COORDINATE-INDEPENDENT CRITERIA FOR HOPF BIFURCATIONS

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Dedicated to Jürgen Scheurle
on the occasion of his retirement from non-mathematical duties.

Abstract. We discuss the occurrence of Poincaré-Andronov-Hopf bifurcations in parameter dependent ordinary differential equations, with no a priori assumptions on special coordinates. The first problem is to determine critical parameter values from which such bifurcations may emanate; a solution for this problem was given by W.-M. Liu. We add a few observations from a different perspective. Then we turn to the second problem, viz., to compute the relevant coefficients which determine the nature of the Hopf bifurcation. As shown by J. Scheurle and co-authors, this can be reduced to the computation of Poincaré-Dulac normal forms (in arbitrary coordinates) and subsequent reduction, but feasibility problems quickly arise. In the present paper we present a streamlined and less computationally involved approach to the computations. The efficiency and usefulness of the method is illustrated by examples.

1. Introduction. Poincaré-Andronov-Hopf bifurcations (briefly called Hopf bifurcations in the present paper) frequently occur in parameter dependent ordinary differential equations, with numerous applications. A classical comprehensive source is Marsden and McCracken [20]. Among the monographs on differential equations and dynamical systems which discuss Hopf bifurcations we mention only a few, viz. Guckenheimer and Holmes [14], Thm. 3.4.2, Amann [1], Thm. 26.21, Aulbach [2], Satz 7.11.1 and Chicone [4], Section 8.3 (in particular Thm. 8.25).

The familiar statement of the Hopf bifurcation theorem assumes that the system is given in a standardized form which includes (i) a distinguished real bifurcation parameter as well as (ii) a convenient choice of coordinates. Given a suitable real bifurcation parameter, one coefficient in the Poincaré-Dulac normal form (which is rational in the coefficients of the Taylor expansion) determines the nature of the bifurcation (subcritical, supercritical or degenerate). The computation of this coefficient is unproblematic in dimension two (and fairly easy in higher dimensions) if the linearization at the stationary point is given in (real) Jordan canonical form.

2010 Mathematics Subject Classification. Primary: 34C20, 34C23, 37G15.
Key words and phrases. Normal form, stability, characteristic polynomial, FitzHugh-Nagumo equation, predator-prey system.

The first author acknowledges support by the DFG Research Training Group GRK 1632 “Experimental and constructive algebra”. Both authors thank an anonymous reviewer for helpful comments.

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But for systems that are not normalized, technical problems in the practical implementation arise in several ways. For multi-parameter systems, prior to determining a single parameter so that (i) is satisfied, one has to address the problem of finding the critical parameter values from which Hopf bifurcations emanate (for some choice of a curve through this point in parameter space). This part of the problem is concerned only with the linearization of the vector field, and has been resolved by W.-M. Liu [19] based on the classical Routh-Hurwitz criteria. For (polynomial) systems which model chemical reaction networks Liu’s ansatz was taken further by Errami et al. [8] who devised an algorithm to find all possible stationary points and critical parameter values for Hopf bifurcations. But Errami et al. did not proceed to determine the nature of the bifurcations. This part of the problem involves nonlinear terms in the Taylor expansion, and this is the part we mainly address in the present paper, with Poincaré-Dulac normal forms as the fundamental tool.

Normal forms, and the ensuing reductions, were discussed in coordinate-independent settings (including an algorithm for their computation) by Scheurle and Walcher [24] and Mayer, Scheurle and Walcher [21]. In principle these results may be used for the necessary computations to determine the nature of a Hopf bifurcation, but this general approach has the drawback of high computational expense.

In the present paper we introduce a more efficient approach to compute the necessary coefficients for the discussion of the nature of a Hopf bifurcation. We first give a brief review of the Hopf bifurcation theorem from the perspective of normal forms and then turn to computations in arbitrary coordinates. We start by recalling Liu’s [19] results to determine critical parameter values for Hopf bifurcations, and add a few observations. Then, in the central part (4.2–4.4), we turn to determining the relevant coefficients for a Hopf scenario without computing a full normal form. Finally we present a variant of the approach by Errami et al. [8] that is applicable to arbitrary polynomial systems. To show applicability, we discuss some examples, including a FitzHugh-Nagumo equation and a three-dimensional predator-prey system. In an appendix we briefly recall some results from [21], for the readers’ convenience, and present one coefficient for the predator-prey system in full size.

2. Notation and preliminaries. We consider a parameter dependent differential equation

\[ \dot{x} = h(x, p) \]  

defined for \((x, p)\) in some nonempty and open subset of \(\mathbb{R}^n \times \mathbb{R}^m\), and \(h\) of class \(C^4\). We assume throughout that \(x = 0\) is a stationary point; thus

\[ h(0, p) = 0 \quad \text{for all } p. \]

Our goal is to identify parameter values \(p^*\) from which Hopf bifurcations emanate, and to determine the nature of these bifurcations. In general the parameter space will have dimension greater than one, while the Hopf bifurcation theorem refers to a single real parameter. Thus we require that for suitable \(v \in \mathbb{R}^m\) the system

\[ \dot{x} = \hat{h}(x, \varepsilon) := h(x, p^* + \varepsilon v) \]  

undergoes a Hopf bifurcation at \(\varepsilon = 0\). (In an apparently more general approach, one could consider curves \(\gamma(\varepsilon) = p^* + \varepsilon v + o(\varepsilon)\) in parameter space, but only first order terms in \(\varepsilon\) turn out to be relevant.)
We recall the notion of Lie derivative: Given an autonomous equation \( \dot{x} = g(x) \) on an open \( U \subseteq \mathbb{R}^n \) and a differentiable \( \sigma : U \to \mathbb{R} \), the Lie derivative \( L_g(\sigma) \) of \( \sigma \) with respect to \( g \) is defined by
\[
L_g(\sigma)(x) = D\sigma(x)g(x).
\]
(For parameter dependent equations such as (1), we will always consider the Lie derivative with respect to \( x \).) Lie derivatives will play a central role in the following.

**Remark 1.** The Lie derivative of a linear vector field is well understood; see e.g. [21]: Given a linear map \( x \mapsto Bx \), the Lie derivative \( L_B \) acts on polynomials \( \psi : \mathbb{R}^n \to \mathbb{R} \), and sends the subspace \( S_k \) of homogeneous polynomials of degree \( k \) to itself, for every \( k \geq 1 \). If \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( B \) (in the complexification, and counted with algebraic multiplicity) then the eigenvalues of \( L_B|_{S_k} \) are
\[
\sum_{j=1}^{k} k_j \lambda_j, \text{ with nonnegative integers } k_j \text{ and } \sum k_j = k
\]
(likewise counted with multiplicity). The eigenspaces of \( L_B|_{S_1} \) are spanned by linear forms \( \mu \) which correspond to the left eigenvectors of \( B \). The eigenspaces of \( L_B|_{S_k} \) for \( k > 1 \) are spanned by products of those linear forms.

In our setting the map \( B \) will be the linearization of (2) at the stationary point 0, hence \( B = B(p) \) is parameter dependent, and in general the eigenvalues cannot be computed explicitly. But the characteristic polynomial of \( B \) can be determined, and it coincides with the characteristic polynomial of \( L_B|_{S_1} \). Starting from this, one can recursively determine polynomials which annihilate \( L_B|_{S_k} \) for \( k > 1 \); see [21], with some improvements in [16]. In turn, knowledge of an annihilating polynomial will allow to solve a system of linear equations, or to compute projections onto certain subspaces, even in the presence of parameters.

3. **The bifurcation scenario in suitable coordinates.** In this section we review some known results. We first consider bifurcations from the perspective of Poincaré-Dulac normal forms, referring to Bibikov [3] and in particular his notion of normal form on an invariant manifold (NFIM, called NFIS in [3]). Proofs of the Hopf bifurcation theorem need not explicitly rely on Poincaré-Dulac normal forms, but these provide insight into their structure and relevant parameters; see Bibikov [3]. We assume that the linearization of (2) at \( \varepsilon = 0 \) has a pair of purely imaginary eigenvalues \( \pm i\omega \neq 0 \), while the remaining eigenvalues have real part < 0. Moreover there exist coordinates such that the matrix has the form
\[
\begin{pmatrix}
0 & -\omega & 0 & \cdots & 0 \\
\omega & 0 & 0 & \cdots & 0 \\
0 & 0 & * & \cdots & * \\
0 & 0 & * & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & * & \cdots & *
\end{pmatrix}
\]
By normal form theory (more precisely, as a consequence of [3], Thm. 3.1), after the introduction of an additional variable \( x_0 = \varepsilon \) and a further (near-identity) coordinate transformation, the subspace defined by \( x_3 = \cdots = x_n = 0 \) is invariant
for the Taylor expansion up to degree three, and the Taylor approximation is in NFIM up to degree three, thus we have

\[
\begin{align*}
\dot{x}_0 &= 0 \\
\dot{x}_1 &= -\omega x_2 + x_0(\alpha x_1 - \mu x_2) + (x_1^2 + x_2^2)(\beta x_1 - \gamma x_2) + x_0^2(\rho x_1 - \sigma x_2) \\
\dot{x}_2 &= \omega x_1 + x_0(\alpha x_2 + \mu x_1) + (x_1^2 + x_2^2)(\beta x_2 + \gamma x_1) + x_0^2(\rho x_2 + \sigma x_1) \\
\dot{x}_3 &= \cdots \\
\vdots \\
\dot{x}_n &= \cdots 
\end{align*}
\]

