A NOTE ON $\sigma$-MODEL WITH THE TARGET $S^n$

M.V. MOVSHEV

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1. Introduction

It is well known that the Schrödinger operator derived from a general quantum field theory has to be defined on an infinite-dimensional space. For example if the theory is a sigma-model with target a sphere

$$S^k = \{ n \in \mathbb{R}^{k+1} | n \cdot n = R^2 \}$$

(in physics jargon the theory of $n$-field or $O(k+1)$-model) then the Schrödinger operator $H$ has to be defined on the loop space $L(S^k) = \text{Maps}(S^1, S^k)$. It seems to be a formidable task to define $H$ acting directly in $L^2(L(S^k))$ because of the famous quantum field theory divergencies.
Much of the work has been done in mathematical community to estimate the difference \( k \Gamma \) of differential geometry of the underlying manifold \([5],[4],[26]\).

Conjecture 1. (C.f. p.1175) Let \( \Gamma_N \) be the difference \( \lambda_2 - \lambda_1 \) of the first two eigenvalues of Neumann (Dirichlet?) problem on \( \mathcal{L}_{N,U}(S^k) \). Then there are constant \( C_1,C_2 > 0 \) such that

\[
\Gamma(N) \geq C_1N^2e^{-C_2R^2}
\]

It is well known that the sets of eigenvalues \( \{\eta_k\} \) of a positive self-adjoint operator \( H_{\mathcal{L}_{N,U}(S^k)} \) can be arranged in a nondecreasing order of magnitude as follows:

\[
\eta_1 \leq \eta_2 \leq \cdots \leq \eta_k \leq \cdots \to \infty
\]

Much of the work has been done in mathematical community to estimate the difference \( \Gamma_k = \eta_k - \eta_1 \) for the Neumann problem for a general Schrödinger operator in terms of differential geometry of the underlying manifold \([5],[4],[26]\).
Here is some relevant definitions: let \((M^m, g_{ij})\) be an \(m\)-dimensional compact Riemannian manifold with metric \(g_{ij}\) and \(\partial M \neq \emptyset\) be the boundary of \(M\). Let \(\Delta\) be the Laplacian operator associated to \(g_{ij}\) on \(M\) then we define on \(M\) a Schrödinger operator by
\[
\Delta - q(x),
\]
where \(q(x) \in C^2(M)\). We consider the following Neumann eigenvalue problem:
\[
\Delta u - qu = -\eta u \quad \text{in} \quad M
\]
\[
\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial M.
\]

**Definition 2.** \(\partial M\) is said to satisfy the "interior rolling \(R\)-ball" condition if for each point \(p \in \partial M\) there is a geodesic ball \(B_q(R/2)\), centered at \(q \in M\) with radius \(R/2\), such that \(p = B_q(R/2) \cap \partial M\) and \(B_q(R/2) \subset M\).

**Theorem 3.** [5] Let \(M^m, m \geq 3\), be an \(m\)-dimensional compact manifold with boundary \(\partial M\) and \(\omega(q) = \sup_q - \inf_q \) denote the oscillation of \(q\) over \(M\). Suppose that the Ricci curvature of \(M\) satisfies \(\text{R}^{ij} + Kg_{ij}\) is positive-definite, \(K \geq 0\) and the second fundamental form \(\alpha_{ij}\) of \(\partial M\) with respect to outward pointing unit normal \(\nu\) satisfies \(\alpha_{ij} + Hg_{ij}|_{\partial M}\) positive-definite \(H \geq 0\). Suppose that \(\partial M\) also satisfies the "interior rolling \(r\)-ball condition" with \(r\) chosen small (see [4], [26] for the choice of \(r\)). Then the gap \(\Gamma_k\) of \(k\)-th Neumann eigenvalues \(\eta_k\) and \(\eta_1\) of \(M\) satisfies \(\Gamma_k \geq \alpha_1 k^2 m\) for all \(k = 2, 3, \ldots\) and some explicitly computable constant \(\alpha_1\) depending on \(m, K, H, r\) diameter of \(M\) and \(\omega(q)\).

Here are some formulations for the problem with zero potential.

**Theorem 4.** [4] Let \(M\) be as in Theorem 3. Then the first nonzero Neumann eigenvalue \(\lambda_1\) of Laplacian operator has a lower bound given by
\[
\lambda_1 \geq C_p + \frac{1}{1 + H^2} \left( \frac{1 - \alpha^2}{4(m - 1)d^2 B_1^2 - B_2} \right) \exp(-B_1)
\]
where \(C_p\) and \(R\) are positive constants less then one,
\[
d = \text{diameter of } M
\]
\[
B_1 = 1 + \left( 1 + \frac{4(m - 1)d^2 B_2}{1 - \alpha^2} \right)^{\frac{1}{2}}
\]
\[
B_2 = (1 + H)B_3 + \frac{(2m - 3)^2 + (4m - 5)\alpha^2 H^2}{(m - 1)R^2 \alpha^2} + (1 + H)^2 K
\]
\[
B_3 = \frac{2(m - 1)H(1 + H)(1 + 3H)}{R} + \frac{H(1 + H)}{R^2}
\]

**Theorem 5.** [18] Let \(n \geq 2\) be an integer and let \(D, a, e, K\) be real numbers with \(D > 0\) and \(a > 0\). Then a continuous function \(C(n, e, a, D, K)\) of these numbers can be constructed, such that for any compact, Riemannian manifold \((M, \partial M, g)\) with diameter \(d\) the inequality
\[
\lambda_1 d^2 \geq C(n, e, a, K, D)
\]
holds, provided that the following conditions are satisfied:
1. \(d \leq D\),
(2) the Ricci curvature is $\geq (n-1)K$, 
(3) the so-called "sectional radius" (defined as the radius of injectivity of the exponential map of the normal bundle of $\partial M$ into $M$ is $\geq a$, and 
(4) the principal curvatures of the boundary are $\leq e$. Where

$$C(n, e, a, K, D) = \alpha(n) \exp \left( -\beta(n)D \max \left( e, \frac{1}{a} \right) \right) \text{ if } K \geq 0$$

$$C(n, e, a, K, D) = \alpha(n) \exp \left( -\beta(n)D \max \left( e, \frac{1}{a} \right) - \gamma(n)D \frac{\sqrt{|K|}}{Th \left( \frac{a\sqrt{|K|}}{2} \right)} \right) \text{ if } K < 0$$

with

$$\alpha(n) = \frac{e^{-2}}{4(n-1)}$$
$$\beta(n) = 16n$$
$$\gamma(n) = 2(n-1)^{\frac{3}{2}}$$

This theorem motivates our interest in differential geometry of $\tilde{L}_{N,U}(S^k)$. In this paper (Section 6) we are going to present explicit formulas for Ricci curvature of $\tilde{L}_{N,U}(S^k)$ the second quadratic form $\alpha$.

We also going to study (Section 7) Schrödinger equation on the space $L_1(S^k)$. We derive asymptotic of radial eigenfunctions near singular locus of $L_1(S^k)$.

## 2. THE CLASSICAL O(k+1)-SIGMA MODEL

In this section we are going to remind the reader the basic definition related to the theory of $n$-field. First of all the field of the theory is the smooth map $n: \mathbb{R} \times S^1 \to \mathbb{R}^{k+1}$ subject to a constraint. To formulate the constraint we choose coordinates $t, \theta$ on $\mathbb{R} \times S^1$. Also we equip $\mathbb{R}^{k+1}$ with the dot-product such that for $x, y \in \mathbb{R}^{k+1}, x \cdot y = \sum_{i=1}^{k+1} x_i y_i$. We will usually use an abbreviation $x^2 := x \cdot x$. We assume that $n$ satisfies

$$n(t, \theta)^2 - R^2 = 0$$

$$n(t, \theta + L) = n(t, \theta)$$

Thus, effectively, $n$ is a map $\mathbb{R} \times S^1 \to S^k$, where $S^1$ is a circle of radius $\frac{L}{2\pi}$ and $S^k$ is a sphere of radius $R$. Lagrangian $L(n)dt\theta$ of the theory is

$$L(n)dt\theta = \frac{1}{2} \left( n_t^2 - n_\theta^2 \right) dt\theta$$

In our finite approximation the space of fields $\text{Maps}(\mathbb{R} \times S^1, S^k) = \text{Maps}(\mathbb{R}, L(S^k))$ gets replaced by $\text{Maps}(\mathbb{R}, L_N(S^k))$. The circle of questions typically discussed when Lagrangian has been written is what are the solutions of the equation of motion and what are spectral properties of the associated Schrödinger operator. In the present situation we have to postpone answering these questions there is more pressing one: what are the singularities
of $\mathcal{L}_N(S^k)$ and is it possible to resolve them. We will answer these question in the next sections.

3. Resolution of Singularities

$\mathcal{L}(S^k)$ is a homogenous space of the loop groups $\mathcal{L}(SO(k+1)) \subset \mathcal{L}(O(k+1))$. It means that we have the action map

$$\mathcal{L}(O(k+1)) \times \mathcal{L}(S^k) \rightarrow \mathcal{L}(S^k)$$

and the product map

$$\mathcal{L}(O(k+1)) \times \mathcal{L}(O(k+1)) \rightarrow \mathcal{L}(O(k+1))$$

Only the action of constants $O(k+1) \subset \mathcal{L}(O(k+1))$ survives after the truncation $\mathcal{L}(S^k) \Rightarrow \mathcal{L}_K(S^k)$. The group $O(k+1)$ is an algebraic submanifold in $Mat_{k+1}(\mathbb{R})$. Following the analogy with $S^k$ we define a finite-dimensional subvariety $\mathcal{L}_M(O(k+1)) \subset \mathcal{L}(O(k+1))$. The maps

$$\mathcal{L}_M(O(k+1)) \times \mathcal{L}_K(S^k) \rightarrow \mathcal{L}_{M+K}(S^k)$$
$$\mathcal{L}_M(O(k+1)) \times \mathcal{L}_K(O(k+1)) \rightarrow \mathcal{L}_{M+K}(O(k+1))$$

are restriction of $\mathcal{L}(S^k)$. 

The set of (smooth) homomorphisms $\text{Hom}(S^1, SO(k+1))$ is a subset of $\mathcal{L}(O(k+1))$. The group $SO(k+1)$ acts on $\text{Hom}(S^1, SO(k+1))$ by conjugations and $\text{Hom}(S^1, SO(k+1))$ is a union of $SO(k+1)$-orbits. The intersection $\text{Hom}(S^1, SO(k+1)) \cap \mathcal{L}(SO_1(k+1))$ contains an element $\lambda_0$. It is a block-sum of standard two-dimensional representation $\tau$ and a trivial $k-1$-dimensional representation. We denote the $SO(k+1)$ orbit of $\lambda_0$ by $G(k+1)$.

The map

$$\mu_N : G(k+1) \times \mathcal{L}_{N-1}(S^k) \rightarrow \mathcal{L}_N(S^k)$$

is a restriction of the top map in (8) with $M = 1$ and $K = N - 1$ from $\mathcal{L}_1(O(k+1))$ to $G(k+1)$.

The goal of this section is to construct an open subset

$$X_N \subset G(k+1) \times (\mathcal{L}_N(S^{k+1})(\mathcal{L}_{N-1}(S^{k+1}))$$

which is isomorphic to $\mathcal{L}_{N+1}(S^{k+1}) \mathcal{L}_{N-1}(S^{k+1})$ and it projects onto $\mathcal{L}_N(S^{k+1}) \mathcal{L}_{N-1}(S^{k+1}), N \geq 1$. The fibers of the projection are affine spaces. Thus $\mu_N$ is binational isomorphism. We can use $\mu_N$ for different $N$ to birationally identify $\mathcal{L}_N(S^k)$ with $G(k+1)^{\times N} \times S^k$. Moreover the map $\mu : G(k+1)^{\times N} \times S^k \rightarrow \mathcal{L}_N(S^{k+1})$ $\mu(\lambda_N(z), \ldots, \lambda_1(z), n) = \lambda_N(z) \ldots \lambda_1(z) n$ is a resolution of singularities.

The reader might wish to compare this with a ideologically similar result [23] Appendix A, in which the authors describe generators and relations in the group polynomial loops in $U(n)$.

First we want to identify the space $G(k+1)$. Representation $\lambda \in G(k+1)$ carries an information about a two-dimensional subspace $V(1) \subset \mathbb{R}^{k+1}$ together with the orientation
or specified by the operator \( W = \lambda(\sqrt{-1} \mathbf{q}) \). Conversely, any point \( V^{or} \) in an oriented Grassmannian \( \widetilde{\text{Gr}}_2(\mathbb{R}^{k+1}) \) determines a homomorphism \( \lambda_{V^{or}} \in G(k+1) \) by the rule
\[
\lambda_{V^{or}}(\theta) = P_V \cos(\theta) + W_V \sin(\theta) + \text{Id} - P_V
\]
where \( P_V \) is an orthogonal projection on \( V \), \( W_V \) is an operator of projection on \( V \) followed by rotation on ninety degree in agreement with orientation \( or \). Note that \( \lambda_{V^{or}}(\theta) = \lambda_{V^{or}}(\theta)^{-1} \) and \( G(k+1) \) is closed under taking inverses.

For construction of \( X_N \) we will use some lemmas.

**Lemma 6.** Let
\[
\{ a, b \} ||a|| = ||b||
\]
be an orthogonal basis for \( V^{or} \in \widetilde{\text{Gr}}_2(\mathbb{R}^{k+1}) \). In the future we will call an admissible basis an orthogonal basis that satisfies \( \lambda_\pi \). The map \( \gamma_\pi : S^1 \to V, \gamma_\pi(\theta) = \cos(\theta)a + \sin(\theta)b \) satisfies
\[
\lambda_{V^{or}} \gamma_\pi = \gamma_{\pi-1}
\]
if \( \{a, b\} \) is a basis of \( V \) that agrees with orientation and
\[
\lambda_{V^{or}} \gamma_\pi = \gamma_{\pi-1}
\]
if \( \{a, b\} \) has the opposite orientation.

**Lemma 7.** Let \( V^{or} \) be a space generated by a positively oriented basis \( \{a[N], b[N]\} \)-the leading coefficients \( \{n(\theta) \} \in \mathcal{L}_N(S^k) \setminus \mathcal{L}_{N-1}(S^k) \). Let \( P \) be the orthogonal projection on \( V \). Vectors \( P(a[N-1]), P(b[N-1]) \) have equal lengths and if nonzero are orthogonal and define the same orientation as \( \{a[N], b[N]\} \).

