Random Sequential Generation of Intervals for the Cascade Model of Food Webs

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Abstract The cascade model generates a food web at random. In it the species are labeled from 0 to \( m \), and arcs are given at random between pairs of the species. For an arc with endpoints \( i \) and \( j \) \((i < j)\), the species \( i \) is eaten by the species labeled \( j \). The chain length (height), generated at random, models the length of food chain in ecological data. The aim of this note is to introduce the random sequential generation of intervals as a Poisson model which gives naturally an analogous behavior to the cascade model.

Keywords food chain, asymptotic length, Poisson approximation, Sequential interval generation

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1 Introduction

The cascade model (Cohen (1990), Cohen and Newman (1985, 1986), Cohen, Briand and Newman (1990), Newman (1992)) is introduced to explain the ecological data on community food webs. The model generates a food web at random. In it the species are labeled from 0 to $m$, and arcs are given at random between pairs of the species. For an arc with endpoints $i$ and $j$ ($i < j$), the species $i$ is eaten by the species labeled $j$. The chain length (height) is compared with the length of food chain in ecological data. The problem of chain length of food webs gives natural questions for random evolution of graphs in which connectedness and other related problems are studied (Erdős and Rényi (1960), Shepp (1989), Durrett and Kesten (1990)).

The random sequential bisection of intervals (Sibuya and Itoh (1987), Janson and Neininger (2008), Dutour Sikiric and Itoh (2011)) gives an analogous asymptotic behavior to the binary search tree (Robson (1979), Flajolet and Odlyzko (1982), Mahmoud and Pittel (1984), Devroye (1986)). Here we introduce a random sequential generation of intervals to understand the chain length for the cascade model, extending the idea of random sequential bisection of intervals. We can naturally obtain a formula for the chain length, as given in section 2 and section 3, which suggests the corresponding formula for the original cascade model as given in section 4. The random sequential interval generation is a Poisson approximation for the cascade model of food webs, in the sense given in equation (12) in section 3, and equation (16) in section 4.

Consider the random oriented graph with vertex set $\{0, 1, 2, ..., m\}$ in which the $\binom{m+1}{2}$ oriented edges $(i, j)$ with $i < j$ occur independently of each other with probability $p = c$ and no edge $j \leq i$ occurs. Let us give a random variable $X_{ij}$ to each oriented edge $(i, j)$ with $i < j$ with the vertex set $\{0, 1, 2, ..., m\}$, where $X_{ij}$ are mutually independent random variable with $X_{ij} = 1$ with probability $p = c$ and $X_{ij} = 0$ with probability $1 - c$.

Consider the random variables $X_{ij}$ with $i = 0$. For sufficiently small $c$, the number $N = \sum_{j=1}^{m} X_{ij}$ with $i = 0$ is approximately distributed by the Poisson distribution $\frac{(cm)^{k}}{k!} e^{-cm}$. Let each of $N$ random variables $X_{0,k(1)}, X_{0,k(2)}, ..., X_{0,k(N)}$ take the value 1. Consider for each $k(l), l = 1, 2, ..., N$, the random variables $X_{k(l),k(l)+1}, X_{k(l),k(l)+2}, ..., X_{k(l),m}$. Then $\sum_{j=k(l)+1}^{m} X_{ij}$ with $i = k(l)$ is approximately distributed by the Poisson distribution $\frac{(cm-k(l))^k}{k!} e^{-(m-k(l))}$. We define the Poisson generation of intervals to define the random sequential generation of intervals for the cascade model as follows.

**Poisson generation of intervals** is defined for the interval $[0, y]$ to take a random variable $N(y)$ distributed by the Poisson distribution with the parameter $cy$ as

$$P(N(y) = k) = \frac{1}{k!}(cy)^k e^{-cy}$$

to generate the intervals $[0, X_1(y)], [0, X_2(y)], ..., [0, X_{N(y)}(y)]$, where $X_i(y)$ is distributed uniformly at random on the interval $[0, y]$, mutually independently,
for each $i$.

At step 1 we apply the Poisson generation to the interval $[0, x]$. At step $1 < j$, for each interval $[0, y]$ generated at the step $j - 1$ apply the Poisson generation to the interval independently from other intervals and independently from the previously generated intervals. Each interval which does not generate any interval at step $j$ does not generate any interval after the step $j$. We continue the steps as long as we have at least one generated interval.

