A CRITERION FOR COMPACTNESS IN $L_p(\mathbb{R})$ OF THE RESOLVENT OF THE MAXIMAL STURM-LIOUVILLE OPERATOR OF GENERAL FORM

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Abstract. We consider the equation

$$-(r(x)y'(x))'+q(x)y(x)=f(x), \quad x \in \mathbb{R}$$

where $f \in L_p(\mathbb{R})$, $p \in (1, \infty)$ and

$$r > 0, \quad q \geq 0, \quad \frac{1}{r} \in L^1_{loc}(\mathbb{R}), \quad q \in L^1_{loc}(\mathbb{R}),$$

$$\lim_{|d| \to \infty} \int_{x-d}^{x} \frac{dt}{r(t)} \cdot \int_{x-d}^{x} q(t) dt = \infty, \quad x \in \mathbb{R}.$$ 

We assume that this equation is correctly solvable in $L_p(\mathbb{R})$. Under these assumptions, we study the problem on compactness of the resolvent $L^{-1}_p: L_p(\mathbb{R}) \to L_p(\mathbb{R})$ of the maximal continuously invertible Sturm-Liouville operator $L_p: D_p(\mathbb{R}) \to L_p(\mathbb{R})$. Here

$$L_p y = -(ry')' + qy, \quad y \in D_p,$$

$$D_p = \{ y \in L_p(\mathbb{R}) : y, ry' \in AC^1_{loc}(\mathbb{R}), \quad -(ry')' + qy \in L_p(\mathbb{R}) \}.$$ 

For the compact operator $L^{-1}_p: L_p(\mathbb{R}) \to L_p(\mathbb{R})$, we obtain two-sided sharp by order estimates of the maximal eigenvalue.

1. Introduction

In the present paper, we consider the equation

$$-(r(x)y'(x))'+q(x)y(x)=f(x), \quad x \in \mathbb{R} \quad (1.1)$$

where $f \in L_p(\mathbb{R})$, $(L_p(\mathbb{R}) := L_p)$, $p \in (1, \infty)$ and

$$r > 0, \quad q \geq 0, \quad r^{-1} \in L^1_{loc}(\mathbb{R}), \quad q \in L^1_{loc}(\mathbb{R}) \quad \left( r^{-1} := \frac{1}{r} \right), \quad (1.2)$$

$$\lim_{|d| \to \infty} \int_{x-d}^{x} \frac{dt}{r(t)} \cdot \int_{x-d}^{x} q(t) dt = \infty, \quad x \in \mathbb{R}. \quad (1.3)$$

Our general goal consists in finding criteria for compactness of the resolvent of equation (1.1). To state the problem more precisely, we need the following definitions and restrictions.

Here and in the sequel, by a solution of equation (1.1), we mean any function $y$ absolutely continuous together with $ry'$ and satisfying (1.1) almost everywhere on $\mathbb{R}$. We say that equation (1.1) is correctly solvable in a given space $L_p$, $p \in [1, \infty)$ if the following assertions hold (see [9] Ch.III, §6, no.2):

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I) for every function \( f \in L^p \), there exists a unique solution of (1.1), \( y \in L^p \);

II) there exists an absolute constant \( c(p) \in (0, \infty) \) such that the solution of (1.1), \( y \in L^p \), satisfies the inequality

\[
\|y\|_p \leq c(p)\|f\|_p, \quad \forall f \in L^p \quad (\|f\|_p := \|f\|_{L^p}).
\]

(1.4)

See [8] and §2 below for precise conditions that guarantee I)–II). In the sequel, for brevity, this is referred to as “problem I)–II)” or “question on I)–II”). It is easy to see that the problem I)–II) can be reformulated in different terms (see [8, 1]).

To this end, let us introduce the set \( D_p \) and the operator \( L_p : \)

\[
D_p = \{ y \in L^p : ry' \in L^p, -(ry')' + qy \in L^p \},
\]

\[
L_p y = -(ry')' + qy, \quad y \in D_p.
\]

(Here \( AC^{\text{loc}}(\mathbb{R}) \) is the set of functions absolutely continuous on every finite segment.) The linear operator \( L_p \) is called the maximal Sturm-Liouville operator, and problem I)–II) is obviously equivalent to the problem on existence and boundedness of the operator \( L_p^{-1} : L^p \rightarrow L^p \) (see [1]).

We can now give a precise statement of the problem studied in the present paper:

To find minimal additional requirements to (1.2) and (1.3) to the functions \( r \) and \( q \) under which, together with I)–II), the following condition III) also holds (“problem I)–III)” or “question on I)–III)”):

III) for a given \( p \in (1, \infty) \) the operator \( L_p^{-1} : L^p \rightarrow L^p \) is compact.

The main goal of the present paper is an answer to the question on I)–III).

For the reader’s convenience we outline the structure of the paper. In §2 we collect the preliminaries necessary for exposition; §3 contains a list of all results of the paper together with comments; §4 contains the proofs; in §5 we present examples of applications of our results to a concrete equation; and, finally, §6 contains the proofs of some technical assertions.

2. Preliminaries

**Theorem 2.1.** [3] Suppose that conditions (1.2) and

\[
\int_{-\infty}^{x} q(t)dt > 0, \quad \int_{x}^{\infty} q(t)dt > 0, \quad x \in \mathbb{R}
\]

(2.1)

hold. Then the equation

\[
(r(x)z'(x))' = q(x)z(x), \quad x \in \mathbb{R}
\]

(2.2)
Theorem 2.3. 

\[ \int_{-\infty}^{0} \frac{dt}{r(t)u^2(t)} < \infty, \quad \int_{0}^{\infty} \frac{dt}{r(t)v^2(t)} < \infty, \quad \int_{-\infty}^{0} \frac{dt}{r(t)v^2(t)} = \int_{0}^{\infty} \frac{dt}{r(t)u^2(t)} = \infty. \]  

Moreover, properties (2.3)--(2.6) determine the FSS \( \{u, v\} \) uniquely up to constant mutually inverse factors.

**Corollary 2.2.** [3] Suppose that conditions (1.2) and (2.1) hold. Then equation (2.2) has no solutions \( z \in L_p \) apart from \( z \equiv 0 \).

The FSS from Theorem 2.1 is denoted below by \( \{u, v\} \).

**Theorem 2.3.** [3] [10] For the FSS \( \{u, v\} \) we have the Davies-Harrell representations

\[ u(x) = \sqrt{\rho(x)} \exp \left( -\frac{1}{2} \int_{x_0}^{x} \frac{d\xi}{r(\xi)\rho(\xi)} \right), \quad v(x) = \sqrt{\rho(x)} \exp \left( \frac{1}{2} \int_{x_0}^{x} \frac{d\xi}{r(\xi)\rho(\xi)} \right) \]  

where \( x \in \mathbb{R} \), \( \rho(x) = u(x)v(x) \), \( x_0 \) is a unique solution of the equation \( u(x) = v(x) \) in \( \mathbb{R} \).

Furthermore, for the Green function \( G(x, t) \) corresponding to equation (1.1):

\[ G(x, t) = \begin{cases} u(x)v(t), & x \geq t \\ u(t)v(x), & x \leq t \end{cases} \]  

and for its “diagonal value” \( G(x, t) \big|_{x=t} = \rho(x) \), we have the following representation (2.9) and equalities (2.10):

\[ G(x, t) = \sqrt{\rho(x)\rho(t)} \exp \left( -\frac{1}{2} \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right). \]  

\[ \int_{-\infty}^{0} \frac{d\xi}{r(\xi)\rho(\xi)} = \int_{0}^{\infty} \frac{d\xi}{r(\xi)\rho(\xi)} = \infty. \]  

**Remark 2.4.** Representations (2.7) and (2.8) are given in [10] for \( r \equiv 1 \) and in [3] for \( r \neq 1 \). See [3] for equalities (2.10). Throughout the sequel conditions (1.2)--(1.3) are assumed to be satisfied (if not stated otherwise) without special mentioning.

**Lemma 2.5.** [3] For every given \( x \in \mathbb{R} \) each of the following equations

\[ \int_{x-d}^{x} \frac{dt}{r(t)} \int_{x-d}^{x} q(t)dt = 1, \quad \int_{x}^{x+d} \frac{dt}{r(t)} \int_{x}^{x+d} q(t)dt = 1 \]  

has a fundamental system of solutions (FSS) with the following properties:

\[ v(x) > 0, \quad u(x) > 0, \quad v'(x) \geq 0, \quad u'(x) \leq 0, \quad x \in \mathbb{R}, \]  

\[ r(x)[v'(x)u(x) - u'(x)v(x)] = 1, \quad x \in \mathbb{R}, \]  

\[ \lim_{x \to -\infty} \frac{v(x)}{u(x)} = \lim_{x \to \infty} \frac{u(x)}{v(x)} = 0, \]  

\[ \int_{-\infty}^{0} \frac{dt}{r(t)u^2(t)} < \infty, \quad \lim_{t \to 0} \frac{dt}{r(t)v^2(t)} < \infty, \quad \lim_{t \to 0} \frac{dt}{r(t)v^2(t)} = \lim_{t \to 0} \frac{dt}{r(t)u^2(t)} = \infty. \]
in \( d \geq 0 \) has a unique finite positive solution. Denote them by \( d_1(x) \) and \( d_2(x) \), respectively. For \( x \in \mathbb{R} \) we introduce the following functions:

\[
\varphi(x) = \int_{x-d_1(x)}^{x} \frac{dt}{r(t)}, \quad \psi(x) = \int_{x}^{x+d_2(x)} \frac{dt}{r(t)},
\]

\[
h(x) = \frac{\varphi(x)\psi(x)}{\varphi(x) + \psi(x)} \left( \left( \int_{x-d_1(x)}^{x+d_2(x)} q(t)dt \right)^{-1} \right) \quad (2.12)
\]

**Theorem 2.6.** For \( x \in \mathbb{R} \) the following inequalities hold:

\[
2^{-1}v(x) \leq (r(x)v'(x))\varphi(x) \leq 2v(x) \quad (2.13)
\]

\[
2^{-1}u(x) \leq (r(x)u'(x))\psi(x) \leq 2u(x)
\]

\[
2^{-1}h(x) \leq \rho(x) \leq 2h(x). \quad (2.14)
\]

**Corollary 2.7.** Let \( r \equiv 1 \). For every given \( x \in \mathbb{R} \) consider the following equation:

\[
d \cdot \int_{x-d}^{x+d} q(t)dt = 2 \quad (2.15)
\]

in \( d \geq 0 \). Equation (2.15) has a unique finite positive solution. Denote it by \( \tilde{d}(x) \). We have the inequalities:

\[
4^{-1} \cdot \tilde{d}(x) \leq \rho(x) \leq 3 \cdot 2^{-1} \tilde{d}(x), \quad x \in \mathbb{R}. \quad (2.16)
\]

**Remark 2.8.** Two-sided sharp by order a priori estimate of type (2.13) first appear in [18] (for \( r \equiv 1 \) and under some additional requirements to \( q \)). Under conditions (1.2) and \( \inf_{x \in \mathbb{R}} q(x) > 0 \), estimates similar to (2.13), with other more complicated auxiliary functions, were given in [16]. Sharp by order estimates of the function \( \rho \) were first obtained in [17] (under some additional requirements to \( r \) and \( q \)). Therefore, we call inequalities of such type Otelbaev inequalities. Note that in [17] auxiliary functions more complicated than \( h \) and \( \tilde{d} \) were used. The function \( \tilde{d} \) was introduced by M. Otelbaev (see [15]).

Throughout the sequel we denote by \( c, c(p), \ldots \) absolute positive constants which are not essential for exposition and may differ even within a single chain of computations. We write \( \alpha(x) \asymp \beta(x) \), \( x \in (a, b) \) if positive functions \( \alpha \) and \( \beta \) defined in \( (a, b) \) satisfy the inequalities

\[
c^{-1} \cdot \alpha(x) \leq \beta(x) \leq c\alpha(x), \quad x \in (a, b).
\]

**Lemma 2.9.** For \( x \in \mathbb{R} \) we have the inequality

\[
r(x)|\rho'(x)| < 1. \quad (2.17)
\]
In addition, the inequality $m < 1$ where
\[
m = \sup_{x \in \mathbb{R}} r(x)|\rho'(x)|
\] (2.18)
holds if and only if $\varphi(x) \asymp \psi(x)$, $x \in \mathbb{R}$.

We also introduce a new auxiliary function $s$ and the function $d$ already known from [3]. The properties of the functions are similar, and therefore for brevity we present them together. See [3] for the proofs for $d$, and §6 below for the proofs for $s$.

**Lemma 2.10.** [3] §6 below] For every $x \in \mathbb{R}$ each of the equations
\[
\int_{x-d}^{x+d} \frac{dt}{r(t)h(t)} = 1, \quad \int_{x-s}^{x+s} \frac{dt}{r(t)\rho(t)} = 1
\] (2.19)
in $d \geq 0$ and $s \geq 0$ has a unique finite positive solution. Denote the solutions of (2.19) by $d(x)$ and $s(x)$, respectively. The functions $d(x)$ and $s(x)$ are continuous for $x \in \mathbb{R}$.

**Lemma 2.11.** [3] §6 below] For $x \in \mathbb{R}$, $t \in [x - \varepsilon d(x), x + \varepsilon d(x)]$ ($t \in [x + \varepsilon s(x), x + \varepsilon s(x)]$)
and $\varepsilon \in [0, 1]$, we have the inequalities:
\[
(1 - \varepsilon)d(x) \leq d(t) \leq (1 + \varepsilon)d(x),
\] (2.20)
\[
((1 - \varepsilon)s(x) \leq s(t) \leq (1 + \varepsilon)s(x)).
\] (2.21)

In addition, we have the equalities:
\[
\lim_{x \to -\infty} (x + d(x)) = -\infty, \quad \lim_{x \to \infty} (x - d(x)) = \infty,
\] (2.22)
\[
\lim_{x \to -\infty} (x + s(x)) = -\infty, \quad \lim_{x \to \infty} (x - s(x)) = \infty.
\] (2.23)

**Definition 2.12.** [7] Suppose we are given $x \in \mathbb{R}$, a positive and continuous function $\varphi(t)$ for $t \in \mathbb{R}$, a sequence $\{x_n\}_{n \in \mathbb{N}'}$, $\mathbb{N}' = \{\pm 1, \pm 2, \ldots\}$. Consider segments $\Delta_n = [\Delta^-_n, \Delta^+_n]$, $\Delta^+_n = x_n \pm \varphi(x_n)$. We say that the segments $\{\Delta_n\}_{n=1}^\infty$ ($\{\Delta_n\}_{n=-\infty}^{1}$) form an $\mathbb{R}(x, \varphi)$-covering of $[x, \infty)$ ($(-\infty, x]$) if the following requirements hold:

1) $\Delta^+_n = \Delta^-_{n+1}$ for $n \geq 1$ ($\Delta^+_n = \Delta^-_n$ for $n \leq -1$),
2) $\Delta_0 = x$ ($\Delta^+_0 = x$), $\bigcup_{n \geq 1} \Delta_n = [x, \infty)$ $\left(\bigcup_{n \leq -1} \Delta_n = (-\infty, x]\right)$.

