HOMEOTOPY GROUPS OF ROOTED TREE LIKE NON-SINGULAR
FOLIATIONS ON THE PLANE

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Abstract. Let $F$ be a non-singular foliation on the plane with all leaves being closed subsets, $H^+(F)$ be the group of homeomorphisms of the plane which maps leaves onto leaves endowed with compact open topology, and $H^+_0(F)$ be the identity path component of $H^+(F)$. The quotient $\pi_0 H^+(F) = H^+(F)/H^+_0(F)$ is an analogue of a mapping class group for foliated homeomorphisms. We will describe the algebraic structure of $\pi_0 H^+(F)$ under an assumption that the corresponding space of leaves of $F$ has a structure similar to a rooted tree of finite diameter.

1. Introduction

Non-singular foliations on the plane were studied by W. Kaplan \cite{Kaplan1, Kaplan2} and H. Whitney \cite{Whitney} in the 40–50 years of the XX century. In particular, W. Kaplan in \cite{Kaplan2} has generalized a theorem of E. Kamke and proved that every non-singular foliation $F$ on the plane admits a continuous function $f: \mathbb{R}^2 \to \mathbb{R}$ such that
1) the leaves of $f$ are connected components of level sets $f^{-1}(c), c \in \mathbb{R}$;
2) near each $z \in \mathbb{R}^2$ there are local coordinates $(u, v)$ in which $f(u, v) = u + f(z)$.
This result was further extended to foliations with singularities by W. Boothby \cite{Boothby}, and J. Jenkins and M. Morse \cite{JenkinsMorse}. Topological classification of different kinds of functions on surfaces was investigated in many papers, see e.g. A. Fomenko and A. Bolsinov \cite{FomenkoBolsinov}, A. Oshemkov \cite{Oshemkov}, V. Sharko \cite{SharkoSharko}, \cite{SharkoSoroka}, O. Prishlyak \cite{Prishlyak}, \cite{PrishlyakSharko}, E. Polulyakh and I. Yurchuk \cite{PolulyakhYurchuk}, E. Polulyakh \cite{Polulyakh}, V. Sharko and Yu. Soroka \cite{SharkoSoroka}.

W. Kaplan in \cite{Kaplan1, Kaplan2} has also mentioned that a non-singular foliation on the plane is glued of countably many strips along open boundary intervals and such that each strip has a foliation by parallel lines. In a recent paper S. Maksymenko and E. Polulyah \cite{MaksymenkoPolulyah} studied non-singular foliations $F$ on arbitrary non-compact surfaces $\Sigma$ glued from strips in a similar way. They proved contractibility of the connected components of groups $H(F)$ of homeomorphisms of $\Sigma$ mapping leaves onto leaves. Thus the homotopy type of $H(F)$ is determined by the quotient group $\pi_0 H(F) = H(F)/H_0(F)$ of path components of $H(F)$, where $H_0(F)$ is the identity path component of $H(F)$.

In the present paper we compute the groups $\pi_0 H(F)$ for a special class of non-singular foliations on the plane whose spaces of leaves have the structure similar to rooted trees of finite diameter, see Theorem \ref{main_theorem}.

2. Striped surfaces

Let $\Sigma_i$ be a surface with a foliation $F_i, i = 1, 2$. Then a homeomorphism $h : \Sigma_1 \to \Sigma_2$ will be called foliated if it maps leaves of $F_1$ onto leaves of $F_2$.

Definition 2.1. A subset $S \subset \mathbb{R}^2$ will be called a model strip if the following two conditions hold:

2010 Mathematics Subject Classification. Primary 57S05; Secondary 57R30, 55Q05.

Key words and phrases. Non-singular foliations, homeotopy groups.
1) $\mathbb{R} \times (-1, 1) \subseteq S \subseteq \mathbb{R} \times [-1, 1]$;
2) $S \cap \mathbb{R} \times \{-1, 1\}$ is a union of open mutually disjoint finite intervals.

Put
$$\partial_- S = S \cap (\mathbb{R} \times \{-1\}), \quad \partial_+ S = S \cap (\mathbb{R} \times \{1\}), \quad \partial S = \partial_- S \cup \partial_+ S.$$ 

Notice that every model strip has an oriented foliation consisting of horizontal arcs $\mathbb{R} \times t, t \in (-1, 1)$, and connected components of $\partial S$.

Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of model strips, and
$$X = \bigcup_{\lambda \in \Lambda} \partial_- S_\lambda, \quad Y = \bigcup_{\lambda \in \Lambda} \partial_+ S_\lambda.$$ 

By Definition 2.1 $X$ and $Y$ are disjoint unions of open intervals, therefore one can also write
$$X = \bigcup_{\alpha \in A} X_\alpha, \quad Y = \bigcup_{\beta \in B} Y_\beta,$$
where $X_\alpha$ and $Y_\beta$ are open boundary intervals of those models strips and $A$ and $B$ are some index sets.

We will now glue model strips $S_\lambda$ by identifying some of the intervals of $X_\alpha$ with some of the intervals of $Y_\beta$. In order to make this let us fix any set of indexes $\Lambda$ and two injective maps $p : C \rightarrow A$ and $q : C \rightarrow B$. Notice that for each $\gamma \in C$ there exists a unique preserving orientation affine homeomorphism $\varphi_\gamma : X_{p(\gamma)} \rightarrow Y_{q(\gamma)}$. Then the quotient space
$$\Sigma := \bigsqcup_{\lambda \in \Lambda} S_\lambda / \{X_{p(\gamma)} \sim Y_{q(\gamma)}\}$$
will be called a striped surface.

Remark 2.2. A unique preserving orientation affine homeomorphism $\phi : (a, b) \rightarrow (c, d)$ is given by $\phi(t) = \frac{d-a}{b-a} (t-a)$.

