Generic Detectability and Isolability of Topology Failures in Networked Linear Systems

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Abstract—This paper studies the possibility of detecting and isolating topology failures (including link failures and node failures) of a networked system from subsystem measurements, in which subsystems are of fixed high-order linear dynamics, and exact interaction weights among them are unknown. We prove that in such class of networked systems with the same network topologies, the detectability and isolability of a given topology failure (set) are generic properties, indicating that it is the network topology that dominates the property of being detectable or isolable for a failure (set). We first give algebraic conditions for detectability and isolability of arbitrary parameter perturbations for a lumped plant, and then derive graph-theoretical necessary and sufficient conditions for generic detectability and isolability for the networked system. On the basis of these results, we consider the problems of deploying the smallest set of sensors for generic detectability and isolability. We reduce the associated sensor placement problems to the hitting set problems, which can be effectively solved by greedy algorithms with guaranteed approximation performances.

Index Terms—Failure detectability and isolability, generic property, graph theory, sensor placement, networked system

I. INTRODUCTION

There exist many large-scale systems consisting of a large number of subsystems in the real world. These subsystems, usually geographically distributed, are interconnected through a network. Such systems are often called networked systems. Many critical infrastructures can be modeled as networked systems, such as power systems [1], the Internet [2], wireless communication network [3] and transportation network [4]. The security and reliability of networked systems have aroused great concern from various aspects [2, 3, 5, 6].

In networked systems, a common type of fault is the perturbation/variant of components of its network structure. For example, links may be blocked or removed, making signals unable to be transmitted normally, and nodes (agents) may not operate normally or even lose communications with their neighbors, leading to loss of system performances. Such type of structure variants can result from either failure of network components (such as links or nodes), or denial-of-service attacks [2, 3, 5, 7, 8]. The failure of a set of links or nodes is collectively called topology failure in this paper. Topology failures may have huge impact on the security and normal functioning of a networked system. One example is the catastrophic power outage in southern Italy in 2003, which was reportedly caused by failures of some high voltage transmission lines [5]. Considering the possible catastrophic cascading consequences caused by topology failures, the timely detection and isolation have become particularly important [6].

Fault detection and isolation (FDI) techniques have long been active in control community [9-14]. The main targets are to determine whether faults occur and to locate them. Many detection and isolation approaches have been proposed, including geometric theory based approaches [11], observer-based approaches [12], data-driven approaches [13], and so on. However, the majority of literature on this topic deals with faults that are linked to either additional external signals or undesired parameter deviations [9, 14]. Topology failures, on the other hand, result in perturbations on the structure of system intrinsic dynamics. Unlike common parameter deviations, topology failures shift the nominal parameters to only some discrete values, which are usually hard to be modeled as external disturbances.

Nevertheless, in literature, the detection of topology failures has drawn on FDI techniques, that is, by comparing the discrepancies between the current system output and the nominal output to determine whether the system has undergone topology failures [9]. Such problems have recently attracted researchers’ attention. In [15, 16], Rahimian et al. studied detectability of single or multiple link failures for multi-agent systems under the agreement protocol. They introduced the concept of distinguishable flow graph and gave sufficient conditions to distinguish faulty links. In [17], Battistelli and Tesi used observability from switching systems theory to characterize indiscernible states in networks of single-integrators, i.e., the initial states that generate exactly the same outputs for the nominal system and the system after failures. The same authors further extended the former work to networked diffusively coupled high-order systems [18], whiles Patil and Tesi considered indiscernible topological variations in networks with descriptor subsystems, where the subsystems can be heterogeneous [19]. In [20], Rahimian and Preciado studied detection and isolation algorithms of single-link failure in networked linear systems. They related the discontinuity of higher-order derivatives of system outputs caused by the removal of a single link to the distance from the end of the removed link to the observed node.

However, all of the above works depend on accurate system parameters, which means accurate parameters are required when applied. In addition, it is usually not easy to extend their results to the case with simultaneous failures of multiple links/nodes. For many practical systems, accurate system pa-
rameters may be hard to obtain, but their zero-nonzero patterns, i.e., which entry of the system matrices is zero and which is not, might be easier accessible. This forms a class of systems sharing the same “structure”. In control theory, some properties will become generic in this class of systems, i.e., either for almost all systems in this class, these properties hold true, or for none these properties hold true. For example, controllability and observability are two well-known generic properties, both for a lumped structured plant \[21\] and a networked system \[22, 23\].

Generic properties are particularly prominent in analyzing large-scale networked systems, not only because they usually can intuitively show how topologies influence the considered properties, but also because they often can be verified efficiently by means of graphical tools \[21, 22\]. In this paper, we study generic detectability and isolability of topology failures for a networked linear system, where subsystem dynamics are given and identical, but the weights of interaction links among them are unknown. We study under what conditions we can give problem formulations and some preliminaries. Section III, we study by means of graphical tools \[21, 22\]. In this paper, we study given topology failure (set) for a class of networked systems are both generic properties. That means, it is how subsystems are interconnected, rather than the mean, the node set and \(G\) is the subsystem interconnection topology, with the node set \(\{1, ..., N\}\), and \(A\) is the state matrix for none these properties hold true. For example, controllability almost all systems in this class, these properties hold true, or for none these properties hold true. For example, controllability and observability are two well-known generic properties. For a networked system, \(M, \rho(M)\) denotes its spectral radius, namely, the maximum absolute value of its eigenvalues.

**II. Problem Formulation and Preliminaries**

A. Preliminaries

Concepts in graph theory: In a directed graph (digraph) \(D = (V, E)\), where \(V\) is the node set and \(E \subseteq V \times V\) is the edge (or link) set, a path from \(v_i \in V\) to \(v_j \in V\) is a sequence of edges \(\{(v_i, v_{i+1}), (v_{i+1}, v_{i+2}), ..., (v_{j-1}, v_j)\}\). The length of a path is the number of edges it contains. The distance from \(v_i\) to \(v_j\), denoted by \(\text{dist}(v_i, v_j)\), is the length of the shortest path from \(v_i\) to \(v_j\). If there is no path from \(v_i\) to \(v_j\), then \(\text{dist}(v_i, v_j) = \infty\).

In this paper, the adjacency matrix of a weighted digraph \(D\) is a matrix \(W \in \mathbb{R}^{|V| \times |V|}\) such that \(W_{ij} \neq 0\) only if \(\{v_j, v_i\} \in E\), where \(W_{ij}\) is the weight of \(\{v_j, v_i\}\). If there is no path from \(v_i\) to \(v_j\), then \(\text{dist}(v_i, v_j) = \infty\).

