QUANTUM SCHUR SUPERALGEBRAS AND KAZHDAN–LUSZTIG COMBINATORICS

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Abstract. We introduce the notion of quantum Schur (or $q$-Schur) superalgebras. These algebras share certain nice properties with $q$-Schur algebras such as base change property, existence of canonical $\mathbb{Z}[v, v^{-1}]$-bases, and the duality relation with quantum matrix superalgebra $A(m|n)$. We also construct a cellular $\mathbb{Q}(v)$-basis and determine its associated cells, called super-cells, in terms of a Robinson–Schensted–Knuth super-correspondence. In this way, we classify all irreducible representations over $\mathbb{Q}(v)$ via super-cell modules.

1. Introduction

The quantum Schur (or $q$-Schur) algebra is the key ingredient of a so-called quantum Schur–Weyl theory. This theory investigates a three-level duality relation which includes: (1) quantum Schur-Weyl reciprocity for the quantum enveloping algebra $U(\mathfrak{gl}_n)$ and Hecke algebras $\mathcal{H}(S_r)$ via the tensor space $V_n^{\otimes r}$ — a $U(\mathfrak{gl}_n)$-$\mathcal{H}(S_r)$-bimodule; (The algebras $S(n, r) := \text{End}_{\mathcal{H}}(V_n^{\otimes r})$ are homomorphic images of $U(\mathfrak{gl}_n)$ and are called quantum Schur algebras.) (2) certain category equivalences between categories of $\mathcal{H}$-modules and $S(n, r)$-modules; (3) the realization and presentation problems in which quantum $\mathfrak{gl}_n$ is reconstructed via quantum Schur algebras as a vector space together with certain explicit multiplication formulas on basis elements, and quantum Schur algebras are presented by generators and relations. We refer the reader to Parts 3 and 5 of [7] and the reference therein for a full account of the quantum Schur–Weyl theory, and to [6] for the affine version of the theory. Naturally, one expects a super version of the quantum Schur–Weyl theory.

Schur superalgebras and their quantum analogue have been investigated in the context of (quantum) general linear Lie superalgebras or supergroups; see, e.g., [19], [4], [8], [17]. For example, Mitsuhashi [17] has established (for a generic $q$) the super version of quantum Schur-Weyl reciprocity, and Brundan and Kujawa have investigated representations for Schur superalgebras and provided a surprising application to the proof of Mullineux conjecture. Thus, like quantum Schur algebras, quantum Schur superalgebras will play a decisive role in a super-version of the quantum Schur–Weyl theory.

In this paper, we will investigate quantum Schur superalgebras in the context of Hecke algebras and Kazhdan–Lusztig combinatorics. We will first define a quantum Schur superalgebra as the endomorphism superalgebra of certain signed $q$-permutation modules for Hecke algebras of type $A$. By introducing standard and...
canonical bases, we establish a cell theory for quantum Schur superalgebras. Thus, a super version of Robinson-Schensted-Knuth correspondence is developed to get the cell decomposition and the classification of (ordinary) irreducible representations.

We organize the paper as follows. After a brief review of Hecke algebras and their Kazhdan–Lusztig combinatorics, we discuss, as preparation, some combinatorial facts, including a description of super-representatives of double cosets and the Robinson–Schensted–Knuth (RSK) super-correspondence. We introduce in §5 the notion of quantum Schur superalgebra by using the q-analogues of the modules $M^{\lambda,\mu}$ given in [20, 1.2] and prove that this is the same algebra as given in [17] defined by the tensor superspace. We construct an integral standard basis which is used to construct a Kazhdan–Lusztig type (or canonical) basis in §6. In particular, we establish the base change property. In order to understand its representations over the field $\mathbb{Q}(\upsilon)$, we further introduce another basis, a cellular type basis, over the field via the Kazhdan–Lusztig basis of the Hecke algebra. Thus, cell relations can be introduced and cells modules form a complete set of non-isomorphic irreducible modules. This result can be considered as a generalization of Theorem 1.4 in [14] to the super case. Finally, we prove that quantum Schur superalgebras are the linear dual of the homogeneous components of the quantum general linear supergroup introduced by Manin [16].

Throughout the paper, we make the following notational convention.

Let $m, n$ be nonnegative integers, not both zero. Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ be the set of integers modulo 2. Fix the map

$$\hat{i} : \{1, 2, \ldots, m, m + 1, \ldots, m + n\} \to \mathbb{Z}_2$$

(1.0.1)

such that $\hat{i} = \begin{cases} 0, & \text{if } 1 \leq i \leq m, \\ 1, & \text{if } m + 1 \leq i \leq m + n. \end{cases}$

Let $\mathcal{Z} = \mathbb{Z}[\upsilon, \upsilon^{-1}]$ be the ring of Laurent polynomials in indeterminate $\upsilon$. If $\mathcal{A}$ denotes a $\mathcal{Z}$-algebra, we shall use the same letter of boldface $\mathbf{A}$ to denote the $\mathbb{Q}(\upsilon)$-algebra obtained by base change to $\mathbb{Q}(\upsilon)$. In other words, $\mathbf{A} = \mathcal{A} \otimes \mathbb{Q}(\upsilon)$.

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2. Hecke algebras and their Kazhdan–Lusztig combinatorics

Assume for the moment that $\mathcal{H} = \mathcal{H}(W)$ is the Hecke algebra associated with a Coxeter system $(W, S)$. Thus, $\mathcal{H}$ is an associative $\mathcal{Z}$-algebra with basis $\{T_w\}_{w \in W}$ subject to the relations (where $q = \upsilon^2$)

$$\begin{cases} T_s^2 = (q - 1)T_s + q, & \text{for } s \in S; \\ T_y T_w = T_{yw}, & l(yw) = l(y) + l(w), \end{cases}$$

(2.0.2)
where \( l \) is the length function relative to \( S \). Clearly, \( \mathcal{H} \) admits an anti-involution \( \tau : \mathcal{H} \to \mathcal{H} \) sending \( T_w \) to \( T_{w^{-1}} \). We first briefly review the construction of the canonical (or Kazhdan–Lusztig) bases of Hecke algebras.

Let \( - : \mathcal{H} \to \mathcal{H} \) be the \( \mathbb{Z} \)-linear involution on \( \mathcal{H} \) such that \( \mathbf{v} = \mathbf{v}^{-1} \) and \( T_w = T_{w^{-1}} \).

In [14], Kazhdan and Lusztig showed that, for any \( w \in W \), there is a unique element \( C_w \in \mathcal{H} \) such that \( C_w = C_w \) and

\[
C_w = \mathbf{v}^{-l(w)} \sum_{y \leq w} P_{y,w} (\mathbf{v}^2) T_y
\]

where \( \leq \) is the Chevalley-Bruhat order on \( W \) and \( P_{y,w} \) is a polynomial in \( q = \mathbf{v}^2 \) with degree less than \( \frac{1}{2}(l(w) - l(y) - 1) \) for \( y < w \) and \( P_{w,w} = 1 \). Moreover, \( \{ C_w \mid w \in W \} \) forms a free \( \mathbb{Z} \)-basis of \( \mathcal{H} \).

Let \( \iota \) be the involution on \( \mathcal{H} \) defined by setting

\[
\iota \left( \sum_{w \in W} a_w T_w \right) = \sum_{w \in W} \varepsilon_w (-1)^{l(w)} T_{\iota w}, \text{ where } \varepsilon_w = (-1)^{l(w)}.
\]

Write \( B_w = \iota (C_w) \). Then \( B_w = \sum_{y \leq w} \varepsilon_y \varepsilon_w \mathbf{v}^{l(w)} \mathbf{v}^{-2l(y)} T_{y,w} \). Both \( \{ C_w \}_{w \in W} \) and \( \{ B_w \}_{w \in W} \) are called canonical or Kazhdan-Lusztig bases for \( \mathcal{H} \).

For \( x, y \in W \), let \( \mu(y, w) \) be the coefficient of \( q^{\frac{1}{2}(l(w) - l(y) - 1)} \) in \( P_{y,w} \). The following formulae are due to Kazhdan and Lusztig [14].

For any \( s \in S \) and \( w \in W \),

\[
C_s C_x = \begin{cases} 
(\mathbf{v} + \mathbf{v}^{-1}) C_w, & \text{if } sw < w, \\
C_{sw} + \sum_{y < x, y < g} \mu(y, w) C_y, & \text{if } sw > w.
\end{cases}
\]

Here, \( \leq \) denote the Bruhat ordering of \( W \).

Canonical bases have important applications to representations of Hecke algebras through the notion of cells. Following [14], we define preorder \( \leq_L \) on \( W \) by declaring that \( x \leq_L y \) if there is a sequence \( z_0 = x, z_1, \ldots, z_k = y \) such that \( C_{z_i} \) appears in the expression of \( C_x C_{z_{i+1}} \) with non-zero coefficient for some \( s \in S \). Define \( x \leq_R y \) by declaring that \( x^{-1} \leq_L y^{-1} \). Let \( \leq_{LR} \) be the preorder generated by \( \leq_L \) and \( \leq_R \). The corresponding equivalence relations are denoted by \( \sim_L, \sim_R \) and \( \sim_{LR} \). Call the equivalence classes of \( W \) with respect to \( \sim_L, \sim_R \) and \( \sim_{LR} \), respectively, left cells, right cells and two-sided cells of \( W \).

Let

\[
\mathcal{R}(w) = \{ s \in S \mid ws < w \} \text{ and } \mathcal{L}(w) = \mathcal{R}(w^{-1}).
\]

The following result is well-known. See [14, 2.4(i)].

**Lemma 2.1.** If \( w_1 \leq_L w_2 \), then \( \mathcal{R}(w_1) \supseteq \mathcal{R}(w_2) \). Hence, \( w_1 \sim_L w_2 \) implies \( \mathcal{R}(w_1) = \mathcal{R}(w_2) \).

Every left cell \( \kappa \) defines a left cell module

\[
E^\kappa := \operatorname{span}\{ C_w \mid w \leq_L \kappa \}/\operatorname{span}\{ C_w \mid w <_L \kappa \}.
\]

\(^1\)In [14], \( C_w \) is denoted by \( C'_w \), while \( B_w \) is denoted by \( C_w \).
where \( w \leq_L \kappa \) means \( w \leq_L y \) for some (equivalently, for all) \( y \in \kappa \), and \( w <_L y \) means \( w \leq_L y \) but \( w \neq_L y \).

It is known from [14] that cells for the symmetric group \( \mathfrak{S}_r \), which is a Coxeter group with \( S = \{(1, 2), (2, 3), \ldots, (r - 1, r)\} \), are completely determined via the Robinson–Schensted map and left cell modules form a complete set of all irreducible \( \mathcal{H}_{Q(w)} \)-modules. We now give a brief description of these facts.

For non-negative integers \( N, r \) with \( N > 0 \), a composition \( \lambda \) of \( r \), denoted by \( \lambda \vdash r \), is a sequence \((\lambda_1, \lambda_2, \ldots, \lambda_N)\) of non-negative integers \( \lambda_i \) with \( N \) parts such that \( |\lambda| = \sum_{i=1}^{N} \lambda_i = r \). If such a sequence decreases weakly, then \( \lambda \) is called a partition of \( r \) (with at most \( N \) parts), denoted by \( \lambda \vdash r \).

The Young diagram \( Y(\lambda) \) for a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \vdash r \) is a collection of boxes arranged in left-justified rows with \( \lambda_i \) boxes in the \( i \)-th row of \( Y(\lambda) \). Thus, if \( \lambda \) has \( N \) parts, we say that \( Y(\lambda) \) has \( N \) rows.

A \( \lambda \)-tableau (or a tableau of shape \( \lambda \)) is obtained by inserting integers into boxes of \( Y(\lambda) \). If the entries of a tableau \( s \) are exactly \( 1, 2, \ldots, r \), then \( s \) is called an exact tableau. The symmetric group \( \mathfrak{S}_r \) acts on exact tableaux \( s \) by permuting its entries.

Let \( t^\lambda \) (resp. \( t^s \)) be the \( \lambda \)-tableau obtained from the Young diagram \( Y(\lambda) \) by inserting \( 1, 2, \ldots, r \) from left to right (resp. top to bottom) along successive rows (resp. columns). For example, for \( \lambda = (4, 3, 1) \),

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & \\
8 \\
\end{array}, \quad \begin{array}{ccc}
1 & 4 & 6 \\
2 & 5 & 7 \\
3 \\
\end{array}.
\]

Let \( \mathcal{R}_i^\lambda \) be the \( i \)-th of \( t^\lambda \), and let \( \mathfrak{S}_\lambda \) be the row stabilizer subgroup of \( \mathfrak{S}_r \). For an exact tableau \( s \), if \( w(t^\lambda) = s \), write \( w = d(s) \). Note that \( d(s) \) is uniquely determined by \( s \).

A \( \lambda \)-tableau \( t \) is row (resp., column) increasing if the entries in each row (resp. column) of \( t \) strictly increase from left to right (resp., from top to bottom). It is well known that there is a bijection between the set of all row increasing exact \( \lambda \)-tableaux and the set \( \mathcal{D}_\lambda^{-1} \) of shortest left coset representatives of \( \mathfrak{S}_\lambda \). (Thus, \( \mathcal{D}_\nu \) is the set of right \( \mathfrak{S}_\nu \)-coset representatives of minimal length.) In particular, \( s \) is a row increasing exact \( \lambda \)-tableau if and only if \( d(s) \in \mathcal{D}_\lambda^{-1} \).

For a partition \( \lambda \), an exact \( \lambda \)-tableau is standard if it is both row increasing and column increasing. Let \( T^s(\lambda) \) be the set of all standard \( \lambda \)-tableaux.

Standard tableaux are used to describe elements of \( \mathfrak{S}_r \) via the well-known *Robinson-Schensted correspondence*. This map sets up a bijection

\[
\mathfrak{S}_r \rightarrow \bigcup_{\lambda \in \lambda^+(r)} T^s(\lambda) \times T^s(\lambda), \quad w \xrightarrow{\text{RS}} (P(w), Q(w)); \quad (2.1.1)
\]

see, e.g., [7 Cor. 8.9]. Here \( P(w) \) is a standard tableau obtained by applying an insertion algorithm to \((j_1, j_2, \ldots, j_r)\), where \( w(i) = j_i \), and \( Q(w) \) is the recording tableau; see, e.g., [7 (8.2.5)]. Moreover, we have \( Q(w) = P(w^{-1}) \).

One of the important applications of the Robinson-Schensted correspondence is the decomposition of symmetric groups into Kazhdan–Lusztig cells which are defined
above Lemma 2.1. The following result is given in [14, Th. 1.4]; see [7, Th. 8.25] for a purely combinatorial proof.

**Theorem 2.2.** Suppose \( x, y \in \mathcal{S}_r \). Then

1. \( x \sim_L y \) if and only if \( Q(x) = Q(y) \).
2. \( x \sim_R y \) if and only if \( P(x) = P(y) \).
3. \( x \sim_{LR} y \) if and only if \( P(x) \) and \( P(y) \) have the same shape.

Moreover, let \( \kappa_\lambda \) denote the left cell containing the longest element \( w_{0, \lambda} \) of \( \mathcal{S}_\lambda \) and \( S_\lambda := (E^{\ast \lambda})^{\ast} \) the corresponding dual left cell module. Then \( \{S_{\lambda, Q(\nu)}\}_{\lambda \in \lambda_r} \) is a complete set of non-isomorphic irreducible right \( H_{Q(\nu)} \)-modules and, for any left cell \( \kappa \), \( S_{\lambda, Q(\nu)} \cong (E^{\ast \lambda}_{Q(\nu)})^{\ast} \) if and only if \( \kappa \) lie in the two-sided cell containing \( w_{0, \lambda} \).

This result has a natural generalization to quantum Schur algebras; see [12, (5.3.3)]. We will develop a super version of this result in \$7.8\$.

### 3. Super-representatives of double cosets

Let \( M(m + n, r) \) be the set of \((m + n) \times (m + n)\) matrices \( A = (a_{i,j}) \) with \( a_{i,j} \in \mathbb{N} \) and \( \sum a_{ij} = r \), and let \( M(m + n) = \bigcup_{r \geq 0} M(m + n, r) \). Let

\[
\begin{align*}
\text{ro}(A) &= (\sum_j a_{1,j}, \sum_j a_{2,j}, \ldots, \sum_j a_{n+m,j}) \\
\text{co}(A) &= (\sum_j a_{j,1}, \sum_j a_{j,2}, \ldots, \sum_j a_{j,n+m}).
\end{align*}
\]

Define

\[
M(m|n, r) = \{(a_{ij}) \in M(m + n, r) : a_{ij} \in \{0, 1\} \text{ if } i + j = 1\},
\]

\[
M(m|n) = \bigcup_{r \geq 0} M(m|n, r). \tag{3.0.1}
\]

Let \( \Lambda(N, r) \) (resp. \( \Lambda^+(N, r) \)) be the set of compositions (resp. partitions) of \( r \) with \( N \) parts. We also write \( \Lambda^+(r, r) \) for \( \Lambda^+(r, r) \), the set of partitions of \( r \), and write \( 0 \) for the unique element in \( \Lambda(N, 0) \).

