Moduli and Brane Intersections

Neil D. Lambert

Department of Physics and Astronomy
Rutgers University
Piscataway NJ 08855
USA

ABSTRACT

We discuss the worldvolume description of intersecting D-branes, including the metric on the moduli space of deformations. We impose a choice of static gauge that treats all the branes on an equal footing and describes the intersection of D-branes as an embedded special Lagrangian three-surface. Some explicit solutions to these equations are given and their interpretation in terms of a superpotential on moduli space is discussed. These surfaces are interpreted in terms of the flat direction of a non-Abelian superpotential and imply the existence of non-compact $G_2$ manifolds.
1 Introduction

D-branes provide a fascinating bridge between the classical geometry of minimal surfaces and quantum non-Abelian gauge theory. The usual set-up consists of several branes intersecting so that their common worldvolume has some Poincaré invariance and preserves some supersymmetry. From the microscopic D-brane perspective the low energy dynamics is governed by open string theory and hence coincides with Yang-Mills theory to lowest order in derivatives. In the macroscopic description branes are minimal volume surfaces their dynamics is essentially determined by geometry. To be more precise the induced metric on the brane defines a metric on the moduli space of deformations of the surface and the low energy effective action is then given by the associated σ-model.

Recently, in connection with compactifications of M-theory on exceptional manifolds, brane intersections with minimal supersymmetry have been studied and related to gauge theory [1, 2, 3]. Supersymmetry imposes the additional condition that the brane worldvolume is not just minimal but calibrated [4]. In [5] Bogomol’nyi equations were derived for various intersections where static gauge was imposed with respect to one of the branes. The resulting equations are known as Harvey-Lawson equations [6] and are also known to be local conditions for the embedded surface to be calibrated. Here we wish to revisit some of this analysis but treat all the branes on an equal footing. In particular we will not impose static gauge for any one particular brane. A benefit of this analysis is that near the intersection of two branes the scalar fields behave smoothly (although they will now diverge linearly at infinity).

This paper is organized as follows. In section 2 we discuss some general features of supersymmetric embeddings of D-branes into spacetime. In particular we obtain a general form for the metric on the moduli space of deformations which yields a simple condition for the normalisability of the moduli. In section 3 we specialise to the case of three-dimensional intersections. Our discussion will be in terms of D3-branes but can readily be applied to other cases, most notably D6-branes. In particular we discuss solutions with preserve eight and four supercharges and correspond to Riemann surfaces and special Lagrangian surfaces respectively. We also include a discussion of the moduli spaces and the relation to flat directions of a non-Abelian superpotential. Finally in section 4 we consider the analogous case of intersecting D6-branes and their lift to new $G_2$ manifolds in M-theory.

2 Supersymmetric Embeddings

We consider Dp-branes where the worldvolume gauge field vanishes and the background spacetime is flat. Most of our analysis can also be readily extended to cases where the background is of special holonomy, although concrete results will be considerably more difficult to obtain. The condition that the brane is supersymmetric is $\Gamma \epsilon_1 = \epsilon_2$ where $\epsilon_1$ and $\epsilon_2$ are the ten-dimensional supersymmetry generators. The projector $\Gamma$ is

$$
\Gamma = \frac{1}{(1 + p)!} \frac{1}{\sqrt{-\det(g)}} \epsilon^{i_0 \ldots i_p} \partial_{i_0} X^{m_0} \ldots \partial_{i_p} X^{m_p} \Gamma_{m_0 \ldots m_p}.
$$

Here $\Gamma_m$, $m = 0, \ldots, 9$ form the ten-dimensional Clifford algebra. The indices $i, j, \ldots = 0, 1, \ldots, p$ label the worldvolume coordinates $\sigma^i$ of the brane and $g_{ij} = \partial_i X^m \partial_j X^m \eta_{mn}$ is the induced metric on the brane. In this paper we will assume that all branes are static and fix $\sigma^0 = X^0$. Thus what follows $i, j = 1, \ldots, p$.  

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The important constraint of this system is that $\epsilon_1$ and $\epsilon_2$ are constant spinors. Thus finding a supersymmetric brane configuration requires finding an embedding such that $\Gamma \epsilon_1$ is constant over the brane. In particular, since $\Gamma^2 = \pm 1$ and $\Gamma^\dagger = \pm \Gamma$, where the sign depends on $p$, we can follow \[4\] and observe that

$$\epsilon_2 \epsilon_2 + \epsilon_1 \epsilon_1 + \epsilon_1^\dagger \Gamma \epsilon_2 - \epsilon_2^\dagger \Gamma \epsilon_1 = (\epsilon_2 - \Gamma \epsilon_1)^\dagger (\epsilon_2 - \Gamma \epsilon_1) \geq 0,$$

with equality iff the embedding is supersymmetric, i.e. $\epsilon_2 = \Gamma \epsilon_1$. If we normalize $\epsilon_2 \epsilon_2 = \epsilon_1 \epsilon_1 = 1$ and multiply by $\sqrt{-\text{det}(g)} d\sigma^1 \wedge ... \wedge d\sigma^p$ we find the inequality

$$\sqrt{-\text{det}(g)} d\sigma^1 \wedge ... \wedge d\sigma^p \geq \omega_{m_1...m_p} \partial_1 X^{m_1} ... \partial_p X^{m_p} d\sigma^1 \wedge ... \wedge d\sigma^p,$$

where

$$\omega_{m_1...m_p} = \frac{1}{2} \epsilon_2^\dagger \Gamma_{0m_1...m_p} \epsilon_1 + \text{c.c.} \quad (2.4)$$

The inequality (2.3) states that the volume form on the brane’s worldvolume, when evaluated on any tangent plane, is bounded below by the pull-back to the worldvolume of the closed spacetime $p$-form $\omega$. Note that the same argument applies to $\bar{D}p$-branes where $\Gamma \epsilon_1 = - \epsilon_2$ but as a consequence the sign of $\omega$ is reversed in (2.3).

