PARABOLIC AND ELLIPTIC EQUATIONS WITH VMO COEFFICIENTS

N.V. KRYLOV

Abstract. An $L^p$-theory of divergence and non-divergence form elliptic and parabolic equations is presented. The main coefficients are supposed to belong to the class $VMO_x$, which, in particular, contains all functions independent of $x$. Weak uniqueness of the martingale problem associated with such equations is obtained.

1. Introduction

The goal of this paper is to expand the $L^p$-theory of parabolic equations to a larger class of operators, having discontinuous coefficients, than previously known. By doing this we also obtain a generalization of a result of Stroock-Varadhan ([15]) about weak uniqueness of solutions of Itô equations, which was our main motivation (see Remark 2.2). For issues related to stochastic processes it is enough to consider the corresponding PDEs in the whole space, and therefore we do not consider boundary-value problems. Uniqueness problem for stochastic equations is an old one. Recently the interest in solving it for discontinuous coefficients reappeared in connection with diffusion approximation (see [11], [12]).

According to the famous counterexample of Nadirashvili there could not exist theory of solvability of equations with general discontinuous coefficients even if they are uniformly bounded and equations are uniformly elliptic. Therefore, much effort was applied to treat particular cases of discontinuity. First came equations with piecewise continuous coefficients, see [13] and [7]. Then, truly remarkable and absolutely unpredictable results about $W^2_p$-estimates for elliptic equations with VMO coefficients appeared in [4]. They were later developed into existence theory for non-divergence form elliptic and parabolic equations in [5] and [1]. The results in [1], [4], and [5] are based on deep versions of the Calderón-Zygmund theorem and estimates of certain commutators.

1991 Mathematics Subject Classification. 35K10, 35J15, 60J60.

Key words and phrases. Second-order equations, vanishing mean oscillation, martingale problem.

The work was partially supported by NSF Grant DMS-0140405.
A different approach to divergence form elliptic and parabolic equations with VMO coefficients is developed in [2] and [3]. These two papers also could be used as a good source of further references on the subject of VMO and equations in divergence and non-divergence forms. One can also consult papers [6] and [14] for various versions and extensions.

In what concerns parabolic equations there is a flaw in the results in [1], [3], and [6]. Namely, these do not contain quite classical results about solvability in Sobolev spaces of equations whose leading coefficients depend only on \( t \) and are just measurable functions (see [15] and the references therein).

We correct this flaw and treat divergence and non-divergence form elliptic and parabolic operators, the main coefficients of which are in VMO. Actually, as in [2] and [3], a slightly more general class of coefficients is allowed (see Remark 2.9). In contrast with many of the above references we do not consider boundary-value problems for the reasons explained in the beginning. This also makes the presentation clearer and allows us to use a unified approach to elliptic and parabolic divergence and non-divergence form equations. We do not treat the \( L_q - L_p \) theory either and only mention that parabolic equations with mixed norms and coefficients constant in time are considered in [6] and with coefficients, which are uniformly continuous in \( x \) and measurable in \( t \), in [10].

In a sense our approach is a combination of the approach in [4], [5], [11] and the one in [2], [3]. On the one hand, we use pointwise estimates of the sharp function of second order derivatives, on the other hand, we do not use integral representations of these derivatives to deal with contributions from “far away”, but deal with these contributions by splitting the function into two parts, one of which is “harmonic”, that is satisfies the homogeneous equation.

The article is organized as follows. Section 2 contains our main results. In Section 3 we discuss some auxiliary results that are later used for non-divergence and divergence form equations. Section 4 is devoted to proving our main results for non-divergence type equations. Then comes Section 5 with few more auxiliary results needed for divergence type equations and the short final Section 6 deals with the proofs of our results for such equations.

Hongjie Dong and Doyoon Kim kindly showed the author few errors in the original version of the article for which the author is sincerely grateful.
2. Main results

Let $\mathbb{R}^d$ be a $d$-dimensional Euclidean space of points $x = (x^1, ..., x^d)$ and

$$\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}.$$ 

We are dealing with four types of operators: parabolic

$$Lu(t, x) = u_t(t, x) + a^{ij}(t, x)u_{x_i x_j}(t, x) + b^i(t, x)u_{x_i}(t, x) + c(t, x)u(t, x),$$

acting on functions given on $\mathbb{R}^{d+1}$ and elliptic

$$Mu(x) = a^{ij}(x)u_{x_i x_j}(x) + b^i(x)u_{x_i}(x) + c(x)u(x),$$

acting on functions given on $\mathbb{R}^d$. We assume that the coefficients of these operators are measurable and by magnitude are dominated by a constant $K < \infty$. We also assume that the matrices $a = (a^{ij})$ are, perhaps, nonsymmetric and satisfy

$$a^{ij} \lambda_i \lambda_j \geq \delta |\lambda|^2$$

for all $\lambda \in \mathbb{R}^d$ and all possible values of arguments. Here $\delta > 0$ is a fixed constant.

To state our last assumption we set $B_r(x)$ to be the open ball in $\mathbb{R}^d$ of radius $r$ centered at $x$, $B_r = B_r(0)$, $Q_r(t, x) = (t, t + r^2) \times B_r(x)$, and $Q_r = Q_r(0, 0)$. Denote

$$\text{osc}_x(a, Q_r(t, x)) = r^{-2}|B_r|^{-2} \int_{t}^{r^2} \int_{y, z \in B_r(x)} |a(s, y) - a(s, z)| \, dy \, dz, ds,$$

$$a^{(x)}_R = \sup_{(t, x) \in \mathbb{R}^{d+1}} \sup_{r \leq R} \text{osc}_x(a, Q_r(t, x)), \quad a^{(x)}_\infty = a^{(x)}_\infty.$$ 

This definition is either naturally modified if $a$ is independent of $t$ as in the elliptic operators or is kept as is. We assume that $a \in VMO_x$, that is

$$\lim_{R \to 0} a^{(x)}_R = 0.$$ 

For convenience of stating our results we take any continuous function $\omega(R)$ on $[0, \infty)$, such that $\omega(0) = 0$ and $a^{(x)}_R \leq \omega(R)$ for all $R \in [0, \infty)$. Obviously, $a \in VMO_x$ if $a$ depends only on $t$.

By $W^1_p$ and $W^2_p$ we denote the usual Sobolev spaces on $\mathbb{R}^d$. Also for $T \in (0, \infty)$ introduce

$$\Omega(T) = (0, T) \times \mathbb{R}^d.$$
and, as usual, define $W^{1,2}_p(T)$ as the closure of the set $C^{1,2}(\Omega(T))$ in the norm
$$
\|u\|_{W^{1,2}_p(T)} = \|u\|_{L^p(\Omega(T))} + \|u_x\|_{L^p(\Omega(T))} + \|u_{xx}\|_{L^p(\Omega(T))} + \|u_t\|_{L^p(\Omega(T))}.
$$
By $\overset{\circ}{W}^{1,2}_p(T)$ we mean the closure in the same norm of the subset of $C^{1,2}(\Omega(T))$ consisting of functions vanishing for $t = T$. Finally,
$$
\mathcal{H}_p(T) = (1 - \Delta)^{1/2}W^{1,2}_p(T), \quad \overset{\circ}{\mathcal{H}}_p(T) = (1 - \Delta)^{1/2}\overset{\circ}{W}^{1,2}_p(T),
$$
where $\Delta$ is the Laplacian in $x$ variables. Needless to say all equations below are understood in the sense of generalized functions.

Now we fix $T \in (0, \infty)$ and $p \in (1, \infty)$ and state our main results.

**Theorem 2.1.** For any $f \in L^p(\Omega(T))$ there exists a unique $u \in \overset{\circ}{W}^{1,2}_p(T)$ such that $Lu = f$. Furthermore, there is a constant $N$, depending only on $d$, $T$, $K$, $\delta$, $p$, and the function $\omega$, such that for any $u \in \overset{\circ}{W}^{1,2}_p(T)$ we have
$$
\|u\|_{\overset{\circ}{W}^{1,2}_p(T)} \leq N\|Lu\|_{L^p(\Omega(T))}.
$$

**Remark 2.2.** The following is aimed at specialists in stochastic processes. There are solutions of the stochastic differential equation associated with the operator $L$. We also know that Itô’s formula is applicable to $u \in W^{1,2}_p(T)$ if $p \geq d + 1$. It follows that the solution $u$ is represented as the expectation of certain integral functional containing $f$. Such expectations are therefore uniquely defined by $L$. This leads to weak uniqueness of solutions of stochastic differential equations with uniformly nondegenerate bounded diffusion of class $VMO_x$ and bounded measurable drift. More details can be found in [15], where the weak uniqueness is proved for equations with uniformly nondegenerate bounded diffusion, which is continuous in $x$ uniformly in $t$, and bounded measurable drift.

**Remark 2.3.** Estimate (2.2) is similar to interior estimates from [1]. However, the space $VMO$ in [1] does not include functions which are independent of $x$ and are measurable in $t$.

**Theorem 2.4.** Let $f = (f^1, ..., f^d)$, $g, f^i \in L^p(\Omega(T))$ for $i = 1, ..., d$. Then there is a unique $u \in \overset{\circ}{\mathcal{H}}_p(T)$ such that
$$
Lu = \text{div} f + g.
$$
Furthermore, there is a constant $N$, depending only on $d$, $T$, $K$, $\delta$, $p$, and the function $\omega$, such that
$$
\|u\|_{L^p(\Omega(T))} + \|u_x\|_{L^p(\Omega(T))} \leq N(\|f\|_{L^p(\Omega(T))} + \|g\|_{L^p(\Omega(T))}).
$$
Remark 2.5. Estimate (2.3) is similar to interior estimates from \[3\]. However, like in \[1\], the space \(VMO\) in \[3\] is defined through approximations by constants and does not include functions which are independent of \(x\) and are measurable in \(t\). Also there are no lower order terms in \(L^p\) in \[3\].