**Lemma 2.13.** [7] Suppose that for a positive and continuous function $\varphi(t)$ for $t \in \mathbb{R}$, we have the relations
\[
\lim_{t \to \infty} (t - \varphi(t)) = \infty \left(\lim_{t \to -\infty} (t + \varphi(t)) = -\infty\right).
\] (2.24)
Then for every $x \in \mathbb{R}$ there is an $\mathbb{R}(x, \varphi)$-covering of $[x, \infty)(\mathbb{R}(x, \varphi)$-covering of $(-\infty, x]$.
Remark 2.14. If for some $x \in \mathbb{R}$ there exist $\mathbb{R}(x, \varkappa)$-coverings of both $[x, \infty)$ and $(-\infty, x]$, then their union will be called an $\mathbb{R}(x, \varkappa)$-covering of $\mathbb{R}$.

Lemma 2.15. [3] §6 below] For every $x \in \mathbb{R}$ there exist $\mathbb{R}(x, d)$ and $\mathbb{R}(x, s)$-coverings of $\mathbb{R}$.

Remark 2.16. Assertions of the type in Lemma 2.15 and estimates of the form (2.20) were introduced by Otelbaev (see [15]).

Lemma 2.17. [3] §6 below] Let $x \in \mathbb{R}$, $t \in [x - d(x), x + d(x)]$ ($t \in [x - s(x), x + s(x)]$). Then the following inequalities hold:

\[
\begin{align*}
\alpha^{-1}v(x) & \leq v(t) \leq \alpha v(x), \\
\alpha^{-1}u(x) & \leq u(t) \leq \alpha u(x), \\
\alpha^{-1}p(x) & \leq p(t) \leq \alpha p(x), \\
(4\alpha)^{-1}h(x) & \leq h(t) \leq 4\alpha h(x).
\end{align*}
\]

Here $\alpha = \exp(2)$.

Theorem 2.18. [8] Suppose that conditions (1.2) and (2.1) hold and $p \in (1, \infty)$. Then equation (1.1) is correctly solvable in $L_p$ if and only if the Green operator $G : L_p \to L_p$ is bounded. In the latter case, for every function $f \in L_p$ the solution $y \in L_p$ of (1.1) is of the form $y = Gf$. In particular, $L_p^{-1} = G$. Here (see (2.8)):

\[
(Gf)(x) \overset{\text{def}}{=} \int_{-\infty}^{\infty} G(x, t)f(t)dt, \quad x \in \mathbb{R}, \quad f \in L_p.
\]

Remark 2.19. If $r^{-1} \notin L_1(-\infty, 0)$ and $r^{-1} \notin L_1(0, \infty)$, then condition (2.1) and, a fortiori, (1.3) are necessary for correct solvability of equation (1.1) in $L_p$, $p \in (1, \infty)$ (see [8]).

Lemma 2.20. [8] Suppose that conditions (1.2) and (2.1) hold and $p \in (1, \infty)$. Consider the integral operators

\[
\begin{align*}
(G_1f)(x) &= u(x) \int_{-\infty}^{x} v(t)f(t)dt, \quad x \in \mathbb{R}, \\
(G_2f)(x) &= v(x) \int_{x}^{\infty} u(t)f(t)dt, \quad x \in \mathbb{R}.
\end{align*}
\]

We have the relations

\[
G = G_1 + G_2,
\]

\[
\frac{\|G_1\|_{p^{-p}} + \|G_2\|_{p^{-p}}}{2} \leq \|G\|_{p^{-p}} \leq \|G_1\|_{p^{-p}} + \|G_2\|_{p^{-p}}.
\]
Theorem 2.21. [8] Equation (1.1) is correctly solvable in $L_p$, $p \in (1, \infty)$ if and only if $B < \infty$. Here
\[ B \overset{\text{def}}{=} \sup_{x \in \mathbb{R}} h(x)d(x). \] (2.33)
Moreover, the following relations hold:
\[ \|G\|_{p \to p} \asymp \|G_1\|_{p \to p} \asymp \|G_2\|_{p \to p} \asymp B. \] (2.34)

Theorem 2.22. Let (1.2) and (2.1) be satisfied. Then equation (1.1) is correctly solvable in $L_p$, $p \in (1, \infty)$ if and only if $S < \infty$. Here
\[ S \overset{\text{def}}{=} \sup_{s \in \mathbb{R}} (\rho(x)s(x)). \] (2.35)

Remark 2.23. Theorems 2.21 and 2.22 are proved in the same way because the properties of the functions $d$ and $s$, $\rho$ and $h$ are quite analogous (see above). Moreover, the proof of Theorem 2.22 is even simpler compared to Theorem 2.21 because there is no need to apply estimates (2.14). In particular, for this reason, in Theorem 2.22 instead of condition (1.3) of Theorem 2.21 there appears a weaker condition (2.1). Thus, since the proof of Theorem 2.22 is reduced to the repetition of the argument from [8], we do not present it here.

Theorem 2.24. [8, 6] Suppose that the conditions (1.2) and $r \equiv 1$ hold. Then equation (1.1) is correctly solvable in $L_p$, $p \in [1, \infty)$ if and only if there exists $a > 0$ such that $m(a) > 0$. Here
\[ m(a) = \inf_{x \in \mathbb{R}} \int_{x-a}^{x+a} q(t)dt. \]

Theorem 2.25. [8, 3] For every $p \in (1, \infty)$ equation (1.1) is correctly solvable in $L_p$ if $A > 0$. Here
\[ A = \inf_{x \in \mathbb{R}} A(x), \quad A(x) = \frac{1}{2d(x)} \int_{x-d(x)}^{x+d(x)} q(t)dt. \] (2.36)

Remark 2.26. In contrast to the condition $B < \infty$, the meaning of the requirement $A > 0$ is quite obvious: some special Steklov average of the function $q$ must be separated from zero uniformly on the whole axis (see [3]). Moreover, the requirement $A > 0$ can be viewed as a weakening of the simplest condition $\inf_{x \in \mathbb{R}} q(x) > 0$ guaranteeing correct solvability of (1.1) in $L_p$, $p \in [1, \infty)$ (see [3, 16]). We continue this comment in the next assertion (Theorem 2.28) by defining a meaningful class of equations (1.1) (see [4]) in which the requirement $B < \infty$ is equivalent to a condition of the form $A > 0$. Towards this end, we need a new auxiliary function.
Lemma 2.27. [4, 5] Let \( \varphi(x) \asymp \psi(x), x \in \mathbb{R} \). For a given \( x \in \mathbb{R} \) consider the equation in \( \mu \geq 0 \):

\[
\int_{x-\mu}^{x+\mu} q(t) h(t) dt = 1. \tag{2.37}
\]

Equation (2.37) has at least one positive finite solution. Let

\[
\mu(x) = \inf_{\mu \geq 0} \left\{ \mu : \int_{x-\mu}^{x+\mu} q(t) h(t) dt = 1 \right\}. \tag{2.38}
\]

The function \( \mu(x) \) is continuous for \( x \in \mathbb{R} \), and, in addition,

\[
\lim_{x \to -\infty} (x + \mu(x)) = -\infty, \quad \lim_{x \to \infty} (x - \mu(x)) = \infty. \tag{2.39}
\]

Theorem 2.28. [4] Let \( \varphi(x) \asymp \psi(x), x \in \mathbb{R} \). Then \( B < \infty \) if and only if \( \tilde{A} > 0 \). Here

\[
\tilde{A} = \inf_{x \in \mathbb{R}} \tilde{A}(x), \quad \tilde{A}(x) = \frac{1}{2\mu(x)} \int_{x-\mu(x)}^{x+\mu(x)} q(t) dt. \tag{2.40}
\]

Remark 2.29. To apply Theorem 2.21 to concrete equations, one has to know the auxiliary functions \( h \) and \( d \). Usually it is not possible to express these functions through the original coefficients \( r \) and \( q \) of equation (1.1). However, it is easy to see that when studying the value of \( B \), one can replace in an equivalent way the functions \( h \) and \( d \) with their sharp by order two-sided estimates. In most cases, such inequalities can be obtained using standard tools of local analysis (see, e.g., [3] and a detailed exposition in [4]; one example of obtaining such estimates is given in §6 below). It is clear that in concrete cases of the question on I)–II), it is particularly convenient to use criteria which either do not use the functions \( h \) and \( d \) at all, or use, say, only the function \( h \). Such assertions are contained in the following theorem.

Theorem 2.30. [8] Suppose that conditions (1.2)–(1.3) hold. Then we have the following assertions:

A) Equation (1.1) is correctly solvable in \( L_p \), \( p \in (1, \infty) \) if any of the following conditions holds:

1) \( B_1 < \infty \), \( \bar{B}_1 = \sup_{x \in \mathbb{R}} B_1(x) \), \( B_1(x) = r(x)h^2(x) \), \( \bar{B}_1(x) = r(x)h^2(x) \), \( B_1(x) = r(x)h^2(x) \).

2) \( B_2 < \infty \), \( \bar{B}_2 = \sup_{x \in \mathbb{R}} B_2(x) \), \( B_2(x) = r(x)\varphi(x)\psi(x) \).

3) \( B_3 < \infty \), \( \bar{B}_3 = \sup_{x \in \mathbb{R}} B_3(x) \), \( B_3(x) = h(x) \cdot |x| \).

B) Suppose that in addition to (1.2) and (1.3) the following conditions hold:

\[
r^{-1} \in L_1, \quad q \notin L_1(-\infty, 0), \quad q \notin L_1(0, \infty). \tag{2.44}
\]
Then equation (1.1) is correctly solvable in $L_p$, $p \in (1, \infty)$ if $\theta < \infty$. Here $\theta = \sup_{x \in \mathbb{R}} \theta(x)$,

$$\theta(x) = \|x| \left( \int_{-\infty}^{x} \frac{dt}{r(t)} \right) \cdot \left( \int_{x}^{\infty} \frac{dt}{r(t)} \right).$$

(2.45)

We also need the following known facts.

**Theorem 2.31.** [11, Ch.IV, §8, Theorem 20] Let $p \in (1, \infty)$. The set $\mathcal{K} \subset L_p$ is precompact if and only if the following conditions hold:

1) $\sup_{f \in \mathcal{K}} \|f\|_p < \infty,$

2) $\lim_{\delta \to 0} \sup_{f \in \mathcal{K}} \|f(\cdot + t) - f(\cdot)\|_p = 0,$

3) $\lim_{x \to \infty} \sup_{f \in \mathcal{K}} \int_{|x| \geq N} |f(x)|^p dx = 0.$

(2.46)

(2.47)

(2.48)

Let $\mu, \theta$ be almost everywhere finite measurable positive functions defined in the interval $(a, b)$, $-\infty \leq a < b \leq \infty$.

We introduce the integral operators

$$(Kf)(x) = \mu(x) \int_{x}^{b} \theta(t)f(t)dt, \quad x \in (a, b),$$

and

$$(\tilde{K}f)(x) = \mu(x) \int_{a}^{x} \theta(t)f(t)dt, \quad x \in (a, b).$$

(2.49)

(2.50)

**Theorem 2.32.** [20] [13, Ch.1, §1.3] For $p \in (1, \infty)$ the operator $K : L_p(a, b) \to L_p(a, b)$ is bounded if and only if $H_p(a, b) < \infty$. Here $H_p(a, b) = \sup_{x \in (a, b)} H_p(x, a, b)$,

$$H_p(x, a, b) = \left[ \int_{a}^{x} \mu(t)^{p'} dt \right]^{1/p'} \cdot \left[ \int_{x}^{b} \theta(t)^{p'} dt \right]^{1/p'}, \quad p' = \frac{p}{p - 1}.$$ 

(2.51)

In addition, the following inequalities hold:

$$H_p(a, b) \leq \|K\|_{L_p(a, b) \to L_p(a, b)} \leq (p^{1/p'}(p')^{1/p'})^{1/p'} H_p(a, b),$$

(2.52)

**Theorem 2.33.** [20] [13, Ch.1, §1.3] For $p \in (1, \infty)$ the operator $\tilde{K} : L_p(a, b) \to L_p(a, b)$ is bounded if and only if $\tilde{H}_p(a, b) < \infty$. Here $\tilde{H}_p(a, b) = \sup_{x \in (a, b)} \tilde{H}_p(x, a, b)$, and

$$\tilde{H}_p(x, a, b) = \left[ \int_{a}^{x} \theta(t)^{p'} dt \right]^{1/p'} \cdot \left[ \int_{x}^{b} \mu(t)^{p'} dt \right]^{1/p'}, \quad p' = \frac{p}{p - 1}.$$ 

(2.53)

In addition, the following inequalities hold:

$$\tilde{H}_p(a, b) \leq \|K\|_{L_p(a, b) \to L_p(a, b)} \leq (p^{1/p'}(p')^{1/p'})^{1/p'} \tilde{H}_p(a, b).$$

(2.54)

Note that some assertions (mainly of a technical nature) will be given in §4–§5 in the course of the exposition.
3. Main Results

Recall that, if conditions I)–II) hold, then $L_p^{-1} = G$, $p \in (1, \infty)$ (see Theorem 2.18). Therefore, in the sequel in the statements of the theorems, we write the operator $G$ instead of the operator $L_p^{-1}$.

Our main result is the following theorem.

**Theorem 3.1.** Let $p \in (1, \infty)$, and suppose that equation (1.1) is correctly solvable in $L_p$. Then the operator $G : L_p \to L_p$ is compact if and only if

$$\lim_{|x| \to \infty} h(x)d(x) = 0. \quad (3.1)$$

**Theorem 3.2.** Suppose that conditions (1.2) and (2.1) hold, $p \in (1, \infty)$, and equation (1.1) is correctly solvable in $L_p$. Thus the operator $G : L_p \to L_p$ is compact if and only if

$$\lim_{|x| \to \infty} \rho(x)s(x) = 0. \quad (3.2)$$

**Remark 3.3.** Theorems 3.1 and 3.2 are related to one another in the same way as Theorems 2.21 and 2.22 (see Remark 2.23). Therefore, we do not present a proof of Theorem 3.2.

**Theorem 3.4.** Suppose that condition (3.1) holds. Then the operator $G : L_2 \to L_2$ is compact, self-adjoint, and positive. Its maximal and eigenvalue $\lambda$ satisfies the estimates (see (2.33)):

$$c^{-1}B \leq \lambda \leq cB. \quad (3.3)$$

**Remark 3.5.** Theorems 3.1 and 3.4 were obtained in [5] under an additional requirement $\varphi \asymp \psi(x), x \in \mathbb{R}$. The meaning of condition (3.1) can be clarified “in terms of the coefficients” of equation (1.1) in the same way as is done in Remark 2.26 for the interpretation of the condition $B < \infty$. In particular, in order to expand on Theorem 3.1, we state the following theorem.

