Separating Bohr Denseness from Measurable Recurrence

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Abstract: We prove that there is a set of integers $A$ having positive upper Banach density whose difference set $A - A := \{a - b : a, b \in A\}$ does not contain a Bohr neighborhood of any integer, answering a question asked by Bergelson, Hegyvári, Ruzsa, and the author, in various combinations. In the language of dynamical systems, this result shows that there is a set of integers $S$ which is dense in the Bohr topology of $\mathbb{Z}$ and which is not a set of measurable recurrence.

Our proof yields the following stronger result: if $S \subseteq \mathbb{Z}$ is dense in the Bohr topology of $\mathbb{Z}$, then there is a set $S' \subseteq S$ such that $S'$ is dense in the Bohr topology of $\mathbb{Z}$ and for all $m \in \mathbb{Z}$, the set $(S' - m) \setminus \{0\}$ is not a set of measurable recurrence.

Key words and phrases: Bohr topology, measurable recurrence

1 Introduction

1.1 Difference sets

As usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the real numbers with the usual topology, and $\mathbb{T}$ denotes $\mathbb{R}/\mathbb{Z}$ with the quotient topology. For $A, B \subseteq \mathbb{Z}$, we let $A + B$ denote the sumset $\{a + b : a \in A, b \in B\}$ and $A - A$ the difference set $\{a - b : a, b \in A\}$. If $c \in \mathbb{Z}$ the translate of $A$ by $c$ is $A - c := \{a - c : a \in A\}$. The Bohr topology of $\mathbb{Z}$ is the weakest topology on $\mathbb{Z}$ making all homomorphisms from $\mathbb{Z}$ into $\mathbb{T}$ continuous. We call neighborhoods in this topology Bohr neighborhoods; see §3 for an explicit description. We say that $S$ is Bohr dense if $S$ is dense with respect to the Bohr topology. We write $d^*(A)$ for the upper Banach density of a set of integers $A$, defined as $d^*(A) := \limsup_{n \to \infty} \sup_{k \in \mathbb{Z}} \frac{|A \cap \{k+1, \ldots, k+n\}|}{n}$.

The following problem was posed first in [8] and subsequently in [3], [9], and [10].
Problem 1.1. Prove or disprove: for all $A \subseteq \mathbb{Z}$ having $d^*(A) > 0$, there is an $n \in \mathbb{Z}$ such that $A - A$ contains a Bohr neighborhood of $n$.

Our main result disproves the statement in Problem 1.1.

Theorem 1.2. For all $\varepsilon > 0$, there are sets $S, A \subseteq \mathbb{Z}$ such that $S$ is dense in the Bohr topology of $\mathbb{Z}$, $d^*(A) > \frac{1}{2} - \varepsilon$, and $(A - A) \cap S = \emptyset$.

The set $A - A$ in Theorem 1.2 does not contain a Bohr neighborhood, since $S \cap B \neq \emptyset$ for every Bohr neighborhood $B$.

If $\delta \geq 0$, we say $S$ is $\delta$-nonrecurrent if there is a set $A \subseteq \mathbb{Z}$ having $d^*(A) > \delta$ and $(A - A) \cap S = \emptyset$. The proof of Theorem 1.2 yields the following stronger statement.

Theorem 1.3. If $S \subseteq \mathbb{Z}$ is Bohr dense and $\delta < \frac{1}{2}$, then there is a Bohr dense $\delta$-nonrecurrent subset $S' \subseteq S$.

Repeatedly applying Theorem 1.3 produces the following corollary, showing that there are Bohr dense sets which are very far from being sets of measurable recurrence—see §2 for definition of this term.

Corollary 1.4. If $S \subseteq \mathbb{Z}$ is Bohr dense then there is a Bohr dense set $S' \subseteq S$ such that for all $m \in \mathbb{Z}$, the set $(S' - m) \setminus \{0\}$ is not a set of measurable recurrence.

Remark 1.5. In [8], Ruzsa defines the difference set topology to be the topology on $\mathbb{Z}$ generated by translates of sets of the form $A - A$, where $A \subseteq \mathbb{Z}$ has positive upper Banach density. A set $S \subseteq \mathbb{Z}$ is a set of measurable recurrence if and only if $0$ lies in the closure of $S$ with respect to this topology, while $S - m$ is a set of measurable recurrence if and only if $m$ lies in the closure of $S$. In these terms, Corollary 1.4 states that every Bohr dense set $S \subseteq \mathbb{Z}$ contains a Bohr dense subset $S'$ which is closed, discrete, and nowhere dense in the difference set topology.

2 Measure preserving systems; outline of proof

2.1 Measure preserving systems

By measure preserving system we mean a triple $(X, \mu, T)$ where $(X, \mu)$ is a probability measure space and $T : X \rightarrow X$ is an invertible transformation preserving $\mu$: for every measurable set $D \subseteq X$, $T^{-1}D$ is measurable and $\mu(T^{-1}D) = \mu(D)$.

We say that $S \subseteq \mathbb{Z}$ is a set of measurable recurrence if for every measure preserving system $(X, \mu, T)$ and every measurable set $D \subseteq X$ with $\mu(D) > 0$ there is an $n \in S$ such that $D \cap T^n D \neq \emptyset$.

Correspondence principles such as [2, Proposition 3.1] or [7, Theorem 3.18] allow us to phrase the concept of $\delta$-nonrecurrence in terms of measure preserving systems. Here is the correspondence principle we need for our proofs.

Lemma 2.1. Let $\delta > 0$ and $S \subseteq \mathbb{Z}$. The following are equivalent:

(i) There is a measure preserving system $(X, \mu, T)$ and $D \subseteq X$ with $\mu(D) > \delta$ such that $\mu(D \cap T^n D) = \emptyset$ for all $s \in S$. 

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(ii) There exists \(A \subseteq \mathbb{Z}\) with \(d^*(A) > \delta\) such that \((A - A) \cap S = \emptyset\).

(iii) There is a \(\delta' > \delta\) such that for all \(n \in \mathbb{N}\) there exists \(A_n \subseteq \{0, \ldots, n - 1\}\) with \(|A_n| \geq \delta'n\) and \((A_n - A_n) \cap S = \emptyset\).

The following lemma is crucial in constructions of \(\delta\)-nonrecurrent sets; it follows from Theorems 1 and 2 of [18]. It is also a consequence of the proof of [6, Theorem 2.1].

**Lemma 2.2.** Let \(\delta > 0\). Let \(S \subseteq \mathbb{Z}\) and \(0 \leq \delta < \delta'\). If every finite subset of \(S\) is \(\delta'\)-nonrecurrent, then \(S\) is \(\delta\)-nonrecurrent.

Lemmas 2.1 and 2.2 are well known but rarely collected together and stated as we have here, so we prove them in \(\S9\).

### 2.2 Torus rotations and Rohlin towers

Fixing \(d \in \mathbb{N}\) and \(\alpha \in \mathbb{T}^d\), the corresponding torus rotation is the measure preserving system \((\mathbb{T}^d, \mu, R)\), where \(Rx = x + \alpha\) and \(\mu\) is Haar probability measure on \(\mathbb{T}^d\). We say that \((\mathbb{T}^d, \mu, R)\) is minimal if \(\{n \alpha : n \in \mathbb{Z}\}\) is dense in \(\mathbb{T}^d\).

A Rohlin tower for a measure preserving system \((X, \mu, T)\) is a collection of mutually disjoint measurable subsets of \(X\) having the form \(\mathcal{T} = \{E, TE, T^2E, \ldots, T^{N-1}E\}\). We say the tower has base \(E\), height \(N\), and we call the elements of \(\mathcal{T}\) the levels of \(\mathcal{T}\). A set \(D \subseteq X\) is \(\mathcal{T}\)-measurable if \(D\) is a union of levels of \(\mathcal{T}\).

From now on we write \([N]\) for the interval \(\{0, \ldots, N - 1\}\) in \(\mathbb{Z}\). If \(S \subseteq \mathbb{Z}\) is a finite \(\delta\)-nonrecurrent set and \(\mathcal{T}\) is a Rohlin tower of height \(N\) and base \(E\), we say that \(\mathcal{T}\) witnesses the \(\delta\)-nonrecurrence of \(S\) if there is a set \(A \subseteq [N]\) such that \(A + S \subseteq [N]\), \(A \cap (A + S) = \emptyset\), and \(|A|\mu(E) > \delta\). Note that this implies \(D := \bigcup_{n \in A} T^nE\) satisfies \(\mu(D) > \delta\) and \(D \cap T^nD = \emptyset\) for all \(s \in S\).

### 2.3 Extending \(\delta\)-nonrecurrent sets with pairs of Rohlin towers

Proposition 4.4 provides a special class of Rohlin towers which are the focal point of our main argument. Lemma 2.3 indicates that such towers can be used to construct \(\delta\)-nonrecurrent sets with prescribed properties.

**Lemma 2.3.** Let \(S \subseteq \mathbb{Z}\) be finite, \(\delta > 0\), and let \((X, \mu, T)\) be a measure preserving system. Let

\[
\mathcal{J} = \{E, TE, \ldots, T^{N-1}E\}, \quad \mathcal{J}' = \{E', TE', \ldots, T^{N-1}E'\}
\]

be Rohlin towers for \(T\) with \(E \subseteq E'\) and define \(S' := \{n \in \mathbb{Z} : T^nE \subseteq E'\}\). If \(\mathcal{J}\) witnesses the \(\delta\)-nonrecurrence of \(S\), then \(S \cup (S + S')\) is \(\delta\)-nonrecurrent.

**Remark 2.4.** The hypothesis \(E \subseteq E'\) in Lemma 2.3 implies \(0 \in S'\), and thus \(S \subseteq S + S'\), so we could simply write “\(S + S'\) is \(\delta\)-nonrecurrent” in the conclusion. Instead, we want to emphasize that the new \(\delta\)-nonrecurrent set contains \(S\).
1.2 is a special case of Theorem 1.3, we address only the latter in the sequel. Corollary 1.4 follows from a

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Bohr-Hamming balls

straightforward diagonalization based on repeated application of Theorem 1.3.

