Strong Approximation of Stochastic Allen-Cahn Equation with White Noise

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Abstract. We establish an optimal strong convergence rate of a fully discrete numerical scheme for second order parabolic stochastic partial differential equations with monotone drifts, including the stochastic Allen-Cahn equation, driven by an additive space-time white noise. Our first step is to transform the original stochastic equation into an equivalent random equation whose solution possesses more regularity than the original one. Then we use the backward Euler in time and spectral Galerkin in space to fully discretize this random equation. By the monotonicity, in combination with the factorization method and stochastic calculus in martingale-type 2 Banach spaces, we derive a uniform $L^\infty$-estimation and a Hölder-type regularity for both stochastic and random equations. Finally, the strong convergence rate of the fully discrete scheme is obtained. Several numerical experiments are carried out to verify the theoretical result.

1. Introduction

Strong approximations for stochastic partial differential equations (SPDEs) with Lipschitz coefficients have been well studied, see, e.g., [ACLW16, CHL17a, CLS13, BJK16] and references therein. For certain types of SPDEs driven by colored noises with non-Lipschitz coefficients, [CHL17b, Dör12, FLZ17] obtained strong convergence rates for numerical approximations by using the monotonicity or exponential integrability and Sobolev embedding to control the $L^\infty$-bound.

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of the exact and numerical solutions. It is an open problem to derive the strong convergence of fully discrete schemes for second order parabolic SPDEs with non-Lipschitz coefficients driven by space-time white noise. In particular, to the best of our knowledge, there exist few works on strong approximations of the stochastic Allen–Cahn equation with space-time white noise. This is the main motivation of the present study.

Our main concern in this paper is to derive the strong convergence rate of a fully discrete scheme for the following parabolic SPDE driven by an additive Brown sheet $W$ in a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$:

$$
\frac{\partial X(t,\xi)}{\partial t} = \frac{\partial^2 X(t,\xi)}{\partial \xi^2} + f(X(t,\xi)) + \frac{\partial^2 W(t,\xi)}{\partial t \partial \xi}, \quad (t,\xi) \in (0, T) \times (0, 1),
$$

(1.1)

with initial value and Dirichlet boundary condition:

$$
X(t,0) = X(t,1) = 0, \quad X(0,\xi) = X_0(\xi), \quad (t,\xi) \in [0, T] \times (0, 1).
$$

(1.2)

Here $f = f_1 + f_2$, where $f_1$ is a Lipschitz function and $f_2$ is a one-sided Lipschitz function with polynomial growth. We remark that if $f(x) = x - x^3$, then Eq. (1.1) is called the stochastic Allen–Cahn equation, which has been extensively studied mathematically and numerically in literatures; see, e.g., [FLP14, Fun16, KKL07, KLL15, LQ17, Pro] and references therein.

For a slightly different version of the stochastic Allen–Cahn equation with space-time white noise, [YZ17, Theorem 3.1] got a convergence rate in probability sense for spectral Galerkin approximations when $X_0 \in \dot{H}^{3/4}(0,1)$. The first result on strong approximations of stochastic partial differential equations with monotone drifts driven by space-time white noise is given in [BJ17, Corollary 6.17] for stochastic Ginzburg-Landau equations. There the authors obtained the strong convergence rate for a nonlinearity-truncated, Euler-type scheme under the assumption that $X_0 \in \dot{H}^{3/4}(0,1)$. Their method was then generalized in [BGJK17] to a nonlinearity-truncated, fully discrete scheme for the stochastic Allen–Cahn equation with space-time white noise. The authors proved that, provided $X_0 \in \dot{H}^{1}(0,1)$, there holds that

$$
\sup_{0 \leq m \leq M} \|X(t_m) - X_N^m\|_{L^2(\Omega \times (0,1))} = O(N^{-\beta} + \tau^{\beta/2}),
$$

(1.3)

for any $\beta \in (0, 1/2)$, where $X_N^m$ denotes the numerical solution and $h, \tau$ are the size of finite element and temporal step, respectively.

Our strong approximation of Eq. (1.1) consists of two steps. The first step is to transform the original stochastic equation (1.1) into an
equivalent random equation (2.5) with random coefficient whose solution possesses more regularity than the original one. The spatial spectral Galerkin approximation of Eq. (1.1) is exactly the sum of the spectral Galerkin approximation of the aforementioned random equation (2.5) and the spectral approximate Ornstein–Uhlenbeck process; see (3.3). Then we use the backward Euler scheme (3.5) to discretize the random spectral Galerkin approximate equation (3.3). To derive the strong convergence rate of this fully discrete approximation, we make fully use of the monotonicity of the random equation, in combination with the factorization method and stochastic calculus in martingale-type 2 Banach spaces, to derive a priori \( L^\infty \)-estimation and a Hölder-type regularity for the solutions of Eq. (1.1) and (2.5) (see Lemmas 2.1 and 2.2).

Our main result shows that the proposed fully discrete scheme possesses the strong convergence rate

\[
\sup_{0 \leq m \leq M} \|X(t_m) - X^N_N\|_{L^2(\Omega \times (0,1))} = O(N^{-\gamma} + \tau^{1/4})
\]

for any \( \gamma \in (0, 1/2) \), provided that \( X_0 \in \dot{H}^{1/2+\epsilon} \) for \( \epsilon > 0 \) (see (3.10) in Theorem 3.1), where \( h, \tau \) are the size of finite element and temporal step, respectively. For less regular initial datum, we also derive a sub-optimal strong convergence rate for the fully discrete scheme with an logarithm factor (see (3.8)–(3.9)). Taking into account of the optimal Sobolev regularity in Lemma 2.1, our convergence rate is also sharp. It should be noted that the proposed scheme is implicit which avoids the truncation of the nonlinearity, and its temporal strong convergence rate is 1/4 which removes an infinitesimal factor of (1.3) as in [BGJK17].

The rest of this article is organized as follows. Some preliminaries and a priori \( L^\infty \)-estimation and a Hölder-type regularity for the solutions of Eq. (1.1) and (2.5) are given in the next section, followed by the strong convergence analysis for the proposed fully discrete scheme in Section 3. Several numerical experiments are given to support the theoretical claims in the last section.

2. Preliminaries

In this section, we give some commonly used notations and the optimal Sobolev and Hölder-type regularity for the solution of Eq. (1.1). They are used in the next section to deduce the sharp strong convergence rate of fully discrete scheme.

