Tannaka Reconstruction for Crossed Hopf Group Algebras

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Abstract

We provide an analog of Tannaka Theory for Hopf algebras in the context of crossed Hopf group coalgebras introduced by Turaev. Following Street and our previous work on the quantum double of crossed structures, we give a construction, via Tannaka Theory, of the quantum double of crossed Hopf group algebras (not necessarily of finite type).

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Introduction

Turaev [25, 26] generalized Reshetikhin-Turaev [19] invariants and the notion of TQFT to the case of 3-manifolds endowed with a homotopy class of maps to $K(G, 1)$, where $G$ is a discrete group (see also Lê and Turaev [16] and Virelizier [28]). One of the key points in [26] is the notion of crossed Hopf $G$-coalgebra. In the same way as one can use categories of representations of modular Hopf algebras to construct Reshetikhin-Turaev invariants of 3-manifolds, one can use categories of representations of modular crossed Hopf $G$-coalgebras to construct homotopy invariants of maps from 3-manifolds to the Eilenberg-Mac Lane space $K(G, 1)$. Similarly, $G$-coalgebras are used by Virelizier [28] to construct homotopy invariants which generalize Hennings invariant of 3-manifolds.

Roughly speaking, a crossed Hopf $G$-coalgebra $H$ is a family $\{H_\alpha\}_{\alpha \in G}$ of algebras endowed with a comultiplication $\Delta_{\alpha, \beta}: H_{\alpha \beta} \to H_\alpha \otimes H_\beta$, a counit $\varepsilon: k \to H_1$ (where 1 is the unit of $G$), and an antipode $s_\alpha: H_\alpha \to H_{\alpha^{-1}}$. It is also required that $H$ is endowed with a family of algebra isomorphisms $\varphi_{\beta}: H_\alpha \to H_{\beta \alpha \beta^{-1}}$, the conjugation, compatible with the above structures and
such that \( \phi_{\beta\gamma} = \phi_{\beta} \circ \phi_{\gamma} \). If \( G = 1 \), then we recover the usual definition of a Hopf algebra. A \textit{universal R-matrix} and a \textit{twist} for a \( H \) are, respectively, families \( R = \{ \zeta(\alpha) \otimes \zeta(\beta) = R_{\alpha,\beta} \in H_\alpha \otimes H_\beta \}_{\alpha,\beta \in G} \) and \( \theta = \{ \theta_\alpha \in H_\alpha \}_{\alpha \in G} \) satisfying axioms that explicitly involve the conjugation. Properties of \( G \)-coalgebra are studied in [27].

In this article, following the constructions given, in the case of a Hopf algebra, by Joyal and Street [10], we consider which conditions a functor \( F \) from a tensor category \( C \) to the category of complex vector spaces should satisfy to obtain a crossed group Hopf algebra \( H \) such that \( C \) will be isomorphic to the category of \( H \)-comodules. In doing that, we extend Tannaka Theory to the crossed case. By using our generalization [29] of Joyal and Street’s center construction [12], we provide a construction of the quantum co-double of a crossed Hopf group coalgebra similar to the construction given by Street in the case of a Hopf algebra [23]. More in detail, starting from a crossed Hopf group coalgebra \( H \), we construct a coquasitriangular crossed Hopf group coalgebra \( D'(H) \) such that the category of comodules of \( D'(H) \) is equivalent to the center of the category of comodules of \( D(H) \). In particular, if \( D'(H) \) is of finite type (i.e. if every \( D'(H)_\alpha \) is finite-dimensional), then it is isomorphic to the dual of \( D(H^*) \), where \( H^* \) is the dual of \( H \) and \( D(H^*) \) is the generalization of Drinfeld quantum double [4] introduced in [30].

We should observe that, once the correct axioms are fixed, then the proofs follow in a relatively simple way by generalizing the standard theory. However, to determinate the precise conditions for a crossed category to be Tannakian is a necessary step preliminary to other projects like, for instance, the quantization of the crossed structures (which is discussed in an article in preparation).

1 Crossed Group Categories

We recall some basic definitions about crossed structures, in particular the definition of a crossed group category.

1.1 Basic Definitions

A \textit{Crossed G-category} \( C \) is given by the following data:

- a tensor category \( C \);
- a family of subcategories \( \{ C_{\alpha} \}_{\alpha \in G} \) such that \( C \) is disjoint union of this family and that \( U \otimes V \in C_{\alpha\beta} \), for any \( \alpha, \beta \in G, U \in C_{\alpha}, \) and \( V \in C_{\beta} \);
- a group homomorphism \( \Phi: G \to \text{Aut}(C); \beta \mapsto \Phi_{\beta} \), the \textit{conjugation} (where \( \text{Aut}(C) \) is the group of invertible strict tensor functors from \( C \) to itself) such that \( \Phi_{\beta}(C_{\alpha}) = C_{\beta\alpha^{-1}} \) for any \( \alpha, \beta \in G \).

Given \( \alpha \in G \), the subcategory \( C_{\alpha} \) is said the \( \alpha \)-th \textit{component} of \( C \) while the functors \( \Phi_{\beta} \) are said \textit{conjugation isomorphisms}. Given \( \beta \in G \) and an object \( V \in C_{\beta} \), the functor \( \Phi_{\beta} \) is denoted by \( V(\cdot) \), as in [20], or even by \( \beta(\cdot) \). The crossed \( G \)-category \( C \) is \textit{strict} when it is strict as a tensor category. When \( G = 1 \) we recover the usual definition of a tensor category.

We say that a crossed \( G \)-category \( C \) is \textit{abelian} if
each component \( \mathcal{C}_\alpha \) is an abelian category,

- for all objects \( U \) in \( \mathcal{C} \) both the functor \( U \otimes \cdot \) and \( \cdot \otimes U \) are both additive and exact, and

- for all \( \beta \in G \), the functor \( \varphi_\beta \) is both additive and exact.

Let \( \mathcal{D} \) be an abelian tensor category. A functor \( F: \mathcal{C} \to \mathcal{D} \) is exact if each functor \( F_\alpha: \mathcal{C}_\alpha \to \mathcal{D} \) is exact.

**Remark 1.1.** Let us recall the definition of the category of crossed \( G \)-sets \([8,9,12]\) (see also \([3]\), Example 9), also called \( G \)-automorphic sets \([2]\) or \( G \)-racks \([7]\).

A \( G \)-set is a set \( X \) together with a left action \( G \times X : (\alpha,x) \mapsto \alpha x \) such that \( (\alpha \beta)x = \alpha(\beta x) \) and \( 1x = x \) for all \( \alpha, \beta \in G \), and \( x \in X \). A crossed \( G \)-set is a \( G \)-set \( X \) together with a function \( | \cdot | : X \to G \) such that \( |\alpha x| = \alpha |x| \alpha^{-1} \), for all \( \alpha \in G \) and \( x \in X \). A morphism of crossed \( G \)-sets \( f: X \to Y \) is a function such that \( f(\alpha x) = \alpha f(x) \) and \( |f(x)| = |x| \), for all \( \alpha \in G \) and \( x \in X \). The category \( \mathcal{E} \) of crossed \( G \)-sets is a tensor category with the Cartesian product of \( G \)-sets \( X \otimes Y \) and by setting \( |(x,y)| = |x||y| \), for all \( x \in X \) and \( y \in Y \). The category \( \mathcal{E} \) is braided with \( c_{X,Y}(x,y) = (|y|x, x) \), for all \( x \in X \) and \( y \in Y \). Finally \( \mathcal{E} \) is balanced with \( \theta_X(x) = |x|x \) for all \( x \in X \). Crossed \( G \)-categories are nothing but monoidal objects in the 2-category \( \text{Cat}(\mathcal{E}) \) of categories in \( \mathcal{E} \) (the 2-category of objects in \( \mathcal{E} \) introduced by Ehresmann \([5,6]\) is discussed, for instance, in \([11]\)). In particular, the definitions of braided (balanced etc.) crossed \( G \)-category provided below agree with the general definitions provided in \([12]\) and \([3]\).

