POLYNOMIAL-SIZE VECTORS ARE ENOUGH FOR THE UNIMODULAR TRIANGULATION OF SIMPLICIAL CONES

MICHAEL VON THADEN

ABSTRACT. In a recent paper, Bruns and von Thaden established a bound for the length of vectors involved in a unimodular triangulation of simplicial cones. The bound is exponential in the square of the logarithm of the multiplicity, and improves previous bounds significantly. The authors mentioned that the next goal would be a bound that is polynomial in the multiplicity but not knowing if such a bound exists. In this paper we will prove that such a bound, which is polynomial in the multiplicity $\mu$, indeed exists. In detail, the bound is of the type $\mu^{f(d)}$ with $f(d) \in \mathcal{O}(d)$.

1. INTRODUCTION

In [3] Bruns and von Thaden established a bound for the length of vectors involved in a unimodular triangulation of simplicial cones. Length is hereby measured by the basic simplex $\Delta_C$ of $C$ that is spanned by the origin and the extreme integral generators of $C$. We are interested in an upper bound for the dilatation factor $c$ for which all subdividing vectors are contained in $c\Delta_C$. Bruns and von Thaden gave an upper bound for $c$, which was exponential in the square of the logarithm of the multiplicity of the cone $C$, hereby improving a result from Bruns and Gubeladze [1, Theorem 4.1] which itself was an improvement of the standard argument applied for the desingularization of toric varieties. Bruns and von Thaden mentioned in [3] that the next goal would be a bound that is polynomial in the multiplicity but not knowing if such a bound exists. In this paper we will prove that such a bound, which is polynomial in the multiplicity $\mu$, indeed exists. In detail, the bound is of the type $\mu^{f(d)}$ with $f(d) \in \mathcal{O}(d)$.

Bruns and Gubeladze used the upper bound for cones [1, Theorem 4.1] to prove a similar bound for lattice polytopes. They showed that multiples $cP$ of lattice polytopes $P$ can be covered by unimodular simplices for all $c \geq g(d)$ with $g \in \mathcal{O}(d^6)$. So, this threshold does only depend on $d$ and not on the multiplicity of $P$.

Of course, a corresponding result for the unimodular triangulation of lattice polytopes—i.e. does there exist a $c_d$ depending only on $d$ and not on $P$ or its multiplicity such that $cP$ admits a unimodular triangulations for all $c \geq c_d$—would be very desirable but seems currently out of reach. The best result so far is the celebrated Knudsen-Mumford-Waterman theorem [5]. It states that a $c'$ exists such that the multiples $cP$ of a lattice polytope $P$ have unimodular triangulations for all $c \geq c'$. But Knudsen, Mumford and Waterman did not provide an explicit bound. Recently, Haase, Paffenholz, Piechnik and Santos closed this gap in [4] and provided an explicit bound, which is doubly exponential in the volume of
the lattice polytope \( P \). Aside from this proof, [4] also gives a comprehensive overview of the topic as does [2, Chapter 3]. Furthermore, we refer the reader to [2] for any unexplained terminology.

2. Auxiliary results

One of the main ideas of the proof in [3] was that a cone whose multiplicity is a power of 2, or, generally speaking, whose multiplicity is a product of small primes could be triangulated using just short vectors. Therefore, in a first step one might wish to triangulate a cone into cones whose multiplicities are exclusively products of small primes while keeping the length of the subdividing vectors as short as possible.

