A UNIMODULAR SOLVABLE LIE ALGEBRA WITH $b_3 = 0$
ADMITTING CLOSED $G_2$-STRUCTURES

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Abstract. In contrast to the symplectic case, we show that there exist unimodular Lie groups admitting left-invariant exact $G_2$-structures. Indeed, we construct a seven-dimensional unimodular completely solvable Lie algebra $g$ with third Betti number $b_3(g^*) = 0$ and with a closed $G_2$-structure. Moreover, we show that the corresponding left-invariant exact $G_2$-structure on the simply connected Lie group with Lie algebra $g$ is an expanding Laplacian soliton.

1. Introduction

Let $M$ be a seven-dimensional smooth manifold. A $G_2$-structure on $M$ is a reduction of the structure group of its frame bundle from $\text{GL}(7, \mathbb{R})$ to the compact exceptional Lie group $G_2$. In [18], Gray proved that a smooth 7-manifold carries $G_2$-structures if and only if it is orientable and spin.

The existence of a $G_2$-structure on $M$ is characterized by the existence of a globally defined 3-form $\varphi \in \Omega^3(M)$ satisfying a certain nondegeneracy condition. The $G_2$-form $\varphi$ gives rise to a Riemannian metric $g_\varphi$ with volume form $dV_\varphi$ via the identity

$$g_\varphi(X, Y) dV_\varphi = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi,$$

for any pair of vector fields $X, Y$ on $M$, where $\iota_X$ denotes the contraction by $X$. Moreover, at each point $x \in M$ there exists a basis $\{e_1, \ldots, e_7\}$ of the cotangent space $T^*_x M$ for which

$$\varphi|_x = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$  

Here, $e^{123}$ stands for $e^1 \wedge e^2 \wedge e^3$, and so on. Such a basis is called an adapted $G_2$-basis.

A $G_2$-structure is said to be closed (or calibrated) if the defining 3-form $\varphi$ satisfies the equation $d\varphi = 0$, while it is called co-closed (or co-calibrated) if $d^* \varphi \equiv 0$, $^* \varphi$ being the Hodge operator defined by $g_\varphi$ and $dV_\varphi$. When both of these conditions hold, the intrinsic torsion of the $G_2$-structure vanishes identically, the Riemannian metric $g_\varphi$ is Ricci-flat, and $\text{Hol}(g_\varphi) \subseteq G_2$ (cf. [3, 14]). In this case, the $G_2$-structure is said to be torsion-free.

The existence of a torsion-free $G_2$-structure on a compact 7-manifold $M$ imposes various constraints on the topology. For instance, the third Betti number satisfies $b_3(M) \geq 1$ [2], and the first Betti number $b_1(M) \in \{0, 1, 3, 7\}$ [19]. Moreover, if $\text{Hol}(g_\varphi) = G_2$, then the fundamental group $\pi_1(M)$ is finite [19], and so $b_1(M) = 0$. However, if $M$ is a non-compact manifold with a torsion-free $G_2$-structure, the third Betti number $b_3(M)$ may be zero. In [5], Bryant and Salamon constructed $G_2$-holonomy metrics, which asymptotically look like the metric cone of the homogeneous nearly-Kähler manifolds $\mathbb{CP}^3$, $\mathbb{F}_{1,2} = \text{SU}(3)/\mathbb{T}^2$ or $S^3 \times S^3$. By [1], if $(S, g_S)$ is a nearly Kähler 6-manifold, then its metric cone
In the literature, all known examples of compact 7-manifolds $M$ admitting a closed $G_2$-structure, but not torsion-free $G_2$-structures, have $b_1(M) \geq 0$ and $b_3(M) > 0$ (see [8, 10, 11, 12, 13]). A longstanding open question concerns the existence of closed $G_2$-structures on compact 7-manifolds with $b_3(M) = 0$. Examples of this type occur in the non-compact case: for instance, on the aforementioned metric cone of the flag manifold $\mathbb{F}_{1,2}$, on simply connected solvable Lie groups (see e.g. [8, 10, 11] and note that in all these cases the corresponding Lie algebra is such that its third Betti number satisfies $b_3 > 0$), and on the non-compact manifold $T(S^2 \times \mathbb{R}) \times \mathbb{R}$, where $T(S^2 \times \mathbb{R})$ is the tangent bundle of $S^2 \times \mathbb{R}$ [14].

