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On consistency and inconsistency of nonparametric tests

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Abstract: For χ²-tests with increasing number of cells, Cramer-von Mises tests, tests generated L²-norms of kernel estimators and tests generated quadratic forms of estimators of Fourier coefficients we find necessary and sufficient conditions of consistency and inconsistency of sequences of alternatives having a given rate of convergence to hypothesis in L²-norm. We show that asymptotic of type II error probabilities of sums of alternatives of consistent and inconsistent sequences coincide with the asymptotic for consistent sequence. We find analytic assignment of consistent sequences that do not have inconsistent components. We point out the largest convex orthosymmetric sets of functions such that any sequence of alternatives belonging to this set and having a given rate of convergence to hypothesis is consistent. We show that these sets are balls in Besov spaces B²∞. We point out asymptotically minimax tests if sets of alternatives are maxiset with deleted "small" L²-balls.

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1. Introduction

Let X₁, . . . , Xn be i.i.d.r.v.’s with c.d.f. F(x), x ∈ (0, 1). Let c.d.f. F(x) have a density p(x) = 1 + f(x) = dF(x)/dx, x ∈ (0, 1). Suppose that f ∈ L²(0, 1) with the norm

\|f\| = \left( \int_0^1 f²(x)dx \right)¹⁄₂ < ∞.

We explore the problem of testing hypothesis

\mathbb{H}_0 : f(x) = 0, \quad x ∈ (0, 1)

versus f belongs to some nonparametric set of alternatives.

The quality of nonparametric tests is well studied if f belongs to parametric family with finite number of parameters.

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If set of alternatives is nonparametric, we can distinguish two directions of the problem exploration: semiparametric approach or distance method and approaches based on assumption that a priori information about density smoothness is provided.

Mann and Wald [24] were pioneers in the development of first approach. Mann and Wald [24] established the optimal order of number of cells for chi-squared tests if Kolmogorov distances of alternatives from the hypothesis are greater some constants. Further development distance method has obtained in papers Horowitz and Spokoiny [13] and Ermakov [7, 8, 10]. For $\chi^2$-tests with increasing number of cells, tests generated $L_2$-norms of kernel estimators and tests generated quadratic forms of estimators of Fourier coefficients asymptotic minimaxity of tests has been established. In these papers nonparametric sets of alternatives were defined by the distance generating nonparametric tests. The distribution function (or signal in the problem of signal detection) belongs to the set of alternatives if its distance is more than given constants.

If a priori information is provided about density smoothness, the test quality is explored usually in the following setup. We have a priori information that density $f$ belongs to a ball $U$ in some functional space $\mathcal{S} \subset L_2$. We wish to test a hypothesis (1.1) versus alternatives

$$
\mathbb{H}_n : f \in V_n = \{ f : \| f \|^2 \geq \rho_n, f \in U \},
$$

with $\rho_n \to 0$ as $n \to \infty$.

The rate of consistency $\rho_n$ allowing to assign consistent sequences of tests has been explored in many papers (see Ingster and Suslina [17], Laurent, Loubes and Marteau [22] and Comminges and Dalalyan [4] and references therein). Asymptotically minimax tests has been proposed for some balls in functional spaces. If $U$ is a ball in Besov space $B^s_{2\infty}$, Ingster [16] pointed out rates of consistency $\rho_n$ for chi-squared tests with increasing number of cells, Kolmogorov and Cramer - von Mises tests.

In section 3 we show that, for this setup, there are consistent tests for some sequence $\rho_n \to 0$ as $n \to \infty$, iff, the set $U$ is compact. Thus the sets of alternatives are very poor in this setup in comparison with semiparametric approach.

In semiparametric approach we establish asymptotic minimaxity of test statistics for wide nonparametric sets of alternatives defined in terms of the distance generating these test statistics. However such a description does not carry any evident information on rate of consistency of alternatives in more strong metrics.

Paper goal is to explore consistency and inconsistency of sequences of alternatives $f_n$ having a given rate of convergence to hypothesis in $L_2$-metrics, $cn^{-r} \leq \| f_n \| \leq Cn^{-r}$, $0 < r < 1/2$. This results are provided for $\chi^2$-tests with increasing number of cells, Cramer-von Mises tests, tests generated $L_2$-norms of kernel estimators and tests generated quadratic forms of estimators of Fourier coefficients.

Thus we explore the problem of hypothesis testing (1.1) with alternatives

$$
\mathbb{H}_n : f = f_n, \quad cn^{-r} \leq \| f_n \| \leq Cn^{-r}.
$$
On the base of analysis of Fourier coefficients of sequence $f_n$ we wish to make a conclusion about consistency or inconsistency of this sequence.

For the problems of hypothesis testing with contiguous alternatives $L_2$-norm is naturally arises in the study of test behaviour. If we consider the problem of testing hypothesis (1.1) versus simple alternatives $H_1^n: f(x) = f_n(x) = cn^{-1/2} h(x), \|h\| < \infty$, then the asymptotic of type II error probabilities of Neymann-Pearson tests is defined by $L_2$- norm $\|h\|^2$. Similar situation takes place also for the problem of signal detection in Gaussian white noise. This explains the choice of $L_2$ - norm as approximation measure in the paper.

In terms of Fourier coefficients we establish necessary and sufficient conditions of consistency and inconsistency of sequences of alternatives. We show that asymptotic of type II error probabilities of sums of alternatives from consistent and inconsistent sequences coincide with the asymptotic for consistent sequence. After that we obtain the result in two directions.

We point out analytic assignment of sequences of alternatives that do not have inconsistent components. We call these sequences purely consistent sequences of alternatives.

We point out the largest closed bounded orthosymmetric convex sets $U$ such that any sequence of alternatives $f_n \in U$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, is consistent. We call sets $U$ maxisets. It turns out that maxisets are balls in Besov space $B^{s}_{2\infty}$ with $r = \frac{2s}{1+4s}$ for $\chi^2$-tests with increasing number of cells, tests generated $L_2$-norms of kernel estimators, tests generated quadratic forms of estimators of Fourier coefficients and $r = \frac{\tau}{2+2\tau}$ for Cramer-von Mises tests.

We explore the relations of inconsistent, consistent, purely consistent sequences alternatives and sequences of alternatives from maxisets if these sequences have a given rate of convergence to hypothesis.

We show that any sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, from maxisets is purely consistent.

We show that, for any $\varepsilon > 0$, for any purely consistent sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, there is maxiset and sequence $f_1^n$ from the maxiset such that $\|f_n - f_1^n\| \leq \varepsilon n^{-r}$.

We show that any consistent sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, contains additive component $f_1^n$, $cn^{-r} \leq \|f_1^n\| \leq C1n^{-r}$, from maxiset, that is, $\|f_n\|^2 = \|f_1^n\|^2 + \|f_n - f_1^n\|^2$. Moreover, for any $\varepsilon > 0$, there is maxiset such that the differences of type II error probabilities of alternatives $f_n$ and additive component $f_1^n$ from this maxiset are smaller than $\varepsilon$.

Therefore, each function $f_n$ of consistent sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, contains sufficiently smooth function $f_1^n \in B^{s}_{2\infty}$, $cn^{-r} \leq \|f_1^n\| \leq Cn^{-r}$ as additive component.

For nonparametric estimation the notion of maxisets has been introduced Kerkyacharian and Picard [20]. The maxisets of widespread nonparametric esti-
mators have been comprehensively explored (see Cohen, DeVore, Kerkyacharian, Picard [3], Kerkyacharian and Picard [21], Rivoirard [25], Bertin and Rivoirard [26], Ermakov [12] and references therein). For nonparametric hypothesis testing completely different definition of maxisets has been introduced Autin, Clausel, Jean-Marc Freyermuth and Marteau [1].

The tests generated $L_2$– norms of kernel estimators and the tests generated quadratic forms of estimators of Fourier coefficients are explored for the problem of signal detection in Gaussian white noise. We observe a realization of random process $Y_n(t)$ defined stochastic differential equation

$$dY_n(t) = f(t)dt + \frac{\sigma}{\sqrt{n}}dw(t), \quad t \in [0,1], \quad \sigma > 0,$$

where $f \in L_2(0,1)$ is unknown signal and $dw(t)$ is Gaussian white noise.

The problem of hypothesis testing (1.1) versus alternatives (1.3) is the same.

Paper is organized as follows. In section 2 main definitions are provided. In section 3, we show that, if set $U$ is absolutely convex, then the existence of consistent tests implies compactness of set $U$. In sections 4, 5, 6 and 7 we explore main properties of consistent and inconsistent sequences of alternatives for test statistics based on quadratic forms of estimators of Fourier coefficients, $L_2$– norms of kernel estimators, $\chi^2$– tests and Cramer– von Mises tests respectively. We point out asymptotically minimax tests if sets of alternatives are maxiset with deleted "small" $L_2$-balls in section 8. Sections 9 contains proofs of all Theorems.

We use letters $c$ and $C$ as a generic notation for positive constants. Denote $\chi(A)$ the indicator of an event $A$. Denote $[a]$ whole part of real number $a$. For any two sequences of positive real numbers $a_n$ and $b_n$, $a_n = O(b_n)$ and $a_n \asymp b_n$ imply respectively $a_n < Cb_n$ and $a_n \leq b_n \leq Ca_n$ for all $n$ and $a_n = o(b_n)$ implies $a_n/b_n \to 0$ as $n \to \infty$.

Denote

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-t^2/2\} dt, \quad x \in \mathbb{R}^1,$$

the standard normal distribution function.

Let $\phi_j$, $1 \leq j < \infty$, be orthonormal system of functions. Define the sets

$$\tilde{B}^s_{2\infty}(P_0) = \left\{ f : f = \sum_{j=1}^{\infty} \theta_j \phi_j, \quad \sup_{\lambda > 0} \lambda^{2s} \sum_{j>\lambda} \theta_j^2 < P_0, \quad \theta_j \in \mathbb{R}^1 \right\}. \tag{1.5}$$

Under some conditions on the basis $\phi_j$, $1 \leq j < \infty$, the space

$$\tilde{B}^s_{2\infty} = \left\{ f : f = \sum_{j=1}^{\infty} \theta_j \phi_j, \quad \sup_{\lambda > 0} \lambda^{2s} \sum_{j>\lambda} \theta_j^2 < \infty, \quad \theta_j \in \mathbb{R}^1 \right\}.$$

is Besov space $B^s_{2\infty}$ (see Rivoirard [25]). In particular, $\tilde{B}^s_{2\infty}$ is Besov space $B^s_{2\infty}$ if $\phi_j$, $1 \leq j < \infty$, is trigonometric basis.
If $\phi_j(t) = \exp\{2\pi i j t\}$, $x \in (0, 1)$, $j = 0, \pm 1, \ldots$, is trigonometric basis, denote

$$\mathbb{H}^s_2(P_0) = \left\{ f : f = \sum_{j=-\infty}^{\infty} \theta_j \phi_j, \sup_{\lambda>0} \lambda^{2s} \sum_{|j|\geq \lambda} |\theta_j|^2 < P_0 \right\}.$$ 

The balls in Nikols’ki classes

$$\int (f^{(l)}(x + t) - f^{(l)}(x))^2 \, dx \leq L|t|^{2(s-l)}, \quad \|f\| < C,$$

with $l = [s]$ are the balls in $\mathbb{H}^s_2$. We also introduce definition of balls in Besov spaces $\mathbb{B}^s_2$ in terms of wavelet basis $\phi_{kj}(x) = 2^{k/2} \phi(2^{k/2} - j)$, $1 \leq j < 2^k$, $1 \leq k < \infty$. Denote

$$\tilde{\mathbb{B}}^s_2(P_0) = \left\{ f : f = 1 + \sum_{k=1}^{\infty} \sum_{j=1}^{2^k} \theta_{kj} \phi_{kj}, \sup_{\lambda>0} 2^{2\lambda s} \sum_{k>\lambda} \sum_{j=1}^{2^k} \theta_{kj}^2 \leq P_0, \theta_{jk} \in \mathbb{R}^1 \right\}.$$ 

2. Main definitions

2.1. Consistency and $n^{-r}$-consistency

For any test $K_n = K_n(X_1, \ldots, X_n)$ denote $\alpha(K_n)$ its type I error probability, and $\beta(K_n, f)$ its type II error probability for alternative $f \in L_2(0, 1)$.

For the problem of testing hypothesis $H_0 : f = 0$ versus alternatives $H_n : f = f_n$, we say that sequence of alternatives $f_n$ is consistent if there is sequence of tests $K_n$ generated test statistics $T_n$ such that

$$\limsup_{n \to \infty} (\alpha(K_n) + \beta(K_n, f_n)) < 1. \quad (2.1)$$

If $cn^{-r} < \|f_n\| < Cn^{-r}$ additionally, we say that sequence of alternatives $f_n$ is $n^{-r}$-consistent (see Tsybakov [28]).

We say that sequence of alternatives $f_n$ is inconsistent if for each sequence of tests $K_n$ generated test statistics $T_n$ there holds

$$\liminf_{n \to \infty} (\alpha(K_n) + \beta(K_n, f_n)) \geq 1. \quad (2.2)$$

If $cn^{-r} < \|f_n\| < Cn^{-r}$ additionally, we say that sequence of alternatives $f_n$ is $n^{-r}$-inconsistent.

Denote

$$\beta(K_n, V_n) = \sup\{\beta(K_n, f) : f \in V_n\}.$$ 

We say that, for test statistics $T_n$, problem of hypothesis testing is $n^{-r}$-consistent on the set $U$ (consistent on the sets $V_n$ respectively) if there is sequence of tests $K_n$ generated test statistics $T_n$ such that

$$\limsup_{n \to \infty} (\alpha(K_n) + \beta(K_n, V_n)) < 1. \quad (2.3)$$
2.2. Purely consistent sequences

We say that \( n^{-r} \)-consistent sequence of alternatives is purely \( n^{-r} \)-consistent if there is not subsequence \( f_n \) such that \( f_{n_j} = f_{1n_j} + f_{2n_j} \) where sequence \( f_{2n_j} \), \( \|f_{2n_j}\| > c_1 n^{-r} \), is inconsistent and \( \|f_{n_j}\|^2 = \|f_{1n_j}\|^2 + \|f_{2n_j}\|^2 \).

For test statistics mentioned above we show that the following holds.

A. Sequence of alternatives \( f_n, cn^{-r} < \|f_n\| < Cn^{-r} \), is purely \( n^{-r} \)-consistent, iff, for any \( n^{-r} \)-inconsistent subsequence of alternatives \( f_{1n_j} \), there holds

\[
\|f_{n_j} + f_{1n_j}\|^2 = \|f_{n_j}\|^2 + \|f_{1n_j}\|^2 + o(n_i^{-r})
\]

as \( i \to \infty \).

2.3. Maxisets

Let \( \phi_j, 1 \leq j < \infty \), be orthonormal basis in \( L_2 \). We say that a set \( U, U \subset L_2 \), is orthosymmetric with respect to this basis if \( f = \sum_{j=1}^{\infty} \theta_j \phi_j \in U \) implies \( f = \sum_{j=1}^{\infty} \bar{\theta}_j \phi_j \in U \) for any \( \bar{\theta}_j = \theta_j \) or \( \bar{\theta}_j = -\theta_j, j = 1, 2, \ldots \).

We call bounded closed sets \( \gamma U, 0 < \gamma < \infty \), maxisets if

i. set \( U \) is convex,

ii. there is basis \( \phi_j, 1 \leq j < \infty \), such that the set \( U \) is orthosymmetric with respect to this basis.

iii. any subsequence of alternatives \( f_{n_j} \in U, cn_j^{-r} < \|f_{n_j}\| < Cn_j^{-r}, n_j \to \infty \) as \( j \to \infty \), is consistent,

iv. if \( f \notin \gamma U \) for all \( \gamma > 0 \), then, in any convex, orthosymmetric set \( U \) that contains \( f \), there is inconsistent subsequence of alternatives \( f_{n_j} \in U \), \( cn_j^{-r} < \|f_{n_j}\| < Cn_j^{-r}, n_j \to \infty \) as \( j \to \infty \).

Test statistics of tests generated \( L_2 \)-norms of kernel estimators and Cramer-von Mises tests admit representation as sum of squares of estimators of Fourier coefficients. Therefore, for these test statistics, consistency of sequence \( f_n = \sum_{j=1}^{\infty} \theta_n \phi_j \) implies consistency of any sequence \( \tilde{f}_n = \sum_{j=1}^{\infty} \tilde{\theta}_n \phi_j \) with any \( \tilde{\theta}_j = \theta_j \) or \( \tilde{\theta}_j = -\theta_j, j = 1, 2, \ldots \). Moreover, type II error probabilities of sequences \( f_n \) and \( f_{1n} \) has the same asymptotic. Thus requirement ii. seems natural for test statistics admitting representation as a sum of squares of estimators of Fourier coefficients.

As follows from Theorems 6.1 and 6.2 given below, for chi-squared tests, consistency of sequence \( f_n = \sum_{j=1}^{\infty} \theta_n \phi_j \) implies consistency of any sequence \( \tilde{f}_n = \sum_{j=1}^{\infty} \tilde{\theta}_n \phi_j \) as well.

2.4. Regular and perfect maxisets

We say that maxisets are regular if there hold
\textit{i.} any sequence of alternatives \( f_n \in \mathbb{L}_2(0,1) \) is \( n^{-r} \)-consistent, iff, there are \( \gamma U, \gamma > 0 \), and sequence \( f_{1n} \in \gamma U, c_2 n^{-r} \leq \| f_{1n} \| \leq C_2 n^{-r} \), such that there holds
\[
\| f_n \| = \| f_{1n} \| + \| f_n - f_{1n} \|. \tag{2.5}
\]

\textit{ii.} Sequence \( f_n \) is purely \( n^{-r} \)-consistent, iff, for any \( \varepsilon > 0 \), there is \( \gamma_\varepsilon \) and sequence \( f_{1n} \in \gamma_\varepsilon U \) such that \( \| f_n - f_{1n} \| \leq \varepsilon n^{-r} \) and (2.5) holds.