If you want to apply stellar subdivision to come up with a triangulation of \( C \) by the desired cones, which form should the vectors \( x \) used for stellar subdivision have? Let us just remember that if the primitive vectors \( v_1, \ldots, v_d \in \mathbb{Z}^d \) generate a simplicial cone \( C \) of dimension \( d \), and if \( U \) denotes the sublattice of \( \mathbb{Z}^d \) spanned by these vectors, then \( \mu(C) \) is the index of \( U \) in \( \mathbb{Z}^d \), and each residue class has a representative in

\[
\text{par}(v_1, \ldots, v_d) = \{q_1v_1 + \cdots + q_dv_d : 0 \leq q_i < 1\}.
\]

If \( p \) divides \( \mu(C) \), then there is an element of order \( p \) in \( \mathbb{Z}^d / U \), and consequently there exists a vector \( x = \frac{1}{p} \sum_{i=1}^{d} z_i v_i \in \mathbb{Z}^d \setminus \{0\}, \ z_i \in \mathbb{Z}, \ 0 \leq z_i < p \).

If we do now apply stellar subdivision with respect to \( x \) to \( C \), then the resulting cones \( E_i \) have multiplicities \( \mu(E_i) = z_i / p \cdot \mu(C) < \mu(C) \). This means that if you could choose \( x \) in a way that the \( z_i \) are composite numbers, you increase the number of prime factors for the triangulating cones thereby ensuring that the prime factors are getting smaller.

But are there always vectors of type (1) such that the \( z_i \) are always composite numbers? In general, this is not the case as one easily sees by looking at the following general example. Let \( d + 1 \) be a prime number and let the cone \( C \subset \mathbb{R}^d \) be generated by the vectors

\[
v_d = \sum_{j=2}^{d} (d + 1 - j)e_j + (d + 1)e_1, \quad v_i = e_i, \quad i = 2, \ldots, d.
\]

Then \( \mu(C) = d + 1 \in \mathbb{P} \). Furthermore, each residue class of \( \mathbb{Z}^d \) modulo the sublattice \( U \) generated by the vectors \( v_i \) has a representative, which is of the form

\[
x_j = \frac{1}{d+1} \sum_{k=1}^{d} (jk \text{rem}(d+1)) v_k, \quad j = 1, \ldots, d + 1,
\]

where \( a \text{rem} b \) shall denote the remainder of \( a \) modulo \( b \). Then, for every \( j \) we have \( \{jk \text{rem}(d+1) : k = 1, \ldots, d\} = \{1, \ldots, d\} \). So, in this case there is no vector \( x \) of the form (1) such that all \( z_i \) are just composite numbers.

But is there a condition which ensures that such a vector \( x \) of type (1) with all \( z_i \) being composite numbers exists? We will now prove quite easily that \( \text{par}(v_1, \ldots, v_d) \) indeed always contains a vector of form (1) such that all the \( z_i \) are composite numbers – as long
as the largest prime factor of $\mu(C)$ is bounded below by $e^{\tau d}$, where $\tau = 1.25506$. This fact is a direct consequence of the following lemma which has already been proved in [3] with the help of an upper bound for the prime number counting function $\pi(n)$ as provided by Rosser & Schoenfeld in [6].

**Lemma 2.1.** With the notation introduced, let $M \subset \{1, \ldots, d\}$ such that $|M| \leq \log(p) / \tau$, $\tau = 1.25506$. Then there exists an element $x$ of order $p$ modulo $U$ such that none of the coefficients $z_i$, $i \in M$, is an odd prime $< p$.

If you take $M = \{1, \ldots, d\}$, the lemma implies that there exists an element $x \in \text{par}(v_1, \ldots, v_d) \setminus \{0\}$ of type (1) such that none of the coefficients $z_i, i = 1, \ldots, d$ in (1) is an odd prime as long as

$$d \leq \frac{\log(p)}{\tau}, \quad \tau = 1.25506,$$

which just translates into

$$p \geq e^{\tau d}.$$  

Therefore, we have

**Theorem 2.2.** Let $C = R_+ v_1 + \cdots + R_+ v_d \subset R^d, d \geq 2$ be a simplicial $d$-cone such that $p_{\text{max}} := p_{\text{max}}(\mu(C)) \geq e^{\tau d}$, where $p_{\text{max}}(n) := \max\{p \in \mathbb{P} : p \mid n\}$. Then there exists a vector

$$x = \frac{1}{p_{\text{max}}} \left( \sum_{i=1}^d z_i v_i \right) \in \text{par}(v_1, \ldots, v_d) \setminus \{0\}.$$

such that $z_i \notin \mathbb{P}_{>2}$ for all $i$.