For \((\lambda, \mu) \in \Lambda(m, r_1) \times \Lambda(n, r_2)\), let

\[\lambda \vee \mu = (\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_n) \in \Lambda(m + n, r_1 + r_2).\]

Every element in \( \Lambda(m + n, r) \) has the form \( \lambda \vee \mu \) for some \((\lambda, \mu) \in \Lambda(m, r_1) \times \Lambda(n, r_2)\) with \( r_1 + r_2 = r \). Let

\[
\begin{align*}
\Lambda(m|n, r) &= \{\lambda|\mu : \lambda \in \Lambda(m, r_1), \mu \in \Lambda(n, r_2), \lambda \vee \mu \in \Lambda(m + n, r)\} \\
\Lambda^+(m|n, r) &= \{\lambda|\mu \in \Lambda(m|n, r) : \lambda_1 \geq \cdots \geq \lambda_m, \mu_1 \geq \cdots \geq \mu_n\}. \tag{3.0.2}
\end{align*}
\]

Thus, we may identify \( \Lambda(m|n, r) \) with \( \Lambda(m + n, r) \) via the map \( \lambda|\mu \mapsto \lambda \vee \mu \). Hence,

\[
\mathcal{S}_{\lambda|\mu} := \mathcal{S}_{\lambda \vee \mu} \cong \mathcal{S}_\lambda \times \mathcal{S}_\mu
\]

is well-defined. We will write \( x|y \in \mathcal{S}_{\lambda|\mu} \) to mean that \( x \in \mathcal{S}_\lambda \) and \( y \in \mathcal{S}_\mu \), where

\[
\lambda^* = \lambda \vee (1^r-|\lambda|) \quad \text{and} \quad \mu^* = (1^r-|\mu|) \vee \mu.
\]

In this notation, \( \mathcal{S}_{\lambda|\mu} = \mathcal{S}_\lambda \times \mathcal{S}_{\mu} \). We will also write, for any \( A \in M(m|n, r) \), \( \text{ro}(A) = \lambda|\mu \) or \( \text{co}(A) = \xi|\eta \) as elements in \( \Lambda(m|n, r) \).
For notational simplicity, we will identify \( \nu \) with the set \( S_\nu \cap S \). Let \( D_\nu \) (resp. \( D^+_\nu \)) be the set of right \( S_\nu \)-coset representatives of minimal (resp., maximal) length. Thus, for \( \rho \models r \), the set
\[
D_{\nu, \rho} = D_\nu \cap D_\rho^{-1} \quad \text{(resp., } D^+_{\nu, \rho} = D^+_\nu \cap D^+_\rho \text{)}
\]
consists of minimal (resp. maximal) double coset representatives of double cosets in \( D_\nu \backslash S_r \backslash D_\rho \). In particular, for \( \lambda | \mu \in \Lambda(m|n, r) \), \( D_{\lambda | \mu} \), \( D^+_{\lambda | \mu} \) and \( D^-_{\lambda | \mu, \xi | \eta} \) are defined.

For \( \lambda | \mu \in \Lambda(m|n, r) \), and \( \xi | \eta \in \Lambda(m'|n', r) \), define
\[
D^+_{\lambda | \mu} = D^+_{\lambda} \cap D^*_\mu \quad \text{resp. } D^-_{\lambda | \mu} = D^-_{\lambda} \cap D^*_{\mu},
\]
and
\[
D_{\lambda | \mu, \xi | \eta} = D^+_{\lambda, \xi} \cap D^*_{\mu, \eta} \quad \text{resp. } D^-_{\lambda | \mu, \xi | \eta} = D^-_{\lambda, \xi} \cap (D^*_{\xi | \eta})^{-1}.
\]

It is clear that we have
\[
D^+_{\lambda | \mu, \xi | \eta} = D^+_{\lambda, \xi} \cap D^*_{\mu, \eta} = \begin{cases}
    & \{x \in S_r : sx < x, tx > x, \forall s \in \lambda, t \in \mu, \\
    & xs < x, xt > x, \forall s \in \xi, t \in \eta \},
\end{cases}
\]
\[
D^-_{\lambda | \mu, \xi | \eta} = D^-_{\lambda, \xi} \cap D^*_{\mu, \eta} = \begin{cases}
    & \{x \in S_r : sx > x, tx < x, \forall s \in \lambda, t \in \mu, \\
    & xs > x, xt < x, \forall s \in \xi, t \in \eta \}.
\end{cases}
\]

Moreover, if \( \emptyset \) denotes the empty subset of \( S \) associated with those \( \xi | \eta \) whose components are 0 or 1, then \( D^+_{\lambda | \mu, \emptyset} = D^+_{\lambda | \mu} \).

The following result links the above sets with certain trivial intersection property.

**Lemma 3.1.** Let \( \lambda | \mu, \xi | \eta \in \Lambda(m|n, r) \). For any \( d \in D^+_{\lambda | \mu, \xi | \eta} \), the following are equivalent.

1. \( D^+_{\lambda | \mu, \xi | \eta} \cap \mathcal{S}_{\lambda, \mu} d \mathcal{S}_{\xi, \eta} \neq \emptyset \);
2. \( D^-_{\lambda | \mu, \xi | \eta} \cap \mathcal{S}_{\lambda, \mu} d \mathcal{S}_{\xi, \eta} \neq \emptyset \);
3. \( \mathcal{S}_{\lambda, \mu} \cap d \mathcal{S}_{\xi, \eta} d^{-1} = \{1\} \) and \( \mathcal{S}_{\mu, \eta} \cap d \mathcal{S}_{\xi, \eta} d^{-1} = \{1\} \).

Moreover, if one of the conditions holds, then

1'. \( D^+_{\lambda | \mu, \xi | \eta} \cap \mathcal{S}_{\lambda, \mu} d \mathcal{S}_{\xi, \eta} = D^+_{\lambda, \xi} \cap \mathcal{S}_{\lambda} d \mathcal{S}_{\xi} = \{d^*\} \);
2'. \( D^-_{\lambda | \mu, \xi | \eta} \cap \mathcal{S}_{\lambda, \mu} d \mathcal{S}_{\xi, \eta} = D^-_{\mu, \eta} \cap \mathcal{S}_{\mu} d \mathcal{S}_{\eta} = \{d^*\} \).

**Proof.** Let \( d^* \) be the unique element in \( D^+_{\lambda, \xi} \cap \mathcal{S}_{\lambda} d \mathcal{S}_{\xi} \). Thus, \( D^+_{\lambda | \mu, \xi | \eta} \cap \mathcal{S}_{\lambda, \mu} d \mathcal{S}_{\xi, \eta} \neq \emptyset \) is equivalent to the condition \( d^* \in D^*_{\mu, \eta} \). However,
\[
\mathcal{S}_{\lambda, \mu} \cap d \mathcal{S}_{\xi, \eta} d^{-1} \neq \{1\} \quad \text{(resp., } \mathcal{S}_{\mu, \eta} \cap d \mathcal{S}_{\xi, \eta} d^{-1} \neq \{1\})
\]
\[
\iff \exists s \in \eta \text{ satisfying } t = s d s^{-1} \in \lambda \quad \text{(resp., } \exists s \in \xi \text{ satisfying } t = s d s^{-1} \in \mu)
\]
\[
\iff d^* s < d^* \quad \text{(resp., } t d^* < d^*)
\]
\[
\iff d^* \notin D^*_{\mu, \eta}.
\]

So (1) and (3) are equivalent. A similar argument shows that (2) is equivalent to the conditions \( d^{-1} \mathcal{S}_{\lambda} d \cap \mathcal{S}_{\eta} = \{1\} \) and \( d^{-1} \mathcal{S}_{\mu} d \cap \mathcal{S}_{\tau} = \{1\} \). Hence, (2) and (3) are equivalent. The last assertion follows from definition. \( \square \)
It is well-known that double cosets of the symmetric group can be described in terms of matrices. More precisely, there is a bijection

$$j : \mathfrak{J}(N, r) := \{(\nu, w, \rho) \mid \nu, \rho \in \Lambda(N, r), w \in \mathfrak{D}_{\nu, \rho}\} \rightarrow M(N, r)$$  \hspace{1cm} (3.1.1)

such that if $j(\nu, w, \rho) = A = (a_{ij})$ then $a_{ij} = |R^i_k \cap wR^j_k|$, where $R^k_j$ is the $k$-th row of $t^\lambda$. In other words, for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ and $1 \leq k \leq N$,

$$R^k_j = \{\lambda_1 + \cdots + \lambda_{k-1} + 1, \lambda_1 + \cdots + \lambda_{k-1} + 2, \ldots, \lambda_1 + \cdots + \lambda_{k-1} + \lambda_k\}.$$

Moreover, if $j(\nu, w, \rho) = A$, then $j(\rho, w^{-1}, \nu) = A^t$, the transpose of $A$. We now describe a “super” version of $j$.

**Proposition 3.2.** Let

$$\mathfrak{J}(m|n, r) = \bigcup_{\lambda|\mu, \xi|\eta \in \Lambda(m|n, r)} \{(\lambda|\mu, d, \xi|\eta) : d \in \mathfrak{D}_{\lambda|\mu, \xi|\eta}, \mathfrak{D}^\prime_{\lambda|\mu, \xi|\eta} \cap \mathfrak{S}_{\lambda|\mu}d\mathfrak{S}_{\xi|\eta} \neq \emptyset\}.$$

By restriction, the map $j$ given in (3.1.1) induces a bijection

$$j : \mathfrak{J}(m|n, r) \rightarrow M(m|n, r).$$  \hspace{1cm} (3.2.1)

**Proof.** For $w, y \in \mathfrak{S}_r$, it is well-known that

$$a_{ij} := |R^\lambda_{ij} \cap wR^\xi_{ij}| = |R^\lambda_{ij} \cap yR^\xi_{ij}|$$

whenever $\mathfrak{S}_{\lambda|\mu}w\mathfrak{S}_{\xi|\eta} = \mathfrak{S}_{\lambda|\mu}y\mathfrak{S}_{\xi|\eta}$.

For $\lambda|\mu \in \Lambda(m|n, r)$, if we put $R^\lambda_i = R^\lambda_{ij} \cap wR^\xi_j$ for $1 \leq i \leq m$ and $R^\mu_j = R^\mu_{ij} \cap wR^\xi_j$ for $1 \leq j \leq n$, then

$$a_{ij} = \begin{cases} |R^\lambda_i \cap wR^\xi_j|, & \text{if } i \leq m, j \leq m, \\
|R^\lambda_i \cap wR^\xi_{j-1}|, & \text{if } i \leq m, j \geq m + 1, \\
|R^\mu_{i-m} \cap wR^\xi_j|, & \text{if } i \geq m + 1, j \leq m, \\
|R^\mu_{i-m} \cap wR^\xi_{j-1}|, & \text{if } i \geq m + 1, j \geq m + 1. \\
\end{cases}$$

Now, $w \in \mathfrak{D}^\lambda_{\lambda|\mu, \xi|\eta}$ if and only if both $\mathfrak{S}_{\lambda|\mu}w\mathfrak{S}_{\xi|\eta} \cdot w^{-1} = \{1\}$ and $\mathfrak{S}_{\mu}w\mathfrak{S}_{\xi} \cdot w^{-1} = \{1\}$. This is equivalent to $|R^\lambda_i \cap wR^\xi_j| \leq 1$ and $|R^\mu_{i-m} \cap wR^\xi_j| \leq 1$ for all $1 \leq i \leq m, m+1 \leq j \leq m+n$, or $m+1 \leq i \leq m+n, 1 \leq j \leq m$. Hence, regarding $\mathfrak{J}(m|n, r)$ as a subset of $\mathfrak{J}(m+n, r)$, $j$ sends $\mathfrak{J}(m|n, r)$ into $M(m|n, r)$. So the restriction is well-defined. The bijectivity follows from that of $j$ and the argument above. (One may also use Proposition 3.6 below to see the surjectivity.) \hfill $\Box$

Let

$$\mathfrak{J}(m|n, r)^{\prime \prime} = \bigcup_{\lambda|\mu, \xi|\eta \in \Lambda(m|n, r)} \{(\lambda|\mu, w, \xi|\eta) : w \in \mathfrak{D}^\prime_{\lambda|\mu, \xi|\eta}\},$$

and define $\mathfrak{J}(m|n, r)^{-\prime \prime}$ similarly. The following can be seen easily from Lemma 3.1 and Proposition 3.2.

**Corollary 3.3.** There are bijections

$$j^{\prime \prime} : \mathfrak{J}(m|n, r)^{\prime \prime} \rightarrow M(m|n, r)$$

and

$$j^{-\prime \prime} : \mathfrak{J}(m|n, r)^{-\prime \prime} \rightarrow M(m|n, r)$$  \hspace{1cm} (3.3.1)

such that, if $A = j^{\prime \prime}(\lambda|\mu, w, \xi|\eta) = j^{-\prime \prime}(\lambda|\mu, w', \xi|\eta)$, then $\text{ro}(A) = \lambda|\mu$, $\text{co}(A) = \xi|\eta$.  

The map \( j^{+,-} \) will be used to introduce cell relations on \( M(m|n, r) \) in \( \S 7 \).

The map \( j \) can be used to explicitly describe the shortest and longest elements in the double coset corresponding to a matrix \( A \in M(N, r) \). Write \( w_A^- \) for \( w \) if \( j(\nu, w, \rho) = A \). Then \( w_A^- \) is the shortest element in the double coset \( \mathfrak{S}_\nu w\mathfrak{S}_\rho \). Let \( w_A^+ \) be the longest element in \( \mathfrak{S}_\nu w\mathfrak{S}_\rho \). By [10] (or [7] Exer. 8.2), \( w_A^- \) (resp. \( w_A^+ \)) can be computed as follows: construct a pseudo-matrix \( A_- \) associated with \( A \) by replacing \( a_{1,1} \) by the sequence consisting of the first \( a_{1,1} \) integers of \( \{1, 2, \ldots, r\} \), \( a_{1,2} \) by the sequence of the next \( a_{1,2} \) integers, etc., from left to right down successive rows, and then form the permutation \( w_A^- \) which is obtained by reading \( A_- \) from left to right inside the sequences and from top to bottom, and followed by left to right along successive columns.

**Example 3.4.** If \( A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \), then \( A_- = \begin{pmatrix} (1, 2) & \emptyset & 3 \\ 4 & (5, 6) & \emptyset \\ 7 & (8, 9) & 10 \end{pmatrix} \) and \( w_A^- = (1, 2, 4, 7, 5, 6, 8, 9, 3, 10) \).

By reversing the integers in each row of \( A_- \) and form a pseudo-matrix \( A_+ \), the permutation \( w_A^+ \) is obtained by reading \( A_+ \) from left to right inside the sequences and from bottom to top, and followed by left to right along successive columns. For the example above, we have

\[
A_+ = \begin{pmatrix} 3 & 2 & \emptyset & 1 \\ 6 & (5, 4) & \emptyset \\ 10 & (9, 8) & 7 \end{pmatrix} \quad \text{and} \quad w_A^+ = (10, 6, 3, 2, 9, 8, 5, 4, 7, 1)
\]

We now generalize this construction to the elements in \( \mathfrak{S}(m|n, r)^{+,-} \) and \( \mathfrak{S}(m|\mu, r)^{+,-} \). Write \( w_A^{+,-} \) for \( w \) if \( j^{+,-}(\nu, w, \rho) = A \) and \( w_A^{-+} \) for \( w' \) if \( j^{-+}(\nu, w', \rho) = A \). Suppose \( A = (a_{ij}) \in M(m|n, r) \). By regarding \( A \) as an element in \( M(m + n, r) \), construct a pseudo-matrix \( A_{+, -} \) (resp., \( A_{-, +} \)) by reversing the integers in each row of \( A_- \) for the first \( m \) rows (resp. last \( n \) rows).

Now, define the permutation \( w_A^{+, -} \) (resp., \( w_A^{-+, +} \)) by reading \( A_{+, -} \) (resp., \( A_{-, +} \)) from left to right inside the sequences and from bottom to top (resp., top to bottom), followed by left to right along the first \( m \) successive columns, and then from top to bottom (resp., bottom to top) for the next \( n \) successive columns.

**Example 3.5.** If \( A \) is the matrix as given in Example 3.4 then \( A \in M(m|n, 10) \) for \( m = 1 \) and \( n = 2 \), \( \text{ro}(A) = \lambda|\mu = (3)|(3, 4) \), \( \text{co}(A) = \xi|\eta = (4)|(4, 2) \), and

\[
A_{+, -} = \begin{pmatrix} (3, 2) & \emptyset & 1 \\ 4 & (5, 6) & \emptyset \\ 7 & (8, 9) & 10 \end{pmatrix}, \quad A_{-, +} = \begin{pmatrix} (1, 2) & \emptyset & 3 \\ 6 & (5, 4) & \emptyset \\ 10 & (9, 8) & 7 \end{pmatrix}
\]

Hence, \( w_A^{+,-} = (7, 4, 3, 2, 5, 6, 8, 9, 1, 10) \) and \( w_A^{-+, +} = (1, 2, 6, 10, 9, 8, 5, 4, 7, 3) \).

**Proposition 3.6.** Maintain the notation introduced above. If \( A \in M(m|n, r) \) with \( \text{co}(A) = \xi|\eta \) and \( \text{ro}(A) = \lambda|\mu \) then \( w_A^{+,-} \in \mathfrak{D}^{+,-}_{\lambda|\mu, \xi|\eta} \) and \( w_A^{-+, +} \in \mathfrak{D}^{-+, +}_{\lambda|\mu, \xi|\eta} \).
it follows that

Since

\[ l(w) = \sum_{j=1}^{r} \#\{(j,k) \mid j < k, i_j > i_k\} \]  

(3.6.1)

it follows that \( l(ws_k) = l(w) + 1 \) if and only if \( i_k < i_{k+1} \), while \( l(s_kw) = l(w) + 1 \) if and only if \( \{k, k+1\} \) is a subsequence of \( \{i_1, i_2, \ldots, i_r\} \). Now, the result follows immediately by taking \( w = w_{A}^{1,0} \) of \( w_{A}^{1,0} \).

\[ \square \]

4. Young supertableaux and RSK super-correspondence

Before generalizing 2.2, we need some combinatorial preparations.

