Normal Representation of Hyperplane Arrangements over an Ordered Field and Convex Positive Bijections

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Abstract. In this article we prove in main Theorem A that any hyperplane arrangement \((\mathcal{H}_m^n)^F\) (Definition 2.2) over an ordered field \(F\) (Definition 2.1) with the associated normal system \(\mathcal{N}\) (Definitions 2.8, 2.10) can be represented isomorphically (Definition 2.13) by a hyperplane arrangement \((\tilde{\mathcal{H}}_m^n)^F\) with a given associated normal system \(\tilde{\mathcal{N}}\) if and only if the normal systems \(\mathcal{N}\) and \(\tilde{\mathcal{N}}\) are isomorphic, that is, there is a convex positive bijection (Definition 2.12) between a pair of associated sets of normal antipodal pairs of \(F\)-vectors (Note 2.9) of \(\mathcal{N}\) and \(\tilde{\mathcal{N}}\). In particular in dimension two any line arrangement can be represented isomorphically by lines with any given set of distinct slopes of the same cardinality. Also as a consequence we have a coarse classification where we show that there is a bijection between isomorphism classes of \(K\)-hyperplane arrangements of a fixed cardinality up to translation of any hyperplane and isomorphism classes on \(K\)-normal systems of the same fixed cardinality for any dense subfield \(K \subset F\) (Theorem 10.2 and Note 10.3).

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1. Introduction and a brief survey

The theory of hyperplane arrangements is a well studied and vast subject. There are many view points and perspectives on this subject. The literature survey of this field consists of a lot of very good open problems. From a theoretical view point A. Dimca [1], P. Orlik & H. Terao [7] give an accessible

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introduction to this subject to those who are interested in algebraic geometry and algebraic topology. In the context of arrangements, matroids form combinatorial abstractions of vector configurations and hyperplane arrangements. E. Katz [4] gives a survey of theory of matroids aimed at algebraic geometers. From a computational view point problem nine in [8], [9] by S. Smale, is the following well known open question in this subject for the past two and a half decades.

**Question 1.1.** Does there exist a strongly polynomial time algorithm to decide the feasibility of the linear system of inequalities

\[ Ax \geq b \]

over the field of rational numbers?

Also Survey [6] by N. Megiddo mentions about the various computational aspects with a footnote mentioning the relevance of the subject to economists as well due to its vast applicability.

Here in this article we look at the classification of hyperplane arrangements (in general position, refer to Definition 2.2) over the field of rationals and real numbers or more generally over an ordered field \( F \) (refer to Definition 2.1). The textbooks S. Lang [5] and N. Jacobson [2], [3] give a basic introduction to such ordered fields. This type of field is also briefly mentioned in Survey [6] on page 229 from an algebraic point of view. The association of invariants to hyperplane arrangements for the purpose of classification of the same is a well established method over various fields like \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \) and finite fields \( \mathbb{F}_q \) where \( q \) is a prime power. By associating invariants to hyperplane arrangements over an ordered field \( F \), here, we give a criterion as to when a hyperplane arrangement is represented isomorphically by a given set of normals, more precisely, by a normal system (refer to Definition 2.8), where we prove main Theorem A which is stated in Section 2. The result that is proved in this article is new and uses techniques from geometry of space and spatial arrangement of points in the field of linear algebra, convex geometry and theory of polytopes. In the next section we state the main result.

2. Definitions and statement of the main result

We begin the section with a few definitions before we can state main Theorem A of this article.

**Definition 2.1.**
A totally ordered field \((F, \leq)\) satisfying the following two properties

- P1: If \( x, y, z \in F \) then \( x \leq y \Rightarrow x + z \leq y + z \).
- P2: If \( x, y \in F \) then \( x \geq 0, y \geq 0 \Rightarrow xy \geq 0 \).

is simply called an ordered field for the sake of convenience. For example any subfield of \( \mathbb{R} \) is an ordered field with the induced ordering from the field of reals.
Definition 2.2 (A Hyperplane Arrangement).
Let \( m, n \) be positive integers. Let \( \mathbb{F} \) be a field. We say a set
\[
(\mathcal{H}^m_n) = \{H_1, H_2, \ldots, H_n\}
\]
of \( n \) hyperplanes in \( \mathbb{F}^m \) forms a hyperplane arrangement if
- Condition 1: For \( 1 \leq r \leq m, 1 \leq i_1 < i_2 < \ldots < i_r \leq n \) we have
  \[
  \dim_{\mathbb{F}}(H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_r}) = m - r \text{ (as an affine space)}.
  \]
- Condition 2: For \( r > m, 1 \leq i_1 < i_2 < \ldots < i_r \leq n \) we have
  \[
  H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_r} = \emptyset.
  \]

By a hyperplane arrangement, we always mean in general position, (that is, with Conditions 1,2), in this article.

Definition 2.3 (A Bounded/An Unbounded Region).
Let \( \mathbb{F} \) be an ordered field. Let \( m, n \) be positive integers. Let
\[
(\mathcal{H}^m_n) = \{H_1, H_2, \ldots, H_n\}
\]
be a hyperplane arrangement where an equation for \( H_i \) is given by
\[
\sum_{j=1}^{m} a_{ij} x_i = b_i, \text{ with } a_{ij}, b_i \in \mathbb{F}, 1 \leq j \leq m, 1 \leq i \leq n.
\]
Then a polyhedral region is defined to be a set of solutions for any choice of \( n \) inequalities as follows.
\[
\{(x_1, x_2, \ldots, x_m) \in \mathbb{F}^m \mid \sum_{j=1}^{m} a_{ij} x_i \leq, \geq b_i, 1 \leq i \leq n\}.
\]
A region \( R \) is unbounded if there exists \( v, u \in R \) such that \( v + t(u - v) \in R \) either for all \( t \geq 0 \) or for all \( t \leq 0 \). Otherwise \( R \) is said to be bounded.

Note 2.4. There are \( 2^n \) choices of inequalities for the regions and however only a few of the regions are actually non-empty as given by the following theorem whose proof is well known in the literature on hyperplane arrangements.

In this article, from now on, a polyhedral region means a non-empty polyhedral region.

Theorem 2.5. Let \( \mathbb{F} \) be an ordered field. Let \( n, m \) be positive integers. Let
\[
(\mathcal{H}^m_n) = \{H_1, H_2, \ldots, H_n\}
\]
be a hyperplane arrangement. Then there are
- \( \sum_{i=0}^{m} \binom{n}{i} \) polyhedral regions,
- \( \binom{n-1}{m-1} \) bounded polyhedral regions and
- \( \sum_{i=0}^{m-1} \binom{n}{i} + \binom{n-1}{m-1} \) unbounded polyhedral regions.

Definition 2.6 (Maximally Linearly Independent Set).
Let \( \mathbb{F} \) be a field. Let \( m, n \) be positive integers. We say a set of vectors \( \mathcal{B} = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{F}^m \) is maximally linearly independent if any subset of cardinality at most \( m \) is linearly independent.
Example 2.7. Let $\mathbb{F}$ be a field. Let $\mathcal{H}_m^n = \{H_i : \sum_{j=1}^{m} a_{ij} x_j = b_i, 1 \leq i \leq n\}$ be a hyperplane arrangement. Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be any set containing a normal for each hyperplane in $\mathcal{H}_m^n$. Then $\mathcal{B}$ is maximally linearly independent.

Definition 2.8 (Normal System).
Let $\mathbb{F}$ be an ordered field. Let $\mathcal{N} = \{L_1, L_2, \ldots, L_n\}$ be a finite set of lines passing through the origin in $\mathbb{F}^m$. Let $\mathcal{U} = \{\pm v_1, \pm v_2, \ldots, \pm v_n\}$ be a set of antipodal pairs of $\mathbb{F}$-vectors on these lines. We say $\mathcal{N}$ forms a normal system if the set $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ of $\mathbb{F}$-vectors is maximally linearly independent.

Note 2.9. By an $\mathbb{F}$-vector we mean a vector with coordinates in the field $\mathbb{F}$.

Definition 2.10 (Normal System Associated to a Hyperplane Arrangement).
Let $\mathbb{F}$ be an ordered field. Let $\mathcal{H}_m^n = \{H_i : \sum_{j=1}^{m} a_{ij} x_j = b_i, 1 \leq i \leq n\}$ be a hyperplane arrangement. Then the normal system $\mathcal{N}$ associated to the hyperplane arrangement is given by $\mathcal{N} = \{L_i = \{t(a_{i1}, a_{i2}, \ldots, a_{im}) \in \mathbb{F}^m \mid t \in \mathbb{F}\} \mid 1 \leq i \leq n\}$ and a set of antipodal pairs of normal $\mathbb{F}$-vectors is given by $\mathcal{U} = \{\pm v_1, \ldots, \pm v_n\}$ where $0 \neq v_i \in L_i, 1 \leq i \leq n$. For example we can choose by default $\mathcal{U} = \{\pm(a_{i1}, a_{i2}, \ldots, a_{im}) \in \mathbb{F}^m \mid 1 \leq i \leq n\}$.

Also for the normal system $\mathcal{N}$ if we fix the coefficient matrix $[a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq m} \in M_{n \times m}(\mathbb{F})$ and write the equations for the hyperplanes $H_i : \sum_{j=1}^{m} a_{ij} x_j = b_i, 1 \leq i \leq n$ for some $(b_1, b_2, \ldots, b_n) \in \mathbb{F}^n$ forming a hyperplane arrangement then we say that $\mathcal{H}_m^n$ is given by the normal system $\mathcal{N}$.

Definition 2.11 (Normal Simple Base).
Let $\mathbb{F}$ be an ordered field. Let $\mathcal{N} = \{L_1, L_2, \ldots, L_n\}$ be a finite set of lines passing through the origin in $\mathbb{F}^m$ forming a normal system. Let $\mathcal{U} = \{\pm v_1, \pm v_2, \ldots, \pm v_n\}$ be a set of antipodal pairs of $\mathbb{F}$-vectors on these lines. We say a subset $\mathcal{B} = \{w_1, w_2, \ldots, w_m\} \subset \mathcal{U}$ is a normal simple base if it is a base for $\mathbb{F}^m$ and the only vectors which can be expressed as a non-negative linear combination of the vectors in $\mathcal{B}$ are the vectors in $\mathcal{B}$ themselves.

