COMMENTS AND REPLIES

Comment on ‘Hawking radiation from fluctuating black holes’

Igor Khavkine

Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands

E-mail: i.khavkine@uu.nl

Received 31 August 2010, in final form 1 September 2010
Published 17 January 2011
Online at stacks.iop.org/CQG/28/038001

Abstract

Takahashi and Soda (2010 Class. Quantum Grav. 27 175008) have recently considered the effect (at lowest non-trivial order) of dynamical, quantized gravitational fluctuations on the spectrum of scalar Hawking radiation from a collapsing Schwarzschild black hole. However, due to an unfortunate choice of gauge, the dominant (even divergent) contribution to the coefficient of the spectrum correction that they identify is a pure gauge artifact. I summarize the logic of their calculation, comment on the divergences encountered in its course and on how they could be eliminated, and thus the calculation be completed.

PACS numbers: 04.70.Bw, 04.62.+v, 04.70.Dy, 05.40.—a

1. Introduction

In their recent work [10], Takahashi and Soda have tackled an interesting and challenging question, that of assessing the influence of dynamical, quantized metric fluctuations on the spectrum of scalar Hawking radiation from a black hole. A natural sibling question that could be directly attacked with essentially the same techniques is that of the back reaction of Hawking radiation on the quantum geometry of the black hole. An answer to either of these questions would provide us with valuable insight into the properties of quantum gravity, as seen through the prism of its reduction to effective quantum field theory on a curved background.

The authors of [10] show that the spectrum of scalar Hawking radiation from a collapsing Schwarzschild black hole is corrected due to the presence of dynamical, quantized gravitational fluctuations interacting with the quantum scalar field at the lowest non-trivial (cubic) order. The choice of a collapsing spacetime instead of an eternal black hole is similar to the choice made in Hawking’s original calculation [3]. The same result could be obtained by using the Unruh vacuum on an eternal black hole spacetime [11]. The corrected expected number of quanta (expected by an asymptotic observer at future null infinity) in a single field mode of...
static frequency $\omega$ and spherical harmonic index $\ell$, according to equations (71), (81) and (169) of [10], is

$$\langle N_{\omega \ell} \rangle = \frac{1}{e^{2\pi \omega/\kappa} - 1} + C_{\ell} \frac{\epsilon^2 \omega_{\text{cut}}^3 \ell_p^2}{\kappa^3 M^2 L} \coth(\pi \omega/\kappa) + \cdots,$$

(1)

where $C_{\ell}$ is a numerical constant, $M$ is the Schwarzschild mass, $\kappa = 1/(4M)$ is the surface gravity, $\ell_p$ is the Planck length, $\epsilon > 0$ is a small distance cutting off radial integration before the horizon, and $\omega_{\text{cut}}$ and $L$ are the parameters regulating the divergence of a 1-loop Feynman integral. Higher order corrections are presumed to be subleading in $\ell_p$ or one of the regulator parameters.

I will summarize some of the key steps in the calculation leading to (1) and discuss the divergences that appear along the way, as well as ways of resolving them. Section 2 reviews some relevant background material and establishes the notation. Section 3 discusses the divergence that prompted the introduction of the $\epsilon$-regulator, how it is related to the authors' choice of gauge and how a better choice of gauge eliminates this divergence and the need for this regulator. Section 4 discusses the divergence that prompted the introduction of the $\omega_{\text{cut}}$ and $L$ regulators and how these regulators could be removed using standard perturbative renormalization. Finally, section 5 emphasizes that, as a consequence of the results of section 3, the dominant contribution, identified in [10], to the coefficient in front of the correction in equation (1) is a pure gauge artifact. It also summarizes the non-trivial steps achieved in the calculation of Takahashi and Soda and how it could be completed to obtain a reliable, parameter-free estimate of the size of the correction to the Hawking spectrum.

2. Background and notation

The calculation in [10] starts out by quantizing free scalar and metric perturbation (graviton) fields on a collapsing Schwarzschild background. The Bogolubov coefficients, which transform between asymptotic in- and out-modes, are estimated using a geometric optics approximation and give the standard Hawking spectrum for the free scalar field. Then, the metric perturbations are gauge fixed and reduced to two physical scalar degrees of freedom, which are quantized as free fields. Finally, the explicit form of the interaction vertex between the scalar field and the metric perturbations is identified and used to compute the correction to the Hawking spectrum of the scalar field at lowest perturbative order. The relevant details of these steps are discussed in the rest of this comment.