(3)

omitting terms of higher degree. This system allows reduction: Letting \(\phi_0 := x_0\) and \(\phi_1 := x_1^2 + x_2^2\), the map \(\Phi = (\phi_0, \phi_1)^T\) sends solutions of this NFIM to solutions of the two-dimensional system

\[
\begin{align*}
\dot{y}_0 &= 0 \\
\dot{y}_1 &= 2(\alpha + \rho y_0)y_0y_1 + 2\beta y_1^2 
\end{align*}
\]

(4)

From the reduced equation one determines the nature of the bifurcation as \(x_0\) crosses 0; see e.g. Marsden and McCracken [20] for proofs. Note that only solutions with \(y_1 \geq 0\) are of interest for (3), since \(\phi_1(x) \geq 0\) for all \(x\). For easy reference we enumerate all possible cases with \(\alpha \neq 0\) and \(\beta \neq 0\). (The list is twice as long as usual, since we do not prescribe “crossing the imaginary axis from left to right” for a pair of conjugate eigenvalues.)

**Lemma 3.1.** Assume that \(\alpha \neq 0\) and \(\beta \neq 0\). Then the following scenarios occur.

(a) Given \(y_0 > 0\) such that \(\alpha\) and \(\alpha + \rho y_0\) have the same signs (which is the case for sufficiently small \(y_0\)), consider the second equation of (4).

(i) In case \(\alpha > 0\) the stationary point 0 is repelling.

- When \(\beta > 0\) there is no further stationary point on the half-line \(y_1 > 0\).
- When \(\beta < 0\) there is a unique attracting stationary point at \(y_1 = -\frac{(\alpha + \rho y_0)y_0}{\beta} > 0\), which gives rise to a unique attracting limit cycle for (3).

(ii) In case \(\alpha < 0\) the stationary point 0 is attracting.

- When \(\beta < 0\) there is no further stationary point on the half-line \(y_1 > 0\).
- When \(\beta > 0\) there is a unique repelling stationary point at \(y_1 = -\frac{(\alpha + \rho y_0)y_0}{\beta} > 0\), which gives rise to a unique repelling limit cycle for (3).

(b) Given \(y_0 < 0\) such that \(\alpha\) and \(\alpha + \rho y_0\) have the same signs (which is the case for sufficiently small \(|y_0|\)), consider the second equation of (4).

(i) In case \(\alpha > 0\) the stationary point 0 is attracting.

- When \(\beta < 0\) there is no further stationary point on the half-line \(y_1 > 0\).
- When \(\beta > 0\) there is a unique attracting stationary point at \(y_1 = -\frac{(\alpha + \rho y_0)y_0}{\beta} > 0\), which gives rise to a unique attracting limit cycle for (3).

(ii) In case \(\alpha < 0\) the stationary point 0 is repelling.

- When \(\beta > 0\) there is no further stationary point on the half-line \(y_1 > 0\).
• When $\beta < 0$ there is a unique attracting stationary point at $y_1 = -\left((\alpha + p y_0)/\beta\right) > 0$, which gives rise to a unique attracting limit cycle for (3).

The Hopf bifurcation theorem (roughly speaking) states that the higher order terms in the NFIM do not affect the local qualitative behavior. We do not state it in full generality here, but just give a basic version:

**Theorem 3.2.** Let (3) be a NFIM of system (2) up to degree 3, with $\alpha \neq 0$, $\beta \neq 0$. Then for sufficiently small $\varepsilon = x_0$, the statements in Lemma 3.1 about stationary points and limit cycles of (3) continue to hold for system (2) in some neighborhood of 0.

Thus, the qualitative study of (2) near $\varepsilon = 0$ amounts to computing the coefficients $\alpha$ and $\beta$ and checking the list in Lemma 3.1 (unless a degenerate case with $\alpha = 0$ or $\beta = 0$ occurs). However, these computations pose a nontrivial obstacle, and this is the motivation for the present work.

For systems in suitable coordinates a very detailed account of the normal form computations was given in Marsden and McCracken, [20], Section 4A (see also Section 5 for a translation of Hopf’s original work); later improvements and shortcuts can be found e.g. in Hassard and Wan [15] (in dimension two), and in [25] (in arbitrary dimension) for the case that the linearization is in (real or complex) Jordan form.

But for general systems, an exact determination of eigenvalues or eigenspaces is not possible (a fortiori so when the system is parameter dependent). In this situation a coordinate-free approach to compute Poincaré-Dulac normal forms (as presented by Scheurle and co-authors in [24] and [21]), followed by a reduction (analogous to passing from (3) to (4); see [21] and Appendix C) opens a path, in principle. An illustrative example is given in [21], Section 6. However, feasibility problems quickly arise in higher dimensions, and moreover the normal form contains more information than necessary to compute the coefficients $\alpha$, $\beta$ in (3). We will therefore show how to streamline the computations. The motivation for the shortcut is the determination of Lyapunov’s focus quantities in the two-dimensional center problem (see e.g. Artes et al. [7] Ch. 4).

4. The bifurcation scenario in unsuitable coordinates.

4.1. Critical parameter values. The first task is to identify those parameter values $p^*$ of (1) from which Hopf bifurcations may emanate. This amounts to conditions on the eigenvalues of

$$B(p) := Dh(0, p),$$

since a Hopf bifurcation emanates from $p = p^*$ only if all eigenvalues of $B(p^*)$ have real part $\leq 0$ and there is exactly one complex conjugate pair with real part zero. This problem has been solved by Liu [19], building on classical results by Routh and Hurwitz.

We first consider the roots of a general normalized polynomial

$$\chi(\tau) := \tau^n + c_1 \tau^{n-1} + \cdots + c_n; \quad n \geq 2$$

with real coefficients $c_i$.

**Remark 2.** We recall a few (classical and more recent) results on the position of the roots of (6) in the complex plane; see Gantmacher [10], Ch. V, §6, and Liu [19].
• If all roots of $\chi$ have real parts $< 0$ then $c_1 > 0, \ldots, c_n > 0$.
• (Routh and Hurwitz): All zeros of $\chi$ have real parts $< 0$ if and only if all Hurwitz determinants $\Delta_1, \ldots, \Delta_{n-1}$ are $> 0$.
• (Extension of the Routh-Hurwitz criterion; Liu [19]): All roots of $\chi$ have real part $\leq 0$ with exactly one complex conjugate pair with real part zero, if and only if
  \[ \Delta_1 > 0, \ldots, \Delta_{n-2} > 0 \text{ and } \Delta_{n-1} = 0. \]
  If the coefficients $c_i$ depend on a real parameter $\varepsilon$, then these conditions at $\varepsilon = 0$, together with $\partial \Delta_{n-1}/\partial \varepsilon|_{\varepsilon=0} \neq 0$, are necessary and sufficient for a Hopf bifurcation at $\varepsilon = 0$. (Liu requires the derivative to be $> 0$, in order to ensure “crossing from left to right”.)

Although the problem is thus solved in principle, it seems worthwhile to consider it from a different perspective.

**Lemma 4.1.** Let $B = B(p)$ as in (5), and assume that its characteristic polynomial is given by $\chi$ in (6).

(a) Then $B(p)$ is invertible and admits a pair of eigenvalues which add up to zero if and only if $c_n \neq 0$ and the characteristic polynomial of $L_{B(p)}|_{S_2}$ has constant coefficient zero.

(b) In this case, there is a factorization
 \[ \chi(\tau) = (\tau^{n-2} + a_1 \tau^{n-3} + \cdots + a_{n-2}) \cdot (\tau^2 + b). \]

All roots have real part $\leq 0$ with exactly one purely imaginary pair if and only if $b > 0$ and the first factor satisfies the Hurwitz-Routh conditions.

(c) $B(p)$ satisfies Liu’s conditions on $\Delta_1, \ldots, \Delta_{n-1}$ from Remark 2 if and only if
   \[ L_{B(p)}|_{S_1} \text{ is injective, } \dim \text{Ker} (L_{B(p)}|_{S_2}) = 1, \]
   and the conditions from part (b) hold. In this case the kernel of $L_{B(p)}|_{S_2}$ is spanned by a positive semidefinite quadratic form $\psi$ of rank two.

**Proof.** This follows from Remarks 1 and 2. In suitable coordinates (see (3)) the quadratic form in the kernel may be chosen as $x_1^2 + x_2^2$, hence it is positive semidefinite of rank two for any choice of coordinates.

