Axion-like Dark Matter from the Type-II Seesaw Mechanism

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Although axion-like particles (ALPs) are popular dark matter candidates, their mass generation mechanisms as well as cosmic thermal evolutions are still unclear. In this letter, we propose a new mass generation mechanism of ALP during the electroweak phase transition in the presence of the type-II seesaw mechanism. As ALP gets mass uniquely at the electroweak scale, there is a cutoff scale on the ALP oscillation temperature irrelevant to the specific mass of ALP, which is a distinctive feature of this scenario. The ALP couples to the active neutrinos, leaving the matter effect of neutrino oscillations in a dense ALP environment as a smoking gun. As a by-product, the recent W-boson mass anomaly observed by the CDF collaboration is also quoted by the TeV-scale type-II seesaw. We explain three kinds of new physics phenomena are with one stroke.

Introduction.— Various cosmological observations have confirmed the existence of cold dark matter (DM), which accounts for about 26.8% [1] of the cosmic energy budget. However, the particle nature of DM still elude us. Axion [2–5] is one of the most popular DM candidates motivated by addressing the strong CP problem, with its mass induced by the QCD instanton and its relic abundance arising from the misalignment mechanism [6–11], which drives the coherent oscillation of axion field around the minimum of the effective potential. Couplings of the axion to the standard model (SM) particles are model-dependent and there are three general types of QCD axion models, PQWW [2, 3], KSVZ [12, 13], and DFSZ [14, 15], of which the PQWW axion is excluded by the beam-dump experiments [16–18] and other axion models can be detected via their couplings to photons or SM fermions.

To relax property constraints to the QCD axions, more general classes of axion-like particle (ALP) DM models [19–32] are proposed, with the mass ranging from 10^−22 eV to O(1) GeV [9, 31], where the lower bound is from the fuzzy DM [33] and the upper bound is from the LHC limits. The mass generation mechanism as well as the relic abundance of axion-like DM are blurred and indistinct since people usually pay more attention to the detection signal of ALP in various experiments via its coupling to photon [34–45], a/f_a FF, where a is the ALP field and f_a is the ALP decay constant. It should be mentioned that the mass generation mechanism of the ALP is highly correlated with its interactions with the SM particles. So one cannot simply ignore these facts and directly apply the strategy of searching for QCD axion to detect the ALP. This issue has been concerned recently and several novel approaches have been proposed to address the relic abundance of the light scalar DM, such as the thermal misalignment mechanism [46, 47], which supposes a feeble coupling between the DM and thermal fermions. These attempts provide novel insights to the origin of ALP in the early Universe.

In this letter, we propose a new mechanism of generating the ALP mass during the electroweak phase transition with the help of a Higgs triplet Δ with Y = 1, which is the seesaw particle in the type-II seesaw mechanism [48–53]. Active neutrinos get Majorana mass as Δ develops a tiny but non-zero vacuum expectation value (VEV). We explicitly show that an ALP, which is the Goldstone boson arising from the spontaneous breaking of them global U(1)_L symmetry, can get tiny mass through the quartic coupling with the Higgs triplet and the SM Higgs doublet Φ whenever the global lepton number is explicitly broken by the term µΦ^T iτ_2 Δ^† Φ + h.c. In such a scenario, symmetries break sequently: the U(1)_L first breaks at high energy scale resulting a massless ALP serving as dark energy, then electroweak symmetry is spontaneously broken leading the mass generation of the ALP, which begins to oscillate as its mass is comparable with the Hubble parameter. We derive the relic density of ALP by investing its thermal evolution and solving its equation of motion (EOM) analytically. To further investigate its signal, we explicitly derive the interactions between ALP and SM particles, which arise from the mixing of ALP with other CP-even particles. We argue that neutrino oscillations in certain specific environment may be a smoking gun. As a by-product, we show that the recent W-boson mass anomaly observed by the CDF collaboration [54–64] can be addressed in the same model without conflicting with the LHC constraints.

Framework.— We assume a complex scalar singlet S carries two units of lepton number charge and the U(1)_L is spontaneously broken at high temperature when S gets VEV. Besides, the type-II seesaw mechanism is required for the origin of neutrino mass and S couples to the Higgs triplet Δ and the SM Higgs doublet Φ via the quartic interaction with a real coupling. The most general scalar potential is

\[ V(S, Φ, Δ) = V(Φ, Δ) - μ_6^2 (S^\dagger S) + λ_6 (S^\dagger S)^2 \]
\[ + λ_7 (S^\dagger S)(Φ^\dagger Φ) + λ_8 (S^\dagger S)Tr(Δ^\dagger Δ) \]
\[ + μΦ^T iτ_2 Δ^† Φ + λSΦ^T iτ_2 Δ^† Φ + h.c. \]  

(1)

where V(Φ, Δ) is the most general potential for the type-II seesaw mechanism given in the Supplemental Mate-
rial. The quartic couplings $\lambda_{T,S}$ are relevant for the thermal mass of $S$. It is obvious that $S$ may get non-zero VEV in the early Universe by assuming the small quartic couplings, which is consistent with experimental observations [65–68], leaving the CP-odd component of $S$ as ALP. ALP is massless at the early time until the temperature drops down to the electroweak scale at which both $\Phi$ and $\Delta$ get non-zero VEVs. Then ALP acquires a tiny mass double suppressed by the VEV of the Higgs triplet and the tiny lepton-number-violating parameter $\mu$, which should be naturally small accorded to the naturalness principle of 't Hooft [69].

To analytically derive the mass of ALP, the $\Phi$, $\Delta$, and $S$ can be parametrized as

$$\Phi = \left[ \frac{\phi^+}{\sqrt{2}}, \frac{\phi_0 + \phi_0 + i \pi}{\sqrt{2}} \right], \quad \Delta = \left[ \frac{\Delta^+}{\sqrt{2}}, \frac{\Delta^+}{\sqrt{2}} - \frac{\Delta^-}{\sqrt{2}} \right], \quad S = \frac{\nu_s + \bar{s} + i a}{\sqrt{2}}$$

where $\Delta^0 = (\nu_\Delta + \delta + i n)/\sqrt{2}$ being the neutral component of the Higgs triplet, the $\nu_\Delta$, $\nu_a$, and $\nu_s$ are the VEVs of $\Phi$, $\Delta$, and $S$, respectively. After the electroweak symmetry breaking (EWSB), the remaining physical scalars are as follows, two charged scalar pairs $H^\pm$ and $H^0$, two CP-odd scalars $A$ and $a$, and three CP-even scalars $h$, $H$, and $s$, whose masses may be obtained by unitary transformations to their squared mass matrices. The detailed procedures of diagonalization of all the scalar mass matrices are given in the Supplemental Material. Then the ALP mass in the CP-odd sector can be written as

$$m_a^2 \approx \frac{\sqrt{2} \nu_{\phi}^2 \nu_\Delta (\nu_\Delta^2 + 4 \nu_\Delta^2)}{\nu_s^2 (\nu_\Delta^2 + \nu_s^2) + 8 \nu_s^2 \nu_\Delta^2}.$$  

(3)

In the limits $\nu_\Delta^2/\nu_s^2 \ll 1$ and $\nu_\Delta^2/\nu_s^2 \ll 1$, one has $m_a^2 \approx \mu_\phi^2 \nu_\Delta^2/\sqrt{2} \nu_s^2$, which is double suppressed by the parameters $\nu_\Delta$ and $\mu$ in the type-II seesaw mechanism. ALP DM. As discussed above, the ALP gets a tiny but non-zero mass via the type-II seesaw mechanism during the electroweak phase transition at the critical temperature $T_C \approx 160$ GeV [70]. Neglecting the radiative corrections, the temperature-dependent ALP mass can be written as

$$m_a^2(T) = \begin{cases} \frac{\nu_\phi^2(T) \nu_\Delta(T)}{\sqrt{2} \nu_s^2} \\ 0 \end{cases}, \quad \begin{array}{ll} T \leq T_C \\ T > T_C \end{array}$$

(4)

where $f_a = v_s$, $v_\phi(T)$ and $v_\Delta(T)$ are the temperature-dependent VEVs of the SM Higgs and Higgs triplet, respectively. The EOM of the homogeneous ALP field $a$ ($a \equiv \theta f_a$) in the FRW Universe can be written as [6–8]

$$\ddot{\theta} + 3 H(T) \dot{\theta} + m_a^2(T) \theta = 0,$$

(5)

where the dot denotes the derivative with respect to time, and $H(T) \equiv \dot{R}/R$ is the Hubble parameter in terms of the scale factor $R$. In the radiation-dominated epoch, we have $H(T) = 1/(2t) \approx 1.66 \sqrt{g_a(T)} T^2/m_{pl}$, where $g_a(T)$ is the effective number of the degrees of freedom, and $m_{pl} = 1.22 \times 10^{19}$ GeV being the Planck mass. The initial conditions are taken as $\dot{\theta}(t_i) = \sqrt{(\theta_{a,i}^2)}$ and $\theta(t_i) = 0$, where the angle brackets denote the initial misalignment angle $\theta(t_i)$ averaged over $[-\pi, \pi]$ [10]. The value of $(\theta_{a,i}^2)$ depends on whether the $U(1)_L$ breaking occurs before the inflation ends or after the inflation [10, 30].

In general, the ALP becomes dynamical and starts to oscillate when $m_a(T_{osc}) = 3H(T_{osc})$ [9–11], where $T_{osc}$ is the oscillation temperature. Before the EWSB, the ALP is massless and the angle $\theta$ remains a constant with the initial value $\theta(t) = \theta(t_i)$. Therefore, there is an upper bound on the oscillation temperature $T_{max}^{osc} \equiv T_C$, which leads to the existence of a critical mass

$$m_{aC} = 1.079 \times 10^{-4} \text{eV}.$$  

(6)

The oscillation temperature can be divided into two cases

$$T_{osc} = \begin{cases} T_s, \\ T_C \end{cases}, \quad \begin{array}{ll} m_a < m_{aC} \\ m_a \geq m_{aC} \end{array}$$

(7)

where $T_s$ is derived from the condition $m_a = 3H(T_s)$. Eq. (7) implies that the traditional oscillation condition is only available to the case $m_a < m_{aC}$. For $m_a \geq m_{aC}$, the oscillation temperature is always equal to the critical temperature $T_C$, as shown in Fig. 1. Note that we use the parameter $3H$ instead of the Hubble parameter $H$ to better show the critical point given by Eq. (7). We now investigate the evolution of the ALP, which is frozen at the initial value by the Hubble friction at early times ($3H > m_a$) and behaves as dark energy. As the temperature $T$ of the Universe drops to $T_{osc}$ given by Eq. (7), the ALP starts to oscillate with damped amplitude, and its energy density scales as $R^{-3}$, which is
similar with the ordinary matter [9, 10], until the angle \( \theta \) oscillates around the potential minimum of the ALP at the late time. The evolution of \( \theta \) can be described by the analytical solution of EOM in the radiation dominated Universe when \( H > H_E \approx 10^{-28} \text{eV} \) [9, 71], where \( H_E \) is the Hubble rate at the matter-radiation equality in \( \Lambda \text{CDM} \). The exact analytical expression is given in Sec. B of the Supplemental Material. Alternatively, we can also numerically solve Eq. (5) with the given initial values. Here we consider the post-inflationary scenario and take the initial value as \( \theta(t_i) = \pi/\sqrt{3} \) [10, 30]. The analytical and numerical results are shown in Fig. 2 with the two benchmark ALP masses. We find that the numerical results of the evolution are consistent with the analytical ones.

