Quantum corrections to static solutions of Nahm equation and Sin-Gordon models via generalized zeta-function

Sergey Leble
Gdansk University of Technology, Faculty of Applied Physics and Mathematics, ul. Narutowicza 11/12, 80-952 Gdansk, Poland,

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Abstract

One-dimensional Yang-Mills Equations are considered from a point of view of a class of nonlinear Klein-Gordon-Fock models. The case of self-dual Nahm equations and non-self-dual models are discussed. A quasiclassical quantization of the models is performed by means of generalized zeta-function and its representation in terms of a Green function diagonal for a heat equation with the correspondent potential. It is used to evaluate the functional integral and quantum corrections to mass in the quasiclassical approximation. Quantum corrections to a few periodic (and kink) solutions of the Nahm as a particular case of the Ginzburg-Landau (phi-in-quadro) and Sin-Gordon models are evaluated in arbitrary dimensions. The Green function diagonal for heat equation with a finite-gap potential is constructed by universal description via solutions of Hermit equation. An alternative approach based on Baker-Akhiezer functions for KP equation is proposed. The generalized zeta-function is studied in both forms; its derivative at zero point, expressed in terms of elliptic integrals is proportional to the quantum corrections to mass.

1 Introduction.

1.1 General remarks.

We consider one-dimensional field theory, based on nonlinear Klein-Gordon equations, arising, for example of Sine-Gordon (SG) case, in kink models for crystal structure dislocations [1] or in a context of a relativistic models [2]. Popular operator constructions of quantum field theory such as Yang-Mills one have reductions to one dimensional models [3]. Some kind of embedding of such theory into the multidimensional one is possible: Atiyah-Drinfeld-Hitchin-Manin-Nahm construction may appear as an equivalence between two sets of self-dual equations, one as described above in one dimension, the other in three dimensions (reduced from a Euclidean four dimensional theory by deleting dependence on a single variable) [4].

A class of nonlinear Klein-Gordon equations in the case of static one-dimensional solutions is reduced to

\[ \phi'' - V'(\phi) = 0, \phi = \phi(x), x \in R. \quad (1.1) \]
Suppose the potential $V(\phi)$ is twice continuously differentiable; it guarantees existence and uniqueness of the equations correspondent to (1.1) Cauchy problem solution. The first integral of the equation (1.1) is given by

$$W = \frac{1}{2} \phi'^2 - V(\phi),$$

(1.2)

where $W$ is the integration constant. The equation (1.2) is ordinary first-order differential equation with separated variables. As the phase method shows, solutions of this equations belong to the following families: constant, periodic, separatrix and the so-called “passing” one [11].

In the case of Nahm equation

$$V_N(\phi) = \frac{\phi^4}{2}; \quad W = -\frac{w^4}{2},$$

(1.3)

and for Ginzburg-Landau the ”potential” is

$$V_{gl}(\phi) = \frac{g}{4} (\phi^2 - \frac{m^2}{g})^2,$$

(1.4)

while in the case of Sin-Gordon (SG) model it is

$$V_{sg}(\phi) = \frac{2m^4}{3g} (1 + \cos\left(\frac{1}{m} \sqrt{\frac{3g}{2}} x\right)).$$

(1.5)

We modified this last model to fit it with the first one at small values of the constant $g$, namely

$$V_{sg} = \frac{13m^4}{12g} + V_{gl} + O(g^2).$$

1.2 One-dimensional reduction of Yang-Mills theory.

Starting with the full Yang-Mills equations in four Euclidean dimensions,

$$D_\mu T_{\mu\nu} = 0,$$

(1.6)

for the gauge fields $T_{\mu} = T^+_{\mu}$, where

$$T_{\mu\nu} = T_{\nu,\mu} - T_{\mu,\nu} - \iota[T_{\mu}, T_{\nu}], \quad D_\mu\Phi = \partial_\mu - \iota[T_{\mu}, \Phi].$$

and demanding all fields be independent of three of the variables $x_k$, k=1,2,3 setting $x_4 = z$,

$$\frac{d^2T_k}{dz^2} = [T_j[T_j, T_k]], \quad [T_k, \frac{dT_k}{dz}] = 0.$$  

(1.7)

One can easily check that the self-dual equations,

$$\frac{dT_i}{dz} = \pm \varepsilon_{ijk} T_j T_k,$$

(1.8)

corresponding to the model, imply Eqs. (1.7). Starting from the simplest solution of the system (1.8)

$$T_i = \phi_i \sigma_i$$

(1.9)
built on a base of Pauli matrices one arrives at the Euler system for \( \dot{\phi}(y) \) that is solved in Jacobi functions. The solutions are dressed by the gauge-Darboux transformations [6].

For the second order equations the ansatz similar to (1.9)

\[
T_i = \phi(z)\alpha_i, \ i = 1, 2, 3,
\]

with a constant matrices \( \alpha_i \) and a convenient choice of scaling leads to the pair of equations

\[
2\alpha_i = \sum_{j=1}^{3}[\alpha_j[\alpha_j, \alpha_i]] \\
\phi''(z) = 2\phi^3.
\]

The second order equation for \( \phi(z) \) enters as a particular case into the class of nonlinear (1.1) with \( V'(\phi) = 2\phi^3 \), or into Lagrangian of GL model with \( V \) from (1.3).

### 1.3 Feynmann quantization of a classical field.

An attention to Feynmann quantization formalism of a classical field was recently attracted in connection with a link to a SUSY quasiclassic quantization condition [8] suggested in [9]; see, however, [10].

Historically, quantum corrections started from a [12], see also [14]. Important development concerns the Jacobi variety structure [13].

In the papers of V.Konoplich [15] quantum corrections to a few classical solutions by means of Riemann zeta-function are calculated in dimensions \( d > 1 \). Most interesting of them are the corrections to the kink - the separatrix solution of the field \( \phi^4 \) (GL) model [16].

The method of [15] is rather complicated and it is desired to simplify it, that was the main target of our previous note [17]. We applied the Darboux transformations technique with some nontrivial details missed in [15].

The suggested approach open new possibilities; for example it allows to calculate the quantum corrections to matrix models of similar structure, Q-balls [18] and periodic solutions of the models. The last problem is posed in the review [11].

