TRIMMED SUMS FOR OBSERVABLES ON THE DOUBLING MAP

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Abstract. We establish a strong law of large numbers under intermediate trimming for a particular example of Birkhoff sums of a non-integrable observable over the doubling map. It has been shown in a previous work by Haynes that there is no strong law of large numbers for the considered system after removing finitely many summands (light trimming) even though i.i.d. random variables and also some dynamical systems with the same distribution function obey a strong law of large numbers after removing only the largest summand.

1. Introduction and statement of results

Considering $T$ an ergodic and measure-preserving transformation of a probability space $(\Omega, \mathcal{B}, \mu)$ and an observable $\varphi: \Omega \to \mathbb{R}_{\geq 0}$, there is a crucial difference in terms of the strong law of large numbers between $\varphi$ being integrable or not. In the integrable case we obtain by Birkhoff’s ergodic theorem that $\mu$-almost surely (a.s.)

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{\varphi \circ T^{k-1}}{n} = \int \varphi d\mu,$$

i.e. the strong law of large numbers is fulfilled, whereas in the case of an observable with infinite expectation, Aaronson showed in [Aar77] that for all positive sequences $(d_n)_{n \in \mathbb{N}}$ we have $\mu$-a.s.

$$\limsup_{n \to \infty} \sum_{k=1}^{n} \frac{\varphi \circ T^{k-1}}{d_n} = +\infty \quad \text{or} \quad \liminf_{n \to \infty} \sum_{k=1}^{n} \frac{\varphi \circ T^{k-1}}{d_n} = 0.$$

However, in certain cases it is possible to obtain a strong law of large numbers after deleting a finite number of maximal terms. One of the first investigated examples for this situation is the unique continued fraction expansion of an irrational $x \in [0, 1)$ given by

$$x := \frac{1}{c_1(x) + \frac{1}{c_2(x) + \ddots}}.$$

In this case we consider the probability space $([0, 1), \mathcal{B}, \mu)$ with $\mu$ the Gauss measure given by $d\mu(x) := 1/ (\log 2 (1 + x)) d\lambda(x)$ with $\lambda$ denoting the Lebesgue measure restricted to $[0, 1)$, together with the Gauss map $G: [0, 1) \to [0, 1)$ defined by

$$G(x) := \begin{cases} 0 & \text{if } x = 0 \\ \{1/x\} & \text{else,} \end{cases}$$

where $\{x\} = x - \lfloor x \rfloor$ and $\lfloor x \rfloor = \max \{n \in \mathbb{Z}: n \leq x \}$. The observable $\chi: [0, 1) \to \mathbb{N} \cup \{\infty\}$ with (1)

$$\chi(x) := \lfloor 1/x \rfloor$$

gives then rise to the stationary (dependent, but $\psi$-mixing) process $\chi \circ G^{n-1} = c_n$, $n \in \mathbb{N}$, of the $n$-th continued fraction digit (with this notation including the case of the finite continued fraction

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expansion of $x \in [0,1) \cap \mathbb{Q}$, for which we set $1/0 := \infty$). For this example Diamond and Vaaler showed in [DV86] that we have $\mu$-a.s.

\[
\lim_{n \to \infty} \sum_{k=1}^{n} c_k - \max_{1 \leq \ell \leq n} C_\ell \over n \log n = \frac{1}{\log 2}.
\]

But what happens if we use another transformation $\tau$ instead of the Gauss map $G$? In this paper we are interested in the doubling map $\tau$ defined as $\tau: [0,1) \to [0,1)$ with

\[
\tau(x) := 2x \mod 1.
\]

It is clear that $\lambda$, the Lebesgue measure restricted to $[0,1)$, is an invariant measure with respect to $\tau$. Our main interest throughout the paper lies on the dynamical system $([0,1), \mathcal{B}, \lambda, \tau)$ together with the observable $\chi$ given in $\mathbb{H}$. For all $n \in \mathbb{N}$ we set

\[
a_n := \chi \circ \tau^{n-1} \quad \text{and} \quad S_n := \sum_{k=1}^{n} a_k.
\]

Haynes showed in [Hay14] that the digits $(a_n)$ show a behavior different to the continued fraction digits $(c_n)$ in terms of strong laws of large numbers. To make this more precise we first define our trimmed sums. For each $n \in \mathbb{N}$ and $x \in [0,1)$ let $\pi \in S_n$ be a permutation of $\{1, \ldots, n\}$ such that

\[
a_{\pi(1)}(x) \geq a_{\pi(2)}(x) \geq \ldots \geq a_{\pi(n)}(x).
\]

It is clear that this choice of $\pi$ depends on $n$ and $x$, but for notational convenience in what follows we will suppress this dependence. For any $b \in \mathbb{N}$ we now define

\[
S_n^b := \sum_{k=b+1}^{n} a_{\pi(k)}.
\]

If $b$ does not depend on $n$, then $S_n^b$ is called a lightly trimmed sum and if there exists a sequence of constants $(d_n)$ such that $\lim_{n \to \infty} S_n^b/d_n = 1$ a.s. we denote this behavior as a lightly trimmed strong law. From here on we always denote by a.s. the almost sure convergence with respect to $\lambda$.

It can be easily concluded from [Hay14] Theorem 4] and its proof that for any sequence of constants $(d_n)$ and any constant $b \in \mathbb{N}$ we have a.s. that

\[
\limsup_{n \to \infty} S_n^b/d_n = +\infty \quad \text{or} \quad \liminf_{n \to \infty} S_n^b/d_n = 0,
\]

implying that no lightly trimmed strong law can hold.

The main difference between the continued fraction expansion and the above example is that $\chi$ obeys the structure of the underlying dynamics $G$, but not of $\tau$, i.e. $\chi$ is constant on each slope of the continued fraction transformation while $\tau$ has only one slope on $[0,1/2]$ on which $\chi$ takes different values. This results into $(c_n)_{n \in \mathbb{N}}$ having stronger mixing properties, i.e. being exponentially $\psi$-mixing, see [Phi88]. However, the digits $(a_n)_{n \in \mathbb{N}}$ are still $\alpha$-mixing, see Section 4.1.

We shall note here that this example does not seem exceptional. In [AN03] Aaronson gave general conditions for a lightly trimmed strong law of exponentially $\psi$-mixing random variables, emended by an example for a mixing dynamical system for which a lightly trimmed strong law does not hold even though it would hold for i.i.d. random variables having the same distribution function.

As there can not be a lightly trimmed strong law for the dynamical system $([0,1), \mathcal{B}, \lambda, \tau)$ with the observable $\chi$, the next step is to ask if there can be a strong law of large numbers if the number of deleted terms depends on $n$. It can be concluded from [KS17a] Corollary 1.5] that there has to be a sequence of natural numbers $(b_n)$ tending to infinity such that $b_n = o(n)$ and a norming sequence $(d_n)$ of positive reals such that $\lim_{n \to \infty} S_n^b/d_n = 1$ a.s. We denote this behavior as an intermediately trimmed strong law. However, this qualitative result does not say anything about a minimal trimming sequence $(b_{\text{min}})$. 
It is the aim of this paper to give precise conditions on the growth of the trimming sequence \((b_n)\) and to give a corresponding norming sequence \((d_n)\) such that \(S_{b_n}^n/d_n\) fulfills an intermediately trimmed strong law.

The studied example can be seen as a toy example, a very similar example has also been studied in [Gou] proving a stable limit law for the system \(([0, 1], \mathcal{B}, \lambda, \tau)\) with the observable \(\tilde{\chi}_\alpha : [0, 1) \to \mathbb{R}_{>1} \cup \{\infty\}, \alpha \geq 1/2, \) with \(\tilde{\chi}_\alpha(x) = 1/x^\alpha\) instead of \(\chi\). As Remark 1.3 will show the behavior remains the same no matter if we consider \(\tilde{\chi}_1\) or \(\chi\).

The results can also be seen as a gap to close in the example of the system \(([0, 1], \mathcal{B}, \lambda, \tau)\). If we consider the observable \(\tilde{\chi}_\alpha\) with \(\alpha > 1\), then \(\tilde{\chi}_\alpha\) is integrable and we can apply Birkhoff’s ergodic theorem. If \(\alpha < 1\), then the optimal trimming sequence \((b_n)\) and the corresponding norming sequence \((d_n)\) for an intermediately trimmed strong law can be calculated using [KS17a, Theorem 1.7] and coincide with those in the i.i.d. case, see [KS17a, Remark 1.9]. The here considered case closes the gap for \(\alpha = 1\) and is exceptional as it is the only case which differs significantly from the i.i.d. case.

It is also worth mentioning that strong laws of large numbers under trimming are a widely studied topic for i.i.d. random variables and many limit theorems have already been established in the 80th and 90th. Most of the above mentioned limit theorems have predecessors as i.i.d. versions, for instance Mori gave conditions for a lightly trimmed strong law of large numbers in [Mor76] and [Mor77]. Haeusler and Mason and subsequently Haeusler developed laws of the iterated logarithm for trimmed sums with regularly varying tail distributions, see [HMS7] and [Hae93]. From these results it is possible to establish an intermediatelytrimmed strong law. An intermediately trimmed strong law for more general distribution functions was also subject in [HM91] and [KS17b]. However, as can be seen from the above explanation, the behavior in this example differs fundamentally from the i.i.d. case and the methods therefore cannot be transfered immediately.

We will now state our main results and then outline the structure of the paper.

1.1. **Statement of main results.** In order to more efficiently state our main theorems, we define two collections of positive real valued functions on the natural numbers,

\[
\Psi := \left\{ u : \mathbb{N} \to \mathbb{R}_{>0} : \sum_{n=1}^{\infty} \frac{1}{u(n)} < \infty \right\}, \quad \text{and} \quad \overline{\Psi} := \left\{ u : \mathbb{N} \to \mathbb{R}_{>0} : \sum_{n=1}^{\infty} \frac{1}{u(n)} = \infty \right\}.
\]

Further, remember our setting of the dynamical system \(([0, 1], \mathcal{B}, \lambda, \tau)\) and the observable \(\chi\) with the subsequent definitions given in (3), (4), and (5).

Our first result is a positive result which demonstrates that, by only intermediately trimming the sums \(S_n\), we can cause the remaining quantities to converge a.s.

**Theorem 1.1.** Suppose that \(\psi \in \Psi\) and that, for each \(n \in \mathbb{N}\),

\[
b_n = \left\lfloor \frac{\log \psi(\lfloor \log n \rfloor) - \log \log n}{\log 2} \right\rfloor.
\]

If

\[
\lim_{n \to \infty} \frac{b_n}{\log n}^{1/4} = 0,
\]

then we have that

\[
\lim_{n \to \infty} \frac{S_{b_n}^n}{n \cdot \log n} = 1 \quad \text{a.s.}
\]
As an example application of the above theorem, let $\epsilon > 0$ and $\psi(n) = n \cdot (\log n)^{1+\epsilon}$. Then it is not difficult to show that $\psi \in \Psi$ and that the sequence $(b_n)$ defined by (6) satisfies the estimate
\[
b_n = \frac{(1+\epsilon)}{\log 2} \cdot \log \log \log n + o(1).
\]
Since $\epsilon$ is arbitrary, we may conclude using the theorem that, for almost every $x$, if we exclude the largest
\[
\left\lfloor \frac{(1+\epsilon)}{\log 2} \cdot \log \log \log n \right\rfloor
\]
terms from the sums $S_n$, the remaining quantities will be asymptotic to $n \log n$, as $n$ tends to infinity. We will see from the next result that this is close to best possible.

**Theorem 1.2.** Suppose that $\psi \in \Psi$ and that, for each $n \in \mathbb{N}$,
\[
b_n = \left\lfloor \frac{\log \psi(\lfloor \log n \rfloor)}{\log 2} - \log \log n \right\rfloor
\]
Then for almost every $x$ we have that
\[
\limsup_{n \to \infty} \frac{S_{b_n}^n}{n \cdot \log n} = \infty
\]
and
\[
\liminf_{n \to \infty} \frac{S_{b_n}^n}{n \cdot \log n} \leq 1.
\]

For comparison with the previous example, let $\psi(n) = n \cdot \log n$. Then we have that $\psi \in \Psi$ and that
\[
b_n = \frac{1}{\log 2} \cdot \log \log \log n + o(1),
\]
which is only slightly smaller than the sequences from before. However, for this choice of trimming sequence both (10) and (11) hold almost surely.

**Remark 1.3.** The previous statements remain unchanged if we consider $\tilde{\chi}_1$ with $\tilde{\chi}_1(x) = 1/x$ instead of $\chi$. Let $\tilde{a}_n = \tilde{\chi}_1 \circ \tau^{n-1}$ and let $\tilde{S}_n^b$ be defined as $S_n^b$ using $(\tilde{a}_n)$ instead of $(a_n)$. Then we particularly have $0 \leq \tilde{\chi}_1(x) - \chi(x) < 1$ for all $x \in [0,1)$ and thus $|\tilde{S}_n^{b_n} - S_n^{b_n}| \leq n$ and the statements in (8), (10), and (11) do not change if we replace $S_n^{b_n}$ by $\tilde{S}_n^{b_n}$.

As a companion to above results, we will also consider the distributional properties of the partial sums $S_n$. In this direction we will prove the following theorem.

**Theorem 1.4.** We have that
\[
\lim_{n \to \infty} \frac{S_n}{n \log n} = 1
\]
in distribution.

**Remark 1.5.** The weak limit theorem is in line with the weak limit law for the continued fractions expansion $\lim_{n \to \infty} \sum_{k=1}^{n} c_k / (n \log n) = 1 / \log 2$ in probability, see [Khi35]. It is likely that the mixing properties of a dynamical system have less influence on weak as on strong convergence.

The paper is structured as follows: We first introduce some truncated random variables in Section 2 which are crucial for the proofs of all three theorems. In Section 3 we give the proof of Theorem 1.1 including a skeleton of the proof in Section 3.1 and the details in the subsequent sections.

As we need the statement of Theorem 1.4 for the proof of Theorem 1.2, we will first give the proof of Theorem 1.4 in Section 4 and conclude the paper with a proof of Theorem 1.2 in Section 5.
2. Truncated random variables

For $i, r \geq 1$ define the truncated random variables

$$a_r^i := a_i \cdot 1_{\{a_i \leq r\}} \quad \text{and} \quad T_n^r := \sum_{i=1}^{n} a_r^i.$$  

Further, denote by $F$ the distribution function of $a_1$, which is given explicitly by

$$F(x) = \begin{cases} 1 - \frac{1}{1+|x|} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$  

With this at hand we are able to compute asymptotic formulas for the expectation of the above random variables as follows.

**Lemma 2.1.** If $(f_n)$ is a sequence which tends to infinity then we have, as $n \to \infty$, that

$$E\left( a_1 f_n \right) = \int_0^{f_n} x dF(x) \sim \log f_n,$$

and

$$E\left( T_n f_n \right) = n \cdot \int_0^{f_n} x dF(x) \sim n \cdot \log f_n.$$

Here and in the following we write $g_n \sim h_n$ for two sequences of reals if $\lim_{n \to \infty} g_n / h_n = 1$.

**Proof.** It is clear that the distribution function of $a_1 f_n$ is given by

$$F_{f_n}(x) = 1_{[0,f_n]}(x) \cdot (1 - F(f_n) + F(x)) + 1_{(f_n,\infty)}(x),$$

therefore we have that

$$E\left( a_1 f_n \right) = \int_\mathbb{R} x dF_{f_n}(x) = \int_0^{f_n} x dF(x) = \sum_{k=1}^{f_n} k \cdot \left( \left(1 - \frac{1}{k+1}\right) - \left(1 - \frac{1}{k}\right) \right)$$

$$= \sum_{k=1}^{f_n} \frac{1}{k+1} \sim \log f_n.$$

The proof of (16) then follows easily from the fact that, since $\lambda$ is invariant with respect to the map $\tau$, the function $F_{f_n}$ is the distribution function of $a_1 f_n$, for any choice of $i, n \geq 1$.  

3. Proof of Theorem 1.1

3.1. Proof of main part of Theorem 1.1. In this section we will give a skeleton of the proof of Theorem 1.1. The proof is based on three main lemmas, Lemma 3.1 Lemma 3.2 and Lemma 3.3 which we will state first.

**Lemma 3.1.** For all $\psi \in \Psi$ and all $\epsilon > 0$ we have that

$$\lambda \left( \# \{ k \leq n : |a_k| > \epsilon \cdot n \cdot \log n \} \geq \frac{\log \psi(\log n) - \log \log n}{\log 2} \right) \quad \text{i.o.} = 0.$$

Here and in the following we abbreviate ”infinitely often” by ”i.o.”

Next, we give a lemma stating that the large digits do not contribute too much to a truncated sum.

**Lemma 3.2.** For $\epsilon > 0$ and $t_n = n \cdot (\log n)^{3/4}$ we have that

$$\lambda \left( \sum_{i=1}^{n} a_i 1_{\{a_i \leq \epsilon \cdot n \cdot \log n\}} \geq 3\epsilon \cdot n \cdot \log n \quad \text{i.o.} \right) = 0.$$
Finally, the third lemma gives a limiting result about the truncated sum defined in (13).

