CAN THE ‘STICK-SLIP’ PHENOMENON BE EXPLAINED BY A BIFURCATION IN THE STEADY SLIDING FRICITIONAL CONTACT PROBLEM?

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Abstract. The ‘stick-slip’ phenomenon is the unsteady relative motion of two solids in frictional contact. Tentative explanations were given in the past by enriching the friction law (for example, introducing static and dynamic friction coefficients). In this article, we outline an approach for the analysis of the ‘stick-slip’ phenomenon within the simple framework of the coupling of linear elasticity with the Coulomb dry friction law. Simple examples, both discrete and continuous, show that the solutions of the steady sliding frictional contact problem may exhibit bifurcations (loss of uniqueness) when the friction coefficient is taken as a control parameter. It is argued that such a bifurcation could account, in some cases, for the ‘stick-slip’ phenomenon. The situations of a single point particle, of a linear elastic bounded body with homogeneous friction coefficient and of the elastic half-space with both homogeneous and piecewise constant friction coefficient are analysed and compared.

1. Introduction.

1.1. The ‘stick-slip’ phenomenon. The ‘stick-slip’ phenomenon is an unsteady relative motion of two solids in contact. It is usually recognized that this phenomenon originates in the friction between the two solids and that there is a coupling with the elasticity of the solids, since vibrations (and often audible noise) are generally observed. Examples of stick-slip can be heard from hinges, hydraulic cylinders, etc. Other examples include the music that comes from bowed instruments, the noise of car brakes and tires, the (sometimes) jumping motion of a wiper blade on a glass windshield or that of a piece of chalk on a blackboard. The behaviour of seismically-active faults is also usually explained by invoking a stick-slip model. Measured frequencies range from 1 Hertz or less (example of the wiper blade) to several thousands of Hertz (brake squeal).

There is very little consensus on the origin of the ‘stick-slip’ phenomenon. One tentative explanation which is often proposed is that the law of friction could be responsible for the phenomenon, involving in particular a ‘kinetic’ friction coefficient which is smaller than the ‘static’ friction coefficient. More sophisticated friction law are the so-called ‘state and rate dependent friction’ laws which have been developed to account in particular for the ‘stick-slip’ phenomenon from a stability analysis of the steady sliding [8]. In this line of research, the normal stress is fixed, so that the

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simple Coulomb law can not account for the occurrence of unsteady sliding. It is therefore necessary to refine the friction law. This line of research can be referred to as the constitutive point of view: how to refine the formulation of the friction law so as to fit as best as possible with experimental observations. In this paper, we shall investigate the opposite standpoint which could be qualified as the structural point of view. It consists first in sticking to the Coulomb law, admitting that although surely being an idealized and approximate friction law, it conveys however the most important features of the phenomenon of friction. Since the Coulomb law with fixed normal stress can not account for the stick-slip phenomenon, normal stress should not be fixed and a systematic investigation of the coupling between (linear) elasticity and Coulomb friction has to be thoroughly performed in view of deciding whether it is necessary or not to refine the friction law to account for the onset of the stick-slip phenomenon.

The relevance of the structural point of view can be confirmed by the analysis of the following very simple system.

1.2. Analysis of the simplest discrete system. The simplest system coupling Coulomb friction with linear elasticity is as follows. A point particle of mass \( m \) evolves in a quadratic potential (with stiffness matrix \( K \) supposed to be symmetric and positive definite). An external given \( f \) force acts also on the particle which is prescribed to remain on one side of a straight obstacle moving with constant velocity \( w \) directed along the obstacle (see figure 1). Unilateral contact with Coulomb friction (with friction coefficient \( \mathcal{F} \)) will be assumed between the particle and the obstacle. The location of the particle will be determined by the vector \( u \in \mathbb{R}^d \) (\( d=2,3 \)), in an orthonormal frame of reference, chosen consistently with the geometry of the obstacle so that the first component is the opposite of the distance of the particle to the plane obstacle.

First, the 2-dimensional case (\( d = 2 \)) is investigated and an equilibrium position (that is, a location \( u \) of the particle that remains constant in time) is sought. Introducing the (unknown) reaction force \( r = [r_n \ r_t] \) and using the following notations for the entries of the stiffness matrix:

\[
K = \begin{pmatrix}
k_{nn} & k_{nt} \\
k_{nt} & k_{tt}
\end{pmatrix},
\]

Figure 1. A simple steady sliding frictional contact problem
the set of equations governing the equilibrium reads as:

\[
\begin{align*}
  k_{nn} u_n + k_{nt} u_t &= f_n + r_n, & \text{(normal equilibrium)} \\
  k_{nt} u_n + k_{tt} u_t &= f_t + r_t, & \text{(tangential equilibrium)} \\
  u_n &\leq 0, \quad r_n \leq 0, \quad u_n r_n = 0, & \text{(unilateral contact)} \\
  r_t &= -\mathcal{F} r_n, & \text{(Coulomb friction)}
\end{align*}
\]

(1)

where it has been assumed that \( \text{sgn}(u_t) = 1 \) without restricting the generality.

Eliminating \( u_t \) and \( r_t \), problem (1) is equivalent to:

\[
\begin{align*}
  (k_{nn} k_{tt} - k_{nt}^2) u_n &= (k_{tt} + \mathcal{F} k_{nt}) r_n + k_{tt} f_n - k_{nt} f_t, \\
  u_n &\leq 0, \quad r_n \leq 0, \quad u_n r_n = 0.
\end{align*}
\]

(2)

The reduced problem (2) has a unique solution for every value of the external force \( f \) if and only if:

\[
\frac{k_{tt} + \mathcal{F} k_{nt}}{k_{nn} k_{tt} - k_{nt}^2} > 0,
\]

that is, if and only if:

\[
-\mathcal{F} k_{nt} < k_{tt}. \tag{3}
\]

In addition, if condition (3) is fulfilled and the external force \( f \) is such that the unique equilibrium position achieves active contact (that is, \( r_n < 0 \)), then the dynamics of small oscillations of the particle along the moving obstacle is governed by the equation:

\[
m \ddot{\delta} + (k_{tt} + \mathcal{F} k_{nt}) \delta = 0, \tag{4}
\]

where \( \delta \) denotes the deviation along the tangential direction from the equilibrium position. Stability with respect to normal perturbations is more complicated to study since it requires to handle impacts, and it will be skipped from this introductory example.

We will now analyse further condition (3). As \( K \) is positive definite, \( k_{tt} \) is positive. Therefore, condition (3) is systematically fulfilled whenever \( k_{nt} \geq 0 \). In the situation where \( k_{nt} < 0 \), the condition is fulfilled provided that the friction coefficient satisfies \( \mathcal{F} < k_{tt}/|k_{nt}| \). In the situation where \( \mathcal{F} = k_{tt}/|k_{nt}| \) and there is no external force \( (f = 0) \), then any \( u \) of the form \( u = \begin{pmatrix} 0 \\ u_t \end{pmatrix} \) with \( u_t \in \mathbb{R}^+ \) is an equilibrium position. Hence, considering the friction coefficient as a parameter that is made increasing from 0, the problem has a unique solution until the friction coefficient reaches the critical value \( \mathcal{F} = k_{tt}/|k_{nt}| \) for which the equilibrium problem admits infinitely many solutions. In other words, the system exhibits a bifurcation at this critical value of the friction coefficient. If \( \mathcal{F} < k_{tt}/|k_{nt}| \), then the equilibrium of the system is essentially governed by elasticity which ‘dominates’ friction. When the friction coefficient reaches the critical value \( k_{tt}/|k_{nt}| \), friction becomes able to play at the same level as elasticity. If \( \mathcal{F} > k_{tt}/|k_{nt}| \) and the system has an equilibrium position achieving active contact with the moving obstacle, then the dynamics of small oscillations of the particle along the moving obstacle is still governed by equation (4), showing an instability by divergence of this equilibrium position.