On the other hand, although closed $G_2$-structures may be regarded as the $G_2$-analogues of almost Kähler structures, there are differences in their behavior when the manifold is a Lie group. For example, a unimodular Lie group admitting a left-invariant exact symplectic form cannot admit any left-invariant closed $G_2$-structures [7, 22]. However, in [15] it is proved that there exist non-solvable unimodular Lie groups having left-invariant closed $G_2$-structures. Furthermore, by [9], a unimodular Lie group cannot admit any left-invariant exact symplectic form. Therefore, a natural question is the following:

Are there unimodular Lie groups admitting left-invariant exact $G_2$-structures?

In this note, we give an affirmative answer to this question, producing a seven-dimensional unimodular solvable Lie algebra $g$, with third Betti number $b_3(g^+) = 0$ and with a closed $G_2$-structure $\varphi$ (see Section 3). In particular, $\varphi$ is an exact 3-form on $g$, and it gives rise to a left-invariant exact $G_2$-form $\varphi$ on the simply connected unimodular solvable Lie group $G$ with Lie algebra $g$. In Section 3 we prove that $g$ is not strongly unimodular (see Definition 2.2 for details). Consequently, the Lie group $G$ does not contain any co-compact discrete subgroup. Indeed, by a result of Garland [16], every solvable Lie group containing a lattice must be strongly unimodular.

Finally, in Section 4 we show that the left-invariant exact $G_2$-structure $\varphi$ on $G$ is an expanding Laplacian soliton, that is, it satisfies the equation

$$\Delta_\varphi \varphi = \lambda \varphi + \mathcal{L}_X \varphi,$$

for some positive real number $\lambda$ and some vector field $X \in \mathfrak{X}(G)$, where $\Delta_\varphi$ denotes the Hodge Laplacian of the Riemannian metric $g_\varphi$ induced by $\varphi$, and $\mathcal{L}_X$ denotes the Lie derivative with respect to $X$. Moreover, we discuss the behaviour of the left-invariant self-similar solution of the $G_2$-Laplacian flow on $G$ starting from $\varphi$.

2. The Lie group $G$

Let $g = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle$ be the seven-dimensional unimodular solvable Lie algebra with the following nonzero Lie brackets

$$
\begin{align*}
[e_1, e_3] &= 4 e_3, & [e_1, e_4] &= -e_4, & [e_1, e_5] &= -5 e_5, & [e_1, e_6] &= -e_6, & [e_1, e_7] &= 3 e_7, \\
[e_2, e_3] &= 3 e_3, & [e_2, e_5] &= -4 e_5, & [e_2, e_6] &= -e_6, & [e_2, e_7] &= 2 e_7, \\
[e_3, e_5] &= -e_6, & [e_3, e_6] &= -e_7.
\end{align*}
$$  \hfill (2.1)
It is easy to check that \( g \) is unimodular and completely solvable, that is, for each \( X \in g \) the map \( \text{ad}_X \in \text{End}(g) \) is traceless and has only real eigenvalues. Moreover, \( g \) is the semidirect product \( \mathbb{R}^2 \ltimes n \), where the abelian Lie algebra \( \mathbb{R}^2 = \langle e_1, e_2 \rangle \) acts on the nilradical \( n = \langle e_3, e_4, e_5, e_6, e_7 \rangle \) of \( g \) via the representation

\[
\rho : \mathbb{R}^2 \to \text{Der}(n), \quad \rho(e_1) = \text{diag}(4, -1, -5, -1, 3), \quad \rho(e_2) = \text{diag}(3, 0, -4, -1, 2).
\]

Furthermore, we have the following.

**Lemma 2.1.** The Lie algebra \( g \) is indecomposable.

**Proof.** Let us suppose that \( g \) is a decomposable solvable Lie algebra. Then, we would have \( g = g_1 \oplus g_2 \), with both \( g_1 \) and \( g_2 \) ideals of \( g \). Since \( g \) is solvable, \( g_1 \) and \( g_2 \) are also solvable, and \( g_i = a_i \ltimes n_i \), where \( n_i \) denotes the nilradical of \( g_i \) and \( a_i \) is an abelian Lie algebra. Consequently, we would have

\[
g = (a_1 \ltimes n_1) \oplus (a_2 \ltimes n_2),
\]

and the nilradical of \( g \) would decompose as \( n = n_1 \oplus n_2 \). Up to ordering, the two summands of \( n \) must be \( n_1 = \langle e_3, e_5, e_6, e_7 \rangle \) and \( n_2 = \langle e_4 \rangle \). Moreover,

\[
a_1 = (a_1 e_1 + a_2 e_2), \quad a_2 = (b_1 e_1 + b_2 e_2),
\]

with \( a_i, b_i \in \mathbb{R} \). Now, the condition \([g_1, g_2] = 0\) forces \( a_1 = a_2 = b_1 = b_2 = 0 \), which is a contradiction. \( \square \)

