OPTIMAL MONOTONICITY OF $L^p$ INTEGRAL OF CONFORMAL INVARIANT GREEN FUNCTION

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Abstract. Both analytic and geometric forms of an optimal monotone principle for $L^p$-integral of the Green function of a simply-connected planar domain $\Omega$ with rectifiable simple curve as boundary are established through a sharp one-dimensional power integral estimate of Riemann-Stieltjes type and the Huber analytic and geometric isoperimetric inequalities under finiteness of the positive part of total Gauss curvature of a conformal metric on $\Omega$. Consequently, new analytic and geometric isoperimetric-type inequalities are discovered. Furthermore, when applying the geometric principle to two-dimensional Riemannian manifolds, we find fortunately that \{0,1\}-form of the induced principle is midway between Moser-Trudinger’s inequality and Nash-Sobolev’s inequality on complete noncompact boundary-free surfaces, and yet equivalent to Nash-Sobolev’s/Faber-Krahn’s eigenvalue/Heat-kernel-upper-bound/Log-Sobolev’s inequality on the surfaces with finite total Gauss curvature and quadratic area growth.

1. Introduction

Given a conformal metric of the form

$$\sigma = e^{2u}ds^2 = e^{2u}|dz|^2 = e^{2u}(dx^2 + dy^2)$$

for $z = x + iy$ in a subdomain $\Sigma$ of the two dimensional Euclidean space $\mathbb{R}^2$, we are mainly inspired by Huber’s 1957 Acta Math. paper “Zur isoperimetrischen Ungleichung auf gekrmmten Flächen” [22] and 1954 Ann. Math. paper “On the isoperimetric inequality on surfaces of variable Gaussian curvature” [21] to establish a sharp monotone principle for the power $p \in [0, \infty)$ integral (as well as its limiting case $p \to \infty$)

$$(\Gamma(1+p))^{-1} \left( 4\pi (1-(2\pi)^{-1} \int_{\Omega} \max\{K_\sigma,0\}dA_\sigma \right)^p \int_{\Omega} (g_{(\Omega,\sigma)}(\cdot,a))^p dA_\sigma (\cdot), \quad a \in \Omega$$

of the conformally invariant Green function $g_{(\Omega,\sigma)}(\cdot,\cdot)$ for the two-dimensional conformal Laplacian

$$\Delta_\sigma u = e^{-2u} \Delta u$$

of a simply-connected domain $(\Omega, \sigma)$ on the surface $(\Sigma, \sigma)$ with a rectifiable simple curve as its boundary – see Theorem 4.2. Here and henceforth

$$K_\sigma(z) = -e^{-2u(z)}\Delta u(z) = -e^{-2u(z)} \left( \frac{\partial^2 u(z)}{\partial x^2} + \frac{\partial^2 u(z)}{\partial y^2} \right)$$

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and
\[ dA_\sigma(z) = e^{2u(z)}d\sigma(z) = e^{2u(z)}dxdy \]
are the Gauss curvature and the area element of the surface \((\Sigma, \sigma)\) respectively. Of course, \(\Gamma(\cdot)\) is the classical gamma function.

To reach this geometric principle we will first consider its equivalent analytic form – Theorem 3.2. This extends sharply the following result of Stanton [40]:

**Theorem 1.1.** Let \(\Phi\) be of class \(C^2\) with \(\Delta \Phi > 0\) on a simply-connected domain \(\Omega \subset \mathbb{R}^2\) with \(\partial \Omega\) being a rectifiable simple curve. If
\[
\int_\Omega \max \left\{ \frac{\Delta \ln (\Delta \Phi(z))^{-1}}{\Delta \Phi(z)}, 0 \right\} dA(z) < 2\pi,
\]
then for \(a \in \Omega\),
\[
\int_\Omega g_\Omega(z, a)\Delta \Phi(z)dA(z) \leq \frac{\int_\Omega \Delta \Phi(z)dA(z)}{4\pi(1 - (2\pi)^{-1}\int_\Omega \max \left\{ \frac{\Delta \ln (\Delta \Phi(z))^{-1}}{\Delta \Phi(z)}, 0 \right\} dA(z))}.
\]

**Remark 1.2.** In the case of \(\Delta \Phi = 1\) the inequality (1.1) is back to the so-called Pólya-Szegő’s “stress” inequality – see also [34, p. 115, (12)]:
\[
\int_\Omega g_\Omega(z, a)dA(z) \leq \frac{1}{4\pi} \int_\Omega dA(z),
\]
which was generalized by Bandle [5, p. 61, Example 1] to the inequality
\[
\int_\Omega (g_\Omega(z, a))^p dA(z) \leq \frac{\Gamma(p + 1)}{(4\pi)^p} \int_\Omega dA(z), \quad p \in [0, \infty).
\]
The constants in (1.3) and (1.2) are sharp since they are attained when \(\Omega\) is any Euclidean disk centered at \(a\). Interestingly, (1.3) becomes a special case of Aulaskar-Chen’s “\(Q_p\)-norm” inequality (cf. [4]):
\[
\int_\Omega (g_\Omega(z, a))^p |f'(z)|^2 dA(z) \leq \frac{(4\pi)^{-q}\Gamma(p + 1)}{(4\pi)^{-q}\Gamma(q + 1)} \int_\Omega (g_\Omega(z, a))^q |f(z)|^2 dA(z)
\]
which is valid for all \(0 \leq q < p < \infty, a \in \Omega\), and holomorphic functions \(f\) on \(\Omega\). It is also worth remarking that the equality in (1.4) holds under convergence of the right-hand integral of (1.4) if and only if \(\Omega\) is a simply-connected domain \(\Lambda \subset \mathbb{R}^2\) minus at most a compact set \(E\) of logarithmic capacity zero –
\[
0 = \text{cap}(E) = \exp \left( -\inf_{\mu} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \ln \frac{1}{|z - w|} \right) d\mu(z) d\mu(w) \right)
\]
(cf. [35] p. 25]) where the infimum ranges over all positive probability Radon measures \(\mu\) supported on \(E\), but also \(f\) can be extended to a conformal mapping from \(\Lambda\) onto an open disk in \(\mathbb{R}^2\) centered at \(f(a)\).

In order to prove the equivalent principle we introduce a process that reduces the desired optimal estimate to a one-dimensional calculus inequality in connection with the so-called Riemann-Stieltjes integral – see Theorem 2.1.

Finally, we apply our ideas, methods and techniques to explore an analogue of the geometric monotone principle on two dimensional simply-connected, complete, noncompact and boundary-free Riemannian manifolds with \(2\pi\)-bounded total Gauss curvature – Theorem 5.2 thereby surprisingly finding that with generic
constants, the $0 = p_1 < p_2 = 1$ setting of Theorem 5.2 lies nicely between Moser-Trudinger’s inequality (cf. Adams’s 1988 Ann. Math. paper “A sharp inequality of J. Moser for higher order derivatives” [11] for an account) and Nash-Sobolev’s inequality (cf. Chavel’s 2001 and Saloff-Coste’s 2002 Cambridge Univ. Press books “Isoperimetric Inequalities” [13] and “Aspects of Sobolev-Type Inequalities” [36] for instance) on complete noncompact surfaces without boundary; but this special case is also equivalent to the generic Nash-Sobolev’s/Faber-Krahn’s eigenvalue inequality/Heat-kernel-upper-bound inequality/Log-Sobolev’s inequality on the surfaces with finite total Gauss curvature and quadratic area growth (cf. Li-Tam’s 1991 J. Diff. Geom. paper “Complete surfaces with finite total curvature” [27] for more information on such a kind of surfaces) – Theorem 5.3.

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2. Optimal Monotonicity – Basic Form

In this section we establish a sharp one-dimensional inequality for the Riemann-Stieltjes $L^p$ integral of the radial function – that is – Theorem 2.1 below. This useful and fundamental result seems to be of independent interest although some basic techniques used to argue its special case $c = 2$ have a root in Aulaskar-Chen’s [4, Lemma 2]. Actually, it is a key step to the principles which will be precisely presented in the subsequent sections.

**Theorem 2.1.** Given a constant $c > 0$ and a nonnegative function $X(\cdot)$ on $(0, \infty)$, suppose

\[
X'(t) = \frac{dX(t)}{dt} \leq 0 \quad \text{and} \quad \frac{d(e^{ct}X(t))}{dt} \leq 0 \quad \text{for} \quad t > 0.
\]

For $p \in [0, \infty)$ let $Y_p(t) = -\int_t^\infty r^p dX(r)$ be defined on $[0, \infty)$ in the sense of Riemann-Stieltjes integration.

(i) If $0 \leq p_1 < p_2 < \infty$, then

\[
\frac{c^{p_2}Y_{p_2}(0)}{\Gamma(p_2 + 1)} \leq \frac{c^{p_1}Y_{p_1}(0)}{\Gamma(p_1 + 1)}.
\]

Here

\[
\frac{c^{p_2}Y_{p_2}(0)}{\Gamma(p_2 + 1)} = \frac{c^{p_1}Y_{p_1}(0)}{\Gamma(p_1 + 1)} < \infty
\]

if and only if

\[
X(0) = \lim_{r \to 0^+} X(r) < \infty \quad \text{and} \quad X(t) = e^{-ct}X(0) \quad \text{for} \quad t \geq 0.
\]

(ii) If $Y_{p_0}(0) < \infty$ is valid for some $p_0 \in [0, \infty)$, then

\[
\lim_{t \to \infty} e^{ct}X(t) = \lim_{p \to \infty} \frac{c^pY_{p}(0)}{\Gamma(p + 1)}.
\]
Proof. (i) The supposition $X'(t) \leq 0$ (where $t > 0$) makes both $Y_{p_1}(0)$ and $Y_{p_2}(0)$ meaningful. Without loss of generality we may assume $Y_{p_1}(0) < \infty$ since $Y_{p_1}(0) = \infty$ implies that (2.2) is trivially true. If $p_1 = 0$ then $Y_{p_1}(t) = X(t)$ follows from $d(e^{ct}X(t))/dt \leq 0$. Consequently,

$$
\frac{dY_{0}(t)}{Y_{0}(t)} \leq -cdt = -\frac{t^0 e^{-ct}dt}{\int_{0}^{\infty} t^0 e^{-ct}dr}, \quad t > 0.
$$

If $p_1 > 0$, then both $d(e^{ct}X(t))/dt \leq 0$ and integration-by-part imply that for $t > 0$,

$$
Y_{p_1}(t) = t^{p_1}X(t) + p_1 \int_{t}^{\infty} r^{p_1-1}X(r)dr \leq X(t) \left( t^{p_1} + p_1 e^{ct} \int_{t}^{\infty} r^{p_1-1}e^{-cr}dr \right) = cX(t)e^{ct} \int_{t}^{\infty} r^{p_1}e^{-cr}dr.
$$

As a result, we read off:

$$
\frac{dY_{p_1}(t)}{Y_{p_1}(t)} \leq \frac{ct^{p_1}X(t)dt}{Y_{p_1}(t)} \leq -\frac{t^{p_1}e^{-ct}dt}{\int_{0}^{\infty} r^{p_1}e^{-cr}dr}, \quad t > 0.
$$

Integrating this inequality from 0 to $t$, we obtain

$$
Y_{p_1}(t) \leq \frac{e^{p_1+1}Y_{p_1}(0)}{\Gamma(p_1 + 1)} \int_{0}^{t} r^{p_1-1}e^{-cr}dr, \quad t \geq 0.
$$

With the help of the above estimates we have that for $0 \leq p_1 < p_2 < \infty$,

$$
Y_{p_2}(0) = (p_2 - p_1) \int_{0}^{\infty} t^{p_2-p_1-1}Y_{p_1}(t)dt \leq \frac{e^{p_1+1}(p_2 - p_1)Y_{p_1}(0)}{\Gamma(p_1 + 1)} \int_{0}^{\infty} t^{p_2-p_1-1} \left( \int_{t}^{\infty} r^{p_1}e^{-cr}dr \right)dt = c^{p_1-p_2} \frac{\Gamma(p_2 + 1)}{\Gamma(p_1 + 1)} Y_{p_1}(0),
$$

thereby getting (2.2).

