Is the $X(3872)$ a molecule? *

S. COITO*, G. RUPP

Centro de Física das Interacções Fundamentais, Instituto Superior Técnico,
Technical University of Lisbon, P-1049-001 Lisboa, Portugal

AND

E. VAN BEVEREN

Centro de Física Computacional, Departamento de Física, Universidade de
Coimbra, P-3004-516 Coimbra, Portugal

Because of the controversial $X(3872)$ meson’s very close proximity to the
$D^0\bar{D}^{*0}$ threshold, this charmonium-like resonance is often considered
a meson-meson molecule. However, a molecular wave function must be
essentially of a meson-meson type, viz. $D^0\bar{D}^{*0}$ in this case, with no other
significant components. We address this issue by employing a simple two-
channel Schrödinger model, in which the $J^{PC} = 1^{++}$ $c\bar{c}$ and $D^0\bar{D}^{*0}$ chan-
nels can communicate via the $^{3}P_0$ mechanism, mimicked by string breaking
at a sharp distance $a$. Thus, wave functions and their probabilities are
computed, for different bound-state pole positions approaching the $D^0\bar{D}^{*0}$
threshold from below. We conclude that at the PDG $X(3872)$ mass and
for reasonable values of $a$, viz. $2.0$–$3.0$ GeV$^{-1}$, the $c\bar{c}$ component remains
quite substantial and certainly not negligible, despite accounting for only
about $6$–$10\%$ of the total wave-function probability, owing to the naturally
long tail of the $D^0\bar{D}^{*0}$ component.

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The $X(3872)$ charmonium-like meson is by now a very well established re-
sonance [1]. It was first observed in 2003, by the Belle Collaboration [2], in the
decay $B^\pm \to K^{\pm}\pi^+\pi^- J/\psi$, with significance in excess of $10\sigma$. Since then,
it has been confirmed by several collaborations, viz. Belle, BaBar, CDF, D0, CLEO, and, more recently, by LHCb (see the 2012 PDG [1] listings

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The PDG summary table lists the $X(3872)$ as an isoscalar state with positive $C$-parity, from the observed $\gamma J/\psi$ decay, but unknown $J$ and $P$, having an average mass $m = 3871.68 \pm 0.17$ MeV/$c^2$ and a width $\Gamma < 1.2$ MeV/$c^2$. The two most likely $J^{PC}$ assignments are $1^{++}$ and $2^{-+}$, while the observed hadronic decay modes are $\rho^0 J/\psi$, $\omega J/\psi$, $D^0 \bar{D}^{*0}$, and $D^0 \bar{D}^{*0} \eta$. Henceforth, we shall denote $D^0 \bar{D}^{*0}$ simply by $D^0 \bar{D}^{*0}$.

Meson spectroscopists have been puzzled by the $X(3872)$, because of its low mass as compared to predictions of conventional quark models, as well as its remarkable proximity to the $D^0 \bar{D}^{*0}$ threshold, being “bound” by only 0.15 MeV [1]. This has led to a plethora of model descriptions of the $X(3872)$, viz. as a $c \bar{c}$ state, meson-meson (MM) molecule, tetraquark, or hybrid meson. For a number of reviews on the many different approaches and the experimental situation, see [3]. Recently, we have described [4] the $X(3872)$ as a regular but “unquenched” $1^{++}$ ($^3P_1$) charmonium meson, whose physical mass is dynamically shifted about 100 MeV downwards from the bare $2^3P_1$ $c\bar{c}$ state due to its strong coupling to the $S$-wave $D^0 D^{*0}$ and $D^\pm D^{\ast \mp}$ channels, besides several other OZI-allowed and OZI-forbidden ($\rho^0 J/\psi$, $\omega J/\psi$) channels. Thus, the observed hadronic $X(3872)$ properties were well reproduced [4].

Nevertheless, the closeness of the $X(3872)$ to the $D^0 D^{*0}$ threshold seems to favour a molecular interpretation [5]. In the latter paper, it is stated that, whatever the original mechanism generating the resonance, a near-threshold bound state will always have a molecular structure. This implies that the MM component of the wave function, i.e., $D^0 D^{*0}$, should be the only relevant one. Here, we shall study this issue in a simplified, coordinate-space version of the model employed in [4], restricted to the most important channels, viz. $c\bar{c}$ and $D^0 D^{*0}$. Note that, even if the $X(3872)$ is essentially a molecule, it will mix with $c\bar{c}$ states having the same quantum numbers.