This is precisely the notion of a calibration i.e. a closed $p$-form $\omega$ in the bulk spacetime whose integral over $p$-dimensional surfaces is bounded above by their volume. A surface which saturates the bound, i.e. a supersymmetric embedding, is said to be calibrated by $\omega$. It is straightforward to show that any smooth calibrated surface is volume minimizing in its homology class. To see this we suppose that a surface $\Sigma$ is calibrated and consider a perturbation to a new manifold $\Sigma'$:

$$\text{Vol}(\Sigma) = \int_\Sigma \omega = \int_{\partial B} \omega + \int_{\Sigma'} \omega = \int_{\Sigma'} \omega \leq \text{Vol}(\Sigma') \quad (2.5)$$

Here $B$ is a compact manifold whose boundary is $\Sigma \cup \Sigma'$ and we have used the fact that

$$\int_\Sigma \omega - \int_{\Sigma'} \omega = \int_{\partial B} \omega = \int_B d\omega = 0 \quad (2.6)$$

Thus a calibrated surface satisfies the equation of motion for a $D_p$-brane (in the absence of other worldvolume fields).

Given a smooth embedding of a brane into space it will generally depend on several moduli labeled by $u^a$, $a = 1, ..., n$. It is natural to consider the low energy effective action of the brane, where its moduli become time-dependent. More generally if the brane has an additional isometric direction, such as two D3-branes intersecting over a line, then one can also introduce dependence on these extra spatial dimensions. The result will then be an effective field theory which is a trivial extension of the action we will derive. In static gauge the induced metric is

$$g_{ij} = \partial_i X^m \partial_j X^n \eta_{mn} = \eta_{ij} + \partial_i X^I \partial_j X^J \delta_{IJ},$$

where $\sigma^i = X^i$ and $I = p+1, ..., 9$ labels the transverse coordinates. The (potential) energy for a static $Dp$-brane simply its spatial volume

$$E = \int d^p \sigma \sqrt{-\text{det}(g)} \quad (2.8)$$

\[1\] Since $\Sigma$ will generically be non-compact we require that $\Sigma'$ is different from $\Sigma$ only on a compact set $\Sigma_c$ and that all the integrals are over $\Sigma_c$.
To obtain an effective Lagrangian for the moduli $u^a$ we allow them to pick up a slow time dependence so that the induced metric (2.7) also has $g_{00}$ and $g_{0i}$ components. The effective Lagrangian $\mathcal{L}_{eff}$ is then just (minus) the potential energy functional $E$ expanded to second order in time derivatives but to all orders in the spatial derivatives. If we use a dot to denote a time derivative then we find that expanding (2.8) to second order gives

$$\mathcal{L}_{eff} = \gamma_{ab} \dot{u}^a \dot{u}^b,$$

where

$$\gamma_{ab} = \int d^p \sigma \sqrt{\det(1 + MM^t)} \frac{\partial X^I}{\partial u^a} \left( \frac{1}{1 + M^t M} \right)_{IJ} \frac{\partial X^J}{\partial u^b} , \quad M_{iJ} = \partial_i X^J ,$$

and we have dropped the constant but infinite volume of a static D$p$-brane. In deriving (2.10) we have used the matrix identity $1 - N^t (1 + NN^t)^{-1} N = (1 + N^t N)^{-1}$. Note also that for any real matrix $N$, $N^t N$ is symmetric with non-negative eigenvalues. Hence $1 + MM^t$ and $1 + M^t M$ are always positive definite and therefore the inverse of the matrix $1 + M^t M$ exists everywhere and is positive semi-definite (it can vanish at points were $M^t M$ diverges).

For some moduli the integral in (2.10) will not converge and these moduli must become frozen at low energy. However for other modes, i.e. those whose effect on the embedding vanishes sufficiently quickly at infinity, the integral converges so that the action will have a finite kinetic term. Such moduli are said to be normalisable and are dynamical at low energy. The effective action is then a $\sigma$-model on the space of normalisable deformations of the surface with the metric (2.10).

In this paper we do not gauge fix to any particular brane. An advantage of this choice is that the solutions are smooth embeddings at all finite values of $\sigma^i$ and asymptotically approach flat branes for large $\sigma^i$. Therefore the only divergences in the integral (2.10) arise from the limit of large $\sigma^i$ where $M$ becomes a constant matrix. Hence a modulus $u^a$ is normalisable if $\partial X^I/\partial u^a$ falls off strictly faster than $|\sigma|^{-p/2}$. This is analogous to a similar condition observed in [1] for normalisable metric moduli.

### 3 Three-Dimensional Intersections

For the rest of this paper we will restrict our attention to the case of intersecting D3-branes which preserve at least four supercharges. Our discussion also applies to various other intersecting D$p$-branes. For example we could replace all the D3-branes by D6-branes which intersect over a $3+1$-dimensional flat space. The restriction to examples which preserve four supercharges guarantees that the wrapped branes enjoy the equivalent of four-dimensional $N = 1$ supersymmetry. We will also discuss cases which preserve more supersymmetry where two D3-branes intersect on a line, which can alternatively replaced by two D2-branes intersecting over a point or two D5-branes intersecting over a $3+1$-dimensional space.

We begin with the most general configuration of perpendicularly intersecting D3-branes which preserves four supercharges

$$\begin{align*}
D3 : & \quad 1 \quad 2 \quad 3 \\
\bar{D}3 : & \quad 1 \quad 4 \quad 5 \\
D3 : & \quad 2 \quad 4 \quad 6 \\
\bar{D}3 : & \quad 3 \quad 5 \quad 6
\end{align*}$$

Note that for intersecting D3-branes there is a more general configuration but it only preserves two supercharges and corresponds to associative three-cycle in $\mathbb{R}^7$, for example see [3]. In particular the
branes in (3.11) preserve the supersymmetries

\[
\begin{align*}
\Gamma_{0123}\epsilon_1 &= \epsilon_2 , \\
\Gamma_{0154}\epsilon_1 &= -\epsilon_2 , \\
\Gamma_{0624}\epsilon_1 &= -\epsilon_2 , \\
\Gamma_{0653}\epsilon_1 &= -\epsilon_2 ,
\end{align*}
\]

(3.12)

respectively. The product of any three of these projectors gives the fourth. Hence given any three of the branes in (3.11) one finds that the fourth can be included without breaking any additional supersymmetries. In addition these preserved supersymmetries also allow for D3-branes which do not intersect at right angles. In total 1/8 of the spacetime supersymmetries are preserved by all the branes. The choice of D3-brane or \( \bar{D}3 \)-brane is arbitrary for any three of these branes, however once this choice is made there is no such freedom for the fourth brane. We have made the above choice to simplify the calculations below.

