Algebraic formulation of higher gauge theory

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ABSTRACT: In this paper, we present a purely algebraic formulation of higher
gauge theory and gauged sigma models based on the abstract theory of graded
commutative algebras and their morphisms. The formulation incorporates natu-
rally BRST symmetry and is also suitable for AKSZ type constructions. It is also
shown that for a full–fledged BV formulation including ghost degrees of freedom,
higher gauge and gauged sigma model fields must be viewed as internal smooth
functions on the shifted tangent bundle of a space time manifold valued in a
shifted $L_\infty$–algebroid encoding symmetry. The relationship to other formulations
where the $L_\infty$–algebroid arises from a higher Lie groupoid by Lie differentiation
is highlighted.

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1 Introduction

Higher gauge theory is an generalization of ordinary gauge theory where gauge fields and field strengths are higher degree forms. Higher gauge theory appears to be the most promising candidate for the description of the dynamics of the higher–dimensional extended objects thought to be the basic constituents of matter and mediators of fundamental interactions. In various forms, higher gauge theory is relevant in supergravity, in string and brane theory and in the loop and spin foam formulations of quantum gravity. Presently, the interest in higher gauge theory rests on the hope that it may eventually provide a Lagrangian formulation of the mysterious $N = (2, 0)$ 6–dimensional superconformal field theory describing the effective dynamics of $M5$–branes. See [1] and references therein for a readable introduction to higher gauge theory and [2] for an updated account of the latest developments.

Quite early in the history of the subject, it was realized that in the non Abelian case the symmetry of higher gauge theory cannot be described by ordinary Lie groups. The solution to this problem proposed in [3–6] and now widely accepted is that higher gauge theory should result from a categorification of ordinary gauge theory by formalizing higher gauge symmetry through the algebraic structures stemming from the categorification of ordinary groups, the so–called 2–groups. These ideas have been developed to an increasing degree of generality in the context of $\infty$–Lie theory in refs. [7,8]. Related approaches to the problem were followed in refs. [9–12].

All these endeavours require rather sophisticated mathematical constructions. Indeed, higher gauge theory meets many areas of contemporary mathematics: higher algebraic structures, such as higher categories, higher groups [13,14] and strong homotopy Lie or $L_\infty$–algebras [15,16], higher geometrical structures, such as gerbes [17,18] and higher topological structures, such as higher knots [19,20]. An illustration of these multiple topics and their relationship to fundamental physics can be found in [21,22].

The expression of the full potential of higher gauge theory has been hindered
so far by the scarcity of interesting physically motivated examples, though some progress has been made on this score. A non exhaustive list of contributions in which models relevant in string theory and higher Chern–Simons theory are worked out is \[23–29\]. It is therefore desirable to devise formulations of higher gauge theory with general templates allowing for broad classes of interesting examples. This is the point of view adopted in this paper.

In this paper, we propose a purely algebraic formulation of higher gauge theory based on the abstract theory of graded commutative algebras and their morphisms that naturally incorporates BRST symmetry and is thus suitable for perturbative quantization. We do so by building higher gauge theory as a generalization of ordinary gauge theory cast in the language of NQ–manifold and \(L_x\)–algebroid geometry \[30\] which has a natural graded commutative algebraic reading.

Our formulation builds on the framework developed by Bojowald, Grötz- mann, Kotov and Strobl in refs. \[31–33\], which is called BGKS theory in the following for short and is briefly reviewed in subsect. \[1.1\]. We stress our indebtedness to these authors. A related approach is provided in ref. \[34\].

A basic element of our formulation is BRST theory (see \[35\] for a review and extensive referencing) in the superfield framework pioneered a long time ago by Baulieu and Thierry–Mieg in refs. \[36–38\] and Bonora, Pasti and Tonin in refs. \[39–41\]. This is reviewed in subsect \[1.2\].

In subsect. \[1.3\] we explain in broad lines our the basic tenets on which our theory is based in order to make the content the main body of the paper more easily understandable. In subsect. \[1.4\] we finally outline the plan of the paper.

1.1 BGKS theory

BGKS theory is a geometrical formulation of higher gauge theory. In the usual way, a higher gauge field is viewed as a higher connection on a background higher principal bundle and its gauge field strength as the connection’s curvature. Everything is however cast in the language of NQ–manifold theory, which renders the generalization of the customary notions of ordinary gauge theory to the higher case particularly simple and elegant.
Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$, $M$ be a manifold and $P$ be a trivial principal $G$–bundle over $M$. A connection of $P$ is then simply a $\mathfrak{g}$–valued 1–form $A$. The curvature of $A$ is the $\mathfrak{g}$–valued 2–form

$$F_A = dA + \frac{1}{2}[A, A]. \quad (1.1.1)$$

Gauge transformations are the symmetry transformations of gauge theory. Geometrically, they are automorphisms of $P$ covering the identity. An infinitesimal gauge transformation is simply a $\mathfrak{g}$–valued 0–form $\epsilon$. It acts on a connection $A$ as

$$\delta A = -d\epsilon - [A, \epsilon]. \quad (1.1.2)$$

The corresponding variation of the curvature $F_A$ of $A$ is

$$\delta F_A = [\epsilon, F_A]. \quad (1.1.3)$$

Following [31–33], we reformulate the above well–known facts as follows.

A differential graded commutative algebra is a graded commutative algebra $C$ endowed with a nilpotent degree 1 derivation $Q_C$. A NQ–manifold is a non negatively graded manifold $X$ equipped with a homological vector field $Q_X$, i.e. a degree 1 nilpotent vector field, so that the smooth function algebra $C^\infty(X)$ together with $Q_X$ constitute a differential graded commutative algebra.

By a standard construction, with the manifold $M$ one can associate the NQ–manifold $(T[1]M, d)$, where $T[1]M$ is the 1–shifted tangent bundle of $M$ and $d$ is the homological vector field

$$d = \xi^i \partial_{x^i}, \quad (1.1.4)$$

$x^i, \xi^i$ being degree 0, 1 base and fiber coordinates of $T[1]M$. Similarly, with the Lie algebra $\mathfrak{g}$ one can associate the NQ–manifold $C[1], Q_\mathfrak{g}$, where $\mathfrak{g}[1]$ is the 1–shifted vector space $\mathfrak{g}$ and $Q_\mathfrak{g}$ is the Chevalley–Eilenberg differential,

$$Q_\mathfrak{g} \pi = -\frac{1}{2}[\pi, \pi] \quad (1.1.5)$$

with $\pi = \pi^a t_a$, $\pi^a$ being degree 1 coordinates of $\mathfrak{g}[1]$ with respect to a given basis $t_a$ of $\mathfrak{g}$ assumed of degree 0.
It is a basic fact that the graded commutative algebra $\Omega^*_p(M)$ of differential forms and the de Rham differential $d_{dR}$ of $M$ can be identified with the graded commutative algebra $C^*(T[1]M)$ and the homological vector field $d$, respectively. The datum of a connection 1–form $A$ is then equivalent to that of a graded manifold morphism $a : T[1]M \to \mathfrak{g}[1]$ if $A$ and $a$ are related as

$$A = a^* \pi, \quad (1.1.6)$$

since in this way $A$ is determined by and determines $a$. Correspondingly, The curvature 2–form $F_A$ of $A$ can be expressed in terms of $a$ as

$$F_A = da^* \pi + \frac{1}{2} [a^* \pi, a^* \pi] = da^* \pi - a^* Q_g \pi. \quad (1.1.7)$$

$F_A$ thus provides a measure of the failure of $a$ to be a NQ–manifold morphism.

To see how gauge transformation can be accommodated in this framework, one proceeds as follows. We replace the map $a$ with its graph $\hat{a} : T[1]M \to T[1]M \times \mathfrak{g}[1]$. Since the datum of $a$ is equivalent to that of $\hat{a}$, it should be possible to reformulate everything in terms of $\hat{a}$. Indeed, $\hat{a}$ is also a graded manifold morphism. $T[1]M \times \mathfrak{g}[1]$ is a NQ–manifold with homological vector field $d + Q_g$. Furthermore, the datum of a connection $A$ amounts to that of $\hat{a}$, since (1.1.6) can be recast as

$$A = \hat{a}^* \pi. \quad (1.1.8)$$

Similarly, we can rewrite the expression (1.1.7) of the curvature $F_A$ of $A$ as

$$F_A = d\hat{a}^* \pi + \frac{1}{2} [\hat{a}^* \pi, \hat{a}^* \pi] = d\hat{a}^* \pi - \hat{a}^* (d + Q_g) \pi. \quad (1.1.9)$$

We call a vector field $w$ on $T[1]M \times \mathfrak{g}[1]$ vertical, if it is everywhere directed along $\mathfrak{g}[1]$. An infinitesimal gauge transformation $\epsilon$ is encoded in a degree $-1$ vertical vector field $w$ on $T[1]M \times \mathfrak{g}[1]$. The relation of $\epsilon$ to $w$ is

$$\epsilon = w \pi. \quad (1.1.10)$$

Taking (1.1.8) into account, the gauge variation (1.1.2) of $A$ is given by

$$\delta A = -dw \pi - [\hat{a}^* \pi, w \pi] = -\hat{a}^* [d + Q_g, w] \pi \quad (1.1.11)$$
where in the last term \([-,-]\) denotes the Lie bracket of the vector field Lie algebra of \(T[1]M \times g[1]\). Similarly, on account of (1.1.9), the gauge variation (1.1.3) of \(F_A\) reads as

\[
\delta F_A = [w \pi, d \hat{a}^* \pi - \hat{a}^* (d + Q_\pi) \pi] = -(d \hat{a}^* - \hat{a}^* (d + Q_\pi))(d + Q_\pi, w) \pi. \tag{1.1.12}
\]

The above reformulation of ordinary Yang–Mills theory immediately points to its higher generalization. One replaces the Lie algebra \(g\) with an \(L_\infty\) algebra \(v\), a graded vector space equipped with a set of brackets satisfying generalized Jacobi identities. Similarly to the ordinary Lie case, the algebraic structure of \(v\) is encoded in the 1–shifted space \(v[1]\) and a Chevalley–Eilenberg differential \(Q_v\) with an action on \(v[1]\) expressed via a coordinate vector \(\pi\). The datum of a higher connection polyform \(A\) is now equivalent to that of a graded manifold morphism \(a : T[1]M \to v[1]\) with the relationship of \(A\) to \(a\) still expressed by (1.1.6). The curvature polyform \(F_A\) is then given by the last term of (1.1.7) with \(Q_g\) replaced by \(Q_v\).

Higher gauge transformations are encoded by degree \(-1\) vertical vector fields on \(T[1]M \times v[1]\). The gauge variations of the higher connection and curvature polyforms \(A\) and \(F_A\) are then given by the last terms of eqs. (1.1.11) and (1.1.12), respectively, again with \(Q_g\) replaced by \(Q_v\).

1.2 Superfield approach to BRST symmetry

The basic idea of BRST theory is to replace a local gauge symmetry by a degree 1 global symmetry acting on an extended field space containing ghost fields in addition to the original gauge fields. BRST symmetry transformation is essentially infinitesimal and is required to satisfy the Wess–Zumino consistency condition.

In BRST theory, fields are bigraded: in addition to form degree, they have ghost degree. While form degree is definite non negative, ghost degree can have any sign. A field \(\phi^{(p,k)}\) of form degree \(p\) and ghost degree \(k\) is said to have bidegree \((p,k)\). The total degree, or simply degree, is the sum of the form and ghost degrees. A field \(\phi^{(n)}\) of degree \(n\) is a sum of fields \(\phi^{(p,n-p)}\) of bidegree \((p,n-p)\) with \(0 \leq p \leq d\), \(d\) being the dimension of the space–time manifold \(M\). We can
then view $\phi^{(n)}$ as a BRST superfield and the $\phi^{(p,n-p)}$ as its components. 

The de Rham differential $d_{dR}$ has bidegree $(1, 0)$. BRST transformation is given by the action of a bidegree $(0, 1)$ derivation $s_{BRST}$, the BRST variation operator, which by virtue of the Wess–Zumino consistency condition is nilpotent,

$$s_{BRST}^2 = 0.$$  \hspace{1cm} (1.2.1)

In BRST theory, we have therefore two odd derivations. $d_{dR}$ and $s_{BRST}$ anticommute, $s_{BRST}d_{dR} + d_{dR}s_{BRST} = 0$.

The above can be stated in the language of NQ–manifold theory employed in the BGKS theory of subsect. 1.1. While ordinary fields belong to the graded commutative function algebra $C^\infty(T[1]M)$, BRST superfields span the algebra $C^\infty(T[1]M) \otimes G_{\mathbb{R}}$, where $G_{\mathbb{R}}$ is the ghost algebra

$$G_{\mathbb{R}} = \bigoplus_{k=-\infty}^{\infty} \mathbb{R}[k]$$  \hspace{1cm} (1.2.2)

with $\mathbb{R}[k]$ the $k$–shifted real line. $C^\infty(T[1]M) \otimes G_{\mathbb{R}}$ has indeed two distinct gradings. The first stems from the grading of $T[1]M$ in the usual way and can thus be identified with form degree. The second originates from the independent grading of $G_{\mathbb{R}}$ and corresponds to ghost degree. The de Rham differential $d_{dR}$ and the BRST variation operator $s_{BRST}$ are expressed as anticommuting nilpotent bidegree $(1, 0)$ and $(0, 1)$ derivations $d$ and $s$ on $C^\infty(T[1]M) \otimes G_{\mathbb{R}}$, respectively.

In the BRST formulation of ordinary gauge theory, the gauge field $A$ is promoted to a $\mathfrak{g}[1]$–valued BRST superfield $A$, the BRST gauge superfield. The total field strength $F_A$ of $A$, a $\mathfrak{g}[2]$–valued BRST superfield, is required to vanish,

$$F_A = (d + s)A + \frac{1}{2}[A, A] = 0.$$  \hspace{1cm} (1.2.3)

This relation defines the BRST variation $sA$ of $A$,

$$sA = -dA - \frac{1}{2}[A, A] \equiv -F_A.$$  \hspace{1cm} (1.2.4)

It is straightforward to check that $s^2 A = 0$ as required by (1.2.1).

It is illuminating to write the BRST gauge superfield $A$ and its BRST variation $sA$ in components. $A$ has a component expansion of the form
\[ \mathcal{A} = c + A + \gamma + \ldots \] (1.2.5)

Here, \( c \) is the component of bidegree \((0,1)\) customarily called Faddeev–Popov ghost. \( A \) is the component of bidegree \((1,0)\) and is nothing but the familiar gauge field, as indicated by the notation. \( \gamma \) is a component of bidegree \((2,-1)\). The ellipses denote components of higher form and lower ghost degree components whose number depends on the space–time dimension \(d\). Inserting (1.2.5) into (1.2.4), the BRST variations \( s \mathcal{A}, s \gamma, \ldots \) of \( c, A, \gamma, \ldots \) are easily obtained.

The BRST variation of the gauge field \( A \) is

\[ sA = -dc - [A, c], \] (1.2.6)

while that of the field strength \( F_A \) of \( A \)

\[ sF_A = -[c, F_A]. \] (1.2.7)

They are formally identical to (1.1.2) and (1.1.3). The Faddeev–Popov ghost field \( c \) behaves therefore as a degree 1 infinitesimal gauge transformation parameter. \( c \) has however a non trivial BRST variation

\[ sc = -\frac{1}{2}[c, c]. \] (1.2.8)

The BRST variation of the ghost field \( \gamma \),

\[ s\gamma = -F_A - [c, \gamma], \] (1.2.9)

is interesting because it yields the field strength \( F_A \) of the gauge field \( A \).

The possibility of a BRST formulation of higher gauge theory on the same lines should now be quite evident. The higher gauge field \( A \) is promoted to a \( \mathfrak{v}[1] \)-valued BRST higher gauge superfield \( \mathcal{A} \). Imposing the vanishing of the total BRST field strength \( \mathcal{F}_A \) of \( \mathcal{A} \), generalizing condition (1.2.3), it is then possible to obtain the BRST variation \( s \mathcal{A} \) of \( \mathcal{A} \), generalizing (1.2.4). More specifically, unfolding \( A \) as a collection of fields \( A_p \) with \( A_p \) a \( p + 1 \)-form valued in the degree \( p \) subspace \( \mathfrak{v}_p \) of \( \mathfrak{v}, \mathcal{A} \) gets expressed as a collection of degree \( p + 1 \) \( \mathfrak{v}_p \)-valued BRST superfield \( \mathcal{A}_p \). For the vanishing of the total BRST field strength \( \mathcal{F}_{A_p} \) of
A_p, one obtains the BRST variations sA_p of A_p.

1.3 BRST reformulation of BGKS theory

The way gauge symmetry emerges in BGKS theory is quite different from that it does in BRST theory, though the two formulations are obviously related. BGKS theory is simple, elegant and geometrical, but in its present form cannot be used in perturbative quantum field theory. BRST theory treats ghost and gauge fields on an equal 'democratic' footing. By design it provides the natural setting for gauge fixing, but it lacks a clear geometrical underpinning.

Consider again the BGKS formulation of ordinary gauge theory expounded in subsect. 1.1. From inspecting (1.1.6) and (1.1.7), it appears that both the connection A and its curvature F_A depend on the map \( a : T[1]M \to g[1] \) through its pull-back operator \( a^* : C^\infty(g[1]) \to C^\infty(T[1]M) \). \( a^* \) is a graded commutative algebra morphism. By (1.1.6), on one hand \( a^* \) determines \( A \), on the other \( a^* \) is determined by \( A \), since \( C^\infty(g[1]) = S(g[1]) \), the symmetric algebra of \( g[1] \), and hence \( a^* \) is fixed by its action on the degree 1 subspace \( g[1] \subset S(g[1]) \).

Therefore, we can identify the connection \( A \) with the morphism \( a^* \). The operator \( F_{a^*} : C^\infty(g[1]) \to C^\infty(T[1]M) \) defined by

\[
F_{a^*} = da^* - a^*Q_g.
\]

appearing in (1.1.7) is a degree 1 derivation of \( C^\infty(g[1]) \) valued in \( C^\infty(T[1]M) \) over \( a^* \). By (1.1.7), on one hand \( F_{a^*} \) determines \( F_A \), on the other \( F_{a^*} \) is determined by \( F_A \), as, being \( C^\infty(g[1]) = S(g[1]) \), \( F_{a^*} \) is fixed by its action on the degree 1 subspace \( g[1] \) once \( a^* \) is given. Therefore, we can identify the curvature \( F_A \) with the derivation \( F_{a^*} \).

The derivation \( F_{a^*} \) encodes a basic property of the algebra morphism \( a^* \). \( F_{a^*} \) vanishes precisely when \( a^* \) is a differential morphism. It thus measures the failure of \( a^* \) to be so.

In higher gauge theory, the Lie algebra \( g \) is replaced by an \( L_\infty \)-algebra \( v \) and the higher connection \( A \) and curvature \( F_A \) depend again on the mapping \( a : T[1]M \to v[1] \) through its pull-back operator \( a^* : C^\infty(v[1]) \to C^\infty(T[1]M) \).
The above considerations extend essentially unchanged.

The maps $a : T[1]M \to \mathfrak{v}[1]$ employed in BGKS theory are ordinary graded manifold morphisms. As we shall endeavour to show in this paper, a geometrical framework allowing for more general *internal* graded manifold morphisms $a : T[1]M \to \mathfrak{v}[1]$ paves the way to a reformulation of BGKS theory that is interesting on its own and moreover subsumes BRST theory from the start in an efficient and elegant way. Let us see how.

Internal morphisms differ from ordinary ones in that their local expression contains smooth functions of the base coordinates $x^i$ of possibly non-zero degree. For an internal $a : T[1]M \to \mathfrak{v}[1]$, one can define an internal pull-back operator $a^\# : C^\infty(\mathfrak{v}[1]) \to C^\infty(T[1]M) \otimes G_\mathbb{R}$ having the *internal* function algebra $C^\infty(T[1]M) \otimes G_\mathbb{R}$ as its range, where $G_\mathbb{R}$ is the graded algebra \textcolor{red}{(1.2.2)}. Since $C^\infty(T[1]M) \otimes G_\mathbb{R}$ is precisely the algebra of BRST superfields, as explained in subsect. \textcolor{red}{1.2}, we are now in the position of merging BGKS and BRST theory.

In the BRST formulation of ordinary gauge theory, the connection $A$ is promoted to a BRST connection superfield $\mathcal{A}$, which we describe as a degree 1 element of the function space $C^\infty(T[1]M) \otimes G_\mathbb{R} \otimes \mathfrak{g}$ containing components of all possible form and ghost degree compatible with it having total degree 1 (cf. subsect. \textcolor{red}{1.2}). Setting

$$\mathcal{A} = a^\# \pi$$

defines an internal graded manifold morphism $a : T[1]M \to \mathfrak{v}[1]$ whose datum is equivalent to that of $\mathcal{A}$. In this way, thinking *ab initio* in terms of internal morphisms allows us to reformulate the BGKS theory in a way that fully incorporates BRST theory, as we anticipated. Upon replacing the Lie algebra $\mathfrak{g}$ with an $L_\infty$–algebra $\mathfrak{v}$, the same conclusion is reached with regard to higher gauge theory.

This reformulation is not as straightforward as it may seem at first glance. As the reader will see, a thorough reconsideration of the part relative to gauge transformation is involved. Further, the resulting theory will allow us also to make contact with BV approaches to higher gauge theory \textcolor{red}{[42][44]} and in particular in
the AKSZ formulation \[49\].

### 1.4 Plan of the paper

In sect. 2, we show how the intuitions about higher gauge theory presented in the previous subsections can be systematically organized into a purely algebraic formulation based on the abstract theory of graded commutative algebras and their morphisms. Differential graded commutative algebras algebras, that is algebras endowed with a nilpotent differential, play a basic role. The formulation incorporates naturally BRST symmetry and is also suitable for an AKSZ type construction of higher gauge theoretic models.

In sect. 3, we work out an explicit formulation of higher gauge theory and gauged sigma models relying on the formal framework developed in sect. 2. The characterizing point of our construction is the realization that for a full–fledged BV formulation incorporating ghost degrees of freedom higher fields must be viewed as internal smooth functions from the shifted tangent bundle of a space time manifold to a shifted $L_{\infty}$–algebroid encoding the higher gauge symmetry. We analyze also the case where the $L_{\infty}$–algebroid arises from a higher Lie groupoid by Lie differentiation and elucidate the relation between higher gauge symmetry expressed through simplicial homotopy and BRST symmetry.

Throughout the paper, we rely heavily on graded differential geometry, which through its multiple ramifications is the most natural framework within which carrying out our program is possible.

### 1.5 Outlook

Our analysis, albeit inspired by geometry, indicates that the most basic features of higher gauge theory are ultimately algebraic. The formulation of sect. 2 captures many of the most basic properties of standard higher gauge theoretic models without committing to the assumption that the relevant differential graded commutative algebras are algebras of smooth functions on appropriate graded manifolds as in BGKS and BRST theory reviewed above. Sometimes in the future, it is our hope, this may allow to work out models with a definite higher
gauge theoretic outlook in contexts quite different from the supergravity, string
and brane theoretic ones where higher gauge theory was elaborated in the first
place, with innovative applications to the study of confining and gapped phases
of gauge theory [45,46], statistical mechanics and topological phases of matter
[47,48] and more.
2 Algebraic higher gauge theory

In this section, we present an abstract purely algebraic formulation of higher gauge theory based on the theory of graded commutative algebras and their differential enhancements. Our framework lends itself to multiple readings, each of which is interesting on its own. On the geometrical side, graded commutative algebras are models for the algebras of smooth functions on $N$–manifolds. Similarly, differential graded commutative algebras are models for the algebras of smooth functions on $NQ$–manifolds. On the physical mathematical side, non-differential morphisms between differential graded commutative algebras formalize higher gauge fields, their defects encode their higher curvatures and the defect identities they obey are the analog of the basic Bianchi identities. The formulation incorporates naturally BRST symmetry and is also suitable for an AKSZ type construction of higher gauge theory.

2.1 Differential algebras, graded algebra morphisms and defect

A graded vector space $V$ is a vector space endowed with a direct sum decomposition $V = \bigoplus_{p \in \mathbb{Z}} V_p$. For $p \in \mathbb{Z}$, a degree $p$ element $f$ of $V$ is just an element of $V_p$. An element $f$ of $V$ is called homogeneous if it is of degree $p$ for some $p$. In that case, we write $|f| = p$.

For $p \in \mathbb{Z}$, a degree $p$ linear map $T : V' \to V$ of graded vector spaces is a linear map such that $TV'_q \subseteq V_{p+q}$ for all $q \in \mathbb{Z}$. A linear map $T : V' \to V$ is said homogeneous if it is of degree $p$ for some $p$. In that case, we write $|T| = p$.

A graded algebra $A$ is a graded vector space equipped with an associative and distributive product such that $A_pA_q \subseteq A_{p+q}$ for all $p, q \in \mathbb{Z}$ and a unit $1$ with $|1| = 0$. A graded commutative algebra $C$ is a graded algebra such that

$$fg = (-1)^{|f||g|}gf \quad (2.1.1)$$

for all homogeneous $f, g \in C$. In the following, we shall consider exclusively graded commutative algebras. A morphism $\Upsilon$ of a graded commutative algebra $C_2$ into another $C_1$ is a degree 0 linear map $\Upsilon : C_2 \to C_1$ such that for $f, g \in C_2$
Graded commutative algebras and their morphisms from a category \textsf{grcAlg}.

For \( p \in \mathbb{Z} \), a degree \( p \) derivation \( D \) of a graded commutative algebra \( C \) is a degree \( p \) linear map \( D : C \to C \) such that for homogeneous \( f, g \in C \)

\[
D(fg) = D(f)g + (-1)^{|f||g|}fDg.
\] (2.1.3)

The degree \( p \) derivations \( D \) of \( C \) form a vector space \( \text{Der}_p(C) \). The derivations \( D \) of all possible degrees \( p \) span the graded vector space \( \text{Der}(C) = \bigoplus_{p \in \mathbb{Z}} \text{Der}_p(C) \) of all derivations of \( C \). A derivation \( D \) of \( \text{Der}(C) \) is called homogeneous if it is of degree \( p \) for some \( p \). In that case, we write \(|D| = p\).

\( \text{Der}(C) \) is a graded Lie algebra with the Lie brackets

\[
[X, Y] = XY - (-1)^{|X||Y|}YX
\] (2.1.4)

for homogeneous derivations \( X, Y \in \text{Der}(C) \). These brackets are graded antisymmetric and satisfy the graded Jacobi identity,

\[
[X, Y] + (-1)^{|X||Y|}[Y, X] = 0,
\] (2.1.5)

\[
(-1)^{|Z||X|}[X, [Y, Z]] + (-1)^{|X||Y|}[Y, [Z, X]]
\]

\[
+ (-1)^{|Y||Z|}[Z, [X, Y]] = 0,
\] (2.1.6)

for homogeneous derivations \( X, Y, Z \in \text{Der}(C) \), as it is straightforward to verify.

