Enumerating 3-generated axial algebras of Monster type

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Abstract

An axial algebra is a commutative non-associative algebra generated by axes, that is, primitive, semisimple idempotents whose eigenvectors multiply according to a certain fusion law. The Griess algebra, whose automorphism group is the Monster, is an example of an axial algebra. We say an axial algebra is of Monster type if it has the same fusion law as the Griess algebra.

The 2-generated axial algebras of Monster type, called Norton-Sakuma algebras, have been fully classified and are one of nine isomorphism types. In this paper, we enumerate and construct the 3-generated axial algebras of Monster type which do not contain a 5A, or 6A subalgebra.

1 Introduction

Axial algebras are a new class of non-associative algebra which were introduced by Hall, Rehren and Shpectorov in [5]. They axiomatise some key properties of VOAs and the Griess algebra. Frenkel, Lepowsky and Meurman constructed the Moonshine VOA $V^\natural$ whose automorphism group is the Monster $M$, the largest sporadic finite simple group, and has the Griess algebra as the weight 2 part. The rigorous theory of VOAs was developed by Borcherds [1] and it played a key role in his proof of the monstrous moonshine conjecture.

One of the key properties which axial algebras axiomatise was first observed in VOAs by Miyamoto [12]. He showed that you could associate involutory automorphisms $\tau_a$ of a VOA $V$, called Miyamoto involutions, to some conformal vectors $a$ in $V$ called Ising vectors [12]. Moreover, in the Moonshine VOA, $\frac{2}{3}$ is an idempotent in the Griess algebra, called a 2A-axis.

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An **axial algebra** is a commutative non-associative algebra which is generated by *axes*, that is, primitive semi-simple idempotents which decompose the algebra into a direct sum of eigenspaces. The eigenvectors with respect to an axis multiply according to a certain fusion law. We say that an axial algebra is of **Monster type** if it has the Monster fusion law (see Section 2 for details). In particular, the Griess algebra is an axial algebra of Monster type.

If the fusion law is $\mathbb{Z}_2$-graded, such as the Monster fusion law is, then to each axis $a$ we may associate an involutory algebra automorphism $\tau_a$ which we call a **Miyamoto involution**. The group generated by all such Miyamoto involutions is called the **Miyamoto group**. For the Griess algebra, the Miyamoto group is the Monster. In this way we generalise a key feature of VOAs and the Griess algebra.

Given an algebra, it is natural to ask what the $k$-generated subalgebras are. The 2-generated subalgebras of the Griess algebra were first studied by Norton [3]. He showed that the isomorphism class of the subalgebra is determined by the conjugacy class of the product $\tau_a \tau_b$ in the Monster, where $\tau_a$ and $\tau_b$ are the involutions associated to the axes $a$ and $b$ which generate the subalgebra. There are nine classes, labelled 1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A and 6A. Amazingly, Sakuma showed that the isomorphism type of the sub VOA in $V^\sharp$ generated by two Ising vectors is also determined by the conjugacy class and is one of the nine types. The result was extended to Majorana algebras (a special type of axial algebra) in [7] and to axial algebras of Monster type in [5]. These nine algebras are known as the **Norton-Sakuma algebras**.

In this paper, we turn our attention to the 3-generated axial algebras. However, we must be careful – the Griess algebra is 3-generated. So, there seems no hope of classifying all 3-generated axial algebras. Hence we restrict our attention to a subclass. We say that an axial algebra is a **$k$-algebra** if it contains only Norton-Sakuma subalgebras of type $nL$, where $n \leq k$. So all axial algebras of Monster type, in particular the Griess algebra, are 6-algebras. In this paper, we enumerate and construct the 4-algebras.

If $A$ is a $k$-algebra, then the product of any two Miyamoto involutions has order at most $k$. So, $A$ being a $k$-algebra implies that its Miyamoto group is a $k$-transposition group. All 3-generated 3-transposition groups are well known and they are quite small. Hence our choice to construct 4-algebras.

We note that the minimal 3-generated axial algebras were constructed in [9] (and also independently in unpublished work by the second author). The minimal 3-generated axial algebras are those which are 3-generated and all of whose subalgebras are 2-generated. Note that necessarily their Miyamoto groups are minimal 3-generated. In the slightly wider class of 3-generated axial algebras with a minimal 3-generated Miyamoto group, there are 161 cases to consider, of which 55 lead to non-trivial algebras.
We begin by constructing all the 3-generated 4-transposition groups up to similarity. Given such a list we may use the MAGMA implementation [211] of the algorithm in [10] to construct the algebras. For each group \( G \), we determine the possible actions on the axes. For each group and action, we then find all the possible configurations of Norton-Sakuma subalgebras, which we call the shape, and try to construct an algebra of that shape.

Our results can be found in Tables 4 and 5 on pages 21 and 25. We find that there are 31 possible actions of the Miyamoto group on a closed set of axes with over 11,000 possible shapes in total (compared to 161 for the minimal 3-generated axial algebras). However, for the vast majority of these, the algebra collapses, showing that there is no axial algebra of that shape. In fact, for 99% of all the possible shapes, the algebra is trivial. This poses the following questions:

**Problem.** Why do so many axial algebras collapse? Can we detect when they do collapse?

The non-trivial algebras that we do construct and also the cases we are not able to complete are given in Table 5. There are 45 non-trivial algebras and 56 shapes which we could not complete. It is likely that for some of these which we could not complete, there are multiple algebras of that shape.

We observe from our list that all the axial algebras we construct have a Frobenius form; that is, a bilinear form which associates with the algebra product. This adds weight to a conjecture in [10], that all axial algebras of Monster type have a Frobenius form. Moreover, we find that all the forms are positive definite, except for two which are positive semi-definite. For these two, we may quotient out by the radical of the form to obtain two more algebras.

Note that if an axial algebra (of Monster type) has a Miyamoto group which is a 2-group, then it is necessarily a 4-algebra. Surveying our list, we see that there are no axial algebras whose Miyamoto group is a 2-group of nilpotency class at least 2. (There are however some cases we could not complete.)

The structure of the paper is as follows. In Section 2 we recall the definition of axial algebras and give some key properties. We describe the shape of an algebra and give a brief overview of the construction algorithm used. Section 4 contains the details of how to calculate the 3-generated 4-transposition groups up to similarity. This allows us, in Section 4 to determine the possible configuration of axes. In Section 5 we show by hand that certain configurations of axes are forbidden, whilst others lead to well known algebras. Finally, in Section 6 we give the outcome of our computations, detailing the algebras we construct, those that cannot exist and those cases we could not complete.
2 Background

We will review the definition and some properties of axial algebras which were first introduced by Hall, Rehren and Shpectorov in [5]. We will pay particular attention to the motivating example coming from the Monster sporadic finite simple group.

**Definition 2.1.** Let $\mathbb{F}$ be a field, $\mathcal{F} \subseteq \mathbb{F}$ a subset, and $\star : \mathcal{F} \times \mathcal{F} \to 2^\mathbb{F}$ a symmetric binary operation. We call the pair $(\mathcal{F}, \star)$ a fusion law over $\mathbb{F}$. A single instance $\lambda \star \mu$ is called a fusion rule.

Abusing notation, we will often just write $\mathcal{F}$ for $(\mathcal{F}, \star)$. We can also extend the operation $\star$ to subsets $I \subseteq \mathcal{F}$ in the obvious way.

Let $A$ be a non-associative (i.e. not-necessarily-associative) commutative algebra over $\mathbb{F}$. We will write $\langle\langle Y \rangle\rangle$ for the subalgebra generated by the set $Y$ of elements of $A$ and we say $A$ is $k$-generated if $A = \langle\langle Y \rangle\rangle$ and $|Y| = k$. For an element $a \in A$, the adjoint endomorphism $\text{ad}_a : A \to A$ is defined by $\text{ad}_a(v) := av$, $\forall v \in A$. Let $\text{Spec}(a)$ be the set of eigenvalues of $\text{ad}_a$, and for $\lambda \in \text{Spec}(a)$, let $A_\lambda(a)$ be the $\lambda$-eigenspace of $\text{ad}_a$. Where the context is clear, we will write $A_\lambda$ for $A_\lambda(a)$. We will also adopt the convention that for subsets $I \subseteq \mathcal{F}$, $A_I := \bigoplus_{\lambda \in I} A_\lambda$.

**Definition 2.2.** Let $(\mathcal{F}, \star)$ be a fusion law over $\mathbb{F}$. An element $a \in A$ is an $\mathcal{F}$-axis if the following hold:

1. $a$ is idempotent (i.e. $a^2 = a$);
2. $a$ is semisimple (i.e. the adjoint $\text{ad}_a$ is diagonalisable);
3. $\text{Spec}(a) \subseteq \mathcal{F}$ and $A_\lambda A_\mu \subseteq A_{\lambda \star \mu}$, for all $\lambda, \mu \in \text{Spec}(a)$.

Furthermore, we say the axis $a$ is primitive if $A_1 = \langle a \rangle$.

**Definition 2.3.** An $\mathcal{F}$-axial algebra is a pair $(A, X)$ such that $A$ is a non-associative commutative algebra and $X$ is a set of $\mathcal{F}$-axes which generate $A$. An axial algebra is primitive if it is generated by primitive axes.

Although an axial algebra has a distinguished generating set $X$, we will abuse the above notation and just write $A$ for the pair $(A, X)$. Where the fusion law is clear from context, we will drop the $\mathcal{F}$ and simply use the term axial algebra. Note that it has been usual in the literature to drop the adjective primitive and consider only primitive axial algebras.

This paper is focused on axial algebras with the Monster fusion law which is defined over $\mathbb{R}$ and is given in Table 1.

