ON REGULARITY THEORY FOR $n/p$-HARMONIC MAPS INTO MANIFOLDS

FRANCESCA DA LIO AND ARMIN SCHIKORRA

Abstract. In this paper we continue the investigation started in the paper [10] of the regularity of the so-called weak $\frac{n}{p}$-harmonic maps in the critical case. These are critical points of the following nonlocal energy

$$\mathcal{L}_s(u) = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{n}} u(x)|^p \, dx,$$

where $u \in H^{s,p}(\mathbb{R}^n, \mathcal{N})$ and $\mathcal{N} \subset \mathbb{R}^N$ is a closed $k$ dimensional smooth manifold and $s = \frac{n}{p}$. We prove Hölder continuity for such critical points for $p \leq 2$. For $p > 2$ we obtain the same under an additional Lorentz-space assumption. The regularity theory is in the two cases based on regularity results for nonlocal Schrödinger systems with an antisymmetric potential.

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1. Introduction

Half-harmonic maps were first studied by Rivièe and the first-named author [9, 8]. The $L^2$-regularity theory has been extended to higher dimension [17, 21, 4, 25], and to $L^p$-energies [10, 23, 24]. Compactness and quantization issues have been addressed [5, 6].

Here we extend our analysis of weak $n/p$-harmonic maps initiated in [10] in the sphere case to general target manifolds.

They are critical points of the energy

\[
\mathcal{L}_p(u) = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{p}} u(x)|^p dx
\]

acting on maps $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^N)$ which pointwise map into a smooth, closed (compact and without boundary) $k$-dimensional manifold $\mathcal{N} \subset \mathbb{R}^N$. This class of maps is commonly denoted by $H^{s,p}(\mathbb{R}^n, \mathcal{N})$. We will refer to Section 2 for the precise definition of such functional spaces. The Euler-Lagrange equation for critical points $u$ can be formulated as follows

\[
\Pi(u)((-\Delta)^{\frac{s}{p}} ((-\Delta)^{\frac{s}{p}} u)^{p-2}(-\Delta)^{\frac{s}{p}} u)) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n)
\]

where $\Pi: U_\delta \to \mathcal{N}$ is the standard nearest point projection of a $\delta$-neighborhood $U_\delta$ of $\mathcal{N}$ onto $\mathcal{N}$. Our first main result is the regularity theory for the case $p < 2$.

**Theorem 1.1.** Assume that $u \in H^{s,\frac{n}{s}}(\mathbb{R}^n, \mathcal{N})$, for $\frac{n}{s} \leq 2$, is a solution to (1.2). Then $u$ is locally Hölder continuous.

The case $p > 2$ presents additional difficulties. Here we show, that under the additional assumption that $(-\Delta)^{\frac{s}{p}} u$ belongs to the smaller Lorentz space $L^{(p,2)}$, regularity theory follows. More precisely we have

**Theorem 1.2.** Assume that $u \in H^{s,\frac{n}{s}}(\mathbb{R}^n, \mathcal{N})$, for $\frac{n}{s} \geq 2$, is a solution to (1.2). If we additionally assume

\[
\|(-\Delta)^{\frac{s}{p}} u\|_{(\frac{n}{s},2)} < \infty,
\]

then $u$ is locally Hölder continuous.

Let us stress that the extra assumption (1.3) is not motivated by geometric arguments, but by pure analytic considerations, and we do not know if (1.3) is a necessary assumption. Indeed this is related to a major open problem, the regularity theory of $n$-harmonic maps into manifolds and generalized $H$-systems, see [27]. Also in that case, regularity can only be proven under additional analytic assumptions that cannot be justified geometrically, see [14, 22]. However, these additional assumptions do not a priori rule out the possible singularities such as $\log \log 1/|x|$, so the geometric structure of the Euler-Lagrange equation plays an important role.
Both theorems follow from a reduction to a system with antisymmetric structure, in the spirit of Rivièr e’s seminal work [18] which was adapted to nonlocal equations first by Rivièr e and the first-named author [8], for related arguments see also [4, 16]. Namely we have

**Proposition 1.3.** Let $u$ satisfy the hypotheses either of Theorem 1.1 or of Theorem 1.2, $p = \frac{n}{s}$. Set $w := \left| (-\Delta)^{\frac{s}{2}} u \right|^{p-2} (-\Delta)^{\frac{s}{2}} u$. Then $w$ satisfies

\[
(-\Delta)^{\frac{s}{2}} w^i = \Omega_{ij} w^j + E_i(w),
\]

where $\Omega_{ij} = -\Omega_{ji}$ belongs to $L^p(\mathbb{R}^n)$ (for $p \leq 2$) or to $L^{(p,2)}(\mathbb{R}^n)$ (for $p > 2$).

Moreover, $E_i$ is so that for any $\varepsilon > 0$ there exists a radius $R = R(\varepsilon)$ and a $K \in \mathbb{N}$ so that for any $k_0 \in \mathbb{N}$, $k_0 > K$, any $x_0 \in \mathbb{R}^n$ and for any radius $r \in (0, 2^{-k_0}R)$ it holds that for any $\varphi \in C_c^\infty(B(x_0, r))$

\[
\int_{\mathbb{R}^n} E_i(w) \varphi \lesssim \varepsilon \left( \| \varphi \|_{\infty} + \| (-\Delta)^{\frac{s}{2}} \varphi \|_{(p,2)} \right) \left( \| w \|_{(p',\infty),B(x_0,2k_0r)} + \sum_{k=k_0}^{\infty} 2^{-k\sigma} \| w \|_{(p',\infty),B(x_0,2^kr)} \right),
\]

if $p > 2$ and

\[
\int_{\mathbb{R}^n} E_i(w) \varphi \lesssim \varepsilon \left( \| \varphi \|_{\infty} + \| (-\Delta)^{\frac{s}{2}} \varphi \|_{p} \right) \left( \| w \|_{(p',\infty),B(x_0,2k_0r)} + \sum_{k=k_0}^{\infty} 2^{-k\sigma} \| w \|_{(p',\infty),B(x_0,2^kr)} \right),
\]

if $p \leq 2$. Here $\sigma > 0$ is a uniform constant only depending on $s$ and $n$.

Then, Theorem 1.1 and Theorem 1.2 follow from the following result on Schrödinger-type equations and the Sobolev embedding for Sobolev-Morrey spaces [1].

**Proposition 1.4.** If $w \in L^{\frac{n}{n-s}}(\mathbb{R}^n, \mathbb{R}^N)$ is a solution of

\[
(-\Delta)^{\frac{s}{2}} w^i = \Omega_{ij} w^j + E_i(w) \quad \text{in } \mathbb{R}^n,
\]

where $\Omega_{ij} = -\Omega_{ji} \in L^{(\frac{n}{s},2)}$ and $E$ is as in Proposition 1.3. Then there exists $\alpha > 0$ so that for every $x_0 \in \mathbb{R}^n$ it holds

\[
\sup_{x \in B(x_0,\rho)} \rho^{-\alpha} \| w \|_{(\frac{n}{n-s},\infty),B(x,\rho)} < \infty.
\]

Proposition 1.4 implies in particular, that solutions of

\[
(-\Delta)^{\frac{s}{2}} w^i = \Omega_{ij} w^j
\]

improve their integrability when $\Omega \in L^{(\frac{n}{s},1)}$ without any antisymmetry assumption. Indeed, then $\Omega_{ij} w^j$ satisfies the conditions of $E_i$. This special case is related to the Lipschitz regularity of solutions of

\[
\text{div}(|\nabla u|^{n-2} \nabla u) = \Omega|\nabla u|^{n-2} \nabla u
\]

under the assumption that $\Omega \in L^{(n,1)}$, which was proven by Duzaar and Mingione, [11].

Let us also remark, that in the local case, i.e. for $s = 2$ and $n = 2$, the assumption of Proposition 1.4 are not optimal: Rivièr e showed in [19] that in that case $\Omega_{ji} \in L^{(\frac{n}{2}, \frac{n}{2})}$
suffices to improve integrability. Nevertheless, observe that for \( n = 1 \) and \( s = \frac{1}{2} \) we recover the regularity Theorem by Rivière and the first author \([8]\). Also, for \( \frac{n}{s} < 2 \) our assumptions are weaker than \( \Omega \in L^{\frac{n}{s}}(\mathbb{R}^n) \).

