A semidiscrete Galerkin scheme for a two-scale coupled elliptic-parabolic system: well-posedness and convergence approximation rates

Martin Lind¹, Adrian Muntean¹, and Omar Richardson*¹

¹Department of Mathematics and Computer Science, Karlstad University, Sweden

August 15, 2018

Abstract

In this paper, we study the evolution of a gas-liquid mixture via a coupled system of elliptic-parabolic equations posed on two separated spatial scales. The model equations describe the interplay between macroscopic and microscopic pressures in an unsaturated heterogeneous medium with distributed microstructures. Besides ensuring the well-posedness of our two-scale model, we design two-scale convergent numerical approximations and prove a priori error estimates and propose an a posteriori error estimator. Finally, we propose a macroscopic mesh refinement strategy that ensures a redistribution of the local macroscopic errors until an overall error reduction is achieved.

Keywords: elliptic-parabolic system, weak solutions, Galerkin approximations, distributed microstructures, error analysis, macroscopic mesh refinement strategy

MSC (2010): 35K58, 65N30, 65N15

1 Introduction

This work is concerned with the design and approximation of evolution equations able to describe multiscale spatial interactions in gas-liquid mixtures. The long term goal and ultimate target is to set the foundation for a rigorous mathematical justification of Richards-like equations. Upscaled equations for the motion of flow in unsaturated porous media are chosen in a rather ad hoc manner by various engineering communities. The main issue is that one lacks a rigorous derivation of Darcy’s law for such flows altering between compressibility and incompressibility. Therefore, the reliability of macroscopically imposed

*email: omar.richardson@kau.se
laws which are exclusively based on first principles and perhaps on single-scale fitting strategies is limited. One the other hand, the fully saturated case is clear. We refer the reader to Chapter 1 of [14] for a derivation of the Darcy’s law in the saturated case using periodic homogenization arguments. Regarding the context of Darcy’s law for unsaturated flows, related work is reported, for instance, in [26], [20] and [8].

If the geometry of the porous media has a dual porosity structure, and hence, characteristic scales can possible be separated, then PDE models with distributed microstructures are in theory able to describe the relevant multi-scale spatial interactions occurring in gas-liquid mixtures. Now, the challenge shifts from multiscale modeling to the implementation of multiscale models. Consequently, we are concerned with the two-scale computability issue – complex systems of evolution equations acting on two spatial scales are notoriously hard to approximate especially if moving boundaries or stochastic dynamics are involved within e.g. the distributed microstructures.

In this paper, we consider a coupled system of partial differential equations describing the evolution of the pressure of a compressible air-liquid mixture on two spatial scales, when the amount of liquid is low and trapped in the internal structure of a porous medium. The derivation of our particular model originates from applying a formal two-scale homogenization to a particular scaling of the level set equation coupled with Stokes equations for fluid flow (see [28]).

2

\[
\begin{align*}
- \Delta_x \pi &= f(\pi, \rho) \quad &\text{in } S \times \Omega, \\
\partial_t \rho - D \Delta_y \rho &= 0 \quad &\text{in } S \times \Omega \times Y, \\
D \nabla_y \rho \cdot n_y &= k(\pi + p_F - R \rho) \quad &\text{in } S \times \Omega \times \Gamma_R, \\
D \nabla_y \rho \cdot n_y &= 0 \quad &\text{in } S \times \Omega \times \Gamma_N, \\
\pi &= 0 \quad &\text{in } S \times \partial \Omega, \\
\rho(0, x, y) &= \rho_0(x, y) \quad &\text{in } \Omega \times \overline{Y},
\end{align*}
\]

where \(\Gamma_R \cup \Gamma_N = \partial Y\), \(\Gamma_R \cap \Gamma_N = \emptyset\) and \(f : S \times \Omega \times Y \to \mathbb{R}\) is a function. We refer to (1)-(6) as \((P_1)\).
Figure 1: The macroscopic domain $\Omega$ and microscopic pore $Y$ at $x \in \Omega$.

Note that ($P_1$) describes the interaction between a compressible viscous fluid (with density $\rho$) in a porous domain $\Omega$, where the pores are partly filled with a gas that exerts an average (macroscopic) pressure $\pi$. The interaction between the fluid and the gas is determined by the right hand side of (1) and the microscopic boundary condition in (3), through the fluid-gas interface represented by $\Gamma_R$. The mathematical problem stated in (1)-(6) (referred to as ($P_1$)), contains a number of dimensional constant parameters: $A$ (gas permeability), $D$ (diffusion coefficient for the gaseous species), $p_F$ (atmospheric pressure) and $\rho_F$ (gas density). In addition, we need the dimensional functions $k$ (Robin coefficient) and $\rho_I$ (initial liquid density). Except for the Robin coefficient $k$, all the model parameters and functions are either known or can be accessed directly via measurements.

If the Neumann part of the boundary $\Gamma_N := \partial Y \setminus \Gamma_R$ is accessible via measurements, then the inaccessibility of the boundary $\Gamma_R$ can be compensated for in such a way that parameters like $k$ entering two-scale transmission conditions can be identified (compare [19]).

In this context, we prove the existence and uniqueness of a discrete-in-space, continuous-in-time finite element element approximation and prove convergence of this approximation of ($P_1$). The main results of this contribution are the well-posedness of the Galerkin approximation (Theorem 1), convergence rates for the approximation (Theorem 2), and controlled error estimators which can then be used to refine the grid (Theorem 3).
The choice of problem and approach is in line with other investigations running for two-scale systems, or systems with distributed microstructures, like [18, 21, 25]. The reader is also referred to the FEM strategies developed by the engineering community to describe the evolution of mechanical deformations in structured heterogeneous materials; see e.g., [16] and references cited therein. Other classes of computationally challenging two-scale problems are mentioned, for instance, in [27], where the pore scale model has a priori unknown boundaries, and in [15] for a smoldering combustion scenario. This paper continues an investigation started in related works. In [19], we study the solvability issue and derive inverse Robin estimates for a variant of this model problem. Two-scale Galerkin approximations have been derived previously for related problem settings; see e.g. our previous attempts [23], [22], [5], and [18]. Ref. [18] stands out since it is for the first time that the issue of feedback estimates is put in the context of computational efficiency of PDEs posed on multiple scales. Unfortunately, the obtained theoretical estimates in loc. cit. are not computable. This aspect is addressed here in Theorem 3.

The rest of this paper is structured as follows. In Section 2, we discuss the technical concepts and requirements we need before starting our analysis. Then, in Section 3, we show the Galerkin approximation is well-posed and converges to the weak solution of the original system. In Section 4, we prove a priori convergence rates for the Galerkin approximation. Next, in Section 5, we design an error estimator, prove upper and lower bounds with respect to the true error, and propose a macroscopic mesh refinement strategy. Finally, in Section 6, we conclude this paper and provide an outlook into future research.

2 Concept of weak solution, assumptions and technical preliminaries

2.1 Weak solutions

Essentially, we look for solutions to \((P_1)\) in the weak sense. This is motivated by the fact that the underlying structured media can be of composite type, allowing for discontinuities in the model parameters. However, already at this stage it is worth mentioning that the solutions to \((P_1)\) are actually more regular than stated, i.e. with minimal adaptations of the working assumptions the regularity of the solutions can be lifted so that they turn out to be strong or even classical. We will lift their regularity only when needed.

**Definition 1** (Weak solution). A weak solution of \((P_1)\) is a pair \((\pi, \rho) \in L^2(S; H_0^1(\Omega)) \times L^2(S; L^2(\Omega; H^1(Y)))\) that satisfies for all test functions \((\varphi, \psi) \in H_0^1(\Omega) \times L^2(\Omega; H^1(Y))\) the identities

\[
A \int_{\Omega} \nabla_{x} \pi \cdot \nabla_x \varphi dx = \int_{\Omega} f(\pi, \rho) \varphi dx,
\]  

(7)
\[
\int_{\Omega} \int_{Y} \partial_t \rho \psi dy dx + D \int_{\Omega} \int_{Y} \nabla_y \rho \cdot \nabla_y \psi dy dx = \kappa \int_{\Omega} \int_{\Gamma_R} (\pi + p_F - R \rho) \psi \sigma d\sigma dx, \tag{8}
\]
for almost every \( t \in S \).

### 2.2 Assumptions

We introduce a set of assumptions that allows us to ensure the weak solvability and approximation of \((P_1)\).

\((A_1)\) The domains \( \Omega \) and \( Y \) are convex polygons.

\((A_2)\) All model parameters are positive; in particular \( D, R, p_F, \kappa \).

\((A_3)\) \( A > \max (C_f, C_\pi) \). The value of \( C_f \) and \( C_\pi \) is given in Section 3.

\((A_4)\) \( \rho_I \in L^2(\Omega; H^1(Y)) \).

\((A_5)\) \( f : S \times \Omega \times Y \) in (1) satisfies the following structural conditions:

(i) \( f \) is once continuously differentiable in \( \pi \) and \( \rho \).

(ii) \( f(s, r) \) is a contraction in \( s \) for all \( r \).

(iii) \( f(0, r) = 0 \) for all \( r \).

(iv) There exists a \( \theta > 0 \) such that \( f(s, r) = 0 \) for all \( s > \theta \).

\((A_2)\) and \((A_4)\) are straightforward assumptions related to the physical setting. \((A_1)\) is a condition to ease the interaction with the finite element mesh. \((A_3)\) and \((A_5)\) are technical conditions required to prove well-posedness of the solution.

Before moving on, we remark that the formal shape of the right hand side in (1) must be of the form \( f(\pi, g(\rho)) \) for some \( g : S \times \Omega \times Y \rightarrow S \times \Omega \).

### 2.3 Technical preliminaries

The rest of the section introduces the notation of the functional spaces and norms used in the paper.

Let \( f, g : D \rightarrow \mathbb{R} \). Then the Lebesgue and Sobolev norms are defined as follows:

\[ ||f||_{L^p(D)} := \begin{cases} (\int_D |f(x)|^p dx)^{1/p} & \text{for } 1 \leq p < \infty, \\ \text{ess sup} \{|f| : x \in D\} & \text{for } p = \infty, \end{cases} \tag{9} \]

\[ ||f||_{H^k(D)} := \left( \sum_{|\alpha| \leq k} \int_D |\partial^\alpha f|^2 dx \right)^{1/2} \tag{10} \]

with \( \partial^\alpha f \) denoting derivatives in the weak sense.
Furthermore, for $\mathcal{L}^2(D)$ and $\mathcal{H}^k(D)$ we have the following inner products.

$$\langle f, g \rangle_{\mathcal{L}^2(D)} := \int_D f(x)g(x)dx,$$

(11)

$$\langle f, g \rangle_{\mathcal{H}^k} := \sum_{|\alpha| \leq k} \langle \partial^\alpha f, \partial^\alpha g \rangle_{\mathcal{L}^2(D)}.$$

(12)

Moreover, we use $\mathcal{H}^1_0(D)$ to denote the following function space:

$$\mathcal{H}^1_0(D) := \{ u \in \mathcal{H}^1(D) : u|_{\partial D} = 0 \},$$

(13)

and $\mathcal{H}^{-1}(D)$ to denote the dual space of (13), equipped with the norm

$$\|T\|_{\mathcal{H}^{-1}(D)} = \sup \left\{ (T, u) : u \in \mathcal{H}^1_0(D), \|u\|_{\mathcal{H}^1_0(D)} = 1 \right\}.$$  

(14)

Let $B$ be a Banach space with norm $\| \cdot \|_B$. Then $u$ belongs to Bochner space $\mathcal{L}^2(S; B)$ if its norm is finite, defined as follows:

$$\|u\|_{\mathcal{L}^2(S,B)} := \begin{cases} \left( \int_S \|u(t)\|^p_B \, dt \right)^{1/p} & \text{for } 1 \leq p < \infty, \\ \text{ess sup}_{t \in S} \left\{ \|u(t)\|_B \right\} & \text{for } p = \infty. \end{cases}$$

(15)

An introduction to the concepts of Lebesgue and Bochner integration as well as on inner products and norms can be found in any functional analysis textbook (e.g. [1]).