2.1. Notations. Let \( p \geq 1, \ r \in [1, \infty], \ q \in [2, \infty], \ \theta \geq 0 \) and \( \delta \in [0, 1] \). Here and after we denote \( \mathbb{L}^q_\xi := \mathbb{L}^q(0,1) \) and \( H := \mathbb{L}^2_\xi \).
Similarly, $L^p_\omega$ and $L^\infty_\omega$ denote the related Lebesgue spaces on $\Omega$ and $[0, T]$, respectively. For convenience, sometimes we use the temporal, sample path and spatial mixed norm $\| \cdot \|_{L^p \cap L^\infty}$ in different orders, such as

$$\|X\|_{L^p \cap L^\infty} := \left( \int_\Omega \left( \int_0^T \left( \int_0^1 |X(t, \xi, \omega)|^q \, d\xi \right)^\frac{p}{q} \, dt \right)^\frac{q}{p} \, d\mathbb{P}(\omega) \right)^{\frac{1}{p}}$$

for $X \in L^p_\omega L^\infty_\omega$, with the usual modification for $r = \infty$ or $q = \infty$.

Denote by $A$ the Dirichlet Laplacian on either $H$ or $L^2_\xi$. Then $A$ is the infinitesimal generator of an analytic $C_0$-semigroup $S(\cdot)$ on $H$ or $L^2_\xi$, and thus one can define the fractional powers $(-A)\theta$ of the operator $-A$. Let $\theta \geq 0$ and $L^{\theta, q}_\xi$ (denote $\mathbb{H}^\theta := L^{\theta, 2}_\xi$) be the domain of $(-A)^{\theta/2}$ equipped with the norm $\| \cdot \|_{\theta, q}$.

For a Banach space $(B, \| \cdot \|_B)$ and a bounded closed subset $\mathcal{C} \subset \mathbb{R}^d$, we use $C(\mathcal{C}; B)$ to denote the Banach space consisting of $B$-valued continuous functions $f$ such that $\|f\|_{C(\mathcal{C}; B)} := \sup_{\xi \in \mathcal{C}} \|f(\xi)\|_B < \infty$, and $C^\delta(\mathcal{C}; B)$ with $\delta \in (0, 1]$ to denote the $B$-valued function $f$ such that

$$\|f\|_{C^\delta(\mathcal{C}; B)} := \sup_{\xi \in \mathcal{C}} \|f(\xi)\|_B + \sup_{\xi, \eta \in \mathcal{C}, \xi \neq \eta} \frac{\|f(\xi) - f(\eta)\|_B}{|\xi - \eta|^\delta} < \infty.$$ 

In the following, when $B = \mathbb{R}$ and $\mathcal{C} = [0, 1]$ we simply denote $C^\delta([0, 1]; \mathbb{R}) = C^\delta$. Similarly, we use $L^p(\Omega; C([0, T]; B))$ to denote the Banach space consisting of $B$-valued a.s. continuous stochastic processes $X = \{X(t) : t \in [0, T]\}$ such that

$$\|X\|_{L^p(\Omega; C([0, T]; B))} := \left( \mathbb{E} \left[ \sup_{t \in [0, T]} \|X(t)\|^p_B \right] \right)^{\frac{1}{p}} < \infty,$$

and $L^p(\Omega; C^\delta([0, T]; B))$ with $\delta \in (0, 1]$ to denote $B$-valued stochastic processes $X = \{X(t) : t \in [0, T]\}$ such that

$$\|X\|_{L^p(\Omega; C^\delta([0, T]; B))} := \left( \mathbb{E} \left[ \sup_{t \in [0, T]} \|X(t)\|^p_B \right] \right)^{\frac{1}{p}} + \left( \mathbb{E} \left[ \sup_{t,s \in [0, T], t \neq s} \frac{\|X(t) - X(r)\|_B^p}{|t - t_1|^\delta} \right] \right)^{\frac{1}{p}} < \infty.$$

The main condition on the nonlinear function $f$ is the following assumption.
ASSUMPTION 2.1. There exist constants $L_f, G_f > 0, K \in \mathbb{N}_+$ and functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ satisfying
\[
|f_1(\xi) - f_1(\eta)| \leq L_f|\xi - \eta|, \quad |f_1(\xi)| \leq L_f(1 + |\xi|), \quad \xi, \eta \in \mathbb{R},
\]
\[
|f_2(\xi)| \leq G_f(1 + |\xi|^K), \quad f_2(\xi) \leq 0, \quad |f_2(\xi)| \leq G_f(1 + |\xi|^{K-1}), \quad \xi \in \mathbb{R},
\]
such that $f = f_1 + f_2$.

In order to apply the theory of stochastic analysis in infinite dimensional settings, we need to transform the original SPDE (1.1) into an infinite dimensional stochastic evolution equation. To this end, let us define
\[
\{ F(t) \}_{t \in [0, T]} : \mathbb{R}^N \to \mathbb{R}
\]
and define $F$ associated with $f$ such that
\[
\begin{align*}
F_1(X)(\xi) &:= f_1(X(\xi)), \\
F_2(X)(\xi) &:= f_2(X(\xi)), \\n\end{align*}
\]
and define $F = F_1 + F_2$. Denote by $W_H$ the $H$-valued cylindrical Wiener process in $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, i.e., there exists an orthonormal basis $\{h_k\}_{k=1}^\infty$ of $H$ and a sequence of mutually independent Brownian motions $\{\beta_k\}_{k=1}^\infty$ such that
\[
W_H(t) = \sum_{k=1}^\infty h_k\beta_k(t), \quad t \in [0, T].
\]

Then Eq. (1.1) with initial-boundary condition (1.2) is equivalent to the following stochastic evolution equation:
\[
\text{(SACE)} \quad dX(t) = (AX(t) + F(X(t)))dt + dW_H(t), \quad t \in (0, T]; \quad X(0) = X_0.
\]

Note that for any $q \geq 2$ and $\theta \geq 0$, the function space $\mathbb{L}_\xi^{\theta, q}$ is a martingale-type 2 Banach space. We need the following Burkholder inequality in martingale-type 2 Banach space (see, e.g., [Brz97, Theorem 2.4]):
\[
\begin{align*}
\| \int_0^t \Phi(r) dW_H(r) \|_{\mathbb{L}_\xi^{p, q}} &\leq C \| \Phi \|_{\mathbb{L}_\xi^{2(p, q)}},
\end{align*}
\]
for $p, q \geq 2$, where $\gamma(H, \mathbb{L}_\xi^q)$ denotes the radonifying operator norm:
\[
\| \Phi \|_{\gamma(H, \mathbb{L}_\xi^q)} := \left\| \sum_{k=1}^\infty \gamma_k \Phi h_k \right\|_{\mathbb{L}_\xi^{2(q, q)}}.
\]

Here $\{h_k\}_{k=1}^\infty$ is any orthonormal basis of $H$ and $\{\gamma_n\}_{n \geq 1}$ is a sequence of independent $\mathcal{N}(0, 1)$-random variables on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, provided that the above series converges. We also note that $\mathbb{L}_\xi^q$ with $q \geq 2$ is a Banach function space with finite cotype, and then
Φ ∈ γ(H; L^q_ξ) if and only if (∑_{k=1}^{∞} (∑_{l=1}^{q} (Φ_{l} h_{k})^2)^{1/2}) belongs to L^q_ξ for any orthonormal basis \{h_k\}_{k=1}^{∞} of H; see [vNVW08, Lemma 2.1]. Moreover, in this situation,

\[(2.3) \quad \|Φ\|_{γ(H; L^q_ξ)}^2 \simeq \left\|\sum_{k=1}^{∞} (Φ_{l} h_{k})^2\right\|_{L^{q/2}_ξ}, \quad Φ \in γ(H; L^q_ξ).\]

For convenience, we frequently use the generic constant C, which may be different in each appearance and is independent of h and τ.

