Categorizations of limits of Grothendieck groups over a Frobenius $P$-category

Lluis Puig
CNRS, Institut de Mathématiques de Jussieu, lluis.puig@imj-prg.fr
6 Av Bizet, 94340 Joinville-le-Pont, France

Abstract: In [9, Ch. 14] and [10] we consider suitable inverse limits of Grothendieck groups of categories of modules in characteristics $p$ and zero, obtained from a folded Frobenius $P$-category $\mathcal{F}(\widehat{P}, \hat{\text{aut}}_{\text{fin}})$ [10, 2.8], which covers the case of the Frobenius $P$-categories associated with blocks; moreover, in [13] we show that a folded Frobenius $P$-category is actually equivalent to the choice of a regular central $k^*$-extension $\mathcal{F}^e$ of $\mathcal{F}$ [9, 11.2]. Here, taking advantage of the existence of a perfect $\mathcal{F}^e$-locality $\mathcal{P}^e$, recently proved in [3], [5] and [11], we exhibit those inverse limits as the true Grothendieck groups of the categories of $K^*\hat{G}$- and $k^*\hat{G}$-modules for a suitable $k^*$-group $\hat{G}$ associated to the $k^*$-category $\mathcal{P}^e$ obtained from $\mathcal{P}$ and $\mathcal{F}^e$.

1. Introduction

1.1. Let $p$ be a prime number and $\mathcal{O}$ a complete discrete valuation ring with a field of quotients $K$ of characteristic zero and a residue field $k$ of characteristic $p$; we assume that $k$ is algebraically closed and that $K$ contains “enough” roots of unity for the finite family of finite groups we will consider. Let $G$ be a finite group, $b$ a block of $G$ — namely a primitive idempotent in the center $Z(\mathcal{O}G)$ of the group $\mathcal{O}$-algebra — and $(P, e)$ a maximal Brauer $(b, G)$-pair [9, 1.16]; recall that the Frobenius $P$-category $\mathcal{F}(b, G)$ associated with $b$ is the subcategory of the category of finite groups where the objects are all the subgroups of $P$ and, for any pair of subgroups $Q$ and $R$ of $P$, the morphisms $\varphi$ from $R$ to $Q$ are the group homomorphisms $\varphi: R \to Q$ induced by the conjugation of some element $x \in G$ fulfilling

$$(R, g) \subset (Q, f)^T$$

1.1.1

where $(Q, f)$ and $(R, g)$ are the corresponding Brauer $(b, G)$-pairs contained in $(P, e)$ [9, Ch. 3].

1.2. In [9, Ch. 14] we consider a suitable inverse limit of Grothendieck groups of categories of modules in characteristic $p$ obtained from $\mathcal{F}(b, G)$, which according to Alperin’s Conjecture should be isomorphic to the Grothendieck group of the category of finitely dimensional $kGb$-modules. In [10] we generalize this construction in two directions. On the one hand, we also consider a suitable inverse limit of Grothendieck groups of categories of modules in characteristic zero which again, according to Alperin’s Conjecture, should be isomorphic to the Grothendieck group of the category of finitely dimensional $KGb$-modules. On the other hand, with the introduction of the folded Frobenius $P$-categories [10, §2], we are able to extend all these constructions to any folded Frobenius $P$-category.
1.3. Let us recall our definitions. Denoting by $P$ a finite $p$-group, by $\mathfrak{G}$ the category formed by the finite groups and the injective group homomorphisms, and by $\mathcal{F}_P$ the subcategory of $\mathfrak{G}$ where the objects are all the subgroups of $P$ and the morphisms are the group homomorphisms induced by the conjugation by elements of $P$, a Frobenius $P$-category $\mathcal{F}$ is a subcategory of $\mathfrak{G}$ containing $\mathcal{F}_P$ where the objects are all the subgroups of $P$ and the morphisms fulfill the following three conditions [9, 2.8 and Proposition 2.11]

1.3.1 For any subgroup $Q$ of $P$ the inclusion functor $(\mathcal{F})_Q \to (\mathfrak{G})_Q$ is full.

1.3.2 $\mathcal{F}_P(P)$ is a Sylow $p$-subgroup of $\mathcal{F}(P)$.

1.3.3 For any subgroup $Q$ of $P$ such that we have $\xi(C_P(Q)) = C_P(\xi(Q))$ whenever $\xi : Q \to C_P(Q)$ is an $\mathcal{F}$-morphism, any $\mathcal{F}$-morphism $\varphi : Q \to P$ and any subgroup $R$ of $N_P(\varphi(Q))$ containing $\varphi(Q)$ such that $\mathcal{F}_P(Q)$ contains the action of $\mathcal{F}_R(\varphi(Q))$ over $Q$ via $\varphi$, there is an $\mathcal{F}$-morphism $\zeta : R \to P$ fulfilling $\zeta(\varphi(u)) = u$ for any $u \in Q$.

1.4. Moreover, we say that a subgroup $Q$ of $P$ is $\mathcal{F}$-selfcentralizing if we have

$$C_P(\varphi(Q)) \subset \varphi(Q)$$

for any $\varphi \in \mathcal{F}(P, Q)$, and we denote by $\mathcal{F}^{sc}$ the full subcategory of $\mathcal{F}$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$. We call $\mathcal{F}^{sc}$-chain any functor $q : \Delta_n \to \mathcal{F}^{sc}$ where the $n$-simplex $\Delta_n$ is considered as a category with the morphisms defined by the order [9, A2.2]; we denote by $\mathfrak{G}(\Delta_n, \mathcal{F}^{sc})$ this set of functors and by $\mathfrak{ch}^*(\mathcal{F}^{sc})$ the category where the objects are all the $\mathcal{F}^{sc}$-chains $(q, \Delta_n)$, where $n$ runs on $\mathbb{N}$, and the morphisms from $q : \Delta_n \to \mathcal{F}^{sc}$ to another $\mathcal{F}^{sc}$-chain $r : \Delta_m \to \mathcal{F}^{sc}$ are the pairs $(\nu, \delta)$ formed by an order preserving map $\delta : \Delta_m \to \Delta_n$ and by a natural isomorphism $\nu : q \circ \delta \cong r$ [9, A2.8]. Recall that we have a canonical functor

$$\text{aut}_{\mathcal{F}^{sc}} : \mathfrak{ch}^*(\mathcal{F}^{sc}) \to \mathfrak{G}$$

mapping any $\mathcal{F}^{sc}$-chain $q : \Delta_n \to \mathcal{F}^{sc}$ to the group of natural automorphisms of $q$ [9, Proposition A2.10].

1.5. Recall that a $k^*$-group $\hat{G}$ is a group endowed with an injective group homomorphism $\theta : k^* \to Z(\hat{G})$ [7, §5], that $G = \hat{G} / \theta(k^*)$ is the $k^*$-quotient of $\hat{G}$ and that a $k^*$-group homomorphism is a group homomorphism which preserves the multiplication by $k^*$; let us denote by $k^* \mathfrak{G}$ the category of $k^*$-groups with finite $k^*$-quotient. Then, a folded Frobenius $P$-category $(\mathcal{F}, \text{aut}_{\mathcal{F}^{sc}})$ is a pair formed by a Frobenius $P$-category $\mathcal{F}$ and, by a functor

$$\text{aut}_{\mathcal{F}^{sc}} : \mathfrak{ch}^*(\mathcal{F}^{sc}) \to k^* \mathfrak{G}$$

lifting the canonical functor $\text{aut}_{\mathcal{F}^{sc}}$ in the case of the Frobenius $P$-category $\mathcal{F}(b, G)$ above, we already know that the situation provides $k^*$-groups $\hat{F}(b, G)(Q)$
lifting \( \mathcal{F}_{(b,G)}(Q) \) for any \( \mathcal{F} \)-selfcentralizing subgroup \( Q \) of \( P \) [9, 7.4] and in [9, Theorem 11.32] we prove the existence of a lifting \( \hat{\text{aut}}_{\mathcal{F}_{(b,G)}} \) of \( \text{aut}_{\mathcal{F}_{(b,G)}} \) extending them; note that in [9, Theorem 11.32] we may assume that \( k \) is just the closure of the prime subfield. But in [13, Theorem 3.7] we prove that any folder structure on \( \mathcal{F} \) comes from an essentially unique regular central \( k^* \)-extension \( \hat{\mathcal{F}}^e \) of \( \tilde{\mathcal{F}}^e \) and, from now on, a folder structure on \( \mathcal{F} \) means a regular central \( k^* \)-extension \( \hat{\mathcal{F}}^e \) of \( \tilde{\mathcal{F}}^e \); if this regular central \( k^* \)-extension comes from a regular central \( \hat{k}^* \)-extension where \( \hat{k} \) is the algebraic closure of the prime subfield then we say that the folder structure is finite.

**Lemme 1.6.** For any finite folder structure \( \hat{k^*} \) of \( \mathcal{F}^e \) there are a finite subfield \( \hat{k} \) of \( k \) and a regular central \( \hat{k}^* \)-extension \( \hat{\mathcal{F}}^e \) of \( \tilde{\mathcal{F}}^e \) such that the extension of \( \hat{\mathcal{F}}^e \) from \( \hat{k^*} \) to \( k^* \) is equivalent to \( \tilde{\mathcal{F}}^e \).

**Proof:** We actually may assume that \( k \) is the algebraic closure of the prime subfield; then, for any \( \mathcal{F}^e \)-morphism \( \varphi : R \to Q \), choose a lifting \( \hat{\varphi} : R \to Q \) of \( \varphi \) in \( \tilde{\mathcal{F}}^e(Q, R) \); thus, for any pair of \( \mathcal{F}^e \)-morphisms \( \varphi : R \to Q \) and \( \psi : T \to R \) we get

\[
\hat{\varphi} \circ \hat{\psi} = \lambda_{\varphi,\psi} \cdot \hat{\varphi} \circ \psi
\]

for a suitable finite family \( \{ \lambda_{\varphi,\psi} \}_{\varphi,\psi} \) of elements of \( k^* \) which are algebraic; hence, the subfield \( \hat{k} \) of \( k \) generated by this family is finite and it is clear that we can define a regular central \( \hat{k}^* \)-extension \( \hat{\mathcal{F}}^e \) of \( \tilde{\mathcal{F}}^e \) contained in \( \hat{\mathcal{F}}^e \) by setting

\[
\hat{\mathcal{F}}^e(Q, R) = \bigcup_{\varphi \in \mathcal{F}^e(Q, R)} \hat{k^*} \cdot \hat{\varphi} \subset \hat{\mathcal{F}}^e(Q, R)
\]

and that this inclusion induces a bijection

\[
k^* \times_{\hat{k^*}} \hat{\mathcal{F}}^e(Q, R) \cong \hat{\mathcal{F}}^e(Q, R)
\]

1.7. Note that a regular central \( k^* \)-extension \( \hat{\mathcal{F}}^e \) of \( \mathcal{F}^e \) induces a regular central \( \hat{k}^* \)-extension \( \hat{\mathcal{F}}^e \) of the exterior quotient \( \tilde{\mathcal{F}}^e \) of \( \mathcal{F}^e \) — namely of the quotient of \( \tilde{\mathcal{F}}^e \) by the inner automorphisms of the objects [9, 6.1]; moreover, assuming that the folder structure \( \hat{\mathcal{F}}^e \) is finite, choosing a finite subfield \( \hat{k} \) of \( k \) and a regular central \( \hat{k}^* \)-extension \( \hat{\mathcal{F}}^e \) of \( \mathcal{F}^e \) as above, and denoting by \( \hat{\mathcal{F}}^e \) regular \( \hat{k}^* \)-extension of the exterior quotient \( \tilde{\mathcal{F}}^e \), if follows easily from [13, Proposition 3.5] that the inclusion \( \hat{\mathcal{F}}^e_p \subset \hat{\mathcal{F}}^e \) can be lifted to a faithful functor \( \hat{\mathcal{F}}^e_p \to \hat{\mathcal{F}}^e \) and then, with the terminology introduced in [12, 2.2], \( \hat{\mathcal{F}}^e \) becomes a \( \hat{\mathcal{F}}^e_p \)-category and fulfills the finiteness condition in [12, 4.1]; at this point, it follows from [12, Proposition 4.6] that \( \hat{\mathcal{F}}^e \) becomes a multiplicative \( \hat{\mathcal{F}}^e_p \)-category since it inherits from \( \tilde{\mathcal{F}}^e \) both conditions in this proposition.
1.8. On the other hand, in [3], [5] and [11] it has been recently proved that there exists a unique perfect $F^\infty$-locality $P^\infty$ [9, 17.4 and 17.13]; more explicitly, denote by $T_P^\infty$ the category where the objects are all the $F$-self-centralizing subgroups of $P$, where the set of morphisms from $R$ to $Q$ is the $P$-transporter $T_P(R, Q)$ for a pair of $F$-self-centralizing subgroups $Q$ and $R$ of $P$, and where the composition is induced by the product in $P$; then, there is a unique Abelian extension $\pi^\infty : P^\infty \rightarrow F^\infty$ of $F^\infty$ endowed with a faithful functor $\tau^\infty : T_P^\infty \rightarrow P^\infty$ in such a way that the composition $\pi^\infty \circ \tau^\infty : T_P^\infty \rightarrow F^\infty$ is the canonical functor defined by the conjugation in $P$, that $P^\infty(Q)$ endowed with $\tau^\infty_Q : N_P(Q) \rightarrow P^\infty(Q)$ and $\pi^\infty_Q : P^\infty(Q) \rightarrow F^\infty(Q)$ is an $F$-localizer of $Q$ for any $F$-self-centralizing subgroup $Q$ of $P$ fully normalized in $F$ [9, Theorem 18.6], and that $Z(R)$ acts regularly over the fibers of the map

$$P^\infty(Q, R) \rightarrow F^\infty(Q, R)$$

induced by $\pi^\infty$ [9, 17.7] for any pair of $F$-self-centralizing subgroups $Q$ and $R$ of $P$.

1.9. Then, the so-called $F$-localizing functor considered in [10, 3.2.1]

$$\text{loc}_{F^\infty} : \text{ch}^*(F^\infty) \rightarrow \tilde{\text{Loc}}$$

is actually just a quotient of the canonical functor

$$\text{aut}_{F^\infty} : \text{ch}^*(F^\infty) \rightarrow \mathfrak{O}$$

mapping any $P$-chain $q : \Delta_n \rightarrow P^\infty$ to the group of natural automorphisms of $q$ [9, Proposition A2.10]; moreover, a regular central $k^*$-extension $\tilde{F}^\infty$ of $F^\infty$ determines via $\pi^\infty$ a regular central $k^*$-extension $\tilde{P}^\infty$ of $P^\infty$ and, once again, the faithful functor $\tau^\infty : T_P^\infty \rightarrow P^\infty$ can be lifted to a faithful functor $\tilde{\tau}^\infty : T_P^\infty \rightarrow \tilde{P}^\infty$ [13, Proposition 3.5]; hence, the corresponding functor

$$\tilde{\text{loc}}_{F^\infty} : \text{ch}^*(F^\infty) \rightarrow k^* \cdot \tilde{\text{Loc}}$$

considered in [10, 3.3.1] is presently just a quotient of the obvious canonical functor

$$\text{aut}_{\tilde{P}^\infty} : \text{ch}^*(\tilde{P}^\infty) \rightarrow k^* \cdot \mathfrak{O}$$

mapping any $\tilde{P}$-chain $\tilde{q} : \Delta_n \rightarrow \tilde{P}^\infty$ to the $k^*$-group of natural automorphisms of $\tilde{q}$ [9, Proposition A2.10], and we simply set $\text{aut}_{\tilde{P}^\infty}(\tilde{q}, \Delta_n) = \tilde{P}^\infty(\tilde{q})$.

1.10. In this situation, considering the contravariant functors

$$\text{g}_C : k^* \cdot \mathfrak{O} \rightarrow \text{O-mod} \quad \text{and} \quad \text{g}_B : k^* \cdot \mathfrak{O} \rightarrow \text{O-mod}$$

mapping any $k^*$-group $\hat{G}$ with finite $k^*$-quotient on the extensions to $O$

$$\mathcal{G}_C(\hat{G}) = \mathcal{O} \otimes \mathbb{Z} \mathcal{G}_C^\infty(\hat{G}) \quad \text{and} \quad \mathcal{G}_B(\hat{G}) = \mathcal{O} \otimes \mathbb{Z} \mathcal{G}_B^\infty(\hat{G})$$
of the respective Grothendieck groups $G^Z_K(\hat{G})$ and $G^Z_K(\hat{G})$ of the categories of finitely dimensional $K_*$-modules and $K_*$-modules, and any $k^*$-group homomorphism $\hat{\theta} : \hat{G} \to \hat{G}'$ on the corresponding restriction maps, with the the notation in [10, 3.5.3] it is quite clear that

$$G_K(F, \text{aut}_{P^w}) = \lim_{\leftarrow} (g_K \circ \text{loc}_{P^w}) \cong \lim_{\leftarrow} (g_K \circ \text{aut}_{P^w})$$

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1.10.3.

Our purpose here is to exhibit these $O$-modules as the $O$-extensions of the very Grothendieck groups of suitable categories; as a matter of fact, we find a $k^*$-group $\hat{G}(\hat{P}^w)$, with finite $k^*$-quotient, such that

$$G_K(F, \text{aut}_{P^w}) \cong G_K(\hat{G}(\hat{P}^w))$$ and $G_K(F, \text{aut}_{P^w}) \cong G_K(\hat{G}(\hat{P}^w))$ 1.10.4.

in a compatible way with the corresponding decomposition maps [10, 3.5.3]. We borrow our notation from [9] and [10].

**Remark 1.11.** Denoting by $g^Z_k$ the contravariant functor mapping any $k^*$-group $\hat{G}$ with finite $k^*$-quotient on the positive part $G^Z_K(\hat{G})$ of the very Grothendieck group of the category of finitely dimensional $K_*$-modules — the part formed by the classes of the $K_*$-modules — note that the inverse limit of the functor $g^Z_K \circ \text{aut}_{P^w}$ still makes sense and actually it generates $\lim_{\leftarrow} (g^Z_K \circ \text{aut}_{P^w})$. Indeed, denoting by $R_q$ the class in $G^Z_K(\hat{P}^w(q))$ of the regular $K_*, \hat{P}^w(q)$-module $K_*, \hat{P}^w(q)$ for any $\hat{P}^w$-chain $q : \Delta_\infty \to \hat{P}^w$, and choosing a multiple $m$ of all the orders $|\hat{P}^w(q)|$ where $q$ runs over the set of $\hat{P}^w$-chains, it is easily checked that the family $R = \left\{ \frac{m}{|\hat{P}^w(q)|} R_q \right\}_q$ belongs to $\lim_{\leftarrow} (g^Z_K \circ \text{aut}_{P^w})$ and that, for any element $X$ of $\lim_{\leftarrow} (g^Z_K \circ \text{aut}_{P^w})$, the sum of $X$ with a suitable multiple of $R$ belongs to $\lim_{\leftarrow} (g^Z_K \circ \text{aut}_{P^w})$. The analogous argument also holds for $g^Z_k$.