Hence, as long as $\mu(C)$ has a prime factor $p \geq e^{\tau d}$, there is also an element $x = \frac{1}{p} \left( \sum_{i=1}^d z_i v_i \right) \in \text{par}(v_1, \ldots, v_d) \setminus \{0\}$ such that all $z_i$ are composite numbers or are equal to 2. These short vectors can then be used for successive stellar subdivision until you arrive at cones $D_i$ for which $p_{\text{max}}(\mu(D_i)) < e^{\tau d}$ and which do constitute a triangulation of the original cone $C$.

The following definition will help us to shorten any further explanations or statements with respect to this kind of triangulation procedures.

**Definition 2.3.** An $f$-triangulation is defined as a triangulation of a cone $C$ by cones $D_i$ for which $p_{\text{max}}(\mu(D_i)) < f$ for all $i$.

### 3. The Algorithm

With the help of Theorem 2.2, we are now ready to formulate an algorithm which provides us with an $e^{\tau d}$-triangulation of a cone $C$ by cones $D_i$. As we will see, the vectors involved in this triangulation are short and the multiplicities of the cones $D_i$ constituting the triangulation are smaller than the multiplicity $\mu(C)$ of the original cone $C$.
Bounded prime factors triangulation – BPFT

Input: The initial cone \( C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d \)

Output: An \( e^{\tau d} \)-triangulation \( \hat{T}(C) \) of \( C \)

1: \( \hat{T}(C) := \{ C \} \)
2: \( \hat{A}(C) := \{ C \} \)
3: \( \xi_C(-i) := v_i \) for \( i = 1, \ldots, d \)
4: \( \xi_C(i) := 0 \) for \( i \in \mathbb{N}_0 \)
5: while \( \hat{T}(C) \) contains a cone \( D = \mathbb{R}_+ \xi_D(i_1) + \cdots + \mathbb{R}_+ \xi_D(i_d) \) (where \( i_1 > i_2 > \cdots > i_d \geq -d \)) such that \( p_{\max}(\mu(D)) \geq e^{\tau d} \) do
6: \( p := p_{\max}(\mu(D)) \)
7: \( \text{FIND} \ x = 1/p \left( \sum_{j=1}^d z_j \xi_D(i_j) \right) \in \text{par}(\xi_D(i_1), \ldots, \xi_D(i_d)) \setminus \{ 0 \} \) such that \( z_j \notin \mathbb{P}_{>2} \)
8: for all \( E \in \hat{T}(C) \) with \( x \in E \) do
9: Apply stellar subdivision to \( E \) by \( x \) (let \( E_j \) (\( j = 1, \ldots, m \)) be the resulting cones)
10: \( \hat{T}(C) := (\hat{T}(C) \setminus \{ E \}) \cup \{ E_j : j = 1, \ldots, m \} \)
11: \( \hat{A}(C) := \hat{A}(C) \cup \{ E_j : j = 1, \ldots, m \} \)
12: end for
13: \( \nu := \max \{ i : \xi_E(i) \neq 0 \} \)
14: for all \( j = 1, \ldots, m \) do
15: for all \( k \leq \nu \) do
16: \( \xi_E(k) := \xi_E(k) \)
17: end for
18: \( \xi_E(\nu + 1) := x \)
19: end for
20: end while
21: Return \( \hat{T}(C) \)

As in [3] the set \( \hat{A}(C) \) contains the original cone \( C \) and all cones being created in the course of the algorithm and the set \( \hat{T}(C) \) is a strict subset of \( \hat{A}(C) \) unless \( \mu(C) \) is not divisible by any primes greater than or equal to \( e^{\tau d} \). \( \hat{A}(C) \) has been introduced out of technical reasons; it will help us to analyze certain properties of the resulting triangulation.

