On some problems of Euclidean Ramsey theory *

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Annotation.

In the paper we prove, in particular, that for any measurable coloring of the euclidian plane into two colours there is a monochromatic triangle with some restrictions on the sides. Also we consider similar problems in finite fields settings.

1 Introduction

Let $\Pi = \mathbb{R}^2$ be the ordinary euclidian plane. Any partition of $\Pi$ onto $k$ disjoint sets $C_1, \ldots, C_k$ is called $k$–coloring of $\Pi$ and the sets $C_1, \ldots, C_k$ are called colors. A well–known unsolved question of Euclidian Ramsey Theory (see [3], [10]) asks us about the existence of a monochromatic (that is belonging to the same color) non–equilateral triangle (that is just any three points from $\Pi$) in any two–coloring of the plane. The problem seems to be difficult and only partial results are known, see [10]. In particular, the question remains open even for the case of a degenerate triangle, having all three points lying on a line. A parallel and even more famous problem in the area is to find the chromatic number of the plane $\chi(\mathbb{R}^2)$, that is the smallest number of colors sufficient for coloring the plane in such a way that no two points of the same color are unit distance apart. It is well–known that $4 \leq \chi(\mathbb{R}^2) \leq 7$. In his beautiful paper [4] Falconer proved that if all colors are measurable sets then the correspondent measurable chromatic number of the plane is at least five. Our paper is devoted to a measurable analog of the considered two–coloring problem. The main result is the following, see Theorem 6 and Theorem 9 from section 3.

**Theorem 1** Let $ABC$ be a nondegenerate triangle such that $|AB|/|AC| = \omega$. Suppose that

$$\min_{t \geq 0} (J_0(t) + J_0(\omega t)) \geq -0.5972406,$$

where $J_0$ is the zeroth Bessel function. Then any measurable coloring of $\mathbb{R}^2$ into two colors contains a monochromatic triangle.

Further, if

$$\min_{t \geq 0} (J_0(t) + J_0(\kappa t) + J_0((1 + \kappa)t)) > -1.$$

Then for any measurable coloring of the plane $\Pi$ into two colors there is a monochromatic collinear triple $\{x, y, z\}$ such that $y \in [x, z]$ and $\|z - y\|/\|y - x\| = \kappa$.

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The proof uses simple Fourier analysis in spirit of paper [9] and hugely relies on the fact that we have deal just with two colors. Also we consider a model situation of the plane over the prime finite field $\mathbb{F}_p \times \mathbb{F}_p$ and prove (a slightly stronger) analog of Theorem 1, see section 2. The proof develops the method from [1], [6], [12].

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2 Finite fields case

Let $p$ be a prime number, and $\mathbb{F}_p$ be the prime field. Let also $\Pi = \mathbb{F}_p \times \mathbb{F}_p$ be the prime plane. If $x \in \Pi$ then we write $x = (x_1, x_2)$. For any $j \neq 0$ define a sphere in $\Pi$, that is the set

$$S_j = \{ x \in \Pi : \|x\| := x_1^2 + x_2^2 = j \}.$$

For any function $f : \Pi \to \mathbb{C}$ denote its Fourier transform as

$$\hat{f}(r) := \sum_{x \in \Pi} f(x) e^{-2\pi i (x_1 r_1 + x_2 r_2) / p} = \sum_{x \in \Pi} f(x) e^{2\pi i \langle x, r \rangle / p} = \sum_{x \in \Pi} f(x) e(\langle x, r \rangle).$$

The inverse formula takes place

$$f(x) = p^{-2} \sum_{r \in \Pi} \hat{f}(r) e(\langle r, x \rangle). \quad (1)$$

For any two functions $f, g : \Pi \to \mathbb{C}$ the Parseval identity holds

$$\sum_{x \in \Pi} f(x) \overline{g(x)} = p^{-2} \sum_{r \in \Pi} \hat{f}(r) \overline{\hat{g}(r)}. \quad (2)$$