2. Categorization for the characteristic zero case

2.1. Let $P$ be a finite $p$-group and $(F, \hat{F}^w)$ a finite folded Frobenius $P$-category; denote by $P$ and $P^w$ the respective perfect $F$- and $\hat{F}^w$-localities [11, §6 and §7] and by $\pi : P \to F$ and $\tau : T_P \to P$ the structural functors [9, 17.3]; then, the regular central $k^*$-extension $\hat{F}^w$ of $F^w$ determines via $\pi^w$ a regular central $k^*$-extension $\hat{P}^w$ of $P^w$ and we set (cf. 1.10.3)

$$G_K(\hat{F}^w) = \lim_{\leftarrow} (g_K \circ \text{aut}_{P^w})$$ 2.1.1;
that is to say, \( \mathcal{G}_K(\hat{P}^{sc}) \) is the subset of elements
\[
\{ X_{(\hat{q},\Delta_n)} \}_{\hat{q} \in \hat{\mathcal{F}}(\Delta_n, \hat{P}^{sc})} \cap \prod_{n \in \mathbb{N}} \prod_{\hat{q} \in \hat{\mathcal{F}}(\Delta_n, \hat{P}^{sc})} \mathcal{G}_K(\hat{P}^{sc}(\hat{q}))
\]
2.1.2

such that, for any \( \hat{P}^{sc} \)-morphism \((\nu, \delta): (\hat{q}, \Delta_n) \to (\hat{r}, \Delta_m)\), they fulfill
\[
\text{res}_{\mathfrak{aut}_{\hat{P}^{sc}}}(\nu, \delta)(X_{(\hat{r},\Delta_m)}) = X_{(\hat{q},\Delta_n)}
\]
2.1.3;

moreover, we assume that, for any extension \( K' \) of \( K \), the scalar extension from \( K \) to \( K' \) induces an isomorphism between \( \mathcal{G}_K(\hat{P}^{sc}(\hat{q})) \) and \( \mathcal{G}_{K'}(\hat{P}^{sc}(\hat{q})) \) for any \( n \in \mathbb{N} \) and any \( \hat{q} \in \hat{\mathcal{F}}(\Delta_n, \hat{P}^{sc}) \). Our purpose in this section is both to exhibit \( \mathcal{G}_K(\hat{P}^{sc}) \) as the extension to \( \mathcal{O} \) of the very Grothendieck group \( \mathcal{G}_K(\hat{P}^{sc}) \) of a suitable subcategory of \( \mathcal{K}-\mathfrak{mod} \)-valued contravariant \( k^* \)-functors over the category \( \hat{P}^{sc} \), and to show that this subcategory is equivalent to the category of \( K, \hat{G} \)-modules for a suitable \( k^* \)-group \( \hat{G} \) with finite \( k^* \)-quotient.

2.2. First of all, it follows from [10, Corollary 8.4] and from 1.10.3 above that
\[
\text{rank}_{\mathcal{O}}(\mathcal{G}_K(\hat{P}^{sc})) = \sum_{(\hat{q},\Delta_n)} (-1)^n \text{rank}_{\mathcal{O}}(\mathcal{G}_K(\hat{P}^{sc}(\hat{q})))
\]
2.2.1

where \( (\hat{q},\Delta_n) \) runs over a set of representatives for the set of isomorphism classes of \( \text{ch}^*(\hat{P}^{sc}) \)-objects such that \( \text{ch}^*(\hat{P}^{sc}) \)-objects \( 1 \leq i \leq n \) — called regular \( \text{ch}^*(\hat{P}^{sc}) \)-objects [9, A5.2]. This formula suggests to consider the following scalar product in \( \mathcal{G}_K(\hat{P}^{sc}) \); with the notation in 2.1.2 above, if \( X = \{ X_{(\hat{q},\Delta_n)} \}_{(\hat{q},\Delta_n)} \) and \( X' = \{ X'_{(\hat{q},\Delta_n)} \}_{(\hat{q},\Delta_n)} \) are two elements of \( \mathcal{G}_K(\hat{P}^{sc}) \) then we define
\[
\langle X, X' \rangle = \sum_{(\hat{q},\Delta_n)} (-1)^n \langle X_{(\hat{q},\Delta_n)}, X'_{(\hat{q},\Delta_n)} \rangle
\]
2.2.2

where \( (\hat{q},\Delta_n) \) is running over the same set and where, for such a \( (\hat{q},\Delta_n) \), \( \langle X_{(\hat{q},\Delta_n)}, X'_{(\hat{q},\Delta_n)} \rangle \) denotes the scalar product of \( X_{(\hat{q},\Delta_n)} \) and \( X'_{(\hat{q},\Delta_n)} \) in the Grothendieck group \( \mathcal{G}_K(\hat{P}^{sc}(\hat{q})) \). Note that there is a canonical bijection between a set of representatives for the set of isomorphism classes of \( \text{regular} \text{ch}^*(\hat{P}^{sc}) \)-objects and a set of representatives for the set of \( \mathcal{F} \)-isomorphism classes of nonempty sets of \( \mathcal{F} \)-selfcentralizing subgroups of \( P \), totally ordered by the inclusion.

2.3. Recall that the canonical group homomorphism \( \mathcal{O}^* \to k^* \) admits a unique section \( k^* \to \mathcal{O}^* \); thus, the category \( \mathcal{K}-\mathfrak{mod} \) admits an evident \( k^* \)-action and a contravariant \( k^* \)-functor \( m: \hat{P}^{sc} \to \mathcal{K}-\mathfrak{mod} \) is a functor such that \( m(\lambda \hat{x}) = \lambda m(\hat{x}) \) for any \( \hat{P}^{sc} \)-morphism \( \hat{x}: R \to Q \) and any \( \lambda \in k^* \).
Moreover, it is clear that any contravariant $k^*$-functor $m : \hat{\mathcal{P}}^c \to \mathcal{K}\text{-mod}$ determines a new contravariant $k^*$-functor \cite[A3.7.3]{ reference }

$$m^{ch} = m \circ v_{\hat{\mathcal{P}}^c} : ch^*(\hat{\mathcal{P}}^c) \to \mathcal{K}\text{-mod}$$

2.3.1

sending any $\hat{\mathcal{P}}^c$-chain $\hat{q} : \Delta_n \to \hat{\mathcal{P}}^c$ to $m(\hat{q}(0))$ and any $ch^*(\hat{\mathcal{P}}^c)$-morphism

$$(\hat{x}, \delta) : (\hat{r}, \Delta_m) \to (\hat{q}, \Delta_n)$$

2.3.2,

where $\delta : \Delta_n \to \Delta_m$ is an order-preserving map and $\hat{x} : \hat{r} \circ \delta \cong \hat{q}$ a natural isomorphism, to the $\mathcal{K}$-linear map

$$m(\hat{x}_0 \circ \hat{r}(0 \bullet \delta(0))) : m(\hat{q}(0)) \to m(\hat{r}(0))$$

2.3.3.

2.4. In particular, in the case where $n = m$ and $\delta$ is the identity map, we get a $k^*$-compatible action over $m(\hat{q}(0))$ of the $k^*$-group $\text{aut}^{\hat{\mathcal{P}}^c}_{\hat{\mathcal{P}}^c} (\hat{q}) = \hat{\mathcal{P}}^c(\hat{q})$; that is to say, $m(\hat{q}(0))$ becomes a $\mathcal{K},\hat{\mathcal{P}}^c(\hat{q})$-module. Similarly, $m(\hat{r}(0))$ becomes a $\mathcal{K},\hat{\mathcal{P}}^c(\hat{r})$-module and, via the $k^*$-group homomorphism

$$\text{aut}^{\hat{\mathcal{P}}^c}_{\hat{\mathcal{P}}^c}(\hat{x}, \delta) : \hat{\mathcal{P}}^c(\hat{r}) \to \hat{\mathcal{P}}^c(\hat{q})$$

2.4.1,

$m(\hat{q}(0))$ also becomes a $\mathcal{K},\hat{\mathcal{P}}^c(\hat{r})$-module $\text{Res}_{\text{aut}^{\hat{\mathcal{P}}^c}_{\hat{\mathcal{P}}^c}(\hat{x}, \delta)}(m(\hat{q}(0)))$ and then the $\mathcal{K}$-linear map 2.3.3 is clearly a $\mathcal{K},\hat{\mathcal{P}}^c(\hat{r})$-module homomorphism

$$m^{ch}(\hat{x}, \delta) : \text{Res}_{\text{aut}^{\hat{\mathcal{P}}^c}_{\hat{\mathcal{P}}^c}(\hat{x}, \delta)}(m(\hat{q}(0))) \to m(\hat{r}(0))$$

2.4.2.

2.5. Similarly, any natural map $\mu : m \to m'$ between contravariant $k^*$-functors $m$ and $m'$ from $\hat{\mathcal{P}}^c$ to $\mathcal{K}\text{-mod}$ determines a new natural map

$$\mu^{ch} = \mu \circ v_{\hat{\mathcal{P}}^c} : m^{ch} \to m'^{ch}$$

2.5.1

sending any $\hat{\mathcal{P}}^c$-chain $\hat{q} : \Delta_n \to \hat{\mathcal{P}}^c$ to the $\mathcal{K}$-linear map

$$\mu(\hat{q}(0)) : m(\hat{q}(0)) \to m'(\hat{q}(0))$$

2.5.2;

then, it follows from the naturalness of $\mu$ that this map is actually a $\mathcal{K},\hat{\mathcal{P}}^c(\hat{q})$-module homomorphism. Let us denote by $\mathfrak{Nat}(m', m)$ the $\mathcal{K}$-module of natural maps from $m$ to $m'$.

2.6. We are interested in the contravariant $k^*$-functors $m : \hat{\mathcal{P}}^c \to \mathcal{K}\text{-mod}$ — called reversible — mapping any $\hat{\mathcal{P}}^c$-morphism $\hat{x} : R \to Q$ on a $\mathcal{K}$-linear isomorphism

$$m(\hat{x}) : m(Q) \cong m(R)$$

2.6.1;

in this case, it is quite clear that the contravariant $k^*$-functor $m^{ch}$ also maps any $ch^*(\hat{\mathcal{P}}^c)$-morphism on a $\mathcal{K}$-linear isomorphism. Conversely, 2.6.2. A contravariant $k^*$-functor $\chi : ch^*(\hat{\mathcal{P}}^c) \to \mathcal{K}\text{-mod}$ mapping any $ch^*(\hat{\mathcal{P}}^c)$-morphism on a $\mathcal{K}$-linear isomorphism comes from a reversible contravariant $k^*$-functor.
2.7. Indeed, for any \( \mathcal{F} \)-selfcentralizing subgroup \( Q \) of \( P \), considering the \( \hat{\mathcal{P}}^\ast -\)chain \( \hat{q}_Q: \Delta_0 \to \hat{\mathcal{P}}^\ast \) sending 0 to \( Q \), we set \( m(Q) = \text{r}(\hat{q}_Q, \Delta_0) \); moreover, for any \( \hat{\mathcal{P}}^\ast -\)morphism \( \hat{x}: R \to Q \), considering the \( \hat{\mathcal{P}}^\ast -\)chain \( \hat{q}_{\hat{x}}: \Delta_1 \to \hat{\mathcal{P}}^\ast \) mapping 0 on \( R \), 1 on \( Q \) and the \( \Delta_1 \)-morphism \( 0 \circ 1 \) on \( \hat{x} \), we have the evident \( \text{ch}^\ast(\hat{\mathcal{P}}^\ast ) \)-morphisms

\[
(id_R, \delta_0^0): (\hat{q}_{\hat{x}}, \Delta_1) \to (\hat{q}_R, \Delta_0) \tag{2.7.1}
\]

and we set \( m(\hat{x}) = \text{r}((id_R, \delta_0^0))^{-1} \circ \text{r}(id_Q, \delta_0^0) \). This correspondence is actually a \( k^\ast \)-functor since, for another \( \hat{\mathcal{P}}^\ast -\)morphism \( \hat{y}: T \to R \), considering the new \( \hat{\mathcal{P}}^\ast -\)chain \( \hat{y}: \Delta_2 \to \hat{\mathcal{P}}^\ast \) sending 0 to \( T \), 1 to \( R \), 2 to \( Q \), \( 0 \circ 1 \) to \( \hat{y} \) and \( 1 \circ 2 \) to \( \hat{x} \), and extending the notation above, we get the evident commutative diagram in the category \( \text{ch}^\ast(\hat{\mathcal{P}}^\ast ) \)

\[
\begin{array}{ccc}
\hat{q}_T & \to & \hat{q}_R \\
| & & | \\
\hat{q}_{\hat{y}} & \to & \hat{q}_{\hat{y}} \\
\downarrow & & \downarrow \\
\hat{x} & \to & \hat{x}
\end{array}
\tag{2.7.2}
\]

which the functor \( \text{r} \) sends to a commutative diagram of isomorphisms in \( \mathcal{K}\text{-mod} \); then, it is easily checked that \( m(\hat{x} \cdot \hat{y}) = m(\hat{y}) \circ m(\hat{x}) \) and that \( m^\text{ch} = \text{r} \).

2.8. If \( m \) and \( m' \) are \textit{reversible contravariant} \( k^\ast \)-functors from the \( k^\ast \)-category \( \hat{\mathcal{P}}^\ast \) to \( \mathcal{K}\text{-mod} \), a \textit{natural map} \( \mu: m \to m' \) is a correspondence sending any \( \mathcal{F} \)-selfcentralizing subgroup \( Q \) of \( P \) to a \( \mathcal{K} \)-linear map \( \mu_Q: m(Q) \to m'(Q) \) in such a way that, for any \( \hat{\mathcal{P}}^\ast -\)morphism \( \hat{x}: R \to Q \), we have the commutative diagram

\[
\begin{array}{ccc}
m(Q) & \xrightarrow{m(\hat{x})} & m(R) \\
\mu_Q \downarrow & & \downarrow \mu_R \\
m'(Q) & \xrightarrow{m'(\hat{x})} & m'(R)
\end{array}
\tag{2.8.1;}
\]

equivalently, considering the \textit{reversible contravariant functor}

\[
m^\ast \otimes_\mathcal{K} m': \hat{\mathcal{P}}^\ast \to \mathcal{K}\text{-mod}
\tag{2.8.2}
\]

mapping \( Q \) on \( \text{Hom}_\mathcal{K}(m(Q), m'(Q)) \) and \( \hat{x}: R \to Q \) on the \( \mathcal{K} \)-linear map

\[
\text{Hom}_\mathcal{K}(m(Q), m'(Q)) \to \text{Hom}_\mathcal{K}(m(R), m'(R))
\tag{2.8.3}
\]

sending any \( \alpha \in \text{Hom}_\mathcal{K}(m(Q), m'(Q)) \) to \( m'(\hat{x}) \circ \alpha \circ m(\hat{x})^{-1} \), the commutativity of the diagrams above means that \( \mu \) belongs to the inverse limit of \( m^\ast \otimes_\mathcal{K} m' \); that is to say, we get

\[
\mathfrak{Nat}(m', m) = \lim_{\longleftarrow} (m^\ast \otimes_\mathcal{K} m')
\tag{2.8.4}
\]
2.9. Moreover, let \( \mathfrak{m} : \hat{\mathcal{P}}^{sc} \to \mathcal{K}\text{-mod} \) be a \textit{reversible contravariant} \( \ast \)-\textit{functor}; with the notation in 2.4 above, homomorphism 2.4.2 becomes an isomorphism and therefore the restriction map induced by homomorphism 2.4.1

\[
\mathfrak{g}_K (\text{aut}_{\hat{\mathcal{P}}^{sc}} (\hat{x}, \delta)) : \mathfrak{g}_K (\hat{\mathcal{P}}^{sc} (\hat{q})) \to \mathfrak{g}_K (\hat{\mathcal{P}}^{sc} (\hat{r}))
\]

2.9.1 sends the class \( X_{\mathfrak{m}(\hat{q}(0))} \) in \( \mathfrak{g}_K (\hat{\mathcal{P}}^{sc} (\hat{q})) \) of the \( \mathcal{K}_s \hat{\mathcal{P}}^{sc} (\hat{q}) \)-module \( \mathfrak{m}(\hat{q}(0)) \) to the class \( X_{\mathfrak{m}(\hat{r}(0))} \) in \( \mathfrak{g}_K (\hat{\mathcal{P}}^{sc} (\hat{r})) \) of the \( \mathcal{K}_s \hat{\mathcal{P}}^{sc} (\hat{r}) \)-module \( \mathfrak{m}(\hat{r}(0)) \). That is to say, the family

\[
\{ X_{\mathfrak{m}(\hat{q}(0))} \}_{(\hat{q}, \Delta_n)} \in \prod_{n \in \mathbb{N}} \prod_{\hat{q} \in \mathfrak{g}(\Delta_n, \hat{\mathcal{P}}^{sc})} \mathfrak{g}_K (\hat{\mathcal{P}}^{sc} (\hat{q}))
\]

2.9.2 fulfills condition 2.1.3 and therefore it belongs to \( \mathfrak{g}_K (\hat{\mathcal{P}}^{sc}) \); let us denote by \( X_{\mathfrak{m}} \) this family which, clearly, only depends on the isomorphism class of \( \mathfrak{m} \).