Now we turn to the two-channel model used in [6], with parameters adjusted for the $X(3872)$. Consider a coupled $q\bar{q} - M_1 M_2$ system, with the $q\bar{q}$ pair confined through a harmonic-oscillator (HO) potential, whereas the two mesons $M_1, M_2$ are free. The correponding 2 $\times$ 2 radial Schrödinger equation is given by Eq. (1), with the Hamiltonians (2) and (3). Here, $\mu_{c,f}$ is the reduced mass in either channel, $m_q = m_{\bar{q}}$ the constituent quark mass, $l_c, l_f$ the orbital angular momenta, and $\omega$ the HO frequency:

$$\begin{pmatrix} h_c & V \\ V & h_f \end{pmatrix} \begin{pmatrix} u_c \\ u_f \end{pmatrix} = E \begin{pmatrix} u_c \\ u_f \end{pmatrix} ;$$

$$h_c = \frac{1}{2\mu_c} \left( -\frac{d^2}{dr^2} + \frac{l_c(l_c + 1)}{r^2} \right) + \frac{1}{2} \mu_c \omega^2 r^2 + m_q + m_{\bar{q}} ;$$

(1)

(2)
Note that we use here relativistic definitions for the MM reduced mass $\mu_f$ and relative momentum $k$, even below threshold contrary to [6], though this is practically immaterial for the $X(3872)$. At some “string-breaking” distance $a$, transitions between the two channels are described by an off-diagonal point-like potential with strength $g$:

$$V = \frac{g}{2\mu_c a} \delta(r - a).$$

Continuity and twice integrating Eqs. (1–3) yields the boundary conditions:

$$u_r(r \uparrow a) - u_r(r \downarrow a) + \lambda a u_f(a) = u_f(r \uparrow a) - u_f(r \downarrow a) + \frac{\lambda \mu_f}{\mu_c} u_c(a) = 0,$$

$$u_c(r \uparrow a) = u_c(r \downarrow a) \quad \text{and} \quad u_f(r \uparrow a) = u_f(r \downarrow a).$$

A general solution to this problem is given by Eqs. (7) and (8) for the confined and the MM state, respectively. The two-component function $u(r) = (u_c(r), u_f(r))$ is related to the radial wave function as $u(r) = rR(r)$:

$$u_c(r) = \begin{cases} A_c F_c(r) & r < a, \\ B_c G_c(r) & r > a; \end{cases}$$

$$u_f(r) = \begin{cases} A_f J_{l_f}(kr) & r < a, \\ B_f \left[ J_{l_f}(kr)k^{2l_f+1} \cot \left( \delta_{l_f}(E) \right) - N_{l_f}(kr) \right] & r > a. \end{cases}$$

Now, $F_c(r)$ vanishes at the origin and $G_c(r)$ falls off exponentially for $r \to \infty$. Defining then $z = \mu \omega r^2$ and

$$\nu = \frac{E - 2m_c}{2\omega} - \frac{l_c + 3/2}{2},$$

we get

$$F(r) = \frac{1}{\Gamma(l + 3/2)} z^{(l+1)/2} e^{-z/2} \phi(-\nu, l + 3/2, z),$$

$$G(r) = -\frac{1}{2} \Gamma(-\nu) r z^{l/2} e^{-z/2} \psi(-\nu, l + 3/2, z).$$

Here, the functions $\phi$ and $\psi$ are the confluent hypergeometric functions of first and second kind, respectively, and the $\Gamma$ function acts as a normalising function. The functions $J$ and $N$ in Eq. (8) are defined in terms of the
spherical Bessel and Neumann functions \( j, n \), i.e., \( J_\ell(kr) = k^{-\ell}r j_\ell(kr) \) and \( N_\ell(kr) = k^{\ell+1}r n_\ell(kr) \). From the boundary conditions \[7\] and the explicit wave-function expressions in Eqs. \[8\] and \[9\], we obtain

\[
G_c(a)F_c(a) - F_c(a)G_c(a) = \frac{2}{a} J_\ell(ka)F_c(a) \frac{A_f}{B_c},
\]

\[
J_\ell(ka)N_\ell(ka) - J_\ell(ka)N'_\ell(ka) = \frac{2}{a} \frac{\mu_f}{\mu_c} J_\ell(ka)F_c(a) \frac{A_f}{B_f}.
\]

Using next the Wronskian relations

\[
W(F_c(a), G_c(a)) = \lim_{r \to a} \left[ F_c(r)G'_c(r) - F'_c(r)G_c(r) \right] = 1,
\]

\[
W(N_\ell(ka), J_\ell(ka)) = \lim_{r \to a} \left[ N_\ell(ka)J'_\ell(ka) - N'_\ell(ka)J_\ell(ka) \right] = -1.
\]

yields

\[
A_fB_f = -\frac{\mu_f}{\mu_c} A_c B_c
\]

and

\[
A_fB_f = -\left[ \frac{g^2 \mu_f}{a^2 \mu_c} j_\ell^2(ka)F_c^2(a) \right]^{-1} B_c \frac{A_c}{A_c}.
\]