2.5 Organization of the article

In §3 we state definitions related to Bohr neighborhoods and prove some standard compactness

properties. Section 4 introduces Bohr-Hamming balls, whose relevant properties are recorded in Lemma 4.2 and Proposition 4.4; these are proved in §8 and §§5-7, respectively.

Proof. Assuming \( \mathcal{T}, \mathcal{T}', \) and \( \delta \) are as in the hypothesis, there is an \( A \subseteq [N] \) such that \( A + S \subseteq [N], \)

\( A \cap (A + S) = \emptyset, \) and \( |A|\mu(E) > \delta. \) Then

\[
D := \bigcup_{n \in A} T^n E, \quad D' := \bigcup_{n \in A} T^n E'
\]

each have measure strictly greater than \( \delta, \) and the disjointness of the levels of \( \mathcal{T} \) implies

\[
D \cap T^s D = \emptyset \quad \text{and} \quad D' \cap T^s D' = \emptyset \quad \text{for all } s \in S.
\]  \( 2.1 \)

To prove that \( S \cup (S + S') \) is \( \delta \)-nonrecurrent it therefore suffices to prove

\[
D \cap T^{s+s'} D = \emptyset \quad \text{for all } s \in S, s' \in S',
\]  \( 2.2 \)

so fix \( s \in S \) and \( s' \in S'. \) Note that \( D \subseteq D' \) since \( E \subseteq E', \) and that \( T^s E \subseteq E', \) by the definition of \( S'. \)

Then \( T^s D = \bigcup_{a \in A} T^{a+s} E \subseteq \bigcup_{a \in A} T^{a} E' = D'. \) Now \( T^{s+s'} D \subseteq T^{s'} D', \) so the containment \( D \subseteq D' \) and the disjointness of \( D' \) from \( T^{s'} D' \) implies \( D \cap T^{s+s'} D = \emptyset. \) We have proved the lemma, as \( 2.1 \) and \( 2.2 \)

imply that \( S \cup (S + S') \) is \( \delta \)-nonrecurrent. \( \square \)

2.4 Outline of the main argument

Lemma 2.3 forms the basis of an inductive construction of a \( \delta \)-nonrecurrent set which is Bohr dense. This construction requires two compactness properties: first, that Bohr denseness can be approximated by

\( k \)-Bohr denseness (Definition 3.3), which in turn can be approximated using finite subsets of \( \mathbb{Z} \) (Lemma 3.5). The corresponding compactness property for measurable recurrence is provided by Lemma 2.2.

Starting with a finite \( \delta \)-nonrecurrent set \( S_1, \) we use Lemma 4.5 to find an \( N \in \mathbb{N} \) and a finite set \( A \subseteq [N] \)

witnessing the \( \delta \)-nonrecurrence of \( S_1, \) meaning \( |A| > \delta N, \) \( A \cap (A + S_1) = \emptyset, \) and \( A + S_1 \subseteq [N]. \) We then use Proposition 4.4 to find a minimal torus rotation \( (T^d, \mu, R) \) and Rohlin towers \( \mathcal{T}, \mathcal{T}' \) as in Lemma 2.3 with \( |A|\mu(E) > \delta \) such that the set \( \{ n : R^n E \subseteq E' \} \) contains a Bohr-Hamming ball \( BH \) (Definition 4.1), which itself is \( k \)-Bohr dense (Lemma 4.2). Lemma 2.3 then implies \( S_1 \cup (S_1 + BH) \) is \( \delta \)-nonrecurrent. Since \( k \)-Bohr denseness is translation invariant, we will get that \( S_1 \cup (S_1 + BH) \) is \( k \)-Bohr dense, and Lemma 3.5 will allow us to chose a finite subset \( S_2 \subseteq S_1 \cup (S_1 + BH) \) which is approximately \( k \)-Bohr dense. Since \( S_1 \) is finite we may include \( S_1 \) in \( S_2. \) Repeating this argument, we produce a sequence of sets \( S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots \) where each \( S_k \) is approximately \( k \)-Bohr dense and \( \delta \)-nonrecurrent. The union \( \bigcup_{k \in \mathbb{N}} S_k \) will be the desired Bohr dense \( \delta \)-nonrecurrent set.

2.5 Organization of the article

The argument outlined in §2.4 is the main one used in the proof of Theorem 1.2; complete details are

provided in §4. A superficial modification of this argument will prove Theorem 1.3 as well. As Theorem

1.2 is a special case of Theorem 1.3, we address only the latter in the sequel. Corollary 1.4 follows from a

straightforward diagonalization based on repeated application of Theorem 1.3.

In §3 we state definitions related to Bohr neighborhoods and prove some standard compactness

properties regarding the Bohr topology. Section 4 introduces Bohr-Hamming balls, whose relevant

properties are recorded in Lemma 4.2 and Proposition 4.4; these are proved in §8 and §§5-7, respectively.
Lemma 4.3 combines Lemma 4.2 and Proposition 4.4 to form the inductive step in the proof of Theorem 1.3. The proofs of Theorem 1.3 and Corollary 1.4 are presented immediately after the proof of Lemma 4.3.

3 Bohr neighborhoods

We identify $\mathbb{T}$ with the interval $[0, 1) \subseteq \mathbb{R}$ when defining elements and subsets of $\mathbb{T}$. For $x \in \mathbb{T}$, let $\tilde{x}$ denote the unique element in $[0, 1)$ such that $x = \tilde{x} + \mathbb{Z}$, and define $\|x\| := \min \{|\tilde{x} - n| : n \in \mathbb{Z}\}$. For $d \in \mathbb{N}$ and $x = (x_1, \ldots, x_d) \in \mathbb{T}^d$, let $\|x\| := \max_{j \leq d} \|x_j\|$.

Fixing $d \in \mathbb{N}$, $\alpha \in \mathbb{T}^d$, and a nonempty open set $U \subseteq \mathbb{T}^d$, the Bohr neighborhood determined by these parameters is

$$B(\alpha; U) := \{n \in \mathbb{Z} : n\alpha \in U\}.$$ 

We say that $B(\alpha; U)$ has rank $d$. Observe that $B(\alpha; U)$ may be empty, as we make no assumptions on $\alpha$. However, when $0 \in U$, $B(\alpha; U)$ is nonempty, as it contains 0. The Bohr topology on $\mathbb{Z}$ is the weakest topology containing $B(\alpha; U)$ for every $\alpha \in \mathbb{T}^d$ and open $U \subseteq \mathbb{T}^d$, for every $d \in \mathbb{N}$.

Given $\alpha \in \mathbb{T}^d$ and $\varepsilon > 0$, we define

$$\text{Bohr}_0(\alpha, \varepsilon) := \{n \in \mathbb{Z} : \|n\alpha\| < \varepsilon\}$$

to be a basic Bohr neighborhood of 0 having rank $d$ and radius $\varepsilon$. These form a neighborhood base around 0 for the Bohr topology. For a given $n \in \mathbb{Z}$, the collection of translates

$$\{B + n : B \text{ is a basic Bohr neighborhood of 0}\}$$

forms a neighborhood base at $n$ in the Bohr topology.

Example 3.1. The set of odd integers $B := 2\mathbb{Z} + 1$ is the Bohr neighborhood $B(\alpha; U)$, determined by $\alpha = \frac{1}{2} \in \mathbb{T}$ and $U = \mathbb{T} \setminus \{0\}$. For every $\delta < \frac{1}{2}$, $B$ is $\delta$-nonrecurrent, since the set $2\mathbb{Z}$ of even integers has upper Banach density $\frac{1}{2}$, while $(2\mathbb{Z} + 2\mathbb{Z}) \cap B = \emptyset$.

Observation 3.2. If $m \in B(\alpha; U)$, then the translate $B(\alpha; U) - m$ is a Bohr neighborhood of 0, and therefore contains a basic Bohr neighborhood of 0. Consequently, every nonempty Bohr neighborhood having rank at most $d$ contains a translate of a basic Bohr neighborhood of 0 having rank at most $d$.

Definition 3.3 (Bohr denseness and its approximations). We say that $S \subseteq \mathbb{Z}$ is

- **Bohr recurrent** if $S \cap B \neq \emptyset$ for every Bohr neighborhood of 0.
- **$d$-Bohr recurrent** if $S \cap B \neq \emptyset$ for every Bohr neighborhood of 0 having rank at most $d$.
- **$(d, \varepsilon)$-Bohr recurrent** if $S \cap B \neq \emptyset$ for every basic Bohr neighborhood of 0 with rank at most $d$ and radius at least $\varepsilon$.
- **Bohr dense** if $S \cap B \neq \emptyset$ for every nonempty Bohr neighborhood $B$. 

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1.3. \(d\)-Bohr dense if \(S \cap B \neq \emptyset\) for every nonempty Bohr neighborhood with rank at most \(d\). Equivalently, \(S\) is \(d\)-Bohr dense if \(S - m\) is \(d\)-Bohr recurrent for all \(m \in \mathbb{Z}\).

The equivalence asserted in the last item above is due to Observation 3.2 and the fact that \((S - m) \cap B\) is a translate of \(S \cap (B + m)\). The next observation follows immediately from the relevant definitions and Observation 3.2.

**Observation 3.4.** Let \(S \subseteq \mathbb{Z}\). Then

(i) \(S\) is \((d, \varepsilon)\)-Bohr recurrent if and only if for all \(\alpha \in \mathbb{T}^d\), there exists \(s \in S\) such that \(\|s \alpha\| < \varepsilon\).

(ii) \(S\) is Bohr dense if and only if for all \(d \in \mathbb{N}\), \(\varepsilon > 0\), and \(m \in \mathbb{Z}\), the set \(S - m\) is \((d, \varepsilon)\)-Bohr recurrent.

The next lemma is an instance of compactness required for our proofs.