The so-called 2A-axes in the Griess algebra satisfy the Monster fusion law. Indeed, noting that these generate the Griess algebra, shows that it is an axial algebra. We say that an axial algebra is of Monster type if it is an axial algebra with the Monster fusion law.
Definition 2.4. A Frobenius form on an axial algebra $A$ is a (symmetric) bilinear form $(\cdot, \cdot) : A \times A \to \mathbb{F}$ such that the form associates with the algebra product. That is, for all $x, y, z \in A$,

$$(x, yz) = (xy, z).$$

Note that an associating bilinear form on an axial algebra is necessarily symmetric [5, Proposition 3.5]. Also, the eigenspaces for an axis in an axial algebra are perpendicular with respect to the Frobenius form. In [8, Proposition 4.5], it is shown that a Frobenius form is uniquely defined by its values on the axes $a \in X$. The Frobenius forms where $(a, a) \neq 0$, for all $a \in X$ are of particular interest. That is, those which are non-zero on the set of axes $X$.

2.1 Gradings and automorphisms

We are most interested in axial algebras where there is a group of automorphisms we can associate to the algebra in a natural way. The property which allows us to associate an automorphism to an axis is a grading.

In general, axial algebras can be graded by any abelian group $T$, but the axial algebras of Monster type, which we are particularly concerned about in this paper, have a $\mathbb{Z}_2$-grading. So here, we give a simplified version of the definition of a $\mathbb{Z}_2$-grading. For the more general $T$-grading see [8]. We will write $\mathbb{Z}_2$ as $\{+,-\}$ with the usual multiplication of signs.

Definition 2.5. The fusion law $\mathcal{F}$ is $\mathbb{Z}_2$-graded, if $\mathcal{F}$ has a partition $\mathcal{F} = \mathcal{F}_+ \cup \mathcal{F}_-$ such that

$$\mathcal{F}_s \star \mathcal{F}_t \subseteq \mathcal{F}_{st}$$

for all $s, t \in \mathbb{Z}_2$.

Note that, in the same way as we allow empty eigenspaces, we also allow empty parts in the partition in the above definition. Note that the Monster fusion law $\mathcal{M}$ is $\mathbb{Z}_2$-graded where $\mathcal{M}_+ = \{1, 0, \frac{1}{4}\}$ and $\mathcal{M}_- = \{\frac{1}{32}\}$. 

Table 1: Monster fusion law

|        | 0 | 1 | $\frac{1}{4}$ | $\frac{1}{32}$ |
|--------|---|---|---------------|---------------|
| 0      | 0 | 1 | $\frac{1}{4}$ | $\frac{1}{32}$ |
| $\frac{1}{4}$ | 1 | $\frac{1}{4}$ | $\frac{1}{32}$ |
| $\frac{1}{32}$ | $\frac{1}{32}$ | $\frac{1}{32}$ | $\frac{1}{32}$ | 1, 0, $\frac{1}{4}$ |
Let $A$ be an algebra and $a \in A$ an $F$-axis (we do not require $A$ to be an axial algebra). If $F$ is $\mathbb{Z}_2$-graded, then this induces a $\mathbb{Z}_2$-grading on $A$ with respect to the axis $a$. Here the $t$-graded subspace $A_t$ of $A$ is

$$A_t = A_{F_t} = \bigoplus_{\lambda \in F_t} A_\lambda$$

Suppose that $F$ is not of characteristic 2. Now, when $F$ is $\mathbb{Z}_2$-graded, this leads to automorphisms of the algebra. We define a map $\tau_a : A \to A$ by

$$v^{\tau_a} = \begin{cases} v & \text{if } v \in A_+ \\ -v & \text{if } v \in A_- \end{cases}$$

and extend linearly to $A$. Since $A$ is $\mathbb{Z}_2$-graded, this map $\tau_a$ is an automorphism of $A$, which we call the Miyamoto involution, associated to $a$. The following is an easy lemma.

**Lemma 2.6.** Suppose that $A$ is a $\mathbb{Z}_2$-graded axial algebra over a field $F$ of characteristic not 2. Let $a \in X$ and $g \in \text{Aut}(A)$. Then $a^g$ is another axis of $A$,

$$A_\lambda(a^g) = A_\lambda(a)^g$$

for all $\lambda \in F$ and $\tau_{a^g} = \tau_{a^g}$.

Since we have a (possibly) different automorphism for each axis $a$, this gives us a larger group of automorphisms.

**Definition 2.7.** Let $A$ be a $\mathbb{Z}_2$-graded axial algebra over a field $F$ of characteristic not 2. Then, the Miyamoto group is the group

$$G(X) := \langle \tau_a : a \in X \rangle$$

We may also abuse the above definition and consider the group $G(Y) := \langle \tau_a : a \in Y \rangle$ generated by the Miyamoto involutions associated with the axes is a subset $Y \subseteq X$. For a subset $Y \subseteq X$ of axes, we define $\bar{Y} = Y_{G(Y)}$. It turns out that $G(\bar{Y}) = G(Y)$ and so $\bar{Y}^{G(Y)} = \bar{Y}$. We call $\bar{Y}$ the closure of $Y$ and we say that $Y$ is closed if $Y = \bar{Y}$. In \cite{X}, it is also shown that $\langle \langle X \rangle \rangle = \langle \langle \bar{X} \rangle \rangle$. In this paper, we will normally assume that the set $X$ of axes is closed as we can always enlarge $X$ to $\bar{X}$ without changing the algebra, or Miyamoto group.

**Lemma 2.8.** Let $A$ be an axial algebra of Monster type with Miyamoto group $G$ and $a$ an axis of $A$. Then, $\tau_a \in Z(G_a)$.

**Proof.** Let $g \in G_a$. Then, $\tau_{a^g} = \tau_{a^g} = \tau_a$ and so $\tau_a \in Z(G_a)$. \qed
| Type | Basis | Products & form |
|------|-------|----------------|
| 2A   | $a_0, a_1, a_\rho$ | $a_0 \cdot a_1 = \frac{1}{3}(a_0 + a_1 - a_\rho)$, $a_0 \cdot a_\rho = \frac{1}{3}(a_0 + a_\rho - a_1)$, $(a_0, a_1) = (a_0, a_\rho) = (a_1, a_\rho) = \frac{1}{3}$ |
| 2B   | $a_0, a_1$ | $a_0 \cdot a_1 = 0$, $(a_0, a_1) = 0$ |
| 3A   | $a_{-1}, a_0, a_1, u_\rho$ | $a_0 \cdot a_1 = \frac{1}{3}(2a_0 + 2a_1 - a_{-1}) - \frac{\sqrt{3}}{2}u_\rho$ |
|      | $a_{-1}, a_0, a_1, v_\rho$ | $a_0 \cdot v_\rho = \frac{1}{3}(2a_0 - a_1 - a_{-1}) + \frac{\sqrt{3}}{2}v_\rho$, $v_\rho \cdot u_\rho = u_\rho$, $(u_\rho, u_\rho) = \frac{\sqrt{3}}{2}$, $(a_0, a_1) = \frac{1}{3}$, $(a_0, u_\rho) = \frac{1}{3}$ |
| 3C   | $a_{-1}, a_0, a_1$ | $a_0 \cdot a_1 = \frac{1}{3}(a_0 + a_1 - a_{-1})$, $(a_0, a_1) = \frac{1}{3}$ |
| 4A   | $a_{-1}, a_0, a_1, a_2, v_\rho$ | $a_0 \cdot a_1 = \frac{1}{3}(3a_0 + 3a_1 - a_{-1} - a_2 - 3v_\rho)$ |
|      | $a_{-1}, a_0, a_1, a_2, v_\rho$ | $a_0 \cdot v_\rho = \frac{1}{3}(5a_0 - 2a_1 - a_2 - 2a_{-1} + 3v_\rho)$ |
|      | $v_\rho \cdot v_\rho = v_\rho$, $a_0 \cdot a_2 = 0$, $(a_0, a_1) = \frac{1}{3}$, $(a_0, a_2) = 0$, $(a_0, v_\rho) = \frac{2}{3}$, $(v_\rho, v_\rho) = 2$ |
| 4B   | $a_{-1}, a_0, a_1, a_2, a_\rho, a_{\rho^2}$ | $a_0 \cdot a_1 = \frac{1}{3}(a_0 + a_1 - a_{-1} - a_2 - a_{\rho^2})$, $(a_0, a_1) = \frac{1}{3}$, $(a_0, a_2) = (a_0, a_{\rho^2}) = \frac{1}{3}$ |
| 5A   | $a_{-2}, a_{-1}, a_0, a_1, a_2, w_\rho$ | $a_0 \cdot a_1 = \frac{1}{3}(3a_0 + 3a_1 - a_2 - a_{-1} - a_{-2}) + w_\rho$, $a_0 \cdot a_2 = \frac{1}{3}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$ |
|      | $a_{-2}, a_{-1}, a_0, a_1, a_2, w_\rho$ | $a_0 \cdot w_\rho = \frac{1}{3}(a_1 - a_{-1} - a_2 - a_{-2}) + \frac{\sqrt{3}}{2}w_\rho$, $w_\rho \cdot w_\rho = \frac{3}{2w_\rho}(a_{-2} + a_{-1} + a_0 + a_1 + a_2)$ |
|      | $w_\rho \cdot w_\rho = w_\rho$, $a_0 \cdot a_2 = 0$, $(a_0, a_1) = \frac{1}{3}$, $(a_0, w_\rho) = 0$, $(w_\rho, w_\rho) = \frac{3}{2w_\rho}$ |
| 6A   | $a_{-2}, a_{-1}, a_0, a_1, a_2, a_{\rho^3}, u_\rho, a_\rho^2$ | $a_0 \cdot a_1 = \frac{1}{3}(a_0 + a_1 - a_{-2} - a_{-1} - a_2 - a_3 - a_{\rho^3}) + \frac{\sqrt{3}}{2}u_\rho^2$, $a_0 \cdot a_2 = \frac{1}{3}(2a_0 + 2a_2 + a_{-2}) - \frac{\sqrt{3}}{2}u_\rho^2$, $a_0 \cdot u_\rho = \frac{1}{3}(2a_0 - a_2 + a_{-2}) + \frac{\sqrt{3}}{2}u_\rho^2$, $a_0 \cdot u_{\rho^2} = \frac{1}{3}(a_0 + a_3 - a_{\rho^2})$, $a_{\rho^2} \cdot u_{\rho^2} = 0$, $(a_\rho^3, u_{\rho^2}) = 0$, $(a_0, a_1) = \frac{1}{3}$, $(a_0, a_2) = \frac{1}{3}$, $(a_0, a_3) = \frac{1}{3}$ |

Table 2: Norton-Sakuma algebras
2.2 Norton-Sakuma algebras

Since the defining property of axial algebras is that they are generated by a set of axes, it is natural to ask: What are the axial algebras which are generated by just two axes? We call such axial algebras 2-generated.