We say that maxisets are \textit{perfect} if, for any \( \varepsilon > 0 \) and for any positive constants \( c \) and \( C, c < C \), there are \( \gamma_\varepsilon \) and \( n_\varepsilon \) satisfying the following requirement:

if sequence of alternatives \( f_n \in \mathbb{L}_2, cn^{-r} \leq \| f_n \| \leq Cn^{-r} \), is consistent then, there is sequence \( f_{1n} \in \gamma_\varepsilon U, c_1 n^{-r} \leq \| f_{1n} \| \leq C_1 n^{-r} \), such that (2.5) holds and, for any \( n > n_\varepsilon \), there hold
\[
| \beta(K_n, f_n) - \beta(K_n, f_{1n}) | \leq \varepsilon \tag{2.6}
\]
and
\[
\beta(K_n, f_n - f_{1n}) \geq 1 - \alpha - \varepsilon. \tag{2.7}
\]
Here \( K_n, \alpha(K_n) = \alpha(1 + o(1)) \) as \( n \to \infty \), is a sequence of tests generated test statistics \( T_n \).

\textbf{2.5. Another approach to definition of maxisets}

The definition of maxisets we begin with preliminary notation.

Let \( \mathcal{S} \subset \mathbb{L}_2(0,1) \) be Banach space with norm \( \| \cdot \|_\mathcal{S} \) and let \( U(\gamma) = \{ f : \| f \|_\mathcal{S} \leq \gamma, f \in \mathcal{S} \}, \gamma > 0 \), be a ball in \( \mathcal{S} \).

Define subspaces \( \Pi_k, 1 \leq k < \infty \), by induction.

Denote \( d_1 = \max \{ \| f \|, f \in U(1) \} \) and denote \( e_1 \) function \( e_1 \in U = U(1) \) such that \( \| e_1 \| = d_1 \). Denote \( \Pi_1 \) linear space generated vector \( e_1 \).

For \( i = 2, 3, \ldots \) denote \( d_i = \max \{ \rho(f, \Pi_{i-1}), f \in U \} \) with \( \rho(f, \Pi_{i-1}) = \min \{ \| f - g \|, g \in \Pi_{i-1} \} \). Define function \( e_i, e_i \in U \), such that \( \rho(e_i, \Pi_{i-1}) = d_i \). Denote \( \Pi_i \) linear space generated functions \( e_1, \ldots, e_i \).

For any \( f \in \mathbb{L}_2(0,1) \) denote \( f_{1n} \), the projection of \( f \) onto the subspace \( \Pi_i \) and denote \( \tilde{f}_i = f - f_{1n} \).

Thus we associate with each \( f \in \mathbb{L}_2(0,1) \) sequence of functions \( \tilde{f}_i, \tilde{f}_i \to 0 \) as \( i \to \infty \). This allows to cover by our consideration the all space \( \mathbb{L}_2(0,1) \).

Suppose that the functions \( e_1, e_2, \ldots \) are sufficiently smooth. Then, considering the functions \( \tilde{f}_i \), we ”in some sense delete the most smooth part \( f_{1n} \) of function \( f \) and explore the behaviour of remaining part.”

For the problem of hypothesis testing on a density we suppose that for all \( f \in \mathcal{S} \) there holds \( \int_0^1 f(s) \, ds = 0 \).

We say that \( U(\gamma), \gamma > 0 \), are maxisets for test statistics \( T_n \) and \( \mathcal{S} \) is maxispace if the following two statements take place.
i. There is sequence of tests $K_n$, $\alpha(K_n) = \alpha(1 + o(1))$, $0 < \alpha < 1$, generated test statistics $T_n$ such that there holds

$$\limsup_{n \to \infty}(\alpha(K_n) + \beta(K_n, V_n)) < 1,$$  \hspace{1cm} (2.8)

ii. For any $f \in L^2(0,1)$, $f \notin \mathcal{I}$, $\int_0^1 f(s) ds = 0$, there are sequences $i_n, j_n$ with $i_n \to \infty$ as $n \to \infty$ such that $c j^{-\tau} i_n < \| \tilde{f}_n \| < C j^{-\tau}$ for some constants $c$ and $C$. If $1 + \tilde{f}_{i_n}(s) \geq 0$ for all $s \in [0,1]$, then any sequence of tests $K_{j_n}$, $\alpha(K_{j_n}) = \alpha(1 + o(1))$, $0 < \alpha < 1$, generated test statistics $T_{j_n}$ satisfies the following inequality

$$\liminf_{n \to \infty}(\alpha(K_{j_n}) + \beta(K_{j_n}, \tilde{f}_n)) \geq 1.$$

(2.9)

3. Necessary conditions of consistency

In order to introduce a priori information on alternative smoothness the sets of alternatives in nonparametric hypothesis testing are usually defined as closed absolutely convex set $U$ with "small $L^2$–balls removed" (see Ingster and Suslina [17], Yu. I. Ingster, T. Sapatinas, I. A. Suslina, [18] and Comminges and Dalalyan [4] (see also references therein)). The set $U$ is supposed compact. Theorem 3.1 provided below shows that this is necessary condition.

We consider the problem of signal detection in Gaussian white noise discussed in Introduction. The problem is explored in terms of sequence model. The stochastic differential equation (1.4) can be rewritten in terms of a sequence model for orthonormal system of functions $\phi_j$, $1 \leq j < \infty$, in the following form

$$y_j = \theta_j + \frac{\sigma}{\sqrt{n}} \xi_j, \hspace{1cm} 1 \leq j < \infty$$  \hspace{1cm} (3.1)

where

$$y_j = \int \phi_j dY_n(t), \hspace{0.5cm} \xi_j = \int \phi_j dw(t) \hspace{0.5cm} \text{and} \hspace{0.5cm} \theta_j = \int f \phi_j dt.$$  

Denote $y = \{y_j\}_{j=1}^\infty$ and $\theta = \{\theta_j\}_{j=1}^\infty$.

We can consider $\theta$ as a vector in Hilbert space $\mathbb{H}$ with the norm $||\theta|| = \sum_{i=1}^{\infty} \theta_i^2$.

In this notation the problem of hypothesis testing can be rewritten in the following form. One needs to test the hypothesis $\mathbb{H}_0 : \theta = 0$ versus alternatives $\mathbb{H}_n : \theta \in V_n = \{ \theta : ||\theta|| \geq \rho_n, \theta \in U, U \subset \mathbb{H} \}$.

We say that set $U$ is center-symmetric if $\theta \in U$ implies $-\theta \in U$.

Theorem 3.1. Suppose that set $U$ is bounded, convex and center-symmetric. Then there is consistent tests for some sequence $\rho_n \to 0$ as $n \to \infty$, iff, the set $U$ is relatively compact.
If set $U$ is relatively compact, there is consistent estimator (see Ibragimov and Khasminskii [15] and Johnstone [19]), and we can choose consistent estimator as consistent test statistics.

If set $U$ is unbounded or is not center-symmetric, one can try to distinguish bounded, convex and center-symmetric part $U_1 \subset U$ and to implement Theorem 3.1 to the set $U_1$. The remaining set $U \setminus U_1$ of alternatives can be analyzed on the base of Theorem 5.3 in Ermakov [11].

Remark. Let $U \subset L_2$ be bounded set. Then there is consistent estimator onto the set $U$, iff, set $U$ is relatively compact (see Ibragimov and Khasminskii [15] and Johnstone [19]). Theorem 3.1 can be considered as an analogue of this statement for the problems of hypothesis testing. Note that compactness requirement also arises in ill-posed problems with deterministic noise (see Engl, Hanke and Neubauer [5]).

4. Quadratic test statistics

We explore problem of signal detection in Gaussian white noise (1.4) discussed in Introduction. The problem is provided in terms of sequence model.

If $U$ is compact ellipsoid in Hilbert space, the asymptotically minimax test statistics are quadratic forms

$$T_n(Y_n) = \sum_{j=1}^{\infty} \kappa_{jn}^2 y_j^2 - \sigma^2 n^{-1} \rho_n,$$

with some specially defined coefficients $\kappa_{jn}^2$ (see Ermakov [6]). Here $\rho_n = \sum_{j=1}^{\infty} \kappa_{jn}^2$.

If coefficients $\kappa_{jn}^2$ satisfy some regularity assumptions, the test statistics $T_n(Y_n)$ are asymptotically minimax (see Ermakov [9]) for the wider sets of alternatives

$$\mathbb{H}_n : f \in Q_n(c) = \{ \theta : \theta = \{\theta_j\}_{j=1}^{\infty}, A_n(\theta) > c, \theta_j \in \mathbb{R} \},$$

with

$$A_n(\theta) = n^2 \sigma^{-4} \sum_{j=1}^{\infty} \kappa_{jn}^2 \theta_j^2.$$

A sequence of tests $L_n, \alpha(L_n) = \alpha(1 + o(1)), 0 < \alpha < 1$, is called asymptotically minimax if, for any sequence of tests $K_n, \alpha(K_n) \leq \alpha$, there holds

$$\liminf_{n \to \infty} (\beta(K_n, Q_n(c)) - \beta(L_n, Q_n(c))) \geq 0. \quad (4.1)$$

Sequence of test statistics $T_n$ is asymptotically minimax if the tests generated test statistics $T_n$ are asymptotically minimax.

Section goal is to describe the structure of $n^{-r}$-consistent and $n^{-r}$-consistent sequences of alternatives for test statistics $T_n(Y_n)$ with coefficients $\kappa_{jn}^2$ satisfying the following assumptions.

A1. For each $n$ sequence $\kappa_{jn}^2$ is decreasing.
A2. There are positive constants $C_1, C_2$ such that, for each $n$, there holds

$$C_1 < A_n = \sigma^{-4} n^2 \sum_{j=1}^{\infty} \kappa_{jn}^4 < C_2. \tag{4.2}$$

A3. There are positive constants $C_1$ and $C_2$ such that $C_1 n^{-2r} \leq \rho_n \leq C_2 n^{-2r}$. Denote

$$k_n = \sup \{ k : \sum_{j<k} \kappa_{jn}^2 \leq \frac{1}{2} \sum_{j=1}^{\infty} \kappa_{jn}^2 \}.$$ 

Denote $\kappa_{2n}^2 = \kappa_{k_n,n}^2$. 

A4. There are $C_1$ and $\lambda > 1$ such that, for any $\delta > 0$,

$$\kappa_{(1+\delta)k_n,n}^2 < C_1 (1 + \delta)^{-\lambda} \kappa_n^2. \tag{4.3}$$

A5. There holds $\kappa_{1n}^2 \asymp \kappa_n^2$. For any $c > 1$ there is $C$ such that $\kappa_{ck_n,n}^2 \geq C \kappa_n^2$ for all $n$.

Example. Let

$$\kappa_{jn}^2 = n^{-\lambda} \frac{1}{j^{\gamma} + cn^\beta}, \quad \gamma > 1,$$

with $\lambda = 2 - 2r - \beta$ and $\beta = (2 - 4r)\gamma$. Then A1 – A5 hold.

Note that A1-A5 implies

$$\kappa_n^2 = \kappa_{k_n,n}^2 \asymp n^{-2} k_n^{-1} \quad \text{and} \quad k_n \asymp n^{2-4r}. \tag{4.4}$$

**Theorem 4.1.** Assume A1-A5. Sequence of alternatives $f_n$ is $n^{-r}$-consistent, iff, there are $c_1$, $c_2$ and $n_0$ such that there holds

$$\sum_{|j| < c_2 k_n} \theta_{jn}^2 > c_1 n^{-2r} \tag{4.5}$$

for all $n > n_0$.

**Theorem 4.2.** Assume A1-A5. Sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, is inconsistent, iff, for all $c_2$, there holds

$$\sum_{|j| < c_2 k_n} \theta_{jn}^2 = o(n^{-2r}). \tag{4.6}$$

**Theorem 4.3.** Assume A1-A5. Let sequence of alternatives $f_n$ be $n^{-r}$-consistent. Then, for any $n^{-r}$-inconsistent sequence of alternatives $f_{1n}$, for tests $K_n$, $\alpha(K_n) = o(1 + o(1))$, generated test statistics $T_n$, there holds

$$\lim_{n \to \infty} (\beta(K_n, f_n) - \beta(K_n, f_n + f_{1n})) = 0. \tag{4.7}$$

**Theorem 4.4.** Assume A1-A5. Sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, is purely $n^{-r}$-consistent, iff, for any $\epsilon > 0$, there is $C_1 = C_1(\epsilon)$ such that there holds

$$\sum_{j > C_1 k_n} \theta_{jn}^2 \leq \epsilon n^{-2r}. \tag{4.8}$$

for all $n > n_0(\epsilon)$. 


Theorem 4.5. Assume A1-A5. Then A holds.

Denote \( s = \frac{r}{2 - 4r} \). Then \( r = \frac{2s}{1 + 4s} \).

Theorem 4.6. Assume A1-A5. Then the balls \( \mathbb{B}_{2\infty}^s(P_0) \) are maxisets for test statistics \( T_n(Y_n) \).

The balls \( \mathbb{B}_{2\infty}^s(P_0) \) in Theorems 4.6 can be replaced with any ball in \( \mathbb{B}_{2\infty}^s \) generated equivalent norm.

Theorem 4.7. Assume A1-A5. Then maxisets are regular.

Theorem 4.8. Assume A1-A5. Then maxisets are perfect.

Remark 4.1. Let \( \kappa_{jn}^2 = 0 \) for \( j > l_n \) and let \( \kappa_{jn}^2 > 0 \) for \( j \leq l_n \) with \( l_n \to \infty \) as \( n \to \infty \). The analysis of the proofs of Theorems shows that Theorems 4.1 - 4.8 remain valid for this setup if A4 and A5 are replaced with A6.

For any \( c, 0 < c < 1 \), there is \( c_1 \) such that \( \kappa_{[cn],n}^2 \geq c_1 \kappa_{1n}^2 \).

In the reasoning we put \( \kappa_{jn}^2 = \kappa_{1n}^2 \).

Theorems 4.2 and 4.4 hold with the following changes. It suffices to put \( c_2 = 1 \) in Theorem 4.2 and to take \( C_1 = 1 \) in Theorem 4.4.

Proof of corresponding versions of Theorems 4.1 - 4.8 is obtained by simplification of provided reasoning and is omitted.

5. Kernel-based tests

We explore the problem of signal detection of previous section and suppose additionally that function \( f \) belongs to \( L_{2\text{per}}(R^1) \) the set of 1-periodic functions such that \( f(t) \in L_2(0,1), t \in [0,1) \). This allows to extend our model on real line \( R^1 \) putting \( w(t + j) = w(t) \) for all integer \( j \) and \( t \in [0,1) \) and to write the forthcoming integrals over all real line.

Define kernel estimator

\[
\hat{f}_n(t) = \frac{1}{h_n} \int_{-\infty}^{\infty} K \left( \frac{t - u}{h_n} \right) dY_n(u), \quad t \in (0,1),
\]

where \( h_n \) is a sequence of positive numbers, \( h_n \to 0 \) as \( n \to 0 \).

The kernel \( K \) is bounded function such that the support of \( K \) is contained in \([-1,1] \), \( K(t) = K(-t) \) for \( t \in R^1 \) and \( \int_{-\infty}^{\infty} K(t) dt = 1 \).

In (5.1) we suppose that, for any \( v, 0 < v < 1 \), we have

\[
\frac{1}{h_n} \int_{j}^{j+1+v} K \left( \frac{t - u}{h_n} \right) dY_n(u) = \frac{1}{h_n} \int_{0}^{v} K \left( \frac{t - 1 - u}{h_n} \right) f(u) du + \frac{\sigma}{\sqrt{nh_n}} \int_{0}^{v} K \left( \frac{t - 1 - u}{h_n} \right) dw(u)
\]

and

\[
\frac{1}{h_n} \int_{-v}^{0} K \left( \frac{t - u}{h_n} \right) dY_n(u) = \frac{1}{h_n} \int_{1-v}^{1} K \left( \frac{t - u + 1}{h_n} \right) f(u) du
\]
\[
\frac{\sigma}{\sqrt{nh_n}} \int_{1-v}^{1} K\left(\frac{t-u+1}{h_n}\right) dw(u).
\]

For hypothesis testing we implement the kernel-based tests (see Bickel and Rosenblatt [2]) with the test statistics

\[
T_n(Y_n) = T_{nh_n}(Y_n) = nh_n^{1/2} \sigma^{-2} \gamma^{-1}(\|f_n\|^2 - \sigma^2 (nh_n)^{-1} ||K||^2),
\]

where

\[
\gamma^2 = 2 \int_0^\infty \left( \int_0^\infty K(t-s)K(s)ds \right)^2 dt.
\]

For this setup we call sequence of alternatives \(f_n, cn^{-r} \leq \|f_n\| \leq Cn^{-r} \), \(n^{-r}\)-consistent if there is \(c\) depending on this sequence such that sequence of alternatives \(\theta_n\) is consistent for test statistics \(T_n\) with \(h_n > cn^{4r-2}\) and \(h_n \asymp n^{4r-2}\).

We call sequence of alternatives \(f_n, cn^{-r} \leq \|f_n\| \leq Cn^{-r} \), \(n^{-r}\)-inconsistent if sequence of alternatives \(\theta_n\) is inconsistent for all test statistics \(T_n\) with \(h_n \asymp n^{4r-2}\).

We shall explore the problem in terms of sequence model.

Let we observe a realization of random process \(Y_n(t)\) with \(f = f_n\).