**Proof.** Nonzero vectors \( a[N], b[N] \) are orthogonal and \( a[N]^2 = b[N]^2 \) (formulas \( d_{2N}, c_{2N} \), equation \( ?? \)). The formula for projection is
\[
P(v) = \frac{a[N] \cdot v}{a[N]^2} a[N] + \frac{b[N] \cdot v}{b[N]^2} b[N]
\]
Then
\[
P(a[N-1]) \cdot P(b[N-1]) = \frac{a[N] \cdot a[N-1]}{a[N]^2} a[N] \cdot b[N-1] + \frac{b[N] \cdot a[N-1]}{b[N]^2} b[N] \cdot b[N-1] = \frac{1}{a[N]^2 a[N]^2} ((a[N] \cdot a[N-1]) (a[N] \cdot b[N-1]) + (b[N] \cdot a[N-1]) (b[N] \cdot b[N-1])) = \frac{a[N] \cdot a[N-1]}{a[N]^2 a[N]^2} (a[N] \cdot b[N-1] + b[N] \cdot a[N-1]) = 0
\]
We use formulas \( d_{2N-1} \) and \( c_{2N-1} \) from \( ?? \):
\[
a[N] \cdot b[N-1] + b[N] \cdot a[N-1] = 0, a[N] \cdot a[N-1] - b[N] \cdot b[N-1] = 0.
\]
A similar computation shows that quantities
\[
P(a[N-1])^2 = (a[N] \cdot a[N-1])^2 + (b[N] \cdot a[N-1])^2
\]
\[ P(b[N-1])^2 = (a[N] \cdot b[N-1])^2 + (b[N] \cdot b[N-1])^2 \]

are equal. Transition matrix from \{a[N], b[N]\} to \{P(a[N-1]), P(b[N-1])\}

\[
\begin{pmatrix}
\frac{a[N] \cdot a[N-1]}{a[N]^2} & \frac{a[N] \cdot b[N-1]}{b[N]^2} \\
\frac{b[N] \cdot a[N-1]}{b[N]^2} & \frac{b[N] \cdot b[N-1]}{b[N]^2}
\end{pmatrix}
\]

has determinant

\[
\frac{a[N] \cdot a[N-1] \cdot b[N] \cdot b[N-1]}{a[N]^2 b[N]^2} - \frac{b[N] \cdot a[N-1] \cdot a[N] \cdot b[N-1]}{b[N]^2 a[N]^2} = \frac{1}{a[N]^2 a[N]^2} ((a[N] \cdot a[N-1])^2 + (a[N] \cdot b[N-1])^2) = \frac{P(a[N-1])^2}{a[N]^2 a[N]^2} \geq 0
\]

This verifies the statement about orientation. □

**Proposition 8.** The map \([\mathcal{P}]\) is onto.

**Proof.** It suffices to show that for any \(N \geq 1\) and \(n(\theta) \in \mathcal{L}_N(S^k) \setminus \mathcal{L}_{N-1}(S^k) \exists \lambda \in G(k+1)\) such that \(\lambda^{-1}n \in \mathcal{L}_{N-1}(S^k)\). Then trivially \(\lambda(\lambda^{-1}n) = n\). Then the general statement will follows because \(\mathcal{L}_N(S^k)\) is filtered by \(\mathcal{L}_l(S^k) \setminus \mathcal{L}_{l-1}(S^k), l = 0, \ldots, N\).

We denote a homomorphism \(\lambda_{V,w}\) based on the space \(V\) from Lemma 7 by \(\lambda\). We would like to show that \(\lambda^{-1}n \in \text{Trig}_{N-1}\). In terms of Fourier harmonics \(n = \sum_{l=0}^{N} n_l \) it suffice to verify that \(\lambda^{-1}n_N, \lambda^{-1}n_{N-1} \in \text{Trig}_{N-1}\). All other terms \(\lambda^{-1}n_l, l \leq N-2\) belong to \(\text{Trig}_{N-1}\) by virtue of basic trigonometric identities. Note that \(\lambda^{-1}n_N \in \text{Trig}_{N-1}\) follows from Lemma 6 and the choice of \(\lambda\). Denote \(P_V\) from Lemma 7 by \(P\). Then \(\lambda^{-1}\) acts trivially on \((\text{Id} - P)n_{N-1}\) so \((\text{Id} - P)n_{N-1} \in \text{Trig}_{N-1}\). By Lemma 7 \(\lambda^{-1}P n_{N-1} \in \text{Trig}_{N-2}\), thus \(\lambda^{-1}n_{N-1} \in \text{Trig}_{N-1}\) and Proposition is proven.

The set \(\mathcal{L}_1(SO(k+1))\) is invariant with respect to taking inverses because \(\phi(\theta)^{-1} = \phi(\theta)^t\), where \(t\) stands for transpose which is a linear operation. Then \(n(\theta)\) is equal to \(\phi(\theta)^{-1}\phi(\theta)n(\theta)\). The statement would follow if for any \(n(\theta) \in \mathcal{L}_N(S^k)\) we will find \(\phi(\theta) \in \mathcal{L}_1(SO(k+1))\) such that \(\phi(\theta)n(\theta) \in \mathcal{L}_{N-1}(S^k)\). Matrix coefficients of \(\phi(\theta) = V + A \cos(\theta) + B \sin(\theta) \in \mathcal{L}_1(SO(k+1))\) have to satisfy

\[
AB^t + BA^t = 0, AA^t - BB^t = 0
\]

\[
VB^t + BV^t = 0, VA^t + AV^t = 0
\]

\[
AA^t +VV^t = Id
\]

We expand \(\phi(\theta)n(\theta)\) into Fourier series \(a'[N+1] \cos((n+1)\theta) + b'[N+1] \sin((n+1)\theta) + a'[N] \cos(n\theta) + b'[N] \sin(n\theta) + \cdots\) whose coefficients satisfy

\[
a'[N+1] = \frac{1}{2}(Aa[N] - Bb[N]), b'[N+1] = \frac{1}{2}(Ba[N] + Ab[N])
\]

\[
a'[N] = \frac{1}{2}(Aa[N-1] - Bb[N-1] + 2Va[N]), b'[N] = \frac{1}{2}(Ba[N-1] + Ab[N-1] + 2Vb[N])
\]

\[
\cdots
\]
Our goal is to find $A, B, V$ that satisfy constraints (12) such that $a'[N+1], b'[N+1], a'[N], b'[N]$ vanish. Introduce notations: $(a, b)_{ij}$ stands for a matrix whose $i$-th row is a vector $a$, $j$-th row is a vector $b$ and all the remaining rows vanish. From equations (??) we know that if one of $a[N], b[N]$ is nonzero then both are nonzero. Let $a = a[N]/\|a[N]\|$, $b = b[N]/\|b[N]\|$ be normalized leading coefficients. We set $A = (a, b)_{k,k+1}$, $B = (b, -a)_{k,k+1}$. To define $V$ we choose $k - 1$ orthonormal vectors $e_1, \ldots, e_{k-1}$ perpendicular to $a$ and $b$. We define $V$ to be the matrix $(e_1, \ldots, e_{k-1}, 0, 0)$, whose first $k - 1$ are formed by vectors $e_i$ and whose remaining two rows are equal to zero. In geometric terms $V$ is an orthogonal projection on orthogonal complement $\langle a, b \rangle$. We leave it to the reader to verify that equations (13,12). The reader should use four equations from the set (??) $a[N]^2 = b[N]^2, a[N] \cdot b[N] = 0, a[N] \cdot a[N - 1] - b[N] \cdot b[N - 1] = 0, a[N] \cdot b[N - 1] + b[N] \cdot a[N - 1] = 0$. We denote the resulting element in $L_1(SO(k + 1))$ by $\phi_{ab}$

\[ \square \]

**Proposition 9.** Pick $n \in L_N(S^k) \setminus L_{N-1}(S^k)$. Suppose $n = \lambda_i n_i, \lambda_i \in G(k + 1), n_i \in L_{N-1}(S^k), i = 1, 2$. Then $\lambda_1 = \lambda_2$ and $n_1 = n_2 \in L_{N-1}(S^k) \setminus L_{N-2}(S^k)$.

**Proof.** We decompose maps into finite Fourier series

\[ n = v + \sum_{k=1}^{N} a[k] \cos(k \theta) + b[k] \sin(k \theta), \quad n_i = v^i + \sum_{k=1}^{N} a^i[k] \cos(k \theta) + b^i[k] \sin(k \theta), i = 1, 2 \]

\[ \lambda_i = P_i \cos(\theta) + W_i \sin(\theta) + Id - P_i, i = 1, 2 \]

Then

\[ a[N+1] = P_i a'[N] - W_i b'[N] \]

\[ b[N+1] = W_i a'[N] + P_i b'[N], i = 1, 2 \]

By assumptions $a[N+1], b[N+1] \neq 0$ and by virtue of equations (10) $W_i a[N+1] = b[N+1], W_i b[N+1] = -a[N+1], W_i b[N+1] = -a[N+1], i = 1, 2$. This implies that

\[ \langle a[N+1], b[N+1] \rangle \subset \text{Im}P_i \subset \text{Im}W_i \subset \text{Im}P_i \]

and because of that $P_1 = P_2, W_1 = W_2$. We conclude that $\lambda_1 = \lambda_2 = \lambda$ and $\lambda^{-1}n = n_1 = n_2$. Were $n_1$ an element of $L_{N-2}(S^k)$, then $\lambda n_1$ would be an element of $L_{N-1}(S^k)$, which is impossible because $\lambda n_1 = n = L_N(S^k) \setminus L_{N-1}(S^k)$.

\[ \square \]

**Remark 10.** The space $G(k + 1)$ is diffeomorphic to a complex quadric $Q \subset \mathbb{P}(\mathbb{C}^{k+1})$: an admissible basis (14) for $V^{or} \in \mathcal{G}_{2}((\mathbb{R}^{k+1})_2)$ generate a line $\langle a + \sqrt{-1}b \rangle \subset \mathbb{P}(\mathbb{C}^{k+1})$. Equation $(a + \sqrt{-1}b, a + \sqrt{-1}b) = |a|^2 - |b|^2 + 2\sqrt{-1}(a, b) = 0$ is automatically satisfied. Complex rescaling of the generator $a + \sqrt{-1}b$ corresponds to different choices of admissible bases for $V^{or}$.

Now we are ready to determine the subset $X_N$

**Definition 11.** Fix $n \in L_N(S^k)$. We call an element $\lambda \in G(k + 1)$ $n$-regular if $\lambda n \in L_{N-1}(S^k) \setminus L_N(S^k)$. If $\lambda n \in L_N(S^k) \setminus L_{N-1}(S^k)$ then we call $\lambda$ an $n$-singular homomorphism. We denote the set of $n$-singular elements in $G(k + 1)$ by $D(n)$. 
Proposition 12. The set $G(k+1)\setminus D(n)$ is algebraically isomorphic to $\mathbb{C}^{k-1} \cong \mathbb{R}^{2k-2}$.

Proof. Equation $\lambda n \in L_N(S^k)$ is equivalent to equations

$$\begin{align*}
Pa[N] - Wb[N] &= 0 \\
Wa[N] + Pb[N] &= 0
\end{align*}$$

Equation (15) are equivalent to

$$\begin{align*}
a \cdot a[N] &= b \cdot b[N] \\
b \cdot a[N] &= -a \cdot b[N]
\end{align*}$$

These equations are equivalent to orthogonality of complex vectors

$$\begin{align*}
A &= a[N] + \sqrt{-1}b[N] \\
A' &= (a + \sqrt{-1}b)
\end{align*}$$

To summarize-the set $Q \setminus D$ is isomorphic to $G(k+1)\setminus D(n)$ is contractible $\square$

We arrive at the following conclusion
Corollary 13. The space $L_{N+1}(S^{k+1}) \setminus L_N(S^{k+1})$ is homotopy equivalent to $L_N(S^{k+1}) \setminus L_{N-1}(S^{k+1})$, $N \geq 1$. Thus by results in Section 3 $L_{N+1}(S^{k+1}) \setminus L_N(S^{k+1})$ is homotopy equivalent to $L_1(S^{k+1}) \setminus L_0(S^{k+1})$, which is homotopy equivalent to the Stiefel manifold $V_2$.

Remark 14. The spaces $L_N(S^3)$ are dense in $L(S^3)$ in $L^\infty$ topology. For the proof see David E Speyer response at https://mathoverflow.net/questions/158105/real-varieties-with-enough-algebraic-loops.

4. Equation of motion

The Lagrangian \([5]\) with $n(t, \theta) \in \text{Maps}(\mathbb{R}, L_N(S^k))$ can be written in terms of unconstrained fields by means of the Lagrange multiplier $c$:

\[
L(n, c) := \frac{1}{2} \left( n_t \cdot n_t - n_\theta \cdot n_\theta + c(n \cdot n - R^2) \right), \quad n \in \text{Trig}_N(\mathbb{R}^{k+1}), c \in \text{Trig}_{2N}(\mathbb{R})
\]

Introduce a notation

\[
D_N(\theta) = \frac{\sin((N + 1/2)\theta)}{\sin(\theta/2)}
\]

Denote by $(\text{Pr}_N n)(\theta) = \frac{1}{2\pi} \int_{S^1} D_N(\theta - \theta') n(\theta')d\theta'$ the orthogonal projection onto $\text{Trig}_N(\mathbb{R}^{k+1})$.

Equations of motion for \([10]\) look like

\[
\text{Pr}_N(-n_{tt} + n_{\theta\theta} + cn) = 0
\]

\[
n \cdot n - R^2 = 0
\]

Note that $\text{Pr}_N(-n_{tt} + n_{\theta\theta}) = -n_{tt} + n_{\theta\theta}$, so \([18]\) is equivalent to

\[
-n_{tt}(t, \theta) + n_{\theta\theta}(t, \theta) + \frac{1}{2\pi} \int_0^{2\pi} c(t, \theta') n(t, \theta') D_N(\theta' - \theta)d\theta' = 0
\]

After taking the dot-product with $n$ we get

\[
n^2 - n_{\theta}^2 + g_{n,N}(c) = 0
\]

Here $g_{n,N}$ is an operator on $\text{Trig}_{2N}(\mathbb{R})$ define by the formula

\[
g_{n,N}(c) = \frac{1}{2\pi} \int_0^{2\pi} c(t, \theta') n(t, \theta') \cdot n(t, \theta) D_N(\theta' - \theta)d\theta'
\]

The inverse to $g_N$ is the operator $G_{n,N}$. we can solve the last equation for $c$:

\[
c = G_{n,N}(n_{\theta}^2 - n^2)
\]

finally equation \([18]\) becomes equation of extremals

\[
-n_{tt}(t, \theta) + n_{\theta\theta}(t, \theta) + \frac{1}{2\pi} \int_0^{2\pi} G_N(n_{\theta}^2 - n^2)(t, \theta') n(t, \theta') D_N(\theta' - \theta)d\theta' = 0
\]

The remarkable fact \([7]\) is that the $O(k + 1)$-model is completely integrable. It is not clear if it admits a sequence of finite integrable approximations. In particular it is not clear whether \([20]\) is integrable.
If we explicitly impose the constraints (19) and \( n \in \text{Trig}_N(\mathbb{R}^{k+1}) \) we get the action for a system that moves on a manifold \( L_N(S^k) \) in the potential \( U = \int_{S^1} n_\theta^2 \). On general grounds we know that classical trajectories satisfy Newton’s law equation \( \nabla n_\theta' = -\ast dU(n) \). \( \nabla \) is the Levi-Civita connection associated with the metric on \( L_N(S^k) \). In the next section we will describe the metric on \( L_1(S^k) \)-the simplest manifold in the family \( L_N(S^k) \).

