A NEW FORMULA FOR THE MASS OF A STATIONARY
AXISYMMETRIC CONFIGURATION

R.M. Avakian and G. Oganessyan*

Dept. of Theoretical Physics,
Yerevan State University,
375049 Yerevan, Armenia.

* Present address: Theoretical Astrophysics Group, Tata Institute of Fundamental
Research, 400 005 Bombay, India. E-mail: gurgen@tifrvax.tifr.res.in
ABSTRACT

It has been shown by one of the authors\textsuperscript{1} that in isotropic spherical coordinates there is a relation between the mass of a static spherical gravitating body and the pressure distribution inside it. In this paper the result is generalized for the case of stationary axisymmetric configurations.
1. It has been shown that in the case of a static spherically symmetric distribution of matter there exists the following relation

\[ M^2 = \frac{32\pi}{k} \int_0^{r_s} P(r) r^3 e^{\nu+2\lambda} dr, \] (1)

where \( M \) is the mass of the spherical configuration, \( P(r) \) is the pressure distribution inside, \( k \) is the gravitational constant and \( \nu \) and \( \lambda \) are the metric functions, with \( r_s \) being the boundary of the configuration, given by \( P(r_s) = 0 \). It should be mentioned that \( r \) is the isotropic radial coordinate, i.e. the metric is written in the form

\[ ds^2 = e^{\nu(r)} c^2 dt^2 - e^{\lambda(r)} [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)]. \] (2)

Taking the Newtonian limit of (1) one gets

\[ M^2 = \frac{32\pi}{k} \int_0^{r_s} P(r) r^3 dr. \] (3)

The same relation can be easily obtained in the framework of Newtonian theory\(^1\).

2. Now we shall try to generalize the formula (1) for the case of axisymmetric gravitational fields, which can be produced either by a stationarily rotating axisymmetric configuration or by a motionless body with a similar distribution of matter. In the latter case one should assume the presence of internal stresses in the matter.

Since (1) can be derived only in isotropic coordinates, the generalization is possible in the coordinates which in the spherically symmetric limit, for instance when the angular velocity \( \Omega \) becomes zero, go over into the isotropic form. It is easy to see that the metric will meet this requirement if written in the form

\[ ds^2 = (e^{\nu(r)} - \omega^2 r^2 \sin^2 \theta e^\mu) c^2 dt^2 - e^{\lambda(r)} (dr^2 + r^2 d\theta^2) - r^2 \sin^2 \theta e^\mu d\phi^2 - 2\omega r^2 \sin^2 \theta e^\mu c d\phi dt. \] (4)

where \( \nu, \mu, \lambda \) and \( \omega \) are functions of \( r, \theta \) and \( \Omega \). When \( \Omega = 0 \) the distribution is spherical, \( \nu, \lambda \) and \( \mu \) depend only on \( r \) and as a consequence of the spherical symmetry \( e^{\lambda} \) and \( e^\mu \) equate to each other, so as to make the angular term proportional to \((d\theta^2 + \sin^2 \theta d\phi^2)\).

It is known\(^2\) that if the components of the metric tensor do not depend on \( x^0 = ct \) the component \( R^0_0 \) of the Ricci tensor can be written as

\[ R^0_0 = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (\sqrt{-g} g^{0i} \Gamma^0_{0i}), \] (5)

where \( g = \det g_{ik} = -r^4 \sin^2 \theta e^{\nu+2\lambda+\mu}, i = 0, 1, 2, 3, \alpha = 1, 2, 3 \). In the axially symmetric case the components of the metric tensor are independent also of \( x^3 = \phi \), and the component \( R^3_3 \) can be written in the same form:
\[ R^3_3 = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (\sqrt{-g} g^{3i} \Gamma^\alpha_{3i}). \]  

(6)

Now we take into account the Einstein equations and consider the combination \( R^0_0 + R^3_3 \)

\[ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} [\sqrt{-g} (g^{0i} \Gamma^\alpha_{0i} + g^{3i} \Gamma^\alpha_{3i})] = - \frac{8\pi k}{c^4} (T^1_1 + T^2_2). \]  

(7)

where \( T^k_i \) and \( T^2_2 \) are components of the energy-momentum tensor which in the case of perfect fluid is given by

\[ T^k_i = (P + \rho c^2) u^i u^k - P \delta^k_i. \]  

(8)

Calculating \( \Gamma^\alpha_{0i} \) and \( \Gamma^\alpha_{3i} \) and feeding them into (7), one gets

\[ \sin \theta \frac{\partial}{\partial r} \left[ r^3 \frac{\partial}{\partial r} (e^{\nu+\mu}) \right] + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \sin^2 \theta \frac{\partial}{\partial \theta} (e^{\nu+\mu}) \right] = - \frac{8\pi k}{c^4} (T^1_1 + T^2_2) \sqrt{-g}. \]  