An important step in quantizing a free field is solving its classical equations of motion. Usually, this is accomplished by decomposing an arbitrary solution into a set of modes. On a Minkowski background, due to the translational symmetry, it is easiest to work with the set where each mode function is proportional to a plane wave:

$$\phi^0_k(x) = \exp(ik_\mu x^\mu),$$

(2)

where $k_\mu$ is the wave vector indexing the mode and $x^\mu$ are the global inertial coordinates. We restrict ourselves only to the case of massless fields; hence, $k^2 = 0$. On a curved background, we can no longer make use of translation symmetry. However, the background used in [10] is presumed to be (at least outside the horizon) spherically symmetric, static in the asymptotic future, and approximately static in the asymptotic past. To make maximal use of the available symmetry, it is convenient to decompose the fields into a set of modes, where each mode function is proportional to a scalar or tensor spherical harmonic as well as $\phi^{\pm}_{\omega \ell j}(t, r)$, where $j$ indexes independent tensor polarizations (if any). These functions have the following asymptotic properties:

$$\phi^{\pm}_{\omega \ell j} \sim e^{-i\omega t} R^{\pm}_{\omega \ell j}(r) \quad \text{for} \quad t \rightarrow \pm \infty,$$

(3)
where, up to normalization, the radial functions $R_{\omega \ell j}^{\pm}$ are uniquely specified by the geometry and the above-listed conditions. The fact that the $\phi_{\omega \ell j}$ and $\phi_{\omega \ell j}^{\ast}$ mode functions do not coincide is ultimately responsible for the Hawking effect [2, section 10.2].

While the decomposition of a scalar field into spherical harmonics is a standard exercise, the case of metric perturbations is more subtle, but is a topic with an extensive literature, starting with the seminal work of Regge and Wheeler [7]. An up-to-date review can be found in [1]. Reference [5] is particularly useful as it presents the formalism of metric perturbations in a spherically symmetric spacetime in a way that is gauge invariant and covariant with respect to the changes of the coordinates in the $(t, r)$-plane. For reference, we establish the correspondence between the notations of [10] and [5].

Due to spatial inversion symmetry, the perturbations naturally decompose into odd- and even-parity sectors. We shall only consider the even-parity ones. Note that, below, the asterisks denote the components that can be deduced from symmetry:

Takahashi and Soda [10]

$$h_{\mu \nu} = \begin{pmatrix} f \bar{H} & H_1 & v_{ia} \\ * & H/f & u_{ia} \\ * & * & r^2 K_{gab} + B_{gab} \end{pmatrix}. \quad (4)$$

The components of the metric perturbation $h_{\mu \nu}$ are given in static Schwarzschild coordinates\(^1\), $f(r) = 1 - 2M/r$, $\gamma_{ab}$ is the standard metric on the unit 2-sphere, and $T_{gab}$ denotes covariant differentiation of the tensor $T$ with respect to $\gamma_{ab}$. The components of the perturbation are parametrized by the scalars $\bar{H}, H_1, H, v, w, K$ and $B$. The components $f \bar{H}$ and $H/f$ correspond to $h_{tt}$ and $h_{rr}$ respectively:

Martel and Poisson [5]

$$p_{\mu \nu} = \sum_{\ell m} \left( h_{\ell m}^{ia} Y_{\ell m}^{ia} + j_{\ell m}^{ia} Y_{\ell m}^{ia} \right). \quad (5)$$

The components of the metric perturbation $p_{\mu \nu}$ are given with respect to the coordinates that respect spherical symmetry but are arbitrary on the $(t, r)$-plane, $\ell$ and $m$ are respectively the orbital and magnetic spherical harmonic indices, and $\Omega_{AB}$ is the standard metric on the unit 2-sphere. The components of the perturbation are parametrized by the scalars $K_{\ell m}$, $G_{\ell m}$, and the $(t, r)$-plane tensors $h_{\ell m}^{ia}$ and $j_{\ell m}^{ia}$. The vector and tensor spherical harmonics are defined from the scalar $Y_{\ell m}$ as follows: $Y_{\ell m}^{ia} = D_A Y_{\ell m}$ and $Y_{\ell m}^{AB} = [D_A D_B + \frac{1}{2}(\ell + 1)\Omega_{AB}] Y_{\ell m}$, where $D_A$ is the covariant derivative with respect to $\Omega_{AB}$. Where no confusion is possible, the spherical harmonics indices may be omitted.