Hence, this simple system exhibits an analogy with the situation of an elastic straight rod submitted to a compressive axial force, for which the straight configuration is the unique equilibrium configuration (and is a stable one) as long as the compressive force does not exceed the so-called Euler critical value, and which admits infinitely many bent equilibrium configurations in addition to the
straight configuration (which is now an unstable equilibrium configuration) when
the compressive force exceeds the Euler critical value. Actually, the frictional system
is more complex than Euler’s elastica. The poverty of the above example does not
enable to feel this whole complexity. But, it is sufficient to consider the situation
d = 3 (particle evolving in the three-dimensional space) instead of d = 2 to see new
features arising. Analysing in the situation d = 3, the stability of the equilibrium of
the particle with respect to tangential perturbations as performed above, shows that
(tangential) stability still holds true for small values of the friction coefficient, but
one can now encounters destabilization either by divergence or flutter, depending
on the rigidity matrix K. There are examples where the destabilization by flutter
is the first one encountered (when increasing the friction coefficient). In this latter
case, the equilibrium position is unique for values of the friction coefficient being in
a neighbourhood of the critical value: flutter instability is not detected by the loss
of uniqueness of the equilibrium configuration.

1.3. The steady sliding frictional contact problem for a linear elastic con-
tinuum. The continuum counterpart of the preceding discrete example is the steady
sliding frictional contact problem which was first studied in [2]. It is the problem of
the equilibrium of a (linearly) elastic body against a moving obstacle, the geometry
of which remaining invariable with respect to time. In $\mathbb{R}^2$, such a situation is met
with a straight obstacle moving along the direction of its boundary or with an
obstacle whose boundary is a circle rotating around its center (in that case, the
obstacle can be either the disk or the region outside the circle). In $\mathbb{R}^3$, this situation
encompasses the case where the boundary of the obstacle is any surface of revolution
rotating around its revolution axis but there are other cases such as the case of a
rotating infinite screw. Dry friction between the moving obstacle and the elastic
body is assumed and a displacement field in the elastic body, that is independent
of time, is sought. Formally, the problem consists in finding a displacement field
$u: \Omega \rightarrow \mathbb{R}^d$ ($d = 2, 3$) such that:

$$\begin{align*}
& \text{div} \sigma(u) + f^p = 0, \quad \text{in } \Omega, \\
& u = u^p, \quad \text{on } \Gamma_u, \\
& t \overset{\text{def}}{=} \sigma \cdot n = t^p, \quad \text{on } \Gamma_t, \\
& u_n - g^p \leq 0, \quad t_n \leq 0, \quad (u_n - g^p)t_n = 0, \quad \text{on } \Gamma_c, \\
& t_{t} = -\mathcal{F} t_{n} w_{t} / |w_{t}|, \quad \text{on } \Gamma_c,
\end{align*}$$

(5)

where $\Omega$ denotes some smooth bounded open set in $\mathbb{R}^d$ (the so-called stress-free
reference configuration), $\Gamma_u \cup \Gamma_t \cup \Gamma_c = \partial \Omega$ denotes a splitting of the boundary into
three disjoint parts, and $n$ is the outward unit normal. As usual, $u$ is the (unknown)
displacement, $\sigma(u)$ is the Cauchy stress associated with this displacement by the
linear elastic constitutive law, and $t = \sigma \cdot n$ denotes the surface traction. Any
vector field $v$ defined on part of the boundary can be split into its normal and
tangential parts: $v = v_n n + v_t$. The loading conditions are defined by $u^p$ (the
surface displacement prescribed on $\Gamma_u$), $t^p$ (the surface tractions prescribed on $\Gamma_t$),
$f^p$ (the prescribed body forces). The geometry of the moving rigid obstacle is coded
through the (initial) gap $g^p$, measured (algebraically) along the outward unit normal
$n$ to the boundary $\partial \Omega$ of the reference configuration. The steady motion of the rigid
obstacle results in a velocity field defined on the surface of the obstacle as follows.
It is assumed that the outward normal line based at each point of $\Gamma_c$ intersects the surface of the moving obstacle, so that a value of the velocity $w$ of the moving obstacle can be associated with each point of $\Gamma_c$. Hence, the motion of the obstacle is coded through a given velocity field $w$ defined on $\Gamma_c$. The friction coefficient is denoted by $\mathcal{F}$. In the particular case where $\mathcal{F} = 0$, problem (5) reduces to the so-called Signorini problem, that is, the equilibrium of a linear elastic body that is pressed against a frictionless obstacle. Therefore, the steady sliding frictional contact problem can formally be seen as a generalization of the Signorini problem in which the unknown reaction force is no longer directed along the normal to the boundary but along a given direction which is possibly slanted with respect to the normal (see figure 2). This generalized Signorini problem is actually encountered in any machine having components in relative motion, and which is expected to run steadily.

There is therefore a strong motivation to extend the discussion performed in section 1.2 about the simple discrete system to the case of a (linear elastic) continuum. Of course, this new situation involves much more complexity and it seems that a discussion of stability is beyond today’s state of art in the understanding of elastodynamics near an obstacle. The discussion will therefore have to be restricted (at least in a first step) to the analysis of the possible occurrence of a bifurcation of the equilibrium when the friction coefficient, taken as a parameter, is varied. In other terms, the object of study will be the (existence and) uniqueness of the equilibrium configuration.

Figure 2. The Signorini problem (left) and the steady sliding frictional contact problem (right)
This line of research has been developed by the author for few years and this article is an occasion of making a synthesis of the results obtained so far.

2. Basic theory for the case of bounded bodies.

2.1. Existence and uniqueness in the case of a small friction coefficient.

The objective of this section is to study the set of solutions for the general steady sliding frictional contact problem (5), in the spirit of the analysis performed in section 1.2 about the simple discrete system. It will be proved that problem (5) has always a unique solution for small values of the friction coefficient and that it may have more than one solution for values of the friction coefficient that are large enough. In other terms, it will be proved that the bifurcation exhibited by the simple discrete system is also encountered in the case of a linear elastic body. The analysis will be performed by deriving a weak formulation for the general steady sliding frictional contact problem (5), in the form of a variational inequality.