Regarding the second conclusion of (i), we consider two aspects. On the one hand, if

$$
X(0) = \lim_{t \to 0^+} X(t) < \infty \quad \text{and} \quad X(t) = X(0)e^{-ct} \quad \text{for} \quad t > 0,
$$

then

$$
Y_{p_1}(0) = c^{-p}\Gamma(p + 1)X(0) < \infty \quad \text{for any} \quad p \in [0, \infty),
$$

and accordingly (2.3) holds. On the other hand, assume (2.3) is valid. From the above treatment it follows that $Y_{p_1}(0) < \infty$ ensures $X(0) = \lim_{t \to 0^+} X(t) < \infty$. If the statement “$X(t) = e^{-ct}X(0)$ for $t \geq 0$” is not true, there are two positive numbers $r_0$ and $t_0$ such that $r_0 > t_0$ and $X(r_0) < e^{-c(r_0-t_0)}X(t_0)$ hold, and hence the continuity of $X(\cdot)$ produces such a constant $\delta > 0$ that $X(r_0) < e^{-c(r_0-t)}X(t)$ whenever $t \in (t_0 - \delta, t_0]$. Therefore $d(e^{ct}X(t))/dt \leq 0$ is applied to derive that $X(r) < e^{-c(r-t)}X(t)$ as $t \in (t_0 - \delta, t_0]$ and $r \geq r_0$. Consequently, we obtain

$$
Y_{p_1}(t) < cX(t)e^{ct} \int_{t}^{\infty} r^{p_1}e^{-cr}dr \quad \text{when} \quad t \in (t_0 - \delta, t_0],
$$

and thereby
whence finding
\[ Y_{p_1}(t) < \frac{c^{p_1+1}Y_{p_1}(0)}{\Gamma(p_1+1)} \int_t^\infty r^{p_1}e^{-cr}dr \quad \text{for} \quad t \in (t_0 - \delta, t_0). \]
This, along with (2.3), yields
\[ Y_{p_2}(0) = (p_2 - p_1) \int_0^\infty t^{p_2-p_1-1}Y_{p_1}(t)dt < \frac{c^{p_1-p_2}\Gamma(p_2+1)}{\Gamma(p_1+1)} Y_{p_1}(0) < \infty, \]
contradicting the previous equality assumption.

(ii) Suppose \( Y_{p_0}(0) < \infty \) holds for some \( p_0 \in [0, \infty) \). From the argument for (i) we see that \( Y_p(0) < \infty \) is valid for all \( p \geq p_0 \) and so that via integration-by-parts and \( d(e^{ct}X(t))/dt \leq 0 \),
\[ Y_p(0) = p \int_0^\infty r^{p-1}X(r)dr = p \int_0^\infty e^{ct}X(r)r^{p-1}e^{-cr}dr = \frac{(e^{ct}X(t) \int_0^t r^{p-1}e^{-cr}dr)}{c^p} \bigg|_0^\infty - p \int_0^\infty \left( \int_0^t r^{p-1}e^{-cr}dr \right) d(e^{ct}X(t)) = \frac{\Gamma(p+1)}{c^p} \lim_{t \to \infty} e^{ct}X(t) - p \int_0^\infty \left( \int_0^t r^{p-1}e^{-cr}dr \right) d(e^{ct}X(t)). \]
Therefore, the desired limit formula (2.4) follows from verifying that
\[ 0 \geq I(p, c) = \frac{c^p}{\Gamma(p+1)} \int_0^\infty \left( \int_0^t r^{p-1}e^{-cr}dr \right) d(e^{ct}X(t)) \to 0 \quad \text{as} \quad p \to \infty. \]
Notice that the condition \( d(e^{ct}X(t))/dt \leq 0 \) deduces that for any \( \epsilon > 0 \) there exists a \( t_0 > 0 \) such that \( -\epsilon < \int_{t_0}^\infty d(e^{ct}X(t)) \leq 0 \). So
\[ I_1(p, c) = \frac{c^p}{\Gamma(p+1)} \int_{t_0}^\infty \left( \int_0^t r^{p-1}e^{-cr}dr \right) d(e^{ct}X(t)) \geq \int_{t_0}^\infty d(e^{ct}X(t)) > -\epsilon. \]
Meanwhile, integrating by parts derives
\[ I_2(p, c) = \frac{c^p}{\Gamma(p+1)} \int_0^{t_0} \left( \int_0^t r^{p-1}e^{-cr}dr \right) d(e^{ct}X(t)) \geq \frac{c^p}{\Gamma(p+1)} \int_0^{t_0} t^p d(e^{ct}X(t)) \geq \frac{c^p e^{ct_0}}{\Gamma(p+1)} \int_0^{t_0} t^{p-1}dX(t) \geq \frac{-c^p e^{ct_0} \Gamma(p-1) Y_{p_0}(0)}{\Gamma(p+1)} \to 0 \quad \text{as} \quad p \to \infty. \]
The estimates on \( I_1(p, c) \) and \( I_2(p, c) \), along with \( d(e^{ct}X(t))/dt \leq 0 \), imply that
\[ -2\epsilon < I(p, c) = I_1(p, c) + I_2(p, c) \leq 0 \]
holds for sufficiently large \( p \). Thus, \( \lim_{p \to \infty} I(p, c) = 0 \), as required. \( \square \)
Remark 2.2. A close look at (2.2)-(2.3)-(2.4) leads us to conjecture that Theorem 2.1 (i) is still valid for \(-1 < p_1 < 0\). This thought is also supported by the following analysis:

(i) Using the Cauchy-Schwarz inequality and (2.2), we find that for \(-1 < p_1 < 0\),
\[
Y_0(0) \leq (Y_{p_1}(0))^{\frac{1}{2}}(Y_{-p_1}(0))^{\frac{1}{2}} \leq (c^{p_1}\Gamma(1-p_1)Y(0))^{\frac{1}{2}},
\]
thereby getting
\[
Y_0(0) \leq c^{p_1}\Gamma(1-p_1)Y_{p_1}(0) = \left(\frac{\pi p_1}{\sin\pi p_1}\right)\left(\frac{c^{p_1}}{\Gamma(1+p_1)}\right)Y_{p_1}(0).
\]
The estimate (2.6) and the Hölder inequality yield that for \(-1 < p_1 < p_2 < 0\),
\[
Y_{p_2}(0) \leq \min\left\{\left(\frac{\pi p_1c^{p_1}}{(\sin\pi p_1)\Gamma(1+p_1)}\right)^{1-\frac{p_2}{p_1}}, \left(\frac{\pi p_2c^{p_2}}{(\sin\pi p_2)\Gamma(1+p_2)}\right)^{\frac{p_2}{p_1}-1}\right\}Y_{p_1}(0).
\]
However, when \(X(t) = e^{-ct}X(0)\), the equalities in (2.6) and (2.7) do not occur.

(ii) Noticing that
\[
\lim_{p \to -1^+} \frac{\Gamma(1+p)}{(1+p)^{1-p}} = 1, \quad 0 \leq -\int_1^\infty \frac{dX(r)}{r^{1-p}} < \infty \quad \text{for} \quad p \in (-1,0); \quad cX(t) \leq -X'(t),
\]
and that \((1+p)r^pdr\), as a measure on \([0,1]\), converges weakly to the point mass at \(r = 0\) as \(p \to -1^+\), we achieve
\[
\lim_{p \to -1^+} \frac{c^pY_p(0)}{\Gamma(1+p)} = -\lim_{p \to -1^+} \frac{c^p}{\Gamma(1+p)} \left(\int_0^1 r^p dX(r) + \int_1^\infty r^p dX(r)\right)
\]
\[
= -c^{-1} \lim_{p \to -1^+} \frac{\int_0^1 X'(r)dr^{1+p}}{(1+p)\Gamma(1+p)}
\]
\[
= -c^{-1} \lim_{t \to 0^+} X'(t) \geq X(0).
\]

3. Optimal Monotonicity – Analytic Form

We first recall a definition of the well-known Green function of a bounded domain and its corresponding Robin function. Suppose \(\Omega\) is a bounded domain of \(\mathbb{R}^3\) with boundary \(\partial\Omega\). Given \(a \in \Omega\), the Green function \(g_{\Omega}(\cdot, a)\) of \(\Omega\) is the solution of the following Dirichlet boundary problem:
\[
\{ \begin{align*}
\Delta g_{\Omega}(z,a) &= -\delta_a(z), \quad z \in \Omega, \\
g_{\Omega}(z,a) &= 0, \quad z \in \partial\Omega.
\end{align*} \}
\]
Here \(\delta_a(\cdot)\) is the Dirac measure at \(a \in \Omega\). Such a solution may be evaluated by
\[
g_{\Omega}(z,a) = -(2\pi)^{-1} \left( H_{\Omega}(z,a) + \ln |z-a| \right),
\]
where \(H_{\Omega}(\cdot,a)\) is a harmonic function (i.e., \(\Delta H(\cdot,a) = 0\)) with the same values as \(-\ln |\cdot-a|\) on \(\partial\Omega\) – this gives the Robin’s function/mass \(H_{\Omega}(a,a)\) and the conformal radius \(R_{\Omega}(a)\) of \(\Omega\) at \(a \in \Omega\):
\[
H_{\Omega}(a,a) = -2\pi \lim_{z \to a} \left( (2\pi)^{-1} \ln |z-a| + g_{\Omega}(z,a) \right)
\]
and
\[
R_{\Omega}(a) = \exp \left( - H_{\Omega}(a,a) \right).
\]
In virtue of the fact that if \( u \) is of class \( C^1 \) on \( \Omega \) and its second-order partial derivatives are piecewise continuous on \( \Omega \) and if \( u \) is continuous on \( \Omega \cup \partial \Omega \) then
\[
u(a) = u_0(a) - \int_{\Omega} g_{\Omega}(z,a) \Delta u(z) dA(z), \quad a \in \Omega
\]
where \( u_0 \) is the solution of the above Dirichlet problem for \( \Omega \) with the same values \( u \) on \( \partial \Omega \), Huber proved the following assertion – \cite{21} Theorem 2:

**Theorem 3.1.** Let \( \Omega \) be the interior of a rectifiable simple curve \( \partial \Omega \) in \( \mathbb{R}^2 \). Suppose \( u \) is continuous on \( \Omega \cup \partial \Omega \) and of class \( C^1 \) as well as its second-order derivatives are piecewise continuous on \( \Omega \). Then

\[
\left( \int_{\partial \Omega} e^{u(z)} dL(z) \right)^2 \geq 4\pi \left( 1 - (2\pi)^{-1} \int_{\Omega} \max\{ -\Delta u(z), 0 \} dA(z) \right) \int_{\Omega} e^{2u(z)} dA(z),
\]
with equality when and only when \( u \) is \( \ln |f'| \) of a conformal map \( f \) from \( \Omega \) onto a Euclidean disk in \( \mathbb{R}^2 \).