Definition 2.12 (Convex Positive Bijection and Isomorphic Normal Systems).
Let $\mathbb{F}$ be an ordered field. Let $\mathcal{N}_1 = \{L_1, L_2, \ldots, L_n\}, \mathcal{N}_2 = \{M_1, M_2, \ldots, M_n\}$
be two finite sets of lines passing through the origin in $\mathbb{F}^m$ both of them have the same cardinality $n$ which form normal systems. Let

$$\mathcal{U}_1 = \{\pm v_1, \pm v_2, \ldots, \pm v_n\}, \mathcal{U}_2 = \{\pm w_1, \pm w_2, \ldots, \pm w_n\}$$

be two sets of antipodal pairs of $\mathbb{F}$-vectors on these lines in $\mathcal{N}_1, \mathcal{N}_2$ respectively. We say a bijection $\delta : \mathcal{U}_1 \rightarrow \mathcal{U}_2$ is a convex positive bijection if

$$\delta(-u) = -\delta(u), u \in \mathcal{U}_1$$

and for any base $\mathcal{B} = \{u_1, u_2, \ldots, u_m\} \subset \mathcal{U}_1$ and a vector $u \in \mathcal{U}_1$ we have

$$u = \sum_{i=1}^{m} a_i u_i \text{ with } a_i > 0, 1 \leq i \leq m, \text{ if and only if},$$

$$\delta(u) = \sum_{i=1}^{m} b_i \delta(u_i) \text{ with } b_i > 0, 1 \leq i \leq m.$$  

We say two normal systems are isomorphic if there exists a convex positive bijection between their corresponding sets of antipodal pairs of normal $\mathbb{F}$-vectors.

**Definition 2.13 (Isomorphism Between Two Hyperplane Arrangements).**

Let $\mathbb{F}$ be an ordered field. Let

$$(\mathcal{H}^m_{n})^\mathbb{F}_1 = \{H^1_1, H^1_2, \ldots, H^1_n\}, (\mathcal{H}^m_{n})^\mathbb{F}_2 = \{H^2_1, H^2_2, \ldots, H^2_n\}$$

be two hyperplane arrangements in $\mathbb{F}^m$. We say a map $\phi : (\mathcal{H}^m_{n})^\mathbb{F}_1 \rightarrow (\mathcal{H}^m_{n})^\mathbb{F}_2$ is an isomorphism between these two hyperplane arrangements if $\phi$ is a bijection between the sets $(\mathcal{H}^m_{n})^\mathbb{F}_1, (\mathcal{H}^m_{n})^\mathbb{F}_2$, in particular on the subscripts, and given $1 \leq i_1 < i_2 < \ldots < i_{m-1} \leq n$ with lines

$$L = H^1_{i_1} \cap H^1_{i_2} \cap \ldots \cap H^1_{i_{m-1}}, M = H^2_{\phi(i_1)} \cap H^2_{\phi(i_2)} \cap \ldots \cap H^2_{\phi(i_{m-1})},$$

the order of vertices of intersection on the lines $L, M$ agree via the bijection induced by $\phi$ again on the sets of subscripts of cardinality $m$ (corresponding to the vertices on $L$) containing $\{i_1, i_2, \ldots, i_{m-1}\}$ and (corresponding to the vertices on $M$) containing $\{\phi(i_1), \phi(i_2), \ldots, \phi(i_{m-1})\}$. There are four possibilities of pairs of orders and any one pairing of orders out of the four pairs must agree via the map induced by $\phi$.

**Note 2.14.** If there is an isomorphism between two hyperplane arrangements $(\mathcal{H}^m_{n})^\mathbb{F}_i, i = 1, 2$ then there exists a piecewise linear bijection of $\mathbb{F}^m$ to $\mathbb{F}^m$ which takes one arrangement to another using suitable triangulation of polyhedrality. For obtaining a piecewise linear isomorphism extension from vertices to the one dimensional skeleton (refer to Definition 11.1) of the arrangements, further subdivision is not needed.

Here we mention a theorem on preservation of central points without proof as this is a standard theorem.

**Theorem 2.15 (A Theorem on Preservation of Central Points).**

Let $\mathbb{F}$ be an ordered field. Let

$$(\mathcal{H}^m_{n})^\mathbb{F}_1 = \{H^1_1, H^1_2, \ldots, H^1_n\}, (\mathcal{H}^m_{n})^\mathbb{F}_2 = \{H^2_1, H^2_2, \ldots, H^2_n\}$$
be two hyperplane arrangements in \( \mathbb{F}^m \). Let \( \phi : (\mathcal{H}_m^m)_{\mathbb{F}_1} \rightarrow (\mathcal{H}_m^m)_{\mathbb{F}_2} \) be a bijection. Using the subscripts let \( \phi^{\text{vert}} : \text{Vert}(\mathcal{H}_m^m)_{\mathbb{F}_1} \rightarrow \text{Vert}(\mathcal{H}_m^m)_{\mathbb{F}_2} \) be the induced map on the vertices of the arrangements. Then \( \phi \) is an isomorphism of the arrangements if and only if for any three vertices on any line of the arrangement in \( (\mathcal{H}_m^m)_{\mathbb{F}_1} \) the map \( \phi^{\text{vert}} \) preserves the central vertex.

We state the main theorem of this article and another equivalent and more symmetric one.

**Theorem A (Normal Representation Theorem: Main Theorem).**
Let \( \mathbb{F} \) be an ordered field. Let \( \mathcal{N}_1 = \{L_1, L_2, \ldots, L_n\} \) be a normal system of cardinality \( n \) and \( \mathcal{U}_1 \) be a set of antipodal pairs of \( \mathbb{F} \)-vectors of this normal system. Let \( (\mathcal{H}_n^m)_{\mathbb{F}} \) be a hyperplane arrangement in \( \mathbb{F}^m \) and \( \mathcal{N}_2 \) be the normal system associated to \( (\mathcal{H}_n^m)_{\mathbb{F}} \) with \( \mathcal{U}_2 \) a set of antipodal pairs of normal \( \mathbb{F} \)-vectors of the normal system \( \mathcal{N}_2 \). Then the hyperplane arrangement \( (\mathcal{H}_n^m)_{\mathbb{F}} \) with normal system \( \mathcal{N}_2 \) can be represented isomorphically by another hyperplane arrangement with normal system \( \mathcal{N}_1 \) if and only if there exists a convex positive bijection \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \).

**Theorem B (Normal Representation Theorem: Symmetric Form).**
Let \( \mathbb{F} \) be an ordered field. Let \( (\mathcal{H}_n^m)_{\mathbb{F}_1}, (\mathcal{H}_n^m)_{\mathbb{F}_2} \) be two hyperplane arrangements with sets \( \mathcal{U}_1, \mathcal{U}_2 \) as antipodal pairs of normal \( \mathbb{F} \)-vectors. Then there exists an isomorphism between the hyperplane arrangements up to translation of any hyperplane if and only if there exists a convex positive bijection between \( \mathcal{U}_1, \mathcal{U}_2 \).

Restating the theorem, we have that, if two hyperplane arrangements \( (\mathcal{H}_n^m)_{\mathbb{F}}, (\mathcal{H}_n^m)_{\mathbb{F}} \) are isomorphic then their associated normal systems \( \mathcal{N} \) and \( \tilde{\mathcal{N}} \) are isomorphic. Conversely, if we have two hyperplane arrangements \( (\mathcal{H}_n^m)_{\mathbb{F}}, (\mathcal{H}_n^m)_{\mathbb{F}} \), whose associated normal systems \( \mathcal{N} \) and \( \tilde{\mathcal{N}} \) are isomorphic, then, there exist translates of each of the hyperplanes in the hyperplane arrangement \( (\mathcal{H}_n^m)_{\mathbb{F}} \), giving rise to a translated hyperplane arrangement \( (\mathcal{H}_n^m)_{\tilde{\mathbb{F}}} \), such that, this arrangement and \( (\mathcal{H}_n^m)_{\mathbb{F}} \) are isomorphic.

**Note 2.16.** As a consequence in dimension two, since there always exists a convex bijection between any two sets of antipodal pairs of \( \mathbb{F} \)-vectors of the same cardinality we have that any line arrangement can be represented by a set of lines with any given set of distinct slopes of the same cardinality.

Now we mention an important observation which is the crux in proving Theorem B later. This observation is a straightforward observation in the plane \( \mathbb{F}^2 \) where \( \mathbb{F} \) is an ordered field.

**Observation 2.17 (On the Central Point of Intersection).**
If we have three pairwise intersecting generic lines, intersecting a fourth generic line at infinity in a plane (refer to Definition 6.1), and if we express a normal of the fourth line as a positive linear combination of any pair of normals of the other three lines then the normal direction of the line corresponding to only the central point of intersection on the fourth line reverses its sign/direction (refer to Figure 1).

3. The structure of the paper

In this section we mention the structure of the paper which includes the main result that is proved. In Section 2 we mention the required definitions, the important definitions being the definition of normal system, convex positive bijections, isomorphism of normal systems and isomorphism of hyperplane arrangements in order to state main Theorem A and an equivalent Theorem B of the article.

Later in Section 4 we summarize the method to prove main Theorem A. In Section 5 using the notion of concurrency arrangements associated to hyperplane arrangements we reduce main Theorem A to Theorem B. This reduction is the initial step towards proving Theorem A.
In Section 6 we introduce hyperplanes at infinity and prove extension Theorem 6.4 for an isomorphism between two hyperplane arrangements when a hyperplane at infinity is added to each one of them where the induced co-dimension one arrangements on the hyperplanes at infinity are isomorphic by an isomorphism induced on the subscripts. The proof of Theorem 6.4 uses the base case Lemma 6.3 and the fact that the skeleton of $k$-dimensional planes for $k \geq 1$ of a hyperplane arrangement is connected (in the point set topological sense) in zariski topology over infinite fields and in particular over an ordered field $\mathbb{F}$ which is proved (refer to Theorem 11.2) in the appendix. Section 7 is an elementary section which gives the existence of orthogonal projections over ordered fields even though square roots of a general positive element need not be in the field. This is useful later to prove Theorem B. In Section 8 we see that the proof of Theorem B relies on the Observation 2.17. In Section 9 we construct examples in three dimensions of normal systems consisting of six lines which are not isomorphic which is contrary to the two dimensional intuition of line arrangements. In Section 10 we prove Theorem 10.2 where we give an equivalent criterion in terms two infinity arrangements (refer to Definition 10.1) for Theorem B. In topology appendix Section 11 we prove some point set topological facts concerning zariski topology of hyperplane arrangements. In fact we prove Theorem 11.2 that the skeleton of $k$-dimensional planes for $k \geq 1$ of a hyperplane arrangement is connected (in the point set topological sense) in zariski topology over infinite fields. This completes the summary about the structure of various sections of this article.

4. Summary of the method to prove main Theorem A

Before the summary, we define another important invariant, the concurrency arrangement associated to a hyperplane arrangement.

Definition 4.1 (Concurrency Arrangement).