**Lemma 3.3.** We set \( t_n = n(\log n)^{3/4} \). Then

\[
\lim_{n \to \infty} \frac{T_{t_n}^n}{\mathbb{E}(T_{t_n}^n)} = 1 \quad \text{a.s.}
\]

The proofs of these lemmas are given in Sections 3.3 and 3.4.

As the last step in this section we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** We set again \( t_n = n \cdot (\log n)^{3/4} \) and note that \( \epsilon \cdot n \cdot \log n > t_n \), for \( n \) sufficiently large. Then we can conclude from Lemma 3.1 and the definition of \((b_n)\) that for all \( \epsilon > 0 \)

\[
\lambda \left( S_{b_n}^n \geq T_{t_n}^n \cdot n \cdot \log n \quad \text{i.o.} \right) = 0.
\]

Since we have by (16) and Lemma 3.3 that

\[
\lambda \left( T_{t_n}^n \geq (1 + \epsilon) \cdot n \cdot \log n \quad \text{i.o.} \right) = 0,
\]

we can conclude from (17) and Lemma 3.2 that

\[
\lambda \left( S_{b_n}^n \geq (1 + 4 \epsilon) \cdot n \cdot \log n \quad \text{i.o.} \right) = 0.
\]

On the other hand we have for all \( x \in [0, 1) \) that

\[
S_{b_n}^n = \sum_{k=1}^{n} a_{\pi(k)} - \sum_{\ell=1}^{b_n} a_{\pi(\ell)} \geq \sum_{k=1}^{n} \left( a_{\pi(k)} \cdot \mathbb{1}_{\{a_{\pi(k)} \leq t_n\}} \right) - \sum_{\ell=1}^{b_n} \left( a_{\pi(\ell)} \cdot \mathbb{1}_{\{a_{\pi(\ell)} \leq t_n\}} \right) \geq T_{t_n}^n - b_n \cdot t_n
\]

and

\[
\frac{b_n \cdot t_n}{\epsilon \cdot n \cdot \log n} = \frac{b_n}{\epsilon \cdot (\log n)^{1/4}}
\]

which tends to zero by (7). Combining this with the statement of Lemma 3.3 yields for all \( \epsilon > 0 \) that

\[
\lambda \left( S_{b_n}^n \leq (1 + \epsilon) \cdot n \log n \quad \text{i.o.} \right) = 0.
\]

Combining (18) and (19) gives the statement of the theorem. □

The rest of Section 3 is structured as follows. In Section 3.2 we will introduce the induced transformation \( \tau_B \) given in (20). Since the random variables \((a_n)\) highly depend on each other, it is difficult to prove statements directly. The induced transformation will partly solve this problem as we will see in later sections. The method to use the induced transformation is classical for piecewise expanding interval maps with an indifferent fixed point. It goes back to Kakutani and Rokhlin dealing with infinite measure preserving measures, see [Kak43] and [Roh48]. However, it is also used in the finite measure case to prove limit results on the doubling map taking advantage of the independence structure, see [Gou].

With these techniques at hand we are able to prove Lemma 3.1 and Lemma 3.2 in Section 3.3 and Lemma 3.3 in Section 3.4.
3.2. Properties of the induced transformation $\tau_B$. We start this section by defining the induced transformation or jump transformation $\tau_B$. Let $B := [1/2, 1)$ and define the first exit time of $B$ by $\phi : [0, 1) \to \mathbb{N}$ as
\[
\phi(x) := \inf \{ n \in \mathbb{N}_0 : r^n x \in B \} + 1.
\]
Furthermore, we define the jump transformation $\tau_B : [0, 1) \to [0, 1)$ by
\[
\tau_B x := \tau^{\phi(x)} x.
\]
Our strategy is to prove limit results for the sequence of random variables $(\chi \circ \tau_B^{-1})_n$ instead of $(\chi \circ \tau^{-1})_n$ and relate the limit results for the first to the latter random variables in the end of Sections 3.3 and 3.4.

The reason for this approach is that the sequence $(\varphi \circ \tau_B^{-1})_{n \in \mathbb{N}}$ is independent for the right choice of $\varphi$ as we will see in Lemma 3.5 and Corollary 3.6.

Our first lemma reads as follows.

**Lemma 3.4.** $\tau_B$ is invariant with respect to $\lambda$.

It will be proven later in this section.

In order to state our next lemma we define the intervals
\[
J_{j,i}^m := [1/2^j - (i + 1)/2^{j+m}, 1/2^j - i/2^{j+m})
\]
with $j \in \mathbb{N}_0$ and $i \in \{0, \ldots, 2^{m-1} - 1\}$. For given $m \in \mathbb{N}$ the intervals $(J_{j,i}^m)_{j,i}$ form a partition of $[0, 1)$. Further, denote by $\mathcal{J}^m$ the $\sigma$-algebra generated by $(J_{j,i}^m)_{j,i}$.

For simplicity we also define $(J_n)_{n \in \mathbb{N}}$ with $J_n = J_{n,0}^1 = [1/2^{n+1}, 1/2^n)$, for all $n \in \mathbb{N}_0$ and $\mathcal{J}$ the $\sigma$-algebra generated by $(J_n)$. Note that $J_0 = [1/2, 1) = B$.

Then our next lemma reads as follows.

**Lemma 3.5.** Let, for all $n \in \mathbb{N}$, $\nu_n : [0, 1) \to \mathbb{R}$ be measurable with respect to $\mathcal{J}^m$. Then, for $u \in \mathbb{N}_0$, the random variables $(\nu_n \circ \tau_B^{m(n-1)})_{n \in \mathbb{N}}$ are mutually independent with respect to $\lambda$.

The next corollary follows immediately from this lemma.

**Corollary 3.6.** Let, for all $n \in \mathbb{N}$, $\nu_n : [0, 1) \to \mathbb{R}$ be measurable with respect to $\mathcal{J}$. Then the random variables $(\nu_n \circ \tau_B^{(n-1)})_{n \in \mathbb{N}}$ are mutually independent with respect to $\lambda$.

**Remark 3.7.** For technical reasons we also introduce $\Omega' \subset [0, 1)$ as
\[
\Omega' := \{ x \in [0, 1) : x \text{ does not have a finite binary expansion} \}.
\]
The points with a finite binary expansion are exceptional on the one hand as they are finally mapped to zero and $\phi(0) = \infty$. On the other hand, we have for $x \in J_n \cap \Omega'$ that $\chi(x) \in [2^n, 2^{n+1} - 1)$ but $2^{-n-1} \in J_n$ and $\chi(2^{-n-1}) = 2^{n+1} \notin [2^n, 2^{n+1} - 1)$.

Still, $\Omega'$ is of full measure and thus the above lemmas, Lemma 3.4, Lemma 3.5, and Corollary 3.6 are still valid if we restrict ourselves to $\Omega'$ and the respective $\sigma$-algebras $\mathcal{J} \cap \Omega'$ and $\mathcal{J}^m \cap \Omega'$.

To clarify our calculations we will sometimes write $\lambda|_{\Omega'}(A) = \lambda(\Omega' \cap A)$ even though $\lambda|_{\Omega'}(A) = \lambda(A)$ holds for all $\lambda$-measurable sets $A$.

We will prove the previous lemmas by a general approach considering interval maps as in the following lemma.

**Lemma 3.8.** Let $[0, 1)$ be partitioned into $(W_i)_{i \in I}$ with $W_i = [c_i, d_i)$ and $I$ a finite or countable index set and let $\mathcal{W}$ be the $\sigma$-algebra generated by those intervals. Further, for all $i \in I$, let $\xi : [0, 1) \to [0, 1)$ be defined on $W_i$ by
\[
\xi|_{W_i} x := -\frac{c_i}{d_i - c_i} + \frac{1}{d_i - c_i} \cdot x.
\]
Then \( \lambda \) is \( \xi \)-invariant.

In other words the map \( \xi \) maps each of the intervals \( W_i \) to the full interval and on each interval the function \( \xi \) has a constant positive gradient. One example is the doubling map itself with the partition \( [0, 1/2) \) and \( [1/2, 1) \). We note here that these maps are generalised Lüroth maps, in this generalised form first studied in \[BBDK96\], but see also \[KMS16, Chapter 1.4.1\], \[DK02, Chapter 2\]. A proof of \[3.8\] is given in \[BBDK96, Theorem 1\]. (In some literature it is assumed that the partitioning intervals are ordered by size, but this assumption does not change the proof.)

Also note that the above definition implies only that \( \xi \) is a.s. defined on \([0, 1)\). Having for example the partition into the intervals \([1/2^k, 1/2^{k-1})\) with \( k \in \mathbb{N} \) gives \( \bigcup_{k=1}^{\infty} \left[ 1/2^k, 1/2^{k-1} \right) = (0, 1) \). For the following we will ignore the nullset of points which might not been defined.

Furthermore, the above defined maps have the following handy property:

**Lemma 3.9.** Let \( \xi \) be given as in \[23\] with the corresponding partition \((W_i)_{i \in I}\). If, for all \( n \in \mathbb{N} \)
the map \( \varphi_n \colon [0, 1) \to \mathbb{R} \) is measurable with respect to \( W \), then the random variables \((\varphi_n \circ \xi^{n-1})_{n \in \mathbb{N}}\)
are mutually independent with respect to \( \lambda \).

**Proof.** \[BBDK96\] Lemma 1] states that the random variables \((\varphi_n \circ \xi^{n-1})_{n \in \mathbb{N}}\) are mutually independent with respect to \( \lambda \), where \( \varphi_n(x) = i \) if \( i \in W_i \).

The proof remains the same if we replace \( \varphi_n \) by a sequence of more general mappings \((\varphi_n)\) which are measurable with respect to \( W \). \(\square\)

With the above two lemmas at hand we are able to prove Lemma 3.4 and Lemma 3.5.

**Proof of Lemma 3.4** For \( x \in J_n \), we only have to show the representation

\begin{equation}
(24) \quad \tau_B x = -\frac{2^{-n-1}}{2^n - 2^{-n-1}} + \frac{1}{2^n - 2^{-n-1}} : x = -1 + 2^{n+1} x.
\end{equation}

Applying Lemma 3.8 immediately gives the statement of Lemma 3.4. We note here that \( \phi(0) = \infty \) and \( \tau_B 0 \) is not defined, i.e. \( \tau_B \) is only almost surely defined on \([0, 1)\). On \( B \) we have that \( \tau x = 2x \mod 1 = -1 + 2x \). Obviously, for \( x \in B \) we have that \( \phi(x) = 1 \) and thus \( \tau_B x = \tau x = -1 + 2x \).

In general, if \( x \in J_n \), then \( \phi(x) = n + 1 \), i.e. we have that \( 2^n \cdot x \in B \) and thus \( \tau_B x = \tau^{\phi(x)} x = \tau^{-n+1} x = \tau^{n+1} x = -1 + 2^{n+1} x \). \(\square\)

**Proof of Lemma 3.5** It is enough to show independence of \((\nu_n \circ \tau_B^{-(n-1)m})^{-1}(A), n \in \mathbb{N}\), where \( A \) is an \( \cap \)-stable set which generates \( J^m \), see for example \[Kle07, Theorem 2.16\], i.e. we might use \( A = (J^n_{j,i})_{j,i} \). If we define \( A_{u,m,j,i}^n := \{ x : \tau_B^{-u} x \in J^n_{j,i} \} \), it is enough to prove independence for sequences of sets \((A_{u,m,v(n)})_{n \in \mathbb{N}}\) for all possible functions \( v \colon \mathbb{N} \to \{ (j, i) : j \in \mathbb{N}_0, i \in \{0, \ldots, 2^{m-1} - 1 \} \} \).

Furthermore, we have by the \( \tau_B \)-invariance of \( \lambda \), see Lemma 3.4, that

\[ \lambda \left( A_{u,m,v(1)} \cap \ldots \cap A_{u,m,v(n)}^n \right) = \lambda \left( \tau_B^{-u} \left( A_{0,m,v(1)} \cap \ldots \cap A_{0,m,v(n)}^n \right) \right) = \lambda \left( A_{0,m,v(1)}^1 \cap \ldots \cap A_{0,m,v(n)}^n \right). \]

The last equation implies that we only have to prove independence of \((A_{0,m,v(n)})_{n \in \mathbb{N}}\) or independence of \((\nu_n \circ \tau_B^{-(n-1)m})_{n \in \mathbb{N}}\).

Our strategy is to apply Lemma 3.9 to the transformation \( \tau_B^m \). For doing so we define for given \( m \in \mathbb{N} \) an auxiliary partition of \([0, 1)\) by the intervals

\[ L_{i_1, \ldots, i_m} := [a_{i_1, \ldots, i_m}, b_{i_1, \ldots, i_m}) \]
Lemma 3.11. We have for all 
and thus 
induction argument assuming 
The first statement follows immediately from the definition of (25) with respect to this partition. 

Proof. We have that 
Lemma 3.10. \(\phi(x)\) and \(\sigma\), and hence, \(\psi\). Then the following lemma gives the relation between (24) and (23). Hence, the representation in (24) gives that (25) gives that \(\tau_B x \in L_{i_1,...,i_m}\) and thus \(\tau_B x \in J_{i_1-1}\). Hence, applying the representation in (24) repeatedly and using an induction argument shows, for \(x \in L_{i_1,...,i_m}\), 

\[
\tau_B^m x = -1 - 2^i m - 2^{i_m+i_{m-1}} - \ldots - 2^{i_m+i_{m-1}+i_2} + 2^{i_1+\ldots+i_m} x.
\]

On the other hand we have that 
and

\[
\frac{a_{i_1,...,i_m}}{b_{i_1,...,i_m} - a_{i_1,...,i_m}} = 2^{i_1+\ldots+i_m}
\]

Hence, the representation in (25) coincides with the representation in (24), allowing us to apply Lemma 3.10. This yields that the random variables \((\nu_n \circ \tau_B^u(n-1)m)\) are independent if, for all \(n \in \mathbb{N}\), \(\nu_n\) is measurable with respect to the \(\sigma\)-algebra \(\mathcal{L}\) generated by \((L_{i_1,...,i_m})\). Noting that \(\mathcal{L}\) is a sub-\(\sigma\)-algebra of \(\mathcal{L}\) gives the statement of the lemma. \(\square\)

3.3. Zero-one laws concerning the number of large entries \(a_n\). In this section we will prove Lemma 3.11 and Lemma 3.13. We will start with a set of definitions and lemmas relevant for the proof of these lemmas. For the following we set 

\[
\beta_n(x) := \chi \circ \tau_B^{n-1}(x)
\]

and

\[
\phi_n(x) := \sum_{k=0}^{n-1} \phi \circ \tau_B^k.
\]

Then the following lemma gives the relation between \((a_n)\) and \((\beta_n)\).

Lemma 3.10. We have that \(a_1 = \beta_1\) and, for \(k \in \mathbb{N}_{\geq 2}\), that 

\[
\beta_k = a_{\phi_{k-1}}(x)+1.
\]

Proof. The first statement follows immediately from the definition of \((a_n)\) and \((\beta_n)\).

By definition \(\tau_B = \tau^{\phi(x)}\). Obviously, \(\beta_2 = \chi \circ \tau_B = \chi \circ \tau^{\phi(x)} = a_{\phi(x)+1} = a_{\phi_1(x)+1}\). Using an induction argument assuming \(\beta_1 = \tau^{\phi(x)}\) gives then 

\[
\beta_k(x) = \tau^{\phi_k(x)}(x) = \tau^{\phi(x)}(\beta_k(x)) = \beta_{k-1}(x).
\]

and thus 

\[
\beta_k(x) = \chi \circ \tau^{\phi_k(x)}(x) = \chi \circ \tau^{\phi_k(x)}(x) = a_{\phi_{k-1}}(x)+1.
\]

\(\square\)

The following two lemmas, Lemma 3.11 and Lemma 3.13 give zero-one laws for large entries \(\beta_i\).

Lemma 3.11. We have for all \(\psi \in \Phi\) that 

\[
\lambda(\# \{i \leq n: \beta_i \geq n \cdot \psi([\log n])\} \geq 1 \text{ i.o.}) = 0.
\]

In order to prove this lemma we will start with a technical lemma.
Lemma 3.12. Let \( \psi \in \Psi \). Then there exists \( \omega \in \Psi \) such that
\[
\omega ([\log_2 n]) \leq \psi ([\log n]).
\]

Proof. We define \( \omega : \mathbb{N} \to \mathbb{R}_{>0} \) as
\[
\omega (n) = \min \left\{ \psi \left( \left\lfloor \frac{n \cdot \log 2} {\log 2} \right\rfloor + j \right) : j \in \{0, 1\} \right\}.
\]
Recall that \( \psi \in \Psi \). Then for the functions \( \tilde{\psi} : \mathbb{N} \to \mathbb{R}_{>0} \) and \( \psi : \mathbb{N} \to \mathbb{R}_{>0} \) given by \( \tilde{\psi}(n) = \psi ([\kappa \cdot n]) \) with \( \kappa > 0 \) and \( \psi(n) = \min \{ \psi(n), \psi(n + 1) \} \) it holds that \( \tilde{\psi}, \psi \in \Psi \). Hence, \( \omega \in \Psi \).

