Next-to-Maximal Helicity Violating Amplitudes in Gauge Theory

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Abstract

Using the novel diagrammatic rules recently proposed by Cachazo, Svrcek, and Witten, I give a compact, manifestly Lorentz-invariant form for tree-level gauge-theory amplitudes with three opposite helicities.

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1. Introduction

The computation of amplitudes in gauge theories is important to future physics analyses at colliders. Tree-level amplitudes in perturbative QCD, for example, provide the leading-order approximation to multi-jet processes at hadron colliders. At tree level, all-gluon amplitudes in pure $SU(N)$ gauge theory are in fact identical to those in the $\mathcal{N} = 4$ supersymmetric theory, since only gluons can appear as interior lines.

This hidden supersymmetry manifests itself in the vanishing of amplitudes for certain helicity configurations, namely those in which all or all but one gluon helicities are identical [1]. (I follow the usual convention where all momenta are taken to be outgoing; recall that flipping a momentum from outgoing to incoming also flips the helicity.) These vanishings are expressed in two of the three Parke–Taylor equations [2],

$$A^{\text{tree}}_n(k_1^+, k_2^+, \ldots, k_n^+) = 0,$$
$$A^{\text{tree}}_n(k_1^-, k_2^+, \ldots, k_n^+) = 0.$$  \hspace{0.5cm} (1.1)

The next amplitude in this sequence, with two opposite-helicity gluons, is the maximally helicity-violating amplitude that does not vanish, and is conventionally called the MHV amplitude. It has a simple form, recalled below, given by the third Parke–Taylor equation. Continuing in the sequence, we find amplitudes with three opposite helicity gluons, which I will call ‘next-to-MHV’ or NMHV for short. These are the subject of this paper.

The $\mathcal{N} = 4$ supersymmetric theory is of great interest for a number of reasons, especially its links with string theories. Witten has recently proposed [3] a novel link between a twistor-space topological string theory and the amplitudes of the $\mathcal{N} = 4$ supersymmetric gauge theory. This proposal generalizes Nair’s earlier construction [4] of MHV amplitudes. A number of authors have investigated issues connected with the derivation of gauge-theory amplitudes from the topological string theory [5,6] as well as alternative approaches [7,8] and related issues [9,10]. Based on investigations in this string theory, Cachazo, Svrcek, and Witten (CSW) proposed [11] a novel construction of tree-level gauge-theory amplitudes. It expresses any amplitude in terms of propagators and basic ‘vertices’ which are off-shell continuations of the Parke-Taylor amplitudes. The construction makes manifest the factorization on multi-particle poles.

Cachazo et. al. used their construction to give a simple form for an amplitude in the NMHV class. (See eqn. (3.7) of ref. [11].) Their form extends straightforwardly to all NMHV amplitudes,
and is sufficient for numerical computations. However, although it is Lorentz invariant, the invariance is not manifest because of apparent (spinor) poles involving an external reference momentum. For the same reason, the CSW form is not convenient for feeding into the unitarity machinery [12] in order to compute loop amplitudes [13]. The purpose of this paper is to transform the CSW form into one which is manifestly Lorentz-invariant, and suitable for use as a building block in computing loop amplitudes. In the next section, I review the off-shell continuation needed to formulate the CSW construction, which I discuss in section 3. I compute the NMHV amplitude in section 4.

2. Off-Shell Continuations

It is convenient to write the full tree-level amplitude using a color decomposition [14],

\[ A_n^{\text{tree}}(\{k_i, \lambda_i, a_i\}) = \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A_n^{\text{tree}}(\sigma(1^{\lambda_1}, \ldots, n^{\lambda_n})) , \]  

where \( S_n/Z_n \) is the group of non-cyclic permutations on \( n \) symbols, and \( j^{\lambda_j} \) denotes the \( j \)-th momentum and helicity \( \lambda_j \). The notation \( j_1 + j_2 \) appearing below will denote the sum of momenta, \( k_{j_1} + k_{j_2} \). I use the normalization \( \text{Tr}(T^a T^b) = \delta^{ab} \). The color-ordered amplitude \( A_n \) is invariant under a cyclic permutation of its arguments. It is the object we will calculate directly.