The approximate quantum corrections to the solutions of the equation (1.1) are obtained via the Feynmann functional integral method evaluated by the stationary phase analog [19]. It gives the following relation

\[
\exp\left[-\frac{S_{qu}}{\hbar}\right] \simeq \frac{A}{\sqrt{\det D}},
\]

where \( S_{qu} \) denotes quantum action, corresponding the potential \( V(\phi) \), \( A \) - some quantity determined by the vacuum state at \( V(\phi) = 0 \), and \( \det D \) is the determinant of the operator

\[
D = -\partial^2_x - \Delta_y + V''(\phi(x)).
\]

The argument \( y \in R^{d-1} \) stands for the transverse variables on which the solution \( \phi(x) \) does not depend. The operator \( D \) appears while the second variational derivative of the quantum action functional (which enter the Feynmann trajectory integral) is evaluated. For the vacuum action \( S_{vac} \) the relation of the form (1.12) is valid if \( S_{qu} \) is changed to \( S_{vac} \) and \( D \) is replaced by the ”vacuum state” operator \( D_0 = -\partial^2_x - \Delta_y + \nu \). Then, the quantum correction

\[
\Delta S_{qu} = S_{qu} - S_{vac},
\]

(1.14)
is obtained by the mentioned twice use of the formula (1.12) as
\[ \Delta S_{qu} = \frac{\hbar}{2} \ln \left( \frac{\det D}{\det D_0} \right). \]  
(1.15)

Hence, the problem of determination of the quantum correction is reduced to one of evaluation of the determinants \(D\) and \(D_0\) ratio. The methodic of the evaluation will be presented in the following section.

1.4 The generalized Riemann zeta-function and Green function of heat equation.

The generalized zeta-function appears in many problems of quantum mechanics and quantum field theories which use the Lagrangian \( L = (\partial \phi)^2/2 - V(\phi) \) and it is necessary to calculate a Feynmann functional integral in the quasiclassical approximation.

The scheme is following. Let the set \( \{ \lambda_n \} = S \) be a spectrum of a linear operator \( L \),

\[ \ln(\det L) = \sum_{\lambda_n \in S} \ln \lambda_n, \]  
(1.16)

where the sum in the r.h.s. is formal one.

The generalized Riemann zeta-function \( \zeta_L(s) \) is defined by the equality

\[ \zeta_L(s) = \sum_{\lambda_n \in S} \lambda_n^{-s}. \]  
(1.17)

This definition should be interpreted as analytic continuation to the complex plane of \( s \) from the half plane \( \text{Re} s > \sigma > 0 \) in which the sum in (1.17) converges. Differentiating the relation (1.17) with respect to \( s \) at the point \( s = 0 \) yields

\[ \ln(\det L) = -\zeta_L'(0). \]  
(1.18)

The generalized zeta-function (3.89) admits the representation via the diagonal \( g_L \) of a Green function of the operator \( \partial_t + L \).

A link to the diagonal Green function (heat kernel formalism) has been used in quantum theory since works by Fock (21). There is a representation in terms of the formal sum over the spectrum

\[ g_L(t, r, r_0) = \sum_n \exp[-\lambda_n t] \psi_n(r) \psi_n^*(r_0), \]  
(1.19)

where the normalized eigenfunctions \( \psi_n(r) \) correspond to eigenvalues \( \lambda_n \) of the operator \( L \).

Let us next define

\[ \gamma_L(t) = \sum_k e^{-\lambda_k t} = \int g_L(t, r, r) \, dr \]  
(1.20)

The Mellin transformation yields the generalized zeta function of the operator \( L \):

\[ \zeta_L(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \gamma_L(t) \, dt. \]  
(1.21)
Returning to the operators (1.13) \( L \to D, D_0: \ D = -\Delta + V''(\phi_0(x)) \) and \( D_0 = - \frac{\partial^2}{\partial x^2} - \Delta_y + \nu \), we pose the main problem for a periodic potential \( u(x) \):

\[
\left( \frac{\partial}{\partial t} + D_1 \right) g_D (t, x, x_0) = \delta (t) \delta (x - x_0) \tag{1.22}
\]

where \( D_1 = - \frac{\partial^2}{\partial x^2} + u(x) \), \( V \leq V_m \), that means a necessary presence of a continuous part of the spectrum \( \lambda \in [\lambda_0, +\infty) \). The basic relation (1.15) points to a necessity of evaluation of the determinants of the operators

\[
D = D_0 + u(x), \quad D_0 = - \partial_x^2 - \Delta_y + \nu \tag{1.23}
\]

where \( \lambda \) is a positive number and \( x \in \mathbb{R} \) is one of variables, while \( y \in \mathbb{R}^{d-1} \) is a set of other variables. The operator \( \Delta_y \) is the Laplace operator in \( d-1 \) dimensions, \( u(x) \) is one-dimensional potential that is defined by the condition

\[
V''(\phi_0(x)) = \lambda + u(x), \tag{1.24}
\]

where \( \phi_0(x) \) is the classical static solution of the equation of motion.

The Green function for a Hermitian operator \( D^+_1 = D_1 \) is:

\[
g_{D_1} (t, x, x_0) = \sum_k e^{-\lambda_k t} \psi_k (x) \psi_k^*(x_0) \Theta (t). \tag{1.25}
\]

The generalized zeta-function, defined by the relations (1.20,1.21), will be referred as the zeta-function of the operator \( D \).