The paper is organized as follows. In Section 2 we introduce some preliminary definitions and notations. Section 3 is devoted to the proof of Proposition 1.3. In Section 4 we show how to perform a change of gauge in a system of the form (1.4). In Section 5 we prove Proposition 1.4.

2. Preliminaries: function spaces and the fractional Laplacian

In this Section we introduce some notations and definitions that are used in the paper.

For \( n \geq 1 \), we denote respectively by \( \mathcal{S}(\mathbb{R}^n) \) and \( \mathcal{S}'(\mathbb{R}^n) \) the spaces of Schwartz functions and tempered distributions.

Given a function \( v \) we will denote either by \( \hat{v} \) or by \( \mathcal{F}[v] \) the Fourier Transform of \( v : \)

\[
\hat{v}(\xi) = \mathcal{F}[v](\xi) = \int_{\mathbb{R}^n} v(x)e^{-i\langle \xi, x \rangle} \, dx.
\]

We introduce the following topological subspace of \( \mathcal{S}(\mathbb{R}^n) \) :

\[
\mathcal{Z}(\mathbb{R}^n) = \{ \varphi \in \mathcal{S}(\mathbb{R}^n) : (D^\alpha \mathcal{F}[v])(0) = 0, \text{ for every multi-index } \alpha \}. \]

Its topological dual \( \mathcal{Z}'(\mathbb{R}^n) \) can be identified with the quotient space \( \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) \) where \( \mathcal{P}(\mathbb{R}^n) \) is the collection of all polynomials, (see e.g. \([28]\)).

Given \( q > 1 \) and \( s \in \mathbb{R} \) we also set

\[
\dot{H}^{s,q}(\mathbb{R}^n) := \{ v \in \mathcal{Z}'(\mathbb{R}^n) : \mathcal{F}^{-1}[|\xi|^s \mathcal{F}[v]] \in L^q(\mathbb{R}^n) \}.
\]

For a submanifold \( \mathcal{N} \) of \( \mathbb{R}^m \) we can define

\[
\dot{H}^{s,q}(\mathbb{R}^n, \mathcal{N}) = \{ u \in \dot{H}^{s,q}(\mathbb{R}^n, \mathbb{R}^m) : u(x) \in \mathcal{N}, \text{a.e.} \}.
\]

Finally we denote \( \mathcal{H}^1(\mathbb{R}^n) \) the homogeneous Hardy Space in \( \mathbb{R}^n \).

We recall that if \( sp = n \) then

\[
\dot{H}^{s,p}(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n),
\]

where \( BMO(\mathbb{R}^n) \) is the space of bounded mean oscillation dual to \( \mathcal{H}^1(\mathbb{R}^n) \).

The \( s \)-fractional Laplacian of a function \( u : \mathbb{R}^n \to \mathbb{R} \) is defined as a pseudo differential operator of symbol \( |\xi|^{2s} \):

\[
(-\Delta)^s u(\xi) = |\xi|^{2s} \hat{u}(\xi).
\]
For every $\sigma \in (0, n)$ we denote by $I^\sigma$ the Riesz Potential, that is
\[ I^\sigma f(x) := c_\sigma \int_{\mathbb{R}^n} \frac{f(z)}{|x-z|^{n-\sigma}} \, dz. \]

Finally we introduce the definition of Lorentz spaces (see for instance Grafakos’s monograph [13] for a complete presentation of such spaces). For $1 \leq p < +\infty$, $1 \leq q \leq +\infty$, the Lorentz space $L^{(p,q)}(\mathbb{R}^n)$ is the set of measurable functions satisfying
\[ \left\{ \begin{array}{ll}
\int_0^{+\infty} (t^{1/p} f^*(t))^{q \frac{dt}{t}} < +\infty, & \text{if } q < \infty, \ p < +\infty \\
\sup_{t>0} t^{1/p} f^*(t) < \infty & \text{if } q = \infty, \ p < \infty,
\end{array} \right. \]
where $f^*$ is the decreasing rearrangement of $|f|$.

We observe that $L^{p,\infty}(\mathbb{R}^n)$ corresponds to the weak $L^p$ space. Moreover for $1 < p < +\infty$, $1 \leq q \leq +\infty$ the dual space of $L^{(p,q)}$ is $L^{(\frac{p'}{p},\frac{q'}{q})}$ if $q > 1$ and it is $L^{(1,\infty)}$ if $q \leq 1$.

Let us define
\[ \dot{H}^{s,(p,q)}(\mathbb{R}^n) = \{ v \in \mathcal{Z}'(\mathbb{R}^n) : \mathcal{F}^{-1} ||\xi|^s \mathcal{F}[v]| \in L^{(p,q)}(\mathbb{R}^n) \}. \]

In the sequel we will often use the H"older inequality in the Lorentz spaces: if $f \in L^{(p_1,q_1)}$, $g \in L^{(p_2,q_2)}$, with $1 \leq p_1,p_2,q_1,q_2 \leq +\infty$. Then $fg \in L^{r,s}$, with $r^{-1} = p_1^{-1} + p_2^{-1}$ and $s^{-1} = q_1^{-1} + q_2^{-1}$, (see for instance [13]).

To conclude we introduce some basic notation.

$B(\bar{x},r)$ is the ball of radius $r$ and centered at $\bar{x}$. If $\bar{x} = 0$ we simply write $B_r$. If $x,y \in \mathbb{R}^n$, $x \cdot y$ is the scalar product between $x,y$.

Given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i$ is a nonnegative integer, we denote by $|\alpha| = \alpha_1 + \ldots + \alpha_n$ the order of $\alpha$.

Given $q > 1$ we denote by $q'$ the conjugate of $q$: $q^{-1} + q'^{-1} = 1$.

In the sequel we will often use the symbols $a \lesssim b$ and $a \simeq b$ instead of $a \leq C b$ and $C^{-1} b \leq a \leq C b$, respectively, whenever the multiplicative constants $C$ appearing in the estimates are not relevant for the computations and therefore they are omitted.

3. Rewriting the Euler-Lagrange equations: Proof of Proposition 1.3

For a fixed manifold $\mathcal{N}$ and $p \in \mathcal{N}$ we denote by $\Pi(p)$ the projection onto the tangent plane $T_p \mathcal{N}$, and by $\Pi^\perp(p) = I - \Pi(p)$ the projection onto the normal space $(T_p \mathcal{N})^\perp$.

For $s > 0$ we first introduce the following three-term commutator
\[
H_s(f,g) = (-\Delta)^{\frac{s}{2}} (fg) - (-\Delta)^{\frac{s}{2}} fg - f (-\Delta)^{\frac{s}{2}} g.
\]
Such a commutator has been used for the first time in [9] in the case $s = \frac{1}{2}$ in the context of 1/2-harmonic maps (see also [21]). It represents the error term of the Leibniz rule for $(-\Delta)\hat{\sigma}$. We recall here some estimates of (3.1) for general $s > 0$ that we will use in the sequel, (see e.g. [25, 15]).

**Lemma 3.1.** Let $s \in (0, 1]$, For any $t \in (0, s)$, $p, p_1, p_2 \in (1, \infty)$, $q, q_1, q_2 \in [1, \infty]$ such that
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},
\]
it holds that
\[
\|H_s(f, \varphi)\|_{L^{(p,q)}(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\frac{s}{2}} f\|_{L^{(p_1,q_1)}(\mathbb{R}^n)} \|(-\Delta)^{\frac{s}{2}} \varphi\|_{L^{(p_2,q_2)}(\mathbb{R}^n)}.
\]

**Lemma 3.2.** Let $s \in (0, 1]$, $p \in (1, \infty)$, $p' = \frac{p}{p-1}$, $q \in [1, \infty)$, $q' = \frac{q}{q-1} \in [1, \infty)$. Then, for any $a, b \in C_c^\infty(\mathbb{R}^n)$,
\[
\int_{\mathbb{R}^n} H_s(a, b) (-\Delta)^{\frac{s}{2}} \varphi \lesssim [\varphi]_{BMO} \|(-\Delta)^{\frac{s}{2}} a\|_{L^{(p,q)}(\mathbb{R}^n)} \|(-\Delta)^{\frac{s}{2}} b\|_{L^{(p',q')} (\mathbb{R}^n)}.
\]
In particular, by the duality of Hardy-space $H^1$ and $BMO$,
\[
\|(-\Delta)^{\frac{s}{2}} (H_s(a, b))\|_{H^1} \lesssim \|(-\Delta)^{\frac{s}{2}} a\|_{L^{(p,q)}(\mathbb{R}^n)} \|(-\Delta)^{\frac{s}{2}} b\|_{L^{(p',q')} (\mathbb{R}^n)}.
\]

We will recall the following result

**Lemma 3.3 (Coifman-Rochberg-Weiss [3]).** For any smooth and compactly supported $f, g \in C_c^\infty(\mathbb{R}^n)$ and any $i = 1, \ldots, n$ we define the commutator
\[
[R_i, f](g) = R_i(fg) - fR_i(g)
\]
Then for $p > 1$ there is constant $C > 0$ (depending on $p, n$) such that
\[
\|R_i, f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{BMO(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}.
\]

We will use the following extension of Lemma 3.3.