We also define the random sequential generation of intervals with an exponentially distributed starting point as follows. At step 0 we generate the interval $[0, x - Z]$ where $Z$ is distributed by the density $e^{-cz}$. If $([0, x - Z] \not\subset [0, x])$, we finish and stop to make the next step. If $([0, x - Z] \subset [0, x])$, we proceed to step 1. At step 1 we apply the Poisson generation to the interval $[0, x - Z]$. At step $1 < j$, for each interval generated at the step $j - 1$ apply the Poisson generation to the interval independently from other intervals and independently from the previously generated intervals. Each interval which does not generate any interval at step $j$ does not generate any interval after the step $j$. We continue the steps as long as we have at least one generated interval. In the original cascade model of food webs, the value $cm$ is assumed to be a constant. Assuming $c$ is a constant independent from $m$, the asymptotic behavior of the longest chain is of mathematical interest, which may have applications for example to task graphs for parallel processing in computer science (Newman (1992)). As shown in sections 2, 3, 4, and 5, the random sequential generation of intervals helps to understand the asymptotic length of the longest chain of the cascade model.

## 2 Random sequential generation by the Poisson distribution

For each stopped interval generated by the above procedure, we count the number of steps to get the stopped interval. Let $K(x, a)$ be the number of intervals which take $a$ steps until stopping. Let $L(x, a)$ be the expectation of $K(x, a)$. We get the asymptotic behavior of $L(x, a)$ as in the case of random sequential bisection for the binary search tree (Sibuya and Itoh (1987)).

We have

$$L(x, 0) = e^{-cx}.$$  \hspace{1cm} (1)

For $1 \leq a$, we have,

$$L(x, a) = \frac{1}{x} \int_0^x \sum_{k=1}^\infty \frac{1}{k!} (cx)^k e^{-cx} k L(y, a - 1) \, dy,$$

$$= \frac{1}{x} \int_0^x \sum_{k=1}^\infty \frac{1}{k!} (cx)^k e^{-cx} k L(y, a - 1) \, dy,$$

\hspace{1cm} 3
\[ cx \int_{0}^{x} \sum_{k=0}^{\infty} \frac{1}{k!} (cx)^k e^{-cx} L(y, a - 1) dy, \]
\[ = c \int_{0}^{x} L(y, a - 1) dy. \]  \(2\)

Hence for \(a = 1\), we have
\[ L(x, 1) = c \int_{0}^{x} L(y, 0) dy, \]
\[ = c \int_{0}^{x} e^{-cy} dy. \]

We have finally
\[ L(x, a) = c \int_{0}^{x} c \int_{0}^{x_{a-1}} \ldots c \int_{0}^{x_1} c^{a-x} dx_1 dx_2 \ldots dx_{a-1} \]
\[ = \sum_{j=0}^{\infty} (-1)^j \frac{(cx)^{a+j}}{(a+j)!}. \]  \(3\)

Hence we have
\[ L(x, a) + L(x, a + 1) = \frac{(cx)^a}{a!}. \]  \(4\)

Put \(a = k x\), and apply the Stirling formula \(n! \sim \sqrt{2\pi n^{n+1/2}} e^{-n}\), then
\[ L(x, a) + L(x, a + 1) = \frac{(cx)^{kx}}{(kx)!} \sim \frac{(cx)^{kx}}{\sqrt{2\pi(kx)^{kx+1/2}} e^{-kx}}. \]  \(5\)

As
\[ \frac{(cx)^{kx}}{\sqrt{2\pi(kx)^{kx+1/2}} e^{-kx}} = \frac{1}{\sqrt{2\pi kx}} (c^{kx} e^{kx}), \]  \(6\)
we have the following theorem, which corresponds to the stronger result by Newman (1992) for the cascade model.

**Theorem 1**

(i) \[ L(x, a) + L(x, a + 1) = \frac{(cx)^a}{a!}. \]  \(7\)

(ii) Put \(a = k x\), for \(x \rightarrow \infty\), if \(e \, c < k\),
\[ L(x, a) + L(x, a + 1) \rightarrow 0, \]  \(8\)

if \(k < e \, c\),
\[ L(x, a) + L(x, a + 1) \rightarrow \infty. \]  \(9\)
3 The chain length with an exponentially distributed starting point

The expected chain length for the case that the starting interval is \([0, x - u]\) \((⊂ [0, x])\) where \(u\) is distributed by the density \(ce^{-cu}\) is given by

\[
M(x, a) = \int_{0}^{x} L(x - u, a) ce^{-cu} du.
\]

\[
M(x, a) + M(x, a + 1) = \int_{0}^{x} (L(x - u, a) + L(x - u, a + 1)) ce^{-cu} du
\]

\[
= \int_{0}^{x} \frac{(c(x - u))a}{a!} ce^{-cu} du. \quad (10)
\]

\[
M(x, a) + M(x, a + 1) = (-1)^a e^{-cx} + \sum_{j=0}^{a} (-1)^j \frac{(cx)^{a-j}}{(a-j)!}. \quad (11)
\]

Hence we have Theorem 2 in the same way to Theorem 1 by using the Stirling formula.