From the relation (1.20) for the function \( \gamma_D(t) \) it follows an important property of multiplicity: if the operator \( D \) is a sum of two differential operators \( D = D_1 + D_2 \), which depend on different variables, the following equality holds

\[
\gamma_D(t) = \gamma_{D_1}(t)\gamma_{D_2}(t). \tag{1.25}
\]

Generally, for a constant potential \( \nu \) (see Sec. 3), for the one-dimensional case,

\[
\gamma_{D_0}(t) = \frac{\exp[\nu t]}{2\sqrt{\pi t}^d}. \tag{1.26}
\]

We will need the value of the function \( \gamma_D(t) \) for the vacuum state, when the operator \( D = -\Delta_y \) is equal to the \( d-1 \)-dimensional Laplacian. In this case \( \nu = 0 \), hence

\[
\gamma_{D_0}(t) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} dk \exp(-|k|^2 t) = (4\pi t)^{-(d-1)/2}. \tag{1.27}
\]

Combining as in (1.25) yields

\[
\gamma_{D_0}(t) = \frac{\exp[\nu t]}{2\sqrt{\pi t}}. \tag{1.28}
\]

Then,

\[
\zeta_{D_0}(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \gamma_{D_0}(t) dt = \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)} \frac{1}{(2\sqrt{\pi})^d} \nu^{\frac{d}{2} - s}. \tag{1.29}
\]

A quantum correction to the action in one-loop approximation for the classical solution \( \phi(x) \) is calculated via zeta-function by the formula

\[
\Delta S_{qu} = -\frac{\hbar}{2} [\zeta'_D(0) - \zeta'_{D_0}(0)]/2. \tag{1.30}
\]
2 The classic static solutions and energy of solitons.

2.1 Static solutions of $\phi^4$.

In the case of static solutions of the $\varphi^4$ model the potential is determined by the formulas

$$V(\varphi) = \frac{g}{4}(\varphi^2 - \frac{m^2}{g})^2,$$

therefore the equation of motion has the form

$$\varphi''(x) + m^2 \varphi - g \varphi^3 = 0,$$

that yields (1.2) in the form

$$(\varphi')^2 = \frac{g}{2}(\varphi^2 - \frac{m^2}{g})^2 + 2W.$$

Its restricted solutions, as it follows from phase plane analysis, exist if

$$-\frac{m^2}{4g} \leq W \leq 0.$$

The separatrix (W=0) solution of (2.33) is the kink/antikink

$$\varphi_0 = \pm \sqrt{\frac{2}{g}} b \tanh(bx), \quad b = \frac{m}{\sqrt{2}}.$$

While inside the interval (2.34) the equation (2.33) is expressed in terms of the elliptic Jacobi sinus

$$\varphi = \pm \sqrt{\frac{2}{g}} k \operatorname{sn}(bx; k), \quad b = \frac{m}{\sqrt{1 + k^2}}, \quad 0 < k < 1.$$

The constant W is given by

$$W = -\frac{1}{2} \left(1 - k^2\right)^2 \frac{m^4}{4g}.$$

The energy in the Nahm case $k = i$ should be studied separately, because the links (2.36) between $k, b$ and $W$ is not valid.

The family of the restricted solutions contains also the constant vacuum ones (W=0)

$$\varphi = \pm \frac{m}{\sqrt{g}}.$$

After the substitution of (2.35) into (1.24) we obtain the following potential $u(x)$:

$$u(x) = -6b^2/ch^2(bx),$$

with the meaning of the constant $b = m/\sqrt{2}$. As a result the two-level reflectionless potential of one-dimensional Schrödinger equation $-\partial_x^2 + u(x)$ appears. Eigenvalues and the normalized eigenfunctions of which are correspondingly (its numeration is chosen from above to lowercase).

$$\lambda_1 = -b^2, \quad \psi_1(x) = \sqrt{3b/2} \sinh(bx)/\cosh^2(bx);$$
$$\lambda_2 = -4b^2, \quad \psi_2(x) = \sqrt{5b/2} \cosh(bx).$$

In the case of the periodic solution of the phi-in-quadro model the potential has a "cnoidal" form

$$u(x) = -6k^2b^2 \text{cn}^2(bx; k) + (5k^2 - 1)b^2.$$

This potential differs from second Lamé equation potential by the constant hence its spectrum is two-gap one.
2.2 SG model

Let us briefly describe SG model \([35]\) we integrate the equation (1.1) with the potential (1.5) arriving at the first-order differential equation with the parameter \(W\) (1.2).

\[
(\varphi')^2 = \frac{4m^4}{3g} \left( 1 + \cos\left( \frac{1}{m} \sqrt{\frac{3g}{2}} \varphi \right) \right) + 2W. \tag{2.41}
\]

The solutions are restricted and hence have the direct physical relevance, if

\[-\frac{4m^4}{3g} \leq W \leq 0, \tag{2.42}\]

that follows from phase plane analysis. If \(W = 0\) the nontrivial solutions are interpreted as kink and antikink

\[
\varphi(x) = \pm \sqrt{\frac{2}{3g}} \arcsin \tanh(mx) (mod \Phi), \quad \Phi = 2m\pi \sqrt{\frac{2}{3g}}, \tag{2.43}
\]

while at the interval

\[-\frac{4m^4}{3g} < W < 0,\]

the solution of (2.41) yields a periodic function expressed via elliptic Jacobi function. To find it one plug \(\varphi = \pm 2m \sqrt{\frac{2}{3g}} \arcsin z\), then the equation (2.41) goes to

\[
(z')^2 = m^2 \left( 1 + \frac{3Wg}{4m^4} - z^2 \right) \left( 1 - z^2 \right), \tag{2.44}
\]

The solution of (2.44) at the interval (2.42) is given by

\[
z = ksn(mx; k), \tag{2.45}
\]

where

\[
k = \sqrt{1 + \frac{3gW}{4m^4}}, \tag{2.46}
\]

is the module of the elliptic function. Hence

\[
W = \frac{4(k^2 - 1)m^4}{3g}. \tag{2.47}
\]

Finally

\[
\varphi = \pm 2m \sqrt{\frac{2}{3g}} \arcsin ksn(mx; k) (mod \Phi). \tag{2.48}
\]

The class of restricted solutions contains also

\[
\varphi = 0 (mod \Phi); W = -\frac{3m^4}{3g}, \tag{2.49}
\]

and

\[
\varphi = \pm \pi m \sqrt{\frac{2}{3g}} (mod \Phi); W = 0. \tag{2.50}
\]
Other static solutions are obtained by shifts

\[ x \rightarrow x + x_0, \quad \varphi \rightarrow \varphi + \Phi, \]

that follows from Klein-Gordon equation invariance and SG equation potential periodicity.