By $\pi_I \in \mathcal{H}^1_0(\Omega)$ we denote the solution of

$$-A\Delta_x \pi = f(\pi, \rho_I) \text{ in } \Omega,$$

$$\pi = 0 \text{ in } \partial \Omega.$$  

(16)

(16) is a stationary elliptic equation giving access to the value of $\pi$ from (1) at time $t = 0$.

Finally, we introduce several constants: $c_i$ refers to constants from the interpolation-trace theorem (see Lemma 4) and $c_p$ to constants arising from Poincaré's inequality. Moreover, we define the following two constants:

$$c_\pi := \max_r |\partial_r f(r, s)|,$$

$$c_p := \max_s |\partial_s f(r, s)|.$$  

3 Well-posedness

In this section we prove that $(P_1)$ has a weak solution by approximating it with a Galerkin projection. We show the projection exists and is unique, and proceed by proving it converges to the weak solution of $(P_1)$. First, we introduce the necessary tools for defining the Galerkin approximation.
We use one mesh partition for each of the two spatial scales. Let $\mathcal{B}_H$ be a mesh partition for $\Omega$ consisting of simplices. We denote the diameter of an element $B \in \mathcal{B}_H$ with $H_B$, and the global mesh size with $H := \max_{B \in \mathcal{B}_H} H_B$. We introduce a similar mesh partition $\mathcal{K}_h$ for $Y$ with global mesh size $h := \max_{K \in \mathcal{K}_h} h_K$.

Our macroscopic and microscopic finite element spaces $V_H$ and $W_h$ are, respectively:

\[
V_H := \{ v \in \mathcal{L}^2(\Omega) \mid v|_B \in \mathcal{H}^1(B) \text{ for all } B \in \mathcal{B}_H \},
\]

\[
W_h := \{ w \in \mathcal{L}^2(Y) \mid w|_K \in \mathcal{H}^1(K) \text{ for all } K \in \mathcal{K}_h \}.
\]

Let $\langle \xi_B \rangle_{\mathcal{B}_H} := \text{span}(V_H)$ and $\langle \eta_K \rangle_{\mathcal{K}_h} := \text{span}(W_h)$, and let $\alpha_B, \beta_B : S \to \mathbb{R}$ denote the Galerkin projection coefficient for a patch $B$ and $B \times K$, respectively.

We introduce the following finite-dimensional Galerkin approximations of the functions $\pi$ and $\rho$:

\[
\pi^H(t, x) := \sum_{B \in \mathcal{B}_H} \alpha_B(t) \xi_B(x),
\]

\[
\rho^{H,h}(t, x, y) := \sum_{B \in \mathcal{B}_H, K \in \mathcal{K}_h} \beta_B(t) \xi_B(x) \eta_K(y),
\]

(17)

where we clamp $\alpha_B(t) = 0$ for all $B \in \mathcal{B}_H$ with $\partial B \cap \Omega \neq \emptyset$ to represent the macroscopic Dirichlet boundary condition.

Reducing the space of test functions to $V_H$ and $W_h$, we obtain the following discrete weak formulation: find a solution pair $(\pi^H(t, x), \rho^{H,h}(t, x, y)) \in \mathcal{L}^2(S; V_H) \times \mathcal{L}^2(S; V_H \times W_h)$ that are solutions to

\[
A \int_{\Omega} \nabla_x \pi^H \cdot \nabla_x \varphi \, dx = \int_{\Omega} f(\pi^H, \rho^{H,h}) \varphi \, dx,
\]

(18)

and

\[
\int_{\Omega} \int_Y \partial_t \rho^{H,h} \psi \, dy \, dx + D \int_{\Omega} \int_Y \nabla_y \rho^{H,h} \cdot \nabla_y \psi \, dy \, dx = \kappa \int_{\Omega} \int_{\Gamma_R} (\pi^H + p_F - R \rho^{H,h}) \psi \, d\sigma_y \, dx,
\]

(19)

for any $\varphi \in V_H$ and $\psi \in V_H \times W_h$ and almost every $t \in S$.

These concepts lead us to the first proposition.

**Proposition 1** (Existence and uniqueness of the Galerkin approximation). There exists a unique solution $(\pi^H, \rho^{H,h})$ to the system in (18)-(19).

**Proof.** The proof is divided in three steps. In step 1, the local existence in time is proven. In step 2, global existence in time is proven. Step 3 is concerned with proving the uniqueness of the system.
We introduce an integer index for $\alpha_B(t)$ and $\beta_{BK}(t)$ to increase the legibility of arguments in this proof. Let $N_1 := \{1, \ldots, |B_H|\}$ and $N_2 := \{1, \ldots, |K_h|\}$. We introduce bijective mappings $n_1 : N_1 \to B_H$ and $n_2 : N_2 \to K_h$, so that each index $j \in N_1$ corresponds to an element $B \in B_H$ and each index $k \in N_2$ corresponds to a $K \in K_h$.

**Step 1: local existence of solutions to (18) - (19):** By substituting $\varphi = \xi_i$ and $\psi = \xi_i \eta_k$ for $i \in N_1$ and $k \in N_2$ in (18)-(19) we obtain the following system of ordinary differential equations coupled with algebraic equations.

\[
\sum_{j \in N_1} P_{ij} \alpha_j(t) = F_i(\alpha, \beta) \quad \text{for} \quad i \in N_1, \tag{20}
\]

\[
\beta'_{ik}(t) + \sum_{j \in N_1, l \in N_2} Q_{ijkl} \beta_{jl}(t) = c_{ik} + \sum_{j \in N_1} E_{ijk} \alpha_j \quad \text{for} \quad i \in N_1 \quad \text{and} \quad k \in N_2, \tag{21}
\]

with

\[
P_{ij} := A \int_{\Omega} \nabla_x \xi_i \cdot \nabla_x \xi_j \, dx,
\]

\[
F_i := \int_{\Omega} f \left( \sum_{j \in N_1} \alpha_j(t) \xi_j, \sum_{j \in N_1, l \in N_2} \beta_{jl}(t) \xi_j \eta_l \right) \xi_i \, dx,
\]

\[
Q_{ijkl} := D \int_{\Omega} \xi_i \xi_j \, dx \int_{Y} \nabla_y \eta_k \cdot \nabla_y \eta_l \, dy + \kappa R \int_{\Omega} \xi_i \xi_j \, dx \int_{\Gamma_R} \eta_k \eta_l \, d\sigma_y, \tag{22}
\]

\[
E_{ijk} := \kappa \int_{\Omega} \xi_i \xi_j \, dx \int_{\Gamma_R} \eta_k \, d\sigma_y,
\]

\[
c_{ik} := \kappa p \int_{\Omega} \xi_i \, dx \int_{\Gamma_R} \eta_k \, d\sigma_y.
\]

Applying (6) to (18) and (19) yields:

\[
\alpha_i(0) = \int_{\Omega} \xi_i \pi_I \, dx,
\]

\[
\beta_{ik}(0) = \int_{\Omega} \int_{Y} \xi_i \eta_k \rho_I \, d\sigma_y. \tag{23}
\]

For all $t \in S$, the coefficients $\alpha_i(t), \beta_{ik}(t)$ of (17) are determined by (20), (21) and (23).

Since the system of ordinary differential equations in (21) is linear, we are able to explicitly formulate the solution representation for $\beta_{ik}$ with respect to $\alpha_i$. Let $\alpha_i$ be given, and let $Q$ and $E$ denote matrices given by:

\[
Q \beta = \sum_{j \in N_1, l \in N_2} Q_{ijkl} \beta_{jl}, \tag{24}
\]

\[
E \alpha = \sum_{j \in N_1} E_{ijk} \alpha_j. \tag{25}
\]
Then $\beta_{ik}$ can be expressed as

$$
\beta_{ik}(t) = \beta_{ik}(0)e^{-Qt} + Q^{-1}(c + E\alpha_i)(I - e^{-Qt}).
$$

(26)

Substituting (26) in (21) results in the expression:

$$
-Q\beta_{ik}(0)e^{-Qt} + (c + E\alpha_j)e^{-Qt} + Q\beta_{ik}(0)e^{-Qt} + (c + E\alpha_j)(I - e^{-Qt}) = (c + E\alpha_j).
$$

(27)

($A_5$) implies that for all $i \in N_1$, $F_i$ are contractions. Pick $\beta_1(t), \beta_2(t)$ that satisfy (26) given some $\alpha_1(t), \alpha_2(t)$. Then it holds that:

$$
|F_1(\alpha_1, \beta_1) - F_1(\alpha_2, \beta_2)|,
\leq |F_1(\alpha_1, \beta_1) - F_1(\alpha_1, \beta_2) + F_1(\alpha_1, \beta_2) - F_1(\alpha_2, \beta_2)|,
$$

$$
= \int_{\Omega} f \left( \sum_{j \in N_1} \alpha_{j,1}(t)\xi_j, \sum_{j \in N_1, i \in N_2} \beta_{j,1}(t)\xi_j\eta_i \right) \xi_i \Bigg| - f \left( \sum_{j \in N_1} \alpha_{j,1}(t)\xi_j, \sum_{j \in N_1, i \in N_2} \beta_{j,1}(t)\xi_j\eta_i \right) \xi_i dx \Bigg|
$$

$$
+ \int_{\Omega} f \left( \sum_{j \in N_1} \alpha_{j,1}(t)\xi_j, \sum_{j \in N_1, i \in N_2} \beta_{j,2}(t)\xi_j\eta_i \right) \xi_i \Bigg| - f \left( \sum_{j \in N_1} \alpha_{j,2}(t)\xi_j, \sum_{j \in N_1, i \in N_2} \beta_{j,2}(t)\xi_j\eta_i \right) \xi_i dx \Bigg|,
$$

$$
\leq \int_{\Omega} e_p \sum_{j \in N_1, i \in N_2} (\beta_{j,1}(t) - \beta_{j,2}(t))\xi_j\eta_i + c_\alpha \sum_{j \in N_1} (\alpha_{j,1}(t) - \alpha_{j,2}(t))\xi_j d\xi
$$

$$
\leq c_\beta \left| \sum_{j \in N_1} \beta_{j,1}(t) - \beta_{j,2}(t) \right| + c_\alpha \left| \sum_{j \in N_1} \alpha_{j,1}(t) - \alpha_{j,2}(t) \right|,
$$

(28)

with $c_\alpha, c_\beta$ defined as

$$
c_\alpha := c_\alpha \max_{j \in N_1} \int_{\Omega} \xi_j d\xi \leq c_\xi, \quad c_\beta := c_p \max_{j \in N_1, i \in N_2} \int_{\Omega} \xi_j \eta_i d\xi \leq c_p.
$$

(29)

Now, we derive a time-dependent continuity estimate for sufficiently small $t$. Again picking $\beta_1(t)$ and $\beta_2(t)$ (not necessarily the same as in (28)):

$$
\left| \beta_1(t) - \beta_2(t) \right| = \left| I - e^{Qt} \right| \cdot \left| Q^{-1}D \right| \cdot \left| \alpha_1(t) - \alpha_2(t) \right|,
$$

$$
= \left| Qt + O(t^2) \right| \cdot \left| Q^{-1}D \right| \cdot \left| \alpha_1(t) - \alpha_2(t) \right|,
$$

$$
\leq tc \left| \alpha_1 - \alpha_2 \right| \text{ for small } t.
$$

(30)
Using (30) we obtain a Lipschitz bound on all $F_i$ in the interval $[0, \tau]$ for any choice of $\tau < T$:
\[
\|F_i(\alpha_1(t), \beta_1(t)) - F_i(\alpha_2(t), \beta_2(t))\| \\
\leq \|F_i(\alpha_1(t), \beta_1(t)) - F_i(\alpha_1(t), \beta_2(t))\| + \|F_i(\alpha_1(t), \beta_2(t)) - F_i(\alpha_2(t), \beta_2(t))\|, \\
\leq c_\alpha \|\alpha_1(t) - \alpha_2(t)\| + c_\beta \|\beta_1(t) - \beta_2(t)\|, \\
\leq (c_\alpha + c_\beta C\tau) \|\alpha_1(t) - \alpha_2(t)\|.
\]

(31)

Choosing $\tau$ small enough to satisfy $c_\alpha + c_\beta C\tau < 1$ makes $F$ a contraction on $[0, \tau]$. By Banach’s fixed point theorem, it follows that the equation $F(\alpha(t), \beta(t)) = \alpha(t)$ has a solution for $\alpha$ in $L^2(S)$. Substitution of $\alpha(t)$ into (26) leads to the corresponding $\beta$. Existence of $\pi^H$ and $\rho^{H,h}$ follows directly.