Theorem 2.6. There exists a constant \(\lambda_0\), depending only on \(d, K, \delta, p\), and the function \(\omega\), such that, for any \(\lambda \geq \lambda_0\) and \(f \in L^p(\mathbb{R}^d)\) there exists a unique \(u \in W^{2,p}\) satisfying \(Mu - \lambda u = f\).

Furthermore, there is a constant \(N\), depending only on \(d, K, \delta, p\), and the function \(\omega\), such that for any \(u \in W^{2,p}\) and \(\lambda \geq \lambda_0\) we have

\[
\lambda \|u\|_{L^p(\mathbb{R}^d)} + \|u\|_{W^{2,p}} \leq N \|(M - \lambda)u\|_{L^p(\mathbb{R}^d)}.
\] (2.4)

Remark 2.7. Without much stretching the truth one can say that Theorem 2.6 belongs to the authors of \[4\]. The following theorem is close to some results of \[2\], in which, however, the lower order terms are not allowed.

Theorem 2.8. There exists a constant \(\lambda_0\), depending only on \(d, K, \delta, p\), and the function \(\omega\), such that, for any \(\lambda \geq \lambda_0\) and \(f = (f^1, \ldots, f^d), g \in L^p(\mathbb{R}^d)\) there exists a unique \(u \in W^{1,p}\) satisfying \(Mu - \lambda u = \text{div} f + g\).

Furthermore, there is a constant \(N\), depending only on \(\lambda, d, K, \delta, p\), and the function \(\omega\), such that

\[
\|u\|_{W^{1,p}} \leq N (\|f\|_{L^p(\mathbb{R}^d)} + \|g\|_{L^p(\mathbb{R}^d)}).
\]

Remark 2.9. As usual in such situations, from our proofs one can see that instead of the assumption that \(a \in VMO_x\) we are, actually, using that there exists \(R \in (0, \infty)\) such that \(a^{\#}(x) \leq \varepsilon\), where \(\varepsilon > 0\) is a constant depending only on \(d, p, \delta, K\).

Remark 2.10. Denote

\[ u_{Q_r(t,x)} = \int_{Q_r(t,x)} u(s,y) \, dy \, ds, \]

the average value of a function \(u(s,y)\) over \(Q_r(t,x)\) and

\[ u_{B_r(x)}(t) = \int_{B_r(x)} u(t,y) \, dy \]

the average value of a function \(u(t,y)\) over \(B_r(x)\).

Also introduce \(A\) as the set of \(d \times d\) matrix-valued measurable functions \(\bar{a} = \bar{a}(t)\) depending only on \(t\), satisfying conditions (2.1) and such that \(|\bar{a}^{ij}| \leq K\). A standard fact to recall is that for any \(\bar{a} \in A\)

\[
\text{osc}_x(a, Q_r) \leq 2 \int_{Q_r} |a(s,x) - \bar{a}(s)| \, dx \, ds
\]
and for $\bar{a}(t) = a_{B_r}(t)$
\[
\int_{Q_r} |a(s, x) - \bar{a}(s)| \, dx \, ds \leq \text{osc}_x (a, Q_r).
\]
This allows one to give obvious equivalent definitions of $VMO_x$.

3. Auxiliary results

In the lemmas of this section
\[
\bar{L}u(t, x) = \bar{a}^{ij}(t) u_{x^i x^j}(t, x) + u_t(t, x),
\]
where $\bar{a} \in A$.

**Lemma 3.1.** Let $p \in [1, \infty)$, $R \in (0, \infty)$, $u \in C^\infty_{loc}(\mathbb{R}^{d+1})$,
\[
f = (f^1, ..., f^d), \quad f^i, g \in L^p_{loc}(\mathbb{R}^{d+1}),
\]
and $\bar{L}u = \text{div } f + g$ in $Q_R$. Then
\[
\int_{Q_R} |u(t, x) - u_{Q_R}|^p \, dx \, dt \leq N R^p \int_{Q_R} (|u_x|^p + |f|^p + R^p |g|^p) \, dx \, dt,
\]
where $N = N(d, K, p)$.

Proof. Assume (3.2) is true for $R = 1$. Substitute $v(t, x) = u(R^2 t, Rx)$ in (3.2) written for $R = 1$ and $v$ in place of $u$. Observe that
\[
v_{Q_1} = u_{Q_R}, \quad \int_{Q_1} |v(t, x) - v_{Q_1}|^p \, dx \, dt = \int_{Q_R} |u(t, x) - u_{Q_R}|^p \, dx \, dt,
\]
\[
\int_{Q_1} |v_x|^p \, dx \, dt = R^p \int_{Q_R} |u_x|^p \, dx \, dt,
\]
\[
\bar{L}^R v(t, x) = R^2 (\bar{L}u)(R^2 t, Rx)
\]
\[
= R(\text{div } (f(R^2 t, R \cdot))(x) + R^2 g(R^2 t, Rx),
\]
where $\bar{L}^R$ is constructed from $\bar{a}(R^2 t)$. Then (3.2) with $R = 1$ and $v$ in place of $u$ yields
\[
\int_{Q_R} |u(t, x) - u_{Q_R}|^p \, dx \, dt \leq N R^p \int_{Q_R} |u_x|^p \, dx \, dt
\]
\[
+ N R^p \int_{Q_R} |f|^p \, dx \, dt + N R^{2p} \int_{Q_R} |g|^p \, dx \, dt.
\]

Hence, we need only prove (3.2) for $R = 1$. In that case take a function $\zeta \in C^\infty_0(B_1)$ with unit integral. Then by Poincaré’s inequality, for any $t \in (0, 1)$ and
\[
\bar{u}(t) = \int_{B_1} \zeta(y) u(t, y) \, dy.
\]
we have
\[ \int_{B_1} |u(t, x) - \bar{u}(t)|^p \, dx = \int_{B_1} |\int_{B_1} [u(t, x) - u(t, y)] \zeta(y) \, dy|^p \, dx \]
\[ \leq N \int_{B_1} \int_{B_1} |u(t, x) - u(t, y)|^p \, dx \, dy \leq N \int_{B_1} |u_x(t, x)|^p \, dx. \quad (3.3) \]

Observe that for any constant \( c \) the left-hand side of (3.2) is less than a constant times (remember \( R = 1 \))
\[ \int_{Q_1} |u(t, x) - c|^p \, dx \, dt \leq 2^p \int_{Q_1} |u(t, x) - \bar{u}(t)|^p \, dx \, dt + 2^p \int_0^1 |\bar{u}(t) - c|^p \, dt. \]

By (3.3) the first term on the right is less than the right-hand side of (3.2). To estimate the second term, take
\[ c = \int_0^1 \bar{u}(t) \, dt. \]

Then by Poincaré’s inequality
\[ \int_0^1 |\bar{u}(t) - c|^p \, dt \leq N \int_0^1 | \int_{B_1} \zeta u_t \, dx|^p \, dt, \]
where \( u_t = -(\bar{a}^{ij} u_{x^i})_{x^j} + \text{div} f + g \). Integrating by parts with respect to \( x \) shows that this term is also less than the right-hand side of (3.2).

The lemma is proved.

Lemma 3.2. There is a constant \( N = N(d) \) such that for any \( R > 0 \) and \( u \in C^\infty_\text{loc}(\mathbb{R}^{d+1}) \) we have
\[ \int_{Q_R} |u_{x^i}(t, x) - (u_{x^i})_{Q_R}| \, dx \, dt \leq N R \int_{Q_R} (|u_{xx}| + |u_t|) \, dx \, dt. \quad (3.4) \]
\[ \int_{Q_R} |u(t, x) - u_{Q_R} - x^i(u_{x^i})_{Q_R}| \, dx \, dt \leq N R^2 \int_{Q_R} (|u_{xx}| + |u_t|) \, dx \, dt. \quad (3.5) \]

Proof. To prove (3.4) it suffices to take \( \bar{a}^{ij} = \delta^{ij} \), introduce \( f = \bar{L}u \), note that \( \bar{L}(u_x) = f_x \), and apply Lemma 3.1 with \( p = 1 \).

To prove (3.5) set \( v(t, x) = u(t, x) - u_{Q_R} - x^i(u_{x^i})_{Q_R} \) and observe that
\[ v_{Q_R} = 0, \quad v_x = u_x - (u_x)_{Q_R}. \]
Hence for \( g := \bar{L}v \ (= \bar{L}u) \) and \( f \equiv 0 \) by Lemma 3.1 we find
\[ \int_{Q_R} |u(t, x) - u_{Q_R} - x^i(u_{x^i})_{Q_R}| \, dx \, dt = \int_{Q_R} |v(t, x) - v_{Q_R}| \, dx \, dt \]
\[
\leq NR \int_{Q_R} (|u_x - (u_x)Q_R| + R| u_t | + R|u_{xx}|) \, dx dt.
\]

It only remains to use (3.4). The lemma is proved.

Define the parabolic boundary of \( Q_r(t, x) \) by

\[
\partial' Q_r(t, x) = ([t, t + r^2] \times \partial B_r(x)) \cup \{(t + r^2, y) : y \in B_r(x)\}.
\]

**Lemma 3.3.** Let \( u \in C^\infty_{\text{loc}}(\mathbb{R}^{d+1}) \) and \( \bar{a} \) be infinitely differentiable. Then there is a unique function \( h \in C^{1,2}_{\text{per}}(\bar{Q}_4) \) such that \( \bar{L}h = 0 \) in \( Q_4 \) and \( h = u \) on \( \partial Q_4 \). Furthermore, \( h \) is infinitely differentiable in \( Q_4 \) and in \( Q_1 \) we have

\[
|h_{xx}| + |h_{tx}| + |h_{xxx}| + |h_{txx}| \leq N(d, K, \delta) \int_{Q_4} (|u_{xx}| + |u_t|) \, dx dt. \tag{3.6}
\]

Proof. The existence, uniqueness, and the stated properties of continuity \( h \) and its derivatives are classical results.