Let us evaluate the energy of the nontrivial static solutions of both models via the energy density definitions

\[ e(x) = \int_{-\infty}^{\infty} \left( (\varphi')^2 / 2 + V(\varphi) \right) dx, \quad (2.51) \]

for kinks and, in a case of periodic solutions,

\[ E = 2 \int_{0}^{l} e(x) dx, \quad (2.52) \]

the constant ”l” is the period of the solution.

For the SG kink:

\[ E_k = \frac{16m^2}{g}, \quad (2.53) \]

and, for a periodic soliton

\[ E_p = \frac{8m^2}{g} [(1 - k^2) K + 2E], \quad (2.54) \]

where \( K(k), E(k) \) - complete elliptic Legendre integrals.

### 2.3 Static solutions of Nahm model as a specific case of \( \varphi^4 \) model

The solution of the (1.11), is a particular but specific case of static GL model \((m=0, g=2)\)

\[ \varphi^2 = \varphi^4 - w^4. \quad (2.55) \]

A solution of (1.11) is expressed via elliptic functions [7], namely while

\[ \int_{0}^{\varphi} \frac{d\phi}{\sqrt{\varphi^4 - w^4}} = \int_{0}^{\varphi} \frac{d\phi}{\sqrt{(\varphi^2 - w^2)(\varphi^2 + w^2)}} = z, \quad (2.56) \]

or

\[ \frac{1}{w} \int_{\frac{w}{w}}^{\frac{w}{w}} \frac{dt}{\sqrt{(t^2-1)(t^2+1)}} = \frac{1}{w} \int_{i}^{i} \frac{dt}{\sqrt{(1-t^2)(1+t^2)}}. \]

Hence the solution is

\[ \varphi = wsn(\imath wz, \imath), \quad (2.57) \]

that may be tested by direct differentiation in (2.55).

The invariants

\[ g_2 = w^4, \]
\[ g_3 = 0, \quad (2.58) \]

alternatively determine the potential in terms of the Weierstrass function,

\[ \phi(z, X) = w + \frac{w^3}{\varphi(z; w^4, 0) - w^2 / 2}. \quad (2.59) \]
or, if take into account he relation
\[
\phi(z; w^4, 0) = e_3 + \frac{e_1 - e_3}{sn^2(\sigma z)}, \quad \sigma = \sqrt{e_1 - e_3}, \quad k = \frac{\sqrt{e_2 - e_3}}{e_1 - e_3}
\] (2.60)
one has for \(e_3 = 0, e_1 = -e_2 = w^2/2\), the parameters of the solutions in Jacobi terms \(k = i, \sigma = w/\sqrt{2}\)

\[
\phi(z) = \sigma \frac{sn[\sigma (z - z_0)]dn[\sigma (z - z_0)]}{cn[\sigma (z - z_0)]}.
\] (2.61)
This form coincideds with one from [5]. We will use the expression of the potential in terms of the solution (2.57)

\[
V''(\phi_0(x)) = \left(\frac{\phi^4}{2}\right)^{''} = 6\phi^2 = -6\sigma^2 \left(1 - cn^2(\sigma z, i)\right).
\] (2.62)

3 The generalized zeta-function via Hermit equation

In a spirit of Hermit approach, see, e.g. [24], the function \(\hat{g}_L(p, x, x) = \int \exp[-pt]g_L(t, x, x)dt\) is a solution of bilinear equation

\[
2GG'' - (G')^2 - 4(u(x) + p)G^2 + 1 = 0, \quad G(p, x) = \hat{g}_L(p, x, x)
\] (3.63)
which in a case of reflectionless and finite-gap solutions is solved more effectively than (4.91). It is possible to cover all necessary classes of solutions of the models ( A,B for SG, C,D for \(\phi^4\) case and \(D_0\) for Nahm, via the universal representation by means of polynomials (in p) \(P, Q\)

\[
G(p, x) = P(p, z)/2\sqrt{Q(p)},
\] (3.64)
where

\[
z = sech^2(bx)
\]
for kinks A,C, and

\[
z = cn^2(bx; k)
\]
for the periodic B,D.

\[
b^2(\rho(z)(2P'' - (P')^2) + \rho'(z)PP' - (p + u(z))P^2 + Q = 0,
\] (3.65)
the primes denote derivatives with respect to z, while

\[
\rho = \begin{cases} 
 z^2(1 - z), cases A, C; \\
 z(1 - z)(1 - k^2 + k^2 z), cases D_0, B, D.
\end{cases}
\] (3.66)

\[
u(z) = \begin{cases} 
b^2(1 - 2z), case (A) \\
b^2(2k^2 - 1 - 2k^2 z), case (B) \\
2b^2(2 - 3z), case (C) \\
b^2(5k^2 - 1 - 6k^2 z), case (D) \\
-6b^2 (1 - z) case (D_0)
\end{cases}
\] (3.67)
Substituting \((3.69)\) into \((3.65)\) gives for each power of \(p = 0, 1, 2\)
\[
-2P_1(z) - u(z) + q_2 = 0, \\
b^2(2\rho(z)P''_1 + \rho'(z)P'_1) - P^2_1 - 2u(z)P_1 + q_1 = 0. \\
b^2(\rho(z)(2P_1P''_1 - P''_1^2) + \rho'(z)P_1P'_1 - u(z)P^2_1 + q_0 = 0.
\] (3.68)
respectively.

Let us start with the cases (A,B). The form of the polynomial is determined from well-known facts of the reflectionless potentials theory. The substitution
\[
P = p + P_1(z), \quad Q = p^3 + q_2p^2 + q_1p + q_0
\] (3.69)
into \((3.68)\) results in
\[
P_1 = k^2b^2z, \quad q_2 = b^2(2k^2 - 1), \quad q_1 = b^4k^2(k^2 - 1), \quad q_0 = 0
\] (3.70)
includes (A) in a sense that for the case \(k = 1\) \((3.70)\) gives
\[
P_1 = b^2z.
\] (3.71)