For \( \lambda \in \Lambda^{+}(n, r) \) and \( \mu \in \Lambda(n, r) \), a \( \lambda \)-tableau \( S \) of content (or type) \( \mu \mid n \) is the tableau obtained from \( Y(\lambda) \) by inserting each box with numbers \( i, 1 \leq i \leq n \), such that the number \( i \) occurring in \( S \) is \( \mu_i \). If the entries in \( S \) are weakly increasing in each row (resp., column) and strictly increasing in each column (resp., row), \( S \) is called a row (resp., column) semi-standard \( \lambda \)-tableau of content \( \mu \). A \( \lambda \)-semistandard tableau is simply called semistandard tableau sometimes. Let \( T(\lambda, \mu) \) (resp., \( T^{ss}(\lambda, \mu) \)) be the set of all \( \lambda \)-tableaux (resp., semi-standard \( \lambda \)-tableau) of content \( \mu \). If \( T^{ss}(\lambda, \mu) \neq \emptyset \), then \( \lambda \supseteq \mu \).

Fix two non-negative integers \( m, n \) with \( m + n > 0 \), define

\[ \Lambda^{+}(r)_{m|n} = \{ \lambda \in \Lambda^{+}(r), \lambda_{m+1} \leq n \} \]  

(4.0.2)

If \( \lambda \in \Lambda^{+}(r)_{m|n} \), then \( Y(\lambda) \) is inside a hook of height \( m \) and base \( n \) and is called a \( (m, n) \)-hook Young diagram. See, e.g., [2, 2.3] where \( \Lambda^{+}(r)_{m|n} \) is denoted as \( H(m, n; r) \).

The set \( \Lambda^{+}(r)_{m|n} \) is in general not a subset of \( \Lambda(m|n, r) \) or \( \Lambda^{+}(m|n, r) \); see [3.0.2]. However, each partition \( \lambda \in \Lambda^{+}(r)_{m|n} \) uniquely determines a pair of partitions \( \lambda' \) and \( \lambda'' \) with

\[ \lambda' = (\lambda_1, \ldots, \lambda_m), \quad \lambda'' = (\lambda_{m+1}, \lambda_{m+2}, \ldots)^t, \]  

(4.0.3)

where, for \( \nu \vdash r, \nu^t \) denotes the partition dual to \( \nu \). (In other words, the Young diagram \( Y(\nu^t) \) is the transpose of \( Y(\nu) \).) The condition \( \lambda_{m+1} \leq n \) implies \( \lambda'\lambda'' \in \Lambda^{+}(m|n, r) \). Thus, we obtain an injective map

\[ \Lambda^{+}(r)_{m|n} \longrightarrow \Lambda^{+}(m|n, r), \quad \lambda \longmapsto (\lambda', \lambda''). \]  

(4.0.4)

The pair \( (\lambda', \lambda'') \) is sometimes called a dominant weight in a representation theory of quantum general linear superalgebra \( U_q(\mathfrak{gl}(m|n)) \); see, e.g., [18] and [17]. Note that, for \( \lambda \in \Lambda^{+}(r)_{m|n} \), \( Y(\lambda) \) is called an \( (m, n) \)-hook diagram in [11 §4.1].

We now introduce, following [20 §1.2] (cf. [11 Def. 4.1]), the notion of semistandard \( \lambda \)-supertableau of content \( \mu|\nu \).

Let \( \lambda \in \Lambda^{+}(r)_{m|n} \), \( \mu|\nu \in \Lambda(m|n, r) \). A \( \lambda \)-tableau \( S \) of content \( \mu \lor \nu \) is called a semi-standard \( \lambda \)-supertableau of content \( \mu|\nu \) if

a) the entries in \( S \) are weakly increasing in each row and each column of \( S \);

b) the numbers in \( \{1, 2, \ldots, m\} \) are strictly increasing in the columns and the numbers in \( \{m + 1, m + 2, \ldots, m + n\} \) are strictly increasing in the rows.
In other words, a semi-standard \( \lambda \)-supertableau of content \( \mu | \nu \) is a tableau of content \( \mu | \nu \) such that the tableau \( T|_{[1,m]} \) obtained by removing entries \( m+1, \ldots, m+n \) is a (row) semi-standard tableau of content \( \mu \) and the tableau obtained from \( T \) by removing \( T|_{[1,m]} \) is a column semi-standard skew-tableau of content \( \nu \).

Let \( T^{\text{ss}}(\lambda, \mu | \nu) \) be the set of all semi-standard \( \lambda \)-supertableaux of content \( \mu | \nu \). Clearly, \( T^{\text{ss}}(\lambda, \mu | \emptyset) = T^{\text{ss}}(\lambda, \mu) \). Moreover, for \( S \in T^{\text{ss}}(\lambda, \mu | \nu) \), the subtableau obtained by removing all \( i \)-th rows from \( S \) with \( 1 \leq i \leq m \) is column semistandard.

**Example 4.1.** For any \( \lambda \in \Lambda^+(r)_{m,n} \), there is a unique \( \lambda \)-tableau \( T_\lambda \) of content \( \lambda' | \lambda'' \). For example, if \( \lambda = (4, 4, 3, 2, 2, 1) \) and \( m = 2, n = 4 \), then \( \lambda' = (4, 4) \), \( \lambda'' = (4, 3, 1) \) and

\[
T_\lambda = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 4 & 4 & 5 \\
3 & 4 \\
3 & 4 \\
3 \\
\end{array}
\]

The following result is known; see [20, Theorem 2] or [1, Lemma 4.2]. For completeness, we include a proof.

**Lemma 4.2.** For a partition \( \lambda \in \Lambda^+(r) \), \( T^{\text{ss}}(\lambda, \mu | \nu) \neq \emptyset \) for some \( \mu | \nu \in \Lambda(m | n, r) \) if and only if \( \lambda \in \Lambda^+(r)_{m,n} \).

**Proof.** If \( \lambda \in \Lambda^+(r)_{m,n} \), then \( T^{\text{ss}}(\lambda, \lambda' | \lambda'') = \{ T_\lambda \} \neq \emptyset \). Conversely, suppose \( T \in T^{\text{ss}}(\lambda, \mu | \nu) \). Then the numbers \( 1, 2, \ldots, m \) do not appear in the rows below row \( m \). Let \( T'' \) be the transpose of the tableau obtained by removing the first \( m \) rows from \( T \). Replacing every entry \( x \) in \( T'' \) by \( x - m \) yields a semistandard \( \lambda'' \)-tableau with content \( \nu^{(2)} \) for some \( \nu^{(2)} \in \Lambda(n, r_2) \). Now, \( \lambda'' \geq \nu^{(2)} \) implies that \( \lambda_{m+1} \), which is the number of parts of \( \lambda'' \), is less than or equal to the number of parts of \( \nu^{(2)} \), which is \( \leq n \). Hence, \( \lambda \in \Lambda^+(r) \). \( \square \)

The RSK super-correspondence is about a bijection between \( M(m | n, r) \) and the pairs of semistandard super tableaux of the same shape. Since the correspondence will be used to describe super-cells and associated modules, our construction relies on the relationship between Kazhdan-Lusztig cells of \( \mathfrak{S}_r \) and their combinatorial characterization.

For a fixed \( w \in \mathfrak{S}_r \) and \( T \in T(\lambda, \mu) \). Define \( w_T \in \mathfrak{S}_r \) by letting \( w_T(t^i) \) be the row standard \( \mu \)-tableau such that the integers in the \( i \)-th row of \( w_T(t^i) \) are the entries of \( w(t^i) \) whose positions are the same as those of the \( \mu_i \) entries \( i \) in \( T \). It is easy to see that the map \( T(\lambda, \mu) \rightarrow \mathcal{D}_\mu^{-1}, T \mapsto w_T \) is bijective. The inverse \( T_w^{\lambda | \mu} \) of this map can be defined as follows: for \( x \in \mathcal{D}_\mu^{-1} \), define \( T_w^{\lambda | \mu}(x) \in T(\lambda, \mu) \) by specifying that, for all \( i, j \), if the entry in \((i, j)\) position of \( w(t^i) \) is \( a \), then the entry in the same position in \( T_w^{\lambda | \mu}(x) \) is the row index of \( a \) in the row standard \( \mu \)-tableau \( x(t^i) \).
Example 4.3. If $\lambda = (431)$, $\mu = (3,2,2,1)$, $w = w_{0,\lambda}$ the longest element in $S\lambda$, and $T = \begin{array}{cccc}
1 & 1 & 1 & 2 \\
2 & 3 & 4 & 3 \\
3 & & & \\
4 & & & 
\end{array}$, then

$$w_{0,\lambda}(t^i) = \begin{array}{cccc}
4 & 3 & 2 & 1 \\
7 & 6 & 5 & 1 \\
8 & & & 6 \\
5 & & & 8
\end{array} \quad \text{and} \quad (w_{0,\lambda})_T(t^j) = \begin{array}{cccc}
2 & 3 & 4 & \\
1 & 7 & & \\
6 & 8 & & \\
5 & & & 
\end{array}$$

If we write $\mu$ as $\xi|\eta = (3,2)|(2,1)$, then

$$(w_{0,\lambda})_T w_{0,\xi}(t^{\xi|\eta}) = \begin{array}{cccc}
4 & 3 & 2 & \\
7 & 1 & & \\
6 & 8 & & \\
5 & & & 
\end{array}$$

Observe from the example that the tableau $(w_{0,\lambda})_T w_{0,\xi}(t^{\xi|\eta})$, where $\mu = \xi|\eta \in \Lambda(m|n,r)$, is obtained from $(w_{0,\lambda})_T(t^{\xi|\eta})$ by reversing the entries in the $i$th-rows for each $i$, $1 \leq i \leq m$.

We say that $i = (i_1, i_2, \ldots, i_k)$ is a subsequence of $j = (j_1, j_2, \ldots, j_l)$ if it is obtained from $j$ by deleting some entries of $j$.

Lemma 4.4. For $\lambda \vdash r, \mu|\nu \in \Lambda(m|n,r)$, and $T \in T(\lambda, \mu \lor \nu)$, let $(w_{0,\lambda})_T w_{0,\mu} = (j_1, j_2, \ldots, j_r)$ be the permutation sending $i$ to $j_i$. Then, $T \in T^{ss}(\lambda, \mu|\nu)$ if and only if the rows $R_i$ and columns $C_j$ of $w_{0,\lambda}(t^i)$ are all subsequence of $j_1, j_2, \ldots, j_r$.

Proof. If $y = (w_{0,\lambda})_T w_{0,\mu} = (j_1, j_2, \ldots, j_r)$, then $(w_{0,\lambda})_T w_{0,\mu}(t^{\nu|\nu})$ has sequence $j_1, \ldots, j_{\mu_1}$ in the first row, $j_{\mu_1+1}, \ldots, j_{\mu_1+\mu_2}$ in the second and so on, and the first $m$ rows are obtained by reversing the first $m$ rows of $(w_{0,\lambda})_T(t^{\nu|\nu})$, while the next $n$ rows are the same as the corresponding rows of $(w_{0,\lambda})_T(t^{\nu|\nu})$. In particular, the first $m$ rows are decreasing, while the next $n$ rows are increasing.

A column of $T$ has the form $a_1 a_2 \ldots a_0 a b b \ldots$ (from top to bottom) with $p$ a’s, $q$ b’s and so on for some $l, p, q, \geq 0$, where $a_1 < a_2 \cdots < a_l \leq m < a < b \cdots \leq m+n$. By definition, the first $l$ members of $C_j$ are placed in different rows of $(w_{0,\lambda})_T(t^{\nu|\nu})$ (and hence of $(w_{0,\lambda})_T w_{0,\mu}(t^{\nu|\nu})$) with row indexes $a_1, a_2, \ldots, a_l$, and then the next $p$ members of $C_j$ are placed (as a whole) in row $a$ and the next $q$ members are in row $b$, and so on. Note that $a, b, \cdots$ are strictly great than $m$. Hence, $C_j$ is a subsequence of $j_1, j_2, \ldots, j_r$. This proves the result for $C_j$ for all $j$’s.

Likewise, the $i$th row of $T$ has the form $a a b b \ldots a_1 a_0 a_1 a_2 \ldots a_0 b b \ldots$ (from top to bottom) with $p$ a’s, $q$ b’s and so on for some $l, p, q, \geq 0$, where $a < b \cdots \leq m < a_1 < a_2 < \cdots < a_l \leq m+n$. Thus, the first $p$ members of $R_i$ (as a whole) form part of the row $a$ of $(w_{0,\lambda})_T(t^{\nu|\nu})$ (and hence of $(w_{0,\lambda})_T w_{0,\mu}(t^{\nu|\nu})$ since they are decreasing), the next $q$ members form part of row $b$, and so on. Then the members of $R_i$ are placed in different rows between row $m+1$ and row $m+n$. Hence, $R_i$ is a subsequences of $j_1, j_2, \ldots, j_r$.

The argument above shows that if either the $i$th row or $j$th column of $T$ is not (weakly) increasing, then either $R_i$ or $C_j$ is not a subsequences of $j_1, j_2, \ldots, j_r$, proving the lemma.

The following result is the key to the establishment of the RSK super-correspondence.
Proposition 4.5. Suppose $\mu|\nu \in \Lambda(m|n,r)$ and $\lambda \in \Lambda(r)^+$. If $\varpi_\lambda$ denotes the right cell of $\mathfrak{S}_r$ containing $w_{0,\lambda}$, then

$$\mathfrak{D}^+_{\lambda(0),\mu|\nu} \cap \varpi_\lambda = \{(w_{0,\lambda})^T w_{0,\mu} \mid T \in \mathfrak{T}^{ss}(\lambda,\mu|\nu)\}.$$ 

Proof. By [9, 3.2] or more precisely, [7, Lem. 8.20], we have

$$\varpi_\lambda = \{(w_{0,\lambda})_t \mid t \in \mathfrak{T}^s(\lambda)\},$$

where $(w_{0,\lambda})_t$ is simply defined by $(w_{0,\lambda})_t(t) = w_{0,\lambda}(t^\lambda)$. Hence,

$$\mathfrak{D}^+_{\lambda(0),\mu|\nu} \cap \varpi_\lambda = \{(w_{0,\lambda})_t \mid t \in \mathfrak{T}^s(\lambda), (w_{0,\lambda})_t \in \mathfrak{D}^+_{\lambda(0),\mu|\nu}\}.$$ 

We now prove that

$$\{(w_{0,\lambda})_t \mid t \in \mathfrak{T}^s(\lambda), (w_{0,\lambda})_t \in \mathfrak{D}^+_{\lambda(0),\mu|\nu}\} = \{(w_{0,\lambda})^T w_{0,\mu} \mid T \in \mathfrak{T}^{ss}(\lambda,\mu|\nu)\}.$$ 

If we put $\omega = (1^r)$, then $\mathfrak{T}^s(\lambda) = \mathfrak{T}^s(\lambda,\omega)$. Suppose $t \in \mathfrak{T}^s(\lambda)$ and $(w_{0,\lambda})_t \in \mathfrak{D}^+_{\lambda(0),\mu|\nu}$. Then by definition $x = (w_{0,\lambda})_t w_{0,\mu} \in \mathfrak{D}^+_{\mu|\nu}$ and $T = (w_{0,\lambda})_t w_{0,\mu} \in \mathfrak{T}^s(\lambda,\mu|\nu)$ so that $x = (w_{0,\lambda})^T$ and $(w_{0,\lambda})_t = (w_{0,\lambda})^T w_{0,\mu}$. We claim that $T \in \mathfrak{T}^{ss}(\lambda,\mu|\nu)$. Indeed, suppose $(w_{0,\lambda})_t = (i_1, i_2, \ldots, i_r)$. By applying Lemma [4] to the case where $\mu|\nu = \omega|0$, $t$ is standard implies that the rows $R_t$ and columns $C_j$ of $w_{0,\lambda}(t^\lambda)$ are subsequences of $i_1, i_2, \ldots, i_r$. Thus, the same lemma (applied to $(w_{0,\lambda})^T w_{0,\mu} = (i_1, i_2, \ldots, i_r)$) implies that $T \in \mathfrak{T}^{ss}(\lambda,\mu|\nu)$.

Conversely, for any $T \in \mathfrak{T}^{ss}(\lambda,\mu|\nu)$, assume $(w_{0,\lambda})^T w_{0,\mu} = (j_1, j_2, \ldots, j_r)$. By Lemma [4], the rows $R_t$ and columns $C_j$ of $w_{0,\lambda}(t^\lambda)$ are subsequences of $j_1, j_2, \ldots, j_r$. Suppose $R_1 = \{j_1, \ldots, j_{i_1}\}$, $R_2 = \{j_{i_1+1}, \ldots\}$ and so on. Then the $\lambda$-tableau $t$ obtained by putting $i_1, \ldots, i_{\lambda_1}, i_{\lambda_1+1}, \ldots$ from left to right down successive rows is standard and $(w_{0,\lambda})_t = (w_{0,\lambda})^T w_{0,\mu}$. \hfill \Box

Corollary 4.6. For $\mu|\nu \in \Lambda(m|n,r)$, $\mathfrak{D}^+_{\theta,\mu|\nu}$ is a union of left cells. For $\lambda \vdash r$, if $K_\lambda \cap \mathfrak{D}^+_{\theta,\mu|\nu}$ is a union of left cells containing $w_{0,\lambda}$, then the number $m_{\lambda,\mu|\nu}$ of left cells in $K_\lambda \cap \mathfrak{D}^+_{\theta,\mu|\nu}$ is $|\mathfrak{T}^{ss}(\lambda,\mu|\nu)|$.

