ALGEBRAIC STRUCTURES OF TROPICAL MATHEMATICS

ZUR IZHAKIAN, MANFRED KNEBUSCH, AND LOUIS ROWEN

Abstract. Tropical mathematics often is defined over an ordered cancellative monoid \( M \), usually taken to be \((\mathbb{R},+)\) or \((\mathbb{Q},+)\). Although a rich theory has arisen from this viewpoint, cf. [L1], idempotent semirings possess a restricted algebraic structure theory, and also do not reflect certain valuation-theoretic properties, thereby forcing researchers to rely often on combinatoric techniques.

In this paper we describe an alternative structure, more compatible with valuation theory, studied by the authors over the past few years, that permits fuller use of algebraic theory especially in understanding the underlying tropical geometry. The idempotent max-plus algebra \( A \) of an ordered monoid \( M \) is replaced by \( R := L \times M \), where \( L \) is a given indexing semiring (not necessarily with 0). In this case we say \( R \) layered by \( L \). When \( L \) is trivial, i.e., \( L = \{1\} \), \( R \) is the usual bipotent max-plus algebra. When \( L = \{1,\infty\} \) we recover the “standard” supertropical structure with its “ghost” layer. When \( L = \mathbb{N} \) we can describe multiple roots of polynomials via a “layering function” \( s : R \to L \).

Likewise, one can define the layering \( s : R^{(n)} \to L^{(n)} \) componentwise; vectors \( v_1, \ldots, v_m \) are called tropically dependent if each component of some nontrivial linear combination \( \sum \alpha_i v_i \) is a ghost, for “tangible” \( \alpha_i \in R \). Then an \( n \times n \) matrix has tropically dependent rows iff its permanent is a ghost.

We explain how supertropical algebras, and more generally layered algebras, provide a robust algebraic foundation for tropical linear algebra, in which many classical tools are available. In the process, we provide some new results concerning the rank of \( d \)-independent sets (such as the fact that they are semi-additive), put them in the context of supertropical bilinear forms, and lay the matrix theory in the framework of identities of semirings.

1. Introduction

Tropical geometry, a rapidly growing area expounded for example in [Gat, ItMS, L1, MS, SS], has been based on two main approaches. The most direct passage to tropical mathematics is via logarithms. But valuation theory has richer algebraic applications (for example providing a quick proof of Kapranov’s theorem), and much of tropical geometry is based on valuations on Puiseux series. The structures listed above are compatible with valuations, and in \( \S 2.4 \) we see how valuations fit in with this approach.

In his overview, Litvinov [L2] describes tropicalization as a process of dequantization. Thus, one is motivated to develop the algebraic tools at the tropical level, in order to provide an intrinsic theory to support tropical geometry and linear algebra. The main mathematical structure of tropical geometry is the max-plus algebra, which is viewed algebraically as an ordered monoid. Considerable recent activity [CHWW, W] concerns geometry over monoids, but the ordering provides extra structure which enables us to draw on classical algebraic structure theory.

The max-plus algebra is fine for answering many combinatoric questions, but it turns out that a more sophisticated structure is needed to understand the algebraic structure connected with valuations. Our overlying objective is to translate ordered monoids into an algebraic theory supporting tropical linear algebra and geometry, using the following approaches:

- Algebraic geometry as espoused by Zariski and Grothendieck, using varieties and commutative algebra in the context of category theory.
- Linear algebra via tropical dependence, the characteristic polynomial, and (generalized) eigenspaces.
- Algebraic formulations for more sophisticated concepts such as resultants, discriminants, and Jacobians.
This approach leads to the use of polynomials and matrices, which requires two operations. Our task has been to pinpoint the appropriate category of semirings in which to work, or equivalently, how far do we dequantize in the process of tropicalization? In this survey we compare four structures, listed in increasing level of refinement:

- The max-plus algebra,
- Supertropical algebra,
- Layered tropical algebras,
- Exploded supertropical algebras.

We review the layered algebra in [K] compare it to the max-plus algebra, and then in [J] survey its linear algebraic theory, especially in terms of different notions of bases, proving a new result (Proposition [25]) about the semi-additivity of the rank of d-independent sets of a layered vector space. In [M] we see how these considerations lead naturally to a theory of identities. Due to lack of space, we often refer the reader to [ZKR4] [ZKR5] for more details.

2. Algebraic Background

We start by reviewing some notions which may be familiar, but are needed extensively in our exposition. The basic tropicalization, or dequantization, involves taking logarithms to \((\mathbb{R}, +)\), which as explained in [L1] replaces conventional multiplication by addition, and conventional addition by the maximum. This is called the max-plus algebra of \((\mathbb{R}, +)\).

2.1. Ordered groups and monoids. Recall that a monoid \((\mathcal{M}, \cdot, 1)\) is a set with an associative operation \(\cdot\) and a unit element 1. We usually work with Abelian monoids, in which the operation is commutative. The passage to the max-plus algebra in tropical mathematics can be viewed algebraically via ordered groups (such as \((\mathbb{R}, +)\)), and, more generally, ordered monoids.

An Abelian monoid \(\mathcal{M} := (\mathcal{M}, \cdot, 1)\) is cancellative if \(ab = ac\) implies \(b = c\). There is a well-known localization procedure with respect to a submonoid \(S\) of a cancellative Abelian monoid \(\mathcal{M}\), obtained by taking \(\mathcal{M} \times S/ \sim\), where \(\sim\) is the equivalence relation given by \((a, s) \sim (a', s')\) iff \(as = a's\). Localizing with respect to all of \(\mathcal{M}\) yields its group of fractions, cf. [Bo] [W]. We say that a monoid \(\mathcal{M}\) is power-cancellative (called torsion-free by [W]) if \(a^n = b^n\) for some \(n \in \mathbb{N}\) implies \(a = b\). A monoid \(\mathcal{M}\) is called \(\mathbb{N}\)-divisible (also called radicalizable in the tropical literature) if for each \(a \in \mathcal{M}\) and \(m \in \mathbb{N}\) there is \(b \in \mathcal{M}\) such that \(b^m = a\). For example, \((\mathbb{Q}, +)\) is \(\mathbb{N}\)-divisible.

Remark 2.1. The customary way of embedding an Abelian monoid \(\mathcal{M}\) into an \(\mathbb{N}\)-divisible monoid, is to adjoin \(\sqrt[n]{a}\) for each \(a \in \mathcal{M}\) and \(m \in \mathbb{N}\), and define

\[
\sqrt[n]{a} \sqrt[m]{b} := \sqrt[\text{lcm}(n,m)]{a^nb^m}.
\]

This will be power-cancellative if \(\mathcal{M}\) is power-cancellative.

An ordered Abelian monoid is an Abelian monoid endowed with a total order satisfying the property:

\[
a \leq b \quad \text{implies} \quad ga \leq gb,
\]

for all elements \(a, b, g\). Any ordered cancellative Abelian monoid is infinite.

One advantage of working with ordered monoids and groups is that their elementary theory is well-known to model theorists. The theory of ordered \(\mathbb{N}\)-divisible Abelian groups is model complete, cf. [M] p. 116] and [Sa] pp. 35, 36], which essentially means that every \(\mathbb{N}\)-divisible ordered cancellative Abelian monoid has the same algebraic theory as the max-plus algebra \((\mathbb{Q}, +)\), which is a much simpler structure than \((\mathbb{R}, +)\).

From this point of view, the algebraic essence of tropical mathematics boils down to \((\mathbb{Q}, +)\). Sometimes we want to study its ordered submonoid \((\mathbb{Z}, +)\), or even \((\mathbb{N}, +)\), although they are not \(\mathbb{N}\)-divisible.

Nevertheless, just as one often wants to study the arithmetic of \(\mathbb{Q}\) by viewing finite homomorphic images of \(\mathbb{Z}\), we want the option of studying finite homomorphic images of the ordered monoid \((\mathbb{N}, +)\). Towards this end, we define the \(q\)-truncated monoid \(\mathcal{M} = [1, q] := \{1, 2, \ldots, q\}\), given with the obvious ordering: the sum and product of two elements \(k, \ell \in L\) are taken as usual, if not exceeding \(q - 1\), and is \(q\) otherwise. In other words, \(q\) could be considered as the infinite element of the finite monoid \(\mathcal{M}\).
2.2. Semirings without zero. So far, dequantization has enabled us to pass from algebras to ordered Abelian monoids, which come equipped with a rich model theory ready to implement, and as noted above, there is a growing theory of algebraic geometry over monoids [CHWW]. But to utilize standard tools such as polynomials and matrices, we need two operations (addition and multiplication), and return to the language of semirings, using [GG] as a general reference. We write † to indicate that we do not require the zero element.

A semiring† $(R, +, \cdot, 1)$ is a set $R$ equipped with binary operations $+$ and $\cdot$ such that:

- $(R, +)$ is an Abelian semigroup;
- $(R, \cdot, 1_R)$ is a monoid with identity element $1_R$;
- Multiplication distributes over addition.

A semifield† is a semiring† in which every element is (multiplicatively) invertible. In particular, the max-plus algebras $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, and $(\mathbb{R}, +)$ are semifields†, since $+$ now is the multiplication.

A semiring is a semiring† with a zero element $0_R$ satisfying

$$a + 0_R = a, \quad a \cdot 0_R = 0_R = 0_R \cdot a, \quad \forall a \in R.$$ 

We use semirings† instead of semirings since the zero element can be adjoined formally, and often is irrelevant. For example, the zero element of the max-plus algebra would be $-\infty$, which requires special attention.

A semifield is a semifield† with a zero element adjoined. Note that under this definition the customary field $\mathbb{Q}$ with the usual operations is not a semifield, since $\mathbb{Q} \setminus \{0\}$ is not closed under addition.

Any ordered Abelian monoid gives rise to a max-plus semiring†, where the operations are written $\circ$ and $\oplus$ and defined by:

$$a \circ b := \max\{a, b\}; \quad a \oplus b := a + b.$$ 

Associativity and distributivity (of $\circ$ over $\oplus$) hold, but NOT negation, since $a \circ b \neq -\infty$ unless $a = b = -\infty$. Although the circle notation is standard in the tropical literature, we find it difficult to read when dealing with algebraic formulae. (Compare $x^4 + 7x^3 + 4x + 1$ with $x \circ x \circ x \circ x + 7 \circ x \circ x \circ 4 \circ x \circ 1$.)