Further, put

$$(f \ast g)(x) := \sum_{y \in \Pi} f(y) g(x-y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in \Pi} f(y) g(y+x).$$

Then

$$\hat{f} \ast \hat{g} = \hat{f} \hat{g} \quad \text{and} \quad \hat{f} \circ \hat{g} = \overline{\hat{f}} \hat{g}, \quad (3)$$

where for a function $f : \Pi \to \mathbb{C}$ we put $f^c(x) := f(-x)$. Clearly, $(f \ast g)(x) = (g \ast f)(x)$ and $(f \circ g)(x) = (g \circ f)(-x)$, $x \in \Pi$. If $A \subseteq \Pi$ is a set then denote by $A(x)$ its characteristic function.

Using Gauss and Kloosterman sums one can prove the following rather standard lemma, see e.g. [6]. Exact formula for the cardinalities of the spheres in $\Pi$ can be obtained as well.

Lemma 2 We have

$$|S_j| = p + 2\theta \sqrt{p}, \quad (4)$$

where $|\theta| \leq 1$, and for all $r \neq 0$ one has

$$|\hat{S}_j(r)| \leq 2 \sqrt{p}. \quad (5)$$
Moreover, for any invertible \( g : \Pi \to \Pi \) and all \( r \neq 0 \) the following holds

\[
|\hat{g}(S_j)(r)| \leq 2\sqrt{p}.
\] (6)

**Proof.** We will prove just (6), the proof of (4), (5) is similar and is contained in [6], Lemma 2. Put \( S = S_j \). We have

\[
\hat{g}(S)(r) = \sum_x S(g^{-1}x)e(-\langle x, r \rangle) = \sum_x S(x)e(-\langle g(x), r \rangle) = p^{-1} \sum_{k \in \mathbb{F}_p} e(k(\|x\| - j) - \langle g(x), r \rangle)
\]

\[
= p^{-1} \sum_{k \neq 0} e(-kj) \sum_x e(k\|x\| - \langle g(x), r \rangle) = p^{-1} \sum_{k \neq 0} e(-kj) \sum_x e(k\|x\| - ax_1 - bx_2),
\]

where \( a, b \in \mathbb{F}_p \) are some constants depending of \( r \). Completing the square and using the well–known formula

\[
G(\alpha) := \sum_x e(\alpha x^2) = \left( \frac{\alpha}{p} \right) G(1),
\]

where \( G(1) = \sum_x e(z^2) \) is the Gauss sum and \( \left( \frac{\alpha}{p} \right) \) is the Legendre symbol, we obtain

\[
\hat{g}(S)(r) = \frac{G^2(1)}{p} \sum_{k \neq 0} e(-kj - ck^{-1}),
\]

where \( c \) is some constant. Now applying \( |G(1)| = \sqrt{p} \) and the estimate for the Kloosterman sums \( [11] \), we get

\[
|\sum_{k \neq 0} e(-kj - ck^{-1})| \leq 2\sqrt{p}.
\]

This completes the proof. \( \square \)

For any set \( A \subseteq \Pi \) denote by \( f_A(x) \) the balanced function of the set \( A \), that is \( f_A(x) = A(x) - |A|/|\Pi| \). Clearly, \( \sum_x f_A(x) = 0 \). By \( I : \Pi \to \Pi \) denote the identity map.

**Theorem 3** Let \( p \) be a sufficiently large prime number. Suppose that \( g \) is an invertible affine transformation of \( \Pi \) such that \( g - I \) is also invertible. Then for any two–coloring of the plane \( \Pi \) and any \( a \neq 0 \) there is a monochromatic triple \( \{x, y, z\} \) such that \( y = x + s, \ s \in S_a \) and \( z = x + g(s) \).