For \(-\infty < j < \infty\), denote

\[
\hat{K}(jh) = \frac{1}{h} \int_{-h}^{h} \exp\{2\pi i j t\} K\left(\frac{t}{h}\right) dt, \quad h > 0,
\]

\[
y_{jn} = \int_0^1 \exp\{2\pi i j t\} dY_n(t),
\]

\[
\xi_j = \int_0^1 \exp\{2\pi i j t\} dw(t),
\]

\[
\theta_{jn} = \int_0^1 \exp\{2\pi i j t\} f_n(t) dt.
\]

Denote \(Y_n = \{y_{jn}\}_{j=-\infty}^{\infty}\) and denote \(\theta_n = \{\theta_{jn}\}_{j=-\infty}^{\infty}\).

In this notation we can write our sequence model in the following form

\[
y_{jn} = \hat{K}(jh_n)\theta_{jn} + \sigma n^{-1/2} \hat{K}(jh_n)\xi_j, \quad -\infty \leq j < \infty.
\]

and

\[
T_n(Y_n) = nh_n^{1/2} \sigma^{-2} \gamma^{-1} \left( \sum_{j=-\infty}^{\infty} |\hat{K}^2(jh_n)y_{jn}^2| - n^{-1} \sigma^2 \sum_{j=-\infty}^{\infty} |\hat{K}^2(jh_n)| \right).
\]

The sequence models (3.1) and (5.2) does not have serious differences for exploration. Thus similar results hold for (5.2) setup with \(k_n \asymp h_n^{-1}\).

Denote \(k_n = [n^{2-4r}]\).
Theorem 5.1. Sequence of alternatives $f_n$ is $n^{-r}$-consistent, iff, there are $c_1$, $c_2$ and $n_0$ such that there holds
\[\sum_{|j| < c_2 k_n} |\theta_{jn}|^2 > c_1 n^{-2r}\] (5.3)
for all $n > n_0$.

Theorem 5.2. Sequence of alternatives $f_n$, $cn^{-r} < \|f_n\| < C n^{-r}$, is inconsistent, iff, for all $c_2$, there holds
\[\sum_{|j| < c_2 k_n} |\theta_{jn}|^2 = o(n^{-2r}).\] (5.4)

Theorem 5.3. Let sequence of alternatives $f_n$ be $n^{-r}$-consistent. Then, for any $n^{-r}$-inconsistent sequence of alternatives $f_{1n}$ for tests $K_n$, $\alpha(K_n) = \alpha(1 + o(1))$, generated test statistics $T_n$, there holds
\[\lim_{n \to \infty} (\beta(K_n, f_n) - \beta(K_n, f_n + f_{1n})) = 0.\] (5.5)

Note that (5.5) holds for tests $K_n$ with $h_n < cn^{4r-2}$ where $c$ depends on sequence $f_n$ and $h_n$ are the same for sequences of alternatives $f_n$ and $f_n + f_{1n}$.

Theorem 5.4. Sequence of alternatives $f_n$, $cn^{-r} < \|f_n\| < C n^{-r}$, is purely $n^{-r}$-consistent, iff, for any $\epsilon > 0$, there is $C_1 = C_1(\epsilon)$ such that there holds
\[\sum_{|j| > C_1 k_n} |\theta_{jn}|^2 \leq c n^{-2r}.\] (5.6)
for all $n > n_0(\epsilon)$.

Theorem 5.5. A holds.

Denote $s = \frac{r}{2 - 4r}$. Then $r = \frac{2s}{1 + 4s}$.

Theorem 5.6. Balls $B_{2\infty}^s(P_0)$, $P_0 > 0$, are maxisets for test statistics $T_n(Y_n)$.

In Theorem 5.6, iv. in definition of maxisets holds for test statistics $T_n$ having arbitrary values $h_n > 0$, $h_n \to 0$ as $n \to \infty$.

Theorem 5.7. Maxisets are regular.

Theorem 5.8. Maxisets are perfect.

6. $\chi^2$-tests

Let $X_1, \ldots, X_n$ be i.i.d.r.v.'s having c.d.f. $F_n(x)$, $x \in (0, 1)$. Let c.d.f. $F_n(x)$ have a density $1 + f_n(x) = dF_n(x)/dx$, $x \in (0, 1)$, $f_n \in L_2^{per}(0, 1)$.

We explore the problem of testing hypothesis (1.1), (1.3) discussed in introduction.
Let \( \hat{F}_n(x) \) be empirical c.d.f. of \( X_1, \ldots, X_n \).

Denote \( \hat{p}_n = \hat{F}_n((i+1)/k_n) - \hat{F}_n(i/k_n), 1 \leq i \leq k_n \).

Test statistics of \( \chi^2 \)-tests equal

\[
T_n(\hat{F}_n) = n k_n \sum_{i=1}^{k_n} (\hat{p}_n - 1/k_n)^2.
\]

Let

\[
f_n = \sum_{j=-\infty}^{\infty} \theta_{jn} \phi_j, \quad \phi_j(x) = \exp\{2\pi i j x\}, \quad x \in (0,1).
\]

Denote \( m_n = \left[ n^{1-\frac{1}{2r}} \right] \times n^{2-4r} \).

**Theorem 6.1.** If sequence of alternatives \( f_n \) is \( n^{-r} \)-consistent, then there are \( c_1 \) and \( c_2 \) such that there holds

\[
\sum_{|j| < c_2 m_n} |\theta_{jn}|^2 > c_1 n^{-2r}.
\]

**Theorem 6.2.** If, for the sequence of alternatives \( f_n \), \( cn^{-r} \leq \|f_n\| \leq Cn^{-r} \), there are \( c_1 \) and \( c_2 \) such that (6.1) holds, then there is sequence \( k_n \approx n^{2-4r} \) such that \( f_n \) is consistent.

In what follows, we call sequence of alternatives \( f_n \), \( cn^{-r} \leq \|f_n\| \leq Cn^{-r} \), \( n^{-r} \)-consistent, if there is \( k_n \approx n^{2-4r} \), such that sequence of alternatives \( f_n \) is consistent for test statistics \( T_n \) with such a choice of \( k_n \).

We call sequence of alternatives \( f_n \), \( cn^{-r} \leq \|f_n\| \leq Cn^{-r} \), \( n^{-r} \)-inconsistent if sequence of alternatives \( f_n \) is inconsistent for all test statistics \( T_n \) with \( k_n \approx n^{2-4r} \).

**Theorem 6.3.** Sequence of alternatives \( f_n \), \( cn^{-r} < \|f_n\| < Cn^{-r} \), is inconsistent, iff, for all \( c_2 \), there holds

\[
\sum_{|j| < c_2 m_n} |\theta_{jn}|^2 = o(n^{-2r})
\]

as \( n \to \infty \).

**Theorem 6.4.** Let sequence of alternatives \( f_n \) be \( n^{-r} \)-consistent. Then, for any \( n^{-r} \)-inconsistent sequence of alternatives \( f_{1n} \) such that \( 1 + f_n + f_{1n} \) are densities, for tests \( K_n, \alpha(K_n) = \alpha(1 + o(1)) \), generated test statistics \( T_n \), there holds

\[
\lim_{n \to \infty} (\beta(K_n; f_n) - \beta(K_n; f_n + f_{1n})) = 0.
\]

For setup of Theorem 6.4, in the case of sequence of alternatives \( f_n + f_{1n} \), number of cells \( k_n \) for test statistics \( T_n \) is the same as in the case of sequence of alternatives \( f_n \).
Theorem 6.5. Sequence of alternatives \( f_n, \ cn^{-r} < \| f_n \| < Cn^{-r} \), is purely \( n^{-r} \)-consistent, iff, for any \( \epsilon > 0 \) there is \( C_1 = C_1(\epsilon) \) such that there holds

\[
\sum_{|j| > C_1m_n} \| \theta_{jn} \|^2 \leq c n^{-2r}.
\]

(6.4)

for all \( n > n_0(\epsilon) \).

Theorem 6.6. \( A \) holds.

Theorem 6.7. Let sequence of alternatives \( f_n, \ cn^{-r} < \| f_n \| < Cn^{-r}, \) be purely \( n^{-r} \)-consistent. Then there is \( c_1 \) such that sequence of alternatives \( f_n \) is consistent for any test statistics \( T_n \) with number of cells \( k_n > c_1 n^{2-4r} \).

The statement of Theorem 6.7 holds if functions of sequence alternatives are sums functions from purely consistent sequence of alternatives and functions from inconsistent sequence of alternatives.

Theorem 6.8. Balls \( B^s_{2\infty}(P_0) \) in Besov spaces \( B^s_{2\infty} \) with \( s = \frac{2}{r-2} \) are maxisets for \( \chi^2 \)-tests with the number of cells \( k_n > n^{2-4r} = n^{1/2s} \).

For setup of Theorem 6.8, in definition of maxisets, iv. holds for test statistics \( T_n \) with arbitrary choice of \( k_n \) with \( k_n \to \infty \) as \( n \to \infty \).

Discussion The definition of \( \chi^2 \)-tests is based on indicator functions. Thus \( \chi^2 \)-tests should detect well distribution functions with stepwise densities. Besov spaces \( B^s_{2\infty}, s \geq 1 \), do not contain stepwise functions. It seems strange.

Let us consider \( \chi^2 \)-test with \( k_n = 2^l n, l_n \to \infty \) as \( n \to \infty \). Then \( \chi^2 \)-test statistics admit representation

\[
T_n(\hat{F}_n) = k_n n \sum_{i=1}^{l_n} \sum_{j=1}^{2^i} \hat{\beta}_{ij}^2,
\]

with

\[
\hat{\beta}_{ij} = \frac{1}{n} \sum_{m=1}^{n} \phi_{ij}(X_m),
\]

where \( \phi_{ij} \) are functions of Haar orthogonal system, \( \phi_{ij}(x) = 2^{i/2} \phi(2^i x - j) \) with \( \phi(x) = 1 \) if \( x \in (0, 1/2) \), \( \phi(x) = -1 \) if \( x \in (1/2, 1) \) and \( \phi(x) = 0 \) otherwise.

Implementing the same reasoning as in the case quadratic test statistics and using Theorem 9.3 given below, we get that \( \chi^2 \)-test statistics with the number of cells \( k = 2^l, l = 1, 2, \infty \) have maxisets

\[
\tilde{B}^s_{2\infty}(P_0) = \left\{ f : f = 1 + \sum_{k=1}^{\infty} \sum_{j=1}^{2^k} \beta_{kj} \phi_{kj}, \ \sup_{\lambda > 0} 2^{k_\lambda} \sum_{k>\lambda} \sum_{j=1}^{2^k} \beta_{kj}^2 \leq P_0 \right\}.
\]

This statement is true as well.
Suppose function $f$ is sufficiently smooth and $\beta_{kj}$ are Fourier coefficients of $f$ for Haar orthogonal system. Since $\beta_{kj} = 2^{-k/2} \frac{df}{dx}(j2^{-k})(1 + o(1))$ as $k \to \infty$, then

$$\sum_{j=1}^{2^k} \beta_{kj}^2 = C2^{-k/2} \int \left( \frac{df}{dx} \right)^2 dx(1 + o(1)).$$

Thus we see that $f$ does not belong usually to $B_{s}^{2}, s > 1$, for such a setup.

Kernel-based tests also detect stepwise densities well. However these densities does not also belong to the maxispaces for kernel-based tests.

**Theorem 6.9.** Maxisets are regular.

In Theorem 6.10 given below we consider slightly modified definition of perfect maxisets. We suppose that (2.5), (2.6) and (2.7) should be hold only for sequence of alternatives $f_n$ satisfying the following requirement.

B. There is $c_0$ such that, for all $c > c_0$,

$$1 + f_{cn} = 1 + \sum_{|j| > cm_n} \theta_j \phi_j \quad \text{and} \quad 1 + f_n - f_{cn} = 1 + \sum_{|j| < cm_n} \theta_j \phi_j$$

are densities.

**Theorem 6.10.** Maxisets are perfect.

In proof of Theorem 6.10 we show that there exist $C_\epsilon = C(\epsilon, c, C, c_0)$ such that, for densities $1 + f_{1n} = 1 + \sum_{|j| > cm_n} \theta_j \phi_j$, (2.5), (2.6) and (2.7) hold. By Lemma 9.3 given below, there is $\gamma_\epsilon$ such that $f_{1n} \in \gamma_\epsilon U$.

7. Cramer – von Mises tests

We consider Cramer – von Mises test statistics as functionals

$$T^2(\hat{F}_n - F_0) = \int_0^1 (\hat{F}_n(x) - F_0(x))^2 dF_0(x)$$

depending on empirical distribution function $\hat{F}_n$. Here $F_0(x) = x, x \in (0, 1)$.

The functional $T$ is a norm on the set of differences of distribution functions. Therefore we have

$$T(\hat{F}_n - F_0) - T(F - F_0) \leq T(\hat{F}_n - F) \leq T(\hat{F}_n - F_0) + T(F - F_0). \quad (7.1)$$

Hence it is easy to see that sequence of alternatives $F_n$ is consistent, iff, there is $c$ such that

$$nT^2(F_n - F_0) > c \quad \text{for all} \quad n > n_0 \quad (7.2)$$

In previous sections functionals $T_n$ depend on $n$. In this setup we explore the unique functional $T$ for all $n$ and $r$, $0 < r < 1/2$. To separate the study of sequences of alternatives for different $r$, we modify the definition of $n^{-r}$-consistency.
Sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, is called $n^{-r}$-consistent, if (2.1) holds, and, for any sequence of tests $K_n$ generated test statistics $T^2(F_n - F_0)$, there does not hold

$$\lim_{n \to \infty} \inf (\alpha(K_n) + \beta(K_n, f_n)) = 0. \tag{7.3}$$

The definition of $n^{-r}$-inconsistency remains the same.

The results are provided in terms of sequence model

$$f_n = \sum_{j=1}^{\infty} \theta_j \phi_j, \tag{7.4}$$

with the orthonormal functions $\phi_j(x) = \sqrt{2} \sin(\pi jx), x \in [0,1], 1 \leq j < \infty$.

Denote $k_n = \lceil n^{(1-2r)/2} \rceil$.

For sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, (7.3) does not hold, iff, for any $\varepsilon > 0$, there is $c_3$ such that there hold

$$\sup_{l < c_3 k_n} n l^{-2} \sum_{|j| < l} \theta_{jn}^2 < \varepsilon, \tag{7.5}$$

for all $n > n_0(\varepsilon, c_3)$.

**Theorem 7.1.** Sequence of alternatives $f_n$ is $n^{-r}$-consistent, iff, there holds (7.5), and there are $c_1, c_2$ and $n_0$ such that there holds

$$\sum_{|j| < c_2 k_n} \theta_{jn}^2 > c_1 n^{-2r} \tag{7.6}$$

for all $n > n_0$.

**Theorem 7.2.** Sequence of alternatives $f_n$ is $n^{-r}$-inconsistent, iff, for all $c_2$, there holds

$$\lim_{n \to \infty} \sup_{l < c_2 k_n} n l^{-2} \sum_{|j| < l} \theta_{jn}^2 = 0. \tag{7.7}$$

**Theorem 7.3.** Let $\kappa > 0$. Let sequence of alternatives $f_n$, $\|f_n\| \leq Cn^{-\kappa}$, be consistent. Let $f_1n$, $\|f_1n\| \leq C_1 n^{-\kappa}$, be any inconsistent sequence of alternatives such that $1 + f_n(x) + f_1n(x)$ are densities. Then, for tests $K_n$, $\alpha(K_n) = \alpha(1 + o(1))$, generated test statistics $T_n$, there holds

$$\lim_{n \to \infty} (\beta(K_n, f_n) - \beta(K_n, f_n + f_1n)) = 0. \tag{7.8}$$

**Theorem 7.4.** Sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, is purely $n^{-r}$-consistent, iff, (7.5) holds and, for any $\varepsilon > 0$, there is $C_1 = C_1(\varepsilon)$ such that there holds

$$\sum_{j > C_1 k_n} \theta_{jn}^2 \leq \varepsilon n^{-2r}. \tag{7.9}$$

for all $n > n_0(\varepsilon)$. 
Theorem 7.5. If sequence of alternatives $f_n$ is purely $n^{-\tau}$-consistent, then, for any $n^{-\tau}$-inconsistent subsequence of alternatives $f_{1n}$, there holds
\[ \|f_{ni} + f_{1n}\|^2 = \|f_{ni}\|^2 + \|f_{1n}\|^2 + o(n_i^{-\tau}). \] (7.10) as $i \to \infty$.

Theorem 7.6. The balls $B_{2n}^s(P_0)$ with $s = \frac{2r}{1-r}$, $r = \frac{s}{2s+2r}$, are maxisets for Cramér – von Mises test statistics. Here the balls $B_{2n}^s(P_0)$ are defined by (1.5) with orthonormal system of functions $\phi_j(t) = \sqrt{2} \sin(\pi jt)$, $t \in [0,1)$, $j = 1,2,\ldots$.

Theorem 7.7. Maxisets are regular.

In Theorem 7.8 given below we consider the same definition of perfect maxisets as in section 6.

Theorem 7.8. Maxisets are perfect.

8. Asymptotically minimax tests for maxisets

Let we observe a random process $Y_n(t)$, $t \in [0,1)$, defined by the stochastic differential equation (1.4) with unknown signal $f$.

Our goal is to point out asymptotically minimax tests for the problem testing of the hypothesis $H_0 : f(t) = 0$, $t \in [0,1)$, versus the alternatives
\[ H_n : \|f\|^2 > \rho_n \asymp n^{-\frac{1}{1+r}} \]
if a priori information is provided that $f \in \tilde{B}_{2\infty}^s(P_0)$.

Denote $V_n = \{ f : \|f\|^2 \geq \rho_n, f \in \tilde{B}_{2\infty}^s(P_0) \}$.

Note that, for Besov balls $\tilde{B}_{2\infty}^s(P_0)$ we get that penalized maximum likelihood estimators are asymptotically minimax [12]. This illustrates the role of such a priori information in statistical inference.