### 1.2 Duals

A **left autonomous crossed \( G \)-category** \( \mathcal{C} = (\mathcal{C}, (\cdot)^*) \) is a crossed \( G \)-category \( \mathcal{C} \) endowed with a choice of left duals \((\cdot)^*\) which are compatible with the conjugation in the sense that if \( U^* \) is a left dual of \( U \) via \( d_U: U^* \otimes U \to I \) and \( b_U: U \otimes U^* \to I \), then we have

\[
\Phi_\beta(b_U) = b_{\Phi_\beta(U)} \quad \text{and} \quad \Phi_\beta(d_U) = d_{\Phi_\beta(U)}
\]

for any \( \beta \in G \) and \( U \in \mathcal{C} \). Notice that a duality \((b_U, d_U)\) for an object \( U \) in \( \mathcal{C}_\alpha \) induces a duality on all \( \Phi_\alpha(U) \) compatible with the previous equation id and only if the equation

\[
(b_U, d_U) = (\Phi_\beta(b_U), \Phi_\beta(d_U))
\]

is satisfied for all \( \beta \) which commute with \( \alpha \).

Similarly, it possible to introduce the notion of a right autonomous crossed \( G \)-category. An **autonomous crossed \( G \)-category** is a crossed category that is both left and right autonomous. When \( G = 1 \) we recover the usual definition of (left/right) autonomous tensor category \([11]\).

### 1.3 Braiding

A **braiding** in a crossed \( G \)-category \( \mathcal{C} \) is a family of isomorphisms

\[
c = \left\{ c_{U,V} \in \mathcal{C} \left[ U \otimes V, (U \otimes V) \otimes U \right] \right\}_{U,V \in \mathcal{C}}
\]
satisfying the following conditions.

- For all arrows \( f \in \mathcal{C}_\alpha(U,U') \) \( g \in \mathcal{C}(V,V') \) we have
  \[
  \left( (\alpha g) \otimes f \right) \circ c_{U,V} = c_{U',V'} \circ (f \otimes g). \tag{1a}
  \]

- For any \( U, V, W \in \mathcal{C} \) we have
  \[
  c_{U \otimes V,W} = a_{U,V,W} \circ (c_{U,V} \otimes V) \circ a_{U,V,W}^{-1} \circ (U \otimes c_{V,W}) \tag{1b}
  \]
  and
  \[
  c_{U,V \otimes W} = a_{U,V,W} \circ (U \otimes c_{V,W}) \circ a_{U,V,W}^{-1}. \tag{1c}
  \]

- For any \( U, V \in \mathcal{C} \) and \( \beta \in G \) we have
  \[
  \Phi_\beta(c_{U,V}) = c_{\Phi_\beta(U),\Phi_\beta(V)}. \tag{1d}
  \]

A crossed \( G \)-category endowed with a braiding is called a \textit{braided crossed} \( G \)-category. When \( G = 1 \), we recover the usual definition of a braided tensor category \cite{12}.

### 1.4 Twist

A \textit{twist} in a braided crossed \( G \)-category \( \mathcal{C} \) is a family of isomorphisms

\[
\theta = \left\{ \theta_U : U \to \mathcal{U}(U) \right\}_{U \in \mathcal{C}}
\]
satisfying the following conditions.

- \( \theta \) is \textit{natural}, i.e. for any \( f \in \mathcal{C}_\alpha(U,V) \) we have
  \[
  \theta_V \circ f = (\alpha f) \circ \theta_U. \tag{2a}
  \]

- For any \( U \in \mathcal{C}_\alpha \) and \( V \in \mathcal{C}_\beta \) we have
  \[
  \theta_{U \otimes V} = c_{U \otimes V,U' \otimes V} \circ \theta_{U \otimes V} \circ (\theta_U \otimes \theta_V). \tag{2b}
  \]

- For any \( U \in \mathcal{C} \) and \( \alpha \in G \) we have
  \[
  \Phi_\alpha(\theta_U) = \theta_{\Phi_\alpha(U)}. \tag{2c}
  \]

A braided crossed \( G \)-category endowed with a twist is called a \textit{balanced crossed} \( G \)-category. When \( G = 1 \) we recover the usual definition of a balanced tensor category \cite{12}.

A \textit{ribbon crossed} \( G \)-category \( \mathcal{C} \) is a balanced crossed \( G \)-category that is also a left autonomous crossed \( G \)-category such that for any \( U \in \mathcal{C}_\alpha \) (with \( \alpha \in G \)),

\[
\left( (\mathcal{U}(U) \otimes \theta_{U^*}) \circ b_{U,U} \right) \circ b_{U,U} = (\theta_U \otimes U^*) \circ b_{U,U}. \tag{3}
\]

When \( G = 1 \) we recover the usual definition of a ribbon category \cite{18, 24}, also called \textit{tortile tensor category} \cite{12, 13, 22}. 
2 Crossed Hopf Group Algebras

We recall the definition of a crossed Hopf group algebra as in \[30\] and we introduce the notion of a cobraided and a coribbon crossed Hopf group algebra.

2.1 Basic Definitions

A crossed Hopf $G$-algebra $H$ is a family \( \{(H_{\alpha}, \Delta_{\alpha}, \eta_{\alpha})\}_{\alpha \in G} \) of coalgebras endowed with the following data.

- A family of coalgebra morphisms $\mu_{\alpha, \beta}: H_{\alpha} \otimes H_{\beta} \to H_{\alpha \beta}$, the multiplication, that is associative in the sense that we have
  \[
  \mu_{\alpha, \beta, \gamma} \circ (\mu_{\alpha, \beta} \otimes H_{\gamma}) = \mu_{\alpha, \beta, \gamma} \circ (H_{\alpha} \otimes \mu_{\beta, \gamma}).
  \]

If, for all $h \in H_{\alpha}$ and $k \in H_{\beta}$, we set $hk = \mu_{\alpha, \beta}(h, k)$, then \( \mu \) can be simply rewritten as the usual associativity law \((hk)l = h(kl)\) for all $h \in H_{\alpha}, k \in H_{\beta}, l \in H_{\gamma}$.

- An algebra morphism $\eta: k \to H_{1}$, the unit, such that, if we set $1 = \eta(1_k)$, then $1h = h1$ for any $h \in H_{\alpha}$ and $\alpha \in G$.

- A set of coalgebra isomorphisms $\varphi_{\beta} =: H_{\alpha} \to H_{\beta \alpha^{-1}}$, the conjugation, such that
  \[
  - \varphi_{\beta}(hk) = \varphi_{\beta}(h)\varphi_{\beta}(k),
  
  \]
  \[
  - \varphi_{\beta}(1) = 1.
  \]

- A set of linear isomorphisms $S_{\alpha}: H_{\alpha} \to H_{\alpha^{-1}}$, the antipode, such that
  \[
  \mu_{\alpha^{-1}, \alpha} \circ (S_{\alpha} \otimes H_{\alpha}) \circ \Delta_{\alpha} = \eta \circ \varepsilon_{\alpha} = \mu_{\alpha, \alpha^{-1}} \circ (H_{\alpha} \otimes S_{\alpha}) \circ \Delta_{\alpha}.
  \]

The coalgebra $H_{\alpha}$ is called the $\alpha$-th component of $H$. We say that $H$ is of finite type if $\dim H_{\alpha} < \infty$ for all $\alpha \in G$.

2.2 Packed Form of a Crossed Group Algebra

An equivalent definition of crossed Hopf $G$-algebra can be obtained as follows.