In the Griess algebra, the 2-generated subalgebras, called Norton-Sakuma algebras, were investigated by Norton and shown to be one of nine different types \[3\]. In particular, for each pair of axes \(a_0, a_1\) in the Griess algebra, the isomorphism class of the subalgebra which they generate is determined by the conjugacy class in the Monster of the product \(\tau_{a_0}\tau_{a_1}\) of the two involutions \(\tau_{a_0}\) and \(\tau_{a_1}\) associated to the axes. The nine different type are: 1A (when \(a_0 = a_1\)), 2A, 2B, 3A, 3C, 4A, 4B, 5A and 6A.

The algebra 1A is just one dimensional, but the remaining eight Norton-Sakuma algebras are given in Table 2 whose content we will now explain. Let \(nL\) be one of the 2-generated algebras. Since its generating axes \(a_0\) and \(a_1\) give involutions \(\tau_{a_0}\) and \(\tau_{a_1}\) in the Monster, we have the dihedral group \(D_{2n} \cong \langle \tau_{a_0}, \tau_{a_1} \rangle\) acting as automorphisms of \(nL\) (possibly with a kernel). In particular, let \(\rho = \tau_{a_0}\tau_{a_1}\). We define

\[a_{\varepsilon + 2k} = a_{\varepsilon}^{\rho^k}\]

for \(\varepsilon = 0, 1\). It is clear that these \(a_i\) are all axes as they are conjugates of \(a_0\) or \(a_1\). The orbits of \(a_0\) and \(a_1\) under the action of \(\rho\) (in fact, under the action of \(D_{2n}\)) have the same size. If \(n\) is even, then these two orbits have size \(\frac{n}{2}\) and are disjoint and if \(n\) is odd, then the orbits coincide and have size \(n\). We define the map \(\tau\) by extending \(\tau_{a_0}\) and \(\tau_{a_1}\) using \(\tau^g = \tau_{a_0^g}\) for all \(g \in \text{Aut}(nL)\). In almost all cases, the axes \(a_i\) are not enough to span the algebra. We index the additional basis elements by powers of \(\rho\). Using the action of \(D_{2n}\), it is enough to just give the products in Table 2 to fully describe each algebra. The axes in each algebra are primitive and each algebra admits a Frobenius form that is non-zero on the set of axes; the values for this are also listed in the table.

Amazingly the classification of 2-generated algebras also holds, and is known as Sakuma’s theorem \[13\], if we replace the Griess algebra by the weight two subspace \(V_2\) of a vertex operator algebra (VOA) \(V = \bigoplus_{n=0}^{\infty} V_n\) over \(\mathbb{R}\) where \(V_0 = \mathbb{R}1\) and \(V_1 = 0\) (those of OZ-type). After Majorana algebras were defined generalising such VOAs, the result was reproved for Majorana algebras by Ivanov, Pasechnik, Seress and Shpectorov in \[7\]. In the paper introducing axial algebras, the result was also shown to hold in axial algebras of Monster type over a field of characteristic 0 which have a Frobenius form \[5\]. It is conjectured that the Frobenius form is not required.

**Conjecture 2.9.** \[1\] A 2-generated axial algebra of Monster type over a field of characteristic 0 is one of the nine Norton-Sakuma algebras.

\[1\]A proof of this conjecture was recently announced by Franchi, Mainardis and Shpecttorov at the Axial Algebra Focused Workshop in Bristol in May 2018.
Since the 2-generated axial subalgebras of an axial algebra \( A \) are just the Norton-Sakuma algebras, we have the following lemma. First, fix notation by defining \( D_{a,b} \) to be the dihedral group generated by \( \tau_a \) and \( \tau_b \) for axes \( a, b \in X \). Define \( X_{a,b} = a^D \cup b^D \). It is clear that \( D_{a,b} = D_{b,a} \) and \( X_{a,b} = X_{b,a} \).

**Lemma 2.10.** Let \( A \) be an axial algebra of Monster type, \( a, b \in X \) and \( D := D_{a,b} \). Then we have the following:

1. \( k := |a^D| = |b^D| \).
2. If \( a \) and \( b \) are in the same orbit, then \( k = 1, 3, \) or \( 5 \).
3. If \( a \) and \( b \) are in different orbits, then \( k = 1, 2, \) or \( 3 \).

Moreover, the Norton-Sakuma algebra generated by \( a \) and \( b \) has type \( nL \), where \( n = |X_{a,b}| \).

**Proof.** A direct proof would be long and computational. So instead we observe that each Norton-Sakuma algebra is contained in the Griess algebra and there we have a bijection between axes and 2A-involutions in the Monster. So, we may take the dihedral subgroup \( H \leq M \) generated by the involutions associated to each axis (in the Griess algebra). In particular, up to the kernel, the action of \( H \) on \( X \) is the same as the action of \( D \) on \( X \).

Since in the Griess algebra we have a bijection between axes and 2A-involutions and \( \tau_x^g = \tau_{x^g} \) for \( g \in H \), we may consider the orbits of involutions in \( H \) rather than the orbits of axes. The result now follows from properties of dihedral groups.

### 2.3 Shapes

In this section, we will give a brief description of the configuration of the Norton-Sakuma subalgebras of an axial algebra, which we call the shape. For a more full discussion, see [10]. Throughout this section, let \( G \) be the Miyamoto group of a \( \mathbb{Z}_2 \)-graded axial algebra \( A \) of Monster type.

Since \( A \) is spanned by products of its axes, we see that \( G \) acts faithfully on the set of axes \( X \). We will consider \( \tau \) to be a map from \( X \) to \( G \), where \( \tau_a \) is the Miyamoto involution associated to \( a \in X \).

Since \( G \) is a group of automorphisms of \( A \), the subalgebra \( B \) generated by \( a, b \in X \) is isomorphic to the subalgebra generated by \( a^g \) and \( b^g \), for all \( g \in G \). So we say that the shape of an algebra is a map from the set of \( G \)-orbits of \( X \times X \) to the set of Norton-Sakuma algebras. Given that the shape of \( \langle\langle a, a \rangle\rangle \) must be 1A, we may ignore the diagonal. We note that there are restrictions on the possible maps which can be the shape of an algebra.

A Norton-Sakuma algebra has type \( nL \). By Lemma 2.10, \( n \) is uniquely determined by the action of the group \( D_{a,b} = \langle \tau_a, \tau_b \rangle \) on \( a \) and \( b \).
We now consider what the possible L can be. If \( a, b, c, d \in X \), then we say \( a, b \) dominates \( c, d \) if \( c, d \in X_{a,b} \). In particular, when this happens, 
\[ X_{c,d} \subseteq X_{a,b} \text{ and } D_{c,d} \leq D_{a,b}. \]
If \( a, b \) dominates \( c, d \), then the choice of 2-generated subalgebra for \( a, b \) determines the choice for \( c, d \). In particular, for the Norton-Sakuma algebras we have the following inclusions:

\[
\begin{array}{c|c}
\langle a, b \rangle & \langle c, d \rangle \\
 4A & 2B \\
 4B & 2A \\
 6A & 3A \\
\end{array}
\]

We also note here that each smaller algebra is contained in exactly one over algebra of a given \( n \). So in fact, if \( a, b \) dominates \( c, d \), not only does the choice of Norton-Sakuma subalgebra for \( a, b \) determine the choice for \( c, d \), the choice for \( c, d \) also determines the choice for \( a, b \).

**Definition 2.11.** The *shape graph* is a directed graph on the \( G \)-orbits of 
\( X \times X - \{(x, x) : x \in X\} \) with edges given by domination.

By the above, there is at most one choice of 2-generated subalgebra for each weakly connected component (i.e. a connected component of the undirected graph). Sometimes there is no choice for a given component. Namely, when that component contains a 6A, or 5A. Note that we do not claim that the shape uniquely defines the algebra. Indeed there are examples of different algebras which have the same shape (we shall see such examples in Proposition 6.1). However it turns out that in many cases it does.

### 2.4 Construction algorithm

In [10], an algorithm is described for constructing an axial algebra of a given shape. In this paper, a MAGMA implementation [11, 2] of this algorithm is used to calculate the 3-generated axial algebras not containing any 5A, or 6A subalgebras. We give a brief description of the inputs needed to run the algorithm here.

Let \( G \) be a group which acts faithfully on a set \( X \). Our putative Miyamoto group is a (subgroup of) \( G \) and \( X \) will be our set of axes. It is clear that we may just consider the action up to isomorphisms of actions. We must now consider what the possible \( \tau \)-maps can be.

**Definition 2.12.** A map \( \tau : X \to G \) is called a \( \tau \)-map if for all \( x \in X \),
\[ g \in G \]
1. \( \tau_x^2 = 1 \)
2. \( \tau_x^g = \tau_{xg} \).
The group \( G_0 := \langle \tau_x : x \in X \rangle \leq G \) is called the Miyamoto group of \( \tau \).