2.10. Actually, any \textit{reversible contravariant} \( \ast \)-\textit{functor} \( \mathfrak{m} : \hat{\mathcal{P}}^{sc} \to \mathcal{K}\text{-mod} \) is \textit{naturally isomorphic} to a \textit{contravariant} \( \ast \)-\textit{functor} \( \mathfrak{n} \) from \( \hat{\mathcal{P}}^{sc} \) to \( \mathcal{K}\text{-mod} \) — called \textit{reduced} — mapping any \( \mathcal{F} \)-selfcentralizing subgroup \( Q \) of \( P \) on the \textit{same} finite dimensional \( \mathcal{K} \)-module \( M \) and any \( \hat{\mathcal{P}}^{sc} \)-morphism \( \hat{\tau}_{Q, R}(1) \) on \( \text{id}_M \); indeed, setting \( M = \mathfrak{m}(P) \), we define \( \mathfrak{n} \) sending any \( \mathcal{F} \)-selfcentralizing subgroup \( Q \) of \( P \) to \( M \) and any \( \hat{\mathcal{P}}^{sc} \)-morphism \( \hat{x} : R \to Q \) to the \( \mathcal{K} \)-linear map

\[
\mathfrak{m}(\hat{\tau}_{P, R}(1))^{-1} \circ \mathfrak{m}(\hat{x}) \circ \mathfrak{m}(\hat{\tau}_{P, Q}(1)) : M \cong M
\]

2.10.1 which, if \( R \) is contained in \( Q \), clearly maps \( \hat{\tau}_{Q, R}(1) \) on \( \text{id}_M \); then, we have the obvious \textit{natural isomorphism} \( \mathfrak{n} \cong \mathfrak{m} \) sending \( Q \) to

\[
\mathfrak{m}(\hat{\tau}_{P, Q}(1)) : M \cong \mathfrak{m}(Q)
\]

2.10.2 Moreover, two of such \textit{reduced contravariant} \( \ast \)-\textit{functors} \( \mathfrak{n} \) and \( \mathfrak{n}' \) mapping \( P \) on the same \( \mathcal{K} \)-module \( M \) are \textit{naturally isomorphic} if and only if there is \( s \in \text{GL}_K(M) \) fulfilling \( \mathfrak{n}'(\hat{x}) = s \mathfrak{n}(\hat{x})^* \) for any \( \hat{\mathcal{P}}^{sc} \)-morphism \( \hat{x} : R \to Q \).

2.11. Finally, we call \( \mathcal{K}_s \hat{\mathcal{P}}^{sc} \)-\textit{module} any \textit{reversible contravariant} \( \ast \)-\textit{functor} \( \mathfrak{m} \) such that, setting \( M = \mathfrak{m}(P) \), the \( \ast \)-subgroup \( G(\mathfrak{m}) \) of \( \text{GL}_K(M) \) generated by \( \mathfrak{m}(\hat{\tau}_{P, R}(1))^{-1} \circ \mathfrak{m}(\hat{x}) \circ \mathfrak{m}(\hat{\tau}_{P, Q}(1)) \), where \( \hat{x} : R \to Q \) runs over the set of \( \hat{\mathcal{P}}^{sc} \)-morphisms, has a finite \( \ast \)-quotient \( G(\mathfrak{m}) \); it is clear that the \textit{direct sum} of \( \mathcal{K}_s \hat{\mathcal{P}}^{sc} \)-modules is a \( \mathcal{K}_s \hat{\mathcal{P}}^{sc} \)-\textit{module}; we denote by \( \mathcal{K}_s \hat{\mathcal{P}}^{sc} \text{-mod} \) the category formed by the \( \mathcal{K}_s \hat{\mathcal{P}}^{sc} \)-modules and by the \textit{natural maps} between them.

\textbf{Proposition 2.12.} If \( \mathfrak{m} : \hat{\mathcal{P}}^{sc} \to \mathcal{K}\text{-mod} \) is a \( \mathcal{K}_s \hat{\mathcal{P}}^{sc} \)-\textit{module} then any \textit{reversible contravariant} \( \ast \)-\textit{functor} \( \mathfrak{n} : \hat{\mathcal{P}}^{sc} \to \mathcal{K}\text{-mod} \) contained in \( \mathfrak{m} \) is a \( \mathcal{K}_s \hat{\mathcal{P}}^{sc} \)-\textit{module} too and admits a complement in \( \mathfrak{m} \). In particular, \( \mathfrak{m} \) is isomorphic to a direct sum of simple \( \mathcal{K}_s \hat{\mathcal{P}}^{sc} \)-\textit{modules}. 
Theorem 2.15. For any $\mathcal{K}\mathcal{P}^\infty$-module $\mathfrak{m}$ and any $n \geq 1$ we have

$$\mathbb{H}^n(\mathcal{P}^\infty, \mathfrak{m}) = \mathbb{H}_n(\mathcal{P}^\infty, \mathfrak{m}) = \mathbb{H}_n^\infty(\mathcal{P}^\infty, \mathfrak{m}) = \{0\}$$  \hspace{1cm} 2.15.1.

Proof: The equality $\mathbb{H}^n(\mathcal{P}^\infty, \mathfrak{m}) = \mathbb{H}_n(\mathcal{P}^\infty, \mathfrak{m})$ follows from [9, Proposition A4.13] and the equality $\mathbb{H}^n_\infty(\mathcal{P}^\infty, \mathfrak{m}) = \mathbb{H}_n^\infty(\mathcal{P}^\infty, \mathfrak{m})$ follows from [9, Proposition A5.7]. More generally, denoting by $I$ the interior structure of $\mathcal{P}^\infty$.
[9, 1.3] mapping any $\mathcal{F}$-selfcentralizing subgroup $Q$ of $P$ on $\tau_Q^*(Q)$, it still follows from [9, Proposition A4.13] that $H^n(\mathcal{P}^e, \mathfrak{m})$ coincides with the $\mathcal{I}$-stable $n$-cohomology group $\mathbb{H}^n_I(\mathcal{P}^e, \mathfrak{m})$ of $\mathcal{P}^e$ over $\mathfrak{m}$ [9, A3.18]; thus, it suffices to prove that, for any $n \geq 1$, $H^n_I(\mathcal{P}^e, \mathfrak{m}) = \{0\}$.

We may assume that $\mathfrak{m}$ is reduced; setting $M = \mathfrak{m}(P)$, since the $k^*$-group $\hat{G}(\mathfrak{m}) \subset \text{GL}_K(M)$ has a finite $k^*$-quotient, it stabilizes an $\mathcal{O}$-submodule $M^\circ$ of $M$ such that $M \cong K \otimes_{\mathcal{O}} M^\circ$; then, $\hat{G}(\mathfrak{m})$ is contained in $\text{GL}_\mathcal{O}(M^\circ)$ and it suffices to define $m^\circ(\hat{x}) = m(\hat{x})$ for any $\hat{\mathcal{P}}$-morphism $\hat{x}: R \to Q$ to get a contravariant functor $m^\circ$ from $\mathcal{P}^e$ to $\mathcal{O}$-$\text{mod}$ such that $m \cong K \otimes_{\mathcal{O}} m^\circ$; hence, it suffices to prove that, for any $n \geq 1$, we have

$$H^n_I(\mathcal{P}^e, m^\circ) = \{0\}$$

2.15.2.

More precisely, setting $\bar{m}^\circ = k \otimes_{\mathcal{O}} m^\circ$ and $\bar{M}^\circ = k \otimes_{\mathcal{O}} M^\circ$, it suffices to prove that

$$H^n_I(\mathcal{P}^e, \bar{m}^\circ) = \{0\}$$

2.15.3;

indeed, as in 2.14 above, for any $n \in \mathbb{N}$ we set

$$C^n(\mathcal{P}^e, m^\circ) = \prod_{q \in \mathcal{G}(\Delta_n, \mathcal{P}^e)} M^\circ$$

2.15.4

and denote by $C^n_I(\mathcal{P}^e, m^\circ)$ the $\mathcal{O}$-submodule of $I$-stable families; then, if equality 2.15.3 holds and $c_0 \in C^n_I(\mathcal{P}^e, m^\circ)$ is an $n$-cocycle, denoting by $\varpi$ a generator of $J(\mathcal{O})$, we already have that

$$c_0 \equiv d_{n-1}(a_0) \pmod{\varpi}$$

2.15.5

for a suitable $a_0 \in C^{n-1}_I(\mathcal{P}^e, m^\circ)$, so that we have $c_0 - d_{n-1}^I(a_0) = \varpi \cdot c_1$ for a unique $c_1 \in C^{n-1}_I(\mathcal{P}^e, m^\circ)$ which is again an $n$-cocycle since $\bar{M}^\circ$ is a free $\mathcal{O}$-module; thus, for any $i \in \mathbb{N}$ we inductively can define $c_i \in C^n_I(\mathcal{P}^e, m^\circ)$ and $a_i \in C^{n-1}_I(\mathcal{P}^e, m^\circ)$ fulfilling

$$c_i \equiv d_{n-1}^I(a_i) \pmod{\varpi} \quad \text{and} \quad c_i - d_{n-1}^I(a_i) = \varpi \cdot c_{i+1}$$

2.15.6

and then, according to the completeness of $\mathcal{O}$, it is quite clear that

$$c_0 = d_{n-1}^I(\sum_{i \in \mathbb{N}} \varpi^i \cdot a_i)$$

2.15.7.

Now, denoting by $\mathfrak{h}_0(\bar{m}^\circ) : \mathcal{P}^e \to k$-$\text{mod}$ the contravariant $k^*$-subfunctor of $\bar{m}^\circ$ sending any $\mathcal{F}$-selfcentralizing subgroup $Q$ of $P$ to the fixed points of $Q$ in $\bar{m}^\circ(Q)$ [9, 14.21], it is quite clear that the inclusion of $\mathfrak{h}_0(\bar{m}^\circ)$ in $\bar{m}^\circ$ induces $\mathcal{O}$-module isomorphisms

$$C^n_I(\mathcal{P}^e, \mathfrak{h}_0(\bar{m}^\circ)) \cong C^n_I(\mathcal{P}^e, \bar{m}^\circ)$$

2.15.8

for any $n \in \mathbb{N}$ which are compatible with the differential maps; thus, it suffices to prove that $H^n_I(\mathcal{P}^e, \mathfrak{h}_0(\bar{m}^\circ)) = \{0\}$ for any $n \geq 1$. 
Moreover, the \( \mathcal{I} \text{-exterior quotient} \) of \( \mathcal{P}^\infty \) [9, 1.3] coincides with the exterior quotient \( \overline{\mathcal{F}}^\infty \) (cf. 1.7) and then the contravariant functor \( \mathfrak{h}_0(\overline{\mathbb{m}}^\infty) \) factorizes through the canonical functor \( \mathcal{P}^\infty \rightarrow \overline{\mathcal{F}}^\infty \). On the other hand, as in 1.7 above, with the terminology introduced in [12, 2.2] \( \mathcal{P}^\infty \) becomes a \( \mathcal{T}^\infty \text{-category} \) and it fulfills the finiteness condition in [12, 4.1]; at this point, it follows from [12, Proposition 4.6] that \( \mathcal{P}^\infty \) is a multiplicative \( \mathcal{T}^\infty \text{-category} \) since, by [9, Propositions 24.2 and 24.4], \( \mathcal{P}^\infty \) fulfills both conditions in this proposition.

That is to say, the additive cover \( \mathfrak{ac}(\mathcal{P}^\infty) \) admits direct products [12, 4.2]; in particular, we have a functor

\[
\text{int}_P : \mathcal{P}^\infty \rightarrow \mathfrak{ac}(\mathcal{P}^\infty)
\]

sending any \( \mathcal{F} \)-selfcentralizing subgroup \( Q \) of \( P \) to the direct product \( Q \times P \) in \( \mathfrak{ac}(\mathcal{P}^\infty) \); hence, setting \( i_Q^P = \tau_{\mathfrak{ac}}^P(1) \) and denoting by \( I_Q \) the finite set of pairs \( (Q', a') \) formed by an \( \mathcal{F} \)-selfcentralizing subgroup \( Q' \) of \( P \) and by a \( \mathcal{P}^\infty \)-morphism \( a' : Q' \rightarrow Q \) belonging to \( \mathcal{P}^\infty(P, Q')_{i_Q^P} \) [12, 4.5.1], we may assume that

\[
Q \times P = \bigoplus_{(Q', a') \in I_Q} Q'
\]

Actually, it follows from [12, 4.10] that we have a functor \( I : \mathcal{P}^\infty \rightarrow \mathfrak{Set} \) mapping any \( \mathcal{F} \)-selfcentralizing subgroup \( Q \) of \( P \) on the finite set \( I_Q \) and any \( \mathcal{P}^\infty \)-morphism \( x : R \rightarrow Q \) on the map \( I_x : I_R \rightarrow I_Q \) determined by the \( \mathfrak{ac}(\mathcal{P}^\infty) \)-morphism

\[
x \times i_R^P : R \times P \rightarrow Q \times P
\]

that is to say, any element \( (R', b') \) in \( I_R \) determines an element \( (Q', a') \) in \( I_Q \) in such a way that \( Q' \) contains \( R' \) and that, setting \( i_{R'}^Q = \tau_{\mathfrak{ac}}^Q(1) \), we have

\[
a' \cdot i_{R'}^Q = x \cdot b'
\]

Now, in order to prove that \( \mathbb{H}_\mathcal{I}^n(\mathcal{P}^\infty, \mathfrak{h}_0(\overline{\mathbb{m}}^\infty)) = \{0\} \) for any \( n \geq 1 \), we will quote [12, Theorem 3.5]; for this purpose, we need to consider a homotopic system \( \mathcal{H} \) — as introduced in [12, 2.6] — associated with \( \mathcal{T}^\infty \), with the \( \mathcal{T}^\infty \text{-category} \) \( \mathcal{P}^\infty \) and with the subcategory \( \mathcal{I} \) of \( \mathcal{P}^\infty \); our homotopic system \( \mathcal{H} \) is the quintuple formed by the interior structure \( \mathcal{I} \) above, by the trivial co-interior structure of \( \mathcal{P}^\infty \), by the functor \( I : \mathcal{P}^\infty \rightarrow \mathfrak{Set} \) above, by the functor [12, 2.5]

\[
\mathfrak{w} : I \times \mathcal{P}^\infty \rightarrow \overline{\mathcal{F}}^\infty \subset \overline{\mathcal{F}}^\infty
\]

mapping any \( I \times \mathcal{P}^\infty \)-object \( (Q', a', Q) \) on \( Q' \) and any \( I \times \mathcal{P}^\infty \)-morphism

\[
(i_{R'}^Q, x) : (R', b', R) \rightarrow (Q', a', Q)
\]
on $i^Q_{R'}: R' \to Q'$, where $Q$ and $R$ are $\mathcal{F}$-selfcentralizing subgroups of $P$, $(Q', a')$ and $(R', b')$ are respective elements of $I_Q$ and $I_R$, and $x: R \to Q$ is a $\mathcal{P}^e$-morphism fulfilling equality 2.15.12 above; finally, denoting by

$$\tilde{p}: I \rtimes \mathcal{P}^e \to \tilde{\mathcal{F}}^e$$ 2.15.15

the forgetful functor mapping $(Q', a', Q)$ on $Q$ and $(i^Q_{R'}, x)$ on the image $\tilde{x}$ of $x$ in $\tilde{\mathcal{F}}^e(Q, R)$, the fifth term in $\mathcal{H}$ is the natural map

$$\omega: \mathfrak{w} \to \tilde{p}$$ 2.15.16

sending any $I \rtimes \mathcal{P}^e$-object $(Q', a', Q)$ to the image

$$\tilde{a}': Q' \to Q$$

$$\mathfrak{w}(Q', a', Q) \to \tilde{p}(Q', a', Q)$$

of $a'$ in $\tilde{\mathcal{F}}^e(Q, Q')$; the naturalness of $\omega$ is then easily checked from equality 2.15.12.

At this point, following [12, 2.9 and 2.10], from the homotopic system $\mathcal{H}$ and from the contravariant functor $h_0(\bar{\mathfrak{m}}^Q)$, which factorizes through the canonical functor $\mathcal{P}^e \to \tilde{\mathcal{F}}^e$, we get a contravariant functor

$$\mathcal{H}(h_0(\bar{\mathfrak{m}}^Q)) : \mathcal{P}^e \to k\text{-mod}$$ 2.15.18

sending any $\mathcal{F}$-selfcentralizing subgroup $Q$ of $P$ to

$$\left(\mathcal{H}(h_0(\bar{\mathfrak{m}}^Q))\right)(Q) = \prod_{(Q', a') \in I_Q} (\bar{\mathcal{M}}^Q)^{Q'}$$ 2.15.19

and a natural map

$$\Delta_\mathcal{H}(h_0(\bar{\mathfrak{m}}^Q)) : h_0(\bar{\mathfrak{m}}^Q) \to \mathcal{H}(h_0(\bar{\mathfrak{m}}^Q))$$ 2.15.20

sending any $\mathcal{F}$-selfcentralizing subgroup $Q$ of $P$ to the $k$-module homomorphism

$$\Delta_\mathcal{H}(h_0(\bar{\mathfrak{m}}^Q))_Q : (\bar{\mathcal{M}}^Q)^Q \to \prod_{(Q', a') \in I_Q} (\bar{\mathcal{M}}^Q)^{Q'}$$ 2.15.21

mapping any $\bar{m} \in (\bar{\mathcal{M}}^Q)^Q$ on

$$\Delta_\mathcal{H}(h_0(\bar{\mathfrak{m}}^Q))_Q(\bar{m}) = \sum_{(Q', a') \in I_Q} (\bar{m}^{Q'})(\bar{a}')(\bar{m})$$ 2.15.22.

Then, it follows from [12, Theorem 3.5] that, for our purpose, it suffices to exhibit a natural section of $\Delta_\mathcal{H}(h_0(\bar{\mathfrak{m}}^Q))$

$$\theta : \mathcal{H}(h_0(\bar{\mathfrak{m}}^Q)) \to h_0(\bar{\mathfrak{m}}^Q)$$ 2.15.23;
explicitly, for any \( \mathcal{F} \)-selfcentralizing subgroup \( Q \) of \( P \) we will define a section

\[
\theta_Q : \left( \prod_{(Q',a') \in I_Q} (M^\circ)^Q \right)^Q \rightarrow (M^\circ)^Q
\]

2.15.24

of \( \Delta_H(\mathfrak{h}_0(\tilde{m}^\circ))_Q \). Note that the action of \( u \in Q \) maps the element

\[
\tilde{m} = \sum_{(Q',a') \in I_Q} \tilde{m}_{(Q',a')} \in \prod_{(Q',a') \in I_Q} (M^\circ)^Q
\]

2.15.25

on \( \sum_{(Q',a') \in I_Q} \tilde{m}_{(Q',\tau^*(u)-a')} \); thus, \( \tilde{m} \) belongs to \( \left( \prod_{(Q',a') \in I_Q} (M^\circ)^Q \right)^Q \) if and only if we have \( \tilde{m}_{(Q',\tau^*(u)-a')} = \tilde{m}_{(Q',a')} \) for any \( u \in Q \); that is to say, \( \tilde{m}_{(Q',a')} \) only depends on the pair \( (Q', \tilde{a}') \).