Therefore, we concentrate on proving (3.6). First, notice that by subtracting an appropriate affine function of \( x \) from \( u \) and \( h \) we reduce the general case to the one that \( u_{Q_4} = (u_{x1})_{Q_4} = \ldots = (u_{xd})_{Q_4} = 0 \). \tag{3.7}

Then, as in the proof of Theorem 8.4.4 of [9] by using Bernstein’s method one proves that for the derivative \( D^\alpha \) of any order with respect to \( x \)

\[
\sup_{Q_1} |D^\alpha h| \leq N(d, K, \delta, \alpha) \sup_{Q_2} |h|.
\]

Since \( h_{tx} = -\bar{a}^{ij}h_{xixj} \) and \( h_{txx} = -\bar{a}^{ij}h_{xxixj} \), it follows that to prove (3.6) it suffices to prove that

\[
|h| \leq N \int_{Q_4} (|u_{xx}| + |u_t|) \, dx dt \tag{3.8}
\]

on \( Q_2 \) under the assumption that (3.7) holds.

Now, take an infinitely differentiable function \( \zeta \) on \( Q_4 \) such that it equals 1 near \( \partial Q_4 \) and zero inside \( Q_3 \). Without loss of generality assume that \( \bar{a} \) is symmetric. Then \( v = h - \zeta u \) satisfies

\[
\bar{L}v = -\zeta \bar{L}u - 2\bar{a}^{ij}\zeta u_{xj} - u\bar{L}\zeta
\]

and \( v = 0 \) on \( \partial Q_4 \). By the maximum principle \(|v|\) is less than the bounded solution \( w \) of the Cauchy problem

\[
\bar{L}w = -(|\zeta \bar{L}u| + 2|\bar{a}^{ij}\zeta u_{xj}| + |u\bar{L}\zeta|) I_{Q_4} =: f \tag{3.9}
\]

in \( \{ t \leq 16 \} \) with zero terminal condition for \( t = 16 \). This solution is written explicitly as the convolution of \( f \) and a kernel admitting Gaussian-like estimates. Since \( f \) vanishes inside \( Q_3 \), the convolution in
Q_2 is estimated by the integral of f over \( \mathbb{R}^{d+1} \). After that (3.8) follows from Lemma 3.2. The lemma is proved.

**Lemma 3.4.** Let \( \bar{a}(t) \) be infinitely differentiable. We assert that there exists a constant \( N = N(d, \delta, K) \) such that for any \( \kappa \geq 4, r > 0, u \in C^{\infty}_{\text{loc}}(\mathbb{R}^{d+1}), \) and the solution of \( \bar{L}h = 0 \) in \( Q_{\kappa r} \) with boundary condition \( h = u \) on \( \partial Q_{\kappa r} \) we have

\[
|h_{xx} - (h_{xx})_{Q_r}|_{Q_r} \leq N \kappa^{-1}(|u_{xx}| + |u_t|)_{Q_{\kappa r}}.
\]

(3.10)

**Proof.** Parabolic dilations allow us to only concentrate on \( r = 1 \). The same argument and Lemma 3.3 show that the inequality

\[
\kappa|h_{xxx}| + \kappa^2|h_{txx}| \leq N(d, K, \delta)(|u_{xx}| + |u_t|)_{Q_{\kappa}}\quad (3.11)
\]

holds in \( Q_{\kappa/4} \). Since \( \kappa \geq 4, (3.11) \) holds in \( Q_1 (= Q_r) \). After that it only remains to observe that the left-hand side of (3.10) with \( r = 1 \) is less than a constant times

\[
\sup_{Q_1}(|h_{xxx}| + |h_{txx}|) \leq \kappa^{-1} \sup_{Q_1}(\kappa|h_{xxx}| + \kappa^2|h_{txx}|) \leq N \kappa^{-1}(|u_{xx}| + |u_t|)_{Q_{\kappa}}.
\]

The lemma is proved.

**Lemma 3.5.** Let \( q \in (1, \infty) \). Then there exists a constant \( N = N(q, d, \delta, K) \) such that for any \( \kappa \geq 4, r > 0, u \in C^{\infty}_{\text{loc}}(\mathbb{R}^{d+1}), \) we have

\[
|u_{xx} - (u_{xx})_{Q_r}|_{Q_r} \leq N \kappa^{-1}(|\bar{L}u| + |u_{xx}|)_{Q_{\kappa r}} + N \kappa^{(d+2)/q}(|\bar{L}u|^q)_{Q_{\kappa r}}^{1/q}.
\]

(3.12)

**Proof.** We may certainly assume that \( \bar{a} \) is infinitely differentiable. In that case by Lemma 3.3

\[
|h_{xx} - (h_{xx})_{Q_r}|_{Q_r} \leq N \kappa^{-1}(|u_{xx}| + |\bar{L}u|)_{Q_{\kappa r}}.
\]

(3.13)

Furthermore, \( \bar{L}(u - h) = \bar{L}u \) in \( Q_{\kappa r} \) and \( u - h = 0 \) on \( \partial Q_{\kappa r} \). If \( \bar{a} \) were constant, then by the standard Sobolev space theory we would have

\[
\int_{Q_{\kappa r}} |u_{xx} - h_{xx}|^q \, dx \, dt \leq N \int_{Q_{\kappa r}} |\bar{L}u|^q \, dx \, dt,
\]

where \( N = N(d, \delta, K, q) \). This estimate is certainly true even if \( \bar{a} \) is not constant. However, we could not find it in the literature and instead with some reluctance we are going to use Theorem 2.10 of \( \mathcal{S} \) (see also Remark 2.9 there), which implies that

\[
\int_{Q_{\kappa r}} (\kappa^2 r^2 - |x|^2)^q |u_{xx} - h_{xx}|^q \, dx \, dt \leq N \int_{Q_{\kappa r}} (\kappa^2 r^2 - |x|^2)^q |\bar{L}u|^q \, dx \, dt,
\]
where \( N \) depends only on \( \kappa r, q, d, K, \) and \( \delta. \) Observe that \( \kappa^2 r^2 - |x|^2 \geq \kappa^2 r^2 / 2 \) in \( Q_{\kappa r/2} \) and \( \kappa^2 r^2 - |x|^2 \leq \kappa^2 r^2 \) in \( Q_{\kappa r/2} \). It follows that

\[
\int_{Q_{\kappa r/2}} |u_{xx} - h_{xx}|^q \, dx \, dt \leq N \int_{Q_{\kappa r}} |\bar{L}u|^q \, dx \, dt.
\]

Parabolic dilations show that \( N \) is independent of \( \kappa r, \) and since \( \kappa r / 2 \geq r \) we get

\[
\int_{Q_r} |u_{xx} - h_{xx}|^q \, dx \, dt \leq N \int_{Q_{\kappa r}} |\bar{L}u|^q \, dx \, dt.
\]

By Hölder’s inequality

\[
|u_{xx} - h_{xx}|_{Q_r} \leq N \kappa^{(d+2)/q} (|\bar{L}u|^q)_{Q_{\kappa r}}^{1/q},
\]

which after being combined with (3.13) shows that there is a constant matrix \( \sigma = (h_{xx})_{Q_r} \) such that \( (|u_{xx} - \sigma|)_{Q_r} \) is less than the right-hand side of (3.12). The discussion in the end of Section 2 shows that this proves the lemma.

Set

\[
L_0 u(t, x) = u_t (t, x) + a^{ij} (t, x) u_{x^i x^j} (t, x)
\]

and introduce the (parabolic) maximal and sharp functions of \( g \) by

\[
M g(t, x) = \sup_{r > 0} \int_{Q_r(t, x)} |g(s, y)| \, dy \, ds,
\]

\[
g^#(t, x) = \sup_{r > 0} \int_{Q_r(t, x)} |g(s, y) - g_{Q_r(t, x)}| \, dy \, ds.
\]

Here is the main result of this section, in which all assumptions of Section 2 are imposed apart from the assumption that \( a \in \mathcal{MO}_x. \)

**Theorem 3.6.** Let \( q, \alpha, \beta \in (1, \infty), \) \( \alpha^{-1} + \beta^{-1} = 1, \) and \( R \in (0, \infty). \) Then there exists a constant \( N = N(d, \delta, K, q, \alpha) \) such that for any \( u \in C^\infty_0 (Q_R) \) we have

\[
(u_{xx})^{#} \leq N \left[ M(|L_0 u|^q) \right]^{\mu/q} \left[ M|u_{xx}| \right]^{1-\mu} + N \hat{a}^{\mu/(\beta q)} \left[ M(|u_{xx}|^{\alpha q}) \right]^{1/(\alpha q)},
\]

on \( \mathbb{R}^{d+1}, \) where \( \mu = q/(q + d + 2), \) \( \hat{a} = a^{#}(x). \)

Proof. First, fix \( \kappa \geq 4, r \in (0, \infty), \) and \( (t_0, x_0) \in \mathbb{R}^{d+1}. \) Introduce

\[
\tilde{a}^{ij} (t) = a_{ij}^{1/2} (t_0, x_0) \quad \text{if} \quad \kappa r < R, \quad \tilde{a}^{ij} (t) = a_{ij}^{1/2} (t) \quad \text{if} \quad \kappa r \geq R,
\]

\[
A^{q}_\rho = (|L_0 u|^q)_{Q_\rho(t_0, x_0)}, \quad B^{q}_\rho = |u_{xx}|_{Q_\rho(t_0, x_0)}, \quad C^{\alpha q}_\rho = (|u_{xx}|^{\alpha q})_{Q_\rho(t_0, x_0)},
\]

\[
A = \sup_{\rho > 0} A^{q}_\rho, \quad B = \sup_{\rho > 0} B^{q}_\rho, \quad C = \sup_{\rho > 0} C^{\alpha q}_\rho.
\]
By Lemma \ref{lem:holder}

\[
|u_{xx} - (u_{xx})_{Q_{r}(t_{0},x_{0})}|_{Q_{r}(t_{0},x_{0})} \leq N \kappa^{-1} (|L u| + |u_{xx}|)_{Q_{r}(t_{0},x_{0})} + N \kappa^{(d+2)/q} (|\bar{L} u|^q)_{Q_{r}(t_{0},x_{0})}.
\]