Thus, when appealing to the abstract theory of semirings we use the usual algebraic notation of $\cdot$ (often suppressed) and $+$ respectively for multiplication and addition.

The max-plus algebra satisfies the property that $a + b \in \{a, b\}$; we call this property bipotence. In particular, the max-plus algebra, viewed as a semiring†, is idempotent in the sense that $a + a = a$ for all $a$. Although idempotence pervades the theory, it turns out that what is really crucial for many applications is the following fact:

**Remark 2.2.** In any idempotent semiring†, if $a + b + c = a$, then $a + b = a$. (Proof: $a = a + b + c = (a + b + c) + b = a + b$.)

Let us call such a semiring† proper. Note that a proper semiring cannot have additive inverses other than 0, since if $c + a = 0$, then $a = a + 0 = a + c + a$, implying $a = a + c = 0$.

Any proper semiring† $R$ gives rise to a partial order, given by $a \leq b$ iff $a + c = b$ for some $c \in R$. This is a total order when the semiring† $R$ is bipotent. Thus, the categories of bipotent semirings† and ordered monoids are isomorphic, and each language has its particular advantages.

2.3. The function semiring†.

**Definition 2.3.** The function semiring† $\text{Fun}(S, R)$ is the set of functions from a set $S$ to a semiring† $R$.

$\text{Fun}(S, R)$ becomes a semiring† under componentwise operations, and is proper when $R$ is proper. Customarily one takes $S = R^{(n)}$, the Cartesian product of $n$ copies of $R$. This definition enables us to work with proper subsets, but the geometric applications lie outside the scope of the present paper.

2.3.1. Polynomials and power series. $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ always denotes a finite set of indeterminates commuting with the semiring† $R$; often $n = 1$ and we have a single indeterminate $\lambda$. We have the polynomial semiring† $R[\Lambda]$. As in [LR1], we view polynomials in $R[\Lambda]$ as functions, but perhaps viewed over some extension $R'$ of $R$. More precisely, for any subset $S \subseteq R^{(n)}$, there is a natural semiring† homomorphism

$$\psi : R[\Lambda] \to \text{Fun}(S, R),$$

(2.2)
obtained by viewing a polynomial as a function on $S$.

When $R$ is a semifield, the same analysis is applicable to Laurent polynomials $R[\Lambda, \Lambda^{-1}]$, since the homomorphism $\lambda_i \mapsto a_i$ then sends $\lambda_i^{-1} \mapsto a_i^{-1}$. Likewise, when $R$ is power-cancellative and divisible, we can also define the semiring of rational polynomials $R[\Lambda, \text{rat}]$, where the powers of the $\lambda_i$ are taken to be arbitrary rational numbers. These can all be viewed as elementary formulas in the appropriate languages, so the model theory alluded to earlier is applicable to the appropriate polynomials and their (tropical) roots in each case.

Other functions over the bipotent semiring $\mathbb{R}$ of an ordered monoid $\mathcal{M}$ can be defined in the same way. For example, if $\mathcal{M}$ is an ordered submonoid of $(\mathbb{R}^+, \cdot)$, then we can define the formal exponential series

$$\exp(a) := \sum_k \frac{a^k}{k!}$$

(2.3)

since $a < m$ implies $\frac{a^{m+1}}{(m+1)!} < \frac{a^m}{m!}$, and thus (2.3) becomes a finite sum. It follows at once that $\exp(\lambda) := \sum \frac{\lambda^k}{k!}$ is defined in $\text{Fun}(R, R)$.

2.4. Puiseux series and valuations. Since logarithms often do not work well with algebraic structure, tropicalists have turned to the algebra of Puiseux series, denoted $K$, whose elements have the form

$$p(t) = \sum_{\tau \in \mathbb{Q}_{\geq 0}, c_\tau \in K} c_\tau t^\tau,$$

where the powers of $t$ are taken over well-ordered subsets of $\mathbb{Q}$. Here $K$ is any algebraically closed field of characteristic 0, customarily $C$. Intuitively, we view $t$ as a “generic element.” In the literature, the powers $\tau$ are often taken in $K$ rather than $\mathbb{Q}$, but it is enough to work with $K$, for which it much easier to compute the powers of $t$. The algebra $K$ is an algebraically closed field.

Now recall that a valuation from an integral domain $W$ to an ordered monoid $(\mathcal{G}, +)$ is a multiplicative monoid homomorphism $v : W \setminus \{0\} \to \mathcal{G}$, i.e., with

$$v(ab) = v(a) + v(b),$$

and satisfying the property $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a, b \in K$. We formally put $v(0) = \infty$. For example, the field of Puiseux series has the order valuation $v$ given by

$$v(p(t)) := \min\{\tau \in \mathbb{Q}_{\geq 0} : c_\tau \neq 0\}.$$

As $t \to 0$, the dominant term in $p(t)$ becomes $c_{v(p(t))} t^{v(p(t))}$.

The following basic observation in valuation theory shows why valuations are relevant to the tropical theory.

**Remark 2.4.** If $v(a) \neq v(b)$, then $v(a + b) = \min\{v(a), v(b)\}$. Inductively, if $v(a_1), \ldots, v(a_m)$ are distinct, then

$$v\left(\sum_{i=1}^m a_i\right) = \min\{v(a_i) : 1 \leq i \leq m\} \in \mathcal{G}.$$

Consequently, if $\sum a_i = 0$, then at least two of the $v(a_i)$ are the same. These considerations are taken much more deeply in [Bia].

When $W$ is a field, the value monoid $\mathcal{G}$ is a group. Much information about a valuation $v : W \to \mathcal{G} \cup \{\infty\}$ can be garnered from the target $v(W)$, but valuation theory provides some extra structure:

- The **valuation ring** $O_v = \{a \in W : v(a) \geq 0\}$,
- The **valuation ideal** $P_v = \{a \in W : v(a) > 0\}$,
- The **residue ring** $\bar{W} = O_v / P_v$, a field if $W$ is a field.

For example, the valuation ring of the order valuation on the field $K$ of Puiseux series is $\{p(t) \in K : c_\tau = 0$ for $\tau \leq 0\}$, and the residue field is $K$.

We replace $v$ by $-v$ to switch minimum to maximum, and $\infty$ by $-\infty$. One can generalize the notion of valuation to permit $W$ to be a semiring $\mathbb{R}$; taking $W = \mathcal{M}$, we see that the identity map is a valuation, which provides one of our main examples.
2.5. The standard supertropical semiring\(^{\dagger}\). This construction, following [IzR1], refines the max-plus algebra and picks up the essence of the value monoid. From now on, in the spirit of max-plus, we write the operation of an ordered monoid \(\mathcal{M}\) as multiplication.

We start with an Abelian monoid \(\mathcal{M} := (\mathcal{M}, \cdot)\), an ordered group \(\mathcal{G} := (\mathcal{G}, \cdot)\), and an onto monoid homomorphism \(v : \mathcal{M} \to \mathcal{G}\). We write \(a^\nu\) for \(v(a)\), for \(a \in \mathcal{M}\). Thus every element of \(\mathcal{G}\) is some \(a^\nu\). We write \(a \cong b\) if \(a^\nu = b^\nu\).

Our two main examples:

- \(\mathcal{M} = \mathcal{G}\) is the ordered monoid of the max-plus algebra (the original example in Izhakian’s dissertation);
- \(\mathcal{M}\) is the multiplicative group of a field \(F\), and \(v : F^\times \to \mathcal{G}\) is a valuation. Note that we forget the original addition on the field \(F\!\!\).

Our objective is to use the order on \(\mathcal{G}\) to study \(\mathcal{M}\). Accordingly we want to define a structure on \(\mathcal{M} \cup \mathcal{G}\).

The standard supertropical semiring\(^{\dagger}\) \(R\) is the disjoint union \(\mathcal{M} \cup \mathcal{G}\), made into a monoid by starting with the given multiplications on \(\mathcal{M}\) and \(\mathcal{G}\), and defining \(a \cdot b^\nu\) and \(a^\nu \cdot b\) to be \((ab)^\nu\) for \(a, b \in \mathcal{M}\). We extend \(v\) to the ghost map \(\nu : R \to \mathcal{G}\) by taking \(\nu|_\mathcal{M} = v\) and \(\nu|_\mathcal{G}\) to be the identity on \(\mathcal{G}\). Thus, \(\nu\) is a monoid projection.

We make \(R\) into a semiring\(^{\dagger}\) by defining

\[
a + b = \begin{cases} 
a & \text{for } a^\nu > b^\nu; \\
b & \text{for } a^\nu < b^\nu; \\
a^\nu & \text{for } a^\nu = b^\nu. \end{cases}
\]

\(R\) is never additively cancellative (except for \(\mathcal{M} = \{1\}\)). \(\mathcal{M}\) is called the tangible submonoid of \(R\). \(\mathcal{G}\) is called the ghost ideal.

\(R\) is called a supertropical domain\(^{\dagger}\) when the monoid \(\mathcal{M}\) is (multiplicatively) cancellative.

Strictly speaking, a supertropical domain\(^{\dagger}\) will not be a semifield\(^{\dagger}\) since the ghost elements are not invertible. Accordingly, we define a 1-semifield\(^{\dagger}\) to be a supertropical domain\(^{\dagger}\) for which \(\mathcal{M}\) is a group.

Motivation: The ghost ideal \(\mathcal{G}\) is to be treated much the same way that one treats the zero element in commutative algebra. Towards this end, we write

\[
a \models b_{\text{gs}} \quad \text{if} \quad a = b \text{ or } a = b + \text{ghost.}
\]

(Accordingly, write \(a \models 0_{\text{gs}}\) if \(a\) is a ghost.) Note that for \(a\) tangible, \(a \models b\) iff \(a = b\). If needed, we could formally adjoin a zero element in a separate component; then the ghost ideal is \(\mathcal{G} \!: = \mathcal{G} \cup \{0\}\). We may think of the ghost elements as uncertainties in classical algebra arising from adding two Puiseux series whose lowest order terms have the same degree.

\(R\) is a cover of the max-plus algebra of \(\mathcal{G}\), in which we “resolve” tangible idempotence, in the sense that \(a + a = a^\nu\) instead of \(a + a = a\).

