Extensions of the Dynamic Programming Framework: Battery Scheduling, Demand Charges, and Renewable Integration

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Abstract—In this paper, we consider dynamic programming problems with non-separable objective functions. We show that for any problem in this class, there exists an augmented-state dynamic programming problem which satisfies the principle of optimality and the solutions to which yield solutions to the original forward separable problem. We further generalize this approach to stochastic dynamic programming problems by extending the definition of the principle of optimality to problems driven by random variables. We then apply the resulting algorithms to the problem of optimal battery scheduling with demand charges using a data-based stochastic model for electricity usage and solar generation by the consumer.

I. INTRODUCTION

Many problems in engineering and economics involve discrete time processes coupled with decision variables and an objective function. These optimization problems are commonly solved using Dynamic Programming (DP) [1]. DP is a class of algorithms that break down complex optimization problems into simpler sequential subproblems, each of which is solved using Bellman’s Equation. For DP to work, however, we require that the optimization problem satisfies the principle of optimality; from any point on an optimal trajectory, the remaining portion of the optimal trajectory is also optimal for the problem initiated at that point [2]. DP problems commonly have an additively separable objective function of the form $J(u, x) = \sum_{t=0}^{T-1} c_t(x(t), u(t)) + c_T(x(T))$. Problems of this form can be shown to satisfy the principle of optimality. However in many problems of practical interest we find non-additively separable objective functions. For example, if the objective is of the form $J(u, x) = \max_{0 \leq k \leq T} d_k(x(k))$ then the problem does not satisfy the principle of optimality. In this paper we propose a general method for solving optimization problems with non-separable objective functions by constructing equivalent optimization problems with additively separable objective functions. Such reformulated problems then satisfy the principle of optimality and can therefore be solved using Bellman’s Equation.

To generalize our methodology to stochastic DP we propose an extension of the definition of the principle of optimality to problems that involve random variables. As discussed in [3] such an extension is non trivial. Inspired by [4], we construct probability measures on the sets the state variable can take at each time stage induced by the underlying random variables, we then propose a stochastic principle of optimality; we say a stochastic problem satisfies the principle of optimality if from any point on a trajectory followed using the optimal policy, $\pi$, the policy $\pi$ is also optimal for the problem initiated from that point with probability one.

Dynamic programming for problems which do not satisfy the principle of optimality has received relatively little attention and there are few results in the literature in which this problem has been addressed. The only generalized approach to the problem seems to be that taken in [5] which considered the use of multi-objective optimization in the case where the objective function is “backward separable”. Our approach differs from [5] as we only consider a class of “forward separable” objective functions. In this paper we show that almost any objective function is forward separable in a certain sense and that for such problems there exists an additively separable augmented-state dynamic programming problem that satisfies the principle of optimality and from which solutions to the original forward separable problem can be recovered - See Section III. However, the resulting augmented-state dynamic programming problem has a higher dimensional state space than the original optimization problem - an issue that can potentially render the augmented problem intractable due to the “curse of dimensionality”. For this reason, we propose a complexity metric for the forward separable representation and show that in certain cases the dimensionality of the augmented system does not significantly exceed the dimensionality of the original problem - a case we refer to as Naturally Forward Separable (NFS).

Using augmented states to solve problems with non-separable variance type objective functions was briefly discussed in [6]; however this method was ultimately rejected due to computational intractability. Instead an approach of searching for equivalent separable objective functions was taken instead; however this approach is not compatible with objective functions which include input arguments or stochastic dynamics. In this paper we therefore consider using an augmented state method, make it rigorous, and extend it to a general class of NFS objective functions including both variance type and maximum type functions. In summary, the technical contributions of this paper are fourfold: 1) We show how augmented states can help solve non-separable optimization problems and how this approach is tractable for DP problems with NFS objective functions. 2) We show how maximum type functions are a special class of NFS functions. 3) We propose an extension of the principle of optimality to stochastic dynamic programming problems and show how state augmentation can be used to solve such stochastic problems. 4) We solve the problem of finding the optimal battery scheduling for electricity usage. 

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Finally, we note that in practice it is rare to be able to analytically solve Bellman’s Equation. Therefore, once the augmented-state dynamic programming problem is formulated we also propose a map to an approximated DP problem based on the work in [7] and [8]. The approximated DP problem can then be analytically solved. Using the optimal solution from the approximated DP problem a feasible sequence of decision variables for the original problem can then be reconstructed. Application to Battery Scheduling with Demand Charges

In this paper we will apply our augmented-state dynamic programming methods to battery scheduling optimization problems. In 2012, 95,000 new distributed solar Photovoltaic (PV) systems were installed nationally, a 36% increase from 2011 and yielding a total of approximately 300,000 installations total [9]. Further, utility-scale PV generating capacity has increased at an even faster rate, with 2012 installations more than doubling that of 2011 [10]. Meanwhile, partially due to the development of energy-efficient appliances and new materials for insulation, US electricity demand has plateaued [11]. As a consequence of these trends, utility companies are faced with the problem that demand peaks continue to grow. Specifically, as per the US EIA [12], the ratio of peak demand to average demand has increased dramatically over the last 20 years.

Fundamentally, the problem faced by utilities is that consumers are typically charged based on total electricity consumption, while utility costs are based both on consumption and for building and maintaining the generating capacity necessary to meet peak demand. Recently, several public and private utilities have moved to address this imbalance by charging residential consumers based on the maximum rate ($ per kW) of consumption - a cost referred to as a demand charge. Specifically, as per Arizona SRP and APS have mandatory demand charges for residential consumers [13].

For consumers, load is relatively inflexible and hence the most direct approach to minimizing the effect of demand charges is the use of battery storage devices such as the Tesla Powerwall [14], [15], [16]. These devices allow consumers to shift electricity consumption away from periods of peak demand, thereby minimizing the effect of demand charges. In this paper, we specifically focus on battery storage coupled with HVAC (Heating, ventilation, and air conditioning) and solar generation. This is due to the fact that load from HVAC and electricity from solar generation can be forecast well a priori.

The use of battery storage has been well documented in the literature [17] and in particular, there have been several results on the optimal use of batteries for residential customers [18], [19], [20], [21]. Within this literature, there are relatively few results which include demand charges. Of those which do treat demand charges, we mention [22] which proposes a heuristic form of dynamic programming, and the recent work in [23], wherein the optimization problem is broken down into several agents, and a Lagrangian approach is used to perform the optimization. Furthermore, in [24] a similar energy storage problem is solved using optimized curtailment and load shedding. An $L_p$ approximation of the demand charge was used in combination with multi-objective optimization in [25] and, in addition, the optimal use of building mass for energy storage was considered in [26], wherein a bisection on the demand charges was used. However, we note that none of these approaches resolve the fundamental mathematical problem of dynamic programming with a non-separable cost function and hence are either inaccurate, computationally expensive, or are not guaranteed to converge. Finally, we note that there has been no work to date on optimization of demand charges coupled with stochastic models of solar generation.

In this paper, we formulate the battery storage problem as a dynamic program with an objective function consisting of both integrated time-of-use charges and a maximum term representing the demand charge. Furthermore, we model solar generation as a Gauss-Markov process and minimize the expected value of the objective. The fundamental mathematical challenge with dynamic programming problems of this form is that, as shown in Section III problems which include maximum terms in the objective do not satisfy the principle of optimality and thus recursive solution of the Bellman equation (11) does not yield an optimal policy.

The rest of this paper is organized as follows. In Section II we propose a precise definition of the principle of optimality and show that if this definition holds, then the Bellman’s equation can be used to define an optimal policy. Next, we consider a class of optimization problems called forward separable optimization problems; here we show that DP problems with summation or maximum terms in the objective function are forward separable. We then show that the principle of optimality does not hold for certain forward separable DP problems. In Section III we show that for any forward separable DP problem, there exists a separable augmented-state DP problem for which the principle of optimality holds and from which solutions to the original forward separable problem can be recovered. In Section IV we introduce the battery scheduling problem and show it is a special case of a forward separable DP problem with a NFS objective function. In Section V we show how to approximate and numerically solve augmented-state dynamic programming problems. In Section VI we show how to numerically solve the battery scheduling problem for given forecast solar data. In Section VII we show that the augmented DP approach can also be used to solve stochastic dynamic programming problems with forward separable objectives. In section VIII we apply our approach to the battery scheduling problem using a Gauss-Markov model of solar generation extracted from data provided by local utility SRP.

II. BACKGROUND: GENERALIZED DYNAMIC PROGRAMMING

In this paper, we propose a generalized class of dynamic programming problems. Specifically, we define a generalized dynamic programming problem as an indexed sequence of optimization problems $G(t_0,x_0)$, defined by an indexed sequence of objective functions $J_{t_0,x_0} : \mathbb{R}^m \times (T-t_0) \times \mathbb{R}^n \times (T-t_0+1)$
where we say that \( u^* \in \mathbb{R}^{m \times (T-t_0)} \) and \( x^* \in \mathbb{R}^{n \times (T-t_0+1)} \) solve \( G(t_0, x_0) \) if,

\[
(u^*, x^*) = \arg \min_{u, x} J_{t_0,x_0}(u, x) \tag{1}
\]

subject to:

\[
x(t+1) = f[x(t), u(t), t] \text{ for } t = t_0, \ldots, T
\]

\[
x(t_0) = x_0, \text{ and } x(t) \in \mathbb{X} \subset \mathbb{R}^n \text{ for } t = t_0, \ldots, T
\]

\[
u(t) \in U \subset \mathbb{R}^m \text{ for } t = t_0, \ldots, T-1
\]

\[
u = (u(t_0), \ldots, u(T-1)) \text{ and } x = (x(t_0), \ldots, x(T))
\]

where \( f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}^n \), \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^m \) for all \( t \). We denote \( J_{t_0,x_0}^* = J_{t_0,x_0}(u^*, x^*) \).

We will call \( \{x(t)\}_{t_0 \leq t \leq T} \) the state variables and \( n = \dim(\mathbb{X}) \) the state space dimension. Similarly we will call \( \{u(t)\}_{t_0 \leq t \leq T-1} \) the input (control) variables and \( m = \dim(U) \) the input (control) space dimension. For cases where the dimension of the state variable, \( x(t) \), varies with time, we slightly abuse notation and define the state space dimension as \( \max_{t_0 \leq t \leq T} \dim(\mathbb{X}) \).