**Definition 4.2.**

(i) We call $p^*$ a weakly critical parameter value of system (1) if condition (7) holds at $p = p^*$.

(ii) If in addition the characteristic polynomial of $B(p^*)$ satisfies the conditions from Lemma 4.1(b) then we call $p^*$ a critical parameter value of system (1).

We illustrate the conditions of Lemma 4.1 for small degrees.

**Example 1.**

• For degree $n = 2$, one just has $\chi(\tau) = \tau^2 + c_2$, with $c_2 > 0$.
• Consider $\chi(\tau) = \tau^3 + c_1 \tau^2 + c_2 \tau + c_3$: thus $n = 3$. A straightforward computation shows that $\chi$ satisfies the conditions from Lemma 4.1(a) if and only if $c_3 = c_1 c_2 \neq 0$ (or $\Delta_2 = 0$). This yields a factorization
 \[ \chi = (\tau + c_1)(\tau^2 + c_2) \]
 and the relevant setting for Hopf bifurcations is determined by $c_1 > 0$ and $c_2 > 0$. 

• For \( n = 4 \), hence \( \chi(\tau) = \tau^4 + c_1 \tau^3 + c_2 \tau^2 + c_3 \tau + c_4 \), the conditions from Lemma 4.1(a) hold only if

\[
c_1 > 0, c_4 > 0 \text{ and } -c_1^2 c_4 + c_1 c_2 c_3 - c_3^2 = 0 \quad (\text{or } \Delta_3 = 0)
\]

with factorization

\[
\chi = (\tau^2 + c_1 \tau + (c_2 - c_3/c_1)) \cdot (\tau^2 + c_3/c_1).
\]

The relevant conditions for Hopf bifurcations hold if and only if the coefficients of both factors are \( > 0 \).

At critical parameter values of (1), the eigenvalues of \( B(p^*) \) ensure that a transformation to NFIM (3) up to degree 3 exists for (2), given any choice of \( v \); there remains to determine \( \alpha \) and \( \beta \), which will be done in the next subsection. There is a geometric interpretation: In parameter space \( \mathbb{R}^m \), the condition from Lemma 4.1(a) defines a hypersurface, and the Hurwitz-Routh inequalities from part (b) determine an open semi-algebraic subset of this hypersurface. The condition \( \alpha \neq 0 \) in (3) means that for some \( v \) the curve \( \varepsilon \mapsto p^* + \varepsilon v \) in parameter space crosses the hypersurface transversally.

Remark 3. (a) We compare Liu’s [19] approach to computing critical parameter values with the one given in Lemma 4.1. Thus let (6) represent the characteristic polynomial of \( B = B(p) \). Liu computes the determinant \( \Theta = \Delta_{n-1} \) of the Hurwitz-Routh matrix

\[
L_n := \begin{bmatrix}
c_{n-1} & c_n & 0 & 0 & 0 & \cdots & 0 \\
c_{n-3} & c_{n-2} & c_{n-1} & c_n & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

of size \( n \times n \), where \( c_0 := 1 \) and \( c_i = 0 \) for all \( i < 0 \) and \( i \geq n + 1 \). Taking the approach from Lemma 4.1 one determines the characteristic polynomial of \( L_B|_{S_2} \) and considers its constant coefficient. In terms of eigenvalues (i.e. roots of \( \chi \)) this constant coefficient, up to sign, is (equal to the determinant of \( L_B|_{S_2} \)) given by

\[
\prod_{1 \leq k < \ell \leq n} (\lambda_k + \lambda_\ell) \cdot \prod_{k=1}^{n} 2\lambda_k,
\]

in view of Remark 1. However, here it suffices to consider the first factor \( \tilde{\Theta} \) because all eigenvalues are nonzero. (Since both factors are symmetric in the \( \lambda_k \), by the argument in Appendix B one also obtains a corresponding factorization when the constant coefficient is rewritten in terms of the \( c_i \).) One verifies that \( \Theta \) and \( \tilde{\Theta} \) (as polynomials in the \( c_i \)) are irreducible, have the same degree, and have the same set of complex zeros; hence they coincide by Hilbert’s Nullstellensatz, up to a constant factor. (Details will be given in [16].)

The approach by Liu is clearly more efficient in terms of computational expense when only critical parameter values are to be determined. However, we will need the annihilating polynomials of \( L_B \) on \( S_k \) for \( k = 2, 3, 4 \) later on. In particular, from the annihilating polynomial of \( L_B \) on \( S_2 \) we find an element in the kernel, which is the quadratic form needed in the reduction.
(b) The condition \( \alpha \neq 0 \) in (3) is equivalent to Liu’s partial derivative condition on \( \Delta_{n-1} \). Indeed, up to a nonzero factor, \( \Delta_{n-1} \) is equal to the expression in (8). We may assume that the system is in NFIM (3) with \( \varepsilon = x_0 \), and let \( \lambda_{1,2} = x_0 \alpha \pm i(\omega + x_0 \mu) + \cdots \)

to obtain
\[
\Delta_{n-1} = 2x_0 \alpha \cdot (\tilde{\Delta} + x_0 \cdot (\cdots)), \quad \frac{\partial \Delta_{n-1}}{\partial x_0}|_{x_0=0} = 2\alpha \tilde{\Delta}
\]
with \( \tilde{\Delta} \neq 0 \).

4.2. Restatement of the Hopf conditions. Given a critical parameter value \( p^* \) we introduce the abbreviation
\[
f(x) := h(x, p^*),
\]
and furthermore for \( (x_0, x)^n \in \mathbb{R} \times \mathbb{R}^n \), and \( v \in \mathbb{R}^m \) as in (2) we set
\[
F_v(x_0, x) := \begin{pmatrix} 0 \\ h(x, p^* + x_0 v) \end{pmatrix}, \quad C(p^*) := DF_v(0, 0).
\]
We will briefly write \( F(x_0, x) \) instead of \( F_v(x_0, x) \) when the context is clear.

**Proposition 1.** There is a Hopf bifurcation of system (1) emanating from a parameter \( p^* \), for system (2), with \( \alpha \neq 0 \) and \( \beta \neq 0 \) in the NFIM (3), if and only if:

(i) The conditions from Lemma 4.1(c) hold.
(ii) With \( f \) given in (9) and \( \psi \) as in Lemma 4.1(c) there is a smooth function
\[
\tilde{\psi} = \psi + \text{higher order terms in } x
\]
such that
\[
L_f(\tilde{\psi}) = 2\beta \psi^2 + \text{h.o.t.}
\]
(iii) Letting \( F = F_v \) as in (10), there is a smooth function
\[
\tilde{\theta} = \psi + \text{higher order terms in } (x_0, x)
\]
such that
\[
L_F(\tilde{\theta}) = 2\alpha x_0 \psi + \text{h.o.t.}
\]

**Proof.** We have
\[
F(0, x) = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}.
\]
Denote by \( \tilde{F}(x_0, x) \) a NFIM (3) of \( F \) up to degree three. Then part (i) is a direct consequence of Lemma 4.1. We proceed to prove (ii) and (iii). After normalization we have in particular
\[
\tilde{F}(0, x) = \begin{pmatrix} 0 \\ \tilde{f}(x) \end{pmatrix}
\]
with
\[
\tilde{f}(x) = \begin{pmatrix}
-\omega x_2 + (x_1^2 + x_2^2)(\beta x_1 - \gamma x_2) \\
\omega x_1 + (x_1^2 + x_2^2)(\beta x_2 + \gamma x_1) \\
\vdots
\end{pmatrix}
\]
and \( \psi(x) = x_1^2 + x_2^2 \) as well as
\[
L_f(\psi)(x) = 2\beta \psi(x)^2 + \cdots, \quad L_F(\psi)(x) = 2\alpha x_0 \psi(x)^2 + \cdots
\]
in view of (4). Since \( \hat{f} \) is a NFIM of \( f \) up to degree three, there is a near-identity transformation

\[
x \mapsto \Gamma(x) = x + \text{h.o.t.}, \quad \Gamma^{-1}(x) = x + \text{h.o.t.}
\]

such that

\[
\hat{f}(x) = D\Gamma(x)^{-1} f(\Gamma(x)).
\]

Defining

\[
\hat{\psi} := \psi \circ \Gamma^{-1},
\]

the behavior of the Lie derivative under transformations implies that

\[
L_f(\hat{\psi}) = L_f(\psi) \circ \Gamma^{-1}
\]

and we obtain

\[
L_f(\hat{\psi}) = (2\beta\psi^2 + \text{h.o.t.}) \circ \Gamma^{-1} = 2\beta\psi^2 + \text{h.o.t.},
\]

as asserted in (ii). The proof of condition (iii) is similar. There is a near-identity normalizing transformation

\[
\begin{pmatrix} x_0 \\ x \end{pmatrix} \mapsto \Phi\left( \begin{pmatrix} x_0 \\ x \end{pmatrix} \right) = \begin{pmatrix} x_0 \\ x + \text{h.o.t.} \end{pmatrix}
\]

such that

\[
\hat{F}(x_0, x) = D\Phi\left( \begin{pmatrix} x_0 \\ x \end{pmatrix} \right)^{-1} F(\Phi\left( \begin{pmatrix} x_0 \\ x \end{pmatrix} \right))
\]