Using spinor products [15], the third Parke–Taylor equation takes the simple form,

\[ A_n^{\text{tree}}(1^+, \ldots, m_1^-, (m_1+1)^+, \ldots, m_2^-, (m_2+1)^+, \ldots, n^+) = i \frac{(m_1 m_2)^4}{(1 2) (2 3) \cdots (n-1) n \langle n 1 \rangle} . \]  

The CSW construction [11] builds amplitudes out of building blocks which are off-shell continuations of the Parke–Taylor amplitudes. We can obtain an off-shell formulation which is equivalent to the CSW one (but slightly more convenient for explicit calculations) by considering first the off-shell continuation of a gluon polarization vector. In sewing an off-shell gluon carrying momentum \( K \), we will want to sum over all (physical) polarization states. We can do this via the identity,

\[ \sum_{\sigma} \varepsilon_\mu^{(\sigma)}(K, q) \varepsilon^*_{\nu}^{(\sigma)}(K, q) = -g_{\mu\nu} + \frac{K_{\mu} q_{\nu} + q_{\mu} K_{\nu}}{q \cdot K} , \]  

where \( q \) is the reference or light-cone vector, satisfying \( q^2 = 0 \).

Observe that we can always decompose the off-shell momentum \( K \) into a sum of two massless momenta, where one is proportional to \( q \),

\[ K = k^\parallel + \eta(K) q. \]  

† As noted in ref. [11], their form can be made Lorentz-invariant by choosing the reference momentum to be one of the momenta of the process after some manipulation; but this still leaves undesirable denominators, ones that do not reconstruct propagators in the unitarity-based sewing method.
The constraint \((k^b)^2 = 0\) yields
\[
\eta(K) = \frac{K^2}{2q \cdot K}.
\] (2.5)

Of course, if \(K\) goes on shell, \(\eta\) vanishes. Also, if two off-shell vectors sum to zero, \(K_1 + K_2 = 0\), then so do the corresponding \(k^b\)’s.

Noting that \(q \cdot K = q \cdot k^b\), we can then rewrite eqn. (2.3) as follows,
\[
\sum_{\sigma} \varepsilon_{\mu}^{(\sigma)}(K, q) \varepsilon^{*}_{\nu}^{(\sigma)}(K, q) = -g_{\mu\nu} + \frac{k^*_{\mu} q^\nu + q^\mu k^*_{\nu}}{q \cdot k^b} + \frac{2\eta(K) q^\mu q^\nu}{q \cdot k^b}
= \sum_{\sigma} \varepsilon_{\mu}^{(\sigma)}(k^b, q) \varepsilon^{*}_{\nu}^{(\sigma)}(k^b, q) + \frac{2\eta(K) q^\mu q^\nu}{q \cdot k^b} \] (2.6)
\[
= \sum_{\sigma} \varepsilon_{\mu}^{(\sigma)}(k^b, q) \varepsilon^{*}_{\nu}^{(\sigma)}(k^b, q) + \frac{K^2}{(q \cdot k^b)^2} q^\mu q^\nu.
\]

In this expression, \(\varepsilon(k^b, q)\) is of course just the polarization vector for a massless momentum, and so can be expressed in terms of spinor products. The power of \(K^2\) in the second term will cancel the \(1/K^2\) in the propagator, leading to an additional contribution to the four-point vertex. One can formulate a light-cone version of the recurrence relations using such a modified four-point vertex, and retaining only the first term for the gluon propagator. This leads to the simple rule of continuing an amplitude off-shell by replacing \(\varepsilon_{\mu}(k, q) \rightarrow \varepsilon_{\mu}(k^b, q)\). For MHV amplitudes, this amounts to the prescription,
\[
\langle jj' \rangle \rightarrow \langle j^b j' \rangle,
\] (2.7)
when \(k_j\) is taken off shell.

The choice of the momentum \(q\) is equivalent to the choice of the constant spinor \(\eta\) in ref. [11]. The continuation given there amounts to taking
\[
\langle jj' \rangle \rightarrow [q j] \langle jj' \rangle \rightarrow \langle q^+ | k_j | j'^+ \rangle;
\] (2.8)
but this is just equal to
\[
\langle q^+ | k_j^b | j'^+ \rangle = [q j^b] \langle j^b j' \rangle.
\] (2.9)
(The extra factors of \([q j^b]\) present in the CSW construction cancel when sewing vertices into an on-shell amplitude.)

