Abstract. The cusped hyperbolic $n$-orbifolds of minimal volume are well known for $n \leq 9$. Their fundamental groups are related to the Coxeter $n$-simplex groups $\Gamma_n$. In this work, we prove that $\Gamma_n$ has minimal growth rate among all non-cocompact Coxeter groups of finite covolume in $\text{Isom} \mathbb{H}^n$. In this way, we extend previous results of Floyd for $n = 2$ and of Kellerhals for $n = 3$, respectively. Our proof is a generalization of the methods developed together with Kellerhals for the cocompact case.

1 Introduction

Let $\mathbb{H}^n$ denote the real hyperbolic $n$-space with its isometry group $\text{Isom} \mathbb{H}^n$.

A hyperbolic Coxeter polyhedron $P \subset \mathbb{H}^n$ is a convex polyhedron of finite volume all of whose dihedral angles are integral submultiples of $\pi$. Associated to $P$ is the hyperbolic Coxeter group $\Gamma \subset \text{Isom} \mathbb{H}^n$ generated by the reflections in the bounding hyperplanes of $P$. By construction, $\Gamma$ is a discrete group with associated orbifold $O^n = \mathbb{H}^n / \Gamma$ of finite volume.

We focus on non-compact hyperbolic Coxeter polyhedra, having at least one ideal vertex $v_\infty \in \partial \mathbb{H}^n$. Notice that the stabilizer of the vertex $v_\infty$ is isomorphic to an affine Coxeter group. The group $\Gamma$ is called non-cocompact, and its quotient space $O^n$ has at least one cusp.

The hyperbolic Coxeter group $\Gamma$ is the geometric realization of an abstract Coxeter system $(W, S)$ consisting of a group $W$ with a finite generating set $S$ together with the relations $s^2 = 1$ and $(ss')^{m_{s's'}} = 1$, where $m_{s's'} = m_{s's} \in \{2, 3, \ldots, \infty\}$ for all $s, s' \in S$ with $s \neq s'$. The growth series $f_S(t)$ of $W = (W, S)$ is given by

$$f_S(t) = 1 + \sum_{k=1} a_k t^k,$$

where $a_k \in \mathbb{Z}$ is the number of elements in $W$ with $S$-length $k$. The growth rate $\tau_W$ of $W = (W, S)$ is defined as the inverse of the radius of convergence of $f_S(t)$.

We are interested in small growth rates of non-cocompact hyperbolic Coxeter groups in $\text{Isom} \mathbb{H}^n$ for $n \geq 2$. For $n = 2$, Floyd [6] showed that the Coxeter group $\Gamma_2 = [3, \infty]$ generated by the reflections in the triangle with angles $\pi/2, \pi/3$, and 0 is the (unique) group of minimal growth rate. For $n = 3$, Kellerhals [13] proved that
Table 1: The hyperbolic Coxeter $n$-simplex group $\Gamma_n$.

| $\Gamma_2$ | $\Gamma_3$ |
|-----------|-----------|
| ![Diagram](image1) | ![Diagram](image2) |
| $\Gamma_4$ | $\Gamma_5$ |
| ![Diagram](image3) | ![Diagram](image4) |
| $\Gamma_6$ | $\Gamma_7$ |
| ![Diagram](image5) | ![Diagram](image6) |
| $\Gamma_8$ | $\Gamma_9$ |
| ![Diagram](image7) | ![Diagram](image8) |

the tetrahedral group $\Gamma_3$ generated by the reflections in the Coxeter tetrahedron with symbol $[6,3,3]$ realizes minimal growth rate in a unique way.

Consider the hyperbolic Coxeter $n$-simplices and their reflection groups $\Gamma_n \subset \text{Isom}\mathbb{H}^n$ depicted in Table 1. For their volumes, we refer to [12]. Observe that $\Gamma_n$ is of minimal covolume among all non-cocompact hyperbolic Coxeter $n$-simplex groups.

The aim of this work is to prove the following result in the context of growth rates.

**Theorem** Let $2 \leq n \leq 9$. Among all non-cocompact hyperbolic Coxeter groups of finite covolume in $\text{Isom}\mathbb{H}^n$, the group $\Gamma_n$ given in Table 1 has minimal growth rate, and as such the group is unique.

