First and second derivative Hölder estimates for generated Jacobian equations

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Abstract
We prove two Hölder regularity results for solutions of generated Jacobian equations. First, that under the A3 condition and the assumption of nonnegative $L^p$ valued data solutions are $C^{1,\alpha}$ for an $\alpha$ that is sharp. Then, under the additional assumption of positive Dini continuous data, we prove a $C^2$ estimate. Thus the equation is uniformly elliptic and when the data is Hölder continuous solutions are in $C^{2,\alpha}$.

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1 Introduction

Generated Jacobian equations are a class of PDE which model problems in geometric optics and have recently seen applications to monopolist problems in economics [17, 22, 27]. These equations are of the form

$$\det DY(\cdot, u, Du) = \psi(\cdot, u, Du)$$ in $\Omega$, (1)

where $Y : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\Omega$ is a domain, and $u : \Omega \to \mathbb{R}$ satisfies a convexity condition that ensures the PDE is elliptic.

The precise form of this condition on $u$ and the structure of $Y$ requires a number of definitions which we introduce in Sect. 2. However, the framework we work in includes, in addition to the just listed applications, two well-known special cases. First, the Monge–Ampère equation (take $Y(\cdot, u, Du) = Du$). Second, the Monge–Ampère type equations from optimal transport (take $Y(\cdot, u, Du) = Y(\cdot, Du)$ as the optimal transport map depending on the corresponding potential function $u$ for the optimal transport problem). Indeed, generated Jacobian equations (GJEs) were introduced to treat the aforementioned new applications using the techniques from optimal transport. That’s what we do here: Two fundamental results for the regularity theory in optimal transport are the $C^{1,\alpha}$ result of Liu [12] and the $C^{2,\alpha}$ results of Liu et al. [13]. These are based on the corresponding results of Caffarelli in the Monge–Ampère case, respectively, [2] and [1]. In this paper we extend the results of [12, 13] to generated Jacobian equations. Our main results are stated precisely at the conclusion of Sect. 2, once the necessary definitions have been introduced. The importance of our results is that, first, the $C^{1,\alpha}$ (or at least some $C^1$) result is a necessary assumption in the derivation of models in geometric optics and, second, the $C^{2,\alpha}$ result puts us in the regime of classical elliptic PDE and lets us bootstrap higher regularity.

We note that the corresponding $C^{1,\alpha}$ result of Loeper in the optimal transport setting [15] has been extended to generated Jacobian equations by Jeong [8]. Thus our key contribution for the $C^{1,\alpha}$ result is improving the value of $\alpha$ so that it is sharp. As we explain at the conclusion of Sect. 2 this yields a corresponding improvement on the $C^{2,\alpha}$ result. An outline of the paper is provided by the table of contents.

2 Generating functions, $g$-convexity, and GJEs

In this section we state the essential definitions. Further introductory material can be found in the expository article of Guillen [6], and more detailed outlines of the whole theory in [7, 18, 22, 23].

The framework for GJEs was introduced by Trudinger [22] and is built around a generalized notion of convexity. The generating function, which we now define, plays a central role, essentially that of affine hyperplanes in classical convexity.

Definition 1 A generating function is a function, which we denote by $g$, satisfying the conditions A0, A1, A1*, and A2.

A0. $g \in C^4(\bar{\Gamma})$ where $\Gamma$ is a bounded domain of the form

$$\Gamma = \{(x, y, z); x \in U, y \in V, z \in I_{x,y}\},$$

for domains $U, V \subset \mathbb{R}^n$ and $I_{x,y}$ an open interval for each $(x, y) \in U \times V$. Moreover we assume there is an open interval $J$ such that $g(x, y, I_{x,y}) \supset J$ for each $(x, y) \in U \times V$. 

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**A1.** For each \((x, u, p) \in \mathcal{U}\) defined by 
\[
\mathcal{U} = \{(x, g(x, y, z), g_x(x, y, z)); (x, y, z) \in \Gamma\},
\]
there is a unique \((x, y, z) \in \Gamma\), whose \(y, z\) components we denote by \(Y(x, u, p), Z(x, u, p)\), satisfying 
\[
g(x, y, z) = u \quad \quad g_x(x, y, z) = p. \tag{2}
\]

**A1\(^*\).** For each fixed \(y, z\) the mapping \(x \mapsto \frac{g_x}{g}(x, y, z)\) is injective on its domain of definition.

**A2.** On \(\bar{T}\) there holds \(g_z < 0\) and the matrix 
\[
E := g_{i,j} - g_z^{-1}g_{i,z}g_{,j}
\]
satisfies \(\det E \neq 0\). Here subscripts before and after a comma denote, respectively, differentiation in \(x\) and \(y\).

Two examples are \(g(x, y, z) = x \cdot y - z\) which generates (in accordance with Definition 3) the Monge–Ampère equation and standard convexity, and \(g(x, y, z) = c(x, y) - z\), where \(c\) is a cost function from optimal transport, which generates the Monge–Ampère type equation from optimal transport, and the cost convexity theory [24, Chapter 5]. By a duality structure, which we do not need and thus don’t introduce here, the condition \(A1^*\) is dual to \(A1\), thereby justifying the name. The \(A0\) condition is weakened by some authors who treat \(C^2\) generating functions [7, 9]. However our interior Pogorelov estimate, an essential tool for the \(C^{2, \alpha}\) result, is obtained by differentiating the PDE twice which relies on a \(C^4\) generating function.

**Definition 2** Let \(\Omega \subset U\) be a domain and \(u \in C^0(\overline{\Omega})\). We call \(u\) \(g\)-convex provided for every \(x_0 \in \Omega\) there is \(y_0 \in V\), \(z_0\) such that 
\[
u(x_0) = g(x_0, y_0, z_0), \tag{3}
\]
\[
u(x) \geq g(x, y_0, z_0) \quad \text{for all } x \in \Omega \text{ satisfying } x \neq x_0, \tag{4}
\]
\[
u(\overline{\Omega}, y_0, z_0) \subset J. \tag{5}
\]

When the inequality in (4) is strict \(u\) is called strictly \(g\)-convex. When the function \(x \mapsto g(x, y_0, z_0)\) satisfies (3),(4), and (5) it is called a \(g\)-support of \(u\) at \(x_0\).

The set of all \(y\) such that there is \(z\) for which \(g(\cdot, y, z)\) is a \(g\)-support at \(x\) is denoted \(Yu(x)\). When \(u\) is differentiable and \(g(\cdot, y, z)\) is a \(g\)-support at \(x\) then \(u - g(\cdot, y, z)\) has a minimum at \(x\) implying by (2) that \(y = Y(x, u, Du)\). Similarly, if \(g(\cdot, y, z)\) is a \(g\)-support at \(x\) and \(u\) is \(C^2\) then \(D^2u(x) - g_{,zz}(x, y, z)\) is a nonnegative definite matrix. When \(u\) is not differentiable at \(x\), \(Yu(x)\) is not a singleton.

**Definition 3** A generated Jacobian equation is an equation of the form (1) where the mapping \(Y\) derives from a generating function as in the A1 condition.

For a GJE to make sense we must have \((\text{Id}, u, Du)(\Omega) \subset \mathcal{U}\). By calculations which are now standard [22], a \(C^2\) solution of (1) satisfies the Monge–Ampère type equation 
\[
\det[D^2u - A(\cdot, u, Du)] = B(\cdot, u, Du) \quad \text{in } \Omega, \tag{6}
\]

where \(A, B\) are defined on \(U\) by 
\[
A_{ij}(x, u, p) = g_{ij}(x, Y(x, u, p), Z(x, u, p)), \tag{7}
\]
\[
B(x, u, p) = \det E(x, u, p)\psi(x, u, p). \tag{8}
\]

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This equation is degenerate elliptic provided \( u \) is \( g \)-convex.

The following definition extends the definition of Aleksandrov solution for the Monge–Ampère equation to generated Jacobian equations.

**Definition 4** A \( g \)-convex function \( u : \Omega \to \mathbb{R} \) is called an Aleksandrov solution of

\[
\det DY(\cdot, u, Du) = \nu \text{ in } \Omega, 
\]

for \( \nu \) a Borel measure on \( \Omega \), provided for every Borel \( E \subset \Omega \)

\[
|Yu(E)| = \nu(E).
\]

Whilst (9) would, classically, require a \( C^2 \) function, we’ve defined Aleksandrov solutions for merely \( g \)-convex functions. However it is a consequence of the change of variables formula and the Lebesgue differentiation theorem that \( C^2 \) Aleksandrov solutions are classical solutions.

We introduce one final condition on the generating function. It was introduced by Ma, Trudinger, and Wang [16] and was extended to GJEs in [22]. The necessity of the weakened form, for even \( C^1 \) regularity, was proved by Loeper [15].

**A3.** There is \( c > 0 \) such that

\[
D_{p_i p_j} A_{ij}(x, u, p) \xi_i \xi_j \eta_k \eta_l \geq c,
\]

for all unit vectors \( \xi, \eta \) satisfying \( \xi \cdot \eta = 0 \).