Step 2: global existence of solutions to (18) - (19): We cover time interval $S$ into $N$ intervals of length at most $\tau$ such that $S \subseteq \bigcup_{n=1}^{N}((n-1)\tau, n\tau]$. From the arguments in the previous paragraph it is clear a solution exists on the first interval $[0, \tau]$. This allows us to provide an induction argument for the existence of a solution on interval $n$:

Given that interval $n$ has local solution $\beta((n-1)\tau, n\tau]$, we can obtain values $\beta(n\tau), \beta'(n\tau), \alpha(n\tau)$ as initial values to the local system on interval $n+1$, and show existence of a solution. This way, we are able to construct a solution satisfying (18) - (19) everywhere on $S$.

Step 3: uniqueness of solutions to (18) - (19): We decouple the system and use a fixed point argument to show that this system has a globally unique solution in time.

Let $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ be two solutions satisfying (18) - (19) with the same initial data. Let $\bar{\beta}(t) := \beta_1(t) - \beta_2(t)$ and $\bar{\alpha}(t) := \alpha_1(t) - \alpha_2(t)$. By starting from (21) and multiplying both equations with $\bar{\beta}(t)$, we obtain
\[
\langle \bar{\beta}(t), \bar{\beta}(t) \rangle = \langle Q\bar{\beta}(t), \bar{\beta}(t) \rangle + \langle E\bar{\alpha}(t), \bar{\beta}(t) \rangle, \\
\frac{1}{2} \frac{d}{dt} \|\bar{\beta}(t)\|^2 \leq ||Q||^2 \|\bar{\beta}(t)\|^2 + ||E||^2 \|\bar{\alpha}(t)\| \|\bar{\beta}(t)\|.
\]

(32)

Since $\bar{\beta}(0) = 0$, by applying Grönwall’s differential inequality, we know that $\bar{\beta}(t) \equiv 0$. Combined with (26), it immediately follows that $\bar{\alpha}(t) \equiv 0$, and therefore, $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$.

Note that showing the stability of the finite element approximation with respect to data and initial conditions follows an analogous argument. The proof is omitted here.

The remaining part of this section is devoted to proving that the system in (18)-(19) converges to the solution of the Galerkin projection converges to the weak solution of (P1) (as stated in Proposition 2). To this end, we first formulate the lemmata that help us prove this statement.
**Lemma 1** (Aubin-Lions lemma). Let $B_0 \hookrightarrow B \subset B_1$ be Banach spaces, i.e. $B_0$ be compactly embedded in $B$ and $B$ be continuously embedded in $B_1$. Let

$$W := \left\{ u \in L^2(S; B_0) \mid \partial_t u \in L^2(S; B_1) \right\}. \quad (33)$$

Then the embedding of $W$ into $L^2(S; B)$ is compact.

**Lemma 2** (Weak maximum principle). Assume $(A_1) - (A_5)$: Then $\pi^H(t, \cdot) \in L^\infty(\Omega)$ and $\rho^{H,h}(t, \cdot) \in L^\infty(\Omega \times Y)$ for all $t \in S$.

We refer the reader to [2] for the original proof of the statement.

**Proof.** We use a weak maximum principle according to Stampacchia ([29]). Consequently, we test the weak formulation with $\varphi = (\pi^H - M_1)^+$ and $\psi = (\rho^{H,h} - M_2)^+$ for suitable $M_1$ and $M_2$. Assumptions $(A_3)$ and $(A_5)$ are used in this context. From (18) we obtain

$$A \int_{\Omega} |\nabla_x \cdot \pi^H \nabla_x (\pi^H - M_1)^+| \, dx = \int_{\Omega} |f(\pi^H, \rho^{H,h})(\pi^H - M_1)^+| \, dx, \quad (34)$$

The left hand side of (34) can be manipulated as:

$$A \int_{\Omega} |\nabla_x \pi^H \cdot \nabla_x (\pi^H - M_1)^+| \, dx = A \int_{\Omega} |\nabla_x (\pi^H - M_1) \cdot \nabla_x (\pi^H - M_1)^+| \, dx,$$

$$= A \int_{\Omega} |\nabla_x (\pi^H - M_1)^+|^2, \quad (35)$$

which can be bounded with the right hand side of (34):

$$A \int_{\Omega} |\nabla_x (\pi^H - M_1)^+|^2 \leq \int_{\Omega} |f(\pi^H, \rho^{H,h}) - f(M_1, \rho^{H,h}) + f(M_1, \rho^{H,h})|(\pi^H - M_1)^+ \, dx,$$

$$\leq \int_{\Omega} c_{\pi} |\pi^H - M_1|(\pi^H - M_1)^+ + |f(M_1, \rho^{H,h})|(\pi^H - M_1)^+ \, dx,$$

$$= c_{\pi} \int_{\Omega} ((\pi^H - M_1)^+)^2 \, dx \leq c_p c_{\pi} \int_{\Omega} (\nabla_x (\pi^H - M_1)^+)^2 \, dx. \quad (36)$$
Proceeding similarly with (19):

\[
\begin{align*}
&\int_{\Omega} \int_{\Gamma_R} \partial_t \rho^{H,h}(\rho^{H,h} - M_2)^+ dy dx + \int_{\Omega} \int_{\Gamma_R} D\nabla_y \rho^{H,h} \cdot \nabla_y (\rho^{H,h} - M_2)^+ dy dx, \\
&= \int_{\Omega} \int_{\Gamma_R} k(\pi^H + p_F - R\rho^{H,h})(\rho^{H,h} - M_2)^+ d\sigma_y dx, \\
&= \kappa \int_{\Omega} \int_{\Gamma_R} (\pi^H - M_1 + p_F + M_1 - RM_2 - R(\rho^{H,h} - M_2)) (\rho^{H,h} - M_2)^+ d\sigma_y dx, \\
&= \kappa \int_{\Omega} \int_{\Gamma_R} ((\pi^H - M_1)^+ - (\pi^H - M_1)^-(\rho^{H,h} - M_2)^+) (\rho^{H,h} - M_2)^+ d\sigma_y dx \\
&\quad + \kappa \int_{\Omega} \int_{\Gamma_R} (p_F + M_1 - RM_2)(\rho^{H,h} - M_2)^+ d\sigma_y dx \\
&\quad + \kappa R \int_{\Omega} \int_{\Gamma_R} ((\rho^{H,h} - M_2)^+ - (\rho^{H,h} - M_2)^-) (\rho^{H,h} - M_2)^+ d\sigma_y dx, \\
&\leq \kappa \int_{\Omega} \int_{\Gamma_R} (\pi^H - M_1)^+ (\rho^{H,h} - M_2)^+ - (\pi^H - M_1)^-(\rho^{H,h} - M_2)^+ d\sigma_y dx \\
&\quad + \kappa (p_F + M_1 - RM_2) \int_{\Omega} \int_{\Gamma_R} (\rho^{H,h} - M_2)^+ d\sigma_y dx + \kappa R \int_{\Omega} \int_{\Gamma_R} (\rho^{H,h} - M_2)^{+2} d\sigma_y dx \\
&\quad - \kappa R \int_{\Omega} \int_{\Gamma_R} (\rho^{H,h} - M_2)^-(\rho^{H,h} - M_2)^+ d\sigma_y dx, \\
&\leq \varepsilon \int_{\Omega} \int_{\Gamma_R} (\pi^H - M_1)^{+2} d\sigma_y dx + \kappa (R + c_\varepsilon) \int_{\Omega} \int_{\Gamma_R} (\rho^{H,h} - M_2)^{+2} d\sigma_y dx.
\end{align*}
\]

(37)

Add (36) and (37) with \(\varepsilon > 0\) small. Applying the trace inequality twice in (37) ensures that \(\pi^H\) and \(\rho^{H,h}\) are uniformly bounded if the pair \((M_1, M_2)\) is chosen such that

\[
\begin{align*}
&\begin{cases}
M_2 = \theta, \\
M_1 < R\theta - p_F, \\
M_1 \geq ||\rho||_{L^2(\Omega \times Y)}, \\
\theta > \frac{p_l}{R}.
\end{cases}
\end{align*}
\]

(38)

Lemma 3 (Regularity lift). Let \((\pi^H, \rho^{H,h})\) be a solution to (18)-(19). Then it must hold that

\[
\begin{align*}
\pi^H \in L^\infty(S \times \Omega) \cap L^\infty(S; H^1_0(\Omega)), \\
\rho^{H,h} \in L^2(S; L^2(\Omega; H^1(Y))) \cap L^\infty(S; L^\infty(\Omega; L^\infty(Y))).
\end{align*}
\]

(39)

Proof. Testing (18) with \(\varphi = \pi^H\) and (19) with \(\psi = \rho^{H,h}\) yields identities

\[
A ||\nabla_x \pi^H||_{L^2(\Omega)}^2 = \int_{\Omega} f(\pi^H, \rho^{H,h}) \pi^H dx,
\]

(40)
To obtain a bound on $\pi$, by applying Grönwall’s inequality we obtain the desired estimates:

$$
\frac{1}{2} \frac{d}{dt} \|\rho^{H,h}\|^2_{L^2(\Omega \times Y)} + D \|\nabla_u \rho^{H,h}\|^2_{L^2(\Omega \times Y)} = \int_{\Omega} \int_{\Gamma_R} \kappa (\pi^H + p_F) \rho^{H,h} \, d\sigma_y \, dx - \kappa R \|\rho^{H,h}\|^2_{L^2(\Omega \times Y)}. \tag{41}
$$

We recall the embedding

$$
H^1(\Omega) \hookrightarrow L^2(\Gamma_R), \tag{42}
$$

which implies that there exists a $c_E$ such that

$$
\|u\|_{L^2(\Omega; L^2(\Gamma_R))} \leq c_E \|u\|_{L^2(\Omega; H^1(\Omega))}, \tag{43}
$$

for all $u \in L^2(\Omega; H^1(\Omega))$. Using Cauchy-Schwarz’ inequality and (43), we can bound the right hand side of (41) as

$$
\int_{\Omega} \int_{\Gamma_R} \kappa (\pi^H + p_F) \rho^{H,h} \, d\sigma_y \, dx \leq \kappa |\Gamma_R| \left( \|\pi^H\|_{L^2(\Omega)} + |\Omega| p_F \right) \|\rho^{H,h}\|_{L^2(\Omega \times \Gamma_R)}.
$$

Then, we add to both sides of (41) a term $D \|\rho^{H,h}\|^2_{L^2(\Omega \times Y)}$ to get

$$
\frac{1}{2} \frac{d}{dt} \|\rho^{H,h}\|^2_{L^2(\Omega \times Y)} + D \|\rho^{H,h}\|^2_{L^2(\Omega; H^1(\Omega))}, \tag{45}
$$

$$
\leq D \|\rho^{H,h}\|^2_{L^2(\Omega \times Y)} + c_E |\Gamma_R| \left( |\pi^H\|_{L^2(\Omega)} + p_F |\Omega| \right) \|\rho^{H,h}\|_{L^2(\Omega; H^1(\Omega))}.
$$