The energy density of ALP is \( \rho_a(t) = \theta^2(t) f_a^2/2 + m_a^2(T) \theta^2(t) f_a^2/2 \). Since the ratio of ALP number density to the entropy density is conserved, the ALP energy density at the present can be written as \( \rho_a(T_0) \simeq \rho_a(T_{\text{osc}})(R_{\text{osc}}/R)^3 = 1/2 m_a(T_{\text{osc}}) m_a(T_0) f_a^2 \langle \theta_{a,i}^2 \rangle g_{ss}(T_0)/s(T_{\text{osc}}) \) [9–11], where \( T_{\text{osc}} \) is the CMB temperature at present, and \( s = 2\pi^2 g_{ss} T^3/45 \) is the entropy density with \( g_{ss} \) the relativistic degrees of freedom of the entropy. The ALP mass is almost temperature-independent, which indicates \( m_a(T_{\text{osc}}) = m_a(T_0) = m_a \), so the ALP energy density at present is

\[
\rho_a(T_0) \simeq \frac{1}{2} m_a^2 f_a^2 \langle \theta_{a,i}^2 \rangle g_{ss}(T_0)/s(T_{\text{osc}}) \left( \frac{T_0}{T_{\text{osc}}} \right)^3. \tag{8}
\]

The relic density of ALP at present is defined as \( \Omega_a h^2 \equiv \rho_a h^2/(\rho_{c,0} h^2) \) [9, 10], where \( \rho_{c,0} \equiv 3m_{\nu}^2 H_0^2/(8\pi) \) is the critical energy density, \( T_0 = 2.4 \times 10^{-4} \text{eV} \), and \( g_{ss}(T_0) = 3.94 \) [72]. Combining these parameters with Eq. (8), the relic density of ALP can be estimated as

\[
\Omega_a h^2 = \left\{ \frac{0.056 \langle \theta_{a,i}^2 \rangle}{\left( \frac{f_a}{10^{13} \text{GeV}} \right)^2 \left( \frac{m_a}{10^{-5} \text{eV}} \right)^2}, m_a < m_{aC} \right. \tag{9}
\]

\[
= \left\{ \frac{0.0146 \langle \theta_{a,i}^2 \rangle}{\left( \frac{f_a}{10^{10} \text{GeV}} \right)^2 \left( \frac{m_a}{10^{-2} \text{eV}} \right)^2}, m_a \geq m_{aC} \right. \]

Since the initial misalignment angle \( \langle \theta_{a,i}^2 \rangle^{1/2} \sim \mathcal{O}(1) \), the relic density is almost determined by the decay constant \( f_a \) and its mass \( m_a \). In Fig. 3, we show the relic density \( \Omega_a h^2 \) as a function of \( m_a \) with the four benchmark values of \( f_a \sim \mathcal{O}(10^{10} - 10^{13}) \text{GeV} \). The vertical black dotted line represents the critical mass \( m_{aC} \), on two sides of which the ALP density evolve differently. We find that there exists the allowed parameter space that may address the observed DM relic abundance, \( \Omega_a h^2 \simeq 0.12 \) [1, 72].

**ALP interactions.**—Now we investigate interactions of the ALP with ordinary matters including the Higgs and active neutrinos. ALP may couple to the SM Higgs as well as active neutrinos in forms \( \lambda_{haa} haa \) and \( \nu^\ast_L \tau \lambda_{a\nu} \nu_L + \text{h.c.} \) with the couplings

\[
\lambda_{haa} : -\lambda U_{11} V_{13} V_{23} f_a + \frac{1}{2} \lambda U_{21} V_{13}^2 f_a, \tag{10}
\]

\[
\lambda_{a\nu} : V_{23} m_\nu/v_\Delta,
\]

where \( U_{ij}, V_{ij} \ (i, j = 1, 2, 3) \) are the orthogonal matrices diagonalizing scalar matrices given in the Supplemental Material, and \( m_\nu \) is the neutrino mass matrix in the flavor basis. The complete interactions of the ALP are listed in Table IV of the Supplemental Material.
the SM Higgs decays into two ALPs \((h \rightarrow aa)\), the constraint of Higgs invisible decay from the LHC set an upper bound on the coupling \(\lambda_{bhaa} < 1.536\) GeV [73]. We have checked that the coupling predicted by this model always satisfy this constraint.

The interaction of ALP with active neutrinos may cause matter effect in neutrino oscillations. Since ALP is the classical field, the effective potential can be directly written as \(V_{\text{eff}} = i\sqrt{2}\rho_a V^{23} m_a^{-1} v_\Delta^{-1} \cos(m_a t)\nu_L^\dagger m_\nu \nu_L + \text{h.c.}\), which contributes an effective mass to active neutrinos and can be diagonalized by the same unitary transformation as that in vacuum. In this case, the three-flavor oscillation amplitude can be written as

\[
A_{\alpha \rightarrow \beta} = \sum_i \hat{U}_{\beta i} \hat{U}_{\alpha i}^\dagger \exp \left[ -i \frac{m_i^2 x}{2E} \left( 1 + \frac{\rho_a V^{23}}{m_\nu v_\Delta^2} \right) + \frac{\rho_a V^{23} \cos(2m_i x)}{2m_i^2 \rho_a} \right],
\]

where \(\hat{U}_{\alpha i}\) is the matrix element of the PMNS matrix [76, 77], \(\alpha, \beta = \{e, \mu, \tau\}, \ i = \{1, 2, 3\}\), and \(m_i\) is the mass of the \(i\)-th neutrino mass eigenstate. Notice that Eq. (11) is same as the formula of neutrino oscillation in the vacuum up to the factor in the bracket.

We find that it is difficult to probe this matter effect with a fixed \(v_a\) in vacuum, because of the low DM energy density \(\rho_a\) and the super-small suppression factor \(V^{23}\). The matter effect induced by this ALP-neutrino interaction becomes important only if the active neutrinos propagate in a dense celestial body, such as an axion star performed in [74, 78]. As an illustration, we show in Fig. 4 the neutrino oscillation probability \(P(\nu_e \rightarrow \nu_\mu)\) as a function of the neutrino energy in an axion star by setting \(\rho_a^{\text{dense}} = 6.97 \times 10^{23} \text{g m}^{-3}\) [74], which corresponds to the axion star of mass \(M_a^{\text{dense}} = 13.6 M_\odot\) and radius \(R_a^{\text{dense}} = 45.2\) km. In Fig. 4, the matter effect induced by a dense axion star makes the neutrino oscillation spectrum different from that in the vacuum.

**W mass anomaly.**— Now we calculate the deviation of W-boson mass from the SM prediction at one-loop level within the framework of this model. In general, the expression of the W-boson mass \(m_W\) can be parameterized as [79, 80]

\[
m_W^2 = m_Z^2 \left[ 1 + \sqrt{1 - \frac{4\alpha_{em}}{\sqrt{2} G_F m_Z^2} (1 + \Delta r)} \right],
\]

where \(G_F\) is the Fermi constant, \(\alpha_{em}\) is the fine-structure constant, and \(\Delta r = \Delta m^2_{em} - \Delta m^2_{H^+} - \Delta m^2_{H^+}\). The explicit expression of \(\Delta r\) is given in the Supplemental Material.

Both the Higgs triplet [81–92] and the scalar singlet [68, 93] may contribute to \(\Delta r\) and thus to the W-boson mass. Given that \(v_\phi\) is much larger than \(v_a\) and \(v_\Delta\), it is reasonable to expect that the mixing angles \(\alpha_2\) and \(\alpha_3\) in CP-even sector are approximately zero \((\alpha_2, \alpha_3 \simeq 0)\) as the scalar singlet is nearly decoupled from the other scalar fields. The remaining \(\alpha_1\) can be written as

\[
\alpha_1 \simeq \arctan \left( \frac{v_\phi v_\Delta (\lambda_4 + \lambda_5) - 2 M^2_{\Delta} v_\Delta v_\phi}{m_h^2 - M^2_{\Delta}} \right).
\]

Here we take the coupling \(\lambda_4 = 0\) [82, 94] for simplicity. The \(M^2_{\Delta}\) and \(\lambda_5\) correlated with the splitting of triplet mass spectrum are obtained from the limits of \(v_\Delta^2 / v_\phi^2, v_\Delta^2 / v_\phi^2 \ll 1\) as

\[
\lambda_5 \simeq \frac{4m^2_{H^+} - 4m^2_{H^+}}{v_\phi^2} \simeq \frac{4m^2_{H^{\pm}} - 4m^2_{H^{++}}}{v_\phi^2},
\]
and $M_A^2 \simeq m_A^2$.

We denote the lightest mass of $m_{H^+}$, $m_{H^+}$, and $m_A$ as the variable $m_L$ and show in Fig. 5 the improved $m_W$ as the function of $m_L$ for various $\eta$, which is defined as $\eta \equiv m_{H^+}^2 - m_{H^+}^2 - m_A^2$. Three typical values of $|\eta| = (100)^2 \text{GeV}^2$, $(150)^2 \text{GeV}^2$, and $(200)^2 \text{GeV}^2$ are selected for comparisons. The dashed and solid lines correspond to the function of $m_L$ for different $\eta$, $m_W$ increases diversely with the decrease of $m_L$, and some curves can reach the range of the CDF measurement [54]. In this case, the CDF anomaly can be explained by taking $|\eta| \in [(150)^2, (200)^2] \text{GeV}^2$ for $m_L \lesssim 500 \text{GeV}$.

**Summary.**— In this letter, we have proposed a new mass generation mechanism of ALP from the type-II seesaw mechanism that give rise to the active neutrino Majorana masses. The typical oscillation temperature of ALP shows a cut-off at the critical temperature of the EWSB, which is the typical trappings of this kind of ALP. Although the ALP does not couple to the diphoton, it might be detected in future neutrino oscillation experiments due to the matter effect induced by the ALP-neutrino interactions. Finally, we show that the $W$-mass anomaly observed by the CDF collaboration can be explained by the TeV-scale type-II seesaw. All these observations make three different kinds of new physics phenomena tightly connected with each other in a single model.