The A3 weak (A3w) condition, is the same but with \( c = 0 \). The A3w condition is not only necessary for the regularity theory, but also essential for the \( g \)-convex structure (and the \( c \)-convex structure in optimal transport) [10, 11, 21, 26].

2.1 Statement of main theorems

**Theorem 1** Let \( g \) be a generating function satisfying A3. Let \( u : \Omega \to \mathbb{R} \) be a \( g \)-convex Aleksandrov solution of (9) with \( \nu = f \, dx \in L^p(U) \) for \( p > (n + 1)/2 \). Then \( u \in C^{1, \alpha}(\Omega) \) with

\[
\alpha = \frac{\beta(n + 1)}{2n^2 + \beta(n - 1)} \text{ where } \beta = 1 - \frac{n + 1}{2p}.
\]

**Remark 1** Liu proved this value, \( \alpha = (2n - 1)^{-1} \) when \( p = \infty \), is sharp. That is, there exists a function \( u \) which solves a GJE satisfying the hypothesis of the theorem, and which is in \( C^{1, \alpha} \) for the stated \( \alpha \), but not in \( C^{1, \alpha+\varepsilon} \) for any \( \varepsilon > 0 \). We note that Loeper [15] and Jeong [8] have proved the Hölder regularity for the right-hand side a measure \( \nu \) satisfying \( \nu(B_r(x)) \leq Ce^{r(1-1/p)} \). Our proof is easily adapted to this condition (which is more general than above but comes at the expense of a smaller \( \alpha \)). We indicate the necessary changes in a remark after the proof of Theorem 1.

Our \( C^2 \) estimate is for \( \nu = f \, dx \) where \( f \) is Dini continuous.

**Definition 5** (Dini continuity) Let \( f : \Omega \to \mathbb{R} \). The oscillation of \( f \) is

\[
\omega_f(r) := \sup\{|f(x) - f(y)|; x, y \in \Omega \text{ with } |x - y| < r\}.
\]

Then \( f \) is called Dini continuous if

\[
\int_0^1 \frac{\omega_f(r)}{r} \, dr < \infty.
\]
Theorem 2 Let \( g \) be a generating function satisfying A3. Let \( u : \Omega \to \mathbb{R} \) be an Aleksandrov solution of (9) with \( v = f \, dx \). If \( \lambda \leq f \leq \Lambda \) and \( f \) is Dini continuous then \( u \in C^2(\Omega) \). If \( f \in C^\alpha(\Omega) \) then \( u \in C^{2,\alpha}_{loc}(\Omega) \).

**Remark 2** Our result can be stated more precisely as follows. Under the above hypothesis for each \( \Omega' \subset \subset \Omega \) there is \( C > 0 \) depending on \( g, \Omega', \Omega, \|u\|_{C^1(\Omega)} \) such that

1. If \( f \) is Dini continuous we have
   \[
   |D^2u(x) - D^2u(y)| \leq C \left[ d + \int_0^d \frac{\omega_f(r)}{r} \, dr + d \int_d^1 \frac{\omega_f(r)}{r^2} \, dr \right],
   \]
   where \( x, y \in \Omega' \) and \( d := |x - y| \).

2. If \( f \in C^\alpha(\Omega) \) for some \( \alpha \in (0, 1) \) then \( u \in C^{2,\alpha}(\Omega) \) with
   \[
   \|u\|_{C^{2,\alpha}(\Omega')} \leq C + \frac{C}{\alpha(1 - \alpha)} \|f\|_{C^\alpha(\Omega)}.
   \]

We do not prove (10) and (11) directly—we just prove a \( C^2 \) estimate. This ensures the equation is uniformly elliptic then (10) and (11) follow from [25, Theorem 3.1] (details in Sect. 6).

**Remark 3** A common form of \( v \) arising in applications is

\[
\nu = \frac{f(x)}{f^*(Y(x, u, Du))} \, dx.
\]

Theorem 1 applies whenever \( f \in L^p \) and \( f^* > \lambda \) for some positive constant \( \lambda \). Then the assumption of Hölder continuous right-hand side in Theorem 2 is satisfied when \( \lambda < f, f^* < \Lambda \) and \( f, f^* \) are Hölder continuous. More precisely if \( f \in C^\beta, f^* \in C^\gamma \) then Theorem 1 implies the sharp exponent with \( p = \infty \) and we subsequently obtain \( u \in C^{2,\alpha'} \) for \( \alpha' = \min\{\beta, \gamma, 1/(2n - 1)\} \).

As is standard we prove apriori estimates for smooth solutions. The results then hold by approximation and uniqueness of the Dirichlet problem in the small [22, Lemma 4.6]. The results above all hold without boundary conditions, that is they are interior local results. This is possible because we are considering Aleksandrov solutions under A3. With A3 weakened to A3w such results are not possible. Moreover for applications to optimal transport to conclude that a potential function is an Aleksandrov solution we require also a boundary condition—the second boundary value problem with a target satisfying a convexity condition [16].

### 3 Background results and a normalization lemma

We will use a number of background results. These originally appeared in [7] though we’ll use the formulation in [20].

We assume \( g \) is a generating function satisfying A3w and \( u \) is a strictly \( g \)-convex function with \( \det DYu =: \nu \) in the Aleksandrov sense. Let \( x_0 \) be given and \( g(\cdot, y_0, z_0) \) a support at \( x_0 \) put \( z_h = z_0 - h \) where \( h > 0 \) and define the section

\[
S_h = S_h(x_0) := \{u < g(\cdot, y_0, z_h)\},
\]

which by the strict \( g \)-convexity is compactly contained in \( \Omega \) for sufficiently small \( h \).

A lemma [20, Theorems 3,5], which we employ repeatedly is the following.
Lemma 1 There exists $C, d > 0$ which depend only on $g$, such that if $\text{diam}(S_h) < d$ and $\nu$ is a doubling measure, then
\[ C^{-1}|S_h|\nu(S_h) \leq \sup_{S_h} |u(\cdot) - g(\cdot, y_0, z_h)|^n \leq C|S_h|\nu(S_h). \]

Note the requirement that $\nu$ is a doubling measure is only necessary for the lower bound. Also, $C^{-1}h \leq \sup_{S_h} |u(\cdot) - g(\cdot, y_0, z_h)| \leq Ch$ for $C > 0$ depending only on $\sup |g_z|$, $\inf |g_z|$. In the special case where $\nu = f \, dx$ and $\lambda \leq f \leq \Lambda$ we have
\[ C^{-1}\lambda|S_h|^2 \leq h^n \leq C\Lambda|S_h|^2. \] (12)

We introduce new coordinates
\[ \tilde{x} := \frac{g_y}{g_z}(x, y_0, z_h), \] (13)

When $A3w$ is satisfied $S_h(x_0)$ is convex in the $\tilde{x}$ coordinates [23, Lemma 2.3]. We often use this result in conjunction with the minimum ellipsoid. The minimum ellipsoid of an arbitrary open convex set $U$ is the unique ellipsoid of minimal volume, denoted $E$, containing $U$. It satisfies $\frac{1}{n}E \subset U \subset E$ where this is dilation with respect to the centre of the ellipsoid [14, Lemma 2.1]. We assume, after a rotation and translation that the minimum ellipsoid of $S_h$ is
\[ E = \left\{ x; \sum_{i=1}^{n} \frac{x_i^2}{r_i^2} \leq 1 \right\}, \text{ where } r_1 \geq \cdots \geq r_n. \] (14)

Then elementary convex geometry implies $c_n r_1 \ldots r_n \leq |S_h| \leq C_n r_1 \ldots r_n$ and (12) becomes
\[ C^{-1}r_1 \ldots r_n \leq h^{n/2} \leq Cr_1 \ldots r_n. \] (15)

We also define here the shape constant. Let $U$ be any convex set, and assume its minimum ellipsoid is given by (14). Then a shape constant is any $C$ satisfying $C \geq r_1/r_n$ and the shape constant is just $r_1/r_n$, explicitly, the infimum of all shape constants. For solutions of generated Jacobian equations the shape constant of $S_h(x_0)$ carries information about $D^2u(x_0)$ (see Lemmas 5 and 6).

When $A3w$ is strengthened to $A3$ we have a particularly strong estimate concerning the geometry of sections and their height. It is used repeatedly throughout this paper. In optimal transport this estimate is due to Liu [12, Lemma 4] and we largely follow his proof.