**Definition 1.** The function \( J_{t_0,x_0} : \mathbb{R}^{m \times (T-t_0)} \times \mathbb{R}^{n \times (T-t_0+1)} \) is said to be additively separable if there exists functions, \( c_T(x) : \mathbb{R}^n \to \mathbb{R} \), and \( c_t(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) for \( t = t_0, \ldots, T-1 \) such that,

\[
J_{t_0,x_0}(u, x) = \sum_{t=t_0}^{T-1} c_t(x(t), u(t)) + c_T(x(T)), \tag{2}
\]

where \( u = (u(t_0), \ldots, u(T-1)) \) and \( x = (x(t_0), \ldots, x(T)) \).

The average scaled magnitude of the state vector over the time interval, \( J(u, x) = \frac{1}{T} \sum_{t=t_0}^{T} a_t(x(t)) \) where \( a_t : \mathbb{R}^n \to \mathbb{R} \), is clearly an example of an additively separable function. However later on we will see that variance type functions are not additively separable.

**Definition 2.** We say the sequence of inputs \( u = (u(t_0), \ldots, u(T-1)) \subset \mathbb{R}^{m \times (T-t_0)} \) is feasible if \( u(t) \in U \) for \( t = t_0, \ldots, T-1 \) and if \( x(t+1) = f[x(t), u(t), t] \) and \( x(t_0) = x_0 \), then \( x(t) \in \mathbb{X} \) for all \( t \). For a given \( x \), we denote by \( \Gamma_x \), the set \( u \in U \) such that \( f[x, u, t] \in \mathbb{X} \). In this paper we only consider problems where \( \Gamma_x \) is nonempty for all \( x \) and \( t \).

Note that for this class of optimization problems, feasibility is inherited. That is, if \( u = (u(t_0), \ldots, u(T-1)) \) is feasible with \( x = (x(t_0), \ldots, x(T)) \) for \( G(t, x(t)) \) and \( v = (v(s_0), \ldots, v(T-1)) \) if feasible with \( h = (h(s_0), \ldots, h(T)) \) for \( G(s, x(s)) \) where \( s > t \), then \( w = (u(t_0), \ldots, u(s-1), v(s), \ldots, v(T-1)) \) with \( z = (x(t_0), \ldots, x(s-1), h(s), \ldots, h(T)) \) is feasible for \( G(t, x(t)) \).

In certain cases, indexed optimization problems of the Form of \( G(t_0, x_0) \) can be solved using an optimal policy.

**Definition 3.** A policy is any map from the present state and time to a feasible input \( (x, t) \mapsto u(t) \in \Gamma_x,t \), as \( u(t) = \pi(x,t) \). We denote the set of policies corresponding to some optimization problem as \( \Pi \). We say that \( \pi^* \) is an optimal policy for Problem (1) if

\[
u^* = (\pi^*(x_0, t_0), \ldots, \pi^*(x(T-1), T-1))
\]

where \( x(t+1)^* = f[x(t)^*, \pi^*(x(t)^*, t), t] \) for all \( t \).

The “Principle of Optimality” defines a class of optimization problems that satisfy Bellman’s equation and from which an optimal policy can be retrieved.

**Definition 4.** We say an optimization problem, \( G(t_0, x_0) \), of the Form (1) satisfies the principle of optimality if the following holds. For any \( s \) and \( t \) with \( t_0 \leq t < s < T \), if \( u^* = (u(t), \ldots, u(T-1)) \) and \( x^* = (x(t), \ldots, x(T)) \) solve \( G(t, x(t)) \) then \( v = (u(s), \ldots, u(T-1)) \) and \( h = (x(s), \ldots, x(T)) \) solve \( G(s, x(s)) \).

The classical form of the dynamic programming algorithm, as originally defined in [1], can be used to solve indexed optimization problems of the Form (1) with an additively separable objective function. We denote this class of optimization problems by \( P(t_0, x_0) \):

\[
\min_{u, x} J_{t_0,x_0}(u, x) = \sum_{t=t_0}^{T-1} c_t(x(t), u(t)) + c_T(x(T)) \tag{3}
\]

subject to:

\[
x(t+1) = f[x(t), u(t), t] \text{ for } t = t_0, \ldots, T
\]

\[
x(t_0) = x_0, \text{ and } x(t) \in \mathbb{X} \subset \mathbb{R}^n \text{ for } t = t_0, \ldots, T
\]

\[
u(t) \in U \subset \mathbb{R}^m \text{ for } t = t_0, \ldots, T-1
\]

\[
u = (u(t_0), \ldots, u(T-1)) \text{ and } x = (x(t_0), \ldots, x(T))
\]

Note that \( J_{t_0,x_0} = c_T(x) \). We will refer to \( x(t_0) \in \mathbb{R}^n \) as the initial state. \( J_{t_0,x_0} \) is the objective function, \( c_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) for \( t = t_0, \ldots, T-1 \), \( c_T : \mathbb{R}^n \to \mathbb{R} \) are given functions and \( f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}^n \) is a given vector field. The following lemma shows that this class of problems satisfies the principle of optimality.

**Lemma 5.** Any problem of Form \( P(t_0, x_0) \) in [3] satisfies the principle of optimality.

**Proof.** Suppose \( u^* = (u(t), \ldots, u(T-1)) \) and \( x^* = (x(t), \ldots, x(T)) \) solve \( P(t, x(t)) \) in (2). Now we suppose by contradiction that there exists some \( s > t \) such that \( v = (u(s), \ldots, u(T-1)) \) and \( h = (x(s), \ldots, x(T)) \) do not solve \( P(s, x(s)) \). We will show that this implies \( u^* \) and \( x^* \) do not solve \( P(t, x) \) in (2), thus verifying the conditions of the Principle of Optimality. If \( v \) and \( h \) do not solve \( P(s, x(s)) \), then there exist feasible \( w, z \) such that \( J_{s,x(s)}(w, z) < J_{s,x(s)}(v, h) \). i.e.

\[
J_{s,x(s)}(w, z) = \sum_{t=s}^{T-1} c_t(z(t), w(t)) + c_T(z(T)) < \sum_{t=s}^{T-1} c_t(x(t), u(t)) + c_T(x(T)) = J_{s,x(s)}(v, h)
\]

Now consider the proposed feasible sequences \( \hat{u} = (u(t), \ldots, u(s-1), w(s), \ldots, w(T-1)) \) and \( \hat{x} = (x(t), \ldots, x(s) \ldots, x(T)) \).
Bellman’s equation provides a state-feedback law or optimum policy \( u^* \) and a function \( S \). The class of optimization problems of the form

\[
J_{t,x}(u, x) = \sum_{k=0}^{T-1} c_k(x(k), u(k)) + c_T(z(T))
\]

and \( J_{t,x}(u^*, x^*) \) which contradicts optimality of \( u^*, x^* \). Therefore, this class of problems satisfies the principle of optimality. \( \square \)

**Proposition 6** (27). For optimization problems of the form \( P(t, x) \) in (3) with optimal objective values \( J_{t,x}^* \), define the function \( F(x, t) = J_{t,x}^* \). Then the following hold for all \( x \) in \( X_t \):

\[
F(x, t) = \min_{u \in \mathcal{U}_x} \{ c_t(x, u) + F(x(t+1), t+1) \} \forall t \in \{0, ..., T-1\}
\]

Then \( \theta(x, t) = \arg \min_{u \in \mathcal{U}_x} \{ c_t(x, u) + F(x(t+1), t+1) \} \).

**Dynamic Programming with Maximum Terms** In this paper we consider the special class of indexed optimization problem, \( S(t_0, x_0) \). In contrast to problems of the form \( P(t_0, x_0) \) in (1), class \( S(t_0, x_0) \) has supremum (or maximum) terms in the objective. Specifically, these problems have the following form.

\[
\min_{u, x} J_{t_0, x_0}(u, x) := \sum_{t=0}^{T-1} c_t(x(t), u(t)) + c_T(z(T)) + \max_{t \leq k \leq T} d_t(x(k))
\]

subject to:

\[
x(t+1) = f[x(t), u(t), t] \text{ for } t = t_0, ..., T-1
\]

\[
x(t_0) = x_0, \text{ } x(t) \in X_t \subset \mathbb{R}^n \text{ for } t = t_0, ..., T
\]

\[
u(t) \in U \text{ for } t = t_0, ..., T-1
\]

\[
u = (u(t_0), ..., u(T-1)) \text{ and } x = (x(t_0), ..., x(T))
\]

where \( c_t(x) : \mathbb{R}^n \rightarrow \mathbb{R}; \ c_t(x, u) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \) for \( t \leq t \leq T - 1; \ d_t(x) : \mathbb{R}^n \rightarrow \mathbb{R} \) for \( t = t_0, ..., T; \ f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \rightarrow \mathbb{R}^n \).

**Lemma 8.** The class of optimization problems of the form \( S(t_0, x_0) \) in (5) does not satisfy the principle of optimality.

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**Table I**

| feasible \( u \) | objective value | feasible \( u \) | objective value |
|------------------|----------------|------------------|----------------|
| \((0, 0, 0)\)    | 0              | \((0, 0, 0)\)    | 0              |
| \((0, h, 0)\)    | \(h/2\)        | \((h, 0, 0)\)    | 0              |
| \((0, h, -h)\)   | \((S + h)\)    | \((h, -h, 0)\)   | \(-h\)         |
| \((0, h, -h)\)   | \((S + h)\)    | \((h, -h, 0)\)   | \(-h\)         |

**Proof.** We give a counterexample. For \( h > 0 \), we consider the following problem \( S(0, 0) \):

\[
\min_{u \in \mathbb{R}, x \in \mathbb{R}^n} \sum_{t=0}^{2} c_t(u(t)) + \max_{0 \leq k \leq 3} x(k)
\]

subject to: \( x(t+1) = x(t) + u(t) \), \( x(0) = 0 \), \( 0 \leq x(t) \leq h \), \( u(t) \in \{ -h, 0, h \} \)

where here we define \( c_0(u(0)) = -u(0) \), \( c_1(u(1)) = u(1) \), \( c_2(u(2)) = -u(2)/2 \).

