Status of heavy quark physics from the lattice
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In this short review, I present a summary of various methods used to simulate heavy quarks on the lattice. I mainly focus on effective theories, and give some physical results.

1. Introduction

Lattice simulations provide a powerful tool to study QCD beyond perturbation theory. In particular, for quark masses around or below that of the strange quark, today’s lattices offer the possibility of very accurate and direct calculations of hadronic observables. The recent feasibility of simulating dynamical fermions (i.e. without neglecting the effects of internal quark loops) provide a significant reduction (and a better control) of the systematic errors. However, simulating heavy quarks on a lattice is much more challenging than simulating the strange quark. This is rather unfortunate, because precise calculations in the heavy-quarks sector are needed to constrain the standard model (see, e.g. [1,2]).

The problem of heavy quarks on the lattice is a long-standing one, and various approaches have been proposed already more than 20 years ago. Of course these ideas have grown up, and together with the improvement of the lattice techniques, they have become more and more predictive. In addition to that, other very interesting strategies have been proposed more recently. Some of them have already given physical results, and others are not in that stage yet. The scope of this talk is to present the various methods (and their limitations) that exist to deal with heavy quarks on the lattice. I present also a (limited) selection of results which have been obtained by different lattice groups. I would like to apologize to all colleagues that I cannot cite, due to space limitations. For an overview of the most recent lattice results see [5].

This paper is organized as follow: In Sect. 2 I review the problem associated with simulating heavy quarks on a lattice. In Sect. 3 I present an overview of the different approaches. In the last section, I report some recent physical results.

2. Heavy quarks on the lattice

Lattice QCD allows for the computation of physical quantities from first principle, i.e. directly from the Lagrangian of QCD. But working in a finite and discrete space-time implies systematic errors, which have to be under control. The main problem with heavy quarks is that the discretization errors (when a traditional Wilson-like fermionic action is used) are proportional to powers of the bare quark mass. Thus one has to require $am_{\text{quark}} \ll 1$. To simulate a b-quark at its physical mass (≈ 5GeV) on a space-time of volume $(aN)^4 = (2 \text{ fm})^4$, one needs $N \gg 50$ points for each space-time dimension, which is impossible with present computers.

One can simulate various quark masses in the regime where the discretization errors are under control (say around the charm quark), and extrapolate the results to the b-quark. The problem is that this extrapolation is done over a large range (because $m_b \approx 4m_c$), and the associated systematic error is then difficult to control. Another possibility is to use effective theories, like heavy quark effective theory (HQET) or non relativistic QCD (NRQCD). Since the heavy quark mass is much larger than the other scales (like its 3-momentum or $\Lambda_{\text{QCD}}$), the idea is to expand the QCD Lagrangian in inverse powers of the heavy quark mass and keep only the leading terms (1

Direct simulations of the c-quark are doable, but some care has to be taken to control the discretization errors [6].

This extrapolation can be replaced by a –more safe– interpolation, by the use of an effective theory. See [7] for a quenched computation of $f_{B_s}$ done in this way.
give more details below). The procedure results an effective theory, which has to be matched with QCD. This matching is a source of uncertainty when it is done perturbatively. Recently, some efforts have been made to remove the largest discretization errors by adding appropriate terms to the Lagrangian. One obtains a Lagrangian which describes both light and heavy quarks. One of the main problem with these methods is to compute the coefficients which come in front of the terms that one adds to the Lagrangian.

In the next section, I present shortly these effective theories, and their lattice discretization (see [8] for a pedagogical review).