The correspondence between the two notations should now be clear, given the equality $h_{\mu \nu} \, dx^\mu \, dx^\nu = p_{\mu \nu} \, dx^\mu \, dx^\nu$. However, note that reference [10] uses lowercase Latin indices, $T_{ia}$ for spherical tensors, while [5] uses lowercase Latin indices for $(t, r)$-plane tensors and uppercase Latin indices, $T_A$, for spherical tensors. Formulas given while referring to a given paper will use the notation of that paper.

### 3. Singularity of ‘convenient’ gauge

For canonical quantization of the metric perturbations, it is necessary to isolate the single physical degree of freedom in the even-parity sector, known in the literature as the Zerilli or Zerilli–Moncrief function; the gauge freedom arising from linearized coordinate transformations must be completely fixed. Takahashi and Soda impose what they call a ‘convenient’ gauge, where $v = 0$, $B = 0$ and $K = 0$. Further, they solve the constraints following from the equations of motion and express all remaining perturbation components

\(^1\) More precisely, as described previously, they are approximately static coordinates on the collapsing spacetime.
in terms of the scalar Zerilli function $\psi_Z$. I will show that gauge transformations required to enforce the ‘convenient’ gauge are singular at the horizon. As a consequence, some remaining non-zero components will be singular at the horizon as well. This singularity becomes obvious when these components are explicitly expressed in terms of the Zerilli function in a coordinate system regular at the horizon.

Let a field be called regular at a point $x$ if it is continuous and smooth in some neighborhood of that point; otherwise it is called singular at $x$. Regularity at $x$ implies that the components of the tensor field must be continuous, smooth functions in a neighborhood of $x$, when expressed in any coordinate system that covers $x$. The converse implication is also true. It is a sensible mathematical and physical requirement that metric perturbations be restricted to everywhere regular tensor fields. Note, however, that regularity in static Schwarzschild coordinates is insufficient to establish global regularity. Recall that Schwarzschild coordinates consist of two coordinate charts (exterior, $r > 2M$, and interior, $0 < r < 2M$), neither of which covers any point on the horizon. To check regularity at the horizon, it is necessary and sufficient to check continuity and smoothness in any coordinate system that does cover the horizon. We shall use the advanced Eddington–Finkelstein (EF) coordinates \[4–6\]; they are regular on the future horizon, which suffices for our purposes. These $(v, r)$ coordinates are related to the static Schwarzschild $(t, r)$ coordinates as follows:

\[
\begin{align*}
 r^* & = \int \frac{dr}{f} = r + 2M \ln \left| \frac{dr}{2M} - 1 \right|, \\
 v & = t + r^*, \quad dv = dt + f^{-1} dr, \quad \partial_t = \partial_v, \\
 r & = r, \quad dr = dr, \quad \partial_r = f^{-1} \partial_v + \partial_r.
\end{align*}
\]

3.1. ‘Convenient’ gauge in EF coordinates

The gauge chosen by the authors of [10] for even-parity perturbations is termed the ‘convenient’ gauge: $K = B = v = 0$. In the notation of [5], the ‘convenient’ gauge is equivalent to $K = 0, G = 0$, and $j_a(\partial_t)^a = 0$. A gauge is called good if an arbitrary perturbation $p_{\mu\nu}$ can be transformed by a unique vector field $\xi_\mu = (\xi_v, D_v \xi)$ that implements the required transformation. Otherwise I call the gauge singular. From appendix E of [5], the explicit form of the gauge transformation is