**Proof.** Let \( S = S_a \) and \( A, B \) be the colors of our coloring. We are interested into the quantity

\[
\sigma(A) := \sum_x \sum_{s \in S} A(x)A(x + s)A(x + g(s)),
\]

and similar for the color \( B \). Let us rewrite the quantity \( \sigma(A) \) in terms of the balanced function of \( A \). Put \( \delta_A = |A|/|\Pi|, \delta_B = |B|/|\Pi| \). Then because of the balanced function has zero mean, we get

\[
\sigma(A) := \sum_x \sum_{s \in S} (\delta_A + f_A)(x)(\delta_A + f_A)(x + s)(\delta_A + f_A)(x + g(s)) =
\]
Let us estimate \( \sigma_1 \). By formulas (2), (3), we obtain

\[
\sigma_1 = \sum_s S(s)(f_A \circ f_A)(s) = p^{-2} \sum_r \widehat{\mathcal{S}}(r)|\widehat{f_A}(r)|^2
\]

and thus, using the Parseval identity once more time as well as Lemma 2 we get

\[
\sigma_1 = p^{-2} \sum_{r \neq 0} \widehat{\mathcal{S}}(r)|\widehat{f_A}(r)|^2 \leq 2\sqrt{p} \cdot p^{-2} \sum_r |\widehat{f_A}(r)|^2 \leq 2\sqrt{p}|\mathcal{A}|.
\]

So, it is negligible comparing the main term in (7). Now by the invertibility of \( g \), we have

\[
\sigma'_1 = \sum_s S(s)(f_A \circ f_A)(g(s)) = \sum_s S(g^{-1}(s))(f_A \circ f_A)(s)
\]

and we can apply the arguments above because of one can use bound (6) of Lemma 2 instead of (5). Finally

\[
\sigma''_1 = \sum_s S(s)(f_A \circ f_A)(g(s) - s) = \sum_s S((g - I)^{-1}(s))(f_A \circ f_A)(s)
\]

and by the invertibility of \( g - I \) and in view of Lemma 2 we can estimate \( \sigma''_1 \) similarly as \( \sigma'_1 \).

It remains to calculate the quantity \( \sigma_2 \). Using the inverse formula (1) it is easy to see that

\[
\sigma_2 = \sigma_2(A) = p^{-4} \sum_{u,v} \widehat{f_A}(u - v)|\widehat{f_A}(u)|\widehat{f_A}(v) \cdot \left( \sum_s S(s)e((s,u) + (g(s),v)) \right).
\]

Because of \( A(x) + B(x) = 1 \) we have \( \widehat{f_A}(r) = -\widehat{f_B}(r) \) for all \( r \in \Pi \). It follows that \( \sigma_2(A) + \sigma_2(B) = 0 \). Another way to see the fact is to check the identity \( f_A(x) + f_B(x) = 0 \). Whence, using Lemma 2 again, we obtain

\[
\sigma(A) + \sigma(B) \geq |\mathcal{S}|p^2(\delta_A^3 + \delta_B) - 6\sqrt{p}|A| - 6\sqrt{p}|B| \geq \frac{|\mathcal{S}|p^2}{4} - 6p^2\sqrt{p} \geq \frac{p^3}{4} - 6.5p^2\sqrt{p} > 0,
\]

provided by \( p > 1000 \), say. This completes the proof. \( \Box \)

Because of two distinct points of \( \Pi \) can be transformed to another pair by a composition of an orthogonal transformation and a dilation (see e.g. [11], [3]) then we obtain two immediate consequences of the theorem above.