We say that a \( \tau \)-map is **admissible** if it satisfies the conclusions of Lemma 2.10. That is, the orbits \( a^D \) and \( b^D \) of \( D = D_{a,b} \) have the same size being either 1, 3, 5, or 1, 2, 3 depending on if \( a \in b^D \), or \( a^D \) and \( b^D \) are disjoint. Since a non-admissible \( \tau \)-map cannot be the \( \tau \) coming from an axial algebra, from now on we only consider admissible \( \tau \)-maps.

The normaliser \( N = N_{\text{Sym}(X)}(G) \) of the action of \( G \) on \( X \) acts on the set of admissible \( \tau \)-maps by

\[
\tau_x \mapsto (\tau_{x^m-1})^m
\]

for \( m \in N, x \in X \). Note that \( G \) itself acts trivially on each \( \tau \), which means that the action of \( N/G \) on \( \tau \)-maps can be defined. Thus, we may just consider admissible \( \tau \)-maps up to the action of \( N/G \).

We define **domination** and the **shape graph** as in the previous section. As observed above, for the Monster fusion law, any one choice of Norton-Sakuma subalgebra for a weakly connected component of the shape graph determines all other Norton-Sakuma algebras in that component.

Given a group \( G \) acting faithfully on a set \( X \) and an admissible \( \tau \)-map \( \tau \), we may consider all the possible shapes. Let \( K = \text{Stab}_N(\tau) \). As noted above, \( G \) acts trivially on each \( \tau \), and in fact it also fixes every shape. On the other hand, \( K \) (or rather \( K/G \)) permutes the \( G \)-orbits of \( X \times X - \{(x,x) : x \in X\} \), and so may act non-trivially on the set of shapes. So, we may consider shapes up to the action of \( K \).

The construction algorithm in [10] takes as its input a group \( G \) acting faithfully on a set \( X \) and an admissible \( \tau \)-map \( \tau \), and a shape. Given such a \( G \), \( X \), \( \tau \), and shape, the algorithm builds an axial algebra \( A \) with axes \( X \) and Miyamoto group \( G_0 \).

Roughly speaking, the algorithm progresses by defining a vector space with partial algebra multiplication. We glue in subalgebras to cover each subalgebra in the shape. The algorithm has three main stages:

1. **Expansion** by adding the products of vectors we do not already know how to multiply.
2. **Work** to discover relations and construct the eigenspaces for the idempotents.
3. **Reduction** by factoring out by known relations.

We continue applying these three stages until all the algebra products are known and the algorithm terminates. Note that there is no guarantee that this process will finish, indeed if the algebra is not finite-dimensional it will not finish! However it does complete in many cases. If it does complete, then either the axial algebra \( A \) has collapsed to a 0-dimensional algebra, indicating that no axial algebra of the given shape exists, or an axial algebra \( A \) of the required shape is constructed.
3 Groups for 3-generated 4-algebras

Recall that an axial algebra is $m$-generated if it can be generated by a set of axes of size $m$.

By analogy with $k$-transposition groups, let us define $k$-algebras as axial algebras where any two axes generate a Norton-Sakuma subalgebra of type $nL$ with $n \leq k$. Then every axial algebra of Monster type is a 6-algebra.

In this paper we enumerate the class of 3-generated 4-algebras. Why are we taking this particular class? Sakuma’s Theorem provides a complete description of the 2-generated case. Hence the 3-generated case is the first one of interest. Our approach to the classification is via first finding the related groups. It immediately follows from Lemma 2.10 that groups associated with $k$-algebras are $k$-transposition groups. The 3-generated 6-transposition groups include, for instance, the Monster group, as well as many of its subgroups. So classifying such groups looks pretty hopeless. Hence we need to restrict $k$. On the opposite end, all 3-generated 3-transposition groups are well known and they are all quite small. We believe that considering 4-transposition groups is more challenging while still doable.

For the remainder of the paper, we fix the following notation. Let $A$ be a 4-algebra generated by axes $a$, $b$, and $c$. Then its Miyamoto group is $G = \langle \tau_a, \tau_b, \tau_c \rangle$ and we let $X = aG \cup bG \cup cG$. So, $X$ is a closed set of axes and $G$ acts faithfully on $X$ by permuting the axes. Since $A$ is a 3-generated 4-algebra, $G$ is a 3-generated 4-transposition group. Hence, we may begin by classifying all such possible permutation groups with this property.

3.1 3-generated 4-transposition groups

In this section we deal exclusively with groups, so it will be convenient for us to write $x, y$, and $z$ instead of $\tau_a, \tau_b$, and $\tau_c$. Note that we do not assume that the three conjugacy classes of $x, y$ and $z$ are pairwise distinct. We write $D := xG \cup yG \cup zG$ for the set of generating involutions.

We approach the problem via presentations. Clearly, every $t \in D$ satisfies $t^2 = 1$. Furthermore, for $t, s \in D$, we have either $(ts)^3 = 1$, or $(ts)^4 = 1$. Note that the last case includes the case where $(ts)^2 = 1$. Since we will later take all quotients to build a full list of groups, it suffices to just add relations of the form $(ts)^3 = 1$, or $(ts)^4 = 1$.

We begin by imposing the relations $(ts)^3 = 1$, or $(ts)^4 = 1$ for each distinct pair $\{s, t\} \subset \{x, y, z\}$. This gives four main cases. Since products of conjugates of the generators must also have the order at most 4, we may also add extra relations of the form $(ts^g)^3 = 1$, or $(ts^g)^4 = 1$. We do this to get the fourteen groups listed in Table 3 (Note that in this table we are omitting the relations $x^2$, $y^2$, and $z^2$.)

In particular, by using MAGMA, we can see that all the groups are finite. It is clear that every 3-generated group of 4-transpositions is a quotient of
Table 3: Possible 4-transposition groups

| $G_i$ | Case                  | Extra                  | Order | 4-trans |
|-------|-----------------------|------------------------|-------|---------|
| $G_1$ | $(xy)^3, (xz)^3, (yx)^3$ | $(xy)^3$               | 54    | y       |
| $G_2$ | $(xy)^4$              |                        | 96    | y       |
| $G_3$ | $(xy)^3, (xz)^3, (yz)^4$ | $(xy)^3$               | 96    | y       |
| $G_4$ | $(xy)^4$              |                        | 336   |        |
| $G_5$ | $(xy)^3, (xz)^4, (yz)^4$ | $(xy)^3, (xz)^4$       | 2     | y       |
| $G_6$ | $(xy)^3, (xz)^4$      |                        | 384   | y       |
| $G_7$ | $(xy)^4, (xz)^3$      |                        | 336   | y       |
| $G_8$ | $(xy)^4, (xz)^4$      |                        | 2304  | n       |
| $G_9$ | $(xy)^4, (xz)^4, (yz)^4$ | $(xy)^3, (xz)^4$       | 336   | y       |
| $G_{10}$ | $(xy)^3, (xz)^4$ |                        | 2304  | n       |
| $G_{11}$ | $(xy)^4, (xz)^4$ |                        | 2304  | n       |
| $G_{12}$ | $(xy)^4, (xz)^4, (yz)^3$ |                        | 2304  | n       |
| $G_{13}$ | $(xy)^4, (xz)^4, (yz)^4, (xx)^3$ | $(xy)^3, (xz)^4$       | 7776  | n       |
| $G_{14}$ | $(xy)^4, (xz)^4, (yz)^4, (xx)^4$ | $(xy)^3, (xz)^4$       | 32768 | n       |

one of the $G_i$.

**Lemma 3.1.** All 3-generated 4-transposition groups are finite.

However, we have still not added enough relations to force all the $G_i$ to be 4-transposition groups yet. This is indicated in the last column of the table. For the groups $G_i$ marked with ‘n’ we need extra relations to identify the largest quotients of $G_i$ that are 4-transposition groups. This can be done easily on the computer using **Magma** and we give brief details here of the relations needed.

For $G_8$, $zz^xy$ has order 6. Adding the relator $(zz^xy)^2$ produces a group $G'_8$ of order 8, which is a group of 4-transpositions. Similarly, adding $(zz^xy)^3$ produces $G''_8$ of order 1152, which is also a group of 4-transpositions.

Similarly for $G_{10}$, $zz^xy$ is of order 6, and this leads to quotients $G'_{10}$ and $G''_{10}$ of orders 8 and 1152 that are both 4-transposition groups.

For the group $G_{11}$, we instead find that $yy^xz$ has order 6. Adding the relators $(yy^xz)^2$ and $(yy^xz)^3$ leads to 4-transposition quotients $G'_{11}$ and $G''_{11}$ of orders 8 and 1152.

For $G_{12}$, the element $xx^yz$ has order 6, and we get two 4-transposition quotients $G'_{12}$ and $G''_{12}$ of orders 8 and 1152.

In the case of $G_{13}$, the element $yy^xz$ has order 6. Adding the extra relators $(yy^xz)^2$ and $(yy^xz)^3$ gives quotients $G'_{13}$ and $G''_{13}$ of orders 32 and 3888, which are both groups of 4-transpositions.
Finally, for $G_{14}$, both $yy^{xz}$ and $zz^{xy}$ have order 8. Adding the two extra relators $(yy^{xz})^4$ and $(zz^{xy})^4$ together gives a 4-transposition quotient $G'_{14}$ of order 8192.

This gives us a list of 19 finite finitely presented groups, $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8', G_8''$, $G_9, G_{10}', G_{10}'', G_{11}', G_{11}'', G_{12}', G_{12}'', G_{13}', G_{13}''$, and $G_{14}'$, such that any 3-generated 4-transposition group is a quotient of one of these groups.