After applying Young’s inequality with the small parameter $\varepsilon > 0$, we get

$$
\frac{1}{2} \frac{d}{dt} \|\rho^{H,h}\|^2_{L^2(\Omega \times Y)} + (D - \varepsilon) \|\rho^{H,h}\|^2_{L^2(\Omega; H^1(\Omega))}, \tag{46}
$$

$$
\leq D \|\rho^{H,h}\|^2_{L^2(\Omega \times Y)} + c_E^2 \varepsilon |\Gamma_R|^2 \left( |\pi^H\|_{L^2(\Omega)} + p^2_F |\Omega|^2 \right).
$$

By applying Grönwall’s inequality we obtain the desired estimates:

$$
\|\rho^{H,h}\|^2_{L^2(\Omega \times Y)} \leq C_p e^{D_1 t}, \tag{47}
$$

$$
\|\nabla \rho^{H,h}\|^2_{L^2(\Omega \times Y)} \leq C_p + \varepsilon \|\rho^{H,h}\|_{L^2(\Omega \times Y)}, \tag{48}
$$

with

$$
C_p = \kappa^2 c_E^2 |\Gamma_R|^2 \left( |\pi^H\|_{L^2(\Omega)} + p^2_F |\Omega|^2 \right).
$$

To obtain a bound on $\pi^H$, we test (18) with $\pi^H$ and then use (A3) and (A5):

$$
A \|\nabla \pi^H\|^2 = \int_{\Omega} f(\pi^H, \rho^{H,h}) \pi^H \, dx
$$

$$
= \int_{\Omega} \left( f(\pi^H, \rho^{H,h}) - f(0, \rho^{H,h}) + f(0, \rho^{H,h}) \right) \pi^H \, dx, \tag{49}
$$

$$
\leq \sqrt{\int_{\Omega} (c_p \pi^H)^2 \, dx} \|\pi^H\|_{L^2(\Omega)} \leq c_p \|\pi^H\|_{L^2(\Omega)} \|\nabla \pi^H\|_{L^2(\Omega)}.
$$

13
This yields the upper bound
\[
||\nabla_x \pi^H||_{L^2(\Omega)} \leq \frac{C}{A} ||\pi^H||_{L^\infty(\Omega)}.
\] (50)

With these lemmata we are ready to state and prove a first convergence result.

**Proposition 2** (Convergence of the Galerkin approximation). Let \((\pi^H, \rho^{H,h}) \in L^2(S; L^2(\Omega)) \times L^2(S; L^2(\Omega; H^1(Y)))\) be a solution to (18)-(19) and let \((\pi, \rho)\) be the weak solution to \((P_1)\). Then
\[
\pi^H \rightarrow \pi, \quad \rho^{H,h} \rightarrow \rho,
\] (51)
for \(H, h \rightarrow 0\).

**Proof.** To apply Lemma 1, we first need to show
\[
\partial_t \pi^H \in L^2(S; L^2(\Omega)).
\] (52)

Then, by choosing
\[
B_0 = H^1_0(\Omega), \quad B_1 = B = L^2(\Omega),
\] (53)
we satisfy the requirements of Aubin-Lions’ lemma to get convergence of \(\pi^H\) in \(L^2(\Omega)\).

Concerning \(\rho^{H,h}\), again we use Aubin-Lions’ lemma to prove convergence, this time for the following spaces:
\[
B_0 = H^1_0(\Omega; H^1(Y)), \quad B_1 = B = L^2(\Omega; L^2(Y)),
\] (54)

Note that \(\rho^{H,h} \in L^2(S; B)\). To conclude the argument, what remains to show are the following steps:
\[
\partial_t \rho^{H,h} \in L^2(S; L^2(\Omega; L^2(Y))),
\] (55)
and
\[
\nabla_x \rho^{H,h} \in L^2(S; L^2(\Omega; H^1(Y))).
\] (56)

We start with the estimates that provide us (52) and (55) and then we handle (56).
Let $f_\pi$ and $f_p$ denote the partial derivatives of $f$ to respectively $\pi$ and $\rho$. We introduce $u^H := \partial_\pi u^H$ and $v^h := \partial_t \rho^{H,h} - \partial_t I$. We differentiate (P1) with respect to $t$ and obtain the following system

\begin{align*}
-A \Delta_x u^H &= f_\rho v^h + f_\pi u^H \quad \text{in } S \times \Omega, \\
\partial_t v^h &= D \Delta_y v^h \quad \text{in } S \times \Omega \times Y, \\
D \nabla_y v^h &= \kappa (u^H - Rv) \quad \text{on } S \times \Omega \times \Gamma_R, \\
D \nabla_y v^h &= 0 \quad \text{on } S \times \Omega \times \Gamma_N, \\
\quad u^H &= 0 \quad \text{on } S \times \partial \Omega \\
\quad v(t = 0) &= 0 \quad \text{on } S \times \Omega \times Y
\end{align*}

By multiplying (58) with $v^h$ and integrating the result over $\Omega \times Y$, we obtain an equation that can be bounded similarly to (44). This gives:

\begin{align*}
\frac{1}{2} & \frac{d}{dt} \|v^h\|_{L^2(\Omega \times Y)}^2 + D \|\nabla_y v^h\|_{L^2(\Omega \times Y)}^2 \\
&= \kappa \int_{\Omega \times \Gamma_R} (u^H - Rv^h)v^h, \\
&\leq \kappa \int_{\Omega \times \Gamma_R} u^H v^h - \kappa R \int_{\Omega \times \Gamma_R} (v^h)^2, \\
&\leq \kappa \epsilon \left( c_\epsilon |\Gamma_R| \|u^H\|_{L^2(\Omega)}^2 + \epsilon \|v^h\|_{L^2(\Omega \times Y)}^2 \right), \\
&\leq \kappa \epsilon \left( c_\epsilon |\Gamma_R| \|u^H\|_{L^2(\Omega)}^2 + \epsilon (c_p + 1) \|\nabla_y v^h\|_{L^2(\Omega \times Y)}^2 \right).
\end{align*}

From (63), by integration in time we obtain

\begin{equation}
\|\partial_t \rho^{H,h}\|_{L^2(\Omega \times Y)}^2 \leq c_\epsilon |\Gamma_R| \|\partial_t \pi^H\|_{L^2(\Omega \times Y)}^2.
\end{equation}

By multiplying (57) with $v^h$ and integrating the result over $\Omega$, we obtain

\begin{equation}
A \|\nabla_x u^H\|_{L^2(\Omega)} = \int_\Omega (f_\pi u^H + f_\rho v^h) u^H \leq c_1 \|u^H\|_{L^2(\Omega)}^2 + c_2 \|v^h\|_{L^2(\Omega)}^2,
\end{equation}

which, by combining this inequality with Poincaré’s inequality for $u^H$, yields

\begin{equation}
\|u^H\|_{L^2(\Omega)} \leq c_p \|\nabla u^H\|_{L^2(\Omega)}.
\end{equation}

Thus, we obtain an upper bound that holds in the interior of $\Omega$, say $\Omega_\delta$.

\begin{equation}
\left( \frac{A}{c_p} - c_1 \right) \|\partial_t \pi^H\|_{L^2(\Omega_\delta)}^2 \leq c_2 \|\partial_t \rho^{H,h}\|_{L^2(\Omega_\delta \times Y)}^2.
\end{equation}

Here, $\Omega_\delta$ is defined as any subset of $\Omega$ such that $d(\partial \Omega, \Omega_\delta) \geq \delta$, ($d(\cdot, \cdot)$ measures the distance between two sets) and where $\delta > 0$. 

15
To handle the integral estimates on the boundary layer $\Omega \setminus \Omega$, we recall (19). For any $\Omega_\delta \subset \Omega$, by testing with $\psi = \partial_t \rho^{H,h}$ and integrating over the time domain, we obtain the identity

$$
2 \int_0^T ||\partial_t \rho^{H,h}||^2_{L^2(\Omega_\delta \times Y)} + ||\nabla_y \rho^{H,h}||^2_{L^2(\Omega_\delta \times Y)} - ||\nabla_y \rho^T_H||^2_{L^2(\Omega_\delta \times Y)} = 2 \kappa \int_0^T \int_{\Omega_\delta \times \Gamma_R} \rho^{H,h} \partial_t \pi^H \\
+ \kappa \rho^T_H \int_{\Omega_\delta \times \Gamma_R} \rho^{H,h}.
$$

Conveniently rearranging the terms of (67) yields:

$$
2 \int_0^T ||\partial_t \rho^{H,h}||^2_{L^2(\Omega \setminus \Omega_\delta)} + ||\nabla_y \rho^{H,h}||^2_{L^2(\Omega \setminus \Omega_\delta \times Y)} \\
\leq 2 \kappa ||\pi^H \partial_t \rho^{H,h}||^2_{L^2(\Omega_\delta \times \Gamma_R)} + \tilde{c} ||\rho^T_H||^2_{L^2(\Omega_\delta \times \Gamma_R)} + \tilde{c} ||\rho^{H,h}||^2_{L^2(\Omega_\delta \times \Gamma_R)} \\
+ ||\nabla_y \rho^T_H||^2_{L^2(\Omega_\delta \times \Gamma_R)} + 2 \kappa \int_0^T \int_{\Omega_\delta \times \Gamma_R} |\rho^{H,h} \partial_t \pi^H| \leq C\delta.
$$

Now we can extend the bound in (66) to hold on the entire domain $\Omega$, i.e.:

$$
\left(\frac{A}{c_p} - c_1\right) ||\partial_t \pi^H||^2_{L^2(\Omega)} = \sup_{\delta > 0} \left(\frac{A}{c_p} - c_1\right) ||\partial_t \pi^H||^2_{L^2(\Omega_\delta)} \leq C_2 ||\partial_t \rho^{H,h}||^2_{L^2(\Omega)}.
$$

Finally, to obtain (56) we adapt an interior regularity argument from [11], Chapter 6. We let $\Omega_\delta \subset W \subset \subset \Omega$ and define a smooth cutoff function $\zeta : \Omega \to [0,1]$ satisfying

$$
\begin{cases}
\zeta(x) = 1 & \text{for } x \in \Omega_\delta, \\
\zeta(x) = 0 & \text{for } x \in \Omega \setminus W.
\end{cases}
$$

We introduce the directional finite difference

$$
D_i^\lambda \rho^{H,h} := \frac{\rho^{H,h}(t, x + \lambda e_i, y) - \rho^{H,h}(t, x, y)}{\lambda}
$$

for $\lambda > 0$.