**Lemma 3.4 (Theorem 6.1 in [15]).** Let $s \in (0, 1]$ and $p \in (1, \infty)$ and $q, q_1, q_2 \in [1, \infty]$ with
\[
\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}. \quad \text{Then, for } f, g \in C_c^\infty(\mathbb{R}^n), \text{ and for } p, p_1, p_2 \in (1, \infty), \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \quad \sigma \in [s, 1],
\]
\[
\|((-\Delta)^{\frac{s}{2}} g)(f)\|_{L^{(p,q)}(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\frac{s}{2}} g\|_{L^{(p_1,q_1)}(\mathbb{R}^n)} \|I^{\sigma-s} f\|_{L^{(p_2,q_2)}(\mathbb{R}^n)}.
\]
(3.3) remains valid if one replaces $(-\Delta)^{\frac{s}{2}}$ by $R_i(-\Delta)^{\frac{s}{2}}$, where $R_i$ is the $i$th Riesz transform.

Also, for $p_1, p_2, p \in (1, \infty)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\sigma \in [0, 1)$,
\[
\|R_i, g\|_{L^{(p,q)}(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\frac{s}{2}} g\|_{L^{(p_1,q_1)}(\mathbb{R}^n)} \|I^{\sigma-s} f\|_{L^{(p_2,q_2)}(\mathbb{R}^n)}.
\]
(3.4) For a map $u : \mathbb{R}^n \to \mathcal{N}$ any derivative $\partial_\alpha u$ is a tangential vector, i.e. $\partial_\alpha u \in T_u \mathcal{N}$. In particular, $\Pi^\perp(p) \nabla u = 0$. If we replace the gradient $\nabla u$ by $(-\Delta)^{\frac{s}{2}} u$ there is no reason for this to be true. However, a certain tangential inclination of $(-\Delta)^{\frac{s}{2}} u$ can be measured in the following sense.
Lemma 3.5. Assume that \( u \in \dot{H}^{s,p}(\mathbb{R}^n, \mathcal{N}) \), where \( p = \frac{n}{s} \in (1, \infty) \). Then
\[
\|\Pi^\perp(u)(-\Delta)^{\frac{s}{2}}u\|_{(p,q)} \lesssim \|(-\Delta)^{\frac{s}{2}}u\|_{(p,q)} \|(-\Delta)^{\frac{s}{2}}u\|_{(p,q)}.
\]
whenever \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \).
Moreover, for \( p \leq 2 \)
\[
\|\Pi^\perp(u)(-\Delta)^{\frac{s}{2}}u\|_{(p,q)} \lesssim \|(-\Delta)^{\frac{s}{2}}u\|_{(p,q)} \|(-\Delta)^{\frac{s}{2}}u\|_{(p,q)}.
\]
Also the localized versions of the above estimates hold: for some \( \sigma = \sigma(s) > 0 \), for every \( k_0 \in \mathbb{N} \),
\[
\|\Pi^\perp(u)(-\Delta)^{\frac{s}{2}}u\|_{(p,q), B(x_0,r)} \lesssim \left( \|(-\Delta)^{\frac{s}{2}}u\|_{(p,q), B(x_0,2^k_0 r)} + \sum_{k=k_0}^{\infty} 2^{-k\sigma} \|(-\Delta)^{\frac{s}{2}}u\|_{(p,q), B(x_0,2^k r)} \right) \cdot \left( \|(-\Delta)^{\frac{s}{2}}u\|_{(p,q), B(x_0,2^k_0 r)} + \sum_{\ell=k_0}^{\infty} 2^{-\ell\sigma} \|(-\Delta)^{\frac{s}{2}}u\|_{(p,q), B(x_0,2^\ell r)} \right),
\]
and
\[
\|\Pi^\perp(u)(-\Delta)^{\frac{s}{2}}u\|_{(p,q), B(x_0,r)} \lesssim \left( \|(-\Delta)^{\frac{s}{2}}u\|_{(p,q), B(x_0,2^k_0 r)} + \sum_{k=k_0}^{\infty} 2^{-k\sigma} \|(-\Delta)^{\frac{s}{2}}u\|_{(p,q), B(x_0,2^k r)} \right) \cdot \left( \|(-\Delta)^{\frac{s}{2}}u\|_{(p,q), B(x_0,2^k_0 r)} + \sum_{\ell=k_0}^{\infty} 2^{-\ell\sigma} \|(-\Delta)^{\frac{s}{2}}u\|_{(p,q), B(x_0,2^\ell r)} \right).
\]

Proof. The localization arguments are by now standard, we only indicate how to prove the global estimates.

The estimate (3.5) follows for \( s \in (0,1] \) from
\[
\|\Pi^\perp(u)(-\Delta)^{\frac{s}{2}}u(x)\| \lesssim |H_s(u,u)|,
\]
(see e.g. [26, Lemma E.1.], and also Proposition 4.1 in [7], for related properties) and by applying Lemma 3.1. For \( s \geq 1 \) we use that \( (-\Delta)^{\frac{s}{2}} = (-\Delta)^{\frac{s-1}{2}} R_\alpha \partial_\alpha \), and thus \( \Pi^\perp(u)\partial_\alpha u = 0 \) implies
\[
\Pi^\perp(u)(-\Delta)^{\frac{s}{2}}u = [\Pi^\perp(u), (-\Delta)^{\frac{s-1}{2}} R_\alpha](\partial_\alpha u) = \Pi^\perp(u)(-\Delta)^{\frac{s-1}{2}} R_\alpha(\partial_\alpha(u)) - (-\Delta)^{\frac{s-1}{2}} R_\alpha(\Pi^\perp(u)\partial_\alpha(u))
\]
The estimate then follows from Lemma 3.4.

For the second estimate (3.6), assume that \( p \in (1,2] \), and observe \( \Pi \in L^\infty(\mathcal{N}, \mathbb{R}^N) \) implies that pointwise
\[
\|(-\Delta)^{\frac{s}{2}}u|^{p-2}\Pi^\perp(u)(-\Delta)^{\frac{s}{2}}u| \lesssim |(-\Delta)^{\frac{s}{2}}u|^{p-1}.
\]
Moreover, in view of (3.7), for \( s \in (0,1) \),
\[
\|(-\Delta)^{\frac{s}{2}}u|^{p-2}\Pi^\perp(u)(-\Delta)^{\frac{s}{2}}u| \lesssim |(-\Delta)^{\frac{s}{2}}u|^{p-2}|H_s(u,u)|.
\]
Pointwise interpolating these two estimates, for any $\beta \in [0, 1]$
\[ |((\Delta)^{\frac{1}{2}} u)^{p-2}(\Delta)^{\frac{1}{2}} u| \lesssim |((\Delta)^{\frac{1}{2}} u)^{\beta(p-1)}((\Delta)^{\frac{1}{2}} u)^{(1-\beta)(p-2)}|H_s(u,u)|^{1-\beta}.\]
Since $p \in (1, 2]$, set $\beta = 2 - p \in [0, 1]$. Thus,
\[ |((\Delta)^{\frac{1}{2}} u)^{p-2}(\Delta)^{\frac{1}{2}} u| \lesssim |H_s(u,u)|^{p-1}.\]
Thus,
\[ |((\Delta)^{\frac{1}{2}} u)^{p-2}(\Delta)^{\frac{1}{2}} u|_{p'} \lesssim |((\Delta)^{\frac{1}{2}} u)^{p-1}||((\Delta)^{\frac{1}{2}} u)^{p-1}|_{(p,\infty)}||((\Delta)^{\frac{1}{2}} u)^{p-1}||_{(p,p)}.\]
The case $s \geq 1$ follows once again from Lemma 3.4.

