A modification of the Anderson-Mirković conjecture for Mirković-Vilonen polytopes in types $B$ and $C$

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Abstract

We give an explicit description of the (lowering) Kashiwara operators on Mirković-Vilonen polytopes in types $B$ and $C$, which provides a simple method for generating Mirković-Vilonen polytopes inductively. This description can be thought of as a modification of the original Anderson-Mirković conjecture, which Kamnitzer proved in the case of type $A$, and presented a counterexample in the case of type $C_3$.

1 Introduction.

Let $G$ be a connected, simply-connected, semisimple algebraic group over $\mathbb{C}$, and $G^\vee$ its Langlands dual group. Mirković and Vilonen ([MV1], [MV2]) discovered a family of closed, irreducible, algebraic subvarieties, called MV cycles, of the affine Grassmannian $G$ associated to $G$, which provide a basis for each finite-dimensional irreducible highest weight representation of $G^\vee$ (or equivalently, of its Lie algebra $g^\vee$).

In order to obtain an explicit combinatorial description of MV cycles, Anderson ([A]) defined MV polytopes for the Lie algebra $g$ of $G$ to be moment map images of these cycles, which are drawn in the real form $h_R := \sum_{j \in I} \mathbb{R}h_j$ of the Cartan subalgebra $h$ of $g$, where the $h_j$, $j \in I$, are the simple coroots of $g$; in [Kam1], Kamnitzer characterized these MV polytopes as pseudo-Weyl polytopes that satisfy “tropical” Plücker relations. Furthermore, inspired by the crystal structure on the set of MV cycles due to Braverman, Finkelberg, and Gaitsgory ([BG], [BFG]), Anderson and Mirković proposed a conjecture (the AM conjecture) describing a crystal structure for $g^\vee$ on the set of MV polytopes; this conjecture gives a method for generating MV polytopes inductively without making use of the tropical Plücker
relations. The AM conjecture above was proved in the case $g = sl_n$ by Kamnitzer ([Kam2]), who also presented a counterexample in the case $g = sp_6$.

The purpose of this paper is to prove a kind of modification of the original AM conjecture for simple Lie algebras of types $B$ and $C$. Let us explain our results more precisely. In this paper, we assume that $g$ is a simple Lie algebra of type $A$ over $\mathbb{C}$. Let $\omega : I \to I$ be a (Dynkin) diagram automorphism of order 2 of the index set $I = \{1, 2, \ldots, \ell\}$. Then it induces a Lie algebra automorphism (also denoted by) $\omega : g \to g$, which stabilizes the Cartan subalgebra $\mathfrak{h}$, and hence induces $\omega \in GL(\mathfrak{h}^*)$ by: $\langle \omega(\lambda), h \rangle = \langle \lambda, \omega(h) \rangle$ for $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$. We set $g^\omega := \{ x \in g \mid \omega(x) = x \}$ and $\mathfrak{h}^\omega := \{ h \in \mathfrak{h} \mid \omega(h) = h \}$. It is known that if $g$ is of type $A_\ell$ with $\ell = 2n - 1$, $n \in \mathbb{Z}_{\geq 2}$, (resp., of type $A_\ell$ with $\ell = 2n$, $n \in \mathbb{Z}_{\geq 2}$) then $g^\omega$ is a simple Lie algebra of type $C_n$ (resp., type $B_n$) with Cartan subalgebra $\mathfrak{h}^\omega$. Moreover, the Weyl group $\widehat{W}$ of $g^\omega$ can be identified with the subgroup $W^\omega$ of the Weyl group $W = \langle s_i \mid i \in I \rangle$ (through a group isomorphism $\Theta : \widehat{W} \to W^\omega$) consisting of the elements of $W$ fixed under the action of the diagram automorphism $\omega : I \to I$ given by: $\omega(s_i) = s_{\omega(i)}$ for $i \in I$.

Following Kamnitzer, let $\mathcal{MV}$ denote the set of MV polytopes $P = P(\mu_\bullet) \subset \mathfrak{h}_R$, with GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$, such that $\mu_{w_0} = 0 \in \mathfrak{h}_R$, where $w_0 \in W$ is the longest element. Here the GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$ of an MV polytope $P$ is a collection (which may have repetition) of elements of $\mathfrak{h}_R := \sum_{j \in I} \mathbb{Z} h_j$, and gives the set of vertices of the convex polytope $P$. Let $P = P(\mu_\bullet) \in \mathcal{MV}$ be an MV polytope with GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$. Then, the image $\omega(P)$ of $P$ (as a set) under $\omega \in GL(\mathfrak{h})$ is identical to the element $P(\mu'_\bullet) \in \mathcal{MV}$ with GGMS datum $\mu'_\bullet = (\mu'_w)_{w \in W}$, where $\mu'_w := \omega(\mu_{\omega(w)})$ for $w \in W$. We set $\mathcal{MV}^\omega := \{ P \in \mathcal{MV} \mid \omega(P) = P \}$, and define the set $\widehat{\mathcal{MV}}$ of MV polytopes for $g^\omega$ in the same manner as we defined $\mathcal{MV}$ for $g$. Now, to each element $P = P(\mu_\bullet)$ of $\mathcal{MV}^\omega$, we assign a convex polytope $\Phi(P) = P \cap \mathfrak{h}^\omega$ in $\mathfrak{h}^\omega \cap \mathfrak{h}_R$, which turns out to be the element $\widehat{\Phi}(\widehat{\mu}_\bullet)$ of $\widehat{\mathcal{MV}}$ with GGMS datum $\widehat{\mu}_\bullet = (\widehat{\mu}_w)_{\widehat{w} \in \widehat{W}}$, where $\widehat{\mu}_\widehat{w} = \mu_{\Theta(\widehat{w})} \in \mathfrak{h}^\omega \cap \mathfrak{h}_R$ for $\widehat{w} \in \widehat{W}$.

One of our main results (Theorem 2.5.6) of this paper asserts that the map $\Phi : \mathcal{MV}^\omega \to \widehat{\mathcal{MV}}$ defined above is a bijection. This result can be thought of as an application of the general idea of realizing crystals for a non-simply-laced Kac-Moody algebra as the fixed point subsets under a diagram automorphism of those for a simply-laced Kac-Moody algebra. Such an idea has often been used since Lusztig’s pioneering work ([L2 Chapter 14]); cf., to name a few, [X], [NS1], [NS2], [S], and also [KLP].

Using the result above, we prove that for each $1 \leq j \leq n$, the (lowering) Kashiwara operator $\widehat{f}_j$ on $\widehat{\mathcal{MV}}$ for the “LBZ” crystal structure due to Lusztig and Berenstein-Zelevinsky
is realized (through the bijection $\Phi : \mathcal{MV}^\omega \to \hat{\mathcal{MV}}$) as the restriction to $\mathcal{MV}^\omega \subset \mathcal{MV}$ of a certain composition $f_j^\omega$ of the Kashiwara operators $f_j$ and $f_{\omega(j)}$ on $\mathcal{MV}$ for the LBZ crystal structure. Moreover, from the original AM conjecture (proved by Kamnitzer) applied to MV polytopes in $\mathcal{MV}^\omega$, we obtain a description (Theorems 3.2.3, 3.2.4, and 3.2.5), in terms of GGMS data, of the (lowering) Kashiwara operators $\hat{f}_j$, $1 \leq j \leq n$, on MV polytopes in $\hat{\mathcal{MV}}$. Here we should mention that our description of the (lowering) Kashiwara operators on MV polytopes for $g^\omega$ in types $B$ and $C$ is rather analogous to the one in the original AM conjecture, and can be thought of as a kind of modification of it.

This paper is organized as follows. In subsections 2.1 and 2.2, following Kamnitzer, we recall the definition and some basic properties of MV polytopes, and also the LBZ crystal structure on them. Next, in subsections 2.3 and 2.4, we introduce a natural action of the diagram automorphism $\omega$ on MV polytopes in type $A$, and then study the set of MV polytopes fixed by this action. In subsection 2.5, we state one of our main results (Theorem 2.5.6), the proof of which occupies subsections 2.6 and 2.7. By making use of this result, in section 3, we present an explicit description (Theorems 3.2.3, 3.2.4, and 3.2.5) of the (lowering) Kashiwara operators on MV polytopes in types $B$ and $C$.

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2 Mirković-Vilonen polytopes and diagram automorphisms.

2.1 Mirković-Vilonen polytopes. Let $g$ be a finite-dimensional semisimple Lie algebra (not necessarily of type $A$) over the field $\mathbb{C}$ of complex numbers associated to the root datum $(A = (a_{ij})_{i,j \in I}, \Pi = \{\alpha_j\}_{j \in I}, \Pi^\vee = \{h_j\}_{j \in I}, h^*, h)$, where $A = (a_{ij})_{i,j \in I}$ is the Cartan matrix, $h$ is the Cartan subalgebra, $\Pi = \{\alpha_j\}_{j \in I} \subset h^* := \text{Hom}_\mathbb{C}(h, \mathbb{C})$ is the set of simple roots, and $\Pi^\vee = \{h_j\}_{j \in I} \subset h$ is the set of simple coroots; note that $\langle \alpha_j, h_i \rangle = a_{ij}$ for $i, j \in I$, where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between $h^*$ and $h$. We denote by $x_j$, $y_j$, $j \in I$, the Chevalley generators of $g$. Let $W = \langle s_i \mid i \in I \rangle$ be the Weyl group of $g$, where $s_i$ is the simple reflection for $i \in I$, and let $e$, $w_0 \in W$ denote the unit element and the longest element of $W$, respectively. Denote by $\Lambda_i \in h^*$, $i \in I$, the fundamental weights, and set

$$\Gamma := \{w \cdot \Lambda_i \mid w \in W, i \in I\} \subset h^*.$$
Let $\mathfrak{g}^\vee$ be the (Langlands) dual Lie algebra of $\mathfrak{g}$, that is, the finite-dimensional semisimple Lie algebra over $\mathbb{C}$ associated to the root datum $(^t\Pi = \{a_{ij}\}_{i,j \in I}, \Pi^\vee = \{\Pi_{ij}\}_{j \in I}; \Pi = \{\Pi_{ij}\}_{j \in I}, \mathfrak{h}, \mathfrak{h}^*)$; note that the Cartan subalgebra of $\mathfrak{g}^\vee$ is not $\mathfrak{h}$, but $\mathfrak{h}^*$.

We recall from [Kam1] the definitions and some basic properties of pseudo-Weyl polytopes and Mirković-Vilonen (MV for short) polytopes. Set $\mathfrak{h}_\mathbb{Z} := \bigoplus_{j \in I} \mathbb{Z}h_j$, and $\mathfrak{h}_\mathbb{R} := \bigoplus_{j \in I} \mathbb{R}h_j$. For each $w \in W$, we define a partial ordering $\geq_w$ on $\mathfrak{h}_\mathbb{R}$ by: $h \geq_w h'$ if $w^{-1} \cdot h - w^{-1} \cdot h' \in \sum_{j \in I} \mathbb{R}_{\geq 0}h_j$. Denote by $\mathcal{V}$ the set of collections $\mu_* = (\mu_w)_{w \in W}$ of elements in $\mathfrak{h}_\mathbb{R}$ such that $\mu_{w'} \geq_w \mu_w$ for all $w, w' \in W$ and $\mu_{w_0} = 0$. Note that if $\mu_* = (\mu_w)_{w \in W} \in \mathcal{V}$, then $\mu_w \in \sum_{j \in I} \mathbb{R}_{\leq 0}h_j$ for all $w \in W$, since $\mu_w \geq_w \mu_{w_0} = 0$ implies $w_0^{-1} \cdot \mu_w \in \sum_{j \in I} \mathbb{R}_{\geq 0}h_j$ and hence $\mu_w \in \sum_{j \in I} \mathbb{R}_{\geq 0} w_0 \cdot h_j$. To each $\mu_* = (\mu_w)_{w \in W} \in \mathcal{V}$, we associate a (convex) polytope $P(\mu_*) \subset \mathfrak{h}_\mathbb{R}$ by:

$$P(\mu_*) = \{h \in \mathfrak{h}_\mathbb{R} \mid h \geq_w \mu_w \text{ for all } w \in W\}, \quad (2.1.1)$$

and call it the pseudo-Weyl polytope with Gelfand-Goresky-MacPherson-Serganova (GGMS for short) datum $\mu_* = (\mu_w)_{w \in W}$. It is easy to see that if $\mu_* = (\mu_w)_{w \in W} \in \mathcal{V}$, then $\mu_{w} = \mu_{w'}$, i.e., $\mu_w = \mu_{w'}$ for all $w \in W$.

Let $\mu_* = (\mu_w)_{w \in W} \in \mathcal{V}$. For each $\gamma \in \Gamma$, we define $M_\gamma \in \mathbb{R}$ by:

$$M_\gamma = \langle w \cdot \Lambda_i, \mu_w \rangle \in \mathbb{R} \quad \text{if } \gamma = w \cdot \Lambda_i \text{ for some } w \in W \text{ and } i \in I; \quad (2.1.2)$$

note that $\langle w \cdot \Lambda_i, \mu_w \rangle \in \mathbb{R}$ does not depend on the expression $\gamma = w \cdot \Lambda_i, w \in W, i \in I$, of $\gamma \in \Gamma$. Then, we have

$$\mu_w = \sum_{i \in I} M_{w \cdot \Lambda_i} w \cdot h_i \quad \text{for } w \in W. \quad (2.1.3)$$

It follows immediately from (2.1.3) that for $w \in W$ and $i \in I$,

$$\mu_{ws_i} - \mu_w = L w \cdot h_i, \quad \text{where}$$

$$L = -M_{w \cdot \Lambda_i} - M_{w s_i \cdot \Lambda_i} - \sum_{j \in I, j \neq i} a_{ij} M_{w \cdot \Lambda_j}, \quad (2.1.4)$$

which we call the length formula (see [Kam1] Eq.(8))). By using the length formula, we see easily that for each $w \in W$ and $i \in I$, the condition $\mu_{ws_i} \geq_w \mu_w$ is equivalent to the edge inequality (see [Kam1] Eq.(6)):

$$M_{ws_i \cdot \Lambda_i} + M_{w \cdot \Lambda_i} + \sum_{j \in I, j \neq i} a_{ji} M_{w \cdot \Lambda_j} \leq 0. \quad (2.1.5)$$

**Remark 2.1.1.** Let $w \in W$. It follows by induction on $W$ with respect to the (weak) Bruhat ordering that $\mu_{ws_i} \geq_w \mu_w$ for all $i \in I$ implies $\mu_w \geq_w \mu_w$ for all $w' \in W$. 


We denote by $\mathcal{E}$ the set of collections $M_\bullet = (M_{\gamma})_{\gamma \in \Gamma}$ of real numbers, with $M_{w_0 \cdot \Lambda_i} = 0$ for all $i \in I$, satisfying the edge inequality (2.1.5) for all $w \in W$ and $i \in I$. Now, it is clear that by (2.1.2) and (2.1.3), the elements of $\mathcal{V}$ and those of $\mathcal{E}$ are in bijective correspondence, which we denote by $D : \mathcal{V} \to \mathcal{E}$, so that if $M_\bullet = D(\mu_\bullet)$, then the pseudo-Weyl polytope $P(\mu_\bullet)$ is identical to

$$P(M_\bullet) := \{ h \in \mathfrak{h}_R \mid \langle \gamma, h \rangle \geq M_\gamma \text{ for all } \gamma \in \Gamma \};$$

(2.1.6)

we call $M_\bullet \in \mathcal{E}$ the edge datum of the pseudo-Weyl polytope $P(\mu_\bullet) = P(M_\bullet)$. We set $\mathcal{P} := \{ P(\mu_\bullet) \mid \mu_\bullet \in \mathcal{V} \} = \{ P(M_\bullet) \mid M_\bullet \in \mathcal{E} \}$.

**Remark 2.1.2.** We know from [Kam1, Proposition 2.2] that the set of vertices of the pseudo-Weyl polytope $P(\mu_\bullet)$ is the collection $\mu_\bullet = (\mu_w)_{w \in W}$ (possibly, with repetitions). In particular, $P(\mu_\bullet)$ is identical to the convex hull in $\mathfrak{h}_R$ of the set of real numbers, with $\mathcal{E}$.