The simply connected solvable unimodular Lie group \( G \) with Lie algebra \( g \) is diffeomorphic to \( \mathbb{R}^7 \) endowed with the following product

\[
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5 \\
y_6 \\
y_7
\end{pmatrix}
\cdot
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{pmatrix}
= \begin{pmatrix}
y_1 + x_1 \\
y_2 + x_2 \\
y_3 + e^{4y_1 + 3y_2} x_3 \\
y_4 + e^{-y_1} x_4 \\
y_5 + e^{-5y_1 - 4y_2} x_5 \\
y_6 + \bar{x}_6 \\
y_7 + \bar{x}_7
\end{pmatrix},
\]

where

\[
\bar{x}_6 := -4 e^{4y_1 + 3y_2} y_5 x_3 - 5 e^{-5y_1 - 4y_2} y_3 x_5 + e^{-y_1 - y_2} x_6,
\]

\[
\bar{x}_7 := 16 e^{4y_1 + 3y_2} y_3 y_5 x_3 + 4 e^{4y_1 + 3y_2} y_6 x_3 - 5 e^{-5y_1 - 4y_2} y_3^2 x_5 + \frac{1}{3} e^{-y_1 - y_2} y_3 x_6 + e^{3y_1 + 2y_2} x_7.
\]

If we denote a system of global coordinates on \( G \) by \((x_1, \ldots, x_7)\), then the basis of \( g \) considered above can be identified with the following basis of left-invariant vector fields on the
Lemma 2.4. The Betti numbers of the Lie algebra $\mathfrak{g}$:

$$
\begin{align*}
e_1 &= \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_2}, \quad e_3 = e^{4x_1+3x_2} \left( \frac{\partial}{\partial x_3} - 4x_5 \frac{\partial}{\partial x_6} + \frac{4}{3} (4x_3 x_5 + x_6) \frac{\partial}{\partial x_7} \right), \\
e_4 &= e^{-x_1} \frac{\partial}{\partial x_4}, \quad e_5 = e^{-5x_1-4x_2} \left( \frac{\partial}{\partial x_5} - 5x_3 \frac{\partial}{\partial x_6} - \frac{5}{6} x_2^2 \frac{\partial}{\partial x_7} \right), \\
e_6 &= e^{-x_1-x_2} \left( \frac{\partial}{\partial x_6} + \frac{1}{3} x_3 \frac{\partial}{\partial x_7} \right), \quad e_7 = e^{3x_1+2x_2} \frac{\partial}{\partial x_7}.
\end{align*}
$$

We claim that $G$ does not contain any lattice, i.e., any co-compact discrete subgroup. Indeed, by [16, Prop. 3.3], if a solvable Lie group contains a lattice, then it must be strongly unimodular according to the following.

**Definition 2.2.** Let $S$ be a simply connected solvable Lie group with Lie algebra $\mathfrak{s}$. Denote by $\mathfrak{m}^{(0)} := \mathfrak{m}$ the nilradical of $\mathfrak{s}$ and, for each positive integer $i \geq 1$, let $\mathfrak{m}^{(i)} := [\mathfrak{m}, \mathfrak{m}^{(i-1)}]$ denote the $i^{\text{th}}$ term in the descending central series of $\mathfrak{m}$. The Lie algebra $\mathfrak{s}$ is **strongly unimodular** if for all $X \in \mathfrak{s}$ the restriction of $\text{ad}_X$ to each space $\mathfrak{m}^{(i)}/\mathfrak{m}^{(i+1)}$ is traceless. In this case, the Lie group $S$ is said to be **strongly unimodular**.

**Lemma 2.3.** The Lie algebra $\mathfrak{g}$ is not strongly unimodular. Therefore, the solvable unimodular Lie group $G$ does not contain lattices.