This modification in the structure permits us to detect corner roots of tropical polynomials in terms of the algebraic structure, by means of ghosts. Namely, we say that \(a \in R^{(n)}\) is a root of a polynomial \(f \in R[\Lambda]\) when \(f(a) \in \mathcal{G}\). This concise formulation enables us to apply directly many standard mathematical concepts from algebra, algebraic geometry, category theory, and model theory, as described in [IzKR1–IzKR3] and [IzR6].

The standard supertropical semiring works well with linear algebra, as we shall see.

2.6. Kapranov’s Theorem and the exploded supertropical structure. Given a polynomial \(f(\Lambda) = \sum_i p_i(\lambda_1^{i_1} \cdots \lambda_n^{i_n}) \in k[\Lambda]\), where \(i = (i_1, \ldots, i_n)\), i.e., with each \(p_i\) a Puiseux series, we define its tropicalization \(\hat{f}\) to be the tropical polynomial \(\sum_i v(p_i) \lambda_1^{i_1} \cdots \lambda_n^{i_n}\). (In the tropical literature, this is customarily written in the circle notation.) By Remark 2.2 if \(a \in k^{(n)}\) is a root of \(f\) in the classical sense, then \(v(a)\) is a tropical root of \(\hat{f}\). Kapranov showed, conversely, that any tropical root of \(\hat{f}\) has the form \(v(a)\) for suitable \(a \in k^{(n)}\), and valuation theory can be applied to give a rather quick proof of this fact, although we are not aware of an explicit reference. (See [R1] Proposition 12.58 for an analogous proof of a related valuation-theoretic result.)

To prove Kapranov’s theorem, one needs more than just the lowest powers of the Puiseux series appearing as coefficients of \(f\), but also their coefficients; i.e., we also must take into account the residue field of the
order valuation on Puiseux series. Thus, we need to enrich the supertropical structure to include this extra information. This idea was first utilized by Parker [Par] in his “exploded” tropical mathematics. Likewise, Kapranov’s Theorem has been extended by Payne [Pay1, Pay2], for which we need the following more refined supertropical structure, initiated by Sheiner [ShSh].

**Definition 2.5.** Given a valuation $v : W \rightarrow \mathcal{G}$, we define the **exploded** supertropical algebra $R = W \times \mathcal{G}$, viewed naturally as a monoid. (Thus we are mixing the “usual” world with the tropical world.)

We make $R$ into a semiring$^\dagger$ by defining

$$(c, a) + (d, b) = \begin{cases} (c, a) \quad & \text{when } a > b; \\ (d, b) \quad & \text{when } a < b; \\ (c + d, a) \quad & \text{when } a = b. \end{cases}$$

Sheiner’s theory parallels the standard supertropical theory, where now the ghost elements are taken to be the 0-layer $\{0\} \times \mathcal{G}$.

3. **The layered structure**

The standard supertropical theory has several drawbacks. First, it fails to detect the multiplicity of a root of a polynomial. For example we would want $3$ to have multiplicity $5$ as a tropical root of the tropical polynomial $(\lambda + 3)^5$; this is not indicated supertropically. Furthermore, serious difficulties are encountered when attempting to establish a useful intrinsic differential calculus on the supertropical structure. Also, some basic supertropical verifications require ad hoc arguments.

These drawbacks are resolved by refining the ghost ideal into different “layers,” following a construction of [WW] Example 3.4 and [AKCC] Proposition 5.1. Rather than a single ghost layer, we take an indexing set $L$ which itself is a partially ordered semiring$^\dagger$; often $L = \mathbb{N}$ under classical addition and multiplication.

Ordered semirings$^\dagger$ can be trickier than ordered groups, since, for example, $a > b$ in $(\mathbb{R}, \cdot)$ does not imply $-a > -b$, but rather $-a < -b$. To circumvent this issue, we require all elements in the indexing semiring$^\dagger$ to be non-negative.

**Construction 3.1 ( [IZKR3] Construction 3.2).** Suppose we are given a cancellative ordered monoid $\mathcal{G}$, viewed as a semiring$^\dagger$ as above. For any partially ordered semiring$^\dagger$ $L$ we define the semiring$^\dagger$ $R := \mathcal{R}(L, \mathcal{G})$ to be set-theoretically $L \times \mathcal{G}$, where we denote the “layer” $\{\ell\} \times \mathcal{G}$ as $R_\ell$ and the element $(\ell, a)$ as $[\ell]a$; we define multiplication componentwise, i.e., for $k, \ell \in L$, $a, b \in \mathcal{G}$,

$$(k)_{a} \cdot [\ell]_{b} = [k\ell]_{(ab)}, \quad (3.1)$$

and addition via the rules:

$$[k]_{a} + [\ell]_{b} = \begin{cases} [k]_{a} & \text{if } a > b, \\ [\ell]_{b} & \text{if } a < b, \\ [k+\ell]_{a} & \text{if } a = b. \end{cases} \quad (3.2)$$

The sort map $s : R \rightarrow L$ is given by $s([1]a) = k$.

$R$ is indeed a semiring$^\dagger$. We identify $a \in \mathcal{G}$ with $[1]_{a} \in R_1$.

In most applications the “sorting” semiring$^\dagger$ $L$ is ordered, and its smallest nonzero element is 1. In this case, the monoid $\{[\ell]a : 0 < \ell \leq 1\}$ is called the tangible part of $R$. The ghosts are $\{[\ell]a : \ell > 1\}$, and correspond to the ghosts in the standard supertropical theory. The ghosts together with $R_0$ comprise an ideal. If there is a zero element it would be $[0]0$.

One can view the various choices of the sorting semiring$^\dagger$ $L$ as different stages of degeneration of algebraic geometry, where the crudest (for $L = \{1\}$) is obtained by passing directly to the familiar max-plus algebra. The supertropical structure is obtained when $L = \{1, \infty\}$, where $R_1$ and $R_\infty$ are two copies of $\mathcal{G}$, with $R_1$ the tangible submonoid of $R$ and $R_\infty$ being the ghost copy. Other useful choices of $L$ include $\{1, 2, \infty\}$ (to distinguish between simple roots and multiple roots) and $\mathbb{N}$, which enables us to work with the multiplicity of roots and with derivatives, as seen below. In order to deal with tropical integration as anti-differentiation, one should consider the sorting semirings$^\dagger$ $\mathbb{Q}_{>0}$ and $\mathbb{R}_{>0}$, but this is outside our present scope.

By convention, $[\ell]\lambda$ denotes $[\ell]\mathbf{1}_R \lambda$. Thus, any monomial can be written in the form $[\ell]_{a_1} \lambda^{i_1} \cdots \lambda^{i_n}$, where $i = (i_1, \ldots, i_n)$. We say a polynomial $f$ is tangible if each of its coefficients is tangible.
Note that the customary decomposition $R = \bigoplus_{\ell \in L} R_{\ell}$ in graded algebras has been strengthened to the partition $R = \bigcup_{\ell \in L} R_{\ell}$. The ghost layers now indicate the number of monomials defining a corner root of a tangible polynomial. Thus, we can measure multiplicity of roots by means of layers. For example,

$$(\lambda + 3)^5 = [1]\lambda^5 + [5]3\lambda^4 + [10]6\lambda^3 + [10]9\lambda^2 + [5]12\lambda + [1]15,$$

and substituting 3 for $\lambda$ gives $[3^2]15 = [2^2]3^5$.

### 3.1. Layered derivatives

Formal derivatives are not very enlightening over the max-plus algebra. For example, if we take the polynomial $f = \lambda^2 + 5\lambda + 8$, which has corner roots 3 and 5, we have $f' = 2\lambda + 5$, having corner root 3, but the common corner root 3 of $f$ and $f'$ could hardly be considered a multiple root of $f$. This difficulty arises from the fact that $1 \neq 2$ in the max-plus algebra. The layering permits us to define a more useful version of the derivative (where now $R$ contains a zero element $0_R$):

**Definition 3.2.** The **layered derivative** $f'_{\text{lay}}$ of $f$ on $R[\lambda]$ is given by:

$$
\left(\sum_{j=0}^{n} [j]_{\text{lay}} \alpha_j \lambda^j\right)'_{\text{lay}} := \sum_{j=1}^{n} [j+1]_{\text{lay}} \alpha_j \lambda^{j-1}.
$$

(3.3)

In particular, for $\alpha = [1]_{\text{lay}}\in R_1$,

$$(\alpha \lambda^j)'_{\text{lay}} := [j]_{\text{lay}} \lambda^{j-1} \quad (j \geq 2), \quad (\alpha \lambda)'_{\text{lay}} := \alpha, \quad \text{and} \quad \alpha'_{\text{lay}} := 0_R.$$

Thus, we have the familiar formulas:

1. $(f + g)'_{\text{lay}} = f'_{\text{lay}} + g'_{\text{lay}}$;
2. $(fg)'_{\text{lay}} = f'_{\text{lay}}g + fg'_{\text{lay}}$

This is far more informative in the layered setting (say for $L = \mathbb{N}$) than in the standard supertropical setting, in which $(\alpha \lambda)'$ is ghost for all $j \geq 2$.

### 3.2. The tropical Laplace transform

The classical technique of Laplace transforms has a tropical analog which enables us to compare the various notions of derivative. Suppose $L$ is infinite, say $L = \mathbb{N}$ . Formally permitting infinite vectors $(a_\ell)_{\ell \in L}$ permits us to define a homomorphism $R[[\Lambda]] \to \mathcal{A}(L, R)$ given by

$$\sum a_{\ell} \lambda^\ell \mapsto ([k]! a_{\ell}).$$

(Strictly speaking, we would want the image to be $([k]! a_{\ell})$, but this would complicate the notation and require us to take $L = \mathbb{Q}^+$. For example, $\exp_{\text{lay}}(a) \mapsto ([k]! a_{\ell})$ where each $a_{\ell} = a$.)

Now we define $([\ell]! a_{\ell})' = ([\ell-1]! a_{\ell})$. Then $\exp'_{\text{lay}} = \exp_{\text{lay}}$. This enables one to handle trigonometric functions in the layered theory.