Remark 2.3. In [8] a wider class of striped surfaces is considered: it is also allowed to identify arbitrary connected components of $\bigsqcup_{\lambda \in \Lambda} \partial S_\lambda$ and the gluing affine homeomorphisms may reverse orientation.

Let also $p : \bigsqcup_{\lambda \in \Lambda} S_\lambda \rightarrow \Sigma$ be the quotient map and $p_\lambda : S_\lambda \rightarrow \Sigma$ be the restriction of $p$ to the model strip $S_\lambda$. Then the pair $(S_\lambda, p_\lambda)$ will be called a chart for the strip $S_\lambda$.

Since the homeomorphism $\varphi_\gamma$ identifies leaves of such foliations, we see that every striped surface has the foliation $F$ consisting of foliations on model strips. This foliation will be called canonical.

Moreover, each leaf of the foliation on the model strip is oriented and the gluing preserves orientation. Therefore all leaves of $F$ are oriented.

**Special leaves.** Let $U \subset \Sigma$ be a subset. Then the union of all leaves of $F$ intersecting $U$ is called the saturation of $U$ with respect to $F$ and denoted by $\text{Sat}(U)$.

A leaf $\omega$ of $F$ will be called special if
$$\omega \neq \bigcap_{N(\omega)} \text{Sat}(N(\omega)),$$
where $N(\omega)$ runs over all open neighborhoods of $\omega$.

For instance each leaf $\omega$ belonging to the interior of a strip is non-special. Moreover, suppose $\omega = X_{p(\gamma)} \sim Y_{q(\gamma)}$ is a leaf such that $\partial_- S_\lambda = X_{p(\gamma)}$ and $\partial_+ S_\gamma = Y_{q(\gamma)}$, see Figure 2.1(a). Then the topological structure of the foliation $F$ near $\omega$ is “similar” to the structure of $F$ near “internal” leaves of strips and such a leaf is non-special as well, see [8] Lemma 3.2].
It also follows from that lemma that \( \omega \) is special if and only if one of the following two conditions hold, see Figure 2.1(b):

1) \( \omega \) is the image of gluing of leaves \( X_{p(\gamma)} \) and \( Y_{q(\gamma)} \) such that either \( X_{p(\gamma)} \subseteq \partial_- S_\lambda \) or \( Y_{q(\gamma)} \subseteq \partial_+ S_{\lambda'} \) for some \( \gamma \in C \), \( \lambda, \lambda' \in \Lambda \);

2) \( \omega \subseteq \partial_- S_\lambda \) or \( \omega \subseteq \partial_+ S_{\lambda} \) for some \( \lambda \in \Lambda \).

**Reduced striped surfaces.** A striped surface \( \Sigma \) will be called reduced whenever a leaf \( \omega \) is special if and only if one of the following conditions holds:

1) \( \omega \) is an image of gluing of some leaves \( X_{p(\gamma)} \sim Y_{q(\gamma)} \) for some \( \gamma \in C \);

2) \( \omega \subseteq \partial_- S_\lambda \) or \( \omega \subseteq \partial_+ S_{\lambda} \) for some \( \lambda \in \Lambda \).

Let \( S \) be a model strip such that \( \partial_- S = (0, 1) \times -1 \) and \( \partial_+ S = (0, 1) \times 1 \). Let also \( \phi : \partial_- S \to \partial_+ S \) be a homeomorphism defined by \( \phi(t, -1) = (t, 1), t \in (0, 1) \), and \( C = S/\phi \) be the quotient space obtained by identifying each \( x \in \partial_- S \) with \( \phi(x) \in \partial_+ S \).

Then \( C \) is a striped surface homeomorphic with a cylinder, and its canonical foliation has no special leaves.

It follows from [8, Theorem 3.7] that every striped surface (in the sense of (2.1), see Remark 2.3) is foliated homeomorphic either to \( C \) or to a reduced surface.

**Graph of a striped surface.** For a reduced striped surface \( \Sigma \) not foliated homeomorphic with \( C \) define an oriented graph \( \Gamma(\Sigma) \) whose vertexes are strips and whose edges are special leaves. More precisely: if \( \omega = X_{p(\gamma)} \sim Y_{q(\gamma)} \) is a special leaf of \( F \), \( X_{p(\gamma)} \subset \partial_- S_{\lambda_0} \), and \( Y_{q(\gamma)} \subset \partial_+ S_{\lambda_1} \), then we assume that \( \omega \) is an edge between vertices \( S_{\lambda_0} \) and \( S_{\lambda_1} \) oriented from \( S_{\lambda_1} \) to \( S_{\lambda_0} \).

If \( \lambda_0 = \lambda_1 \), then \( \omega \) correspond to a loop in \( \Gamma(\Sigma) \) at \( S_{\lambda_0} = S_{\lambda_1} \) being a vertex of \( \Gamma(\Sigma) \).

Since a model strip may have infinitely many boundary components, we see that a graph \( \Gamma(\Sigma) \) can be not locally finite. On the other hand, it can have a finite diameter \( \text{diam}(\Gamma(\Sigma)) \), being the minimal non-negative integer \( d \) such that every two vertices \( v_1 \) and \( v_2 \) are connected in \( \Gamma(\Sigma) \) by a path consisting at most \( d \) edges.

**Admissible striped surfaces.** Recall that a family \( \mathcal{V} = \{V_i\}_{i \in \Lambda} \) of subsets in a topological space \( X \) is called locally finite whenever for each \( x \in X \) there exists an open neighborhood intersecting only finitely many elements from \( \mathcal{V} \).

It is well known and is easy to see that a union of a locally finite family of closed subsets is closed, e.g. [7, Chapter 1, § 5.VIII].