Let $H$ be a crossed Hopf $G$-algebra. We obtain a Hopf algebra $H_{pk}$, which we call the packed form of $H$. As a $G$-graded coalgebra, $H_{pk}$ is the direct sum of the components of $H$. Let $i_{\alpha}$ be the inclusion of $H_{\alpha} \hookrightarrow H_{pk}$. The multiplication $\mu_{pk}$ of $H_{pk}$ is, by definition, the colimit of linear morphisms $\lim_{\alpha, \beta \in G} (i_{\alpha \beta} \circ \mu_{\alpha, \beta})$, i.e., the unique linear map from $H_{pk} \otimes H_{pk}$ to $H_{pk}$ such that the restriction to $H_{\alpha} \otimes H_{\beta} \subset H_{pk} \otimes H_{pk}$ coincides with $\mu_{\alpha, \beta}$. The unit $H_{pk}$ is $1_{pk} = 1 \in H_{1} \subset H_{pk}$. The antipode of $H_{pk}$ is the sum $S_{pk} = \sum_{\alpha \in G} S_{\alpha}$. The Hopf algebra $H_{pk}$ is endowed with a group morphism $\varphi: G \to \text{Aut}(H_{pk})$: $\alpha \mapsto \varphi_{pk, \alpha}$, where $\varphi_{pk, \alpha} = \sum_{\beta \in G} \varphi_{\beta}^{\alpha}: \bigoplus_{\beta \in G} H_{\beta} \hookrightarrow \bigoplus_{\beta \in G} H_{\beta}$.

Conversely, let $H_{\text{tot}} = (H_{\text{tot}}, \varphi_{\text{tot}})$ be a Hopf algebra (with multiplication $\mu_{\text{tot}}$, unit 1, and antipode $S_{\text{tot}}$) endowed with a group homomorphism $\varphi_{\text{tot}}: G \to \text{Aut}(H_{\text{tot}})$: $\alpha \mapsto \varphi_{\text{tot}, \alpha}$. Suppose that the following conditions are satisfied.

- There exists a family of sub-coalgebras $\{H_{\alpha}\}_{\alpha \in G}$ of $H_{\text{tot}}$ such that $H_{\text{tot}} = \bigoplus_{\alpha \in G} H_{\alpha}$. 
• $H_\alpha \cdot H_\beta \subset H_{\alpha\beta}$ for all $\alpha, \beta \in G$.

• $1 \in H_1$.

• $\varphi_{\alpha,\beta}$ sends $H_\alpha \subset H_{\text{tot}}$ to $H_{\beta\alpha \beta^{-1}} \subset H_{\text{tot}}$.

• $S_{\text{tot}}(H_\alpha) \subset H_{\alpha^{-1}}$.

In the obvious way $H_{\text{tot}}$ gives rise to a crossed Hopf $G$-algebra $H$ such that $H_{pk} = H_{\text{tot}}$. We say that $H_{pk}$ is the packed form of $H$.

Remark 2.1. By dualizing in obvious way the axioms for a crossed Hopf $G$-algebra one get the dual notion of crossed Hopf $G$-coalgebra, see [26]. In particular, if $H$ is a crossed Hopf $G$-algebra of finite type (i.e. dim $H_\alpha < +\infty$ for all $\alpha \in G$), then, by taking the linear duals of the components of $H$, we get in obvious way a crossed Hopf $G$-coalgebra $H^*$, the dual of $H$. Notice, however, that, in general, it is not possible to describe a crossed Hopf $G$-coalgebra as a graded Hopf algebra.

We say that a crossed Hopf $G$-algebra $H$ is cosemisimple if every component of $H$ is cosemisimple as a coalgebra. It is proved in [27] that, if $K$ is a finite type crossed Hopf $G$-algebra, then the components of $K$ are semisimple algebras in and only if the component $K_1$ is a semisimple Hopf algebra. Therefore, by duality, we obtain that a crossed Hopf $G$-algebra $H$ of finite type is cosemisimple if and only if its component $H_1$ is a cosemisimple Hopf algebra.

2.3 Categories of Comodules

Let $H$ be a crossed Hopf $G$-algebra. We define the crossed $G$-category $\mathcal{Com} H$ of left $H$-comodules as follow.

• $\mathcal{Com}_\alpha H$ is the category of $H_\alpha$-comodules.

• The tensor unit $I$ is the ground field $\mathbb{C}$ with the comodule structure given by the the unit of $H$ (i.e. $c \mapsto c \otimes 1_H$ for all scalars $c$).

• Let $M$ be a $H_\alpha$-comodule with coaction $\Delta_M : M \to H_\alpha \otimes M : m \mapsto m_\alpha \otimes m_M$ and let $N$ be a $H_\beta$-comodule with coaction $\Delta_N : N \to H_\beta \otimes N : n \mapsto n_\beta \otimes n_N$. Define $M \otimes N$ as the vector space $M \otimes \mathbb{C} N$ with the $H_{\alpha\beta}$-comodule structure given by

$$\Delta_{M \otimes N} : m \otimes n \mapsto m_\alpha n_\beta \otimes m_M \otimes n_N.$$  

• The functors $\Phi_\beta$ are defined as follows. Let $M$ be a $H_\alpha$-comodule. The $H_{\beta\alpha \beta^{-1}}$-comodule $\beta M = \Phi_\beta(M)$ is isomorphic to $M$ as a vector space and the element in $\beta M$ corresponding to $m \in M$ is denoted $\beta m$. The coaction of $H_{\beta\alpha \beta^{-1}}$ is obtained by setting

$$(\beta m)_{\beta\alpha \beta^{-1}} \otimes (\beta m)_M = \varphi_\beta(m_\alpha) \otimes (m_M).$$

If $f : M \to N$ is a morphism of $H_\alpha$-comodules, then set

$$\beta f = \Phi_\beta(f) : \beta(M) \to \beta(N) : \beta m \mapsto \beta(f(m)).$$
Lemma 2.2. For all $\alpha, \beta \in G$ the map
\[ \hat{\varphi}_\beta : \beta(H_\alpha) \to H_{\beta\alpha\beta^{-1}} : \beta h \mapsto \varphi_\beta(h) \]
is an isomorphism of $H_{\beta\alpha\beta^{-1}}$-comodules.

The proof is straightforward and left to the reader.

A **cobraided crossed Hopf G-algebra** is a crossed Hopf G-algebra $H$ endowed with a family of linear maps
\[ \gamma_H = \{ \gamma_{\alpha,\beta} : H_\alpha \otimes H_\beta \to \mathbb{C} \} \]
satisfying the following conditions.
- There exists a family of linear maps $\tilde{\gamma}_{\alpha,\beta} : H_\alpha \otimes H_\beta \to \mathbb{C}$ such that, for all $h \in H_\alpha$ and $k \in H_\beta$,
  \[ \tilde{\gamma}_{\alpha,\beta}(h' \otimes k')\gamma_{\alpha,\beta}(h'' \otimes k'') = \gamma_{\alpha,\beta}(h' \otimes k')\tilde{\gamma}_{\alpha,\beta}(h'' \otimes k'') = \varepsilon(h)\varepsilon(k). \quad (5a) \]
- For all $h \in H_\alpha$ and $k \in H_\beta$, we have
  \[ k' h' \gamma_{\alpha,\beta}(h'' \otimes k'') = \gamma_{\alpha,\beta}(h' \otimes k') \varphi_\beta(h'')k''. \quad (5b) \]
- For all $h_1 \in H_{\alpha_1}$, $h_2 \in H_{\alpha_2}$, and $k \in H_\beta$, we have
  \[ \gamma_{\alpha_1\alpha_2,\beta}(h_1 h_2) \otimes k = \gamma_{\alpha_1,\beta}(h_1 \otimes k')\gamma_{\alpha_2,\beta}(h_2 \otimes k''). \quad (5c) \]
- For all $h \in H_\alpha$, $k_1 \in H_{\beta_1}$, and $k_2 \in H_{\beta_2}$, we have
  \[ \gamma_{\alpha,\beta_1\beta_2}(h \otimes (k_1 k_2)) = \gamma_{\beta_2\alpha,\beta_1^{-1}}(\varphi_\beta_2(h'') \otimes k_1)\gamma_{\alpha,\beta_2}(h' \otimes k_2). \quad (5d) \]
- For all $h \in H_\alpha$, $k \in H_\beta$, and $\lambda \in G$, we have
  \[ \gamma_{\lambda^{-1}\alpha,\beta}(\varphi_\lambda(h) \otimes \varphi_\lambda(k)) = \gamma_{\alpha,\beta}(h \otimes k). \quad (5e) \]

**Theorem 2.3.** There is a 1–1 correspondence between braidings in $\mathcal{C}omH$ (as a crossed G-category) and cobraided crossed Hopf G-algebras structures on $H$.