Since \( u \in \{ -h, 0, h \} \), there are 27 input sequences, only 8 of which are feasible. In Table I we calculate the objective value of each feasible input sequence and deduce the optimal input is \( u^* = (h, -h, h) \), yielding an optimal trajectory of \( x^* = \{ 0, h, 0, h \} \). Following this input sequence until \( t = 2 \) we examine the problem \( S(2, 0) \).

\[
\min_{u \in \mathbb{R}, 0 \leq x(3) \leq h} c_2(u(2)) + \max_{2 \leq k \leq 3} x(k)
\]

subject to: \( x(t+1) = x(t) + u(t) \), \( x(2) = 0 \), \( 0 \leq x(t) \leq h \), \( u(t) \in \{ -h, 0, h \} \)

For this sub-problem, there are two feasible inputs: \( u(2) \in \{ 0, h \} \). Of these, the first is optimal (objective value \( h/2 \) vs 0). Thus we see that although \( u^* = \{ h, -h, h \} \) and \( x^* = \{ 0, h, 0, h \} \) solve \( S(0, 0) \), \( v = \{ h \} \) and \( h = \{ 0, h \} \) do not solve \( S(2, 0) \). \( \square \)

### III. How State Augmentation Can Be Used to Solve Dynamic Programming Problems

In this section we will define the class of forward separable objective functions and show that the maximum function is an example of such a function. We will show that for dynamic programming problems with a forward separable objective function, augmenting the state variables allows us to use standard dynamic programming techniques to solve the problem.

Forward separable functions were first defined in (28). In the next definition we will build upon the concept of forward separability by introducing the notion of augmented dimension. Later in Section III-B we will see that the augmented dimension of a forward separable objective function relates to the complexity of solving the associated optimization problem.

**Definition 9.** The function \( J : \mathbb{R}^m \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R} \) is said to be forward separable if there exists functions \( \phi_0 : \mathbb{R}^n \to \mathbb{R}^d_0, \phi_t : \mathbb{R}^n \times \mathbb{R}^{d_t} \to \mathbb{R}, \text{ and } \phi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}^d \) for \( i = t_0 + 1, \cdots, T - 1 \) such that

\[
J(u, x) = \phi_T(x(T), \phi_{T-1}[x(T-1), u(T-1), \phi_{T-2} \{ \ldots, \phi_{t_0+1} \{ x(t_0+1), u(t_0+1), \phi_{t_0+1} \{ x(t_0), u(t_0) \} \} \}, \ldots)]).
\]
Lemma 10. Consider the forward separable functions, $J_1 : \mathbb{R}^{m_1 \times (T_1 - 1)} \times \mathbb{R}^{m_2 \times (T_1 - 1)} \to \mathbb{R}$ and $J_2 : \mathbb{R}^{m_2 \times (T_2 - 1)} \times \mathbb{R}^{m_2 \times (T_2 - 1)} \to \mathbb{R}$ with representation dimensions $l_1$ and $l_2$ respectively. If $G = J_1 + J_2$ then $G$ is a forward separable function and has a representation dimension less than or equal to $l_1 + l_2$.

Proof. For simplicity consider the case $t_1 = t_2$ and $T_1 = T_2$: other cases follow by the same argument. Suppose $J_1$ and $J_2$ are forward separable and there exists functions $\{g_i\}$ and $\{h_i\}$ such that $J_1$ and $J_2$ can be written in the form (6) with associated representation dimensions $l_1$ and $l_2$ respectively. We now show that $G$ is forward separable by defining the functions $\{f_i\}$ such that $G$ can be written in the form (6).

\begin{align*}
\phi_1(x,u) &= g_1(x,u), \\
\phi_1(x,u,w) &= \left[\begin{array}{c} g_1(x,u) \\ h_1(x,u,w^{d_1 - 1}) \end{array} \right] \quad \text{for } i \in \{t_1 + 1, \ldots, T_1 - 1\} \\
\phi_T(x,u,w) &= g_T(x,u,w^{d_1 - 1}) + h_T(x,u,w^{d_1 - 1}),
\end{align*}

where $d_1 = \dim(\text{Im}\{g_1\})$ and $s_i = \dim(\text{Im}\{h_i\})$ for $i \in \{t_1, \ldots, T_1 - 1\}$.

We conclude that $G$ has a representation dimension, denoted $l_G$, such that

\begin{align*}
l_G &= \max_{i \in \{t_1, \ldots, T_1 - 1\}} \{d_i + s_i\} \\
&\leq \max_{i \in \{t_1, \ldots, T_1 - 1\}} \{d_i\} + \max_{i \in \{t_0, \ldots, T - 1\}} \{s_i\} \\
&= l_1 + l_2.
\end{align*}
However, we now observe
\[
z_2(T + 1) = \phi_T(z_1(T)), z_2(T)
\]
\[
z_2(T) = \phi_T - 1(z_1(T - 1), w(T - 1), z_2(T - 1))
\]
\[
\vdots
\]
\[
z_2(t_0 + 1) = \phi_{t_0}(z_1(t_0), w(t_0)).
\]
\[
z_2(t_0) = 0.
\]
Hence we have,
\[
L(w, x) = z_2(T + 1)
\]
\[
= \phi_T(z_1(T), \phi_T - 1(z_1(T - 1), w(T - 1), \phi_T - 2(\ldots, \phi_{t_0 + 1}(z_1(t_0 + 1), w(t_0 + 1), \phi_{t_0}(z_1(t_0), w(t_0))) \ldots)).
\]
\[
= J(u, x).
\]
Hence if \( w \) and \( z \) solve \( A(t_0, x_0) \) with objective \( L_{t_0, x_0} = z_2(T + 1) \), then \( w \) and \( z_1 \) solve \( H(t_0, x_0) \) with objective value \( J_{t_0, x_0} \).

**Proposition 12.** The augmented optimization problem \( A(t_0, x_0) \) in (10) satisfies the Principle of Optimality.

**Proof.** \( A(t_0, x_0) \) is a special case of \( P(t_0, x_0) \) (3) where \( c_i = 0 \) for \( i \neq T \) and \( c_{T}(z_1, z_2)^T = z_2 \). Lemma 5 shows optimization problems of the form \( P(t_0, x_0) \) satisfy the principle of optimality.

Lemma 11 tells us that for any forward separable problem of the form \( H(t_0, x_0) \) there exists an equivalent optimization problem of the form \( A(t_0, x_0) \) (10). Furthermore Proposition 12 shows that \( A(t_0, x_0) \) satisfies the principle of optimality. Therefore a solution for \( H(t_0, x_0) \) can be found by recursively solving Bellman’s equation (4) for \( A(t_0, x_0) \).

To understand the augmented approach intuitively, we note that dynamic programming breaks a multi-period planning problem into simpler optimization problems at each stage. However, for non-separable problems, to make the correct decision at each stage we need past information about the system. In this context, the augmented state contains the historic information necessary to make the correct decision at the present time. However by adding augmented states we increase the state space dimension and the complexity of the optimization problem.

**Corollary 13.** Suppose the forward separable function, \( J : \mathbb{R}^{m \times (T - t_0)} \times \mathbb{R}^{n \times (T + 1 - t_0)} \rightarrow \mathbb{R} \) is the objective function for the optimization problem \( H(t_0, x_0) \) (9) and has a representation dimension of \( I \). Then the associated augmented optimization problem with this representation, \( A(t_0, x_0) \) (10), has a state space of dimension \( I + n \) and input space of dimension \( m \).

**B. Examples of Forward Separable Functions**

Next we will show that it is possible to represent any function as a forward separable function. To do this we recall some notation. For a vector \( v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n \) we define \( v_i^j = (v_i, \ldots, v_j) \) for some \( 1 \leq i < j \leq n \).

**Lemma 14.** Any function \( J : \mathbb{R}^{m \times (T - t_0)} \times \mathbb{R}^{n \times (T + 1 - t_0)} \rightarrow \mathbb{R} \) can be shown to be forward separable with a representation of dimension \( l(n, m, T - t_0) = (T - t_0)(n + m) \).

**Proof.** Consider some function \( J : \mathbb{R}^{m \times (T - t_0)} \times \mathbb{R}^{n \times (T + 1 - t_0)} \rightarrow \mathbb{R} \). To show \( J \) is forward separable we will give functions \( \{ \phi_i \}_{i=0}^T \) that satisfy (6).

The function \( \phi_0 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n + m} \) is defined by,
\[
\phi_0(x, u) = [x^T, u^T] = [x_1, \ldots, x_n, u_1, \ldots, u_m].
\]

For \( i \in \{t_0 + 1, \ldots, T - 1\} \) the function \( \phi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{(i+1-t_0)(n+m)} \) is defined by,
\[
\phi_i(x, u) = [w^{(i-t_0)}_1, x^T_1, w^{(i-t_0)(n+m)}_{n(i-t_0)+1}, u^T].
\]

The function \( \phi_T : \mathbb{R}^n \times \mathbb{R}^{T - t_0} \rightarrow \mathbb{R}^{(n + m)} \) is defined by,
\[
\phi_T(x, w) = J([w^{(T - t_0)}_1, x_1], [w^{(n + m)}_{n(T - t_0)+1}]).
\]

Moreover it can be seen that the maximum dimension of the images of the maps \( \{ \phi_i \}_{i=0}^T \) is \( (T - t_0)(n + m) \) showing the dimension of this representation of \( J \) is \( l(n, m, T - t_0) = (T - t_0)(n + m) \).

In the above approach to show that \( J(u, x) \) is forward separable we naively took the strategy of using the functions \( \{ \phi_i \}_{i=0}^T \) to act like memory functions; that is to store the entire historic state trajectory and input sequence used. If \( J(u, x) \) is the objective function for some optimization problem \( H(t_0, x_0) \) (9) then this approach would result in the associated augmented optimization problem, \( A(t_0, x_0) \) (10), having a very large state space dimension. Corollary 13 shows taking this naive approach results in the problem \( A(t_0, x_0) \) having state space dimension \( (T - t_0)(n + m) + n \). For a large number of time-steps, \( T - t_0 \), \( A(t_0, x_0) \) will be intractable. For this reason we next define a special class of forward separable functions that have a representation with dimension independent of the number of time-steps.

**Definition 15.** We say a function \( J : \mathbb{R}^{m \times (T - t_0)} \times \mathbb{R}^{n \times (T + 1 - t_0)} \rightarrow \mathbb{R} \) is a Naturally Forward Separable Function (NFSF) if there exists maps, \( \{ \phi_i \}_{i=0}^T \), that satisfy (6) with associated representation dimension independent of \( n, m \) and \( T \).