### 3.3. Layered domains with symmetry, and patchworking

Akian, Gaubert, and Guterman [AkGG] Definition 4.1] introduced an involutory operation on semirings, which they call a **symmetry**, to unify the supertropical theory with classical ring theory. One can put their symmetry in the context of $\mathcal{A}(L, \mathcal{G})$.

**Definition 3.3.** A **negation map** on a semiring $L$ is a function $\tau : L \to L$ satisfying the properties:

| N1. | $\tau(k\ell) = \tau(k)\ell = k\tau(\ell)$; |
| N2. | $\tau^2(k) = k$; |
| N3. | $\tau(k + \ell) = \tau(k) + \tau(\ell)$. |

Suppose the semiring $L$ has a negation map $\tau$ of order $\leq 2$. We say that $R := \mathcal{A}(L, \mathcal{G})$ has a **symmetry** $\sigma$ when $R$ is endowed with a map

$$\sigma : R \to R$$

and a negation map $\tau$ on $L$, together with the extra axiom:

| S1. | $s(\sigma(a)) = \tau(s(a))$, $\forall a \in R$. |
Example 3.4. Suppose $L$ is an ordered semiring$^\dagger$. We mimic the well-known construction of $\mathbb{Z}$ from $\mathbb{N}$. Define the doubled semiring$^\dagger$

$$D(L) = L_1 \times L_{-1},$$

the direct product of two copies $L_1$ and $L_{-1}$ of $L$, where addition is defined componentwise, but multiplication is given by

$$(k, \ell) \cdot (k', \ell') = (kk' + \ell\ell', kk' + \ell k').$$

In other words, $D(L)$ is multiplicatively graded by $\{\pm 1\}$.

$D(L)$ is endowed with the product partial order, i.e., $(k', \ell') \geq (k, \ell)$ when $k' \geq k$ and $\ell' \geq \ell$.

Here is an example relating to “patchworking.” [ILMS].

Example 3.5. Suppose $\mathcal{G}$ is an ordered Abelian monoid, viewed as a semiring$^\dagger$ as in Construction 3.4. Define the doubled layered domain$^\dagger$

$$R = \mathcal{R}(D(L), \mathcal{G}) = \{(k, \ell), a : (k, \ell) \neq (0, 0), a \in \mathcal{G}\},$$

but with addition and multiplication given by the following rules:

$$(k, \ell), a + (k', \ell'), b = \begin{cases} (k, \ell), a & \text{if } a > b, \\ (k', \ell'), b & \text{if } a < b, \\ (k + k', \ell + \ell'), a & \text{if } a = b. \end{cases}$$

$$(k, \ell), a \cdot (k', \ell'), b = (kk' + \ell\ell', kk' + \ell k', ab).$$

Remark 3.6. In $R = \mathcal{R}(D(L), \mathcal{G})$, the symmetry $\sigma : R \to R$ given by $\sigma : ((k, \ell), a) \mapsto ((\ell, k), a)$ is analogous to the one described in [AKGG], and behaves much like negation.

For example, when $L = \{1, \infty\}$, we note that $D(L) = \{(1, 1), (1, \infty), (\infty, 1), (\infty, \infty)\}$, which is applicable to Viro’s theory of patchworking, where the “tangible” part could be viewed as those elements of layer $(1, 1), (1, \infty),$ or $(\infty, 1)$. Explicitly, comparing with Viro’s use of hyperfields in [VI, § 3.5], we identify these three layers respectively with $0, 1,$ and $-1$ in his terminology, and the element $(\infty, \infty)$ with the set $\{0, 1, -1\}$.

4. Matrices and linear algebra

As an application, the supertropical and layered structures provide many of the analogs to the classical Hamilton-Cayley-Frobenius theory. $M_n(R)$ denotes the semiring$^\dagger$ of $n \times n$ matrices over a semiring $R$. (Note that the familiar matrix operations do not require negation.)

Although one of the more popular and most applicable aspects of idempotent mathematics, idempotent matrix theory is handicapped by the lack of an element $-1$ with which to construct the determinant. Many ingenious methods have been devised to circumvent this difficulty, as surveyed in [AKBG]; also cf. [AkGG] and many interesting papers in this volume. Unfortunately these give rise to many different notions of rank of matrix, and often are difficult to understand. The layered (and more specifically, supertropical) theories give a unified and relatively straightforward notion of rank of a matrix, eigenvalue, adjoint, etc.

4.1. The supertropical determinant. This discussion summarizes [ZR3]. We define the supertropical determinant $|A|$ of a matrix $A = (a_{i,j})$ to be the permanent:

$$|A| = \sum_{\pi \in S_n} a_{1, \pi(1)} \cdots a_{n, \pi(n)}.$$  \hspace{1cm} (4.1)

Defining the transpose matrix $(a_{i,j})^t$ to be $(a_{j,i})$, we have

$$|(a_{i,j})^t| = |(a_{i,j})|.$$  \hspace{1cm} (4.2)

$|A| = 0_R$ iff “enough” entries are $0_R$ to force each summand in Formula (4.1) to be $0_R$. This property, which in classical matrix theory provides a description of singular subspaces, is too strong for our purposes. We now take the natural supertropical version. Write $\mathcal{T}$ for the tangible elements of our supertropical semiring $R$, and $\mathcal{T} = \mathcal{T} \cup \{0\}$.

Definition 4.1. A matrix $A$ is nonsingular if $|A| \in \mathcal{T}$; $A$ is singular when $|A| \in \mathcal{G}$.
The standard supertropical structure often is sufficient for matrices, since it enables us to distinguish between nonsingular matrices (in which the tropical $n \times n$ determinant is computed as the unique maximal product of $n$ elements in one track) and singular matrices.

The tropical determinant is not multiplicative, as seen by taking the nonsingular matrix $A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$. Then $A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is singular and $|A^2| = 5^2 \neq 2 \cdot 2$. But we do have:

**Theorem 4.2.** For any $n \times n$ matrices over a supertropical semiring $R$, we have

$$|AB| \leq |A| |B|_{\text{gs}}.$$

In particular, $|AB| = |A| \cdot |B|$ whenever $|AB|$ is tangible.

We say a permutation $\sigma \in S_n$ attains $|A|$ if $|A| \cong a_{\sigma(1)} \cdots a_{\sigma(n)}$.

- By definition, some permutation always attains $|A|$.
- If there is a unique permutation $\sigma$ which attains $|A|$, then $|A| = a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$.
- If at least two permutations attain $|A|$, then $A$ must be singular. Note in this case that if we replaced all nonzero entries of $A$ by tangible entries of the same $\nu$-value, then $A$ would still be singular.

### 4.2. Quasi-identities and the adjoint.

**Definition 4.3.** A quasi-identity matrix $I_G$ is a nonsingular, multiplicatively idempotent matrix equal to $I + Z_G$, where $Z_G = 0_R$ on the diagonal, and whose off-diagonal entries are ghosts or $0_R$.

$$|I_G| = 1_R$$

by the nonsingularity of $I_G$. Also, for any matrix $A$ and any quasi-identity, $I_G$, we have $AI_G = A + AG$, where $AG = AZ_G \in M_n(G)$.

There is another notion to help us out.

**Definition 4.4.** The $(i,j)$-minor $A'_{i,j}$ of a matrix $A = (a_{i,j})$ is obtained by deleting the $i$ row and $j$ column of $A$. The adjoint matrix $\text{adj}(A)$ of $A$ is defined as the transpose of the matrix $(a'_{i,j})$, where $a'_{i,j} = |A'_{i,j}|$.

**Remark 4.5.**

(i) Suppose $A = (a_{i,j})$. An easy calculation using Formula (4.1) yields

$$|A| = \sum_{j=1}^{n} a_{i,j} a'_{i,j}, \quad \forall i. \quad (4.2)$$

Consequently, $a_{i,j} a'_{i,j} \leq |A|$ for each $i,j$.

(ii) If we take $k \neq i$, then replacing the $i$ row by the $k$ row in $A$ yields a matrix with two identical rows; thus, its determinant is a ghost, and we thereby obtain

$$\sum_{j=1}^{n} a_{i,j} a'_{k,j} \in G, \quad \forall k \neq i; \quad (4.3)$$

Likewise

$$\sum_{j=1}^{n} a_{j,i} a'_{j,k} \in G, \quad \forall k \neq i.$$

One easily checks that $\text{adj}(B) \text{adj}(A) = \text{adj}(AB)$ for any $2 \times 2$ matrices $A$ and $B$. However, this fails for larger $n$, cf. [ZSR Example 4.7]. We do have the following fact, which illustrates the subtleties of the supertropical structure, cf. [ZSR Proposition 5.6]:

**Proposition 4.6.** $\text{adj}(AB) = \text{adj}(B) \text{adj}(A) + \text{ghost}.$

**Definition 4.7.** For $|A|$ invertible, define

$$I_A = A \frac{\text{adj}(A)}{|A|}, \quad I'_A = A \frac{\text{adj}(A)}{|A|}.$$
The matrices $I_A$ and $I'_A$ are quasi-identities, as seen in [LZR3, Theorem 4.13]. The main technique of proof is to define a string (from the matrix $A$) to be a product $a_{i_1,j_1} \cdots a_{i_k,j_k}$ of entries from $A$ and, given such a string, to define its digraph to be the graph whose edges are $(i_1,j_1), \ldots, (i_k,j_k)$, counting multiplicities. A $k$-multicycle in a digraph is the union of disjoint simple cycles, the sum of whose lengths is $k$; thus every vertex in an $n$-multicycle appears exactly once. A careful examination of the digraph in conjunction with Hall’s Marriage Theorem yields the following major results from [LZR3 Theorem 4.9 and Theorem 4.12]:

**Theorem 4.8.**

(i) $|A| \text{adj}(A) = |A|^n$.

(ii) $|\text{adj}(A)| = |A|^{n-1}$.

In case $A$ is a nonsingular, we define $A^\nabla = \frac{\text{adj}(A)}{|A|}$. Thus $AA^\nabla = I_A$, and $A^\nabla A = I'_A$. Note that $I'_A$ and $I_A$ may differ off the diagonal, although $I_A A = AA^\nabla A = AI'_A$.

