Mirror Symmetry in
Three Dimensional Gauge Theories

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We discuss non-trivial fixed points of the renormalization group with dual descriptions in $N = 4$ gauge theories in three dimensions. This new duality acts as mirror symmetry, exchanging the Higgs and Coulomb branches of the theories. Quantum effects on the Coulomb branch arise classically on the Higgs branch of the dual theory. We present examples of dual theories whose Higgs/Coulomb branch are the ALE spaces and whose Coulomb/Higgs branches are the moduli space of instantons of the corresponding $ADE$ gauge group. In particular, we show that in three dimensions small $E_8$ instantons in string theory are described by a local quantum field theory.
1. Introduction

Motivated by [1,2] three dimensional gauge theories with \( N = 4 \) supersymmetry were recently studied from the viewpoint of string theory [3]. One of the main new results is a new connection between \( ADE \) groups and \( ADE \) singularities. A \( U(1) \) gauge theory with \( n + 1 \) electrons has a global symmetry \( SU(n+1) = A_n \). Its moduli space has two branches: a Higgs branch which is the same as the moduli space of \( A_n \) instantons and a Coulomb branch which has an \( A_n \) singularity. Similarly, an \( SU(2) \) gauge theory with \( n \) quarks has a global symmetry \( SO(2n) = D_n \). Its moduli space also has two branches: a Higgs branch which is the same as the moduli space of \( D_n \) instantons and a Coulomb branch with a \( D_n \) singularity.

These theories were further studied in [4] where many more details of their Coulomb branches were analyzed from a field theoretic viewpoint. Also, new relations between these theories and their four dimensional counterparts [5, 6] were uncovered. These theories are thus a natural setup for studying various physical phenomena and relations between mathematics and physics. In this note we continue the investigation of these theories and find a surprising duality, further strengthening the link between \( ADE \) groups and \( ADE \) singularities.

\( N = 4 \) theories in three dimensions have a global \( SO(4) \cong SU(2)_L \times SU(2)_R \), with \( SU(2)_R \) the \( R \) symmetry seen in six dimensional \( N = 1 \) theories and \( SU(2)_L \) (denoted \( SU(2)_{R2} \) in [3] and \( SU(2)_N \) in [4]) associated with rotations in the three directions reduced in going to three dimensions. The Higgs branch is a hyper-Kahler manifold with Kahler forms transforming in the adjoint of \( SU(2)_R \) and invariant under \( SU(2)_L \). The Coulomb branch is also a hyper-Kahler manifold with Kahler forms transforming in the adjoint of \( SU(2)_L \) and invariant under \( SU(2)_R \).

As in [7,8], we can easily prove a few non-renormalization theorems by promoting some of the coupling constants to background superfields:

1. When the gauge coupling constant is promoted to a superfield, the scalars in that superfield transform as \((1, 3, 1)\) of \( SU(2)_L \times SU(2)_R \). Therefore, they can only appear in the Coulomb branch. Hence, the Higgs branch is not renormalized by quantum effects [8].

2. Mass terms are background vector fields [8,4] which transform as \((3, 1)\) of \( SU(2)_L \times SU(2)_R \). As with the gauge couplings, they affect the metric on the Coulomb branch but not the Higgs branch.
3. Fayet-Iliopoulos $D$ terms for $U(1)$ factors of the gauge group transform as $(1, 3)$ of $SU(2)_L \times SU(2)_R$. Therefore, they can affect only the metric on the Higgs branch but not the Coulomb branch.

Many of these $N = 4$ theories flow in the IR to new non-trivial fixed points of the renormalization group. In particular, the theories at the singularities mentioned above are at such interacting fixed points \[3\]. As in four dimensional gauge theories, there could be dual descriptions of the fixed points \[3\]. The duality which we will present exchanges:

1. $SU(2)_L$ and $SU(2)_R$,
2. The Coulomb branch and the Higgs branch,
3. Mass terms and Fayet-Iliopoulos $D$ terms.