The bounded open set Ω is assumed to be of class $C^1$, that is, to be a Lipschitz set with outward unit normal $n$ to the boundary $\partial \Omega$ being Lipschitz-continuous on $\partial \Omega$. Let $\Gamma_u$, $\Gamma_c$, $\Gamma_t$ be three nonintersecting Lipschitz open subsets of the submanifold $\partial \Omega$, such that $\partial \Omega = \Gamma_u \cup \Gamma_c \cup \Gamma_t$. The sets $\Gamma_c$ or $\Gamma_t$ may be empty but not $\Gamma_u$. The constitutive law of the linear elastic body will be written as:

$$\sigma = L : \varepsilon,$$

where $\sigma$ is the Cauchy stress tensor, $\varepsilon = (\nabla u + (\nabla u)^T)/2$ is the linearized strain tensor, and $L(x)$ is the fourth order tensor of the elastic moduli, which possibly varies in the body and is assumed to satisfy the usual symmetry and positivity assumptions:

$$\varepsilon : L : \varepsilon' = \varepsilon' : L : \varepsilon, \quad \varepsilon : L : \varepsilon \geq \alpha \varepsilon : \varepsilon,$$

for some constant $\alpha > 0$, and all $\varepsilon, \varepsilon'$. The elastic energy naturally defines the symmetric bilinear form:

$$a(u, v) \overset{\text{def}}{=} \int_{\Omega} \varepsilon(u) : L : \varepsilon(v),$$

on the Hilbert space $H^1(\Omega)$. We will assume $u^0 \in H^{1/2}(\Gamma_u)$ which is a necessary and sufficient condition to ensure that:

$$V \overset{\text{def}}{=} \left\{ u \in H^1(\Omega) \mid u = u^0 \text{ on } \Gamma_u \right\},$$

is nonempty. It is therefore a closed (affine) subspace of $H^1(\Omega)$ with associated Hilbert space:

$$V_0 \overset{\text{def}}{=} \left\{ u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_u \right\}.$$

The bilinear form $a(\cdot, \cdot)$ is trivially continuous on $V$. The fact that it is also coercive in the sense:

$$\exists \alpha > 0, \quad \forall u, v \in V, \quad a(u - v, u - v) \geq \alpha \|u - v\|^2_{H^1},$$

is also true, although it is difficult to prove, and this is known as the Korn inequality.

Starting with the local equations (5) for some solution $u \in V$ and some test function $v \in V$, we obtain formally (that is, assuming as much regularity as necessary
to perform the algebra):

\[ a(u, v - u) = \int_{\Omega} f_p \cdot (v - u) + \int_{\Gamma_t} t_p \cdot (v - u) + \int_{\Gamma_c} t \cdot (v - u), \]

\[ \geq \int_{\Omega} f_p \cdot (v - u) + \int_{\Gamma_t} t_p \cdot (v - u) + \int_{\Gamma_c} t \cdot (v_t - u_t), \]

\[ \geq \int_{\Omega} f_p \cdot (v - u) + \int_{\Gamma_t} t_p \cdot (v - u) - \int_{\Gamma_t} F t_n (v_t - u_t) \cdot w_t/|w_t|, \]

where it has been used that the test function \( v \) satisfies the condition of non-penetration into the obstacle, that is, \( v_n - g_p \leq 0 \) on \( \Gamma_c \). From now on, we will restrict ourselves to the situation where \( \mathcal{F} = 0 \). To extend this result to nonzero values of the friction coefficient, we have to design a Hilbert space where the bilinear form \( a + \mathcal{F} b \) will be both continuous and coercive. Coercivity requires the \( H^1 \) topology, but \( b \) has no chance to be well-defined all over \( H^1 \), so that we are actually driven towards a closed subspace of \( H^1(\Omega) \). The following theorem provides the precise answer to this problem.

**Theorem 2.1.** Assume the following regularity assumptions:

\( f_p \in L^2(\Omega) \), \( t_p \in L^2(\Gamma_t) \), \( \frac{w_i}{|w_i|} \in W^{1,\infty}(\Gamma_c) \).

Then,

\[ H \overset{\text{def}}{=} \left\{ u \in V \mid \forall v \in H^1(\Omega) \text{ with } v = 0 \text{ on } \Gamma_t \cup \Gamma_u, \ a(u, v) = \int_{\Omega} f_p \cdot v + \int_{\Gamma_t} t_p \cdot v \right\} \]

is a closed (affine) subspace of \( H^1(\Omega) \) with associated Hilbert space:

\[ H_0 \overset{\text{def}}{=} \left\{ u \in V_0 \mid \forall v \in H^1(\Omega) \text{ with } v = 0 \text{ on } \Gamma_c \cup \Gamma_u, \ a(u, v) = 0 \right\}. \]

In addition, the bilinear form defined by:

\[ b(u, v) \overset{\text{def}}{=} \int_{\Gamma_t} t_n(u) v_t \cdot w_t/|w_t|, \]

(for those \( u \in H \) such that \( t_n(u) \in L^2(\Sigma) \)) can be uniquely extended to be continuous on \( H \times H_0 \).

**Proof.** The unique solution \( u_0 \in V \) of the elastic equilibrium problem:

\[ \forall v \in V, \quad a(u_0, v - u_0) = \int_{\Omega} f_p \cdot (v - u_0) + \int_{\Gamma_t} t_p \cdot (v - u_0), \]

belongs to \( H \). The vector space:

\[ V_0 \overset{\text{def}}{=} \left\{ u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_u \right\}, \]
endowed with the scalar product \( a(\cdot, \cdot) \) is a Hilbert space which can be identified with a closed subspace of \( H^1(\Omega) \). The same is therefore true of:

\[
H_0 \overset{\text{def}}{=} \left\{ u \in V_0 \mid \forall v \in H^1(\Omega) \text{ with } v = 0 \text{ on } \Gamma_e \cup \Gamma_u, \quad a(u, v) = 0 \right\}.
\]

Hence \( H = \{ u_0 \} + H_0 \) is a closed affine subspace of \( H^1(\Omega) \), as claimed.

We recall that the trace operator is a linear continuous operator from \( H^1(\Omega) \) onto \( H^{1/2}(\partial \Omega) \). There are basic difficulties with the fractional Sobolev space \( H^{1/2}(\partial \Omega) \) that were discovered and overcome in \([6]\). They will be briefly recalled here. Let \( \Sigma \) be a Lipschitz open subset of \( \partial \Omega \). For \( v \in H^{1/2}(\Sigma) \), the extension \( \tilde{v} \) of \( v \) by zero on \( \partial \Omega \setminus \Sigma \) may fail to be in \( H^{1/2}(\partial \Omega) \). On the dual side, the restriction \( t_{\Sigma} \) to \( \Sigma \) of some \( t \in H^{-1/2}(\partial \Omega) \) may also fail to be in \( H^{-1/2}(\Sigma) \). This led to formulating the following definition \(([6]\)). Set:

\[
H_{00}^{1/2}(\Sigma) = \left\{ v \in H^{1/2}(\Sigma) \mid \tilde{v} \in H^{1/2}(\partial \Omega) \right\}.
\]

To explain why the notation \( 00 \) is used (instead of \( 0 \)), it should be recalled here that \( C^\infty_c(\Sigma) \) is dense in \( H^{1/2}(\Sigma) \). Since \( \Sigma \) is a Lipschitz subset of \( \partial \Omega \), one can find a positive Lipschitz-continuous function \( \rho : \Sigma \to \mathbb{R} \) which vanishes at the boundary \( \partial \Sigma \) of \( \Sigma \) at the same rate as the distance function to the boundary:

\[
\forall x_0 \in \partial \Sigma, \quad \lim_{x \to x_0} \frac{\rho(x)}{d(x, \partial \Sigma)} = 1.
\]

The definition of \( H_{00}^{1/2}(\Sigma) \) is equivalent (for a proof, see \([6]\)) to:

\[
H_{00}^{1/2}(\Sigma) = \left\{ v \in H^{1/2}(\Sigma) \mid \rho^{-1/2}v \in L^2(\Sigma) \right\}.
\]  

(6)

The space \( H_{00}^{1/2}(\Sigma) \) is a Hilbert space for the norm:

\[
\|v\|_{H_{00}^{1/2}} \overset{\text{def}}{=} \left( \|v\|^2_{H^{1/2}} + \|\rho^{-1/2}v\|^2_{L^2} \right)^{1/2},
\]

(two different possible functions \( \rho \) yield equivalent norms). The dual space \( H_{00}^{1/2}(\Sigma) \) is larger than \( H^{-1/2}(\Sigma) \) and the restriction to \( \Sigma \) defines a continuous linear mapping \( H^{-1/2}(\partial \Omega) \to H_{00}^{1/2}(\Sigma) \).