Let $w \in W$ and $i, j \in I$ be such that $ws_i > w$, $ws_j > w$, and $i \neq j$, where $>$ denotes the (weak) Bruhat ordering on $W$. We say that an element $M_\bullet = (M_{i})_{\gamma \in \Gamma} \in \mathcal{E}$ satisfies the tropical Plücker relation at $(w, i, j)$ if $a_{ij} = a_{ji} = 0$, or one of the following holds:

1. $a_{ij} = a_{ji} = -1$, and

$$M_{ws_i \cdot \Lambda_i} + M_{ws_j \cdot \Lambda_j} = \min\left( M_{w \cdot \Lambda_i} + M_{ws_i s_j \cdot \Lambda_i}, M_{ws_i s_j \cdot \Lambda_i} + M_{w \cdot \Lambda_j} \right);$$

(2.1.7)

2. $a_{ij} = -1$, $a_{ji} = -2$, and

$$M_{ws_j \cdot \Lambda_j} + M_{ws_i s_j \cdot \Lambda_j} + M_{ws_i \cdot \Lambda_i} = \min\left( \begin{array}{c} 2M_{ws_i s_j \cdot \Lambda_j} + M_{w \cdot \Lambda_i} \\ 2M_{w \cdot \Lambda_j} + M_{ws_i s_j \cdot \Lambda_i} \\ M_{w \cdot \Lambda_j} + M_{ws_i s_j \cdot \Lambda_j} + M_{ws_i \cdot \Lambda_i} \end{array} \right),$$

(2.1.8)

$$M_{ws_j s_i \cdot \Lambda_i} + 2M_{ws_i s_j \cdot \Lambda_j} + M_{ws_i \cdot \Lambda_i} = \min\left( \begin{array}{c} 2M_{w \cdot \Lambda_j} + 2M_{ws_i s_j \cdot \Lambda_i} \\ 2M_{ws_j s_i \cdot \Lambda_j} + 2M_{ws_i \cdot \Lambda_i} \\ M_{ws_j s_i \cdot \Lambda_i} + 2M_{ws_i s_j \cdot \Lambda_j} + M_{w \cdot \Lambda_i} \end{array} \right);$$

(2.1.9)

3. $a_{ij} = -2$, $a_{ji} = -1$, and

$$M_{ws_j s_i \cdot \Lambda_i} + M_{ws_i \cdot \Lambda_i} + M_{ws_i s_j \cdot \Lambda_j} = \min\left( \begin{array}{c} 2M_{ws_i \cdot \Lambda_i} + M_{ws_j s_i \cdot \Lambda_j} \\ 2M_{ws_i s_j \cdot \Lambda_i} + M_{w \cdot \Lambda_j} \\ M_{ws_j s_i \cdot \Lambda_i} + M_{w \cdot \Lambda_i} + M_{ws_i s_j \cdot \Lambda_j} \end{array} \right),$$

(2.1.10)

$$M_{ws_j \cdot \Lambda_j} + 2M_{ws_i \cdot \Lambda_i} + M_{ws_i s_j \cdot \Lambda_j} = \min\left( \begin{array}{c} 2M_{ws_i \cdot \Lambda_i} + 2M_{ws_i s_j \cdot \Lambda_j} \\ M_{w \cdot \Lambda_j} + 2M_{ws_i s_j \cdot \Lambda_i} \end{array} \right).$$

(2.1.11)
We omit the tropical Plücker relations for the case \(a_{ij}a_{ji} = 3\), since we do not use them in this paper.

We say that an element \(M_\bullet \in \mathcal{E}\) satisfies the tropical Plücker relations if it satisfies the tropical Plücker relation at \((w, i, j)\) for each \(w \in W\) and \(i, j \in I\) such that \(ws_i > w\), \(ws_j > w\), and \(i \neq j\).

**Definition 2.1.3.** An element \(M_\bullet = (M_\gamma)_{\gamma \in \Gamma} \in \mathcal{E}\) is called a Berenstein-Zelevinsky (BZ for short) datum if \(M_\gamma \in \mathbb{Z}\) for all \(\gamma \in \Gamma\), and if it satisfies the tropical Plücker relations. In this case, the pseudo-Weyl polytope \(P(M_\bullet)\) with edge datum \(M_\bullet\) is called a Mirković-Vilonen (MV for short) polytope for \(g\).

Let \(\mathcal{E}_{MV}\) denote the subset of \(\mathcal{E}\) consisting of all BZ data, and \(\mathcal{V}_{MV}\) the corresponding subset of \(\mathcal{V}\) under the bijection \(D : \mathcal{V} \to \mathcal{E}\). We set

\[
\mathcal{M}_{\mathbb{V}} := \{P(M_\bullet) \mid M_\bullet \in \mathcal{E}_{\mathbb{V}}\} = \{P(\mu_\bullet) \mid \mu_\bullet \in \mathcal{V}_{\mathbb{V}}\} \subset \mathcal{P}.
\]

**Remark 2.1.4.** If \(\mu_\bullet = (\mu_w)_{w \in W} \in \mathcal{V}\) corresponds to \(M_\bullet = (M_\gamma)_{\gamma \in \Gamma} \in \mathcal{E}\) under the bijection \(D : \mathcal{V} \to \mathcal{E}\), then, by (2.1.2)

\[
\mu_w \in \mathfrak{h}_\mathbb{Z} \text{ for all } w \in W \iff M_\gamma \in \mathbb{Z} \text{ for all } \gamma \in \Gamma.
\]

Hence, if \(\mu_\bullet = (\mu_w)_{w \in W} \in \mathcal{V}_{\mathbb{V}}\), then \(\mu_w \in \mathfrak{h}_\mathbb{Z}\) for all \(w \in W\).

Now, let \(\mathcal{B}\) denote the canonical basis of the negative part \(U_q^-(g^\vee)\) of the quantized universal enveloping algebra \(U_q(g^\vee)\) associated to the (Langlands) dual Lie algebra \(g^\vee\) (see [L1, Part 4]). For each reduced word \(i = (i_1, i_2, \ldots, i_m)\) for the longest element \(w_0 \in W\), where \(m\) denotes the length of \(w_0\), there exists a bijection \(b_i : \mathbb{Z}_{\geq 0}^m \to \mathcal{B}\), which is called a Lusztig parametrization of \(\mathcal{B}\) (see [L3, Proposition 8.2]). Also, by [Kam1, Theorem 7.1], there exists a bijection \(\psi_1 : \mathcal{M}_{\mathbb{V}} \to \mathbb{Z}_{\geq 0}^m\) given by: \(\psi_1(P(\mu_\bullet)) = (L_1, L_2, \ldots, L_m)\), where the \(L_k \in \mathbb{Z}_{\geq 0}, 1 \leq k \leq m,\) are determined via the length formula (see (2.1.4)): \(\mu_{w_k^1} - \mu_{w_{k-1}^1} = L_k w_{k-1}^i \cdot h_{i_k}\), with \(w_k^1 := s_{i_1}s_{i_2} \cdots s_{i_k}\), for \(1 \leq k \leq m\). Furthermore, we know from [Kam1, Theorem 7.2] that there exists a bijection \(\Psi' : \mathcal{M}_{\mathbb{V}} \to \mathcal{B}\) such that \(\Psi' = b_i \circ \psi_1\) holds for all reduced words \(i\) for \(w_0\). Thus, we define a bijection \(\Psi : \mathcal{M}_{\mathbb{V}} \to \mathcal{B}(\infty)\) to be the composition of the bijection \(\Psi' : \mathcal{M}_{\mathbb{V}} \to \mathcal{B}\) with the canonical bijection from the canonical basis \(\mathcal{B}\) onto the crystal basis \(\mathcal{B}(\infty)\) for the negative part \(U_q^-(g^\vee)\).

We endow \(\mathcal{M}_{\mathbb{V}}\) with a crystal structure (due to Lusztig and Berenstein-Zelevinsky) for \(g^\vee\) through the bijection \(\Psi : \mathcal{M}_{\mathbb{V}} \to \mathcal{B}(\infty)\) above so that \(\Psi : \mathcal{M}_{\mathbb{V}} \to \mathcal{B}(\infty)\) is an isomorphism of crystals for \(g^\vee\). Let us recall from [Kam2, §§3.5 and 3.6] a description of this crystal structure.
on $\mathcal{MV}$. Let $P = P(\mu_\ast) \in \mathcal{MV}$ be an MV polytope with GGMS datum $\mu_\ast = (\mu_w)_{w \in W} \in \mathcal{V}_{\mathcal{MV}}$. The weight $\text{wt}(P)$ of $P$ is, by definition, equal to the vertex $\mu_e \in \sum_{j \in I} \mathbb{Z}_{\leq 0} h_j = \left( \sum_{j \in I} \mathbb{R}_{\leq 0} h_j \right) \cap \mathfrak{h}_\mathbb{Z}$. For each $j \in I$, let $f_j$ (resp., $e_j$) denote the lowering (resp., raising) Kashiwara operator on $\mathcal{MV}$. Then, $e_j P$ and $f_j P$ for each $j \in I$ are given as follows (see [Kam2, Theorem 3.5]). If $\mu_e = \mu_{s_j}$, then $e_j P = 0$, where $0$ is an additional element, which is not contained in $\mathcal{MV}$. Otherwise, $e_j P$ is a unique MV polytope $P(\mu'_e) \in \mathcal{MV}$ with GGMS datum $\mu'_e = (\mu'_w)_{w \in W}$ such that $\mu'_e = \mu_e + h_j$, and $\mu'_w = \mu_w$ for all $w \in W$ with $s_j w < w$. Similarly, $f_j P$ is a unique MV polytope $P(\mu'_e) \in \mathcal{MV}$ with GGMS datum $\mu'_e = (\mu'_w)_{w \in W}$ such that $\mu'_e = \mu_e - h_j$, and $\mu'_w = \mu_w$ for all $w \in W$ with $s_j w < w$. Note that since $s_j w_0 < w_0$ for all $j \in I$, $\mu_{w_0} = 0$ implies $\mu'_{w_0} = 0$. It is understood that $e_j 0 = f_j 0 = 0$. In addition, we set $\varepsilon_j(P) := \max \{ e_j^k P \mid e_j^k P = 0 \}$ and $\varphi_j(P) := \langle \alpha_j, \text{wt}(P) \rangle + \varepsilon_j(P)$.

Remark 2.1.5. Define an element $\mu_0^0 = (\mu_w)_{w \in W}$ of $\mathcal{V}$ by: $\mu_w = 0 \in \mathfrak{h}$ for all $w \in W$. It is obvious that $\mu_0^0 \in \mathcal{V}$ is contained in $\mathcal{V}_{\mathcal{MV}}$, and the weight of the MV polytope $P^0 := P(\mu_0^0) \in \mathcal{MV}$ is equal to $0 \in \mathfrak{h}_\mathbb{Z}$. Therefore, under the isomorphism $\Psi : \mathcal{MV} \cong \mathcal{B}(\infty)$ of crystals for $\mathfrak{g}^\vee$, the MV polytope $P^0 \in \mathcal{MV}$ is sent to the element $u_\infty \in \mathcal{B}(\infty)$ corresponding to the identity element $1 \in U_q^-(\mathfrak{g}^\vee)$.

2.2 Transition map between Lusztig parametrizations. In this subsection, we keep the notation and assumptions of [2.1]. For two reduced words $i$ and $i'$ for the longest element $w_0 \in W$ of length $m$, we define the transition map $R_i^m : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{Z}_{\geq 0}^m$ between Lusztig parametrizations by: $R_i^m = b_{i'}^{-1} \circ b_i$. Note that the transition map $R_i^m : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ is identical to the bijection $\psi_i \circ \psi_i^{-1} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{Z}_{\geq 0}^m$ since $b_i \circ \psi_i = \psi_i \circ \psi_i (= \Psi')$.

In this subsection, we briefly review the theory of “geometric lifting” of the transition map between Lusztig parametrizations of the canonical basis, which plays a key role in our proof of Proposition 2.5.3 below. Let $G = G(\mathbb{C})$ be a connected, simply-connected, semisimple algebraic group (or rather, Lie group) over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. For $j \in I$, we denote by $x_j(t)$ (resp., $y_j(t)$), $t \in \mathbb{C}$, the one-parameter subgroup of $G$ given by: $x_j(t) = \exp(t x_j)$ (resp., $y_j(t) = \exp(t y_j)$) for $t \in \mathbb{C}$, where $\exp : \mathfrak{g} \rightarrow G$ denotes the exponential map. Now, let $N_{\geq 0}$ denote the multiplicative semigroup generated by all $x_j(t)$ for $j \in I$ and $t \geq 0$, and set $N_{>0} := N_{\geq 0} \cap (B_- w_0 B_-)$, where $B_-$ is the Borel subgroup of $G$ generated by all $y_j(t) = \exp(t y_j)$ for $j \in I$ and $t \in \mathbb{C}$, together with the maximal torus $T$ of $G$ with Lie algebra $\mathfrak{h}$. Each reduced word $i = (i_1, i_2, \ldots, i_m)$ for $w_0$ gives rise to a bijection $x_i : \mathbb{R}_{\geq 0}^m \rightarrow N_{>0}$ by:

$$x_i(t_1, t_2, \ldots, t_m) = x_{i_1}(t_1)x_{i_2}(t_2) \cdots x_{i_m}(t_m)$$

for $(t_1, t_2, \ldots, t_m) \in \mathbb{R}_{>0}^m$ (see [L2]). The following is one of the main results of [BZ2] (for
the “tropicalization” procedure, we refer the reader to [BFZ §2.1, BK §2.4, and also NY §1.3].

**Theorem 2.2.1 (BFZ Theorem 5.2).** Let $i, i'$ be two reduced words for $w_0 \in W$.

1. Each component of the transition map $R_1^i(t_1, t_2, \ldots, t_m) := x_i^{-1} \circ x_i : \mathbb{R}_0^m \to \mathbb{R}_0^m$ is a subtraction-free rational expression in $t_1, t_2, \ldots, t_m$.

2. Each component of the transition map $R_1^{i'} = b_{i'}^{-1} \circ b_i : \mathbb{Z}_0^m \to \mathbb{Z}_0^m$ is the tropicalization of the corresponding component of $R_1^i(t_1, t_2, \ldots, t_m)$.

**Remark 2.2.2.** For later use, we record explicit formulas for the transition map $R_i^i : \mathbb{R}_0^d \to \mathbb{R}_0^d$ from [BZ1, Theorem 3.1], where $i$ and $i'$ have the form $i = (i, j, i, \ldots), i' = (j, i, j, \ldots)$ of length $d$. We use the notation $R_i^i(t_1, t_2, \ldots, t_d) = (t_1', t_2', \ldots, t_d')$. Note that explicit formulas for the transition map $R_i^i : \mathbb{Z}_0^d \to \mathbb{Z}_0^d$ are also obtained from these formulas through the tropicalization procedure by Theorem 2.2.1(2).

1. If $a_{ij} = a_{ji} = 0$, then $d = 2$ and $t_1' = t_2 = t_1$.

2. If $a_{ij} = a_{ji} = -1$, then $d = 3$ and

$$t_1' = \frac{t_2t_3}{\pi}, \quad t_2' = t_1 + t_3, \quad t_3' = \frac{t_1t_2}{\pi},$$

where $\pi = t_1 + t_3$.

3. If $a_{ij} = -1, a_{ji} = -2$, then $d = 4$ and

$$t_1' = \frac{t_2t_3t_4}{\pi_1}, \quad t_2' = \frac{\pi_2^2}{\pi_1}, \quad t_3' = \frac{\pi_2}{\pi_1}, \quad t_4' = \frac{t_1t_2t_3}{\pi_1},$$

where $\pi_1 = t_1t_2 + (t_1 + t_3)t_4$, $\pi_2 = t_1(t_2 + t_4)^2 + t_3t_4^2$.

4. If $a_{ij} = -2, a_{ji} = -1$, then $d = 4$ and

$$t_1' = \frac{t_2t_3^2t_4}{\pi_2}, \quad t_2' = \frac{\pi_2}{\pi_1}, \quad t_3' = \frac{\pi_2}{\pi_1}, \quad t_4' = \frac{t_1t_2t_3}{\pi_1},$$

where $\pi_1 = t_1t_2 + (t_1 + t_3)t_4$, $\pi_2 = t_1^2t_2 + (t_1 + t_3)^2t_4$.

### 2.3 Diagram automorphism for $A_\ell$

For the remainder of this paper, we assume that $\mathfrak{g}$ is of type $A_\ell, \ell \geq 3$, and $I := \{1, 2, \ldots, \ell\}$. Let $\omega : I \to I$ be the Dynkin diagram automorphism of order 2 given by: $\omega(j) = \ell - j + 1$ for $j \in I$. Then, the $\omega : I \to I$ induces a Lie algebra automorphism $\omega \in \text{Aut}(\mathfrak{g})$ of order 2 such that $\omega(x_j) = x_{\omega(j)}, \omega(y_j) = y_{\omega(j)}, \omega(h_j) = h_{\omega(j)}$ for $j \in I$. Note that the Cartan subalgebra $\mathfrak{h}$ is stable under $\omega \in \text{Aut}(\mathfrak{g})$, and hence induces $\omega \in \text{GL}(\mathfrak{h}^*)$ by: $\langle \omega(\lambda), h \rangle = \langle \lambda, \omega(h) \rangle$ for $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$. We set

$$\mathfrak{g}^\omega := \{ x \in \mathfrak{g} \mid \omega(x) = x \} \quad \text{and} \quad \mathfrak{h}^\omega := \{ h \in \mathfrak{h} \mid \omega(h) = h \}.$$
Furthermore, the $\omega : I \to I$ induces a group automorphism $\omega \in \text{Aut}(W)$ of order 2 such that $\omega(s_i) = s_{\omega(i)}$ for $i \in I$. We set $W^\omega := \{w \in W \mid \omega(w) = w\}$.

Remark 2.3.1. (1) We see easily from the definition of $\omega \in \text{Aut}(W)$ that if $w > w'$, then $\omega(w) > \omega(w')$ for $w, w' \in W$, and $\ell(\omega(w)) = \ell(w)$ for $w \in W$, where $\ell : W \to \mathbb{Z}_{\geq 0}$ denotes the length function on $W$. In particular, we have $\omega(e) = e$ and $\omega(w_0) = w_0$.

(2) It follows immediately from the definition of $\omega \in \text{GL}(\mathfrak{h}^*)$ that $\omega(\Lambda_j) = \Lambda_{\omega(j)}$ and $\omega(\alpha_j) = \alpha_{\omega(j)}$ for $j \in I$.

(3) It is easy to show that

\begin{align*}
\omega(w\lambda) &= \omega(w)\omega(\lambda) \quad \text{for } w \in W \text{ and } \lambda \in \mathfrak{h}^*, \quad (2.3.1) \\
\omega(wh) &= \omega(w)\omega(h) \quad \text{for } w \in W \text{ and } h \in \mathfrak{h}. \quad (2.3.2)
\end{align*}

In particular, it follows from (2.3.2) that $\mathfrak{h}_\omega \subset \mathfrak{h}$ is stable under the action of $W^\omega \subset W$.

(4) It follows from part (2) and (2.3.1) that $\omega(w\Lambda_i) = \omega(w)\Lambda_{\omega(i)}$ for $w \in W$ and $i \in I$. Therefore, the set $\Gamma$ is stable under the action of $\omega \in \text{GL}(\mathfrak{h}^*)$.

(5) We see easily that $h \geq_w h'$ if and only if $\omega(h) \geq_{\omega(w)} \omega(h')$ for $w \in W$ and $h, h' \in \mathfrak{h}_\mathbb{R}$.