**Theorem 3.6.** [5] Let $\varphi \asymp \psi(x), x \in \mathbb{R}, \ p \in (1, \infty)$, and suppose that equation (1.1) is correctly solvable in $L_p$. Then the operator $G : L_p \to L_p$ is compact if and only if

$$\lim_{|x| \to \infty} \hat{A}(x) = \infty \quad (3.4)$$

(see (2.40)).

Thus, if $\varphi(x) \asymp \psi(x), x \in \mathbb{R}$, requirement (3.1) means that some special Steklov average value of the function $q$ must tend to infinity at infinity.
We can now present several consequences of Theorem 3.1. Their significance consists in the fact that they allow us to clarify the question on I)–III) either not using at all the functions \(k\) and \(d\), or with the help of only \(h\) (see Remark 2.29).

**Corollary 3.7.** Let \(p \in (1, \infty)\) and \(\mathcal{A} > 0\) (see (2.36)). Then the operator \(G : L^p \rightarrow L^p\) is compact if \(\mathcal{A}(x) \rightarrow \infty\) as \(|x| \rightarrow \infty\).

**Corollary 3.8.** Let \(p \in (1, \infty)\) and \(q(x) \rightarrow \infty\) as \(|x| \rightarrow \infty\). Then the operator \(G : L^p \rightarrow L^p\) is compact.

**Corollary 3.9.** Suppose that conditions (1.2) hold, \(r(x) \equiv 1, x \in \mathbb{R}\), and \(m(a_0) > 0\) for some \(a_0 \in (0, \infty)\) (see Theorem 2.24). Then the operator \(G : L^p \rightarrow L^p\) is compact if and only if the Molchanov condition (see [8]) holds:

\[
\lim_{|x| \rightarrow \infty} \int_{x-a}^{x+a} q(t)\,dt = \infty, \quad \forall a \in (0, \infty).
\]  

(3.5)

**Corollary 3.10.** Let \(p \in (1, \infty)\). Then assertions I)–III) hold if and only if any of the following conditions is satisfied:

1) \(B_1 < \infty\) (see (2.41)), \(r(x)h^2(x) \rightarrow 0\) as \(|x| \rightarrow \infty\)  
2) \(B_2 < \infty\) (see (2.42)), \(r(x)\varphi(x)\psi(x) \rightarrow 0\) as \(|x| \rightarrow \infty\)  
3) \(B_3 < \infty\) (see (2.43)), \(h(x) \cdot |x| \rightarrow 0\) as \(|x| \rightarrow \infty\)  

(3.6)  
(3.7)  
(3.8)

**Corollary 3.11.** Denote

\[
r_0 = \sup_{x \in \mathbb{R}} r(x), \quad h_0 = \sup_{x \in \mathbb{R}} h(x).
\]  

(3.9)

Let \(r_0 < \infty\). Then (1.1) is correctly solvable in \(L^p, p \in (1, \infty)\) if \(h_0 < \infty\). In addition, the operator \(G : L^p \rightarrow L^p, p \in (1, \infty)\) is compact if \(h(x) \rightarrow 0\) as \(|x| \rightarrow \infty\).

**Remark 3.12.** Note that the requirement

\[
q(x) \rightarrow \infty \quad \text{as} \quad |x| \rightarrow \infty
\]  

(3.10)

is so strong that the answer to the question on I)–III) is no dependent on the behaviour (within the framework of (1.2)) of the function \(r\). In this connection, look at the opposite situation and find out what requirements on the function \(r\) is the positive solution of the behaviour (within a certain framework) of the function \(q\). See Theorems 3.13 and 3.14 below for possible answers to these questions.
We emphasize that these assertions have been obtained from Theorems 2.22 and 3.2 where (1.3) is not used. Therefore, in Theorems 3.13 and 3.14 requirements on the function \( q \) are weakened to conditions (1.2) and (2.1).

**Theorem 3.13.** Suppose that together with (1.2) condition (2.1) holds and \( \theta < \infty \) (see Theorem 2.30). Then equation (1.1) is correctly solvable in \( L_p, p \in (1, \infty) \). In addition, the operator \( G : L_p \to L_p, p \in (1, \infty) \) is compact if \( \theta(x) \to 0 \) as \( |x| \to \infty \) (see (2.45)).

**Theorem 3.14.** Suppose that conditions (1.2), (2.1) hold and \( \nu < \infty \). Here \( \nu = \sup_{x \in \mathbb{R}} \nu(x) \),

\[
\nu(x) = r(x) \left( \int_{-\infty}^{x} \frac{dt}{r(t)} \right)^{2} \cdot \left( \int_{x}^{\infty} \frac{dt}{r(t)} \right)^{2}, \quad x \in \mathbb{R}. \tag{3.11}
\]

Then equation (1.1) is correctly solvable in \( L_p, p \in (1, \infty) \). If, in addition, \( \nu(x) \to 0 \) as \( |x| \to \infty \), then the operator \( G : L_p \to L_p, p \in (1, \infty) \) is compact.

4. Proofs

*Proof of Theorem 3.1.* Necessity.

Let us check (3.1) as \( x \to \infty \). (The case \( x \to -\infty \) is treated in a similar way.) Let \( \{\Delta_n\}_{n \in \mathbb{N}'} \) be an \( \mathbb{R}(0, d) \)-covering of \( \mathbb{R} \), \( F = \{f_n(t)\}_{n \in \mathbb{N}'} \) and

\[
f_n(t) = \begin{cases} 
d(x_n)^{-1/p} & \text{if } t \in \Delta_n \\
0 & \text{if } t \notin \Delta_n
\end{cases}, \quad n \in \mathbb{N}'.
\]

Then \( \|f_n\|_p^p = 2, n \in \mathbb{N}' \) and the set \( \{Gf_n\}_{n \in \mathbb{N}'} \) is precompact in \( L_p \). Let \( x \in \Delta_n, n \in \mathbb{N}' \). In the following relations we apply (2.14) and (2.25)–(2.26):

\[
(Gf_n)(x) = u(x) \int_{\Delta_n^-}^x v(t)f_n(t)dt + v(x) \int_{-\infty}^{\Delta_n^+} u(t)f_n(t)dt
\]

\[
= \frac{u(x)}{u(x_n)} \rho(x_n) \int_{\Delta_n^-}^x \frac{v(t)}{v(x_n)} \cdot \frac{dt}{d(x_n)^{1/p}} + \frac{v(t)}{v(x_n)} \cdot \frac{dt}{u(x_n) d(x_n)^{1/p}}
\]

\[
\geq c^{-1} \rho(x_n) d(x_n)^{1/p'} \geq c^{-1} h(x_n) d(x_n)^{1/p'}, \quad n \in \mathbb{N}'. \tag{4.1}
\]

By Theorem 2.31 for a given \( \varepsilon > 0 \) there exists \( N(\varepsilon) \gg 1 \) such that

\[
\sup_{f_n \in F} \int_{|x| \geq N(\varepsilon)} |(Gf_n(t))|^{p} dt \leq \varepsilon.
\]

From the properties of an \( \mathbb{R}(0, d) \)-covering of \( \mathbb{R} \), it follows that there exists \( n_0 = n_0(\varepsilon) \in \mathbb{N} = \{1, 2, 3, \ldots\} \) such that \( N(\varepsilon) \in \Delta_{n_0} \). Set \( n_1 = n_1(\varepsilon) = n_0(\varepsilon) + 1 \). Since \( N(\varepsilon) \leq \Delta_{n_1}^- \), we
have
\[ \sup_{f_k \in F} \int_{\Delta_{n_1}} |(Gf_k)(t)|^p dt \leq \varepsilon. \] (4.2)

Let \( k \geq n_1(\varepsilon) \). Then from (4.1)–(4.2), it follows that
\[ \varepsilon \geq \int_{\Delta_{n_1}} |(Gf_k)(t)|^p dt \geq \int_{\Delta_k}^{\Delta_{k+1}} |(Gf_k)(t)|^p dt \geq c^{-1}(h(x_k)d(x_k))^p. \]

Therefore, \( \lim_{k \to \infty} (h(x_k)d(x_k)) = 0 \).

Proof of Theorem 3.1. Sufficiency. Assume that the hypotheses of the theorem are satisfied. Then by Theorem 2.18 the operator \( G : L_p \to L_p \) is bounded, and by Lemma 2.20 so are the operators \( G_1 : L_p \to L_p \) and \( G_2 : L_p \to L_p \). Clearly, if \( G_1 \) and \( G_2 \) are compact, then so is \( G \) (see (2.31)). Furthermore, compactness of \( G_1 \) and \( G_2 \) is checked in the same way, and therefore below we only consider \( G_2 \).

Let \( F = \{ f \in L_p : \| f \|_p \leq 1 \} \). Compactness of \( G_2 : L_p \to L_p \) will be established as soon as we check that the set \( W = \{ g \in L_p : g = G_2f, f \in F \} \) is precompact in \( L_p \). Below we show that the set \( W \) satisfies conditions 1), 2), and 3) of Theorem 2.31 and thus proves Theorem 3.1.

Verification of condition 1). The above arguments (together with the definition of the set \( F \) and Theorems 2.18 and 2.21) imply the inequality 1),
\[ \sup_{g \in W} \| g \|_p = \sup_{f \in F} \| G_2f \|_p \leq \| G_2 \|_{p \to p} \cdot \| f \|_p \leq \| G_2 \|_{p \to p} \leq cB < \infty \Rightarrow 1). \]

Verification of condition 3). We need some auxiliary assertions.

Lemma 4.1. Let \( x \in \mathbb{R} \) and let \( \{ \Delta_n \}_{n \in \mathbb{N}} \) be an \( \mathbb{R}(x,d) \)-covering of \( \mathbb{R} \). Then
\[ \int_{\Delta_k}^{\Delta_{k+1}} \frac{d\xi}{r(\xi)h(\xi)} = \frac{\xi}{n} - 1, \quad if \quad n \leq -1 \]
\[ \int_{\Delta_k}^{\Delta_{k+1}} \frac{d\xi}{r(\xi)h(\xi)} = \frac{\xi}{n} - 1, \quad if \quad n \geq 1. \] (4.3)

Lemma 4.2. Let \( B < \infty \) (see (2.33)), \( x \in \mathbb{R} \), \( p \in (1, \infty) \) and
\[ \theta_p(x) = \left[ \int_{-\infty}^{x} v(t)^p dt \right]^{1/p} \cdot \left[ \int_{x}^{\infty} u(\xi)^{p'} d\xi \right]^{1/p'}, \quad p' = \frac{p}{p-1}. \] (4.4)
Then we have the inequalities
\[ c^{-1}h(x)d(x) \leq \theta_p(x) \leq \begin{cases} B^{1/p} \text{sup}_{t \leq x} (h(t)d(t))^{1/p'}, & \text{if } x \geq 0 \\ B^{1/p'} \text{sup}_{t \leq x} (h(t)d(t))^{1/p}, & \text{if } x \leq 0 \end{cases} \] (4.5)

Proof. Let \( p \in (1, 2], \gamma \in (0, 1] \) (the number \( \gamma \) will be chosen later). Now we apply Theorems 2.1 and 2.3.

\[ \theta_p(x) \leq \left[ \int_{-\infty}^{x} v(t)^p \, dt \right]^{1/p} \cdot u(x)^\gamma \cdot \left[ \int_{x}^{\infty} u(\xi)^{(1-\gamma)p'} \, d\xi \right]^{1/p'} \]
\[ \leq \left[ \int_{-\infty}^{x} \rho(t)^\gamma \cdot v(t)^{(1-\gamma)p} \, dt \right]^{1/p'} \cdot \left[ \int_{x}^{\infty} u(\xi)^{(1-\gamma)p'} \, d\xi \right]^{1/p'} \]
\[ = \left[ \int_{-\infty}^{x} \rho(t)^{1+\gamma-p} \exp \left( -\frac{1-\gamma}{2} \int_{t}^{x} \frac{ds}{r(s)\rho(s)} \right) \cdot \exp \left( -\frac{1-\gamma}{2} \int_{x}^{\infty} \frac{ds}{r(s)\rho(s)} \right) \, dt \right]^{1/p'} \]
\[ \cdot \left[ \int_{x}^{\infty} \rho(\xi)^{1-2p'} \exp \left( -\frac{1-\gamma}{2} \int_{x}^{\infty} \frac{ds}{r(s)\rho(s)} \right) \, d\xi \right]^{1/p'} \]
\[ = \left[ \int_{-\infty}^{x} \rho(t)^{1+\gamma-p} \exp \left( -\frac{1-\gamma}{2} \int_{t}^{x} \frac{ds}{r(s)\rho(s)} \right) \, dt \right]^{1/p} \]
\[ \cdot \left[ \int_{x}^{\infty} \rho(\xi)^{1-2p'} \exp \left( -\frac{1-\gamma}{2} \int_{x}^{\infty} \frac{ds}{r(s)\rho(s)} \right) \, d\xi \right]^{1/p'} \] (4.6)

Let \( \gamma_1 \) be the solution of the equation
\[ \frac{1+\gamma}{2} p = \frac{1-\gamma}{2} p' \quad \Rightarrow \quad \gamma := \gamma_1 = \frac{p' - p}{p' + p} \]

For \( \gamma = \gamma_1 \) inequality (4.6) takes the form
\[ \theta_p(x) \leq \left[ \int_{-\infty}^{x} \rho(t) \exp \left( -(p - 1) \int_{t}^{x} \frac{ds}{r(s)\rho(s)} \right) \, dt \right]^{1/p} \]
\[ \cdot \left[ \int_{x}^{\infty} \rho(\xi) \exp \left( -\int_{x}^{\infty} \frac{ds}{r(s)\rho(s)} \right) \, d\xi \right]^{1/p'} := \left( J_1(x) \right)^{1/p} \cdot \left( J_2(x) \right)^{1/p'}. \] (4.7)

Let us estimate \( J_1(x) \) and \( J_2(x) \). We only consider the case \( x \geq 0 \) because the case \( x \leq 0 \) is treated in a similar way. Below we use the properties of an \( \mathbb{R}(x, d) \)-covering of \( \mathbb{R} \), inequalities
and (2.14), and equalities (4.3):

\[ J_1(x) = \int_{-\infty}^{x} \rho(t) \exp \left( -\left( p - 1 \right) \int_{t}^{x} \frac{ds}{r(s)\rho(s)} \right) dt \]

\[ = \sum_{h=-\infty}^{-1} \int_{\Delta_n} \rho(t) \exp \left( -\left( p - 1 \right) \int_{t}^{x} \frac{ds}{r(s)\rho(s)} \right) dt \]

\[ \leq c \sum_{h=-\infty}^{-1} h(x_n)d(x_n) \exp \left( -\frac{p}{2} \int_{\Delta_n}^{+1} \frac{ds}{r(s)h(s)} \right) \]

\[ \leq cB \sum_{n=-\infty}^{-1} \exp \left( -\frac{p}{2} (|n| - 1) \right) = cB, \]  