A graded algebra possesses a distinguished degree 0 derivation \( E \), the Euler derivation, characterized by the property that

\[
Ef = |f|f
\] (2.1.7)

for homogeneous \( f \in C \). If \( D \in \text{Der}(C) \) is a homogeneous derivation of \( C \), then

\[
[E, D] = |D|D.
\] (2.1.8)

The notion of derivation has the following generalization that will turn out to be important in our analysis. A degree \( p \) derivation \( D \) of a graded commutative
algebra $C_2$ valued in another $C_1$ over a morphism $\Upsilon : C_2 \to C_1$ is a degree $p$ linear map $D : C_2 \to C_1$ such that

$$D(fg) = Df\Upsilon g + (-1)^{|f||g|}\Upsilon fDg \quad (2.1.9)$$

for homogeneous $f, g \in C_2$. The degree $p$ derivations $D$ of $C_2$ valued in $C_1$ over $\Upsilon$ form a vector space $\text{Der}_{p\Upsilon}(C_2, C_1)$. The derivations $D$ of all possible degrees $p$ span the graded vector space $\text{Der}_\Upsilon(C_2, C_1) = \bigoplus_{p \in \mathbb{Z}} \text{Der}_{p\Upsilon}(C_2, C_1)$ of all derivations of $C_2$ valued in $C_1$ over $\Upsilon$. A derivation $D$ of $\text{Der}_\Upsilon(C_2, C_1)$ is called homogeneous if it is of degree $p$ for some $p$. In that case, we write $|D| = p$. A degree $p$ derivation $D$ of a graded commutative algebra $C$ is just a degree $p$ derivation of $C$ valued in $C$ over the identity $\text{id}_C$.

A *differential graded commutative algebra* $C$ is a graded commutative algebra equipped with a *differential*, a degree 1 derivation $Q$ of $C$ that is nilpotent,

$$Q^2 = 0. \quad (2.1.10)$$

With $C$, there are associated two cochain complexes and their cohomologies. The first is $(C, Q)$. The second is $(\text{Der}(C), \text{ad}Q)$, where $\text{ad}Q = [Q, -]$. A morphism $\Upsilon : C_2 \to C_1$ of a differential graded commutative algebra $C_2$ into another $C_1$ is a graded commutative algebra morphism such that

$$Q_1\Upsilon - \Upsilon Q_2 = 0. \quad (2.1.11)$$

Differential graded commutative algebras and their morphisms constitute a category $\text{dgrcAlg}$ that is a non full subcategory of $\text{grcAlg}$.

Let $C_1, C_2$ be differential graded commutative algebras with differentials $Q_1, Q_2$, respectively. For any morphism $\Upsilon : C_2 \to C_1$ of graded commutative algebras, we define its *defect* $F_\Upsilon : C_2 \to C_1$ to be the degree 1 linear map

$$F_\Upsilon = Q_1\Upsilon - \Upsilon Q_2. \quad (2.1.12)$$

$F_\Upsilon$ is in fact a degree 1 derivation of $C_2$ valued in $C_1$ over the $\Upsilon$. Intuitively, $F_\Upsilon$ measures the failure of $\Upsilon$ to be a morphism of differential graded commutative algebras: $\Upsilon$ is one precisely when its defect $F_\Upsilon = 0$. 

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The defect $F_{Y}$ of $Y$ satisfies the defect identity

$$Q_1 F_Y + F_Y Q_2 = 0,$$  \hspace{1cm} (2.1.13)

which follows from the nilpotence of $Q_1$, $Q_2$.

As we shall see, any higher gauge theory belongs to the realm of differential graded commutative algebras. The theory’s field space is a graded commutative algebra $C_1$ and the theory’s field type space is another $C_2$. In the Lagrangian, the kinetic term is determined by a differential $Q_1$ on $C_1$, while the gauge field self–interaction is by another $Q_2$ on $C_2$. The theory’s higher gauge field content is encoded in a graded commutative algebra morphism $Y : C_2 \to C_1$ in the sense that, for any type $f \in C_2$, $Yf \in C_1$ is the gauge field of type $f$. The defect $F_Y$ of $Y$ encodes the higher gauge field curvatures and the defect identity the higher Bianchi identities these obey meaning that, for $f \in C_2$, $F_Y f \in C_1$ is the gauge curvature of the gauge field of type $f$ and the corresponding defect identity is the Bianchi identity this obeys. It is therefore important to study these objects in greater detail.

2.2 The graded morphism manifold and the defect vector field

We have found an elegant geometric interpretation of the defect of a graded commutative algebra morphism and defect identity, which now we illustrate.

Let $C_1$, $C_2$ be graded commutative algebras. We assume that the set of (non differential) graded commutative algebra morphisms

$$M(C_2, C_1) = \text{Hom}_{\text{grAlg}}(C_2, C_1)$$ \hspace{1cm} (2.2.1)

from $C_2$ to $C_1$ is an ordinary manifold (at least at a formal level). We aim to study its geometry.

$M(C_2, C_1)$ is characterized by certain distinguished vector bundles. For $p \in \mathbb{Z}$, the vector bundle $\text{Der}_p(C_2, C_1)$ of degree $p$ derivations $D$ of $C_2$ valued in $C_1$ is the vector bundle over $M(C_2, C_1)$ whose fiber at a point $Y$ is the vector space $\text{Der}_p(Y)(C_2, C_1)$ of degree $p$ derivations $D$ of $C_2$ valued in $C_1$ over $Y$. Similarly, the vector bundle $\text{Der}(C_2, C_1)$ of derivations $D$ of $C_2$ valued in $C_1$ is the vector
bundle over $M(C_2, C_1)$ whose fiber at a point $\Upsilon$ is the vector space $\text{Der}_\Upsilon(C_2, C_1)$ of all derivations $D$ of $C_2$ valued in $C_1$ over $\Upsilon$. Clearly, one has $\text{Der}(C_2, C_1) = \bigoplus_{p \in \mathbb{Z}} \text{Der}_p(C_2, C_1)$.

To study the geometry of the manifold $M(C_2, C_1)$, it is necessary to describe its tangent bundle $TM(C_2, C_1)$. The tangent space $T\Upsilon M(C_2, C_1)$ to $M(C_2, C_1)$ at a point $\Upsilon$ can be characterized as follows. Since $\Upsilon_p(fg) = \Upsilon(f\Upsilon g)$ \hspace{1cm} (2.2.2) for $f, g \in C_2$, a tangent vector $\dot{\Upsilon} \in T\Upsilon M(C_2, C_1)$ is just a degree 0 linear map $\dot{\Upsilon} : C_2 \rightarrow C_1$ obeying the condition $\dot{\Upsilon}(fg) = \dot{\Upsilon}f\Upsilon g + \Upsilon f\dot{\Upsilon} g$ \hspace{1cm} (2.2.3) that is a degree 0 derivation of $C_2$ valued in $C_1$ over $\Upsilon$ (cf. subsect. 2.1). The tangent bundle $TM(C_2, C_1)$ is therefore identified with the vector bundle $\text{Der}_0(C_2, C_1)$ of degree 0 derivations of $C_2$ valued in $C_1$.

In a graded geometric description of $M(C_2, C_1)$, one needs also to consider the degree shifted forms of the tangent bundle $TM(C_2, C_1)$. Let $p \in \mathbb{Z}$. The $p$–shifted tangent space $T\Upsilon[p]M(C_2, C_1)$ to $M(C_2, C_1)$ at a point $\Upsilon$ can be characterized by the natural graded generalization of condition (2.2.3). A tangent vector of $T\Upsilon[p]M(C_2, C_1)$ is a degree $p$ linear map $\dot{\Upsilon} : C_2 \rightarrow C_1$ obeying $\dot{\Upsilon}(fg) = \dot{\Upsilon}f\Upsilon g + (-1)^{p[f]}\Upsilon f\dot{\Upsilon} g$ \hspace{1cm} (2.2.4) that is a degree $p$ derivation of $C_2$ valued in $C_1$ over $\Upsilon$. The $p$–shifted tangent bundle $T[p]M(C_2, C_1)$ is therefore identified with the vector bundle $\text{Der}_p(C_2, C_1)$ of degree $p$ derivations of $C_2$ valued in $C_1$.

Suppose now that $C_1, C_2$ are differential graded commutative algebras with differentials $Q_1, Q_2$, respectively. As we saw in subsect. 2.1, the defect $F\Upsilon$ of a morphism $\Upsilon \in M(C_1, C_2)$ is a degree 1 derivation of $C_2$ valued in $C_1$ over $\Upsilon$, hence a tangent vector of $T\Upsilon[1]M(C_2, C_1)$. Therefore, the defect map $\Phi \rightarrow F\Phi$.
encodes geometrically a degree 1 vector field $F$ over $M(C_2, C_1)$, that is a section of the 1–shifted tangent bundle $T[1]M(C_2, C_1)$. We shall call $F$ the defect vector field for obvious reasons. $F$ can be written as

$$F = \langle F_\Phi, \partial_\Phi \rangle,$$  \hfill (2.2.5)

where $\partial_\Phi$ denotes derivation with respect to $\Phi$ and $\langle - , - \rangle$ denotes the tangent–cotangent duality pairing (index contraction). $F$ is in fact nilpotent

$$F^2 = 0$$  \hfill (2.2.6)

as a consequence of the defect identity \((2.1.13)\).

Proof. From \((2.1.12)\), \((2.2.5)\) and \((2.1.13)\), we have indeed

$$F^2 = -\langle Q_1 F_\Phi + F_\Phi Q_2, \partial_\Phi \rangle = 0$$  \hfill (2.2.7)

showing \((2.2.6)\).

$M(C_2, C_1)$ equipped with the homological defect vector field $F$ is in this way an NQ–manifold (cf. subsect. 3.2).

The shifted tangent bundle $T[1]M(C_2, C_1)$ of $M(C_2, C_1)$ has itself a rich geometry stemming from the defect map and identity. As a manifold, $T[1]M(C_2, C_1)$ is described by the base coordinate $\Phi$ and degree 1 fiber coordinate $\delta \Phi$ viewed as a formal vector. $T[1]M(C_2, C_1)$ is characterized by two distinguished degree 1 vector fields, that is sections of the twice iterated 1–shifted tangent bundle $T[1]^2M(C_2, C_1)$. The first one is the canonical vector field

$$\delta = \langle \delta \Phi, \partial_\Phi \rangle.$$  \hfill (2.2.8)

The second one is the derivative vector field $l_F$ of the defect vector field $F$. Explicitly, it reads as

$$l_F = \langle F_\Phi, \partial_\Phi \rangle - \langle \delta F_\Phi, \partial_\delta \Phi \rangle,$$  \hfill (2.2.9)

where $\delta F_\Phi$ is given by the expression

$$\delta F_\Phi = -Q_1 \delta \Phi - \delta \Phi Q_2.$$  \hfill (2.2.10)
and \( \partial_{\delta \Phi} \) denotes derivation with respect to \( \delta \Phi \) of degree \(-1\). \( \delta \) and \( l_F \) have both degree \(1\), are both nilpotent and mutually anticommute,

\[
\begin{align*}
\delta^2 &= 0, \\
(l_F)^2 &= 0, \\
l_F \delta + \delta l_F &= 0.
\end{align*}
\]

Proof. (2.2.11) is obvious. From (2.1.12), (2.2.9), (2.2.10) and (2.1.13), we have

\[
l_F^2 = -\langle Q_1 F_{\Phi} + F_{\Phi} Q_2, \partial_{\Phi} \rangle - \langle \delta (Q_1 F_{\Phi} + F_{\Phi} Q_2), \partial_{\Phi} \rangle = 0,
\]

showing (2.2.12). Finally, one has

\[
l_F \delta + \delta l_F = \langle \delta F_{\Phi}, \partial_{\Phi} \rangle - \langle \delta F_{\Phi}, \partial_{\Phi} \rangle = 0
\]

leading to (2.2.13).

So, \( T[1]M(C_2, C_1) \) has two compatible NQ–manifold structures associated with the homological canonical and defect Lie derivative vector fields \( \delta \) and \( l_F \).

In general, the algebra \( \Omega^*(X) \) of exterior differential forms of a manifold \( X \) can be identified with the graded commutative algebra \( C^{\infty}(T[1]X) \) of functions of the shifted tangent bundle \( T[1]X \), the de Rham differential \( d_X \) of \( X \) with the canonical homological vector field \( \delta \) of \( T[1]X \) and the de Rham cohomology of \( X \) with the \( \delta \) cohomology of \( T[1]X \). In the case of \( M(C_2, C_1) \), the algebra \( \Omega^*(M(C_2, C_1)) \) is acted upon by two compatible differentials, the de Rham and the defect Lie derivative ones, and features the associated cohomologies.

As explained in subsect. 2.1 in a higher gauge theory characterized by a differential graded commutative algebra \( C_1 \) of gauge fields and another \( C_2 \) of field types, a graded commutative algebra morphism \( \Upsilon : C_2 \to C_1 \) represents a collection of higher gauge fields, its defect \( F_\Upsilon \) the associated collection of higher gauge curvatures and the defect identity of \( F_\Upsilon \) the Bianchi identities these obey. The morphism manifold \( M(C_2, C_1) \) can so be regarded as the space of all higher gauge field configurations, the vector field \( F \) of \( M(C_2, C_1) \) as the higher gauge curvature map and its nilpotence as a condition encoding the Bianchi identities.
2.3 BRST theory of graded morphisms

We have just seen in subsect. 2.2 that in higher gauge theory the higher gauge field configuration space can be identified with the manifold $M(C_2, C_1)$ of non-differential morphisms of two differential graded commutative algebras $C_2, C_1$. The BRST analysis of higher gauge symmetry thus reduces to that of the symmetry of $M(C_2, C_1)$. The resulting BRST theory turns out to be very rich.

Let $C_1, C_2$ be graded commutative algebras and let $\Phi \in M(C_2, C_1)$ be a graded commutative algebra morphism variable. Then,

$$\Phi(fg) = \Phi f \Phi g$$  \hspace{1cm} (2.3.1)

for $f, g \in C_2$.

On general grounds, the BRST symmetry of $M(C_2, C_1)$ is encoded in degree 1 derivations $s$ of $M(C_2, C_1)$. By consistency with (2.3.1), $s\Phi$ must satisfy

$$s\Phi(fg) = s\Phi f \Phi g + (-1)^{|f|} \Phi f s\Phi g$$  \hspace{1cm} (2.3.2)

for $f, g \in C_2$. $s\Phi$ is therefore a degree 1 derivation of $C_2$ valued in $C_1$ over $\Phi$.

The twice iterated variation $s^2\Phi$ is a degree 2 derivation of $C_2$ valued in $C_1$ over $\Phi$ as it is readily checked from (2.3.2). Complying with general principles of gauge theory, we impose the Wess–Zumino consistency condition

$$s^2 \Phi = 0.$$  \hspace{1cm} (2.3.3)

Following Kotov and Strobl [32], we consider the case when $s\Phi$ has the form

$$s\Phi = -U_1 \Phi + \Phi U_2,$$  \hspace{1cm} (2.3.4)

where $U_1 \in \text{Der}(C_1), U_2 \in \text{Der}(C_2)$ are degree 1 derivation variables. It is immediately checked that $s\Phi$ is a degree 1 derivation of $C_2$ valued in $C_1$ over $\Phi$, as required.

To impose the nilpotence condition (2.3.3), we have to extend the degree 1 derivative action of $s$ to $U_1, U_2$. A sufficient condition for (2.3.3) to hold is that

$$sU_1 = -\frac{1}{2}[U_1, U_1], \quad sU_2 = -\frac{1}{2}[U_2, U_2].$$  \hspace{1cm} (2.3.5)
Proof. A simple calculation furnishes

\[ s^2 \Phi = -(sU_1 + U_1^2)\Phi + \Phi(sU_2 + U_2^2). \]  \hspace{1cm} (2.3.6)

Thus, if the (2.3.5) hold, \( s^2 \Phi = 0 \) as required.

The nilpotence condition (2.3.3) extends to \( U_1, U_2, \)

\[ s^2 U_1 = 0, \quad s^2 U_2 = 0, \]  \hspace{1cm} (2.3.7)

Proof. As the proof of the statement is identical for \( U_1, U_2, \) we suppress indexes.

Using the (2.3.5) and the graded Jacobi identity (2.1.6), we obtain

\[ s^2 U = -\frac{1}{2}[U, [U, U]] = 0, \]  \hspace{1cm} (2.3.8)

showing the (2.3.7).

Next, we assume that a differential structure is added to our graded commutative algebras. We thus consider two differential graded commutative algebras \( C_1, C_2 \) with differentials \( Q_1, Q_2 \). We want to compute the variation \( sF_\Phi \) of the defect \( F_\Phi \) of \( \Phi \). As it is reasonable, we shall assume that

\[ sQ_1 = 0, \quad sQ_2 = 0, \]  \hspace{1cm} (2.3.9)

since \( Q_1, Q_2 \) are fixed data of our construction. From (2.1.12), we obtain,

\[ sF_\Phi = -U_1 F_\Phi - F_\Phi U_2 + [Q_1, U_1] \Phi - \Phi [Q_2, U_2]. \]  \hspace{1cm} (2.3.10)

Proof. By (2.1.12) and (2.3.9),

\[ sF_\Phi = -Q_1 s \Phi - s \Phi Q_2. \]  \hspace{1cm} (2.3.11)

Inserting (2.3.4) into (2.3.11) and using (2.1.12), we obtain (2.3.10) through a simple rearrangement.

Inspired by gauge theory, it is natural to require that the variation \( sF_\Phi \) of \( F_\Phi \) should depend on \( \Phi \) only through \( F_\Phi \) itself \hspace{1cm} [32]. For this reason, the last two terms of the right hand side of (2.3.10) should be absent. A sufficient condition
for this to be the case is that

\[ [Q_1, U_1] = 0, \quad [Q_2, U_2] = 0. \tag{2.3.12} \]

When these hold, (2.3.10) takes indeed the required form

\[ sF_\phi = -U_1F_\phi - F_\phi U_2. \tag{2.3.13} \]

The restrictions (2.3.12) are compatible with the variations (2.3.5), as required by consistency.

**Proof.** Again, since the proof of the statement is identical for \( U_1, U_2 \), we suppress indexes. A simple calculation using only the (2.3.5) furnishes

\[ s[Q, U] = [[Q, U], U]. \tag{2.3.14} \]

The relations (2.3.5) and the conditions (2.3.12) are therefore compatible.

The expressions (2.3.4) and (2.3.5) and the conditions (2.3.12) together define the appropriate BRST variation \( s \) of \( M(C_2, C_1) \) when the graded commutative algebras \( C_1, C_2 \) are differential.

(2.3.12) entails that \( U_1, U_2 \) are 1–cocycles of the derivation cochain complexes \( \text{Der}(C_1), \text{ad} Q_1 \), \( \text{Der}(C_2), \text{ad} Q_2 \), respectively (cf. subsect. 2.1). Thus suggests that the cohomology of these should play a basic role in the analysis of the BRST symmetry of \( M(C_2, C_1) \).

Requiring \( U_1, U_2 \) to be cohomologically trivial yields a restricted variant of the BRST symmetry with special properties. If \( U_1, U_2 \) are 1–coboundaries, then

\[ U_{\text{res}1} = [Q_1, X_1], \quad U_{\text{res}2} = [Q_2, X_2]. \tag{2.3.15} \]

where \( X_1 \in \text{Der}(C_1) \), \( X_2 \in \text{Der}(C_2) \) are degree 0 derivation variables. \( X_1, X_2 \) are however defined only modulo 0–cocycles of \( \text{Der}(C_1), \text{Der}(C_2) \), respectively. In order (2.3.5) to be satisfied, it is sufficient that

\[ sX_1 = \frac{1}{2}[X_1, [Q_1, X_1]], \quad sX_2 = \frac{1}{2}[X_2, [Q_2, X_2]] \tag{2.3.16} \]

modulo 1–cocycles of \( \text{Der}(C_1), \text{Der}(C_2) \), respectively.
Proof. Again, we suppress all indexes as they are inessential to the argument. Recalling (2.3.9) and using (2.3.10), we find

\begin{align*}
s[Q, X] &= -[Q, sX] \\
&= -\frac{1}{2} [Q, [X, [Q, X]]] = -\frac{1}{2} [[Q, X], [Q, X]],
\end{align*}

where we used the graded Jacobi identity (2.1.6) and the nilpotence relation 

\([Q, [Q, -]] = 0\). Substituting the relation (2.3.15) into (2.3.17), we recover the

the relations (2.3.5).

As observed in [32], in the left hand sides of the first relation (2.3.16) there appears the degree 1 derived brackets \([Q, -], -\) on the derivation space \(\text{Der}(C_1)\) associated with the differentials \(Q_1\) of \(C_1\) and similarly for the other relation.

The BRST variations (2.3.5) are nilpotent in the appropriate sense, namely

\begin{align*}
s^2 X_1 &= 0, \\
s^2 X_2 &= 0
\end{align*}

modulo 2–coboundaries of \(\text{Der}(C_1), \text{Der}(C_2)\), respectively.

Proof. Again, we suppress all indexes as they are inessential to the argument. From relations (2.3.5) and (2.3.16), written as \(sX = [X, U_{\text{res}}]/2\) using (2.3.15) for brevity, we have

\begin{align*}
s^2 X &= \frac{1}{2} \{[sX, U_{\text{res}}] + [X, sU_{\text{res}}]\} \\
&= \frac{1}{4} \{[[X, U_{\text{res}}], U_{\text{res}}] - [X, [U_{\text{res}}, U_{\text{res}}]]\} = -\frac{1}{4} [[X, U_{\text{res}}], U_{\text{res}}],
\end{align*}

where the graded Jacobi identity (2.1.6) was used. From (2.3.12) and (2.3.15), we also have

\begin{align*}
[Q, [[X, U_{\text{res}}], X]] &= [[[Q, X], U_{\text{res}}], X] - [[X, U_{\text{res}}], [Q, X]] \\
&= [[U_{\text{res}}, U_{\text{res}}], X] - [[X, U_{\text{res}}], U_{\text{res}}]] = -3[[X, U_{\text{res}}], U_{\text{res}}].
\end{align*}

From (2.3.19), (2.3.20), it follows that

\begin{align*}
s^2 X &= \frac{1}{12} [Q, [[X, U_{\text{res}}], X]].
\end{align*}
In the above calculation, we did not take into account the indeterminacy of $X$ and $sX$ modulo 0– and 1–cocycles of $\text{Der}(C)$, which may affect the expression found. Suppose that we shift $X$ and $sX$ by a 0– and a 1–cocycles $\Delta X$ and $\Delta sX$, respectively. Then, by (2.3.5), (2.3.12) (2.3.15) and (2.3.19), $s^2 X$ is shifted by the amount

$$
\Delta s^2 X = \frac{1}{2}[[\Delta sX, [Q, X]]] - \frac{1}{4}[\Delta X, [Q, [X, U_{\text{res}}]]]
$$

where we used the cocycle relations $[Q, \Delta X] = 0$ and $[Q, \Delta sX] = 0$, the nilpotence relation $[Q, [Q, -]] = 0$ and the graded Jacobi identity (2.1.6). By (2.3.21) and (2.3.22), it follows that $s^2 X$ vanishes modulo a 2–coboundary of $\text{Der}(C)$.

A restricted variation operation $s_{\text{res}}$ is yielded from $s$ when $U_1, U_2$ are given by (2.3.15). From (2.3.16), the restricted variation $s_{\text{res}} \Phi$ takes the form

$$
s_{\text{res}} \Phi = X_1 F_\Phi - F_\Phi X_2 - Q_1 (X_1 \Phi - \Phi X_2) + (X_1 \Phi - \Phi X_2)Q_2,
$$

as follows form a straightforward calculation. Similarly, from (2.3.13), the restricted variation $s_{\text{res}} F_\Phi$ reads as

$$
s_{\text{res}} F_\Phi = -Q_1 (X_1 F_\Phi - F_\Phi X_2) - (X_1 F_\Phi - F_\Phi X_2)Q_2.
$$

Note that these variations are not affected by the indetermination of $X_1, X_2$ by 0–cocycles of $\text{Der}(C_1), \text{Der}(C_2)$, since they are ultimately expressible in terms of $U_{\text{res}1}, U_{\text{res}2}$, which are not.

We have thus two forms of BRST symmetry in $M(C_2, C_1)$, the primary and the restricted. The primary BRST complex is the graded commutative algebra

$$
C^\infty(M(C_2, C_1)) \otimes \text{Poly}(\text{Der}_1(C_1)) \otimes \text{Poly}(\text{Der}_1(C_2))
$$

with a degree 1 derivative action of the primary BRST variation operator $s$. The restricted BRST complex is the graded commutative algebra

$$
C^\infty(M(C_2, C_1)) \otimes \text{Poly} \left( \frac{\text{Der}_0(C_1)}{Z \text{Der}_0(C_1)} \right) \otimes \text{Poly} \left( \frac{\text{Der}_0(C_2)}{Z \text{Der}_0(C_2)} \right)
$$
with a degree 1 derivative action of the restricted BRST variation operator \( s_{\text{res}} \), where \( Z \text{ Der}_0(C_1) \), \( Z \text{ Der}_0(C_2) \) denote the vector spaces of 0–cocycles of the derivation cochain complexes \( (\text{Der}(C_1), \text{ad } Q_1), (\text{Der}(C_2), \text{ad } Q_2) \), respectively.

A determination \( s_D \) of \( s \) is the degree 1 variation operation resulting from an assignment of particular values \( U_{1D}, U_{2D} \) to the degree 1 derivation variables \( U_1, U_2 \). While \( s \) is nilpotent, a generic determination \( s_D \) of it is not in general, because \( U_{1D}, U_{2D} \) do not satisfy (2.3.5).

A restricted determination \( s_{\text{res}D} \) of \( s_{\text{res}} \) is similarly the degree 1 restricted variation operation resulting from an assignment of particular values \( X_{1D}, X_{2D} \) to the degree 0 derivation variables \( X_1, X_2 \). Again, albeit \( s_{\text{res}} \) is nilpotent, a generic determination \( s_{\text{res}D} \) of it is not in general.

An important case considered in the following is the determination \( s_Q \) of \( s \), which we shall call \textit{canonical}, specified by the value assignment

\[
U_{1Q} = Q_1, \quad U_{2Q} = Q_2. \tag{2.3.27}
\]

By (2.1.12) and (2.3.4), \( s_Q \) acts on \( \Phi \) as

\[
s_Q \Phi = -F_\Phi \tag{2.3.28}
\]

while, by the defect identity (2.1.13) and (2.3.13), \( s_Q \) is inert on \( F_\Phi \),

\[
s_Q F_\Phi = 0. \tag{2.3.29}
\]

By (2.3.29) the determination \( s_Q \) is actually nilpotent

\[
s_Q^2 \Phi = -s_Q F_\Phi = 0. \tag{2.3.30}
\]

The determination \( s_Q \) equals secretly the restricted determination \( s_{\text{res}Q} \) specified by the value assignment

\[
X_{1Q} = -E_1, \quad X_{2Q} = -E_2, \tag{2.3.31}
\]

where \( E_1, E_2 \) are the Euler derivations of \( C_1, C_2 \), respectively, as follows readily from (2.3.15) recalling that \( Q_1 = [E_1, Q_1], Q_2 = [E_2, Q_2] \).
The primary and restricted BRST symmetries are distinct if the degree 1 cohomology of at least one of the derivation cochain complexes \((\text{Der}(C_1), \text{ad} Q_1), (\text{Der}(C_2), \text{ad} Q_2)\) is non trivial. In such a case, the natural question arises about their different interpretation and role in higher gauge theory. At the present level of abstractness, it seems unlikely that such a question can be answered.

The determination \(s_Q\) of \(s\) given by (2.3.28) is to be compared with the familiar BRST transformations of ordinary gauge theory described in subsect. 1.2 and expressed by (1.2.4) in BRST superfield form. The formal correspondence is manifest. We have given in this way a graded geometrical foundation to the BRST formulation of higher gauge theory.

2.4 BRST symmetry and the differential cone

Let \(C\) be a graded commutative algebra. A differential structure on \(C\) is a choice of a differential \(Q\) on \(C\) making \(C\) a differential graded commutative algebra (cf. subsect. 2.1). The differential structures of \(C\) form a manifold \(\mathcal{Q}_C\).

