DOUBLE LOGARITHMIC INEQUALITY WITH A SHARP CONSTANT IN FOUR SPACE DIMENSIONS

MOHAMED MAJDOUB AND TAREK SAANOUNI

Abstract. We prove a Log Log inequality with a sharp constant in four dimensions for radially symmetric functions. We also show that the constant in the Log estimate is almost sharp.

1. Introduction and statement of the results

The Sobolev embeddings in four dimensions [1],

\[ W^{2,p} \hookrightarrow L^{\frac{8}{8-2p}} \quad \text{for} \quad 1 \leq p < 2 \quad \text{and} \quad W^{2,p} \hookrightarrow C^{2-\frac{4}{p}} \quad \text{for} \quad 2 < p < \infty \]

fails in the limiting case \( p = 2 \). In the setting of a bounded domain we have the injection \( W^{2,2} \subset L^q \) for any \( q < \infty \). The function \( \log(1 - \log |x|) \) is a counterexample if the domain is a subset of the unit ball. Moreover, \( H^2 := W^{2,2} \) functions are in a so-called Orlicz space [5], i.e. their exponential powers are integrable functions. Precisely, we have the following Adams’ type inequality.

**Theorem 1.1** ([13], Theorem 2.2). For any \( \alpha \in (0, 32\pi^2) \) there exists a constant \( C(\alpha) > 0 \) such that

\[ \int_{\mathbb{R}^4} \left( e^{\alpha |u(x)|^2} - 1 \right) dx \leq C(\alpha)\|u\|_{L^2}^2 \quad \forall \; u \in W^{2,2}(\mathbb{R}^4) \quad \text{with} \quad \|\Delta u\|_{L^2} \leq 1, \]

and this inequality is false for \( \alpha > 32\pi^2 \).

We stress that \( \alpha = 32\pi^2 \) becomes admissible if we require \( \|u\|_{W^{2,2}} \leq 1 \) rather than \( \|\Delta u\|_{L^2} \leq 1 \), where

\[ \|u\|^2_{W^{2,2}} = \|\Delta u\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2}. \]

**Theorem 1.2** ([16], Theorem 1.4). There exists a constant \( C > 0 \) such that for any domain \( \Omega \subset \mathbb{R}^4 \)

\[ \sup_{u \in H^2_0(\Omega), \; \|u\|_{H^2} \leq 1} \int_{\Omega} \left( e^{32\pi^2|u(x)|^2} - 1 \right) dx \leq C \]

and this inequality is sharp.

In this work, we prove that in the radial case we can control the \( L^\infty \) norm with \( \dot{H}^2 \) norm and a stronger norm with Logarithmic growth or double logarithmic growth. The
We also define the ratio $N(\alpha) := C^\alpha(\mathbb{R}^4)$ the space of $\alpha$-Hölder continuous functions endowed with the norm
\[ \|u\|_{C^\alpha} = \|u\|_{C^\alpha(\mathbb{R}^4)} := \|u\|_{L^\infty(\mathbb{R}^4)} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}. \]
Moreover, $\hat{C}^\alpha := \hat{C}^\alpha(\mathbb{R}^4)$ denotes the homogenous space of $\alpha$-Hölder continuous functions endowed with the semi norm
\[ \|u\|_{\hat{C}^\alpha} = \|u\|_{\hat{C}^\alpha(\mathbb{R}^4)} := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}. \]
We also define the ratio $N_\alpha(u) := \frac{\|u\|_{C^\alpha}}{\|\Delta u\|_{L^2}}$. For any positive real number $r$, $B_r$ is the ball of $\mathbb{R}^4$ centered at the origin with radius $r$ and $B := B_1$. The space $H^2_0(\Omega)$ stands for the completion in the Sobolev space $H^2$ of smooth and compactly supported functions. $H^2_{0,\text{rad}}(\Omega)$ (respectively $H^2_{\text{rad}}(\Omega)$) is the space of radially symmetric functions of $H^2_0(\Omega)$ (respectively $H^2(\Omega)$).

Our first result reads

**Theorem 1.3. (Double Log estimate)** Let $\alpha \in (0, 1)$. A positive constant $C_\alpha$ exists such that for any function $u \in (H^2_{0,\text{rad}} \cap \hat{C}^\alpha)(B)$, we have
\[ \|u\|_{L^\infty}^2 \leq \frac{1}{8\pi^2 \alpha} \|\Delta u\|_{L^2}^2 \log \left( e^3 + C_\alpha N_\alpha(u) \sqrt{\log(2e + N_\alpha(u))} \right). \]
Moreover, the constant $\frac{1}{8\pi^2 \alpha}$ in the above inequality is sharp.

The second result of this paper is the following

**Theorem 1.4. (Log estimate)** Let $\alpha \in (0, 1)$. For any $\lambda > \frac{1}{8\pi^2 \alpha}$ there exists $C_\lambda > 0$ such that for any function $u \in (H^2_{0,\text{rad}} \cap \hat{C}^\alpha)(B)$, we have
\[ \|u\|_{L^\infty}^2 \leq \lambda \|\Delta u\|_{L^2}^2 \log \left( C_\lambda + N_\alpha(u) \right). \]
Moreover, the above inequality is false for $\lambda = \frac{1}{8\pi^2 \alpha}$.

We derive the following global estimate.

**Corollary 1.5. (Global Log estimate)** Let $\alpha \in (0, 1)$. For any $\lambda > \frac{1}{8\pi^2 \alpha}$ and any $\mu \in (0, 1]$, there exists $C_\mu > 0$ such that for any function $u \in (H^2(\mathbb{R}^4) \cap C^\alpha)(\mathbb{R}^4)$, we have
\[ \|u\|_{L^\infty}^2 \leq \lambda \|u\|_{H^2}^2 \log \left( C_\lambda + \frac{8\pi \mu^{-\alpha} \|u\|_{C^\alpha}}{\|u\|_{H^2}} \right). \]
Where $\|u\|_{H^2} := (1 + 3\mu) \|\Delta u\|_{L^2}^2 + 3\mu \|u\|_{H^1}^2$.