B. Detectability and Isolability of Topology Failures

Consider a networked system consisting of \(N\) linear time invariant subsystems. Let \(G = (V, E)\) be a digraph describing the subsystem interconnection topology, with the node set \(V = \{1, ..., N\}\), and a directed edge \((i, j) \in E_{\text{sys}}\) from node \(i\) to node \(j\) exists if the \(j\)th subsystem is directly influenced by the \(i\)th one. Dynamics of the \(i\)th system is

\[
\dot{x}_i(t) = Ax_i(t) + B \sum_{j=1}^{N} w_{ij} \Gamma x_j(t), \quad y_i(t) = Cx_i(t) \quad (1)
\]

where \(A \in \mathbb{R}^{n \times n}\) is the state transition matrix, \(B \in \mathbb{R}^{n \times m}\) is the input matrix, \(\Gamma \in \mathbb{R}^{m \times n}\) is the internal coupling matrix between subsystems, \(x_i(t) \in \mathbb{R}^n\) is the state vector, \(y_i(t) \in \mathbb{R}^p\) is the subsystem output vector, and \(w_{ij} \in \mathbb{R}\) is the weight of edge (link) from the \(j\)th subsystem to the \(i\)th one satisfying \(w_{ij} \neq 0\) only if \((j, i) \in E\), for \(i, j \in \{1, ..., N\}\). Denote the set of all weights \(w_{ij}\) by \(\{w_{ij}\}\). Notice that self-loops could be contained in \(E\), which could result from self-feedbacks, consensus-based agreement protocols, etc[1].

In this paper, we focus on how the network topology plays its role in failure detectability and isolability. Hence, we do not take the external inputs into consideration (i.e., the external inputs are fixed to be zero). However, our approaches can be extended to the case with known external inputs.
Suppose that subsystems indexed by the set $S \subseteq \{1, ..., N\}$ are directed measured. Define

$$S \doteq \text{col}\{[e_i]_t\}^{T}_{i \in S}.$$  

Let $x(t) = [x_1^T(t), ..., x_N^T(t)]^T$, $y(t) = \text{col}\{y_i(t)\}^{T}_{i \in S}$, and $W = [w_{ij}]$ be the adjacency matrix of $G$. The lumped state-space representation of (1) then is

$$\dot{x}(t) = \Phi x(t), y(t) = Q x(t)$$  

(2)

where $\Phi = I_N \otimes A + W \otimes H, Q = S \otimes C$,  

(3)

with $H = \beta G \in \mathbb{R}^{n \times n}$. Let $n_x = Nn, n_y = |S|p$, then $\Phi \in \mathbb{R}^{n_x \times n_x}$, $Q \in \mathbb{R}^{n_y \times n_x}$.

Equation (1) models a networked system with multi-input-multi-output subsystems, which arises in modeling interacted liquid tanks [24, 25], synchronizing networks of linear oscillators [22, 25], electrical systems [26], power networks [1], etc.

In practical engineering, common topology failures include link failures and node (or agent) failures. That is, the failure of a set of links $E_f \subseteq E$ corresponds to that all edges in $E_f$ are removed from $G$. The failure of a set of nodes $V_f \subseteq V$ corresponds to that, for each node $i \in V_f$, all edges adjacent to $i$, including ingoing, outgoing edges and self-loops, i.e., \{(i, j) : (j, i) \in E\} $\cup$ \{(j, i) : (i, j) \in E\}, are removed from $G$. Obviously, node failures are special cases of link failures. Hence, we shall focus on link failures in the rest of this paper, and we will use failure $E_f \subseteq E$ to denote the failure of removing all links of $E_f$ from $G$. With the failure $E_f$, the topology of the resulting networked system becomes $\hat{G} = (V, \hat{E}(E_f))$, with its adjacency matrix being denoted by $\hat{W}$, which is obtained from $W$ by setting the entries corresponding to $E_f$ to zero. We express dynamics of (1) after failure $E_f$ as

$$\dot{x}(t) = \bar{\Phi} x(t), y(t) = Q x(t)$$  

(4)

with $\bar{\Phi} = I_N \otimes A + \hat{W} \otimes H$.

The above formulation rises an interesting problem: Is it possible to detect and isolate topology failures from system outputs given the faultless nominal network dynamics (1)? Let $y(x_0, \hat{G}, t)$ (respectively, $y(x_0, \hat{G}, t)$) be the output vector of the networked system (1) with topology $G$ ($\hat{G}$) and initial state $x_0$ at time $t \geq 0$. Following [15, 16, 18], the detectability of failure $E_f$ is defined as follows.

**Definition 1:** For the networked system (1), failure $E_f \subseteq E$ is detectable if there exists $x_0 \in \mathbb{R}^{n_x}$ such that $y(x_0, \hat{G}, t) - y(x_0, \hat{G}, t) \neq 0$.

In the isolation problem, it is often the case that the exact failure is not known, but we may have prior knowledge of the possible failure candidates [10]. Suppose that the emerging failure belongs to a known prior topology failure set $E = \{E_1, ..., E_r\}$, where $E_i \subseteq E$, and $r$ is finite. For example, if at most one link is removed (namely, single-link failure), then $E = E$. Since $E$ is a combinatorial set of links in $E$, we have $r \leq 2^{|E|}$. Let $E_0 = \emptyset$. For each $E_i, 0 \leq i \leq r$, let $G_i = (V, E \backslash E_i)$. Failure isolation is possible from a prior failure set $E$, only if there is a unique topology $G_i$ that can explain the output response of the resulting networked system.

**Definition 2:** For the networked system (1), a failure set $E = \{E_1, ..., E_r\}$ is isolable if for any two integers $i, j \in \{0, ..., r\}$ with $i \neq j$, there exists $x_{0ij} \in \mathbb{R}^{n_x}$ such that $y(x_{0ij}, G_i, t) - y(x_{0ij}, G_j, t) \neq 0$.

We will show in the next section that, if a failure set $E = \{E_1, ..., E_r\}$ is isolable, then for almost all $x_0 \in \mathbb{R}^{n_x}$ except a proper subspace of $\mathbb{R}^{n_x}$, $y(x_0, G_i, t) - y(x_0, G_j, t) \neq 0$ for any two integers $i, j \in \{0, ..., r\}$ with $i \neq j$.

In many practical scenarios, while parameters $A, B, \Gamma, C$ for subsystem dynamics are often known from physically modeling (one of the most common dynamics is the high-order integrator) or system identification, the exact weights $\{w_{ij}\}$ among subsystems might be hard to know due to parameter uncertainties or geographical distance between subsystems. However, the knowledge about which $w_{ij}$ is zero or not may be easily accessible. We will show failure detectability and isolability are generic properties. In other words, either for almost all weights $\{w_{ij}\}$ with the corresponding zero-nonzero patterns, a given failure (set) is detectable (isolable), or for all weights $\{w_{ij}\}$ with the corresponding zero-nonzero patterns, the answers to the same problems are NO. The purpose of this paper is to find conditions under which such generic properties hold true, and apply them to the associated sensor placement problems.

**Remark 1:** In literature, the assumption of knowing subsystem dynamics but with little/no knowledge on the subsystem interaction weights is common in many aspects on networked systems, including topology reconstruction [27], system identification [28], as well as structural controllability [22, 23].

### III. ALGEBRAIC CONDITIONS FOR FAILURE DETECTABILITY AND ISOABILITY

In this section, we will give necessary and sufficient algebraic conditions for failure detectability and isolability. We assume that all parameters for the nominal dynamics (1) are known, including the weights $\{w_{ij}\}$. Our conditions are in terms of the lumped state-space parameters (3) and the corresponding parameter perturbations. In other words, our results can be seen as conditions for either networks of single-integrators, or state-space modeled plants where the parameter perturbations do not necessarily result from topology failures.