### 2.2. Well-posedness and A Priori Estimation.

Recall that a predictable stochastic process \(X : [0, T] \times Ω → ℋ\) is called a mild solution of Eq. (SACE) if \(X \in L^∞(0, T; H)\) a.s. and it holds a.s. that

\[(2.4) \quad X(t) = S(t)X_0 + \int_0^t S(t - r)F(X(r))dr + W_A(t), \quad t ∈ [0, T],\]

where \(W_A = \{W_A(t) = \int_0^t S(t - r)dW_H(r) : t ∈ [0, T]\}\) is the so-called Ornstein–Uhlenbeck process. The uniqueness of the mild solution of Eq. (SACE) is understood in the sense of stochastical equivalence.

Set \(Y(t) := X(t) - W_A(t), t ∈ [0, T]\). Then X is the unique solution of Eq. (SACE) if and only if Y is the unique mild solution of the following random partial differential equation (PDE):

\[(2.5) \quad Y'(t) = AY(t) + F(Y(t) + W_A(t)), \quad t ∈ [0, T]; \quad Y(0) = X_0.\]

The mild solution of the above Eq. (2.5) is equivalent to its variational solution, i.e., for any subdivision \(\{0 = t_0 < t_1 < ⋯ < t_m < t_{m+1} < ⋯ < t_M = T\}\) with \(M ∈ \mathbb{N}_+\) of the time interval \([0, T]\) and \(v ∈ \mathbb{H}^1\) it holds a.s. that

\[(2.6) \quad \langle Y(t_{m+1}) - Y(t_m), v⟩ + \int_{t_m}^{t_{m+1}} ⟨∇Y, ∇v⟩dr = \int_{t_m}^{t_{m+1}} ⟨F(X), v⟩dr,\]

for any \(m ∈ \mathbb{Z}_{M-1} := \{0, 1, ⋯, M - 1\}\).

The existence of a unique (mild or variational) solution of Eq. (2.4) and its moments’ estimation have been established in [LQ17]. To make it complete, we give a self-contained proof of uniform moments’ estimation by using essentially the same idea of [LQ17, Proposition 3.1]. For simplicity, we assume that the initial datum \(X_0\) is a deterministic function; the case \(X_0\) is random possessing certain bounded p-moments can also be handled by similar arguments as in [LQ17].

As in [LQ17, Lemma 2.1] where we have shown that the Sobolev and Hölder regularity of the Ornstein–Uhlenbeck process \(W_A\), our main
tool is the following factorization formula which is valid by deterministic and stochastic Fubini theorems:
\[
\int_0^t S(t-r)F(X(r))dW_H(r) = \frac{\sin(\pi \alpha)}{\pi} \int_0^t (t-r)^{\alpha-1}S(t-r)F_\alpha(r)dr,
\]
\[
\int_0^t S(t-r)dW_H(r) = \frac{\sin(\pi \alpha)}{\pi} \int_0^t (t-r)^{\alpha-1}S(t-r)W_\alpha(r)dr,
\]
where \(\alpha \in (0, 1)\) and
\[
F_\alpha(t) := \int_0^t (t-r)^{-\alpha}S(t-r)F(X(r))dW_H(r),
\]
\[
W_\alpha(t) := \int_0^t (t-r)^{-\alpha}S(t-r)dW_H(r), \quad t \in [0, T].
\]
It was proved in [Brz97, Lemma 3.3] that, when \(p > 1\) and \(1/p < \alpha < 1\), the linear operator \(R_\alpha\) defined by
\[
R_\alpha f(t) := \int_0^t (t-r)^{\alpha-1}S(t-r)f(r)dr, \quad t \in [0, T],
\]
is bounded from \(L^p(0, T; L^q_\omega)\) to \(C^\delta([0, T]; L^\theta_\xi)\) with \(\delta < \alpha - 1/p\) when \(\theta = 0\) or \(\delta = \alpha - 1/p - \theta/2\) when \(\theta > 0\) and \(\alpha > \theta/2 + 1/p\).

**Lemma 2.1.** Let \(\beta \in (0, 1/2)\). Assume that \(X_0 \in \mathcal{H}^\beta \cap L^\infty\). Then for any \(p \geq 1\) and \(\beta \in (0, 1/2)\), there exists a constant \(C = C(T, p, \epsilon, \beta, X_0)\) such that
\[
\|X\|_{L^p_\omega L^{\infty}_\xi} + \|X\|_{L^p_\omega L^{\infty}_\xi H^\beta} + \|Y\|_{L^p_\omega L^{\infty}_\xi} + \|Y\|_{L^p_\omega L^{\infty}_\xi H^\beta} \leq C,
\]
and that
\[
\|X(t) - X(s)\|_{L^p(\Omega; H)} \leq C|t - s|^{\beta/2}, \quad t, s \in [0, T].
\]
Moreover, if \(X_0 \in \mathcal{H}^{1/2} \cap L^\infty\). Then
\[
\|X(t) - X(s)\|_{L^p(\Omega; H)} \leq C|t - s|^{1/4}, \quad t, s \in [0, T].
\]
**Proof.** For the initial term in Eq. (2.4), by the property of \(S\),
\[
\|S(t)X_0\|_{L^\infty_\xi} + \|S(t)X_0\|_\beta \leq C(\|X_0\|_{L^\infty_\xi} + \|X_0\|_\beta),
\]
\[
\|S(t)X_0 - S(s)X_0\| \leq C|t - s|^{\beta/2}\|X_0\|_\beta.
\]
Let \(p, q \geq 2\) and \(t \in (0, T]\). Applying Fubini theorem and the Burkholder inequality (2.2), we have
\[
\|W_\alpha\|_{L^p_\omega L^{\infty}_\xi}^p = \int_0^T \mathbb{E}\left[\left\| \int_0^t (t-r)^{-\alpha}S(t-r)dW_H(r) \right\|_{L^q_\xi}^p \right] dt
\]
\[ \text{Then by } (2.3) \text{ and the uniform boundedness of } \{e_k = \sqrt{2\sin(k\pi \cdot)}\}_{k=1}^\infty, \text{ we get} \]