First, we will prove that the algorithm really provides us with an \( e^{\tau d} \)-triangulation. Then we will show that the generators of the resulting cones \( E_i \in \hat{T}(C) \) are short. Building on these results we will finally introduce new bounds for the length of vectors involved in unimodular triangulations of simplicial cones in section [5].

The next lemma has already been stated implicitly in line 7 of the BPFT algorithm.

Lemma 3.1. Let \( D = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_d \) be a cone to which we apply a stellar subdivision by a vector \( x \) in the BPFT algorithm. Then \( x \) is of the form

\[
x = \frac{1}{p_{\max}} \left( \sum_{j=1}^d z_j w_j \right), \quad p_{\max} = \max \{ p \in \mathbb{P} : p | \mu(D) \},
\]
such that either
(1) \( z_j = 0 \) or
(2) \( z_j < p_{\text{max}} \) is a composite number or
(3) \( z_j = 2 < p_{\text{max}} \)

Based upon the previous algorithm we get the following result.

**Lemma 3.2.** Let \( D, E \in \hat{\mathcal{A}}(C) \) such that \( E \) results from \( D \) by stellar subdivision in the course of the BPFT algorithm. Then

\[
\mu(E) < \mu(D).
\]

**Proof.** Let \( D = \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_d \subset \mathbb{R}^d \). Due to lines 7 and 9 of the algorithm \( E \) is generated from \( D \) by stellar subdivision with a vector \( x \in \text{par}(w_1, \ldots, w_d) \setminus \{0\} \). Therefore, we have \( \mu(E) = \frac{1}{p} \mu(D) \) with natural numbers \( z < p \), which implies that \( \mu(E) < \mu(D) \). \( \Box \)

**Theorem 3.3.** For a simplicial \( d \)-cone \( C \) the BPFT algorithm computes an \( e^{rd} \)-triangulation of \( C \).

**Proof.** The algorithm applies successive stellar subdivisions to the initial cone \( C \). It stops when all multiplicities only have prime factors smaller than \( e^{rd} \), and that it stops after finitely many iterations follows from Lemma 3.2. \( \Box \)

### 4. Bounds for \( e^{rd} \)-triangulation

**Theorem 4.1.** Let \( D \in \hat{T}(C) \). Then, for all \( s \geq 0 \)

\[
\xi_D(s) \in \left( d \cdot 2^k \right) \Delta_C.
\]

**Proof.** The proof of this theorem is similar to the proof of Theorem 4.1 in [1] respectively the proof of Theorem 2.1 in [2]. We consider the following sequence:

\[
h_k = 1, \quad k \leq -1, \quad h_k = h_{k-1} + \cdots + h_{k-d}, \quad k \geq 0.
\]

Because

\[
h_k - h_{k-1} = h_{k-1} - h_{k-d-1}
\]

for \( k \geq 1 \) and \( h_0 \geq h_l \) for \( l \leq -1 \), it follows by induction that this sequence is increasing. Since for \( k \geq 1 \)

\[
h_k = h_{k-1} + (h_{k-2} + \cdots + h_{k-d-1}) - h_{k-d-1} = 2h_{k-1} - h_{k-d-1} < 2h_{k-1},
\]

and because \( h_0 = d \), we arrive at

\[
h_k \leq d \cdot 2^k
\]

for \( k \geq 0 \). This inequality will be needed in the following.

Now, we will prove via induction on \( s \) that

\[
\xi_D(s) \in h_s \Delta_C
\]

So, let \( s = 0 \). If \( \xi_D(0) = 0 \), there is nothing to prove.
So, suppose that $\xi_D(0) \neq 0$. By the construction of $\xi_D(0)$ it follows that this vector was used for the stellar subdivision of the initial cone $C$. Hence, $\xi_D(0)$ is of the form

$$\xi_D(0) = \sum_{i=1}^{d} z_i v_i \in \mathbb{Z}^d \setminus \{0\},$$

where $z_i < 1$ for all $i$ (Lemma 3.1). Therefore, $x \in d\Delta_C$, which finishes the case $s = 0$.