**Definition 3:** Consider $(\Phi, Q), (\bar{\Phi}, Q)$ in (2) and (4) respectively. Let $y(x_0, \Phi, t)$ and $y(x_0, \bar{\Phi}, t)$ be the output signals of system (2) and system (4), respectively, with initial state $x_0$. We say $(\Phi, Q)$ and $(\bar{\Phi}, Q)$ are distinguishable (also say $\Phi$ and $\bar{\Phi}$ are distinguishable if $Q$ is implicitly known), if there exists $x_0 \in \mathbb{R}^{n_x}$, such that $y(x_0, \Phi, t) - y(x_0, \bar{\Phi}, t) \neq 0$.

For the networked system (1), link failure $E_f$ is detectable, if and only if $(\Phi, Q)$ and $(\bar{\Phi}, Q)$ are distinguishable. The following theorem gives necessary and sufficient conditions for distinguishability of $(\Phi, Q)$ and $(\bar{\Phi}, Q)$.

**Theorem 1:** Given $(\Phi, Q)$ and $(\bar{\Phi}, Q)$ in (2) and (4) respectively, let the perturbation matrix $\Delta \Phi = \Phi - \bar{\Phi} \in \mathbb{R}^{n_x \times n_x}$. The following statements are equivalent:

1. $(\Phi, Q)$ and $(\bar{\Phi}, Q)$ are distinguishable;

   $$Q \Delta \Phi \
   Q \Phi \Delta \Phi \
   : \
   Q \Phi^{n_x - 1} \Delta \Phi$$

   $\neq 0$;

2. $(\Phi, Q)$ and $(\bar{\Phi}, Q)$ are distinguishable.
(3) The transfer function $Q(\lambda I - \Phi)^{-1} \Delta \Phi \neq 0$.

**Proof:** (1) $\iff$ (2): For a given $x_0 \in \mathbb{R}^{n_x}$, $y(x_0, G_i, t) = Q e^{\Phi_i t} x_0$, $y(x_0, G_j, t) = Q e^{\Phi_j t} x_0$. To make $Q e^{\Phi_i t} x_0 - Q e^{\Phi_j t} x_0 \equiv 0$ for arbitrary $x_0 \in \mathbb{R}^{n_x}$, $Q e^{\Phi_i t} - Q e^{\Phi_j t} \equiv 0$ must hold. Notice that,

$$Q e^{\Phi_i t} = Q(\Phi + \Phi i + \frac{1}{2} \Phi^2 i^2 + \frac{1}{6} \Phi^3 i^3 + \cdots)$$

$$Q e^{\Phi_j t} = Q(\Phi + \Phi j + \frac{1}{2} \Phi^2 j^2 + \frac{1}{6} \Phi^3 j^3 + \cdots).$$

Hence,

$$Q e^{\Phi_i t} - Q e^{\Phi_j t} = Q(\Phi - \Phi i + \frac{1}{2} (\Phi^2 - \Phi^2) i^2 + \cdots).$$

Therefore, $Q e^{\Phi_i t} - Q e^{\Phi_j t} \equiv 0$ requires that $Q(\Phi i - \Phi j) = 0$, for $i = 1, \ldots, n_x$. Notice that, if $Q(\Phi i - \Phi j) = 0$ for some $i \geq 1$ (in fact it holds for $i = 1$), then $Q\Phi i = Q\Phi j = Q\Phi i - Q\Phi j = 0$. According to the Cayley-Hamilton theorem, there exists $\sum_{i=1}^{n_x} a_i \Phi^i = 0$ in $\mathbb{R}^{n_x}$, such that $Q\Phi^i \Delta \Phi = \sum_{i=1}^{n_x} a_i \Phi^i Q\Phi^i \Delta \Phi = 0$. Hence, this proves that, (1) and (2) are equivalent.

(2) $\iff$ (3): Suppose that (2) holds, equivalently, if $Q(\lambda I - \Phi)^{-1} \Delta \Phi \equiv 0$, then $Q(\lambda I - \Phi)^{-1} \Delta \Phi = 0$ for $i = 1, \ldots, n_x$. In fact, when $\lambda > \rho(\Phi)$, it holds that

$$Q(\lambda I - \Phi)^{-1} \Delta \Phi = Q\lambda^{-1} I + \lambda^{-1} \Phi + \lambda^{-2} \Phi^2 + \cdots \Delta \Phi.$$

To make $Q(\lambda I - \Phi)^{-1} \Delta \Phi = 0$, each coefficient of $\lambda^{-i}$ must be zero. That is, $Q\Phi^i \Delta \Phi = 0$ for $i = 1, \ldots, n_x$, which is equivalent to that $Q\Phi^i \Delta \Phi = 0$ for $i = 1, \ldots, n_x$.

We are now proving (3) $\iff$ (2). Consider the converse-negative direction. Suppose that (2) is not true. If $\lambda > \rho(\Phi)$, by the Cayley-Hamilton theorem, there exists $(a_0, \ldots, a_{n_x-1}) \in \mathbb{R}^{n_x}$, such that

$$(\lambda I - \Phi)^{-1} \Delta \Phi = \lambda^{-1} \sum_{i=0}^{\infty} \lambda^{-i} a_i \Phi^i.$$ 

Hence, $Q(\lambda I - \Phi)^{-1} \Delta \Phi = \sum_{i=0}^{n_x-1} \lambda^{-1-i} a_i Q\Phi^i \Delta \Phi = 0$ holds for all $\lambda > \rho(\Phi)$. This further means that $Q(\lambda I - \Phi)^{-1} \Delta \Phi = 0$ for all $\lambda \in \mathbb{C}$. Hence we have (3) $\Rightarrow$ (2), which finishes the proof.

Condition (3) of Theorem 1 suggests the distinguishability of $(\Phi, Q)$ and $(\Phi, \bar{Q})$ requires that, the perturbation $\Delta \Phi$ in the system state transition matrices can be inferred in the system output response.

Consider the failure set $E = \{E_1, \ldots, E_r\}$. Let $\Phi_i$ be the lumped state transition matrix of the networked system after the link failure $E_i$, $1 \leq i \leq r$, which is defined in the same way as $\Phi$ for $E_i$, and let $\Phi_0 \equiv \Phi$. From Definitions 2 and 3, $E$ is isomorphic, if and only if for any two integers $i, j \in \{0, \ldots, r\}$ with $i \neq j$, $(\Phi_i, Q)$ and $(\Phi_j, Q)$ are distinguishable. Combined with Theorem 1 this immediately leads to the following proposition.

**Proposition 1:** For the networked system 1, a failure set $E = \{E_1, \ldots, E_r\}$ is isomorphic, if for any two integers $i, j \in \{0, \ldots, r\}$, $i \neq j$, $Q(\lambda I - \Phi_i)^{-1} \Delta \Phi_{ij} \neq 0$ holds, where $\Delta \Phi_{ij} = \Phi_i - \Phi_j$.

From their derivations, Theorem 1 and Proposition 1 are valid for arbitrary parameter perturbations $\Delta \Phi$ (or $\Delta \Phi_{ij}$) not necessarily resulting from topology failures. On the basis of Proposition 1 we give a property of an isolable failure set as follows.