**Remark 1.6.** When we deal with higher order derivatives, we cannot reduce the problem to the radial case as in dimension two for example. The reason is that, for a given function $u \in W^{2,2}$, we do not know wether or not $u^2$ (the Schwarz symmetrization of $u$) still belongs to $W^{2,2}$. Even if this is the case, no inequality of the form $\|\Delta u^2\|_{L^2} \leq \|\Delta u\|_{L^2}$ is known to hold. To overcome this difficulty, one can try to apply a suitable comparison principle as
Define the frequency projectors by Proposition 2.1. Recall that we have

\[ \| \lambda > \] instead of any \( \lambda > \) function

and choose two nonnegative radial functions \( \chi \in C^\infty_0(B_\frac{4}{3}) \) and \( \varphi \in C^\infty_0(C_0) \) such that

\[ \chi + \sum_{j \in \mathbb{N}} \varphi(2^{-j}) = 1 \text{ on } \mathbb{R}^4 \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}) = 1 \text{ on } \mathbb{R}^4 \setminus \{0\}. \]

Define the frequency projectors by

\[ \mathcal{F}(\hat{\Delta}_j u) := \varphi(2^{-j}) F u \quad \text{for} \quad j \in \mathbb{Z}, \]

\[ \Delta_j u = 0 \quad \text{if} \quad j \leq -2, \quad \mathcal{F}(\Delta_{-1} u) = \chi F u \quad \text{and} \quad \Delta_j u := \hat{\Delta}_j u \quad \text{for} \quad j \geq 0. \]

Recall that

\[ \| u \|_{L^2}^2 \approx \left( \sum_{j \in \mathbb{Z}} 2^{4j} \| \hat{\Delta}_j u \|_{L^2}^2 \right) \quad \text{and} \quad \| u \|_{C^m} \approx \sup_{j \in \mathbb{Z}} \left( 2^{j\alpha} \| \hat{\Delta}_j u \|_{L^\infty} \right). \]

We have the following result in the whole space

**Proposition 2.1.** Let \( \alpha \in (0, 1) \). There exists a positive constant \( C := C_\alpha \) such that for any function \( u \in (C^\alpha \cap H^2)(\mathbb{R}^4) \), one has

\[ \| u \|_{L^\infty} \leq C \| u \|_{L^2}^2 + C \| \Delta u \|_{L^2}^2 \log \left( e + N_\alpha(u) \right). \]

**Proof.** We have

\[ u = \Delta_{-1} u + \sum_{j \in \mathbb{N}} \Delta_j u = \Delta_{-1} u + \sum_{j=0}^{m-1} \Delta_j u + \sum_{j=m}^{\infty} \Delta_j u \]

where \( m \) is an integer to fix later. Using Bernstein inequality, we get

\[ \| u \|_{L^\infty} \leq C \| \Delta_{-1} u \|_{L^2} + C \sum_{j=0}^{m-1} 2^{2j} \| \Delta_j u \|_{L^2} + \sum_{j=m}^{\infty} 2^{-j\alpha} \| \Delta_j u \|_{L^\infty} \]

\[ \leq C \| u \|_{L^2} + C \sqrt{m} \left( \sum_{j=0}^{m-1} 2^{j\alpha} \| \Delta_j u \|_{L^2}^2 \right)^{1/2} + C \left( \sum_{j=m}^{\infty} 2^{-j\alpha} \right) \| u \|_{C^\alpha} \]

\[ \leq C \left( \| u \|_{L^2} + \sqrt{m} \| \Delta u \|_{L^2} + \frac{2^{-ma}}{1 - 2^{-\alpha}} \| u \|_{C^\alpha} \right). \]

So

\[ \| u \|_{L^\infty} \leq C \left( \| u \|_{L^2}^2 + m \| \Delta u \|_{L^2}^2 + \frac{2^{-2ma}}{(1 - 2^{-\alpha})^2} \| u \|_{C^\alpha}^2 \right). \]
Taking for $E(x)$ the integer part of any real $x$,
\[ m := \max \left( 1, 1 + E(2 \log_2(N_\alpha(u)^2)) \right), \]
the proof is achieved.

Clearly, if $u$ is supported in the unit ball, then by Poincaré inequality and Proposition 2.1, we get
\[ \|u\|_\infty^2 \leq C_\alpha \|\Delta u\|_{L^2}^2 \log \left( C_0 + N_\alpha(u) \right) \]
for some constant $C_0$ big enough.

3. PROOF OF THEOREM 1.3

To prove (1.3) and the fact that the constant is sharp, it is sufficient to show that
\[ \inf_{u \in (H^2_{0,\text{rad}} C^{\alpha})(B)} \frac{\|\Delta u\|_{L^2}^2 \log \left[ e^3 + C_0 N_\alpha(u) \sqrt{\log(2e + N_\alpha(u))} \right]}{\|u\|_\infty^2} = 8\pi^2 \alpha. \]

Let prove, first, the optimality of the constant $8\pi^2 \alpha$ in the previous equality. Define for $\varepsilon > 0$, the functions
\[ v_\varepsilon(x) := \begin{cases} \sqrt{\frac{1}{32\pi^2} \log(\frac{1}{\varepsilon})} - \frac{|x|^2}{\sqrt{8\pi^2 \log(\frac{1}{\varepsilon})}} + \frac{1}{\sqrt{8\pi^2 \log(\frac{1}{\varepsilon})}} & \text{if } |x| \leq \varepsilon^{\frac{1}{4}}, \\ \frac{1}{\sqrt{2\pi^2 \log(\frac{1}{\varepsilon})}} & \text{if } \varepsilon^{\frac{1}{4}} \leq |x| \leq 1, \\ \frac{(-|x|+1)(|x|-2)^2}{\sqrt{2\pi^2 \log(\frac{1}{\varepsilon})}} & \text{if } 1 < |x| < 2, \\ 0 & \text{if } |x| \geq 2. \end{cases} \]