The proof, in main features, repeats the reasoning in Ermakov [6]. The main difference is the solution of another extremal problem caused by another definition of sets of alternatives. Other differences have technical character and are also caused the differences of definition of sets of alternatives.

The results will be provided in terms of sequence model (see section 3).

Define $k = k_n$ and $\kappa^2 = \kappa_n^2$ as the solution of two equations
\[ \frac{1}{2s} k_n^{1+2s} \kappa_n^2 = P_0 \] (8.1)
and
\[ k_n \kappa_n^2 + k_n^{-2s} P_0 = \rho_n. \]  
\( (8.2) \)

Denote \( \kappa_j^2 = \kappa_n^2 \), for \( 1 \leq j \leq k_n \) and \( \kappa_j^2 = 2sP_0j^{-2s-1} \), for \( j > k_n \).

Define test statistics
\[ T_n^a(Y_n) = \sigma^{-2} n \sum_{j=1}^{\infty} \kappa_j^2 y_j^2. \]

and put
\[ A_n = \sigma^{-4} n^2 \sum_{j=1}^{\infty} \kappa_j^4, \]
\[ C_n = \sigma^{-2} n \rho_n. \]

For type I error probabilities \( \alpha, 0 < \alpha < 1 \), define the critical regions
\[ S_n^a = \{ y : (T_n^a(y) - C_n)(2A_n)^{-1/2} > x_\alpha \} \]
with \( x_\alpha \) defined by equation \( \alpha = 1 - \Phi(x_\alpha) \).

**Theorem 8.1.** Let
\[ 0 < \liminf_{n \to \infty} A_n \leq \limsup_{n \to \infty} A_n < \infty. \]  
\( (8.3) \)

Then the tests \( L_n^a \) with critical regions \( S_n^a \) are asymptotically minimax with \( \alpha(L_n^a) = \alpha(1 + o(1)) \) and
\[ \beta(L_n^a, V_n) = \Phi(x_\alpha - (A_n/2)^{1/2})(1 + o(1)) \]  
\( (8.4) \)
as \( n \to \infty \).

**Example.** Let \( \rho_n = R(\sigma^2/n)^{1+4s}(1 + o(1)) \) as \( n \to \infty \). Then
\[ A_n = \sigma^{-4} n^2 \rho_n^{1+4s} \frac{8s^2}{(1+4s)(1+2s)}((1 + 2s)P_0)^{-1/2s}(1 + o(1)) \]
\[ = R^2 \frac{8s^2}{(1+4s)(1+2s)}((1 + 2s)P_0)^{-1/2s}(1 + o(1)). \]

Ingster, Sapatinas, Suslina [18] and Laurent, Loubes, Marteau [22] have explored the problem of signal detection for linear inverse ill-posed problems. The setup was treated in terms of sequence model
\[ y_j = \lambda_j \theta_j + \frac{\sigma}{\sqrt{n}} \xi_j, \quad 1 \leq j < \infty \]
where \( \xi_j \) are i.i.d.r.v.’s having standard normal distribution and \( \lambda_j \) is sequence of eigenvalues of linear operator.

It is easy to see that, if \( |\lambda_j| \asymp j^{-\gamma} \), then the maxisets for tests statistics defined as quadratic forms of \( y_j, 1 \leq j < \infty \), are the balls in \( \mathbb{B}_r^{2s} \) with \( r = \)
Thus it is of interest to point out asymptotically minimax test statistics for the problem of testing of hypothesis $H_0 : \theta = 0$ versus alternatives $H_n : \theta \in V_n$.

Define test statistics

$$T_n(Y_n) = \sigma^{-2n} \sum_{j=1}^{\infty} \kappa_j^2 y_j^2,$$

with $\kappa_j^2$ defined the equations $\kappa_j^2 = a\lambda_j^{-2}$ for $j \leq k_n$ and $\kappa_j^2 = 2sP_0\lambda_j^{-1-2s}$ for $j > k_n$, where constants $a = a_n$ and $k_n$ are the solutions of equations

$$a_n \sum_{j=1}^{k_n} \lambda_j^{-4} + P_0 k_n^{-2s} = \rho_n(1 + o(1)) \quad \text{and} \quad a_n \lambda k_n^{-4} = 2sP_0 k_n^{-1-4s}(1 + o(1)).$$

In this notation the definition of $A_n$ and the critical regions $S_n^\alpha$ is the same as in Theorem 8.1.

**Theorem 8.2.** Let $|\lambda_j| > j^{-\gamma}$. Then for the above setup and for above notation the statement of Theorem 8.1 holds.

**Example.** Let $\lambda^2_j = Aj^{-2\gamma}$ and let $\rho_n \asymp n^{-\frac{1}{\gamma}}$. Then

$$A_n = \sigma^{-4} n^2 \rho_n^{-\frac{1+4\gamma}{2}} A^2 \frac{8s^2(1 + 4\gamma)}{(1 + 2s + 4\gamma)(1 + 4s + 4\gamma)} \left(1 + 2s + 4\gamma\right) P_0 \frac{1+4\gamma}{2} (1 + o(1)).$$

Proof of Theorem 8.2 is akin to that of Theorem 8.1 and is omitted.

9. **Proof of Theorems**

9.1. **Proof of Theorem 3.1**

For any vectors $\theta_1 \in \mathbb{H}$ and $\theta_2 \in \mathbb{H}$ define segment $\text{int}(\theta_1, \theta_2) = \{\theta : \theta = (1-\lambda)\theta_1 + \lambda\theta_2, \lambda \in [0, 1]\}$.

Proof of Theorem 3.1 is based on the following Lemma 9.1

**Lemma 9.1.** For any vectors $\theta_1 \in U$ and $\theta_2 \in U$ we have $\text{int}\left(\frac{\theta_1-\theta_2}{2}, \frac{\theta_1+\theta_2}{2}\right) \subset U$. There hold $0 \in \text{int}\left(\frac{\theta_1-\theta_2}{2}, \frac{\theta_1+\theta_2}{2}\right)$ and segment $\text{int}\left(\frac{\theta_1-\theta_2}{2}, \frac{\theta_1+\theta_2}{2}\right)$ is parallel to segment $\text{int}(\theta_1, \theta_2)$.

**Remark 3.1.** Let we have segment $\text{int}(\theta_1, \theta_2) \subset U$. Let $\eta$ and $-\eta$ be points of intersection of line $L = \{\theta : \theta = \lambda(\theta_1 - \theta_2), \lambda \in \mathbb{R}^1\}$ and boundary of set $U$. Then, by Lemma 9.1, we have $||\theta_1 - \theta_2|| \leq 2||\eta||$.

**Proof of Lemma 9.1.** The segments $\text{int}(\theta_1, \theta_2) \subset U$ and $\text{int}(-\theta_1, -\theta_2) \subset U$ are parallel. For each $\lambda \in [0, 1]$ we have $(1-\lambda)\theta_1 + \lambda\theta_2 \in \text{int}(\theta_1, \theta_2)$ and $-\lambda\theta_1 - (1-\lambda)\theta_2 \in \text{int}(-\theta_1, -\theta_2)$. The middle $\theta_\lambda = \frac{(1-2\lambda)\theta_1 - (1-2\lambda)\theta_2}{2}$ of segment $\text{int}((1-\lambda)\theta_1 + \lambda\theta_2, -\lambda\theta_1 - (1-\lambda)\theta_2) \subset U$ belongs to segment $\text{int}\left(\frac{\theta_1-\theta_2}{2}, \frac{\theta_1+\theta_2}{2}\right)$ and, for
each point $\theta$ of segment $\text{int}\left(\theta_{1}, \theta_{2}\right)$, there is $\lambda \in [0, 1]$ such that $\theta = \theta_{\lambda}$.
Therefore $\text{int}\left(\frac{\theta_{1} - \theta_{2}}{2}, \frac{\theta_{2} - \theta_{1}}{2}\right) \subset U$.

Proof of Theorem 3.1. Without loss of generality we can suppose that the set $U$ is closed. Define sequence of orthogonal vectors $e_{i}$ by induction.

Let $e_{1}, e_{2} \in U$, be such that $\|e_{i}\| = \sup\{\|\theta\|, \theta \in U\}$. Denote $\Pi_{1}$ linear subspace generated $e_{1}$. Denote $\Pi_{1}$ subspace orthogonal to $\Pi_{1}$.

Let $e_{i} \in U \cap \Pi_{i-1}$ be such that $\|e_{i}\| = \sup\{\|\theta\| : \theta \in U \cap \Pi_{i-1}\}$. Denote $\Pi_{i}$ linear subspace generated vectors $e_{1}, \ldots, e_{i}$. Denote $\Pi_{i}$ subspace orthogonal to $\Pi_{i}$.

For all natural $i$ denote $d_{i} = \|e_{i}\|$. Note that $d_{i} \rightarrow 0$ as $i \rightarrow \infty$. Otherwise, by Theorem 5.3 in Ermakov [11], there does not exist consistent test for the problem testing hypothesis $H_{0} : \theta = 0$ versus alternatives $H_{n} : \theta = e_{i}$, $i = 1, 2, \ldots$.

For any $\varepsilon \in (0, 1)$ denote $l_{\varepsilon} = \min\{j : d_{j} < \varepsilon, j = 1, 2, \ldots\}$.

Denote $B_{r}(\theta)$ ball having radius $r$ and center $\theta$.

It suffices to show that, for any $\varepsilon_{1} > 0$ there is finite coverage of set $U$ by balls $B_{\varepsilon_{1}}(\theta)$.

Denote $\varepsilon = \varepsilon_{1}/6$.

Denote $U_{\varepsilon}$ projection of set $U$ onto subspace $\Pi_{l_{\varepsilon}}$.

Denote $\bar{B}_{r}(\theta)$ ball in $\Pi_{l_{\varepsilon}}$ having radius $r$ and center $\theta \in \Pi_{l_{\varepsilon}}$. There is ball $\bar{B}_{r}(0)$ such that $\bar{B}_{r}(0) \subset U$. Denote $\delta = \min\{\varepsilon, \varepsilon_{1}\}$.

Let $\theta_{1}, \ldots, \theta_{k}$ be $\delta$-net in $U_{\varepsilon}$.

Let $\eta_{1}, \ldots, \eta_{k}$ be points of $U$ such that $\theta_{i}$ is projection of $\eta_{i}$ onto subspace $\Pi_{l_{\varepsilon}}$ for $1 \leq i \leq k$.

Let us show that $B_{\varepsilon_{1}}(\eta_{1}), \ldots, B_{\varepsilon_{1}}(\eta_{k})$ is coverage of set $U$.

Let $\eta \in U$ and let $\theta$ be projection of $\eta$ onto $\Pi_{l_{\varepsilon}}$. Let $\|\theta_{i} - \theta\| \leq \delta$. It suffices to show that $\eta \in B_{\varepsilon_{1}}(\eta_{i})$.

By Lemma 9.1, $\text{int}\left(\frac{\eta_{i} - \eta}{2}, \frac{\eta - \eta_{i}}{2}\right) \subset U$. We have $\theta_{i} - \theta \in \bar{B}_{r}(0)$. Therefore $\left((\eta_{i} - \eta_{1}) - (\eta - \theta)\right)/2 \in U$ and vector $(\eta_{i} - \theta_{i}) - (\eta - \theta)$ is orthogonal subspace $\Pi_{l_{\varepsilon}}$. Therefore, by Remark 3.1, $\|(\eta_{i} - \theta_{1}) - (\eta - \theta))\|/2 \leq 2\varepsilon$. Therefore $\|\eta - \eta_{i}\| \leq 4\varepsilon + \|\theta_{i} - \theta\| < 5\varepsilon$. This implies $\eta \in B_{\varepsilon_{1}}(\eta_{i})$. This completes proof of Theorem 3.1.

9.2. Proof of Theorems of section 4

The reasoning is based on Theorem 9.1 on asymptotic minimaxity of test statistics $T_{n}$.

Theorem 9.1. Assume A1-A5. Then sequence of tests $K_{n}(Y_{n}) = \chi\{n^{-1}T_{n}(Y_{n}) > (2A_{n})^{1/2}x_{n}\} \text{ is asymptotically minimax for the sets } Q_{n}(c) \text{ of alternatives.}$

There holds
$$\beta(K_{n}, \theta) = \Phi(x_{\alpha} - A_{n}(\theta)(2A_{n})^{-1/2})(1 + o(1))$$
uniformly in all $\theta$ such that $A_{n}(\theta) < C$. Here $x_{\alpha}$ is defined by the equation $\alpha = 1 - \Phi(x_{\alpha})$. 


A version of Theorem 9.1 for the problem of signal detection with heteroscedastic white noise has been proved in Ermakov [8].

Proof of Theorem 9.1. Theorem 9.1 and its version for Remark 4.1 setup can be deduced straightforwardly from Theorem 1 in Ermakov [6].

The lower bound follows from Theorem 1 in [6] straightforwardly.

The upper bound follows from the following reasoning. We have

\[
\sum_{j=1}^{\infty} \kappa_{jn}^2 \theta_j^2 = \sum_{j=1}^{\infty} \kappa_{jn}^2 \theta_j^2 + 2 \frac{\sigma}{\sqrt{n}} \sum_{j=1}^{\infty} \kappa_{jn}^2 \theta_j \xi_j + \frac{\sigma^2}{n} \sum_{j=1}^{\infty} \kappa_{jn}^2 \xi_j^2 \quad (9.2)
\]

with

\[
E[J_{3n}] = \frac{\sigma^2}{n} \rho_n, \quad \text{Var}[J_{3n}] = 2 \frac{\sigma^4}{n^4} A_n, \quad (9.3)
\]

\[
\text{Var}[J_{2n}] = \frac{\sigma^2}{n} \sum_{j=1}^{\infty} \kappa_{jn}^4 \theta_j^2 \leq \frac{\sigma^2 \kappa^2}{n} \sum_{j=1}^{\infty} \kappa_{jn}^2 \theta_j^2 = o(J_{1n}). \quad (9.4)
\]

By Chebyshev inequality, it follows from (9.2) - (9.4), that, if \( A_n n^{-2} = o(J_{1n}) \), that \( A_n n^{-2} = o\left(\sum_{j=1}^{\infty} \kappa_{jn}^2 \theta_j^2\right) \) as \( n \to \infty \), then \( \beta(L_n, \theta_n) \to 0 \) as \( n \to \infty \). Thus it suffices to explore the case

\[
A_n^2 \asymp A_n(\theta) = n^2 \sum_{j=1}^{\infty} \kappa_{jn}^2 \theta_j^2. \quad (9.5)
\]

If (9.5) holds, then implementing the reasoning of proof of Lemma 1 in [6] we get that (9.1) holds. This completes the proof of Theorem 9.1.

Proof of Theorem 4.1. Let (4.5) hold. Then, by A5 and (4.4), we have

\[
A_n(\theta_n) = n^2 \sum_{j=1}^{\infty} \kappa_{jn}^2 \theta_j^2 \geq C n^2 \kappa_n^2 \sum_{j=1}^{c^2 K_n} \theta_j^2 \asymp n^2 \kappa_n^2 n^{-2r} \asymp 1. \quad (9.6)
\]

By Theorem 9.1, this implies sufficiency.

Necessary conditions follows from sufficiency conditions in Theorem 4.2.

Proof of Theorem 4.2. Let (4.6) hold. Then, by (4.4) and A2, we have

\[
A_n(\theta_n) \leq C n^2 \kappa_n^2 \sum_{j=1}^{c^2 K_n} \theta_j^2 + C n^2 \kappa_{[c^2 n]}^2 \sum_{j>c^2 n} \theta_j^2 \asymp o(1) + O(\kappa_{[c^2 n]}^2 / \kappa_n^2). \quad (9.7)
\]

By A4, we have

\[
\lim_{c^2 \to \infty} \lim_{n \to \infty} \kappa_{[c^2 n]}^2 / \kappa_n^2 \to 0, \quad (9.8)
\]

By Theorem 9.1, (9.7) and (9.8), we get sufficiency.

Necessary conditions follows from sufficiency conditions in Theorem 4.1.
Proof of Theorem 4.3. Let \( f_n = \sum_{j=1}^{\infty} \theta_j \phi_j \) and let \( f_{1n} = \sum_{j=1}^{\infty} \eta_j \phi_j \).

Denote \( \eta_n = \{ \eta_j \}_{j=1}^{\infty} \).

We have

\[
|A_n(\theta) - A_n(\theta + \eta_n)| = n^2 \left| \sum_{j=1}^{\infty} \kappa_j^2 \theta_j^2 - \sum_{j=1}^{\infty} \kappa_j^2 (\theta_j + \eta_j)^2 \right|
\]

\[
\leq n^2 \left| \sum_{j=1}^{\infty} \kappa_j^2 \theta_j \eta_j \right| + A_n(\eta_n) = J_n + A_n(\eta_n).
\]

By Cauchy inequality, we have

\[
J_n \leq A_n^{1/2}(\theta) A_n^{1/2}(\eta).
\]

By inconsistency of sequence \( f_{1n} \) and Theorem 9.1, we get \( A_n^{1/2}(\eta_n) = o(1) \) as \( n \to \infty \). Therefore, by (9.10), \( J_n + A_n(\eta_n) = o(1) \) as \( n \to \infty \). Hence, by Theorem 9.1 and (9.9), we get Theorem 4.3.

Proof of Theorem 4.6. iii. The statement follows from Theorem 4.1 and Lemma 9.2 provided below.