**Lemma 3.5.** Let \(d \in \mathbb{N}\).

(i) If \(S \subseteq \mathbb{Z}\) is \(d\)-Bohr recurrent, then for all \(\varepsilon > 0\), there is a finite set \(S' \subseteq S\) such that \(S'\) is \((d, \varepsilon)\)-Bohr recurrent.

(ii) If \(S \subseteq \mathbb{Z}\) is \(d\)-Bohr dense, then for all \(M \in \mathbb{N}\) and all \(\varepsilon > 0\), there exists a finite set \(S' \subseteq S\) such that for all \(m \in \mathbb{Z}\) with \(|m| \leq M\), the translate \(S' - m\) is \((d, \varepsilon)\)-Bohr recurrent.

**Proof.** We prove Part (i) by proving its contrapositive: assuming \(\varepsilon > 0\) and that for every finite \(S' \subseteq S\) there is an \(\alpha \in \mathbb{T}^d\) with \(\|s \alpha\| \geq \varepsilon\) for all \(s \in S'\), we will find an \(\alpha \in \mathbb{T}^d\) such that \(\|s \alpha\| \geq \varepsilon\) for all \(s \in S\). Enumerate \(S\) as \((s_j)_{j \in \mathbb{N}}\), and for each \(n\) choose \(\alpha_n \in \mathbb{T}^d\) such that \(\|s_j \alpha_n\| > \varepsilon\) for all \(j \leq n\); this is possible due to our hypothesis on finite subsets of \(S\). Choose a convergent subsequence \((\alpha_{n_k})_{k \in \mathbb{N}}\) and call the limit \(\alpha\). Now for all \(s \in S\), we have \(\|s \alpha_{n_k}\| \to \|s \alpha\|\), and our choice of \(\alpha_n\) means that \(\|s \alpha_{n_k}\| > \varepsilon\) for all but finitely many \(k\). Thus \(\|s \alpha\| \geq \varepsilon\) for all \(s \in S\).

Part (ii) follows from Part (i), Observation 3.4, and the definition of “\(d\)-Bohr dense”.

The next lemma is essentially Lemma 5.11 of [1]. We use it to derive Corollary 1.4 from Theorem 1.3.

**Lemma 3.6.** Let \((S_n)_{n \in \mathbb{N}}\) be a sequence of Bohr dense subsets of \(\mathbb{Z}\). Then there is a sequence of finite sets \(R_n \subseteq S_n\) such that \(\bigcup_{n \in \mathbb{N}} R_n\) is Bohr dense.

**Proof.** The Bohr denseness of \(S_n\) and Lemma 3.5 allow us to choose, for each \(n\), a finite subset \(R_n \subseteq S_n\) such that \(R_n - m\) is \((n, 1/n)\)-Bohr recurrent for each \(m\) with \(|m| < n\). Observation 3.4 then implies that \(\bigcup_{n \in \mathbb{N}} R_n\) is Bohr dense.

4 Bohr-Hamming balls; proof of Theorem 1.3 and Corollary 1.4

4.1 Bohr-Hamming Balls

For \(\varepsilon > 0\), \(d \in \mathbb{N}\), and \(x = (x_1, \ldots, x_d) \in \mathbb{T}^d\), let

\[w_\varepsilon(x) := \left| \{ j : \|x_j\| \geq \varepsilon \} \right|\]
So \( w_\varepsilon(x) \) is the number of coordinates of \( x \) differing from 0 by at least \( \varepsilon \). Following [11], we call \( \alpha \in \mathbb{T}^d \) a generator if \( \{n\alpha : n \in \mathbb{Z}\} \) is dense in \( \mathbb{T}^d \).

**Definition 4.1.** Let \( k < d \in \mathbb{N} \), \( \varepsilon > 0 \), and \( \alpha \in \mathbb{T}^d \). The **Bohr-Hamming ball with rank \( d \) and radius \( (k, \varepsilon) \)** around 0 determined by \( \alpha \) is

\[
BH(\alpha; k, \varepsilon) := \{ n \in \mathbb{Z} : w_\varepsilon(n\alpha) \leq k \}.
\]

So \( n \in BH(\alpha; k, \varepsilon) \) if at most \( k \) coordinates of \( n\alpha \) differ from 0 by at least \( \varepsilon \). If \( \alpha \) is a generator, we say that \( BH(\alpha; k, \varepsilon) \) is proper.

The next lemma, implicit in Section 2 of [11], asserts a useful relation between Bohr-Hamming balls and Bohr neighborhoods.

**Lemma 4.2.** Let \( k < d \in \mathbb{N} \) and \( \varepsilon > 0 \). If \( BH \) is a proper Bohr-Hamming ball with rank \( d \) and radius \( (k, \varepsilon) \) and \( B \) is a nonempty Bohr neighborhood with rank \( k \), then \( BH \cap B \) contains a nonempty Bohr neighborhood with rank \( k \).

Consequently, if \( S \subseteq \mathbb{Z} \) is \( d \)-Bohr dense, then \( S \cap BH \) is \( k \)-Bohr dense.

We prove Lemma 4.2 in §8, following closely the proof of Lemmas 2.2 and 2.3 of [11].

The proof of Theorem 1.3 is mostly contained in the following lemma, which is an easy consequence of the subsequent proposition.

**Lemma 4.3.** Let \( \delta > 0 \) and \( k \in \mathbb{N} \). If \( S \subseteq \mathbb{Z} \) is finite and \( \delta \)-nonrecurrent, then there is an \( \eta > 0 \) and a proper Bohr-Hamming ball \( BH \) of radius \( (k, \eta) \) such that \( S + BH \) is \( \delta \)-nonrecurrent.

**Proposition 4.4.** For every \( k \in \mathbb{N} \), \( \varepsilon > 0 \), and prime \( p \), there exist \( d \in \mathbb{N} \), a minimal torus rotation \( (\mathbb{T}^d, \mu, R) \) by \( \alpha \in \mathbb{T}^d \), \( \eta > 0 \), a proper Bohr-Hamming ball \( BH = BH(\alpha; k, \eta) \) with rank \( d \), and Rohlin towers

\[
\mathbb{T} = \{ R^nE : 0 \leq n \leq p - 1 \}, \quad \mathbb{T}' = \{ R^nE' : 0 \leq n \leq p - 1 \}
\]

such that \( \mu(E) \geq \frac{1 - \varepsilon}{p} \), \( E \subseteq E' \), and \( R^nE \subseteq E' \) for all \( n \in BH \).

The proof of Proposition 4.4 occupies §§5-7.

We need one more standard lemma for the proof of Lemma 4.3.

**Lemma 4.5.** If \( S \subseteq \mathbb{Z} \) is a finite \( \delta \)-nonrecurrent set, then for all sufficiently large \( N \), there exists \( A \subseteq [N] \) with \( |A| > \delta N \), \( A + S \subseteq [N] \), and \( A \cap (A + S) = \emptyset \).

The condition \( A \cap (A + S) = \emptyset \) is equivalent to \((A - A) \cap S = \emptyset\); we will use this from time to time without comment.

**Proof.** Assume \( S \) is finite and \( \delta \)-nonrecurrent and let \( M = \max\{|s| + 1 : s \in S\} \). We choose, by Part (iii) of Lemma 2.1, a \( \delta' > \delta \) such that for all \( N \in \mathbb{N} \), there is an \( A_N \subseteq [N] \) with \( |A_N| \geq \delta'N \) and \( A_N \cap (A_N + S) = \emptyset \). Choose \( N \) large enough that \( \delta'N - 2M > \delta N \). Letting \( A = A_N \cap [M - 1, N - M - 1] \), we have \( |A| \geq |A_N| - 2M > \delta'N - 2M > \delta N \), \( A + S \subseteq [N] \), and \( A \cap (A + S) = \emptyset \). \( \square \)
Proof of Lemma 4.3. To prove Lemma 4.3, we will apply Lemma 2.3 to the Rohlin towers provided by Proposition 4.4. Let \( k \in \mathbb{N} \) and assume \( S \subseteq \mathbb{Z} \) is finite and \( \delta \)-nonrecurrent. We will find a minimal torus rotation \((\mathbb{T}^d, \mu, R)\) by an \( \alpha \in \mathbb{T}^d \), a measurable set \( D \subseteq \mathbb{T}^d \) having \( \mu(D) > \delta \), and a Bohr-Hamming ball \( BH = BH(\alpha; k, \eta) \) such that
\[
D \cap R^dD = \emptyset \quad \text{for all } n \in S + BH. \tag{4.1}
\]

To construct \( D \), we first apply Lemma 4.5 to find a prime \( p \) and \( A \subseteq [p] \) having \( |A| > \delta p \) such that \( A \cap (A + S) = \emptyset \) and \( A + S \subseteq [p] \); this is possible due to our assumptions on \( S \). Fix \( \varepsilon > 0 \) so that \( |A|/p > \delta \) and invoke Proposition 4.4 with this \( \varepsilon \). We form \( D \) by copying \( A \) into levels of the tower \( \mathcal{T} \) provided by Proposition 4.4:
\[
D := \bigcup_{a \in A} R^dE.
\]

By our choice of \( \varepsilon \) and the mutual disjointness of the levels of \( \mathcal{T} \), we have
\[
\mu(D) = |A|\mu(E) > \frac{|A|(1-\varepsilon)}{p} > \delta.
\]

To prove that \( D \) satisfies (4.1), observe that the hypotheses of Lemma 2.3 hold with \( p \) in place of \( N \). Proposition 4.4 states that \( BH \subseteq \{ n : R^dE \subseteq E' \} \), so we may cite Lemma 2.3 with \( BH \) in place of \( S' \) and conclude that (4.1) holds.

4.2 Proof of Theorem 1.3 and Corollary 1.4

Recall the statement of Theorem 1.3: if \( S \subseteq \mathbb{Z} \) is Bohr dense and \( \delta < \frac{1}{2} \), then there is Bohr dense \( \delta \)-nonrecurrent subset \( S' \subseteq S \).