Clearly $u_\varepsilon(x) := v_\varepsilon(2x) \in H^2_0(B)$. Moreover, for small $\varepsilon > 0$, we have
\[ \|u_\varepsilon\|_{L^\infty(B)} = \|v_\varepsilon\|_{L^\infty(B_2)} = \sqrt{\frac{1}{32\pi^2} \log(\frac{1}{\varepsilon})} + \frac{1}{\sqrt{8\pi^2 \log(\frac{1}{\varepsilon})}} \]
and
\[ \|u_\varepsilon\|_{\text{Lip}(B)} = 2\|v_\varepsilon\|_{\text{Lip}(B_2)} \leq \frac{2}{\pi \sqrt{2\varepsilon^{\frac{1}{4}} \log(\frac{1}{\varepsilon})}}. \]

Since $\|u_\varepsilon\|_{C^\alpha} \leq \|u_\varepsilon\|_{L^\infty}^{1-\alpha} \|u_\varepsilon\|_{\text{Lip}}^\alpha$, we get
\[ \|u_\varepsilon\|_{C^\alpha} \leq C_\alpha \log(\frac{1}{\varepsilon})^{\frac{1}{2} - \alpha}. \]

Using the fact that $\|\Delta u_\varepsilon\|_{L^2(B)}^2 = \|\Delta v_\varepsilon\|_{L^2(B_2)}^2 = 1 + O(1/\log(1/\varepsilon))$, we have
\[ N_\alpha(u_\varepsilon) \leq C_\alpha \log(\frac{1}{\varepsilon})^{\frac{1}{2} - \alpha}. \]

So, for $C_0 > 0$,
\[ \lim_{\varepsilon \to 0} \frac{\|\Delta u_\varepsilon\|_{L^2}^2 \log \left[ e^3 + C_0 N_\alpha(u_\varepsilon) \sqrt{\log(2e + N_\alpha(u_\varepsilon))} \right]}{\|u_\varepsilon\|_\infty^2} \leq 8\pi^2 \alpha. \]
Finally
\[
\inf_{u \in (H^2_{0,\text{rad}} \cap C^\infty)(B)} \frac{\|\Delta u\|_{L^2}^2 \log \left[ e^3 \sqrt{C_0 N(u)} \sqrt{\log(2e + N(u))} \right]}{\|u\|_{L^\infty}^2} \leq 8\pi^2 \alpha.
\]

Let us prove the opposite inequality. Without loss of generality we can normalize \(\|u\|_{L^\infty} = 1\). Moreover, using a translation argument we may assume that \(u(0) = 1\). Since \(u\) vanishes on the boundary, we deduce that
\[
\|u\|_{C^\alpha} \geq \left| u\left(\frac{x}{|x|}\right) - u(0) \right| = 1.
\]

Moreover, if \(\|u\|_{C^\alpha} = 1\) then \(u(x) = 1 - |x|^\alpha\) and the inequality is evident. In fact \(1 - u(x) = |u(x) - 1| \leq |x|^\alpha\) thus \(u(x) \geq 1 - |x|^\alpha\), moreover if \(u(x_0) > 1 - |x_0|^\alpha\) then \(\frac{|u(x_0) - u(1)|}{1 - |x_0|^\alpha} > 1\) and \(\|u\|_{C^\alpha} > 1\), which is absurd. In the sequel we assume that \(\|u\|_{C^\alpha} > 1\).

For \(D > 1\), we denote the space
(3.7) \[ K_D := \{ u \in H^2_{0,\text{rad}}(B), \quad u(r) \geq 1 - Dr^\alpha, \quad \text{for any} \quad 0 < r \leq 1 \}. \]

It is sufficient to prove that for some \(C_\alpha > 0\), we have
\[
8\pi^2 \alpha \leq \inf_{D \geq 1} \inf_{u \in K_D} \|\Delta u\|_{L^2}^2 \log \left[ e^3 + \frac{C_\alpha D}{\|\Delta u\|_{L^2}} \sqrt{\log(2e + \frac{D}{\|u\|_{L^2}}} \right].
\]

Consider the minimizing problem
(3.8) \[ I[u] := \|\Delta u\|_{L^2}^2(B) \]

among the functions belonging to the set \(K_D\). This is a variational problem with obstacle. It has a unique minimizer \(u^*\) which is variationally characterized by
(3.9) \[ \int_B \Delta v \Delta u^* \geq \|\Delta v\|_{L^2(B)}^2, \quad \forall v \in K_D. \]

Moreover \(u^* \in W^{3,\infty}(B)\), (see [12]). Hence we have an open radially symmetric set \(\mathcal{O} := \{ x \in B, u^*(x) > 1 - D|x|^\alpha \}\).