\[ \|S(t)\|^2_{\gamma(H;L^q_\xi)} \preceq \left( \sum_{k=1}^\infty \|S(t)e_k\|^2_{L^{q/2}_\xi} \right) \leq \sum_{k=1}^\infty e^{-2\lambda_k t} \|e_k\|^2_{L^q_\xi} \leq Ct^{-\frac{1}{2}}, \]

where the elementary inequality \[\sum_{k=1}^\infty e^{-2\lambda_k t} \leq Ct^{-\frac{1}{2}}\] is used. Then

\[ \|W_\alpha\|_{L^p_tL^q_x} \leq C \left( \int_0^T \left( \int_0^t r^{-(2\alpha + \frac{1}{2})} dr \right)^{\frac{2}{p}} dt \right)^{\frac{1}{p}}, \]

which is finite if and only if \( \alpha \in (0, 1/4) \). As a result of the Hölder continuity characterization, \( W_A \in L^p(\Omega; C^\delta([0, T]; L^q_\xi)) \) for any \( \delta, \theta \geq 0 \) with \( \delta + \theta / 2 < 1/4 \). By the Sobolev embedding \( L^p_\xi \hookrightarrow \mathbb{H}^\beta \) with sufficiently large \( q \) and \( \beta \leq \theta < 1/2 \), we conclude that

\[ (2.12) \quad E\left[ \sup_{t \in [0, T]} \|W_A(t)\|^p_{L^\infty_\xi} \right] + E\left[ \sup_{t \in [0, T]} \|W_A(t)\|_{L^\theta_\xi}^p \right] \leq C, \]

\[ (2.13) \quad \|W_A(t) - W_A(s)\|_{L^p(\Omega; H)} \leq C|t - s|^\gamma, \quad t, s \in [0, T], \]

for any \( p \geq 1, \beta \in (0, 1/2) \) and \( \gamma \in (0, 1/4) \).

Testing both sides of Eq. (2.5) by \( Y^{2LK-1} \) with \( L \in \mathbb{N}_+ \) and integrating by parts yield that

\[ \frac{1}{2LK} \frac{d}{dt} \|Y(t)\|_{L^{2LK}_\xi}^2 + (2LK - 1) \int_0^1 |Y(t)|^{2LK-2} |\nabla Y(t)|^2 d\xi \]

\[ = \int_0^1 (F(Y(t) + W_A(t)) - F(W_A(t))) Y^{2LK-1}(t) d\xi \]

\[ + \int_0^1 F(W_A(t)) Y^{2LK-1}(t) d\xi. \]

Due to Assumption 2.1 and Hölder and Young inequalities, we have

\[ \int_0^1 (F(Y(t) + W_A(t)) - F(W_A(t))) Y^{2LK-1}(t) d\xi \leq L_f \|Y(t)\|_{L^{2LK}_\xi}^2. \]

Then by Hölder and Young inequalities, we have

\[ \frac{d}{dt} \|Y(t)\|_{L^{2LK}_\xi}^2 \leq C \left( 1 + \|Y(t)\|_{L^{2LK}_\xi}^2 + \|W_A(t)\|_{L^{2LK}_\xi}^2 \right), \]

from which by Grönwall inequality and (2.12) we conclude that

\[ \|Y(t)\|_{L^p_tL^q_x L^{2LK}_\xi} \leq C \left( 1 + \|X_0(t)\|_{L^p(\Omega; L^{2LK}_\xi)} \right). \]
Consequently, for any $\alpha \in (0, 1)$ we get
\[
\|F_\alpha\|_{L_\xi^{pK}L_\xi^{2L}}^p = \int_0^T \mathbb{E} \left[ \left\| \int_0^t (t-r)^{-\alpha} S(t-r) F(X(r)) \, dr \right\|_{L_\xi^{2L}}^p \right] \, dt \\
\leq \int_0^T \mathbb{E} \left[ \left( \int_0^t (t-r)^{-\alpha} \|S(t-r) F(X(r))\|_{L_\xi^{2L}} \, dr \right)^p \right] \, dt \\
\leq C \left( 1 + \|X\|_{L_\xi^{pK}L_\xi^{2L}}^p \right) \leq C \left( 1 + \|X_0\|_{L_\xi^{2L}}^p \right).
\]
Therefore, $\int_0^T S(\cdot - r) F(X(r)) \, dr \in L^p(\Omega; C^\delta([0,T]; L_\xi^{\theta,2}L))$ for any $\delta, \theta \geq 0$ with $\delta + \theta/2 < 1$. Then by Sobolev embedding we have
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \int_0^t S(t-r) F(X(r)) \, dr \right\|_{L_\xi^\infty}^p \right] \\
+ \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \int_0^t S(t-r) F(X(r)) \, dr \right\|_{L_\xi^\beta}^p \right] \leq C,
\]
and
\[
\left\| \int_0^t S(t-r) F(X(r)) \, dr - \int_0^s S(s-r) F(X(r)) \, dr \right\|_{L^p(\Omega; H)} \\
\leq C |t-s|^{\gamma}, \quad t, s \in [0,T],
\]
for any $p \geq 1$, $\beta \in (0, 1/2)$ and $\gamma \in (0, 1)$.

Combining (2.10)-(2.14) and the relation that $X = Y + W_A$, we get (2.7) and (2.8). To show the last inequality (2.9), we only need to give a refined estimation of (2.13):
\[
\|W_A(t) - W_A(s)\|_{L^p(\Omega; H)} \leq C |t-s|^{1/2}, \quad t, s \in [0,T].
\]
Due to the fact that $W_A$ is Gaussian, we only need to show (2.13) for $p = 2$. Without loss of generality, assume that $0 \leq s \leq t \leq T$. By Itô isomtery, we have
\[
\mathbb{E} \left[ \|W_A(t) - W_A(s)\|^2 \right] \\
= \mathbb{E} \left[ \left\| \int_s^t S(t-r) \, dW_H(r) \right\|^2 \right] \\
+ \mathbb{E} \left[ \left\| \int_s^t (S(t-r) - S_N(s-r)) \, dW_H(r) \right\|^2 \right] \\
= \int_0^{t-s} \left[ \sum_{k=1}^{\infty} e^{-2\lambda_k r} \right] \, dr + \sum_{k=1}^{\infty} \frac{1 - e^{-2\lambda_k s}}{2\lambda_k} \left( 1 - e^{-\lambda_k (t-s)} \right)^2
\]
\[ \leq \int_0^{t-s} \left[ \sum_{k=1}^{\infty} e^{-2\lambda_k r} \right] dr + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1 - e^{-\lambda_k(t-s)}}{\lambda_k} \]
\[ = \int_0^{t-s} \left[ \sum_{k=1}^{\infty} e^{-2\lambda_k r} \right] dr + \frac{1}{2} \int_0^{t-s} \left[ \sum_{k=1}^{\infty} e^{-\lambda_k r} \right] dr \leq C(t-s). \]

This completes the proof of (2.16). \( \square \)

Next we use the uniform estimation in Lemma 2.1 to derive the following H"older-type regularity of the solution \( X \) of Eq. (1.1).