Here and later on, \( dL(z) \) stands for the arc-length element. Below is our optimal analytic principle for monotonicity of the \( L^p \) integral of Green’s function with respect to the conformal area measure \( e^{2u} dA \).

**Theorem 3.2.** Let \( \Omega \) be the interior of a rectifiable simple curve \( \partial \Omega \) in \( \mathbb{R}^2 \). Suppose \( u \) is continuous on \( \Omega \cup \partial \Omega \) and of class \( C^1 \) as well as its second-order derivatives are piecewise continuous on \( \Omega \). Set
\[
p \in [0, \infty), \quad a \in \Omega, \quad \kappa(\Omega) = 1 - (2\pi)^{-1} \int_{\Omega} \max\{ -\Delta u(z), 0 \} dA(z) > 0,
\]
and
\[
\mathcal{F}(p,a,\kappa(\Omega)) = \left( \frac{4\pi \kappa(\Omega)}{p} \right)^{p} \int_{\Omega} (g_{\Omega}(z,a))^p e^{2u(z)} dA(z).
\]
Then
\[
0 \leq p_1 < p_2 < \infty \Rightarrow \mathcal{F}(p_2,a,\kappa(\Omega)) \leq \mathcal{F}(p_1,a,\kappa(\Omega)).
\]
The second equality in (3.2) occurs when and only when there exists a conformal map \( f \) from \( \Omega \) onto a Euclidean disk centered at \( f(a) \) in \( \mathbb{R}^2 \) such that \( u = \ln |f'| \).

(ii)
\[
\lim_{p \to \infty} \mathcal{F}(p,a,\kappa(\Omega)) = \begin{cases} \pi e^{u(a)} R_{\Omega}(a) \mu_1, & \kappa(\Omega) < 1, \\ 0, & \kappa(\Omega) = 1, \end{cases}
\]
where
\[
e^{u(a)} R_{\Omega}(a) = R_{f(\Omega)}(f(a))
\]
whenever \( u = \ln |f'| \) for a conformal mapping \( f \) from \( \Omega \) onto \( f(\Omega) \).

**Proof.** (i) For \( t \geq 0 \) and \( a \in \Omega \) let
\[
\Omega_t = \{ z \in \Omega : g_{\Omega}(z,a) > t \}.
\]
Then
\[
\partial \Omega_t = \{ z \in \Omega : g_{\Omega}(z,a) = t \}.
\]
If \( \psi \) is a conformal map from \( \Omega \) onto the unit disk \( \mathbb{D} \subset \mathbb{R}^2 \) with \( \psi(a) = 0 \) then \( g_\Omega(\cdot, \cdot) \) can be expressed by the following formula (cf. [15, p. 172]):

\[
g_\Omega(z, a) = -(2\pi)^{-1} \ln |\psi(z)|, \quad z \in \Omega,
\]

and hence a direct computation yields

\[
\frac{\partial g_\Omega(z, a)}{\partial n} = (2\pi)^{-1} |\psi'(z)|, \quad z \in \Omega.
\]

In the above and below, \( \partial/\partial n \) is the inner normal derivative. Putting

\[
X(t) = \int_{\Omega_t} e^{2u(z)} dA(z) \quad \text{and} \quad X_0(t) = \int_{\partial\Omega_t} \left| \frac{e^{u(z)}}{\psi'(z)} \right|^2 \left( \frac{\partial g_\Omega(z, a)}{\partial n} \right)^2 dL(z),
\]

we get that for \( p \in [0, \infty) \),

\[
Y_p(t) = \int_{\Omega_t} (g_\Omega(z, a))^p e^{2u(z)} dA(z)
\]

\[
= \int_{\Omega_t} \left( \frac{e^{u(z)}}{\psi'(z)} \right)^2 (g_\Omega(z, a))^p |\psi'(z)|^2 dA(z)
\]

\[
= \int_t^\infty \left( \int_{\partial\Omega_t} (g_\Omega(z, a))^p \left| \frac{e^{u(z)}}{\psi'(z)} \right|^2 \left( \frac{\partial g_\Omega(z, a)}{\partial n} \right)^2 dL(z) d\nu(z) \right) dr
\]

\[
= \int_t^\infty r^p X_0(r) dr,
\]

whence finding

\[
(3.5) \quad X(t) = \int_t^\infty X_0(r) dr, \quad t > 0.
\]

The formula (3.5) indicates that \( X(\cdot) \) satisfies the first condition of (2.1). Moreover, letting \( t \to 0^+ \) we achieve

\[
Y_p(0) = \int_{\Omega} (g_\Omega(z, a))^p e^{2u(z)} dA(z)
\]

\[
= \int_0^\infty r^p X_0(r) dr
\]

\[
= -\int_0^\infty r^p dX(r).
\]

Since \( \partial\Omega_t \) is a real analytic curve and

\[
\int_{\partial\Omega_t} \left( \frac{\partial g_\Omega(z, a)}{\partial n} \right) dL(z) = (2\pi)^{-1} \int_{\partial\Omega_t} |\psi'(z)| dL(z) = 1
\]

for almost all \( t \geq 0 \), we may apply Huber’s inequality in Theorem 3.1 to \( \Omega_t \cup \partial\Omega_t \) and then use the Cauchy-Schwarz inequality to deduce
\[4\pi\kappa(\Omega)X(t) \leq 4\pi \left(1 - (2\pi)^{-1}\right) \int_{\Omega_t} \max\{-\Delta u(z), 0\} dA(z) \int_{\Omega_t} e^{2u(z)} dA(z) \]

\[
\leq \left( \int_{\partial\Omega_t} e^{u(z)} dL(z) \right)^2
\]

\[= \left( \int_{\partial\Omega_t} \left| \frac{e^{u(z)}}{\psi(z)} \right| \left| \frac{\partial g_{\Omega}(z, a)}{\partial n} \right| dL(z) \right)^2 \]

\[\leq \left( \int_{\partial\Omega_t} \left| \frac{e^{u(z)}}{\psi(z)} \right|^2 \left| \frac{\partial g_{\Omega}(z, a)}{\partial n} \right| dL(z) \right) \left( \int_{\partial\Omega_t} \left| \frac{\partial g_{\Omega}(z, a)}{\partial n} \right| dL(z) \right)
\]

\[= X_0(t). \]

The estimate \([3.6]\) ensures

\[\frac{d(e^{4\pi\kappa(\Omega)t} X(t))}{dt} = e^{4\pi\kappa(\Omega)t} (4\pi\kappa(\Omega)X(t) - X_0(t)) \leq 0,\]

and then makes the second condition in \([2.1]\) available for \(c = 4\pi\kappa(\Omega)\). An easy application of Theorem \([2.1]\) (i) implies that \([3.2]\) is true for all \(a \in \Omega\).

Next, let us handle the equality of \([3.2]\). If \(u\) equals \(\ln |f'|\) for a conformal map \(f\) from \(\Omega\) onto \(D(f(a), R) = \{w \in \mathbb{R}^2 : |w - f(a)| < R\}\) for which the Green function is

\[g_{D(f(a), R)}(w_1, w_2) = (2\pi)^{-1} \ln \left| \frac{R^2 - (w_1 - f(a))(w_2 - f(a))}{R(w_1 - w_2)} \right|\]

where \(w_1, w_2 \in D(f(a), R)\), then \(\kappa(\Omega) = 1\), and hence from the conformal invariance of Green’s functions it follows that for \(t \geq 0\),

\[X(t) = \int_{\{z \in \Omega : g_{\Omega}(z, a) > t\}} |f'(z)|^2 dA(z) \]

\[= \int_{\{w \in D(f(a), R) : g_{D(f(a), R)}(w, f(a)) > t\}} dA(w) \]

\[= \int_{\{w \in D(f(a), R) : \ln(R/|w - f(a)|) > 2\pi t\}} dA(w) \]

\[= e^{-4\pi t} R^2. \]

Accordingly, the equality part of Theorem \([2.1]\) (i) is used to derive the validity of the equality of \([3.2]\).

Conversely, the equality part of Theorem \([2.1]\) (i) suggests us to show only that \(X(t) = e^{-4\pi\kappa(\Omega)t} X(0)\) implies \(u = \ln |f'|\) where \(f\) is a conformal map from \(\Omega\) onto a Euclidean disk centered at \(f(a)\) in \(\mathbb{R}^2\). Now, suppose \(X(t) = e^{-4\pi\kappa(\Omega)t} X(0)\). By \([3.5]\) we have

\[e^{-4\pi\kappa(\Omega)t} X(0) = \int_{t}^{\infty} X_0(r) dr = e^{-4\pi\kappa(\Omega)t} \int_{0}^{\infty} X_0(r) dr.\]

Differentiating the left-hand equality and using the hypothesis, we obtain

\[X_0(t) = 4\pi\kappa(\Omega)e^{-4\pi\kappa(\Omega)t} X(0) = 4\pi\kappa(\Omega)X(t), \quad t \geq 0,\]

and consequently, the first inequality in \([3.3]\) becomes an equality for \(t \geq 0\). According to the equality case of Theorem \([3.3]\) we know that \(u\) is the same as \(\ln |f'|\)
for a conformal map \( f \) from \( \Omega \) onto a Euclidean disk \( D(b, R) \) with center \( b \) and radius \( R \). So, \( \kappa(\Omega) = 1 \). This in turn yields

\[
X(t) = \int_{\{z \in \Omega : g_\Omega(z,a) > t\}} |f'(z)|^2 dA(z) = e^{-4\pi t} X(0).
\]

The formula (3.7) must enforce \( b = f(a) \). To see this point, suppose \( b \neq f(a) \), then \( \delta = \overline{f(a) - b} \) meets \( 0 < |\delta| < R \) and \( 0 < \lambda = (R^2 - |\delta|^2)/R < R \). Because of (3.7) and the conformal invariance of Green’s functions, we get

\[
e^{-4\pi t} \pi R^2 = e^{-4\pi t} \int_{\{w \in D(b,R) : g_{D(b,R)}(w,f(a)) > t\}} dA(w)
= \int_{\{w \in D(b,R) : g_{D(b,R)}(w,f(a)) > t\}} dA(w)
= \int_{\{w \in D(b,R) : |\lambda/(w-f(a)) - \delta/R| > e^{2\pi t}\}} dA(w)
\leq \lambda^2 \pi e^{-4\pi t} < R^2 \pi e^{-4\pi t} \text{ as } t \to \infty,
\]

thereby reaching a contradiction.