In the following, we assume that $\mathfrak{g}$ is either of type $A_\ell$ with $\ell = 2n - 1$, $n \in \mathbb{Z}_{\geq 2}$, or of type $A_\ell$ with $\ell = 2n$, $n \in \mathbb{Z}_{\geq 2}$. If $\ell = 2n - 1$, $n \in \mathbb{Z}_{\geq 2}$, then we know (see, for example, [Kac §8.3]) that the fixed point subalgebra $\mathfrak{g}^\omega$ is the finite-dimensional simple Lie algebra of type $C_n$ (see the figure below); the Cartan subalgebra of $\mathfrak{g}^\omega$ is $\mathfrak{h}^\omega$, and the Chevalley generators $\{x_j^\omega, y_j^\omega, h_j^\omega \mid 1 \leq j \leq n\}$ of $\mathfrak{g}^\omega$ are as follows:

\begin{align*}
x_j^\omega &= x_j + x_{\omega(j)} \quad \text{for } 1 \leq j \leq n - 1, \\
y_j^\omega &= y_j + y_{\omega(j)} \quad \text{for } 1 \leq j \leq n - 1, \\
h_j^\omega &= h_j + h_{\omega(j)} \quad \text{for } 1 \leq j \leq n - 1,
\end{align*}

(2.3.3)

If $\ell = 2n$, $n \in \mathbb{Z}_{\geq 2}$, then we know (see, for example, [Kac §8.3]) that the fixed point subalgebra $\mathfrak{g}^\omega$ is the finite-dimensional simple Lie algebra of type $B_n$ (see the figure below):
the Cartan subalgebra of $\mathfrak{g}^\omega$ is $\mathfrak{h}^\omega$, and the Chevalley generators $\{x_j^\omega, y_j^\omega, h_j^\omega \mid 1 \leq j \leq n\}$ of $\mathfrak{g}^\omega$ are as follows:

$$
\begin{align*}
    x_j^\omega &= x_j + x_{\omega(j)} \quad \text{for} \ 1 \leq j \leq n-1, \\
    y_j^\omega &= y_j + y_{\omega(j)} \quad \text{for} \ 1 \leq j \leq n-1, \\
    h_j^\omega &= h_j + h_{\omega(j)} \quad \text{for} \ 1 \leq j \leq n-1, \\
    x_n^\omega &= \sqrt{2}(x_n + x_{\omega(n)}), \\
    y_n^\omega &= \sqrt{2}(y_n + y_{\omega(n)}), \\
    h_n^\omega &= 2(h_n + h_{\omega(n)}). 
\end{align*} 
$$

(2.3.4)

Let $\tilde{\mathfrak{A}} = (a_{ij})_{i,j \in \tilde{I}}$ denote the Cartan matrix of $\mathfrak{g}^\omega$, with index set $\tilde{I} := \{1, 2, \ldots, n\}$. Let $\tilde{\mathfrak{W}} = \langle \tilde{s}_i \mid i \in \tilde{I} \rangle$ be the Weyl group of $\mathfrak{g}^\omega$, where $\tilde{s}_i$, $i \in \tilde{I}$, are the simple reflections, and let $\tilde{e}$, $\tilde{w}_0 \in \tilde{\mathfrak{W}}$ denote the unit element and the longest element of $\tilde{\mathfrak{W}}$, respectively. Set

$$
\tilde{\Gamma} := \{ \tilde{w} \cdot \tilde{\Lambda}_i \mid \tilde{w} \in \tilde{\mathfrak{W}}, i \in \tilde{I} \},
$$

where $\tilde{\Lambda}_i \in (\mathfrak{h}^\omega)^*$, $i \in \tilde{I}$, are the fundamental weights for $\mathfrak{g}^\omega$ given by: $\tilde{\Lambda}_i = a_i \Lambda_i|_{\mathfrak{h}^\omega}$ for $i \in \tilde{I}$, with

$$
a_i := \begin{cases} 
    \frac{1}{2} & \text{if } \ell = 2n, \ n \in \mathbb{Z}_{\geq 2}, \ \text{and } i = n, \\
    1 & \text{otherwise.} 
\end{cases} 
$$

(2.3.5)

We define $\tilde{\mathfrak{V}}$ (resp., $\tilde{\mathfrak{E}}$) for $\mathfrak{g}^\omega$ in the same manner as we defined $\mathfrak{V}$ (resp., $\mathfrak{E}$) for $\mathfrak{g}$, and denote by $\tilde{\mathcal{P}}$ the set of pseudo-Weyl polytopes $\tilde{\mathcal{P}}(\tilde{\mu}_\bullet) \subset \mathfrak{h}^\omega \cap \mathfrak{h}_\mathbb{R}$ with GGMS datum $\tilde{\mu}_\bullet = (\tilde{\mu}_\tilde{w})_{\tilde{w} \in \tilde{\mathfrak{W}}} \in \tilde{\mathfrak{V}}$. Also, we define a bijection $\tilde{D} : \tilde{\mathfrak{V}} \rightarrow \tilde{\mathfrak{E}}$ as in (2.11) if $\tilde{D}(\tilde{\mu}) = \tilde{M}_\bullet = (\tilde{M}_\tilde{\gamma})_{\tilde{\gamma} \in \tilde{\Gamma}} \in \tilde{\mathfrak{E}}$, then

$$
\tilde{\mathcal{P}}(\tilde{\mu}_\bullet) = \{ h \in \mathfrak{h}^\omega \cap \mathfrak{h}_\mathbb{R} \mid h \geq_{\tilde{\mathfrak{W}}} \tilde{\mu}_\tilde{w} \text{ for all } \tilde{w} \in \tilde{\mathfrak{W}} \} 
$$

$$
= \{ h \in \mathfrak{h}^\omega \cap \mathfrak{h}_\mathbb{R} \mid (\tilde{\gamma}, h) \geq \tilde{M}_\tilde{\gamma} \text{ for all } \tilde{\gamma} \in \tilde{\Gamma} \}, 
$$

(2.3.6)

where the partial ordering $\geq_{\tilde{\mathfrak{W}}}$ on $\mathfrak{h}^\omega \cap \mathfrak{h}_\mathbb{R}$ for each $\tilde{w} \in \tilde{\mathfrak{W}}$ is defined by: $h \geq_{\tilde{\mathfrak{W}}} h'$ if $\tilde{w}^{-1} \cdot h - \tilde{w}^{-1} \cdot h' \in \sum_{j \in \tilde{I}} \mathbb{R}_{\geq 0} h_j^\omega$. Now, let $\tilde{\mathfrak{E}}_{\text{MV}}$ denote the subset of $\tilde{\mathfrak{E}}$ consisting of all elements (called BZ data for $\mathfrak{g}^\omega$) $\tilde{M}_\bullet = (\tilde{M}_\tilde{\gamma})_{\tilde{\gamma} \in \tilde{\Gamma}} \in \tilde{\mathfrak{E}}$, with $\tilde{M}_\tilde{\gamma} \in \mathbb{Z}$ for $\tilde{\gamma} \in \tilde{\Gamma}$, which satisfy
the tropical Plücker relation at \((\hat{w}, i, j)\) for each \(\hat{w} \in \hat{W}\) and \(i, j \in \hat{I}\) such that \(\hat{w}s_i > \hat{w}, \hat{w}s_j > \hat{w}\), and \(i \neq j\), where \(>\) denotes the (weak) Bruhat ordering on \(\hat{W}\). Also, let \(\hat{V}_{MV}\) denote the subset of \(\hat{V}\) corresponding to \(\hat{E}_{MV}\) under the bijection \(\hat{D} : \hat{V} \rightarrow \hat{E}\). Set
\[
\hat{M}_V := \{ \hat{P}(\hat{\mathcal{M}}) | \hat{\mathcal{M}} \in \hat{E}_{MV} \} = \{ \hat{P}(\hat{\mu}_\bullet) | \hat{\mu}_\bullet \in \hat{V}_{MV} \},
\]
and call an element of \(\hat{M}_V\) an MV polytope for \(\hat{g}^\omega\). We endow \(\hat{M}_V\) with a crystal structure in the same manner as we did for \(M_V\), so that we have an isomorphism of crystals \(\hat{\Psi} : \hat{M}_V \sim \hat{B}(\infty)\), where \(\hat{B}(\infty)\) denotes the crystal basis for the negative part \(U_q^-((\hat{g}^\omega)^\vee)\) of the quantized universal enveloping algebra \(U_q((\hat{g}^\omega)^\vee)\) associated to the (Langlands) dual Lie algebra \((\hat{g}^\omega)^\vee\) of \(g^\omega\). For each \(j \in \hat{I}\), we denote by \(\hat{f}_j\) (resp., \(\hat{e}_j\)) the lowering (resp., raising) Kashiwara operator on the crystal \(\hat{M}_V\). Let \(\hat{u}_\infty \in \hat{B}(\infty)\) denote the element of \(\hat{B}(\infty)\) corresponding to the identity element \(1 \in U_q^-((\hat{g}^\omega)^\vee)\), and \(\hat{P}_0 \in \hat{M}_V\) the MV polytope which is sent to \(\hat{u}_\infty\) under the isomorphism \(\hat{\Psi} : \hat{M}_V \sim \hat{B}(\infty)\) (see Remark [2.1.5]).

It is well-known (for a proof, see, e.g., [FRS Corollary 3.4]) that there exists a group isomorphism \(\Theta : \hat{W} \sim W^\omega\) such that \(\Theta(\hat{s}_i) = s_i^\omega\) for all \(i \in \hat{I}\), where
\[
s_i^\omega := \begin{cases} s_is_{\omega(i)} = s_{\omega(i)}s_i & \text{if } 1 \leq i \leq n - 1, \\ s_n & \text{if } \ell = 2n - 1, n \in \mathbb{Z}_{\geq 2}, \text{ and } i = n, \\ s_ns_{\omega(n)}s_n = s_{\omega(n)}s_n s_{\omega(n)} & \text{if } \ell = 2n, n \in \mathbb{Z}_{\geq 2}, \text{ and } i = n. \end{cases} \tag{2.3.7}
\]

**Remark 2.3.2.** (1) Recall that \(\mathfrak{h}^\omega\) is stable under the action of \(W^\omega\) (see Remark 2.3.1(3)), and that \(\mathfrak{h}^\omega\) is the Cartan subalgebra of \(\hat{g}^\omega\). It is easy to check that
\[
\Theta(\hat{w}) \cdot h = \hat{w} \cdot h \quad \text{for all } \hat{w} \in \hat{W} \text{ and } h \in \mathfrak{h}^\omega. \tag{2.3.8}
\]

(2) It follows from (2.3.8) that for \(h, h' \in \mathfrak{h}^\omega\) and \(\hat{w} \in \hat{W}\),
\[
h \geq_{\hat{w}} h' \quad \text{if and only if } h \geq_{\Theta(\hat{w})} h'. \tag{2.3.9}
\]

(3) Let \(\hat{w} \in \hat{W}\), and set \(w := \Theta(\hat{w}) \in W^\omega\). We deduce from [NS1 Lemma 3.2.1] that for each \(j \in \hat{I}\),
\[
\hat{s}_j \hat{w} < \hat{w} \iff s_j^\omega w < w \iff s_j w < w \text{ and } s_{\omega(j)} w < w.
\]

### 2.4 Action of the diagram automorphism \(\omega\) on \(M_V\). We keep the notation and assumptions of 2.3. For an element \(\mu_\bullet = (\mu_w)_{w \in W} \in V\), we define \(\omega(\mu_\bullet)\) to be a collection \((\mu_w')_{w \in W}\) of elements in \(\mathfrak{h}_\mathbb{R}\) given by: \(\mu_w' = \omega(\mu_{\omega(w)})\) for \(w \in W\). Then, using Remark 2.3.1(1) and (5), we can easily check that \(\omega(\mu_\bullet) \in V\) for all \(\mu_\bullet \in V\).
Remark 2.4.1. Let $\mu_\bullet = (\mu_w)_{w \in W} \in \mathcal{V}$. Set $(M_\gamma)_{\gamma \in \Gamma} := D(\mu_\bullet) \in \mathcal{E}$ and $(M'_\gamma)_{\gamma \in \Gamma} := D(\omega(\mu_\bullet)) \in \mathcal{E}$. Then we have $M'_\gamma = M_{\omega(\gamma)}$ for all $\gamma \in \Gamma$. Indeed, using Remark 2.3.1(4), we have

$$M'_{w \cdot \Lambda_i} = \langle w \cdot \Lambda_i, \mu'_w \rangle = \langle w \cdot \Lambda_i, \omega^{-1}(\mu_{\omega(w)}) \rangle = \langle \omega(w) \cdot \Lambda_i, \mu_{\omega(w)} \rangle = M_{\omega(w) \cdot \Lambda_i} = M_{\omega(w) \cdot \Lambda_i}$$

for each $w \in W$ and $i \in I$.

Now we set

$$\mathcal{V}^\omega := \{ \mu_\bullet \in \mathcal{V} \mid \omega(\mu_\bullet) = \mu_\bullet \} \quad \text{and} \quad \mathcal{E}^\omega := D(\mathcal{V}^\omega) \subset \mathcal{E}.$$  

The next lemma follows immediately from the definition of the action of $\omega$ on $\mathcal{V}$ and Remark 2.4.1.

Lemma 2.4.2. (1) Let $\mu_\bullet = (\mu_w)_{w \in W} \in \mathcal{V}$. Then, $\mu_\bullet \in \mathcal{V}^\omega$ if and only if $\omega(\mu_w) = \mu_{\omega(w)}$ for all $w \in W$. In particular, if $\mu_\bullet = (\mu_w)_{w \in W} \in \mathcal{V}^\omega$, then $\mu_w \in \mathfrak{h}^\omega$ for all $w \in W^\omega$.

(2) Let $M_\bullet = (M_\gamma)_{\gamma \in \Gamma} \in \mathcal{E}$. Then, $M_\bullet \in \mathcal{E}^\omega$ if and only if $M_{\omega(\gamma)} = M_\gamma$ for all $\gamma \in \Gamma$.

Let $P = P(\mu_\bullet)$ be a pseudo-Weyl polytope with GGMS datum $\mu_\bullet = (\mu_w)_{w \in W}$. Then it follows from (2.1.1) and Remark 2.3.1(5) that the image $\omega(P) = \{ \omega(h) \mid h \in P \}$ of $P$ (as a set) under $\omega \in \text{GL}(\mathfrak{h})$ is identical to the pseudo-Weyl polytope $P(\omega(\mu_\bullet)) \in \mathcal{P}$. For this reason, we define an action of $\omega$ on the set $\mathcal{P} = \{ P(\mu_\bullet) \mid \mu_\bullet \in \mathcal{V} \}$ of pseudo-Weyl polytopes by: $\omega(P(\mu_\bullet)) = P(\omega(\mu_\bullet))$ for $\mu_\bullet \in \mathcal{V}$. Since $\omega(P(\mu_\bullet)) = P(\omega(\mu_\bullet))$ for $\mu_\bullet \in \mathcal{V}$, it follows that $\omega(P(\mu_\bullet)) = P(\mu_\bullet)$ if and only if $\omega(\mu_\bullet) = \mu_\bullet$. Therefore, we have

$$\mathcal{P}^\omega := \{ P \in \mathcal{P} \mid \omega(P) = P \} = \{ P(\mu_\bullet) \mid \mu_\bullet \in \mathcal{V}^\omega \} = \{ P(M_\bullet) \mid M_\bullet \in \mathcal{E}^\omega \}.$$  

Using Remark 2.4.1 along with Remark 2.3.1(1),(4), we can check that the subset $\mathcal{V}^{\mathcal{M}V}$ of $\mathcal{V}$ is stable under the action of $\omega$ on $\mathcal{V}$, which implies that the set $\mathcal{M}V \subset \mathcal{P}$ of MV polytopes for $\mathfrak{g}$ is stable under the action of $\omega$ on $\mathcal{P}$. We set

$$\mathcal{V}^{\mathcal{M}V} := \mathcal{V}^{\mathcal{M}V} \cap \mathcal{V}^\omega \quad \text{and} \quad \mathcal{E}^{\mathcal{M}V} := \mathcal{E}^{\mathcal{M}V} \cap \mathcal{E}^\omega = D(\mathcal{V}^{\mathcal{M}V})$$

$$\mathcal{M}V^{\omega} := \mathcal{M}V \cap \mathcal{P}^\omega = \{ P(\mu_\bullet) \mid \mu_\bullet \in \mathcal{V}^{\mathcal{M}V} \} = \{ P(M_\bullet) \mid M_\bullet \in \mathcal{E}^{\mathcal{M}V} \}.$$  

2.5 MV polytopes for $\mathfrak{g}$ fixed by $\omega$ and MV polytopes for $\mathfrak{g}^\omega$. Recall that $\mathfrak{g}$ is of type $A_\ell$, $\ell \geq 3$. Namely, $\mathfrak{g}$ is either of type $A_\ell$ with $\ell = 2n - 1$, $n \in \mathbb{Z}_{\geq 2}$, or of type $A_\ell$ with
ℓ = 2n, n ∈ Z_{≥2}. If ℓ = 2n − 1, n ∈ Z_{≥2} (resp., ℓ = 2n, n ∈ Z_{≥2}), then g^w is of type C_n (resp., of type B_n).

For μ∗ = (μ_w)_{w ∈ W} ∈ V^w, we define Φ(μ∗) to be a collection (μ̂_w)_{w ∈ W} of elements in h^w ∩ h_R given by: μ̂_w = μ_{θ(w)} for w ∈ W. Using Remark 2.3.2(2), along with Lemma 2.4.2(1) and the fact that Θ(μ̂_0) = μ̂_0, we obtain the following lemma.