3. Effective theories

A number of simplifications can be made when one deals with one or several heavy quarks. To make this explicit, one can derive an effective Lagrangian, which is much simpler than the original. The starting point is the observation that the momentum of a heavy quark inside a hadron can be written as \( p = m_Q v + k \). In that decomposition, \( v \) is the velocity, and \( k \) the residual momentum, which is zero if the heavy quark is on-shell. For a heavy quark interacting with light degrees of freedom, the components of the residual momentum \( k \) are of order \( \Lambda_{QCD} \), and therefore much smaller than \( m_Q \). One can separate the higher and lower components (this refers to the case where the \( \gamma \) matrices are written in the Dirac basis) of the heavy quark field \( Q(x) \) in \( h_\pm(x) = e^{i m_Q v \cdot x} P_\pm Q(x) \), where \( P_\pm = \frac{1 \pm \gamma_5}{2} \). Then one finds the effective tree-level Lagrangian\(^3\) (see e.g. [11] for a simple derivation) which reads

\[
\mathcal{L}_{\text{eff}} = h_+(x) \left[ i v. D + (i D_\pm)^2 + \frac{g \sigma \cdot G}{4 m_Q} + \ldots \right] h_+(x)
\]

where the ellipse represent higher order terms.

The values of the coefficients given in eq. 1 only hold at tree level\(^4\). They have to be renormalized to include loops effects. This is usually done though a perturbative matching with QCD. One computes a physical process at a certain order of the effective theory, and imposes the result to be equal to its QCD value. However, on the lattice, a perturbative computation of these coefficients will lead to a result which is power divergent in the continuum limit. Thus a non-perturbative matching with QCD is needed, this has been done in the case of HQET [11] (I will give an explicit example in Sect. 4.2), but not for NRQCD.

3.1. Heavy quark effective theory

HQET describes a situation where only one heavy quark is present, like heavy-light mesons. It is then more natural to write everything in the rest frame of this heavy quark. The previous effective Lagrangian is seen as an expansion in \( \Lambda_{QCD}/m_Q \). With \( \Lambda_{QCD} \sim 500 \text{ MeV} \) and \( m_Q \sim 5 \text{ GeV} \) the theory is expected to have roughly a 10% precision at the leading order, and 1% at the next to leading order. At the leading order, when \( m_Q \to \infty \), the Lagrangian is then \( \mathcal{L}_0 = h_+(x) [i D_0] h_+(x) \). It represents a heavy quark acting only as a static color source. One can see that, at this order, the light quarks are independent of the flavor and of the spin of the heavy quark. The second and the third terms appear at the next to leading order, and represent respectively the interaction due to the motion and to the spin of the heavy quark:

\[
\mathcal{L}_{\text{kin}} = -\bar{h}_+(x) \left[ \frac{D^2}{2 m_Q} \right] h_+(x) \quad (2)
\]

\[
\mathcal{L}_{\text{spin}} = -\bar{h}_+(x) \left[ \frac{g \bar{\sigma} \cdot B}{4 m_Q} \right] h_+(x). \quad (3)
\]

A lattice formulation of HQET at the leading order is the so-called Eichten-Hill action [11]. It is important to note that the higher order terms appear only as insertion in the static green functions. Explicitly, at the \( 1/m_Q \) order, one writes

\[
\exp (-S_{\text{light}} - S_{\text{HQET}}) = \exp (-S_{\text{light}} - S_{\text{stat}}) \times \left( 1 + a^4 \sum_x \left[ \omega_{\text{kin}} O_{\text{kin}} + \omega_{\text{spin}} O_{\text{spin}} \right] \right).
\]
For a green function of an operator $O$, this means:
\[
\langle O \rangle = \langle O \rangle_{\text{stat}} + \omega_{\text{kin}} a^d \sum_x \langle O O_{\text{kin}}(x) \rangle_{\text{stat}} + \omega_{\text{spin}} a^d \sum_x \langle O O_{\text{spin}}(x) \rangle_{\text{stat}},
\]
where $\langle O \rangle_{\text{stat}}$ is the expectation value of $O$ given by the Lagrangian $L_{\text{light}} + L_{\text{stat}}$. In that way, the continuum limit is well defined, and the theory is (power-counting) renormalizable. In the previous expressions $O_{\text{kin}}$ and $O_{\text{spin}}$ are the lattice version of the operators corresponding to (2) and (4) respectively. The coefficients $\omega_{\text{kin}}$ and $\omega_{\text{spin}}$ are fixed by the matching with QCD.