Going to the cases (C,D), generally
\[
P = p^2 + P_1(z)p + P_2(z), \quad Q = p^5 + q_4p^4 + q_3p^3 + q_2p^2 + q_1p + q_0
\] (3.72)
A substitution of \((3.72)\) into the Hermite equation splits in the system
\[
-2P_1 - u + q_1 = 0, \\
-2P_2 - P^2_1 - 2uP_1 + b^2(2\rho P''_1 + \rho' P'_1) + q_3 = 0, \\
b^2(\rho(2P''_2 + 2P_1P''_1 - (P'_1)^2) + \rho'(P'_2 + P'_1P'_1) - 2P_1P_2) - u(2P_2 + P^2_1) + q_2 = 0, \\
b^2(2\rho(2P''_1P_2 - P'_1P'_2 + P'_1P''_2) + \rho'(P'_1P'_2 + P'_1P'_2) - P^2_2 - 2uP_1P_2 + q_1 = 0, \\
b^2(\rho(2P''_2P_2 - P''_2^2) + \rho'P''_2P'_2) - uP^2_2 + q_0 = 0.
\] (3.73)
The potentials \(u(z) = c - 6b^2k^2z\), yields for the case D
\[
P_1(z) = \alpha z + \beta, \quad P_2(z) = a'z^2 + b'z + c',
\] (3.74)
where
\[
\alpha = 3k^2b^2, \\
\beta = 3b^2, \\
a' = 18b^4k^4, \\
b' = -3b^2k^2(11b^2k^2 - q_4 - b^2), \\
c' = 63b^4k^4/4 - 5b^2k^2q_4/2 + b^2q_4/4 - q_3^2/4 + 9b^4k^2/2 + 3b^4/4,
\] (3.75)
are functions only of \(k, b\), where
\[
q_0 = 0, q_1 = -27k^2(1 - k^2)^2b^8, \\
q_2 = -9b^6(k^2 + 1)(k^4 - 4k^2 + 1), q_3 = 3b^4(1 + 9k^2 + k^4), q_4 = 5b^2(1 + k^2).
\] (3.76)
Finally,
\[
Q = \prod_{i=1}^{i=5}(p - p_i),
\] (3.77)

where the polynomial \( Q \) simple roots \( p_i \) are ordered so that \( 0 < k < 1 \)
\[-(2\sqrt{1-k^2} + k^4 + 1 + k^2)b^2 < -3b^2 < -3k^2b^2 < 0 < -(2\sqrt{1-k^2} + k^4 - 1 - k^2)b^2. \quad (3.78)\]

For the cases \( (D_0, C, D) \) we begin from that it is the result of substitution of \( P_1 \) from the first equation of \( (3.68) \) into the second one. Next, the third equation yields
\[ Q = p(p^2 + (1 - k^2)b^2)(p - k^2b^2), \quad (3.79) \]
with the simple roots \( p_i \). while in the particular case of \( \phi^4 \) \( (C) \)
\[ q_0 = 0, \quad q_1 = 0, q_2 = 36b^6, \quad q_3 = 33b^4, q_5 = 10b^2. \]
The arguments in \( (3.73) \) are omitted.

Let us pick up the expressions determining \( \hat{\gamma}(p) \):
\[ \hat{\gamma}(p) = \int P(z) \frac{dz}{2\sqrt{Q}} = \frac{p^2}{2\sqrt{Q}} \int \alpha z + \beta \frac{dx}{2\sqrt{Q}} + 1 \int (p'z^2 + b'z + c') \frac{dx}{2\sqrt{Q}}. \quad (3.80) \]

The case of Nahm equation yields \( q_4 = 0, \quad q_3 = -21b^4, q_2 = 0, q_1 = 108b^8, q_0 = 0, \) hence \( P_1(z) = -3b^2(z - 1), P_2 = 18b^6z^2 - 36b^6z \).
\[ Q(p) = \prod_{i=1}^{i=5} (p - p_i) = p(p + 3b^2)(-p + 3b^2)(2\sqrt{3b^2} - p)(2\sqrt{3b^2} + p), \quad (3.81) \]

where the polynomial \( Q \) simple roots \( p_i \) are easily ordered for real \( b \).

So we need three integrals over the period \( K \).
\[ \int_0^K dx, \int_0^K z dx, \int_0^K z^2 dx. \quad (3.82) \]

Let us go to the variable \( z, \) \( dz = d(cn^2(x)) = -2cn(x)sn(x)dn(x)dx, \) a bit more convenient to put \( y = 1 - z, dy = -dz \)
\[ \int_0^K dx = \int_0^1 \frac{dy}{y(1-y)(1-k^2y)} = 2K(k), \]
\[ \int_0^K sn^2(x; k)dx = \frac{1}{2} \int_0^1 \frac{y^2 dy}{\sqrt{y(1-y)(1-k^2y)}} = \frac{K(k) - E(k)}{k^2}, \]
\[ \int_0^K sn^4(x; k)dx = \int_0^1 \frac{y^2 dy}{\sqrt{y(1-y)(1-k^2y)}} = \frac{1}{3k}((2 + k^2)K(k) - 2(1 + k^2)E(k)). \quad (3.83) \]

hence
\[ \hat{\gamma}(p) = 2K(k) \frac{p^2}{2\sqrt{Q}} + \alpha(-\frac{K(k) - E(k)}{k^2} + (\beta - \alpha)2K(k) \frac{p}{2\sqrt{Q}} + (a'(-2 + k^2)K(k) - 2(1 + k^2)E(k))) + (b' - 2a') \frac{K(k) - E(k)}{k^2} + (a' - b' + c')2K(k) \frac{1}{2\sqrt{Q}}. \quad (3.84) \]

or, finally
\[ \zeta(s) = \int_0^\infty (\int \hat{\gamma}(p)e^{pt} dp) t^{s-1} dt / \Gamma(s); \quad (3.85) \]

which is expressed via integrals
\[ \int_0^{s\sqrt{Q}} dp \]
\[ \int_0^{2\sqrt{Q}} dp \]
\[ \int_0^{3\sqrt{Q}} dp \]
\[ \int_0^{4\sqrt{Q}} dp \quad (3.86) \]
by a contour that contains all branch points of the integrands (inverse Laplace transform). Or, via
\[
\int_0^\infty e^{pt} t^{s-1} dt = \int_0^\infty e^{t(p)^{s-1} \frac{d}{dp} t} = -\frac{1}{(-p)^{s-2}} \int_0^\infty e^{-t} t^{s-1} dt = -\frac{1}{(-p)^{s}} \Gamma(s),
\] (3.87)

Imp < 0, one arrives at
\[
\zeta(s) = -\int_l \frac{\hat{\gamma}(p)}{(-p)^{s}} dp.
\] (3.88)

In the case of Nahm the mass depends on b:
\[
\zeta(s) = -\int_l \frac{1}{(-p)^{s}} \frac{2K(i)p^2 + 3b^2(K(i) - E(i))p - 48b^4K(i)}{2\sqrt{p+3b^2}(-p+3b^2)(2\sqrt{3b^2} - p)(2\sqrt{3b^2} + p)} dp.
\] (3.89)

The result is given by hyperelliptic integral that can be evaluated numerically or via elliptic functions if the symmetry of integrand is taken into account.