Proof. Since

$$\mathfrak{D}^+_{\theta,\mu|\nu} = \{w \in \mathfrak{S}_r \mid R(w) \supseteq \mu, R(w) \cap \nu = \emptyset\} = \{(\mathfrak{D}^+_{\mu})^{-1} \mid R(w) \cap \nu = \emptyset\},$$

and $(\mathfrak{D}^+_{\mu})^{-1}$ is a union of left cells $\kappa$ satisfying $R(\kappa) \supseteq \mu$, it follows that $\mathfrak{D}^+_{\theta,\mu|\nu}$ is a union of left cells $\kappa$ in $(\mathfrak{D}^+_{\mu})^{-1}$ satisfying $R(\kappa) \cap \nu = \emptyset$. Hence, by Proposition 4.5

$$m_{\lambda,\mu|\nu} = |\mathfrak{D}^+_{\theta,\mu|\nu} \cap K_\lambda \cap \varpi_\lambda| = |\mathfrak{D}^+_{\lambda,\mu|\nu} \cap \varpi_\lambda| = |\mathfrak{T}^{ss}(\lambda,\mu|\nu)|,$$

as required. \hfill \Box

Assume $\mu|\nu \in \Lambda(m|n,r)$. For $T \in \mathfrak{T}^{ss}(\lambda,\mu^*)$, replacing $\nu_1$ entries $m+1, \ldots, m+\nu_1$ of $T$ by $m+1$, $\nu_2$ entries $m+\nu_1+1, \ldots, m+\nu_1+1+\nu_2$ by $m+2$, and so on, yields a $\lambda$-tableau $T^\circ$ of type $\mu \lor \nu$, which may not be in $\mathfrak{T}^{ss}(\lambda,\mu|\nu)$. Let

$$\mathfrak{T}^{ss}(\lambda,\mu^*)^\circ = \{T \in \mathfrak{T}^{ss}(\lambda,\mu^*) : T^\circ \in \mathfrak{T}^{ss}(\lambda,\mu|\nu)\}.$$ 

2 A right action was used for the symmetric group $\mathfrak{S}_r$ in [9]. Thus, the left cell containing $w_{0,\lambda}$ was used there.
Thus, we may identify \( T^{ss}(\lambda, \mu|\nu) \) as the subset \( T^{ss}(\lambda, \mu|\nu) \circ \) of \( T^{ss}(\lambda, \mu^*) \). This identification is compatible with the inclusion \( D_{\lambda,\mu|\nu}^+ \cap \varpi_\lambda \subseteq D_{\lambda,\mu^*|\nu}^+ \cap \varpi_\lambda \).

We are now ready to describe RKS super-correspondence. Suppose \( w \in D_{\lambda,\mu|\nu}^+ \). Let \( (P(w), Q(w)) = (s, t) \) be the image of \( w \) under the Robinson-Schensted map, i.e., \( w \xrightarrow{RS} (s, t) \). Let \( \nu' \) be the shape of \( s \) where \( \nu' \) is the partition dual to \( \nu \). Define \( x, y \in S_r \) such that \( P(x^{-1}) = s, Q(x^{-1}) = t, P(y) = t, \) and \( Q(y) = t \). Since \( P(w_{0,\nu}) = Q(w_{0,\nu}) = t, \) by Theorem 2.2, \( w_{0,\nu} \sim_L x^{-1} \sim_R w \) and \( w_{0,\nu} \sim_R y \sim_L w \).

Thus, by Lemma 2.1 \( R(x) = L(w), R(y) = L(w) \) and \( L(x) = L(y) = \nu \). This implies that \( x \in D_{\nu,0,\lambda|\nu}^+ \cap \varpi_\nu \) and \( y \in D_{\nu,0,\lambda|\nu}^+ \cap \varpi_\nu \), where \( \varpi_\nu \) is the right cell of \( S_r \) which contains \( w_{0,\nu} \). By Proposition 4.5, there is a pair of semi-standard \( \nu \)-tableaux \( (S_w, T_w) \in T^{ss}(\nu, |\lambda, \mu| \times T^{ss}(\nu, |\xi| \eta), \) which are determined uniquely by \( x \) and \( y \), respectively. In particular, \( \nu \in \Lambda^+(r)_{m|n} \). Thus, we obtain a map

\[
\partial = \partial_{\lambda,\mu,\xi|\eta} : D_{\lambda,\mu,\xi|\eta}^+ \rightarrow \bigcup_{\nu \in \Lambda^+(r)_{m|n}} T^{ss}(\nu, |\lambda, \mu| \times T^{ss}(\nu, |\xi| \eta), \ w \mapsto (S_w, T_w). (4.6.1)
\]

The symmetry of the Robinson-Schensted correspondence implies that the map \( \partial \) satisfies a similar property:

\[
\partial(w) = (S_w, T_w) \implies \partial(w^{-1}) = (T_w, S_w).
\]

**Theorem 4.7.** The maps \( \partial_{\lambda,\mu,\xi|\eta} \) for any \( \lambda|\mu, \xi|\eta \in \Lambda(m|n, r) \), are bijection which induce a bijective correspondence

\[
M(m|n, r) \xrightarrow{\text{RSK}} \bigcup_{\nu \in \Lambda^+(r)_{m|n}} T^{ss}(\nu, |\lambda, \mu| \times T^{ss}(\nu, |\xi| \eta), \ A \xrightarrow{\text{RSK}} (S(A), T(A)).
\]

Moreover, if \( A \xrightarrow{\text{RSK}} (S, T) \) then \( A' \xrightarrow{\text{RSK}} (S, T) \).

**Proof.** By Proposition 5.2 and (4.6.1), we need only construct the inverse map \( \partial^{-1} \) of \( \partial = \partial_{\lambda,\mu,\xi|\eta} \) for the first assertion. By Proposition 4.5 each pair \( (S, T) \in T^{ss}(\nu, |\lambda, \mu| \times T^{ss}(\nu, |\xi| \eta) \) defines two elements \( x = (w_{0,\nu})_s w_{0,\lambda} \in D_{\nu,0,\lambda|\nu}^+ \cap \varpi_\nu \) and \( y = (w_{0,\nu})_t w_{0,\xi} \in D_{\nu,0,\xi|\nu}^+ \cap \varpi_\nu \). By [9, 3.2] (cf. footnote 2), \( P(x) = P(y) = t, \) \( x \sim_L w_{0,\nu} \sim_R y \). Thus, \( L(w) = \mathcal{R}(x) \) and \( \mathcal{R}(w) = \mathcal{R}(y) \). Hence, \( w \in D_{\lambda,\mu,\xi|\eta}^+ \) and \( \partial^{-1}(S, T) = w \). The last assertion is clear.

We give an example to illustrate the proof.

**Example 4.8.** Let \( \nu = (3, 3, 1) \) and \( m = 1 \) and \( n = 3 \). Then \( \nu' = (3) \), \( \nu'' = (2, 1, 1) \) and \( T^{ss}(\nu, \nu'|\nu'') \) = \( \{T\} \) with

\[
T = \begin{array}{ccc}
1 & 1 & 1 \\
2 & 3 & 4 \\
2 &
\end{array}
\]
Thus, 

\[
\begin{align*}
    w_{0,\nu}(t') & = \begin{pmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 7 \end{pmatrix}, \\
    (w_{0,\nu})^T(t'' \nu t'') & = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 4 \\ 5 \end{pmatrix}.
\end{align*}
\]

Hence, \( x = y = (w_{0,\nu})_T w_{0,\nu'} = (3, 2, 1, 6, 7, 5, 4) \in \mathcal{D}^{\nu \nu'}_{\nu' \nu} \) and \( Q(x) = Q(y) = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 6 \\ 3 & 7 \end{pmatrix} \). Hence, \( \partial^{-1}(\mathcal{T}, \mathcal{T}) = w \in \mathcal{D}^{\nu \nu'}_{\nu' \nu} \) where \( w \xrightarrow{RS} (Q(x), Q(y)) \).

This bijective correspondence is called the Robinson–Schensted–Knuth super-correspondence. We will write, for any \( A \in M(m|n, r), A \xrightarrow{\text{RSKs}} (S, T) \) if \( S(A) = T \) and \( T(A) = T \).

**Remark 4.9.** (1) This correspondence is the super version of the correspondence given in [12, §5.3]; cf. [7, Remark 9.26]. This correspondence is different from the so-called \((m, n)\)-RoSch correspondence described in [2, 2.5].

5. Signed \( q \)-permutation modules and Quantum Schur superalgebras

The Hecke algebra \( \mathcal{H} = \mathcal{H}(r) \) associated to the symmetric group \( \mathfrak{S}_r \) is an associative \( \mathcal{Z} \)-algebra generated by \( T_i, 1 \leq i \leq r-1 \) subject to the relations (where \( q = \nu^2 \))

\[
\begin{cases}
    T_i^2 = (q-1)T_i + q, & \text{for } 1 \leq i \leq r-1, \\
    T_i T_j = T_j T_i, & \text{for } 1 \leq i < j \leq r-1, \\
    T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & \text{for } 1 \leq i \leq r-2.
\end{cases}
\]

(5.0.1)

For any commutative ring \( R \) which is a \( \mathcal{Z} \)-algebra, let \( \mathcal{H}_R \) be the algebra obtained by base change to \( R \). Let \( v, q \) be the images of \( \nu, q \) in \( R \), respectively.

For each \( \lambda|\mu \in \Lambda(m|n, r) \), define

\[
x_{\lambda} = \sum_{w \in \mathfrak{S}_{\lambda^*}} T_w, \quad y_{\mu} = \sum_{w \in \mathfrak{S}_{\mu^*}} (-q)^{-l(w)} T_w
\]

where \( l(w) \) is the length of \( w \). The \( \mathcal{H} \)-module \( x_{\lambda} \mathcal{H} \) is called a \( q \)-permutation module. We call \( x_{\lambda} y_{\mu} \mathcal{H} \) a signed \( q \)-permutation module. These modules share certain nice properties with \( q \)-permutation modules; cf. e.g., [7, §7.6]. We continue to follow the notation used in §3. Thus, a composition \( \lambda|\mu \) is identified with the set \( \mathfrak{S}_\lambda \cap S \).

**Lemma 5.1.** Let \( \lambda|\mu \in \Lambda(m|n, r) \).

(1) The right \( \mathcal{H}_R \)-module \( x_{\lambda} y_{\mu} \mathcal{H}_R \) is free with basis \( \{ x_{\lambda} y_{\mu} T_d \}_{d \in \mathfrak{S}_{\lambda|\mu}} \).

(2) \( x_{\lambda} y_{\mu} \mathcal{H}_R = \{ h \in \mathcal{H} : T_i h = q h, T_i h = -h, \forall s \in \lambda^*, t \in \mu^* \} \).

(3) \( (\mathcal{H}_R x_{\lambda} y_{\mu})^* := \text{Hom}_R(\mathcal{H}_R x_{\lambda} y_{\mu}, R) \cong x_{\lambda} y_{\mu} \mathcal{H}_R \).

**Proof.** Statement (1) is clear. For \( h = \sum_w f_w T_w \in \mathcal{H}_R, T_i h = q h, T_i h = -h \) imply \( f_w = f_{sw} \) and \( f_{tw} = -q^{-1} f_w \) for all \( s \in \lambda, t \in \mu \) with \( tw > w \), which force

\[
h = \sum_{x \in \mathfrak{S}_{\lambda^*}, y \in \mathfrak{S}_{\mu^*}, d \in \mathfrak{S}_{\lambda|\mu}} (-q)^{-l(y)} f_d T_x T_y T_d = x_{\lambda} y_{\mu} \sum_{d \in \mathfrak{S}_{\lambda|\mu}} f_d T_d.
\]
The converse inclusion is clear, proving (2). For (3), consider the “trace form”
\[
\langle , \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{Z}, \quad \langle a, b \rangle = \text{tr}(ab),
\]
where \(\text{tr}(\sum_w f_w T_w) = f_1\). A direct computation shows that \(\langle x_\lambda y_{\mu} T_u, T_v x_\lambda y_{\mu} \rangle = \delta_{u,v} - 1) q^{(u)} P_{\lambda, \mu}(q)\), where \(P_{\lambda, \mu}(q)\) is the Poincare Polynomial of \(\mathcal{S}_{\lambda, \mu}\). Thus, we obtain a perfect paring
\[
\langle , \rangle : x_\lambda y_{\mu} \mathcal{H} \times x_\lambda y_{\mu} \rightarrow \mathbb{Z}, \quad (x_\lambda y_{\mu} T_u, T_v x_\lambda y_{\mu}) = \delta_{u,v} - 1) q^{(u)}.
\]
Now base change gives the required perfect paring for the isomorphism. \(\square\)

For a composition \(\mu \vdash r\), let \(\tilde{\mu}\) be the partition obtained by rearranging the parts of \(\mu\). If \(\mu \in \Lambda(m, r_1)\) with \(r_1 \leq r\), define \(\tilde{\mu} = \mu \cup (1^{r-r_1})\). Then \(\tilde{\mu} = \Lambda(m+r-r_1, r)\).

The following result can be considered as the quantum version of [20, Lem 3]. Recall, for \(\lambda \vdash r\), the Specht module \(S_{\lambda}\) of \(\mathcal{H}\) associated with \(\lambda\) defined as a dual cell module in Theorem 2.2.

**Proposition 5.2.** For any \(\mu, \nu \in \Lambda(m, n, r)\), we have
\[
x_\mu y_\nu \mathcal{H}_{Q(\nu)} \cong \bigoplus_{\lambda \in \Lambda^+(r)_{m,n}, \lambda \supseteq \tilde{\mu}^*} m_{\lambda, \mu} S_{\lambda, Q(\nu)}.
\]

If \(\tilde{\mu}^* \in \Lambda^+(r)_{m,n}\), then \(S_{\mu, Q(\nu)}\) is a direct summand of \(x_\mu y_\nu \mathcal{H}_{Q(\nu)}\) with multiplicity 1.

**Proof.** Consider the basis \(\{C_w \mid R(w) \supseteq \mu\}\) and a left cell filtration for \(\mathcal{H}_{x_\mu}\):
\[
\mathcal{H}_{x_\mu} = E_{0}^\mu \supseteq E_{1}^\mu \supseteq \cdots \supseteq E_{n-1}^\mu \supseteq E_{n}^\mu = 0.
\]

Since the set \(\{C_w y_\nu \mid R(w) \supseteq \mu\}\) \{0\} forms a basis for \(\mathcal{H}_{x_\mu y_\nu}\), this filtration induces a filtration of \(\mathcal{H}_{x_\mu y_\nu}\):
\[
\mathcal{H}_{x_\mu y_\nu} = E_{0}^{\mu \nu} \supseteq E_{1}^{\mu \nu} \supseteq \cdots \supseteq E_{n}^{\mu \nu} = 0
\]
with subfactors isomorphic to left cell modules. Applying Lemma 5.1 yields a dual left cell filtration for \(x_\mu y_\nu \mathcal{H}\):
\[
0 = F_{0}^{\mu \nu} \subseteq F_{1}^{\mu \nu} \subseteq \cdots \subseteq F_{n}^{\mu \nu} = x_\mu y_\nu \mathcal{H}
\]
where \(F_{j}^{\mu \nu} = (\mathcal{H}_{x_\mu y_\nu} / E_{j}^{\mu \nu})^*\). The required isomorphism follows from base change to \(Q(\nu)\), Corollary 4.6 and Proposition 2.2. The last equality follows from the fact that \(m_{\mu, \nu} = |T^{ass}(\tilde{\mu}^*, \mu, \nu)| = 1\) if \(\tilde{\mu}^* \in \Lambda^+(r)_{m,n}\). \(\square\)

**Remark 5.3.** When \(\mu, \nu \in \Lambda(m, n, r)\) with \(r - |\mu| > n\), \(S_{\nu}^{\mu} S_{\nu}^{\mu}\) is not a direct summand of \(x_\mu y_\nu \mathcal{H}_{Q(\nu)}\). However, when \(\mu \in \Lambda^+(r)_{m,n}\), \(S_{\nu}^{\mu}\) is a direct summand of \(x_\mu y_\nu \mathcal{H}_{Q(\nu)}\) with multiplicity 1 since \(|T^{ass}(\mu, \nu)| = 1\).

For \(D = \lambda, \mu, d, \xi, \eta) \in M(m, n, r)\) (see (3.3.1)), we identify \(D\) with the double coset \(D = \mathcal{G}_{\mu, \xi} d \mathcal{G}_{\xi, \eta}\). Since \(d \in \mathcal{D}_{\alpha, \beta} \cap D\) is the shortest element in \(D\), every \(w \in D\) can be uniquely written as \(w = x.y.d.u.v\) with \(x \in \mathcal{G}_{\lambda, \mu}\) and \(u | v \in \mathcal{S}_{\xi, \eta} \cap \mathcal{D}_{\alpha, \beta}\), where \(\alpha = \alpha(D), \beta = \beta(D)\) are compositions of \(|\xi|\) and \(|\eta|\), respectively, defined by
\[
\mathcal{G}_{\alpha} = d^{-1} \mathcal{G}_{\lambda, \mu} d \cap \mathcal{G}_{\xi, \eta} \quad \text{and} \quad \mathcal{G}_{\beta} = d^{-1} \mathcal{G}_{\rho} d \cap \mathcal{G}_{\eta}.
\] (5.3.1)
By Lemma 3.1, \( S_{\lambda\mu\xi} = d^{-1}\mathcal{S}_{\lambda\mu}d \cap \mathcal{S}_{\xi\eta} = \mathcal{S}_\alpha \cap \mathcal{S}_\beta \). Define
\[
T_D = \sum_{u[v] \in \mathcal{S}_{\xi\eta} \cap \mathcal{D}_{\alpha\beta}} (-q)^{-l(v)} x_\lambda y_\mu T_d T_u T_v.
\]
(5.3.2)

It is clear from the definition that
\[
T_D = x_\lambda y_\mu h_1 = h_2 x_\xi y_\eta = h'_1 x_\lambda T_d y_\eta h''_1 = h'_2 x_\lambda T_d y_\xi h''_2
\]
for some \( h_1, h_2, h'_1, h''_1, h'_2, h''_2 \in \mathcal{H}_R \).