If we introduce complex coordinates\(^2\)
\[
Z^1 = X^1 + iX^6, \quad Z^2 = X^2 + iX^5 \quad \text{and} \quad Z^3 = X^3 + iX^4
\]
then we can construct the three-form \( dZ^1 \wedge dZ^2 \wedge dZ^3 \) of \( C^3 = R^6 \) and also the calibration

\[
\omega = \text{Re}(dZ^1 \wedge dZ^2 \wedge dZ^3) = dX^1 \wedge dX^2 \wedge dX^3 - dX^1 \wedge dX^5 \wedge dX^4 - dX^6 \wedge dX^2 \wedge dX^4 - dX^6 \wedge dX^5 \wedge dX^3 .
\]

(3.13)

Note that, after imposing the constraints (3.12) this expression for \( \omega_{mnp} \) agrees with (2.4). Clearly \( \omega \) calibrates each of the brane worldvolumes in (3.11), i.e. the pull back of \( \omega \) to the worldvolume is the volume form (or minus the volume form for the anti-branes).

Following [8] we introduce new coordinates \( Y^m \)
\[
Y^1 = \frac{1}{\sqrt{2}}(X^1 + X^6), \quad Y^2 = \frac{1}{\sqrt{2}}(X^2 + X^5), \quad Y^3 = \frac{1}{\sqrt{2}}(X^3 + X^4), \\
Y^4 = \frac{1}{\sqrt{2}}(X^1 - X^6), \quad Y^5 = \frac{1}{\sqrt{2}}(X^2 - X^5), \quad Y^6 = \frac{1}{\sqrt{2}}(X^3 - X^4).
\]

(3.14)

The point of the \( Y^m \) coordinates is that they do not favour any one brane over the others. Our next step is to impose the static gauge \( \sigma^i = Y^i, \ i = 1, 2, 3 \). The coordinates \( Y^I, \ I = 4, 5, 6 \) are now viewed as functions of \( \sigma^i \). In effect we have introduced an imaginary brane which lies at the same angle with each of the branes in (3.11).

Next we need to solve \( \Gamma\epsilon_1 = \epsilon_2 \), where \( \Gamma \) is given in (2.1), for constant spinors \( \epsilon_1 \) and \( \epsilon_2 \) which satisfy the constraints (3.12). It is helpful to introduce the matrix \( M^j_i = \partial_i Y^{j+3} \) and also let \( \hat{\Gamma}^i = \Gamma^{i+3} \) (so that \( \Gamma_i \) and \( \hat{\Gamma}^i \) form two anti-commuting copies of the three-dimensional Clifford algebra). The supersymmetry projections (3.12) are now equivalent to

\[
\begin{align*}
\Gamma^1\hat{\Gamma}_1\epsilon_1 &= \Gamma^2\hat{\Gamma}_2\epsilon_1 = \Gamma^3\hat{\Gamma}_3\epsilon_1 \\
\frac{1}{\sqrt{2}}\Gamma_0(\Gamma_1 + \hat{\Gamma}_1)\Gamma_{23}\epsilon_1 &= \frac{1}{\sqrt{2}}\Gamma_{01}(\Gamma_2 + \hat{\Gamma}_2)\Gamma_{3}\epsilon_1 = \frac{1}{\sqrt{2}}\Gamma_{012}(\Gamma_3 + \hat{\Gamma}_3)\epsilon_1 = \epsilon_2 .
\end{align*}
\]

(3.15)

\(^2\) We will have no need for the remaining three coordinates \( X^7, X^8, X^9 \) in this paper.
Expanding out $\Gamma \epsilon_1 = \epsilon_2$ leads to

$$
\epsilon_2 = \frac{\Gamma_{0123}}{\sqrt{-\det(g)}} \left\{ 1 + \frac{1}{2} \sum_{i<j} (M_i^j - M_j^i) \Gamma^i \hat{\Gamma}_j + \text{tr}(M) \Gamma^1 \hat{\Gamma}_1 
- \det(M) \Gamma^1 \hat{\Gamma}_1 - \frac{1}{2} \epsilon^{ijk} \epsilon_{lmn} \Gamma^k \hat{\Gamma}_l \Gamma^1 \hat{\Gamma}_1 M_i^j M_j^m \right\} \epsilon_1.
$$

(3.16)

Terms involving $\Gamma^i \hat{\Gamma}_j$ and $\Gamma^i \hat{\Gamma}_j \Gamma^1 \hat{\Gamma}_1$ with $i \neq j$ must vanish to ensure that $\Gamma \epsilon_1 = \epsilon_2$. The vanishing of the $\Gamma^i \hat{\Gamma}_j$, $i \neq j$ terms imply that

$$
M_i^j = M_j^i.
$$

(3.17)

In addition (3.17) also ensures the vanishing of the $\Gamma^i \hat{\Gamma}_j \Gamma^1 \hat{\Gamma}_1$, $i \neq j$ terms. Once this is imposed the remaining terms can be rearranged and the condition $\Gamma \epsilon_1 = \sqrt{2} \Gamma^0 (\Gamma^1 + \hat{\Gamma}_1) \epsilon_1$ becomes

$$
\text{tr}(1 + M) = \frac{1}{2} \det(1 + M) + 2.
$$

(3.18)

In general the condition (3.17) is solved by writing $Y^{j+3} = \partial_j F$ for some potential $F(\sigma^i)$. The first condition then becomes a second order non-linear differential equation for $F$. We note that $M = 0$, i.e. a brane lying in the $Y^1, Y^2, Y^3$ plane, is not supersymmetric (with respect to the supersymmetries (3.12)). With these conventions the branes in (3.11) correspond to the solutions

$$
\begin{align*}
M &= \text{diag}(+1, +1, +1), \\
M &= \text{diag}(+1, -1, -1), \\
M &= \text{diag}(-1, +1, -1), \\
M &= \text{diag}(-1, -1, +1),
\end{align*}
$$

(3.19)

respectively.