Let

$$(\mathcal{H}_n^m)^F = \{H_1, H_2, \ldots, H_n\}$$

be a hyperplane arrangement of $n$ hyperplanes in an $m$-dimensional space over the ordered field $\mathbb{F}$. Let the equation for $H_i$ be given by

$$\sum_{j=1}^{m} a_{ij}x_j = b_i, \text{ with } a_{ij}, b_i \in \mathbb{F}, 1 \leq j \leq m, 1 \leq i \leq n.$$

For every $1 \leq i_1 < i_2 < \ldots < i_{m+1} \leq n$ consider the hyperplane $M_{\{i_1, i_2, \ldots, i_{m+1}\}}$ passing through the origin in $\mathbb{F}^n$ in the variables $y_1, y_2, \ldots, y_n$ whose equation
is given by

$$\text{Det} \begin{pmatrix} a_{i11} & a_{i12} & \cdots & a_{i1(m-1)} & a_{i1m} & y_{i1} \\ a_{i21} & a_{i22} & \cdots & a_{i2(m-1)} & a_{i2m} & y_{i2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{im-11} & a_{im-12} & \cdots & a_{im-1(m-1)} & a_{im-1m} & y_{im-1} \\ a_{im1} & a_{im2} & \cdots & a_{im(m-1)} & a_{imm} & y_{im} \\ a_{im+11} & a_{im+12} & \cdots & a_{im+1(m-1)} & a_{im+1m} & y_{im+1} \end{pmatrix} = 0$$

Then the associated concurrency arrangement of hyperplanes passing through the origin in \(\mathbb{F}^n\) is given by

$$(C_n^m)^F = \{M_{i_1,i_2,\ldots,i_{m+1}} \mid 1 \leq i_1 < i_2 < \ldots < i_{m+1} \leq n\}.$$  

**Note 4.2.** Even though the definition of hyperplanes of the concurrency arrangement involves the coefficients of the variables \(x_i, 1 \leq i \leq m\) we can pick and fix any one set of equations for the hyperplanes \(H_i, 1 \leq i \leq n\) of the hyperplane arrangement to define the concurrency arrangement.

**Note 4.3.** In general the normal lines of these hyperplanes need not form a normal system. However they will be distinct as they correspond to different subsets of \(\{1,2,\ldots,n\}\) of cardinality \(m+1\) with these \((m+1)\)-coefficients non-zero and the remaining \((n-m-1)\) are zero coefficients.

**Note 4.4 (Convention: Fixing the coefficient matrix of any hyperplane arrangement for a fixed given normal system).** Let \(\mathcal{N} = \{L_1,L_2,\ldots,L_n\}\) be a normal system in \(\mathbb{F}^m\). Let \(U = \{\pm(a_{i1},a_{i2},\ldots,a_{im}) \mid (a_{i1},a_{i2},\ldots,a_{im}) \in L_i, 1 \leq i \leq n\}\) be a set of antipodal pairs of vectors of the normal system \(\mathcal{N}\). We fix the coefficient matrix \([a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq m} \in M_{n \times m}(\mathbb{F})\). Let \((\mathcal{H}_n^m)^F = \{H_1,H_2,\ldots,H_n\}\) be any hyperplane arrangement with the normal system \(\mathcal{N}\). When we write equations for the hyperplane \(H_i\), we use the fixed coefficient matrix and write

\[H_i : \sum_{j=1}^{m} a_{ij}x_j = b_i \text{ for some } b_i \in \mathbb{F}.\]

With this coefficient matrix we define the concurrency arrangement which depends only on the normal system. Two hyperplane arrangements with the same normal system gives two points \((b_1,b_2,\ldots,b_n),(c_1,c_2,\ldots,c_n)\). If these vectors lie in the same cone of the concurrency arrangement then the hyperplane arrangements are isomorphic by an isomorphism which is trivial on subscripts. In general if the arrangements are isomorphic by such an isomorphism we say \((b_1,b_2,\ldots,b_n)\) is isomorphic to \((c_1,c_2,\ldots,c_n)\). For example \((b_1,b_2,\ldots,b_n)\) is isomorphic to \(-(b_1,b_2,\ldots,b_n)\) even though they lie in opposite cones.

**Note 4.5.** We note that regions of the concurrency arrangement \((C_n^m)^F\) are all convex conical, unbounded and there are at most \(\sum_{i=0}^{n} (\binom{n+i}{m+1}) - (\binom{n+i}{m+1}-1)\) such regions using Theorem 2.5.
We summarize the steps involved to prove main Theorem A.

1. First we consider a hyperplane arrangement 
\[ (\mathcal{H}_m^n)^F \]
and associate to it a concurrency arrangement (refer to Definition 4.1) 
\[ (\mathcal{C}^n_{\binom{n}{m+1}})^F \]
of hyperplanes through origin in \( F^n \) whose set of \( \binom{n}{m+1} \) normals are just only distinct. This concurrency arrangement has all its convex conical regions unbounded and whose cardinality is bounded above by 
\[
\sum_{i=0}^{n} \left( \binom{n}{i} \right) - \left( \frac{n+1}{m+1} \right).
\]

2. Let 
\[ (\mathcal{H}_n^m)_1, (\mathcal{H}_n^m)_2 \]
be two isomorphic hyperplane arrangements. Let 
\[ (\mathcal{C}^n_{\binom{n}{m+1}})^F_1, (\mathcal{C}^n_{\binom{n}{m+1}})^F_2 \]
be their corresponding concurrency arrangements. We prove (refer to Theorem 5.2) that there exists a bijection of the regions of the concurrency arrangements which takes the conical region corresponding to \( (\mathcal{H}_n^m)_1 \) to the conical region corresponding to \( (\mathcal{H}_n^m)_2 \) such that under the bijection corresponding regions give rise to isomorphic hyperplane arrangements in \( F^m \).

3. We reduce the Normal Representation Theorem A(NRT) to the symmetric one Theorem B by proving that NRT holds if there exists an isomorphism between some two hyperplane arrangements with \( U_1, U_2 \) as their sets of antipodal pairs of normal \( F \)-vectors.

4. Finally we prove that such an isomorphism exists if and only if there exists a convex positive bijection between their sets of antipodal pairs of normal \( F \)-vectors. The proof of this final step is somewhat involved. It is given in Sections \([6-8]\).

This completes the summary of the method to prove main Theorem A.

5. Concurrency arrangement of a hyperplane arrangement

In this section we mainly prove Corollary 5.3 with some preliminary observations and Theorem 5.2 using concurrency arrangements associated to a hyperplane arrangement.
5.1. Passing to an adjacent cone by moving through a hyperplane which gives rise to an \( m \)-dimensional simplex polyhedrality in the concurrency arrangement

We begin with a definition.

**Definition 5.1 (Simplex Polyhedrality).**

Let

\[
\left( \mathcal{H}_n^m \right)^F = \{ H_1, H_2, \ldots, H_n \}
\]

be a hyperplane arrangement of \( n \) hyperplanes in an \( m \)-dimensional space over the ordered field \( \mathbb{F} \). We say a set of \( m + 1 \) hyperplanes

\[
\{ H_{i_1}, H_{i_2}, \ldots, H_{i_{m+1}} \mid 1 \leq i_1 < i_2 < \ldots < i_m < i_{m+1} \leq n \}
\]

give rise to an \( m \)-dimensional simplex polyhedrality of the arrangement if the equations of these \( m + 1 \) hyperplanes gives rise to a bounded polyhedral region (refer to Definition 2.3) of the arrangement.

Now we prove the following theorem.

**Theorem 5.2.** Let \( \mathbb{F} \) be an ordered field. Let \( \left( \mathcal{H}_n^m \right)^F_1 = \{ H_1^1, H_2^1, \ldots, H_n^1 \}, \left( \mathcal{H}_n^m \right)^F_2 = \{ H_1^2, H_2^2, \ldots, H_n^2 \} \) be two arrangements which are isomorphic by an isomorphism which is identity on the subscripts. Let \( \left( C_{(m+1)}^n \right)^F_1, \left( C_{(m+1)}^n \right)^F_2 \) be their associated concurrency arrangements respectively and let the constant coefficients be given by

\[
(b_1, b_2, \ldots, b_n), (c_1, c_2, \ldots, c_n)
\]

respectively which lie in the interior of two cones of the concurrency arrangements \( \left( C_{(m+1)}^n \right)^F_1, \left( C_{(m+1)}^n \right)^F_2 \) respectively. Suppose the subscripts \( 1 \leq i_1 < i_2 < \ldots < i_m < i_{m+1} \leq n \) gives rise to an \( m \)-dimensional simplex polyhedrality (refer to Definition 5.1) of both the arrangements. Let the constant coefficients \( (b_1, b_2, \ldots, b_n), (c_1, c_2, \ldots, c_n) \) which lie in the interior of the two cones be moved to the interior of their adjacent cones passing through single boundary hyperplanes (co-dimension one) corresponding to \( 1 \leq i_1 < i_2 < \ldots < i_m < i_{m+1} \leq n \) to new coefficients \( (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n), (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n) \) giving rise to two new hyperplane arrangements \( \left( \mathcal{H}_n^m \right)^F_1, \left( \mathcal{H}_n^m \right)^F_2 \) respectively. Then the hyperplane arrangements \( \left( \mathcal{H}_n^m \right)^F_1, \left( \mathcal{H}_n^m \right)^F_2 \) are isomorphic by an isomorphism which is also identity on the subscripts.

**Proof.** Here in both the arrangements the following similar change in the order of vertices of intersection on the one dimensional lines occur.

Let \( A = \{ j_1, j_2, \ldots, j_{m-1} \} \subset \{ i_1 < i_2 < \ldots < i_m < i_{m+1} \} \) be any subset of cardinality \( (m - 1) \). Let \( A^c = \{ i_1, i_2, \ldots, i_m, i_{m+1} \} \setminus A = \{ j_1, j_m+1 \} \) be the corresponding subset of cardinality two. Then for each \( j = 1, 2 \), on the line \( \bigcap_{i \in A} H_i^j \), there is a swap of points

\[
\bigcap_{i \in A} H_i^j \cap H_{j_m}^j, \bigcap_{i \in A} H_i^j \cap H_{j_{m+1}}^j.
\]
Hence the two new hyperplane arrangements $(\mathcal{H}_n^m)^1_F, (\mathcal{H}_n^m)^2_F$ are isomorphic by an isomorphism which is also identity on the subscripts. This proves the theorem.

We state the corollary below.

**Corollary 5.3.** With the notations in Theorem A, Theorem B, Theorem 5.2 we have that in order to prove Theorem A, it is enough to prove Theorem B.

**Proof.** Let $((b_1, b_2, \ldots, b_n), (c_1, c_2, \ldots, c_n)$ be the constant coefficients of the two arrangements $(\mathcal{H}_n^m)^1_F = \{H_1^1, H_1^2, \ldots, H_1^n\}, (\mathcal{H}_n^m)^2_F = \{H_2^1, H_2^2, \ldots, H_2^n\}$ which are isomorphic by an isomorphism which is identity on the subscripts. Let $(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n)$ be the constant coefficients of the new arrangement $(\mathcal{H}_n^m)^1_F$. Then this arrangement is obtained by applying repeated applications of swaps of points on lines in the proof of Theorem 5.2 to suitable $m$-dimensional simplex polyhedralities at each stage. We apply similar sequence of changes to the arrangement $(\mathcal{H}_n^m)^2_F$ to obtain a new arrangement $(\mathcal{H}_n^m)^2_F$ with constant coefficients $(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n)$ preserving the property that at every stage in the sequence, the pair of arrangements are isomorphic by an isomorphism which is also identity on the subscripts. This proves the corollary.