Actually, denoting by \( \mathcal{T}_Q \) the set of pairs \( (Q', \tilde{a}') \) when \( (Q', \tilde{a}') \) runs over \( \mathcal{T}_Q \), it follows from [12, Corollary 4.7] that the direct product in \( \mathfrak{ac}(\mathcal{P}^\circ) \) induces a direct product in \( \mathfrak{ac}(\mathcal{F}') \) — noted \( \hat{\times} \) — and that we may assume that

\[
Q \hat{\times} P = \bigoplus_{(Q',\tilde{a}') \in \mathcal{T}_Q} Q'
\]

we denote by \( \mathcal{T}_Q^\circ \) the set of pairs \( (Q', \tilde{a}') \in \mathcal{T}_Q \) — called extremal — where \( \tilde{a}' \) is an isomorphism and it is easily checked that we have a canonical bijection

\[
\mathcal{T}_Q^\circ \cong \tilde{\mathcal{F}}(P,Q)
\]

2.15.27

With all this notation, for any element \( \tilde{m} = \sum_{(Q',a') \in I_Q} \tilde{m}_{(Q',a')} \) belonging to

\[
\left( \mathcal{H}(\mathfrak{h}_0(\tilde{m}^\circ)) \right)(Q) = \left( \prod_{(Q',a') \in I_Q} (M^\circ)^Q \right)^Q
\]

2.15.28

we define

\[
\theta_Q(\tilde{m}) = \frac{1}{|\mathcal{F}^\circ(P,Q)|} \sum_{(Q',\tilde{a}') \in \mathcal{T}_Q^\circ} (\tilde{m}^\circ(a'))^{-1}(\tilde{m}_{(Q',a')})
\]

2.15.29

where, for any \( (Q', \tilde{a}') \in \mathcal{T}_Q^\circ \), \( a' \) denotes a representative of \( \tilde{a}' \) in \( \mathcal{P}^\circ(Q,Q') \); this makes sense since it follows from [9, 6.6.4 and Proposition 6.7] and from condition 1.3.2 above that \( p \) does not divide \( |\tilde{\mathcal{F}}(P,Q)| \) and, since the element \( (\tilde{m}^\circ(a'))^{-1}(\tilde{m}_{(Q',a')}) \) belongs to \( (M^\circ)^Q \), it is clear that \( \theta_Q(\tilde{m}) \) does not depend on the choice of the representative \( a' \); moreover, it follows from definition 2.15.22 above that \( \theta_Q \) is indeed a section of \( \Delta_H(\mathfrak{h}_0(\tilde{m}^\circ))_Q \).
It only remains to prove that the correspondence sending any $\mathcal{F}$-selfcentralizing subgroup $Q$ of $P$ to $\theta_Q$ is natural; that is to say, for any $\mathcal{P}^\ast$-morphism $x : R \to Q$ we have to prove the commutativity of the following diagram

\[
\begin{array}{ccc}
(H(\hat{h}_0(\m^\circ)))&(Q) \xrightarrow{\theta_Q} \m^\circ(Q) \\
\downarrow & & \downarrow \m^\circ(x) \\
(H(\hat{h}_0(\m^\circ)))&(R) \xrightarrow{\theta_R} \m^\circ(R)
\end{array}
\]

2.15.30.

Explicitly, it follows from 2.15.11 and 2.15.12 above that the $k$-module homomorphism

\[
(H(\hat{h}_0(\m^\circ)))_(x) : \left( H(\hat{h}_0(\m^\circ)))_(Q) \to \left( H(\hat{h}_0(\m^\circ)))_(R) \right)
\]

sends the element $\tilde{m} = \sum_{(Q',a') \in I_Q} \tilde{m}(Q',a')$ above to

\[
\sum_{(R',b') \in I_R} (\m^\circ(i_{Q'}))^{-1}(\tilde{m}(Q',a'))
\]

2.15.32.

where, for any $(R',b') \in I_R$, $(Q',a')$ is the unique element of $I_Q$ such that $R' \subset Q'$ and $a' \cdot i_{R'} = x \cdot b'$ (cf. 2.15.12); then, $\theta_R$ maps this element on

\[
\frac{1}{|\mathcal{F}^\ast(P,R)|} \sum_{(R',b') \in \mathcal{F}^\ast(R,R')} (\m^\circ(b'))^{-1}(\tilde{m}(Q',a'))
\]

2.15.33.

where, for any $(R',b') \in \mathcal{F}^\ast(R,R')$, $b'$ denotes a representative of $\tilde{b}'$ in $\mathcal{P}^\ast(R,R')$.

But, it follows from [12, Lemma 4.4] that the category $\mathcal{AC}(\mathcal{F}^\ast)$ admits pull-backs and, more precisely, that we have the pull-back

\[
\begin{array}{ccc}
\hat{z} & \nearrow \hat{x} & Q \\
\downarrow & & \downarrow \\
R & \nearrow Q \times P & \stackrel{\hat{z}}{R} \times P
\end{array}
\]

2.15.34;

explicitly, for any $(Q',a') \in \mathcal{F}^\ast_Q$, choosing a representative $a'$ of $\hat{a}'$ and a set of representatives $W_{Q',a'}$ for the set of double classes $\varphi_x(R \setminus Q)/\varphi_{a'}(Q')$ where $\varphi_x \in \mathcal{F}(Q,R)$ and $\varphi_{a'} \in \mathcal{F}(Q,Q')$ are the respective images of $x$ and $a'$, it is well-known that we get the pull-back in $\mathcal{AC}(\mathcal{F}^\ast)$

\[
\begin{array}{ccc}
\hat{z} & \nearrow \hat{a}' & Q' \\
\downarrow & \nearrow & \downarrow \\
R & \nearrow Q' & \bigoplus_{w \in W_{Q',a'}} R_w'
\end{array}
\]

2.15.35.
where $W_{(Q', \tilde{a}')}$ is the subset of $w \in W_{(Q', \tilde{a}')}^e$ such that the subgroup of $Q'$

$$R_w^e = \varphi_{a'}^{-1}(\varphi_x(R)^w)$$

2.15.36

is still $\mathcal{F}$-selfcentralizing; moreover, denote by $b_w'$ the element in $P_{w}^{\infty}(R, R_w)$ fulfilling $a' \cdot t_{W_{w}}^{Q'} = x \cdot b_w'$. At this point, in the category ac($\mathcal{F}_{w}^{\infty}$) we get

$$R \times P \cong \bigoplus_{(Q', \tilde{a}') \in \mathcal{L}_Q} \bigoplus_{w \in W_{(Q', \tilde{a}')}^{e}} R_w^e$$

2.15.37

and we actually may assume that

$$\mathcal{L}_R = \bigcup_{(Q', \tilde{a}') \in \mathcal{L}_Q} \{(W_w', b_w') \in W_{(Q', \tilde{a}')}^{e})$$

2.15.38.

Consequently, denoting by $W_{(Q', \tilde{a}')}^{\circ}$ the subset of $w \in W_{(Q', \tilde{a}')}^e$ such that $R_w'$ is $\mathcal{F}$-isomorphic to $R$, we clearly have $W_{(Q', \tilde{a}')}^{\circ} \subset W_{(Q', \tilde{a}')}^{e}$ and from 2.15.32 and 2.15.33 we get

$$|\mathcal{F}_{w}^{\infty}(P, R)| \cdot \left(\theta_R \circ \left(\mathcal{H}(b_w(\tilde{m}^{\circ}))\right)(x)\right)(\tilde{m})$$

2.15.39

$$= \sum_{(Q', \tilde{a}') \in \mathcal{L}_Q} \sum_{w \in W_{(Q', \tilde{a}')}^{\circ}} \left(\tilde{m}^{\circ}(b_w')^{-1} \circ \tilde{m}^{\circ}(t_{W_{w}}^{Q'})\right)(\tilde{m}(Q', a'))$$

where, for any $(Q', \tilde{a}') \in \mathcal{L}_Q$ and any $w \in W_{(Q', \tilde{a}')}^{\circ}$, we have $a' \cdot t_{W_{w}}^{Q'} = x \cdot b_w'$; in particular, we still have

$$\tilde{m}^{\circ}(t_{W_{w}}^{Q'}) \circ \tilde{m}^{\circ}(a') = \tilde{m}^{\circ}(b_w') \circ \tilde{m}^{\circ}(x)$$

2.15.40

and therefore the composition

$$\tilde{m}^{\circ}(b_w')^{-1} \circ \tilde{m}^{\circ}(t_{W_{w}}^{Q'}) = \tilde{m}^{\circ}(x) \circ \tilde{m}^{\circ}(a')^{-1}$$

2.15.41

does not depend on $w \in W_{(Q', \tilde{a}')}^{\circ}$.

On the other hand, from definition 2.15.29 we get

$$|\mathcal{F}_{w}^{\infty}(P, Q)| \cdot \left(\tilde{m}^{\circ}(x) \circ \theta_Q\right)(\tilde{m})$$

2.15.42.

$$= \sum_{(Q', \tilde{a}') \in \mathcal{L}_Q} \left(\tilde{m}^{\circ}(x) \circ \tilde{m}^{\circ}(a')^{-1}\right)(\tilde{m}(Q', a'))$$

Hence, in order to prove the commutativity of diagram 2.15.30, it suffices to show that, for any $(Q', \tilde{a}') \in \mathcal{L}_Q$, either $(Q', \tilde{a}')$ belongs to $\mathcal{L}_Q$ and we have $|W_{(Q', \tilde{a}')}^{\circ}| = 1$, or otherwise $p$ divides $|W_{(Q', \tilde{a}')}^{\circ}|$; but, the set $W_{(Q', \tilde{a}')}^{\circ}$ is actually a set of representatives for $\varphi_{a'}(Q') \setminus \mathcal{T}_Q(\varphi_{a'}(Q'), \varphi_x(R))$ and if this quotient is not empty then $N_Q(\varphi_{a'}(Q'))/\varphi_{a'}(Q')$ acts freely on it. We are done.
Corollary 2.16. For any pair of $K_{sc}$-modules $m$ and $m'$ we have
\[ \langle X_m, X_{m'} \rangle = \dim_K(\mathfrak{Nat}(m', m)) \] 2.16.1.

**Proof:** According to 2.8.4 we have
\[ \mathfrak{Nat}(m', m) = \varprojlim (m^* \otimes_K m') = \bigcap_{q \in \mathfrak{Nat}(\Delta_n, P^{nc})} (m^* \otimes_K m') \] 2.16.2

and therefore, setting $C^n = \mathbb{C}_r(P^{nc}, m^* \otimes_K m')$ for any $n \in \mathbb{N}$ (cf. 2.6), it follows from Theorem 2.15 that we have an infinite exact sequence
\[ 0 \rightarrow \mathfrak{Nat}(m', m) \rightarrow C^0 \rightarrow \ldots \rightarrow C^n \rightarrow C^{n+1} \rightarrow \ldots \] 2.16.3;

actually, we can identify $C^n$ with the set of elements
\[ (m_q)_{q \in \mathfrak{Nat}(\Delta_n, P^{nc})} \in \prod_{q \in \mathfrak{Nat}(\Delta_n, P^{nc})} (m^* \otimes_K m')(q(0)) \] 2.16.4

such that, for any natural isomorphism $x : q \cong q'$ between regular $P^{nc}$-valued $n$-chains $q$ and $q'$, $(m^* \otimes_K m')(x_0)$ maps $m_{q'}$ on $m_q$; that is to say, we actually have
\[ \mathbb{C}_r(P^{nc}, m^* \otimes_K m') \cong \prod_{q} (m^* \otimes_K m')(q(0))^{P^{nc}(q)} \] 2.16.5

where $q$ runs over a set of representatives for the set of isomorphism classes in $\mathfrak{Nat}(\Delta_n, P^{nc})$ [9, A5.3].

On the other hand, it is clear that for $n$ big enough there are no regular $P^{nc}$-valued $n$-chains and therefore, in the exact sequence above, only finitely many terms are not zero; thus, we still get
\[ \dim_K(\mathfrak{Nat}(m', m)) = \sum_{(q, \Delta_n)} (-1)^n \dim_K \left( (m^* \otimes_K m')(q(0))^{P^{nc}(q)} \right) \] 2.16.6

where $(q, \Delta_n)$ runs over a set of representatives for the isomorphism classes of regular $\chi^*(P^{nc})$-objects (cf. A5.3). Moreover, for any functor $\hat{q} : \Delta_n \rightarrow P^{nc}$ lifting $q$ we have
\[ (m^* \otimes_K m')(q(0)) = \text{Hom}_K\left(\hat{m}(\hat{q}(0)), m'(\hat{q}(0))\right) \] 2.16.7

and, in particular, we get
\[ (m^* \otimes_K m')(q(0))^{P^{nc}(q)} = \text{Hom}_{K_{sc}}(\hat{m}(\hat{q}(0)), m'(\hat{q}(0))) \] 2.16.8;

thus, denoting by $X_m(\hat{q}(0))$ and $X_{m'}(\hat{q}(0))$ the respective classes of $m(\hat{q}(0))$ and $m'(\hat{q}(0))$ in $G_K(P^{nc}((\hat{q}))$, we obtain (cf. 2.1 and 2.4)
\[ \dim_K (m^* \otimes_K m')(q(0))^{P^{nc}(q)} = \langle X_m(\hat{q}(0)), X_{m'}(\hat{q}(0)) \rangle \] 2.16.9.

Now, equality 2.16.1 follows from equality 2.16.6.
Corollary 2.17. The $O$-module homomorphism

$$\text{cat}_K : \mathcal{G}(\mathcal{K}, \hat{P}^{sc} - \text{mod}) \rightarrow \mathcal{G}_K(\hat{P}^{sc})$$

is injective. In particular, there are finitely many isomorphism classes of simple $K, \hat{P}^{sc}$-modules.

**Proof:** It is clear that if $m$ and $m'$ are nonisomorphic simple $\hat{P}^{sc}$-modules then we have $\mathfrak{Mat}(m, m') = \{0\}$; consequently, if $\{X_i\}_{i \in I}$ is a finite family in $\mathcal{G}(\mathcal{K}, \hat{P}^{sc} - \text{mod})$ of classes of simple $K, \hat{P}^{sc}$-modules and for a family $\{\lambda_i\}_{i \in I}$ in $O$ we have $\sum_{i \in I} \lambda_i \text{cat}_K(X_i) = 0$, it suffices to perform the scalar product by $\text{cat}_K(X_j)$ to get $\lambda_j = 0$ for any $j \in I$.

2.18. In order to prove that $\text{cat}_K : \mathcal{G}(\mathcal{K}, \hat{P}^{sc} - \text{mod}) \rightarrow \mathcal{G}_K(\hat{P}^{sc})$ is also surjective, following Lemma 1.6 above we consider a suitable finite subfield $\hat{k}$ of $k$ and a regular central $\hat{k}^*$-extension $\hat{F}^{sc}$ of $F^{sc}$ such that the extension of $\hat{F}^{sc}$ from $\hat{k}^*$ to $k^*$ is equivalent to $\hat{F}^{sc}$; denote by $\hat{O}$ the converse image of $\hat{k}$ in $O$ and by $\hat{K}$ the field of quotients of $\hat{O}$. Then, denoting by $\hat{P}^{sc}$ the converse image of $\hat{F}^{sc}$ in $\hat{P}^{sc}$, we may choose $\hat{k}$ big enough to get

$$\mathcal{G}_K(\hat{P}^{sc}(\hat{q})) \cong \mathcal{G}_K(\hat{P}^{sc}(\hat{q}))$$

for any $\hat{P}^{sc}$-chain $\hat{q}$.

2.19. Let $\ell$ be a prime number not dividing neither $|\hat{k}^*|$ nor $|F(Q)|$ for any $F$-selfcentralizing subgroup $Q$ of $P$, and denote by $O_\ell$ a complete discrete valuation ring with a quotient field $K_\ell$ of characteristic zero and a finite residue field $k_\ell$ of characteristic $\ell$; we can choose $O_\ell$ in such a way that $(k_\ell)^*$ would contain a (unique) subgroup isomorphic to $\hat{k}^*$; in particular, choosing an inclusion $\hat{k}^* \subset (k_\ell)^*$, any $\hat{k}^*$-group $\hat{G}$ induces a $(k_\ell)^*$-group $(k_\ell)^* \times_{\hat{k}^*} \hat{G}$ and, since this correspondence is functorial, from $\hat{P}^{sc}$ we actually get a $(k_\ell)^*$-category $\hat{P}^{sc, \ell}$ containing $\hat{P}^{sc}$. Moreover, via a suitable field $\hat{K}$ containing $\hat{K}$ and $K_\ell$, it is clear that choosing $O_\ell$ big enough for any $\hat{P}^{sc}$-chain $\hat{q} : \Delta_n \rightarrow \hat{P}^{sc}$ we can get an $O$-module isomorphism

$$\mathcal{G}_K(\hat{P}^{sc}(\hat{q})) \cong \mathcal{G}_{K_\ell}(\hat{P}^{sc, \ell}(\hat{q}'))$$

where $\hat{q}'$ is determined by $\hat{q}$ and by the inclusion $\hat{P}^{sc} \subset \hat{P}^{sc, \ell}$. Finally, according to our choice of $\ell$, we know that the Brauer decomposition map determines an $O$-module isomorphism

$$\mathcal{G}_{K_\ell}(\hat{P}^{sc, \ell}(\hat{q}')) \cong \mathcal{G}_{K_\ell}(\hat{P}^{sc, \ell}(\hat{q}'))$$

for any $\hat{P}^{sc}$-chain $\hat{q} : \Delta_n \rightarrow \hat{P}^{sc}$. 
2.20. Consequently, for any $\hat{\mathcal{P}}^\varphi$-chain $\hat{q}: \Delta_n \to \hat{\mathcal{P}}^\varphi$, the composition of all these $\mathcal{O}$-module isomorphisms supplies an $\mathcal{O}$-module isomorphism

$$G_\mathcal{K}(\hat{\mathcal{P}}^\varphi(\hat{q})) \cong G_{k_t}(\hat{\mathcal{P}}^{\varphi,\ell}(\hat{q}^\ell))$$

and therefore, since all of them are functorial, we get a natural isomorphism

$$g_\mathcal{K} \circ \text{aut}_{\mathcal{P}^{\varphi,\ell}} \cong g_{k_t} \circ \text{aut}_{\hat{\mathcal{P}}^{\varphi,\ell}}$$

so that we still get an $\mathcal{O}$-module isomorphism

$$G_\mathcal{K}(\hat{\mathcal{P}}^\varphi) = \lim_{\leftarrow} (g_\mathcal{K} \circ \text{aut}_{\mathcal{P}^{\varphi,\ell}}) \cong \lim_{\leftarrow} (g_{k_t} \circ \text{aut}_{\hat{\mathcal{P}}^{\varphi,\ell}}) = G_{k_t}(\hat{\mathcal{P}}^{\varphi,\ell})$$

Moreover note that, as in 1.9 above, the faithful functor $\tau^\varphi: T^\varphi_P \to \mathcal{P}^{\varphi,\ell}$ can be lifted to a faithful functor $\hat{\tau}^\varphi: \hat{T}^\varphi_P \to \hat{\mathcal{P}}^{\varphi,\ell}$ [13, Proposition 3.5], and we set $i^Q_{Q,R} = \hat{\tau}^\varphi_{Q,R}(1)$ for any pair of $\mathcal{F}$-selfcentralizing subgroups $Q$ and $R$ of $P$ fulfilling $R \subset Q$; in particular, $\hat{\mathcal{P}}^\varphi$ and $\hat{\mathcal{P}}^{\varphi,\ell}$ become divisible $\mathcal{F}^\varphi$-localities and therefore, for another such a pair $Q'$ and $R'$, we have a restriction map

$$i_{R',R}^Q: \hat{\mathcal{P}}^{\varphi,\ell}(Q',Q)_{R',R} \to \hat{\mathcal{P}}^{\varphi,\ell}(R',R)$$

fulfilling $i_{R',R}^Q \cdot i_{R',R}^Q(\hat{x}) = \hat{x} \cdot i_{R,R}^Q$ for any $\hat{x} \in \hat{\mathcal{P}}^{\varphi,\ell}(Q',Q)$ sending $R$ to $R'$.