In many respects this duality is reminiscent of the duality of \[9\] in four dimensional gauge theories and to the presentation of mirror symmetry in two dimensions given in \[10\]. In all these cases the duality applies only to the long distance theory (the theories at short distance are free) and the matter fields and gauge fields mix in a non-trivial way. Furthermore, as in \[10\], terms in the superpotential like mass terms are exchanged with Fayet-Iliopoulos $D$ terms.

Because the Coulomb branch gets quantum corrections while the Higgs branch does not, the duality exchange (2.) means that quantum effects in one theory arise classically in the dual and visa-versa.

Another interesting aspect of the duality is that the fixed point can have global symmetries which are manifest in one description but arise quantum-mechanically in the dual. To see that, note that the mass terms can be regarded as expectation values of background gauge fields transforming in the adjoint of any global flavor symmetry; $N = 4$ supersymmetry requires them to be in the Cartan subalgebra. Fayet-Iliopoulos (FI) terms, on the other hand, are not associated with any visible global symmetry. Therefore, the exchange (3.) means that visible global symmetries are exchanged with hidden ones.

Indeed, we can exhibit part of the hidden global symmetries associated with FI terms. For every $U(1)$ factor in the gauge group\[9\], there is a conserved current, $* F$, which could be coupled to a background gauge field $a_\mu$ as $a \wedge F$. Supersymmetrizing this coupling, i.e. turning $a$ into a vector superfield, and giving nonzero expectation value to the scalar in the

\[1\] As found in \[11\], the corresponding conservation law for the non-Abelian part of the gauge group is violated by instantons.
a multiplet, we find the FI term. Therefore, for every gauge $U(1)$ factor there is a global $U(1)$ symmetry which is not manifest in the Lagrangian. The FI term can be thought of as background vector superfield coupled to the corresponding conserved current. If there are $r$ $U(1)$ factors, the above $U(1)^r$ global symmetry is the Cartan part of, a generally non-Abelian, hidden global symmetry of rank $r$.

Concretely, we consider gauge theories constructed by Kronheimer [12], which are based on the Dynkin diagrams of the $ADE$ gauge groups. The Higgs branch gives the ALE spaces [12]. We argue that the Coulomb branch in three dimensions gives the moduli space of a small $ADE$ instanton. This gives a new connection between the ALE spaces and the corresponding $ADE$ gauge group. At the point where the Higgs and Coulomb branches intersect, we argue that there is an interacting fixed point. For the $A_{n-1}$ and $D_n$ cases, we argue that the theories are dual to the $U(1)$ and $SU(2)$ theories with $n$ hypermultiplets discussed in [34]. The theories based on $E_{6,7,8}$ lead to new interesting fixed points which we identify. The duality shows that ALE gravitational instantons and the moduli space of $ADE$ instantons are naturally dual to each other: the dual theories have one as the Higgs branch and the other has the Coulomb branch.

It is worth stressing that the $E_{6,7,8}$ fixed points give a local field theory realization of the small $E_{6,7,8}$ instantons, with the corresponding $E_{6,7,8}$ global symmetry arising as a hidden symmetry. In $d = 4, 5, 6$ dimensions it is not known if small $E_n$ instantons correspond to local field theories. Going to three dimensions, we exhibit local field theories which flow to these interacting theories.

The relation between our duality and mirror symmetry of Calabi-Yau spaces becomes obvious in the context of string compactification. Consider a pair of mirror Calabi-Yau spaces $\mathcal{M}$ and $\mathcal{M}'$. The map between them exchanges vector multiplets and hypermultiplets in four dimensions. Compactifying further to three dimensions on $\mathcal{M} \times S^1$ and $\mathcal{M}' \times S^1$, the two theories become indistinguishable. The precise relation between them is that the two radii are inverse of each other. Then, the duality discussed in this note becomes identical to mirror symmetry on the Calabi-Yau space.