**Proof.** It is sufficient to prove that $\mathfrak{g}$ is not strongly unimodular. Let $\mathfrak{n} = \mathfrak{n}^{(0)} = [\mathfrak{g}, \mathfrak{g}]$ be the nilradical of $\mathfrak{g}$. Then, the only nonzero terms in the descending central series of $\mathfrak{n}$ are

$$
\begin{align*}
n^{(0)} &= n = \langle e_3, e_4, e_5, e_6, e_7 \rangle, \\
n^{(1)} &= [\mathfrak{n}, \mathfrak{n}^{(0)}] = \langle e_6, e_7 \rangle, \\
n^{(2)} &= [\mathfrak{n}, \mathfrak{n}^{(1)}] = \langle e_7 \rangle.
\end{align*}
$$

From (2.1), it follows that $\text{ad}_{e_1}|_{n^{(2)}} = 3 \text{Id}$. This implies that $\mathfrak{g}$ is not strongly unimodular.

Let $(e^1, \ldots, e^7)$ be the dual basis of $(e_1, \ldots, e_7)$. Then, the structure equations (2.1) of $\mathfrak{g}$ can also be written by means of the Chevalley-Eilenberg differentials of the covectors $e^i$. In detail:

$$
\begin{align*}
d e^1 &= 0 = d e^2, \\
d e^3 &= -4 e^{13} - 3 e^{23}, \\
d e^4 &= e^{14}, \\
d e^5 &= 5 e^{15} + 4 e^{25}, \\
d e^6 &= e^{16} + e^{26} + e^{35}, \\
d e^7 &= -3 e^{17} - 2 e^{27} + e^{36}.
\end{align*}
$$

Using these equations, we compute the cohomology groups of the Chevalley-Eilenberg complex $(\Lambda^*(\mathfrak{g}^*), d)$ of $\mathfrak{g}$:

$$
\begin{align*}
H^1(\mathfrak{g}^*) &= \langle e^1, e^2 \rangle, \\
H^2(\mathfrak{g}^*) &= \langle e^1 \wedge e^2 \rangle, \\
H^3(\mathfrak{g}^*) &= H^4(\mathfrak{g}^*) = \{0\}, \\
H^5(\mathfrak{g}^*) &= \langle e^{34567} \rangle, \\
H^6(\mathfrak{g}^*) &= \langle e^{234567}, e^{134567} \rangle, \\
H^7(\mathfrak{g}^*) &= \langle e^{1234567} \rangle.
\end{align*}
$$

**Lemma 2.4.** The Betti numbers of the Lie algebra $\mathfrak{g}$ are the following

$$
\begin{align*}
b_1(\mathfrak{g}^*) &= b_6(\mathfrak{g}^*) = 2, \\
b_2(\mathfrak{g}^*) &= b_5(\mathfrak{g}^*) = b_7(\mathfrak{g}^*) = 1, \\
b_3(\mathfrak{g}^*) &= b_4(\mathfrak{g}^*) = 0.
\end{align*}
$$
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3. A LEFT-INVARlANT EXACT $G_2$-STRUCTURE ON THE LIE GROUP $G$

We describe an explicit example of a left-invariant exact $G_2$-structure $\varphi$ on $G$. Since $G$ is unimodular, this shows in particular that exact $G_2$-structures behave differently from exact symplectic structures (cf. [9]).

Let us consider the following basis of $\mathfrak{g}^*$:

$$E^1 = e^1 + \frac{3}{4} e^2, \quad E^2 = e^3, \quad E^3 = \frac{1}{4\sqrt{3}} e^2, \quad E^4 = -\sqrt{3} e^4 + 2\sqrt{3} e^6,$$

$$E^5 = e^5, \quad E^6 = e^4 + 2e^6, \quad E^7 = 8\sqrt{3} e^7.$$ 

The structure equations of $\mathfrak{g}$ with respect to this new basis are:

\[
\begin{align*}
\text{d}E^1 &= 0, \\
\text{d}E^2 &= -4 E^{12}, \\
\text{d}E^3 &= 0, \\
\text{d}E^4 &= E^{14} + 2\sqrt{3} E^{25} - \sqrt{3} E^{34} + 6 E^{36}, \\
\text{d}E^5 &= 5 E^{15} + \sqrt{3} E^{35}, \\
\text{d}E^6 &= E^{16} + 2 E^{25} + 2 E^{34} - \sqrt{3} E^{36}, \\
\text{d}E^7 &= -3 E^{17} + 2 E^{24} + 2\sqrt{3} E^{26} + \sqrt{3} E^{37}.
\end{align*}
\]