Applying \( [\log_2 n] \) on \( \omega \) in (28) yields
\[
\omega ([\log_2 n]) = \min \left\{ \psi \left( \left\lfloor \frac{\log n} {\log 2} \right\rfloor + j \right) : j \in \{0, 1\} \right\}.
\]
Since on the one hand we have
\[
\left\lfloor \frac{\log n} {\log 2} \right\rfloor \cdot 2 \geq \left\lfloor \frac{\log n} {\log 2} - 1 \right\rfloor \cdot 2 \geq \log n - 1
\]
and on the other hand
\[
\left\lfloor \frac{\log n} {\log 2} \right\rfloor \cdot 2 \leq \left\lfloor \frac{\log n} {\log 2} \right\rfloor \cdot 2 = \log n,
\]
we have that
\[
\min \left\{ \psi \left( \left\lfloor \frac{\log n} {\log 2} \right\rfloor + j \right) : j \in \{0, 1\} \right\} \leq \psi ([\log n])
\]
and (27) follows.

Proof of Lemma 3.11. Let \( \psi \in \Psi \) be given. By Lemma 3.12 there exists \( \tilde{\psi} \in \Psi \) such that \( \tilde{\psi} ([\log_2 m]) \leq \psi ([\log m]), \) for all \( m \in \mathbb{N} \). Let for the following \( \tilde{\psi} \) fulfill this inequality. Since \( \lambda \) is \( \tau_2 \)-invariant, see Lemma 3.3, we have, using the distribution function in (14), for all \( i, k \in \mathbb{N} \),
\[
\lambda \left( \beta_i \geq 2^{k + [\log_2 \tilde{\psi}(k)]} \right) = \lambda \left( \lambda \geq 2^{k + [\log_2 \tilde{\psi}(k)]} \right) = \frac{1}{2^{k + [\log_2 \tilde{\psi}(k)]}} < 2^{-k + 1} \cdot \tilde{\psi}(k).
\]
Next we notice that
\[
\lambda \left( \# \left\{ i \leq 2^{k + 1} : \beta_i \geq 2^{[k + \log_2 \tilde{\psi}(k)]} \right\} \geq 1 \right) \leq \sum_{i=1}^{2^{k + 1}} \lambda \left( \beta_i \geq 2^{[k + \log_2 \tilde{\psi}(k)]} \right) < \frac{4}{\psi(k)}.
\]
Since \( \tilde{\psi} \in \Psi \) this implies
\[
\sum_{k=1}^{\infty} \lambda \left( \# \left\{ i \leq 2^{k + 1} : \beta_i \geq 2^{[k + \log_2 \tilde{\psi}(k)]} \right\} \geq 1 \right) < \infty
\]
and applying the first Borel-Cantelli lemma yields
\[
\lambda \left( \# \left\{ i \leq 2^{k + 1} : \beta_i \geq 2^{[k + \log_2 \tilde{\psi}(k)]} \right\} \geq 1 \text{ i.o.} \right) = 0.
\]
If we define the sequence of sets \( (I_k)_{k \in \mathbb{N}} \)
\[
I_k := [2^k, 2^{k + 1} - 1] \cap \mathbb{N},
\]
then we have for every \( n \in I_k \) that
\[
\lambda \left( \# \left\{ i \leq n : \beta_i \geq 2^{[k + \log_2 \tilde{\psi}(k)]} \right\} \geq 1 \text{ i.o.} \right) = 0.
\]
On the other hand, if \( n \in I_k \), we obtain by our choice of \( \tilde{\psi} \) that
\[
2^{[k + \log_2 \tilde{\psi}(k)]} \leq 2^{k + [\log_2 \tilde{\psi}(k)]} \leq n \cdot \tilde{\psi} ([\log_2 n]) \leq n \cdot \psi ([\log n]).
\]
Applying this estimation on (22) yields the statement of the lemma.
Lemma 3.13. For \( t_n = n \cdot (\log n)^{3/4} \) we have that
\[
\lambda\left( \# \{ i \leq n : \beta_i \geq t_n \} \geq 2 \text{ i.o.} \right) = 0.
\]

Proof of Lemma 3.13. For \( n \in I_k \) with \( I_k \) as in (31) we have
\[
t_n = n \cdot (\log n)^{3/4} \geq 2^k \cdot (\log 2^k)^{3/4} = 2^{k+3/4 \cdot \log_2 (\log 2 \cdot k)} \geq 2^{k+[3/4 \cdot \log_2 (\log 2 \cdot k)]}.
\]
This implies
\[
\bigcup_{n \in I_k} \# \{ i \leq n : \beta_i \geq t_n \} < \# \left\{ i \leq 2^{k+1} : \beta_i \geq 2^{k+[3/4 \cdot \log_2 (\log 2 \cdot k)]} \right\}.
\]

For the following we will restrict our space to \( \Omega' \) defined in (22). Our strategy is to consider, for \( k \in \mathbb{N} \),
\[
\lambda\left( \bigcup_{n \in I_k} \# \{ i \leq n : \beta_i \geq t_n \} \geq 2 \right) \leq \lambda|_{\Omega'}\left( \# \left\{ i \leq 2^{k+1} : \beta_i \geq 2^{k+[3/4 \cdot \log_2 (\log 2 \cdot k)]} \right\} \geq 2 \right)
\]
(33)
\[
= \sum_{\ell=2}^{2^{k+1}} \lambda|_{\Omega'}\left( \# \left\{ i \leq 2^{k+1} : \beta_i \geq 2^{k+[3/4 \cdot \log_2 (\log 2 \cdot k)]} \right\} = \ell \right)
\]
and to calculate the summands independently. We have, for each \( i \in \mathbb{N} \), \( q \in \mathbb{N}_0 \), \( \{ \beta_i \geq 2^q \} = \{ 1_{0,2^{-q}} \circ \tau_B^{i-1} = 1 \} \). Clearly, \( 1_{0,2^{-q}} \) is \( \mathcal{F}' \)-measurable and thus, by Corollary 3.6 and Remark 3.7 the events \( \{ \beta_i \geq 2^q \} \) are independent. Since, by Lemma 3.13 \( \lambda \) is additionally \( \tau_B \)-invariant, we obtain for these summands
\[
\lambda|_{\Omega'}\left( \# \left\{ i \leq 2^{k+1} : \beta_i \geq 2^{k+[3/4 \cdot \log_2 (\log 2 \cdot k)]} \right\} = \ell \right) \leq \left( 2^{{k+1}} \ell \right)^{-1} \cdot \lambda \left( \chi \geq 2^{k+[3/4 \cdot \log_2 (\log 2 \cdot k)]} \right)^{\ell}.
\]
Using the distribution function of \( \chi \) given in (14) gives
\[
\lambda|_{\Omega'}\left( \# \left\{ i \leq 2^{k+1} : \beta_i \geq 2^{k+[3/4 \cdot \log_2 (\log 2 \cdot k)]} \right\} = \ell \right) < 2^{(k+1)\ell} \cdot \left( \frac{1}{2^{k+[3/4 \cdot \log_2 (\log 2 \cdot k)]}} \right)^{\ell}
< 2^{(k+1)\ell} \cdot \left( \frac{2}{2^{k+3/4 \cdot \log_2 (\log 2 \cdot k)}-1} \right)^{\ell}
= \left( \frac{8}{(\log 2 \cdot k)^{3/4}} \right)^{\ell}.
\]
Hence, applying this on (33) and using the geometric series formula implies
\[
\lambda\left( \bigcup_{n \in I_k} \# \{ i \leq n : \beta_i \geq t_n \} \geq 2 \right) \leq \sum_{\ell=2}^{2^{k+1}} \left( \frac{8}{(\log 2 \cdot k)^{3/4}} \right)^{\ell} < \sum_{\ell=2}^{\infty} \left( \frac{8}{(\log 2 \cdot k)^{3/4}} \right)^{\ell}
\]
\[
= \left( \frac{8}{(\log 2 \cdot k)^{3/4}} \right)^{2} \cdot \left( 1 - \frac{8}{(\log 2 \cdot k)^{3/4}} \right)^{-1}.
\]
We have that \( 1 - 8/(\log 2 \cdot k)^{3/4} \geq 1/2 \) if \( k \) is sufficiently large. This implies
\[
\lambda\left( \bigcup_{n \in I_k} \# \{ i \leq n : \beta_i \geq t_n \} \geq 2 \right) < \frac{64}{(\log 2 \cdot k)^{3/2}},
\]
for \( k \in \mathbb{N} \) sufficiently large, which implies
\[
\sum_{k=1}^{\infty} \lambda \left( \bigcup_{n \in I_k} \# \{ i \leq n : \beta_i \geq t_n \} \geq 2 \right) < \infty.
\]
Applying the first Borel-Cantelli lemma yields
\[
\lambda \left( \bigcup_{n \in I_k} \{ i \leq n : \beta_i \geq t_n \} \geq 2 \text{ i.o.} \right) = 0.
\]
Noting that \((I_k)\) is a partition of the natural numbers gives the statement of the lemma.

As a last step before we can start with the proof of the main lemmas we need the following technical lemma.

**Lemma 3.14.** Assume, for some \(i \in \mathbb{N}\), \(x \in \Omega'\), that \(a_i(x) = r \geq 2\). Then we have for all \(j \in \mathbb{N}_{\leq \lfloor \log_2 r \rfloor}\) that \(a_i + j(x) = \lfloor r/2^j \rfloor\).

**Proof.** The statement \(a_i(x) = r\) is equivalent to \(\chi \circ \tau^{i-1}(x) = r\). The definition of \(\chi\) implies \(\tau^{i-1}(x) \in 1/(r+1), 1/r\) and \(\tau^{i-1}(x) \in J_{\lfloor \log_2 r \rfloor - 1}\) (taking into account that \(x\) we restrict the space to \(\Omega'\)). If \(j \leq \lfloor \log_2 r \rfloor\), then \(\tau^j \circ \tau^{i-1}(x) = 2^j \cdot \tau^{i-1}(x)\), see the proof of Lemma 3.4. Hence,
\[
a_{i+j} = \chi \left( 2^j \cdot \tau^{i-1}(x) \right) \in \left( \frac{r}{2^j}, \frac{r+1}{2^j} \right).
\]
Since \(\left( (r+1)/2^j \right) - r/2^j \leq 1\) and \(a_{i+j}\) can only attain natural values we have \(a_{i+j} = \lfloor r/2^j \rfloor\). □

Finally, we prove the two main lemmas of this section.

**Proof of Lemma 3.14** For ease of notation set \(i(1) := 1\) and \(i(k) := i(k, x) := \phi_{k-1}(x) + 1\), for all \(k \in \mathbb{N}_{\geq 2}\). Lemma 3.10 implies \(\beta_k(x) = a_i(k)(x)\). Since \(\phi(x) \geq 1\), we have that \(i(k) \geq k\) implying
\[
\# \{ k \leq n : a_k > \epsilon \cdot n \cdot \log n \} \leq \# \{ k \leq i(n) : a_k > \epsilon \cdot n \cdot \log n \}.
\]
If we set \(Y_{k,n} := \sum_{j=0}^{i(k+1)-i(k)-1} \mathbb{1}_{a_{i(k)+j} > \epsilon \cdot n \cdot \log n}\), then Lemma 3.11 implies
\[
\# \{ k \leq n : a_k > \epsilon \cdot n \cdot \log n \} \leq \sum_{\ell=1}^{n} Y_{k,n}
\]
and by Lemma 3.11 we have eventually almost surely, for each \(k \leq n\), \(\bar{\psi} \in \Psi\),
\[
Y_{k,n} = \sum_{j=0}^{i(k+1)-i(k)-1} \mathbb{1}_{\{ \epsilon \cdot n \cdot \log n < a_{i(k)+j} \leq n \bar{\psi}(\lfloor \log n \rfloor) \}}
\]
Let us restrict ourselves again to \(\Omega'\) given in 22. If, on \(\Omega'\), \(a_n = r\) with \(r \geq 2\), then by Lemma 3.14
\[
a_{i+\lfloor \log_2 r/\ell \rfloor + 1} \leq \frac{r}{2^{\lfloor \log_2 (r/\ell) \rfloor + 1}} < \frac{r}{2^{\log_3 (r/\ell)}} \leq \ell.
\]
Setting \(r = n \cdot \bar{\psi}(\lfloor \log n \rfloor)\) and \(\ell = \epsilon \cdot n \cdot \log n\) and applying this on 36 yields, for all \(k \leq n\),
\[
Y_{k,n} \leq \sum_{j=0}^{i(k+1)-i(k)-1} \mathbb{1}_{\{ \epsilon \cdot n \cdot \log n < a_{i(k)+j} \leq n \bar{\psi}(\lfloor \log n \rfloor) \}}
\]
eventually almost surely.

Furthermore, applying Lemma 3.13 and noting that \(t_n = n \cdot (\log n)^{3/4} < cn \log n\), for \(n\) sufficiently large, yields that eventually almost surely at most one summand on the righthand side of 35 can be non-zero. Combining this with 37 yields for all \(\bar{\psi} \in \Psi\)
\[
\lambda \left( \# \{ k \leq n : a_k > \epsilon \cdot n \cdot \log n \} > \frac{n \cdot \bar{\psi}(\lfloor \log n \rfloor)}{\log n} \right) \text{ i.o.} \) = 0.
\]
If we set \(\bar{\psi}(n) := \psi(n)/\epsilon\) for given \(\psi \in \Psi\), we obtain \(\bar{\psi} \in \Psi\) and
\[
\frac{n \cdot \bar{\psi}(\lfloor \log n \rfloor)}{\log n} = \left[ \frac{\log \Psi(\lfloor \log n \rfloor) - \log \log n}{\log 2} \right].
\]
Combining this consideration with (38) gives the statement of the lemma.

Proof of Lemma 3.2. We use the same notation of \( i \) introduced at the beginning of the proof of Lemma 3.1, giving \( \beta_k(x) = a_i(k)(x) \).

Since \( i \) is strictly increasing we have in particular \( i(k) \geq k \) and thus we have on \( \Omega' \) that

\[
\sum_{k=1}^{n} a_k \cdot \mathbb{1}_{\{t_n \leq a_k \leq \epsilon \cdot n \cdot \log n\}} \leq \sum_{k=1}^{i(n)} a_k \cdot \mathbb{1}_{\{t_n \leq a_k \leq \epsilon \cdot n \cdot \log n\}}
\]

(39)

\[
= \sum_{k=1}^{n} \sum_{j=0}^{i(k+1)-i(k)-1} a_{i(k)+j} \cdot \mathbb{1}_{\{t_n \leq a_{i(k)+j} \leq \epsilon \cdot n \cdot \log n\}} = \sum_{k=1}^{n} Z_{k,n}.
\]

Furthermore, for \( j \in [0, i(k+1) - i(k))] \cap \mathbb{N} \) and \( x \in \Omega' \) we obtain from Lemma 3.14

\[
a_{i(k)+j}(x) = \lfloor a_{i(k)}(x)/2^j \rfloor = \lfloor \beta_k(x)/2^j \rfloor.
\]

Let \( 1 \leq k \leq n \) and let \( \ell \in \mathbb{N} \) be minimally chosen such that \( \lfloor \beta_k/2^\ell \rfloor \leq \epsilon \cdot n \cdot \log n \). Then we have on \( \Omega' \)

\[
Z_{k,n} = \sum_{j=0}^{i(k+1)-i(k)-1} a_{i(k)+j} \cdot \mathbb{1}_{\{t_n \leq a_{i(k)+j} \leq \epsilon \cdot n \cdot \log n\}} \leq \sum_{j=0}^{i(k+1)-i(k)-1} \left\lfloor \frac{\beta_k}{2^j} \right\rfloor \cdot \mathbb{1}_{\{t_n \leq \lfloor \beta_k/2^j \rfloor \leq \epsilon \cdot n \cdot \log n\}}
\]

(40)

\[
< \sum_{j=\ell}^{\infty} \frac{\beta_k}{2^j} = \beta_k \cdot 2^{-\ell+1}.
\]

By the choice of \( \ell \) we have that

\[
Z_{k,n} \leq 2 \left( \epsilon \cdot n \cdot \log n + 1 \right) \leq 3 \epsilon \cdot n \cdot \log n,
\]

for \( n \) sufficiently large.

Furthermore, Lemma 3.13 implies that eventually almost surely \( \#\{k \leq n: Z_{k,n} > 0\} \leq 1 \). Combining this with (39) and (40) gives the statement of the lemma.

3.4. Limit results for the truncated sum \( T_n^m \). This section is devoted to the proof of Lemma 3.3. For the following we define \( \eta: [0,1) \rightarrow \mathbb{R}_{\geq 0} \) and its truncated version \( \eta^r \), for \( r \geq 1 \), by

\[
\eta(x) := \sum_{k=0}^{\phi(x)-1} \chi \circ \tau^k (x) \quad \text{and} \quad \eta^r := \sum_{k=0}^{\phi(x)-1} \chi^r \circ \tau^k (x).
\]

Furthermore, we define for \( m, j \in \mathbb{N} \), and \( i \in \{0, \ldots, 2^j-1\} \)

\[
y_{j,i}(m) := \frac{2^j+1}{2^m-i-1} \quad \text{and} \quad z_{j,i}(m) := \frac{2^j+1}{2^m-i} - j - 1.
\]

Further, we define the observables \( v_m, w_m: [0,1) \rightarrow \mathbb{R}_{\geq 0} \) as well as their truncated versions \( v_m^r, w_m^r: [0,1) \rightarrow \mathbb{R}_{\geq 0} \) for \( r \geq 1 \) as functions piecewise constant on \( J_{j,i}^m \) such that for \( x \in J_{j,i} \)

\[
v_m(x) = y_{j,i}, \quad w_m(x) = z_{j,i}, \quad v_m^r(x) = \min_{\left\lfloor \log_2 r \right\rfloor, i}, \quad w_m^r(x) = \min_{\left\lfloor \log_2 r \right\rfloor, j, i}.
\]

Those observables obey the \( \sigma \)-algebra \( \mathcal{F}^m \) defined in (21) and thereafter. Since, for given \( u \in \mathbb{N}_0 \), each of the sequences of random variables \( (\nu \circ \tau^m)^{(n-u)} \) are independent, see Lemma 3.3, it is easier to prove statements for those sequences of random variables instead of \( (\eta \circ \tau^m_{B})_{\infty} \), see the proof of Lemma 3.3.
In the next lemmas, Lemma 3.15, Lemma 3.16, and Lemma 3.17, we will work out the relation between the observables $v_m^r$, $w_m^r$, and $\eta^r$, showing that the first two give an approximation of the latter one.

