Abstract. We determine conditions under which a generic gauge invariant nonautonomous and inhomogeneous nonlinear partial differential equation in the two-dimensional space-time continuum can be transformed into standard autonomous forms. In addition to the nonlinear Schrödinger equation, important examples include the derivative nonlinear Schrödinger equation, the quintic complex Ginzburg–Landau equation, and the Gerdjikov–Ivanov equation. This approach provides a mathematical description of nonstationary media supporting unidimensional signal propagation and/or total field trapping. In particular, we study self-similar and nonspreading wave packets for Schrödinger equations. Some other important coherent structures are also analyzed and applications to nonlinear waves in inhomogeneous media such as atmospheric plasma, fiber optics, hydrodynamics, and Bose–Einstein condensation are discussed.

1. Introduction

The nonlinear Schrödinger equation is a universal mathematical model for many physical systems — in a variety of nonlinear problems, the one-dimensional nonlinear Schrödinger equation with a cubic nonlinearity describes the propagation of an envelope wave riding over a carrier when the lowest order contribution to the dispersion relation is proportional to the square of a local amplitude of the wave envelope. This equation is generic in consideration of the lowest effects of dispersion and nonlinearity on a wave packet, including nonlinear optics, deep water waves, acoustics, plasma physics and nonlinear propagation of plane diffracted wave beams in the focusing regions of the ionosphere [9], [16], [18], [27], [28], [74], [84], [85], [91], [98], [105], [166], [147], [203], and [204].

The inverse scattering method is a standard approach to several completely integrable nonlinear partial differential equations, including the Korteweg-de Vries, nonlinear Schrödinger or Gross–Pitaevskii, Sine-Gordon and Kadomtsev–Petviashvili equations [2], [9], [143], [154], [165], and [166]. At the same time, the physical situations in which these equations arise tend to be highly idealized. The inclusion of effects such as damping, external forces, an inhomogeneous medium with variable density and a higher order of the nonlinearity may provide a more realistic model. However, the addition of these perturbation effects could mean that the system is no longer completely integrable (see an example in Refs. [2], [31], [64], and [180]). Hence it is of interest to determine under what conditions the perturbation preserves the completely integrability. Some nonintegrable systems...
in question possess important in practice classes of exact solutions (see, for instance, [39], [40], [128], [137] and the references therein). These solutions may serve as a natural starting point of perturbation methods and also provide a useful testing ground for numerical investigation of more complicated models.

In this paper, we show that the following nonlinear PDE in the two-dimensional space-time continuum:

\[
\frac{\partial \psi}{\partial t} + Q \left( \frac{\partial}{\partial x}, x \right) \psi = P \left( \psi, \psi^*, \frac{\partial \psi}{\partial x}, \frac{\partial \psi^*}{\partial x} \right),
\]

(1.1)

where \( Q \) is a quadratic of two (noncommuting) operators \( \partial/\partial x \) and \( x \) with time-dependent coefficients, the asterisk denotes the complex conjugation, and \( P \) is certain gauge invariant nonlinearity, can be reduced by a simple change of variables to the autonomous form:

\[
\frac{\partial \chi}{\partial \tau} + \frac{\partial^2 \chi}{\partial \xi^2} = R \left( \chi, \chi^*, \frac{\partial \chi}{\partial \xi}, \left( \frac{\partial \chi}{\partial \xi} \right)^* \right),
\]

(1.2)

which is somehow easier to analyze by a variety of available tools (in general, the new time variable \( \tau \) may be complex-valued and the new space variable \( \xi \) is a linear complex-valued function of \( x \); the precise form of this transformation is given by Theorem 1 below). Among important in applications examples are the derivative nonlinear Schrödinger equation, which describes the propagation of circular polarized nonlinear Alfvén waves in plasma physics, the quintic complex Ginzburg–Landau equation, the Gerdjikov–Ivanov equation, and others. Here, we mainly concentrate on variants of the nonlinear Schrödinger equation for which the unperturbed models are known to be completely integrable and/or have explicit solutions.

The paper is organized as follows. In the next section, we transform (1.1) into the standard form (1.2). An overview of integrable autonomous cases and some exact solutions of important nonintegrable PDE models are given in sections 3 to 6 in order to make our presentation as self-contained as possible (a detailed bibliography is provided). Some applications are discussed in section 7. In particular, we study self-similar and nonspreading wave packets for cubic nonlinear Schrödinger equations and quintic complex Ginzburg–Landau equations when some other interesting coherent structures are also available. A self-focusing wave packet for the linear Schrödinger equation, an “Airy gun”, which self-accelerates to an infinite velocity in a finite time, is found in analogy to the well-known blow up property of nonlinear PDE solutions. A natural hierarchy of these wave packets is discussed from a “hidden” symmetry viewpoint and applications to inhomogeneous media including fiber optics, atmospheric plasma, hydrodynamics, and Bose condensation are briefly reviewed.

2. Transforming Nonlinear Schrödinger Equation into Autonomous Forms

We start from the nonautonomous nonlinear Schrödinger equation

\[
i \frac{\partial \psi}{\partial t} = H \left( t \right) \psi + h \left( t \right) |\psi|^p \psi
\]

(2.1)

on \( \mathbb{R} \), where the variable Hamiltonian \( H = Q \left( p, x, t \right) \) is an arbitrary quadratic form of two operators \( p = -i\partial/\partial x \) and \( x \), namely,

\[
i \psi_t = -a \left( t \right) \psi_{xx} + b \left( t \right) x^2 \psi - ic \left( t \right) x \psi_x - id \left( t \right) \psi - f \left( t \right) x \psi + ig \left( t \right) \psi_x + h \left( t \right) |\psi|^p \psi
\]

(2.2)
(a, b, c, d, f, and g are suitable real-valued functions of time only) under the following integrability condition [174], [180]:

\[
h = h_0 a(t) \beta^2(t) \mu_{p/2}(t) = h_0 \beta^2(0) \mu^2(0) \frac{a(t) \lambda^2(t)}{(\mu(t))^{2-p/2}}
\]  

(2.3)

\(h_0\) is a constant, functions \(\beta, \lambda,\) and \(\mu\) will be defined below.

(The corresponding linear quantum systems, when \(h = 0\), are known as the generalized (driven) harmonic oscillators. Some examples, a general approach and known elementary solutions can be found in Refs. [43], [44], [45], [46], [51], [123], [198], [201], and [209].)

For completeness we combine the results established in [116] and [180].

**Lemma 1.** The substitution

\[
\psi = e^{i(\alpha(t)x^2 + \delta(t)x + \kappa(t))} \sqrt{\mu(t)} \chi(\xi, \tau), \quad \xi = \beta(t)x + \varepsilon(t), \quad \tau = \gamma(t)
\]

transforms the non-autonomous and inhomogeneous Schrödinger equation (2.2) into the autonomous form

\[
i\chi_{\tau} + h_0 |\chi|^p \chi = \chi_{\xi \xi} - c_0 \xi^2 \chi \quad (c_0 = 0, 1)
\]

(2.5)

provided that

\[
\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = c_0 a\beta^4,
\]

(2.6)

\[
\frac{d\beta}{dt} + (c + 4a\alpha) \beta = 0,
\]

(2.7)

\[
\frac{d\gamma}{dt} + a\beta^2 = 0
\]

(2.8)

and

\[
\frac{d\delta}{dt} + (c + 4a\alpha) \delta = f + 2g\alpha + 2c_0 a\beta^3 \varepsilon,
\]

(2.9)

\[
\frac{d\varepsilon}{dt} = (g - 2d\delta) \beta,
\]

(2.10)

\[
\frac{d\kappa}{dt} = g\delta - a\delta^2 + c_0 a\beta^2 \varepsilon^2.
\]

(2.11)

Here

\[
\alpha = \frac{1}{4a} \frac{\mu'}{\mu} - \frac{d}{2a}.
\]

(2.12)

(A Mathematica based proof of Lemma 1 is given by Christoph Koutschan [108].)

The substitution (2.12) reduces the inhomogeneous equation (2.6) to the second order ordinary differential equation:

\[
\mu'' - \tau(t) \mu' + 4\sigma(t) \mu = c_0 (2a)^2 \beta^4 \mu,
\]

(2.13)

that has the familiar time-varying coefficients

\[
\tau(t) = \frac{a'}{a} - 2c + 4d, \quad \sigma(t) = ab - cd + d^2 + \frac{d}{2} \left( \frac{a'}{a} - \frac{d'}{d} \right).
\]

(2.14)
The time-dependent coefficients \( \alpha_0, \beta_0, \gamma_0, \delta_0, \varepsilon_0, \kappa_0 \) satisfy the Riccati-type system (2.6)–(2.11) with \( c_0 = 0 \) and are given as follows [43], [179]:

\[
\alpha_0 (t) = \frac{1}{4a(t)} \frac{\mu_0(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)},
\]

(2.15)

\[
\beta_0 (t) = -\frac{\lambda(t)}{\mu_0(t)}, \quad \lambda(t) = \exp\left(-\int_0^t (c(s) - 2d(s)) \, ds\right),
\]

(2.16)

\[
\gamma_0 (t) = \frac{1}{2\mu_1(0)} \frac{\mu_1(t)}{\mu_0(t)} + \frac{d(0)}{2a(0)}
\]

(2.17)

and

\[
\delta_0 (t) = \frac{\lambda(t)}{\mu_0(t)} \int_0^t \left[ (f(s) - \frac{d(s)}{a(s)} g(s)) \mu_0(s) + \frac{g(s)}{2a(s)} \mu_0'(s) \right] \frac{ds}{\lambda(s)},
\]

(2.18)

\[
\varepsilon_0 (t) = -\frac{2a(t)}{\mu_0(t)} \delta_0(t) + 8 \int_0^t \frac{a(s) \sigma(s) \lambda(s)}{(\mu_0(s))^2} (\mu_0(s) \delta_0(s)) \, ds
\]

\[
+ 2 \int_0^t \frac{a(s) \lambda(s)}{\mu_0'(s)} \left( f(s) - \frac{d(s)}{a(s)} g(s) \right) \, ds,
\]

(2.19)

\[
\kappa_0 (t) = \frac{a(t)}{\mu_0(t)} \delta_0^2(t) - 4 \int_0^t \frac{a(s) \sigma(s)}{(\mu_0(s))^2} (\mu_0(s) \delta_0(s))^2 \, ds
\]

\[
- 2 \int_0^t \frac{a(s)}{\mu_0'(s)} (\mu_0(s) \delta_0(s)) \left( f(s) - \frac{d(s)}{a(s)} g(s) \right) \, ds.
\]

(2.20)

\( (\delta_0(0) = -\varepsilon_0(0) = g(0)/(2a(0)) \) and \( \kappa_0(0) = 0 \) provided that \( \mu_0 \) and \( \mu_1 \) are the standard (real-valued) solutions of equation (2.13) when \( c_0 = 0 \) corresponding to the initial conditions \( \mu_0(0) = 0, \) \( \mu_0'(0) = 2a(0) \neq 0 \) and \( \mu_1(0) \neq 0, \mu_1'(0) = 0 \). (Proofs of these facts are outlined in Refs. [43] and [47]. Here, the integrals are treated in the most possible general way which may include stochastic calculus.)

