Morse functions to graphs and topological complexity for hyperbolic 3-manifolds

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Abstract

Scharlemann and Thompson define the width of a 3-manifold $M$ as a notion of complexity based on the topology of $M$. Their original definition had the property that the adjacency relation on handles gave a linear order on handles, but here we consider a more general definition due to Saito, Scharlemann and Schultens, in which the adjacency relation on handles may give an arbitrary graph. We show that for compact hyperbolic 3-manifolds, this is linearly related to a notion of metric complexity, based on the areas of level sets of Morse functions to graphs, which we call Gromov area.
1 Introduction

Mostow rigidity implies that for hyperbolic 3-manifolds, the hyperbolic metric is a topological invariant, so one might hope that the topological and metric complexities are related. We shall show that this is indeed the case for certain definitions of topological and metric complexity. We first describe the notions of complexity we shall use, and then give a brief outline of the arguments used to relate topological and metric complexity in the subsequent sections. In [HM16] we considered the linear version of these invariants, while in this paper we consider the more general case of invariants constructed from maps to graphs. It will be convenient to work with the collection of hyperbolic 3-manifolds which are complete, but do not contain cusps, and are not necessarily of finite volume. The reason for not considering manifolds with cusps is that in the cusped case the surfaces separating the 3-dimensional regions in the topological decomposition we construct from the metric might have essential intersection with the cusps. In other words, if the cusps are truncated to form a hyperbolic manifold with torus boundary components, then the dividing surfaces may be surfaces with essential boundary components on the boundary tori. However, the currently available versions of the topological decomposition results we use, due to Scharlemann-Thompson and Saito-Scharlemann-Schultens, assume that the dividing surfaces are closed.

This paper is not entirely self-contained, and relies on the results of [HM16], however we review the main definitions and results from [HM16] for the convenience of the reader.

1.1 Metric complexity

In [HM16], we considered the following definition of metric complexity. Let $M$ be a closed Riemannian 3-manifold, and let $f: M \to \mathbb{R}$ be a Morse function, i.e. $f$ is a smooth function, all critical points are non-degenerate, and distinct critical points have distinct images in $\mathbb{R}$. We define the area of $f$ to the maximum area of any level set $F_t = f^{-1}(t)$ over all points $t \in \mathbb{R}$. We define the Morse area of $M$ to be the infimum of the area of all Morse functions $f: M \to \mathbb{R}$.

More generally, we may consider maps $f: M \to X$, where $X$ is a trivalent graph. Recall that for a Morse function $f: M \to \mathbb{R}$ there are singularities of index 0, 1, 2 and 3. The singularities of index 0 and 3 are known as birth or death singularities respectively, and the level set foliation near the singular point in $M$ is locally homeomorphic to the level sets of the function $x^2 + y^2 + z^2$ close to the origin in $\mathbb{R}^3$. For singularities of index 2 and 3, the level sets near the singular point in $M$ are locally homeomorphic to the level sets of the function $x^2 + y^2 - z^2$ close to the origin in $\mathbb{R}^3$.

In the case of index 2 or 3, there is a map from a small open ball containing the singular point to the leaf space of the level set foliation. As the singular leaf divides a small ball about the singular point into three connected components, the leaf space is a trivalent graph with a single vertex and three edges, and we call such a map a trivalent singularity. If $X$ is a trivalent tree, we say a
map $f : M \to X$ is Morse if it is a Morse function on the interior of each edge of $X$, and at each trivalent vertex $v$ of $X$ the pre-image under $f$ is locally homeomorphic to a trivalent singularity. We say that the area of $f$ is the maximum area of $F_t$, as $t$ runs over all points $t \in X$. The Gromov area of $M$ is the infimum of the area of $f : M \to X$ over all trivalent graphs $X$, and all Morse functions $f : M \to X$.

This definition of metric complexity is a variant of Uryson width, studied by Gromov in [Gro88], though we consider the area of the level sets instead of the diameter. Alternatively, one may consider it to be a variant of the definition of the waist of a manifold, but we prefer to call it area, as the dimension of our spaces is fixed, and the fibers have dimension two.

### 1.2 Topological complexity

We now describe the notions of topological complexity we shall consider. A handlebody is a compact 3-manifold with boundary, homeomorphic to a regular neighborhood of a graph in $\mathbb{R}^3$. Up to homeomorphism, a handlebody is determined by the genus $g$ of its boundary surface. Every 3-manifold $M$ has a Heegaard splitting, which is a decomposition of the manifold into two handlebodies. This immediately gives a notion of complexity for a 3-manifold, called the Heegaard genus, which is the smallest genus of any Heegaard splitting of the 3-manifold.

There is a refinement of this, due to Scharlemann and Thompson [ST94], which we now describe. A compression body $C$ is a compact 3-manifold with boundary, constructed by gluing some number of 2-handles to one side of a compact (but not necessarily connected) surface cross interval and capping off any resulting 2-sphere components with 3-balls. The side of the surface cross interval with no attached 2-handles is called the top boundary of the compression body, and denoted by $\partial_+ C$, and any other boundary components are called the lower boundary of the compression body, and denoted by $\partial_- C$. A linear generalized Heegaard splitting, which we shall abbreviate to linear splitting, is a decomposition of a 3-manifold $M$ into a linearly ordered sequence of (not necessarily connected) compression bodies $C_1, \ldots, C_{2n}$, such that the top boundary of an odd numbered compression body $C_{2i+1}$ is equal to the top boundary of the subsequent compression body $C_{2i+2}$, and the lower boundary of $C_{2i+1}$ is equal to the lower boundary of the previous compression body $C_{2i}$. Let $H_i$ be the sequence of surfaces consisting of the top boundaries of the compression bodies $C_{2i-1}$ and $C_{2i}$. The complexity $c(H_i)$ of the surface $H_i$ is the sum of the genera of each connected component, and the complexity of the linear splitting is the collection of integers $\{c(H_i)\}$, arranged in decreasing order. We order these complexities with the lexicographic ordering. The width of the linear splitting is the maximum value (i.e. the first value) of $c(H_i)$ in the collection $\{c(H_i)\}$.

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1We warn the reader that these are often referred to as generalized Heegaard splittings in the literature; however we wish to distinguish them from a more general notion described subsequently, which is also occasionally referred to in the literature as a generalized Heegaard splitting.
The linear width of a 3-manifold \( M \) is the minimum width over all possible linear generalized Heegaard splittings. As a Heegaard splitting is a special case of a linear splitting, the Heegaard genus of \( M \) is an upper bound for the linear width of \( M \). A linear splitting which gives the minimum complexity of all possible linear splittings is called the thin position linear splitting.