With these results at hand we are able to prove the subsequent Lemma 3.18, an analogous statement to Lemma 3.3 for the Birkhoff sum $\sum_{k=0}^{n-1} \eta^{2n} \circ \tau^k$ instead of $\sum_{k=0}^{n-1} \chi^{2n} \circ \tau^k$.

In the last part of this section we will then relate the limiting results for the Birkhoff sums $\sum_{k=0}^{n-1} \eta^{2n} \circ \tau^k$ and $\sum_{k=0}^{n-1} \chi^{2n} \circ \tau^k$ proving Lemma 3.3.

**Lemma 3.15.** We have for all $x \in \Omega'$, all $m \in \mathbb{N}$, and all $r \geq 1$ that

$$w_m^r (x) \leq \eta^r (x) \leq v_m^r (x).$$

**Proof.** We start by showing

$$w_m (x) \leq \eta (x) \leq v_m (x), \quad (43)$$

for all $x \in \Omega'$. We will first give a connection between $\chi(x)$ and $\eta(x)$. Let $x \in J_n \cap \Omega'$, then we have on the one hand $\phi(x) = n + 1$, see the proof of Lemma 3.4. On the other hand, $|\log \chi(x)| = |\log_2 \{1/x\}| = n$ giving $\phi(x) = |\log_2 \chi(x)| + 1$.

Using Lemma 3.14 gives

$$\chi(x) + \sum_{k=1}^{\phi(x)-1} (\chi(x) \cdot 2^{-k} - 1) \leq \eta(x) \leq \sum_{k=0}^{\phi(x)-1} \chi(x) \cdot 2^{-k}. \quad (45)$$

Since $\phi(x) - 1 = |\log_2 \chi|$ using the geometric series estimate gives

$$\chi(x) + \sum_{k=1}^{\phi(x)-1} (\chi(x) \cdot 2^{-k} - 1) \geq \sum_{k=0}^{\log_2 \chi} \chi \cdot 2^{-k} - (|\log_2 \chi| - 1) \geq 2\chi - |\log_2 \chi| + 1. \quad (46)$$

Applying this on (45) and using the geometric series formula also for the righthand side of (45) gives

$$2\chi - |\log_2 \chi| + 1 \leq \eta \leq 2\chi. \quad (47)$$

Given this estimate and assuming that $x \in J_{J_n}^m$ yields

$$\chi(x) \leq \left( \frac{1}{2r} - \frac{i + 1}{2^{i+m}} \right)^{-1} = \left( \frac{2^{j+m}}{2m - i - 1} \right) \leq \frac{2^{j+m}}{2m - i - 1}. \quad (48)$$

Applying this on the second inequality of (46) gives the second estimate of (44).

On the other hand, if $x \in J_{J_n}^m$, we have that

$$\chi(x) \geq \left( \frac{1}{2r} - \frac{i}{2^{i+m}} \right)^{-1} = \left( \frac{2^{j+m}}{2m - i} \right) \geq \frac{2^{j+m}}{2m - i} - 1 \quad (49)$$

and using (47) gives

$$|\log_2 \chi| \leq \max_{i \in \{0, \ldots, 2^{m-1} - 1\}} \left[ \log_2 \frac{2^{j+m}}{2m - i - 1} \right] \leq \log_2 2^{j+1} = j + 1. \quad (50)$$

Applying (48) and (49) on (46) gives the first estimate in (44).

In order to investigate $\eta^r(x)$ further, we notice that on the one hand

$$\{\chi \leq r\} = \left\{ x: \left[ \frac{1}{x} \right] \leq r \right\} = \left\{ x: \log_2 \left[ \frac{1}{x} \right] \leq \log_2 r \right\} \supset \left\{ x: \log_2 \left[ \frac{1}{x} \right] \leq \left[ \log_2 r \right] \right\} = \left\{ x: x \geq 2^{-\left[ \log_2 r \right]} \right\},$$

and using (48), we have

$$\left[ \log_2 \frac{1}{x} \right] \leq \log_2 \left[ \frac{1}{x} \right] \leq \log_2 \left[ \frac{1}{x} \right] \leq \log_2 \left[ \frac{1}{x} \right] \leq \left[ \log_2 r \right].$$

Thus, we have

$$\left[ \log_2 \frac{1}{x} \right] \leq \left[ \log_2 \frac{1}{x} \right] \leq \left[ \log_2 \frac{1}{x} \right] \leq \left[ \log_2 \frac{1}{x} \right] \leq \left[ \log_2 r \right].$$

Applying this on (46) gives

$$2\chi - |\log_2 \chi| + 1 \leq \eta \leq 2\chi.$$
For \( x \) and applying (50) and (51) gives Lemma 3.16. For all \\
\( \{ \chi \leq r \} = \left\{ x : \log_2 \left\lfloor \frac{1}{x} \right\rfloor \leq \log_2 r \right\} \subset \left\{ x : \log_2 \left\lfloor \frac{1}{x} \right\rfloor \leq \left\lfloor \log_2 r \right\rfloor \right\} = \left\{ x : \log_2 \left\lfloor \frac{1}{x} \right\rfloor \leq \log_2 r \right\} \\
(51) \quad \subset \left\{ x : \log_2 \frac{1}{x} \leq \log_2 r + 1 \right\} = \left\{ x : x \geq 2^{-\left\lfloor \log_2 r \right\rfloor - 1} \right\}.

Furthermore, using the definition \\
\( \eta^r(x) = \sum_{k=0}^{\phi(x)-1} \chi^r \circ \tau^k(x) = \sum_{k=0}^{\phi(x)-1} \left( \chi \cdot 1_{\{\chi \leq r\}} \right) \circ \tau^k(x) \)
and applying (50) and (51) gives \\
(52) \quad \sum_{k=0}^{\phi(x)-1} \left( \chi \cdot 1_{\{\chi \leq r\}} \right) \circ \tau^k(x) \leq \eta^r(x) \leq \sum_{k=0}^{\phi(x)-1} \left( \chi \cdot 1_{\{\chi \leq r\}} \right) \circ \tau^k(x).

Moreover, if \( \phi(x) = n \), then \( x \in J_{n-1} \) and for all \( j \in [0, n-1] \cap \mathbb{N}_0 \) we have that \( \tau^j x = 2^j x \in J_{n-j-1} \). On the other hand, for \( q \in \mathbb{N}_0 \), we have that \( \{ x \geq 2^{-q} \} \cap \Omega' = \bigcup_{j=0}^{q} J_{j} \cap \Omega' \). This implies for all \( x \in \Omega' \) \\
\[ \sum_{k=0}^{\phi(x)-1} \left( \chi \cdot 1_{\{\chi \leq r\}} \right) \circ \tau^k(x) = \sum_{k=0}^{\phi(x)-1} \left( \chi \cdot 1_{\{\chi \leq r\}} \right) (2^k x) \]
\[ = \sum_{k=\max\{0, \phi(x)-q\}}^{\phi(x)-1} \left( \chi \cdot 1_{\{\chi \leq r\}} \right) (2^k x) \]
\[ = \sum_{k=\max\{0, \phi(x)-q\}}^{\phi(x)-1} \chi (2^k x) \]
\[ = \sum_{k=0}^{\phi(x)-\max\{0, \phi(x)-q\}-1} \chi (2^{k+\max\{0, \phi(x)-q\}} x) \]
\[ = \sum_{k=0}^{\phi(x)-\max\{0, \phi(x)-q\}-1} \chi \circ \tau^k (2^{\max\{0, \phi(x)-q\}} x). \]
Moreover, \\
\( \phi \left(2^{\max\{0, \phi(x)-q\}} x\right) = \phi(x) - \max\{0, \phi(x) - q\} \),
and using the definition of \( \eta \) implies \\
\[ \sum_{k=0}^{\phi(x)-1} \left( \chi \cdot 1_{\{\chi \leq r\}} \right) (\tau^k(x)) = \sum_{k=\max\{0, \phi(x)-q\}}^{\phi(x)-1} \chi (\tau^k(x)) = \sum_{k=0}^{\phi(x)-\max\{0, \phi(x)-q\}-1} \chi (\tau^k(x)) \]
\[ = \eta \left(2^{\max\{0, \phi(x)-q\}} x\right). \]
Applying (52) gives for all \( x \in \Omega' \) \\
\[ \eta \left(2^{\max\{0, \phi(x)-\left\lfloor \log_2 r \right\}} x\right) \leq \eta^r(x) \leq \eta \left(2^{\max\{0, \phi(x)-\left\lfloor \log_2 r \right\}-1} x\right). \]
Furthermore, if \( x \in J_{j+1} \), then for all \( q \in [0, j] \cap \mathbb{N}_0 \), it holds that \( 2^{\max\{j-q, 0\}} x \in J_{\max\{j, \left\lfloor \log_2 r \right\}\}} \). For \( x \in \Omega' \), applying the first/second inequality of (44) on (52) and (13) gives the first/second inequality of (13). \( \square \)

**Lemma 3.16.** For all \( \epsilon > 0 \) there exist \( M \in \mathbb{N} \), \( R > 0 \) such that for all \( m > \mathbb{N}_0 \) and \( r \geq R \) \\
\[ \mathbb{E}(\nu_m) \leq (2 + \epsilon) \cdot \log r. \]
**Proof.** Since \( v_r^m \) is a piecewise constant function attaining the value \( y_{\min(j, \lfloor \log_2 r \rfloor), i} \) on the interval \( J_{j,i}^m \) we have that

\[
E(v_r^m) = \sum_{i=0}^{2^{m-1}-1} \sum_{j=0}^{\infty} \lambda(J_{j,i}^m) \cdot y_{\min(j, \lfloor \log_2 r \rfloor), i}
\]

(53)

\[
= \sum_{i=0}^{2^{m-1}-1} \left( \sum_{j=0}^{\lfloor \log_2 r \rfloor} \lambda(J_{j,i}^m) \cdot y_{j,i} + \sum_{j=\lfloor \log_2 r \rfloor+1}^{\infty} \lambda(J_{j,i}^m) \cdot y_{\lfloor \log_2 r \rfloor, i} \right).
\]

From the definition of \( J_{j,i}^m \) it follows that

\[
\lambda(J_{j,i}^m) = \frac{1}{2^{2m-2i-1} - 2^{m-i-1} + 1}.
\]

(54)

Hence, inserting this value and the value of \( y_{j,i} \) given in (41) into (53) yields

\[
E(v_r^m) = \sum_{i=0}^{2^{m-1}-1} \left( \sum_{j=0}^{\lfloor \log_2 r \rfloor} \frac{2}{2^{2m-2i-1} - 2^{m-i-1} + 1} + \sum_{j=\lfloor \log_2 r \rfloor+1}^{\infty} \frac{1}{2^{2m-2i-1} - 2^{m-i-1} + 1} \right) = 2^{\lfloor \log_2 r \rfloor + 2} \sum_{i=0}^{2^{m-1}-1} \frac{1}{2^{2m-2i-1} - 2^{m-i-1} + 1}.
\]

(55)

Since \( 1/x \) is monotonically decreasing, the last sum can be estimated by the integral

\[
\sum_{i=0}^{2^{m-1}-1} \frac{1}{2^{2m-2i-1} - 2^{m-i-1} + 1} \leq \int_{0}^{2^{m-1}} \frac{1}{2^{2m} - x - 1} dx = \log(2^{2m} - 1) - \log(2^{2m-1} - 1)
\]

and for each \( \epsilon > 0 \) we can find \( M \) such that for \( m \geq M \)

\[
\sum_{i=0}^{2^{m-1}-1} \frac{1}{2^{2m-2i-1} - 2^{m-i-1} + 1} \leq (1 + \epsilon/3) \cdot \log 2.
\]

Combining this with (55) gives the statement of the lemma. \( \square \)

The next lemma is the analogous statement to the previous lemma for \( w_r^m \).

**Lemma 3.17.** For all \( \epsilon > 0 \) there exist \( M \in \mathbb{N} \), \( R > 0 \) such that for all \( m \geq M \) and \( r \geq R \)

\[
E(w_r^m) \geq (2 + \epsilon) \cdot \log r.
\]

**Proof.** Analogously as in the proof of Lemma 3.16 we have that

\[
E(w_r^m) = \sum_{i=0}^{2^{m-1}-1} \left( \sum_{j=0}^{\lfloor \log_2 r \rfloor-1} \lambda(J_{j,i}^m) \cdot z_{j,i} + \sum_{j=\lfloor \log_2 r \rfloor}^{\infty} \lambda(J_{j,i}^m) \cdot z_{\lfloor \log_2 r \rfloor, i} \right).
\]

(56)
From (54) and (51) it follows that
\[
\sum_{j=0}^{[\log_2 r] - 1} \lambda(J_{j,i}) \cdot z_{j,i} + \sum_{j=[\log_2 r]}^{\infty} \lambda(J_{j,i}) \cdot z_{j[\log_2 r],i} = \sum_{j=0}^{[\log_2 r] - 1} \frac{2}{2m-i} + \sum_{j=[\log_2 r]}^{\infty} \frac{1}{2m-i} \cdot \frac{1}{2^{j-[\log_2 r]-1}} - \sum_{j=0}^{\infty} \frac{j+1}{2^{j+m}} > \sum_{j=0}^{[\log_2 r] - 1} \frac{2}{2m-i} - \sum_{j=0}^{\infty} \frac{j+1}{2^{j+m}} \geq \frac{2[\log_2 r]}{2m-i} - 4,
\]
for \( m \) sufficiently large, where the last inequality follows from
\[
\sum_{j=0}^{\infty} \frac{j+1}{2^{j+m}} \leq \sum_{j=0}^{\infty} \frac{1}{2^{j/2}} = \frac{1}{1 - 2^{-1/2}} < 4,
\]
for \( m \) sufficiently large. Combining (56) and (57) yields
\[
\mathbb{E}(w_m^r) \geq \sum_{i=0}^{2^{m-1}-1} \left( \frac{2[\log_2 r]}{2m-i} - 4 \right) = 2[\log_2 r] \cdot \sum_{i=0}^{2^{m-1}-1} \left( \frac{1}{2m-i} \right) - 2^{m+1},
\]
for \( m \) sufficiently large, where the sum can be estimated by an integral as follows
\[
\sum_{i=0}^{2^{m-1}-1} \frac{1}{2m-i} \geq \int_{-1}^{2^{m-1}-1} \frac{1}{2m-x} \, dx = \log(2m + 1) - \log(2^{m-1} + 1) \geq (1 - \epsilon/3) \log 2,
\]
for \( m \) sufficiently large. Combining this with (58) gives
\[
\mathbb{E}(w_m^r) \geq (1 - \epsilon/3) \log 2 \cdot 2[\log_2 r] - 2^{m+1} \geq (2 - \epsilon) \cdot \log_2 r,
\]
for \( r \) and \( m \) sufficiently large (if \( \epsilon \) was chosen sufficiently small) and thus the statement of the lemma follows. \( \square \)

In the following lemma we give a statement related to Lemma 3.3 using \((\eta \circ \tau_B^{n-1})\) instead of \((\chi \circ \tau_B^{n-1})\). With the previously attained properties of \( \eta \) we are able to prove this lemma.

To formulate this lemma let, for any function \( \varphi : [0,1) \to \mathbb{R}_{>0} \) and \( r > 0 \),
\[
\mathbb{T}_n^r \varphi = \sum_{k=1}^{n} \varphi \circ \tau_B^{k-1} \cdot 1_{\{\varphi \circ \tau_B^{k-1} \leq r\}}
\]
and as in Lemma 3.3 we set \( t_n = n \cdot (\log(n))^{3/4} \).

**Lemma 3.18.** We have that
\[
\lim_{n \to \infty} \frac{\mathbb{T}_n^{t_n \eta}}{\mathbb{E}(\mathbb{T}_n^{t_n \eta})} = 1 \quad a.s.
\]

In order to prove this lemma we first need an exponential inequality. The following lemma generalizes Bernstein’s inequality and can be found for example in [Hoe63].