The systems (2.6)–(2.11) can be solved by the following variants of a nonlinear superposition principle [116] and [174].

**Lemma 2.** The solution of the Riccati-type system (2.6)–(2.11) when \( c_0 = 0 \) is given by

\[
\mu(t) = 2\mu(0) \mu_0(t) (\alpha(0) + \gamma_0(t)),
\]

(2.21)

\[
\alpha(t) = \alpha_0(t) - \frac{\beta_0(t)}{4(\alpha(0) + \gamma_0(t))},
\]

(2.22)

\[
\beta(t) = -\frac{\beta(0) \beta_0(t)}{2(\alpha(0) + \gamma_0(t))} = \frac{\beta(0) \mu(0)}{\mu(t)} \lambda(t),
\]

(2.23)

\[
\gamma(t) = \gamma(0) - \frac{\beta^2(0)}{4(\alpha(0) + \gamma_0(t))}
\]

(2.24)

and

\[
\delta(t) = \delta_0(t) - \frac{\beta_0(t) (\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))},
\]

(2.25)
\[ \varepsilon(t) = \varepsilon(0) - \frac{\beta(0) (\delta(0) + \varepsilon_0(t))}{2 (\alpha(0) + \gamma_0(t))}, \quad (2.26) \]
\[ \kappa(t) = \kappa(0) + \kappa_0(t) - \frac{(\delta(0) + \varepsilon_0(t))^2}{4 (\alpha(0) + \gamma_0(t))} \quad (2.27) \]

in terms of the fundamental solution (2.15)–(2.20) subject to the arbitrary initial data \( \mu(0), \alpha(0), \beta(0) \neq 0, \gamma(0), \delta(0), \varepsilon(0), \kappa(0) \).

**Lemma 3.** The solution of the Ermakov-type system (2.6)–(2.11) when \( c_0 = 1 (\neq 0) \) is given by
\[ \mu = \mu(0) \mu_0 \sqrt{\beta^4(0) + 4 (\alpha(0) + \gamma_0)^2}, \quad (2.28) \]
\[ \alpha = \alpha_0 - \beta_0^2 \frac{\alpha(0) + \gamma_0}{\beta^4(0) + 4 (\alpha(0) + \gamma_0)^2}, \quad (2.29) \]
\[ \beta = \frac{\beta(0) \beta_0}{\sqrt{\beta^4(0) + 4 (\alpha(0) + \gamma_0)^2}} = \frac{\beta(0) \mu(0)}{\mu(t)} \lambda(t), \quad (2.30) \]
\[ \gamma = \gamma(0) - \frac{1}{2} \arctan \left( \frac{\beta^2(0)}{2 (\alpha(0) + \gamma_0)} \right), \quad \alpha(0) > 0 \quad (2.31) \]
and
\[ \delta = \delta_0 + \beta_0^2 \frac{\varepsilon(0) \beta^3(0) + 2 (\alpha(0) + \gamma_0) (\delta(0) + \varepsilon_0)}{\beta^4(0) + 4 (\alpha(0) + \gamma_0)^2}, \quad (2.32) \]
\[ \varepsilon = \frac{2 \varepsilon(0) (\alpha(0) + \gamma_0) - \beta(0) (\delta(0) + \varepsilon_0)}{\sqrt{\beta^4(0) + 4 (\alpha(0) + \gamma_0)^2}}, \quad (2.33) \]
\[ \kappa = \kappa(0) + \kappa_0 - \varepsilon(0) \beta^3(0) \frac{\delta(0) + \varepsilon_0}{\beta^4(0) + 4 (\alpha(0) + \gamma_0)^2} \]
\[ + (\alpha(0) + \gamma_0) \frac{\varepsilon^2(0) \beta^2(0) - (\delta(0) + \varepsilon_0)^2}{\beta^4(0) + 4 (\alpha(0) + \gamma_0)^2} \quad (2.34) \]

in terms of the fundamental solution (2.15)–(2.20) subject to the arbitrary initial data \( \mu(0), \alpha(0), \beta(0) \neq 0, \gamma(0), \delta(0), \varepsilon(0), \kappa(0) \).

Here we would like to present a new compact form of these solutions. In order to do that, let us introduce the following complex-valued function:
\[ z = \left( 2 \alpha(0) + \frac{d(0)}{a(0)} \right) \mu_0(t) + \frac{\mu_1(t)}{\mu_1(0)} + i \beta^2(0) \mu_0(t) \quad (2.35) \]
(a complex parametrization of Green’s function and linear invariants of generalized harmonic oscillators are also discussed in Refs. [52] and [75]). Then
\[ z = c_1 E(t) + c_2 E^*(t), \quad (2.36) \]
where the complex-valued solutions are given by
\[ E(t) = \frac{\mu_1(t)}{\mu_1(0)} + i \mu_0(t), \quad E^*(t) = \frac{\mu_1(t)}{\mu_1(0)} - i \mu_0(t) \quad (2.37) \]
and the corresponding complex-valued parameters are defined as follows
\[
c_1 = \frac{1 + \beta^2 (0)}{2} - i \left( \alpha (0) + \frac{d (0)}{2a (0)} \right), \quad c_2 = \frac{1 - \beta^2 (0)}{2} + i \left( \alpha (0) + \frac{d (0)}{2a (0)} \right)
\]
with
\[
c_1 + c_2 = 1, \quad |c_1|^2 - |c_2|^2 = c_1 - c_2^* = \beta^2 (0).
\]
In addition,
\[
z (0) = c_1 + c_2 = 1, \quad z' (0) = 2ia (0) (c_1 - c_2).
\]
The inverses are given by
\[
E = \frac{c_1^* z - c_2 z^*}{|c_1|^2 - |c_2|^2}, \quad E^* = \frac{c_1 z^* - c_2^* z}{|c_1|^2 - |c_2|^2}
\]
and
\[
\mu_0 = \frac{z - z^*}{2i (c_1 - c_2^*)}, \quad \mu_1 = \frac{(c_1^* - c_2^*) z + (c_1 - c_2) z^*}{2 (c_1 - c_2^*)}
\]
in terms of our complex vector \( \mathbf{z} \).

One can readily verify that
\[
\alpha_0 = \frac{1}{4a} \frac{(z - z^*)'}{z - z^*} + \frac{d}{2a}, \quad \beta_0 = -2i \lambda \frac{c_1 - c_2^*}{z - z^*},
\]
\[
\gamma_0 = \frac{(c_1^* - c_2^*) z + (c_1 - c_2) z^*}{2i (z - z^*)} + \frac{d (0)}{2a (0)},
\]
and equations (2.48)–(2.20) can be rewritten in terms of vector \( z \) in view of (2.42).

Finally, we introduce the second complex vector:
\[
\zeta = \varepsilon (0) \beta (0) + i (\delta (0) + \varepsilon_0) = c_3 + i \varepsilon_0, \quad c_3 = \varepsilon (0) \beta (0) + i \delta (0)
\]
and indicate the inverse relations between the essential, real and complex, parameters:
\[
\alpha (0) = \frac{c_1^* - c_1}{2i}, \quad \beta^2 (0) = c_1 - c_2^* = |c_1|^2 - |c_2|^2
\]
and
\[
\delta (0) = \frac{c_3 - c_3^*}{2i}, \quad \varepsilon (0) = \pm \frac{c_3 + c_3^*}{2 \sqrt{|c_1|^2 - |c_2|^2}}.
\]

Then solutions of the initial value problem for the Riccati and Ermakov-type systems can be found in the following complex forms.

**Lemma 4.** In terms of our time-dependent complex vectors \( \zeta \) and \( z \), the solution of Riccati-type system is given by: \( \mu = \mu (0) \) \( \text{Re} \ z = \mu (0) \) \( (z + z^*)/2 \) and
\[
\alpha = \alpha_0 - \frac{\lambda^2}{2 |z|^2} \text{Im} \ z = \alpha_0 - \frac{\lambda^2}{2i |z|^2} z - z^*, \quad \beta = \pm 2 \lambda \sqrt{|c_1|^2 - |c_2|^2} \frac{z - z^*}{z + z^*},
\]
\[
\gamma = \gamma (0) - \frac{1}{2i} \frac{z - z^*}{z + z^*}, \quad \delta = \delta_0 + \lambda \frac{\text{Im} \ z}{\text{Re} \ z} = \delta_0 - i \lambda \frac{\zeta - \zeta^*}{z + z^*},
\]
\[
\varepsilon = \varepsilon (0) \pm i \sqrt{|c_1|^2 - |c_2|^2} \frac{\zeta - \zeta^*}{z + z^*}, \quad \kappa = \kappa (0) + \kappa_0 + \frac{(\zeta - \zeta^*)^2}{8i (c_1 - c_2^*)} \frac{z - z^*}{z + z^*}.
\]
Lemma 5. The Ermakov-type system can be solved as follows: \( \mu = \mu (0) \ |z| \) and

\[
\alpha = \alpha_0 + \lambda \frac{c_1 - c_2^*}{2i |z|^2} \frac{z + z^*}{z - z^*}, \quad \beta = \pm \lambda \sqrt{|c_1|^2 - |c_2|^2} \frac{z + z^*}{|z|}, \quad \gamma = \gamma (0) - \frac{1}{2} \arg z, \tag{2.50}
\]

\[
\delta = \delta_0 + \lambda \frac{\zeta z - \zeta^* z^*}{2i |z|^2}, \quad \varepsilon = \pm \frac{\zeta z + \zeta^* z^*}{2 |z| \sqrt{|c_1|^2 - |c_2|^2}}, \tag{2.51}
\]

\[
\kappa = \kappa (0) + \kappa_0 + \frac{(\zeta^2 z + \zeta^* z^*)(z - z^*)}{8i (c_1 - c_2^*) |z|^2}. \tag{2.52}
\]

(The proofs are rather straightforward and left to the reader.)