(3.1)

According to (1.2), we consider the $G_2$-structure $\varphi$ on $\mathfrak{g}$ with adapted $G_2$-basis $(E^1, \ldots, E^7)$, that is,

$$\varphi = E^{123} + E^{145} + E^{167} + E^{246} - E^{257} - E^{347} - E^{356}. \quad (3.2)$$

Then, $\varphi$ is an exact 3-form on $\mathfrak{g}$. Indeed,

$$\varphi = d \left( -\frac{1}{4} E^{23} + \frac{1}{6} E^{45} - \frac{1}{2} E^{67} \right).$$

The 3-form $\varphi$ gives rise to a left-invariant closed $G_2$-structure on the Lie group $G$, which we shall denote by the same letter. Clearly, $\varphi$ is exact and it induces the left-invariant Riemannian metric $g_\varphi = \sum_{i=1}^7 (E^i)^2$ on $G$. This proves the following.

**Proposition 3.1.** Let $G$ be the simply connected unimodular solvable Lie group of dimension seven, whose Lie algebra $\mathfrak{g}$ is defined by the equations (3.1). Then, the 3-form $\varphi$ given in (3.2) defines a left-invariant exact $G_2$-structure on $G$. Moreover, $\varphi$ induces the left-invariant Riemannian metric $g_\varphi = \sum_{i=1}^7 (E^i)^2$ on $G$.

Recall that the intrinsic torsion of a closed $G_2$-structure $\varphi$ can be identified with the unique 2-form $\tau$ satisfying $d \ast_\varphi \varphi = \tau \wedge \varphi = -\ast_\varphi \tau$ (cf. [11]). In our case, this intrinsic torsion form $\tau$ is given by

$$\tau = -4 E^{23} + 12 E^{45} - 8 E^{67}.$$ 

Since $G$ is a simply connected unimodular completely solvable Lie group, it follows from [17] that the isometry group of the Riemannian manifold $(G, g_{\varphi})$ is the algebraic isometry group, i.e., $\text{Isom}(G, g_{\varphi}) = G \rtimes C$, where $C := \text{Aut}(\mathfrak{g}) \cap O(g_{\varphi})$ denotes the set of orthogonal
automorphisms of \((G, g_\varphi)\) (see also \cite{20} for more details). Now, the Lie algebra of \(C\) is given by \(\text{Lie}(C) = \text{Der}(g) \cap \mathfrak{so}(g_\varphi)\). We have

\[
\text{Lie}(C) = \{0\}.
\]

Indeed, a generic derivation \(D \in \text{Der}(g)\) has the following matrix representation with respect to the basis \(\{E_1, \ldots, E_7\}\)

\[
D = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & 0 & 0 \\
\frac{2\alpha_2 \alpha_4}{\sqrt{3}} & \frac{\alpha_2}{\sqrt{3}} & \frac{\alpha_3}{\sqrt{3}} & \frac{\alpha_5}{\sqrt{3}} & \frac{\alpha_6}{\sqrt{3}} & \frac{\alpha_7}{\sqrt{3}} + \frac{\alpha_2}{12} & \frac{1}{2} \\
\frac{\alpha_5}{\sqrt{3}} & \frac{\alpha_5}{\sqrt{3}} & \frac{\alpha_5 + 3\alpha_2}{\sqrt{3}} & \frac{\alpha_5}{\sqrt{3}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\alpha_8 & \frac{\alpha_5 + 3\alpha_2}{\sqrt{3}} & \frac{\alpha_5}{\sqrt{3}} & \frac{\alpha_5}{\sqrt{3}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
\]

Thus, \(D\) does not belong to \(\mathfrak{so}(g_\varphi) = \mathfrak{so}(7)\) unless \(D = 0\).

4. The \(G_2\)-Laplacian flow for \((G, \varphi)\)

In this section, we study the \(G_2\)-Laplacian flow on the solvable unimodular Lie group \(G\) starting from the left-invariant exact \(G_2\)-structure \(\varphi\) given in (3.2).