By using Hölder’s inequality and the fact that \(\kappa^{-1} \leq 1\) we obtain

\[
\kappa^{-1}|\bar{L} u|_{Q_{r}(t_{0},x_{0})} \leq \kappa^{(d+2)/q} (|\bar{L} u|^q)_{Q_{r}(t_{0},x_{0})},
\]

\[
|u_{xx} - (u_{xx})_{Q_{r}(t_{0},x_{0})}|_{Q_{r}(t_{0},x_{0})} \leq N \kappa^{-1} B_{\kappa r} + N \kappa^{(d+2)/q} (|\bar{L} u|^q)_{Q_{r}(t_{0},x_{0})}.
\]

Here

\[
\int_{Q_{r}(t_{0},x_{0})} |Lu|^q \, dx \, dt \leq 2q (I + J),
\]

where

\[
I = \int_{Q_{r}(t_{0},x_{0})} |L_0 u|^q \, dx \, dt \leq N (\kappa r)^{d+2} A_{\kappa r}^q,
\]

\[
J = \int_{Q_{r}(t_{0},x_{0})} |(L_0 - \bar{L}) u|^q \, dx \, dt = \int_{Q_{r}(t_{0},x_{0}) \cap Q_R} \ldots \leq N J_1^{1/\alpha} J_2^{1/\beta},
\]

\[
J_1 = \int_{Q_{r}(t_{0},x_{0})} |u_{xx}|^{q_0} \, dx \, dt \leq N (\kappa r)^{d+2} C_{\kappa r}^{q_0},
\]

\[
J_2 = \int_{Q_{r}(t_{0},x_{0}) \cap Q_R} |a(t,x) - \bar{a}(t)|^{q_3} \, dx \, dt.
\]

If \(\kappa r \geq R\), then we estimate \(J_2\) by the integral over \(Q_R\), which is less than

\[
NR^{d+2} \int_{Q_R} |a(t,x) - \bar{a}(t)| \, dx \, dt \leq N (\kappa r)^{d+2} a_R^{\#(x)}.
\]

In case \(\kappa r < R\) we estimate \(J_2\) by

\[
N (\kappa r)^{d+2} \int_{Q_{r}(t_{0},x_{0})} |a(t,x) - \bar{a}(t)| \, dx \, dt \leq N (\kappa r)^{d+2} a_{\kappa r}^{\#(x)} \leq N (\kappa r)^{d+2} a_R^{\#(x)}.
\]

It follows that

\[
J \leq N (\kappa r)^{d+2} a_{1/\beta}^{1/\beta} C_{\kappa r}^{q}
\]

and

\[
\int_{Q_{r}(t_{0},x_{0})} |\bar{L} u|^q \, dx \, dt \leq N (\kappa r)^{d+2} A_{\kappa r}^q + N (\kappa r)^{d+2} a_{1/\beta}^{1/\beta} C_{\kappa r}^q.
\]

Coming back to (3.15) we get

\[
|u_{xx} - (u_{xx})_{Q_{r}(t_{0},x_{0})}|_{Q_{r}(t_{0},x_{0})} \leq N \kappa^{-1} B_{\kappa r} + N \kappa^{(d+2)/q} (A_{\kappa r} + a_{1/(\beta q)} C_{\kappa r})
\]
\[
\leq N\kappa^{-1}B + N\kappa^{(d+2)/q}(A + \hat{a}^{1/(\beta q)}C). \tag{3.16}
\]

So far \(\kappa \geq 4\) and \(r > 0\) were fixed. Now we allow them to vary and observe that \eqref{3.16} is also true for \(\kappa \in (0, 4)\) since \(B\) is present on the right. After that upon taking supremums with respect to \(r > 0\) and then minimizing with respect to \(\kappa > 0\) we come to
\[
\left(u_{xx}\right)^\#(t_0, x_0) \leq N\left[\hat{a}^{1/(\beta q)}C + A\right]^\mu B^{1-\mu}
\]
\[
\leq N\hat{a}^{\mu/(\beta q)}C^\mu B^{1-\mu} + NA^\mu B^{1-\mu}.
\]

By noting that \(B \leq C\) and replacing \(B\) with \(C\) in the first term on the right we come to what is precisely \eqref{3.14} at point \((t_0, x_0)\). The theorem is proved.

Set
\[
L_p = L_p(\mathbb{R}^{d+1}). \tag{3.17}
\]

**Corollary 3.7.** For any \(p \in (1, \infty)\) there exists a constant \(\varepsilon > 0\), depending only on \(p, d, K, \) and \(\delta\), such that if \(a_R^{\#(x)} \leq \varepsilon\) for an \(R > 0\), then for any \(u \in W^{1,2}_p\) we have
\[
\|u_{xx}\|_{L_p} \leq N\left(\|Lu\|_{L_p} + \|u_x\|_{L_p} + \|u\|_{L_p}\right), \tag{3.18}
\]
where \(N = N(R, p, d, K, \delta)\).

Indeed, one the account of the presence of \(\|u_x\|_{L_p}\) and \(\|u\|_{L_p}\) on the right, one may certainly assume that \(b \equiv 0\) and \(c \equiv 0\). The assumption: \(u \in C^\infty_0(\mathbb{R}^{d+1})\) also does not restrict generality.

Next, if \(u \in C^\infty_0(Q_R)\), then by \eqref{3.14}, Fefferman-Stein theorem on sharp functions, and the Hardy-Littlewood maximal function theorem
\[
\|u_{xx}\|_{L_p} \leq N\left(\|u_{xx}\|^\#_{L_p} \leq N\left(\|Lu\|^q_{L_p}\right)^{1/q} \|u_{xx}\|^{1-\mu}_{L_p}
\]
\[
+ N\left(a_R^{\#(x)}\right)^{\mu/(\beta q)} \|M(|u_{xx}|^{(\alpha q)})\|^{(1/(\alpha q))}_{L_p}
\]
\[
\leq N\|Lu\|^\mu_{L_p} \|u_{xx}\|^{1-\mu}_{L_p} + N\left(a_R^{\#(x)}\right)^{\mu/(\beta q)} \|u_{xx}\|_{L_p},
\]
provided that \(p > q\alpha\), that can easily be arranged.

It follows that if \(a_R^{\#(x)}\) is small enough, then
\[
\|u_{xx}\|_{L_p} \leq N(p, d, K, \delta)\|Lu\|_{L_p}.
\]

After that \eqref{3.18} is derived by a standard procedure using partitions of unity. We say a little bit more about this procedure in the proof of Theorem 5.7.
4. Proof of Theorems 2.1 and 2.6

We suppose that the assumptions of Section 2 are satisfied and take a $p \in (1, \infty)$. We recall notation (3.17) and introduce $W^{1,2}_p$ as the Sobolev space of functions $u(t, x)$ on $\mathbb{R}^{d+1}$ such that $u, u_x, u_{xx}, u_t \in L^p$ with natural norm.

**Theorem 4.1.** There are constants $\lambda_0$ and $N$, depending only on $p, K, \delta, d$, and $\omega$, such that for any $\lambda \geq \lambda_0$ and $u \in W^{1,2}_p$ we have

$$\lambda \|u\|_{L^p} + \sqrt{\lambda} \|u_x\|_{L^p} + \|u_{xx}\|_{L^p} + \|u_t\|_{L^p} \leq N \|(L - \lambda)u\|_{L^p}. \quad (4.1)$$

Furthermore, for any $\lambda \geq \lambda_0$ and $f \in L^p$ there exists a unique $u \in W^{1,2}_p$ such that $(L - \lambda)u = f$.

Proof. The second assertion is derived from the first one by the method of continuity. To prove (4.1) observe that

$$\|u_t\|_{L^p} \leq \|Lu\|_{L^p} + N\|u_{xx}\|_{L^p} + N\|u_x\|_{L^p} + \|u\|_{L^p},$$

$$\|Lu\|_{L^p} \leq \|Lu - \lambda u\|_{L^p} + \lambda \|u\|_{L^p}.$$  

Therefore, Corollary 3.7 shows that we need only prove that for large $\lambda$

$$\lambda \|u\|_{L^p} + \sqrt{\lambda} \|u_x\|_{L^p} \leq N \|(L - \lambda)u\|_{L^p}. \quad (4.2)$$

We derive (4.2) again from (3.18) by employing an old Agmon’s idea.