2.21. At this point, let us call reduced $(k_t)_*\hat{\mathcal{P}}^{\varphi,\ell}$-module any contravariant $(k_t)_*\hat{\mathcal{P}}^{\varphi,\ell}$-module $\mathcal{M} \to k_t$-$\text{mod}$ mapping any $\mathcal{F}$-selfcentralizing subgroup $Q$ of $P$ on the same finite dimensional $k_t$-module $M_t$ and, for any $\mathcal{F}$-selfcentralizing subgroup $R$ of $Q$, the $\hat{\mathcal{P}}^{\varphi,\ell}$-morphism $\hat{i}_{R,R}^Q$ on $\text{id}_{M_t}$; this is coherent with our terminology above since the group $\text{GL}_{k_t}(M_t)$ and therefore the $(k_t)_*$-subgroup $G(\mathcal{M})$ of $\text{GL}_{k_t}(M_t)$ generated by $\mathcal{M}(\hat{x})$, where $\hat{x}: R \to Q$ runs over the set of $\hat{\mathcal{P}}^{\varphi,\ell}$-morphisms, are finite.

**Theorem 2.22.** With the notation above, let $\{X_{\hat{q}}\}_{\hat{q}}$ where $\hat{q}$ runs over the set of $\hat{\mathcal{P}}^{\varphi,\ell}$-chains be a family which belongs to $G_{k_t}(\hat{\mathcal{P}}^{\varphi,\ell})$ and fulfills that, for such a $\hat{q}$, $X_{\hat{q}}$ is the class of a $(k_t)_*\hat{\mathcal{P}}^{\varphi,\ell}$-module $M_{\hat{q}}$. Then, there exists a reduced $(k_t)_*\hat{\mathcal{P}}^{\varphi,\ell}$-module $\mathcal{M}_t: \hat{\mathcal{P}}^{\varphi,\ell} \to k_t$-$\text{mod}$ such that $X_{\hat{q}}$ is the class of $\mathcal{M}_t(\hat{q}(0))$ in $G_{k_t}(\hat{\mathcal{P}}^{\varphi,\ell}(\hat{q}))$ for any $\hat{\mathcal{P}}^{\varphi,\ell}$-chain $\hat{q}$.

**Proof:** Let $\mathcal{X}$ be a nonempty set of $\mathcal{F}$-selfcentralizing subgroups of $P$ which contains any subgroup of $P$ admitting an $\mathcal{F}$-morphism from some subgroup in $\mathcal{X}$, and respectively denote by $T^\varphi_P$, $\mathcal{F}^\varphi$, $\mathcal{P}^\varphi$ and $\hat{T}^{\varphi,\ell}$ the full subcategories of $T^\varphi_P$, $\mathcal{F}^\varphi$, $\mathcal{P}^\varphi$ and $\hat{\mathcal{P}}^{\varphi,\ell}$ over $\mathcal{X}$ as the set of objects; arguing by induction on $|\mathcal{X}|$, we prove that there is a reduced $(k_t)_*\hat{\mathcal{P}}^{\varphi,\ell}$-module $\mathcal{M}_t: \hat{\mathcal{P}}^{\varphi,\ell} \to k_t$-$\text{mod}$ such that $X_{\hat{q}}$ is the class of $\mathcal{M}_t(\hat{q}(0))$ in $G_{k_t}(\hat{\mathcal{P}}^{\varphi,\ell}(\hat{q}))$ for any $\hat{\mathcal{P}}^{\varphi,\ell}$-chain $\hat{q}$.
If $X = \{P\}$ then $\hat{\mathcal{P}}^{x,t}$ has just one object $P$; in this situation, still denoting by $P$ the $\hat{\mathcal{P}}^{x,t}$-chain mapping $0$ on $P$, $M_P$ is a $(k_t)_*\hat{\mathcal{P}}^{x,t}(P)$-module and the structural $(k_t)^*$-group homomorphism $\hat{\mathcal{P}}^{x,t}(P) \to \text{GL}_{k_t}(M_P)$ induces the reduced $(k_t)_*\hat{\mathcal{P}}^{x,t}$-module $m^x_t : \hat{\mathcal{P}}^{x,t} \to k_t\text{-mod}$.

Otherwise, choose a minimal element $U$ in $X$ fully normalized in $\mathcal{F}$ and set

$$\mathcal{X} = \mathcal{X} - \{(\theta(U)) | \theta \in \mathcal{F}(P,U)\}$$

thus, it follows from our induction hypothesis that there exists a reduced $(k_t)_*\hat{\mathcal{P}}^{x,t}$-module $m^x_U : \hat{\mathcal{P}}^{x,t} \to k_t\text{-mod}$ such that $X_\hat{q}$ is the class of $m_\hat{x}(\hat{q}(0))$ in $G_{k_t}(\hat{\mathcal{P}}^{x,t}(\hat{q}))$ for any $\hat{\mathcal{P}}^{x,t}$-chain $\hat{q}$; let us set $M = m^x_U(P)$.

If $N_{\mathcal{F}}(U) = \mathcal{F}$ [9, Proposition 2.16], we also have $N_{\mathcal{F}}(U) = \mathcal{P}$ [9, 17.5] and then $\hat{\mathcal{P}}^{x,t}$ actually coincides with the category $\mathcal{T}^{x}_{\hat{\mathcal{P}}^{x,t}(U)}$ where $X$ is the set of objects and where, for a pair of subgroups $Q$ and $R$ in $X$, the $(k_t)^*$-set of morphisms from $R$ to $Q$ is the $\hat{\mathcal{P}}^{x,t}$-transporter

$$\mathcal{T}^{x}_{\hat{\mathcal{P}}^{x,t}(U)}(Q,R) = \{\hat{x} \in \hat{\mathcal{P}}^{x,t}(U) | \hat{x} \cdot \hat{\tau}^U(R) \cdot \hat{x}^{-1} \subset \hat{\tau}^U(Q)\}$$

Indeed, since $N_{\mathcal{F}}(U) = \mathcal{P}$, any $\hat{\mathcal{P}}^{x,t}$-morphism $\hat{x} : R \to Q$ comes from a $\hat{\mathcal{P}}^{x,t}$-morphism $\hat{x}^U$ from $R \cdot U$ to $Q \cdot U$ stabilizing $U$ and fulfilling (cf. 2.20)

$$\hat{x}^U \cdot \hat{\tau}^U_R = \hat{\tau}^U_Q \cdot \hat{x}$$

Moreover, since $\hat{\tau}^U_R$ is an epimorphism, this equality determines $\hat{x}^U$ and the element $\hat{x}^U(\hat{x})$ of $\hat{\mathcal{P}}^{x,t}(U)$ induced by $\hat{x}^U$ clearly belongs to $\mathcal{T}^{x}_{\hat{\mathcal{P}}^{x,t}(U)}(Q,R)$; thus, this correspondence defines a functor

$$\iota^x_U : \hat{\mathcal{P}}^{x,t} \to \mathcal{T}^{x}_{\hat{\mathcal{P}}^{x,t}(U)}$$

compatible with the structural functors to $\hat{\mathcal{P}}^{x,t}$, and then it is easily checked that this functor is an equivalence of categories.

Now, still denoting by $U$ the $\hat{\mathcal{P}}^{x,t}$-chain mapping $0$ on $U$ and considering the $(k_t)_*\hat{\mathcal{P}}^{x,t}(U)$-module $M_U$ above and the structural $(k_t)^*$-group homomorphism

$$\rho_U : \hat{\mathcal{P}}^{x,t}(U) \to \text{GL}_{k_t}(M_U)$$

it suffices to set $m^x_U(Q) = M_U = m^x_U(R)$ and $m^x_U(\hat{x}) = \rho_U(\iota^x_U(\hat{x}))$, for any $\hat{x} \in \hat{\mathcal{P}}^{x,t}(Q,R)$, to get a reduced $(k_t)_*\hat{\mathcal{P}}^{x,t}$-module

$$m^x_U : \hat{\mathcal{P}}^{x,t} \to k_t\text{-mod}$$

we claim that it fulfills the announced condition.
Indeed, any \( \xi \)-chain \( \hat{q} : \{ n \} \to \hat{\mathcal{P}}_{\xi} \) can be extended to a \( \hat{\mathcal{P}}_{\xi} \)-chain \( \hat{q}^U : \{ n+1 \} \to \hat{\mathcal{P}}_{\xi} \) sending \( n+1 \) to \( \hat{q}(n) \cdot U \) and \( n \cdot n+1 \) to \( \hat{\mathcal{P}}_{\xi} \), and we have an obvious \( \mathfrak{h}_* \)-morphism

\[
(id_{\hat{q}}, \delta_{\Delta(n+1)}^{\hat{q}}) : (\hat{q}^U, \Delta_{n+1}) \to (\hat{q}, \Delta_n)
\]

consequently, we may assume that every \( \hat{\mathcal{P}}_{\xi} \)-chain \( U \) can be extended to a \( \hat{\mathcal{P}}_{\xi} \)-chain \( U \hat{q} : \{ 0 \} \to \hat{\mathcal{P}}_{\xi} \) and we have obvious \( \mathfrak{h}_* \)-morphism

\[
(\hat{q}^U, \Delta_{n+1}) \to (\hat{q}(n) \cdot U, \Delta_0) \to (U \hat{q}(n), \Delta_1) \to (U, \Delta_0)
\]

and the family \( \{ \hat{\mathcal{P}}_{\xi} \} \) belongs to \( \mathcal{G}_{\mathfrak{h}}(\hat{\mathcal{P}}_{\xi}) \), it follows from our choice of \( \ell \) that we still have the \( \hat{\mathcal{P}}_{\xi} \)-module isomorphisms (cf. 2.4.2)

\[
M_{\hat{q}} \cong M_{\hat{q}^U} \cong \text{Res}_{\hat{\mathcal{P}}_{\xi}(\hat{q})}(M_{\hat{q}(n), U}) \cong \text{Res}_{\hat{\mathcal{P}}_{\xi}(\hat{q})}(M_{U \hat{q}(n)})
\]

and of \( \ell \)-sublocality

\[
\text{Res}_{\hat{\mathcal{P}}_{\xi}(\hat{q})}(M_U) = m_\ell(\hat{q}(0))
\]

Actually, any reduced \( \hat{\mathcal{P}}_{\xi} \)-module \( n_\ell^\xi : \hat{\mathcal{P}}_{\xi} \to k_\ell \text{-mod} \) fulfilling this condition determines a \( \hat{\mathcal{P}}_{\xi} \)-module structure on \( n_\ell^\xi(U) \) which, according to our choice of \( \ell \), is isomorphic to \( M_U \); then, identifying to each other the \( \hat{\mathcal{P}}_{\xi}(U) \)-locality of \( \hat{\mathcal{P}}_{\xi}(U) \) and of \( M_\ell \), it is easily checked from 2.22.4 above that we have the equality \( n_\ell^\xi = m_\ell^\xi \).

From now on, we assume that \( N_P(U) \neq F \); then, arguing by induction on the size of \( F \), we may assume that there exists a unique isomorphism class of reduced \( \hat{\mathcal{P}}_{\xi}(U) \)-modules \( m_\ell^\xi : \hat{\mathcal{P}}_{\xi}(U) \to k_\ell \text{-mod} \) fulfilling the announced condition; moreover, \( N_{\hat{\mathcal{P}}_{\xi}(U)} \) is a \( N_{\hat{\mathcal{P}}_{\xi}(U)} \)-sublocality of \( \hat{\mathcal{P}}_{\xi}(U) \) and the restrictions to \( N_{\mathfrak{p}}(U) \) of \( m_\ell^\xi \) and of \( m_\ell^\xi \) define two reduced \( N_{\hat{\mathcal{P}}_{\xi}(U)} \)-modules, both fulfilling the announced condition; consequently, we may assume that \( m_\ell^\xi(N_P(U)) = M \) and then, up to a conjugation by a suitable element of \( \text{GL}_{k_\ell}(M) \), that \( m_\ell^\xi(U) = m_\ell(\hat{y}) \) for any \( N_{\hat{\mathcal{P}}_{\xi}(U)} \)-morphism \( \hat{y} : R \to Q \).

For any \( V \in \mathcal{I} - \mathfrak{g} \) fully normalized in \( F \), by [9, Corollary 2.13] there is a \( \hat{\mathcal{P}}_{\xi}(V) \)-isomorphism \( \hat{y} : N_P(U) \to N_P(V) \) fulfilling \( \psi(\hat{y}) = V \), where \( \hat{y} \) is the image of \( \hat{y} \) in \( F(N_P(V), N_P(U)) \); then, we get a reduced \( N_{\hat{\mathcal{P}}_{\xi}(U)} \)-module \( m_\ell^\xi \cdot N_{\hat{\mathcal{P}}_{\xi}(U)} \begin{maplelatex}
\hat{y} : R \to Q \end{maplelatex}
\]

\[
m_\ell^\xi(\hat{y})^{-1} \circ m_\ell^\xi(U, \phi_{\ell}^{-1}(Q)) \circ m_\ell^\xi(\hat{y})^{-1} \cdot \hat{y} : R, \phi_{\ell}^{-1}(U) \to R, \phi_{\ell}^{-1}(Q)
\]

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and it is clear that \( m_{V,t}^{s} \) still fulfills the announced condition; moreover, it does not depend on our choice of \( \hat{y} \) since, for another choice \( \hat{y}' = \hat{y} \cdot \hat{s} \), the element \( \hat{s} \) belongs to \( \hat{\mathcal{P}}_{V}^{x,\ell} \left( N_P(U) \right)_{U} \) and therefore we have \( m_{V,t}^{\hat{s}} = m_{V,t}^{\hat{y}} \). Similarly, it is easily checked that, for any \( N_{\hat{\mathcal{P}}_{V}^{x,\ell}}(V) \)-morphism \( \hat{x} : R \to Q \), we have \( m_{V,t}^{\hat{x}} = m_{V,t}^{\hat{y}} \).

At this point, for any \( V, V' \in \mathfrak{X} - \mathfrak{Y} \) fully normalized in \( \mathcal{F} \), setting \( N = N_P(V) \) and \( N' = N_P(V') \), it follows from [9, condition 2.8.2] that any \( \hat{\mathcal{P}}_{V}^{x,\ell} \)-morphism \( \hat{x} : V \to V' \) factorizes as \( \hat{x} = \hat{\tau}_{V,V'}^{N,N'}(\hat{y}) \cdot \hat{s} \) for suitable \( \hat{y} \) in \( \hat{\mathcal{P}}_{V}^{x,\ell}(N', N)_{V',V} \) and \( \hat{s} \) in \( \hat{\mathcal{P}}_{V}^{x,\ell}(V) \); then, in \( \text{GL}_{k}(M) \) we define

\[
m_{V,t}^{\hat{x}} = m_{V,t}^{\hat{y}} \circ m_{V,t}^{\hat{s}}
\]

this definition does not depend on our choice since for such another decomposition \( \hat{x} = \hat{\tau}_{V,V'}^{N,N'}(\hat{y}) \cdot \hat{s} \), we get \( \hat{y} = \hat{y} \cdot \hat{t} \) and \( \hat{s} = \hat{\tau}_{V}^{\hat{t},N}(\hat{t})^{-1} \cdot \hat{s} \) for a suitable \( \hat{t} \) in \( \hat{\mathcal{P}}_{V}^{x,\ell}(N)_{V} \), so that we have

\[
m_{V,t}^{\hat{x}}(\hat{s}) \circ m_{V,t}^{\hat{s}}(\hat{y}) = m_{V,t}^{\hat{y}}(\hat{s}) \circ m_{V,t}^{\hat{y}}(\hat{t})^{-1} \circ m_{V,t}^{\hat{y}}(\hat{t}) \circ m_{V,t}^{\hat{s}}(\hat{y})
\]

\[
= m_{V,t}^{\hat{y}}(\hat{s}) \circ m_{V,t}^{\hat{y}}(\hat{y})
\]

In particular, for any \( \hat{y} \in \hat{\mathcal{P}}_{V}^{x,\ell}(N', N)_{V',V} \) we have

\[
m_{V,t}^{\hat{y}}(\hat{y}) = m_{V,t}^{\hat{y}}(\hat{y})
\]