**Corollary 4** Let \( p \) be a sufficiently large prime number, \( p \equiv -1 \pmod{4} \), and three points \( A, B, C \in \Pi \) form a non-equilateral triangle. Then for any two-coloring of the plane \( \Pi \) there is a monochromatic triangle congruent to \( \triangle ABC \).
Proof. First of all note that any triangle has a pair of sides such that the quotient of its "lengths" is a quadratic residue. Let $\|A - B\| = a$, $\|A - C\| = b$, and $a/b$ be a quadratic residue. Then there is an affine transformation $g$ (which is a composition of an orthogonal map of $\Pi$ and a dilation, see [5]) such that $g(A) = A$, $g(B) = C$. It is easy to see that both maps $g$ and $g - I$ are invertible. Indeed, if $g$ is not invertible then there is $x \neq 0$ such that $\|x\| = 0$. But in view of the assumption $p \equiv -1 \pmod{4}$, we derive $x = 0$ with a contradiction (note that in the case of $p \equiv 1 \pmod{4}$ there is $i \in \mathbb{F}_p$ such that $i^2 \equiv -1 \pmod{p}$ and hence there are $x \neq 0$ with $\|x\| = 0$). Finally, if $g - I$ is not invertible then $\triangle ABC$ is equilateral and for some $x$ one has $gx = x$ and hence $g$ is a mirror symmetry (in the case there are some restrictions on the length of the side $a$ of $\triangle ABC$ as $a$ is nonresidual and $a + 1$ is residual but we miss them). This completes the proof.

Corollary 5 Let $p$ be a sufficiently large prime number. Then for any two–coloring of the plane $\Pi$ and any $a, b \neq 0$ such that $a/b$ is a quadratic residue there is a monochromatic collinear triple $\{x, y, z\}$ with $\|y - x\| = a$, $\|z - y\| = b$.

Problem. Is it possible to find larger monochromatic configurations from $\Pi$ in the spirit of papers [2], [12]?

3 Euclidian plane

In the section we consider the case of the usual euclidian plane and try to obtain an analog of Theorem 3. The proof follows the arguments from [9] as well as the approach (and the notation) from the previous section.

Put $\Pi = \mathbb{R}^2$ and let $A \subseteq \Pi$ be a measurable set. By the upper density of $A$ define

$$\overline{d}_A := \limsup_{T \to +\infty} \frac{\text{Vol}(A \cap [-T, T]^2)}{(2T)^2}. \tag{8}$$

A measurable, complex valued function $f : \Pi \to \mathbb{C}$ is called periodic if there is a basis $b_1, b_2 \in \Pi$ such that for all $\alpha_1, \alpha_2 \in \mathbb{Z}$ one has $f(x + \alpha_1 b_1 + \alpha_2 b_2) = f(x)$. The set $L = \{\alpha_1 b_1 + \alpha_2 b_2 : \alpha_1, \alpha_2 \in \mathbb{Z}\}$ is called the period lattice of $f$ and $L^* = \{u \in \Pi : \langle u, x \rangle \in \mathbb{Z}, \forall x \in L\}$ is called the dual lattice of $L$. Here $\langle \cdot, \cdot \rangle$ is the usual scalar product onto $\Pi$. If the characteristic function of a measurable set $A$ is periodic then it is easy to see that $\limsup$ in (8) can be replaced by a simple limit.

We have a scalar product on the space of periodic functions

$$\langle f, g \rangle = \lim_{T \to +\infty} \frac{1}{(2T)^2} \int_{[-T, T]^2} f(x) \overline{g(x)} \, dx. \tag{8}$$

The Fourier transform of a (periodic) function $f$ is given by the formula $\hat{f}(u) = \langle f(x), e^{i\langle u, x \rangle} \rangle$. It is easy to check that the support of Fourier transform of a periodic function $f$ belongs to $2\pi L^*$.

In particular, the support is a discrete set.
The Bessel function of the first kind $J_\nu(z)$ is the series (see e.g. [8])

$$J_\nu(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}.$$ (9)

It is well–known that

$$J_0(\|u\|) = \frac{1}{2\pi} \langle S_1(x), e^{iux} \rangle,$$ (10)

where $J_0$ is the zeroth Bessel function and

$$S_a = \{ x \in \Pi : \|x\| := \sqrt{x_1^2 + x_2^2} = a \}$$ is a circle of radios $a$.

**Theorem 6** Let $a > 0$ and $\kappa > 0$ be real numbers. Suppose that for all $t \geq 0$ one has

$$J_0(t) + J_0(\kappa t) + J_0((1 + \kappa)t) > -1.$$ (11)

Then for any measurable coloring of the plane $\Pi$ into two colors there is a monochromatic collinear triple $\{x, y, z\}$ such that $y \in [x, z]$ and $\|y - x\| = a$, $\|z - y\| = \kappa a$.