Define \( \Sigma \) as the interior of \( \partial \Omega \setminus \Gamma_u \) in \( \partial \Omega \). It is a Lipschitz open subset of \( \partial \Omega \). Take \( u \in H \) and \( v \in H_{00}^{1/2}(\Sigma) \) arbitrary. Since the trace operator is linear continuous and surjective from \( V_0 \) onto \( H_{00}^{1/2}(\Sigma) \), \( v \) is the trace on \( \Sigma \) of some \( \tilde{v} \in V_0 \). Since \( u \in H \), the expression:

\[
a(u, \tilde{v}) - \int_\Omega f^p \cdot \tilde{v},
\]

does not depend on the particular choice of \( \tilde{v} \in V_0 \) and depends only on \( v \in H_{00}^{1/2}(\Sigma) \). Since this expression is linear continuous with respect to \( v \in H_{00}^{1/2}(\Sigma) \), it defines an element \( t \in H_{00}^{1/2}(\Sigma) \) to which the generalized Green’s formula applies:

\[
\forall v \in V_0, \quad \langle t, v \rangle_{H_{00}^{1/2}, H_{00}^{1/2}} = a(u, v) - \int_\Omega f^p \cdot v.
\]

With \( u \in V \), it is not possible to define \( t = \sigma(u) \cdot n \) on \( \partial \Omega \), in general. However, with \( u \in H \), we have \( \text{div} \sigma(u) = 0 \), and the use of Green’s formula makes it possible to define \( t = \sigma(u) \cdot n \) on \( \Sigma \), as an element of the dual space \( t \in H_{00}^{1/2}(\Sigma) \), as well as
the normal and tangential parts \( t_n, t_t \). Of course, the restriction of \( t \) to \( \Gamma_t \subset \Sigma \) is nothing but \( t^p \), and for those \( u \in H \) such that \( t(u) \in L^2(\Sigma) \), we have:

\[
\forall z \in V_0, \quad \int_{\Gamma_t} t(u) \cdot z = \langle t(u), z \rangle_{H^{1/2'}_0, H_0^{1/2}} - \int_{\Gamma_t} t^p \cdot z = a(u, z) - \int_\Omega f^p \cdot z - \int_{\Gamma_t} t^p \cdot z,
\]

which is therefore continuous with respect to the norm of \( V_0 \).

To obtain the conclusion of the theorem, it is now only needed to prove that if \( v \in H_0 \), then:

\[
\left( v_t \cdot w_t, |w_t| \right)_n,
\]

is the restriction to \( \Gamma_c \) of some \( z \in H^{1/2}_0(\Sigma) \). But this should be clearly apparent from the fact that \( w_t, |w_t| \) and \( n \) are assumed to be Lipschitz-continuous on \( \Gamma_c \) and the definition (6) of \( H^{1/2}_0(\Sigma) \).

Given \( g^p \in H^{1/2}(\Gamma_c) \), it is readily checked that:

\[
K \overset{\text{def}}{=} \left\{ v \in H \mid v_n \leq g^p \text{ on } H^{1/2}(\Gamma_c) \right\},
\]

is a closed convex subset of \( H \). It can be empty in the case where the non-penetration condition on \( \Gamma_c \) is incompatible with the prescribed displacement on \( \Gamma_u \). Hence, the solution of the steady sliding frictional contact problem (5) has to be sought in \( K \).

**Problem 1.** Find \( u \in K \) such that:

\[
\forall v \in K, \quad a(u, v - u) + \mathcal{F} b(u, v - u) \geq \int_\Omega f^p \cdot (v - u) + \int_{\Gamma_t} t^p \cdot (v - u).
\]

A standard argument shows that any solution of problem 1 satisfies equations (5) in an appropriate weak sense (that is, in the sense of distributions for the partial differential equations and in the sense of the theory of traces for the boundary conditions).

To be able to discuss the existence and uniqueness of solutions for problem 1, we must first ensure that the contact condition on \( \Gamma_c \) is compatible with the boundary condition on \( \Gamma_u \), that is, that \( K \) is nonempty. Denoting by \( u_0 \) the unique solution of the elastic problem:

\[
\forall v \in V, \quad a(u_0, v - u_0) = \int_\Omega f^p \cdot (v - u_0) + \int_{\Gamma_t} t^p \cdot (v - u_0),
\]

a sufficient condition to guarantee that \( K \) is nonempty is:

\[
g^p \in H^{1/2}(\Gamma_c), \quad \text{and} \quad \text{supp} \langle u_{0n} - g^p \rangle^+ \subset \Gamma_c,
\]

where \( \langle x \rangle^+ = \max\{x, 0\} \) denotes the positive part and where it should be recalled that \( \Gamma_c \) was supposed open in \( \partial \Omega \). Indeed, condition (7) ensures that there is an element in \( H^{1/2}(\Gamma_c \cup \Gamma_u) \) equalling \( u^p \) on \( \Gamma_u \) and \( \langle u_{0n} - g^p \rangle^- = \min\{u_{0n} - g^p, 0\} \) on \( \Gamma_c \). This fact can be used to construct an element in \( K \) as the solution of a linear elastic problem.

**Theorem 2.2.** Assume the following regularity assumptions:

\[
f^p \in L^2(\Omega), \quad t^p \in L^2(\Gamma_t), \quad \frac{w_t}{|w_t|} \in W^{1,\infty}(\Gamma_c),
\]

and that the compatibility condition (7) holds true. Then, there exists \( \mathcal{F}_c > 0 \), such that, for all \( \mathcal{F} \in [0, \mathcal{F}_c] \), problem 1 has one and only one solution.
Proof. Theorem 2.1 yields the existence of some constant $M \geq 0$ such that:

$$\forall u, v \in H, \quad |b(u - v, u - v)| \leq M\|u - v\|_{H^1(\Omega)}^2.$$ 

In addition, the bilinear form $a$ is coercive:

$$\forall u, v \in H, \quad a(u - v, u - v) \geq \alpha\|u - v\|_{H^1(\Omega)}^2.$$ 

Setting:

$$F_c = \alpha/M,$$

it is readily seen that the bilinear form $a + \mathcal{F}b$ is continuous and coercive on $H$ for all $\mathcal{F} \in [0, F_c]$. Since $K$ is a nonempty closed convex subset of $H$, the Lions-Stampacchia theorem [7] yields a unique solution of problem 1 in the case $\mathcal{F} \in [0, F_c]$.

Theorem 2.2 was first proved in [2].

2.2. Example of multiple solutions for large friction coefficient. The example given in this section is on similar lines to that presented by Hild [5], in a slightly different context.