We let $\lambda$ be small and we test (19) with

$$
\psi = -D_i^{-\lambda} \zeta^2 D_i^\lambda \rho,
$$

which gives us:

$$
- \int \partial_t \rho D_i^{-\lambda} \zeta^2 D_i^\lambda \rho - \int \nabla_y \rho \cdot \nabla_y D_i^{-\lambda} \zeta^2 D_i^\lambda \rho = -\kappa \int (\pi + \rho F - R \rho) D_i^{-\lambda} \zeta^2 D_i^\lambda \rho.
$$
Because of the properties of the support of $\zeta$, it holds that for any $f \in \Omega$
\[ \int_{\Omega} \psi D_i^{-\lambda} f = - \int_{\Omega} f D_i^\lambda \psi. \] (74)
Applying the property in (74) to (73) yields
\[ \int_{\Omega \times Y} \zeta D_i^\lambda \partial_t \rho^{H,h} D_i^\lambda \rho^{H,h} + D \int_{\Omega \times Y} \zeta^2 D_i^\lambda \nabla_y \rho^{H,h} \cdot D_i^\lambda \nabla_y \rho^{H,h} \]
\[ = \kappa \int_{\Omega \times \Gamma_R} \zeta^2 D_i^\lambda (\pi^H + p_F - \nabla_y \rho^{H,h}) D_i^\lambda \rho^{H,h}, \]
leaving to
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega \times Y} |\zeta D_i^\lambda \rho^{H,h}|^2 + D \int_{\Omega \times Y} |\zeta D_i^\lambda \nabla_y \rho^{H,h}|^2 \]
\[ = \kappa \int_{\Omega \times \Gamma_R} \zeta^2 D_i^\lambda \pi^H D_i^\lambda \rho^{H,h} - \kappa R \int_{\Omega \times \Gamma_R} |\zeta D_i^\lambda \rho^{H,h}|^2. \] (76)
Using Young’s inequality combined with the inequality, we estimate the third term of (76) as follows:
\[ \kappa \int_{\Omega \times \Gamma_R} \zeta^2 D_i^\lambda \pi^H D_i^\lambda \rho^{H,h} \]
\[ \leq \kappa|\Gamma_R| \||\nabla \pi^H||_{L^2(\Omega \times \Gamma_R)} \||\nabla_y \rho^{H,h}||_{L^2(\Omega \times \Gamma_R)}^2, \]
\[ \leq C_\varepsilon \kappa|\Gamma_R| \||\nabla \pi^H||_{L^2(\Omega \times \Gamma_R)} \||\nabla_y \rho^{H,h}||_{L^2(\Omega \times \Gamma_R)}^2, \]
\[ \leq C_\varepsilon \kappa|\Gamma_R| \||\nabla \pi^H||_{L^2(\Omega \times \Gamma_R)} \||\nabla_y \rho^{H,h}||_{L^2(\Omega \times Y)} \]
\[ \leq C_\varepsilon \kappa|\Gamma_R| \||\nabla \pi^H||_{L^2(\Omega \times \Gamma_R)} \|D_i^\lambda \rho^{H,h}||_{L^2(\Omega \times Y)} \]
\[ \leq C_\varepsilon \kappa|\Gamma_R| \||\nabla \pi^H||_{L^2(\Omega \times Y)} \|D_i^\lambda \rho^{H,h}||_{L^2(\Omega \times Y)} \]
\[ + \frac{\varepsilon^2}{2} \||\nabla \pi^H||_{L^2(\Omega \times Y)} \|
\]
Now, combining (77) with (76), we obtain the required estimate for (56):
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega \times Y} |\zeta D_i^\lambda \rho^{H,h}|^2 \]
\[ \leq C_\varepsilon \kappa|\Gamma_R| \||\nabla \pi^H||_{L^2(\Omega \times Y)} + \frac{\varepsilon^2}{2} \||\nabla \rho^{H,h}||_{L^2(\Omega \times Y)} \]
Using Grönwall’s inequality, we conclude that $D_i^\lambda \rho^{H,h} \in L^2(\Omega \times Y)$, and by letting $\lambda \to 0$, we obtain
\[ \nabla_x \rho^{H,h}, \nabla_x \nabla_y \rho^{H,h} \in L^2(S \times \Omega \times Y). \] (79)
With the newly found estimates (71), (79) and (64), we are able to apply Lemma 1 and we obtain that
\[ W \hookrightarrow L^2(S \times \Omega \times Y), \]
which proves: \[(π^H, ρ^{H,h}) \to (π, ρ)\]
for \(h, H \to 0\).

The preliminary work allows us to state the first main result of this paper.

**Theorem 1** (Well-posedness of the system). *The system in (18)-(19) has a unique solution \(π^H \in L^2(S; V^H)\) and \(ρ^{H,h} \in L^2(S; V^H \times W^h)\).*

*Proof.* The proof of this theorem is a direct result of Proposition 1 and Proposition 2. \(\square\)

4 Convergence rates for semidiscrete Galerkin approximations

In this section, we obtain convergence rates of the numerical approximations (18) – (19). The following argument is largely based on standard arguments from [17], adapted to multiscale systems.

**Proposition 3** (Regularity lift). *Recall \((A_4)\) and \((A_1)\). If \((π^H, ρ^{H,h})\) is a solution to (7)-(8), then

\[π^H \in L^2(S; H^2(Ω)),\]
\[ρ^{H,h} \in L^2(S; H^2(Ω; H^2(Y))).\]*

*Proof.* We omit the proof and refer to [13]. \(\square\)

**Lemma 4** (Interpolation-trace inequality). *Let \(u \in L^2(Ω; L^2(Γ_R))\), and let \(Γ_R \subset \partial Y\). Then

\[||u||^2_{L^2(Ω; L^2(Γ_R))} \leq ε||\nabla_y u||^2_{L^2(Ω; L^2(Y))} + c_ε(c_ε + 1)||u||^2_{L^2(Ω; L^2(Y))},\] (80)

with trace constant \(c_ε\) independent of \(ε\) and \(c_ε = (\sqrt{2}ε)^{-1}\).

*Proof.* The proof follows from applying Young’s inequality with a small parameter \(ε\) to the standard trace inequality. \(\square\)

Let \(R_h\) and \(R_H\) be the microscopic and macroscopic Ritz projection operator respectively.

**Lemma 5** (Projection error estimates). *Then there exists strictly positive constants \(γ_l\) \((with \ l \in \{1, 2, 3, 4\}\) ), independent of \(h\) and \(H\), such that projections \(R_h π\) and \(R_H ρ\) that satisfy

\[||π - R_H π||_{L^2(Ω)} \leq γ_1 H^2||π||_{H^2(Ω)},\] (81)
\[||π - R_H π||_{H^1(Ω)} \leq γ_2 H||π||_{H^2(Ω)},\] (82)
\[||ρ - R_H R_H ρ||_{L^2(Ω; L^2(Y))} \leq γ_3(H^2 + h^2)||ρ||_{L^2(Ω; H^2(Y) \cap L^2(Y; H^2(Ω))},\] (83)

for all \((π, ρ) \in H^2(Ω) \times [L^2(Ω; H^2(Y)) \cap L^2(Y; H^2(Ω))].\)
Proof. (81) and (82) are standard Ritz projection error estimates. For details on the proof, see for instance [30] and [17]. Specific to this context, (83) is a two-scale estimate which accounts for the presence of the microscopic Robin boundary condition (3) and therefore requires some tuning. See e.g. [22] for similar estimates. Here, we only present the proof of (83).

Let $\omega := R_h \rho - \rho$. Let $\varphi \in L^2(\Omega; H^2(\gamma))$ be the weak solution to

\[
(P_2) \begin{cases}
-\Delta \varphi = \omega & \text{in } \Omega \times Y, \\
-\nabla \varphi \cdot n = \alpha \varphi & \text{on } \Omega \times \Gamma_R, \\
-\nabla \varphi \cdot n = 0 & \text{on } \Omega \times \Gamma_N.
\end{cases}
\]

(84)

We denote the Ritz projection error of $\varphi$ with $e_\varphi$. By testing with $\psi$ and integrating over $\Omega \times Y$, we obtain

\[
\langle \omega, \psi \rangle_{L^2(\Omega; L^2(\gamma))} = \langle \nabla \varphi, \nabla \psi \rangle_{L^2(\Omega; L^2(\gamma))} + \langle \nabla \varphi \cdot n, \psi \rangle_{L^2(\Omega; L^2(\Gamma_R))}.
\]

(85)

Testing with $\psi = \omega$ specifically, subtracting the Galerkin approximation from the weak solution and using $(R_h \Delta \varphi, \omega) = 0$, we obtain:

\[
||\omega||_{L^2(\Omega; L^2(\gamma))}^2 = \langle \nabla \omega, \nabla \varphi \rangle_{L^2(\Omega; L^2(\gamma))} + \langle \alpha \varphi, \omega \rangle_{L^2(\Omega; L^2(\Gamma_R))},
\]

\[
= \langle \nabla e_\varphi, \nabla \varphi \rangle_{L^2(\Omega; L^2(\gamma))} + \langle \alpha e_\varphi, \omega \rangle_{L^2(\Omega; L^2(\Gamma_R))},
\]

\[
\leq c_\varepsilon ||\nabla e_\varphi||_{L^2(\Omega; L^2(\gamma))} ||\nabla \varphi||_{L^2(\Omega; L^2(\gamma))} + \varepsilon ||e_\varphi||_{L^2(\Omega; L^2(\gamma))} ||\omega||_{L^2(\Omega; L^2(\gamma))}.
\]

(86)

Applying the Ritz projection estimates (81) and (82), we obtain the following bound:

\[
||\omega||_{L^2(\Omega; L^2(\gamma))}^2 \leq c_\varepsilon h^2 ||\varphi||_{H^2(\Omega; \gamma)}^2 + \varepsilon h^2 ||\omega||_{L^2(\Omega; L^2(\gamma))}.
\]

Using Friedrich’s inequality $||\varphi||_{H^2(\Omega; \gamma)} \leq C ||\Delta \varphi||_{L^2(\Omega; L^2(\gamma))} = C ||\omega||_{L^2(\Omega; L^2(\gamma))}$ for some $C$ and choosing $\varepsilon < c$ we obtain

\[
(1 - \varepsilon)||\omega||_{L^2(\Omega; L^2(\gamma))}^2 \leq C h^2 ||\omega||_{L^2(\Omega; L^2(\gamma))}.
\]

(86) yields:

\[
||\omega||_{L^2(\Omega; L^2(\gamma))} \leq ||R_h \rho - \rho||_{L^2(\Omega; L^2(\gamma))} \leq \gamma_3 h^2.
\]

(87)

Finally, we can derive (83) as follows:

\[
||\psi - R_H R_h \psi||_{L^2(\Omega; L^2(\gamma))} = ||\psi - R_h \psi + R_h \psi - R_H R_h \psi||_{L^2(\Omega; L^2(\gamma))},
\]

\[
\leq ||\psi - R_h \psi||_{L^2(\Omega; L^2(\gamma))} + ||R_h \psi - R_H R_h \psi||_{L^2(\Omega; L^2(\gamma))},
\]

\[
\leq \gamma_3 h^2 ||\psi||_{L^2(\Omega; H^2(\gamma))} + \tilde{\gamma}_4 H^2 ||R_h \psi||_{L^2(\gamma; H^2(\gamma))},
\]

\[
\leq \gamma_3 (H^2 + h^2) ||\psi||_{L^2(\Omega; H^2(\gamma)) \cap L^2(\gamma; H^2(\gamma))}.
\]

(88)
By applying Lemma 4 and Lemma 5, we can finally obtain the desired convergence rates. Let us denote the errors of the Galerkin projection as
\[
e_{\pi} := \pi - \pi^{H},
\]
\[
e_{\rho} := \rho - \rho^{H,h}.
\]

**Theorem 2** (Convergence rates). Let \((\pi^{H}, \rho^{H,h})\) be a solution to (7)-(8). Then the following statement holds: there exist constants \(M_1, M_2 > 0\) independent of \(h\) and \(H\), such that
\[
||e_{\pi}||_{L^{\infty}((0,T);L^{2}(\Omega))} \leq C(H^2 + h^2),
\]
\[
||e_{\rho}||_{L^{\infty}((0,T);L^{2}(\Omega;L^{2}(Y)))} \leq C(H^2 + h^2).
\]