\(\mathcal{Q}_C\) is contained in the linear space \(\text{Der}_1(C)\) of degree 1 derivation of \(C\), but it is not itself a linear space because of the quadratic nature of the nilpotence condition \(Q^2 = 0\) (cf. eq. (2.1.10)). While nilpotence is broken by addition (if \(Q, Q'\) are differentials on \(C\) their sum \(Q + Q'\) in general is not), it is preserved by rescaling (if \(Q\) is a differential on \(C\), so is \(tQ\) for \(t \in \mathbb{R}\)). This shows that \(\mathcal{Q}_C\) is a cone contained in \(\text{Der}_1(C)\). We thus call \(\mathcal{Q}_C\) the differential cone of \(C\).

Let \(Q\) be a fixed differential in \(\mathcal{Q}_C\). We want to deform \(Q\) within \(\mathcal{Q}_C\). This involves adding to \(Q\) a deformation \(W\) such that

\[ Q_W = Q + W \]

is another differential in \(\mathcal{Q}_C\). In order this to be the case, \(W\) must be a degree 1 derivation of \(C\), so that \(Q_W\) also is, and must satisfy the deformation equation

\[ [Q, W] + \frac{1}{2}[W, W] = 0, \]

so that \(Q_W\) is nilpotent as required. For infinitesimal deformations, we can drop the term \(\frac{1}{2}[W, W]\) and (2.4.2) reduces to the infinitesimal deformation equation
\[ [Q, W] = 0 \]  \hspace{1cm} (2.4.3)

stating that \( W \) is a 1–cocycle in the \( \text{ad} \, Q \) cohomology. (2.4.3) admits in particular the cohomologically trivial restricted solution

\[ W_{\text{res}} = [Q, N], \]  \hspace{1cm} (2.4.4)

with \( N \) a degree 0 derivation of \( C \).

A special solution of the deformation equation (2.4.2) as well as its infinitesimal form (2.4.3) is

\[ W_Q = uQ, \]  \hspace{1cm} (2.4.5)

with \( u \in \mathbb{R} \). This corresponds to a rescaling \( Q \to (1 + u)Q \) of \( Q \). \( W_Q \) is actually of the restricted form (2.4.4) with

\[ N_Q = -uE, \]  \hspace{1cm} (2.4.6)

\( E \) being the Euler derivation of \( C \).

Next, suppose that \( C_1, C_2 \) are graded commutative algebras and that \( Q_{C_1}, Q_{C_2} \) are their respective differential cones. Pick reference differentials \( Q_1, Q_2 \) for \( C_1, C_2 \) and consider infinitesimal deformations \( W_1, W_2 \) of \( Q_1, Q_2 \), so that \( W_1, W_2 \) satisfy eq. (2.4.3). If \( \Phi : C_2 \to C_1 \) is a graded algebra morphism, \( \Phi \) is not affected by the deformations, but its defect \( F_\Phi \) is: (2.4.7) implies indeed that

\[ F_{\Phi W} = F_\Phi + W_1 \Phi - \Phi W_2. \]  \hspace{1cm} (2.4.7)

Now, by (2.4.3), the deformation terms in the right hand side are just the variation \(-s_W \Phi\) of \( \Phi \) yielded by the determination \( s_W \) of the BRST variation operator \( s \) specified by the value assignment

\[ U_{1W} = W_1, \quad U_{2W} = W_2 \]  \hspace{1cm} (2.4.8)

(cf. subsect. 2.3). We can thus write (2.4.7) suggestively as

\[ F_{\Phi W} = F_\Phi - s_W \Phi. \]  \hspace{1cm} (2.4.9)
In this way, there exists a one–to–one correspondence between infinitesimal deformations \( W_1, W_2 \) of the differential structures \( Q_1, Q_2 \) of \( C_1, C_2 \) and the determinations \( s_W \) of the BRST variation operator \( s \) such that (2.4.9) holds. Further, by (2.3.15) and (2.4.4), the correspondence maps restricted deformations \( W_{\text{res}1}, W_{\text{res}2} \) to restricted determinations \( s_{\text{res}W} \).

When \( W_1, W_2 \) are of the special form (2.4.5),

\[
W_{1Q} = uQ_1, \quad W_{2Q} = uQ_2, \tag{2.4.10}
\]

the variation \( s_W \Phi \) takes the form

\[
s_{WQ} \Phi = us_Q \Phi, \tag{2.4.11}
\]

where \( s_Q \) is the canonical determination of \( s \) (cf. eq. (2.3.28)). A concurrent rescaling \( Q_1 \to (1 + u)Q_1, Q_2 \to (1 + u)Q_2 \) of the differentials \( Q_1, Q_2 \) therefore corresponds essentially to \( s_Q \) under the aforementioned one–to–one correspondence.

In higher gauge theory, the differentials \( Q_1, Q_2 \) of the differential graded commutative algebras \( C_1, C_2 \) yield the kinetic and the self-interaction terms of the higher gauge fields, respectively. They constitute the basic data upon which the theory rests. This suggests that the theory space consists of the Cartesian product \( Q_{C_1} \times Q_{C_2} \) of the differential cones of \( C_1, C_2 \) or, equivalently, the differential cone \( Q_{C_1 \oplus C_2} \) of the differential graded commutative algebra direct sum \( C_1 \oplus C_2 \) of \( C_1, C_2 \). This claim however is not completely correct as we explain next.

The relative normalization of \( Q_1, Q_2 \) determines the value of the higher gauge fields’ self-interaction coupling strength and is thus physically relevant. The overall normalization of \( Q_1, Q_2 \) is instead physically irrelevant, as a change of its value amounts to a change of the overall normalization of the Lagrangian. For this reason, the theory space is not quite the differential cone \( Q_{C_1 \oplus C_2} \) but the projective cone \( \mathbb{P}Q_{C_1 \oplus C_2} = Q_{C_1} \times Q_{C_2} / \mathbb{R}^\times \), where \( \mathbb{R}^\times \) is the multiplicative group of non zero reals acting on \( Q_{C_1} \times Q_{C_2} \) by simultaneous rescaling.

Therefore, deformations \( W_1, W_2 \) of the differential structures \( Q_1, Q_2 \) of \( C_1, C_2 \) modulo those of the form (2.4.10) yield infinitesimal translations in the theory.
space $FQ_{C_1 \oplus C_2}$. The ineffective deformations (2.4.10) are precisely those which correspond to the canonical determination $s_Q$ of $s$.

There are two archetypal higher gauge theories whose gauge symmetry is described by $s_Q$: higher BF gauge theory and higher Chern–Simons gauge theory. These are studied next after a brief review of classical BV theory.

### 2.5 Classical BV theory

In this subsection, we recall briefly the basic definitions of formal BV theory used in the following. See ref. [44] for a physically motivated introduction.

Let $A$ be a graded commutative algebra and $n \in \mathbb{Z}$. A degree $n$ Poisson–Gerstenhaber structure on $A$ is a bilinear bracket $(-,-): A \times A \to A$ with the following properties. $|u| = |v| + n$ for homogeneous $u,v \in A$. Further,

\begin{align*}
(u,v) + (-1)^{|u|+n}(|v|+n)(v,u) &= 0, \\
(-1)^{|v|+(n+1)}(u,(v,w)) + (-1)^{|u|+n}(|v|+n)(v,(w,u)) + (-1)^{|v|+n}(|w|+n)(w,(u,v)) &= 0, \\
(u,vw) &= (u,v)w + (-1)^{|u|+(n+1)}v(u,w)
\end{align*}

for homogeneous $u,v,w \in A$. A graded commutative algebra $A$ equipped with a degree $n$ Poisson–Gerstenhaber structure $(-,-)$ is called a degree $n$ Poisson–Gerstenhaber algebra. $(-,-)$ is also called Poisson–Gerstenhaber brackets. One usually employs the term Poisson for $n$ even and Gerstenhaber for $n$ odd.

Let $A$ be a graded commutative algebra and $n \in \mathbb{Z}$ as before. A degree $n$ classical Batalin-Vilkovisky or BV structure on $A$ consists of a degree $n$ Poisson–Gerstenhaber structure $(-,-)$ on $A$ and a distinguished degree $1-n$ element $S \in A$ satisfying the condition

\[(S,S) = 0.\] (2.5.4)

A graded commutative algebra $A$ equipped with a degree $n$ classical BV structure $((-,-),S)$ is called a degree $n$ classical BV algebra. $S$ is called classical BV master action and (2.5.4) classical BV master equation.
If $A$ is a degree $n$ classical BV algebra, the operator $\delta : A \to A$ defined by

$$\delta u = (S, u)$$  \hspace{1cm} (2.5.5)

has degree 1 and is nilpotent, so that

$$\delta^2 = 0.$$  \hspace{1cm} (2.5.6)

$\delta$ is called \textit{classical BV differential}. A classical BV algebra is in this way a differential graded commutative algebra whose differential $\delta$ is Hamiltonian with respect the underlying Poisson–Gerstenhaber structure with Hamiltonian element $S$. As a cochain complex, $(A, \delta)$ is called the \textit{classical BV complex} of $A$. Its cohomology is the \textit{classical BV cohomology} of $A$.

In applications of BV theory to field theoretic models, $A$ is as a rule a graded commutative algebra $A_{\mathcal{F}}$ of field functionals on a field space $\mathcal{F}$ endowed with a degree $-1$ symplectic form $\omega$, called \textit{BV form}. Just as in ordinary Hamiltonian mechanics the canonical symplectic form of phase space yields the canonical Poisson brackets, $\omega$ yields a degree 1 Gerstenhaber structure $(-, -)_\omega$ on $A_{\mathcal{F}}$, called \textit{BV antibrackets}, rendering in this way $A_{\mathcal{F}}$ a degree 1 Gerstenhaber algebra. Further, the data which define $\mathcal{F}$ and $\omega$ contain also elements allowing for the construction of a natural BV master action $S_{\mathcal{F}, \omega}$ and the associated BV differential $\delta_{\mathcal{F}, \omega}$, turning $A_{\mathcal{F}}$ into a degree 1 classical BV algebra. However, the datum of the symplectic form $\omega$ is not always strictly necessary. In certain cases it is possible to define a degree 1 Gerstenhaber structure on an algebra $A_{\mathcal{F}}$ of field functionals by directly and consistently assigning the value of the brackets $(z_i, z_j)$ for a set of basic field functionals $z_i$ of $A_{\mathcal{F}}$ in terms of which all other field functionals can be expressed compatibly with the properties (2.5.1), (2.5.2). Further, it is possible to construct a master action $S_{\mathcal{F}}$ as a distinguished function of the $z_i$ and obtain the associated differential $\delta_{\mathcal{F}}$. A BV algebra of field functionals is so obtained again. The AKSZ formulation of the classical BV theory of ref. [49,50] provides an elegant geometrical implementation of this program.

The field functionals $f$ composing the graded algebra $A_{\mathcal{F}}$ are characterized by ghost degree and so can be viewed as maps $f : \mathcal{F} \to G_\mathbb{R}$, where $G_\mathbb{R}$ is the
graded commutative algebra

\[ G_\mathbb{R} = \bigoplus_{k=-\infty}^{\infty} \mathbb{R}[k] \]  

(cf. eq. (1.2.2)), called ghost algebra. In general, however, not all such maps \( f \) belong to \( A_\mathcal{F} \). Restrictions on the content of \( A_\mathcal{F} \) may be required.

\( L_\infty \) algebras are extensions of ordinary Lie algebras. An \( L_\infty \)–algebra is a graded vector space \( L \) equipped with a collection of multiple argument brackets satisfying generalized Jacobi identities. The \( L_\infty \)–algebra structure is encoded in a degree 1 nilpotent Chevalley–Eilenberg differential \( Q_L \) acting on the 1–shift \( L[1] \) of \( L \). A classical BV algebra \( A_\mathcal{F} \) of functionals on a field space \( \mathcal{F} \) of a field theory with BV differential \( \delta_\mathcal{F} \) therefore secretly supports an infinite dimmensional \( L_\infty \)–algebra structure on the \( -1 \)–shift \( A_\mathcal{F}[-1] \) of \( A_\mathcal{F} \). This describes the full symmetry of the field theory at the most basic level.

### 2.6 Higher BF gauge theory

The first basic higher gauge theoretic model whose symmetry is described by the canonical determination \( s_Q \) of the BRST variation operation \( s \) is higher BF theory. Below, we present an abstract BV formulation of the model based on the BRST framework worked out in subsects. 2.3, 2.4.

Consider the \(-1\)–shifted cotangent bundle \( T^*[-1]M(C_2, C_1) \) of the non differential morphism manifold \( M(C_2, C_1) \) of two differential graded commutative algebras \( C_1, C_2 \). Then, \( T^*[-1]M(C_2, C_1) \) is equipped with the canonical degree \(-1\) BV symplectic form \( \Omega_{BV} \)

\[ \Omega_{BV} = \langle \delta \Phi^*, \delta \Phi \rangle, \]  

where \( \Phi \in M(C_2, C_1) \) and \( \Phi^* \in T^*[\mathcal{F}][-1]M(C_2, C_1) \) are the base and fiber variables of \( T^*[-1]M(C_2, C_1) \) and \( \langle -, - \rangle \) stands for the natural cotangent–tangent pairing. With \( \Omega_{BV} \), in turn, there are associated the canonical BV antibrackets. These can be written compactly as

\[ (\langle A^*, \Phi \rangle, \langle \Phi^*, B \rangle)_{BV} = \langle A^*, B \rangle, \]  

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where \( A^* \in T^* \Phi M(C_2, C_1), B \in T_\Phi M(C_2, C_1) \) and the pairing with \( A^* \) and \( B \) is tacitly assumed to follow the BV antibracketing.

We can extend the canonical determination \( s_Q \) of the BRST variation operation \( s \) from \( M(C_2, C_1) \) to \( T^*[1]M(C_2, C_1) \) by setting

\[
s_Q \Phi = -F_\Phi, \quad s_Q \Phi^* = \hat{F}^{\vee} \Phi^*,
\]

where \( \hat{F}^{\vee} \Phi^* \) is defined by the relation

\[
\langle \hat{F}^{\vee} \Phi^*, B \rangle = -\langle \Phi^*, F_B \rangle
\]

with \( B \in T_\Phi M(C_2, C_1) \) and \( F_B = Q_1 B - B Q_2 \). Since \( (2.6.3) \) is just \( (2.3.28) \), \( (2.6.3), (2.6.4) \) do indeed define an augmentation of \( s_Q \). The extension keeps the property of nilpotence.

**Proof.** We have \( s_Q^2 \Phi = 0 \) by \( (2.3.30) \). By iterating \( (2.6.4) \) twice, we get

\[
s_Q^2 \langle \Phi^*, B \rangle = s_Q \langle \hat{F}^{\vee} \Phi^*, B \rangle = -s_Q \langle \Phi^*, \hat{F}_B \rangle = -\langle \hat{F}^{\vee} \Phi^*, \hat{F}_B \rangle = \langle \Phi^*, \hat{F}_B \rangle.
\]

It is now immediately verified that \( \hat{F}_{\hat{F}_B} \) vanishes,

\[
\hat{F}_{\hat{F}_B} = Q_1(Q_1B - BQ_2) + (Q_1B - BQ_2)Q_2 = 0.
\]

by the nilpotence of \( Q_1, Q_2 \). So, we have \( s_Q^2 \Phi^* = 0 \) as well.

The action of the extended canonical determination \( s_Q \) on \( M(C_2, C_1) \) is Hamiltonian with degree 0 BV master action

\[
S_{BV} = -\langle \Phi^*, F_\Phi \rangle.
\]

Indeed, \( S_{BV} \) satisfies the classical master equation

\[
(S_{BV}, S_{BV})_{BV} = 0.
\]

Further, a straightforward calculation shows that

\[
(S_{BV}, \langle A^*, F_\Phi \rangle)_{BV} = -\langle A^*, F_\Phi \rangle = \langle A^*, s_Q \Phi \rangle,
\]
\[(S_{BV}, \langle \Phi^*, B \rangle)_{BV} = \langle \tilde{F}^\vee_{\Phi^*}, B \rangle = \langle s_Q \Phi^*, B \rangle, \quad (2.6.11)\]

where \(A^* \in T^*_\phi[u]M(C_2, C_1), \quad B \in T_\phi[0]M(C_2, C_1)\)

**Proof.** We show only \((2.6.9)\). Using \((2.6.2)\), we have

\[
(S_{BV}, S_{BV})_{BV}
\]

\[
= \langle \langle \Phi^*, F_\phi \rangle, \langle \Phi^*, B \rangle \rangle_{BV}|_{B=F_\phi} + \langle \langle A^*, F_\phi \rangle, \langle \Phi^*, F_\phi \rangle \rangle_{BV}|_{A^*=\phi^*}
\]

\[
= \langle \Phi^*, \tilde{F}_B \rangle|_{B=F_\phi} + \langle A^*, \tilde{F}_{F_\phi} \rangle|_{A^*=\phi^*}
\]

\[
= 2\langle \Phi^*, \tilde{F}_{F_\phi} \rangle
\]

Here, \(\tilde{F}_{F_\phi}\) vanishes by the defect identity \((2.1.13)\),

\[
\tilde{F}_{F_\phi} = Q_1 F_\phi + F_\phi Q_2 = 0. \quad (2.6.13)
\]

\((2.6.9)\) is thus shown.

On account of \((2.6.10), (2.6.11)\), we shall also denote the variation operation \(s_Q\) by \(\delta_{BV}\).

It should be now apparent that our analysis does indeed provide an abstract higher gauge theoretic generalization of BF gauge theory, as we anticipated above. The BV master action \(S_{BV}\) given by \((2.6.8)\) has the standard form of that of BF gauge theory. Furthermore, the Euler–Lagrange equation ensuing from \(S_{BV}\), which in the BV framework can be formally expressed as the vanishing of the BV variations \(\delta_{BV} \Phi, \delta_{BV} \Phi^*\) of \(\Phi, \Phi^*\), have too by \((2.6.10), (2.6.11)\) the standard form of those of BF gauge theory. We have worked out in this way a general formal framework suitable for the construction of BF type higher gauge theoretic models.

### 2.7 Higher Chern–Simons gauge theory

The second basic higher gauge theoretic model whose symmetry is described by the canonical determination \(s_Q\) of the BRST variation operation \(s\) is higher Chern–Simons theory. We present now an abstract formulation of this model.
based again on the BRST framework of subsects. 2.3 2.4 Our approach follows closely the AKSZ paradigm of ref. 49.

Consider once more two differential graded commutative algebras \( C_1, C_2 \) together with the manifold \( M(C_2, C_1) \) of their non differential morphisms. The formulation rests on a set of assumptions which we state next.

1. A degree \(-n \) \( G_\mathbb{R} \)–linear map \( \mu : C_1 \to G_\mathbb{R} \), where \( n \in \mathbb{N}, n > 0 \) and \( G_\mathbb{R} \) is the ghost algebra defined in (1.2.2), with the following properties is given.

   \( \mu \) is non singular, that is

   \[
   \mu(uv) = 0 \quad \text{for all } v \in C_1 \Rightarrow u = 0 \quad (2.7.1)
   \]

   for any \( u \in C_1 \). Furthermore, the kernel of \( \mu \) contains the range of \( Q_1 \),

   \[
   \mu \circ Q_1 = 0. \quad (2.7.2)
   \]

   To get an intuitive understanding of these conditions, think \( C_1 \) as the algebra of differential forms on a \( n \)–dimensional manifold without boundary and \( Q_1 \) as the de Rham differential. Then, \( \mu \) can be regarded as the integration operation and (2.7.2) as the statement of Stokes’ theorem.

2. A degree \(-n + 1 \) classical BV structure on \( C_2 \) with Poisson–Gerstenhaber brackets \( (-,-)_2 \) and classical BV master action \( S_2 \) (cf. subsect. 2.5 eqs. (2.5.1)–(2.5.3) and (2.5.4)) such that

   \[
   Q_2 f = (S_2, f)_2 \quad (2.7.3)
   \]

   for \( f \in C_2 \) is given (cf. eq. (2.5.5)).

   Again, for the sake of intuition, one can think of \( C_2 \) as a graded commutative algebra of functions on a graded phase space with graded Poisson brackets \( (-,-)_2 \)

   and a distinguished degree \( n \) function \( S_2 \) satisfying \( (S_2, S_2)_2 = 0 \) which is the Hamiltonian for a homological vector field \( Q_2 = (S_2,-) \).

\(^2\) \( G_\mathbb{R} \)–linearity is conventionally defined as the property that \( \mu(\theta u) = (-1)^{|	heta|} \theta \mu(u) \) and \( \mu(u \theta) = \mu(u) \theta \) with \( u \in C_1 \) and \( \theta \in G_\mathbb{R} \).
The AKSZ formulation of higher Chern–Simons gauge theory involves further data, which are required for the construction of the classical BV master action.

3. There are a degree \( n \) map \( K_1 : M(C_2, C_1) \to C_1 \) and a degree \( n - 1 \) fiberwise linear map \( B_1 : TM(C_2, C_1) \to C_1 \) enjoying the following properties. The left tangent map \( TK_1 : TM(C_2, C_1) \to C_1 \) of \( K_1 \) is compatible with left multiplication in \( C_1 \) in the sense that

\[
TK_1(\Phi)(u\dot{\Phi}) - Q_1B_1(\Phi)(u\dot{\Phi}) = u(TK_1(\Phi)(\dot{\Phi}) - Q_1B_1(\Phi)(\dot{\Phi})) \quad (2.7.4)
\]

for \( \Phi \in M(C_2, C_1), \dot{\Phi} \in \bigoplus_{k=-\infty}^{\infty} T\Phi[k]M(C_2, C_1) \) and \( u \in C_1 \). Furthermore, \( K_1 \) obeys

\[
TK_1(\Phi)(\Phi(f, -)_{2}) - Q_1B_1(\Phi)(\Phi(f, -)_{2}) = (-1)^{|f|-n+1}Q_1\Phi f \quad (2.7.5)
\]

for \( \Phi \in M(C_2, C_1) \) and homogeneous \( f \in C_2 \).

As we shall see, \( \mu(K_1(\Phi)) \) constitutes the ‘kinetic’ term of the higher gauge fields in the BV action. \( Q_1B_1(\Phi)(\dot{\Phi}) \) is an exact term that is produced when \( K_1(\Phi) \) is varied and that vanishes upon integration. \( (2.7.4) \) is a minimal condition that \( K_1 \) must satisfy in a local Lagrangian field theory. \( (2.7.5) \) is a restriction on the general form of \( K_1 \) that ensures that the action has the required properties. The BV action’s ‘self–interaction’ term of the higher gauge fields turns out to be \( \mu(\Phi S_2) \) and so does not involve new data.

With the above data, it is possible to define degree 1 Poisson–Gerstenhaber brackets on the graded commutative algebra \( C^\infty(M(C_2, C_1)) \otimes G_\mathbb{R} \) of \( G_\mathbb{R} \)–valued functions on the morphism manifold \( M(C_2, C_1) \) by setting

\[
(\mu(u \Phi f), \mu(v \Phi g))_{BV} = (-1)^{(|f|-n+1)|v|} \mu(uv \Phi(f, g)_2) \quad (2.7.6)
\]

with homogeneous \( u, v \in C_1, f, g \in C_2 \).

Proof. To begin with, we notice that by the non singularity of \( \mu \), eq. \( (2.7.1) \), any function in \( C^\infty(M(C_2, C_1)) \otimes G_\mathbb{R} \) can be expressed in terms of the basic functions of \( \mu(u \Phi f) \) with \( u \in C_1, f \in C_2 \). To show that the brackets \( (2.7.6) \) define a degree
Gerstenhaber structure on $C^\infty(M(C_2, C_1)) \otimes G_\mathbb{R}$ it is sufficient to show that they have degree 1 and enjoy the properties (2.5.1), (2.5.2).

Since $|(f, g)_2| = -n + 1 + |f| + |g|$ for $f, g \in C_2$ and $|\mu(u \Phi f)| = -n + |u| + |f|$ for $u \in C_1, f \in C_2$, one has

$$|\mu(uv \Phi(f, g)_2)| = 1 + |\mu(u \Phi f)| + |\mu(v \Phi g)|$$ (2.7.7)

for $u \in C_1, f, g \in C_2$. The brackets (2.7.6) have therefore degree 1. By straightforward calculations, one verifies that

$$(\mu(u \Phi f), \mu(v \Phi g))_{BV}$$ (2.7.8)

$$+ (-1)^{(-n+|u|+|f|+1)(-n+|v|+|g|+1)}(\mu(v \Phi g), \mu(u \Phi f))_{BV} = 0,$$

(2.7.9)

$$(-1)^{(-n+|u|+|h|+1)(-n+|v|+|g|+1)}(\mu(u \Phi f), (\mu(v \Phi g), \mu(w \Phi h)))_{BV}$$

$$+ (-1)^{(-n+|v|+|g|+1)(-n+|u|+|h|+1)}(\mu(w \Phi h), (\mu(u \Phi f), \mu(v \Phi g)))_{BV} = 0,$$

for $u, v, w \in C_1, f, g, h \in C_2$, showing that the brackets (2.7.6) enjoy the properties (2.5.1), (2.5.2). The statement follows.

Hence, $C^\infty(M(C_2, C_1)) \otimes G_\mathbb{R}$ is equipped with degree 1 Gerstenhaber brackets $(-, -)_{BV}$ induced naturally by the Poisson-Gerstenhaber brackets of $C_2$. These are in fact the BV brackets in our AKSZ formulation of higher Chern–Simons gauge theory.

Using $K_1$ and $S_2$ as basic building blocks, we can construct a degree 0 Hamiltonian $S_{BV}$ for the nilpotent degree 1 defect vector field of $M(C_2, C_1)$ (cf. subsect. 2.2), here expressed as the map $\mu(u \Phi f) \rightarrow \mu(u F \Phi f)$ with $u \in C_1, f \in C_2$:

$$S_{BV}(\Phi) = (-1)^n \mu(K_1(\Phi) + \Phi S_2).$$ (2.7.10)

Indeed, $S_{BV}$ satisfies the master equation

$$(S_{BV}(\Phi), S_{BV}(\Phi))_{BV} = 0$$ (2.7.11)

and moreover one has
\[ \mu(u \, F_f) = -(-1)^{-n+|u|}(S_{BV}(\Phi), \mu(u \, \Phi f))_{BV} \] (2.7.12)

for \( u \in C_1, \ f \in C_2 \).