\[
\begin{align*}
 h'_{vv} & = h_{vv} - 2\partial_r \xi_v - \partial_v \xi_r, \\
 h'_{vr} & = h_{vr} - \partial_r \xi_v + \frac{2M f}{r^2} \xi_r, \\
 h'_{rr} & = h_{rr} - \frac{2M f}{r^2} \xi_r, \\
 j'_v & = j_v - \partial_r \xi_v + \xi_r, \\
 j'_r & = j_r - \partial_r \xi_v - \xi_r + \frac{2}{r} \xi, \\
 K' & = K - \frac{2 f}{r} \xi_r - \frac{2}{r} \xi_v + \frac{\ell (\ell + 1)}{r^2} \xi, \\
 G' & = G - \frac{2}{r^2} \xi.
\end{align*}
\]
The conditions \( G' = 0 \) and \( j'_a(\partial_t)^a = j'_a(\partial_v)^a = j'_v = 0 \) are easily obtained by setting \( \xi = r^2 G/2 \) and \( \xi_v = j_v - r^2 \partial_v G/2 \), independent of any other requirements. On the other hand, setting \( K' = 0 \) requires

\[
\xi_r = \frac{r}{2f} \left( K - \frac{2}{r} j_v + r \partial_v G + \frac{\ell(\ell + 1)}{2} G \right).
\] (15)

Note that \( f(r = 2M) = 0 \); therefore, since \( K, G \) and \( j_v \) were assumed to be arbitrary regular functions, \( \xi_r \) must be singular at \( r = 2M \), the future horizon, which is covered by our choice of advanced EF coordinates. Hence, the ‘convenient’ gauge is necessarily singular. Moreover, this singularity appears explicitly in the \( j'_r, h'_{rr} \) and \( h'_{vr} \) components of the metric perturbation, though not in the \( h'_{vv} \) one.

Note that the above result would have been difficult, though not impossible, to obtain directly in static Schwarzschild coordinates. Since no point of the horizon is covered by \((t, r)\) coordinates, even regular tensor fields may have components that diverge as powers of \(1/f\) as \( r \to 2M \), though such divergences will have a specific structure. For instance, this structure can be identified by transforming a regular tensor field from EF to Schwarzschild coordinates. Thus, the regularity of a tensor field could be checked in static coordinates by examining the structure of the divergences of its components as \( r \to 2M \), though with some effort. In fact, a variant of this technique was unsuccessfully used in section 4.4 of [10]. Unfortunately, the authors ultimately failed to note the singular nature of their gauge choice. Also note that the same procedure can be used to check that the standard Regge–Wheeler gauge \( ^2 (j_a = 0, G = 0) \) and the more recently proposed light-cone gauge of Preston and Poisson [6] are both regular.

### 3.2. Zerilli function

After imposing the ‘convenient’ gauge, the authors of [10] proceed to solve the constraints among the remaining independent even-parity components of \( h_{\mu\nu} \), isolate the single physical degree of freedom (the Zerilli function \( \psi^Z \)), and express \( h_{\mu\nu} \) in terms of \( \psi^Z \). The Zerilli function is quantized as a normal scalar field. The expression for \( h_{\mu\nu} \) in terms of \( \psi^Z \) is necessary to obtain the correct (lowest non-trivial order) coupling between \( \psi^Z \) and a Klein–Gordon field \( \phi \). This coupling is obtained from the cubic term in the expansion of the standard, massless Einstein–Hilbert–Klein–Gordon Lagrangian.

The derivation of the explicit expressions for \( h_{\mu\nu} \) in terms of \( \psi^Z \) in [10] is fairly involved and specific to static Schwarzschild coordinates. However, given that the Zerilli function is already known in a gauge-independent form, cf equation (4.23) of [5], these expressions may be obtained algorithmically in any coordinate system, if the gauge-fixed equations of motion in that coordinate system are given. This opens the possibility of using computer algebra software to reduce the manual labor necessary to reproduce their derivation.

For instance, the equations of motion for the even-parity metric perturbations in EF coordinates are given in appendix E of [5]. In ‘convenient’ gauge they reduce to the following. First, we write down the so-called gauge-invariant combinations:

\[
\tilde{h}_{vv} = h_{vv} + \frac{2M f}{r^2} j_r, \quad (16)
\]

\[
\tilde{h}_{vr} = h_{vr} - \partial_v j_r - \frac{2M}{r^2} j_r, \quad (17)
\]

\[
\tilde{h}_{rr} = h_{rr} - 2\partial_v j_r. \quad (18)
\]

\(^2\) Referred to as Zerilli gauge in [10].
The Zerilli function, from appendix E of [5], is given by
\[ \tilde{K} = -\frac{2f}{r} j_r. \] (19)