Our Theorem should be compared with the volume minimality results for cusped hyperbolic $n$-orbifolds $O^n$ for $2 \leq n \leq 9$. These results are due to Siegel [16] for $n = 2$, Meyerhoff [15] for $n = 3$, Hild and Kellerhals [10] for $n = 4$, and Hild [9] for $n \leq 9$. Indeed, the fundamental group of $O^n$ is related to $\Gamma_n$ in all these cases.

The work is organized as follows. In Section 2.1, we set the background about hyperbolic Coxeter polyhedra and their associated reflection groups. Furthermore, we present a result of Felikson and Tumarkin about their combinatorics as given by [5, Theorem B], which will play a crucial role in our proof. In fact, we will exploit the (non-)simplicity of the Coxeter polyhedra in a most useful way. In Section 2.2, we discuss growth series and growth rates of Coxeter groups and introduce the notion of extension of a Coxeter graph. We also provide some illustrating examples. The monotonicity result of Terragni [18] for growth rates, presented in Theorem 2.2, will be another major ingredient in our proof. Finally, Section 3 is devoted to the proof of our result. We perform it in two steps by assuming that the Coxeter graph under consideration has an affine component of type $\tilde{A}_1$ or not.

## 2 Hyperbolic Coxeter groups and growth rates

### 2.1 Coxeter polyhedra and their reflection groups

Let $X^n$ denote one of the geometric $n$-spaces of constant curvature, the unit $n$-sphere $S^n$, the Euclidean $n$-space $E^n$, or the real hyperbolic $n$-space $\mathbb{H}^n$. As usual, we embed $X^n$ in a suitable quadratic space $Y^{n+1}$. In the Euclidean case, we take the affine model and write $E^n = \{ x \in E^{n+1} \mid x_{n+1} = 0 \}$. In the hyperbolic case, we interpret $\mathbb{H}^n$ as the
upper sheet of the hyperboloid in \( \mathbb{R}^{n+1} \), that is,

\[
\mathbb{H}^n = \{ x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle_{n,1} = -1, x_{n+1} > 0 \},
\]

where \( \langle x, x \rangle_{n,1} = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 \) is the standard Lorentzian form. Its boundary \( \partial \mathbb{H}^n \) can be identified with the set

\[
\partial \mathbb{H}^n = \{ x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle_{n,1} = 0, \sum_{k=1}^{n+1} x_k^2 = 1, x_{n+1} > 0 \}.
\]

In this picture, the isometry group of \( \mathbb{H}^n \) is isomorphic to the group \( O^+(n,1) \) of positive Lorentzian matrices leaving the bilinear form \( \langle \cdot, \cdot \rangle_{n,1} \) and the upper sheet invariant.

It is well known that each isometry of \( \mathbb{X}^n \) is a finite composition of reflections in hyperplanes, where a hyperplane \( H = H_\nu \) in \( \mathbb{X}^n \) is characterized by a normal unit vector \( \nu \in \mathbb{S}^{n+1} \). Associated to \( H_\nu \) are two closed half-spaces. We denote by \( H^\nu_\nu \) the half-space in \( \mathbb{X}^n \) with outer normal vector \( \nu \).

A (convex) \( n \)-polyhedron \( P = \bigcap_{i \in I} H_i^- \subset \mathbb{X}^n \) is the non-empty intersection of a finite number of half-spaces \( H_i^- \) bounded by the hyperplanes \( H_i = H_{i\nu} \) for \( i \in I \). A facet of \( P \) is of the form \( F_i = P \cap H_i \), for some \( i \in I \). In the sequel, for \( \mathbb{X}^n \neq \mathbb{S}^n \), we always assume that \( P \) is of finite volume. In the Euclidean case, this implies that \( P \) is compact, and in the hyperbolic case, \( P \) is the convex hull of finitely many points \( \nu_1, \ldots, \nu_k \in \mathbb{H}^n \cup \partial \mathbb{H}^n \).

If \( \nu_i \in \mathbb{H}^n \), then \( \nu_i \) is an ordinary vertex, and if \( \nu_i \in \partial \mathbb{H}^n \), then \( \nu_i \) is an ideal vertex of \( P \), respectively.

If all dihedral angles \( \alpha_{ij} = \angle(H_i, H_j) \) formed by intersecting hyperplanes \( H_i, H_j \) in the boundary of \( P \) are either zero or of the form \( \frac{\pi}{m_{ij}} \) for an integer \( m_{ij} \geq 2 \), then \( P \) is called a Coxeter polyhedron in \( \mathbb{X}^n \). Observe that the Gram matrix \( \text{Gr}(P) = (\langle \nu_i, \nu_j \rangle_{\mathbb{S}^{n+1}})_{i,j \in I} \) is a real symmetric matrix with 1's on the diagonal and non-positive coefficients off the diagonal. In this way, the theory of Perron–Frobenius applies. For further details and references about Coxeter polyhedra in \( \mathbb{X}^n \), we refer to [4, 19, 20].