Lemma 2 Assume $g$ is a generating function satisfying $A3$ and $u : \Omega \to \mathbb{R}$ is a $C^2$ $g$-convex function. Assume that $S_h(x_0) \subset \subset \Omega$ and that the minimum ellipsoid of $S_h(x_0)$ (in the $\tilde{x}$ coordinates) is (14). Then there is $C$ depending only on $g$ such that
\[ \frac{hr_1^2}{r_n^2} \leq C. \]

Proof We work in the $\tilde{x}$ coordinates, though keep the notation $x$. Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by
\[ T(x) = (x_1/r_1, \ldots, x_n/r_n). \] (16)

Note
\[ B_{1/n} \subset U := TS_h \subset B_1. \] (17)

Let $\tilde{x}$ be the boundary point of $U$ on the negative $x_n$ axis. In a neighbourhood of $\tilde{x}$, denoted $\mathcal{N}$, we represent $\partial \Omega$ as a graph of some function $\rho$, that is
\[ \partial U \cap \mathcal{N} = \{(x', \rho(x')); x' = (x_1, \ldots, x_{n-1})\}. \]
Using (17) we may assume \( \rho \) is defined for \( |x'| < 1/n \). Similarly by (17) and the convexity of \( U \) we conclude \( |D\rho| \leq c(n) \) when \( |x'| \leq 1/2n \). Let \( \gamma \) be the curve

\[
\gamma = \{ \gamma(t) = (t, 0, \ldots, 0, \rho(te_1)); -1/4n < t < 1/4n \}.
\]

Because (17) implies \( |\rho_{11}(\gamma(t))| \leq C(n) \) at some \( t \in (-1/4n, 1/4n) \) the proof will be complete provided we can show that

\[
C|\rho_{11}(\gamma(t))| \geq \frac{h}{r_1^2/r_n^2}, \tag{18}
\]

for every \( t \in (-1/4n, 1/4n) \).

Let \( v : U \rightarrow \mathbb{R} \) be defined on \( \bar{x} \in U \) by

\[
v(\bar{x}) = u(T^{-1}\bar{x}) - g(T^{-1}\bar{x}, y_0, z_h).
\]

Differentiating \( v(\gamma(t)) = 0 \) once, then twice, with respect to \( t \) gives

\[
D_k v \dot{\gamma}_k = 0, \tag{19}
\]

\[
D_{\dot{\gamma}} v = -D_k v \ddot{\gamma}_k = -D_n v \rho_{11}. \tag{20}
\]

To estimate \( D_{\dot{\gamma}}^2 v = D_{kl} v \dot{\gamma}_k \dot{\gamma}_l \) we compute at \( \bar{x} = Tx \)

\[
D_{kl} v = [u_{kl} - g_{kl}(x, y_0, z_h)] r_k r_l - [u_{kl} - g_{kl}(x, Yu(x), Zu(x))] r_k r_l + [g_{kl}(x, Yu(x), Zu(x)) - g_{kl}(x, y_0, z_h)] r_k r_l \geq |g_{kl}(x, Yu(x), Zu(x)) - g_{kl}(x, y_0, z_h)| r_k r_l.
\]

The inequality is by \( g \)-convexity of \( u \). Put \( \rho_{gh}(\cdot) := g(\cdot, y_0, z_h) \). Then use the definition of \( A_{ij} \) (equation (7)), \( u(x) = g_h(x) \) on \( \partial S_h \), and a Taylor series to obtain

\[
D_{kl} v \geq [A_{kl}(x, u(x), D_u(x)) - A_{kl}(x, u(x), Dg_h(x))] r_k r_l = A_{kl,pm}(x, u(x), Dg_h(x)) D_m (u - g_h) r_k r_l + A_{kl,pm} p_n(x, u(x), p_r) D_m (u - g_h) D_n (u - g_h) r_k r_l r_l, \tag{21}
\]

where \( p_r = \tau D_u(x) + (1 - \tau) Dg_h(x) \) for some \( \tau \in (0, 1) \). A direct, but involved, calculation which we relegate to Appendix 1 implies

\[
A_{kl,pm}(x, u(x), Dg_h(x)) D_m (u - g_h) r_k r_l \dot{\gamma}_k \dot{\gamma}_l = 0.
\]

Thus, using also \( D_m (u - g_h) = D_m v/r_m \), (21) becomes

\[
D_{kl} v \geq A_{kl,pm} p_n D_m v D_n v \frac{r_k r_l}{r_m r_n}. \tag{22}
\]

Now returning to \( D_{\dot{\gamma}}^2 v \) we have

\[
D_{\dot{\gamma}}^2 v \geq A_{kl,pm} p_n \frac{D_m v}{r_m} \frac{D_n v}{r_n} (\dot{\gamma}_k r_k) (\dot{\gamma}_l r_l). \tag{23}
\]

Since by (19) \( \dot{\gamma} \) is orthogonal to \( Dv \) we also have orthogonality of \( \xi := (r_1 \dot{\gamma}_1, \ldots, r_n \dot{\gamma}_n) \) and \( \eta := (D_1 v/r_1, \ldots, D_n v/r_n) \). Thus employing the A3 condition in (23) yields

\[
D_{\dot{\gamma}}^2 v \geq c |\xi|^2 |\eta|^2 \geq c \frac{|D_n v|^2 r_1^2}{r_n^2}.
\]
Now we substitute this into (20) to obtain

\[ \rho_{11} \geq c \frac{|D_n v|^2}{r_n^2}. \]

If we can show \(|D_n v| \geq C h\) then we’ve obtained (18).

For this final estimate fix \(x_1 \in \partial S_h\) and set

\[ h(\theta) := g(x_0, y_1, z_1) - g(x_0, y_0, z_h) \] where \(x_0 = \theta x_1 + (1 - \theta) x_0\),

where \(g(\cdot, y_1, z_1)\) supports \(u\) at \(x_1\) so \(Du(x_1) = g_x(x_1, y_1, z_1)\). A standard argument [18, Eq. A.14] using the A3w condition implies that for \(K\) depending only on \(g\),

\[ h''(\theta) \geq -K |h'(\theta)|. \]

Then we follow [19, Eq. 19] (there are similar arguments in [7, 22–24]) to obtain

\[ h'(t_1) \leq e^{K(t_2-t_1)} h'(t_2), \]

for \(t_1 < t_2\). Choosing \(t_2 = 1\) and integrating from \(t_1 = 0\) to 1 we have

\[ -h(0) \leq C(K) h'(1). \quad (24) \]

Now \(-h(0) = g(x_0, y_0, z_h) - g(x_0, y_1, z_1)\) \(\geq \inf |g_z|h\) and

\[ |h'(1)| = |D_i u(x_1 - x_0)| = \left| D_i v \frac{(x_1 - x_0)}{r_i} \right| \leq |Dv|, \]

where we’ve used that by the minimum ellipsoid \(|(x_1 - x_0)|/r_i \leq 2\). Thus (24) becomes \(|Dv| \geq ch\). To conclude we control \(|Dv|\) by \(|D_n v|\). Indeed, by (19) \(D_1 v + \rho_1 D_n v = 0\), similar reasoning implies \(D_i v + \rho_i D_n v\) for \(i = 2, 3, \ldots, n - 1\). Thus

\[ Dv = (-\rho_1 D_n v, \ldots, -\rho_{n-1} D_n v, D_n v). \]

Recalling \(|D\rho| \leq C\) we complete the proof with the observation \(|Dv| \leq C|D_n v|\). \(\square\)

### 4 \(C^{1,\alpha}\) regularity

The \(C^{1,\alpha}\) regularity is essentially an immediate consequence of Lemmas 1 and 2. The proof of Theorem 1 is as follows.

**Proof** Step 1. [Proof for strictly g-convex functions] Fix \(x_0\), without loss of generality equal to 0, and then \(h\) sufficiently small to ensure \(S_h(x_0) \subset \subset \Omega\). By Lemma 1

\[ h^n \leq C|S_h|v(S_h). \quad (25) \]

Then assuming the minimum ellipsoid of \(S_h(x_0)\) is given by (14), (25) becomes

\[ h^n \leq C(r_1 \ldots r_n) \int_{S_h} f \, dx. \]

Using Hölder’s inequality (with the second function equal to 1) we have

\[ h^n \leq C(r_1 \ldots r_n)|S_h|^{1-1/p} \|f\|_{L^p} \leq C(r_1 \ldots r_n)^{2-1/p} \|f\|_{L^p}. \quad (26) \]
Now we conclude as in [12] (which is where Lemma 2 is used). More precisely (26) is the 5th inequality on [12, pg. 446], so the rest of this step is exactly as given there.

**Step 2. [Proof for g-convex functions]** When \( u \) is not strictly \( g \)-convex, we may consider on a small enough neighbourhood of \( x_0 \) the function \( u + \varepsilon |x - x_0|^2 \). Indeed by the proof of [18, Theorem 2.22] this function is strictly \( g \)-convex on a neighbourhood of \( x_0 \) depending only on \( g \) (in particular, independent of \( u \) and \( \varepsilon \)). Moreover it is an Aleksandrov solution of a generated Jacobian equation with right-hand side in the original \( L^p \) space. This is an consequence of the identity \( Y u(x) = Y(x, u, (x), \partial u(x)) \) for \( \partial u(x) \) the subgradient. Thus by the previous proof

\[
0 \leq u(x) - g(x, y_0, z_0) \leq u(x) + \varepsilon |x - x_0|^2 - g(x, y_0, z_0) \leq C|x - x_0|^{1+\alpha},
\]

as required.