**Lemma 9.2.** Let \( cn^{-r} \leq \| f_n \| \leq Cn^{-r} \) and \( f_n \in c_1 U \). Then, for \( k_n = C_1 n^{2-r} (1 + o(1)) = C_1 n^{2-r} (1 + o(1)) \) with \( C_1 > 2c_1/c \), there holds

\[
\sum_{j=1}^{k_n} \theta_j^2 > \frac{c}{2} n^{-2r}.
\]

Proof. If \( k_n^2/C_1 = C_1^{2-s} n^{2r} (1 + o(1)) \) and \( f_n \in c_1 U \), then we have

\[
k_n^2 \sum_{j=k_n}^{\infty} \theta_j^2 \leq C_1^{2-s} n^{2r} \sum_{j=k_n}^{\infty} \theta_j^2 (1 + o(1)) \leq c_1.
\]

Hence

\[
\sum_{j=k_n}^{\infty} \theta_j^2 \leq c_1 C_1^{-2-s} n^{-2r} \leq \frac{c}{2} n^{-2r}.
\]

Therefore (9.11) holds.

Proof of Theorem 4.6. iv. Suppose the opposite. Then there are \( f = \sum_{j=1}^{\infty} \tau_j \phi_j \notin \mathbb{B}_2^{\infty} \), and a sequence \( m_l, m_l \to \infty \) as \( l \to \infty \), such that

\[
m_l^{2s} \sum_{j=m_l}^{\infty} \tau_j^2 = C_l,
\]

with \( C_l \to \infty \) as \( l \to \infty \).

Define a sequence \( \eta_l = \{ \eta_{jl} \}_{j=1}^{\infty} \) such that \( \eta_{jl} = 0 \) if \( j < m_l \) and \( \eta_{jl} = \tau_j \) if \( j \geq m_l \).

Since \( U_f \) is convex and orthosymmetric we have \( \hat{f}_l = \sum_{j=1}^{\infty} \eta_{jl} \phi_j \in U_f \).
For alternatives \( \tilde{f} \) we define sequence \( n_\ell \) such that

\[
\| \eta_\ell \|^2 \asymp n_\ell^{-2r} \asymp m_\ell^{-2s} C_l.
\]

(9.15)

Then

\[
n_\ell \asymp C_l^{-1/(2r)} m_\ell^{s/r} = C_l^{-1/(2r)} m_\ell^{s/r}. \]

(9.16)

Therefore we get

\[
m_\ell \asymp C_l^{(1-2r)/r} n_\ell^{-2-4r}. \]

(9.17)

By A4, (9.17) implies

\[
\kappa_{m_\ell n_\ell}^2 = o(\kappa_{n_\ell}^2) \quad \text{(9.18)}
\]

Using (4.4), A2 and (9.18), we get

\[
\sum_{j=1}^{\infty} \kappa_{m_\ell n_\ell}^2 \leq \sum_{j=m_\ell}^{\infty} \kappa_{m_\ell n_\ell}^2 \leq C_1 n_\ell^{2-2r} \kappa_{m_\ell n_\ell}^2 = O(\kappa_{m_\ell n_\ell}^2) = o(1).
\]

(9.19)

By Theorem 9.1, (9.19) implies \( n^{-r} \)-inconsistency of subsequence of alternatives \( \tilde{f} \).

**Proof of Theorem 4.7.** Theorem 4.7 follows from Lemmas 9.3 – 9.5.

**Lemma 9.3.** For any \( c \) and \( C \) there is \( \gamma \) such that, if \( \| f_n \| \leq C n^{-r} \) and \( f_n = \sum_{j=1}^{c n} \theta_{jn} \phi_j \), then \( f_n \in \gamma U \).

Proof. We have

\[
k_n^{2-2r} \sum_{j=1}^{\infty} \theta_{jn}^2 \leq C_1 n_\ell^{2-2r} \sum_{j=m_\ell}^{\infty} \theta_{jn}^2 = O(\kappa_{m_\ell n_\ell}^2) = o(1).
\]

(9.20)

This implies Lemma 9.3.

**Lemma 9.4.** Let (2.5) hold. Then sequence \( f_n \) is \( n^{-r} \)-consistent.

Let \( f_n = \sum_{j=1}^{\infty} \theta_{jn} \phi_j \) and let

\[
f_1 = \sum_{j=1}^{\infty} \eta_{jn} \phi_j, \quad f_n = f_1 + \sum_{j=1}^{\infty} \zeta_{jn} \phi_j.
\]

For any \( \delta > 0 \) there is \( c \) such that

\[
\sum_{j>cn} \eta_{jn}^2 < \delta n^{-2r} \quad \text{(9.21)}
\]

for each \( f_1 \in c_1 U \), \( \| f_1 \| \leq C_2 n^{-r} \).

We have

\[
J_n = \left| \sum_{j>cn} \theta_{jn}^2 - \sum_{j>cn} \zeta_{jn}^2 \right| \leq \sum_{j>cn} |\eta_{jn}(2\theta_{jn} - \eta_{jn})| \leq \left( \sum_{j>cn} \eta_{jn}^2 \right)^{1/2} \left( \sum_{j>cn} \theta_{jn}^2 \right)^{1/2} \leq C_1^{1/2} n^{-2r}.
\]

(9.22)
By (2.5), we have \( \|f_n\| \geq \|f_{1n}\| \). Hence, by (9.21) and (9.22), we get
\[
\sum_{j < c_k n} \theta_{jn}^2 \geq \sum_{j < c_k n} \eta_{jn}^2 - \sum_{j > c_k n} \eta_{jn}^2 - J_n
\geq \sum_{j < c_k n} \eta_{jn}^2 - \delta n^{-2r} - C \delta^{1/2} n^{-2r}
\geq \|f_{1n}\|^2 - 2 \delta n^{-2r} - C \delta^{1/2} n^{-2r}.
\] (9.23)

By Theorem 4.1, (9.23) implies consistency of sequence \( f_n \).

**Lemma 9.5.** Let sequence \( f_n, cn^{-r} \leq \|f_n\| \leq Cn^{-r} \), be consistent. Then (2.5) holds.

**Proof.** By Theorem 4.1, there are \( c_1 \) and \( c_2 \) such that sequence \( f_{1n} = \sum_{j=1}^{c_2 k_n} \theta_{jn} \phi_j \) is consistent and \( \|f_{1n}\| \geq c_1 n^{-r} \). By Lemma 9.3, there is \( \gamma > 0 \) such that \( f_{1n} \in \gamma U \).

For proof of ii. it suffices to put
\[
f_{1n} = \sum_{j < C_1(\varepsilon) k_n} \theta_{jn} \phi_j.
\]

By Lemma 9.3, there is \( \gamma_\varepsilon \) such that \( f_{1n} \in \gamma_\varepsilon U \).

**Proof of Theorem 4.4.** Sufficiency. Let \( f_{ni} = f_{1ni} + f_{2ni} \), and \( \|f_{ni}\|^2 = \|f_{1ni}\|^2 + \|f_{2ni}\|^2 \), \( \|f_{1ni}\| > c_1 n_i^{-r} \) and \( \|f_{2ni}\| > c_2 n_i^{-r} \), \( f_{1ni} = \sum_{j=1}^{\infty} \theta_{1jn} \phi_j \) and \( f_{2ni} = \sum_{j=1}^{\infty} \theta_{2jn} \phi_j \). Then, by (4.8), estimating similarly to (9.21) - (9.23), we get
\[
\sum_{j < c_1 k_n} \theta_{1jn}^2 > \frac{1}{2} c_1^2 n_i^{-2r}
\] (9.24)
for sufficiently small \( \varepsilon \) and \( n > n_0(\varepsilon) \)

Similar estimate holds for the sequence \( f_{2ni} \). Hence, by Theorem 4.1, we get \( n_i^{-r} \)-consistency of sequences \( f_{1ni} \) and \( f_{2ni} \).

**Proof of Theorem 4.4.** Necessary conditions. Let (4.8) do not valid. Then there are \( \varepsilon > 0 \) and sequences \( C_i \to \infty \), \( n_i \to \infty \) as \( i \to \infty \) such that
\[
\sum_{j > C_i k_{ni}} \theta_{jn_i}^2 > \varepsilon n_i^{-2r}.
\] (9.25)

Then, by A4 and (4.4), we get
\[
n_i^2 \sum_{j > C_i k_{ni}} \kappa_{jn_i}^2 \theta_{jn_i}^2 = o(1).
\] (9.26)

Therefore, by Theorem 9.1, subsequence \( f_{1ni} = \sum_{j > C_i k_{ni}} \theta_{jn} \phi_j \) is inconsistent.

**Proof of Theorem 4.5.** Necessary conditions are rather evident and proof is omitted. The proof of sufficiency is also simple.
Lemma 9.6. Let for sequence $f_n$, $cn^{-r} < \| f_n \| < Cn^{-r}$, (2.4) hold. Then sequence $f_n$ is purely $n^{-r}$-consistent.

Suppose $f_n = \sum_{j=1}^{\infty} \theta_j n \phi_j$ is not purely $n^{-r}$-consistent. Then, by Theorem 4.4, there are sequence $n_i$, $c_1$ and $c_{n_i} \to \infty$ as $i \to \infty$ such that

$$\sum_{j > c_{n_i} k_{n_i}} \theta_j^2 n_i > c_1 n_i^{-r}.$$  \hfill (9.27)

Therefore, if we put $f_{1n} = \sum_{j > c_{n_i}} \theta_j n_i \phi_j$, then (2.4) does not hold.

Proof of Theorem 4.8. By A4, for any $\delta > 0$ there is $c$ such that

$$n^2 \sum_{j > c n} \theta_j^2 n \phi_j^2 \leq \delta.$$  \hfill (9.28)

By Lemma 9.3, there is $\gamma > 0$ such that $f_{1n} = \sum_{j < c n} \theta_j n \phi_j \in \gamma U$. By Theorem 9.1 and (9.28), for sequence of alternatives $f_{1n}$, (2.6) and (2.7) hold.

9.3. Proof of Theorems of section 5

Denote

$$T_{1n}(f) = \int_0^1 \left( \frac{1}{h_n} \int K \left( \frac{t-s}{h_n} \right) f(s) \, ds \right)^2 \, dt.$$  

Define the sets

$$Q_{nh_n} = \{ f : T_{1n}(f) > \rho_n, f \in L_{2, \text{per}}^2(R^1) \}.$$  

Proof of Theorems is based on the following Theorem 9.2 on asymptotic minimaxity of kernel-based tests [8].

Theorem 9.2. Let $h_n^{-1/2} n^{-1} \to 0$, $h_n \to 0$ as $n \to \infty$. Let

$$0 < \liminf_{n \to \infty} n \rho_n h_n^{1/2} \leq \limsup_{n \to \infty} n \rho_n h_n^{1/2} < \infty.$$  \hfill (9.29)

Then the family of kernel-based tests $L_n = \chi_{\{ T_n(Y_n) \geq x_\alpha \}}, \alpha(L_n) = \alpha(1 + o(1))$, is asymptotically minimax for the sets of alternatives $Q_{nh_n}$.

There holds

$$\beta(L_n, Q_{nh_n}) = \Phi(x_\alpha - \kappa^{-1} \sigma^{-2} n h_n^{1/2} \rho_n)(1 + o(1)).$$  \hfill (9.30)

Here $x_\alpha$ is defined the equation $\alpha = 1 - \Phi(x_\alpha)$.

Moreover, there holds

$$\beta(L_n, f_n) = \Phi(x_\alpha - \gamma^{-1} \sigma^{-2} n h_n^{1/2} \rho_n)(1 + o(1))$$  \hfill (9.31)

uniformly on sequences $f_n \in L_{2, \text{per}}^2(R^1)$ such that $T_{1n}(f_n) = \rho_n(1 + o(1))$. 

On consistency and inconsistency

We have

$$T_{1n}(f_n) = \sum_{j=-\infty}^{\infty} |\hat{K}(jh_n)|^2 |\theta_{jn}|^2. \quad (9.32)$$

Thus, if we put $|\hat{K}(jh_n)|^2 = \kappa_{jn}^2$, we get that asymptotic (9.1) in Theorem (9.1) and asymptotic (9.31) coincide.

The function $\hat{K}(\omega), \omega \in \mathbb{R}$, may have zeros. This cause the main differences in the statement of Theorems and in the reasoning. To clarify the differences we provide the proof of Theorem 5.1 and iv. in Theorem 5.8. Other proofs will be omitted.

In what follows we denote $\rho_n(\theta_n) = T_{1n}(f_n)$.

The function $\hat{K}(\omega), \omega \in \mathbb{R}$, is analytic and $\hat{K}(0) = 1$. Therefore there is an interval $(-b, b), 0 < b < \infty$, such that $|\hat{K}(\omega)| > c$ for all $\omega \in (-b, b)$ for some positive constant $c$.

Proof of Theorem 5.1. Let (5.3) hold. We have

$$\rho_n(\theta_n) = \sum_{j=-\infty}^{\infty} |\hat{K}(jh_n)|^2 |\theta_{jn}|^2 \geq \sum_{|j| < c_2 k_n} |\hat{K}(jh_n)|^2 |\theta_{jn}|^2 \quad (9.33)$$

for $c_2 k_n < b_0 h_n^{-1}$. By Theorem 9.2 this implies consistency.

Proof of Theorem 5.8 iv. Let $f = \sum_{j=1}^{\infty} \tau_j \phi_j \notin cU$ for all $c > 0$. Then there is sequence $m_l, m_l \to \infty$ as $l \to \infty$, such that

$$m_l^{2s} \sum_{|j| \geq m_l} |\tau_j|^2 = C_l \quad (9.34)$$

with $C_l \to \infty$ as $l \to \infty$.

It is clear that we can define a sequence $m_l$ such that

$$m_l^{2s} \sum_{m_l \leq |j| \leq 2m_l} |\tau_j|^2 > \delta C_l \quad (9.35)$$

where $\delta, 0 < \delta < 1/2$, does not depend on $l$. Otherwise, we have

$$2^{2s(i-1)}m_l^{2s} \sum_{j=2^{i-1}m_l}^{2im_l} \tau_j^2 < \delta C_l$$

for all $i = 1, 2, \ldots$, that implies that the left hand-side of (9.34) does not exceed $2\delta C_l$.

Define a sequence $\eta_l = \{\eta_{jl}\}_{j=-\infty}^{\infty}$ such that $\eta_{jl} = \tau_j, |j| \geq m_l$, and $\eta_{jl} = 0$ otherwise.

Denote

$$\hat{f}_l(x) = f_l(x, \eta_l) = \sum_{j=-\infty}^{\infty} \eta_{jl} \exp\{2\pi i j x\}.$$
For alternatives \( \eta \) we define \( n_l \) such that \( \| \eta_l \| \asymp n_l^{-r} \).

Then

\[
n_l \asymp C_l^{-1/(2r)} m_l^{1/r}
\]

(9.36)

We have \( |\hat{K}(\omega)| \leq \hat{K}(0) = 1 \) for all \( \omega \in \mathbb{R}_1 \) and \( |\hat{K}(\omega)| > c > 0 \) for \( |\omega| < b \).

Hence, if we put \( h_l = h_{n_l} = 2^{-1} b^{-1} m_l^{-1} \), then there is \( C > 0 \) such that, for all \( h > 0 \), there holds

\[
T_{1n_l}(\hat{f}_l, h_l) = \sum_{j=-\infty}^{\infty} |\hat{K}(jh_l)\eta_{jl}|^2 > C \sum_{j=-\infty}^{\infty} |\hat{K}(jh)\eta_{jl}|^2 = CT_{1n_l}(\hat{f}_l, h_l).
\]

(9.37)

Thus we can choose \( h = h_l \) for further reasoning.

We have

\[
\rho_{n_l} = \sum_{|j|>m_l} |\hat{K}(jh_l)\eta_{jl}|^2 \asymp \sum_{j=m_l}^{2m_l} |\eta_{jl}|^2 \asymp n_l^{-2r}.
\]

(9.38)

If we put in estimates (9.16), (9.17), \( k_l = [h_{n_l}^{-1}] \) and \( k_l = m_l \), then we get

\[
h_l^{-1/2} \asymp C_l^{(2r-1)/2} n_l^{2r-1}.
\]

(9.39)

By (9.38) and (9.39), we get

\[
n_l \rho_{n_l} h_l^{-1/2} \asymp C_l^{-(1-2r)/2}.
\]

(9.40)

By Theorem 9.2, this implies inconsistency of sequence of alternatives \( \hat{f}_l \).

9.4. Proof of Theorems of section 6

Proof of Theorems is based on Theorem 9.3 provided below. Theorem 9.3 is a summary of results of Theorems 2.1 and 2.4 in Ermakov [7].

Denote \( p_{in} = F_n(i/k_n) - F_n((i-1)/k_n), 1 \leq i \leq k_n \).

Denote \( \mathcal{S} \) the set of all distribution functions.

Define functionals \( T_n : \mathcal{S} \to \mathbb{R}_1 \),

\[
T_n(F) = nk_n \sum_{i=1}^{k_n} (p_{in} - 1/k_n)^2.
\]

Define sets of alternatives

\[
Q_n(b_n) = \left\{ F : T_n(F) \geq b_n, F \in \mathcal{S} \right\}.
\]

The definition of asymptotic minimaxity of tests is the same as in section 4.

Define the tests

\[
K_n = \chi(2^{-1/2} k_n^{-1/2}(T_n(\hat{F}_n) - k_n + 1) > x_\alpha)
\]

where \( x_\alpha \) is defined the equation \( \alpha = 1 - \Phi(x_\alpha) \).
**Theorem 9.3.** Let $k_n^{-1}n^2 \to \infty$ as $n \to \infty$. Let
\[ 0 < \lim \inf_{n \to \infty} k_n^{-1/2}b_n \leq \lim \sup_{n \to \infty} k_n^{-1/2}b_n < \infty. \] (9.41)

Then $\chi^2$-tests $K_n$ are asymptotically minimax for the sets of alternatives $Q_n(b_n)$.

There holds
\[ \beta(K_n, F) = \Phi(x_\alpha - 2^{-1/2}k_n^{-1/2}T_n(F))(1 + o(1)) \] (9.42)
uniformly in $F$ such that $T_n(F) \leq Ck_n^{1/2}$.

For any complex number $a = b + id$ denote $\bar{a} = b - id$.