**Proposition 2:** For the networked system 1, if a failure set $E = \{E_1, \ldots, E_r\}$ is isomorphic, then there exists a common $x_0 \in \mathbb{R}^{n_x}$, such that for any $i, j \in \{0, \ldots, r\}$, $i \neq j$, $y(x_0, G_i, t) - y(x_0, G_j, t) \neq 0$ holds. Moreover, denote the set of all $x_0$ satisfying the aforementioned condition by $X_0 \subseteq \mathbb{R}^{n_x}$. Then, $\mathbb{R}^{n_x} \setminus X_0$ has zero Lebesgue measure in $\mathbb{R}^{n_x}$.

**Proof:** Notice that

$$y(x_0, G_i, t) - y(x_0, G_j, t) = Q e^{\Phi_i t} x_0 - Q e^{\Phi_j t} x_0$$

$$= Q, Q e^{\text{diag}(\Phi_i, \Phi_j)t} \left[ \begin{array}{c} I \end{array} \right] x_0.$$

Following the proof of Theorem 1 substitute the Taylor expansion of $e^{\text{diag}(\Phi_i, \Phi_j)t}$ into the above formula, use the Cayley-Hamilton theorem, and we obtain that $Q e^{\Phi_i t} x_0 - Q e^{\Phi_j t} x_0 \neq 0$, if and only if

$$\text{col} \left\{ (Q, -Q) \begin{bmatrix} \Phi_i^k & 0 \\ 0 & \Phi_j^k \end{bmatrix} \left[ \begin{array}{c} I \\ I \end{array} \right]_{k=1}^{2n_x-1} \right\} x_0 \neq 0.$$

Hence, if $x_0 \notin \text{ker}(F_{ij})$, then $Q e^{\Phi_i t} x_0 - Q e^{\Phi_j t} x_0 \neq 0$. If $E$ is isolable, by Proposition 1 $F_{ij} \neq 0$. Hence, $\text{ker}(F_{ij})$ is a proper subspace of $\mathbb{R}^{n_x}$. In addition, $\bigcup_{0 \leq i < j \leq r} \text{ker}(F_{ij})$ is also a proper subspace of $\mathbb{R}^{n_x}$ and has zero Lebesgue measure in $\mathbb{R}^{n_x}$, since the union of any finite number of proper subspaces of $\mathbb{R}^{n_x}$ is a proper subspace of $\mathbb{R}^{n_x}$. Therefore, any $x_0$ in $\mathbb{R}^{n_x} \setminus \bigcup_{0 \leq i < j \leq r} \text{ker}(F_{ij})$ makes $y(x_0, G_i, t) - y(x_0, G_j, t) \neq 0$, for $i, j \in \{0, \ldots, r\}$, $i \neq j$.

As a byproduct of Proposition 2 a naive off-line procedure for failure isolation may be built using the obtained formulas. With the knowledge of the faultless dynamics 1 and the prior failure set $E$, randomly generate a common initial state $x_0$, compute and record the output response for each link failure in $E$ (forming an off-line lookup table, possibly with enormous storage and computational complexity). When an unidentified failure in $E$ occurs, record the output response and find the nearest one in the lookup table, whose associated failure then could be isolated.

**IV. GENERICITY OF FAILURE DETECTABILITY AND ISOLABILITY**

From now on, we deal with the situation where the exact values of $\{w_{ij}\}$ are unknown, but their zero-nonzero patterns are accessible. We call a set of real values for $\{w_{ij}\}$ with the corresponding zero-nonzero patterns a weight realization. A property is called *generic*, if either for almost all weight realizations of $\{w_{ij}\}$ except for a set with zero Lebesgue measure in the corresponding parameter space, this property holds true, or for all weight realizations of $\{w_{ij}\}$, this property does not hold. In this section, we will prove that, failure detectability and
isolation are generic properties for the considered networked systems.

**Proposition 3**: For the networked system (1) with known \((A, H, C)\) and zero-nonzero patterns of the weights \(\{w_{ij}\}\), detectability of a failure \(\mathcal{E}_f \subseteq \mathcal{E}\) is a generic property.

**Proof**: Let \(z_1, \ldots, z_{|E|}\) be free parameters in \(\{w_{ij}\}\) that can take nonzero real values independently. Assume that \(\mathcal{E}_f\) is undetectable, which requires that \(Q \Phi^{k-1} \Delta \Phi = 0\) for \(k = 1, \ldots, n_x\), by Theorem 1. Each \(Q \Phi^{k-1} \Delta \Phi = 0\) induces at most \(n_x^2\) scalar equations, and assume that through \(k = 1, \ldots, n_x\), there are in total \(q\) informative constraints (meaning that none of these constraints is a combination of the rest), denoted by
\[
\begin{cases}
f_1(z_1, \ldots, z_{|E|}) = 0 \\
\vdots \\
f_q(z_1, \ldots, z_{|E|}) = 0,
\end{cases}
\]
where each \(f_i(z_1, \ldots, z_{|E|})\) is a polynomial of \((z_1, \ldots, z_{|E|})\) with real coefficients. These constraints are equivalent to \(F(z_1, \ldots, z_{|E|}) = \sum_{i=1}^{n_x} f_i^T(z_1, \ldots, z_{|E|}) = 0\). As \(F(z_1, \ldots, z_{|E|})\) is a polynomial of \((z_1, \ldots, z_{|E|})\), if it is not identically zero, then for almost all values of \((z_1, \ldots, z_{|E|})\) except for the algebraic variety \(\{(z_1, \ldots, z_{|E|}) \in \mathbb{R}^{|E|} : F(z_1, \ldots, z_{|E|}) = 0\}\), there is zero Lebesgue measure in \(\mathbb{R}^{|E|}\), \(F(z_1, \ldots, z_{|E|}) \neq 0\); otherwise, for all values of \((z_1, \ldots, z_{|E|})\) in \(\mathbb{R}^{|E|}\), \(F(z_1, \ldots, z_{|E|}) = 0\). This proves the proposed statement. □

An immediate result from Propositions 1 and 3 is that, distinguishability of \((\Phi_0, I)\) and \((\Phi_0, Q)\) is a generic property for the networked system (1), \(i, j \in \{0, \ldots, r\}\).

**Proposition 4**: For the networked system (1), isolability of a failure set \(\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}\) is a generic property.