By construction \( x_0 \circ \Phi^{-1} = x_0 \). Defining

\[
\hat{\theta} = \psi \circ \Phi^{-1},
\]

the same arguments as above show that

\[
L_F(\hat{\theta}) = L_F(\psi) \circ \Phi^{-1} = 2\alpha x_0\psi + \text{h.o.t.}
\]

4.3. Computational matters I: Basic observations. In this subsection we turn to critical parameter values and to the computation of their associated coefficients \( \alpha \) and \( \beta \). For a fixed critical parameter value \( p^* \) we consider Taylor expansions

\[
\begin{align*}
 f(x) & = B(p^*)x + f^{(2)}(x) + f^{(3)}(x) + \cdots \\
 F(x_0, x) & = C(p^*) \begin{pmatrix} x_0 \\ x \end{pmatrix} + F^{(2)}(x_0, x) + \cdots
\end{align*}
\]

with \( f(i) \), resp. \( F(i) \) homogeneous of degree \( i \). Note the block structure of

\[
C(p^*) = \begin{pmatrix} 0 & 0 \\ 0 & B(p^*) \end{pmatrix}.
\]

Proposition 2. Let \( p^* \) be a critical parameter value and let \( \psi \neq 0 \) be a positive semidefinite quadratic form in the kernel of \( L_B(p^*) \). Then \( L_B(p^*) \) is invertible on \( S_3 \) and admits a one-dimensional kernel on \( S_2 \) and on \( S_4 \). The kernel is spanned by \( \psi \) resp. by \( \psi^2 \).

- There are homogeneous \( \hat{\psi}_j \) of degree \( j \in \{3, 4\} \) and a scalar \( \beta \) such that with \( \hat{\psi} = \psi + \hat{\psi}_3 + \hat{\psi}_4 \) one has

\[
L_f(\hat{\psi}) = 2\beta\psi^2 + \text{h.o.t.}
\]
• This identity amounts to
\[ L_{B(p^* \cdot)}(\psi) = 0 \text{ in degree 2} \]
(which holds by definition),
\[ L_{B(p^* \cdot)}(\hat{\psi}_3) + L_f(\psi) = 0 \text{ in degree 3,} \tag{14} \]
and
\[ L_{B(p^* \cdot)}(\hat{\psi}_4) + L_f(\hat{\psi}_3) + L_f(\psi) = 2\beta \psi^2 \text{ in degree 4.} \tag{15} \]
• Equation (14) is a linear equation for \( \hat{\psi}_3 \) which has a unique solution. In equation (15) the right-hand side \( 2\beta \psi^2 \) is the kernel component of the kernel-image decomposition of \( L_f(\hat{\psi}_3) + L_f(\psi) \) with respect to \( L_{B(p^* \cdot)} \).

Proof. The NFIM of \( F \) is given by (3). The statements on invertibility and the dimension of the kernel follow from Remark 1 and Lemma 4.1. The remainder of the proof is a straightforward consequence of Proposition 1 and elementary computations.

Remark 4. There is no need to compute \( \hat{\psi}_4 \) in equation (15); it suffices to determine the kernel component.

Note that the computation of \( \beta \) does not depend on \( v \in \mathbb{R}^m \) in (2). One obtains \( \alpha \) in a similar manner, but in this step the choice of \( v \) is relevant.

Proposition 3. Let \( p^\ast \) be a critical parameter value such that the eigenvalue conditions from Lemma 4.1(b) hold, and let \( \psi \) be as in Proposition 2.
(a) There is a homogeneous \( \hat{\theta}_3 \) of degree 3 and a scalar \( \alpha \) such that with \( \hat{\theta} = \psi + \hat{\theta}_3 \) one has
\[ L_f(\hat{\theta}) = 2\alpha x_0 \psi + h.o.t. \]
(b) This identity amounts to
\[ L_{C(p^* \cdot)}(\psi) = 0 \text{ in degree 2} \]
(which holds by definition) and
\[ L_{C(p^* \cdot)}(\hat{\theta}_3) + L_F(\psi) = 2\alpha x_0 \psi \text{ in degree 3.} \tag{16} \]
(c) Here \( 2\alpha x_0 \psi \) is the kernel component of the kernel-image decomposition of \( L_F(\psi) \) with respect to \( L_{C(p^* \cdot)} \).

Proof. The proof is analogous to that of Proposition 2. We have an additional eigenvalue \( \lambda_0 = 0 \) here, therefore the kernel of \( L_{C(p^* \cdot)|S_3} \) is two-dimensional, being spanned by \( x_3^0 \) and \( x_0 \psi \). But the kernel component of \( L_F(\psi) \) cannot contain the monomial \( x_3^0 \). Indeed, since 0 is a stationary point of system (2) for all \( \varepsilon \), in a representation
\[ F^{(2)}(x_0, x) = \begin{pmatrix} 0 \\ f^{(2)}(x) \\ x_0 \end{pmatrix} + x_0 \begin{pmatrix} 0 \\ A_x \\ 0 \end{pmatrix} + x_0^2 \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \tag{17} \]
(with linear \( A \) and constant \( c \)) one has necessarily \( c = 0 \).

Remark 5. It is not necessary to compute \( \hat{\theta}_3 \) in equation (16); only the kernel component of \( L_{F(2)}(\psi) \) is required. The latter task can be simplified further, since from (17) (noting \( c = 0 \)) and (16) one sees that only the Lie derivative of \( \psi \) with respect to \( x_0 \begin{pmatrix} 0 \\ A_x \end{pmatrix} \) can contribute to the kernel component, and therefore it suffices to compute the kernel component \( 2\alpha \psi \) of \( L_A(\psi) \).
Remark 6. The determination of parameter values which give rise to a Hopf bifurcation starts with critical parameter values, which lie on a hypersurface in parameter space that is given by an equation $\omega(p) = 0$ in parameter space; see the paragraph following Example 1. Here $\omega$ could be the Hurwitz determinant $\Delta_{n-1}$ or the constant term of the characteristic polynomial of $L_B|_{S_2}$. Given $p^*$ with $\omega(p^*) = 0$ and $v \in \mathbb{R}^m$, we have

$$\omega(p^* + \varepsilon v) = \varepsilon (\text{grad} \omega(p^*), v) + o(\varepsilon)$$

for any (small) $\varepsilon \in \mathbb{R}$. Hence there exist $v$ such that $\alpha \neq 0$ for system (2) whenever $\text{grad} \omega(p^*) \neq 0$.

4.4. Computational matters II: The procedure. We start again with system (1), with $h(0,p) = 0$ and $B(p) = Dh(0,p)$ for all $p$.

1. Annihilating polynomials for $L_B(p)|_{S_k}$ for $k \in \{1, 2, 3, 4\}$: The case $k = 1$ amounts to finding the characteristic polynomial of $B(p)$; from this one successively determines annihilating polynomials on $S_2$, $S_3$ and $S_4$, as described in the Appendix, Section B and in [21], Section 3.

   Note: If one starts from the characteristic polynomial $\mu$ for $L_C$ (with any linear $C$) on $S_1$ then the coefficients of the corresponding annihilating polynomials for $L_C$ on any $S_k$ are polynomials in the coefficients of $\mu$.

2. Basic tasks (see [21], Prop. 2.1): We abbreviate $V = S_k$ and $T := L_B(p)|_{S_k}$, and let

$$q(\tau) = \tau^{\ell} + \sum_{i=1}^{\ell} \beta_i \tau^{\ell-i}$$

be a polynomial such that $q(T) = 0$, with the additional condition that either $\beta_\ell \neq 0$ (whenever $T$ is invertible) or $\beta_\ell = 0 \neq \beta_{\ell-1}$.

- If $\beta_\ell \neq 0$ then the solution of the equation $Tv = w$ is given by

$$w = -\frac{1}{\beta_\ell}(T^{\ell-1} + \sum_{i=1}^{\ell-1} \beta_i T^{\ell-1-i})v.$$

- If $\beta_\ell = 0$ and $\beta_{\ell-1} \neq 0$ then the kernel component of $v$ is given by

$$\frac{1}{\beta_{\ell-1}}(T^{\ell-2} + \sum_{i=1}^{\ell-2} \beta_i T^{\ell-2-i})v.$$

3. Critical parameter values: Note that all coefficients of annihilating polynomials are themselves polynomials in the parameters $p$.

- Choose an annihilating polynomial for $L_B(p)|_{S_1}$ with generically nonzero constant coefficient $\gamma_1(p)$. (If the constant coefficient $\gamma_1$ is zero for any $p$ then no Hopf bifurcation can occur.)