**5.2. Number of $m$-dimensional simplex polyhedralities of a hyperplane arrangement**

Here we mention a note that given a hyperplane arrangement how many $m$-dimensional simplex polyhedralities exist in the arrangement.

**Note 5.4 (Number of Simplex Polyhedralities of a Hyperplane Arrangement).**

Let $(\mathcal{H}_n^m)^F = \{H_i : \sum_{j=1}^{m} a_{ij}x_j = c_i, 1 \leq i \leq n, a_{ij}, c_i \in \mathbb{F}\}$ be a hyperplane arrangement in $\mathbb{F}^m$. Let $(\mathcal{C}^n_{\binom{n}{m+1}})^F$ be its associated concurrency arrangement in $\mathbb{F}^m$. Let $C$ denote the convex cone containing the point $(c_1, c_2, \ldots, c_n)$ of $(\mathcal{C}^n_{\binom{n}{m+1}})^F$ in its interior.

The number of simplex polyhedralities of the hyperplane arrangement $(\mathcal{H}_n^m)^F$ is precisely equal to the number of co-dimension one boundary hyperplanes of $\mathbb{F}^m$ in the concurrency arrangement of the convex cone $C$ containing $(c_1, c_2, \ldots, c_n)$.

**6. Hyperplanes at infinity and an extension theorem**

Here in this section we prove an extension theorem for an isomorphism between two hyperplane arrangements when a hyperplane at infinity is added to each arrangement. We start with the definition of a hyperplane at infinity.

**Definition 6.1.** Let $\mathbb{F}$ be an ordered field. Let $(\mathcal{H}_n^m)^F$ be a hyperplane arrangement. We say a hyperplane $H \subset \mathbb{F}^m$ is a hyperplane at infinity with respect to $(\mathcal{H}_n^m)^F$ if all the bounded intersections of the arrangement $(\mathcal{H}_n^m)^F$, that is,
the zero dimensional vertices of intersection of the arrangement lie only on one side of \( H \) (possibly including \( H \) itself).

**Note 6.2.** Given a generic normal pair of antipodal \( \mathbb{F} \)-vectors there exist two parallel hyperplanes at infinity with given normal vectors on either side of the bounded intersections of the hyperplane arrangement.

We prove below an extension theorem for an isomorphism between two hyperplane arrangements which allows us to extend isomorphisms when we add hyperplanes at infinity under certain conditions. Theorem 6.4 is used later in proving Theorem B.

First we state a lemma before stating Theorem 6.4.

**Lemma 6.3.** Let \( \mathbb{F} \) be an ordered field. Let \( \mathcal{L}_3^2 = \{L_1, L_2, L_3\}, \tilde{\mathcal{L}}_3^2 = \{\tilde{L}_1, \tilde{L}_2, \tilde{L}_3\} \) be two line arrangements, (that is, sets of three generic lines) in the plane \( \mathbb{F}^2 \). Let \( L_4, L'_4 \) be two parallel lines at infinity in \( \mathbb{F}^2 \) on either side of the bounded set of points of intersection \( \{L_1 \cap L_2, L_2 \cap L_3, L_1 \cap L_3\} \), giving rise to line arrangements \( \mathcal{L}_4^2 = \{L_1, L_2, L_3, L_4\} \) and \( \tilde{\mathcal{L}}_4^2 = \{\tilde{L}_1, \tilde{L}_2, \tilde{L}_3, \tilde{L}_4\} \) respectively. Let \( \hat{L}_4, \hat{L}'_4 \) be two parallel lines at infinity in \( \mathbb{F}^2 \) on either side of the bounded set of points of intersection \( \{\hat{L}_1 \cap \hat{L}_2, \hat{L}_2 \cap \hat{L}_3, \hat{L}_1 \cap \hat{L}_3\} \), giving rise to line arrangements \( \hat{\mathcal{L}}_4^2 = \{\hat{L}_1, \hat{L}_2, \hat{L}_3, \hat{L}_4\} \) and \( \hat{\tilde{\mathcal{L}}}_4^2 = \{\tilde{\hat{L}}_1, \tilde{\hat{L}}_2, \tilde{\hat{L}}_3, \tilde{\hat{L}}_4\} \) respectively. Then

1. For any \( 1 \leq i \leq 3 \), \( L_i \cap L_4 \) is the central point of intersection on \( L_4 \) among the three points \( L_1 \cap L_4, L_2 \cap L_4, L_3 \cap L_4 \) if and only if \( L_i \cap L'_4 \) is the central point of intersection on \( L'_4 \) among the three points \( L_1 \cap L'_4, L_2 \cap L'_4, L_3 \cap L'_4 \).

2. For any \( 1 \leq i \leq 3 \), \( \tilde{L}_i \cap \tilde{L}_4 \) is the central point of intersection on \( \tilde{L}_4 \) among the three points \( \tilde{L}_1 \cap \tilde{L}_4, \tilde{L}_2 \cap \tilde{L}_4, \tilde{L}_3 \cap \tilde{L}_4 \) if and only if \( \tilde{L}_i \cap \tilde{L}'_4 \) is the central point of intersection on \( \tilde{L}'_4 \) among the three points \( \tilde{L}_1 \cap \tilde{L}'_4, \tilde{L}_2 \cap \tilde{L}'_4, \tilde{L}_3 \cap \tilde{L}'_4 \).

3. Suppose for some particular \( 1 \leq j \leq 3 \) we have \( L_j \cap L_4 \) is the central point of intersection on \( L_4 \) among the three points \( L_1 \cap L_4, L_2 \cap L_4, L_3 \cap L_4 \) and also \( \tilde{L}_j \cap \tilde{L}_4 \) is the central point of intersection on \( \tilde{L}_4 \) among the three points \( \tilde{L}_1 \cap \tilde{L}_4, \tilde{L}_2 \cap \tilde{L}_4, \tilde{L}_3 \cap \tilde{L}_4 \). Then we have

- either \( \mathcal{L}_4^2 \cong \tilde{\mathcal{L}}_4^2 \) and \( \mathcal{L}_4^2 \cong \tilde{\mathcal{L}}_4^2 \) by isomorphisms \( \phi, \psi \) preserving the subscripts \( \{1, 2, 3, 4\} \) of the lines in the arrangements,

- or \( \mathcal{L}_4^2 \cong \tilde{\mathcal{L}}_4^2 \) and \( \mathcal{L}_4^2 \cong \tilde{\mathcal{L}}_4^2 \) by isomorphisms \( \phi, \psi \) preserving the subscripts \( \{1, 2, 3, 4\} \) of the lines in the arrangements.

or equivalently under the isomorphisms, for \( 1 \leq i \leq 3 \), \( L_i \xrightarrow{\phi, \psi} \tilde{L}_i \) pairings occur and

- either \( (L_4 \xrightarrow{\phi} \tilde{L}_4) \) and \( (L'_4 \xrightarrow{\psi} \tilde{L}'_4) \) pairings occur,

- or \( (L_4 \xrightarrow{\phi} \tilde{L}_4) \) and \( (L'_4 \xrightarrow{\psi} \tilde{L}'_4) \) pairings occur.

**Proof.** The proof of the lemma is an immediate observation about central points on all the four lines of each of the line arrangements \( \mathcal{L}_4^2, \mathcal{L}_4^2, \tilde{\mathcal{L}}_4^2, \tilde{\mathcal{L}}_4^2 \).
We illustrate the proof of the lemma with Figure 2. In this figure Fig:I can be taken as the line arrangements $L_4^2, L_4^2'$ and any one of Fig:II, Fig:III, Fig:IV can be taken as line arrangements $\tilde{L}_4^2, \tilde{L}_4^2'$. Finally we obtain isomorphisms as

either $4 \leftrightarrow 4', i \leftrightarrow i, 1 \leq i \leq 3, Fig : IV$

or $4 \leftrightarrow 4', 4' \leftrightarrow 4, i \leftrightarrow i, 1 \leq i \leq 3, Fig : II, Fig : III$

The possibilities can be taken as the base case for extension Theorem 6.4.

Here we state and prove extension Theorem 6.4.

**Theorem 6.4 (Extension Theorem).**

Let $\mathbb{F}$ be an ordered field. Let $m > 1, n > m + 1$ be two positive integers. Let

$$(\mathcal{H}_m^{n-1})^{\mathbb{F}} = \{H_1, H_2, \ldots, H_{n-1}\}, (\mathcal{H}_m^{n-1})^{\bar{\mathbb{F}}} = \{\bar{H}_1, \bar{H}_2, \ldots, \bar{H}_{n-1}\}$$

be two isomorphic hyperplane arrangements by an isomorphism $\phi$ which takes $H_i \rightarrow \bar{H}_i, 1 \leq i \leq n - 1$. Now suppose

$H^1_n, H^2_n$ and $\bar{H}^1_n, \bar{H}^2_n$

are two pairs of parallel hyperplanes at either infinities with respect to the arrangements $(\mathcal{H}_m^{n-1})^{\mathbb{F}}$ and $(\mathcal{H}_m^{n-1})^{\bar{\mathbb{F}}}$. (Hence all the bounded intersections lie
in the hyperplane strip spaces which is in between the two pairs of parallel hyperplanes.) If the induced map $\phi|_{\text{induced}}$ of the isomorphism $\phi$ on the following two hyperplane arrangements
\[
\mathcal{M}_{n-1}^{m-1} = \{H_n^r \cap H_1, H_n^r \cap H_2, \ldots, H_n^r \cap H_{n-1}\},
\]
\[
\mathcal{M}_{n-1}^{m-1} = \{\tilde{H}_n^s \cap \tilde{H}_1, \tilde{H}_n^s \cap \tilde{H}_2, \ldots, \tilde{H}_n^s \cap \tilde{H}_{n-1}\}
\]
is an isomorphism for any one $(r, s) \in \{1, 2\} \times \{1, 2\}$ then the isomorphism $\phi$ extends to an isomorphism $\tilde{\phi}$ on the following two arrangements
\[
\mathcal{H}_{n-1}^m \cup \{H_n^{r_1}\}, \tilde{\mathcal{H}}_{n-1}^m \cup \{\tilde{H}_n^{s_1}\}
\]
which takes
\[
H_i \leftrightarrow \tilde{H}_i, 1 \leq i \leq n-1, H_n^{r_1} \leftrightarrow \tilde{H}_n^{s_1}
\]
for some choice of $(r_1, s_1) \in \{1, 2\} \times \{1, 2\}$. Moreover the isomorphism also extends to an isomorphism for the complementary ordered pair $(r_2, s_2)$ of $(r_1, s_1)$ where $\{r_1, r_2\} = \{s_1, s_2\} = \{1, 2\}$.

Proof. Using the fact that the induced map $\phi|_{\text{induced}}$ is an isomorphism and the original map $\phi$ is an isomorphism we prove that the order of intersections agree on all the one dimensional lines for some choice of $(r_1, s_1) \in \{1, 2\} \times \{1, 2\}$ in the hyperplane arrangements
\[
\mathcal{H}_{n-1}^m \cup \{H_n^{r_1}\}, \tilde{\mathcal{H}}_{n-1}^m \cup \{\tilde{H}_n^{s_1}\}
\]
under the map $\tilde{\phi}$ for a suitable definition of $\tilde{\phi}$. Now we make an important observation.