In the notation employed here, an amplitude is manifestly Lorentz invariant (or equivalently manifestly gauge invariant) when it is manifestly free of \(q\), whether present explicitly or implicitly via \(k^b\)’s.
3. Amplitudes from MHV Building Blocks

The CSW construction replaces ordinary Feynman diagrams with diagrams built out of MHV vertices and ordinary propagators. Each vertex has exactly two lines carrying negative helicity (which may be on or off shell), and any number of lines carrying positive helicity. The propagator takes the simple form $i/K^2$, because the physical state projector (2.3) is now effectively supplied by the vertices. The simplest vertex is an amplitude with one leg taken off shell,

$$A_{\text{tree}}^n(1^+, \ldots, m_1^-, (m_1+1)^+, \ldots, m_2^-, (m_2+1)^+, \ldots, (n-1)^+, (-K_{1\ldots(n-1)})^+),$$

where $K_{j\ldots l} \equiv k_j + \cdots + k_l$.

It will be convenient to denote the projected $k^\flat$ momentum built out of $-K_{1\ldots n}$ by $\{1\cdots n\}$, for example $\langle j k^\flat(-K_{1\ldots n}, q) \rangle = \langle j \{1\cdots n\} \rangle$. Because

$$k^\flat(K + k^\flat, q) = K + K' - \eta(K')q - \eta(K + k^\flat)q = K + K' - \left[\eta(K') + \frac{K^2 + 2K \cdot K' - \eta(K')2q \cdot K}{2q \cdot (K + K')}\right]q \approx K + K' - \frac{K^2 + 2K \cdot K' + \eta(K')2q \cdot K'}{2q \cdot (K + K')}q = k^\flat(K + K', q),$$

it does not matter whether we feed in the original off-shell momentum or the corresponding massless projection.

The simplest vertices then have the explicit expression,

$$A_{\text{tree}}^n(1^+, \ldots, m_1^-, (m_1+1)^+, \ldots, m_2^-, (m_2+1)^+, \ldots, (n-1)^+, (-K_{1\ldots(n-1)})^+) = i \langle m_1 m_2 \rangle^4 \frac{\langle 12 \rangle \langle 23 \rangle \cdots \langle (n-1) \{1\cdots(n-1)\} \rangle \{1\cdots(n-1)\} \{1\cdots(n-1)\} \{1\cdots(n-1)\}}{\{1\cdots(n-1)\} \{1\cdots(n-1)\} \{1\cdots(n-1)\} \{1\cdots(n-1)\}},$$

$$A_{\text{tree}}^n(1^+, \ldots, m_1^-, (m_1+1)^+, \ldots, (-K_{1\ldots(n-1)})^-) = i \langle m_1 \{1\cdots(n-1)\} \rangle^4 \frac{\langle 12 \rangle \langle 23 \rangle \cdots \langle (n-1) \{1\cdots(n-1)\} \rangle \{1\cdots(n-1)\} \{1\cdots(n-1)\}}{\{1\cdots(n-1)\} \{1\cdots(n-1)\} \{1\cdots(n-1)\} \{1\cdots(n-1)\}}.$$

The CSW rules then instruct us to write down all tree diagrams with MHV vertices, subject to the constraints that each vertex have exactly two negative-helicity gluons attached, and that each propagator connect legs of opposite helicity. For amplitudes with two negative-helicity gluons, the vertex with all legs taken on shell is then the amplitude; for amplitudes with three negative-helicity gluons, we must write down all diagrams with two vertices. One of the vertices has two of the external negative-helicity gluons attached to it, while the other has only one. An example of such a diagram is shown in fig. 1.
Figure 1. A term in the CSW representation of an NMHV amplitude. The black dot represents the multiparticle pole multiplying the two on-shell amplitudes.

