \) in \( N = 3 \) BRST formulations.

\[ (2.32) \text{ and for local form in } (2.33), \text{ with numbers of fictitious Grassmann-odd and Grassmann-even fields (with the same number of physical degrees of freedom as compared to the case of space-time extended SUSY gauge theories) in the spectrum of the total configuration spaces larger than those for } N = 1, 2 \text{ BRST symmetry cases. It is shown in the Statement 1, that to realize the } N = m \text{ BRST symmetry transformations with more than two Grassmann-odd parameters, } \lambda_p, p = 1, 2, ..., m \text{ (in substituting instead of the infinitesimal gauge parameters } \xi = C^p \lambda_p \text{ the } m \text{-plet of Grassmann-odd ghost fields) when formulating the corresponding quantum actions, } S_{\Phi_k(N(k))}^{(N(k))}(2.36) \text{ with the gauge-fixing terms (respecting } N(k) = m \text{ BRST invariance) to be added to the classical Yang–Mills action the spectrum of } k = k(N) \text{ should obey to the relation (2.38), whereas to perform the gauge-fixing procedure without using an odd non-degenerate transformation changing the Grassmann parities for some fictitious fields its spectrum } k = k_u(N) \text{ is described by (2.39).}

An irreducible representation space, \( M_{min}^{(3)} \), for the 3-parametric abelian superalgebra \( G(3) \) of anti-commuting generators \( \xi^p \) with triplet of Grassmann-odd constant parameters, \( \lambda_p \), with its action on the local coordinates, fields \( \Phi_{(3)} \), has been explicitly constructed by Eqs. (2.45). To formulate a local quantum action with appropriate gauge-fixing procedure, we have followed two ways. First, that proves the condition (2.38) of the Statement 1, is based on the original using of Grassmann-odd non-degenerate operator II which changes the Grassmann parities and acts on \( G(3) \)-irreducible space of initial Yang–Mills fields \( A^\mu, \) triplets \( C^p, B^{pq}, \) odd singlet \( \hat{B} \), in such a way, that the respective change of variables in the subspace of part of fictitious fields, \( \phi^M \) (2.8) has permitted to make possible to pass to a new basis of fictitious fields in which the local quantum action (2.22) and path integral (2.13) in Landau gauge with a new form of \( N = 3 \) BRST symmetry transformations, (2.23), (2.28), have been constructed. Second, the non-minimal sector of the fields \( \Phi_{(3)} \) containing antighost field (as a connection) \( \overline{C} \) to incorporate the usual gauge condition, \( \chi(A, B) \), into a gauge fermionic functional \( \Psi_{(3)} \) (2.75) has been introduced, on which a new \( N = 3 \) representation of \( G(3) \)-superalgebra is explicitly realized (2.69). The sector contains two \( G(3) \)-singlets, \( \overline{C}, B \) with usual Nakanishi-Lautrup field and two \( G(3) \)-triplets, \( B^p, \hat{B}^{pq} \) and together
with the fields, \( \Phi_{(3)} \) composes the fields \( \Phi_{(3)} \) of reducible \( G(3) \)-superalgebra representation parameterizing the total configuration space \( \mathcal{M}_{tot}^{(3)} \), on which the quantum action \( S_{\Phi_{(3)}(\xi)}(\Phi_{(3)}) \) \((2.78) - (2.81)\) and path integral \( Z_{1|\Psi(0)}(\xi) \) \((2.73)\) in \( R_{\xi} \)-like gauges, determined by the gauge fermionic functional \( \Psi_{(3)}(\xi)(\Phi_{(3)}) \) \((2.75)\), have been explicitly constructed. The set of the transformations in \( \mathcal{M}_{tot}^{(3)} \), \( \delta_{\lambda} \Phi_{(3)}(\xi) = \Phi_{(3)}(\xi) \overline{\delta}^r \lambda_p \), determined by \((2.45), (2.60)\), leaving both the quantum action and the integrand of the respective path integral by invariant, we call \( N = 3 \) BRST transformations. The quantum (non-renormalized) action \( S_{\Psi_{(3)}(\xi)} \) contains the terms quadratic in fictitious fields leading to the same one-loop contribution for the effective action as one for the quantum actions constructed according to \( N = 1 \) and \( N = 2 \) BRST symmetry principles in smaller configuration space, whereas for more than quadratic in powers of ghost fields terms in \( S_{\Psi_{(3)}(\xi)} \), described by the \( S_{add(3)}(2.81) \) which generates the ghost vertexes to be different than ones derived from the former actions.

We have established with the help of \( G(1) \)-superalgebra with nilpotent generator \( \overline{s} \) and parameter \( \lambda \) being additional to \( G(3) \)-superalgebra, but acting on the fields of \( \mathcal{M}_{tot}^{(3)} \) by the rule \((3.7)\), the fact that the \( G(1) \)-invariant path integral \( Z_{1|\Psi(0)} \) \((3.7)\) with the quantum action \( S_{\Psi(1)} \) and, at least for special quadratic gauge fermionic functional \( \Psi_{(1)} \) \((3.8)\) given on \( \mathcal{M}_{tot}^{(3)} \) is equivalent to the \( N = 1 \) antiBRST invariant path integral \((3.12)\) with the quantum action \( S_{\Psi(1)}(\xi) \) \((3.13)\) constructed by the standard Faddeev-Popov method with use of \( N = 1 \) antiBRST symmetry transformations acting in the standard configuration space, \( \mathcal{M}_{tot} \) of fields \( \lambda_p, C, C^\tau, B \). We call the transformations \((3.9)\) with parameter \( \lambda \) which led to the \( G(1) \)-invariance of \( S_{\Psi(1)} \) and integrand of \( Z_{1|\Psi(0)} \) by \( N = 1 \) antiBRST symmetry transformations in \( \mathcal{M}_{tot}^{(3)} \).

It was shown the Grassmann-odd parameters: \( G(3) \)-triplet \( \lambda_p \) and \( G(1) \)-singlet \( \lambda \), of \( G(3) \) and \( G(1) \) superalgebras acting on the space \( \mathcal{M}_{tot}^{(3)} \), are uniquely combined within quartet of parameters \( \lambda_p, \lambda \), as well as the quartet of the generators \( \overline{s} \tau^r = (\overline{s}_p, \overline{s}_\lambda) \) to form a \( G(4) \)-superalgebra whose irreducible representation contains the same fields as reducible one \( \Phi_{(3)} \) for \( G(3) \)-superalgebra in \( \mathcal{M}_{tot}^{(3)} \) but organized in \( G(4) \)-antisymmetric tensors, \( \Phi_{(4)} = (A^\mu, C^\tau, B^r, r^2, r^3, B) \) according to the rule \((3.15)\), which parameterize \( N = 4 \) total configuration space \( \mathcal{M}_{tot}^{(4)} \). The explicit action of the generators \( \overline{s} \tau^r \) on each component from \( \Phi_{(4)} \) was constructed by Eqs. \((3.20)\) with preservation of the \( G(4) \)-superalgebra: \( \{ \overline{s} \tau^r, \overline{s} \tau^s \} = 0 \). The respective \( N = 4 \) SUSY transformations, \( \delta_{\lambda} \Phi_{(4)} = \Phi_{(4)} \overline{\delta}^r \lambda_p \), have appeared, according to their definition, by \( N = 4 \) BRST transformations for the quantum action \( S_{\lambda(4)} \), and local path integral \( Z_{qY(0)}(\xi) \) \((4.1)\) with help of addition to the classical action of the \( N = 4 \) BRST exact term generated by the quartic powers in \( \overline{s} \tau^r \) applied to the gauge Bosonic functional, \( Y(4) \Phi_{(4)} \) \((4.8)\). For \( R_{\xi} \)-like family of gauges determined by the functional \( Y_0(4)\Phi_{(4)} \) \((4.16)\) the quantum action \( S_{Y_0(4)\Phi_{(4)}} \) \((4.17)\) was exactly calculated for the Landau gauge \( (\xi = 0) \), whereas for the Feynman gauge \( (\xi = 1) \) the additional summand \( S_{add(4)} \) \((4.10)\) to the standard gauge-fixed and quadratic \( \{4 \) Grassmann-odd \( C^r (C^1, B^{234}), (C^2, B^{134}), (C^3, B^{124}), (C^4, B^{123}) \) and 3 Grassmann-even \( B^{12}, B^{13}, B^{24}, B^{14}, B^{25} \) pairs of ghost fields \} parts \( S_{gf(4)}, S_{gbh(4)} \) of the quantum action contains the 8-th powers in odd \( C^r \) and 4-th powers in even \( B^{r_1 r_2} \) fields. For any \( \xi \) classical action and the functionals \( S_{gf(4)}, S_{gbh(4)} \) lead to the same contribution into one-loop effective action as those for the known and above quantum actions constructed according to the \( N = 1, N = 2 \) and both \( N = 3 \) BRST symmetry recipes. It was explicitly shown on the level of the non-renormalized path integrals the equivalence among \( N = 3 \) BRST invariant path integral evaluated in the \( R_{\xi} \)-like gauges and usual \( N = 1 \) BRST invariant path integral in the \( R_{\xi} \)-gauges \((2.34)\). For \( N = 4 \) BRST invariant path integral its equivalence with \( N = 1 \) BRST invariant path integral was found in case of Landau gauge in \((4.13)\).