**Proof.** Let $H$ be a cobraided crossed Hopf G-algebra. We obtain a braiding in $\mathcal{C}omH$ by setting, for every $H_\alpha$-comodule $M$ and every $H_\beta$-comodule $M$,
\[ c_{M,N}^\gamma(m \otimes n) = \gamma_{\beta,\alpha}(n_\beta \otimes m_\alpha) \left( \alpha(n_N) \right) \otimes m_M. \]

Conversely, let $c$ be a braiding in $\mathcal{C}omH$. We obtain a cobraided structure of $H$ by setting, for all $\alpha, \beta \in G$,
\[ \gamma_{\alpha,\beta}^c : H_\alpha \otimes H_\beta \stackrel{c_{H_\alpha,H_\beta}}{\longrightarrow} \left( \alpha H_\beta \right) \otimes H_\alpha \stackrel{\varphi_\beta \otimes \varepsilon_\alpha}{\longrightarrow} H_{\alpha\beta\alpha^{-1}} \otimes H_\alpha \stackrel{\varepsilon_{\alpha\beta\alpha^{-1}} \otimes \varepsilon_\alpha}{\longrightarrow} \mathbb{C}, \]
where $\gamma_\alpha$ is defined as in Lemma 2.2. The proof of the theorem is an adaptation of the standard proof for Hopf algebras (see [14] and [10]). \qed
A coribbon crossed Hopf $G$-algebra is a crossed Hopf $G$-algebra $H$ endowed with a cobraided structure $\gamma$ and a family

$$\tau_H = \{ \tau_\alpha : H_\alpha \to \mathbb{C} \},$$
called cotwisted structure, satisfying the following conditions.

- There exists a family of linear maps $\tilde{\tau}_\alpha : H_\alpha \to \mathbb{C}$ such that, for all $h \in H_\alpha$,
  $$\tau_\alpha(h')\tilde{\tau}_\alpha(h'') = \tilde{\tau}_\alpha(h')\tau_\alpha(h'') = \varepsilon_\alpha(h).$$

- For all $h \in H_\alpha$ we have
  $$\tau_\alpha(h')\varphi_\alpha(h'') = h' \tau_\alpha(h'').$$

- For all $h \in H_\alpha$ and $k \in H_\beta$, we have
  $$\tau_{\alpha\beta}(hk) = \gamma_{\beta,\alpha}(k'_{\beta} \otimes h'_{\alpha})\tau_\alpha(h''_{\alpha})(\tau_{\alpha\beta}^{-1} \circ \varphi_\alpha)(k''_{\beta})(\varphi_\alpha \circ \gamma_{\alpha,\beta})(h''_{\alpha} \otimes \varphi_\beta(k''_{\beta})).$$

- For all $\alpha \in G$, we have
  $$\tau_{\alpha}^{-1} \circ S_{\alpha} = \tau_{\alpha}.$$

- For all $\alpha, \beta \in G$,
  $$\tau_{\beta\alpha}^{-1} \circ \varphi_\beta = \tau_\alpha.$$

**Theorem 2.4.** There is a 1–1 correspondence between twist in $\text{Com} H$ and cotwisted structures in $H$.

**Proof.** Let $H$ be a cotwisted crossed Hopf $G$-algebra. We obtain a twist in $\text{Com} H$ by setting, for every $H_\alpha$-comodule $M$,

$$\theta^*_M(m) = \tau_\alpha(m_\alpha)^\alpha(m_M).$$

Conversely, let $\theta$ be a twist in $\text{Com} H$. We obtain a cotwisted structure of $H$ by setting, for all $\alpha \in G$,

$$\tau^\theta_\alpha : H_\alpha \xrightarrow{\theta_{H_\alpha}} H_\alpha \xrightarrow{\hat{\varphi}_\alpha} H_\alpha \xrightarrow{\varepsilon_\alpha} \mathbb{C}.$$

Again, the proof of the theorem is an adaptation of the standard one. \qed

### 2.4 Modularity

We give a definition of modular coribbon Hopf algebra by transposing the usual axioms of a modular (ribbon) Hopf algebra, see e.g. [24] or [14]. Then we introduce the notion of modular coribbon Hopf $G$-algebra.

Let $H_1$ be a coribbon Hopf algebra, let $U$ be a finite-dimensional $H_1$-comodule, and let $f : U \to U$ an endomorphism of $H_1$-comodules. We define the quantum trace of $f$ as the scalar

$$qtr(f) = \sum_{i=1}^{n} \tau_1((e_i)_{H_1})\tilde{\gamma}_{1,1}((e_i')_{H_1} \otimes (e_i')_{H_1})f((e_i)_U)(e_i)_U^*,$$
being \( \{ e_i \}_{i=1}^n \) a basis of \( U \), \( \{ e^i \}_{i=1}^n \) the dual basis \( \gamma_{1,1} \) the inverse of the cobrading of \( H_1 \), and \( \tau_1 \) the coribbon of \( H_1 \) (notice that this is nothing but the usual quantum trace of \( f \) inside the category \( \mathcal{C}omH_1 \)). We say that \( U \) is negligible if \( \text{qtr}(\text{id}_U) = 0 \).

We say that a coribbon Hopf algebra \( H_1 \) is modular if it is endowed with a finite family of simple finite-dimensional \( H_1 \)-comodules \( \{ V_i \}_{i \in I} \) satisfying the following conditions.

- There exists an element \( 0 \in I \) such that \( V_0 = \mathbb{C} \) (with the structure of \( H_1 \)-comodule given by the multiplication).
- For any \( i \in I \), there exists \( i^* \in I \) such that \( V_{i^*} \) is isomorphic to \( V_i^* \).
- For any \( j, k \in I \), the \( H_1 \)-comodule \( V_j \otimes V_k \) is isomorphic to a finite sum of elements of \( \{ V_i \}_{i \in I} \), possibly with repetitions, and a negligible \( H_1 \)-comodule.
- Let \( s_{ij} \) be the quantum trace of the endomorphism

\[
c_{V_{i^*}, V_{j^*}}: V_i \otimes V_{j^*} \to V_i \otimes V_{j^*}, \quad x \otimes y \mapsto \gamma_{1,1}(y_{H_1} \otimes x_{H_1}^*) \gamma_{1,1}(x_{H_1'} \otimes y_{H_1'}),
\]

The matrix \( (s_{ij})_{i,j \in I} \) is invertible.

A coribbon Hopf \( G \)-algebra \( H \) is modular if its component \( H_1 \) is modular. We observe that, if \( \dim H_1 < +\infty \), then \( H_1 \) is modular if and only if \( H_1^* \) is modular in the usual sense. Therefore, if \( H \) is of finite type, then it is modular if and only if \( H^* \) is modular in the usual sense [26, 27]. Also, \( H_1 \) is modular if and only if the category \( \mathcal{C}omH_1 \) is modular in the usual sense.

## 3 Tannaka Theory for Tensor Categories

We recall some basic facts about Tannaka Theory. For a classical reference, see [20]. For a reference about Tannaka Theory and braided tensor categories see [10], of which we follow the approach, or [21]. In a couple of cases we also reproduce a short sketch of the proof since the notation introduced will be used in the sequel.

### 3.1 Dinatural Transformations

Let both \( \mathcal{C} \) and \( \mathcal{B} \) be two small categories and let both \( S \) and \( T \) be functors from \( \mathcal{C}^{\text{op}} \times \mathcal{C} \) to \( \mathcal{B} \). A dinatural transformation \( \alpha: S \to T \) (see, e.g., [17]) is a function that assigns to an object \( C \in \mathcal{C} \) an arrow \( \alpha_C: S(C, C) \to T(C, C) \) in \( \mathcal{B} \) such that, for all arrows \( f: C \to D \) in \( \mathcal{C} \), the diagram

\[
\begin{array}{ccc}
S(C, C) & \xrightarrow{\alpha_C} & T(C, C) \\
S(C) \downarrow \quad & & \downarrow \\
S(D) & \xrightarrow{\alpha_D} & T(D, D)
\end{array}
\]

\[
\begin{array}{ccc}
S(D, C) & \xrightarrow{\alpha_C} & T(D, C) \\
S(D) \downarrow \quad & & \downarrow \\
S(D) & \xrightarrow{\alpha_D} & T(D, D)
\end{array}
\]

\[
\begin{array}{ccc}
S(C, D) & \xrightarrow{\alpha_C} & T(C, D) \\
S(D) \downarrow \quad & & \downarrow \\
S(D) & \xrightarrow{\alpha_D} & T(D, D)
\end{array}
\]

\[
\begin{array}{ccc}
S(D, C) & \xrightarrow{\alpha_C} & T(D, C) \\
S(D) \downarrow \quad & & \downarrow \\
S(D) & \xrightarrow{\alpha_D} & T(D, D)
\end{array}
\]
commutes.