Recall that if \(M\) is a 7-manifold endowed with a closed \(G_2\)-structure \(\varphi\), the Laplacian flow starting from \(\varphi\) is the initial value problem

\[
\begin{align*}
\frac{\partial}{\partial t} \varphi(t) &= \Delta_\varphi(t) \varphi(t), \\
d\varphi(t) &= 0, \\
\varphi(0) &= \varphi,
\end{align*}
\]

where \(\Delta_\varphi\) denotes the Hodge Laplacian of the Riemannian metric \(g_\varphi\) induced by \(\varphi\). This flow was introduced by Bryant in \cite{4} as a potential tool to study the existence of \(G_2\)-holonomy metrics on compact manifolds. Short-time existence and uniqueness of the solution of (4.1) when \(M\) is compact were proved in \cite{6}. The same results for the class of invariant closed \(G_2\)-structures on homogeneous spaces were discussed in \cite{21}.

Before studying the flow (4.1) in our case, it is worth recalling the notion of algebraic soliton introduced by Lauret in \cite{21}. Let \(\varphi\) be a closed \(G_2\)-structure on a seven-dimensional Lie algebra \(\mathfrak{h}\), and let \(\tau\) be the corresponding intrinsic torsion form. Using the inner product \(g_\varphi\) on \(\mathfrak{h}\) induced by \(\varphi\), one can define a skew-symmetric operator

\[
\tau_\varphi: \mathfrak{h} \to \mathfrak{h}
\]

associated with the 2-form \(\tau\) via the identity

\[
\tau(\cdot, \cdot) = g_\varphi(\tau_\varphi \cdot, \cdot).
\]

**Definition 4.1** (\cite{21}). The left-invariant closed \(G_2\)-structure induced by \(\varphi\) on the simply connected Lie group \(H\), with Lie algebra \(\mathfrak{h}\), is called an algebraic soliton of the \(G_2\)-Laplacian flow (4.1) if there exist a derivation \(D \in \text{Der}(\mathfrak{h})\) and a real number \(c\) such that

\[
\text{Ric}(g_\varphi) - \frac{1}{12} \text{tr}(\tau_\varphi^2) \text{Id} + \frac{1}{2} \tau_\varphi^2 = c \text{Id} + D.
\]
In [21], Lauret proved the following result.

**Theorem 4.2** ([21]). Let \( \varphi \) be an algebraic soliton on a simply connected Lie group \( H \). Then,

\[
\Delta \varphi = -3c \varphi - \mathcal{L}_{X_D} \varphi, \tag{4.3}
\]

where \( X_D \) is the vector field on \( H \) defined by the one-parameter subgroup of \( \text{Aut}(H) \) associated with the derivation \( D \). In particular, \( \varphi \) is a Laplacian soliton of the \( G_2 \)-Laplacian flow on \( H \). Moreover, the left-invariant solution of the \( G_2 \)-Laplacian flow starting from \( \varphi \) at \( t = 0 \) is given by

\[
\varphi(t) = (-2ct + 1)^{3/2} h(t)^* \varphi, \tag{4.4}
\]

where \( h(t) = \exp \left(-\frac{1}{2c} \log(-2ct + 1)D\right) \) if \( c \neq 0 \), and \( h(t) = \exp(tD) \), otherwise.

**Remark 4.3.** Recall that the action of an automorphism \( a \in \text{Aut}(g) \) on a \( k \)-form \( \gamma \in \Lambda^k (g^*) \) is given by

\[
a^* \gamma (X_1, \ldots, X_k) = \gamma (a^{-1} X_1, \ldots, a^{-1} X_k),
\]

for all \( X_1, \ldots, X_k \in g \).

We now prove that the exact \( G_2 \)-structure considered in Section 3 is an algebraic soliton.

**Theorem 4.4.** The left-invariant exact \( G_2 \)-structure \( \varphi \) (given in Proposition 3.1) on the unimodular solvable Lie group \( G \) is an algebraic soliton with \( c < 0 \). In particular, it is an expanding Laplacian soliton of the \( G_2 \)-Laplacian flow.