For both \( N = 3 \) and \( N = 4 \) BRST invariant formulations of the quantum actions the generating functionals of Greens functions, including effective actions were determined and Ward identities \((2.91), (4.14)\) for them, which follow from the respective algebraic \( N = m, m = 3, 4 \) BRST invariance, were derived as well as the independence on the choice of the gauge condition for the respective path integral under the corresponding small variation of the gauge: \( \Psi_{(3)} \rightarrow \Psi_{(3)} + \delta \Psi_{(3)} \) and \( Y(4) \rightarrow Y(4) + \delta Y(4) \) were established by means of infinitesimal FD \( N = m \) BRST transformations.
The finite $N = 3$ and $N = 4$ BRST transformations were restored to form respectively the Abelian supergroups $G(m) = \exp[\sum p \lambda_p]$, $p = 1, 2, ..., m$ acting on the respective configuration space $\mathcal{M}_{\text{tot}}^{(m)}$ by means of two ways: first, by continuation of the invariance of any regular functional under algebraic $N = m$, $m = 3, 4$ BRST transformations to full invariance under finite transformations, second by means of resolution of the Lie equations. The sets $G(m)$ (5.3), (5.22) of finite FD $N = m$ BRST transformations were introduced and the respective Jacobians of the change of variables in $\mathcal{M}_{\text{tot}}^{(m)}$ generated by these transformations were calculated in (5.3), (5.22). For functionally-dependent Grassmann-odd parameters, $\tilde{\lambda}_p = (-1)^m \frac{1}{(m-1)!} \Lambda_{\text{tot}}(\Phi_{(m)}(\epsilon_{\text{e}})) \rightarrow \sum p \lambda_p$ with a some potential functional $\Lambda_{\text{tot}}$ Grassmann-odd(even) for $m = 3$ $(m = 4)$ (5.11), (5.21) the Jacobians above are transformed to the respective $N = m$ BRST exact terms (5.21), (5.26). The latter Jacobians were applied, first, to the establishing of the independence upon the choice of the gauge condition for finite variation of the respective path integral, $Z_{\Phi(3)}(0) = Z_{\Phi(3)}(\tilde{J}(3))$, $Z_{\Phi(4)}(J(4))$ depending on the functionally-dependent FD parameters $\tilde{\lambda}_p$, $p = 1, 2, ..., m$, and therefore on the finite variation of the gauge $\Phi_{(3)}$, $\Phi_{(4)}$ respectively. Third, they have permitted to establish gauge independence of $Z_{\Phi(3)}(\tilde{J}(3))$, $Z_{\Phi(4)}(J(4))$ upon the respective choice of the gauge condition $\Phi_{(3)} \rightarrow \Phi_{(3)} + \tilde{\Psi}_{(3)} + \tilde{\Psi}_{(3)}$ and $\Phi_{(4)} \rightarrow \Phi_{(4)} + \tilde{\Psi}_{(4)} + \tilde{\Psi}_{(4)}$ on the corresponding mass-shell: $\tilde{J}(3) = 0$, $J(4) = 0$.

The new Ward identities (5.3), (5.22) for the extended (by means of sources $K_p, K_{pq}, \mathcal{K}$ to the $N = 3$ BRST variations $\Phi_{(3)}$, $\Phi_{(3)}$, and $\Phi_{(3)}$, generating functional of vertex Green’s functions, $\Gamma((\Phi_{(3)}), K_p, K_{pq}, \mathcal{K})$, obtained from the part of extended $N = 3$ BRST transformations (6.12) in the space of $\Phi_{(3)}$, $K_p, K_{pq}, \mathcal{K}$ for constant $\lambda_p$, reproduced the new differential-geometric objects. i.e., $G(3)$-triplets of antibrackets: $(\bullet, \bullet)^p$ and odd-valued first-order differential operators $V^p$ (5.20).

The gauge-independent Gribov-Zwanziger model of Yang–Mills fields without residual Gribov ambiguity in the infrared region of the field $A^\mu$ configurations described by gauge-invariant, and therefore $N = m$ BRST invariant, for $m = 3, 4$, horizon functional $H(A^\mu)$ (5.20) in terms of gauge-invariant transverse fields $A^\mu_{\mu}$ (6.28) (63), firstly proposed in (37) within $N = 1$ BRST symmetry realization but with non-local BRST transformations was suggested in non-local form but with local $N = 3, N = 4$ BRST invariance by the Eqs. (6.32), (6.33). The partially local, (in view of residual presence of non-local vector field $A^\mu_{\mu}$) Gribov-Zwanziger model was proposed with non-local $N = 1$ BRST symmetry (5.3), (5.35), due to inverse gauge-invariant Faddeev-Popov matrix $(M^{-1}) (A^\mu)$ presence for auxiliary fields in (5.36).

The extension of the basics for the diagrammatic Feynman technique within perturbation theory for the $N = 3$ and $N = 4$ BRST invariant quantum actions for the Yang–Mills theory were proposed due to the presence of additional both Grassmann-odd and Grassmann-even fictitious fields.

Concluding, let us present the spectrum of irreducible representations for a $G(l)$ Abelian superalgebra with $l = 0$ (non-gauge theories), $l = 1$ (BRST symmetry algebra), $l = 2$ (BRST-antiBRST symmetry algebra), $l = 3$ (superalgebra with 3 BRST symmetries), and so on according to the chain (2.41) – (2.43), by a numeric pyramid partially similar to the Pascal triangle (51), which contains in its left-hand side the symbol "$d|A^\mu$" relating to number of degrees of freedom of the classical Yang–Mills fields $A^\mu$ with suppressed $su(N)$ indices: where the $l$-th row, corresponding to the field content $\Phi_{(l)}$ of an irreps space for the $G(l)$ superalgebra, is constructed from the symbols of $d|A$, $l|C^{[1]}$, $C_{1}^{1}|B^{[2]}$, ..., $B_{1}(l)(C_{1}^{1} = k!/(l!(k - l)!))$, corresponding to the degrees of freedom (modulo the dimension of $su(N)$) for $A^{\mu}_{\mu}$, $C^{[1]}_{1}$, $B^{[2]}_{1}$, ..., $B_{1}(l)$, $p_1, 1, 2, ..., l$, whose sum is equal to $(2l + d - 1)$. The symbols related by an arrow: $d|A \rightarrow l|C^{[1]}$ meaning the part of the chain generated by the $N = l$-BRST generator $\hat{s}^{l_1}$, $r_1 = 1, 2, ..., l$ by the rule: $A^{\mu}_{\mu} \hat{s}^{l_1} = D^{\mu}_{\nu} C^{\nu}$ with omitting the arrow over $\hat{s}^{l_1}$ for the readability in the Table (51). From the second row ($N = 2$), the rule of filling the triangle starts to work, whereas for $N = 0$ there is no
transformation, $\Pi$, intended to present the path integral with the Grassmann-even Nakanishi–Lautrup canonical formalism. Then, it is intended to examine the case of irreducible dynamical systems subject to a generalization, in a manifest way, of the Faddeev–Popov rules in Yang–Mills theories to the case of reducible $N = 1$ BRST algebra without an additional trivial BRST doublet, $\bar{C}, B$ necessary to construct quantum action and local path integral which as the fields from the non-minimal sector, answering for the reducible representation of $G(1)$-superalgebra, selected into another Table. Notice, that the second left-hand side only contains the numbers $1, 2, 3, ..., 2K$ of Grassmann-odd fictitious fields, $C, C^{p_2}, C^{p_3}, ..., C^{p_{2K}}$; the third left-hand side (starting from $N = 2$) only contains the numbers $1, 3, 6, ..., C^{2K}_2$ of Grassmann-even fictitious fields $B, B^{p_3q_1}, B^{p_4q_1}, ..., B^{p_{2K}q_{2K}}$, etc. The final right-hand side of the triangle is composed of the Nakanishi–Lautrup $G(l)$-singlet fields $B \equiv \hat{B}_2, \hat{B}_3 \equiv \hat{B}, \hat{B}_4, ..., \hat{B}_l$, with alternating Grassmann parity, $\epsilon(\hat{B}_l) = l$, respectively for $l = 2, 3, ..., 2K$.