**Proof.** By (2.7.1), (2.7.6), for \( u \in C_1, \ f \in C_2 \) one has
\[ (\mu(u \, \Phi f), \Phi)_{BV} = (-1)^{n(|u|+|f|)}u \, \Phi(f, -)_2. \] (2.7.13)

Exploiting (2.7.2), (2.7.4), (2.7.5) and (2.7.13), we find
\[ (\mu(K_1(\Phi)), \mu(u \, \Phi f))_{BV} \] (2.7.14)
\[ = (-1)^{|u|+|f|+n}(\mu(u \, \Phi f), \mu(K_1(\Phi)))_{BV} \]
\[ = (-1)^{(n+1)|u|+|f|+n}\mu(TK_1(\Phi)((\mu(u \, \Phi f), \Phi)_{BV})) \]
\[ = (-1)^{|u|+|f|+n}\mu(TK_1(\Phi)(\phi_f, -)_2) \]
\[ = (-1)^{|u|+|f|+n}\mu(Q_1B_1(\Phi)(\phi_f, -)_2) \]
\[ + u(TK_1(\Phi)(\phi_f, -)_2 - Q_1B_1(\Phi)(\phi_f, -)_2)) \]
\[ = -(-1)^{|u|}\mu(u \, \Phi f), \]

from which on account of (2.7.1) it follows readily that
\[ (\mu(K_1(\Phi)), \Phi)_{BV} = (-1)^{n-1}Q_1\Phi. \] (2.7.15)

Using (2.7.15) together with (2.7.2), we obtain
\[ (\mu(K_1(\Phi)), \mu(K_1(\Phi)))_{BV} = (-1)^n\mu(TK_1(\Phi)((\mu(K_1(\Phi)), \Phi)_{BV})) \] (2.7.16)
\[ = -\mu(TK_1(\Phi)(Q_1(\Phi)) = -\mu(Q_1K_1(\Phi)) = 0. \]

(2.7.16) is a first basic relation. Employing (2.7.14) and (2.7.2), one has further
\[ (\mu(K_1(\Phi)), \mu(\Phi S_2))_{BV} = -\mu(Q_1\Phi S_2) = 0. \] (2.7.17)

(2.7.17) is a second basic relation. By (2.7.6) and (2.7.3), one has
\[ (\mu(\Phi S_2), \mu(u \, \Phi f))_{BV} = (-1)^{|u|}\mu(u \, \Phi(S_2, f)_2) = (-1)^{|u|}\mu(u \, \Phi Q_2 f) \] (2.7.18)
for $u \in C_1$, $f \in C_2$. By (2.5.4) and (2.7.18), we have then

$$\langle \mu(\Phi S_2), \mu(\Phi S_2) \rangle_{BV} = \mu(\Phi(S_2, S_2)_2) = 0. \quad (2.7.19)$$

(2.7.19) is a third basic relation.

On account of (2.7.10), using (2.7.17), (2.7.17) and (2.7.19), we find

$$\langle S_{BV}(\Phi), S_{BV}(\Phi) \rangle_{BV} = (\mu(K_1(\Phi)), \mu(K_1(\Phi)))_{BV} \quad (2.7.20)$$

$$+ 2(\mu(K_1(\Phi)), \mu(\Phi S_2))_{BV} + (\mu(\Phi S_2), \mu(\Phi S_2))_{BV} = 0.$$

(2.7.11) is thus proven.

Finally, by (2.1.12), (2.7.10), (2.7.14) and (2.7.18), we have

$$- (-1)^{-n+|u|}(S_{BV}(\Phi), \mu(u \Phi f))_{BV} \quad (2.7.21)$$

$$= -(-1)^{|u|}[(\mu(K_1(\Phi)), \mu(u \Phi f))_{BV} + (\mu(\Phi S_2), \mu(u \Phi f))_{BV}].$$

$$= \mu(u Q_1 \Phi f) - \mu(u \Phi Q_2 f) = \mu(u F_{\Phi} f).$$

(2.7.12) is thus shown.

By virtue of (2.7.12), $S_{BV}$ is the Hamiltonian function for the canonical determination $s_Q$ of the BRST variation $s$. Indeed,

$$s_Q \mu(u \Phi f) = (S_{BV}, \mu(u \Phi f))_{BV} \quad (2.7.22)$$

for $u \in C_1$, $f \in C_2$.

**Proof.** One has

$$s_Q \mu(u \Phi f) = (-1)^{-n+|u|} \mu(u s_Q \Phi f) = -(-1)^{-n+|u|} \mu(u F_{\Phi} f). \quad (2.7.23)$$

The statement follows then immediately from (2.7.12).}

For this reason, the variation operation $s_Q$ is just the BV differential $\delta_{BV}$.

It should be now apparent that our analysis does indeed provide an abstract higher gauge theoretic generalization of Chern–Simons gauge theory, as we anticipated. The BV action $S_{BV}$ given by (2.7.10) has the standard form of that
of Chern–Simons gauge theory upon viewing the contributions of $K_1$ and $S_2$ as the action’s kinetic and interaction terms, respectively. Furthermore, the Euler–Lagrange equation yielded by $S_{BV}$, equivalent to the vanishing of the BV variation $\delta_{BV}\Phi$ of $\Phi$, also have by (2.7.12), (2.7.22) the standard form of those of Chern–Simons gauge theory. We have worked out in this way a general formal framework suitable for the construction of Chern–Simons type higher gauge theoretic models.

In the next section, we shall provide examples of higher gauge and gauge sigma models that can be constructed using the axiomatic framework developed in subsects. 2.6 2.7.
3 Higher gauge theory and gauged sigma models

In this section, we shall work out a formulation of higher gauge theory and gauged sigma models relying on the formal framework developed in sect. 2. The characterizing point of our construction is the realization that for a full–fledged BV formulation incorporating ghost degrees of freedom higher gauge fields viewed as smooth functions on the shifted tangent bundle of a space time manifold are not sufficient. A more general kind of gauge fields are required, which are internal rather than simply ordinary smooth functions.

The mathematical framework appropriate for the attainment of this goal is furnished by graded differential geometry, which we review to set our terminology and notation and highlight those points which are relevant in the subsequent analysis. In particular, we go through the theory of internal graded manifold morphisms. We also dwell on the theory NQ–manifold and \( L_\infty \) algebroids and the closely related theory of Lie quasi–groupoid differentiation. With this graded geometrical set–up in place, the beauty and naturalness of our formulation of higher gauge theory and gauged sigma models fully emerges.

3.1 Graded manifolds and manifold morphisms

In this subsection, we review the basic notions of graded geometry relevant in the following. We follow mainly ref. [30]. See also ref. [51] for a readable updated introduction.

A graded manifold \( M \) is a locally ringed space \((M_0, \mathcal{O}_M)\), where \( M_0 \) is a smooth manifold and \( \mathcal{O}_M \) is a sheaf of graded commutative algebras over \( M_0 \) which is locally isomorphic to \( \mathbb{C}_{\mathbb{R}^{\dim M_0}} \otimes S(V) \) for some fixed finite-dimensional graded real vector space \( V \) of vanishing degree 0 component \( V_0 \). \( M_0 \) and \( \mathcal{O}_M \) are called respectively the body and the structure sheaf of \( M \). A morphism \( \varphi : M_1 \to M_2 \) of two graded manifolds \( M_1, M_2 \) is a morphism \( \varphi : (M_1, \mathcal{O}_{M_1}) \to (M_2, \mathcal{O}_{M_2}) \)

\[^3\] Here, \( C_N^{\infty} \) is the sheaf of smooth realvalued functions on a smooth manifold \( N \) and \( S(E) \) is the graded symmetric algebra of a graded vector space \( E \). See the cited references for further details.
of their associated locally ringed spaces. The associated ordinary manifold morphism \( \varphi_0 : M_{10} \to M_{20} \) and sheaf morphism \( \varphi^* : \mathcal{O}_{M_2} \to \varphi_0^* \mathcal{O}_{M_1} \) are called respectively the body of and the pull-back by \( \varphi \). Graded manifolds and graded manifold morphisms form a category \( \text{grMf} \).

An \( N \)-manifold is a graded manifold \( M \) for which the vector space \( V \) entering the definition of the local model is non negatively graded. \( N \)-manifolds form a full subcategory \( \text{N-Mf} \) of \( \text{grMf} \). In this paper, we consider mainly though non exclusively \( N \)-manifolds.

The structure sheaf \( \mathcal{O}_M \) of a graded manifold \( M \) decomposes according to degree as a direct sum \( \mathcal{O}_M = \bigoplus_k \mathcal{O}_M^k \). For each \( k \), \( \mathcal{O}_M^k \) is a sheaf of \( C_{M_0} \) modules. Letting \( \mathcal{O}_{Mk} \) be the subsheaf of graded commutative algebras generated by \( \mathcal{O}_{Mk} = \bigoplus_{l \leq k} \mathcal{O}_M^l \) for \( k \in \mathbb{Z} \), one has a filtration of sheaves \( \cdots \subset \mathcal{O}_{Mk} \subset \mathcal{O}_{Mk+1} \subset \cdots \). For a \( N \)-manifold \( M \), \( \mathcal{O}_M^k = \mathcal{O}_{Mk} = 0 \) for \( k < 0 \) and \( \mathcal{O}_M^0 = \mathcal{O}_M = C_{M_0} \). Further, for \( k \geq 0 \), \( (M_0, \mathcal{O}_{Mk}) \) is a locally ringed space defining a graded manifold \( M_k \). The \( M_k \) fit in a sequence of fibrations \( M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \). \( M \) is the projective limit of the sequence. If \( M = M_n \) for some \( n \), \( M \) is said of finite degree \( n \). Else, \( M \) is said of infinite degree. The \( N \)-manifolds considered in this paper are mostly finite degree.

In a graded manifold \( M \), the local isomorphism between the structure sheaf \( \mathcal{O}_M \) and the local model \( C_{\mathbb{R}^{\dim M_0}} \otimes S(V) \) defines a set of local coordinates of \( M \): if \( \xi^i = (\eta_0^a, \zeta^r) \) are a set of coordinates of \( \mathbb{R}^{\dim M_0} \times V \), respectively, then their preimages \( x^i = (y_0^a, z^r) \) constitute a set of local coordinates of \( M \). The \( y_0^a \) are local coordinates for the body \( M_0 \) of \( M \) and are called thus body coordinates. They all have 0 degree. The \( z^r \) are called vector coordinates. They can have any degree, but for \( M \) an \( N \)-manifold they are all non negatively graded.

We consider next the space of morphisms of two graded manifolds because of its importance in the analysis of later subsections. Before doing that, however, the following remarks are in order. In geometry, one often encounters spaces of maps between manifolds and one would like to handle these as manifolds of some sort. This is not straightforward at all because of their infinite dimensional nature. The best approach to the subject combining mathematical rigour and practical
usability is by employing functional diffeology (see e.g. [32] for background). The same problem arises when dealing with spaces of morphisms between graded manifolds. In order to treat these as infinite dimensional graded manifold, one has to resort to a graded generalization of functional diffeology. Following this path, however, would bring us to far afield. Here, we prefer to proceed at a more modest heuristic level by formally enlarging the categories $\text{Mf}$, $\text{grMf}$ in such a way to include (graded) functional diffeological spaces.

The morphism set $\text{Hom}_{\text{grMf}}(M_1, M_2)$ of two graded manifolds $M_1$, $M_2$ is an infinite-dimensional manifold called the *hom manifold* of $M_1$, $M_2$. As such, $\text{Hom}_{\text{grMf}}(M_1, M_2)$ contains $\text{Hom}_{\text{Mf}}(M_{10}, M_{20})$ as a submanifold. The construction of a special set of local coordinates of $\text{Hom}_{\text{grMf}}(M_1, M_2)$ illustrates this. Let $\varphi \in \text{Hom}_{\text{grMf}}(M_1, M_2)$ be a morphism of $M_1$, $M_2$. Then, $\varphi$ is described locally by the pull–backs $\varphi^* x_2^{i_2}$ of the local coordinates $x_2^{i_2}$ of $M_2$. These in turn can be expressed as functions of the local coordinates $x_1^{i_1}$ of $M_1$ as

$$\varphi^* x_2^{i_2}(x_1) = \sum_{R_1,|z_1^{R_1}|=|x_2^{i_2}|} \varphi^{i_2 R_1}(y_{10}) z_1^{R_1},$$

where $R_1$ denotes a multi–index $r_1 \ldots r_{1h}$, $z_1^{R_1}$ stands for the product $z_1^{r_1} \ldots z_1^{r_{1h}}$ of the corresponding vector coordinates $z_1^{r_1}$ of $M_1$ and $\varphi^{i_2 R_1}(y_{10})$ is a degree 0 smooth function of the body coordinates $y_{10}^{a_1}$ of $M_1$ and $\varphi^{i_2 R_1} \neq 0$ for finitely many values of $R_1$. For varying $\varphi \in \text{Hom}_{\text{grMf}}(M_1, M_2)$, the $\varphi^{i_2 R_1}$ constitute a set of local coordinates of $\text{Hom}_{\text{grMf}}(M_1, M_2)$. Since the $\varphi^{i_2 R_1}$ are all functions and have all degree 0, $\text{Hom}_{\text{grMf}}(M_1, M_2)$ is an infinite dimensional manifold as stated. The submanifold of $\text{Hom}_{\text{grMf}}(M_1, M_2)$ defined by the conditions $\varphi^{i_2 R_1} = 0$ for either $i_2 = r_2$ or $i_2 = a_2$ and $R_1 \neq \emptyset$ clearly is $\text{Hom}_{\text{Mf}}(M_{10}, M_{20})$.

For two fixed graded manifolds $M_1$, $M_2$, the functor from $\text{grMf}^{\text{op}}$ to $\text{Mf}$ defined by the assignment $N \to \text{Hom}_{\text{grMf}}(M_1 \times N, M_2)$ is representable, so that there exists a graded manifold $\text{Hom}_{\text{grMf}}(M_1, M_2)$, unique up to a unique isomorphism, having the property that $\text{Hom}_{\text{grMf}}(M_1 \times N, M_2) = \text{Hom}_{\text{grMf}}(N, \text{Hom}_{\text{grMf}}(M_1, M_2))$

[30]. $\text{Hom}_{\text{grMf}}(M_1, M_2)$ is called the *internal hom manifold* of $M_1$, $M_2$. Its body $\text{Hom}_{\text{grMf}}(M_1, M_2)_0$ is just $\text{Hom}_{\text{grMf}}(M_1, M_2)$. The construction of a suitable set of local coordinates of $\text{Hom}_{\text{grMf}}(M_1, M_2)$ highlights the difference between the
ordinary hom manifold $\text{Hom}_{\text{grMf}}(M_1, M_2)$ considered above and the internal hom manifold $\text{Hom}_{\text{grMf}}(M_1, M_2)$. Consider a morphism $\varphi \in \text{Hom}_{\text{grMf}}(M_1 \times N, M_2)$ of $M_1 \times N, M_2$. Then, similarly to before, $\varphi$ is described locally by the pull-backs $\varphi^*x^i_2$ of the local coordinates $x^i_2$ of $M_2$. These are functions of the local coordinates $x^i_1$ of $M_1$ and $u^\kappa$ of $N$ of the form

$$
\varphi^*x^i_2(x_1, u) = \sum_{R_1, |\varphi^*x^i_2(x_1)|=|x^i_2|} \varphi^2 R_1(y_{10}, u)z_1^{R_1},
$$

(3.1.2)

where $R_1$ and $z_1^{R_1}$ are defined as before and $\varphi^2 R_1(y_{10}, u)$ is a smooth function of the body coordinates $y_{10}^a$ of $M_1$ depending on the $u^\kappa$ of generally non zero degree and non vanishing for finitely many values of $R_1$. Now, we can regard $\varphi$ as a graded manifold morphism $\phi$ from $N$ to $\text{Hom}_{\text{grMf}}(M_1, M_2)$ by setting formally $\phi(u)^\#x^i_2(x_1) = \varphi^*x^i_2(x_1, u)$. The $\varphi^2 R_1(-, u)$ then are the local coordinates of $\phi(u)$ in $\text{Hom}_{\text{grMf}}(M_1, M_2)$. This shows that in $\text{Hom}_{\text{grMf}}(M_1, M_2)$ coordinates are smooth functions of the $y_{10}^a$ as earlier but with degrees varying in certain possibly infinite ranges including 0. $\text{Hom}_{\text{grMf}}(M_1, M_2)$ is so as an infinite dimensional graded manifold with body $\text{Hom}_{\text{grMf}}(M_1, M_2)_0 = \text{Hom}_{\text{grMf}}(M_1, M_2)$.

The algebra of smooth functions of a graded manifold $M$ is the graded commutative algebra $C^\infty(M) = \mathcal{O}_M(M_0)$ of global sections of the structure sheaf $\mathcal{O}_M$. $C^\infty(M)$ is an infinite-dimensional manifold called the graded smooth function manifold of $M$. As such, $C^\infty(M)$ contains $C^\infty(M_0)$ as a submanifold. A set of local coordinates of $C^\infty(M)$ showing this can be constructed as follows. A function $f \in C^\infty(M)$ reads as a function of the local coordinates $x^i$ of $M$ as

$$
f(x) = \sum_{R, |x_R|=|f|} f_R(y)z^R
$$

(3.1.3)

analogously to $\text{(3.1.1)}$, where $f_R(y)$ is a degree 0 smooth function of the body coordinates $y^a$ of $M$ and $f_R \neq 0$ for finitely many values of $R$. The $f_R$ are then the local coordinates of $f$ in $C^\infty(M)$. Since they all are degree 0 functions, $C^\infty(M)$ is an infinite dimensional ordinary manifold as claimed. The submanifold $C^\infty(M_0)$ is defined by the conditions $f_R(y) = 0$ for $R \neq \emptyset$. The above discussion shows further that $C^\infty(M) = \text{Hom}_{\text{grMf}}(M, G_{\mathbb{R}})$, where $G_{\mathbb{R}}$ is the graded algebra defined in $\text{(2.5.7)}$, analogously to ordinary differential geometry.
A morphism \( \varphi : M_1 \to M_2 \) of two graded manifolds \( M_1, M_2 \) induces a morphism \( \varphi^* : C^\infty(M_2) \to C^\infty(M_1) \) of their associated graded commutative algebras of graded smooth functions via its sheaf theoretic pull-back, called pull-back of \( \varphi \). \( \varphi^* \) fully characterizes \( \varphi \).

For a fixed graded manifolds \( M \), the functor from \( \text{grMf}^{\text{op}} \) to \( \text{Mf} \) defined by the assignment \( N \to C^\infty(M \times N) \) is representable, so that there exists a graded manifold \( C^\infty(M) \), unique up to a unique isomorphism, having the property that \( C^\infty(M \times N) = \text{Hom}_{\text{grMf}}(N, C^\infty(M)) \). \( C^\infty(M) \) is called the internal graded smooth function manifold of \( M \). Its body \( C^\infty(M)_0 \) is just \( C^\infty(M) \). A natural set of local coordinates of \( C^\infty(M) \) elucidating this can be constructed as follows. A function \( f \in C^\infty(M \times N) \) reads as a function of the local coordinates \( x^i \) of \( M \) as

\[
f(x, u) = \sum_{R:|R|+|z_R|=|f|} f_R(y, u) z^R,
\]

analogously to (3.1.2), where \( f_R(y, u) \) is a smooth function of the body coordinates \( y^a \) of \( M \) depending on the coordinates \( u^k \) of \( N \) of generally non zero degree and non vanishing for finitely many values of \( R \). Now, we can regard \( f \) as a graded manifold morphism \( \phi_f \) from \( N \) to \( C^\infty(M) \) by setting formally \( \phi_f(u)(x) = f(x, u) \). The \( f_R(\cdot, u) \) then are the local coordinates of \( \phi_f(u) \) in \( C^\infty(M) \). This shows that in \( C^\infty(M) \) coordinates are smooth functions of the \( y^a \) as earlier but with degrees varying in certain possibly infinite ranges including 0. \( C^\infty(M) \) is in this way an infinite dimensional graded manifold with body \( C^\infty(M)_0 = C^\infty(M) \). The above discussion shows further that \( C^\infty(M) = \text{Hom}_{\text{grMf}}(M, G_\mathbb{R}) \), as expected. Note that \( C^\infty(M) \) is also a graded algebra. In fact, one has

\[
C^\infty(M) \simeq C^\infty(M) \otimes G_\mathbb{R}.
\] (3.1.5)

A internal morphism \( \phi : M_1 \to M_2 \) of two graded manifolds \( M_1, M_2 \) induces a morphism \( \phi^# : C^\infty(M_2) \to C^\infty(M_1) \) of the graded commutative algebras of graded smooth functions of \( M_2 \) and internal graded smooth functions of \( M_1 \) called the internal pull-back of \( \phi \). To see this more explicitly, consider a graded manifold \( N \) and a graded manifold morphism \( \varphi : M_1 \times N \to M_2 \) which we may view equivalently as a morphism \( \phi : N \to \text{Hom}_{\text{grMf}}(M_1, M_2) \) into the internal hom
manifold of $M_1$, $M_2$. For $f \in C^\infty(M_2)$, $\varphi^*f \in C^\infty(M_1 \times N)$ can be regarded in equivalent fashion as a graded manifold morphism $\phi^#f$ from $N$ to $C^\infty(M_1)$ by setting formally $\phi(u)^#f(x) = \varphi^*f(x_1, u)$. This defines the required internal pull–back of $f$ by $\phi$.

A graded manifold $M$ is a locally ringed space $(M_0, \mathcal{O}_M)$ with a special local model. The rings attached by the structure sheaf $\mathcal{O}_M$ to the open sets of the body $M_0$ are not in general rings of functions of a certain kind. For this reason, for a morphism $\varphi : M_1 \to M_2$ of two graded manifolds $M_1$, $M_2$, which is a morphism of the underlying locally ringed spaces $(M_{10}, \mathcal{O}_{M_1})$, $(M_{20}, \mathcal{O}_{M_2})$, the associated pull–back $\varphi^* : C^\infty(M_2) \to C^\infty(M_1)$ is not given in general by a straightforward generalization of the well–known expression of elementary differential geometry.

Let us examine this point in more detail. Suppose that a graded function $f \in C^\infty(M_2)$ is given by an expansion of the form \eqref{3.1.3} as a function $f(x_2)$ of the local coordinates $x_2^{i_2}$ of $M_2$. Suppose further that the components $\varphi^{i_2}$ of $\varphi$ are given by expansions of the form \eqref{3.1.1} as functions $\varphi^{i_2}(x_1)$ of the local coordinates $x_1^{i_1}$ of $M_1$. Then, as a function of the $x_1^{i_1}$, $\varphi^*f(x_1)$ is not given by $f(x_2)|_{x_2^{i_2} = \varphi^{i_2}(x_1)}$ in general. In fact, such an object may not be polynomial in the vector coordinates $z_1^{r_1}$ of $M_1$ as it should even if the $\varphi^{i_2}(x_1)$ are. In particular, the assignment of the components $\varphi^{i_2}(x_1)$ cannot by itself specify the pull–back operation $\varphi^*$ by setting $\varphi^*f(x_1) = f(x_2)|_{x_2^{i_2} = \varphi^{i_2}(x_1)}$. In two instances this however can be done, namely when either $a)$ $M_1$ is an N–manifold or $b)$ the vector coordinates $z_1^{r_1}$ of $M_1$ are all odd, since then the aforementioned problems with polynomiality do not arise. Similar conclusions are reached for an internal morphism $\phi : M_1 \to M_2$ and the associated internal pull–back operation $\phi^#$.

The assignment of the components $\phi^{i_2}(x_1)$ cannot by itself specify the pull–back operation $\phi^#$ by setting $\phi^#f(x_1) = f(x_2)|_{x_2^{i_2} = \phi^{i_2}(x_1)}$, but it can when the vector coordinates $z_1^{r_1}$ of $M_1$ are all odd. Happily, these are precisely the cases which will occur in the following analysis, where the pull–back operators $\varphi^*$ or $\phi^#$ will always be tacitly assumed to be defined in the above standard fashion.

**Example 3.1.** If $E \to X$ is a vector bundle over a manifold $X$, then the 1–shifted vector bundle $M = E[1]$ is an N–manifold with $M_0 = X$ and $M_1 = E[1]$, hence
of degree 1. Conversely, every degree 1 N–manifold is of the form $E[1]$ for some vector bundle $E$.

**Example 3.2.** The $-1$–shifted real line $\mathbb{R}[-1]$ is a graded manifold with point body concentrated in degree $-1$. For any manifold $M$, the internal hom manifold $\text{Hom}_{grMf}(\mathbb{R}[-1], M)$ is isomorphic to the $1$–shifted tangent bundle $T[1]M$ of $M$ and hence is a degree 1 N–manifold (see eg. 3.1). More generally, the internal hom manifold $\text{Hom}_{grMf}(\mathbb{R}^q[-1], M)$ is isomorphic to the $q$-fold $1$–shifted tangent bundle $T[1]^qM$ of $M$. We shall refer to this remarkable fact in later subsections.

**Example 3.3.** The $1$–shifted real line $\mathbb{R}[1]$ is a point body degree 1 N–manifold. For this, the graded smooth function manifold $C^\infty(\mathbb{R}[1])$ is isomorphic to $\mathbb{R}^2$ while the internal graded smooth function manifold $C^\infty(\mathbb{R}[1])$ is isomorphic to $G_\mathbb{R} \mathbb{R}^2$. As graded commutative algebras, $C^\infty(\mathbb{R}[1])$ and $C^\infty(\mathbb{R}[1])$ are respectively $\mathbb{R} \oplus \mathbb{R}$ and $G_\mathbb{R} \oplus \mathbb{R}$, where $\mathbb{R}$ is a formal degree 1 parameter.

In the BGKS formulation of higher gauge theory reviewed in subsect. 1.1 higher gauge fields are morphisms of suitable graded manifolds. In the BRST formulation of the theory, graded manifold morphisms are not sufficient for a complete description of BRST gauge superfields. To incorporate such superfields, internal morphisms are required. Indeed, as shown in the analysis carried out above, while local coordinates for $\text{Hom}_{grMf}(M_1, M_2)$ are degree 0 smooth functions of the body coordinates of $M_1$, local coordinates for $\text{Hom}_{grMf}(M_1, M_2)$ are smooth functions of the body coordinates of all possible degrees. Since in BRST theory fields of non zero ghost degree are involved, $\text{Hom}_{grMf}(M_1, M_2)$ rather than $\text{Hom}_{grMf}(M_1, M_2)$ is the natural functional manifold for describing BRST fields in a BRST extension of BGKS theory.

### 3.2 NQ–manifolds

In this subsection, we review briefly the theory of NQ–manifolds because these are the kind of N–manifolds which are the natural target spaces of higher gauge and gauged sigma models. Our discussion will be kept at the level of local graded geometry focusing on those properties that are most relevant in the following.
See again ref. [51] for further background and complete referencing.

A degree $p$ vector field $X$ on a graded manifold $M$ is a degree $p$ derivation of the graded commutative algebra $C^\infty(M)$. It has a local coordinate expression

$$X = X^i \partial_i.$$  \hfill (3.2.1)

Here, the local functions $X^i$ are the components of $X$. They are homogeneous of degree $|X^i| = |x^i| + p$. The vector fields of all integer degrees form a graded Lie algebra $\mathfrak{X}(M)$. In local coordinates, the Lie brackets of two homogeneous vector fields $X, Y \in \mathfrak{X}(M)$ of degrees $p, q$ are given by

$$[X, Y] = (X^j \partial_j Y^i - (-1)^{pq} Y^j \partial_j X^i) \partial_i.$$  \hfill (3.2.2)

A NQ–manifold $M$ is an N–manifold equipped with a homological vector field $Q$, that is a degree 1 vector field on $M$ such that

$$Q^2 = [Q, Q]/2 = 0.$$  \hfill (3.2.3)

It is useful to write (3.2.3) in local coordinates using (3.2.2),

$$Q^j \partial_j Q^i \partial_i = 0.$$  \hfill (3.2.4)

For a NQ–manifold $M$, the graded commutative algebra $C^\infty(M)$ of smooth functions of $M$ is differential with differential $Q$. The algebraic theory of these type of algebras expounded in subsect. 2.1 thus applies. In particular, $M$ is characterized by the $Q$–cohomology of $C^\infty(M)$.