(4.8)

\[ J_2(x) = \int_{x}^{\infty} \rho(\xi) \exp \left( -\int_{x}^{\xi} \frac{ds}{r(s)\rho(s)} \right) d\xi = \sum_{n=1}^{\infty} \int_{\Delta_n} \rho(\xi) \exp \left( -\int_{x}^{\xi} \frac{ds}{r(s)\rho(s)} \right) d\xi \]

\[ \leq c \sum_{n=1}^{\infty} h(x_n)d(x_n) \exp \left( -\frac{1}{2} \int_{\Delta_n}^{1} \frac{ds}{r(s)\rho(s)} \right) \]

\[ \leq c \sup_{t \geq x} \left( h(t)d(t) \right) \sum_{n=1}^{\infty} \exp \left( -\frac{n-1}{2} \right) = c \sup_{t \geq x} \left( h(t)d(t) \right). \]  

(4.9)

Thus, for \( p \in (1, 2] \), the upper estimate in (4.5) follows from (4.8)–(4.9). Let \( p \in (2, \infty) \), \( \gamma \in (0, 1) \) (the number \( \gamma \) will be chosen later). Now we apply Theorems 2.1 and 2.3.

\[
\theta_p(x) = \left[ \int_{-\infty}^{x} v(t)^p dt \right]^{1/p} \cdot \left[ \int_{x}^{\infty} u(\xi)^{p'} d\xi \right]^{1/p'}
\]

\[
\leq \left[ \int_{-\infty}^{x} v(t)^{(1-\gamma)p} dt \right]^{1/p} \cdot v(x)^{\gamma} \cdot \left[ \int_{x}^{\infty} u(\xi)^{p'} d\xi \right]^{1/p'}
\]

\[
\leq \left[ \int_{-\infty}^{x} v(t)^{(1-\gamma)p} dt \right]^{1/p} \cdot \left[ \int_{x}^{\infty} \rho(\xi)^{\gamma p'} \cdot u(\xi)^{(1-\gamma)p'} d\xi \right]^{1/p'}
\]

\[
\leq \left[ \int_{-\infty}^{x} \rho(t)^{(1-\gamma)p} \cdot \exp \left( -\frac{1-\gamma}{2} \int_{t}^{x} \frac{ds}{r(s)\rho(s)} \right) \cdot \exp \left( \frac{1-\gamma}{2} \int_{x}^{\infty} \frac{ds}{r(s)\rho(s)} \right) dt \right]^{1/p}
\]

\[
\cdot \left[ \int_{x}^{\infty} \rho(\xi)^{(1-\gamma)p'} \cdot \exp \left( -\frac{1-\gamma}{2} \int_{\xi}^{\infty} \frac{ds}{r(s)\rho(s)} \right) \cdot \exp \left( \frac{1-\gamma}{2} \int_{x}^{\infty} \frac{ds}{r(s)\rho(s)} \right) d\xi \right]^{1/p'}
\]

\[
\leq \left[ \int_{-\infty}^{x} \rho(t)^{(1-\gamma)p} \exp \left( -\frac{1-\gamma}{2} \int_{t}^{x} \frac{ds}{r(s)\rho(s)} \right) dt \right]^{1/p}
\]

\[
\cdot \left[ \int_{x}^{\infty} \rho(\xi)^{(1-\gamma)p'} \exp \left( -\frac{1-\gamma}{2} \int_{\xi}^{\infty} \frac{ds}{r(s)\rho(s)} \right) d\xi \right]^{1/p'}
\]

(4.10)

Let now \( \gamma \) be the solution \( \gamma_2 \) of the equation

\[
\frac{1-\gamma}{2} p = \frac{1+\gamma}{2} p' \quad \Rightarrow \quad \gamma := \gamma_2 = \frac{p - p'}{p + p'}.
\]
For $\gamma = \gamma_2$ inequality (4.10) takes the form

$$
\theta_p(x) \leq \left[ \int_{-\infty}^{x} \rho(t) \exp \left( - \int_{t}^{x} \frac{ds}{r(s)\rho(s)} \right) dt \right]^{1/p} \cdot \left[ \int_{x}^{\infty} \rho(\xi) \exp \left( -(p' - 1) \int_{x}^{\xi} \frac{ds}{r(s)\rho(s)} \right) d\xi \right]^{1/p'}.
$$

(4.11)

That (4.11) implies the upper estimate in (4.5) can be proved similarly to the proof of the same estimate from (4.7), and therefore we omit the proof. It remains to obtain the lower estimate in (4.5). The following inequality follows from (2.14) and (2.26):

$$
\theta_p(x) \geq \left[ \int_{x-d(x)}^{x} v(t)^p dt \right]^{1/p} \cdot \left[ \int_{x}^{\infty} u(t)^{p'} dt \right]^{1/p'},
$$

$$
\geq c^{-1} v(x) d(x)^{1/p} \cdot c^{-1} u(x) d(x)^{1/p'} = c^{-1} \rho(x) d(x) \geq c^{-1} h(x) d(x).
$$

Corollary 4.3. Let $p \in (1, \infty)$ and $B < \infty$ (see (2.33)). Then $\theta_p(x) \to 0$ as $|x| \to \infty$ if and only if condition (3.1) holds.

Proof. This is an immediate consequence of (4.5). \hfill \Box

Corollary 4.4. Let $p \in (1, \infty)$ and $B < \infty$ (see (2.33)). Suppose that condition (3.1) holds, $N \geq 1$ and

$$
\theta_p^{(+)}(x, N) = \left[ \int_{x}^{N} v(t)^p dt \right]^{1/p} \cdot \left[ \int_{x}^{\infty} u(\xi)^{p'} d\xi \right]^{1/p'}, \quad x \geq N,
$$

(4.12)

$$
\theta_p^{(-)}(x, N) = \left[ \int_{-\infty}^{x} v(t)^p dt \right]^{1/p} \cdot \left[ \int_{x}^{-N} u(\xi)^{p'} d\xi \right]^{1/p'}, \quad x \leq -N.
$$

Then

$$
\theta_p^{(-)}(x, N) \to 0, \quad \theta_p^{(+)}(x, N) \to 0 \quad \text{as} \quad N \to \infty.
$$

(4.14)

Proof. Now we use (4.5):

$$
0 < \theta_p^{(+)}(x, N) \leq \sup_{x \geq N} \theta_p^{(+)}(x, N) \leq \sup_{x \geq N} \theta_p(x)
$$

$$
\leq cB^{1/p} \sup_{t \geq N} (h(t) d(t))^{1/p'} \to 0 \quad \text{as} \quad N \to \infty \quad \Rightarrow \quad \text{(4.14)}.
$$

The second relation of (4.14) can be checked in a similar way. \hfill \Box
Let us now check 3). The following relations are obvious:

\[
\sup_{g \in W} \int_{|x| \geq N} |g(t)|^p dt = \sup_{f \in F} \int_{|x| \geq N} |(G_2 f)(x)|^p dx \\
\leq 2 \sup_{f \in F} \max \left\{ \int_{-\infty}^{-N} |(G_2 f)(x)|^p dx, \int_{N}^{\infty} |(G_2 f)(x)|^p dx \right\}.
\]

Denote

\[
T_1(N) = \sup_{f \in F} \int_{-\infty}^{-N} |(G_2 f)(x)|^p dx,
\]

\[
T_2(N) = \sup_{f \in F} \int_{N}^{\infty} |(G_2 f)(x)|^p dx.
\]

To prove 3), it is enough to verify that

\[
T_1(N) \to 0, \quad T_2(N) \to 0 \quad \text{as} \quad N \to \infty.
\] (4.17)

Let us check (4.17) for \(T_2(N)\). Now we use the definition of the set \(F\), Theorem 2.32 and Corollary 4.14.

\[
T_2(N) = \sup_{f \in F} \int_{N}^{\infty} |(G_2 f)(x)|^p dx \leq \|G_2\|_{L_p(N, \infty) \to L_p(N, \infty)}^p \sup_{f \in F} \|f\|_p^p
\]

\[
\leq c(p) \sup_{x \geq N} \left[ \left( \int_{N}^{\infty} v(t)^p dt \right)^{1/p} \cdot \left( \int_{x}^{\infty} u(\xi)^p d\xi \right)^{1/p} \right]^p \sup_{f \in F} \|f\|_p^p
\]

\[
\leq c(p) \sup_{x \geq N} \|\theta_p^{(+)}(x, N)\|^p \to 0 \quad \text{as} \quad N \to \infty.
\]

Let us go to \(T_1(N)\). First consider the value \((G_2 f)(x)\) for \(x \leq -N\) and \(f \in F\):

\[
(G_2 f)(x) = v(x) \int_{x}^{\infty} u(\xi) f(\xi) d\xi = v(x) \int_{x}^{-N} u(\xi) f(\xi) d\xi + v(x) \int_{-N}^{\infty} u(\xi) f(\xi) d\xi
\]

\[
:= (P_N f)(x) + (\hat{P}_N f)(x).
\] (4.18)

Here

\[
(P_N f)(x) = v(x) \int_{x}^{-N} u(\xi) f(\xi) d\xi, \quad x \leq -N, \quad f \in F,
\]

\[
(\hat{P}_N f)(x) = v(x) \int_{-N}^{\infty} u(\xi) f(\xi) d\xi, \quad x \leq -N, \quad f \in F.
\] (4.20)

The following relations are obvious:

\[
T_1(N) = \sup_{f \in F} \int_{-\infty}^{-N} |(G_2 f)(x)|^p dx = \sup_{f \in F} \int_{-\infty}^{-N} |(P_N f)(x) + (\hat{P}_N f)(x)|^p dx
\]

\[
\leq 2^p \sup_{f \in F} \left[ \int_{-\infty}^{-N} |(P_N f)(x)|^p dx + \int_{-\infty}^{-N} |(\hat{P}_N f)(x)|^p dx \right]
\]

\[
\leq c(p) \left[ \sup_{f \in F} \int_{-\infty}^{-N} |(P_N f)(x)|^p dx + \sup_{f \in F} \int_{-\infty}^{-N} |(\hat{P}_N f)(x)|^p dx \right]
\]

\[
= c(p)[\tilde{T}_1(N) + \hat{T}_1(N)].
\] (4.21)
Here
\[
\hat{T}_1(N) = \sup_{f \in F} \int_{-\infty}^{-N} |(\hat{P}_N f)(x)|^p dx, \quad (4.22)
\]
\[
\hat{T}_1(N) = \sup_{f \in F} \int_{-\infty}^{-N} |(\hat{P}_N f)(x)|^p dx. \quad (4.23)
\]
Clearly, \(T_1(N)\) satisfies (4.17) if
\[
\tilde{T}_1(N) \to 0, \quad T_2(N) \to 0 \quad \text{as} \quad N \to \infty. \quad (4.24)
\]

To prove the first relation of (4.24), we use the definition of the set \(F\), Theorem 2.32 and Corollary 4.4:
\[
\tilde{T}_1(N) = \sup_{f \in F} \int_{-\infty}^{-N} |(\tilde{P}_N f)(x)|^p dx \leq \|\tilde{P}_N\|_{L^p(-\infty,-N)} \cdot \sup_{f \in F} \|f\|^p_{L^p(-\infty,-N)} \leq \varepsilon.
\]

To prove the second relation of (4.24), we use the definition of the set \(F\), H"older’s inequality and Corollary 4.3:
\[
\hat{T}_1(N) = \sup_{f \in F} \int_{-\infty}^{-N} |(\hat{P}_N f)(x)|^p dx \leq \sup_{f \in F} \left( \int_{-\infty}^{\infty} u(\xi)^p d\xi \right)^{1/p} \cdot \sup_{f \in F} \|f\|^p_{L^p(-\infty,-N)} \leq \varepsilon.
\]

Thus relation (4.17) holds, and therefore condition 3) is satisfied.

Verification of condition 2). According to (2.31), it is enough to show that
\[
\lim_{\delta \to 0} \sup_{f \in \mathcal{K}} \|((G_i f)(\cdot + t) - (G_i f)(\cdot))\|_p = 0, \quad i = 1, 2. \quad (4.25)
\]
Both equalities of (4.25) are checked in the same way; therefore, below we only consider the case \(i = 2\). Furthermore, equality (4.25) will be prove as soon as we find \(\delta = \delta(\varepsilon) \in (0, 1]\) for a given \(\varepsilon > 0\) such that
\[
\sup_{f \in \mathcal{K}} \|((G_2 f)(\cdot + t) - (G_2 f)(\cdot))\|_p \leq \varepsilon. \quad (4.26)
\]
Thus, let \( \varepsilon > 0 \) be given. Set \( N \geq 1 \) (the choice of \( N \) will be made more precise later). Then for \( f \in \mathcal{K} \) we have

\[
\|(G_2f)(\cdot + t) - (G_2f)(\cdot)\|_p^p = \|(G_2f)(\cdot + t) - (G_2f)(\cdot)\|_{L^p(-N,N)}^p
\]

\[
+ \|(G_2f)(\cdot + t) - (G_2f)(\cdot)\|_{L^p(-\infty,-N)} + \|(G_2f)(\cdot + t) - (G_2f)(\cdot)\|_{L^p(N,\infty)}^p
\]

\[
\leq \|(G_2f)(\cdot + t) - (G_2f)(\cdot)\|_{L^p(-N,N)}^p + 2\|G_2f\|_{L^p(-\infty,-N+1)}^p
\]

\[
+ 2\|G_2f\|_{L^p(N-1,\infty)}^p.
\]  \tag{4.27}

By 3), for the given \( \varepsilon > 0 \) there exists \( N_0 = N_0(\varepsilon) \) such that

\[
\sup_{f \in \mathcal{K}} \|G_2f\|_{L^p(-\infty,-N_0+1)}^p + \sup_{f \in \mathcal{K}} \|G_2f\|_{L^p(N_0-1,\infty)}^p \leq \frac{\varepsilon_p}{4}.
\]

Therefore, for \( N = N_0 \) inequality \((4.27)\) can be continued as follows:

\[
\|(G_2f)(\cdot + t) - (G_2f)(\cdot)\|_p^p \leq \|(G_2f)(\cdot + t) - (G_2f)(\cdot)\|_{L^p(-N_0,N_0)}^p + \frac{\varepsilon_p^p}{2}.
\]  \tag{4.28}

Throughout the sequel, \( |x| \leq N_0 \) and \( |t| \leq \delta \) (the number \( \delta \) will be chosen later). Let us continue estimate \((4.28)\). We have

\[
|(G_2f)(x + t) - (G_2f)(x)| = \left| v(x + t) \int_{x + t}^{\infty} u(\xi) f(\xi) d\xi - v(x) \int_{x}^{\infty} u(\xi) f(\xi) d\xi \right|
\]

\[
\leq |v(x + t) - v(x)| \cdot \left| \int_{x}^{\infty} u(\xi) f(\xi) d\xi \right| + v(x + t) \left| \int_{x}^{x + t} u(\xi) f(\xi) d\xi \right|
\]

\[
:= (Af)(x,t) + (Bf)(x,t).
\]  \tag{4.29}

Here

\[
(Af)(x,t) = |v(x + t) - v(x)| \cdot \left| \int_{x}^{\infty} u(\xi) f(\xi) d\xi \right|, \quad f \in \mathcal{K},
\]  \tag{4.30}

\[
(Bf)(x,t) = v(x + t) \cdot \left| \int_{x}^{x + t} u(\xi) f(\xi) d\xi \right|, \quad f \in \mathcal{K}.
\]  \tag{4.31}

Let us introduce the numbers

\[
\delta_1 = \min_{x \in [-N_0,N_0]} d(x), \quad \eta = \sup_{x \in [-N_0,N_0]} \sup_{|t| \leq \delta} \left| \int_{x}^{x + t} \frac{d\xi}{r(\xi) h(\xi)} \right|.
\]  \tag{4.32}

From absolute continuity of the Lebesgue integral, it follows that given \( \varepsilon > 0 \), one can choose \( \delta = \delta(\varepsilon) \) so small that the following inequalities hold:

\[
\delta \leq \delta_1, \quad \eta \leq \frac{\varepsilon}{\alpha}.
\]  \tag{4.33}

(Here \( \alpha \) is a positive number to be chosen later.)