Note that the relations on $x$, $y$, and $z$, in the group $G_{14}'$ are satisfied in every 3-generated 4-transposition group that is a 2-group. Hence all such groups are quotients of $G_{14}'$. This observation allows us to remove from our list the six groups: $G_5$, of order 2; $G_7$, $G_{10}'$, $G_{11}'$, and $G_{12}'$, of order 8; and $G_{13}'$, of order 32. Thirteen groups remain.

### 3.2 Similar groups

Using Magma, we find all quotients of the above 13 groups. However, this new list, surely, contains many of the same groups. Let us consider which groups give the same algebra.

Suppose that $x'$ is conjugate $x = \tau_a$, $y'$ to $y$, and $z'$ to $z$. Then, by Lemma 2.6, there is an axis $b' \in G'$ in the orbit $a^G$ such that $x' = \tau_{a'}$. Similarly, we find $b' \in b^G$ and $c' \in c^G$ such that $y' = \tau_{b'}$ and $z' = \tau_{c'}$. Suppose that $G' := \langle x', y', z' \rangle \cong G$. Then the algebra $A' := \langle \langle a', b', c' \rangle \rangle$ is invariant under the action of $G = G'$ and so contains $a, b, c$. Since $X$ is a closed set of axes and $G = G'$, we have $A' = A$.

Recall that a multiset is a set where we allow repeated elements. We make the following definition:

**Definition 3.2.** Two 3-generated groups $G$ and $G'$, viewed together with distinguished triples of generators $x, y, z$ and $x', y', z'$ respectively, are called similar if there is an isomorphism $\varphi : G \to G'$ such that the multiset $\{x^G, y^G, z^G\}$ coincides with the multiset $\{\varphi(x)^G, \varphi(y)^G, \varphi(z)^G\}$.

By the above argument, similar groups (considered with the same set of axes) will give isomorphic algebras. Note that if $G$ and $G'$ are similar via $\varphi$ then $G/N$ is similar to $G'/N'$, where $N \leq G$ and $N' = N\varphi$. Hence, we may consider our list of 13 groups up to similarity. This reduces our list to 7 groups. In fact, all groups of equal order among our 13 groups are similar; that is, $G_2$ and $G_3$ are similar; $G_4, G_7$, and $G_9$ are similar; and $G_8, G_{10}', G_{11}'$, and $G_{12}'$ are also similar. Using Magma to take all quotients of groups in this list up to similarity, we find 55 3-generated 4-transposition groups up to similarity.

Many of these groups will be ruled out in the next section where we compute possible actions of $G$ on the set of axes.
4 Configuration of axes

We now consider what the possible configurations of axes are. We give several results which will help us to reduce the number of cases to be considered. We begin by giving some general lemmas for an arbitrary axial algebra.

**Lemma 4.1.** Let $A$ be an axial algebra of Monster type and $a$ an axis of $A$. Then, $\tau_a = 1$ if and only if $a$ is fixed by the entire Miyamoto group $G$.

**Proof.** Let $b \in X$ be another axis of $A$. Suppose that $\tau_a = 1$. By Lemma 2.10 the orbits of $D = \langle \tau_a, \tau_b \rangle$ on $a$ and on $b$ have the same size. However, both $\tau_a = 1$ and $\tau_b$ fix $b$, so $|a^D| = |b^D| = 1$ and hence $\tau_b$ fixes $a$. Since this is true for every axis $b$ and $G$ is generated by the Miyamoto involutions, $G$ fixes $a$.

Conversely, suppose that $G$ fixes $a$. Then, for every axis $b \in X$, the orbit of $D = \langle \tau_a, \tau_b \rangle$ on $a$ has size one. Since $|a^D| = |b^D| = 1$, $\tau_a$ fixes every axis $b \in X$. However, $G$ acts faithfully on the axes, so $\tau_a = 1$.

We now consider the case where a Miyamoto involution $x$ is not the identity.

**Definition 4.2.** A non-trivial Miyamoto involution $x$ has the uniqueness property if there exists a unique axis $a \in X$ such that $x = \tau_a$. We say an axis $a \in X$ has the uniqueness property if $\tau_a$ has the uniqueness property.

It is easy to see that when $x$ has the uniqueness property, the stabilizer $G_a$ of $a$ in $G$ coincides with the centraliser $C_G(x)$ and, furthermore, there is a natural $G$-invariant bijection between the $G$-orbit $a^G$ and the conjugacy class $x^G$.

**Lemma 4.3.** Let $A$ be an axial algebra of Monster type and $a \in X$. If there exists $b \in X$ such that the order of $\tau_a \tau_b$ is 5 then $\tau_a$ has the uniqueness property. If $A$ has no subalgebras of type 6A and there exists $b \in X$ such that the order of $\tau_a \tau_b$ is 3 then $\tau_a$ has the uniqueness property.

**Proof.** Suppose $c \in X$ and $x := \tau_a = \tau_c$ and let $D = D_{a,b} = D_{c,b} = \langle \tau_a, \tau_b \rangle$. If the order of $\tau_a \tau_b$ is 5, then $\langle \langle a, b \rangle \rangle$ and $\langle \langle c, b \rangle \rangle$ must both be 5A algebras. Since these only have one orbit of axes under the action of $D$, $a$, $b$ and $c$ all lie in the same orbit of $D$ and so $\langle \langle a, b \rangle \rangle = \langle \langle c, b \rangle \rangle$. A simple computation in the 5A algebra shows that all the five axes have distinct Miyamoto involutions.

If $A$ contains no 6A subalgebras and the order of $\tau_a \tau_b$ is 3, then $\langle \langle a, b \rangle \rangle$ and $\langle \langle c, b \rangle \rangle$ are both algebras of type either 3A, or 3C. As above, these only have one orbit of axes under $D$ so the subalgebras are equal and again the involutions are distinct. In both cases, $c = a$ and so $\tau_a$ has the uniqueness property.

In this paper, we are particularly interested in the 4-algebra case where there are no 6A, or 5A subalgebras.
Corollary 4.4. If $A$ has no 6A subalgebras and the Miyamoto involution $x$ does not lie in $O_2(G)$ then $x$ has the uniqueness property.

Proof. Suppose that the order of the product $xy$ is even for all $y \in x^G$. Then, $(x,y)$ is nilpotent for all $y \in x^G$ and, by Baer’s Theorem, $x \in O_2(G)$, a contradiction. So, there exists some $y \neq x$ in $x^G$ such that the order of $xy$ is odd. Since these are Miyamoto involutions, this order is either 3, or 5. As we assume there are no 6A subalgebras, by Lemma 4.3, $x$ has the uniqueness property.

What can be said about the stabiliser $G_a$ of an axis when it does not have the uniqueness property? We introduce a property that is slightly weaker than uniqueness.

Definition 4.5. Let $x = \tau_a$ be a non-trivial Miyamoto involution. We say $x$ is strong if $x = \tau_a \neq \tau_b$ for any $b \in a^G$, $b \neq a$. We say an axis $a \in X$ is strong if $\tau_a$ is strong.

Clearly, if an axis $a$, or Miyamoto involution $x = \tau_a$ has the uniqueness property, then it is strong. We have the following easy lemma.

Lemma 4.6. Let $a$ be an axis with a non-trivial Miyamoto involution $x = \tau_a$. Then, the following are equivalent.

1. $a$ is strong.
2. $G_a = C_G(x)$.
3. There is a natural $G$-invariant bijection between $a^G$ and $x^G$.

What can be said about the stabiliser $G_a$ of an axis which is not unique, or strong? Clearly, $G_a$ is always contained in $C_G(x)$. However, a priori, it can be any subgroup of $C_G(x)$. We wish to have better control over the stabiliser. Note that, if we have no 6A subalgebras and $a$ is not unique, by Lemma 4.3, the order of $xy$ is 1, 2, 4 for all other Miyamoto involutions $y$.

Lemma 4.7. Let $x$ and $y$ be two Miyamoto involutions, where $x = \tau_a \neq 1$.

1. If the order of $xy$ is 2 and $y$ is strong then $y \in G_a$.
2. If the order of $xy$ is 4 then $[x,y] = (xy)^2 \in G_a$.

Proof. Let $D := \langle x, y \rangle$ and $b \in X$ such that $\tau_b = y$. Consider the action of $D$ on $B := \langle \langle a, b \rangle \rangle$, noting that $D$ may well act on $B$ with a non-trivial kernel.

If the order of $xy$ is 2, then $B$ is either a 2L, or a 4L dihedral algebra, where L can be either A, or B. If in addition $y$ is strong, then $b$ is the unique axis in its orbit with the Miyamoto involution $y$. Suppose $B \cong 4L$. Then, $D$ has two orbits of length 2 on the axes. In particular, $x = \tau_a$ conjugates
b to the other axis c in $b^D$. However, $x$ and $y$ commute, so $\tau_b = \tau_c = y$, contradicting the assumption that $y$ is strong. Hence, $B \cong 2L$ and $y$ fixes $a$.

If the order of $xy$ is 4, then $x$ and $y$ don’t commute and so the $D$-orbit containing $a$ cannot be of length 1. So it is of length 2 or 4. Again, looking at the list of dihedral algebras, it can never be length 4, so it must be length 2. Hence, $(xy)^2 \in G_a$ (in fact $B \cong 4L$ and $\langle (xy)^2 \rangle$ is the kernel of the action on $B$).

4.1 Configurations of axes for 3-generated 4-transposition groups

Using the results on unique and strong axes, we may now compute all the possible actions we must consider to enumerate the 3-generated 4-algebras. From Section 3, there are 55 groups to consider. Let $G = \langle x, y, z \rangle$ be one of these, where the generators $x$, $y$, and $z$ are the Miyamoto involutions corresponding to the axes $a$, $b$, and $c$ generating the 3-generated 4-algebra $A$. We also let $X = a^G \cup b^G \cup c^G$, but recall that we do not assume that the orbits are disjoint.