In terms of these combinations, the equations of motion for the metric perturbations, where for brevity we use \( \lambda = \ell(\ell + 1) \) and \( \mu = (\ell - 1)(\ell + 2) \), are
\begin{align*}
0 &= -\tilde{\alpha}_r^2 \tilde{K} - \frac{2}{r} \partial_r \tilde{K} - \frac{1}{r} \partial_r \tilde{h}_{rr,rr} + \frac{f}{r^2} \partial_r \tilde{h}_{rr} + \frac{\lambda r + 4M}{2r^3} \tilde{h}_{rr}, \\
0 &= \partial_r \partial_r \tilde{K} + \frac{2}{r} \partial_r \tilde{K} + \frac{r-M}{r^2} \partial_r \tilde{h}_{rr} - \frac{f}{r^2} \partial_r \tilde{h}_{rr} - \frac{2}{r} \partial_r \tilde{h}_{rr,rr} + \frac{\lambda r + 4M}{2r^3} \tilde{h}_{rr}, \\
0 &= \partial_r \partial_r \tilde{h}_{rr} - \partial_r \tilde{h}_{rr} - \partial_r \tilde{K} + \frac{2}{r^2} \tilde{h}_{rr} - \frac{r-M}{r^2} \tilde{h}_{rr,rr}, \\
0 &= -\tilde{\alpha}_r^2 \tilde{h}_{rr,rr} + 2\partial_r \partial_r \tilde{h}_{rr} - \partial_r \tilde{h}_{rr} + 2\partial_r \partial_r \tilde{K} - \frac{r-M}{r^2} \partial_r \tilde{h}_{rr,rr} + \frac{2}{r^2} \partial_r \tilde{h}_{rr,rr} - \frac{\lambda r^2}{2} \tilde{h}_{rr} \tilde{h}_{rr,rr} \tilde{h}_{rr,rr}.
\end{align*}

The Zerilli function, from appendix E of [5], is given by
\[ \Psi = \frac{2r}{\lambda} \left[ \tilde{K} - \frac{2}{\Lambda} (\tilde{h}_{rr,rr} - r \partial_r \tilde{K}) \right], \] (27)
\[ \Lambda = (\ell - 1)(\ell + 2) + \frac{6M}{r} = \mu + \frac{6M}{r}, \quad \ell \geq 2. \] (28)

The restriction on the spherical harmonic index stems from the fact that the \( \ell = 0, 1 \) modes are not dynamical. The denominator \( \Lambda \) is clearly non-vanishing either in the exterior or interior black hole regions. Note that the above expression may differ by an \( \ell \)-dependent multiplicative constant from \( \psi^Z \) defined in [10], which is immaterial for the purposes of this discussion.

The algorithm for expressing \( p_{\mu \nu} \) in terms of \( \Psi \) consists of the following steps. Recall that each of the above equations is linear in the components of \( p_{\mu \nu} \) and their partial derivatives. Also, let \( p_n \) denote the set of all \( n \)th partial derivatives of the \( p_{\mu \nu} \) components \( h_{v,v}, h_{v,v}, h_{v,v}, \) and \( j_r \). Note that we are not including \( \Psi \) or its derivatives in \( p_n \).

This algorithm is inspired by the study of the formal integrability of partial differential equations [8, 9]. It is not too difficult to see how the simplified version presented here is equivalent to solving the constraint equations 'by hand'.

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6
(1) **Initialization.** Let \( E \) be a list of expressions whose vanishing is equivalent to equations (20)–(27), e.g., the right-hand sides of those equations. Further, divide this list into subsets \( E_n \), each containing no more than \( n \) partial derivatives acting on the components of \( p_{\mu\nu} \) (that is, variables from \( P_0 \) up to \( P_n \) only). Lastly, define \( E_{-1} \) to be the subset of expressions that depend on \( \Psi \) and its derivatives only; it starts out empty.

(2) **Iteration.** Repeat for \( n = 1, 0, \) and \(-1\), in that order. Apply \( \partial_{t} \) and \( \partial_{r} \) to each element of \( E_n \) and collect the results in \( E'_{n+1} \). Using linear operations, eliminate the variables \( P_{n+1} \) (being the highest order derivatives) from \( E_{n+1} \cup E'_{n+1} \). Replace \( E_{n+1} \) by the eliminated expressions and add the remaining independent expressions to \( E_n \) which is possible since the remaining expressions will have no more than \( n \) derivatives acting on each component of \( p_{\mu\nu} \).