In the following estimate of \((Af)(x,t)\), we use \((4.33)\), the equalities (see \([II]\))

\[
\frac{v'(x)}{v(x)} = \frac{1 + r(x) \rho'(x)}{2 r(x) \rho(x)}, \quad \frac{u'(x)}{u(x)} = \frac{1 - r(x) \rho'(x)}{2 r(x) \rho(x)}, \quad x \in \mathbb{R},
\]  \tag{4.34}
and estimates (2.17), (2.25) and (2.26):

\[(Af)(x,t) = |v(x + t) - v(x)| \cdot \left| \int_{x}^{\infty} u(\xi)f(\xi)d\xi \right| \leq \int_{x}^{x+t} \left| \frac{u(\xi)f(\xi)}{v(x)} \cdot \frac{v'(s)}{v(s)} \cdot ds \right| \cdot |(G_2 f)(x)| \]

\[= \left| \int_{x}^{x+t} \frac{2}{\rho(s)} \cdot \frac{\varepsilon^2}{r(s)} \cdot ds \cdot |(G_2 f)(x)| \right| \leq \frac{c\varepsilon}{\alpha} \cdot |(G_2 f)(x)|. \quad (4.35)\]

Furthermore, in the estimate of \((Bf)(x,t)\) we use (2.25), (2.26), (4.33), Hölder’s inequality and the definition of the set \(K\):

\[(Bf)(x,t) = v(x + t) \left| \int_{x}^{x+t} u(\xi)f(\xi)d\xi \right| \leq v(x + t)u(x + t) \cdot \rho(x) \left| \int_{x}^{x+t} \frac{u(\xi)}{u(x)} \cdot \frac{u(x)}{u(x + t)} \cdot |f(\xi)|d\xi \right| \]

\[\leq cp(x) \left| \int_{x}^{x+t} |f(\xi)|d\xi \right| \leq cp(x)|t|^{1/p'} \cdot \|f\|_p \]

\[\leq cp(x)\delta^{1/p'} \leq c\left( \max_{|x| \leq N_0} \rho(x) \right) \delta^{1/p'}. \quad (4.36)\]

The following estimates are derived from (4.35), (4.36), the definition of the set \(K\) and (2.34):

\[|(G_2 f)(x + t) - (G_2 f)(x)| \leq (Af)(x,t) + (Bf)(x,t) \]

\[\leq \frac{c\varepsilon}{\alpha} |(G_2 f)(x)| + c\left( \max_{|x| \leq N_0} \rho(x) \right) \delta^{1/p'} \Rightarrow \]

\[\|(G_2 f)(\cdot + t) - (G_2 f)(\cdot)\|_{L_p(-N_0,N_0)} \leq \frac{c\varepsilon}{\alpha} \|(G_2 f)\|_p + c\left( \max_{|x| \leq N_0} \rho(x) \right) N_0^{1/p} \cdot \delta^{1/p'} \]

\[\leq \frac{cB}{\alpha} \varepsilon + c\left( \max_{|x| \leq N_0} \rho(x) \right) N_0^{1/p} \delta^{1/p'}. \]

Set \(\alpha = 2^{1+1/p} \cdot cB\) and, if necessary, choose a smaller \(\delta\) so that the following inequality holds:

\[c\left( \max_{|x| \leq N_0} \rho(x) \right) \cdot N_0^{1/p} \delta^{1/p'} \leq \frac{\varepsilon}{2^{1+1/p}}. \]

Then we get the estimates

\[\|(G_2 f)(\cdot + t) - (G_2 f)(\cdot)\|_{L_p(-N_0,N_0)} \leq \frac{\varepsilon}{2^{1/p}} \Rightarrow (\text{see (4.23)}),\]

\[\|(G_2 f)(\cdot + t) - (G_2 f)(\cdot)\|_p \leq \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p \Rightarrow (\text{1.25}) \Rightarrow 2).\]

The theorem is proved.

\[\square\]

**Proof of Theorem 3.4** We need the following assertion.
Lemma 4.5. Suppose that condition (3.1) holds. Then $B < \infty$ (see (2.33)).

Proof. From (3.1) and (2.14) it follows that $\rho(x)d(x) \to 0$ as $|x| \to \infty$. Hence there is $x_0 \gg 1$ such that $\rho(x)d(x) \leq 1$ for $|x| \geq x_0$. By Lemma 4.10, the function $\rho(x)d(x)$ is continuous for $x \in \mathbb{R}$ and is therefore bounded on $[-x_0, x_0]$. Hence $S < \infty$ (see (2.35)), and therefore $B < \infty$ (see (2.33)).

Let us now go to the assertion of the theorem. Since $G(x, t) = G(t, x)$ for all $t, x \in \mathbb{R}$ (see (2.9)), the operator $G : L_2 \to L_2$ is symmetric and bounded (see Lemma 4.5 and (2.34)). Hence the operator $G$ is self-adjoint and, by Theorem 3.1, compact. Furthermore, estimates (3.3) follow from positivity of $G$ which, in turn, will be proved below. Towards this end, we need the following two lemmas.

Lemma 4.6. The equalities

$$\lim_{|x| \to \infty} \frac{u(x)}{v(x)} \cdot \int_{-\infty}^{x} v(t)^2 dt = 0,$$

$$\lim_{|x| \to \infty} \frac{v(x)}{u(x)} \cdot \int_{x}^{\infty} u(t)^2 dt = 0$$

hold if and only if condition (3.1) is satisfied.

Proof of Lemma 4.6. Necessity. Both equalities are checked in the same way, and therefore below we only consider (4.38). Below $x \in \mathbb{R}$, and we apply estimates (2.25) and (2.14):

$$I(x) = \frac{v(x)}{u(x)} \int_{x}^{\infty} u^2(t) dt \geq \frac{v(x)}{u(x)} \cdot \int_{x}^{x+d(x)} u^2(t) dt$$

$$= \frac{v(x)}{u(x)} \int_{x}^{x+d(x)} \left( \frac{u(t)}{u(x)} \right)^2 \cdot u^2(x) dt \geq c^{-1} \rho(x) d(x) \geq c^{-1} h(x) d(x) > 0.$$ 

It remains to refer to (4.38).

Proof of Lemma 4.6. Sufficiency. From (2.7) we obtain the equality

$$I(x) = \frac{v(x)}{u(x)} \cdot \int_{x}^{\infty} u^2(t) dt = \int_{x}^{\infty} \rho(t) \exp \left( - \int_{x}^{t} \frac{d\xi}{r(\xi) \rho(\xi)} \right) dt.$$ (4.39)
Let \( x \to \infty \). Below we use (4.39), properties of an \( \mathbb{R}(x, d) \)-covering of \([x, \infty)\), (2.26) and (4.3):

\[
I(x) = \sum_{n=1}^{\infty} \int_{\Delta_n} \rho(t) \exp \left( -\int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \leq c \sum_{n=1}^{\infty} \rho(x_n) d(x_n) \exp \left( -\int_{\Delta_i}^{\Delta_i^+} \frac{d\xi}{r(\xi)\rho(\xi)} \right)
\]

\[
\leq c \sum_{n=1}^{\infty} h(x_n) d(x_n) \exp \left( -\frac{1}{2} \int_{\Delta_i}^{\Delta_i^+} \frac{d\xi}{r(\xi)h(\xi)} \right)
\]

\[
\leq c \sup_{t \geq x} (h(t) d(t)) \sum_{n=1}^{\infty} \exp \left( -\frac{n-1}{2} \right) = c \sup_{t \geq x} (h(t) d(t)).
\]

The latter inequality and (3.1) imply (4.38) (as \( x \to \infty \)). Let now \( x \to -\infty \). Fix \( \varepsilon > 0 \) and choose \( \ell = \ell(\varepsilon) \gg 1 \) so that the following estimate will hold:

\[
4c_0 B \ell^2 \cdot \exp \left( -\frac{\ell - 1}{2} \right) \leq \varepsilon, \quad c_0 = \sum_{k=1}^{\infty} \exp \left( -\frac{k-1}{2} \right).
\]  

(4.40)

Consider the segments \( \{\Delta_k\}_{k=1}^{\infty} \) from an \( \mathbb{R}(x, d) \)-covering of \([x, \infty)\). Let us show that

\[
\lim_{x \to -\infty} \Delta_i^+ = -\infty.
\]  

(4.41)

Assume the contrary: there exists \( c > -\infty \) such that \( \Delta_i^+ \geq c \) as \( x \to -\infty \). Then by (2.10) and (4.3), we have

\[
\ell = \int_{\Delta_i^+}^{\Delta_i} \frac{d\xi}{r(\xi)h(\xi)} \geq \int_{\Delta_i}^{c} \frac{d\xi}{r(\xi)h(\xi)} \geq \frac{1}{2} \int_{\Delta_i}^{c} \frac{d\xi}{r(\xi)\rho(\xi)} \to \infty \quad \text{as} \quad x \to -\infty,
\]

a contradiction, so (4.41) is proved.

Let us now choose \( x_1(\varepsilon) \) and \( x_2(\varepsilon) \) so that the following inequalities will hold:

\[
4e^2 c_0 \cdot \ell \cdot (h(t) d(t)) \leq \varepsilon \quad \text{for} \quad t \leq -x_1(\varepsilon),
\]

\[
\Delta_i^+ \leq -x_1(\varepsilon) \quad \text{for} \quad x \leq -x_2(\varepsilon).
\]  

(4.42)

(4.43)

Let \( x_0 = \max\{x_1(\varepsilon), x_2(\varepsilon)\} \). Below for \( x \leq x_0 \) we use (4.39), properties of an \( \mathbb{R}(x, d) \)-covering of \( \mathbb{R} \), (2.26), (4.42), (4.43) and (4.40):

\[
I(x) = \sum_{n=1}^{\infty} \int_{\Delta_n} \rho(t) \exp \left( -\int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right)
\]

\[
\leq 2e^2 \left\{ \sum_{n=1}^{\ell} h(x_n) d(x_n) \exp \left( -\frac{n-1}{2} \right) + \sum_{n=\ell+1}^{\infty} h(x_n) d(x_n) \exp \left( -\frac{n-1}{2} \right) \right\}
\]

\[
\leq 2e^2 c_0 \sup_{t \leq \Delta_i^+} (h(t) d(t)) + 2e^2 c_0 B \exp \left( -\frac{\ell - 1}{2} \right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

The obtained estimates lead to (4.38). \( \square \)
**Lemma 4.7.** Let \( p \in (1, \infty) \), \( f \in L_p \) and \( y = Gf \). Then, if condition (3.1) holds, we have
\[
\lim_{|x| \to \infty} r(x)y'(x)y(x) = 0. \tag{4.44}
\]

**Proof.** From (3.1) and Lemma 4.5 it follows that \( B < \infty \). Let, for example, \( x \to \infty \) (the case \( x \to -\infty \) is treated in a similar way). Below we use the definition and properties of the operator \( G : L_2 \to L_2 \) (see (2.28)), (2.17), (2.29)–(2.32) and the Schwarz inequality:
\[
\begin{align*}
  r(x)|y'(x)| \cdot |y(x)| & \leq (G[f])(x) \cdot \left[ r(x) \frac{d}{dx}(Gf)(x) \right] \\
  & \leq (G[f])(x) \cdot \left[ r(x) \frac{|u'(x)|}{u(x)} \cdot (G_1[f])(x) + \frac{r(x)v'(x)}{v(x)}(G_2[f])(x) \right] \\
  & \leq \frac{[(G[f])(x)]^2}{\rho(x)} \leq c \frac{[(G_1[f])(x)]^2 + [(G_2[f])(x)]^2}{\rho(x)} \\
  & \leq c \left\{ \frac{u(x)}{v(x)} \cdot \int_{-\infty}^{x} u^2(t)dt + \frac{v(x)}{u(x)} \cdot \int_{x}^{\infty} u^2(t)dt \right\} \cdot \|f\|_2^2.
\end{align*}
\]
It remains to apply Lemma 4.6.
\[\square\]

Let us now complete the proof of the theorem. Below we assume that \( f \in L_2 \) and \( y := Gf \).