(ii) Owing to \( X(0) = \int_\Omega e^{2u(z)} dA(z) < \infty \), the preceding argument and (2.4) yield that

\[
\lim_{t \to \infty} \mathcal{F}(p, a, \kappa(\Omega)) = \lim_{t \to \infty} e^{4\pi \kappa(\Omega)t} X(t)
\]
exists for every \( a \in \Omega \). Fix a point \( \zeta_0 \in \Omega \) and suppose \( f \) is a Riemann mapping associated with \( \zeta_0 \) — that is, a conformal map \( f \) from \( \Omega \) onto the unit open disk \( \mathbb{D} \) such that \( f(\zeta_0) = 0 \) and \( R_\Omega(\zeta_0) = |f'(\zeta_0)|^{-1} \). Via the superposition \( F(z,a) = \phi_{f(a)}(f(z)) \) of \( f \) with a standard Möbius transform from \( \mathbb{D} \) onto itself:

\[
\phi_w(z) = \frac{w-z}{1-w\bar{z}}, \quad z, w \in \mathbb{D},
\]

we find that \( F(\cdot, a) \) is a Riemann mapping associated with \( a \in \Omega \), and so that

\[
R_\Omega(a) = \left| \frac{dF(z,a)}{dz} \right|_{z=a}^{-1} = 1 - \left| f(a) \right|^2/|f'(a)|^2.
\]

Now, if \( h \) is the inverse map of \( f \) and \( a = h(b) \), then

\[
X(t) = \int_{\Omega} e^{2u(z)} dA(z)
= \int_{\{|w| < e^{-2\pi t}\}} e^{2u_{\phi_{f(a)}}(w)} \left| (h \circ \phi_w)'(w) \right|^2 dA(w)
= \int_{\{|w| < e^{-2\pi t}\}} e^{2u_{\phi_{f(a)}}(w)} \left| (h \circ \phi_w)'(w) \right|^2 dA(w),
\]
and hence
\[
\lim_{t \to \infty} \frac{X(t)}{e^{-4\pi \kappa(\Omega)t}} = \pi \lim_{t \to \infty} e^{4\pi (\kappa(\Omega)-1)t} \int_{\{w \in \mathbb{D} : e^{2\pi |w|} < 1\}} \left( \frac{[(h \circ \phi_0)'(w)]}{e^{-u_0'f \circ \phi_0(w)}} \right)^2 dA(w).
\]

\[
= \pi \lim_{t \to \infty} e^{4\pi (\kappa(\Omega)-1)} e^{2u_0(h)} |(f \circ \phi_0)'(0)|^2
\]
\[
= \pi \lim_{t \to \infty} e^{4\pi (\kappa(\Omega)-1)} \left( \frac{e^{u(a)} |h'(b)| (1 - |b|^2)}{|f'(a)|} \right)^2
\]
\[
= \pi \lim_{t \to \infty} e^{4\pi (\kappa(\Omega)-1)} \left( \frac{e^{u(a)} (1 - |f(a)|^2)}{|f'(a)|} \right)^2
\]
\[
= \pi \lim_{t \to \infty} e^{4\pi (\kappa(\Omega)-1)} t \left( \frac{e^{u(a)} R_{\Omega}(a)}{|f'(a)|} \right)^2.
\]

Now, if there is a conformal mapping \( f \) from \( \Omega \) onto \( f(\Omega) \) such that \( e^u = |f'| \), then the conformal transformation law for the Robin function/mass (cf. [6]) derives
\[
e^{u(a)} R_{\Omega}(a) = |f'(a)||R_{\Omega}(a)| = R_{f(\Omega)}(f(a)),
\]
as desired. \( \square \)

**Remark 3.3.** In accordance with Remark 2.2, we strongly feel that Theorem 3.2 (i) is also true for \(-1 < p_1 < 0\). The coming-up-next estimates, corresponding to ones in Remark 2.2, are in support of this feeling.

(i) When \(-1 < p_1 < 0\),
\[
\int_{\Omega} e^{2u(z)} dA(z) \leq \left( \frac{\pi p_1}{\sin \pi p_1} \right)^{p_1} \left( \frac{4\pi \kappa(\Omega)}{\Gamma(1 + p_1)} \right)^{p_1} \int_{\Omega} (g_{\Omega}(z, a))^p e^{2u(z)} dA(z).
\]
The inequality (3.8), along with Hölder’s inequality, gives that if \(-1 < p_1 < p_2 < 0\) then
\[
\int_{\Omega} (g_{\Omega}(z, a))^{p_2} e^{2u(z)} dA(z) \leq c(p_1, p_2, \kappa(\Omega)) \int_{\Omega} (g_{\Omega}(z, a))^{p_1} e^{2u(z)} dA(z),
\]
where
\[
c(p_1, p_2, \kappa(\Omega)) = \min \left\{ \left( \frac{\pi p_1}{\sin \pi p_1} \Gamma(1 + p_1) \right)^{1 - \frac{p_2}{p_1}}, \left( \frac{\pi p_2}{\sin \pi p_2} \Gamma(1 + p_2) \right)^{\frac{p_2}{p_1} - 1} \right\},
\]
and (4.9) is not optimal.

(ii) When \(-1 < p < 0\),
\[
\int_{\Omega} e^{2u(z)} dA(z) \leq -4\pi \kappa(\Omega) \lim_{t \to 0^+} \frac{d}{dt} \left( \int_{\Omega_t} e^{2u(z)} dA(z) \right)
\]
\[
= \lim_{p \to -1^+} \frac{\Gamma(1 + \frac{p}{p})}{\Gamma(1 + p)} \int_{\Omega} (g_{\Omega}(z, a))^p e^{2u(z)} dA(z)
\]
\[
= (4\pi \kappa(\Omega))^{-1} \int_{\partial \Omega} \left( \frac{e^{2u(z)}}{\partial \phi_0(z, a)} \right) dL(z).
\]

(iii) From (2.4) and (3.3) we see
\[
\lim_{t \to \infty} e^{4\pi \kappa(\Omega)t} \int_{\Omega_t} e^{2u(z)} dA(z) = \left\{ \begin{array}{ll}
0, & \kappa(\Omega) < 1, \\
\pi \left( e^{u(a)} R_{\Omega}(a) \right)^2, & \kappa(\Omega) = 1,
\end{array} \right.
\]
whose special case \( u = 0 \) produces the corresponding limit formula in [10] Lemma 10 (cf. [30] Lemma 1(c)).

More interestingly, a combination of Theorems 3.1-3.2 and Remark 3.3 implies a chain of inequalities linking the integrals on a domain and its boundary.

**Corollary 3.4.** Let \( \Omega \) be the interior of a rectifiable simple curve in \( \mathbb{R}^2 \). Suppose \( u \) is continuous on \( \Omega \cup \partial \Omega \) and of class \( C^1 \) as well as its second-order derivatives are piecewise continuous on \( \Omega \). Suppose 

\[
p \in (0, \infty), \quad a \in \Omega, \quad \kappa(\Omega) = 1 - (2\pi)^{-1} \int_{\Omega} \max\{-\Delta u(z), 0\} dA(z) > 0,
\]

and

\[
F(p, a, \kappa(\Omega)) = \left(\frac{4\pi \kappa(\Omega)}{\Gamma(p + 1)}\right)^p \int_{\Omega} (g_{\Omega}(z, a))^p e^{2u(z)} dA(z).
\]

Then

\[
4\pi \kappa(\Omega) F(p, a, \kappa(\Omega)) \leq \left( \int_{\partial \Omega} e^{u(z)} dL(z) \right)^2 \leq \int_{\partial \Omega} \left( \frac{e^{2u(z)}}{\frac{\partial g_{\Omega}(z, a)}{\partial n}} \right) dL(z),
\]

where the left- (right-) hand equality in (3.12) occurs when and only when there is a conformal map \( f \) from \( \Omega \) onto a Euclidean disk centered at \( f(a) \) in \( \mathbb{R}^2 \) such that \( u = \ln |f'| \) (there is a positive number \( \lambda \) such that \( u = \ln (\lambda \frac{\partial g_{\Omega}(z, a)}{\partial n}) \)).

**Proof.** Since the setting \( 0 = p_1 < p_2 = p < \infty \) of Theorem 3.2 (i) tells us that

\[
F(p, a, \kappa(\Omega)) \leq \int_{\Omega} e^{2u(z)} dA(z)
\]

holds for every \( a \in \Omega \), the corollary follows from Theorems 3.1 and 3.2, the foregoing inequality and the following Cauchy-Schwarz’s inequality-based estimate:

\[
\left( \int_{\partial \Omega} e^{u(z)} dL(z) \right)^2 \leq \int_{\partial \Omega} \left( \frac{e^{2u(z)}}{\frac{\partial g_{\Omega}(z, a)}{\partial n}} \right) dL(z) \int_{\partial \Omega} \left( \frac{\partial g_{\Omega}(z, a)}{\partial n} \right) dL(z)
\]

\[
= \int_{\partial \Omega} \left( \frac{e^{2u(z)}}{\frac{\partial g_{\Omega}(z, a)}{\partial n}} \right) dL(z),
\]

where the inequality becomes an equality when and only when

\[
e^{2u(z)} \left( \frac{\partial g_{\Omega}(z, a)}{\partial n} \right)^{-1} = \lambda \left( \frac{\partial g_{\Omega}(z, a)}{\partial n} \right)
\]

holds for some constant \( \lambda > 0 \). \( \square \)

4. **Optimal Monotonicity – Geometric Form**

The monotonicity established in the last section may be extendable to an optimal geometric monotone principle for the \( L^p \)-integral of Green’s function of a simply-connected domain on any abstract surface (cf. [5] for more information).

To see this, suppose \( S \) is a surface which has such an isothermic representation \((\Sigma, \sigma)\) that \( \Sigma \) is a subdomain of \( \mathbb{R}^2 \) and has the positive definite quadratic form (i.e., Riemannian metric):

\[
\sigma = e^{2u} ds^2 = e^{2u} |dz|^2 = e^{2u(z)}(dx^2 + dy^2), \quad z = x + iy \in \Sigma.
\]
Of course, \( u \) is here assumed to be continuous on \( \Sigma \) and its boundary \( \partial \Sigma \), be of class \( C^1 \), and have piecewise continuous second-order partial derivatives on \( \Sigma \).