**Theorem 2**

(i)

\[
M(x, a) + 2M(x, a + 1) + M(x, a + 2) = \frac{(cx)^{a+1}}{(a+1)!}. \quad (12)
\]

(ii) Put \(a = kx\), for \(x \to \infty\), if \(ec < k\),

\[
M(x, a) + 2M(x, a + 1) + M(x, a + 2) \to 0, \quad (13)
\]

if \(k < ec\),

\[
M(x, a) + 2M(x, a + 1) + M(x, a + 2) \to \infty. \quad (14)
\]

4 On Cascade model

Putting \(m = S - 1\), the expected number of chains with length \(n\) on the cascade model (Cohen and Newman (1986)) is given by

\[
E(C_n) = p^n \sum_{k=n}^{S-1} (S - k) \binom{k-1}{n-1} q^{S-k-1}. \quad (15)
\]

By using this formula, we can extend equation (12) to the cascade model. The following theorem will show that our Poisson interval tree is a natural continuous model for the cascade model of food webs and gives an elementary approach to
the theorems for the cascade model (Newman (1992)) on the expected length of a chain and the asymptotic length of the longest chain.

**Theorem 3.**

(i) 
\[ E(C_n) + 2E(C_{n+1}) + E(C_{n+2}) = \binom{S}{n+1} p^n. \]  
(16)

(ii) Put \( n = kS \), for \( S \to \infty \), if \( e < k \),
\[ E(C_n) + 2E(C_{n+1}) + E(C_{n+2}) \to 0, \]  
(17)
if \( k < e \),
\[ E(C_n) + 2E(C_{n+1}) + E(C_{n+2}) \to \infty. \]  
(18)

**Proof.** By using
\[ E\left(\frac{C_n}{p^n}\right) = \sum_{k=n}^{S-1} (S-k) \binom{k-1}{n-1} q^{S-k-1}, \]  
(19)

\[ E\left(\frac{C_{n+1}}{p^n}\right) = \sum_{k=n+1}^{S-1} (1-q)(S-k) \binom{k-1}{n} q^{S-k-1} \]
\[ = \sum_{k=n+1}^{S-1} (S-k) \binom{k-1}{n} q^{S-k-1} \]
\[ - \sum_{k=n+1}^{S-1} (S-k) \binom{k-1}{n} q^{S-k}, \]  
(20)

and
\[ E\left(\frac{C_{n+2}}{p^n}\right) = \sum_{k=n+2}^{S-1} (1-q)^2(S-k) \binom{k-1}{n+1} q^{S-k-1} \]
\[ = \sum_{k=n+2}^{S-1} (S-k) \binom{k-1}{n+1} q^{S-k-1} \]
\[ - \sum_{k=n+2}^{S-1} 2(S-k) \binom{k-1}{n+1} q^{S-k} \]
\[ + \sum_{k=n+2}^{S-1} (S-k) \binom{k-1}{n+1} q^{S-k+1}, \]  
(21)

we can easily obtain eq. (19).

(ii) is obtained by using the Stirling formula as in the proof of Theorem 1 and Theorem 2.
5 On the height of the generated tree

For each stopped interval generated by the above procedure in the section, we count the number of steps $a$ to get the stopped interval. Consider the maximum $H(x)$ of the numbers and let us call it the height of the generated tree. For the probability $F(x, h) \equiv Pr(H(x) \leq h)$ of the height of Poisson interval tree $H(x)$, we have for $h = 0$, 

$$F(x, h) = e^{-cx},$$

for $1 \leq h$,

$$F(x, h) = e^{-cx} + \sum_{k=1}^{\infty} \frac{(cx)^k}{k!} e^{-cx} \frac{1}{x^k} \int_0^x \cdots \int_0^x F(y_1, h-1) \cdots F(y_k, h-1) \, dy_1 \cdots dy_k,$$

$$= e^{-cx} \sum_{k=0}^{\infty} \frac{1}{k!} (c \int_0^x F(y, h-1) \, dy)^k$$

$$= e^{-cx} e^{c \int_0^x F(y, h-1) \, dy}. \quad (23)$$

We can sequentially integrate at each step and obtain $F(x, h)$ starting from $F(x, 0) = e^{-cx}$. The expansion at $x = 0$ is obtained by Mathematica up to $(cx)^2$ for example for $h = 0, 1, 2, ..., 7$, which gives reasonable numerical values of $F(x, h) - F(x, h-1)$ when $cx \leq 2.1$, while it does not give reasonable numerical values for $2.2 \leq cx$. For $4 \leq h$ the first five terms of the probability $F(x, h) - F(x, h-1)$ that the the height is $h$ is given by

$$\frac{(cx)^h}{h!} - 2 \frac{(cx)^{h+1}}{(h+1)!} - \frac{(cx)^{h+2}}{(h+2)!} + \frac{(cx)^{h+3}}{(h+3)!} - 8 \frac{(cx)^{h+4}}{(h+4)!}. \quad (24)$$

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