We have all the ingredients for Proposition 1.3.

**Proof of Proposition 1.3.** Recall the definition of projections $\Pi(u)$ and $\Pi^\perp(u)$ above. Observe that these are symmetric matrices.

Also observe that for $p < 2$, $|((\Delta)^{\frac{1}{2}} u)^{p-1}|_{(p,2)} \lesssim |((\Delta)^{\frac{1}{2}} u)^{p-1}|.$

From (1.2) and $\Pi(u) + \Pi^\perp(u) = Id$ we have for any $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^N)$
\[
\int_{\mathbb{R}^n} |((\Delta)^{\frac{1}{2}} u)^{p-2}((\Delta)^{\frac{1}{2}} u \cdot (\Delta)^{\frac{1}{2}} \varphi
\]
\[= \int_{\mathbb{R}^n} |((\Delta)^{\frac{1}{2}} u)^{p-2}(\Pi(u)((\Delta)^{\frac{1}{2}} u \cdot (\Delta)^{\frac{1}{2}} (\Pi^\perp(u)\varphi)
\]
\[+ \int_{\mathbb{R}^n} |((\Delta)^{\frac{1}{2}} u)^{p-2}(\Pi^\perp(u)((\Delta)^{\frac{1}{2}} u \cdot (\Delta)^{\frac{1}{2}} (\Pi^\perp(u)\varphi).\]

Then with $\Pi(u)\Pi^\perp(u) = 0$ we find
\[\int_{\mathbb{R}^n} |((\Delta)^{\frac{1}{2}} u)^{p-2}((\Delta)^{\frac{1}{2}} u \cdot (\Delta)^{\frac{1}{2}} \varphi
\]
\[= \int_{\mathbb{R}^n} |((\Delta)^{\frac{1}{2}} u)^{p-2}(\Delta)^{\frac{1}{2}} u \cdot \varphi
\]
\[+ \int_{\mathbb{R}^n} |(\Delta)^{\frac{1}{2}} u|^p u \cdot H_s(\Pi^\perp(u), \varphi)
\]
\[+ \int_{\mathbb{R}^n} |(\Delta)^{\frac{1}{2}} u|^p u \cdot (\Delta)^{\frac{1}{2}} (\Pi^\perp(u)\varphi).\]

Now we define $\Omega \in L^p(\mathbb{R}^n, so(N))$,
\[\Omega := ((\Delta)^{\frac{1}{2}} \Pi^\perp(u) \Pi(u) - \Pi(u)(\Delta)^{\frac{1}{2}} \Pi^\perp(u).\]
Observe that for $p > 2$ the extra assumption (1.3) implies $\Omega \in L^{(p,2)}(\mathbb{R}^n, so(N))$. Then we have

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} u|^{p-2} (-\Delta)^{\frac{1}{2}} u (-\Delta)^{\frac{1}{2}} \phi$$

$$= \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} u|^{p-2} \Omega (-\Delta)^{\frac{1}{2}} u \phi$$

$$+ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} u|^{p-2} \Pi(u) (-\Delta)^{\frac{1}{2}} \Pi^\perp(u) (-\Delta)^{\frac{1}{2}} u \phi$$

$$+ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} u|^{p-2} \Pi(u) (-\Delta)^{\frac{1}{2}} H_s(\Pi^\perp(u), \phi)$$

$$+ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} u|^{p-2} (\Pi^\perp(u)) (-\Delta)^{\frac{1}{2}} \Pi^\perp(u) \phi$$

And again by $\Pi(u)\Pi^\perp(u) = 0$,

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} u|^{p-2} (-\Delta)^{\frac{1}{2}} u (-\Delta)^{\frac{1}{2}} \phi$$

$$= \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} u|^{p-2} \Omega (-\Delta)^{\frac{1}{2}} u \phi + \int_{\mathbb{R}^n} E \varphi$$

where

$$\int_{\mathbb{R}^n} E \varphi := - \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} u|^{p-2} H_s(\Pi(u), \Pi^\perp(u)) (-\Delta)^{\frac{1}{2}} u \varphi$$

$$+ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} u|^{p-2} \Pi(u) (-\Delta)^{\frac{1}{2}} u H_s(\Pi^\perp(u), \varphi)$$

$$- \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} u|^{p-2} (-\Delta)^{\frac{1}{2}} \Pi(u) \Pi^\perp(u) (-\Delta)^{\frac{1}{2}} u \varphi$$

$$+ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} u|^{p-2} (\Pi^\perp(u)) (-\Delta)^{\frac{1}{2}} \Pi^\perp(u) \varphi.$$
This is the crucial point where our assumption $\|(−Δ)^{1/2}u\|_{(p,2)} < ∞$ enters (which is only a nontrivial assumption for $p > 2$). In the same spirit,

$$\int_{\mathbb{R}^n} |(−Δ)^{1/2}u|^p \Pi(−Δ)^{1/2} u \ H_s(\Pi^\perp(u), \varphi)$$

$$\lesssim \|(−Δ)^{1/2}u\|_{(p,∞)}^{p-1} \|(−Δ)^{1/2}u\|_{(p,2)} \|(−Δ)^{1/2} \varphi\|_{(p,2)}.$$ 

The remaining terms of $E$ can be estimated by Lemma 3.5.

For $p \leq 2$:

$$\left| \int_{\mathbb{R}^n} |(−Δ)^{1/2}u|^p \Pi(−Δ)^{1/2}u \ (−Δ)^{1/2} u \ \varphi \right|$$

$$\lesssim \|(−Δ)^{1/2}u\|_{(p,∞)}^{p-1} \|(−Δ)^{1/2}u\|_{p}^{p-1} \|(−Δ)^{1/2} \Pi(u)\|_{p} \|\varphi\|_{∞}$$

$$\lesssim \|(−Δ)^{1/2}u\|_{(p,∞)}^{p-1} \|(−Δ)^{1/2}u\|_{p} \|\varphi\|_{∞}$$

and

$$\int_{\mathbb{R}^n} |(−Δ)^{1/2}u|^p \Pi(−Δ)^{1/2}u \ (−Δ)^{1/2} u \ (−Δ)^{1/2} \Pi^\perp(u) \varphi$$

$$\lesssim \|(−Δ)^{1/2}u\|_{(p,∞)}^{p-1} \|(−Δ)^{1/2}u\|_{p}^{p-1} \|(−Δ)^{1/2} \Pi(u)\|_{p} \|\varphi\|_{∞}$$

$$\lesssim \|(−Δ)^{1/2}u\|_{(p,∞)}^{p-1} \|(−Δ)^{1/2}u\|_{p} \|\varphi\|_{∞}$$

where to estimate $\|(−Δ)^{1/2} \Pi^\perp(u)\|_{p}$ we use the fact that

$$(−Δ)^{1/2} \Pi^\perp(u) \varphi = H_s(\Pi^\perp(u), \varphi) + (−Δ)^{1/2} (\Pi^\perp(u)) \varphi + \Pi^\perp(u) (−Δ)^{1/2} \varphi$$

Lemma 3.1 and Sobolev embeddings.

For $p > 2$:

$$\left| \int_{\mathbb{R}^n} |(−Δ)^{1/2}u|^p \Pi(−Δ)^{1/2}u \ (−Δ)^{1/2} u \ \varphi \right|$$

$$\lesssim \|(−Δ)^{1/2}u\|_{(p,∞)}^{p-2} \|(−Δ)^{1/2} \Pi(u)\|_{(p,∞)} \Pi^\perp(u) \ (−Δ)^{1/2} u \|_{(p,1)} \|\varphi\|_{∞}$$

$$\lesssim \|(−Δ)^{1/2}u\|_{(p,∞)}^{p-2} \|(−Δ)^{1/2}u\|_{p} \|\varphi\|_{∞}$$

and

$$\int_{\mathbb{R}^n} |(−Δ)^{1/2}u|^p \Pi(−Δ)^{1/2}u \ (−Δ)^{1/2} u \ (−Δ)^{1/2} \Pi^\perp(u) \varphi$$

$$\lesssim \|(−Δ)^{1/2}u\|_{(p,∞)}^{p-2} \|(−Δ)^{1/2} \varphi \Pi^\perp(u)\|_{(p,2)} \Pi^\perp(u) \ (−Δ)^{1/2} u \|_{(p,2)}.$$

$$\lesssim \|(−Δ)^{1/2}u\|_{(p,∞)}^{p-1} \|(−Δ)^{1/2}u\|_{(p,2)} \|(−Δ)^{1/2} \varphi\|_{(p,2)} + \|(−Δ)^{1/2} u\|_{(p,2)} \|(−Δ)^{1/2} \varphi\|_{(p,2)} .$$