Proof of Theorem 1.3. Let \( S \subseteq \mathbb{Z} \) be Bohr dense and let \( \delta < \frac{1}{2} \). By Lemma 2.2 it suffices to find \( \delta' > \delta \) and a Bohr dense set \( S' \subseteq S \) such that every finite subset \( S'' \subseteq S' \) is \( \delta'' \)-nonrecurrent. Fixing \( \delta' \) with \( \delta < \delta' < \frac{1}{2} \), we will construct an increasing sequence \( S_1 \subseteq S_2 \subseteq \ldots \) of subsets of \( S \) such that each \( S_k \) is \( \delta' \)-nonrecurrent and satisfies the following condition:
\[
\text{for all } m \in \mathbb{Z} \text{ with } |m| \leq k, \text{ the translate } S_k - m \text{ is } (k, 1/k)\text{-Bohr recurrent.} \tag{4.2}
\]

To construct \( S_1 \), we find an odd integer \( s_1 \in S \), and let \( S_1 = \{s_1\} \). Such \( s_1 \) exists, as the odd integers form a Bohr neighborhood (Example 3.1) and \( S \) is Bohr dense. Now \( S_1 \) is \( \delta' \)-nonrecurrent, as the set of odd numbers is \( \delta' \)-nonrecurrent for every \( \delta' < \frac{1}{2} \).

For the inductive step of the construction, we assume \( S_{k-1} \) is a finite \( \delta' \)-nonrecurrent subset of \( S \). We apply Lemma 4.3 to find a proper Bohr-Hamming ball \( BH \) with radius \((k, \eta)\) such that \( S_{k-1} + BH \) is \( \delta' \)-nonrecurrent. Lemma 4.2 implies \( S \cap (S_{k-1} + BH) \) is \( k \)-Bohr dense, and Lemma 3.5 provides a finite subset \( S_k \) of \( S \cap (S_{k-1} + BH) \) satisfying (4.2). Since \( 0 \in BH \) we have \( S_{k-1} \subseteq S_{k-1} + BH \). The finiteness of \( S_{k-1} \) and the latter containment means we can choose \( S_k \) to satisfy \( S_{k-1} \subseteq S_k \) as well.

Letting \( S' := \bigcup_{k \in \mathbb{N}} S_k \), we have that every finite subset of \( S' \) is contained in one of the sets \( S_k \), and each \( S_k \) is \( \delta' \)-nonrecurrent, so Lemma 2.2 implies \( S' \) is \( \delta \)-nonrecurrent. The Bohr denseness of \( S' \) follows from (4.2) and Observation 3.4. 

\[\square\]
The next lemma records two elementary facts for the proof of Corollary 1.4.

**Lemma 4.6.** Let $R, S \subseteq \mathbb{Z}$.

(i) If neither $R$ nor $S$ is a set of measurable recurrence then $R \cup S$ is not a set of measurable recurrence.

(ii) If $S \subseteq \mathbb{Z}$ is finite then $S \setminus \{0\}$ is not a set of measurable recurrence.

Part (i) is proved by taking the cartesian product of measure preserving systems witnessing the nonrecurrence of $R$ and $S$. Part (ii) follows from considering a group rotation on $\mathbb{Z}/N\mathbb{Z}$, where $N = 1 + \max\{|s| : s \in S\}$.

We now prove Corollary 1.4, which says that if $S \subseteq \mathbb{Z}$ is Bohr dense then there is a Bohr dense set $S' \subseteq S$ such that for all $m \in \mathbb{Z}$, the set $(S' - m) \setminus \{0\}$ is not a set of measurable recurrence.

**Proof of Corollary 1.4.** Let $S \subseteq \mathbb{Z}$ be Bohr dense. We begin by constructing a decreasing sequence $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$ of Bohr dense subsets of $S$ such that for each $n$,

$$
\text{neither } S_n - n \text{ nor } S_n + n \text{ is a set of measurable recurrence.}
$$

We begin with $n = 0$ and apply Theorem 1.3 to find a Bohr dense subset $S_0 \subseteq S$ which is not a set of measurable recurrence. Supposing $S_{n-1}$ is defined and is Bohr dense, then each of its translates is Bohr dense as well, and we may apply Theorem 1.3 to $S_{n-1} - n$ to find a Bohr dense subset $S_{n,0} \subseteq S_{n-1}$ such that $S_{n,0} - n$ is not a set of measurable recurrence. Repeating this process with $S_{n,0} + n$ in place of $S_{n-1} - n$ produces a Bohr dense set $S_n \subseteq S_{n-1}$ satisfying (4.3). Having constructed $S_n$, Lemma 3.6 provides finite sets $R_n \subseteq S_n$ such that $S' := \bigcup_{n \in \mathbb{N}} R_n$ is Bohr dense.

To complete the proof we fix $m \in \mathbb{Z}$ and will show that $m \in \mathbb{Z}$, $(S' - m) \setminus \{0\}$ is not a set of measurable recurrence. Observe that $(S' - m) \setminus (S_m - m)$ is finite, as all but finitely many of the $R_n$ are contained in $S_m$. Thus $S' - m$ can be written as $E \cup (S_m - m)$, where $E$ is finite. Since $S_m - m$ is not a set of measurable recurrence, Lemma 4.6 implies that $(S' - m) \setminus \{0\}$ is also not a set of measurable recurrence.

**5 Rohlin towers in $(\mathbb{Z}/p\mathbb{Z})^d$**

In §7 we prove Proposition 4.4 by constructing certain Rohlin towers for minimal torus rotations. In this section we prove Lemma 5.1, establishing much of the structure of the towers while working in $(\mathbb{Z}/p\mathbb{Z})^d$, where $p$ is a fixed prime. Section 6 explains the routine process of copying this structure into $\mathbb{T}^d$.

**5.1 Hamming balls in $\mathbb{Z}/N\mathbb{Z}$**

For $N, d \in \mathbb{N}$ we let $G_N^d$ denote the group $(\mathbb{Z}/N\mathbb{Z})^d$. We write elements of $G_N^d$ as $x = (x_1, \ldots, x_N)$, where $x_j \in \mathbb{Z}/N\mathbb{Z}$. In general we write $0 := (0, \ldots, 0)$ and $1 := (1, \ldots, 1) \in G_N^d$. If $n \in \mathbb{Z}$ we write $n1$ for $(n, \ldots, n)$. For $x \in G_N^d$, define

$$
w(x) := |\{j : x_j \neq 0\}|,
$$

so that $w(x)$ is the number of coordinates of $x$ which are not equal to 0. Given $k \in \mathbb{N}$, let

$$
H_k := \{x \in G_N^d : w(x) \leq k\}.
$$
So $H_k$ is the set elements of $G^d_p$ which are nonzero in at most $k$ coordinates, otherwise known as the Hamming ball of radius $k$ around $0$.

**Lemma 5.1.** Let $p \in \mathbb{N}$ be prime. For all $k \in \mathbb{N}$ and all $\varepsilon > 0$, there exists $d \in \mathbb{N}$ and sets $A, A_1 \subseteq G^d_p$ such that $|A| > \frac{1-\varepsilon}{p}|G^d_p|$, $A \subseteq A_1$, and $A + H_k \subseteq A_1$, while the translates

$$A_1, A_1 + 1, \ldots, A_1 + (p-1)1,$$

are mutually disjoint.

The proof of Lemma 5.1 occupies the remainder of this section. To construct $A$ and $A_1$ we need sets which are very nearly invariant under translation by elements of $H_k$, and whose translates by $1, \ldots, (p-1)1$ are mutually disjoint. Such sets are defined in §5.2, and assembled to form $A$ and $A_1$ in §5.3.

### 5.2 Bias cells

Fix a prime $p$ for the remainder of this section. For $t \in \mathbb{Z}/p\mathbb{Z}$ and $y = (y_1, \ldots, y_d) \in G^d_p$, let

$$w(y; t) := |\{j: y_j = t\}|,$$

so that $w(y; t)$ is the number of coordinates of $y$ which are equal to $t$. We let $\mathcal{P}$ denote the collection of nonempty proper subsets of $\mathbb{Z}/p\mathbb{Z}$. For each $C \in \mathcal{P}$ and $k, d \in \mathbb{N}$, let

$$\text{Bias}(C, k, d) := \{y \in G^d_p: w(y; t) > \frac{d}{p} + k \text{ if } t \in C, w(y; t) < \frac{d}{p} - k \text{ if } t \notin C\}.$$  

For example, with $p = 3$ and $C = \{0, 1\}$, $\text{Bias}(C, 5, 3000)$ is the set of $y \in G^d_3$ such that more than 1005 coordinates of $y$ are equal to 0, more than 1005 coordinates of $y$ are equal to 1, and fewer than 995 coordinates of $y$ are equal to 2.

The following lemma records some elementary properties of $\text{Bias}(C, k, d)$.

**Lemma 5.2.** Let $C, C' \in \mathcal{P}$. For all $d, k \in \mathbb{N}$

(i) $\text{Bias}(C, k, d) + 1 = \text{Bias}(C + 1, k, d),$

(ii) if $C \neq C'$ then $\text{Bias}(C, k, d) \cap \text{Bias}(C', k, d) = \emptyset$.

If $l < k$ then

(iii) $\text{Bias}(C, k, d) \subseteq \text{Bias}(C, l, d),$

(iv) $\text{Bias}(C, k, d) + H_l \subseteq \text{Bias}(C, k - l, d).$

**Proof.** To prove Part (i), observe that $w(x + 1; t) = w(x; t - 1)$ for all $x \in G^d_p$ and all $t \in \mathbb{Z}/p\mathbb{Z}$. If $x$ satisfies the inequalities defining $\text{Bias}(C, k, d)$, it follows that $x + 1$ satisfies the inequalities defining $\text{Bias}(C + 1, k, d)$.

To prove Part (ii) note that if $y$ lies in the intersection written in (ii) and $t \in C \triangle C'$, then $w(y; t)$ is both strictly greater than and strictly less than $\frac{d}{p}$. This is impossible, so the intersection is empty.
Part (iii) follows immediately from the relevant definition.