In the next section we introduce and discuss the gauge groups associated with Kronheimer’s “hyper-Kahler quotient” of the ALE spaces [12]. In sect. 3 we discuss the duality between the $A_{n-1}$ case of these theories and $U(1)$ with $n$ electrons. In sect. 4 we discuss the duality between the $D_n$ case and $SU(2)$ with $n$ quarks. In sect. 5 we discuss the $E_{6,7,8}$ cases.
2. The $K_G$ gauge theories

We consider $N = 4$ gauge theories in three dimensions which are naturally associated with a group $G$ which, for the moment, we take to be simply laced. The gauge group is $K_G \equiv (\prod_{i=0}^{r} U(n_i))/U(1)$, where $i$ runs over the nodes of the extended Dynkin diagram of the group $G$ of rank $r$ and $n_i$ is the Dynkin index of the node, with $n_0 = 1$ corresponding to the extended node. The overall $U(1)$ which is not gauged is the sum of the $U(1)$ generators over all $U(n_i)$. For the matter content, we take $\oplus_{ij} a_{ij}(n_i,n_j)$, where $a_{ij}$ is one if there is a link in the extended Dynkin diagram connecting nodes $i$ and $j$ and zero otherwise. Note from inspection of the Dynkin diagrams that every $U(n_i)$ factor has $2n_i$ fundamental flavors.

These gauge theories were introduced by Kronheimer [12] in his “hyper-Kahler quotient” construction of the ALE spaces. In physical terms: these theories have a Higgs branch on which the gauge group is completely broken, with one hypermultiplet left massless. As discussed in [12], this Higgs branch is the ALE space $C^2/\Gamma_G$, with $\Gamma_G$ the discrete $SU(2)$ subgroup corresponding to gauge group $G$.

The orbifold singularity in $C^2/\Gamma_G$ can be resolved by introducing Fayet-Iliopoulos $D$ terms in the $K_G$ gauge theory, corresponding directly to the blowing up modes of [12]. The $r$ FI parameters $\vec{\zeta}_a, a = 1 \ldots r$, are naturally considered as being in the Cartan subalgebra of $G$: turning on some $\vec{\zeta}_a$ blows up $C^2/\Gamma_G$ to $C^2/\Gamma_H$, with $H$ related to $G$ by adjoint Higgsing. In fact, putting the IIA theory on this space, $G$ is promoted to a gauge symmetry and the $\vec{\zeta}_a$ really are the flat-directions of a $G$ adjoint matter field. In the $K_G$ gauge theory, the added $\vec{\zeta}_a$ lead to matter expectation values which Higgs $K_G$ down to the corresponding $K_H$.

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2 For example, $|\Gamma_G| = \sum_i n_i^2$, the nodes $i$ of the extended Dynkin diagram correspond to the irreducible representations $R_i$ of $\Gamma_G$, with $|R_i| = n_i$, and $R_F \times R_i = \sum_j a_{ij} R_j$, with $R_F$ the fundamental two dimensional representation and $a_{ij}$ as above.

3 Each $\vec{\zeta}_a$ is a real triple, which transforms in the adjoint of $SU(2)_R$. 


The blown up ALE spaces can be written as surfaces in $C^3$ given by $\mathcal{B}_G(\zeta, X)$:

$$
\begin{align*}
G = SU(r + 1) & \quad X_1^2 + X_2^2 + X_3^{r+1} = B_G(\zeta, X) \\
G = SO(2r) & \quad X_1^2 + X_2^2 X_3 + X_3^{r-1} = B_G(\zeta, X) \\
G = E_6 & \quad X_1^2 + X_2^3 + X_3^4 = B_G(\zeta, X) \\
G = E_7 & \quad X_1^2 + X_2^3 + X_3 X_4^3 = B_G(\zeta, X) \\
G = E_8 & \quad X_1^2 + X_2^5 + X_3^5 = B_G(\zeta, X).
\end{align*}
$$

The blowing up polynomials are $B_G = \sum_{a=1}^r P_{ca(G)}(\zeta) R_{C_2(G)-ca(G)}(X)$, where the subscripts are the degrees of the polynomials under the scaling where the $\zeta_a$ have degree one and \ref{1} has degree $C_2(G)$, the dual Coxeter number of $G$. $R_{C_2(G)-ca(G)}(X)$ are the non-trivial chiral ring deformations of the LHS of \ref{1} and $ca(G)$ are the degrees of the Casimirs of $G$. The Casimir dependence on $\vec{\zeta}_i$ again reflects that they are in the CSA of $G$.