We have
\[ n^{-1}k_n^{-1}T_n(F) = \sum_{l=0}^{k_n-1} \left( \int_{l/k_n}^{(l+1)/k_n} f(x)dx \right)^2. \] (9.43)

Using representation $f(x)$ in terms of Fourier coefficients
\[ f(x) = \sum_{j=-\infty}^{\infty} \theta_j \exp\{2\pi ijx\}, \] (9.44)
we get
\[ \int_{l/k_n}^{(l+1)/k_n} f(x)dx = \sum_{j=-\infty}^{\infty} \frac{\theta_j}{2\pi i j} \exp\{2\pi ijl/k_n\}(\exp\{2\pi ij/k_n\} - 1), \] (9.45)
for $1 \leq l < k_n$.

**Lemma 9.7.** There holds
\[ n^{-1}k_n^{-1}T_n(F) = k_n \sum_{m=-\infty}^{\infty} \sum_{j \neq mk_n} \frac{\theta_j \bar{\theta}_{j-mk_n}}{4\pi^2 j(j - mk_n)} (2 - 2 \cos(2\pi j/k_n)). \] (9.46)

Proof. We have
\[ n^{-1}k_n^{-1}T_n(F) = \sum_{l=0}^{k_n-1} \left( \sum_{j \neq 0} \frac{\theta_j}{2\pi i j} \exp\{2\pi ijl/k_n\}(\exp\{2\pi ij/k_n\} - 1) \right) \times \left( \sum_{j \neq 0} \frac{-\bar{\theta}_j}{-2\pi i j} \exp\{-2\pi ijl/k_n\}(\exp\{-2\pi ij/k_n\} - 1) \right) = J_1 + J_2, \] (9.47)
with
\[ J_1 = \sum_{l=0}^{k_n-1} \sum_{m=-\infty}^{\infty} \sum_{j \neq mk_n, j_1 = j - mk_n} \frac{\theta_{j_1} \bar{\theta}_{j_1}}{4\pi^2 j j_1} \exp\{2\pi i lm\} \times (\exp\{2\pi ij/k_n\} - 1)(\exp\{-2\pi ij/k_n\} - 1) \] (9.48)
and
\[ J_2 = k_n \sum_{m=-\infty}^{\infty} \sum_{j \neq mk_n} \frac{\theta_j \bar{\theta}_{j-mk_n}}{4\pi^2 j(j - mk_n)} (2 - 2 \cos(2\pi j/k_n)). \]
and
\[ J_2 = \sum_{l=0}^{k_n-1} \sum_{j \neq j_1, j \neq -mk_n} \frac{\theta_j \bar{\theta}_{j_1}}{4\pi^2 j_1} \exp\{2\pi i(j - j_1)l/k_n\} \times (\exp\{2\pi ij/k_n\} - 1)(\exp\{-2\pi ij_1/k_n\} - 1) = 0. \] (9.49)

In the last equality of (9.49) we make use of the identity
\[ \sum_{l=0}^{k_n-1} \exp\{2\pi i(j - j_1)l/k_n\} = \frac{\exp\{2\pi i(j - j_1)k_n/k_n\} - 1}{\exp\{2\pi i(j - j_1)/k_n\} - 1} = 0, \] (9.50)
if \( j - j_1 \neq mk_n \) for all integer \( m \).

For any c.d.f \( F \) denote \( \tilde{F} \) c.d.f. having the density
\[ 1 + \tilde{f}_m(x) = 1 + \sum_{|j| > m} \theta_j \exp\{2\pi i j x\}. \]

For \( d, d > 1 \), denote \( i_n = \lfloor dk_n \rfloor \).

Denote \( \eta_j = \theta_j \) if \( |j| > i_n \) and \( \eta_j = 0 \) if \( |j| < i_n \).

**Lemma 9.8.** There holds
\[ n^{-1} k_n^{-2} T_n(\tilde{F}_n) \leq C k_n^{-1} i_n^{-1} \sum_{|j| > i_n} |\theta_j|^2. \] (9.51)

Proof. Using the agreement 0/0 = 0, we have
\[
n^{-1} k_n^{-2} T_n(\tilde{F}_n) = \sum_{m=-\infty}^{\infty} \sum_{j \neq mk_n} \frac{\eta_j \bar{\eta}_{j-mk_n}}{4\pi^2 j(j-mk_n)} (2 - 2 \cos(2\pi j/k_n))
\]
\[
\leq C \sum_{|j| > i_n} \sum_{m=-\infty}^{\infty} \frac{\eta_j \bar{\eta}_{j-mk_n}}{j+mk_n} (2 - 2 \cos(2\pi j/k_n))
\]
\[
= C \sum_{j=1}^{k_n} \sum_{m=-\infty}^{\infty} \frac{\eta_j + mk_n}{j + mk_n} \sum_{m=-\infty}^{\infty} \frac{\eta_j + (m + m_1)k_n}{j + (m + m_1)k_n}
\]
\[
= C \sum_{j=1}^{k_n} \left( \sum_{m=-\infty}^{\infty} \left| \frac{\eta_j + mk_n}{j + mk_n} \right|^2 \right)
\]
\[
\leq C \sum_{j=1}^{k_n} \left( \sum_{|m| > d} |\eta_j + mk_n|^2 \right) \left( \sum_{|m| > d} (j + mk_n)^{-2} \right)
\]
\[
\leq C \sum_{j=-\infty}^{\infty} |\eta_j|^2 \sum_{|m| > d} (mk_n)^{-2} \leq C k_n^{-1} i_n^{-1} \sum_{|j| > i_n} |\theta_j|^2.
\]
This completes proof of Lemma 9.8.
Proof of Theorem 6.2. Let (6.1) hold. For any $a > 0$, denote
\[ \tilde{f}_{n,a,m} = \sum_{|j| > am} \theta_j \phi_j \]
and denote
\[ f_{n,c_1,k_n,C_1,m} = \tilde{f}_{n,c_1,m} - \tilde{f}_{n,C_1,m}, \quad \bar{f}_n = \tilde{f}_{n,c_1,m} = f_n - \tilde{f}_{n,c_1,m}, \]
with $C_1 > c_1$.
Denote $\tilde{F}_{n,C_1,m}, F_{n,c_1,m,C_1,m}$ and $\hat{F}_n = \hat{F}_{n,C_1,m}$ functions having derivatives $1 + \tilde{f}_{n,C_1,m}$, $1 + f_{n,c_1,m,C_1,m}$ and $1 + \tilde{f}_n$ respectively and such that $\tilde{F}_{n,C_1,m}(1) = 1$, $F_{n,c_1,m,C_1,m}(1) = 1$ and $\hat{F}_n(1) = 1$.
Let $T_n$ be chi-squared test statistics with a number of cells $k_n = [c_3 m_n]$ where $c_1 < c_3 < C_1$.
We have
\[ T_n^{1/2}(\tilde{F}_n) - T_n^{1/2}(F_{n,c_1,m,C_1,m}) - T_n^{1/2}(\hat{F}_{n,C_1,m}) \leq T_n^{1/2}(F_n). \quad (9.53) \]
Denote
\[ \bar{p}_j = \frac{1}{k_n} \int_{(j-1)/k_n}^{j/k_n} \tilde{f}_n(x) dx. \]
By Lemmas 3 and 4 in section 7 of Ulyanov [29], we have
\[ S_n(\tilde{F}_n) = k_n \sum_{j=1}^{k_n} \int_{(j-1)/k_n}^{j/k_n} (\tilde{f}_n(x) - \bar{p}_j)^2 dx \leq 2\omega^2 \left( \frac{1}{k_n}, \tilde{f}_n \right). \quad (9.54) \]
Here
\[ \omega^2(h,f) = \int_0^1 (f(t+h) - f(t))^2 dt, \quad h > 0, \]
for any $f \in L^2_{per}$. If $f = \sum_{j=-\infty}^{\infty} \theta_j \phi_j$, then
\[ \omega^2(h,f) = 2 \sum_{j=1}^{\infty} |\theta_j|^2 (2 - 2 \cos(jh)). \quad (9.55) \]
Since $1 - \cos(x) \leq x^2$, then, by (9.54) and (9.55), we have
\[ \|f_n\| - n^{-1/2} T_n^{1/2}(\tilde{F}_n) \leq S_n^{1/2}(\tilde{F}_n) \leq c_1 c_3^{-1} \|f_n\|. \quad (9.56) \]
By Lemma 9.8, we get
\[ n^{-1} T_n(\tilde{F}_{n,C_1,k_n}) < C_1^{-1} c_3 \|\tilde{f}_{n,C_1,k_n}\|^2 < CC_1^{-1} c_3 m_n^{-2r}. \quad (9.57) \]
We have
\[ n^{-1/2} T_n^{1/2}(F_{n,c_1,m,C_1,m}) \leq \|f_{n,c_1,m,C_1,m}\|. \quad (9.58) \]
Fix $\delta$, $0 < \delta < 1$, and fix $c_2$. There are at most $2[\delta^{-1}]$ intervals $[c_2\delta^{-2i}, C_2\delta^{-2i-2}]$, $0 \leq i \leq 2\delta^{-1}$ such that for at least one of them, for $c_1 = c_2\delta^{-2i}$ and $C_1 = C_2\delta^{-2i-2}$ there holds

$$\sum_{c_1 m_n < |j| < C_1 m_n} |\theta_{jn}|^2 = \| f_{n,c_1 m_n,c_1 m_n} \|^2 < C\delta n^{-r}. \quad (9.59)$$

Note that the choice of $i$ depends on $n$.

For any $c_1$ and $C_1$ such that $(9.59)$ holds, we put $c_3 = C_1\delta$.

Since the choice of $\delta$ was arbitrary, then, by $(9.53)$, $(9.56)$, $(9.57)$ and $(9.59)$ together, we get $k_n^{-1/2}T_n(F_n) \asymp 1$. By Theorem 9.3, this implies sufficiency.

**Proof of Theorem 6.3.** Sufficiency. Let $k_n = \lfloor n^{\frac{1}{4-r}} \rfloor$. It is clear that we can always make additional partitions of cells and test statistics with these additional partitions of cells will be also consistent if the number of cells will have the same order $n^{2-4r}$.

For $C_1 > 2c_1$, we have

$$T_n^{1/2}(F_n) \leq T_n^{1/2}(\tilde{F}_{n,c_1 m_n}) + T_n^{1/2}(\tilde{F}_{n,c_1 m_n}). \quad (9.60)$$

By Lemma 9.8, we have

$$n^{-1}T_n(\tilde{F}_{n,c_1 m_n}) \leq C_1^{-1}kn_n^{-1}\|\tilde{f}_n\|^2 \leq C_1^{-1}Cn^{-2r}. \quad (9.61)$$

We have

$$\|\tilde{f}_{n,c_1 m_n}\|^2 \geq n^{-1/2}T_n^{1/2}(\tilde{F}_{n,c_1 m_n}). \quad (9.62)$$

Since one can take arbitrary value $C_1$, $C_1 > 2c_1$, then, by Theorem 9.3, $(6.2)$ and $(9.60)$ - $(9.62)$ together, we get inconsistency of sequence $f_n$.

If $f_n$ is inconsistent, then $(6.2)$ follows from Theorem 6.2.

Theorem 6.1 follows from sufficiency statement of Theorem 6.3.

**Proof of Theorem 6.4.** Denote $F_0(x)$ c.d.f. of uniform distribution, $x \in [0,1]$. Denote $F_{1n}$ c.d.f. having the density $1 + f_{1n}$.

Let $f_{1n} = \sum_{j=-\infty}^{\infty} \eta_{jn} \phi_j$. For any $a > 0$ denote

$$\tilde{f}_{a1n} = \sum_{|j| < am_n} \eta_{jn} \phi_j \quad \text{and} \quad \tilde{f}_{a1n} = f_{1n} - \tilde{f}_{a1n}.$$ Define functions $\tilde{F}_{a,1n}(x)$, $\tilde{F}_{a,1n}(x)$ with $x \in [0,1]$ such that $\tilde{f}_{a,1n}(x) = d\tilde{F}_{a,1n}(x)/dx$, $\tilde{f}_{a,1n}(x) = d\tilde{F}_{a,1n}(x)/dx$ and $\tilde{F}_{a,1n}(1) = 1$, $\tilde{F}_{a,1n}(1) = 1$.

We have

$$T_n^{1/2}(F_n + F_{1n} - F_0) \leq T_n^{1/2}(F_n) + T_n^{1/2}(F_{1n})$$

$$\leq T_n^{1/2}(F_n) + T_n^{1/2}(\tilde{F}_{a,1n}) + T_n^{1/2}(\tilde{F}_{a,1n}) \quad (9.63)$$

and

$$T_n^{1/2}(F_n + F_{1n} - F_0) \geq T_n^{1/2}(F_n) - T_n^{1/2}(\tilde{F}_{a,1n}) - T_n^{1/2}(\tilde{F}_{a,1n}). \quad (9.64)$$
Therefore, by Theorem 9.3, it suffices to estimate $T_n^{1/2}(\tilde{F}_{a,n})$ and $T_n^{1/2}(\tilde{F}_{a1n})$.

By Theorem 6.3, we have

$$k_n^{-1/2}T_n(\tilde{F}_{a,1n}) \leq nk_n^{-1/2}\|\tilde{f}_{a,1n}\|^2 = o(n^{1-2r}k_n^{-1/2}) = o(1). \quad (9.65)$$

By Lemma 9.8, we have

$$k_n^{-1/2}T_n(\tilde{F}_{a,1n}) \leq Ca^{-1}nk_n^{-1/2}\|\tilde{f}_{a,1n}\|^2 = O(a^{-1}n^{1-2r}k_n^{-1/2}) = O(a^{-1}). \quad (9.66)$$

By Theorem 9.3 and (9.63) - (9.66) together, we get Theorem 6.4.

Proof of Theorem 6.7. Let $f_n = \sum_{j=-\infty}^{\infty} \theta_{jn}\phi_j$. For any $a > 0$ denote

$$\tilde{f}_{an} = \sum_{|j|<am_n} \theta_{jn}\phi_j \quad \text{and} \quad \tilde{f}_{an} = f_n - \tilde{f}_{an}.$$ 

Fix $\epsilon > 0$. Let $C_1 = C_1(\epsilon)$ satisfies (6.4). Let $a > C_1$. Define functions $\tilde{F}_{a,n}(x)$, $\tilde{F}_{a,n}(x)$ with $x \in [0, 1]$ such that $dF_{a,n}(x)/dx = 1 + \tilde{f}_{a,n}(x)$, $d\tilde{F}_{a,n}(x)/dx = 1 + \tilde{f}_{a,n}(x)$ and $\tilde{F}_{a,n}(1) = 1$, $\tilde{F}_{a,n}(1) = 1$.

We have

$$T_n^{1/2}(F_n) \geq T_n^{1/2}(\tilde{F}_{a,n}) - T_n^{1/2}(\tilde{F}_{an}) \quad (9.67)$$

By (9.56), we have

$$n^{-1/2} T_n^{1/2}(\tilde{F}_{an}) \geq \|\tilde{f}_{an}\|\,(1 - C_1a^{-1}) \quad (9.68)$$

We have

$$n^{-1} T_n(\tilde{F}_{an}) \leq \|\tilde{f}_{an}\|^2. \quad (9.69)$$

Since choice of $\epsilon > 0$ and $a$ was arbitrary, by Theorem 9.3 and (9.67) - (9.69) together, we get Theorem 6.7.

Proof of Theorem 6.8 iv. Suppose the opposite. Then there is sequence $i_l$, $i_l \to \infty$ as $l \to \infty$, such that

$$i_l^2\|\tilde{f}_{i_l}\|^2 = C_l, \quad (9.70)$$

with $C_l \to \infty$ as $l \to \infty$. Here $f = \sum_{j=\infty}^{\infty} \tau_j\phi_j$ and $f_{i_l} = \sum_{|j|>i_l} \tau_j\phi_j$.

Let $n_l$ be such that $n_l^{-r} < \|\tilde{f}_{i_l}\|$. Let $\eta_l$ be such that $\eta_l^{-r} < \|\tilde{f}_{i_l}\|$.

Then, estimating similarly to (9.16) and (9.17), we get $i_l^{-1/2} \leq C_l^{(2r-1)/2}n_l^{2r-1}$.

If $k_l = o(i_l)$, then, by Lemma 9.8, we get

$$k_l^{-1/2}T_{n_l}(\tilde{F}_{i_l}) \leq k_l^{1/2}i_l^{-1}n_l \sum_{|j|>i_l} |\tau_j|^2 \leq k_l^{1/2}i_l^{-1}n_l^{1-2r} = o(C_l^{(2r-1)/2}). \quad (9.71)$$

Let $k_l \approx i_l$ or $i_l = o(k_l)$. We have

$$n_l^{-2r} \leq \|\tilde{f}_{i_l}\|^2 \geq n_l^{-1}T_{n_l}(\tilde{F}_{i_l}). \quad (9.72)$$

Therefore

$$k_l^{-1/2}T_{n_l}(\tilde{F}_{i_l}) \leq Ck_l^{-1/2}n_l^{1-2r} = Ck_l^{-1/2}i_l^{1/2}C_l^{(2r-1)/2} = o(1). \quad (9.73)$$
By Theorem 9.3, (9.71) and (9.73) imply iv.

**Proof of Theorem 6.10.** Let
\[ f_{1n} = \sum_{|j|<cm_n} \theta_j \phi_j. \]  

(9.74)

Then, by Lemma 9.3 there is \( \gamma \) such that \( f_{1n} \in \gamma U \).

