Regularity of the correctors and local gradient estimate of the homogenization for the elliptic equation: linear periodic case

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Abstract $C^{\alpha}$ and $W^{1,\infty}$ estimates for the first-order and second-order correctors in the homogenization are presented based on the translation invariant and Li-Vogelius's gradient estimate for the second order linear elliptic equation with piecewise smooth coefficients. If the data are smooth enough, the error of the first-order expansion for piecewise smooth coefficients is locally $O(\varepsilon)$ in the Hölder norm; it is locally $O(\varepsilon)$ in $W^{1,\infty}$ when coefficients are Lipschitz continuous. It can be partly extended to the nonlinear parabolic equation.

Keywords: gradient estimate, homogenization, translation invariant, De Giorgi-Nash estimate

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1 Introduction

Consider the homogenization of the following elliptic problem: find $u_\varepsilon \in H^1_0(\Omega)$,

$$\mathcal{A}_\varepsilon u_\varepsilon = -\frac{\partial}{\partial x_i} \left( a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial u_\varepsilon}{\partial x_j} \right) = f(x), \quad \text{in } \Omega.$$  \hspace{1cm} (1.1)

Here $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and the summation convention is used. $\mathcal{A} = (a_{ij})$ is symmetric and positive definite; $a_{ij}(y)$ is 1-periodic in $y$, $1 \leq i, j \leq n$; $a_{ij}(y)$ is at least piecewise $C^{\mu}$ to obtain the error estimate in $C^{\beta}, W^{1,\infty}$.

Assume all of the data are smooth enough, the $O(\varepsilon)$ error estimate in $L^\infty$ was presented by A. Bensoussan, J. L. Lions and G. Papanicolaou [1]; also see M. Avellaneda and Lin FangHua [2]. O. A. Oleinik, A. S. Shamaev and G. A. Yosifian [3] proved the $O(\varepsilon^{1/2})$ estimate in $H^1$. Cao and Cui [4] studied the spectral properties and the numerical algorithms in perforated domains. Su et al [5] investigated the quasi-periodic problems; Zhang and Cui [6] gave a numerical example for the Rosseland equation. All of these were based on the multiple-scale expansion method [7]. There are also some other famous methods, such as periodic unfolding method [8], Multiscale Finite Element Method(MFEM [9]) and Heterogeneous Multiscale Method(HMM [10]).

The second-order expansion in Section 2 is classical which can be found in Chapter 1 [1] or Chapter 7 [7]. Translation invariant in Section 3 implies the equivalence between the boundary and the interior estimate for an abstract periodic problem. The $C^{\alpha}, W^{1,\infty}$ estimates for correctors in Section 4 follow from the De Giorgi-Nash estimate and Li Yanyan-M. Vogelius's work for piecewise smooth coefficients, respectively. In Section 5, more than the traditional $L^\infty$ estimate $(a_{ij}(y) \in C^{\gamma}([0,1]^n), [2])$, we obtain the $C^{\beta}$ error estimate $(a_{ij}(y))$ piecewise $C^{\mu}$

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in $C^{1,\alpha}$ subdomains, Corollary 5.4). At the end, we prove the main result: the error of the first-order expansion is $O(\varepsilon)$ in $W_{loc}^{1,\infty}$ for Lipschitz continuous coefficients (Corollary 6.3) based on M. Avellaneda-Lin FangHua’s gradient estimate. As far as we know, there are not such kinds ($C^0, W^{1,\infty}$) of error estimates in the homogenization.

2 Second-order two-scale expansion

**Definition 2.1.** The periodic cell $Y = (0, 1)^n$. Let $C^\infty_{\text{per}}(Y)$ be the subset of $C^\infty(\mathbb{R}^n)$ of $Y$-periodic functions (restricted on $Y$). Denote by $H^1_{\text{per}}(Y)$ the closure of $C^\infty_{\text{per}}(Y)$ in the $H^1$ norm. $W^1_{\text{per}}(Y) = \{ \varphi \in H^1_{\text{per}}(Y) : \int_Y \varphi = 0 \}$. $\|u\|_{W^1_{\text{per}}(Y)} \equiv \|\nabla u\|_{L^2(Y)}$. In the same way, we can define $W^2_{\text{per}}(Y)$ where $Y = Y + z, z \in \mathbb{R}^n$.