The above description of the Higgs branch of the $K_G$ theories holds in any dimension upon reduction from $N = 1$ in six dimensions. As discussed in the introduction, the Higgs branch is uncorrected by quantum effects. The hyper-Kahler structure completely fixes the metrics on the spaces \ref{1}.

In three dimensions, the above gauge theories have a moduli space of vacua consisting of two branches: the one dimensional Higgs branch described above and a rank($K_G$) = $C_2(G) - 1$, where $C_2(G)$ is the dual Coxeter number of $G$, dimensional Coulomb branch. These two branches intersect at a point, where we claim there is a non-trivial renormalization group fixed point. At the fixed point, there is an accidental $G$ global symmetry which is visible only at long distance.

Unlike the Higgs branch, the Coulomb branch is corrected by quantum effects. Classically, the Coulomb branch is $(R^3 \times S^1)^{C_2(G)-1}$. We argue that quantum effects correct it to be the moduli space of a $G$ instanton, with the point at the origin corresponding to an instanton of zero size. Note that the Coulomb branch has the right dimension, $C_2(G) - 1$, to be the moduli space of a $G$ instanton (eliminating $R^4$ translations). Further, along the Coulomb branch the accidental global $G$ symmetry is broken exactly corresponding to the breaking of $G$ by a $G$ instanton. Turning on non-zero $\vec{\zeta}_i$ lifts components of the Coulomb branch.

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4 When we write such expressions, two components of $\vec{\zeta}$ have been combined into a complex parameter $\zeta$ and the third has been set to zero.

5 We count dimensions in quaternionic units, corresponding to four real scalars each.
branch corresponding to the adjoint breaking $G \rightarrow H$: the remaining components of the Coulomb branch correspond to $K_H$.

In addition to the possibility of turning on the FI parameters, we can consider turning on masses for the hypermultiplets appearing above. However, a linear combination of the masses can be eliminated for every $U(1)$ factor in $K_G$ by shifting the origin of the Coulomb branch. For $G = A_r$ there are $r + 1$ hypermultiplets, associated with the $r + 1$ links of the extended Dynkin diagrams, and $r$ different $U(1)$ factors, so there is a single mass parameter $\vec{m}$ which can not be eliminated. For $\vec{m} \neq 0$ the Higgs branch is lifted. For $G = D_r$ or $G = E_r$ there are $r$ hypermultiplets and $r$ different $U(1)$ factors, so all masses can be set to zero in these cases.

To summarize, the dimension $d_H$ of the Higgs branch, the dimension $d_C$ of the Coulomb branch, the number $\# \vec{m}$ of possible mass terms, and the number $\# \vec{\zeta}$ of possible FI terms are given by:

| groups   | $d_H$ | $d_C$ | $\# \vec{m}$ | $\# \vec{\zeta}$ |
|----------|-------|-------|---------------|------------------|
| $K_{A_{n-1}}$ | 1     | $n - 1$ | 1             | $n - 1$          |
| $K_{D_n}$    | 1     | $2n - 3$ | 0             | $n$              |
| $K_{E_6}$    | 1     | 11    | 0             | 6                |
| $K_{E_7}$    | 1     | 17    | 0             | 7                |
| $K_{E_8}$    | 1     | 29    | 0             | 8                |

The corresponding quantities for $U(1)$ with $n$ electrons or $SU(2)$ with $n$ quarks are

| groups   | $d_H$ | $d_C$ | $\# \vec{m}$ | $\# \vec{\zeta}$ |
|----------|-------|-------|---------------|------------------|
| $U(1)$   | $n - 1$ | 1     | $n - 1$       | 1                |
| $SU(2)$  | $2n - 3$ | 1     | $n$           | 0                |

We now discuss the situation for various $G$ in more detail.