4 The generalized zeta-function as a combination of Baker-Achiezer functions.

Let us consider the Green function defined in the Sec.1 by the relation (1.19). The method we use here is based on the technique of finite-gap integration of the KP equation [25, 26, 27]. The source part of the problem is one-dimensional version of the heat equation (1.22) with the potential \( u(x) = V''(\phi(x)) \)
\[
\left( \frac{\partial}{\partial t} - \frac{d^2}{dx^2} + u(x) \right) G(t, x, x_0) = \delta(t) \delta(x - x_0).
\] (4.90)

Let us integrate the equation (4.90) by x over \([x_0 - \epsilon, x_0 + \epsilon]\), hence
\[
\lim_{\epsilon \to 0} \int_{x_0 - \epsilon}^{x_0 + \epsilon} \left( \frac{\partial}{\partial t} - \frac{d^2}{dx^2} + u(x) \right) G(t, x, x_0) dx = \lim_{\epsilon \to 0} \frac{dG(t,x_0 + \epsilon,x_0)}{dx} \delta(t) \delta(x - x_0) dx = \delta(t).
\] (4.91)

The function \( G(t, x, x_0) \) is supposed to be continuous at \( t > 0 \).

Solutions of the equation (4.90) with the zero r.h.s. are expressed in terms of the Baker-Achiezer functions \( \psi_s(x, t; P) \) built by the polynomial \( q(s) = sx + s^2t \) of local parameter \( s \) [25, 27]. Consider a Riemann surface of genus \( g \) and a theta function on it. Then there are holomorphic differentials \( \omega_k, \ k = 1, ..., g \), normalized as
\[
\int_{a_i} \omega_k = 2\pi i \delta_{ik}.
\] (4.92)

The non-special poles divisor on the surface is denoted as \( D = P_1 + ... + P_g \) and the matrix of periods is
\[
B_{jk} = \int_{b_j} \omega_k,
\] (4.93)
that define the general Riemann Theta:

$$\Theta(z) = \Theta(z|B) = \sum_{N \in \mathbb{Z}^g} \exp[(BN \cdot N)/2 + (N \cdot z)]. \quad (4.94)$$

The parametrization by the polynomial \(q(s)\) yields

$$\psi_s(x, t; P) = C(P) \exp(\int_{P_{\infty}}^{P} [d\Omega^{(1)} x + d\Omega^{(2)} t]) \frac{\Theta(xU + tV + A(P) + D)}{\Theta(xU + tV + D)}, \quad (4.95)$$

where \(A(P) = \int_{P_{\infty}}^{P} \omega\) is Abelian map, \(C(P)\) is a constant and the vectors of \(b\)-periods

$$U_i = \int_{b_i} d\Omega^{(1)}, \quad V_i = \int_{b_i} d\Omega^{(2)} \quad (4.96)$$
define the argument of the \(\Theta\)-function. The link between the potential \(u(x)\) and \(\psi\) is given by

$$u(x) = -2 \frac{\partial^2}{\partial x^2} \ln \Theta(xU + tV + D) \quad (4.97)$$

which is recognized as general Matveev-Its formula [25].

So the parameters of the potential and the Green function arise from (4.96) as the position of the cycles \(b_i\) depends on parameters of the potential via the curve that determine the Riemann surface.

Applying the classical method, one builds the Green function which has the form prescribed by (4.91) via the linear independent solution \(\psi_{-s}(x, t; P)\) as:

$$G_0(t, x, x_0) = \frac{1}{M} \psi_s(x, t; P) \psi_{-s}(x_0, t; P) \quad (4.98)$$

that account the unit jumps of the first derivative with respect to \(x\) at \(x = x_0\) and the Wronskian \(M = \psi_s \psi_s'_{-s} - \psi_{-s} \psi_s'\) is determined by a normalization at the borders of the period.

The boundary condition at \(t = 0\) is satisfied if integrate (4.98) with respect to \(s\) along a contour \(C\) determined by the spectrum of the operator \(D_1\)

$$\int_C ds \frac{1}{M} \psi_s(x, t; P) \psi_{-s}(x_0, t; P). \quad (4.99)$$

The zero value at \(t < 0\) is guaranteed by the theorem on Laplace transform [7].

The diagonal values of the Green function are given by

$$G(t, x, x) = \int_C \frac{dM}{M} \psi_s(x, t; P) \psi_{-s}(x, t; P) = \int ds \exp(\int_{P_{\infty}}^{P} d\Omega + \int_{P_{\infty}}^{P_-} d\Omega) \left( \frac{\Theta(xU + tV + A(P) + D)}{\Theta(xU + tV + D)} \right) \left( \frac{\Theta(xU + tV + A(P_-) + D)}{\Theta(xU + tV + D)} \right) \quad (4.100)$$

\(P_-\) corresponds to \(q_{-s} = -sx + s^2 t\). The function \(\gamma(t)\) and, next, the zeta-function are immediately written as [1.21].

The case of SG illustrates the idea in the simplest manner. We start from the Lame equation,

$$\left(-\frac{d^2}{du^2} + 2\varphi(u)\right) \Psi = H \Psi, \quad (4.101)$$
which solutions
\[
\Psi = \frac{\sigma(u + \varphi^{-1}(-H))}{\sigma(u)} \exp[\varsigma(\varphi^{-1}(-H))u], \tag{4.102}
\]
form a complete set (see e.g. \[27, 31\]). The linear independent solution is obtained by the change of the sign before \(\varphi^{-1}(-H))\).