**Definition 2.4.** A model strip \( S \) will be called admissible if the closures of intervals in \( \partial_- S \) and \( \partial_+ S \) are mutually disjoint and constitute a locally finite family in \( \mathbb{R}^2 \).

**Example 2.5.** A model strip with

\[
\partial_+ S = \bigcup_{n \in \mathbb{Z}, \{-1, 0\}} \left( \frac{1}{n+1}, \frac{1}{n} \right) \times 1
\]

is not admissible, since condition 2) of Definition 2.1 fails.
It will be convenient to use the following notation:

\[ [0] = \emptyset, \quad [n] = \{1, 2, \ldots, n\}, \quad -N = \{-1, -2, \ldots\}. \]

Let also \( J_i = (i, i + 0.5), \) \( i \in \mathbb{Z}, \) and for a subset \( \Delta \subset \mathbb{Z} \) denote

\[ A_\Delta = \bigcup_{i \in \Delta} J_i. \]

In particular, consider the following collections of mutually disjoint open intervals:

\[
A_{[n]} = \bigcup_{i=1}^{n} (i, i + 0.5), \quad n = 0, 1, \ldots, \quad A_N = \bigcup_{i \in \mathbb{N}} (i, i + 0.5), \\
A_{-N} = \bigcup_{-i \in \mathbb{N}} (i, i + 0.5), \quad A_Z = \bigcup_{i \in \mathbb{Z}} (i, i + 0.5),
\]

which will be called standard. The following easy lemma is left for the reader.

**Lemma 2.6.** Let \( S \) be an admissible model strip. Then there exists a homeomorphism \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) preserving each line \( \mathbb{R} \times t, \) \( t \in (-1, 1), \) with its orientation, and such that \( h(S) \) is a model strip with \( \partial_-h(S) = A_\alpha \times \{-1\} \) and \( \partial_+h(S) = A_\beta \times \{1\}, \) where \( A_\alpha \) and \( A_\beta \) are standard collections of intervals, i.e. \( \alpha, \beta \in \{[0], [1], \ldots, N, -N, \mathbb{Z}\}; \) see Figure 2.2. Moreover, \( \alpha \) and \( \beta \) do not depend on a particular choice of such \( h. \)

Thus for an admissible model strip \( S \) its foliated topological type is determined by the ordinal type of collections of boundary intervals in \( \partial_-S \) and \( \partial_+S. \)

### 3. Wreath products

Let \( H \) and \( S \) be two groups. Denote by \( Map(H, S) \) the group of all maps (not necessarily homomorphisms) \( \varphi : H \to S \) with respect to the point-wise multiplication. Then the group \( H \) acts on \( Map(H, S) \) by the following rule: the result of the action of \( \varphi \in Map(H, S) \) on \( h \in H \) is the composition map:

\[ \varphi \circ h : H \to H \to S. \]

The semidirect product \( Map(H, S) \ltimes H \) corresponding to this action will be denoted by \( S \ltimes H \) and called the wreath product of \( S \) and \( H. \) Thus

\[
S \ltimes H = Map(H, S) \ltimes H
\]

is the Cartesian product \( Map(H, S) \times H \) with the multiplication given by the formula

\[
(\varphi_1, h_1) \cdot (\varphi_2, h_2) = \left( (\varphi_1 \circ h_2) \cdot \varphi_2, h_1 \cdot h_2 \right)
\]

for \( (\varphi_1, h_1), (\varphi_2, h_2) \in Map(H, S) \times H. \)
Let $\varepsilon : H \to S$ be the constant map into the unit of $S$. Then the pair $(\varepsilon, \text{id}_H)$ is the unit element of $S \cap H$. Moreover, if $(\varphi, h) \in S \cap H$ and $\varphi^{-1} \in \text{Map}(H, S)$ is the point-wise inverse of $\varphi$, then $(\varphi^{-1} \circ h^{-1}, h^{-1})$ is the inverse of $(\varphi, h)$ in $S \cap H$.

We also have the following exact sequence:

$$1 \to \text{Map}(H, S) \xrightarrow{i} S \cap H \xrightarrow{\pi} H \to 1,$$

where $i(\varphi) = (\varphi, e)$, $e$ is the unit of $H$, and $\pi(\varphi, h) = h$. Moreover, $\pi$ admits a section $s : H \to S \cap H$ defined by $s(h) = (\varepsilon, h)$.

### 4. Main result

**Homeotopy group of a canonical foliation.** Let $\Sigma$ be a striped surface with a canonical foliation $F$. Denote by $H(F)$ the groups of all foliated homeomorphisms $h : \Sigma \to \Sigma$, i.e. homeomorphisms mapping leaves of $F$ onto leaves. We will endow $H(F)$ with the corresponding compact open topology.

Recall that all leaves of $F$ are oriented. Then we denote by $H^+(F)$ the subgroup of $H(F)$ consisting of homeomorphisms $h : \Sigma \to \Sigma$ such that for each leaf $\omega$ the restriction map $h : \omega \to h(\omega)$ is orientation preserving.

Let $H^+_0(F)$ be the identity path component of $H^+(F)$. It consists of all $h \in H^+(F)$ isotopic to id$_C$ in $H^+(F)$. Then $H^+_0(F)$ is a normal subgroup of $H^+(F)$, and the corresponding quotient

$$\pi_0 H^+(F) = H^+(F)/H^+_0(F)$$

will be called the *homeotopy* group of $F$.