**Remark 5.4.** The element \( T_D \) is also defined for any \( D = j(\lambda|\mu, \xi|\eta) \in M(m+n, r) \). However, if \( d \in \mathcal{D}_{\lambda\mu, \xi|\eta} \) does not satisfy the two trivial intersection properties in Lemma 3.2(3), then \( T_D = 0 \).

We will continue to make the following identification in the sequel.
\[
M(m|n, r)_{\lambda|\mu, \xi|\eta} := \{ D \in M(m|n, r) : ro(D) = \lambda|\mu, co(D) = \xi|\eta \}
\]
\[
= \{ D \in \mathcal{S}_{\lambda\mu} \cap \mathcal{S}_{\xi\eta} : D \cap \mathcal{D}_{\lambda\mu, \xi|\eta} \neq \emptyset \}.
\]
(5.4.1)

**Proposition 5.5.** If \( \mathcal{H}^{+,-}_{\lambda|\mu, \xi|\eta} \) denotes the free \( R \)-submodule of \( \mathcal{H}_R \) spanned by \( T_D \) for all \( D \in M(m|n, r)_{\lambda|\mu, \xi|\eta} \), then
\[
\mathcal{H}^{+,-}_{\lambda|\mu, \xi|\eta} = x_\lambda y_\mu \mathcal{H}_R \cap \mathcal{H}_R x_\xi y_\eta
\]
\[
= \{ h \in \mathcal{H} : T_s h = h T_1 = q h, T_s h = h T_2 = -h, \forall s_1 \in \lambda^*, s_2 \in \mu, t_1 \in \xi^*, t_2 \in \eta \}.
\]

**Proof.** When \( \mu = \eta = (0) \), it is Curtis’ result in [3]. In general, the proof is similar. We leave the reader to verify \( T_s T_D = T_D T_{11} = q T_D, T_{22} T_D = T_D T_{12} = -T_D \) for all \( s_1 \in \lambda^*, s_2 \in \mu, t_1 \in \xi^*, t_2 \in \eta \). This proves \( \subseteq \) part of the result.

Conversely, Suppose \( h \in \mathcal{H}_R \) with \( T_s h = q h \) for all \( s \in \lambda^* \). By [3] 1.9, we have \( a_w = a_{sw} \) for any \( s \in \lambda^* \). Similarly, we have \( a_w = a_{ws} \) for any \( s \in \xi^* \). Therefore, \( a_w = a_{y_1, y_2} \) if \( w = x_1 \cdot x_2 \cdot d \cdot y_1 \cdot y_2 \) with \( x_1 \in \mathcal{S}_{\lambda^*}, x_2 \in \mathcal{S}_\mu, y_1 \in \mathcal{S}_\xi, \) and \( y_2 \in \mathcal{S}_\eta \). Similarly, we have \( a_w = -q a_{tw} \) (resp., \( a_w = -q a_{wt} \)) if \( tw > w \) and \( t \in \mu \) (resp., \( wt > w \) and \( t \in \eta \)). Consequently, for \( w = x_1 \cdot x_2 \cdot d \cdot y_1 \cdot y_2 \) as given above, \( a_w = (-q)^{-l(x_2)} a_{d, y_2} = a_d(-q)^{-l(x_2)+l(y_2)} \). Hence, \( h \in \mathcal{H}^{+,-}_{\lambda|\mu, \xi|\eta} \).

**Definition 5.6.** Let \( \mathcal{T}(m|n, r, \mathcal{R}) = \oplus_{\lambda|\mu, \xi|\eta} \mathcal{S}_{\lambda\mu} \mathcal{S}_{\xi\eta} \mathcal{H}_R \) and
\[
\mathcal{S}(m|n, r, \mathcal{R}) := \text{End}_{\mathcal{H}_R}(\mathcal{T}(m|n, r, \mathcal{R}))
\]
and define a \( \mathbb{Z}_2 \)-grading by setting, for \( i = 0, 1 \),
\[
\mathcal{S}(m|n, r)_i = \bigoplus_{\lambda|\mu, \xi|\eta \in M(m|n, r)_{\lambda|\mu, \xi|\eta}} \text{Hom}_{\mathcal{H}_R}(x_\xi y_\eta \mathcal{H}_R, x_\lambda y_\mu \mathcal{H}_R).
\]
(5.6.1)

We call the \( \mathcal{R} \)-algebra \( \mathcal{S}(m|n, r, \mathcal{R}) \) with supermultiplication (see (5.8.1) below) the **quantum Schur superalgebra** (or \( q \)-Schur superalgebras) over \( \mathcal{R} \). We will simply write \( \mathcal{S}(m|n, r) \) for \( \mathcal{S}(m|n, r, \mathcal{R}) \) and \( \mathcal{T}(m|n, r) \) for \( \mathcal{T}(m|n, r, \mathcal{R}) \).
Note that there is also a $\mathbb{Z}_2$-grading on $\mathfrak{T}(m|n; r; R)$ with
\[
\mathfrak{T}(m|n; r; R)_0 = \bigoplus_{\lambda|\mu \equiv 0(\text{mod } 2)} x\lambda y_\mu \mathcal{H}_R, \quad \mathfrak{T}(m|n; r; R)_1 = \bigoplus_{\lambda|\mu \equiv 1(\text{mod } 2)} x\lambda y_\mu \mathcal{H}_R.
\]

**Remark 5.7.** For the convenience of later use, our definition of $S(m|n; r)$ is taken over
the ring $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$. However, it is clear that the quantum Schur superalgebras
is well defined over $\mathbb{Z}[q, q^{-1}]$, where $q = v^2$. Thus, specializing $q$ to $q \in R$
yields the quantum Schur superalgebras over $R$ (without assuming $\sqrt{q}$ exists in $R$).

The quantum Schur superalgebras share some nice properties with the quantum Schur algebras.

Recall the bijection introduced in (3.2.1). For $A = j(\lambda|\mu, d, \xi|\eta) \in M(m|n; r)$, define $\phi_A = \phi_{\lambda|\mu, \xi|\eta} \in S(m|n; r; R)$ by
\[
\phi_{\lambda|\mu, \xi|\eta}^d(x_\alpha y_\beta h) = \delta_{\xi|\eta|\alpha|\beta} T_{\lambda|\mu} \delta_{\xi|\eta} h,
\]
for all $\alpha|\beta \in \Lambda(m|n; r)$ and $h \in \mathcal{H}_R$. Clearly, $\phi_{\lambda|\mu, \xi|\eta} \in S(m|n; r; R)$ if $|\mu| + |\eta| \equiv i(\text{mod } 2)$.

**Theorem 5.8.** For any commutative ring $R$ which is a $\mathbb{Z}$-module, the set \{\phi_A \mid A \in M(m|n; r)\} forms an $R$-basis for $S(m|n; r; R)$. In particular, $S(m|n; r; R) \cong S(m|n; r) \otimes R$ has rank
\[
|M(m|n; r)| = \sum_{k=0}^{r} \binom{m^2 + n^2 + k - 1}{k} \binom{2mn}{r - k}.
\]
Moreover, there is an algebra anti-involution $\tau : S(m|n; r; R) \rightarrow S(m|n; r; R)$ satisfying $\tau(\phi_A) = \phi_A^{T}$, where $A^T$ denotes the transpose of $A$.

**Proof.** Since $S(m|n; r) = \bigoplus_{\lambda|\mu, \xi|\eta} \text{Hom}_{\mathcal{H}_R}(x_\xi y_\eta \mathcal{H}_R, x\lambda y_\mu \mathcal{H}_R)$, and
\[
\text{Hom}_{\mathcal{H}_R}(x_\xi y_\eta \mathcal{H}_R, x\lambda y_\mu \mathcal{H}_R) \cong x\lambda y_\mu \mathcal{H}_R \cap \mathcal{H}_R x_\xi y_\eta
\]
as $R$-modules, the first assertion follows from Proposition 5.5. The rank assertion
follows from a base change to a field by specializing $v$ to 1 and [2 Th.4.18]. The rest
of the proof is clear.

For $A = j(\lambda|\mu, d, \xi|\eta)$, by the $\mathbb{Z}_2$-grading (5.6.1), set $\hat{A} = |\mu| + |\eta|(\text{mod } 2)$. Then
the supermultiplication is given by
\[
\phi_A \phi_B = (-1)^{\hat{A}\hat{B}} \phi_A \circ \phi_B, \quad \text{for all } A, B \in M(m|n; r).
\]
It is clear that the associativity holds with respect to the supermultiplication. Moreover, it is clear $\phi_A T(m|n; r; R)_i \subseteq T(m|n; r; R)_{\hat{A}+i}$ for all $A$. Hence, $T(m|n; r; R)$ is an $S(m|n; r; R)$-supermodule.
6. Canonical bases for quantum Schur superalgebras

We now introduce canonical bases for quantum Schur superalgebras. Recall the Kashiwara–Lusztig bases \( \{ C_w \} \) and \( \{ B_w \} \) for the Hecke algebra \( \mathcal{H} = \mathcal{H}(\mathfrak{S}_r) \).

For \( D, D' \in M(m|n, r)_{\lambda|\mu, \xi|\eta} \) regarded as double cosets as in (5.4.1), let \( w^+_D \) (resp. \( w^-_D \)) be the longest (resp., shortest) element in \( D \) and define
\[
D \leq D' \text{ if and only if } w^+_D \leq w^-_{D'},
\]
which is equivalent to \( w^+_D \leq w^-_{D'} \); see, e.g., [7, Lem. 4.35]. Clearly,
\[
l(w^+_D) = l(w_{0, \lambda}) + l(w_{0, \mu}) + l(d) + l(w_{0, \xi}) - l(w_{0, \alpha}) + l(w_{0, \eta}) - l(w_{0, \beta}),
\]
where \( d \in D \cap D_{\lambda|\mu, \xi|\eta} \), and \( \alpha, \beta \) are defined as in (5.3.1). For \( d \in D_{\lambda|\mu, \xi|\eta} \), let \( d^\ast, \ast d \) be defined as in Lemma 3.1 and
\[
\mathcal{T}_D = \mathbf{v}^{-l(d^\ast)} \mathbf{v}^{l(d^\ast) - l(d)} T_D.
\]
(6.0.2)

If \( D = \mathfrak{S}_{\lambda|\mu} \), then \( \mathcal{T}_D = \mathbf{v}^{-l(w_{0, \lambda})} \mathbf{v}^{l(w_{0, \mu})} T_{\mathfrak{S}_{\lambda|\mu}} = (-1)^{l(w_{0, \mu})} C_{w_{0, \lambda}, w_{0, \mu}} B_{w_{0, \mu}} \). Note that
\[
l(w^+_D) = l(d^\ast) + l(d) - l(d).
\]

**Lemma 6.1.** The restriction of the bar involution \( \cdot \) on \( \mathcal{H} \) induces a bar involution \( \cdot \) on \( \mathcal{H}^+_{\lambda|\mu, \xi|\eta} \). Moreover, for \( D, C \in M(m|n, r)_{\lambda|\mu, \xi|\eta} \), there exist \( r_{C, D} \in \mathcal{Z} \) such that \( r_{D, D} = 1 \) and
\[
\overline{\mathcal{T}}_D = \sum_{C \in M(m|n, r)_{\lambda|\mu, \xi|\eta} \subseteq D} r_{C, D} T_C.
\]
(6.1.1)

**Proof.** Since \( \overline{\pi}_\lambda = q^{-l(w_{0, \lambda})} x_\lambda \) and \( \overline{\pi}_\mu = q^{l(w_{0, \mu})} y_\mu \), it follows that \( \overline{\mathcal{T}}_D \in \mathcal{H}^+_{\lambda|\mu, \xi|\eta} \). By Proposition 5.5 the restriction yields a bar involution on \( \mathcal{H}^+_{\lambda|\mu, \xi|\eta} \). On the other hand, since
\[
\mathcal{T}_D = \mathbf{v}^{l(d^\ast)} \mathbf{v}^{-l(d^\ast) + l(d)} \sum_{u|v \in \mathfrak{S}_{\xi|\eta} \cap \mathfrak{S}_{\alpha|\beta}} (-q)^{l(u)} \overline{\pi}_\lambda \overline{\pi}_\mu \mathcal{T}_D \mathcal{T}_u \mathcal{T}_v,
\]
and \( T^{-1}_s = q^{-1} T_s + (q^{-1} - 1) \). Proposition 5.5 implies that \( \overline{\mathcal{T}}_D \) can be written as in (6.1.1). It remains to prove that \( r_{D, D} = 1 \). We write \( \overline{\mathcal{T}}_D \) as a linear combination of \( x_\lambda y_\mu T_z, z \in \mathfrak{S}_{\lambda|\mu} \). By (6.1.1), as the leading term of \( \mathcal{T}_D \), \( x_\lambda y_\mu T_{T_{w_{0, \xi}w_{0, \alpha}} T_{w_{0, \eta}w_{0, \beta}}} \) has coefficient \( r_{D, D} \mathbf{v}^{l(d^\ast)} \mathbf{v}^{-l(d^\ast) + l(d)} (-q)^{l(w_{0, \eta}w_{0, \beta})} \). On the other hand, since
\[
(-q)^{l(w_{0, \eta}w_{0, \beta})} \overline{\pi}_\lambda \overline{\pi}_\mu \mathcal{T}_{T_{w_{0, \xi}w_{0, \alpha}} T_{w_{0, \eta}w_{0, \beta}}}
=q^{-l(d^\ast)} q^{l(d^\ast) - l(d)} (-q)^{-l(w_{0, \eta}w_{0, \beta})} x_\lambda y_\mu T_{T_{w_{0, \xi}w_{0, \alpha}} T_{w_{0, \eta}w_{0, \beta}}} + \text{lower terms},
\]
the same coefficient is equal by (6.1.2) to \( \mathbf{v}^{l(d^\ast)} \mathbf{v}^{-l(d^\ast) + l(d)} q^{-l(d^\ast)} q^{l(d^\ast) - l(d)} (-q)^{l(w_{0, \eta}w_{0, \beta})} \). Hence, \( r_{D, D} = 1 \). \( \square \)

By this lemma, a standard construction (see, e.g., [7, §0.5]) gives the following.

**Proposition 6.2.** There exists a unique \( \mathcal{Z} \)-basis \( \{ C_D \} \) for \( \mathcal{H}^+_{\lambda|\mu, \xi|\eta} \) such that \( \mathcal{C}_D = C_D \) and \( C_D = \sum_{C \subseteq D} p_{C, D} \mathcal{T}_C \), where \( p_{D, D} = 1 \) and \( p_{C, D} \in \mathbf{v}^{-1} \mathbb{Z} \mathbf{v}^{-1} \) if \( C < D \). Moreover, if \( D = \mathfrak{S}_{\lambda|\mu} \), then \( C_D = \mathcal{T}_{\mathfrak{S}_{\lambda|\mu}} \).
For any $D \in M(m|n, r)$, if we put
\[ \varphi_D = \sum_{C} r_{C,D} \varphi_C, \]
where $\co(D) = \xi|\eta$ and $\varphi_D$ is defined in (6.7), then
\[ \varphi_D(\mathcal{T}_{\psi_{\alpha|\beta}}) = \delta_{\co(D),\alpha|\beta} \mathcal{T}_D; \]
cf. (6.0). We now have the following.

**Theorem 6.3.** The bar involution $\bar{\cdot} : \mathcal{Z} \to \mathcal{Z}$ can be extended to a ring homomorphism $\bar{\cdot} : \mathcal{S}(m|n, r) \to \mathcal{S}(m|n, r)$ defined by linearly extending the action:
\[ \Theta = \sum_{C,D} r_{C,D} \Theta_{\alpha|\beta}, \]
where the scalars $r_{C,D}$ are defined in (6.0). In particular, there is a unique basis $\{\Theta_{D}\}_{D \in M(m|n, r)}$ satisfying
\[ \bar{\Theta}_D = \Theta_D, \quad \Theta_D - \varphi_D \in \sum_{C < D} \varphi^{-1} \mathbb{Z}[\varphi^{-1}] \varphi_C. \]

**Proof.** We first observe that $\varphi_D(\mathcal{T}_{\psi_{\alpha|\beta}}) = \varphi_D(\mathcal{T}_{\psi_{\alpha|\beta}}^{\omega})$ and the bar involution preserves the $\mathbb{Z}_2$-grading. Thus, for $C, D \in M(m|n, r)$, $\varphi_C \varphi_D = (-1)^{C_{D}} \varphi_C \circ \varphi_D$ and $\varphi_C \bar{\varphi}_D = (-1)^{C_{D}} \bar{\varphi}_C \circ \bar{\varphi}_D$. Hence, to prove that the bar involution is a ring homomorphism, it suffices to prove that $\varphi_C \circ \varphi_D = \varphi_C \circ \bar{\varphi}_D$, for all $C, D$ with $\co(C) = \ro(D)$. This is clear since
\[
\varphi_C \circ \bar{\varphi}_D(\mathcal{T}_{\psi_{\alpha|\beta}}) = \bar{\varphi}_C(\mathcal{T}_{\psi_{\alpha|\beta}}) = \bar{\varphi}_C(\mathcal{T}_{\psi_{\alpha|\beta}}^{h_D}) \quad \text{where} \quad \mathcal{T}_D = \mathcal{T}_{\psi_{\alpha|\beta}}^{h_D},
\]
proving the first assertion. For the last assertion, the construction of the basis is standard; see, e.g., [7] §0.5.