This leads to the following form (without loss of generality, we may take the first leg to have negative helicity),

\[
A_{\text{tree}}^{n}(1^{-}, 2^{+}, \ldots, m_{2}^{-}, (m_{2} + 1)^{+}, \ldots, m_{3}^{-}, (m_{3} + 1)^{+}, \ldots, n^{+}) = \\
\sum_{j_{1} = m_{2} + 1}^{m_{3}} \sum_{j_{2} = 2}^{m_{2}} i A_{\text{tree}}^{j_{1} - j_{2} + 1}(j_{2}^{+}, \ldots, m_{2}^{-}, \ldots, (j_{1} - 1)^{+}, \{j_{2} \cdots (j_{1} - 1)\}^{-}) \\
\times A_{\text{tree}}^{n - j_{1} + j_{2} + 1}(1^{-}, 2^{+}, \ldots, (j_{2} - 1)^{+}, \{1 \cdots (j_{2} - 1); j_{1} \cdots n\}^{+}, j_{1}^{+}, \ldots, m_{3}^{+}, \ldots, n^{+}) \\
+ \sum_{j_{1} = m_{3} + 1}^{m_{2} + 1} \sum_{j_{2} = m_{2} + 1}^{m_{3} + 1} i A_{\text{tree}}^{j_{1} - j_{2} + 1}(j_{2}^{+}, \ldots, m_{3}^{-}, \ldots, (j_{1} - 1)^{+}, \{j_{2} \cdots (j_{1} - 1)\}^{-}) \\
\times A_{\text{tree}}^{n - j_{1} + j_{2} + 1}(1^{-}, 2^{+}, \ldots, m_{2}^{-}, \ldots, (j_{2} - 1)^{+}, \{1 \cdots (j_{2} - 1); j_{1} \cdots n\}^{+}, j_{1}^{+}, \ldots, n^{+}) \\
+ \sum_{j_{1} = 2}^{m_{2}} \sum_{j_{2} = m_{3} + 1}^{n + 1} i A_{\text{tree}}^{j_{2} - j_{1} + 1}(j_{1}^{+}, \ldots, m_{2}^{-}, \ldots, m_{3}^{-}, \ldots, (j_{2} - 1)^{+}, \{j_{1} \cdots (j_{2} - 1)\}) \\
\times A_{\text{tree}}^{n - j_{2} + j_{1} + 1}(j_{2}^{+}, \ldots, n^{+}, 1^{-}, 2^{+}, \ldots, (j_{1} - 1)^{+}, \{1 \cdots (j_{1} - 1); j_{2} \cdots n\}^{-})
\]  
(3.4)

where \( s_{j\cdots l} = K_{j\cdots l}^{2} \equiv (k_{j} + \cdots + k_{l})^{2} \), and all indices are to be understood mod \( n \). The three double sums correspond to the three different choices ((1, \( m_{2} \)), (\( m_{2}, m_{3} \)), or (\( m_{3}, 1 \))) we can make for the pair of negative-helicity gluons which enter the same MHV vertex. In the next section, I will evaluate this expression explicitly.
4. Next-to-MHV Amplitudes

Begin the evaluation of eqn. (3.4) by substituting the explicit forms of the vertices (3.3), and then remove an overall factor of $i(\{1\} \langle 2 \rangle \langle 3 \rangle \cdots \langle n \rangle)^{-1}$.

A generic term in the second double sum then has the form,

$$\frac{(1 m_2)^4 \langle (j_1 - 1) j_1 \rangle \langle (j_2 - 1) j_2 \rangle \langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle^4}{\langle (j_2 - 1) \{ j_1 \cdots (j_2 - 1) \} \rangle \langle (j_1 \cdots (j_2 - 1) \} j_1 \rangle} \times \frac{1}{\langle (j_1 - 1) \{ j_1 \cdots (j_2 - 1) \} \rangle \langle (j_2 \cdots (j_1 - 1) \} j_2 \rangle s_{j_2 \cdots (j_1 - 1)}} \quad (4.1)$$

where $\{ j_1 \cdots (j_2 - 1) \} \equiv \{1 \cdots (j_2 - 1); j_1 \cdots n \}$. We can use momentum conservation, followed by the Schouten identity $\langle a b \rangle \langle c d \rangle = \langle a d \rangle \langle c b \rangle + \langle a c \rangle \langle b d \rangle$ to rewrite this as,