Let $B$ be an object of $\mathcal{C}$, and $S = B$, i.e. the constant functor fixed by $B$. We can rewrite the commutativity of (7) as

$$T(C, f) \circ \alpha_C = T(f, D) \circ \alpha_D.$$ 

We say that $B$ is an end of $T$ if $\alpha$ is universal for the property that, for any dinatural transformation $\alpha': B' \to T$, there exists an arrow $b \in \mathcal{B}(B', B)$ such that the diagram

$$\begin{array}{ccc}
B & \xrightarrow{\alpha} & B' \\
\downarrow b \downarrow & & \downarrow \alpha' \\
T(C, C) & \xrightarrow{\alpha} & T(C, C)
\end{array}$$

commutes for all objects $C \in \mathcal{C}$. If $T$ has an end, then this end is unique up to canonical isomorphism.

Let us consider again diagram (7) and suppose this time that $T = B$ for a fixed object $B \in \mathcal{C}$. We can rewrite the commutativity of (7) as

$$\alpha_C \circ S(f, C) = \alpha_D \circ S(D, f).$$

We say that $B$ is a coend of $S$ if $\alpha$ is universal for the property that for any $\alpha': S \to B'$, there exists an arrow $b \in \mathcal{B}(B, B')$ such that the diagram

$$\begin{array}{ccc}
B & \xrightarrow{\alpha} & B' \\
\downarrow b \downarrow & & \downarrow \alpha' \\
S(C, C) & \xrightarrow{\alpha} & S(C, C)
\end{array}$$

commutes for all $C \in \mathcal{C}$. Again, if $S$ has a coend, then this coend is unique up to canonical isomorphism.

### 3.2 Ends and Coends as Vector Spaces

Let both $X$ and $Y$ be functors from a small category $\mathcal{C}$ to the category $\mathcal{V}ect$ of complex vector spaces.

**Lemma 3.1.** The bifunctor $\text{Hom}_k(X(\underline{C}), Y(\underline{C}))$ has an end $\text{Hom}(X, Y)$.

**Proof (sketch).** To construct $\text{Hom}(X, Y)$, for every morphism $f: A \to B$ in $\mathcal{C}$ define two linear maps

$$p_f, q_f: \prod_{C \in \mathcal{C}} \text{Hom}_C(X(C), Y(C)) \to \text{Hom}_k(X(A), Y(B))$$

by

$$p_f: u = \{u_C|C \in \mathcal{C}\} \mapsto Y(f) \circ u_A,$$

$$q_f: u = \{u_C|C \in \mathcal{C}\} \mapsto u_B \circ X(f),$$

and $\text{Hom}(X, Y)$ as the equalizer of all pairs $(p_f, q_f)$. When $X = Y$ we will use the notation $\text{End}(X) = \text{Hom}(X, Y)$. 

\[\square\]
Lemma 3.2. If both $X$ and $Y$ send all objects to finite dimensional vector spaces, then the bifunctor $\text{Hom}_\mathcal{C}(X(\cdot), Y(\cdot))$ has a coend $\text{Hom}^\ast(X, Y)$ and $\text{Hom}(X, Y)$ is the dual of $\text{Hom}^\ast(X, Y)$.

Proof (sketch). Define $\text{Hom}^\ast(X, Y)$ as the coequalizer of all pairs $(^{t}p_f, ^{t}q_f)$, where $f: A \to B$ is an arrow in $\mathcal{C}$ and

$^{t}p_f, ^{t}q_f: \text{Hom}_\mathcal{C}(X(A), Y(B))^* \longrightarrow \sum_{C \in \mathcal{C}} \text{Hom}_\mathcal{C}(X(C), Y(C))^*$.

When $X = Y$ we will use the notation $\text{End}^\ast(X) = \text{Hom}(X, Y)$. □

Let $U$ and $V$ be two finite dimensional vector spaces. By means of the canonical isomorphisms $\text{Hom}_\mathcal{C}(U, V)^* \cong V^* \otimes_U U \cong \text{Hom}_\mathcal{C}(V, U)$ we have a pairing

$\langle \cdot, \cdot \rangle: \text{Hom}_\mathcal{C}(U, V) \times \text{Hom}_\mathcal{C}(V, U) \to \mathbb{C}: (h, k) \mapsto \text{tr}(hk)$.

By using this, $\text{Hom}^\ast(X, Y)$ can be defined as the common coequalizer of all maps $i_f, j_f: \text{Hom}(Y(B), X(A)) \to \sum_{C \in \mathcal{C}} \text{Hom}(Y(C), X(C))$, where, for any $h \in \text{Hom}_\mathcal{C}(Y(B), X(A))$, we set

$i_f(h) = (h \circ Y(f), A)$ and

$j_f(h) = (X(f) \circ h, B)$.

Here the second component of a pair $(h \circ Y(f), A)$ indicates to which component of the direct sum this element belongs.

For any object $C$ in $\mathcal{C}$ and any map $h \in \text{Hom}_\mathcal{C}(Y(C), X(C))$, let $[h]$ be the image of $h$ under the canonical map $\text{Hom}_\mathcal{C}(Y(C), X(C)) \to \text{Hom}^\ast(X, Y)$. The space $\text{Hom}^\ast(X, Y)$ is generated as a vector space by the symbols $[h]$ subject to the relations

$\bullet \left[ c_1 h_1 + c_2 h_2 \right] = c_1 [h_1] + c_2 [h_2] \text{ for all } h_1, h_2 \in \text{Hom}(Y(C), X(C)), \text{ and } c_1, c_2 \in \mathbb{C}$,

$\bullet \left[ k \circ Y(f) \right] = \left[ X(f) \circ k \right] \text{ for all } f: A \to B \text{ in } \mathcal{C} \text{ and } k \in \text{Hom}_\mathcal{C}(Y(B), X(A))$.

The pairing between $\text{Hom}(X, Y)$ and $\text{Hom}^\ast(X, Y)$ is given by

$\langle \cdot, \cdot \rangle: \text{Hom}(X, Y) \times \text{Hom}^\ast(X, Y) \to \mathbb{C}: (u, [h]) \mapsto \text{tr}(uC \circ h)$,

where $h \in \text{Hom}_\mathcal{C}(Y(C), X(C))$.

3.3 The Coalgebra $\text{End}^\ast(X)$

Let $X: \mathcal{C} \to \mathcal{Vect}$ be a functor whose values are finite dimensional vector spaces. The space $\text{End}^\ast(X) = \text{Hom}^\ast(X, X)$ is a coalgebra as follows.

Recall that for any vector space $V$ the space $\text{End}(V)$ is a coalgebra via

$\delta: \text{End}(V) \to \text{End}(V) \otimes \text{End}(V): e_i^j \mapsto e_i^k \otimes e_j^k$,
where \( e_1, \ldots, e_n \) is a basis of \( V \), \( e_j^i = e_i^* \otimes e_j \), and on the right hand side the sum runs on \( k = 1, \ldots, n \). The counit of \( \text{End}(V) \) is the trace \( \text{tr}: \text{End}(V) \to \mathbb{C} \).