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[1] N. Aghanim et al. (Planck), “Planck 2018 results. VI. Cosmological parameters,” Astron. Astrophys. 641, A6 (2020), [Erratum: Astron.Astrophys. 652, C4 (2021)], arXiv:1807.06209 [astro-ph.CO].
[2] R. D. Peccei and Helen R. Quinn, “Constraints Imposed by CP Conservation in the Presence of Instantons,” Phys. Rev. D 16, 1791–1979 (1977).
[3] R. D. Peccei and Helen R. Quinn, “CP Conservation in the Presence of Instantons,” Phys. Rev. Lett. 38, 1440–1443 (1977).
[4] Steven Weinberg, “A New Light Boson?” Phys. Rev. Lett. 40, 223–226 (1978).
[5] Frank Wilczek, “Problem of Strong $P$ and $T$ Invariance in the Presence of Instantons,” Phys. Rev. Lett. 40, 279–282 (1978).
[6] Michael Dine and Willy Fischler, “The Not So Harmless Axion,” Phys. Lett. B 120, 137–141 (1983).
[7] John Preskill, Mark B. Wise, and Frank Wilczek, “Cosmology of the Invisible Axion,” Phys. Lett. B 120, 127–132 (1983).
[8] L. F. Abbott and P. Sikivie, “A Cosmological Bound on the Invisible Axion,” Phys. Lett. B 120, 133–136 (1983).
[9] David J. E. Marsh, “Axion Cosmology,” Phys. Rept. 643, 1–79 (2016), arXiv:1510.07633 [astro-ph.CO].
[10] Luca Di Luzio, Maurizio Giannotti, Enrico Nardi, and Luca Visinelli, “The landscape of QCD axion models,” Phys. Rept. 870, 1–117 (2020), arXiv:2003.01100 [hep-ph].
[11] Raymond T. Co, Lawrence J. Hall, and Keisuke Harigaya, “Axion Kinetic Misalignment Mechanism,” Phys. Rev. Lett. 124, 251802 (2020), arXiv:1910.14152 [hep-ph].
[12] Jihn E. Kim, “Weak Interaction Singlet and Strong CP Invariance,” Phys. Rev. Lett. 43, 103 (1979).
[13] Mikhail A. Shifman, A. I. Vainshtein, and Valentin I. Zakharov, “Can Confinement Ensure Natural CP Invariance of Strong Interactions?” Nucl. Phys. B 166, 493–506 (1980).
[14] Michael Dine, Willy Fischler, and Mark Srednicki, “A Simple Solution to the Strong CP Problem with a Harmless Axion,” Phys. Lett. B 104, 199–202 (1981).
[15] A. R. Zhitnitsky, “On Possible Suppression of the Axion Hadron Interactions. (In Russian),” Sov. J. Nucl. Phys. 31, 260 (1980).
[16] Jihn E. Kim, “Light Pseudoscalars, Particle Physics and Cosmology,” Phys. Rept. 150, 1–177 (1987).
[17] H. Faissner, E. Frenzel, W. Heinrigs, A. Preussger, D. Sann, and U. Sann, “LIMIT ON AXION DECAY INTO AN ELECTRON PAIR,” Phys. Lett. B 96, 201–205 (1980).
[18] H. Faissner, E. Frenzel, W. Heinrigs, A. Preussger, D. Sann, and U. Sann, “Observation of the Two - Photon Decay of a Light Penetrating Particle,” Phys. Lett. B 103, 234–240 (1981).
[19] Y. Chikashige, Rabindra N. Mohapatra, and R. D. Peccei, “Are There Real Goldstone Bosons Associated with Broken Lepton Number?” Phys. Lett. B 98, 265–268 (1981).
[20] G. B. Gelmini and M. Roncadelli, “Left-Handed Neutrino Mass Scale and Spontaneously Broken Lepton Number,” Phys. Lett. B 99, 411–415 (1981).
[21] Frank Wilczek, “Axions and Family Symmetry Breaking,” Phys. Rev. Lett. 49, 1549–1552 (1982).
[22] Z. G. Berezhiani and M. Yu. Khlopov, “The Theory of broken gauge symmetry of families. (In Russian),” Sov. J. Nucl. Phys. 51, 739–746 (1990).
[23] Joerg Jaeckel, “A Family of WISPy Dark Matter Candidates,” Phys. Lett. B 732, 1–7 (2014), arXiv:1311.0880 [hep-ph].
[24] Edward Witten, “Some Properties of O(32) Superstrings,” Phys. Lett. B 149, 351–356 (1984).
[25] Joseph P. Conlon, “The QCD axion and moduli stabilisation,” JHEP 05, 078 (2006), arXiv:hep-th/0602233.
[26] Michele Cicoli, Mark Goodsell, and Andreas Ringwald, “The type IIB string axiverse and its low-energy phenomenology,” JHEP 10, 146 (2012), arXiv:1206.0819 [hep-th].
[27] Howard M. Georgi, Lawrence J. Hall, and Mark B. Wise,
“Grand Unified Models With an Automatic Peccei-Quinn Symmetry,” Nucl. Phys. B 192, 409–416 (1981).

[28] A. G. Dias, A. C. B. Machado, C. C. Nishi, A. Ringwald, and P. Vaudrevange, “The Quest for an Intermediate-Scale Accidental Axion and Further ALPs,” JHEP 06, 037 (2014), arXiv:1403.5760 [hep-ph].

[29] Kang-Sin Choi, Hans Peter Nilles, Saul Ramos-Sanchez, and Patrick K. S. Vaudrevange, “Accions,” Phys. Lett. B 675, 381–386 (2009), arXiv:0902.3070 [hep-th].

[30] A. Ringwald, “Axions and Axion-Like Particles,” in 49th Rencontres de Moriond on Electroweak Interactions and Unified Theories (2014) pp. 223–230, arXiv:1407.0546 [hep-ph].

[31] Martin Bauer, Mathias Heiles, Matthias Neubert, and Andrea Thamm, “Axion-Like Particles at Future Colliders,” Eur. Phys. J. C 79, 74 (2019), arXiv:1808.10323 [hep-ph].

[32] Matías M. Reynoso, Oscar A. Sampayo, and Agustín M. Carulli, “Neutrino interactions with ultralight axion-like dark matter,” Eur. Phys. J. C 82, 274 (2022), arXiv:2203.11642 [hep-ph].

[33] Wayne Hu, Rennan Barkana, and Andrei Gruzinov, “Cold and fuzzy dark matter,” Phys. Rev. D 85, 115016 (2000), arXiv:astro-ph/0003365.

[34] P. Sikivie, “Experimental Tests of the Invisible Axion,” Phys. Rev. Lett. 51, 1415–1417 (1983), [Erratum: Phys.Rev.Lett. 52, 605 (1984)].

[35] Georg Raffelt and Leo Stodolsky, “Mixing of the Photon with Low Mass Particles,” Phys. Rev. D 37, 1237 (1988).

[36] S. Andriamonje et al. (CAST), “An Improved limit on the axion-photon coupling from the CAST experiment,” JCAP 04, 010 (2007), arXiv:hep-ex/0702006.

[37] Klaus Ehret et al., “New ALPS Results on Hiddden-Sector Lightweights,” Phys. Lett. B 689, 149–155 (2010), arXiv:1004.1313 [hep-ex].

[38] A. Abramowski et al. (H.E.S.S.), “Constraints on axionlike particles with H.E.S.S. from the irregularity of the PKS 2155-304 energy spectrum,” Phys. Rev. D 88, 102003 (2013), arXiv:1311.3148 [astro-ph.HE].

[39] M. Ajello et al. (Fermi-LAT), “Search for Spectral Irregularities due to Photon–Axionlike-Particle Oscillations with the Fermi Large Area Telescope,” Phys. Rev. Lett. 116, 161101 (2016), arXiv:1603.06978 [astro-ph.HE].

[40] V. Anastassopoulos et al. (CAST), “New CAST Limit on the Axion-Photon Interaction,” Nature Phys. 13, 584–590 (2017), arXiv:1705.02290 [hep-ex].

[41] Alexander V. Gramolin, Deniz Aybas, Dorian Johnson, Janos Adam, and Alexander O. Sushkov, “Search for axion-like dark matter with ferromagnets,” Nature Phys. 17, 79–84 (2021), arXiv:2003.03348 [hep-ex].

[42] Hai-Jun Li, Jun-Guang Guo, Xiao-Jun Bi, Su-Jie Lin, and Peng-Fei Yin, “Limits on axion-like particles from Mrk 421 with 4.5-year period observations by ARGO-YBJ and Fermi-LAT,” Phys. Rev. D 103, 083003 (2021), arXiv:2008.09464 [astro-ph.HE].

[43] Chiara P. Salemi et al., “Search for Low-Mass Axion Dark Matter with ABRACADABRA-10 cm,” Phys. Rev. Lett. 127, 081801 (2021), arXiv:2102.06722 [hep-ex].

[44] Julia Sisk Reynés, James H. Matthews, Christopher S. Reynolds, Helen R. Russell, Robyn N. Smith, and M. C. David Marsh, “New constraints on light axion-like particles using Chandra transmission grating spectroscopy of the powerful cluster-hosted quasar A1821+643,” Mon. Not. Roy. Astron. Soc. 510, 1264–1277 (2021), arXiv:2109.03261 [astro-ph.HE].

[45] Hai-Jun Li, “Probing photon-ALP oscillations from the flat spectrum radio quasar 4C+21.35,” Phys. Lett. B 829, 137047 (2022), arXiv:2203.08573 [astro-ph.HE].

[46] Brian Batell and Akshay Ghalasai, “Thermal Misalignment of Scalar Dark Matter,” (2021), arXiv:2109.04476 [hep-ph].

[47] Eung Jin Chun, “Bosonic dark matter in a coherent state driven by thermal fermions,” Phys. Lett. B 825, 136880 (2022), arXiv:2109.07423 [hep-ph].

[48] George Lazarides, Q. Shafi, and C. Wetterich, “Proton Lifetime and Fermion Masses in an SO(10) Model,” Nucl. Phys. B 181, 287–300 (1981).

[49] Rabindra N. Mohapatra and Goran Senjanovic, “Neutrino Masses and Mixings in Gauge Models with Spontaneous Parity Violation,” Phys. Rev. D 23, 165 (1981).

[50] W. Konetschny and W. Kummer, “Nonconservation of Total Lepton Number with Scalar Bosons,” Phys. Lett. B 70, 433–435 (1977).

[51] T. P. Cheng and Ling-Fong Li, “Neutrino Masses, Mixings and Oscillations in SU(2) x U(1) Models of Electroweak Interactions,” Phys. Rev. D 22, 2860 (1980).

[52] M. Magg and C. Wetterich, “Neutrino Mass Problem and Gauge Hierarchy,” Phys. Lett. B 94, 61–64 (1980).

[53] J. Schechter and J. W. F. Valle, “Neutrino Masses in SU(2) x U(1) Theories,” Phys. Rev. D 22, 2227 (1980).