There is a further refinement of this, described in Saito, Scharlemann and Schultens \cite{SSS16}. A graph generalized Heegaard splitting, which we shall abbreviate to graph splitting, and is called a fork complex in \cite{SSS16}, is a decomposition of a compact 3-manifold \( M \) into compression bodies \( \{C_i\} \), such that for each compression body \( C_i \), there is a compression body \( C_j \) such that the top boundary of \( C_i \) is equal to the top boundary of \( C_j \). Furthermore, for each component of the lower boundary of \( C_i \), there is a compression body \( C_k \), such that that component of the lower boundary of \( C_i \) is equal to a component of the lower boundary of \( C_k \). We emphasize that different components of the lower boundary of \( C_i \) may be attached to lower boundary components of different compression bodies. Let \( \{H_i\} \) be the collection of top boundary surfaces. The complexity of the graph splitting is the collection of integers \( \{c(H_i)\} \), arranged in decreasing order. Again, we put the lexicographic ordering on these complexities. A graph splitting which realizes the minimum complexity is called a thin position graph splitting. The width of the graph splitting is the maximum integer (i.e the first integer) that appears in the complexity. The graph width of a 3-manifold \( M \) is the minimum width over all possible graph splittings of \( M \). As a linear splitting is a special case of a graph splitting, the linear width of \( M \) is an upper bound for the graph width of \( M \). The graph corresponding to the graph splitting is the graph whose vertices are compression bodies, with edges connecting pairs of compression bodies with common boundary components.

1.3 Results

In order to bound metric complexity in terms of topological complexity we shall assume the following result announced by Pitts and Rubinstein \cite{PR86} (see also Rubinstein \cite{Rub05}).

**Theorem 1.1.** \cite{PR86,Rub05} Let \( M \) be a Riemannian 3-manifold with a strongly irreducible Heegaard splitting. Then the Heegaard surface is isotopic to a minimal surface, or to the boundary of a regular neighborhood of a non-orientable minimal surface with a small tube attached vertically in the I-bundle structure.

A full proof of this result has not yet appeared in the literature, though recent progress has been made by Colding and De Lellis \cite{CDL03}, De Lellis and Pellandrini \cite{DLP10}, and Ketover \cite{Ket13}.

In \cite{HM16} we showed:

**Theorem 1.2.** There is a constant \( K > 0 \), such that for any closed hyperbolic 3-manifold,

\[
K(\text{linear width}(M)) \leq \text{Morse area}(M) \leq 4\pi(\text{linear width}(M)),
\]

(1)

where the right hand bounds hold assuming Theorem 1.1.
In this paper we show:

**Theorem 1.3.** There is a constant $K > 0$, such that for any closed hyperbolic 3-manifold,

$$K(\text{graph width}(M)) \leq \text{Gromov area}(M) \leq 4\pi(\text{graph width}(M)),$$

(2)

where the right hand bounds hold assuming Theorem 1.1.

We also expect there to be upper and lower bounds on topological complexity in terms of Uryson width, i.e. using diameter instead of area, but we do not expect them to be linear.

### 1.4 Related work in 3-manifolds

It may be of interest to compare our results with recent work of Brock, Minsky, Namazi and Souto [BMNS16] on manifolds with bounded combinatorics. Let $C_1, \ldots, C_n$ be a finite collection of homeomorphism types of compact 3-manifolds with marked boundary, which we shall refer to as *model pieces*, and fix a metric on each one. A 3-manifold $M$ is said to have *bounded combinatorics* if it is a union of (possibly infinitely many) model pieces glued together by homeomorphisms along their boundaries, with certain restrictions on the gluing maps, which we do not describe in detail here. In particular, a manifold with bounded combinatorics is a manifold of bounded topological width. They show that such a manifold $M$ is hyperbolic, with a lower bound on the injectivity radius, and the hyperbolic metric is $K$-bilipschitz homeomorphic to the induced metric on $M$ arising from the metrics on the model pieces. A choice of foliation with compact leaves, containing the boundary leaves, on each model piece then shows that the metric complexity is linearly related to the topological complexity for this class of manifolds, where the linear constants depend on the collection of model pieces.

Note that in our context, a bound on the topological width of the manifold implies that the manifold is a union of compression bodies of bounded genus, and there are finitely many of these up to homeomorphism. Their result assumes restrictions on the gluing maps, but then shows the resulting manifold is hyperbolic, but the bilipschitz constant $K$ depends on the width of $M$, i.e the genus of the compression bodies. We assume that the manifold $M$ is compact and hyperbolic, and make no restriction on the gluing maps between the compression bodies, but we show that the linear constants relating topological and metric complexities are independent of the genus of the compression bodies.

### 1.5 Outline

In [HM16] we considered the linear case, in which the range of the Morse function $f: M \to \mathbb{R}$ is $\mathbb{R}$. Such a Morse function has the property that for each $t \in \mathbb{R}$, the pre-image $f^{-1}(t)$ is compact and separating. For the case in which the range of the Morse function $f: M \to X$ is a graph, one may consider the lifted
Morse function $\tilde{f} : \tilde{M} \rightarrow \tilde{X}$, where $\tilde{X}$ is the universal cover of $X$, and $\tilde{M}$ is the corresponding cover of $M$. This lifted Morse function has the property that for each $t \in \tilde{X}$, each pre-image $\tilde{f}^{-1}(x)$ is compact and separating, and so many of the arguments from [HM16] go through directly in this case. In particular, we construct polyhedral approximations to the level sets of $\tilde{f}$, and show that they have bounded topological complexity, as we now describe.

A choice of Margulis constant $\mu$ determines a thick-thin decomposition for $M$, in which the thin part is a disjoint union of Margulis tubes. We also choose a Voronoi decomposition determined by a maximal $\epsilon$-separated collection of points in $M$. This implies that every Voronoi cell has diameter at most $\epsilon$, and, given $\mu$, we may choose $\epsilon$ small enough such that every Voronoi cell that intersects the thick part contains an embedded ball of radius $\epsilon/2$. The thick-thin decomposition of $M$, and the Voronoi decomposition of $M$, lift to thick-thin decompositions and Voronoi decompositions of the cover $\tilde{M}$. We give the details of this construction in Sections 2.1, 2.2 and 2.3.

A separating surface $F$ in $\tilde{M}$ determines a partition of the Voronoi cells, depending on which side of the surface the majority of the volume of the (metric) ball of radius $\epsilon/2$ inside the Voronoi cell lies. We will call the boundary between these two sets of Voronoi cells a polyhedral surface $S$, which is a union of faces of Voronoi cells, and we can think of this as a combinatorial approximation to the original surface $F$.

A key observation from [HM16] is that the number of faces of the polyhedral surface in the thick part is bounded by the area of $F$. This is because in the thick part of $M$, the metric ball of radius $\epsilon/2$ in each Voronoi cell is embedded, so moving the ball along a geodesic connecting the centers of the two Voronoi produces at some point a metric ball whose volume is divided exactly in two, giving a lower bound to the area of $F$ near that point. There are bounds on the number of vertices and edges of any Voronoi cell in terms of $\epsilon$, so a bound on the number of faces of $S$ in the thick part gives a bound on the Euler characteristic of $S$. We are unable to control the number of faces in the thin part, so we cap off the part of $S$ in the thick part with surfaces of bounded Euler characteristic contained in the thin part. This produces surfaces of bounded genus, which we call capped surfaces.