**Proof.** It is sufficient to show that the closed \( G_2 \)-structure \( \varphi \) on \( g \) given by (3.2) satisfies the equation (4.2). Let \((E_1, \ldots, E_7)\) be the basis of \( g \) whose dual basis \((E^1, \ldots, E^7)\) is the adapted \( G_2 \)-basis for \( \varphi \). The matrix associated with the Ricci operator of \( g_\varphi \) with respect to this basis is

\[
\text{Ric}(g_\varphi) = \text{diag}(-52, -16, -44, 20, -8, -20, 8).
\]

Moreover, the square of the operator \( \tau_\varphi \) is

\[
\tau_\varphi^2 = \text{diag}(0, -16, -16, -144, -144, -64, -64).
\]

Now, it is easy to check that the equation (4.2) is satisfied for \( c = -\frac{44}{3} \) and for the following derivation of \( g \)

\[
D = \text{diag}(0, 28, 0, 0, -28, 0, 28).
\]

Since \(-3c = 44 > 0\), the left-invariant \( G_2 \)-structure \( \varphi \) is an expanding Laplacian soliton. \( \square \)

**Corollary 4.5.** On the simply connected solvable unimodular Lie group \( G \), the Laplacian flow starting from the left-invariant exact \( G_2 \)-structure \( \varphi \), defined in (3.2), has the left-invariant immortal solution

\[
\varphi(t) = \left(\frac{88}{3} t + 1\right)^{\frac{2}{3}} (E^{123} + E^{167} + E^{246} - E^{257} - E^{347}) + \left(\frac{88}{3} t + 1\right)^{\frac{2}{3}} (E^{145} - E^{356}),
\]

where \( t \in \left(-\frac{3}{88}, +\infty\right) \).

**Proof.** By Theorem 4.3, we know that \( \varphi \) is an algebraic soliton with \( c = -\frac{44}{3} \) and \( D = \text{diag}(0, 28, 0, 0, -28, 0, 28) \). Then, the result follows from the identity (4.4). \( \square \)
Recall that the $G_2$-Laplacian flow evolving a left-invariant closed $G_2$-structure $\varphi$ on a simply connected solvable Lie group $H$ is equivalent to a flow evolving the Lie bracket $\{\cdot,\cdot\}$ of the Lie algebra $\mathfrak{h}$ of $H$. By [21] Thm. 3.8, $\varphi$ is an algebraic soliton of the form (4.2) if and only if the solution of the corresponding bracket flow on $\mathfrak{h}$ is given by

$$\mu(t) = (-2ct + 1)^{-1/2} [\cdot,\cdot].$$

From this, we immediately obtain the following.

**Corollary 4.6.** Let $\varphi(t)$ be the solution (given in Corollary 4.5) of the $G_2$-Laplacian flow on the simply connected solvable Lie group $G$. Then, the solution of the corresponding bracket flow on $\mathfrak{g}$ is $\mu(t) = (\frac{88}{3} t + 1)^{-1/2} [\cdot,\cdot]$. In particular, the Lie algebra $(\mathfrak{g}, \mu(t))$ tends to the abelian Lie algebra as $t \to +\infty$.

For the sake of brevity, from now on we let

$$f(t) := \frac{88}{3} t + 1.$$

Using the relation (1.1), we can compute the left-invariant metric $g_{\varphi(t)}$ induced by $\varphi(t)$ on $G$, obtaining

$$g_{\varphi(t)} = f(t) \left[ (E^1)^2 + (E^3)^2 + (E^4)^2 + (E^6)^2 \right] + f(t)^{-\frac{11}{2}} \left[ (E^2)^2 + (E^7)^2 \right] + f(t)^{\frac{3}{2}} (E^5)^2.$$

From this, we see that an adapted $G_2$-basis for $\varphi(t)$ is given by

$$\left( f(t)^{\frac{3}{2}} E^1, f(t)^{-\frac{5}{2}} E^2, f(t)^{\frac{5}{2}} E^3, f(t)^{\frac{5}{2}} E^4, f(t)^{\frac{3}{2}} E^5, f(t)^{\frac{3}{2}} E^6, f(t)^{-\frac{5}{2}} E^7 \right).$$

We now study the asymptotic behaviour of the curvature tensors $R(t)$, $\text{Ric}(t)$ and $\text{Scal}(t)$ of the left-invariant metric $g_{\varphi(t)}$. First, we compute the norm of the Riemann curvature tensor $R(t)$ with respect to $g_{\varphi(t)}$, and we have

$$|R(t)|_{\varphi(t)} = \left( R_{ijkl} g_{\varphi}^{ja} g_{\varphi}^{jb} g_{\varphi}^{kc} g_{\varphi}^{ld} R_{abcd} \right)^{1/2} = 18208 f(t)^{-1}.$$