[54] T. Aaltonen et al. (CDF), “High-precision measurement of the W boson mass with the CDF II detector,” Science 376, 170–176 (2022).

[55] Chih-Ting Lu, Lei Wu, Yongcheng Wu, and Bin Zhu, “Electroweak precision fit and new physics in light of the W boson mass,” Phys. Rev. D 106, 035034 (2022), arXiv:2204.03796 [hep-ph].

[56] J. de Blas, M. Pierini, L. Reina, and L. Silvestrini, “Impact of the recent measurements of the top-quark and W-boson masses on electroweak precision fits,” (2022), arXiv:2204.04204 [hep-ph].

[57] Alessandro Strumia, “Interpreting electroweak precision data including the W-mass CDF anomaly,” JHEP 08, 248 (2022), arXiv:2204.04191 [hep-ph].

[58] Yi-Zhong Fan, Tian-Peng Tang, Yue-Lin Sming Tsai, and Lei Wu, “Inert Higgs Dark Matter for CDF II W-Boson Mass and Detection Prospects,” Phys. Rev. Lett. 129, 091802 (2022), arXiv:2204.03693 [hep-ph].

[59] Peter Athron, Andrew Fowlie, Chih-Ting Lu, Lei Wu, Yongcheng Wu, and Bin Zhu, “The W boson Mass and Muon g – 2: Hadronic uncertainties or new physics?” (2022), arXiv:2204.03996 [hep-ph].

[60] Henning Bahl, Johannes Braathen, and Georg Weiglein, “New physics effects on the W-boson mass from a doublet extension of the SM Higgs sector,” Phys. Lett. B 833, 137295 (2022), arXiv:2204.05269 [hep-ph].

[61] K. S. Babu, Sudip Jana, and Vishnu P. K., “Correlating W-Boson Mass Shift with Muon g-2 in the Two Higgs Doublet Model,” Phys. Rev. Lett. 129, 121803 (2022), arXiv:2204.05303 [hep-ph].

[62] Pouya Asadi, Cari Cesarotti, Katherine Fraser, Samuel Homiller, and Aditya Parikh, “Oblique Lessons from the W Mass Measurement at CDF II,” (2022), arXiv:2204.05283 [hep-ph].

[63] Luca Di Luzio, Ramona Gröber, and Paride Paradisi, “Higgs physics confronts the MW anomaly,” Phys. Lett. B 832, 137250 (2022), arXiv:2204.05284 [hep-ph].

[64] Peter Athron, Markus Bach, Douglas H. J. Jacob, Wo-
jcich Kotlar斯基, Dominik Stöckinger, and Alexander Voigt, “Precise calculation of the W boson pole mass beyond the Standard Model with FlexibleSUSY,” (2022), arXiv:2204.05285 [hep-ph].

Robert M. Schabinger and James D. Wells, “A Minimal spontaneously broken hidden sector and its impact on Higgs boson physics at the large hadron collider,” Phys. Rev. D 72, 093007 (2005), arXiv:hep-ph/0509209.

Brian Patt and Frank Wilczek, “Higgs-field portal into hidden sectors,” (2006), arXiv:hep-ph/0605188.

Giovanni Marco Pruna and Tania Robens, “Higgs singlet extension parameter space in the light of the LHC discovery,” Phys. Rev. D 88, 115012 (2013), arXiv:1303.1150 [hep-ph].

D. López-Val and T. Robens, “Δr and the W-boson mass in the singlet extension of the standard model,” Phys. Rev. D 90, 114018 (2014), arXiv:1406.1043 [hep-ph].

Gerard ’t Hooft, “Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking,” NATO Sci. Ser. D 59, 135–157 (1980).

Michela D’Onofrio, Kari Rummukainen, and Anders Tranberg, “Sphaleron Rate in the Minimal Standard Model,” Phys. Rev. Lett. 113, 141602 (2014), arXiv:1404.3565 [hep-ph].

Daniel Baumann, “Inflation,” in Theoretical Advanced Study Institute in Elementary Particle Physics: Physics of the Large and the Small (2011) pp. 523–680, arXiv:0907.5424 [hep-th].

Martin Bauer and Tilman Plehn, “Inflation,” in Lecture Notes in Physics, Vol. 959 (Springer, 2019)

Giovanni Pruna and Tania Robens, “∆r and the W-boson mass in the singlet extension of the standard model,” Phys. Rev. D 90, 114018 (2014), arXiv:1406.1043 [hep-ph].

Pierre-Henri Chavanis, “Phase transitions between dilute and dense axion stars,” Phys. Rev. D 98, 023009 (2018), arXiv:1705.01987 [hep-th].

P. A. Zyla et al. (Particle Data Group), “Review of Particle Physics,” PTEP 2020, 083C01 (2020).

Eric Braaten, Abhishek Mohapatra, and Hong Zhang, “Dense Axion Stars,” Phys. Rev. Lett. 117, 121801 (2016), arXiv:1512.00108 [hep-ph].

H. Hollik, “Precision tests of the electroweak theory,” J. Phys. G 23, 1503–1537 (1997).

M. Awramik, M. Czakon, A. Freitas, and G. Weiglein, “Precise prediction for the W boson mass in the standard model,” Phys. Rev. D 69, 053006 (2004), arXiv:hep-ph/0311148.

Shinya Kanemura and Kei Yagyu, “Radiative corrections to electroweak parameters in the Higgs triplet model and implication with the recent Higgs boson searches,” Phys. Rev. D 85, 115009 (2012), arXiv:1201.6287 [hep-ph].

Mayumi Aoki, Shinya Kanemura, Mariko Kikuchi, and Kei Yagyu, “Radiative corrections to the Higgs boson couplings in the triplet model,” Phys. Rev. D 87, 015012 (2013), arXiv:1211.6029 [hep-ph].

Shinya Kanemura and Kei Yagyu, “Implication of the W boson mass anomaly at CDF II in the Higgs triplet model with a mass difference,” Phys. Lett. B 831, 137217 (2022), arXiv:2204.07511 [hep-ph].

Yu Cheng, Xiao-Gang He, Zhong-Lv Huang, and Ming-Wei Li, “Type-II seesaw triplet scalar effects on neutrino trident scattering,” Phys. Lett. B 831, 137218 (2022), arXiv:2204.05031 [hep-ph].

Debasish Borah, Satyabrata Mahapatra, Dibyendu Nanda, and Narendra Sahu, “Type II Dirac seesaw with observable ∆Neff in the light of W-mass anomaly,” Phys. Lett. B 833, 137297 (2022), arXiv:2204.08266 [hep-ph].

Julian Heeck, “W-boson mass in the triplet seesaw model,” Phys. Rev. D 106, 015004 (2022), arXiv:2204.10274 [hep-ph].

Lorenzo Calibbi and Xiyuan Gao, “Lepton Flavour Violation in minimal grand-unified type II seesaw models,” (2022), arXiv:2206.10682 [hep-ph].

Oleg Popov and Rahul Srivastava, “The Triplet Dirac Seesaw in the View of the Recent CDF-II W Mass Anomaly,” (2022), arXiv:2204.08568 [hep-ph].

Nabarun Chakrabarty, “The muon g − 2 and W-mass anomalies explained and the electroweak vacuum stabilised by extending the minimal Type-II seesaw,” arXiv:2206.11771 [hep-ph].

J. T. Penedo, Yakefu Reyimuaji, and Xinyi Zhang, “Axionic Dirac seesaw and electroweak vacuum stability,” (2022), arXiv:2208.03329 [hep-ph].

Yu Cheng, Xiao-Gang He, Fei Huang, Jin Sun, and Zhi-Peng Xing, “Electroweak precision tests for triplet scalars,” (2022), arXiv:2208.06760 [hep-ph].

Maximilian Berbig, “The Type II Dirac Seesaw Portal to the mirror sector: Connecting neutrino masses and a solution to the strong CP problem,” (2022), arXiv:2209.14246 [hep-ph].

Kodai Sakurai, Fuminobu Takahashi, and Wen Yin, “Singlet extensions and W boson mass in light of the CDF II result,” Phys. Lett. B 833, 137324 (2022), arXiv:2204.04770 [hep-ph].

Henning Bahl, Wen Han Chiu, Christina Gao, Lian-Tao Wang, and Yi-Ming Zhong, “Tripling down on the W boson mass,” (2022), arXiv:2207.04059 [hep-ph].

Michael E. Peskin and Tatsu Takeuchi, “Estimation of oblique electroweak corrections,” Phys. Rev. D 46, 381–409 (1992).

A. Arhrib, R. Benbrik, M. El Kacimi, L. Rahili, and S. Semlali, “Extended Higgs sector of 2HDM with real singlet facing LHC data,” Eur. Phys. J. C 80, 13 (2020), arXiv:1811.12431 [hep-ph].

T. Blank and W. Hollik, “Precision observables in SU(2) x U(1) models with an additional Higgs triplet,” Nucl. Phys. B 514, 113–134 (1998), arXiv:hep-ph/9703392.

Kaoru Hagiwara, Satyabrata Mahapatra, and C. S. Kim, “A Novel approach to confront electroweak data and theory,” Z. Phys. C 64, 559–620 (1994), [Erratum: Z. Phys. C 68, 352 (1995)], arXiv:hep-ph/9409380.

G. Passarino and M. J. G. Veltman, “One Loop Corrections for e+ e− Annihilation Into mu+ mu− in the Weinberg Model,” Nucl. Phys. B 160, 151–207 (1979).
Axion-like Dark Matter from the Type-II Seesaw Mechanism

Supplemental Material

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The Supplemental Material is organized as follows. In Sec., we present the model in detail. The analytical solution of the EOM for ALP is given in Sec.. In Sec., we calculate the branching fraction of the decay process $h \rightarrow aa$. In Sec., we discuss the calculation of $W$-boson mass anomaly. Finally, the gauge-scalar interactions, the ALP interactions, and the gauge boson self-energies are listed in Secs.,, and , respectively.