We have
\[ |T_n^{1/2}(F_n) - T_n^{1/2}(F_{1n})| \leq T_n^{1/2}(F_n - F_{1n} + F_0). \]  

(9.75)

If \( k_n = [c_n m_n] \) with \( c_n < c_0 \) and \( c > 2c_0 \), then, by Lemma 9.8, we have
\[ n^{-1/2} T_n^{1/2}(F_n - F_{1n} + F_0) \leq c_0 c^{-1} \| f_n - f_{1n} \|. \]  

(9.76)

Since the choice of \( c \) is arbitrary, by Theorem 9.3, (9.75) and (9.76) imply (2.6) and (2.7).

9.5. **Proof of Theorems of section 7**

We can write the functional \( T^2(F - F_0) \) in the following form (see Ch.5, Shorack and Wellner [27])
\[ T^2(F - F_0) = \int_0^1 \int_0^1 (\min\{s, t\} - st) f(t) f(s) ds \, dt \]  

(9.77)

with \( f(t) = d(F(t) - F_0(t))/dt \).

If we consider the expansion of function
\[ f(t) = \sqrt{2} \sum_{j=1}^{\infty} \theta_j \sin(\pi j t), \quad \theta = \{\theta_j\}_{j=1}^{\infty} \]  

(9.78)

on eigenfunctions of operator with the kernel \( \min\{s, t\} - st \), then we get
\[ nT^2(F - F_0) = n \sum_{j=1}^{\infty} \frac{\theta_j^2}{\pi^2 j^2}. \]  

(9.79)

**Proof of Theorem 7.1.** If sequence of alternatives \( f_n, cn^{-r} \leq \|f_n\| \leq Cn^{-r} \), is consistent, then (7.2), (9.79), (7.3) together implies (7.5).

This allows to analyze in the reasoning sequences \( \tilde{f}_n = \sum_{|j|<c_t k_n} \theta_{jn} \phi_j \) instead of sequences \( f_n \). Condition (7.2) and (9.79) reduces analysis of consistency of sequences \( f_n \) to the reasoning of subsection 9.2 with another parameters \( r \) and \( s \). We omit the most part of the reasoning.

Let (7.6) hold. Then we have
\[ n \sum_{j=1}^{\infty} \frac{\theta_j^2}{\pi^2 j^2} \geq n \sum_{j<c_2 k_n} \frac{\theta_j^2}{\pi^2 j^2} \geq c_2^{-2} nk_n^{-2} \sum_{j<c_2 k_n} \theta_j^2 \geq 1. \]  

(9.80)
By (7.2), this implies sufficiency.

Necessary conditions follows from sufficiency conditions in Theorem 7.2.

Proof of Theorem 7.2. Let (7.7) hold. Then we have

\[ \sum_{j=1}^{\infty} \frac{\theta^2_j}{\pi^2 j^2} = \sum_{j < c_2 k_n} \frac{\theta^2_j}{\pi^2 j^2} + n \sum_{j > c_2 k_n} \frac{\theta^2_j}{\pi^2 j^2} \leq o(1) + C k_n^2 \sum_{j > c_2 k_n} \theta^2_{jn} = o(1) + (c_2 k_n)^{-2} n^{1-2r} = O(c_2^2). \] (9.81)

Since \( c_2 \) is arbitrary, then, by (7.2), (9.81) implies sufficiency.

Necessary conditions in Theorems 7.2 follow from sufficiency statement in Theorems 7.1.

Proof of Theorem 7.6 iii. The reasoning are akin to proof of Theorem 4.6 iii. The statement follows from (7.2) and Lemma 9.9 provided below.

Lemma 9.9. Let \( c n^{-r} \leq \|f_n\| \leq C n^{-r} \) and \( f_n \in c_1 U \). Then, for \( k_n = C_1 n^{(1-2r)/2} (1 + o(1)) \) with \( C_1 > 2c_1/c \), there holds

\[ \sum_{j=1}^{k_n} \theta^2_{jn} > \frac{c}{2} n^{-2r}. \] (9.82)

Proof of Lemma 9.9 is akin to proof of Lemma 9.2 and is omitted.

Proof of Theorem 7.6 iv. The reasoning are akin to proof of Theorem 4.6 iv. Suppose the opposite. Then there are \( f = \sum_{j=1}^{\infty} \tau_j \phi_j \notin B^2_2 \), and a sequence \( m_l, m_l \to \infty \) as \( l \to \infty \), such that (9.14) holds. Define sequences \( \eta_l, m_l \) and \( \tilde{f}_l \) by the same way as in the proof of Theorem 4.6. Then

\[ n_l \geq C_l^{-1/(2r)} m_l^{s/r} = C_l^{-1/(2r)} m_l^{1-2r}. \] (9.83)

Therefore we get

\[ m_l \geq C_l^{(1-2r)/(4r)} n_l^{(1-2r)/2}. \] (9.84)

Hence we get

\[ n_l \sum_{j=1}^{m_l} \frac{\eta^2_{jl}}{j^2} \leq m_l m_l^{-2} \sum_{j=m_l}^{\infty} \theta^2_{jl} \geq n_l^{-1-2r} m_l^{-2} \approx C_l^{2-2r} = o(1). \] (9.85)

By (7.2), (9.85) implies \( n^{-r} \)-inconsistency of sequence of alternatives \( \tilde{f}_l \).

Proof of Theorem 7.3. To implement Hungary construction to the study of asymptotics of type II error probabilities we need some statement on uniform continuity of limit distributions of statistics \( T_n \). This statement is provided in Lemma 9.10.

Denote \( b(t) \) Brownian bridge, \( t \in [0, 1] \).

Lemma 9.10. The densities of \( T_n^2(b(F_n(t)) + \sqrt{n}(F_n(t) - t)) \) are uniformly bounded onto the set of c.d.f.’s \( F_n \) such that \( \int_0^1 (dF_n(x)/dx)^2 dx < C \).
Proof. We have

\[ T^2(b(F_n(t)) + \sqrt{n}(F_n(t) - F_0(t))) = \int_0^1 (b(F_n(t)) + \sqrt{n}(F_n(t) - F_0(t)))^2 dt \]

\[ = \int_0^1 \left( \sqrt{2} \sum_{k=1}^{\infty} \xi_k \frac{\sin(\pi k F_n(t))}{k \pi} + n^{1/2}(F_n(t) - t) \right)^2 dt, \]

(9.86)

where \( \xi_k = \sqrt{2} \int_0^1 b(t) \sin(\pi k t) dt \).

Hence, we have

\[ T^2(\xi_1, \xi_2, J_n) = a_n \xi_1^2 + 2b_n \xi_1 \xi_2 + c_n \xi_2^2 + d_1 \xi_1 + d_2 \xi_2 + e_n, \]

(9.87)

with

\[ a_n = 2\pi^{-2} \int_0^1 \sin^2(\pi F_n(t)) dt, \]
\[ b_n = \pi^{-2} \int_0^1 \sin(\pi F_n(t)) \sin(2\pi F_n(t)) dt, \]
\[ c_n = \frac{1}{2} \pi^{-2} \int_0^1 \sin^2(2\pi F_n(t)) dt, \]
\[ d_{1n} = \sqrt{2}\pi^{-1} \int_0^1 \sin(\pi F_n(t)) J_n(t) dt, \quad d_{2n} = \frac{1}{\sqrt{2}} \pi^{-1} \int_0^1 \sin(2\pi F_n(t)) J_n(t) dt, \]
\[ e_n = \int_0^1 J_n^2(t) dt, \]

where

\[ J_n(t) = \sqrt{2} \sum_{k=3}^{\infty} \xi_k \frac{\sin(\pi k F_n(t))}{k \pi} + n^{1/2}(F_n(t) - t). \]

We can write

\[ \mathbf{P}(T^2(\xi_1, \xi_2, J_n) < c) = \int \chi_{\{T_n^2(x, y, \omega) < c\}} dG_n(x, y|\omega) d\mu_n(\omega), \]

(9.88)

where \( G_n(x, y|\omega) \) is conditional p.m. of \( \xi_1, \xi_2 \) given \( J_n \) and \( \mu_n \) is p.m. of \( J_n \).

Thus, for the proof of Lemma 9.10 it suffices to show that distribution functions

\[ H_n(c|\omega) = \int \chi_{\{T_n(x, y, \omega) < c\}} dG_n(x, y|\omega) \]

(9.89)

have uniformly bounded densities \( h_n(c|\omega) \) w.r.t. Lebesgue measure.

Define matrix \( R_n = \{u_{ijn}\}_{i,j=1}^2 \) with \( u_{11n} = a_n, u_{22n} = c_n \) and \( u_{12n} = u_{21n} = b_n \). Denote \( I \) the unit matrix.

The distribution function \( H_n(c|\omega) \) has characteristic function

\[ c \left( \det(I - 2itR) \right)^{-1/2} \exp\{itq(a_n, b_n, c_n, d_{1n}, d_{2n}, \omega)\}, \]

(9.90)
where \( q(a_n, b_n, c_n, d_{1n}, d_{2n}, e_n, \omega) \) is some function.

Note that \( c (\det(I - 2itR))^{-1/2} \) is characteristic function of quadratic form of two Gaussian independent r.v.'s. Therefore, if \( \det(R_n) > c_1 \), then the densities \( h_n \) of these quadratic forms are uniformly bounded.

We have

\[
\det(R_n) = \int_0^1 \int_0^1 (\sin^2(\pi F_n(x)) \sin^2(2\pi F_n(y)) - \sin(\pi F_n(x)) \sin(\pi F_n(y)) \sin(2\pi F_n(x)) \sin(2\pi F_n(y))) \, dx \, dy
\]

\[
= 4 \int_0^1 \int_0^1 \sin^2(\pi F_n(x)) \sin^2(\pi F_n(y)) \cos^2(\pi F_n(y)) - \cos(\pi F_n(x)) \cos(\pi F_n(y))) \, dx \, dy.
\]

The right-hand side of (9.91) is positive by Cauchi inequality.

For any \( c_1, c_2 > c_3 \) there is \( c \) such that there holds

\[
\int_0^1 f^2 g^2 d\mu - \left( \int_0^1 f g d\mu \right)^2 > c
\]

for any \( f, g \in L^2(d\mu) \) such that

\[
f(x) > c_2, \quad x \in \Omega_1, \quad \mu(\Omega_1) > c_1,
\]

\[
g(x) < c_3, \quad x \in \Omega_2, \quad \mu(\Omega_2) > c_1.
\]

Here \( \mu \) is probability measure.

Thus \( \det(R_n) > c_4 \) if there is \( \delta > 0 \) and measurable sets \( \Omega_n \) such that

\[
1 + f_n(x) > \delta \text{ for } x \in \Omega_n, \int_{\Omega_n} dx > \delta.
\]

Suppose the opposite. Then there are \( \delta_i \to 0 \) as \( i \to \infty \) and sequence \( n_i \) such that

\[
1 + f_{n_i}(x) < \delta_i \text{ for } x \in [0, 1) \setminus \Omega_{n_i} \text{ and } \int_{\Omega_{n_i}} dx < 2\delta_i. \text{ This implies that there is measurable sets } \Psi_i, \Psi_i \subset [0, 1), \int_{\Psi_i} dx > \delta_i \text{ such that } 1 + f_{n_i} > \delta_i^{-1}/8.
\]

Therefore we have

\[
\int_0^1 f_{n_i}^2(x) \, dx \geq \int_{\Psi_i} f_{n_i}^2(x) \, dx > c\delta_i^{-1}.
\]

We come to contradiction. This completes the proof of Lemma 9.10.

Since \( T \) is a norm, by Hungary construction (see Th. 3, Ch. 12, section 1, Schorack and Wellner [27]) and by Lemma 9.10, the proof of Theorem 7.3 is reduced to the proof of following inequality

\[
|P(T^2(b(F_n(t) + F_{1n}(t) - F_0(t)) + \sqrt{n}(F_n(t) + F_{1n}(t) - 2F_0(t))) > x_\alpha) - P(T^2(b(F_n(t)) + \sqrt{n}(F_n(t) - F_0(t))) > x_\alpha)| < \varepsilon.
\]

Since \( T \) is a norm, the proof of (9.94) is reduced to the proof that, for any \( \delta_1 > 0 \), there hold

\[
P(|T(b(F_n(t) + F_{1n}(t) - F_0(t))) - T(b(F_{1n}(t))))| > \delta_1) = o(1),
\]

\[
P(|b(F_n(t) + F_{1n}(t) - F_0(t)) - b(F_{1n}(t))| > \delta_1) = o(1),
\]

\[
P(|b(F_n(t) + F_{1n}(t) - F_0(t))| > \delta_1) = o(1).
\]
\[ n^{1/2} |T(F_n(t) + F_{1n}(t) - 2F_0(t)) - T(F_{1n}(t) - F_0(t))| < \delta_n, \quad (9.96) \]

where \( \delta_n \to 0 \) as \( n \to \infty \).

Note that

\[ |T(b(F_n(t) + F_{1n}(t) - F_0(t))) - T(b(F_n(t)))| \leq T(b(F_n(t) + F_{1n}(t) - F_0(t)) - b(F_{1n}(t))) \leq T(F_n(t) - F_0(t)). \quad (9.97) \]

and

\[ |T(F_n(t) + F_{1n}(t) - 2F_0(t)) - T(F_n(t) - F_0(t))| \leq T(F_{1n}(t) - F_0(t)). \quad (9.98) \]

We have

\[ \mathbf{E}T^2(b(F_n(t) + F_{1n}(t) - F_0(t)) - b(F_n)) \]
\[ = \int_0^1 \mathbf{E}(b(F_n(t) + F_{1n}(t) - F_0(t)) - b(F_n(t)))^2 \, dt \]
\[ = \int_0^1 (F_n(t) + F_{1n}(t) - F_0(t) - (F_n(t) + F_{1n}(t) - F_0(t))^2 + F_n(t) - F_{1n}(t) - 2F_0(t)) + 2(F_n(t) + F_{1n}(t) - F_0(t))F_n(t) \, dt \]
\[ = \int_0^1 \max\{F_{1n}(t) - F_0(t), F_0(t) - F_{1n}(t)\} - (F_{1n}(t) - F_0(t))^2 \, dt \]
\[ \leq C \max_{0 < t < \eta} |F_{1n}(t) - F_0(t)|. \quad (9.99) \]

Let \( f_{1n} = \sum_{j=1}^{\infty} \theta_{1jn} \phi_j \). We have

\[ \max_{0 < t < \eta} |F_{1n}(t) - F_0(t)| \leq C \sum_{j=1}^{\infty} \left| \frac{\theta_{1jn}}{j} \right| \]
\[ \leq C \left( \sum_{j=1}^{\infty} \theta_{1jn}^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} j^{-2} \right)^{1/2} = o(n^{-\kappa}). \quad (9.100) \]

By (9.97), (9.99) and (9.100), we get (9.95).

Since sequence \( f_{1n} \) is inconsistent, we have

\[ nT^2(F_{1n}(t) - F_0(t)) = o(1) \quad (9.101) \]

as \( n \to \infty \). By (9.98) and (9.101), we get (9.96).

By Lemma 9.10 and (9.95), (9.96), we get (9.94).

Proof of Theorem 7.8. Since \( T \) is a norm, by Hungary construction (see Th. 3, Ch. 12, section 1, Schorack and Wellner [27]) and by Lemma 9.10, the proof of (2.6) and (2.7) is reduced to the proof of two following inequalities.

\[ |\mathbf{P}(T^2(b(F_n(t)) + \sqrt{n}(F_n(t) - F_0(t))) > x_\alpha) - \mathbf{P}(T^2(b(F_{1n}(t)) + \sqrt{n}(F_{1n}(t) - F_0(t))) > x_\alpha)| < \epsilon \quad (9.102) \]
and
\[
\mathbf{P}(T^2(b(F_n(t) - F_{1n}(t) + F_0(t)) + \sqrt{n}(F_n(t) - F_{1n}(t))) < x_\alpha) > 1 - \alpha - \epsilon. \quad (9.103)
\]
Since \( T \) is a norm, the proof of (9.102) and (9.103) is reduced to the proof that, for any \( \delta_1 > 0 \), there hold
\[
\mathbf{P}(|T(b(F_n(t)) - T(b(F_{1n}(t)))| > \delta_1) = o(1), \quad (9.104)
\]
\[
\mathbf{P}(|T(b(F_0(t) + F_n(t) - F_{1n}(t))) - T(b(F_0(t)))| > \delta_1) = o(1), \quad (9.105)
\]
and
\[
n^{1/2}|T(F_n(t)) - T(F_{1n}(t))| < \delta_n, \quad (9.106)
\]
\[
n^{1/2}|T(F_0(t) + F_n(t) - F_{1n}(t)) - T(F_0(t))| < \delta_n, \quad (9.107)
\]
where \( \delta_n \to 0 \) as \( n \to \infty \).

Note that
\[
|T(b(F_n(t))) - T(b(F_{1n}(t)))| \leq T(b(F_n(t)) - b(F_{1n}(t))) \quad (9.108)
\]
and
\[
|T(F_n(t)) - T(F_{1n}(t))| \leq T(F_n(t) - F_{1n}(t)). \quad (9.109)
\]

We have
\[
\mathbf{E}T^2(b(F_n) - b(F_{1n})) = \int_0^1 \mathbf{E}(b(F(t)) - b(F_{1n}(t)))^2 dt
\]
\[
= \int_0^1 ((F_n(t) - \min(F_n(t), F_{1n}(t)) + (F_{1n}(t) - \min(F_n(t), F_{1n}(t))
\]
\[
- (F_n(t) - F_{1n}(t))^2 dt \leq C \max_{0<\theta<1} |F_n(t) - F_{1n}(t)|. \quad (9.110)
\]

By Lemma 9.3, one can take \( f_{1n} = \sum_{j>C_{1k}n} \theta_j n \phi_j \) with arbitrary value \( C_1 \).