3. $G = SU(n)$

3.1. $G = SU(2) – a self-dual example$

The gauge theory $K_{SU(2)}$ constructed as in the previous section is $U(1)$ with $n = 2$ electrons. The moduli space is a one dimensional Higgs branch and a one dimensional

$^6$ $\vec{m}$ is a real triple, transforming in the adjoint of $SU(2)_L$. 
Coulomb branch, intersecting at the origin. It was argued in \cite{3} that the theory at the origin is a non-trivial fixed point. There are two types of coupling constants which can be added: a mass difference $\vec{m} = \vec{m}_1 - \vec{m}_2$ and a Fayet-Iliopoulos $D$ term $\vec{\zeta}$. For $\vec{m} \neq 0$ and $\vec{\zeta} = 0$, the $SU(2)_F$ flavor symmetry which rotates the two electrons is explicitly broken, the Higgs branch is lifted, and there is no non-trivial fixed point. For $\vec{\zeta} \neq 0$ and $\vec{m} = 0$, the $SU(2)_F$ flavor symmetry is spontaneously broken, the Coulomb branch is lifted, and there is no non-trivial fixed point. For both $\vec{m}$ and $\vec{\zeta} \neq 0$, the moduli space is zero dimensional.

The structure of the Higgs branch is exact at the classical level and is given by (2.1) for $G = SU(2)$; the gauge invariant fields $X_i = E_f \tilde{E}^g(\sigma_i)^f_g$, have (2.1) as a classical constraint with $B_{SU(2)} = \zeta^2$. The metric on the Higgs branch is determined by the hyper-Kahler structure to be that of Eguchi-Hanson:

$$ds^2 = g^2(\vec{x})(dt + \omega \cdot d\vec{x})^2 + g^{-2}(\vec{x})d\vec{x} \cdot d\vec{x},$$  \hspace{1cm} (3.1)

where

$$g^{-2}(\vec{x}) = \sum_{i=1}^{2} \frac{1}{|\vec{x} - \vec{\zeta}_i|}, \hspace{1cm} \tilde{\nabla}(g^{-2}) = \tilde{\nabla} \times \omega,$$  \hspace{1cm} (3.2)

and $\vec{\zeta}_1 - \vec{\zeta}_2 = \vec{\zeta}$ and $i$ labels the two $U(1)$ gauge groups before we mod out by their sum. See \cite{14} for a review with references on these spaces.

Classically the Coulomb branch is $R^3 \times S^1$ but, as discussed in \cite{3,4}, quantum corrections (essentially given at one loop) lead to a hyper-Kahler space with the Taub-NUT metric:

$$ds^2 = g^2(\vec{x})(dt + \omega \cdot d\vec{x})^2 + g^{-2}(\vec{x})d\vec{x} \cdot d\vec{x},$$  \hspace{1cm} (3.3)

with

$$g^{-2}(\vec{x}) = g^{-2}_{cl} + \sum_{i=1}^{2} \frac{1}{|\vec{x} - \vec{m}_i|}, \hspace{1cm} \tilde{\nabla}(g^{-2}) = \tilde{\nabla} \times \omega,$$  \hspace{1cm} (3.4)

where $g^{-2}_{cl}$ is the classical $U(1)$ gauge coupling. In the limit $g^{-2}_{cl} \rightarrow 0$, (3.3) agrees with (3.1).

$U(1)$ with $n = 2$ electrons has a self duality which exchanges the classically exact Higgs branch with the purely quantum part of the Coulomb branch and the FI term $\vec{\zeta}$ with the mass $\vec{m}$. The fact that it is necessary to take $g_{cl} \rightarrow \infty$ in (3.3) is standard for duality in theories which are not-finite: duality is a property of the long-distance physics \cite{9}. Further, the Higgs (Coulomb) branch is the moduli space of an $SU(2)$ instanton: the
hyper-Kahler $SU(2)_R \times SU(2)_L$ action corresponds to $SU(2)$ rotations of the instanton and the distance from the origin corresponds to the instanton size.

For $\vec{m} = \vec{\zeta} = 0$, $SU(2)_F \times SU(2)_R$ is spontaneously broken along the Higgs branch to a diagonal subgroup. It is unbroken along the Coulomb branch. The duality says that there is a new $\tilde{SU}(2)_F$ symmetry at long distance. $\tilde{SU}(2)_F \times SU(2)_L$ is spontaneously broken to a diagonal subgroup along the Coulomb branch but unbroken along the Higgs branch.