The Singlet-Triplet Model

The relevant Lagrangian is given by

$$
\mathcal{L}_{SHT} = (\partial_{\mu} S)^\dagger (\partial^{\mu} S) + (D_{\mu} \Phi)^\dagger (D^{\mu} \Phi) + \text{Tr}[(D_{\mu} \Delta)^\dagger (D^{\mu} \Delta)] - V(S, \Phi, \Delta) + \mathcal{L}_{\text{Yukawa}} ,
$$

where the covariant derivatives are defined as

$$
D_{\mu} \Phi = \left( \partial_{\mu} + i \frac{g}{2} \tau^a W^a_{\mu} + i \frac{g'}{2} B_{\mu} \right) \Phi , \quad D_{\mu} \Delta = \partial_{\mu} \Delta + i \frac{g}{2} \tau^a W^a_{\mu} \Delta + i g' B_{\mu} \Delta .
$$

The general form of the scalar potential $V(S, \Phi, \Delta)$ is given by

$$
V(S, \Phi, \Delta) = -\mu_5^2 (\Phi \dagger \Phi) + \mu_6^2 \text{Tr}(\Delta \dagger \Delta) - \mu_7^2 (S\dagger S) + \lambda_1 (\Phi \dagger \Phi)^2 + \lambda_2 \left[ \text{Tr}(\Delta \dagger \Delta) \right]^2 + \lambda_3 \text{Tr}[(\Delta \dagger \Delta)^2] + \lambda_4 (\Phi \dagger \Phi) \text{Tr}(\Delta \dagger \Delta) + \lambda_5 \Phi \dagger \Delta \dagger \Phi \dagger \Phi + \mu_\Phi T_{\tau} \tau_2 \Delta \dagger \Phi + \lambda_\Phi S \Phi T_{\tau} \tau_2 \Delta \dagger \Phi + \text{h.c.} ,
$$

and $\mathcal{L}_{\text{Yukawa}}$ is the Yukawa interaction of left-handed lepton doublets [48–53],

$$
-\mathcal{L}_{\text{Yukawa}} = y_{\alpha \beta} T_{\tau} \tau_2 \Delta \dagger \ell_\alpha^\beta + \text{h.c.} ,
$$

where $y_{\alpha \beta}$ denotes the $3 \times 3$ complex symmetric matrix, and $\ell_\alpha^\beta = (\nu^\alpha_L, e^\alpha_L)^T$ is the left-handed lepton doublet with $\alpha = \{ e, \mu, \tau \}$. From Eq. (S18) we know that $\Delta$ carries a lepton number charge of $-2$. The scalar fields $\Phi$, $\Delta$, and $S$ can be parametrized as

$$
\Phi = \left( \frac{\phi^+}{\sqrt{2}} \right) , \quad \Delta = \left( \begin{array}{c} \Delta^+ \\ \Delta^0 \\ -\Delta^- \end{array} \right) , \quad S = v_s + \hat{s} + i \hat{a} ,
$$

where $\Delta^0 = (v_\Delta + \delta + i \eta)/\sqrt{2}$, and $\delta^2 = v_\phi^2 + 2v_\Delta^2 \simeq (246 \text{ GeV})^2$. The $v_\phi$, $v_\Delta$, and $v_s$ are the VEVs of the Higgs doublet, the Higgs triplet, and the scalar singlet, respectively. After the Higgs triplet acquires a VEV $v_\Delta$, Eq. (S18) gives rise to the mass matrix of active neutrinos,

$$
(m_\nu)_{\alpha \beta} = y_{\alpha \beta} v_\Delta / \sqrt{2} .
$$

At the tree level, the $W$ boson and the $Z$ boson obtain masses through Higgs mechanism,

$$
\begin{align*}
\frac{m_W^2}{4} &= \frac{g^2}{4} \left( v_\phi^2 + 2v_\Delta^2 \right) , \\
\frac{m_Z^2}{4 \cos^2 \theta_W} &= \frac{g^2}{4 \cos^2 \theta_W} \left( v_\phi^2 + 4v_\Delta^2 \right) .
\end{align*}
$$

The electroweak $\rho$ parameter can slightly deviate from 1, i.e. [95],

$$
\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = \frac{1 + \frac{2 v_\Delta^2}{v_\phi^2}}{1 + \frac{4 v_\Delta^2}{v_\phi^2}} .
$$
Actually, the experimental measurement of the $\rho$ parameter gives $\rho^{exp} = 1.0002 \pm 0.0009$ [73], which implies that $v_\Delta \lesssim 7 \text{GeV}$ [83] according to Eq. (S22). The physical scalar sectors are obtained by rotating the weak eigenstates of the scalar fields with the following orthogonal transformations

$$
\begin{align*}
\left( \begin{array}{c} G^+ \\ H_+^{\pm} \end{array} \right) &= \mathcal{R}(\beta) \left( \begin{array}{c} \phi^+ \\ \Delta^\pm \end{array} \right), \\
\left( \begin{array}{c} G_a \\ A \end{array} \right) &= \mathcal{V}(\beta'_1, \beta'_2, \beta'_3) \left( \begin{array}{c} \chi \\ \eta \end{array} \right), \\
\left( \begin{array}{c} h \\ H_s \end{array} \right) &= \mathcal{U}(\alpha_1, \alpha_2, \alpha_3) \left( \begin{array}{c} \phi \\ \delta \end{array} \right),
\end{align*}
$$

where the expressions of the orthogonal matrices $\mathcal{R}(\beta)$, $\mathcal{V}(\beta'_1, \beta'_2, \beta'_3)$, and $\mathcal{U}(\alpha_1, \alpha_2, \alpha_3)$ can be found in Ref. [96]. The mixing angles $\beta$ and $\beta_i$ are

$$
\tan \beta = \frac{\sqrt{2}v_\Delta}{v_\phi}, \quad \tan \beta'_1 = \frac{2v_\Delta}{v_\phi}, \quad \tan \beta'_2 = 0, \quad \tan 2\beta'_3 = \frac{-2\lambda v_\Delta v_s v_\phi \sqrt{v_\phi^2 + 4v_\Delta^2}}{v_\phi^2 (-\lambda v_\Delta^2 + \lambda v_s^2 + \sqrt{2\mu} v_s) + 4v_\Delta^2 v_s (\sqrt{2\mu} + \lambda v_s)}.
$$

The masses of charged and CP-odd physical states are

$$
\begin{align*}
m_{H^{++}}^2 &= M_\Delta^2 - \lambda_3 v_\Delta^2 - \frac{\lambda_5}{2} v_\phi^2 \approx M_\Delta^2 - \frac{\lambda_5}{2} v_\phi^2, \\
m_{H^+}^2 &= \left( M_\Delta^2 - \frac{\lambda_5}{4} v_\phi^2 \right) \left( 1 + \frac{2v_\Delta^2}{v_\phi^2} \right) \approx M_\Delta^2 - \frac{\lambda_5}{4} v_\phi^2, \\
m_A^2 &= M_\Delta^2 \left( 1 + \frac{4v_\Delta^2}{v_\phi^2} + \frac{v_\phi^2}{v_s^2} \right) \approx M_\Delta^2, \\
m_S^2 &= \frac{\sqrt{2\mu} v_\phi^2 v_\Delta^2 (v_\phi^2 + 4v_\Delta^2)}{2v_\phi^2 (v_\phi^2 + v_s^2) + 8v_\Delta^2 v_s^2} \approx \frac{\mu v_\phi^2 v_\Delta}{\sqrt{2} v_s^2},
\end{align*}
$$

where $M_\Delta^2 = \frac{\mu v_\phi^2}{\sqrt{2} v_s} + \frac{\lambda v_s v_\phi^2}{2v_\phi^2} \approx \frac{\lambda v_s v_\phi^2}{2v_\phi^2}$. Here we defines $\eta = m_{H^{++}}^2 - m_{H^+}^2 - m_A^2 - m_S^2$.

Given that the value of $v_s$ is much larger than that of $v_\phi$ and $v_\Delta$, it is reasonable to expect that the mixing angles $\alpha_2$ and $\alpha_3$ in CP-even sector are approximately zero ($\alpha_2, \alpha_3 \simeq 0$) because the scalar singlet is nearly decoupled from the other scalar fields. The remaining $\alpha_1$ can be written as

$$\alpha_1 \simeq \arctan \left[ \frac{v_\phi v_\Delta (\lambda_4 + \lambda_5) - 2M_\Delta^2 v_\Delta/v_\phi}{m_\Delta^2 - M_\Delta^2} \right].$$

Here we take $\lambda_4 = 0$ [82, 94], and $\lambda_5$ which is correlated with the splitting of triplet mass spectrum is given as

$$\lambda_5 \simeq \frac{4m_\Delta^2 - 4m_{H^+}^2}{v_\phi^2} \approx \frac{4m_{H^+}^2 - 4m_{H^{++}}^2}{v_\phi^2}.$$ 

In this case ($\alpha_2, \alpha_3 \simeq 0$), the mass eigenvalues of CP-even physical states are

$$
\begin{align*}
m_h^2 &= v_\phi \left( 2\lambda_1 v_\phi - \tan \alpha_1 \left( 2M_\Delta^2 v_\Delta/v_\phi^2 - v_\Delta (\lambda_4 + \lambda_5) \right) \right), \\
m_H^2 &= v_\phi \left( 2\lambda_1 v_\phi + \cot \alpha_1 \left( 2M_\Delta^2 v_\Delta/v_\phi^2 - v_\Delta (\lambda_4 + \lambda_5) \right) \right), \\
m_s^2 &= \frac{\lambda v_\Delta v_\phi^2}{2v_s} + 2\lambda_6 v_s^2.
\end{align*}
$$

**Analytical solution of the EOM**

To solve the EOM for ALP given by Eq. (5) analytically, we take the initial conditions as [10, 30]

$$\theta(t_i) = \frac{\pi}{\sqrt{3}}, \quad \dot{\theta}(t_i) = 0,$$

then the analytical solution is

$$
\theta(t) = -\pi \left[ -2m_{t_1} J_{\frac{1}{4}}(m_{t_1}) Y_{\frac{1}{4}}(m_{t_1}) + 2m_{t_1} J_{\frac{1}{4}}(m_{t_1}) Y_{\frac{1}{4}}(m_{t_1}) - Y_{\frac{1}{4}}(m_{t_1}) J_{\frac{1}{4}}(m_{t_1}) - 2m_{t_1} J_{\frac{1}{4}}(m_{t_1}) Y_{\frac{1}{4}}(m_{t_1}) + J_{\frac{1}{4}}(m_{t_1}) Y_{\frac{1}{4}}(m_{t_1}) - J_{\frac{1}{4}}(m_{t_1}) Y_{\frac{1}{4}}(m_{t_1}) \right] / \left[ 2\sqrt{3} t_i^2 \left[ J_{\frac{1}{4}}(m_{t_1}) Y_{\frac{1}{4}}(m_{t_1}) - J_{\frac{1}{4}}(m_{t_1}) Y_{\frac{1}{4}}(m_{t_1}) \right] \right],
$$

(S30)
where $J_n$ and $Y_n$ are the Bessel functions of rank-$n$. By taking $t_i \to 0$ [47] and $t_i = t_C$ (the critical time when ALP starts to oscillate at $T_C$), the analytical evolution is shown in Fig. 2 as the red curves, as a comparison of the numerical simulations (blue curves).