More generally, if \( Q \) and \( Q' \) are a pair of subgroups of \( P \) respectively contained in \( N \) and \( N' \), and strictly containing \( V \) and \( V' \), for any \( \hat{x} \in \hat{\mathcal{P}}_{V}^{x,\ell}(Q', Q)_{V',V} \), we claim that

\[
m_{V,t}^{\hat{x}}(\hat{y}) = m_{V,t}^{\hat{y}}(\hat{x})
\]

indeed, it follows from [9, condition 2.8.2] that \( \hat{\tau}_{V,V'}^{Q',Q}(\hat{x}) = \hat{\tau}_{V}^{N',\hat{y}}(\hat{y}) \cdot \hat{z} \) for suitable elements \( \hat{y} \in \hat{\mathcal{P}}_{V}^{x,\ell}(N', N)_{V',V} \) and \( \hat{z} \in \hat{\mathcal{P}}_{V}^{x,\ell}(V) \); consequently, setting \( Q'' = \varphi_{\hat{x}^{-1}}(Q') \in N \), we get

\[
\hat{z} = \hat{\tau}_{V,V'}^{Q''}(\varphi_{\hat{x}^{-1}}(\hat{y})^{-1} \cdot \hat{x})
\]

and therefore, setting \( \hat{s} = \hat{\tau}_{V,V'}^{N,N'}(\hat{y})^{-1} \cdot \hat{x} \) which belongs to \( \hat{\mathcal{P}}_{V}^{x,\ell}(Q'', Q)_{V,V} \), we still get \( \hat{x} = \hat{\tau}_{V}^{N',\hat{y}}(\hat{y}) \cdot \hat{s} \); hence, we obtain

\[
\hat{\tau}_{V,V'}^{Q',Q}(\hat{x}) = \hat{\tau}_{V}^{N',\hat{y}}(\hat{y}) \cdot \hat{\tau}_{V,V'}^{Q'',Q}(\hat{s}) \quad \text{and} \quad m_{V,t}^{\hat{x}}(\hat{y}) = m_{V,t}^{\hat{s}}(\hat{y}) \circ m_{V,t}^{\hat{y}}(\hat{y})
\]

and, since \( \hat{\tau}_{V,V'}^{Q'',Q}(\hat{s}) \) belongs to \( \hat{\mathcal{P}}_{V}^{x,\ell}(V) \) and \( m_{V,t}^{\hat{s}} \) is a reduced \( N_{\hat{\mathcal{P}}_{V}^{x,\ell}(V)} \)-module, we finally have (cf. 2.22.12)

\[
m_{V,t}^{\hat{x}}(\hat{y}) = m_{V,t}^{\hat{s}}(\hat{y}) \circ m_{V,t}^{\hat{y}}(\hat{y}) = m_{V,t}^{\hat{y}}(\hat{s}) \circ m_{V,t}^{\hat{y}}(\hat{y}) = m_{V,t}^{\hat{y}}(\hat{y})
\]
For another $V'' \in \mathfrak{X} - \mathfrak{Y}$ fully normalized in $\mathcal{F}$, setting $N'' = N_P(V'')$ and considering a $\hat{\mathcal{P}}^{x,\ell}$-morphism $\hat{x} : V' \to V''$, we claim that
\[
m^*_i(\hat{x} \cdot \hat{x}) = m^*_i(\hat{x}) \circ m^*_i(\hat{x}')
\]
indeed, assuming that
\[
\hat{x} = \hat{r}^{N',N}(\gamma') \cdot \hat{s} \quad \text{and} \quad \hat{x}' = \hat{r}^{N'',N'}(\gamma') \cdot \hat{s}'
\]
for suitable $\hat{\gamma} \in \hat{\mathcal{P}}^{\gamma,\ell}(N',N)_{V' \cdot V}$, $\hat{\gamma}' \in \hat{\mathcal{P}}^{\gamma,\ell}(N'',N')_{V' \cdot V}$, $\hat{s} \in \hat{\mathcal{P}}^{x,\ell}(V)$ and $\hat{s}' \in \hat{\mathcal{P}}^{x,\ell}(V')$, we get
\[
m^*_i(\hat{x}) \circ m^*_i(\hat{x}') = m^*_i(\hat{s}) \circ m^*_i(\hat{\gamma}) \circ m^*_i(\hat{s}') \circ m^*_i(\hat{\gamma}')
\]
\[
= m^*_i(\hat{\gamma}) \circ (m^*_i(\hat{s}) \circ m^*_i(\hat{s}') \circ m^*_i(\hat{\gamma}') \circ m^*_i(\hat{\gamma})^{-1}) \circ m^*_i(\hat{\gamma}' \\ \hat{\gamma})
\]
\[
\hat{x} \cdot \hat{x} = \hat{r}^{N',N}(\hat{\gamma}) \cdot \hat{s} \cdot \hat{r}^{N',N}(\hat{\gamma}') \cdot \hat{s}' \cdot \hat{r}^{N',N}(\hat{\gamma}) \cdot \hat{s}
\]
\[
\hat{x} \cdot \hat{s} = \hat{r}^{N',N}(\hat{\gamma}) \cdot \hat{s} \cdot \hat{r}^{N',N}(\hat{\gamma}') \cdot \hat{s}' \cdot \hat{r}^{N',N}(\hat{\gamma}) \cdot \hat{s}
\]
Moreover, it is clear that $\hat{\gamma}' \cdot \hat{\gamma}$ belongs to $\hat{\mathcal{P}}^{\gamma,\ell}(N'',N)_{V' \cdot V}$ and it follows easily from the very definition of $m^*_i(\hat{x})$ in 2.22.11 above that the element
\[
\hat{s}'' = \hat{r}^{N',N}(\hat{\gamma})^{-1} \cdot \hat{s} \cdot \hat{r}^{N',N}(\hat{\gamma}) \quad \text{in} \quad \hat{\mathcal{P}}^{x,\ell}(V)
\]
fulfills
\[
m^*_i(\hat{s}'') = m^*_i(\hat{\gamma}) \circ m^*_i(\hat{s}') \circ m^*_i(\hat{\gamma})^{-1}
\]
consequently, from 2.22.21 we obtain
\[
m^*_i(\hat{x}) \circ m^*_i(\hat{x}') = m^*_i(\hat{s}) \circ m^*_i(\hat{s}') \circ m^*_i(\hat{\gamma}') \circ m^*_i(\hat{\gamma})
\]
\[
= m^*_i(\hat{\gamma}) \circ (m^*_i(\hat{s}) \circ m^*_i(\hat{s}') \circ m^*_i(\hat{\gamma}') \circ m^*_i(\hat{\gamma})^{-1}) \circ m^*_i(\hat{\gamma}' \hat{\gamma})
\]
which proves our claim.

We are ready to consider any pair of subgroups $V$ and $V'$ in $\mathfrak{X} - \mathfrak{Y}$.
We clearly have $N = N_P(V) \neq V$ and it follows from [9, Proposition 2.7] that there is an $\mathcal{F}$-morphism $\nu : N \to P$ such that $\nu(V)$ is fully normalized in $\mathcal{F}$; moreover, we choose $\hat{n} \in \hat{\mathcal{P}}^{\mathfrak{Y},\ell}(\nu(N),N)$ lifting the $\mathcal{F}$-isomorphism $\nu_*$ determined by $\nu$. That is to say, we may assume that

2.22.24 There is a pair $(N,\hat{n})$ formed by a subgroup $N$ of $P$ which strictly contains and normalizes $V$, and by an element $\hat{n}$ in $\hat{\mathcal{P}}^{\mathfrak{Y},\ell}(\nu(N),N)$ lifting $\nu_*$ for a $\mathcal{F}$-morphism $\nu : N \to P$ such that $\nu(V)$ is fully normalized in $\mathcal{F}$.

We denote by $\mathfrak{R}(V)$ the set of such pairs and often we write $\hat{n}$ instead of $(N,\hat{n})$, setting $\hat{n}N = \varphi_N(N)$ and $\hat{n}V = \varphi_N(V)$. Then, for any $\hat{\mathcal{P}}^{x,\ell}$-morphism $\hat{x} : V \to V'$, we consider pairs $(N,\hat{n})$ in $\mathfrak{R}(V)$ and $(N',\hat{n}')$ in $\mathfrak{R}(V')$ and, since $\hat{n}V$ and $\hat{n}'V'$ are both fully normalized in $\mathcal{F}$, we can define
\[
m^*_i(\hat{x}) = m^*_i(\hat{n}) \circ m^*_i(\hat{r}^{N,N,V}_{\nu \nu',V}(\hat{n}) \cdot \hat{x} \cdot \hat{r}^{N,N}(\hat{n})^{-1}) \circ m^*_i(\hat{n}')^{-1}
\]
This definition is independent of our choices; indeed, for another pair \((\tilde{N}, \tilde{n})\) in \(\mathfrak{M}(V)\), setting \(\tilde{N} = \langle N, \bar{N} \rangle\) and considering a new \(F\)-morphism \(\psi : \tilde{N} \to P\) such that \(\psi(V)\) is fully normalized in \(F\), we can obtain a third pair \((\tilde{N}, \tilde{m})\) in \(\mathfrak{M}(V)\); then, \(\tilde{r}_{m,N,N}(\tilde{m})\cdot\tilde{\bar{N}}^{-1}\) and \(\tilde{r}_{m,N,N}(\tilde{m})\cdot\tilde{\bar{N}}^{-1}\) respectively belong to \(\tilde{P}^{m,t}(\bar{N}, \bar{N})\) and to \(\tilde{P}^{m,t}(\bar{N}, \bar{N})\); in particular, since \(\tilde{m}V, \tilde{m}V\) and \(\tilde{m}V\) are fully normalized in \(F\), we get
\[
\begin{align*}
\hat{m}_V^m(\tilde{m}) &= \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) = 22.26, \\
&= \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) = 22.27.
\end{align*}
\]
Consequently, we obtain
\[
\begin{align*}
\hat{m}_V^m(\tilde{m}) &= \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) = 22.28.
\end{align*}
\]
Symmetrically, we can replace \((\tilde{N}', \tilde{n}')\) for another pair \((\tilde{N}', \tilde{n}')\) in \(\mathfrak{M}(V')\).

Moreover, equality 22.21.15 still holds with this general definition; indeed, for any pair of subgroups \(Q\) and \(Q'\) of \(P\) respectively normalizing and strictly containing \(V\) and \(V'\), we claim that
\[
\begin{align*}
\hat{m}_V^m(\tilde{m}) &= \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) = 22.29;
\end{align*}
\]
indeed, it is clear that we have pairs \((Q, \bar{n})\) in \(\mathfrak{M}(V)\) and \((Q', \bar{n}')\) in \(\mathfrak{M}(V')\), and by the very definition 22.21.5 and by equality 22.21.5 we have
\[
\begin{align*}
\hat{m}_V^m(\tilde{m}) &= \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) \circ \hat{m}_V^m(\tilde{m}) = 22.30.
\end{align*}
\]
Once again, for another \(V'' \in \mathfrak{X} - \mathfrak{Q}\), setting \(N'' = N_P(V'')\) and considering a \(\tilde{P}^{x,t}\)-morphism \(\tilde{x}' : V' \to V''\), we claim that
\[
\begin{align*}
\hat{m}_V^m(\tilde{x}') &= \hat{m}_V^m(\tilde{x}') \circ \hat{m}_V^m(\tilde{x}') = 22.31;
\end{align*}
\]
indeed, considering a pair \((N'', \hat{n}'')\) in \(\Omega(V'')\) and setting \(\hat{x}'' = \hat{x}'\cdot \hat{x}\), from the very definition 2.22.25 we get

\[
m_{\ell}^y(\hat{x}) = m_{\ell}^y(\hat{n}) \circ m_{\ell}^x(\hat{r}_{\ell}^N N' (\hat{n}')) \cdot \hat{x} \cdot \hat{r}_{\ell}^N N (\hat{n}^{-1}) \circ m_{\ell}^y(\hat{n}^{-1})
\]

\[
m_{\ell}^y(\hat{x}') = m_{\ell}^y(\hat{n}) \circ m_{\ell}^x(\hat{r}_{\ell}^N N'' (\hat{n}'')) \cdot \hat{x}' \cdot \hat{r}_{\ell}^N N (\hat{n}^{-1}) \circ m_{\ell}^y(\hat{n}^{-1})
\]

\[
m_{\ell}^y(\hat{x}'') = m_{\ell}^y(\hat{n}) \circ m_{\ell}^x(\hat{r}_{\ell}^N N'' (\hat{n}'')) \cdot \hat{x}'' \cdot \hat{r}_{\ell}^N N (\hat{n}^{-1}) \circ m_{\ell}^y(\hat{n}^{-1})
\]

and it follows from equality 2.22.19 that the composition of the first and the second equalities above coincides with the third one.

At this point, we are able to complete the definition of the reduced \((k_\ell)_\ast \hat{\cal P}^{x,\ell}\)-module \(m_{\ell}^x : \hat{\cal P}^{x,\ell} \to k_\ell\)-mod fulfilling the announced condition. For any \(\hat{\cal P}^{x,\ell}\)-morphism \(\hat{x} : R \to Q\) either \(R\) belongs to \(\cal Q\) and we simply set \(m_{\ell}^x(\hat{x}) = m_{\ell}^y(\hat{x})\), or \(R\) belongs to \(\mathfrak{X} - \cal Q\) and, denoting by \(R_{\ast}\) the image of \(R\) in \(Q\) and by \(\hat{x}_\ast : R_{\ast} \cong R_{\ast}\) the \(\hat{\cal P}^{x,\ell}\)-isomorphism determined by \(\hat{x}\), we set \(m_{\ell}^x(\hat{x}) = m_{\ell}^y(\hat{x}_\ast)\) (cf. 2.22.25); note that if \(Q\) contains \(R\) then we have \(m_{\ell}^x(\hat{\cal R}_{\ast}) = \text{id}_{R_{\ast}}\). Moreover, we claim that for another \(\hat{\cal P}^{x,\ell}\)-morphism \(\hat{y} : T \to R\) we have

\[
m_{\ell}^y(\hat{x} \cdot \hat{y}) = m_{\ell}^y(\hat{y}) \circ m_{\ell}^x(\hat{x})
\]

indeed, if \(T\) belongs to \(\cal Q\) then we just have

\[
m_{\ell}^y(\hat{x} \cdot \hat{y}) = m_{\ell}^y(\hat{x} \cdot \hat{y}) = m_{\ell}^y(\hat{y}) \circ m_{\ell}^y(\hat{x}) = m_{\ell}^y(\hat{y}) \circ m_{\ell}^y(\hat{x})
\]

If \(R\) belongs to \(\mathfrak{X} - \cal Q\) then \(\hat{y}\) is a \(\hat{\cal P}^{x,\ell}\)-isomorphism and, with the notation above, we have \(T_{\ast} = R_{\ast}\) and \((\hat{x} \cdot \hat{y})_{\ast} = \hat{x}_{\ast} \cdot \hat{y}_{\ast}\); in this case, from equality 2.22.19 we get

\[
m_{\ell}^y(\hat{x} \cdot \hat{y}) = m_{\ell}^y((\hat{x} \cdot \hat{y})_{\ast}) = m_{\ell}^y(\hat{x}_{\ast} \cdot \hat{y}_{\ast}) = m_{\ell}^y(\hat{y}_{\ast}) \circ m_{\ell}^y(\hat{x}_{\ast}) = m_{\ell}^y(\hat{y}) \circ m_{\ell}^y(\hat{x})
\]

Finally, assume that \(T \in \mathfrak{X} - \cal Q\) and \(R \in \cal Q\), denote by \(T_{\ast}\) and \(T_{\ast \ast} \subset R_{\ast}\) the respective images of \(T\) in \(R\) and \(Q\), and by \(\hat{x}_{\ast \ast} : T_{\ast} \to T_{\ast \ast}\) the \(\hat{\cal P}^{x,\ell}\)-isomorphism fulfilling

\[
\hat{x}_{\ast \ast} \cdot \hat{R}_{\ast} = \hat{T}_{\ast \ast} \cdot \hat{x}_{\ast \ast}
\]

then, setting \(\bar{R} = N_R(T_{\ast})\) and \(\bar{T}_{\ast} = N_R(T_{\ast \ast})\), it follows from 2.23.16 and 2.22.31 that we have

\[
m_{\ell}^y(\hat{x} \cdot \hat{y}) = m_{\ell}^y(\hat{x}_{\ast \ast} \cdot \hat{y}_{\ast \ast}) = m_{\ell}^y(\hat{y}_{\ast}) \circ m_{\ell}^y(\hat{x}_{\ast \ast})
\]

\[
= m_{\ell}^y(\hat{y}) \circ m_{\ell}^y(\hat{R}_{\ast} R_{\ast \ast} (\hat{x}_{\ast \ast}))
\]

\[
= m_{\ell}^y(\hat{y}) \circ m_{\ell}^y(\hat{R}_{\ast} R_{\ast \ast} (\hat{x}_{\ast \ast})) = m_{\ell}^y(\hat{y}) \circ m_{\ell}^y(\hat{x})
\]

It is not difficult to check that the functor \(m_{\ell}^x : \hat{\cal P}^{x,\ell} \to k_\ell\)-mod is a reduced \((k_\ell)_\ast \hat{\cal P}^{x,\ell}\)-module which fulfills the announced condition. We are done.
Corollary 2.23. The homomorphism cat\(_K\) : \(G(K, \hat{\mathcal{P}}^{\infty} \text{-mod}) \rightarrow G_K(\hat{\mathcal{P}}^{\infty})\) is bijective.

Proof: According to Corollary 2.17, it suffices to prove the surjectivity. Let \(\{X_q\}_{\hat{q}}\) where \(\hat{q}\) runs over the set of \(\hat{\mathcal{P}}^{\infty}\)-chains be a family belonging to \(G_K(\hat{\mathcal{P}}^{\infty})\); according to Remark 1.11, we actually may assume that \(X_q\) is the class of a \(K, \hat{\mathcal{P}}^{\infty}(\hat{q})\)-module \(M_q\) for any \(\hat{\mathcal{P}}^{\infty}\)-chain \(\hat{q}\). Since the isomorphisms 2.18.1, 2.19.1 and 2.19.2 all preserve the classes of modules, it follows from isomorphism 2.20.3 that \(\{X_q\}_{\hat{q}}\) determines a family \(\{X_{q'}\}_{\hat{q}'}\) in \(G_{k_{\hat{q}}} (\hat{\mathcal{P}}^{\infty}_{\hat{q}'}\text{-chains})\) where \(\hat{q}'\) runs over the set of \(\hat{\mathcal{P}}^{\infty}_{\hat{q}'}\)-chains and, for such a \(\hat{q}'\), \(X_{q'}\) is the class of a \((k_{\hat{q}}), \hat{\mathcal{P}}^{\infty}_{\hat{q}'}(\hat{q}')\)-module \(M_{q'}\). Consequently, it follows from Theorem 2.22 above that there exists a reduced \((k_{\hat{q}}), \hat{\mathcal{P}}^{\infty}_{\hat{q}'}\)-module \(\hat{m}_{\hat{q}'} : \hat{\mathcal{P}}^{\infty}_{\hat{q}'} \rightarrow k_{\hat{q}}\text{-mod}\) such that \(X_{q'}\) is the class of \(\hat{m}_{\hat{q}'}(\hat{q}'(0))\) in \(G_{k_{\hat{q}}} (\hat{\mathcal{P}}^{\infty}_{\hat{q}'}(\hat{q}'))\) for any \(\hat{\mathcal{P}}^{\infty}_{\hat{q}'}\)-chain \(\hat{q}'\).