In this way, the lift of a Morse function $\tilde{f} : \tilde{M} \rightarrow \tilde{X}$ gives rise to a collection of polyhedral surfaces in $\tilde{M}$ of bounded genus. These surfaces are constant except at finitely many points of $\tilde{X}$, which we call cell splitters, where a level set divides the ball contained in a Voronoi cell exactly in half. We give the details of the construction of the capped surfaces and the properties of the cell splitters in Sections 2.4 and 2.5.

The key step, in Section 2.6, is to show that we may construct these surfaces equivariantly in $\tilde{M}$, so they project down to embedded surfaces in $M$, with the same bounds on their topological complexity.

Finally, in Section 2.7, by considering the local configuration near a cell splitter, we show that the regions between the capped surfaces may be constructed using a number of handles bounded in terms of the area of the level sets $\tilde{f}^{-1}(t)$,
and so this the bounds topological complexity of the decomposition of $M$ given by the capped surfaces in terms of metric complexity of $M$.

The bound in the other direction is a direct consequence of the bound from [HM16], though we review the argument in the Section for the convenience of the reader.

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2 Gromov area bounds graph width

In this section we show that we can bound the topological complexity of the manifold in terms of its metric complexity, i.e. we show that graph width is bounded in terms of Gromov area.

**Theorem 2.1.** There is a constant $K$, such that for any closed hyperbolic 3-manifold $M$,

$$\text{graph width}(M) \leq K(\text{Gromov area}(M)).$$

Let $f: M \to X$ be a Morse function onto a graph $X$, such that the Gromov area of $f$ is arbitrarily close to the Gromov area of $M$. Any metric graph is arbitrarily close to a trivalent metric graph, so we may assume the graph is trivalent. We now show that we may assume the level sets of $f$ are connected.

**Proposition 2.2.** Let $M$ be a Riemannian manifold, and let $f: M \to X$ be a Morse function onto a trivalent graph $X$. Then there is a trivalent graph $X'$, and a Morse function $f': M \to X'$ with connected level sets, with $\text{Gromov area}(f') \leq \text{Gromov area}(f)$.

**Proof.** The level sets of the function $f$ give a singular foliation of $M$ with compact leaves, which we shall call the *level set foliation*, and the leaves of this foliation are precisely the connected components of the pre-images of points in $M$. Consider the leaf space $L$ of the level set foliation, i.e. the space obtained from $M$ by identifying points in the same leaf. As all leaves are compact, the leaf space is Hausdorff. The leaf space is a trivalent graph, with vertices corresponding to vertex singularities, and the maximum area of the pre-images of the quotient map is less than or equal to the maximum area of the pre-images of $f$. Therefore, we may choose $f'$ to be the leaf space quotient map $f': M \to L$, which is a Morse function onto a trivalent graph, and has connected level sets,
with the property that the area of the level sets of \( f' \) is bounded by the area of the level sets of \( f \).

In particular, this means that the vertices of \( X \) are precisely the critical points of the Morse function \( f \) in which a connected level set splits into two connected components.

### 2.1 Morse functions to trees

We would like to work in the cover \( \tilde{M} \) of \( M \) corresponding to the universal cover \( \tilde{X} \) of the graph \( X \), which will have the key advantage that all pre-image surfaces are separating in \( M \). In fact, the induced map on fundamental groups \( f_* : \pi_1 M \to \pi_1 X \) is surjective, but as we do not use this property, we omit the proof.

Let \( p : \tilde{M} \to M \) be the cover of \( M \) corresponding to the kernel of the induced map \( f_* : \pi_1 M \to \pi_1 X \), and let \( c : \tilde{X} \to X \) be the universal cover of \( X \), so \( \tilde{X} \) is a tree. Then the map \( f \circ p : \tilde{M} \to X \) lifts to a map \( h = \tilde{f} \circ \tilde{p} : \tilde{M} \to \tilde{X} \). Since each leaf \( F_t \) in \( M \) maps to a single point in \( X \), the fundamental group of each leaf is contained in \( \ker(f) \). Therefore, each leaf in \( M \) lifts to a leaf in \( \tilde{M} \), and as the cover is regular, the pre-image of a point \( t \in \tilde{X} \) is a disjoint union of homeomorphic copies of \( F_t \). In particular, the area bound for the leaves \( F_t \) in \( M \) is also an area bound for the leaves \( H_t = h^{-1}(t) \) in \( \tilde{M} \).

\[
\begin{array}{c}
\tilde{M} \xrightarrow{h = \tilde{f} \circ \tilde{p}} \tilde{X} \\
p \\
M \xrightarrow{f} X
\end{array}
\]

As \( \tilde{X} \) is a tree, every point is separating, and so every pre-image surface \( H_t = h^{-1}(t) \) is also separating.

### 2.2 Voronoi cells

We will approximate the level sets of \( f \) by surfaces consisting of faces of Voronoi cells. We now describe in detail the Voronoi cell decompositions we shall use, and their properties. The definitions in this section are taken verbatim from [HM16], but we include them in this section for the convenience of the reader.

A **polygon** in \( \mathbb{H}^3 \) is a bounded subset of a hyperbolic plane whose boundary consists of a finite number of geodesic segments. A **polyhedron** in \( \mathbb{H}^3 \) is a convex topological 3-ball in \( \mathbb{H}^3 \) whose boundary consists of a finite collection of polygons. A **polyhedral cell decomposition** of \( \mathbb{H}^3 \) is a cell decomposition in which which every 3-cell is a polyhedron, each 2-cell is a polygon, and the edges
are all geodesic segments. We say a cell decomposition of a complete hyperbolic manifold \( M \) is polyhedral if its pre-image in the universal cover \( \mathbb{H}^3 \) is polyhedral.

Let \( X = \{ x_i \} \) be a discrete collection of points in 3-dimensional hyperbolic space \( \mathbb{H}^3 \). The Voronoi cell \( V_i \) determined by \( x_i \in X \) consists of all points of \( M \) which are closer to \( x_i \) than any other \( x_j \in X \), i.e.

\[
V_i = \{ x \in \mathbb{H}^3 \mid d(x, x_i) \leq d(x, x_j) \text{ for all } x_j \in \tilde{X} \}.
\]

We shall call \( x_i \) the center of the Voronoi cell \( V_i \), and we shall write \( V = \{ V_i \} \) for the collection of Voronoi cells determined by \( X \). Voronoi cells are convex sets in \( \mathbb{H}^3 \), and hence topological balls. The set of points equidistant from both \( x_i \) and \( x_j \) is a totally geodesic hyperbolic plane in \( \mathbb{H}^3 \). A face \( \Phi \) of the Voronoi decomposition consists of all points which lie in two distinct Voronoi cells \( V_i \) and \( V_j \), so \( \Phi \) is contained in a geodesic plane. An edge \( e \) of the Voronoi decomposition consists of all points which lie in three distinct Voronoi cells \( V_i, V_j, V_k \), which is a geodesic segment, and a vertex \( v \) is a point lying in four distinct Voronoi cells \( V_i, V_j, V_k \) and \( V_l \). By general position, we may assume that all edges of the Voronoi decomposition are contained in exactly three distinct faces, the collection of vertices is a discrete set, and there are no points which lie in more than four distinct Voronoi cells. We shall call such a Voronoi decomposition a regular Voronoi decomposition, and it is a polyhedral decomposition of \( \mathbb{H}^3 \). As each edge is 3-valent, and each vertex is 4-valent, this implies that the dual cell structure is a simplicial triangulation of \( \mathbb{H}^3 \), which we shall refer to as the dual triangulation. The dual triangulation may be realised in \( \mathbb{H}^3 \) by choosing the vertices to be the centers \( x_i \) of the Voronoi cells and the edges to be geodesic segments connecting the vertices, and we shall always assume that we have done this. In this case the triangles and tetrahedra are geodesic triangles and tetrahedra in \( \mathbb{H}^3 \).