This result is refined in [LZR4 Theorem 2.18]. One might hope that $A \text{adj}(A) A = |A| A$, but this is false in general! The difficulty is that one might not be able to extract an $n$-multicycle from

(a) $a_{i,j} \alpha_{k,l} a_{k,l}$. (4.4)

For example, when $n = 3$, the term $a_{1,1}(a_{1,3} a_{3,2}) a_{2,2} = a_{1,1} a'_{2,1} a_{2,2}$ does not contain an $n$-multicycle. We do have the following positive result from [LZR4 Theorem 4.18]:

**Theorem 4.9.** $\text{adj}(A) \text{adj}(\text{adj}(A)) \text{adj}(A) \cong |A|^{n-1} \text{adj}(A)$ for any $n \times n$ matrix $A$.

4.3. The supertropical Hamilton-Cayley theorem.

**Definition 4.10.** Define the characteristic polynomial $f_A$ of the matrix $A$ to be $f_A = |\lambda I + A|$, and the tangible characteristic polynomial to be a tangible polynomial $f_A^\nabla = \lambda^n + \sum_{i=1}^{n} \alpha_i \lambda^{n-i}$, where $\alpha_i$ are tangible and $\alpha_i \cong \alpha_i$, such that $f_A = \lambda^n + \sum_{i=1}^{n} \alpha_i \lambda^{n-i}$.

Under this notation, we see that $\alpha_k \in R$ arises from the dominant $k$-multicycles in the digraph of $A$. We say that a matrix $A$ satisfies a polynomial $f \in R[\lambda]$ if $f(A) \in M_n(G)$. A

**Theorem 4.11.** (Supertropical Hamilton-Cayley, [LZR3 Theorem 5.2]) Any matrix $A$ satisfies both its characteristic polynomial $f_A$ and its tangible characteristic polynomial $f_A^\nabla$.

4.4. Tropical dependence. Now we apply supertropical matrix theory to vectors. As in classical mathematics, one defines a module (often called semi-module in the literature) analogously to module in classical algebra, noting again that negation does not appear in the definition. It is convenient to stipulate that the module $V$ has a zero element $0_V$, and then we need the axiom:

$$a0_V = 0_V \text{ for all } a \in R.$$  

Also, if $0 \in R$ then we require that $0v = 0_V$ for all $v \in V$.

In what follows, $F$ always denotes a 1-semifield. In this case, a module over $F$ is called a (supertropical) vector space. The natural example is $F(n)$, with componentwise operations. As in the classical theory, there is the usual familiar correspondence between the semiring $M_n(F)$ and the linear transformations of $F(n)$.

For $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in F(n)$, we write $v \equiv w$ when $v_i \equiv w_i$ for all $1 \leq i \leq n$.

Here is an application of the adjoint matrix, used to solve equations.

**Remark 4.12.** Suppose $A$ is nonsingular, and $v \in F(n)$. Then the equation $Aw = v$ has the solution $w = A^\nabla v$. Indeed, writing $I_A = I + Z_G$ for a ghost matrix $Z_G$, we have

$$Aw = AA^\nabla v = I_A v = (I + Z_G)v \equiv v.$$  

This leads to the supertropical analog of Cramer’s rule [LZR4 Theorem 3.5]:
Theorem 4.13. If $A$ is a nonsingular matrix and $v$ is a tangible vector, then the equation $Ax = v$ has a solution over $F$ which is the tangible vector having value $A^\gs v$.

Our next task is to characterize singularity of a matrix $A$ in terms of “tropical dependence” of its rows. In some ways the standard supertropical theory works well with matrices, since we are interested mainly in whether or not this matrix is nonsingular, i.e., if its determinant is tangible; at the outset, at least, we are not concerned with the precise ghost layer of the determinant.

Definition 4.14. A subset $W \subset F^{(n)}$ is tropically dependent if there is a finite sum $\sum \alpha_i w_i \in W^{(n)}$, with each $\alpha_i \in T$, but not all of them 0; otherwise $W \subset F^{(n)}$ is called tropically independent. A vector $v \in F^{(n)}$ is tropically dependent on $W$ if $W \cup \{v\}$ is tropically dependent.

By [IzKR2] Proposition 4.5, we have:

Proposition 4.15. Any $n + 1$ vectors in $F^{(n)}$ are tropically dependent.

Theorem 4.16. ([IZR3] Theorem 6.5.) Vectors $v_1, \ldots, v_n \in F^{(n)}$ are tropically dependent, iff the matrix whose rows are $v_1, \ldots, v_n$ is singular.

Corollary 4.17. The matrix $A \in M_n(F)$ over a supertropical domain $F$ is nonsingular iff the rows of $A$ are tropically independent, iff the columns of $A$ are tropically independent.

Proof. Apply the theorem to $|A|$ and $|A^t|$, which are the same. \hfill \Box

There are two competing supertropical notions of base of a vector space, that of a maximal independent set of vectors, and that of a minimal spanning set, but this is unavoidable since, unlike the classical theory, these two definitions need not coincide.

4.5. Tropical bases and rank. The customary definition of tropical base, which we call s-base (for spanning base), is a minimal spanning set (when it exists). However, this definition is rather restrictive, and a competing notion provides a richer theory.

Definition 4.18. A d-base (for dependence base) of a vector space $V$ is a maximal set of tropically independent elements of $V$. A d,s-base is a d-base which is also an s-base. The rank of a set $B \subseteq V$, denoted $\text{rank}(B)$, is the maximal number of d-independent vectors of $B$.

Our d-base corresponds to the “basis” in [MS] Definition 5.2.4. In view of Proposition 4.15 all d-bases of $F^{(n)}$ have precisely $n$ elements.

This leads us to the following definition.

Definition 4.19. The rank of a vector space $V$ is defined as:

$$\text{rank}(V) := \max \{ \text{rank}(B) : B \text{ is a d-base of } V \}.$$

We have just seen that $\text{rank}(F^{(n)}) = n$. Thus, if $V \subseteq F^{(n)}$, then $\text{rank}(V) \leq n$.

We might have liked $\text{rank}(V)$ to be independent of the choice of d-base of $V$, for any vector space $V$. This is proved in the classical theory of vector spaces by showing that dependence is transitive. However, transitivity of dependence fails in the supertropical theory, and, in fact, different d-bases may contain different numbers of elements, even when tangible. An example is given in [MS] Example 5.4.20, and reproduced in [IzKR2] Example 4.9 as being a subspace of $F^{(4)}$ having d-bases both of ranks 2 and 3.

Example 4.20. The matrix $A = \begin{pmatrix} 4 & 4 & 0 \\ 4 & 4 & 1 \\ 4 & 4 & 2 \end{pmatrix}$ has rank 2, but is “ghost annihilated” by the tropically independent vectors $v_1 = (1, 1, 0)^t$ and $v_2 = (1, 1, 1)^t$; i.e., $Av_1 = Av_2 = (5^v, 5^v, 5^v)^t$, although $2 + 2 > 3$.

We do have some consolations.

Proposition 4.21 ([IzKR2] Proposition 4.11]). For any tropical subspace $V$ of $F^{(n)}$ and any tangible $v \in V$, there is a tangible d-base of $V$ containing $v$ whose rank is that of $V$.

Proposition 4.22 ([IzKR2] Proposition 4.13]). Any $n \times n$ matrix of rank $m$ has ghost annihilator of rank $\geq n - m$. 
4.5.1. Semi-additivity of rank.

**Definition 4.23.** A function \( \text{rank}_S : S \to \mathbb{N} \) is monotone if for all \( S_2 \subseteq S_1 \subseteq S \) we have

\[
\text{rank}_S(S_2 \cup \{s\}) - \text{rank}_S(S_1 \cup \{s\}) \geq \text{rank}_S(S_2) - \text{rank}_S(S_1)
\]

for all \( s \in S \).

Note that \[12\] says that \( \text{rank}_S(S_1) - \text{rank}_S(S_2) \geq \text{rank}_S(S_1 \cup \{s\}) - \text{rank}_S(S_2 \cup \{s\}) \). Also, taking \( S_2 = \emptyset \) yields \( \text{rank}_S(S_1 \cup \{s\}) - \text{rank}_S(S_1) \leq 1 \).

**Lemma 4.24.** If \( \text{rank}_S : S \to \mathbb{N} \) is monotone, then

\[
\text{rank}_S(S_1) + \text{rank}_S(S_2) \geq \text{rank}_S(S_1 \cup S_2) + \text{rank}_S(S_1 \cap S_2)
\]

for all \( S_1, S_2 \subset S \).

**Proof.** Induction on \( m = \text{rank}_S(S_2 \setminus S_1) \). If \( m = 0 \), i.e., \( S_2 \subseteq S_1 \), then the left side of \[12\] equals the right side. Thus we may assume that \( m \geq 1 \). Pick \( s \) in a d-base of \( S_2 \setminus S_1 \). Let \( S_2' = S_2 \setminus \{s\} \). Noting that \( \text{rank}_S(S_2' \setminus S_1) = m - 1 \), we see by induction that

\[
\text{rank}_S(S_1) + \text{rank}_S(S_2') \geq \text{rank}_S(S_1 \cup S_2') + \text{rank}_S(S_1 \cap S_2'),
\]

or (taking \( S_1 \cup S_2' \) instead of \( S_2 \) in \[12\]),

\[
\text{rank}_S(S_1) - \text{rank}_S(S_1 \cap S_2) = \text{rank}_S(S_1) - \text{rank}_S(S_1 \cap S_2') \geq \text{rank}_S(S_1 \cup S_2') - \text{rank}_S(S_2') \geq \text{rank}_S(S_1 \cup S_2) - \text{rank}_S(S_2).
\]

yielding \[12\]. \( \square \)

**Proposition 4.25.** \( \text{rank}(S_1) + \text{rank}(S_2) \geq \text{rank}(S_1 \cup S_2) + \text{rank}(S_1 \cap S_2) \) for all \( S_1, S_2 \subset S \).

**Proof.** rank is a monotone function, since each side of \[12\] is 0 or 1, depending on whether or not \( s \) is independent of \( S_i \), and only decreases as we enlarge the set. \( \square \)

4.6. Supertropical eigenvectors. The standard definition of an eigenvector of a matrix \( A \) is a vector \( v \), with eigenvalue \( \beta \), satisfying \( Av = \beta v \). It is well known \[BR\] that any (tangible) matrix has an eigenvector.