**Lemma 2.5.1.** We have Φ(μ∗) ∈ Ṽ for all μ∗ ∈ V^w.

**Remark 2.5.2.** Let μ∗ = (μ_w)_{w ∈ W} ∈ V^w, and set (μ̂_w)_{w ∈ W} := Φ(μ∗) ∈ Ṽ. Also, we set (M_γ)_{γ ∈ Γ} := D(μ∗) ∈ E^w and (M̂_γ)_{γ ∈ Γ} := Ḑ(Φ(μ∗)) ∈ ḅ. Then, for each w ∈ W and i ∈ I, we have M̂_w :∼ Λ_i = a_i M_{θ(w)} :∼ Λ_i, where a_i is as defined in (2.3.5). Indeed, we have

\[ M̂_w :∼ Λ_i = \langle w : Λ_i, w : Λ_i \rangle = \langle ∑_{i ∈ I} w_i : Λ_i, \rangle Λ_i \cdot w : Λ_i = \langle Λ_i, \Theta(μ̂_w) \cdot Λ_i \rangle \]

Therefore, noting that M̂_w :∼ Λ_i = a_i Λ_i | h^w, we obtain

\[ M̂_w :∼ Λ_i = \langle ∑_{i ∈ I} w_i : Λ_i, \rangle Λ_i \cdot w : Λ_i = \langle ∑_{i ∈ I} a_i Λ_i \cdot w : Λ_i \rangle = a_i Λ_i \cdot w : Λ_i = a_i M_{θ(w)} :∼ Λ_i. \]

By Lemma 2.5.1, we can define a map (also denoted by) Φ : P^w → Ḑ by: Φ(P(μ∗)) = Ḑ(Φ(μ∗)) for μ∗ ∈ V^w. If μ∗ = (μ_w)_{w ∈ W} ∈ V^w and Φ(μ∗) = (μ̂_w)_{w ∈ W} ∈ Ṽ, then it follows from (2.3.9) that

\[ Φ(μ∗) = \{ h ∈ h^w \mid h ≥_w μ̂_w \} \]

\[ = \{ h ∈ h^w \mid h ≥_w μ̂_w \} \]

\[ = \{ h ∈ h^w \mid h ≥_w μ_w \} \] for all w ∈ W^w. (2.5.1)

**Remark 2.5.3.** Let μ∗ = (μ_w)_{w ∈ W} ∈ V^w and Φ(μ∗) = (μ̂_w)_{w ∈ W} ∈ Ṽ. Then we see from (2.5.1) that P(μ∗) ∩ h^w ⊂ Φ(P(μ∗)). Also, since Φ(P(μ∗)) = Ḑ(Φ(μ∗)) is the convex hull in h^w ∩ h_R of the collection Φ(μ∗) = (μ̂_w)_{w ∈ W} (see Remark 2.1.2) and μ̂_w = μ_{θ(w)} ∈ P(μ∗) ∩ h^w for all w ∈ W, it follows that Φ(P(μ∗)) = Ḑ(Φ(μ∗)) ⊂ P(μ∗) ∩ h^w. Therefore, we conclude that Φ(P(μ∗)) = P(μ∗) ∩ h^w. In addition, if μ_w ∈ h^w for some w ∈ W, then μ_w is a vertex of the convex polytope P(μ∗) ∩ h^w = Φ(P(μ∗)), so that μ_w = μ̂_w = μ_{θ(w)} for some w ∈ W.

**Proposition 2.5.4.** We have Φ(μ∗) ∈ Ṽ_{MV} for all μ∗ ∈ V^w_{MV}.

The proof of this proposition will be given in 2.6. It follows from this proposition that Φ(MV^w) ⊂ M̂V. Hence the restriction of the map Φ : P^w → Ḑ to MV^w gives rise to a map Φ : MV^w → M̂V.
Now we define operators \( f_j^\omega \), \( j \in \hat{I} \), on \( \mathcal{MV} \) by:

\[
f_j^\omega = \begin{cases} 
  f_j f_{\omega(j)} & \text{if } 1 \leq j \leq n - 1, \\
  f_n & \text{if } \ell = 2n - 1, n \in \mathbb{Z}_{\geq 2}, \text{ and } j = n, \\
  f_n f_{\omega(n)}^2 f_n & \text{if } \ell = 2n, n \in \mathbb{Z}_{\geq 2}, \text{ and } j = n.
\end{cases}
\]  

(2.5.2)

Remark 2.5.5. Since \( \mathcal{MV} \) is isomorphic to \( B(\infty) \) as a crystal for \( \mathfrak{g}^\vee \), we deduce from [Kas] Proposition 7.4.1 that \( f_j f_{\omega(j)} = f_{\omega(j)} f_j \) if \( 1 \leq j \leq n - 1 \), and that \( f_n f_{\omega(n)}^2 f_n = f_{\omega(n)} f_n f_{\omega(n)} \) if \( \ell = 2n \), \( n \in \mathbb{Z}_{\geq 2} \).

Theorem 2.5.6. (1) The subset \( \mathcal{MV}^\omega \) of \( \mathcal{MV} \) is stable under the operators \( f_j^\omega \) for all \( j \in \hat{I} \).

(2) Each element \( P \in \mathcal{MV}^\omega \) is of the form \( P = f_{j_1}^\omega f_{j_2}^\omega \cdots f_{j_n}^\omega P^0 \) for some \( j_1, j_2, \ldots, j_k \in \hat{I} \).

(3) The map \( \Phi : \mathcal{MV}^\omega \to \hat{MV} \) is a unique bijection such that \( \Phi(P^0) = \hat{P}^0 \), and such that \( \Phi \circ f_j^\omega = \hat{f}_j \circ \Phi \) for all \( j \in \hat{I} \).

The proof of this theorem will be given in §2.7.

Remark 2.5.7. The existence of a bijection \( \mathcal{MV}^\omega \to \hat{MV} \) satisfying the conditions of part (3) of Theorem 2.5.6 follows immediately from [NS2] Theorem 3.4.1 (see also [L1] Theorem 14.4.9) for the case in which \( \mathfrak{g}^\vee \) is not of type \( A_{2n} \); note that the orbit Lie algebra associated to \( \mathfrak{g}^\vee \) is precisely the dual Lie algebra \( (\mathfrak{g}^\omega)^\vee \) of \( \mathfrak{g}^\omega \). However, for our purpose, we need a more explicit description of the bijection in terms of polytopes, such as the one given in this subsection.

2.6 Proof of Proposition 2.5.4. This subsection is devoted to the proof of Proposition 2.5.4. We keep the notation and assumptions of §2.5. We know from Lemma 2.5.1 that if \( \mu_\bullet = (\mu_w)_{w \in W} \in \mathcal{V}^\omega \), then \( \hat{\mu}_\bullet := \Phi(\mu_\bullet) = (\hat{\mu}_{\omega})_{\omega \in \hat{W}} \) is an element of \( \hat{\mathcal{V}} \). In this subsection, by setting \( \hat{M}_\bullet := \hat{D}(\hat{\mu}_\bullet) = (\hat{M}_{\hat{\gamma}})_{\hat{\gamma} \in \hat{\Gamma}} \in \hat{\mathcal{E}} \), we first prove that \( \hat{M}_{\hat{\gamma}} \in \mathbb{Z} \) for all \( \hat{\gamma} \in \hat{\Gamma} \), and then prove that \( \hat{M}_\bullet \) satisfies the tropical Plücker relations.

We begin with the following simple lemma.

Lemma 2.6.1. Let \( P = P(\mu_\bullet) \in \mathcal{MV} \) be an MV polytope with GGMS datum \( \mu_\bullet = (\mu_w)_{w \in W} \in \mathcal{V}_{\mathcal{MV}} \), and \( \mathbf{i} = (i_1, i_2, \ldots, i_m) \) a reduced word for \( w_0 \in W \). Then we have \( \psi_{\omega(\mathbf{i})}(\omega(P)) = \psi_1(P) \), where \( \omega(\mathbf{i}) := (\omega(i_1), \omega(i_2), \ldots, \omega(i_m)) \) is also a reduced word for \( w_0 \in W \).

Proof. If we write \( \psi_1(P) \in \mathbb{Z}_{\geq 0}^m \) as \( \psi_1(P) = (L_1, L_2, \ldots, L_m) \in \mathbb{Z}_{\geq 0}^m \), then by the definition, we have \( \mu_{w_{i_k}} - \mu_{w_{i_{k-1}}} = L_k w_{i_{k-1}} \cdot h_{i_k} \), with \( w_{i_k} = s_{i_1} s_{i_2} \cdots s_{i_k} \), for \( 1 \leq k \leq m \).
Similarly, if we write $\psi_{\omega(i)}(\omega(P)) \in \mathbb{Z}^m_{\geq 0}$ as $\psi_{\omega(i)}(\omega(P)) = (L'_1, L'_2, \ldots, L'_m) \in \mathbb{Z}^m_{\geq 0}$, and denote $\omega(\mu_\ast) \in \mathcal{V}_{\text{MV}}$ by $\mu'_\ast = (\mu'_w)_{w \in W}$, then we have $\mu'_{w_k^{(i)}} - \mu'_{w_{k-1}^{(i)}} = L'_k w_k^{(i)} \cdot h_{\omega(i_k)}$, with $w_k^{(i)} = s_{\omega(i_1)} s_{\omega(i_2)} \cdots s_{\omega(i_k)}$, for $1 \leq k \leq m$. Because $\mu'_{w_k^{(i)}} = \omega(\mu_{\omega(w_k^{(i)})}) = \omega(\mu_{w_k^{(i)})})$ for $1 \leq k \leq m$ by the definition of $\omega(\mu_\ast)$, we have

$$L'_k w_k^{(i)} \cdot h_{\omega(i_k)} = \mu'_{w_k^{(i)}} - \mu'_{w_{k-1}^{(i)}} = \omega(\mu_{w_k^{(i)}}) - \omega(\mu_{w_{k-1}^{(i)}}) = \omega(\mu_{w_k^{(i)}}) - \omega(\mu_{w_{k-1}^{(i)}}) = L_k w_k^{(i)} \cdot h_{\omega(i_k)} = L_k w_k^{(i)} \cdot h_{\omega(i_k)},$$

from which it follows that $L_k = L'_k$ for all $1 \leq k \leq m$. This proves the lemma. 

Let $P = P(\mu_\ast) \in \mathcal{M}_\text{MV}$ be an MV polytope with GGMS datum $\mu_\ast = (\mu_w)_{w \in W} \in \mathcal{V}_{\text{MV}}$. Then, by Lemma 2.6.1, we have $\psi_{\omega(i)}(\omega(P)) = \psi(P)$ for a reduced word $i$ for $w_0 \in W$. Since $\psi_{\omega(i)} : \mathcal{M}_\text{MV} \to \mathbb{Z}^m_{\geq 0}$ is a bijection, it follows that

$$\omega(P) = P \iff \psi_{\omega(i)}(P) = \psi(P). \quad (2.6.1)$$

Now we recall from [NSI, Lemma 3.2.1] that if $\hat{w}_0 = \hat{s}_{j_1} \hat{s}_{j_2} \cdots \hat{s}_{j_\tilde{m}}$, $j_1, j_2, \ldots, j_\tilde{m} \in I$, is a reduced decomposition of the longest element $\hat{w}_0$ of $\hat{W}$, then $w_0 = s_{j_1}^{s_{j_2}^{s_{j_2}} \cdots s_{j_\tilde{m}}}^{s_{j_\tilde{m}}} \in W$, is a reduced decomposition of the longest element $w_0$ of $W$, where $s_{j}^{s_{j'}}, j \in I$, are as defined in (2.3.7). Using this fact, to each reduced word $j = (j_1, j_2, \ldots, j_\tilde{m})$ for $\hat{w}_0 \in \hat{W}$, we associate a reduced word $i = (i_1, i_2, \ldots, i_m)$ for $w_0 \in W$ as follows. For each $1 \leq k \leq \tilde{m}$, we define elements $i_l^{(k)} \in I$, $1 \leq l \leq N_k$, where $N_k = \ell(s_{j_k}),$ by:

$$\begin{cases}
i_1^{(k)} = j_k, i_2^{(k)} = \omega(j_k), \text{ with } N_k = 2, \text{ if } 1 \leq j_k \leq n - 1, \\
i_1^{(k)} = j_k, \text{ with } N_k = 1, \text{ if } \ell = 2n - 1, n \in \mathbb{Z}_{\geq 2}, \text{ and } j_k = n, \\
i_1^{(k)} = j_k, i_2^{(k)} = \omega(j_k), i_3^{(k)} = j_k, \text{ with } N_k = 3, \text{ if } \ell = 2n, n \in \mathbb{Z}_{\geq 2}, \text{ and } j_k = n.
\end{cases}$$

Then we set

$$\begin{align*}
i = (i_1, i_2, \ldots, i_m) \\
:= (i_1^{(1)}, \ldots, i_1^{(N_1)}, i_1^{(2)}, \ldots, i_2^{(N_2)}, \ldots, i_1^{(m)}, \ldots, i_1^{(m)}) \in \mathbb{Z}^m_{\geq 0},
\end{align*}$$

and call it the canonical reduced word for $w_0 \in W$ associated to $j$. Recall that $\omega(i) = (\omega(i_1), \omega(i_2), \ldots, \omega(i_m))$ is also a reduced word for $w_0 \in W$.

**Proposition 2.6.2.** Let $P = P(\mu_\ast) \in \mathcal{M}_\text{MV}$ be an MV polytope with GGMS datum $\mu_\ast = (\mu_w)_{w \in W} \in \mathcal{V}_{\text{MV}}$. Let $j = (j_1, j_2, \ldots, j_\tilde{m})$ be a reduced word for $\hat{w}_0 \in \hat{W}$, and $i = (i_1, i_2, \ldots, i_m)$ be the canonical reduced word for $w_0 \in W$. If $\omega(i) = (\omega(i_1), \omega(i_2), \ldots, \omega(i_m))$, then $\psi_{\omega(i)}(\omega(P)) = \psi(P)$.
\((i_1, i_2, \ldots, i_m)\) the associated canonical reduced word for \(w_0 \in W\). If we write \(\psi_1(P) \in \mathbb{Z}_{\geq 0}^m\) as
\[
\psi_1(P) = (L_1, L_2, \ldots, L_m) = (L_1^{(1)}, \ldots, L_{N_1}^{(1)}, L_1^{(2)}, \ldots, L_{N_2}^{(2)}, \ldots, L_1^{(\tilde{m})}, \ldots, L_{N_{\tilde{m}}}^{(\tilde{m})}) \in \mathbb{Z}_{\geq 0}^m,
\]
then we have \(L_1^{(k)} = \cdots = L_{N_k}^{(k)}\) for all \(1 \leq k \leq \tilde{m}\).

**Proof.** We prove the equalities \(L_1^{(k)} = \cdots = L_{N_k}^{(k)}\) in the case that \(\ell = 2n, n \in \mathbb{Z}_{\geq 2}\), and \(j_k = n\) (hence \(N_k = 3\)); the proofs for the other cases are similar (or, even simpler). For simplicity of notation, we further assume that \(k = 1\) and \(n = 2\); we have
\[
j = (2, 1, 2, 1) \quad \text{and} \quad i = (2, 3, 2, 1, 4, 2, 3, 2, 1, 4),
\]
with \(\tilde{m} = 4\) and \(m = 10\), and \((L_1, L_2, L_3) = (L_1^{(1)}, L_2^{(1)}, L_3^{(1)})\). If we take a reduced word
\[
i' = (3, 2, 3, 1, 4, 2, 3, 2, 1, 4)
\]
for \(w_0 \in W\), then the bijection \(\psi_1 \circ \psi_1^{-1} : \mathbb{Z}_{\geq 0}^{10} \to \mathbb{Z}_{\geq 0}^{10}\) is identical to the transition map \(R_i' : \mathbb{Z}_{\geq 0}^{10} \to \mathbb{Z}_{\geq 0}^{10}\). Therefore, by setting
\[
\psi_1(P) = (L_1', L_2', \ldots, L_{10}') \in \mathbb{Z}_{\geq 0}^{10},
\]
we obtain from Remark 2.2.2 the following relations (note that \(a_{23} = a_{32} = -1\) in our case):
\[
L'_1 = L_2 + L_3 - p, \quad L'_2 = p,
L'_3 = L_1 + L_2 - p, \quad \text{where } p = \min(L_1, L_3), \quad (2.6.2)
L'_k = L_k \text{ for } 4 \leq k \leq 10.
\]

Also, since \(\omega(P) = P\) and \(\omega(i) = (3, 2, 3, 4, 1, 3, 2, 3, 4, 1)\), by setting
\[
\psi_{\omega(i)}(P) = (L_1'', L_2'', \ldots, L_{10}'') \in \mathbb{Z}_{\geq 0}^{10},
\]
we obtain from (2.6.1) the relation \(L''_k = L_k\) for all \(1 \leq k \leq 10\). Since \(L''_1 = L'_1, L''_2 = L'_2, L''_3 = L'_3\) by the definitions (see (2.1.4)), we have
\[
L_1 = L'_1, \quad L_2 = L'_2, \quad L_3 = L'_3. \quad (2.6.3)
\]

By combining (2.6.2) and (2.6.3), we get
\[
L_1 = L_2 + L_3 - p, \quad L_2 = p,
L_3 = L_1 + L_2 - p, \quad \text{where } p = \min(L_1, L_3).
\]
Hence we deduce that \(L_1 = L_3\), and then that \(L_2 = \min(L_1, L_3) = L_1\). This proves the proposition. \(\square\)
The argument in the proof of Proposition 2.6.2 also shows the following proposition.