In turn, for the reducible representation space of $G(2K - 1)$-superalgebra, for integer $K$ determining the non-minimal sector of fields to be necessary to provide gauge-fixing procedure without odd supermatrix, the spectrum of additional fields is described by the Table corresponding to the exact sequence of the $G(N)$ superalgebra, the generalized Faddeev–Popov rules must be described by odd non-degenerate transformation, $\bar{C}, B$ intended to present the path integral with the Grassmann-even Nakanishi–Lautrup field $B_{2(K-1)} \equiv \Pi \hat{B}_{2K-1}$ exponentiating the standard gauge condition, added to the classical action using an $N = (2K - 1)$ BRST-exact form.

It follows from the both Tables that the generalization of the Faddeev-Popov quantizations for the case of $N = 2K - 1$ BRST invariance without using of an odd non-degenerate transformation, when formulating the local quantum action and path integral leads to the dimension of the total configuration space $\mathcal{M}_{tot}^{2K - 1}$ to coinciding with the one for $\mathcal{M}_{tot}^{2K}$ realizing $N = 2K$ BRST symmetry for the same purpose.

There are various directions to extend the results of the present study. Let us mention some of them. First, to develop the case of $N = 3, 4$ BRST symmetries transformations in a Yang–Mills theory as a dynamical system with first-class constraints in the generalized canonical formalism. Second: to develop the case of $N = 3, 4$ BRST symmetry transformations for irreducible general gauge theories in Lagrangian formalism, including theories with a closed algebra of rank 1. Third: to generalize, in a manifest way, the Faddeev–Popov rules in Yang–Mills theories to the case of $N = 2K - 1$ and $N = 2K$, $K > 2$ BRST symmetry transformations in Lagrangian formalism and in generalized canonical formalism. Then, it is intended to examine the case of irreducible dynamical systems subject to a generalization, in a manifest way, of the Faddeev–Popov rules in Yang–Mills theories to the case of reducible $N = 1$ BRST algebra without an additional trivial BRST doublet, $\bar{C}, B$ necessary to construct quantum action and local path integral which as the fields from the non-minimal sector, answering for the reducible representation of $G(1)$-superalgebra, selected into another Table.

| $N = 0$: | $d|A$ | $\rightarrow$ | $d|A$ | $\rightarrow$ | $1|C$ | $\rightarrow$ |
| $N = 1$: | $d|A$ | $\rightarrow$ | $2|C^0$ | $\rightarrow$ | $1|B$ |
| $N = 2$: | $d|A$ | $\rightarrow$ | $3|B|^{p_1}$ | $\rightarrow$ | $1|B$ |
| $N = 3$: | $d|A$ | $\rightarrow$ | $6|B|^{r_2}$ | $\rightarrow$ | $1|B$ |
| $N = 4$: | $d|A$ | $\rightarrow$ | $10|B|^{r_3}$ | $\rightarrow$ | $1|B$ |
| $N = 2K$: | $d|A$ | $\rightarrow$ | $2K|C|^{r_2}$ | $\rightarrow$ | $1|B$ |

Table 1: Numbers of fictitious fields in addition to $A_\mu$ for each $N = 0, 1, 2, ..., 2K$.

| $N = 1$: | $1|C$ | $\rightarrow$ | $3|B|^{p_1}$ | $\rightarrow$ | $1|B$ |
| $N = 3$: | $1|C$ | $\rightarrow$ | $5|B|^{r_2}_{(N)}$ | $\rightarrow$ | $1|B$ |
| $N = 5$: | $1|C$ | $\rightarrow$ | $10|B|^{r_3}_{(N)}$ | $\rightarrow$ | $1|B$ |
| $N = 2K - 1$: | $1|C$ | $\rightarrow$ | $N|B|^{p}_{(N)}$ |

Table 2: Numbers of fictitious fields from the non-minimal sectors for each odd $N = 1, 3, 5, ..., 2K - 1$. 


to \( N = 2K - 1, \ N = 2K, \ K > 2 \) BRST symmetry transformations and to compare the results with superfield formulations with \( N \) BRST charges in [52]. Next, it is planned to consider an irreducible general gauge theory subject to \( N = 2K - 1, \ N = 2K, \ K > 2 \) BRST symmetry transformations in the Lagrangian formalism. The problem of study of the renormalizability for the suggested \( N = 3, 4 \) BRST invariant formulations of the quantum actions so as to have completely renormalized respective general gauge theory subject to \( \xi = 0 \), when \( \xi = 0 \), hence, the \( N = m \) BRST symmetry, regularization by higher-derivatives [55], recently developed for \( N = 2 \) superfield formulation of Abelian and super Yang–Mills theories [56] on a basis of \( \mathcal{N} = 2 \) harmonic superspace approach. We intend to study these problems in forthcoming works.

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Appendix

A On \( N = 3 \) BRST invariant gauge-fixing in \( N = 3 \) irreducible superspace

Here, we will prove that it is impossible to perform \( N = 3 \) BRST invariant gauge-fixing procedure within the set of fields \( \Phi_{(3)}^{A_3} \) parameterizing the superspace of irreducible representation of \( \mathcal{G}(3) \)-superalgebra without using of non-degenerate odd-valued change of variables among the components of \( \Phi_{(3)} \) to explicitly construct such a gauge-fixing.

Indeed, it is easy to see that in the basis of additional to \( \mathcal{A}^\mu \) fields in \( \Phi_{(3)}^{A_3} = (A^\mu, C^p, B^{pq}, \hat{B}) \) composing the irreducible representation space of \( \mathcal{G}(3) \)-superalgebra, on which due to Lemma 1 the \( N = 3 \) SUSY transformations is realized (2.43), there are enough coordinates to reach a non-local Faddeev–Popov path integral (2.31) with preservation of the symmetry above. The terms in the functional \( S_L^{(3)}(\Phi_{(3)}) \) for \( (2.36) \) for \( N(k) = 3, k = 1 \), with the fermionic gauge-fixing functional, \( \frac{1}{2} F_{(3)} \{ (\Phi_{(3)}) \} [ \delta p_1 \delta p_2 \delta p_3 \delta p_4 \gamma_{pq} \] are calculated following to the rules (2.82), (2.86) similar to the \( N = 2 \) BRST symmetry case (2.17) for \( \xi = 0 \), when \( F_{(3)} \{ (\Phi_{(3)}) \} = F_{(3)} \{ (A) \} \):

\[
F_{(3)} \{ (A) \}^{(2)} p = \int d^4x Tr \frac{\delta F_{(3)}^{(3)}}{\delta A^\mu} D_\mu C^p = - \int d^4x Tr D_\mu \left( \frac{\delta F_{(3)}^{(3)}}{\delta A^\mu} \right) C^p = - \int d^4x Tr \chi_F(A) C^p, \quad (A.1)
\]

\[
F_{(3)} \{ (A) \}^{(2)} p S^{(2)} q_{pqr} \varepsilon_{pqr} = \int d^4x Tr \left( \int d^4y C^p(x) M^F(A,x;y) C^q(y) - \chi_F(A) B^{pq} \right) \varepsilon_{pqr}, \quad (A.2)
\]

for \( M^F(A,x;y) = \frac{\delta \chi_F(A,x)}{\delta A^\mu(y)} D_\mu(y), \quad (A.3)\)

\[
F_{(3)} \{ (A) \} = \int d^4x Tr \left\{ \int d^4y \{ 2 B^{pq}(x) M^F(A,x;y) C^r(y) + C^p(x) M^F(A,x;y) B^{pq} \} \varepsilon_{pqr} - \chi_F(A) \left[ \hat{B} + \frac{1}{2} \left[ B^{pq}, C^r \right] \varepsilon_{pqr} \right] \right\}. \quad (A.4)
\]