**Lemma 3.19** (Generalized Bernstein inequality). For \( n \in \mathbb{N} \) let \( Y_1, \ldots, Y_n \) be independent random variables such that \( \|Y_i - \mathbb{E}(Y_i)\|_\infty \leq M < \infty \) for \( i = 1, \ldots, n \). Let \( Z_n = \sum_{i=1}^{n} Y_i \). Then we have for all \( t > 0 \) that
\[
\mathbb{P}\left( \max_{k \leq n} |Z_k - \mathbb{E}(Z_k)| \geq t \right) \leq 2 \exp\left( -\frac{t^2}{2\mathbb{V}(Z_n) + \frac{2}{3}Mt} \right).
\]
With the help of Lemma 3.19 we are able to prove the following Lemma 3.20 for the special case of non-negative random variables.

**Lemma 3.20.** For \( n \in \mathbb{N} \) let \( Y_1, \ldots, Y_n \) be i.i.d. non-negative random variables such that \( Y_1 \leq K < \infty \). Let \( Z_n = \sum_{i=1}^{n} Y_i \). Then we have for all \( \kappa > 0 \) that

\[
P \left( \max_{1 \leq i \leq n} |Z_i - \mathbb{E}(Z_i)| \geq \kappa \cdot \mathbb{E}(Z_n) \right) \leq 2 \exp \left( -\frac{3\kappa^2}{6 + 2\kappa} \cdot \frac{\mathbb{E}(Z_n)}{K} \right).
\]

**Proof.** First note that we may chose \( M = K \) in Lemma 3.19 to obtain

\[
\max_{1 \leq i \leq n} |Y_i - \mathbb{E}(Y_i)| \leq K = M.
\]

Since

\[
\forall (Z_n) = n \left( \int_0^K x^2dF(x) - \left( \int_0^K x dF(x) \right)^2 \right) < n \int_0^K x^2dF(x)
\]

it follows by Lemma 3.19 that

\[
P \left( \max_{1 \leq i \leq n} |Z_i - \mathbb{E}(Z_i)| \geq \kappa \cdot \mathbb{E}(Z_n) \right) \leq 2 \exp \left( -\frac{\kappa^2 \cdot \mathbb{E}(Z_n)^2}{2\overline{V}(Z_n) + \frac{2}{\kappa} \cdot K \cdot \mathbb{E}(Z_n)} \right)
\]

\[
< 2 \exp \left( -\frac{\kappa^2}{2 + \frac{4}{\kappa}} \cdot \frac{\mathbb{E}(Z_n)}{K} \right)
\]

\[
= 2 \exp \left( -\frac{3\kappa^2}{6 + 2\kappa} \cdot \frac{\mathbb{E}(Z_n)}{K} \right).
\]

\[\square\]

Now we are able to start with the proof of Lemma 3.18.

**Proof of Lemma 3.18.** The proof can be summarized into two main steps. First we fix \( \epsilon > 0 \) and find sets \( \{A_i\}_{i \in I} \) and a corresponding index set \( I \) such that there exists \( M \in \mathbb{N} \) fulfilling

\[
\bigcup_{n \in \mathbb{N} \geq M} \{T_n^{2\epsilon n} \cdot \eta - \mathbb{E}(T_n^{2\epsilon n} \cdot \eta) > \epsilon \cdot \mathbb{E}(T_n^{2\epsilon n} \cdot \eta)\}
\]

\[
\cup \bigcup_{n \in \mathbb{N} \geq M} \{\mathbb{E}(T_n^{2\epsilon n} \cdot \eta) - T_n^{2\epsilon n} \cdot \eta > \epsilon \cdot \mathbb{E}(T_n^{2\epsilon n} \cdot \eta)\} \subset \bigcup_{i \in I} A_i.
\]

Afterwards we calculate \( \lambda(A_i) \) for each \( i \in I \) and show that \( \sum_{i \in I} \lambda(A_i) < \infty \). Applying then the first Borel-Cantelli lemma gives the statement of the lemma.

We start by introducing the following notation:

\[
\overline{v}_m := v_m^r - \int v_m^r d\lambda \quad \text{and} \quad \overline{m}_m := w_m^r - \int w_m^r d\lambda.
\]

Using the second inequality of Lemma 3.15 we first obtain the following inclusion

\[
\{T_n^{2\epsilon n} \cdot \eta - \mathbb{E}(T_n^{2\epsilon n} \cdot \eta) > \epsilon \cdot \mathbb{E}(T_n^{2\epsilon n} \cdot \eta)\}
\]

\[
\subset \{T_n^{2\epsilon n} \cdot v_m - \mathbb{E}(T_n^{2\epsilon n} \cdot v_m) > \epsilon \cdot \mathbb{E}(T_n^{2\epsilon n} \cdot v_m)\}
\]

\[
\subset \{T_n^{2\epsilon n} \cdot v_m > \epsilon \cdot \mathbb{E}(T_n^{2\epsilon n} \cdot v_m) - (1 + \epsilon) \cdot \mathbb{E}(T_n^{2\epsilon n} \cdot w_m) - \mathbb{E}(T_n^{2\epsilon n} \cdot v_m)\}.
\]

Combining Lemma 3.15, Lemma 3.16 and Lemma 3.17 yields that for all \( \overline{\epsilon} > 0 \) there exist \( M, N \in \mathbb{N} \) such that, for all \( m \geq M \) and \( n \geq N \),

\[
\mathbb{E}(T_n^{2\epsilon n} \cdot v_m) - \mathbb{E}(T_n^{2\epsilon n} \cdot \eta) \leq \mathbb{E}(T_n^{2\epsilon n} \cdot v_m) - \mathbb{E}(T_n^{2\epsilon n} \cdot w_m) \leq 2\overline{\epsilon} \cdot \mathbb{E}(T_n^{2\epsilon n} \cdot v_m).
\]
Setting $\bar{\epsilon} = \epsilon / 6$ yields, for $\epsilon \in (0, 1)$,
\[
\epsilon \cdot \mathbb{E} \left( T_{2n}^{2n} v_m \right) - (1 + \epsilon) \cdot \left( \mathbb{E} \left( T_{2n}^{2n} v_m \right) - \mathbb{E} \left( T_{2n}^{2n} \eta \right) \right) \geq \left( \epsilon - (1 + \epsilon) \cdot 2 \cdot \frac{\epsilon}{6} \right) \cdot \mathbb{E} \left( T_{2n}^{2n} v_m \right) 
\geq \frac{\epsilon}{3} \cdot \mathbb{E} \left( T_{2n}^{2n} v_m \right).
\]
Inserting this in (60) yields
\[
\left\{ T_{2n}^{2n} \eta - \mathbb{E} \left( T_{2n}^{2n} \eta \right) > \epsilon \cdot \mathbb{E} \left( T_{2n}^{2n} \eta \right) \right\} \subset \left\{ \left| T_{2n}^{2n} v_m \right| > \epsilon / 3 \cdot \mathbb{E} \left( T_{2n}^{2n} v_m \right) \right\},
\]
for $m, n$ sufficiently large. Analogously, we obtain by the first inequality of Lemma 3.15
\[
\left\{ \mathbb{E} \left( T_{2n}^{2n} \eta \right) - T_{2n}^{2n} \eta > \epsilon \cdot \mathbb{E} \left( T_{2n}^{2n} \eta \right) \right\} 
\subset \left\{ \mathbb{E} \left( T_{2n}^{2n} \eta \right) - T_{2n}^{2n} w_m > \epsilon \cdot \mathbb{E} \left( T_{2n}^{2n} w_m \right) \right\} 
\subset \left\{ \mathbb{E} \left( T_{2n}^{2n} w_m \right) - T_{2n}^{2n} w_m > \epsilon \cdot \mathbb{E} \left( T_{2n}^{2n} w_m \right) \right\} 
\subset \left\{ \left| T_{2n}^{2n} w_m \right| > \epsilon / 3 \cdot \mathbb{E} \left( T_{2n}^{2n} w_m \right) \right\},
\]
sufficiently large, where the forth line follows from a similar calculation as in (60).

For the following let us always assume that $m$ is large enough that the above inclusions hold. We first proceed with the estimation of (61). The estimation of (62) follows very much analogously, as we will see later on.

We define $I_j^m = [m^j, m^{j+1}] \cap \mathbb{N}$ for $j, m \in \mathbb{N}$ (a generalisation of $I_j = I_j^2$ defined in §3.1) and for $n \in I_j^m$ we have
\[
T_{2n}^{2n} v_m = \sum_{u=0}^{m-1} \sum_{\ell=0}^{\gamma(n,u)} T_{2n}^{2n} \circ \tau_B^{f_{m+u}},
\]
where $\gamma(n, u)$ can be uniquely determined and takes values in the interval $I_{j-1}^{m+u}$ if $n \in I_j^m$. However, given our following estimations there is no need to further investigate which exact value $\gamma(n, u)$ attains for given $n$ and $u$. With this in mind we obtain the following inclusion
\[
\left\{ \left| T_{2n}^{2n} v_m \right| > \epsilon / 3 \cdot \mathbb{E} \left( T_{2n}^{2n} v_m \right) \right\} \subset \bigcup_{u=0}^{m-1} \left\{ \sum_{\ell=0}^{\gamma(n,u)} T_{2n}^{2n} \circ \tau_B^{f_{m+u}} \right\} > \frac{\epsilon}{3 \cdot m} \cdot \mathbb{E} \left( T_{2n}^{2n} v_m \right).
\]
The reason we make this last estimate is that, by Lemma 3.5 $\sum_{\ell=0}^{\gamma(n,u)} T_{2n}^{2n} \circ \tau_B^{f_{m+u}}$ is a sum of independent random variables. This will later facilitate to estimate the probability of the single events.

In the next steps we aim to combine all events for $n \in I_j^m$. For doing so we notice that for $n \in I_j^m$ we have that
\[
\mathbb{E} \left( T_{2n}^{2n} v_m \right) = n \cdot \mathbb{E} \left( v_{2n}^{2n} \right) \geq m^j \cdot \mathbb{E} \left( v_{2n}^{2n} \right) = \frac{1}{m} \cdot \mathbb{E} \left( \sum_{\ell=0}^{m^{j-1}} v_{2n}^{2n} \circ \tau_B^{f_{m+u}} \right).
\]
For the following we set $\epsilon_1 = \epsilon / (3m)$. Hence, $n \in I_j^m$ and thus $\gamma(n, u) \in I_j^{m+u}$ implies
\[
\left\{ \sum_{\ell=0}^{\gamma(n,u)} T_{2n}^{2n} \circ \tau_B^{f_{m+u}} \right\} > \epsilon_1 \cdot \mathbb{E} \left( T_{2n}^{2n} v_m \right)
\subset \bigcup_{k \in I_j^{m+u}} \left\{ \sum_{\ell=0}^{\gamma(k,u)} T_{2n}^{2n} \circ \tau_B^{f_{m+u}} \right\} > \epsilon_1 \cdot \mathbb{E} \left( \sum_{\ell=0}^{m^{j-1}} v_{2n}^{2n} \circ \tau_B^{f_{m+u}} \right).
\]
In order to keep our notation short we introduce the set $\Gamma$ for an index set $J$, a non-negative integrable observable $\varphi$, and a transformation $\xi$ as

$$\Gamma (J, \varphi, \xi) = \left\{ \max_{k \in J} \left| k \sum_{\ell=0}^{k} (\varphi - \E (\varphi)) \circ \xi^\ell \right| > \epsilon_1 \cdot \E \left( \min_{k \in J} \left| \sum_{\ell=0}^{k-1} \varphi \circ \xi^\ell \right| \right) \right\}.$$  

Then the righthand side of (64) writes as $\Gamma \left( I_{j-1}^m, v_{2t_n}^2 \circ \tau_B^u, \tau_B^m \right)$.

For different $n \in I_j^m$ the term $2t_n$ takes different values. In the next steps we aim to tackle this problem in order to obtain a concise expression of the righthand side of (64) which only depends on $m$ and $j$. Since $2t_n$ is monotonically increasing, we have that $2t_n \in [2t_{mj}, 2t_{mj+1})$ if $n \in I_j^m$.

We set

$$r_j = \lfloor \log_2 (2t_{mj}) \rfloor \quad \text{and} \quad s_j = \lfloor \log_2 (2t_{mj+1}) \rfloor.$$  

Note that there is a dependence on $m$ in $r_j$ and $s_j$ which we omit for brevity. Keeping in mind that by its definition $v_{2t} = v_{1}^k$ if $\lfloor \log_2 r \rfloor = \lfloor \log_2 k \rfloor$ and using the above notation we obtain from (64)

$$\bigcup_{n \in I_j^m} \left\{ \gamma(n,u) \sum_{\ell=0}^{2t_n^k} x_{B}^{e \cdot m + u} \right\} > \epsilon_1 \cdot \E \left( T_{n}^{2t_n^k} v_m^1 \right) \subset \bigcup_{p=r_j}^{s_j} \Gamma \left( I_{j-1}^m, v_{2t_n^k}^p \circ \tau_B^u, \tau_B^m \right),$$

(65)

The reason we do this estimate is that in stead of considering $# I_{j-1}^m = m^j - 1 - m^{j-1}$ summands we later only consider $# [r_j, s_j] \cap N$ summands (estimated in (63)) in the Borel-Cantelli sum yielding a better result.

Combining (61) with (63), (64), and (65) gives the existence of $N, J \in \N$ such that

(66)

$$\bigcup_{n \in \N \geq N} \left\{ T_{n}^{2t_n^k} - \E \left( T_{n}^{2t_n^k} \right) > \epsilon \cdot \E \left( T_{n}^{2t_n^k} \right) \right\} \subset \bigcup_{j \in \N \geq j} \bigcup_{u=0}^{m} \bigcup_{p=r_j}^{s_j} \Gamma \left( I_{j-1}^m, v_{2t_n^k}^p \circ \tau_B^u, \tau_B^m \right).$$

The case starting with (62) can be done analogously resulting in the existence of $N, J \in \N$ such that

(67)

$$\bigcup_{n \in \N \geq N} \left\{ \E \left( T_{n}^{2t_n^k} \right) - T_{n}^{2t_n^k} > \epsilon \cdot \E \left( T_{n}^{2t_n^k} \right) \right\} \subset \bigcup_{j \in \N \geq j} \bigcup_{u=0}^{m} \bigcup_{p=r_j}^{s_j} \Gamma \left( I_{j-1}^m, v_{2t_n^k}^p \circ \tau_B^u, \tau_B^m \right).$$

We now start the second part of the proof by estimating

$$\sum_{j=J}^{\infty} \sum_{u=0}^{m} \sum_{p=r_j}^{s_j} \left( \lambda \left( \Gamma \left( I_{j-1}^m, v_{2t_n^k}^p \circ \tau_B^u, \tau_B^m \right) \right) + \lambda \left( \Gamma \left( I_{j-1}^m, v_{2t_n^k}^p \circ \tau_B^u, \tau_B^m \right) \right) \right).$$

As mentioned earlier in this section, by Lemma 3.20 $\sum_{\ell=0}^{k} v_{2t_n^k}^p \circ \tau_B^e \cdot m + u$ is a sum of independent random variables and we can apply Lemma 3.20.