For the induction step assume the statement is true for $s$ replaced by $s - 1 \geq 0$. Again there is nothing to prove if $\xi_D(s) = 0$. Otherwise $\xi_D(s) \neq 0$ is a vector used for stellar subdivision. With the same notation as above, it follows by construction of $\xi_D(s)$ that

$$\xi_D(s) = \sum_{i=1}^{d} z_i \xi_D(j_i) \in \mathbb{Z}^d \setminus \{0\}$$

such that $s > j_1 > j_2 > \ldots > j_d$ and again $z_i < 1$. So, it follows by induction that

$$\xi_D(s) \in (h_{j_1} + \cdots + h_{j_d})\Delta_C.$$ 

Because the $h_i$ are increasing, this means that $\xi_D(s) \in (h_{s-1} + \cdots + h_{s-d})\Delta_C = h_s\Delta_C$, which finishes the proof. □

The next definition will be helpful in showing that the length of every chain of cones

$$E_0 = D \subset E_1 \subset E_2 \ldots \subset E_L = C,$$

where $E_i$ is generated from $E_{i+1}$ by stellar subdivision and $D$ belongs to the resulting $e^\cdot\Delta$-triangulation of $C$, is relatively short.

**Definition 4.2.** Let $n$ be a natural number, $n = \prod_{i=1}^{\infty} p_i^{\alpha_i}$ be its prime decomposition. Then for $r \in \mathbb{R}$ we define $\phi_r(n) = r(\log(n) - \eta(n))$, where $\eta(n) = \sum_{i=1}^{\infty} \alpha_i$. Hence, $\phi_r(n) = \sum_{i=1}^{\infty} \alpha_i(r\log(p_i) - r)$.

The functions $\phi_r$ have some obvious nice properties, which we will need in the following.

**Lemma 4.3.**

1. $\phi_r(ab) = \phi_r(a) + \phi_r(b)$ for $a, b \in \mathbb{N}$, $r \in \mathbb{R}$
2. $\phi_r(a/b) = \phi_r(a) - \phi_r(b)$ for $a, b \in \mathbb{N}$, $b \mid a$, $r \in \mathbb{R}$
3. $\phi_r(a) \geq 0$ for $a \in \mathbb{N}$, $r \in \mathbb{R}_+$

**Lemma 4.4.** Let $D, E \in \hat{A}(C)$ such that $E$ results from $D$ by stellar subdivision in the course of the BPFT algorithm. Then

$$\phi_1(\mu(E)) \leq \phi_1(\mu(D)) - 1.$$ 

**Proof.** Due to lines 7 and 9 of the algorithm,

$$\mu(E) = \mu(D) \frac{f}{p_{\text{max}}},$$

where $p_{\text{max}} = \max\{p \in \mathbb{P} : p | \mu(D)\}$ and $f \in \mathbb{N}_{>0}$ is either
(1) a composite number smaller than $p_{\text{max}}$, i.e. $f = uv$ with natural numbers $u, v > 1$

or

(2) $f = 2 < p_{\text{max}}$.

For the first case we have by Lemma 4.3 and because $p_{\text{max}} | \mu(D)$

$$\phi_1(\mu(E)) = \phi_1(\mu(D)) - \phi_1(p_{\text{max}}) + \phi_1(f) =$$

$$\phi_1(\mu(D)) + \phi_1(u) + \phi_1(v) - \ln(p_{\text{max}}) - 1 \leq \phi_1(\mu(D)) - 1.$$  

For the second case it follows that

$$\phi_1(\mu(E)) = \phi_1(\mu(D)) - \phi_1(p_{\text{max}}) + \phi_1(2) =$$

$$\phi_1(\mu(D)) - \phi_1(p_{\text{max}}) \leq \phi_1(\mu(D)) - 1,$$

because $p_{\text{max}} \geq e^{\tau d} \geq 3.5$ for all $d \geq 1$, which implies that $p_{\text{max}} \geq 5$, because $p_{\text{max}}$ is a prime. Therefore, $\phi_1(p_{\text{max}}) \geq \phi_1(5) = \ln(5) - 1 > 1$. This proves the lemma.