The equation (4.101) is directly connected with one with the cnoidal potential \[1, 24\]
\[
\left(- \frac{d^2}{dz^2} + 2k^2 - 2k^2 \text{cn}(z, k)^2\right) \Psi = h \Psi, \tag{4.103}
\]
where \(z = iK' + u\sqrt{e_1 - e_3}, H = (e_1 - e_3)h + 2e_3, k^2 = \frac{e_2 - e_3}{e_1 - e_3}, e_1 = (2 - k^2), e_2 = (2k^2 - 1), e_3 = -(1 + k^2), \) and v.v. \(e_i = \varphi(\omega_i), \omega_1 = \omega, \omega_2 = \omega - \omega', \omega_3 = \omega'. \) The link between Weierstrass and Jacobi functions
\[
\text{sn}^2(w, k) = \frac{e_1 - e_3}{\varphi(z) - e_3}, \quad w = z\sqrt{e_1 - e_3}, \tag{4.104}
\]
is obviously used. It allows to express a solution of (4.103) as
\[
\Psi = \frac{\sigma(z - iKe_1/e_3 + \varphi^{-1}((e_1 - e_3)h + 2e_3))}{\sigma(z - iKe_1/e_3)} \exp[\frac{z - iK'}{\sqrt{e_1 - e_3}}\varsigma(\varphi^{-1}((e_1 - e_3)h + 2e_3))], \tag{4.105}
\]
e_1 - e_3 = 3, \varphi^{-1}(H) = \rho, h = \frac{\varphi(\rho) - 2e_3}{e_1 - e_3}, u = \frac{z - iKe_1/e_3}{\sqrt{e_1 - e_3}} \quad K'(k') \) is the complete elliptic integral. The parameters are expressed in terms of half-periods of the Weierstrass function, defined by the curve with the uniformization \((\varphi'(u))^2 = 4(\varphi(u))^3 - g_2\varphi(u) - g_3\).

It is convenient to perform numerical calculations using Jacobi theta-function, namely
\[
\sigma(u) = 2\omega \exp[\frac{\eta u^2}{2\omega}] \vartheta(\frac{u}{2\omega}) \quad \eta = \varsigma(\omega), \tag{4.106}
\]
that leads to
\[
\Psi = \exp[\frac{\eta(2u\varphi + \rho^2)}{2\omega}] \vartheta(\frac{u + \rho}{2\omega}).
\]
The particular elliptic solution of the heat equation with the potential as in (4.103) is the product \(\psi(z, t, h) = \exp[-ht]\Psi(z)\).

5 BA function and ”twisted BPS monopoles”

The BA function (Its-Matveev formula) for SG case is expressed in terms of Jacobi theta functions and solves the correspondent Lame equation with \(u_L(x) = 2k^2(1 - cn^2(x, k))\) and the spectral parameter \(h\) \[7\]
\[
\psi_+(x; h) = \exp[\frac{\eta(2u\varphi + \rho^2)}{2\omega}] \vartheta(\frac{u + \rho}{2\omega}), \tag{5.107}
\]
where \(\rho = \varphi^{-1}(3h - 2(1 + k^2)), \eta = \varsigma(\omega), \omega = \varphi^{-1}(2 - k^2), v = (x - iK')/\sqrt{3}, \quad K'(k')\) is the complete elliptic integral, \(\varphi\) is the Weierstrass function, \(\varsigma(\omega)\) is the Weierstrass zeta function \((\varsigma' = -\varphi)\) \[7\]. The potential \(u_L(x)\) of the standard Lamé equation differs from one \(u(x)\) by the
constant factor and shift, namely \( u_L = b^2u + b^2 \), compare with (3.66). In this section we choose for simplicity of formulas \( b = 1 \). Hence the spectral parameters are connected by \( h = p + 1 \).

The basic theta function of the representation (5.107) is defined by

\[
\vartheta(w) = \vartheta(w|\tau) = \sum_{m=-\infty}^{\infty} \exp[i\pi(m^2\tau + 2mw)],
\]

(5.108)

where \( \exp[i\pi\tau] = \eta \), \( Im\ \tau \) should be positive for the series convergence. The series convergence is rapid, therefore the representation (5.107) is convenient for numeric evaluation of the integrals in the zeta formalism.

The Green function \( g_h \) of the spectral Lamé problem may be constructed as a product of two independent solutions \( \psi_+, \psi_- \) of the spectral equation with the same \( h \):

\[
g_h(x, x_0) = \frac{1}{\mathbb{W}} \{ \psi_+(x; h) \psi_-(x_0; h), \quad x < x_0 \\
\psi_-(x; h) \psi_+(x_0; h), \quad x > x_0 \}.
\]

(5.109)

The Wronskian factor \( \mathbb{W} \) is chosen to normalize (5.111) so that account the unit jumps of the first derivative with respect to \( x \):

\[
\lim_{\epsilon \to 0} \frac{d g_h(x_0 + \epsilon, x_0, x_0)}{dx} - \frac{d g_h(x_0 - \epsilon, x_0)}{dx} = -1.
\]

(5.110)

The independent solution \( \psi_-(x, t; h) \) may be chosen antisymmetric with respect to the reflection \( x \to -x \), e.g. defined via \( \vartheta(w + \frac{1}{2}) \).

The boundary condition at \( t = 0 \) may be account via the integral by spectrum (see also (1.19)) of the linear independent solutions product

\[
g(t, x, x_0) = \int \frac{1}{\mathbb{W}} \{ \Psi_+(x, t; h), \Psi_-(x_0, t; h), \quad x < x_0 \}
\Psi_+(x, t; h), \Psi_-(x, t; h), \quad x > x_0 \} dh.
\]

(5.111)

where \( \Psi_+(x, t; h) = \exp[-ht] \psi_+(x; h) \).

Finally we integrate the diagonal values of the Green function \( \gamma_D(t, x) = \int g(t, x, x) \) from (5.111) by the period of a solution and, after that, one can use the definition (3.89) of the generalized zeta-function via (5.111). Integrals dependence on \( z_i \) are expressed via the Weierstrass function of \( w \). It is also known, that

\[
\theta(z + B) = \exp[-\frac{B}{2} - z]\theta(z),
\]

(5.112)

so the ratio of the theta-functions has the period \( B \).