Now, for any \(v \in C_0^\infty(\mathcal{O})\) and any real number \(|\tau|\) small enough, we have \(u^* + \tau v \geq 1 - Dr^\alpha\) thus \(u^* + \tau v \in K_D\). So by (3.9) we have
\[
\int_B \Delta (u^* + \tau v) \Delta u^* \geq \|\Delta u^*\|_{L^2(B)}^2.
\]

Taking \(\tau\) positive then negative and \(v^* := \Delta u^*\), we have
\[
\int_{\mathcal{O}} v^* \Delta v = 0, \quad \forall v \in C_0^\infty(\mathcal{O}).
\]

Thus \(u^*\) is biharmonic on \(\mathcal{O}\),
\[
\frac{1}{r^3} \frac{d}{dr} (r^3 \frac{dv^*}{dr}) = \Delta v^* = \Delta^2 u^* = 0 \quad \text{and} \quad u^* \in C^\infty(\mathcal{O}).
\]

So, there exists two real numbers \(a\) and \(b\) such that
\[
v^*(r) = \frac{1}{r^3} \frac{d}{dr} (r^3 \frac{du^*}{dr}) = \frac{a}{r^2} + b.
\]
With a straightforward computation, and using the boundary condition, there exists a real number $c$ such that

$$u^*(r) = -b - c + \frac{c}{r^2} + a \log(r) + br^2.$$ 

Now, by the boundary condition $\frac{du^*}{dr}(1) = 0$, we have

$$u^*(r) = -b - c + \frac{c}{r^2} + 2(c - b) \log(r) + br^2.$$ 

Moreover, $u^*$ cannot start to be biharmonic at $r = 0$ because of boundary condition. So there exists a real number $r_0 \in (0, 1)$ such that

$$u^*(r) = \begin{cases} 
1 - Dr^\alpha & \text{if } 0 \leq r \leq r_0, 

b(r^2 - 1) + c\left(\frac{1}{r^2} - 1\right) + 2(c - b) \log(r) & \text{if } r_0 < r \leq 1.
\end{cases}$$

Since $u^* \in C^2(B)$, we have

$$\begin{align*}
1 - Dr^\alpha &= (r^2 - 1 - 2 \log(r))b + \left(\frac{1}{r_0^2} - 1 + 2 \log(r_0)\right)c, \\
-\alpha Dr^\alpha &= 2(r_0 - \frac{1}{r_0})b + \frac{2c}{r_0}(1 - \frac{1}{r_0}), \\
-\alpha(\alpha - 1)Dr^\alpha &= 2\left(1 + \frac{1}{r_0}\right)b + \frac{2c}{r_0}\left(\frac{3}{r_0} - 1\right). 
\end{align*}$$

We consider the two last equations

$$\begin{pmatrix}
-\alpha Dr^\alpha - 1 \\
-\alpha(\alpha - 1)Dr^\alpha - 2
\end{pmatrix} = \begin{pmatrix}
2(r_0 - \frac{1}{r_0}) \\
2\left(1 + \frac{1}{r_0}\right)
\end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = A \begin{pmatrix} b \\ c \end{pmatrix}.$$ 

Let $x := r_0^2$. With a simple computation, we obtain

$$\det(A) = -\frac{8}{r_0^2}(r_0^2 - 1)^2,$$

$$b = \frac{\alpha D}{4} \frac{x^{\frac{\alpha}{2}}}{1 - x} \left[\alpha + \frac{2}{1 - x}\right],$$

$$c = \frac{\alpha D}{4} \frac{x^{\frac{\alpha}{2} + 1}}{1 - x} \left[\alpha - 2 + \frac{2}{1 - x}\right].$$

Substituting in the first equation of the precedent system, we obtain

$$D(x) = \frac{4(x - 1)^2}{x^{\frac{\alpha}{2}}[4(x - 1)^2 + (\alpha(2 + \alpha)(1 - x))(x - 1 - \log(x)) + \alpha((\alpha - 2)(1 - x) - 2)(1 - x + x \log(x))]}.$$ 

Now, let us compute $\|\Delta u^*\|_{L^2(B)}^2$. Since

$$\Delta u^*(r) = \begin{cases} 
-\alpha D(\alpha + 2)r^{\alpha - 2} & \text{if } 0 \leq r \leq r_0, 

\frac{4(c - b)}{r^2} + 8b & \text{if } r_0 \leq r \leq 1,
\end{cases}$$

we obtain

$$\begin{align*}
\|\Delta u^*\|_{L^2(B)}^2 &= 2\pi^2 \left[(D(\alpha + 2))^2 \int_0^{r_0} r^{2\alpha - 2} \, dr + \int_{r_0}^{1} r^{3} \left(\frac{4(c - b)}{r^2} + 8b\right)^2 \, dr\right] \\
&= (\pi(\alpha + 2))^2 \alpha D^2 x^\alpha - 16\pi^2 (c - b)^2 \log(x) + 32\pi^2 b^2(1 - x^2) + 64\pi^2 b(c - b)(1 - x).
\end{align*}$$

It follows that

$$\|\Delta u^*\|_{L^2(B)}^2 = \pi^2 x^\alpha D^2 \left[\alpha(\alpha + 2)^2 - \alpha^2 \log \left(\frac{1 + 2 - (\alpha - 2)x^2}{1 - x^2}\right) + \frac{2\alpha^2}{(1 - x)^3}(\alpha + 2 - (\alpha - 2)x)((\alpha - 4)x^2 + 2x - (\alpha - 2))\right].$$
We denote
\[ g(x) = \pi^2 x^\alpha \left[ (\alpha + 2)^2 - \alpha^2 \log x \frac{(\alpha - 2)(\alpha - x)^2}{(1 - x)^2} + \frac{2\alpha^2}{(1 - x)}(\alpha + 2 - \alpha x)((\alpha - 4)x^2 + 2x - \alpha - 2) \right], \]
\[ D(x) = \frac{4}{x^2 \pi^2 \left[ 4(x - 1)^2 + \alpha(2 + \alpha(1 - x)) (x - 1 - \log x) + \alpha((\alpha - 2)(1 - x) - 2)(1 - x + x \log x) \right]}, \]
\[ F_C(x) := \|\Delta u^*\|_{L^2(B)}^2 \log \left[ e^3 + C_N(u^*) \sqrt{\log(2e + N_u(u^*))} \right], \]
\[ = D(x) g(x) \log \left[ e^3 + C \sqrt{\frac{\log(2e + 1/\sqrt{g(x)})}{g(x)}} \right]. \]

It is sufficient to prove that a constant \( C_\alpha \) exists such that
\[ F_{C_\alpha} \geq 8\pi^2 \alpha \quad \text{on} \quad (0, 1]. \]