**Corollary 16.** Consider the naturally forward separable functions, \( J_1 : \mathbb{R}^{m_1 \times (T_1 - t_1)} \times \mathbb{R}^{n_1 \times (T_1 + 1 - t_1)} \rightarrow \mathbb{R} \) and \( J_2 : \mathbb{R}^{m_2 \times (T_2 - t_2)} \times \mathbb{R}^{n_2 \times (T_2 + 1 - t_2)} \rightarrow \mathbb{R} \). If \( G = J_1 + J_2 \) then \( G \) is also a naturally forward separable function.

**Proof.** Suppose \( J_1 \) and \( J_2 \) have a representation of dimensions \( l_1 \) and \( l_2 \), respectively. Lemma 10 shows there exists a representation of \( G \) with dimension \( l_2 \leq l_1 + l_2 \). Since \( J_1 \) and \( J_2 \) are NFSF’s \( l_1 \) and \( l_2 \) are independent of \( n_1, m_1 \) and \( T_1 \) for \( i = 1, 2 \). Therefore using the functions \( \{ \phi_i \} \) from (6) the resulting augmented dimension, \( l_2 \), will be independent of \( n_i, m_i \) and \( T_i \) for \( i = 1, 2 \), making \( G \) a NFSF.

In (7) we saw that additively separable functions are examples of NFSF’s. We next present several more examples of NFSF’s. One important example is in Lemma 19 where a forward separable representation of the coordinate-wise maximum function is given. The coordinate-wise maximum function is important because it appears as the objective function for the optimal battery scheduling problem for consumers subject to demand charges.
Lemma 17. Consider the function \( J : \mathbb{R}^{m \times (T-1)} \times \mathbb{R}^n \to \mathbb{R} \) such that,

\[
J(u, x) = \{ i \in \{0, \ldots, T \} : \|x(i)\|_2 > M \}
\]

where \( u = (u(0), \ldots, u(T-1)) \), \( u(i) \in \mathbb{R}^m \), \( x = (x(0), \ldots, x(T)) \), \( x(i) \in \mathbb{R}^n \), \( M \in \mathbb{R} \), \( \| \cdot \|_2 \) is the Euclidean norm and for \( B \subseteq \mathbb{N} \) we denote \( |B| \) to be the cardinality of the set \( B \). Then \( J \) is a NFSF and has a representation dimension of 1.

Proof. We present functions such that \( J(u, x) \) can be written in the form (9).

The function \( \phi_0 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) is defined by,

\[
\phi_0(x, u) = \begin{cases} 1 & \text{if } \|x\|_2 > M \\ 0 & \text{otherwise} \end{cases}
\]

The function \( \phi_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) is defined by,

\[
\phi_i(x, u, w) = \begin{cases} w + 1 & \text{if } \|x\|_2 > M \\ w & \text{otherwise} \end{cases}
\]

for \( 1 \leq t \leq T - 1 \).

The function \( \phi_T : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is defined by,

\[
\phi_T(x, w) = \begin{cases} w + 1 & \text{if } \|x\|_2 > M \\ w & \text{otherwise} \end{cases}
\]

Moreover it can be seen that the maximum dimension of the images of the maps \( \{ \phi_i \}_{i=0}^{T-1} \) is 1 showing the dimension of this representation of \( J \) is 1.

Lemma 18. Consider the variance type function, \( J : \mathbb{R}^{m \times T} \times \mathbb{R}^{n \times (T-1)} \to \mathbb{R} \) defined by,

\[
J(u, x) = \sum_{t=0}^{T} \left( a_t(x(t)) - \frac{1}{T} \sum_{s=0}^{T} a_s(x(s)) \right)^2 \tag{11}
\]

where \( u = (u(0), \ldots, u(T-1)) \), \( u(i) \in \mathbb{R}^m \), \( x = (x(0), \ldots, x(T)) \), \( x(i) \in \mathbb{R}^n \), \( a : \mathbb{R}^n \to \mathbb{R} \). Then \( J \) is a NFSF and has a representation dimension of 2.

Proof. Expanding the right hand side of (11) as in (9) we get,

\[
J(u, x) = \sum_{t=0}^{T} \left[ a_t^2(x(t)) - \frac{2}{T} a_t(x(t)) \sum_{s=0}^{T} a_s(x(s)) + \frac{1}{T^2} \left( \sum_{s=0}^{T} a_s(x(s)) \right)^2 \right] = \sum_{t=0}^{T} a_t^2(x(t)) - \frac{1}{T} \left( \sum_{s=0}^{T} a_s(x(s)) \right)^2.
\]

We now present functions such that \( J(u, x) \) can be written in the form (9). The function \( \phi_0 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^2 \) is defined by,

\[
\phi_0(x, u) = \begin{bmatrix} a_1^2(x) \\ a_1(x) \end{bmatrix}.
\]

The function \( \phi_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^2 \to \mathbb{R}^2 \) is defined by,

\[
\phi_i(x, u, [w_1, w_2]^T) = \begin{bmatrix} w_1 + a_i^2(x) \\ w_2 + a_i(x) \end{bmatrix} \quad \text{for } 1 \leq i \leq T - 1.
\]

The function \( \phi_T : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R} \) is defined by,

\[
\phi_T(x, [w_1, w_2]^T) = (w_1 + a_T^2(x)) - \frac{1}{T} (w_2 + a_T(x))^2.
\]

Moreover it can be seen that the maximum dimension of the images of the maps \( \{ \phi_i \}_{i=0}^{T-1} \) is 2 showing the dimension of this representation of \( J \) is 2.

We now show that the maximum function, that appears in the objective function of the battery scheduling problem in Section IV is a NFSF.

Lemma 19. Consider the function \( J : \mathbb{R}^{m \times T} \times \mathbb{R}^{n \times (T+1)} \to \mathbb{R} \) such that,

\[
J(u, x) = \max_{0 \leq k \leq T-1} \{ c_k(u(k), x(k)) , c_T(x(T)) \}
\]

where \( u = (u(0), \ldots , u(T-1)) \), \( u(i) \in \mathbb{R}^m \), \( x = (x(0), \ldots , x(T)) \), \( x(i) \in \mathbb{R}^n \), \( c_k : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R} \) for \( 0 \leq k \leq T-1 \) and \( c_T : \mathbb{R}^n \to \mathbb{R} \). Then \( J \) is a NFSF and has a representation dimension of 1.

Proof.

\[
J(u, x) = \max_{0 \leq k \leq T-1} \{ c_k(u(k), x(k)) , c_T(x(T)) \}
\]

\[
= \max \{ c_T(x(T)) , \{ c_{T-1}(u(T-1), x(T-1)) , \cdots , \\max \{ \cdots , \{ \cdots , \{ c_0(u(0), x(0)) \} \} \} \}.
\]

It is now clear that we can write \( J \) in the form (9) as follows. The function \( \phi_0 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) is defined by,

\[
\phi_0(x, u) = c_0(x, u).
\]

The function \( \phi_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R} \) is defined by,

\[
\phi_i(x, u, w) = \max(c_i(x, u), w) \quad \text{for } 0 \leq i \leq T - 2.
\]

The function \( \phi_T : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is defined by,

\[
\phi_T(x, w) = \max(c_T(x), w).
\]

Moreover it can be seen that the maximum dimension of the images of the maps \( \{ \phi_i \}_{i=0}^{T-1} \) is 1 showing the dimension of this representation of \( J \) is 1.

Corollary 20. \( S(t_0, x_0) \) is a special case of \( H(t_0, x_0) \) with a NFSF objective function.

Proof. Consider the objective function from Problem \( S(t_0, x_0) \),

\[
J_{t_0, x_0}(u, x) = \sum_{t=0}^{T-1} c_t(x(t), u(t)) + c_T(x(T)) + \max_{0 \leq k \leq T} d_k(x(k)).
\]

\( J_{t_0, x_0} \) is the sum of an additively separable function, shown to be a NFSF with representation dimension 1 in (7), and a pointwise maximum function, shown to be a NFSF with representation dimension 1 in Lemma 19. Therefore by Corollary 16 we deduce \( J_{t_0, x_0} \) is a NFSF with representation dimension less than or equal to 2.

We have now shown \( S(t_0, x_0) \) is of the same form as problems in the class \( H(t_0, x_0) \); which we know are equivalent to problems of the class \( A(t_0, x_0) \). Since problems of class \( A(t_0, x_0) \) satisfy the principle of optimality, they can be solved using dynamic programming and a solution to the original Problem \( S(t_0, x_0) \) can be retrieved. In the following section, we will apply this technique to optimal battery scheduling in the presence of demand charges.
IV. APPLICATION TO THE ENERGY STORAGE PROBLEM

In this section, we apply the augmented dynamic programming methodology to optimal scheduling of batteries in the presence of demand charges. We first propose a simple model for the dynamics of battery storage. We then formulate the objective function using electricity pricing plans which include demand charges. We see that the system described becomes an optimization problem of the form $S(t_0, x_0)$ which can be tractably solved as it has a NSF as an objective function (Corollary 20).

A. Battery Dynamics

We will model the energy stored in the battery by the difference equation:

$$e(k + 1) = \alpha(e(k) + \eta u(k) \Delta t)$$

(12)

where $e(k)$ denotes the energy stored in the battery at time step $k$, $\alpha$ is the bleed rate of the battery, $\eta$ is the efficiency of the battery, $u(k)$ denotes the charging/discharging (+/-) at time step $k$ and $\Delta t$ is the amount of time passed between each time step. Moreover we denote the maximum charge and discharge rate by $\bar{u}$ and $\bar{g}$ respectively. Thus we have the constraint that $u(k) \in [\bar{u}, \bar{g}] := U$ for all $k$. Similarly we also add the constraint $e(k) \in [\bar{e}, \bar{e}] := X$ for all $k$ where $\bar{e}$ and $\bar{e}$ are the capacity constraints of the battery (typically $\bar{e} = 0$).

B. The objective function

Let us denote $q(k)$ as the power supplied by the grid at time step $k$.

$$q(k) = q_o(k) - q_s(k) + u(k)$$

(13)

where $q_o(k)$ is the power consumed by HVAC/appliances at time step $k$ and $q_s(k)$ is the power supplied by solar photovoltaics at time step $k$. For now, it is assumed that both are known apriori.