Hence,

\[
S_{F_{(3)}}(\Phi_{(3)}) = S_{F_{(3)}}(\Phi_{(3)}) = S_0 + \frac{1}{2} F_{(3)}(A)(S)\varepsilon = S_0 + S_{F_{(3)}}(\text{rf}) + S_{F_{(3)}}(\text{gh}) + S_{F_{(3)}}(\text{add}), \quad (A.5)
\]
\[
S_{F(3)\text{[gf]}} + S_{F(3)\text{[gh]}} = \int d^dx \text{tr} \left\{ \tilde{B} \chi^F(A) + \frac{1}{3!} \int d^dy \left( 2B^{pq}(x)M^F(A, x; y)C^r(y) + C^p(x)M^F(A, x; y)B^{qr}(y) \right) \varepsilon_{pqr} \right\},
\]
\[
S_{F(3)\text{[add]}} = -\frac{\varepsilon_{pqr}}{3!} \int d^dx \left\{ \frac{1}{2} \chi^F(A) [B^{pq}, C^r] + \int d^dyd^dz C^p(x) \frac{\delta M^F(A, x; z)}{\delta A_\mu(z)} D_\mu(z)C^r(z)C^q(y) \right\},
\]
where \( \chi^F(A) \) may be interpreted as a Grassmann-odd analog of gauge conditions (2.70), (2.114) used in the \( N = 1, 2 \) BRST symmetry realizations for the quantum action, and therefore \( M^F(A, x; y) \) should be considered as a Grassmann-odd analog of the Faddeev–Popov matrix (1.4).

### A.1 Non-degenerate odd-valued change of fictitious fields

To provide a satisfactory description, we must deal neither with the appearance in \( Z_{\delta}^F \) of the \( \delta \)-function \( \delta(\chi^F) \) from odd-valued functions, nor with the superdeterminant \( \text{sdet}M^F(A) \) from an odd-valued matrix \( M^F(A) \), we may pass to another basis of auxiliary fields, \( \Phi(3) \), in the representation space \( M^{(3)} = M^{(3)}_{\text{min}} \), of the \( N = 3 \) superalgebra \( \mathcal{G}(3) \) with the same number of Grassmann-odd and Grassmann-even fields. To this end, we introduce a non-degenerate transformation in \( M^{(3)}: \Phi(3) \rightarrow \tilde{\Phi}(3) = \Xi\Phi(3) \), with unaffected Yang–Mills fields \( A_\mu \), ghost fields \( C^1, C^3 \), bosonic fields \( B^{13} \), and to be transformed fictitious fields \( \hat{\phi}^M = (B^{23}, B^{12}, C^2, \hat{B}) \), by introducing a Grassmann-odd non-degenerate matrix \( N = ||N_{MN}|| \) (analogous to the odd supermatrix \( \omega = ||\omega^{AB}|| = ||(\Gamma^A, \Gamma^B)||, \epsilon(\omega) = 1 \), resulting from the odd Poisson bracket, \( \{, \} \), calculations with respect to the field-antifield variables \( \Gamma^A \) in the field-antifield formalism [11, 15], composed from the unit matrices \( 1_{(N^2-1)} \) with suppressed \( su(N) \)-indices, as follows:

\[
\phi^M \rightarrow \hat{\phi}^M = \mathcal{N}^{MN} \hat{\phi}^N : \begin{pmatrix} B_2 \\ B \\ C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \Pi & 0 \\ 0 & 0 & 0 & \Pi \\ \Pi & 0 & 0 & 0 \\ 0 & \Pi & 0 & 0 \end{pmatrix} \begin{pmatrix} B^{23} \\ B^{12} \\ C^2 \\ \hat{B} \end{pmatrix},
\]

with the odd non-degenerate supermatrix \( \Pi \), which turns the only fields of definite parity into new fields with the same properties but with opposite parity: \( \Pi(B^{23}, B^{12}, C^2, \hat{B}) = (C^1, C^3, B_2, B) \), so that by definition, the property to be idempotent for \( \Pi \) holds: \( \Pi^2 = 1 \). Notice that the separation of the (un)transformed fields in \( \Phi(3) \) is not unique for unaffected \( A^p \). Note, that in the usual sense [11, 15], \( \text{sdet}N = 0 \).

The supermatrix \( \mathcal{N} \) plays the role of an inverse for itself, which make it possible to express the initial fictitious fields \( \phi^N \) from (A.8) as functions of new fictitious fields \( \hat{\phi}^N \):

\[
\phi^M = \mathcal{N}^{MN} \hat{\phi}^N, \quad \epsilon(\mathcal{N}) = 1, \quad \text{because of } N^2 = 1_{(N^2-1)}.
\]

10If one attempts to exponentiate the non-local path integral (2.21) over \( M^{(3)} = \{ \phi^{(\mathcal{M})} \} \) in the basis of, first, the auxiliary fields \( \{ C^p, B^{pq}, \hat{B} \} \) by means of the one Lie-group \( G \)-valued field \( B^{12} \) from the triplet of Grassmann-even fields \( B^{pq} \), to exponentiate \( \delta(\chi) \), second, the pair, \( C^1, C^2 \) from the triplet of Grassmann-odd fields \( C^p \), to exponentiate \( \text{det}M \), third, the pair \( B^{13}, B^{23} \) from Grassmann-even fields \( B^{pq} \), and the remaining pair of Grassmann-odd fields, \( C^1, \hat{B} \), to exponentiate, respectively, \( \text{det}^{-1}M \) and \( \text{det}M \), we get:

\[
Z_{\delta}^F = \int d\delta(\chi) \text{det}^2(M(A)) \text{det}^{-1}M(A) \exp \left\{ \frac{1}{h} S_0(A) \right\} = \int d\Phi(3) \exp \left\{ \frac{1}{h} \tilde{S}_L(\Phi(3)) \right\},
\]

for \( \tilde{S}_L(\Phi(3)) = S_0(A) + \int d^dx \text{tr} \left( \chi(A)B^{12} + C^1 M(A)C^2 + B^{23} M(A)B^{13} + C^3 M(A)\hat{B} \right) \).

However, to provide \( N = 3 \) BRST invariance of the local action \( \tilde{S}_L(\Phi(3)) \) for Yang–Mills theory one must impose additional requirement: \( \delta_A B^{12} = 0 \), being rather restrictive one.
A.2 $N = 3$ BRST-invariance and path integral in new fictitious fields

The following step is based on a definition of the gauge fermion $F_{(3)0}(A)$ with help of the odd matrix $\Pi$ in quadratic form consistent for the Landau gauge:

$$F_{(3)0}^L(A) = -\frac{1}{2} \int d^4x \text{tr} \, A^\mu \Pi A_\mu = -\frac{1}{2} \int d^4x \, A^{\mu m} \Pi^{mn} A_\mu^n, \quad \text{for } \epsilon(F_{(3)0}) = 1. \quad (A.10)$$

Because the map $\Pi$ acts linearly, turning the points (with coordinates) in a fiber of the respective bundle into the same points (with coordinates) in a fiber of another bundle, but with opposite parity, then the respective infinitesimal gauge for $A^\mu$ and $N(3)$ SUSY transformations for $\Phi_{(3)}$ make by natural the properties:

$$\Pi \delta A^\mu = \delta(\Pi A^\mu) \Rightarrow \Pi \delta \lambda A^\mu = \delta \lambda (\Pi A^\mu) = \Pi D^\mu(A) C^p \lambda_p, \quad (A.11)$$

$$\Pi \partial_\mu = \partial_\mu \Pi, \quad \Pi D^\mu(A) C^p = D^\mu(A) \Pi C^p \Leftrightarrow \Pi[A, C^p] = [A, \Pi C^p], \quad (A.12)$$

where the last relation maybe considered as the continuation of the commutativity property of $\Pi$ with partial derivative $\partial_\mu$.