We note

$$v_{2t_n^k}^p \leq \max_{i \in \{0, \ldots, 2m^k - 1 \}} \nu_{\lfloor \log_2 v_{2t_n^k}^p \rfloor, i} = \max_{i \in \{0, \ldots, 2m^k - 1 \}} \nu_{p, i}$$

$$= \max_{i \in \{0, \ldots, 2m^k - 1 \}} \frac{2p^m + 1}{2m^k - 1} = \frac{2p^m + 1}{2m^k - (2m^k - 1) - 1} = 2p^m + 1.$$  

(68)

Note that by its definition $v_{2t}^1 = v_{1}^k$ if $\lfloor \log_2 r \rfloor = \lfloor \log_2 k \rfloor$ and using the above notation we obtain from (64)

$$\bigcup_{n \in I_j^m} \left\{ \gamma(n,u) \sum_{\ell=0}^{2t_n^k} x_{B}^{e \cdot m + u} \right\} > \epsilon_1 \cdot \E \left( T_{n}^{2t_n^k} v_m^1 \right) \subset \bigcup_{p=r_j}^{s_j} \Gamma \left( I_{j-1}^m, v_{2t_n^k}^p \circ \tau_B^u, \tau_B^m \right),$$

(65)
This yields, for all \( u \in \mathbb{N}_0 \),

\[
\lambda \left( \Gamma \left( I_{j-1}^m, v_{m}^{2^p} \circ \tau_B^m \right) \right)
\leq \lambda \left( \max_{k \in I_{j-1}^m} \left| \sum_{\ell=0}^{k-1} v_{m}^{2^p} \circ \tau_B^{\ell} \right| > \epsilon_1 \cdot \mathbb{E} \left( \min\{ k \in I_{j-1}^m \} - 1 \right) \right)
\leq 2 \exp \left( -\epsilon_2 \cdot m^{j-1} \cdot \frac{\mathbb{E} (v_m^{2^p})}{2^{p+2}} \right)
\]  

(69)

with \( \epsilon_2 := (3\epsilon_1^2) / (6 + 2\epsilon_1) \). Further note that by Lemma 3.15 and (68) we have \( w_m^{2^p} \leq v_m^{2^p} \leq 2^{p+2} \). Thus, an analogous calculation as above yields

\[
\lambda \left( \Gamma \left( I_{j-1}^m, w_m^{2^p} \circ \tau_B^m \right) \right) \leq 2 \exp \left( -\epsilon_2 \cdot m^{j-1} \cdot \frac{\mathbb{E} (v_m^{2^p})}{2^{p+2}} \right).
\]

(70)

Next note that by Lemma 3.47 for \( \epsilon \in (0, 2 - 1 / \log 2) \), there exists \( M, L \) such that, for \( m \geq M \) and \( 2^p > L \),

\[
\mathbb{E} (w_m^{2^p}) > (2 - \epsilon) \log 2^p \geq p.
\]

(71)

Combining (69) and (70) with (71) and noting that, by Lemma 3.15 \( \mathbb{E} (v_m^{2^p}) \geq \mathbb{E} (w_m^{2^p}) \) yields

\[
\lambda \left( \Gamma \left( I_{j-1}^m, v_m^{2^p} \circ \tau_B^m \right) \right) + \lambda \left( \Gamma \left( I_{j-1}^m, w_m^{2^p} \circ \tau_B^m \right) \right) \leq 4 \exp \left( -\epsilon_2 \cdot m^{j-1} \cdot \frac{p}{2^{p+2}} \right).
\]

(72)

Since \( p/2^{p+2} \) is monotonically decreasing for \( p \in [r_j, s_j] \) and \( j \) sufficiently large, we have for \( p \in [r_j, s_j] \)

\[
\frac{p}{2^{p+2}} \geq \frac{s_j}{2^{s_j+2}} \geq \frac{\log_2 (2t_{m^{j+1}})}{2 \log_2 (2t_{m^{j+1}})} \geq \frac{t_{m^{j+1}}}{24 \cdot t_{m^{j+1}}} \geq \frac{(j + 1) \log_2 m}{24 \cdot m^{j+1} \cdot ((j + 1) \log m)^{3/4}} \geq \frac{j^{1/4}}{2^{4} \cdot m^{j+1}}.
\]

Combining this with (72) yields, for all \( p \in [r_j, s_j] \cap \mathbb{N} \),

\[
\lambda \left( \Gamma \left( I_{j-1}^m, v_m^{2^p} \circ \tau_B^m \right) \right) + \lambda \left( \Gamma \left( I_{j-1}^m, w_m^{2^p} \circ \tau_B^m \right) \right) \leq 4 \exp \left( -\epsilon_2 \cdot m^{j-1} \cdot \frac{s_j}{2^{s_j+2}} \right) \leq 4 \exp \left( -\epsilon_2 \cdot m^{j-1} \cdot \frac{j^{1/4}}{2^{4} \cdot m^{j+1}} \right)
\]

\[
= 4 \exp \left( -\epsilon_2 \cdot m^{j-1} \cdot \frac{j^{1/4}}{2^{4} \cdot m^{j+1}} \right) \leq 4 \exp \left( -j^{1/5} \right),
\]

for \( j \) sufficiently large.

Furthermore,

\[
s_j - r_j = \left| \log_2 \left( 2t_{m^{j+1}} \right) \right| - \left| \log_2 \left( 2t_{m^j} \right) \right| \leq \log_2 \left( \frac{2t_{m^{j+1}}}{2t_{m_j}} \right) + 1
\]

\[
= \log_2 \left( \frac{m^{j+1} \cdot (\log m^{j+1})^{3/4}}{m^j \cdot (\log m^j)^{3/4}} \right) + 1 \leq \log_2 m + 2,
\]

(73)
for $j$ sufficiently large. This implies
\[
\sum_{p=r_j}^{s_j} \left( \lambda \left( \Gamma \left( I_{j-1}^m, v^p_m \circ \tau_B^m \right) \right) + \lambda \left( \Gamma \left( I_{j-1}^m, w^p_m \circ \tau_B^m \right) \right) \right) 
\leq 4 \sum_{p=r_j}^{s_j} \exp \left( -j^{1/5} \right) \leq 4 (\log m + 2) \exp \left( -j^{1/5} \right),
\]
for $j$ sufficiently large.

Since $\lambda \left( \Gamma \left( I_{j-1}^m, v^p_m \circ \tau_B^m \right) \right)$ and $\lambda \left( \Gamma \left( I_{j-1}^m, w^p_m \circ \tau_B^m \right) \right)$ do not differ for different $u \in \mathbb{N}_0$, see (60), we obtain
\[
\sum_{u=0}^{m-1} \sum_{p=r_j}^{s_j} \left( \lambda \left( \Gamma \left( I_{j-1}^m, v^p_m \circ \tau_B^m \right) \right) + \lambda \left( \Gamma \left( I_{j-1}^m, w^p_m \circ \tau_B^m \right) \right) \right) 
\leq 4m (\log m + 2) \exp \left( -j^{1/5} \right),
\]
for $j$ sufficiently large. Finally, if we choose $J$ sufficiently large, then
\[
\sum_{j=J}^{\infty} \sum_{u=0}^{m-1} \sum_{p=r_j}^{s_j} \left( \lambda \left( \Gamma \left( I_{j-1}^m, v^p_m \circ \tau_B^m \right) \right) + \lambda \left( \Gamma \left( I_{j-1}^m, w^p_m \circ \tau_B^m \right) \right) \right) 
\leq \sum_{j=J}^{\infty} 4m (\log m + 2) \exp \left( -j^{1/5} \right) < \infty.
\]

An application of the first Borel-Cantelli lemma on (66) and (67) gives the statement of the lemma.

The next lemma gives a statement about average hitting times and will later give us the possibility to compare $T^{2n}_n$ with $T^{2m}_n$.

**Lemma 3.21.** We have
\[
\lambda \left( \left| \sum_{k=0}^{n-1} \phi \circ \tau_{\mathbb{N}}^k - 2m \right| > n^{3/4} \ i.o. \right) = 0.
\]

**Proof.** First note, that since $\phi$ is measurable on $\mathcal{F}$, by Corollary 3.6, the sequence of random variables $(\phi \circ \tau_{\mathbb{N}}^k)_{k \in \mathbb{N}}$ is independent. We define the following sets
\[
\Upsilon_n = \left\{ \left| \sum_{k=0}^{n-1} \phi \circ \tau_{\mathbb{N}}^k x - 2n \right| > n^{3/4} \right\} \quad \text{and} \quad \Xi_n = \# \left\{ k \leq n : \phi \circ \tau_{\mathbb{N}}^{k-1} > 2 \log n \right\} = 0.
\]

If we denote by $A^c$ the complementary event of an event $A$, then we have
\[
\lambda \left( \limsup_{n \to \infty} \Upsilon_n \right) \leq \lambda \left( \limsup_{n \to \infty} (\Upsilon_n \cap \Xi_n) \cup \Xi_n^c \right) = \lambda \left( \limsup_{n \to \infty} (\Upsilon_n \cap \Xi_n) \cup \limsup_{n \to \infty} \Xi_n^c \right) 
\leq \lambda \left( \limsup_{n \to \infty} (\Upsilon_n \cap \Xi_n) \right) + \lambda \left( \limsup_{n \to \infty} \Xi_n^c \right).
\]
In order to estimate the first summand of (74) we set $\phi^r = \phi \cdot 1_{\{\phi \leq r\}}$. Then we have

$$\mathcal{Y}_n \cap \Xi_n \subset \left\{ \sum_{k=0}^{n-1} \phi^{2 \log n \circ \tau_B^k} - 2n \right\}^{n^{3/4}}$$

$$= \left\{ \sum_{k=0}^{n-1} \left( \phi^{2 \log n} - \mathbb{E} \left( \phi^{2 \log n} \right) \right) \circ \tau_B^k - \left(2n - n \cdot \mathbb{E} \left( \phi^{2 \log n} \right) \right) \right\}^{n^{3/4}}$$

(75)

$$\subset \left\{ \sum_{k=0}^{n-1} \left( \phi^{2 \log n} - \mathbb{E} \left( \phi^{2 \log n} \right) \right) \circ \tau_B^k \right\}^{n^{3/4}} \setminus \left\{ \sum_{k=0}^{n-1} \left( \phi^{2 \log n} - \mathbb{E} \left( \phi^{2 \log n} \right) \right) \circ \tau_B^k \right\}^{n^{3/4}} \setminus \left(2n - n \cdot \mathbb{E} \left( \phi^{2 \log n} \right) \right)$$

We aim to apply Lemma 3.19 for which we need to calculate $\gamma_n = \left( \mathbb{E} \left( \phi^{2 \log n} \right) \right) \cdot n$ first. Obviously,

$$\mathbb{E} \left( \phi^{2 \log n} \right) = \sum_{k=1}^{\lfloor 2 \log n \rfloor} k \cdot \lambda(\phi = k) = \sum_{k=1}^{\lfloor 2 \log n \rfloor} k \cdot 2^{-k}.$$

This can be easily seen, as $\phi(x) = k$ if $x \in J_{k-1}$ and $\lambda(J_{k-1}) = 2^{-k}$. Remember that $\sum_{k=1}^{\infty} k / 2^k = 2$. This gives

$$2 - \mathbb{E} \left( \phi^{2 \log n} \right) = 2 - \sum_{k=1}^{\lfloor 2 \log n \rfloor} k \cdot 2^{-k} = \sum_{k=\lfloor 2 \log n \rfloor + 1}^{\infty} k \cdot 2^{-k}.$$

Calculating the remainder term with $j = \lfloor 2 \log n \rfloor + 1$ and applying the geometric series formula gives

$$2 - \mathbb{E} \left( \phi^{2 \log n} \right) = \sum_{k= \lfloor 2 \log n \rfloor + 1}^{\infty} \frac{k}{2^k} = j \cdot \sum_{k= \lfloor 2 \log n \rfloor + 1}^{\infty} \frac{1}{2^k} + \sum_{k= \lfloor 2 \log n \rfloor + 1}^{\infty} \frac{k - j}{2^k} = j \cdot 2^{-j+1} + 2^{-j} \sum_{k=0}^{\infty} \frac{k}{2^k}$$

$$= j \cdot 2^{-j+1} + 2^{-j+1} = \frac{2 \log n}{2^{\lfloor 2 \log n \rfloor}} + 2.$$ 

This yields

$$\gamma_n \leq \frac{2 \cdot (2 \log n + 2)}{n^{2 \log 2}} \cdot n = 4 \cdot (\log n + 1) \cdot n^{1-2 \log 2} \leq n^{1/2},$$

for $n$ sufficiently large. Thus, $n^{3/4} - \gamma_n \geq n^{5/8}$, for $n$ sufficiently large. Combining this with 75 and applying Lemma 3.19 yields

$$\lambda(\mathcal{Y}_n \cap \Xi_n) \leq \lambda \left( \sum_{k=0}^{n-1} \left( \phi^{2 \log n} - \mathbb{E} \left( \phi^{2 \log n} \right) \right) \circ \tau_B^k \right) \geq \frac{n^{5/4}}{2 \mathcal{V} \left( \sum_{k=0}^{n-1} \phi^{2 \log n} \circ \tau_B^k \right) + \frac{2}{3} \cdot n^{5/8} \cdot 2 \log n},$$

(76)

for $n$ sufficiently large. We have, using independence of $\phi \circ \tau_B^k$ and a similar approach as in the calculation of $\mathbb{E} \left( \phi^{2 \log n} \right)$, that

$$\mathcal{V} \left( \sum_{k=0}^{n-1} \phi^{2 \log n} \circ \tau_B^k \right) = n \cdot \mathbb{V} \left( \phi^{2 \log n} \right) \leq n \cdot \int \left( \phi^{2 \log n} \right)^2 \lambda = n \cdot \sum_{k=1}^{\lfloor 2 \log n \rfloor} k^2 / 2^k$$

$$\leq n \cdot \sum_{k=1}^{\infty} k^2 / 2^k = 6n.$$

Applying this on 76 gives

$$\lambda(\mathcal{Y}_n \cap \Xi_n) \leq \exp \left( -\frac{n^{5/4}}{12n + \frac{2}{3} \cdot n^{5/8} \cdot 2 \log n} \right) \leq \exp \left( -\frac{n^{5/4}}{13n} \right) \leq \exp \left( -n^{1/8} \right).$$
for $n$ sufficiently large. Since $\sum_{n=1}^{\infty} \exp(-n^{1/8}) < \infty$, applying the first Borel-Cantelli lemma yields

(77) \quad \lambda(\Upsilon_n \cap \Xi_n \ i.o.) = 0.

In the next steps we calculate the second summand of (74). We have, for $x \in \Omega'$, that $\phi \circ \tau_B^{-1}\uparrow n$ is equivalent to $\tau_B^{-1}\uparrow n \circ \psi \circ \tau_B^{-1}\uparrow n \in \bigcup_{j=0}^{\lfloor \log n \rfloor-1} J_j$, see the proof of Lemma 3.13 and this is equivalent to $\beta_k(x) \geq 2^{\lfloor \log n \rfloor-1}$. As $n^{2\log 2/4} \leq 2^{\lfloor \log n \rfloor-1}$, we obtain

$$\Xi_n \subset \left\{ \# \left\{ k \leq n : \beta_k \geq n^{2\log 2/4} \right\} \geq 1 \right\} \subset \left\{ \# \left\{ k \leq n : \beta_k \geq n \cdot n^{2\log 2-1/4} \right\} \geq 1 \right\}. $$

If we set $\psi(n) = e^{n(2\log 2-1/4)}$, then $\psi([\log n]) = e^{[\log n](2\log 2-1)}$ and $\psi \in \Psi$. Hence, we can apply Lemma 3.11 which yields that the second summand of (74) equals zero.

Combining this with (77) and (74) yields the statement of the lemma. \hfill \Box

**Proof of Lemma 3.3** We remember the definition of $\phi_n$ in (20). The strategy is to compare $\sum_{k=1}^{n} \eta^{2\phi_k} \circ \tau_B^{-1}\uparrow n$ with $T_{\phi_n}^{\infty}$ with $\ell$ to be determined and finally use Lemma 3.18 to obtain the statement of the lemma.

Lemma 3.21 implies that we have eventually almost surely

$$2n - n^{3/4} \leq \phi_n \leq 2n + n^{3/4}. $$

This yields that we have for every $\epsilon \in (0, 1/2)$ eventually almost surely

$$\phi_{[n(1/2-\epsilon)]} \leq n \cdot (1-2\epsilon) + (n \cdot (1/2-\epsilon))^{3/4} \leq n$$

and

$$\phi_{[n(1/2+\epsilon)]} \geq 2 \cdot n \cdot (1/2+\epsilon) - (2n \cdot (1/2+\epsilon))^{3/4} \geq n.$$

Thus, we have for every $\epsilon \in (0, 1/2)$ eventually almost surely

$$T_{\phi_{[n(1/2-\epsilon)]}}^{\infty} \leq T_{\phi_{[n(1/2+\epsilon)]}}^{\infty} \leq T_{\phi_{[n(1/2-\epsilon)]}}^{\infty}.$$

An easy calculation shows, for all $\epsilon \in (0, 1/2)$, that $2t_{[n(1/2-\epsilon)]} \leq t_n \leq 2t_{[n(1/2+\epsilon)]}$ for $n$ sufficiently large. This implies

$$T_{\phi_{[n(1/2-\epsilon)]}}^{\infty} \leq T_{\phi_{[n(1/2+\epsilon)]}}^{\infty} \leq T_{\phi_{[n(1/2-\epsilon)]}}^{\infty}$$

eventually almost surely. We remember that

$$T_{\phi_n}^{\infty} = \sum_{k=0}^{\phi(n)-1} \chi^r \circ \tau_k = \sum_{k=0}^{n-1} \eta^r \circ \tau_k \circ T_{\phi_n}^{\infty},$$

which implies

$$T_{\phi_{[n(1/2-\epsilon)]}}^{\infty} \eta \leq T_{\phi_{[n(1/2-\epsilon)]}}^{\infty} \eta \leq T_{\phi_{[n(1/2-\epsilon)]}}^{\infty} \eta$$

eventually almost surely. Using Lemma 8.18 implies, for all $\epsilon \in (0, 1/2)$, eventually almost surely

$$\mathbb{E}\left(T_{\phi_{[n(1/2-\epsilon)]}}^{\infty} \eta \right) \cdot (1-\epsilon) \leq T_{\phi_{[n(1/2-\epsilon)]}}^{\infty} \eta \leq \mathbb{E}\left(T_{\phi_{[n(1/2+\epsilon)]}}^{\infty} \eta \right) \cdot (1+\epsilon).$$

Furthermore, since by Lemma 8.3 $\lambda$ is $\tau_B$-invariant, Lemma 8.15 implies that we have for every $\epsilon \in (0, 1/2)$ and $m \in \mathbb{N}$ sufficiently large eventually almost surely

$$T_{\phi_{[n(1/2-\epsilon)]}}^{\infty} \geq \left| n \cdot (1/2-\epsilon) \right| \cdot \mathbb{E}\left(\eta^{2\phi_{[n(1/2-\epsilon)]}} \right) \cdot (1-\epsilon) \geq n \cdot \mathbb{E}\left(\eta^{2\phi_{[n(1/2-\epsilon)]}} \right) \cdot (1/2-2\epsilon).$$

Choosing $m$ sufficiently large and applying Lemma 8.17 for every $\epsilon \in (0, 1/2)$ eventually almost surely

$$T_{\phi_{[n(1/2-\epsilon)]}}^{\infty} \geq n \cdot \log(2t_{[n(1/2-\epsilon)]}) \cdot (1-3\epsilon) \geq n \cdot \log(n \cdot (1-2\epsilon)) \cdot (1-3\epsilon) \geq n \cdot \log n \cdot (1-4\epsilon).$$

(78)
On the other hand, by an analogous combination of Lemma 3.13 and Lemma 3.16 we have for every \( \epsilon \in (0, 1/2) \) eventually almost surely
\[
T_n^\epsilon \leq \lfloor n \rfloor \cdot \mathbb{E} \left( \eta_{\lfloor n \rfloor (1/2 + \epsilon)} \right) \cdot (1 + \epsilon) \leq n \cdot \mathbb{E} \left( \eta_{\lfloor n \rfloor (1/2 + \epsilon)} \right) \cdot (1/2 + 2\epsilon) \\
\leq n \cdot \log(2t_{\lfloor n \rfloor (1/2 + \epsilon)}) \cdot (1 + 2\epsilon) \leq n \cdot \log(2t_{\lfloor n \rfloor (1/2 + \epsilon)}) \cdot (1 + 6\epsilon) \\
\leq n \cdot \log n^{1+\epsilon} \cdot (1 + 7\epsilon) \leq n \cdot \log n \cdot (1 + 12\epsilon).
\]

Combining (78) and (79) gives the statement of the lemma. \( \square \)

4. Proof of the weak convergence Theorem 1.4

4.1. Mixing properties of the digits \((a_n)\). In contrast to the proof of Theorem 1.1 we don’t use the independence properties of the induced transformation but show that the digits \((a_n)\) are \(\alpha\)-mixing which enables us to prove Theorem 1.4. So we will first give the definition of \(\alpha\)-mixing random variables.