**Remark 4** Loeper [15] and Jeong [8] proved the \( C^{1,\alpha} \) regularity for Aleksandrov solutions of

\[
\det D Y u = \nu,
\]

where \( \nu \) satisfies that for some \( p \in (n, +\infty) \) and \( C_\nu > 0 \) there holds

\[
\nu(B_\varepsilon(x)) \leq C_\nu \varepsilon^{n((1-1/p))}.
\]

Our proof is easily adapted to this condition. In (25) we use

\[
\nu(S_h) \leq \nu(B_{2r_1}(x_0)) \leq Cr_1^{n((1-1/p))},
\]

and combine with Lemma 2.

## 5 Interior Pogorelov estimate for constant right-hand side

Now we start work on the \( C^{2,\alpha} \) estimate. Recall we will prove this by establishing a \( C^2 \) (i.e. uniform ellipticity) estimate when the right-hand side is Dini continuous. Then we obtain the \( C^{2,\alpha} \) estimate from the elliptic theory [25]. The \( C^2 \) estimate for Dini continuous right hand side is a perturbation of the same result for the special case of constant right-hand side, the proof of which is the goal of this section. First we introduce a strengthening of the \( C^{1,\alpha} \) result and a strict \( g \)-convexity estimate that holds by duality.

**Lemma 3** Let \( g \) be a generating function satisfying A3 and \( u : \Omega \to \mathbb{R} \) be a \( g \)-convex solution of \( \lambda \leq \det D Y u \leq \Lambda \). For each \( \Omega' \subset \subset \Omega \) there is \( C, d, \beta, \gamma > 0 \) depending on \( \lambda, \Lambda, g, \Omega', \Omega \) for which the following holds. Whenever \( x_0 \in \Omega' \), \( g(\cdot, y_0, z_0) \) is the \( g \)-support at \( x_0 \) and \( x \in B_d(x_0) \) there holds

\[
C^{-1}|x - x_0|^{1+\gamma} \leq u(x) - g(x, y_0, z_0) \leq C|x - x_0|^{1+\beta}.
\] (27)

The right-hand inequality is the \( C^{1,\alpha} \) estimate of Guillen and Kitagawa [7] which follows from the strict convexity in [23, Lemma 4.1]. The left inequality follows from the right by duality. We give the proof in Appendix 1.

**Lemma 4** Assume that \( u \in C^4(\Omega) \cap C^2(\overline{\Omega}) \) is a \( g \)-convex solution of

\[
\det[D^2 u - A(\cdot, u, D u)] = f_0 \text{ in } \Omega,
\]

\[
u(\cdot, y, z - h) \text{ on } \partial \Omega,
\] (28)

\( g \) is a generating function satisfying A3 and \( u \) is a \( g \)-convex solution of

\[
\lambda \leq \det D Y u \leq \Lambda \text{ in } \Omega.
\]
where \( g(\cdot, y, z) \) is a support at some \( \overline{x} \in \Omega \) and \( f_0 > 0 \) is a real number. We assume \( g \) satisfies A3w, \( h > 0 \) is sufficiently small (determined in the proof), and \( C_0 \) is the shape constant of \( \Omega \). For each \( \tau \in (0, 1) \) there is \( C > 0 \) depending on \( \|A\|_{C^2}, \|u\|_{C^1(\Omega)}, \tau, C_0, h \) such that in

\[
S_{\tau h} := \{ x; u(x) < g(x, y, z - \tau h) \},
\]

we have

\[
\sup_{S_{\tau h}} |D^2u| \leq C. \tag{29}
\]

**Proof**

This is essentially the estimate

\[
(g(\cdot, y, z - h) - u(\cdot))^{\beta} |D^2u(\cdot)| \leq C, \tag{30}
\]

which was given in [23]. Since we need to ensure the constant only depends on \( \|A\|_{C^2} \) we will provide full details. The proof is via a Pogorelov type estimate: We consider a certain test function which attains a maximum in \( \Omega \). Our choice of test function ensures it both controls the second derivatives and is controlled at its maximum point.

We let \( \phi := |x - \overline{x}|^2 \) and introduce for \( x \in \Omega, \xi \in S^{n-1} \) (the unit sphere) both the function

\[
v(x, \xi) = \kappa \phi + \tau |Du|^2/2 + \log(w_{\xi\xi}) + \beta \log[g(\cdot, y, z - h) - u],
\]

and the differential operator

\[
L(v) := w^{ij}[D_{ij}v - D_{pk}A_{ij}D_{vk}], \tag{31}
\]

where \( w_{ij} = u_{ij} - A_{ij}(x, u, Du) \) and \( D_{pk}A_{ij} \) is evaluated at \( (x, u, Du) \).

We use the notation \( u_0 = g(\cdot, y, z - h) \) and \( \eta = u_0 - u \). Because the nonnegative function \( e^v \) is 0 on \( \partial \Omega \), \( v \) attains an interior maximum at \( x_0 \in \Omega \) and some \( \xi \) assumed without loss of generality to be \( e_1 \). We also assume, again without loss of generality, that at \( x_0 \), \( w \) is diagonal.

At an interior maximum \( Du = 0 \) and \( D^2v \leq 0 \) so that

\[
0 \geq Lv = \kappa L\phi + \tau L(|Du|^2/2) + L(log(w_{11})) + \beta L \log \eta. \tag{32}
\]

We will compute each term in (32) and from this obtain (30).

**Term 1:** \( L\phi \). This one’s immediate—provided the domain (and subsequently \( |D\phi| \)) is chosen sufficiently small depending on \( |A_{ij,p}| \) we have

\[
L\phi \geq w^{ii} - C. \tag{33}
\]

**Term 2:** \( L(|Du|^2) \) We compute

\[
L(|Du|^2/2) = w^{ii}[u_{ki}u_{ki} + u_k[w^{ii}(u_{kii} - D_{pi}A_{ii}u_{ik})]]. \tag{34}
\]

We note (using \( u_{ij} = w_{ij} + A_{ij} \)) that

\[
w^{ij}u_{ki}u_{ki} = w^{ii}(w_{ki} + A_{ki})(w_{ki} + A_{ki}) \geq w_{ii} - C(1 + w^{ii}), \tag{35}
\]

and by differentiating the PDE in the direction \( e_k \) at \( x_0 \)

\[
w^{ij}[u_{ijk} - A_{ij,p}u_{ik}] = w^{ij}(A_{ij,k} + A_{ij,u}u_k). \tag{36}
\]

Hence (34) becomes

\[
L(|Du|^2/2) \geq w_{ii} - C(1 + w^{ii}). \tag{37}
\]

---

1 Here differentiation is with respect to \( x \), never \( \xi \).
Term 3: $L(\log(w_{11}))$. To begin, we differentiate the PDE twice in the $e_1$ direction and obtain, with the notation $w_{ij,k} := D_{x_k}w_{ij}$, that

$$w_{ij}^i[u_{1i11} - D_{p_k}A_{ii,ku_{1k1}}] = w_{ij}^i w_{ij,j,1} + w_{ij}^i[A_{ii,11} - 2A_{ii,1u_{1u}} + 2A_{ii,1p_{1p}u_{111}} + 2A_{ii,u_{1u}11} + 2A_{ii,p_{1p}u_{111}}]
\geq w_{ij}^i w_{ij,j,1} + w_{ij}^i [A_{ii,1p_{1p}u_{111}} - C(w_{ii} + w_{ii} + w_{ii} w_{ij})]. \quad (38)$$

We use A3 with $\xi = e_i$, $\eta = e_j$ for $i \neq j$ we see $A_{ii,p_{1p}u_{111}} \geq 0$ so that

$$w_{ij}^i A_{ii,p_{1p}u_{111}1} w_{11}^2 \geq -C w_{11}.$$ 

Thus (38) becomes

$$L(u_{11}) \geq w_{ij}^i w_{ij,j,1} + w_{ij}^i w_{ij,j,1} - C(w_{ii} + w_{ii} + w_{ii} w_{ij}).$$

We perform similar calculations for $L A_{11}$ to obtain

$$L(u_{11}) \geq w_{ij}^i w_{ij,j,1} - C(w_{ii} + w_{ii} + w_{ii} w_{ij}). \quad (39)$$

We use this to compute $L(\log w_{11})$ as follows. First,

$$L(\log w_{11}) = -\frac{w_{ij}^i w_{11,11}^2}{w_{11}^2} + L(w_{11}) \frac{w_{11}^2}{w_{11}}.$$ 

Hence by (39)

$$L(\log w_{11}) \geq \frac{w_{ij}^i w_{ij,j,1}^2}{w_{11}^2} - \frac{w_{ij}^i w_{11,11}^2}{w_{11}^2} - \frac{C}{w_{11}} (w_{ii} + w_{ii} + w_{ii} w_{ij}). \quad (40)$$