**Proof**: By Proposition 1 the statement follows from Proposition 3 and the fact that the union of a finite number of algebraic varieties in \(\mathbb{R}^{|E|}\) also has zero Lebesgue measure in \(\mathbb{R}^{|E|}\). □

**Example 1 (Genericity of Detectability and Isolability)**: Consider a networked system of single-integrators. Let
\[
\Phi = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0, 1, 0 \end{bmatrix}.
\]
Denote \(Z = (a_1, \ldots, a_5)\), and \(\Phi_0 = \Phi\). Consider two failures \(\mathcal{E}_1 = (1, 4)\) and \(\mathcal{E}_2 = (1, 3)\). We obtain
\[
Q(\Lambda I - \Phi_0)^{-1} \Delta \Phi_0 = \begin{bmatrix} a_5(a_1 a_2 + a_3 a_4) \\ -a_5(a_1 a_2 + a_3 a_4) \end{bmatrix}, \quad Q(\Lambda I - \Phi_0)^{-1} \Delta \Phi_{12} = [0, 0, 0, -a_3 a_4].
\]
Hence, \(\mathcal{E}_1\) is detectable in the set \(\{Z \in \mathbb{R}^5 : a_5(a_1 a_2 + a_3 a_4) \neq 0\}\). And \(\{\mathcal{E}_1, \mathcal{E}_2\}\) is isolable in \(\{Z \in \mathbb{R}^5 : a_5(a_1 a_2 + a_3 a_4) \neq 0, a_3 a_4 \neq 0\}\). Both sets are everywhere dense in \(\mathbb{R}^5\). □

The above two propositions reveal that, it is the topology of the faultless networked system, rather than the exact weights of the subsystem links, that dominates detectability and isolability of a given failure (set). We say that a failure \(\mathcal{E}_f\) is **generically detectable**, if for almost all weight realizations of \(\{w_{ij}\}\), \(\mathcal{E}_f\) is detectable for the corresponding networked systems. Similarly, a failure set \(\mathcal{E}\) is **generically isolable**, if for almost all weight realizations of \(\{w_{ij}\}\), \(\mathcal{E}\) is isolable for the corresponding networked systems. From Propositions 3 and 4 if there exists one weight realization for \(\{w_{ij}\}\) such that a given failure is detectable for the corresponding system, then this failure is generically detectable. Such property holds true for generic isolability.

**V. GRAPH-THEORETIC CONDITIONS FOR GENERIC DETECTABILITY AND ISOLABILITY**

In this section, graph-theoretic conditions for generic detectability and isolability of a failure (set) are given for the networked systems.

**A. Conditions for Generic Detectability**

To present the conditions for generic detectability, we first introduce some definitions. For a failure \(\mathcal{E}_f \subseteq \mathcal{E}\), let \(V_{\mathcal{E}}(\mathcal{E}_f)\) denote the set of ending nodes of \(\mathcal{E}_f\). Recall that \(\mathcal{S}\) is the set of locations of sensors. Define a distance index \(d_{\text{min}}\) in \(\mathcal{G}\) as
\[
d_{\text{min}} = \min_{v \in V_{\mathcal{E}}(\mathcal{E}_f), u \in \mathcal{S}} \text{dist}(v, u).
\]
That is, \(d_{\text{min}}\) is the shortest distance between the ending nodes of \(\mathcal{E}_f\) and nodes that are directly measured (i.e., sensor nodes).

**Lemma 1** ([22]): Let \(M\) be an adjacency matrix of a digraph \(\mathcal{D}\) with node set \(\{1, \ldots, N\}\). Then, i) \([M^k]_{ij} = 0\) if \(k < \text{dist}(j, i)\); ii) \([M^k]_{ij} \neq 0\) only if there is path from \(j\) to \(i\) with length \(k\).

**Lemma 2**: Given \(A, H, C \in \mathbb{R}^{n \times n}\), let \(H_s(\lambda) = (\Lambda I - A)^{-1}H\). Let \(\{n_{i \lambda}^\text{max}\}\) be any (infinite or finite) subsequence of \(\{1, 2, \ldots, \infty\}\).

Then, there exists a dense set \(\Lambda \subseteq \mathbb{C}\), such that when \(\lambda \in \Lambda\), \(I + \sum_{i=1}^{\text{max}} H_s^n(\lambda)\) is invertible.

**Proof**: Let \(\{\lambda_k\}_{k=1}^{\infty}\) be the eigenvalues of \(H_s(\lambda)\). Then, the eigenvalues of \(I + \sum_{i=1}^{\text{max}} H_s^n(\lambda)\) are \(\{1 + \sum_{i=1}^{\text{max}} \lambda_k^i\}_{k=1}^{\infty}\). Hence, there exists some dense set \(\Lambda\) such that \((H_s(\lambda))\) is small enough if \(\lambda \in \Lambda\) making \(\text{abs}(\sum_{i=1}^{\text{max}} \lambda_k^i) \leq \sum_{i=1}^{\text{max}} \rho(H_s(\lambda))^{n_i} < 1\). Consequently, all eigenvalues of \(I + \sum_{i=1}^{\text{max}} H_s^n(\lambda)\) are nonzero. □

**Theorem 2**: For the networked system (1) with known \((A, H, C)\) and zero-nonzero patterns of the weights \(\{w_{ij}\}\), a failure \(\mathcal{E}_f \subseteq \mathcal{E}\) is generically detectable, if and only if
\[
d_{\text{min}} \leq r_{\text{max}} - 1. \tag{5}
\]
\(\lambda \geq \frac{1}{2} \rho(A + A^T)\) when \(\lambda\) is large enough, \(\rho(H_s(\lambda))\) is small enough.

If \(r_{\text{max}} = \infty\) and \(d_{\text{min}} = \infty\), this inequality does not hold.
Proof: Note that Condition (3) of Theorem 1 can be used to prove this theorem. We first derive a formula which is used for proving both necessity and sufficiency. Let $W$ be the adjacency matrix of $(\mathcal{V}, \mathcal{E}\setminus \mathcal{E}_f)$. Recall that $\Phi = I_N \otimes A + W \otimes H$. Define $\Delta W = W - W$. Then, $\Phi = \Phi - \Phi = \Delta W \otimes H$. The transfer function $G_f(\lambda) = Q(\lambda I - \Phi)^{-1} \Delta \Phi$

$= S \otimes C(\lambda I - A) - I_N \otimes (\lambda I - A)^{-1} W \otimes H)^{-1}$

$= S \otimes C[I_N \otimes (\lambda I - A) - I_N \otimes (\lambda I - A)^{-1} W \otimes H)^{-1}$

$\cdots \times \Delta W \otimes H$

$= S \otimes C[I_N - W \otimes (\lambda I - A)^{-1}H^1 - I_N \otimes (\lambda I - A)^{-1}$

$\cdots \Delta W \otimes H$ \hspace{1cm} (6)

When $\lambda \in \Lambda \triangleq \{ \lambda \in \mathbb{C} : \rho(W)\rho((\lambda I - A)^{-1}H) < 1 \}$, which is dense in $\mathbb{C}$, (6) can be rewritten as

$G_f(\lambda) = S \otimes C \sum_{k=0}^{r_{\max}} (S \otimes C)^{k} \Delta W \otimes (\lambda I - A)^{-1} H = 0 \hspace{1cm} (7)$

Necessity: Suppose that (5) is not true. Then, either i) $d_{\min} = 0$, $r_{\max} = 0$ or ii) $d_{\min} \geq 1$, and $d_{\min} > r_{\max} - 1$. In case i), as $r_{\max} = 0$, we have $C[(\lambda I - A)^{-1}H]^k = 0$ for $k \geq 1$. Hence, $G_f(\lambda) = 0$ for $\lambda \in \Lambda$, which means $G_f(\lambda) \equiv 0$, leading to the undetectability of $\mathcal{E}_f$. In case ii), without losing generality, suppose that the sensor nodes are indexed as $1, ..., |S|$, and the ending nodes of failure $\mathcal{E}_f$ as $q + 1, ..., N$, $q \geq |S|$. Then, if $k < d_{\min}$, we have the following partitions:

$S = [I_\mathcal{E}], W_k = \begin{bmatrix} W_{k,1}^{11} & W_{k,1}^{12} & 0 \\ W_{k,2}^{11} & W_{k,2}^{12} & W_{k,2}^{13} \\ W_{k,3}^{11} & W_{k,3}^{12} & W_{k,3}^{13} \end{bmatrix}, \Delta W = \begin{bmatrix} 0 \\ 0 \\ \Delta W_3 \end{bmatrix}$ \hspace{1cm} (8)

where $W_{k,1}^{21}, W_{k,2}^{21}, W_{k,3}^{21}$, and $\Delta W_3$ have dimensions respectively $|S| \times |S|, (q-|S|) \times (q-|S|), (N-q) \times (N-q)$, and $(N-q) \times N$, and the rest have compatible dimensions. Note that the $(1,3)$th block of $W_{k}$ is zero due to Lemma 4, and the fact that $k < d_{\min}$. It is easy to see that

$SW_k \Delta W = 0, \forall k \in \{0, ..., d_{\min}\}$. \hspace{1cm} (9)

Hence, $G_f(\lambda) = 0$ for $\lambda \in \Lambda$ from (7), making $G_f(\lambda) \equiv 0$. Thus, $\mathcal{E}_f$ is undetectable.

Sufficiency: By generlicity of detectability, to show sufficiency it is enough to construct a weight realization $\{w_{ij}\}$ associated with which $\mathcal{E}_f$ is detectable. By reordering nodes, suppose that the shortest path from $\mathcal{V}_R(\mathcal{E}_f)$ to $S$ in $\mathcal{G}$ is $\mathcal{P} = \{(d, d_{\min}), (d_{\min} - 1, d_{\min} - 2), \cdots, (2, 1)\}$, and $(i^*, \hat{d}) \in \mathcal{E}_f$, where $d \equiv d_{\min} + 1$. Let the weights of links in $\mathcal{E}_f \setminus \{(i^*, \hat{d}) \cup \mathcal{P}\}$ be zero (then links in $\mathcal{E}_f \setminus \{(i^*, \hat{d}) \}$ have zero weights), while links in $\{(i^*, \hat{d}) \cup \mathcal{P}\}$ have weight 1.

Then,

$W = e_{d_{\min} + 1}^{[N]} + \sum_{i=1}^{d_{\min}} e_i \Delta W = e_{d_{\min} + 1}^{[N]}$. \hspace{1cm} (10)

We consider two cases. See Fig. 2.

Case 1), $i^* = d_{\min} + 1$ (Fig. 2(a)). Without losing generality, let $i^* = d_{\min} + 2$. Since each nonzero entry of $W$ is 1, from Lemma 1, $[W^k]_{ij} = 1$ for $k \in \{0, 1, \cdots, d_{\min}\}$. Hence, considering $S = \{1\}$, we have $SW^k \Delta W = [e_1^{[N]}]^{T}W^k e_{i^*}^{[N]} = 0$ if $k \in \{0, 1, 2, \cdots, d_{\min}\}$, and $SW^k \Delta W [e_1^{[N]}]^{T}$ if $k = d_{\min}$. Consequently, $G_f(\lambda) = [e_{1}^{[N]}]^{T} \otimes CH_s(\lambda)^{d_{\min}+1} \neq 0$ from (7), making $\mathcal{E}_f$ detectable.

Case 2), $i^* \in \{1, 2, \cdots, d_{\min} + 1\}$ (Fig. 2(b)). In this case, from Lemma 1 we have

$[W^k]_{ij} = \begin{cases} 1, \text{ if there is a path from } \hat{d} \text{ to } 1 \text{ with length } k \\ 0, \text{ otherwise} \end{cases}$

Suppose $[W^k]_{ij} = 1$ for $k \in \{n_1, n_2, \cdots\}$, where $n_1 = d_{\min}$. Considering $S = \{1\}$, we have $SW^k \Delta W = [e_1^{[N]}]^{T}W^k e_{i^*}^{[N]} = [e_1^{[N]}]^{T}$ for $k \in \{n_1, n_2, \cdots\}$, and otherwise $SW^k \Delta W = 0$. Substituting these into (7), we get

$G_f(\lambda) = [e_1^{[N]}]^{T} \otimes \left( C \sum_{i=1}^{d_{\max}} [(\lambda I - A)^{-1}H]^{n_i+1} \right)$

$= [e_1^{[N]}]^{T} \otimes CH_s^{d_{\min}+1}(\lambda) \{ I + \sum_{i=2}^{k_{\max}} H_s^{n_i-n_1} \}$

where $k_{\max} = \max \{ k : n_k \leq r_{\max} - 1 \}$. As (5) holds, the value of $\lambda$ making $CH_s^{d_{\min}+1}(\lambda) \neq 0$ is everywhere dense in $\mathbb{C}$. Together with Lemma 2 we know there exists a dense set $\Lambda \subseteq \mathbb{C}$, such that for $\lambda \in \Lambda$, $CH_s^{d_{\min}+1}(\lambda) \neq 0$ and $I + \sum_{i=2}^{k_{\max}} H_s^{n_i-n_1} \lambda$ is invertible, making $G_f(\lambda) \neq 0$. This proves the detectability of $\mathcal{E}_f$ by Theorem 1.

Theorem 2 gives a graph-theoretic condition for generic failure detectability. Notice that $H_s(\lambda)$ is a transfer function from the internal input to the internal output of a subsystem. A deep insight of Theorem 2 indicates that, the necessary and sufficient condition for generic detectability of failure $\mathcal{E}_f$ is that, at least one sensor should receive signals from at least one ending node of the faulty links.

When $A = 0 \in \mathbb{R}^{1 \times 1}$, $B = 0 \in \mathbb{C} = 1$, system (1) collapses to a networked system of single-integrators, or alternatively speaking, the conventional structured system where every entry in the system matrices is either fixed zero or a free parameter (2). In this case, $r_{\max} = \infty$. Theorem 2 immediately leads to the following result.

Corollary 1: For a networked system of single-integrators (or a structured system), a failure $\mathcal{E}_f$ is generically detectable, if and only if there exists a path from one ending node of $\mathcal{E}_f$ to one of the sensor nodes in $\mathcal{G}$.

Example 2: Consider a networked system with 5 subsystems. The parameters for subsystem dynamics are respectively

$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $H = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $C = [1, 0, 0]$. 


which do not appear in $W_i$ (see (8) and (10) respectively). In the proof for necessity, this difference does not violate (9), as the corresponding partitions like (8) still hold. In the proof for sufficiency, such difference leads to that $W$ may not contain $e_{d_i}^{[N]}$ in (10). It is an easy manner to validate such difference does not violate the validity of the remaining arguments.

**Theorem 3:** Consider the networked system (1) with known $(A, H, C)$ and zero-nonzero patterns of weights $\{w_{ij}\}$. A failure set $E = \{E_1, ..., E_r\}$ is generically isolable, if and only if

$$d_{\text{min}}^E \leq r_{\text{max}} - 1,$$

where the transfer index $r_{\text{max}}$ is defined in Section V.A.