Therefore we get for $p \leq 2$

$$\int_{\mathbb{R}^n} E \varphi \lesssim \|(−Δ)^{1/2}u\|_{(p,∞)}^{p-1} (1 + \|(−Δ)^{1/2}u\|_{p} ) \|(−Δ)^{1/2}u\|_{(p,2)} \|\varphi\|_{∞} + \|(−Δ)^{1/2} \varphi\|_{p} .$$
and for $p > 2$

$$\int_{\mathbb{R}^n} E \varphi \lesssim \|(-\Delta)^{\frac{1}{2}} u\|_{p,\infty}^{p-1} (1 + \|(-\Delta)^{\frac{1}{2}} u\|_p) \|(-\Delta)^{\frac{1}{2}} u\|_{p,2} \left( \|\varphi\|_{\infty} + \|(-\Delta)^{\frac{1}{2}} \varphi\|_{p,2} \right).$$

Consequently if we assume that $\varphi \in C^\infty_c(B(x_0, r))$, the above estimates can be localized and we find for $p \leq 2$

$$\int_{\mathbb{R}^n} E \varphi \lesssim C \left( 1 + \|(-\Delta)^{\frac{1}{2}} u\|_{p,\mathbb{R}^n} \right) \left( \|(-\Delta)^{\frac{1}{2}} u\|_{p,\infty, B(x_0, 2^{k_0} r)} + \sum_{k=k_0}^{\infty} 2^{-\sigma k} \|(-\Delta)^{\frac{1}{2}} u\|_{p,\infty, B(x_0, 2^{k} r)} \right)$$

and for $p > 2$ if we additionally assume (1.3),

$$\int_{\mathbb{R}^n} E \varphi \lesssim C \left( 1 + \|(-\Delta)^{\frac{1}{2}} u\|_{p,2} \right) \left( \|(-\Delta)^{\frac{1}{2}} u\|_{p,\infty, B(x_0, 2^{k_0} r)} + \sum_{k=k_0}^{\infty} 2^{-\sigma k} \|(-\Delta)^{\frac{1}{2}} u\|_{p,\infty, B(x_0, 2^{k} r)} \right) \left( \|\varphi\|_{\infty} + \|(-\Delta)^{\frac{1}{2}} \varphi\|_{p,2} \right).$$

For all $k_0$ sufficiently large and $2^{k_0} r$ sufficiently small, we can assume by absolute continuity of the integral that

$$\|(-\Delta)^{\frac{1}{2}} u\|_{p, B(x_0, 2^{k_0} r)} + \sum_{k=k_0}^{\infty} 2^{-\sigma k} \|(-\Delta)^{\frac{1}{2}} u\|_{p, B(x_0, 2^{k} r)} < \varepsilon.$$

and under the assumption (1.3) also

$$\|(-\Delta)^{\frac{1}{2}} u\|_{p, B(x_0, 2^{k_0} r)} + \sum_{k=k_0}^{\infty} 2^{-\sigma k} \|(-\Delta)^{\frac{1}{2}} u\|_{p, B(x_0, 2^{k} r)} < \varepsilon.$$

This proves the localized estimate for $E$ and we conclude the proof of Proposition 1.3. \(\square\)

4. Construction of a good gauge

The next theorem is an adaptation of [8, Theorem 1.2] of Rivière and the first-named author, which is a choice of a good gauge. It follows the strategy developed by Rivière in [18] which was itself inspired by Uhlenbecks construction of Coulomb gauges [29]. For extensions and relations to the moving frame method by Hélein see also [20, 4, 25, 12].

**Theorem 4.1** (Choice of gauge). There exists $\varepsilon > 0$ so that the following holds.

Whenever $\Omega \in L^{(\frac{2}{2})}(\mathbb{R}^n, so(N))$ satisfies

$$\|\Omega\|_{(\frac{2}{2},2)} < \varepsilon,$$
then there exists $P \in \dot{H}^{s, (\frac{n}{2}, 2)}(\mathbb{R}^n, SO(N))$ so that
\[ \|(-\Delta)^{\frac{s}{2}} P + P\Omega\|_{L^{(\frac{n}{2}, 1)}(\mathbb{R}^n, \mathbb{R}^N \times \mathbb{R}^N)} \lesssim \|\Omega\|_{(\frac{n}{2}, 2)}. \]
Moreover,
\[ \|(-\Delta)^{\frac{s}{2}} P\|_{L^{(\frac{n}{2}, 2)}(\mathbb{R}^n, \mathbb{R}^N \times \mathbb{R}^N)} \lesssim \|\Omega\|_{(\frac{n}{2}, 2)}. \]

Theorem 4.1 is a consequence of the following

**Theorem 4.2.** For any $q \in [1, \infty]$ there exist $\varepsilon > 0$ so that if
\[ \|\Omega\|_{(\frac{n}{2}, q)} < \varepsilon, \]
then there exists $P \in \dot{H}^{s, (p, q)}(\mathbb{R}^n, SO(N))$ so that
\[ \tag{4.1} P^T (-\Delta)^{\frac{s}{2}} P - (-\Delta)^{\frac{s}{2}} P P^T + 2\Omega = 0 \quad \text{in} \quad \mathbb{R}^n. \]

**Proof of Theorem 4.1.** From (4.1)
\[ (-\Delta)^{\frac{s}{2}} P + P\Omega = \frac{1}{2} \left( (-\Delta)^{\frac{s}{2}} PP^T + P(-\Delta)^{\frac{s}{2}} P^T \right) P, \]
that is, since $(-\Delta)^{\frac{s}{2}} (PP^T) \equiv (-\Delta)^{\frac{s}{2}} I \equiv 0$,
\[ (-\Delta)^{\frac{s}{2}} P + P\Omega = \frac{1}{2} H_s(P, P^T) P, \]
Since $(-\Delta)^{\frac{s}{2}} P \in L^{(\frac{n}{2}, 2)}(\mathbb{R}^n)$ from the three commutator estimates (see Lemma 3.1) we find that
\[ H_s(P, P^T) \in L^{(\frac{n}{2}, 1)}(\mathbb{R}^n, \mathbb{R}^N), \]
and have consequently shown that
\[ (-\Delta)^{\frac{s}{2}} P + P\Omega \in L^{(\frac{n}{2}, 1)}(\mathbb{R}^n, \mathbb{R}^N). \]

4.1. **Construction of the optimal gauge: Proof of Theorem 4.2.** In order to establish (4.1) we adapt the strategy from [8, Theorem 1.2]. For notational simplicity we prove this theorem only for $L^{(\frac{n}{2}, 2)}$ (i.e. $q = 2$) the case we need.

For the rest of this section fix $1 < q_1, q_2 < \infty$ exponents so that $1 < q_1 < \frac{n}{s} < q_2 < \infty$.

As in [8, Proof of Theorem 1.2, Step 4], by an approximation argument it suffices to prove the claim under the stronger assumption that $\Omega \in L^{q_1} \cap L^{q_2}(\mathbb{R}^n)$ with good estimates. More precisely, for $\varepsilon > 0$ let
\[ U_\varepsilon := \left\{ \Omega \in L^{q_1} \cap L^{q_2}(\mathbb{R}^n, so(N)) : \|\Omega\|_{(\frac{n}{2}, 2)} \leq \varepsilon \right\}, \]
and for constants $\varepsilon, \Theta > 0$ let $\mathcal{V}_{\varepsilon, \Theta} \subset U_\varepsilon$ be the set where we have the decomposition (4.1) with the estimates
\[ \|(-\Delta)^{\frac{s}{2}} P\|_{(p, 2)} \leq \Theta \|\Omega\|_{(p, 2)} \]
(4.3) \[ \|(-\Delta)^{\frac{1}{2}} P\|_{q_1} \leq \Theta \|\Omega\|_{q_1}, \quad \|(-\Delta)^{\frac{1}{2}} P\|_{q_2} \leq \Theta \|\Omega\|_{q_2}. \]

That is,

\[ \mathcal{V}_{\varepsilon, \Theta} := \left\{ \Omega \in \mathcal{U}_\varepsilon : \begin{array}{l} \text{there exists } P \in \dot{H}^{s,q_1} \cap \dot{H}^{s,q_2}(\mathbb{R}^n, SO(N)), \text{ so that} \\ P - I \in L^{\frac{n q}{n-q}}(\mathbb{R}^n, \mathbb{R}^{N \times N}) \text{ and (4.2), (4.3),} \\
\text{and (4.1) holds.} \end{array} \right\} \]

Let us remark a technical detail. The condition \( P - I \in L^{\frac{n q}{n-q}}(\mathbb{R}^n, \mathbb{R}^{N \times N}) \) corresponds to prescribing Dirichlet data at infinity. For our purpose, there is no advantage above, the proof below works also without this Dirichlet assumption which essentially corresponds to a Neumann-type condition at infinity. For our purpose, there is no advantage to either choice. We then need to prove the following

**Proposition 4.3.** There exist \( \Theta > 0 \) and \( \varepsilon > 0 \) so that \( \mathcal{V}_{\varepsilon, \Theta} = \mathcal{U}_\varepsilon \).