To prove Part (iv), let \( x \in \text{Bias}(C, k, d) \) and \( y \in H_t \), with the aim of showing \( x + y \in \text{Bias}(C, k - l, d) \). Then \( x \) satisfies \( w(x; t) > \frac{d}{p} + k \) for every \( t \in C \) and \( w(x; t) < \frac{d}{p} - k \) for every \( t \notin C \), while \( y \) has at most \( l \) nonzero entries. Thus \( x + y \) differs from \( x \) in at most \( l \) coordinates, so that \( |w(x; t) - w(x + y; t)| \leq l \) for each \( t \in \mathbb{Z}/p\mathbb{Z} \). The conditions on \( w(x; t) \) then imply \( w(x + y; t) < \frac{d}{p} - k + l \) for each \( t \in C \) and \( w(x + y; t) > \frac{d}{p} + k - l \) for each \( t \notin C \). Thus \( x + y \in \text{Bias}(C, k - l, d) \).

5.3 Assembling bias cells

Note that \( \mathbb{Z}/p\mathbb{Z} \) acts on \( P \) by translation; call this action \( \tau \). Every \( C \in P \) belongs to a \( \tau \)-orbit of cardinality \( p \), since every \( C \in P \) satisfies \( C \neq C + 1 \), and the cardinality of an orbit divides the order of the acting group; this is the only place where we use the primeness of \( p \). Choose a collection of sets \( P_0 \) representing each \( \tau \)-orbit (i.e. every \( \tau \)-orbit contains exactly one element of \( P_0 \)), so that

\[
\{P_0, P_0 + 1, \ldots, P_0 + (p - 1)\}
\]

is a partition of \( P \). We fix this choice of \( P_0 \) for the remainder of the section.

For example, when \( p = 3 \), we have

\[
P = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\},
\]

and we choose \( P_0 = \{\{0\}, \{0, 1\}\} \). Then \( P_0 + 1 = \{\{1\}, \{1, 2\}\} \) and \( P_0 + 2 = \{\{2\}, \{0, 2\}\} \).

Lemma 5.1 will be proved by taking \( A \) to be the following:

\[
E_0(k, d) := \bigcup_{C \in P_0} \text{Bias}(C, k, d).
\]  

(5.1)

We write \( E(k, d) \) for the union of all the bias cells:

\[
E(k, d) := \bigcup_{C \in P} \text{Bias}(C, k, d).
\]  

(5.2)

We will see in Lemma 5.4 that \( E(k, d) \) is the disjoint union of the translates \( E_0(k, d) + n1, 0 \leq n \leq p - 1 \), so the following lemma will let us estimate \( |E_0(k, d)| \).

Lemma 5.3. For fixed \( k \in \mathbb{N}, \varepsilon > 0, \) and \( d \) sufficiently large depending on \( k \) and \( \varepsilon \), we have

\[
|E(k, d)| > (1 - \varepsilon)|G_d^d|.
\]

Proof. We will prove that \( E'(k, d) := G_d^d \setminus E(k, d) \) satisfies \( \lim_{d \rightarrow \infty} |E'(k, d)|/|G_d^d| = 0 \), which is equivalent to the statement of the lemma. Note that \( E'(k, d) \) is the set of elements \( (x_1, \ldots, x_d) \) such that \( |w(x; t) - \frac{d}{p}| \leq k \) for some \( t \in \mathbb{Z}/p\mathbb{Z} \). To estimate \( |E'(k, d)| \) it therefore suffices to fix \( t \in \mathbb{Z}/p\mathbb{Z} \) and \( m \in \left[\frac{d}{p} - k, \frac{d}{p} + k\right] \) and count the number of \( x \) with \( w(x; t) = m \). The collection of such \( x \) can be enumerated by choosing \( m \) coordinates of \( x \) to be equal to \( t \), and filling in the remaining \( d - m \) coordinates with any of
the \( p - 1 \) elements of \( \mathbb{Z}/p\mathbb{Z} \) besides \( t \). The number of \( x \) such that \( w(x; t) = m \) is therefore \( (p - 1)^d - m \binom{d}{m} \). Summing over the relevant values of \( m \) and \( t \), we find that

\[
|E'(k, d)| \leq p(2k + 1)(p - 1)^d M_d,
\]

(5.3)

where \( M_d = \max\{d \binom{d}{m} : m \leq \frac{d}{p} + k\} \). Estimating the binomial coefficients in \( M_d \) with Stirling’s formula, we have \( \lim_{d \to \infty} (p - 1)^d M_d / p^d = 0 \) (remembering that \( p \) and \( k \) are fixed). Inequality (5.3) then implies \( \lim_{d \to \infty} |E'(k, d)| / |G_p^d| = 0 \), as desired.

**Lemma 5.4.** With \( E_0(k, d) \) and \( E(k, d) \) as defined in (5.1) and (5.2),

(i) for all \( k, l, d \in \mathbb{N} \) with \( l < k \), we have

\[
E_0(k, d) + H_l \subseteq E_0(k - l, d).
\]

(ii) The translates \( E_0(k, d), E_0(k, d) + 1, \ldots, E_0(k, d) + (p - 1)1 \) partition \( E(k, d) \).

(iii) For fixed \( k \in \mathbb{N} \), \( \varepsilon > 0 \), and sufficiently large \( d \), we have

\[
|E_0(k, d)| > \frac{1 - \varepsilon}{p} |G_p^d|.
\]

**Proof.** Part (i) follows from the definition of \( E_0(k, d) \) and Part (iv) of Lemma 5.2.

Now to prove Part (ii). To show that the sets \( E_0(k, d), E_0(k, d) + 1, \ldots, E_0(k, d) + (p - 1)1 \) are mutually disjoint, fix \( n \neq m \in \{0, 1, \ldots, p - 1\} \). We will show that \( E_0(k, d) + n1 \) is disjoint from \( E_0(k, d) + m1 \). It suffices to prove that if \( C, C' \in \mathcal{P}_0 \) (not necessarily distinct), then \( \text{Bias}(C + n, k, d) \) is disjoint from \( \text{Bias}(C' + m, k, d) \), as Part (i) of Lemma 5.2 implies

\[
E_0(k, d) + n = \bigcup_{C \in \mathcal{P}_0} \text{Bias}(C + n, k, d), \quad E_0(k, d) + m = \bigcup_{C \in \mathcal{P}_0} \text{Bias}(C + m, k, d).
\]

Our choice of \( \mathcal{P}_0 \) implies that if \( C, C' \in \mathcal{P}_0 \) and \( m \neq n \), we have \( C + n \neq C' + m \). Part (ii) of Lemma 5.2 then implies \( \text{Bias}(C + n, k, d) \cap \text{Bias}(C' + m, k, d) = \emptyset \), as desired.

To see that the union of the translates \( E_0(k, d) + m1 \) is \( E(k, d) \), it suffices to prove that for each \( C \in \mathcal{P} \), there is an \( m \) such that \( \text{Bias}(C, k, d) \subseteq E_0 + m1 \). Our choice of \( \mathcal{P} \) means that for all \( C \in \mathcal{P} \), there exists \( m \in \mathbb{Z}/p\mathbb{Z} \) such that \( C - m \in \mathcal{P} \), and the definition of \( E_0 \) means that \( \text{Bias}(C - m, k, d) \subseteq E_0 \). We then have \( \text{Bias}(C - m, k, d) + m1 \subseteq E_0 + m1 \), and Lemma 5.2 simplifies the left hand side of this containment to \( \text{Bias}(C, k, d) \). We have therefore shown \( \text{Bias}(C, k, d) \subseteq E_0 + m1 \), as desired.

Finally, the estimate in Part (iii) follows from the estimate on \( E(k, d) \) in Lemma 5.3 and the fact that the translates \( E_0(k, d) + m1, 0 \leq m \leq p - 1 \), partition \( E(k, d) \) and all have the same cardinality.

**Proof of Lemma 5.1.** To prove Lemma 5.1, we fix \( k \in \mathbb{N} \), a prime \( p \), and \( \varepsilon > 0 \). Use part (iii) of Lemma 5.4 to choose \( d \) sufficiently large that \( |E_0(k + 1, 1)| > \frac{1 - \varepsilon}{p} |G_p^d| \). Let \( A = E_0(k + 1, d) \), and let \( A_1 = E_0(1, d) \). Now Part (i) of Lemma 5.4 implies \( A + H_k \subseteq A_1 \), and Part (ii) of Lemma 5.4 implies that the sets \( A_1 + n1, 0 \leq n \leq p - 1 \), are mutually disjoint. The containment \( A \subseteq A_1 \) follows from the containment \( A + H_k \subseteq A_1 \) and the fact that \( 0 \in H_k \). This completes the proof of Lemma 5.1.
6 Copying sets from $G_N^d$ into $\mathbb{T}^d$

Fix $N, d \in \mathbb{N}$. As in the previous section, $G_N^d$ is the group $(\mathbb{Z}/N\mathbb{Z})^d$. In this section we present a standard way of associating subsets of $\mathbb{T}^d$ to subsets of $G_N^d$. Under this association, the containment $A + H_k \subseteq A_1$ in Lemma 5.1 will yield near invariance of the associated sets under translation by elements of an approximate Hamming ball, which we describe in Definition 6.2. The near invariance mentioned here is proved in Lemma 6.3.

Write $\mu$ for Haar probability measure on $\mathbb{T}^d$. Let $\phi : G_N^d \to \mathbb{T}^d$ be the homomorphism given by

$$\phi(x_1, \ldots, x_d) := (x_1/N, \ldots, x_d/N).$$

For a given $\varepsilon \geq 0$, let

$$Q_{N,\varepsilon} := [\varepsilon, 1/N - \varepsilon]^d \subseteq \mathbb{T}^d.$$  

This is simply a half-open cube of side length $1/N - 2\varepsilon$. If $A \subseteq G_N^d$, define $A_{\varepsilon}^\square \subseteq \mathbb{T}^d$ by

$$A_{\varepsilon}^\square := \phi(A) + Q_{N,\varepsilon},$$

so that $A^\square_{\varepsilon}$ is a disjoint union of translates of $Q_{N,\varepsilon}$. The cubes $Q_{N,0}$ tile $\mathbb{T}^d$: we have $\mathbb{T}^d = \phi(G_N^d) + Q_{N,0}$.