3.2. $G = SU(n)$, $n > 2$

The gauge theory is $K_{SU(n)} = U(1)^n/U(1) \cong U(1)^{n-1}$, with $n$ hypermultiplets $Q_i$ charged under $U(1)_i$ and $U(1)_{i+1}$ for $i = 0 \ldots n-1$, with $U(1)_n \equiv U(1)_0$. There is a one dimensional Higgs branch, which intersects an $n-1$ dimensional Coulomb branch at the origin. There are $n-1$ independent FI terms: $\vec{\zeta}_i$, $i = 0 \ldots n-1$, with $\sum_i \vec{\zeta}_i = 0$, and one independent mass term, $\vec{m}$. For $\vec{m} \neq 0$, the Higgs branch is lifted. For $\vec{\zeta}_i \neq 0$ some of the Coulomb branch is lifted.

We argue that these theories have non-trivial fixed points where they are dual to the $U(1)$ theories with $n$ electrons discussed in [3]. This dual theory has a one dimensional Coulomb branch which intersects a $n-1$ dimensional Higgs branch at the origin. The Higgs branch of the $U(1)^{n-1}$ theory is mapped to the Coulomb branch of the $U(1)$ dual and visa-versa.

Following the discussion in the introduction, the $K_{SU(n)} \cong U(1)^{n-1}$ theory has a hidden global $U(1)^{n-1}$ symmetry which couples to the FI terms $\vec{\zeta}_i$. The duality shows that this global symmetry is promoted to a global $SU(n)$ symmetry at the fixed point, which is the manifest flavor symmetry of the dual theory. Under the duality, the $n-1$ FI terms of the $K_{SU(n)}$ theory are mapped to the $n-1$ independent masses of the $U(1)$ dual, which we write as $\vec{m}'_i$, $i = 0 \ldots n-1$, with $\sum_i \vec{m}'_i = 0$. The single mass $\vec{m}$ of the $K_{SU(n)}$ theory is mapped to the FI parameter $\vec{\zeta}'$ of the $U(1)$ theory.

Turning on $\vec{\zeta}_i$ FI terms leads to hypermultiplet expectation values which Higgs $K_{SU(n)} \to K_H$, with $H$ related to $SU(n)$ by adjoint breaking. In the dual $U(1)$ theory, this corresponds to added mass terms, which leads to the dual of $K_H$. For example, taking $\vec{\zeta}_i \neq 0$ with $n-1$ equal values breaks $U(1)^n/U(1) \to U(1)^{n-1}/U(1)$. This is mapped to giving an electron a mass in the $U(1)$ dual, leaving $n-1$ light flavors. The duality is thus preserved in the low energy theory. Similarly, for $\vec{m} \neq 0$ in the $K_{SU(n)}$ theory the
Higgs branch is lifted, which is mapped to the lifting of the Coulomb branch for $ζ' \neq 0$ in the $U(1)$ dual.

The Higgs branch of $U(1)$ with $n$ electrons was interpreted in [15], as the moduli space of $SU(n)$ instantons (modulo $R^4$ translations). This is consistent with the fact that the global $SU(n)$ symmetry is broken to $SU(n-2) \times U(1)$ on the Higgs branch with light hypermultiplets transforming as $(n-2)_1 \oplus 1_0$. In the dual theory, the hidden global $SU(n)$ theory is so broken along the Coulomb branch.

As further evidence for the duality, we now compare the Higgs and the Coulomb metrics in the dual theories, showing that they are interchanged.