We have
\[ g(x) \sim \pi^2 \alpha^2 (\alpha + 2)x^\alpha \log \left( \frac{1}{x} \right) \quad \text{and} \quad D(x) \sim \frac{4}{\alpha(2 + \alpha)x^2 \pi \log \left( \frac{1}{x} \right)}, \]
where \( \sim \) is used to indicate that the ratio of the two sides goes to 1 as \( x \) goes to zero. Thus
\[ F_C(x) \sim \left( \frac{4}{\alpha(2 + \alpha)x^2 \pi (-\log(x))} \right)^2 \pi^2 \alpha^2 (\alpha + 2)x^\alpha \log \left( \frac{1}{x} \right) \log(e^3 + \frac{C}{\sqrt{g(x)}}) \]
\[ \sim \frac{16\pi^2 \log \left( e^3 + \frac{C}{\sqrt{g(x)}} \right)}{\log \left( \frac{1}{x} \right)} \]
\[ \sim 8\pi^2 \alpha. \]

Consequently, there exists \( x_\alpha \in (0, 1) \) such that
\[ F_{C_\alpha}(x) \geq 8\pi^2 \alpha \quad \text{for all} \quad x \in [0, x_\alpha]. \]

Now, to study the behaviour of \( D(x) \) for \( x \to 1 \), we denote
\[ y := x - 1, \quad h(y) := 4y^2 + \alpha(2 - \alpha y)(y - \log(y + 1)) - \alpha((\alpha - 2)y + 2)(-y + (y + 1) \log(1 + y)). \]

An easy computation yields to
\[ h(y) = 4y^2 + \alpha(2 - \alpha y)(y - \log(y + 1)) - \alpha((\alpha - 2)y + 2)(-y + (y + 1) \log(1 + y)) \]
\[ = 4y^2 + \alpha(2 - \alpha y)(\frac{y^2}{2} + o(y^2)) - \alpha((\alpha - 2)y + 2)(\frac{y^2}{2} + o(y^2)) \]
\[ = 4y^2 + o(y^2), \quad \text{as} \quad y \to 0. \]

Hence,
\[ D(1^-) = 1 \quad \text{and} \quad \inf_{x_\alpha, 1} D = D(x_\alpha) > 0. \]

Moreover, \( g \geq 0 \) and \( g(x) \neq 0 \) for any \( x \in (0, 1] \) because if \( g(x) = 0 \) then \( u^* \) is harmonic on \( B \), which is absurd. Thus \( g \geq y_\alpha > 0, \frac{1}{g} \leq y_\alpha \) on \([x_\alpha, 1]\) and
\[ F_C(x) \geq D^2(x_\alpha) y_\alpha \log(e^3 + C \sqrt{y_\alpha}). \]

Taking \( C_\alpha = 1 + \frac{8\pi^2 \alpha}{C \sqrt{y_\alpha}} \), we have
\[ F_{C_\alpha}(x) \geq 8\pi^2 \alpha \quad \text{for all} \quad x \in [0, 1]. \]
Taking $C$ which is absurd. Thus $B$ have

$$\inf_{u \in (H^2_{0,\text{rad}} \cap C^\alpha)(B)} \frac{\|\Delta u\|_{L^2}^2 \log (C\lambda + N_\alpha(u))}{\|u\|_{L^\infty}^2} \geq \frac{1}{\lambda}.$$ 

Arguing as previously, it is sufficient to prove that for some $C_\lambda > 0$, we have

$$\frac{1}{\lambda} \leq \inf_{D \geq 1} \inf_{u \in K_D} \|\Delta u\|_{L^2}^2 \log \left( C + \frac{D}{\|\Delta u\|_{L^2}} \right),$$

where the set $K_D$ is already defined in (3.7). Since for all $C > 1$, the function

$$t \mapsto t^2 \log(C + \frac{1}{t})$$

is increasing, it is sufficient to minimize $I[u]$ among the functions belonging to the set $K_D$. Consider $u^*$ a such minimizer. Recall that with previous computations, we have

$$\|\Delta u^*\|_{L^2(B)}^2 = \pi^2 \alpha^2 D^2 \left[ \alpha(\alpha + 2)^2 - \alpha^2 \log x \frac{(\alpha + 2 - (\alpha - 2)x^2)^2}{(1-x)^4} + \frac{2\alpha^2}{(1-x)^3}((\alpha + 2 - \alpha)x((\alpha - 4)x^2 + 2x - \alpha - 2)) \right],$$

$$H(x) := \|\Delta u^*\|_{L^2(B)}^2 \log (C + N_\alpha(u^*)) = D^2(x)g(x) \log \left( C + \frac{1}{\sqrt{g(x)}} \right),$$

$$g(x) := \pi^2 \alpha^2 \left[ \alpha(\alpha + 2)^2 - \alpha^2 \log x \frac{(\alpha + 2 - (\alpha - 2)x^2)^2}{(1-x)^4} + \frac{2\alpha^2}{(1-x)^3}((\alpha + 2 - \alpha)x((\alpha - 4)x^2 + 2x - \alpha - 2)) \right].$$

$$D(x) = \frac{4(x-1)^2}{x^5 \left[ 4(x-1)^2 + \alpha(2 + \alpha(1-x))(x-1 - \log(x)) + \alpha((\alpha - 2)(1-x) - 2)(1-x + x \log(x)) \right]}.$$ 

Recall also that

$$g(x) \sim \pi^2 \alpha^2 (\alpha + 2)^2 x^\alpha \log \left( \frac{1}{x} \right), \quad D(x) \sim \frac{4}{\alpha(2 + \alpha)x^\frac{\alpha}{2} \log \left( \frac{1}{x} \right)} \quad \text{and} \quad H(x) \sim 8\pi^2 \alpha.$$ 