For each generator $u$ of $G$ corresponding to an axis $d$, we begin by finding the possible stabilisers $G_d$ of the axis $d$. If $u = 1$, then $G_a = G$. Otherwise, we use Lemma 4.3 and Corollary 4.4 to try to identify if $d$ has the uniqueness property. If it does, then $G_d = C_G(u)$. We find that for thirteen of the 55 groups, all the axes have the uniqueness property (including the trivial group), a further seven have two unique axes, three have one unique axis, while the remaining 32 have no unique axes.

For an axis $d$ without the uniqueness property, we use Lemma 4.7 to build the largest possible group $H$ such that $H \leq G_d \leq C_G(u)$. We find for several axes which didn’t initially have the uniqueness property in fact the only possible stabiliser is indeed the full centraliser $C_G(u)$. The largest index of $H$ in $C_G(u)$ is 8, which occurs just once. At this stage, 29 groups have $H = G_d = C_G(u)$ for all their axes, 11 have for 2 axes and 4 have for one axis.

Now, for every group $G$ and for each axis $d$, we have a list of possible stabilisers $G_d$. For a single axis $d$ and possible stabiliser $G_d$, the action of $G$ on the axis $d$ is isomorphic to the coset action of $G$ on $G_d$. In this way, for each axis $d$ and possible stabiliser $G_d$, we may construct an orbit $O_d$ of axes.

We consider the group $G$, together with three possible stabilisers $G_a$, $G_b$, $G_c$ of the three axes $a$, $b$ and $c$, respectively. We must be careful in building the possible sets of axes $X$ as we did not assume that the conjugacy classes of the involutions were distinct. If two axes, say $a$ and $b$, have Miyamoto involutions such that $x^G = y^G$ and $G_a$ is conjugate in $G$ to $G_b$, then the orbits $O_a$ and $O_b$ have isomorphic actions of $G$ on them. So, there are two possibilities to combine them. Either $a^G \cup b^G$ is the disjoint union of $O_a$ and
\( O_b \), or is equal to a single copy of \( O_a \cong O_b \).

For the disjoint union, we must additionally check that we haven’t inadvertently introduced a 6A subalgebra. Indeed, suppose that \( d, e' \in aG \) such that \( \langle \langle d, e' \rangle \rangle \sim 3L \). Let \( e \in bG \) be the corresponding axis to \( e' \); so \( \tau_e = \tau_{e'} \).

Now \( D = \langle \tau_d, \tau_e \rangle = \langle \tau_d, \tau_{e'} \rangle \) has an orbit of length 3 on \( d \) and an orbit of length 3 on \( e \). However, since \( d \in O_a \) and \( e \in O_b \) and these are disjoint, \( \langle \langle d, e \rangle \rangle \sim 6A \). So, if a disjoint union would result in such a 6A subalgebra, we discard this option.

Now, for every group \( G \) and all possible stabilisers \( G_a, G_b, G_c \) we build the possible sets of axes \( X \), considering joining two or three orbits where possible. If the resulting set of axes \( X \) generates an algebra which is in fact 2-generated, then it is one of the known Norton-Sakuma algebras and we may discard it. Otherwise we find all such sets of axes \( X \) with the action of the group \( G \) up to isomorphism of actions. There are 31 such actions and they are given in Table 5 on page 25. There we record the group, the size of the orbits on the axes and the number of possible shapes.

We note that for each action, the number of admissible \( \tau \)-maps is exactly one. That is, the \( \tau \)-map defined by \( \tau_a = x, \tau_b = y \) and \( \tau_c = z \) and extended by conjugation is the only admissible one.

## 5 Forbidden configurations of axes

In this section, we consider some specific configurations of axes for an arbitrary 3-generated axial algebra of Monster type and show that they lead to easy direct sums of axial algebras, the 6A Norton-Sakuma algebra, or collapse.

Let \( A \) be an axial algebra of Monster type generated by three axes \( a, b, c \). We further suppose that the Norton-Sakuma algebras generated by \( \{a, b\} \) is of type 2L, \( \{a, c\} \) is of type 2L' and \( \{b, c\} \) is of type 3K, or 5A, where L, L', K \( \in \{A, B\} \).

First of all, note that the Miyamoto involution \( \tau_a \) fixes \( a, b \) and \( c \), and hence it is trivial. Therefore, the Miyamoto group \( G \) of \( A \) is \( G = \langle \tau_b, \tau_c \rangle \), which is conjugate to either \( S_3 \) or \( D_{10} \). In both cases, the group conjugates \( b \) to \( c \) while fixing \( a \). Hence, \( \langle \langle a, b \rangle \rangle \cong \langle \langle a, c \rangle \rangle \) and so we can talk about the shape \( 3K2L \) or \( 5A2L \).

If \( 2L = 2B \) then it is easy to see that \( A \) is isomorphic to the direct sum algebra \( 3K \oplus 2B \) or \( 5A \oplus 2B \). Hence it is 5-dimensional if \( 3K = 3A \), 4-dimensional if \( 3K = 3C \), and 7-dimensional for \( 5A \).

**Theorem 5.1.** Let \( A = \langle \langle a, b, c \rangle \rangle \) be a 3-generated axial algebra of Monster type where \( \langle \langle a, b \rangle \rangle \cong \langle \langle a, c \rangle \rangle \cong 2A \) and \( \langle \langle b, c \rangle \rangle \) is one of \( 3A \), \( 3C \), or \( 5A \). Then, \( \langle \langle b, c \rangle \rangle \cong 3A \) and \( A \) is the 8-dimensional Norton-Sakuma algebra of type 6A. Consequently, there exists no algebra \( A \) of shape \( 3C2A \), or \( 5A2A \).
Proof. We begin by fixing some notation. The algebra $B = \langle\langle a, b \rangle\rangle$ is of type 2A, so we may pick a basis $\{a, b, b'\}$, where $b'$ is the extra axis in $B$.

Recall that $\tau_a$ acts trivially on $A$. So, $A_{132}$ is the trivial subspace. So, with respect to $a$, $A$ is still $\mathbb{Z}_2$-graded but with the grading $A_+(a) = A_1(a) \oplus A_0(a)$ and $A_-(a) = A_+(a)$. The associated involution that negates $A_-$ we call $\sigma_a$ to distinguish it from $\tau_a$. By a calculation in the 2A algebra, $b' = b^{\sigma_a}$.

We claim that $C = \langle\langle b', c \rangle\rangle$ coincides with the whole of $A$.

Note that $\tau_b$ fixes $a$ and so, by Lemma 2.6, $\sigma_a^b = \sigma_a \tau_b = \sigma_a$ and $\tau_b$ and $\sigma_a$ commute. However, $\tau_b = \tau_b \tau_a = \tau_b \sigma_a = \tau_b$ and hence $\langle \tau_b, \tau_c \rangle = \langle \tau_b, \tau_c \rangle = G$.

By Lemma 2.10 all the axes of $\langle\langle b, c \rangle\rangle$ under $G$ are in one orbit. Hence, $C$, which is invariant under $\langle \tau_b, \tau_c \rangle = G$ also contains $b$. Furthermore, since $C$ contains $b$ and $b'$, it must contain the whole of $B = \langle\langle b, b' \rangle\rangle$, and so it contains $a$. Therefore, $C = A$, as claimed.

In particular, $A$ is generated by two axes, and so it must be one of the Norton-Sakuma algebras. Since $G = \langle \tau_b, \tau_c \rangle$ does not conjugate $b'$ to $c$, the algebra $A$ can only be of type 6A. It is well known that this algebra contains the algebra 3A and not 3C or 5A. Hence $\langle\langle b, c \rangle\rangle$ is of type 3A.

If we consider just 4-algebras, then there are no 6A subalgebras. So the above theorem may be used to rule out several possible shapes. In particular, if for a putative shape we see a subset of axes with a 3C2A induced shape, then that shape does not lead to a non-trivial algebra. We will use this in the next section to rule out some cases.

6 Results

Using the MAGMA implementation of the algorithm described in [10], we construct all the 3-generated 4-algebras. For many of the cases, particularly for the larger 2-groups, the algebras collapse and hence there are no axial algebras of this shape. We list the number of each of these for the larger groups in Table 4. We describe the remaining algebras in Table 5, excluding the Norton-Sakuma algebras. Note that here we will omit the 0-dimensional algebras except where there are only a few for a given case.

The columns in Table 5 are

- Miyamoto group.
- Axes, where we give the size decomposed into the sum of orbit lengths.
- Shape. If an algebra contains a 4A, or 4B, we omit to mention the 2B, or 2A, respectively, which is contained in it.
- Dimension of the algebra. A question mark indicates that our algorithm did not complete and a 0 indicates that the algebra collapses.
• The minimal $m$ for which $A$ is $m$-closed. Recall that an axial algebra is $m$-closed if it is spanned by products of length at most $m$ of the axes.

• Whether the algebra has a Frobenius form that is non-zero on the set of axes $X$. If it is additionally positive definite or positive semi-definite, we mark this with a pos, or semi, respectively.

For the larger groups, we use the following method to quickly show that the algebras for most of the shapes collapse. We search for a subset $Y \subset X$ of axes where $B = \langle \langle Y \rangle \rangle$ is an algebra with Miyamoto group $H$ which we have already computed and for which most shapes collapse. Then, necessarily, the algebra $A = \langle \langle X \rangle \rangle$ collapses. For example, in a putative algebra for $2 \times D_8$ acting on $4 + 4 + 8$ axes, we find a subalgebra on $2 + 2 + 4$ axes with Miyamoto group $2^3$. However, we have previously shown that most of these shapes do not lead to non-trivial axial algebras, hence this rules out many cases for $2 \times D_8$. Note also that we do not have to restrict ourselves to 3-generated subalgebras.