**Example 4.26.** The characteristic polynomial \( f_A \) of

\[
A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}
\]

is \((\lambda + 4)(\lambda + 1) + 0 = (\lambda + 4)(\lambda + 1)\), and the vector \((4, 0)\) is a eigenvector of \( A \), with eigenvalue 4. However, there is no eigenvector having eigenvalue 1.

In general, the lesser roots of the characteristic polynomial are “lost” as eigenvalues. We rectify this deficiency by weakening the standard definition.

**Definition 4.27.** A tangible vector \( v \) is a generalized supertropical eigenvector of a (not necessarily tangible) matrix \( A \), with generalized supertropical eigenvalue \( \beta \in T \), if \( A^m v \models \beta^m v \) for some \( m \); the minimal such \( m \) is called the multiplicity of the eigenvalue (and also of the eigenvector). A supertropical eigenvector is a generalized supertropical eigenvector of multiplicity 1.

**Example 4.28.** The matrix \( A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \) of Example \[20\] also has the tangible supertropical eigenvector \( v = (0, 4) \), corresponding to the supertropical eigenvalue 1, since

\[
Av = (4^v, 5) = 1v + (4^v, -\infty).
\]

**Proposition 4.29.** If \( v \) is a tangible supertropical eigenvector of \( A \) with supertropical eigenvalue \( \beta \), the matrix \( A + \beta I \) is singular (and thus \( \beta \) must be a (tropical) root of the characteristic polynomial \( f_A \) of \( A \)).

Conversely, we have:

**Theorem 4.30** (\[ZR3\] Theorem 7.10). Assume that \( v|_T : T \to G \) is 1:1. For any matrix \( A \), the dominant tangible root of the characteristic polynomial of \( A \) is an eigenvalue of \( A \), and has a tangible eigenvector. The other tangible roots are precisely the supertropical eigenvalues of \( A \).
Let us return to our example \( A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \). Its characteristic polynomial is \( \lambda^2 + 2\lambda + 2 = (\lambda + 0)(\lambda + 2) \), whose roots are 2 and 0. The eigenvalue 2 has tangible eigenvector \( v = (0, 2) \) since \( Av = (2, 4) = 2v \), but there are no other tangible eigenvalues. \( A \) does have the tangible supertropical eigenvalue 0, with tangible supertropical eigenvector \( w = (2, 1) \), since \( Aw = (2, 3w) = 0w + (-\infty, 3w) \). Note that \( A + 0I = \begin{pmatrix} 0^\nu & 0 \\ 1 & 2 \end{pmatrix} \) is singular, because \( |A + 0I| = 2^\nu \).

Furthermore, \( A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \) is a root of \( \lambda^2 + 4A \), and thus \( A \) is a root of \( g = \lambda^4 + 4\lambda^2 = (\lambda(\lambda + 2))^2 \), but 0 is not a root of \( g \) although it is a root of \( f_A \). This shows that the naive formulation of Frobenius’ theorem fails in the supertropical theory, and is explained in the work of Adi Niv \cite{N}.

4.7. Bilinear forms and orthogonality. One can refine the study of bases by introducing angles, i.e., orthogonality, in terms of bilinear forms. Let us quote some results from \cite{IzKR2}.

**Definition 4.31.** A (supertropical) bilinear form \( B \) on a (supertropical) vector space \( V \) is a function \( B : V \times V \to F \) satisfying

\[
B(v_1 + v_2, w_1 + w_2) = B(v_1, w_1) + B(v_1, w_2) + B(v_2, w_1) + B(v_2, w_2),
\]

for all \( \alpha \in F \) and \( v_i \in V \) and \( w_j \in V' \).

We work with a fixed bilinear form \( B = \langle \ , \ \rangle \) on a (supertropical) vector space \( V \subseteq F^n \). The **Gram matrix** of vectors \( v_1, \ldots, v_k \in F^n \) is defined as the \( k \times k \) matrix

\[
\tilde{G}(v_1, \ldots, v_k) = \begin{pmatrix}
\langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_k \rangle \\
\langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_2, v_k \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle v_k, v_1 \rangle & \langle v_k, v_2 \rangle & \cdots & \langle v_k, v_k \rangle
\end{pmatrix}.
\] (4.8)

The set \( \{v_1, \ldots, v_k\} \) is **nonsingular** (with respect to \( B \)) when its Gram matrix is nonsingular.

In particular, given a vector space \( V \) with s-base \( \{b_1, \ldots, b_k\} \), we have the matrix \( \tilde{G} = \tilde{G}(b_1, \ldots, b_k) \), which can be written as \( (g_{i,j}) \) where \( g_{i,j} = \langle b_i, b_j \rangle \). The singularity of \( \tilde{G} \) does not depend on the choice of s-base.

**Definition 4.32.** For vectors \( v, w \) in \( V \), we write \( v \perp w \) when \( \langle v, w \rangle \in G \), that is \( \langle v, w \rangle \in \mathcal{G} \), and say that \( v \) is left ghost orthogonal to \( w \). We write \( W \perp \) \( \{v \in V : v \perp w \text{ for all } w \in W\} \).

**Definition 4.33.** A subspace \( W \) of \( V \) is called **nondegenerate** (with respect to \( B \)), if \( W \perp \cap W = \text{ghost} \). The bilinear form \( B \) is **nondegenerate** if the space \( V \) is nondegenerate.

**Lemma 4.34.** Suppose \( \{w_1, \ldots, w_m\} \) tropically spans a subspace \( W \) of \( V \), and \( v \in V \). If \( \sum_{i=1}^m \beta_i \langle v, w_i \rangle \in \mathcal{G} \) for all \( \beta_i \in \mathcal{T} \), then \( v \in W \perp \).

**Theorem 4.35.** (\cite{IzKR2} Theorem 6.7) Assume that vectors \( w_1, \ldots, w_k \in V \) span a nondegenerate subspace \( W \) of \( V \). If \( \langle \tilde{G}(w_1, \ldots, w_k) \rangle \notin \mathcal{G} \), then \( w_1, \ldots, w_k \) are tropically dependent.

**Corollary 4.36.** If the bilinear form \( B \) is nondegenerate on a vector space \( V \), then the Gram matrix (with respect to any given supertropical d,s-base of \( V \)) is nonsingular.

**Definition 4.37.** The bilinear form \( B \) is **supertropical alternate** if \( \langle v, v \rangle \in \mathcal{G} \) for all \( v \in V \). \( B \) is **supertropical symmetric** if \( \langle v, w \rangle + \langle w, v \rangle \in \mathcal{G} \) for all \( v, w \in V \).

We aim for the supertropical version (\cite{IzKR2} Theorem 6.19) of a classical theorem of Artin, that any bilinear form in which ghost-orthogonality is symmetric must be a supertropically symmetric bilinear form.

**Definition 4.38.** The (supertropical) bilinear form \( B \) is **orthogonal-symmetric** if it satisfies the following property for any finite sum, with \( v_i, w_i \in V \):
\[ \sum_{i} \langle v_i, w \rangle \in \mathcal{G} \iff \sum_{i} \langle w, v_i \rangle \in \mathcal{G}, \quad (4.9) \]

\( B \) is **supertropically orthogonal-symmetric** if \( B \) is orthogonal-symmetric and satisfies the additional property that \( \langle v, w \rangle \cong_{\nu} \langle w, v \rangle \) for all \( v, w \in V \) satisfying \( \langle v, w \rangle \in \mathcal{T} \).

The symmetry condition extends to sums, and after some easy lemmas we obtain ([IZKR2 Theorem 6.19]):

**Theorem 4.39.** Every orthogonal-symmetric bilinear form \( B \) on a vector space \( V \) is supertropically symmetric.

5. **Identities of semirings, especially matrices**

The word “identity” has several interpretations, according to its context. First of all, there are well-known matrix identities such as the Hamilton-Cayley identity which says that any matrix is a root of its characteristic polynomial.

Since the classical theory of polynomial identities is tied in with invariant theory, we also introduce layered polynomial identities (PIs), to enrich our knowledge of layered matrices.

5.1. **Polynomial identities of semirings**\( ^{\dagger} \). We draw on basic concepts of polynomial identities, i.e., PIs, say from [R2 Chapter 23]. Since semirings\( ^{\dagger} \) do not involve negatives, we modify the definition a bit.

**Definition 5.1.** The free \( \mathbb{N} \)-semiring \( \mathbb{N}\{x_1, x_2, \ldots \} \) is the monoid semiring\( ^{\dagger} \) of the free (word) monoid \( \{x_1, x_2, \ldots \} \) over the semiring\( ^{\dagger} \) \( \mathbb{N} \).

**Definition 5.2.** A (semiring\( ^{\dagger} \) ) polynomial identity (PI) of a semiring\( ^{\dagger} \) \( R \) is a pair \((f, g)\) of (noncommutative) polynomials \( f(x_1, \ldots, x_m), g(x_1, \ldots, x_m) \in \mathbb{N}\{x_1, \ldots, x_m\} \) for which

\[ f(r_1, \ldots, r_m) = g(r_1, \ldots, r_m), \quad \forall r_1, \ldots, r_m \in R. \]

We write \((f, g) \in \text{id}(R)\) when \((f, g)\) is a PI of \( R \).

**Remark 5.3.** A **semigroup identity** of a semigroup \( S \) is a pair \((f, g)\) of (noncommutative) monomials \( f(x_1, \ldots, x_m), g(x_1, \ldots, x_m) \in \mathbb{N}\{x_1, \ldots, x_m\} \) for which \( f(s_1, \ldots, s_m) = g(s_1, \ldots, s_m), \forall s_1, \ldots, s_m \in S \). If \( S \) is contained in the multiplicative semigroup of a semiring\( ^{\dagger} \) \( R \), the semigroup identities of \( S \) are precisely the semiring\( ^{\dagger} \) PIs \((f, g)\) where \( f \) and \( g \) are monomials.

Akian, Gaubert and Guterman [AKGG] Theorem 4.21] proved their **strong transfer principle**, which immediately implies the following easy but important observation:

**Theorem 5.4.** If \( f, g \in \mathbb{N}\{x_1, \ldots, x_n\} \) have disjoint supports and \( f - g \) is a PI of \( M_n(\mathbb{Z}) \), then \( f = g \) is also a semiring\( ^{\dagger} \) PI of \( M_n(\mathbb{R}) \) for any commutative semiring\( ^{\dagger} \) \( R \).