Proposition 4.3 follows from a continuity method, once we show the following four properties

(i) \( \mathcal{U}_\varepsilon \) is connected.

(ii) \( \mathcal{V}_{\varepsilon, \Theta} \) is nonempty.

(iii) For any \( \varepsilon, \Theta > 0 \), \( \mathcal{V}_{\varepsilon, \Theta} \) is a relatively closed subset of \( \mathcal{U}_\varepsilon \).

(iv) There exist \( \Theta > 0 \) and \( \varepsilon > 0 \) so that \( \mathcal{V}_{\varepsilon, \Theta} \) is a relatively open subset of \( \mathcal{U}_\varepsilon \).

Property (i) is clear, since \( \mathcal{U}_\varepsilon \) is starshaped with center 0: for any \( \Omega \in \mathcal{U}_\varepsilon \) we have \( t\Omega \in \mathcal{U}_\varepsilon \) for all \( t \in [0,1] \). Property (ii) is also obvious since \( P := I \) is an element of \( \mathcal{V}_{\varepsilon, \Theta} \). The closedness property (iii) follows almost verbatim from [8, Proof of Theorem 1.2, Step 1, p.1315]: there one replaces \((-\Delta)^{\frac{1}{2}} \) by \((-\Delta)^{\frac{q}{2}} \), \( q \) by \( q_2, q' \) by \( q_1 \), and the \( L^2 \)-norm by the \( L^{(q,2)} \)-norm (for which we still can use the lower semicontinuity). Observe that a uniform bound of the \( L^n \)-norm as in (4.3) implies by Sobolev embedding in particular a uniform bound \( P - I \in L^{\frac{n q}{n-q}}(\mathbb{R}^n, \mathbb{R}^{N \times N}) \).

The main point is to show the openness property (iv). For this let \( \Omega_0 \) be arbitrary in \( \mathcal{V}_{\varepsilon, \Theta} \), for some \( \varepsilon, \Theta > 0 \) chosen below. Let \( P_0 \in \dot{H}^{s,q_1} \cap \dot{H}^{s,q_2}(\mathbb{R}^n, SO(N)) \), \( P_0 - I \in L^{\frac{n q_1}{n-q_1}}(\mathbb{R}^n, \mathbb{R}^{N \times N}) \) so that the decomposition (4.1) as well as the estimates (4.2), (4.3) are satisfied for \( \Omega_0 \).

We introduce the map

\[ F(U) := (P_0 \exp(U))^{-T} (-\Delta)^{\frac{1}{2}} (P_0 \exp(U)) - (-\Delta)^{\frac{q}{2}} (P_0 \exp(U))^{-T} (P_0 \exp(U)) \]

Observe that for \( U \in L^{\frac{n q_1}{n-q_1}}(\mathbb{R}^n, so(N)) \),

\[ P_0 \exp(U) - I = (P_0 - I) \exp(U) + I - \exp(U) \in L^{\frac{n q_1}{n-q_1}}(\mathbb{R}^n, \mathbb{R}^{N \times N}). \]

Indeed, observe that \( U \in L^\infty \) and thus \( (P_0 - I) \exp(U) \in L^{\frac{n q_1}{n-q_1}}(\mathbb{R}^n, \mathbb{R}^{N \times N}). \) Moreover,

\[ |I - \exp(U)| \lesssim (1 + \|U\|_\infty)|U|, \]
and thus $I - \exp(U) \in L^{\frac{nq}{n-q_1}}(\mathbb{R}^n, \mathbb{R}^{N\times N})$.

As in [8, Proof of Theorem 1.2, Step 2, p.1316] we can conclude that $F$ is $C^1$ as a map from

$$F : L^{\frac{nq}{n-q_1}} \cap \dot{H}^{s,q_1} \cap \dot{H}^{s,q_2}(\mathbb{R}^n, so(N)) \to L^{q_1} \cap L^{q_2}(\mathbb{R}^n, so(N)).$$

and that we can compute $DF(0)$ as

$$\frac{d}{dt} \bigg|_{t=0} F(t\eta) = L(\eta),$$

where for $\eta \in L^{\frac{nq}{n-q_1}} \cap \dot{H}^{s,q_1} \cap \dot{H}^{s,q_2}(\mathbb{R}^n, so(N))$,

$$L(\eta) := -\eta P_0^T (-\Delta)^{\frac{1}{2}} P_0 + (-\Delta)^{\frac{1}{2}} (\eta P_0^T) P_0 + P_0^T (-\Delta)^{\frac{1}{2}} P_0 \eta - (-\Delta)^{\frac{1}{2}} P_0^T P_0 \eta$$

In order to use a fixed-point argument for $F$, we need to show that $L$ is an isomorphism.

**Lemma 4.4.** For any $\Theta > 0$ there exists a $\varepsilon > 0$ so that the following holds for any $\Omega_0$ and $P_0$ as above.

For any $\omega \in L^{q_1} \cap L^{q_2}(\mathbb{R}^n, so(N))$ there exists a unique $\eta \in L^{\frac{nq}{n-q_1}} \cap \dot{H}^{s,q_1} \cap \dot{H}^{s,q_2}(\mathbb{R}^n, so(N))$ so that

$$\omega = L(\eta)$$

and for some constant $C = C(\Omega_0, \Theta) > 0$ it holds

$$\|\eta\|_{L^{\frac{nq}{n-q_1}}} + \|(-\Delta)^{\frac{1}{2}} \eta\|_{L^{q_1}(\mathbb{R}^n)} + \|(-\Delta)^{\frac{1}{2}} \eta\|_{L^{q_2}(\mathbb{R}^n)} \leq C \left( \|\omega\|_{L^{q_1}(\mathbb{R}^n)} + \|\omega\|_{L^{q_2}(\mathbb{R}^n)} \right)$$

**Proof.** We follow the strategy of [8, Lemma 4.1]. First we find $\eta$ in some $\dot{H}^{s,r}$ for $r \in (q_1, \frac{n}{s})$ and then that it belongs to the right spaces.

**Step 1:** For $\eta \in L^{\frac{nq}{n-q_1}} \cap \dot{H}^{s,q_1} \cap \dot{H}^{s,q_2}(\mathbb{R}^n, so(N))$ we rewrite

$$L(\eta) = 2(-\Delta)^{\frac{1}{2}} \eta + H(\eta)$$

where

$$H(\eta) := \eta \left( -P_0^T (-\Delta)^{\frac{1}{2}} P_0 + (-\Delta)^{\frac{1}{2}} (P_0^T) P_0 - (-\Delta)^{\frac{1}{2}} P_0^T P_0 \eta \right) + H_s(\eta, P_0).$$

In particular, for any $r \in (1, \frac{n}{s})$, by Hölder’s inequality,

$$\|H(\eta)\|_{L^{(r,2)}(\mathbb{R}^n)} \lesssim \|\eta\|_{L^{\frac{nr}{n+r-4}}(\mathbb{R}^n)} \|(-\Delta)^{\frac{1}{2}} P_0\|_{L^{(r,2)}(\mathbb{R}^n)} + \|H_s(\eta, P_0)\|_{L^{(r,2)}(\mathbb{R}^n)}.$$

By Sobolev embedding,

$$\|\eta\|_{L^{\frac{nr}{n+r-4}}(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\frac{1}{2}} \eta\|_{L^{(r,\infty)}(\mathbb{R}^n)}.$$

By the three-commutator estimates,

$$\|H_s(\eta, P_0)\|_{L^{(r,2)}(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\frac{1}{2}} \eta\|_{L^{(r,\infty)}(\mathbb{R}^n)} \|(-\Delta)^{\frac{1}{2}} P_0\|_{(r,2)}.$$
Consequently, in view of (4.2),
\[ \|H(\eta)\|_{L^{(r,2)}(\mathbb{R}^n)} \leq C \Theta \varepsilon \left\| (-\Delta)^{\frac{2}{n}} \eta \right\|_{L^{(r,\infty)}(\mathbb{R}^n)} \leq C \Theta \varepsilon \|(-\Delta)^{\frac{2}{n}} \eta\|_{L^{(r,2)}(\mathbb{R}^n)} \]
Choosing \( \varepsilon \) small enough (depending on \( \Theta \)), we obtain that \( L(\eta) \) is invertible as a map from \( L^{(r,2)}(\mathbb{R}^n, so(N)) \) to \( L^{(\frac{rn}{n+r^2},2)}(\mathbb{R}^n, so(N)) \), whenever \( r \in (q_1, \frac{n}{\theta}) \).