**Proof:** This theorem is based on Propositions 1, 5, and Theorem 2. For necessity, if (11) is not true, then there exist two integers $i, j \in \{0, ..., r\}$ such that $d_{ij} > r_{\text{max}} - 1$. From Proposition 5, $\Phi_i$ and $\Phi_j$ are not generically distinguishable, which means that $E$ is not generically isolable.

For sufficiency, let $Z = (z_1, ..., z_\ell)$ be free parameters in $\{w_{ij}\}$ that can take values independently. For each pair $i, j \in \{0, ..., r\}$, $i < j$, following Proposition 5 a numerical realization for $Z$ exists so that $\Phi_i$ and $\Phi_j$ are distinguishable. From Proposition 3 the set of values for $Z$ making $\Phi_i$ and $\Phi_j$ not distinguishable, denoted by $P_{ij}$, has zero Lebesgue measure in $\mathbb{R}^{|E|}$. Hence, any $Z$ in $\mathbb{R}^{|E|}\backslash(\bigcup_{0 \leq i < j \leq r} P_{ij})$ (which is everywhere dense in $\mathbb{R}^{|E|}$) makes $\Phi_i$ and $\Phi_j$ distinguishable, for each pair $(i, j)$ with $0 \leq i < j \leq r$. With Proposition 1 this proves the sufficiency.

**Remark 2:** In Theorem 2 determining $d_{\text{min}}^E$ requires computing $d_{ij}$ for $(r + 1)^2$ times, which grows quadratically with $|E|$. When $|E|$ grows exponentially with $|E|$, this is still a huge computation cost. It is excepted that, exploring the inherent structures of $E$ may sometimes avoid computing all $d_{ij}$ (c.f., Proposition 5).

Theorems 2 and 3 give some fundamental structural limitations for the networked system to support detectability and isolability of a failure (set). These conditions must be satisfied before whatever detection and isolation algorithms are valid.

Using Theorems 2 and 3 the following proposition points out that sensor placement for generic detectability of every single-link failure is equivalent to that for generic isolability of the set of all single-link failures for the networked system, which is a little surprising.

**Proposition 6:** In the networked system (1), if every single-link failure of $G$ is generically detectable for a sensor placement $S$, then the set of all single-link failures (i.e., $E = \tilde{E}$) is generically isolable.

**Proof:** For any link $e_1 = (i, j) \in \tilde{E}$ to be generically detectable, it holds in $G$ that $\min_{j' \in S} \text{dist}(j, j') \leq r_{\text{max}} - 1$. Considering any link $e_2 = (k, l) \in \tilde{E} \backslash \{e_1\}$, in $(\mathcal{V}, \mathcal{E} \backslash \{e_1\})$, we have $\min_{j' \in S} \text{dist}(j, j') \leq r_{\text{max}} - 1$. As $j \in \mathcal{V}(e_1 \cup e_2)$, from Proposition 5, $e_1$ and $e_2$ are generically distinguishable. Hence, by Theorem 3 the set of all single-link failures is generically isolable.
VI. SENSOR PLACEMENT FOR GENERIC DETECTABILITY AND ISOLABILITY

In this section, on the basis of results in Section V we explore the problems of determining the minimum number of sensors to ensure generic detectability and isolability. We will reduce these problems to the hitting set problems and use greedy algorithms to approximate them with guaranteed performances.

A. Sensor Placement Problems

We consider two sensor placement problems.

**Problem 1 (sensor placement for detectability of every single-link failure):** For the networked system (1), determine the minimum number of sensors such that the failure of every single-link of \( E \) is generically detectable.

**Problem 2 (sensor placement for failure isolability):** For the networked system (1), determine the minimum number of sensors such that a given failure set \( E = \{ E_1, ..., E_r \} \) is generically isolable.

B. Hitting Set Problem

Both Problems 1 and 2 are combinatorial problems. To further solve them, we introduce the hitting set problem.

**Definition 4 (Hitting set problem):** Let \( \Sigma = \{ S_1, ..., S_q \} \) be a collection of subsets of \( V \), i.e., \( S_i \subseteq V \), \( \forall i \). The hitting set problem is to find the smallest subset \( \bar{S} \subseteq V \) that intersects (hits) every set in \( \Sigma \), i.e., \( S_i \cap \bar{S} \neq \emptyset \), \( \forall i \).

Hitting set problem is known to be NP-hard. The greedy algorithm (Algorithm 1) can return a solution with a multiplicative factor \( O(\ln q) \), more precisely, \( 1 + \ln q \), of the optimal solution, which is the best approximation performance that could be achieved in polynomial time. The greedy algorithm for solving a hitting set problem is given as Algorithm 1, in which the function \( f(\bar{S}) \) is defined as \( f(\bar{S}) = \sum_{i=1}^{q} \mathbb{1}(S_i \cap \bar{S}) \) for \( \bar{S} \subseteq V \), where function \( \mathbb{1}(x) = 1 \) if \( x \neq \emptyset \), otherwise \( \mathbb{1}(x) = 0 \).

**Algorithm 1:** Greedy Algorithm for Hitting Set Problem

**Input:** \((\Sigma, V)\)

1. Initialize \( \bar{S} = \emptyset \).
2. while \( f(\bar{S}) < q \) do
   3. \( \bar{s} \leftarrow \arg \max_{s \in V \setminus \bar{S}} f(\bar{S} \cup \{s\}) - f(\bar{S}) \)
   4. \( \bar{S} \leftarrow \bar{S} \cup \{ \bar{s} \} \)
3. end while
4. Output: \( \bar{S} \)

C. Analysis and Algorithms

An analytical result is first given as follows, which, immediate from Theorem 2, is the basis of the subsequent derivations.

**Proposition 7:** For the networked system (1), the minimum number of sensors for generic detectability of arbitrary given failure \( E_j \) is 1. Moreover, any node in \( \bigcup_{i \in V \setminus E_j} \{ j \in V : \text{dist}(i, j) \leq r_{\text{max}} - 1 \} \) can be the sensor node.

Consider Problem 1. Denote the set of nodes which has at least one ingoing link (including self-loop) from other nodes in \( G \) by \( V_s \), i.e., \( V_s = \{ i \in V : (j, i) \in E, j \in V \} \). For each \( i \in V_s \), denote the set of nodes whose distance from \( i \) is not greater than \( r_{\text{max}} - 1 \) by \( S_i \), i.e., \( S_i = \{ j \in V : \text{dist}(i, j) \leq r_{\text{max}} - 1 \} \). From Theorem 2 a sensor location \( S \subseteq V \) making every single-link failure generically detectable, if and only if \( S \) intersects every \( S_i \), i.e., \( S_i \cap S \neq \emptyset \), \( \forall i \in V_s \). Let

\[
\Sigma = \{ S_i : i \in V_s \}.
\]

Then, finding the smallest \( S \) is equivalent to solving the hitting set problem on \((\Sigma, V)\). Hence, the greedy algorithm (Algorithm 1) could be adopted to approximate Problem 1.