(3) **Termination.** Iterate step (2) until the number of independent expressions in \( E_0 \) is the same as the number of variables in \( P_0 \). Optionally, keep iterating until \( E_{-1} \) is non-empty.

(4) **Explicit Solution.** Set each expression in \( E_0 \) to zero and solve the resulting linear equations for the variables in \( P_0 \). Each of the \( p_{\mu\nu} \) components will then be explicitly expressed in terms of \( \Psi \) and its derivatives.

If \( E_{-1} \) is non-empty, then setting each of its elements to zero is equivalent to the explicit equation of motion for \( \Psi \). This algorithm is not guaranteed to terminate for an arbitrary set of partial differential equations with constraints (though a generalized version of it is guaranteed to terminate under fairly general conditions). However, if it is known to terminate for a set of partial differential equations expressed in one coordinate system, then it will terminate for the same set of equations expressed in any other coordinate system. The results of section 2.1 of [10] essentially show that the algorithm terminates for equations (20)–(27), when expressed in static Schwarzschild coordinates.

Applying this algorithm, we can obtain explicit expressions for the non-zero components of \( p_{\mu\nu} \) in EF coordinates. The corresponding expressions in static Schwarzschild coordinates can be obtained by applying the same algorithm to the equations of motion explicitly given in appendix C of [5] or by applying the usual coordinate transformation rules of tensor calculus. The results agree with section 2.1 of [10].

In these explicit expressions, the singularity of the ‘convenient’ gauge is apparent from the presence of terms proportional to inverse powers of \( 1/f \) in \( j_r \), \( h_{rr} \), and \( h_{\nu\nu} \), which diverge as \( r \to 2M \), presuming that \( \Psi \) is itself regular at the horizon. These divergences can be removed by the following explicit (singular) gauge transformation: \( \xi = 0, \xi_{\nu} = 0, \) and

\[
\xi_{r} = -\frac{M(4M\partial_{r}\Psi + \lambda \Psi)}{2\lambda r f}.
\]

After this gauge transformation, the explicit expressions for the components of the metric perturbation become

\[
h_{\nu\nu} = -\frac{2\mu \Psi M(\lambda + 1)^2}{3\lambda^2 r^2} - \frac{(\lambda + 1)[12\partial_{r}\Psi M(\lambda + 1) + \mu \Psi(\lambda - 8) + 36\partial_{\nu}\Psi M]}{18\lambda r}
- \frac{\Psi M^2\lambda + 4\partial_{\nu}\Psi M^3}{r^5} - \frac{\Psi M(\lambda - 8) + \mu \Psi(\lambda + 4) + 12\mu(\partial_{\nu}\Psi)M}{6r^2} + \frac{\mu \Psi(\lambda + 4) + 12\mu(\partial_{\nu}\Psi)M}{18r}
+ r\partial_{r}^2\Psi + \partial_{\nu}\Psi + \partial_{\nu}\Psi
\]

(30)
The expressions are manifestly regular at the horizon. Finally, the explicit equation of motion for $\Psi$ is

$$2\partial_r \partial_t \Psi + f \partial_r^2 \Psi + f' \partial_r \Psi = \frac{1}{\Lambda^2} \left[ \mu^2 \left( \frac{\mu + 2}{r^2} + \frac{6M}{r^3} \right) + \frac{36M^2}{r^4} \left( \mu + \frac{2M}{r} \right) \right] \Psi, \quad (34)$$

where the left-hand side is simply the d’Alambertian, $\Box \Psi$, on the $(t, r)$-plane. This is the well-known equation of motion for the Zerilli function, derived in both [10] and [5].

4. Interaction and divergences

Once both the scalar field and the metric perturbations are quantized, and their cubic coupling is explicitly derived, the calculation in [10] proceeds as follows (though this logic is only implicit in its technical details). The lowest order correction to the scalar 2-point function is computed, which amounts to taking into account the single loop diagram shown in figure 1.