Let $\Omega \subset \mathbb{R}^2$ be the triangle having the vertices $A = (0, 0)$, $B = (1, 0)$ and $C = (x_c, y_c)$ (where $x_c \in ]0, 1[$ and $y_c > 0$ will be fixed later on). We take $AB$ as $\Gamma_c$, $BC$ as $\Gamma_u$ and $AC$ as $\Gamma_t$. This geometry does not meet the regularity assumption made at the beginning of section 2.1, but it can readily be checked that all the results given in section 2.1 apply to this particular geometry. It will be assumed that the body remains free of body forces $f_p \equiv 0$ and that $\Gamma_t$ remains free of surface traction $t_p \equiv 0$. The rigid obstacle will be the half-space $y < 0$, so that $g_p \equiv 0$, and is assumed to move at a constant velocity $w > 0$ along the $x$ axis. The material is assumed to be linearly elastic and homogeneous isotropic with Young’s modulus $E = 1$ and Poisson ratio $\nu \in ]-1, 1/2[.$

The aim here is to analyse the corresponding steady sliding frictional contact problem. It can be easily checked that the null displacement field always gives a solution to the problem. From theorem 2.2, it is known that if $\mathcal{F} < \mathcal{F}_c$, then there exist no solutions other than the null one. Therefore, if we are able to find a nonzero solution to the steady sliding frictional contact problem, this will provide us with an example where the uniqueness result in theorem 2.2 does not extend to all values of the friction coefficient $\mathcal{F}$, that is, where problem 1 exhibits a bifurcation with respect to the parameter $\mathcal{F}$.

![Figure 3. Geometry of the example with multiple solutions](image-url)
In line with Hild [5], only linear displacement fields (that is, displacement fields whose both components are linear functions of the space variables \(x\) and \(y\)) will be considered. Since \(BC\) is clamped, such a linear displacement field is fully determined by the displacement of point \(A\). Only cases where \(AB\) remains in contact with the moving obstacle will be considered, so that the displacement of point \(A\) will be taken to be of the form \((\delta, 0)\), and the displacement field in \(\Omega\) will be fully determined by \(\delta\):

\[
\begin{align*}
u_x(x, y) &= \delta \left[(1 - x) - (1 - x_c)y/y_c\right], \\
u_y(x, y) &= 0.
\end{align*}
\]

The corresponding stress field \(\sigma(u)\) is constant in \(\Omega\), and has the following components:

\[
\begin{align*}
\sigma_{xx} &= -\frac{\delta(1 - \nu)}{(1 - 2\nu)(1 + \nu)}, \\
\sigma_{xy} &= -\frac{(1 - x_c)\delta}{2y_c(1 + \nu)}, \\
\sigma_{yy} &= -\frac{\delta \nu}{(1 - 2\nu)(1 + \nu)},
\end{align*}
\]

so that its divergence vanishes. The outward unit normal on \(\Gamma_t\) has the following components:

\[
\begin{align*}
 n_x &= -y_c/\sqrt{x_c^2 + y_c^2}, \\
 n_y &= x_c/\sqrt{x_c^2 + y_c^2}.
\end{align*}
\]

Hence, the surface traction vanishes identically on \(\Gamma_t\) if and only if \(\nu > 0\) and:

\[
x_c = 1 - 2\nu, \quad y_c = (1 - 2\nu)\sqrt{\frac{\nu}{1 - \nu}},
\]

which will be assumed from now on. The surface traction on \(\Gamma_c\) which has \((0, -1)\) as outward unit normal, reads as:

\[
\begin{align*}
t_x &= \frac{(1 - x_c)\delta}{2y_c(1 + \nu)}, \\
t_y &= \frac{\delta \nu}{(1 - 2\nu)(1 + \nu)}.
\end{align*}
\]

This is consistent with sliding in the Coulomb friction law if and only if:

\[
\delta \geq 0, \quad \mathcal{F} = \frac{(1 - 2\nu)(1 - x_c)}{2\nu y_c} = \sqrt{\frac{1 - \nu}{\nu}}.
\]

In conclusion, if (8) and (9) are assumed to hold, then the steady sliding frictional contact problem will have infinitely many solutions: \(\delta \geq 0\) can be chosen arbitrarily.

2.3. Discussion. Let us first make a synthesis of the results obtained in subsection 2.1. It was proved that under the regularity hypotheses:

\[
f^p \in L^2(\Omega), \quad t^p \in L^2(\Gamma_t), \quad \frac{w_t}{|w_t|} \in W^{1, \infty}(\Gamma_c), \quad g^p \in H^{1/2}(\Gamma_c),
\]

and the compatibility condition (7), there exists a positive value \(\mathcal{F}_c > 0\), such that for all \(\mathcal{F} < \mathcal{F}_c\) all loading parameters \(f^p, t^p\), the steady sliding frictional contact problem (5) has one and only one solution \(u \in H^1(\Omega)\). This result has been obtained by reformulating the problem as a variational inequality of the form:

\[
u \in K, \quad \text{and}
\]

\[
\forall v \in K, \quad a(u, v - u) + \mathcal{F}b(u, v - u) \geq \int_{\Omega} f^p \cdot (v - u) + \int_{\Gamma_t} t^p \cdot (v - u).
\]

Here, \(K\) denotes a nonempty closed convex subset of a closed subspace \(H\) of \(H^1(\Omega)\) and \(a + \mathcal{F}b\) is a bilinear form, which is continuous and coercive on \(H\) for \(\mathcal{F} < \mathcal{F}_c\). The existence and uniqueness result was therefore a consequence of the Lions-Stampacchia theorem.

This result raises the following issues.
1. Critical review of the regularity hypotheses. Looking at the regularity assumptions (10), the only one that could be restrictive in practical cases is the third one, namely \( \frac{w_t}{|w_t|} \) is required to be Lipschitz-continuous. Indeed, consider the example of a plane rigid obstacle in the three-dimensional space that is rotating around a line that is orthogonal to itself, and the case of a bounded elastic body that is pressed against that plane at the center of rotation (the intersection of the plane with the line it rotates around). Then, the normalised orthoradial vector field \( \frac{w_t}{|w_t|} \) is not even continuous at the center of rotation. This situation is therefore out of the scope of the regularity hypothesis \( \frac{w_t}{|w_t|} \in W^{1,\infty} \). In the case of a plane problem, \( \frac{w_t}{|w_t|} \) is either the constant +1 or the constant −1, so it could be thought at first sight that the Lipschitz-continuity hypothesis encompasses every possible practical plane situation. This is not completely true if we consider composite bodies for which it will be desirable to have a non-constant friction coefficient (more precisely, a stepwise constant friction coefficient). In that case, the constant \( F \) in the preceding analysis should be replaced by \( F_{\text{max}}\xi(x) \), where \( \xi : \Gamma_{\text{c}} \rightarrow [0, 1] \) is a (generally stepwise constant) given function. This amounts to replace \( \frac{w_t}{|w_t|} \) by \( \xi \frac{w_t}{|w_t|} \) in the preceding analysis and the Lipschitz-continuity assumption is now clearly unsustainable.