Therefore, there exists $x_\lambda \in (0, 1)$ such that

$$\lambda H(x) \geq 1 \quad \text{for all} \quad x \in [0, x_\lambda].$$

Moreover, via previous calculus

$$D(1^-) = 1 \quad \text{and} \quad \inf_{[x_\lambda, 1]} D = D(x^\lambda) > 0.$$ 

Note also that $g \geq 0$ and $g(x) \not\equiv 0, \forall x \in (0, 1]$ because if $g(x) = 0$ then $u^*$ is harmonic on $B$ which is absurd. Thus $g \geq g_\lambda > 0$ on $[x_\lambda, 1]$. So

$$\lambda H(x) \geq \lambda D^2(x^\lambda)g_\lambda \log(C\lambda).$$ 

Taking $C_\lambda = 1 + e^{\pi \alpha \sqrt{x_\lambda^\lambda/\lambda \lambda}},$ we have

$$\lambda H(x) \geq 1 \quad \text{for all} \quad x \in [0, 1].$$

Now, let us prove that (1.4) is false for $\lambda = \frac{1}{8\pi^2 \alpha}$ which means that it is sharp. Precisely, we show that a sequence of functions $u_n \in (H^2_{0,\text{rad}} \cap C^\alpha)(B)$ exists such that for $n$ big enough the following holds

$$\|u_n\|_{L^\infty}^2 > \frac{1}{8\pi^2 \alpha} \|\Delta u_n\|_{L^2}^2 \log \left( n^\frac{\alpha}{2} + N_\alpha(u_n) \right).$$
Similarly, a simple scaling argument in Theorem 1.4 yields
\[ b_n = \frac{\alpha D_n}{4} x_n^\beta [\alpha + \frac{2}{1 - x_n}], \]
\[ c_n = \frac{\alpha D_n}{4} x_n^{\beta + 1} [\alpha - 2 + \frac{2}{1 - x_n}]. \]

Using previous computations it is sufficient to prove that
\[ H_n := \|\Delta u_n\|_{L^2}^2 \log \left( n^\beta + N_\alpha(u_n) \right) \]
\[ = \|\Delta u_n\|_{L^2}^2 \log \left( n^\beta + \frac{D_n}{\|\Delta u_n\|_{L^2}} \right) \]
\[ = D_n^2 g_n \log \left( n^\beta + \frac{1}{\sqrt{g_n}} \right) < 8\pi^2 \alpha, \]
where \( g_n := g(x_n) \) and
\[ g(x) = \pi^2 x^\alpha \left[ \alpha(\alpha+2)-\alpha^2 \log x \left( \frac{\alpha + 2 - (\alpha - 2)x^2}{1 - x^4} \right) + \frac{2\alpha^2}{(1 - x)^3} \left( \alpha+2-\alpha x \right) \left( \alpha-4 \right) x^2 + 2x - \alpha - 2 \right]. \]

We have, for some sequence of positive real numbers \( \beta_n \) vanishing at infinity,
\[ g_n < \pi^2 \alpha^2 (2 + \alpha)^2 x_n^\alpha \log \left( \frac{1}{x_n} \right)(1 + \beta_n) \quad \text{and} \quad D_n \sim \frac{4}{\alpha(2 + \alpha)x_n^\beta \log \left( \frac{1}{x_n} \right)}. \]

Where \( \sim \) is used here to indicate that the ratio of the two sides goes to 1 when \( n \) goes to infinity. Thus, for some sequence \( \beta_n \) vanishing at infinity,
\[ H_n < D_n^2 \pi^2 \alpha^2 (2 + \alpha)^2 x_n^\alpha \log \left( \frac{1}{x_n} \right)(1 + \beta_n) \log \left( n^\beta + \frac{1}{\sqrt{2\alpha^2(2 + \alpha)^2 x_n^\alpha \log \left( \frac{1}{x_n} \right)(1 + \beta_n)}} \right) \]
\[ < \frac{16\pi^2}{\log \left( \frac{1}{x_n} \right)}(1 + \beta_n) \log \left( n^\beta + \frac{1}{\pi\alpha(2 + \alpha) \sqrt{x_n^\alpha \log \left( \frac{1}{x_n} \right)(1 + \beta_n)}} \right) \]
\[ < \frac{16\pi^2}{\log \left( \frac{1}{x_n} \right)}(1 + \beta_n) \left[ \frac{\alpha}{2} \log \left( \frac{1}{x_n} \right) + \log \left( n^\beta x_n^\beta + \frac{1}{\pi\alpha(2 + \alpha) \sqrt{\log \left( \frac{1}{x_n} \right)(1 + \beta_n)}} \right) \right] \]

To conclude, it is sufficient to take the limit as \( n \) goes to infinity.

5. CASE OF THE WHOLE SPACE

Theorems 1.3 and 1.4 were stated in the unit ball. If the function \( u \) is supported in a \( B_R \), a simple scaling argument gives
\[ \|u\|_{L^2(B_R)} \leq \frac{1}{8\pi^2 \alpha} \|\Delta u\|^2_{L^2(B_R)} \log \left[ e^3 + C_0 R^\alpha N_\alpha(u) \sqrt{\log(2e + R^\alpha N_\alpha(u))} \right]. \]

Similarly, a simple scaling argument in Theorem 1.4 yields
Corollary 5.1. (Log estimate) Let $\alpha \in (0, 1)$. For any $\lambda > \frac{1}{8\pi^2 \alpha}$ there exists $C_\lambda > 0$ such that for any $R > 0$ and any radial function $u \in (H^4 \cap C^\alpha)(B_R)$, we have

$$\|u\|_{L^\infty}^2 \leq \lambda \|\Delta u\|_{L^2}^2 \log \left(C_\lambda + R^\alpha N_2(u)\right).$$

(5.11)

Now, in the whole space we have the following result.