A morphism $\varphi : M_1 \to M_2$ of NQ–manifolds is a morphism of graded manifolds with the property that the associated morphism $\varphi^* : C^\infty(M_2) \to C^\infty(M_1)$ of graded commutative algebras satisfies

$$Q_1 \varphi^* - \varphi^* Q_2 = 0.$$  \hfill (3.2.5)

that is that $\varphi^* : C^\infty(M_2) \to C^\infty(M_1)$ is a morphism of differential graded commutative algebras. In local coordinates, (3.2.5) takes the form

$$(Q_1^j \partial_j \varphi^r - Q_2^r \circ \varphi) \partial_{2r}|_{\varphi} = 0.$$  \hfill (3.2.6)
Suppose that $M_1, M_2$ are NQ–manifolds and that $\varphi : M_1 \to M_2$ is a morphism of graded not necessarily NQ manifolds. Then, $\varphi$ is characterized by the defect of the algebra morphism $\varphi^*$

$$F_\varphi = Q_1\varphi^* - \varphi^*Q_2$$ (3.2.7)

satisfying the defect identity

$$Q_1F_\varphi + F_\varphi Q_2 = 0$$ (3.2.8)

(cf. eq. (2.1.12), (2.1.13)). In local coordinates, $F_\varphi$ is given by

$$F_\varphi = (Q_1^i\partial_{i1}\varphi^* - Q_2^r\circ \varphi)\partial_{2r}|_{\varphi} = 0$$ (3.2.9)

and the defect identity takes the form

$$(Q_1^i\partial_{i1}F_\varphi^* + F_\varphi^*\partial_{2r}Q_2^r \circ \varphi)\partial_{2r}|_{\varphi}$$

$$- (-1)^{|x^i|} (F_\varphi^*Q_1^i\partial_{i1}\varphi^* + Q_2^r \circ \varphi F_\varphi^*)\partial_{2r}\partial_{2r}|_{\varphi} = 0$$ (3.2.10)

**Example 3.4.** The 1–shifted tangent bundle $T[1]N$ of a manifold $N$ is an NQ–manifold. As a graded manifold, $T[1]N$ is described locally by degree 0 base coordinates $x^i$ and degree 1 fiber coordinates $\xi^i$. The graded commutative algebra $C^\infty(T[1]N)$ consists of the functions of the form

$$\alpha = \sum_{h\geq 0} \frac{1}{h!} \alpha_{i_1\ldots i_h}(x)\xi^{i_1}\ldots\xi^{i_h}$$ (3.2.11)

and is isomorphic to the graded commutative algebra $\Omega^*(N)$ of differential forms $\alpha = \sum_{h\geq 0} \frac{1}{h!} \alpha_{i_1\ldots i_h}(x)dx^{i_1}\ldots dx^{i_h}$. The homological vector field of $T[1]N$ is

$$d = \xi^i\partial_{xi}$$ (3.2.12)

Under the isomorphism $C^\infty(T[1]N) \simeq \Omega^*(N)$, $d$ answers to the de Rham differential $d_N = dx^i\partial_{xi}$. The cohomology of the cochain complex $(C^\infty(T[1]N), d)$ is therefore isomorphic to the familiar de Rham cohomology of the complex $(\Omega^*(N), d_N)$.

**Example 3.5.** The 1–shifted cotangent bundle $T^*[1]N$ of a manifold $N$ is an N–
manifold. $T^*[1]N$ is described locally by degree 0 base coordinates $x^i$ and degree 1 fiber coordinates $\xi_i$. The graded algebra $C^\infty(T^*[1]N)$ consists therefore of the functions of the form

$$ U = \sum_{h \geq 0} \frac{1}{h!} U^{i_1 \ldots i_h}(x) \xi_{i_1} \cdots \xi_{i_h} \quad (3.2.13) $$

and so is isomorphic to the graded commutative algebra $\mathfrak{X}^*(N)$ of multivectors $U = \sum_{h \geq 0} \frac{1}{h!} U^{i_1 \ldots i_h}(x) \partial_{i_1} \cdots \partial_{i_h}$. $T^*[1]N$ is equipped with the canonical degree 1 symplectic form

$$ \omega = d\xi_i dx^i. \quad (3.2.14) $$

Under the isomorphism $C^\infty(T^*[1]N) \simeq \mathfrak{X}^*(N)$, the Poisson brackets $\{-,-\}$ associated with $\omega$ correspond to the classical Schouten brackets $[-,-]_S$.

Let $N$ be a Poisson manifold. Then, the graded vector bundle $T^*[1]N$ is an NQ–manifold. Indeed, $T^*[1]N$ is equipped with the degree 1 vector field

$$ \delta = P^{ij}(x) \xi_j \partial_{x_i} + \frac{1}{2} \partial_i P^{jk}(x) \xi_j \xi_k \partial_{\xi^i}, \quad (3.2.15) $$

where the $P^{ij}$ are the Poisson structure functions. Further, the relations the $P^{ij}$ satisfy are equivalent to $\delta$ being homological. Remarkably, under the isomorphism $C^\infty(T^*[1]N) \simeq \mathfrak{X}^*(N)$, $\delta$ corresponds to the Poisson–Lichnerowicz differential $\delta_{PL} = [P,-]_S$. The cohomology of the cochain complex $(C^\infty(T^*[1]N), \delta)$ is therefore isomorphic to the Poisson–Lichnerowicz cohomology of the complex $(\mathfrak{X}^*(N), \delta_{PL})$.

The homological vector field $\delta$ turns out to be symplectic actually Hamiltonian: one has $Q = \{S,-\}$, where $S$ is the degree 2 function

$$ S = \frac{1}{2} P^{ij}(x) \xi_i \xi_j. \quad (3.2.16) $$

satisfying the master equation

$$ \{S,S\} = 0. \quad (3.2.17) $$

A degree $k$ PQ–manifold is a degree $k$ NQ–manifold equipped with a degree $k$
symplectic 2–form with respect to which the homological vector field is symplectic. $T^*[1]N$ is therefore a degree 1 PQ manifold. It can be shown that the most general degree 1 PQ–manifold is of the form $T^*[1]N$ for some Poisson manifold $N$.

**Example 3.6.** A *Lie algebroid* is a vector bundle $A \to N$ over a manifold $N$, with a structure of NQ–manifold on the 1–shifted bundle $A[1]$ necessarily of degree 1 (cf. eg. 3.1). If we denote by $x^i$ and $\xi^a$ the base and fiber coordinates of $A[1]$, the homological vector field $Q$ of $A[1]$ has the form

$$Q = \rho^i_a(x)\xi^a \partial_{x^i} - \frac{1}{2} f^a_{bc}(x)\xi^b \xi^c \partial_{\xi^a}. \quad (3.2.18)$$

The coefficients $\rho^i_a$ are the local coordinate representation of a bundle map $\rho : A \to TN$ called the *anchor* of $A$, viz.

$$\rho^i(e) = \rho^i_a e^a \quad (3.2.19)$$

for $e \in A$. The coefficients $f^a_{bc}$ together with the $\rho^i_a$ define an antisymmetric bracket structure $[-, -]_A$ on $\Gamma(A)$ explicitly

$$[s, t]^a = \rho^i_b s^b \partial_{t^i} - \rho^i_b t^b \partial_{s^i} + f^a_{bc} s^b t^c \quad (3.2.20)$$

with $s, t \in \Gamma(A)$. The nilpotence of $Q$ ensures that the brackets $[-, -]_A$ are Lie and that the linear map $\Gamma(A) \to \mathfrak{X}^1(N)$ induced by $\rho$ is Lie. Examples 3.4 and 3.5 show that the tangent bundle of a manifold and the cotangent bundle of a Poisson manifold are naturally Lie algebroids. Indeed, every degree 1 NQ–manifold is of the form $A[1]$ for some Lie algebroid $A$.

The above are all examples of $L_\infty$–algebroids, a broad class of NQ–manifolds to which the next subsection is devoted.

### 3.3 $L_\infty$–algebras and algebroids

$L_\infty$–algebroids are NQ–manifolds representing a far reaching generalization of Lie algebroids. They encode the symmetry of higher gauge theory and constitute the

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4 Here and in the following, we denote by $\Gamma(V)$ the space of sections of a vector bundle $V$. 51
natural target spaces of higher gauged sigma models as will be shown later below. We illustrate their theory in this subsection.

A graded vector bundle over an ordinary manifold $N$ is a vector bundle with a direct sum decomposition of the form

$$E = \bigoplus_{p=0}^{m} E_p$$

with $m \in \mathbb{N}$. The subbundles $E_p$ are conventionally assigned degree $p$. More general gradings are possible but they will not be considered here. An $L_{\infty}$–algebroid is a graded vector bundle $E$ with an assignment of a homological vector field $Q_E$ on the 1–shifted bundle $E[1]$,

$$Q_E^2 = 0.$$  

If $m = 0$, $E$ is called a Lie algebroid. For generic $m$, $E$ is called a Lie $m + 1$–algebroid or a $m + 1$–term $L_{\infty}$–algebroid. When $N$ is a point, one uses the term algebra instead of algebroid. In particular, for $m = 0$ we have an ordinary Lie algebra and for a generic $m$ a Lie $m + 1$–algebra or a $m + 1$–term $L_{\infty}$–algebra. The nilpotence relation encodes a very rich geometrical structure on $E$, which we shall make more explicit next.

The shifted bundle $E[1]$ is described locally by base coordinates $x^i$ and fiber coordinates $\xi^a$. The $x^i$ form a vector $x \in \mathbb{R}^d$, where $d = \text{dim } N$. The $\xi^a$ form a vector $\xi$ of the 1–shift $V_E[1]$ of the graded vector space

$$V_E = \bigoplus_{p=0}^{m} V_{Ep} \quad V_{Ep} = \mathbb{R}^{r_p}[p],$$

where $r_p = \text{rank } E_p$. $\xi$ decomposes accordingly in degree $p + 1$ components $\xi_p \in \mathbb{R}^{r_p}[p + 1]$ with $p = 0, \ldots, m$.

On a trivializing neighbourhood $U \subset N$, $Q_E$ has the following structure,

$$Q_E = \rho'(\xi_0) \partial_{x_i} + \left\langle \partial \xi - \sum_{\kappa \geq 2} \frac{(-1)^\kappa}{\kappa!} [\xi, \ldots, \xi, \xi], \partial \xi \right\rangle.$$  

In this expression, $\rho \in C^\infty(U, \text{Hom}(V_{E0}, \mathbb{R}^d))$, $\partial \in C^\infty(U, \text{Hom}_{-1}(V_E, V_E))$ and $[-, \ldots, -]_\kappa \in C^\infty(U, \text{Hom}_{\kappa-2}(\wedge^\kappa V_E, V_E))$, where $\wedge^\lambda V_E$ denotes the graded $\lambda$–th
exterior power of $V_E$. $\langle -, - \rangle$ denotes the $V_E - V_E^\vee$ duality pairing. For a morphism $\varphi \in \text{Hom}_\lambda(\wedge^\lambda V_E; V_E)$, the notation $\langle \varphi(\xi, \ldots, \xi), \xi^* \rangle$ is a shorthand for $\varphi^a_{a_1 \ldots a_n} \xi^{a_1} \ldots \xi^{a_n} \xi^{a_n}_*$, where the reals $\varphi^a_{a_1 \ldots a_n}$ are the components of $\varphi$ with respect to the canonical basis of $V_E$. The defining properties of $\rho$, $\partial$ and the $[\cdot, \ldots, \cdot]_\kappa$ ensure that $Q_E$ has degree 1 as required. The expansion (3.3.4) can be cast in a more explicit form as

$$Q_E = \rho^i(\xi_0) \partial_{x_i} + \langle \partial \xi_1 - \frac{1}{2}[\xi_0, \xi_0], \partial \xi_0 \rangle$$  \hspace{1cm} (3.3.5)

$$+ \langle \partial \xi_2 - [\xi_0, \xi_1] + \frac{1}{4}[\xi_0, \xi_0, \xi_0], \partial \xi_1 \rangle$$

$$+ \langle \partial \xi_3 - [\xi_0, \xi_2] - \frac{1}{2}[\xi_1, \xi_1] + \frac{1}{8}[\xi_0, \xi_0, \xi_1] - \frac{1}{16}[\xi_0, \xi_0, \xi_0, \xi_0], \partial \xi_2 \rangle + \ldots,$$

where we have suppressed the suffixes $\kappa$ form the brackets $[\cdot, \ldots, \cdot]_\kappa$ since its value is evident from the number of its arguments.

Enforcing the nilpotence relation (3.3.2) yields a host of relations involving $\rho$, $\partial$ and the $[\cdot, \ldots, \cdot]_\kappa$.

$$2\rho^i(\xi_0) \partial_{x_j} \rho^j(\xi_0) - \rho^i([\xi_0, \xi_0]) = 0,$$  \hspace{1cm} (3.3.6)

$$\rho^i(\partial \xi_1) = 0,$$  \hspace{1cm} (3.3.7)

$$\left( \rho^i(\xi_0) \partial_{x_j} + \left( \partial \xi - \sum_{\lambda \geq 2} \frac{(-1)^\lambda}{\lambda!} [\xi, \ldots, \xi]_\kappa, \partial \xi \right) \right)$$

$$+ \left( \partial \xi - \sum_{\kappa \geq 2} \frac{(-1)^\kappa}{\kappa!} [\xi, \ldots, \xi]_\kappa \right) = 0.$$  \hspace{1cm} (3.3.8)

The first two relations express specific properties of the local function $\rho^i$. The third one summarizes the algebraic relations of the local functions $[\cdot, \ldots, \cdot]_\kappa$. These can be made more explicit in terms of the components $\xi_\rho$ of $\xi$.

$$3\rho^i(\xi_0) \partial_{x_i} [\xi_0, \xi_0] + 3[\xi_0, [\xi_0, \xi_0]] - \partial [\xi_0, \xi_0, \xi_0] = 0,$$  \hspace{1cm} (3.3.9)

5 For any integer $k$ and any two graded vector spaces $V, W$, $\text{Hom}_k(V, W)$ is the set of all degree $k$ linear mappings $T : V \to W$. Notice that $\text{Hom}_k(V, W) = \text{Hom}_0(V, W_{-k})$. Usually, one writes $\text{Hom}(V, W) = \text{Hom}_0(V, W)$.

6 Recall that for a graded vector space $V$, $V^\vee$ is a graded vector space with $V^\vee_k = V_{-k}$. Further, the duality pairing $\langle -, - \rangle$ of $V, V^\vee$ pairs $V_k$ and $V^\vee_{-k}$.
\[
\rho^i(\xi_0) \partial_{x_1} \xi_1 + [\xi_0, \partial \xi_1] - \partial [\xi_0, \xi_1] = 0, 
\]
(3.3.10)
\[
\partial \partial \xi_2 = 0, 
\]
(3.3.11)
\[
2\rho^i(\xi_0) \partial_{x_1} [\xi_0, \xi_1] + 2[\xi_0, [\xi_0, \xi_1]] - [[\xi_0, \xi_0], \xi_1] 
- [\xi_0, \xi_0, \partial \xi_1] - \partial [\xi_0, \xi_0, \xi_1] = 0, 
\]
(3.3.12)
\[
4\rho^i(\xi_0) \partial_{x_1} [\xi_0, \xi_0, \xi_0] + 4[\xi_0, [\xi_0, \xi_0]] 
- 6[\xi_0, \xi_0, [\xi_0, \xi_0]] - \partial [\xi_0, \xi_0, \xi_0, \xi_0] = 0, 
\]
(3.3.13)
\[
2[\partial \xi_1, \xi_1] + \partial [\xi_1, \xi_1] = 0, 
\]
(3.3.14)
\[
\rho^i(\xi_0) \partial_{x_1} \xi_2 + [\xi_0, \partial \xi_2] - \partial [\xi_0, \xi_2] = 0, 
\]
(3.3.15)
\[
\partial \partial \xi_3 = 0, 
\]
(3.3.16)

Up to this point, we have adhered to the usual convention that the fiber coordinates \(\xi_p\) of the subbundles \(E_p[1]\) are those induced by a choice of a local frame of \(E_p[-p]\) and so of degree \(p+1\). An equivalent description can be achieved if we use instead the fiber coordinates \(\bar{\xi}_p\) induced by a choice of frame of \(E_p[1]\) and thus of degree 0. We denote by \(\bar{E}\) and \(\bar{E}_p\) the vector bundles \(E\) and \(E_p\) when the latter coordinate convention is adopted.

On any trivializing neighbourhood \(U \subset N\), a section \(s \in \Gamma(\bar{E})\) is represented simply by a function \(s^\sim \in C^\infty(U, \mathbb{R}^r)\), where \(r = \text{rank} E\). Similarly, a section \(s \in \Gamma(\bar{E}_p)\) is represented by a function \(s^\sim \in C^\infty(U, \mathbb{R}^{r_p})\) with \(r_p = \text{rank} E_p\).

The function \(\rho\) entering the local expansion (3.3.4) of \(Q_E\) is the local representation of a bundle morphism \(\rho_E : \bar{E}_0 \to TN\), the anchor of \(\bar{E}\). At the section level, \(\rho_E\) induces a linear map \(\rho_E : \Gamma(\bar{E}_0) \to \mathfrak{X}^1(N)\) defined by
\[
\rho_E^i(s)^\sim = \rho^i(s^\sim) 
\]
(3.3.17)
with \(s \in \Gamma(\bar{E}_0)\).

The functions \(\partial\) and \([-\ldots,-\]_\kappa\) also appearing in the local expansion (3.3.4) of \(Q_E\) define a linear boundary map \(\partial_E : \Gamma(\bar{E}) \to \Gamma(\bar{E})\) and a collection of multilinear brackets \([-\ldots,-]_{E\kappa} : \bigotimes^\kappa \Gamma(\bar{E}) \to \Gamma(\bar{E})\) of \(\bar{E}\) with \(\kappa \geq 2\), respectively. Explicit local expressions for these can be written down:
\[(\partial_E s)^\sim = \partial s^\sim \] (3.3.18)

and

\[([s_1, s_2]_E)^\sim = \rho^i(\pi_0(s_1)^\sim)\partial_{x^i}s_2^\sim - \rho^i(\pi_0(s_2)^\sim)\partial_{x^i}s_1^\sim + [s_1^\sim, s_2^\sim], \] (3.3.19)

\[([s_1, s_2, \ldots, s_\kappa]_E)^\sim = [s_1^\sim, s_2^\sim, \ldots, s_\kappa^\sim], \quad \kappa \geq 3, \] (3.3.20)

with any \(s, s_1, \ldots, s_\kappa \in \Gamma(\bar{E}).\) Above, \(\pi_0 : \Gamma(\bar{E}) \to \Gamma(\bar{E}_0)\) is the obvious projection. Furthermore, for \(s \in \Gamma(\bar{E})\) expressed as \(s = \bigoplus_{p=0}^{m} s_p\) with \(s_p \in \Gamma(\bar{E}_p),\) \(\hat{s} \in \Gamma(\bar{E})\) is defined through \(\hat{s} = \bigoplus_{p=0}^{m} (-1)^ps_p.\) \(\partial_E\) and the \([-,-,-]_\kappa\) are all compatible with the gradation of \(\bar{E}\) in the sense that \(\partial_E : \Gamma(\bar{E}_p) \to \Gamma(\bar{E}_{p-1})\) and \([-,-,-]_\kappa : \Gamma(\bar{E}_{p_1}) \otimes \cdots \otimes \Gamma(\bar{E}_{p_\kappa}) \to \Gamma(\bar{E}_{\Sigma_{p_1+\kappa-2}}).\) The gradation determines also the symmetry properties of the \([-,-,-]_\kappa.\) In particular, one has \([-,-] : \wedge^2 \Gamma(\bar{E}_0) \to \Gamma(\bar{E}_0).\) Note however that \([-,-]_E\) is not a Lie bracket in general, since the Jacobi identity may not be satisfied.

The relations \((3.3.6)-(3.3.8)\) determine various structural properties of \(\partial_E\) and relations obeyed by the anchor \(\rho_E,\) the boundary \(\partial_E\) and the brackets \([-,-,-]_\kappa\) of \(\bar{E}.\) We mention only the most basic ones. The anchor \(\rho_E\) has the property that for any two sections \(s, t \in \Gamma(\bar{E}_0)\)

\[\{\rho_E(s), \rho_E(t)\}_{X^1(N)} = \rho_E([s, t]_E), \] (3.3.21)

where the brackets in the left hand side are the Lie brackets of \(X^1(N)\) analogously to Lie algebroids. Further, the boundary map \(\partial_E\) is nilpotent

\[\partial_E^2 = 0. \] (3.3.22)

\(\Gamma(\bar{E})\) is a thus chain complex with boundary operator \(\partial_E,\) justifying the name given to this latter. Many other relations involving the anchor, boundary and brackets of \(\bar{E}\) generalizing the classical Lie theoretic Jacobi identity follow from \((3.3.9)-(3.3.16).\)

**Example 3.7.** Let \(\mathfrak{g}\) be a simple Lie algebra of compact type and \(c \in \mathbb{R}.\) The string Lie 2–algebra \(\text{string}_c(\mathfrak{g})\) is defined as follows. As a graded vector space,
\begin{equation}
\text{string}_c(g) = g \oplus \mathbb{R}[1]
\end{equation}

The boundary $\partial : \mathbb{R}[1] \to g$ vanishes. The brackets are

\begin{align}
[x, y] &= [x, y]_g, \\
[x, y, z] &= c \langle x, [y, z] \rangle_g,
\end{align}

for $x, y, z \in g$ and vanish in all other cases, where $[\cdot, \cdot]_g$ and $\langle \cdot, \cdot \rangle$ are the Lie brackets and a suitably normalized invariant symmetric non singular bilinear form of $g$. $\text{string}_c(\mathfrak{so}(n))$ is relevant in string theory.

\textbf{Example 3.8.} The tangent bundle $TN$ of a manifold $N$ (cf. eg. (3.4)), the cotangent bundle $TN$ of a Poisson manifold $N$ (cf. eg. (3.5)) and in general any Lie algebroid $A$ (cf. eg. 3.6) are all examples of $L_\infty$–algebroids.

\textbf{Example 3.9.} Let $V \to N$ be a metric vector bundle. Then, with $V$ there is associated a degree 2 symplectic $N$–manifold $L$ as follows. Consider the 2–shifted cotangent bundle $T^*[2]V[1]$ of the 1–shifted bundle $V[1]$. Then, $T^*[2]V[1]$ is a degree 2 symplectic $N$–manifold. Indeed, $T^*[2]V[1]$ is described locally by degree 0 base coordinates $x^i$ and degree 1 fiber coordinates $\xi^a$ of $V[1]$ and by corresponding cotangent degree 2 base coordinates $p_i$ and degree 1 fiber coordinates $\eta_a$. Further, $T^*[2]V[1]$ is equipped with the canonical degree 2 symplectic 2–form $\omega_0 = dp_idx^i + d\eta_a d\xi^a$. Assume for simplicity that the chosen local trivializations of $V$ are such that the coefficients $g_{ab}$ of the metric of $V$ are constant. Then, the covariant constraint $\eta_a = \frac{1}{2}g_{ab}\xi^b$ defines a submanifold $M$ of $T^*[2]V[1]$. $M$ is a degree 2 symplectic $N$–manifold. Indeed, $M$ is described by the degree 0, 1, 2 coordinates $x^i, \xi^a, p_i$ and is equipped with the degree 2 symplectic 2–form

\begin{equation}
\omega = dp_idx^i + \frac{1}{2}d\xi^a g_{abc} d\xi^c
\end{equation}

yielded by the pull-back of $\omega_0$ by the embedding $M \to T^*[2]V[1]$. It can be shown that conversely every degree 2 symplectic $N$–manifold $M$ stems from a metric vector bundle $V$ by the above construction \[30\].

Since the constraint defining the embedding $M$ into $T^*[2]V[1]$ is linear, $M$
can be identified with the 1–shift $L[1]$ of a graded vector bundle $L$ over $N$. The identification is non canonical, depending as it does on an arbitrary choice of a metric connection of $V$. Below, we assume that a choice has been made.

The metric vector bundle $V$ is a Courant algebroid if the graded vector bundle $L$ is an $L_x$–algebroid with the homological vector field $Q_L$ of $L[1]$ Hamiltonian with respect to the Poisson bracket structure associated with the symplectic form $\omega$ [30]. In that case, $Q_L$ can be shown to read as

$$Q_L = \rho^i_a(x)\xi^a \partial_{x^i} + (-\partial_{x^i}\rho^i_a(x)\xi^a)p_j 
+ \frac{1}{6} \partial^i_{x^j} f_{abc}(x)\xi^a \xi^b \xi^c \partial_{p^i} + \frac{1}{2} \delta_{abc}(x)\xi^b \xi^c \partial_{\xi^a}$$ (3.3.27)

for certain local functions $\rho^i_a$ and $f_{abc}$. Indeed, since $Q_L$ is Hamiltonian, $Q_L = \{S, -\}$ for some degree 3 function $S$ on $L[1]$ locally of the form

$$S = -\rho^i_a(x)\xi^a p_i + \frac{1}{6} f_{abc}(x)\xi^a \xi^b \xi^c,$$ (3.3.28)

leading to (3.3.27). The nilpotence of $Q_L$ is equivalent to the master equation

$$\{S, S\} = 0.$$ (3.3.29)

The structure functions $\rho^i_a$ and $f_{abc}$ define the Courant anchor and brackets of $V$, respectively. They obey a number of distinguished relations consequent to (3.3.29).

For a Courant algebroid $V$, $L[1]$ is therefore a degree 2 PQ–manifold. In [30], it is shown that conversely every degree 2 PQ–manifold stems from a Courant algebroid $V$ through the above construction.

### 3.4 Lie quasi–groupoid and $L_x$–algebroids

Lie quasi–groupoids are groupoid–like geometrical structures yielding $L_x$–algebroids via Lie differentiation much as Lie group yield Lie algebras. For this reason, they have attracted much interest in recent years [12, 54]. Lie quasi–groupoids are Kan simplicial manifolds and so belong to the realm of simplicial theory. We give now a concise review of the theory of Lie quasi–groupoid focused
on Lie differentiation referring the interested reader to the above papers for a more comprehensive treatment. Some new results are also presented.

A simplicial set $\mathcal{X}$ is a non negatively graded set equipped a collection of degree $-1$ face maps $f_{X_i}: \mathcal{X} \to \mathcal{X}$ and degree $1$ degeneracy maps $d_{X_i}: \mathcal{X} \to \mathcal{X}$ indexed by $i \geq 0$ and satisfying distinguished simplicial identities. More explicitly, a simplicial set $\mathcal{X}$ is a collection of sets $X_p$ indexed by $p \geq 0$ together with maps $f_{X_i}^p: X_p \to X_{p-1}$ with $p \geq i \geq 0$, $p > 0$ and $d_{X_i}^p: X_p \to X_{p+1}$ with $p \geq i \geq 0$ obeying the relations

$$f_{X_i}^{p+1} \circ f_{X_j}^p = f_{X_{j-1}}^{p+1} \circ f_{X_i}^p \text{ for } i < j, \quad p > 0,$$

$$d_{X_i}^{p+1} \circ d_{X_j}^p = d_{X_{j+1}}^{p+1} \circ d_{X_i}^p \text{ for } i \leq j,$$

$$f_{X_i}^{p+1} \circ d_{X_j}^p = d_{X_{j+1}}^{p+1} \circ f_{X_i}^p \text{ for } i < j, \quad p > 0,$$

$$f_{X_i}^{p+1} \circ d_{X_j}^p = d_{X_{j-1}}^{p+1} \circ f_{X_i}^p \text{ for } j > i + 1, \quad p > 0,$$

$$d_{X_i}^{p+1} \circ d_{X_i}^p = id_X = f_{X_{i+1}}^{p+1} \circ d_{X_i}^p.$$

An element $s_p \in \mathcal{X}_p$ is called a $p$-simplex of $\mathcal{X}$. A simplicial set morphism $g$ of the simplicial sets $\mathcal{X}, \mathcal{Y}$ is a degree $0$ map $g: \mathcal{X} \to \mathcal{Y}$ compatible with the face and degeneracy maps $f_{X_i}, d_{X_i}, f_{Y_i}, d_{Y_i}$ of $\mathcal{X}, \mathcal{Y}$. Stated explicitly, a simplicial morphism $g$ of the simplicial sets $\mathcal{X}, \mathcal{Y}$ is a collection of maps $g^p: \mathcal{X}_p \to \mathcal{Y}_p$ indexed by $p \geq 0$ such that

$$g^{p-1} \circ f_{X_i}^p = f_{Y_i}^p \circ g^p \text{ for } p \geq i \geq 0, \quad p > 0,$$

$$g^{p+1} \circ d_{X_i}^p = d_{Y_i}^p \circ g^p \text{ for } p \geq i \geq 0.$$

Simplicial sets and morphisms form a category $\text{sSet}$.