- Choose an annihilating polynomial for $L_B(p)|_{S_2}$ with constant coefficient $\gamma_2(p)$ and degree one coefficient $\gamma_3(p)$. The critical parameters are those zeros of $\gamma_2$ which are not zeros of $\gamma_1$ or $\gamma_3$. (If those conditions cannot be satisfied for any annihilating polynomial then no Hopf bifurcation can occur.)

4. Check conditions: Given a critical parameter value $p^*$, verify that the conditions from Lemma 4.1(c) hold, and determine a positive semidefinite quadratic form $\psi$ in the kernel of $L_B(p)|_{S_2}$. 

5. Determining $\beta$ for a given $p^*$: Solve equation (14) for $\hat{\psi}_3$, then determine the kernel component of $L_{f(3)}(\hat{\psi}_3) + L_{f(3)}(\psi)$ and divide by $2\psi^2$. (To reduce the expenditure, in the last step one may specialize $x$ to any $x \in \mathbb{R}^n$ with $\psi(x) \neq 0$.)

6. Determining $\alpha$ for a given $p^*$: With the notation of (13), first define $A$ via

$$\begin{pmatrix} 0 \\ Ax \end{pmatrix} = F^{(2)}(1, x) - \begin{pmatrix} 0 \\ f^{(2)}(x) \end{pmatrix}$$

(compare (16)). Then determine the kernel component of $L_A(\psi)$ and divide by $2\psi$. (Again one may specialize $x$ to $x$ to reduce expenditure.)

We add some comments on practical matters and the size of the problems. The ultimate goal is to determine $\alpha$ and $\beta$, and essentially this amounts to linear algebra over the rational function field $\mathbb{R}(p)$, thus the presence of parameters complicates the problem. We work with the annihilating polynomials instead of using more direct methods; this seems to be most appropriate for finding projections onto the kernel.

Thus it is necessary to first obtain annihilating polynomials of $L_{B|S_k}$ for $k \in \{2, 3, 4\}$. The procedure is outlined in Appendix B. Some simplifications and shortcuts are noted in [16], and furthermore, this work needs to be done only once (see the Example below). Starting from the characteristic polynomial, the degree of an annihilating polynomial of $L_{B|S_k}$ is given by the dimension \( n + k - 1 \) of the space $S_k$, which is tolerable for reasonably small $n$. As noted in the Appendix, one may first determine annihilating polynomials on $S_k$ for a “generic” characteristic polynomial and then substitute for the coefficients.

**Example 2.** In dimension three, consider the characteristic polynomial $\chi = \tau^3 + c_1\tau^2 + c_2\tau + c_3$ with $c_1c_2 = c_3$; thus Liu’s condition $\Delta_2 = 0$ holds. By the procedure in Appendix B we obtain annihilating polynomials $\chi_k$ (in a factorized form) of $L_{B|S_k}$ for $k = 2, 3, 4$, as follows.

\[
\begin{align*}
\chi_2 &= \tau(\tau^2 + 4c_2)(\tau + 2c_1)(\tau^2 + 2c^2_1\tau + c^2_2) \\
\chi_3 &= (\tau^2 + 9c_2)(\tau + 3c_1)(\tau + c_1)(\tau^2 + c_2) \\
&\quad \cdot (\tau^4 + 6c_1\tau^3 + (13c^2_2 + 5c_2)\tau^2 + (12c^3_1 + 18c_1c_2)\tau + 4c^4_1 + 17c^2_1c_2 + 4c^2_2) \\
\chi_4 &= \tau(\tau^2 + 16c_2)(\tau + 4c_1)(\tau^2 + 4c_1\tau + 4c^2_1 + 4c_2)(\tau + 2c_1) \\
&\quad \cdot (\tau^2 + 4c_2)(\tau^2 + 2c_1\tau + c^2_1 + c_2) \\
&\quad \cdot (\tau^4 + 8c_1\tau^3 + (22c^2_1 + 10c_2)\tau^2 + (24c^3_1 + 56c_1c_2)\tau + 9c^4_1 + 82c^2_1c_2 + 9c^2_2)
\end{align*}
\]

A problem occurs with item 3 above: The characterization of critical parameter values itself from Liu’s condition $\Delta_{n-1} = 0$ (or by the approach via $S_2$) is only the first step; subsequently one has to work with the relations determining critical parameter values. The factorization of $\Delta_{n-1}$ (as a polynomial in $p$) is helpful in this respect, because it yields irreducible components of the variety of critical parameters. In any case, relations which define critical parameter values have to be considered from here on. Symbolic computation software is capable of incorporating relations, but at some expense. Modulo this problem, and modulo positivity conditions, the remaining steps are straightforward. As for item 4 above, one may determine a quadratic form $\psi$ with $L_{B(p)}(\psi) = 0$ by an ansatz with undetermined coefficients (solving a system of linear equations over $\mathbb{R}(p)$) or by using a projection onto the kernel as in item 2 (starting with any quadratic form).
As can be seen, the procedure in its present stage is far from automatic. The examples below will illustrate the applicability of the method in nontrivial settings as well as limitations in terms of feasibility.

5. A general approach for polynomial systems. For system (1) we imposed the restriction that 0 is stationary for any value of \( p \). For general parameter dependent systems this restriction is not welcome, and one would prefer an approach which allows to determine those parameter values for which a Hopf bifurcation occurs at some stationary point. We will outline here how this can be done at least for polynomial systems. The fundamental idea in this respect is due to Errami et al. [8] who focused attention on chemical reaction networks and used methods from real algebra. Here we will present a variant for arbitrary polynomial systems which yields explicit conditions for the parameter values (and is similar to the determination of Tikhonov-Fenichel parameter values in [13]). We consider a system

\[
\dot{x} = q(x, p) = \begin{pmatrix}
q_1(x, p) \\
\vdots \\
q_n(x, p)
\end{pmatrix}
\] (18)

with polynomial right hand side (in variables and parameters), but no fixed stationary point. We are interested in parameter values such that a Hopf bifurcation occurs at some stationary point of (18), and by elimination theory we can determine necessary conditions for these parameter values. For background and terminology see Cox et al. [5] (in particular Chapter 3).

Proposition 4. For \( y \in \mathbb{R}^n \), \( p \in \mathbb{R}^m \) let \( \omega(y, p) = \Delta_{n-1}(Dq(y, p)) \) be the Hurwitz determinant of the Jacobian of \( q \).

(a) If \( y^* \) is a stationary point of \( \dot{x} = q(x, p^*) \) from which a Hopf bifurcation emanates then \((y^*, p^*)\) is a zero of the ideal

\[
I := \langle q_1(y, p), \ldots, q_n(y, p), \omega(y, p) \rangle \subseteq \mathbb{R}[y, p].
\]

(b) Moreover \( p^* \) is then a zero of the elimination ideal

\[
I \cap \mathbb{R}[p].
\]

Proof. In order for \( y^* \) to be stationary for (18) at \( p = p^* \), \( (y^*, p^*) \) must be a common zero of \( q_1, \ldots, q_n \). The condition \( \omega(y^*, p^*) = 0 \) is due to Lemma 4.1. Thus part (a) holds, and part (b) is a direct consequence.

Remark 7. • The underlying idea is that \( q_1(y, p) = \cdots = q_n(y, p) = \omega(y, p) = 0 \), for fixed \( p \), is an overdetermined system for \( y \), and its solvability will in general impose nontrivial conditions on \( p \) which are given by the elimination ideal.

• In the applications below, symbolic computation software (MAPLE and SINGULAR) was used to obtain elimination ideals. Here, increasing the number of parameters or the dimension of the phase space may create a bottleneck in general.

• One may alternatively consider an annihilating polynomial for the action of \( L_{Dq(y, p)} \) on \( S_2 \) and take \( \omega(y, p) \) as its constant coefficient; see Remark 3.

• One should keep in mind that computing the elimination ideal and investigating the polynomial conditions in the parameters \( p_1, \ldots, p_m \) is just the first step. For instance, if we consider the case \( n = 3 \) we have to ensure that
c_1 > 0, c_2 > 0 (in the notation of (6)) which imposes further conditions on
the parameters.

6. Applications. In this section we discuss some examples which, in addition to il-
illustrating the algorithm, are of interest for applications. The systems are dependent
on parameters. While the computations go through with the full set of parameters,
the output in some cases is too large to be conveniently readable; for this reason we
will then specialize parameter values.