To define the cost of electricity we divide the day $t \in [0, T]$ into on-peak and off-peak periods. We define an off peak period starting from 12am till $t_{on}$ and $t_{off}$ till 12am. We define an on-peak period between $t_{on}$ till $t_{off}$. The Time-of-Use (TOU, $\$ per kWh) electricity cost during on-peak and off-peak is denoted by $p_{on}$ and $p_{off}$ respectively. We further simplify this as $p_{on}$ if $k \in T_{on}$ and $p_{off}$ if $k \in T_{off}$ where $T_{on}$ and $T_{off}$ are the on-peak and off-peak hours, respectively. These TOU charges define the first part of the objective function as:

$$J_E(\mathbf{u}, \mathbf{e}) = p_{off} \sum_{k=0}^{t_{on}-1} q(k) \Delta t + p_{on} \sum_{k=t_{on}}^{t_{off}-1} q(k) \Delta t + p_{off} \sum_{k=t_{off}}^{T} q(k) \Delta t = \sum_{k \in [0, T)} p_k (q_o(k) - q_s(k)) \Delta t + \sum_{k \in [0, T]} p_k u(k) \Delta t$$

(14)

where the daily terminal timestep is $T = 24 / \Delta t$. Clearly, only the second term in this objective function is significant for the purposes of optimization.

We also include a demand charge, which is a cost proportional to the maximum rate of power taken from the grid during on-peak times. This cost is determined by $p_d$ which is the price in $\$ per kW. Thus it follows the demand charge will be:

$$J_D(\mathbf{u}, \mathbf{e}) = p_d \max_{k \in [t_{on}, \ldots, t_{off}-1]} \{q_o(k) - q_s(k) + u(k)\}$$

C. 24 hr Optimal Residential Battery Storage Problem

We may now define the problem of optimal battery scheduling in the presence of demand and Time-of-Use charges, denoted $D(0, x_0)$.

$$\min_{\mathbf{u}, \mathbf{e}} \{J_E(\mathbf{u}, \mathbf{e}) + J_D(\mathbf{u}, \mathbf{e})\} \quad \text{subject to}$$

$$e(k+1) = \alpha(e(k) + \eta u(k) \Delta t) \quad \text{for} \quad k = 0, \ldots, T$$

$$e_0 = e_0, \quad e(k) \in X, \quad u(k) \in U \quad \text{for} \quad k = 0, \ldots, T,$$

$$\mathbf{u} = (u(0), \ldots, u(T - 1))$$

and $\mathbf{e} = (e(0), \ldots, e(T))$

where recall $U := [\bar{u}, \bar{g}]$ and $X := [\bar{e}, \bar{e}]$.

Proposition 21. Problem $D(0, x_0)$ is a special case of $S(t_0, x_0)$

Proof. Let $c_i = p_i(q_o(i) - q_s(i) + u(i)) \Delta t$

$$d_i = \begin{cases} p_d(q_o(k) - q_s(k) + u_k) & k \in T_{on} \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that our algorithmic approach to forward separable dynamic programming can be applied to this problem as per Corollary 20. That is, it can be represented as an augmented dynamic programming problem of Form $A(t_0, x_0)$.

V. NUMERICAL IMPLEMENTATION FOR GENERAL DP PROBLEMS

In Section III we showed that all forward separable problems of the form $H(t_0, x_0)$ have an equivalent optimization problem of the form $A(t_0, x_0)$. Problems of the form $A(t_0, x_0)$ are special cases of problems of the form $P(t_0, x_0)$. In this section we show how to numerically solve problems of the form $P(t_0, x_0)$.

For implementation, we use an approximation scheme that maps our class of dynamic programming problems to a much simpler class of dynamic programming problems with finite state and control spaces. It is known for dynamic programming problems with countable state and control spaces the infimum in Bellman’s equation (3) is attained and the optimal cost to go function, $F(x, t)$, can be computed by enumeration. Similar numerical schemes with convergence proofs can be found in [8] [7].

A. Construction of Approximated Tractable Optimization Problems

Consider the optimization problem $P(t_0, x_0)$ with compact state and control spaces of the form $X = [\bar{x}, \bar{x}]^n$ and $U = [\bar{u}, \bar{u}]^m$. For optimization problems of this form it is generally possible to solve Bellman’s Equation (3). We thus need to consider a sequence of “close” optimization problems with countable state and control spaces. We define a sequence of approximated optimization problems indexed by $k$ and denoted by $P_k(t_0, x_0)$.

$$\min_{\mathbf{u}, \mathbf{x}} \sum_{t=0}^{T-1} c_t(x(t), u(t)) + c_T(x(T))$$

subject to:

$$x(t + 1) = \arg \min_{x \in \mathcal{X}} \{ ||y - f(x(t), u(t), t)||^2 \}$$

$$x(t_0) = x_0, \quad x(t) \in \mathcal{X} \subset \mathbb{R}^n, \quad u(t) \in \mathcal{U} \subset \mathbb{R}^m \quad \text{for} \quad t = t_0, \ldots, T,$$

$$\mathbf{u} = (u(t_0), \ldots, u(T - 1))$$

and $\mathbf{x} = (x(t_0), \ldots, x(T))$. 

Figure 1. The resulting state trajectories from using the policy constructed from $P_k(t_0, x_0)$ in the Optimization Problem (3).

Where $X_k = \{x_1, ..., x_k\}^n$ such that $x = x_1 < x_2 < ... < x_k = \bar{x}$ and $|x_{i+1} - x_i|_2 = \frac{\bar{x}}{k}$ for $1 \leq i \leq k - 1$, and $U_k = \{u_1, ..., u_k\}^m$ such that $u = u_1 < u_2 < ... < u_k = \bar{u}$ and $|u_{i+1} - u_i|_2 = \frac{\bar{u}}{k}$ for $1 \leq i \leq k - 1$.

B. Constructing a Feasible Policy from the Solution of the Approximated Optimization Problem

By iteratively solving Bellman’s equation (4) we can find an optimal solution to $P_k(t_0, x_0)$ which we denote as $(x^*_k, u^*_k)$. Because the vector fields that define the underlying dynamics of $P(t_0, x_0)$ and $P_k(t_0, x_0)$ are different, the solution $(x^*_k, u^*_k)$ is only approximately feasible for $P(t_0, x_0)$. However using the optimal policy for $P_k(t_0, x_0)$, $x_0^*$, we can construct a feasible policy for $P(t_0, x_0)$ in the following way,

$$\theta_k(x, t) = \arg\min_{u \in \Gamma_{x,t}} \|x^*_k - \arg\min_{y \in x_k} \|y - x\|_2\| - u\|_2 \in \Pi$$

where we recall $\Gamma_{x,t}$ is the set of feasible inputs such that $u \in \Gamma_{x,t}$ then $u \in U$ and $f(x, u, t) \in X$ for the optimization problem $P(t_0, x_0)$ (3).

C. Convergence of our Constructed Policy

Suppose $\theta_k(x, t)$, from (16), is a feasible policy for $P(t_0, x_0)$ from the optimal policy of $P_k(t_0, x_0)$ using (16). Let $u_k = (\theta_k(x_0, t), ..., \theta_k(x_k(T-1), T-1))$ and $x_k = (x_k(t_0), ..., x_k(T))$ where $x_k(t_0) = x_0$, $x_k(t+1) = f(x_k(t), \theta_k(x_k(t), t), t)$ and $f$ is the vector field from $P_k(t_0, x_0)$. If $P(t_0, x_0)$ satisfies assumptions (A1) to (A4) then it is known,

$$\lim_{k \to \infty} \|J_{x_0, t_0}(u_k, x_k) - J^*_t|_{x_0, t_0} = 0$$

where $J_{x_0, t_0}(u_k, x_k)$ is the resulting value objective function of $P(t_0, x_0)$ when the policy $\theta_k$ is used and $J^*_t|_{x_0, t_0}$ is the optimal value of the objective function.

D. Illustrative Example

To illustrate how we use state augmentation we consider a dynamic programming problem from (5). During this example we will,

1) Show the objective function is forward separable.
2) Construct the associated augmented optimization problem of form $A(t_0, x_0)$.
3) Approximate the augmented optimization problem, $A(t_0, x_0)$, with an associated optimization problem of Form $P_k(t_0, x_0)$.
4) Numerically solve $P_k(t_0, x_0)$ for different discretization levels, $k \in \mathbb{N}$.
5) Construct a feasible policy for $A(t_0, x_0)$ from the optimal policy of $P_k(t_0, x_0)$ for different values of $k \in \mathbb{N}$ using (16).
6) Show graphically how the value of the objective function $A(t_0, x_0)$ under the feasible policy constructed from the optimal policy of $P_k(t_0, x_0)$ approaches optimality as the discretization parameter, $k$, is increased.

Let us consider the optimization problem,

$$\min J = x(3)^2u(0)^2 + u(1)^2 + u(1)u(2)^2$$

$$+ |u(0)^2 + u(1)^2 + u(1)u(2)^2|^2$$

subject to, $x(t+1) = x(t)$ for $t \in \{1, 2, 3\}$.

$x(0) = 10, u(0), u(1), u(2) \geq 0$.

In [5] an analytic solution for (18) was found to be:

$$x^* = \begin{bmatrix} 10 \\ 6.3943938 \\ 5.782475 \\ 3.8882658 \end{bmatrix}, u^* = \begin{bmatrix} 1.5638699 \\ 1.105823 \\ 1.4871604 \end{bmatrix}, J^* = 74.767439.$$

The objective function $J$ in (18) is a NNF and has a representation dimension of 2. This can be shown by writing $J$ in the form (6) using the functions,

$$\phi_0(x, u) = \begin{bmatrix} u^2 \\ 0 \end{bmatrix}, \phi_1 \left( x, u, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) = \begin{bmatrix} w_1 + u^2 \\ u \end{bmatrix}$$

$$\phi_2 \left( x, u, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) = \begin{bmatrix} w_1^2 + w_2^2 \\ 0 \end{bmatrix}$$

$$\phi_3 \left( x, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) = \begin{bmatrix} x^2 \sqrt{w_1} + w_1^2 \\ 0 \end{bmatrix}.$$

The Optimization Problem (18) can then be written in the form $A(t_0, x_0)$ using state augmentation,

$$\min z_3(4)$$

subject to,

$$z_1(t+1) = \frac{z_1(t)}{u(t)}, z_2(t+1) = \begin{cases} u(t) & \text{if } t=1 \\ 0 & \text{otherwise} \end{cases} \forall t \in \{1, 2, 3\}, z_3(1) = u(1)^2, z_3(2) = z_3(1) + u(2)^2, z_3(3) = z_3(2) + z_2(2)^2 u(2), z_3(4) = z_3(3) + \sqrt{z_3(3)} + z_3(3)^2, z_3(1) = 10, z_2(2) = 0, z_3(0) = 0, u(0), u(1), u(2) \geq 0.$$

The Optimization Problem (19) is now a special case of $P(t_0, x_0)$ and equivalent to the original Problem (18). The associated approximated optimization problem of the form $P_k(t_0, x_0)$ (15) can now be found by selecting appropriate compact state and control spaces; $X \subset \mathbb{R}^3$ and $U \subset \mathbb{R}$. A feasible policy for (18) is then constructed from the optimal policy of the associated $P_k(t_0, x_0)$ using (16). Figure 1 shows the state trajectories by following different constructed policies for various values of $k$. It is seen that for $k = 200$ the algorithm
produces a solution within three significant figures of the analytic optimal objective function for \((15)\).