As a consequence, for the Ermakov-type system one gets

\[
\frac{2i (\alpha - \alpha_0)}{\beta^2} = \frac{z + z^*}{z - z^*}, \tag{2.53}
\]

\[
i (\alpha - \alpha_0) - \frac{\beta^2}{2} = \beta^2 \frac{z - z^*}{z - z^*}, \quad i (\alpha - \alpha_0) - \frac{\beta^2}{2} = \beta^2 \frac{z^*}{z - z^*} \tag{2.54}
\]

\[
\varepsilon + i \frac{\delta - \delta_0}{\beta} = \frac{\zeta z}{\beta (0) |z|}, \quad \varepsilon - i \frac{\delta - \delta_0}{\beta} = \frac{\zeta^* z^*}{\beta (0) |z|}, \tag{2.55}
\]

\[
\varepsilon^2 + \left( \frac{\delta - \delta_0}{\beta} \right)^2 = \varepsilon^2 (0) + \left( \frac{\delta (0) + \varepsilon_0}{\beta (0)} \right)^2 \tag{2.56}
\]

and

\[
\kappa = \kappa (0) + \kappa_0 + \frac{\delta - \delta_0}{2\beta} \varepsilon - \frac{\varepsilon_0 + \delta (0)}{2\beta (0)} \varepsilon (0) \tag{2.57}
\]

These “quasi-invariants” can be useful, for example, when making comparison of calculations done by different approximation methods.

In this paper, we extend “The Simple Three Lemmas Approach”, which is summarized and modified above for the linear problem, by the complexification of all time-dependent coefficients and by a classification of the simplest nonautonomous nonlinear terms. For a generic nonautonomous (derivative) nonlinear Schrödinger equation of the form

\[
i \psi_t = H \psi + R (\psi), \tag{2.58}
\]

where \( H \) is an arbitrary variable quadratic Hamiltonian and \( R (\psi) = P (\psi, \psi^*, \psi_x, \psi_x^*) \) is a polynomial in four variables, we assume the natural gauge invariance condition:

\[
P \left( \psi e^{iS}, \psi^* e^{-iS}, (\psi e^{iS})_x, (\psi^* e^{-iS})_x \right) = C e^{iS} P \left( \psi, \psi^*, \psi_x, \psi_x^* \right). \tag{2.59}
\]

The lowest terms that satisfy this condition are given by

\[
P (\psi, \psi^*, \psi_x, \psi_x^*) = h_0 \psi + (h_1 x + h_2) |\psi|^2 \psi + i h_3 |\psi|^2 \psi_x + i h_4 \psi^2 \psi_x^* + h_5 |\psi|^4 \psi \tag{2.60}
\]

where \( h_k = h_k (x, t), k = 0, ..., 5 \) are some real or complex-valued functions. Our result is as follows.

Theorem 1. The substitution \((2.7)\) transforms the non-autonomous and inhomogeneous equation \((2.58)\) with the lowest gauge invariant nonlinearities \((2.60)\), namely,

\[
i \psi_t + a (t) \psi_{xx} - b (t) x^2 \psi + ic (t) x \psi_x + id (t) \psi + f (t) x \psi - ig (t) \psi_x \tag{2.61}
\]


\[= h_0 \psi + (h_1 x + h_2) |\psi|^2 \psi + i h_3 |\psi|^2 \psi_x + i h_4 \psi^2 \psi_x^* + h_5 |\psi|^4 \psi,\]

into the autonomous form

\[- i \chi_x + \chi_{\xi\xi} - c_0 \xi^2 \chi = d_0 \chi + (d_1 \xi + d_2) |\chi|^2 \chi + i d_3 |\chi|^2 \chi_{\xi} + i d_4 \chi^2 (\chi_{\xi})^* + d_5 |\chi|^4 \chi \quad (2.62)\]

provided that

\[h_1 = a_2 \beta^2 |\mu| \left( d_1 \beta + \frac{2 \alpha}{\beta} d_3 - \frac{2 \alpha^*}{\beta^*} d_4 \right) e^{2 \text{Im} S},\]

\[h_2 = a_3 \beta^2 |\mu| \left[ d_1 \varepsilon + d_2 + \frac{\delta}{\beta} d_3 - \frac{\delta^*}{\beta^*} d_4 \right] e^{2 \text{Im} S},\]

\[h_3 = d_3 a_2 \beta |\mu| e^{2 \text{Im} S}, \quad h_4 = d_4 a_2 \beta^2 |\mu| e^{2 \text{Im} S},\]

\[h_5 = d_5 a_2 \beta^2 |\mu|^2 e^{4 \text{Im} S}, \quad h_0 = d_0 a_2 \beta^2.\]

Here \(d_0, d_1, d_2, d_3, d_4, d_5\) are constants and \(S = \alpha x^2 + \delta x + \kappa\).

**Proof.** For the case of complex-valued coefficients, the transformation of the linear part is similar to [116] and [180] (with the use of contour integration if needed). Changing the variables in the nonlinear part:

\[\mu^{1/2} e^{-iS} P (2.67)\]

\[= h_0 \chi + i h_3 \beta |\mu| e^{-2 \text{Im} S} \chi_{\xi} + i h_4 e^{\text{Im} S} \chi^2 (\chi_{\xi})^* + \frac{h_5}{|\mu|^2} e^{-4 \text{Im} S} |\chi|^4 \chi\]

and substituting into (2.61), one completes the proof with the aid of our conditions (2.63)–(2.66). \(\square\)

When \(c_0 = d_0 = d_1 = d_3 = d_4 = d_5 = 0, d_2 = h_0 \neq 0\) and \(d_0 = d_1 = d_2 = d_3 = d_4 = 0, d_5 = h_0 \neq 0\), we reproduce the results of Lemma 1 for \(p = 2\) and \(p = 4\), respectively. The derivative nonlinear Schrödinger equation, which was first derived for the propagation of circular polarized nonlinear Alfvén waves in plasma physics [132], [133], and [192], appears when \(c_0 = d_0 = d_1 = d_2 = d_5 = 0\) and \(d_3 = 2d_4\) (see [3], [26], [34], [91], [96], [95], [140] and the references therein for methods of solution of this equation). The amplitude equations of cubic-quintic type have been derived via asymptotic analysis of the governing Navier-Stokes equations of fluid mechanics in the limit of long spatial and slow temporal oscillations near the onset of instability [139]. Among other special cases are the Chen–Lee–Lui derivative nonlinear Schrödinger equation [26] and the Gerdjikov–Ivanov equation [65], [202].

Generic autonomous equations of the type (2.62) and some of their extensions such as quintic complex Ginzburg–Landau equation are discussed in [19], [22], [31], [32], [34], [60], [61], [62], [63], [64], [83], [89], [99], [103], [113], [114], [128], [135], [136], [149], [162], [163], [190], and [193] (see also the references therein).

If conditions (2.63)–(2.66) are not satisfied in certain application, one yet can find \(c_k\) as functions of time and spatial variables thus moving the time dependence into the nonlinear part only. Then
one may use perturbation methods and/or some parameter control when possible. The Feshbach resonance in Bose–Einstein condensation provides a classical example of such nonlinearity control [50], [88], [152], [153], [154], [156], and [157]. A justification of the so-called local density approximation (see [19], [104] and the references therein) can be obtained from Lemma 1, \( c_0 = 0 \), under the following adiabatic condition:

\[
\frac{d}{dt} \left( \frac{h}{a(t)^{2} \beta(t) \mu^{\nu/2}(t)} \right) \ll 1,
\]

when a classical motion of the corresponding quadratic system is already taken into account (in general, the derivative should be taken with respect to a small parameter of the system under consideration).

Moreover, in some important autonomous special cases, our Theorem 1 allows to determine the maximum symmetry groups of the particular equations; see, for example, [141], [142], [124] and the references therein. An extension to random-valued coefficients \( f \) and \( g \) is also possible, cf. [58]. Certain variations of initial data can be analyzed too.

### 3. Integrability of Generalized Autonomous Nonlinear Schrödinger Equations

In the case \( c_0 = d_0 = d_1 = 0 \), a detailed Painlevé analysis of equation (2.62) is performed by Clarkson and Cosgrove [34] (see also [32] for the extension to the case of complex parameters). They have shown that this equation possesses the Painlevé property for partial differential equations only if \( d_5 = d_4 (2d_4 - d_3) / 4 \). When this relation holds, this is equivalent under a gauge transformation [113], [89]:

\[
\chi = \phi \exp \left( -i \nu \int_{\xi_0}^{\xi} |\phi|^2 \, d\xi \right),
\]

(3.1)

to a hybrid of the nonlinear Schrödinger equation and the derivative nonlinear Schrödinger equation,

\[
- i \phi_{\tau} + \phi_{\xi \xi} + \lambda |\phi|^2 \phi + i \mu (|\phi|^2 \phi)_{\xi} = 0
\]

(3.2)

(\( \lambda \), \( \mu \) and \( \nu \) are constants), where one can assume that \( \lambda = 0 \) without loss of generality [193]. The corresponding derivative nonlinear Schrödinger equation is known to be completely integrable [3], [5], [26], [94], [89], [95], [119], [120], and [140] (see also the references therein). Then solutions of the original equation are constructed by the gauge transformation (3.1) in principle (see, for example, Refs. [34], [32], and [89] for more details). Explicit solitary wave solutions can be found in [59], [89], [97], [99], [117], [128], and [163] (see also the references therein). Our Theorem 1 allows to extend these results to a larger class of nonautonomous and inhomogeneous nonlinear Schrödinger equations. In the next sections, we will apply these general results to a nonautonomous solitons and other coherent structures in inhomogeneous medias including atmospheric plasmas.