Given a collection of points \( X = \{ x_i \} \) in a hyperbolic 3-manifold \( M \), let \( \tilde{X} \) be the pre-image of \( X \) in the universal cover of \( M \), which is isometric to \( \mathbb{H}^3 \). As \( \tilde{X} \) is equivariant, the corresponding Voronoi cell decomposition \( V \) of \( \mathbb{H}^3 \) is also equivariant. The distance condition implies that the interior of each Voronoi cell \( V \) is mapped down homeomorphically by the covering projection, though the covering projection may identify faces, edges or vertices of \( V \) under projection into \( M \). By abuse of notation, we shall refer to the resulting polyhedral decomposition of \( M \) as the Voronoi decomposition \( V \) of \( M \). By general position, we may assume that \( V \) is regular. The dual triangulation is also equivariant, and projects down to a triangulation of \( M \), which we will also refer to as the dual triangulation, though this triangulation may no longer be simplicial.

We shall write \( B(x, r) \) for the closed metric ball of radius \( r \) about \( x \) in \( M \), i.e.

\[
B(x, r) = \{ y \in M \mid d(x, y) \leq r \}.
\]

A metric ball in \( M \) need not be a topological ball in general. We shall write \( \text{inj}_M(x) \) for the injectivity radius of \( M \) at \( x \), i.e. the radius of the largest embedded ball in \( M \) centered at \( x \). Then the injectivity radius of \( M \), denoted
\text{inj}(M)$, is defined to be

$$\text{inj}(M) = \inf_{x \in M} \text{inj}_M(x).$$

We say a collection \( \{x_i\} \) of points in \( M \) is \( \epsilon \)-separated if the distance between any pair of points is at least \( \epsilon \), i.e. \( d(x_i, x_j) \geq \epsilon \), for all \( i \neq j \). Let \( \{x_i\} \) be a maximal collection of \( \epsilon \)-separated points in \( M \), and let \( V \) be the corresponding Voronoi cell division of \( M \). Since the collection \( \{x_i\} \) is maximal, each Voronoi cell is contained in a metric ball of radius \( \epsilon \) about its center. Furthermore, if the injectivity radius at the center \( x_i \) is at least \( 2\epsilon \), then as the points \( x_i \) are distance at least \( \epsilon \) apart, each Voronoi cell contains a topological ball of radius \( \epsilon/2 \) about its center, i.e.

$$B(x_i, \epsilon/2) \subset V_i \subset B(x_i, \epsilon).$$

**Definition 2.3.** Let \( M \) be a complete hyperbolic 3-manifold. We say a Voronoi decomposition \( V \) is \( \epsilon \)-regular, if it is regular, and it arises from a maximal collection of \( \epsilon \)-separated points.

A simple arc in the boundary of a tetrahedron is a properly embedded arc in a face of the tetrahedron with endpoints in distinct edges. A triangle in a tetrahedron is a properly embedded disc whose boundary is a union of three simple arcs, and a quadrilateral is a properly embedded disc whose boundary is the union of four simple arcs. A normal surface in a triangulated 3-manifold is a surface that intersects each tetrahedron in a union of normal triangles and quadrilaterals.

One useful property of \( \epsilon \)-regular Voronoi decompositions is that the boundary of any union of Voronoi cells is an embedded surface, in fact an embedded normal surface in the dual triangulation.

**Proposition 2.4.** [HM16, Proposition 2.2] Let \( M \) be a complete hyperbolic manifold without cusps, and let \( V \) be an \( \epsilon \)-regular Voronoi decomposition. Let \( P \) be a union of Voronoi cells in \( V \), and let \( S \) be the boundary of \( P \). Then \( S \) is an embedded surface in \( M \).

In [HM16] this result is stated for compact hyperbolic 3-manifolds, but the proof works for complete hyperbolic 3-manifolds without cusps.

We shall say a Voronoi cell \( V_i \) with center \( x_i \) is an \( \epsilon \)-deep Voronoi cell if the injectivity radius at \( x_i \) is at least \( 4\epsilon \), i.e. \( \text{inj}_M(x_i) \geq 4\epsilon \), and in particular this implies that the metric ball \( B(x_i, 3\epsilon) \) is a topological ball. We shall also call centers, faces, edges and vertices of \( \epsilon \)-deep Voronoi cells \( \epsilon \)-deep. In the next section we will choose a fixed \( \epsilon \) independent of the manifold \( M \), and we will just say deep instead of \( \epsilon \)-deep. We shall write \( \mathcal{W} \) for the subset of \( V \) consisting of deep Voronoi cells. If \( \epsilon < \frac{1}{4} \text{inj}(M) \), then \( V = \mathcal{W} \) and all Voronoi cells are deep.

Finally, we recall that there are bounds, which only depend on \( \epsilon \), on the number of vertices, edges and faces of a deep Voronoi cell.
Proposition 2.5. \[HM16\] Proposition 2.3 Let \( M \) be a complete hyperbolic 3-manifold with an \( \epsilon \)-regular Voronoi decomposition \( \mathcal{V} \), and let \( \mathcal{W} \) be the collection of deep Voronoi cells. Then there is a number \( J \), which only depends on \( \epsilon \), such that each deep Voronoi cell \( W_i \in \mathcal{W} \) has at most \( J \) faces, edges and vertices.

Again, in \[HM16\], these results are stated for compact hyperbolic 3-manifolds, but the proofs work for complete hyperbolic 3-manifolds without cusps.

2.3 Margulis tubes

We will use the Margulis Lemma and the thick-thin decomposition for finite volume hyperbolic 3-manifolds, and we now review these results.

Given a number \( \mu > 0 \), let \( X_\mu = M_{[\mu, \infty)} \) be the thick part of \( M \), i.e. the union of all points \( x \) of \( M \) with \( \text{inj}_M(x) \geq \mu \). We shall refer to the closure of the complement of the thick part as the thin part and denote it by \( T_\mu = M \setminus X_\mu \).

The Margulis Lemma states that there is a constant \( \mu_0 > 0 \), such that for any compact hyperbolic 3-manifold, the thin part is a disjoint union of solid tori, called Margulis tubes, and each of these solid tori is a regular metric neighborhood of an embedded closed geodesic of length less than \( \mu_0 \). In the case in which \( M \) is complete without cusps, there is an extra possibility, as a component of the thin part may also be the universal cover of such a solid torus, and we shall refer to such a component as an infinite Margulis tube. We shall call a number \( \mu_0 \) for which this result holds a Margulis constant for \( \mathbb{H}^3 \).