**Lemma 4.5.** Let $D \in \hat{\Delta}(C)$ be an arbitrary cone resulting from the BPFT algorithm. Furthermore, we define

$$\chi(D) = \max\{i : \xi_D(i) \neq 0\}.$$  

Then

$$\chi(D) \leq \phi_1(\mu(C)) - 1.$$  

**Proof.** Let $D \in \hat{\Delta}(C)$. By the algorithm, there is a chain of cones

$$E_0 = D \subset E_1 \subset E_2 \ldots \subset E_L = C$$

such that $E_i$ is generated from $E_{i+1}$ by stellar subdivision. Lemma 4.4 implies that $\phi_1(\mu(D)) \leq \phi_1(\mu(C)) - L$. On the other hand, by construction, $\chi(D) = \chi(C) + L$, where $\chi(C) = -1$. Therefore

$$\chi(D) = L - 1 \leq \phi_1(\mu(C)) - \phi_1(\mu(D)) - 1.$$  

This proves the lemma, because $\phi_1(a) \geq 0$ for all $a \in \mathbb{N}$.  

**Corollary 4.6.** Every simplicial $d$-cone $C = \mathbb{R}_+ v_1 + \ldots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$, $d \geq 2$, which is not already unimodular (i.e., $\mu(C) > 1$) has an $e^{\tau d}$-triangulation $C = D_1 \cup \ldots \cup D_i$ such that

$$\operatorname{Hilb}(D_i) \subset \left(d \cdot 2^{\phi_1(\mu(C)) - 1}\right) \Delta_C$$

for all $i$. Furthermore, taking into account the definition of $\phi_1$, we have

$$\operatorname{Hilb}(D_i) \subset \left(\frac{d}{4} \cdot \mu(C)\right) \Delta_C$$

for all $i$.

**Proof.** Due to 4.1 and 4.5 it follows that

$$\operatorname{Hilb}(D_i) \subset \left(d \cdot 2^{\phi_1(\mu(C)) - 1}\right) \Delta_C.$$  

Because due to 4.2 and $\mu(C) > 1$ we have $\phi_1(\mu(C)) \leq \ln(\mu(C)) - 1$. It follows that $2^{\phi_1(\mu(C)) - 1} \leq \mu(C)/4$, which finally proves the corollary.
5. Bounds for unimodular triangulation

Building on the previous bound we will now introduce new bounds for the length of vectors involved in unimodular triangulations of simplicial cones. This will be done with the help of the following corollary from [3].

Corollary 5.1. Let \( \varepsilon = 5 + 3/2 \cdot \text{ld}(3/2) \) and \( p = 1/2 \cdot \text{ld}(3/2) \). So, \( \varepsilon \approx 5.88 \) and \( p \approx 0.29 \).

Then every simplicial d-cone \( C = \mathbb{R}_{+}v_1 + \cdots + \mathbb{R}_{+}v_d \subset \mathbb{R}^d, \ d \geq 2 \), which is not already unimodular (i.e., \( \mu(C) > 1 \)) has a unimodular triangulation \( C = D_1 \cup \ldots \cup D_t \) such that for all \( i \)

\[
\text{Hilb}(D_i) \subset \left( \frac{d^2}{64} \cdot \mu(C)^{p + \text{ld}(\mu(C))} + \varepsilon \right) \Delta_C.
\]

Furthermore, we will need the following lemma, which will help us with connecting the previous corollary and Corollary 4.6 to achieve our main result of a new upper bound for the length of vectors involved in the unimodular triangulation of simplicial cones.