### 4. Cubic Nonlinear Schrödinger Equation

The cubic nonlinear Schrödinger equation has a plane wave solution with an amplitude dependent phase. Under a simple condition (Lighthill’s criterion), the modulation instability leads to self-focusing, namely, the parts of the wave front with the larger amplitude will propagate slower and this effect will enhance itself by the focusing from adjacent parts (Benjamin–Feir instability [18], [166], [177], [182], and [194]). If the pressure of waves can balance the attraction force, the new
stationary solutions exist, which cannot be obtained by small nonlinear perturbation of the linear problem. These solitary waves, preserve their shapes during the time evolution. If the nonlinearity cannot balance the dispersion, a finite time collapse or dispersion that leads to decay occur. A detailed study of the exact solutions by different methods and many applications of the nonlinear Schrödinger equations can be found in [2], [5], [30], [50], [63], [76], [77], [86], [87], [102], [112], [115], [138], [143], [151], [174], and [180] (see also the references therein).

Equation (2.5) is integrable when $c_0 = 0$ and $p = 2$ [204], [205], and [206]. It has two standard forms, namely, focusing and defocusing. In this section, we concentrate on nonspreading and self-similar solitary wave solutions.

4.1. A Similarity Reduction to the Second Painlevé. In the traditional notations, the corresponding defocusing and focusing nonlinear Schrödinger equations,

$$i\psi_t + \psi_{xx} = \pm 2|\psi|^2\psi$$ (4.1)

by the following substitution

$$\psi(x, t) = e^{ig(x - 2gt^2/3)t} g^{1/3} F(g^{1/3}(x - gt^2)), \quad g = a/2$$ (4.2)

($a$ is the acceleration) can be reduced to the (modified) second Painlevé equations:

$$F'' = zF \pm 2F^3, \quad z = g^{1/3}(x - gt^2),$$ (4.3)

whose (bounded) solutions are the nonlinear Airy functions with known asymptotics as $z \to \pm \infty$ [33], [174].

Combining with the familiar Galilei transformation [165],

$$\psi(x, t) = e^{i(x - vt^2/2)v/2} \chi(x - vt, t)$$ (4.4)

($v$ is the velocity), one obtains a more general solution of this type

$$\psi(x, t) = e^{i(x - vt/2)v/2 + ig(x - vt - 2gt^2/3)t} \times g^{1/3} F(g^{1/3}(x - vt - gt^2))$$ (4.5)

in terms of the second Painlevé Airy functions. (A similarity reduction of the nonlinear Schrödinger equation to the second Painlevé equation was also discussed in Refs. [63], [66], [171], and [181]. The linear case is investigated in Ref. [17] where self-accelerating Airy beams were introduced.) With the help of Lemma 1 we extend these results to the most general nonautonomous and inhomogeneous integrable system of this kind.

4.2. Self-Similar Solutions. The substitution

$$\Psi(X, T) = \chi \left(\sqrt{\frac{2}{\pm ho}} X, \frac{2}{\pm ho} T\right)$$ (4.6)

transforms equation (2.5) into the focusing and defocusing forms in the new coordinates $X$ and $T$:

$$i\Psi_T + \Psi_{XX} \pm 2|\Psi|^2\Psi = 0.$$ (4.7)

By Lemma 1, these transformations give a similarity reduction of the nonautonomous nonlinear Schrödinger equation (2.2) to the second Painlevé equations (4.3). As the result,

$$\psi(x, t) = \frac{1}{\sqrt{\mu(t)}} e^{i(\alpha(t)x + \delta(t)x + \kappa(t))} \Psi \left(\sqrt{\pm ho} (\beta(t)x + \varepsilon(t)), \frac{\pm ho}{2} \gamma(t)\right),$$ (4.8)
where $\Psi (X, T)$ is the solution, say in terms of the second Painlevé transcendent in (4.5), with a trivial change of the notation.

As is known, the nonautonomous Schrödinger equation (2.2) under the integrability condition (2.3) has also the following solution:

$$
\psi (x, t) = e^{i\phi} \sqrt{\mu} \exp \left( i \left( \alpha x^2 + \beta xy + \gamma \left( y^2 - g_0 \right) + \delta x + \varepsilon y + \kappa \right) \right) \times G (\beta x + 2\gamma y + \varepsilon),
$$

(4.9)

where the elliptic function $G$ satisfies equation

$$
\left( \frac{dG}{dz} \right)^2 = C_0 + g_0 G^2 + \frac{1}{2} h_0 G^4 \quad (C_0 \text{ is a constant of integration}).
$$

(4.10)

and $\phi$, $y$, $g_0$ and $h_0$ are real parameters (see also Ref. [174] for a direct derivation of this solution). Examples include bright and dark solitons, and Jacobi elliptic transcendental solutions for nonlinear wave profiles [2], [112], [143], [165], [174]. In the original case (4.1), setting $C_0 = y = 0$, gives the stationary breather, which is located about $x = 0$ and oscillates at a frequency equal to $g_0$ [164, 165].

4.3. Asymptotics and Connection Problems. In the defocusing case, the nonlinear Schrödinger equation (4.1) has the bounded solution,

$$
\psi (x, t) = e^{i(x-vt/2)v/2+ig(x-vt-2gt^2/3)t} g_1^{1/3} A_{k_0} \left( g_1^{1/3} (x-vt-gt^2) \right),
$$

(4.11)

in terms of the nonlinear Airy function $A_{k_0}$, when $-1 < k_0 < 1$ and $k_0 \neq 0$. The corresponding asymptotics are investigated in [4], [5], [12], [13], [14], [16], [40], [48], [49], [130], [131], [158], [167], and [183] (see also the references therein for study of this nonlinear Airy function). The end results are

$$
A_{k_0} (z) = \begin{cases} 
  k_0 \text{Ai} (z), & z \to +\infty \\
  r |z|^{-1/4} \sin (s (z) - \theta_0) + o \left( |z|^{-1/4} \right), & z \to -\infty.
\end{cases}
$$

(4.12)

(the parameter $k_0$ represents amplitude of this nonlinear wave). Here, the following relations hold

$$
s (z) = \frac{2}{3} |z|^{3/2} - \frac{3}{4} r^2 \ln |z|, \quad r^2 = -\pi^{-1} \ln \left( 1 - k_0^2 \right)
$$

(4.13)

and

$$
\theta_0 = \frac{3}{2} r^2 \ln 2 + \arg \Gamma \left( 1 - \frac{i}{2} r^2 \right) + \frac{\pi}{4} \left( 1 - 2\text{sign} (k_0) \right).
$$

(4.14)

If $|k_0| = 1$, then

$$
F (z) \sim \text{sign} (k_0) \sqrt{|z|/2}, \quad z \to -\infty.
$$

(4.15)

If $|k_0| > 1$, then $F (z)$ has a pole at a finite point $z = c_0$, dependent on $k_0$, and

$$
F (z) \sim \text{sign} (k_0) (z - c_0)^{-1}, \quad z \to c_0^+.
$$

(4.16)

In the focusing case, the reduction (4.2) results in the modified second Painlevé equation,

$$
F'' = zF - 2F^3,
$$

(4.17)
and any nontrivial real solution satisfies [33]:
\[
F(z) = r\, |z|^{-1/4} \sin (s(z) - \theta_0) + O \left( |z|^{-5/4} \ln |z| \right), \quad z \to -\infty,
\] (4.18)
where
\[
s(z) = \frac{2}{3} |z|^{3/2} + \frac{3}{4} r^2 \ln |z|
\] (4.19)
with \( r \neq 0 \) and \( \theta_0 \) arbitrary real constants.

The second asymptotic is as follows. If
\[
\theta_0 + \frac{3}{2} r^2 \ln 2 - \frac{1}{4} \pi - \arg \Gamma \left( \frac{i}{2} r^2 \right) = \pi n, \quad n = 0, \pm 1, \pm 2, \ldots,
\] (4.20)
we have
\[
F(z) \sim k_0 \text{Ai} (z), \quad z \to +\infty,
\] (4.21)
where \( k_0 \) is a nonzero real constant. The connection formulas are
\[
r^2 = \pi^{-1} \ln \left( 1 + k_0^2 \right), \quad \text{sign} (k_0) = (-1)^n.
\] (4.22)

For the generic case, when the condition (4.20) is not satisfied, one gets [33]:
\[
F(z) = \alpha \sqrt{z/2} + \alpha \beta |2z|^{-1/4} \cos (s(z) + \theta) + O \left( z^{-1} \right), \quad z \to +\infty,
\] (4.23)
where \( \alpha, \beta > 0 \) and \( \theta \) are real constants, and
\[
s(z) = \frac{1}{3} (2z)^{3/2} - \frac{3}{2} \beta^2 \ln z.
\] (4.24)
The connection formulas for \( \alpha, \beta \) and \( \theta \) are given by
\[
\alpha = -\text{sign} (\text{Im} \xi), \quad \beta^2 = \frac{1}{\pi} \ln \frac{1 + |\xi|^2}{2 |\text{Im} \xi|},
\] (4.25)
\[
\theta = -\frac{3}{4} \pi - \frac{7}{2} \beta^2 \ln 2 + \arg (1 + \xi^2) + \arg \Gamma (i\beta^2),
\] where
\[
\xi = \left( \exp \left( \pi r^2 \right) - 1 \right)^{1/2} \exp \left( i \left( \frac{3}{2} r^2 \ln 2 - \frac{1}{4} \pi + \theta - \arg \Gamma \left( \frac{i}{2} r^2 \right) \right) \right).
\] (4.26)
(See [33] and the references therein for more details.)

In sections 7.1–7.2, we discuss applications of the second Painlevé transcendents to nonlinear waves in fiber optics, plasmas and ocean.

5. Derivative Nonlinear Schrödinger Equations

The derivative nonlinear Schrödinger equation describes the propagation of circular polarized nonlinear Alfvén waves in plasma physics [132], [133], and [192]. This equation is completely integrable [3], [5], [26], [94], [89], [95], [119], [120], and [140] (see also the references therein). Its different forms are related through the gauge transformations [113], [193]. Explicit solutions can be found, for example, in Refs. [59], [95], [96], [97], [99], [117], and [178].
6. Quintic Complex Ginzburg–Landau Equation

A number of nonintegrable nonlinear dissipative PDEs are known to display a wide variety of complex behavior, where the global time evolution is governed by the dynamics of spatially localized structures. This was shown in particular for the family of exact solutions of the one-dimensional supercritical complex Ginzburg–Landau equation [7], [14], [15], [24].