**Class $\mathfrak{F}$.** Denote by $\mathfrak{F}$ the class of striped surfaces

$$\Sigma = \bigsqcup_{\lambda \in \Lambda} S_\lambda / \{X_\varphi(\gamma) \sim Y_q(\gamma)\}$$

satisfying the following conditions:

1) each $S_\lambda, \lambda \in \Lambda$, is admissible,

$$\partial_- S_\lambda = J_1 \times \{-1\} = A_1 \times \{-1\}, \quad \partial_+ S_\lambda = A_{\Delta_\lambda} \times \{1\},$$

where $\Delta_\lambda$ coincides with one of the standard collections $A_{[n]}$, $A_{N}$, $A_{-N}$, or $A_Z$;

2) the graph $\Gamma(\Sigma)$ is connected and has a finite diameter and no cycles.

In particular, if $\Sigma \in \mathfrak{F}$, then each model strip $S_\lambda$ of $\Sigma$ regarded as a vertex of $\Gamma(\Sigma)$ has at most one incoming edge and at most countably many outcoming edges linearly ordered with respect to $\Delta_\lambda$.

Since $\Gamma(\Sigma)$ is connected and has a finite diameter and no cycles, it follows that there exists a unique vertex having no incoming edges. We will call this vertex a *root* and the corresponding strip a *root* strip.

Thus every surface $\Sigma \in \mathfrak{F}$ of diameter $d$ can be represented as follows, see Figure 4.1.

$$\Sigma = S \cup_{\partial_- S} \left( \bigcup_{i \in \Delta} \Sigma_i \right),$$

(4.1)

where

- $S$ is a root strip of $\Sigma$,

$$\partial_- S = J_1 \times \{-1\}, \quad \partial_+ S = \bigcup_{i \in \Delta} J_1 \times \{1\},$$

where $\Delta \in \{[0], [1], \ldots, N, -N, Z\}$.

- $\Sigma_i$ is either empty or it is a striped surface belonging to $\mathfrak{F}$ and its graph $\Gamma(\Sigma_i)$ has diameter less than $d$. 

Suppose $\Sigma_i$ is non-empty and let $S_i$ be the root strip of $\Sigma_i$. Then $\partial_- S_i = J_i \times \{-1\}$ is glued to the boundary interval $J_i \times \{1\}$ of $\partial_+ S$ by the homeomorphism $\varphi : J_i \equiv (1, 1.5) \rightarrow J_i \equiv (i, i + 0.5), \quad \varphi(t) = t + i - 1$.

![Figure 4.1. A striped surface $\Sigma \in \mathcal{F}$ whose graph $\Gamma(\Sigma)$ has diameter 3](image)

Obviously, $\Sigma \in \mathcal{F}$ is a connected and simply connected non-compact surface. Therefore it follows from [3] that the interior of $\Sigma$ is homeomorphic to $\mathbb{R}^2$.

The class of homeotopy groups of foliations on striped surfaces which belongs to the class $\mathcal{F}$ will be denoted by $\mathcal{P}$, i.e.

$$\mathcal{P} = \{ \pi_0 H^+(F) \mid F \text{ is a canonical foliation of some striped surface } \Sigma \in \mathcal{F} \}.$$  

We will also define another class of groups $\mathcal{G}$.

**Definition 4.1.** Let $\mathcal{G}$ be the minimal class of groups satisfying the following conditions:

1) $\{1\} \in \mathcal{G}$;
2) if $A_i \in \mathcal{G}$ for $i \in \mathbb{N}$, then $\prod_{i \in \mathbb{N}} A_i \in \mathcal{G}$;
3) if $A \in \mathcal{G}$, then $A \wr \mathbb{Z} \in \mathcal{G}$.

**Lemma 4.2.** A group $G$ belongs to $\mathcal{G}$ if and only if it can be obtained from the unit group $\{1\}$ by a composition of finitely many operations of the following types:

(a) countable direct products;
(b) wreath product with the group $\mathbb{Z}$.

**Proof.** Let $\mathcal{G}_0$ be the class of groups $G$ which can be obtained from the unit group $\{1\}$ by a composition of finitely many operations of types (a) and (b). Then any class of groups satisfying conditions 1)–3) of Definition [4.1] contains $\mathcal{G}_0$, whence $\mathcal{G}_0 \subset \mathcal{G}$. On the other hand, $\mathcal{G}_0$ also satisfies conditions 1)–3) of Definition [4.1], whence $\mathcal{G} \subset \mathcal{G}_0$ as well.  

Every representation $\xi(G)$ of $G$ as a composition of operations (a) and (b) will be called a representation of $G$ in the class $\mathcal{G}$. Such a representation is not unique. For example,

$$Z \cong \{1\} \wr \mathbb{Z} \cong 1 \times (1 \wr \mathbb{Z}) \cong (1 \times 1 \wr \mathbb{Z}).$$

**Definition 4.3.** The **height** of a representation $\xi(G)$ of $G$ in the class $\mathcal{G}$ is a non-negative integer defined inductively as follows:

1) $h(\{1\}) = 0$;
2) $h(\xi(G) \wr \mathbb{Z}) = 1 + h(\xi(G))$;
3) $h\left( \prod_{i \in A} \xi(A_i) \right) = 1 + \max_i \{h(\xi(A_i))\}$. 