Note that $\Theta_D$ is the element satisfying $\Theta_D(\mathcal{T}_{\psi_{\alpha|\beta}}) = C_D$ and, if $D = j(\lambda|\mu, 1, \lambda|\mu)$, then $\Theta_D = \varphi_D$ is an idempotent.

The basis $\{\Theta_D\}$ does not seem to have a direct connection with the canonical bases $\{C_{w}\}$ for Hecke algebras. However, there is a $\mathbb{Q}(\varphi)$-basis which is defined via the $C$-basis.

Let $y'_{\mu} = \varphi^{l(\omega_0, \mu)} y_{\mu} = (-1)^{l(\omega_0, \mu)} B_{\omega_0, \mu}$ so that $\tau_{\mu} = y'_{\mu}$. For $D = j(\lambda|\mu, d, \xi|\eta)$, let
\[ \mathcal{T}_D' = y'_{\mu} \mathcal{T}_D y'_{\eta}; \]
where $D^* = \mathfrak{S}_{\lambda} \cdot d \mathfrak{S}_{\tau}$ and $\mathcal{T}_{D^*} = \varphi^{-l(d')} \sum_{x \in D^*} T_x$. Clearly,
\[ \mathcal{T}_D = \varphi^{l(\omega_0, \mu)} P_{\psi_{\alpha|\beta}}(q^{-1}) \mathcal{T}_D, \quad (6.3.1) \]
where $P_{\mathcal{S}_{\beta}}(q)$ is the Poincaré polynomial of $\mathcal{S}_{\beta}$; see (5.3.1) for the definition of $\beta = \beta(D)$. This is because, by the definitions of (5.3.2) and (6.0.2) of $T_D$ and $\mathcal{T}_D$, 

$$\mathcal{T}_D = \sum_{v \in \mathcal{E}_{\eta^*} \cap \mathcal{D}_{\alpha^*}} v^{l(\nu) - l(d)} y_\nu \mathcal{T}_{D^*}(-q)^{-l(v)} \mathcal{T}_v = \frac{v^{l(\nu) - l(d)}}{P_{\mathcal{S}_{\beta}}(q^{-1})} y_\nu \mathcal{T}_{D^*} y_\eta,$$  

and $l(\nu) = l(w_{0,\nu}) + l(w_{0,\beta})$.

Let $\mathcal{S}_{\lambda\mu,\xi\eta}^+\mathcal{J}^-\mathcal{S}_{\lambda\mu,\xi\eta}^+$ be the $Z$-span of $\mathcal{T}_D$, $D \in M(m|n, r)_{\lambda\mu,\xi\eta}$. This is a $Z$-submodule of $\mathcal{H}_{\lambda\mu,\xi\eta}$ satisfying

$$\mathcal{S}_{\lambda\mu,\xi\eta}^+\mathcal{J}^-\mathcal{S}_{\lambda\mu,\xi\eta}^+ \otimes \mathbb{Q}(v) = \mathcal{H}_{\lambda\mu,\xi\eta}^+ \otimes \mathbb{Q}(v).$$

Proposition 6.4. For any $C, D \in M(m|n, r)_{\lambda\mu,\xi\eta}$, there exist $r_{C,D} \in Z$ such that $r_{D,D}^* = 1$ and

$$\mathcal{T}_D = \sum_{C \in M(m|n, r)_{\lambda\mu,\xi\eta}} r_{C,D}^* \mathcal{T}_C.$$

Moreover, if $\{C_D\}_{D \in M(m|n, r)_{\lambda\mu,\xi\eta}}$ denotes the associated canonical basis for $\mathcal{S}_{\lambda\mu,\xi\eta}^+\mathcal{J}^-\mathcal{S}_{\lambda\mu,\xi\eta}^+$, then $C_D := y'_D C^* D^* y'_D$, where $d^*$ is the longest element in $D^*$.

Proof. Let $D = f(\lambda|\mu, d, \xi|\eta)$. Since $\mathcal{T}_{D^*} = \sum_{B^* \in \mathcal{E}_{\alpha}\mathcal{S}_{\lambda\mu,\xi\eta}} r_{B^*,D^*}^* \mathcal{T}_{B^*}$, it follows that

$$\mathcal{T}_D = y'_D \mathcal{T}_{D^*} y'_D = \sum_{B^* \in \mathcal{E}_{\alpha}\mathcal{S}_{\lambda\mu,\xi\eta}} r_{B^*,D^*}^* y'_D \mathcal{T}_{B^*} y'_D.$$

Here, $B^* \leq D^*$. In other words, if $d_{B^*}$ denotes the shortest element in $B^*$, then $d_{B^*} \leq d$. By Remark 5.4, $y'_D \mathcal{T}_{B^*} y'_D \neq 0$ implies that $f(B^*) := \mathcal{S}_{\lambda\mu,\xi\eta}^+ B^* \mathcal{S}_{\lambda\mu,\xi\eta}^+$ for some $d_B \in \mathcal{D}_{\lambda\mu,\xi\eta}^+$ and $d_B \leq d_{B^*} \leq d$, and hence, $f(B^*) \in M(m|n, r)_{\lambda\mu,\xi\eta}$ and $f(B^*) \leq D$. Thus, if $C \in M(m|n, r)_{\lambda\mu,\xi\eta}$ and define $r_{C,D}^* = \sum_{B^* \in \mathcal{E}_{\alpha}|f(B^*) = C} r_{B^*,D^*}^*$, then $r_{D,D}^* = r_{D^*,D^*}^* = 1$. This proves the first assertion.

On the other hand, $C_{D^*} = \mathcal{T}_{D^*} = \sum_{C \in \mathcal{E}_{\alpha}|f(B^*) = C} r_{C,D^*}^* \mathcal{T}_{C^*}$ for some $p_{C,D^*} \in \mathbb{V}^{-1} \mathbb{Z}[\mathbb{V}^{-1}]$. Putting $b_D = y'_D C^* D^* y'_D$, we have $b_D = b_{D^*}$, and a similar argument shows that $b_D = \sum_{C \in \mathcal{E}_{\alpha}} c_{D,C}^* \mathcal{T}_D$ where $C \in M(m|n, r)_{\lambda\mu,\xi\eta}$ with $p_{C,D} = 1$ and $C_{C,D} = \mathbb{V}^{-1} \mathbb{Z}[\mathbb{V}^{-1}]$ for $C < D$. Now, the uniqueness of the canonical basis forces $C_D = b_D = y'_D C^* D^* y'_D$, proving the last statement. \hfill $\square$

Taking bar involution on both sides of (6.3.1), we obtain the following relation on the entries of “$R$-matrices” $(r_{C,D})$ and $(r_{C^*,D^*})$:

$$r_{C,D}^* = r_{C,D} \frac{v^{l(w_{0,\beta}(D))} P_{\mathcal{S}_{\alpha}(D)}(q)}{v^{l(w_0,\alpha(C))} P_{\mathcal{S}_{\beta}(C)}(q^{-1})} \quad \text{for all } C, D \in M(m|n, r).$$  

Thus, no obvious relation between the $C$-basis and $C^*$-basis is seen. However, when restrict to the tensor space, the two bases coincide.

Remark 6.5. If $m + n \geq r$, then there exist unique $\omega_1 \omega_2 \in \Lambda(m|n, r)$ such that

$$\omega_1 \vee \omega_2 = \omega := (1, \ldots, 1, 0, \ldots).$$
Thus, if $\xi|\eta = \omega_1|\omega_2$, then $\mathcal{S}_\beta = \{1\}$ and hence, $\mathfrak{h}_{\lambda|\mu,\omega_1|\omega_2}^\pm = \mathcal{H}_{\lambda|\mu,\omega_1|\omega_2}^\pm$ and $r_{C,D} = r_{C,D}$. Consequently, $\mathcal{C}_D = \mathcal{C}_D'$ in this case. Thus, if we put

$$M(m|n,r)_{\text{tsp}} = \begin{cases} \{A \in M(m|n,r) \mid \text{co}(A) = \omega_1|\omega_2\}, & \text{if } m+n \geq r, \\ \{A \in M(m'|n',r) \mid \text{ro}(A) \in \Lambda(m|n,r), \text{co}(A) = \omega_1|\omega_2\}, & \text{if } m+n < r, \end{cases}$$

where $m \leq m'$, $n \leq n'$ and $m' + n' \geq r$, then

$$\{\mathcal{C}_D \mid D \in M(m|n,r)_{\text{tsp}}\} = \{\mathcal{C}_D' \mid D \in M(m|n,r)_{\text{tsp}}\}$$

forms a basis for $\mathfrak{T}(m|n,r)$. Call it the canonical basis of $\mathfrak{T}(m|n,r)$.

Let $\mathcal{S}(m|n,r) = \mathcal{S}(m|n,r) \otimes \mathbb{Q}(x)$. For every $D = j(\lambda|\mu,\xi|\eta) \in M(m|n,r)$, define $\Theta_D' \in \mathcal{S}(m|n,r)$ by setting

$$\Theta_D'(x_\xi^\prime y_\eta^\prime) = C_D' = y_\mu^\prime C_D y_\eta^\prime,$$

where $x_\xi = C_{w_0,\xi} = v^{-l(\omega_0,\xi)}x_\xi$.

**Corollary 6.6.** The set $\{\Theta_D'\}_{D \in M(m|n,r)}$ forms a $\mathbb{Q}(x)$-basis for $\mathcal{S}(m|n,r)$.

We will prove by using cell theory that this basis gives rise to all simple modules of $\mathcal{S}(m|n,r)$ in Section 7. Such a result can be considered as a generalization of [14, Theorem 1.4].

7. **Supercells and their associated cell representations**

We now use the basis $\{\Theta_D'\}_{D}$ given at the end of §6 to construct irreducible representations of $\mathcal{S}(m|n,r)$. Recall the map defined in (4.4.1). We also write $\lambda|\mu = \xi|\eta$ if $\lambda = \xi$ and $\mu = \eta$.

**Definition 7.1.** For $A, B \in M(m|n,r)$ with $A = j^{\pm-}(\alpha|\beta, y, \gamma|\delta)$ and $B = j^{\pm-}(\lambda|\mu, w, \xi|\eta)$, define

$$A \leq_L B \iff y \leq_L w \text{ and } \xi|\eta = \gamma|\delta \text{ (or } \text{co}(A) = \text{co}(B)).$$

Define $A \leq_R B$ if $A^T \leq_L B^T$. Let $\leq_{LR}$ be the preorder generated by $\leq_L$ and $\leq_R$. The relations give rise to three equivalence relations $\sim_L, \sim_R$ and $\sim_{LR}$. Thus, $A \sim_B B$ if and only if $A \leq_X B \leq_X A$ for all $X \in \{L, R, LR\}$. The corresponding equivalence classes in $M(m|n,r)$ with respect to $\sim_L, \sim_R$ and $\sim_{LR}$ are called left cells, right cells and two-sided cells, respectively.

In particular, for $A, B$ as above, we have

1. $A \sim_L B \iff y \sim_L w$ and $\xi|\eta = \gamma|\delta$;
2. $A \sim_R B \iff y \sim_R w$ and $\lambda|\mu = \alpha|\beta$;
3. $A \leq_L B$ and $A \sim_{LR} B \iff A \sim_L B$;
4. $A \leq_R B$ and $A \sim_{LR} B \iff A \sim_R B$;

Statements (3) and (4) follows from the fact that if $y \leq_L w$ and $y \sim_{LR} w$ then $y \sim_L w$; see [15, Cor. 6.3(c)].

**Lemma 7.2.** For $A, B \in M(m|n,r)$, if $\Theta_A' \Theta_B' = \sum_{C \in M(m|n,r)} f_{A,B,C} \Theta_C$, then $f_{A,B,C} \neq 0$ implies $C \leq_L B$ and $C \leq_R A$. 


Proof. Let \( A = j^{+,-}(\alpha|\beta, y, \gamma|\delta) \) and \( B = j^{+,-}(\lambda|\mu, w, \xi|\eta) \). If \( \lambda|\mu \neq \gamma|\delta \), then \( f_{A,B,C} = 0 \) for all \( C \). Suppose \( \lambda|\mu = \gamma|\delta \) and let \( h_\lambda \in \mathbb{Z}[v, v^{-1}] \) be defined by \( x'_\lambda x'_\lambda = h_\lambda x'_\lambda \). We have by \((5.3.2)\)

\[
\Theta'_A \Theta'_B (x'_\lambda y'_n) = \Theta'_A (y'_\mu C_w y'_n) = h_\lambda^{-1} \Theta'_A (x'_\lambda y'_\mu) C_w y'_n = h_\lambda^{-1} y'_\beta C_y y'_\mu C_w y'_n = \sum_{z \in D_{\alpha|\beta, \xi|\eta}^{+,-}} h_\lambda^{-1} h_{y,w,z} y'_\beta C_z y'_n = \sum_{z \in D_{\alpha|\beta, \xi|\eta}^{+,-}} h_\lambda^{-1} h_{y,w,z} \Theta'_C (x'_\xi y'_n),
\]

where \( h_{y,w,z} \in \mathbb{Z}[v, v^{-1}] \) satisfy \( C_y y'_\beta C_w = \sum_z h_{y,w,z} C_z \), and \( C = j^{+,-}(\alpha|\beta, z, \xi|\eta) \). Here we have used the fact that \( y'_\beta C_z y'_n \neq 0 \implies z \in D_{\alpha|\beta, \xi|\eta}^{+,-} \) (see Remark 5.4). Hence,

\[
f_{A,B,C} = \begin{cases} h_\lambda^{-1} h_{y,w,z}, & \text{if } y'_\beta C_z y'_n \neq 0, \\ 0, & \text{otherwise.} \end{cases}
\]

Since \( h_{y,w,z} \neq 0 \) implies \( z \leq_L w \), it follows that \( f_{A,B,C} \neq 0 \) implies \( z \leq_L w, co(C) = co(B) \), proving the first assertion. The second assertion follows from the anti-involution \( \tau \) given in Theorem 5.8. \( \square \)

For each \( A \in M(m|n, r) \), let \((S(A), T(A))\) be the image of \( A \) under the RSK super-correspondence in Theorem 4.7. The following result can be considered as a generalization of Theorem 2.2(1)–(3).

**Lemma 7.3.** Suppose \( A, B \in M(m|n, r) \). Then

1. \( A \sim_L B \) if and only if \( T(A) = T(B) \).
2. \( A \sim_R B \) if and only if \( S(A) = S(B) \).
3. \( A \sim_{LR} B \) if and only if \( T(A), T(B) \) have the same shape.

**Proof.** Suppose \( w_1 \in D_{\alpha|\beta, \xi|\eta}^{+,-} \) and \( w_2 \in D_{\alpha|\beta, \gamma|\delta}^{+,-} \) which have images \((S_{w_1}, T_{w_1})\) and \((S_{w_2}, T_{w_2})\) under the map \( \partial \) defined in \((4.6.1)\). By the constructions of \( \partial \) and its inverse (see proof of Theorem 4.7), we see easily the following:

1. \( w_1 \sim_L w_2 \) and \( \xi|\eta = \gamma|\delta \) if and only if \( T_{w_1} = T_{w_2} \).
2. \( w_1 \sim_R w_2 \) and \( \lambda|\mu = \alpha|\beta \) if and only if \( S_{w_1} = S_{w_2} \).
3. \( w_1 \sim_{LR} w_2 \) if and only if \( T_{w_1} = T_{w_2} \) have the same shape.

Now the assertions follow immediately. \( \square \)

For \( \nu \in \Lambda^+(r)_{m|n} \), let

\[
I(\nu) = \bigcup_{\lambda|\mu \in \Lambda(m|n, r)} T^{ess}(\nu, \lambda|\mu).
\]

By the RSK super-correspondence, if \( A \xrightarrow{RSK} (S, T) \in I(\nu) \), we relabel the basis element \( \Theta'_A \) as

\[
\Theta'_S, T := \Theta'_A.
\]

**Proposition 7.4.** The \( \mathbb{Q}(\nu) \)-basis for \( S(m|n, r) \)

\[
\{ \Theta'_S, T | \nu \in \Lambda^+(r)_{m|n}, S, T \in I(\nu) \} = \{ \Theta'_A | A \in M(m|n, r) \}
\]

is a cellular basis in the sense of [13].
Proof. Recall from [13] the ingredients for a cellular basis. We have a poset \( \Lambda^+(r)_{m|n} \) together with the dominance order \( \triangleright \), index sets \( I(\nu) \) of the basis, and an anti-involution \( \tau \) satisfying \( \tau(\Theta^\nu_{S,T}) = \Theta^\nu_{T,S} \) by Theorems 5.8 and 5.7. It remains to check the triangular relations.