Let \( P = \bigcap_{i=1}^N H_i^- \subset \mathbb{X}^n \) be a Coxeter \( n \)-polyhedron. Denote by \( r_i = r_{H_i} \) the reflection in the bounding hyperplane \( H_i \) of \( P \), and let \( G = G_P \) be the group generated by \( r_1, \ldots, r_N \). It follows that \( G \) is a discrete subgroup of finite covolume in \( \text{Isom}\mathbb{X}^n \), called a geometric Coxeter group.

A geometric Coxeter group \( G \subset \text{Isom}\mathbb{X}^n \) with generating system \( S = \{ r_1, \ldots, r_N \} \) is the geometric realization of an abstract Coxeter system \( (W, S) \). In fact, we have \( r_i^2 = 1 \) and \( (r_i r_j)^{m_{ij}} = 1 \) with \( m_{ij} = m_{ji} \in \{ 2, 3, \ldots, \infty \} \) as above. Here, \( m_{ij} = \infty \) indicates that \( r_i r_j \) is of infinite order.

For \( \mathbb{X}^n = \mathbb{S}^n \), \( G \) is a spherical Coxeter group and as such finite. For \( \mathbb{X}^n = \mathbb{E}^n \), \( G \) is a Euclidean or affine Coxeter group and of infinite order. By a result of Coxeter [3], the irreducible spherical and Euclidean Coxeter groups are entirely classified. In contrast to this fact, hyperbolic Coxeter groups are far from being classified. For a survey about partial classification results, we refer to [4].

For the description of abstract and geometric Coxeter groups, one commonly uses the language of weighted graphs and Coxeter symbols. Let \( (W, S) \) be an abstract Coxeter system with generating system \( S = \{ s_1, \ldots, s_N \} \) and relations of the form \( s_i^2 = 1 \) and \( s_i s_j m_{ij} = 1 \) with \( m_{ij} = m_{ji} \in \{ 2, 3, \ldots, \infty \} \). The Coxeter graph of the Coxeter
system \((W, S)\) is the non-oriented graph \(\Sigma\) whose nodes correspond to the generators \(s_1, \ldots, s_N\). If \(s_i\) and \(s_j\) do not commute, their nodes \(n_i, n_j\) are connected by an edge with weight \(m_{ij} \geq 3\). We omit the weight \(m_{ij} = 3\) since it occurs frequently. The number \(N\) of nodes is the order of \(\Sigma\). A subgraph \(\sigma \subset \Sigma\) corresponds to a special subgroup of \((W, S)\), that is, a subgroup of the form \((W_T, T)\) for a subset \(T \subset S\).

Observe that the Coxeter graph \(\Sigma\) is connected if \((W, S)\) is irreducible.

In the case of a geometric Coxeter group \(G = (W, S) \subset \text{Isom} \mathbb{X}^n\), we call its Coxeter graph \(\Sigma\) spherical, affine, or hyperbolic, if \(\mathbb{X}^n = S^n, \mathbb{E}^n,\) or \(\mathbb{H}^n\), respectively.

In Table 2, we reproduce all the connected affine Coxeter graphs, using the classical notation, with the exception of the three groups \(\tilde{E}_6, \tilde{E}_7, \tilde{E}_8\) (they will not appear in the following).

An abstract Coxeter group with a simple presentation can conveniently be described by its Coxeter symbol. For example, the linear Coxeter graph with edges of successive weights \(k_1, \ldots, k_N \geq 3\) is abbreviated by the Coxeter symbol \([k_1, \ldots, k_N]\).

Let us specify the context and consider a Coxeter polyhedron \(P = \bigcap_{i=1}^N H^-_i \subset \mathbb{H}^n\). Denote by \(\Gamma = G_P \subset \text{Isom} \mathbb{H}^n\) its associated Coxeter group and by \(\Sigma\) its Coxeter graph. Since \(P\) is of finite volume, the graph \(\Sigma\) is connected. Furthermore, if \(P\) is not compact, then \(P\) has at least one ideal vertex.