The coalgebra structure on \( \text{End}^\dagger(X) \) is given by

\[
\Delta[e^i_j] = [e^i_k] \otimes [e^k_j],
\]

with counit

\[
\varepsilon[h] = \text{tr}(h),
\]

that is, \( \text{End}^\dagger(X) \) is a quotient of direct sums of coalgebras via the canonical map

\[
\sum_{C \in \mathcal{C}} \text{End}(X(C)) \to \text{End}(X).
\]

The following lemma will be crucial in the next section. Let both \( X: \mathcal{C} \to \mathcal{V}ect \) and \( Y: \mathcal{D} \to \mathcal{V}ect \) be functors whose values are finite dimensional vector spaces. Set

\[
X \otimes Y: \mathcal{C} \times \mathcal{D} \to \mathcal{V}ect: (U, V) \to U \otimes V.
\]

**Lemma 3.3.** The map

\[
m: \text{End}^\dagger(X) \otimes \text{End}^\dagger(Y) \to \text{End}^\dagger(X \otimes Y)
\]

\[
[S] \otimes [T] \mapsto [S \otimes T]
\]

(for any object \( A \) in \( \mathcal{C} \), and \( B \) in \( \mathcal{D} \), and any \( S \in \text{End}(X(A)) \) and \( T \in \text{End}(Y(B)) \)) is a coalgebras isomorphism.

Let both \( X \) and \( Y \) as above. We say that \( F: \mathcal{C} \to \mathcal{D} \) is an equivalence [resp. an isomorphism] of Tannakian categories it is an equivalence [resp. an isomorphism] of categories and \( X = Y \circ F \).

**Lemma 3.4.** The map \( \text{End}^\dagger(X) \to \text{End}^\dagger(Y) : [h] \to [h] \) (for all objects \( C \) in \( \mathcal{C} \) and all \( h \in \text{End}(X(C)) = \text{End}((Y \circ F)(C)) \)) is a morphism of coalgebras and it is an isomorphism if and only if \( F \) is an equivalence of Tannakian categories.

We also need a small variant of the previous lemma.

**Lemma 3.5.** Let \( F: \mathcal{C} \to \mathcal{D} \) be an isomorphism if categories and let \( f: X \to Y \circ F \) be a natural isomorphism. The map \( \text{End}^\dagger(X) \to \text{End}^\dagger(Y) : [h] \to [ f_C \circ h \circ f_C^{-1} ] \) is an isomorphism of coalgebras.

The proof is straightforward and left to the reader.

The main theorem of Tannaka Theory is the following one.

**Theorem 3.6.** Let \( \mathcal{C} \) be an abelian category endowed with a functor \( F: \mathcal{C} \to \mathcal{V}ect \) whose values are finite dimensional vector spaces and which is both exact and faithful. The category \( \mathcal{C} \) is equivalent to the category \( \text{ComEnd}^\dagger(F) \) of finite dimensional \( \text{End}^\dagger(F) \)-comodules. Moreover, if \( \mathcal{C} \) is equivalent as a Tannakian category to \( \text{ComC} \) for another coalgebra \( C \), then \( C \) is canonically isomorphic to \( \text{End}^\dagger(F) \).
4 Tannaka Reconstruction for Crossed Structures

We introduce now the analog of Tannaka Theory in the context of crossed structures.

Let $\mathcal{C}$ be an autonomous abelian crossed $G$-category (again we suppose $\mathcal{C}$ strict, but the results can obviously be generalized to the case of a category equivalent to a strict one). A fiber crossed $G$-functor is a couple $(F, \{\varphi_\beta\}_{\beta \in G})$ such that $F: \mathcal{C} \to \text{Vect}$ is an autonomous tensor functor (i.e., a tensor functor with preserves dualities) whose values are finite dimensional vector spaces and $\varphi_\beta$ satisfying the following conditions.

- $\varphi_{\beta_1} \circ \varphi_{\beta_2} = \varphi_{\beta_1 \beta_2}$ for all $\beta_1, \beta_2 \in G$ and $\varphi_1_G = \text{Id}$.
- If we set $F_\alpha = F|_{\mathcal{C}_\alpha}$, then $\varphi_\beta$ induces a natural isomorphism $F_\alpha \cong F_{\beta \alpha^{-1}} \circ \Phi_\beta$.

We say that $\mathcal{C}$ is a Tannakian crossed $G$-category if it is endowed with a fiber functor. We say that two Tannakian crossed $G$-categories $\mathcal{C}$ with fiber functor $\mathcal{X} \to \mathcal{D}$ are equivalent [resp. isomorphic] if there is an equivalence [resp. an isomorphism] of crossed $G$-categories $F: \mathcal{C} \to \mathcal{D}$ such that $X = Y \circ F$.

**Theorem 4.1.** Let $\mathcal{C} = (\mathcal{C}, F, \{\varphi_\beta\}_{\beta \in G})$ be a Tannakian crossed $G$-category. There exists a crossed Hopf $G$-algebra $H = H(\mathcal{C})$, unique up to isomorphism, such that $\mathcal{C} \cong \text{Com}H$ as Tannakian crossed $G$-categories.

**Proof (sketch).** For all $\alpha \in G$, let us set $H_\alpha = \text{End}(F_\alpha)$. The functor $F_\alpha$ is obviously exact and faithful, so we have $\mathcal{C}_\alpha \cong \text{Com}H_\alpha$. By Lemma 3.6, the conjugation $\Phi$ of $\mathcal{C}$ and the $\varphi$ give rise to a family of isomorphism $\varphi_\beta: H_\alpha \to H_{\beta \alpha^{-1}}$ with the obvious property that $\varphi_1_G = \text{Id}$ and $\varphi_{\beta_1 \beta_2} = \varphi_{\beta_1} \circ \varphi_{\beta_2}$.

Suppose, for simplicity, that $\mathcal{C}$ is strict. Then, by Lemma 3.5, with $X = F_\alpha$ and $Y = F_\beta$, and observing that $F_\alpha(U) \otimes F_\beta(V) = F_{\alpha \beta}(U \otimes V)$ for all objects $U$ and $V$ in $\mathcal{C}$, we get a morphism (which is actually also an isomorphism)

$$m_{\alpha, \beta}: H_\alpha \otimes H_\beta \to H_{\alpha \beta}.$$

It is easy to prove that in that way we obtained a multiplication for $H$ with unit $[1]$, being $1$ the element of $F_{1_G}(I)$ corresponding to $1 \in I$ under the isomorphism $F_{1_G}(I) \cong I$. If $\mathcal{C}$ is not strict, then one can define the product of $[h]$ and $[k]$ (where $h \in \text{End}(F(U))$ and $k \in \text{End}(F(V))$) as the class of the map

$$F(U \otimes V) \cong F(U) \otimes F(V) \xrightarrow{h \otimes k} F(U) \otimes F(V) \cong F(U \otimes V)$$

(notice that exactly the way used by Joyal and Street [10] to endow the coalgebra $H = \bigoplus_{\alpha \in G} H_\alpha$ of a structure of bialgebra).

Now, let $U$ be an object in $\mathcal{C}_\alpha$ and let $U^*$ be its left dual. Since tensor functors preserve the dual pairings, for all $h \in \text{End}(F_\alpha(U))$ there is a transposed endomorphism $^t h \in \text{End}(F_{\alpha^{-1}}(U^*))$. We set

$$s_\alpha([h]) = [^t h].$$

The proof that $s$ is an antipode is then an adaptation of that in [10] (Proposition 5, page 468) and is omitted here.
Finally, by using Lemma 3.5, we get a family of isomorphisms \( \varphi_\beta : H_\alpha \rightarrow H_{\beta_\alpha\beta_1} \). To check that \( \varphi \) satisfies our axioms is routine.

**Corollary 4.2.** If \( \mathcal{C} \) is braided, then \( H \) is coquasitriangular and \( \mathcal{C} \cong \text{Com}H \) as braided crossed \( G \)-categories. If \( \mathcal{C} \) is ribbon, then \( H \) is coribbon and \( \mathcal{C} \cong \text{Com}H \) as ribbon crossed \( G \)-categories.