Consider the space $\mathbb{R}^{d+2} = \{(t, z) = (t, x, y) : t, y \in \mathbb{R}, x \in \mathbb{R}^d\}$ and the function

$$\tilde{u}(t, z) = u(t, x)\zeta(y) \cos(\mu y), \quad (4.3)$$

where $\mu = \sqrt{\lambda}$ and $\zeta$ is a $C_0^{\infty}(\mathbb{R})$-function, $\zeta \not\equiv 0$. Also introduce the operator

$$\tilde{L}u(t, z) = L(t, x)u(t, z) + u_{yy}(t, z). \quad (4.4)$$

Finally, set

$$\tilde{B}_r(z_0) = \{|z - z_0| < r\}, \quad \tilde{Q}_r(t_0, z_0) = (t_0, t_0 + r^2) \times \tilde{B}_r(z_0).$$

For any $r \in (0, \infty)$, $(t_0, z_0) \in \mathbb{R}^{d+2}$, and appropriate $\tilde{a}(t)$ we have

$$\int_{\tilde{Q}_r(t_0, z_0)} |a(t, x) - \tilde{a}(t)| \, dz \, dt \leq \int_{(t_0, t_0 + r^2)} \int_{|x-x_0| < r} \int_{|y-y_0| < r} |a(t, x) - \tilde{a}(t)| \, dz \, dt$$

$$= 2r \int_{\tilde{Q}_r(t_0, z_0)} |a(t, x) - \tilde{a}(t)| \, dz \, dt \leq Nr^{d+3}a_R^\#(x). \quad (4.5)$$
Since \( a \in VMO_x \), it follows that (3.18) holds with \( \tilde{u}, \tilde{L} \), and \( \mathbb{R}^{d+2} \) in place of \( u, L \), and \( \mathbb{R}^{d+1} \), respectively. Now, observe that

\[
\int_{\mathbb{R}} |\zeta(y) \sin(\mu y)|^p dy
\]

is bounded above and away from zero for \( \mu \geq 1 \), so that

\[
\|u_x\|_{L^p}^p = \left( \int_{\mathbb{R}} |\zeta(y) \sin(\mu y)|^p dy \right)^{-1} \int_{\mathbb{R}^{d+2}} |u_x(t, x) \zeta(y) \sin(\mu y)|^p dzdt
\]

\[
\leq N\mu^{-p} \int_{\mathbb{R}^{d+2}} |u_x(t, x) [(\zeta(y) \cos(\mu y))' - \zeta'(y) \cos(\mu y)]|^p dzdt
\]

\[
\leq N\mu^{-p} \int_{\mathbb{R}^{d+2}} |\tilde{u}_{xy}(t, z)|^p dzdt + N_1\mu^{-p} \int_{\mathbb{R}^{d+2}} |u_x(t, x) \zeta'(y)|^p dzdt.
\]

The last term can be absorbed by what we started with if

\[
N_1\mu^{-p} \int_{\mathbb{R}} |\zeta'(y)|^p dy \leq 1/2,
\]

in which case

\[
\mu \|u_x\|_{L^p} \leq N \|\tilde{u}_{zz}\|_{L^p(\mathbb{R}^{d+2})}, \tag{4.6}
\]

Similarly,

\[
\|u\|_{L^p}^p \leq N\mu^{-2p} \int_{\mathbb{R}^{d+2}} |\tilde{u}_{yy}(t, z) - u(t, x) [2\mu\zeta'(y) \sin(\mu y) + \zeta''(y) \cos(\mu y)]|^p dzdt,
\]

\[
\mu^2 \|u\|_{L^p} \leq N \|\tilde{u}_{zz}\|_{L^p(\mathbb{R}^{d+2})} + N(\mu + 1) \|u\|_{L^p},
\]

which along with (4.6) yield

\[
\mu^2 \|u\|_{L^p} + \mu \|u_x\|_{L^p} \leq N \|\tilde{u}_{zz}\|_{L^p(\mathbb{R}^{d+2})}.
\]

Thus, the left-hand side of (4.2) is estimated through the left-hand side of (3.18) written for \( \tilde{u}, \tilde{L} \), and \( \mathbb{R}^{d+2} \) in place of \( u, L \), and \( \mathbb{R}^{d+1} \), respectively. In turn, the right-hand side of the latter is easily shown to be less than a constant times

\[
\|\tilde{L}\tilde{u}\|_{L^p(\mathbb{R}^{d+2})} + \|u_x\|_{L^p} + (\mu + 1) \|u\|_{L^p}
\]

\[
\leq N \|Lu - \lambda u\|_{L^p} + \|u_x\|_{L^p} + (\mu + 1) \|u\|_{L^p}.
\]

This proves (4.2) and the theorem.

**Proof of Theorem 2.6** As usual, it suffices to prove the apriori estimate (2.4). In turn, to do this it suffices to substitute \( v(t, x) = \zeta(t/n)u(x) \), where \( \zeta \in C_0^\infty(\mathbb{R}) \), into (4.1) with \( Lv = Mv + v_t \), let \( n \to \infty \), and observe that

\[
\|v\|_{L^p}^p = na\|u\|_{L^p(\mathbb{R}^d)}^p, \quad \|v_x\|_{L^p}^p = na\|u_x\|_{L^p(\mathbb{R}^d)}^p;
\]

\[
\|v_{xx}\|_{L^p}^p = na\|u_{xx}\|_{L^p(\mathbb{R}^d)}^p, \quad \|(L - \lambda)v\|_{L^p}^p \leq \|v_t\|_{L^p}^p.
\]
+na\|(M-\lambda)u\|_{L_p(\mathbb{R}^d)}^p = n^{1-p}\beta\|u\|_{L_p(\mathbb{R}^d)}^p + na\|(M-\lambda)u\|_{L_p(\mathbb{R}^d)}^p,$
where
$$\alpha = \int_{\mathbb{R}} |\zeta|^p \, dt, \quad \beta = \int_{\mathbb{R}} |\zeta'|^p \, dt.$$

The theorem is proved.

**Proof of Theorems 2.1** We take $\lambda_0$ from Theorem 4.1. The method of continuity and the properties of the heat equation show that if $g \in L_p$ and $g(t, x) = 0$ for $t \geq T$, then the solution $v$ of $(L-\lambda_0)v = g$ also vanishes for $t \geq T$, and, therefore, satisfies $(L-\lambda_0)v = g$ in $\Omega(T)$ with zero condition at $t = T$. We have constructed a solution from $W^{1,2}_p(T)$ not of $Lu = f$ but of $(L-\lambda_0)v = g$. One gets rid of $\lambda_0$ by substitution $u \exp(\lambda_0 t) = v$. One gets estimate (2.2) from (4.1) by taking $g = 0$ not only for $t \geq T$ but also for $t \leq 0$.

It only remains to show uniqueness of solution in $W^{1,2}_p(T)$ of $Lu = 0$, which is equivalent to showing uniqueness for $(L-\lambda_0)u = 0$. We extend $u$ for $t \geq T$ as zero, obtaining a function on $(0, \infty) \times \mathbb{R}^d$, which we then extend to negative $t$ to become an even function of $t$. Call the resulting function $\tilde{u}$ and denote $f = (L-\lambda_0)\tilde{u}$. Obviously, $\tilde{u} \in W^{1,2}_p$, and since $f = 0$ for $t \geq 0$ the (unique) solution of $(L-\lambda_0)v = f$ should also vanish for $t \geq 0$ as is explained in the beginning of the proof. Hence, $u = 0$ in $\Omega(T)$ and the theorem is proved.

5. Auxiliary results for divergence type equations

In this section we discuss some properties of the operator
$$\mathcal{L}u(t, x) = u_t(t, x) + (a^{ij}(t, x)u_{x_i}(t, x) + b^i(t, x)u(t, x))_{x_j} + b^i(t, x)u_{x_i}(t, x) + c(t, x)u(t, x).$$

All assumptions of Section 2 are imposed apart from the assumption that $a \in VMO_x$. We take the operator $\hat{L}$ from (3.1) with an $\tilde{a} \in A$.

**Lemma 5.1.** Let $u \in C^\infty(\mathbb{R}^d)$, $f = (f^1, ..., f^d)$, $f^i, g \in L_{2,loc}$ and assume that $\hat{L}u = \text{div} f + g$ and $\tilde{a}$ is infinitely differentiable. Let $R > 0$, $\kappa \geq 4$ and let $h$ be the solution of $\hat{L}h = 0$ in $Q_{\kappa R}$ with boundary condition $h = u$ on $\partial Q_{\kappa R}$. Then

$$\left(\left| h_x - (h_x)_{Q_R} \right|^2 \right)_{Q_{\kappa R}} \leq NR^2\left(\left| g \right|^2 \right)_{Q_{\kappa R}} + N\kappa^{-2}\left(\left| u_x \right|^2 + \left| f \right|^2 \right)_{Q_{\kappa R}}, \quad (5.1)$$

where $N = N(d, \delta, K)$.

Proof. By self-similarity we may assume that $R = 1$. Then, by Lemma 3.1 applied to $h_x$ in place of $u$, we see that to prove (5.1) it
suffices to show that
\[
\left| h_{xx} \right|^2_{Q_1} \leq N \left( |g|^2 \right)_{Q_\kappa} + N\kappa^{-2} \left( |u_x|^2 + |f|^2 \right)_{Q_\kappa}.
\] (5.2)
The left-hand side of (5.2) will increase if we replace \( Q_1 \) with \( Q_{\kappa/4} \) since \( \kappa \geq 4 \). After that one more application of parabolic dilations shows that we need only prove that
\[
\int_{Q_1} \left| h_{xx} \right|^2 \, dx \, dt \leq N \int_{Q_4} \left( |u_x|^2 + |f|^2 + |g|^2 \right) \, dx \, dt.
\] (5.3)
Furthermore, adding a constant to \( u \) results in adding the same constant to \( h \) and does not affect the equation and (5.3). Therefore, we may assume that \( u_{Q_4} = 0 \).