2. Critical review of the coercivity requirement. In the light of the bifurcation exhibited by the simple system analysed in section 1.2, the following question naturally arises. Coercivity of the bilinear form \( a + Fb \) ensures the uniqueness of solution for the variational inequality, whatever the load \( (f_p, t_p) \) is. But, it is natural to question further the link between loss of coercivity and bifurcation. Insight can be gained by examining the finite-dimensional situation which is known as the linear complementarity problem. Given a squared matrix \( A \) of order \( n \geq 1 \), \( f \in \mathbb{R}^n \) and:

\[
K \overset{\text{def}}{=} \left\{ v \in \mathbb{R}^n \mid \forall i, \quad v_i \leq 0 \right\},
\]

the linear complementarity problem is that of finding \( u \in K \) such that:

\[
\forall v \in K, \quad ^t u A (v - u) \geq ^t f (v - u). \tag{11}
\]

Hence, the linear complementarity problem can be seen as a finite-dimensional counterpart of the variational inequality governing the steady sliding frictional contact problem. A squared matrix is said to be a \( P \)-matrix if all its principal minors are positive. The central result in the theory of linear complementarity problem is that the linear complementarity problem (11) has one and only one solution for all \( f \in \mathbb{R}^n \) if and only if \( A \) is a \( P \)-matrix (for a proof, see [4]). In the case where the matrix \( A \) is symmetric, it is well-known that it is positive definite (which is the same as coercivity in the finite-dimensional framework) if and only if it is a \( P \)-matrix. In the case of a possibly non-symmetric matrix, a positive definite matrix is always a \( P \)-matrix, but the reverse statement may fail to be true. Therefore, for symmetric matrices, a bifurcation (loss of uniqueness of solution) can be detected by a loss of coercivity, but, in the case of a non-symmetric matrix, the loss of coercivity can occur for a value of the control parameter which is smaller than the critical value at which bifurcation (loss of uniqueness of solution) happens. The case of a symmetric bilinear form is the situation met in Euler’s buckling and is characteristic of the existence of an underlying energy. Friction is a dissipative phenomenon, and,
unsurprisingly, the bilinear form $a + \mathcal{F}b$ that arises in the weak formulation is non-symmetric. Hence, the value $\mathcal{F}_c$ arising in the above analysis of the steady sliding frictional contact problem and which is associated with the loss of coercivity of the bilinear form $a + \mathcal{F}b$ should be thought of only as a lower bound of the critical value of the friction coefficient for which a bifurcation happens.

Further insight into these two issues can be obtained by studying the particular case where the elastic body is an isotropic homogeneous half-space undergoing plane strains only. This is the object of the following section.

3. Analysis of the particular case of the elastic half-space.

3.1. The formal problem. We consider an isotropic homogeneous linearly elastic two-dimensional half-space defined by $z > 0$. The Poisson ratio is denoted by $\nu \in [-1, 1/2]$ and the force unit is chosen so that the Young modulus $E = 1$. We denote by $x$ the space variable along the boundary and by $t(x)$ the surface traction distribution on the boundary and the normal and tangential components will be addressed as $t_n(x)$ and $t_t(x)$. The following conditions:

$$\lim_{r \to \infty} \sigma(u) = 0, \quad u = O(\log(r)),$$

are prescribed at infinity, where $u$ denotes the displacement field and $\sigma$ the stress field. Setting:

$$\bar{u} = \frac{u}{2(1-\nu^2)}, \quad \text{and} \quad \gamma = \frac{1 - 2\nu}{2(1-\nu)} \in [0, 3/4],$$

the surface displacement resulting from a given surface traction distribution $t(x)$ is given (see for example [1] for a proof) by:

$$\bar{u}'_n(x) = \frac{1}{\pi} \oint_{-1}^{1} \frac{t_n(x')}{x' - x} \, dx' - \gamma t_t(x),$$

$$\bar{u}'_t(x) = \frac{1}{\pi} \oint_{-1}^{1} \frac{t_t(x')}{x' - x} \, dx' + \gamma t_n(x),$$

where the sign $\oint$ recalls that the integral should be understood in terms of the Cauchy principal value. The surface displacement is obtained up to an arbitrary additive constant, which is interpreted as being a rigid motion. This arbitrary rigid motion cannot be fixed by prescribing appropriate conditions at infinity since the displacement field is generally infinite at infinity.

Consider some rigid obstacle, the geometry of which is defined by the equation $-z = \psi(x) \, (x \in [-1, 1])$, moving at a constant velocity $v > 0$ along $x$, which is assumed to be parallel to the boundary of the half-space (see figure 4). Set:

$$\quad \bar{\psi} = \frac{\psi}{2(1-\nu^2)}.$$

The steady sliding frictional contact problem was introduced and studied in [2]. It is formally that of finding $t(x), \bar{u}(x) : [-1, 1] \to \mathbb{R}$ such that:

- $\frac{1}{\pi} \oint_{-1}^{1} \frac{t_n(x')}{x' - x} \, dx' - \gamma t_t(x) = \bar{u}'_n(x),$
where \( P > 0 \) is the given normal component of the prescribed total force exerted on the moving obstacle, \( \mathcal{F} \geq 0 \) is the friction coefficient. Note that if \( \bar{\psi} \) is changed into \( \bar{\psi} + C \), then we get a solution for the new problem by just changing \( \bar{u}_n \) into \( \bar{u}_n + C \) in the solution. This means that the penetration of the indentor into the half-space is undefined and this is due to the fact the displacement field is infinite at infinity. The problem can be parametrized by the total force \( P \) only, and not by the height of the moving obstacle, because it is undetermined. This fact is intimately connected with the fact that the stress field in the half-space is not square integrable: the elastic energy of the solution is infinite and this is the reason why the problem has to be brought to the boundary by use of the fundamental solution of the Neumann problem for the half-space (the so-called Boussinesq solution). Focusing on the normal components, this formal problem reduces to that of finding

\[
\begin{align*}
&\frac{1}{\pi} \int_{-1}^{1} t_n(x') \, dx' + \gamma t_n(x) = \bar{u}_n'(x), \\
&\bar{u}_n \leq \bar{\psi}, \quad t_n \leq 0, \quad (\bar{u}_n - \bar{\psi}) t_n \equiv 0,
\end{align*}
\]

\[
\int_{-1}^{1} t_n(x') \, dx' = -P,
\]

where we have set \( s(x) = \gamma \mathcal{F}(x) \).

In the sequel, two particular cases will be studied.

1. The case of homogeneous friction, that is, the case where \( s(x) \) is a constant function. It will be proved that the corresponding steady sliding frictional contact problem has always one and only one solution (\( \mathcal{F} = +\infty \)).
2. The case where of a discontinuous friction coefficient, namely $s(x)$ taking the value $s_-$ on $]-1,0[$ and $s_+$ on $]0,1[$. It will be seen that distinct features arise whether $s_- < s_+$ or $s_- > s_+$.

3.2. The case of homogeneous friction. In this section, the function $s(x)$ is supposed to be identically equal to the constant $s \geq 0$. The weak formulation of the formal problem (12) was first derived in [2]. As it takes the normal traction $t_n$ as main unknown, this weak formulation is a dual formulation, unlike the variational inequality derived in section 2.1 in view of analysing the steady sliding frictional contact problems for bounded bodies. An equivalent primal weak formulation was obtained in [3]. Here, only the dual weak formulation will be recalled.

Some useful facts, detailed proof of which can be found in [1], will first be recalled. With arbitrary $\hat{t} \in H^{-1/2}(-1,1)$, the extension by zero on the whole real line (still denoted by $\hat{t}$) defines a distribution $\hat{t} \in H^{-1/2}(\mathbb{R})$. The following convolution products:

$$\hat{t} \ast \log |\cdot|, \quad \hat{t} \ast \text{sgn}(\cdot),$$

(where $\text{sgn}(\cdot)$ is the sign function) define distributions over $\mathbb{R}$ whose restrictions to the interval $]-1,1[$ are in $H^{1/2}(-1,1)$. In addition, the bilinear form defined by:

$$a^* (\hat{t}_1, \hat{t}_2) \overset{\text{def}}{=} -\big\langle \hat{t}_1 \ast \log |x|, \hat{t}_2 \big\rangle_{H^{1/2},H^{-1/2}},$$

is symmetric and is also positive definite. It therefore defines a scalar product on the space $H^{-1/2}(-1,1)$, and this scalar product induces a norm that is equivalent to that of $H^{-1/2}$ (see [1] for a proof). The bilinear form:

$$b^*(\hat{t}_1, \hat{t}_2) \overset{\text{def}}{=} \big\langle \hat{t}_1 \ast \text{sgn}(x), \hat{t}_2 \big\rangle_{H^{1/2},H^{-1/2}},$$

can easily be seen to be skew-symmetric. It is continuous on $H^{-1/2} \times H^{-1/2}$. These facts can be used to obtain the weak formulation of the steady sliding frictional contact problem. The first equation in the formal problem (12) can be rewritten under the form:

$$-\frac{1}{\pi} t_n \ast \log |\cdot| + \frac{s}{2} t_n \ast \text{sgn}(\cdot) = \bar{u}_n + C, \quad \text{in } ]-1,1[,$$

where $C$ is an arbitrary constant.