We can also find the condition that $\Gamma \epsilon_1 = -\epsilon_2$. From (3.16) we now find

$$
\det(1 + M) = 2 \det(M) + 2, \quad M_i^j = M_j^i.
$$

(3.20)

Flipping all the signs in (3.19) gives solutions to (3.20) and describes the intersection

$$
\begin{align*}
\bar{D}3 &: 4 \ 5 \ 6 \\
D3 &: 2 \ 3 \\
D3 &: 1 \ 3 \ 5 \\
D3 &: 1 \ 2 \ 4
\end{align*}
$$

(3.21)

However we will not devote any more time here to these embeddings.

Let us compare these embedding conditions with the equation that results if we had instead imposed static gauge with respect to the first D3-brane, i.e. $\sigma_1 = X^1, \sigma_2 = X^2, \sigma_3 = X^3$. The Bogomolnyi condition was found in (3.22) to be

$$
\det(\tilde{M}) = \text{tr}(\tilde{M}), \quad \tilde{M}_i^j = \tilde{M}_j^i.
$$

(3.22)
where $\tilde{M}_i^j = \partial_i X^j + 3$. This is the original Harvey-Lawson equation for a special Lagrangian embedding [3]. While this equation is somewhat simpler than (3.18) or (3.20), it has the disadvantage that only the first brane in (3.11) appears as a solution; namely $\tilde{M} = 0$.

Finally, if (3.18) is satisfied then

$$\sqrt{-\det(g)} = 2|\text{tr}(M) - \det(M)|,$$

which is a total derivative. Note that the condition $\det(g) = 0$ is the same as the the embedding equation (3.22) and corresponds to a surface that is calibrated with respect to $\tilde{\omega} = \text{Re}(dY^1 + i dY^6) \wedge (dY^2 + i dY^5) \wedge (dY^3 + i dY^4)$. However, as we saw above, $1 + MM^t$ is positive definite and therefore $\det(g)$ never vanishes. Hence surfaces which are calibrated with respect to $\omega$ and $\tilde{\omega}$ have no common tangents.

### 3.1 Two D3-branes intersecting over a line

By describing intersecting branes as a single brane embedded into spacetime we are implicitly assuming that the intersection point has been smoothed-out so that a manifold structure exists. Thus the general configuration of intersecting branes is realised as a smooth solution $M(\sigma^i)$ to (3.18) which tends to the constant solutions (3.19) in various limits. That such solutions exist is less clear. Let us therefore start by discussing the simpler case of two intersecting D3-branes over a line. Here we can exploit the underlying holomorphic structure to construct all the necessary solutions.

If we remove the first two D3-branes from (3.11) we obtain the brane configuration

$$\begin{align*}
\bar{D}3 : & \quad 2 \quad 4 \quad 6 \\
D3 : & \quad 3 \quad 5 \quad 6
\end{align*}$$

(3.24)

These branes preserve the supersymmetries

$$\Gamma_{0624} \epsilon_1 = -\epsilon_2, \quad \Gamma_{0653} \epsilon_1 = -\epsilon_2,$$

(3.25)

respectively. Hence they preserve a common 1/4 of the spacetime supersymmetries.

From the worldvolume of our imaginary brane this is achieved by fixing $Y^4 = -\sigma^1$ and restricting $Y^5$ and $Y^6$ to be independent of $\sigma^1$. The Bogomol'nyi conditions (3.18) then reduce to

$$\partial_2 Y^5 = -\partial_3 Y^6, \quad \partial_2 Y^6 = \partial_3 Y^5.$$

(3.26)

These equations are readily recognized as the Cauchy-Riemann equations which are solved by introducing complex coordinates

$$Z = Y^6 + i Y^5, \quad w = \sigma^2 + i \sigma^3.$$

(3.27)

In this case the full non-linear Bogomol'nyi equations reduce to [9]

$$\bar{\partial} Z = 0.$$

(3.28)

Solutions to (3.28) that represent intersecting branes are of course easy to construct following [10]. They may be implicitly described by a holomorphic equation $E(Z, w) = 0$. We are interested two intersecting branes so that for each $w$ (except for some isolated points) there should be two solutions for $Z$. Thus we write $E = A(w)Z^2 + B(w)Z + C(w)$. Furthermore, for $w \to \infty$, the branes should
Figure 1: A holomorphic embedding of 2 D3-branes

resemble flat planes, \( i.e. \ Z \sim \lambda w \) in which case the angle between the two branes is \( \theta = 2 \arctan |\lambda| \). In addition there should be no other points where \( Z \to \infty \). Thus \( A = 1, \ B \) should be linear in \( w \) and \( C \) quadratic. Without loss of generality we may shift \( Z \) to set \( B = 0 \) and write \( C(w) = \lambda^2(w - w_0)^2 - \alpha^2 \) so that

\[
Z = \pm \sqrt{\alpha^2 - \lambda^2(w - w_0)^2}.
\]

(3.29)

An artist’s impression of part of this curve is in figure 1, the full curve is a two-dimensional surface in four dimensions and is connected. The solid line represents the curve and the dashed lines represent the undeformed intersection. The two branches, \( i.e. \) choices of sign, correspond to the top and bottom half of the curve. There are two branch points \( w = w_0 \pm \alpha/\lambda \) and so the full solution consists of two copies of the complex \( w \) plane. Clearly \( \alpha \) plays the role of smoothing out the intersection, \( w_0 \) describes the location of the intersection and \( \lambda \) controls the angle between the two branes. As \( w \to \infty \) we find that

\[
M \to \begin{pmatrix}
-1 & 0 & 0 \\
0 & \pm \text{Re}(\lambda) & \mp \text{Im}(\lambda) \\
0 & \mp \text{Im}(\lambda) & \pm \text{Re}(\lambda)
\end{pmatrix}.
\]

(3.30)

Thus for \( \lambda^2 = 1 \) these are indeed smooth curves that interpolate between two of the constant solutions in (3.19).