3.2. Non relativistic QCD

In a heavy-heavy hadron, the dynamics is rather different than with heavy-light, and HQET is not an appropriate theory. Starting from the same effective Lagrangian $\mathcal{L}$, it is possible to derive another effective theory, called NRQCD\footnote{NRQCD can also be used for heavy light mesons and 0.3 for the c-quark). Then the leading discretization errors are of order $O(a^2 \vec{p}^2)$ and $O(a \frac{\vec{r}^2}{m_Q})$, which are the same order of magnitude as the relativistic corrections. These corrections to the Lagrangian introduce new coefficients, which contain power law divergences (in $g^2/\alpha m_Q$), because of the non-renormalizability of the theory. They are computed in perturbation theory, implying that the lattice spacing should not be too small (typically $a \sim 1/m_Q$). To reduce the lattice spacing (and increase the precision), one should add more and more terms to the Lagrangian. Naively, these radiative corrections can contribute for 20 or 40% errors, so these perturbative calculations with care. A lot of work has been done is the last years to obtain lattice results for the hadron spectrum with a very high accuracy, and the inclusion of dynamical fermion plays a crucial role, as discussed in Sect. 4.1.

3.3. Relativistic heavy quarks

The lattice formulations of both HQET and NRQCD are obtained from their continuum version. One first writes down an effective Lagrangian in an euclidian space-time at a certain order, then this Lagrangian is discretized. Of course the resulting theory is only valid for heavy quarks (and low velocity for NRQCD).

This is rather different in the Fermilab approach (see e.g. [14,15,16]). The aim is to find a lattice Lagrangian able to describe both small and large masses. The idea is to use on-shell Symanzik improvement, treating both $a$ and $1/m_Q$ as short distances, in such a way that the theory still makes sense when $a m_Q > 1$. Starting from the standard Wilson-Dirac operator, one can write
\[
\mathcal{L}_{\text{lat}} = \mathcal{L}_{\text{Wilson}} + \sum_{O} a^{\text{dim}O-4} c_O \bar{O} \text{ lat}.
\]

The (dimension 5) kinetic and spin operators appear in $\mathcal{O}_{\text{lat}}$, in the the previous expression. Their coefficients $c_O$ are functions of $a m_Q$ which are tuned to reproduce their continuum values (taken e.g. from NRQCD). Unfortunately this matching is done done (at least partially) in perturbation theory. The claim is that when $a \to 0$, the action reduces to the (improved) Wilson action. Then contrary to NRQCD, the continuum limit
of (4) exists. Once all the coefficients have been computed at a sufficient order, the Lagrangian (4), has the properties of NRQCD (or HQET) for $am_Q > 1$, and reduces to the light action for $am_Q < 1$. In the same spirit, the authors of [17] reduce the discretizations errors up to order $a\Lambda_{\text{QCD}}$, by using 4 dimension 5 operators (with 4 different coefficients). In a very recent work [18], it is shown that the coefficients of these operators are not independent, and that, indeed, all errors of order $O(a|\vec{p}|)$ and $O((am_Q)^n)$ can be removed (for arbitrary n) by a proper choice of 3 coefficients (including the bare quark mass $m_0$). Moreover, contrary to NRQCD or to the Fermilab method, these coefficients can be computed non-perturbatively [19], and then would represent a great improvement.