The simplicial set category $\text{sSet}$ is Cartesian closed. Thus, for any two simplicial sets $\mathcal{X}, \mathcal{Y}$, there is a simplicial set $\text{Hom}_{\text{sSet}}(\mathcal{X}, \mathcal{Y})$, unique up to unique simplicial isomorphism, such that, for any simplicial set $\mathcal{Z}$, $\text{Hom}(\mathcal{Z} \times \mathcal{X}, \mathcal{Y}) \simeq \text{Hom}(\mathcal{Z}, \text{Hom}_{\text{sSet}}(\mathcal{X}, \mathcal{Y}))$. $\text{Hom}_{\text{sSet}}(\mathcal{X}, \mathcal{Y})$ is called the internal simplicial hom set of $\mathcal{X}, \mathcal{Y}$.

A simplicial set $\mathcal{X}$ is called Kan if every horn has a simplex that fills it, i.e.
if for every \( p > 0 \) and \( k \) with \( p \geq k \geq 0 \) and every collection of \( p - 1 \)-simplexes \( s_0, \ldots, s_{k-1}, s_{k+1}, \ldots, s_p \in \mathcal{K}_{p-1} \) satisfying the Kan compatibility condition

\[
f^{-1}_{\mathcal{K}_i}(s_j) = f^{-1}_{\mathcal{K}_{j-1}}(s_i) \quad \text{for} \quad i < j, \ i \neq k, \ j \neq k,
\]

when \( p > 1 \), there exists a \( p \)-simplex \( s \in \mathcal{K}_p \) such that

\[
s_i = f^p_{\mathcal{K}_i}(s) \quad \text{for} \quad i \neq k.
\]

If \( s \) is unique for \( p > q \), \( \mathcal{K} \) is called a \( q \) Kan simplicial set.

It is possible to define a simplicial object in a category \( C \) by replacing sets and maps by objects and morphisms of \( C \) in the above definitions. One obtains in this way a simplicial category conventionally denoted by \( sC \). In particular, one can define simplicial groups, manifolds etc. as well as their graded counterparts. All these are simplicial sets with extra structures. However, Cartesian closedness does not hold in general.

A \((q)\) Kan simplicial set \( \mathcal{G} \) is called a \((q)\) quasi-groupoid. Similarly, a \((q)\) Kan simplicial manifold is called a \((q)\) Lie quasi-groupoid. These are the kind of geometrical structures yielding infinitesimally \( L_{\mathcal{K}} \)-algebras and algebroids as we show next.

Let \( \mathbb{N}\mathbb{R}[\cdot] \) denote the nerve of the pair groupoid of the \(-1\)-shifted real line \( \mathbb{R}[-1] \). Concretely, \( \mathbb{N}\mathbb{R}[\cdot] \) is the simplicial graded manifold which in degree \( p \) features the graded manifold

\[
\mathbb{N}\mathbb{R}[\cdot]_p = \mathbb{R}^{p+1}[-1]
\]

and whose face and degeneracy maps are given by

\[
\begin{align*}
f^p_{\mathbb{N}\mathbb{R}[\cdot]_i}(\theta_0, \ldots, \theta_p) &= (\theta_0, \ldots, \bar{i}, \ldots, \theta_p), \\
d^p_{\mathbb{N}\mathbb{R}[\cdot]_i}(\theta_0, \ldots, \theta_p) &= (\theta_0, \ldots, \theta_i, \theta_i, \ldots, \theta_p).
\end{align*}
\]

For \( k \geq 0 \), let \( \mathbb{N}\mathbb{R}[\cdot](k) \) be the simplicial graded submanifold of \( \mathbb{N}\mathbb{R}[\cdot] \) generated by \( \mathbb{R}^k[-1] \times \{0\} \) in degree \( k \). As \( f_{\mathbb{N}\mathbb{R}[\cdot](k)}^{k+1}((\mathbb{R}^{k+1}[-1] \times \{0\}) = \mathbb{R}^k[-1] \times \{0\}, \mathbb{N}\mathbb{R}[\cdot](k) \subset \mathbb{N}\mathbb{R}[\cdot](k + 1) \). We have so a sequence \( \mathbb{N}\mathbb{R}[\cdot](k) \) of simplicial
graded manifolds indexed by \( k \geq 0 \) organized in a filtration

\[
\mathbb{N}\mathbb{R}[-1](0) \subset \mathbb{N}\mathbb{R}[-1](1) \subset \mathbb{N}\mathbb{R}[-1](2) \subset \cdots \subset \mathbb{N}\mathbb{R}[-1].
\]  

Let \( Z \) be any graded manifold. With \( Z \) there is associated the constant simplicial graded manifold \( C_Z \). Concretely, \( C_Z \) is the simplicial graded manifold which in every degree \( p \) exhibits the graded manifold \( Z \) itself

\[
C_{zp} = Z
\]  
and whose face and degeneracy maps are all the identity map \( \text{id}_Z \)

\[
f^p_{cZ}(z) = z, \quad d^p_{cZ}(z) = z.
\]

For a graded manifold \( Z \) and simplicial graded manifold \( \mathcal{E}' \), the simplicial graded manifold hom set \( \text{Hom}_{\text{grMf}}(\mathbb{N}\mathbb{R}[-1] \times C_Z, \mathcal{E}') \) can be identified with a submanifold of the infinite dimensional manifold

\[
H(Z, \mathcal{E}') = \prod_{p=0}^{\infty} \text{Hom}_{\text{grMf}}(\mathbb{N}\mathbb{R}[-1], C_{zp})
\]

and so it is itself a manifold. Each of the factors in the right hand side of (3.4.17) can be written in terms of the internal hom manifold \( \text{Hom}_{\text{grMf}}(\mathbb{N}\mathbb{R}[-1], \mathcal{E}') \) of \( \mathbb{N}\mathbb{R}[-1], \mathcal{E}' \)

\[
\text{Hom}_{\text{grMf}}(\mathbb{N}\mathbb{R}[-1], C_{zp}) = \text{Hom}_{\text{grMf}}(C_{zp}, \text{Hom}_{\text{grMf}}(\mathbb{N}\mathbb{R}[-1], \mathcal{E}')).
\]

(c.f. subsect. 3.1). This suggests defining a \textit{simplicial internal graded manifold hom set} \( \text{Hom}^*_\text{grMf}(\mathbb{N}\mathbb{R}[-1], \mathcal{E}') \): an internal simplicial morphism is defined as an ordinary morphism but with an internal rather than an ordinary graded manifold morphism \( \mathbb{N}\mathbb{R}[-1] \rightarrow \mathcal{E}' \) at degree \( p \). \( \text{Hom}^*_\text{grMf}(\mathbb{N}\mathbb{R}[-1], \mathcal{E}') \) is identifiable with a graded submanifold of the infinite dimensional graded manifold

\[\text{(3.4.13)}\]  

\[\text{(3.4.14)}\]  

\[\text{(3.4.15)}\]  

\[\text{(3.4.16)}\]  

\[\text{(3.4.17)}\]  

\[\text{(3.4.18)}\]
\[ H(\mathcal{X}) = \prod_{p=0}^{\infty} \text{Hom}_{\text{grMf}}(N\mathbb{R}[-1], \mathcal{X}_p) \]  

(3.4.19)

and so it is itself a graded manifold. In this way, by the triviality of \( C_Z \), the study of \( \text{Hom}_{\text{grMf}}(N\mathbb{R}[-1] \times C_Z, \mathcal{X}) \) can be reduced to that of \( \text{Hom}_{\text{grMf}}(N\mathbb{R}[-1], \mathcal{X}) \).

A similar analysis clearly can be carried out also for the \( N\mathbb{R}r\epsilon \) space. Let \( G \) be a Lie quasi–groupoid viewed as a simplicial graded manifold. The following theorem, originally proven in ref. [54] and rederived in ref. [12], holds.

Let \( g \) be a simplicial internal graded manifold morphism. Then, \( g \) can be extended to a sequence \( g(l) \in \text{Hom}^*_\text{grMf}(N\mathbb{R}[-1](l), \mathcal{X}) \) of internal simplicial morphisms indexed by \( l \geq 0 \) such that

\[ g(l) = g(m)
\]

for \( l \leq m \). Furthermore, if \( k \geq q \), the sequence \( g(l) \) is unique. The proof relies in an essential way on the property that \( \mathcal{X} \) is Kan and uniquely Kan in degree \( q \) or larger. It follows from the theorem that every simplicial internal graded manifold morphism \( g(q) \in \text{Hom}^*_\text{grMf}(N\mathbb{R}[-1](q), \mathcal{X}) \) determines a unique simplicial internal morphism \( g \in \text{Hom}^*_\text{grMf}(N\mathbb{R}[-1], \mathcal{X}) \) such that

\[ g(q) = g|_{N\mathbb{R}[-1](q)} \]  

(3.4.21)

\( g \) is completely defined in terms of the unique sequence of simplicial internal morphisms \( g(l) \in \text{Hom}^*_\text{grMf}(N\mathbb{R}[-1](l), \mathcal{X}) \) extending \( g(q) \) by

\[ g|_{N\mathbb{R}[-1](l)} = g(l). \]  

(3.4.22)

Each simplicial internal graded manifold morphism \( g \in \text{Hom}^*_\text{grMf}(N\mathbb{R}[-1], \mathcal{X}) \), conversely, is determined by the restriction of \( g \) to \( N\mathbb{R}[-1](q) \) and, consequently, by the restriction of the component \( g^a \) of \( g \) to \( \mathbb{R}^q[-1] \times \{0\} \). The internal hom manifold \( \text{Hom}_{\text{grMf}}(\mathbb{R}^q[-1] \times \{0\}, \mathcal{X}_q) \) of \( \mathbb{R}^q[-1] \times \{0\}, \mathcal{X}_q \) is isomorphic to the total space of the graded vector bundle \( T[1]^q_{\mathcal{X}_q} \rightarrow \mathcal{X}_q \). So, the graded manifold \( \text{Hom}^*_\text{grMf}(N\mathbb{R}[-1], \mathcal{X}) \) can be identified with a graded submanifold of this latter.

A point \( \gamma \in \mathcal{X}_0 \) defines a mapping \( g(0)_0 : \{0\} \rightarrow \mathcal{X}_0 \) by setting \( g(0)_0(0) = \gamma \) and therefore determines a unique simplicial internal graded manifold morphism.
$g(0) \in \text{Hom}_{\text{sgmFf}}^*(\mathbb{N}[1](0), \mathcal{G})$. This can be extended non uniquely to a sequence $g(l) \in \text{Hom}_{\text{sgmFf}}^*(\mathbb{N}[1](l), \mathcal{G})$ of simplicial internal morphisms which determines in turn a simplicial internal morphism $g \in \text{Hom}_{\text{sgmFf}}^*(\mathbb{N}[1], \mathcal{G})$ via (3.4.22). The non uniqueness of the sequence $g(l)$ entails that of the resulting morphism $g$. In [12], it is shown that the simplicial morphisms $g$ so yielded are parametrized by the total space of the 1–shift $\text{Lie}\mathcal{G}[1]$ of the graded vector bundle

$$\text{Lie}\mathcal{G} = \bigoplus_{p=1}^{q} \bigotimes_{i=0}^{p-1} \mathcal{G}_{\gamma} \ker f_{\gamma i+1}^{p} [p - 1] \to \mathcal{G}_0,$$  

(3.4.23)

where $\mathcal{G}_{\gamma} = d_{\gamma 0}^{p-1} \circ \cdots \circ d_{\gamma 0}^{0} : \mathcal{G}_0 \to \mathcal{G}_p$ with $d_{\gamma 0}^{0} = \text{id}_{\mathcal{G}_0}$ and here and below $f_{* \cdots *|m}$ with $k$–fold $*$ denotes the $k$–fold tangent map of a map $f$ evaluated at a point $m$ of its domain. Notice that $\text{Lie}\mathcal{G} \subset \bigoplus_{p=1}^{q} \mathcal{G}_{\gamma} \ker \mathcal{G}_{\gamma} [p - 1] \mathcal{G}_p$. Remarkably, a rather explicit expression of this parametrization can be furnished. Its elaboration is a bit lengthy but in fact completely algorithmic. One writes a general ansatz for the maps $g^p(\theta_0, \ldots, \theta_{p-1}, 0)$ of the form

$$g^p(\theta_0, \ldots, \theta_{p-1}, 0) = \gamma^p_0 + \sum_{r=0}^{p-1} \sum_{0 \leq i_0 < \cdots < i_r \leq p-1} \theta_{i_0} \cdots \theta_{i_r} \gamma^p_{i_0 \cdots i_r},$$  

(3.4.24)

where $\gamma^p_0$ is a degree 0 and the $\gamma^p_{i_1 \cdots i_r}$ are degree $r$ local functions on $\mathcal{G}_0$, and derives all relations $\gamma^p_0$ and the $\gamma^p_{i_1 \cdots i_r}$ must obey in order the $g^p$ to be the components of a simplicial morphism $g \in \text{Hom}_{\text{sgmFf}}^*(\mathbb{N}[1], \mathcal{G})$ starting from $g^0$.

$g^0$ is parametrized by a point $\gamma \in \mathcal{G}_0$ and a vector $\xi_0$ of the fiber $T[1]_{\mathcal{G}_{\gamma}} \mathcal{G}_1$ of the vector bundle $\mathcal{G}_{\gamma} \mathcal{G}_1$ satisfying

$$f_{\mathcal{G}_{\gamma} \mathcal{G}_1}^1 (\xi_0) = 0.$$  

(3.4.25)

$\xi_0$ can hence be identified with a vector $\alpha_1$ of the fiber $\ker f_{\mathcal{G}_{\gamma} \mathcal{G}_1}^1 [1]$ of the vector subbundle $\mathcal{G}_{\gamma} \ker f_{\mathcal{G}_{\gamma} \mathcal{G}_1}^1 \subset \text{Lie}\mathcal{G}[1]$. In terms of $\gamma$, $\xi_0$, $g^0$ reads

$$g^0(\theta_0) = \xi_0^0(\gamma) + \theta_0 \rho_{\gamma} (\xi_0),$$  

(3.4.26)

where $\rho_{\gamma} (\xi_0)$ is given by

$$\rho_{\gamma} (\xi_0) = f_{\mathcal{G}_{\gamma} \mathcal{G}_1}^1 (\xi_0).$$  

(3.4.27)
$g^1$ is parametrized by $\gamma$, $\xi_0$ and the degree 2 component $\xi_1$ of a vector of the fiber $T[1]^2\zeta_{\gamma}^3(\gamma)G_2$ of the vector bundle $\zeta_{\gamma}^3*T[1]^2G_2$ satisfying

$$f_{\gamma^*|_{\gamma}}^2(\xi_1) + f_{\gamma^*|_{\gamma}}^2(\xi_0)(d_{\gamma^*|_{\gamma}}^1(\xi_0), d_{\gamma^*|_{\gamma}}^1(\xi_0)) = 0$$  \hspace{1cm} (3.4.28)

for $i = 0, 1$, the lowest degree components of which are determined by $\xi_0$. The Kan nature of $\mathcal{G}$ ensures that (3.4.28) has a solution. The solution is unique up to the degree 2 component of a vector $\alpha_2$ of the fiber $\bigcap_{i=0}^1 \ker f_{\gamma^*|_{\gamma}}^2[2]$ of the vector subbundle $\bigcap_{i=0}^1 \zeta_{\gamma}^2 \ker f_{\gamma^*}[2] \subset \text{Lie} \mathcal{G}[1]$. In terms of $\gamma$, $\xi_0$, $\xi_1$, $g^1$ has the expansion

$$g^1(\theta_0, \theta_1) = \zeta_{\gamma}^1(\gamma) - (\theta_1 - \theta_0)\xi_0$$  \hspace{1cm} (3.4.29)

$$+ \theta_1 d_{\gamma^*|_{\gamma}}^0(\xi_0) + \theta_0(\zeta_{\gamma}^1(\gamma) - \frac{1}{2}[\xi_0, \xi_0]_{\gamma}),$$

where $\zeta_{\gamma^*|_{\gamma}}$ satisfies

$$\zeta_{\gamma^*|_{\gamma}}^1 = f_{\gamma^*|_{\gamma}}^2(\xi_1),$$  \hspace{1cm} (3.4.30)

$$\frac{1}{2}[\xi_0, \xi_0]_{\gamma} = -f_{\gamma^*|_{\gamma}}^2(\xi_1)(d_{\gamma^*|_{\gamma}}^1(\xi_0), d_{\gamma^*|_{\gamma}}^1(\xi_0)).$$  \hspace{1cm} (3.4.31)

g^2 is parametrized by $\gamma$, $\xi_0$, $\xi_1$ and the degree 3 component $\xi_2$ of a vector of the fiber $T[1]^3\zeta_{\gamma}^3(\gamma)G_3$ of the vector bundle $\zeta_{\gamma}^3*T[1]^3G_3$ satisfying

$$f_{\gamma^*|_{\gamma}}^3(\xi_2)$$  \hspace{1cm} (3.4.32)

$$+ f_{\gamma^*|_{\gamma}}^3(\xi_0)(d_{\gamma^*|_{\gamma}}^1(\xi_0), d_{\gamma^*|_{\gamma}}^0(\xi_0))$$

$$- f_{\gamma^*|_{\gamma}}^3(\xi_0)(d_{\gamma^*|_{\gamma}}^1(\xi_0), d_{\gamma^*|_{\gamma}}^0(\xi_0))$$

$$+ f_{\gamma^*|_{\gamma}}^3(\xi_0)(d_{\gamma^*|_{\gamma}}^1(\xi_0), d_{\gamma^*|_{\gamma}}^0(\xi_0))$$

$$+ f_{\gamma^*|_{\gamma}}^3(\xi_0)(d_{\gamma^*|_{\gamma}}^1(\xi_0), d_{\gamma^*|_{\gamma}}^0(\xi_0))$$

$$- f_{\gamma^*|_{\gamma}}^3(\xi_0)(d_{\gamma^*|_{\gamma}}^1(\xi_0), d_{\gamma^*|_{\gamma}}^0(\xi_0))$$

$$+ f_{\gamma^*|_{\gamma}}^3(\xi_0)(d_{\gamma^*|_{\gamma}}^1(\xi_0), d_{\gamma^*|_{\gamma}}^0(\xi_0)),$$
\[
\begin{align*}
&\left( d^2_{\mathcal{G}2^*|\xi_0^*(\gamma)} (d^1_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\xi_0), d^1_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\xi_0)) \\
&\quad - f^3_{\mathcal{G}1^*|\xi_0^*(\gamma)} ((d^2_{\mathcal{G}1} \circ d^1_{\mathcal{G}1})_*|\xi_0^*(\gamma)) (\xi_0), \\
&\quad (d^2_{\mathcal{G}0} \circ d^1_{\mathcal{G}1})_*|\xi_0^*(\gamma)) (\xi_0), (d^2_{\mathcal{G}0} \circ d^1_{\mathcal{G}0})_*|\xi_0^*(\gamma)) (\xi_0)) = 0.
\end{align*}
\]

for \( i = 0, 1, 2 \), the lowest degree components of which are determined by \( \xi_0, \xi_1 \).

Again, the Kan nature of \( \mathcal{G} \) ensures that \((3.4.30)\) has a solution unique up to the degree 3 component \( \alpha_3 \) of a vector of the fiber \( \bigcap_{i=0}^2 \ker f^3_{\mathcal{G}1^*|\xi_0^*(\gamma)} [3] \) of the vector subbundle \( \bigcap_{i=0}^2 \ker f^3_{\mathcal{G}1^*|\xi_0^*(\gamma)} [3] \subset \text{Lie} \mathcal{G}[1] \). In terms of \( \gamma, \xi_0, \xi_1, \xi_2, \gamma^2 \) reads

\[
g^2(\theta_0, \theta_1, \theta_2) = \xi_2^2(\gamma) - (\theta_1 - \theta_0) d^1_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\xi_0) - (\theta_2 - \theta_1) d^1_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\xi_0) \quad (3.4.33)
\]

\[
+ \theta_2(d^1_{\mathcal{G}0} \circ d^0_{\mathcal{G}0})*|\xi_0^*(\gamma)) (\rho_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\xi_0)) + (\theta_1 \theta_2 - \theta_0 \theta_0) \xi_1
\]

\[
- (\theta_1 \theta_2 - \theta_0 \theta_2) [d^1_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\partial_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\xi_0) - \frac{1}{2}[\xi_0, \xi_0])_g] + d^1_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\partial_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\xi_0), \xi_0) (3.4.34)
\]

\[
+ \theta_1 \theta_2 [d^1_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\partial_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\xi_0) + d^1_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\partial_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\xi_0), \xi_0)
\]

\[
+ \theta_0 \theta_1 (\partial_{\mathcal{G}2^*|\xi_0^*(\gamma)} - [\xi_0, \xi_1]_g + \frac{1}{2}[\xi_0, \xi_0]_g)
\]

where \( \partial_{\mathcal{G}1^*|\xi_0^*(\gamma)} [\xi_0, \xi_1]_g, [\xi_0, \xi_0]_g \) are given by

\[
\partial_{\mathcal{G}2^*|\xi_0^*(\gamma)} = f^3_{\mathcal{G}3^*|\xi_0^*(\gamma)} (\xi_2),
\]

\[
[\xi_0, \xi_1]_g = - f^3_{\mathcal{G}3^*|\xi_0^*(\gamma)} ((d^2_{\mathcal{G}1} \circ d^1_{\mathcal{G}1})_*|\xi_0^*(\gamma)) (\xi_0), d^2_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\xi_1)) \quad (3.4.35)
\]

\[
+ f^3_{\mathcal{G}3^*|\xi_0^*(\gamma)} ((d^1_{\mathcal{G}0} \circ d^1_{\mathcal{G}1})_*|\xi_0^*(\gamma)) (\xi_0), d^2_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\xi_1))
\]

\[
- f^3_{\mathcal{G}3^*|\xi_0^*(\gamma)} ((d^2_{\mathcal{G}0} \circ d^1_{\mathcal{G}0})_*|\xi_0^*(\gamma)) (\xi_0), d^2_{\mathcal{G}2^*|\xi_0^*(\gamma)} (\xi_1)),
\]

\[
\frac{1}{2}[\xi_0, \xi_0]_g = f^3_{\mathcal{G}3^*|\xi_0^*(\gamma)} ((d^2_{\mathcal{G}1} \circ d^1_{\mathcal{G}1})_*|\xi_0^*(\gamma)) (\xi_0),
\]

\[
d^2_{\mathcal{G}2^*|\xi_0^*(\gamma)} (d^1_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\xi_0), d^1_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\xi_0))
\]

\[
- f^3_{\mathcal{G}3^*|\xi_0^*(\gamma)} ((d^2_{\mathcal{G}0} \circ d^1_{\mathcal{G}1})_*|\xi_0^*(\gamma)) (\xi_0),
\]

\[
d^2_{\mathcal{G}1^*|\xi_0^*(\gamma)} (d^1_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\xi_0), d^1_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\xi_0))
\]

\[
+ f^3_{\mathcal{G}3^*|\xi_0^*(\gamma)} ((d^2_{\mathcal{G}0} \circ d^1_{\mathcal{G}0})_*|\xi_0^*(\gamma)) (\xi_0),
\]

\[
d^2_{\mathcal{G}2^*|\xi_0^*(\gamma)} (d^1_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\xi_0), d^1_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\xi_0))
\]

\[
= f^3_{\mathcal{G}3^*|\xi_0^*(\gamma)} ((d^2_{\mathcal{G}1} \circ d^1_{\mathcal{G}1})_*|\xi_0^*(\gamma)) (\xi_0),
\]

\[
d^2_{\mathcal{G}0^*|\xi_0^*(\gamma)} (d^1_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\xi_0), d^1_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\xi_0))
\]

\[
- f^3_{\mathcal{G}3^*|\xi_0^*(\gamma)} ((d^2_{\mathcal{G}0} \circ d^1_{\mathcal{G}1})_*|\xi_0^*(\gamma)) (\xi_0),
\]

\[
d^2_{\mathcal{G}1^*|\xi_0^*(\gamma)} (d^1_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\xi_0), d^1_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\xi_0))
\]

\[
+ f^3_{\mathcal{G}3^*|\xi_0^*(\gamma)} ((d^2_{\mathcal{G}0} \circ d^1_{\mathcal{G}0})_*|\xi_0^*(\gamma)) (\xi_0),
\]

\[
d^2_{\mathcal{G}2^*|\xi_0^*(\gamma)} (d^1_{\mathcal{G}0^*|\xi_0^*(\gamma)} (\xi_0), d^1_{\mathcal{G}1^*|\xi_0^*(\gamma)} (\xi_0))
\]

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The components \( g^p \) of \( g \) with \( p \geq 3 \) can be obtained in the same way, although the amount of computation required increases very rapidly with \( p \).

Via the degeneracy maps of \( G \), the data \( \gamma \) and \( \xi_p \) with \( 0 \leq p \leq q - 1 \) coordinate a graded submanifold of the graded vector bundle \( T[1]^q G \rightarrow G \). Conversely, via the face maps of \( G \), a point of the submanifold gives rise to a set of data \( \gamma \) and \( \xi_p \). At the same time the data \( \gamma \) and \( \xi_p \) coordinate the vector bundle \( \text{Lie} G \).

In ref. [12], it is shown \( \text{Lie} G \) is an \( L\infty \)-algebroid. It is not difficult to see the reason why. The parametrization of the simplicial internal graded manifold \( \text{Hom}_{\text{srMf}}^*(\mathbb{N}[1], G) \) described above can be thought of as defining a higher exponential map \( \exp_g : \text{Lie} G \rightarrow \text{Hom}_{\text{srMf}}^*(\mathbb{N}[1], G) \), which with the data \( \gamma \) and \( \xi_p \) associates the simplicial internal morphism

\[
g = \exp_g(\gamma, \xi_0, \ldots, \xi_{q-1}).
\]

(3.4.37)

The Chevalley–Eilenberg differential \( Q_{\text{Lie} G} \) of \( \text{Lie} G \) is defined implicitly by

\[
Q_{\text{Lie} G} \exp_g(\gamma, \xi_0, \ldots, \xi_{q-1}) = D \exp_g(\gamma, \xi_0, \ldots, \xi_{q-1}),
\]

(3.4.38)

where \( D \) is the formal operator acting as

\[
Du_p(\theta_0, \ldots, \theta_p) = \frac{d}{d\epsilon} u_p(\theta_0 + \epsilon, \ldots, \theta_p + \epsilon).
\]

(3.4.39)

on a simplicial morphism \( u \in \text{Hom}_{\text{srMf}}^*(\mathbb{N}[1], G) \). As \( D^2 = 0 \), \( Q_{\text{Lie} G} \) is nilpotent by construction. It is straightforward to verify that \( Q_{\text{Lie} G} \) is given by an expression of the form

\[
Q_{\text{Lie} G} = \rho_g, \partial_g, [-, -]_g, [-, -, -]_g, \ldots
\]

being precisely the anchor, the boundary and the brackets of an \( L\infty \)-algebroid structure on \( \text{Lie} G \). In this sense, \( \text{Lie} G \) describes \( G \) infinitesimally and is the quasi Lie groupoid analog of the Lie algebra of a Lie group. The classic Lie case is recovered by considering the 1 Lie quasi–groupoid \( G \) that is the nerve of the single object delooping groupoid \( \mathcal{B} G \) of an ordinary Lie groupoid \( G \).