We could also try to find smooth solutions which interpolate between other pairs of branes in (3.11). For the middle two branes we perform an analogous construction but now with the \( \sigma_3 \) dependence constrained to be \( Y^6 = -\sigma_3 \). The second and fourth branes are also analogous but we impose \( Y^5 = -\sigma_2 \). However if we try to consider the intersection of the first and second branes in a similar manner then we must set \( Y^4 = \sigma_1 \). As a result the Bogomol'n'yi equation (3.18) does not simplify to a complex embedding condition. Instead we find the two-dimensional Monge-Ampere equation

\[
\partial_2^2 U \partial_3^2 U - (\partial_2 \partial_3 U)^2 = 1,
\]

(3.31)

where as before we set \( Y^{i+3} = \partial_i F \) and take \( F = \frac{1}{2} \sigma_1^2 + U(\sigma_2, \sigma_3) \). Analogous equations arise for the other intersections involving the first brane.

From the symmetry of the original problem it is clear that there should be no physical difference between smooth intersections of the first brane with the others and the complex intersections constructed
above. Indeed let us change coordinates to \( \tilde{w} = \text{Im}(Z) - i\text{Re}(w) \), \( \tilde{Z} = -\text{Re}(Z) - i\text{Im}(w) \). \( \text{(3.32)} \)

Note that now neither \( \tilde{w} \) or \( \tilde{Z} \) are simply worldvolume coordinates or embedding coordinates. After a little calculation however one can show that the condition \( \tilde{\partial} \tilde{Z} = 0 \) becomes

\[
\partial Z = \tilde{\partial} \tilde{Z} , \quad |\partial Z|^2 - |\tilde{\partial} Z|^2 = 1.
\]

The first condition implies that \( Z = 2\tilde{\partial}U \) where \( U \) is a real function. We now find that the second condition in \( \text{(3.33)} \) is precisely the Monge-Ampere equation \( \text{(3.31)} \).

To illustrate this point one can see that taking \( \tilde{Z}(\tilde{w}) \) to have the form \( \text{(3.29)} \) (and setting \( w_0 = 0, \alpha^2 \) real and \( \lambda^2 = 1 \) for simplicity) produces the solution

\[
Z = \pm \sqrt{w^2 + \alpha^2 \frac{w}{\bar{w}}},
\]

under the change in static gauge given in \( \text{(3.32)} \). In this case the corresponding function \( U \) only depends on \( \sigma^2_1 + \sigma^2_2 \) and can be readily seen to solve \( \text{(3.31)} \). Note that near \( w = 0 \), \( Z \sim |\alpha| \sqrt{w/\bar{w}} \) is non-vanishing although \( \partial Z \) and \( \tilde{\partial} Z \) diverge, i.e. at \( w = 0 \) there is a circle of radius \( |\alpha| \). This is a failure of static gauge and is easily resolved by reflecting the solution across this circle and corresponds to choosing the opposite sign for the embedding in \( \text{(3.34)} \) (recall that in the holomorphic case this corresponds to going onto the second sheet of the Riemann surface). A sketch of this intersection is therefore identical to figure 1 but rotated by \( \pi/2 \).

Hence solutions to the Monge-Ampere equation can be found by taking any holomorphic embedding \( \tilde{Z}(\tilde{w}) \). We note that the converse is not true. For example one can check that starting with \( \tilde{Z}(\tilde{w}, \tilde{\bar{w}}) \) of the form \( \text{(3.34)} \) also leads to a solution \( Z(w, \bar{w}) \) of \( \text{(3.31)} \) (this time \( U \) only depends on \( \sigma^2_2 - \sigma^2_3 \)). Although again one can find coordinates in which this embedding is holomorphic. Therefore the non-linearity in the Monge-Ampere equation merely reflects an awkward choice of static gauge.

### 3.2 Three D3-branes Intersecting Over a Point

In the previous section we explicitly constructed solutions to the embedding equation \( \text{(3.18)} \) that described the resolution of any pairs of branes in \( \text{(3.11)} \) in single smooth complex surface. Next we would like to obtain solutions to the embedding equation \( \text{(3.18)} \) which depend non-trivially on all three coordinates and describe smooth deformations of three intersecting branes.

#### 3.2.1 \( SO(3) \) symmetry

We consider the simplest possibility of a spherically symmetric ansatz, namely \( Y^{i+3} = \partial_i F(r) \) where \( r = \sqrt{\sigma^2_1 + \sigma^2_2 + \sigma^2_3} \). Substituting this into \( \text{(3.18)} \) leads to a second order differential equation which can be integrated once to give

\[
G^3 + 3G^2 - 3G - 1 = \frac{w^3}{r^3},
\]

\( \text{(3.35)} \)

\( ^{3}I \) thank N. Constable and W. Taylor for suggesting this change of coordinates.
where $G = F'/r$ and $u$ is an arbitrary constant. A plot of the cubic equation is given in figure 2. The embedding is then found by $Y^{i+3} = \sigma_i G(r)$. Thus for a generic value of $r$ we find three solutions which can be interpreted as three D3-branes. As $r \to \pm\infty$ the solutions are $G = 1, -2 - \sqrt{3}, -2 + \sqrt{3}$. This corresponds to three flat D3-branes. In particular $G = 1$ describes a D3-brane in the $X^1, X^2, X^3$-plane while the other two solutions correspond to two D3-branes in the planes $(X^4, X^5, X^6) = \pm \sqrt{3}(X^1, X^2, X^3)$. This is precisely the intersection considered in [3]. Just as in the Monge-Ampere example, $r = 0$ is not a singularity but an artifact of static gauge. Instead we see that, as $r \to 0$, $Y^{i+3} = u\sigma_i/r$, i.e. the coordinates lie on a 2-sphere. We can construct a smooth manifold by attaching another copy of the solution, where $r$ is negative and gluing it to this 2-sphere. Our resident artist’s impression of this curve is given in figure 3, but some explanation is required. First we start at on the D3-brane corresponding to $G = 1$ at $r = +\infty$. As $r$ decreases (without loss of generality we assume that $u > 0$) but is still positive we move left on figure 3 until $r$ goes to zero. However we can smoothly pass here to $r < 0$ and we continue on the left of figure two until we reach $G = -2 - \sqrt{3}$ as $r \to -\infty$. This is plotted as the top part of the curve in figure 3. Next we construct another branch of the solution by starting at $G = -2 - \sqrt{3}$ and $r = \infty$ and then moving further right on figure 2 passing through a minimum value of $r$ (i.e. a maximum value of $G$) and finally out to $G = -2 + \sqrt{3}$ as $r \to \infty$ again. This is plotted as part of the curve on the right and below the horizontal in figure 3. Finally a third branch is constructed if we continue on moving right on figure two from $G = -2 + \sqrt{3}$ as $r \to -\infty$, through a maximum value of $|r|$ (i.e. a minimum value of $G$) again until we get to $G = 1$ at $r = -\infty$. This is plotted as the part of the curve on the left and below the horizontal axis in figure three. The rest of the curve arises as a result of the spherical symmetry. In particular sections of the curve which are reflected into each other through the origin are in fact connected by the $SO(3)$ symmetry in the full six dimensions.