**Proof.** By testing (7) with \(\pi\) and \(\pi^{H}\) and testing (18) with \(\pi^{H}\), we obtain the following identities:
\[
A \int_{\Omega} \nabla x \pi \cdot \nabla x \pi dx = \int_{\Omega} f(\pi, \rho)\pi dx,
\]
\[
- A \int_{\Omega} \nabla x \pi \cdot \nabla x \pi^{H} dx = \int_{\Omega} f(\pi, \rho)\pi^{H} dx,
\]
\[
+ A \int_{\Omega} \nabla x \pi \cdot \nabla x \pi^{H} dx = \int_{\Omega} f(\pi, \rho)\pi^{H} dx,
\]
\[
- A \int_{\Omega} \nabla x \pi^{H} \cdot \nabla x \pi^{H} dx = \int_{\Omega} f(\pi^{H}, \rho^{H})\pi^{H} dx.
\]
which, summed up, results in the following identity:
\[
A \int_{\Omega} \nabla x e_{\pi} \cdot \nabla x e_{\pi} = \int_{\Omega} f(\pi, \rho)e_{\pi} - (f(\pi, \rho) - f(\pi^{H}, \rho^{H,h}))\pi^{H} dx.
\]

Let \(\phi^{H} \in V^{H}\) be arbitrary. By applying the identity
\[
A \int_{\Omega} \nabla x \pi^{H} \cdot \nabla x \phi^{H} - \nabla x \pi \cdot \nabla x \phi^{H} dx = \int_{\Omega} (f(\pi, \rho) - f(\pi^{H}, \rho^{H,h}))\phi^{H} dx,
\]
and using (95), we obtain the following estimate (omitting \(L^{2}\) norm indications

20
in their respective spaces for clarity):

\[ \frac{A}{c_p} ||e_{\pi}||^2 \leq A ||\nabla_x e_{\pi}||^2 \]

\[ = A \int_{\Omega} \nabla_x (\pi - \pi^H) \cdot \nabla_x (\pi - \pi^H) \, dx \]

\[ = A \int_{\Omega} \nabla_x (\pi - \pi^H) \cdot \nabla_x (\pi - \phi^H) \, dx \]

\[ + A \int_{\Omega} \nabla_x (\pi - \pi^H) \cdot \nabla_x (\phi^H - \pi^H) \, dx, \quad (97) \]

\[ \leq A ||\nabla_x e_{\pi}|| ||\nabla_x (\pi - \phi^H)|| + \int_{\Omega} (f(\pi, \rho) - f(\pi^H, \rho^{H,h})(\phi^H - \pi^H) \, dx, \]

\[ \leq A ||\nabla_x e_{\pi}|| ||\nabla_x (\pi - \phi^H)|| + \left| \int_{\Omega} (f(\pi, \rho) - f(\pi^H, \rho^{H,h})) (\phi^H - \pi) \, dx \right|, \]

\[ \leq \frac{A}{2} \left( ||\nabla_x e_{\pi}||^2 + ||\nabla_x (\pi - \phi^H)||^2 \right) \]

\[ + (c_{\pi} ||e_{\pi}|| + c_\rho ||e_\rho||) ||\phi^H - \pi|| + c_\rho ||e_\rho|| ||e_{\pi}|| + c_{\pi} ||e_{\pi}||^2. \]

Moving the first and last term of (97) to the left hand side, we obtain the following inequality, which can be further bounded as follows:

\[ \left( \frac{A}{2c_{\pi}} - c_{\pi} \right) ||e_{\pi}||^2 \leq \frac{A}{2} ||\nabla_x (\pi - \phi^H)||^2 + (c_{\pi} ||e_{\pi}|| + c_\rho ||e_\rho||) ||\phi^H - \pi|| \]

\[ + c_\rho ||e_\rho|| ||e_{\pi}|| \]

\[ \leq \frac{A}{2} ||\nabla_x (\pi - \phi^H)||^2 + 2\epsilon ||e_{\pi}||^2 + \left( \frac{c_{\pi}^2}{2} + \frac{c_\rho^2}{4\epsilon} \right) ||e_\rho||^2 \]

\[ + \left( \frac{c_{\pi}^2}{2} + \frac{c_\rho^2}{4\epsilon} \right) ||\phi^H - \pi||^2. \quad (98) \]

Finally, by compensating the small terms and using the finite element approximation property:

\[ \min_{\chi \in V_h} ||\phi^H - \chi||_{H^1(\Omega)} \leq CH ||\phi||_{H^2(\Omega)}, \quad (99) \]
applied to (98) we obtain the inequality
\[
\left( \frac{A}{2c_p} - 2 \varepsilon \right) \| e \|_{L^2(\Omega)}^2 \leq \left( \frac{c_p^2}{2} + \frac{c_p^2}{4\varepsilon} \right) \| e \|_{L^2(\Omega; L^2(Y))}^2 \\
+ \max \left\{ \frac{c_p^2}{2} + \frac{c_p^2}{4\varepsilon}, \frac{A}{2} \right\} \| \varphi^H - \pi \|_{H^1(\Omega)}^2 \\
\leq \left( \frac{c_p^2}{2} + \frac{c_p^2}{4\varepsilon} \right) \| e \|_{L^2(\Omega; L^2(Y))}^2 + \frac{CH^2 \| \pi \|_{H^2(\Omega)}}{2}.
\]
(100)

with C a generic constant independent of H.

Continuing, from (19), we get
\[
e^\rho = \rho^{H,h} - \rho = (\rho^{H,h} - R_hR_h\rho) + (R_hR_h\rho - \rho) =: \theta + \psi.
\]
(101)

We bound \( \psi \) by using Lemma 5:
\[
\| \psi(t) \|_{L^2(\Omega; L^2(Y))} \leq \gamma_3 (H^2 + h^2) \| \rho \|_{L^2(\Omega; H^2(\Omega)) \cap L^2(Y; H^2(\Omega))},
\]
\[
= \gamma_3 (H^2 + h^2) \left| \rho I + \int_0^t \partial_t \rho ds \right|_{L^2(\Omega; H^2(\Omega)) \cap L^2(Y; H^2(\Omega))},
\]
(102)

and bound \( \theta \) from (101) using the formulation: for all \( \varphi \in V^h \) we have that
\[
\langle \partial_t \theta, \varphi \rangle_{L^2(\Omega; L^2(Y))} + D(\nabla \theta, \nabla \varphi)_{L^2(\Omega; L^2(Y))} \\
= -\langle R_h \partial_t \rho, \varphi \rangle_{L^2(\Omega; L^2(Y))} - D(\nabla \rho, \nabla \varphi)_{L^2(\Omega; L^2(Y))},
\]
\[
= \langle \partial_t \rho - R_h \partial_t \rho, \varphi \rangle_{L^2(\Omega; L^2(Y))},
\]
\[
= \langle \partial_t \psi, \varphi \rangle_{L^2(\Omega; L^2(Y))}.
\]
(103)

Substituting \( \varphi = \theta \) in (103) yields:
\[
\frac{1}{2} \left\| \theta \right\|_{L^2(\Omega; L^2(Y))}^2 + D \left\| \nabla \theta \right\|_{L^2(\Omega; L^2(Y))}^2 \\
= \langle \partial_t \rho - R_h \partial_t \rho, \theta \rangle, \\
\leq \left\| \partial_t \rho - R_h \partial_t \rho \right\|_{L^2(\Omega; L^2(Y))} \left\| \theta \right\|_{L^2(\Omega; L^2(Y))}, \\
\leq \gamma_3 (h^2 + H^2) \left\| \partial_t \rho \right\|_{L^2(\Omega; L^2(Y))} \left\| \theta \right\|_{L^2(\Omega; L^2(Y))},
\]
(104)
Dividing the left and right hand side of (104) by $||\theta||$, we obtain:

$$\frac{d}{dt}||\theta||_{L^2(\Omega;L^2(Y))} \leq \gamma_3(h^2 + H^2)||\partial_t \rho||_{L^2(\Omega;H^2(Y))},$$

$$||\theta(t)||_{L^2(\Omega;L^2(Y))} \leq ||\theta(0)||_{L^2(\Omega;L^2(Y))} + \gamma_3(h^2 + H^2) \int_0^t ||\partial_t \rho||_{L^2(\Omega;H^2(Y))}dx,$$

$$\leq ||\rho_{H,h}^I - \rho_I||_{L^2(\Omega;L^2(Y))} + ||\rho_I - \mathcal{R}_H \mathcal{R}_h \rho_I||_{L^2(\Omega;L^2(Y))}$$

$$+ \gamma_3(h^2 + H^2) \int_0^t ||\partial_t \rho||_{L^2(\Omega;H^2(Y))}dx,$$

$$\leq \gamma_3(h^2 + H^2) \left( c_I + C + \int_0^t ||\partial_t \rho||_{L^2(\Omega;H^2(Y))}dx \right).$$

(105)

Because of $(A_4)$, the Galerkin projection error of the initial condition satisfies:

$$||\rho_I - \rho_{H,h}^I||_{L^2(\Omega;L^2(Y))} \leq c_I(H^2 + h^2).$$

(106)

Combining (102) and (105) proves the desired estimate in (90).

$$||\rho_{H,h} - \rho||_{L^2(\Omega;L^2(Y))} = ||\theta + \psi||_{L^2(\Omega;L^2(Y))} \leq C(H^2 + h^2)||\partial_t \rho||_{L^2(\Omega;H^2(Y))}.$$ 

(107)  

Finally, (89) follows by combining (107) with (100). 

## 5 A posteriori refinement strategy

In this section we develop a computable error estimator which we will use to refine the finite element grid and obtain a lower overall error for the macroscopic equation (18). In this strategy, we aim for reliable error estimators, i.e. estimators which provide an upper and lower bound on the error.

There is a difference in usability between (a priori) error bounds and (a posteriori) error estimators. Where error bounds guarantee the upper bound of the error, often they are not sharp, not computable, and mainly useful for proving well-posedness of the numerical approximation. Error estimators, on the other hand, should be computable quantities approximating the true value of the error, and preferably provide both an upper bound and an lower bound on the error. An upper bound is required to guarantee the maximum error satisfies a certain tolerance. A lower bound makes sure that the error estimator does not overestimate the true error too much, ensuring the efficiency of the refinement strategy.

For a review on the different strategies in error control, we refer to e.g. [4] and [12]. The line of arguments we present, is based on [32].

In this section, we only describe how to obtain error estimators for the macroscopic equation. Strategies to obtain error estimators for parabolic equations can be found in e.g. [3], [24], [10] and [9].
5.1 Mesh-related notation

We assume the mesh partition $B_H$ with diameter $H$ as defined in Section 3. For each element $B \in B_H$, we denote the set of vertices with $\mathcal{V}(B)$ and the set of edges with $\mathcal{E}(B)$. The complete set of edges is denoted with $\mathcal{E}$. Where necessary, we differ between vertices (edges) in $\Omega$ and on $\partial \Omega$ by denoting them as $B_{H,\Omega}$ ($\mathcal{V}_{H,\Omega}$) and $B_{H,\partial \Omega}$ ($\mathcal{V}_{H,\partial \Omega}$), respectively. To denote patches in $\Omega$ with certain structures, we use the following symbols:

- $\omega_B$ denotes the union of all elements that share an edge with $B$.
- $\tilde{\omega}_B$ denotes the union of all elements that share a point with $B$.
- $\omega_E$ denotes the union of all elements adjacent to $E$.
- $\tilde{\omega}_E$ denotes the union of all elements that share a point with $B$.
- $\omega_x$ denotes the union of all elements that have $x$ as a vertex.

Furthermore, for any $E \in \mathcal{E}$, $n_E$ denotes the unit vector orthogonal to $E$ and $J_E(v)$ denotes the jump across $E$ of some piece-wise continuous function $v$ in the direction of $n_E$.