**The decay of SM Higgs into ALPs**

The SM Higgs can decay into ALPs where the relevant coupling is listed in Table IV, and the decay width is estimated to be

$$\Gamma_{h \rightarrow aa} = \frac{\lambda_{haa}^2}{8\pi m_h} \left(1 - \frac{4m_a^2}{m_h^2}\right)^{1/2},$$

(S31)

where $\lambda_{haa} \equiv -\lambda U_{11} V_{13} v_s + \frac{1}{2} \lambda U_{21} V_{13}^2 v_s$ with $\lambda \approx \frac{2v_D M_Z^2}{v_s v_\phi}$, and we neglect the $v_\Delta$ and $v_\phi$ terms. The total width of the SM Higgs is $\Gamma_h = 3.2$ MeV [73]. As discussed in Fig. 5, we take $\mu = 10^{-5}$ GeV, $v_\Delta = 1$ MeV, $v_s = 10^{13}$ GeV, and $m_s = 1000$ GeV. Then the branching fraction of the decay $h \rightarrow aa$ as a function of $m_L$ is shown in Fig. S1. Note that the branching fraction is of order $\sim \mathcal{O}(10^{-78})$, which implies that this decay process is almost impossible to be detected by current experiments.

![FIG. S1. The branching fraction of $h \rightarrow aa$ as a function of $m_L$.](image)

**One-loop radiative corrections to $W$-boson mass**

In this section, we calculate the $W$-boson mass at the one-loop level. First of all, the expression of $m_W$ is given by [79, 80]

$$m_W^2 = \frac{m_Z^2}{2} \left[1 + \frac{4\pi \alpha_{em}}{\sqrt{2} G_F m_Z^2} (1 + \Delta r)\right],$$

(S32)

where $\Delta r$ is defined as

$$\Delta r = \Delta \alpha_{em} - \frac{c_W^2}{s_W} \Delta \rho_{\text{loop}} + \Delta \rho_{\text{rem}},$$

(S33)

with [81–83]

$$\Delta \alpha_{em} = \Gamma_{\gamma\gamma}'(0) - \Pi_{\gamma\gamma}(m_Z^2),$$

(S34)

$$\Delta \rho_{\text{loop}} = \frac{\Pi_{ZZ}(0)}{m_Z^2} - \frac{\Pi_{WW}(0)}{m_W^2} + \frac{2s_W}{c_W} \frac{\Pi_{Z\gamma}(0)}{m_Z^2},$$

(S35)
\[ \Delta r_{\text{rem}} = \frac{c_W^2}{s_W^2} \left[ \frac{\Pi_{ZZ}(0)}{m_Z^2} - \text{Re} \left[ \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} \right] \right] + \left( 1 - \frac{c_W^2}{s_W^2} \right) \left[ \frac{\Pi_{WW}(0)}{m_W^2} - \text{Re} \left[ \frac{\Pi_{WW}(m_W^2)}{m_W^2} \right] \right] + \Pi'_{\gamma\gamma}(m_Z^2) + \delta_{\nu B}. \] (S36)

The explicit expressions for the gauge boson self-energies \( \Pi_{WW}, \Pi_{\gamma\gamma}, \) and \( \Pi_{Z\gamma} \) are listed in Sec. The \( \delta_{\nu B} \) denotes the contribution from the vertex and box radiative corrections, which are calculated in Refs. [97, 98]. Other input experimental values related to electroweak parameters are [54, 73]

\[
\begin{align*}
\alpha^{-1}_{\text{em}} &= 137.03599, \quad G_F = 1.1663787 \times 10^{-5} \text{ GeV}^{-2}, \\
s_W^2 &= 0.23121, \quad m_h = 125.10 \text{ GeV}, \\
m_Z &= 91.1876 \text{ GeV}, \quad m_{\text{CDF}} = 80.4335 \pm 0.0094 \text{ GeV}.
\end{align*}
\]

To better show the correlation between \( m_W \) and the CP-even Higgs mixing angles \( \alpha_i = \{\alpha_1, \alpha_2, \alpha_3\} \), we show scattering plots of \( m_W \) versus \( \sin \alpha_1 \) in Fig. S2, by setting mixing angles \( \alpha_i \) is random values in the range \((-0.4, 0.4)\). For other numerical inputs, we set \( m_L = 300 \text{ GeV}, \eta = -200^2 \text{ GeV}^2, \mu = 10^{-5} \text{ GeV}, \nu = 1 \text{ MeV}, v_s = 10^{13} \text{ GeV} \) and \( m_s = 1000 \text{ GeV} \). It can be easily seen that the allowed parameter space of \( m_W \) measured by the CDF collaboration [54] are displayed as the arched areas in both panels. However, \( \sin \alpha_2 \) and \( \sin \alpha_3 \) show the significantly different distributions. In the left panel, the allowed value of \( \sin \alpha_2 \) has a relatively uniform distribution in the range \((-0.4, 0.4)\), while the right panel shows that the allowed points for \( \sin \alpha_3 \) are almost smaller than 0.1, which implies that \( \sin \alpha_2 \) is usually irrelevant to the corrected \( m_W \).

![FIG. S2. \( m_W \) and \( \sin \alpha_1 \) as a function of \( \sin \alpha_2 \) (left) vs. \( \sin \alpha_3 \) (right).](image)

**The gauge-scalar interactions**

In order to express the weak eigenstates on the right side of Eq. (S23) in terms of mass eigenstates on the left side, it’s useful to get the transposed form of the orthogonal matrices, which are listed as follows,

\[
R(\beta) = \left[ R(\beta) \right]^T = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix},
\]

\[
V(\beta_1', \beta_2', \beta_3') = \left[ V(\beta_1', \beta_2', \beta_3') \right]^T = \begin{pmatrix} \cos \beta_1' \cos \beta_2' - \cos \beta_1' \sin \beta_2' \sin \beta_3' - \sin \beta_1' \cos \beta_2' \sin \beta_3' - \sin \beta_1' \sin \beta_2' \cos \beta_3' + \sin \beta_1' \sin \beta_2' \cos \beta_3' \\ \sin \beta_1' \cos \beta_2' - \sin \beta_1' \sin \beta_2' \sin \beta_3' - \sin \beta_1' \sin \beta_2' \cos \beta_3' + \sin \beta_1' \sin \beta_2' \cos \beta_3' \\ \cos \beta_2' \sin \beta_3' \end{pmatrix},
\]

\[
U(\alpha_1, \alpha_2, \alpha_3) = \left[ U(\alpha_1, \alpha_2, \alpha_3) \right]^T = \begin{pmatrix} \cos \alpha_1 \cos \alpha_2 - \cos \alpha_1 \sin \alpha_2 \sin \alpha_3 - \sin \alpha_1 \cos \alpha_3 - \cos \alpha_1 \sin \alpha_2 \cos \alpha_3 + \sin \alpha_1 \sin \alpha_3 \\ \sin \alpha_1 \cos \alpha_2 + \cos \alpha_1 \cos \alpha_3 - \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 - \cos \alpha_1 \sin \alpha_3 + \sin \alpha_1 \sin \alpha_2 \cos \alpha_3 \\ \sin \alpha_2 \cos \alpha_3 \end{pmatrix}.
\]
In the following, we list the vertices and coefficients for the corresponding interactions in Tables I, II, and III. For simplicity, we use the abbreviations $s_\beta(\sin \beta), c_3(\cos \beta), s_W(\sin \theta_w),$ and $c_W(\cos \theta_w)$ for the mixing angle $\beta$ and the Weinberg angle $\theta_w$, respectively. In addition, the matrix elements of the matrices $U(\alpha_1, \alpha_2, \alpha_3)$ and $V(\beta_1', \beta_2', \beta_3')$ are denoted as $U_{ij}, V_{ij}(i, j = 1, 2, 3)$, respectively.

| Vertices | Coefficient | Vertices | Coefficient |
|----------|-------------|----------|-------------|
| $hW_\mu^+ W^-\nu$ | $g^2(U_{21}\nu_\Delta + \frac{1}{2}U_{11}\nu_\phi)g_{\mu \nu}$ | $hZ_\mu Z_\nu$ | $\frac{g^2}{2\sqrt{W}}(2U_{21}\nu_\Delta + \frac{1}{2}U_{11}\nu_\phi)g_{\mu \nu}$ |
| $HW_\mu^+ W^-\nu$ | $g^2(U_{21}\nu_\Delta + \frac{1}{2}U_{11}\nu_\phi)g_{\mu \nu}$ | $H Z_\mu Z_\nu$ | $\frac{g^2}{2\sqrt{W}}(2U_{21}\nu_\Delta + \frac{1}{2}U_{11}\nu_\phi)g_{\mu \nu}$ |
| $sW_\mu^+ W^-\nu$ | $g^2(U_{21}\nu_\Delta + \frac{1}{2}U_{11}\nu_\phi)g_{\mu \nu}$ | $s Z_\mu Z_\nu$ | $\frac{g^2}{2\sqrt{W}}(2U_{21}\nu_\Delta + \frac{1}{2}U_{11}\nu_\phi)g_{\mu \nu}$ |
| $H^{\pm} W_\mu^+ Z_\nu$ | $\sqrt{2}u_\Delta c_\beta (-3 + c^2_W) \frac{1}{2} - 2u_\phi s_W s_W t_W g_{\mu \nu}$ | $G^{\pm} W_\mu^+ Z_\nu$ | $\sqrt{2}u_\Delta c_\beta (-3 + c^2_W) \frac{1}{2} - 2u_\phi s_W s_W t_W g_{\mu \nu}$ |
| $H^{\pm} W_\mu^+ W_\nu^\mp$ | $(g^2 u_\Delta / \sqrt{2}) g_{\mu \nu}$ | $G^{\pm} W_\mu^+ A_\nu$ | $\frac{g^2}{2\sqrt{W}}(u_\phi c_\beta + \sqrt{2}u_\Delta s_\beta) g_{\mu \nu}$ |