**Proof.** Suppose that \( \mathcal{C} \) is braided. The equivalence \( E: \text{Com}H \rightarrow \mathcal{C} \) induces a braiding in \( \text{Com}H \) by setting, for all objects \( M \) and \( N \) in \( \text{Com}H \), \( c_{M,N} = E^{-1}(c_{E(M),E(N)}) \). Thus, by Theorem 2.3, \( H \) is coquasitriangular. The proof for the ribbon is similar.

## 5 Quantum Co-double Construction

In [29] the author generalized the center construction for tensor categories [12] to the case of crossed group categories. Stating from any crossed group category \( \mathcal{C} \), the center construction provides a braided tensor category \( Z(\mathcal{C}) \). In similar way, by means of the generalization in [29] of the construction in [15] and [23], starting from a braided crossed \( G \)-category \( \mathcal{D} \), one can construct a ribbon crossed \( G \)-category \( \theta(\mathcal{D}) \). By means of the Tannaka theory for crossed \( G \)-categories, we can use these constructions to recover in the crossed case the results about the quantum co-double of a Hopf algebra discussed in [23]. Let us recall the definition of both the center \( Z(\cdot) \) and the ribbon extension \( \theta(\cdot) \) as in [29].

### 5.1 The Center

Let \( \mathcal{C} \) be a crossed \( G \)-category. The component \( Z_\alpha(\mathcal{C}) = \big( Z(\mathcal{C}) \big)_\alpha \) of \( Z(\mathcal{C}) \) is the category whose objects are couples \((U, \zeta)\), where \( U \) is an object in \( \mathcal{C}_\alpha \) and \( \zeta \) is a natural isomorphism of endomorphic functors in \( \mathcal{C} \)

\[
\zeta_{\cdot} : (U \otimes \cdot) \rightarrow U \otimes U
\]

such that, for all \( V,W \in \mathcal{C} \),

\[
c_{V \otimes W} = \left( (U \otimes V) \otimes \zeta_{\cdot} \right) \otimes (\zeta_{\cdot} \otimes Y).
\]

If both \((U, \zeta)\) and \((V, \vartheta)\) are objects in \( Z_\alpha(\mathcal{C}) \), then an arrow \( f : (U, \zeta) \rightarrow (V, \vartheta) \) is a map \( f : U \rightarrow V \) such that

\[
\left( (W \otimes V) \otimes f \right) \circ \zeta_{\cdot} = \vartheta_{\cdot} \circ (f \otimes W),
\]

for all \( W \in \mathcal{C} \). The tensor product of \((U, \zeta)\) in \( Z_\alpha(\mathcal{C}) \) and \((U', \zeta')\) in \( Z_\alpha'(\mathcal{C}) \) is obtained by setting

\[
(U, \zeta) \otimes (U', \zeta') = (U \otimes U', \zeta \overline{\otimes} \zeta'),
\]

where, for all \( W \in \mathcal{C} \),

\[
(\zeta \overline{\otimes} \zeta')_W = (\zeta_{\cdot} (\zeta_{\cdot} \otimes V) \circ (U \otimes \zeta_{\cdot})).
\]
Finally $\Phi_\beta(U, c_u) = (\Phi_\beta(U), c^\beta_u)$, where, for all $V \in \mathcal{C}$,

$$c^\beta_v = \Phi_\beta(c_{\beta^{-1}(X)}).$$

The crossed $G$-category $\mathcal{Z}(\mathcal{C})$ is braided by setting

$$c_{(U, c_u)(U', c'_u)} = c_{U'}.$$

**Lemma 5.1.** If $\mathcal{C}$ is abelian, then $\mathcal{Z}(\mathcal{C})$ is abelian and the forgetful functor $Z: \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$: $(U, c_u) \mapsto U$ is exact.

The following proof is adapted from the one for the center of a tensor category given by Street [23].

**Proof.** For all $\alpha \in G$, let $0_\alpha$ be the object $0$ of $\mathcal{C}_\alpha$. For all objects $X$ in $\mathcal{C}_\beta$, we have

$$0_\alpha \otimes X = 0_{\alpha\beta} = (\alpha X) \otimes 0_\beta.$$

So, if $0_{\alpha}[X]$ is the identity of $0_{\alpha\beta}$, then we get an object $(0_\alpha, 0_{\alpha\beta})$ in $\mathcal{Z}(\mathcal{C})$.

Let both $(U, c_u)$ and $(U', c'_u)$ be objects in $\mathcal{Z}(\mathcal{C})$ and let $X$ be an object in $\mathcal{C}$. By setting

$$(\epsilon \oplus c')_X: (U \oplus U') \otimes X \cong (U \otimes X) \oplus (U' \otimes X) \xrightarrow{\epsilon \otimes c_X} \left(\alpha X\right) \otimes U \oplus \left(\alpha X\right) \otimes U' \cong \left(\alpha X\right) \otimes (U \oplus U'),$$

we get an object $(U, c_u) \oplus (U', c'_u) = (U \oplus U', (\epsilon \oplus c'_u))$ in $\mathcal{C}_\alpha$.

Let $f: (U, c_u) \to (U', c'_u)$ be an arrow in $\mathcal{Z}(\mathcal{C})$ and let $k(f): K(f) \to U$ and $c(f): U' \to C(f)$ be the kernel and the cokernel of $f$ in $\mathcal{C}_\alpha$. Since the following diagram in exact

$$
\begin{array}{cccccc}
K(f) \otimes X & \xrightarrow{k(f) \otimes X} & U \otimes X & \xrightarrow{f \otimes X} & U' \otimes X & \xrightarrow{c(f) \otimes X} & C(f) \otimes X \\
\downarrow \mathcal{R}_X[f] & & \downarrow \mathcal{E}_X[f] & & \downarrow \mathcal{E}_X[f] & & \downarrow \mathcal{E}_X[f] \\
(\alpha X) \otimes K(f) & \xrightarrow{(\alpha X) \otimes k(f)} & (\alpha X) \otimes U & \xrightarrow{(\alpha X) \otimes f} & (\alpha X) \otimes U' & \xrightarrow{(\alpha X) \otimes c(f)} & (\alpha X) \otimes C(f)
\end{array}
$$

there are unique $\mathcal{R}_X[f]$ and $\mathcal{E}_X[f]$ making it commutative. One can prove that $k(f): (K(f) \otimes U, \mathcal{R}_X[f]) \to (U, c_u)$ and $c(f): (U', c'_u) \to (C(f), \mathcal{E}_X[f])$ are the kernel and the cokernel of $f$ in $\mathcal{Z}(\mathcal{C})$. The rest follows easily. \qed

### 5.2 The Ribbon Extension

Let $\mathcal{D}$ be a braided crossed $G$-category. To define $\theta(\mathcal{D})$ let us firstly define the balanced crossed $G$-category $\mathcal{D}^Z$. The component $\mathcal{D}^Z_\alpha = (\mathcal{D}^Z)_\alpha$ of $\mathcal{D}^Z$ is the category whose objects are couples $(U, t)$, where $U$ is an object in $\mathcal{D}_\alpha$ and $t: U \to U$ is an isomorphism in $\mathcal{D}$. An arrow $f$: $(U, t) \to (U', t')$ in $\mathcal{D}^Z$ is a map $f: U \to U'$ in $\mathcal{D}_\alpha$ such that

$$(U f) \circ t = t' \circ f.$$
We set 
\[(U, t) \otimes (U', t') = (U \otimes U', t \hat{\otimes} t'),\]
where 
\[t \hat{\otimes} t' = c\left((v \otimes v')_U\right) \circ c\left((v_U)_U\right) \circ (t \otimes t').\]
Finally, we set 
\[\Phi_{\beta}(U, t) = \left(\Phi_{\beta}(U), \Phi_{\beta}(t)\right).\]
The braiding of \(\mathcal{D}\) induces a braiding in \(\mathcal{D}^Z\) and \(\mathcal{D}^Z\) is balanced with the twist 
\[\theta_{(U, t)} = t,\]
The following lemma follows by the proof of Proposition 4 in [23].

**Lemma 5.2.** If \(\mathcal{D}\) is abelian, then \(\mathcal{D}^Z\) is abelian and the forgetful functor 
\(Z_1: \mathcal{D}^Z \to \mathcal{D}: (U, t) \mapsto U\) is exact.