(9)

Multiplying both sides by \( r \sin \theta \) and integrating over the whole 3-dimensional space we obtain

\[ \int_0^\infty \int_0^\pi \int_0^{2\pi} \sin^2 \theta \frac{\partial}{\partial r} \left[ r^3 \frac{\partial}{\partial r} (e^{\nu+\mu}) \right] d\theta d\phi + \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\partial}{\partial \theta} \left[ r \sin \theta \frac{\partial}{\partial \theta} (e^{\nu+\mu}) \right] d\theta d\phi = - \frac{8\pi k}{c^4} \int \int_V (T^1_1 + T^2_2) r \sin \theta \sqrt{-g} d\theta d\phi. \]  

(10)

The integration in the RHS of (10) is over the volume \( V \) of the body, since \( T^k_i = 0 \) outside the matter.

One can easily take the second integral in the LHS of (10) with respect to \( \theta \) and see that it is zero. Taking the first integral with respect to \( r \) one gets

\[ \int_0^{2\pi} \int_0^\pi \sin^2 \theta d\theta d\phi \left[ r^3 \frac{\partial}{\partial r} (e^{\nu+\mu}) \right] \bigg|_0^\infty = 2\pi \int_0^\pi \sin \theta d\theta [r^3 \frac{\partial}{\partial r} (e^{\nu+\mu})]_{r \to \infty} \]

\[ = - \frac{8\pi k}{c^4} \int \int_V (T^1_1 + T^2_2) r \sin^2 \theta d\theta d\phi. \]  

(11)

It is notable that the expansion of \( e^{\nu+\mu} \) in terms of \( \frac{1}{r^2} \) should start from the \( \frac{1}{r^4} \) term, otherwise the integral will diverge.
In order to calculate the integral in (11) one has to know the expansions of $e^\nu$ and $e^\mu$ up to $\frac{1}{r^2}$ order for the metric (4) in the external domain. One can start from the known expansions in the harmonic coordinates\(^3\)

\[
\begin{align*}
\text{ds}^2 &= (1 - \frac{r_g}{R} + \frac{r_g^2}{2R^2})c^2 dt^2 - (1 + \frac{r_g}{R} + \frac{r_g^2}{2R^2})dR^2 - R^2(1 + \frac{r_g}{R} + \frac{r_g^2}{4R^2})(d\theta^2 + \sin^2 \theta d\phi^2) \\
&+ 2\frac{2kJ}{c^2R}d\phi dt,
\end{align*}
\]

where $J$ is the angular momentum, $r_g$ the gravitational radius and $M$ the mass of the body. One can see that in this approximation the metric coefficients do not depend on the angular coordinates. Thus, the transition from (12) to the form (4) can be made by a scale transformation

\[
R = r(1 + \frac{C}{r} + \frac{D}{r^2}),
\]

where $C$ and $D$ are unknown constants. Inserting (13) into (12) and demanding that (12) go over into the form (4), we get $C = 0$, $D = \frac{r_g^2}{8}$.

Now we can easily find the metric coefficients written in $\frac{1}{r^2}$ approximation in the "isotropic" coordinates

\[
\begin{align*}
e^\nu &= 1 - \frac{r_g}{r} + \frac{r_g^2}{2r^2}, \\
e^\lambda &= 1 + \frac{r_g}{r} + \frac{3r_g^2}{8r^2}, \\
e^{\nu + \mu} &= 1 - \frac{r_g^2}{16r^2}.
\end{align*}
\]

Inserting (14) into (11) and integrating with respect to we obtain the formula we have been after:

\[
M^2 = -\frac{64}{k} \int_0^\pi \int_0^{r_s(\theta)} (T_1^1 + T_2^2) r^3 \sin^2 \theta e^{\frac{\nu + 2\lambda + \mu}{2}} dr d\theta,
\]

where $r_s(\theta)$ is the boundary of the configuration, $P[r_s(\theta)] = 0$.

In the static case, when $P, \nu$ and $\lambda = \mu$ do not depend on the angular coordinate, an elementary integration with respect to $\theta$ immediately leads to (1).

\textbf{Acknowledgments.}
We would like to thank the participants of the theoretical seminar at the Yerevan State University and the members of the Theoretical Astrophysics Group, Tata Institute of Fundamental Research, for useful discussions.

References

1. Avakian R.M., *Astrofizika*, 33, 429, 1990.
2. L.D. Landau and E.M. Lifshitz, *Classical Theory of Fields* (Pergamon, New York, 1975).
3. G.S. Saakyan, *Space-Time and Gravitation* (Yerevan State University, 1982, in Russian).