Proposition 2.6.3. Let \( P = P(\mu_{\bullet}) \in \mathcal{MV} \) be an MV polytope with GGMS datum \( \mu_{\bullet} = (\mu_w)_{w \in W} \in \mathcal{V}_{\text{MV}} \). Let \( j = (j_1, j_2, \ldots, j_{\hat{m}}) \) be a reduced word for \( \hat{w}_0 \in \hat{W} \), and \( i = (i_1, i_2, \ldots, i_{\hat{m}}) \) the associated canonical reduced word for \( w_0 \in W \). We write \( \psi_1(P) \in \mathbb{Z}_0^m \) and \( \psi_\omega(i)(P) \in \mathbb{Z}_0^m \) as

\[
\psi_1(P) = (L_1, L_2, \ldots, L_m)
\]
\[
= (L_1^{(1)}, \ldots, L_N^{(1)}, L_1^{(2)}, \ldots, L_N^{(2)}, \ldots, L_1^{(\hat{m})}, \ldots, L_N^{(\hat{m})}) \in \mathbb{Z}_0^m.
\]
\[
\psi_\omega(i)(P) = (L_1, L_2, \ldots, L_m)
\]
\[
= (L_1^{(1)}, \ldots, L_N^{(1)}, L_1^{(2)}, \ldots, L_N^{(2)}, \ldots, L_1^{(\hat{m})}, \ldots, L_N^{(\hat{m})}) \in \mathbb{Z}_0^m.
\]

If \( L_1^{(k)} = L_2^{(k)} = \cdots = L_N^{(k)} \) for all \( 1 \leq k \leq \hat{m} \), then we have \( L_k = L_k' \) for all \( 1 \leq k \leq m \).

Corollary 2.6.4. Keep the notation and assumptions of Proposition 2.6.2. Let \( \hat{P} := \Phi(P(\mu_{\bullet})) \) be a pseudo-Weyl polytope with GGMS datum \( \hat{\mu} = \Phi(\mu_{\bullet}) = (\hat{\mu}_{\hat{w}})_{\hat{w} \in \hat{W}} \in \hat{\mathcal{V}} \). Then, for a reduced word \( j = (j_1, j_2, \ldots, j_{\hat{m}}) \) for \( \hat{w}_0 \in \hat{W} \), we have \( \hat{\mu}_{\hat{w}_1} - \hat{\mu}_{\hat{w}_{k-1}} = L_t^{(k)} \hat{w}_{k-1} \cdot h_{jk}^{(1)} \) for every \( 1 \leq l \leq N_k \), \( 1 \leq k \leq \hat{m} \), with \( \hat{w}_k^{(1)} := s_{j_1} s_{j_2} \cdots s_{j_k} \), \( 1 \leq k \leq \hat{m} \).

Proof. Again, we assume that \( \ell = 2n \), \( n \in \mathbb{Z}_0^2 \), \( j_k = n \), and further that \( k = 1 \) and \( n = 2 \). By the definition of \( \hat{\mu}_{\bullet} = \Phi(\mu_{\bullet}) \), we have \( \hat{\mu}_{\hat{w}_1} - \hat{\mu}_{\hat{w}_{k-1}} = \mu_{\Theta(\hat{w}_1)} - \mu_{\Theta(\hat{w}_{k-1})} \), where \( \Theta(\hat{w}_1) = \Theta(s_{j_1}) = s_{j_1}^2 = s_2 s_3 s_2 \) in our case. Also, recall that \( (L_1, L_2, L_3) = (L_1^{(1)}, L_2^{(1)}, L_3^{(1)}) \) are determined via the length formula:

\[
\begin{cases}
\mu_{s_2} - \mu_e = L_1 h_2, \\
\mu_{s_2 s_3} - \mu_{s_2} = L_2 s_2 \cdot h_3, \\
\mu_{s_2 s_3 s_2} - \mu_{s_2 s_3} = L_3 s_2 s_3 \cdot h_2.
\end{cases}
\]

Therefore, we have

\[
\hat{\mu}_{\hat{w}_1} - \hat{\mu}_{\hat{e}} = \mu_{s_2 s_3 s_2} - \mu_e
\]
\[
= (\mu_{s_2 s_3 s_2} - \mu_{s_2 s_3}) + (\mu_{s_2 s_3} - \mu_{s_2}) + (\mu_{s_2} - \mu_e)
\]
\[
= L_3 s_2 s_3 \cdot h_2 + L_2 s_2 \cdot h_3 + L_1 h_2
\]
\[
= L_3 h_3 + L_2 (h_2 + h_3) + L_1 h_2
\]
\[
= (L_1 + L_2) h_2 + (L_2 + L_3) h_3.
\]

Here we know from Proposition 2.6.2 that \( L_1 = L_2 = L_3 \). Hence we conclude that

\[
\hat{\mu}_{\hat{w}_1} - \hat{\mu}_{\hat{e}} = 2L_1 (h_2 + h_3) = L_1 h_2^{\omega} = L_2 h_2^{\omega} = L_3 h_2^{\omega}.
\]

This proves the corollary.
Let \( \mu_\bullet = (\mu_w)_{w \in \mathcal{W}} \in \mathcal{V}^\omega \), and set \( \hat{\mu}_\bullet = \Phi(\mu_\bullet) = (\hat{\mu}_w)_{\hat{w} \in \hat{\mathcal{W}}} \). Since \( \hat{\mu}_{\hat{w}_0} = \mu_{w_0} = 0 \), we can show that \( \hat{\mu}_\hat{w} \in \sum_{j \in \mathbb{Z}} \mathbb{Z}h_j^\mathbb{Z} \) for all \( \hat{w} \in \hat{\mathcal{W}} \) by repeated use of Corollary 2.6.4: take a reduced word \( j = (j_1, j_2, \ldots, j_m) \) for \( \hat{w}_0 \in \hat{\mathcal{W}} \) such that \( \hat{w}_k^1 = \hat{w} \) for some \( 0 \leq k \leq m \). Hence, for \( \hat{M}_\bullet = \hat{D}(\hat{\mu}_\bullet) = (\hat{M}_\gamma)_{\gamma \in \hat{\Gamma}} \), we have \( \hat{M}_\gamma \in \mathbb{Z} \) for all \( \gamma \in \hat{\Gamma} \).

Thus, it remains to prove that \( \hat{M}_\bullet = \hat{D}(\hat{\mu}_\bullet) = (\hat{M}_\gamma)_{\gamma \in \hat{\Gamma}} \) satisfies the tropical Plücker relations. We prove the tropical Plücker relation at \( \hat{w}, n - 1, n \) for \( \hat{w} \in \hat{\mathcal{W}} \) in the case \( \ell = 2n, n \in \mathbb{Z}_{\geq 2} \); note that \( \hat{a}_{n - 1, n} = -1 \) and \( \hat{a}_{n, n - 1} = -2 \). Since the proofs of the other tropical Plücker relations are similar (or, even simpler), we leave them to the reader. For simplicity of notation, we further assume that \( \hat{w} = \hat{e} \) and \( n = 2 \). Namely, we will prove

\[
\begin{align*}
\hat{M}_{\hat{s}_2 \hat{\Lambda}_2} + \hat{M}_{\hat{s}_1 \hat{s}_2 \hat{\Lambda}_2} + \hat{M}_{\hat{s}_1 \hat{\Lambda}_1} &= \min \left( \frac{2\hat{M}_{\hat{s}_1 \hat{s}_2 \hat{\Lambda}_2} + \hat{M}_{\hat{\Lambda}_1}}, \frac{2\hat{M}_{\hat{\Lambda}_2} + \hat{M}_{\hat{s}_1 \hat{s}_2 \hat{\Lambda}_2} + \hat{M}_{\hat{s}_1 \hat{\Lambda}_1}} \right), \tag{2.6.4} \\
\hat{M}_{\hat{s}_2 \hat{s}_1 \hat{\Lambda}_1} + 2\hat{M}_{\hat{s}_1 \hat{s}_2 \hat{\Lambda}_2} + \hat{M}_{\hat{s}_1 \hat{\Lambda}_1} &= \min \left( \frac{2\hat{M}_{\hat{\Lambda}_2} + 2\hat{M}_{\hat{s}_2 \hat{s}_1 \hat{\Lambda}_1}}, \frac{2\hat{M}_{\hat{s}_2 \hat{s}_1 \hat{s}_2 \hat{\Lambda}_2} + 2\hat{M}_{\hat{s}_1 \hat{\Lambda}_1}} \right). \tag{2.6.5}
\end{align*}
\]

(see (2.1.8) and (2.1.9)).

We consider reduced words \( j \) and \( j' \) for \( \hat{w}_0 \in \hat{\mathcal{W}} \) of the form

\[
\begin{align*}
j &= (j_1, j_2, j_3, j_4) := (1, 2, 1, 2), \quad j' &= (j'_1, j'_2, j'_3, j'_4) := (2, 1, 2, 1).
\end{align*}
\]

As in the proof of [Kam1, Proposition 5.4], we see by use of the length formula (2.1.4) that \( \hat{M}_\bullet = \hat{D}(\hat{\mu}_\bullet) = (\hat{M}_\gamma)_{\gamma \in \hat{\Gamma}} \) satisfies (2.6.4) and (2.6.5) if the following relations hold (cf. Remark 2.2.2):

\[
\begin{align*}
\hat{L}'_1 &= \hat{L}_2 + \hat{L}_3 + \hat{L}_4 - \hat{p}_1, \quad \hat{L}'_2 = 2\hat{p}_1 - \hat{p}_2, \\
\hat{L}'_3 &= \hat{p}_2 - \hat{p}_1, \quad \hat{L}'_4 = \hat{L}_1 + 2\hat{L}_2 + \hat{L}_3 - \hat{p}_2, \quad \text{ where} \\
\hat{p}_1 &= \min(\hat{L}_1 + \hat{L}_2, \hat{L}_1 + \hat{L}_4, \hat{L}_3 + \hat{L}_4), \\
\hat{p}_2 &= \min(\hat{L}_1 + 2\hat{L}_2, \hat{L}_1 + 2\hat{L}_4, \hat{L}_3 + 2\hat{L}_4);
\end{align*}
\]

here the \( \hat{L}_k, 1 \leq k \leq 4 \), and the \( \hat{L}'_k, 1 \leq k \leq 4 \), are determined via the length formula;

\[
\begin{align*}
\hat{\mu}_{\hat{w}_k} - \hat{\mu}_{\hat{w}'_k} &= \hat{L}_k \hat{w}_k^1 \cdot h_j^\mathbb{Z}, \quad \text{with } \hat{w}_k^1 = \hat{s}_j \hat{s}_j \cdots \hat{s}_j, \\
\hat{\mu}_{\hat{w}_k} - \hat{\mu}_{\hat{w}'_k} &= \hat{L}_k \hat{w}'_k \cdot h_j^\mathbb{Z}, \quad \text{with } \hat{w}'_k = \hat{s}_j \hat{s}_j \cdots \hat{s}_j.
\end{align*}
\]

(Equation 2.6.4 follows from the first one: \( \hat{L}'_1 = \hat{L}_2 + \hat{L}_3 + \hat{L}_4 - \hat{p}_1 \), and equation 2.6.5 follows from the last one: \( \hat{L}'_4 = \hat{L}_1 + 2\hat{L}_2 + \hat{L}_3 - \hat{p}_2 \).) In other words, \( \hat{M}_\bullet = \hat{D}(\hat{\mu}_\bullet) = (\hat{M}_\gamma)_{\gamma \in \hat{\Gamma}} \).
satisfies (2.6.4) and (2.6.5) if
\[
\tilde{R}^j_i(L_1, L_2, L_3, L_4) = (\tilde{L}_1', \tilde{L}_2', \tilde{L}_3', \tilde{L}_4'),
\]
where \( \tilde{R}^j_i : \mathbb{Z}_{\geq 0}^4 \to \mathbb{Z}_{\geq 0}^4 \) is the transition map between Lusztig parametrizations of the canonical basis of \( U_q^-(g^\omega)^\vee \). We know from Theorem 2.2.1 and [BZ11] Proposition 7.4 that the transition map \( \tilde{R}^j_i : \mathbb{R}_{>0}^4 \to \mathbb{R}_{>0}^4 \) is the tropicalization of the transition map \( \tilde{R}^j_i : \mathbb{R}_{>0}^4 \to \mathbb{R}_{>0}^4 \) defined by: \( \tilde{R}^j_i = (x_j^\omega)^{-1} \circ x_i^\omega \). Here we should remark that the bijections \( x_j^\omega, x_j^\omega : \mathbb{R}_{>0}^4 \to N_{>0} \cap G^\omega \) are given by:
\[
x_j^\omega(u_1, u_2, u_3, u_4) = x_{j_1}^\omega(u_1)x_{j_2}^\omega(u_2)x_{j_3}^\omega(u_3)x_{j_4}^\omega(u_4)
x_j^\omega(u_1, u_2, u_3, u_4) = x_{j_1}^\omega(u_1)x_{j_2}^\omega(u_2)x_{j_3}^\omega(u_3)x_{j_4}^\omega(u_4)
\]
for \((u_1, u_2, u_3, u_4) \in \mathbb{R}_{>0}^4\), where \( N_{>0} \cap G^\omega \) is the set of fixed points of \( N_{>0} \) under the action of \((\text{the lifting to } G) \) of \( \omega \), and \( x_j^\omega(u) := \exp(u x_j^\omega) \) for \( u \in \mathbb{C}, j \in \hat{I} \).

Now, to the reduced words \( j = (1, 2, 1, 2) \) and \( j' = (2, 1, 2, 1) \) for \( \hat{w}_0 \in \hat{W} \), we associate canonical reduced words
\[
i = (1, 4, 2, 3, 2, 1, 4, 2, 3, 2, 1, 4)
i' = (2, 3, 2, 1, 4, 2, 3, 2, 1, 4)
\]
for \( w_0 \in W \), and set
\[
\psi_i(P(\mu_\bullet)) = (L_1, L_2, \ldots, L_{10}) \in \mathbb{Z}_{\geq 0}^{10} \quad \text{and} \quad \psi_i(P(\mu_\bullet)) = (L'_1, L'_2, \ldots, L'_{10}) \in \mathbb{Z}_{\geq 0}^{10}.
\]
Then we have
\[
R^j_i(L_1, L_2, \ldots, L_{10}) = (L'_1, L'_2, \ldots, L'_{10}).
\]
Also, since \( \omega(P(\mu_\bullet)) = P(\mu_\bullet) \), it follows from Proposition 2.6.2 and Corollary 2.6.4 that
\[
\begin{align*}
L_1 = L_2 &= \tilde{L}_1, \\
L_3 = L_4 = L_5 &= \tilde{L}_2, \\
L_6 = L_7 &= \tilde{L}_3, \\
L_8 = L_9 = L_{10} &= \tilde{L}_4, \\
L'_1 = L'_2 = L'_3 &= \tilde{L}'_1, \\
L'_4 = L'_5 &= \tilde{L}'_2, \\
L'_6 = L'_7 = L'_8 &= \tilde{L}'_3, \\
L'_9 = L'_{10} &= \tilde{L}'_4.
\end{align*}
\]
(2.6.7)
Thus, summarizing the above, what we must show is that if \((L_1, L_2, \ldots, L_{10}) \in \mathbb{Z}_{\geq 10} \) and \((L'_1, L'_2, \ldots, L'_{10}) \in \mathbb{Z}_{\geq 10} \) are related by the transition map as
\[
R^j_i(L_1, L_2, \ldots, L_{10}) = (L'_1, L'_2, \ldots, L'_{10}),
\]
and if the relations (2.6.7) hold, then the relation
\[
\tilde{R}^j_i(L_1, L_2, L_3, L_4) = (\tilde{L}'_1, \tilde{L}'_2, \tilde{L}'_3, \tilde{L}'_4)
\]
19
holds. To show this statement, in view of the functoriality of the tropicalization (see [BK, Corollary 2.10], and also [NY, Propositions 1.9 and 1.10]), it suffices to show the following lemma.

Lemma 2.6.5. Define a map $H_1: \mathbb{R}^4_{>0} \to \mathbb{R}^{10}_{>0}$ by:

$$H_1(u_1, u_2, u_3, u_4) = \left( u_1, u_1 \frac{u_2}{\sqrt{2}}, \frac{u_2}{\sqrt{2}}, u_3, u_3 \frac{u_4}{\sqrt{2}}, \frac{u_4}{\sqrt{2}} \right),$$

and a map $H_2: \mathbb{R}^4_{>0} \to \mathbb{R}^{10}_{>0}$ by:

$$H_1(t_1, t_2, \ldots, t_{10}) = (\sqrt{2}t_1, \sqrt{2}t_2, t_3, t_3, \sqrt{2}t_4, t_4, \sqrt{2}t_5, \sqrt{2}t_6, t_7, t_7).$$

Then, the composition $H_2 \circ \mathcal{R}_1^j \circ H_1: \mathbb{R}^4_{>0} \to \mathbb{R}^4_{>0}$ is identical to the transition map $\hat{\mathcal{R}}_j^j: \mathbb{R}^4_{>0} \to \mathbb{R}^4_{>0}$.

**Proof.** Let $(u_1, u_2, u_3, u_4) \in \mathbb{R}^4_{>0}$, and set $(u'_1, u'_2, u'_3, u'_4) := \hat{\mathcal{R}}_j^j(u_1, u_2, u_3, u_4) \in \mathbb{R}^4_{>0}$. Then it follows from the definition of $\hat{\mathcal{R}}_j^j$ that

$$x_j^\omega(u_1, u_2, u_3, u_4) = x_j^\omega(u'_1, u'_2, u'_3, u'_4).$$  \hfill (2.6.8)

Here we know from [BZ1 Proposition 7.4(b)] that

$$x_j^\omega(u_1, u_2, u_3, u_4) = x_1\left( u_1, \frac{u_2}{\sqrt{2}}, \sqrt{2}u_2, \frac{u_2}{\sqrt{2}}, u_3, u_3, \frac{u_4}{\sqrt{2}}, \frac{u_4}{\sqrt{2}} \right),$$

$$x_j^\omega(u'_1, u'_2, u'_3, u'_4) = x_1\left( u'_1, \frac{u'_2}{\sqrt{2}}, \sqrt{2}u'_2, \frac{u'_2}{\sqrt{2}}, u'_2, u'_2, \frac{u'_3}{\sqrt{2}}, \frac{u'_3}{\sqrt{2}}, u'_3, u'_3, \sqrt{2}u'_4, \frac{u'_4}{\sqrt{2}} \right).$$  \hfill (2.6.9)

By combining (2.6.8) and (2.6.9), we conclude that

$$(H_2 \circ \mathcal{R}_1^j \circ H_1)(u_1, u_2, u_3, u_4) = (u'_1, u'_2, u'_3, u'_4),$$

which is the desired equality. This proves the lemma. \hfill \Box

Thus, we have completed the proof of the tropical Plücker relations for $\widetilde{M}_\bullet = \hat{D}(\hat{\mu}_\bullet) = (\widetilde{M}_g)_{g \in \tilde{\Gamma}}$, and hence the proof of Proposition 2.5.4.

