Quantum Game Theory

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Abstract

We pursue a general theory of quantum games. We show that quantum games are more efficient than classical games, and provide a saturated upper bound for this efficiency. We demonstrate that the set of finite classical games is a strict subset of the set of finite quantum games. We also deduce the quantum version of the Minimax Theorem and the Nash Equilibrium Theorem.

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The field of quantum games is currently attracting much attention within the physics community [1,2,3,4]. In addition to their own intrinsic interest, quantum games offer a new vehicle for exploring the fascinating world of quantum information [3,4,5]. So far, research on quantum games has tended to concentrate on finding interesting phenomena when a particular classical game is quantized. As a result, studies of quantum games have centered on particular special cases rather than on the development of a general theoretical framework.

This paper aims to pursue the general theory of quantum games. We are able to identify a definite sense in which quantum games are 'better' than classical games, in terms of their efficiency. We show explicitly that, in terms of the number of (qu)bits required, there can be a factor of 2 increase in efficiency if we play games quantum mechanically. Hence we are able to quantify a distinct advantage of quantum games as compared to their classical counterparts. Furthermore, through the formalism developed, we are able to demonstrate that the set of finite classical games is a strict subset of finite quantum games. Namely, all finite classical games can be played by quantum rules but not vice versa. We also deduce the quantum version of two of the most important theorems in classical game theory: the minimax theorem for zero-sum games and the Nash theorem for general static games.

We start by defining what is meant by a game. In game theory, a game consists of a set of players, a set of rules which dictate what actions a player can take, and a payoff function specifying the reward for a given set of played strategies. In other words, it is a triple $hN; P_i$ where $N$ is the number of players, $P$ is a set of strategies for the $k$-th player, and $P_i: R^N$ such that each $P_k(i)$ with $1 \leq k \leq N$ is the payoff function for the $k$-th player. Without loss of generality, we can imagine the existence of a referee who computes the corresponding payoff function after he receives the strategies being played by each of the players. This formal structure includes all classical games and all quantum games. In other words any game, whether classical or quantum, is fully described by the corresponding triple $hN; P_i$. On the other hand, given any triple $hN; P_i$, it is not hard to imagine a purely classical game that might be associated to it.

So in what sense are quantum games any 'better' than classical games? We believe that the answer lies in the issue of efficiency. Although any game could be played classically, the physically feasible ones form a very restricted subset. As we have seen, playing a game is ultimately about information exchange between the players and the referee—hence if a particular game requires you to submit an infinite
amount of information before the payoffs functions can be computed, it will not be a playable game. In other words, we are interested in those games which require only a finite amount of resources and time to play. Hence a connection can be made between this consideration and the study of algorithms. In the study of computation, we learn that there are computable functions which may however not be computed efficiently. Shor's great contribution to information theory was to advance the boundary of the set of efficiently computable functions \[6\]. This naturally begs the question as to whether it is more efficient to play games quantum mechanically than classically. We will show shortly that, in terms of efficiency, some quantum games can indeed outperform classical games.

The quantum game protocol that we study is a generalization of that described in \[2\]. We use the term static quantum games to reflect the similarity of the resulting games to static classical games. To play a static quantum game, we start with an initial state which is represented by qubits. The referee then divides the state into \(N\) sets of qubit parts, sending the \(k\)-th set to player \(k\). The players separately operate on the qubits that they receive, and then send them back to the referee. The referee then determines the payoffs for the players with regard to the measured outcome of a collection of POVM operators \(f_{M_{g}}\). Anticipating the focus on efficiency, we choose the dimension of \(N\) to be \(2^q\) where \(qN\) is the number of qubits, and we set \(n = 2^q\). We also assume that the players share the initial qubits equally, i.e., each one of them will receive \(q\) qubits. This assumption is inessential and is for ease of exposition only. Following the game's protocol, the players operate independently on the states, and hence it is natural to allow them access to all possible physical maps. Specifically, we allow each player to have access to the set of trace-preserving completely positive maps, i.e., for each \(k\), we set \(k\) to be the set of trace-preserving completely positive maps. Indeed, the only way to restrict the players' strategy sets in this protocol is to perform some measurement at the referee's end - this is incorporated into our formalism by allowing the referee the set of all POVM operators.