Exact solitary wave solutions of the one-dimensional quintic complex Ginzburg–Landau equation are obtained in Refs. [128] and [163]. In their notation and terminology,

$$\frac{\partial A}{\partial t} = \varepsilon A + (b_1 + ic_1) \frac{\partial^2 A}{\partial x^2} - (b_3 - ic_3) |A|^2 A - (b_5 - ic_5) |A|^4 A,$$

(6.1)

where $\varepsilon$, $b_1$, $c_1$, $b_3$, $c_3$, $b_5$, $c_5$ are real constants and the field $A(x,t)$ is complex-valued (see also [29], [32], [172], [173]). These solutions are expressed in terms of hyperbolic functions, and include coherent structures with a strong spatial localization such as pulses and fronts, as well as, sources and sinks. Equation (6.1) is a one-dimensional model of the large-scale behavior of many nonequilibrium pattern-forming systems (see, for example, [7], [15], [24], [23], [128], [162], [163] and the references therein).

A systematic method for obtaining analytic solitary wave solutions of nonintegrable PDEs has been introduced by Conte and Musette [40], [137] and further developed by Hone [79], [80] and Vernov [187], [188] (see also [32] for another approach and [41], [42], [60], [63], [100] for exact solutions). The unique elliptic traveling wave solution of (6.1) is found in Ref. [188]. Our approach allows to extend some of these results to nonautonomous systems under consideration.

7. Some Applications

For applications of Schrödinger equations in a variety of nonlinear problems, see Refs. [25], [28], [56], [60], [71], [74], [76], [77], [81], [82], [98], [84], [85], [91], [110], [129], [132], [136], [135], [155], [165], [200], [192], [191], [203], and [204].

7.1. Accelerating Airy-Type Packets in Optics. In a Kerr medium with cubic nonlinearity, for which the dependence of the index of refraction on intensity is given by

$$n = n_0(\omega) + i\chi(\omega) + n_2 |E|^2,$$

(7.1)

the propagation of optical pulses in single mode dispersive fibers is described by the nonlinear Schrödinger equation of the form [25], [76], [77], [177]:

$$i \left( \frac{\partial \psi}{\partial t} + \omega_0^2 \frac{\partial^2 \psi}{\partial x^2} + \nu_0 \psi \right) + \frac{1}{2} \omega_0^2 \frac{\partial^2 \psi}{\partial x^2} + \nu_0 \psi^n = 0,$$

(7.2)

which is derived under assumptions that the complex envelope amplitude function $\psi(x,t)$ varies slowly compared to the carrier and that the nonlinear and dispersion terms are weak. Here,

$$\omega_0 = \frac{\partial \omega_0}{\partial k_0}, \quad \omega_0'' = \frac{\partial^2 \omega_0}{\partial k_0^2}, \quad \nu_0 = \chi(\omega_0) \frac{\omega_0}{n_0},$$

(7.3)

and $\nu$ is a geometric factor which depends on the radial variation of the guided electric field (we use the notation of Ref. [76]). In equation (7.2), the second term describes the envelope propagation with group velocity $\omega_0$ (in the absence of the rest of the terms), the third term describes the effect of
absorption, the fourth term describes the effect of dispersion, and the fifth term describes the effect of nonlinearity (further details can be found in [76]). This equation is usually called a nonlinear parabolic equation in plasma physics [74], [84], and [85].

In most of applications, equation (7.2) has constant coefficients and, therefore, is not integrable if $\nu_0 \neq 0$. A similar special case of our generic equation (2.2), when $a = \omega''_0/2$, $b = c = f = 0$, $g = -\omega'_0$, and $h = -\nu_0 n_2/n_0$ can be transformed into the standard forms (4.1) by Lemma 1 with $c_0 = 0$. In our notations, the standard solutions are given by $\mu_0 = 2a e^{2at}$ and $\mu_1 = (1 - 2at) e^{2at}$. By (2.15)–(2.20), $\lambda = e^{2at}$ and

$$\alpha_0(t) = -2\beta_0(t) = \gamma_0(t) = \frac{1}{4at}, \quad \delta_0 = -\varepsilon_0 = \frac{g}{2a}, \quad \kappa_0(t) = \frac{g^2t}{4a}. \quad (7.4)$$

Then $\mu = \mu(0)(1 + 4\alpha(0)at)e^{2at}$ and

$$\alpha = \frac{\alpha(0)}{1 + 4\alpha(0)at}, \quad \beta = \frac{\beta(0)}{1 + 4\alpha(0)at}, \quad \gamma = \gamma(0) - \frac{\beta^2(0)at}{1 + 4\alpha(0)at},$$

$$\delta = \frac{\delta(0) + 2\alpha(0)gt}{1 + 4\alpha(0)at}, \quad \varepsilon = \varepsilon(0) + \beta(0) \frac{g - 2a\delta(0)}{1 + 4\alpha(0)at}t, \quad \kappa = \kappa(0) + \delta(0) \frac{g - \delta(0)}{1 + 4\alpha(0)at}t \quad (7.5)$$

by Lemma 2, thus providing the general transformation (4.8) into the standard forms (4.7). Finally, the integrability condition (2.3) takes the form

$$h = h_0 \beta^2(0) \mu(0) \frac{ae^{2at}}{1 + 4\alpha(0)at} = a (1 + 2t(d - 2\alpha(0)a)t) + O(t^2), \quad t \to 0, \quad (7.6)$$

which indicates the most general integrable system of this kind and may be achieved in fibers, say, through the geometrical factor $\nu$ in order to support nonspreading signals. Moreover, our equations (7.5) describe explicit evolution of variations of initial data. The choice of initial condition $d = 2\alpha(0)a$ allows to include effects of absorption into the zero approximation. Our modification of initial condition allows to compensate some intensity losses.

The bright and dark soliton solutions are given in Refs. [76] and [77] (see also [21], [136], [204]–[206] and the references therein). But here, we mainly concentrate on solutions related to the second Painlevé transcendent.

Explicit solutions (4.2) and (4.5) in terms of the nonlinear Airy functions reveal several remarkable features. In quantum mechanical terms, the “probability density” $|\psi(x,t)|^2$ not only remains unchanged in form but also continually accelerates in empty space. Therefore, these solutions extend to nonlinear cases the nonspreading Airy packet introduced by Berry and Balazs [17] for the unidimensional linear Schrödinger equation without potential (see also [72] and [186]). There is no violation of Ehrenfest’s theorem because the Airy function is not square integrable and cannot represent the probability density for a single particle. What accelerates in the Airy packet is not any individual particle but the caustic (i.e., the envelope, or focus) of the family of orbits (see [17] for more details on this geometrical approach). These nonspreading and freely accelerating wave packets have recently been demonstrated in both one- and two-dimensional configurations in optics [169], [170] and we would like to elaborate on this topic in detail.

It is worth noting that an expansion transformation from the Schrödinger group, e.g. formula (2.8) of [124] with $m = -1/t_1$, gives another accelerating solution of the free particle equation,
\[ i\psi_t + \psi_{xx} = 0, \]
in terms of Airy function as follows:

\[
\psi(x,t) = \sqrt{\frac{|t_1|}{t_1 - t}} \exp \left( ig \left( x - \frac{2g}{3} \frac{t_1 t^2}{t_1 - t} \right) \frac{t_1^2 t}{(t_1 - t)^2} - \frac{i x^2}{4 (t_1 - t)} \right)
\times g^{1/3} \text{Ai} \left( g^{1/3} \left( x - \frac{gt_1 t^2}{t_1 - t} \right) \frac{t_1}{t_1 - t} \right),
\]

which is convenient for \( t < t_1 \). The degenerate case, when \( t_1 = 0 \), can be analyzed with the help of transformation (2.9) of [124]:

\[
\psi(x,t) = \frac{1}{\sqrt{2t}} \exp \left( \frac{i}{4t} \left( x^2 + \left( x + \frac{g}{12t} \frac{9}{2t} \right) \right) \right) g^{1/3} \text{Ai} \left( -\frac{g^{1/3}}{2t} \left( x + \frac{g}{8t} \right) \right).
\]

From now on, we choose \( t_1 > 0 \) for the sake of simplicity. (The most general solution of this kind can be obtained by transformation (2.5) of [124].)

An arbitrary point \( \text{Ai}(x_0 = \text{constant}) \) on the Airy function graph accelerates in empty space according to the law:

\[
x(t) = \frac{gt_1^3}{t_1 - t} - 2gt_1^2 + \left( gt_1 + \frac{x_0}{g^{1/3} t_1} \right) (t_1 - t),
\]

\[
\frac{dx}{dt} = \frac{gt_1^3}{(t_1 - t)^2} - \left( gt_1 + \frac{x_0}{g^{1/3} t_1} \right),
\]

\[
\frac{d^2x}{dt^2} = \frac{2gt_1^3}{(t_1 - t)^3}
\]

in the laboratory frame of reference. This “kinematics” can be obtained as a unique solution of the following singular boundary value problem on \((-\infty, t_1)\):

\[
(t_1 - t)^2 \frac{d^2x}{dt^2} - (t_1 - t) \frac{dx}{dt} - x = 2gt_1^2
\]

(analog of Newton’s second law of motion),

\[
\lim_{t \to -\infty} \frac{x(t)}{t_1 - t} = C_1 = gt_1 + \frac{x_0}{g^{1/3} t_1}
\]

\[
\left( \frac{dx}{dt} \sim -C_1, \ \text{an asymptotic value of constant velocity at } -\infty \right),
\]

\[
\lim_{t \to t_1^-} (t_1 - t) x(t) = C_2 = gt_1^3 \neq 0
\]

\[
(x \sim C_2/(t_1 - t), \ \text{a given residue of the simple pole at } t_1).
\]

The corresponding family of orbits in phase space is given by the equation:

\[
\frac{Q^2}{C_2} = 2C_1 + P + \frac{C_1^2}{P},
\]

where \( P = dx/dt + C_1 \) and \( Q = x + 2gt_1^2 \). By the quadratic formula,

\[
P = \frac{Q^2}{2C_2} - C_1 \pm \sqrt{\frac{Q^2}{2C_2} \left( \frac{Q^2}{2C_2} - 2C_1 \right)}.
\]
This curve degenerates to a parabola when $C_1 = 0$.