Then, obviously, \( f = L_2 y \), and we have the relations
\[
\begin{align*}
  \int_{-\infty}^{\infty} (Gf)(x) \cdot \bar{f}(x)dx &= \int_{-\infty}^{\infty} y(x)\overline{(L_2y)(x)}dx = \lim_{b \to \infty} \int_{a}^{b} y(x)[-(r(x)y'(x))' + q(x)y(x)]dx \\
  &= \lim_{b \to \infty} \int_{a}^{b} y(x)[-(r(x)y'(x))'] + q(x)y(x)dx \\
  &= \lim_{b \to \infty} \int_{a}^{b} \left[-r(x)y''(x)y(x) + \int_{a}^{b} (r(x)y'(x))^2 + q(x)|y(x)|^2 \right]dx \\
  &= \int_{-\infty}^{\infty} (r(x)y'(x))^2 + q(x)|y(x)|^2 dx \geq 0.
\end{align*}
\]
\[\square\]

**Proof of Corollary 3.7.** The following relations are based on Theorem 2.1
\[
\begin{align*}
  r(x)v'(x) - r(t)v'(t) &= \int_{t}^{x} q(\xi)v(\xi)d\xi, \quad t \leq x \in \mathbb{R} \\
  r(t)u'(t) - r(x)u'(x) &= \int_{x}^{t} q(\xi)u(\xi)d\xi, \quad t \geq x \in \mathbb{R} \\
  r(x)v'(x) &\geq \int_{-\infty}^{x} q(\xi)v(\xi)d\xi, \quad -r(x)u'(x) \geq \int_{x}^{\infty} q(\xi)u(\xi)d\xi
\end{align*}
\]
\[ 1 = r(x)[v'(x)u(x) - u'(x)v(x)] \geq u(x) \int_{-\infty}^{x} q(t)v(t)dt + v(x) \int_{x}^{\infty} q(t)u(t)dt \]

\[ = \int_{-\infty}^{\infty} q(t)G(x,t)dt. \]

Below we continue the last inequality using (2.9), (2.14), (2.19) and (2.26):

\[ 1 \geq \int_{x-d(x)}^{x+d(x)} q(t)G(x,t)dt = \int_{x-d(x)}^{x+d(x)} \sqrt{\rho(t)\rho(x)} \exp \left( -\frac{1}{2} \left| \int_{x}^{t} \frac{d\xi}{r(\xi)\rho(\xi)} \right| \right) dt \]

\[ \geq c^{-1}h(x)\exp \left( -\frac{1}{4} \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{r(\xi)h(\xi)} \right) \cdot \int_{x-d(x)}^{x+d(x)} q(t)dt = c^{-1}h(x) \int_{x-d(x)}^{x+d(x)} q(t)dt. \quad (4.45) \]

Equation (1.1) is correctly solvable in \( L_p, p \in (1, \infty) \) since \( A > 0 \) (see Theorem 2.25), and from (4.45) it follows that

\[ c(A(x))^{-1} \geq h(x)d(x), \quad x \in \mathbb{R} \Rightarrow (3.1). \]

The assertion now follows from Theorem 3.1.

**Proof of Corollary 3.3.** Since \( q(x) \to \infty \) as \( |x| \to \infty \), condition (1.3) holds, and therefore all auxiliary functions are defined (see Lemmas 2.5 and 2.10). Let \( q(x) \geq 1 \) for \( |x| \geq x_1 \). Then by (2.22), there exists \( x_2 \gg x_1 \) such that for \( |x| \geq x_2 \) we have

\[ [x - d(x), x + d(x)] \cap [-x_1, x_1] = \emptyset. \quad (4.46) \]

Then for \( |x| \geq x_2 \) from (4.45) it follows that

\[ c \geq h(x) \int_{x-d(x)}^{x+d(x)} q(t)dt \geq h(x) \int_{x-d(x)}^{x+d(x)} 1dt = 2h(x)d(x) \Rightarrow \]

\[ \sup_{|x| \geq x_2} (h(x)d(x)) \leq 2c < \infty. \]

Since the function \( h(x)d(x) \) is bounded on \( [-x_2, x_2] \) (see the proof of Lemma 4.5), we have \( B < \infty \), and by Theorem 2.21 equation (1.1) is correctly solvable in \( L_p, p \in (1, \infty) \). Further, from (2.22) it follows that

\[ c(A(x))^{-1} \geq h(x)d(x), \quad |x| \geq x_2; \quad A(x) \to \infty \quad \text{as} \quad |x| \to \infty. \]

Hence condition (3.1) holds, and the assertion of the corollary follows from Theorem 3.1.

**Proof of Corollary 3.4.** We need the following fact whose proof is presented for the sake of completeness.

**Lemma 4.8.** Suppose that conditions (1.2)–(1.3) hold and \( r \equiv 1 \). Then equality (3.5) holds if and only if \( d(x) \to 0 \) as \( |x| \to \infty \) (see (2.15)).
Proof of Lemma 4.8. Necessity. Assume the contrary: equality (3.5) holds but \( \tilde{d}(x) \to 0 \) as \( |x| \to \infty \). This means that there exist \( \varepsilon > 0 \) and points \( \{x_n\}_{n=1}^\infty \) such that \( |x_n| \to \infty \) as \( n \to \infty \) and \( \tilde{d}(x_n) \geq \varepsilon \). This implies

\[
\frac{1}{\varepsilon} \geq \frac{1}{d(x_n)} = \frac{1}{2} \int_{x_n-\tilde{d}(x_n)}^{x_n+\tilde{d}(x_n)} q(t)dt \geq \frac{1}{2} \int_{x_n-\varepsilon}^{x_n+\varepsilon} q(t)dt, \quad n \geq 1.
\]

Thus equality (3.5) breaks down for \( a = \varepsilon \), a contradiction.

Proof of Lemma 4.8. Sufficiency. If \( \tilde{d}(x) \to 0 \) as \( |x| \to \infty \), then for any \( a \in (0, \infty) \) and for all \( |x| \gg 1 \), we have

\[
\frac{1}{2} \int_{x-a}^{x+a} q(t)dt \geq \frac{1}{2} \int_{x-\tilde{d}(x)}^{x+\tilde{d}(x)} q(t)dt = \frac{1}{d(x)} \Rightarrow (3.5).
\]

Let us now go to the corollary. For \( r \equiv 1 \), from (2.10) and (2.26) we obtain

\[
1 = \int_{x-d(x)}^{x+d(x)} \frac{dt}{h(t)} \leq c \frac{d(x)}{h(x)}
\]

\[
1 = \int_{x-d(x)}^{x+d(x)} \frac{dt}{h(t)} \leq c^{-1} \frac{d(x)}{h(x)} \quad \Rightarrow \quad h(x) \asymp d(x), \quad x \in \mathbb{R}.
\]

On the other hand, from (2.10) and (2.14), it follows that \( h(x) \asymp \rho(x) \asymp \tilde{d}(x), \quad x \in \mathbb{R} \). Putting this together, we obtain the main relations: \( h(x) \asymp d(x) \asymp \tilde{d}(x), \quad x \in \mathbb{R} \). Further, as \( m(a_0) > 0 \) for some \( a_0 \in (a, \infty) \), we conclude that equation (1.1) is correctly solvable in \( L_p, \quad p \in (1, \infty) \) by Theorem 2.24. We have \( h(x)d(x) \to 0 \) as \( |x| \to \infty \) if and only if \( \tilde{d}(x) \to 0 \) as \( |x| \to \infty \) since \( h(x)d(x) \asymp \tilde{d}^2(x), \quad x \in \mathbb{R} \). The assertion of the corollary now follows from Lemma 4.8.

Proof of Corollary 3.10. By Theorem 2.30 in all the following cases 1)–3), equation (1.1) is correctly solvable in \( L_p, \quad p \in (1, \infty) \). Let us show that in the same cases condition (3.1) holds, and thus by Theorem 3.1 our assertion will then be proved.

1) Let \( x \in \mathbb{R}, \Delta(x) = [x-d(x), x+d(x)] \). Below we use the Schwarz inequality and (2.19):

\[
2d(x) = \int_{\Delta(x)} \sqrt{\frac{r(t)h(t)}{r(t)h(t)}} dt \leq \left( \int_{\Delta(x)} r(t)h(t)dt \right)^{1/2} \cdot \left( \int_{\Delta(x)} \frac{dt}{r(t)h(t)} \right)^{1/2} = \left( \int_{\Delta(x)} r(t)h(t)dt \right)^{1/2} \Rightarrow
\]

\[
4d^2(x) \leq \int_{\Delta(x)} r(t)h(t)dt, \quad x \in \mathbb{R},
\]

(4.47)
Let \( \eta(x) = \sup_{t \in \Delta(x)} (r(t)h^2(t)) \). From (2.22) and (3.6) it follows that \( \eta(x) \to 0 \) as \( |x| \to \infty \).

Further, from (1.47) using (2.26), we obtain
\[
4d^2(x) \leq \int_{\Delta(x)} r(t)h^2(t) \cdot \frac{h'(x)}{h(t)} \frac{dt}{h(x)} \leq c\eta(x) \frac{d(x)}{h(x)} \Rightarrow 0 < h(x)d(x) \leq c\eta(x), \quad x \in \mathbb{R} \quad \Rightarrow (3.1).
\]

2) This assertion follows from 1) and (2.12), (3.7) and (3.6):
\[
r(x)h^2(x) = r(x)\varphi(x)\psi(x) \frac{\varphi(x)\psi(x)}{(\varphi(x) + \psi(x))^2} \leq r(x)\varphi(x)\psi(x), \quad x \in \mathbb{R}.
\]

3) From (2.22) it follows that \( d(x) \leq |x| \) for all \( |x| \gg 1 \). Hence \( 0 < h(x)d(x) \leq h(x)|x| \) for all \( |x| \geq 1 \), and therefore (3.1) holds because of (3.8).

Proof of Corollary 3.11. For \( x \in \mathbb{R} \), according to (2.19) and (2.20), we have
\[
1 = \int_{\Delta(x)} \frac{dt}{r(t)h(t)} \geq c^{-1}h(x) \int_{\Delta(x)} \frac{dt}{r(t)} \geq c^{-1} \frac{d(x)}{r_0}h(x) \Rightarrow 0 < h(x)d(x) \leq ch^2(x).
\]

Then \( B \leq ch_0^2 < \infty \), and condition (3.1) holds. The assertion follows from Theorems 2.21 and 3.1.

Proof of Theorem 3.13. Since \( \theta(x) \to 0 \) as \( |x| \to \infty \), we have \( r^{-1} \in \mathbb{L}_1(\mathbb{R}) \) and \( \theta < \infty \) (see (2.45)). We now need the following lemma.

Lemma 4.9. Suppose that conditions (1.2) and (2.1) hold and \( r^{-1} \in \mathbb{L}_1 \). Then we have the equality
\[
\rho(x) \leq \tau \int_{-\infty}^{x} \frac{dt}{r(t)} \cdot \int_{x}^{\infty} \frac{dt}{r(t)}, \quad x \in \mathbb{R}.
\]

Here
\[
\tau = \max \left\{ \left( \int_{-\infty}^{0} \frac{dt}{r(t)} \right)^{-1}, \left( \int_{0}^{\infty} \frac{dt}{r(t)} \right)^{-1} \right\}.
\]

Proof. From Theorem 2.1 it easily follows that
\[
u(x) = u(x) \int_{x}^{\infty} \frac{dt}{r(t)v^2(t)}; \quad v(x) = u(x) \int_{-\infty}^{x} \frac{dt}{r(t)u^2(t)}; \quad x \in \mathbb{R}.
\]

From (4.50) and (2.3) we now obtain
\[
\rho(x) = v^2(x) \int_{x}^{\infty} \frac{dt}{r(t)v^2(t)} \leq \int_{x}^{\infty} \frac{dt}{r(t)}, \quad x \in \mathbb{R},
\]
\[
\rho(x)u^2(x) \int_{-\infty}^{x} \frac{dt}{r(t)u^2(t)} \leq \int_{-\infty}^{x} \frac{dt}{r(t)}, \quad x \in \mathbb{R}.
\]
Hence
\[
\rho(x) = \begin{cases} 
\int_{x}^{\infty} \frac{dt}{r(t)}, & \text{if } x \geq 0 \\
\int_{-\infty}^{x} \frac{dt}{r(t)}, & \text{if } x \leq 0
\end{cases}
\] (4.51)

Estimate (4.48) follows from (4.51) and (4.49).

Further, from (2.23) we conclude that \( s(x) \leq |x| \) for all \(|x| \geq 1\), and therefore
\[
\rho(x)s(x) \leq \tau|x| \cdot \int_{-\infty}^{x} \frac{dt}{r(t)} \cdot \int_{x}^{\infty} \frac{dt}{r(t)} = \tau \theta(x), \quad |x| \geq 1.
\]
The latter inequality means that \( S \leq \tau \theta < \infty \). Hence equation (1.1) is correctly solvable in \( L_p, p \in (1, \infty) \) by Theorem 2.22. If \( \theta(x) \to 0 \) as \( |x| \to \infty \), then condition (3.2) holds, and the operator \( G : L_p \to L_p, p \in (1, \infty) \), is compact by Theorem 3.2.

**Proof of Theorem 3.14.** Below we follow the scheme of the proof of Corollary 3.10.1. Let \( x \in \mathbb{R}, \hat{\Delta}(x) = |x - s(x) + s(x)| \) (see (2.19)). From the Schwarz inequality and (2.19), we get
\[
2s(x) = \int_{\hat{\Delta}(x)} \sqrt{\frac{r(t)}{r(t)}} \frac{dt}{r(t)} \leq \left( \int_{\hat{\Delta}(x)} r(t) \frac{dt}{r(t)} \right)^{1/2} \left( \int_{\hat{\Delta}(x)} \frac{dt}{r(t)} \right)^{1/2} = \left( \int_{\hat{\Delta}(x)} r(t) \frac{dt}{r(t)} \right)^{1/2}, \quad x \in \mathbb{R} \Rightarrow
\]
\[
4s^2(x) \leq \int_{\hat{\Delta}(x)} r(t) \frac{dt}{r(t)}, \quad x \in \mathbb{R}. \quad (4.52)
\]
Further, since \( \nu < \infty \), we have \( r^{-1} \in L_1 \). Therefore by Lemma 4.19 we have estimate (4.48). This implies the inequality
\[
r(x)\rho^2(x) \leq cv(x), \quad x \in \mathbb{R}. \quad (4.53)
\]
Since \( \nu < \infty \), from (4.52), (4.53) and (4.54), we get
\[
4s^2(x) \leq \int_{\hat{\Delta}(x)} r(t) \frac{dt}{r(t)} \leq cv(x) \Rightarrow
\]
\[
s(x)\rho(x) \leq cv, \quad x \in \mathbb{R} \Rightarrow S \leq cv. \quad (4.54)
\]
From (4.54) and Theorem 2.22 it follows that equation (1.1) is correctly solvable in \( L_p, p \in (1, \infty) \). Let \( \nu(x) \to 0 \) as \( |x| \to \infty \). Then by (4.53) and (2.23), we also have \( \tilde{\eta}(x) \to 0 \) as \( |x| \to \infty \), where \( \tilde{\eta}(x) = \sup_{t \in \hat{\Delta}(x)} r(t) \rho^2(t) \). Hence
\[
4s^2(x) \leq \int_{\hat{\Delta}(x)} r(t) \frac{dt}{r(t)} \leq c\tilde{\eta}(x) s(x) \Rightarrow
\]
\[
\rho(x)s(x) \leq c\tilde{\eta}(x), \quad x \in \mathbb{R} \Rightarrow \lim_{|x| \to \infty} \rho(x)s(x) = 0.
\]
Thus the operator $G : L^p \to L^p$, $p \in (1, \infty)$, is compact by Theorem 3.2.

5. ADDITIONAL ASSERTIONS. EXAMPLE

Below we consider equation (1.1) with coefficients

$$r(x) = e^{\alpha|x|}, \quad q(x) = e^{\beta|x|}, \quad x \in \mathbb{R}$$  \hspace{1cm} (5.1)

where $\alpha$ and $\beta$ are any given real numbers. In what follows, for brevity we refer to it as equation (5.1).