In particular, setting \(M_{\hat{q}} = \hat{m}_{\hat{q}}(P)\) and denoting by \(\hat{G}(m_{\hat{q}})\) the \((k_{\hat{q}})^*\)-subgroup of the finite group \(\text{GL}_{k_{\hat{q}}}(M_{\hat{q}})\) generated by \(m_{\hat{q}}(\hat{x})\) where \(\hat{x} : R \rightarrow Q\) runs over the set of \(\hat{\mathcal{P}}^{\infty}_{\hat{q}'}\)-morphisms, \(M_{\hat{q}}\) becomes a \((k_{\hat{q}}), \hat{G}(m_{\hat{q}})\)-module, determining an element \(X_{\hat{q}}\) in \(G_{k_{\hat{q}}} (\hat{G}(m_{\hat{q}}))\); since the Brauer decomposition map

\[
G_{K_{\hat{q}}} (\hat{G}(m_{\hat{q}})) \rightarrow G_{k_{\hat{q}}} (\hat{G}(m_{\hat{q}}))
\]

is surjective, \(X_{\hat{q}}\) can be lifted to some element \(\hat{X}_{\hat{q}} \in G_{K_{\hat{q}}} (\hat{G}(m_{\hat{q}}))\) which is then the difference of the classes of suitable \((K_{\hat{q}}), \hat{G}(m_{\hat{q}})\)-modules \(\hat{M}_{\hat{q}}\) and \(\hat{M}_{\hat{q}}''\).

But, since we have an obvious \((k_{\hat{q}})^*\)-functor from the \((k_{\hat{q}})^*\)-category \(\hat{\mathcal{P}}^{\infty}_{\hat{q}'}\) to the \((k_{\hat{q}})^*\)-category with a unique object \(\emptyset\) and \((k_{\hat{q}})^*\)-group of automorphisms \(\hat{G}(m_{\hat{q}})\), \(\hat{M}_{\hat{q}}\) and \(\hat{M}_{\hat{q}}''\) determine respective reduced \((K_{\hat{q}}), \hat{\mathcal{P}}^{\infty}_{\hat{q}'}\)-modules \(\hat{m}_{\hat{q}} : \hat{\mathcal{P}}^{\infty}_{\hat{q}'} \rightarrow K_{\hat{q}}\text{-mod}\) and \(\hat{m}_{\hat{q}}'' : \hat{\mathcal{P}}^{\infty}_{\hat{q}'} \rightarrow K_{\hat{q}}\text{-mod}\); then, since \(\hat{\mathcal{P}}^{\infty}_{\hat{q}'} \subset \hat{\mathcal{P}}^{\infty}_{\hat{q}''}\) and \(K_{\hat{q}} \subset \hat{K}\) (cf. 2.19), \(\hat{m}_{\hat{q}}\) and \(\hat{m}_{\hat{q}}''\) determine respective reduced \((\hat{K}), \hat{\mathcal{P}}^{\infty}\)-modules \(\hat{m} : \hat{\mathcal{P}}^{\infty} \rightarrow \hat{K}\text{-mod}\) and \(\hat{m}'' : \hat{\mathcal{P}}^{\infty} \rightarrow \hat{K}\text{-mod}\); finally, since we actually may assume that \(\hat{K}\) contains \(K\) and that \(K\) is big enough, \(\hat{m}\) and \(\hat{m}''\) come from respective reduced \((K), \hat{\mathcal{P}}^{\infty}\)-modules \(m : \hat{\mathcal{P}}^{\infty} \rightarrow K\text{-mod}\) and \(m'' : \hat{\mathcal{P}}^{\infty} \rightarrow K\text{-mod}\), and it is easily checked that the difference between their images in \(G_K(\hat{\mathcal{P}}^{\infty})\) coincides with the starting family \(\{X_q\}_{\hat{q}}\).

Corollary 2.24. There exists a \(K, \hat{\mathcal{P}}^{\infty}\)-module \(m : \hat{\mathcal{P}}^{\infty} \rightarrow K\text{-mod}\) such that, for any \(\hat{\mathcal{P}}^{\infty}\)-chain \(\hat{q} : \Delta_n \rightarrow \hat{\mathcal{P}}^{\infty}\), the class of \(m(\hat{q}(0))\) in \(G_K(\hat{\mathcal{P}}^{\infty}(\hat{q}))\) is a multiple of the regular \(K, \hat{\mathcal{P}}^{\infty}(\hat{q})\)-module. In particular, the \(k^*\)-functor from \(\hat{\mathcal{P}}^{\infty}\) to the \(k^*\)-category over one object with automorphism \(k^*\)-group \(\hat{G}(m)\) is faithful.
Proof: From Remark 1.11 we already know that, denoting by $R_\delta$ the class in $G_{k\ell}(\hat{\mathcal{P}}^{v,e}(\hat{q}))$ of the regular $(k\ell)_s\hat{\mathcal{P}}^{v,e}(\hat{q})$-module $(k\ell)_s\hat{\mathcal{P}}^{v,e}(\hat{q})$, and choosing a multiple $m$ of all the orders $|\mathcal{P}(\hat{q})|$ where $\hat{q}$ runs over the set of $\hat{\mathcal{P}}^{v,e}$-chains, it is easily checked that the family $R = \{\frac{m}{|\mathcal{P}(\hat{q})|}R_\delta\}_{\hat{q}}$ belongs to $G_{k\ell}(\hat{\mathcal{P}}^{v,e})$; hence, it follows from Theorem 2.22 that there exists a reduced $(k\ell)_s\hat{\mathcal{P}}^{v,e}$-module $n_\ell: \hat{\mathcal{P}}^{v,e} \to k\ell\mod$ such that $\frac{m}{|\mathcal{P}(\hat{q})|}R_\delta$ is the class of $n_\ell(\hat{q}(0))$ in $G_{k\ell}(\hat{\mathcal{P}}^{v,e}(\hat{q}))$ for any $\hat{\mathcal{P}}^{v,e}$-chain $\hat{q}$.

As above, setting $N_\ell = n_\ell(P)$ and denoting by $\hat{G}(n_\ell)$ the $(k\ell)_s$-subgroup of the finite group $GL_{k\ell}(N_\ell)$ generated by $n_\ell(x)$ where $x: R \to Q$ runs over the set of $\hat{\mathcal{P}}^{v,e}$-morphisms, $N_\ell$ becomes a $(k\ell)_s\hat{G}(n_\ell)$-module determining an element in $G_{k\ell}(\hat{\mathcal{P}}(n_\ell))$; then, we know that a multiple of this element comes, via the Brauer decomposition map, from a true $(K\ell)_s\hat{G}(n_\ell)$-module and, via a field $\hat{K}$ containing $K\ell$ and $K$, it comes from a $K\ell\hat{G}(n_\ell)$-module $M$; finally, the reduced $K\ell\hat{\mathcal{P}}^{v,e}$-module $m: \hat{\mathcal{P}}^{v,e} \to K\ell\mod$ determined by $M$ fulfills the announced condition.

2.25. Let $\sigma_{\hat{\mathcal{P}}^{v,e}}: \hat{\mathcal{P}}^{v,e} \to K\ell\mod$ be the direct sum of a set of reduced representatives for the set of isomorphism classes of simple $K\ell\hat{\mathcal{P}}^{v,e}$-modules and denote by $\hat{G}(\hat{\mathcal{P}}^{v,e})$ the $k^*$-subgroup of $GL_K(\sigma_{\hat{\mathcal{P}}^{v,e}}(P))$ generated by $\sigma_{\hat{\mathcal{P}}^{v,e}}(x)$ where $x: R \to Q$ runs over the set of $\hat{\mathcal{P}}^{v,e}$-morphisms and by $\tau_{\hat{\mathcal{P}}^{v,e}}$ the functor determined by $\sigma_{\hat{\mathcal{P}}^{v,e}}$ from $\hat{\mathcal{P}}^{v,e}$ to the $k^*$-category over one object with the automorphism $k^*$-group $\hat{G}(\hat{\mathcal{P}}^{v,e})$, which is faithful by Corollary 2.24 above.

Corollary 2.26. With the notation above, the functor $\tau_{\hat{\mathcal{P}}^{v,e}}$ induces an equivalence of categories from $K\ell\hat{G}(\hat{\mathcal{P}}^{v,e})\mod$ to $K\ell\hat{\mathcal{P}}^{v,e}\mod$. Moreover, the regular representation of $\hat{G}(\hat{\mathcal{P}}^{v,e})$ induces a $K\ell\hat{\mathcal{P}}^{v,e}$-module $\tau_{\hat{\mathcal{P}}^{v,e}}: \hat{\mathcal{P}}^{v,e} \to K\ell\mod$ such that, for any $\hat{\mathcal{P}}^{v,e}$-chain $\hat{q}: \Delta_n \to \hat{\mathcal{P}}^{v,e}$, the class of $\tau_{\hat{\mathcal{P}}^{v,e}}(\hat{q}(0))$ in $G_{K\ell}(\hat{\mathcal{P}}^{v,e}(\hat{q}))$ is a multiple of the regular $K\ell\hat{\mathcal{P}}^{v,e}(\hat{q})$-module.

Proof: It is clear that, for any simple $K\ell\hat{\mathcal{P}}^{v,e}$-module $\sigma: \hat{\mathcal{P}}^{v,e} \to K\ell\mod$, the $k^*$-group $\hat{G}(\hat{\mathcal{P}}^{v,e})$ acts on $\sigma(P)$ and then $\sigma(P)$ becomes a simple $K\ell\hat{G}(\hat{\mathcal{P}}^{v,e})$-module; more generally, the restriction from $\hat{G}(\hat{\mathcal{P}}^{v,e})$ to $\hat{\mathcal{P}}^{v,e}$ via $\tau_{\hat{\mathcal{P}}^{v,e}}$ clearly determines a functor from $K\ell\hat{G}(\hat{\mathcal{P}}^{v,e})\mod$ to $K\ell\hat{\mathcal{P}}^{v,e}\mod$ which induces a bijection between the sets of isomorphism classes of simple $K\ell\hat{G}(\hat{\mathcal{P}}^{v,e})$- and $K\ell\hat{\mathcal{P}}^{v,e}$-modules; since both categories are semisimple, this functor is an equivalence of categories. Moreover, since the functor $\tau_{\hat{\mathcal{P}}^{v,e}}$ is faithful, for any $\hat{\mathcal{P}}^{v,e}$-chain $\hat{q}: \Delta_n \to \hat{\mathcal{P}}^{v,e}$ the $k^*$-group $\hat{\mathcal{P}}^{v,e}(\hat{q})$ is $k^*$-isomorphic to a $k^*$-subgroup of $G(\hat{\mathcal{P}}^{v,e})$. 
3. Categorization for the characteristic $p$ case

3.1. With the notation in 2.1 above, this time we set (cf. 1.10.3)

$$G_k(\hat{P}^\infty) = \lim_{\to} (g_k \circ \text{aut}_{p^\infty})$$

that is to say, $G_k(\hat{P}^\infty)$ is the subset of elements

$$\{Z(\hat{q}, \Delta, n) \mid \hat{q} \in \hat{\Delta}(\Delta, \hat{P}^\infty)\} \in \prod_{n \in \mathbb{N}} \prod_{\hat{q} \in \hat{\Delta}(\Delta, \hat{P}^\infty)} G_k(\hat{P}^\infty)$$

such that, for any $\hat{P}^\infty$-morphism $(\nu, \delta) : (\hat{q}, \Delta_n) \to (\hat{r}, \Delta_m)$, they fulfill

$$\text{res}_{\text{aut}_{p^\infty}(\nu, \delta)}(Z(\hat{r}, \Delta_m)) = Z(\hat{q}, \Delta_n)$$

Our purpose in this section is to show that $G_k(\hat{P}^\infty)$ is the extension to $O$ of the very Grothendieck group of the category of $k_*G(\hat{P}^\infty)$-modules for the $k^*$-group $G(\hat{P}^\infty)$ introduced in 2.25 above.

3.2. As in 2.3 above, we call contravariant $k^*$-functor $m : \hat{P}^\infty \to k\text{-mod}$ any functor such that $m[\lambda, x] = \lambda \cdot m(\hat{x})$ for any $\hat{P}^\infty$-morphism $\hat{x} : R \to Q$ and any $\lambda \in k^*$; once again, any contravariant $k^*$-functor $m : \hat{P}^\infty \to k\text{-mod}$ determines a new contravariant $k^*$-functor $[9, A3.7.3]

$$m^\text{ch} = m \circ \nu_{p^\infty} : \text{ch}^*(\hat{P}^\infty) \to k\text{-mod}$$

sending any $\hat{P}^\infty$-chain $\hat{q} : \Delta_n \to \hat{P}^\infty$ to $m(\hat{q}(0))$ and any $\text{ch}^*(\hat{P}^\infty)$-morphism

$$(\hat{x}, \delta) : (\hat{r}, \Delta_m) \to (\hat{q}, \Delta_n)$$

where $\delta : \Delta_n \to \Delta_m$ is an order-preserving map and $\hat{x} : \hat{r} \circ \delta \cong \hat{q}$ a natural isomorphism, to the $k$-linear map

$m(\hat{x}_0 \circ \hat{r}(0 \bullet \delta(0)) : m(\hat{q}(0)) \to m(\hat{r}(0))$}

3.3. As in 2.4 above, in the case where $n = m$ and $\delta$ is the identity map, we get a $k^*$-compatible action of the $k^*$-group $\text{aut}_{p^\infty}(\hat{q}) = \hat{P}^\infty(\hat{q})$ over $m(\hat{q}(0))$, so that $m(\hat{q}(0))$ becomes a $k_*\hat{P}^\infty(\hat{q})$-module; similarly, $m(\hat{r}(0))$ becomes a $k_*\hat{P}^\infty(\hat{r})$-module and, via the $k^*$-group homomorphism $\text{aut}_{p^\infty}(\hat{x}, \delta)$, $m(\hat{q}(0))$ also becomes the $k_*\hat{P}^\infty(\hat{r})$-module $\text{Res}_{\text{aut}_{p^\infty}(\hat{x}, \delta)}(m(\hat{q}(0)))$; then the $k$-linear map 3.2.3 is clearly a $k_*\hat{P}^\infty(\hat{r})$-module homomorphism

$m^\text{ch}(\hat{x}, \delta) : \text{Res}_{\text{aut}_{p^\infty}(\hat{x}, \delta)}(m(\hat{q}(0))) \to m(\hat{r}(0))$

Similarly, any natural map $\mu : m \to m'$ between contravariant $k^*$-functors $m$ and $m'$ from $\hat{P}^\infty$ to $k\text{-mod}$ determines a new natural map

$$\mu^\text{ch} = \mu * \nu_{p^\infty} : m^\text{ch} \to m'^\text{ch}$$

sending any $\hat{P}^\infty$-chain $\hat{q} : \Delta_n \to \hat{P}^\infty$ to the $k$-linear map $\mu_{\hat{q}(0)}$ and it follows from the naturalness of $\mu$ that this map is actually a $k_*\hat{P}^\infty(\hat{q})$-module homomorphism.
3.4. Once again, we are interested in the contravariant $k^*$-functors $m: \hat{\mathcal{P}}^\text{ev} \to k\text{-mod}$ — called reversible — mapping any $\hat{\mathcal{P}}^\text{ev}$-morphism $\hat{x}: R \to Q$ on a $k$-linear isomorphism $m(\hat{x}): m(Q) \cong m(R)$; in this case, it is quite clear that the contravariant $k^*$-functor $m^\text{ch}$ also maps any $\text{ch}^*(\hat{\mathcal{P}}^\text{ev})$-morphism on a $k$-linear isomorphism and, as in 2.6.2 above, the converse is also true. Moreover, if $m: \hat{\mathcal{P}}^\text{ev} \to k\text{-mod}$ is a reversible contravariant $k^*$-functor, homomorphism 3.3.1 becomes an isomorphism and therefore the restriction map

$$g_k(\text{aut}_{\mathcal{P}}^\text{ev}((\hat{x}, \delta))): G_k(\hat{\mathcal{P}}^\text{ev}((\hat{q}))) \to G_k(\hat{\mathcal{P}}^\text{ev}((\hat{r})))$$ 3.4.1

sends the class $Z_m(\hat{q}(0))$ in $G_k(\hat{\mathcal{P}}^\text{ev}((\hat{q})))$ of the $k^*$-$\hat{\mathcal{P}}^\text{ev}$-module $m(\hat{q}(0))$ to the class $Z_m(\hat{r}(0))$ in $G_k(\hat{\mathcal{P}}^\text{ev}((\hat{r})))$ of the $k^*$-$\hat{\mathcal{P}}^\text{ev}$-module $m(\hat{r}(0))$. That is to say, the family

$$\{Z_m(\hat{q}(0))\}_{\hat{q}, \Delta_n} = \prod_{n \in \mathbb{N}} \prod_{\hat{q} \in \hat{G}(\Delta_n, \hat{\mathcal{P}}^\text{ev})} G_k(\hat{\mathcal{P}}^\text{ev}((\hat{q})))$$ 3.4.2

fulfills condition 3.1.3 and therefore it belongs to $G_k(\hat{\mathcal{P}}^\text{ev})$; let us denote by $Z_m$ this family which, clearly, only depends on the isomorphism class of $m$.

3.5. This time we call $k^*$-$\hat{\mathcal{P}}^\text{ev}$-module $m: \hat{\mathcal{P}}^\text{ev} \to k\text{-mod}$ just any contravariant functor obtained by restriction from a $k^*$-$\hat{G}(\hat{\mathcal{P}}^\text{ev})$-module via the functor $\mathcal{I}_{\hat{\mathcal{P}}^\text{ev}}$ introduced in 2.15; thus, denoting by $k^*$-$\hat{\mathcal{P}}^\text{ev}$-mod the category formed by these $k^*$-$\hat{\mathcal{P}}^\text{ev}$-modules and by the natural maps between them, we clearly get a faithful functor

$$k^*\hat{G}(\hat{\mathcal{P}}^\text{ev})\text{-mod} \to k^*\hat{\mathcal{P}}^\text{ev}\text{-mod}$$ 3.5.1

moreover, it is easily checked that any natural map $\mu: m \to m'$ between $k^*$-$\hat{\mathcal{P}}^\text{ev}$-modules $m$ and $m'$ induces a $k^*\hat{G}(\hat{\mathcal{P}}^\text{ev})$-morphism $\mu_P: m(P) \to m'(P)$; consequently, the functor above is actually an equivalence of categories. Thus, denoting by $\mathcal{G}(k^*\hat{\mathcal{P}}^\text{ev}\text{-mod})$ the extension to $\mathcal{O}$ of the Grothendieck group of $k^*\hat{\mathcal{P}}^\text{ev}\text{-mod}$, the equivalence 3.5.1 induces an $\mathcal{O}$-module isomorphism

$$\mathcal{G}(k^*\hat{\mathcal{P}}^\text{ev}\text{-mod}) \cong G_k(\hat{G}(\hat{\mathcal{P}}^\text{ev}))$$ 3.5.2.