Under this parameter system the Gauss curvature at every point of \((\Sigma, \sigma)\) is determined by

\[
K_\sigma = -e^{-2u} \Delta u = -\Delta_\sigma u,
\]

where \( \Delta_\sigma \) is the Laplacian operator associated with the planar conformal metric \( \sigma \). Here it is perhaps appropriate to mention the following open problem of Berger type: Find a conformal metric \( e^{2u} ds^2 \) on a domain \( \Sigma \subseteq \mathbb{R}^2 \) with prescribed Gaussian curvature \( K \); equivalently find a solution \( u \) to the semi-linear elliptic equation \( Ke^{2u} + \Delta u = 0 \) for a given function \( K \) on \( \Sigma \). It is well-known that if \( K = -4 \) and \( \Sigma = \Omega \) (considered in the last section) then \( \Delta u = 4e^{2u} \) is the so-called Liouville’s equation and takes the Robin function/mass \( H_{\Omega}(\cdot, \cdot) \) as the solution (see e.g. [10]). Furthermore, it is proved in [37] that if \( K \) is of class \( C^2 \) and bounded on a bounded domain \( \Sigma \) then the Liouville equation has a solution on \( \Sigma \). Additionally, on the unbounded domain \( \Sigma = \mathbb{R}^2 \), searching for a solution of the equation under the condition \( \int_{\mathbb{R}^2} K dA_\sigma < \infty \) is of particular interest; see [12], and [24] (showing that \( K \in C^\infty(\mathbb{R}^2) \) is the Gauss curvature of some Riemannian metric on \( \mathbb{R}^2 \)), as well as [11] (for more information on nonlinear elliptic equations in conformal geometry).

Given a bounded and open subset \((O, \sigma)\) of \((\Sigma, \sigma)\) with boundary \((\partial O, \sigma)\), we denote by \( g_{(O, \sigma)}(\cdot, a) \) the Green function of \((O, \sigma)\) with pole \( a \in O \) for \( \Delta_\sigma \) provided that this function is determined by the Dirichlet boundary problem:

\[
\begin{cases}
\Delta_\sigma g_{(O, \sigma)}(z, a) = -\delta_a(z), & z \in O, \\
g_{(O, \sigma)}(z, a) = 0, & z \in \partial O.
\end{cases}
\]

Note that the first equation is understood under the distribution with respect to the area element \( dA_\sigma \). So, this Green function \( g_{(O, \sigma)}(z, a) \) coincides with the Green function \( g_\sigma(z, a) \) (i.e., \( g_{(O, ds^2)}(z, a) \)) for \( \Delta \) discussed in the last section. Usually, the definition of the Green function \( g_{(O, \sigma)}(\cdot, \cdot) \) can be extended to the surface \((\Sigma, \sigma)\) through setting \( g_{(O, \sigma)}(z, a) = 0 \) for \( z \in \Sigma \setminus O \).

On the surface \((\Sigma, \sigma)\) the length and area elements are defined by

\[
dL_\sigma(z) = e^{u(z)} dL(z) \quad \text{and} \quad dA_\sigma(z) = e^{2u(z)} dA(z) \quad \text{for} \quad z \in \Sigma
\]

respectively. This gives the length of a rectifiable simple curve \( C = (\partial \Omega, \sigma) \) on \((\Sigma, \sigma)\) and the area of a simply-connected domain \( D = (\Omega, \sigma) \):

\[
L_\sigma(C) = \int_C dL_\sigma = \int_{\partial \Omega} e^{u(z)} dL(z) \quad \text{and} \quad A_\sigma(D) = \int_D dA_\sigma = \int_{\Omega} e^{2u(z)} dA(z).
\]

As a result, the distance \( d_\sigma(z, a) \) between \( z \) and \( a \) in \((\Sigma, \sigma)\) is defined by \( \inf_{\gamma} L_\sigma(\gamma) \) where the infimum is taken over all rectifiable simple curves \( \gamma \) connecting \( z \) and \( a \). In terms of the Green function and the distance function, we introduce a concept of the Robin function/mass \( H_{(\Omega, \sigma)}(a, a) \) and the conformal radius \( R_{(\Omega, \sigma)}(a) \) of \((\Omega, \sigma)\) below:

\[
H_{(\Omega, \sigma)}(a, a) = -2\pi \lim_{z \to a} \left( (2\pi)^{-1} \ln d_\sigma(z, a) + g_{(\Omega, \sigma)}(z, a) \right)
\]

and

\[
R_{(\Omega, \sigma)}(a) = \exp \left( -H_{(\Omega, \sigma)}(a, a) \right).
\]

Furthermore, let

\[
K_\sigma^\pm(z) = \max\{\pm K_\sigma(z), 0\} = \max\{\mp \Delta_\sigma u(z), 0\}.
\]
Then the surface version of the Huber’s Theorem 3.1 is the following assertion (cf. [21 Theorem 3]).

**Theorem 4.1.** Let \( \sigma = e^{2u} ds^2 \) be a conformal metric on a domain \( \Sigma \subseteq \mathbb{R}^2 \) for which \( u \) is continuous on \( \Sigma \cup \partial \Sigma \) but also is of class \( C^1 \) and piecewise continuous second-order partial derivatives on \( \Sigma \). If a rectifiable simple curve \( \partial D \) of length \( L_\sigma(\partial D) \) encloses a simply-connected domain \( D \) of area \( A_\sigma(D) \) on the surface \( (\Sigma, \sigma) \), then

\[
(L_\sigma(\partial D))^2 \geq 4\pi A_\sigma(D) \left( 1 - (2\pi)^{-1} \int_D K_\sigma^+ dA_\sigma \right).
\]

The equality in (4.2) holds when and only when \( K_\sigma \) vanishes on \( D \) and \( \partial D \) is a geodesic circle on \( (\Sigma, \sigma) \).

With the help of Theorems 2.1 and 4.1 we obtain a geometric description of Theorem 3.2.

**Theorem 4.2.** Let \( \sigma = e^{2u} ds^2 \) be a conformal metric on a domain \( \Sigma \subseteq \mathbb{R}^2 \) for which \( u \) is continuous on \( \Sigma \cup \partial \Sigma \) but also is of class \( C^1 \) and piecewise continuous second-order partial derivatives on \( \Sigma \). Suppose \( D = (\Omega, \sigma) \) is a simply-connected domain with \( \partial D = (\partial \Omega, \sigma) \) being a rectifiable simple curve on \( (\Sigma, ds^2) \). If

\[
p \in [0, \infty), \quad (a, \sigma) \in D, \quad \kappa_\sigma(D) = 1 - (2\pi)^{-1} \int_D K_\sigma^+ dA_\sigma > 0,
\]

and

\[
G(p, a, \kappa_\sigma(D)) = \left( \frac{4\pi \kappa_\sigma(D)}{\Gamma(p + 1)} \right)^p \int_D (g_D(\cdot, a))^p dA_\sigma(\cdot),
\]

then

(i) \( 0 \leq p_1 < p_2 < \infty \Rightarrow G(p_1, a, \kappa_\sigma(D)) \leq G(p_2, a, \kappa_\sigma(D)) \),

(ii) \[
\lim_{p \to \infty} G(p, a, \kappa_\sigma(D)) = \begin{cases} 0, & \kappa_\sigma(D) < 1, \\ \pi (R(\Omega, \sigma)(a))^2, & \kappa_\sigma(D) = 1, \end{cases}
\]

where

\[
R(\Omega, \sigma)(a) = R_{f(\Omega)}(f(a))
\]

whenever \( u = \ln |f'| \) for a conformal mapping \( f \) from \( \Omega \) onto \( f(\Omega) \).

**Proof.** Since \( G(p, a, \kappa_\sigma(D)) \) actually coincides with \( F(p, a, \kappa(\Omega)) \), (4.3) follows from (3.2) right away. Moreover, the right-hand equality in (4.3) holds if and only if the right-hand equality in (3.2) holds. This amounts to \( u = \ln |f'| \) where \( w = f(z) \) is a conformal mapping from \( \Omega \) onto a Euclidean disk centered at \( f(a) \) in \( \mathbb{R}^2 \). Note that for such a conformal mapping \( f \),

\[
|dw| = |f'(z)||dz| = e^u ds = dL_\sigma.
\]

Thus we see that there is an isometry from \( D \) onto a Euclidean disk centered at \( f(a) \), thereby getting that \( K_\sigma = 0 \) on \( D \) but also the boundary \( \partial D \) becomes a geodesic circle with center \( (a, \sigma) \).
Corollary 4.4. Let $\sigma = e^{2u}ds^2$ be a conformal metric on a domain $\Sigma \subseteq \mathbb{R}^2$ for which $u$ is continuous on $\Sigma \cup \partial \Sigma$ but also is of class $C^1$ and piecewise continuous

Next, (4.4) and (4.5) follow from (3.3), (3.4) and a series of calculations:

$$
(2\pi)^{-1} \log \left( R(\Omega, \sigma)(a) \right) = \lim_{z \to a} \left( 2\pi)^{-1} \log d_{\sigma}(z, a) + g(\Omega, \sigma)(z, a) \right)
$$

$$
= \lim_{z \to a} \left( 2\pi)^{-1} \log d_{\sigma}(z, a) + g_\Omega(z, a) \right)
$$

$$
= \lim_{z \to a} \left( 2\pi)^{-1} \log (e^{u(a)}|z - a|) + g_\Omega(z, a) + O(|z - a|) \right)
$$

$$
= (2\pi)^{-1} \left( u(a) + \log R_\Omega(a) \right).
$$

In the last second equality we have used a readily-checked fact (cf. [41, Lemma 1]) that there are two positive constants $c_1, c_2$ to ensure the implication:

$$
|z - a| < c_1 \Rightarrow \left| \frac{\log d_{\sigma}(z, a)}{|z - a|} - u(a) \right| \leq c_2 |z - a|.
$$

$$
\square
$$

Remark 4.3. Like Remark 3.3, we have (4.6)-(4.7)-(4.8) parallel to (3.8)-(3.9)-(3.10): (i) When $-1 < p_1 < 0$,

$$
A_\sigma(D) \leq \left( \frac{\pi p_1}{\sin \pi p_1} \right) \left( \frac{4\pi \kappa_\sigma(D)}{\Gamma(1 + p_1)} \right)^{p_1} \int_D \left( g_\Omega(\cdot, a) \right)^{p_1} dA_\sigma(\cdot).
$$