Lemma 5.2. Let \( C = \mathbb{R}_{+}v_1 + \cdots + \mathbb{R}_{+}v_d \subset \mathbb{R}^d, \ d \geq 2 \), be a simplicial d-cone and let \( p \in \mathbb{P} \) such that \( p | \mu(C) \).

Furthermore, let us assume that every simplicial d-cone \( E \) with \( \mu(E) = p \) admits a unimodular triangulation \( E = F_1 \cup \ldots \cup F_t \) such that

\[
\text{Hilb}(F_i) \subset k_p \Delta_E.
\]

for all \( i \) for a certain \( k_p \in \mathbb{R} \). Then \( C \) admits a triangulation \( C = D_1 \cup \ldots \cup D_t \) such that

1. \( \text{Hilb}(D_i) \subset k_p \Delta_C \) and
2. \( \mu(D_i) = \mu(C)/p \)

for all \( i \).

Proof. Let \( E = \mathbb{R}_{+}v_1' + \cdots + \mathbb{R}_{+}v_d' \subset \mathbb{R}^d, \ d \geq 2 \) be a simplicial d-cone with \( \mu(E) = p \).

For each of the cones \( F_i = \mathbb{R}_{+}w_1' + \cdots + \mathbb{R}_{+}w_d' \subset \mathbb{R}^d \) constituting the unimodular triangulation of \( E \) we have that every vector \( w_j' \) must have the following form

\[
w_j' = \frac{1}{p} \left( \sum_{k=1}^{d} a_{jk}' v_k' \right),
\]

with \( a_{jk}' \in \mathbb{N} \) for all \( i, j, k \).

Let now \( W_{ij}' \in \mathbb{R}^{d \times d} \) be the matrix formed by the row vectors \( w_j' \), let \( A^i := (a_{jk}')_{j=1, \ldots, d; k=1, \ldots, d} \) and \( V' \in \mathbb{R}^{d \times d} \) be the matrix formed by the row vectors \( v_k' \).

Then we have that

\[
det \left( \frac{1}{p} A^i V' \right) = det \left( W_{ij}' \right) = \mu(F_i) = 1
\]

for all \( i \), which implies that

\[
det \left( \frac{1}{p} A^i \right) = \frac{1}{det(V')} = \frac{1}{\mu(E)} = \frac{1}{p}.
\]

Therefore, the triangulation of \( C = D_1 \cup \ldots \cup D_t \) given by \( D_i = \mathbb{R}_{+}w_1' + \cdots + \mathbb{R}_{+}w_d' \subset \mathbb{R}^d \) – where \( w_j' = \frac{1}{p} \left( \sum_{k=1}^{d} a_{jk}' v_k' \right) \) – has the desired properties, because, first, we have that
for $V \in \mathbb{R}^{d \times d}$ formed by the row vectors $v_k$
\[
\mu(D_i) = \det \left( \frac{1}{p} \cdot A^i V \right) = \det \left( \frac{1}{p} \cdot A^i \right) \cdot \mu(C) = \frac{\mu(C)}{p}
\]
And second, it is obvious that $\text{Hilb}(D_i) \subset k_p \Delta_C$ for all $i$, if $\text{Hilb}(F_i) \subset k_p \Delta_E$. □

**Corollary 5.3.** Let $\gamma = \rho \tau \log(d)$ and $\kappa = \varepsilon - 5$. So, $\gamma \approx 0.53$ and $\kappa \approx 0.88$. Then every simplicial $d$-cone $C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d$, $d \geq 2$, which is not already unimodular (i.e., $\mu(C) > 1$) has a unimodular triangulation $C = D_1 \cup \ldots \cup D_t$ such that for all $i$
\[
\text{Hilb}(D_i) \subset \left( \frac{d}{4} \cdot \mu(C)^{\gamma d + 2 \log(d) + \kappa} \right) \Delta_C.
\]

**Proof.** Due to 4.6 $C$ has an $e^{\tau d}$-triangulation $C = D_1 \cup \ldots \cup D_t$ such that for all $i$
\[
\text{Hilb}(D_i) \subset \left( \frac{d}{4} \cdot \mu(C) \right) \Delta_C.
\]
Furthermore,
\[
(2) \quad \mu(D_i) \leq \mu(C)
\]
for all $i$.