If \( \mu_0 \) is a Margulis constant for \( \mathbb{H}^3 \), then so is \( \mu \) for any \( 0 < \mu < \mu_0 \), and furthermore, given \( \mu \) and \( \mu_0 \) there is a number \( \delta > 0 \) such that \( N_\delta(T_\mu) \subseteq T_{\mu_0} \).

For the remainder for this section we shall fix a pair of numbers \( (\mu, \epsilon) \) such that there are Margulis constants \( 0 < \mu_1 < \mu < \mu_2 \), a number \( \delta \) such that \( N_\delta(T_\mu) \subseteq T_{\mu_2} \setminus T_{\mu_1} \), and \( \epsilon = \frac{1}{4} \min\{\mu_1, \delta\} \). We shall call \((\mu, \epsilon)\) a choice of MV-constants for \( \mathbb{H}^3 \).

Let \((\mu, \epsilon)\) be a choice of MV-constants, and consider an \( \epsilon \)-regular Voronoi decomposition of \( M \). The fact that \( N_\delta(T_\mu) \subseteq T_{\mu_2} \setminus T_{\mu_1} \) means that we may adjust the boundary of \( T_\mu \) by an arbitrarily small isotopy so that it is transverse to the Voronoi cells, and we will assume that we have done this for the remainder of this section. Our choice of \( \epsilon \) implies that the thick part \( X_\mu \) is contained in the Voronoi cells in the deep part, i.e. \( X_\mu \subset \bigcup_{W_i \in \mathcal{W}} W_i \), so in particular \( \partial X_\mu = \partial T_\mu \) is contained in the deep part. Furthermore, each deep Voronoi cell hits at most one component of \( T_\mu \).

2.4 Cell splitters

The polyhedral surfaces we construct will be constant, except for a discrete collection of points in \( Y \), which roughly speaking correspond to points \( t \in Y \) for which the level set \( f^{-1}(t) \) divide a Voronoi cell in half. For technical reasons, we use points which divide a ball of fixed size in the Voronoi cell in half, as we now describe.

Let \( t \) be a point in a trivalent tree \( Y \). We shall write \( Y_t^+ \) for the closures of the connected components of \( Y \setminus \{t\} \), and we shall call these the complements of
there is at least one complementary region \( Y \) of \( v \).

**Proof.** We first show existence. Let \( Y \to t \) a tree with an \( \epsilon \)-regular Voronoi decomposition \( V \). Let \( h: M \to Y \) be a Morse function to a tree, and let \( V \) be a Voronoi cell with center \( x \). Suppose that a point \( t \in Y \) has the property that for each complementary region \( H_t^{c_1} \), the volume of \( H_t^{c_1} \cap B(x, \epsilon/2) \cap V \) is at most half the volume of the topological ball \( B(x, \epsilon/2) \cap V \). Then we say that \( t \) is a cell splitter for the Voronoi cell \( V \).

**Proposition 2.7.** Let \( M \to Y \) be a complete hyperbolic 3-manifold without cusps, with an \( \epsilon \)-regular Voronoi decomposition \( V \). Let \( h: M \to Y \) be a Morse function to a tree, and let \( V \) be a Voronoi cell with center \( x \). Then there is a unique cell splitter \( t \in Y \) for \( V \).

**Proof.** We first show existence. Let \( B \) be the topological ball \( B(x, \epsilon/2) \cap V \), and let \( v \) be the volume of this ball. Consider \( h(B) \subset Y \). If \( Y \) is a vertex of \( Y \), which is a cell splitter, then we are done. Otherwise, suppose no vertex of \( h(B) \) is a cell splitter. If \( t \) is a vertex in \( h(B) \) which is not a cell splitter, then there is at least one complementary region \( Y_t^{c_1} \) such that \( H_t^{c_1} \cap B(x, \epsilon/2) \cap V \) has volume more than \( \frac{1}{2}v \) and \( Y_t^{c_1} \cap h(B) \) has at least one fewer vertex. So proceeding by induction, we may reduce to the case in which \( h(B) \) contains an interval \( I \) with no vertices such that \( h^{-1}(I) \cap B(x, \epsilon/2) \cap V \) has volume at least \( \frac{1}{2}v \). In this case, let \( t_0 \) and \( t_1 \) be the endpoints of \( I \), and consider \( h^{-1}([t_0, s]) \), for \( s \in I \). When \( s = t_0 \), this has volume less than \( \frac{1}{2}v \), and has volume greater than \( \frac{1}{2}v \) when \( s = t_1 \). As the volume changes continuously with \( s \), there is a point \( t' \) such that \( h_{t'} \) divides \( B \) into two regions, each of which has volume exactly \( \frac{1}{2}v \), so \( t' \) is a cell splitter for \( V \).

We now show uniqueness. First suppose \( t \) is a cell splitter which is not a vertex. Then there are precisely two complementary regions \( H_t^{c_1} \) and \( H_t^{c_2} \), each of which must have exactly half the volume of \( B(x, \epsilon/2) \cap V \), and we shall denote this volume by \( v \). Any other point \( t' \) has a complementary region which contains at least one of these complements, and so has volume greater than \( \frac{1}{2}v \), and so can not be a cell splitter.

Finally suppose \( t \) is a cell splitter which is a vertex. Then there are three complements \( H_t^{c_1}, H_t^{c_2} \) and \( H_t^{c_3} \), each of which has volume at most \( \frac{1}{2}v \). As each region has volume at most \( \frac{1}{2}v \), any two regions must have total volume at least \( \frac{1}{2}v \). Any other point \( t' \in Y \) must have a complementary region which contains at least two of the complements of \( H_t \), and so has a complement with volume strictly greater than \( \frac{1}{2}v \), and so can not be a cell splitter.

**Definition 2.8.** We say that a Morse function \( f: M \to Y \) to a tree \( Y \) is generic with respect to a Voronoi decomposition \( V \) if the cell splitters for distinct Voronoi
cells $V_i$ correspond to distinct points $t_i \in Y$. We say a point $t \in Y$ is \emph{generic} if it is not a critical point for the Morse function, and is not a cell splitter.

We may assume that $f$ is generic by an arbitrarily small perturbation of $f$, and we shall always assume that $f$ is generic from now on. Finally, we remark that a trivalent vertex in $Y$ is not necessarily a cell splitter.

### 2.5 Polyhedral and capped surfaces

Let $Q$ be a 3-dimensional submanifold of a complete hyperbolic 3-manifold $M$ without cusps, with boundary an embedded separating surface $F$. In this section we show how to approximate $Q$ by a union of Voronoi cells, which in turn gives an approximation to $F$ by an embedded surface $S$ which is a union of faces of Voronoi cells.

We say a region $R$ is \emph{generic} if for every Voronoi cell $V_i$ with center $x_i$, the region consisting of the intersection of $B(x_i, \epsilon/2)$ with the interior of $V_i$ does not have exactly half its volume lying in $R$. We say a separating surface $F$ in $M$ is \emph{generic} if it bounds a generic region.