The inequality (3.8), plus H"older’s inequality, gives that for $-1 < p_1 < p_2 < 0$,

$$
\int_D \left( g_\Omega(\cdot, a) \right)^{p_2} dA_\sigma(\cdot) \leq c(p_1, p_2, \kappa_\sigma(D)) \int_D \left( g_\Omega(\cdot, a) \right)^{p_1} dA_\sigma(\cdot).
$$

(ii) When $-1 < p < 0$,

$$
A_\sigma(D) \leq -(4\pi \kappa_\sigma(D))^{-1} \lim_{t \to 0^+} \frac{d}{dt} \left( \int_{\Omega_t} dA_\sigma(\cdot) \right)
$$

$$
= \lim_{p \to -1^+} \left( \frac{4\pi \kappa_\sigma(D)}{\Gamma(1 + p)} \right)^{p} \int_D \left( g_\Omega(\cdot, a) \right)^{p} dA_\sigma(\cdot)
$$

$$
= \left( \frac{4\pi \kappa_\sigma(D)}{\Gamma(1 + p)} \right)^{p} \int_{\Omega} \left( \frac{\partial g_\Omega(\cdot, a)}{\partial n_\sigma} \right)^{-1} dL_\sigma(\cdot).
$$

In the above and below,

$$
\frac{\partial g_\Omega(\cdot, a)}{\partial n_\sigma} = e^{-u(\cdot)} \frac{\partial g_\Omega(\cdot, a)}{\partial n}
$$

is the inner normal derivative of $g_\Omega(\cdot, a)$ with respect to the metric $\sigma = e^{2u}ds^2$. (ii) From (2.3) and (4.4) we see the counterpart of (3.11) below:

$$
\lim_{t \to \infty} A_\sigma(\{z \in \Omega : g_\Omega(z, a) > t \}) = \begin{cases} 0, & \kappa_\sigma(D) < 1, \\ \pi(R(\Omega, \sigma)(a))^{2}, & \kappa_\sigma(D) = 1. \end{cases}
$$

Needless to say, the newfound optimal isoperimetric-type inequality in the following corollary is of particular interest.

Corollary 4.4. Let $\sigma = e^{2u}ds^2$ be a conformal metric on a domain $\Sigma \subseteq \mathbb{R}^2$ for which $u$ is continuous on $\Sigma \cup \partial \Sigma$ but also is of class $C^1$ and piecewise continuous
second-order partial derivatives on \( \Sigma \). Suppose \( D = (\Omega, \sigma) \) is a simply-connected domain on \( (\Sigma, \sigma) \) with \( \partial D = (\partial \Omega, \sigma) \) being a rectifiable simple curve. If \( p \in (0, \infty), \ (a, \sigma) \in D, \ \kappa_\sigma(D) = 1 - (2\pi)^{-1} \int_D K_\sigma^+ dA_\sigma > 0, \) and

\[
G(p, a, \kappa_\sigma(D)) = \frac{(4\pi \kappa_\sigma(D))^p}{\Gamma(p + 1)} \int_D (g_D(\cdot, a))^p dA_\sigma(\cdot),
\]

then

\[
4\pi \kappa_\sigma(D) G(p, a, \kappa_\sigma(D)) \leq (L_\sigma(\partial D))^2 \leq \int_{\partial D} \left( \frac{\partial g_D(\cdot, a)}{\partial n_\sigma} \right)^{-1} dL_\sigma(\cdot),
\]

where the left- (right-) hand inequality in (4.10) happens when and only when \( K_\sigma \) vanishes on \( D \) and \( \partial D \) is a geodesic circle centered at \((a, \sigma) \in D \) (there is a positive number \( \lambda \) such that \( u = \ln (\lambda \partial g_D(\cdot, a)/\partial n) \)).

Proof. This follows immediately Corollary 5.4. \( \Box \)

5. Application

In this final section we are concerned about how to apply the previous ideas, methods and techniques to settling some problems on complete noncompact surfaces without boundary.

In accordance with the definition adapted by [26] and [27], we say that \((M^2, g)\) is a complete noncompact boundary-free surface provided that \((M^2, g)\) is a two-dimensional complete noncompact manifold \(M^2\) without boundary, equipped with a Riemannian metric \(g\). On such a surface, we always employ

\[
d(\cdot, \cdot); \ K_g(\cdot); \ K_g^\pm(\cdot) = \max\{\pm K_g, 0\}; \ \chi(\cdot); \ dA_g(\cdot); \ d\lambda_g(\cdot); \ \Delta_g(\cdot); \ \nabla_g(\cdot),
\]

to denote the distance function; the Gauss curvature; the positive or negative part of the Gauss curvature; the Euler characteristic; the area element; the length element; the Laplacian operator; the gradient, respectively – see also Shiohama-Shioya-Tanaka’s monograph [39] for some related materials. The following celebrated Gaussian-Bonnet type results (i) and (ii) are due to Cohn-Vossen [14] and Huber [23], and Hartman [19] and Shiohama [38], in the above-mentioned order.

**Theorem 5.1.** Let \((M^2, g)\) be a complete noncompact boundary-free surface with \( K_g^- \) being integrable with respect to \( dA_g \). Then

(i) \(M^2\) is conformally equivalent to a compact Riemann surface minus finitely many points. Moreover

\[
\int_{M^2} K_g dA_g \leq 2\pi \chi(M^2) \quad \text{and} \quad \int_{M^2} |K_g| dA_g < \infty.
\]

Especially, \(M^2\) is conformally equivalent to \(\mathbb{R}^2\) whenever \(M^2\) is simply-connected.

(ii) For any geodesic ball \(B(a, r) = \{z \in M^2 : d_g(z, a) < r\}\) centered at \(a \in M^2\) with radius \(r > 0\) and its boundary \(\partial B(a, r) = \{z \in M^2 : d_g(z, a) = r\}\) on \((M^2, g)\),

\[
\chi(M^2) - (2\pi)^{-1} \int_{M^2} K_g dA_g = \lim_{r \to \infty} \frac{\left( L_g(\partial B(a, r)) \right)^2}{4\pi A_g(B(a, r))}.
\]
Given a bounded and open subset $O$ of $\mathbb{M}^2$ with boundary $\partial O$, we denote by $g_{(O,\mathbb{g})}(\cdot,a)$ the Green function of $O$ with pole at $a \in O$ for $\Delta_\mathbb{g}$ provided this function is decided by the Dirichlet boundary problem:

\[
\begin{cases}
\Delta_\mathbb{g} g_{(O,\mathbb{g})}(z,a) = -\delta_a(z) & , \quad z \in O, \\
g_{(O,\mathbb{g})}(z,a) = 0 & , \quad z \in \partial O. 
\end{cases}
\]

The first equation is clearly understood under the distribution with respect to the area element $dA_\mathbb{g}$. Moreover, the definition of this Green's function can be extended to the surface $(\mathbb{M}^2,\mathbb{g})$ via letting $g_{(O,\mathbb{g})}(z,a) = 0$ for $z \in \mathbb{M}^2 \setminus O$. From [3, Theorem 4.13] it turns out that there exist a small number $\epsilon > 0$ and a function $H_{(O,\mathbb{g})}(\cdot,\cdot)$

(which is continuous symmetric on $O \times O$ and $C^\infty$-smooth on $O \times O \setminus \{(a,a)\}$) such that $d_\mathbb{g}(z,a) < \epsilon$ implies

\[
g_{(O,\mathbb{g})}(z,a) = -(2\pi)^{-1} \left( \ln d_\mathbb{g}(z,a) + H_{(O,\mathbb{g})}(z,a) \right).
\]

Consequently, a combined use of the Green function and the distance function induces the Robin function/mass $H_{(O,\mathbb{g})}(a,a)$ and the conformal radius $R_{(O,\mathbb{g})}(a)$ at $a \in O$ under the metric $\mathbb{g}$:

\[
H_{(O,\mathbb{g})}(a,a) = -2\pi \lim_{z \to a} \left( (2\pi)^{-1} \ln d_\mathbb{g}(z,a) + g_{(O,\mathbb{g})}(z,a) \right)
\]

and

\[
R_{(O,\mathbb{g})}(a) = \exp \left( -H_{(O,\mathbb{g})}(a,a) \right).
\]

As an immediate application of Theorems 3.1-3.2, we have the following assertion whose (i) has slightly stronger hypothesis and conclusion than Li-Tam's ones in [27, Theorem 5.1].

**Theorem 5.2.** Let $(\mathbb{M}^2,\mathbb{g})$ be a simply-connected complete noncompact boundary-free surface with

\[
\int_{\mathbb{M}^2} K_\mathbb{g}^- dA_\mathbb{g} < \infty \quad \text{and} \quad \int_{\mathbb{M}^2} K_\mathbb{g}^+ dA_\mathbb{g} < 2\pi.
\]

Then

(i) For $(\mathbb{M}^2,\mathbb{g})$, the best isoperimetric constant:

\[
\tau_\mathbb{g}(\mathbb{M}^2) = \inf \left\{ \left( \frac{L_\mathbb{g}(\partial O)}{4\pi A_\mathbb{g}(O)} \right)^2 : O \in BRD(\mathbb{M}^2) \right\}
\]

satisfies

\[
1 - (2\pi)^{-1} \int_{\mathbb{M}^2} K_\mathbb{g}^+ dA_\mathbb{g} \leq \tau_\mathbb{g}(\mathbb{M}^2) \leq 1 - (2\pi)^{-1} \int_{\mathbb{M}^2} K_\mathbb{g}^- dA_\mathbb{g},
\]

where the infimum is taken over all relatively compact domains $O \subseteq \mathbb{M}^2$ (written as $O \in RCD(\mathbb{M}^2)$). Obviously, the equalities in \((5.1)\) occur when $K_\mathbb{g}$ is nonnegative on $\mathbb{M}^2$.

(ii) For $a \in O$, $O \in BRD(\mathbb{M}^2)$ with $C^\infty$ boundary $\partial O$, $\partial g_{(O,\mathbb{g})}(\cdot,a)/\partial n_\mathbb{g}$-the inner normal derivative of $g_{(O,\mathbb{g})}(\cdot,a)$ under $\mathbb{g}$, and $0 \leq p < \infty$, the $L^p$-integral of the Green's function $g_{(O,\mathbb{g})}(\cdot,a)$:

\[
\mathcal{H}(p,a,O,\tau_\mathbb{g}(\mathbb{M}^2)) = \frac{4\pi \tau_\mathbb{g}(\mathbb{M}^2)^p}{\Gamma(1+p)} \int_O (g_{(O,\mathbb{g})}(\cdot,a))^p dA_\mathbb{g}(\cdot)
\]
where the second/third/fourth equality in (5.2) holds when $K$ also enjoys $\chi$(5.3) $\lim$ (5.2) and $B$. In particular, if $\chi$ holds readily from (5.1) follows readily from whence verifying the left-hand inequality in (5.1). Clearly, the right-hand inequality of (5.5) follows mutuactionally disjoint simply-connected domains $O_1,\ldots,O_k$ each of which is homeomorphic to the unit disk $D$. Using Theorem 4.1 we obtain

$$A_g(O) \leq A_g(O_0) \leq \left(4\pi (1 - 2\pi)^{-1} \int_{O_0} K_g^+ d\mathcal{A}_g \right)^{-1} \left(L_g(\partial O_0)\right)^2 \leq \left(4\pi (1 - 2\pi)^{-1} \int_{M^2} K_g^+ d\mathcal{A}_g \right)^{-1} \left(L_g(\partial O)\right)^2,$$

whence verifying the left-hand inequality in (5.1). Clearly, the right-hand inequality of (5.1) follows readily from

$$\tau_g(M^2) \leq \frac{\left(L_g(\partial B(a,r))\right)^2}{4\pi A_g(B(a,r))}$$

and Theorem 5.2 (ii) thanks to $\chi(M^2) = 1$ for the simply-connected surface $M^2$ and $B(a,r) \in BRD(M^2)$.