When $i, j = 1$ in the first term and $i = 1$ in the second these terms cancel. At the expense of an inequality we discard terms with neither $i$ nor $j = 1$. Thus

$$\frac{w_{ij}^i w_{ij,j,1}^2}{w_{11}^2} - \frac{w_{ij}^i w_{11,11}^2}{w_{11}^2} \geq \sum_{i > 1} w_{ij}^i w_{11,11}^2 + \sum_{j > 1} w_{ij}^i w_{11,11}^2 - \sum_{i > 1} w_{ij}^i w_{11,11}^2
\geq \frac{1}{w_{11}^2} \sum_{i > 1} w_{ij}^i [2w_{11,11}^2 - w_{11,11}^2]
\geq \frac{1}{w_{11}^2} \sum_{i > 1} w_{ij}^i w_{11,11}^2 + \frac{2}{w_{11}^2} \sum_{i > 1} w_{ij}^i [w_{11,11}^2 - w_{11,11}^2]
\geq \frac{1}{w_{11}^2} \sum_{i > 1} w_{ij}^i w_{11,11}^2.$$
Rewriting the second sum in terms of the $A$ matrix yields
\[
\frac{w^{ii} w^{jj} w_{jj,1}^2}{w_{11}^2} - \frac{w^{ii} w_{11,i}^2}{w_{11}^2} = \frac{1}{w_{11}^2} \sum_{i>1} w^{ii} w_{11,i}^2 \\]
\[+ \frac{2}{w_{11}^2} \sum_{i>1} w^{ii} (D_i A_{11} - D_1 A_{i1})(2w_{11,i} + D_i A_{11} - D_1 A_{i1})\]
\[= \frac{1}{w_{11}^2} \sum_{i>1} w^{ii} [w_{11,i}^2 + 4w_{11,i}(D_i A_{11} - D_1 A_{i1}) + 2(D_i A_{11} - D_1 A_{i1})^2].\]

Now, Cauchy’s inequality implies
\[4w_{11,i}(D_i A_{11} - D_1 A_{i1}) \geq -\frac{w_{11,i}^2}{2} - 8(D_i A_{11} - D_1 A_{i1})^2.\]
Thus
\[\frac{w^{ii} w^{jj} w_{jj,1}^2}{w_{11}^2} - \frac{w^{ii} w_{11,i}^2}{w_{11}^2} \geq \frac{1}{2w_{11}^2} \sum_{i=2}^n w^{ii} w_{11,i}^2 - C w^{ii},\]
and on returning to (40)
\[L(\log(w_{11})) \geq \frac{1}{2w_{11}^2} \sum_{i=2}^n w^{ii} w_{11,i}^2 - C(1 + w_{ii} + w^{ii}). \tag{42}\]

Now, using (33),(37), and (42) in (32) implies
\[0 \geq \kappa (w^{ii} - C) + \tau [w_{ii} - C(1 + w^{ii})] + \frac{1}{2w_{11}^2} \sum_{i=2}^n w^{ii} w_{11,i}^2 - C(1 + w_{ii} + w^{ii}) + \beta L(\log \eta).\tag{43}\]

**Term 3: $L(\log \eta)$**. First, write
\[L(\log \eta) = \frac{L\eta}{\eta} - \sum_{i=1}^n w^{ii} \left( \frac{D_i \eta}{\eta} \right)^2. \tag{44}\]

We compute (using $D_{ii} u_0 = A_{ii}(\cdot, u_0, Du_0)$)
\[L\eta = w^{ii} [D_{ii} u_0 - D_{ii} u - D_{pk} A_{ii}(\cdot, u, Du) D_k \eta] \\]
\[= w^{ii} [-w_{ii} + A_{ii}(\cdot, u_0, Du_0) - A_{ii}(\cdot, u, Du) - D_{pk} A_{ii}(\cdot, u, Du) D_k \eta] \\]
\[\geq w^{ii} [A_{ii,u} \eta + A_{ii}(\cdot, u, Du_0) - A_{ii}(\cdot, u, Du) - D_{pk} A_{ii}(\cdot, u, Du) D_k \eta] - C \\]
\[\geq w^{ii} D_{pk, pi} A_{ii} D_k \eta D_l \eta - C - C w^{ii} \eta. \tag{45} \]

The final line is by Taylor series for $A_{ii}(\cdot, u, (1-t) Du_0 + t Du)$. For each $i$ write
\[D_{pk, pi} A_{ii} D_k \eta D_l \eta = \sum_{k, l \neq i} D_{pk, pi} A_{ii} D_k \eta D_l \eta + 2 \sum_{l \neq i} D_{pi, pi} A_{ii} D_i \eta D_l \eta + D_{pi, pi} A_{ii} D_i \eta D_i \eta \]
\[+ D_{pi, pi} A_{ii} D_i \eta D_i \eta \]
\[\sum_{k, l \neq i} D_{pk, pi} A_{ii} D_k \eta D_l \eta \]
\[+ 2 \sum_{l \neq i} D_{pi, pi} A_{ii} D_i \eta D_l \eta + D_{pi, pi} A_{ii} D_i \eta D_i \eta. \]
Then by A3w the first term is nonnegative, so that
\[ D_{pk,p} A_{ii} D_k \eta D_l \eta \geq -C \eta - C(D(\eta))^2. \]

Returning to (45) we see
\[ L \eta \geq -C(1 + w^{ii} \eta) - Cw^{ii} D_i \eta - Cw^{ii} (D_i \eta)^2. \]

Which into (44) implies
\[ L(\log \eta) \geq -\frac{C}{\eta} - Cw^{ii} - C \sum_{i=1}^{n} w^{ii} \left( \frac{D_i \eta}{\eta} \right)^2. \]  

(46)

Here we’ve used that we can assume \( \eta < 1 \), and also used Cauchy’s to note
\[ w^{ii} D_i \eta \eta = \sqrt{w^{ii}} \sqrt{w^{ii}} \frac{D_i \eta}{\eta} \leq w^{ii} + w^{ii} \left( \frac{D_i \eta}{\eta} \right)^2. \]

Now we deal with the final term in (46). We can assume that \( w^{11}(D_1 \eta/\eta)^2 \leq 1 \), for if not we have (29) with \( \beta = 2 \). Since we are at a maximum \( D_i v = 0 \), that is
\[ \frac{D_i \eta}{\eta} = -\frac{1}{\beta} \left[ \frac{w_{11,i}}{w_{11}} + \kappa D_i \varphi + \tau D_k w_{ik} + \tau D_k u_{ik} \right]. \]

This implies
\[ \sum_{i=2}^{n} w^{ii} \left( \frac{D_i \eta}{\eta} \right)^2 \leq C \left[ \sum_{i=2}^{n} \frac{w^{ii} w_{11,i}^2}{w_{11}^2 \beta^2} + \frac{\kappa^2}{\beta^2} w^{ii} |D_i \varphi|^2 + \frac{\tau^2}{\beta^2} (w_{ii} + w^{ii}) \right]. \]

Choosing \( \beta \geq 1,2C \) and returning to (46) we obtain
\[ \beta L(\log \eta) \geq -\frac{C \beta}{\eta} - \kappa^2 |D\varphi|^2 w^{ii} - \frac{C \tau^2}{\beta} [w^{ii} + w_{ii}] - C \beta w^{ii} - \frac{1}{2} w_{11} w_{11,i}^2. \]

Substituting into (43) completes the proof:
\[ 0 \geq \kappa (w^{ii} - C) + \tau [w_{ii} - C(1 + w^{ii})] - C(1 + w_{ii} + w^{ii}) - \frac{C \beta}{\eta} \]
\[ - C \kappa |D\varphi|^2 w^{ii} - \frac{C \tau^2}{\beta} [w^{ii} + w_{ii}] - C \beta w^{ii} \]
\[ = w^{ii} [\kappa - \tau C - C \kappa^2 |D\varphi|^2 - \frac{C \tau^2}{\beta} - C \beta] \]
\[ + w_{ii} [\tau - C - \frac{C \tau^2}{\beta}] - C[\kappa + \tau + \frac{\beta}{\eta}]. \]

Take \( \text{diam}(\Omega) \), and subsequently \( |D\varphi| \), small enough to ensure \( \kappa^2 |D\varphi|^2 \leq 1 \) (our choice of \( \kappa \) will only depend on allowed quantities). A further choice of \( \beta \geq \tau^2, \tau \) large depending only on \( C \), and finally \( \kappa \) large depending on \( \tau, C, \beta \) implies
\[ 0 \geq w^{ii} + w_{ii} - \frac{C}{\eta}. \]

This implies \( \eta w_{ii} \leq C \) at the maximum point, and the proof is complete. \( \square \)
6 $C^{2,\alpha}$ regularity

In this section we will prove the $C^{2,\alpha}$ estimate via a $C^2$ estimate. We adapt the method of proof used by Liu, Trudinger, and Wang in the optimal transport case [13]. We also use some details from Figalli’s exposition in the Monge–Ampère case [3, Section 4.10]. Here’s how we obtain the $C^2$ estimate. When the right hand side is constant the $C^2$ estimate is true by the interior Pogorelov’s estimate of the previous section. Then the argument for Dini-continuous $f$ is to perturb the argument for constant $f$. That is we zoom in, treating a series of normalized approximating problems with constant right hand side.