Finally, with the expression for the MM scattering wave function \( u_f(r) \) (second line in Eq. \[8\]), the final result for \( \cot \delta_f(E) \) is obtained, reading

\[
\cot \left( \delta_f(E) \right) = -\left[ g^2 \frac{\mu_f}{\mu_c} k j_\ell^2(ka)F_c(a)G_c(a) \right]^{-1} + \frac{n_\ell(ka)}{j_\ell(ka)}.
\]

Now, in the present \( X(3872) \) model, there is only one scattering channel, viz. for the \( D^0 D^{*0} \) system. Thus, poles in the \( S \)-matrix, which represent possible resonances, bound states, or virtual states, are given by the simple relation \( \cot \delta_f(E) = i \). On the other hand, the solutions to the two-component radial wave function \[7\] are then fully determined by relations \[14\] and \[15\], up to an overall normalisation constant.

Next we apply this formalism to the coupled \( c\bar{c} - D^0 D^{*0} \) system. In the confined channel, the \( c\bar{c} \) system is in a \( 2^3P_1 \) state, and so \( l_c = 1 \), whereas the \( D^0 D^{*0} \) channel has \( l_f = 0 \). In Table \[1\] we give the fixed parameters of the model, with the HO frequency \( \omega \) and the constituent charm mass as in

| Parameter | \( \omega \) | \( m_c \) | \( m_{D^0} \) | \( m_{D^{*0}} \) | \( m_{D^0} + m_{D^{*0}} \) |
|-----------|-------------|----------|-------------|-------------|-----------------|
| Value (MeV) | 190 | 1562 | 1864.84 | 2006.97 | **3871.81** |

\[7\], being unaltered ever since. However, the radial quantum number \( \nu \) in
Table 2. Bound and virtual states near the $D^0D^{*0}$ threshold.

| $a$ (GeV$^{-1}$) | $g$   | Pole (MeV) | Type of bound state |
|-----------------|-------|------------|---------------------|
| 2.0             | 1.133 | 3871.68    | virtual             |
| 2.0             | 1.150 | 3871.81    | virtual             |
| 2.0             | 1.153 | 3871.81    | real                |
| 2.0             | 1.170 | 3871.68    | real                |
| 3.0             | 2.097 | 3871.68    | virtual             |
| 3.0             | 2.144 | 3871.81    | virtual             |
| 3.0             | 2.150 | 3871.81    | real                |
| 3.0             | 2.199 | 3871.68    | real                |

Eq. (9) varies as a function of the energy, and therefore will generally be non-integer, becoming even complex for resonance poles. The parameter that determines such variations is the coupling $g$. In the uncoupled case, i.e., for $g = 0$, one recovers the bare $^3P_1$ HO spectrum, with energies $(3599 + 2n\omega)$ MeV $(n = 0, 1, 2, \ldots)$. The only other free parameter is the string-breaking distance $a$. Now we try to find $S$-matrix poles as a function of the coupling $g$ and for two reasonable values of $a$, viz. 2.0 and 3.0 GeV$^{-1}$ ($\approx 0.4$ and 0.6 fm). Searching near the $D^0D^{*0}$ threshold, a dynamical pole is found, either on the first Riemann sheet, corresponding to a bound state, or on the second one, which represents a virtual state (see Ref. [4], second paper). These results are presented in Table 2 and Fig. 1.

Fig. 1. Dynamical real (solid) and virtual (dashed) pole trajectories for $a = 2.0$ GeV$^{-1}$ (left) and $a = 3.0$ GeV$^{-1}$ (right). The arrows indicate pole movement for increasing $g$. The PDG $X(3872)$ mass is labelled by *. Also see Table 2.
Fig. 2. Radial wave-functions for $E = 3871.68$ MeV and $g = 1.170$, $g = 2.199$ for $a = 2.0$ GeV$^{-1}$ (left) and $a = 3.0$ GeV$^{-1}$ (right). Also see Table 2.

Note that the dynamical pole arises from the $D^0D^{*0}$ continuum and is not connected to the bare $2^3P_1$ $c\bar{c}$ state at 3979 MeV, contrary to the situation in [4] (first paper). For our study here, this is of little consequence.

Finally, we depict the normalised two-component wave-function $R(r)$ in Fig. 2, evaluated for the PDG [1] $X(3872)$ mass of 3872.68 MeV. One clearly sees the $P$-wave behaviour of the $c\bar{c}$ component, whereas the $D^0D^{*0}$ is in an $S$-wave. Moreover, the $c\bar{c}$ admixture is certainly not negligible, despite the low total probabilities of 6.13% and 10.20%, for $a = 2$ GeV$^{-1}$ and $a = 3$ GeV$^{-1}$, respectively, which are logical because of the very long tail of the $D^0D^{*0}$ component; also see [8]. Soon we will publish more detailed work.

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