Matrix Theory from Schild Action

Ichiro Oda

Edogawa University, 474 Komaki, Nagareyama City, Chiba 270-01, JAPAN

Abstract

Starting from the Schild action for membrane, we present an alternative formulation of Matrix Theory. First of all, we construct the Schild action for general bosonic p-brane which is classically equivalent to the Nambu-Goto action for p-brane. Next, based on the constraint obtained from the variational equation for the auxiliary field in the case of \( p = 2 \) (membrane), we construct a new matrix model which is closely related to the matrix model of M-theory as developed by Banks, Fischler, Shenker and Susskind (BFSS). Our present formulation is a natural extension of the construction of type IIB matrix model by Yoneya to the case of M-theory.

\(^1\) E-mail address: ioda@edogawa-u.ac.jp
1 Introduction

Among some dramatic developments in string theory in recent years maybe the most exciting one has been the discovery of Matrix Theory [1]. It has long been a mystery to understand the microscopic degrees of freedom of string theory at the short distance regime. For the first time in Matrix Theory it was conjectured and further confirmed that D-particles [2] may be the fundamental building block for M-theory [3]. Indeed, it was shown that M-theory is equivalent to the $N \to \infty$ limit of the non-relativistic quantum mechanics of $N$ D-particles in weakly coupled region of IIA superstring [1]. Thus in Matrix Theory the starting action for the matrix degrees of freedom was derived from IIA superstring.

On the other hand, there is another matrix model as a candidate for the non-perturbative formulation of type IIB superstring, what we call, IIB matrix model [4]. The action in this model has the form of the large $N$ matrix model of ten dimensional super Yang-Mills theory reduced to a point and was constructed by starting from the Schild action [5] for the Green-Schwarz IIB superstring [6]. Furthermore, a similar model to the original IIB matrix model was built out of a different method where the constraint derived from the variational equation with respect to the auxiliary field in the Schild action for string has played a critical role [7]. Afterwards, it was also shown that this model can be understood from the viewpoint of breakdown of topological symmetry [8].

Then one is naturally led to ask whether Matrix Theory can be also derived from the Schild action for membrane by the strategy adopted in the reference [7]. This is the main problem that I want to study in this paper. We will show that this is indeed the case. However, in constrast to the case of IIB matrix model, we have to fix the gauge symmetries in the light-cone gauge completely, which makes the physically interesting interpretation of the constraint as the space-time uncertainty relation [7, 9, 10] very vague.

The paper is organized as follows. In section 2 it is shown that it is possible to make the Schild action for general bosonic p-brane that is classically equivalent to the Nambu-Goto action for p-brane, where special attention is paid to the constraints in the Schild action. In section 3, on a basis of the contraint in the Schild action for membrane we construct a new matrix model of M-theory in the light-cone gauge. It is mentioned that this new Matrix Theory has $N = 2$ supersymmetry and yields Matrix Theory by Banks et al. [1] as the low energy effective theory of many distant clusters of D-branes. The final section is devoted to discussions.

2 The Schild action for p-brane

In this section, we construct the Schild action [5] for general bosonic p-brane that is equivalent to the Nambu-Goto action for p-brane and then analyse the structure of the constraints in the Hamiltonian formalism. In this paper, we confine ourselves to only the equivalence at the classical level since it currently seems to be very difficult to prove the quantum equivalence among various formulations of p-brane except in string theory ($p = 1$). Thus, in the following
we shall neglect the contribution stemming from the functional measures, the ghosts and the normal orderings.

First of all, let us recall the Schild action \(^{[5]}\) for bosonic string \((p = 1)\), which is of the form

\[
S_p^{n=1} = -\frac{1}{n} \int d^2 \xi \, e \left[ \frac{1}{e^n} \left\{ -\frac{1}{2\lambda_1^2} (\sigma^{\mu_1 \mu_2})^2 \right\}^{\frac{p}{2}} + n - 1 \right],
\]

where \(e(\xi)\) is a positive definite scalar density defined on the string world sheet parametrized by \(\xi^0\) and \(\xi^1\), \(\lambda_1 = 2\pi\alpha'\), and \(\sigma^{\mu_1 \mu_2}\) is defined as \(\varepsilon^{\alpha_1 \alpha_2} \partial_{\alpha_1} X^{\mu_1} \partial_{\alpha_2} X^{\mu_2}\). Here \(X^\mu(\xi)\) \((\mu = 0, 1, \ldots, D - 1)\) are space-time coordinates and the index \(\alpha\) runs over the world sheet indices 0 and 1. Throughout this paper, we assume that the space-time metric takes the flat Minkowskian form defined as \(\eta_{\mu \nu} = \text{diag}(- + + \cdots +)\).