So, let $\mu(D_i) = \prod_{j=1}^{n_i} p_j^\alpha_{j,i}$ be the prime decomposition of $\mu(D_i)$, where $p_1 < \ldots < p_{n_i} < e^{\tau d}$. Then, due to successive application of Lemma 5.2 and Corollary 5.1 it follows that each of the cones $D_i$ admits a unimodular triangulation $D_i = F_1^i \cup \ldots \cup F_s^i$ such that
\[
\text{Hilb}(F_k^i) \subset \prod_{j=1}^{n_i} \left( \frac{d^2}{64} p_j \log(p_j) + \varepsilon \right)^{\alpha_{j,i}} \Delta_{D_i}
\]
for all $i, k$. Keeping in mind that the $D_i$ constitute an $e^{\tau d}$-triangulation of $C$, we get that $C$ has a unimodular triangulation
\[
C = \bigcup_{i=1}^{t} \bigcup_{k=1}^{s_i} F_k^i
\]
such that for all $i, k$ we have
\[
\text{Hilb}(F_k^i) \subset \left( \frac{d}{4} \cdot \mu(C) \cdot \prod_{j=1}^{n_i} \left( \frac{d^2}{64} p_j \log(p_j) + \varepsilon \right)^{\alpha_{j,i}} \right) \Delta_C.
\]
Because
\[
\sum_{j=1}^{n_i} \alpha_{j,i} \leq \log(\mu(D_i))
\]
it follows that
\[
\prod_{j=1}^{n_i} \left( \frac{d^2}{64} \right)^{\alpha_{j,i}} \leq \left( \frac{d^2}{64} \right)^{\log(\mu(D_i))} = \mu(D_i)^{2 \log(d) - 6} \leq \mu(C)^{2 \log(d) - 6}.
\]
Furthermore, we have

\[
\prod_{j=1}^{n_i} \left( p_j^{\rho \text{ld}(p_j) + \varepsilon} \right)^{a_{j,i}} \leq \left( \prod_{j=1}^{n_i} p_j^{a_{j,i}} \right)^{\rho \text{ld}(p_{n_i}) + \varepsilon} = \mu(D_i)^{\rho \text{ld}(p_{n_i}) + \varepsilon} \leq \mu(C)^{\rho \tau \text{ld}(\varepsilon) d + \varepsilon},
\]

where the last inequality follows from \( p_j < \varepsilon^{\tau d} \) for all \( j \) and equation (2).

Putting it all together, we get that

\[
\text{Hilb}(F^i_k) \subset \left( \frac{d}{4} \mu(C)^{\gamma d + 2\text{ld}(d) + \kappa} \right) \Delta_C,
\]

where \( \gamma = \rho \tau \text{ld}(\varepsilon) \approx 0.53 \) and \( \kappa = \varepsilon - 5 \approx 0.88 \). \( \square \)

Via simplification of the above notation we finally get

**Corollary 5.4.** Every simplicial \( d \)-cone \( C = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_d \subset \mathbb{R}^d, d \geq 2 \), which is not already unimodular (i.e., \( \mu(C) > 1 \)) has a unimodular triangulation \( C = D_1 \cup \cdots \cup D_t \) such that for all \( i \)

\[
\text{Hilb}(D_i) \subset \mu(C)^{f(d)} \Delta_C
\]

with \( f(d) \in O(d) \).

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ISM, INTERNATIONAL SCHOOL OF MANAGEMENT, 60598 FRANKFURT AM MAIN, GERMANY

E-mail address: michael.vonthaden@ism.de