This diagram is evaluated using an optical theorem-like identity, which is also schematically illustrated in the figure. The correction to the scalar Hawking radiation spectrum is encoded in this correction to the 2-point function.
an on-shell 4-momentum \( k \) and a tensor polarization index \( j \) (if any), while the vertex factor is proportional to the triple mode function overlap integral

\[
V_{k,k',k''} = \int d^4 x \, \phi_0^0(x) \phi_0^0(x) \phi_0^0(x) \tag{35}
\]

\[
\approx \int d^4 x \, \exp[-i(k + k' + k'')_\mu x^\mu] \sim \delta(k + k' + k''). \tag{36}
\]

It is apparent, since each mode function is regular everywhere, that the integrand defining the vertex factor is \textit{locally integrable}, that is, its integral over any bounded region exists and is finite. It is also apparent that, nonetheless, \( V_{k,k',k''} \) is a distribution, which follows from the global convergence properties of the above integral. Since these properties rely only on the local regularity of mode functions, they are expected to hold in curved spacetimes as well.

On the black hole background, it is most convenient to perform diagrammatic calculations in angular momentum–frequency space. Each line of a diagram is then labeled by a frequency \( \omega \), a pair of spherical harmonics indices \( \ell, m \), and a tensor polarization index \( j \). Roughly speaking, the vertex factor is once again proportional to the triple mode function overlap integral

\[
V_{\omega \ell, \omega' \ell'} = \int w(r) \phi^\pm_\omega \ell(t, r) \phi^\pm_\omega' \ell'(t, r), \tag{37}
\]

where \( w(r) \) is a 2-form on the \((t, r)\)-plane, which takes into account the invariant volume measure and \( r \)-dependent coefficients that come from the expression for \( h_{\mu \nu} \) in terms of \( \psi^2 \).

Our discussion up to this point shows that, in a regular gauge, each of the \( w, \phi^\pm_\omega \ell \), as well as their products, should be locally integrable, including in a neighborhood of any point on the horizon.

More precisely, to take into account all kinds of vertices that couple the scalar field to metric perturbations, we must also consider derivative couplings. In that case, some of the \( \phi^\pm_\omega \ell \)'s in (37) will be acted upon by partial derivatives. If each \( \phi^\pm_\omega \ell \) is regular, then any scalars made up of its derivatives will also be regular; therefore, the local integrability argument is unmodified. Note, however, due to the presence of the black hole horizon, that the \( \phi^\pm_\omega \ell \) fail to be regular at the horizon. They are still bounded, but become highly oscillatory in the vicinity of the future horizon, with the oscillation phase diverging at the horizon itself (see section 10.2 of [2], for instance). Hence, derivatives of \( \phi^\pm_\omega \ell \) may become unbounded, though highly oscillatory, in a neighborhood of the horizon. Nonetheless, despite being unbounded, due to the oscillatory behavior of the integrands, their integrals should be evaluable in a distributional sense. Hence, even on a black hole background, and even with derivative couplings, the integrand in the triple mode function overlap (37) should be locally integrable, though perhaps only distributionally.

4.1. Divergence in the triple mode function overlap

The expressions \( K^{\text{even}}_{\omega \ell, \omega' \ell'} \) (for even-parity modes) and \( H^{\text{odd}}_{\omega \ell, \omega' \ell'} \) (for odd-parity modes), introduced in equation (128) of [10], are closely related to these kinds of triple mode function overlap integrals. Despite the expectations expressed above, equations (160–162) of [10] show that the integrand defining \( K^{\text{even}}_{\omega \ell, \omega' \ell'} \) fails to be locally integrable. This failure of local integrability can be traced to the singularity of the ‘convenient’ gauge, which introduces terms of the form \( \int \text{vol} / r^n \) with \( n = 1, 2 \) into the integral in (37), making it diverge in the vicinity of the horizon. In fact, as follows from the results of section 3, the only such locally non-integrable terms that contribute to the \( K^{\text{even}}_{\omega \ell, \omega' \ell'} \) are precisely the ones that can be removed by
the explicit gauge transformation (29) and hence are pure gauge artifacts. On the other hand, the integrand defining \( V_{\omega,\omega'} \) would be locally integrable, as expected, in any regular gauge.

Unfortunately, the authors of [10] have mistakenly identified these divergent contributions to \( K_{\omega,\omega',\omega''} \) as the dominant ones, have arbitrarily regulated them using a principal value prescription in the radial integral over an interval of size \( \epsilon \) about the horizon, and have dropped all other terms, including \( H_{\omega,\omega',\omega''} \). Since this \( \epsilon \)-regulator appears in the multiplicative coefficient of the spectrum correction in (1), the only conclusion to be drawn from the preceding discussion is that the size of the correction has not been correctly estimated and that what has been estimated is a pure gauge artifact. To obtain a reliable estimate, the triple mode function overlap integrals would have to be analyzed anew, once they are rewritten in a regular gauge.