The next lemma records the basic properties of this construction.

Lemma 6.1. Let $A, B \subseteq G_N^d$ and $\varepsilon \geq 0$. Then

(i) $(A \cap B)^\square_{\varepsilon} = A^\square_{\varepsilon} \cap B^\square_{\varepsilon},$

(ii) $\mu(A^\square_{\varepsilon}) = |A|(1/N - 2\varepsilon)^d,$

(iii) $\lim_{\varepsilon \to 0^+} \mu(A^\square_{\varepsilon}) = \mu(A^\square_0) = |A|N^{-d},$

(iv) If $\delta \leq \varepsilon$ and $y \in \mathbb{T}^d$ has $\|y\| < \delta$, then $A^\square_{\varepsilon} + y \subseteq A^\square_{\varepsilon - \delta}$.

Proof. Part (i) follows immediately from the definitions. We get Part (ii) by observing that $A^\square_{\varepsilon}$ is a disjoint union of $|A|$ cubes in $\mathbb{T}^d$ having side length $1/N - 2\varepsilon$. Part (iii) follows immediately from Part (ii). Part (iv) follows from the observation that if $\|y\| < \delta$, then $Q_{N,\varepsilon} + y \subseteq Q_{N,\varepsilon - \delta}$. \qed

The important consequence of Part (i) in Lemma 6.1 is that when $A_1, \ldots, A_j \subseteq G_N^d$ are mutually disjoint, the corresponding sets $(A_1)^\square_0, \ldots, (A_j)^\square_0 \subseteq \mathbb{T}^d$ are mutually disjoint.

Recall from §4 that for $x \in \mathbb{T}^d$ and $\varepsilon > 0$, we defined $w_\varepsilon(x) := \{j : \|x_j\| \geq \varepsilon\}$. 

Definition 6.2. For $k < d \in \mathbb{N}$ and $\varepsilon > 0$, we define the approximate Hamming ball of radius $(k, \varepsilon)$ around $0_{\mathbb{T}^d}$ as

$$\text{Hamm}(k, \varepsilon) := \{x \in \mathbb{T}^d : w_\varepsilon(x) \leq k\}.$$  

So $\text{Hamm}(k, \varepsilon)$ is the set of $x = (x_1, \ldots, x_d) \in \mathbb{T}^d$ where at most $k$ coordinates differ from 0 by at least $\varepsilon$.

The following lemma is crucial in deriving the containment $R^nE_0 \subseteq E_1$ (for $n \in BH$) in Proposition 4.4 from the containment $A + H_k \subseteq A_1$ in Lemma 5.1. For a set $B \subseteq \mathbb{T}^d$ we use $\overline{B}$ to denote its topological closure.
Lemma 6.3. Let \( \varepsilon \geq \eta > 0, k < d \in \mathbb{N} \), and \( A \subseteq G_N^d \). Let \( U := \text{Hamm}(k, \eta) \subseteq \mathbb{T}^d \) and \( H := H_k \subseteq G_N^d \), as in §5.1. Then

(i) \( A^\varepsilon + U \subseteq (A + H)^0 \).

(ii) If \( \varepsilon > \eta \), then \( \overline{A^\varepsilon + U} \subseteq (A + H)^0 \).

Proof. To prove Part (i), note that the left hand side therein is \( \phi(A) + Q_{N, \varepsilon} + U \), and the right hand side simplifies as \( \phi(A + H) + Q_{N, 0} = \phi(A) + \phi(H) + Q_{N, 0} \). It therefore suffices to prove that

\[
Q_{N, \varepsilon} + U \subseteq \phi(H) + Q_{N, 0}.
\]

To prove this containment, let \( u \in U \) with the aim of showing \( Q_{N, \varepsilon} + u \subseteq \phi(H) + Q_{N, 0} \). This \( u \) can be written as \( y + z \), where \( \|y\| < \eta \) and \( z = (z_1, \ldots, z_d) \) has at most \( k \) nonzero coordinates. Part (iv) of Lemma 6.1 implies \( Q_{N, \eta} + y + z \subseteq Q_{N, 0} + z \), so we must show that

\[
Q_{N, 0} + z \subseteq \phi(H) + Q_{N, 0}.
\]

The left hand side above is the set of \( x \) in \( \mathbb{T}^d \) where at most \( k \) coordinates of \( x \) lie outside \([0, \frac{1}{N})\). Fixing such an \( x \) as \((x_1, \ldots, x_d)\), we will show that \( x \not\in \phi(H) + Q_{N, 0} \). For each \( j \), choose \( h_j \in \{0, \ldots, N - 1\} \) so that \( x_j \in \left[\frac{h_j}{N}, \frac{h_j + 1}{N}\right) \). Then \( h_j \neq 0 \) for at most \( k \) indices \( j \), since at most \( k \) coordinates of \( x \) lie outside \([0, \frac{1}{N})\). Setting \( h = (h_1 \mod N, \ldots, h_d \mod N) \), we have \( h \in H \), and \( x \not\in \phi(h) + Q_{N, 0} \). This proves the containment (6.2), and therefore establishes (6.1), concluding the proof of Part (i).

To prove Part (ii), assume \( \varepsilon > \eta > 0 \), and choose \( \varepsilon' \) and \( \eta' \) so that \( \varepsilon > \varepsilon' > \eta' > \eta \). Let \( U' := \text{Hamm}(k, \eta') \). Our choice of \( \varepsilon' \) and \( \eta' \) means that \( \overline{A^\varepsilon} \subseteq A^\varepsilon' \) and \( \overline{U} \subseteq U' \). We therefore have \( \overline{A^\varepsilon + U} = \overline{A^\varepsilon} + \overline{U} \subseteq A^\varepsilon' + U' \subseteq (A + H)^0 \), where the last containment is an instance of Part (i). \( \square \)

7 Rohlin towers for torus rotations; proof of Proposition 4.4

The following lemma is a restatement of Proposition 4.4. It is proved by associating the sets provided by Lemma 5.1 to subsets of \( \mathbb{T}^d \) using the machinery of §6.

Lemma 7.1. For all \( k \in \mathbb{N} \), every prime \( p \), and all \( \varepsilon > 0 \), there exist \( d \in \mathbb{N} \), \( \eta > 0 \), sets \( E, E' \subseteq \mathbb{T}^d \), a generator \( \alpha \in \mathbb{T}^d \), and an approximate Hamming ball \( U := \text{Hamm}(k, \eta) \subseteq \mathbb{T}^d \) such that

(i) the translates \( E', E' + \alpha, \ldots, E' + (p - 1)\alpha \), are mutually disjoint,

(ii) \( \mu(E) > \frac{1 - \varepsilon}{p} \), and

(iii) \( E + U \subseteq E' \).

Consequently, the Bohr-Hamming ball \( BH := BH(\alpha; k, \eta) \) satisfies \( E + BH \alpha \subseteq E' \), and thus \( E \subseteq E' \).

Proposition 4.4 follows from Lemma 7.1, as Part (i) here asserts that \( \{R^nE' \colon 0 \leq n \leq p - 1\} \) is a Rohlin tower for the torus rotation on \( \mathbb{T}^d \) by \( \alpha \), and the containment \( E \subseteq E' \) then implies \( \{R^nE \colon 0 \leq n \leq p - 1\} \) is a Rohlin tower as well. The containment \( E + BH \alpha \) here is the part of Proposition 4.4 asserting \( R^nE \subseteq E' \) for all \( n \in BH \).
Proof. Fix \( k \in \mathbb{N} \), a prime \( p \), and \( \varepsilon > 0 \). By Lemma 5.1, choose \( d \) sufficiently large and \( A, A_1 \subseteq (\mathbb{Z}/p\mathbb{Z})^d \) such that the translates

\[
A_1, A_1 + 1, \ldots, A_1 + (p - 1)1 \text{ are mutually disjoint,} \tag{7.1}
\]

\(|A| > \frac{1 - \varepsilon/2}{p} p^d\), \( A \subseteq A_1 \), and \( A + H_k \subseteq A_1 \). We fix these choices of \( A \) and \( A_1 \subseteq (\mathbb{Z}/p\mathbb{Z})^d \) and use them to select \( \alpha \in \mathbb{T}^d \), \( E \subseteq \mathbb{T}^d \), and \( E' \subseteq \mathbb{T}^d \). We use the definitions of \( \phi \) and \( (\cdot \circ) \) established in §6, so that \( \phi : (\mathbb{Z}/p\mathbb{Z})^d \to \mathbb{T}^d \) and \( \phi(1) = (1/p, \ldots, 1/p) \in \mathbb{T}^d \).

The disjointness in (7.1) and Part (i) of Lemma 6.1 imply that the sets \( (A_1)_{0} + (p - 1)\phi(1) \) are mutually disjoint. By our definition of \( (\cdot \circ) \) and \( \phi \), this means that the translates

\[
(A_1)_{0}, (A_1)_{0} + \phi(1), \ldots, (A_1)_{0} + (p - 1)\phi(1) \text{ are mutually disjoint.} \tag{7.2}
\]

Now we specify the sets \( E, U, \) and \( E' \). Part (iii) of Lemma 6.1 provides an \( \eta > 0 \) satisfying \( \mu(A_{2\eta}^\circ) > (1 - \varepsilon/2)|A|p^{-d} \); our choices of \( \eta \) and \( A \) then guarantee that \( \mu(A_{2\eta}^\circ) > \frac{1 - \varepsilon}{p} \). Let

\[
E := A_{2\eta}^\circ, \quad U := \text{Hamm}(k, \eta), \quad E' := A_{2\eta}^\circ + U.
\]

Parts (ii) and (iii) of the present lemma are evidently satisfied by this \( E \). To find \( \alpha \) satisfying Part (i), we first observe that Lemma 6.3 implies \( E' \subseteq (A + H_k)_{0}^{\circ} \). Our choice of \( A \) and \( A_1 \) then implies \( E' \subseteq (A_1)_{0}^{\circ} \).