To find an extension $\tilde{\phi}$ and to prove this theorem we restrict our space of attention to two dimensional planes of interest as follows. Let
\[
1 \leq i_1 < i_2 < \ldots < i_{m-2} \leq n-1 < n, 1 \leq i_{m-1} < i_m < i_{m+1} \leq n-1 < n
\]
and
\[
\{i_{m-1} < i_m < i_{m+1}\} \cap \{i_1 < i_2 < \ldots < i_{m-2}\} = \emptyset.
\]
Let
\[
L_{\{i_1, i_2, \ldots, i_m\}} = H_i \cap \ldots \cap H_{i_m} \cap H_{i_{m-1}}
\]
\[
L_{\{i_1, i_2, \ldots, i_m\}} = H_i \cap \ldots \cap H_{i_m} \cap H_{i_{m-1}}
\]
\[
L_{\{i_1, i_2, \ldots, i_m\}} = H_i \cap \ldots \cap H_{i_m} \cap H_{i_{m-1}}
\]
\[
L_{\{i_1, i_2, \ldots, i_m\}} = H_i \cap \ldots \cap H_{i_m} \cap H_{i_{m-1}}
\]
\[
L_{\{i_1, i_2, \ldots, i_m\}} = H_i \cap \ldots \cap H_{i_m} \cap H_{i_{m-1}}
\]
be the corresponding triples of lines with the corresponding two dimensional planes of interest being
\[
H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_{m-2}}, \tilde{H}_{i_1} \cap \tilde{H}_{i_2} \cap \ldots \cap \tilde{H}_{i_{m-2}}
\]
For \((r, s) \in \{1, 2\} \times \{1, 2\}\) the pairs of parallel hyperplanes at infinity gives rise to lines at infinity in the two dimensional planes of interest respectively. The lines are given by

\[
L^r_{i_1, i_2, \ldots, i_{m-2}, n} = H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_{m-2}} \cap H_r, r \in \{1, 2\}
\]
\[
L^s_{i_1, i_2, \ldots, i_{m-2}, n} = \tilde{H}_{i_1} \cap \tilde{H}_{i_2} \cap \ldots \cap \tilde{H}_{i_{m-2}} \cap \tilde{H}_s, s \in \{1, 2\}
\]

Now consider Figure 3. As we have both isomorphisms \(\phi\) for points of the lines not on the new hyperplanes and \(\phi_{induced}\) for points on the lines of the new hyperplanes \(H_n, \tilde{H}_n\) we can use Lemma 6.3 on each of the two dimensional planes of interest to obtain an isomorphic pairing \((r_1, s_1) \in \{1, 2\} \times \{1, 2\}\) and its complementary pair \((r_2, s_2) \in \{1, 2\} \times \{1, 2\}\).

The only possible ambiguity is whether the pair \((r_1, s_1)\) and the complementary pair \((r_2, s_2)\) is the same for all two dimensional planes of interest. There are two possible choices. \(\{(r_1, s_1), (r_2, s_2)\} = \{(1, 1), (2, 2)\}\) or \(\{(r_1, s_1), (r_2, s_2)\} = \{(1, 2), (2, 1)\}\). To remove this ambiguity we use continuity and connectedness arguments. We use usual topology over the field of reals. Otherwise we use zariski topology over infinite fields in particular over an ordered field \(\mathbb{F}\).
We obtain the same pair \((r_1, s_1)\) and its complementary pair \((r_2, s_2)\) for all two dimensional planes of interest, because, the map which takes union of zariski planes of interest, the skeleton of two dimensional planes of the hyperplane arrangement to the set with discrete topology containing two elements which are complementary extension pairs

\[
\{\{(1, 1), (2, 2)\}, \{(1, 2), (2, 1)\}\}
\]
is continuous. This is because
- it is easy for the reader to note that the map agrees (patches up) on the intersection of two such planes (if they intersect) which is a line of the arrangement, since the isomorphic pairing \((r_1, s_1), (r_2, s_2)\) is the same for all two dimensional zariski planes of interest \(\{i_1 < i_2 < \ldots < i_{m-2}\}\)
- which contains a fixed line of the arrangement say \(\{j_1 < j_2 < \ldots < j_{m-1}\} \supset \{i_1 < i_2 < \ldots < i_{m-2}\}\).
- Inverse image of a single point is a finite union of two dimensional zariski planes and hence it is closed.
- Moreover the union of planes of interest of the arrangement is connected (point set topological sense) in zariski topology (refer to Theorem 11.2 with \(k = 2\)).

In Figure 3 for any choice of plane of interest we have either \((r_1, s_1) = (1, 1)\) or \((r_1, s_1) = (2, 2)\) and not \((1, 2), (2, 1)\). This proves exactly the statement of the theorem.

7. Existence of orthogonal projections over ordered fields

In this section we prove the existence of certain projections which will be useful to the proof of the main theorem in the next section. We note orthogonal projections exist over ordered fields even though square roots of a general positive element need not be in the field. Let \(\mathbb{F}\) be an ordered field. Let \(v^i = (x^i_1, x^i_2, \ldots, x^i_n)^t, 1 \leq i \leq k\) be any finite set of linearly independent vectors in \(\mathbb{F}^n\) for \(k \leq n\) spanning a given subspace. Define a linear transformation \(T\) given as follows.

\[
T : \mathbb{F}^n \longrightarrow \mathbb{F}^k \text{ where } [T]_{k \times n} = [x^j_i]_{1 \leq i \leq n, 1 \leq j \leq k}
\]

Now we have \(\ker(T) = \langle v^i : 1 \leq i \leq k \rangle^\perp\). We have row rank of \(T\) is \(k\).

Since \(\text{row-rank}(T) = \text{col-rank}(T), \text{Rank } + \text{ Nullity} = n\)

we have \(\dim(\ker(T)) = n - k\). Define on \(\mathbb{F}^m\) with \(m > 0\) a positive integer,

\[
\langle v = (x_1, x_2, \ldots, x_m), w = (y_1, y_2, \ldots, y_m) \rangle_{\mathbb{F}^m} = \sum_{i=1}^{m} x_i y_i.
\]

This is a symmetric bilinear form with the property that
- \(\langle v, v \rangle_{\mathbb{F}^m} \geq 0\) for \(v \in \mathbb{F}^m\).
- \(\langle v, v \rangle_{\mathbb{F}^m} = 0 \iff v = 0\).
Then for \( w_1 \in \mathbb{F}^n, w_2 \in \mathbb{F}^k \)
\[
<Tw_1, w_2>_{\mathbb{F}^k} = w_2^tTw_1 = w_1^tT^tw_2 =< w_1, T^tw_2>_{\mathbb{F}^n}.
\]

Now we observe that if \( w_1 \in Ker(T) \iff < w_1, T^tw_2>_{\mathbb{F}^n} = 0 \) for all \( w_2 \in \mathbb{F}^k \).
So we conclude that
\[
Ker(T)^\bot = Range(T^t) = Span\left< v^i : 1 \leq i \leq k \right>.
\]
So we conclude that
\[
Ker(T) \bigoplus Range(T^t) = \mathbb{F}^n.
\]

We define the orthogonal projections as \( P, Q : \mathbb{F}^n \to \mathbb{F}^n \) such that
\[
P_{|_{Ker(T)}} = 0, P_{|_{Range(T^t)}} = I, Q_{|_{Ker(T)}} = I, Q_{|_{Range(T^t)}} = 0.
\]
These projections satisfy the following relations.
\[
I = P + Q, P^2 = P, Q^2 = Q, P^t = P, Q^t = Q,
\]
that is
\[
<w_1, w_2>_{\mathbb{F}^n} =< w_1, Pw_2>_{\mathbb{F}^n}, <Qw_1, w_2>_{\mathbb{F}^n} =< w_1, Qw_2>_{\mathbb{F}^n}
\]
for \( w_1, w_2 \in \mathbb{F}^n \). This proves the existence of orthogonal projections.

8. Proof of the main theorem

In this section we prove main Theorem A by proving Theorem B. Here we prove both the following implications.

Normal Systems are isomorphic \( \iff \) Theorem B holds.

Proof \( \Rightarrow \). Suppose the normal systems \( \mathcal{U}_1, \mathcal{U}_2 \) are isomorphic. Let \( \delta : \mathcal{U}_1 \to \mathcal{U}_2 \) be a convex positive bijection. Without loss generality let us assume that \( \delta \) induces trivial permutation, that is, identity on subscripts. Suppose using the bijection we have constructed isomorphic arrangements
\[
(\mathcal{H}^n_{l-1})^1_1 = \{H^1, H^2, \ldots, H^1_{l-1}\}, (\mathcal{H}^n_{l-1})^2_2 = \{H^2, H^2, \ldots, H^2_{l-1}\}
\]
for \( l > m + 1 \). Note that for \( l - 1 \leq m + 1 \) the hyperplane arrangements are isomorphic by an isomorphism which is identity on the subscripts. Then we add hyperplanes at infinity \( H^1_l \) and \( H^2_l \), whose subscripts correspond to each other under the bijection \( \delta \), to the arrangements \( (\mathcal{H}^m_{l-1})^1_1, (\mathcal{H}^m_{l-1})^2_2 \) respectively. Now we prove the following. The induced hyperplane arrangements
\[
(\mathcal{M}^m_{l-1})^1_1 = \{H^1 \cap H^1, H^1 \cap H^1, \ldots, H^1 \cap H^1\}
\]
\[
(\mathcal{M}^m_{l-1})^1_2 = \{H^2 \cap H^2, H^2 \cap H^2, \ldots, H^2 \cap H^2\}
\]
are isomorphic again by an isomorphism which is identity on the subscripts. Before we prove this we define the following. First we observe that for \( i = 1, 2 \) any zero dimensional vertex on a line of the arrangement in the hyperplane \( H^i_l \) is an intersection of a line which is contained in \( H^i_l \) and a line which is not contained in the hyperplane \( H^i_l \). For every line
\[
H^i_{k_1} \cap H^i_{k_2} \cap \ldots \cap H^i_{k_{m-1}}, 1 \leq k_1 < k_2 < \ldots < k_{m-1} \leq l - 1
\]
not in the hyperplane $H^i_l$ we associate by choosing a direction an $F$-vector

\[ n^i_{\{k_1, k_2, ..., k_{m-1}\}} \]

which is outward pointing on the other side of the bounded intersections of $(H^m_{l-1})^F_i$ and which makes a positive dot product with an outward normal of $H^i_l$ which is also on the other side. The positive dot product is obtained by evaluating the linear functional of the outward normal of $H^i_l$ at normals $n^i_{\{k_1, k_2, ..., k_{m-1}\}}$. Now we prove the following claim.