**Proof.** We follow the arguments of the proof of Theorem 3. Let $S = S_a$ be the circle of radios $a$ and $A, B$ be the colors of upper densities $\overline{\sigma}_A, \overline{\delta}_B$. We suppose that $A$ and $B$ do not contain collinear triples $\{x, y, z\}$ such that $y \in [x, z]$ and $\|y - x\| = a$, $\|z - y\| = \kappa a$.

One can assume that $A(x)$ and $B(x)$ are periodic functions. Indeed, choose $T$ is large enough that $[-T + (1 + \kappa)a, T - (1 + \kappa)a] / (2T)^2$ is sufficiently close to 1 and such that $\text{Vol}(A \cap [-T, T])/ (2T)^2$ is sufficiently close to $\overline{\delta}_A$. After that construct a periodic tiling of $\mathbb{R}^2$ with copies of $A \cap [-T + (1 + \kappa)a, T - (1 + \kappa)a]$ and $B \cap [-T + (1 + \kappa)a, T - (1 + \kappa)a]$, translating the copies by the points of lattice $2TZ$. Denote the obtained new colors as $A_\ast$ and $B_\ast$. Clearly, $\delta_{A_\ast}$ can be chosen close to $\overline{\delta}_A$ and that $\delta_{A_\ast} + \delta_{B_\ast} = 1$. Note also that $A_\ast, B_\ast$ do not contain collinear triples $\{x, y, z\}$ such that $y \in [x, z]$ and $\|y - x\| = a$, $\|z - y\| = \kappa a$.

As in the proof of Theorem 3 consider the quantity $\sigma(h_1, h_2, h_3)$, which is trilinear by three arguments $h_1, h_2, h_3$, namely,

$$\sigma(A_\ast, A_\ast, A_\ast) = \sigma(A_\ast) := \lim_{T \to +\infty} \frac{1}{(2T)^2} \int \int_{s \in S} (\delta_{A_\ast} + f_{A_\ast})(x)(\delta_{A_\ast} + f_{A_\ast})(x + s)(\delta_{A_\ast} + f_{A_\ast})(x - \kappa s) \, dxds,$$

where again $f_{A_\ast}(x) = A_\ast(x) - \delta_{A_\ast}$ is the balanced function of $A_\ast$. We have

$$\sigma(A_\ast) := 2\pi a \delta_{A_\ast}^3 + \delta_{A}(\sigma(f_{A_\ast}, f_{A_\ast}, 1) + \sigma(f_{A_\ast}, 1, f_{A_\ast}) + \sigma(1, f_{A_\ast}, f_{A_\ast})) + \sigma(f_{A_\ast}).$$ (12)

As in the proof of Theorem 3 the following holds $\sigma(f_{A_\ast}) + \sigma(f_{B_\ast}) = 0$ and thus we need to bound the remain three quantities in (12). Clearly,