**Lemma 2.2.** Let \( \beta \in (0, 1/2] \). Assume that \( X_0 \in \mathbb{H}^\beta \cap L^\infty \). Then for any \( p \geq 1 \), there exists a constant \( C = C(T, p, K, \beta, X_0) \) such that for any \( 0 \leq s \leq t \leq T \) there holds that

\( (2.17) \quad \int_s^t \mathbb{E} \left[ \| F(X(t)) - F(X(r)) \|^2 \right] dr \leq C(t-s)^{1+\beta}, \)

\( (2.18) \quad \int_s^t \mathbb{E} \left[ \| \nabla Y(t) - \nabla Y(r) \|^2 \right] dr \leq C(t-s)^{1+\beta}. \)

**Proof.** We start with the first estimation (2.17). By Young inequality, we get
\[
F(X(t)) - F(X(r)) = f_1(X(t)) - f_1(X(r)) + f_2(X(t)) - f_2(X(r))
\leq C \left( 1 + |X(t)|^{K-1} + |X(r)|^{K-1} \right) \cdot |X(t) - X(r)|.
\]

The moments’ boundedness and H"older-type regularity of \( X \) in Lemma 2.1 yield that
\[
\int_s^t \mathbb{E} \left[ \| F(X(t)) - F(X(r)) \|^2 \right] dr 
\leq C \left( 1 + \sup_{t \in [0,T]} \| X(t) \|_{L_2^{K-1}(\Omega; L_\infty)}^{2(K-1)} \right) \times \left[ \int_s^t \| X(t) - X(r) \|_{L_2^{K}(\Omega; L_\infty)}^2 dr \right]
\leq C(t_2 - t_1)^{1+\beta},
\]
which proves (2.17).

Next we prove the last inequality (2.18). Let \( t_0 \in (0, T] \) be fixed and take \( s \in [0, t_0] \). Then by Hölder inequality,
\[
\frac{1}{2} \| Y(t_0) - Y(r) \|^2_{L_2} + \int_s^{t_0} \| \nabla (Y(t_0) - Y(r)) \|^2 dr
\leq \int_s^{t_0} \langle F(X(t_0)) - F(X(r), Y(t_0) - Y(r)) \rangle dr
\]
Then the spectral approximation of Eq. (1.1) is to find an 
\[ F \]
\[ \{ \]
where 
\[ S \]

Thus we get (2.18) by (2.17) and the regularity (2.7) of \( X \). \( \square \)

3. Fully Discrete Approximation

In this section, we study the fully discrete scheme of Eq. (1.1) and derive its optimal strong convergence rate.

3.1. Backward Euler–Spectral Galerkin Approximation. Let 
\[ M, N \in \mathbb{N}_+ \]. Denote by \( P_N \) the orthogonal projection operator from 
\( H \) to its finite dimensional subspace \( V_N \) spanned by the eigenvectors 
\[ \{ \epsilon_k = \sqrt{2}\sin(k\pi\cdot) \}_{k=1}^N \] corresponding to the first \( N \) eigenvalues 
\[ \{ \lambda_k = (k\pi)^2 \}_{k=1}^N \] of negative Dirichlet Laplacian \(-A\):
\[ \langle P_N u, v_N \rangle = \langle u, v_N \rangle, \quad u \in H, v_N \in V_N. \]

Then the spectral approximation of Eq. (1.1) is to find an \( \mathcal{F}_t \)-adapted \( V_N \)-valued process \( X_N = \{ X_N(t) : t \in [0, T] \} \) such that
\[ \begin{aligned}
\frac{dX_N(t)}{dt} &= (A_N X_N(t) + P_N F(X_N(t))) dt + P_N dW_H(t), \quad t \in [0, T]; \\
X_N(0) &= P_N X_0.
\end{aligned} \] (3.2)

The mild solution of Eq. (3.2) is given by
\[ X_N(t) = S_N(t) P_N X_0 + \int_0^t S_N(t-r) P_N F(X_N(r)) dr + W_A^N(t), \quad t \in [0, T], \]
where \( S_N = \{ S_N(t) := e^{A_N t} : t \in [0, T] \} \) is the analytic \( C_0 \)-semigroup

\[ \frac{dN(t)}{dt} = A_N Y_N(t) + P_N F(Y_N(t) + W_A^N(t)), \quad t \in [0, T]; \quad Y_N(0) = P_N X_0. \] (3.3)

Let \( M \in \mathbb{N}_+ \) and denote \( Z_M := \{ 0, 1, \cdots, M \} \). Similarly to (2.6), it is clear that the spectral approximation (3.2) of Eq. (1.1) is equivalent to find an \( V_N \)-valued process \( X_N \) such that for all subdivision \( \{ t_m : m \in Z_M \} \) of \( [0, T] \) and \( v_N \in V_N \) it holds a.s. that
\[ \langle Y_N(t_{m+1}) - Y_N(t_m), v_N \rangle + \int_{t_m}^{t_{m+1}} \langle \nabla Y_N, \nabla v_N \rangle dr = \int_{t_m}^{t_{m+1}} \langle F(X_N), v_N \rangle dr. \] (3.4)
The backward Euler approximation of (3.4) is to find a $V_N$-valued discrete process $\{Y^m_N : N \in \mathbb{N}_+, m \in \mathbb{Z}_M\}$ such that for all $v_N \in V_N$ it holds a.s. that
\begin{equation}
(Y^{m+1}_N - Y^m_N, v_N) + \tau \langle \nabla Y^{m+1}_N, \nabla v_N \rangle = \tau \langle F^{m+1}_N, v_N \rangle,
\end{equation}
where $F^{m+1}_N := F(Y^{m+1}_N + W^N_A(t_{m+1})), m \in \mathbb{Z}_{M-1}$. We call the fully discrete scheme (3.5) the backward Euler–spectral Galerkin scheme of Eq. (2.5). Set
\begin{equation}
X^m_N = Y^m_N + W^N_A(t_m), m \in \mathbb{Z}_M.
\end{equation}
Then $X^m_N$ is an approximation of the solution $X$ of Eq. (1.1) at $t_m, m \in \mathbb{Z}_{M-1}$. In this sense, (3.5)–(3.6) can be seen as the backward Euler–Galerkin scheme of Eq. (1.1).

3.2. Strong Convergence Rate. This section is devoted to establish the strong convergence rate for the backward Euler–spectral Galerkin scheme (3.5) and (3.6) of Eq. (1.1).

We begin with the following essentially optimal error estimation between the Ornstein–Uhlenbeck process $W_A$ and its approximation $W^N_A$.