If \( \zeta \in C_0^\infty(\mathbb{R}^{d+1}) \) is such that \( \zeta = 1 \) on \( Q_1 \) and \( \zeta = 0 \) near \( \partial Q_4 \), then by observing that
\[
\bar{L}(\zeta h) = 2\bar{a}^{ij}\zeta_x h + \bar{h} \zeta
\] in \( \{0 < t < 16\} \) and applying Theorem 2.1, we see that the left-hand side of (5.3) is less than
\[
\int_{0<t<16} \left| (\zeta h)_{xx} \right|^2 \, dx \, dt \leq N \int_{0<t<16, \zeta \neq 0} \left( |h_x|^2 + |h|^2 \right) \, dx \, dt.
\]
On the account of taking appropriate \( \zeta \) we get that
\[
\int_{Q_1} \left| h_{xx} \right|^2 \, dx \, dt \leq N \int_{Q_2} \left( |h_x|^2 + |h|^2 \right) \, dx \, dt.
\] (5.4)
Then we take a smooth \( \eta \) such that \( \eta = 1 \) near \( \partial Q_4 \) and \( \eta = 0 \) on \( Q_2 \) and observe that the function \( v = h - u \eta \) vanishes on \( \partial Q_4 \) and in \( Q_4 \) satisfies
\[
\bar{L}v = -\eta (\text{div } f + g) - u \bar{L} \eta - 2\bar{a}^{ij}\eta_x u_x.
\]
The usual energy estimate yields
\[
\int_{Q_4} \left( |v_x|^2 + |v|^2 \right) \, dx \, dt \leq N \int_{Q_4} \left( |u|^2 + |f|^2 + |g|^2 \right) \, dx \, dt,
\]
which along with (5.4) lead to
\[
\int_{Q_1} \left| h_{xx} \right|^2 \, dx \, dt \leq N \int_{Q_4} \left( |u|^2 + |f|^2 + |g|^2 \right) \, dx \, dt.
\]
Finally, Poincaré’s inequality (recall that \( u_{Q_4} = 0 \)) allows us to obtain (5.3). The lemma is proved.
Let \( u \in C^\infty_{loc}(\mathbb{R}^{d+1}), f = (f^1, ..., f^d), f^i \in L_{2,loc}, \kappa \geq 4, \) \( r > 0. \) Assume that \( \bar{L}u = \text{div} f \) in \( Q_{kr} \). Then there exists a constant \( N = N(d, \delta, K) \) such that

\[
|u_x - (u_x)_{Q_r}|_{Q_r} \leq N\kappa^{-1}(|u_x|^2)_{Q_{kr}}^{1/2} + N\kappa^{d+2}/2(|f|^2)_{Q_{kr}}^{1/2}.
\]

(5.5)

Proof. We may assume that \( \bar{a} \) is infinitely differentiable. This follows from the fact that if \( \bar{a}_n \in A \) are such that \( \bar{a}_n \to \bar{a} \) (a.e.) as \( n \to \infty \) and the operators \( \bar{L}_n \) are constructed from \( \bar{a}_n \), then \( \bar{L}_n u = \text{div} f_n \), where

\[
f_n = f^i + (\bar{a}_n^{ij} - \bar{a}^{ij})u_{xj} \to f^i \in L_2(Q_{kr}).
\]

Assuming that \( \bar{a} \) is infinitely differentiable we introduce \( h \) as the solution of

\[
\bar{L}h(t, x) := \bar{a}^{ij}(t)h_{x^j}(t, x) + h_t(t, x) = 0
\]

in \( Q_{kr} \) with boundary condition \( h = u \) on \( \partial Q_{kr} \). By using Lemma 5.1 we get (remember \( \kappa \geq 4 \))

\[
(|h_x - (h_x)_{Q_r}|^2)_{Q_r} \leq N\kappa^{-2}(|u_x|^2 + |f|^2)_{Q_{kr}}
\]

\[
\leq N\kappa^{-2}(|u_x|^2)_{Q_{kr}} + N\kappa^{d+2}/2(|f|^2)_{Q_{kr}}.
\]

(5.6)

Furthermore, \( \bar{L}(u - h) = \text{div} f \) in \( Q_{kr} \) and \( u - h = 0 \) on \( \partial Q_{kr} \). By the energy estimate it follows that

\[
k^{-(d+2)}(|u_x - h_x|^2)_{Q_r} \leq (|u_x - h_x|^2)_{Q_{kr}} \leq N(|f|^2)_{Q_{kr}}.
\]

By combining this with (5.5) and using Hölder’s inequality we see that for the constant vector \( \sigma := (h_x)_{Q_r} \) the expression \( |u_x - \sigma|_{Q_r} \) is less than the right-hand side of (5.5). This proves the lemma.

Now we set

\[
\mathcal{L}_0u(t, x) = u_t(t, x) + (a^{ij}(t, x)u_{x^i}(t, x))_{x^j}
\]

and state the central result of this section.

Theorem 5.3. Let \( \alpha, \beta \in (1, \infty), \alpha^{-1} + \beta^{-1} = 1, \) and \( R \in (0, \infty). \) Let

\[
u \in C^\infty_0(Q_R), \ f = (f^1, ..., f^d), \ f^i \in L_{2,loc}. \) Assume that \( \mathcal{L}_0u = \text{div} f. \)

Then there exists a constant \( N = N(d, \delta, K, \alpha) \) such that

\[
(u_x)^\# \leq N\left[M(|f|^2)\right]^{u/2}\left[M(|u_x|^2)\right]^{(1-\mu)/2} + N\hat{a}^{\mu/(2\beta)}\left[M(|u_x|^{2\alpha})\right]^{1/(2\alpha)}
\]

on \( \mathbb{R}^{d+1}, \) where \( \mu = 2/(d+4) \) and \( \hat{a} = a_R^\#(x). \)

Proof. First, fix \( \kappa \geq 4, \) \( r \in (0, \infty), \) and \( (t_0, x_0) \in \mathbb{R}^{d+1}. \) Introduce

\[
A = \left[M(|f|^2)(t_0, x_0)\right]^{1/2}, \quad B = \left[M(|u_x|^2)(t_0, x_0)\right]^{1/2},
\]

\[
C = \left[M(|u_x|^{2\alpha})(t_0, x_0)\right]^{1/(2\alpha)}.
\]
Also take $\bar{a}(t)$ as in the proof of Theorem 3.6 and note that
\[ \bar{L}u = \text{div } f + ((\bar{a}^{ij} - \bar{a}^{ij})u_{x^i})_{x^j}. \]

Then by Lemma 5.2
\[ |u_x - (u_x)_{Q_r(t_0,x_0)}|_{Q_r(t_0,x_0)} \leq N\kappa^{-1} B + N\kappa^{(d+2)/2}(A + D), \]
where
\[ D^2 := (|\bar{a} - a|^2|u_x|^2)_{Q_{\kappa r}(t_0,x_0)} \leq \left( \int_{Q_{\kappa r}(t_0,x_0)} I_{Q_R}|\bar{a} - a|^{2\beta} dx dt \right)^{1/\beta} C^2. \]

We estimate the first factor on the right in the same way as in the proof of Theorem 3.6 and find $D \leq N\bar{a}^{1/(2\beta)} C$, which leads to
\[ |u_x - (u_x)_{Q_r(t_0,x_0)}|_{Q_r(t_0,x_0)} \leq N\kappa^{-1} B + N\kappa^{(d+2)/2}(A + \bar{a}^{1/(2\beta)} C). \]

Having $B$ on the right allows us to assert that this inequality obtained for $\kappa \geq 4$ is actually true for all $\kappa > 0$. Then maximizing with respect to $r > 0$ and minimizing with respect to $\kappa > 0$ shows that $(u_x)^\#(t_0,x_0)$ is less than
\[ N(A^\mu + \bar{a}^{\mu/(2\beta)} C^\mu) B^{1-\mu}. \]

Observing that $B \leq C$ leads to (5.7) at $(t_0, x_0)$ and proves the theorem.

Similarly to Corollary 3.7 we have the following.

**Corollary 5.4.** Let $p \in (2, \infty)$ and $R \in (0, \infty]$. Then there exist constants $\varepsilon > 0$ and $N < \infty$, depending only on $p$, $d$, $K$, and $\delta$, such that if $a_R^{\#(x)} \leq \varepsilon$, then for any $u \in C^\infty_0(Q_R)$ we have
\[ \|u_x\|_{L^p} \leq N\|f\|_{L^p}, \quad (5.8) \]
provided that $L_0u = \text{div } f$ and $f = (f^1, ..., f^d)$, $f^i \in L^p$.

To extend this result to functions not necessarily vanishing outside $Q_R$ we need to introduce the parameter $\lambda$.

**Lemma 5.5.** Let $p \in (2, \infty)$, $R \in (0, \infty]$, $f = (f^1, ..., f^d)$, $f^i, g \in L^p(Q_R)$, $u \in C^\infty_0(Q_{R/2})$, $\lambda \in \mathbb{R}$, and
\[ Lu - \lambda u = \text{div } f + g. \]

We assert that there exist constants $\varepsilon \in (0, \infty)$, depending only on $p$, $d$, $K$, and $\delta$, and $\lambda_0$, $N \in (0, \infty)$, depending only on the same parameters and $R$, such that if $a_R^{\#(x)} \leq \varepsilon$, then we have
\[ \sqrt{\lambda}\|u_x\|_{L^{p}} + \lambda\|u\|_{L^{p}} \leq N(\sqrt{\lambda}\|f\|_{L^p} + \|g\|_{L^p}), \quad (5.9) \]
provided that $\lambda \geq \lambda_0$. 

Proof. First, we observe that the terms \((\tilde{b}^i u)_x\) and \(b^i u_x\) in \(Lu\) can be included in \(\text{div} f\) and \(g\), respectively. Then it is seen that without losing generality we may assume that \(L = L_0\).