Then it is quite straightforward to build the Schild action for general bosonic p-brane by generalizing \((1)\). The concrete expression is given by

\[
S_p^{n} = -\frac{1}{n} \int d^{p+1} \xi \, e \left[ \frac{1}{e^n} \left\{ -\frac{1}{(p+1)!\lambda_p^2} (\sigma^{\mu_1 \cdots \mu_{p+1}})^2 \right\}^{\frac{p}{2}} + n - 1 \right],
\]

where \(\sigma^{\mu_1 \cdots \mu_{p+1}} = \varepsilon^{\alpha_1 \cdots \alpha_{p+1}} \partial_{\alpha_1} X^{\mu_1} \cdots \partial_{\alpha_{p+1}} X^{\mu_{p+1}}\) and the world volume index \(\alpha\) takes the values 0, 1, \ldots, \(p\).

In fact, we can demonstrate that \((4)\) is equivalent to the Nambu-Goto action for p-brane as follows. Taking the variation with respect to the auxiliary field \(e(\xi)\), one obtains the constraint

\[
e(\xi) = \frac{1}{\lambda_p} \sqrt{-\frac{1}{(p+1)!}(\sigma^{\mu_1 \cdots \mu_{p+1}})^2}.
\]

Plugging the constraint \((3)\) into the Schild action \((4)\), one obtains

\[
S_p^{n} = -\int d^{p+1} \xi \, e
= -\frac{1}{\lambda_p} \int d^{p+1} \xi \sqrt{-\det \partial_\alpha X^\mu \partial_\beta X_\mu}
= S_{\text{Nambu-Goto}},
\]

where the identity

\[
\det \partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{(p+1)!}(\sigma^{\mu_1 \cdots \mu_{p+1}})^2
\]

was used. Hence the Schild action \((4)\) becomes at least classically equivalent to the famous form of the Nambu-Goto action \(S_{NG}\).
In order to understand the constraint (3) more closely, it is useful to make use of the Hamiltonian formalism. The canonical conjugate momenta to the $X^\mu$ are given by

$$P_\mu = \frac{1}{e^{n-1} p! \lambda^2_p} \left\{ -\frac{1}{(p+1)! \lambda^2_p (\sigma^{\mu_1 \cdots \mu_{p+1}})^2} \right\}^{\frac{1}{2}} \times \sigma_{\mu_1 \cdots \mu_p} \epsilon^{i_1 \cdots i_p} \partial_{i_1} X^{\mu_1} \cdots \partial_{i_p} X^{\mu_p},$$

(6)

where the index $i$ takes the values from 1 to $p$. From (6), it is easy to see that the momenta satisfy the primary constraints

$$P_\mu \partial_i X^\mu = 0,$$

(7)

$$P^2 + \frac{1}{\lambda^2_p} \det \partial_i X^\mu \partial_j X_\mu = 0,$$

(8)

where the lapse (Hamiltonian) constraint (8) is a consequence of the constraint (3) while the shift (momentum) constraints (7) come from the definition (3) trivially. In this sense, the constraint (3) encodes all the dynamical informations of the Schild action for p-brane.

At this stage, it is valuable to point out that in the case of string theory the constraint (3) expresses the space-time uncertainty principle of string [9] when the Poisson bracket is replaced by a commutator in the large $N$ matrix model, and is utilized as the first principle for constructing a type IIB supersymmetric matrix model [7]. Then it is quite natural to ask whether one can also construct a new matrix theory of M-theory if one starts with the constraint (3) in the case of membrane $p = 2$. In the next section, we shall show that this conjecture is indeed true, but to this aim we have to fix the gauge symmetries in terms of the light-cone gauge completely.

3 Matrix Theory from Schild action for membrane

The Schild action for string [3] has provided us with a useful starting point for constructing the matrix models of the type IIB superstring [4, 7]. Since it was shown in the previous section that we can also construct the Schild action for general p-brane, a natural question arises as to whether it is possible to implement an analogous procedure in the M-theory, which amounts to making Matrix Theory by starting with the Schild action for membrane. This is exactly the problem that I want to address in this article.

Hence let us start with the Schild action for bosonic membrane which is obtained by putting $p = 2$ in (2), in eleven space-time dimensions $D = 11$. Then the action (2) becomes

$$S_n^{p=2} = -\frac{1}{n} \int d^3 \xi \ e \left[ \frac{1}{e^n} \left\{ -\frac{1}{3! \lambda^2_p (\sigma^{\mu\nu\rho})^2} \right\} \right]^{\frac{1}{2}} + n - 1 \right],$$