In ref. [12] it is claimed that higher gauge symmetry can be described math-
ematically in terms of simplicial homotopies of simplicial maps. For this reason, we briefly review this topic next. A simplicial homotopy of two simplicial sets \(\mathcal{X}, \mathcal{Y}\) is simply a simplicial set morphism \(h \in \text{Hom}_{sSet}(\mathcal{X} \times \Delta^1, \mathcal{Y})\), where \(\Delta^1\) is the so-called standard simplicial 1-simplex, a simplicial set that is a combinatorial analogue of the geometric 1-simplex. There are source and target maps \(\sigma, \tau : \text{Hom}_{sSet}(\mathcal{X} \times \Delta^1, \mathcal{Y}) \to \text{Hom}_{sSet}(\mathcal{X}, \mathcal{Y})\) so that a homotopy \(h\) of \(\mathcal{X}, \mathcal{Y}\) connects two simplicial maps \(\sigma, \tau \in \text{Hom}_{sSet}(\mathcal{X}, \mathcal{Y})\), \(h : \sigma(h) \Rightarrow \tau(h)\).

As \(\text{Hom}_{sSet}(\mathcal{X} \times \Delta^1, \mathcal{Y}) = \text{Hom}_{sSet}(\mathcal{X}, \text{Hom}_{sSet}(\Delta^1, \mathcal{Y}))\) for any pair of simplicial sets \(\mathcal{X}, \mathcal{Y}\), the study of simplicial homotopies can be reduced to that of the internal simplicial hom set \(\text{Hom}_{sSet}(\Delta^1, \mathcal{Y})\). As we shall explain shortly, \(\text{Hom}_{sSet}(\Delta^1, \mathcal{Y})\) admits a rather explicit description. Furthermore, it has an important property: when \(\mathcal{G}\) is a quasi-groupoid, \(\text{Hom}_{sSet}(\Delta^1, \mathcal{G})\) also is.

Let \(\mathcal{G}\) be a quasi-groupoid. For \(p \geq 0\), an element \(w \in \text{Hom}_{sSet}(\Delta^1, \mathcal{G})_p\) can be encoded in a \(p + 1\)-tuple \(w_i \in \mathcal{G}_{p+1}, 0 \leq i \leq p\), satisfying

\[
 f_{\mathcal{G}_i}^{p+1}(w_i) = f_{\mathcal{G}_i}^{p+1}(w_{i-1}) \quad \text{for } 0 < i \leq p. \tag{3.4.40}
\]

The face and degeneracy maps of \(\text{Hom}_{sSet}(\Delta^1, \mathcal{G})\) are defined as

\[
 f_{\text{Hom}_{sSet}(\Delta^1, \mathcal{G})_i}^p(w)_j = f_{\mathcal{G}_i}^{p+1}(w_{j+1}) \quad \text{for } i \leq j, \tag{3.4.41}
\]

\[
 f_{\mathcal{G}_i}^{p+1}(w) \quad \text{for } i > j, \text{ with } 0 \leq i \leq p, 0 \leq j \leq p - 1, \tag{3.4.42}
\]

\[
 d_{\text{Hom}_{sSet}(\Delta^1, \mathcal{G})_i}^p(w)_j = d_{\mathcal{G}_i}^{p+1}(w_{j-1}) \quad \text{for } i < j, \tag{3.4.42}
\]

\[
 d_{\mathcal{G}_i}^{p+1}(w) \quad \text{for } i \geq j, \text{ with } 0 \leq i \leq p, 0 \leq j \leq p + 1, \tag{3.4.42}
\]

where \(w \in \text{Hom}_{sSet}(\Delta^1, \mathcal{G})_p\).

Source and target simplicial maps \(\sigma, \tau \in \text{Hom}_{sSet}(\text{Hom}_{sSet}(\Delta^1, \mathcal{G}), \mathcal{G})\) exist. The maps \(\sigma^p, \tau^p \in \text{Hom}_{sSet}(\text{Hom}_{sSet}(\Delta^1, \mathcal{G})_p, \mathcal{G}_p)\) describing \(\sigma, \tau\) at degree \(p\) are

\[
 \sigma^p(w) = f_{\mathcal{G}_0}^{p+1}(w_0), \tag{3.4.43}
\]

\[
 \tau^p(w) = f_{\mathcal{G}_{p+1}}^{p+1}(w_p) \tag{3.4.44}
\]

with \(w \in \text{Hom}_{sSet}(\Delta^1, \mathcal{G})_p\).
The above consideration extend essentially unchanged to simplicial objects in relevant categories such as $\text{MF}$ and $\text{grMF}$.

It turns out that, when $\mathcal{G}$ is a Lie quasi–groupoid, $\text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})$ is as well. However, when $\mathcal{G}$ is a $q$ Lie quasi–groupoid for some $q$, $\text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})$ is not in general a $q^1$ Lie quasi–groupoid for any finite $q^1$.

Let $\mathcal{G}$ be a $q$ Lie quasi–groupoid. Then, the exponential parametrization of the simplicial internal graded manifold hom manifold $\text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})$ associated with an $r$ Lie quasi–groupoid $\mathcal{K}$ we described earlier in this subsection can be extended to $\text{Hom}_{\text{sgrMf}}^*(\mathbb{N}[\mathbb{R}][1], \text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G}))$ upon replacing $\mathcal{K}$ with $\text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})$. Since $\text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})$ may not be a finite $q^1$ Lie quasi–groupoid, the parametrization is generally infinite dimensional. Let us spell this out in some detail.

By (3.4.23), $\text{Hom}_{\text{sgrMf}}^*(\mathbb{N}[\mathbb{R}][1], \text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G}))$ is parametrized by the total space of the 1–shift Lie $\text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})[1]$ of the graded vector bundle

$$\text{Lie Hom}_{\text{sSet}}(\Delta^1, \mathcal{G}) = \bigoplus_{p=1}^{\infty} \cap_{0 \leq i, j \leq p, i \neq j, j+1} \mathcal{G}_{p,j} \text{ker} f_{\mathcal{G}_1}^p \to \mathcal{G}_1. \quad (3.4.45)$$

In the above expression and the following, $\mathcal{G}_{p,j} = \mathcal{G}_{p-1} \circ \cdots \circ \mathcal{G}_1 : \mathcal{G}_1 \to \mathcal{G}_p$ with $\mathcal{G}_0 = \text{id}_{\mathcal{G}_1}$ by convention. $l_{p,j}$ is a notational shorthand for the index string $0, \ldots, 0, 1, \ldots, 1$ with 0, 1 occurring $j$ and $p - j$ times, respectively.

Let $h \in \text{Hom}_{\text{sgrMf}}^*(\mathbb{N}[\mathbb{R}][1], \text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G}))$ be a simplicial internal graded manifold morphism. Then, by our earlier analysis, $h^0$ is parametrized by a point $\beta \in \mathcal{G}_1$ and two vectors $\chi^0$, $\chi^1$ lying in the fibers $T[1]_{\mathcal{G}_1}^\beta \mathcal{G}_2$, $T[1]_{\mathcal{G}_0}^\beta \mathcal{G}_2$ of the vector bundles $\mathcal{G}_1^\beta \mathcal{G}_2$, $\mathcal{G}_0^\beta \mathcal{G}_2$, respectively, satisfying by (3.4.40)

$$f_{\mathcal{G}_1}^2 (\chi^1_1) = f_{\mathcal{G}_1}^2 (\chi^1_0). \quad (3.4.46)$$

Owing to (3.4.25), using (3.4.41), (3.4.42), we have further

$$f_{\mathcal{G}_0}^2 (\chi^1_1) = 0. \quad (3.4.47)$$

From (3.4.26), (3.4.27), using again (3.4.41), (3.4.42), we find that

$$h^0_0 (\theta_0) = \chi^1_0 (\beta) + \theta_0 f_{\mathcal{G}_2}^2 (\chi^1_0). \quad (3.4.48)$$

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is parametrized by \( \beta, \chi^0, \chi^1 \) and the degree 2 components \( \chi^2_0, \chi^2_1, \chi^2_2 \) of three vectors belonging to the fibers \( T[1]^2 \otimes_{\mathcal{G}_3} T[1]^2 \) of the vector bundles \( \mathcal{G}_3 \), respectively, satisfying on account of (3.4.41)

\[
f^3_{g|jk} (\beta)^2 \chi^2_{2-j-k} + f^3_{g|jk} (\beta)^2 (\chi^1_{1-k}) + d^2_{g|jk} (\beta)^2 (\chi^1_{1-j}) + d^2_{g|jk+1} (\beta)^2 (\chi^1_{1-j}) = 0
\]

with \((i, j, k) = (0, 0, 1), (0, 0, 1), (1, 0, 0), (2, 1, 1)\). By (3.4.28), on account of (3.4.41), (3.4.42), we have further

\[
f^3_{g|jk} (\beta)^2 \chi^2_{2-j-k} + f^3_{g|jk} (\beta)^2 (\chi^1_{1-k}) + d^2_{g|jk} (\beta)^2 (\chi^1_{1-j}) + d^2_{g|jk+1} (\beta)^2 (\chi^1_{1-j}) = 0
\]

with \((i, j, k) = (0, 0, 0), (0, 0, 1), (1, 0, 0), (2, 1, 1)\). By (3.4.29)−(3.4.32), employing (3.4.41), (3.4.42), we obtain then

\[
h^1_i (\theta_0, \theta_1) = \zeta^2_{g|jk} (\beta)^2 (\chi^2_i) + f^3_{g|jk} (\beta)^2 (\chi^1_{1-k}) + d^2_{g|jk} (\beta)^2 (\chi^1_{1-j})
\]

with \(i = 0, 1\). The components \( h^p \) of \( h \) with \( p \geq 2 \) can be obtained in the same way with an amount of computation increasing very rapidly with \( p \).

For any simplicial internal graded manifold morphism \( h \in \text{Hom}_{s\text{GgM}}^* (\Delta^1, \mathcal{G}) \), the source and target of \( h \) are the simplicial internal morphisms \( \sigma \circ h, \tau \circ h \in \text{Hom}_{s\text{GgM}}^* (\mathcal{N}[0, 1], \mathcal{G}) \), because the source and target maps \( \sigma, \tau \) are simplicial as recalled earlier. Their components \( \sigma^p \circ h^p, \tau^p \circ h^p \) should so be expressible by means of the exponential parametrization found earlier, eqs. (3.4.26), (3.4.29), (3.4.33) etc. This is indeed the case: there exists bundle maps \( \hat{\sigma}, \hat{\tau} : \text{Lie Hom}_{s\text{GgM}}^*(\Delta^1, \mathcal{G})[1] \to \text{Lie} \mathcal{G}[1] \), such that

\[
\sigma \circ \text{exp}_{\text{Hom}_{s\text{GgM}}^*(\Delta^1, \mathcal{G})} = \exp_{\mathcal{G}} \circ \hat{\sigma},
\]

(3.4.52)
\[ \tau \circ \text{Set} \exp_{\text{Hom}_{\text{Set}}(\Delta^1, \mathcal{G})} = \exp_{\mathcal{G}} \circ \tilde{\tau}. \] (3.4.53)

The source map \( \tilde{\sigma} \) is given by

\[ \tilde{\sigma}^* \gamma = f^1_{g0}(\beta), \] (3.4.54)

\[ \tilde{\sigma}^* \xi_0 = f^2_{g0*|g^1_0}(\chi^1_0), \] (3.4.55)

\[ \tilde{\sigma}^* \xi_1 = f^3_{g0*|g^2_1}(\chi^2_0) + f^3_{g0*|g^3_1}(\chi^1_0), \] (3.4.56)

e tc. Similarly, the target map \( \tilde{\tau} \) is given by

\[ \tilde{\tau}^* \gamma = f^1_{g1}(\beta), \] (3.4.57)

\[ \tilde{\tau}^* \xi_0 = f^2_{g1*|g^1_0}(\chi^1_1), \] (3.4.58)

\[ \tilde{\tau}^* \xi_1 = f^3_{g1*|g^2_0}(\chi^2_1) + f^3_{g1*|g^3_0}(\chi^1_1), \] (3.4.59)

e tc.

There is a subtle relationship of the maps \( \tilde{\sigma}, \tilde{\tau} \) and the Chevalley–Eilenberg differential \( Q_{\text{Lie}_{\mathcal{G}}} \) and this shows that simplicial homotopy secretly incarnates the symmetry of simplicial mapping. Specifically, there is a family of graded manifold morphisms \( p_t : \text{Lie}_{\mathcal{G}}[1] \to \text{Lie}_{\text{Hom}_{\text{Set}}(\Delta^1, \mathcal{G})}[1] \) depending on an odd parameter \( t \in \mathbb{R}[-1] \) (that is a mapping \( p : \text{Lie}_{\mathcal{G}}[1] \to \mathcal{T}[1] \text{Lie}_{\text{Hom}_{\text{Set}}(\Delta^1, \mathcal{G})}[1] \)) such that

\[ \tilde{\sigma} \circ p_t = \text{id}_{\text{Lie}_{\mathcal{G}}[1]}, \] (3.4.60)

\[ \tilde{\tau} \circ p_t = \text{id}_{\text{Lie}_{\mathcal{G}}[1]} + tQ_{\text{Lie}_{\mathcal{G}}}. \] (3.4.61)

We do not have a general proof of this fact, but we have been able to construct the map \( p_t \) up to simplicial degree 2,

\[ p_t^* \beta = \zeta^1_{g^1_0}(\gamma) + t\xi_0, \] (3.4.62)

\[ p_t^* \chi^1_i = d^1_{g1*|g^1_0}(\chi^1_0) - (-1)^i t\xi_1, \] (3.4.63)

\[ p_t^* \chi^2_i = d^2_{g1*|g^2_0}(\chi^1_0), \] (3.4.64)

\[ + d^2_{g1*|g^2_0}(\chi^1_0), d^1_{g1*|g^2_0}(\chi^1_0) + (-1)^i t\xi_2 \]

e tc. Using (3.4.25), (3.4.28), (3.4.30), (3.4.31), (3.4.32), (3.4.34)–(3.4.36), it is

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straightforward though lengthy to verify that the conditions 3.4.46, 3.4.47, 3.4.49, 3.4.50 as well as the basic relations 3.4.60, 3.4.61 are all identically satisfied.

3.5 Higher gauged sigma models

We now expound a formulation of higher gauged sigma model theory based on the abstract algebraic framework of sect. 2. Its basic data are therefore two differential graded commutative algebras $C_1$, $C_2$ and higher gauged sigma model fields are modeled as non differential morphisms from $C_2$ to $C_1$. Both $C_1$ and $C_2$ are algebras of functions on appropriate graded manifolds. While $C_2$ is the algebra of smooth functions on the 1–shifted $L_\infty$–algebroid encoding the higher gauge symmetry, $C_1$ is the algebra of internal smooth functions of 1–shifted tangent bundle of the relevant space–time manifold. The higher fields are in this way the pull–backs of internal graded manifold morphisms from the latter to the former. The incorporation of ghost degrees of freedom is so possible and a complete BRST formulation is reached.

$C_1$ is the graded commutative algebra $C^\infty(T[1]Z) \otimes G_R$ of functions on the 1–shifted tangent bundle $T[1]Z$ of an ordinary manifold $Z$ (cf. eg. 3.4), the source manifold, valued in the graded vector space $G_R$ defined by (2.5.7). The differential $Q_1$ of $C_1$ is the canonical homological vector field $d$ of $T[1]Z$, as is natural. Describe locally $T[1]Z$ by degree 0 base coordinates $z^\alpha$ and degree 1 fiber coordinates $\zeta^\alpha$. Then, a homogeneous degree $s$ element $f \in C^\infty(T[1]Z) \otimes G_R$ has an expansion of the form

$$f = \sum_{0 \leq \nu \leq n} \frac{1}{h!} f_{\alpha_1...\alpha_h}(z) \zeta^{\alpha_1} \cdots \zeta^{\alpha_h}$$

(3.5.1)

(cf. eq. 3.2.11), where $n = \dim Z$ and the local functions $f_{\alpha_1...\alpha_h}$ have degree $s - h$. Further, $d$ is given by given by

$$d = \zeta^{\alpha} \partial_{z^\alpha}$$

(3.5.2)

(cf. eq. 3.2.12). It is important to realize that the algebra $C^\infty(T[1]Z) \otimes G_R$ is in fact bigraded. The two gradings are the form grading, that is the polynomial
degree in the odd coordinates $\zeta^a$, and the **ghost grading**, that is the $G_\mathbb{R}$-degree. The grading of $C^\infty(T[1]Z) \otimes G_\mathbb{R}$ is just the total grading of the bigrading. So, while a degree $s$ element $f \in C^\infty(T[1]Z)$ can be identified with an $s$-form of $Z$, a degree $s$ element $f \in C^\infty(T[1]Z) \otimes G_\mathbb{R}$ can be viewed as a non homogeneous form–ghost whose total form plus ghost degree is $s$,

$$f = \sum_{0 \leq h \leq n} f^{(h,s-h)}$$

where $f^{(h,g)}$ has form–ghost bidegree $(h, g)$. Notice that the degree 1 vector field $d$ has form–ghost bidegree $(1, 0)$.

$C_2$ is the graded commutative algebra $C^\infty(E[1])$ of functions on the 1–shifted bundle $E[1]$ of an $L_\infty$–algebroid $E$ over an ordinary manifold $N$ (cf. subsect. 3.3). The differential $Q_2$ of $C_2$ is the homological vector field $Q_E$ on $E[1]$ associated with the $L_\infty$ structure of $E$ and is given by eq. (3.3.5) in terms of the local representations $\rho, \partial$ and $[-,\ldots,-]_\kappa$ of the anchor, boundary map and multiple argument brackets of $E$, respectively, when $E[1]$ is described locally by degree 0 base coordinates $x^i$ and degree 1 fiber coordinates $\xi^p$.

A higher gauge field is a graded commutative algebra morphism $\Phi : C_2 \to C_1$, its curvature is the associated defect $F_\Phi$ defined according to (2.1.12), and the Bianchi identity this obeys is the defect identity (2.1.13) as we anticipated in subsect. 2.1. Adopting a somewhat more suitable notation and terminology, we define a **BRST higher gauged sigma model field** as a graded commutative algebra morphisms $A : C^\infty(E[1]) \to C^\infty(T[1]Z) \otimes G_\mathbb{R}$. The **BRST higher gauged sigma model curvature** $F_A$ of $A$ is then,

$$F_A = dA - A Q_E.$$  \tag{3.5.4}

The Bianchi identity correspondingly is

$$dF_A + F_A Q_E = 0.$$  \tag{3.5.5}

The higher gauged sigma model field components are

$$\varphi^i = Ax^i.$$  \tag{3.5.6}
\[ A_p = A \xi_p. \] (3.5.7)

\( \varphi \) is the customary sigma model field describing the embedding of \( Z \) into \( N \). The \( A_p \) are the higher gauge fields \( \varphi \) couples to. The higher gauged sigma model curvature components are

\[ v^i = d \varphi^i - \rho^i(A_0), \] (3.5.8)

\[ F_p = dA_p + P_p(A_0, A_1, \ldots), \] (3.5.9)

where \( P_p(A_0, A_1, \ldots) \) is a polynomial in the gauge field components \( A_0, A_1, \ldots \) constructed using the \( L_x \)-algebroid boundary and brackets. Specifically,

\[ F_0 = dA_0 + \frac{1}{2}[A_0, A_0] - \partial A_1, \] (3.5.10)

\[ F_1 = dA_1 + [A_0, A_1] - \frac{1}{6}[A_0, A_0, A_0] - \partial A_2, \] (3.5.11)

\[ F_2 = dA_2 + [A_0, A_2] + \frac{1}{2}[A_1, A_1] - \frac{1}{2}[A_0, A_0, A_1] \]
\[ + \frac{1}{24}[A_0, A_0, A_0] - \partial A_3, \ldots. \] (3.5.12)

We remind that it is tacitly understood that in (3.5.8) \( \rho^i\) and in (3.5.9) and (3.5.10)-(3.5.12) the brackets \([-,-,-]\) are all formally evaluated at \( x^j = \varphi^j \).

The Bianchi identities obeyed by the sigma model field components are

\[ dv^i + v^i \partial_x \rho^j(A_0) + \rho^j(F_0) = 0, \] (3.5.13)

\[ dF_p + Q_p(A_0, A_1, \ldots; F_0, F_1, \ldots) - v^i \partial_x P_p(A_0, A_1, \ldots) = 0, \] (3.5.14)

where \( Q_p(A_0, A_1, \ldots; F_0, F_1, \ldots) \) is a polynomial in the gauge field and gauge curvature components \( A_0, A_1, \ldots \) and \( F_0, F_1, \ldots \) of degree 1 in the latter constructed again using the \( L_x \)-algebroid boundary and brackets. Explicitly,

\[ dF_0 + [A_0, F_0] + \partial F_1 - v^i \partial_x (\frac{1}{2}[A_0, A_0] - \partial A_1) = 0, \] (3.5.15)

\[ dF_1 + [A_0, F_1] - [F_0, A_1] + \frac{1}{2}[A_0, A_0, F_0] + \partial F_2 \]
\[ - v^i \partial_x ([A_0, A_1] - \frac{1}{6}[A_0, A_0, A_0] - \partial A_2) = 0, \] (3.5.16)

\[ dF_2 + [A_0, F_2] - [F_0, A_2] - [A_1, F_1] - [A_0, F_0, A_1] \] (3.5.17)
\[ + \frac{1}{2} [A_0, A_0, F_1] + \frac{1}{2} [A_0, A_0, A_0, F_0] + \partial F_3 - \varphi_i \partial_{\varphi_i} ([A_0, A_2] \\
+ \frac{1}{2} [A_1, A_1] - \frac{1}{2} [A_0, A_0, A_1] + \frac{1}{2} [A_0, A_0, A_0] - \partial A_3) = 0, \ldots \]

All the above relations can be made more explicit by expanding the fields in subcomponents of given form–ghost bidegree as in (3.5.3). For the higher gauged sigma model field components, this expansion has the form

\[ \varphi^i = \sum_{0 \leq h \leq n} \varphi^{(h,-h)i}, \quad (3.5.18) \]

\[ A_p = \sum_{0 \leq h \leq n} A_p^{(h,p+1-h)}. \quad (3.5.19) \]

For the corresponding higher curvature components, the expansion reads as

\[ \varphi^i = \sum_{0 \leq h \leq n} \varphi^{(h,-h)i}, \quad (3.5.20) \]

\[ F_p = \sum_{0 \leq h \leq n} F_p^{(h,p+2-h)}. \quad (3.5.21) \]

Substituting the (3.5.18), (3.5.19), (3.5.20), (3.5.21) into the (3.5.8), (3.5.9) or (3.5.10)–(3.5.12), (3.5.13), (3.5.14) or (3.5.15)–(3.5.17), we obtain rather explicit expressions of the curvature subcomponents and the Bianchi identities they obey in terms of the field subcomponents. For a given \( L_x \)-algebroid \( E \), the structure of these expressions depends explicitly on the dimension \( n \) of the base manifold \( Z \). However, if we truncate the expansions by setting all fields of negative ghost degree to 0, we are left with expressions of a universal form independent from \( n \). In particular, if we set all components with non zero ghost degree to zero, we recover the familiar expressions of the higher gauged sigma model curvatures and Bianchi identities.

**Example 3.10.** Semistrict higher gauge theory is a higher gauge theory for which the \( L_x \)-algebroid \( E \) is a Lie 2–algebra. It has been studied by different approaches in refs. [11, 26]. In semistrict higher gauge theory, neglecting all negative ghost degree contributions, the subcomponents of the higher gauge fields \( A_0, A_1 \) are

\[ A_0^{(0,1)} = \gamma, \quad A_0^{(1,0)} = \alpha, \quad (3.5.22) \]

\[ A_1^{(0,2)} = \Gamma, \quad A_1^{(1,1)} = C, \quad A_1^{(0,1)} = B. \quad (3.5.23) \]
The subcomponents of the corresponding higher gauge curvatures $F_0, F_1$ are

$$
F_0^{(0,2)} = \phi = \frac{1}{2}[\gamma, \gamma] - \partial \Gamma, \tag{3.5.24}
$$
$$
F_0^{(1,1)} = g = d\gamma + [a, \gamma] - \partial C, \tag{3.5.25}
$$
$$
F_0^{(2,0)} = f = da + \frac{1}{2}[a, a] - \partial B, \tag{3.5.26}
$$
$$
F_1^{(0,3)} = \Psi = [\gamma, \Gamma] - \frac{1}{6}[\gamma, \gamma, \gamma], \tag{3.5.27}
$$
$$
F_1^{(1,2)} = K = d\Gamma + [a, \Gamma] + [\gamma, C] - \frac{1}{2}[a, \gamma, \gamma], \tag{3.5.28}
$$
$$
F_1^{(2,1)} = H = dC + [a, C] + [\gamma, B] - \frac{1}{2}[a, a, \gamma], \tag{3.5.29}
$$
$$
F_1^{(3,0)} = G = dB + [a, B] - \frac{1}{2}[a, a],
$$

These obey the Bianchi identities

$$
d\phi + [a, \phi] - [g, \gamma] + \partial K = 0, \tag{3.5.26}
$$
$$
dg + [a, g] + [\gamma, f] + \partial H = 0, \tag{3.5.27}
$$
$$
df + [a, f] + \partial G = 0, \tag{3.5.28}
$$
$$
d\Psi + [a, \Psi] - [g, \Gamma] - [\phi, C] + [\gamma, K] + \frac{1}{2}[g, \gamma, \gamma] + [a, \gamma, \phi] = 0, \tag{3.5.29}
$$
$$
dK + [a, K] - [g, C] - [\phi, B] - [f, \Gamma]
$$
$$
+ [\gamma, H] + \frac{1}{2}[a, a, \phi] + [g, a, \gamma] + \frac{1}{2}[f, \gamma, \gamma] = 0, \tag{3.5.30}
$$
$$
dH + [a, H] - [g, B] - [f, C] + [\gamma, G] + \frac{1}{2}[a, a, g] + [f, a, \gamma] = 0. \tag{3.5.31}
$$

These identities were obtained by another route in [26, 27, 29].

For reasons explained in subsect. 3.1, the components $\phi^i, A_p$ of the BRST higher gauged sigma model field $A$ given by the expansions (3.5.18), (3.5.19) specify an internal graded manifold morphism $a : T[1]Z \to E[1]$. Since the vector coordinates of $T[1]Z$ are all odd, a pull–back operator $a^\# : C^\infty(E[1]) \to C^\infty(T[1]Z)$ of the standard differential geometric form such that

$$
a^\# x^i = \varphi^i, \tag{3.5.29}
$$
$$
a^\# \xi_p = A_p \tag{3.5.30}
$$
is defined. By eqs. (3.5.6), (3.5.7), recalling that $C^\infty(T[1]Z) \simeq C^\infty(T[1]Z) \otimes G_R$ (cf. eq. (3.1.5)), it is apparent that $a^\#$ equals precisely the BRST field $A$ one started with. Viceversa, an internal morphism $a : T[1]Z \to E[1]$ of components $\varphi^i$, $A_p$ specifies a BRST field $A$ such that

$$Ax^i = \varphi^i,$$  

$$A\xi_p = A_p.$$  

by setting $A = a^\#$. The manifold $M(C^\infty(E[1]), C^\infty(T[1]Z) \otimes G_R)$ of BRST higher gauged sigma model fields can so be identified with the internal hom manifold $\text{Hom}_{\text{grMf}}(T[1]Z, E[1])$ of $T[1]Z, E[1]$. In this sense, the present theory is a BRST extension of the BGKS theory of refs. [31–33].