6.1 Normalization of sections

Here we explain the procedure for normalizing a solution on a section. We assume we are given a strictly $g$-convex function $u : \Omega \to \mathbb{R}$ which is an Aleksandrov solution of

$$\det D^2u = f,$$

as well as a point $x_0$ and corresponding $g$-support $g(\cdot, y_0, z_0)$. As usual, we consider the section $S_h(x_0)$. The definition of Aleksandrov solution is coordinate independent so we may assume we are in the coordinates given by (13) and $S_h$ is convex. Assume the minimum ellipsoid is given by (14) and $T$ is given by (16). We want to consider the PDE solved by

$$v(\bar{x}) = \frac{1}{h}[u(T^{-1}\bar{x}) - g(T^{-1}\bar{x}, y_0, z_0 - h)].$$

(47)

on $U := TS_h$. Importantly $B_{1/n} \subset U \subset B_1$, $v|_{\partial U} = 0$, and $C^{-1} \leq |\inf_U v| \leq C$ for $C$ depending only on $g$. Thus this is a natural generalization of the normalization procedure for Monge–Ampère equations. We show that $v$ solves a MATE

$$\det[D^2v - \bar{\mathcal{A}}(\cdot, v, Dv)] = \bar{\mathcal{B}}(\cdot, v, Dv),$$

(48)

for $\bar{\mathcal{A}}, \bar{\mathcal{B}}$ satisfying $\|\bar{\mathcal{A}}\|_{C^2} \leq C\|A\|_{C^2}$ and $C^{-1}B \leq \bar{B} \leq CB$ for $C$ depending only on $g$. When $v$ is defined by (47) we use the notation

$$\bar{\mathcal{A}}_{ij}(\bar{x}, v, Dv) = \frac{r_ir_j}{h}A_{ij}(T^{-1}\bar{x}, u, Du),$$

$$Yv = Y(T^{-1}x, u, Du)$$

and similarly for $Zv$. We compute directly $v$ solves the equation (48) for

$$\bar{\mathcal{A}}_{ij}(\bar{x}, v, Dv) = \frac{r_ir_j}{h}[g_{ij}(T^{-1}\bar{x}, Yv, Zv) - g_{ij}(T^{-1}\bar{x}, y_0, z_0 - h)]$$

$$= \frac{r_ir_j}{h}[g_{ij}(T^{-1}\bar{x}, Yv, Zv) - g_{ij}(T^{-1}\bar{x}, y_0, z_0) + O(h)]$$

$$= \bar{\mathcal{A}}_{ij}(\bar{x}, v(\bar{x}), Du(\bar{x})) - \bar{\mathcal{A}}_{ij}(\bar{x}, v(\bar{x}_0), Du(\bar{x}_0)) + O(h)$$

$$= \bar{\mathcal{A}}_{ij}(\bar{x}, v(\bar{x}), Du(\bar{x})) - \bar{\mathcal{A}}_{ij}(\bar{x}, \bar{v}(\bar{x}), Du(\bar{x}_0)) + O(h).$$

In this form we can argue exactly as in [12, pg. 440] to obtain

$$\bar{\mathcal{A}}_{ij}(\bar{x}, v, Dv) = \frac{hr_ir_j}{r_kr_l}A_{ij, pkpl}(T^{-1}\bar{x}, hv, \frac{h}{r_i}v_kv_l) + O(h).$$

Then $\bar{\mathcal{A}}$ is bounded by Lemma 2 when $A_3$ is satisfied. For the $C^2$ estimates for $\bar{\mathcal{A}}$ note $\frac{h}{r_i}$ is bounded by the strict convexity and differentiability estimate (27). Finally the pinching

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estimate on $B$ follows from $\overline{B}(\cdot, v, Du) = h^u(\det T)^2 B(\cdot, u, Du)$, $(\det T)^{-1} = r_1 \ldots r_n$, and (15).

### 6.2 Lemmas for the $C^{2,\alpha}$ estimate

Here we show we can estimate the shape constant of sections by the second derivatives of the function and vice versa. A subtlety is that sections of different height are convex in different coordinates. We assume at the outset some initial coordinates are fixed. We will say a section has a shape constant $C$ if after performing the change of variables (13) the section has a shape constant $C$. Note because the Jacobian of the transformation (13) is $E$ (from A2) if $S_h$ has shape constant $C$ then it is still true, in the initial coordinates with respect to which $S_h$ may not be constant, that $S_h$ contains a ball of radius $r$ and is contained in a ball of radius $R$ for $R/r \leq C |\Lambda|/|\lambda|$ where $\Lambda, \lambda$ are respectively the minimum and maximum eigenvalues of $E$ over $\Gamma$.

**Lemma 5** Let $u$ be a strictly $g$-convex solution of

$$\det[D^2u - A(\cdot, u, Du)] = f \text{ in } \Omega. \quad (49)$$

Fix $x_0 \in \Omega$ and a support $g(\cdot, y_0, z_0)$ at $x_0$. Assume $S_h(x_0) \subset \subset \Omega$, and $|D^2u(x_0)| \leq M$ for some $M$. Then $S_h(x_0)$ has a shape constant which depends only on $M, h, f, g$ and the constant in the Pogorelov Lemma 4.

**Proof** We normalize the section $S_h$ and solution as in Sect. 6.1 and let the normalized solution be denoted by $v$. By the Pogorelov interior estimates we have

$$cI \leq \overline{w}(Tx_0) \leq CI,$$

for $\overline{w} = D^2v - \overline{A}(x, v, Du)$. Similarly by the PDE and $|D^2u(x_0)| \leq M$ we have $cI \leq w(x_0) \leq CI$ for $w(x) = D^2u(x_0) - A(x_0, u(x_0), Du(x_0))$. Using in addition

$$\frac{1}{h} \overline{w}(x_0) = T^T \overline{w}(Tx_0) T,$$

we obtain (see [3, Eq. 4.26]) a bi-Lipschitz estimate $\|T\|, \|T^{-1}\| \leq C$. Since we can assume $T$ is of the form (16) we obtain $r_1/r_n \leq C^2$, the desired estimate on the shape constant of $S_h(x_0)$.

**Lemma 6** Assume $u \in C^3(\Omega)$ is a strictly $g$-convex solution of

$$\det[D^2u - A(\cdot, u, Du)] = f \text{ in } \Omega.$$

Fix $x_0$ and assume $g(\cdot, y_0, z_0)$ is a g-support at $x_0$. If there is a sequence $h_k \to 0$ such that each

$$S_{h_k} := \{ u < g(\cdot, y_0, z_0 - h_k) \},$$

has a shape constant less than some $C_0$ then $|D^2u(x_0)| \leq C$ for $C$ depending on $C_0, \sup |f|$ and $g$.

**Proof** Without loss of generality $x_0 = 0$ and the minimum ellipsoid of $S_{h_k}$ has axis $r^{(k)}_1 \geq \ldots \geq r^{(k)}_n$. Our assumption is $r^{(k)}_1 \leq C_0 r^{(k)}_n$. Then, by (15)

$$C_0^{n+1} (r^{(k)}_1)^n \leq r^{(k)}_1 \ldots r^{(k)}_n \leq C |S_{h_k}| \leq C h_k^{n/2},$$

$\square$ Springer
that is \( r_1^{(k)} \leq C \sqrt{h_k} \). Moreover, because \( r_1^{(k)} \) is the largest axis of the minimum ellipsoid there holds \( S_h \subset B_{r_1^{(k)}}(0) \). Thus
\[
S_h \subset B_{C \sqrt{h_k}}(0).
\] (50)

Let \( w_{ij} = u_{ij}(0) - g_{ij}(0, y_0, z_0) \) and denote the minimum eigenvalue of \( w \) by \( \lambda \) and corresponding normalized eigenvector by \( x_\lambda \). Using a Taylor series we have
\[
u(tx_\lambda) - g(tx_\lambda, y_0, z_0 - h) \leq t^2 \lambda + O(t^3) - \inf g_h h_k.
\]
Thus provided \( k \) is taken sufficiently large we have \( tx_\lambda \in S_h \) for \( t = \sqrt{\inf g_h h_k} \). That is,
\[
|x| \geq C \sqrt{\frac{h_k}{\lambda}}.
\] (51)
Combining (50) and (51) we obtain a lower bound on \( \lambda \) which implies an upper bound on the largest eigenvalue of \( w \) by the PDE.

Before proving the \( C^2 \) estimate we state one final lemma.