VI. Numerically Solving the Deterministic Battery Scheduling Problem

Our proposed approximation scheme can be applied to solve the battery scheduling problem, \(D(0,e_0)\). This is done by creating an augmented state variable based on the maximum function in the objective function, as in Section III and thus constructing an equivalent optimization problem of the form \(A(0,x_0)\) \((10)\); which is a special case of \(P(t_0, x_0)\). Figure 2 shows how the monthly cost decreases when we use policies constructed from the associated discretized optimization problems, \(P_k(t_0,x_0)\), and \(k\) is increased. Although we do not get a monotonically decreasing sequence of costs, the error does decrease as \(k \to \infty\). Figure 3 also shows that augmenting and then following our proposed discretization scheme for the battery scheduling problem results in a policy that reduces the consumption demand peak as \(k\) is increased. Figure 4 shows that the computational time used to solve the optimization problem of the form \(P_k(t_0,x_0)\) associated with \(D(0,e_0)\) increases with respect to \(k\).

We used solar and usage data obtained by local utility Salt River Project in Tempe, AZ, for power variables \(q_s\) and \(q_a\). We also use pricing data from SRP for the parameters \(p_{on}\), \(p_{off}\) and \(p_d\). Battery data obtained for the Tesla Powerwall was used to get the parameters \(\alpha\), \(\eta\), \(\bar{u}\), \(\bar{u}_d\) and \(\bar{e}\). The results of the simulation are shown in Figure 5. The policy used for this simulation was created using our augmentation and approximation scheme with \(k = 20\). Interpolation was used to aid solving Bellam’s equation \((21)\) and decrease the approximation error. These results show an improvement in accuracy over results obtained based on the approach to a similar problem in \((25)\) (approximately $0.98 savings). As expected, we see the battery charges during off-peak and then discharges during on peak times to reduce ToU charges, while maintaining a reserve which it uses to keep consumption flat during on peak times, thereby minimizing the demand charge. As a result the power stabilizes during on peak times - becoming constant.

VII. Extension to Stochastic Models

Our state-augmentation approach in Section III can be applied to general stochastic forward separable optimization problems. Analogous to the deterministic case for any stochastic dynamic programming problem with forward separable
objective \((H_i(t_0, x_0))\) \(\text{(20)}\), there exists a stochastic dynamic programming problem with additively separable objective function of the form \((Q(t_0, x_0))\) \(\text{(21)}\) whose solution yields a solution to the original forward separable problem. Before we introduce the problem \(H_i(t_0, x_0)\) we define a map from a chosen policy, initial condition and random inputs to the trajectory, \(x\), followed by the underlying dynamics of this problem; this will clarify which random variables the expectation in the objective function is respect to.

**Definition 22.** For a vector field \(f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \times \mathbb{R}^q \rightarrow \mathbb{R}^n\), a set of optimal polices \(\Pi\) associated with some optimization problem, a starting time \(t_0 \in \mathbb{N}\), and terminal time \(T \in \mathbb{N}\), let us denote the state map by \(\psi_{f,t_0} : \Pi \times \mathbb{R}^n \times \mathbb{N} \times \mathbb{R}^q \rightarrow \mathbb{R}^n\). We say that \(x = \psi_{f,t_0}(\pi, x_0, T, [v]_{t_0}^{-1})\) if \(x = x(T)\) where \(x(T)\) is a solution to the following recursion equations \(x(t_0) = x_0\), \(x(t + 1) = f(x(t), \pi(x(t)), t, v(t))\) for \(t \in \{t_0, ..., T - 1\}\) and \([v]_{t_0}^{-1} = [v(t_0), ..., v(T - 1)] \in \mathbb{R}^q\). We denote the image of the state vector under a set of instantiations \(Y \subset \mathbb{R}^q\) for \(t \in \{t_0, ..., T - 1\}\) by \(\psi_{f,t_0}(\pi, x_0, T, Y) = \{\psi_{f,t_0}(\pi, x_0, T, [v]_{t_0}^{-1}) \in \mathbb{R}^n : [v]_{t_0}^{-1} \in Y\}\).

We also denote the trajectory map by \(\Phi_{f,t_0} : \Pi \times \mathbb{R}^n \times \mathbb{N} \times \mathbb{R}^q \times (T - t_0) \rightarrow \mathbb{R}^n \times (T - t_0)\). We say that \((u, x) = \Phi_{f,t_0}(\pi, x_0, T, [v]_{t_0}^{-1})\) if \(u = (\pi(x(t_0), t_0), ..., \pi(x(T - 1), T - 1))\), and \(x = (x(t_0), ..., x(T))\) is such that \(x(t) = \psi_{f,t_0}(\pi, x_0, t, [v]_{t_0}^{-1})\) for \(t \in \{t_0, ..., T - 1\}\).

We define the class of general stochastic dynamic programming problems with forward separable objective as \(H_i(t_0, x_0)\),

\[
\pi^{H_i} = \arg \min_{\pi \in \Pi} \mathbb{E}_{v} \left[ J_{H_i, t_0}^{f}(\Phi_{f,t_0}(\pi, x_0, T, [v]_{t_0}^{-1})) \right] \tag{20}
\]

subject to: \(\psi_{f,t_0}(\pi, x_0, t, [v]_{t_0}^{-1}) \in X_t\) for \(t = t_0, ..., T\)
\(\pi(x, t) \in U_t\) and \(v(t) \in \mathbb{R}^q \sim \mathcal{N}(0, \Sigma_{q \times q})\) \(\forall x \in X_t, \forall t = t_0, ..., T - 1\),

where \(J_{H_i, t_0}^{f} : \mathbb{R}^m \times (T - t_0) \times \mathbb{R}^n \times (T - t_0 + 1) \rightarrow \mathbb{R}\) is a forward separable function with associated representation \(\{\Phi_{t_0, t}^{i} \}_{t_0 = t_0}^{T}\).
separable objective function defined in Definition 1 \( f : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{N} \times \mathbb{R}^q \rightarrow \mathbb{R}^m \); \( \Psi_{f,0} \) and \( \Phi_{f,0} \) are the state and trajectory map respectively defined in Definition 22. \( U_t \) is assumed to be some compact subset of \( \mathbb{R}^n \times (0,b) \); \( X_t \subset \mathbb{R}^n \times (0,b+1) \); \( \{ v(T_t) \} = \{ v(t) : v(T - 1) \} \in \mathbb{R}^n \times (T_0 - T - 1) \). Define \( J_{t_0,x_0}^{**} = E_{v_{t_0}} \left( J_{t_0,x_0}^{**}(\Phi_{f,0}(\pi_{x_0},x_0,T,v(T_0-T-1))) \right) \) as the expected cost of the optimal policy when applied to \( Q(t_0,x_0) \).

An optimization problem of form \( Q(t_0,x_0) \) is also of form \( H_0(t_0,x_0) \) because the objective function of \( Q(t_0,x_0) \) is an additively separable function which was shown in (7) to be forward separable. Moreover \( Q(t_0,x_0) \) is of the classical form commonly solved using Bellman’s equation 29.

A. Principle of Optimality for Stochastic Problems

As discussed in 3 the extension of the principle of optimality to the stochastic case is non-trivial. We next give an example from 29 of a stochastic dynamic programming problem which shows that an optimal policy may not be optimal for every instantiation of the random variables at future time steps.

Let us consider the following stochastic dynamic programming problem \( W(0,0) \),

\[
\pi^* = \arg\min_{\pi \in \Pi} \mathbb{E}_{v(0)} \left( J_{0,0}(\Phi_{f,0}(\pi,0,1,[v(0)])) \right) \tag{22}
\]

subject to: \( v(0) \sim U[0,1] \), \( x(0) = x_0 \).

Here \( J_{0,0}(u,x) = -\sum_{n=1}^T u(n), f(x,u,t,v) = v \), and \( \pi \in \Pi \iff \pi(x,t) \in \{ 0,1 \} \forall x \in \mathbb{R}, t = 0,1 \).

**Lemma 23.** The policy \( \pi(x,t) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{if } x = 1 \end{cases} \) is optimal for the problem \( W(0,0) \) but not optimal for the problem \( W(1,1) \).

**Proof.** Clearly \( J_{0,0}(u,x) \geq -2 \) for all \((u,x) \in \{ 0,1 \}^2 \times \mathbb{R} \) and \( J_{0,0}(u,x) = -2 \) attainable using the input \((u(0),u(1)) = (1,1) \); therefore any solution of \( W(0,0) \) will minimize the objective function to a value of -2. Now using the law of total expectation we get,

\[
\mathbb{E}_{v(0)} \left( J_{0,0}(\Psi_{f,0}(\pi,0,[v(0)])) \right) \\
= -\mathbb{E}_{v(0)}(\pi(0,0) + \pi(v(0),1)) \\
= -\pi(0,0) - \mathbb{E}_{v(0)}(\pi(v(0),1)v(0) \in \{ 0,1 \} \mathbb{P}_v(v(0) \in \{ 0,1 \})) \\
= -\pi(0,0) - \mathbb{E}_{v(0)}(\pi(v(0),1)v(0) = 0) \mathbb{P}_v(v(0) = 0) \\
= -2,
\]

since the probability of a continuous random variable (such as a uniformly distributed random variable) taking a particular value is 0. Thus it follows the policy \( \pi \) is optimal for \( W(0,0) \). Trivially \( \pi \) is not optimal for \( W(1,1) \) as the objective functions becomes 0 under \( \pi \) whereas the input \( u(1) = 1 \) produces a smaller objective function value of -1.