$$- \frac{(1 m_2)^4 \langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle^2}{\langle m_3 j_1 \rangle \langle m_3 j_2 \rangle} \frac{\langle m_3 j_1 \rangle}{\langle (j_1 \cdots (j_2 - 1) \} j_1 \rangle} \frac{\langle m_3 (j_1 - 1) \rangle}{\langle (j_1 \cdots (j_2 - 1) \} (j_1 - 1) \rangle} \times \left( \frac{\langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle^2}{\langle m_3 \{ j_1 \cdots (j_2 - 1) \} \} \rangle \langle j_1 \cdots (j_2 - 1) \} j_2 \rangle s_{j_1 \cdots (j_2 - 1)} \right) \quad (4.2)$$

If we now gather all terms in a double sum containing $\langle m_3 j_1 \rangle$ and $\langle m_3 j_2 \rangle$ (for generic values of $j_1$ and $j_2$), we find

$$- \frac{(1 m_2)^4 \langle m_3 j_1 \rangle \langle m_3 j_2 \rangle}{\langle j_1 j_2 \rangle} \times \left[ \frac{\langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle^2}{\langle j_1 \{ j_1 \cdots (j_2 - 1) \} \rangle \langle j_1 \cdots (j_2 - 1) \} j_2 \rangle s_{j_1 \cdots (j_2 - 1)} \right] - \frac{\langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle^2}{\langle j_1 \{ j_1 \cdots (j_2 - 1) \} \rangle \langle j_1 \cdots (j_2 - 1) \} j_2 \rangle s_{j_1 \cdots (j_2 - 1)} \langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle^2} - \frac{\langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle^2}{\langle j_1 \{ j_1 \cdots (j_2 - 1) \} \rangle \langle j_1 \cdots (j_2 - 1) \} j_2 \rangle s_{j_1 \cdots (j_2 - 1)} \langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle^2} + \frac{\langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle^2}{\langle j_1 \{ j_1 \cdots (j_2 - 1) \} \rangle \langle j_1 \cdots (j_2 - 1) \} j_2 \rangle s_{j_1 \cdots (j_2 - 1)} \langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle^2} \right) \quad (4.3)$$

Multiply and divide by $\langle j_1 j_2 \rangle$, and use the Schouten identity again to split denominators,

$$- \frac{(1 m_2)^4 \langle m_3 j_1 \rangle \langle m_3 j_2 \rangle}{\langle j_1 j_2 \rangle} \times \left[ \frac{\langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle \langle j_1 \{ j_1 \cdots (j_2 - 1) \} \rangle s_{j_1 \cdots (j_2 - 1)}}{\langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle \langle j_1 \{ j_1 \cdots (j_2 - 1) \} \rangle \langle j_1 \cdots (j_2 - 1) \} j_2 \rangle s_{j_1 \cdots (j_2 - 1)} \right] + \frac{\langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle \langle (j_1 + 1) \{ j_1 \cdots (j_2 - 1) \} \rangle s_{(j_1 + 1) \cdots (j_2 - 1)}}{\langle m_3 \{ j_1 \cdots (j_2 - 1) \} \rangle \langle (j_1 + 1) \{ j_1 \cdots (j_2 - 1) \} \rangle \langle (j_1 + 1) \cdots (j_2 - 1) \} s_{(j_1 + 1) \cdots (j_2 - 1)}} \right] \quad (4.4)$$
We must next use a partial-fractioning identity,

\[
\frac{\langle m \{K, k_j \} \rangle}{\langle j \{K, k_j \} \rangle (K + k_j)^2} = \frac{\langle m^- \{K, k_j \} \mid j^- \rangle}{2 k_j \cdot \{K, k_j \}(K + k_j)^2} = \frac{-4 q \cdot (K + k_j)k_j \cdot K + 2 (K + k_j)^2q \cdot k_j \langle K \rangle}{K^2(K + k_j)^2 + 2 k_j \cdot \{K \} K^2}
\]

which in turn relies on the identity

\[
q \cdot (K + k_j) k_j \cdot \{K, k_j \} = q \cdot K k_j \cdot \{K \}.
\]