We may also use Theorem 5.1 to show that some of the shapes do not lead to algebras. In particular, 2 shapes for $S_3 \wr 2$, 16 for $A_2 \wr D_8$ and 12 for $3^4.D_8 : 2$ all contain an induced $3C2A$ shape on $1 + 3$ axes, so the algebras for these shapes all collapse.

We now comment on our results. Firstly, observe that the trivial group acting on three axes with shape $(2A)^3$ has three possible algebras of dimension 3, 6 and 9. Indeed, we can prove this (almost) by hand.

**Proposition 6.1.** Let $A$ be a 3-generated axial algebra of Monster type with trivial Miyamoto group $G$ and shape $(2A)^3$. Then, $A$ is in fact an axial algebra of Jordan type $\frac{1}{4}$. Moreover, $A$ is isomorphic to one of the following:

1. The Matsuo algebra of dimension 3 for the group $S_3$,
2. The Matsuo algebra of dimension 6 for the group $S_4$,
3. The Matsuo algebra of dimension 9 for the group $3 : S_3$.

**Proof.** Since the Miyamoto group $G$ (with respect to the Monster fusion law) is the trivial group, each axis has trivial $\frac{1}{32}$-eigenspace. So we may restrict the fusion law to the Jordan fusion law of type $\frac{1}{4}$ and hence $A$ is an axial algebras of Jordan type $\frac{1}{4}$. The Miyamoto group $H$ (with respect to the Jordan fusion law) is no longer trivial and so the set of axes $X$ is no longer closed. In fact, in each 2A subalgebra, the third basis vector can be taken to be an axis. We introduce $a'$, $b'$ and $c'$ so that the three 2A subalgebras are $\langle\langle a, b, c' \rangle\rangle$, $\langle\langle a, b', c \rangle\rangle$ and $\langle\langle a', b, c \rangle\rangle$. By inspection in the first two subalgebras, the Miyamoto involution $\sigma_a$ fixes $a$, swaps $b$ and $c'$ and also swaps $b'$ and $c$. Similarly for $\sigma_b$ and $\sigma_c$. 
| Group | Axes | Number of Shapes | Number of 0-dim algebras | Number of non-trivial algebras | Number of incomplete algebras |
|-------|------|------------------|--------------------------|-------------------------------|-------------------------------|
| 1     | 1 + 1 + 1 | 4 | 0 | 4 |
| 2^2   | 1 + 2 + 2 | 6 | 0 | 6 |
| 2^2   | 2 + 2 + 2 | 2 | 0 | 1 | 1 |
| 2^2   | 2 + 2 + 2 | 6 | 2 | 3 | 1 |
| S_3   | 1 + 3 | 4 | 1 | 3 |
| 2^3   | 2 + 2 + 4 | 4 | 4 | 0 |
| 2^3   | 2 + 2 + 4 | 18 | 12 | 4 | 2 |
| 2^3   | 2 + 4 + 4 | 12 | 11 | 1 |
| 2^3   | 4 + 4 + 4 | 20 | 16 | 1 | 3 |
| 2 × D_8 | 4 + 4 + 8 | 80 | 80 | 0 |
| 3^2 : 2 | 9 | 5 | 1 | 2 | 2 |
| S_4   | 6 | 4 | 0 | 4 |
| 2^2 : 2 | 4 + 4 + 4 | 24 | 18 | 0 | 6 |
| 2^2 : 2 | 4 + 8 + 8 | 216 | 214 | 0 | 2 |
| 2^2 : 2 | 4 + 4 + 8 | 24 | 21 | 0 | 3 |
| 2 : 2 | 8 + 8 + 8 | 288 | 284 | 0 | 4 |
| 2^4 : D_8 | 8 + 8 + 8 | 364 | 357 | 0 | 7 |
| S_3 : 2 | 6 + 6 | 6 | 4 | 0 | 2 |
| 4^2 : S_3 | 12 | 4 | 1 | 2 | 1 |
| 2^2 : S_4 | 6 + 12 | 16 | 12 | 1 | 3 |
| 2^2 : S_4 | 12 + 12 | 16 | 13 | 3 |
| 2^4 · 2^3 | 8 + 8 + 16 | 1560 | 1558 | 0 | 2 |
| PSL(2, 7) | 21 | 4 | 0 | 3 | 1 |
| 2^5 · 2^3 | 8 + 16 + 16 | 2520 | 2514 | 0 | 6 |
| 2^5 · 2^3 | 16 + 16 + 16 | 1540 | 1535 | 0 | 5 |
| 2^5 · D_8, 2 | 16 + 16 + 32 | 2520 | 2520 | 0 |
| 2^4 · 2^4 · 2^2 | 16 + 32 + 32 | 1560 | 1560 | 0 |
| 2^4 · 2^4 · 2^2 | 32 + 32 + 32 | 364 | 363 | 0 | 1 |
| A_4 · D_8 | 12 + 24 | 32 | 30 | 0 | 2 |
| 3^3 · D_8 : 2 | 18 + 18 + 18 | 26 | 24 | 0 | 2 |

Table 4: Summary

Using the action of the σ involutions, observe that \(\langle a, a' \rangle \cong \langle b, b' \rangle \cong \langle c, c' \rangle\). This subalgebra can be either 1A, 2B, or 2A and the order of \(\sigma_a \sigma_{a'}\) is 1, 2 and 3, respectively. We construct a group presentation for \(H\) with
respect to the three generators $\sigma_a$, $\sigma_b$ and $\sigma_c$. In the first case, it is $S_3$ and in the second it is $S_4$ with all three generators conjugate in an orbit of size 6. In the third case, it is $3^2 : S_3$ with all three generators conjugate in an orbit of size 9.

By [6], $A$ is Matsuo algebras for some 3-transposition group. Taking all the possibilities for $H$, we see the only possible options are the three listed.

We note that the first case in the above lemma is in fact the Norton-Sakuma algebra 2A, which is 2-generated, and so we omit it from Table 5.

Secondly, we note that all the axial algebras constructed have Frobenius forms. This supports a conjecture in [10], that all axial algebras of Monster type have a Frobenius form. Furthermore, we note that all the forms are positive semi-definite, with the vast majority being positive definite. In the two cases where the form is semi-definite, the radical of the form is an ideal $\mathfrak{N}$, so we may quotient by this ideal to get another axial algebra which has a positive definite Frobenius form. In both cases, the radical of the form is 3-dimensional and hence there is an 11- and 13-dimensional axial algebra, respectively. Note that all Norton-Sakuma algebras except for 2B are simple [8]. Hence, as no axes are contained in the radical, the two new quotients have the same shape as before.

Thirdly, there are no axial algebras in our list with a Miyamoto group which is a 2-group and has nilpotency class 2 or more. (We should note that there were cases which we could not complete.) Observe that if $A$ is an axial algebra of Monster type whose Miyamoto group is a 2-group, then it is necessarily a 4-algebra. So, the above comment suggests that if $A$ is an axial algebra with a Miyamoto group $G$ which is a 2-group, then $G$ is in fact elementary abelian.

| $G$ | axes | shape | dim | $m$ | form |
|-----|------|-------|-----|-----|------|
| 1   | 1+1+1| $(2A)^3$ | 6,9 | 2,3 | pos  |
| 1   | 1+1+1| $(2A)^2$ | 6   | 3   | pos  |
| 1   | 1+1+1| $2A (2B)^2$ | 4   | 2   | pos  |
| 1   | 1+1+1| $(2B)^3$ | 3   | 1   | pos  |
| $2^2$ | 1+2+2| $4A (2A)^2$ | 14  | 3   | semi |
| $2^2$ | 1+2+2| $4A 2A 2B$ | 10  | 3   | pos  |
| $2^2$ | 1+2+2| $4A (2B)^2$ | 6   | 2   | pos  |
| $2^2$ | 1+2+2| $4B (2A)^2$ | 5   | 1   | pos  |
| $2^2$ | 1+2+2| $4B 2A 2B$ | 8   | 2   | pos  |
| $2^2$ | 1+2+2| $4B (2B)^2$ | 6   | 2   | pos  |
| $2^2$ | 2+2+2| $(4A)^3$ | ?   |      |      |

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| $2^2$ | $2+2+2$ | $(4B)^3$ | 7 | 2 | pos |
|-------|--------|----------|----|---|-----|
| $2^2$ | $2+2+2$ | $(4A)^2 (2A)^2$ | ? |   |     |
| $2^2$ | $2+2+2$ | $(4A)^2 2A 2B$ | 0 | 0 | -   |
| $2^2$ | $2+2+2$ | $(4A)^2 (2B)^2$ | 9 | 3 | pos |
| $2^2$ | $2+2+2$ | $(4B)^2 (2A)^2$ | 11 | 2 | pos |
| $2^2$ | $2+2+2$ | $(4B)^2 2A 2B$ | 8 | 2 | pos |
| $2^2$ | $2+2+2$ | $(4B)^2 (2B)^2$ | 0 | 0 | -   |
| $S_3$ | $1+3$  | $3A 2A$   | 8 | 2 | pos |
| $S_3$ | $1+3$  | $3A 2B$   | 5 | 2 | pos |
| $S_3$ | $1+3$  | $3C 2A$   | 0 | 0 | -   |
| $S_3$ | $1+3$  | $3C 2B$   | 4 | 1 | pos |
| $2^3$ | $2+2+4$ | $(4A)^2 2A (2B)^2$ | ? |   |     |
| $2^3$ | $2+2+4$ | $(4A)^2 (2B)^3$ | 13 | 3 | pos |
| $2^3$ | $2+2+4$ | $4A 4B (2A)^2 2B$ | 15 | 3 | pos |
| $2^3$ | $2+2+4$ | $4A 4B 2A (2B)^2$ | 12 | 2 | pos |
| $2^3$ | $2+2+4$ | $(4B)^2 (2A)^2 2B$ | ? |   |     |
| $2^3$ | $2+2+4$ | $(4B)^2 (2B)^3$ | 10 | 2 | pos |
| $2^3$ | $2+4+4$ | $(4A)^2 (4B)^2 (2A)^2$ | 16 | 2 | semi |
| $2^3$ | $4+4+4$ | $(4A)^6 (2B)^3$ | ? |   |     |
| $2^3$ | $4+4+4$ | $(4A)^4 (4B)^2 (2A)^2 2B$ | ? |   |     |
| $2^3$ | $4+4+4$ | $(4A)^2 (4B)^4 (2A)^2 2B$ | ? |   |     |
| $2^3$ | $4+4+4$ | $(4B)^6 (2B)^3$ | 15 | 2 | pos |