**Claim 8.1.** If there exists a convex positive bijection $\delta : U_1 \rightarrow U_2$ which is identity on the subscripts then the map of the $F$-directions

\[ \{n^1_{\{k_1, k_2, ..., k_{m-1}\}} \mid 1 \leq k_1 < k_2 < \ldots < k_{m-1} \leq l - 1 \} \]

to the directions

\[ \{n^2_{\{k_1, k_2, ..., k_{m-1}\}} \mid 1 \leq k_1 < k_2 < \ldots < k_{m-1} \leq l - 1 \} \]

taking

\[ n^1_{\{k_1, k_2, ..., k_{m-1}\}} \mapsto n^2_{\{k_1, k_2, ..., k_{m-1}\}} \]

with the respective subscript satisfies the convexity triple property for the lines, that is, if $A, B, C$ denote three subsets of $\{1, 2, \ldots, l - 1\}$ each of cardinality $m - 1$ given by

- $A = \{j_1 < j_2 < \ldots < j_{m-2} \} \cup \{j_{m-1}\}$,
- $C = \{j_1 < j_2 < \ldots < j_{m-2} \} \cup \{j_m\}$,
- $B = \{j_1 < j_2 < \ldots < j_{m-2} \} \cup \{j_{m+1}\}$

then

\[ n^1_C = a_1 n^1_A + b_1 n^1_B, \text{ for some } a_1 > 0, b_1 > 0 \]

\[ \iff n^2_C = a_2 n^2_A + b_2 n^2_B, \text{ for some } a_2 > 0, b_2 > 0. \]

**Proof of Claim.** We observe that for $i = 1, 2$ the $F$-vectors $n^i_A, n^i_B, n^i_C$ span a two dimensional space. Hence there exist coefficients $a_i, b_i \in F^* = \mathbb{F}\{0\}$ such that $a_i n^i_A + b_i n^i_B = n^i_C$. For $i = 1, 2$ let

\[ U'_i = \{\pm n^i_1, \pm n^i_2, \ldots, \pm n^i_l\} \subset U_i, \delta : U'_i \rightarrow U'_2, \delta(n^i_j) = n^2_j, 1 \leq j \leq l \]

be the sets of antipodal pairs of normal vectors for the hyperplane arrangements $\{H^i_1, H^i_2, \ldots, H^i_l\}, i = 1, 2$ respectively with the bijection $\delta$ being identity on subscripts. Let $n^i_i$ be the outward normal of $H^i_i$. Consider the following system of equations for $i = 1, 2$.

\[ n^i_i = \sum_{k=1}^{m-2} x^k_i n^i_{j_k} + x^{m-1}_i n^i_{j_{m-1}} + x^m_i n^i_{j_m} \]

\[ n^i_i = \sum_{k=1}^{m-2} y^k_i n^i_{j_k} + y^{m-1}_i n^i_{j_{m-1}} + y^m_i n^i_{j_{m+1}} \]

\[ n^i_i = \sum_{k=1}^{m-2} z^k_i n^i_{j_k} + z^{m-1}_i n^i_{j_{m-1}} + z^m_i n^i_{j_{m+1}} \]
Now we have for

\[ 1 \leq k \leq m-2, \quad \text{sign}(x_k^i) = \text{sign}(x_k^1), \ \text{sign}(y_k^i) = \text{sign}(y_k^1), \ \text{sign}(z_k^i) = \text{sign}(z_k^1) \]

and also signs are the same for the remaining respective coefficients as well for \( i = 1, 2 \).

For \( i = 1, 2 \) let \( P^i_{\{j_1, j_2, \ldots, j_{m-2}\}} \) be the orthogonal projection onto the two dimensional \( \mathbb{F} \)-space \( V^i \) with \( \mathbb{F} \) - bases given by

\[ \{n_A^i, n_B^i\} \text{ or } \{n_B^i, n_C^i\} \text{ or } \{n_A^i, n_C^i\} \]

and whose \( \mathbb{F} \)-kernel \( U^i \) is spanned by \( \{n_{j_1}^i, n_{j_2}^i, \ldots, n_{j_{m-2}}^i\} \). We have

\[ \mathbb{F}^m = U^i \oplus V^i, \ i = 1, 2, \quad P^i_{\{j_1, j_2, \ldots, j_{m-2}\}} : \mathbb{F}^m \to V^i \]

We have the following projected equations of \( \mathbb{F} \)-vectors.

\[ P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}(n_j^i) = \]

\begin{align*}
\quad & x_i^{m-1} P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}(n_{j_{m-1}}^i) + x_i^m P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}(n_{j_m}^i) = \\
\quad & y_i^m P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}(n_{j_{m-1}}^i) + y_i^{m+1} P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}(n_{j_{m+1}}^i) = \\
\quad & z_i^{m-1} P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}(n_{j_{m-1}}^i) + z_i^{m+1} P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}(n_{j_{m+1}}^i) =
\end{align*}

which have the coefficients with the same sign for \( i = 1, 2 \). We refer to Figure 4 for an illustrative example. Here we observe the following orthogonality conditions (perpendicularity conditions in Figure 4).

\[ n_A^i n_{j_{m-1}}^i = 0 \Rightarrow n_A^i P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}(n_{j_{m-1}}^i) = 0, \]

\[ n_C^i n_{j_m}^i = 0 \Rightarrow n_C^i P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}(n_{j_m}^i) = 0, \]

\[ n_B^i n_{j_{m+1}}^i = 0 \Rightarrow n_B^i P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}(n_{j_{m+1}}^i) = 0 \]

In the right hand side of three equations 8.1 each of the vectors \( n_{j_{m-1}}^i, n_{j_m}^i, n_{j_{m+1}}^i \) appear twice. Now we write the three equations individually with a suitable choice of signs \( (\text{sign})_t \in \{\pm 1\}, 1 \leq t \leq 6 \) with

\begin{align*}
| x_i^{m-1} | = (\text{sign})_1 x_i^{m-1}, & \quad | x_i^m | = (\text{sign})_2 x_i^m, & \quad | y_i^m | = (\text{sign})_3 y_i^m, \\
| y_i^{m+1} | = (\text{sign})_4 y_i^{m+1}, & \quad | z_i^{m-1} | = (\text{sign})_5 z_i^{m-1}, & \quad | z_i^{m+1} | = (\text{sign})_6 z_i^{m+1}
\end{align*}

for \( n_{j_{m-1}}^i, n_{j_m}^i, n_{j_{m+1}}^i \) such that the coefficients are positive, that is,

\[ P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}(n_j^i) = \]

\begin{align*}
| x_i^{m-1} | P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}((\text{sign})_1 n_{j_{m-1}}^i) + | x_i^m | P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}((\text{sign})_2 n_{j_m}^i) = \\
| y_i^m | P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}((\text{sign})_3 n_{j_m}^i) + | y_i^{m+1} | P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}((\text{sign})_4 n_{j_{m+1}}^i) = \\
| z_i^{m-1} | P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}((\text{sign})_5 n_{j_{m-1}}^i) + | z_i^{m+1} | P^i_{\{j_1, j_2, \ldots, j_{m-2}\}}((\text{sign})_6 n_{j_{m+1}}^i)
\end{align*}

We look for which of these three vectors \( n_{j_{m-1}}^i, n_{j_m}^i, n_{j_{m+1}}^i \) changes its sign in \( \{(\text{sign})_1 \mapsto (\text{sign})_5\}, \{(\text{sign})_2 \mapsto (\text{sign})_3\}, \{(\text{sign})_4 \mapsto (\text{sign})_6\} \) while appearing twice in these three equations 8.2. Here more importantly since \( \delta \) is a convex positive bijection consistency is maintained for \( i = 1, 2 \), that is,
we have the same suitable choice of signs of vectors for the three equations as they correspond bijectively by $\delta$.

We have reduced to the two dimensional scenario in the plane of interest $H_{j_1}^i \cap H_{j_2}^i \cap \ldots \cap H_{j_{m-2}}^i (= D^i$ in Figure 4) for $i = 1, 2$. Here we use Observation 2.17 to conclude that only the sign of the vector corresponding to central point changes when $i = 1$ and when $i = 2$ These corresponding subscripts agree for $i = 1, 2$. More elaborately for $i = 1, 2$, the lines are given by

$$L_A^i = L_{\{j_1, j_2, \ldots, j_{m-2}, j_{m-1}\}}^i = H_{j_1}^i \cap H_{j_2}^i \cap \ldots \cap H_{j_{m-2}}^i \cap H_{j_{m-1}}^i$$

$$L_B^i = L_{\{j_1, j_2, \ldots, j_{m-2}, j_{m+1}\}}^i = H_{j_1}^i \cap H_{j_2}^i \cap \ldots \cap H_{j_{m-2}}^i \cap H_{j_{m+1}}^i$$

and the points of intersection of these three lines are given by

$$H_{j_1}^i \cap H_{j_2}^i \cap \ldots \cap H_{j_{m-2}}^i \cap H_{j_{m-1}}^i \cap H_{j_{m+1}}^i$$

$$H_{j_1}^i \cap H_{j_2}^i \cap \ldots \cap H_{j_{m-2}}^i \cap H_{j_{m-1}}^i \cap H_{j_{m+1}}^i$$

$$H_{j_1}^i \cap H_{j_2}^i \cap \ldots \cap H_{j_{m-2}}^i \cap H_{j_{m}}^i \cap H_{j_{m+1}}^i$$

Figure 4. Two dimensional zariski plane of interest $\{j_1 < j_2 < \ldots < j_{m-2}\}$
and with the same notation as in the claim, the line at infinity is
\[ L^i_{\{j_1,j_2,\ldots,j_{m-2},L\}} = H^i_{j_1} \cap H^i_{j_2} \cap \ldots \cap H^i_{j_{m-2}} \cap H^i_L (= L^i_L \text{ in Figure 4}) \]
which contains the points
\[ P^i_{A \cup \{L\}} = H^i_{j_1} \cap H^i_{j_2} \cap \ldots \cap H^i_{j_{m-2}} \cap H^i_{j_{m-1}} \cap H^i_L (= A^i \text{ in Figure 4}) \]
\[ P^i_{C \cup \{L\}} = H^i_{j_1} \cap H^i_{j_2} \cap \ldots \cap H^i_{j_{m-2}} \cap H^i_{j_{m}} \cap H^i_L (= C^i \text{ in Figure 4}) \]
\[ P^i_{B \cup \{L\}} = H^i_{j_1} \cap H^i_{j_2} \cap \ldots \cap H^i_{j_{m-2}} \cap H^i_{j_{m+1}} \cap H^i_L (= B^i \text{ in Figure 4}) \]
in the plane of interest
\[ H^i_{j_1} \cap H^i_{j_2} \cap \ldots \cap H^i_{j_{m-2}} (= D^i \text{ in Figure 4}), i = 1, 2. \]

We observe that
\[ P^1_{C \cup \{L\}} \text{ is in between } P^1_{A \cup \{L\}} \text{ and } P^2_{B \cup \{L\}} \]
\[ \iff P^2_{C \cup \{L\}} \text{ is in between } P^2_{A \cup \{L\}} \text{ and } P^2_{B \cup \{L\}}. \]

We also have that \( n^i_{C} = a_i n^i_A + b_i n^i_B \) with \( a_i > 0, b_i > 0 \) if and only if \( P^i_{C \cup \{L\}} \)
is in between \( P^i_{A \cup \{L\}} \) and \( P^i_{B \cup \{L\}} \). Hence the claim follows.