Thus the curve plotted in figure 3 apparently smoothly interpolates between all three pairs of branes which are are $2\pi/3$ angles with each other. However one can see from figure 3 that each of the three D3-branes is deformed in two directions. Clearly a single D3-brane can only choose one such way to bend. This can be overcome by considering two of each of the D3-branes, which are asymptotically parallel and coincident. The full curve then represents the blow up of six D3-branes into a smooth Special Lagrangian.

![Figure 2: $u^3/r^3$ as a function of $G$](image)
Figure 3: 6 D3-branes embedded as a special Lagrangian

Figure 4: 3 D3-branes embedded as a special Lagrangian

Figure 5: 4 D3-branes embedded as a special Lagrangian
If we recall that $X^i = (Y^i + Y^{i+3})/\sqrt{2}$ and $X^{i+3} = (Y^i - Y^{i+3})/\sqrt{2}$ then we can define

$$\phi^i_1 = \sqrt{\frac{1 - G}{3 + G}}X^i, \quad \phi^i_2 = \sqrt{\frac{1 + G}{1 - G}}X^{i+3}.$$ (3.36)

Now (3.35) becomes

$$\vec{\phi}_1 \cdot \vec{\phi}_2 = -\text{sgn}(G^2 - 1)|\vec{\phi}_1||\vec{\phi}_2|, \quad |\vec{\phi}_1|(3|\vec{\phi}_1|^2 - |\vec{\phi}_2|^2) = \rho,$$ (3.37)

where $\rho = u^3/\sqrt{6}$. For $|G| > 1$ this is the equation for the curve first discussed in [3] and more recently in [2]. In particular the construction of [2] consists of keeping only one section of the curve, which smooths out the intersection between two D3-branes, while the third brane remains flat and shoots through the hole created by the others. The resulting embedding is sketched in figure 4. Another construction of [2] is to simply delete the flat brane that shoots through the middle, i.e. just to consider the intersection of two D3-branes.

In a sense the region $|G| \geq 1$ contains one and a half copies of the curve discussed in [2] since, in addition to a patch which interpolates between one D3-brane with $G = 1$ and another with $G = -2 - \sqrt{3}$, the region $|G| \geq 1$ also includes a portion of the curve that approaches the brane at $G = -2 - \sqrt{3}$ and then, apparently, abruptly ends at $G = -1$. To resolve this we may extend to all values of $G$ to get the embedding sketched in figure 3. In effect this contains three copies of the curve in [2], whereby all three branes are smoothed out. Alternatively we may only extend as far as $G = -2 + \sqrt{3}$ which corresponds to keeping only two sections of the curve in figure 3 and again all three D3-branes are smoothed out. This is sketched in figure 5. However here again the middle D3-brane is deformed in both directions and thus must be replaced by a pair of asymptotically parallel and coincident D3-branes. Hence it really represents the embedding of four separate D3-branes.

### 3.2.2 $U(1)$ symmetry

Next we would like to discuss intersections which are invariant only under a $U(1)$ symmetry. In particular recently special Lagrangian surfaces in $\mathbb{C}^3$ which are invariant under $(Z^1, Z^2, Z^3) \to (Z^1, e^{i\theta}Z^2, e^{-i\theta}Z^3)$ have been discussed in relation to mirror symmetry [11].

If we consider the coordinates that we introduced in (3.32) then, under this action,

$$\tilde{w} \to e^{-i\theta}\tilde{w}, \quad \tilde{Z} \to e^{-i\theta}\tilde{Z}.$$ (3.38)

Therefore our task is to generalise the embeddings in section 3.1 to include a non-trivial dependence on $\sigma_1$ and a non-holomorphic dependence on $\tilde{w}$ while respecting the symmetry (3.38). Therefore the most general ansatz is to write $\tilde{Z} = \tilde{w}A(\tilde{w}\tilde{\bar{w}}, \sigma_1)$, where $A$ is an arbitrary function. Examining the infinitesimal version of the transformation (3.38), and using the fact that $\partial_2Y^6 = \partial_3Y^5$, we learn that $|\tilde{Z}|^2 - |\tilde{w}|^2 = 2a(\sigma_1)$. Hence we find

$$\tilde{Z} = \pm\tilde{w}\sqrt{1 + \frac{2a}{\tilde{w}\tilde{\bar{w}}}e^{i\theta}},$$ (3.39)

where $\theta$ is real and depends on $\sigma_1$ and $\tilde{w}\tilde{\bar{w}}$. Note that the dependence of $\tilde{Z}$ on $\tilde{w}$ at a fixed value of $\sigma_1$ is similar to the embedding (3.34). One simple ansatz for a solution is to take $\theta$ and $a$ to be constant.
and $Y^4$ linear in $\sigma_1$. This gives example 5.2 in [11] (with $\gamma = 0$) and is essentially the holomorphic embedding (3.29) written in peculiar coordinates. Taking the limit $|\tilde{w}| \to \infty$ and solving (3.18) we find

$$M \to \begin{pmatrix} \frac{1+\tan(\theta)}{1-\tan(\theta)} & 0 & 0 \\ 0 & \pm \frac{\cos(\theta)}{\sin(\theta)} & \pm \frac{1}{\sin(\theta)} \\ 0 & \pm \frac{1}{\sin(\theta)} & \pm \frac{\cos(\theta)}{\sin(\theta)} \end{pmatrix}. \quad (3.40)$$

This corresponds to a smooth intersection of two D3-branes which asymptotically lie in the planes defined by

$$X^4 = \frac{1}{\cos(\theta)} X^2 \pm \tan(\theta) X^3, \quad X^5 = \frac{1}{\cos(\theta)} X^3 \pm \tan(\theta) X^2, \quad X^6 = \tan(\theta) X^1. \quad (3.41)$$

Less trivial solutions are more difficult to construct. However an existence proof has been given in [11] (in addition we learn that $a$ is always constant). Nevertheless the fact that these solutions take the form (3.39) allows us to make some remarks about the D3-brane interpretation. Just as above the apparent divergence at $\tilde{w} = 0$ is an artifact of static gauge which is cured by adding a copy of the curve corresponding to the opposite choice of sign in (3.39). Assuming that $\theta$ is independent of $|\tilde{w}|$ as $\tilde{w} \to \infty$ we see that for a fixed value of $\sigma_1$ there are two flat D3-branes which asymptotically take the form (3.41). Thus these special Lagrangians involve two intersecting D3-branes which are in a sense twisted and tilted along $\sigma_1$.