4. Physical results

4.1. Unquenched results

A few years ago, in a common paper [20], the MILC, HPQCD and Fermilab collaborations presented a comparison of lattice computation of hadronic observables with the experimental results. In the quenched approximation, the results agree within 10%, and when the effects of quark vacuum polarization are included, the agreement increases to 3%, as one can see in Fig 1. Last year, the same collaborations achieved the computation of various quantities which were experimentally unknown or poorly known at that time, like the decay constant of the $D_s$ and the $D_+$ mesons, or the mass of the $B_c^+$ mesons. After precise experimental measurement of these quantities, the main conclusion is again that the inclusion of dynamical fermions is very important for the agreement between lattice and experimental results [21] (see e.g. Fig 2). The b-quark was simulated with a $O(a^2, v^4)$-improved NRQCD action, and the c-quark with the Fermilab action. For the light dynamical fermions, the staggered action was used. The problem with these fermions, is the use of fourth root trick, to eliminate the non-physical ‘tastes’. Since this can introduce non-locality, this can be potentially dangerous (for a recent review about this, see [22]). The advantage of using staggerd fermion resides in the fact that they are numerically very cheap. Simulating light quarks is then easier than than with other actions (in this work the sea quark masses go down to $\sim m_s/6$).

![Comparison of quenched and unquenched results with experiments.](image1.png)

Figure 1. Comparison of quenched and unquenched results with experiments.

![Comparison of quenched ($n_f = 0$) and unquenched ($n_f = 2, 3$) results of $f_{D_s}$ with its experimental value.](image2.png)

Figure 2. Comparison of quenched ($n_f = 0$) and unquenched ($n_f = 2, 3$) results of $f_{D_s}$ with its experimental value.

4.2. Quenched results in HQET

I report here some calculations in the b-sector, which present some theoretical advantages, despite the use of the quenched approximation.

A computation of the bag parameter of $B_s - \bar{B}_s$ mixing in the static limit has been presented at this conference [23]. The main advantage of this
computation is that the light quark is simulated with the overlap action [24], which presents a chiral symmetry at finite lattice spacing. This simplifies the renormalization of the 4-quark operators, and reduces the systematic errors. The result is $B_{\text{BS}} = 0.92(3)^6$.

The two last examples concern the mass of the b-quark and the decay constant $f_{B_s}$. One common point is the use of a small volume (here of space extent $L_1 \sim 0.4$ fm) to simulate a relativistic b-quark with discretization errors under control. The authors of [28] have used HQET together with the step scaling method, to compute these quantities [26,27]. One starts by the computation of a finite-volume observable $\Phi(L_1)$. The evolution to a larger volume $L_{i+1} = kL_i$ is given by a step scaling function (ssf), defined by $\sigma(L_2) = \Phi(L_2)/\Phi(L_1)$. If the volume of space extent $L_\infty = L_{\text{final}}$ is large enough to be considered as an infinite volume, one has

$$\Phi(L_\infty) = \sigma(L_{\text{final}})\ldots\sigma(L_3)\sigma(L_2)\Phi(L_1) \quad (5)$$

In the r.h.s. of (5), $\Phi(L_1)$ is computed in QCD, and the ssf are interpolated to the b-quark mass by using QCD with lower masses together with a static calculation. The l.h.s. can be an HQET prediction or an experimental input. One choice of observable is the pseudo-scalar mesons mass. In the infinite volume, it is fixed to the experimental value of the $B_s$ mass. The b-quark mass is then extracted from the r.h.s. of (5). Of course, in the case of the decay constant, the l.h.s. of (5) is a prediction. The 'static' results are $m_{B_s}^{\text{MS}}(m_b) = 4.421(67) \text{ GeV}$ and $F_{B_s} = 191(6) \text{ MeV}$.