Finally, for legibility we use the following notation to refer to norms on elements or edges.

\[
\|u\|_B := \|u\|_{L^2(B)},
\]
\[
\|u\|_E := \|u\|_{L^2(E)}.
\]

5.2 Auxiliary results

For any $x \in \mathcal{V}(B)$, we denote the piece-wise linear basis function that takes value 1 in $x$ and 0 in the other nodes by $\lambda_x$. This allows us to define the following cutoff functions for any $B \in B_H$:

\[
\psi_B := \alpha_B \Pi_{x \in \mathcal{V}(B)},
\]

and for any $E \in \mathcal{E}$

\[
\psi_E := \alpha_E \Pi_{x \in \mathcal{V}(E)}.
\]

Here, the coefficients $\alpha_E$, $\alpha_B$ are chosen such that:

\[
\max_{x \in B} \psi_B(x) = \max_{x \in E} \psi_E(x) = 1.
\]

With the cutoff functions defined in (108) and (109), we can obtain the
following bounds for any \( v \in \mathcal{H}^1(B) \) and \( \varphi \in \mathcal{H}^1(E) \):

\[
\begin{align*}
\theta_1 \|v\|_B^2 &\leq \int_B \psi_B v^2 \, dx \leq \|v\|_B^2, \\
\|\phi_B v\|_{\mathcal{H}^1(B)} &\leq \frac{\theta_2}{H_B} \|v\|_B, \\
\theta_3 \|\varphi\|_E^2 &\leq \int_E \psi_E \varphi^2 \, dx \leq \|\varphi\|_E^2, \\
\|\psi_E \varphi\|_{\mathcal{H}^1(\omega_E)} &\leq \frac{\theta_4 \sqrt{H}}{\|\varphi\|_E}, \\
\|\psi_E \varphi\|_{L^2(\omega_E)} &\leq \theta_5 \sqrt{H} \|\varphi\|_E,
\end{align*}
\] (111)

for any element \( B \in \mathcal{B} \), edge \( E \in \mathcal{E} \).

In order to obtain error estimates on both elements and edges, we introduce a quasi-interpolation operator \( I_H \) defined as

\[
I_H \varphi := \sum_{x \in \mathcal{V}_H} \frac{\int_{\omega_x} \varphi \, dx}{|\omega_x|}.
\] (112)

This allows for the following estimates:

\[
\begin{align*}
\|v - I_H v\|_B &\leq c_{I1} H_B \|v\|_{\mathcal{H}^1(\omega_B)}, \\
\|v - I_H v\|_E &\leq c_{I2} \sqrt{H} \|v\|_{\mathcal{H}^1(\omega_E)}.
\end{align*}
\] (113)

We omit the derivation of (111) and (113) and refer the reader to [7] and Section 3.1 of [31], respectively.

### 5.3 Macroscopic error estimator

The error estimator is composed from the jump discontinuities on each edge and the residuals in each patch. Formally defined, for any \( B \in \mathcal{B}_H \) and \( E \in \mathcal{E} \), we define

\[
\begin{align*}
R_B(\pi^H) &:= A \Delta \pi^H + f(\pi^h, \rho^{H,h}), \quad B \in \mathcal{B}, \\
R_E(\pi^H) &:= \begin{cases} 
- J_E(n_E \cdot A \nabla u_h), & E \in \mathcal{E}_{H,\Omega}, \\
0, & E \in \mathcal{E}_{H,\partial\Omega},
\end{cases}
\end{align*}
\] (114)

which can be combined into a complete residual operator \( R : V_H \rightarrow \mathbb{R} \), defined implicitly in the following identity:

\[
\langle R(\pi^H), \varphi \rangle = \sum_{B \in \mathcal{B}_H} \int_B R_B(\pi^H) \varphi \, dx + \sum_{E \in \mathcal{E}_H} \int_E R_E(\pi^H) \varphi \, dx.
\] (115)

(115) must hold for all \( \varphi \in V_H \).
Recall $e_x = \pi - \pi^H$ denotes the finite element error of $\pi$. There is an equivalence between the norm of residual $R$ and the norm of true error. Subtracting (18) from (7), we obtain

$$\int_{\Omega} \nabla e_x \cdot \nabla \varphi dx = \int_{\Omega} f(\pi, \rho)\varphi dx - \int_{\Omega} A\nabla \pi^H \cdot \nabla \varphi dx,$$

$$= \sum_{B \in B_H} \int_{B} f(\pi, \rho)\varphi - A\nabla \pi^H \cdot \nabla \varphi dx,$$

$$= \sum_{B \in B_H} \int_{B} f(\pi, \rho)\varphi + A\Delta \pi^H \varphi dx - \int_{\partial B} A\varphi n_B \cdot \nabla \pi^H dx,$$

$$= \sum_{B \in B_H} \left( f(\pi, \rho) + A\Delta \pi^H \right) dx + \sum_{E \in E} \int_{E} J_E(-n_E \cdot \nabla \pi^H)\varphi dx,$$

$$= \langle R(\pi^H), \varphi \rangle. \quad \text{(116)}$$

Picking a suitable $c_*$ and $c_1$, by applying Poincaré’s inequality and Cauchy-Schwarz’ inequality shows that:

$$c_* ||e_x||_{L^2(\Omega)}^2 \leq ||\nabla e_x||_{L^2(\Omega)}^2 = \int_{\Omega} \nabla e_x \cdot \nabla e_x dx \leq ||R(\pi^H)||_{H^{-1}(\Omega)}^2. \quad \text{(117)}$$

On the other hand, recalling (14) and picking $\varphi$ such that

$$\langle R(\pi^H), \phi \rangle = ||R(\pi^H)||_{H^{-1}(\Omega)},$$

we obtain

$$\langle R(\pi^H), \varphi \rangle \leq ||\nabla \pi||_{L^2(\Omega)} ||\nabla \varphi||_{L^2(\Omega)},$$

$$\leq e^* ||e_x||_{H^1(\Omega)} ||\varphi||_{H^1(\Omega)},$$

$$= e^* ||e_x||_{H^1(\Omega)}, \quad \text{(118)}$$

showing equivalence of the norms.

Although (115) is a reliable estimator, it is not a computable quantity. Therefore, we introduce a new quantity $\eta_R$, which is based only on computable values:

$$\eta^2_{R,B} := H_B^2 ||R_B\pi^H||_{L^2(\Omega)}^2 + \sum_{E \in E(B)} \beta_E H ||R_E(\pi^H)||_{E}^2,$$

$$\eta^2_R := \sum_{B \in B} \eta^2_{R,B}. \quad \text{(119)}$$

Here, $\beta_E = \frac{1}{2}$ if $E \in \mathcal{E}_\Omega$, and $\beta_E = 1$ if $E \in \mathcal{E}_{\partial \Omega}$.
Theorem 3 (Reliable error estimation). Assume \((A_1)\) and \((A_5)\) For every \(t \in S\), the error norm \(\|e_\pi\|\) can be approximated by the error estimator \(\eta_R\). The following bounds hold:

\[
c_* \|e_\pi\|_{H^1(\Omega)} \leq \eta_R \leq c^* \|e_\pi\|_{H^1(\Omega)} + O(H, h^2). \tag{120}
\]

Proof. For any \(\varphi \in V^H\) with \(\|\varphi\|_{L^2(\Omega)} = 1\), the following estimate holds:

\[
\langle R(\pi^H), \varphi \rangle = \langle R(\pi^H), \varphi - I_H \varphi \rangle, \\
= \sum_{B \in B_H} \int_B R_B(\pi^H)(\varphi - I_H \varphi) + \sum_{E \in \mathcal{E}} \int_E R_E(\pi^H)(\varphi - I_H \varphi), \\
= \sum_{B \in B_H} ||R_B(\pi^H)||_B \|\varphi - I_H \varphi\|_B + \sum_{E \in \mathcal{E}} ||R_E(\pi^H)||_E \|\varphi - I_H \varphi\|_E, \\
\leq \sum_{B \in B_H} c_{11} H ||R_B(\pi^H)||_B \|\varphi\|_{H^1(\bar{\omega}_B)} + \sum_{E \in \mathcal{E}} c_{12} \sqrt{H} ||R_E(\pi^H)||_E \|\varphi\|_{H^1(\bar{\omega}_E)}, \\
\leq \max(c_{11}, c_{12}) \left( \sum_{B \in B_H} H^2 ||R_B(\pi^H)||_B^2 + \sum_{E \in \mathcal{E}} H ||R_E(\pi^H)||_E^2 \right)^{1/2}, \\
\times \left( \sum_{B \in B_H} \|\varphi\|_{H^1(\bar{\omega}_B)}^2 + \sum_{E \in \mathcal{E}} \|\varphi\|_{H^1(\bar{\omega}_E)}^2 \right),
\leq c \left( \sum_{B \in B_H} H^2 ||R_B(\pi^H)||_B^2 + \sum_{E \in \mathcal{E}} H ||R_E(\pi^H)||_E^2 \right)^{1/2} \|\varphi\|_{H^1(\Omega)}, \\
\leq c \sum_R \eta_R. \tag{121}
\]

This provides us with the first inequality in (120).

The second inequality requires some auxiliary definitions. Let \(f^H\) denote the Galerkin projection of \(f\). We use \(\bar{R}_B\) and \(\bar{R}_E\) to denote the residuals where \(f\) is replaced with \(f^H\). Additionally, we introduce

\[
w_B = \psi_B \bar{R}_B(\pi^H), \\
w_E = \psi_E \bar{R}_E(\pi^H), \tag{122}
\]

to conveniently manipulate the residual norm. Finally, let \(C_f\) be a constant defined as

\[
C_f := \frac{C}{\theta_1} \left( \|\pi\|_{H^2(\Omega)} + \|\rho\|_{L^2(\Omega; H^2(Y))} \right). \tag{123}
\]

Here, \(C\) is a constant independent of \(\pi, \rho, h\) and \(H\).
Next, we bound each of the terms of $\eta$ to obtain a lower bound for the error.

\[
\theta_1 \left\| \tilde{R}_B(\pi^H) \right\|_B^2 \\
\leq \int_B \tilde{R}_B(\pi^H)^2 w_B = \int_B \tilde{R}_B(\pi^H) w_B \\
= \langle R(\pi^H), w_B \rangle + \int_B (f^H(\pi^H, \rho^{H,h}) - f(\pi, \rho)) w_B, \\
= \int_B \nabla \psi \cdot \nabla w_B + \int_B (f^H(\pi^H, \rho^{H,h}) - f(\pi, \rho)) w_B, \\
\leq c^* \| \psi \|_{B^1(B)} \| w_B \|_{B^1(B)} + \| f^H(\pi^H, \rho^{H,h}) - f(\pi, \rho) \|_B \| w_B \|_B, \\
\leq \frac{c^* \theta_2}{H} \left\| \tilde{R}_B(\pi^H) \right\|_B \| \psi \|_{B^1(B)} + \| f^H(\pi, \rho) \|_B \left\| \tilde{R}_B(\pi^H) \right\|_B \\
+ \theta_1 C_f(H^2 + h^2) \left\| \tilde{R}_B(\pi^H) \right\|_B. \\
\text{(124)}
\]

Dividing (124) once by its common factor and rearranging terms results in

\[
H \left\| \tilde{R}_B(\pi^H) \right\|_B \leq \frac{c^* \theta_2}{\theta_1} \| \psi \|_{B^1(B)} + \frac{1}{\theta_1} H \left\| f^H(\pi^H, \rho^{H,h}) - f(\pi, \rho) \right\|_B \\
+ H C_f(H^2 + h^2), \\
\text{(125)}
\]

which, after applying the triangle inequality results in

\[
H \left\| R_B(\pi^H) \right\|_B \leq \frac{c^* \theta_2}{\theta_1} \| \psi \|_{B^1(B)} + \left(1 + \frac{1}{\theta_1}\right) H \left\| f^H(\pi^H, \rho^{H,h}) - f(\pi, \rho) \right\|_B \\
+ H C_f(H^2 + h^2), \\
\text{(126)}
\]