**Lemma 7** Assume \( u_i \) for \( i = 1, 2 \) is a \( C^{2,\alpha} \) solution of
\[
\det[D^2 u_i - A(\cdot, u_i, Du_i)] = f_i \text{ in } \Omega
\]
where \( f_i \) is a Hölder continuous function and \( ||u_i||_{C^4} \leq K \). For any \( \Omega' \subset \subset \Omega \) there is \( C > 0 \) depending on \( K, g, \text{dist}(\partial \Omega, \Omega') \) such that
\[
||u_1 - u_2||_{C^3(\Omega')} \leq C ||f_1 - f_2||_{C^\alpha(\Omega)} + \sup_{\Omega} |u_1 - u_2|.
\]

**Proof** We linearise the PDE (as in [13, Lemma 4.2]), obtaining \( L(u_1 - u_2) = f_1 - f_2 \) for a suitable linear operator with coefficients and ellipticity constants controlled by estimates \( ||u_i||_{C^4} \leq K \). The lemma then follows from the classical Schauder theory [5, Theorem 6.2].

\[\square\]

### 6.3 Proof of the \( C^{2,\alpha} \) estimate

As stated at the start of this section, Theorem 2 follows from an interior \( C^2 \) estimate. This is what we now prove.

**Theorem 3** Let \( g \) be a generating function satisfying A3 and \( u \in C^2(\Omega) \) be a \( g \)-convex solution of
\[
\det[D^2 u - A(\cdot, u, Du)] = f \text{ in } \Omega.
\]
If \( f \) is Dini-continuous with \( 0 < \lambda < f \leq \Lambda < \infty \) then for each \( \Omega' \subset \subset \Omega \) we have an estimate \( \sup_{\Omega'} |D^2 u| \leq C(f, g, \Omega, \Omega') \).

**Proof** Step 1. [Setup of approximating problems] At the outset we fix \( x_0 \in \Omega \), where without loss of generality \( x_0 = 0 \). Consider
\[
S_h = \{x; u(x) < g(x, y_0, z_0 - h)\},
\]
where \( g(\cdot, y_0, z_0) \) is the support at 0 and \( h > 0 \) is chosen small enough to ensure Lemma 4 applies. We normalize so that \( B_{1/n} \subset S_h \subset B_1 \) with \( B_1 \) the minimum ellipsoid. Moreover, we assume \( h \) is chosen small enough to ensure that after this normalization

\[
\int_0^1 \frac{\omega(r)}{r} < \varepsilon,
\]

for an \( \varepsilon \) to be chosen in the proof (recall \( \omega \) is from Definition 5). Note such a choice of \( h \) is controlled by (27) and thus up to rescaling we assume \( h = 1 \).

We introduce a sequence of approximating problems. Define the domains

\[
U_k = \{ x; u(x) < g(x, y_0, z_0 - 1/4^k) \},
\]

and let \( f_k = \inf_{U_k} f \).

Let \( u_k \in C^4(U_k) \) be the solution (whose existence is guaranteed by the Perron method) of

\[
\text{det}[D^2 u_k - A(\cdot, u_k, Du_k)] = f_k \text{ in } U_k, \quad u_k = u \text{ on } \partial U_k.
\]

In addition put

\[
v_k = \sup_{x, y \in U_k} |f(x) - f(y)|.
\]

Using (15) the section \( U_k \) is contained in a ball of of radius \( C2^{-k} \) for \( C \) depending on the shape constant of \( U_k \). Thus when the shape constant of \( U_k \) is controlled \( C v_k \leq \omega(2^{-k}) =: \omega_k \) for \( C \) dependent on the shape constant. We’ll use this use at the conclusion of the proof. Lemmas 5 and 6 suggest the structure of our proof: It suffices to show a shape constant of each \( U_k \) is controlled by a fixed constant independent of \( k \) and this, in turn, follows from a uniform estimate on each \( |D^2 u_k| \) inside a subsection of \( U_k \). More precisely, we’ll prove by induction that for each \( k = 0, 1, 2, \ldots \) we have

\[
|D^2 u_k(x)| \leq \sup_{V_{k_0}^\tau} |D^2 u_0| + 1 =: M,
\]

for any \( x \in V_{k_0}^\tau \) which is defined by

\[
V_k^\tau = \{ x; u_k(x) < g(x, y_0, z_0 - 1/4^k) \}.
\]

By the uniform estimates in [20]\(^2\) there is a choice of \( \tau_0 \) sufficiently close to 0 and independent of \( k \) such that

\[
x_0 \in V_k^{\tau_0}, \quad U_{k+1} \subset V_k^{\tau_0}.
\]

**Step 2.** [Induction base case: \( k = 0 \)] It is clear that (55) holds for \( k = 0 \). However we note here that \( M \) is controlled by the interior Pogorelov estimate, that is in terms of \( \tau_0, h \) and our initial normalizing transformation which is, in turn, controlled by (15).

**Step 3.** [Inductive step] Now we assume (55) up to some fixed \( k \). We rescale our solution and domain by introducing

\[
\bar{u}_k(\bar{x}) := 4^k u_k \left( \frac{\bar{x}}{2^k} \right), \quad \bar{u}_{k+1}(\bar{x}) := 4^k u_{k+1} \left( \frac{\bar{x}}{2^k} \right).
\]

\(^2\) Use the upper bound [20, Theorem 4] and corresponding lower bound obtained as in [4, Theorem 6.2].
where \( x = 2^k x \). The function \( \overline{u}_k \) solves
\[
det[D^2 \overline{u}_k - \overline{A}(\cdot, \overline{u}_k, D\overline{u}_k)] = f_k \text{ in } \overline{U}_k
\]
\[
\overline{u}_k = 4^{-k} g(\cdot, y_0, z_0 - h/4^k) \text{ on } \partial \overline{U}_k,
\]
for \( \overline{U}_k := 2^k U_k \) and \( \overline{A}(x, u, p) = A(2^{-k} x, 4^{-k} u, 2^{-k} p) \) and similarly for \( \overline{u}_{k+1} \). Note this transformation does not change the magnitude of the second derivatives of \( u_k \) and \( u_{k+1} \). Thus the inductive hypothesis (55) and Lemma 5 implies \( \overline{U}_k \) has a shape constant depending only on \( M, f \) and the constant in the Pogorelov lemma. We claim that \( \overline{U}_{k+1} \) has a shape constant depending on the same parameters and the constants in (15). To see this assume the minimum ellipsoids of \( \overline{U}_k \) and \( \overline{U}_{k+1} \) have axis \( R_1 \geq \cdots \geq R_n \) and \( r_1 \geq \cdots \geq r_n \) respectively. By (15) applied to the section \( \overline{U}_{k+1} \) we have
\[
C 4^{-n(k+1)/2} \leq C |\overline{U}_{k+1}| \leq r_1 \cdots r_n \leq r^{n-1}_n r_n.
\]
Using this to compute an upper bound on \( 1/r_n \), we obtain
\[
\frac{r_1}{r_n} \leq C 4^{n(k+1)/2} r_n^2 \leq C 4^{n(k+1)/2} R_1^n,
\]
where the final inequality is because \( \overline{U}_k \) is the larger section. Now, let \( c_0 \) be the shape constant of \( \overline{U}_k \), that is \( R_1 \leq c_0 R_n \). Using (15) again, this time applied to the section \( \overline{U}_{k+1} \), we have
\[
C 4^{-nk/2} \geq R_1 \cdots R_n \geq R_1 R_{n-1}^{-1} \geq c_0^{(n-1)} R_1^n.
\]
Combining (58) and (59) implies the claimed fact that the shape constant of \( \overline{U}_{k+1} \) may be estimated in terms of the shape constant of \( \overline{U}_k \) and the constant in (15).\(^3\)

Noting that \( \|\overline{A}\|_{C^2} \leq \|A\|_{C^2} \) we obtain by Lemma 4 a \( C^2 \) estimate depending on allowed quantities and \( M \) in \( V_{k_0}^{70/4} \). Then the Evans–Krylov interior estimates imply \( C^{2,\alpha} \) estimates in \( V_{k_0}^{70/2} \) and subsequently higher estimates by the elliptic theory. Similarly for \( \overline{u}_{k+1} \) in the corresponding sections \( V_{k+1}^{70/4}, V_{k+1}^{70/2} \).

Linearising and using the maximum principle on small domains(see [18, Lemma A.3]) we obtain
\[
|\overline{u}_k - \overline{u}|, |\overline{u}_{k+1} - \overline{u}| \leq C v_k.
\]
Thus \( |\overline{u}_k - \overline{u}_{k+1}| \leq C v_k \) in \( V_{k+1}^{70/2} \). Then, by Lemma 7, in \( V_{k+1}^{70} \) we have
\[
|D^2 u_k(x) - D^2 u_{k+1}(x)| \leq C v_k.
\]
(Note we first obtain this for \( \overline{u}_k, \overline{u}_{k+1} \) then use \( D^2 \overline{u}_k = D^2 u_k \).) We’ve used that by (57) the estimates for \( \overline{u}_k \) in \( V_{k_0}^{70} \) hold on the entirety of \( \overline{U}_{k+1} \). Since all we’ve used is the induction hypothesis we can conclude the same inequality for \( k \) replaced by \( i = 0, 1, 2, \ldots, k \). Moreover because these sections have a controlled shape constant we have
\[
|D^2 u_i(x) - D^2 u_{i+1}(x)| \leq C \omega_i = C \omega(2^{-i}),
\]
for \( i = 0, 1, 2, \ldots, k \) and \( x \in V_{k+1}^{70} \).