Now, we can write the path integral related to (2.31) in a local form, (2.33) for $(3)$ superalgebra, as follows:

$$Z_{F_{(3)0}^L} = \int d\Phi_{(3)} \exp \left\{ \frac{i}{\hbar} S_{F_{(3)0}^L}(\Phi_{(3)}) \right\}, \quad \text{with } S_{F_{(3)0}^L} = S_0(A) + \frac{1}{3!} F_{(3)0}^L(A) \left( \frac{\epsilon}{\hbar} \right)^3 \left( \Phi_{(3)} \right)_M \rightarrow \Phi_{(3)}^M, \quad (A.13)$$

$$S_{F_{(3)0}^L} + S_{F_{(3)0}^L} \mid_{\text{gh}} = \frac{1}{2} \int d^4x \text{tr} \left\{ \left( (\Pi \chi(A)) + \chi(A)\Pi \right) - B - \frac{1}{2} (M(A) C^p) \right\} (\Phi_{(3)} M \rightarrow \Phi_{(3)}^M), \quad (A.14)$$

$$S_{F_{(3)0}^L} \mid_{\text{lad}} = \frac{\epsilon_{pqr}}{2! 3!} \int d^4x \text{tr} \left\{ \frac{1}{2} \left( (\Pi \chi(A)) + \chi(A)\Pi \right) B_{pq}, C^r \right\} + \left\{ [M(A) C^r, \Pi C^q] + [D^\mu(A) C^r, \partial_\mu C^q] \right\} \Pi C^p \right\} (\Phi_{(3)} M \rightarrow \Phi_{(3)}^M), \quad (A.15)$$

with usual Faddeev-Popov matrix, $M = M(A)$ and with taken account for the relations

$$F_{(3)0}^L(A) \left( \frac{\epsilon}{\hbar} \right)^3 = \frac{1}{2} \int d^4x \text{tr} \left\{ (\Pi \chi(A)) + \chi(A)\Pi \right\} C^p, \quad (A.16)$$

$$F_{(3)0}^L(A) \left( \frac{\epsilon}{\hbar} \right)^3 \epsilon_{pqr} = \frac{1}{2} \int d^4x \text{tr} \left\{ (\Pi \chi(A)) + \chi(A)\Pi \right\} B_{pq} - (M(A) C^p) \epsilon_{pqr}, \quad (A.17)$$

$$F_{(3)0}^L(A) \left( \frac{\epsilon}{\hbar} \right)^3 = \frac{1}{2} \int d^4x \text{tr} \left\{ (\Pi \chi(A)) + \chi(A)\Pi \right\} - (M(A) C^p) \epsilon_{pqr}, \quad (A.18)$$

Here the relations (2.32), (2.33), (A.11), (A.12) and (B.9) for Landau gauge $\chi(A) = 0$ were used as well as the vanishing of the terms, $[C^p, C^q] \epsilon_{pqr} = 0$. 43
Note, first, the terms proportional to the $\Pi(A)$ in (A.18) may be easily elaborated by the rule

$$tr (\Pi(A)) \hat B = \Pi^{mn} \chi^n(A) \hat B^m = \chi^n(A) \Pi^{mn} \hat B^m = tr \chi(A) \Pi \hat B,$$

(A.19)

$$tr (\Pi(A)) [B^{pq}, C^r] = tr \chi(A) \Pi [B^{pq}, C^r],$$

(A.20)

by virtue of the properties (A.11), (A.12). Second, the quadratic in the fictitious fields with Faddeev–Popov matrix summands, we can present due to the same properties as follows:

$$tr (M(A) \Pi C^q) B^{pr} = tr B^{pr} M(A) \Pi C^q = tr (B^{pr} \Pi M(A) C^q) = tr (\Pi B^{pr}) M(A) C^q.$$  

(A.21)

Expressing the fields $\phi^M$ in terms of $\hat \phi^M$, according to the change of variables (A.8) in $\mathcal{M}^{(3)}$, we get for the action $S_{F^{(3)}} (A.13)$ and $S_{F^{(3)}} (A.15)$ with use of (A.19)–(A.21) and with use of dual field $B_2 = -B^{13} = \varepsilon_{132} B^{13}$:

$$S_{F^{(3)}} \left( \hat \phi \right) = S_0(A) + \int d^4x tr \left( \chi(A) B + \overline{C}^3 M(A) C^3 + \overline{C}^1 M(A) C^1 \right. \left. + \overline{B}_2 M(A) B_2 \right)$$

$$+ \frac{1}{3!} \int d^4x tr \left\{ \chi(A) \Pi \left[ [B_2, \Pi B_2] + \frac{1}{2!} [\Pi C^{2k+1}, C^{2k+1}] \right] \right.$$ 

$$+ \frac{1}{2!} \int d^4x tr \left\{ \left[ (M(A) C^r, \Pi C^q) C^p + [D^\mu (A) C^r, \partial_\mu \Pi C^q] C^p \right.$$ 

$$- \left( [M(A) C^r, C^q] + [D^\mu (A) C^r, \partial_\mu C^q] \right) \Pi C^p \right\} \left[ \Pi C^{(2,3)} \rightarrow \Pi (B_2, \Pi B_2) \right].$$

(A.22)

Here, the role of Faddeev–Popov ghosts is a mixed one, in comparison with the initial basis of fictitious fields $C^p, B^{pq}, \hat B$. For example, in the first row of (A.22) for the fields $C, \overline{C}$, used within the original Faddeev–Popov quantization as ghost and antighost fields, we have, respectively, $C^1, \Pi B^{23}$ and $C^3, \Pi B^{12}$.

Therefore, as far as the last condition in (A.12) holds true, the functional $S_{F^{(3)}} (\hat \phi)$, with the gauge functional (A.10) which determines the path integral $Z_{F^{(3)}} (A.13)$ in the Landau gauge with a local quantum action solving the problem of generalization of the Faddeev–Popov rules in the case of the irreducible representation $N = 3$-parametric $\mathcal{G}(3)$ superalgebra.

The latter local action (as well as the measure $d\hat \Phi^{(3)}$ corresponding to the Landau gauge is invariant under $N = 3$ (therefore called as $N = 3$ BRST transformations, which, at the algebraic level in a new basis of fields, $\hat \phi^{(3)}$, are written with allowance for (2.45), (A.8), (A.9), as follows:

$$A^\mu_\nu = D^\mu (C^1 \delta^{1p} + \Pi B_2 \delta^{2p} + C^3 \delta^{3p}),$$

(A.23)

$$C^1 \delta^{1p} = \frac{1}{2} \left[ C^1, C^1 \right] \delta^{1p} + \Pi C \delta^{2p} + \left\{ - B_2 \left[ C^1, C^3 \right] \right\} \delta^{3p},$$

$$C^2 \delta^{2p} = \left\{ - B_2 \left[ C^1, C^1 \right] \right\} \delta^{3p} + \left\{ - \Pi C \right\} \delta^{2p} + \left\{ \overline{C} \right\} \delta^{3p},$$

$$C^3 \delta^{3p} = \left\{ \left[ \Pi C^1, C^1 \right] \right\} \delta^{1p} + \Pi \left\{ \Pi C^1, C^2 \right\} \delta^{2p} + \Pi \left\{ \Pi C^1, C^3 \right\} \delta^{3p},$$

$$B_2 \delta^{2p} = \frac{1}{2} \left[ B_2, C^1 \right] + \frac{1}{\varepsilon} \left[ C^1, C^3 \right] \delta^{3p} + \left\{ \Pi B \left[ C^1, C^2 \right] \right\} \delta^{2p} + \frac{1}{\varepsilon} \left[ B_2, C^3 \right] \delta^{3p} + \frac{1}{\varepsilon} \left[ B_2, C^3 \right] \delta^{3p},$$