### 4.2. Divergence in the summations over modes

The logic outlined at the beginning of this section culminates in equation (144) of [10], which expresses the correction to the Hawking radiation spectrum in terms of the triple mode function overlaps, \( K_{\omega,\omega',\omega''} \) and \( H_{\omega,\omega',\omega''} \), the Bogolubov coefficients relating the \( \phi^{\pm}_{\omega\ell j} \) modes, \( \alpha_{\omega,\omega'} \) and \( \beta_{\omega,\omega'} \), and some Clebsch–Gordan coefficients coming from the integration of products of spherical harmonics, \( C_{\ell,\ell',\ell''}^{m, m', m''} \). This expression for the correction is schematically, as illustrated in figure 1,

\[
\sum_{\omega,\omega'} \left| V_{\omega,\omega',\omega''} \right|^2,
\]

where each \( \sum_{\omega} \) compactly represents a combined sum-integral over all mode indices, also including spherical harmonic and polarization indices. Once \( V_{\omega,\omega',\omega''} \) is estimated, the outer mode sums are seen to be divergent. In equations (163) and (164) of [10], this divergence is regulated by essentially introducing lower and upper frequency cutoffs, respectively \( 1/L \) and \( \omega_{\text{cut}} \), cf also equation (82) in [10].

Leaving aside the fact that the estimates of the size of \( V_{\omega,\omega',\omega''} \) cannot be completely trusted due to gauge artifacts, a divergence in (38) is to be expected, as in any generic 1-loop perturbative calculation. The standard way to deal with such a divergence is to introduce a local counter-term in the original Lagrangian density. Since this divergence appears in a correction to the scalar self-energy, as illustrated on the lhs in figure 1, such a counter-term would only renormalize the kinetic and mass parts of the scalar field Lagrangian density. Thus, renormalization would allow the regulator dependence of the final result to be removed, such that the coefficient in front of the correction in (1) would not depend on \( \omega_{\text{cut}} \) and \( L \).

### 5. Discussion

The authors of [10] have tackled an interesting and challenging question. Unfortunately, their calculation suffers from a few problems. The so-called ‘convenient’ gauge chosen by the authors for the even-parity metric perturbations turns out to be singular, unlike the standard Regge–Wheeler gauge. The singularities introduced by their choice of gauge result in spurious divergences, which mask all other contributions in the vertex factors characterizing the coupling of metric perturbations to the scalar field. Moreover, another divergence, corresponding to the expected 1-loop divergence of perturbative quantum field theory, is not removed via renormalization. Both kinds of divergences are regulated, introducing arbitrary parameters into the calculation. As clearly seen in (1), the final result for the correction to the spectrum of scalar Hawking radiation depends on the regulators \( \epsilon \), \( \omega_{\text{cut}} \) and \( L \). Their presence makes the given estimate for the size of the correction unreliable.
Despite these problems, the authors have successfully addressed major, necessary parts of the calculation: (a) quantization of a scalar field and metric perturbations in a black hole background (via gauge fixing and explicit reduction to physical degrees of freedom), (b) explicit evaluation of the cubic scalar-graviton coupling (modulo gauge issues), (c) estimation of the in–out mode Bogolubov coefficients via a geometric optics approximation and (d) an explicit expression for the spectrum correction in terms of Bogolubov coefficients, triple mode function overlaps and Clebsch–Gordan coefficients. It would suffice only minor modifications and a careful application of standard quantum field theoretic techniques to complete this calculation and obtain a definite, parameter-free estimate for the correction to the spectrum of scalar Hawking radiation. Moreover, the same techniques are readily applicable to the problem of the perturbative back reaction of Hawking radiation on the quantum geometry of the black hole.

Acknowledgments

The author would like to thank Tomohiro Takahashi, Erik Plauschinn, Alessandro Torielli, Paul Reska and Tomislav Prokopec for helpful discussions. The author was supported by a Postdoctoral Fellowship from the National Science and Engineering Research Council (NSERC) of Canada.

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