Then
\[
\max_{0<\theta<1} |F_n(t) - F_{1n}(t)| \leq C \sum_{j>C_{1k}n} \frac{|\theta_j n \phi_j|}{j}
\]
\[
\leq C \left( \sum_{j>C_{1k}n} \frac{\theta_j^2 n \phi_j^2}{j} \right)^{1/2} \left( \sum_{j>C_{1k}n} \frac{\phi_j^2}{j} \right)^{1/2}
\]
\[
\leq C_1^{-1} n^{-r} k_n^{-1/2} \leq cC_1^{-1} n^{-1/4-r/2}. \quad (9.111)
\]

By (9.108) and (9.111), we get (9.104).

We have
\[
nT^2(F_n(t) - F_{1n}(t)) \leq C \sum_{j>C_{1k}n} \frac{\theta_j^2 n \phi_j^2}{j^2}
\]
\[
\leq CC_1^{-2} nk_n^{-2} \sum_{j>C_{1k}n} \theta_j^2 \leq CC_1^{-2} nk_n^{-2} n^{-2r} \leq CC_1^{-2}. \quad (9.112)
\]

By (9.109) and (9.112), we get (9.106).

By Lemma 9.10 and (9.104), (9.106), (9.110) together, we get (9.102).

Proof of (9.105) and (9.107) is similar and is omitted.
Proof of Theorem 8.1

Fix $\delta, 0 < \delta < 1$. Denote $\kappa_j^2(\delta) = 0$ for $j > \delta^{-1}k_n$. Define $\kappa_j^2(\delta), 1 \leq j < k_n\delta = \delta^{-1}k_n$, the equations (8.1) and (8.2) with $P_0$ and $\rho_0$ replaced with $P_0(1 - \delta)$ and $\rho_n(1 + \delta)$ respectively. Similarly to [6], we find Bayes test for a priori distribution $\theta_j = \eta_j = \eta_j(\delta), 1 \leq j < \infty$, with Gaussian independent random variables $a_0, E\eta_j = 0, E\eta_j^2 = \kappa_j^2(\delta)$, and show that these tests are asymptotically minimax for some $\delta = \delta_n \to 0$ as $n \to \infty$.

Lemma 9.11. For any $\delta, 0 < \delta < 1$, there holds

\[ P(\eta(\delta) = \{ \eta_j(\delta) \}_{j=1}^{\infty} \in V_n) = 1 + o(1) \quad (9.113) \]

as $n \to \infty$.

Denote

\[ A_{n,\delta} = \sigma^{-4}n^2 \sum_{j=1}^{\infty} \kappa_j^4(\delta). \]

By straightforward calculations, we get

\[ \lim_{\delta \to 0} \lim_{n \to \infty} A_n A_n^{-1}(\delta) = 1. \quad (9.114) \]

Denote $\gamma_j^2(\delta) = \kappa_j^2(\delta)(n^{-1}\sigma^2 + \kappa_j^2(\delta))^{-1}$.

By Neymann-Pearson Lemma, Bayes critical region is defined the inequality

\[
C_1 < \prod_{j=1}^{k_n,\delta} (2\pi)^{-1/2} \kappa_j^{-1}(\delta) \int \exp \left\{ - \sum_{j=1}^{k_n,\delta} (2\gamma_j^2(\delta))^{-1} (u_j - \gamma_j^2(\delta)y_j)^2 \right\} du \exp \{-T_{n,\delta}(y)\} = C \exp \{-T_{n,\delta}(y)\} (1 + o(1))
\]

(9.115)

where

\[ T_{n,\delta}(y) = n\sigma^{-2} \sum_{j=1}^{\infty} \gamma_j^2(\delta)y_j^2. \]

Define critical region

\[ S_{n,\delta} = \{ y : R_{n,\delta}(y) = (T_{n,\delta}(y) - C_{n,\delta})(2A_n(\delta))^{-1/2} > x_\alpha \} \]

with

\[ C_{n,\delta} = E_0 T_{n,\delta}(y) = \sigma^{-2} \sum_{j=1}^{\infty} \gamma_j^2(\delta). \]

Denote $L_{n,\delta}$ the tests with critical regions $S_{n,\delta}$.

Denote $\gamma_j^2 = \kappa_j^2(n^{-1}\sigma^2 + \kappa_j^2)^{-1}, 1 \leq j < \infty$. Define test statistics $T_n, R_n$, critical regions $S_n$ and constants $C_n$ by the same way as test statistics $T_{n,\delta}, R_{n,\delta}$, critical regions $S_{n,\delta}$ and constants $C_{n,\delta}$ respectively with $\gamma_j^2(\delta)$ replaced with $\gamma_j^2$ respectively. Denote $L_n$ the test having critical region $S_n$. 


Lemma 9.12. Let $H_0$ hold. Then the distributions of tests statistics $R_n^a(y)$ and $R_n(y)$ converge to the standard normal distribution. 

For any family $\theta_n = \{\theta_{jn}\} \in \mathfrak{T}_n$ there holds

$$P_{\theta_n} \left( \left( T_n^a(y) - C_n - \sigma_n^{-4} n^2 \sum_{j=1}^{\infty} \kappa_j^2 \theta_{jn}^2 \right) (2A_n)^{-1/2} < x_\alpha \right) = \Phi(x_\alpha)(1 + o(1))$$

(9.116)

and

$$P_{\theta_n} \left( \left( T_n^a(y) - C_n - \sigma_n^{-4} n^2 \sum_{j=1}^{\infty} \kappa_j^2 \theta_{jn}^2 \right) (2A_n)^{-1/2} < x_\alpha \right) = \Phi(x_\alpha)(1 + o(1))$$

(9.117)

as $n \to \infty$.

Hence we get the following Lemma.

Lemma 9.13. There holds

$$\beta(L_n, V_n) = \beta(L_n^a, V_n)(1 + o(1))$$

(9.118)

as $n \to \infty$.

Lemma 9.14. Let $H_0$ hold. Then the distributions of tests statistics $(T_n^\delta(y) - C_n^\delta)(2A_n)^{-1/2}$ converge to the standard normal distribution.

There holds

$$P_{\eta^\delta} \left( \left( T_n^\delta(y) - C_n^\delta - A_n^\delta \right) (2A_n^\delta)^{-1/2} < x_\alpha \right) = \Phi(x_\alpha)(1 + o(1))$$

(9.119)

as $n \to \infty$.

Lemma 9.15. There holds

$$\lim_{\delta \to 0} \lim_{n \to \infty} \mathbb{E}_{\eta^\delta} \beta_{\eta^\delta}(L_n^\delta) = \lim_{n \to \infty} \mathbb{E}_{\eta_0} \beta_{\eta_0}(L_n)$$

(9.120)

where $\eta_0 = \{\eta_{0j}\}_{j=1}^{\infty}$ and $\eta_{0j}$ are i.i.d. Gaussian random variables, $\mathbb{E}[\eta_{0j}] = 0$, $\mathbb{E}[\eta_{0j}^2] = \kappa_j^2$, $1 \leq j < \infty$.

Define Bayes a priori distribution $P_y$ as a conditional distribution of $\eta$ given $\eta \in V_n$. Denote $K_n = K_n^\delta$ Bayes test with Bayes a priori distribution $P_{\eta^\delta}$. Denote $W_n$ critical region of $K_n^\delta$.

For any sets $A$ and $B$ denote $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Lemma 9.16. There holds

$$\lim_{\delta \to 0} \lim_{n \to \infty} \int_{V_n} P_{\theta} (S_n^\delta \Delta V_n^\delta) dP_y = 0$$

(9.121)

and

$$\lim_{\delta \to 0} \lim_{n \to \infty} P_0 (S_n^\delta \Delta V_n^\delta) = 0.$$
In the proof of Lemma 9.16 we show that the integrals in the right hand-side of (9.115) with integration domain $V_n$ converge to one in probability as $n \to \infty$. This statement is proved both for hypothesis and Bayes alternative (see [6]).

Lemmas 9.11-9.16 implies that, if $\alpha(K_n) = \alpha(L_n)$, then

$$\int_{V_n} \beta_\theta(K_n) \, dP_y = \int_{V_n} \beta_\theta(L_n) \, dP_y(1 + o(1)) = \int \beta_{\eta_0}(L_n) \, dP_{\eta_0}(1 + o(1)).$$

(9.123)

**Lemma 9.17.** There holds

$$E_{\eta_0} \beta_{\eta_0}(L_n) = \beta_n(L_n)(1 + o(1)).$$

(9.124)

Lemmas 9.12, 9.15, (9.114), (9.123) and Lemma 9.17, imply Theorem 8.1.

**9.7. Proof of Lemmas**

Proofs of Lemmas 9.12, 9.13 and 9.15 are akin to the proofs of similar statements in [6] and are omitted.

**Proof of Lemma 9.11.** By straightforward calculations, we get

$$\sum_{j=1}^{\infty} E\eta_j^2(\delta) \geq \rho \epsilon (1 + \delta/2)$$

(9.125)

and

$$\text{Var} \left( \sum_{j=1}^{\infty} \eta_j^2(\delta) \right) < Cn^2 A_n \leq \rho_n^2 k^{-1}. (9.126)$$

Hence, by Chebyshev inequality, we get

$$P \left( \sum_{j=1}^{\infty} \eta_j^2(\delta) > \rho_n \right) = 1 + o(1)$$

(9.127)

as $n \to \infty$. It remains to estimate

$$P_\mu(\eta \notin B_{2\infty}(P_0)) = P \left( \max_{l_1 \leq i \leq l_2} i^{2s} \sum_{j=i}^{l_2} \eta_j^2 - P_0(1 - \delta_1/2) > P_0 \delta_1/2 \right) \leq \sum_{i=l_1}^{l_2} J_i$$

(9.128)

with

$$J_i = P \left( \left( \sum_{j=i}^{l_2} \eta_j^2 - P_0(1 - \delta_1/2) > P_0 \delta_1/2 \right) \right)$$

To estimate $J_i$ we implement the following Proposition (see [14]).

**Proposition 9.1.** Let $\xi = \{\xi_i\}_{i=1}^{\infty}$ be Gaussian random vector with i.i.d.r.v.’s $\xi_i$, $E[\xi_i] = 0$, $E[\xi_i^2] = 1$. Let $A \in R^d \times R^d$ and $\Sigma = A^T A$. Then

$$P \left( ||A\xi||^2 > \text{tr}(\Sigma) + 2 \sqrt{\text{tr}(\Sigma^2)} t + 2 \|\Sigma\| t \right) \leq \exp \{ -t \}. \quad (9.129)$$
We put $\Sigma_i = \{ \sigma_{ij} \}_{l,j = i}^{k_i}$ with $\sigma_{jj} = j^{-2s-1}i^{-2s} \frac{P_0}{2s}$ and $\sigma_{ij} = 0$ if $l \neq j$.

Let $i \leq k_n$. Then

$$\text{tr}(\Sigma_i^2) = i^{4s} \sum_{j=i}^{\infty} \sigma_j^2(\delta) < i^{4s}((k_n - i)\kappa(\delta) + k_n^{-4s-1}P_0) < Ck_n^{-1}. \quad (9.130)$$

and

$$\| \Sigma_i \| \leq i^{2s} < Ck_n^{-1}. \quad (9.131)$$

Therefore

$$2\sqrt{\text{tr}(\Sigma_i^2)t} + 2||\Sigma_i||t \leq C(\sqrt{k_n^{-1}} t + k_n^{-1}t) \quad (9.132)$$

Hence, putting $t = k_n^{1/2}$, by Proposition 9.1, we get

$$\sum_{i=1}^{k_n} J_i \leq Ck_n \exp\{-Ck_n^{1/2}\}. \quad (9.133)$$

Let $i \geq k_n$. Then

$$\text{tr}(\Sigma_i^2) < Ci^{-1}, \quad \text{and} \quad ||\Sigma_i|| \leq Ci^{-1} \quad (9.134)$$

Hence, putting $t = i^{1/2}$, by Proposition 9.1, we get

$$\sum_{i=k_n+1}^{k_n} J_i \leq \sum_{i=k_n+1}^{k_n} \exp\{-C_i^{1/2}\} < \exp\{-C_1k_n^{1/2}\}. \quad (9.135)$$

Now (9.128), (9.133), (9.135) together implies Lemma 9.11.

Proof of Lemma 9.16. By reasoning of the proof of Lemma 4 in [6], Lemma 9.16 will be proved, if we show, that

$$P\left(\sum_{j=1}^{\infty} (\eta_j(\delta) + y_j\gamma_j(\delta)\sigma^{-1}n^{1/2})^2 > \rho_n\right) = 1 + o(1) \quad (9.136)$$

and

$$P\left(\sup_i i^{2s} \sum_{j=1}^{\infty} (\eta_j(\delta) + y_j\gamma_j(\delta)\sigma^{-1}n^{1/2})^2 > \rho_n\right) = 1 + o(1) \quad (9.137)$$

where $y_j, 1 \leq j < \infty$ are distributed by hypothesis or Bayes alternative.

We prove only (9.137) in the case of Bayes alternative. In other cases the reasoning are similar.

We have

$$i^{2s} \sum_{j=i}^{\infty} (\eta_j(\delta) + y_j\gamma_j(\delta)\sigma^{-1}n^{1/2})^2 = i^{2s} \sum_{j=i}^{\infty} \eta_j^2(\delta) \quad (9.138)$$

$$+ i^{2s} \sum_{j=i}^{\infty} \eta_j(\delta)y_j\gamma_j(\delta)\sigma^{-1}n^{1/2} + i^{2s} \sum_{j=i}^{\infty} y_j^2\gamma_j^2(\delta)\sigma^{-2}n = J_{1i} + J_{2i} + J_{3i}. \quad (9.138)$$
The required probability for $J_{1i}$ is provided Lemma 9.11.

We have
\[
J_{2i} \leq J_{1i}^{1/2} J_{3i}^{1/2}.
\] (9.139)

Thus it remains to show that, for any $C$,
\[
P_{\eta(\delta)} \left( \sup_i \sum_{j=1}^{\infty} \gamma_j^2(\delta) \sigma^{-2} \gamma_j^2(\delta) \sigma^{-2} n > C \delta \right) = o(1)
\] (9.140)
as $n \to \infty$.

Note that $y_j = \zeta_j + \sigma n^{-1/2} \xi_j$ where $\zeta_j, y_j, 1 \leq j < \infty$ are i.i.d. Gaussian random variables, $E \zeta_j = 0, E \xi_j^2 = \kappa_j^2(\delta), E \xi_j = 0, E \xi_j^2 = 1$.

Hence, we have
\[
\sigma^{-2} n \sum_{j=1}^{\infty} \gamma_j^4(\delta) = \sigma^{-2} n \sum_{j=1}^{\infty} \gamma_j^4(\delta) \zeta_j^2 + \sigma^{-1} n^{1/2} \sum_{j=1}^{\infty} \gamma_j^4(\delta) \zeta_j \xi_j
\]
\[
+ \sum_{j=1}^{\infty} \gamma_j^4(\delta) \xi_j^2 = I_{1i} + I_{2i} + I_{3i}.
\] (9.141)

Since $n \gamma_j^2 = o(1)$, the estimates for probability of $i^2 s I_{1i}$ are evident. It suffices to follow the estimates of (9.128). We have $I_{2i} \leq I_{1i}^{1/2} I_{3i}^{1/2}$. Thus it remains to show that, for any $C$
\[
P_{\eta(\delta)} \left( \sup_i \sum_{j=1}^{\infty} \gamma_j^4(\delta) \xi_j^2 > \delta / C \right) = o(1)
\] (9.142)
as $n \to \infty$. Since $\gamma_j^2 = \kappa_j^2(1 + o(1)) = o(1)$, this estimate is also follows from estimates (9.128).

\textbf{Proof of Lemma 9.17.} By Lemmas 9.12, 9.13 and 9.15, it suffices to show that
\[
\inf_{\theta \in \mathcal{V}_n} \sum_{j=1}^{\infty} \kappa_j^2 \theta_j^2 = \sum_{j=1}^{\infty} \kappa_j^2.
\] (9.143)

Denote $u_k = k^2 \sum_{j=k}^{\infty} \theta_j^2$. Note that $u_k \leq P_0$.

Then $\theta_j^2 = u_j j^{-2s} - u_{j+1} (j+1)^{-2s}$. Hence we have
\[
A_n(\theta) = \sum_{j=1}^{\infty} \kappa_j^2 \theta_j^2 = \kappa^2 \sum_{j=1}^{k_n} \theta_j^2 + \sum_{j=k_n}^{\infty} \kappa_j^2 (u_j j^{-2s} - u_{j+1} (j+1)^{-2s})
\]
\[
= \kappa^2 \sum_{j=1}^{k_n} \theta_j^2 + \kappa u_k k_n^{-2s} + 2s P_0 \sum_{j=k_n+1}^{\infty} u_j (j^{-4s-1} - (j-1)^{-2s-1} j^{-2s})
\]
\[
= \kappa^2 \rho_n + 2s P_0 u_j (j^{-4s-1} - (j-1)^{-2s-1} j^{-2s}).
\] (9.144)
Since \( j^{-4s-1} - (j-1)^{-2s-1} j^{-2s} \) is negative, then \( \inf A(\theta) \) is attained for \( u_j = P_0 \) and therefore \( \theta_j^2 = \kappa_j^2 \) for \( j > k \).

Thus the problem is reduced to the solution of the following problem

\[
\kappa^2 \inf_{\theta_j} \sum_{j=1}^{k_n} \theta_j^2 + \sum_{j=k_n+1}^{\infty} \kappa_j^4 \quad (9.145)
\]

if

\[
\sum_{j=1}^{k_n} \theta_j^2 + \sum_{j=k_n+1}^{\infty} \kappa_j^2 = \rho_n
\]

and

\[
k_n^{2s} \sum_{j=k_n}^{\infty} \theta_j^2 < P_0, \quad 1 \leq j < \infty,
\]

with \( \theta_j^2 = \kappa_j^2 \) for \( j \geq k_n \).

It is easy to see that this infimum is attained if \( \theta_j^2 = \kappa_j^2 = \kappa^2 \) for \( j \leq k_n \).

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