This containment and (7.2) imply that the translates

\[
E', E' + \phi(1), \ldots, E' + (p - 1)\phi(1) \tag{7.3}
\]

are mutually disjoint. They are all compact, as well, so for every \( \alpha \) sufficiently close to \( \phi(1) \), the translates

\[
E', E' + \alpha, \ldots, E' + (p - 1)\alpha \text{ are mutually disjoint.} \tag{7.4}
\]

In particular, we can choose such an \( \alpha \) to be a generator, and with this \( \alpha \) the disjointness in (7.4) implies the disjointness asserted in Part (i) of the lemma.

Finally, the containment \( E + BH \alpha \subseteq E' \) follows from the containment \( E + U \subseteq E' \) and the fact that \( n \in BH(\alpha; k, \eta) \) if and only if \( n\alpha \in U \). \( \square \)

8 Proof of Lemma 4.2

As in §3, for a given \( \alpha \in \mathbb{T} \) we write \( \bar{\alpha} \) for its representative in \([0, 1) \subseteq \mathbb{R} \). Given \( E \subseteq \mathbb{R} \), we write \( \text{span} E \) for the set of rational linear combinations of elements of \( E \) (i.e. the \( \mathbb{Q} \)-linear span). We will abbreviate sets of indexed elements \( \{x_1, \ldots, x_d\} \) as \( \{x_i\} \); for example, \( \text{span}\{x_1, \ldots, x_d\} \) may be written as \( \text{span}\{x_i\} \).

We also suppress the index of summation in sums, where it will cause no confusion.

We need the following standard facts from harmonic analysis, presented in references such as [4, 12, 16, 17]. An additive character of \( \mathbb{T}^d \) is a continuous homomorphism from \( \mathbb{T}^d \) to \( \mathbb{T} \).

- Every additive character of \( \mathbb{T}^d \) has the form \( (x_1, \ldots, x_d) \to \sum n_ix_i \) for some fixed \( (n_1, \ldots, n_d) \in \mathbb{Z}^d \).
We prove the contrapositive of Part (i). Assuming

\[ \text{Proof.} \]

adjoining elements of \( B \) which vanishes on \( K \) and not on \( L \).

- (Kronecker’s characterization) \( \alpha = (\alpha_1, \ldots, \alpha_d) \in T^d \) is a generator if and only if \( \{ \tilde{\alpha}_1, \ldots, \tilde{\alpha}_d, 1 \} \) is linearly independent over \( \mathbb{Q} \).

The next lemma is a variant of Kronecker’s characterization, and is proved in essentially the same way. Given an element \( g \) of a group, we use \( \langle g \rangle \) to denote the subgroup generated by \( g \). We write \( \bar{A} \) for the topological closure of a subset \( A \) of a topological space.

**Lemma 8.1.** Let \( r, k \in \mathbb{N} \), and suppose \( \alpha = (\alpha_1, \ldots, \alpha_r) \in T^r \) and \( \beta = (\beta_1, \ldots, \beta_k) \in T^k \) are such that \( \text{span}\{\tilde{\alpha}_i\} \cap \text{span}\{\tilde{\beta}_j\} \cup \{1\} = \{0\} \). Then

(i) \( \text{in } T^r \times T^k \) we have \( \langle (\alpha, \beta) \rangle = \langle \alpha \rangle \times \langle \beta \rangle \).

(ii) If \( U \subseteq T^k \), \( V \subseteq T^r \) are open and \( B(\alpha; U) \), \( B(\beta; V) \) are nonempty Bohr neighborhoods, then \( B((\alpha, \beta); U \times V) \) is a nonempty Bohr neighborhood of rank \( r + k \).

**Proof.** We prove the contrapositive of Part (i). Assuming \( \langle (\alpha, \beta) \rangle \neq \langle \alpha \rangle \times \langle \beta \rangle \), the former must be properly contained in the latter, since \( \langle \alpha, \beta \rangle \subseteq \langle \alpha \rangle \times \langle \beta \rangle \). This means there is an additive character of \( T^r \times T^k \) vanishing on \( \langle (\alpha, \beta) \rangle \) and not on \( \langle \alpha \rangle \times \langle \beta \rangle \). In other words, there exists \( (n_1, \ldots, n_r, m_1, \ldots, m_k) \in \mathbb{Z}^{r+k} \) such that \( \sum_{i,j} n_i \alpha_i + m_j \beta_j = 0 \in T \), and at least one of \( \sum n_i \alpha_i, \sum m_j \beta_j \) is nonzero. Together this implies both \( \sum n_i \alpha_i \) and \( \sum m_j \beta_j \) are nonzero. Thus there is an integer \( N \) such that \( \sum n_i \alpha_i = N - \sum m_j \beta_j \), meaning the respective spans of \( \{ \tilde{\alpha}_1, \ldots, \tilde{\alpha}_r \} \) and \( \{ \tilde{\beta}_1, \ldots, \tilde{\beta}_d, 1 \} \) intersect nontrivially.

To prove Part (ii), assume \( B(\alpha; U) \) and \( B(\beta; V) \) are both nonempty. Note that \( B((\alpha, \beta); U \times V) \neq \emptyset \) if and only if the intersection \( Y := (U \times V) \cap \langle (\alpha, \beta) \rangle \) is nonempty (since \( U \times V \) is open). The same observation implies \( \langle \alpha \rangle \cap U \) and \( \langle \beta \rangle \cap V \) are both nonempty, by the hypothesis of Part (ii). The set \( Y \) is therefore nonempty, and this yields the conclusion.

**Proof of Lemma 4.2.** The first assertion of the lemma states that if \( k < d \in \mathbb{N} \), \( \varepsilon > 0 \), and \( BH \) is a proper Bohr-Hamming ball with rank \( d \) and radius \( (k, \varepsilon) \) and \( B \) is a nonempty Bohr neighborhood with rank \( k \), then \( BH \cap B \) contains a Bohr neighborhood with rank \( d \). To prove this, fix such \( BH \) and \( B \), write \( \alpha = (\alpha_1, \ldots, \alpha_d) \) for the frequency determining \( BH \), and write \( B \) as \( \{ n \in \mathbb{Z} : n\beta \in V \} \), where \( V \subseteq T^k \) is open and \( \beta = (\beta_1, \ldots, \beta_k) \in T^k \).

Let \( W, Y, \) and \( Z \) denote \( \text{span}\{\{\tilde{\alpha}_i\} \cup \{1\}\} \), \( \text{span}\{\{\tilde{\beta}_j\} \cup \{1\}\} \), and \( \text{span}\{\{\tilde{\alpha}_i\} \cup \{\tilde{\beta}_j\} \cup \{1\}\} \), respectively. Our assumption that \( BH \) is proper means that \( \alpha \) is a generator, so \( W \) has dimension \( d + 1 \), and the dimension of \( Z \) is at least \( d + 1 \).

We select \( d - k \) elements \( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{d-k} \) of the \( \tilde{\alpha}_i \), so that \( \text{span}\{\tilde{\alpha}_i\} \cap \text{span}\{\tilde{\beta}_j\} \cup \{1\} = \{0\} \). To make this selection, first choose a basis \( B \) for \( Y \) from among the \( \tilde{\beta}_j \), then extend \( B \) to a basis \( B' \) for \( Z \) by adjoining elements of \( \{\tilde{\alpha}_i\} \cup \{1\} \). This basis must contain at least \( d + 1 \) elements, since \( Z \) has dimension \( \geq d + 1 \), so \( B' \) contains at least \( d - k \) elements \( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{d-k} \) of the \( \tilde{\alpha}_i \). Since \( B' \) is linearly independent and the \( \tilde{\alpha}_i \) are disjoint from \( B \cup \{1\} \), we get that \( \text{span}\{\tilde{\alpha}_i\} \cap \text{span}\{\tilde{\beta}_j\} \cup \{1\} \) is trivial.

Let \( \alpha' = (\alpha_1, \ldots, \alpha_{d-k}) \in T^{d-k} \), and let \( C \) be the basic Bohr neighborhood \( B(\alpha'; U) \), where \( U \) is the \( \varepsilon \)-ball around 0 in \( T^{d-k} \). Comparing their respective definitions, we see that \( BH \) contains \( C \); furthermore
C is nonempty, as 0 ∈ C. Thus, \( BH \cap B \) contains \( C \cap B = B((\alpha', \beta'); U \times V) \). The latter set is a Bohr neighborhood with rank \( d \) which, by Lemma 8.1, is nonempty.

The second assertion of Lemma 4.2 follows from the first: if \( S \) is \( d \)-Bohr dense and \( B \) is a Bohr neighborhood with rank \( k \), then \( S \cap (BH \cap B) \neq \emptyset \), by virtue of the fact that \( BH \cap B \) contains a nonempty Bohr neighborhood of rank \( d \). Thus \( (S \cap BH) \cap B \neq \emptyset \) for every Bohr neighborhood \( B \) with rank \( k \), so \( S \cap BH \) is \( k \)-Bohr dense. \( \square \)

9 Proof of Lemmas 2.1 and 2.2

Lemma 2.1 says that for \( \delta > 0 \) and \( S \subseteq \mathbb{Z} \), the following conditions are equivalent.

(i) There is a measure preserving system \((X, \mu, T)\) and \( D \subseteq X \) with \( \mu(D) > \delta \) such that \( \mu(D \cap T^n D) = \emptyset \) for all \( s \in S \).

(ii) There exists \( A \subseteq \mathbb{Z} \) with \( d^s(A) > \delta \) such that \((A - A) \cap S = \emptyset\).