Let \( S(m|n,r)^{\nu} \) be the \( \mathbb{Q}(v) \)-subspace spanned by \( \Theta^\alpha_{S_1,T_1} \) for all \( \alpha \triangleright \nu \) and \( S_1,T_1 \in I(\alpha) \). For \( \lambda, \nu \in \Lambda^+(r)_{m|n} \) and \( S,T \in I(\lambda), S',T' \in I(\nu) \), Lemmas 7.2 and 7.3(3) imply that:

\[
\Theta^\lambda_{S,T} \Theta^\nu_{S',T'} = \sum_{C \in M(m|n,r), C \sim L B} f_{A,B,C} \Theta^\nu_C \mod S(m|n,r)^{\nu},
\]

where \( A \xrightarrow{\text{RSK}} (S,T), B \xrightarrow{\text{RSK}} (S',T') \) and \( C \xrightarrow{\text{RSK}} (S'',T'') \). Since \( C \sim L B \), it follows from 7.3(1), \( T'' = T' \). If \( \lambda \triangleright \nu \), then all \( f_{A,B,C} = 0 \). If \( \lambda = \nu \) and \( f_{A,B,C} \neq 0 \), then \( C \sim_R A \). Hence, \( S'' = S \) and \( f_{A,B,C} = f(T,S') \) is independent of \( T' \). Finally, if \( \lambda \triangleleft \nu \) and \( f_{A,B,C} \neq 0 \), then \( f_{A,B,C} = f(S,T,S') \) is also independent of \( T' \), as required. \( \square \)

For each \( \nu \in \Lambda^+(r)_{m|n} \) and \( T \in I(\nu) \), let

\[
L(\nu)_T = S(m|n,r)^{\nu,T} / S(m|n,r)^{\nu},
\]

where \( S(m|n,r)^{\nu,T} \) is the \( \mathbb{Q}(v) \)-space spanned by \( S(m|n,r)^{\nu} \) and \( \Theta^\nu_{S,T} \). These are called left cell modules. Let \( T_\nu \) be the unique element in \( T^{ssss}(\nu,\nu'|\nu'') \) as described in Example 11 and let \( L(\nu) = L(\nu)_{T_\nu} \). The following result generalizes the second part of Theorem 2.2.

**Theorem 7.5.** For each \( \nu \in \Lambda^+(r)_{m|n} \) and \( T \in I(\nu) \), we have \( L(\nu)_{T} \cong L(\nu) \) as \( S(m|n,r) \)-supermodules. Moreover, the set \( \{ L(\nu) \mid \nu \in \Lambda^+(r)_{m|n} \} \) is a complete set of pair-wise non-isomorphic irreducible \( S(m|n,r) \)-supermodules.

**Proof.** The first assertion follows from the cellular property. Thus, Proposition 7.3 implies \( \dim S(m|n,r) = \sum_{\nu \in \Lambda^+(r)_{m|n}} (\dim L(\nu))^2 \). Since \( v \) is an indeterminate, \( \mathcal{H} \) is semisimple. Hence, \( S(m|n,r) \) is also semisimple as the super product does not change the radical of the endomorphism algebra with a usual product. By the Wedderburn-Artin Theorem, \( \{ L(\nu) \mid \nu \in \Lambda^+(r)_{m|n} \} \) is a complete set of pair-wise non-isomorphic irreducible \( S(m|n,r) \)-modules. Finally, it is routine to check that \( L(\nu) \)'s are \( S(m|n,r) \)-supermodules. In fact, they are the absolute irreducible supermodules in the sense of [3, 2.8]. \( \square \)

We end this section with a second look at the canonical basis for \( \mathfrak{S}(m|n,r) \) described in Remark 6.5. Recall the \( \phi \)-basis defined in (5.7.1).}

**Lemma 7.6.** If \( m + n \geq r \), then there is an \( S(m|n,r) \)-\( \mathcal{H} \)-bimodule isomorphism between \( S(m|n,r) \phi_{\omega_1|\omega_2} \) and \( \mathfrak{S}(m|n,r) \), where \( \omega_1, \omega_2 \) are defined in 6.5 and \( \phi_{\omega_1|\omega_2} := \phi_{\omega_1|\omega_2,\omega_2|\omega_2} \).

**Proof.** Consider the evaluation map

\[
ev : S(m|n,r) \phi_{\omega_1|\omega_2} \xrightarrow{\sim} \mathfrak{S}(m|n,r), \phi \mapsto \phi(1),
\]

which is clearly an \( S(m|n,r) \)-\( \mathcal{H} \)-bimodule isomorphism. \( \square \)
If \( m + n < r \), we choose \( m', n' \) with \( m \leq m' \), \( n \leq n' \) and \( m' + n' \geq r \). Then \( \Lambda(m|n, r) \) can be regarded as a subset of \( \Lambda(m'|n', r) \). Let \( e = \sum_{\lambda|\mu \in \Lambda(m|n, r)} \phi_{\text{diag}}(\lambda|\mu) \).

Then \( S(m|n, r) \cong eS(m'|n', r)e \) is a centralizer subalgebra of \( S(m'|n', r) \), and the map \( ev \) above induces \( S(m|n, r) - H \)-bimodule isomorphism \( eS(m'|n', r)e \phi_{\omega_1|\omega_2} \cong \mathcal{T}(m|n, r) \).

Let

\[
\mathcal{E}(m|n, r) = \begin{cases} 
S(m|n, r) \phi_{\omega_1|\omega_2}, & \text{if } m + n \geq r, \\
eS(m'|n', r) \phi_{\omega_1|\omega_2}, & \text{if } m + n < r,
\end{cases}
\]

where \( m \leq m' \), \( n \leq n' \) and \( m' + n' \geq r \). By the lemma and Remark 6.3, we have the following.

**Proposition 7.7.** By identifying \( \mathcal{E}(m|n, r) \) with \( \mathcal{T}(m|n, r) \), the basis \( 6.5.1 \) for \( \mathcal{T}(m|n, r) \) identifies the basis

\[
\{ \Theta_D = \Theta_D' \mid A \in M(m|n, r) \}_{\mathcal{E}} \}
\]

(for \( \mathcal{E}(m|n, r) \)), which is canonically related (in the sense of Theorem 6.3) to the standard basis \( \{ \varphi_A \mid A \in M(m|n, r) \}_{\mathcal{E}} \) for \( \mathcal{E}(m|n, r) \).

By definition, \( T^{\text{ess}}(\nu, \omega_1|\omega_2) = T^s(\nu) \). Thus, by 4.7 restriction gives a bijection:

\[
M(m|n, r) \rightarrow \bigcup_{\lambda|\mu \in \Lambda^+(m|n, r)} T^{\text{ess}}(\nu, \lambda|\mu) \times T^s(\nu).
\]

Fix a linear ordering on \( \Lambda^+(m|n, r) = \{ \nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(N)} \} \) which refines the opposite dominance ordering \( \succeq \), i.e., \( \nu^{(i)} \succeq \nu^{(j)} \) implies \( i < j \). For each \( 1 \leq i \leq N \), let \( \mathcal{E}_i \) denote the \( \mathcal{Z} \)-free submodule of \( \mathcal{E}(m|n, r) \) spanned by all \( \Theta^{(i)}_{\mathcal{E},\mathcal{E}} \) with \( (S, t) \in T^{\text{ess}}(\nu^{(i)}, \lambda|\mu) \times T^s(\nu^{(i)}) \). Then we obtain a filtration by \( S(m|n, r) - H \)-subbimodules:

\[
0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_N = \mathcal{E}(m|n, r).
\]

Let \( \mathcal{E}_i = \mathcal{E}_i \otimes \mathbb{Q}(\nu) \). By the cellular property established in Proposition 7.4, each section \( \mathcal{E}_i/\mathcal{E}_{i-1} \) is isomorphic to a direct sum of \( |T^s(\nu^{(i)})| \) copies of left cell modules \( L(\nu^{(i)}) \) and to a direct sum of \( |I(\nu^{(i)})| \) copies of right \( \mathcal{H} \)-modules \( S_{\mathbb{Q}(\nu)}^{(i)} \). Hence, \( \mathcal{E}_i/\mathcal{E}_{i-1} \cong L(\nu^{(i)}) \otimes S_{\mathbb{Q}(\nu)}^{(i)} \) as \( S(m|n, r) - H \)-bimodules.

**Corollary 7.8.** There is an \( S(m|n, r) - H \)-bimodule decomposition:

\[
\mathcal{E}(m|n, r) \cong \bigoplus_{\nu \in \Lambda^+(m|n, r)} L(\nu) \otimes S_{\mathbb{Q}(\nu)}^{(i)}.
\]

**Remark 7.9.** For \( \nu = \nu^{(i)} \), let \( L(\nu) \) be the submodule of \( \mathcal{E}_i/\mathcal{E}_{i-1} \) spanned by all \( \Theta^{(i)}\mathcal{E}_i\mathcal{E} \). Since \( \Theta^{\nu}_{\mathcal{E},\mathcal{E}} = \Theta^{\nu^{(i)}}_{\mathcal{E},\mathcal{E}} \) by Remark 6.3, one checks directly that \( L(\nu) \) is an \( S(m|n, r) \)-module. In other words, \( L(\nu) \) is closed under the action of the canonical basis \( \Theta_A \). Base change allows us to investigate representations at roots of unity. We hope to classify the irreducible \( S(m|n, r)_R \)-supermodules elsewhere when \( \nu^2 \) is specialized to a root of unity in a field \( R \).

Note that the right cell module \( S^\lambda \) defined by the right cell containing \( w_{0, \lambda} \) is a homomorphic image of \( x_{\lambda} \mathcal{H} \), while the dual left cell module \( S^\lambda \) defined in 2.2 is a submodule of \( x_{\lambda} \mathcal{H} \).
8. A super analogue of the quantum Schur–Weyl reciprocity

In this section, we first establish a double centralizer property. Then we prove that the algebra \( \mathcal{S}(m|n,r) \) is isomorphic to the endomorphism algebra of a tensor space considered in [17]. Thus, we reproduced the super analogue of the quantum Schur–Weyl reciprocity established in [17].

Let \( \mathfrak{T}(m|n,r) = \mathfrak{T}(m|n,r) \otimes \mathcal{Q}(\mathfrak{v}) \).

**Theorem 8.1.** The \( \mathcal{S}(m|n,r) \)-mod structure \( \mathfrak{T}(m|n,r) \) satisfies the following double centralizer property

\[
\mathcal{S} = \text{End}_{\mathfrak{H}}(\mathfrak{T}(m|n,r)) \text{ and } \overline{\mathcal{H}} = \text{End}_{\mathcal{S}}(\mathfrak{T}(m|n,r)),
\]

where \( \overline{\mathcal{H}} \) is the image of \( \mathcal{H} \) in \( \text{End}_{\mathcal{Q}(\mathfrak{v})}(\mathfrak{T}(m|n,r)) \) and \( \mathcal{S} = \mathcal{S}(m|n,r) \). Moreover, there is a category equivalence

\[
\text{Hom}_{\mathfrak{H}}(-, \mathfrak{T}(m|n,r)) : \text{mod-}\mathfrak{H} \to \mathcal{S}-\text{mod}.
\]

**Proof.** First, as a quotient of a semisimple algebra, \( \overline{\mathcal{H}} \) is semisimple. By Corollary 7.8, \( \mathcal{S}_{\mathcal{Q}(\mathfrak{v})}^\rho \), \( \rho \in \Lambda^+(r)_{m|n} \), are non-isomorphic irreducible \( \overline{\mathcal{H}} \)-modules. Thus, \( \dim \overline{\mathcal{H}} \geq d := \sum_{\rho \in \Lambda^+(r)_{m|n}} (\dim \mathcal{S}_{\mathcal{Q}(\mathfrak{v})}^\rho)^2 \). On the other hand, Corollary 7.8 implies that \( \dim \text{End}_{\mathcal{S}}(\mathfrak{T}(m|n,r)) = d \). Hence, a dimensional comparison forces \( \overline{\mathcal{H}} = \text{End}_{\mathcal{S}}(\mathfrak{T}(m|n,r)) \). The rest of the proof is clear by noting that the inverse functor of \( \text{Hom}_{\mathfrak{H}}(-, \mathfrak{T}(m|n,r)) \) is \( \text{Hom}_{\mathcal{S}}(-, \mathfrak{T}(m|n,r)) \). \( \square \)

We now relate the quantum Schur superalgebras with the quantum enveloping superalgebra \( U^\rho_{\mathfrak{q}}(\mathfrak{gl}(m|n)) \). We use the quantum superspace \( V(m|n) \) considered in [16] and [17].

Let \( V(m|n) \) be a free \( \mathbb{Z} \)-module of rank \( m + n \) with basis \( e_1, e_2, \ldots, e_{m+n} \). The map by setting \( \hat{i} = 0 \) if \( 1 \leq i \leq m \), and \( \hat{i} = 1 \) otherwise, as given in (1.1.1) yields a \( \mathbb{Z}_2 \)-grading on \( V(m|n) = V_0 \oplus V_1 \) where \( V_0 \) is spanned by \( e_1, e_2, \ldots, e_m \) and \( V_1 \) by \( e_{m+1}, e_{m+2}, \ldots, e_{m+n} \). Thus, \( V(m|n) \) becomes a “superspace”.

Let \( \overline{\mathcal{R}} : V(m|n)^{\otimes 2} \to V(m|n)^{\otimes 2} \) be defined by

\[
(e_c \otimes e_d)\overline{\mathcal{R}} = \begin{cases} ce_c \otimes e_c, & \text{if } c = d \leq m, \\
-v^{-1}e_c \otimes e_c, & \text{if } m + 1 \leq c = d, \\
(-1)^{d-1}e_d \otimes e_c + (v - v^{-1})e_c \otimes e_d, & \text{if } c > d, \\
(-1)^{c-1}e_d \otimes e_c, & \text{if } c < d. \end{cases} \tag{8.1.1}
\]

The following result is proved in [17], Th2.1.

**Lemma 8.2.** If we define linear operator

\[
\overline{\mathcal{R}}_i = \text{id}^{\otimes i-1} \otimes \overline{\mathcal{R}} \otimes \text{id}^{\otimes r-i-1} : V(m|n)^{\otimes r} \to V(m|n)^{\otimes r},
\]

then

1. \( (\overline{\mathcal{R}}_i - v)(\overline{\mathcal{R}}_i + v^{-1}) = 0. \)
2. \( \overline{\mathcal{R}}_i \overline{\mathcal{R}}_j = \overline{\mathcal{R}}_j \overline{\mathcal{R}}_i \) if \( 1 \leq i < j \leq r - 1. \)
3. \( \overline{\mathcal{R}}_i \overline{\mathcal{R}}_{i+1} \overline{\mathcal{R}}_i = \overline{\mathcal{R}}_{i+1} \overline{\mathcal{R}}_i \overline{\mathcal{R}}_{i+1} \) for any \( 1 \leq i \leq r - 2. \)
Consider a new basis for $H$ by setting $T_w = v^{-l(w)}T_w$. Then, $H$ is an associative $Z$-algebra generated by $T_i = \nu^{-1}T_i$, $1 \leq i \leq r - 1$ subject to the relations
\[
\begin{align*}
(T_i - v)(T_i + v^{-1}) &= 0, \\
T_j T_i &= T_i T_j, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1},
\end{align*}
\] (8.2.1)

Let
\[
I(m|n, r) = \{i = (i_1, i_2, \cdots, i_r) \in \mathbb{N}^r \mid 1 \leq i_j \leq m + n \forall j\},
\] (8.2.2)

and, for $i \in I(m|n, r)$, let
\[
e_1 = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}.
\]

Clearly, the set $\{e_i\}_{i \in I(m|n, r)}$ form a basis for $V(m|n)^{\otimes r}$.

For each $i \in I(m|n, r)$, define $\lambda|\mu \in \Lambda(m|n, r)$ to be the weight $\text{wt}(i)$ by setting
\[
\begin{align*}
\lambda_k &= \#\{k: i_j = k, 1 \leq j \leq r\}, \\
\mu_k &= \#\{m + k: i_j = m + k, 1 \leq j \leq r\}, \forall 1 \leq k \leq n.
\end{align*}
\]

For each $\lambda|\mu \in \Lambda(m|n, r)$, define $i_{\lambda|\mu} \in I(m|n, r)$ by
\[
i_{\lambda|\mu} = (1, \cdots, 1, \cdots, m, \cdots, m, m + 1, \cdots, m + n, \cdots, m + n)
\]

The symmetric group $\mathfrak{S}_r$ acts on $I(m|n, r)$ by place permutation:
\[
i_{i}w = (i_{w(1)}, i_{w(2)}, \cdots, i_{w(r)}).
\] (8.2.3)

Clearly, the weight function $\text{wt}$ induces a bijection between the $\mathfrak{S}_r$-orbits and $\Lambda(m|n, r)$.

**Proposition 8.3.** The tensor superspace $V(m|n)^{\otimes r}$ is a right $H$-module, and is isomorphic to the $H$-module $\mathfrak{T}(m|n, r) = \bigoplus_{\lambda|\mu \in \Lambda} x_{\lambda|\mu} H$.

**Proof.** By defining an action of $T_i$ on $V(m|n)^{\otimes r}$ via $\mathcal{R}_i$, the first assertion follows from Lemma 8.2.