5. Variety \( L_1(S^k) \)

The space \( L_1(S^k) \) is the simplest nontrivial case of our construction. It consists of a triple of vectors \((v, a, b) = (v, a[1], b[1])\) that satisfy

\[
\begin{align*}
  v \cdot a &= v \cdot b = a \cdot b = 0 \\
  a^2 - b^2 &= 0 \\
  v^2 + a^2 - R^2 &= 0
\end{align*}
\]

This data has a simple geometric interpretation (cf. [14] Section 2.3). Nonzero vectors \( a \) and \( b \) span a two-plane \( W_{a,b} \subset \mathbb{R}^{k+1} \), which intersects \( S^k \) by a big circle. We see that the Grassmannian \( \widetilde{Gr}_2(\mathbb{R}^{k+1}) \) coincides with the space of such circles. A choice of an orthogonal basis \( \{a, b\} \) for \( W \) encodes a parametrization of \( W \cap S^k \). A shift \( W \to v + W \) of \( W, v \in W^\perp \) produces a family of smaller circles \( (W + v) \cap S^k \), which degenerate to points of \( S^k \) when \( v \cdot v = R^2 \). Thus for a fixed \( W \) the set of admissible shifts is a \( k - 1 \)-dimensional disk \( D^{k-1} = \{v \in W^\perp | v \cdot v \leq R^2\} \). From this we deduce that the complement \( L_1(S^k) \backslash L_0(S^k) \) is a \( D^{k-1} \times S^1 \)-bundle over oriented Grassmannian \( \widetilde{Gr}_2(\mathbb{R}^{k+1}) \): under projection

\[
p : L_1(S^k) \backslash L_0(S^k) \to \widetilde{Gr}_2(\mathbb{R}^{k+1})
\]

the triple \((v, a, b)\) get mapped to \( W_{a,b} \in \widetilde{Gr}_2(\mathbb{R}^{k+1}) \). The following two vector bundles over \( \widetilde{Gr}_2(\mathbb{R}^{k+1}) \) were extensively studied in the theory of characteristic classes (cf [19]):

\[
\gamma_2 = \{(v, W) | v \in W, W \in \widetilde{Gr}_2(\mathbb{R}^{k+1}) \} \to \{W | W \in \widetilde{Gr}_2(\mathbb{R}^{k+1}) \}
\]

\[
\gamma_2^+ = \{(v, W) | v \in W^\perp, W \in \widetilde{Gr}_2(\mathbb{R}^{k+1}) \} \to \{W | W \in \widetilde{Gr}_2(\mathbb{R}^{k+1}) \}
\]

Our bundle is a fiber product of the disk and the circle bundles \( D^{k-1}(\gamma_2^+) \times S(\gamma_2) \).

It is convenient to introduce vectors \( e_v = \frac{v}{|v|}, e_a = \frac{a}{|a|}, e_b = \frac{b}{|b|} \). A triple \((e_v, e_a, e_b)\) defines a point in a Stiefel manifold \( V_3(\mathbb{R}^{k+1}) \) of orthonormal 3-frames in \( \mathbb{R}^{k+1} \). In this paper we will use two local parametrizations:

\[
\begin{align*}
  \text{trig} : (0, \pi R/2) \times V_3(\mathbb{R}^{k+1}) &\to L_1(S^k) \\
  \text{alg} : (0, 1) \times V_3(\mathbb{R}^{k+1}) &\to L_1(S^k).
\end{align*}
\]

They are defined by the formulae:

\[
\begin{align*}
  \text{trig}(\tau, e_v, e_a, e_b) &= (R \sin (\tau/R) e_v, R \cos (\tau/R) e_a, R \cos (\tau/R) e_b) \\
  \text{alg}(t, e_v, e_a, e_b) &= \left(R t^{1/2} e_v, R(1-t)^{1/2} e_a, R(1-t)^{1/2} e_b\right)
\end{align*}
\]
The inverses
\[ \text{trig}^{-1}(v,a,b) = (R \arcsin (|v|/R), v/|v|, a/|a|, b/|b|) \]
\[ \text{alg}^{-1}(v,a,b) = (|v|/R, v/|v|, a/|a|, b/|b|) \]
are defined away from the sets \( S = \{(Re_v,0,0)|e_v \in S^k\} \) and \( S' = \{(0,Re_a,Re_b)|(e_a,e_b) \in V_2(\mathbb{R}^{k+1})\} \).

**Proposition 15.** The singular locus \( \mathcal{L}_1(S^k)^{\text{sing}} \) coincides with \( S \).

**Proof.** The set \( S \) is a subset of \( \mathcal{L}_1(S^k)^{\text{sing}} \) because the differentials \( d(a^2 - b^2) \) and \( d(a \cdot b) \) vanish on \( S \). Let \( S' \) be \( \{(0,Re_a,Re_b)|(e_a,e_b) \in V_2(\mathbb{R}^{k+1})\} \). The complement \( \mathcal{L}_1(S^k) \setminus (S \cup S') \) is smooth because the map \( \text{alg} \) defines a diffeomorphism of the set with a smooth manifold.

A direct inspection shows that differentials of \( \text{trig} \) are linearly independent along \( S' \). By inverse function theorem \( \mathcal{L}_1(S^k) \) is smooth near \( S' \). \( \square \)

As a corollary we see that \( \dim \mathcal{L}_1(S^k) = 3k - 2 \)

The space \( \text{Trig}_1(\mathbb{R}^{k+1}) \) is equipped with metrics
\[ g = dv \cdot dv + \frac{1}{2}(da \cdot da + db \cdot db). \]

Our next goal is to compute restrictions \( g = \text{pol}^* g' \).

For this purpose we need to have a description of a tangent space \( T_x \) to a point \( x \in V_3(\mathbb{R}^{k+1}) \). \( V_3(\mathbb{R}^{k+1}) \equiv \text{SO}(k+1)/\text{SO}(k-2) \) is a homogeneous \( \text{SO}(k+1) \times \text{SO}(3) \)-space. Suppose the point \( x \) is represented by a coset \( \text{SO}(k-2) \). Then the tangent space to \( x \) is \( \mathfrak{so}(k+1)/\mathfrak{so}(k-2) \). Using the isomorphism \( \mathfrak{so}(k+1) \cong \Lambda^2 \mathbb{R}^{k+1} \) and the inner product we identify the tangent space to \( x = (e_v,e_a,e_b) \) with \( \Lambda^2 \langle e_v,e_a,e_b \rangle \) \( \in \mathbb{R}^{k+1} \). The space \( T_x \) being a subspace in \( \mathbb{R}^{k+1} \) \( \times \mathbb{R}^{k+1} \) carries its own inner product induced from the metric \( (l \otimes m, l' \otimes m') = (l \cdot l') \times (m \cdot m') \). It is convenient to use the corresponding metric, which we denote by \( h \), as a reference point in the space of metrics on \( V_3(\mathbb{R}^{k+1}) \).

We will need a description of an \( h \)-orthonormal basis for \( T_x \). For this purpose we complete \( x \) to an orthonormal basis \( \{e_v,e_a,e_b,e_1,\ldots,e_{k-2}\} \) for \( \mathbb{R}^{k+1} \). We define a basis \( B_x \) for \( T_x \) as \( \{e_{va},e_{vb},e_{ab},e_{vi},e_{ai},e_{bi}|i = 1,\ldots,k-2\} \), where
\[ e_{ij} = \frac{1}{\sqrt{2}}(e_i e_j - e_j e_i) \]
i, j = v, a, b, 1, \ldots, k - 3. Indeed each \( e_{ij} \in B_x \) defined a variation of the frame:
\[ x \rightarrow x + \epsilon \frac{1}{\sqrt{2}}((e_i \cdot e_v)e_j - (e_j \cdot e_v)e_i, (e_i \cdot e_a)e_j - (e_j \cdot e_a)e_i, (e_i \cdot e_b)e_j - (e_j \cdot e_b)e_i) \]
where \( \epsilon \) is an infinitesimal parameter. The metric \( h \) in this frame is
\[ g_{ref} = de_{va}^2 + de_{vb}^2 + de_{ab}^2 + \sum_{i=1}^{k-2} de_{vi}^2 + de_{ai}^2 + de_{bi}^2 \]
In the above notations the pullback of the metric $g_{\Omega}$ is equal to

$$\text{alg}^* g = \frac{R^2}{4} \left( \frac{1}{t(1-t)} dt^2 + g_{\Omega}(t) \right), \quad \text{trig}^* g = d\tau^2 + \frac{R^2}{4} g_{\Omega}(\sin^2(\tau/R))$$

$$g_{\Omega}(t) = (1 + t)de_{va}^2 + (1 + t)de_{vb}^2 + 2(1 - t)de_{ab}^2 +$$

$$+ \sum_{i=1}^{k-2} 2tde_{vi}^2 + (1 - t)de_{ai}^2 + (1 - t)de_{bi}^2$$

$$\text{trig}^* g = \frac{R^2}{4} \left( \frac{4}{R^2} d\tau^2 + \left(1 + \sin^2 \left(\frac{\tau}{R}\right)\right) de_{va}^2 + \left(1 + \sin^2 \left(\frac{\tau}{R}\right)\right) de_{vb}^2 + 2 \cos^2 \left(\frac{\tau}{R}\right) de_{ab}^2 +$$

$$\sum_{i=1}^{k-2} \left(2 \sin^2 \left(\frac{\tau}{R}\right) de_{vi}^2 + \cos^2 \left(\frac{\tau}{R}\right) de_{ai}^2 + \cos^2 \left(\frac{\tau}{R}\right) de_{bi}^2\right)\right)$$

$$\text{alg}^* g = \frac{R^2}{4} \left( \frac{1}{t(1-t)} dt^2 + (1 + t)de_{va}^2 + (1 + t)de_{vb}^2 + 2(1 - t)de_{ab}^2 +$$

$$+ \sum_{i=1}^{k-2} 2tde_{vi}^2 + (1 - t)de_{ai}^2 + (1 - t)de_{bi}^2\right)$$

Proposition 16.

(1) $s(t) = \sqrt{\det G_{\Omega}(t)} = \text{dvol}_{g_{\Omega}} / \text{dvol}_{g_{\text{ref}}} = 2^{k-2} t^{k-2} (1 - t)^{k-1} (1 + t)$

$\sqrt{\det G_{\text{ref}}(t)} = \text{dvol}_{g_{\Omega}} / \text{dvol}_{g_{\text{ref}}} = 2^{k-2} t^{k-2} (1 - t)^{k-1}$

(2) Let $G = d\tau^2 + g_{\text{ref}}$ be the product metric on $(0, \frac{\pi R}{2}) \times V_3(\mathbb{R}^{k+1})$. The quantity $\text{dvol}_{\text{alg}^* g} / \text{dvol}_{G}$ is equal to

$w_{trig}^k(\tau) = \frac{R^{3k-3}}{2^{\frac{3k-1}{2}}} \sin^{k-2} (\tau/R) \cos^{2k-3} (\tau/R) (1 + \sin^2 (\tau/R))$

The volume of $(0, \frac{\pi R}{2}) \times V_3(\mathbb{R}^{k+1})$ with respect to $\text{dvol}_{\text{alg}^* g}$ is equal to

$$\frac{2^{3k-1} R^{3k-2} \Gamma(k - 1) \Gamma \left(\frac{k+1}{2}\right)}{\Gamma \left(\frac{3k}{2} - \frac{1}{2}\right)} \times \frac{\pi}{2} \times \text{Vol}_{g_{\text{ref}}}(V_3(\mathbb{R}^{k+1}))$$

(3) Let $G' = dt^2 + g_{\text{ref}}$ be the product metric on $(0, 1) \times V_3(\mathbb{R}^{k+1})$. The quantity $\text{dvol}_{\text{alg}^* g} / \text{dvol}_{G}$ is equal to
\[ w_{\text{alg}}^k(t) = c_k t^{\frac{k-3}{2}} (1 - t)^{k-2} (1 + t) \text{ where} \]
\[ c_k = \frac{R^{3k-2}}{2^{\frac{3k-3}{2}}} \]

In the following we will omit explicit \( k \)-dependence of \( w_{\text{trig}}^k, w_{\text{alg}}^k \) when the value of \( k \) is clear from the context.

The volume of a unit \( k \)-sphere is equal to (see e.g. [10])
\[ \text{Vol}(S^k) = \frac{2\pi^{\frac{k+1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)} \]

6. Ricci curvature

Such information could be useful because general theorems concerning Schrödinger operator (like [5]) rely on it.

6.1. Some explicit computations for \( L_1(S^k) \). In this section we present results of computation of Ricci curvature of \( L_1(S^k) \) carried out with a help of Mathematica. Here we present some partial but explicit results.

Ricci operator \( \text{Ric}_i^j = \text{Ric}_{ik} g^{kj} \) be diagonalized in a \( g_{ij} \)-onthonormal basis.

For \( k = 2 \) one can compute all relevant tensors explicitly using RGTC Mathematica pakage. The eigen values for the space \( L_1(S^2) \) are
\[ \frac{3t^2 + 2t + 3}{R^2 (t+1)^2} \text{ with multiplicity one. In particular, the scalar curvature is } \frac{4t(3t^2 + 5)}{R^2 (t+1)^2}. \]

From this we see that for \( L_1(S^2) \)
\[ \text{Ric}_{ij} \geq -\frac{1}{R^2 g_{ij}}. \]

The map
\[ p : L_1(S^k) \to (0, R), (v, a, b) \to |v|/R \]
is a submersion because it commutes with the action of \( \text{SO}(k + 1) \). It is defined on \( L_k \subset L_1(S^k) \) which consists of \( a, b, v \) such that \( ||a|| \neq 0, R \).

It is natural to use O’Neill formulas [22] for computation of the Ricci curvature of \( L_1(S^k) \).

The fibers of the map \( p \) are Stiefel manifolds that carry \( \text{SO}(k + 1) \)-homogeneous metric.