**Lemma 3.1.** Let $p \geq 1$. There exists a constant $C = C(p)$ such that
\begin{equation}
\sup_{t \in [0,T]} \left( \mathbb{E} \left[ \|W_A(t) - W^N_A(t)\|^p \right] \right)^{\frac{1}{p}} \leq CN^{-1/2}.
\end{equation}

**Proof.** The difference of the Ornstein–Uhlenbeck processes can be rewritten as
\begin{equation*}
W_A(t) - W^N_A(t) = \int_0^t (S(t-r) - S_N(t-r)P_N)\,dW_H(r), \quad t \in [0,T].
\end{equation*}
Since $W_A - W^N_A$ is Gaussian, we only need to show (3.7) for $p = 2$. By Itô isometry and elementary calculations, we get
\begin{align*}
\sup_{t \in [0,T]} \mathbb{E} \left[ \|W_A(t) - W^N_A(t)\|^2 \right] &\leq \sum_{k=1}^{\infty} \int_0^T \| (S(r) - S_N(r)P_N)e_k \|^2 \,dr \\
&\leq \sum_{k=N+1}^{\infty} \frac{1 - e^{-2\lambda_k T}}{2\lambda_k} \leq \frac{1}{2\pi^2} N^{-1}.
\end{align*}
This completes the proof of (3.7).
Remark 3.1. The estimation (3.7) is sharp in the sense that
\[ \mathbb{E}[\|W_A(t) - W_A^N(t)\|^2] = \sum_{k=N+1}^{\infty} \frac{1 - e^{-2\lambda_k t}}{2\lambda_k} \geq \frac{t}{2(1 + 2\pi^2 t)} N^{-1}, \]
for \( t > 0 \), where we have used the elementary estimation \( e^x \geq 1 + x \) for any \( x \geq 0 \).

Theorem 3.1. Let \( \epsilon, \beta \in (0, 1/2), \zeta > 0 \) and \( \tau < \frac{1}{4(L_f + \zeta)} \).

1. Assume that \( X_0 \in \mathbb{H}^\beta \cap L^\infty \). There exists a constant \( C = C(T, \beta, \zeta, X_0) \) such that
   \[ \sup_{m \in \mathbb{Z}_M} \|X(t_m) - X_N^m\|_{L^2(\Omega; H)} \leq C((\ln N)^2 N^{-\beta} + \tau^\beta/2). \]

2. Assume that \( X_0 \in \mathbb{H}^{1/2} \cap L^\infty \). For any \( \gamma \in (0, 1/2) \), there exists a constant \( C = C(T, \zeta, \gamma, X_0) \) such that
   \[ \sup_{m \in \mathbb{Z}_M} \|X(t_m) - X_N^m\|_{L^2(\Omega; H)} \leq C((\ln N)^2 N^{-\gamma} + \tau^{1/4}). \]

3. Assume that \( X_0 \in \mathbb{H}^{1/2+\epsilon} \). For any \( \gamma \in (0, 1/2) \), there exists a constant \( C = C(T, \beta, \zeta, \gamma, X_0) \) such that
   \[ \sup_{m \in \mathbb{Z}_M} \|X(t_m) - X_N^m\|_{L^2(\Omega; H)} \leq C(N^{-\gamma} + \tau^{1/4}). \]

Proof. Let \( X_0 \in \mathbb{H}^\beta \) with \( \beta \in (0, 1/2) \). Define \( e_N^m := \mathcal{P}_N Y(t_m) - Y_N^m, m \in \mathbb{Z}_M \). Then \( e_N^m \in V_N \) and \( X(t_m) - X_N^m = (\text{Id}_H - \mathcal{P}_N) X(t_m) + e_N^m, m \in \mathbb{Z}_M \). Then by triangle inequality and the moment’s estimation (2.7), we get
\[
\sup_{m \in \mathbb{Z}_M} \|X(t_m) - X_N^m\|_{L^2(\Omega; H)} \\
\leq \sup_{m \in \mathbb{Z}_M} \|(\text{Id}_H - \mathcal{P}_N) X(t_m)\|_{L^2(\Omega; H)} + \sup_{m \in \mathbb{Z}_M} \|e_N^m\|_{L^2(\Omega; H)} \\
\leq C N^{-\beta} \sup_{t \in [0,T]} \|X(t)\|_{L^2(\Omega; \mathbb{H}^\beta)} + \sup_{m \in \mathbb{Z}_M} \|e_N^m\|_{L^2(\Omega; H)}.
\]