(4.2)  

$$Z \cong \{1\} \wr \mathbb{Z} \cong 1 \times (1 \wr \mathbb{Z}) \cong (1 \times 1 \wr \mathbb{Z}).$$
Example 4.4. Below are examples of representations of groups \{1\}, \mathbb{Z} and \mathbb{Z} \wr \mathbb{Z} in the class \mathcal{G} and their heights:

\[
\begin{align*}
\eta(\{1\}) &= 0, & \eta(\{1\} \times \{1\}) &= 1, \\
\eta(\{1\} \times \mathbb{Z}) &= 1, & \eta(\{1\} \times (\{1\} \times \mathbb{Z})) &= 2, \\
\eta((\{1\} \times \mathbb{Z}) \times (\{1\} \times \{1\} \times \mathbb{Z})) &= 3.
\end{align*}
\]

\[
\begin{align*}
\eta(\{1\}) &= 0, & \eta(\{1\} \times \{1\}) &= 1, \\
\eta(\{1\} \times \mathbb{Z}) &= 1, & \eta(\{1\} \times (\{1\} \times \mathbb{Z})) &= 2, \\
\eta(((\{1\} \times \mathbb{Z}) \times (\{1\} \times \{1\} \times \mathbb{Z})) &= 3.
\end{align*}
\]

Let \mathcal{G}' \subset \mathcal{G} be a subclass of \mathcal{G} consisting of groups admitting a representation of finite height in \mathcal{G}. The aim of the present paper is to prove the following theorem:

**Theorem 4.5.** Classes \mathcal{P} and \mathcal{G}' coincide.

In other words, a group \mathcal{G} is isomorphic with a homeotopy group \(H^+(F)\) of some striped surface \(\Sigma \in \mathcal{F}\) with a canonical foliation \(F\) if and only if \(\mathcal{G}\) can be obtained from the unit group \(\{1\}\) by a composition of finitely many operations of types (a) and (b) of Lemma 4.2.

5. Preliminaries

Let \(\Sigma\) be a striped surface belonging to \(\mathcal{F}\) presented in the form (4.1), and \(S\) be the root strip of \(\Sigma\). We will use coordinates \((x, y)\) from the chart for \(S\), so we can assume that \(\partial_+ S = \bigcup_{i \in \Delta} J_i \times \{1\}\).

Notice that if \(h \in H^+(F)\), then \(h(S) = S\), whence there exists a unique number \(\eta(h) \in \mathbb{Z}\) such that in the chart for \(S\) we have that

\[
h(J_i \times \{1\}) = J_{i + \eta(h)} \times \{1\}
\]

for all \(i \in \Delta\). One can easily check that the correspondence \(h \mapsto \eta(h)\) is a homomorphism

\[
\eta : H^+(F) \to \mathbb{Z}.
\]

Obviously, \(\eta\) can be a non-zero homomorphism only when \(\Delta = \mathbb{Z}\).

Consider the following two subgroups of \(H^+(F)\):

\[
\begin{align*}
Q_S &= \{ h \in H^+(F) \mid h(\omega) = \omega, \text{ for each leaf } \omega \text{ of } F \subset S \}, \\
H^+(F, S) &= \{ h \in H^+(F) \mid h|_S = \text{id}|_S \}.
\end{align*}
\]

It is evident that

\[
(5.2) \quad H^+(F, S) \subset Q_S \subset \ker(\eta).
\]

**Lemma 5.1.** Embeddings (5.2) are homeotopy equivalences.

**Proof.** First we will construct a deformation of \(\ker(\eta)\) into \(Q_S\). Let \(h \in \ker(\eta)\). Since \(h(S) = S\), it follows that \(h\) interchanges leaves of \(F\). In the coordinates \((x, y)\) in the chart for \(S\) these leaves are the lines \(y = \text{const}\), whence

\[
h(x, y) = (\alpha(x, y), \beta(y)),
\]

where \(\alpha : S \to \mathbb{R}\) and \(\beta : [-1, 1] \to [-1, 1]\) are continuous functions such that for each \(y \in (0, 1)\) the correspondence \(x \mapsto \alpha(x, y)\) is a preserving orientation homeomorphism \(\mathbb{R} \to \mathbb{R}\).

Then \(h \in Q_S\) iff \(\beta(y) = y\) for all \(y \in [0, 1]\). Define the map \(H : \ker(\eta) \times [0, 1] \to \ker(\eta)\) by the formula

\[
H(h, t)(z) = \begin{cases} 
(\alpha(x, y), (1 - t)\beta(y) + ty), & z = (x, y) \in S, \\
& z \in \Sigma \setminus S.
\end{cases}
\]

One can easily check that \(H_0 = \text{id}_{\ker(\eta)}, H_t(Q_S) \subset Q_S\) for all \(t \in [0, 1]\), and \(H(h, 1) \in Q_S\).

Hence \(H\) is a deformation of \(\ker(\eta)\) into \(Q_S\), and so the inclusion \(Q_S \subset \ker(\eta)\) is a homotopy equivalence.
Similarly, let \( h \in Q_S \), so
\[
    h(x, y) = (\alpha(x, y), y)
\]
for all \((x, y) \in S\). Notice that \( h \in H^+(F, S) \) iff \( \alpha(x, y) = x \) and \( \beta(y) = y \) for all \((x, y) \in S\).

Let
\[
    h(x, y) = (\alpha_i(x, y), \beta_i(y))
\]
be the restriction of \( h \) onto root strip \( S_i \) of \( \Sigma_i \) in the corresponding chart of \( S_i \). Since \( \partial_- S_i = J_i \times \{-1\} \), we see that if \( h \in H^+(F, S) \), then \( \alpha_i(x, -1) = x \) for all \( x \in J_1 \) and \( i \in \Delta \).

Fix a continuous function \( \varepsilon : [-1, 1] \to [0, 1] \) such that
\[
    \varepsilon(y) = \begin{cases} 
        0, & y \in (-1, -0.8), \\
        1, & y \in (0, 1) 
    \end{cases}
\]
and define the following homotopy \( G : Q_S \times [0, 1] \to Q_S \) by
\[
    G(h, t)(z) = \begin{cases} 
        ((1 - t)\alpha(x, y) + tx, y), & z = (x, y) \in S, \\
        (1 - tz(y))\alpha_i(x, y) + t\varepsilon(y)x, \beta(y)), & z = (x, y) \in S_i, \\
        z, & z \notin S \cup (\cup_{j \in \Delta} S_i).
    \end{cases}
\]
Since \( \partial_- S_i \) is glued to the boundary component \( J_i \times \{1\} \) by an affine homeomorphism, and the formulas for \( G \) are affine for each fixed \( t \) and \( y \), it follows that those formulas agree on \( J_i \times \{1\} \) and \( \partial_- S_i \), c.f. [8]. This implies that \( G \) is a continuous map.