**Proof.** Since \( \mathbb{Z} \) is an infinite integral domain, \( f - g \) is also a PI of \( M_n(\mathbb{C}) \), where \( C = \mathbb{Z}[\xi_1, \xi_2, \ldots] \) denotes the free commutative ring in countably many indeterminates, implying \((f, g)\) is a semiring\( ^{\dagger} \) PI of \( M_n(\mathbb{N}[\xi_1, \xi_2, \ldots]) \). But the semiring\( ^{\dagger} \) \( M_n(\mathbb{R}) \) is a homomorphic image of \( M_n(\mathbb{N}[\xi_1, \xi_2, \ldots]) \), implying \((f, g) \in \text{id}(M_n(\mathbb{R})). \)

**Corollary 5.5.** Any PI of \( M_n(\mathbb{Z}) \) yields a corresponding semiring\( ^{\dagger} \) PI of \( M_n(\mathbb{R}) \) for all commutative semirings\( ^{\dagger} \) \( R \).

**Proof.** Take \( f \) to be the sum of the terms having positive coefficient, and \( g \) to be the sum of the terms having negative coefficient, and apply the theorem.

Many (but not all) matrix PIs can be viewed in terms of Theorem 5.4, although semiring versions of basic results such as the Amitsur-Levitzki Theorem and Newton’s Formulas often are more transparent here.

We say that polynomials \( f(x_1, \ldots, x_m) \) and \( g(x_1, \ldots, x_m) \) are a \( t \)-**alternating pair** if \( f \) and \( g \) are interchanged whenever we interchange a pair \( x_i \) and \( x_j \) for some \( 1 \leq i < j \leq t \). For example, \( x_1x_2 \) and \( x_2x_1 \) are a 2-alternating pair. Sometimes we write the non-alternating variables as \( y_1, y_2, \ldots \); we write \( y \) as shorthand for all the \( y_j \).
Definition 5.6. We partition the symmetric group $S_t$ of permutations in $t$ letters into the even permutations $S^+_t$ and the odd permutations $S^-_t$. Given a $t$-linear polynomial $h(x_1, \ldots, x_t; y)$, we define the $t$-alternating pair

$$h^+_{alt}(x_1, \ldots, x_t; y) := \sum_{\sigma \in S^+_t} h(x_{\sigma(1)}, \ldots, x_{\sigma(t)}; y)$$

and

$$h^-_{alt}(x_1, \ldots, x_t; y) := \sum_{\sigma \in S^-_t} h(x_{\sigma(1)}, \ldots, x_{\sigma(t)}; y).$$

The standard pair is $Stn_t := (h^+_{alt}, h^-_{alt})$, where $h = x_1 \cdots x_t$. Explicitly,

$$Stn_t := \left( \sum_{\sigma \in S^+_t} x_{\sigma(1)} \cdots x_{\sigma(t)}, \sum_{\sigma \in S^-_t} x_{\sigma(1)} \cdots x_{\sigma(t)} \right).$$

The Capelli pair is $Cap_t := (h^+_{alt}, h^-_{alt})$, where $h = x_1 y_1 x_2 y_2 \cdots x_t y_t$. Explicitly,

$$Cap_t := \left( \sum_{\sigma \in S^+_t} x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots y_{t-1} x_{\sigma(t)} y_t, \sum_{\sigma \in S^-_t} x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots y_{t-1} x_{\sigma(t)} y_t \right).$$

Proposition 5.7. Any t-alternating pair $(f, g)$ is a PI for every semiring $R$ spanned by fewer than $t$ elements over its center.

Proof. Suppose $R$ is spanned by $\{b_1, b_2, \ldots, b_{t-1}\}$. We need to verify

$$f \left( \sum \alpha_{i_1} b_{i_1}, \ldots, \sum \alpha_{i_t} b_{i_t} \right) = g \left( \sum \alpha_{i_1} b_{i_1}, \ldots, \sum \alpha_{i_t} b_{i_t} \right).$$

Since $f$ and $g$ are linear in these entries, it suffices to verify

$$f(b_{i_1}, \ldots, b_{i_t}) = g(b_{i_1}, \ldots, b_{i_t})$$

for all $i_1, \ldots, i_t$. But by hypothesis, two of these must be equal, say $i_k$ and $i_{k'}$, so switching these two yields (5.1) by the alternating hypothesis. \qed

Let $e_{i,j}$ denote the matrix units. The semiring version of the Amitsur-Levitzki theorem, that $Stn_{2n} \in \text{id}(M_n(N))$, is an immediate consequence of Theorem 5.4, and its minimality follows from:

Lemma 5.8. Any pair of multilinear polynomials $f(x_1, \ldots, x_m)$ and $g(x_1, \ldots, x_m)$ having no common monomials do not comprise a PI of $M_n(R)$ unless $m \geq 2n$.

Proof. Rewriting indices we may assume that $x_1 \cdots x_m$ appears as a monomial of $f$, but not of $g$, and we note (for $\ell = \left\lceil \frac{m}{2} \right\rceil + 1$) that

$$f(e_{1,1}, e_{1,2}, e_{2,2}, e_{2,3}, \ldots, e_{k-1,k}, e_{k,k}, \ldots) = e_{1,\ell} \neq 0,$$

but $g(e_{1,1}, e_{1,2}, e_{2,2}, e_{2,3}, \ldots, e_{k-1,k}, e_{k,k}, \ldots) = 0$. \qed

Likewise, the identical proof of [R2, Remark 23.14] shows that the Capelli pair $Cap_{n^2}$ is not a PI of $M_n(C)$, and in fact $(e_{1,1}, 0) \notin Cap_{n^2}(M_n(R))$ for any semiring $R$.

5.2. Surpassing identities. The surpassing identity $f \mid g$ holds when $f(a_1, \ldots, a_m) \mid g(a_1, \ldots, a_m)$ for all $a_1, \ldots, a_m \in R$.

Example 5.9. Take the general $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\text{tr}(A) = a + d$ and $|A| = ad + bc$.

$$A^2 = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & bc + d^2 \end{pmatrix},$$

so

$$A^2 + adI = \begin{pmatrix} a(a + d) + bc & b(a + d) \\ c(a + d) & bc + d(a + d) \end{pmatrix} = \text{tr}(A)A + bcI,$$

implying

$$A^2 + |A| I = \text{tr}(A)A + bc^*I,$$
yielding the surpassing identity $A^2 + |A| I \models \text{tr}(A)A$ for $2 \times 2$ matrices.

We might hope for a surpassing identity involving alternating terms in the Hamilton-Cayley polynomial, but a cursory examination of matrix cycles dashes our hopes.

**Example 5.10.** Let $A = \begin{pmatrix} -d & a \\ c & -b \\ - \end{pmatrix}$. Then $A^2 = \begin{pmatrix} cd & ab & - \\ -cd & ac & - \\ bc & - & - \end{pmatrix}$ and $A^3 = \begin{pmatrix} abc & cd^2 & acd \\ c^2 & abc & - \\ - & bc & -abcd \end{pmatrix}$, implying

$$A^3 = \alpha A + |A|$$

in this case, where $\alpha$ denotes the other coefficient in $f_A$. But for $A = \begin{pmatrix} a & - & - \\ - & b & - \\ - & - & c \end{pmatrix}$ we have

$$A^3 + \alpha A + 2 \begin{pmatrix} - & - & - \\ - & abc & - \\ - & - & abc \end{pmatrix} = \text{tr}(A)A^2 + |A|,$$

so neither $A^3 + \alpha A$ nor $\text{tr}(A)A^2 + |A|$ necessarily surpasses the other.

5.3. **Layered surpassing identities.** Since we want to deal with general layers, we write $2a$ (instead of $a^\nu$) for $a + a$, but note that $s(2a) = 2s(a)$. When working with the layered structure, we can extend the notion of PI from Definition 5.2 by making use of the following relations that arise naturally in the theory.

**Definition 5.11.** The *L-surpassing relation* $\models_L$ is given by

$$a \models_L b \text{ iff either } \begin{cases} a = b + c & \text{with } c \text{ (b)-ghost,} \\ a = b, \\ a \equiv_\nu b & \text{with } a \text{ (b)-ghost.} \end{cases} \quad (5.2)$$

It follows that if $a \models_L b$, then $a + b$ is (b)-ghost. When $a \not\models_L b$, this means $a \geq_\nu b$ and $a$ is (b)-ghost.

**Definition 5.12.** The *surpassing* $(L, \nu)$-*relation* $\models_\nu$ is given by

$$a \models_\nu b \text{ iff } a \models_L b \text{ and } a \equiv_\nu b. \quad (5.3)$$

The *surpassing* $L$-*identity* $f \models_L g$ holds for $f, g \in \text{Fun}(R^{(a)}, R)$ if $f(a_1, \ldots, a_n) \models_L g(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in R$.

The *surpassing* $(L, \nu)$-*identity* $f \models_\nu g$ holds for $f, g \in \text{Fun}(R^{(a)}, R)$ if $f(a_1, \ldots, a_n) \models_\nu g(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in R$.

5.3.1. **Layered surpassing identities of commutative layered semirings.** Just as the Boolean algebra satisfies the PI $x^2 = x$, we have some surpassing identities for commutative layered domains.

**Proposition 5.13.** (Frobenius identity) $\prod_L x_1^m + x_2^m$.  

*Proof.* This is just a restatement of [IzK5 Remark 5.2].

**Proposition 5.14.** $(x_1 + x_2 + x_3)(x_1x_3 + x_2x_3 + x_1x_2) \equiv_\nu (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$. More generally, let $g_1 = \sum_i x_i, g_2 = \sum_{i<j} x_i x_j, \ldots$, and $g_{m-1} = \sum_i \prod_{j \neq i} x_j$. Then

$$g_1 \cdots g_{m-1} \equiv_\nu \prod_{i<j} (x_i + x_j). \quad (5.4)$$

*Proof.* This is just a restatement of [IzR1 Theorem 8.51].
5.4. Layered surpassing identities of matrices.