First, we restrict ourselves to two-player games. If a payoff of \(\langle \psi \rangle_k\) for player \(k\) is associated with the measured outcome \(m\), the payoff for player \(k\) is then \(\text{tr}(R^k)\) where \(R^k = a_k M^{\overrightarrow{Y}} M^Y + \frac{1}{n} M^L\), and \(M^Y\) is the resulting state. For example, if player I decides to use operation \(E = fE_k g\) and player II \(F = fF_k g\), then \(P_k (E; F) = \text{tr}(R^k (E_k \otimes F_k))^Y)\). Hence \(P_k (E; F) = \text{tr}(R^k (E_k \otimes F_k))\).

We now consider a set of operators \(E\) for which will form a basis for the set of operators in the state space. If \(E = fE_k = e_k E^g\) and
F = \sum_{k=1}^{P} f_{k} E g,\text{ then the payo is}
\begin{align*}
x \in \sigma_{k} \in \mathbb{K} f_{k} f_{k} A^{k}
\end{align*}
where A^{k} = \text{tr}R^{k}(E^{*} E) (E^{y} E^{y})].\text{ Letting }\sum_{k=1}^{P} f_{k} = \sigma_{k} \text{ and } \sum_{k=1}^{P} f_{k} f_{k}, \text{ then and are positive hermitian matrices with } 16^{3} 4^{3} \text{ independent real parameters. Note that the number } 16^{3} \text{ comes from the fact that } 16^{3} \text{ real parameters are needed to specify a } 4^{3} 4^{3} \text{ positive hermitian matrix, while } 4^{3} \text{ comes from the fact that } E^{y} E^{y} = I \text{ with the assumptions that } E \text{ is positive and hermitian. This procedure is the same as the so-called chimatrix representation [7]. For a general matrix, we now observe that } \text{tr}( ) = \text{tr}( ) \text{. Hence we have } A = A \text{. Therefore, the payo is actually } ; ; ; \text{ Re}[ A ], \text{ which is always real as expected. To recap, the strategy sets for the players are } f; g; f; g; \text{ these are subsets of the set of positive hermitian matrices such that}
\begin{align*}
x \in \sigma_{k} \in \mathbb{K} E^{y} E^{y} = I ; \quad x \in \sigma_{k} \in \mathbb{K} E^{y} E^{y} = I ;
\end{align*}
The payo is given by } P ; ; ; \text{ Re}[ A ] \text{. As shown above, we may now identify } \sigma_{k} \text{ to be the set of positive semidefinite hermitian matrices satisfying condition } (1). \text{ It then follows that } A_{k} \text{ is a convex, compact Euclidean space.}

The above analysis can easily be generalized to } N \text{-player games. For a particular } N \text{-player static game, } P_{k} ( ) = \sum_{k=1}^{N} A_{k} \text{ where } A_{k} = \text{tr}R^{k}(E^{*} E) (E^{y} E^{y}) \} \text{(index summation omitted for clarity).}

We can now see a striking similarity between static quantum games and static classical finite games. The payo for a classical finite two-player game has the form } P_{ij}x_{i}A_{i}y_{j} \text{ where } x_{i}y_{j} \text{ belong to some multidimensional simplices and } A \text{ is a general matrix, the payo for a static quantum game is }
\begin{align*}
\sum_{k=1}^{N} A_{k} ; ; ; A \text{ where } ; \text{ belong to some multidimensional compact and convex sets } k. \text{ Indeed the multi-linear structure of the payo function and the convexity and compactness of the strategy sets, are the essential features underlying both classical and quantum games. And one may exploit these similarities to extend some classical results into the quantum domain. Two immediate examples are the Nash equilibrium Theorem and the Minimax Theorem. To show this, we first need to recall some relevant definitions. For a vector } v = (v_{i})_{i \in N}, \text{ we set } v_{k} \text{ to be } (v_{1}; v_{k+1}; ; ; v_{n}) \text{ and we denote } (v_{1}; ; ; v_{k-1}; v_{k+1}; ; ; v_{n}) \text{ by } (v_{k} v_{k}^{(0)}). \text{ We also define the set of best replies for player } k \text{ to be } B_{k} ( ) = \sum_{k=2}^{N} f_{k} 2 \text{.}
B_{k} ( k) = f_{k} 2 \text{.}
\text{ For player } k \text{ to be } B_{k} ( ) = \sum_{k=2}^{N} f_{k} 2 \text{.}
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note that for each \( k \), \( B_k(\sim k) \) is convex and closed in \( k \) and hence compact. Using the notion of best reply, we can easily define what a Nash equilibrium is: an operator profile \( \sim \) is a Nash equilibrium if \( k \in 2 B_k(\sim k) \) for all \( k \).