Besse in [3] gives formulas for curvature tensors of homogenous manifold that we are going to use. We have to set notations first. Let \( G \) be a Lie group and \( F \) be a closed connected Lie subgroup. We denote by \( \mathfrak{g} \) and \( \mathfrak{f} \) the Lie algebras of \( G \) and \( F \) respectively. We assume that \( \mathfrak{g} \) splits
\[ \mathfrak{g} = \mathfrak{f} + \mathfrak{p} \]
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into a direct sum of $\mathfrak{f}$ representations. We identify the tangent space to $eF \in G/F$ with $\mathfrak{g}/\mathfrak{f} \cong \mathfrak{p}$. The $G$-invariant metric on $G/F$ defines an inner product on $\mathfrak{p}$ and is completely characterized by it. All curvature tensors are $G$-invariant and can be written purely in terms of the inner product $(.,.)$ on $\mathfrak{p}$, bracket in $\mathfrak{g}$ and decomposition (31). We denote by $[a, b]_f$ projection of $[a, b]$ onto $\mathfrak{f}$ and by $[a, b]_p$ the corresponding projection on $\mathfrak{p}$. According to [3] the formula for Riemann tensor $R(X, Y)$ reads as

$$\begin{align*}
(R(X, Y)X, Y) &= \frac{3}{4}||[X, Y]_\mathfrak{p}||^2 - \frac{1}{2}([X, [X, Y]]_\mathfrak{p}, Y) - \frac{1}{2}([Y, [Y, X]]_\mathfrak{p}, X) \\
+ ||U(X, Y)||^2 - (U(X, X), U(Y, Y)), & X, Y \in \mathfrak{p}
\end{align*}$$

The map $U: \mathfrak{p} \otimes \mathfrak{p} \rightarrow \mathfrak{p}$ is defined by the formula:

$$2(U(X, Y), Z) = ([Z, X]_\mathfrak{p}, Y) + (X, [Z, Y]_\mathfrak{p})$$

In order to write a formula for Ricci curvature $Ric$ we have to fix orthogonal basis $\{X_i\}$ for $\mathfrak{p}$. Then

$$\begin{align*}
Ric(X, X) &= -\frac{1}{2} \sum_j ||[X, X_j]_\mathfrak{p}||^2 - \frac{1}{2} \sum_j ([X, [X, X_j]_\mathfrak{p}, X_j) - \frac{1}{2} \sum_j ([X, [X, X_j]]_\mathfrak{p}, X_j) \\
+ \frac{1}{4} \sum_{i,j} ([X_i, X_j]_\mathfrak{p}, X)^2 - ([Z, X]_\mathfrak{p}, X)
\end{align*}$$

where $Z = \sum_i U(X_i, X_i)$. It is known that $(Z, X) = \text{tr}(adX)$. This is why $Z = 0$, when the group $G$ is $\text{SO}(k + 1)$.

The formula for the Ricci tensor of the fiber of the projection $p$ is

$$\begin{align*}
Ric^p &= \frac{(2(k + 1) - 4)t^2 + (4(k + 1) - 16)t + 2(k + 1) - 4}{(1 + t)^2} dv_{ab}^2 + \\
&\quad \frac{2(k + 1) - 4}{t + 1} (dv_{va}^2 + dv_{vb}^2) \\
&\quad + (k - 3) \sum_{i=1}^{k-2} (dv_{ai}^2 + dv_{ai}^2 + dv_{bi}^2), k \geq 3
\end{align*}$$

$$\begin{align*}
Ric^p &= \frac{2t^2 - 4t + 2}{(1 + t)^2} dv_{ab}^2 + \frac{2t}{t + 1} (dv_{va}^2 + dv_{vb}^2), \quad k = 2
\end{align*}$$

$Ric^p$ is positive definite when $n \geq 3, t \geq 0$

In order to compute second fundamental form of the fibers of the projection $p$ we choose a vector field $e' = \frac{\partial}{\partial t}$ which is orthogonal to the fibers of $p$. The formula for the form reads

$$T(\eta, \xi) = \frac{1}{|e'|} \nabla_\eta \xi \cdot e'$$
where $\nabla$ is the Levi-Civita connection. We can compute the same quantity if we extend $\eta, \xi, e'$ from $L_1(S^k)$ to $\text{Trig}_1(\mathbb{R}^{k+1})$ compute it there and then restrict back to $L_1(S^k)$. For connection we take Levi-Civita connection on $\text{Trig}_1(\mathbb{R}^{k+1})$.

If vector fields $\eta, \xi$ are elements of $\mathfrak{so}_{k+1}$, then they have a canonical extension to $\text{Trig}_1(\mathbb{R}^{k+1})$, produced by $\mathfrak{so}_{k+1}$ action. The vector field $e'$ is a restriction of the vector field

$$\frac{R^2}{2|v|^2} v \cdot \partial_v - \frac{R^2}{2(R^2 - |v|^2)} a \cdot \partial_a - \frac{R^2}{2(R^2 - |v|^2)} b \cdot \partial_b$$

which is defined on $\text{Trig}_1(\mathbb{R}^{k+1})$, for which we will keep the same notation. Components $v \cdot \partial_v, a \cdot \partial_a, b \cdot \partial_b$ are dilation operators acting on separately on $v, a$ and $b$ components. Advantage of working on $\text{Trig}_1(\mathbb{R}^{k+1})$ is that covariant derivative associated with Levi-Civita connection defined by partial derivatives. In order to simplify result of coordinate computation it is convenient to evaluate $T$ at a point with $v = (R t^{1/2}, 0, 0, \ldots, 0), a = (0, R \sqrt{1-t}, 0, \ldots, 0), b = (0, 0, R \sqrt{1-t}, 0, \ldots, 0)$. The result of the computation is

$$T = \frac{R}{2} \sqrt{t(1-t)} \left( dv^2_{va} + dv^2_{vb} - 2 dv^2_{ab} + \sum_{i=1}^{k-2} (2 dv^2_{vi} - dv^2_{ai} - dv^2_{bi}) \right) k \geq 3$$

$$T = \frac{R}{2} \sqrt{t(1-t)}(dv^2_{va} + dv^2_{vb} - 2 dv^2_{ab}), \quad k = 2$$

The trace of $T$ is equal to

$$\text{tr}T = \frac{2 \left( (7-4k)t^2 + (7-3k)t + k - 2 \right)}{R(t+1) \sqrt{t(1-t)}} \quad k \geq 3$$

$$\text{tr}T = \frac{2 \sqrt{t(1-t)}}{R(t+1)}, \quad k = 2$$

The tensor $CT = T_{ik} g^{kk'} T_{kj}$ is equal to

$$CT = t(1-t) \left( \frac{1}{1+t} dv^2_{va} + \frac{1}{1+t} dv^2_{vb} + \frac{2}{1-t} dv^2_{ab} + \sum_{i=1}^{k-2} \frac{2}{t} dv^2_{vi} + \frac{1}{1-t} dv^2_{ai} + \frac{1}{1-t} dv^2_{bi} \right) k \geq 3$$

$$CT = t(1-t) \left( \frac{1}{1+t} dv^2_{va} + \frac{1}{1+t} dv^2_{vb} + \frac{2}{1-t} dv^2_{ab} \right) k = 2$$

Finally the vertical component of the Ricci tensor is equal to

$$- \left( (8k-7)t^2 + (5k-6)t - 3k + 5 \right) \left( de^2_{ai} + de^2_{bi} \right) \frac{t+1}{t+1}$$

$$+ \left( 16kt^2 + 13kt - 3k - 14t^2 - 19t + 3 \right) de^2_{vi} \frac{t+1}{t+1}$$

$$- 2 de^2_{ab} \left( (8k-7)t^2 + (13k-15)t^2 + (2k-1)t - 3k + 3 \right) (t+1)^2 + ((8k-7)t - 2) \left( de^2_{va} + de^2_{vb} \right)$$
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We will not attempt to compute the horizontal and mixed components here and just notice that if $k \geq 4$ then the mixed component must be zero by symmetry reasons. The general for the Ricci curvature for arbitrary $L_N(S^k)$ will be presented in the next section.

6.2. The curvature of a submanifold $M \subset \mathbb{R}^n$. A review. The space $L_N(S^k)$ are defined as a submanifold inside the linear space $\text{Trig}_N(\mathbb{R}^{k+1})$. Curvature of such submanifold can be computed from fundamental equations of differential geometry. In this section we are going to remind how this is done and set up notations. Let $(f_1(v), \ldots, f_k(v))$ be components of a smooth map $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$. Let us assume that $f$ is transversal at $0 \in \mathbb{R}^k$, that is collection of smooth $df_1, \ldots, df_k$ are linearly independent on $M = \{v \in \mathbb{R}^n | f(v) = 0\}$.

By inverse function theorem $M$ is a smooth submanifold in $\mathbb{R}^n$. Introduce notation: $T_M$ and $N_M$ stands for the tangent and normal bundles of $M \subset \mathbb{R}^n$, $\Gamma(M,L)$ is a space of $C^\infty$-sections of a vector bundle $L$ over $M$. Let $R'$ be the curvature of a metric tensor $g'_ab$ on $\mathbb{R}^n$.

The curvature $R$ of the induced metric $g_{cd}$ on $M$ can be computed via Gauss'-Weingarten equation

$$\langle R'(X,Y)Z,W \rangle = \langle R(X,Y)Z,W \rangle + \langle \alpha(X,Z), \alpha(Y,W) \rangle - \langle \alpha(Y,Z), \alpha(X,W) \rangle$$

where $\alpha \in \Gamma(M, \text{Sym}^2 T_M \otimes N_M)$ is the second quadratic form. We can compute it by the formula

$$\alpha(X,Y) = -\langle X, \nabla_Y e_i \rangle e_i.$$

Here $e_i$ is an orthonormal basis for $N_M$. Obviously we get the same answer if we use the formula

$$\alpha(X,Y) = \langle \nabla_X Y, h_i \rangle h^i$$

where

$$\{h_i | i = 1, \ldots, k\}$$

is a basis for $N_M$ and $\{h^i\}$ is the $g'$-dual basis.

In the following indices labelled by latin letters $i, j, k, l, q, r$ will run through this range. Then

$$\langle \alpha(X,Z), \alpha(Y,W) \rangle = \langle \nabla_X Z, h_i \rangle \langle \nabla_Y W, h_k \rangle s_{ik}$$

$s_{ik} = \langle h_i, h_k \rangle$ and $(s_{ik})$ is the inverse to $(s^{ik})$. In our case collection

$$\{h_i\} = \{\text{grad} f_i\}$$

defines a local basis for $N_M$. Furthermore

$$\frac{\partial}{\partial x^a} \text{written in standard coordinates is a constant matrix.}$$

Then the covariant derivative $\nabla'_a$ associated with Levi-Civita connection is $\frac{\partial}{\partial x^a}$, $R' = 0$ and

$$\langle R(X,Y)Z,W \rangle = \langle \nabla'_Y Z, \text{grad} f_i \rangle \langle \nabla'_X W, \text{grad} f_j \rangle s^{ij} - \langle \nabla'_X Z, \text{grad} f_i \rangle \langle \nabla'_Y W, \text{grad} f_j \rangle s^{ij}$$

$s_{ij} = \langle \text{grad} f_i, \text{grad} f_j \rangle g'$.

We just proved the following proposition.
Proposition 17. The curvature of the submanifold $M \subset \mathbb{R}^n$ (32), where $\mathbb{R}^n$ is equipped with a metric $g'_{ab}$ (30) is given by the formula (37).

For $M$-tangential vectors $\langle Z, \text{grad} f_i \rangle = 0$. Then $\langle \nabla'_Y Z, \text{grad} f_i \rangle = -\langle Z, \nabla'_Y \text{grad} f_i \rangle$ and the formula can be written in a form that is more suitable for computations:

\[
\langle R(X, Y) Z, W \rangle = (38) = \langle \nabla'_Y Z, \text{grad} f_i \rangle \langle \nabla'_X W, \text{grad} f_j \rangle s^{ij} - \langle \nabla'_X Z, \text{grad} f_i \rangle \langle \nabla'_W Y, \text{grad} f_j \rangle s^{ij}
\]

Let $\{l_i\}$ be a local basis of the tangent bundle $T_M$, and $\{l^i\}$ be the $g'$-dual basis. The tensor $g'^{ab} \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b}, 1 \leq ab \leq n$ is a sum of two orthogonal components $l_i \otimes l^i + h_r \otimes h^r$. Then

\[
l_i \otimes l^i = g'^{ab} \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b} - h_r \otimes h^r.
\]

Recall that the Ricci tensor is a contraction

\[
\text{Ric}(X, Y) = \langle R(X, l_i) Y, l^i \rangle = -\langle R(X, l_i) l^i, Y \rangle
\]

We use the second sum in the above formula and (39) to derive $\text{Ric}(X, W)$:

\[
\text{Ric}(X, W) = g'^{ab} \langle \nabla'_X \frac{\partial}{\partial x^b}, \text{grad} f_i \rangle \langle \nabla'_Y \text{grad} f_j \rangle s^{ij} - g'^{ab} \langle \nabla'_X \frac{\partial}{\partial x^b}, \text{grad} f_i \rangle \langle \nabla'_W \frac{\partial}{\partial x^a}, \text{grad} f_j \rangle s^{ij} - \langle \nabla'_X h^r, \text{grad} f_i \rangle \langle \nabla'_W \text{grad} f_j \rangle s^{ij} + \langle \nabla'_X h^r, \text{grad} f_i \rangle \langle \nabla'_W h_r, \text{grad} f_j \rangle s^{ij}
\]

We know that $\frac{\partial}{\partial x^a}$ is covariantly constant so $\nabla'_X \frac{\partial}{\partial x^a} = 0$ for all $X$. Additionally

\[
h^q = s^{qr} \text{grad} f_r
\]

and

\[
\langle \nabla'_X h^r, \text{grad} f_i \rangle = -\langle h^r, \nabla'_X \text{grad} f_i \rangle \text{ because } \langle h^r, \text{grad} f_i \rangle = \delta^r_i
\]

Thus

\[
\text{Ric}(X, W) = -\langle \nabla'_{\text{grad} f_r} h^r, \text{grad} f_i \rangle \langle \nabla'_X \text{grad} f_j \rangle s^{ij} + \langle \nabla'_X h^r, \text{grad} f_i \rangle \langle \nabla'_W \text{grad} f_r, \text{grad} f_j \rangle s^{ij} - \langle \nabla'_{\text{grad} f_r} h^r, \text{grad} f_i \rangle \langle \nabla'_W \text{grad} f_j \rangle s^{qr} s^{ij} - \langle \nabla'_X h^r, \text{grad} f_i \rangle \langle \text{grad} f_r, \nabla'_W \text{grad} f_j \rangle s^{qr} s^{ij}
\]