Let \(v \in \mathbb{H}^n\) be an ordinary vertex of \(P\). Then, its link \(L_v\) is the intersection of \(P\) with a small sphere of center \(v\) that does not intersect any facet of \(P\) not incident to \(v\). It corresponds to a spherical Coxeter polyhedron of \(S^{n-1}\) and therefore to a spherical Coxeter subgraph \(\sigma\) of order \(n\) in \(\Sigma\).

Let \(v_\infty \in \partial \mathbb{H}^n\) be an ideal vertex of \(P\). Then, its link, denoted by \(L_\infty\), is given by the intersection of \(P\) with a sufficiently small horosphere centered at \(v_\infty\) as above. The link \(L_\infty\) corresponds to a Euclidean Coxeter polyhedron in \(\mathbb{E}^{n-1}\) and is related to an affine Coxeter subgraph \(\sigma_\infty\) of order \(\geq n\) in \(\Sigma\).

More precisely, if \(v_\infty\) is a simple ideal vertex, that is, \(v_\infty\) is the intersection of exactly \(n\) among the \(N\) bounding hyperplanes of \(P\), the Coxeter graph \(\sigma_\infty\) is connected and of order \(n\). Otherwise, \(\sigma_\infty\) has \(n_c(\sigma_\infty) \geq 2\) affine components, and we have the following formula:

\[
(1) \quad n - 1 = \text{order}(\sigma_\infty) - n_c(\sigma_\infty).
\]

Recall that a polyhedron is simple if all of its vertices are simple.
Figure 1: The Coxeter polyhedron $P_0 \subset \mathbb{H}^4$.

As in the spherical and Euclidean cases, hyperbolic Coxeter simplices in $\mathbb{H}^n$ are all known, and they exist for $n \leq 9$ (see [1] or [20]). A list of their Coxeter graphs, Coxeter symbols, and volumes can be found in [12]. Among the related Coxeter $n$-simplex groups, the group $\Gamma_n$, as given in Table 1, is of minimal covolume.

The following structural result for simple hyperbolic Coxeter polyhedra due to Felikson and Tumarkin [5, Theorem B] will be a corner stone for the proof of our Theorem.

**Theorem 2.1** Let $n \leq 9$, and let $P \subset \mathbb{H}^n$ be a non-compact simple Coxeter polyhedron. If all facets of $P$ are mutually intersecting, then $P$ is either a simplex or isometric to the polyhedron $P_0$ whose Coxeter graph is depicted in Figure 1.

**2.2 Growth rates and their monotonicity**

Let $(W, S)$ be a Coxeter system and denote by $a_k \in \mathbb{Z}$ the number of elements $w \in W$ with $S$-length $k$. The growth series $f_S(t)$ of $(W, S)$ is defined by

$$f_S(t) = 1 + \sum_{k \geq 1} a_k t^k.$$  

In the following, we list some properties of $f_S(t)$. For references, we refer to [11].

There is a formula due to Steinberg expressing the growth series $f_S(t)$ of a Coxeter system $(W, S)$ in terms of its finite special subgroups $W_T$ for $T \subseteq S$,

$$\frac{1}{f_S(t^{-1})} = \sum_{[W_T] < \infty} \left(\frac{-1}{|T|}\right) f_T(t),$$  

where $W_\emptyset = \{1\}$. By a result of Solomon, the growth polynomial of each term $f_T(t)$ in (2) can be expressed by means of its exponents $\{m_1, m_2, \ldots, m_p\}$ according to the formula

$$f_T(t) = \prod_{i=1}^p [m_i + 1],$$  

where $[k] = 1 + t + \ldots + t^{k-1}$ and, more generally, $[k_1, \ldots, k_r] := [k_1] \cdots [k_r]$. A complete list of the irreducible spherical Coxeter groups together with their exponents can be found in [14]. For example, the exponents of the Coxeter group $A_n$ with Coxeter graph $\cdots$ are $\{1, 2, \ldots, n\}$ so that

$$f_{A_n}(t) = [2, \ldots, n + 1].$$
Furthermore, the growth series of a reducible Coxeter system \((W, S)\) with factor groups \((W_1, S_1)\) and \((W_2, S_2)\) such that \(S = (S_1 \times \{1\}_{W_2}) \cup \{1\}_{W_1} \times S_2\) satisfies the product formula

\[ f_S(t) = f_{S_1}(t) \cdot f_{S_2}(t). \]

In its disk of convergence, the growth series \(f_S(t)\) is a rational function, which can be expressed as the quotient of coprime monic polynomials \(p(t), q(t) \in \mathbb{Z}[t]\) of the same degree. The growth rate \(\tau_W = \tau_{(W,S)}\) is defined by the inverse of the radius of convergence of \(f_S(t)\) and can be expressed by

\[ \tau_W = \limsup_{k \to \infty} a_k^{1/k}. \]

It is the inverse of the smallest positive real pole of \(f_S(t)\) and hence an algebraic integer.