**TABLE I.** The Higgs-gauge-gauge type interactions and the corresponding coefficients.

| Vertices | Coefficient | Vertices | Coefficient |
|----------|-------------|----------|-------------|
| $H^{\pm} H^{\pm} W_\mu^\mp$ | $g c_\beta (p_1 - p_2)_\mu$ | $H^{\pm} H^{\mp} A_\mu$ | $2\epsilon (p_1 - p_2)_\mu$ |
| $H^{\pm} G^{\mp} W_\mu^\pm$ | $g s_\beta (p_1 - p_2)_\mu$ | $H^{\mp} H^{\pm} A_\mu$ | $\epsilon (p_1 - p_2)_\mu$ |
| $H^{\pm} A W_\mu^\mp$ | $-i\frac{g}{2}(V_{12}s_\beta - \sqrt{2}V_{12}c_\beta)(p_1 - p_2)_\mu$ | $G^{\mp} G^{-} A_\mu$ | $\epsilon (p_1 - p_2)_\mu$ |
| $H^{\pm} W W_\mu^\pm$ | $\frac{g}{2}(-U_{11}s_\beta + \sqrt{2}U_{21}c_\beta)(p_1 - p_2)_\mu$ | $H^{\pm} H^{-} Z_\mu$ | $\frac{g^2}{2\sqrt{W}}(c^2_W - s^2_W)(p_1 - p_2)_\mu$ |
| $H^{\pm} W W_\mu^\pm$ | $\frac{g}{2}(-U_{11}s_\beta + \sqrt{2}U_{21}c_\beta)(p_1 - p_2)_\mu$ | $H^{\pm} H^{-} Z_\mu$ | $\frac{g^2}{2\sqrt{W}}(c^2_W - s^2_W - s^2_\phi)(p_1 - p_2)_\mu$ |
| $H^{\pm} A W_\mu^\mp$ | $i\frac{g}{2}(V_{13}s_\beta - \sqrt{2}V_{23}c_\beta)(p_1 - p_2)_\mu$ | $H^{\pm} A W_\mu^\mp$ | $\frac{g^2}{2\sqrt{W}}(c^2_W - s^2_W - s^2_\phi)(p_1 - p_2)_\mu$ |
| $H^{\pm} A W_\mu^\mp$ | $i\frac{g}{2}(V_{13}s_\beta - \sqrt{2}V_{23}c_\beta)(p_1 - p_2)_\mu$ | $H^{\pm} A W_\mu^\mp$ | $\frac{g^2}{2\sqrt{W}}(c^2_W - s^2_W - s^2_\phi)(p_1 - p_2)_\mu$ |
| $G^{\pm} A W_\mu^\mp$ | $-i\frac{g}{2}(V_{12}s_\beta + \sqrt{2}V_{22}s_\beta)(p_1 - p_2)_\mu$ | $H^{\pm} G^{\pm} Z_\mu$ | $\frac{g^2 s_\beta s_\phi}{2\sqrt{W}}(p_1 - p_2)_\mu$ |
| $G^{\pm} H W_\mu^\mp$ | $\frac{g}{2}(U_{12}c_\beta + \sqrt{2}U_{22}s_\beta)(p_1 - p_2)_\mu$ | $G^{\pm} H W_\mu^\mp$ | $\frac{g^2 s_\beta s_\phi}{2\sqrt{W}}(p_1 - p_2)_\mu$ |
| $G^{\pm} H W_\mu^\mp$ | $\frac{g}{2}(U_{11}c_\beta + \sqrt{2}U_{21}s_\beta)(p_1 - p_2)_\mu$ | $G^{\pm} H W_\mu^\mp$ | $\frac{g^2 s_\beta s_\phi}{2\sqrt{W}}(p_1 - p_2)_\mu$ |
| $G^{\pm} G W_\mu^\mp$ | $-i\frac{g}{2}(V_{13}s_\beta + \sqrt{2}V_{23}c_\beta)(p_1 - p_2)_\mu$ | $G^{\pm} G W_\mu^\mp$ | $\frac{g^2 s_\beta s_\phi}{2\sqrt{W}}(p_1 - p_2)_\mu$ |
| $G^{\pm} a W_\mu^\mp$ | $-i\frac{g}{2}(V_{13}s_\beta + \sqrt{2}V_{23}c_\beta)(p_1 - p_2)_\mu$ | $G^{\pm} a W_\mu^\mp$ | $\frac{g^2 s_\beta s_\phi}{2\sqrt{W}}(p_1 - p_2)_\mu$ |
| $G^{\pm} s W_\mu^\mp$ | $-i\frac{g}{2}(U_{12}s_\beta + \sqrt{2}U_{22}c_\beta)(p_1 - p_2)_\mu$ | $G^{\pm} s W_\mu^\mp$ | $\frac{g^2 s_\beta s_\phi}{2\sqrt{W}}(p_1 - p_2)_\mu$ |
| $A W_\mu^\mp$ | $-i\frac{g}{2}(U_{13}s_\beta + \sqrt{2}U_{23}c_\beta)(p_1 - p_2)_\mu$ | $A W_\mu^\mp$ | $\frac{g^2 s_\beta s_\phi}{2\sqrt{W}}(p_1 - p_2)_\mu$ |
| $G s Z_\mu$ | $-i\frac{g}{2\sqrt{W}}(2U_{23}V_{12} + U_{13}V_{11})(p_1 - p_2)_\mu$ | $G s Z_\mu$ | $\frac{g^2 s_\beta s_\phi}{2\sqrt{W}}(p_1 - p_2)_\mu$ |

**TABLE II.** The Higgs-Higgs-gauge type interactions and the corresponding coefficients.

**The ALP-scalar/neutrino interactions**

We list the vertices and the corresponding coefficients for ALP-Higgs and ALP-neutrino interactions in Table IV.
The analytic expressions for gauge boson self-energies are listed as follows. Here we use the Passarino-Veltman functions defined in Ref. [99]. First, the fermion-loop (F) contributions to all the two-point correlation functions are

\[
\Pi^{\mu
u}_{(F)} = \frac{2}{\pi} \left[ \alpha_s \frac{2}{3} c_w^2 - \frac{1}{3} s_w^2 \right] g_{\mu
u}
\]

The coefficients for the self-energies are

\[
\Pi^{\mu
nu}_{(F)} = \frac{2}{\pi} \left[ \alpha_s \frac{2}{3} c_w^2 - \frac{1}{3} s_w^2 \right] g_{\mu
nu}
\]

The gauge boson self-energies

The gauge boson self-energies are given by

\[
\Pi^{\mu
nu}_{(G)} = \frac{2}{\pi} \left[ \alpha_s \frac{2}{3} c_w^2 - \frac{1}{3} s_w^2 \right] g_{\mu
nu}
\]

The coefficients for the self-energies are

\[
\Pi^{\mu
nu}_{(G)} = \frac{2}{\pi} \left[ \alpha_s \frac{2}{3} c_w^2 - \frac{1}{3} s_w^2 \right] g_{\mu
nu}
\]

The coefficients for the self-energies are

\[
\Pi^{\mu
nu}_{(G)} = \frac{2}{\pi} \left[ \alpha_s \frac{2}{3} c_w^2 - \frac{1}{3} s_w^2 \right] g_{\mu
nu}
\]

The coefficients for the self-energies are

\[
\Pi^{\mu
nu}_{(G)} = \frac{2}{\pi} \left[ \alpha_s \frac{2}{3} c_w^2 - \frac{1}{3} s_w^2 \right] g_{\mu
nu}
\]

The coefficients for the self-energies are

\[
\Pi^{\mu
nu}_{(G)} = \frac{2}{\pi} \left[ \alpha_s \frac{2}{3} c_w^2 - \frac{1}{3} s_w^2 \right] g_{\mu
nu}
\]

The coefficients for the self-energies are

\[
\Pi^{\mu
nu}_{(G)} = \frac{2}{\pi} \left[ \alpha_s \frac{2}{3} c_w^2 - \frac{1}{3} s_w^2 \right] g_{\mu
nu}
\]

The coefficients for the self-energies are

\[
\Pi^{\mu
nu}_{(G)} = \frac{2}{\pi} \left[ \alpha_s \frac{2}{3} c_w^2 - \frac{1}{3} s_w^2 \right] g_{\mu
nu}
\]
given by

\[
\Pi_{WW}^{1P}(p^2)_F = \frac{g^2}{16\pi^2} N_f \left[ -B_4 + 2p^2 B_3 \right] (p^2, m_f, m'_f),
\]

\[
\Pi_{Z\gamma}^{1P}(p^2)_F = \frac{g^2}{16\pi^2 \cos^2 \theta_w} N_f \left[ 2p^2 (4\sin^4 \theta_w Q_f^2 - 4\sin^2 \theta_w Q_f^2) B_3 - 2I^2_3 m_2^2 B_0 \right] (p^2, m_f, m_f),
\]

\[
\Pi_{Z\gamma}^{1P}(p^2)_F = -\frac{e^2}{16\pi^2} N_f Q_f^2 \left[ 8p^2 B_3 \right] (p^2, m_f, m_f),
\]

\[
\Pi_{Z\gamma}^{1P}(p^2)_F = -\frac{eg}{16\pi^2 \cos \theta_w} N_f \left[ 2p^2 (-4\sin^2 \theta_w Q_f^2 + 2I_f Q_f) B_3 \right] (p^2, m_f, m_f),
\]

where [81, 82]

\[
B_3(p^2, m_1, m_2) = -B_1(p^2, m_1, m_2) - B_{21}(p^2, m_1, m_2),
B_4(p^2, m_1, m_2) = -m_1^2 B_1(p^2, m_1, m_2) - m_2^2 B_1(p^2, m_1, m_2).
\]

The scalar-loop (S1, S2), the gauge boson-loop (V), and the SM contributions to W-W self-energy are

\[
\Pi_{WW}^{1P}(p^2)_{S1} = -\frac{g^2}{16\pi^2} \left[ B_{22}(p^2, m_{G+}, m_a) \left( V_{13} \cos \beta + \sqrt{2} V_{23} \sin \beta \right)^2 + B_{22}(p^2, m_{H+}, m_a) \left( V_{13} \sin \beta - \sqrt{2} V_{23} \cos \beta \right)^2 \\
+ B_{22}(p^2, m_{G+}, m_A) \left( V_{12} \cos \beta + \sqrt{2} V_{22} \sin \beta \right)^2 + B_{22}(p^2, m_{H+}, m_A) \left( V_{12} \sin \beta - \sqrt{2} V_{22} \cos \beta \right)^2 \\
+ B_{22}(p^2, m_{G+}, m_G) \left( V_{11} \cos \beta + \sqrt{2} V_{21} \sin \beta \right)^2 + B_{22}(p^2, m_{H+}, m_G) \left( V_{11} \sin \beta - \sqrt{2} V_{21} \cos \beta \right)^2 \\
+ 4 \sin^2 \beta B_{22}(p^2, m_{H++}, m_{G+}) + B_{22}(p^2, m_{G+}, m_h) \left( U_{11} \cos \beta + \sqrt{2} U_{21} \sin \beta \right)^2 \\
+ B_{22}(p^2, m_{G+}, m_H) \left( U_{12} \cos \beta + \sqrt{2} U_{22} \sin \beta \right)^2 + B_{22}(p^2, m_{G+}, m_s) \left( U_{13} \cos \beta + \sqrt{2} U_{23} \sin \beta \right)^2 \\
+ B_{22}(p^2, m_{H+}, m_h) \left( \sqrt{2} U_{21} \cos \beta - U_{11} \sin \beta \right)^2 + B_{22}(p^2, m_{H+}, m_H) \left( \sqrt{2} U_{22} \cos \beta - U_{12} \sin \beta \right)^2 \\
+ 4 \cos^2 \beta B_{22}(p^2, m_{H++}, m_{H+}) + B_{22}(p^2, m_{H+}, m_s) \left( \sqrt{2} U_{23} \cos \beta - U_{13} \sin \beta \right)^2 \right];
\]