Let $P$ be the collection of Voronoi cells for which at least half of the volume of $B(x_i, \epsilon/2) \cap \text{interior}(V_i)$ lies in $Q$. We say the $P$ is the \emph{polyhedral region} determined by $Q$. The polyhedral region $P$ may be empty, even if $Q$ is non-empty. The boundary of $P$ is a polyhedral surface $S$, which we shall call the \emph{polyhedral surface} associated to $F = \partial Q$, and is a normal surface in the dual triangulation. We will use the following bound on the number of faces and boundary components of the intersection of the polyhedral surface $S$ with the thick part of the manifold, in terms of the area of the corresponding surface $F$. If $S$ is a surface, we will write $|\partial S|$ for the number of boundary components of $S$, and if $S'$ is a subset of a polyhedral surface $S$, we will write $|S'|$ for the number of faces of $S$ which intersect $S'$.

**Proposition 2.9.** \cite{HM16} Proposition 2.10, 2.13] \textit{Let $(\mu, \epsilon)$ be MV-constants, and let $M$ be a complete hyperbolic 3-manifold without cusps, with an $\epsilon$-regular Voronoi decomposition $V$ with deep part $W$ and thick part $X_\mu$. Then there is a constant $K$, which only depends on the MV-constants, such that for any generic embedded separating surface $F$ in $M$, the corresponding polyhedral surface $S$ satisfies:}

$$\|S \cap X_\mu\| \leq K \text{area}(F),$$

\textit{and}

$$|\partial(S \cap X_\mu)| \leq K \text{area}(F).$$

In \cite{HM16}, this result is stated for the level set of a Morse function $F$ on a compact hyperbolic manifold, and one may then observe that every separating surface is the level set of some Morse function, though in fact, the proof only uses the fact that $F$ is separating. In \cite{HM16} Proposition 2.10] the bound is stated in terms of $S \cap W$. However, as $S \cap X_\mu \subset S \cap W$, the stated bound follows immediately.
For a polyhedral surface $S$, each boundary component of the surface $S \cap X_\mu$ is contained in $T_\mu$, so $S \cap X_\mu$ is a properly embedded surface in $X_\mu$. We now wish to cap off the properly embedded surfaces $S \cap X_\mu$ with properly embedded surfaces in $T_\mu$ to form closed surfaces. We warn the reader that the following definition differs slightly from the definition in [HM16], as we extend the definition to include the case in which $T_\mu$ has infinite components.

**Definition 2.10.** A separating surface $F$ in $M$ gives rise to a polyhedral surface $S$, which meets $\partial T_\mu$ transversely, and intersects $\partial T_\mu$ in a collection of simple closed curves which is separating in $\partial T_\mu$. We replace $S$ inside the thin part by surfaces $\{U_i\}$ which we now describe. For each torus component $T_i$ in $\partial T_\mu$ choose a subsurface $U_i$ bounded by $S \cap \partial T_i$. For each infinite component $T_i$, choose a not necessarily connected surface $U_i$ as follows: for each essential curve in the annulus $\partial T_i$ choose a disc it bounds in $T_i$, and then let $U_i$ be the union of these discs with the planar surface bounded by the remaining inessential curves. We call the resulting surface a capped surface $S^+ = (S \cap X_\mu) \cup \bigcup_i U_i$.

We will use the following property of the capped surfaces.

**Proposition 2.11.** Let $(\mu, \epsilon)$ be MV-constants, and let $M$ be a complete hyperbolic 3-manifold without cusps, with thin part $T_\mu$, and with with an $\epsilon$-regular Voronoi decomposition $V$. Then there is a constant $K$, which only depends on $\epsilon$, such that for any generic embedded separating surface $F$ in $M$, the corresponding capped surface $S^+$ satisfies:

$$\text{genus}(S^+) \leq K \text{area}(F).$$

The proof of this result is essentially the same as the proof of [HM16 Proposition 2.14], and instead of repeating the entire argument, we explain the minor extension needed. The only difference is that [HM16 Proposition 2.14] is stated for closed hyperbolic manifolds, whereas Proposition 2.11 is stated for complete hyperbolic manifolds without cusps, so the thin part of $M$ may have infinite Margulis tubes. This makes no difference to the estimates of the number of faces and boundary components of the resulting polyhedral surface in terms of the area of the original surface. The extension of the definition of capped surface to the infinite case only involves capping off with planar surfaces, so the same genus bounds hold.

### 2.6 Disjoint equivariant surfaces

Each collection of points $t_i$ in $Y$ corresponds to a collection $S_i^+$ of capped surfaces. In this section we show that if the collection of points is equivariant, then we may arrange for the capped surfaces to be disjoint and equivariant.

Let $M$ be a 3-manifold which admits a group of covering translations $G$. We say a subset $U \subset M$ is equivariant if it is preserved by $G$. We say a Voronoi decomposition $V$ of $M$ is equivariant if the centers of the Voronoi cells form an equivariant set in $M$. 
Let $W$ be an equivariant collection of points in $\tilde{X}$, none of which are either cell splitters or critical points of the Morse function $h$. We say two points $t_i$, $t_j$ in $W$ are adjacent if the geodesic connecting them in the tree $\tilde{X}$ does not contain any other point of $W$. We may choose $W$ such that the geodesic in $\tilde{X}$ connecting any pair of adjacent points in $W$ contains either a single cell splitter, a single trivalent trivalent vertex of $\tilde{X}$, or neither of these two types of points.

Consider the collection $S$ of polyhedral surfaces $S_t$, as $t$ runs over $W$. As the collection $W$ is equivariant, $S$ is also equivariant. Note that although each surface in $S$ is individually embedded, each surface in $S$ will share many common faces with other surfaces in $S$. We will now make this collection simultaneously equivariantly disjoint, so that we may push them down to $M$ to obtain a collection of disjoint surfaces which will act as our splitting surfaces in a graph splitting of $M$.

**Proposition 2.12.** Let $M$ be a closed hyperbolic 3-manifold of injectivity radius at least $2\epsilon$, with an $\epsilon$-regular Voronoi decomposition $V$, and a generic Morse function $f : M \to X$ onto a trivalent graph $X$ with connected level sets. Let $p : \tilde{M} \to M$ be the cover of $M$ corresponding to the kernel of the induced map $f_* : \pi_1 M \to \pi_1 X$, and let $c : \tilde{X} \to X$ be the universal cover of $X$. Let $W$ be a discrete equivariant collection of points in $\tilde{X}$. Then the collection of polyhedral surfaces $\{S_w \mid w \in W\}$ in $\tilde{M}$ is equivariantly isotopic to a disjoint collection of surfaces $\{\Sigma_w \mid w \in W\}$, and furthermore this equivariant isotopy may be chosen to be supported in a neighborhood of the 2-skeleton of the induced Voronoi decomposition of $\tilde{M}$.

**Proof.** We now give a recipe for constructing surfaces $\Sigma_t$, for $t \in W$. Each individual surface $\Sigma_t$ will be isotopic to the original $S_t$, but the union of the surfaces $\Sigma_t$ will be equivariantly disjointly embedded in $\tilde{M}$.