(ii) At this time, no conformal mapping is taken into account; yet Theorem 2.4 and the key idea proving Theorem 5.2 will be used. To do so, assume $a \in O$ and $O \in BRD(M^2)$ with $C^\infty$ boundary $\partial O$. For $t \geq 0$ set

$$O_t = \{z \in O : g(O,g)(z,a) > t\}.$$ Then $g(O,g)(\cdot,a)$ is of class $C^\infty$ on $O \setminus \{a\}$, and for almost all $t > 0$ one has

$$\partial O_t = \{z \in O : g(O,g)(z,a) = t\}.$$ In the sequel, by $A_g(O_t)$ we mean $\int_{O_t} d\mathcal{A}_g$ for $t \geq 0$. As a function of $t$, $A_g(O_t)$ is decreasing and satisfied with the differential formula

$$\frac{dA_g(O_t)}{dt} = \int_{\partial O_t} \left(\frac{\partial g(O,g)(z,a)}{\partial n_g} \right)^{-1} dL_g(z) \geq 0.$$
Using the Cauchy-Schwarz inequality, (5.1), (5.5) and the easily-verified formula (through [3, p. 112, (22)] for example)

\[(5.6) \int_{\partial \Omega_t} \left( \frac{\partial g(O, \cdot, a)}{\partial n} \right)_\Omega \, dL_g(z) = 1, \]

we get that for almost every \( t > 0 \),

\[
\left(- \frac{d A_g(O_t)}{dt}\right)^{\frac{1}{2}} = \left( \int_{\partial \Omega_t} \frac{dL_g(z)}{\partial n} \right)^{\frac{1}{2}} \left( \int_{\partial \Omega_t} \left( \frac{\partial g(O, \cdot, a)}{\partial n} \right)_\Omega \, dL_g(z) \right)^{\frac{1}{2}} \\
\geq \int_{\partial \Omega_t} dL_g = L_g(\partial \Omega_t) \\
\geq (4\pi \tau_g(M^2))^{\frac{1}{2}} (A_g(O_t))^{\frac{1}{2}}.
\]

These equalities and inequalities yield

\[
\frac{d}{dt} \left( \exp \left( 4\pi \tau_g(M^2) t \right) A_g(O_t) \right) = 4\pi \tau_g(M^2) A_g(O_t) + \frac{d A_g(O_t)}{dt} \leq 0.
\]

Note that if

\[
X(t) = A_g(O_t); \quad Y_p(t) = -\int_t^\infty r^p dA_g(O_r) \quad \text{for} \quad p \in [0, \infty); \quad c = 4\pi \tau_g(M^2)
\]

then by the layer cake representation (cf. [29, p. 26, Theorem 1.13]) and the integration-by-part,

\[
Y_p(t) = \int_{O_t} g(O, \cdot, a) dA_g(z) = -\int_t^\infty r^p dX(r).
\]

Therefore, using Theorems (4.1(i)-4.2(i)-5.2(i)) as well as (4.9) we derive (5.2) and its equality case whenever \( 0 \leq p_1 < p_2 < \infty \), as well as (5.3) and (5.4). \( \Box \)

Evidently, we can obtain the estimates similar to ones in Remark 4.3 – the details are left to the interested readers. However, an important observation about the above argument is that on a complete noncompact boundary-free surface the sharp isoperimetric inequality must imply the optimal monotone principle for the \( L^p \)-integral of Green’s function. On the other hand, according to the well-known Federer-Fleming type theorem for \((M^2, g)\), the isoperimetric inequality

\[(5.7) \quad 4\pi \tau_g(M^2) A_g(O) \leq L_g(\partial O) \quad \text{for} \quad O \in BRD(M^2) \quad \text{with} \quad C^\infty \ \text{boundary} \ \partial O,
\]

is equivalent to the Sobolev inequality

\[(5.8) \quad 4\pi \tau_g(M^2) \int_{M^2} |f|^2 dA_g \leq \left( \int_{M^2} |\nabla_g f| dA_g \right)^2 \quad \text{for} \quad f \in C^\infty_0(M^2),
\]

where \( C^\infty_0(M^2) \) represents the class of all \( C^\infty \) functions with compact support in \( M^2 \). In particular, if \( K_g \geq 0 \) and \( (5.7)/(5.8) \) holds for \( \tau_g(M^2) = 1 \) then \((M^2, g)\) is isometric to \((\mathbb{R}^2, ds^2)\) (cf. [20, p. 244]). Thus, a very natural question is “What is an equivalent analytic representation of the monotonicity for the \( L^p \)-integral of Green’s function?”. Surprisingly but also naturally, the answer to this question is related to both the Moser-Trudinger inequality and the Nash-Sobolev inequality on \((M^2, g)\).
Theorem 5.3. Let \((M^2,g)\) be a complete, noncompact, and boundary-free surface. Then the following implications \((i)\Rightarrow (ii)\Rightarrow (iii)\) are valid:

(i) There are two positive constants \(c_1\) and \(C_1\) such that Moser-Trudinger’s inequality

\[
(5.9) \quad \int_O \exp \left(c_1 |f(z)|^2 \right) dA_g(z) \leq C_1 A_g(O)
\]

holds for all \(O \in BRD(M^2)\) with \(C^\infty\) boundary and all \(f \in C^\infty_0(O)\) with \(\int_O |\nabla_g f|^2 dA_g \leq 1\).

(ii) There is a constant \(C_2 > 0\) such that the \(0 = p_1 < p_2 = 1\) monotonicity of Green’s function integral

\[
(5.10) \quad \int_O g_{(o,g)}(z,a) dA_g(z) \leq C_2 A_g(O), \quad a \in O
\]

holds for all \(O \in BRD(M^2)\) with \(C^\infty\) boundary.

(iii) There is a constant \(C_3 > 0\) such that Nash-Sobolev’s inequality

\[
(5.11) \quad \left( \int_{M^2} |f|^2 dA_g \right)^\frac{1}{2} \leq C_3 \left( \int_{M^2} |\nabla_g f|^2 dA_g \right)^\frac{1}{2} \left( \int_{M^2} |f| dA_g \right)^\frac{1}{2}
\]

holds for all \(f \in C^\infty_0(M^2)\).

Moreover, if there are two positive constants \(c_0\) and \(C_0\) such that for any \(a \in O\) and \(O \in BRD(M^2)\) with \(C^\infty\) boundary one has

\[
(5.12) \quad A_g(B(a,r)) \geq c_0 r^2 \quad \text{and} \quad L_g(\partial B(a,r)) \leq C_0 r \quad \text{for} \quad 0 < r < \infty,
\]

then the implication \((iii)\Rightarrow (ii)\) is valid too.

Proof. (i)⇒(ii) Suppose (i) is true. For \(t > 0\), \(a \in O\) and \(O \in BRD(M^2)\) with \(C^\infty\) boundary, choose \(f_t(z) = \min\{g_{(o,g)}(z,a), t\}\) and set \(Q_t = \{z \in O : g_{(o,g)}(z,a) < t\}\). Then by Green’s formula and the identity \((5.6)\),

\[
\int_O |\nabla_g f_t|^2 dA_g = \int_{Q_t} |\nabla_g g_{(o,g)}(z,a)|^2 dA_g(z)
= \int_{Q_t} (\Delta_g g_{(o,g)}(z,a)) g_{(o,g)}(z,a) dA_g(z)
+ t \int_{\{z \in O : g_{(o,g)}(z,a) = t\}} \left( \frac{\partial g_{(o,g)}(z,a)}{\partial n_g} \right) dL_g(z)
= t.
\]

Meanwhile, we have

\[
\int_O \exp \left( c_1 |f_t/\sqrt{t}|^2 \right) dA_g(z) \geq \int_{O \setminus Q_t} \exp \left( c_1 |f_t/\sqrt{t}|^2 \right) dA_g(z)
\geq \exp(c_1 t) A_g(O \setminus Q_t).
\]

Via a \(C^\infty\) approximation of \(f_t\), we see that \((5.9)\) is valid for \(f_t/\sqrt{t}\), and so that

\[
A_g(O \setminus Q_t) \leq C_1 A_g(O) \exp(-c_1 t).
\]
This inequality implies
\[
\int_{O} g_{(O,g)}(z,a)A_{g}(z) = \int_{0}^{\infty} A_{g}(O \setminus Q_{t}) dt \\
\leq C_{1}A_{g}(O) \int_{0}^{\infty} \exp(-c_{1}t) dt \\
= C_{1}c_{1}^{-1}A_{g}(O).
\]

Thus (ii) holds with \( C_{2} = C_{1}c_{1}^{-1} \).

(ii) \( \Rightarrow \) (iii) Suppose (ii) is valid. To prove (iii), let \( O \in BRD(M^{2}) \) with \( C^{\infty} \) boundary, and \( \lambda_{1,g}(O) \) be the first nonzero eigenvalue of the Laplacian operator \( \Delta_{g} \) for the Dirichlet problem on \( O \). So, if \( u \neq 0 \) solves
\[
\begin{cases}
(\Delta_{g} - \lambda_{1,g}(O))u(z) = 0, & z \in O, \\
u(z) = 0, & z \in \partial O,
\end{cases}
\]
then for each \( a \in O \) we have
\[
u(a) = \int_{O} g_{(O,g)}(z,a)\Delta_{g}u(z) dA_{g}(z) \\
\leq \lambda_{1,g}(O) \int_{O} g_{(O,g)}(z,a)u(z) dA_{g}(z) \\
\leq \lambda_{1,g}(O) (\sup_{z \in O} u(z)) \int_{O} g_{(O,g)}(z,a) dA_{g}(z)
\]
whence getting
\[
1 \leq \lambda_{1,g}(O) \sup_{a \in O} \int_{O} g_{(O,g)}(z,a) dA_{g}(z) \leq C_{2}\lambda_{1,g}(O)A_{g}(O).
\]

Namely, Faber-Krahn’s eigenvalue inequality
\[
(\lambda_{1,g}(O))^{-1} = \sup \left\{ \int_{O} |f|^{2} dA_{g} : f \in C_{0}^{\infty}(O), f \neq 0 \right\} \leq C_{2}A_{g}(O)
\]
holds for all \( O \in BRD(M^{2}) \). Now, (5.13) and [18, Lemma 6.3] yield (iii) with \( C_{3} = 2(\epsilon(1 - \epsilon)C_{2})^{-1} \leq 8(C_{2})^{-1} \) where \( \epsilon \) is any given constant in \((0,1)\).