Important for the proof of our Theorem is the following result of Terragni [17] about the growth monotonicity.

**Theorem 2.2** Let \((W, S)\) and \((W', S')\) be two Coxeter systems such that there is an injective map \(\iota : S \to S'\) with \(m_s \leq m'_{\iota(s)}(t)\) for all \(s, t \in S\). Then, \(\tau_{(W,S)} \leq \tau_{(W',S')}\).

For \(n \geq 2\), consider a Coxeter group \(\Gamma \subset \text{Isom}\mathbb{H}^n\) of finite covolume. By results of Milnor and de la Harpe, we know that \(\tau_{\Gamma} > 1\). More precisely, and as shown by Terragni [17], \(\tau_{\Gamma} \geq \tau_{\Gamma_9} \approx 1.1380\), where \(\Gamma_9\) is the Coxeter simplex group given in Table 1.

Next, we introduce another tool in the proof of our result, the (simple) extension of a Coxeter graph.

**Definition 2.1** Let \(\Sigma\) be an abstract Coxeter graph. A (simple) extension of \(\Sigma\) is a Coxeter graph \(\Sigma'\) obtained by adding one node linked with a (simple) edge to the Coxeter graph \(\Sigma\).

As a direct consequence of Theorem 2.2, if \(W\) is a Coxeter group with Coxeter graph \(\Sigma\), any extension \(\Sigma'\) of \(\Sigma\) encodes a Coxeter group \(W'\) such that \(\tau_W \leq \tau_{W'}\).

**Example 2.3** Consider an irreducible affine Coxeter graph of order 3 as given in Table 2. Up to symmetry, the graph \(\tilde{A}_2\) has a unique extension given by the Coxeter graph at the top left in Figure 2. This graph describes the Coxeter tetrahedron \([3, 3^{(3)}]\) of finite volume. The Coxeter graphs \(\tilde{C}_2\) and \(\tilde{G}_2\) give rise to the remaining five extensions depicted in Figure 2. By a result of Kellerhals [13], these six Coxeter graphs describe Coxeter tetrahedral groups \(\Lambda\) of finite covolume in \(\text{Isom}\mathbb{H}^3\) whose growth rates satisfy \(\tau_{\Lambda} \geq \tau_{\Gamma_9}\).

**Example 2.4** In a similar way, any extension of an irreducible affine Coxeter graph of order 4 yields a Coxeter simplex group of finite covolume in \(\text{Isom}\mathbb{H}^4\). They are given in Figure 3. Notice that \(\Gamma_4 = [4, 3^{2,1}]\) is part of them.

**Remark 2.5** When considering irreducible affine Coxeter graphs of order greater than or equal to 5, the resulting extensions do not always relate to hyperbolic Coxeter...
$n$-simplex groups of finite covolume. For example, among the extensions of $\tilde{F}_4$, the graph depicted in Figure 4 describes an infinite volume simplex in $\mathbb{H}^5$.

3 Proof of the Theorem

Let $2 \leq n \leq 9$, and consider the Coxeter simplex group $\Gamma_n \subset \text{Isom} \mathbb{H}^n$ whose Coxeter graph is depicted in Table 1. In this section, we provide the proof of our main result stated as follows.

**Theorem** For any $2 \leq n \leq 9$, the group $\Gamma_n$ has minimal growth rate among all non-cocompact hyperbolic Coxeter groups of finite covolume in $\text{Isom} \mathbb{H}^n$, and as such the group is unique.

For $n = 2$ and for $n = 3$, the result has been established by Floyd [6] and Kellerhals [13]. Therefore, it suffices to prove the Theorem for $4 \leq n \leq 9$.