**Theorem 1 (Quantum Nash Equilibrium Theorem)**
For all static quantum games, at least one Nash equilibrium exists.

**Proof:** We know that \( B_k \) is a convex compact subset of a Euclidean space. Since \( B = \cup B_k \) is an upper semicontinuous point-to-set map which takes each \( k \) to a convex set \( B(k) \), the theorem follows from Kakutani’s fixed point theorem [8]. Q.E.D.

We now restrict ourselves to two-player zero-sum games, i.e., \( a^I_m = a^II_m \) for all \( m \). Trivially, given any strategy, player I’s payoff is bounded above by

\[
v(\sim) = \max_{A} \min_{X} v(\sim; A; X)
\]

Similarly, given any strategy, player II’s payoff is bounded below by

\[
v(\sim) = \min_{A} \max_{X} v(\sim; A; X)
\]

We therefore define

\[
v_I = \max_{A} \min_{X} v(\sim; A; X)
\]

\[
v_{II} = \min_{A} \max_{X} v(\sim; A; X)
\]

**Theorem 2 (Quantum Minimax Theorem)**

\[
v_I = v_{II};
\]

**Proof:** We have seen that for each \( k \), \( B_k \) is compact and convex as a Euclidean space. Also the payoff is linear and continuous in and . Therefore, the theorem follows from the Minimax theorem in Ref. [9]. Q.E.D.

We note that the proofs of theorems 1 and theorem 2 are completely analogous to the corresponding classical proofs. This is because the underlying theorems involved—Kakutani’s theorem and the Minimax theorem—are general enough to allow for compact and convex strategy sets without restricting them to only be simplices. We also note that although quantum games can profit from some nice classical results,
problematic issues in classical game theory also carry over to quantum games. For example, one would expect multiple Nash equilibria in general quantum games. Classical game theorists have invented evolutionary game theory [10] to deal with this problem and its quantum analogy awaits full development [11].

If quantum games were merely some replicas of classical games, or vice versa, the subject of quantum games would not be very interesting. Here we show that quantum games are more than that, by showing that in comparison with classical games, not only the set of nilpotent quantum games is strictly larger, quantum games can also be played more efficiently. Before we do this, however, we need to discuss how to quantify efficiency in both classical and quantum cases, and how we then compare the resulting efficiencies. We have seen that the quantum strategy set of $k$ is a compact and convex Euclidean space. Since any compact and convex Euclidean space lies inside some $m$-dimensional simplex, yet at the same time contains a smaller $m'$-dimensional simplex as a subset where $m'$ is unique and equals $16^q$ for $k$. The dimensionality will be the quantity we use to gauge efficiency, because it is well-defined and reflects the number of qubits needed. For example, in order to play a two-player quantum game we need to exchange $4q$ qubits in total. This is because the referee needs to send qubits to the two players and they then need to send them back. The strategy set for each player has dimension $16^q - 1$. If the same number of bit-transfers is allowed in a classical game, then the strategy set for each player will be a simplex of dimension $4^q - 1$. Therefore, does it mean that we have a factor of 2 increase in efficiency by playing quantum games? It is true in general, but is not immediately obvious: although the quantum strategy set has a higher dimension, we do not yet know whether many of the strategies are redundant or not.

In order to show that general quantum games are indeed more efficient, we first perform some concrete calculations. Since the choice of $fE^*$ $g$ is arbitrary, we take $fE^* g = f n_{ij} g$ where $n_{ij}$ denotes a $n \times n$ square matrix such that $(ij)$-entry $= 1$ and all other entries are equal to 0. Denoting by $ij$ and $ijl$ using condition (1), we have the following restrictive conditions on $i$, $ij$, $ijl$, and $ijll$: where the first two sub-indices represent while the latter two represent . A further calculation shows that $n_{ijl} = n_{ij}^2 (i + 1)n + (j + 1)n + 1$. For arbitrary $R$ and $k$, we find the following:

$$A \begin{bmatrix} p_{ij} \\ p_{ijl} \\ n_{ij} \\ n_{ijl} \\ n_{ijll} \end{bmatrix} = R \begin{bmatrix} (i + 1)n + k; (j + 1)n + l \\ (i + 1)n + j; (j + 1)n + l \end{bmatrix}.$$  

We are now ready to prove the following theorem:
Theorem 3 A is diagonal with non-zero diagonal entries for some e and R.