$$\sigma(f_{A_\ast}, f_{A_\ast}, 1) = \langle f_{A_\ast} \circ f_{A_\ast}, S \rangle.$$
Using the Fourier transform, we get
\[
\sigma(f_A, f_A, 1) = \sum_{u \in \mathbb{R}^2} |\hat{f}_A(u)|^2 \hat{S}(u). \tag{13}
\]
As we noted before the sum in (13) is actually taking over a discrete set. Putting
\[
\alpha(t) := \sum_{u \in \mathbb{R}^2 : \|u\|=t} |\hat{f}_A(u)|^2 \geq 0,
\]
we obtain by (10)
\[
\sigma(f_A, f_A, 1) = 2\pi a \sum_{t \geq 0} J_0(at)\alpha(t).
\]
Here we have used the formula
\[
\hat{S}_b(u) = b\hat{S}_1(bu) = 2\pi b J_0(\|bu\|),
\]
where \(b > 0\) is an arbitrary. Similarly,
\[
\sigma(f_A, 1, f_A) = (f_A \circ f_A)((\kappa s), S(s)) = 2\pi a \sum_{t \geq 0} J_0((1 + \kappa)at)\alpha(t),
\]
and
\[
\sigma(1, f_A, f_A) = (f_A \circ f_A)((1 + \kappa)s), S(s)) = 2\pi a \sum_{t \geq 0} J_0((1 + \kappa)at)\alpha(t).
\]
Thus
\[
\sigma(f_A, f_A, 1) + \sigma(f_A, 1, f_A) + \sigma(1, f_A, f_A) \geq 2\pi a \sum_{t \geq 0} \alpha(t)(J_0(at) + J_0((1 + \kappa)at) + J_0((1 + \kappa)at).
\]
By \(J\) define the quantity \(J = \min_{t \geq 0}(J_0(at) + J_0((\kappa at) + J_0((1 + \kappa)at).\) Applying the Parseval identity and the observation \(\hat{A}_s(0) = \delta_{A_s}\), we have
\[
\sum_{t \geq 0} \alpha(t) = \sum_u |\hat{A}_s(u)|^2 - \delta_{A_s} = \delta_{A_s} - \delta_{A_s}. \tag{14}
\]
Returning to (12) and combining it with the last formula, we obtain
\[
(2\pi a)^{-1}(\sigma(A_s) + \sigma(B_s)) \geq \delta_{A_s}^2 + \delta_{B_s}^2 + J(\delta_{A_s}^2 - \delta_{A_s}^2) + J(\delta_{B_s}^2 - \delta_{B_s}^2) = (\delta_{A_s}^2 + \delta_{B_s}^2)(1-J) + (\delta_{A_s}^2 + \delta_{B_s}^2)J.
\]
Because of \(\delta_{A_s} + \delta_{B_s} = 1\) the optimization gives us
\[
(2\pi a)^{-1}(\sigma(A_s) + \sigma(B_s)) \geq \frac{J + 1}{4} > 0.
\]
Here we have used condition (11). This completes the proof. \(\square\)

**Corollary 7** Let \(a > 0\) be a real number. Then for any measurable coloring of the plane \(\Pi\) into two colors there is a monochromatic collinear triple \(\{x, y, z\}\) such that \(y \in [x, z]\) and \(\|y - x\| = \|z - y\| = a\).
Proof. By Theorem[6] we need to estimate $\min_{t \geq 0}(2J_0(at) + J_0(2at)) = \min_{t \geq 0}(2J_0(t) + J_0(2t))$. Using Maple, say, one can calculate $\min_{t \in [0,50]}(2J_0(t) + J_0(2t)) \geq -0.74$. For $t > 50$, applying a crude upper bound $|J_\nu(t)| \leq |t|^{-1/3} \nu \geq 0$ (see e.g. [7]), we insure that the minimum is strictly greater than $-1$ for all $t \geq 0$. This concludes the proof. 

Below we will deal with affine transformations $g$ of the form $g = D_\omega \circ R$, where $R$ be a rotation and $D_\omega$ be a dilation by some $\omega > 0$. Let us note a simple lemma about such $g$.

Lemma 8 Let $g = D_\omega \circ R$, where $D_\omega$ be a dilation by $\omega$ and $R$ be a rotation by $\varphi$. Then $g - I$ has the same form $D_{\omega'} \circ R'$, where $\omega' = \sqrt{\omega^2 - 2\omega \cos \varphi + 1}$ and $R'$ is another rotation.

Proof. To obtain the result we need to solve the system of equations $\omega \cos \varphi - 1 = \omega' \cos \varphi'$, $\omega' \sin \varphi' = \omega \sin \varphi$ in variables $\omega', \varphi'$. Taking a square and a summation give us

$$(\omega')^2 = \omega^2 - 2\omega \cos \varphi + 1 \geq 0$$

and thus $\sin \varphi' = \omega/\omega' \cdot \sin \varphi$, $\cos \varphi' = (\omega \cos \varphi - 1)/\omega'$. On can check that the modules of $\omega/\omega' \cdot \sin \varphi$ as well as $(\omega \cos \varphi - 1)/\omega'$ do not exceed 1 and hence $\varphi'$ exists. This completes the proof.