2.7 Proof of Theorem 2.5.6. This subsection is devoted to the proof of Theorem 2.5.6. We keep the notation and assumptions of §2.5. We first prove the following proposition, which is a part of Theorem 2.5.6(3).

**Proposition 2.7.1.** The map $\Phi: \mathcal{MV}^\omega \to \mathcal{M}\hat{\mathcal{V}}$ is a bijection.
Proof. First we show the injectivity of \( \Phi : M \mathcal{V}^\omega \to \hat{M} \mathcal{V} \). Let \( P \in M \mathcal{V}^\omega \) and \( P' \in M \mathcal{V}^\omega \) be such that \( \Phi(P) = \Phi(P') \). Take a reduced word \( j = (j_1, j_2, \ldots, j_m) \) for \( \hat{w}_0 \in \hat{W} \), and the associated canonical reduced word \( i = (i_1, i_2, \ldots, i_m) \) for \( w_0 \in W \). Then we see from Proposition 2.6.2 and Corollary 2.6.4 that \( \psi_1(P) = \psi_1(P') \), which implies that \( P = P' \) since \( \psi_1 : M \mathcal{V} \to \mathbb{Z}_{\geq 0}^m \) is a bijection. Thus, the injectivity of \( \Phi : M \mathcal{V}^\omega \to \hat{M} \mathcal{V} \) follows.

Next we show the surjectivity of \( \Phi : M \mathcal{V}^\omega \to \hat{M} \mathcal{V} \). Let \( \hat{P} = \hat{P}(\mu_\bullet) \) be an element of \( \hat{M} \mathcal{V} \) with GGMS datum \( \hat{\mu}_\bullet = (\hat{\mu}_w)_{\hat{w} \in \hat{W}} \in \hat{V}_{MV} \). Take a reduced word \( j = (j_1, j_2, \ldots, j_m) \) for \( \hat{w}_0 \in \hat{W} \), and the associated canonical reduced word \( i = (i_1, i_2, \ldots, i_m) \) for \( w_0 \in W \). By the length formula, we have \( \hat{\mu}_w \cdot \hat{\mu}_{w_{k-1}} = \hat{L}_k \hat{w}_{k-1} \cdot \hat{h}_j \), with \( \hat{w}_k = \hat{s}_{j_1} \hat{s}_{j_2} \cdots \hat{s}_{j_k} \), for \( 1 \leq k \leq \hat{m} \). Now we define an element \( P = P(\mu_\bullet) \) of \( M \mathcal{V} \) to be a unique preimage under \( \psi_1 : M \mathcal{V} \to \mathbb{Z}_{\geq 0}^m \) of the element

\[
(L_1, L_2, \ldots, L_m) := \left(\frac{\hat{L}_1}{N_1 \text{ times}}, \frac{\hat{L}_2}{N_2 \text{ times}}, \ldots, \frac{\hat{L}_{\hat{m}}}{N_{\hat{m}} \text{ times}}\right)
\]

of \( \mathbb{Z}_{\geq 0}^m \). Then, from Proposition 2.6.3 together with (2.6.1), we deduce that \( \omega(P) = P \). Moreover, it follows from Corollary 2.6.4 that \( \Phi(P) = \hat{P} \). Thus, the surjectivity of \( \Phi : M \mathcal{V}^\omega \to \hat{M} \mathcal{V} \) follows. This proves the proposition.

To prove part (1) of Theorem 2.5.6, we need the following lemma.

Lemma 2.7.2. Let \( P = P(\mu_\bullet) \in M \mathcal{V} \) be an MV polytope with GGMS datum \( \mu_\bullet = (\mu_w)_{w \in W} \in V_{MV} \). Then, we have \( \omega(f_j P) = f_{\omega(j)} \omega(P) \) for all \( j \in I \).

Proof. Fix \( j \in I \), and take a reduced word \( i = (i_1, i_2, \ldots, i_m) \) for \( w_0 \in W \) such that \( i_1 = j \). If we write \( \psi_1(P) \in \mathbb{Z}_{\geq 0}^m \) as \( \psi_1(P) = (L_1, L_2, \ldots, L_m) \in \mathbb{Z}_{\geq 0}^m \), then we know from [Kam2, Proposition 3.4] that \( \psi_1(f_j P) = (L_1 + 1, L_2, \ldots, L_m) \) also. We know from Lemma 2.6.1 that \( \psi_{\omega(i)}(\omega(P)) = \psi_1(P) = (L_1, L_2, \ldots, L_m) \). Because \( \omega(i) = (\omega(i_1), \omega(i_2), \ldots, \omega(i_m)) \) is a reduced word for \( w_0 \in W \) such that \( \omega(i_1) = \omega(j) \), it follows again from [Kam2, Proposition 3.4] that \( \psi_{\omega(i)}(f_{\omega(j)} \omega(P)) = (L_1 + 1, L_2, \ldots, L_m) \). Therefore, we obtain \( \psi_{\omega(i)}(f_{\omega(j)} \omega(P)) = \psi_1(f_j P) \), which is equal to \( \psi_{\omega(i)}(\omega(f_j P)) \) again by Lemma 2.6.1. From this fact, we conclude that \( f_{\omega(j)}(\omega(P)) = \omega(f_j P) \) since \( \psi_{\omega(i)} : M \mathcal{V} \to \mathbb{Z}_{\geq 0}^m \) is a bijection. This proves the lemma.

The following proposition is precisely part (1) of Theorem 2.5.6.

Proposition 2.7.3. The subset \( M \mathcal{V}^\omega \) of \( M \mathcal{V} \) is stable under the operators \( f_j^\omega \) for all \( j \in \hat{I} \).
Proof. Let $P \in \mathcal{MV}^\omega$ be an MV polytope such that $\omega(P) = P$. We prove that $f_j^\omega P \in \mathcal{MV}^\omega$, i.e., $\omega(f_j^\omega P) = f_j^\omega P$ in the case that $\ell = 2n$, $n \in \mathbb{Z}_{\geq 2}$, and $j = n$; the proofs for the other cases are simpler. Repeated application of Lemma 2.7.2 shows that

$$\omega(f_j^\omega P) = \omega(f_n^\omega P) = \omega(f_n f_{\omega(n)}^2 f_n P) = f_{\omega(n)} f_n^2 f_{\omega(n)} \omega(P) = f_{\omega(n)} f_n^2 f_{\omega(n)} P,$$

since $\omega(P) = P$ by assumption. Here we recall from Remark 2.5.5 that as operators on $\mathcal{MV}$, $f_n f_{\omega(n)}^2 f_n = f_{\omega(n)} f_n^2 f_{\omega(n)}$. Therefore, we obtain

$$\omega(f_n^\omega P) = f_{\omega(n)} f_n^2 f_{\omega(n)} P = f_n f_{\omega(n)}^2 f_n P = f_n^\omega P.$$

This proves the proposition. \(\square\)

The following proposition is a part of Theorem 2.5.6(3).

**Proposition 2.7.4.** We have $\Phi \circ f_j^\omega = \widehat{f}_j \circ \Phi$ for all $j \in \widehat{I}$.

Proof. Let $P \in \mathcal{MV}^\omega$, and let $\widehat{\mu}_\bullet = (\widehat{\mu}_{\Theta(\omega)})_{\omega \in \widehat{W}} \in \widehat{\mathcal{VMV}}$ and $\widehat{\rho}_\bullet = (\widehat{\rho}_{\Theta(\omega)})_{\omega \in \widehat{W}} \in \widehat{\mathcal{VMV}}$ be the GGMS data of $\Phi(P) \in \widehat{\mathcal{MV}}$ and $\Phi(f_j^\omega P) \in \widehat{\mathcal{MV}}$, respectively. Recall that $\widehat{f}_j \Phi(P) \in \widehat{\mathcal{MV}}$ is defined to be the unique MV polytope in $\widehat{\mathcal{MV}}$ with GGMS datum $\widehat{\mu}_\bullet = (\widehat{\mu}_{\Theta(\omega)})_{\omega \in \widehat{W}} \in \widehat{\mathcal{VMV}}$ such that $\widehat{\mu}_{\hat{e}} = \widehat{\mu}_e - h_j^\omega$, and $\widehat{\rho}_{\hat{w}} = \widehat{\rho}_{\hat{w}}$ for all $\hat{w} \in \widehat{W}$ with $\hat{s}_j \hat{w} < \hat{w}$. Hence, in order to prove that $\widehat{f}_j \Phi(P) = \Phi(f_j^\omega P)$, it suffices to show that

$$\widehat{\mu}_{\hat{e}} = \widehat{\mu}_e - h_j^\omega, \quad \text{and} \quad \widehat{\rho}_{\hat{w}} = \widehat{\rho}_{\hat{w}} \quad \text{for all} \quad \hat{w} \in \widehat{W} \text{ with } \hat{s}_j \hat{w} < \hat{w}. \quad (2.7.1)$$

Let $\mu_\bullet = (\mu_w)_{w \in W} \in \mathcal{V}_\mathcal{MV}^\omega$ and $\mu''_\bullet = (\mu''_w)_{w \in W} \in \mathcal{V}_\mathcal{MV}^\omega$ be the GGMS data of $P \in \mathcal{MV}^\omega$ and $f_j^\omega P \in \mathcal{MV}^\omega$, respectively. Note that $\widehat{\mu}_{\hat{w}} = \mu_{\Theta(\hat{w})}$ and $\widehat{\rho}_{\hat{w}} = \mu''_{\Theta(\hat{w})}$ for $\hat{w} \in \widehat{W}$ by the definition of the map $\Phi$. Also, we deduce from the definitions of the lowering Kashiwara operators $f_j$, $j \in I$, and the operators $f_j^\omega$, $j \in \widehat{I}$, that $\mu''_{\hat{w}} = \mu_{\hat{w}} - h_j^\omega$. Hence we obtain $\widehat{\mu}_{\hat{e}} = \mu_{\hat{e}} - h_j^\omega = \mu_e - h_j^\omega$. Next, let $\hat{w} \in \widehat{W}$ be such that $\hat{s}_j \hat{w} < \hat{w}$, and set $w := \Theta(\hat{w})$. Then it follows from Remark 2.3.2 that $s_j w < w$ and $s_{\omega(j)} w < w$. Therefore, we deduce again from the definitions of the lowering Kashiwara operators $f_j$, $j \in I$, and the operators $f_j^\omega$, $j \in \widehat{I}$, that $\mu''_w = \mu_w$, and hence $\widehat{\rho}_{\hat{w}} = \mu''_w = \mu_w = \widehat{\rho}_{\hat{w}}$. This proves (2.7.1), which completes the proof of the proposition. \(\square\)

Now we prove the remaining parts of Theorem 2.5.6. Recall from Remark 2.1.3 that the MV polytope $P^0 = P(\mu_\bullet^0) \in \mathcal{MV}$ corresponding to $u_\infty \in \mathcal{B}(\infty)$ under the isomorphism $\Psi: \mathcal{MV} \rightarrow \mathcal{B}(\infty)$ of crystals for $\mathfrak{g}^\vee$. It is obvious that $\omega(P^0) = P^0$, i.e., $P^0 \in \mathcal{MV}^\omega$. Also, it
follows from the definition of the map \( \Phi : MV^\omega \to \widehat{MV} \) that \( \Phi(P^0) = \widehat{P}^0 \), where \( \widehat{P}^0 \in \widehat{MV} \) is the MV polytope corresponding to \( \widehat{u}^\infty \in \widehat{B}(\infty) \) under the isomorphism \( \widehat{\Psi} : \widehat{MV} \to \widehat{B}(\infty) \) of crystals for \((g^\omega)^\vee\). Moreover, because \( \widehat{MV} \) is isomorphic to \( \widehat{B}(\infty) \), each element \( \widehat{P} \in \widehat{MV} \) is of the form \( \widehat{P} = \widehat{f}_{j_1} \widehat{f}_{j_2} \cdots \widehat{f}_{j_k} \widehat{P}^0 \) for some \( j_1, j_2, \ldots, j_k \in \widehat{I} \).

Let \( P \in MV^\omega \), and set \( \widehat{P} := \Phi(P) \in \widehat{MV} \). Then, as above, there exist \( j_1, j_2, \ldots, j_k \in \widehat{I} \) such that \( \widehat{P} = \widehat{f}_{j_1} \widehat{f}_{j_2} \cdots \widehat{f}_{j_k} \widehat{P}^0 \). Here we note that \( f_{j_1} f_{j_2} \cdots f_{j_k} P^0 \in MV^\omega \) by Proposition 2.7.3. Therefore, by using Proposition 2.7.4, we obtain

\[
\Phi(f_{j_1} f_{j_2} \cdots f_{j_k} P^0) = \widehat{f}_{j_1} \widehat{f}_{j_2} \cdots \widehat{f}_{j_k} \Phi(P^0) = \widehat{f}_{j_1} \widehat{f}_{j_2} \cdots \widehat{f}_{j_k} \widehat{P}^0 = \widehat{P} = \Phi(P),
\]

which implies that \( P = f_{j_1} f_{j_2} \cdots f_{j_k} P^0 \) by Proposition 2.7.1. This proves part (2) of Theorem 2.5.6 and hence the uniqueness assertion in Theorem 2.5.6 (3). Thus, we have completed the proof of Theorem 2.5.6.

3 Descriptions of the lowering Kashiwara operators \( \widehat{f}_j \).

We maintain the assumption that \( g \) is either of type \( A_\ell \) with \( \ell = 2n - 1, n \in \mathbb{Z}_{\geq 2} \), or of type \( A_\ell \) with \( \ell = 2n, n \in \mathbb{Z}_{\geq 2} \).

3.1 Description of \( \widehat{f}_j \) in terms of BZ data. First, let us recall from [Kam2, §5.2] the description of the lowering Kashiwara operators \( f_j, j \in I \), on \( MV \) in terms of BZ data. Note that

\[
\langle \gamma, h_j \rangle \in \{-1, 0, 1\} \quad \text{for all } \gamma \in \Gamma \text{ and } j \in I,
\]

since every fundamental weight for \( g \) (of type \( A_\ell \)) is minuscule. The next theorem follows immediately from [Kam2, §5.1] and (3.1.1).

**Theorem 3.1.1.** Let \( P \in MV \), and \( j \in I \). Let \( M_\gamma = (M_{\gamma})_{\gamma \in \Gamma} \in E_{MV} \) and \( M'_\gamma = (M'_{\gamma})_{\gamma \in \Gamma} \in E_{MV} \) be the BZ data of \( P \) and \( f_j P \), respectively. Then, for each \( \gamma \in \Gamma \),

\[
M'_{\gamma} = \begin{cases} 
\min(M_{\gamma}, M_{s_j, \gamma} + c_j^{M_\gamma}) & \text{if } \langle \gamma, h_j \rangle = 1, \\
M_{\gamma} & \text{otherwise,}
\end{cases}
\]

where \( c_j^{M_\gamma} := M_{\Lambda_j} - M_{s_j, \Lambda_j} - 1 \).

The following corollary will be needed in the next subsection.

**Corollary 3.1.2.** Keep the notation of Theorem 3.1.1. Then we have \( f_j P \supset P \).
Proof. It follows from \((2.1.6)\) that
\[
P = \{ h \in \mathfrak{h}_R \mid \gamma(h) \geq M_\gamma \text{ for all } \gamma \in \Gamma \},
\]
\[
f_j P = \{ h \in \mathfrak{h}_R \mid \gamma(h) \geq M'_\gamma \text{ for all } \gamma \in \Gamma \}.
\]
Also, we see from Theorem \(3.1.1\) that \(M_\gamma \geq M'_\gamma\) for all \(\gamma \in \Gamma\). Hence we obtain \(f_j P \supseteq P\), as desired. \(\Box\)

By Theorem \(2.5.6\), giving a description of \(\hat{f}_j, \ j \in \hat{I}\), on \(\mathcal{M}\mathcal{V}\) is equivalent to giving a description of \(f'_j, \ j \in \hat{I}\), on \(\mathcal{M}\mathcal{V}^\omega\). Applying Theorem \(3.1.1\) successively, we can prove the following proposition.

**Proposition 3.1.3.** Let \(P \in \mathcal{M}\mathcal{V}^\omega\) and \(j \in \hat{I}\). Let \(M_\bullet = (M_\gamma)_{\gamma \in \Gamma} \in \mathcal{E}_\mathcal{M}\mathcal{V}^\omega\) and \(M'_\bullet = (M'_\gamma)_{\gamma \in \Gamma} \in \mathcal{E}_\mathcal{M}\mathcal{V}^\omega\) be the BZ data of \(P\) and \(f'_j P\), respectively.