$3^2 : 2$

| $3^2 : 2$ | 9 | $(3A)^4$ | ? |   |     |
| $3^2 : 2$ | 9 | $(3A)^3 3C$ | 0 | 0 | -   |
| $3^2 : 2$ | 9 | $(3A)^2 (3C)^2$ | ? |   |     |
| $3^2 : 2$ | 9 | $3A (3C)^3$ | 12 | 2 | pos |
| $3^2 : 2$ | 9 | $(3C)^4$ | 9 | 1 | pos |

| $S_4$ | 6 | $3A 2A$ | 13 | 2 | pos |
| $S_4$ | 6 | $3A 2B$ | 13 | 3 | pos |
| $S_4$ | 6 | $3C 2A$ | 9 | 2 | pos |
| $S_4$ | 6 | $3C 2B$ | 6 | 1 | pos |

| $S_4$ | 3+6 | $4A 3A 2A$ | 23 | 3 | pos |
| $S_4$ | 3+6 | $4A 3A 2B$ | 25 | 3 | pos |
| $S_4$ | 3+6 | $4A 3C 2A$ | 0 | 0 | -   |
| $S_4$ | 3+6 | $4A 3C 2B$ | 12 | 2 | pos |
| $S_4$ | 3+6 | $4B 3A 2A$ | 13 | 2 | pos |
| $S_4$ | 3+6 | 4B 3A 2B | 16 | 2 | pos |
|-------|------|----------|----|---|-----|
| $S_4$ | 3+6 | 4B 3C 2A | 9  | 1 | pos |
| $S_4$ | 3+6 | 4B 3C 2B | 12 | 2 | pos |

| $2^2 \wr 2$ | 4 + 4 + 4 | $(4A)^3 (2A)^2$ | ? |
| $2^2 \wr 2$ | 4 + 4 + 4 | $(4A)^3 2A 2B$ | ? |
| $2^2 \wr 2$ | 4 + 4 + 4 | $(4A)^2 4B 2A 2B$ | ? |
| $2^2 \wr 2$ | 4 + 4 + 4 | $(4A)^2 4B (2B)^2$ | ? |
| $2^2 \wr 2$ | 4 + 4 + 4 | $4A (4B)^2 (2A)^2$ | ? |
| $2^2 \wr 2$ | 4 + 4 + 4 | $4A (4B)^2 2A 2B$ | ? |

| $2^2 \wr 2$ | 4 + 8 + 8 | $(4A)^6 2A (2B)^4$ | ? |
| $2^2 \wr 2$ | 4 + 8 + 8 | $(4A)^6 (2B)^5$ | ? |

| $2 \wr 2^2$ | 4 + 4 + 8 | $(4A)^3 4B 2B$ | ? |
| $2 \wr 2^2$ | 4 + 4 + 8 | $(4A)^2 (4B)^2 2B$ | ? |
| $2 \wr 2^2$ | 4 + 4 + 8 | $4A (4B)^3 2B$ | ? |

| $2 \wr 2^2$ | 8 + 8 + 8 | $(4A)^6 4B (2A)^4$ | ? |
| $2 \wr 2^2$ | 8 + 8 + 8 | $(4A)^6 4B (2A)^2 (2B)^2$ | ? |
| $2 \wr 2^2$ | 8 + 8 + 8 | $(4A)^4 (4B)^3 (2A)^3 2B$ | ? |
| $2 \wr 2^2$ | 8 + 8 + 8 | $(4A)^2 (4B)^5 (2A)^2 (2B)^2$ | ? |

| $2.4 : D_8$ | 8 + 8 + 8 | $(4A)^6 (2A)^5 (2B)^3$ | ? |
| $2.4 : D_8$ | 8 + 8 + 8 | $(4A)^6 (2A)^2 (2B)^4$ | ? |
| $2.4 : D_8$ | 8 + 8 + 8 | $(4A)^6 2A (2B)^3$ | ? |
| $2.4 : D_8$ | 8 + 8 + 8 | $(4A)^6 (2B)^6$ | ? |
| $2.4 : D_8$ | 8 + 8 + 8 | $(4A)^4 (4B)^2 2A (2B)^5$ | ? |
| $2.4 : D_8$ | 8 + 8 + 8 | $(4A)^2 (4B)^4 2A (2B)^5$ | ? |
| $2.4 : D_8$ | 8 + 8 + 8 | $(4B)^6 (2A)^3 (2B)^3$ | ? |

| $S_3 \wr 2$ | 6+6 | $4A (3A)^2$ | ? |
| $S_3 \wr 2$ | 6+6 | $4B (3A)^2$ | ? |

| $4^2 : S_3$ | 12 | 4A 3A | ? |
| $4^2 : S_3$ | 12 | 4A 3C | 15 | 2 | pos |
| $4^2 : S_3$ | 12 | 4B 3A | 0 | 0 | - |
| $4^2 : S_3$ | 12 | 4B 3C | 15 | 2 | pos |

| $2^2 : S_4$ | 6+12 | $(4A)^3 3A 2A$ | ? |
| $2^2 : S_4$ | 6+12 | $(4A)^3 3C 2A$ | ? |
| $2^2 : S_4$ | 6+12 | $4A (4B)^2 3A 2B$ | ? |
| $2^2 : S_4$ | 6+12 | $4A (4B)^2 3C 2B$ | 30 | 2 | pos |

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| Algebra          | Generators | Order | Dimension | Description |
|------------------|------------|-------|-----------|-------------|
| $2^2 : S_4$      | 12+12     | 4A    | 3        | $4B \cdot 3C$ | 60 | 3 | pos |
| $2^2 : S_4$      | 12+12     | 4A(4B)$^3$ | 3A       | 4B $\cdot$ | 59 | 3 | pos |
| $2^2 : S_4$      | 12+12     | 4A(4B)$^3$ | 3C       | 4B $\cdot$ | 42 | 2 | pos |
| $2^4 \cdot 2^3$ | 8+8+16    | $4A^6$ | $4B \cdot (2A)^2 \cdot (2B)^4$ | 7 | \? |
| $2^4 \cdot 2^3$ | 8+8+16    | $4A^5$ | $4B \cdot (2A)^2 \cdot (2B)^4$ | 6 | \? |
| $PSL(2,7)$       | 21        | 4A    | 3A       | 6 | \? |
| $PSL(2,7)$       | 21        | 4A    | 3C       | 57 | 3 | pos |
| $PSL(2,7)$       | 21        | 4B    | 3A       | 49 | 2 | pos |
| $PSL(2,7)$       | 21        | 4B    | 3C       | 21 | 1 | pos |
| $2^5 \cdot 2^3$ | 8+16+16   | $(4A)^7$ | $4B \cdot (2A)^2 \cdot (2B)^4$ | 7 | \? |
| $2^5 \cdot 2^3$ | 8+16+16   | $(4A)^6$ | $4B \cdot 2A \cdot (2B)^5$ | 6 | \? |
| $2^5 \cdot 2^3$ | 8+16+16   | $(4A)^5$ | $(4B)^2 \cdot (2A)^4 \cdot (2B)^2$ | 6 | \? |
| $2^5 \cdot 2^3$ | 8+16+16   | $(4A)^4$ | $(4B)^4 \cdot (2A)^4 \cdot (2B)^2$ | 5 | \? |
| $2^5 \cdot 2^3$ | 8+16+16   | $(4A)^3$ | $(4B)^3 \cdot 2A \cdot (2B)^5$ | 5 | \? |
| $2^5 \cdot 2^3$ | 8+16+16   | $(4A)^2$ | $(4B)^5 \cdot (2A)^2 \cdot (2B)^4$ | 5 | \? |
| $2^5 \cdot 2^3$ | 16+16+16  | $(4A)^9$ | $(2A)^6$ | 5 | \? |
| $2^5 \cdot 2^3$ | 16+16+16  | $(4A)^8$ | $(4B)^2 \cdot (2A)^6$ | 4 | \? |
| $2^5 \cdot 2^3$ | 16+16+16  | $(4A)^7$ | $(4B)^3 \cdot (2B)^6$ | 4 | \? |
| $2^5 \cdot 2^3$ | 16+16+16  | $(4A)^6$ | $(4B)^4 \cdot (2A)^6$ | 4 | \? |
| $2^5 \cdot 2^3$ | 16+16+16  | $(4A)^5$ | $(4B)^5 \cdot (2A)^6$ | 4 | \? |
| $2^4 \cdot 2^2$ | 32+32+32  | $(4A)^{12}$ | $(4B)^6$ | 3 | \? |
| $A_2^2 \cdot D_8$ | 12+24    | $(4A)^2 \cdot (3A)^2 \cdot 2A$ | 2 | \? |
| $A_2^2 \cdot D_8$ | 12+24    | $(4A)^2 \cdot 4A \cdot 4B \cdot (3A)^2 \cdot 2A$ | 2 | \? |
| $3^4 \cdot D_8 : 2$ | 18+18+18 | $(4A)^3 \cdot (3A)^6$ | 2 | \? |
| $3^4 \cdot D_8 : 2$ | 18+18+18 | $(4B)^3 \cdot (3A)^6$ | 2 | \? |

Table 5: 3-generated 4-algebras

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