If $u \in H^1_{\text{per}}(Y)$, $u$ has the same trace on the opposite faces of $Y$. We look for a formal asymptotic expansion of the form

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + ... \quad (2.1)$$

where $u_1(\cdot, y), u_2(\cdot, y)$ are $Y$-periodic in $y$. Let $y = \frac{x}{\varepsilon}$, then

$$\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial}{\partial y_i}. \quad (2.2)$$

Substituting (2.1) into (1.1) and equating the power-like terms of $\varepsilon$, we introduce $N_m(y) \in W^1_{\text{per}}(Y), 1 \leq m \leq n$, to make the terms of order $\varepsilon^{-1}$ equal zero,

$$\int_Y a_{ij}(y) \frac{\partial N_m}{\partial y_i} \frac{\partial \varphi}{\partial y_j} = - \int_Y a_{mij}(y) \frac{\partial \varphi}{\partial y_j}, \quad \forall \varphi(y) \in W^1_{\text{per}}(Y). \quad (2.3)$$

Then let $u_1 = N_m \partial_n u_0$. The problem for $u_2$ (the part of order $\varepsilon^0$) admits a unique solution if and only if there exists a $u_0 \in H^1_0(\Omega)$ such that (a compatibility condition, see Theorem 4.26 [7])

$$- \frac{\partial}{\partial x_i} [a_{ij}(y) \frac{\partial u_0}{\partial x_j}] = f, \quad a_{ij} = \int_Y [a_{ij}(y) + a_{il}(y) \frac{\partial N_i}{\partial y_l}] dy. \quad (2.4)$$

This equation called the homogenization equation is well-posed because $(a^0_{ij})$ is elliptic (Proposition 6.12 [7]).

Find $M_{kl} \in W^1_{\text{per}}(Y), 1 \leq k, l \leq n$, such that

$$\int_Y a_{ij}(y) \frac{\partial M_{kl}}{\partial y_i} \frac{\partial \varphi}{\partial y_j} = \int_Y \left[ a_{kl} + a_{km} \frac{\partial N_l}{\partial y_m} - a^0_{kl} \right] \varphi - \int_Y a_{ik} N_i \frac{\partial \varphi}{\partial y_l}, \quad \forall \varphi \in W^1_{\text{per}}(Y). \quad (2.5)$$

If $\varphi(y) = 1$, the righthand side of the above equation equals zero (a compatibility condition). So let $u_2 = M_{kl} \partial^2_{kl} u_0$ to make the $O(\varepsilon^0)$ terms equal zero. Note that $N_m, M_{kl}, u_0$ are independent of $\varepsilon$. We will use this fact again and again.

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**Corollary 2.2.** Under the hypotheses of Theorem 6.2,

$$\sup_{\overline{\Omega}} |\nabla(u_\varepsilon - u_0 - \varepsilon u_1)| \leq C\varepsilon, \quad \Omega' \subset \subset \Omega; \quad (2.6)$$
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\[
\sup_{\Omega'} |A^{x}(\varepsilon)\nabla(u_\varepsilon - u_0 - \varepsilon u_1)| \leq C\varepsilon, \quad \Omega' \subset\subset \Omega. \tag{2.7}
\]

**Proof.**  (1) We only need to prove that \(|\varepsilon \partial_i u_2| = |\partial_i (\varepsilon M_{k\ell}(\frac{x}{\varepsilon}) \partial^2_{k\ell} u_0)| \leq C\varepsilon. \]

\[
\varepsilon \frac{\partial u_2}{\partial x_i} = \frac{\partial M_{k\ell}(\frac{x}{\varepsilon})}{\partial y_i} \partial^2_{k\ell} u_0 + \varepsilon M_{k\ell}(\frac{x}{\varepsilon}) \partial^3_{ik\ell} u_0. \tag{2.8}
\]

\(M_{k\ell}, \frac{\partial M_{k\ell}}{\partial y_i}\) are bounded from Theorem 4.4; \(\partial^3_{ik\ell} u_0 \in W^{1,q}(\Omega') \hookrightarrow C^{0,\alpha}(\Omega') \subset L^\infty(\Omega').\)

(2) Note that \(A(\frac{x}{\varepsilon}) = (a_{ij})\) is bounded. \(\square\)

**Remark 2.3.** We give the estimate (2.7) because the flux \((A\nabla u)\) is very important in physics. One can consider the tensor case where the flux may be the stress in linear elasticity.

### 7 Some problems

It is possible to partly extend the above results to the following cases:

(1) tensor case: Avellaneda-Lin’s Lemma 6.1 is true for the tensor case \cite{2} and elliptic systems with Neumann boundary conditions \cite{13}; Li-Vogelius’s work was extended in \cite{14}.

(2) nonlinear case: Fusco and Moscariello \cite{15} studied the homogenization of quasilinear divergence structure operators. For the second-order expansion, see \cite{16}.

(3) parabolic case: the parabolic \(C^{\alpha,\alpha/2}\) estimate under mixed boundary conditions was presented in \cite{17}; Li-Vogelius’s gradient estimate was extended to parabolic systems in \cite{14}.

(4) nonsmooth case: if the domain is only convex or the righthand side is piecewise smooth, there are many interesting problems. One problem is that the hypotheses in Theorem 5.3 are very strong: \(\partial \Omega \in C^{2,1}, f \in W^{1,q}(\Omega), q > n\). This is a common difficulty for the multiple-scale method (see \cite{1}, \cite{2}).

(5) How can we get a global \(W^{1,\infty}\) error estimate with a proper boundary corrector?

Some results will appear elsewhere.

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