We applied the strong transfer principle of Akian, Gaubert, and Guterman [AkGG602] Theorem 4.21] to the (standard) supertropical matrix semiring in [LZ4148]. We would like to make a similar argument in the layered case, but must avoid the following kind of counterexamples, pointed out by Adi Niv:

Example 5.15. Suppose $A = \begin{pmatrix} [1]^{10} & [2]^{4} \\ [2]^{4} & [10]^{0} \end{pmatrix}$. Then $A^2 = \begin{pmatrix} [1]^{20} & [2]^{14} \\ [2]^{14} & [8]^{8} \end{pmatrix}$, so $|A| = |10|^{10}$ whereas $|A^2| = |8|^{28}$, which does not $\mathbb{N}$-surpass $|A|^2$ (and does not even $\mathbb{N}$-surpass $|A|$).

The difficulty in the example was that some $\nu$-small entry of $A$ has a high layer which provides $|A|$ a high layer but does not affect the powers of $A$. There is a version of surpassing which is useful in this context.

Definition 5.16. An element $c \in R$ is a strong $\ell$-ghost (for $\ell \in L_+$) if $s(c) \geq 2\ell$.

The strong $\ell$-surpassing relation $\odot_{\ell} b$ holds in an $L$-layered domain $^R$, if either

$$\begin{cases} a = b + c \quad &\text{with } c \text{ a strong } \ell\text{-ghost} \\ a = b \end{cases}$$

or if $a = b$.

We often take $\ell = s(b)$. In this case $b + b \odot_{\ell} b$ (as well as $b + b \odot_{\ell} b$).

The strong $\ell$-surpassing relation $\odot_{\ell} (a_{i,j})$ holds for matrices $(a_{i,j})$ and $(b_{i,j})$, if $a_{i,j} \odot_{\ell} b_{i,j}$ for each $i, j$.

We say that a matrix $A$ is $\ell$-layered if each entry has layer $\geq \ell$. We are ready for our other two versions of layered identities.

Definition 5.17. The strong $(\ell, d)$-surpassing identity $f \odot_{\ell,d} g$ holds for $f, g \in \text{Fun}(M_n(R(m), M_n(R))$ if $f(A_1, \ldots, A_m) \odot_{\ell,d} g(A_1, \ldots, A_m)$ with $\ell = d$, for all $\ell$-layered matrices $A_1, \ldots, A_m \in M_n(R)$.

In the standard supertropical theory we take $\ell = 1$, but in the general layered theory we may need to consider other $\ell$. Formally set $P(x_1, \ldots, x_t) = P^+ - P^-$ and $Q(x_1, \ldots, x_t) = Q^+ - Q^-$. We say $Q$ is admissible if the monomials of $Q^+$ and $Q^-$ are distinct, for each pair $(i,j)$.

We then obtain the following metatheorem, along the lines of [AkGG602] (just as in [LZ4148] Theorem 2.41):

Theorem 5.18. Suppose $P = Q$ is a homogeneous matrix identity of $M_n(\mathbb{Z})$ of degree $d$, with $Q$ admissible. Then the matrix semiring $^R M_n(R)$ satisfies the strong $(\ell, d)$-surpassing identity

$$P^+ + P^- \odot_{\ell,d} Q^+ + Q^-.$$
Q1. What are all the semiring\(^*\) PIs of \(M_n(R)\)?
Specifically, we have the Specht-like question:

Q2. Are all semiring\(^*\) PIs of \(M_n(R)\) a consequence of a given finite set?

**Example 5.22.** It is shown in [IzM] that the semiring of \(2 \times 2\) matrices over the max-plus algebra satisfies the semigroup identity

\[
AB^2A AB AB^2A = AB^2A BA AB^2A.
\]

(5.6)

The way of proving this identity is essentially based on showing that pairs of polynomials corresponding to compatible entries in the right and the left product above define the same function. This identification is performed by using the machinery of Newton polytopes, and thus is valid also for supertropical polynomials.

From the results of [IzM], we also conclude that this identity is minimal.

### References

[AkBG] M. Akian, R. Bapat, and S. Gaubert. Max-plus algebra, In: Hobgen, L., Brualdi, R., Greenbaum, A., Mathias, R. (eds.) *Handbook of Linear Algebra*. Chapman and Hall, London, 2006.

[AkGG] M. Akian, S. Gaubert, and A. Guterman. Linear independence over tropical semirings and beyond. In *Tropical and Idempotent Mathematics*, G.L. Litvinov and S.N. Sergeev, (eds.), *Contemp. Math*. 495:1–38, 2009.

[AmL] S.A. Amitsur and J. Levitzki. Minimal identities for algebras. Proc. American Mathematical Society 1, 449–463 1950.

[BrR] R. A. Brualdi and H. J. Ryser. *Combinatorial matrix theory*. Cambridge University Press, 1991.

[CHWW] G. Cortinas, C. Haesemeyer, M. Walker, and C. Weibel, Toric varieties, monoid schemes, and descent. Preprint, 2010.

[DeS] M. Devlin and B. Sturmfels, *Tropical convexity*. Documenta Mathematica 9 (2004), 1–27, Erratum 205–6.

[Gat] A. Gathmann, Tropical algebraic geometry. *Jahresbericht der DMV* 108:3–32, 2006.

[Go] J. Golan, *The theory of semirings with applications in mathematics and theoretical computer science*, Vol. 54, Longman Sci & Tech., 1992.

[Iz] Z. Izhakian, *Tropical arithmetic and matrix algebra*. Chapman and Hall/CRC, Boca Raton, 2009.

[IzKR1] Z. Izhakian, M. Knebusch, and L. Rowen. A Glimpse at Supertropical Valuation theory, An. St. Univ. Ovidius Constanța 19, no. 2, 131–142, 2011.

[IzKR2] Z. Izhakian, M. Knebusch, and L. Rowen, Supertropical linear algebra, Pacific J. of Math., to appear. (Preprint at arXiv:1008.0025.)

[IzKR3] Z. Izhakian, M. Knebusch, and L. Rowen, Dual spaces and bilinear forms in supertropical linear algebra, Linear and Mult. Algebra, to appear. (Preprint at arXiv:1201.6481.)

[IzKR4] Z. Izhakian, M. Knebusch, and L. Rowen. Layered tropical mathematics, preprint at arXiv:0912.1398 (Submitted May 2012)

[IzKR5] Z. Izhakian, M. Knebusch, and L. Rowen. Categorical layered mathematics. *Contemporary Mathematics*, proceedings of the CIEM Workshop on Tropical Geometry, to appear. (Preprint at arXiv:1207.3487)

[IzM] Z. Izhakian and S. W. Margolis. Semigroup identities in the monoid of 2-by-2 tropical matrices. *Semigroup Forum* 80(2), 191–218, 2010.

[IzR1] Z. Izhakian and L. Rowen, Supertropical algebra. *Adv. in Math* 225(4), 2222–2286, 2010.

[IzR2] Z. Izhakian and L. Rowen, The tropical rank of a tropical matrix. *Commun. in Algebra* 37(11), 3912–3927, 2009.

[IzR3] Z. Izhakian and L. Rowen, Supertropical matrix algebra. Israel J. Math., 182(1), 383–424, 2011.

[IzR4] Z. Izhakian and L. Rowen, Supertropical matrix algebra II: Solving tropical equations. *Israel J. Math*. 186(1), 69–97, 2011.

[IzR5] Z. Izhakian and L. Rowen, Supertropical matrix algebra III: Powers of matrices and generalized eigenspaces. *J. of Algebra*, 341(1), 125–149, 2011.

[IzR6] Z. Izhakian and L. Rowen, Supertropical polynomials and resultants, *J. Algebra* 324, 1860–1886, 2010.

[J] Jacobson, N., *Basic Algebra*, Freeman, 1980.

[Ko] V.M. Kopytov, Lattice ordered groups (Russian), Nauka, Moscow (1984)

[L1] G. Litvinov, The Maslov dequantization, idempotent and tropical mathematics: a very brief introduction. *J. of Math. Sciences*, 140(3), 426–444, 2007.

[L2] G. Litvinov, Dequantization of mathematical structures and tropical idempotent mathematics: An introductory lecture. (See next reference), 2012

[L3] G. Litvinov and V.P. Maslov, Tropical and idempotent mathematics: International Workshop. Pneulet Laboratory and Moscow Center for Continuous Mathematical Education, Independent University of Moscow, 2012.

[MS] D. Maclagan and B. Sturmfels, *Tropical Geometry*, Preprint, 2009.

[M] D. Marker, *Model theory: An introduction*, Springer Graduate texts in mathematics; 201, 2002.

[N] A. Niv, Characteristic polynomials of supertropical matrices, *Commun. in Algebra*, to appear, 2012.

[Par] B. Parker. Exploded fibrations, preprint at arXiv: 0705.2408v1, 2007.

[Pay1] S. Payne. Fibers of tropicalizations, Arch. Math., 2010 Correction: preprint at arXiv: ?? [math.AG], 2012.
S. Payne. Analytification is the limit of all tropicalizations, preprint at arXiv: 0806.1916v3 [math.AG], 2009.
L.H. Rowen. Graduate Algebra: Commutative View. Pure and Applied Mathematics 73, Amer. Math. Soc., 2006.
L.H. Rowen. Graduate algebra: A noncommutative view Amer. Math. Soc., 2008.
G.E. Sacks, Saturated Model Theory, Mathematical Lecture Notes 80 Benjamin, 1972.
E. Sheiner and S. Shnider, An exploded-layered version of Payne’s generalization of Kapranov’s theorem, preprint, 2012.
D. E. Speyer and B. Sturmfels, Tropical mathematics, Math. Mag., 82 (2009), 1631/2-173.
O. Viro, Hyperfields for tropical geometry I. Hyperfields and dequantization Preprint at arXiv:math.AG/1006.3034v2
C. Weibel, EGA for monoids. Preprint, 2010.
H.J. Weinert and R. Wiegandt, On the structure of semifields and lattice-ordered groups, Periodica Mathematica Hungaria 32 (1-2) (1996), 129–147.

DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, RAMAT-GAN 52900, ISRAEL
E-mail address: zzur@math.biu.ac.il

DEPARTMENT OF MATHEMATICS, NWF-I MATHEMATIK, UNIVERSITÄT REGENSBURG 93040 REGENSBURG, GERMANY
E-mail address: manfred.knebusch@mathematik.uni-regensburg.de

DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, 52900 RAMAT-GAN, ISRAEL
E-mail address: rowen@math.biu.ac.il