**Step 2:** For given \( \omega \in L^{q_1} \cap L^{q_2}(\mathbb{R}^n, so(N)) \) and \( r_0 \in (q_1, \frac{n}{\theta}) \) let \( \eta \in L^{(\frac{rn}{n+r^2},2)} \) \( \cap H^{s,(r,2)}(\mathbb{R}^n, so(N)) \) so that
\[ \omega = L(\eta). \]
We will show that \((-\Delta)^{\frac{2}{n}} \eta \in L^{q_2}(\mathbb{R}^n)\). Indeed, since \((-\Delta)^{\frac{2}{n}} P_0 \in L^{q_2}(\mathbb{R}^n)\), we can estimate for \( t_1 = \frac{1}{n} - \frac{n}{q_2} + \frac{1}{r_0} \)
\[ \|H(\eta)\|_{L^{(1,2)}(\mathbb{R}^n)} \lesssim \left\| (-\Delta)^{\frac{2}{n}} \eta \right\|_{L^{(r,2)}(\mathbb{R}^n)} \left\| (-\Delta)^{\frac{2}{n}} P_0 \right\|_{L^{q_2}(\mathbb{R}^n)}, \]
which itself follows from Sobolev embedding and the following estimate from Lemma 3.1
\[ \|H_s(\eta, P_0)\|_{L^{(1,2)}(\mathbb{R}^n)} \lesssim \|\eta\|_{L^{(\frac{rn}{n+r^2},2)}} \left\| (-\Delta)^{\frac{2}{n}} P_0 \right\|_{L^{q_2}(\mathbb{R}^n)}. \]
Since
\[ (-\Delta)^{\frac{2}{n}} \eta = \frac{1}{2} \omega - H(\eta) \]
Now either \( t_1 > q_2 \), in which case we use that then \( L^{(1,2)} \cap L^{r_0}(\mathbb{R}^n) \subset L^{q_2}(\mathbb{R}^n) \) (that follows from Step 1) and thus
\[ (-\Delta)^{\frac{2}{n}} \eta \in L^{q_2} \cap L^{r_0}(\mathbb{R}^n). \]
Otherwise, we know that \( \frac{1}{r_0} - \frac{1}{t_1} = \frac{s}{n} - \frac{1}{q_2} > 0 \). In this case we repeat the above argument for \( r_1 := t_1 \) and find \( t_2 \) which either is larger than \( q_2 \) or where \( \frac{1}{r_1} - \frac{1}{t_2} = \frac{s}{n} - \frac{1}{q_2} > 0 \). Possible repeating this procedure finitely many times we find that eventually some \( t_i > q_2 \).

**Step 3** It remains to show that \((-\Delta)^{\frac{2}{n}} \eta \in L^{q_1}(\mathbb{R}^n)\). Since we already know that \((-\Delta)^{\frac{2}{n}} \eta \in L^{(r,2)} \cap L^{q_2}(\mathbb{R}^n)\) for some \( r \in (q_1, \frac{n}{\theta}) \) arbitrarily small, we find that \( \eta \in L^{\infty}(\mathbb{R}^n) \). In particular,
\[ \|\eta (P_0^{T} (-\Delta)^{\frac{2}{n}} P_0 + (-\Delta)^{\frac{2}{n}} P_0^{T} P_0) + (P_0^{T} (-\Delta)^{\frac{2}{n}} P_0 - (-\Delta)^{\frac{2}{n}} P_0^{T} P_0) \eta\|_{q_1, \mathbb{R}^n} \lesssim \|\eta\|_{\infty, \mathbb{R}^n} \|(-\Delta)^{\frac{2}{n}} P_0\|_{q_1, \mathbb{R}^n} < \infty. \]
Moreover, \((-\Delta)^{\frac{2}{n}} \eta \in L^{(r,2)} \cap L^{q_2}(\mathbb{R}^n) \subset L^{\frac{n}{\theta}}(\mathbb{R}^n) \). Thus Lemma 3.1 implies
\[ \|H_s(\eta, P_0)\|_{L^{n}(\mathbb{R}^n)} \lesssim \left\| (-\Delta)^{\frac{2}{n}} \eta \right\|_{L^{\frac{n}{\theta}}(\mathbb{R}^n)} \left\| (-\Delta)^{\frac{2}{n}} P_0 \right\|_{L^{\infty}(\mathbb{R}^n)}. \]
Consequently,
\[ \omega - H(\eta) \in L^{q_1}(\mathbb{R}^n), \]
and thus \((-\Delta)^{\frac{2}{n}} \eta \in L^{q_1} \cap L^{q_2}(\mathbb{R}^n)\). Moreover, by interpolation, \( \eta \in L^{\frac{mn}{n+1}}_{\infty, \mathbb{R}^n} \). The estimates follow by the above considerations. Lemma 4.4 is proven. \( \square \)
We continue with the proof of Proposition 4.3.

Thus, by Implicit Function Theorem applied to $F$, if $\varepsilon = \varepsilon(\Theta) > 0$ is chosen small enough, we find for any $\Omega_0 \in V_{\varepsilon, \Theta}$ some $\delta > 0$ such that for any $\Omega \in U_\varepsilon$ with

$$\|\Omega - \Omega_0\|_{L^q(R^n)} + \|\Omega - \Omega_0\|_{L^2(R^n)} < \delta$$

we find $P = P_0 e^U \in \dot{H}^{s,q_1} \cap \dot{H}^{s,q_2}(R^n, SO(N))$, so that $P - I \in L^{n q_1 - q_2}(R^n, R^{N \times N})$ and (4.1) is satisfied. By continuity of the inverse, we can make $\delta$ possibly smaller to guarantee that

$$\|(-\Delta)^{\frac{s}{2}} (P - P_0)\|_{q_1, R^n} \leq \|\Omega\|_{q_1, R^n}, \quad \|(-\Delta)^{\frac{s}{2}} (P - P_0)\|_{q_2, R^n} \leq \|\Omega\|_{q_2, R^n}.$$

Observe that this does not right away imply (4.2), (4.3). However the above estimate and the fact that $\Omega \in U_\varepsilon$ imply that for any $\sigma > 0$ we can choose $\varepsilon$ small enough so that

$$\|(-\Delta)^{\frac{s}{2}} P\|_{(\frac{q_1}{2}, R^n)} \leq \sigma.$$

The next Lemma shows us that this implies for a small enough choice of $\sigma > 0$ that (4.2), (4.3) hold for a uniform constant $\Theta$.

**Lemma 4.5.** There exists a $\Theta > 0$ and a $\sigma > 0$ so that whenever $P \in \dot{H}^{s,q_1} \cap \dot{H}^{s,q_2}(R^n, SO(N))$ and $P - I \in L^{n q_1 - q_2}(R^n, R^{N \times N})$ so that (4.1) is satisfied and it holds

(4.4) $$\|(-\Delta)^{\frac{s}{2}} P\|_{(\frac{q_1}{2}, R^n)} \leq \sigma,$$

then (4.2), (4.3) hold.