Consider Problem 2. For each pair \( i, j \in \{0, 1, ..., r\} \) with \( i < j \), define a set \( S_{ij} \subseteq V \) as the set of sensor nodes associated with which \( \Phi_i \) and \( \Phi_j \) are generically distinguishable. From Corollary 1

\[
S_{ij} = \bigcup_{k \in V_{R}(E_{ij})} \{ l \in V : \text{dist}(k, l) \leq r_{\text{max}} - 1 \},
\]

where \( \text{dist}(\cdot) \) is defined on \( G_i \), i.e., \( S_{ij} \) is the set of nodes whose distance from one node of \( V_{R}(E_{ij}) \) is no more than \( r_{\text{max}} - 1 \) in \( G_i \). Afterwards, define a collection \( \bar{\Sigma} \) as

\[
\bar{\Sigma} = \{ S_{01}, S_{02}, ..., S_{0r}, S_{12}, ..., S_{1r}, ..., S_{r-1,r} \}.
\]

From Theorem 3 a sensor location \( \bar{S} \subseteq V \) making \( E \) generically isolable, if and only if \( \bar{S} \) intersects every set in \( \bar{\Sigma} \). Hence, finding the smallest \( \bar{S} \) is equivalent to solving the hitting set problem on \((\bar{\Sigma}, V)\), which could also be approximated via the greedy algorithm.

We summarize the above analysis as follows, along with some guaranteed performances of the associated algorithms.

**Proposition 8:** Problem 1 is equivalent to the hitting set problem on \((\Sigma, V)\). The greedy algorithm (Algorithm 1) can return an \( O(\ln |\bar{\Sigma}_s|) \) approximation of the optimal solution.

**Proposition 9:** Problem 2 is equivalent to the hitting set problem on \((\bar{\Sigma}, V)\). Algorithm 1 can return an \( O(\ln \frac{1}{2}(r+1)r) \) approximation of the optimal solution.

VII. SIMULATIONS AND EXAMPLES

We present some simulations and examples to illustrate the main results of this paper.

A. The Five-Node Networked System in Example 2

Consider the five-node networked system in Example 2. Let all links shown in Fig. 2(a) have weight 1. First, in line with Example 2, to show the detectability of each single-link failure with sensor node \( S = \{1\} \), we collect the output responses of the corresponding systems after each single-link failure with a common random initial state \( x_0 \in \mathbb{R}^{15} \) in Fig. 3. From this figure, the output response after failure \( (1, 2), (2, 3) \), or \( (3, 4) \) is the same as that of the original system, while the output response after failure \( (2, 5), (4, 5), \) or \( (5, 1) \) is different from that of the original system, which means each failure of the former three links is undetectable, and the contrary for the latter three links. This is consistent with the claim made in Example 2 based on Theorem 2. Moreover, suppose our goal is to make every single-link failure detectable using as less sensors.
The greedy algorithm returns sensors so that are consistent with Theorem 3. The output responses of the resulting systems after each failure with a common random initial state $x_0 \in \mathbb{R}^{15}$ are shown in Fig. 4. From this figure, we know that both $\{(4,5),(3,4)\}$ and $\{(4,5)\}$ are detectable. However, $\Xi$ is not isolable because its two elements always generate the same outputs. These results are consistent with Theorem 8.

Finally, suppose our goal is to deployment the smallest sensors so that $\Xi$ is isolable. According to Proposition 9, this problem is equivalent to the hitting set problem defined as

$$\Sigma = \{\{1\}, \{1,5\}, \{4\}, \{5\}\}, \mathcal{V} = \{1, \ldots, 5\}.$$ 

The greedy algorithm returns $\mathcal{S} = \{1,4\}$. Through exhaustive search, this solution is optimal. The isolability of $\Xi$ is validated by the output responses of the corresponding systems after failures; see Fig. 5.

Consider a power network consisting of $N$ generators. The dynamics of each generator around its equilibrium state could be described by the following linearized Swing equation [1]:

$$m_i \ddot{\theta}_i + d_i \dot{\theta}_i = -\sum_{j=1}^{N} k_{ij}(\theta_i - \theta_j), \quad y_i = \theta_i, \quad (12)$$

$i \in \{1, \ldots, N\}$, where $\theta_i$ is the phase angle, $m_i$ and $d_i$ are respectively the inertia and damping coefficients, and $k_{ij}$ is the susceptance of the power line from the $j$th generator to the $i$th one. Rewrite (12) as

$$\begin{bmatrix} \ddot{\theta}_i \\ \dot{\theta}_i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -d_i/m_i & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_i \\ \theta_j \\ \theta_j \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{j=1}^{N} w_{ij}[1,0] [\theta_j \theta_j] \end{bmatrix},$$

$$y_i = [1,0] [\theta_j \theta_j], \quad (13)$$

where $w_{ij} = k_{ij}/m_i$ if $j \neq i$, and $w_{ii} = -\sum_{j=1, j \neq i}^{N} k_{ij}/m_i$, which can be seen as weight of the self-loop $(i,i)$. A typical power network topology is the IEEE-9 bus system shown in Fig. 6 which consists of 9 buses and whose link set is denoted by $\mathcal{E}$. In our analysis, each bus is simplified as a generator [30].

Consider the failure of one bus from the IEEE-9 bus system. For example, suppose that bus 1 is removed from this power network, i.e., $\mathcal{E}' = \{(1,4),(4,1),(1,1)\}$ (it should be noted that, the influence on the self-loops of other nodes is neglected in the current analysis). It can be seen that, $r_{\max} = \infty$ for the dynamics (13) whatever value $-d_i/m_i$ takes. According to Theorem 2, deploying one sensor on an arbitrary bus can detect this failure.

Furthermore, suppose we have the prior knowledge that at most one bus is removed from the power network. Then, in this situation the failure set can be formulated as $\Xi = \{\mathcal{E}_{fi}\}_{i=1}^{n}$, where $\mathcal{E}_{fi} = \{(i,j): (i,j) \in \mathcal{E}\} \cup \{(j,i): (j,i) \in \mathcal{E}\}$, i.e., $\mathcal{E}_{fi}$ collects all ingoing and outgoing links of node $i$. By the greedy algorithm described in Algorithm 1 a sensor placement solution is obtained as $\mathcal{S} = \{4\}$ (in fact, deploying one sensor at an arbitrary bus is feasible for failure isolability). Letting $-d_i/m_i = -1$, $\forall i$, and $w_{ij} = 1$ for any links except the self-loops, we collect in Fig. 7 the output responses of the corresponding systems after every single-node failure with a common random initial state $x_0$. It validates that, indeed, the set of every single-node failure is isolable by the proposed sensor deployment.
Fig. 7. Output responses of the IEEE-9 bus power system after every single-node failure with $S = \{4\}$.

VIII. CONCLUSIONS

In this paper, we study generic detectability and isolability of topology failures for a networked linear system, where subsystem dynamics are given and identical, but the weights of interaction links among them are unknown. We give necessary and sufficient graph-theoretical conditions for generic detectability and isolability. These conditions reveal fundamental structural limitations for the networked systems to support detectability and isolability of a given failure (set), which are irrespective of the exact detection and isolation algorithms adopted. These results are further used to deploy the smallest set of sensors to achieve generic detectability and isolability of a given failure (set). For further topics, it is interesting to extend these results to networked heterogeneous systems, to consider parameter dependencies in the interaction weights, and to develop detection and isolation algorithms like [20].

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