Our goal in connection to (5.1) is to obtain for this equation a complete solution of problems 1)–II) and I)–III). As mentioned above, to study concrete equations (1.1), one needs assertions that allow us to obtain sharp by order two-sided estimates of the functions $h$ and $d$ (see Remark 2.29). Below we will see that getting such inequalities is a certain technical problem of local analysis. However, the statement of such a problem depends on the properties of the coefficients of equation (1.1). Therefore, here we restrict ourselves to considering statements “sufficient” for investigation of (5.1). (Cf. [4] where estimates of $h$ and $d$ were obtained for equations (1.1) with nonsmooth and oscillating coefficients $r$ and $q$.)

The next theorem contains a general method that guarantees obtaining estimates for $h$ and $d$. Note that this statement is a formalization of certain devices which were first used by Otelbaev for estimating his auxiliary functions (see [15]).

**Theorem 5.1.** Suppose that conditions (1.2) and (1.3) hold. For a given $x \in \mathbb{R}$ introduce functions in $\eta \geq 0$:

$$F_1(\eta) = \int_{x-\eta}^{x} \frac{dt}{r(t)} \cdot \int_{x-\eta}^{x} q(t)dt,$$

$$F_2(\eta) = \int_{x}^{x+\eta} \frac{dt}{r(t)} \cdot \int_{x}^{x+\eta} q(t)dt,$$

$$F_4(\eta) = \int_{x-\eta}^{x+\eta} \frac{dt}{r(t)h(t)}.$$  \hspace{1cm} (5.2) (5.3) (5.4)

Then the following assertions hold (see Lemmas 2.5 and 2.10):

1) the inequality $\eta \geq d_1(x)$ $(0 \leq \eta \leq d_1(x))$ holds if and only if $F_1(\eta) \geq 1$ ($F_1(\eta) \leq 1$);

2) the inequality $\eta \geq d_2(x)$ $(0 \leq \eta \leq d_2(x))$ holds if and only if $F_2(\eta) \geq 1$ ($F_2(\eta) \leq 1$);

3) the inequality $\eta \geq d(x)$ $(0 \leq \eta \leq d(x))$ holds if and only if $F_4(\eta) \geq 1$ ($F_4(\eta) \leq 1$).

The next theorem is an example of using Theorem 5.1.
Theorem 5.2. Suppose that the following conditions hold:

\[ r > 0, \quad q > 0, \quad r \in A^\text{loc}(\mathbb{R}), \quad q \in A^\text{loc}(\mathbb{R}) \]  

(5.5)

(here \( A^\text{loc}(\mathbb{R}) \) is the set of functions absolutely continuous on every finite interval of the real axis). Let, in addition,

\[ \kappa_1(x) \to 0, \quad \kappa_2(x) \to 0 \quad \text{as} \quad |x| \to \infty \]

where

\[ \kappa_1(x) = r(x) \sup_{|t| \leq 8\hat{d}(x)} \left| \int_x^{x+t} \frac{r'(\xi)}{r^2(\xi)} d\xi \right|, \quad x \in \mathbb{R}, \]  

\[ (5.6) \]

\[ \kappa_2(x) = \frac{1}{q(x)} \cdot \sup_{|t| \leq 8\hat{d}(x)} \left| \int_x^{x+t} q'(\xi) d\xi \right|, \quad x \in \mathbb{R}, \]  

\[ (5.7) \]

\[ \hat{d}(x) = \sqrt{\frac{r(x)}{q(x)}}, \quad x \in \mathbb{R}. \]  

\[ (5.8) \]

Then for all \( |x| \gg 1 \) each of the equations (2.11) has a unique finite positive solution \( d_1(x) \) and \( d_2(x) \), respectively, and we have (see (2.12), (2.19)):

\[ \lim_{|x| \to \infty} \frac{d_1(x)}{\hat{d}(x)} = \lim_{|x| \to \infty} \frac{d_2(x)}{\hat{d}(x)} = 1, \]  

\[ (5.9) \]

\[ \lim_{|x| \to \infty} \varphi(x)\sqrt{r(x)q(x)} = \lim_{|x| \to \infty} \psi(x)\sqrt{r(x)q(x)} = 1, \]  

\[ (5.10) \]

\[ \lim_{|x| \to \infty} h(x)\sqrt{r(x)q(x)} = \frac{1}{2}, \]  

\[ (5.11) \]

\[ e^{-1}\hat{d}(x) \leq d(x) \leq c\hat{d}(x), \quad x \in \mathbb{R}. \]  

\[ (5.12) \]

In addition, \( B < \infty \) (see (2.33) if and only if \( \inf_{x \in \mathbb{R}} q(x) > 0 \), and equality (3.1) holds if and only if \( q(x) \to \infty \) as \( |x| \to \infty \).

Proof. Both relations (5.9) are proved in the same way, and therefore we only consider, say, the second equality. Below we use some properties of the function \( F_2(\eta) \). It is convenient to list these properties as a separate statement.

Lemma 5.3. Under conditions (5.5), the function \( F_2(\eta) \) satisfies the following relations:

1) \( F_2(\eta) \in A^\text{loc}(\mathbb{R}_+), \quad R_+ = (0, \infty); \)

2) \( F_2(\eta) > 0 \) for \( \eta > 0; \)

3) \( F_2(\eta) > 0 \) for \( \eta > 0. \)
Proof. Property 2) is an obvious consequence of (5.5). Further,

\[ F'_2(\eta) = \frac{1}{r(x + \eta)} \int_x^{x+\eta} q(t) dt + q(x + \eta) \int_x^{x+\eta} \frac{dt}{r(t)}. \]

This equality together with (5.5) imply properties 1) and 3). \( \square \)

Lemma 5.4. Let \( \eta(x) = \alpha \hat{d}(x), \ x \in \mathbb{R}, \ \alpha \in (0, 80]. \) Then we have the inequalities

\[ \frac{r(x)}{\eta(x)} \left| \int_0^{\eta(x)} \left( \int_x^{x+s} \frac{r'(\xi)}{r^2(\xi)} d\xi \right) ds \right| \leq \kappa_1(x), \ \ x \in \mathbb{R}, \quad (5.13) \]

\[ \frac{1}{q(x)\eta(x)} \left| \int_0^{\eta(x)} \left( \int_x^{x+s} q'(\xi) d\xi \right) ds \right| \leq \kappa_2(x), \ \ x \in \mathbb{R}. \quad (5.14) \]

Proof. Inequalities (5.13)–(5.14) are obvious. Say,

\[ \frac{r(x)}{\eta(x)} \left| \int_0^{\eta(x)} \left( \int_x^{x+s} \frac{r'(\xi)}{r^2(\xi)} d\xi \right) ds \right| \leq \frac{r(x)}{\eta(x)} \cdot \eta(x) \sup_{|s| \leq 80\hat{d}(x)} \left| \int_x^{x+s} \frac{r'(\xi)}{r^2(\xi)} d\xi \right| = \kappa_1(x). \]

Let us now go to (5.9). Let \( \eta \geq 0. \) The following relations are obvious:

\[ \int_x^{x+\eta} \frac{d\xi}{r(\xi)} = \int_0^{\eta} \frac{ds}{r(x + s)} = \frac{\eta}{r(x)} - \int_0^{\eta} \left( \int_x^{x+s} \frac{r'(\xi)}{r^2(\xi)} d\xi \right) ds 
= \frac{\eta}{r(x)} \left[ 1 - \frac{r(x)}{\eta} \int_0^{\eta} \left( \int_x^{x+s} \frac{r'(\xi)}{r^2(\xi)} d\xi \right) ds \right], \ \ x \in \mathbb{R}, \quad (5.15) \]

\[ \int_x^{x+\eta} q(t) dt = \int_0^{\eta} q(x + s) ds = q(x)\eta + \int_0^{\eta} \left( \int_x^{x+s} q'(\xi) d\xi \right) ds 
= q(x)\eta \left[ 1 + \frac{1}{q(x)\eta} \int_0^{\eta} \left( \int_x^{x+s} q'(\xi) d\xi \right) ds \right], \ \ x \in \mathbb{R}. \quad (5.16) \]

Denote

\[ \delta(x) = \kappa_1(x) + \kappa_2(x), \ \ x \in \mathbb{R}, \]

\[ \eta(x) = \hat{d}(x)(1 + \delta(x)), \ \ x \in \mathbb{R}. \quad (5.17) \]

Then for all \( |x| \gg 1, \) from (5.17), (5.16), (5.15), (5.13) and (5.14), it follows that

\[ F_2(\eta(x)) = \int_x^{x+\eta(x)} \frac{dt}{r(t)} \cdot \int_x^{x+\eta} q(t) dt 
= \eta^2(x) q(x) \frac{r(x)}{r(x)} \left[ 1 - \frac{r(x)}{\eta(x)} \int_0^{\eta(x)} \left( \int_x^{x+s} \frac{r'(\xi)}{r^2(\xi)} d\xi \right) ds \right] 
\cdot \left[ 1 + \frac{1}{q(x)\eta(x)} \int_0^{\eta(x)} \left( \int_x^{x+s} q'(\xi) d\xi \right) ds \right] 
\geq (1 + \delta(x))^2 (1 - \kappa_1(x))(1 - \kappa_2(x)) 
\geq (1 + 2\delta(x))(1 - \delta(x)) = 1 + \delta(x) - 2\delta^2(x) \geq 1. \quad (5.18) \]
Since $F_2(0) = 0$, from (5.18) and Lemma 5.3 it follows that the equation $F_2(d) = 1$ has a unique finite positive solution. Denote it $d_2(x)$. From (5.18) and Theorem 5.1 we obtain the estimate

$$d_2(x) \leq \eta(x) = \tilde{d}(x)(1 + \delta(x)), \quad |x| \gg 1. \quad (5.19)$$

Let now

$$\eta(x) = \tilde{d}(x)(1 - \delta(x)) \gg 1. \quad (5.20)$$

Clearly, $\eta(x) > 0$ for all $|x| \gg 1$. The following relations are similar to (5.18):

$$F_2(\eta(x)) = \int_x^{x + \eta(x)} \frac{dt}{r(t)} \cdot \int_x^{x + \eta(x)} q(t)dt$$

\[= \eta^2(x) \cdot \frac{q(x)}{r(x)} \bigg[ 1 - \frac{r(x)}{\eta(x)} \int_0^{\eta(x)} \left( \int_x^{x+s} \frac{r'(\xi)}{r^2(\xi)} d\xi \right) ds \bigg] \]

\[\cdot \left[ 1 + \frac{1}{q(x)\eta(x)} \int_0^{\eta(x)} \left( \int_x^{x+s} q(\xi)d\xi \right) ds \right] \leq (1 - \delta(x))^2(1 + \kappa_1(x))(1 + \kappa_2(x)) \]

\[= [1 - 2\delta(x) + \delta^2(x)][1 + \kappa_1(x) + \kappa_2(x) + \kappa_1(x)\kappa_2(x)]. \]

It is easy to see that for all $|x| \gg 1$, we have the inequalities:

$$1 - 2\delta(x) + \delta^2(x) \leq 1 - \frac{5}{3}\delta(x)$$

$$\kappa_1(x) \cdot \kappa_2(x) \leq \frac{\kappa_1(x) + \kappa_2(x)}{2} = \frac{\delta(x)}{2}$$

that allow us to continue the estimate

$$F_2(\eta(x)) \leq \left( 1 - \frac{5}{3}\delta(x) \right) \left( 1 + \frac{3}{2}\delta(x) \right) \leq 1 - \frac{\delta(x)}{6} \leq 1. \quad (5.21)$$

From (5.21) and Theorem 5.1 we obtain the inequality

$$d_2(x) \geq \eta(x) = \tilde{d}(x)(1 - \delta(x)), \quad |x| \gg 1. \quad (5.22)$$

From (5.19) and (5.22) we obtain (5.19). Let us now go to (5.10). These inequalities are a consequence of (5.9). Indeed, as above, we get

$$\psi(x) = \int_x^{x+d_2(x)} \frac{dt}{r(t)} = \frac{d_2(x)}{r(x)} - \int_0^{d_2(x)} \left( \int_x^{x+s} \frac{r'(\xi)}{r^2(\xi)} d\xi \right) ds$$

\[= \frac{d_2(x)}{r(x)} \left[ 1 - \frac{r(x)}{d_2(x)} \int_0^{d_2(x)} \left( \int_x^{x+s} \frac{r'(\xi)}{r^2(\xi)} d\xi \right) ds \right] \]

\[\Rightarrow \psi(x) \sqrt{r(x)q(x)} = \frac{d_2(x)}{\tilde{d}(x)} \cdot (1 + \gamma(x)), \quad x \in \mathbb{R}. \quad (5.23)\]

Here (5.23) it easily follows that $|\gamma(x)| \leq \kappa_1(x)$ for $|x| \gg 1$. This proves (5.10) and hence, in view of (2.12), also (5.11). Let us verify (5.12). Let us show that $d(x) \leq 80\tilde{d}(x)$ for all
\(|x| \gg 1\). Assume the contrary. This means that \(d(x) > \eta(x) = 80\hat{d}(x)\) for some \(|x| \gg 1\). In the following relations, apart from the above assumption, we use (2.19), (2.26), (2.22). Theorem 5.1 and the part of the theorem that has already been proved:

\[
1 = \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)h(t)} + \frac{1}{4e^2} \cdot \frac{1}{h(x)} \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} \\
\geq \frac{1}{80} \sqrt{r(x)q(x)} \left[ 2\eta(x) - \int_0^{\eta(x)} \left( \int_x^{x+s} \frac{r'(\xi)}{r^2(\xi)} d\xi \right) d\eta(x) \right] \\
+ \frac{1}{2} \cdot \frac{r(x)}{\eta(x)} \left( \int_x^{x+s} \frac{r'(\xi)}{r^2(\xi)} d\xi \right) \geq 2(1 - \kappa_1(x)) > 1.
\]

Contradiction. Hence

\[
d(x) \leq 80\hat{d}(x) \quad \text{for} \quad |x| \gg 1.
\]

To get the lower estimate of \(d(x)\) for \(|x| \gg 1\), we use

\[
1 = \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)h(t)} \leq \frac{4e^2}{h(x)} \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} \\
\leq 80\sqrt{r(x)q(x)} \left[ \frac{2d(x)}{r(x)} - \int_0^{d(x)} \int_x^{x+s} \frac{r'(\xi)}{r^2(\xi)} d\xi ds \right] \\
+ \int_0^{d(x)} \frac{r'(\xi)}{r^2(\xi)} d\xi ds \leq 160 \frac{d(x)}{d(x)} (1 + \kappa_1(x)) \leq 320 \frac{d(x)}{d(x)}.
\]

Hence

\[
d(x) \geq \frac{\hat{d}(x)}{320} \quad \text{for} \quad |x| \gg 1.
\]

Choose \(x_0 \gg 1\) so that for \(|x| \geq x_0\) inequalities (5.24) and (5.25) would hold together. Let

\[
f(x) = \frac{d(x)}{d(x)} , \quad x \in [-x_0, x_0].
\]

By Lemma 2.10, the function \(f(x)\) is positive and continuous on \([-x_0, x_0]\) and therefore attains on this segment a positive minimum \(m\) and a finite maximum \(M\).