Corollary 5.2. (Global Log estimate) Let $\alpha \in (0, 1)$. For any $\lambda > \frac{1}{8\pi^2 \alpha}$ and any $\mu \in (0, 1]$, there exists $C_\lambda > 0$ such that for any radial function $u \in (H^2 \cap C^\alpha)(\mathbb{R}^4)$, we have

$$\|u\|_{L^\infty}^2 \leq \lambda \|u\|_{H^1}^2 \log \left(C_\lambda + \frac{8\alpha \mu^{-\alpha} \|u\|_{C^\alpha}}{\|\Delta u\|_{L^2}^2}\right),$$

(5.12)

where $\|u\|_{H^1}^2 := (1 + 3\mu)\|\Delta u\|_{L^2}^2 + 3\mu \|u\|_{H^1}^2$.

Proof. Let $\alpha \in (0, 1)$, $\lambda > \frac{1}{8\pi^2 \alpha}$, $\mu \in (0, 1]$ and a radial function $u \in (H^2 \cap C^\alpha)(\mathbb{R}^4)$. Fix a radially symmetric function $\phi \in C_0^\infty(B_1)$ such that $0 \leq \phi \leq 1$, $\phi = 0$ near zero and $|\nabla \phi| \leq 1, |\Delta \phi| \leq 1$. Let $\phi \mu := \phi(\frac{\mu}{2})$ and $u \mu := \phi \mu u$. Assume (without loss of generality) that $\|u\|_{L^\infty} = |u(0)|$. Then,

$$\|u \mu\|_{L^\infty} = \|u\|_{L^\infty} \quad \text{and} \quad \|u \mu\|_{C^\alpha} \leq \|u\|_{C^\alpha}.$$

Applying Corollary 5.1 we obtain

$$\|u\|_{L^\infty}^2 \leq \lambda \|\Delta u \mu\|_{L^2}^2 \log \left(C_\lambda + \frac{8\alpha \mu^{-\alpha} \|u\|_{C^\alpha}}{\|\Delta u \mu\|_{L^2}^2}\right).$$

Now,

$$\|\Delta u \mu\|_{L^2}^2 = \|\Delta \phi \mu u \|_{L^2}^2 + \|\Delta u \phi \mu \|_{L^2}^2 + 4\|\nabla \phi \mu \nabla u\|_{L^2}^2 + 2 \int \Delta \phi \mu u \phi \mu \Delta u + 4 \int \Delta \phi \mu u \nabla \phi \mu \nabla u + 4 \int \phi \mu \Delta u \nabla \phi \mu \nabla u$$

$$\leq \frac{\mu^4}{16} \|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \mu^2 \|\nabla u\|_{L^2}^2.$$

where

$$\begin{align*}
(I) &= \int \Delta \phi \mu u \phi \mu \Delta u \leq \frac{\mu^2}{8} (\|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2), \\
(II) &= \int \Delta \phi \mu u \nabla \phi \mu \nabla u \leq \frac{\mu^4}{16} (\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2), \\
(III) &= \int \phi \mu \Delta u \nabla \phi \mu \nabla u \leq \frac{\mu^2}{4} (\|\Delta u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2).
\end{align*}$$

The proof is achieved because $x \to x^2 \log(C_\lambda + \frac{\mu^2}{2})$, $C > 0$ is increasing. \hfill \Box

We also have the following result

Corollary 5.3. Let $\alpha \in (0, 1)$. For any $\lambda > \frac{1}{8\pi^2 \alpha}$, a constant $C_\lambda > 0$ exists such that for any radial function $u \in (H^2 \cap C^\alpha)(\mathbb{R}^4)$, we have

$$\|u\|_{L^\infty} \leq \|u\|_{L^2} + \|\Delta u\|_{L^2} \sqrt{\lambda \log \left(e + C_\lambda \frac{\|u\|_{C^\alpha}}{\|\Delta u\|_{L^2}}\right)}.$$
Proof. Take the Littlewood-Paley decomposition
\[ u = \Delta_{-1} u + \sum_{j \in \mathbb{N}} \Delta_j u := \Delta_{-1} u + v. \]
Then, applying the previous Corollary via \( \|v\|_{C^\alpha} \leq \|u\|_{C^\alpha} \), yields, for any \( \mu_1 \in [0, 1) \) and any \( \lambda_1 > \frac{1}{8\pi^2 \alpha} \),
\[
\|u\|_{L^\infty} \leq \|\Delta_{-1} u\|_{L^\infty} + \|v\|_{L^\infty} \leq \|u\|_{L^2} + \|v\|_{L^\infty} + \|v\|_{H^1} \sqrt{\lambda_1 \log \left( C\lambda_1 + \frac{8\alpha \mu_1}{\|v\|_{\mu_1}} \right)}. \]
Now, since \( \|v\|_{H^1} \leq C\|\Delta v\|_{L^2} \), we have
\[
\|v\|_{\mu_1}^2 := (1 + 3\mu_1)^2\|\Delta v\|_{L^2}^2 + 3\mu_1 \|v\|_{H^1}^2 \leq (1 + 3\mu_1(1 + C^2))\|\Delta v\|_{L^2}^2.
\]
To conclude the proof, we take \( \lambda_1 \) and \( \mu_1 \) such that \( \lambda > \lambda_1(1 + 3\mu_1)(1 + C^2) \).

Remark 5.4. Of course we have similar results for the log log inequality (1.3) in \( \mathbb{R}^4 \) with the sharp constant \( \frac{1}{8\pi^2 \alpha} \).

6. Appendix

In this section, following ideas of [12], we prove a regularity result of the minimizing function \( u^* \) of the problem (3.8). Recall some notations. Take the radial function \( \psi(r) := \psi_{D, \alpha}(r) = 1 - Dr^\alpha \) and the convex closed set
\[ K_D := \{v \in H^2_0, \text{rad}(B) \text{ s. th } v \geq \psi \text{ on } B\}. \]
Consider the minimizing problem \( I[u] := \|\Delta u\|_{L^2(B)}^2 \) among the functions belonging to the set \( K_D \). This is a variational problem with obstacle. It has a unique minimizer \( u^* \) which is variationally characterized by
\[
\int_B \Delta v \Delta u^* \geq \|\Delta v\|_{L^2(B)}^2, \quad \forall v \in K_D.
\]
We give the following regularity result.