Proposition 18. Ricci curvature of the submanifold $M \subset \mathbb{R}^n$ (32), where $\mathbb{R}^n$ is equipped with a metric $g'_{ij}$ (30) is equal to

\[
\text{Ric}(X, W) = \langle \text{grad} f_q, \nabla'_{\text{grad} f_r} \text{grad} f_i \rangle \langle \nabla'_X \text{grad} f_j \rangle s^{qr} s^{ij} - \langle \text{grad} f_q, \nabla'_X \text{grad} f_i \rangle \langle \text{grad} f_r, \nabla'_{\text{grad} f_j} \rangle s^{qr} s^{ij}
\]

where $X, W \in \Gamma(M, T_M)$. 
Corollary 19. The scalar curvature $Sc$ of $M$ is equal to

\[ Sc = \]

\[
- \langle \text{grad} f_q, \text{grad} \left( \frac{\partial f_i}{\partial x^a} \right) \rangle \langle \text{grad} f_r, \text{grad} \left( \frac{\partial f_j}{\partial x^a} \right) \rangle g^{ab} s^{r} s^{ij} \\
+ \langle \text{grad} f_q, \nabla'_{\text{grad} f_r} \text{grad} f_i \rangle \langle \text{grad} f_k, \nabla'_{\text{grad} f_l} \text{grad} f_j \rangle s^{kl} s^{r} s^{ij} \\
+ \langle \text{grad} f_q, \nabla'_{\text{grad} f_r} \text{grad} f_i \rangle \langle \text{grad} f_l, \nabla'_{\text{grad} f_k} \text{grad} f_j \rangle s^{kl} s^{r} s^{ij}
\]

(42)

Proof. Follows from the formula $R = R(l_i, l^i)$, (39) and (40) \qed

The mean curvature $H$ is equal to

\[ \alpha(l_i, l^i) = \langle \nabla_l j^i, h_i \rangle h^i = g^{ab} \langle \nabla_{\frac{\partial}{\partial x^a}} h_i, h_i \rangle h^i - \langle \nabla h, h^r, h_i \rangle h^i = -\langle \nabla h, h^r, h_i \rangle h^i = \]

\[
\langle \text{grad} f_q, \nabla_{\text{grad} f_r} \text{grad} f_i \rangle \text{grad} f_j s^{r} s^{ij}
\]

It square is equal to

\[ H^2 = s^{r} s^{ij} \langle \text{grad} f_q, \nabla_{\text{grad} f_r} \text{grad} f_i \rangle s^{kl} s^{r} s^{ij} \]

(43)

Note that it is the middle term in the formula for $Sc$.

Proposition 20. Let $M^n$ be an immersed submanifold in the Euclidean space $\mathbb{R}^{n+p}$ where $p$ denotes the codimension. Let $\text{Ric}_{\text{min}}, Sc$, and $H$ denote the functions that assign to each point of $M$ the minimum Ricci curvature, the scalar curvature, and the mean curvature respectively of $M$ at the point. Then we have

\[ \text{Ric}_{\text{min}} \geq Sc - \frac{(n - 1)H^2}{4} + \frac{1}{4n^2} \left( \sqrt{n - 1}(n - 2)|H| - 2\sqrt{(n - 1)H^2 - nSc} \right)^2 \]

(44)

6.3. Curvature tensors of $L_N(S^k)$. So far discussion was very general. Suppose now $\mathbb{R}^n \cong \text{Trig}_N(\mathbb{R}^{k+1})$. It carries a metric metric $\langle \delta n_1(\theta), \delta n_2(\theta) \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} \delta n_1(\theta) \cdot \delta n_2(\theta) d\theta$, which in our coordinates is

\[ dv \cdot dv + \frac{1}{2} \sum_{s=1}^{N} (da[s] \cdot da[s] + db[s] \cdot db[s]) \]

(45)

A trigonometric polynomial $\phi(\theta) \in C_{2N}(S^1)$ defines a quadratic functional

\[ n(\theta) \rightarrow \frac{1}{2\pi} \int_{S^1} (n(\theta) \cdot n(\theta) - R^2) \phi(\theta) d\theta = f_\phi(n) \]

(46)

on $\text{Trig}_N(\mathbb{R}^{k+1})$. We defined the real algebraic variety $L_N(S^k) \subset \text{Trig}_N(\mathbb{R}^{k+1})$ as $\{n(\theta) \in \text{Trig}_N(\mathbb{R}^{k+1}) | f_\phi(n(\theta)) = 0 \ \forall \phi \in C_{2N}(S^1) \}$.

It will be useful to rewrite metric (45) by using Dirichlet kernel (17)

\[ \langle \delta_1 n, \delta_2 n \rangle = \frac{1}{4\pi^2} \int_{S^1 \times S^1} \delta_1 n(\theta_1) \cdot \delta_2 n(\theta_1) D_N(\theta_1 - \theta_2) \]

(47)

The tangent bundle to $\mathbb{R}^{k+1}$ is canonically trivialized. This is why we can identify tangent vectors $T_n(\theta)(\text{Trig}_N(\mathbb{R}^{k+1}))$ with elements $e(\theta) \in \text{Trig}_N(\mathbb{R}^{k+1})$. 


We start curvature computations with writing down the formula for $\text{grad} f_\phi$.

**Proposition 21.** (1)

\begin{equation}
(48) \quad \text{grad} f_\phi(\theta) = \frac{1}{\pi} \int_{S^1} D_N(\theta - \theta') n(\theta') \phi(\theta') d\theta'
\end{equation}

(2) Consider the bilinear form

\begin{equation}
(49) \quad 4g_N(\phi, \psi) := \langle \text{grad} f_\phi, \text{grad} f_\psi \rangle = \frac{1}{4\pi^2} \int_{S^1 \times S^1} 4g_N(\theta, \theta') \phi(\theta) \psi(\theta') d\theta d\theta'
\end{equation}

on $C_{2N}(S^1)$. The kernel $g_N$ is equal to

\begin{equation}
(50) \quad g_N(\theta, \theta') = D_N(\theta - \theta') n(\theta) \cdot n(\theta')
\end{equation}

**Proof.** (1) Let $n_t(\theta)$ be a smooth path $t \in [0, \epsilon)$, $n_0(\theta) = n(\theta)$ Follows from the formula

\begin{equation}
(51) \quad \frac{\partial f_\phi}{\partial t} \bigg|_{t=0} = \frac{2}{2\pi} \int_{S^1} n(\theta) \cdot \frac{\partial n_t(\theta)}{\partial t} \bigg|_{t=0} \phi(\theta) d\theta = \frac{1}{\pi} \int_{S^1 \times S^1} n(\theta') \cdot \frac{\partial n_t(\theta)}{\partial t} \bigg|_{t=0} \phi(\theta') D_N(\theta - \theta') d\theta d\theta'
\end{equation}

(2)

\begin{equation}
(52) \quad 4g(\phi, \psi) = \frac{4}{8\pi^2} \int_{S^1 \times S^1} D_N(\theta - \theta') D_N(\theta - \theta'') n(\theta') \cdot n(\theta'') \phi(\theta') \phi(\theta'') d\theta d\theta' d\theta'' = \frac{4}{4\pi^2} \int_{S^1 \times S^1} D_N(\theta' - \theta'') n(\theta') \cdot n(\theta'') \phi(\theta') \phi(\theta'') d\theta' d\theta''
\end{equation}

We used

\begin{equation}
(53) \quad \frac{1}{2\pi} \int_{S^1} D_N(\theta - \theta') D_N(\theta - \theta'') d\theta = D_N(\theta' - \theta'')
\end{equation}

\[\square\]

Besides inner product $g(\cdot, \cdot)$ the space $C_{2N}(S^1)$ carries the standard $L^2$-inner product $(\phi, \psi) = \frac{1}{2\pi} \int_{S^1} \phi(\theta) \psi(\theta) d(\theta)$. Then $g(\phi, \psi) = (g(\phi), \psi)$, where $g$ is some symmetric operator on $C_{2N}(S^1)$. The inverse by

\begin{equation}
(54) \quad G(\psi)(\theta) = \frac{1}{2\pi} \int_{S^1} G(\theta, \theta') \psi(\theta') d\theta'
\end{equation}

has the kernel that satisfies

\begin{equation}
(55) \quad \frac{1}{2\pi} \int_{S^1} D_N(\theta - \theta') n(\theta) \cdot n(\theta') G(\theta', \theta'') d\theta' = D_{2N}(\theta - \theta'')
\end{equation}

Let $\nabla$ be the covariant derivative associated with $\langle \cdot, \cdot \rangle$-compatible Levi-Civita connection on $T\text{Trig}_l(\mathbb{R}^{k+1})$.

**Proposition 22.** Let $e(\theta)$ be a tangent vector to $\text{Trig}_l(\mathbb{R}^{k+1})$. Then

\begin{equation}
(56) \quad (\nabla_b \text{grad} f_\psi)(\theta) = \frac{1}{2\pi} \int_{S^1} 2D_N(\theta - \theta') e(\theta') \psi(\theta') d\theta'
\end{equation}
Proof. The action of the operator \( \nabla_\theta \) on tensor fields has a very simple description in our coordinates. It acts as \( \frac{\partial}{\partial e} \) component-wise. The vector field \( \text{grad} f_\phi \) has linear coefficients. Introduce temporally coordinates \( \{n^\alpha\} T\text{Trig}(\mathbb{R}^{k+1}) \) in which metric has the identity Gram matrix. \( \nabla_\theta \text{grad} f_\phi \) is obtained by replacing all occurrences of \( n^\alpha \) by \( e^\alpha \). Then the formula immediately follows from \([15]\).

\[ \square \]

**Corollary 23.** The vector field \( (\nabla_{\text{grad} f_\phi} \text{grad} f_\psi)(\theta) \) is equal

\[
\frac{1}{4\pi^2} \int_{S^1 \times S^1} 4D_N(\theta - \theta')D_N(\theta' - \theta'')n(\theta'')\phi(\theta'')\psi(\theta'') d\theta' d\theta''
\]

The formula for Ricci curvature \([11]\) involves contraction of several tensors. In the following corollary we list results of a computation of various components of Ricci tensor:

**Corollary 24.**

Fix in addition two vector fields are \( X(\theta), W(\theta) \). Then

1. The kernel of tri-linear form
   \[
   \langle \text{grad} f_\eta, \nabla_{\text{grad} f_\phi} \text{grad} f_\psi \rangle = \frac{1}{8\pi^3} \int_{S^1 \times S^1 \times S^1} K(\theta, \theta', \theta'')\eta(\theta)\phi(\theta')\psi(\theta'') d\theta d\theta' d\theta''
   \]
   is equal to
   \[
   K(\theta, \theta', \theta'') = 8D_N(\theta - \theta')D_N(\theta' - \theta'')n(\theta) \cdot n(\theta')
   \]

2. For a fixed \( W \) and \( X \) the functional
   \[
   \frac{1}{2\pi} \int_{S^1} T_{X,T}(\theta)\phi(\theta) d\theta = \langle W, \nabla_X \text{grad} f_\phi \rangle
   \]
   has the kernel
   \[ T_{X,T}(\theta) = 2W(\theta) \cdot X(\theta) \]

3. For a fixed \( X \) the kernel of the bilinear form \( \frac{1}{4\pi^2} \int_{S^1 \times S^1} R_X(\theta, \theta')\phi(\theta)\psi(\theta') d\theta d\theta' \) is
   \[
   R_X(\theta, \theta') = 4D_N(\theta - \theta')n(\theta) \cdot X(\theta')
   \]

4. For a fixed \( W \) the kernel of the bilinear form \( \frac{1}{4\pi^2} \int_{S^1 \times S^1} S_W(\theta, \theta')\phi(\theta)\psi(\theta') d\theta d\theta' \) is
   \[
   S_X(\theta, \theta') = 4D_N(\theta - \theta')n(\theta) \cdot W(\theta')
   \]

**Proof.**

1. \[
\langle \text{grad} f_\eta, \nabla_{\text{grad} f_\phi} \text{grad} f_\psi \rangle =
\]

   \[
   = \langle \frac{1}{2\pi} \int_{S^1} 2D_N(\mu - \theta)n(\theta)\eta(\theta) d\theta, \frac{1}{4\pi^2} \int_{S^1 \times S^1} 4D_N(\mu - \theta')D_N(\theta' - \theta'')n(\theta'')\phi(\theta'')\psi(\theta') d\theta' d\theta'' \rangle, \mu \in S^1
   \]

   \[ = \frac{1}{8\pi^3} \int_{S^1 \times S^1 \times S^1} 8D_N(\theta - \theta'')D_N(\theta' - \theta'')n(\theta) \cdot n(\theta')\eta(\theta)\phi(\theta')\psi(\theta'') d\theta d\theta' d\theta'' \]

We used \([50]\).
\[ (W, \nabla_X \text{grad} f_\phi) = \frac{1}{4\pi^2} \int_{S^1 \times S^1} 2W(\theta) \cdot X(\theta') D_N(\theta - \theta') \phi(\theta') d\theta d\theta' = \frac{1}{2\pi} \int_{S^1} 2W(\theta) \cdot X(\theta) \phi(\theta) d\theta \]

(3)

\[ \langle \text{grad} f_\phi, \nabla_X \text{grad} f_\psi \rangle = \frac{1}{8\pi^3} \int_{S^1 \times S^1 \times S^1} 4[D_N(\mu - \theta)n(\theta)\phi(\theta)\psi(\theta)]d\mu d\theta d\theta' = \]

We used (50).