This claim also proves that there is an isomorphism between the co-dimension one arrangements on the hyperplanes at infinity \( H^i_{l}, i = 1, 2 \) which is identity on the subscripts using Theorem 2.15. Now we use extension Theorem 6.4 to conclude the induction step.

\[ \Leftarrow \text{ Proof.} \text{ We prove the other way implication. Suppose there exists an isomorphism between the hyperplane arrangement } (H^m_n)_{1} = \{H^1_1,H^2_1,\ldots,H^1_n\}, (H^m_n)_{2} = \{H^2_1,H^2_2,\ldots,H^2_n\} \text{ which is identity on the subscripts. Let } U_1 = \{\pm v^1_1,\pm v^1_2,\ldots,\pm v^1_n\}, U_2 = \{\pm v^2_1,\pm v^2_2,\ldots,\pm v^2_n\} \text{ be the corresponding sets containing a pair of normal antipodal } \mathbb{F} \text{-vectors then there exists } \delta : U_1 \rightarrow U_2 \text{ which is identity on the subscripts and which is a convex positive bijection.} \text{ To prove this we do the following.} \]

First we assume that by using Theorem 5.2 that both the hyperplane arrangements are obtained by adding a plane at infinity to the earlier arrangement inductively and the arrangements are isomorphic by an isomorphism which is again identity on the subscripts. Let us choose an outward pointing normal \( \mathbb{F} \)-vector \( n^i_j, 1 \leq j \leq m+1 \) for \( H^i_1,H^i_2,\ldots,H^i_{m+1} \) with respect to the \( m \)-dimensional simplex polyhedrality \( \Delta^m H^i_1 H^i_2 \ldots H^i_{m+1} \) and then an outward pointing normal \( \mathbb{F} \)-vector \( n^i_l \) for \( H^i_l, n \geq l \geq m+2 \) on the other side of the zero dimensional vertices of the earlier arrangement for \( i = 1, 2 \) at each stage.

We prove the following claim.

**Claim 8.2.** Consider for \( 1 \leq k_1 < k_2 < \ldots < k_m < k_{m+1} \leq l \) a set of \( m+1 \) hyperplanes
\[ H^i_{k_1}, H^i_{k_2}, \ldots, H^i_{k_m}, H^i_{k_{m+1}}, i = 1, 2 \]
and their normal vectors
\[ n^i_{k_1}, n^i_{k_2}, \ldots, n^i_{k_m}, n^i_{k_{m+1}}, i = 1, 2 \]
respectively. Then for \(1 \leq j \leq m+1\) we have \(n_{k_j}^1\) is an outward pointing normal of the simplex \(\Delta^m H_{k_1}^1 H_{k_2}^1 \ldots H_{k_m}^1 H_{k_{m+1}}^1\) if and only if \(n_{k_j}^2\) is an outward pointing normal of the simplex \(\Delta^m H_{k_1}^2 H_{k_2}^2 \ldots H_{k_m}^2 H_{k_{m+1}}^2\).

**Proof of Claim.** The proof is trivial if \(k_j = j, 1 \leq j \leq m + 1\) because of the choice of the normals. Assume \(\{k_1, k_2, \ldots, k_{m+1}\} \neq \{1, 2, \ldots, m + 1\}\). Fix \(1 \leq j_0 \leq m + 1\). Consider the points

\[
T^i = H_{k_1}^i \cap \ldots \cap H_{k_{j_0}-1}^i \cap H_{k_{j_0}+1}^i \cap \ldots \cap H_{k_{m+1}}^i, \quad i = 1, 2.
\]

If \(n_{k_{j_0}}^i\) points towards the point \(T^i\) then it is an inward pointing normal of the simplex \(\Delta^m H_{k_1}^1 H_{k_2}^1 \ldots H_{k_m}^1 H_{k_{m+1}}^1\). Otherwise it is an an outward pointing normal of the simplex \(\Delta^m H_{k_1}^1 H_{k_2}^1 \ldots H_{k_m}^1 H_{k_{m+1}}^1\).

Let

\[
S^i_r = H_{1}^i \cap \ldots \cap H_{r-1}^i \cap H_{r+1}^i \cap \ldots \cap H_{m+1}^i, 1 \leq r \leq m + 1, i = 1, 2.
\]

\(\{S^i_r | 1 \leq r \leq m+1\}\) are the vertices of the initial simplex \(\Delta^m H_{1}^1 H_{2}^1 \ldots H_{m}^1 H_{m+1}^1\), \(i = 1, 2, m\).

Now \(l \geq m+2\) and there are at least \(m+2\) hyperplanes. Since \(\{k_1, k_2, \ldots, k_{m+1}\} \neq \{1, 2, \ldots, m + 1\}\) there exists \(1 \leq r \leq m + 1\) such that the point \(S^i_r\) is not on the plane \(H_{k_{j_0}}^i\) and different from \(T^i\) for each \(i = 1, 2\). We note that for \(i = 1, 2\), existence of two such different points \(T^i \neq S^i_r\), not on the plane \(H_{k_{j_0}}^i\), does not hold if \(\{k_1, k_2, \ldots, k_{m+1}\} = \{1, 2, \ldots, m + 1\}\).

Now the normals \(n_{k_{j_0}}^i\) of \(H_{k_{j_0}}^i\) point to the other side of \(S^i_r\) by choice for \(i = 1, 2\). Since the arrangements are isomorphic, \(S^1_r, T^1\) are on the same side of \(H_{k_{j_0}}^1\) if and only if \(S^2_r, T^2\) are on the same side of \(H_{k_{j_0}}^2\). Hence we conclude that \(n_{k_{j_0}}^1\) is an outward pointing normal of the simplex \(\Delta^m H_{k_1}^1 H_{k_2}^1 \ldots H_{k_m}^1 H_{k_{m+1}}^1\) if and only if \(n_{k_{j_0}}^2\) is an outward pointing normal of the simplex \(\Delta^m H_{k_1}^2 H_{k_2}^2 \ldots H_{k_m}^2 H_{k_{m+1}}^2\). This proves the claim. 

Now we use the following fact. We have that for an \(m\)-dimensional simplex \(\Delta^m\) if we choose all the normals of the planes as outward pointing say

\[
u_1, u_2, \ldots, u_m, u_{m+1}
\]

then we can express for every \(1 \leq i \leq m + 1\)

\[
-u_i = \sum_{j=1, j \neq i}^{m+1} \alpha_j u_j \text{ with } \alpha_i > 0.
\]

Here the convex positive bijection

\[
\delta : \{\pm n_{1}^1, \pm n_{2}^1, \ldots, \pm n_{m+1}^1\} \rightarrow \{\pm n_{1}^2, \pm n_{2}^2, \ldots, \pm n_{m+1}^2\},
\]

\[
\delta(n_{j}^1) = n_{j}^2, \delta(-n_{j}^1) = -n_{j}^2, 1 \leq j \leq m + 1
\]
is an isomorphism between the truncated normal systems and inductively for 
\( l > m + 1 \) extends to a convex positive bijection
\[
\delta : \{ \pm n_1^1, \pm n_1^2, \ldots, \pm n_{l-1}^1 \} \longrightarrow \{ \pm n_1^2, \pm n_2^2, \ldots, \pm n_{l-1}^2 \},
\]
\[
\delta(n_j^1) = n_j^2, \delta(-n_j^1) = -n_j^2, 1 \leq j \leq l
\]
This proves that there exists a convex positive bijection between \( U_1 \) and \( U_2 \) given by
\[
\delta(n_j^1) = n_j^1, \delta(-n_j^1) = -n_j^2, 1 \leq j \leq n.
\]
There is also another one given by \( -\delta \).
This completes the proof of Theorem B and hence the proof of main Theorem A of the article. 

9. Graphs of compatible pairs associated to normal systems in three dimensions

In this section we associate an invariant namely the graph of compatible pairs for a normal system in three dimensions. Then we observe that this invariant determines a normal system in three dimensions up to an isomorphism. First we need a few definitions.

**Definition 9.1 (Graph of Compatible Pairs).**
Let \( \mathcal{N} = \{ L_1, L_2, \ldots, L_n \} \) be a normal system in three dimensions. Let \( \mathcal{U} = \{ \pm u_1, \pm u_2, \ldots, \pm u_n \} \) be the corresponding set of a pair of antipodal \( \mathbb{F} \)-vectors on these lines of \( \mathcal{N} \). We associate a graph \( G = (V, E) \) as follows. The vertex set of the graph is given by
\[
V = \{ \{ x, y \} \mid x, y \in \mathcal{U}, x \neq \pm y \}.
\]
We say a vertex \( \{ x_1, y_1 \} \) is compatible with another vertex \( \{ x_2, y_2 \} \neq \{ x_1, y_1 \} \) if there exist positive constants \( a > 0, b > 0, c > 0, d > 0 \) in \( \mathbb{F} \) such that
\[
a x_1 + b y_1 = c x_2 + d y_2.
\]
This automatically means that the set
\[
\{ x_1, y_1, x_2, y_2 \} \subset \mathcal{U}
\]
is maximally linearly independent. The edge set \( E \) of the graph is defined as follows. There is an edge between two vertices \( v_1, v_2 \in V \) if they are compatible.

The following theorem is an immediate consequence from the definitions whose proof is straightforward.

**Theorem 9.2.** Let \( \mathcal{N}_1, \mathcal{N}_2 \) be two normal systems in three dimensions. Let \( \mathcal{U}_1, \mathcal{U}_2 \) be the sets of antipodal pairs of \( \mathbb{F} \)-vectors respectively. Let \( \delta : \mathcal{N}_1 \longrightarrow \mathcal{N}_2 \) be a convex positive bijection. Then the graphs \( G_1, G_2 \) of compatible pairs of normal systems respectively are isomorphic by an isomorphism induced by \( \delta \). Conversely if \( \delta \) is a bijection between \( \mathcal{U}_1, \mathcal{U}_2 \) which preserves antipodal pairs such that \( \delta \) induces an isomorphism of the graphs \( G_1, G_2 \) of compatible pairs then \( \delta \) is a convex positive bijection.
Now using the graph invariant of the normal system we give examples of two non-isomorphic normal systems below.

### 9.1. Examples of two non-isomorphic normal systems in three dimensions

In this section we give two examples of normal systems consisting of six lines in three dimensions which are not isomorphic to each other by showing that their graphs of compatible pairs are not isomorphic which is again proved by showing vertices of degree 1 and degree 5 exist in one but not in the other.