In view of (7.9), any two points on the graph, say $\text{Ai}(x_0)$ and $\text{Ai}(y_0)$ with $x_0 < y_0$ taken at the same initial moment of time $t_0$, are contracting to each other:

$$x(t) - y(t) = (x_0 - y_0) \frac{t_1 - t}{g^{1/3} t_1} \rightarrow 0 \quad \text{as} \quad t \rightarrow t_1$$  \hspace{1cm} (7.15)

and their trajectories have two different slopes as $t \rightarrow \pm \infty$, when dispersion that leads to decay occurs. In other words, these analogs of classical trajectories have slanted asymptotes and linearly diverge from each other as $t \rightarrow \pm \infty$:

$$\lim_{t \rightarrow \pm \infty} \left( x(t) - \left( gt_0 + \frac{x_0}{t_1} \right)(t_1 - t) + 2gt_1^2 \right) = 0, \quad \lim_{t \rightarrow t_1} \left( (t_1 - t)x(t) - gt_1^3 \right) = 0$$  \hspace{1cm} (7.16)

but both of them approach to a common vertical asymptote at $t \rightarrow t_1$. With the aid of Airy function asymptotic [146], for any fixed spatial point $x$ one gets

$$|\psi(x,t)| \sim g^{1/3} \sqrt{\frac{2t_1}{\pi}} [(t_1 - t)^2 z]^{-1/4} \left| \sin \left( \frac{2}{3} z^{3/2} \right) \right| \quad \text{as} \quad t \rightarrow t_1^- > 0,$$  \hspace{1cm} (7.17)

where

$$z = g^{1/3} \left( \frac{gt_1 t^2}{t_1 - t} - x \right) \frac{t_1}{t_1 - t} > 0, \quad \lim_{t \rightarrow t_1} (t_1 - t)^2 z = g^{4/3} t_1^4.$$  \hspace{1cm} (7.18)

Thus the leading term remains bounded and does not depend on the spatial coordinate $x$ when $t \rightarrow t_1^-$. As a result, a possible “blow up” of this solution cannot occur at any finite point in space as expected in the linear case.

On the contrary, the cubic nonlinear Schrödinger equation is no longer preserved under the expansion transformation (2.8) of [124]. But the same symmetry, say by our Theorem 1, holds for the quintic nonlinear Schrödinger equation, which is thus invariant under the action of this group of transformations. This is where the blow up solutions do exist and some of them are analyzed in section 7.5 (see, for example, equation (7.39)).

Our solution (7.7), can, in principle, be generated from the Airy packet of Berry and Balazs [17] in the following fashion. Let us consider a generalized harmonic oscillator with the variable Hamiltonian (2.2), which evolves from the initial free particle coefficients and terminates its evolution to the original case of a free particle. Solution of the corresponding Riccati-type system is provided by Lemma 1. By the end of the cycle, the original Airy packet may possess the initial data of (7.7) and this solution will be engaged by the continuity of the wave function. In addition, we provide an example of instability for the linear Airy beam, namely, the corresponding variation of the initial configuration:

$$\psi(x,t_0) = \sqrt{\frac{|t_1|}{t_1 - t_0}} g^{1/3} \text{Ai} \left( g^{1/3} \left( x - \frac{gt_1 t_0^2}{t_1 - t_0} \right) \frac{t_1}{t_1 - t_0} \right)$$

$$\times \exp \left( ig \left( x - \frac{2g}{3} \frac{t_1 t_0^2}{t_1 - t_0} \right) \frac{t_1^2 t_0}{(t_1 - t_0)^2} - \frac{ix^2}{4(t_1 - t_0)} \right),$$  \hspace{1cm} (7.19)

results in a “collapse” of the packet in a finite time $t_0 \leq t < t_1$. In this process, any point from the original beam attains an infinite velocity and acceleration as $t \rightarrow t_1^-$ in the laboratory frame of reference due to deterministic evolution of our solution (one may call this exotic dynamic an “Airy gun”; a Mathematica animation is available on the article website).
The study of propagating nonlinear Airy–Painlevé optical pulses in dispersive fibers was initiated in [66] (see also Ref. [171] for an earlier application of the second Painlevé transcendent in hydrodynamics) and has been continued in recent publications [13], [90], [126], and [161]. Our solution (4.8) incorporates the simplest nonhomogeneous effects of a nonlinear time-varying media in a unified form. The corresponding asymptotics, connection problems, and bibliography are given in section 4.3 for the reader’s convenience. This summary may facilitate further use of these results. An important example of motion of the nonlinear Airy packet in a time-varying spatially uniform force is left to the reader (see [17] for discussion of the classical case).

It is worth addressing a few new features of the nonlinear Airy beams under consideration. According to (4.12), in the defocusing case, which is related to positive group velocity dispersion (dark pulse), a bounded on the entire real line solution corresponds to the real parameter \(|k_0| < 1\), \(k_0 \neq 0\) (a finite range of the ratio of the nonlinearity to dispersion [66]). In the focusing case (anomalous dispersion, bright pulse), the bounded solution can only exist under condition (4.20), which may be thought of as a “quantization rule” in this nonlinear problem. The connection relation take the form

\[
\theta_n = \frac{1}{4} - \frac{3 \ln 2}{2\pi} \ln (1 + k_0^2) + \arg \Gamma \left( \frac{i}{2\pi} \ln (1 + k_0^2) \right) + \pi n, \quad k_0 \neq 0
\]

and the asymptotics of bounded solutions of the modified second Painlevé equation are given by

\[
F(z) = \begin{cases} 
\sim k_0 \text{Ai}(z), & z \to +\infty, \\
\sqrt{\frac{\ln (1 + k_0^2)}{\pi}} |z|^{-1/4} \sin (s(z) - \theta_n) + O \left( |z|^{-5/4} \ln |z| \right), & z \to -\infty
\end{cases}
\]

with

\[
s(z) = s(z) = \frac{2}{3} |z|^{3/2} - \frac{3}{4\pi} \ln (1 + k_0^2) \ln |z|
\]

for \(n = 0, \pm 1, \pm 2, \ldots\). For the bright pulse, there is no restriction on the nonlinear wave amplitude and this case deserves an experimental study.

In our opinion, a joint consideration of (breaking/restoring) symmetry of the linear and (cubic/quintic) nonlinear Schrödinger equations provides a natural hierarchy of accelerating, non-spreading, and blowing up wave packets.

7.2. Nonlinear Airy Waves in Plasma and Ocean. In plasma physics, evolution of weakly nonlinear, strongly dispersive, short wavelength electron Langmuir waves described by their modulating envelope is governed by the Schrödinger equation with cubic nonlinearity. The corresponding solitons are called Langmuir solitons [135], [143] in the case of a homogeneous media. In laser plasma experiments, the plasma is both inhomogeneous and nonlinear to the electromagnetic waves and the large-amplitude plasma waves (see, for example, [27] and the references therein for more details). We demonstrate our approach in the simplest inhomogeneous situation.

Solitons moving with acceleration in linearly inhomogeneous plasma were studied in Refs. [27] and [28] (see also [9] and [184]). Single solitons and multisolitons are found accelerated whereas maintaining their shapes when moving around even upon emerging from collisions with other solitons.
Here we analyze another interesting “one soliton” scenario that is somewhat complementary to recently discovered Airy beams in optics. In the defocusing case,

\[
i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} - 2kx\psi = 2|\psi|^2 \psi,
\]

the following simple substitution

\[
\psi(x,t) = e^{-2i(ktx+2k^2t^3/3)}\chi(\xi,\tau), \quad \xi = x + 2kt^2, \quad \tau = t,
\]

which is originally due to Tappert [27] (see also [180] for a more general case), results in

\[
i \frac{\partial \chi}{\partial \tau} + \frac{\partial^2 \chi}{\partial \xi^2} = 2|\chi|^2 \chi.
\]

When \( g = 2k \), the composition of this transformation with solution (4.11) predicts that, despite of the presence of the external constant force, the following wave packet:

\[
\psi(x,t) = e^{i(x-vt/2)v/2-igvt^2/2g^{1/3}A_k_0 (g^{1/3}(x- vt))}
\]

takes the constant speed \( v \) (the standing wave solution, originally found in Ref. [171], occurs if \( v = 0 \)). Here, the soliton profiles are defined as the Airy-type solutions of the second Painlevé equations and the corresponding asymptotic properties of these nonlinear Airy functions are discussed in section 4 for the both, defocusing and focusing, cases (see also [33] for more details). It is worth noting once again that in the focusing case the “quantization rule” (4.20) should hold for the bounded solution.

For an application of the second Painlevé transcendent, in the case of the nonlinear Schrödinger equation with the linear potential (7.23), to giant waves, as observed on the Agulhas Current of the southwest Indian Ocean, see Ref. [171]. The operation of an electrostatic spherical probe in a slightly ionized, collision-dominated plasma can also be described by the second Painlevé transcendent [36], [92], and [93]. A similar behavior occurs in numerical simulations of the shock waves in plasma and corresponding experiments (see [84], [85], [91] and the references therein).

7.3. Nonlinear Alfvén waves. The propagation of a nonlinear Alfvén wave with a small but not-vanishing wave number along the magnetic field in cold plasmas is governed by the derivative nonlinear Schrödinger equation [132], [133], [192], [191]. The Alfvén solitons are discussed in [134], [155], [200]. The derivative nonlinear Schrödinger equation is also used for modeling of wave processes in nonlinear optics, Stokes waves in fluids of finite depth, etc. Our approach give an opportunity to incorporate the inhomogeneous effects of the media.

Freak waves in plasmas and optical rogue waves are discussed in recent publications [8], [118], [160], [168], and [199].

7.4. Ion Acoustic Envelope Solitons in Explosive Ionospheric Experiments. The theory of nonlinear wave modulation in cohesionless plasma on the basis of the Vlasov description [189] with the nonlinear Landau dumping is developed in [81] and [82] (see also [196]). The corresponding nonlinear Schrödinger equation that describes the amplitude of the plasma density fluctuations has a nonlocal-nonlinear term [82].

The stable ion acoustic envelopes solitons propagating perpendicular to the magnetic field lines can exist in the ionospheric plasma. These solitons were identified by processing the data from the North Star active explosive ionospheric experiment [109], [110]. The parameters of the soliton have been
first estimated under assumption that the coefficients in the Schrödinger equation are constant [109]. The additional ion dissipating terms have been used in order to obtain the stable soliton solutions in a plasma with steep density and temperature gradients [110] (see also the references therein).