**Proof.** In view of (4.1)

$$P^T (-\Delta)^{\frac{s}{2}} P = \frac{1}{2} H_s(P^T, P) - \Omega.$$

In particular, by the three-commutator estimates in Lemma 3.1, for a uniform constant $C$ for any $p \in [q_1, q_2], q \in [1, \infty]$,

$$\|(-\Delta)^{\frac{s}{2}} P\|_{(p, q)} \leq C_1 \|(-\Delta)^{\frac{s}{2}} P\|_{(\frac{q_1}{2}, \infty)} \|(-\Delta)^{\frac{s}{2}} P\|_{(p, q)} + \|\Omega\|_{(p, q)}.$$

Moreover,

$$\|(-\Delta)^{\frac{s}{2}} P\|_{(\frac{q_1}{2}, \infty)} \leq C_2 \|(-\Delta)^{\frac{s}{2}} P\|_{(\frac{q_1}{2}, \infty)} \leq C_2 \sigma$$

for $\sigma$ small enough we can absorb and find

$$\|(-\Delta)^{\frac{s}{2}} P\|_{(p, q)} \leq \frac{1}{1 - C_1 C_2 \sigma} \|\Omega\|_{(p, q)}.$$

Choosing $\Theta := \frac{1}{1 - C_1 C_2 \sigma}$ we conclude. \(\Box\)

Thus the openness property (iv) is proven, Proposition 4.3 is established, and with the approximation argument in [8, Proof of Theorem 1.2, Step 4] Theorem 4.2 is proven. \(\Box\)
5. The improved Morrey space estimate: Proof of Proposition 1.4

Let \( w \in L^{n/s} (\mathbb{R}^n, \mathbb{R}^N) \) be a solution of
\[
(-\Delta)^{\frac{2}{s}} w^j = \Omega_{ij} w^i + E_i (w) \quad \text{in} \ \mathbb{R}^n,
\]
where \( \Omega_{ij} = -\Omega_{ji} \in L^{(\frac{n}{s}, 2)} \) and \( E \) is as above.

By absolute continuity of the integral there exists \( R > 0 \) so that
\[
\sup_{x_0 \in \mathbb{R}^n} \| \Omega_{ij} \|_{L^{(\frac{n}{s}, 2)} (B(x_0, 10R))} < \delta < \varepsilon
\]
for the \( \varepsilon > 0 \) from Theorem 4.1, and \( \delta \) chosen later. It suffices to prove the claim (1.6) in \( B(x_0, \rho) \subset B(y_0, R) \), where \( y_0 \subset \mathbb{R}^n \) is arbitrary (and the constants will not depend on \( y_0 \), but may depend on \( R \)). Let \( \eta_{B(y_0, 2R)} \in C^\infty_c (B(y_0, 10R)) \) be the generic smooth cutoff function which is constantly one in \( B(y_0, 5R) \). Applying Theorem 4.1 to \( \eta_{B(y_0, 2R)} \Omega \) we find \( P \in \dot{H}^{n/2} (\mathbb{R}^n) \) so that
\[
\| (-\Delta)^{\frac{2}{s}} P \|_{L^{(\frac{n}{s}, 2)}} + \| (-\Delta)^{\frac{2}{s}} P + P \eta_{B(y_0, 2R)} \Omega \|_{L^{(\frac{n}{s}, 1)} (\mathbb{R}^n \times \mathbb{R}^{N \times N})} \lesssim \| \eta_{B(y_0, R)} \Omega \|_{L^{(\frac{n}{s}, 2)}}.
\]

We have
\[
(-\Delta)^{\frac{2}{s}} (P w) = -((-\Delta)^{\frac{2}{s}} P + P \Omega) w + P E (w) + ((-\Delta)^{\frac{2}{s}} (P w) + (-\Delta)^{\frac{2}{s}} P w - P (-\Delta)^{\frac{2}{s}} w)
\]
\[
= -((-\Delta)^{\frac{2}{s}} P + P \eta_{B(y_0, R)} \Omega) w - (1 - \eta_{B(y_0, R)}) \Omega w + P E (w)
\]
\[
+ ((-\Delta)^{\frac{2}{s}} (P w) + (-\Delta)^{\frac{2}{s}} P w - P (-\Delta)^{\frac{2}{s}} w)
\]

In particular, for any \( \varphi \in C^\infty_c (B(x_0, \rho)) \), for \( B(x_0, \rho) \subset B(y_0, R) \), possibly choosing \( R \) even smaller for the estimate of \( E_i \) to take effect (in the following we write the estimates for the case \( n/s > 2 \), the case \( n/s \leq 2 \) is analogous), since \( \Omega \varphi = \eta_{B(y_0, 2R)} \Omega \varphi \), for all sufficiently large \( k_0 \), for some \( \sigma > 0 \)
\[
\int_{\mathbb{R}^n} Pw (-\Delta)^{\frac{2}{s}} \varphi \lesssim \| w \|_{L^{(\frac{n}{s}, \infty)} (\mathbb{R}^n), B(x_0, \rho)} \| \varphi \|_{L^{\infty}}
\]
\[
+ \varepsilon \left( \| P \varphi \|_{L^{\infty}} + \| (-\Delta)^{\frac{2}{s}} (P \varphi) \|_{L^{(\frac{n}{s}, 2)}} \right) \frac{1}{\| w \|_{L^{(\frac{n}{s}, \infty)} (\mathbb{R}^n), B(x_0, 2^k \rho)}} + \sum_{k=0}^{\infty} 2^{-k\sigma} \| w \|_{L^{(\frac{n}{s}, \infty)} (\mathbb{R}^n), B(x_0, 2^{k+1} \rho)}
\]
\[
+ \| w \|_{H^s (P, \varphi)} \|_{L^1}.
\]

Firstly,
\[
\left( \| P \varphi \|_{L^{\infty}} + \| (-\Delta)^{\frac{2}{s}} (P \varphi) \|_{L^{(\frac{n}{s}, 2)}} \right) \lesssim \| (-\Delta)^{\frac{2}{s}} \varphi \|_{L^{(\frac{n}{s}, 1)}} \left( 1 + \| (-\Delta)^{\frac{2}{s}} P \|_{L^{(\frac{n}{s}, 2)}} \right).
\]

Moreover, by the three commutator estimates and after localization,
\[
\| w \|_{H^s (P, \varphi)} \|_{L^1} \lesssim \delta \| (-\Delta)^{\frac{2}{s}} \varphi \|_{L^{(\frac{n}{s}, 1)}} \left( \| w \|_{L^{(\frac{n}{s}, \infty)} (\mathbb{R}^n), B(x_0, 2^k \rho)} + \sum_{k=k_0}^{\infty} 2^{-k\sigma} \| w \|_{L^{(\frac{n}{s}, \infty)} (\mathbb{R}^n), B(x_0, 2^{k+1} \rho)} \right)
\]
That is, for any $\varphi \in C^\infty_c(B(x_0, \rho))$ so that $\|(-\Delta)^{\frac{s}{2}} \varphi\|_{(\frac{n}{n-2s}, 1)} \leq 1$, we have

$$\int_{\mathbb{R}^n} P w (-\Delta)^{\frac{s}{2}} \varphi \lesssim (\varepsilon + \delta) \left( \|w\|_{(\frac{n}{n-2s}, \infty), B(x_0, 2k_0 \rho)} + \sum_{k=k_0}^{\infty} 2^{-k\sigma} \|w\|_{(\frac{n}{n-2s}, \infty), B(x_0, 2^k \rho)} \right)$$

Taking the supremum over all such $\varphi$, see e.g. [2, Proposition A.3.], we obtain, possibly for a larger $k_0$,

$$\|w\|_{(\frac{n}{n-2s}, \infty), B(x_0, 2^{-k_0} \rho)} \lesssim (\varepsilon + \delta) \left( \|w\|_{(\frac{n}{n-2s}, \infty), B(x_0, 2k_0 \rho)} + \sum_{k=k_0}^{\infty} 2^{-k\sigma} \|w\|_{(\frac{n}{n-2s}, \infty), B(x_0, 2^k \rho)} \right).$$

Choosing $\varepsilon$ and $\delta$ small enough this is a decay estimate that can be iterated on smaller and smaller balls, and gives the claim. See e.g. [2, Lemma A.8].

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(Francesca Da Lio) DEPARTMENT OF MATHEMATICS, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland.

E-mail address: fdalio@math.ethz.ch

(Armin Schikorra) DEPARTMENT OF MATHEMATICS, 301 THACKERAY HALL, UNIVERSITY OF PITTSBURGH, PA 15260, USA

E-mail address: armin@pitt.edu