Observe that the growth rates of all Coxeter simplex groups in $\text{Isom} \mathbb{H}^n$ are known. Their list can be found in [17]. In particular, one deduces the following strict inequalities:

$$(5) \quad \tau_{\Gamma_4} \approx 1.1380 < \cdots < \tau_{\Gamma_5} \approx 1.2481 < \tau_{\Gamma_6} \approx 1.3717,$$

$$(6) \quad \tau_{\Gamma_3} < \tau_{\Gamma_4} \approx 1.2964.$$

For a fixed dimension $n$, one also checks that $\Gamma_n$ has minimal growth rate among (all the finitely many) non-cocompact Coxeter simplex groups $\Lambda \subset \text{Isom} \mathbb{H}^n$.

As a consequence, we focus on hyperbolic Coxeter groups $\Gamma \subset \text{Isom} \mathbb{H}^n$ generated by at least $N \geq n + 2$ reflections in the facets of a non-compact finite volume Coxeter
polyhedron \( P \subset \mathbb{H}^n \). We have to show that \( \tau_{\Gamma_n} < \tau_{\Gamma} \), which yields unicity of the group \( \Gamma_n \) with this property.

Suppose that the Coxeter polyhedron \( P \) is simple. By Theorem 2.1, \( P \) is either isometric to the polyhedron \( P_0 \subset \text{Isom} \mathbb{H}^4 \) depicted in Figure 1, or \( P \) has a pair of disjoint facets. For the growth rate \( \tau \) of the Coxeter group associated to \( P_0 \), one easily checks with help of the software CoxIter \( [7, 8] \) that \( \tau_{\Gamma_4} < \tau \approx 2.8383 \). Hence, we can assume that \( P \) is not isometric to \( P_0 \). If \( P \) has a pair of disjoint facets, then the Coxeter graph \( \Sigma \) of \( P \) and its associated group \( \Gamma \) contains a subgraph \( \tilde{A}_1 \).

The property that the Coxeter graph \( \Sigma \) contains such a subgraph of type \( \tilde{A}_1 = [\infty] \) allows us to conclude the proof, whether the polyhedron \( P \) is simple or not. In the following, we first look at this property and analyze it more closely.

3.1 In the presence of \( \tilde{A}_1 \)

We start by considering particular Coxeter graphs of order 4 containing \( \tilde{A}_1 \). Their related growth rates will be useful when comparing with the one of \( \Gamma \). This approach is similar to the one developed in [2].

Let \( W_0 = [\infty, 3, 3] \) be the abstract Coxeter group depicted in Figure 5. By means of the software CoxIter, one checks that

\[
\tau_{\Gamma_4} < \tau_{W_0} \approx 1.4655. \tag{7}
\]

Furthermore, consider the two abstract Coxeter groups \( W_1 = [3, \infty, 3] \) and \( W_2 = [\infty, 3^{1,1}] \) given in Figure 6.

For their growth rates, we prove the following auxiliary result.

**Lemma 3.1** \( \tau_{W_0} < \tau_{W_1} \) and \( \tau_{W_0} < \tau_{W_2} \).

**Proof** For \( 0 \leq i \leq 2 \), denote by \( f_i := f_{W_i} \) the growth series of \( W_i \) and by \( R_i \) its radius of convergence. Recall that \( R_i \) is the smallest positive pole of \( f_i \), and that \( \tau_{W_i} = \frac{1}{R_i} \).

We establish the growth functions \( f_i \) according to Steinberg’s formula (2). They are given as follows:

\[
\frac{1}{f_0(t^{-1})} = 1 - \frac{4}{2} + \frac{3}{[2,2]} + \frac{2}{[2,3]} - \frac{1}{[2,2,3]} - \frac{1}{[2,3,4]},
\]

\[
\frac{1}{f_1(t^{-1})} = 1 - \frac{4}{2} + \frac{3}{[2,2]} + \frac{2}{[2,3]} - \frac{2}{[2,2,3]},
\]

\[
\frac{1}{f_2(t^{-1})} = 1 - \frac{4}{2} + \frac{3}{[2,2]} + \frac{2}{[2,3]} - \frac{1}{[2,2,2]} - \frac{1}{[2,3,4]}.\]
Hence, for any \( t > 0 \), one has the positive difference functions given by

\[
\frac{1}{f_0(t^{-1})} - \frac{1}{f_i(t^{-1})} = \frac{1}{[2,2,3,4]} - \frac{1}{[2,2,3,4]} = t^2 + t^3 > 0,
\]

\[
\frac{1}{f_0(t^{-1})} - \frac{1}{f_i(t^{-1})} = \frac{1}{[2,2,2]} - \frac{1}{[2,2,3]} = t^2 > 0.
\]

Therefore, for \( i = 1, 2 \), and for \( u = t^{-1} \in (0,1) \), the smallest positive root \( R_0 \) of \( \frac{1}{f_0(u)} \) is strictly bigger than the one of \( \frac{1}{f_i(u)} \). This finishes the proof.