Clearly, for the stochastic DP problems of form \( H(t_0,x_0) \) 22, such as \( W(0,0) \) 22, the optimal policy \( \pi^* \) does not always result in the same trajectory \( x = (x(t_0),...,x(T)) \) being followed; as this is dependent on the instantiations of the underlying random variables, \( v(T_0-T-1) \). As Lemma 23 has shown there exists stochastic DP problems, with additively separable objective functions, that have optimal policies that are no longer optimal for future time steps if certain instantiations of the underlying random variables are realized. It is too restrictive to extend Definition 4 the principle of optimality for the deterministic case to the stochastic case by requiring stochastic problems satisfying the principle of optimality to be such that their optimal policy is also optimal for each instantiation at any future time step. This is because for stochastic DP problems of form \( H(t_0,x_0) \) we only require the expectation of the function \( J_{t_0,x_0}^{**} \) to be minimized and not the value of \( J_{t_0,x_0}^{**} \) under each instantiation; we do not require the optimal policy to remain optimal for future time steps when instantiations, that have zero probability of occurring, are realized by the underlying random variables. With this in mind and motivated by the work of 4 we now give a probabilistic definition of the principle of optimality for stochastic dynamic programming problems.

**Definition 24.** For an optimization problem \( H(t_0,x_0) \) with optimal policy \( \pi^* \in \Pi \) and associated state map \( \Psi_{f,0} \), defined in definition 22 let us denote the set indexed by \( k \geq 0 \),

\[
Y_k \coloneqq \{ v^{k-1} \in \mathbb{R}^n \times (k-\mathbb{R}) : \pi^* \text{ does not solve } H_0(k,\Psi_{f,0}(\pi^*,x_0,k,[v^{k-1}])) \}
\]

where \( \{ v^{k-1} \} = \{ v(t_0),...,v(k-1) \} \in \mathbb{R}^n \times (k-\mathbb{R}) \). We say stochastic optimization problems of the form \( H(t_0,x_0) \) 20 satisfy the principle of optimality if for any \( k \geq 0 \) we have

\[
\mathbb{P}_{v^{k-1}}(v^{k-1} \in Y_k) = 0.
\]

Where \( \mathbb{P}_{v^{k-1}} \) is the probability measure associated with the random variable \( v^{k-1} \in \mathbb{R}^n \times (k-\mathbb{R}) \). \( v(t) \sim \mathcal{N}(0,I_{d \times d}) \) for \( t \in \{ t_0,...,T-1 \} \).

The next Lemma, which we will require to prove that problems of the form \( Q(t_0,x_0) \) 21 satisfy the principle of optimality from Definition 24, shows that for any policy that uses the entire state space history there exists a “Markovian policy” that uses only current state space information and achieves the same objective value for optimization problems of the form \( Q(t_0,x_0) \).

**Lemma 25. Consider an optimization problem of the form \( Q(t_0,x_0) \) 21 with additively separable objective function \( J_{t_0,x_0}^{Q} \). For any family of functions of the form \( \hat{\pi} : \mathbb{R}^n \times (T_0-T-1) \rightarrow \mathbb{R}^m \) such are \( \hat{\pi}_t(\{ (x(t_0),...,x(t)) \}) \in U_t \) and \( f(x(t),\hat{\pi}_t(\{ (x(t_0),...,x(t)) \}),t,v(t) \in X_{t+1} \) for all \( x(t) \in X_t \), \( i \in \{ t_0,...,t \} \), \( v(t) \in \mathbb{R}^q \) and \( t \in \{ t_0,...,T-1 \} \) there exists \( \alpha \in \Pi \) such that

\[
\mathbb{E}_{v^{T-1}} \left( J_{t_0,x_0}^{Q}(\Phi_{f,0}(\hat{\pi},x_0,T,[v^{T-1}])) \right) \\
= \mathbb{E}_{v^{T-1}} \left( J_{t_0,x_0}^{Q}(\Phi_{f,0}(\hat{\pi},x_0,T,[v^{T-1}])) \right)
\]

where we make a small abuse of notation to extend the trajectory map \( \Phi_{f,0} \) to policies that use the entire state space history.

**Proof.** Proposition 8.1 30 or alternatively Theorem 6.2 4. □
Lemma 26. A problem of Form $Q(t_0, x_0)$ satisfies the Principle of Optimality defined in Definition 24.

Proof. Suppose $\pi^*$ solves $Q(t_0, x_0)$. For $k > t_0$ and the state map $\psi_{t_0, t_0}$ associated with $Q(t_0, x_0)$ let us recall the set defined in Definition 24

$$Y_k := \{v^{k-1}_{t_0} \in \mathbb{R}^n \cap (k-n) : \pi^* \text{ does not solve } Q(k, x_{t_0}^{-1}(v^{k-1}_{t_0}))\}.$$ 

Where $v^{k-1}_{t_0} := [v(t_0), \ldots, v(k-1)] \in \mathbb{R}^n \cap (k-t_0)$, and we use the short-hand $x_{t_0}^{-1}(v^{k-1}_{t_0}) := \psi_{t_0, t_0}(\pi^*, x_0, k, v^{k-1}_{t_0})$.

Now for contradiction suppose there exists $k \in \{t_0, \ldots, T\}$ such that $P[v^{k-1}_{t_0} \in Y_k] > 0$; where $v(t) \sim \mathcal{N}(0, t_{t_0} \sigma_t^2)$ for $t \in \{t_0, \ldots, k-1\}$. For $v^{k-1}_{t_0} \in Y_k$ we know the policy $\pi^*$ is not optimal for $Q(k, x_{t_0}^{-1}(v^{k-1}_{t_0}))$ and thus there exists a feasible policy $\theta \in \Pi$ such that,

$$E[v^{-1}_{t_0}] \left( J_{k,x_{t_0}^{-1}}^Q(\pi^*, x_{t_0}^{-1}(v^{k-1}_{t_0}), T, \pi_{t_0}^{-1}(v^{k-1}_{t_0})) \right) |v^{k-1}_{t_0} \in Y_k$$

$$<$$

$$E[v^{-1}_{t_0}] \left( J_{k,x_{t_0}^{-1}}^Q(\pi^*, x_{t_0}^{-1}(v^{k-1}_{t_0}), T, \pi_{t_0}^{-1}(v^{k-1}_{t_0})) \right) |v^{k-1}_{t_0} \in Y_k.$$ 

Now let us consider the map,

$$\tilde{\pi}_t(x(t_0), \ldots, x(t)) = \begin{cases} \theta(x(t), t) & \text{if } t \geq k, x(k) \in \psi_{t_0, t_0}(\pi^*, x_0, k, Y_k) \\ \pi^*(x(t), t) & \text{otherwise}. \end{cases}$$

Using Lemma 24 there exists a policy $\alpha \in \Pi$ such that (24) holds for $\tilde{\pi}_t$ defined in (25). We will now show $\alpha$ contradicts that $\pi^*$ is the optimal policy for $Q(t_0, x_0)$. We first note using (23) and the law of total probabilities,

$$E[v^{-1}_{t_0}] \left( J_{k,x_{t_0}^{-1}}^Q(\pi^*, x_{t_0}^{-1}(v^{k-1}_{t_0}), T, \pi_{t_0}^{-1}(v^{k-1}_{t_0})) \right)$$

$$= E[v^{-1}_{t_0}] \left( J_{k,x_{t_0}^{-1}}^Q(\pi^*, x_{t_0}^{-1}(v^{k-1}_{t_0}), T, \pi_{t_0}^{-1}(v^{k-1}_{t_0})) \right)$$

$$= E[v^{-1}_{t_0}] \left( J_{k,x_{t_0}^{-1}}^Q(\pi^*, x_{t_0}^{-1}(v^{k-1}_{t_0}), T, \pi_{t_0}^{-1}(v^{k-1}_{t_0})) \right)$$

$$= E[v^{-1}_{t_0}] \left( J_{k,x_{t_0}^{-1}}^Q(\pi^*, x_{t_0}^{-1}(v^{k-1}_{t_0}), T, \pi_{t_0}^{-1}(v^{k-1}_{t_0})) \right)$$

$$= E[v^{-1}_{t_0}] \left( J_{k,x_{t_0}^{-1}}^Q(\pi^*, x_{t_0}^{-1}(v^{k-1}_{t_0}), T, \pi_{t_0}^{-1}(v^{k-1}_{t_0})) \right)$$

We recall the additive structure of $J_{k,x_{t_0}^{-1}}^Q$.

$$J_{k,x_{t_0}^{-1}}^Q(u, x, T) = \sum_{t=0}^{T-1} c_t(p(t), u(t)) + c_T(x(T)).$$

where $u = (u(t_0), \ldots, u(T - 1))$ and $x = (x(t_0), \ldots, x(T))$ and $c_T : \mathbb{R}^n \to \mathbb{R}$, $c_T(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ for $t = t_0, \ldots, T-1$.

Now using the fact $\tilde{\pi}_t(x(t_0), \ldots, x(t)) = \pi^*(x(t), t)$ for all $t < k$, $\tilde{\pi}_t(x(t_0), \ldots, x(t)) = \theta(x(t), t)$ if $t \geq k$ and $x(k) \in$
\(Q(t_0, x_0)\). Since \(Q(t_0, x_0)\) has an additively separable objective function we can use Corollary 28 to find a solution for \(Q(t_0, x_0)\) by solving the stochastic version of Bellman’s equation (29). Unfortunately in some cases it can be hard to analytically solve (29). We now discuss ways to construct an approximate optimization problem that can be solved and has an optimal policy that is “close” to the optimal solution of \(Q(t_0, x_0)\).

To approximately solve \(Q(t_0, x_0)\) we use an approximation scheme that maps our class of dynamic programming problems to a much simpler class of dynamic programming problems with countable and finite state and control spaces. In 38 we discuss in detail our approximation scheme for problems of a similar type.

Unlike in the deterministic case the dynamics are stochastic with underlying randomness \(v \sim \mathcal{N}(0, 1)\) possibly inducing a non compact state space. Therefore before discretizing the state space we must first construct an approximate compact state space.