6.1. Dimension two. We first consider (as a kind of benchmark problem) a general
two-dimensional vector field
\[ h(x) = \begin{pmatrix} 0 & -d \\ 1 & s \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + h_2(x) + h_3(x) + \cdots \]
with stationary point 0 and linearization B in Frobenius normal form. (Note that
a Frobenius normal form can be obtained by rational operations.) From the char-
acteristic polynomial
\[ \chi_B(\tau) = \tau^2 - s\tau + d \]
the conditions \( d > 0 \) and \( s = 0 \) for critical parameter values can be read off directly.
We allow general quadratic and cubic terms in the Taylor expansion, thus
\[ h_2(x) = \begin{pmatrix} c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2 \\ d_1 x_1^2 + d_2 x_1 x_2 + d_3 x_2^2 \end{pmatrix} \]
and
\[ h_3(x) = \begin{pmatrix} a_1 x_1^3 + a_2 x_1^2 x_2 + a_3 x_1 x_2^2 + a_4 x_2^3 \\ b_1 x_1^3 + b_2 x_1^2 x_2 + b_3 x_1 x_2^2 + b_4 x_2^3 \end{pmatrix} \]
where \( a_i, b_i, c_i, d_i \in \mathbb{R} \). With \( d > 0 \) and \( s = 0 \) the characteristic polynomial of \( L_B|S_2 \)
is equal to
\[ \chi_2(\tau) = (\tau^2 + 4d) \cdot \tau, \]
and the kernel of \( L_B|S_2 \) is spanned by the positive definite invariant \( \psi_2 := x_1^2 + dx_2^2 \).
Applying our algorithm, we get the reduced system 
\[ \dot{z} = \beta z^2, \]
with
\[ \beta = \frac{-2c_1 d_1 d_2 - d_1 d_2 d_3 + 3a_1 d_2 + d_2 d_3 a_3 + 2c_2 d_3 + 2c_3 d_3}{4d^2} \]
In order to illustrate the computation of the coefficient \( \alpha \) for a perturbation of
the critical parameter value \( p^* \) we keep the \( a_i, b_i, c_i, d_i \) constant and consider only
\( p^* = (d, 0) \), with \( v = (v_1, v_2) \). Thus from
\[ F_v(x_0, x) = \begin{pmatrix} 0 \\ h(x, p^* + x_0v) \end{pmatrix} \]
we obtain
\[ A = v_2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \]
The kernel component of \( L_A(\psi_2) = 2y(v_1 x_1 + dv_2 x_2) \) with respect to \( L_B \) is equal
to \( v_2 \cdot \psi_2 \). Consequently, we have \( \alpha = v_2 \).
6.2. The FitzHugh-Nagumo system. This system (see FitzHugh [9], Nagumo et al. [23], and Murray [22], Ch. 7.5) is a simplified version of the Hodgkin-Huxley system which models the firing of a neuron. From several versions we choose

\[\begin{align*}
\dot{x}_1 &= x_1 - \frac{x_1^3}{3} - x_2 + I \\
\dot{x}_2 &= cx_1 - bx_2 + a
\end{align*}\]

with parameters \(a, b, c, I\). (This is FitzHugh’s [9] original system (1),(2), with \(y\) replaced by \(-cy\) and variables and some parameters renamed). By Proposition 4 we determine necessary conditions on those parameter values for which a Hopf bifurcation emanates from some stationary point. Thus we have

\[q_1(y,p) = q_2(y,p) = 0\]

and the further condition \(0 = \text{trace}(Dq(y,p)) = 1 - y_1^2 - b = \omega(y,p)\) to impose on the parameter values. Eliminating the variables \(y_1, y_2\) in the ideal

\[J = \langle q_1, q_2, \omega \rangle\]

yields the necessary algebraic condition

\[b^5 + 3b^4 - 6b^3c + 9b^2I^2 - 18abI - 6b^2c + 9a^2 - 4b^2 + 12bc - 9c^2 = 0.\]

(The computation using standard packages in MAPLE takes less than a second on a PC.) From here on we restrict attention to the case with

\[0 < b < 1 \text{ and } b < c.\]

Thus, in contrast to most discussions (e.g. in the references cited above) we focus on a parameter region where the \(x_1\)-nullcline is the graph of a strictly decreasing function, and no slow-fast oscillations are to be expected. We rewrite the condition as

\[I = \frac{(3a \pm \sqrt{-b^3 - 3b^4 + 6b^3c + 9b^2c^2 + 9a^2 - 4b^2 + 12bc - 9c^2})}{3b}\]

With the above restriction on the parameters the unique stationary point is given by

\[y_1^* = \sqrt{1-b} \text{ and } y_2^* = \frac{1}{b} \left( a + c \cdot \sqrt{1-b} \right).\]

For both choices of sign in the expression for \(I\) we obtain the quadratic invariant \(\psi = -2bxy + cx^2 + y^2\) which is positive definite (and thus yields a pair of purely imaginary eigenvalues) whenever \(b^2 < c\). In this case we obtain the reduced system

\[\dot{z} = -\frac{(b^2 - 2b + c)}{(4(b^2 - c)^2)} z^2.\]

Note that the sign of \(\beta = -(b^2 - 2b + c)/(4(b^2 - c)^2)\) may be positive or negative. For \(c \geq 1\) it is certainly negative and we have an attracting limit cycle.

6.3. A predator-prey system. The three dimensional system

\[\begin{align*}
\dot{x}_1 &= rx_1(1 - x_1) - \gamma x_1 x_3 \\
\dot{x}_2 &= -\eta x_2 + \delta x_1 x_3 \\
\dot{x}_3 &= bx_2 - dx_3 + \eta x_2 - \delta x_1 x_3
\end{align*}\]

was introduced in [17]. It models a predator-prey population with one prey \(x_1\) and two stages \(x_2, x_3\) for the predator. All parameters are \(> 0\), and only nonnegative solutions are of interest.
We first use Proposition 4 to determine parameter values for which a Hopf bifurcation may occur at some stationary point. The elimination ideal (computed with the help of SINGULAR [6]) is generated by one element

\[
h = h_1 \cdot h_2 \cdot h_3^2 \cdot h_4 \cdot h_5 \cdot h_6 \cdot h_7^2 \cdot h_8 \cdot h_9^2
\]

with factors

\[
\begin{align*}
h_1 &= \delta + d + \eta \\
h_2 &= d + \eta \\
h_3 &= d \\
h_4 &= -\delta^2 b^3 - \delta^2 b^2 d + \delta b^2 d^2 - \delta^2 b^2 \eta - \delta^2 b d \eta + 3 \delta b^2 d \eta \\
    &\quad + 3 \delta b d^2 \eta + \delta b^2 \eta^2 + 3 \delta b d \eta^2 + 2 \delta d^2 \eta^2 - \delta b d \eta r + b d^2 \eta r + b d \eta^2 r + 2 d^2 \eta^2 r \\
h_5 &= -\delta b + d \eta + \delta r + dr + \eta r + r^2 \\
h_6 &= -d + r \\
h_7 &= -\eta + r \\
h_8 &= r \\
h_9 &= \eta.
\end{align*}
\]

(The computation of \(h\) as well as the factorization take about a second.) The equation \(h = 0\) defines a hypersurface on parameter space and for each \(i\), \(h_i = 0\) defines an irreducible component of this hypersurface. Hopf bifurcations correspond to curves in parameter space which transversally cross some component. The nonnegativity requirement for parameters implies that \(h_1, h_2, h_3, h_8\) and \(h_9\) are irrelevant for potential Hopf bifurcations, and one verifies that \(h_6\) and \(h_7\) correspond to the stationary point 0 (at which no Hopf bifurcation can take place). This leaves \(h_4\) and \(h_5\) for possible Hopf bifurcations at the interior stationary point

\[
\begin{align*}
y_1 &= \frac{d \eta}{b \delta} \\
y_2 &= \frac{d r (b \delta - d \eta)}{b \gamma} \\
y_3 &= \frac{r (b \delta - d \eta)}{\gamma (b \delta)}
\end{align*}
\]

which exists whenever \(b \delta - d \eta > 0\). One verifies that \(h_4 = 0\) indeed corresponds to Liu’s condition \(\Delta_2 = 0\) at the interior stationary point. Moreover we may set \(\gamma = 1\) with no loss of generality (since one may scale system (19) via \(x_2 \mapsto \gamma x_2, x_3 \mapsto \gamma x_3\)), and we will do so in the following.

At the interior stationary point, the characteristic polynomial of the Jacobian is given by \(\chi_\beta(\tau) = \tau^4 + c_1 \tau^2 + c_2 \tau + c_3\) with

\[
\begin{align*}
c_1 &= \frac{b d \delta + b d \eta + d b \eta + d \eta r}{b \delta} \\
c_2 &= \frac{d \eta r (b d - b \delta + b \eta + 2 d \eta)}{b^2 \delta} \\
c_3 &= \frac{d \eta r (b \delta - d \eta)}{b \delta}.
\end{align*}
\]
The condition $h_4 = 0$ is linear in the parameter $r$ and equivalent to

$$r = \frac{\delta}{dy(bd - b\delta + b\eta + 2d\eta)}$$

as long as the denominator does not vanish; this allows to substitute the right hand side for $r$.