(A.25)

$$B_2 \delta^{2p} = \frac{1}{2} \left[ B_2, C^1 \right] + \frac{1}{\varepsilon} \left[ C^1, C^2 \right] \delta^{2p} + \left\{ \Pi B \left[ C^2, C^2 \right] \right\} \delta^{2p} + \frac{1}{\varepsilon} \left[ B_2, C^3 \right] \delta^{2p} + \frac{1}{\varepsilon} \left[ B_2, C^3 \right] \delta^{2p},$$

(A.26)
\[C^i \equiv \{ B + \frac{1}{2} \Pi \left( [\Pi C^1, C^1] - \frac{1}{6} [\Pi \Pi B^2, [C^3, C^1]] \right) \} \delta^{1p} + \frac{1}{2} \Pi \left( [\Pi \Pi B^2, C^3] - \frac{1}{6} [\Pi \Pi B^2, [C^3, \Pi B_2]] \right) \delta^{2p}, \tag{A.27}\]

\[B^i \equiv \frac{1}{2} \Pi \left( \{ B_2, C^1 \delta^{1p} + \Pi \Pi B_2 \delta^{2p} + C^3 \delta^{3p} \} - \frac{1}{2} \left( \{ [\Pi C^1, C^3] \} + [B_2, \Pi \Pi B_2] \right) \right) + \{ [\Pi C^1, C^1], \} \{ C^1 \delta^{1p} + \Pi \Pi B_2 \delta^{2p} + C^3 \delta^{3p} \} \right\} + \frac{1}{2} \left\{ \left( \Pi C^1 \delta^{1p} + \Pi C^3 \delta^{3p} \right), [C^3, C^1] \right\} + \left\{ \left( B_2 \delta^{1p} - \Pi \Pi C^1 \delta^{2p} \right), C^3 \right\} \Pi B^2 \right\}, \tag{A.28}\]

where we introduced the formal identification \( B_2 = \bar{B}^2 \) to use the antisymmetry of: \( C^1 \Pi B^2 = C^1 \Pi B^2 - \Pi \Pi \Pi B^2 C^1 \) being inherited from one for \( C^1 \Pi C^2 \).

Thus, we see, that the preservation of the explicit \( N = 3 \) BRST symmetry for the quantum action \( S_L^L(\Phi(3)) \) in the space of \( G(3) \)-irreducible representation \( M(3) \) requires the introduction of odd non-degenerate supermatrix \( N \) with destroying of \( G(3) \)-covariance of the fields \( \Phi(3) \) to get local path integral \( (A.13) \) with \( N = 3 \) BRST invariance \( (A.23) - (A.28) \).

This fact proves the validity condition (2.38) of the Statement 1 concerning gauge-fixing procedure for odd \( N \).

### B \( N = 4 \) BRST Invariant Yang–Mills Action in \( R_\xi \)-like Gauges

In this Appendix, we present the details of calculations used in Section 4 to find \( N = 4 \) BRST invariant quantum action \( (4.7) - (4.11) \) and establish a correspondence between the gauge-fixing procedures in the Yang–Mills theory described by a gauge-fixing function \( \chi(A, B) = 0 \) from the class of \( R_\xi \)-gauges in \( N = 1 \) BRST formulation and by a gauge-fixing functional \( Y^0_{(4\xi)} \) in the suggested \( N = 4 \) BRST quantization.

To calculate \( S_{Y(4\xi)}(\Phi(4)) \) we have used the results of applications \( (2.82) - (2.86), (2.87), (2.88) \) adapted for \( N = 4 \) case, as well as the property \( (4.11) \) for differentiation of the product and commutator of any two functions by products of the generators \( \xi^p \) up to 4-th order.

Thus, for the quadratic gauge bosonic functional, \( Y^0_{(4\xi)}(\Phi(4)) = Y^0_{(4\xi)}(A) + Y^0_{(4\xi)}(B^{0\eta, \rho}) \), \( (4.6) \) we need the preliminary calculations with action of the first and second powers of \( \xi^p \) on \( Y^0_{(4\xi)}(A) \) with use of the notation for the compact writing, \( \varepsilon_{r_1 r_2 r_3 r_4} \equiv \varepsilon_{[r_1 r_4]} \):

\[Y^0_{(4\xi)}(A) \xi^{r_1} = \int d^dx \ tr A_\mu D^\mu (A) C^{r_1} = - \int d^dx \ tr (\partial^\mu A_\mu) C^{r_1}, \tag{B.1}\]

\[\left( (\partial^\mu A_\mu) C^{r_1} \right) \xi^{r_1} \xi^{r_2} \xi^{r_3} \xi^{r_4} = \left( C^{r_1} M(A) C^{r_2} + (\partial^\mu A_\mu) B^{r_1 r_2} \right) \varepsilon_{[r_1 r_4]}, \tag{B.2}\]

of the third powers, with account for the identities \( (B.9) \) below and equalities \( \int d^dx tr C^{r_1} \ M(A) B^{r_2 r_3} = \int d^dx tr B^{r_2 r_3} \{ M(A) - (\partial^\mu A_\mu), \} C^{r_1} \), obtained with help of the integration by parts:

\[Y^0_{(4\xi)}(A) \prod_{k=1}^3 \xi^{r_k} \varepsilon_{[r_4]} = - \int d^dx tr \left( - B^{r_1 r_2} M(A) C^{r_2} + C^{r_1} M(A) B^{r_2 r_3} - C^{r_1} \left\{ M(A) C^{r_3}, C^{r_2} \right\} + \left\{ D^\mu (A) C^{r_2}, \partial^\mu C^{r_2} \right\} + B^{r_1 r_2} M(A) C^{r_3} + (\partial^\mu A_\mu) \left\{ B^{r_1 r_2 r_3} + \frac{1}{2} \left[ B^{r_1 r_2}, C^{r_3} \right] \right\} \varepsilon_{[r_4]} \right) \]

\[= - \int d^dx tr \left( 3 B^{r_1 r_2} M(A) C^{r_3} + C^{r_1} \partial^\mu \left[ D^\mu C^{r_2}, C^{r_3} \right] + \left\{ B^{r_1 r_2 r_3} + \frac{1}{2} \left[ B^{r_1 r_2}, C^{r_3} \right] \right\} \partial^\mu A_\mu \right) \varepsilon_{[r_4]}, \tag{B.3}\]

\[\text{11The action of the Grassmann-odd operator } \Pi \text{ may be determined on the } su(\hat{N}) \text{ commutator } [A, B] \text{ of any Grassmann-homogeneous quantities } A, B \text{ as } \Pi [A, B] = [\Pi A, B] = (-1)^{\varepsilon(A)} [A, \Pi B], \text{ in such a way that } \Pi \text{ should act only on the fields } \hat{\Phi}^M, \hat{\Phi}^M. \text{ E.g. } \Pi [\Pi B, C^1] = [B, C^1] \text{ and } \Pi [\Pi B, \Pi \Pi B_2] = [B, \Pi \Pi B_2] = [- [\Pi B, \Pi B_2].} \]
and of the fourth power:

\[
Y^0_4(A) \prod_{k=1}^{4} \frac{1}{S^{r_k}} \xi_{[r_k]} = - \int d^4x tr \left( -3 \left\{ B^{r_1r_2r_3} + \frac{3}{4} \left[ B^{r_1r_2}, C^{r_3} \right] \right\} M(A) C^{r_3} + 3 B^{r_1r_2} M(A) B^{r_3r_4} + (\partial^\mu A_\mu) \left\{ \epsilon^{[r_1} B + \frac{1}{4} \left[ [B^{r_1r_2}, C^{r_3}], C^{r_4} \right] - \sum_P (-1)^{P(r_1,r_2,r_3)} \left\{ \frac{1}{4} \left[ [B^{r_1r_2}, C^{r_3}], C^{r_4} \right] \right\} + \frac{1}{2} \left[ [B^{r_1r_2}, C^{r_3}], C^{r_4} \right] \right\} - \frac{3}{2} \left[ B^{r_1r_2}, C^{r_3} \right] - \frac{3}{4} \left[ [B^{r_1r_2}, C^{r_3}], C^{r_4} \right] \right\} + 3 B^{r_1r_2} \partial_\mu \left[ [D^{\mu} C^{r_2}, C^{r_3}], C^{r_4} \right] \right\} \xi_{[r_k]} \right)
\]