Our approach give an alternative opportunity to incorporate nonautonomous dissipative terms. Exact solutions of the cubic complex Ginzburg–Landau equation are to be taken as a nonperturbed approximation. These solutions include coherent structures with a strong spatial localization such as pulses and fronts, as well as, sources and sinks [7], [39], [14], [15], [144], [145], [150], and [163]. More details will be discussed elsewhere.

7.5. Oscillating Coherent Structures. Let us consider the autonomous generalized nonlinear Schrödinger equation (7.27) and address an intriguing question — can self-oscillating solutions exist? By our Theorem 1, the following special case

\[ i \psi_t + \psi_{xx} - x^2 \psi = id_3 |\psi|^2 \psi_x + \psi^2 \psi^*_x + d_5 |\psi|^4 \psi, \]  

(7.27)

(we replace \( \tau = -\gamma \) and use the original notation for convenience) is invariant under the action of the Schrödinger group, which was originally introduced by Niederer [142] for the linear harmonic oscillator, when \( d_3 = d_5 = 0 \) and space-oscillating solutions exist [125], [127]. Therefore the nonlinear equation may possess some interesting oscillating solutions too! The direct action of the Schrödinger group is given by

\[ \psi (x, t) = \sqrt{\frac{\beta (0)}{|z(t)|}} e^{i (\alpha(t)x^2 + \delta(t)x + \kappa(t))} \chi (\beta (t) x + \varepsilon (t) , -\gamma (t)), \]  

(7.28)

where, according to Lemma 5, we define \( z(t) = c_1 e^{2it} + c_2 e^{-2it} \) and find everything in terms of this complex-valued function as follows

\[ \alpha (t) = \frac{i (c_1 c_2^*) z^2 - c_1 c_2 (z^*)^2}{2 (c_1 - c_2^*) |z|^2}, \quad \beta (t) = \pm \frac{\sqrt{|c_1|^2 - |c_2|^2}}{2 |z|^2}, \quad \gamma (t) = \frac{1}{2} \arg z, \]  

(7.29)

\[ \delta (t) = \frac{c_3 z - c_3^* z^*}{2i |z|^2}, \quad \varepsilon (t) = \pm \frac{c_3 z + c_3^* z^*}{2 |z| \sqrt{|c_1|^2 - |c_2|^2}}, \quad \kappa (t) = \frac{(c_3^2 z + c_3^2 z^*) (z - z^*)}{8i (c_1 - c_2^*) |z|^2}. \]

The complex parameters:

\[ c_1 = \frac{1 + \beta^2 (0)}{2} - i \alpha (0), \quad c_2 = \frac{1 - \beta^2 (0)}{2} + i \alpha (0), \quad c_3 = \varepsilon (0) \beta (0) + i \delta (0) \]  

(7.30)

are defined in terms of ‘essential’ real initial data (here we choose \( \gamma (0) = \kappa (0) = 0 \) for the sake of simplicity). The real form of transformation (7.28) and visualization of the corresponding oscillating “missing” solutions for the linear harmonic oscillator can be found in Ref. [125]. In view of

\[ |z|^2 = |c_1|^2 + |c_2|^2 + c_1 c_2^* e^{4it} + c_1^* c_2 e^{-4it}, \quad c_1 c_2^* = \frac{1 - \beta^{-4} (0)}{4} - \alpha ^2 (0) - i \alpha (0), \]  

(7.31)

the shape-preserving configurations, that are somewhat similar to the coherent states of the linear harmonic oscillator, may occur through this transformation only when \( \alpha (0) = 0 \) and \( \beta (0) = \pm 1 \).

Our formulas (7.28)–(7.30) provide a new time-periodic solution of equation (7.27) from any given solution. Although explicit solutions of (7.27) are not readily available in the literature (see, for
example, [38] and the references therein for \( d_3 = 0 \), Bose condensation and/or nonlinear effects in “non-Kerr materials”, e.g. fiber optics beyond the cubic nonlinearity, provide important examples.

7.5.1. Example 1. Mean-field theory has been remarkable successful at description both static and dynamic properties of Bose-Einstein condensates [50], [154]. The macroscopic wave function obeys a 3D cubic nonlinear Schrödinger equation, which is usually called the Gross–Pitaevskii equation in this model. There are several reasons to consider higher order nonlinearity in the Gross–Pitaevskii equation [154]. The quintic case is of particular importance because near Feshbach resonance one may turn the scattering length to zero when the dominant interaction among atoms is due to three-body effects (see [19], [103], [136], [153], [156], [157] and the references therein; on \(^7\)Li, for example, the scattering length is reported as small as 0.01 Bohr radii [157]). Then the nonlinear term in the mean-field equation has the quintic form. Another interesting example is a 1D Bose gas in the limit of impenetrable particles [68], [104], [185].

The 1D-quintic nonlinear Schrödinger equation without potential in dimensionless units,

\[
iA_t + A_{xx} + \frac{3}{4} |A|^4 A = 0,
\]

has the following explicit solutions adapted from [128] (we use the notation and terminology from [128] and [163]; see also [60] and [63]).

Pulses:

\[
A(x, t) = e^{i\phi} \left[ \frac{k}{\cosh k (x - vt)} \right]^{1/2} \times \exp i \left( \frac{vx}{2} + \left( k^2 - v^2 \right) \frac{t}{4} \right)
\]

(\( \phi, v \) and \( k \) are real parameters, the upper sign of the nonlinear term should be taken in (7.32); see also [149] and the references therein). We have

\[
\int_{-\infty}^{\infty} |A(x, t)|^2 \, dx = \pi
\]

and the corresponding plane wave expansion,

\[
A(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} B(p, t) \, dp,
\]

can be found in terms of gamma functions:

\[
B(p, t) = \frac{e^{i\phi}}{2\pi \sqrt{k}} \exp i \left( \frac{v^2 + k^2}{4} - pv \right) \, t \times \Gamma \left( \frac{1}{4} + \frac{i}{2k} \left( p - \frac{v}{2} \right) \right) \Gamma \left( \frac{1}{4} - \frac{i}{2k} \left( p - \frac{v}{2} \right) \right)
\]

with the help of integral (A.3) from Appendix (the case \( \omega = 0 \) allows to evaluate the \( L^1 \)-norm of solution (7.33)). The energy functional is given by

\[
E = -\frac{\psi_x^2}{4} - \frac{1}{4} |\psi|^4 = \frac{v^2}{4} \geq 0
\]

and its positivity provides a sufficient condition for developing of a blow up, namely a singularity such that the wave amplitude tends to infinity in finite time [23], [177], and [208].
Sources and sinks:

\[ A(x,t) = e^{i\phi} r^{1/2} \left[ \frac{\cosh(\sqrt{3}r(x-vt)) + 1}{\cosh(\sqrt{3}r(x-vt)) + 2} \right]^{1/2} \times \exp \left( \frac{vx}{2} - \frac{(v^2 + 3r^2)t}{4} \right) \]  

(7.38)

(\phi, v and r are real parameters; see also [104]). Equation (7.32) has also a class of (double) periodic solutions in terms of elliptic functions [60], [63]. They will be discussed elsewhere.

In addition, direct action of the Schrödinger group [124], [141] on (7.33) produces a six-parameter family of square integrable solutions:

\[ \psi(x,t) = \sqrt{\beta(0)} \exp \left( \frac{\alpha(0)x^2 + \delta(0)x - \delta^2(0)t}{1 + 4\alpha(0)t} + \kappa(0) \right) \]

\[ \times A \left( \beta(0)x - 2\delta(0)t + \varepsilon(0), \frac{\beta^2(0)t}{1 + 4\alpha(0)t} - \gamma(0) \right) \]  

(7.39)

(one can choose \( v = 0 \) without loss of generality; see also [6], [70], [73] and the references therein regarding the Schrödinger group). All of them blow up at the point \( x_0 = -\delta(0)/2\alpha(0) \) in a finite time, when \( t \to t_0 = -1/4\alpha(0) \) and \( \alpha(0) \neq 0 \). (We use real-valued initial data for the corresponding Riccati-type system; see [124] for more details.) Blow up of these solutions can be naturally studied in phase space. The corresponding Wigner function [78], [197] is evaluated with the help of integral (A.1).

A finite time blow up of solutions of the unidimensional quintic nonlinear Schrödinger equation (7.32) is a classical result discussed in many publications; see, for example, [20], [23], [149], [159], [207], [208] and the references therein. This case is critical because any decrease of the power of nonlinearity results in the global existence of solutions [67], [177], [195], [208] (see also Refs. [105] and [104] for a renormalization approach). Here, for the family of solutions (7.39), we have shown that the blow up occurs due to a “hidden symmetry” of this nonlinear PDE. This property holds for all solutions which exponentially decay at infinity.

7.5.2. Example 2. The quintic nonlinear Schrödinger equation in a parabolic confinement,

\[ i\psi_t + \psi_{xx} - x^2\psi + \frac{3}{4} |\psi|^4 \psi = 0, \]  

(7.40)

describes a mean-field model of strongly interacting 1D Bose gases for the practically important case of a harmonic trap [19], [63], [69], [104], [105], [153], [159] and, in particular the so-called Tonks–Girardeau gas of impenetrable bosons [68], [183]; see [101] and [148] for experimental observations. (The time-independent version of the quintic nonlinear Schrödinger equation has been rigorously derived from the many-body problem [122]; see also [121] for a rigorous derivation of the Gross-Pitaevskii energy functional.)

Our observation is as follows. By the gauge transformation (e. g. [6], [73], [124] and the references therein for the linear problem, the quintic nonlinearity is invariant under this transformation by our Theorem 1), equation (7.40) has the following solution:

\[ \psi(x,t) = \frac{e^{-i(\gamma/2)x^2\tan 2t}}{\sqrt{\cos 2t}} A \left( x, \tan 2t, \frac{\tan 2t}{2} \right), \]  

(7.41)
where \( A(x,t) \) is any solution of (7.32), in particular, the pulses and sources (7.33) and (7.38).

Oscillating pulses:

\[
\psi(x,t) = e^{i\phi} \sqrt{\frac{2k}{\cos 2t}} \text{sech}^{1/2}(\frac{2k}{\cos 2t}(x - v \sin 2t)) \times \exp\left(\frac{2vx + (k^2 - v^2 - x^2) \sin 2t}{2 \cos 2t}\right) (7.42)
\]

\( \phi, v, \) and \( k \) are real parameters, the upper sign should be taken in the nonlinear term). They are square integrable at all times:

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx = 1 (7.43)
\]

and

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} |x\psi(x,t)|^2 \, dx = \frac{\pi^2}{(4k)^2} \cos^2 2t + v^2 \sin^2 2t, (7.44)
\]

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} |\psi_x(x,t)|^2 \, dx = \frac{\pi^2}{(4k)^2} \sin^2 2t + v^2 \cos^2 2t + \frac{k^2}{2 \cos^2 2t}. (7.45)
\]

The corresponding elementary integrals are collected in Appendix for convenience; see (A.4).