### 6 SG kinks

The results of the previous sections allow to evaluate corrections to actions for all four (A,B,C,D) cases. The results for \( \phi^2 \) model kinks are well-known, see, e.g. [17], where a table for the dimensions \( d=1,2,3,4 \) is listed. These results fit the case B, which hence was strictly verified.

Let us present the formulas for the kinks of the SG model. The substitution of the expressions from (3.64,3.81,3.69,3.71) yields a divergence of the integral for (3.80) \( \hat{\gamma}(p) \). To regularize the Green function let us divide itd Laplace transform into two parts as

\[
G(p, x) = G_c(p) + G_k(p, x),
\]

(6.113)
where the part $G_c(p)$ is the Green function diagonal (the solution of (3.63) for a constant potential:

$$G_c = \frac{1}{2\sqrt{p + b^2}}.$$ (6.114)

The kink part is easily constructed via (3.64, 3.81, 3.69, 3.71):

$$G_k = \frac{b^2 \text{sech}^2(bx)}{2p\sqrt{p + b^2}}.$$ (6.115)

The representation (6.113) results in two contributions of the quantum corrections to the kink (antikink) mass.

The first one coincides with (6.116) for $\nu = m^2$.

$$\gamma_{D_0}(t) = (4\pi)^{\frac{d}{2}} \frac{\Gamma(s + \frac{1-d}{2})}{\Gamma(s)} m^{d-1-2s},$$ (6.116)

and the second one is obtained by the general scheme, namely

$$\gamma_k(p) = \int_{-\infty}^{\infty} G_k(p, x)dx.$$ (6.117)

Plugging (6.115) into (6.117) yields

$$\gamma_k(p) = \frac{b}{p\sqrt{p + k^2}}.$$ (6.118)

The value of the integral $\int_{-\infty}^{\infty} \text{sech}^2(bx) = 2/b$ is taken into account. The corresponding function (1.21) or, more exactly (??) will be denoted $\gamma_R^k$, the index R label the result of a renormalization provided by the division (6.113). The function may be found directly from a table ([29]) but we would explain the result as an example for further development of the renormalization procedure. Note that the cut for the radical $\sqrt{p + b^2}$ is made along the Re p-axis from $-\infty$ to $-b^2$, the branch with $\sqrt{p + b^2} = i\sqrt{|p + b^2|}$ on the upper and $\sqrt{p + b^2} = -i\sqrt{|p + b^2|}$ on the lower bounds of the cut is chosen. After the deformation of the integral contour one has for $t > 0$

$$\gamma_L^R = 1 - \frac{b}{\pi} \int_0^{\infty} \frac{\exp(\xi + b^2)t}{(\xi + b^2)\sqrt{\xi}} d\xi.$$ (6.119)

The formula (6.119) gives an expansion of the resolvent of the operator $\partial_t + L$ by the operator $L$ spectrum. The direct substitution of (6.119) into (?) leads to a divergent integrals. However in this case a renormalization is not necessary, the integral in the r.h.s. of (6.119) is expressed in terms of the error function

$$\text{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\tau^2} d\tau.$$

namely, after some change of variables

$$\gamma_k(t) = \text{Erf}(b\sqrt{t}) = \frac{2b\sqrt{t}}{\sqrt{\pi}} \int_0^1 \exp[-b^2\tau^2]d\tau.$$ (6.120)
The same result gives [29]. The Mellin transform of this representation gives the following expression of zeta function of the operator $L$.

$$\zeta^R_L(s) = \frac{2b^{-2s} \Gamma(s + 1/2)}{\sqrt{\pi}} \int_0^1 \tau^{-2s-1}d\tau = -\frac{b^{-2s} \Gamma(s + 1/2)}{\sqrt{\pi} \Gamma(s + 1)}.$$  \hfill (6.121)

The integral in (6.121) converged only at $\text{Res} < 0$ but gives the analytical continuation for all $s \in C$ excluding the poles of $\Gamma(s + 1)$.

Substituting the result (6.121) into one arising from (1.25) with account of (6.116) yields

$$\zeta^D_L(s) = -4(4\pi)^{\frac{d}{2}} m^{d-2s-1} \frac{\Gamma(s + 1 - d/2)}{(2s + 1 - d)\Gamma(s)}.$$ \hfill (6.122)

The condition $b = m$ is taken into account. Differentiation of (6.122) by $s$ at the point $s = 0$ gives the desired correction

$$-\frac{1}{2} \frac{d(\zeta^D_L(s))}{ds} =$$

$$-2(4\pi)^{\frac{d}{2}} m^{d-2s-1} \frac{\Gamma(s - \frac{1}{2}d + 1)}{\Gamma(s)(2s - d + 1)^2} \left((d - 2s - 1) \left(\text{Psi}\left(-\frac{1}{2}d + s + 1\right) - 2 \ln m - \text{Psi}(s)\right) + 2\right)$$ \hfill (6.123)

next we plot the dependence of the correction $\frac{d(\zeta^D_L(0))}{ds}$ on $m$ (Fig 1).

We would remind about the choice of the constant $g=2$ in the last sections.

### 7 Conclusion

The integrals in the elliptic case are evaluated numerically by means of rapidly converging series for theta-functions.
It is known [35] that it is possible to form a periodic solution of a soliton model by a shift operation in the complex plane of the soliton (kink) parameter.

In [37] the authors study the diffusion of kinks. The Ginzburg-Landau (GL) model is very popular in different aspects of solid state physics, e.g. for magnetics [36]. Some recent papers open new field for applications [32, 16, 34].

The investigation of dislocations dynamics by means of FK model gives a direct possibility to check quantum soliton effects separating kink and elliptic solitons contribution via the energy dependence on parameters [37]. It is interesting to incorporate our results in a real crystal thermodynamics via statistical physics approach [38] or a direct echo response evaluation with correspondent measurements [39]. In a review [38] a quantization contribution is already discussed. There also direct simulations of kink-antikink pairs [40].

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