As a first consequence, combining (5), (7), and Lemma 3.1, one obtains that

\[
\tau_{\Gamma_n} < \tau_{W_i},
\]

for all \( 4 \leq n \leq 9 \) and \( 0 \leq i \leq 2 \).

Next, suppose that the Coxeter graph \( \Sigma \) of \( \Gamma \) contains a subgraph \( \widetilde{A}_1 \). Since \( \Sigma \) is connected of order \( N \geq n + 2 \geq 6 \), the subgraph \( \widetilde{A}_1 \) is contained in a connected subgraph \( \sigma \) of order 4 in \( \Sigma \), which is related to a special subgroup \( W \) of \( \Gamma \). By Theorem 2.2, one has that \( \tau_{W_i} \leq \tau_{W} \) for some \( 0 \leq i \leq 2 \). By combining (8) with these findings, and by Theorem 2.2 and Lemma 3.1, one deduces that

\[
\tau_{\Gamma_n} < \tau_{W_0} \leq \tau_{W} \leq \tau_{\Gamma}.
\]

This finishes the proof of the Theorem in the presence of a subgraph \( \widetilde{A}_1 \) in \( \Sigma \).

### 3.2 In the absence of \( \widetilde{A}_1 \)

Suppose that the Coxeter graph \( \Sigma \) with \( N \geq n + 2 \) nodes does not contain a subgraph of type \( \widetilde{A}_1 \). In particular, by Theorem 2.1, the corresponding Coxeter polyhedron \( P \subset Isom\mathbb{H}^n \) is not simple, and it follows that \( 5 \leq n \leq 9 \).

Consider a non-simple ideal vertex \( v_\infty \in P \). Its link \( L_\infty \subset \mathbb{E}^{n-1} \) is described by a reducible affine subgraph \( \sigma_\infty \) with \( n_\epsilon = n_\epsilon(\infty) \geq 2 \) components which satisfies \( n - 1 = \text{order}(\sigma_\infty) - n_\epsilon \) by (1). In Table 3, we list all possible realizations for \( \sigma_\infty \) by using the following notations.

Let \( \overline{\sigma}_k \) be a connected affine Coxeter graph of order \( k \geq 3 \) as listed in Table 2, and denote by \( \bigsqcup_k \overline{\sigma}_k \) the Coxeter graph consisting of the components of type \( \overline{\sigma}_k \).

Observe that for any graph \( \bigsqcup_k \overline{\sigma}_k \) in Table 3, one has \( 3 \leq \min_k k \leq 5 \), and that the case \( \min_k k = 5 \) appears only when \( n = 9 \).

Among the different components of \( \sigma_\infty \), we consider the ones of smallest order \( \geq 3 \) together with their extensions.

- Assume that the graph \( \sigma_\infty \) of the vertex link \( L_\infty \) contains an affine component \( \overline{\sigma} \) of order 3. By Example 2.3, we know that any extension of \( \overline{\sigma} \) encodes a Coxeter tetrahedral group \( \Lambda \subset Isom\mathbb{H}^3 \) of finite covolume. The graph \( \Sigma \) itself contains a subgraph \( \sigma \) of order 4 which in turn comprises \( \overline{\sigma} \). The Coxeter graph \( \sigma \) corresponds to a special subgroup \( W \) of \( \Gamma \), and by Theorem 2.2, we deduce that \( \tau_{\Lambda} \leq \tau_{W} \).
Table 3: Reducible affine Coxeter graphs $\sigma_\infty$ with $n_\infty \geq 2$ components $\tilde{\sigma}_k$ of order $k \geq 3$ such that $n = \text{order}(\sigma_\infty) - n_\infty + 1$.