The Higgs branch of the $K_{SU(n)}$ theory is given exactly by the classical result in (2.1), with metric given by

$$ds^2 = g^2(\vec{x})(dt + \vec{\omega} \cdot d\vec{x})^2 + g(\vec{x})^{-2}d\vec{x} \cdot d\vec{x},$$ (3.5)

with

$$g^{-2}(\vec{x}) = \sum_{i=0}^{n-1} \frac{1}{|\vec{x} - \vec{\zeta}_i|}, \quad \vec{\nabla}(g^{-2}) = \vec{\nabla} \times \vec{\omega}. \tag{3.6}$$

Classically the Coulomb branch of the $K_{SU(n)}$ theory is $(R^3 \times S^1)^{n-1}$. Quantum mechanically, the metric is corrected to be the multi-dimensional version of Taub-NUT:

$$ds^2 = (g^{-2})_{ii}d\vec{x}_i \cdot d\vec{x}_j + (g^2)_{ij}dq_i dq_j,$$ (3.7)

with $dq_i = dt_i + \vec{W}_{ij} \cdot d\vec{x}_j$ and

$$(g^{-2})_{ii} = g^{-2}_{i,cl} + \sum_{j \neq i} \frac{a_{ij}}{|\vec{x}_i - \vec{x}_j - \vec{m}_i|},$$

$$(g^{-2})_{ij} = -\frac{a_{ij}}{|\vec{x}_i - \vec{x}_j - \vec{m}_i|}, \quad i \neq j, \tag{3.8}$$

with $a_{ij}$ the Dynkin diagram adjacency matrix and $\vec{\nabla}_i (g^{-2})_{ij} = \vec{\nabla}_i \times \vec{W}_{ij}$. This metric coincides with the multi-monopole metrics found, for example, in [16]. In the present context, this metric is obtained by essentially the same one-loop calculation which entered in the $n = 2$ case (3.4): each $U(1)$, has precisely two electrons carrying its charge and the electrons get an additional effective mass contribution from the coupling to another gauge field. As mentioned before, by shifting the $\vec{x}_i$, the masses $\vec{m}_i$ can be eliminated up to a single linear combination $\vec{m}$.  

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The Higgs branch of $U(1)$ with $n$ electrons is given exactly by the classical result: the metric the same as (3.7) with $g^{-2}_{i,cl} \to 0$ and $\vec{m}_i \to \vec{\zeta}_i$, where the FI term $\vec{\zeta}_i$ is related to the $\vec{\zeta}_i'$ so that $\vec{m} \to \vec{\zeta}'$.

The Coulomb branch of $U(1)$ with $n$ electrons is classically $\mathbb{R}^3 \times S^1$ but, owing to quantum corrections, is given by

$$ds^2 = g^2(\vec{x})(dt + \vec{\omega} \cdot d\vec{x})^2 + g(\vec{x})^{-2}d\vec{x} \cdot d\vec{x}, \quad (3.9)$$

with

$$g^{-2}(\vec{x}) = g_{cl}'^{-2} + \sum_{i=0}^{n-1} \frac{1}{|\vec{x} - \vec{m}'_i|} \quad \vec{\nabla}(g^{-2}) = \vec{\nabla} \times \vec{\omega}. \quad (3.10)$$

This Coulomb branch metric coincides with the Higgs branch metric (3.8) upon taking $g_{cl}'^{-2} = 0$ and replacing $\vec{\zeta}_i \to \vec{m}'_i$.

4. $G = SO(2n)$

The gauge group is $K_{SO(2n)} = U(1)^4 \times U(2)^{n-3}/U(1) \cong U(1)^3 \times U(2)^{n-3}$. Corresponding to the links in the extended $SO(2n)$ Dynkin diagram, there are two doublets under the first $U(2)$ and two under the last $U(2)$, each charged under a $U(1)$, and matter fields transforming as $(2, 2)$ under each adjacent $U(2) \times U(2)$. We argue that these theories have non-trivial fixed points, and that they are dual to the $SU(2)$ with $n$ quark flavors discussed in [3]. Under the duality, the one dimensional Higgs branch of the $K_{SO(2n)}$ theory is mapped to the one dimensional Coulomb branch of the $SU(2)$ dual and the $2n - 3$ dimensional Coulomb branch of the $K_{SO(2n)}$ theory is mapped to the $2n - 3$ dimensional Higgs branch of $SU(2)$ with $n$ quarks. At the fixed point, the $K_{SO(2n)}$ theory has a hidden global $SO(2n)$ symmetry which is mapped to the manifest $SO(2n)$ of the $SU(2)$ dual. The $n$ independent FI deformations of the $K_{SO(2n)}$ theory are mapped to the $n$ independent masses for the quark flavors of the $SU(2)$ dual. The fact that the $SU(2)$ theory has no FI term deformation corresponds to the fact, noted in sect. 2, that the $K_{SO(2n)}$ theory has no hypermultiplet mass term deformation.