**Definition 4.1.** Let \((\Omega', \mathcal{A}, \mathbb{P})\) be a probability measure space and \(\mathcal{B}, \mathcal{C} \subset \mathcal{A}\) two \(\sigma\)-fields, then the following measure of dependence is defined.
\[
\alpha(\mathcal{B}, \mathcal{C}) = \sup_{B, C, D} |\mathbb{P}(B \cap C) - \mathbb{P}(B) \cdot \mathbb{P}(C)|
\]

Furthermore, let \((X_n)_{n \in \mathbb{N}}\) be a (not necessarily stationary) sequence of random variables. For \(-\infty \leq J \leq L \leq \infty\) we can define a \(\sigma\)-field by
\[
\mathcal{A}_j^L = \sigma(X_k, J \leq k \leq L, k \in \mathbb{Z}).
\]

With that the dependence coefficient is defined by
\[
\alpha(n) = \sup_{k \in \mathbb{N}} \alpha(\mathcal{A}_1^k, \mathcal{A}_{k+n}^\infty),
\]

The sequence \((X_n)\) is said to be \(\alpha\)-mixing if \(\lim_{n \to \infty} \alpha(n) = 0\).

For further properties of mixing random variables see [Bra05].

**Lemma 4.2.** The sequence \((a_n)\) is \(\alpha\)-mixing.

**Proof.** The proof is based on a decay of correlation argument for the transfer operator going back to classical results, see for example [Bal00]. Since the proof in this case is reasonably short we redo it as a special case. In order to proceed we first need the notion of bounded variation.

**Definition 4.3.** For \(\varphi : [0, 1) \to \mathbb{R}_{\geq 0}\) the variation \(V(\varphi)\) is given by
\[
V(\varphi) = \sup \left\{ \sum_{i=1}^n |\varphi(x_i) - \varphi(x_{i-1})| : n \geq 1, x_i \in [0, 1), x_0 < x_1 < \ldots < x_n \right\}.
\]

By \(BV\) we denote the Banach space of functions of bounded variation, i.e. of functions \(\varphi\) fulfilling \(V(\varphi) < \infty\). It is equipped with the norm \(\|\varphi\|_{BV} = V(\varphi) + \|\varphi\|_\infty\).

For further properties of functions of bounded variation see for example [BG97, Chapter 2].

We define the transfer operator \(\hat{T}\) as the uniquely up to a.s. equivalence defined operator such that for all \(\varphi \in L^\infty\) and all \(\zeta \in L^1\) it holds that
\[
\int (\varphi \circ \tau) \cdot \zeta d\lambda = \int \varphi \cdot \hat{T}\zeta d\lambda.
\]

For every \(\varphi \in L^\infty\) we have that
\[
\int_0^1 \varphi(\tau x) \cdot \zeta(x) d\lambda(x) = \int_0^{1/2} \varphi(2x) \cdot \zeta(x) d\lambda(x) + \int_{1/2}^1 \varphi(2x - 1) \cdot \zeta(x) d\lambda(x).
\]
Setting $s = 2x$ in the first summand and $t = 2x - 1$ in the second summand yields
\[
\int_0^1 \varphi(tx) \cdot \zeta(x) \, d\lambda(x) = \int_0^1 \varphi(s) \cdot \zeta \left( \frac{s}{2} \right) \, d\lambda \left( \frac{s}{2} \right) + \int_0^1 \varphi(t) \cdot \zeta \left( \frac{t+1}{2} \right) \, d\lambda \left( \frac{t+1}{2} \right)
\]
\[
= \int_0^1 \varphi(x) \cdot \frac{1}{2} \left( \zeta \left( \frac{x}{2} \right) + \zeta \left( \frac{x+1}{2} \right) \right) \, d\lambda(x).
\]
Thus,
\[
(\tilde{\tau} \zeta)(x) = \frac{1}{2} \left( \zeta \left( \frac{x}{2} \right) + \zeta \left( \frac{x+1}{2} \right) \right).
\]
To obtain some decay of correlation we first estimate now
\[
\mathcal{V}(\tilde{\tau} \zeta) = \sup \sum_{i=1}^n |(\tilde{\tau} \zeta)(x_i) - (\tilde{\tau} \zeta)(x_{i-1})|
\]
\[
= \sup \sum_{i=1}^n \frac{1}{2} \left| \zeta \left( \frac{x_i}{2} \right) + \zeta \left( \frac{x_i+1}{2} \right) - \zeta \left( \frac{x_{i-1}}{2} \right) - \zeta \left( \frac{x_{i-1}+1}{2} \right) \right|,
\]
where the supremum is taken over $n \in \mathbb{N}_0$ and $x_i \in [0,1)$ such that $x_0 < \ldots < x_n$. By renaming $x_i/2 = y_i$ and $(x_i + 1)/2 = y_{i+1}$, we obtain
\[
\sum_{i=1}^n \left| \zeta \left( \frac{x_i}{2} \right) + \zeta \left( \frac{x_i+1}{2} \right) - \zeta \left( \frac{x_{i-1}}{2} \right) - \zeta \left( \frac{x_{i-1}+1}{2} \right) \right|
\]
\[
= \sum_{i=1}^n |\zeta(y_i) + \zeta(y_{i+1}) - \zeta(y_{i-1}) - \zeta(y_{i+1})|
\]
\[
\leq \sum_{i=1}^{2n} |\zeta(y_i) - \zeta(y_{i-1})|.
\]
Thus,
\[
\mathcal{V}(\tilde{\tau} \zeta) = \sup \left\{ \sum_{i=1}^n |(\tilde{\tau} \zeta)(x_i) - (\tilde{\tau} \zeta)(x_{i-1})| : n \in \mathbb{N}, x_i \in [0,1), x_1 < \ldots < x_n \right\}
\]
\[
\leq \frac{1}{2} \sup \left\{ \sum_{i=1}^m |\zeta(z_i) - \zeta(z_{i-1})| : m \in \mathbb{N}, z_i \in [0,1), z_1 < \ldots < z_m \right\} = \frac{1}{2} \mathcal{V}(\zeta).
\]

Furthermore, we can decompose the space of $BV$-functions in $BV = P \oplus H$, where $P$ is the projective space $\mathbb{C}$ and $H = \{ \zeta \in BV : \int \zeta \, d\lambda = 0 \}$. Each $\zeta \in BV$ can be written as $\zeta = \int \zeta \, d\lambda + \zeta_H$, where $\int \zeta_H \, d\lambda = 0$. The decomposition of $BV$ is invariant under $\tilde{\tau}$ since $\tilde{\tau} \mathbb{1} = \mathbb{1}$ and for $\int \zeta \, d\lambda = 0$ it holds that $\int \tilde{\tau} \zeta_H \, d\lambda = \int (\mathbb{1} \circ \tau) \zeta_H \, d\lambda = \int \zeta_H \, d\lambda = 0$.

To obtain a decay of correlation result we notice that an iterated application of the definition of the transfer operator yields
\[
\int (\varphi \circ \tau^k) \cdot \zeta \, d\lambda = \int \varphi \cdot (\tilde{\tau}^k \zeta) \, d\lambda.
\]
The decay of correlation is then estimated by
\[
\text{Cor}_n(\varphi, \zeta) = \left| \int (\varphi \circ \tau^n) \cdot \zeta \, d\lambda - \int \varphi \, d\lambda \cdot \int \zeta \, d\lambda \right| = \left| \int \varphi \cdot (\tilde{\tau}^n \zeta) \, d\lambda - \int \varphi \, d\lambda \cdot \int \zeta \, d\lambda \right|
\]
\[
= \left| \int \varphi \cdot \tilde{\tau}^n \left( \int \zeta \, d\lambda + \zeta_H \right) \, d\lambda - \int \varphi \, d\lambda \cdot \int \zeta \, d\lambda \right| = \left| \int \varphi \cdot \tilde{\tau}^n \zeta_H \, d\lambda \right|
\]
\[
\leq \|\varphi\|_1 \cdot \|\zeta_H\|_\infty.
\]
where we used the fact that \( \hat{\tau}^n 1 = 1 \). Since \( \hat{\tau}^n \zeta_H \in H \), it follows that its range has a diameter less or equal to \( V(\zeta) \) and contains zero in the convex hull. Thus, \( \|\hat{\tau}^n \zeta_H\|_\infty \leq V(\hat{\tau}^n \zeta_H) \) and by (80) it follows that
\[
\|\hat{\tau}^n \zeta_H\|_\infty \leq V(\hat{\tau}^n \zeta_H) \leq 2^{-n} \cdot V(\zeta_H) = 2^{-n} \cdot V(\zeta).
\]
Combining this with (81) yields for all \( \varphi \in L^1 \) and all \( \zeta \in BV(1) \)
\[
\text{Cor}_n(\varphi, \zeta) \leq 2^{-n} \cdot \| \varphi \|_1 \cdot V(\zeta).
\]
We further note that each \( a_i \) can only take values in the natural numbers. To prove \( \alpha \)-mixing we first notice that for all \( i, k, n \in \mathbb{N} \) and \( A \in \sigma(\mathbb{N}) \) we have that
\[
\lambda(\{a_k = i\} \cap \{a_{n+k} \in A\}) = \int \mathbf{1}_{\{a_k = i\}} \cdot (\mathbb{I}_A \circ \tau^{n+k-1}) \, d\lambda
\]
\[
= \int (\mathbf{1}_{\{a_k = i\}} \circ \tau^{k-1}) \cdot (\mathbb{I}_A \circ \tau^{n+k-1}) \, d\lambda
\]
\[
= \int (\mathbf{1}_{a_k = i}) \cdot (\mathbb{I}_A \circ \tau^{n-1}) \, d\lambda.
\]
Obviously, \( \| \mathbb{I}_A \|_\infty \leq 1 \) and thus \( \mathbb{I}_A \in L^1 \) and further \( V(\mathbb{I}_{\{a_k = i\}}) \leq 2 \) and thus \( \mathbb{I}_{\{a_k = i\}} \in BV \). Applying (82) and the fact that \( \parallel A \parallel = \lambda(A) \) yields
\[
\lambda(\{a_k = i\} \cap \{a_{n+k} \in A\}) - \lambda(a_k = i) \lambda(a_{n+k} \in A) \leq 2^{-n}.
\]
Since every \( B \in \sigma(\mathbb{N}) \) is just any subset \( I \subset \mathbb{N} \), we have for each \( k, n \in \mathbb{N} \)
\[
\lambda(\{a_k = b\} \cap \{a_{n+k} \in A\}) - \lambda(a_k = b) \lambda(a_{n+k} \in A) \leq \left| \sum_{i \in I} \left( \lambda(\{a_k = i\} \cap \{a_{n+k} \in A\}) - \lambda(a_k = i) \lambda(a_{n+k} \in A) \right) \right|
\]
\[
\leq \sum_{i \in \hat{I} : i \in [2^{n/2}]} \left( \lambda(\{a_k = i\} \cap \{a_{n+k} \in A\}) - \lambda(a_k = i) \lambda(a_{n+k} \in A) \right)
\]
\[
+ \lambda \left( a_k \geq \left\lfloor \frac{2^{n/2}}{2} \right\rfloor \right).
\]
By (83) we can estimate the sum in (84), for each \( k, n \in \mathbb{N} \), by
\[
\sum_{i \in I : i \in [2^{n/2}]} \left( \lambda(\{a_k = i\} \cap \{a_{n+k} \in A\}) - \lambda(a_k = i) \lambda(a_{n+k} \in A) \right) \leq \left\lfloor \frac{2^{n/2}}{2} \right\rfloor \cdot 2^{-n} \leq 2^{-n/2}.
\]
Using the distribution function in (14) the summand in (85) can, for each \( k \in \mathbb{N} \), be estimated by
\[
\lambda(a_k \geq \left\lfloor \frac{2^{n/2}}{2} \right\rfloor) = \frac{1}{\left\lfloor \frac{2^{n/2}}{2} \right\rfloor} \leq 2^{-n/2+1}.
\]
Combining (80) and (87) yields, for all \( k, n \in \mathbb{N} \),
\[
\left| \lambda(\{a_k = b\} \cap \{a_{n+k} \in A\}) - \lambda(a_k = b) \lambda(a_{n+k} \in A) \right| \leq 2^{-n/2+2}.
\]
\[
\square
\]
4.2. Proof of Theorem 1.4. To prove Theorem 1.4 we need additionally to the mixing properties of the digits some auxiliary definitions and lemmas.

**Definition 4.4** (Property B). Let \( (Y_n) \) be a sequence of strictly stationary random variables and \( Z_n = \sum_{k=1}^n Y_k \). We say that Property B is fulfilled for a sequence of constants \( (B_n)_{n \in \mathbb{N}} \) if
\[
\lim_{n \to \infty} \max_{k, \ell} \left| \mathbb{E} \left( \exp \left( it \cdot \frac{Z_{k+\ell}}{B_n} \right) \right) - \mathbb{E} \left( \exp \left( it \cdot \frac{Z_k}{B_n} \right) \right) \cdot \mathbb{E} \left( \exp \left( it \cdot \frac{Z_{\ell}}{B_n} \right) \right) \right| = 0
\]
for all \( t \in \mathbb{R} \), where the maximum is taken over \( k, \ell \in \mathbb{N} \) fulfilling \( 1 \leq k + \ell \leq n \).
The following lemma will give a criterion for convergence in probability for a sum of truncated normed random variables.

**Lemma 4.5 ([Sze01, Theorem 2]).** Let \((Y_n)_{n \in \mathbb{N}}\) be a sequence of non-negative, identically distributed random variables and for \(r > 0\) set \(U_n^{f_n} := \sum_{k=1}^{n} Y_k \cdot 1\{Y_k \leq r\}\). Furthermore, assume that the following hold:

(a) There exists a positive valued sequence \((f_n)\) tending to infinity such that Property B is fulfilled for \(E(U_n^{f_n})\),

(b) we have that
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} Y_k \cdot 1\{Y_k > f_n\}}{E(U_n^{f_n})} = 0
\]

in probability, and

(c) \((U_n^{f_n}/E(U_n^{f_n}))\) is uniformly integrable.

Then
\[
\lim_{n \to \infty} \frac{U_n^{f_n}}{E(U_n^{f_n})} = 1
\]
in probability.

The next two lemmas will enable us to apply Lemma 4.5.

**Lemma 4.6 ([Jak93, Lemma 5.2]).** If \((Y_n)_{n \in \mathbb{N}}\) is a strictly stationary, \(\alpha\)-mixing process, then Property B holds for all sequences \((B_n)\) tending to infinity.

**Lemma 4.7** (de la Vallé-Poissin’s criterion (see [DM78, II, 22])). The family of random variables \((Y_n)_{n \in \mathbb{N}}\) is uniformly integrable if and only if there exists a non-decreasing, convex, continuous function \(h : \mathbb{R}_+ \to \mathbb{R}_+\) fulfilling \(\lim_{x \to \infty} h(x)/x = \infty\) and
\[
\sup_{n \in \mathbb{N}} E(h(Y_n)) < \infty.
\]

With this information at hand we are able to prove Theorem 1.4.

**Proof of Theorem 1.4.** The proof of this theorem will be done in two steps. Setting \(r_n = n \log n\), the first step is to prove that \(\lim_{n \to \infty} \lambda(S_n > T_n^{r_n}) = 0\). In the second step we will prove with the help of Lemma 4.5 that \(\lim_{n \to \infty} T_n^{r_n}/(n \log n) = 1\) in probability.