Subtracting (2.6) from (3.5) with \( v = v_N = e_N^{m+1} \in V_N \subset \mathbb{H}^1 \), we get
\[
\langle (\text{Id}_H - \mathcal{P}_N)(Y(t_{m+1} - Y(t_m)), e_N^{m+1}), e_N^{m+1} - e_N^m, e_N^{m+1} \rangle \\
= - \int_{t_m}^{t_{m+1}} \langle \nabla(Y - Y_N^{m+1}), \nabla e_N^{m+1} \rangle dr + \int_{t_m}^{t_{m+1}} \langle F(X) - F_N^{m+1}, e_N^{m+1} \rangle dr.
\]
The property of $L^2$-projection of $P_N$ yields that
\[ E\left[ \langle (\text{Id}_H - P_N)(Y(t_{m+1} - Y(t_m)), e_{N+1}^m) \rangle \right] = 0. \]
By the elementary identity $(a - b)a = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a - b)^2$, we get
\[ E\left[ \langle e_{N+1}^m - e_N^m, e_{N+1}^m \rangle \right] = \frac{1}{2}\left( E\left[ \|e_{N+1}^m\|^2 \right] - E\left[ \|e_N^m\|^2 \right] \right) + \frac{1}{2}E\left[ \|e_{N+1}^m - e_N^m\|^2 \right]. \]
(3.13)
Applying the fact that $\langle \nabla (\text{Id}_H - P_N)u, \nabla v \rangle = 0$ for any $u \in \mathcal{H}^1$ and $v \in V_N$, Cauchy-Schwarz inequality and (2.18), we obtain
(3.14)
\[ E\left[ -\int_{t_m}^{t_{m+1}} \langle \nabla(Y(r) - Y_{N+1}^{m+1}), \nabla e_{N+1}^{m+1} \rangle dr \right] \]
\[ = -\int_{t_m}^{t_{m+1}} E\left[ \langle \nabla(Y(r) - Y(t_{m+1})), \nabla e_{N+1}^{m+1} \rangle \right] dr - E\left[ \|\nabla e_{N+1}^{m+1}\|^2 \right] \tau \]
\[ \leq \frac{1}{2} \int_{t_m}^{t_{m+1}} E\left[ \|\nabla(Y(r) - Y(t_{m+1}))\|^2 \right] dr - \frac{1}{2}E\left[ \|\nabla e_{N+1}^{m+1}\|^2 \right] \tau \]
\[ \leq C \tau^{1+\beta} - \frac{1}{2}E\left[ \|\nabla e_{N+1}^{m+1}\|^2 \right] \tau. \]
For the third term in Eq. (3.12), the monotonicity of $f$, Hölder and Young inequalities and the relation (3.6) imply that
\[ E\left[ \int_{t_m}^{t_{m+1}} \langle F(X(r)) - F(X_{N+1}^{m+1}), e_{N+1}^{m+1} \rangle dr \right] \]
\[ = \int_{t_m}^{t_{m+1}} E\left[ \langle F(X(r)) - F(X(t_{m+1})), e_{N+1}^{m+1} \rangle \right] dr \]
\[ + E\left[ \langle F(X(t_{m+1})) - F(P_NX(t_{m+1})), e_{N+1}^{m+1} \rangle \right] \tau \]
\[ + E\left[ \langle F(P_NX(t_{m+1})) - F(X_{N+1}^{m+1}), e_{N+1}^{m+1} \rangle \right] \tau \]
\[ \leq \frac{C}{\zeta} \int_{t_m}^{t_{m+1}} E\left[ \|F(X(r)) - F(X(t_{m+1}))\|^2 \right] dr \]
\[ + \frac{C}{\zeta} E\left[ \|F(X(t_{m+1})) - F(P_NX(t_{m+1}))\|^2 \right] \tau + (L_f + \zeta)E\left[ \|e_{N+1}^{m+1}\|^2 \right] \tau, \]
where $\zeta$ is an arbitrary positive number. By the estimation (2.17), we get
\[ \frac{C}{\zeta} \int_{t_m}^{t_{m+1}} E\left[ \|F(X(r)) - F(X(t_{m+1}))\|^2 \right] dr \leq \frac{C}{\zeta} \tau^{1+\beta}. \]
By Assumption 2.1 and the sharp non-uniform $\mathbb{L}^\infty$-estimation of $\mathcal{P}_N$ that
\[
\sup_{t \in [0,T]} \| \mathcal{P}_N u \|_{\mathbb{L}^\infty} \leq C (\ln N) \| u \|_{\mathbb{L}^\infty}, \quad u \in \mathbb{L}^\infty,
\]
as well as the moments' estimation (2.7), we have
\[
C \zeta \mathbb{E} \left[ \| F(X(t_{m+1})) - F(\mathcal{P}_N X(t_{m+1})) \|^2 \right] \tau \\
\leq \frac{C}{\zeta} \left( \mathbb{E} \left[ \| (\text{Id}_H - \mathcal{P}_N) X(t_{m+1}) \|_{2^K}^2 \right] \right)^{1/K} \\
\left[ 1 + \left( \mathbb{E} \left[ \| X(t_{m+1}) \|_{2^K}^2 \right] \right)^{K-1} + \left( \mathbb{E} \left[ \| \mathcal{P}_N X(t_{m+1}) \|_{2^K}^2 \right] \right)^{K-1} \right] \tau \\
\leq \frac{C}{\zeta} (\ln N)^4 N^{-2\beta} \tau.
\]

Consequently, we have
\[
\mathbb{E} \left[ \int_{t_m}^{t_{m+1}} \langle F(X(r)) - F(X^m_N), e^{m+1}_N \rangle dr \right] \\
\leq \frac{C}{\zeta} \left( (\ln N)^4 N^{-2\beta} + \tau^\beta \right) \tau + \left( L_f + \zeta \right) \mathbb{E} \left[ \| e^{m+1}_N \|^2 \right] \tau.
\]

Combining the above estimations (3.13)–(3.15), we derive
\[
\frac{1}{2} \left( \mathbb{E} \left[ \| e^{m+1}_N \|^2 \right] - \mathbb{E} \left[ \| e^m_N \|^2 \right] \right) + \frac{1}{2} \mathbb{E} \left[ \| \nabla e^{m+1}_N \|^2 \right] \tau \\
\leq \left( C + \frac{C}{\zeta} \right) \left( (\ln N)^4 N^{-2\beta} + \tau^\beta \right) \tau + \left( L_f + \zeta \right) \mathbb{E} \left[ \| e^{m+1}_N \|^2 \right] \tau.
\]

Then we deduce that
\[
\left( 1 - 2(L_f + \zeta) \tau \right) \mathbb{E} \left[ \| e^{m+1}_N \|^2 \right] + \mathbb{E} \left[ \| \nabla e^{m+1}_N \|^2 \right] \tau \\
\leq \mathbb{E} \left[ \| e^0_N \|^2 \right] + \left( C + \frac{C}{\zeta} \right) \left( (\ln N)^4 N^{-2\beta} + \tau^\beta \right) \tau.
\]

Summing over $m = 0, 1, \cdots, l - 1$ with $1 \leq l \leq M$, we obtain
\[
\left( 1 - 2(L_f + \zeta) \tau \right) \mathbb{E} \left[ \| e^l_N \|^2 \right] + \sum_{m=0}^{l} \mathbb{E} \left[ \| \nabla e^m_N \|^2 \right] \tau \\
\leq \left( C + \frac{C}{\zeta} \right) \left( (\ln N)^4 N^{-2\beta} + \tau^\beta \right) \tau + 2(L_f + \zeta) \tau \sum_{m=0}^{l-1} \mathbb{E} \left[ \| e^m_N \|^2 \right].
\]
Assume that $\zeta$ is sufficiently small and $\tau < \frac{1}{4(L_f + \zeta)}$. By the discrete Gronwall inequality, we conclude that

$$
\sup_{m \in \mathbb{Z}_M} \mathbb{E} \left[ \|e_N^m\|^2 \right] + \sum_{m=0}^{M} \mathbb{E} \left[ \|\nabla e_N^m\|^2 \right] \tau \\
\leq \left( C + \frac{C}{\zeta} \right) e^{2(L_f + \zeta)T} \left( (\ln N)^4 N^{-2\beta} + \tau^\beta \right).
$$

This, in combination with (3.11), shows (3.8). In the case $X_0 \in \mathbb{H}^{1/2} \cap \mathbb{L}^\infty$, by Lemma (2.2) and the above argument we get the temporal convergence rate $\mathcal{O}(\tau^{1/4})$, which shows (3.9).