Let \(c \gg 1\) be such that

\[
c^{-1} \leq \min \left\{ \frac{1}{320}, m \right\} \leq \max \{80, M\} \leq c.
\]

With such a choice of \(c\), taking into account the fact proven above, we obtain (5.12). The remaining assertions of the theorem follow from (5.11)–(5.12). \(\square\)

We also need the following facts.
Theorem 5.5. [12, Ch.XI, §6]. Suppose that conditions (1.2) and (2.1) hold, and, in addition,
\[ \int_{-\infty}^{0} \frac{dt}{r(t)} = \int_{0}^{\infty} \frac{dt}{r(t)} = \infty. \] (5.26)
Then the relations
\[ v(x) \to 0 \quad \text{as} \quad x \to -\infty, \quad u(x) \to 0 \quad \text{as} \quad x \to \infty \] (5.27)
that hold if and only if
\[ \int_{-\infty}^{x} q(t) \int_{-\infty}^{t} \frac{d\xi}{r(\xi)} dt = \int_{x}^{\infty} q(t) \int_{t}^{\infty} \frac{d\xi}{r(\xi)} dt = \infty, \quad x \in \mathbb{R}. \] (5.28)

Theorem 5.6. Suppose that conditions (1.2), (2.1) and (5.27) hold, and, in addition, equation (1.1) is correctly solvable in \( L_p \), \( p \in (1, \infty) \). Then equalities of (5.28) hold.

Proof. The operator \( G : L_p \to L_p \), \( p \in (1, \infty) \) is bounded by Theorem 2.15. From (2.32) it follows that then so is the operator \( G_2 : L_p \to L_p \), \( p \in (1, \infty) \) (see (2.30)). Then by Theorem 2.32 we have
\[ \left( \int_{-\infty}^{x} v(t)^p dt \right)^{1/p} \left( \int_{x}^{\infty} u(t)^{p'} dt \right)^{1/p'} < \infty. \] (5.29)
Further, by Theorem 2.1 there exist the limits
\[ \lim_{x \to -\infty} v(x) = \epsilon_1 \geq 0, \quad \lim_{x \to \infty} u(x) = \epsilon_2 \geq 0. \]
If here \( \epsilon_1 > 0 \) or \( \epsilon_2 > 0 \), then (5.29) does not hold. Hence \( \epsilon_1 = \epsilon_2 = 0 \). Then (5.28) holds by Theorem 5.5. \( \square \)

Let us now go to equation (5.1). Denote by \( S_p \) the set of linear bounded operators acting from \( L_p \), \( p \in (1, \infty) \), and by \( S_0^{(0)} \) the subset of \( S_p \) consisting of the compact operators. Thus writing \( G \in S_p \) \( (G \in S_0^{(0)}) \) we mean that the operator \( G : L_p \to L_p \) is bounded (compact).

Theorem 5.7. Let \( G \) be the Green operator corresponding to equation (5.1) (see (2.28)). Then, regardless of \( p \in (1, \infty) \) and depending on the numbers \( \alpha \) and \( \beta \), the operator has the properties presented in the following table.

| \( \alpha \) \( \backslash \) \( \beta \) | \( \beta < 0 \) | \( \beta = 0 \) | \( \beta > 0 \) |
|---|---|---|---|
| \( \alpha < 0 \) | \( G \notin S_p \) | \( G \in S_p \) \( G \notin S_0^{(0)} \) | \( G \in S_0^{(0)} \) |
| \( \alpha = 0 \) | \( G \notin S_p \) | \( G \in S_p \) \( G \notin S_0^{(0)} \) | \( G \in S_0^{(0)} \) |
| \( \alpha > 0 \) | \( G \in S_0^{(0)} \) | \( G \in S_0^{(0)} \) | \( G \in S_0^{(0)} \) |

(5.30)
Proof. Let us numerate the entries of matrix (5.30) in the usual way and view them as instances of relations between \( \alpha \) and \( \beta \). We move along the rows of the matrix. Since in the case of (5.1) the functions \( r \) and \( q \) are even, in all the relations in the sequel, we only consider the case \( x \geq 0 \) \((x \gg 1)\).

Case (1, 1) \((\alpha < 0, \beta < 0)\)

Under conditions (1, 1) the hypotheses of Theorem 5.6 hold. Therefore, the operator \( G : L_p \to L_p, p \in (1, \infty) \) can be bounded only if (5.28) holds. In particular, we must have the equality

\[
\infty = \int_{0}^{\infty} e^{\beta t} \left( \int_{x}^{t} e^{-\alpha \xi} d\xi \right) dt \Rightarrow \beta \geq \alpha.
\]

(5.31)

Below we consider cases a) \( \beta > \alpha \) and b) \( \beta = \alpha \) separately.

a) Let \( \alpha < \beta < 0 \). In this case the hypotheses of Theorem 5.2 hold, and therefore \( B = \infty \) because \( \inf_{x \in \mathbb{R}} q(x) = \lim_{x \to \infty} e^{\beta |x|} = 0 \). Thus \( G \notin S \) by Theorems 2.21 and 2.28.

b) Let \( \beta = \alpha < 0 \). In this case \( r = q \), and one can compute \( h \) and \( d \) directly. Thus the equation for \( d_2(x) \) is of the form (see (2.11))

\[
1 = \int_{x}^{\infty} e^{-\alpha \xi} d\xi \cdot \int_{x}^{\infty} e^{\alpha \xi} d\xi = \frac{(e^{d} - 1)(1 - e^{-|\alpha|d})}{\alpha^2}, \quad d \geq 0.
\]

Hence \( d_2(x) = c \). To find \( d_1(x) \), we will first check that \( d_1(x) \leq x \) for all \( x \gg 1 \). Indeed, the function

\[
F(x) = \int_{0}^{x} e^{-\alpha \xi} d\xi \cdot \int_{0}^{x} e^{\alpha \xi} d\xi = \frac{e^{|\alpha|x} - e^{-|\alpha|x} - 2}{\alpha^2} \to \infty
\]
as \( x \to \infty \), and therefore \( d_1(x) \leq x \) for \( x \gg 1 \). Then equation (2.11) for \( d_1(x) \) is of the form:

\[
1 = \int_{-d}^{x} e^{-\alpha \xi} d\xi \cdot \int_{-d}^{x} e^{\alpha \xi} d\xi = \frac{(1 - e^{-|\alpha|d})(e^{|\alpha|d} - 1)}{\alpha^2}, \quad d \geq 0.
\]

Hence \( d_1(x) = d_2(x) = c \). This easily implies the equalities

\[
\varphi(x) = ce^{|\alpha|x}, \quad \psi(x) = ce^{|\alpha|x}, \quad h(x) = ce^{|\alpha|d}, \quad d(x) = c.
\]

Hence \( B = \infty \) and \( G \notin S_p, p \in (1, \infty) \) by Theorems 2.21 and 2.18.

Case (1, 2) \((\alpha < 0, \beta = 0)\)

In this case \( q(x) \equiv 1 \), and therefore \( G \in S_p \) by Theorem 2.25. We will use Theorem 5.2 to answer a more subtle question on the inclusion \( G \in S_p^{(0)} \). It is easy to see that in this case its hypotheses are satisfied, and equation (3.1) does not hold because \( q(x) \to \infty \) as \( |x| \to \infty \). Then \( G \in S_p^{(0)}, p \in (1, \infty) \) by Theorem 3.1.
Case (1, 3) ($\alpha < 0$, $\beta > 0$)

In this situation conditions (1.2)–(1.3) hold and $q(x) \to \infty$ as $|x| \to \infty$. Then $G \in S_p^{(0)}$, $p \in (1, \infty)$ by Corollary 3.8.

Case (2, 1) ($\alpha = 0$, $\beta < 0$)

Since $r \equiv 1$ and $m(a) = 0$, for any $a \in (0, \infty)$, we have $G \notin S_p$, $p \in (1, \infty)$ (see Theorem 2.24).

Case (2, 2) ($\alpha = \beta = 0$)

Since $r \equiv 1$ and $m(a) = 0$, we have $G \notin S_p$, $p \in (1, \infty)$ by Corollary 3.9.

Case (2, 3) ($\alpha = 0$, $\beta > 0$)

We have $r \equiv 1$, $q(x) \to \infty$ as $|x| \to \infty$. Hence $G \in S_p^{(0)} \in (1, \infty)$ by Corollary 3.8 or Corollary 3.9.

Cases (3, 1); (3, 2); (3, 3) ($\alpha > 0$ and $\beta = 0$; $\alpha > 0$ and $\beta > 0$, respectively)

All cases are treated in the same way. Clearly, $r^{-1} \in L_1$, $q > 0$. Then $G \in S_p^{(0)}$, $p \in (1, \infty)$ by Theorem 3.13 or Theorem 3.14.

6. Proofs of Otelbaev’s Lemmas

In this section we present the proofs of Lemmas 2.10, 2.11, 2.15 and 2.17 for the function $s(x)$ (see (2.19)). Assertions of such type (except for Lemma 2.17 and with other auxiliary functions) were first applied by Otelbaev, and therefore we call them Otelbaev’s Lemmas (see [15]).

Proof of Lemma 2.10. Consider the function

$$F(\eta) = \int_{x-\eta}^{x+\eta} \frac{dt}{r(t)\rho(t)}, \quad \eta \geq 0. \quad (6.1)$$

Clearly, the function $F(\eta)$ is continuous for $\eta \in [0, \infty)$; $F(0) = 0$, $F(\infty) = \infty$ (see (2.10)), and

$$F'(\eta) = \frac{1}{r(x+\eta)\rho(x+\eta)} + \frac{1}{r(x-\eta)\rho(x-\eta)} > 0.$$

Therefore the second equation of (2.19) has a unique finite positive solution. Denote it by $s(x)$ and check that the function $s(x)$, $x \in \mathbb{R}$ is continuous. Towards this end, we show that the following inequality holds:

$$|s(x+t) - sx| \leq |t|, \quad |t| \leq s(x), \quad x \in \mathbb{R}. \quad (6.2)$$

To check (6.2), we have to consider two cases: 1) $t \in [0, s(x)]$ and 2) $t \in [-s(x), 0]$.
They are treated in a similar way, and therefore below we only consider Case 1). Thus let \( t \in [0, s(x)] \). Then we have the obvious inclusions

\[
[x - s(x), x + s(x)] \subseteq [(x + t) - (t + s(x)), (x + t) + (t + s(x))],
\]

\[
[(x + t) - (s(x) - t), (x + t) + (s(x) - t)] \subseteq [x - s(x), x + s(x)],
\]

and therefore the following inequalities hold:

\[
1 = \int_{x-s(x)}^{x+s(x)} d\xi \frac{r(\xi)}{r(\xi)\rho(\xi)} \leq \int_{(x+t) - (t + s(x))}^{(x+t) + (t + s(x))} d\xi \frac{r(\xi)}{r(\xi)\rho(\xi)} \Rightarrow s(x + t) \geq s(x) - t \quad \Rightarrow (6.2)
\]

From (6.2) it follows that \( s(x), x \in \mathbb{R} \) is continuous.

**Proof of Lemma 2.11.** Let us rewrite (6.2) in a different way:

\[
s(x) - |t| \leq s(x + t) \leq s(x) + |t| \quad \text{if} \quad |t| \leq s(x). \quad (6.3)
\]

Let \( \xi = x + t \). Then \( t = \xi - x \), and in this notation we obtain inequalities equivalent to (6.3):

\[
s(x) - |\xi - x| \leq s(\xi) \leq s(x) + |\xi - x| \quad \text{if} \quad |\xi - x| \leq s(x). \quad (6.4)
\]

Let \( \varepsilon \in [0, 1] \) and \( |\xi - x| \leq \varepsilon s(x) \). Then, evidently, \( |\xi - x| \leq \varepsilon s(x) \leq s(x) \), and (2.21) follows from (6.4):

\[
s(x) - \varepsilon s(x) \leq s(x) - |\xi - x| \leq s(\xi) \leq s(x) + |\xi - x| \leq s(x) + \varepsilon s(x).
\]

Further, equalities (2.23) are checked in the same way, and therefore below we only consider the second one. We show that \( \lim_{x \to \infty} (x - s(x)) = \infty \). Assume the contrary. Then there exist a sequence \( \{x_n\}_{n=1}^{\infty} \) such that \( x_n \to \infty \) as \( n \to \infty \) and a number \( c \) such that

\[
x_n - s(x_n) \leq c < \infty, \quad n = 1, 2, \ldots
\]

From these assumptions we obtain

\[
1 = \int_{x_n-s(x_n)}^{x_n+s(x_n)} \frac{dt}{r(t)\rho(t)} \geq \int_{c}^{x_n} \frac{dt}{r(t)\rho(t)} \to \infty \quad \text{as} \quad n \to \infty
\]

(see (2.10)). Contradiction. Hence

\[
\lim_{x \to \infty} (x - s(x)) = \infty \Rightarrow \infty \leq \lim_{x \to \infty} (s - s(x)) \leq \lim_{x \to \infty} (x - s(x)) \leq \infty \Rightarrow \lim_{x \to \infty} (x - s(x)) = \lim_{x \to \infty} (x - s(x)) = \infty \Rightarrow (2.23)
\]

\( \square \)
Proof of Lemma 2.15. The assertion of the lemma immediately follows from Lemmas 2.10, 2.11 and 2.13.

Proof of Lemma 2.17. Below for \( t \in [x, x + s(x)] \), we use Theorem 2.1, (4.34), (2.17) and (2.19):

\[
\frac{v'(t)}{v(t)} = \frac{1 + r(t)\rho(t)}{2r(t)\rho(t)} \leq \frac{1}{r(t)\rho(t)}, \quad t \in [x, x + s(x)] \Rightarrow
\]

\[
\ln \frac{v(x + s(x))}{v(x)} \leq \int_{x}^{x+s(x)} \frac{dt}{r(t)\rho(t)} < \int_{x-s(x)}^{x+s(x)} \frac{dt}{r(t)\rho(t)} = 1.
\]

Similarly,

\[
\ln \frac{v(x)}{v(x - s(x))} \leq \int_{x-s(x)}^{x} \frac{dt}{r(t)\rho(t)} < \int_{x-s(x)}^{x+s(x)} \frac{dt}{r(t)\rho(t)} = 1.
\]

This gives the inequalities of (2.27), for example:

\[
e^{-1} \leq \frac{v(x - s(x))}{v(x)} \leq \frac{v(x + s(x))}{v(x)} \leq e, \quad |t - x| \leq s(x).
\]

Inequalities (2.27) for the function \( \rho \) is a consequence of the following inequalities for \( u \) and \( v \):

\[
e^{-1} \leq \frac{\rho(t)}{\rho(x)} = \frac{u(t)}{u(x)} \frac{v(t)}{v(x)} \leq c, \quad |t - x| \leq s(x).
\]

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