1. If \(1 \leq j \leq n - 1\), then for \(\gamma \in \Gamma\), we have

\[
M'_\gamma = \begin{cases} 
M_\gamma & \text{if } \langle \gamma, h_j \rangle \leq 0 \text{ and } \langle \gamma, h_{\omega(j)} \rangle \leq 0, \\
\min(M_\gamma, M_{s_{\omega(j)} \cdot \gamma} + c_j^{M_\bullet}) & \text{if } \langle \gamma, h_j \rangle \leq 0 \text{ and } \langle \gamma, h_{\omega(j)} \rangle = 1, \\
\min(M_\gamma, M_{s_j \cdot \gamma} + c_j^{M_\bullet}) & \text{if } \langle \gamma, h_j \rangle = 1 \text{ and } \langle \gamma, h_{\omega(j)} \rangle \leq 0, \\
\min\left( M_\gamma, M_{s_j \cdot \gamma} + c_j^{M_\bullet}, M_{s_{\omega(j)} \cdot \gamma} + 2c_j^{M_\bullet} \right) & \text{if } \langle \gamma, h_j \rangle = \langle \gamma, h_{\omega(j)} \rangle = 1.
\end{cases}
\]  

(3.1.2)

2. If \(\ell = 2n - 1\), \(n \in \mathbb{Z}_{\geq 2}\), and \(j = n\), then for \(\gamma \in \Gamma\), we have

\[
M'_\gamma = \begin{cases} 
\min(M_\gamma, M_{s_n \cdot \gamma} + c_n^{M_\bullet}) & \text{if } \langle \gamma, h_n \rangle = 1, \\
M_\gamma & \text{otherwise.}
\end{cases}
\]

(3.1.3)

3. If \(\ell = 2n\), \(n \in \mathbb{Z}_{\geq 2}\), and \(j = n\), then for \(\gamma \in \Gamma\), we have

\[
M'_\gamma = \begin{cases} 
\min\left( M_\gamma, M_{s_{\omega(n)} \cdot \gamma} + c_n^{M_\bullet}, M_{s_n \cdot \gamma} + 2c_n^{M_\bullet} \right) & \text{if } \langle \gamma, h_n \rangle = 0 \text{ and } \langle \gamma, h_{\omega(n)} \rangle = 1, \\
\min(M_\gamma, M_{s_n \cdot \gamma} + c_n^{M_\bullet}) & \text{if } \langle \gamma, h_n \rangle = 1 \text{ and } \langle \gamma, h_{\omega(n)} \rangle = -1, \\
\min\left( M_\gamma, M_{s_n \cdot \gamma} + c_n^{M_\bullet}, M_{s_{\omega(n)} \cdot \gamma} + 2c_n^{M_\bullet} \right) & \text{if } \langle \gamma, h_n \rangle = 1 \text{ and } \langle \gamma, h_{\omega(n)} \rangle = 0, \\
\min(M_\gamma, M_{s_{\omega(n)} \cdot \gamma} + c_n^{M_\bullet}) & \text{if } \langle \gamma, h_n \rangle = -1 \text{ and } \langle \gamma, h_{\omega(n)} \rangle = 1, \\
M_\gamma & \text{otherwise.}
\end{cases}
\]

(3.1.4)
Proof. Since the proofs of these formulas are rather straightforward, we only sketch them, leaving the details to the reader. In the proof of part (1), we need the equations \( c_{\omega(j)}^{M*} = c_j^{M*} \) and \( c_j^{M''} = c_j^{M*} \), where \( M'' \in E_{MV} \) is the BZ datum of \( f_{\omega(j)}P \); recall that \( f_{\omega} = f_jf_{\omega(j)} \). The first one \( c_{\omega(j)}^{M*} = c_j^{M*} \) follows immediately from Lemma 2.4.2 (2), and the second one \( c_j^{M''} = c_j^{M*} \) is easily shown by using Theorem 3.1.1 along with Remark 2.3.1 (4) and Lemma 2.4.2 (2). Also, in the proof of part (3), we need the following equations: \( c_{\omega(n)}^{M*(1)} = c_n^{M*} + 1 \), \( c_{\omega(n)}^{M*(2)} = c_n^{M*} \), and \( c_n^{M*(3)} = c_n^{M*} \), where \( M^{(1)} \in E_{MV} \) (resp., \( M^{(2)}, \ M^{(3)} \in E_{MV} \)) is the BZ datum of \( f_nP \) (resp., \( f_{\omega(n)}f_nP, f_{\omega(n)}^{2}f_nP \)); recall that \( f_{\omega} = f_nf_{\omega(n)}f_n \). These equations are easily shown by using Theorem 3.1.1 along with Remark 2.3.1 (4), Lemma 2.4.2 (2), and Lemma 3.1.4 below. □

Lemma 3.1.4. Assume that \( \ell = 2n, n \in \mathbb{Z}_{\geq 2} \). Let \( M_{\bullet} = (M_{\gamma})_{\gamma \in \Gamma} \in E_{MV} \), and let \( w \in W'^{w} \) be such that \( ws_{n} > w \) and \( ws_{n+1} > w \). Then, we have

\[
2M_{ws_{n}}A_{n} = M_{w}A_{n} + M_{ws_{n+1}s_{n}}A_{n}.
\]

(3.1.5)

Proof. By the tropical Plücker relation at \((w, n, n + 1)\) (see (2.1.7)), we have

\[
M_{ws_{n}}A_{n} + M_{ws_{n+1}}A_{n+1} = \min(M_{w}A_{n} + M_{ws_{n}s_{n+1}}A_{n+1}, M_{ws_{n+1}s_{n}}A_{n} + M_{w}A_{n+1}).
\]

(3.1.6)

Since \( M_{\bullet} = (M_{\gamma})_{\gamma \in \Gamma} \in E_{MV} \) and \( w \in W'^{w} \) by assumption, it follows immediately from Lemma 2.4.2 (2) along with Remark 2.3.1 (4) that

\[
\begin{align*}
&M_{ws_{n+1}s_{n+1}}A_{n+1} = M_{ws_{n}}A_{n}, \\
&M_{w}A_{n} = M_{w}A_{n+1}, \\
&M_{ws_{n}s_{n+1}}A_{n+1} = M_{ws_{n+1}s_{n}}A_{n}.
\end{align*}
\]

(3.1.7)

Combining (3.1.6) and (3.1.7), we obtain (3.1.5), as desired. □

3.2 Description of \( \hat{f}_j \) in terms of GGMS data. First, let us recall from [Kam2, §5.1] the description of the lowering Kashiwara operators \( f_j, j \in I, \) on \( \mathcal{MV} \) in terms of GGMS data. Fix \( j \in I \) and \( P \in \mathcal{MV} \). Let \( \mu_{\bullet} = (\mu_{w})_{w \in W} \in V_{MV} \) be the GGMS datum of \( P \), and set \( M_{\bullet} = (M_{\gamma})_{\gamma \in \Gamma} := D(\mu_{\bullet}) \in E_{MV} \). Define a reflection \( \sigma : \mathfrak{h} \to \mathfrak{h} \) by: \( \sigma(h) = s_{j} \cdot h + c_{j}h \) for \( h \in \mathfrak{h} \), where \( c := c_{j}^{M*} = M_{A_j} - M_{s_{j}A_j} - 1 \). Also, we set

\[
W_{+} := \{ w \in W \mid s_{j}w > w \}, \quad W_{-} := \{ w \in W \mid s_{j}w < w \}.
\]

note that \( W = W_{+} \cup W_{-} \). The following was conjectured by Anderson-Mirković, and proved by Kamnitzer.
Theorem 3.2.1 (Kam2 Theorem 5.5). Keep the notation above. Then, \( f_j P \in MV \) is the smallest pseudo-Weyl polytope \( P' \in P \) with GGMS datum \( \mu'_w = (\mu'_w)_{w \in W} \in V \) such that

(i) \( \mu'_w = \mu_w \) for all \( w \in W_- \),

(ii) \( \mu'_e = \mu_e - h_j \),

(iii) \( P' \) contains \( \mu_w \) for all \( w \in W_+ \),

(iv) if \( w \in W_- \) is such that \( \langle \alpha_j, \mu_w \rangle \geq c \), then \( P' \) contains \( \sigma(\mu_w) \).

The aim of this subsection is to give a description of the lowering Kashiwara operators \( \tilde{f}_j, j \in \tilde{I} \), on \( \tilde{MV} \) in terms of GGMS data. For this aim, we introduce some additional notation. For each \( M_\bullet = (M_\gamma)_{\gamma \in \Gamma} \in \mathcal{E}^\omega \), we define a convex polytope \( \tilde{P}(M_\bullet) \) in \( \mathfrak{h}_R \) by:

\[
\tilde{P}(M_\bullet) = \left\{ h \in \mathfrak{h}_R \mid \langle \gamma, h \rangle \geq M_\gamma \text{ for all } \gamma \in \tilde{\Gamma} \right\},
\]

(3.2.1)

where \( \tilde{\Gamma} := \left\{ w\Lambda_i \mid w \in W^\omega, i \in I \right\} \). Also, for each \( \mu_\bullet = (\mu_w)_{w \in W} \in V^\omega \), we define a convex polytope \( \tilde{P}(\mu_\bullet) \) in \( \mathfrak{h}_R \) by:

\[
\tilde{P}(\mu_\bullet) = \left\{ h \in \mathfrak{h}_R \mid h \geq_w \mu_w \text{ for all } w \in W^\omega \right\}.
\]

(3.2.2)

Then, it is obvious that \( \tilde{P}(\mu_\bullet) = \tilde{P}(D(\mu_\bullet)) \) for all \( \mu_\bullet \in V^\omega \). Moreover, it follows from this equality that the set \( \tilde{P}(\mu_\bullet) = \tilde{P}(D(\mu_\bullet)) \) is indeed a convex polytope (but, not necessarily a pseudo-Weyl polytope) in \( \mathfrak{h}_R \), since it is clearly a bounded polyhedral set (see [E] Chapters I and II]). We set \( \tilde{P} := \left\{ \tilde{P}(M_\bullet) \mid M_\bullet \in \mathcal{E}^\omega \right\} = \left\{ \tilde{P}(\mu_\bullet) \mid \nu_\bullet \in V^\omega \right\} \).

Remark 3.2.2. (1) For each \( \mu_\bullet \in V^\omega \), we have \( \tilde{P}(\mu_\bullet) \supset P(\mu_\bullet) \).

(2) We see from Remark 2.3.1(5) and Lemma 2.4.2(1) that the set \( \omega(\tilde{P}(\mu_\bullet)) = \left\{ \omega(h) \mid h \in \tilde{P}(\mu_\bullet) \right\} \) is identical to \( \tilde{P}(\mu_\bullet) \) for all \( \mu_\bullet \in V^\omega \).

(3) It follows from (2.5.1) and (3.2.2) that

\[
\Phi(P(\mu_\bullet)) = \tilde{P}(\Phi(\mu_\bullet)) = \tilde{P}(\mu_\bullet) \cap h^\omega \quad \text{for all } \mu_\bullet \in V^\omega.
\]

(3.2.3)

From (3.2.3), we deduce that if \( \tilde{P}(\mu_\bullet) = \tilde{P}(\mu'_\bullet) \) for \( \mu_\bullet, \mu'_\bullet \in V^\omega \), then \( \mu_\bullet = \mu'_\bullet \) since \( \Phi : MV^\omega \to \tilde{MV} \) is a bijection. Equivalently, if \( \tilde{P}(M_\bullet) = \tilde{P}(M'_\bullet) \) for \( M_\bullet, M'_\bullet \in \mathcal{E}^\omega \), then \( M_\bullet = M'_\bullet \). Thus, by abuse of terminology, we say that \( \mu_\bullet \in V^\omega \) (resp., \( M_\bullet \in \mathcal{E}^\omega \)) is the GGMS (resp., BZ) datum of the convex polytope \( \tilde{P}(\mu_\bullet) \) (resp., \( \tilde{P}(M_\bullet) \)).

Fix \( j \in \tilde{I} \) and \( \tilde{P} \in \tilde{MV} \). Set \( P := \Phi^{-1}(\tilde{P}) \in MV^\omega \). Let \( \mu_\bullet = (\mu_w)_{w \in W} \in V_{MV}^\omega \) and \( \mu'_w = (\mu'_w)_{w \in W} \in V_{MV}^\omega \) be the GGMS data of \( P \) and \( f_j P \), respectively, and set \( M_\bullet = (M_\gamma)_{\gamma \in \Gamma} := D(\mu_\bullet) \in \mathcal{E}_{MV}^\omega \) and \( M'_\bullet = (M'_\gamma)_{\gamma \in \Gamma} := D(\mu'_w) \in \mathcal{E}_{MV}^\omega \). We define reflections \( \sigma : \mathfrak{h} \to \mathfrak{h} \) and \( \tau : \mathfrak{h} \to \mathfrak{h} \) by:

\[
\sigma(h) = s_j \cdot h + ch_j \quad \text{and} \quad \tau(h) = s_{\omega(j)} \cdot h + ch_{\omega(j)},
\]
for \( h \in \mathfrak{h} \), where \( c := c^M_j = M_{\Lambda_j} - M_{s_j \Lambda_j} - 1 \); note that \( c^M_j = c^M_{\omega(j)} \), and that

\[
\begin{cases}
\sigma \tau = \tau \sigma & \text{if } 1 \leq j \leq n - 1, \\
\sigma = \tau & \text{if } \ell = 2n - 1, n \in \mathbb{Z}_{\geq 2}, \text{ and } j = n, \\
\sigma \tau \sigma = \tau \sigma \tau & \text{if } \ell = 2n, n \in \mathbb{Z}_{\geq 2}, \text{ and } j = n.
\end{cases}
\]

Also, we set

\[ W_\omega^+ := \{ w \in W^\omega \mid s_j^\omega w > w \}, \quad W_\omega^- := \{ w \in W^\omega \mid s_j^\omega w < w \}; \]

note that \( W^\omega = W_\omega^+ \cup W_\omega^- \) by Remark 2.3.2(3). We deduce that

\[
\begin{align*}
\widehat{f}_j \widehat{P} &= \widehat{f}_j \Phi(P) = \Phi(f_j^\omega P) \quad \text{by Theorem 2.5.6} \\
&= \Phi(P(\mu'_\omega)) = \widehat{P}(\Phi(\mu'_\omega)) = \widehat{P}(\mu'_\omega) \cap h^\omega \quad \text{by (3.2.3)}.
\end{align*}
\]

Thus, it suffices to give a description of the convex polytope \( \widehat{P}(\mu'_\omega) \subset h_R \).

**Theorem 3.2.3.** Keep the notation above. Assume that \( \ell = 2n - 1, n \in \mathbb{Z}_{\geq 2}, \) or \( \ell = 2n, n \in \mathbb{Z}_{\geq 2}, \) and \( 1 \leq j \leq n - 1 \). Then, \( \widehat{P}(\mu'_\omega) \) is the smallest convex polytope \( \widetilde{P} \) in \( \widehat{P} \) with GGMS datum \( \mu'_\omega = (\mu'_w)_{w \in W} \in V^\omega \) satisfying the following conditions (i)-(v):

(i) If \( w \in W_\omega^+ \), then \( \mu'_w = \mu_w \).

(ii) \( \mu'_e = \mu_e - h_j^\omega \).

(iii) If \( w \in W_\omega^+ \), then \( \mu_w \in \widehat{P} \).

(iv) If \( w \in W \) is such that \( s_j w < w \) and \( \langle \alpha_j, \mu_w \rangle \geq c \), then \( \sigma(\mu_w) \in \widehat{P} \). Also, if \( w \in W \) is such that \( s_{\omega(j)} w < w \) and \( \langle \alpha_{\omega(j)}, \mu_w \rangle \geq c \), then \( \tau(\mu_w) \in \widehat{P} \).

(v) If \( w \in W^- \) is such that \( \langle \alpha_j, \mu_w \rangle \geq c \), then \( \sigma \tau(\mu_w) \in \widehat{P} \).

