Mathematical description of a bounded oil reservoir with a horizontal well: late time flow period

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INTRODUCTION

For horizontal wells, the many aspects of the flow can affect the transient flow behavior. The oil reservoir boundaries based on the well location and the anisotropic permeabilities are parameters that have to be considered when developing a model for pressure distribution. To enhance productivity it’s important to develop detailed mathematical models that can be used to approximate pressure distribution at any given time of production. The objective of carrying out this study thus is to develop an in depth mathematical model that considers a horizontal well in a completely bounded oil reservoir that can be used to approximate pressure distribution when a well has produced for a sometime and the effects of the oil reservoir boundaries are evident.

Different literature provides models for horizontal wells. Carslaw and Jaeger (1959) came up with results that could be applied in transient flow which has continued to influence fluid flow modelling. Advances in well test analysis have continued to be based on the monograph developed by Mathews and Russell (1967). (Gringarten & Ramey, 1973) detailed the use of source and Green's functions to model pressure in a bounded oil reservoir during early time. Over the years different authors used these source functions to model pressure (Daviau et al., 1985; Clonts & Ramey, 1986; Ozkan et al., 1987; Odeh & Babu, 1989; Carvalho & Rosa, 1988). Adewole (2009, 2010) used source
and Green’s functions to develop solutions for pressure distribution for bounded oil reservoirs. Al Rbeawi and Tiab (2013) considered a multi-boundary system and applied the source functions to analyze transient behavior in horizontal wells. Most of these studies have developed mathematical models that consider isotropic permeability which is not always the case for practical horizontal wells and where anisotropy is considered, many assumptions limit the effectiveness of the model. We intend to derive a mathematical model that accounts for anisotropic permeability for a horizontal well considering a bounded oil reservoir for late time flow period.

**METHODOLOGY**

**Description of the Physical Model**

The physical model used to develop this model is as shown in Figure 1. The drainage volume considered assumes an oil reservoir whose length is \( x_s \) taken in the x-axis, the reservoir width is considered to be \( y_s \) taken along y-axis and a reservoir thickness of h along the z-axis. We consider a horizontal well having a length \( L \) parallel to the x-axis. The well is centrally located at \( (x_w, y_w, z_w) \) stretching a length of \( L/2 \) from \( x_w \) in both directions along x.

**Mathematical Description**

To develop the required mathematical model we start by considering the diffusivity equation (Lee, 1982),

\[
\nabla^2 P = \frac{\phi \mu c}{k} \frac{\partial P}{\partial t}
\]

(1)

Where \( P \) is pressure, \( \mu \) is fluid viscosity, \( c \) is compressibility, \( \phi \) is porosity, \( k \) is the formation permeability and \( t \) is time.

Keeping porosity constant, ignoring gravitational forces and considering a slightly compressible fluid, when we consider permeability anisotropy, the heterogeneous three dimensional diffusivity equation becomes (Odeh & Babu, 1989),

\[
k_x \frac{\partial^2 P}{\partial x^2} + k_y \frac{\partial^2 P}{\partial y^2} + k_z \frac{\partial^2 P}{\partial z^2} = \Phi \mu c_t \frac{\partial P}{\partial t}
\]

(2)

Where \( k_x, k_y \) and \( k_z \) are the respective permeabilities along the x-axis, y-axis and z-axis respectively and \( c_t \) is the total formation compressibility.

**Dimensionless Pressure**

The Dimensionless pressure is given by Mathews and Russell (1967)

\[
P_D = \frac{k h \Delta P}{141.2 q \mu B}
\]

(3)

Which can be written as

\[
P_D = \frac{2\pi kh}{q \mu} \Delta P
\]

(4)

Or

\[
\Delta P = \frac{q}{\Phi c_t L} \int_0^t s(x, y, z, t) dt
\]

(5)

With;

\[
\Delta P = \frac{q}{\Phi c_t L} \int_0^t s(x, y, z, t) dt
\]

(6)

Where \( q \) is the flow rate, \( B \) is the formation volume factor and \( \tau \) is a dummy variable for time, \( t \).

The product \( s(x, y, z, t) \) is obtained using the Newman product rule (Gringarten & Ramey, 1973) and is given by

\[
s(x, y, z, t) = s(x, t) \cdot s(y, t) \cdot s(z, t)
\]

(7)

Where \( s(x, t), s(y, t) \) and \( s(z, t) \) are appropriate source and Green’s functions.
Source Functions

As shown in the physical model in Figure 1, the horizontal well indicated can be considered to be a horizontal line source of length $L$ at $(x_w, y_w, z_w)$ and thus the source function is a result of intersection of three sources, (Gringarten & Ramey, 1973) namely;

(a) A source function located at $y = y_w$ in a reservoir having a width $y_e$ as shown in Figure 2.
(b) A source function located at $x = x_w$ in a reservoir having a length $x_e$ as shown in Figure 3.
(c) A source function located at $z = z_w$ in a reservoir having a thickness $h$ as shown in Figure 4.