Lemma 6.1. The minimizing function \( u^* \) of the problem (3.8) satisfies
\[ u^* \in (W^{4, p} \cap H^2_0)(B), \text{ for any } 1 \leq p \leq \frac{4}{4 - \alpha}. \]

The next result is known [3, 9].

Lemma 6.2. Consider the equation
\[ \Delta^2 u = f \text{ in } B, \quad \text{with } u_{|\partial B} = \Delta u_{|\partial B} = 0. \]
If \( f \in L^p(B) \) for some \( 1 < p < \infty \), then the previous equation has a unique strong solution \( u \in W^{4, p}(B) \) which satisfies the boundary condition in the trace sense, moreover
\[ \|u\|_{W^{4, p}(B)} \leq C_p \|f\|_{L^p(B)}. \]
Proof. Take for $\varepsilon > 0$ the function
\[
\theta_\varepsilon(t) := \begin{cases} 
1 & \text{if } t \leq 0, \\
1 - \frac{t}{\varepsilon} & \text{if } 0 \leq t \leq \varepsilon, \\
0 & \text{if } t \geq \varepsilon.
\end{cases}
\]
Clearly, the previous function is uniformly Lipschitz, non-increasing and satisfies $0 \leq \theta_\varepsilon \leq 1$. Let now the penalized problem
\[
\Delta^2 u_\varepsilon = \Delta^2 \psi \theta_\varepsilon(u_\varepsilon - \psi) \quad \text{on } B.
\]
Taking the operator on $H^2_0(B)$,
\[
< Lw, v > := \int_B \left( \Delta w \Delta v - \Delta^2 \psi \theta_\varepsilon(w - \psi)v \right) dx.
\]
We compute, using the fact that $\theta_\varepsilon$ is nonincreasing and $\Delta^2 \psi(r) = \alpha^2(4 - \alpha^2)Dr^\alpha - 2 \geq 0$,
\[
< Lw - Lv, w - v > = \int_B \left( [\Delta(w - v)]^2 - \Delta^2 \psi[\theta_\varepsilon(w - \psi) - \theta_\varepsilon(v - \psi)](w - v) \right) dx
\]
\[
\geq \int_B [\Delta(w - v)]^2 dx \geq C\|w - v\|^2_{H^2_0(B)}.
\]
Which implies that $L$ is strictly monotone and coercive. Moreover, if $w_n \rightharpoonup w$ in $H^2_0(B)$ then $Lw_n \rightharpoonup Lw$ weakly in $H^{-2}(B)$. Thus $L$ is continuous on finite dimensional subspaces of $H^2_0(B)$. Applying Corollary 1.8 of Chapter III in [12], we have the existence of a unique $u_\varepsilon \in H^2_0(B)$ satisfying (6.13). Furthermore, with Lemma 6.2
\[
\|u_\varepsilon\|_{W^{4,p}(B)} \leq C_p\|\Delta^2 \psi\|_{L^p(B)} \quad \text{for any } 1 < p < \frac{4}{4 - \alpha}.
\]
We claim that $u_\varepsilon \in K_D$, which is equivalent to prove that $\zeta = 0$, with $\zeta := u_\varepsilon - \max(u_\varepsilon, \psi) \leq 0$. Since
\[
\int_B \left( \Delta u_\varepsilon \Delta \zeta - \Delta^2 \psi \theta_\varepsilon(u_\varepsilon - \psi)\zeta \right) dx = 0,
\]
we have
\[
\int_B \Delta(u_\varepsilon - \psi) \Delta \zeta dx = \int_B \Delta^2 \psi(\theta_\varepsilon(u_\varepsilon - \psi) - 1)\zeta dx.
\]
Which implies that
\[
\int_B (\Delta \zeta)^2 dx = \int_{\zeta < 0} \Delta^2 \psi(\theta_\varepsilon(u_\varepsilon - \psi) - 1)\zeta dx.
\]
Now, $\zeta < 0$ implies that $u_\varepsilon - \psi < 0$ and $\theta_\varepsilon(u_\varepsilon - \psi) = 1$. Thus $\zeta = 0$ and $u_\varepsilon \in K_D$.
With (6.13), $u_\varepsilon \rightharpoonup \tilde{u}$ in $W^{4,p}(B)$ for any $1 < p < \frac{4}{4 - \alpha}$. Moreover, $\tilde{u} \in K_D$ because $u_\varepsilon \in K_D$.
Let prove that $\tilde{u}$ is solution to (3.8). Let $v \in K_D$ such that $v \geq \psi + \delta$ for some $\delta > 0$. Recall that $< L u_\varepsilon, v - u_\varepsilon > \geq 0$. Applying a Minty’s argument (see for example Lemma 1.5 of chapter III in [12]), yields $< L v, v - u_\varepsilon > \geq 0$. Which implies that
\[
\int_B \Delta v \Delta(v - u_\varepsilon) dx \geq \int_B \Delta^2 \psi \theta_\varepsilon(v - \psi)(v - u_\varepsilon) dx.
\]
If $\varepsilon < \delta$, then $\theta_\varepsilon(v - \psi) = 0$. So, taking $\varepsilon \to 0$ then $\delta \to 0$, we have
\[
\int_B \Delta v \Delta(v - \tilde{u}) dx \geq 0 \quad \text{for any } v \geq \psi.
\]
Applying a second time the same argument of Minty, we conclude that \( \tilde{u} = u^* \) is the solution to the minimizing problem \([6],[7]\).

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