Moreover, one can easily check that \( G_0 = \text{id}_{Q_S} \), \( G_t(H^+(F, S)) \subset H^+(F, S) \) for all \( t \in [0, 1] \), and \( G_1(Q_S) \subset H^+(F, S) \). Hence \( G \) is a deformation of \( Q_S \) into \( H^+(F, S) \), and therefore the inclusion \( H^+(F, S) \subset Q_S \) is a homotopy equivalence as well.

Suppose \( \Sigma_i \) is non-empty for some \( i \in \Delta \). Let \( F_i \) be the canonical foliation on \( \Sigma_i \) and \( S_i \) be the root strip of \( \Sigma_i \). We will denote by \( H^+(F_i, \partial_- S_i) \) the subgroup of \( H^+(F_i) \) consisting of homeomorphisms fixed on \( \partial_- S_i \).

If \( \Sigma_i = \emptyset \), then we will assume that \( H^+(F_i, \partial_- S_i) = \{1\} \).

**Lemma 5.2.** We have an isomorphism
\[
    \pi_0 \ker(\eta) \cong \prod_{i \in \Delta} \pi_0 H^+(F_i, \partial_- S_i).
\]

**Proof.** Evidently, we have a canonical isomorphism
\[
    \alpha : H^+(F, S) \cong \prod_{i \in \Delta} H^+(F_i, \partial_- S_i), \quad \alpha(h) = (h|_{\Sigma_i})_{i \in \Delta}.
\]
Then from Lemma 5.1 we get the following sequence of isomorphisms:
\[
    \pi_0 \ker(\eta) \cong \pi_0 H^+(F, S) \cong \pi_0 \prod_{i \in \Delta} H^+(F_i, \partial_- S_i) = \prod_{i \in \Delta} \pi_0 H^+(F_i, \partial_- S_i).
\]
Lemma is proved.

**Theorem 5.3.** 1) If \( \eta \) is zero homomorphism, then the group \( \pi_0 H^+(F) \) is isomorphic to \( \prod_{i \in \Delta} \pi_0 H^+(F_i, \partial_- S_i) \).

2) Suppose the image of \( \eta \) is \( k\mathbb{Z} \) for some \( k \geq 1 \), so \( \Delta = \mathbb{Z} \). Then the group \( \pi_0 H^+(F) \) is isomorphic to \( \left( \prod_{i=0}^{k-1} \pi_0 H^+(F_i, \partial_- S_i) \right) \mathbb{Z} \).
We claim that \( \pi_0 H^+(F) \cong \prod_{i \in \Delta} \pi_0 H^+(F_i, \partial_- S_i). \)

1) The assumption that \( \eta \) is zero homomorphism means that \( H^+(F) = \ker(\eta) \), whence we get from Lemma 5.2 that \( \pi_0 H^+(F) \cong \prod_{i \in \Delta} \pi_0 H^+(F_i, \partial_- S_i). \)

2) Suppose \( \text{Im} \eta = k\mathbb{Z} \). Then we have an epimorphism \( \tilde{\eta} : H^+(F) \to \mathbb{Z} \) defined by \( \tilde{\eta}(h) = \eta(h)/k \) and such that \( h(\Sigma_r) = \Sigma_r + k \cdot \tilde{\eta}(h), \quad r = 0, 1, \ldots, k - 1. \)

Let \( X = \bigcup_{i=0}^{k-1} \Sigma_i, \quad \partial_- X = \bigcup_{i=0}^{k-1} \partial_- S_i, \)

and \( F_X \) be the oriented foliation on \( X \) induced by \( F \). Denote by \( H^+(F_X, \partial_- X) \) the group of homeomorphisms of \( X \) fixed on \( \partial_- X \) and mapping leaves of \( F_X \) onto leaves and preserving their orientation. Then we have a natural isomorphism \( \prod_{i=0}^{k-1} H^+(F_i, \partial_- S_i) \cong H^+(F_X, \partial_- X) \)

which yields an isomorphism \( \prod_{i=0}^{k-1} \pi_0 H^+(F_i, \partial_- S_i) \cong \pi_0 H^+(F_X, \partial_- X). \)

Therefore for the proof of Theorem 5.3 we should construct an isomorphism \( \beta : \pi_0 H^+(F) \longrightarrow \pi_0 H^+(F_X, \partial_- X) \times \mathbb{Z} \equiv \text{Map}(\mathbb{Z}, \pi_0 H^+(F_X, \partial_- X)) \times \mathbb{Z}. \)

Fix any \( g \in H^+(F) \) with \( \tilde{\eta}(g) = 1 \). Then \( g^{-\tilde{\eta}(h)} \circ h(\Sigma_i) = \Sigma_i, \)

for all \( h \in H^+(F) \) and \( i \in \mathbb{Z} \), whence \( g^{-\tilde{\eta}(h)} \circ h \in \ker(\eta) \). Thus we get a well-defined function \( \varphi_h : \mathbb{Z} \to \pi_0 H^+(F_X, \partial_- X), \quad \varphi_h(j) = \left[ g^{-\tilde{\eta}(h)} \circ h \circ g^j \big|_{X} \right]. \)