In that case we use the same method as in the proof of Theorem 4.1. We take an odd function \(\zeta \in C_0^\infty((-R/2, R/2))\) and introduce \(\tilde{u}\) and \(\tilde{L}\) by formulas (4.3) and (4.4), of course, taking in the latter \(\tilde{L}\) in place of \(L\). Also set \(\mu = \sqrt{\lambda}\), \(\tilde{f}^i(t, z) = f^i(t, x)\zeta(y) \cos(\mu y)\) for \(i = 1, \ldots, d\) and

\[
\tilde{f}^{d+1}(t, z) = g(t, x)\zeta_1(y) - 2u(t, x)\zeta_2(y) + u(t, x)\zeta_3(y),
\]

where

\[
\zeta_1(y) = \int_{-\infty}^y \zeta(s) \cos(\mu s) \, ds, \quad \zeta_3(y) = \int_{-\infty}^y \zeta''(s) \cos(\mu s) \, ds
\]

\[
\zeta_2(y) = \mu \int_{-\infty}^y \zeta'(s) \sin(\mu s) \, ds = -\zeta'(y) \cos(\mu y) + \zeta_3(y).
\]

Observe that \(\zeta_i \in C_0^\infty(\mathbb{R})\) since \(\zeta\) is odd and has compact support. Furthermore, as is easy to check,

\[
\tilde{L}\tilde{u}(t, z) = (\tilde{f}^1(t, z))_{x^1} + \ldots + (\tilde{f}^d(t, z))_{x^d} + (\tilde{f}^{d+1}(t, z))_y.
\]

The computations in (4.4) and the fact that \(\tilde{u}\) has support in \((0, R^2) \times \{|z| < R\}\) convince us that (5.8) holds for \(\tilde{u}\) and \(\tilde{f}\) as long as \(a(#y)\) is small enough. In other words,

\[
\|\tilde{u}_z\|_{L_p(\mathbb{R}^{d+2})} \leq N \sum_{i=1}^{d+1} \|\tilde{f}^i\|_{L_p(\mathbb{R}^{d+2})}, \quad (5.10)
\]

Since

\[
\kappa_0 := \int_{\mathbb{R}^d} |\zeta(y) \sin(\mu y)|^p \, dy, \quad \kappa_1 := \int_{\mathbb{R}^d} |\zeta(y) \cos(\mu y)|^p \, dy
\]

are bounded away from zero for \(\mu \geq 1\), we get

\[
\|u_x\|_{L_p}^p = \kappa_1^{-1} \int_{\mathbb{R}^{d+2}} |u_x\zeta(y) \cos(\mu y)|^p \, dz \, dt \leq \kappa_1^{-1} \|\tilde{u}_z\|_{L_p(\mathbb{R}^{d+2})}^p,
\]

\[
\|u\|_{L_p}^p = \kappa_0^{-1} \mu^{-1} \int_{\mathbb{R}^{d+2}} |\tilde{u}_y - u\zeta'(y) \cos(\mu y)|^p \, dz \, dt
\]

\[
\leq N \mu^{-1}(\|\tilde{u}_z\|_{L_p(\mathbb{R}^{d+2})}^p + \|u\|_{L_p}^p).
\]

It follows from here and (5.10) that for \(\mu\) large enough

\[
\mu \|u\|_{L_p}^p + \|u_x\|_{L_p}^p \leq N\|\tilde{u}_z\|_{L_p(\mathbb{R}^{d+2})}^p \leq N \sum_{i=1}^{d+1} \|\tilde{f}^i\|_{L_p(\mathbb{R}^{d+2})}.
\]

(5.11)
Now we estimate the right-hand side of (5.11). Obviously, for \( i = 1, \ldots, d \)
\[
\| \tilde{f}^i \|_{L^p(\mathbb{R}^{d+2})} \leq N \| f^i \|_{L^p}.
\]
Furthermore,
\[
\zeta_1 = \mu^{-1} [\zeta(y) \sin(\mu y) - \int_{-\infty}^y \zeta'(s) \sin(\mu s) \, ds],
\]
which shows that \( \zeta_1 \) equals \( \mu^{-1} \) times a uniformly bounded function with support not wider than that of \( \zeta \). Hence,
\[
\| g \zeta_1 \|_{L^p(\mathbb{R}^{d+2})} \leq N \mu^{-1} \| g \|_{L^p}.
\]
Also \( \zeta_2 \) and \( \zeta_3 \) are uniformly bounded with support not wider than that of \( \zeta \). Therefore,
\[
\| 2u \zeta_2 + u \zeta_3 \|_{L^p(\mathbb{R}^{d+2})} \leq N \| u \|_{L^p},
\]
\[
\| \tilde{f}^{d+1} \|_{L^p(\mathbb{R}^{d+2})} \leq N \mu^{-1} \| g \|_{L^p} + N \| u \|_{L^p}.
\]
This and (5.11) lead to (5.9) and the lemma is proved.

**Remark 5.6.** If \( p = 2 \), then under no restrictions on \( a^{#(x)} \) estimate (5.9) holds for \( \lambda \) large, generally, or for all \( \lambda > 0 \) under the additional assumption that \( \mathcal{L} = \mathcal{L}_0 \). This is easily proved by integration by parts.

For \( n \in \mathbb{R} \) set
\[
H^n_p = (1 - \Delta)^{-n/2} L^p_p(\mathbb{R}^d), \quad \mathbb{H}^n_p = L^p_p(\mathbb{R}, H^n_p).
\]

**Theorem 5.7.** Let \( p \in (2, \infty) \), \( f = (f^1, \ldots, f^d) \), \( f^i, g \in L^p_p \), \( u \in C^\infty_0(\mathbb{R}^{d+1}) \), \( \lambda \in \mathbb{R} \), and
\[
\mathcal{L} u - \lambda u = \text{div} f + g.
\]
Take \( \varepsilon = \varepsilon(p, d, K, \delta) \) from Lemma 5.5 and assume that \( a^{#(x)}_R \leq \varepsilon \) for an \( R \in (0, \infty) \). Then there exist constants \( \lambda_0, N \in (0, \infty) \), depending only on \( p, d, K, \delta \), and \( R \), such that
\[
\| u_t \|_{H^{p^{-1}}_p} + \sqrt{\lambda} \| u_x \|_{L^p} + \lambda \| u \|_{L^p} \leq N(\sqrt{\lambda} \| f \|_{L^p} + \| g \|_{L^p}) \quad (5.13)
\]
whenever \( \lambda \geq \lambda_0 \).

Proof. By the same reasons as before we may assume that \( \mathcal{L} = \mathcal{L}_0 \). Furthermore,
\[
u_t = \text{div} f + g + \lambda u - (a^{ij} u_{x_i})_{x_j}
\]
and the operators \( (1 - \Delta)^{-1/2} \) and \( (1 - \Delta)^{-1/2}(\partial / (\partial x^i)) \) are bounded in \( L^p_p(\mathbb{R}^d) \), which implies that
\[
\| u_t \|_{H^{p^{-1}}_p} \leq N(\| f \|_{L^p} + \| g \|_{L^p} + \lambda \| u \|_{L^p} + \| u_x \|_{L^p}).
\]
It follows that to prove (5.13) it suffices to prove
\[ \sqrt{\lambda} \| u_x \|_{L^p} + \lambda \| u \|_{L^p} \leq N(\sqrt{\lambda} \| f \|_{L^p} + \| g \|_{L^p}) \] (5.14)
and make sure that \( \lambda_0 \geq 1 \), which is always possible.

Then, we use partitions of unity. Take a \( \zeta \in C_0^\infty(Q_{R/2}) \) with unit integral, introduce
\[ \zeta_{t_0,x_0}(t, x) = \zeta(t - t_0, x - x_0), \quad u_{t_0,x_0}(t, x) = u(t, x)\zeta_{t_0,x_0}(t, x) \]
and observe that
\[ (\mathcal{L} - \lambda)u_{t_0,x_0} = \text{div} f_{t_0,x_0} + g_{t_0,x_0}, \]
where
\[ f_{t_0,x_0} = f^i\zeta_{t_0,x_0} + a^{ij}u(\zeta_{t_0,x_0})_{x^i}, \]
\[ g_{t_0,x_0} = g\zeta_{t_0,x_0} - f^i(\zeta_{t_0,x_0})_{x^i} + a^{ij}(\zeta_{t_0,x_0})_{x^i}u_{x^j} + u(\zeta_{t_0,x_0}). \]
It follows from Lemma \( 5.5 \) applied to \( Q_{R/2}(t_0, x_0) \) in place of \( Q_{R/2} \) that
\[ \lambda^{p/2}\| (u_{t_0,x_0})_x \|_{L^p}^p + \lambda \| u_{t_0,x_0} \|_{L^p}^p \leq N(\lambda^{p/2}\| f_{t_0,x_0} \|_{L^p}^p + \| g_{t_0,x_0} \|_{L^p}^p). \]
By assuming without losing generality that \( \lambda_0 \geq 1 \) we estimate the right-hand side by a constant times
\[ \lambda^{p/2}\| f_{Q_{R}(t_0,x_0)} \|_{L^p}^p + \lambda^{p/2}\| u_{Q_{R}(t_0,x_0)} \|_{L^p}^p + \| g_{\zeta_{t_0,x_0}} \|_{L^p}^p + \| u_x I_{Q_{R}(t_0,x_0)} \|_{L^p}^p. \]
On the left
\[ \| u_x \zeta_{t_0,x_0} \|_{L^p} \leq \| (u_{t_0,x_0})_x \|_{L^p} + \| u(\zeta_{t_0,x_0})_x \|_{L^p}. \]
Hence
\[ \lambda^{p/2}\| u_x \zeta_{t_0,x_0} \|_{L^p}^p + \lambda \| u \zeta_{t_0,x_0} \|_{L^p}^p \leq N(\lambda^{p/2}\| f_{Q_{R}(t_0,x_0)} \|_{L^p}^p + \| g_{\zeta_{t_0,x_0}} \|_{L^p}^p + \| u_x I_{Q_{R}(t_0,x_0)} \|_{L^p}^p) \]
After integrating with respect to \( (t_0, x_0) \) over \( \mathbb{R}^{d+1} \) we conclude
\[ \lambda^{p/2}\| u_x \|_{L^p}^p + \lambda \| u \|_{L^p}^p \leq N(\lambda^{p/2}\| f \|_{L^p}^p + \| g \|_{L^p}^p + \| u_x \|_{L^p}^p) \]
and (5.9) follows. The theorem is proved.
6. Proof of Theorems 2.4 and 2.8