The crossed \(G\)-category \(\mathcal{D}^Z\) is not necessarily ribbon. However, let \(\mathcal{E}\) balanced crossed \(G\)-category and let \(\mathcal{N}(\mathcal{E})\) the full subcategory of \(\mathcal{E}\) of objects \(U\) such that there exists a left dual \(U^*\)

- compatible with \(\Phi\), i.e. such that eq. (12) is satisfied,
- compatible with \(\theta\), i.e. such that eq. (3) is satisfied, and
- we have 
  \[\theta_{U^*} = \omega_{(b_U, d_U)},\]
where, by definition, \(\theta_{U^*} = (U \theta_U) \circ \theta_U, \theta_{U^{-2}} = (\theta_{U^2})^{-1},\) and
  \[\omega_{(b_U, d_U)} = (d_U \otimes_U U \otimes_U U) \circ \left(\left(U \otimes_U U^*\right) \otimes_U \hat{c}_{U, U} \otimes_U U\right) \circ \left((c_U U_U \otimes_U b_U) \otimes U \otimes_U U\right),\]
  where, for all objects \(X\) in \(\mathcal{E}_\alpha\) and \(Y\) in \(\mathcal{E}_\beta\), we set \(\hat{c}_{X, Y} = (c_{Y, -1} X)^{-1}\).

The category \(\mathcal{N}(\mathcal{E})\) is a ribbon crossed \(G\)-category. Thus we set
\[\theta(\mathcal{D}) = \mathcal{N}\left(\mathcal{D}^Z\right).\]

**Lemma 5.3.** If \(\mathcal{E}\) is abelian, then \(\mathcal{N}(\mathcal{E})\) is abelian and the inclusion \(N: \mathcal{N}(\mathcal{E}) \to \mathcal{E}\) is exact. So, when \(\mathcal{E} = \mathcal{D}^Z\) we get an exact functor \(Z_1 = Z_1 \circ N: \theta(\mathcal{D}) \to \mathcal{D}.\)

The following proof is again an adaptation of that given in Street [23] in the case of a tensor category.

**Proof.** We only need to show that \(\mathcal{N}(\mathcal{E})\) is closed under finite limits and finite colimits. Since \((U \oplus V)^* = U^* \oplus V^*\) and \(\theta_{(U \oplus V)} = \theta_{U \oplus V}\), the category \(\mathcal{N}(\mathcal{E})\) is closed under direct sums. Now, if \(f: U \to V\) is a map in an autonomous category and both \(U\) and \(V\) has left duals, then the cokernel object \(C(f^*)\) of \(f^*\) is a left dual of the kernel object \(K(f)\) of \(f\) while the kernel object \(K(f^*)\) of \(f^*\) is a right dual of the cokernel object \(C(f)\) of \(f\). Further, if \(f\) is a map in an abelian crossed \(G\)-category and eq. (12) holds for both \(U^*\) and \(V^*\) then, it holds also for both \(C(f^*)\) and \(K(f^*)\), since the \(\Phi_{\alpha}\) are exact. If we are in a braided crossed \(G\)-category, then a right dual is also a left dual, and so \(K(f^*)\) is a left dual of \(C(f)\). So, if \(f\) is a map in \(\mathcal{N}(\mathcal{E})\), both \(K(f)\) and \(C(f)\) have left dual in \(\mathcal{E}\). By using the naturality of the twist, one can prove that both \(K(f)\) and \(C(f)\) also satisfy eq. (3). \(\square\)
5.3 The Quantum Co-double

**Theorem 5.4.** Let $H$ be a crossed Hopf $G$-algebra. There exists a coquasitriangular crossed Hopf $G$-algebra $D^*(H)$ (unique up to isomorphism) such that

$$\mathcal{C}om(D^*(H)) = \mathcal{Z}(\mathcal{C}om H)$$

as braided Tannakian crossed $G$-categories. Let $H'$ be a coquasitriangular crossed Hopf $G$-algebra. There exists a coribbon crossed Hopf $G$-algebra $R(H')$ (again unique up to isomorphism) such that

$$\mathcal{C}om(R(H')) = \theta(\mathcal{C}om H')$$

as ribbon Tannakian crossed $G$-categories. In particular, for $H = D^*(H)$ we get

$$\mathcal{C}om\left(\mathcal{C}om(D^*(H))\right) = \theta\left(\mathcal{Z}(\mathcal{C}om H)\right).$$

**Proof.** Let $\mathcal{C}om H$ and let $|\cdot| : \mathcal{C}om H \to \mathcal{C}at$ and $Z : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ be the trivial forgetful functors. By Lemma 5.1, the functor $Z \circ |\cdot| : \mathcal{Z}(\mathcal{C}om H) \to \mathcal{C}at$ is a fiber crossed $G$-functor. Thus, we are allowed to apply Tannaka theory getting the Hopf $G$-algebra $D(H)$ such that $\mathcal{Z}(\mathcal{C}om H) \cong \mathcal{C}om(D(H))$.

The proof for the ribbon structures is similar and left to the reader. \qed

We recall that, given a crossed Hopf $G$-coalgebra $K$ of finite type, one can construct a quasitriangular crossed Hopf $G$-coalgebra $\bar{D}(K)$ such that $\mathcal{Z}(\mathcal{M}od K) = \mathcal{M}od(\bar{D}(K))$, being $\mathcal{M}od K$ the category of finite dimensional $K$-modules, see [29, 30]. In particular, $\bar{D}(K)$ is of finite type if and only if $H$ is totally finite (i.e. $\sum_{\alpha \in G} \dim K_{\alpha} < +\infty$). In that case, both the dual of $K$ and the dual of $\bar{D}(K)$ are crossed Hopf $G$-algebras. It is easy to prove that we have an equivalence of crossed $G$-categories $\mathcal{M}od K \cong \mathcal{C}om K^*$. We deduce the following chain of equivalence of braided crossed group categories

$$\mathcal{C}om\left(\mathcal{C}om(\bar{D}(K))^*\right) = \mathcal{M}od(\bar{D}(K)) \cong \mathcal{Z}(\mathcal{M}od K) \cong \mathcal{Z}(\mathcal{C}om(K^*)) \cong \mathcal{C}om(D^*(K^*)).$$

(8)

**Corollary 5.5.** Let $H$ be a totally finite crossed Hopf $G$-algebra. The coquasitriangular crossed Hopf $G$-algebras $D^*(H)$ and $(\bar{D}(H^*))^*$ are isomorphic.

**Proof.** By setting $K = H^*$ in (8) we get an equivalence of braided $G$-categories

$$\mathcal{C}om\left(\mathcal{C}om(\bar{D}(H^*))^*\right) = \mathcal{C}om(D^*(H)).$$

By Tannakka theory, the two crossed Hopf $G$-algebras must be isomorphic. \qed

Starting from a braided crossed Hopf $G$-coalgebra $L$ of finite type, there exists a ribbon crossed Hopf $G$-coalgebra $RT(L)$ such that $\mathcal{M}od(RT(L)) \cong \theta(\mathcal{M}od L)$, see [29, 30]. By means of a chain of equivalence of ribbon $G$-categories similar to the previous one we get the following result.

**Corollary 5.6.** Let $H'$ be a Hopf $G$-algebra of finite type. The coribbon crossed Hopf $G$-algebras $R(H')$ and $(RT(H'))^*$ are isomorphic.

By combining the two previous corollaries, we finally get the following result.
Corollary 5.7. Let $H$ be a totally finite crossed Hopf $G$-algebra. The coribbon crossed Hopf $G$-algebras $R(D^*(H))$ and $\left(RT(D(H^*))\right)^*$ are isomorphic.

The quantum double of a totally finite semisimple crossed Hopf $G$-coalgebra is modular with $\theta_\alpha = 1_\alpha$ \[^{29}\]. One can easily prove the following result.

Corollary 5.8. The quantum co-double $D^*(H)$ of a totally finite crossed Hopf $G$-algebra is modular with $\tau_\alpha = \varepsilon_\alpha$.

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