Define the following map: \( \beta : \pi_0 H^+(F) \longrightarrow \pi_0 H^+(F_X, \partial_- X) \)

by the formula \( \beta(h) = (\varphi_h, \tilde{\eta}(h)), \quad h \in \pi_0 H^+(F). \)

We claim that \( \beta \) is an isomorphism. First notice that the composition operation in \( H^+(F_X, \partial_- X) \times \mathbb{Z} \) is given by the following rule:

\[
(\varphi_{h_1}, n) \cdot (\varphi_{h_2}, m) = (\varphi_{h_1}^m \cdot \varphi_{h_2}, n + m),
\]

where \( \varphi_{h_2}^m(j) = \varphi_{h_2}(j + m). \)

**Proof that \( \beta \) is a homomorphism.** Let \( h_1, h_2 \in H^+(F). \) Then \( \beta(h_1) \circ \beta(h_2) = (\varphi_{h_1}, \tilde{\eta}(h_1)) \cdot (\varphi_{h_2}, \tilde{\eta}(h_2)) \)

\[
= (\varphi_{h_1} \cdot \varphi_{h_2}, \tilde{\eta}(h_1) + \tilde{\eta}(h_2))
= \left( [g^{-\tilde{\eta}(h_1)} \circ h_1 \cdot g^j \circ \tilde{\eta}(h_2) \circ h_2 \circ g^j \big|_{X}], \tilde{\eta}(h_1) + \tilde{\eta}(h_2) \right)
= \left( (g^{-\tilde{\eta}(h_1)h_2} \circ h_1 \circ h_2 \circ g^j \big|_{X}, \tilde{\eta}(h_1) + \tilde{\eta}(h_2) \right)
= (\varphi_{h_1h_2}, \tilde{\eta}(h_1 + h_2)) = \beta(h_1 \circ h_2).
\]
Proof that $\beta$ is injective. Let $h \in H^+(F)$ be such that $[h] \in \ker \beta$. We should prove that $h$ is isotopic in $H^+(F)$ to $\text{id}_\Sigma$.

The assumption $[h] \in \ker \beta$ means that $\beta(h) = (\varphi, n) \in \pi_0H^+(F, \partial_-X)$, where $\varphi : \mathbb{Z} \to \{\text{id}_X\}$ is the constant map into the unit of $\pi_0H^+(F, \partial_-X)$. In particular, since $\eta(h) = 0$, we get from Lemma 4.4 that $h$ is isotopic in $H^+(F)$ to a homeomorphism fixed on $S$. Therefore we can assume that $h$ itself is fixed on $S$, that is $h \in H^+(F, S)$. Then

\[(5.3)\]
\[\varphi_h(j) = [g^{-j} \circ h \circ g^j|_X] = \varepsilon(j) = [\text{id}_X] \in \pi_0H^+(F, \partial_-X)\]

for each $j \in \mathbb{Z}$. In other words, $g^{-j} \circ h \circ g^j|_X$ is isotopic to $\text{id}_X$ relatively $\partial_-X$.

It suffices to prove that for each $i \in \mathbb{Z}$ the restriction $h|_{\Sigma_i}$ is isotopic in $H^+(F_i, \partial_-S_i)$ to $\text{id}_{\Sigma_i}$ relatively to $\partial_-S_i$.

Write $i = r + jk$ for a unique $r \in \{0, k - 1\}$. Then we have the following commutative diagram:

Therefore, we get from $\{5.3\}$ that $[h|_{\Sigma_i}] = [\text{id}_{\Sigma_i}] \in H^+(F_i, \partial_-S_i)$. Hence $h$ is isotopic to $\text{id}_{\Sigma_i}$ in $H^+(F)$.

Proof that $\beta$ is surjective. Let $(\varphi, n) \in \pi_0H^+(F_X, \partial_-X)$ \text{mod} $\mathbb{Z}$. For each $j \in \mathbb{Z}$ fix a homeomorphism $h_j \in H^+(F_X, \partial_-X)$ such that $[h_j] = \phi(j) \in \pi_0H^+(F_X, \partial_-X)$. Now define the following homeomorphism $\hat{h}$ of $\Sigma$ by the formula:

\[\hat{h} = \begin{cases} 
\text{id}_{\Sigma}, & \text{on } S, \\
[g^j \circ h_j \circ g^{-j}] & \text{on } g^j(X)
\end{cases}\]

and put $h = g^n \circ \hat{h}$. Then it is easy to see that $\beta([h]) = (\phi, n)$, whence $\beta$ is surjective. Thus $\beta$ is an isomorphism. \qed

6. Proof of Theorem 5.3

We should prove that $\mathcal{P} = \mathcal{G}'$.

1. First we will show that $\mathcal{G}' \subset \mathcal{P}$.

Let $G \in \mathcal{G}'$, so $G$ has a representation $\xi(G)$ in the class $\mathcal{G}$ of finite height $k = h(\xi(G))$. We have to show that there exists a striped surface $\Sigma \in \mathfrak{S}$ with canonical foliation $F$ such that $G \cong \pi_0H^+(F)$.

If $k = h(\xi(G)) = 0$, then $G$ is the unit group $\{1\}$ and $\xi(G) = \{1\}$. Let $S$ be an admissible model strip with $\partial_-S = A_{[1]} \times \{-1\}$ and $\partial_+S = \emptyset$. Then $S \in \mathfrak{S}$. Let also $F$ be the canonical foliation on $S$. Then

\[\pi_0H^+(F) = \{1\} = G,\]

i.e. $G \in \mathcal{P}$.

Suppose that we have established our statement for all $k$ being less than some $\tilde{k} > 0$.