(iii) There is a \( \delta' > \delta \) such that for all \( n \in \mathbb{N} \) there exist \( A_n \subseteq \{0, \ldots, n - 1\} \) with \( |A_n| \geq \delta'n \) and \((A_n - A_n) \cap S = \emptyset\).

Proof. To prove (i) \( \implies \) (ii), let \( \delta > 0 \), \( S \subseteq \mathbb{Z} \), \((X, \mu, T)\), and \( D \) satisfy (i). We will find \( A \subseteq \mathbb{Z} \) with \( d^s(A) > \delta \) and \( A \cap (A + S) = \emptyset \).

Write \( 1_D \) for the characteristic function of \( D \). By the pointwise ergodic theorem, the limit

\[
F(x) := \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} 1_D(T^n x)
\]

exists for \( \mu \)-almost every \( x \). The dominated convergence theorem implies \( \int F \, d\mu = \int 1_D \, d\mu \), so the limit on the right hand side of (9.1) is greater than \( \delta \) for some \( x \in X \). Fixing such \( x \) and setting

\[
A := \{n : T^n x \in D\},
\]

we have \( \lim_{N \to \infty} \frac{|A \cap \{0, \ldots, N-1\}|}{N} = F(x) \), so \( d^s(A) > \delta \). To prove that \((A - A) \cap S = \emptyset \), note that \( m \in A \cap (A + s) \) if and only if \( T^m x \in D \) and \( T^{m-s} x \in D \), so \( T^m x \in D \cap T^s D \). Our hypothesis that \( D \cap T^s D = \emptyset \) for all \( s \in S \) then implies \( A \cap (A + s) = \emptyset \) for all \( s \in S \), meaning \((A - A) \cap S = \emptyset \).

To prove (ii) \( \implies \) (iii), assume \( A \subseteq \mathbb{Z} \) has \( d^s(A) = \delta'' > \delta' > \delta \). Then there are intervals \( I_k \) with \( |I_k| \to \infty \) such that \( |A \cap I_k| \geq \delta'|I_k| \) for all sufficiently large \( k \).

Fix \( n \in \mathbb{N} \), and write \( I_k \) as a union of mutually disjoint intervals \( J_{k,1}, \ldots, J_{k,r} \) of length \( n \) together with one (possibly empty) interval \( J_{k,0} \) of length at most \( n \). Observe that \( |I_k|/r \geq n \) under this arrangement. Then

\[
|A \cap I_k| = \sum_{i=0}^{r} |A \cap J_{k,i}| \geq \delta'|I_k|,
\]

so \( \sum_{i=0}^{r} |A \cap J_{k,i}| \geq \delta'|I_k| - n \). This implies that for some \( i \leq r \), \( |A \cap J_{k,i}| \geq \frac{1}{r}(\delta'|I_k| - n) \geq \delta'n - \frac{n}{r} \). Since \( |A \cap J_{k,i}| \) is integer valued and \( n/r \to 0 \) as \( k \to \infty \), we get that \( |A \cap J_{k,i}| \geq \delta''n \) for sufficiently large \( k \). We have thus found an interval \( J = J_{k,i} \) of length \( n \) with \( |A \cap J| \geq \delta''n \). We let \( A_n = (A \cap J) - \min(J) \), so that
\( A \subseteq \{0, \ldots, n-1\} \), and \( A_n - A_n \subseteq A - A \). The assumption that \((A - A) \cap S = \emptyset\) allows us to conclude (iii).

Finally we prove (iii) \(\implies\) (i). Let \( \delta' > \delta > 0 \) and let \( A_n \subseteq \{0, \ldots, n-1\} \) have \( |A_n| \geq \delta'n \). Consider the topological space \( X := \{0, 1\}^Z \) with the product topology, so that \( X \) is a compact metrizable space. Let \( T \) be the continuous transformation \( T : X \to X \) given by \((Tx)(m) = x(m + 1)\) (commonly known as the left shift).

Let \( E \) be the clopen set \( \{x \in X : x(0) = 1\} \). For each \( n \), let \( y_n = 1_{A_n} \in X \), meaning \( y_n \) is the characteristic function of \( A_n \), viewed as an element of \( X \). Note that

(a) \( T^m y_n \in E \) if and only if \( m \in A_n \), and

(b) \( T^m y_n \in E \cap T^k E \) if and only if \( m \in A_n \) and \( m - k \in A_n \), meaning \( k \in A_n - A_n \). In particular, if \( k \notin A_n - A_n \), then \( T^m y_n \notin E \cap T^k E \) for all \( m \in \mathbb{Z} \).

For \( x \in X \), let \( \delta_x \) denote the Dirac probability measure concentrated at \( x \).

Let \( \mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k y_n} \), so that each \( \mu_n \) is a Borel probability measure on \( X \). Note that \( \mu_n(E) := \frac{1}{n} |\{k \in \mathbb{N} : T^k y_n \in E\}| = |A_n|/n \), by (a) above, so \( \mu_n(E) \geq \delta' \) for each \( n \).

Let \( \mu \) be a weak\(^*\) limit of the \( \mu_n \). It is easy to verify that \( \mu \) is \( T \)-invariant, by proving \( \int f d\mu = \int f \circ T d\mu \) for every continuous \( f \): write \( \int f d\mu = \int f \circ T d\mu \) as a limit of a subsequence of

\[
I_n := \frac{1}{n} \sum_{k=0}^{n-1} f(T^k y_n) - f(T^{k+1} y_n),
\]

which converges to 0 by cancellation and boundedness of \( f \).

We claim that \( (X, \mu, T) \) is a measure preserving system satisfying (i) in the statement of the lemma. To see this, first note that \( \mu(E) \geq \delta' \), since \( \mu_n(E) \geq \delta' \) for each \( n \). Fixing \( s \in S \) (which is disjoint from \( A_n - A_n \) by hypothesis), we have \( \mu_n(E \cap T^s E) \neq 0 \) for each \( n \), by observation (b). Thus \( \mu(E \cap T^s E) = 0 \) for all \( s \in S \). Now let \( D = E \setminus \bigcup_{s \in S} T^s E \), so that \( \mu(D) = \mu(E) \geq \delta' \). Then \( D \cap T^s D \subseteq D \cap T^s E \) since \( D \subseteq E \). Also \( D \) is disjoint from \( T^s E \), so we have \( D \cap T^s D = \emptyset \) for every \( s \in S \).

Recall Lemma 2.2: if \( 0 \leq \delta < \delta' \) and every finite subset of \( S \subseteq \mathbb{Z} \) is \( \delta' \)-nonrecurrent, then \( S \) is \( \delta \)-nonrecurrent.

**Proof of Lemma 2.2.** Suppose \( S \subseteq \mathbb{Z} \), \( 0 \leq \delta < \delta' \), and that every finite subset of \( S \) is \( \delta' \)-nonrecurrent. Write \( S \) as an increasing union \( \bigcup_{k \in \mathbb{N}} S_k \) of finite sets \( S_k \). For each \( k \), choose a set \( C_k \subseteq \mathbb{Z} \) with \( d^*(C_k) > \delta' \) such that \( (C_k - C_k) \cap S_k = \emptyset \).

We will find a sequence of sets \( A_n \subseteq \{0, \ldots, n-1\} \) such that \( |A_n| \geq \delta'n \) and for each \( n \), infinitely many of the \( C_k \) contain a translate \( A_n + t_n \), of \( A_n \). Under these conditions, we have \( A_n - A_n \subseteq C_k - C_k \) for such \( k \), so that \( (A_n - A_n) \cap S_k = \emptyset \) for infinitely many \( k \). Since the \( S_k \) are increasing and exhaust \( S \), this implies \((A_n - A_n) \cap S = \emptyset\), whereby part (iii) of Lemma 2.1 implies \( S \) is \( \delta \)-nonrecurrent.

We find the sets \( A_n \) by fixing \( n \) and choosing, for each \( k \), an interval \( I_k = [t_k, t_k + n - 1] \) with length \( n \) such that \( |I_k \cap C_k| \geq \delta'n \), just as in the proof of (ii) \(\implies\) (iii) in Lemma 2.1. Letting \( C'_k = (I_k \cap C_k) - t_k \), we see that \( C'_k \subseteq \{0, \ldots, n-1\} \). There are only finitely many subsets of \( \{0, \ldots, n-1\} \), so there is an infinite collection of indices \( k \) such that the \( C'_k \) are mutually identical for these \( k \). We let \( A_n \) be one of these \( C'_k \), so that \( A_n + t_k \subseteq C_k \) for infinitely many \( k \), as desired.
10 Remarks and a problem

Følner [5] proved that if $A \subseteq \mathbb{Z}$ has $d^*(A) > 0$, then $A - A$ contains a set $B \setminus \mathbb{Z}$, where $B$ is a Bohr neighborhood of 0 and $d^*(\mathbb{Z}) = 0$. Kriz [13] constructed the first example of a set $A \subseteq \mathbb{Z}$ having $d^*(A) > 0$ such that $A - A$ does not contain a Bohr neighborhood of 0. Theorem 1.2 shows that Følner’s theorem cannot be improved to say that $A - A$ contains a Bohr neighborhood, even with the modification that the Bohr neighborhood may be around some nonzero $n$.

Our method is very similar to Kriz’s, and to Ruzsa’s simplified version of Kriz’s method presented in [14, 15]: the Bohr-Hamming balls we consider are closely analogous to the embeddings of Kneser graphs used in [13], and our Proposition 4.4 is an extreme modification of Lemma 3.2 in [13]. Katznelson in [11] showed that translates of Bohr-Hamming balls (absent the nomenclature) are $k$-Bohr recurrent but not $(k + 1)$-Bohr recurrent.

Theorem 1.3 suggests the following problem.

**Problem 10.1.** Prove that if $S \subseteq \mathbb{Z}$ is Bohr recurrent, then there is a set $S' \subseteq S$ such that $S'$ is Bohr recurrent and is not a set of measurable recurrence.

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