A non-perturbative calculation of the b-quark mass including the $1/m_Q$ terms has been done in [28]. This is a direct application of the method introduced in [10], where a calculation of the b-quark mass in the static approximation is also performed. Here I just give a quick overview of the strategy. For simplicity, I start with the static case. At this order the meson mass is given in terms of the bare b-quark mass by

$$m_B = m_{\text{bare}} + E^{\text{stat}} \quad (6)$$

We can define an observable $\Phi(L)$ both in QCD and in HQET, such that $\Phi(L_\infty) = m_B$. In a finite volume $L_1$, the equivalent of (6) is

$$\Phi(L_1, M_b)^{\text{QCD}} = m_{\text{bare}} + \Gamma^{\text{stat}}(L_1) \quad (7)$$

where $\Gamma^{\text{stat}}(L_\infty) = E^{\text{stat}}$. Since the bare parameters are independent of the volume, one can use (7) to substitute $m_{\text{bare}}$ in (6). This gives an expression where $[E^{\text{stat}} - \Gamma^{\text{stat}}(L_1)]$ appears. The terms $E^{\text{stat}}$ and $\Gamma^{\text{stat}}$ contain divergences linear in the inverse lattice spacing, but they cancel in the difference. However, since $L_\infty$ is much larger than $L_1$, it is in practice very difficult to find a lattice spacing common to these volumes. Instead, one can introduce an intermediate volume $L_2 = 2L_1$, and the step scaling function

$$\sigma^{\text{stat}} = L_2 \left[ \Gamma^{\text{stat}}(L_2) - \Gamma^{\text{stat}}(L_1) \right] . \quad (8)$$

Then one can write:

$$m_B = \Phi(L_1, M_b)^{\text{QCD}} + [E^{\text{stat}} - \Gamma^{\text{stat}}(L_2)] + \frac{E^{\text{stat}}}{L_2} \quad (9)$$

Now the cancellations of the divergences occur in $E^{\text{stat}} - \Gamma^{\text{stat}}(L_2)$ and in $\sigma^{\text{stat}}$. Of course the procedure can be repeated, but it appears sufficient in practice to do the computation with 3 different volumes $L_\infty \sim 1.4$ fm $> L_2 \sim 0.7$ fm $> L_1 \sim 0.35$ fm. Finally, the mass of the B meson is fixed to its experimental value and one extracts the RGI b-quark mass by an interpolation ($\Phi(L_1, M_b)$ has to be computed for a few quark masses around the b-quark mass). The whole procedure can be generalized at the next order of HQET. The starting point is to rewrite (6) including the $1/m_Q$ term\(^6\)

$$m_B^{\text{av}} = m_{\text{bare}} + E^{\text{stat}} + \omega_{\text{kin}} E^{\text{kin}} \quad (10)$$

Then one has to find an observable to eliminate $\omega_{\text{kin}}$, and applies the same techniques than in the static case (e.g. two other ssf are needed). The\(^7\)

\(^6\)Here and in the following, the errors do not take into account the use of the quenched approximation.

\(^7\)In principle there should be a term $\omega_{\text{kin}} E^{\text{spin}}$, but it can be eliminated by considering a spin-average B meson $m_B^{\text{av}} = \frac{1}{4}(m_B + 3m_{B^*})$. Then the spin-splitting becomes a separate issue.
result is $m_{bS}(m_b) = 4.337(48)$ GeV.
This non-perturbative method presents several good features. It is based on general hypotheses, and so can be applied to various quantities, like heavy-light decay constants. The cost is rather moderate, so it should be possible to generalize it to full QCD. Since the problem of power-law divergences is cured, the theory is well-defined in the continuum limit. Because calculations can be done at the NLO of HQET (the NNLO corrections are expected to be very small), a very good precision can be reached.

5. Outlook

Heavy flavor physics on the lattice is a very active field, and lot of progress has been achieved recently. Some of them are technical (all-to-all propagators [29], noise reductions [30], new algorithms), and other are conceptual (new strategies, new effective actions, non-perturbative renormalization). Results with simulations done with dynamical quarks are already available. Although a lot of them are done with the controversial staggered quarks, simulation of dynamical fermions with other actions are on their way. They have a major role to play in constraining the standard model and in the search for new physics.

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