The Ricci tensor $\text{Ric}(t)$ of $g_{\varphi(t)}$ is given by

$$\text{Ric}(t) = -52 (E^1)^2 - 16 f(t)^{-\frac{11}{2}} (E^2)^2 - 44 (E^3)^2 + 20 (E^4)^2$$

$$- 8 f(t)^{-\frac{11}{2}} (E^5)^2 - 20 (E^6)^2 + 8 f(t)^{\frac{3}{2}} (E^7)^2,$$

and its norm is $|\text{Ric}(t)|_{\varphi(t)} = 5824 f(t)^{-1}$. Finally, the scalar curvature $\text{Scal}(t)$ of $g_{\varphi(t)}$ is

$$\text{Scal}(t) = 112 f(t)^{-1}.$$

We immediately see that the functions $|R(t)|_{\varphi(t)}$, $|\text{Ric}(t)|_{\varphi(t)}$ and $\text{Scal}(t)$ tend to zero as $t$ tends to $+\infty$, while they tend to $+\infty$ as $t$ goes to $-\frac{3}{88}$.

From this last observation, it is also possible to deduce the behaviour of the intrinsic torsion form $\tau(t)$ of the closed $G_2$-structure $\varphi(t)$. Indeed, by [21] the following identity holds:

$$\text{Scal}(t) = -\frac{1}{2} |\tau(t)|_{\varphi(t)}^2.$$

In particular, $\varphi(t)$ tends to a torsion-free $G_2$-structure inducing the flat metric as $t$ goes to $+\infty$. 

Remark 4.7. A left-invariant closed $G_2$-structure $\varphi$ on a simply connected Lie group $H$ is called a semi-algebraic soliton if it satisfies the equation (4.3) for some real number $c$ and some derivation $D \in \text{Der}(\mathfrak{h})$. By Theorem 4.2 every algebraic soliton is semi-algebraic, while the converse is true whenever $D^t \in \text{Der}(\mathfrak{h})$, where $D^t$ denotes the transpose of $D$.

An example of a semi-algebraic soliton, which is not algebraic soliton, is given also in [21].

It is worth pointing out that the Lie group $G$ considered in this note admits left-invariant closed $G_2$-structures that are not semi-algebraic solitons. For instance, on the Lie algebra $\mathfrak{g} = \langle E_1, \ldots, E_7 \rangle$ with the structure equations (3.1), the 3-form

$$\tilde{\varphi} = 2\sqrt{3} E^{367} + E^{347} + 2 E^{356} + 3 E^{257} - \sqrt{3} E^{345} + \frac{\sqrt{3}}{7} E^{237} + E^{246} - \frac{2}{\sqrt{3}} E^{147}$$

$$+3\sqrt{3} E^{156} - E^{167} - E^{235} + \frac{1}{2} E^{136} - 2 E^{145}$$

defines a closed $G_2$-structure with associated metric

$$g_{\tilde{\varphi}} = \frac{14}{\sqrt{8470}} \left[ (E^1)^2 + (E^2)^2 + \frac{2}{7} (E^3)^2 + 4 (E^4)^2 + 15 (E^5)^2 + 20 (E^6)^2 + 9 (E^7)^2ight.$$}

$$+\frac{\sqrt{3}}{42} E^1 \circ E^3 - 4 E^1 \circ E^5 - \frac{18\sqrt{3}}{7} E^1 \circ E^7 - \frac{41\sqrt{3}}{21} E^2 \circ E^4 - \frac{81}{14} E^2 \circ E^6$$

$$+\frac{3\sqrt{3}}{14} E^3 \circ E^5 + \frac{5}{14} E^3 \circ E^7 + \frac{28}{\sqrt{3}} E^4 \circ E^6 + 2\sqrt{3} E^5 \circ E^7 \right].$$

A straightforward computation shows that the left-invariant closed $G_2$-structure induced by $\tilde{\varphi}$ on $G$ does not satisfy the equation (4.3) for any choice of $c \in \mathbb{R}$ and $D \in \text{Der}(\mathfrak{g})$.

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