Similarly to Theorem[6] as well as Theorem[3] one can obtain the following general result, which is however not so wide as Theorem[3].

Theorem 9 Let $a > 0$ and $\omega > 0$ be real numbers. Let also $g = D_\omega \circ R$ be an affine transformation of $\Pi$, where $R$ be a rotation and $D_\omega$ be a dilation by $\omega$. Suppose that for all $t \geq 0$ one has

$$J_0(t) + J_0(\omega t) + J_0 > -1, \quad (15)$$

where $J_0 = \min_{t \geq 0} J_0(t) = -0.4027593957...$. Then for any measurable coloring of the plane $\Pi$ into two colors there is a monochromatic collinear triple $\{x, y, z\}$ such that $y = x + s$, $s \in S_a$ and $z = x + g(s)$. More precisely, if $R$ is a rotation by $\varphi$ then condition (15) can be replaced by

$$J_0(t) + J_0(t\omega) + J_0(t\sqrt{\omega^2 - 2\omega \cos \varphi + 1}) > -1. \quad (16)$$

Proof. We use the notation and the arguments of the proof of Theorem[6] Then

$$\sigma(A_+, A_+, A_+) = \sigma(A_+) := \lim_{T \to +\infty} \frac{1}{(2\pi T)^2} \int \int_{S \times S} (\delta_{A_+} + f_{A_+})(x)(\delta_{A_+} + f_{A_+})(x+s)(\delta_{A_+} + f_{A_+})(x + g(s)) \, dx \, ds,$$

$$= 2\pi a \delta_{A_+} + \delta_A (\sigma(f_{A_+}, f_{A_+}, 1) + \sigma(f_{A_+}, 1, f_{A_+}) + \sigma(1, f_{A_+}, f_{A_+})) + \sigma(f_{A_+}).$$

Again, we need to estimate $\sigma(f_{A_+}, f_{A_+}, 1), \sigma(f_{A_+}, 1, f_{A_+}), \sigma(1, f_{A_+}, f_{A_+})$. The first quantity is the same as in the proof of Theorem[6] The second one equals

$$\sigma(f_{A_+}, 1, f_{A_+}) = ((f_{A_+} \circ f_{A_+})(g(s)), S(s)) = ((f_{A_+} \circ f_{A_+})(s), S(g^{-1}(s))) \cdot \det(g)^{-1}. \quad (17)$$
As we know for any $b > 0$ one has
\[ \hat{S}_b(u) = b \hat{S}_1(bu) = 2\pi b J_0(\|bu\|). \] (18)

Hence
\[ \langle S(g^{-1}(s)), e^{i(u,s)} \rangle = 2\pi a J_0(\omega a \|u\|) \cdot \det(g). \] (19)

In terms of quantities $\alpha(t)$ it follows that
\[ \sigma(f_A, 1, f_A) = 2\pi a \sum_{t \geq 0} \alpha(t) J_0(\omega a t). \]

Finally, in the estimation of the third term $\sigma(1, f_A, f_A)$ the quantity $(g - I)^{-1}$ appears. Hence (see the proof of Theorem 3), we get
\[ \sigma(f_A, 1, f_A) = \sum_{u \in \mathbb{R}^2} |\hat{f}_A(u)|^2 \langle S((g - I)^{-1}(s)), e^{i(u,s)} \rangle. \] (20)

Unfortunately, if $u$ runs over a circle then $(g - I)^{-1}(u)$ do not belong to a circle in the case of general transformation $g$ (but it is so in the case of Theorem 6 when $\{x, y, z\}$ are collinear). Nevertheless we estimate (20) with help of (14) crudely as
\[ \sigma(f_A, 1, f_A) \geq 2\pi a J_0 \cdot (\delta_A - \delta_A^2). \]

Combining all bounds, we obtain
\[ (2\pi a)^{-1}(\sigma(A) + \sigma(B)) \geq \frac{J + J_0 + 1}{4} > 0, \]
where $J = \min_{t \geq 0} (J_0(at) + J_0(\omega at)) = \min_{t \geq 0} (J_0(t) + J_0(\omega t))$. Thus, we have proved (15) and it remains to obtain (16). In the case apply Lemma 8, combining with formula (20) and calculations in (17)—(19). This completes the proof.