(9)

and the constraint (3) is given by

$$e(\xi) = \frac{1}{\lambda^2_p} \sqrt{-\frac{1}{3! (\sigma^{\mu\nu\rho})^2}}$$
where the constant \( \lambda_2 = l_{11}^3 \) with the eleven dimensional Planck length \( l_{11} \) is the inverse of the membrane tension, which will set to be a unity from now on for simplicity. Here for later convenience, it is useful to cast the above constraint (10) into an alternative form

\[
e(\xi) = \sqrt{g} \Delta,
\]

where we introduced the notations [11]

\[
g = \det_{i,j=1,2} g_{ij} = \det_{i,j=1,2} \partial_i X^\mu \partial_j X_\mu,
\]

\[
\Delta = -g_{00} + u^i u_i = -\left( (\partial_0 - u^i \partial_i) X^\mu \right)^2,
\]

\[
u_i = g_{0i}, \quad u^i = g^{ij} u_j.
\]

(12)

At this point, let us give up the Lorentz covariance and take the light-cone gauge. This is due to the fact that it seems to be difficult to apply the Goldstone-Hoppe map between representation theory of the algebra of the area-preserving diffeomorphisms and the large \( N \) matrix theory [12], whose map will be crucially utilized later in constructing the matrix theory. The light-cone gauge in the membrane theory is chosen to be [11]

\[
\partial_\alpha X^+ = \delta_\alpha^0, \quad u^i = 0,
\]

(13)

where the light-cone coordinates are defined as \( X^\pm = \frac{1}{\sqrt{2}}(X^{10} \pm X^0) = X_\mp \). Note that in the light-cone gauge, \( g \) and \( \Delta \) in (12) become

\[
g = \det_{i,j=1,2} \partial_i X^a \partial_j X^a = \frac{1}{2} \{ X^a, X^b \}^2,
\]

\[
\Delta = -g_{00} = -\left( \partial_0 X^a \right)^2,
\]

(14)

where \( a, b = 1, \cdots, 9 \) stand for the indices of the transverse coordinates, and the curly bracket denotes the Poisson bracket defined by \( \{ A, B \} = \varepsilon^{ij} \partial_i A \partial_j B \).

It is well known that the membrane theory has the area-preserving diffeomorphisms as residual symmetries of the world volume diffeomorphisms in the light-cone gauge [11]. Since our idea in this paper is to fix the gauge symmetries completely, the area-preserving diffeomorphisms are also used to pick a gauge

\[
e(\xi) = \sqrt{\Delta (\partial_0 X^a)^2}.
\]

(15)

Under these gauge conditions, the constraint (11) in which we are mainly interested is simply reduced to

\[
\frac{1}{2} (\partial_0 X^a)^2 - \frac{1}{4} \{ X^a, X^b \}^2 = 0.
\]

(16)
Our basic idea for the construction of the matrix model of M-theory, that is, Matrix Theory is to take the condition (16) as the fundamental condition. This strategy was adopted by Yoneya [7] to build the matrix model of type IIB superstring theory, IIB matrix model, since the corresponding constraint in string theory realizes his space-time uncertainty relation [7, 9].

Here it is worthwhile to comment on the difference between string and membrane. In string theory, it is transparent that the constraint (3) with \( p = 1 \) describes the space-time uncertainty relation \( \Delta T \Delta X \geq l_s \) in the matrix theory [7], while in membrane theory it is at present unclear whether the corresponding constraint (10) (or (11)) or its gauge fixed version (16) expresses the space-time uncertainty relation or not. This problem is closely related to our gauge choice where the light-cone gauge kills the dynamical degrees of freedom associated with \( X^+ \) or \( X^0 \) by choosing the gauge \( \partial_0 X^+ = \delta_{a0} \). The space-time uncertainty relation advocated in [7, 9] is the statement about the relation between the time component \( X^0 \) or \( X^+ \) and one space component \( X^a \) so that it is obvious that we cannot understand the space-time uncertainty relation within the present formulation. Perhaps it might be an important step towards the covariant formulation of Matrix Theory that we try to understand the relationship between the constraint (16) and the space-time uncertainty relation of membrane without taking the light-cone gauge. In this note, without worrying such a difficult problem, let us follow the strategy mentioned above and ask if we can make Matrix Theory on a basis of the gauge-fixed constraint (16).

Before doing so, let us use the Goldstone-Hoppe prescription given by [12]

\[ \{ X^a, X^b \} \rightarrow \frac{1}{i} [X^a, X^b], \]

\[ \int d\xi^1 d\xi^2 \rightarrow \text{Tr}. \]  

(17)

Then the constraint (16) becomes

\[ \frac{1}{2} (\partial_0 X^a)^2 + \frac{1}{4} [X^a, X^b]^2 = 0. \]  

(18)

Provided that we require that a weaker form of this constraint

\[ \int d\xi^0 \text{Tr} \left( \frac{1}{2} (\partial_0 X^a)^2 + \frac{1}{4} [X^a, X^b]^2 \right) = 0 \]  