We first show that there is a canonical ordering of the polyhedral surfaces $\Sigma_t$ which share a common face. Let $\Phi$ be a face of a Voronoi cell in $M$, and let $V(x_1)$ and $V(x_2)$ be the adjacent Voronoi cells. Let $t_1$ and $t_2$ be cell splitters for $V(x_1)$ and $V(x_2)$, so that $H_{t_i} = h^{-1}(t_i)$ is the surface which divides $B_{\epsilon/2}(x_i)$ precisely in half, for $i = 1, 2$.

We say a point in $\tilde{X}$ is regular if it is not a cell splitter, and not a critical point for the Morse function $h$.

**Claim 2.13.** The collection of regular points in $\tilde{X}$ corresponding to polyhedral surfaces $\Sigma_t$ which contain the face $\Phi$ is precisely the regular points contained in the geodesic in $\tilde{X}$ from $t_1$ to $t_2$.

**Proof.** The two embedded surfaces $H_{t_1}$ and $H_{t_2}$ divide $\tilde{M}$ into three parts; call them $A, B$ and $C$, with $A$ the part only hitting $H_{t_1}$, and $B$ the part hitting both $H_{t_1}$ and $H_{t_2}$.

Let $\gamma$ be the geodesic in $\tilde{X}$ from $t_1$ to $t_2$. Each point $t$ in $\gamma$ corresponds to a surface $H_t$ dividing $\tilde{M}$ at most 3 parts, one of which contains $A$, and another containing $C$. Let $P_t$ be the part containing $A$. Then, writing $|A|$ for the volume
of a region $A$, 

$$|B_{t/2}(x_1) \cap P_t| \geq |B_{t/2}(x_1) \cap A| \geq \frac{1}{2} |B_{t/2}(x_1)|$$

and 

$$|B_{t/2}(x_2) \setminus P_t| \geq |B_{t/2}(x_2) \setminus C| \geq \frac{1}{2} |B_{t/2}(x_2)|.$$ 

Therefore the two Voronoi cells $V(x_1)$ and $V(x_2)$ lie in different partitions of the Voronoi cells determined by $t$, and so $\Phi$ lies in the polyhedral surface $\Sigma_t$.

Conversely, suppose $t$ does not lie on the path $\gamma$, then $t$ divides $\tilde{X}$ into at most three parts, and $\gamma$ is contained in exactly one of these parts. This means that $H_{t_1}$ and $H_{t_2}$ are contained in the same complementary component of $H_{t_3}$, and so $\Phi$ cannot be a face of $\Sigma_t$. 

It suffices to show that we can isotope the normal surfaces, preserving the fact that they are normal, so that they have disjoint intersection in the 2-skeleton of the dual triangulation.

Let $e$ be an edge of the dual triangulation, with vertices $x_1$ and $x_2$, with corresponding cell splitters $t_1$ and $t_2$. A normal surface $S_i$ intersects $e$ if and only if the corresponding point $w_i$ lies in the geodesic $[t_1, t_2]$ in $\tilde{X}$ connecting $t_1$ and $t_2$. The points $w_i$ in $e$ therefore inherit an order from $[t_1, t_2]$, and we may isotope the normal surfaces by a normal isotopy so that they intersect the edge $e$ in the same order. As the interiors of each edge have disjoint images under the covering translations, and the collection of edges is equivariant, we may do this normal isotopy equivariantly.

Let $\Phi$ be a triangle in the dual triangulation, with vertices $x_1, x_2$ and $x_3$, and corresponding cell splitters $t_1, t_2$ and $t_3$. As above, the collection of normal surfaces which intersect an edge $[x_i, x_j]$ of $\Phi$ corresponds to those $w_i$ lying in the geodesic $[t_i, t_j]$ in $\tilde{X}$. The union of the three geodesics $[t_i, t_j]$ forms a minimal spanning tree for the three cell splitters in $\tilde{X}$. Let $t_0$ be the center of this tree, i.e. the unique point that lies in all three geodesics. Note that the tree may be degenerate, so $t_0$ may be equal to one of the other vertices.
Normal arcs parallel to the edge \([x_2, x_3]\) correspond to surfaces which hit both of the edges \([x_1, x_2]\) and \([x_1, x_3]\), so correspond points \(w_i\) which lie in both \([t_1, t_2]\) and \([t_1, t_3]\), and similarly for the other two cases. The intersection of these two geodesics in \(\tilde{X}\) is \([t_1, t_0]\), and so the corresponding surfaces appear in the same order on each of the edges in \(\Phi\), and so the arcs are disjoint. The same argument applies to each vertex of \(\Phi\).

As the resulting surfaces in \(\tilde{M}\) are disjoint and equivariant, they project down to disjoint surfaces in \(M\).

We now show that the polyhedral surfaces, and their complements, project down homeomorphically into \(\tilde{M}\). As the level set surfaces lift homeomorphically to \(\tilde{M}\), the area bound for the level sets of \(f\) is also an area bound for the level sets of \(h\). Therefore, each polyhedral surface contains a bounded number of faces. The deck transformation group of the universal cover of a graph is equal to the fundamental group of the graph, which is a free group, so the orbit of any face consists of infinitely many disjoint translates. If two lie in the same connected component of a polyhedral surface, then that path corresponds to a covering translation, which has infinite order, so in fact the connected component contains infinitely many faces, which contradicts the fact that there is a bound on the number of faces in each component.

Each complementary region is compact, so the same argument applied to the complementary regions shows that they are all mapped down homeomorphically as well.

### 2.7 Bounded handles

We now bound the number of handles in a complementary region of the capped surfaces, which contains a single cell splitter. The following result will complete the proof of the left hand inequality in Theorem 1.3.
Proposition 2.14. Let \((\mu, \epsilon)\) be MV-constants, and let \(M\) be a complete hyperbolic 3-manifold without cusps, with an \(\epsilon\)-regular Voronoi decomposition \(V\), and a generic Morse function \(h: M \to Y\), where \(Y\) is a tree. Let \(\{u_i\}\) be a collection of points in \(Y\), which separate the cell splitters in \(Y\), and let \(\{S_i^+\}\) be the corresponding collection of capped surfaces. If \(P\) is a complementary component of the capped surfaces in \(M\), the region \(P\) has at most three boundary components, \(S_{i_1}^+, S_{i_2}^+, S_{i_3}^+\) say, where the final surface may be empty. Then \(P\) is homeomorphic to a manifold with a handle decomposition containing at most

\[K \text{ Gromov area}(M)\]

handles, where \(K\) depends only on the MV-constants.

We start with the observation that attaching a compression body \(P\) to a 3-manifold \(Q\) by a subsurface \(S\) of the upper boundary component of \(P\), requires a number of handles which is bounded in terms of the Heegaard genus of \(P\), and the number of boundary components of the attaching surface.

Proposition 2.15. [HM16, Proposition 2.16] Let \(Q\) be a compact 3-manifold with boundary, and let \(R = Q \cup P\), where \(P\) is a compression body of genus \(g\), attached to \(Q\) by a homeomorphism along a (possibly disconnected) subsurface \(S\) contained in the upper boundary component of \(P\) of genus \(g\). Then \(R\) is homeomorphic to a 3-manifold obtained from \(Q\) by the addition of at most \((4g + 2|\partial S|)\) 1- and 2-handles, where \(|\partial S|\) is the number of boundary components of \(S\).