(4)

\[ \langle W, \nabla_{\text{grad} f_\phi} \text{grad} f_\psi \rangle = \frac{1}{8\pi^3} \int_{S^1 \times S^1 \times S^1} 4W(\mu) [D_N(\mu - \theta')D_N(\theta - \theta')n(\theta)\phi(\theta')\psi(\theta')]d\mu d\theta d\theta' = \]

\[ = \frac{1}{4\pi^2} \int_{S^1 \times S^1} 4n(\theta) \cdot W(\theta') D_N(\theta - \theta') \phi(\theta)\psi(\theta') d\theta d\theta' \]

\[ = 0 \]

An immediate corollary of the formulas (38) and (52) and of the definition of \( G \) (51) is that the full curvature tensor is equal to

\[ \langle R_N(X,Y)Z,W \rangle = \frac{1}{4\pi^2} \int_{S^1 \times S^1} \left( Z(\theta) \cdot Y(\theta) X(\theta') \cdot W(\theta') - Z(\theta) \cdot X(\theta) W(\theta') \cdot Y(\theta') \right) G_N(\theta,\theta') d\theta d\theta' \]

We can put together results of the computations and obtain from equation (41) and Corollary 24 a formula for the Ricci curvature:

\[ \text{Ric}(W,X) = \frac{1}{16\pi^4} \int_{S^1 \times S^1 \times S^1} \left( D_N(\mu - \theta) G_N(\theta,\theta') n(\theta) \cdot n(\theta') D_N(\mu' - \theta') G_N(\mu,\mu') W(\mu) \cdot X(\mu) \right. \]

\[ - D_N(\theta - \theta') n(\theta) \cdot X(\theta') D_N(\mu - \mu') n(\mu) \cdot W(\mu') G_N(\theta,\mu) G_N(\theta',\mu') \left. \right) d\theta d\theta' d\mu d\mu' \]

In order to use inequality (44) we compute the square of mean curvature \( H^2 \) and the scalar curvature. For computation of \( H^2 \) we use (43) and (52):

\[ H^2 = \]

(53)

\[ = \frac{1}{2\pi^6} \int_{(S^1)^6} G(\theta,\theta') n(\theta) \cdot n(\theta') D_N(\theta - \theta'') D_N(\theta' - \theta'''') G(\theta'',\mu''') D_N(\mu'' - \mu') n(\mu) \cdot n(\mu') G(\mu',\mu) d\theta \cdots d\mu'' \]

We would like contract \( \text{Ric}(X,W) \) further and find scalar curvature.
Proof. This is a straightforward adaptation of formula (42) to the case of $L_N(S^k)$}

On manifolds $L_N(S^k)$ there are several ways to gauge the closeness of a point $n$ to a singular locus $L_{N-1}(S^k)$. Besides the most obvious way to do it with the distance function $\rho(n, L_{N-1}(S^k))$ we can use the function $n(\theta) \to f_N(n) = A^2_N$. The square of the last Fourier coefficients $A^2_N = B^2_N$ have the same zero locus as $\rho(n, L_{N-1}(S^k))$ and algebraically more simple. This is why we use it in the definition of tubular neighborhood

$$U_\epsilon L_{N-1}(S^k) = \{ n(\theta) \in L_N(S^k) | f(n) \leq \epsilon \}$$

Proposition 26. The set $U_\epsilon L_{N-1}(S^k)$ has a convex boundary $\{ n(\theta) \in L_N(S^k) | f(n) = \epsilon \}$.

Proof. The second fundamental form is the Hessian of $A^2_N$, evaluated on two tangent vectors $X, Y \in T_n(L_N(S^k))$. We used that connection $\nabla$ on $T(L_N(S^k))$ is induced from the trivial connection on $\text{Trig}_N(\mathbb{R}^{k+1})$. Then $\alpha(X, Y) = 2X \cdot Y \nu h$, where $h = \frac{\text{grad} f_N}{\text{grad} f_N}$ is a vector normal to $\partial U_\epsilon L_{N-1}(S^k)$ such that $\langle \text{grad}, h \rangle = 1$ The form $\alpha(X, Y)$ is semidefinite and $\partial U_\epsilon L_{N-1}(S^k)$ is convex.

7. Schrödinger operator on $\mathcal{L}_1(S^k)$

Our goal is to define the Schrödinger operator on $L^2(\mathcal{L}_1(S^k)^{smooth})$. It is the sum $-\Delta + U$, where $U$ is a restriction of the potential defined on $\text{Trig}_1(\mathbb{R}^{k+1})$. Keep in mind that $U$ has an intrinsic meaning for $L_1(S^k)$. The vector field $\rho$ is tangential to $M_1(S^k) \subset \text{Trig}_1(\mathbb{R}^{k+1})$ and $\text{pol}^* g(\rho, \rho) = \langle \rho, \rho \rangle = U$. Equations for $L_1(S^k)$ imply that

$$U = \frac{1}{2L^2}(a^2 + b^2) = \frac{a^2}{L^2} = \frac{R^2}{L^2} \cos^2 \left( \frac{\tau}{R} \right) = \frac{R^2}{L^2} (1 - t)$$

We use a densely defined quadratic form

$$Q(f, g) = \int_{M_1(S^k)^{smooth}} (\nabla f \cdot \nabla g + U fg) d\text{vol}$$

on the $L^2(M_1(S^k)^{smooth}) \cap C^\infty_c(M_1(S^k)^{smooth})$ to define Schrödinger operator $H$ as $Q(f, g) = \langle Hf, g \rangle$ with

$$\langle f, g \rangle = \int_{M_1(S^k)^{smooth}} fg d\text{vol}$$
The fibers of projection (50) are Stiefel manifolds we will start with analysis of the angular part of the Schrödinger operator that acts on functions on the fibers.

7.1. Harmonic analysis on Stiefel manifolds. We let \( V_m(\mathbb{R}^{k+1}) \) denote the Stiefel manifold of real matrices \( V \in M_{k+1,m} \) such that \( V^tV = I_m \). The rotation group \( \text{SO}(k+1) \) acts on \( V_m(\mathbb{R}^{k+1}) \) by left matrix multiplication so that \( V_m(\mathbb{R}^{k+1}) \) is isomorphic to \( \text{SO}(k+1)/\text{SO}(k+1-m) \). We would like to think about \( L^2(\mathbb{R}^{k+1}) \) as \( L^2(\text{SO}(k+1))^{\text{SO}(k+1-m)} \). The later space has an inner product \( \langle f, g \rangle = \int_{\text{SO}(k+1)} f\overline{g}d\mu \), where \( d\mu \) is a normalized bi-invariant Haar measure on \( \text{SO}(k+1) \). According to [5] if \( k > 2m \)

\[
L^2(\mathbb{R}^{k+1}) = \bigoplus_{\omega} H^{k+1,m}_\omega \otimes G_\omega
\]

In this sum \( H^{k+1,m}_\omega \) is an irreducible representation of \( \text{SO}(k+1) \) of highest weight \( \omega = [m_1, \ldots, m_{k+1}] \), with \( m_1 \geq \cdots \geq m_{k+1} \). The linear space \( G_\omega \) coincides with the linear space of some irreducible representation of \( \text{GL}(m) \) if \( m_i = 0 \) for \( i > m \). Otherwise \( G_\omega = 0 \).

In our application we are interested in the case \( m = 3 \). Then

\[
\dim G_{[m_1,m_2,m_3]} = 1/2(m_1 - m_2 + 1)(m_2 - m_3 + 1)(m_1 - m_3 + 2)
\]

7.2. Case \( k = 2 \). In this case \( V_3(\mathbb{R}^3) \) is isomorphic to the group \( \text{O}(3) \). Then \( L^2(V_3(\mathbb{R}^3)) = L^2(\text{SO}(3)) + L^2(\text{SO}(3)) \)

\[
L^2(\text{SO}(3)) = \bigoplus_{l\geq 0} W_{2l} \otimes \overline{W}_{2l}
\]

The space \( W_l \) are highest weight \( l \) representation of \( \mathfrak{so}_3(\mathbb{R}) \otimes \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C}) \). The inverse to \( t \)-transverse part of the metric \( \frac{R^2}{4} g_{\Omega}(t) \) is

\[
\frac{4}{R^2}((1 + t)^{-1}\partial_{e_{va}}^2 + (1 + t)^{-1}\partial_{e_{vb}}^2 + (2(1 - t))^{-1}\partial_{e_{ab}}^2) = \\
\frac{4}{R^2}((1 + t)^{-1}(\partial_{e_{va}}^2 + \partial_{e_{vb}}^2 + \partial_{e_{ab}}^2) + \frac{3t - 1}{2(1 - t^2)}\partial_{e_{ab}}^2)
\]

The group \( \text{O}(3) \) is a \( \text{O}(3) \times \text{SO}(2) \)-homogeneous space, where \( \text{O}(3) \) acts freely from the left and \( \text{SO}(2) \) acts freely from the right. The bi-tensor (56) is a \( \text{O}(3) \times \text{SO}(2) \)-invariant. We use the Levi-Civita connection associated with a \( \text{O}(3) \)-bi-invariant metric \( g_h = de_{va}^2 + de_{vb}^2 + de_{ab}^2 \) on \( \text{O}(3) \) to lift the tensor, interpreted as a symbol, to a differential operator.

We omit a trivial verification that the resulting operator is

\[
H_\Omega = -\frac{4}{R^2(1 + t)}\Delta - \frac{2(3t - 1)}{R^2(1 - t^2)}L_{e_{ab}}^2,
\]

where \( \Delta \) is the Laplace operator associated with the metric \( g_h \) and \( L_{e_{ab}} \) is the Lie derivative along the vector field \( \partial_{e_{ab}} \).

The operator \( \Delta \) up to suitable rescaling coincides with the operator defined by the Casimir element \( C \in U(\mathfrak{so}_3) \).
Lemma 27. Let \( T \in U(\mathfrak{so}_{k+1}) \) be \( \sum_{1 \leq s < t \leq k+1} e_{st}e_{st} \) with \( e_{st} \) as in (25). The operator \( \rho(\Theta) \) in \( \mathbb{R}^{k+1} \) acts as a scalar multiplication on \(-k/2\).

Let \( \rho_l \) be representation of \( U(\mathfrak{so}_3) \) in \( W_1 \).

Lemma 28. The operator \( \Delta \) acts on \( W_{2l} \) by multiplication on \(-\frac{l(2l+1)}{3}\).

Proof. It is well known that \( \rho_l(C) = l(l+1)\text{Id} \) (see e.g. [11]). By \( \mathfrak{so}_3 \)-invariance \( T = \text{const} \times C \). Note that the \( \mathfrak{so}_3 \) representation \( \mathbb{R}^3 \) is isomorphic to \( W_2 \). We find constant \( \text{const} \) equal to \(-1/6\) by comparing \( \text{const} \times l(l+1)_{l=2} \) with the scalar \(-k/2\) for Lemma 27. Under regular representation of \( \mathfrak{so}_3 \) in \( L^2(\text{SO}(3)) \) \( \rho_{L^2(\text{SO}(3))}(T) = \Delta \). □

Lemma 29. Operator \( L^2_{e_{ab}} \) in \( W = W_{2l} \) has a spectral decomposition \( \bigoplus_\lambda W^\lambda \). The eigenvalues satisfy

\[
\lambda = -\frac{s^2}{2}, -l \leq s \leq l
\]

For \( s = 0 \) \( \dim(W^0) = 1 \), for \( s \neq 0 \) \( \dim(W^{-s}) = 2 \)

Proof. The regular element \( e_{ab} \) generates a Cartan subalgebra in \( \mathfrak{so}_3(\mathbb{R}) \otimes \mathbb{C} \). It is known that the operator \( \rho_l(e_{ab}) \) in \( W_l \) can be diagonalized in a weight basis. After a suitable normalization \( e_{ab} \rightarrow c e_{ab} \) the eigenvalues of \( c e_{ab} \) in \( W_{2l} \) become \( 2s, -l \leq s \leq l \). We find normalization constant from the condition that \( e_{ab}^2 \) acts in \( \mathbb{R}^3 \) by the formula \( \rho_{2}(e_{ab}^2)e_a = -1/2e_a, \rho_{2}(e_{ab}^2)e_b = -1/2e_b, \rho_{2}(e_{ab}^2)e_v = 0 \). On the other hand \( c e_{ab} \) has a standard weight basis \( e, h, f \) such that \( \rho_{2}(c e_{ab})e = 2e, \rho_{2}(c e_{ab})h = 0, \rho_{2}(c e_{ab})f = -2e \). From this \( c^2 = -1/8 \) and the eigenvalues are given by (38). The spectral decomposition follows from the weight decomposition

\[
V^0 = W_{2l}^0 \otimes \overline{W}_{2l}, V^{-\frac{s^2}{2}} = (W_{2l}^{2s} + W_{2l}^{-2s}) \otimes \overline{W}_{2l}.
\]

□

Our previous results enable us to compute the eigenvalues of \( \Delta_{\Omega_l} \), which are

\[
\frac{4l(2l+1)}{3R^2(1+t)} + \frac{s^2(3t-1)}{R^2(1-t^2)} = \frac{4l(2l+1)}{3R^2(\sin^2(\tau)+1)} + \frac{s^2(3\sin^2(\tau)-1)}{R^2(1-\sin^2(\tau))} + R^2(1-\sin^2(\tau)), -l \leq s \leq l
\]

7.3. Case \( k = 3 \). In this case \( V_3(\mathbb{R}^4) \) is isomorphic to the group \( \text{SO}(4) \). Then \( L^2(V_3(\mathbb{R}^4)) = L^2(\text{SO}(4)) \) and

\[
L^2(\text{SO}(4)) = \bigoplus_{l \geq 0, l+m \equiv 0 \mod 2} W_l \otimes W_m \otimes \overline{W}_l \otimes \overline{W}_m
\]
Let \( \rho_{eq}U(\mathfrak{so}_4) \to Diff(f(\mathfrak{so}(4))) \) be a identification of the universal enveloping algebra with the algebra of left-invariant differential operators. The operator \( \Delta_{\Omega} \) is equal to image of

\[
-\frac{1}{2} \left( \frac{1}{1+t} c^2_{12} + \frac{1}{1+t} c^2_{13} + \frac{1}{2(1-t)} c^2_{23} + \frac{1}{2t} c^2_{14} + \frac{1}{1-t} c^2_{24} + \frac{1}{1-t} c^2_{34} \right)
\]

The eigenvalues in representations \( W_2 \otimes W_0 \) and \( W_0 \otimes W_2 \otimes W_0 \) are described in the previous subsection. The eigenvalues in \( W_1 \otimes W_1 \) are

\[
\left\{ \frac{t+5}{4(t-1)(t+1)}, \frac{t+5}{4(t-1)(t+1)} \right\}
\]

The formulas for eigenvalues in representations of greater highest weight become significantly more complicated because involve roots of algebraic equations of degree increasing with \( l \) and \( m \). For example eigenvalues in \( W_3 \otimes W_1 \) are

\[
\left\{ \frac{3t^2+12t+1}{4(t-1)t(t+1)}, \frac{3t^2+12t+1}{4(t-1)t(t+1)}, \frac{7t^2+16t+1}{4(t-1)t(t+1)}, \frac{9t^2-16t-1}{4(t-1)t(t+1)}, \frac{t^2+2\sqrt{13}t^4+4t^3+2t^2-4t+1}{4(t-1)t(t+1)} \right\}
\]

7.4. Radial component of the Shrödinger operator. In this section we assume that \( k \geq 2 \) is an integer. The form \( Q(f,g) \) simplifies significantly when \( f, g \) are \( (e_v,e_a,e_b) \)-independent functions:

\[
Q(f,g) = \int_0^1 \left( \frac{4t(1-t)}{R^2} f'(t)g'(t) + R^2(1-t)f(t)g(t) \right) w(t)dt,
\]

where \( w = w^k_{alg} \) as in \(^{(29)}\) written in a chart defined by the map \( alg \). We assume that \( f, g \in C_\infty^0(0,1) \). Integration by parts lead to the operator \( H_{rad} \), which satisfies \( Q_{rad}(f,g) = (H_{rad}f,g)_w \), where \( (f,g)_w = \int_0^1 f(t)g(t)w_{alg}(t)dt \). The operator \( H_{rad} \) is defined by the formula

\[
H_{rad}(f) = -w^{-1}_{alg}(p_k f')' + w^{-1}_{alg}q_k f
\]

\[
p_k(t) = \frac{4}{R^2} t(1-t)w^k_{alg}(t), q_k(t) = R^2(1-t)w^k_{alg}(t)
\]