| $n$ | 5   | 6   | 7   | 8   | 9   |
|-----|-----|-----|-----|-----|-----|
|     | $\tilde{\sigma}_3 \sqcup \tilde{\sigma}_3$ | $\tilde{\sigma}_3 \sqcup \tilde{\sigma}_4$ | $\tilde{\sigma}_3 \sqcup \tilde{\sigma}_5$ | $\tilde{\sigma}_3 \sqcup \tilde{\sigma}_6$ | $\tilde{\sigma}_3 \sqcup \tilde{\sigma}_7$ |
|     | $\tilde{\sigma}_4 \sqcup \tilde{\sigma}_4$ | $\tilde{\sigma}_4 \sqcup \tilde{\sigma}_5$ | $\tilde{\sigma}_4 \sqcup \tilde{\sigma}_6$ | $\tilde{\sigma}_4 \sqcup \tilde{\sigma}_7$ | $\tilde{\sigma}_4 \sqcup \tilde{\sigma}_8$ |
|     | $\tilde{\sigma}_3 \sqcup \tilde{\sigma}_3 \sqcup \tilde{\sigma}_3$ | $\tilde{\sigma}_3 \sqcup \tilde{\sigma}_4 \sqcup \tilde{\sigma}_3$ | $\tilde{\sigma}_3 \sqcup \tilde{\sigma}_3 \sqcup \tilde{\sigma}_4$ | $\tilde{\sigma}_3 \sqcup \tilde{\sigma}_3 \sqcup \tilde{\sigma}_5$ | $\tilde{\sigma}_3 \sqcup \tilde{\sigma}_3 \sqcup \tilde{\sigma}_6$ |

\[\begin{align*}
\tau_{\Delta_1} &\approx 1.678 \\
\tau_{\Delta_2} &\approx 1.599 \\
\tau_{\Delta_3} &\approx 1.668 \\
\tau_{\Delta_4} &\approx 1.702
\end{align*}\]

*Figure 7*: The Coxeter groups $\Delta_i$, $i = 1, \ldots, 4$.

Since $\tau_{\Gamma_5} \leq \tau_{\Lambda}$, and in view of (5) and (6), Theorem 2.2 yields the desired inequality

\[(10) \quad \tau_{\Gamma_n} < \tau_{\Gamma_5} \leq \tau_{\Lambda} \leq \tau_{W} \leq \tau_{\Gamma},\]

which finishes the proof in this case, and for $n = 5$ and $n = 6$; see Table 3.

- Assume that the graph $\sigma_\infty$ contains an affine component $\tilde{\sigma}$ of order 4. We apply the same reasoning as above. By Example 2.4, any extension of $\tilde{\sigma}$ corresponds to a Coxeter 4-simplex group $\Lambda$ of finite covolume, and $\tau_{\Gamma_4} \leq \tau_{\Lambda}$. Again, $\Sigma$ contains a subgraph $\sigma$ comprising $\tilde{\sigma}$. Hence, there exists a special subgroup $W$ of $\Gamma$ described by $\sigma$ so that

\[(11) \quad \tau_{\Gamma_n} < \tau_{\Gamma_4} \leq \tau_{\Lambda} \leq \tau_{W} \leq \tau_{\Gamma}.\]

By (10) and (11), the proof is finished in this case, and for $n = 7$ and $n = 8$; see Table 3.

- Assume that $\sigma_\infty$ contains an affine component $\tilde{\sigma}$ of order 5. By Table 3, one has $7 \leq n \leq 9$. It is not difficult to list all possible extensions of $\tilde{\sigma}$. There are exactly 15 such extensions. It turns out that there are 11 extensions that encode Coxeter 5-simplex groups of finite covolume, whereas the remaining 4 extensions describe 5-simplex groups $\Delta_i$, $i = 1, \ldots, 4$, of infinite covolume. These last four simplices arise by extending $B_4$, $C_4$, and $F_4$. They are given in Figure 7, together with their associated growth rates computed with CoxIter.

In view of (5), it turns out that

\[(12) \quad \tau_{\Gamma_5} < \tau_{\Delta_i}, \quad \text{for} \ i = 1, \ldots, 4.\]

As above, the component $\tilde{\sigma}$ lies in a subgraph $\sigma$ of order 6 in $\Sigma$, and the latter corresponds to a special subgroup $W$ of $\Gamma$ so that

either $\tau_{\Lambda} \leq \tau_{W}$ or $\tau_{\Delta_i} \leq \tau_{W}, \quad 1 \leq i \leq 4$
where $\Lambda$ is a Coxeter 5-simplex group of finite covolume. Since $\tau_{\Gamma_{5}} \leq \tau_{\Lambda}$, and by (5) and (12), one deduces that

$$
\tau_{\Gamma_{a}} < \tau_{\Gamma_{5}} \leq \tau_{W} \leq \tau_{\Gamma}.
$$

This finishes the proof of this case.

Finally, all the above considerations allow us to conclude the proof of the Theorem.

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