Proof: We first note that the diagonal elements of $A$ are

$$A_{abcd} = R_{(n+1)d(01);(n+1)e(10)} = R_{(n+1)b(11);(n+1)c(00)};$$

Therefore for all $a; b; c; d$, we set

$$R_{(n+1)d(01);(n+1)e(10)} = R_{(n+1)b(11);(n+1)c(00)} = 1 = n;$$

We then set all the other entries of $R$ and to be $0$. Q.E.D.

[We note that the above construction still holds true in multiplayer games. A will be a tensor with vanishing entries, except for entries with identical indices.] Any two operations by player 1, and 0, are redundant if $P_1(0; 0) = P_1(0; 0)$ for all. However in the above game, $P_1(0; 0) = 0$, $\Re[\bar{0}] = 0$, $\Re[0] = 0$ in general. Therefore, the payoff depends on all of the independent parameters and there are $16^3$ 4^3 of them. Hence, the upper bound on efficiency is indeed saturated. One could also envisage varying and R in nimesimally to provide a continuum of quantum games with superior efficiency.

We now provide an example. We consider a two-qubit two-player zero-sum game $e$, and take

$$R_{c} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; R_{c} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix};$$

This means that $j = j_1$, where $j = j_1$. For example, the referee may do a von Neumann measurement on the state with respect to the orthonormal basis $fj_1; ji_1; j_1j_1; j_1j_1; j_1j_1; j_1j_1; j_1j_1; j_1j_1$; and then award a payoff of 1 for each player if the outcome is $j_1$, and a payoff of 0 for any other outcomes. The strategy set in this game is of dimension 12, which corresponds to 13 independent strategies classically, therefore at least 8 bits have to be transferred. This is in contrast with the fact that only 4 qubits needed to be transferred in the quantum version. The above game, although reasonably simple, does therefore highlight the potential of quantum games.

Besides efficiency, we will now show that quantum games are indeed strictly more general as claimed. Firstly of all, we note that the
classical strategy set and the quantum strategy set cannot be made identical if the linearity of the payoff function is to be preserved. This is because there is no linear homomorphism that maps $k$ to a simplex of any dimension. In essence, the positivity of $k$, i.e., the conditions $\sum_j f_{ij} = 1$ for all $2^k$, spoils this possibility. Therefore, if we identify $k$ as some multi-dimensional simplex, we must lose linearity of the payoff function. On the other hand, we have seen that every strategy in $k$ is non-redundant in the above games, it is therefore in possible to play the above game classically no matter how you enlarge the strategy set (still finite, of course).

Conversely, we show by an example how classical games can be played within the formalism of quantum games. We consider a general two-player two-move game, the game being classical naturally suggests that the initial qubits are not entangled and the payoffs are determined by measuring the resulting qubits with respect to the computational basis. Hence, without loss of generality,

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & C \\
0 & 0 & 0 & A
\end{bmatrix}
\]

and

\[
R^k = \begin{bmatrix}
r_{11} & 0 & 0 & 0 \\
0 & r_{12} & 0 & 0 \\
0 & 0 & r_{21} & 0 \\
0 & 0 & 0 & r_{22}
\end{bmatrix}
\]

As a result,

\[
A_{abcdijkl}^k = \begin{cases}
R_{2(\leq 1)+k\geq 2\geq 1)+i}^k & \text{if } b = d = j = l = 1; a = c; i = k \\
0 & \text{otherwise}.
\end{cases}
\]

So the only relevant dimensions for player I are $1111$ and $2121$. We now recall the conditions imposed on $i$, which are $i_1 = 1$ and $i_{ij} = 0$ and $i_{ij1} = i_{ij1} = 0$. Therefore, the above quantum game is the same as playing the classical game with the game matrix below:

\[
A^k = \begin{bmatrix}
r_{11}^k & r_{12}^k \\
r_{21}^k & r_{22}^k
\end{bmatrix}
\]

In summary, we have shown that playing games quantum mechanically can be more efficient, and have given a saturated upper bound on this efficiency. In particular, there is a factor of 2 increase in efficiency. We have also deduced the quantum version of the minimax theorem for zero-sum games and the Nash theorem for general static games. In addition, we have shown that finite classical games consist of a strict subset of finite quantum games. We have also pointed out the essential characteristics shared by static quantum and static classical games: these are the linearity of the payoff function and the
convexity and compactness of the strategy sets. Indeed, the success of using linear programming to search for Nash equilibria in classical two-player zero-sum games relies on these characteristics; one would suspect the same method could be applicable to the quantum version as well [5]. Our final words would be a cautious speculation on the possibility that the physical and natural world might already be exploiting this efficiency advantage on the microscopic scale [12].

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