For brevity sake we denote \( p_k(t) \) and \( q_k(t) \) by \( p \) and \( q \). The same formula defines an operator, which we denote by the same symbol, in the extended domain \( H_{rad} : \mathcal{O}_{an}(\mathbb{C}\setminus\{0,1,-1\}) \to \mathcal{O}_{an}(\mathbb{C}\setminus\{0,1,-1\}) \). Here \( \mathcal{O}_{an} \) stands for complex analytic functions of parameter \( z \in \mathbb{C}\setminus\{0,1,-1\} \). The eigenvalue problem \( H_{rad}f = \lambda R^2f \) in this space becomes an ODE:

\[
f''(t) + \left( \frac{k-1}{t-1} + \frac{k-1}{2t} + \frac{1}{t+1} \right) f'(t) + \eta^2 \left( \frac{\lambda-1}{4t} - \frac{\lambda}{4(t-1)} \right) f(t) = 0, \quad \eta = \frac{R^2}{L}
\]

At infinity it has the form

\[
f''(t) + \left( \frac{k-1}{t-1} + \frac{5-3k}{2t} + \frac{1}{t+1} \right) f'(t) + f(t)\eta^2 \left( -\frac{1}{4t^3} - \frac{\lambda}{4t^2} + \frac{\lambda}{4(t-1)} - \frac{\lambda}{4t} \right) f(t) = 0
\]
A more general equation

\[ f''(t) + \left(1 - \mu_2 + \alpha + \frac{1 - \mu_0}{t} + \frac{1 - \mu_1}{t - 1}\right) f'(t) + \frac{\beta_0 + \beta_2 t^2 + \beta_1 t}{t(t-1)(t-a)} f(t) = 0 \]

have been studied in It reduced to our equation after substitution \(a \to -1, \alpha \to 0, \mu_0 \to \frac{3}{2-k}, \mu_1 \to 2-k, \mu_2 \to 0, \beta_0 \to \frac{1}{4}\eta(1-\lambda), \beta_1 \to -\frac{1}{4}\eta\lambda, \beta_2 \to -\frac{\eta}{4} \)

7.5. **Local solutions.** The method of Frobenius [12] enables us two find a series solutions at the relevant singular points. We find that \(z = 0\) is a regular singularity with characteristic exponents \((3-k)/2\) and 0. These series provide us with a general solution of the form

\[ f(z) = c_1 y_1(z) + c_2 \left(\frac{1}{z^{3/2}} y_2(z) + \beta(k, R, \lambda) \ln(z) y_1(z)\right) \quad (k \equiv 1 \mod 2) \]

where \(y_1, y_2\) are analytic functions near zero.

A similar analysis at \(z = 1\) give characteristic exponents \(2-k\) and 0 and

\[ f(z) = c_1 \tilde{y}_1(z - 1) + c_2 \left(\frac{1}{(z-1)^{3/2}} \tilde{y}_2(z - 1) + \alpha(k, R, \lambda) \ln(z - 1) \tilde{y}_1(z - 1)\right) \]

\(\tilde{y}_1, \tilde{y}_2\) are analytic at \(z = 0\)

Differential equation (60) is equivalent to

\[ M_k f = -(p_k f')' + q_k f \]

\[ M_k f - \lambda w^k f = -(p_k f')' + q_k f - \lambda w^k f = 0 \]

whose local solutions also have asymptotics (61,62). As usual, when it is clear from the context we abbreviate \(M_k\) to \(M\).

Let \(AC_{\text{loc}}(0, 1)\) be the space of all complex-valued functions on \((0, 1)\) that are absolutely continuous on any compact interval \([\alpha, \beta] \subset (0, 1)\). We define \(L^2_w(0, 1)\) as a space of Lebesgue measurable functions on \((0, 1)\) with a finite \(\int_0^1 |f|^2 w dt = (f, f)_w\). Let

\[ D_k = \{ f \in AC_{\text{loc}}(0, 1) | p_k f' \in AC_{\text{loc}}(0, 1) \} \]

Following [21] we define the maximal domain of \(M\) as the subspace

\[ \Delta_k = \{ f \in D_k | f, \frac{1}{w^k} M_k(f) \in L^2_w(0, 1) \} \]

7.6. **Endpoint classification of the domain of the operator \(M\).** We follow [21] classification of singularities of (63).

**Proposition 30.**

The following classification of end-points of the domain \((0, 1)\) of the equation (63) holds

1. The point \(t = 0\)

   (a) is a regular point if \(k = 2\),
(b) is limit circle singularity if \( k = 3, 4 \),
(c) is limit point singularity if \( k \geq 5 \).

\( \text{(2)} \) The point \( t = 1 \)
(a) is limit circle singularity if \( k = 2 \),
(b) is limit point singularity if \( k \geq 3 \).

Proof. The quantities \(|q|, |w|\) are finite on \((0, 1)\), therefore only \(|p^{-1}|\) can contribute to singularity. The function \( p^{-1} \) is never integrable near \( t = 1 \) and \( p^{-1} \in L^1(0, 1/2) \) only when \( k = 2 \).

Expression \(|f|^2(t)w(t)\) is locally integrable at \( 0 \in (0, 1) \) \( \forall f \) as in (61) if \( k \leq 4 \), which takes care of the cases (1b,1c).

Likewise, the function \(|f|^2(t)w(t)\) is locally integrable near \( 1 \in (0, 1) \) \( \forall f \) as in (62) if \( k \leq 2 \). □

Definition 31. Define the operator \( T_k : \Omega_k \rightarrow L^2_w(0, 1), k = 2, 3, \ldots \) such that

1. \( \Omega_2 = \{ f \in \Delta_2 \mid \lim_{t \to 0^+} (pf')(t) = \lim_{t \to 1^-} (pf')(t) = 0 \} \)
2. \( \Omega_3 = \{ f \in \Delta_3 \mid \lim_{t \to 0^+} (pf')(t) = 0 \} \)
3. \( \Omega_4 = \{ f \in \Delta_4 \mid \lim_{t \to 0^+} (pf')(t) = 0 \} \)
4. \( \Omega_k = \Delta_k, k \geq 5 \)

and \( T_k f = \frac{1}{w^k} M_k f \) for \( f \in \Omega_k \).

For self-consistency of this definition the reader should check [1]. When confusion can’t arise we abbreviate \( T_k \) to \( T \) and \( \Omega_k \) to \( \Omega \).

Theorem 32. The operator \( T \) (see the definition (35)) is a self-adjoint in the Hilbert space \( L^2_w(0, 1) \)

Proof. See [21] Section 18. □

Theorem 33. Let \( T \) be the self-adjoint operator as defined in Definition 35. Then \( T \) has the following spectral properties:

1. The spectrum \( \sigma(T) \) is real simple and discrete; that is the spectrum consists solely of real simple eigenvalues, say \( \sigma(T_k) = \{ \lambda_n \in \mathbb{R}, n \in \mathbb{N} \} \) with the property
   \[ \lambda_0 \geq 0 \]
   \[ \lambda_n < \lambda_{n+1}, n \in \mathbb{N} \]
   \[ \lim_{n \to +\infty} \lambda_n = +\infty \]

The operator \( T \) is bounded below in \( L^2_w(0, 1) \) with
\[ (Tf, f)_w \geq 0, f \in \Omega \]

Since each eigenvalue is simple the eigenspaces satisfy
\[ \dim\{ f \in \Omega : Tf = \lambda_n f \} = 1, n \in \mathbb{N} \]
Proof. The proof of the similar statement for Heun equation can be found in [1]. It can be used almost verbatim to justify our theorem. Furthermore all other theorems of that paper remain valid in our context. The key moment in the proof in [1] specific to ordinary Heun equation are inequalities (83) and (84). Their analogues in our case are the following. Suppose 0 \leq t \leq 1/2 then
\[ p(t) \geq k_1 t^{k-1} \frac{k-1}{2^{k-2} R^2} t^{\frac{k-1}{2}} \]
\[ q(t) \leq k_2 t^{\frac{k-3}{2}} = c_k R^2 t^{\frac{k-3}{2}} \]
\[ w(t) \leq k_3 t^{\frac{k-3}{2}}, k_3 = c_k \text{ if } k \geq 3 \text{ or } k_3 = 3c_k/2 \text{ if } k = 2 \]
If 1/2 \leq t \leq 1 then
\[ p(t) \geq l_1(1-t)^{k-1} = \frac{3c_k}{2^{k-2} R^2}(1-t)^{k-1} \]
\[ q(t) \leq l_2(1-t)^{k-1}, l_2 = 2R^2c_k \text{ if } k \geq 3 \text{ or } l_2 = 3c_kR^2/\sqrt{2} \text{ if } k = 2 \]
\[ w(t) \leq l_3(1-t)^{k-2}, l_3 = 2c_k \text{ if } k \geq 3 \text{ or } l_3 = 3c_k/\sqrt{2} \text{ if } k = 2 \]
Following arguments of Theorem 23 [1] that uses Hardy type inequalities we deduce that for k \neq 3
\[ \int_a^b (pf'^2 + qf^2 - \lambda wf^2) \, dt > \int_a^b \left( k_1 \frac{(k-3)^2}{16} t^{\frac{k-5}{2}} - k_2 t^{\frac{k-3}{2}} - k_3 |\lambda| t^{\frac{k-3}{2}} \right) f^2 \, dt = \]
\[ = \int_a^b \left( k_1 \frac{(k-3)^2}{16} - (k_2 + k_3 |\lambda|) t \right) t^{\frac{k-5}{2}} f^2 \, dt \]
The last integral is positive if 0 < a < b < \frac{k_1(k-3)^2}{16(k_2 + k_3 |\lambda|)}. If k = 3 then
\[ \int_a^b (pf'^2 + qf^2 - \lambda wf^2) \, dt > \int_a^b \left( \frac{k_1}{4|\ln(t)|^2} - (k_2 + |\lambda| k_3) \right) f^2 \, dt \]
The integrand is obviously positive if 0 < a < b < \delta where \delta is sufficiently small. Suppose that 1/2 < a < b < 1. Then
\[ \int_a^b (pf'^2 + qf^2 - \lambda wf^2) \, dt > \int_a^b \left( l_1 \frac{(k-2)^2}{4} (1-t)^{k-3} - l_2(1-t)^{k-1} - l_3 |\lambda|(1-t)^{k-2} \right) f^2 \, dt = \]
\[ = \int_a^b \left( l_1 \frac{(k-2)^2}{4} - (l_2(1-t)^2 + l_3 |\lambda|(1-t)) \right) (1-t)^{k-3} f^2 \, dt \]
If $k = 2$ then
\[
\int_a^b (pf'^2 + qf^2 - \lambda w f^2) \, dt > \int_a^b \left( \frac{l_1}{4(1-t) \ln(1-t)^2} - (l_2(1-t) + |\lambda|l_3) \right) f^2 \, dt = \\
= \int_a^b \left( \frac{l_1}{4} - (l_2(1-t)^2 \ln(1-t)^2 + |\lambda|l_3(1-t) \ln(1-t)^2) \right) \frac{1}{(1-t) \ln(1-t)^2} f^2 \, dt
\]
The rest of the arguments repeats [1].

7.7. Incorporating harmonics at $k = 2$. Finally we would like to incorporate harmonics of angular Laplacian. Our equation becomes
\[
-f''(t) - \left( \frac{1}{2t} + \frac{1}{t+1} + \frac{1}{t-1} \right) f'(t) + f(t) \left( \frac{4l(2l+1)}{3R^2(1+t)} + \frac{s^2(3t-1)}{R^2(1-t^2)} + \frac{R^2(2\lambda)}{4t} + \frac{\lambda R^2(2\lambda)}{4(t-1)} \right) = 0
\]
\[
Q_t(f, g) = c_2 \int_0^1 \left( \frac{4l(1-t)}{R^2} f'(t) \bar{g}'(t) + \left( R^2(1-t) + \frac{4l(2l+1)}{3R^2(1+t)} + \frac{s^2(3t-1)}{R^2(1-t^2)} \right) f(t) \bar{g}(t) \right) (1+t)t^{-\frac{3}{2}} \, dt,
\]
Suppose $0 \leq t \leq 1/2$ then
\[
p(t) \geq \frac{k_1 t^{\frac{3}{2}}}{R^2} = \frac{3c_2}{R^2} \frac{3}{t^{\frac{3}{2}}}
\]
\[
q(t) \leq \frac{k_2 t^{-\frac{3}{2}}}{R^2} = c_2 \left( \frac{4l(2l+1)}{3R^2} + \frac{s^2}{R^2} + R^2 \right) t^{-\frac{3}{2}}
\]
\[
w(t) \leq k_3 t^{-\frac{3}{2}}, \quad k_3 = \frac{3c_2}{2} \text{ if } k = 2
\]
\[
\int_a^b (pf'^2 + qf^2 - \lambda w f^2) \, dt > \int_a^b \left( \frac{k_1}{16} t^{-\frac{3}{2}} - (k_2 + |\lambda|k_3) t^{-\frac{3}{2}} \right) f^2 \, dt = \\
= \int_a^b \left( \frac{k_1}{16} - (k_2 + |\lambda|k_3) t \right) t^{-\frac{3}{2}} f^2 \, dt
\]
If $1/2 \leq t \leq 1$ then
\[
p(t) \geq \frac{l_1(1-t)}{R^2} = \frac{3\sqrt{2}c_2}{R^2} \frac{2}{1-t}
\]
\[
q(t) \leq \frac{l_2}{(1-t)^{-1}}, \quad l_2 = c_2 \frac{\sqrt{2l(2l+1)}}{3R^2} + \frac{2s^2}{R^2} + \frac{3R^2}{8\sqrt{2}}
\]
\[
w(t) \leq l_3, \quad l_3 = \frac{3c_2}{\sqrt{2}}
\]
\[
\int_a^b (pf'^2 + qf^2 - \lambda w f^2) \, dt > \int_a^b \left( \frac{k_1}{4(1-t) \ln(1-t)^2} - \frac{l_2}{1-t} + |\lambda|l_3) \right) f^2 \, dt = \\
= \int_a^b \left( \frac{l_1}{4} - (l_2 + |\lambda|l_3(1-t) \ln(1-t)^2) \right) \frac{1}{(1-t) \ln(1-t)^2} f^2 \, dt
\]

Additional terms do not change characteristic exponents. The local solutions are
A NOTE ON $\sigma$-MODEL WITH THE TARGET $S^n$

\[ f(z) = c_1 y_1(z) + c_2 z^\frac{1}{2} y_2(z) \]
where $y_1, y_2$ are analytic functions near zero.