For any $i \in I(m|n, r)$ with $\text{wt}(i) = \lambda|\mu$, we have $i = i_{\lambda|\mu} d$ where $d$ is the unique element in $D_{\lambda|\mu}$. Write $i = (i_1, i_2, \cdots, i_r)$ and $(j_1, j_2, \cdots, j_r) = i_{\lambda|\mu}$. Then $i_k = j_{d(k)}$ for all $k$. By definition, we have $\hat{j}_k = 0$ if $k \leq |\lambda|$ and $\hat{j}_k = 1$ if $k > |\lambda|$. Also, $j_k \leq j_l$ whenever $k \leq l$. For any $d \in D_{\lambda|\mu}$ with $i = i_{\lambda|\mu} d$, define
\[
d = \sum_{k=1}^{r} \sum_{1 \leq k < l_{d(i_k)}} \hat{i}_{k} \hat{j}_{l_k}.
\] (8.3.1)

Thus, (8.1.1) implies
\[
(-1)^d e_i T_k = \begin{cases}
(-1)^d(-1)^{\hat{j}_{k}+1} e_{i_k}, & \text{if } i_k < i_{k+1}; \\
v(-1)^d e_i, & \text{if } i_k = i_{k+1} \leq m; \\
-v^{-1}(-1)^d e_i, & \text{if } i_k = i_{k+1} \geq m + 1; \\
(-1)^d e_{i_k} + (v - v^{-1})(-1)^d e_i, & \text{if } i_k > i_{k+1},
\end{cases}
\] (8.3.2)
where \( s_k = (k, k + 1) \). On the other hand,

\[
x_{\lambda\mu} T_d T_k = \begin{cases} 
  x_{\lambda\mu} T_{d s_k}, & \text{if } d s_k \in \mathcal{O}_{\lambda\mu}, \\
  v x_{\lambda\mu} T_d, & \text{if } d s_k = s_d, s_l \in \mathcal{S}_\lambda, \\
  -v^{-1} x_{\lambda\mu} T_{d k}, & \text{if } d s_k = s_d, s_l \in \mathcal{S}_\mu, \\
  x_{\lambda\mu} T_{d s_k} + (v - v^{-1}) x_{\lambda\mu} T_d, & \text{if } d s_k < d. 
\end{cases}
\] (8.3.3)

Since \((-1)^d(-1)^{k+1} = \tilde{d}s_k\) in the first and last case of (8.3.2), it follows that the \( \mathbb{Z} \)-linear map

\[
f : V(m|n)^{\otimes r} \to \bigoplus_{(\lambda,\mu) \in \Lambda} x_{\lambda\mu} \mathcal{H}_R : (-1)^d e_{i_{\lambda\mu} d} \mapsto x_{\lambda\mu} T_d
\] (8.3.4)

is a right \( \mathcal{H} \)-module homomorphism. \( \square \)

**Corollary 8.4.** There is a superalgebra isomorphism

\[
\mathcal{S}(m|n, r) \cong \text{End}_{\mathcal{H}}(V(m|n)^{\otimes r}).
\]

Hence, the quantum Schur superalgebra defined in [8.1.3] is the same algebra considered in [17].

**Remark 8.5.** Let \( U(m|n) = U_q^e(gl(m, n)) \) be the quantum enveloping superalgebra defined in [17] §3. Then \( U(m|n) \) acts naturally on \( V(m|n)^{\otimes r} \), where \( V(m|n)^{\otimes r} = V(m|n)^{\otimes r} \otimes \mathbb{Q}(v) \). By [17] Th. 4.4, \( U(m|n) \) maps onto the algebra \( \text{End}_{\mathcal{H}}(V(m|n)^{\otimes r}) \). Now, Corollary 8.4 and Theorem 8.1 implies the Schur–Weyl reciprocity between \( U(m|n) \) and \( \mathcal{H} \) as described in [17] Th. 4.4].

9. Relation with Quantum Matrix Superalgebras

Like quantum Schur algebras, quantum Schur superalgebras \( \mathcal{S}(m|n, r) \) can also be interpreted as the dual algebra of the \( r \)th homogeneous component \( \mathcal{A}(m|n, r) \) of the quantum matrix superalgebra \( \mathcal{A}(m|n) \). We first recall the following definition which is a special case of quantum superalgebras with multiparameters defined by Manin [10] 1.2. For simplicity, we assume throughout the section that \( F \) is a field of characteristic \( \text{char}(F) \neq 2 \) and \( v \in F \).

**Definition 9.1.** Let \( \mathcal{A}(m|n) \) be the associative superalgebra over \( F \) generated by \( x_{ij} \), \( 1 \leq i, j \leq m + n \) subject to the following relations:

1. \( x_{ii, j}^2 = 0 \), for \( i + j = 1 \);
2. \( x_{ij} x_{ik} = (-1)^{(i+j)(i+k)} v^{(i+1)} x_{ik} x_{ij} \), for \( j < k \);
3. \( x_{ij} x_{kj} = (-1)^{(i+j)(k+j)} v^{(i+1)} x_{kj} x_{ij} \), for \( i < k \);
4. \( x_{ij} x_{kl} = (-1)^{(i+j)(k+l)} x_{kl} x_{ij} \), for \( i < k \) and \( j > l \);
5. \( x_{ij} x_{kl} = (-1)^{(i+j)(k+l)} x_{kl} x_{ij} + (-1)^{(k+j)(i+l)} (v^{-1} - v) x_{il} x_{kj} \), for \( i < k \) and \( j < l \).

Manin [10] proved that \( \mathcal{A}(m|n) \) has also a supercoalgebra structure with co-multiplication \( \Delta : \mathcal{A}(m|n) \to \mathcal{A}(m|n) \otimes \mathcal{A}(m|n) \) and co-unit \( \varepsilon : \mathcal{A}(m|n) \to F \) defined by

\[
\Delta(x_{ik}) = \sum_{j=1}^{m+n} x_{ij} \otimes x_{jk}, \text{ and } \varepsilon(x_{ij}) = \delta_{ij}, \forall i, j, k \leq m + n. \] (9.1.1)
Further, the $\mathbb{Z}_2$ grading degree of $x_{ij}$ is $\hat{i} + \hat{j} \in \mathbb{Z}_2$. The following result is a special case of [16, Th. 1.14].

**Theorem 9.2.** Suppose $v^2 \neq -1$ in $F$. Then $\mathcal{A}(m|n)$ has basis

$$B = \left\{ \prod_{i,j} x_{i,j}^{a_{i,j}} : a_{i,j} \in \mathbb{N}, \text{ and } a_{i,j} \in \{0,1\} \text{ whenever } \hat{i} + \hat{j} = 1 \right\},$$

where the order of $x_{i,j}$ is arranged such that $x_{i,j}$ is the left to $x_{k,l}$ if either $i < k$ or $i = k$ and $j < l$.

For each $A = (a_{i,j}) \in M(m+n)$, define

$$x^A = x_{1,1}^{a_{1,1}} x_{1,2}^{a_{1,2}} \cdots x_{m,n+1}^{a_{m,n+1}} x_{m+1,1}^{a_{m+1,1}} \cdots x_{m+n,m+n}^{a_{m+n,m+n}} \quad (9.2.1)$$

By Definition [9.1](a) and Theorem [9.2], $x^A \neq 0$ if and only if $A \in M(m|n)$. Thus, $B = \{ x^A \mid A \in M(m|n) \}$.

The bialgebra $\mathcal{A}(m|n)$ is an $\mathbb{N}$-graded algebra such that each $x_{ij}$ has degree 1. Let $\mathcal{A}(m|n,r)$ be the subspace of $\mathcal{A}(m|n)$ spanned by monomials of degree $r$. The following result follows immediately.

**Corollary 9.3.** Suppose $v^2 \neq -1$ in $F$. The set $\mathcal{B}_r = \{ x^A : A \in M(m|n,r) \}$ forms an $F$-basis for the coalgebra $\mathcal{A}(m|n,r)$.

We will realize the linear dual $\mathcal{A}(m|n,r)^*$ of $\mathcal{A}(m|n,r)$ as the endomorphism algebra of the tensor space over the Hecke algebra $\mathcal{H}_F$ associated to the symmetric group $\mathfrak{S}_r$. We start by recalling some notations.

Let $I(m|n,r)$ be the set defined in [8.2.2]. The group $\mathfrak{S}_r$ acts on $I(m|n,r) \times I(m|n,r)$ diagonally by $(i,j)w = (iw,jw)$ for any $w \in \mathfrak{S}_r$ and $(i,j) \in I(m|n,r) \times I(m|n,r)$. Then there is a bijection between the set of $\mathfrak{S}_r$-orbits and $M(m+n,r)$. This is seen easily from the map $j$ defined in [8.2.3]: if $j(\lambda|\mu,w,\xi|\mu) = A$, where $w \in \mathcal{D}_{\lambda|\mu,\xi|\mu}$, then $A$ is mapped to the orbit containing $(i_{\lambda|\mu}w, i_{\xi|\mu})$ or $(i_{\lambda|\mu}w, i_{\xi|\mu}w^{-1})$.

Let $x_{i,j} = x_{i_{1,1}} x_{i_{1,2}} \cdots x_{r,j_r}$. Since $x_{i,j}$ and $x_{k,l}$ do not commute each other, we do not have $x_{i,j} = x_{i_{1,k}} x_{i_{2,l}}$ for $w \in \mathfrak{S}_r$, in general. However, by [7, 8.6.9.6] or a direct argument, we have the following.

**Lemma 9.4.** If $A = (a_{i,j}) = j(\lambda|\mu,w,\xi|\eta) \in M(m|n,r)$, then

$$x^w_{i_{\lambda|\mu}, i_{\xi|\eta}} := x_{i_{\lambda|\mu}, i_{\xi|\eta}w^{-1}} = x^A.$$ 

Moreover, $x_{i_{\xi|\eta}w^{-1}, i_{\lambda|\mu}} = (-1)^{w^{-1}} x^A$, where $w^{-1}$ is defined in [8.3.1].

**Proof.** To see the last assertion, note that, if $A = (a_{i,j})$, then

$$x_{i_{\xi|\eta}w^{-1}, i_{\lambda|\mu}} = x_{1,1}^{a_{1,1}} x_{1,2}^{a_{1,2}} \cdots x_{m,n+1}^{a_{m,n+1}} x_{m+1,1}^{a_{m+1,1}} \cdots x_{m+n,m+n}^{a_{m+n,m+n}}.$$ 

The assertion follows from the relation [9.1](4). \qed

Recall from §3 that we wrote $w^-_A$ for $w$ if $j(\lambda|\mu,w,\xi|\mu) = A$. For notational simplicity, we will write $w_A$ for $w^-_A$ in the rest of the section. Note that $A \in M(m|n,r)$ if and only if $w_A$ satisfies the trivial intersection property [3.1](3):

$$\mathfrak{S}_r \cap w_A \mathfrak{S}_\eta w_A^{-1} = \{1\}, \text{ and } \mathfrak{S}_n \cap w_A \mathfrak{S}_\xi w_A^{-1} = \{1\}.$$
Let $\mathcal{A}(m|n, r)^*$ be the dual space of $\mathcal{A}(m|n, r)$. It is well-known that $\mathcal{A}(m|n, r)^*$ is a superalgebra with multiplication given by the following rule
\[
(fg)(v) = (f \otimes g)\Delta(v), \text{ for all } v \in \mathcal{A}(m|n, r)^*.
\]
Note that the action of $f \otimes g$ on $\Delta(v) = \sum v(1) \otimes v(2)$ is given by
\[
(f \otimes g)(v(1) \otimes v(2)) = (-1)^{ij}f(v(1)) \otimes g(v(2))
\]
if the degree of $g$ (resp. $v(1)$) is $i$ (resp. $j$).

For $A \in M(m|n, r)$, let $f_A \in \mathcal{A}(m|n, r)^*$ be defined by $f_A(x_B) = \delta_{AB}$, for $B \in M(m|n, r)$. Then \(\{f_A\}_{A \in M(m|n, r)}\) is the dual basis of $\mathcal{B}_r$.

Since the $\mathbb{Z}_2$-grading degree of the monomial $x_{i,j}m_{k,l}$ is $\sum_{k=1}^{r}(i_k + j_k) \in \mathbb{Z}_2$, it is natural to set the $\mathbb{Z}_2$-grading degree $\hat{f}_A$ of $f_A$ to be
\[
\hat{f}_A = \sum_{k=1}^{r}(i_k + j_k) = |\mu| + |\eta|(\text{mod } 2) = \hat{A},
\]
where $i = i_{\mu|\nu}$, $j = j_{\xi|\mu}$, and $A = j(\lambda|\mu, w, \xi|\mu)$.

Manin [10] proved that the $F$-space $V(m|n)_F$, regarded as the specialization of the $\mathbb{Z}$-free module $V(m|n)$ in §8, is a (right) $\mathcal{A}(m|n)$-comodule with structure map
\[
\delta : V(m|n)_F \to V(m|n)_F \otimes \mathcal{A}(m|n), \quad e_i \mapsto \sum_{j} e_j \otimes x_{j,i}
\]
Since $\mathcal{A}(m|n)$ is a superbialgebra, $V(m|n)^{\otimes r}_F$ is also an $\mathcal{A}(m|n)$-comodule and the structure map is induced by the structure map $\delta$ on $V(m|n)_F$. By abuse of notation, we still use $\delta$ to denote the structure map. Thus, for any $e_i = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}$ with $i \in I(m|n)$, $r$
\[
\delta(e_i) = \sum_{j \in I(m|n)} (-1)^{\sum_{k \leq l \leq r} j_k(i_k + i_l)} e_j \otimes x_{j,i}.
\]
Restriction makes $V(m|n)^{\otimes r}_F$ into an $\mathcal{A}(m|n, r)$-comodule, and hence a left $\mathcal{A}(m|n, r)^*$-module with the action given by
\[
f \cdot e = (\text{id}_{V(m|n)^{\otimes r}_F} \otimes f)\delta(e), \quad \forall f \in \mathcal{A}(m|n, r)^*, e \in V(m|n)^{\otimes r}_F.
\]

**Lemma 9.5.** The action of $\mathcal{A}(m|n, r)^*$ on $V(m|n)^{\otimes r}_F$ is faithful.

**Proof.** Suppose $f \cdot e_i = 0$ for all $i \in I(m|n, r)$. By (9.4.2), $f(x_{j,i}) = 0$ for all $i, j \in I(m|n, r)$. In particular, $f(x_{j,1_{\xi|\mu}}) = 0$ for all $\lambda|\mu \in \Lambda(m|n, r)$ and all $j \in I(m|n, r)$. By Corollary 9.3 and Lemma 9.4, $f = 0$. \qed

With the definition of $\hat{f}_A$, it would be possible to explicitly describe the action $f_A \cdot e_{i_{\xi|\mu}}$, and hence, to make a comparison between bases $\{f_A\}$ and $\{\phi_A\}$ under the isomorphism in Theorem 9.7.

**Proposition 9.6.** The linear map $\mathcal{R} : V(m|n)^{\otimes 2}_F \to V(m|n)^{\otimes 2}_F$ defined in (8.1.1) is an $\mathcal{A}(m|n, r)$-comodule homomorphism. Moreover, the actions of $\mathcal{A}(m|n, r)^*$ and $\mathcal{H}_F$ on $V(m|n)^{\otimes r}_F$ commute.
Proof. We need verify
\[ \delta \circ \bar{\mathcal{R}} = (\text{id}_{\mathcal{A}(m|n,r)} \otimes \bar{\mathcal{R}}) \circ \delta \quad (9.6.1) \]
where \( \delta \) is the comodule structure map on \( V(m|n)_{\bar{\mathcal{F}}}^{\otimes 2} \). We verify the case \( e_i \otimes e_j \) with \( i > j \). One can verify the other cases similarly. We have
\[
(\text{id}_{\mathcal{A}(m|n,r)} \otimes \bar{\mathcal{R}})\delta(e_i \otimes e_j) = v \sum_{k \leq m} x_{i,k} x_{j,k} \otimes e_k \otimes e_k \\
+ (-1)^i v^{-1} \sum_{k \geq m+1} x_{i,k} x_{j,k} \otimes e_k \otimes e_k + \sum_{l < k} (-1)^j x_{ik} x_{jl} \otimes e_l \otimes e_k \\
+ \sum_{l > k} \left\{ (-1)^j x_{ik} x_{jl} + (-1)^{j+l} (v - v^{-1}) x_{il} x_{jk} \right\} \otimes e_l \otimes e_k
\]
On the other hand,
\[
\delta \circ \bar{\mathcal{R}}(e_i \otimes e_j) = \delta((-1)^i e_j \otimes e_i + (v - v^{-1}) e_i \otimes e_j) = \]
\[= (-1)^i \sum_{k,l} (-1)^{j+l} x_{jk} x_{il} e_k \otimes e_l \\
+ (v - v^{-1}) \sum_{k,l} x_{ik} x_{jl} (-1)^{j+l} e_k \otimes e_l
\]
Comparing the coefficients of \( e_k \otimes e_l \) via Definition \[9.1\] yields \( \delta \circ \bar{\mathcal{R}}(e_i \otimes e_j) = (\text{id}_{\mathcal{A}(m|n,r)} \otimes \bar{\mathcal{R}})\delta(e_i \otimes e_j) \). This proves \(9.6.1\). Further, it implies that the actions of \( \mathcal{A}(m|n,r)^* \) and \( \mathcal{H}_{\bar{F}} \) on \( V(m|n)_{\bar{\mathcal{F}}}^{\otimes r} \) commute. (One can also verify it by the definition of the action of the linear dual of a cosuperalgebra \( \mathcal{A} \) on an \( \mathcal{A} \)-cosuperalgebra. See the definition given in [3, p.45].)

The following result is the quantum version of [4, Th. 5.2].

Theorem 9.7. The quantum Schur superalgebra \( \mathcal{S}(m|n,r)_{\bar{F}} \) is isomorphic to the algebra \( \mathcal{A}(m|n,r)^* \). In other words, we have an algebra isomorphism

\[ \mathcal{A}(m|n,r)^* \cong \text{End}_{\mathcal{H}_{\bar{F}}}(\oplus_{\lambda \mu \in \Lambda(m|n,r)} x_{\lambda y_{\mu}} \mathcal{H}_{\bar{F}}) \]

Proof. We have already proved that \( \mathcal{A}(m|n,r)^* \) acts faithfully on \( V(m|n)_{\bar{\mathcal{F}}}^{\otimes r} \). So, \( \mathcal{A}(m|n,r)^* \) is a subalgebra of \( \text{End}_{\mathcal{F}}(V(m|n)_{\bar{\mathcal{F}}}^{\otimes r}) \). By Proposition \[9.6\], \( \mathcal{A}(m|n,r)^* \) is a subalgebra of \( \text{End}_{\mathcal{H}_{\bar{F}}}(V(m|n)_{\bar{\mathcal{F}}}^{\otimes r}) \). A dimensional comparison (see Theorem 5.8 and Corollary 9.3) gives the required isomorphism.

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