The expectation values and variances of the position \( x \) and momentum \( p = i^{-1}\partial/\partial x \) operators are given by

\[
\bar{x} = \langle x \rangle = v \sin 2t, \quad \bar{p} = \langle p \rangle = v \cos 2t (7.46)
\]

and

\[
(\delta x)^2 = \bar{x}^2 - \bar{x}^2 = \frac{\pi^2}{(4k)^2} \cos^2 2t, \quad (\delta p)^2 = \bar{p}^2 - \bar{p}^2 = \frac{\pi^2}{(4k)^2} \sin^2 2t + \frac{k^2}{2 \cos^2 2t}, (7.47)
\]

respectively. [It is worth noting that

\[
\bar{x} = vt, \quad \bar{p} = \frac{v}{2}, \quad (\delta x)^2 = \frac{\pi^2}{4k^2}, \quad (\delta p)^2 = \frac{k^2}{8} (7.48)
\]

with

\[
(\delta p)^2 (\delta x)^2 = \frac{\pi^2}{32} > \frac{1}{4} (7.49)
\]

for the original traveling wave solution (7.33)]. The energy functional is given by

\[
E = \mathcal{H} = \bar{p}^2 + \bar{x}^2 - \frac{1}{4} |\psi|^4 = \frac{\pi^2}{(4k)^2} + v^2 > 0 (7.50)
\]

by the direct evaluation.

A remarkable feature of the oscillating solution (7.42) is that the corresponding probability density converges, say as a sequence, periodically in time, to the Dirac delta function at the turning points: \( |\psi(x,t)|^2 \rightarrow \pi \delta (x \mp v) \) as \( t \rightarrow \pm \pi/4 \) etc., when an “absolute squeezing” and/or localization, namely \( \delta x = 0 \), occurs with \( \delta p = \infty \). The fundamental Heisenberg uncertainty principle holds

\[
(\delta p)^2 (\delta x)^2 = \frac{\pi^2}{32} \left( 1 + \frac{\pi^2}{32k^4} \sin^2 4t \right) \geq \frac{\pi^2}{32} > \frac{1}{4} (7.51)
\]
at all times. (It is worth noting that $\pi^2/8 \approx 1.2337$. The minimum-uncertainty squeezed states for a linear harmonic oscillator, when the absolute minimum of the product can be achieved, are discussed in Ref. [111].)

The corresponding Wigner function [78], [197]:

$$W(x, p, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi^*(x + y/2, t) \psi(x - y/2, t) e^{ipy} dy,$$

(7.52)

can be evaluated in terms of hypergeometric function:

$$W(x, p, t) = \text{sech} \omega F_1 \left( \frac{1}{2} + i\omega, 1/2 - i\omega; -\sinh^2 \vartheta \right)$$

(7.53)

with the aid of integral (A.1). Here

$$\omega = \frac{1}{2k} (p \cos 2t + x \sin 2t - v), \quad \vartheta = \frac{2k}{\cos 2t} (x - v \sin 2t).$$

(7.54)

A visualization of Wigner’s function in phase space can be found at the article website.

Our mathematical example reveals a surprising result that a medium described by the quintic nonlinear Schrödinger equation (7.40) may allow, in principle, to measure the coordinate of a “particle” with any accuracy, below the so-called vacuum noise level and without violation of the Heisenberg uncertainty relation, which is a major obstacle, for example, in the direct detection of gravitational waves [11, 55].

Oscillating sources and sinks:

$$\psi(x, t) = e^{i\phi} \sqrt{\frac{2r}{3^{1/2} \cos 2t}} \left[ 1 - \frac{3}{\cosh \left( \frac{2r}{\cos 2t} (x - v \sin 2t) \right) + 2} \right]^{1/2}$$

(7.55)

$$\times \exp i \frac{2vx - (v^2 + r^2 + x^2) \sin 2t}{2 \cos 2t}$$

($\phi$, $v$ and $r$ are real parameters, we have chosen the lower sign of the nonlinear term in (7.40)). Their detailed investigation will be given elsewhere.

Then the action of Schrödinger group, say in our complex form (7.28)–(7.30), on (7.42) and/or (7.55) produces a six-parameter family of new oscillating solutions of equation (7.40). For example, the following extension of (7.42) holds:

$$\psi(x, t) = e^{i\phi} \sqrt{\frac{2k\beta(0)}{2\alpha(0) \sin 2t + \cos 2t}}$$

(7.56)

$$\times \text{sech}^{1/2} 2k \left( \beta(0) \frac{x - (\delta(0) + v\beta(0)) \sin 2t}{2\alpha(0) \sin 2t + \cos 2t} + \varepsilon(0) \right)$$

$$\times \exp i \frac{(2\alpha(0) \cos 2t - \sin 2t) x^2 + \delta(0) (2x - \delta(0) \sin 2t)}{2 (2\alpha(0) \sin 2t + \cos 2t)}$$

$$\times \exp i \frac{2v (x - \delta(0) \sin 2t) + (k^2 - v^2) \beta(0) \sin 2t}{2 (2\alpha(0) \sin 2t + \cos 2t)} + v\varepsilon(0),$$
which presents the most general solution of this kind. (We assume that \( \gamma(0) = \kappa(0) = 0 \) for the sake of simplicity. Although the breather/pulsing solution, when \( \alpha(0) = \delta(0) = \varepsilon(0) = v = 0 \) and \( \beta(0) = 1 \), was found in Ref. [159], our discussion of the uncertainty relation and Wigner function seems to be missing in the available literature.) The blows up occur periodically in time at the points

\[
x_0 = \pm \frac{\delta(0) + v\beta(0)}{\sqrt{4\alpha^2(0) + 1}} \quad \text{when} \quad \cot 2t = -2\alpha(0).
\] (7.57)

The corresponding Wigner function is given by our formula (7.53) with the following values of parameters:

\[
\omega = \frac{1}{2k\beta(0)} [(p - 2\alpha(0))x \cos 2t + (2\alpha(0)p + x) \sin 2t - \delta(0) - v\beta(0)],
\] (7.58)

\[
\vartheta = 2k \left( \frac{\beta(0) x - (\delta(0) + v\beta(0)) \sin 2t}{2\alpha(0) \sin 2t + \cos 2t} + \varepsilon(0) \right).
\]

(One can put \( v = 0 \) in (7.56)–(7.58) without loss of generality because the general action of the Schrödinger group already includes the Galilei transformation [141], [142].)

7.6. A Nonlinear Harmonic Oscillator and Painlevé IV. The following special case of equation (2.62):

\[
i\psi_t + \psi_{xx} - \frac{1}{4} x^2 \psi = 2x |\psi|^2 \psi + 3 |\psi|^4 \psi,
\] (7.59)

by the substitution

\[
\psi(x,t) = e^{-i(n+1/2)t} u(x)
\] (7.60)

can be reduced to the ordinary differential equation for a nonlinear harmonic oscillator studied in Refs. [10], [11], and [33]:

\[
u'' = 3u^5 + 2xu^3 + \left( \frac{1}{4}x^2 - n - \frac{1}{2} \right) u.
\] (7.61)

There are exact ‘bounded state’ solutions for any integer \( n = 0, 1, 2, \ldots \) that asymptotically approach the wave functions of the linear harmonic oscillator as \( |x| \to \infty \) (see [33] for a summary of these results). The probability density \( |\psi(x,t)|^2 = u^2(x) \) can be found in terms of a special fourth Painlevé transcendent. Equation (7.59) arises also as a symmetry reduction of the derivative nonlinear Schrödinger equation, which is solvable by inverse scattering techniques [3], [10], and [94]. (A symmetry reduction of the cubic-quintic nonlinear Schrödinger equation to fourth Painlevé transcendents is discussed in [63].) There are a very few nonlinear ordinary differential equations which obey the Painlevé property and have bounded solutions and this is one of those. It would be interesting to find a connection of these solutions investigated in Refs. [10]–[11] in details with the theory of Bose–Einstein condensation and fiber optics, say for a stable signal propagation.

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Appendix A. Integral Evaluations

The following integral
\[ \int_{-\infty}^{\infty} \frac{e^{i\omega s}}{\sqrt{\cosh s + \cosh c}} \, ds = \frac{\sqrt{2\pi}}{\cosh \pi \omega} \, _2F_1 \left( \frac{1}{2} + i\omega, \frac{1}{2} - i\omega ; 1 \right), \quad \left| \sinh \frac{c}{2} \right| < 1, \quad (A.1) \]

which Mathematica fails to evaluate in a compact form\(^1\), can be derived as a special case of integral representation (2) on page 82 of Ref. [57]. The hypergeometric function is related to the Legendre associated functions, which are a special case of Jacobi functions, see [54], [106], [107]:
\[ P_{1/2, -i\omega} (\cosh c) = _2F_1 \left( \frac{1}{2} + i\omega, \frac{1}{2} - i\omega ; 1 \right) \quad (A.2) \]

and Mehler conical functions. An important special case is given by
\[ \int_{-\infty}^{\infty} \frac{e^{i\omega s}}{\sqrt{\cosh s}} \, ds = \frac{\Gamma (1/4 + i\omega/2) \Gamma (1/4 - i\omega/2)}{\sqrt{2\pi}}. \quad (A.3) \]

Useful elementary integrals,
\[ \int_{-\infty}^{\infty} \frac{du}{\cosh u} = \pi, \quad \int_{-\infty}^{\infty} \frac{u^2 \, du}{\cosh u} = \frac{3\pi}{4}, \quad \int_{-\infty}^{\infty} \frac{\sinh^2 u \, du}{\cosh^3 u} = \frac{3\pi}{8}, \quad \int_{-\infty}^{\infty} \frac{du}{\cosh^3 u} = \frac{\pi}{2}, \quad (A.4) \]
can be verified by Mathematica.

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\(^1\) Oleksandr Pavlyk gave a Mathematica evaluation of this integral.
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