Let us prove it for $k = \tilde{k}$. It follows from Definition 4.3 that either

(i) $\xi(G) = \prod_{i \in \mathbb{N}} A_i$ where each group $A_i$ has a representation $\xi(A_i)$ in the class $\mathcal{G}$ of height $h(\xi(A_i)) < k$, or

(ii) $\xi(G) = A\mathbb{Z}$, and $A$ has a representation $\xi(A)$ in the class $\mathcal{G}$ of height $h(\xi(A)) < k$.

In the case (i) due to the inductive assumption for each $i \in \mathbb{N}$ there exists a striped surface $\Sigma_i \in \mathfrak{S}$ with foliations $F_i$ such that $A_i = \pi_0H^+(F_i)$.

Let $S$ be an admissible model strip with $\partial_-S = A_{[1]} \times \{-1\}$ and $\partial_+S = A_{\mathbb{N}} \times \{1\}$, and $S_i$ be the root strip of $\Sigma_i$, $i \in \mathbb{N}$. Define the striped surface

\[\Sigma = S \cup_{\partial_-S} \left( \bigcup_{i \in \mathbb{N}} \Sigma_i \right)\]

obtained by identifying $\partial_-S_i \subset \Sigma_i$ with $J_i \times \{1\} \subset \partial_+S$. Then by Theorem 5.3 $\eta$ is a trivial homomorphism, and $\pi_0H^+(F) \cong \prod_{i \in \mathbb{N}} \pi_0H^+(F_i) \cong \prod_{i \in \mathbb{N}} A_i \cong G$. So $G \in \mathcal{P}$. 

\[\square\]
In the case (ii) again by inductive assumption there exists a striped surface \( \hat{\Sigma} \in \tilde{\mathcal{F}} \) with a canonical foliation \( \hat{F} \) such that \( A = \pi_0 H^+(\hat{F}) \).

Take countably many copies \( \hat{\Sigma}_i, i \in \mathbb{Z} \), of \( \hat{\Sigma} \). Let \( \hat{S}_i \) be the root strip of \( \hat{\Sigma}_i \) and \( \hat{F}_i \) be the canonical foliation on \( \hat{\Sigma}_i \).

Let also \( S \) be an admissible model strip with \( \partial_- S = A_1 \times \{-1\} \) and \( \partial_+ S = A_2 \times \{1\} \).

Define the following striped surface:

\[
\Sigma = S \cup_{\partial_+ S} \left( \cup_{i \in \mathbb{Z}} \hat{\Sigma}_i \right).
\]

Obtained by gluing each \( \hat{\Sigma}_i \) to \( S \) by identifying \( \partial_- \hat{S}_i \subset \hat{\Sigma}_i \) with \( J_i \times \{1\} \subset \partial_+ S, i \in \mathbb{Z} \).

Then for every pair \( i, j \in \mathbb{Z} \) there exists \( h \in H^+(F) \) such that \( h(\hat{\Sigma}_i) = \hat{\Sigma}_j \), whence the homomorphism \( \eta \), see \( \text{(5.1)} \) is surjective. Hence by Theorem \( 5.3 \)

\[
\pi_0 H^+(F) \cong \pi_0 H^+(\hat{F}) \cap \mathbb{Z} \cong A \cap \mathbb{Z} \cong G.
\]

Thus, \( G \in \mathcal{P} \) and so \( \mathcal{G}' \subset \mathcal{P} \).

2. Conversely, let us show that \( \mathcal{P} \subset \mathcal{G}' \).

Let \( \Sigma \in \tilde{\mathcal{F}} \) be a striped surface presented in the form \( \text{(2.1)} \) with canonical foliation \( F \) and such that \( \text{diam } \Gamma(\Sigma) = k \). We should prove that \( \pi_0 H^+(F) \) has a finite presentation in the class \( G \), which means that \( \pi_0 H^+(F) \in \mathcal{G}' \).

If \( k = 0 \), then \( \Sigma \) is an admissible model strip with

\[
\partial_- \Sigma = A_1 \times \{-1\}, \quad \partial_+ \Sigma = A_\alpha, \quad \alpha \in \{0, 1, \ldots, N, -N, \mathbb{Z}\}.
\]

Then it easily follows from Theorem \( 5.3 \) that \( \pi_0 H^+(F) \cong \mathbb{Z} \cong \{1\} \cap \mathbb{Z} \) if \( \alpha = \mathbb{Z} \), and \( \pi_0 H^+(F) \cong \{1\} \) otherwise. In both cases \( \pi_0 H^+(F) \in \mathcal{G} \).

Suppose that we have established our statement for all \( k \) being less than some \( \bar{k} > 0 \). We should prove it for \( k = \bar{k} \). Let

\[
\Sigma = S \cup_{\partial_+ S} \left( \cup_{i \in \Delta} \hat{\Sigma}_i \right) \in \tilde{\mathcal{F}}
\]

be such that \( \Gamma(\Sigma) \) has diameter \( \bar{k} \). Then \( \Gamma(\Sigma) \) has diameter less than \( \bar{k} \), and so by inductive assumption \( \pi_0 H^+(F_i, \partial_- S_i) \in \mathcal{G} \). Moreover, according to Theorem \( 5.3 \) we have that

(i) if \( \text{image}(\eta) = 0 \), then \( \pi_0 H^+(F) \cong \prod_{i \in \Delta} \pi_0 H^+(F_i, \partial_- S_i) \in \mathcal{G} \),

(ii) if \( \text{image}(\eta) = k \mathbb{Z} \), then \( \pi_0 H^+(F) \cong \left( \prod_{i=0}^{k-1} \pi_0 H^+(F_i, \partial_- S_i) \right) \cap \mathbb{Z} \in \mathcal{G} \).

Thus \( \mathcal{P} \subset \mathcal{G}' \), and so \( \mathcal{P} = \mathcal{G}' \). Theorem \( 5.3 \) completed.

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Received 31/03/2016