The above construction applies in particular when $E$ is the $L_\infty$–algebroid $\text{Lie} \mathcal{G}$ of a Lie quasi–groupoid $\mathcal{G}$ (cf. subsect. 3.4). As long as we are concerned with higher gauge field and curvature components, as we have done so far, there is not much more we can say in this particular case. The relationship of $\text{Lie} \mathcal{G}$ to $\mathcal{G}$ is expected to be relevant in a formulation of a model of higher parallel transport and in the analysis of finite higher gauge symmetry. While the former lies beyond the scope of the present work, the latter will be examined in the next subsection.

### 3.6 Higher gauged sigma model field BRST variations

In this subsection, building upon the results of subsect. 2.3, we shall compute the canonical determination BRST variations of the higher gauged sigma model fields for a general $L_\infty$–algebroid. For the $L_\infty$–algebroid of a Lie quasi–groupoid, we provide evidence that the definition of higher gauge transformations through simplicial homotopy reproduces infinitesimally the BRST variations.

From the general relation (2.3.28), using (3.5.4), we find

$$s_QA = -F_A = -dA + AQ_E.$$  

(3.6.1)

By (3.5.8), (3.5.9), we have

$$s_Q\varphi^i = -d\varphi^i + \rho^i(A_0),$$  

(3.6.2)
The variations $s_Q A_p$ can be written down more explicitly using the expressions (3.5.10)–(3.5.12) of the higher gauged sigma model curvatures,

\[ s_Q A_0 = -dA_0 - \frac{1}{2} [A_0, A_0] + \partial A_1, \]

\[ s_Q A_1 = -dA_1 - [A_0, A_1] + \frac{1}{6} [A_0, A_0, A_0] + \partial A_2, \]

\[ s_Q A_2 = -dA_2 - [A_0, A_2] - \frac{1}{2} [A_1, A_1] + \frac{1}{2} [A_0, A_0, A_1] \]

\[ - \frac{1}{24} [A_0, A_0, A_0, A_0] + \partial A_3, \quad \ldots. \]

Substituting the expansions (3.5.18), (3.5.19), (3.5.20), (3.5.21) of the fields and curvatures in subcomponents of given form–ghost bidegree into the (3.6.2), (3.6.3) and (3.6.4)–(3.6.6), we obtain rather explicit expressions of these BRST variations. The non zero ghost degree components are necessary for the nilpotence of $s_Q$ (cf. subsect. 2.3).

**Example 3.11.** In semistrict higher gauge theory, introduced in eg. 3.10 neglecting all the negative ghost degree fields, the canonical determination BRST variations are given by

\[ s_Q \gamma = -\phi = -\frac{1}{2} [\gamma, \gamma] + \partial \Gamma, \]

\[ s_Q a = -g = -d\gamma - [a, \gamma] + \partial C, \]

\[ s_Q \Gamma = -\Psi = -[\gamma, \Gamma] + \frac{1}{6} [\gamma, \gamma, \gamma], \]

\[ s_Q C = -K = -d\Gamma - [a, \Gamma] - [\gamma, C] + \frac{1}{2} [a, \gamma, \gamma], \]

\[ s_Q B = -H = -dC - [a, C] - [\gamma, B] + \frac{1}{2} [a, a, \gamma]. \]

These expressions were also obtained in [26, 27, 29]. Since they are truncated to non negative ghost degree, nilpotence of $s_Q$ holds only if certain on–shell conditions are satisfied.

When $E$ is the $L_\infty$–algebroid $\text{Lie} \mathcal{G}$ of a Lie quasi–groupoid $\mathcal{G}$, we can exploit the simplicial description of $\text{Lie} \mathcal{G}$ worked out in subsect. 3.4. In particular,
we can write the canonical determination BRST variation \((3.6.1)\) in terms of the simplicial homotopy source and target maps \(\sigma, \tau\) using \((3.4.60), (3.4.61)\),

\[
s_Q A = -dA + AD_t \dot{\tau} \circ p_t,
\]

where the maps \(p_t : \text{Lie} \mathcal{G}[1] \to \text{Lie Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})[1], t \in \mathbb{R}[-1],\) were introduced in subsect. 3.4 and \(D_t = d/dt\). This shows rather explicitly the relation between infinitesimal higher gauge symmetry and simplicial homotopy hypothesized in ref. [12]. One may wonder whether the analysis of subsect. 3.4 can tell us something about finite higher gauge symmetry.

The exponential map \(\exp_{\mathcal{G}}\) of \(\mathcal{G}\) is a graded manifold morphism from \(\text{Lie} \mathcal{G}[1]\) to \(\text{Hom}^*_{\text{grMf}}(\mathbb{N}\mathbb{R}[-1], \mathcal{G})\). By virtue of this, a higher gauged sigma model field \(a : T[1]Z \to \text{Lie} \mathcal{G}[1]\), viewed here as an internal graded manifold morphism, can be equivalently encoded in the composition

\[
\exp_{\mathcal{G}} \circ a : T[1]Z \to \text{Hom}^*_{\text{grMf}}(\mathbb{N}\mathbb{R}[-1], \mathcal{G}).
\]

Substituting the Lie quasi-groupoid \(\mathcal{G}\) with the simplicial homotopy Lie quasi-groupoid \(\text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})[1]\), we can view a finite higher gauge transformation as a higher gauged sigma model field \(\gamma : T[1]Z \to \text{Lie Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})[1]\) encoded in the composition

\[
\exp_{\text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})} \circ \gamma : T[1]Z \to \text{Hom}^*_{\text{grMf}}(\mathbb{N}\mathbb{R}[-1], \text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})).
\]

As shown in subsect. 3.4, the source and target maps \(\sigma, \tau\) are simplicial manifold morphisms of \(\text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})\) into \(\mathcal{G}\). They thus induce internal graded manifold morphisms

\[
\omega \circ \text{Set} \circ \exp_{\text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})} \circ \gamma : T[1]Z \to \text{Hom}^*_{\text{grMf}}(\mathbb{N}\mathbb{R}[-1], \mathcal{G}), \quad \omega = \sigma, \tau,
\]

by simplicial composition. By relations \((3.4.52), (3.4.53)\), these can be cast as

\[
\omega \circ \text{Set} \circ \exp_{\text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{G})} \circ \gamma = \exp_{\mathcal{G}} \circ \dot{\omega} \circ \gamma.
\]

A comparison of \((3.6.10)\) and \((3.6.13)\) indicates that \(\gamma\) may represent a finite gauge transformation relating the higher gauge fields.
\[ a = \hat{\sigma} \circ \gamma, \quad a' = \hat{\tau} \circ \gamma. \quad (3.6.14) \]

We might substantiate to some extent this claim if we were able to find a one parameter family \( \gamma_t : T[1]Z \to \text{Lie Hom}_{\text{Set}}(\Delta^1, \mathcal{G})[1], \ t \in \mathbb{R}[-1], \) of fields such that \( \hat{\sigma} \circ \gamma_0 = a \) and that

\[ D_t \hat{\tau} \circ \gamma_t = s_Q a = -da + Q_{\text{Lie}} \circ a. \quad (3.6.15) \]

Indeed, \( (3.6.15) \) would precisely reproduce \( (3.6.1) \) with \( E = \text{Lie} \mathcal{G} \) upon switching to the graded algebra morphism \( A = a^\#: C^\infty(\text{Lie} \mathcal{G}[1]) \to C^\infty(T[1]Z) \) associated to \( a \). Relation \( (3.4.61) \) suggests that \( \gamma_t = p_t \circ a \) might do the job. This seems reasonable in the light of the relationship of simplicial homotopy to infinitesimal higher gauge symmetry observed earlier. However, it does not quite work because the differential term \( -da \) in \( (3.6.15) \) cannot be generated in this way. To remedy for this, we introduce a family of translations of \( T[1]Z \) in the form of one of internal graded manifold morphisms \( u_t : T[1]Z \to T[1]Z, \ t \in \mathbb{R}[-1], \) defined as

\[ u_t = \text{id}_{T[1]Z} + td. \quad (3.6.16) \]

By its means, we then set

\[ \gamma_t = p_t \circ a \circ u_{-t}. \quad (3.6.17) \]

It is now straightforward to verify that \( (3.6.15) \) holds true. Further investigation is required about this issue. Our analysis however already shows that simplicial homotopy is related to BRST symmetry in a deep way.

### 3.7 Some explicit constructions

All the main examples of higher gauge theories can be formulated within the abstract algebraic framework worked out in sect. 2 in which higher gauge fields and their curvatures are modeled as non differential morphisms of differential graded commutative algebras and their defects. In this final subsection, we revisit in detail a few of these along the lines of subsects. 2.6 and 2.7 to illustrate the formalism and test its range of applicability.
All the models considered below are instances of higher gauged sigma models and so are covered by the analysis of subsect. 3.5. Therefore, the differential graded commutative algebra $C_1$ is the algebra $C^\infty(T[1]Z)$ of internal graded functions of the 1–shifted tangent bundle $T[1]Z$ of an $n$–fold $Z$ and the differential $Q_1$ is the canonical homological vector field $d$ of $T[1]Z$ given by (3.5.2). Further, the differential graded commutative algebra $C_2$ is the algebra $C^\infty(E[1])$ of graded functions of the 1–shift $E[1]$ of an $L_\infty$ algebroid $E$ on a manifold $N$ and the differential $Q_2$ is the associated homological vector field $Q_E$ of $E[1]$ given by eq. (3.3.5). Finally, the manifold $M(C_2,C_1)$ of BRST higher gauged sigma model fields is the internal hom manifold $\text{Hom}_{\text{grMf}}(T[1]Z,E[1])$.

When the manifold $Z$ is closed, as we assume henceforth, the function algebra $C^\infty(T[1]Z)$ is characterized by the existence of the Berezin integration map

$$\int_{T[1]Z} \varrho : C^\infty(T[1]Z) \to G_\mathbb{R}. \quad (3.7.1)$$

Here, $\varrho$ denotes the Berezin integration measure locally given by $d^n z^n \zeta$. The Berezin map is linear, has degree $-n$, is non singular, that is

$$\int_{T[1]Z} \varrho uv = 0 \quad \text{for all } v \in C^\infty(T[1]Z), \Rightarrow u = 0 \quad (3.7.2)$$

for any $v \in C^\infty(T[1]Z)$, and satisfies the Stokes’ theorem, so that

$$\int_{T[1]Z} \varrho du = 0 \quad (3.7.3)$$

for $u \in C^\infty(T[1]Z)$. The Berezin map allows the construction of BV master actions as space–time integrals of suitable local Lagrangians.

We present now an example of a higher BF gauge theory following the axiomatic approach of subsect. 2.6.

**Example 3.12. Semistrict higher BF gauge theory.** Semistrict higher BF gauge theory is the simplest example of higher BF gauge theory.

The source manifold $Z$ is a generic closed $n$–fold. The target $L_\infty$–algebroid $E$ is a cyclic Lie 2–algebra. Thus, the base manifold $N$ of $E$ is a point and $E$ is just a 2 term graded vector space $v = v_0 \oplus v_1[1]$ equipped with a boundary
map \( \partial : \mathfrak{v}_1 \to \mathfrak{v}_0 \), a set of 2– and 3–argument brackets \([-,-] : \mathfrak{v}_0 \wedge \mathfrak{v}_0 \to \mathfrak{v}_0 \), \([-,-] : \mathfrak{v}_0 \otimes \mathfrak{v}_1 \to \mathfrak{v}_1 \) and \([-,-,-] : \mathfrak{v}_0 \wedge \mathfrak{v}_0 \wedge \mathfrak{v}_0 \to \mathfrak{v}_1 \) with certain properties and a non singular bilinear pairing \((-,-) : \mathfrak{v}_0 \times \mathfrak{v}_1 \to \mathbb{R} \) satisfying
\[
(\partial x_1, y_1) - (\partial y_1, x_1) = 0, \\
([u_0, x_0], x_1) + (x_0, [u_0, x_1]) = 0, \\
(x_0, [u_0, v_0, y_0]) + (y_0, [u_0, v_0, x_0]) = 0
\]
for \( x_0, y_0, u_0, v_0 \in \mathfrak{v}_0, x_1, y_1 \in \mathfrak{v}_1 \). Note that here \( \mathfrak{v}_0, \mathfrak{v}_1 \) are conventionally assumed to both have degree 0. Recall also that the non singularity of \((-,-)\) implies that \( \dim \mathfrak{v}_0 = \dim \mathfrak{v}_1 = \dim \mathfrak{v}/2 = r/2 \).

To construct semistrict higher BF gauge theory, it is necessary to study the geometry of higher BF gauge field manifold \( \text{Hom}_\text{grMf}(T[1]Z, \mathfrak{v}[1]) \). A point \( a \in \text{Hom}_\text{grMf}(T[1]Z, \mathfrak{v}[1]) \) is specified by its components
\[
A_p = a^\# \xi_p, 
\]
with \( p = 0, 1 \). Albeit the \( A_p \) belong to the spaces \( C^\infty(T[1]Z)_{p+1} \otimes \mathbb{R}^{r/2} \), it is possible to regard them as elements of the spaces \( C^\infty(T[1]Z)_{p+1} \otimes \mathfrak{v}_p \). A can therefore be identified with a pair \( A_p \in C^\infty(T[1]Z)_{p+1} \otimes \mathfrak{v}_p \). A tangent vector \( \dot{a} \in T_a[k] \text{Hom}_\text{grMf}(T[1]Z, \mathfrak{v}[1]) \) is so a pair \( \dot{A}_p \in C^\infty(T[1]Z)_{p+1+k} \otimes \mathfrak{v}_p \) and a cotangent vector \( \dot{a}^* \in T^* a[k] \text{Hom}_\text{grMf}(T[1]Z, \mathfrak{v}[1]) \) a pair \( \dot{A}_p^* \in C^\infty(T[1]Z)_{n-p-1+t} \otimes \mathfrak{v}_{1-p} \). The canonical cotangent–tangent pairing of \( \dot{a}, \dot{a}^* \) is
\[
\langle \dot{a}^*, \dot{a} \rangle = \int_{T[1]Z} \theta \left[ (-1)^{(n+1+k)(n+1+t)}(\dot{A}_0, \dot{A}_0^*) + (-1)^{n(n+t)}(\dot{A}_1, \dot{A}_1^*) \right].
\]

As we explained in subsect. 2.6 the field manifold of higher BF gauge theory is the \(-1\)-shifted cotangent bundle \( T^*[1-1] \text{Hom}_\text{grMf}(T[1]Z, \mathfrak{v}[1]) \), which we describe through its base and fiber coordinates \( A_p \in C^\infty(T[1]Z)_{p+1} \otimes \mathfrak{v}_p \) and \( B_p \in C^\infty(T[1]Z)_{n-p-2} \otimes \mathfrak{v}_{1-p} \). The BV symplectic form \( \Omega_{BV} \) is given by (2.6.1) in terms of the cotangent–tangent pairing. By (3.7.8), \( \Omega_{BV} \) reads explicitly as
\[
\Omega_{BV} = \int_{T[1]Z} \theta \left[ -(\delta A_0, \delta B_0) + (\delta B_1, \delta A_1) \right].
\]
The ensuing BV antibrackets can be obtained just by a straightforward application of (2.6.2) and can be cast compactly as

\[
\left( \int_{T[1]Z} \Theta(A_0, U_0^*), \int_{T[1]Z} \Theta(V_0, B_0) \right)_{\text{BV}} = \int_{T[1]Z} \Theta(V_0, U_0^*), \quad (3.7.10)
\]

\[
\left( \int_{T[1]Z} \Theta(U_1^*, A_1), \int_{T[1]Z} \Theta(B_1, V_1) \right)_{\text{BV}} = \int_{T[1]Z} \Theta(U_1^*, V_1) \quad (3.7.11)
\]

with \( V_p \in \mathcal{C}^\infty(T[1]Z)_{p+1} \otimes \mathfrak{v}_{p} \) and \( U_p^* \in \mathcal{C}^\infty(T[1]Z)_{n-p-1} \otimes \mathfrak{v}_{1-p} \). The BV master action \( S_{BV} \) is given eq. (2.6.8) in terms of cotangent–tangent pairing and the curvature map. In the present case, the former is given by (3.7.8), the latter is specified by the pair \( F_p \in \mathcal{C}^\infty(T[1]Z)_{p+2} \otimes \mathfrak{v}_{p} \) given by (3.5.10), (3.5.11). We find in this way

\[
S_{BV} = -\int_{T[1]Z} \Theta \left( (-1)^n \left( dA_0 + \frac{1}{2}[A_0, A_0] - \partial A_1, B_0 \right) \right. \\
\left. + (B_1, dA_1 + [A_0, A_1] - \frac{1}{6}[A_0, A_0, A_0]) \right). \quad (3.7.12)
\]

The BV variations of the \( A_p \) and \( B_p \) are given by the expressions (2.6.3), (2.6.4) with the canonical determination BRST variation \( s_Q \) identified with the BV variation \( \delta_{BV} \). After a straightforward computation, we find

\[
\delta_{BV} A_0 = -dA_0 - \frac{1}{2}[A_0, A_0] + \partial A_1, \quad (3.7.13)
\]

\[
\delta_{BV} A_1 = -dA_1 - [A_0, A_1] + \frac{1}{6}[A_0, A_0, A_0], \quad (3.7.14)
\]

\[
\delta_{BV} B_0 = dB_0 + [A_0, B_0] + [B_1, A_1] - \frac{1}{2}[A_0, A_0, B_1], \quad (3.7.15)
\]

\[
\delta_{BV} B_1 = -dB_1 - [A_0, A_1] + \partial B_0. \quad (3.7.16)
\]

Of course, higher BF gauge theories based on general Lie \( m+1 \)-algebras can be formulated by means of an obvious generalization of the above construction.

There exists a broad variety of higher Chern–Simons like gauged sigma models. They can all be elegantly formulated along the lines of subsect. 2.7 as we shall show momentarily by illustrating a couple of examples.

In all these models, the map \( \mu : C_1 \rightarrow G_{\mathbb{R}} \) is just the Berezin integration map (3.7.1). This is indeed linear, has degree \(-n\) and satisfies by virtue of the properties (3.7.2), (3.7.3) the conditions (2.7.1) and (2.7.2). The degree \(-1\) Ger-
stenhaber brackets \((-,-)_2\) on \(C_2\) and the Hamiltonian \(S_2\) of the differential \(Q_2\) appear respectively as Gerstenhaber brackets \((-,-)\) on \(C^\infty(E[1])\) and a Hamiltonian \(S\) for the homological vector field \(Q_E\) of \(E[1]\). Likewise, the kinetic and boundary maps \(K_1 : M(C_2, C_1) \to C_1\) and \(B_1 : TM(C_2, C_1) \to C_1\) with the properties \((2.7.4)\), \((2.7.5)\) appear as maps \(K : \text{Hom}_{grMf}(T[1]Z, E[1]) \to C^\infty(T[1]Z)\) and \(B : T\text{Hom}_{grMf}(T[1]Z, E[1]) \to C^\infty(T[1]Z)\) of the same nature. The definition of all these data must be provided on a case by case basis.

**Example 3.13.** *The Poisson sigma model.* The Poisson sigma model was introduced long ago in refs. [55, 56]. Its defining data are the following.

The source manifold \(Z\) is a closed 2-fold. The target \(L_8\)-algebroid \(E\) is the cotangent Lie algebroid \(T\breve{\mathcal{N}}\) of a Poisson manifold \(N\) (cf. eg. 3.5). The homological vector field \(Q_E\) is taken to be \(-\delta\), where \(\delta\) is the Poisson–Lichnerowicz homological vector field of the shifted algebroid \(T\breve{\mathcal{N}}\) given by \((3.2.15)\).

The degree \(-1\) Gerstenhaber brackets \((-,-)\) on \(C^\infty(T^*\{1\}N)\) are the odd Poisson brackets \{\(-,-\)\} associated with the canonical degree 1 symplectic 2-form \(\omega\) of eq. \((3.2.14)\). With respect to these, \(Q_E\) is Hamiltonian with Hamiltonian \(-S\), where \(S\) is the function \((3.2.16)\) satisfying the master equation \((3.2.17)\), as follows from the analogous properties enjoyed by \(\delta\).

A higher gauged sigma model field \(a \in \text{Hom}_{grMf}(T[1]Z, T\breve{\mathcal{N}})^{[1]}\) is fully specified by the sigma model fields

\[
\varphi^i = a^\# x^i, \quad (3.7.17)
\]
\[
A_{0i} = a^\# \xi_i \quad (3.7.18)
\]

associated with the base and fiber coordinates \(x^i, \xi_i\) of \(T^*\{1\}N\). Their BV antibrackets defined according to \((2.7.6)\) can be cast as

\[
\left( \int_{T^*[1]} \partial u \varphi^i, \int_{T^*[1]} \partial v A_{0j} \right)_{BV} = \delta^i_j \int_{T^*[1]} \partial u v \quad (3.7.19)
\]

for any \(u \in C^\infty(T[1]Z)\), \(v \in C^\infty(T[1]Z)\) as follows readily from the relation \(\{x^i, \xi_j\} = \delta^i_j\).

In terms of \(\varphi^i\) and \(A_{0i}\), the kinetic map \(K\) is given by

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The boundary map $B$ reads as

$$B(\varphi, A_0)(\dot{\varphi}, \dot{A}_0) = \dot{\varphi}^i A_{0i}. \quad (3.7.21)$$

The fulfilment of the conditions (2.7.4) and (2.7.5) is straightforwardly checked.

We can now write down the model’s BV master action $S_{BV}$ using expression (2.7.10). We obtain

$$S_{BV} = - \int_{\mathcal{T}[\varphi]} \theta \left[ A_0 d\varphi^i + \frac{1}{2} P^{ij}(\varphi) A_{0i} A_{0j} \right]. \quad (3.7.22)$$

We have recovered in this way up to an overall sign the well–known expression of the Poisson sigma model BV action first obtained in [57].

Recalling that the canonical determination BRST variation $s_Q$ equals the BV variation $\delta_{BV}$, we can readily compute the BV variations of $\varphi^i$ and $A_{0i}$ using (3.6.2) and (3.6.4),

$$\delta_{BV} \varphi^i = - d\varphi^i - P^{ij}(\varphi) A_{0j}, \quad (3.7.23)$$

$$\delta_{BV} A_{0i} = - dA_{0i} - \frac{1}{2} \partial_{x_1} P^{jk}(\varphi) A_{0j} A_{0k}. \quad (3.7.24)$$

They coincide with the BV variations of the Poisson sigma model fields computed by other means in [57].

**Example 3.14.** *The Courant sigma model.* The Courant sigma model was first laid forward in ref. [58]. The data defining it are the following.

The source manifold $\mathcal{Z}$ is a closed 3–fold. The target $L_\infty$–algebroid $E$ is is the $L_\infty$–algebroid $L$ associated with a Courant algebroid $V$ through the construction of eq. (3.9). The homological vector field $Q_E$ is the homological vector field $Q_L$ of the shifted algebroid $L[1]$ given by (3.3.27). The degree $-1$ Gerstenhaber brackets $(-,-)$ on $\mathcal{C}^\infty(E[1])$ are the Poisson brackets $\{-,-\}$ associated with the degree 2 symplectic 2–form $\omega$ of eq. (3.3.26). With respect to these, $Q_E$ is Hamiltonian with Hamiltonian $S$, where $S$ is the function given in (3.3.28) satisfying the master equation (3.3.29), reflecting the analogous properties enjoyed by $Q_L$.
A higher gauged sigma model field \( a \in \text{Hom}_{\text{GrMf}}(T[1]Z, E[1]) \) is fully specified by the sigma model fields

\[
\varphi^i = a^# x^i, \quad (3.7.25) \\
A_0^a = a^# \xi^a, \quad (3.7.26) \\
A_1^i = a^# p_i \quad (3.7.27)
\]
corresponding the base and fiber coordinates \( x^i, p_i \xi^a \) of \( E[1]N \). We are abusing notation a bit here, since \( p_i \) is an affine rather than a vector coordinate. \( p_i \) can be turned into a vector coordinate by covariantizing it by means of a metric connection for \( V \). At the end, nothing can depend on the choice of the connection and so it does not make any difference working directly with \( p_i \). With this granted, the BV antibrackets defined according to (2.7.6) can be cast as

\[
\hat{\mathcal{z}}^T r_1 s \mathcal{Z} \hat{u} \varphi^i \quad \hat{\mathcal{z}}^T r_1 s \mathcal{Z} \hat{v} A_0^a \quad \hat{\mathcal{z}}^T r_1 s \mathcal{Z} \hat{w} A_1^i \quad \hat{\mathcal{z}}^T r_1 s \mathcal{Z} \hat{w} A_0^a \quad \hat{\mathcal{z}}^T r_1 s \mathcal{Z} \hat{v} A_1^i 
\]

\[
\delta^i \rho (x^i, p_i) \quad \delta^i \rho (x^i, A_0^a) \quad \delta^i \rho (x^i, A_1^i) \quad \delta^i \rho (x^i, A_0^a) A_0^b \quad \delta^i \rho (x^i, A_1^i) A_0^b 
\]

\[ (3.7.28) \]

\[ (3.7.29) \]

with \( u \in C^\infty(T[1]Z)_3, v \in C^\infty(T[1]Z)_0, w, z \in C^\infty(T[1]Z)_2, \) using that \( \{ x^i, p_j \} = \delta^i \) and \( \{ \xi^a, \xi^b \} = g^{ab} \).

The kinetic map \( K \) is now given by

\[
K(\varphi, A_0, A_1) = A_{1i} d\varphi^i - \frac{1}{2} A_0^a g_{ab} A_0^b 
\]

\[ (3.7.30) \]
in terms of sigma model fields. The boundary map \( B \) reads

\[
B_1(\varphi, A_0, A_1)(\dot{\varphi}, \dot{A}_0, \dot{A}_1) = \dot{\varphi}^i A_{1i} - \frac{1}{2} \dot{A}_0^a g_{ab} A_0^b. 
\]

\[ (3.7.31) \]

It is straightforward to check that (2.7.4) and (2.7.5) are fulfilled.

We can write the model’s BV action using expression (2.7.10)

\[
S_{\text{BV}} = \int_{T[1]Z} \theta \left[ A_{1i} d\varphi^i - \frac{1}{2} A_0^a g_{ab} A_0^b \right. \\
- \rho^i (\varphi) A_0^a A_{1i} + \frac{1}{6} f_{abc} (\varphi) A_0^a A_0^b A_0^c \]

\[ (3.7.32) \]

Up to sign conventions, this is the expression found in ref. [58].
Identifying again the canonical determination BRST variation $s_Q$ with the BV variation $\delta_{BV}$, we can compute the BV variations of $\varphi^i$, $A_0^a$ and $A_0^i$ using (3.6.2) and (3.6.4),

\[
\delta_{BV}\varphi^i = -d\varphi^i + \rho^i_a(\varphi)A_0^a, \tag{3.7.33}
\]
\[
\delta_{BV}A_0^a = -dA_0^a + g^{ad}(-\rho^d(\varphi)A_1 + \frac{1}{2}f_{abc}(\varphi)A_0^bA_0^c), \tag{3.7.34}
\]
\[
\delta_{BV}A_1^i = -dA_1 - \partial_{x^i}^a(\varphi)A_0^aA_1 + \frac{1}{6}\partial_{x^i}f_{abc}(\varphi)A_0^aA_0^bA_0^c. \tag{3.7.35}
\]

Other examples include the ordinary Chern–Simons gauge theory and its semistrict higher counterpart.
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