\[
\Pi_{WW}^{1P}(p^2)_{S2} = \frac{1}{16\pi^2} \left[ A_0(m_a) \left( V_{13}^2 + 2V_{23}^2 \right) + A_0(m_A) \left( V_{12}^2 + 2V_{22}^2 \right) + 4A_0(m_{H++}) + A_0(m_G) \left( V_{11}^2 + 2V_{21}^2 \right) \\
+ A_0(m_{G+}) \left[ 5 - 3 \cos(2\beta) \right] + A_0(m_h) \left( U_{11}^2 + 2U_{21}^2 \right) + A_0(m_H) \left( U_{12}^2 + 2U_{22}^2 \right) + A_0(m_{H+}) \left[ 3 \cos(2\beta) + 5 \right] \\
+ A_0(m_s) \left( U_{13}^2 + 2U_{23}^2 \right) \right];
\]

\[
\Pi_{WW}^{1P}(p^2)_V = \frac{g^4}{16\pi^2} \left[ \frac{1}{16} \left( \frac{\sqrt{2} v_\Delta \sin \beta (\cos(2\theta_w) - 3)}{\cos \theta_w} - 2v_\phi \cos \beta \sin \theta_w \tan \theta_w \right)^2 B_0(p^2, m_{G+}, m_Z) \\
+ \frac{1}{16} \left( \frac{\sqrt{2} v_\Delta \cos \beta (\cos(2\theta_w) - 3)}{\cos \theta_w} + 2v_\phi \cos \beta \sin \theta_w \tan \theta_w \right)^2 B_0(p^2, m_{H+}, m_Z) \\
+ \frac{1}{4} \sin^2 \theta_w \left( \sqrt{2} v_\Delta \sin \beta + v_\phi \cos \beta \right)^2 B_0(p^2, m_{G+}, 0) + \frac{4}{2} v_\Delta^2 B_0(p^2, m_{H++}, m_W) \\
+ \left( \frac{U_{12} v_\phi}{2} + U_{22} v_\Delta \right)^2 B_0(p^2, m_H, m_W) + \left( \frac{U_{13} v_\phi}{2} + U_{23} v_\Delta \right)^2 B_0(p^2, m_s, m_W) \\
+ \left( \frac{U_{11} v_\phi}{2} + U_{21} v_\Delta \right)^2 B_0(p^2, m_h, m_W) \right];
\]
\[ \Pi_{WW}^{1\text{PI}}(p^2)_{\text{SM}} = \frac{g^2}{16\pi^2} \left\{ -\cos^2 \theta_w \left[ (6D - 8)B_{22} + p^2(2B_{21} + 2B_1 + 5B_0) \right] \right. \\
+ \left. - \sin^2 \theta_w \left[ (6D - 8)B_{22} + p^2(2B_{21} + 2B_1 + 5B_0) \right] \right\} - \frac{4g^2}{16\pi^2} \left( p^2 - m_W^2 \right) \cos^2 \theta_w B_0(p^2, m, m_W) \\
+ \sin^2 \theta_w B_0(p^2, 0, m_W). \]  

The scalar-loop (S1, S2), the gauge boson-loop (V), and the SM contributions to Z-Z self-energy are

\[ \Pi_{ZZ}^{1\text{PI}}(p^2)_{S1} = -\frac{g^2}{16\pi^2 \cos^2 \theta_w} \left\{ \left( U_{11} V_{13} + 2U_{21} V_{23} \right)^2 B_{22}(p^2, m, m_a) + \left( U_{12} V_{13} + 2U_{22} V_{23} \right)^2 B_{22}(p^2, m_H, m_a) \\
+ \left( U_{13} V_{13} + 2U_{23} V_{23} \right)^2 B_{22}(p^2, m_H, m_a) + \left( U_{11} V_{12} + 2U_{21} V_{22} \right)^2 B_{22}(p^2, m, m_A) \\
+ \left( U_{12} V_{12} + 2U_{22} V_{22} \right)^2 B_{22}(p^2, m_H, m_A) + \left( U_{13} V_{12} + 2U_{23} V_{22} \right)^2 B_{22}(p^2, m, m_A) \\
+ 4 \left( \cos^2 \theta_w - \sin^2 \theta_w \right) B_{22}(p^2, m, m_{H++}) \right\} - (\cos^2 \beta - \sin^2 \theta_w)^2 B_{22}(p^2, m_G^+, m_G^+) \]  

\[ \Pi_{ZZ}^{1\text{PI}}(p^2)_{S2} = \frac{1}{16\pi^2} \frac{g^2}{\cos^2 \theta_w} \left\{ -\frac{1}{4} A_0(m_a) \left( V_{13}^2 + 4V_{23}^2 \right) + \frac{1}{4} A_0(m_A) \left( V_{12}^2 + 4V_{22}^2 \right) + 2A_0(m_{H++}) \left( \cos^2 \theta_w - \sin^2 \theta_w \right)^2 \\
+ \frac{1}{4} A_0(m_G) \left( V_{11}^2 + 4V_{21}^2 \right) \right\} - 4\sin^2 \beta \cos(2\theta_w) \cos(4\theta_w) + 2] \]  

\[ \Pi_{ZZ}^{1\text{PI}}(p^2)_{V} = \frac{g^4}{16\pi^2} \left\{ \frac{2}{16} B_0(p^2, m_G^+, m_W) \left( \frac{\sqrt{2} v_\Delta \sin \beta \sin(2\theta_w)}{\cos \theta_w} - 2v_\phi \cos \beta \sin \theta_w \tan \theta_w \right)^2 \\
+ \frac{2}{16} B_0(p^2, m_{H++}, m_W) \left( \frac{\sqrt{2} v_\Delta \cos \beta \sin(2\theta_w - 3)}{\cos \theta_w} + 2v_\phi \sin \beta \sin \theta_w \tan \theta_w \right)^2 + \frac{1}{\cos^2 \theta_w} \times \right\} \]  

\[ \Pi_{ZZ}^{1\text{PI}}(p^2)_{\text{SM}} = \frac{g^2}{16\pi^2 \cos^2 \theta_w} \left\{ -\cos^4 \theta_w \left[ (6D - 8)B_{22} + p^2(2B_{21} + 2B_1 + 5B_0) \right] \right. \\
+ \left. - \frac{4g^2}{16\pi^2} \left( p^2 - m_Z^2 \right) B_0(p^2, m, m_W) \right\} - \frac{4g^2}{16\pi^2} \cos^2 \theta_w \left( p^2 - m_Z^2 \right) B_0(p^2, m, m_W). \]  

The scalar-loop (S1, S2), the gauge boson-loop (V), and the SM contributions to γ-γ self-energy are

\[ \Pi_{\gamma\gamma}^{1\text{PI}}(p^2)_{S1} = -\frac{\epsilon^2}{16\pi^2} \left( 16B_{22}(p^2, m_{H++}, m_{H++}) + 4B_{22}(p^2, m_G^+, m_G^+) + 4B_{22}(p^2, m_{H++}, m_{H++}) \right); \]  

\[ \Pi_{\gamma\gamma}^{1\text{PI}}(p^2)_{S2} = \frac{\epsilon^2}{16\pi^2} \left( 8A_0(m_{H++}) + 2A_0(m_G^+) + 2A_0(m_{H++}) \right); \]
\[
\Pi^{\text{PI}}_{\gamma\gamma}(p^2)_{V} = \frac{e^4}{(16\pi^2)} \frac{1}{2 \sin^2 \theta_w} B_0(p^2, m_{G^+, m_W}) \left( \sqrt{2} v_\Delta \sin \beta + v_\phi \cos \beta \right)^2 ; \tag{S56}
\]

\[
\Pi^{\text{PI}}_{\gamma\gamma}(p^2)_{\text{SM}} = -\frac{e^2}{16\pi^2} \left[ (6D - 8)B_{22}(p^2, m_W, m_W) + p^2 (2B_{21} + 2B_1 + 5B_0)(p^2, m_W, m_W) \right.
- 2(D - 1)A_0(m_W) - 2m_W^2 B_0(p^2, m_{G^+, m_W}) \bigg] - \frac{4e^2}{16\pi^2} p^2 B_0(p^2, m_W, m_W). \tag{S57}
\]

The scalar-loop (S1, S2), the gauge boson-loop (V), and the SM contributions to \( Z-\gamma \) self-energy are

\[
\Pi^{\text{PI}}_{\gamma\gamma}(p^2)_{S1} = \frac{1}{16\pi^2} \frac{e g}{\cos \theta_w} \left[ 8 \left( \cos^2 \theta_w - \sin^2 \theta_w \right) B_{22}(p^2, m_{H^+, m_{H^+}}) + 2 \left( - \sin^2 \beta - \sin^2 \theta_w + \cos^2 \theta_w \right) B_{22}(p^2, m_{G^+, m_{G^+}}) \right.
\]

\[
+ 2 \left( - \cos^2 \beta - \sin^2 \theta_w + \cos^2 \theta_w \right) B_{22}(p^2, m_{H^+, m_{H^+}}) \bigg] ; \tag{S58}
\]

\[
\Pi^{\text{PI}}_{\gamma\gamma}(p^2)_{S2} = -\frac{1}{16\pi^2} \frac{e g}{\cos \theta_w} \Bigg[ 4A_0(m_{H^+}) \left( \cos^2 \theta_w - \sin^2 \theta_w \right) + A_0(m_{G^+}) \left( - \sin^2 \beta - \sin^2 \theta_w + \cos^2 \theta_w \right) \Bigg]
\]

\[
+ A_0(m_{H^+}) \left( - \cos^2 \beta - \sin^2 \theta_w + \cos^2 \theta_w \right) \bigg] ; \tag{S59}
\]

\[
\Pi^{\text{PI}}_{\gamma\gamma}(p^2)_{V} = -\frac{1}{16\pi^2} \frac{e^2 g^2}{4 \sin \theta_w} B_0(p^2, m_{G^+, m_W}) \left( \sqrt{2} v_\Delta \sin \beta + v_\phi \cos \beta \right) \left( \frac{\sqrt{2} v_\Delta \sin \beta (\cos(2\theta_w) - 3)}{\cos \theta_w} - 2v_\phi \cos \beta \sin \theta_w \tan \theta_w \right) ; \tag{S60}
\]

\[
\Pi^{\text{PI}}_{\gamma\gamma}(p^2)_{\text{SM}} = \frac{e g}{16\pi^2} \frac{\cos \theta_w}{\cos \theta_w} \left[ \cos^2 \theta_w (6D - 8)B_{22}(p^2, m_W, m_W) + \cos^2 \theta_w p^2 (2B_{21} + 2B_1 + 5B_0)(p^2, m_W, m_W) \right.
\]

\[
- 2 \cos^2 \theta_w (D - 1)A_0(m_W) + 2m_W^2 \left( \sin^2 \theta_w + \sin^2 \beta \right) B_0(p^2, m_{G^+, m_W}) \bigg] + \frac{4eg}{16\pi^2} \frac{\cos^2 \theta_w}{\cos \theta_w} \left( p^2 - \frac{1}{2} m_Z^2 \right) B_0(p^2, m_W, m_W). \tag{S61}
\]