![Figure 1: Physical Model of a Horizontal Well in a Rectangular Drainage Volume](image)

![Figure 2: Infinite-plane Source along y-axis](image)

![Figure 3: Infinite-slab Source along x-axis](image)

![Figure 4: Infinite-plane source along the z-axis](image)
RESULTS AND DISCUSSION

When all the boundaries have started affecting the flow then we consider a pseudosteady state. In this flow period, we consider the following source functions as given by Gringarten & Ramey (1973).

(a) The instantaneous source function along x-axis given as:

\[ s(x, t) = \frac{L}{x_e} \left\{ 1 + \frac{4x_e}{\pi L} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left( -\frac{n^2 \pi^2 x t}{x_e^2} \right) \sin \frac{n \pi L}{2x_e} \cos \frac{n \pi x_w}{x_e} \cos \frac{n \pi x}{x_e} \right\} \]

(b) The instantaneous source function along y-axis defined given as:

\[ s(y, t) = \frac{1}{y_e} \left\{ 1 + 2 \sum_{m=1}^{\infty} \exp \left( -\frac{m^2 \pi^2 y t}{y_e^2} \right) \cos \frac{m \pi y_w}{y_e} \cos \frac{m \pi y}{y_e} \right\} \]

(c) And the instantaneous source function along z-axis given as:

\[ s(z, t) = \frac{1}{h} \left\{ 1 + 2 \sum_{l=1}^{\infty} \exp \left( -\frac{l^2 \pi^2 z t}{h^2} \right) \cos \frac{l \pi z_w}{h} \cos \frac{l \pi z}{h} \right\} \]

Substituting in equation (7) we obtain;

\[ S(x, y, z, t) = \frac{L}{x_e y_e h} \left\{ 1 + \frac{4x_e}{\pi L} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left( -\frac{n^2 \pi^2 x t}{x_e^2} \right) \sin \frac{n \pi L}{2x_e} \cos \frac{n \pi x_w}{x_e} \cos \frac{n \pi x}{x_e} \right\} \]

\[ \frac{1}{y_e} \left\{ 1 + 2 \sum_{m=1}^{\infty} \exp \left( -\frac{m^2 \pi^2 y t}{y_e^2} \right) \cos \frac{m \pi y_w}{y_e} \cos \frac{m \pi y}{y_e} \right\} \cdot \frac{1}{h} \left\{ 1 + 2 \sum_{l=1}^{\infty} \exp \left( -\frac{l^2 \pi^2 z t}{h^2} \right) \cos \frac{l \pi z_w}{h} \cos \frac{l \pi z}{h} \right\} \]

Or

\[ S(x, y, z, t) = \frac{L}{x_e y_e h} \left\{ 1 + \frac{4x_e}{\pi L} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left( -\frac{n^2 \pi^2 x t}{x_e^2} \right) \sin \frac{n \pi L}{2x_e} \cos \frac{n \pi x_w}{x_e} \cos \frac{n \pi x}{x_e} \right\} \cdot \left\{ 1 + 2 \sum_{m=1}^{\infty} \exp \left( -\frac{m^2 \pi^2 y t}{y_e^2} \right) \cos \frac{m \pi y_w}{y_e} \cos \frac{m \pi y}{y_e} \right\} \cdot \left\{ 1 + 2 \sum_{l=1}^{\infty} \exp \left( -\frac{l^2 \pi^2 z t}{h^2} \right) \cos \frac{l \pi z_w}{h} \cos \frac{l \pi z}{h} \right\} \]

Substituting in the \( P_D \) expression, and simplifying equation (4) we obtain;

\[ P_D = \frac{2\pi}{x_e y_e h} \int_0^{x_D} \left\{ \left[ 1 + \frac{x_e}{x_e^2} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left( -\frac{n^2 \pi^2 x_D t}{x_e^2} \right) \sin \frac{n \pi x_w}{x_e} \cos \frac{n \pi x}{x_e} \right] \cdot \left[ 1 + 2 \sum_{m=1}^{\infty} \exp \left( -\frac{m^2 \pi^2 y_D t}{y_e^2} \right) \cos \frac{m \pi y_w}{y_e} \cos \frac{m \pi y}{y_e} \right] \cdot \left[ 1 + 2 \sum_{l=1}^{\infty} \exp \left( -\frac{l^2 \pi^2 z_D t}{h_D^2} \right) \cos \frac{l \pi z_w}{h_D} \cos \frac{l \pi z}{h_D} \right] \right\} dx_D \]
To be able to analyse the slope and thus identify flow regimes, the dimensionless pressure derivative for equation (13) is given by:

\[
P_D' = \frac{2\pi}{x_{eD}y_{eD}} \left\{ \left[ 1 + \frac{2x_{eD}}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \exp\left(-\frac{n^2\pi^2 t_D}{x_{eD}^2}\right) \sin\frac{n\pi}{x_{eD}} \cos\frac{n\pi x_{WD}}{x_{eD}} \cos\frac{n\pi x_{zD}}{x_{eD}} \right] \cdot \left[ 1 + 2 \sum_{m=1}^{\infty} \exp\left(-\frac{m^2\pi^2 t_D}{y_{eD}^2}\right) \cos\frac{m\pi y_{WD}}{y_{eD}} \cos\frac{m\pi y_{zD}}{y_{eD}} \right] \cdot \left[ 1 + 2 \sum_{l=1}^{\infty} \exp\left(-\frac{l^2\pi^2 t_D}{h_{D}^2}\right) \cos\frac{l\pi z_{WD}}{h_{D}} \cos\frac{l\pi z_{zD}}{h_{D}} \right] \right\} t_D \tag{14}
\]

The dimensionless variables used in this derivation are defined as follows:

\[
i_D = \frac{2i}{L} \sqrt{\frac{k}{k_i}} \tag{15}
\]

\[
i_{wD} = \frac{2i_{w}}{L} \sqrt{\frac{k}{k_i}} \tag{16}
\]

\[
i_{eD} = \frac{2i_{e}}{L} \sqrt{\frac{k}{k_i}} \tag{17}
\]

Where \(i = x, y, z\) and \(z_e = h\)

\[
t_D = \frac{4kt}{\phi \mu c_t L^2} \tag{18}
\]

And

\[
\eta_i = \frac{k_i}{\phi \mu c_t} \tag{19}
\]

Where \(\eta_i\) is the diffusivity constant in the \(i\)-axial directions, \(x, y\) and \(z\).

This is a mathematical model that can be applied to approximate the pressure response when we consider a horizontal well inside a completely bounded oil reservoir. This applies when all the boundaries of the reservoir start influencing the flow and thus attaining a pseudosteady state flow. The model derived shows an inverse proportionality between the pressure response and both the width and length of the well. Thus, for any given oil reservoir, if the permeability is high, then an increased pressure response will be attained from short and thinner reservoirs and for low permeability reservoirs the same will be achieved from wide and longer reservoirs.

**CONCLUSION**

From the above discussion, it's evident that the model developed can;

(a) Approximate pressure response during late times,

(b) Approximate the reservoir geometry.

This will require a careful substitution of the oil reservoir parameters. The length of the well as it compares to the length, width and thickness of the reservoir will determine how long this flow period lasts for meaningful analysis. Since the model considers late times, the duration in which a well is considered to be productive is a factor for the pressure response. Thus, this model can be applied to enhance the performance of a well at late times.

**Nomenclature**

- \(B\) Oil formation volume factor, rbbl/stb
- \(c_t\) Formation compressibility, 1/psi
- \(h\) Reservoir pay thickness, ft
- \(h_D\) Dimensionless reservoir pay thickness
- \(i\) Axial flow directions \(x, y\) and \(z\)
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| Symbol | Description |
|--------|-------------|
| $k$    | Reservoir permeability, md |
| $k_x$  | Permeability, x-axis, md |
| $k_y$  | Permeability y-axis, md |
| $k_z$  | Permeability z-axis, md |
| $l$    | Well length, ft |
| $P_D$  | Dimensionless pressure |
| $q$    | Flow rate, bbl/day |
| $s$    | Source |
| $t$    | Time, hrs |
| $t_D$  | Dimensionless form of time |
| $x$    | Length, x-axis, ft |
| $x_D$  | Dimensionless form of length |
| $x_e$  | Reservoir length, ft |
| $x_{eD}$ | Dimensionless reservoir length |
| $x_w$  | Source coordinate, x-axis, ft |
| $x_{wD}$ | Dimensionless form of source coordinate, x-axis |
| $y$    | Width, y-axis, ft |
| $y_D$  | Dimensionless form of width |
| $y_e$  | Reservoir width, ft |
| $y_{eD}$ | Dimensionless reservoir width |
| $y_w$  | Source coordinate, y-direction, ft |
| $y_{wD}$ | Dimensionless form of source coordinate, y-axis |
| $z$    | Thickness, z-axis, ft |
| $z_D$  | Dimensionless form of thickness |
| $z_e$  | Source coordinate, z-axis, ft |
| $z_{wD}$ | Dimensionless form of source coordinate, z-axis |
| $\eta_i$ | Diffusivity constant, i-axis, md-psi/cp |
| $\Phi$ | Porosity |
| $\mu$  | Reservoir fluid viscosity, cp |
| $\tau_D$ | Dimensionless form of dummy variable for time |

**Superscripts**

- $^r$ Derivative

**Subscripts**

- $D$ Dimensionless
- $e$ External
- $i$ Axial direction
- $w$ Wellbore

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