Provided $P \geq 0$, the set:

$$K^* \overset{\text{def}}{=} \big\{ \tilde{t} \in H^{-1/2}(-1,1) \mid \tilde{t} \leq 0 \text{ and } \langle \tilde{t}, \chi \rangle_{-1,1} = -P \big\}, \quad (13)$$

(where the inequality refers to the dual ordering and $\chi_{]-1,1[}$ denotes the function identically equal to 1 on $]-1,1[)$ is nonempty, convex and closed in $H^{-1/2}(-1,1)$. Hence, given $\psi \in H^{1/2}(-1,1)$, the steady sliding frictional contact problem for the half-space admits the following formulation.

**Problem 2.** Find $t_n \in K^*$ such that:

$$\forall \hat{t} \in K^*, \quad a^*(t_n, \hat{t} - t_n) + s b^*(t_n, \hat{t} - t_n) \geq \langle \psi, \hat{t} - t_n \rangle.$$

Since $b^*$ is skew-symmetric, $b^*(\hat{t}, \tilde{t}) = 0$ and $a^* + s b^*$ is coercive for all $s \in \mathbb{R}$. Thanks to the Lions-Stampacchia theorem, problem 2 has therefore one and only one solution for all value of the friction coefficient. The following conclusion can be drawn: the steady sliding frictional contact problem for the half-space exhibits no bifurcation.
Problem 2 admits an equivalent primal formulation in terms of the unknown normal displacement \( \tilde{u}_n \). The natural space for this primal formulation is the quotient space \( H^{1/2}(-1, 1) / (R \chi_{-1,1}) \). It is extensively analysed in [3] and has the advantage to be more parallel to problem 1 of section 2.1. However, the corresponding bilinear forms \( a \) and \( b \) are less pleasant to define and express than their dual counterparts \( a^* \) and \( b^* \). The interested reader will also find in [3] some explicit exact solutions of problem 2, as well as an inventory of all the universal singularities that the solution of problem 2 may experience.

3.3. The case of a discontinuous friction coefficient. In this section, the case where the friction coefficient has a jump is considered. More precisely, the model problem where \( s(x) \) takes the constant value \( s_- \) on \([-1,0[ \) and \( s_+ \) on \([0,1[ \) is going to be analysed.

In order to get insight in this case, the first task is to identify the counterpart of the bilinear form \( a^* + s b^* \) in this case, and to study its coercivity. Picking some \( p \in C^\infty([-1,1[) \) with \( \int_{-1}^{1} p = 0 \), it is readily checked that:

\[
- \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} p(t) p(x) \log |x-t| \, dt \, dx + \int_{-1}^{1} \int_{-1}^{x} s(t) p(t) p(x) \, dt \, dx
= - \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} p(x) p(t) \log |x-t| \, dt \, dx + \frac{s_+ - s_-}{2} \left( \int_{0}^{1} p \right)^2.
\]

The two cases \( s_+ > s_- \) and \( s_+ < s_- \) must therefore be considered separately.

3.3.1. The case where \( s_+ > s_- \). We define the Hilbert space \( \overline{H} \) as the completion of \( C^\infty([-1, +1[) \) in the norm:

\[
\| p \|_{\overline{H}} = \sqrt{\| p \|_{H^{-1/2}}^2 + \left( \int_{0}^{1} p \right)^2}.
\]

The space \( \overline{H} \) is strictly smaller than \( H^{-1/2}(-1, 1) \). The bilinear form defined by:

\[
c^*(p,q) \overset{\text{def}}{=} - \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} p(t) q(x) \log |x-t| \, dt \, dx + \int_{-1}^{1} \int_{-1}^{x} s(t) p(t) q(x) \, dt \, dx,
\]

with \( p,q \in C^\infty([-1, +1[) \) can be uniquely extended as a continuous bilinear form on \( \overline{H} \). This fact is not obvious at all and a precise proof can be found in [3]. Setting:

\[
\overline{H}_0' \overset{\text{def}}{=} \left\{ t \mid \langle t, \chi_{[-1,+1]} \rangle = 0 \right\},
\]

formula (14) shows that \( c^* \) is also coercive on \( \overline{H}_0' \).

Note that \( K^* \), as defined in formula (13), contains only measures on \([-1,1] \) so that \( K^* \) is actually contained in \( \overline{H} \). Hence, given \( \tilde{\psi} \in H^{1/2}(-1, 1) \), the steady sliding frictional contact problem for the half-space admits the following formulation.

**Problem 3.** Find \( t_n \in K^* \) such that:

\[
\forall \hat{t} \in K^*, \quad c^*(t_n, \hat{t} - t_n) \geq \langle \tilde{\psi}, \hat{t} - t_n \rangle.
\]

Problem 3 admits a unique solution, thanks to the Lions-Stampacchia theorem applied in the Hilbert space \( \overline{H}_0' \). Hence, the analysis performed for the case of a constant friction coefficient readily extends to the situation where the friction coefficient has an increasing jump. The only difference is that the Hilbert space
arising in the weak formulation is no longer only determined by the elasticity operator, but is now influenced by the friction operator.

3.3.2. The case where \( s_+ < s_- \). In that case, the bilinear form \( c^* \), although continuous on \( \overline{H}_0^\prime \), is not coercive on \( \overline{H}_0^\prime \). A more thorough study of the weak formulation of the steady sliding frictional contact problem is needed in that case. The reader is referred to [3] where the unconditional existence of a solution is proved for that case, by use of a refined weak formulation. The question of uniqueness remains open there and will require further investigation. In the meanwhile, such a uniqueness of solution is now going to be made plausible by studying more thoroughly the particular case of the rigid flat punch of finite width in the next subsection.

3.3.3. The particular case of the rigid flat punch. We now consider the situation of a rigid flat punch of finite width that is steady sliding at the surface of an elastic half-plane. This is the case where the function \( \psi \) in problem 2, vanishes identically on the interval \([-1, 1]\). It is assumed that there are two different coatings on each of the regions \([-1, 0[\) and \([0, 1[, so that two different constant friction coefficients can be expected on each of those regions.

We first consider the case where the friction coefficient in \([0, 1[\) is larger than that in \([-1, 0[, which falls into the scope of subsection 3.3.1. It is natural to look for the unique solution of the corresponding problem 3 by considering active contact all over the flat punch: \( \bar{a}_n \equiv 0 \) on \([-1, 1[\). This amount to study the solutions \( t_n \) of the homogeneous singular integral equation of the Carleman type:

\[
\frac{1}{\pi} \int_{-1}^{1} \frac{t_n(x')}{x' - x} \, dx' + s(x) t_n(x) = 0, \quad \text{in } ]-1, 1[. \tag{15}
\]

There is an extensive theory for those singular integral equation of the Carleman type in the case of a Lipschitz-continuous function \( s(x) \) (see for example [9]). This theory was extended to the case of a piecewise Lipschitz-continuous function \( s(x) \), possibly having a finite number of jumps, in the appendix A of [1].