### 3.3 Moduli and Superpotentials

The existence of smooth solutions that represent intersecting branes is a signal that there is a Higgs’s branch in the gauge theory living on the intersection. Atiyah and Witten [1] have argued that, at least for the case of three D-branes intersecting at $2\pi/3$ angles, there is a superpotential

$$W \equiv \Phi_{12} \Phi_{23} \Phi_{31}, \quad (3.42)$$

where, for example, $\Phi_{12}$ is the chiral multiplet associated to the open strings that stretch from the first to the second D3-brane. Thus the only flat directions along the Higgs’s branch are when one and only one scalar field is non-vanishing, in which case the other two scalar fields become massive.

The existence of such a potential implies that the space of classical smooth solutions to the calibration equation (3.18) has several branches. Choosing to give a vev to a particular multiplet, say $\Phi_{12}$, is naturally interpreted as smoothing-out the intersection of the first and second branes. However, due to the potential, once this is done the other smoothing-out modes are massive. A result of this interpretation is that there should be no smooth solution that interpolates between all three D3-branes, but smooth solutions between any two pairs of branes should exist. We note that the superpotential (3.42) is supposed to arise as a non-perturbative disk instanton effect [1] and therefore it may seem unlikely that its effects can be visible in the classical description of special Lagaranian surfaces presented here. However the disk instanton in question has zero-size and the superpotential accordingly is not suppressed by any factor of $\alpha'$. Thus one may expect that the classical geometry is sensitive to the superpotential. In any case we will see that the solutions we found above are naturally classified in terms of the flat directions of a non-Abelian superpotential on the moduli space.

For completeness we first consider the case of two D3-branes over a line. Since these configurations in fact preserve eight supercharges a superpotential is forbidden. The moduli space is then simply that
of complex curves with the relevant boundary conditions. The holomorphic condition $\bar{\partial}Z = 0$ implies that the effective Lagrangian \((2.9)\) is simply

$$\mathcal{L}_{\text{eff}} = \int d^2w \frac{\partial Z}{\partial u^a} \frac{\partial \bar{Z}}{\partial u^b} \partial^\mu u^a \partial_\mu u^b .$$ \hspace{1cm} (3.43)

Note that since two D3-branes intersect over a line, it is natural to allow the moduli $u^a$ to depend on this spatial direction as well as time. Hence $\mathcal{L}_{\text{eff}}$ is a two-dimensional Lagrangian and we have accordingly introduced $\mu = 0,1$ to label these directions. It follows from our discussion in section 2 that \((3.43)\) diverges and hence none of the moduli in \((3.23)\) are normalisable. However by introducing additional branes in various ways one can find examples where some of the moduli can be made normalisable and the $\sigma$-model produces a non-trivial effective action \cite{[12, 13, 14]}.

Returning to the case of D3-branes at $2\pi/3$ angles and the solution \((3.35)\) we find that, in contrast to perpendicularly intersecting branes, the modulus $u$ is normalisable. Furthermore one can show that \((2.33)\) becomes

$$\mathcal{L}_{\text{eff}} = \kappa u^3 \bar{u}^2 ,$$ \hspace{1cm} (3.44)

where $\kappa$ is a finite, positive constant. We noted above that \((3.18)\) and hence the associated modulus $u$ led to three interpretations, although this required having parallel pairs of D3-branes rather than three single D3-branes with Abelian gauge fields.

We can understand the appearances of these interpretation from the gauge theory potential \((3.42)\) as follows. If we have $N$, $N'$ and $N''$ of each D3-brane then the moduli coming from stretched strings are in bifundamental representations of the $U(N) \times U(N') \times U(N'')$ gauge group of the D3-branes. In particular the analogue of \((3.42)\) is

$$W = \Phi_{12}^{ab} \Phi_{23}^{b'} c'' \Phi_{31}^{c''a} ,$$ \hspace{1cm} (3.45)

where $a = 1, ..., N$, $b' = 1, ..., N'$ and $c'' = 1, ..., N''$. The supersymmetric vacuum moduli space \((i.e.\) extrema of $W$) is now much larger than the Abelian case and satisfies

$$\Phi_{12}^{ab} \Phi_{23}^{b'} = \Phi_{23}^{b'} c'' \Phi_{31}^{c''a} = \Phi_{31}^{c''a} \Phi_{12} = 0 .$$ \hspace{1cm} (3.46)

If we write $\Phi_{12} = V_1 \otimes V_1'$, $\Phi_{23} = V_{23}' \otimes V_{23}'$ and $\Phi_{31} = V_{31}' \otimes V_{31}$ then \((3.46)\) becomes

$$(V_{12}' \cdot V_{23}') V_{12} \otimes V_{23}' = (V_{23}' \cdot V_{31}') V_{23} \otimes V_{31} = (V_{12} \cdot V_{23}') V_{31}' \otimes V_{12}' = 0 .$$ \hspace{1cm} (3.47)

Therefore we can identify three types of branches of the supersymmetric vacuum moduli space. Firstly the branches which existed in the Abelian case are still present. Here only one scalar field, say $\Phi_{12}$, is non-vanishing. Up to a gauge transformation there are three, one-dimensional branches of these vacua. Secondly there is a symmetric branch where none of the scalar fields vanish. In these cases $V_{12} \cdot V_{23} = V_{12}' \cdot V_{23}' = V_{23}'' \cdot V_{31}'' = 0$. Up to gauge transformations there is an $(N-1)(N'-1)(N''-1)$-dimensional branch of such vacua. Lastly there will also be three mixed branches where only one scalar field vanishes. For example if $\Phi_{31} = 0$ then we find $V_{12}' \cdot V_{23}' = 0$ and hence, up to gauge transformations, there is an $(N'-1)$-dimensional branch of the vacuum moduli space. Similarly there are also $(N''-1)$-dimensional and $(N-1)$-dimensional branches for $\Phi_{12} = 0$ and $\Phi_{23} = 0$ respectively. Note that the existence of one of these mixed branches requires that only one of gauge groups is non-Abelian.