**Proof.** First we see that the convex polytope \( \widehat{P}(\mu'_\omega) \) satisfies conditions (i)-(v). We see from the proof of Proposition 2.7.4 along with Remark 2.3.2(3) that \( \mu'_e = \mu_e - h_j^\omega \), and \( \mu'_w = \mu_w \) for all \( w \in W_\omega^+ \), i.e., that \( \widehat{P}(\mu'_\omega) \) satisfies conditions (i) and (ii). Furthermore, by Remark 3.2.2(1) and Corollary 3.1.2, we have

\[ \widehat{P}(\mu'_\omega) \supset P(\mu'_\omega) = f_j^\omega P = f_j f_{\omega(j)} P \supset f_{\omega(j)} P. \]  

(3.2.4)

Also, we see from Remark 2.3.2(3) that if \( w \in W_\omega^+ \), then \( s_{\omega(j)} w > w \). Hence, by Theorem 3.2.1 \( f_{\omega(j)} P \) contains \( \mu_w \) for all \( w \in W_\omega^+ \). Therefore, it follows from (3.2.4) that \( \widehat{P}(\mu'_\omega) \) contains \( \mu_w \) for all \( w \in W_\omega^+ \), i.e., that \( \widehat{P}(\mu'_\omega) \) satisfies condition (iii). If \( w \in W \) is such that \( s_{\omega(j)} w < w \) and \( \langle \alpha_{\omega(j)}, \mu_w \rangle \geq c \), then by Theorem 3.2.1 \( \tau(\mu_w) \in f_{\omega(j)} P \). Therefore, \( \tau(\mu_w) \in \widehat{P}(\mu'_\omega) \) again by (3.2.4). Similarly, using the equation \( f_j^\omega = f_j f_{\omega(j)} = f_{\omega(j)} f_j \) (see
Remark \ref{2.5.5}, we can show that if $w \in W$ is such that $s_j w < w$ and $\langle \alpha_j, \mu_w \rangle \geq c$, then $\sigma(\mu_w) \in \widetilde{P}(\mu'_j)$. Namely, we have shown that $\widetilde{P}(\mu'_j)$ satisfies condition (iv). It remains to show that $\widetilde{P}(\mu'_j)$ satisfies condition (v). Let $w \in W^\omega$ be such that $\langle \alpha_j, \mu_w \rangle \geq c$. By \ref{3.2.2}, it suffices to show that

$$\sigma(\mu_w) \geq_v \mu'_j \quad \text{for all } v \in W^\omega.$$

\textbf{Claim 1.} If $v \in W^\omega$, then $\sigma(\mu_v) \geq_{s_j^v} \mu'_{s_j^v}$. \hspace{1cm} \hfill (3.2.5)

We set $\gamma_i := s_j^v \cdot \Lambda_i$ for $i \in I$. Since $v \in W^\omega$, and hence $s_j^v \in W^\omega_v$, it follows from Remark \ref{2.3.2}(3) and \cite{MP}, Proposition 4 (i) in §5.2 that $(s_j^v)^{-1} \cdot h_j$ and $(s_j^v)^{-1} \cdot h_{\omega(j)}$ are positive coroots of $g$. Therefore, we have

$$\langle \gamma_i, h_j \rangle = \langle \Lambda_i, (s_j^v)^{-1} \cdot h_j \rangle \geq 0,$$

and

$$\langle \gamma_i, h_{\omega(j)} \rangle = \langle \Lambda_i, (s_j^v)^{-1} \cdot h_{\omega(j)} \rangle \geq 0.$$

Hence, by \ref{3.1.1}, we deduce that

$$\langle \gamma_i, h_j \rangle, \langle \gamma_i, h_{\omega(j)} \rangle = (0,0), (0,1), (1,0), \text{ or } (1,1).$$

Now, recall that $\sigma(\mu_v) \geq_{s_j^v} \mu'_{s_j^v}$ if and only if

$$\langle \gamma_i, \sigma(\mu_v) \rangle \geq \langle \gamma_i, \mu'_{s_j^v} \rangle = M'_\gamma_i \quad \text{for all } i \in I.$$

Also, by direct calculation, we obtain

$$\langle \gamma_i, \sigma(\mu_v) \rangle = \langle \gamma_i, s_j^v \cdot \mu_v + c h_j \rangle = \langle \gamma_i, s_j^v \cdot \mu_v \rangle + c \langle \gamma_i, h_j \rangle$$

$$= \langle v \Lambda_i, \mu_v \rangle + c \langle \gamma_i, h_j \rangle = M_v \cdot \Lambda_i + c \langle \gamma_i, h_j \rangle$$

$$= M_{s_j^v} \cdot \gamma_i + c \langle \gamma_i, h_j \rangle.$$

If $\langle \gamma_i, h_j \rangle = \langle \gamma_i, h_{\omega(j)} \rangle = 1$, then $M_{s_j^v} \cdot \gamma_i + c \langle \gamma_i, h_j \rangle = M_{s_j^v} \cdot \gamma_i + 2c$. Therefore, in this case, we deduce from Proposition \ref{3.1.3}(1) that $\langle \gamma_i, \sigma(\mu_v) \rangle = M_{s_j^v} \cdot \gamma_i + 2c \geq M'_\gamma_i$. Similarly, we can show that $\langle \gamma_i, \sigma(\mu_v) \rangle \geq M'_\gamma_i$ in all other cases of \ref{3.2.7}. This proves Claim 1.

\textbf{Claim 2.} Inequality (3.2.5) holds for all $v \in W^\omega_+$. \hspace{1cm} \hfill (3.2.7)

Since $v \in W^\omega_+$, and hence $s_j^v \in W^\omega$, it follows from Claim 1 that

$$\sigma(\mu_{s_j^v} \gamma_i) \geq_v \mu'_j \quad \text{for all } i \in I.$$

Also, since $\mu_* = (\mu_w)_{w \in W} \in \mathcal{V}$, it follows that $\mu_w \geq_{s_j^v} \mu'_{s_j^v}$, from which we deduce by direct calculation that $\tau(\mu_w) \geq_{s_j^v} \tau(\mu'_{s_j^v})$, and then that $\sigma(\mu_w) \geq_v \sigma(\mu'_{s_j^v})$. Combining the last inequality with \ref{3.2.8}, we get $\sigma(\mu_w) \geq_v \mu'_v$, as desired. This proves Claim 2.
Claim 3. Inequality (3.2.5) holds for all \( v \in W^\omega \).

Since \( w \in W^\omega \subset W^\omega \), it follows from Lemma \[2.4.2\](1) that \( \omega(\mu_w) = \mu_w \), and hence \( \langle \alpha_j, \mu_w \rangle = \langle \alpha_j(\omega), \mu_w \rangle \). Using this, we have

\[
\begin{align*}
\sigma \tau(\mu_w) - \mu_w &= (\varphi_j^\omega \cdot \mu_w + ch_j^\omega) - \mu_w \\
&= \mu_w - \langle \alpha_j, \mu_w \rangle h_j - \langle \alpha_j(\omega), \mu_w \rangle h_{\omega(\gamma)} + ch_j^\omega - \mu_w \\
&= (c - \langle \alpha_j, \mu_w \rangle) h_j^\omega.
\end{align*}
\] (3.2.9)

Since \( v \in W^\omega \), it follows from Remark \[2.3.2\](3) and [MP] Proposition 4(i) in §5.2] that \( v^{-1}(h_j^\omega) \) is a negative coroot of \( g^\omega \), and hence \( h_j^\omega \leq_v 0 \). But, since \( c - \langle \alpha_j, \mu_w \rangle \leq 0 \) by assumption, we see from (3.2.9) that

\[
\sigma \tau(\mu_w) \geq_v \mu_v.
\] (3.2.10)

Also, it follows that \( \mu_w \geq_v \mu_v \) since \( \mu_* = (\mu_w)_{w \in V} \in \mathcal{V} \), and that \( \mu_v' = \mu_v \) since \( v \in W^\omega \) and \( \tilde{P}(\mu_*') \) satisfies condition (i) as shown above. Combining these facts with (3.2.10), we deduce that \( \sigma \tau(\mu_v) \geq_v \mu_w \geq_v \mu_v = \mu_v' \). This proves Claim 3.

By Claims \[2\] and \[3\] inequality (3.2.5) holds for all \( v \in W^\omega = W^\omega_+ \cup W^\omega_- \), that is, \( \tilde{P}(\mu_*') \) satisfies condition (v). Thus we have proved that the convex polytope \( \tilde{P}(\mu_*') \) satisfies conditions (i)-(v).

Next, we prove that if \( \tilde{P}'' \in \tilde{P} \) satisfies conditions (i)-(v), then \( \tilde{P}'' \) must contain \( \tilde{P}(\mu_*') \). Let \( \tilde{P}'' = \tilde{P}(\mu_*'') \in \tilde{P} \) be a convex polytope with GGMS datum \( \mu_*'' = (\mu_w'')_{w \in V} \in \mathcal{V} \) satisfying conditions (i)-(v), and set \( (M''_\gamma)_{\gamma \in \Gamma} := D(\mu_*'') \in \mathcal{E}^\omega \). In order to prove that \( \tilde{P}'' \supset \tilde{P}(\mu_*') \), it suffices to show that \( M'_\gamma \geq M''_\gamma \) for all \( \gamma \in \tilde{\Gamma} \) (see (3.2.1)).

Claim 4. The inequality \( M_\gamma \geq M''_\gamma \) holds for all \( \gamma \in \tilde{\Gamma} \).

Since \( \tilde{P}'' = \tilde{P}(\mu_*'') \) satisfies conditions (i) and (iii), it follows that \( \mu_w \in \tilde{P}'' \) for all \( w \in W^\omega = W^\omega_+ \cup W^\omega_- \). Hence, by (3.2.2), we have \( \mu_w \geq_v \mu_w'' \) for all \( w \in W^\omega \), which implies that \( M_\gamma \geq M''_\gamma \) for all \( \gamma \in \tilde{\Gamma} \), as desired. This proves Claim 4.

Claim 5. Let \( \gamma \in \tilde{\Gamma} \) be such that \( \langle \gamma, h_j \rangle = 1 \). Then, we have \( M_{s_j, \gamma} + c \geq M''_\gamma \).

Write the \( \gamma \in \tilde{\Gamma} \) in the form \( \gamma = s_j w \cdot \Lambda_i \), with \( w \in W \) and \( i \in I \). Since \( \langle \gamma, h_j \rangle = 1 > 0 \), it follows from [MP] Proposition 4(i) in §5.2] that \( s_j w < w \). Also, we have

\[
\begin{align*}
\langle \gamma, \sigma(\mu_w) \rangle &= \langle \gamma, s_j \cdot \mu_w + ch_j \rangle = \langle \gamma, s_j \cdot \mu_w \rangle + c \langle \gamma, h_j \rangle = \langle s_j \cdot \gamma, \mu_w \rangle + c \\
&= \langle w \cdot \Lambda_i, \mu_w \rangle + c = M_{w, \Lambda_i} + c = M_{s_j, \gamma} + c.
\end{align*}
\] (3.2.11)
Assume first that \( \langle \alpha_j, \mu_w \rangle \geq c \). Since \( s_j w < w \) as seen above, we have \( \sigma(\mu_w) \in \tilde{P}'' \) by condition (iv). Therefore, \( \langle \gamma, \sigma(\mu_w) \rangle \geq M''_\gamma \) (see (3.2.11)), and hence \( M_{s_j \gamma} + c \geq M''_\gamma \) by (3.2.11). Assume next that \( \langle \alpha_j, \mu_w \rangle < c \). Recall that \( \mu_w \geq s_j w \mu_{s_j w} \) since \( \mu_* = (\mu_w)_{w \in W} \in \mathcal{V} \). Hence it follows that

\[
\langle \gamma, \mu_w \rangle \geq \langle \gamma, \mu_{s_j w} \rangle = M_\gamma.
\]

Because \( \langle \gamma, h_j \rangle = 1 \) and \( c - \langle \alpha_j, \mu_w \rangle > 0 \) by assumption, we have

\[
\langle \gamma, \sigma(\mu_w) \rangle = \langle \gamma, s_j \cdot \mu_w + ch_j \rangle
= \langle \gamma, \mu_w \rangle + \left( c - \langle \alpha_j, \mu_w \rangle \right) \langle \gamma, h_j \rangle > \langle \gamma, \mu_w \rangle
\geq M_\gamma \quad \text{by (3.2.12)}
\geq M''_\gamma \quad \text{by Claim 4}.
\]

Combining this with (3.2.11), we obtain \( M_{s_j \gamma} + c \geq M''_\gamma \). This proves Claim 5.

**Claim 6.** Let \( \gamma \in \tilde{\Gamma} \) be such that \( \langle \gamma, h_{\omega(j)} \rangle = 1 \). Then, we have \( M_{s_{\omega(j)} \gamma} + c \geq M''_\gamma \).

Note that \( \omega(\gamma) \in \tilde{\Gamma} \) by Remark 2.3.1(4), and that \( \langle \omega(\gamma), h_j \rangle = \langle \gamma, h_{\omega(j)} \rangle = 1 \). Hence, by Claim 5, \( M_{s_j \omega(\gamma)} + c \geq M''_{\omega(\gamma)} \). Since \( M_* \in \mathcal{E}_M^\omega \subset \mathcal{E}_w^\omega \) and \( M''_* \in \mathcal{E}_w^\omega \), it follows from Lemma 2.4.2(2) that \( M_{s_j \omega(\gamma)} = M_{s_{\omega(j)} \gamma} \) and \( M''_{\omega(\gamma)} = M''_\gamma \). Therefore, we obtain \( M_{s_{\omega(j)} \gamma} + c \geq M''_\gamma \), as desired. This proves Claim 6.

**Claim 7.** Let \( \gamma \in \tilde{\Gamma} \) be such that \( \langle \gamma, h_j \rangle = \langle \gamma, h_{\omega(j)} \rangle = 1 \). Then, \( M_{s_j \gamma} + 2c \geq M''_\gamma \).

Write the \( \gamma \in \tilde{\Gamma} \) in the form \( \gamma = s_j^\omega w \cdot \Lambda_i \), with \( w \in W^\omega \) and \( i \in I \). Since \( \langle \gamma, h_j \rangle = \langle \gamma, h_{\omega(j)} \rangle = 1 \), it follows from [MP] Proposition 4(i) in §5.2] that \( s_j w < w \) and \( s_{\omega(j)} w < w \). Hence, by Remark 2.3.2(3), \( w \in W^\omega \). Also, we have

\[
\langle \gamma, \sigma \tau(\mu_w) \rangle = \langle \gamma, s_j^\omega \cdot \mu_w \rangle + c \langle \gamma, h_j^\omega \rangle = \langle s_j^\omega \cdot \gamma, \mu_w \rangle + 2c
= \langle w \cdot \Lambda_i, \mu_w \rangle + 2c = M_{w \cdot \Lambda_i} + 2c = M_{s_j \gamma} + 2c.
\]

Assume first that \( \langle \alpha_j, \mu_w \rangle \geq c \). Since \( w \in W^\omega \) as shown above, we have \( \sigma(\mu_w) \in \tilde{P}'' \) by condition (v). Therefore, \( \langle \gamma, \sigma(\mu_w) \rangle \geq M''_\gamma \) (see (3.2.11)), and hence \( M_{s_j \gamma} + 2c \geq M''_\gamma \) by (3.2.13). Assume next that \( \langle \alpha_j, \mu_w \rangle < c \). Recall that \( \mu_w \geq s_j^\omega w \mu_{s_j^\omega w} \) since \( \mu_* = (\mu_w)_{w \in W} \in \mathcal{V} \). Hence it follows that

\[
\langle \gamma, \mu_w \rangle \geq \langle \gamma, \mu_{s_j^\omega w} \rangle = M_\gamma.
\]

Note that since \( w \in W^\omega \subset W^\omega \), we have \( \omega(\mu_w) = \mu_w \) by Lemma 2.4.2(1), and hence \( \langle \alpha_j, \mu_w \rangle = \langle \alpha_{\omega(j)}, \mu_w \rangle \). From the assumptions that \( \langle \gamma, h_j \rangle = \langle \gamma, h_{\omega(j)} \rangle = 1 \) and \( c -
\[ \langle \alpha_j, \mu_w \rangle > 0 \), using the equality \( \langle \alpha_j, \mu_w \rangle = \langle \alpha_{\omega(j)}, \mu_w \rangle \), we have

\[
\langle \gamma, \sigma \tau(\mu_w) \rangle = \langle \gamma, s^\omega_j \cdot \mu_w + ch^\omega_j \rangle \\
= \langle \gamma, \mu_w - \langle \alpha_j, \mu_w \rangle h_j - \langle \alpha_{\omega(j)}, \mu_w \rangle h_{\omega(j)} + ch^\omega_j \rangle \\
= \langle \gamma, \mu_w - \langle \alpha_j, \mu_w \rangle h_j^\omega + ch^\omega_j \rangle \\
= \langle \gamma, \mu_w \rangle + \left( c - \langle \alpha_j, \mu_w \rangle \right) \frac{\langle \gamma, h_j^\omega \rangle - \langle \gamma, \mu_w \rangle}{2} \geq M_\gamma \tag{3.2.14} \]

\[
\geq M_\gamma' \tag{3.2.13} \]

Combining this inequality with \( (3.2.13) \), we obtain \( M_{s^\omega_j \cdot \gamma} + 2c \geq M_\gamma'' \). This proves Claim 7.

By using Claims 4-7, we can show that \( M_\gamma' \geq M_\gamma'' \) for all \( \gamma \in \tilde{\Gamma} \). As an example, let us consider the case in which \( \gamma(h_j) = 1 \) and \( \gamma(h_{\omega(j)}) \leq 0 \). Then,

\[
M_\gamma' = \min( M_\gamma, M_{s^\omega_j \cdot \gamma} + c) \tag{3.2.14} \]

\[
\geq M_\gamma'' \tag{3.2.13} \]

The proofs for the other cases are similar. Thus we have proved that \( \tilde{P}'' \supset \tilde{P}(\mu'_*) \). This completes the proof of Theorem 3.2.3.

The proof of the following theorem is similar to (and even simpler than) that of Theorem 3.2.3.

**Theorem 3.2.4.** Keep the notation above. Assume that \( \ell = 2n - 1 \), \( n \in \mathbb{Z}_{\geq 2} \), and \( j = n \). Then, the convex polytope \( \tilde{P}(\mu'_*) \) is the smallest convex polytope \( \tilde{P} \) in \( \tilde{P} \) with GGMS datum \( \mu''_* = (\mu''_w)_{w \in W} \in V^\omega \) such that

(i) \( \mu''_w = \mu_w \) for all \( w \in W^\omega \),

(ii) \( \mu''_e = \mu_e - h_n^\omega \),

(iii) \( \tilde{P} \) contains \( \mu_w \) for all \( w \in W^\omega \), and

(iv) if \( w \in W^\omega \) is such that \( \langle \alpha_j, \mu_w \rangle \geq c \), then \( \tilde{P} \) contains \( \sigma(\mu_w) \).

The proof of the following theorem is similar to (but, a little more complicated than) that of Theorem 3.2.3; we leave it to the reader.

**Theorem 3.2.5.** Keep the notation above. Assume that \( \ell = 2n \), \( n \in \mathbb{Z}_{\geq 2} \), and \( j = n \). Then, the convex polytope \( \tilde{P}(\mu'_*) \) is the smallest convex polytope \( \tilde{P} \) in \( \tilde{P} \) with GGMS datum \( \mu''_* = (\mu''_w)_{w \in W} \in V^\omega \) satisfying the following conditions (i)-(v):

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(i) If \( w \in W^\omega \), then \( \mu''_w = \mu_w \).

(ii) \( \mu''_e = \mu_e - h^n \).

(iii) If \( w \in W^- \), then \( \mu_w \in \tilde{P} \).

(iv) If \( w \in W \) is such that \( s_j w < w \) and \( \langle \alpha_j, \mu_w \rangle \geq c \), then \( \sigma(\mu_w) \in \tilde{P} \). Also, if \( w \in W \) is such that \( s_{\omega(j)} w < w \) and \( \langle \alpha_{\omega(j)}, \mu_w \rangle \geq c \), then \( \tau(\mu_w) \in \tilde{P} \).

(v) If \( w \in W^- \) is such that \( \langle \alpha_j, \mu_w \rangle \geq c \), then \( \sigma \tau \sigma(\mu_w) \in \tilde{P} \).

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