\[ f(z) = c_1 y_1(z - 1) + c_2 (y_2(z - 1) + \alpha(k, R, \lambda) \ln(z - 1) y_1(z - 1)) \]

7.8. Estimates of the spectral gap of $M$. R. Lavine in [16] showed that one-dimensional Schrödinger operators with a convex potential admits an estimate for the spectral gap:

**Theorem 34.** Let $U$ be convex on $[0, r]$, and let $\lambda_1$ and $\lambda_2$ be the first two eigenvalues for the Dirichlet Schrödinger operator $-\frac{d^2}{dx^2} + U$ on $[0, r]$. Then $\lambda_2 - \lambda_1 \geq \Gamma_0$ where $\Gamma_0$ is the gap for constant $U$ for the Dirichlet operator with equality only if $U$ is constant. Thus

$\lambda_2 - \lambda_1 \geq \frac{3\pi}{r^2}$

We are going to use this theorem to find the spectral gap for $H_{rad} k \geq 5$. For this purpose we rewrite bilinear form (54) in the chart associated with the map $\text{trig}$ and restrict it to the space of $\text{SO}(k + 1)$-invariant functions. We get

\[ Q_{\text{trig}}(f, g) = \int_0^{\pi R/2} \left( f' \bar{g}' + R^2 \cos^2 (\tau/R) f \bar{g} \right) w d\tau \]
where $w = w_{\text{trig}}$ as in (28). A substitution $f \rightarrow f w^{-1/2}$ transform $Q_{\text{trig}}(f, g)$ to

\[ E_{\text{trig}}(f, g) = \int_0^{\pi R/2} \left( f' \bar{g}' + V_{\text{eff}}(\tau) f \bar{g} \right) d\tau. \]
The substitution simultaneously trivializes the inner product $\int_0^{\pi R/2} f \bar{g} w d\tau \sim \int_0^{\pi R/2} f \bar{g} d\tau$. The differential equation associated is

\[ N_k(f) = -f'' + V_{\text{eff},k} f = \lambda f. \]

Here we introduce $k$-dependence of the potential $V_{\text{eff}}$, which was implicit in the previous formulas of this section. The reader can find a formula for $V_{\text{eff}}$ and discussion of its properties in Appendix ???. The given formulation of the spectral problem is unitary equivalent to the formulation given in Definition 35 if we define

\[ D'(a, b) = \{ f \in AC_{\text{loc}}(a, b) | f' \in AC_{\text{loc}}(a, b) \} \]
\[ \Delta'_k(a, b) = \{ f \in D'(a, b) | f, N_k(f) \in L^2(a, b) \}, 0 \leq a < b \leq \pi R/2 \]
\[ D' = D'(0, \pi R/2) \]
\[ \Delta'_k = \Delta'_k(0, \pi R/2) \]

**Definition 35.** Define the operator $S_k : \Omega'_k \rightarrow L^2(0, \pi R/2), k = 5, 6, \ldots$ such that $\Omega'_k = \Delta'_k$, and $S_k f = N_k f$ for $f \in \Omega'_k$. 

\[ \text{(67)} \]
Unfortunately Lavine’s theorem is valid only for a regular Sturm-Liouville. Following [27] we approximate a singular Sturm-Liouville problem on an interval $(0, \pi R/2)$ by a sequence of regular Sturm-Liouville problems on truncated intervals $(a_r, b_r)$ where

$$0 < a_r < b_r < \pi R/2 \quad a_r \geq a_{r+1}, a_r \to 0 \quad b_{r+1} \geq b_r, b_r \to \pi R/2 \text{ as } r \to \infty$$

By $S_r = S_{r,k}$ we denote self-adjoint realizations of $N_k$ on the intervals $(a_r, b_r)$. Thus $S_r$ are self-adjoint operators in the Hilbert spaces $H_r = L^2((a_r, b_r))$. The spectrum and eigenvalues are denoted by $\sigma(S_r), \lambda_n(S_r), n \in \mathbb{N}$. Among possible $S_r$ there are "inherited" operators $S^i_r$ that are defined by "inherited" boundary conditions. Here is an adaptation of the general Definition 10.8.1 from [27] to our situation.

**Definition 36.** The domain of the self-adjoint $S^i_{r,k}$ is $\Omega_{r,k} = \{f \in \Delta'_k(a_r, b_r) | \lim_{t \to a_r^+} f(t) = \lim_{t \to b_r^-} f(t) = 0\}$ - the Dirichlet condition.

We will also need the following approximation theorem adapted to our needs.

**Theorem 37.** [2] Suppose

$$My = (py')' + qy = \lambda wy \text{ on } J = (a, b), \infty < a < b < \infty,$$

under the conditions

$$\frac{1}{p}, q, w \in L_{loc}(J, \mathbb{R}), w > 0 \text{ a.e. on } J$$

in the Hilbert space $H = L^2(J, w)$ and

$$-\infty < a < a_r < b_r < b < \infty$$

hold. Let $S$ be a self-adjoint realization of $(M, w)$ on $(a, b)$ where $a, b$ limit point singularity of $M$. Let $S^i_r$ be the inherited operator on $(a_r, b_r)$. Assume that the spectrum $\sigma(S) = \{\lambda_n(S) | n \in \mathbb{N}\}$ is bounded below and discrete. Then

$$\lambda_n(S^i_r) \to \lambda_n(S), \text{ as } r \to \infty, \text{ for each } n \in \mathbb{N}$$

**Corollary 38.** The operator $S_k$ (Definition [35]) has a discrete simple real spectrum. The eigenvalues $\lambda_1$ and $\lambda_2$ satisfy $\lambda_2 - \lambda_1 \geq \frac{12}{k}$ if $k \geq 5$ and $R < 59049(k - 4)(k - 2)/4096$.

**Proof.** The operators $S^i_r$ (see Definition [36]) define a regular Sturm-Liouville problem. They have discrete spectrum (see e.g. [27]). Let $\Gamma(A)$ be the difference $\lambda_2(A) - \lambda_1(A)$. By Theorem 37 the inequality $\Gamma(S^i_r) \geq \frac{3\pi}{R} \text{ (bound)}$ implies the statement of corollary. Convexity of $V_{eff}$ has been verified in Proposition 34. Therefore corollary follows from Theorem 34. \qed

The same form in $\tau$-coordinate is

$$Q(f, g) - \lambda(f, g) = \int_0^{\pi R/2} (g' f' - R^2 \cos^2(\tau/R) - \lambda f g) \omega d\tau$$

The fraction $Q(f, f)$ for function $f(x) = 1$ is equal to $\frac{1 - \frac{2k}{1 - 3k} R^2}{\frac{1}{2} - \frac{1}{3k}}$ we conclude that the first eigenvalue satisfies the inequality

$$0 \leq \lambda_1 \leq \frac{1 - \frac{2k}{1 - 3k} R^2}{\frac{1}{2} - \frac{1}{3k}}$$
We transform the form to Liouville form by a change of variables $f = w^{-\frac{1}{2}} F$, where $w = \sin^{k-2} \left( \frac{\tau}{R} \right) \cos^{2k-3} \left( \frac{\tau}{R} \right) \left( 1 + \sin^2 \left( \frac{\tau}{R} \right) \right)$. The result is

$$Q(F, G) = \frac{R^{3k-3}}{2^{\frac{3k-1}{2}}} \int_0^{\frac{\pi R}{2}} F'(\tau) \bar{G}'(\tau) + V(\tau) F(\tau) \bar{G}(\tau) d\tau$$

Normalization of the inner product is $$\frac{R^{3k-3}}{2^{\frac{3k-1}{2}}} \int_0^{\frac{\pi R}{2}} F(\tau) \bar{G}(\tau) d\tau$$ We set $V_{\text{eff}} = V + R^2 \cos^2 (\tau/R) - \lambda$. Then $V_{\text{eff}} = \frac{A(\csc(\tau/R))}{B(\csc(\tau/R))}$, where $A$ and $B$ are the following polynomials:

$$A(x) = (k^2 - 6k + 8) x^{10} + (-4k^2 + 12k + 4R^4 - 4\lambda R^2 - 6) x^8 +$$
$$(-2k^2 - 8k - 4\lambda R^2 + 5) x^6 + (12k^2 - 44k - 8R^4 + 4\lambda R^2 + 44) x^4 +$$
$$(9k^2 - 18k + 4\lambda R^2 + 9) x^2 + 4R^4$$
$$B(x) = 4R^2 x^2 (x^2 - 1) (x^2 + 1)^2$$

Function $V_{\text{eff}}(\tau)$ has poles at $z = \frac{\pi R}{2} s, iR \log (1 + \sqrt{2}) + \pi Rs, iR \log (\sqrt{2} - 1) + \pi Rs, s \in \mathbb{Z}$. 

$$V_{\text{eff}} \sim \frac{k^2 - 6k + 8}{4(\tau - \pi Rs)^2} + \ldots, \mu = 2 - k/2, k/2 - 1$$
$$V_{\text{eff}} \sim \frac{4k^2 - 16k + 15}{4(\tau - (\pi/2 - \pi s) R)^2} + \ldots, \mu = 5/2 - k, k - 3/2$$
$$V_{\text{eff}} \sim -\frac{1}{4 \left( \tau - R(i \log (1 + \sqrt{2}) + \pi s) \right)^2} + \ldots, \mu = 1/2$$
$$V_{\text{eff}} \sim -\frac{1}{4 \left( \tau - R(i \log (\sqrt{2} - 1) + \pi s) \right)^2} + \ldots \mu = 1/2$$
In the above formulas $\mu$ are the characteristic exponents of the singularity. Local solutions for various values of $k$ $\tau' = \tau - \pi R/2$

$k = 2$

$\tau y_1(\tau^2)$

$y_2(\tau^2)$

$\tau^{1/2} y_1(\tau'^2)$,

$\ln(\tau')\tau^{1/2} y_1(\tau'^2) + \tau^{5/2} y_2(\tau'^2)$

$k = 3$

$\tau^{1/2} y_1(\tau^2)$,

$\ln(\tau)\tau^{1/2} y_1(\tau^2) + \tau^{5/2} y_2(\tau^2)$,

$\tau^{3/2} y_1(\tau'^2)$,

$\tau^{3/2} (\ln(\tau') y_1(\tau'^2) + \tau'^{-1/2} y_2(\tau'^2))$

$k = 4$

$y_1(\tau^2)$

$\tau y_2(\tau^2)$

$\tau^{5/2} y_1(\tau'^2)$

$\ln(\tau')\tau^{5/2} y_2(\tau'^2) + \tau'^{-3/2} y_1(\tau'^2)$

$k \geq 5$

$\tau^{k/2-1} y_1(\tau^2)$

$\ln(\tau)\tau^{k/2-1} y_1(\tau^2) + \tau^{2-k/2} y_2(\tau^2)$

$\tau^{k-3/2} y_1(\tau'^2)$

$\ln(\tau')\tau^{k-3/2} y_2(\tau'^2) + \tau^{5/2-k} y_1(\tau'^2)$

If $k \geq 5$ and $R$ is sufficiently small the function $V$ is convex. From this we conclude that

$$\lambda_2 - \lambda_1 \geq \frac{12}{R^2}$$

**References**

[1] P. B. Bailey, W. N. Everitt, D. B. Hinton, and A. Zettl. Some spectral properties of the heun differential equation. In *Operator methods in ordinary and partial differential equations*, volume 132 of Oper. Theory Adv. Appl. Birkhäuser, Basel, 2000.

[2] P. B. Bailey, W. N. Everitt, J. Weidmann, and A. Zettl. Regular approximations of singular sturm-liouville problems. *Results in Mathematics*, 23:3–22, 1993.

[3] A. Besse. *Einstein manifolds*. Springer, 2007.

[4] R. Chen. Neumann eigenvalue estimate on a compact riemannian manifold. *Proc. AMS.*, 1990:961–970, 1990.
[5] R. Chen. Gap estimates of schrödinger operator. *Pacific Journal Of Mathematics*, 178(2), 1997.
[6] P. Deligne, D. Kazhdan, and P. Etingof, editors. *Quantum fields and strings A course for mathematicians*, volume Vol1.2. AMS, 1991.
[7] H. Eichenherr and M. Forger. On the dual symmetry of the nonlinear sigma models. *Nucl. Phys.*, B155(381), 1979.
[8] S. S. Gelbart. Harmonics on stiefel manifolds and generalized hankel transforms. *Bulletin Of The American Mathematical Society*, 78(3), May 1972.
[9] M. N. Hounkonnou and A. Ronveaux. Factorization of generalized lame IA and heun’s differential equations. *Communications in Mathematical Analysis*, 11(1):121–136, 2011.
[10] G. Huber. Gamma function derivation of n-sphere volumes. *Am. Math. Monthly*, 89(5):301–302, 1982.
[11] J. Humphreys. *Introduction to Lie algebras and Representation Theory*. Springer, 1972.
[12] E.L. Ince. *Ordinary Differential Equations*. Dover Publications Inc., New York, 1956.
[13] V. Kac. *Vertex algebras for beginners*, volume 10 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2 edition, 1998.
[14] W. Klingenberg. *Lectures on Closed Geodesics*. Springer, 1978.
[15] S.E. Kozlov. Geometry of real grassmannian manifolds. i, ii, iii. *J. Math. Sci.*, 100(3), 2000.
[16] R. Lavine. The eigenvaluegap for one-dimensionalconvexpotentials. *Proceedings Of The American Mathematical Society*, 121(3), 1994.
[17] P.-F. Leung. An estimate on the ricci curvature of a submanifold and some applications. *Proceedings of the american mathematical society*, 114(4), 1992.
[18] D. Meyer. Minoration de la première valeur propre non nulle du problème de neumann sur les variétés riemanniennes à bord. *Annales de l’institut Fourier*, 36(2):113–125, 1986.
[19] J.W. Milnor and J.D. Stasheff. *Characteristic classes*, volume 76 of *AMS*. PUP, 1974.
[20] M.V. Movshev. On quasimaps to quadrics, 2010. [arXiv:1008.0804v1 [math.QA]].
[21] M.A. Naimark. *Linear Differential Operators II*. Ungar, 1967.
[22] B. O’Neill. The fundamental equations of a submersion. *Michigan Math. J.*, 13:459–469, 1966.
[23] A. Pressley and G.Segal. *Loop Groups*. Oxford Mathematical Monographs. Clarendon Press, 1988.
[24] R. Schäfke and Dieter Schmidt. The connection problem for general linear ordinary differential equations at two regular singular points with applications in the theory of special functions. *Siam J. Math. Anal.*, 11(5), 1980.
[25] K. Uhlenbeck and S.-T. Yau. On the existence of hermitian-yang-mills connections in stable vector bundles. *Comm. Pure Appl. Math.*, 39(S), 1986.
[26] J. Wang. Global heat kernel estimates, preprint, 1994.
[27] A. Zettl. *Sturm-Liouville theory*. AMS, 2005.