For further investigation and application of the results from Section 4, we perform a translation, replacing $x_i$ by $y_i + x_i$ such that $(0, 0, 0)$ is a stationary point of the transformed system. (We keep old names here for new variables.) The following computations are carried out with the help of the software system MAPLE. A quadratic invariant $\psi$ of the linearization (see Lemma 4.1) is given by

$$\psi = a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_1 x_3 + a_4 x_2^2 + a_5 x_2 x_3 + a_6 x_3^2$$

with

$$a_1 = \delta^2 (b^2 \delta^2 - b^4 d^2 \delta + 2b^4 d^2 \delta^2 - 4b^4 d\delta \eta + b^4 \delta^2 \eta - b^4 \delta \eta^2)$$

$$- b^3 d^2 \delta + b^3 d^3 \eta + b^3 d^2 \delta^2 - 8b^3 d^2 \delta \eta + 3b^3 d^2 \delta \eta^2 + 2b^3 d^2 \delta \eta^2 - 5b^3 d \delta \eta^2$$

$$+ b^3 d \eta^3 + b^2 d^4 \eta - 4b^2 d^3 \delta \eta + 6b^2 d^3 \delta \eta^2 + b^2 d^2 \delta^2 \eta - 7b^2 d^2 \delta^2 \eta^2 + 4b^2 d^2 \eta^3$$

$$+ 3bd^4 \eta^2 - 3bd^3 \delta \eta^2 + 5bd^3 \delta \eta^3 + 2d^4 \eta^3),$$

$$a_2 = 2bd\delta(b^3 \delta - b^2 d^2 + b^2 d \delta - 3b^2 d \eta)$$

$$+ b^2 d \eta - b^2 \eta^2 - 3bd^2 \eta + bd \delta \eta - 3bd \eta^2 - 2d^2 \eta^2),$$

$$a_3 = -2bd \delta (b^3 \delta - b^2 d^2 + b^2 d \delta - 3b^2 d \eta + b^2 d \eta)$$

$$- b^2 \eta^2 - 3bd^2 \eta + bd \delta \eta - 3bd \eta^2 - 2d^2 \eta^2),$$

$$a_4 = (b + \eta) b^2 \eta^2 d^2,$$

$$a_5 = 2b^2 \eta^3 d^2,$$

$$a_6 = (b^2 \delta - bd^2 + bd \delta - 3bd \eta - 2d^2 \eta) \eta^2 d^2.$$

(Again, the computation takes less than a second.) Using this invariant, we compute a truncated reduced system (corresponding to (11)) of the form $\dot{z} = \beta \cdot z^2 + \cdots$ where $\beta$ depends on the parameters $b, d, \delta, \eta$. The general expression for the coefficient $\beta$ is given in the Appendix; one sees that the output is manageable if a bit unwieldy (which is caused by the nature of the problem, not by some arbitrary choice).

The missing conditions from Liu [19] for a Hopf bifurcation are $c_1 > 0, c_2 > 0$. We only pick one special case here, setting $b = d = r$, and

$$\delta = \frac{d^2 + 5d\eta + 6\eta^2 + \sqrt{d^4 + 18d^3 \eta + 69d^2 \eta^2 + 84d \eta^3 + 36\eta^4}}{4d + 4\eta}$$

from $h_4 = 0$. Choosing $\eta = d = 1$, which guarantees existence of the interior stationary point due to $bd = \frac{1}{2} (3 + \sqrt{13}) > 1 = d\eta$, and also guarantees $c_1 > 0$ and $c_2 > 0$ (as well as positive semidefiniteness of $\psi$), one finds the coefficient $\beta \approx -0.03 < 0$. Altogether, there is a Hopf bifurcation with an attracting limit cycle. (By Remark 6 we have $\alpha \neq 0$ for some direction $v$.)

7. **Summary and concluding remarks.** In parameter dependent systems with a fixed stationary point, Liu’s [19] approach to finding critical parameter values for Hopf bifurcations was extended by Errami et al. [8]. We augmented this by an
alternative approach, considering the characteristic polynomial of $L_{B(p)}|_{S_2}$. This approach by itself would be of restricted interest, but we need the annihilating polynomial on $S_2$ anyway on our path to finding annihilating polynomials of $L_{B(p)}$ on $S_3$ and $S_4$. The new contribution in the present paper is the computation of the coefficients relevant for the nature of the Hopf bifurcation, yielding a substantial shortcut of the previously known (but rarely feasible) method in [21]. Moreover, taking up ideas from Errami et al. [8] for reaction networks and using methods introduced in [13], we developed an approach for polynomial systems to find all parameter values for which a Hopf bifurcation occurs at some stationary point.

The principal purpose of the present paper was to introduce the approach and show by example that it yields nontrivial results for relevant applications. The paper does not (and was not intended to) address complexity estimates, and makes use only of standard software packages (MAPLE and SINGULAR) in symbolic computations. In this respect, improvements (some of which we hinted at) are clearly possible, and may be taken up in further work. Moreover, we worked with inequalities in an ad-hoc manner, rather than employing real algebra in a systematic fashion. This, and in particular a synthesis of the methods in Errami et al. [8] and the methods devised here, is a topic for future research.

Appendix A. A coefficient. We did not write down the parameter dependent coefficient $\beta = \beta(b, d, \delta, \eta)$ in the predator-prey system (19) above, since it takes up some space. It has the form $\beta = \frac{\mu}{\delta^7}$ with numerator

\[
\begin{align*}
\mu &= 3\delta^2(b + \eta) \cdot (b^{10}d^4 + b^{10}d^5 + b^{10}d^6\eta - 3b^9d^3\delta^3 + 6b^9d^2\delta^4 - 13b^9d^2\delta^3\eta \\
&- 3b^9d^6\delta^4 + 12b^9d^5\delta^3\eta - 13b^9d^5\delta^3\eta^2 - 3b^9d^5\delta^2\eta + 3b^9d^5\delta^2\eta^3 \\
&+ 2b^9d^6\delta^3 - 6b^8d^6\delta^3 + 3b^8d^6\delta^2\eta + 6b^8d^6\delta^4 - 21b^8d^6\delta^3\eta + 4b^8d^6\delta^2\eta^2 \\
&- 2b^8d^7\delta^5 + 19b^8d^7\delta^4\eta - 45b^8d^7\delta^3\eta^2 + 4b^8d^7\delta^2\eta^3 - b^8d^7\delta^2\eta^4 \\
&+ 196d^6\delta^7\eta - 216d^6\delta^6\eta^2 + 36d^6\delta^5\eta^3 - 24d^5\delta^7\eta^2 + 6d^5\delta^6\eta^3 \\
&- 6d^5\delta^6\eta^4 + 2d^5\delta^5\eta^5 + 16d^7d^6\delta\eta - 42d^7d^6\delta^2\eta + 97d^7d^6\delta^3\eta^2 \\
&+ 42d^7d^6\delta^4\eta - 256d^7d^6\delta^5\eta^2 + 2096d^7d^6\delta^6\eta^3 - 2256d^7d^6\delta^7\eta^4 + 2216d^7d^6\delta^8\eta^5 \\
&- 410d^6^7\delta^8\eta^3 + 2096d^6^7\delta^9\eta^4 + 6b^7d^6\delta^5\eta - 84d^7d^6\delta^7\eta^2 + 2216d^7d^6\delta^8\eta^3 \\
&+ 256d^7d^6\delta^9\eta^4 + 976d^7d^6\delta^10\eta^5 + 6d^7d^6\delta^11\eta^6 - 2256d^7d^6\delta^12\eta^7 + 42d^7d^6\delta^13\eta^8 \\
&- 42d^7d^6\delta^14\eta - 16d^7d^6\delta^15\eta - 8b^6d^7\delta\eta + 36b^6d^7\delta^2\eta - 58b^6d^7\delta^3\eta - 64b^6d^7\delta^4\eta \\
&+ 63b^6d^7\delta^5\eta - 185b^6d^7\delta^6\eta + 56b^6d^7\delta^7\eta - 6805b^6d^7\delta^8\eta^2 + 1273b^6d^7\delta^9\eta^3 \\
&- 321b^6d^7\delta^10\eta^4 + 246b^6d^7\delta^11\eta^5 + 52d^6d^7\delta^12\eta^6 - 20196b^6d^7\delta^13\eta^7 + 18725b^6d^7\delta^14\eta^8 \\
&- 321b^6d^7\delta^15\eta^9 + 46b^6d^7\delta^16\eta^10 - 166b^6d^7\delta^17\eta^11 + 1105b^6d^7\delta^18\eta^12 - 20196b^6d^7\delta^19\eta^13 \\
&+ 1273b^6d^7\delta^20\eta^14 - 185b^6d^7\delta^21\eta^15 + 126b^6d^7\delta^22\eta^16 - 166b^6d^7\delta^23\eta^17 + 524b^6d^7\delta^24\eta^18 \\
&- 6805b^6d^7\delta^25\eta^19 + 36b^6d^7\delta^26\eta^20 - 58b^6d^7\delta^27\eta^21 + 4b^6d^7\delta^28\eta^22 - 24b^6d^7\delta^29\eta^23 \\
&+ 56b^6d^7\delta^30\eta^24 - 64b^6d^7\delta^31\eta^25 + 36b^6d^7\delta^32\eta^26 - 8b^6d^7\delta^33\eta^27 + 308b^6d^7\delta^34\eta^28 \\
& - 52b^6d^7\delta^35\eta^29 - 42b^6d^7\delta^36\eta^30 + 21106b^6d^7\delta^37\eta^31 - 14025b^6d^7\delta^38\eta^32 + 264b^6d^7\delta^39\eta^33 \\
&- 26585b^6d^7\delta^40\eta^34 + 50966b^6d^7\delta^41\eta^35 - 1914b^6d^7\delta^42\eta^36 - 685b^6d^7\delta^43\eta^37 + 12706b^6d^7\delta^44\eta^38 \\
& - 477b^6d^7\delta^45\eta^39 + 50966b^6d^7\delta^46\eta^40 - 14025b^6d^7\delta^47\eta^41 + 46b^6d^7\delta^48\eta^42 - 182b^6d^7\delta^49\eta^43 \\
&+ 12706b^6d^7\delta^50\eta^44 - 26585b^6d^7\delta^51\eta^45 + 21106b^6d^7\delta^52\eta^46 - 528b^6d^7\delta^53\eta^47 + 4b^6d^7\delta^54\eta^48 \\
& - 68b^6d^7\delta^55\eta^49 - 264b^6d^7\delta^56\eta^50 - 42b^6d^7\delta^57\eta^51 + 308b^6d^7\delta^58\eta^52 - 84b^6d^7\delta^59 \n\end{align*}
\]
The computation with MAPLE of the cofactor $\beta$ and the element $\tilde{\psi}_3$ (see Proposition 2) take about 10 seconds.