First we note that by the \(\tau\)-invariance of \(\lambda\)

\[
\lambda(S_n > T_n^{r_n}) = \lambda\left(\bigcup_{k=1}^{n} \{a_k > r_n\}\right) \leq \sum_{k=1}^{n} \lambda(a_k > r_n) = n \cdot \lambda(a_1 > r_n).
\]

Using the distribution of \(a_1\) given in (14) gives
\[
\lim_{n \to \infty} \lambda(S_n > T_n^{r_n}) \leq \lim_{n \to \infty} \frac{n}{r_n} = \lim_{n \to \infty} \frac{1}{\log n} = 0.
\]

In order to prove the second part of the theorem we aim to apply Lemma 4.5 on \(Y_k = a_k\) and \((f_n) = (r_n)\). First we notice that \((a_n)\) are strictly stationary and by Lemma 4.6 and Lemma 4.2 we have that Condition B is fulfilled since \((r_n)\) tends to infinity. This gives us (a) of Lemma 4.5.

Furthermore, by (13) we have for \(n\) sufficiently large that
\[
\lambda\left(\frac{\sum_{k=1}^{n} a_k \cdot 1\{a_k > r_n\}}{E(T_n^{r_n})} > \epsilon\right) \leq \lambda\left(\sum_{k=1}^{n} a_k \cdot 1\{a_k > r_n\} > \epsilon/2 \cdot n \cdot \log r_n\right)
\]
\[
\leq \lambda\left(\sum_{k=1}^{n} 1\{a_k > r_n\} \geq 1\right) = \lambda(S_n > T_n^{r_n}).
\]
which by (90) tends to zero and hence (b) holds.

Finally, to prove the uniform integrability of \((T_n^r/E(T_n^r))\) we use Lemma 4.7 for \(Y_n = T_n^r/E(T_n^r)\) and choose \(h\) as \(h(x) = x^2\). We have that

\[
E\left((T_n^r)^2\right) = \sum_{i,j=1}^{n} E\left(a_i^r \cdot a_j^r\right) = \sum_{k=1}^{n} E\left((a_k^r)^2\right) + 2 \sum_{1 \leq i < j \leq n} E\left(a_i^r \cdot a_j^r\right).
\]

For the first summands in (90) we have by (15) for the second summand of (92) we have by (82) and (15)

\[
E\left((a_k^r)^2\right) = \sum_{i=1}^{\lfloor r_n \rfloor} \lambda(a_k = i) \cdot i^2 = \sum_{i=1}^{\lfloor r_n \rfloor} \left(\frac{1}{i} - \frac{1}{i+1}\right) \cdot i^2 = \sum_{i=1}^{\lfloor r_n \rfloor} \frac{i}{i+1} < r_n
\]

and the choice of \((r_n)\) yields

\[
\sum_{k=1}^{n} E\left((a_k^r)^2\right) < n^2 (\log n).
\]

To estimate the second sum in (90) we notice that

\[
E\left(a_i^r \cdot a_j^r\right) = \int (\chi^{r_n} \circ \tau^{i-1}) \cdot (\chi^{r_n} \circ \tau^{j-1}) \, d\lambda = \int \chi^{r_n} \cdot (\chi^{r_n} \circ \tau^{j-i}) \, d\lambda \leq \left(\int \chi^{r_n} \, d\lambda\right)^2 + \text{Cor}_{j-i}(\chi^{r_n}, \chi^{r_n}).
\]

For the first summand we have by (15)

\[
\left(\int \chi^{r_n} \, d\lambda\right)^2 \leq 2 (\log r_n)^2 = (\log (n \cdot \log n))^2 \leq 4 (\log n)^2,
\]

for \(n\) sufficiently large and for the second summand of (92) we have by (82) and (15)

\[
\text{Cor}_{j-i}(\chi^{r_n}, \chi^{r_n}) \leq 2^{-j+i} \cdot \|\chi^{r_n}\|_1 \cdot V(\chi^{r_n}) \leq 2^{-j+i+1} \cdot \log r_n \cdot r_n \leq 2^{-j+i+1} \cdot (\log n + \log \log n) \cdot n \cdot \log n \leq 2^{-j+i+2} \cdot n \cdot (\log n)^2,
\]

for \(n\) sufficiently large. Combining (92) with (93) and (94) yields

\[
\sum_{1 \leq i < j \leq n} E\left(a_i^r \cdot a_j^r\right) \leq \sum_{i=1}^{n} \left(4 (\log n)^2 + \sum_{j>i} 2^{-j+i+2} \cdot n \cdot (\log n)^2\right)
\]

\[
= 4n \cdot (\log n)^2 + 4n^2 \cdot (\log n)^2 \leq 8n^2 \cdot (\log n)^2,
\]

for \(n\) sufficiently large. Combining (90) with (91) and (95) yields \(E\left((T_n^r)^2\right) \leq 9n^2 (\log n)^2\), for \(n\) sufficiently large.

On the other hand, applying (16) yields

\[
E\left(T_n^r\right) \sim n \cdot \log r_n = n \cdot (\log n + \log \log n) \sim n \cdot \log n
\]

and thus

\[
\lim_{n \to \infty} E\left(\frac{T_n^r}{E(T_n^r)}\right)^2 \leq 9 < \infty.
\]

Hence, (S8) follows and by Lemma 4.7 (c) holds. Hence, Lemma 4.5 is applicable giving the weak convergence \(\lim_{n \to \infty} T_n^r/E(T_n^r) = 1\) in probability.

Lastly, (96) gives the denominator in (12).
5. Proof of Theorem 1.2

Proof of Theorem 1.2. Assume that \((b_n)\) is as in (9) with \(\psi \in \mathcal{V}\). The strategy of the proof is to show for arbitrary \(u \in \mathbb{N}\) that

\[
\lambda(\# \{ i \leq n : a_i \geq n \cdot \log n \} \geq b_n + u \text{ i.o.}) = 1
\]

which implies \(\lambda(S_{b_n}^\infty \geq n \cdot \log n \cdot u \text{ i.o.}) = 1\). Noting that \(u\) can be chosen arbitrarily large gives the statement of the theorem.

To show (97) we set \((I_k)\) as in (31). We note that for \(n \in I_j\), we firstly have \(n \geq 2^j\) and secondly \(n \cdot \log n < 2^{j+1} \cdot \log 2^{j+1} < 2^{j+2} \cdot 2 j\), for \(j\) sufficiently large. For \(n \in I_j\) and \(j\) sufficiently large the definition of \((b_n)\) in (9) and this calculation yields

\[
\{ \# \{ i \leq n : a_i \geq n \cdot \log n \} \geq b_n + u \} = \{ \# \{ i \leq n : a_i \geq n \cdot \log n \} \geq \left\lfloor \frac{\log \psi(\lfloor \log n \rfloor) - \log n}{\log 2} \right\rfloor + u \}
\]

\[
\geq \{ \# \{ i \leq 2^j : a_i \geq 2^{j+2} \cdot j \} \geq \left\lfloor \frac{\log \psi(\lfloor \log n \rfloor) - \log n}{\log 2} \right\rfloor \}
\]

In order to proceed we need the following lemma which is an analog of Lemma 3.12.

Lemma 5.1. Let \(\psi \in \mathcal{V}\). Then there exists \(\omega \in \mathcal{V}\) such that

\[
\omega(\lfloor \log_2 n \rfloor) \geq \psi(\lfloor \log n \rfloor).
\]

Proof. The proof can be done analogously to the proof of Lemma 3.12. We define \(\omega : \mathbb{N} \to \mathbb{R}_{>0}\) as

\[
\omega(n) = \max \{ \psi([n \cdot \log 2] + j) : j \in \{0, 1\} \}.
\]

Recall that \(\psi \in \mathcal{V}\). Then for the functions \(\overline{\psi} : \mathbb{N} \to \mathbb{R}_{>0}\) and \(\bar{\psi} : \mathbb{N} \to \mathbb{R}_{>0}\) given by \(\overline{\psi}(n) = \psi([\kappa \cdot n])\) with \(\kappa > 0\) and \(\bar{\psi}(n) = \max \{ \psi(n), \psi(n + 1) \}\) it holds that \(\overline{\psi}, \bar{\psi} \in \mathcal{V}\). Hence, \(\omega \in \mathcal{V}\). Applying \(\lfloor \log_2 n \rfloor\) on \(\omega\) in (100) yields

\[
\omega(\lfloor \log_2 n \rfloor) = \max \left\{ \psi \left( \left\lfloor \frac{\log n}{\log 2} \cdot \log 2 \right\rfloor + j \right) : j \in \{0, 1\} \right\}.
\]

Using (29) and (30) gives

\[
\max \left\{ \psi \left( \left\lfloor \frac{\log n}{\log 2} \cdot \log 2 \right\rfloor + j \right) : j \in \{0, 1\} \right\} \geq \psi(\lfloor \log n \rfloor)
\]

and (99) follows. \(\square\)

Noting that

\[
\log \log n \geq \log (\lfloor \log_2 n \rfloor \cdot \log 2) = \log \lfloor \log_2 n \rfloor + \log \log 2 \geq \log \lfloor \log_2 n \rfloor - \log 2
\]

and applying Lemma 5.1 yields

\[
\left\lfloor \frac{\log (2^{u+1} \psi(\lfloor \log n \rfloor)) - \log n}{\log 2} \right\rfloor \leq \lfloor \log_2 (2^{u+2} \omega(\lfloor \log_2 n \rfloor)) - \log_2 \lfloor \log_2 n \rfloor \rfloor.
\]

We note that \(\{1, \ldots, 2^j\} \supset I_{j-1}\) and for \(n \in I_j\) we have \(j = \lfloor \log_2 n \rfloor\) which implies Inserting then (101) in (98) yields for \(j\) sufficiently large

\[
\{ \# \{ i \leq n : a_i \geq n \cdot \log n \} \geq b_n + u \}
\]

\[
\supset \{ \# \{ i \in I_{j-1} : a_i \geq 2^{j+2} \cdot j \} \geq \lfloor \log_2 (2^{u+2} \omega(j)) - \log_2 j \rfloor \}.
\]
For the following let
\[
\mathcal{T}_j = \left\{ \left[ 2^j, 2^{j+1} - \lfloor \log_2 \left( 2^{u+2} \omega (j+1) \right) \rfloor - 1 \right] \right\}
\]
and set \( \Gamma = \{ j \in \mathbb{N} : 2^{-j-1} \geq \lfloor \log_2 \left( 2^{u+2} \omega (j+1) \right) \rfloor \}. \) Note here that \( \mathcal{T}_j \) and \( \Gamma \) implicitly depend on \( \omega \) and \( u. \)

Assume now that there exists \( i \in I_{j-1} \) fulfilling \( a_i \geq 2^{j+u+5} \cdot \omega (j). \) Then, the fact that \( \log_2 \left( 2^{j+u+5} \cdot \omega (j) \right) \geq \lfloor \log_2 \left( 2^{u+2} \omega (j) \right) \rfloor - 1 \) and Lemma 5.2 imply that we have for all \( k \in \{ i, \ldots, i + \lfloor \log_2 \left( 2^{u+2} \omega (j) \right) - \log_2 j \rfloor - 1 \} \) and \( x \in \Omega' \) that
\[
\begin{align*}
& a_k \geq 2^{j+u+5 - \lfloor \log_2 \left( 2^{u+2} \omega (j) \right) - \log_2 j \rfloor - 1} \cdot \omega (j) - 1 = 2^{-j+u} \lfloor \log_2 \left( 2^{u+2} \omega (j) \right) - \log_2 j \rfloor + 6 \cdot \omega (j) - 1 \\
& \geq 2^{-j+u} \lfloor \log_2 \left( 2^{u+2} \omega (j) \right) - \log_2 j \rfloor + 5 \cdot \omega (j) - 1 = 2^{-j+u+3} - 1
\end{align*}
\]
(103) 
for \( j \) sufficiently large. We further note that
\[
\# \{ i, \ldots, i + \lfloor \log_2 \left( 2^{u+2} \omega (j) \right) - \log_2 j \rfloor - 1 \} = \lfloor \log_2 \left( 2^{u+2} \omega (j) \right) - \log_2 j \rfloor,
\]
and \( \{ i, \ldots, i + \lfloor \log_2 \left( 2^{u+2} \omega (j) \right) - \log_2 j \rfloor - 1 \} \subset I_{j-1}, \) if \( i \in I_{j-1} \) and \( j \) sufficiently large. Applying this on (103) gives
\[
\# \{ i \in I_{j-1} : a_i \geq 2^{j+2} \cdot \omega (j) \} \geq \# \{ i \in I_{j-1} : a_i \geq 2^{j+u+5} \cdot \omega (j) \} \geq 1.
\]
Combining this with (102) gives
\[
\bigcup_{n \in I_j} \{ \# \{ i \leq n : a_i \geq n \cdot \log n \} \geq b_n \} \supset \bigcup_{i \in I_{j-1}} \{ a_i \geq 2^{j+u+5} \cdot \omega (j) \}
\]
and thus, for \( k \in \mathbb{N}, \)
\[
(104) \quad \bigcup_{n \geq k} \{ \# \{ i \leq n : a_i \geq n \cdot \log n \} \geq b_n \} \supset \bigcup_{j \geq \lfloor \log_2 k \rfloor + 1} \bigcup_{i \in I_{j-1}} \{ a_i \geq 2^{j+u+5} \cdot \omega (j) \}.
\]

In the next steps we will make use of the following dynamical Borel-Cantelli lemma.

**Lemma 5.2** ([Kim07 Special case of Theorem 2.1]). Let \( [0,1) \) be partitioned into a finite set of intervals \( W_i = \{ a_i, d_i \}. \) Further, let \( \xi : [0,1) \to [0,1) \) be such that \( \xi \) is derivable on the interior of each \( W_i \) and \( \xi|_{W_i} \in BV. \)

Assume that \( \xi \) has a uniquely absolutely continuous invariant measure \( d\mu = hd\lambda \) and \( h \) is bounded away from 0. If \( A_n \) is a sequence of intervals with \( \sum_{n=1}^{\infty} \mu (A_n) = \infty, \) then
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} 1_{A_n} \circ \xi^{k-1}}{\sum_{k=1}^{n} \mu (A_n)} = 1 \quad a.s.
\]
It follows easily that Lemma 5.2 is applicable. The interval \([0,1)\) is partitioned into the intervals \([0,1/2)\) and \([1/2,1)\) and \( \tau|_{[0,1/2)} = \tau'|_{[1/2,1)} = 2 \cdot 1 \in BV. \) Further, the absolutely continuous measure for this transformation is the Lebesgue measure itself.

We have by (14) that
\[
\lambda \left( a_i \geq 2^{j+u+5} \cdot \omega (j) \right) = \frac{1}{\left( 2^{j+u+5} \cdot \omega (j) \right)}
\]
Thus,
\[
\sum_{j=1}^{\infty} \sum_{i \in T_{j-1}} \lambda \left( a_i \geq 2^{j+u+5} \cdot \omega(j) \right) = \sum_{j=1}^{\infty} \sum_{i \in T_{j-1}} \frac{1}{\left[ 2^{j+u+5} \cdot \omega(j) \right]} = \sum_{j=1}^{\infty} \frac{\#T_{j-1}}{\left[ 2^{j+u+5} \cdot \omega(j) \right]}
\]
\[
= \sum_{j \in \Gamma} \frac{\#T_{j-1}}{\left[ 2^{j+u+5} \cdot \omega(j) \right]} = \sum_{j \in \Gamma} \frac{2^{j+1} - \left\lfloor \log_2 2\omega(j) \right\rfloor}{\left[ 2^{j+u+5} \cdot \omega(j) \right]}.
\]
By the definition of \( \Gamma \) we have for \( j \in \Gamma \) that
\[
\sum_{j=1}^{\infty} \sum_{i \in T_{j-1}} \lambda \left( a_i \geq 2^{j+u+5} \cdot \omega(j) \right) \geq \sum_{j \in \Gamma} \frac{1}{2^{u+7} \cdot \omega(j)}.
\]
Furthermore, we note that \( j \notin \Gamma \) implies \( \omega(j) > 2^{2j-2} \cdot 2^{-u-2} > 2^{2j-3} \), for \( j \) sufficiently large, say larger than \( J \in \mathbb{N} \). Thus,
\[
\sum_{j \notin \Gamma \cap \mathbb{N} \leq J} \frac{1}{\omega(j)} < \sum_{j \notin \Gamma \cap \mathbb{N} \leq J} \frac{1}{2^{2j-3}} < \sum_{j=1}^{\infty} \frac{1}{2^{2j-3}} < \infty
\]
\[
\text{implying that } \sum_{j \in \Gamma} 1/\omega(j) = \infty \text{ since } \omega \in \overline{\Psi}. \text{ Combining this consideration with (105) yields}
\]
\[
\sum_{j=1}^{\infty} \sum_{i \in T_{j-1}} \lambda \left( a_i \geq 2^{j+u+5} \cdot \omega(j) \right) = \infty.
\]
Since
\[
\left\{ a_i \geq 2^{j+u+5} \cdot \omega(j) \right\} = \left\{ \mathbb{1}_{[0,2^{j+u+5} \cdot \omega(j)]^{-1}} \circ \tau^{j-1} = 1 \right\},
\]
where \( \lceil x \rceil = \min \{ n \in \mathbb{Z} : n \geq x \} \), the conditions of Lemma 5.2 are fulfilled and we can apply the second Borel-Cantelli lemma proving
\[
\lambda \left( \bigcap_{n \in \mathbb{N}} \bigcup_{j \geq n} \bigcup_{i \in T_j} \left\{ a_i \geq 2^{j+u+5} \cdot \omega(j) \right\} \right) = 1.
\]
Combining this with (101) gives (37) and thus the statement of the theorem. \( \square \)

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