C. Constructing an Approximated Dynamic Programming Problem with Compact State Space

Consider the optimization problem \(Q(t_0, x_0)\) with compact control space \(U = \{u, \tilde{u}\}^m\) and underlying random variables \(v \sim \mathcal{N}(0, \gamma \Sigma)\). As in (7) we assume \(\forall \varepsilon > 0\) there exists a compact set \(H_{\varepsilon} = [\tilde{x}_f, x_f]^m \subset X\) (that depends on \(\varepsilon\) and \(t\)) such that \(x_0 \in H_{\varepsilon,0}\) and,

\[
\sup_{x_0 \in H_{\varepsilon,0}} \mathbb{P}_x f(x, u, v, t) \notin H_{\varepsilon,1} < \varepsilon. \tag{30}
\]

We then construct the associated approximated optimization problem to \(Q(t_0, x_0)\) denoted by \(Q_{\varepsilon,k}(t_0, x_0)\),

\[
\arg\min_{\pi \in \Pi} \mathbb{E}_{x_0}[\beta_0(t_0) \left( J^O_{\varepsilon,k}(\pi, x_0, T, [v^T_{t-1}]) \right) f(x, u, t, v)] \tag{31}
\]

subject to:

\[
\psi_{f,0}(\pi, x_0, t, [v^T_{t-1}]) \in \tilde{X}_{\varepsilon,k} f(t = 0, ..., T)
\]

\[
\pi(x, t) \in \tilde{U}_k \text{ and } v(t) \in \mathbb{R}^q \sim \mathcal{N}(0, I_{q \times q}) \forall x \in X, \forall t = 0, ..., T - 1,
\]

where \(\tilde{f}(x, u, t, v) = \arg\min_{u \in \tilde{U}_k} \{||y - \tilde{f}(x, u, t, v)||_2\} \), \(\tilde{X}_{\varepsilon,k} = \{x_1, ..., x_k\}^n\) such that \(x_{i-1} = x_{i-2} < ... < x_{k-1} = x_k\) and \(x_{i+1} = x_{i+2} < ... < x_{k-1} = x_k\) \(\forall \leq i \leq k - 1\), \(\tilde{U}_k = \{u_1, ..., u_k\}^m\) such that \(u_1 < u_2 < ... < u_k = \tilde{u}\) and \(||u_{i+1} - u_i||_2 \leq \beta_{k-1}^2\) for \(1 \leq i < k - 1, \) and \(v(T-1) = v(t) \forall t \in T - 1 \in \mathbb{R}^{q \times (T-1)}\).

Analogous to the deterministic case the optimal policy \(\pi^*_{\varepsilon,k}\) for \(Q_{\varepsilon,k}(t_0, x_0)\) can be solved exactly by iteratively solving Bellman’s equation (29). One can then construct a feasible policy for \(Q(t_0, x_0)\) using,

\[
\theta_{\varepsilon,k}(x, t) = \arg\min_{u \in \Gamma_{x,t}} \|\pi^*_{\varepsilon,k}(x, t) - u\|_2 \in \Pi \tag{32}
\]

where \(\Gamma_{x,t}\) is the set of feasible controls at time \(t \in \{0, ..., T - 1\}\) and state position \(x \in \mathbb{R}^n\) for \(Q(t_0, x_0)\) (32) and \(X_{\varepsilon,k}\) is the state constraint in the problem \(Q_{\varepsilon,k}(t_0, x_0)\) (31).

If \(Q(t_0, x_0)\) satisfies assumptions (A1) to (A4) in (7) then

\[
\lim_{\varepsilon \to 0} \mathbb{E}_{x_0}[v_{t-1} J^O_{\varepsilon,k}(\pi_{\varepsilon,k}(x_0, x_0, T, [v^T_{t-1}]) - J^O_{\varepsilon,k}) = 0, \tag{33}
\]

where \(J^O_{\varepsilon,k}(x_0) = \mathbb{E}_{x_0}[v_{t-1} J^O_{\varepsilon,k}(\pi_{\varepsilon,k}(x_0, x_0, T, [v^T_{t-1}]) - J^O_{\varepsilon,k}) \) is the expected cost of the optimal policy when applied to \(Q(t_0, x_0)\).

VIII. Solving the Stochastic Battery Scheduling Problem

To evaluate the effect of stochastic uncertainty on battery scheduling, we identified a Gauss-Markov model of solar generation based on SRP data. We construct the battery scheduling problem in the form \(H_{\varepsilon}(t_0, x_0)\) (27) and then use our proposed state augmentation approach to construct an equivalent optimization problem of form \(Q(t_0, x_0)\) (31). The problem of form \(Q(t_0, x_0)\) is then solved approximately using the methodology of Section VII-C.

A. Solar Generation Model

Our approach to modeling the dynamics of solar generation is based on 31. Our Markov type model can be used to generate high resolution data over large time horizons. The Markov property of the model results in deviation from the mean being correlated time to time, helping represent the physical phenomena of clouds gradually passing over rather than instantaneously appearing.

Our model is a type of autoregressive-moving-average model (ARMAX) 32. In 33 it is seen ARMAX models preform better than auto-regressive integrated moving average (ARIMA) and in 34 it is shown ARMAX models can produce data similar to real data for local sites in California and Colorado.

Exogenous variables, temperature, and humidity, are included as state variables in addition to the primary variable - solar radiance. Cross correlations between state variables are computed from data. Specifically, we take time-series data of these quantities, denoted \(W(t)\) and normalize this data as,

\[
w_j(t) = \frac{W_j(t) - \mu_j(t)}{\sigma_j(t)},
\]

where \(\mu_j(t)\) is the average historic and clear-sky mean of the variable \(W_j\) at time step \(t\) and \(\sigma_j(t)\) is the standard deviation of variable \(W_j\) at time step \(t\).

The generating process is then given by:

\[
w(t) = A w(t - 1) + B v(t - 1) \quad t = 1, ..., T \tag{34}
\]

where \(w(t) \in \mathbb{R}^2\), \(w(0) = 0\) \(v(t) \sim \mathcal{N}(0, I_{3 \times 3})\),

where the matrices \(A\) and \(B\) are chosen to preserve the lag 0 and lag 1 cross-correlations seen in the collected data. Specifically, we can compute these matrices as (31)

\[
A = M_1 M_0^{-1} \quad B B^T = M_0 - M_1 M_0^{-1} M_1^T, \tag{35}
\]

where \(M_i\) is the i-lag cross correlation matrix. So \((M_i)_{m,n} = \rho_i(m, n)\) where \(\rho_i(m, n)\) is the cross-correlation coefficient between variables \(m\) and \(n\) with variable \(n\) lagged by \(i\) time steps. Then, adding back in the mean and deviation, we obtain the power supplied by solar at time step \(k\) as

\[q_k = w_1(k) \sigma_1(k) + \mu_1(k)\]

Figure 7 shows simulated irradiance data from our solar model when compared to actual recorded irradiance data. For this numerical implementation the mean and standard deviation, \((\mu_i(t))_{0 \leq t \leq T}\) and \((\sigma_i(t))_{0 \leq t \leq T}\), were calculated using data from Wunderground for a weather station in Tempe, AZ on October
this model into our battery scheduling optimization problems. Next we incorporate the solar generation model giving an output similar to what is observed in real data. Figure 6 demonstrates a simulation of using the feasible policy obtained via augmenting the state and trajectory map respectively defined in Definition 22. $\psi_{f,0}(\pi, [e_0,0], T, [\nu]_{T-1})$ can be calculated from weather data using equations (35); and all constants are found in Table II.

B. Numerically Solving the Stochastic Battery Scheduling Problem

Using the state augmentation procedure in Section III on the stochastic battery scheduling problem $D_s(0,[e_0,0])$ (36), we may find an optimization problem of the form $Q(t_0,x_0)$ (21) such that the optimal policy for $D_s(0,[e_0,0])$ can be constructed form the optimal policy of $Q(t_0,x_0)$. We may then construct the approximated optimization problem $Q_{s,k}(t_0,k)$ (31) and solve it using Bellman’s equation (29). From the optimal policy of $Q_{s,k}(t_0,k)$ we then construct a feasible policy for $Q(t_0,x_0)$ using (32). Figure 6 demonstrates a simulation of using the feasible policy obtained via augmenting and approximating the stochastic battery scheduling problem with a reasonably selected family of compact state spaces, $\{H_{s,k}\}_{0\leq s\leq T}$, and discretization level $k = 10$. To simply computation we used a one state version of our solar model (34) and used interpolation while solving Bellman’s equation. As seen in the figure this solar generation model gives an output similar to what is observed in real data. Next we incorporate this model into our battery scheduling optimization problems.

Stochastic Battery Scheduling

We now modify Problem $D(0,e_0)$ (14) to give a stochastic version of the battery scheduling problem $D_s(0,[e_0,0])$,

$$\text{arg min}_{\pi \in \Pi} \mathbb{E}_{[\nu]_{T-1}}\left[ J_E(\Phi_{f,0}(\pi, [e_0,0], T, [\nu]_{T-1})) + J_D(\Phi_{f,0}(\pi, [e_0,0], T, [\nu]_{T-1})) \right]$$

subject to: $\psi_{f,0}(\pi, [e_0,0], T, [\nu]_{T-1}) \in E_t \times \mathbb{R}^2$ for $t = 0,\ldots,T$

$\pi(x,t) \in U_t$ and $\nu(t) \in \mathbb{R}^2 \sim \mathcal{N}(0,I_{3x3}) \forall x \in X, \forall t = 0,\ldots,T - 1$,

where $J_E$ is the ToU cost function and $J_D$ is the demand charge found in Section IV-A, $f([e,w],u,t,v) = \frac{\alpha(e + \eta u \Delta t)}{A w + B v}$; $E_t = [\xi,\bar{x}]$ and $U_t = [\underline{u},\bar{u}]$ for all $t \in \{0,\ldots,T\}$; $\psi_{f,0}$ and $\Phi_{f,0}$ are the state and trajectory map respectively defined in Definition 22. $[\nu]_{T-1} = [\nu(0),\ldots,\nu(T-1)] \in \mathbb{R}^{3 \times (T)}$; matrices $A$ and $B$ are calculated from weather data using equations (35); and all constants are found in Table II.
expected the battery charges during the on peak times and conservatively discharges during the off-peak times. The solar data generated from this run were then used as input to the deterministic algorithm in order to compare performance. As anticipated, the deterministic case performs better than the stochastic case.

IX. CONCLUSION

In this paper we have proposed a generalized formulation of the dynamic programming problem and shown that if the objective function is forward separable, such problems may reformulated using state augmentation with an equivalent DP problem with additively separable objective function. Furthermore, we have defined a class of functions, called naturally forward separable functions, such that DP problems with an objective function of this class can be tractably solved using state augmentation. Moreover, we have shown that the problem of optimal scheduling of battery storage in the presence of combined demand and time-of-use charges is a special case of this class of forward separable dynamic programming problems. We have further extended these results to stochastic dynamic programming with a forward separable objective. The proposed algorithms were demonstrated on a battery scheduling problem using first a deterministic and then Gauss-Markov model for solar generation and load.

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