3.6. Moreover, as above the correspondence sending any $k^*$-$\hat{\mathcal{P}}^\text{ev}$-module $m$ to $Z_m$ induces an $\mathcal{O}$-module homomorphism

$$\text{cat}_k: \mathcal{G}(k^*\hat{\mathcal{P}}^\text{ev}\text{-mod}) \to G_k(\hat{\mathcal{P}}^\text{ev})$$ 3.6.1

and it is quite clear that suitable Brauer decomposition maps [10, 3.4.2] yield the following commutative diagram (cf. Corollary 2.23)

$$\begin{array}{ccc}
G_k(\hat{\mathcal{P}}^\text{ev}) & \cong & G_k(\hat{\mathcal{P}}^\text{ev}\text{-mod}) \\
\downarrow^{\delta_{G(\hat{\mathcal{P}}^\text{ev})}} & & \downarrow^{\text{cat}_k} \\
G_k(\hat{G}(\hat{\mathcal{P}}^\text{ev})) & \cong & G_k(\hat{\mathcal{P}}^\text{ev}) \\
\downarrow^{\delta_{\hat{G}(\hat{\mathcal{P}}^\text{ev})}} & & \downarrow^{\text{cat}_{\hat{G}}} \\
G_k(\hat{G}(\hat{\mathcal{P}}^\text{ev})) & \cong & G_k(\hat{\mathcal{P}}^\text{ev})
\end{array}$$ 3.6.2.
But we already know that, for any finite group $H$, the Brauer decomposition map $\delta_H : G_K(H) \rightarrow G_k(H)$ is surjective and even admits a natural section — namely, extending any Brauer character $\varphi$ of $H$ to the central function over $H$ mapping $y \in H$ on $\varphi(y_p)$; moreover, it is easy to check that all this still holds for $k^*$-groups. Consequently, the vertical homomorphisms in the diagram above admit sections and, in particular, they are surjective.

3.7. More generally, for any finite group $H$ and any $p$-element $v$ in $H$, assuming that $K$ is big enough for $H$ we have the Brauer general decomposition map

$$\delta^v_H : G_K(H) \longrightarrow G_k(C_H(v))$$

which, following [2, Appendice], can be defined as the composition

$$\delta^v_H = \delta_{C_H(v)} \circ \omega^v_{C_H(v)} \circ \text{Res}^H_{C_H(v)}$$

where we consider the $v$-twist

$$\omega^v_{C_H(v)} : G_K(C_H(v)) \longrightarrow G_K(C_H(v))$$

determined by the $v$-translation map induced by the multiplication by $v$ in the $O$-valued functions $\text{Fct}(C_H(v), O)$ [10, 9.2]; then, recall that we have [2, Appendice]

$$K \otimes_O G_K(H) \cong \prod_v K \otimes_O G_k(C_H(v))$$

3.8. In order to get a similar result for a $k^*$-group $\hat{H}$ with finite $k^*$-quotient $H$, we have to replace $p$-element by local element; we say that a $p$-element $v$ of $\hat{H}$ is local if either $v = 1$ or the relative trace map

$$t_{(v)}^{(v)} : (k^* \hat{H})^{(v)} \longrightarrow (k^* \hat{H})^{(v)}$$

is not surjective. Then, from the standard results on $k^*$-groups [7, Proposition 5.15], it is easily checked from isomorphism 3.7.4 that

$$K \otimes_O G_K(\hat{H}) \cong \prod_v K \otimes_O G_k(C_{\hat{H}}(v))$$

3.9. The point is that in [10, Theorem 9.3] we prove an analogous result for $G_K(\hat{P})$; explicitly, choose a set of representatives $U \subset P$ for the set of $\mathcal{F}$-isomorphism classes of the elements of $P$ in such a way that, for any $u \in P$, the subgroup $\langle u \rangle$ is fully centralized in $\mathcal{F}$ [9, Proposition 2.7].
For any \( u \in \mathcal{U} \), we have the Frobenius \( C_F(u) \)-category \( C_F(u) \) \cite[Proposition 2.16]{9} and, since a \( C_F(u) \)-selfcentralizing subgroup of \( C_F(u) \) contains \( u \), so that it is also a \( F \)-selfcentralizing subgroup of \( P \), we write \( C_F(u) \) instead of \( C_F(u) \). Then, with obvious notation, it is easily checked that \( C_F(u) \) and \( \hat{C}_F(u) \) are the respective perfect \( C_F(u) \)- and \( C_F(u) \)-localities, and we clearly have the finite folded Frobenius \( C_F(u) \)-category \( (C_F(u), \hat{C}_F(u)) \); thus, as in 2.1 above, via the structural functor \( C_F(u) \rightarrow C_F(u) \) we get a regular central \( k^* \)-extension \( C_{\hat{C}_F(u)} \) of \( C_{\hat{C}_F(u)} \) and, in \cite[9.2.5]{10}, we have defined a general decomposition map

\[
\delta_{\hat{C}_F}^u : \mathcal{G}_K(\hat{P}^e) \rightarrow \mathcal{G}_k(C_{\hat{C}_F(u)})
\]

3.9.1;
finally, in \cite[Theorem 9.3]{10} we state that the set of these general decomposition maps when \( u \) runs over \( \mathcal{U} \) determines a \( \mathcal{K} \)-module isomorphism

\[
\mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k(C_{\hat{C}_F(u)}) \cong \prod_{u \in \mathcal{U}} \mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k(C_{\hat{C}_F(u)})
\]

3.9.2.

**Theorem 3.10.** With the notation above, any local element \( v \) of \( \hat{G}(\hat{P}^e) \) has a \( \hat{G}(\hat{P}^e) \)-conjugate in the image of \( P \) and the restriction induces a \( \mathcal{K} \)-module isomorphism

\[
\mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k(C_{\hat{G}(\hat{P}^e)}(v)) \cong \prod_{u \in \mathcal{U}_v} \mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k(C_{\hat{C}_F(u)})
\]

3.10.1

where \( \mathcal{U}_v \) denotes the set of \( u \in \mathcal{U} \) such that the image is \( \hat{G}(\hat{P}^e) \)-conjugate to \( v \).

**Proof:** With notation in 1.9 and 2.25, for any \( u \in \mathcal{U} \) let us denote by \( u^* \) the \( p \)-element \( f_{\hat{C}_F}(u) \) in \( \hat{G}(\hat{P}^e) \); it is quite clear that the \( k^* \)-functor \( \hat{I}_{\hat{C}_F} : \hat{P}^e \rightarrow \hat{G}(\hat{P}^e) \) still induces a \( k^* \)-functor from \( C_{\hat{C}_F}(u) \) to \( C_{\hat{G}(\hat{P}^e)}(u^*) \); then, by restriction, we get \( \mathcal{O} \)-module homomorphisms

\[
F_u : \mathcal{G}_K(C_{\hat{G}(\hat{P}^e)}(u^*)) \rightarrow \mathcal{G}(\mathcal{K}, C_{\hat{C}_F}(u))_{\mathcal{O} \rightarrow \mathcal{O}}
\]

and

\[
f_u : \mathcal{G}_k(C_{\hat{G}(\hat{P}^e)}(u^*)) \rightarrow \mathcal{G}(\mathcal{K}, C_{\hat{C}_F}(u))
\]

3.10.2.

At this point, we claim that the following diagram is commutative

\[
\mathcal{G}_K(C_{\hat{G}(\hat{P}^e)}) \cong \mathcal{G}(\mathcal{K}, \hat{P}^e)_{\mathcal{O} \rightarrow \mathcal{O}} \cong \mathcal{G}_k(C_{\hat{G}(\hat{P}^e)})
\]

3.10.3,

\[
\mathcal{G}_k(C_{\hat{G}(\hat{P}^e)}(u^*)) \rightarrow \mathcal{G}(k, C_{\hat{C}_F}(u)) \xrightarrow{\text{cat}} \mathcal{G}_k(C_{\hat{C}_F}(u))
\]

the isomorphisms in the top line coming from Corollaries 2.23 and 2.26. First of all, since \( C_{\hat{C}_F}(u) \) is a \( k^* \)-subcategory of \( \hat{P}^e \), it is quite clear that the restriction determines the following commutative diagram

\[
\mathcal{G}_K(C_{\hat{G}(\hat{P}^e)}) \cong \mathcal{G}(\mathcal{K}, \hat{P}^e)_{\mathcal{O} \rightarrow \mathcal{O}} \cong \mathcal{G}_k(C_{\hat{G}(\hat{P}^e)})
\]

3.10.4.
Moreover, it follows from [10, 4.6] that we also have restriction maps

$$G_K(\hat{P}^{se}) \rightarrow G_K(C_{\hat{P}^{se}}(u))$$

and

$$\delta_{C_{\hat{P}^{se}}(u^*)}$$

then, from the definition of the left-hand restriction, it is not difficult to check that we also get a commutative diagram

$$G(K_+ \hat{P}^{se} \text{-mod}) \xrightarrow{\text{cat}_c} G_K(\hat{P}^{se})$$

$$G(K_+ \hat{P}^{se}(u) \text{-mod}) \xrightarrow{\text{cat}_c} G_K(C_{\hat{P}^{se}}(u))$$

hence, we finally get the commutative diagram

$$G_K(\hat{P}^{se}_{\mathcal{G}^{se}}(u^*)) \xrightarrow{F_{\mathcal{G}^{se}}} G(K_+ \hat{P}^{se} \text{-mod}) \xrightarrow{\text{cat}_c} G_K(\hat{P}^{se})$$

$$\delta_{C_{\hat{P}^{se}}(u^*)}$$

$$G_k(C_{\hat{P}^{se}}(u^*)) \xrightarrow{f_{\mathcal{G}^{se}}} G(K_+ \hat{P}^{se}(u) \text{-mod}) \xrightarrow{\text{cat}_c} G_K(C_{\hat{P}^{se}}(u))$$

On the other hand, as in 3.6 above, it follows from the definition of the Brauer decomposition map in [10, 3.4.2] that we have the following commutative diagram

$$G_K(\hat{P}^{se}_{\mathcal{G}^{se}}(u^*)) \xrightarrow{F_{\mathcal{G}^{se}}} G(K_+ \hat{P}^{se} \text{-mod}) \xrightarrow{\text{cat}_c} G_K(\hat{P}^{se})$$

$$\delta_{C_{\hat{P}^{se}}(u^*)}$$

$$G_k(C_{\hat{P}^{se}}(u^*)) \xrightarrow{f_{\mathcal{G}^{se}}} G(K_+ \hat{P}^{se}(u) \text{-mod}) \xrightarrow{\text{cat}_c} G_K(C_{\hat{P}^{se}}(u))$$

But, the vertical left-hand arrow in diagram 3.10.3 is the composition on the right- and on the left-hand of $$\omega_{C_{\hat{P}^{se}}(u^*)}$$ with the left-hand arrows of diagrams 3.10.7 and 3.10.8 (cf. 3.7.2); analogously, the vertical right-hand arrow in diagram 3.10.3 is the composition on the right- and on the left-hand of the following $$\mathcal{O}$$-module automorphism defined in [10, 9.2.4]

$$\Omega_{C_{\hat{P}^{se}}(u)} : G_K(C_{\hat{P}^{se}}(u)) \cong G_K(C_{\hat{P}^{se}}(u))$$

with the right-hand arrows of diagrams 3.10.7 and 3.10.8.

Consequently, in order to show the commutativity of diagram 3.10.3 it remains to prove the commutativity of the following diagram

$$G_K(C_{\hat{P}^{se}}(u^*)) \xrightarrow{f_{\mathcal{G}^{se}}} G(K_+ \hat{P}^{se}(u) \text{-mod}) \xrightarrow{\text{cat}_c} G_K(C_{\hat{P}^{se}}(u))$$

$$\omega_{C_{\hat{P}^{se}}(u^*)}$$

$$\Omega_{C_{\hat{P}^{se}}(u)}$$

$$G_K(C_{\hat{P}^{se}}(u^*)) \xrightarrow{f_{\mathcal{G}^{se}}} G(K_+ \hat{P}^{se}(u) \text{-mod}) \xrightarrow{\text{cat}_c} G_K(C_{\hat{P}^{se}}(u))$$

$$\omega_{C_{\hat{P}^{se}}(u^*)}$$

$$\Omega_{C_{\hat{P}^{se}}(u)}$$
for this purpose, recall that \( G_k(C_{\widehat{P}^{\aleph}}(u)) \) is the inverse limit of the family 
\[ \{ G_k\left((C_{\widehat{P}^{\aleph}}(u))(\hat{q})\right) \} \], and then that the \( O \)-module automorphism \( \Omega_{C_{\widehat{P}^{\aleph}}(u)}^u \) is the inverse limit of the family of \( O \)-module automorphisms 
\[ \omega^n_{(C_{\widehat{P}^{\aleph}}(u))(\hat{q})} : G_k\left((C_{\widehat{P}^{\aleph}}(u))(\hat{q})\right) \cong G_k\left((C_{\widehat{P}^{\aleph}}(u))(\hat{q})\right) \]
where \( \hat{q} : \Delta_n \to C_{\widehat{P}^{\aleph}}(u) \) runs over all the \( C_{\widehat{P}^{\aleph}}(u) \)-chains (cf. 2.1); that is to say, this family defines a natural automorphism [10, 9.2.3]
\[ \omega^n_{C_{\widehat{P}^{\aleph}}(u)} : g_k \circ \text{aut}_{C_{\widehat{P}^{\aleph}}(u)} \cong g_k \circ \text{aut}_{C_{\widehat{P}^{\aleph}}(u)} \]
and we set [10, 9.2.4]
\[ \Omega_{C_{\widehat{P}^{\aleph}}(u)}^u = \lim_{\leftarrow} (\omega^n_{C_{\widehat{P}^{\aleph}}(u)}) \]

But, for any \( C_{\widehat{P}^{\aleph}}(u) \)-chain \( \hat{q} : \Delta_n \to C_{\widehat{P}^{\aleph}}(u) \) we have the structural \( O \)-module homomorphism
\[ \ell^u_{\hat{q}} : G_k\left((C_{\widehat{P}^{\aleph}}(u))(\hat{q})\right) \to G_k\left((C_{\widehat{P}^{\aleph}}(u))(\hat{q})\right) \]
and, moreover, the \( k^\ast \)-functor from \( C_{\widehat{P}^{\aleph}}(u) \) to \( C_{\widehat{G}(\widehat{P}^{\aleph})}(u^*) \) induced by \( \ell_{\widehat{P}^{\aleph}} \) still induces a \( k^\ast \)-group homomorphism

\[ \theta^u_{\hat{q}} : (C_{\widehat{P}^{\aleph}}(u))(\hat{q}) \to C_{\widehat{G}(\widehat{P}^{\aleph})}(u^*) \]

then, it is easily checked that we have the following commutative diagram
\[ \begin{array}{ccc}
G_k\left((C_{\widehat{G}(\widehat{P}^{\aleph})}(u^*))(\hat{q})\right) & \xrightarrow{\ell_{\hat{q}}^u} & G_k\left((C_{\widehat{P}^{\aleph}}(u))(\hat{q})\right) \\
\underset{\text{Res}_{g_k}}{\downarrow} & & \downarrow \omega^n_{(C_{\widehat{P}^{\aleph}}(u))(\hat{q})}
\end{array} \]

Finally, the commutativity of diagram 3.10.10 follows from definition 3.10.13 and the obvious commutativity of the following diagram

\[ \begin{array}{ccc}
G_k\left((C_{\widehat{G}(\widehat{P}^{\aleph})}(u^*))(\hat{q})\right) & \xrightarrow{\text{Res}_{g_k}^u} & G_k\left((C_{\widehat{P}^{\aleph}}(u))(\hat{q})\right) \\
\omega^u_{(C_{\widehat{G}(\widehat{P}^{\aleph})}(u^*))(\hat{q})} & & \downarrow \omega^n_{(C_{\widehat{P}^{\aleph}}(u))(\hat{q})}
\end{array} \]

Now, the commutativity of diagram 3.10.3 for any \( u \in U \) determines the following commutative diagram (cf. 3.9.2)
\[ \begin{array}{ccc}
K \otimes_O G_k(\widehat{G}(\widehat{P}^{\aleph})) & \cong & K \otimes_O G_k(\widehat{P}^{\aleph}) \\
\downarrow & & \downarrow^i \\
\prod_{u \in U} K \otimes_O G_k\left(C_{\widehat{G}(\widehat{P}^{\aleph})}(u^*)\right) & \to & \prod_{u \in U} K \otimes_O G_k\left(C_{\widehat{P}^{\aleph}}(u)\right)
\end{array} \]

3.10.18
which forces the vertical left-hand arrow to be injective and the bottom arrow to be surjective; then, according to isomorphism 3.8.2, the injectivity of the vertical left-hand arrow implies that the set $\{u^*\}_{u \in \mathcal{U}}$ contains a representative for any conjugacy class of local elements of $\tilde{G}(P^v)$; moreover, the surjectivity of the bottom arrow implies that the restriction to the image of the vertical left-hand arrow is an isomorphism and then the announced isomorphism 3.10.1 follows easily.

**Corollary 3.11.** The $\mathcal{O}$-module homomorphism $\text{cat}_k: \mathcal{G}(k, \tilde{\mathcal{P}}^e_{sc}) \to \mathcal{G}_{k}(\mathcal{P}^e)$ is bijective.

**Proof:** Since the the vertical homomorphisms in diagram 3.6.2 above are surjective, this homomorphism is surjective. Moreover, it follows from isomorphism 3.10.1 for $v = 1$ that we have the $\mathcal{K}$-module isomorphism

$$\mathcal{K} \otimes \mathcal{O} \mathcal{G}_k(\tilde{G}(\mathcal{P}^e)) \cong \mathcal{K} \otimes \mathcal{O} \mathcal{G}_k(\tilde{G}(\mathcal{P}^e))$$

and therefore, according to the $\mathcal{O}$-module isomorphism 3.5.2, $\text{cat}_k$ is also injective.

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