Using the identity (4.5), we can rewrite eqn. (4.4); doing so, separating out terms without a \( j_1 \parallel j_2 \) singularity, and collecting terms, we obtain

\[
- \frac{\langle 1 m_2 \rangle^4 \langle m_3 j_1 \rangle \langle m_3 j_2 \rangle}{\langle j_1 j_2 \rangle} \times \left[ \frac{\langle m_3 j_1 \rangle \langle m_3 j_2 \rangle \langle j_1 j_2 \rangle}{s_{j_1 \cdots j_2} s_{(j_1+1) \cdots j_2}} + \frac{\langle m_3 j_2 \rangle \langle m_3 j_1 \rangle \langle j_1 j_2 \rangle}{s_{j_1 \cdots (j_2-1) s_{j_1 \cdots j_2}} \right]
\]

\[
+ \frac{\langle m_3 j_1 \rangle \langle m_3^- \mid K_{(j_1+1) \cdots (j_2-1)} \mid j_{j_2}^- \rangle}{s_{j_1 \cdots (j_2-1) s_{(j_2+1) \cdots j_2} s_{(j_1+1) \cdots (j_2-1)}}} \left( s_{j_1 \cdots j_2} s_{(j_1+1) \cdots j_2} - s_{j_1 \cdots (j_2-1) s_{(j_1+1) \cdots (j_2-1)}} \right)
\]

\[
+ \frac{\langle m_3 j_2 \rangle \langle m_3^- \mid K_{(j_1+1) \cdots (j_2-1)} \mid j_{j_2}^- \rangle}{s_{(j_1+1) \cdots (j_2-1) s_{(j_2+1) \cdots j_2}} s_{(j_1+1) \cdots (j_2-1) s_{j_1 \cdots j_2}}} \left( s_{(j_1+1) \cdots (j_2-1) s_{(j_1+1) \cdots (j_2-1)}} - s_{j_1 \cdots (j_2-1) s_{j_1 \cdots j_2}} \right).
\]

Note that all \( q \) dependence has disappeared.

Rewriting the differences of invariants, using the Schouten identity twice (on the sandwich product \( \langle m_3^- \mid K_{(j_1+1) \cdots (j_2-1)} \mid j_{j_2}^- \rangle \langle j_{j_2}^- \mid K_{(j_1+1) \cdots (j_2-1)} \mid j_{j_1}^- \rangle \), and then on the product \( \langle m_3 j_1 \rangle \times \langle j_{j_1}^- \mid K_{(j_1+1) \cdots (j_2-1)} \mid j_{j_1}^- \rangle \), and combining terms, we obtain two final equivalent forms,

\[
\langle 1 m_2 \rangle^4 \left( \langle m_3^- \mid K_{(j_1+1) \cdots (j_2-1) s_{j_1 \cdots j_2}} \mid m_3^+ \rangle \langle m_3^- \mid K_{(j_1+1) \cdots (j_2-1) s_{j_1 \cdots j_2}} \mid m_3^+ \rangle \right)
\]

\[
+ \langle m_3^- \mid K_{(j_1+1) \cdots (j_2-1) s_{j_1 \cdots j_2}} \mid m_3^+ \rangle \langle m_3^- \mid K_{(j_1+1) \cdots (j_2-1) s_{j_1 \cdots j_2}} \mid m_3^+ \rangle \right)
\]

\[
= \langle 1 m_2 \rangle^4 \left( \langle m_3^- \mid K_{(j_1+1) \cdots (j_2-1) s_{j_1 \cdots j_2}} \mid m_3^+ \rangle \langle m_3^- \mid K_{(j_1+1) \cdots (j_2-1) s_{j_1 \cdots j_2}} \mid m_3^+ \rangle \right)
\]

\[
+ \langle m_3^- \mid K_{(j_1+1) \cdots (j_2-1) s_{j_1 \cdots j_2}} \mid m_3^+ \rangle \langle m_3^- \mid K_{(j_1+1) \cdots (j_2-1) s_{j_1 \cdots j_2}} \mid m_3^+ \rangle \right)
\]