We suppose that the assumptions of Section 2 are satisfied. First, we restate Theorem 5.7 in terms of appropriate Banach spaces. We take $H^{-1}_p$ from (5.14), recall that $W^{1,2}_p$ is introduced in the beginning of Section 4 and set

$$\mathcal{H}_p^1 = (1 - \Delta)^{1/2}W^{1,2}_p$$

with natural norm. It is easy to see that $C_0^\infty(\mathbb{R}^{d+1})$ is dense in $H^p$ and $\mathcal{H}_p^1$. Furthermore, for $u \in C_0^\infty(\mathbb{R}^{d+1})$

$$\|u\|_{\mathcal{H}_p^1} = \|(1 - \Delta)^{-1/2}u\|_{W^{1,2}_p} \sim \|(1 - \Delta)^{-1/2}u\|_{L_p}$$

$$+ \|(1 - \Delta)^{-1/2}u_x\|_{L_p} + \|(1 - \Delta)^{-1/2}u_t\|_{L_p} + \|(1 - \Delta)^{-1/2}u_{xx}\|_{L_p},$$

Since the operators $(1 - \Delta)^{-1/2}$ and $(1 - \Delta)^{-1/2}(\partial/\partial x^i)$ are bounded in $L_p$, it follows that

$$\|u\|_{\mathcal{H}_p^1} \leq N(\|u_t\|_{H^{-1}_p} + \|u\|_{L_p} + \|u_x\|_{L_p}). \quad (6.1)$$

On the other hand, we know that

$$\|u\|_{L_p(\mathbb{R}^d)} + \|u_x\|_{L_p(\mathbb{R}^d)} \leq N\|(1 - \Delta)^{1/2}u\|_{L_p(\mathbb{R}^d)}$$

$$= N\|(1 - \Delta)^{-1/2}(1 - \Delta)u\|_{L_p(\mathbb{R}^d)} \leq N\|(1 - \Delta)^{-1/2}u\|_{L_p(\mathbb{R}^d)}$$

$$+ N\|(1 - \Delta)^{-1/2}u_{xx}\|_{L_p(\mathbb{R}^d)},$$

which shows that the right-hand side of (6.1) is also dominated by a constant times its left-hand side. In other words,

$$\|u\|_{L_p} + \|u_x\|_{L_p} + \|u_t\|_{H^{-1}_p} \quad \text{and} \quad \|u\|_{H^1_p} + \|u_t\|_{H^{-1}_p}$$

define equivalent norms in $\mathcal{H}_p^1$, which are dominated by the left-hand side of (5.13) if $\lambda \geq 1$.

It turns out that its right-hand side can be replaced by a constant times

$$\|\text{div } f + g\|_{H^{-1}_p}. \quad (6.2)$$

Indeed, on the one hand, (6.2) is, obviously, dominated by the right-hand side of (5.13). On the other hand, denote $h = \text{div } f + g$, $\hat{g} = (1 - \Delta)^{-1}h$, $\hat{f}^i = -\hat{g}_{x^i}$. Then

$$\text{div } \hat{f} + \hat{g} = -\Delta \hat{g} + \hat{g} = h.$$ 

Hence we can replace the right-hand side of (5.13) with

$$N(\sqrt{\lambda}\|\hat{f}\|_{L_p} + \|\hat{g}\|_{L_p}),$$

which is less than a constant (also depending on $\lambda$) times $\|h\|_{H^{-1}_p}$. By the way, the above relations between $h$, $\hat{g}$, and $\hat{f}$ show that each $h \in H^{-1}_p$ is written as $\text{div } \hat{f} + \hat{g}$ with $\hat{f}^i, \hat{g} \in L_p$. 

N.V. KRYLOV
It is also worth noting that almost obviously $L$ is a bounded operator from $\mathcal{H}_p^1$ into $\mathbb{H}_p^{-1}$.

This argument allows us to claim that the following result is a corollary of Theorem 5.7.

**Theorem 6.1.** Let $p \in (2, \infty)$. Then there is a constant $\lambda_0$ depending only on $p, d, \delta, K,$ and $\omega$ such that for any $\lambda \geq \lambda_0$ and any $u \in \mathcal{H}_p^1$ we have

$$\|u\|_{\mathcal{H}_p^1} \leq N(\lambda, p, d, \delta, K, \omega)\|(L - \lambda)u\|_{\mathbb{H}_p^{-1}}.$$  

One derives Theorems 2.4 and 2.8 from the following one by repeating the proofs of Theorems 2.1 and 2.6 almost word for word.

**Theorem 6.2.** Let $p \in (1, \infty)$. Then there exists $\lambda_0 = \lambda_0(p, d, K, \delta, \omega)$ such that, for any $u \in \mathcal{H}_p^1$ and $\lambda \geq \lambda_0$, we have

$$\|u_t\|_{\mathbb{H}_p^{-1}} + \|u\|_{\mathcal{H}_p^1} \leq N(p, d, K, \delta, R, \lambda)\|(L - \lambda)u\|_{\mathbb{H}_p^{-1}}. \quad (6.3)$$

Furthermore, for each $\lambda \geq \lambda_0$ and $f \in \mathbb{H}_p^{-1}$ there is a unique $u \in \mathcal{H}_p^1$ such that $(L - \lambda)u = f$.

**Proof.** The second assertion is a standard consequence of the first one and the method of continuity.

Estimate (6.3) is stated in Theorem 5.7 for $p > 2$.

We consider $p \in (1, 2)$ by using duality. Set $q = p/(p - 1)$, take $\lambda_0$ corresponding to $q$ and the operator $L^*$ formally adjoint to $L$ and take $\lambda \geq \lambda_0$. The reader should not be uncomfortable with the fact that the derivative in time enters $L^*$ with a negative sign unlike $L$. Our results are applicable to such operators as well, which is seen after changing variables $t \to -t$.

By the above for any $h \in \mathbb{H}_q^{-1}$ we can find $v \in \mathcal{H}_q^1$ such that

$$(L^* - \lambda)v = h, \quad \|v\|_{\mathcal{H}_q^1} \leq N\|h\|_{\mathbb{H}_q^{-1}}.$$ 

For $u \in \mathcal{H}_p^1$ write

$$|\langle u, h \rangle| = |\langle (L - \lambda)u, v \rangle| \leq \|(L - \lambda)u\|_{\mathbb{H}_p^{-1}}\|v\|_{\mathcal{H}_q^1} \leq N\|(L - \lambda)u\|_{\mathbb{H}_p^{-1}}\|h\|_{\mathbb{H}_q^{-1}}.$$ 

Since $h$ was arbitrary, it follows that

$$\|u\|_{\mathbb{H}_p^{-1}} \leq N\|(L - \lambda)u\|_{\mathbb{H}_p^{-1}}.$$ 

This estimate and the formula

$$u_t = (L - \lambda)u + \lambda u - (a^{ij}u_{x^i} + \hat{b}^i u)_{x^j} - b^i u_{x^i}$$

allow us to get the remaining part of (6.3), which is thus proved for $p \in (1, \infty)$, $p \neq 2$. Once the resolvent operator is constructed for $p \in (1, \infty)$, $p \neq 2$, the case $p = 2$ is covered by interpolation. The theorem is proved.
References

[1] Bramanti, M., and Cerutti, M.C., $W^{1,2}_p$ solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients, Comm. Partial Differential Equations, Vol. 18 (1993), No. 9-10, 1735-1763.

[2] Byun, Sun-Sig, Elliptic equations with BMO coefficients in Lipschitz domains, Trans. Amer. Math. Soc., Vol. 357 (2005), No. 3, 1025-1046.

[3] Byun, Sun-Sig, Parabolic equations with BMO coefficients in Lipschitz domains, J. Differential Equations, Vol. 209 (2005), No. 2, 229–265.

[4] Chiarenza, F., Frasca, M., and Longo, P., Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients, Ricerche Mat., Vol. 40 (1991), No. 1, 149-168.

[5] Chiarenza, F., Frasca, M., and Longo, P., $W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients, Trans. Amer. Math. Soc., Vol. 336 (1993), No. 2, 841-853.

[6] Haller-Dintelmann, R., Heck, H., and Hieber, M., $L^p - L^q$-estimates for parabolic systems in non-divergence form with VMO coefficients, Preprint.

[7] Kim, Doyoon, Second order elliptic equations in $\mathbb{R}^d$ with piecewise continuous coefficients, submitted to Comm in PDEs.

[8] Kim, Kyeong-Hun, and Krylov, N.V., On the Sobolev space theory of parabolic and elliptic equations in $C^1$ domains, SIAM J. Math. Anal., Vol. 36 (2004), No. 2, 618-642.

[9] Krylov, N.V., “Lectures on elliptic and parabolic equations in Hölder spaces”, Amer. Math. Soc., Providence, RI, 1996.

[10] Krylov, N.V., Parabolic equations in $L^p$-spaces with mixed norms, Algebra i Analiz., Vol. 14 (2002), No. 4, 91-106 in Russian; English translation in St. Petersburg Math. J., Vol. 14 (2003), No. 4, 603-614.

[11] Krylov, N.V., On weak uniqueness for some diffusions with discontinuous coefficients, Stoch. Proc. and Appl., Vol. 113 (2004), No. 1, 37-64.

[12] Krylov, N.V., and Liptser R.S., On diffusion approximation with discontinuous coefficients. Stoch. Proc. Appl., Vol. 102 (2002), No. 2, 235-264.

[13] Lorenzi, A., On elliptic equations with piecewise constant coefficients. II, Ann. Scuola Norm. Sup. Pisa (3), Vol. 26 (1972), 839-870.

[14] Lieberman, G., A mostly elementary proof of Morrey space estimates for elliptic and parabolic equations with VMO coefficients, J. Functional Analysis, Vol. 201 (2003), 457-479.

[15] Stroock, D.W., and Varadhan, S.R.S., “Multidimensional diffusion processes”, Springer Verlag, New York etc., 1979.

E-mail address: krylov@math.umn.edu

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455