Proof (of Proposition 2.14). If \(P\) has two boundary components, then the argument is exactly the same as [HM16 Proposition 2.15], so we now consider the case in which \(P\) has three boundary components, which, without loss of generality we may relabel \(S_{i_1}^+, S_{i_2}^+\) and \(S_{i_3}^+\). Let \(t\) be the cell splitter corresponding to the region \(P\), and let \(V\) be the corresponding Voronoi cell. As \(P\) has three boundary components, \(t\) must be a vertex of \(Y\).

We first consider the case in which the Voronoi region \(V\) corresponding to the cell splitter \(t\) in \(h(P)\) is disjoint from the thin part \(T_\mu\). Consider the three polyhedral surfaces \(S_1, S_2, S_3\), corresponding to the three capped surfaces, and let \(\Sigma = \cup S_i \cup V\) be the union of the polyhedral surfaces, together with the Voronoi cell \(V\). By Proposition 2.9 there is a constant \(K\), which only depends on the MV-constants, such that the number of faces of \(\Sigma\) in the thick part is at most \(3K_1\text{Gromov area}(M)\), i.e.

\[\|\Sigma \cap X_\mu\| \leq 3K_1\text{Gromov area}(M),\]

where \(K_1\) is the constant from Proposition 2.9. The number of boundary components of each surface \(S_i \cap X_\mu\) is also bounded by Proposition 2.9, and by Proposition 2.9 the Voronoi cell \(V\) has a bounded number \(J\) of vertices, edges and faces, where \(J\) depends only on the MV-constants. In particular, there is a constant \(A\), depending only on the MV-constants, such that \(P \cap X_\mu\) has a handle structure with at most \(A(\text{Gromov area}(M))\) handles.
To bound the number of handles contained in $P$, we observe that $P$ is a regular neighbourhood of the 3-complex obtained from capping off the boundary components of $\Sigma \cap X_\mu$, using the parts of the capped surfaces in the thin part, i.e. the union of the components of $S_i^+ \cap T_\mu$ over all three capped surfaces. Each component of $S_i^+ \cap T_\mu$ has genus at most one, and the number of boundary components of $\Sigma \cap X_\mu$ is bounded linearly in terms of Gromov area($M$), therefore, there is a constant $B$, depending only on the $MV$-constants, such that the number of handles in $P$ is at most $B$(Gromov area($M$)), as required.

We now consider the case in which the region $P$ has image $h(P)$ in $Y$ which contains the cell splitter $t$, and the corresponding Voronoi cell $V$ intersects $T_\mu$. In this case, the connected components of $V \cap X_\mu$ need not be topological balls, and there may be connected components of $P \cap T_\mu$ whose boundary components are not parallel.

The connected components of $V \cap X_\mu$ are handlebodies of bounded genus, as show in the following result of Kobayashi and Rieck [KR11]. We state a simplified version of their result which suffices for our purposes, see [HM16] for further details.

**Proposition 2.16.** [KR11] Let $\mu$ be a Margulis constant for $\mathbb{H}^3$, $M$ be a finite volume hyperbolic 3-manifold, let $0 < \epsilon < \mu$, and let $V$ be a regular Voronoi decomposition of $M$ arising from a maximal collection of $\epsilon$-separated points. Then there is a number $G$, depending only on $\mu$ and $\epsilon$, such that for any Voronoi cell $V_i$, there are at most $G$ connected components of $V_i \cap X_\mu$, each of which is a handlebody of genus at most $G$, attached to $T_\mu$ by a surface with at most $G$ boundary components.

Recall that attaching a handlebody of genus $G$ to a 3-manifold along a subsurface of the boundary with at most $G$ boundary components requires at most $6G^2$ handles:

**Proposition 2.17.** [HM16, Proposition 2.16] Let $Q$ be a compact 3-manifold with boundary and let $R = Q \cup P$, where $P$ is a compression body of genus $g$, attached to $Q$ by a homeomorphism along a (possibly disconnected) subsurface $S$ contained in the upper boundary component of $P$ of genus $g$. Then $R$ is homeomorphic to a 3-manifold obtained from $Q$ by the addition of at most $(4g + 2|\partial S|)$ 1- and 2-handles, where $|\partial S|$ is the number of boundary components of $S$.

Therefore, adding a Voronoi cell which intersects $\partial T_\mu$ may be realized by at most $6G^2$ handles.

If the Voronoi cell intersects $T_\mu$, then there may be components of $P \cap T_\mu$ whose boundary surfaces are not parallel. This case is considered in the proof of [HM16, Proposition 2.15], when the manifold has no infinite Margulis tubes, so it suffices to consider the case of a component of $P$ contained in an infinite Margulis tube. However, the case of an infinite Margulis tube in which neither surface is an essential disc is the same as the ordinary Margulis tube case, and if both surfaces essential discs then they are parallel. Finally, if exactly one surface is an essential disc, then the other surface lies in the
same homology class, via the component of $P$ in the infinite Margulis tube, and so, after surgering inessential boundary components, is also an essential disc. However, the number of boundary components is at most $K_1 Gromov\ area(M)$, and so the total number of extra handles over all components of $P$ in the infinite Margulis tubes is also bounded by $K_1 Gromov\ area(M)$.

We may choose the constant $K$ to be the maximum of the constants arising from the two cases considered above, thus completing the proof of Proposition 2.15.

3 Topological complexity bounds metric complexity

In this section we will show bounds for metric complexity in terms of topological complexity, i.e. the right hand inequality in Theorem 1.3 assuming the Pitts and Rubinstein result, Theorem 1.4.

We start by reminding the reader of the topological properties of thin position for generalized Heegaard splittings, as shown by Scharlemann and Thompson [ST94] for the linear case and Saito, Scharlemann and Schultens [SSS16] for the graph case.

**Theorem 3.1.** [ST94, SSS16] Let $H$ be a graph splitting that is in thin position. Then every even surface is incompressible in $M$ and the odd surfaces form strongly irreducible Heegaard surfaces for the components of $M$ cut along the even surfaces.

We will use the following result due to Gabai and Colding [CG14, Appendix A], building on recent work of Colding and Minicozzi [CM16]. It is not stated explicitly in their paper, but see [HM16, Theorem 3.2] for further details.

**Theorem 3.2.** [CG14] Let $M$ be a hyperbolic manifold, with (possibly empty) least area boundary, with a minimal Heegaard splitting $H$ of genus $g$. Then, assuming Theorem 1.1, the manifold $M$ has a (possibly singular) foliation by compact leaves, containing the boundary surfaces as leaves, such that each leaf has area at most $4\pi g$.

By Theorem 3.1 we may consider the compression bodies in the graph splitting in pairs, glued along strongly irreducible Heegaard splittings, and then Theorem 3.2 guarantees that each pair has a foliation with each leaf having area at most $4\pi g$. These foliations contain the boundary surfaces as leaves, and so the foliations on each pair extend to foliations of the entire manifold, as required.

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