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The ‘boundary’ terms in the double sums will not contribute all of the terms in eqn. (4.3); but the remaining $q$ dependence cancels against that in the other double sums. The resulting computation leads one to define,

$$H(m_0, m_1, m_2) =$$

$$- \langle m_1 m_2 \rangle^4 \sum_{j_1 = m_2 + 1}^{m_0} \sum_{j_2 = m_0 + 1}^{m_1} \left( \langle m_0^- | K_{j_1 \cdots j_2} k_{j_2} | m_0^+ \rangle \langle m_0^- | K_{j_1 \cdots j_2} k_{j_2} | m_0^+ \rangle \right)_{s_{j_1 \cdots (j_2-1)} s_{(j_1+1) \cdots j_2}}$$

$$+ \langle m_0^- | K_{(j_1+1) \cdots (j_2-1)} k_{j_1} | m_0^+ \rangle \langle m_0^- | K_{(j_1+1) \cdots (j_2-1)} k_{j_2} | m_0^+ \rangle \right)_{s_{(j_1+1) \cdots (j_2-1)} s_{j_1 \cdots j_2}}$$

$$- \langle m_0 m_1 \rangle \langle m_1 m_2 \rangle \langle m_2 m_0 \rangle \sum_{j = m_0 + 1}^{m_1} \frac{1}{s_{(j+1) \cdots m_2} s_{j \cdots m_0}}$$

$$\times \left( (m_2 m_0)^2 \langle m_1^- | K_{(j+1) \cdots m_2} k_{j} | m_1^+ \rangle + \langle m_1 m_2 \rangle^2 \langle m_0^- | K_{(j+1) \cdots m_2} k_{j} | m_0^+ \rangle$$

$$- \langle m_1 m_2 \rangle \langle m_2 m_0 \rangle \langle m_0^- | K_{(j+1) \cdots m_2} k_{j} | m_1^+ \rangle \right)$$

$$+ \delta_{m_0 \neq m_2 + 1} \frac{\langle m_0 m_1 \rangle^2 \langle m_1 m_2 \rangle^2 \langle m_2 m_0 \rangle^2}{s_{(m_2 + 1) \cdots m_0}}$$

$$- \delta_{m_1 + 1, m_2} \langle m_0 m_1 \rangle \langle m_1 m_2 \rangle^2 \langle m_2 m_0 \rangle \langle m_0^- | K_{m_1 \cdots m_2} | (m_2 + 1)^- \rangle$$

$$\times \left( (\langle m_1 m_2 \rangle \langle m_0 (m_2 + 1) \rangle - \langle m_0 m_1 \rangle \langle m_2 (m_2 + 1) \rangle) \right)_{s_{m_1 \cdots m_2} s_{m_2 (m_2 + 1)}}$$

(4.9)

where the prime signifies that any term with a vanishing denominator is to be omitted (as does the $\delta_{m_0 \neq m_2 + 1}$), and where the sums over indices are understood to run along the cyclic order; that is, for example,

$$\sum_{j = n-4}^{3} \equiv \sum_{j = (n-4) \cdots n-1}$$

(4.10)

With this function, we can then write the desired amplitude in the compact form,

$$A_n^{\text{tree}}(1^-, 2^+, \ldots, m_2^-, (m_2 + 1)^+, \ldots, m_3^-, (m_3 + 1)^+, \ldots, n^+) =$$

$$i(H(1, m_2, m_3) + H(m_2, m_3, 1) + H(m_3, 1, m_2))$$

$$\langle 1 | 2 | 3 | \cdots | (n-1) | n \rangle$$

(4.11)

This formula contains as a special case, an expression equivalent to an earlier result for the amplitude with three adjacent negative helicities [16]. I have verified that it agrees numerically with amplitudes computed via a set of light-cone recurrence relations through $n = 10$.

In the unitarity-based method for loop amplitudes [12], four-dimensional amplitudes such as the one computed above suffice for computing one-loop amplitudes in the supersymmetric gauge theories. For non-supersymmetric theories, additional amplitudes with gluon momenta continued to $D = 4 - 2\epsilon$ (or equivalently massive scalars) are required.
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