Let $X_0 \in \mathbb{H}^{1/2 + \epsilon}$. It is clear that

$$
\mathcal{P}_N X(t) = S(t)\mathcal{P}_N X_0 + \mathcal{P}_N \left[ \int_0^t S(t-r)F(X(r))dr \right] + W^N_{\alpha}(t).
$$

By stochastic Fubini theorem, the approximate Ornstein–Uhlenbeck process $W^N_{\alpha}$ possesses the following factorization formula:

$$
\int_0^t S_N(t-r)\mathcal{P}_N dW_H(r) = \frac{\sin(\pi \alpha)}{\pi} \int_0^t (t-r)^{\alpha-1}S_N(t-r)W^N_{\alpha}(t)dr,
$$

where $\alpha \in (0,1)$ and $W^N_{\alpha}(t) := \int_0^t (t-r)^{-\alpha}S_N(t-r)\mathcal{P}_N dW_H(r)$, $t \in [0,T]$. Let $p, q \geq 2$ and $t \in (0,T]$. Similarly to Lemma 2.1, applying Fubini theorem and the Burkholder inequality (2.2) as well as the equivalence (2.3) of $\gamma$-norm, we get

$$
\|W^N_{\alpha}\|_{\mathbb{L}^p_L\mathbb{L}^q_L} = \int_0^T \mathbb{E} \left[ \left\| \int_0^t (t-r)^{-\alpha}S_N(t-r)\mathcal{P}_N dW_H(r) \right\|_{\mathbb{L}^q_L}^p \right] dt \\
\leq C \int_0^T \left( \int_0^t r^{-2\alpha}\|S_N(r)\mathcal{P}_N\|_{\gamma(H^{1/2+\epsilon})}^2 dr \right)^{\frac{p}{2}} dt \\
\leq C \int_0^T \left( \int_0^t r^{-2\alpha}\sum_{k=1}^{\infty} (S_N(r)\mathcal{P}_N e_k)^2 \right)^{\frac{p}{2}} dr dt \\
\leq C \int_0^T \left( \int_0^t r^{-(2\alpha+1)} dr \right)^{\frac{p}{2}} dt.
$$

The last integral is finite if and only if $\alpha \in (0,1/4)$. As a result of the Hölder continuity characterization and Sobolev embedding, $W^N_{\alpha} \in \mathbb{L}^p(\Omega; C^\delta([0,T]; C^\kappa))$ for any $\delta, \kappa \geq 0$ with $\delta + \kappa/2 < 1/4$ uniformly with respect to $N$. In particular, there exists a constant $C = C(T,p)$ such that

$$
(3.16) \quad \sup_{N \in \mathbb{N}_+} \mathbb{E} \left[ \sup_{t \in [0,T]} \|W^N_{\alpha}\|_{\mathbb{L}^p_L}^p \right] \leq C.
$$
It is shown in Lemma 2.1 that \( \int_0^\cdot S(\cdot-r)F(X(r))dr \in L^p(\Omega; C^\delta([0,T]; L^2_\xi)) \) for any \( \delta, \theta \geq 0 \) with \( \delta + \theta/2 < 1 \). In particular, \( \int_0^\cdot S(\cdot-r)F(X(r))dr \in L^p(\Omega; C([0,T]; \mathbb{H}^\gamma)) \) for any \( p \geq 1 \) and \( \gamma \in (0,2) \). Therefore, by the Sobolev embedding \( \dot{H}^{1/2+\epsilon} \subset L^\infty_\xi \) there exists a constant \( C = C(T,p,\epsilon,X_0) \) such that

\[
\sup_{N \in \mathbb{N}_+} \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \mathcal{P}_N \left[ \int_0^t S(t-r)F(X(r))dr \right] \right\|^p \right] \leq C.
\]

Similarly,

\[
\sup_{N \in \mathbb{N}_+} \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| S(t) \mathcal{P}_N X_0 \right\|^p \right] \leq \|X_0\|^p_{1/2+\epsilon}.
\]

Therefore, there exists a constant \( C = C(T,p,\epsilon,X_0) \) such that

\[
\sup_{t \in [0,T]} \|\mathcal{P}_N X(t)\|_{L^\infty} \leq C.
\]

Consequently, for any \( \gamma \in (0,1/2) \) we have

\[
\mathbb{E} \left[ \| F(X(t_{m+1})) - F(\mathcal{P}_N X(t_{m+1})) \|^2 \right] \leq CN^{-2\gamma}.
\]

Following the previous procedure, we get (3.10).

4. Numerical Experiments

In this section, we give several numerical tests to verify the optimality of the strong convergence rate in Theorem 3.1 for the backward Euler–spectral Galerkin scheme (3.5).

Due to Lemma 3.1 and Remark 3.1, the spatial convergence rate of the backward Euler–spectral Galerkin scheme (3.5) is sharp. Our main concern is to simulate the temporal strong convergence rate of the fully discrete scheme (3.5) for the following stochastic Allen–Cahn equation driven by an additive Brown sheet \( W \):

\[
\frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial \xi^2} + (X - X^3) + \frac{\partial^2 W}{\partial t \partial \xi}, \quad (t, \xi) \in (0,T] \times (0,1),
\]

with the initial value and Dirichlet boundary condition (1.2).
To check the optimality of the temporal strong convergence rate of the scheme (3.5) with explicit dependence on \( \beta \), we choose the initial datum \( X_0 \) to be

\[
X_0(\xi) = \sum_{k=1}^{\infty} \frac{e_k(\xi)}{k^{\beta+1/2}}, \quad e_k(\xi) = \sqrt{2} \sin(k\pi \xi), \quad \xi \in (0, 1),
\]

where \( \beta \in (0, 1/2] \). One can show that \( X_0 \in \mathcal{H}^\gamma \) for any \( \gamma \in (0, \beta) \) and that \( X_0 \notin \mathcal{H}^\beta \). We use the backward Euler–spectral Galerkin scheme (3.5) with \( F(x) = x - x^3 \) and initial datum \( Y_0 = \mathcal{P}_N X_0 = \sum_{k=1}^{N} k^{-\beta - 1/2} \cdot e_k \) to fully discrete Eq. (4.1). To simulate the approximate Ornstein–Uhlenbeck process \( W^N_\beta \), it is clear that

\[
W^N_\beta(t_m) = \int_0^{t_m} S(t_m - r) \mathcal{P}_N dW_H(r) = \sum_{k=1}^{N} \left[ \int_0^{t_m} e^{-\lambda_k (t_m - r)} d\beta_k(r) \right] e_k,
\]

where \( \int_0^{t_m} e^{-\lambda_k (t_m - r)} d\beta_k(r) \) is a centered Gaussian random variable with the variance \( \frac{1 - e^{-2\lambda_k t_m}}{2\lambda_k} \). Thus

\[
W^N_\beta(t_m) = \sum_{k=1}^{N} \sqrt{\frac{1 - e^{-2\lambda_k t_m}}{2\lambda_k}} \zeta_k e_k,
\]

where \( \{\zeta_k\}_{k \in \mathbb{Z}_N} \) is a sequence of independent normally distributed random variables.

Figure 4 displays the mean-square convergence rates of the backward Euler–spectral Galerkin scheme (3.5) with three values of \( \beta \): 0.3, 0.4, 0.5. The expected temporal mean-square convergence \( \mathcal{O}(\tau^{\beta/2}) \) of the scheme (3.5), as expected from Theorem 3.1, can be confirmed.

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