Now, using (60), the calculations are standard:
\[
|D^2 u_{k+1}(x)| \leq |D^2 u_0(x)| + \sum_{i=0}^k |D^2 u_{i+1}(x) - D^2 u_i(x)|
\]

\(^3\) To be explicit, because we are proving (55) by induction (not a shape estimate by induction) it does not matter that the shape constant of \( \overline{U}_{k+1} \) is worse than \( \overline{U}_k \).
\[ \leq |D^2 u_0(x)| + C \sum_{i=0}^{k} 2^{-i} \frac{\omega(2^{-i})}{2^{-i}} \]
\[ \leq |D^2 u_0(x)| + C \int_{2^{-k}}^{1} \frac{\omega(r)}{r} \, dr. \]

Thus provided \( \varepsilon \) in (52) is taken sufficiently small we conclude by induction that (55) holds for all \( k \). By the rescaling used in the proof this implies an estimate on the shape constant of each \( U_k \) which is independent of \( k \). Then the desired \( C^2 \) estimate holds by Lemma 6. \( \Box \)

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**A Omitted calculations**

**A.1 Proof that** \( A_{kl,pn} \dot{Y}_k \dot{Y}_l \dot{Y}_m = 0 \) **for** \( \dot{Y} \cdot \eta = 0 \)

Let some initial coordinates, denoted \( x \), be given and define \( \bar{x} = \frac{g_z}{g} (x, y_0, z_0) \). For notation put \( g_h(\cdot) = g(\cdot, y_0, z_h) \). We will show here that for any \( \xi, \eta \) in \( \mathbb{R}^n \)

\[
\frac{\partial^2 x_k}{\partial \bar{x}_i \partial \bar{x}_j} \xi_i \xi_j \eta_k = g_z^2 D_{pk} A_{ij}(x, g_h(x), Dg_h(x)) (E^{-1})_i (E^{-1})_j \eta_h + g_z g_{r,z} [E^{ir} E^{jk} + E^{ik} E^{jr}] \xi_i \xi_j \eta_k
\]  

(61)

So that if our initial coordinates are the \( \bar{x} \) coordinates and \( \xi \cdot \eta = 0 \)

\[ D_{pk} A_{ij}(x, g_h(x), Dg_h(x)) \xi_i \xi_j \eta_k = 0. \]  

(62)

Now to show (61) we recall from [22, Eq. 2.3] that

\[ D_{pk} g_{ij} = E^{r,k} [g_{ij,r} - g_{r,j} g_{ij} g_{z}^{-1}]. \]  

(63)

From the definition of \( \bar{x} \) we compute

\[
\frac{\partial x_k}{\partial \bar{x}_j} = g_z E^{jk},
\]
\[
\frac{\partial}{\partial \bar{x}_i} = g_z E^{ir} \frac{\partial}{\partial x_r}.
\]  

(64)

Applying (64) twice and using the identity for differentiating an inverse matrix gives

\[
\frac{1}{g_z} \frac{\partial^2 x_k}{\partial \bar{x}_i \partial \bar{x}_j} = E^{ir} E^{jk} g_{r,z} - g_z E^{ir} E^{ja} (D_r E_{ab}) E^{bk}.
\]  

(65)

Now, by direct calculation

\[
D_r E_{ab} = -g_a, z \frac{g_z}{g} E_{rb} + g_{ar,b} - g_{ar,z} g_{b} g_{z}^{-1}.
\]
which when substituted into (65) implies
\[
\frac{1}{g_z} \frac{\partial^2 x_k}{\partial x_i \partial x_j} = E^{ij} E^{jk} g_{r,z} + E^{jk} E^{jr} g_{r,z} + g_z E^{ja} E^{bk} [g_{ar,b} - g_{ar,z} g_{b} g_z^{-1}].
\]
Combined with (63) this implies (61). We note if our initial coordinates are the \(\bar{x}\) coordinates then
\[
E_{ij}(x, g_h(x), Dg_h(x)) = g_{ij}(x, y_0, z_h) - \frac{g_{i,z} g_{j}}{g_z} (x, y_0, z_h) = g_z \frac{\partial \bar{x}_j}{\partial \bar{x}_i} = \delta_{ij}.
\]
This implies (62).

### A.2 Quantitative convexity via duality

Here we prove the first inequality in (27) assuming the second inequality. We follow the duality argument in [13] simplified by the transformation in [20]. Indeed by [20, Lemma 3] it suffices to prove the result at the origin for the generating function
\[
g(x, y, z) = x \cdot y - z + a_{ij,k} y_i y_j y_k - f(x, y, z) z, \tag{66}
\]
\[
f(x, y, z) = b_{ij}^{(1)} x_i x_j + b_{ij}^{(2)} x_i y_j + b_{ij}^{(3)} y_i y_j + f^{(2)}(x, y, z) z,
\]
where \(a_{ij,k}, b_{ij}^{(2)}\), \(f^{(2)}\) are \(C^1\) functions. Moreover we assume \(u\) is a strictly \(g\)-convex function satisfying \(u \geq 0\) and \(Du(0), u(0) = 0\). We need to prove \(u(x) \geq C |x|^{1+\gamma}\). Throughout the proof we use the notation \(O(y^p)\) to denote any function \(h(x, y, z)\) satisfying an estimate \(|h(x, y, z)| \leq C |x|^p\) on a neighbourhood of the origin for \(C\) depending only on \(\|g\|_{C^4}\). Similarly for the notation \(O(y^p)\).

The \(g^*\)-transform of \(u\) is \(v : Yu(\Omega) \to \mathbb{R}\) defined by
\[
v(y) = g^*(x, y, u(x)),
\]
for \(x\) satisfying \(y = Yu(x)\) and \(g^*\) the dual generating function (see [20]). The function \(v\) is \(g^*\)-convex with \(g^*\)-support \(g^* (0, \cdot, 0) \approx 0\) and the \(C^{1,\alpha}\) result, which holds by duality, implies \(|v(y)| \leq \overline{C} |y|^{1+\alpha}\). Thus, by duality,
\[
u(x) = \sup_y g(x, y, v(y)) \geq \sup_y g(x, y, \overline{C} |y|^{1+\alpha}). \tag{67}
\]
We also note by the \(C^{1,\alpha}\) estimate for \(u, |Yu(x, u(x), Du(x))| \leq C |x|^{\alpha}\), so we can assume throughout that the neighbourhood over which the supremum is taken, and subsequently \(|y|\), is sufficiently small. The supremum is obtained for \(y\) satisfying
\[
g_y(x, y, \overline{C} |y|^{1+\alpha}) + \overline{C} (1 + \alpha) g_z(x, y, \overline{C} |y|^{1+\alpha}) |y|^{\alpha - 1} y = 0. \tag{68}
\]
Now using (66)
\[
u(x) \geq g(x, y, \overline{C} |y|^{1+\alpha}) = x \cdot y - \overline{C} |y|^{1+\alpha} + O(y^2) - \overline{C} f(x, y, z)|y|^{\alpha + 1},
\]
\[
\geq x \cdot y - \overline{C} |y|^{1+\alpha} + O(y^2) + [O(x) + O(y)] |y|^{\alpha + 1}. \tag{69}
\]
Now we use (68) to estimate \(x \cdot y\) from below. First note
\[
g_y(x, y, \overline{C} |y|^{1+\alpha}) = x + O(x) O(y) + [O(x) + O(y)] |y|^{1+\alpha}. \tag{70}
\]
Thus substituting into (68) and taking an inner product with \( y \) we obtain (near the origin in \( x, y \))

\[
x \cdot y = \overline{C}(1 + \alpha)|g_z(x, y, \overline{C}|y|^{1+\alpha})||y|^{\alpha+1} + O(y^2) + [O(x) + O(y)]|y|^{\alpha+1},
\]

and subsequently returing to (69) implies

\[
u(x) \geq \overline{C}|y|^{\alpha+1} \left[ (1 + \alpha)|g_z(x, y, \overline{C}|y|^{1+\alpha})| - 1 \right] + O(y^2) + [O(x) + O(y)]|y|^{\alpha+1}.
\]

Since \( |g_z| \) is as close to 1 as desired on a sufficiently small neighbourhood of the origin we have

\[
u(x) \geq C(\alpha)|y|^{\alpha+1}.
\]

However also from (68) and (70)

\[
|x| = C(1 + \alpha)|g_z||y|^\alpha + O(x)O(y) + O(y^{\alpha+1})
\leq C|y|^{\alpha}.
\]

This completes the proof.

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