To provide an upper bound on the second term of (120), we use the equivalence of the error and residual norms:

\[
\theta_3 \left\| R_E(\pi^H) \right\|_E^2, \\
\leq \int_B R_E(\pi^H) w_E, \\
= \langle R(\pi^H), w_E \rangle - \sum_{B \in \omega_E} \int_B R_B(\pi^H) w_E, \\
= \int_{\omega_E} \nabla \psi \cdot \nabla w_E - \sum_{B \in \omega_E} \int_B R_B(\pi^H) w_E, \\
\leq c^* \| \psi \|_{B^1(\omega_E)} \| w_E \|_{B^1(\omega_E)} + \sum_{B \in \omega_E} \left\| R_B(\pi^H) \right\|_B \| w_E \|_B, \\
\leq \frac{c^* \theta_1}{\sqrt{H}} \| \psi \|_{B^1(\omega_E)} \left\| R_E(\pi^H) \right\|_E \\
+ \sum_{B \in \omega_E} \theta_3 \sqrt{H} \left\| R_B(\pi^H) \right\|_B \left\| R_E(\pi^H) \right\|_E. \\
\text{28}
\]
Dividing both sides of (127) by its common factor results in:

\[
\sqrt{H} \||R_E(\pi^H)||_E \leq \frac{c^*}{\theta_3} \||e_\pi||_{\mathcal{H}^1(\omega_E)} + \sum_{B \in \omega_E} \theta_5 \sqrt{H} \||R_B(\pi^H)||_B.
\] (128)

Combining (127) with (128) results in

\[
\sqrt{H} \||R_E(\pi^H)||_E \leq \left( \frac{c^*}{\theta_3} + \frac{c^* \theta_2 \theta_3}{\theta_3 \theta_1} \right) \||e_\pi||_{\mathcal{H}^1(\omega_E)} + \frac{\theta_5}{\theta_3} \left( 1 + \frac{1}{\theta_1} \right) \sum_{B \in \omega_E} \||f^H(\pi^H, \rho^{H,h}) - f(\pi, \rho)||_B \]
\[
+ \frac{\theta_5}{\theta_3} C_f (H^2 + h^2),
\] (129)

which yields the following lower bound on the error:

\[
\eta_{R,B} \leq c \left( \||e_\pi||_{\mathcal{H}^1(\omega_E)} + H \sum_{B' \in \omega_B} \||f^H(\pi^H, \rho^{H,h}) - f(\pi, \rho)||_{B'} + \mathcal{O} (H^2, h^2) \right).
\] (130)

We remark that the presence of \( H \) and \( h^2 \) error terms is a typical feature of two-scale models.

### 5.4 Macroscopic mesh refinement strategy

The strong separation of space scales in our setting (microscopic vs. macroscopic) allows us to propose a macroscopic mesh refinement strategy only weakly biased by the distribution of microscopic errors, based on local error indicator \( \eta_{R,B} \) and global error indicator \( \eta_R \). Inspired by the popular and intuitive approach presented in e.g. [33], the goal of this refinement is to reduce the global approximation error and keep the error locally below a prescribed tolerance, i.e. select a refinement strategy satisfying

\[ \eta_R < \tilde{\eta}, \]

where we denote the desired tolerance with \( \tilde{\eta} \).

Our refinement strategy relies on the double sided estimate (120) stated in Theorem 3. This inequality gives us a satisfied upper bound on the global error. We solve (18) on \( \mathcal{B}_H \) for some \( t \), compute the error estimator, and evaluate if refinement is necessary. If so, we repeat this process until the error estimator has reduced to a satisfactory level.

The set of triangles to be refined on each iteration follows directly:

\[
Q_B = \left\{ B \in \mathcal{B}_H \mid \eta_{R,B} > \frac{\tilde{\eta}}{|B_H|} \right\}.
\] (131)
as well as boundary triangles

\[ Q'_B = \left\{ B' \in \bigcup_{B \in Q_B} \omega_B \setminus Q_B \right\} \]  \hspace{1cm} (132)

Each element \( B \in Q_B \) is partitioned into \( 2^{d_1} \) new elements (with \( d_1 \) the dimension of \( \Omega \)), while each element \( B' \in Q'_B \) is refined \( 2^{d_1-1} \) to ensure no vertices collide with edges. An illustration of this process in two dimensions is given in Figure 2 and Figure 3.

![Figure 2: Subset of \( B_H \). The subset \( Q_B \) is indicated in gray.](image)

![Figure 3: Subset of \( B_H \) after mesh refinement.](image)

The strategy can be summarized as follows:

(Step 1) Solve (18) on \( B_H \).
(Step 2) Compute \( \eta_{R,B} \) and \( \eta_R \).
(Step 3) Refine the mesh in \( Q_B \) and \( Q'_B \).
(Step 4) Repeat (Step 1 – Step 3) until \( \eta_R < \bar{\eta} \).

The convergence estimates from Theorem 3 ensure that this procedure will indeed halt for any fixed \( \bar{\eta} \).

6 Conclusion

We constructed a semidiscrete Galerkin approximation of our elliptic-parabolic two scale system (\( P_1 \)) and showed that this approximation is well-posed and that the obtained sequence of Galerkin approximants converges in suitable spaces to the weak solution to the continuous system. Furthermore, we derived \( a \ priori \) rates of convergence and proposed an \( a \ posteriori \) grid refinement strategy at the macroscopic scale.

As natural next steps, future work will address the fully discrete two-scale Galerkin approximation as well as the numerical implementation of the method.
so that the proven convergence rates can be confirmed and the macroscopic refinement strategy can be tested. Additionally, a mesh refinement strategy on the microscopic level could also be considered for either this problem setting, or for its elliptic-elliptic variant (obtained by letting $t \to \infty$ in $(P_1)$).

At this stage, we would like to remark that the interaction between $H_B$ and $h^2$ in the error structure gives an indication on how to choose the mesh size $h$ based on the error estimators in $H_B$. Without going into details, it is worth mentioning that in principle, one can choose the microscopic mesh size to correspond to the macroscopic grid. This way one can ensure that the macroscopic and microscopic errors are roughly of the same order.

## Acknowledgements

The authors acknowledge fruitful discussions with Prof. M. Asadzadeh (Chalmers University, Gothenburg, Sweden). OR thanks Dr. D. Tagami (Kyushu University, Japan) for valuable feedback and acknowledges partial support from Kungl. Vetenskapsakademien, Sweden. ML and AM thank Dr. O. Lakkis and Dr. C. Venkataraman (both with University of Sussex, UK) for the intensive interactions during the Hausdorff Trimester Program “Multiscale Problems: Algorithms, Numerical Analysis and Computation” (Bonn, January 2017).

## References

[1] R. A. Adams and J.J.F. Fournier. *Sobolev Spaces*, volume 140. Academic Press, 2003.

[2] J.P. Aubin. Un théorème de compacité. *CR Acad. Sci. Paris*, 256(24):5042–5044, 1963.

[3] I. Babuška and S. Ohnimus. A posteriori error estimation for the semidiscrete finite element method of parabolic differential equations. *Computer Methods in Applied Mechanics and Engineering*, 190(35):4691–4712, 2001.

[4] C. Carstensen and S. A. Funken. Fully reliable localized error control in the FEM. *SIAM Journal on Scientific Computing*, 21(4):1465–1484, 1999.

[5] V. Chalupecký and A. Muntean. Semi-discrete finite difference multiscale scheme for a concrete corrosion model: a priori estimates and convergence. *Japan Journal of Industrial and Applied Mathematics*, 29(2):289–316, Jun 2012.

[6] G.A. Chechkin and A.L. Piatnitski. Homogenization of boundary-value problem in a locally periodic perforated domain. *Applicable Analysis*, 71(1-4):215–235, 1998.
[7] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems*, volume 40 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics, 2002.

[8] K. R. Daly and T. Roose. Homogenization of two fluid flow in porous media. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 471(2176), 2015.

[9] K. Eriksson and C. Johnson. Adaptive finite element methods for parabolic problems i: A linear model problem. *SIAM Journal on Numerical Analysis*, 28(1):43–77, 1991.

[10] D. J. Estep, M. G. Larson, and R. D. Williams. *Estimating the Error of Numerical Solutions of Systems of Reaction-Diffusion Equations*, volume 696. American Mathematical Soc., 2000.

[11] L.C. Evans. *Partial Differential Equations*, volume 19. American Mathematical Society, 2010.

[12] T. Grätsch and K. Bathe. A posteriori error estimation techniques in practical finite element analysis. *Computers & structures*, 83(4-5):235–265, 2005.

[13] P. Grisvard. *Second-Order Elliptic Boundary Value Problems in Convex Domains*. SIAM, 2011.

[14] U. Hornung. *Homogenization and Porous Media*, volume 6. Springer Science & Business Media, 2012.

[15] E. R. Ijioma and S. E. Moore. Multiscale Galerkin approximation scheme for a system of quasilinear parabolic equations. *ArXiv e-prints*, 2018.

[16] V.G. Kouznetsova, M.G.D. Geers, and W.A.M. Brekelmans. Multi-scale second-order computational homogenization of multi-phase materials : a nested finite element solution. *Computer Methods in Applied Mechanics and Engineering*, 193(48-51):5525–5550, 2004.

[17] S. Larsson and V. Thomée. *Partial Differential Equations with Numerical Methods*, volume 45. Springer Science & Business Media, 2008.

[18] M. Lind and A. Muntean. A priori feedback estimates for multiscale reaction-diffusion systems. *Numerical Functional Analysis and Optimization*, 39(4):413–437, 2018.

[19] M. Lind, A. Muntean, and O. M. Richardson. Well-posedness and inverse Robin estimates for a multiscale elliptic/parabolic system. *Applicable Analysis*, 97(1):89–106, 2018.

[20] K.-F. Liu, Y.-H. Wu, and Y.-C. Hsu. Homogenization theory applied to unsaturated solid-liquid mixture. *Journal of Mechanics*, 28(2):329–335, 2012.
[21] S. A. Meier. Two-scale models for reactive transport and evolving microstructure. PhD thesis, Universität Bremen, Germany, 2008.

[22] A. Muntean and O. Lakkis. Rate of convergence for a Galerkin scheme approximating a two-scale reaction-diffusion system with nonlinear transmission condition. RIMS Kōkyūroku, 1693:85–98, 2010.

[23] A. Muntean and M. Neuss-Radu. A multiscale Galerkin approach for a class of nonlinear coupled reaction–diffusion systems in complex media. Journal of Mathematical Analysis and Applications, 371(2):705–718, 2010.

[24] R. Nochetto, A. Schmidt, and C. Verdi. A posteriori error estimation and adaptivity for degenerate parabolic problems. Mathematics of Computation of the American Mathematical Society, 69(229):1–24, 2000.

[25] M. Peszynska and R. E. Showalter. Multiscale elliptic-parabolic systems for flow and transport. Electr. J. Diff. Eqs., 2007(147):1–30, 2007.

[26] L. Preziosi and A. Farina. On Darcy’s law for growing porous media. International Journal of Non-Linear Mechanics, 37(3):485 – 491, 2002.

[27] M. Redeker, C. Rohde, and I. S. Pop. Upscaling of a tri-phase phase-field model for precipitation in porous media. IMA Journal of Applied Mathematics, 81(5):898–939, 2016.

[28] O.M. Richardson. Mathematical analysis and approximation of a multiscales elliptic-parabolic system. Licentiate thesis, Karlstad University, Sweden, 2018.

[29] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. Ann. Inst. Fourier (Grenoble), 15(1):189–258, 1965.

[30] V. Thomée. Galerkin Finite Element Methods for Parabolic Problems, volume 1054. Springer, 1984.

[31] R. Verfürth. A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques. John Wiley & Sons Inc, 1996.

[32] R. Verfürth. A review of a posteriori error estimation techniques for elasticity problems. Computer Methods in Applied Mechanics and Engineering, 176(1-4):419–440, 1999.

[33] O. C. Zienkiewicz and J. Z. Zhu. A simple error estimator and adaptive procedure for practical engineering analysis. International Journal for Numerical Methods in Engineering, 24(2):337–357, 1987.