(B.4)

Here, we have used that, \([C^{r_1}, C^{r_2}], \xi_{[r_1r_2]}, C^{r_3}, C^{r_4}] = 0\), definition of the Faddeev-Popov operator (1.4), integration by parts, relations (2.82), (2.83), its analog, \((M(A) B^{r_1r_2}) \xi_{[r_k]}\), and easily checked Leibnitz rule of the commutator differentiation for covariant derivative, \(D_\mu(A)\):

\[
(M(A) B^{r_1r_2}) \xi_{[r_k]} = \partial_\mu \left[ D^{\mu} (A) C^{r_3}, B^{r_1r_2} \right] + M(A) (B^{r_1r_2} \xi_{[r_k]})
\]

(B.6)

as well as the relations, first, for the terms with permutation, \(P(r_1, r_2, r_3)\), and second, for \(su(N)\)-valued functions \(F, G\):

\[
- \left( \sum_P (-1)^{P(r_1,r_2)} \frac{1}{4} \left\{ \frac{1}{4} \left[ [B^{r_1r_2}, C^{r_3}], C^{r_4} \right] + \frac{3}{4} \left[ [B^{r_1r_2}, C^{r_3}], C^{r_4} \right] \right\} + \frac{3}{4} \left[ [B^{r_1r_2}, C^{r_3}], C^{r_4} \right] \right\} \xi_{[r_k]}
\]

(B.8)

\[
D_\mu A^\mu = \partial_\mu A^\mu , \quad \int d^4x tr (D_\mu F) G = - \int d^4x tr FD_\mu G .
\]

(B.9)

In turn, the input from the gauge boson part \(Y^B_4(B^{q_1q_2})\), (4.6) into the quantum action (4.7) may be presented as:

\[
Y^B_4(B^{q_1q_2}) \prod_{k=1}^{4} \frac{1}{S^{r_k}} \xi_{[r_k]} = - \frac{2 \xi_{B}^2}{4l} \int d^4x tr \left\{ B^{q_1q_2} \prod_{k=1}^{4} \sum_{r_k} B^{s_{r_k}} B^{q_1q_2} B^{q_1q_2} + 4 B^{q_1q_2} \prod_{k=1}^{3} \sum_{r_k} B^{s_{r_k}} B^{q_1q_2} + 3 B^{q_1q_2} B^{s_{r_1}} B^{s_{r_2}} \right\} \xi_{[r_k]} \xi_{[r_k]},
\]

(B.10)

so that to derive the quadratic in the fields \(B\) terms, which should determine the gauge-fixed action for the Feynman-like gauge it is sufficient to calculate the last summand above, because of, \(B^{q_1q_2} B^{s_{r_1}} B^{s_{r_2}} = \xi^{q_1q_2} B + o(B,C)\), according to \(N = 4\) BRST transformations (3.20).
Let us find the action of the operators \( S^{-r_1} \varepsilon_{[r_4]} \) and \( S^{-r_2} \varepsilon_{[r_4]} \) on \( B^{q_1 q_2} \varepsilon_{[q_4]} \):

\[
B^{q_1 q_2} S^{-r_1} \varepsilon_{[r_4]} \varepsilon_{[q_4]} = \left\{ B^{q_1 q_2} + \frac{1}{2} \left[ \left[ B^{q_1 q_2}, C^{r_1} \right] - \frac{1}{6} \left[ C^{q_1}, \left[ C^{q_2}, C^{r_1} \right] \right] \right] \right\} \varepsilon_{[r_4]} \varepsilon_{[q_4]}, \tag{B.11}
\]

\[
B^{q_1 q_2} S^{-r_2} \varepsilon_{[r_4]} \varepsilon_{[q_4]} = \left\{ S^{-q_1 q_2} + \frac{1}{2} \left[ \left[ B^{q_1 q_2}, C^{r_2} \right] - \frac{1}{6} \left[ C^{q_1}, \left[ C^{q_2}, C^{r_2} \right] \right] \right] \right\} \varepsilon_{[r_4]} \varepsilon_{[q_4]}, \tag{B.12}
\]

Then, for the last term in (B.10) we have

\[
- \frac{\xi g^2}{4} \int d^4 x \, tr \left\{ B^{q_1 q_2} S^{-r_1} \varepsilon_{[r_4]} \varepsilon_{[q_4]} \left( B^{q_1 q_4} S^{-r_3} \varepsilon_{[r_4]} \varepsilon_{[q_4]} \right) \right\} = - \frac{\xi g^2}{4} \int d^4 x \, tr \left\{ B^{q_1 q_3} + B^{q_4 q_2} \right\} \varepsilon_{[q_4]} \varepsilon_{[q_4]} \\]

\[
= - \frac{\xi g^2}{4} \int d^4 x \, tr \left\{ B^{q_1 q_3} + B^{q_4 q_2} \right\} \varepsilon_{[q_4]} \varepsilon_{[q_4]}, \tag{B.13}
\]

where we have used the Fierz-like identities for the products of Levi-Civita tensors:

\[
\varepsilon^{q_1 q_2 r_1 r_2} \varepsilon_{[r_4]} \varepsilon^{q_3 q_4 r_3 r_4} = 4 \varepsilon_{[q_4]}, \quad \text{and} \quad \varepsilon^{q_1 q_2 r_1 r_2} \varepsilon_{[r_4]} \varepsilon^{q_3 q_4 r_3 r_4} \varepsilon_{[q_4]} = 4 \cdot 4!, \tag{B.14}
\]

and its normalization \((2.37), (3.31)\).

Now, we are waiting that the first and second terms in (B.10) of the third and fourth orders in \( S^{-r} \) when acting on \( B^{q_1 q_2} \) will not produce new summands to the gauge-fixed and quadratic in the fictitious fields parts of the action \((4.7)\). Their role concerns only to exclude non-diagonal terms from the last quantity in (B.10) given explicitly in (B.13). To justify the proposal let us show, that the terms linear in \( B \) in (B.13) are absent in \( S_{Y(4)} \left( \Phi_{(4)} \right) \) \((4.7)\). To do so we need the product of three antisymmetrized
generators, $\sum_{k=1}^{3} S_{r_1}^{r_2} S_{r_2}^{r_3} \xi_{[r_4]}$ applied to $B^{q_1 q_2}$:

$$B^{q_1 q_2} \sum_{k=1}^{3} S_{r_1}^{r_2} S_{r_2}^{r_3} \xi_{[r_4]} = \left\{ \varepsilon^{q_1 q_2 r_1 r_2} \left( \frac{1}{2} [B, C^{r_2}] - \frac{1}{12} [B^{s_1 s_2 s_3}, C^{r_2}] \varepsilon_{s_1 s_2 s_3 s_4} \right) + [B^{q_1 q_2 r_1}, B^{r_2 r_3}] - \left( \varepsilon^{q_1 q_2 r_1 r_2} B + \frac{1}{3} [B^{q_1 q_2 r_1}, C^{r_2}] - \sum_{P} (-1)^{P(q_1 q_2 r_1)} \left( \frac{1}{4} [[B^{q_1 q_2}, C^{r_1}], C^{r_2}] + \frac{1}{3} \left([[B^{q_1 q_2}, C^{q_2}], C^{r_2}] - \frac{1}{3} \left([[B^{q_1 q_2}, C^{r_1}], C^{q_2}], C^{r_2} \right) - \frac{1}{3} \left([[B^{q_1 q_2}, C^{q_2}], C^{r_1}], C^{r_2} \right) \right) \right) \right\} \xi^{[r_4]} \xi_{[q_4]}.$$  \hspace{1cm} (B.15)