Next, we prove the second part of the conclusion. Note first that if (iii) holds then according to [36, Theorem 4.2.6] there is a constant \( C_{4} > 0 \) such that the heat-kernel-upper-bound inequality
\[
H(t, z, a) \leq C_{4}t^{-1} \exp \left( -\frac{(d_{g}(z, a))^{2}}{8t} \right)
\]
holds for all \((z, a, t) \in M^{2} \times M^{2} \times (0, \infty)\). Here and henceforth, \( H(t, z, a) \) stands for the heat kernel on \( M^{2} \) – that is – the smallest positive solution to the heat equation
\[
\begin{cases}
(\frac{\partial}{\partial t} - \Delta_{g})H(t, z, a) = 0, & (t, z, a) \in (0, \infty) \times M^{2} \times M^{2}, \\
H(0, z, a) = \delta_{a}(z), & (z, a) \in M^{2} \times M^{2}.
\end{cases}
\]

Even more interestingly, this heat kernel indeed describes the probability of reaching \( z \) at time \( t \) starting from \( a \). Consequently, when \( a \in O \) and \( O \in BRD(M^{2}) \) the integration of \( H(t, z, a) \) over \( O \) against \( dA_{g}(z) \) is the probability \( P_{a}[B_{t} \in O] \) of the Brownian motion \( B_{t} \) reaching \( O \) at \( t \) starting from \( a \) on \((M^{2}, g)\), namely,
\[
P_{a}[B_{t} \in O] = \int_{O} H(t, z, a) dA_{g}(z).
\]
If
\[ t_0(w) = \inf\{t > 0 : B_t(w) \notin O\} \quad \text{and} \quad P_a[t < t_0] \]
represent the first exit-time at \( w \) and the probability that the Brownian motion begins with \( a \) and hits \( O \) by \( t_0 \) respectively, then the corresponding expectation \( E_a[t_0] \) can be formulated as

\[ (5.15) \quad \int_O g_{\{a,t\}}(z,a) dA(z) = E_a[t_0] = \int_0^\infty P_a[t < t_0] dt. \]

In light of the study done in [7, Theorem 1.6], we continue our proof as follows. The condition (5.14) and the layer cake representation (see [29, p. 26, Theorem 1.13] again) yield

\[ P_a[t < t_0] \leq \int_O H(t, z, a) dA(z) \]

\[ \leq C_4 t^{-1} \int_O \exp\left(- (8t)^{-1}(d_g(z,a))^2\right) dA(z) \]

\[ = C_4 t^{-1} \int_0^\infty A_g(\{z \in O : d_g(z,a) > \tau\}) \left(\frac{d}{d\tau} \exp\left(- (8t)^{-1}\tau^2\right)\right) d\tau. \]

The foregoing inequality, plus choosing \( r_0 > 0 \) such that \( A_g(O) = A_g(B(a, r_0)) \), further gives

\[ P_a[t < t_0] \leq C_4 t^{-1} \int_0^{r_0} A_g(\{z \in O : d_g(z,a) > \tau\}) \left(\frac{d}{d\tau} \exp\left(- (8t)^{-1}\tau^2\right)\right) d\tau \]

\[ = C_4 t^{-1} \int_{B(a, r_0)} \exp\left(- (8t)^{-1}(d_g(z,a))^2\right) dA(z) \]

\[ = C_4 t^{-1} \int_0^{r_0} \exp\left(- (8t)^{-1} \tau^2\right) L_g(\partial B(a, r)) d\tau \]

\[ \leq C_0 C_4 \int_0^{r_0} \exp\left(- (8t)^{-1} \tau^2\right) r d\tau \]

\[ \leq 4C_0 C_4 \left(1 - \exp\left(- (8t)^{-1} r_0^2\right)\right). \]

This estimation, along with (5.15) and (5.12), now derives

\[ P_a[2t < t_0] \leq \left(\sup_{z \in O} P_z[t < t_0]\right)^2 \]

\[ \leq \left(4C_0 C_4 \left(1 - \exp\left(- (8t)^{-1} r_0^2\right)\right)\right)^2. \]

This immediately produces

\[ \int_O g_{\{a,t\}}(z,a) dA(z) = A_g(O) \int_0^\infty P_a[\{s \in O : A_g(s) < t_0\}] ds \]

\[ \leq (2r_0 C_0 C_4)^2 \int_0^\infty \left(1 - \exp(-t^{-1})\right)^2 dt \]

\[ = \left(c_0^{-1}(2C_0 C_4)^2 \int_0^\infty \left(1 - \exp(-t^{-1})\right)^2 dt\right) A_g(O), \]

namely, (i) holds.

\[ \square \]

Remark 5.4. Several more comments on the last theorem are in order:
(i) In the case of \((\mathbb{M}^2, g) = (\mathbb{R}^2, ds^2)\), the maximal value of \(c_1\) in (5.9) is \(4\pi\) – this is due to Moser; see also [32]. Moreover, from [31 Proposition 2] and [17 (2.10)] we see that (5.9) with \(c_1 = 4\pi\) amounts to
\[
A_g(E) \leq A_g(O) \exp \left( -4\pi \text{mod}_O(E) \right)
\]
for any compact \(E \subset O\), where
\[
\text{mod}_O(E) = \sup \left\{ \left( \int_O |\nabla g|^2 f^2 dA_g \right)^{-1} : f \in C^\infty_0(O), f \geq 1 \text{ on } E \right\}.
\]

(ii) Under the hypotheses of Theorem 5.3 if \(K_g \geq 0\) and \(C_3 = 4(\pi \lambda_{1,N})^{-1}\) (the Carlen-Loss’s sharp constant in [10]) where \(\lambda_{1,N}\) is the first non-zero Neumann eigenvalue of \(\Delta\) on radial functions on \(D\), then \((\mathbb{M}^2, g)\) is isometric to \((\mathbb{R}^2, ds^2)\) – this result is proved in [22 Theorem 1.4]. Similarly, if \(K_g \geq 0\) and \(\tau_g(\mathbb{M}^2) = 1\), then (5.7) / (5.8) holds with the best Euclidean constant, and hence \((\mathbb{M}^2, g)\) is isometric to \((\mathbb{R}^2, ds^2)\) – see also [20 p. 244]. Accordingly, it is our conjecture that this isometry follows also from the conditions \(C_2 = (4\pi)^{-1}\) and \(K_g \geq 0\). Despite being unable to verify this conjecture, we can obtain a weaker result as follows.

Suppose \(K_g \geq 0\). Then (5.10) yields \(H(t, z, a) \leq C_4 t^{-1}\) and so by Li-Yau’s maximal volume growth theorem in [28],
\[
\liminf_{r \to \infty} A_g(B(a, r)) \left( \pi r^2 \right)^{-1} \geq l_0 \quad \text{for some constant } l_0 > 0.
\]

A use of Gromov’s comparison theorem (cf. [20 p. 11]) gives
\[
l_0 \leq A_g(B(a, r)) \left( \pi r^2 \right)^{-1} \leq 1 \quad \text{for all } r > 0.
\]
Of course, if \(l_0 = 1\) then \((\mathbb{M}^2, g)\) is isometric to \((\mathbb{R}^2, ds^2)\). But, if \(l_0 < 1\) then a result of Cheeger-Colding in [9] produces that \((\mathbb{M}^2, g)\) is diffeomorphic to \((\mathbb{R}^2, ds^2)\).

(iii) From [8 Theorem 3] and its proof it follows that the above Nash-Sobolev’s inequality holds whenever there exists a constant \(C_5 > 0\) such that the Log-Sobolev inequality
\[
\exp \left( \int_{\mathbb{M}^2} |f|^2 \ln |f|^2 dA_g \right) \leq C_5 \int_{\mathbb{M}^2} |\nabla g f|^2 dA_g
\]
holds for all \(f \in C^\infty_0(\mathbb{M}^2)\) with \(\int_{\mathbb{M}^2} |f|^2 dA_g = 1\). As well, it is known that (5.11) implies (5.16) – see [18] for example. Moreover, if \(K_g \geq 0\) and (5.16) holds with \(C_5 = (e\pi)^{-1}\), then \((\mathbb{M}^2, g)\) is isometric to \((\mathbb{R}^2, ds^2)\) – see also [33 Corollary 1.5].

(iv) When compared with the setting on the flat surface \((\mathbb{R}^2, ds^2)\), the requirement \((5.12)\) is not artificial – see also [27] once again. In fact, if \(u\) is a bounded \(C^\infty\) function on \(\mathbb{R}^2\) then (5.12) holds on the manifold \((\mathbb{R}^2, e^{2u} ds^2)\), and hence the previously-stated five inequalities: (5.10); (5.11); (5.13); (5.14); (5.16) are equivalent. Of course, this equivalence is new even for \(u = 0\). Besides, the condition (5.12) is closely related to the following conclusion.

(a) Li’s criterion for the finite total curvature in [25] tells us that if \((\mathbb{M}^2, g)\) is a complete noncompact surface with: finite topological type, at most quadratic area growth – \(\lim_{r \to \infty} r^{-2} A_g(B(a, r))\) existing, and the Gauss curvature being of one sign at each end, then \(\int_{\mathbb{M}^2} |K_g| dA_g < \infty\).

(b) Conversely, if \((\mathbb{M}^2, g)\) is a complete noncompact surface with \(\int_{\mathbb{M}^2} |K_g| dA_g < \infty\) then
(b$_1$) Hartman’s area-length domination in [19] induces two positive constants $c_0^*, C_0^*$ ensuring
\[ A_g(B(a,r)) \leq c_0^* r^2 \quad \text{and} \quad L_g(\partial B(a,r)) \leq C_0^* r \quad \text{for} \quad 0 < r < \infty; \]

(b$_2$) Shiohama’s minimal-area principle in [38] gives
\[
\inf_{g \in \mathcal{M}(M^2)} A_g(M^2) = \left\{ \begin{array}{ll}
4\pi, & \chi(M^2) = 1, \\
-2\pi \chi(M^2), & \chi(M^2) \leq 0,
\end{array} \right.
\]

where $\mathcal{M}(M^2)$ stands for all complete Riemannian metrics $g$ on $M^2$ with the next constraint:
\[
\left\{ \begin{array}{ll}
K_g \leq 1 & \text{as} \quad \chi(M^2) \geq 0, \\
K_g \geq -1 & \text{as} \quad \chi(M^2) < 0.
\end{array} \right.
\]

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