Remark 10 It is well-known that there is a measurable two-coloring of the plane having no monochromatic equilateral triangle of an arbitrary side $a > 0$, see [3]. If we try to apply Theorem 9 in the case then a Maple calculation gives us
\[ \min_{t \geq 0} (2J_0(t)) = 3J_0 = -1.208278187 \ldots \]

It is rather close to the required $-1$.

There is a series of results, see e.g. [10] where the existence of monochromatic triangle with some restrictions on the lengths of the sides and the angles was obtained for any (non-necessary measurable) coloring. For example, in [3] the authors proved that any monochromatic triangle with the smallest side 1 and the angles in the ratio $1 : 2 : 3$, more generally, in the ratio $n : (n + 1) : (2n + 1)$, $1 : 2n : (2n + 1)$ and so on can be found. Concluding the section we note that in our Theorem 9 one does not need to know any angles but just the ratio of the lengths of an arbitrary two sides of the triangle. For example, one can show that for $\omega = 2$ the minimum in (13) is greater that $-0.86$ and hence any monochromatic triangle with the ratio of the sides $1 : 2$ appears.
References

[1] M. Bennett, D. Hart, A. Iosevich, J. Pakianathan and M. Rudnev, Group actions and geometric combinatorics in $\mathbb{F}_p^d$, arXiv:1311.4788v1 [math.CO] 19 Nov 2013.

[2] P. Cameron, J. Cilleruelo, O. Serra, On monochromatic solutions of equations in groups, Rev. Mat. Iberoamericana 23 (2007), no. 1, 385–395.

[3] P. Erdös, R.L. Graham, P. Montgomery, B.L. Rothschild, J. Spencer, E.G. Straus, Euclidean Ramsey theorems I, II, and III, J. Combin. Theory Ser. A, 14 (1973), 341–363; Coll. Math. Soc. Janos Bolyai, 10 (1973) North Holland, Amsterdam (1975), 529–557 and 559–583.

[4] K. J. Falconer, The realization of distances in measurable subsets covering $\mathbb{R}^n$, J. Combin. Theory A, 31 (1981), 187–189.

[5] D. Hart, A. Iosevich, Ubiquity of simplices in subsets of vector spaces over finite fields, Analysis Mathematica, 34 (2008), 29–38.

[6] A. Iosevich, D. Koh, Extension theorems for spheres in the finite field setting, Forum Mathematicum 22 (2010), no.3, 457–483.

[7] L. Landau, Monotonicity and bounds on Bessel functions, Mathematical Physics and Quantum Field Theory, Electronic Journal of Differential Equations, Conf. 04, (2000), 147–154.

[8] A.F. Nikiforov, V.B. Uvarov, Special functions of mathematical physics, Basel : Birkhäuser, 1988.

[9] F.M. de Oliveira Filho, F. Vallentin, Fourier analysis, linear programming, and densities of distance avoiding sets in $\mathbb{R}^n$, arXiv:0808.1822v2 [math.CO] 2 Dec 2008.

[10] A. Soifer, The mathematical coloring book: mathematics of coloring and the colorful life of its creators, Springer Science & Business Media, 2008.

[11] A. Weil, On some exponential sums, Proc. Nat. Acad. Sci. U.S.A., 34, (1948), 204–207.

[12] J. Wolf, The minimum number of monochromatic 4-term progressions in $\mathbb{Z}_p$, Journal of Combinatorics 1 (2010), no. 1, 53–68.

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