Consider, e.g. the summand, $B [B^{q_1 q_2 r_1}, C^{r_2}]$ in (B.13). The only second term in (B.10) gives similar contribution from (B.15), so that their sum is equal to:

$$\int d^4 x \, tr \left( 4 \cdot 3 \cdot 2 B [B^{q_1 q_2 r_1}, C^{r_2}] \varepsilon_{q_1 q_2 r_1 r_2} + 4 \cdot 3 \cdot 2 \varepsilon^{q_1 q_2 r_1 r_2} \left[ B, C^{r_2} \right] B^{q_1 q_2 r_4} \xi_{[r_4]} \xi_{[q_4]} \right) = 4! \int d^4 x \, tr \left( B [B^{q_1 q_2 r_1}, C^{r_2}] \varepsilon_{q_1 q_2 r_1 r_2} + \left[ B, C^{r_2} \right] B^{q_1 q_2 r_4} \xi_{q_1 q_2 r_4 r_3} \right) \xi_{[r_4]} \xi_{[q_4]} = 0,$$

due to the antisymmetry in $r_1, r_2$ of $[B^{q_1 q_2 r_1}, C^{r_2}] \varepsilon_{q_1 q_2 r_1 r_2} = -[B^{q_1 q_2 r_1}, C^{r_2}] \varepsilon_{q_1 q_2 r_1 r_2}$ and the property for $su(N)$-valued functions with definite Grassmann parities:

$$tr F[G, H] = F^{m n} m^{n l} G^{l} H = tr [F, G] H = -tr G[F, H] (-1)^{\varepsilon(F)\varepsilon(G)}.$$  \hspace{1cm} (B.17)

The checking that the remaining terms linear in $B$ in (B.13) do not contribute in the quantum action (1.7) may be fulfilled analogously, but we leave out of the paper scope the proof of this fact.

The $\xi$-dependent part of $N = 4$ BRST invariant quantum action (1.7) take the form:

$$\xi \frac{\partial}{\partial \xi} S_{Y_{\alpha \xi}} (\Phi_{(4)}) = \xi g^2 \int d^4 x \, tr \left\{ B^2 + \frac{1}{12} \left[ B^{q_1 q_2}, B^{r_1 r_2} \right] B^{q_3 q_4} B^{r_3 r_4} \right\} \xi_{[r_4]} \xi_{[q_4]} + \tilde{S}_\xi,$$

without terms linear in $B$ in $\tilde{S}_\xi$, which should be determined from (B.10), (B.13), (B.15) and the results of the product of four antisymmetrized generators, $\sum_{k=1}^{3} S_{r_1}^{r_2} S_{r_2}^{r_3} S_{r_3}^{r_4} \xi_{[r_4]}$ applied to $B^{q_1 q_2}$. 

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Therefore, combining (B.4), (B.18) we have

\begin{equation}
S_{Y(4)}(\Phi(4)) = S_0 + \int d^4x \operatorname{tr}\left\{ \left[ \partial^\mu A_\mu + \xi g^2 B \right] B + \left\{ \frac{1}{3!} B^{r_1 r_2 r_3} M(A) C^{r_4} + \frac{1}{8} B^{r_1 r_2} M(A) B^{r_3 r_4} \right\} \xi_{[r_4]} \right. \\
+ \frac{1}{4!} \left\{ (\partial^\mu A_\mu) \left( 2 [B^{r_1 r_2 r_3}, C^{r_4}] - [[B^{r_1 r_2}, C^{r_3}], C^{r_4}] \right) - B^{r_1 r_2} \left( [C^{r_3}, M(A) C^{r_4}] \right) \\
+ 4 \{ \xi_\mu C^{r_3}, D^{\mu} C^{r_4} \} \right\} C^{r_4} \partial_\mu \left( \left[ D^\mu C^{r_2}, B^{r_3 r_4} \right] - [C^{r_2}, D^\mu B^{r_3 r_4}] + [[D^\mu C^{r_2}, C^{r_3}], C^{r_4}] \right) \xi_{[r_4]} \right. \\
+ \frac{2g^2}{4!} \left( \frac{1}{4!} \left[ B^{q_1 q_2}, B^{r_1 r_2} \right] B^{q_3 q_4}, B^{r_3 r_4} \right) + \frac{1}{4!} \left[ C^{q_1}, \left[ [C^{q_2}, C^{r_2}], C^{r_1} \right] \right] \times \\
\times \left. \left[ C^{q_3}, \left[ [C^{q_4}, C^{r_4}], C^{r_3} \right] \right] \xi_{[r_4]} \xi_{[q_4]} \right) \right\} + \tilde{S}_\xi,
\end{equation}

(B.19)

for \( \tilde{S}_\xi |_{\xi=0} = 0 \), that proves the representation (4.7)–(4.10) for the quantum action. Determining the dual fields (with lower \( G(4) \)-indices) for Grassmann-even \( B^{r_1 r_2} \) and Grassmann-odd \( B^{r_1 r_2 r_3} \) fields:

\begin{equation}
B_{r_1 r_2} = \frac{1}{2!} B^{r_3 r_4} \xi_{r_1 r_2 r_3 r_4} = (B^{34}, -B^{24}, B^{23}, B^{14}, -B^{13}, B^{12}), \tag{B.20}
\end{equation}

\begin{equation}
C_{r_1} = \frac{1}{3!} B^{r_1 r_2 r_3} \xi_{r_1 r_2 r_3 r_4} = (-B^{234}, B^{34}, -B^{234}, B^{123}), \tag{B.21}
\end{equation}

the action (B.19) can be equivalently presented as follows

\begin{equation}
S_{Y(4)} = S_0 + \int d^4x \operatorname{tr}\left\{ \left[ \partial^\mu A_\mu + \xi g^2 B \right] B + \left\{ \xi_\mu C^{r_3}, D^\mu C^{r_4} \right\} \right. \\
+ \frac{1}{4!} \left\{ (\partial^\mu A_\mu) \left( 2 [C^{r_1}, C^{r_2}] - [[B^{r_1 r_2}, C^{r_3}], C^{r_4}] \right) - 2B^{r_1 r_2} \left( [C^{r_3}, M(A) C^{r_4}] \right) \\
+ 4 \{ \xi_\mu C^{r_3}, D^{\mu} C^{r_4} \} \right\} C^{r_4} \partial_\mu \left( \left[ D^\mu C^{r_2}, B^{r_3 r_4} \right] - [C^{r_2}, D^\mu B^{r_3 r_4}] + \frac{1}{2} [[D^\mu C^{r_2}, C^{r_3}], C^{r_4}] \right) \xi_{[r_4]} \right. \\
+ \frac{2g^2}{4!} \left( \frac{1}{4!} \left[ B^{q_1 q_2}, B^{r_1 r_2} \right] B^{q_3 q_4}, B^{r_3 r_4} \right) + \frac{1}{4!} \left[ C^{q_1}, \left[ [C^{q_2}, C^{r_2}], C^{r_1} \right] \right] \times \\
\times \left. \left[ C^{q_3}, \left[ [C^{q_4}, C^{r_4}], C^{r_3} \right] \right] \xi_{[r_4]} \xi_{[q_4]} \right) \right\} + \tilde{S}_\xi, \tag{B.22}
\end{equation}

by virtue of easily checked identity, \( \sum_{1 \leq r_1 < r_2 \leq 3} B^{r_1 r_2} M(A) B_{r_1 r_2} = \frac{1}{4} B^{r_1 r_2} M(A) B_{r_1 r_2} \), justifying the representation (2.23) for the quantum action for the case \( N = 4 \) (k(4)=3) in the Landau gauge (\( \xi = 0 \)) with the identification:

\begin{equation}
(C^0, C^{[3]}, C^{(3)}, C^{[3]}, B^{[3]}, B^{(3)}) = (C^r, C^r, B_{r_1 r_2}, B^{r_1 r_2}) \text{ for } r = 1, ..., 4; \text{ } 1 \leq r_1 < r_2 \leq 3. \tag{B.23}
\end{equation}

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