For the case at hand where $N = N' = N'' = 2$ all of these vacuum branches are one-dimensional and we can identify them with the explicit curves found in section 3.2. In particular the Abelian branches, where only one scalar is chosen to be non-vanishing, correspond to the curve studied in
holonomy. The existence of the large vacuum moduli space discussed in section 3.3 for bigger gauge groups, although they might be related to resolutions of the orbifold constructions in [19]. Furthermore the $X_N$ field strength vanishes. The fact that four-dimensional upon lifting these D6-branes to M-theory we find a solution which is pure geometry (i.e. the four-form field strength vanishes). The second case is known to correspond to an $\alpha \beta \gamma$ mode (corresponding to the symmetric vacuum branch. Finally we note that the pairs of parallel D3-branes coincide at infinity. It is natural then to ask if there is an associated solution where they are separated. To answer this we note that separating a particular pair of D3-branes corresponds to giving a non-zero vev to one of the adjoint scalar fields. Since the $\Phi_{\alpha \beta}$ modes are in bifundamental representations of the gauge group they will become massive via a Higg’s effect. Therefore in such a vacuum the relevant $\Phi_{\alpha \beta}$ scalars must vanish. Hence there should be no moduli corresponding to separating the branes.

4 Relation to non-compact $G_2$ manifolds

The discussion above focused on D3-branes. However nothing substantial changes if we instead consider D6-branes which are also extended along $X^7, X^8, X^9$. As first pointed out in [8] (see also [6, 7]), upon lifting these D6-branes to M-theory we find a solution which is pure geometry (i.e. the four-form field strength vanishes). The fact that four-dimensional $N = 1$ supersymmetry is preserved along the $X^0, X^7, X^8, X^9$ directions further implies that the remaining seven-dimensional geometry $M$ has $G_2$ holonomy.

From the point of view of special Lagrangian surfaces the three types of $SO(3)$ invariant solutions that we have described seem rather trivially related. In particular they simply correspond to various truncations of the curve plotted in figure 3. However upon lifting to M-theory these curves are associated to different $G_2$ manifolds. To see this we can determine the topology of the $M$ using the elegant formula derived in [6],

$$h_2(M) = h_0(L) - 1, \quad h_{\alpha+2}(M) = h_{\alpha}(L), \quad \alpha = 0, 1, 2, 3,$$

(4.48)

where $L$ is the special Lagrangian. The three special Lagrangian embeddings $L$ that we discussed in section 3.2.1 have $(h_0(L), h_2(L)) = (3, 3), (2, 1)$ and $(2, 2)$ for figures 3, 4 and 5 respectively, with all other Betti numbers vanishing. Therefore, upon lifting to M-theory, they correspond to $G_2$ manifolds $M$ such that $(h_0(M), h_2(M), h_4(M)) = (1, 2, 3), (1, 1, 1)$ and $(1, 1, 2)$ respectively, with all other Betti numbers vanishing. The second case is known to correspond to an $\mathbb{R}^3$ bundle over $\mathbb{CP}^2$. The other manifolds have $h_4 > 1$ and thus cannot be $\mathbb{R}^3$ bundles over a smooth four-dimensional surface, although they might be related to resolutions of the orbifold constructions in [9]. Furthermore the existance of the large vacuum moduli space discussed in section 3.3 for bigger gauge groups, i.e. many parallel D3-branes, suggests that there is a wealth of additional $G_2$ manifolds related to the special Lagrangian curve (5.18). However it is less clear what the geometrical picture of these additional deformations is.

The $U(1)$ invariant solutions we discussed also lift to $G_2$ manifolds. These have $(h_0(L), h_2(L)) = (1, 1)$ and hence lift to manifolds with $(h_0(M), h_2(M), h_4(M)) = (1, 0, 1)$. This suggests that their lift is topologically an $\mathbb{R}^4$ bundle over $S^4$. Indeed a $G_2$ manifold with this form was discussed in [1, 2] and is the lift of two intersecting D6-branes at $2\pi/3$ angles (corresponding to the $SO(3)$ invariant curve in
Since the $U(1)$ invariant special Lagrangians describe two D6-branes lying in the planes defined by \((3.41)\) it is natural to suppose that their associated $G_2$ manifolds can be continuously connected to the $\mathbb{R}^4$ bundle over $S^4$ of \([1, 2]\).

## 5 Conclusions

In this paper we revisited the worldvolume analysis for intersecting D-branes, including a discussion of the moduli space metric. We then focused our discussion on D3-brane intersections which preserve four supercharges. In particular we found a new embedding condition which treats all D3-branes on an equal footing. We also discussed some solutions to this condition and identified their moduli space with the vacuum branches of a non-Abelian superpotential. Upon lifting to M-theory these solutions imply the existence on new $G_2$ manifolds.

It would be of interest to find more explicit solutions to the special Lagrangian embedding equation \((3.18)\). Since we have found all the solutions that are predicted by the superpotential \((3.42)\) for the case of D3-branes at $2\pi/3$ angles, it seems natural to assume that any new solutions will satisfy different boundary conditions. It would also be interesting to find solutions which represent all four intersecting D3-branes. However we note that in the solution \((3.33)\) the three distinct D3-branes at infinity correspond to the three solutions to a cubic equation. Since \((3.18)\) is at most cubic in the embedding coordinates it seems that solutions with more than three D3-branes are impossible. Finally it would be interesting to derive embedding equations which do not prefer one D-brane over the others for configurations with less supersymmetry, such as associative, co-associative and spin(7) calibrations.

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