(19)

is the fundamental condition for the construction of Matrix Model, the partition function can be defined as

\[ Z = \int DX^a J[X] \delta \left( \int d\xi^0 \text{Tr} \left( \frac{1}{2} (\partial_0 X^a)^2 + \frac{1}{4} [X^a, X^b]^2 \right) \right), \]  

(20)

where \( J[X] \) is a certain measure factor which will be determined by following the similar consideration as done in the reference [7] whose result is given by

\[ J[X] = \int D\theta \exp \int d\xi^0 \text{Tr} \left( -\theta^T \partial_0 \theta - \theta^T \gamma_a [\theta, X^a] \right). \]  

(21)
After all, the partition function has the form

$$ Z = \int Dc DX^a D\theta \exp \left[ c \int d\xi^0 Tr \left( \frac{1}{2} (D_0 X^a)^2 + \frac{1}{4} [X^a, X^b]^2 \right) 
- \int d\xi^0 Tr \left( \theta^T D_0 \theta + \theta^T \gamma_a [\theta, X^a] \right) \right], $$

where the covariant derivative $D_0 = \partial_0 + iA_0$ was introduced to guarantee the closure under the $N = 2$ supersymmetry

$$\delta X^a = -2\epsilon^T \gamma^a \theta,$$
$$\delta \theta = \frac{c}{2} \left( D_0 X^a \gamma_a + \gamma_- + \frac{1}{2} [X^a, X^b] \gamma_{ab} \right) \epsilon + \epsilon', $$
$$\delta A_0 = -2\epsilon^T \theta, $$
$$\delta c = 0, $$

where $\epsilon$ and $\epsilon'$ are two independent spinorial parameters. In this way, we have succeeded in deriving a Matrix Theory with $N = 2$ supersymmetry by starting with the Schild action for membrane. Of course, following the line of similar arguments to in the reference [7], we can show that the original Matrix Theory [1] can be interpreted as the low energy effective theory of many distant clusters of D-branes from this Matrix Theory.

## 4 Discussions

In this paper, we have pursued the possibility of formulating Matrix Theory in terms of the Schild action for membrane. We have seen that in order to carry out this idea we have to fix the gauge symmetries in the light-cone gauge and furthermore fix the residual area-preserving diffeomorphisms completely. The constraint in the Schild action for membrane, which we have adopted as the fundamental condition, encodes all the dynamical informations of the system, but it is not clear that this constraint has something to do with the space-time uncertainty principle advocated by Yoneya [7, 9] because of the light-cone gauge choice. To fully understand the details of it, we have to wait for completion of the covariant formulation of Matrix Theory.

However, we should remark that no Lorentz covariant formulation can be more than a step on the road to the non-perturbative formulation of M-theory. This is because whatever it is, M-theory cannot have its most fundamental formulation on the fixed, flat Minkowski background mainifold. In other words, M-theory should be independent of the background metric in the most fundamental form, and the background metric must emerge from its fundamental theory through some unknown dynamical mechanism. Recently, such matrix models which do not depend on the background metric have been pushed forward [13, 14].
Acknowledgement

The author thanks A.Sugamoto for valuable discussions and continuous encouragement. This work was supported in part by Grant-Aid for Scientific Research from Ministry of Education, Science and Culture No.09740212.

References

[1] T.Banks, W.Fischler, S.H.Shenker and L.Susskind, Phys.Rev.D55 (1997) 5112.
[2] J.Polchinski, Phys.Rev.Lett.74 (1995) 4724.
[3] C.M.Hull and P.K.Townsend, Nucl.Phys.B438 (1995) 109; E.Witten, Nucl.Phys.443 (1995) 85.
[4] N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, Nucl.Phys.B498 (1997) 467.
[5] A.Schild, Phys.Rev.D16 (1977) 1722.
[6] M.B.Green and J.H.Schwarz, Phys.Lett.B136 (1984) 367.
[7] T.Yoneya, Prog.Theor.Phys.97 (1997) 949.
[8] I.Oda, hep-th/9709005, Mod.Phys.Lett.A (in press); hep-th/9710030, Nucl.Phys.B (in press).
[9] T.Yoneya, Mod.Phys.Lett.A4 (1989) 1587; M.Li and T.Yoneya, Phys.Rev.Lett.78 (1997) 1219.
[10] G. Amelino-Camelia, J.Lukierski and A. Nowicki, hep-th/9706031 and references therein.
[11] B.de Witt, J.Hoppe and N.Nicolai, Nucl.Phys.B305 [FS23] (1988) 545.
[12] J.Hoppe, MIT Ph.D Thesis, 1982, published in Soryuushiron Kenkyu (Kyoto) 80 (1989) 145.
[13] L.Smolin, hep-th/9710191.
[14] I.Oda, hep-th/9801051.