Turning on FI terms in the $K_{SO(2n)}$ theory leads to matter field expectation values which break $K_{SO(2n)}$ to $K_H$, with $H$ related to $G$ by adjoint Higgsing, spontaneously breaking the hidden global $SO(2n)$ symmetry to a hidden global $H$ symmetry. In the dual, the corresponding mass terms explicitly break the manifest $SO(2n)$ symmetry to $H$. As a particular example, turning on equal FI parameters breaks $K_{SO(2n)}$ to $K_{SU(n)}$, with a
remaining $SU(n)$ global symmetry. In the $SU(2)$ dual, the common mass for the $n$ quarks leads to a vacuum with $SU(2)$ Higgsed to $U(1)$ with $n$ massless electrons. The low energy theories are again dual.

The Higgs branch of the $K_{SO(2n)}$ theories is given by the classical result (2.1). The Coulomb branch of $SU(2)$ with $n$ quarks receives quantum corrections, becoming the same $D_n$ ALE space [3] in the $g_{cl} \to \infty$ limit. Similarly, the quantum Coulomb branch of the $K_{SO(2n)}$ theory is expected to coincide with the Higgs branch of $SU(2)$ with $n$ quarks.

The Coulomb branch of the $K_{SO(2n)}$ theory or the Higgs branch of the $SU(2)$ theory gives the $SO(2n)$ instanton moduli. This is compatible with the fact that the global symmetry in the $SU(2)$ theory is broken as $SO(2n) \to SO(2n - 4) \times SU(2)$ on this space with massless hypermultiplets $\frac{1}{2}(2n - 4, 2) \oplus (1, 1)$ [3]. In the $K_{SO(2n)}$ theory the hidden global $SO(2n)$ symmetry must be so broken along the Coulomb branch. The Cartan part of that can be seen as in the introduction.

5. $G = E_{6,7,8}$

For these cases, we argue that the $K_G$ gauge theory again leads to a non-trivial fixed point, with a Coulomb branch which gives the moduli space for $G$ instantons. In these cases, it is not known what the dual theories are whose Higgs branch is the Coulomb branch of the $K_G$ theory. Again, we stress that the $K_{E_{6,7,8}}$ gives a local field theory description of the interesting phenomenon associated with small $E_{6,7,8}$ instantons in string theory.

For $E_6$, we want the Coulomb branch to reflect the global symmetry breaking $E_6 \to SU(6)$, with the massless Coulomb moduli transforming like $\frac{1}{2}(20) \oplus 1$. The dimension agrees with the rank, $C_2(E_6) - 1$, of $K_{E_6}$. Again, there should be a hidden global $E_6$ symmetry at the origin which is so broken along the Coulomb branch. As explained above, its Cartan subgroup is dual to the $U(1)^6$ gauge symmetry.

The situation for $E_7$ and $E_8$ is similar. For $E_7$, we want the Coulomb branch to reflect the global symmetry breaking $E_7 \to SO(12)$, with the massless Coulomb moduli transforming like $\frac{1}{2}(32) \oplus 1$. For $E_8$, we want the Coulomb branch to reflect the global symmetry breaking $E_8 \to E_7$, with the massless Coulomb moduli transforming like $\frac{1}{2}(56) \oplus 1$.

Acknowledgments

We would like to thank G. Moore, S. Shenker, and E. Witten for discussions. KI thanks the Aspen Center for Physics, where this work was partially completed. This work was supported in part by DOE grant #DE-FG02-96ER40559, NSF grant PHY-9513835, and the W.M. Keck Foundation.
Extended Dynkin diagrams and indices.
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