In the case where the function \( s(x) \) takes the constant value \( s_- \) all over \([-1, 0[\) and \( s_+ \) over \([0, 1[, with \( s_+ > s_- \), it is proved in the appendix A of [1] that the analysis of equation (15) is essentially the same as in the case where \( s(x) \) is Lipschitz-continuous: all the solutions in \( \cup_{p>1} L^p(-1, 1) \) of the homogeneous equation (15) are given by:

\[
t_n(x) = -\frac{C}{\sqrt{1 + s^2(x)}} \frac{1}{(1 + x)^{\frac{1}{2} + \alpha_-} |x|^\alpha_- - \alpha_- (1 - x)^{\frac{1}{2} - \alpha_-}},
\]

where:

\[
\alpha_- \overset{\text{def}}{=} \frac{1}{\pi} \arctan(s_-) = \frac{1}{\pi} \arctan \left( \frac{(1 - 2\nu)F_-}{2(1 - \nu)} \right), \quad (x \in [0, 1/2[), \tag{16}
\]

\[
\alpha_+ \overset{\text{def}}{=} \frac{1}{\pi} \arctan(s_+) = \frac{1}{\pi} \arctan \left( \frac{(1 - 2\nu)F_+}{2(1 - \nu)} \right), \quad (x \in [0, 1/2[), \tag{17}
\]

and \( C \in \mathbb{R} \) is an arbitrary constant. Since:

\[
\forall s_-, s_+ \in \mathbb{R}, \quad \int_{-1}^{+1} \frac{1}{\sqrt{1 + s^2(x)}} \frac{dx}{(1 + x)^{\frac{1}{2} + \alpha_-} |x|^\alpha_- - \alpha_- (1 - x)^{\frac{1}{2} - \alpha_-}} = \pi, \tag{18}
\]
and that the above function takes only nonpositive values, it turns out that the unique solution of problem 2 in the case $\psi \equiv 0$ is given by:

$$t_n(x) = -\frac{P}{\pi \sqrt{1 + s^2(x)}} \frac{1}{(1 + x)^{\frac{1}{2} + \alpha} - (1 - x)^{\frac{1}{2} - \alpha}}.$$  

This exact solution was first obtained in [3]. It exhibits power singularities at $x = -1$, $x = 0$ and $x = 1$ (see figure 5). It is proved in [3] that the values of these exponents, as given by formulae (16) and (17), are universal.

In the case where $s_+ < s_-$, if we suppose in addition that the normal surface traction $t_n(x)$ is in $\bigcup_{p>1} L^p(-1, 1)$, then, by mimicking the reasoning in the proof of proposition 2 of [1], we obtain that the solution achieves active contact everywhere below the punch: $\bar{u}_n \equiv 0$ on $[-1, 1]$. Hence, the normal surface traction must solve the homogeneous singular integral equation (15). But, the analysis performed in appendix A of [1] shows that all the solutions in $\bigcup_{p>1} L^p(-1, 1)$ of equation (15) in the case $s_+ < s_-$ are now given by:

$$t_n(x) = -\frac{1}{\sqrt{1 + s^2(x)}} \frac{|x|^{\alpha_+ - \alpha_+} (C + D/x)}{(1 + x)^{\frac{1}{2} + \alpha} - (1 - x)^{\frac{1}{2} - \alpha_+}},$$

where $C$ and $D$ are two arbitrary real constants. But, among these functions, only one is a nonpositive function of total mass $-P$. It is given by:

$$t_n(x) = -\frac{P}{\pi \sqrt{1 + s^2(x)}} \frac{|x|^{\alpha_+ - \alpha_+}}{(1 + x)^{\frac{1}{2} + \alpha} - (1 - x)^{\frac{1}{2} - \alpha_+}},$$

thanks to formula (18). The normal component of the surface traction now goes to zero at the jump in the friction coefficient (see figure 5), which is qualitatively very different from the singularity of the case $s_+ > s_-$. The fact that the surface traction goes to zero at an increasing jump of the friction coefficient is proved to be universal with respect of the shape of the moving indentor (provided it is smooth) in [3].

---

**Figure 5.** Normal component of the surface stress when the larger friction coefficient is front (left) or rear (right).
3.4. **Discussion.** The case of the elastic half-space enlightens the analysis of the steady sliding frictional contact problems performed for bounded bodies in section 2.1.

First, the case of homogeneous friction provides an example of a situation where the bifurcation exhibited in section 2.2 does not happen, that is, where the critical friction coefficient $\mathcal{F}_c$ is infinite.

Second, the case of a jump in the friction coefficient, with largest value being front, shows interesting differences with the classical analysis of Euler’s buckling, where the weak formulation of the equilibrium equations linearized around a prestressed configuration contains the so-called geometric rigidity operator whose role is, at first sight, parallel to that of the friction operator in our analysis. In the analysis of Euler’s buckling, the topology with respect to which, the loss of coercivity, and therefore of uniqueness, is analysed, is that induced by the elastic rigidity operator, namely that of $H^1$. The geometric rigidity operator is always continuous with respect to that topology and does not influence the choice of the natural topology at all. The story is completely different in the case of the steady sliding frictional contact problem with discontinuous friction coefficient, since the analysis shows that coercivity can be achieved only by taking into account the friction operator for the choice of the topology. Making this appropriate choice for the topology then enables us to generalize the analysis of the situation of a constant friction coefficient to the case of a jump in the friction coefficient, with largest value being front.

Finally, the case of a jump in the friction coefficient shows that non-coercivity should not always be thought of as a criterion of non-uniqueness of solution. Indeed, coercivity in that case can only be achieved by considering an appropriate nontrivial topology. Also, in the case where the largest value of the friction coefficient is rear, the consideration of the particular case of the rigid flat punch makes it plausible that some kind of uniqueness can be encountered when coercivity in the appropriate topology is not met.

**REFERENCES**

[1] P. Ballard and J. Janušek, Indentation of an elastic half-space by a rigid flat punch as a model problem for analyzing contact problems with coulomb friction, *Journal of Elasticity*, 103 (2011), 15–52.

[2] P. Ballard, Steady sliding frictional contact problems in linear elasticity, *Journal of Elasticity*, 110 (2013), 33–61.

[3] P. Ballard, Steady sliding frictional contact problem for an elastic half-space with a discontinuous friction coefficient and related stress singularities, to appear in the special issue of *Journal of the Mechanics and Physics of Solids* in honour of Pierre Suquet, (2015).

[4] R. W. Cottle, J.-S. Pang and R. E. Stone, *The Linear Complementarity Problem*, Society for Industrial and Applied Mathematics, Philadelphia, 2009.

[5] P. Hild, Non-unique slipping in the Coulomb friction model in two-dimensional linear elasticity, *The Quarterly Journal of Mechanics and Applied Mathematics*, 57 (2004), 225–235.

[6] J. L. Lions and E. Magenes, *Problèmes aux Limites non Homogènes et Applications*, Volume 1, Dunod, Paris, 1968.

[7] J. L. Lions and G. Stampacchia, *Variational Inequalities*, *Communications on Pure and Applied Mathematics*, 20 (1967), 493–519.

[8] J. R. Rice and A. L. Ruina, Stability of steady frictional slipping, *Journal of Applied Mechanics*, 50 (1983), 343–349.

[9] F. G. Tricomi, *Integral Equations*, Interscience Publishers, New York, 1957.

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