Appendix B. Finding an annihilating polynomial. A procedure to determine an annihilating polynomial for $L_B$ on $S_r$ without explicit knowledge of the eigenvalues of $B$ was outlined in [21], together with a procedure to find an annihilating polynomial for $\text{ad} B_\epsilon$ on $P_r$, the space of homogeneous vector valued polynomials of degree $r$. We provide a variant here, for the reader’s convenience.

Let $B$ have eigenvalues $\lambda_1, \ldots, \lambda_n$, and hence characteristic polynomial

$$
(t - \lambda_1) \cdots (t - \lambda_n) = t^n - \sigma_1 t^{n-1} + \sigma_2 t^{n-2} + \ldots + (-1)^n \sigma_n,
$$

with

$$
\sigma_1 = \lambda_1 + \ldots + \lambda_n, \quad \ldots, \quad \sigma_n = \lambda_1 \cdots \lambda_n
$$

the elementary symmetric polynomials in the $\lambda_j$. (This corresponds to (6), with $\sigma_j = (-1)^j c_j$.) Note that the $\sigma_j$ are known, but not necessarily the $\lambda_k$. The procedure is as follows.
• By Remark 1, in terms of the eigenvalues of $B$, the eigenvalues of $L_B$ on $S_r$ are given (with multiplicities) by all the
$$d_1\lambda_1 + \ldots + d_n\lambda_n,$$
with nonnegative integers $d_1, \ldots, d_n$ such that $\sum d_j = r$.

• This leads to the following step-by-step construction of annihilating polynomials, starting with the annihilating polynomial (20) on $S_1$. If the eigenvalues of $L_B$ on $S_r$ are given by $\alpha_1, \ldots, \alpha_t$ then the eigenvalues of $L_B$ on $S_{r+1}$ are given by all the $\lambda_j + \alpha_k$ with $1 \leq j \leq n$ and $1 \leq k \leq t$. Thus, if the polynomial
$$p(\tau) = \sum_{j=0}^{t^r} \gamma_j \tau^j$$
annihilates $L_B$ on $S_r$ then the polynomial
$$\hat{p}(\tau) = \prod_{i=1}^{n} \left( \sum_{j=0}^{t^r} \gamma_j (\tau - \lambda_i)^j \right)$$
annihilates $L_B$ on $S_{r+1}$.

• The polynomial $\hat{p}(\tau)$ can be determined without explicit knowledge of the roots $\lambda_j$. To see this, let $X_1, \ldots, X_n$ be indeterminates, and write
$$p(\tau) = \sum_{j=0}^{t^r} \gamma_j \tau^j.$$
Then
$$\tilde{p}(\tau) := \prod_{i=1}^{n} \left( \sum_{j=0}^{t^r} \gamma_j (\tau - X_i)^j \right)$$
is obviously symmetric in $X_1, \ldots, X_n$, hence can be expressed in the form
$$\tilde{p}(\tau, X_1, \ldots, X_n) = \sum_{j=0}^{n \cdot t^r} \rho_j(\Sigma_1, \ldots, \Sigma_n) \tau^j$$
with polynomials $\rho_i$ in the elementary symmetric polynomials
$$\Sigma_1 = X_1 + \ldots + X_n, \quad \ldots, \quad \Sigma_n = X_1 \cdots X_n.$$
Therefore
$$\hat{p}(\tau) = \tilde{p}(\tau, \lambda_1, \ldots, \lambda_n)$$
can be expressed as a polynomial in $\tau$ with coefficients that depend only on the $\gamma_j$ and the $\sigma_k$.

• Expressing a symmetric polynomial through elementary symmetric polynomials is a standard task; see the proof given in Lang [18], Ch. IV, Thm. 6.1, or the approach via invariant theory e.g. in Gatermann [11].

• In general the coefficients of the characteristic polynomial of $B(p)$ will be polynomials in the parameters. But it is possible (and advisable) for any given dimension $n$ to start with a “generic” characteristic polynomial (6), determine the annihilating polynomials on $S_r$, $2 \leq r \leq 4$ for this generic polynomial (which needs to be done only once for every $n$), and then substitute. The procedure is facilitated by the fact that the annihilating polynomials on $S_r$ admit a factorization for $r \geq 2$ and the procedure of passing to higher degrees
may be performed for each factor. More details about this will be given in the upcoming thesis [16].

Appendix C. The approach via normal forms. We briefly recall some results from [21], in order to facilitate comparison with the direct approach taken in the present paper.

• Consider a vector field
  \[ f(x) = Bx + f^{(2)}(x) + f^{(3)}(x) + \ldots \]
  in \( \mathbb{R}^n \), with \( B \) linear and each \( f^{(j)} \) homogeneous of degree \( j \), with \( f \) sufficiently differentiable. Moreover let \( B = B_s + B_n \) be the decomposition into semisimple and nilpotent part. (We suppress any parameter dependence in the notation.) One says that \( f \) is in normal form up to degree \( r \) if the Lie bracket \( [B_s x, f^{(j)}(x)] \) vanishes for all \( j \leq r \). (The Lie bracket is defined, as usual, by \[ [p, q](x) = Dq(x)p(x) - Dp(x)q(x). \])

• Any \( C^{r+1} \) vector field \( f \) may be put in normal form \( f^* \) up to degree \( r \) by an invertible analytic diffeomorphism, via a degree-by-degree procedure. Thus, if a vector field is already normalized up to degree \( k - 1 \) then for further normalization one has to solve the equation
  \[ [B, \Gamma^{(k)}] = f^{(k)} - f^*^{(k)} \]
  in the space \( \mathcal{P}_k \) of homogeneous vector polynomials of degree \( k \). Here \( \Gamma^{(k)} \) denotes the lowest order nontrivial term in a near-identity normalizing transformation \( \Gamma(x) = x + \cdots \), and \( f^*^{(k)} \) is the degree \( k \) term in a normalized vector field.

• As shown in [21], the equations for parameter dependent systems may be solved with the help of annihilating polynomials, similar to subsection 4.4. But roughly speaking, due to \( \dim \mathcal{P}_k = n \cdot \dim S_k \), the dimensions of the vector spaces (over the field \( \mathbb{R}(p) \)) can be reduced by taking the direct approach in section 4. (The most significant improvement can be seen from \( \dim S_4 = \dim \mathcal{P}_3 \cdot \frac{n+3}{4n} \).) In addition, no computation of order three terms following the first normalizing transformation (which is carried out via Lie series in [21]) is necessary here.

• The final step in [21] is to reduce the truncated vector field
  \[ f^*(x) = Bx + f^*^{(2)}(x) + \cdots + f^*^{(r)}(x) \]
  by invariants of \( B_s \); such a symmetry reduction is possible due to \( [B_s x, f^*(x)] = 0 \). This corresponds to section 3 when passing from (3) to (4). As can be seen from this, four of the six coefficients in the normal form (which nevertheless have to be computed when proceeding as in [21]) are irrelevant for the reduction. This observation indicates one more advantage of the direct approach.

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Received August 2017; revised January 2018.

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