We consider the following sets $\mathcal{U}_i, i = 1, 2$ of antipodal unit vectors on the two normal systems $\mathcal{N}_i, i = 1, 2$ respectively over the field of rational numbers $\mathbb{Q}$ which is contained in all ordered fields $\mathbb{F}$. Let

$$u_1 = (1, 0, 0) = v_1, u_2 = (0, 1, 0) = v_2, u_3 = (0, 0, 1) = v_3,$$

$$u_4 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = v_4, u_5 = \left(\frac{1}{9}, \frac{4}{9}, \frac{8}{9}\right) = v_5, u_6 = \left(\frac{6}{11}, \frac{6}{11}, \frac{7}{11}\right), v_6 = \left(\frac{2}{11}, \frac{6}{11}, \frac{9}{11}\right).$$

Let

$$\mathcal{U}_1 = \{\pm u_i | 1 \leq i \leq 6\}, \mathcal{U}_2 = \{\pm v_i | 1 \leq i \leq 6\}.$$  

We have $\mathcal{U}_1 \cap \mathcal{U}_2 = \{\pm u_1, \pm u_2, \pm u_3, \pm u_4, \pm u_5\} = \{\pm v_1, \pm v_2, \pm v_3, \pm v_4, \pm v_5\}.$

Now we obtain the following $\binom{6}{1}$ = 15 equations for $\mathcal{U}_1$.

1. $3u_4 = u_1 + 2u_2 + 2u_3 = (1, 2, 2)$.
2. $9u_5 = u_1 + 4u_2 + 8u_3 = (1, 4, 8)$.
3. $11u_6 = 6u_1 + 6u_2 + 7u_3 = (6, 6, 7)$.
4. $12u_4 = 3u_1 + 4u_2 + 9u_5 = (4, 8, 8)$.
5. $5u_1 + 21u_4 = 2u_2 + 22u_6 = (12, 14, 14)$.
6. $88u_6 = 41u_1 + 20u_2 + 63u_5 = (48, 48, 56)$.
7. $u_1 + 9u_5 = 4u_3 + 6u_4 = (2, 4, 8)$.
8. $11u_6 = 3u_1 + u_3 + 9u_4 = (6, 6, 7)$.
9. $9u_1 + 9u_5 = 10u_3 + 22u_6 = (12, 12, 24)$.
10. $9u_5 = 2u_2 + 6u_3 + 3u_4 = (1, 4, 8)$.
11. $18u_4 = 6u_2 + 5u_3 + 11u_6 = (6, 12, 12)$.
12. $54u_5 = 18u_2 + 41u_3 + 11u_6 = (6, 24, 48)$.
13. $44u_6 = 13u_1 + 30u_4 + 9u_5 = (24, 24, 28)$.
14. $123u_4 = 26u_2 + 45u_5 + 66u_6 = (41, 82, 82)$.
15. $13u_3 + 27u_4 = 27u_5 + 11u_6 = (9, 18, 31)$.

Also we obtain the following $\binom{6}{1}$ = 15 equations for $\mathcal{U}_2$.

1. $3v_4 = v_1 + 2v_2 + 2v_3 = (1, 2, 2)$.
2. $9v_5 = v_1 + 4v_2 + 8v_3 = (1, 4, 8)$.
3. $11v_6 = 2v_1 + 6v_2 + 9v_3 = (2, 6, 9)$.
4. $12v_4 = 3v_1 + 4v_2 + 9v_5 = (4, 8, 8)$.
5. $27v_4 = 5v_1 + 6v_2 + 22v_6 = (9, 18, 18)$.
6. $88v_6 = 7v_1 + 12v_2 + 81v_5 = (16, 48, 72)$.
7. $v_1 + 9v_5 = 4v_3 + 6v_4 = (2, 4, 8)$.
8. $v_1 + 11v_6 = 3v_3 + 9v_4 = (3, 6, 9)$.
9. $v_1 + 27v_5 = 6v_3 + 22v_6 = (4, 12, 24)$.
10. $v_5 = 2v_2 + 6v_3 + 3v_4 = (1, 4, 8)$.
11. $11v_6 = 2v_2 + 5v_3 + 6v_4 = (2, 6, 9)$.
12. \(18v_5 = 2v_2 + 7v_3 + 11v_6 = (2, 8, 16)\).
13. \(v_1 + 44v_6 = 18v_4 + 27v_5 = (9, 24, 36)\).
14. \(66v_6 = 2v_2 + 21v_4 + 45v_5 = (12, 6, 54)\).
15. \(v_3 + 11v_6 = 3v_4 + 9v_5 = (2, 6, 10)\).

We observe that the graph of compatible pairs \(G_1 = (V_1, E_1)\) has edges of degree 1 and 5 associated to \(N_1\). For example the degree of the vertex \{-u_1, u_2\} is one in \(G_1\) and the only edge with this vertex is with vertex \{u_4, -u_6\} (equation (5) in the first set) and there is no vertex of degree one in the graph \(G_2 = (V_2, E_2)\) of compatible pairs associated to \(N_2\). Similarly the vertex \{-u_1, -u_5\} has degree 5 in the graph \(G_1\) with edges to the vertices \{u_2, -u_4\}, \{u_2, -u_6\}, \{-u_4, -u_3\}, \{-u_6, -u_3\}, \{u_4, -u_6\} given by equations (4), (6), (7), (9), (13) respectively. There are no vertices of degree 5 in the graph \(G_2\). This shows that not all normal systems of the same cardinality are isomorphic in dimension three unlike dimension two.

10. Isomorphism classes of arrangements up to translation of any hyperplane

Here in this section we prove the following theorem regarding a characterization of in terms of infinity arrangements. Before we state the theorem we need a definition.

Definition 10.1 (Infinity Arrangement).
Let \(\mathbb{F}\) be an ordered field. Let \((\mathcal{H}^n_m)^\mathbb{F}\) be a hyperplane arrangement. We say \((\mathcal{H}^n_m)^\mathbb{F}\) is an infinity arrangement if there exists a permutation \(\sigma \in S_n\) such that the hyperplane \(H_{\sigma(l)}\) is a hyperplane at infinity with respect to the arrangement \(\{H_{\sigma(1)}, H_{\sigma(2)}, \ldots, H_{\sigma(1-l)}\}\), \(2 \leq l \leq n\).

Theorem 10.2. Let \(\mathbb{F}\) be an ordered field. Let
\[(\mathcal{H}^n_m)^\mathbb{F}_1 = \{H_1^1, H_2^1, \ldots, H_n^1\}, (\mathcal{H}^n_m)^\mathbb{F}_2 = \{H_1^2, H_2^2, \ldots, H_n^2\}\]
be two arrangements. Let \(U_1, U_2\) be two corresponding sets of normal antipodal \(\mathbb{F}\)-vectors. The following are equivalent.
1. The arrangements \((\mathcal{H}^n_m)^\mathbb{F}_1, (\mathcal{H}^n_m)^\mathbb{F}_2\) are isomorphic up to translation of any hyperplane which is identity on subscripts.
2. There exists a convex positive bijection \(\delta : U_1 \rightarrow U_2\) which is identity on subscripts.
3. There exist two infinity arrangements \((\mathcal{H}^n_m)^\mathbb{F}_1, (\mathcal{H}^n_m)^\mathbb{F}_2\) which are isomorphic by an isomorphism which is identity on subscripts.

Proof. The prove of this Theorem 10.2 is now straight forward.

Now we have the following bijection for any fixed cardinality of the hyperplane arrangements and the normal systems over an ordered field \(\mathbb{F}\).
Note 10.3.

\[ \text{Isomorphism Classes of } \mathbb{F}-\text{Hyperplane Arrangements} \]

\[ \text{up to translation of any hyperplane} \]

\[ \text{Isomorphism Classes of } \mathbb{F}-\text{Normal Systems} \]

In the above bijection we can replace the field \( \mathbb{F} \) by a dense field \( \mathbb{K} \) in the following sense that

\[ \mathbb{K} \cap (a, b) \neq \emptyset \text{ for every } a < b, a, b \in \mathbb{F}. \]

as there is a bijection between \( \mathbb{K} \)-isomorphism classes and \( \mathbb{F} \)-isomorphism classes.

11. Topology appendix

In this section we prove zariski connectedness (in the sense of point set topology) of positive dimensional skeletons of a hyperplane arrangement over infinite fields. We start with a definition.

**Definition 11.1 (k-Dimensional Skeleton).**

Let \( \mathbb{F} \) be an infinite field. Let \( (H_n^m)_{\mathbb{F}} = \{H_1, H_2, \ldots, H_n\} \) be a hyperplane arrangement. For \( 0 \leq k \leq n \), the skeleton \( S_{n-k} \) of \( (n-k) \)-dimensional planes is defined to be

\[ S_{n-k} = \bigcup_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_k}. \]

Now we state the theorem of this section.

**Theorem 11.2 (Zariski Connectedness of Positive Dimensional Skeletons of the Hyperplane Arrangement).**

Let \( \mathbb{F} \) be an infinite field. Let \( (H_n^m)_{\mathbb{F}} = \{H_1, H_2, \ldots, H_n\} \) with \( H_i \subset \mathbb{A}_{\mathbb{F}}^m, 1 \leq i \leq n \) be a hyperplane arrangement. For \( m-1 \geq k \geq 1 \) the skeleton \( S_k \) is connected in the point set topological sense in the zariski topology on the affine space \( \mathbb{A}_{\mathbb{F}}^m \).

**Note 11.3.** Theorem 11.2 is used in proving extension Theorem 6.4.

Before we prove the theorem we mention the following.

**Note 11.4.**

1. The affine space \( \mathbb{A}_{\mathbb{F}}^m, m \geq 1 \) is irreducible if \( \mathbb{F} \) is infinite.
2. A union of two intersecting \( k \)-dimensional planes is zariski connected but not zariski irreducible.
3. The union of two skew hyperplanes arrangements \((\mathcal{H}_{n_1}^r)^F, (\mathcal{H}_{n_2}^r)^F\) embedded (skewly) in an \(r\)-dimensional affine space with \(r \leq m - 1\) is neither zariski irreducible nor zariski connected.

4. Let \(X\) be a topological space and let \(U \subset X\) be a clopen (closed and open) set. If \(Y \subset X\) is connected or irreducible then we have \(U \cap Y \neq \emptyset \Rightarrow Y \subset U\).

Now we prove the theorem.

**Proof of Theorem 11.2.** Let \(U \subset S_1\) be a non-empty clopen set. Since \(U\) is non-empty \(U\) contains a zariski line as it is irreducible. We can change the \(m - 1\) subscripts of a zariski line one by one to move from one zariski line to another inside \(U\) using Note 11.4(4). Hence \(U = S_1\). Thus the one dimensional skeleton is zariski connected. Similarly we have that for any \(1 \leq i \leq m - 1\), if \(U \subset S_i\) is a non-empty clopen set then \(U = S_i\). Thus the \(i\)-dimensional skeleton is zariski connected. We use Note 11.4(4) in the proof. This proves Theorem 11.2.

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