QUANTIZATION IN THE PRESENCE OF GRIBOV AMBIGUITIES

M. SCHADEN
New York University, 4 Washington Place, New York,
NY 10003, USA
E-mail: ms68@scires.nyu.edu

The non-perturbative validity of covariant BRST-quantization of gauge theories on compact Euclidean space-time manifolds is reviewed. BRST-quantization is related to the construction of a Topological Quantum Field Theory (TQFT) of Witten type on the gauge group. The criterion for the non-perturbative validity of the quantization is that the partition function of the corresponding TQFT does not vanish and that its (equivariant) BRST-algebra is free of anomalies. I sketch the construction of a TQFT whose partition function is proportional to the generalized Euler-characteristic of the coset space $SU(n)_{gauge}/SU(n)_{global}$ with an associated equivariant BRST-algebra that manifestly preserves translational symmetry. Some non-perturbative consequences of this approach are discussed.

1 Introduction

In the best of worlds, restricting a field theory with first class constraints to a (gauge fixing) surface uniquely determines the fields and their conjugate momenta and enables one to proceed with the quantization of the theory as described by Dirac. As originally observed by Gribov and proven inevitable under certain circumstances by Singer, this is unfortunately not the situation one encounters in non-abelian gauge theories. Due to the topological structure of the non-abelian gauge group, the solution to local gauge conditions for the connection never is unique on compact space-times. An analogous situation already occurs if one considers bounded functions on a circle, $f : S_1 \to \mathbb{R}$. (1)

Uniqueness requires that these are periodic and there are therefore always an even number of solutions to $f = 0$ or none. The fact that solutions to $f = 0$ appear and disappear in pairs when $f$ is continuously deformed, suggests that one can construct a topological invariant of the circle that does not depend on the function $f$. Indeed,

$$\chi = \sum_{x_0 : f(x_0) = 0, f'(x_0) \neq 0} \frac{f'(x_0)}{|f'(x_0)|}$$  (2)

where $f'$ denotes the derivative of $f$, is constant under continuous deformations of $f$. Accordingly, we can evaluate Eq. (2) at a function $f = V'$ which is the...
derivative of a bounded Morse potential

\[ V : S_1 \to \mathbb{R} . \] (3)

Eq. (3) in this case is just Morse’s definition of the Euler number \( \chi(S_1) = 0 \) of the circle. The example above can be generalized to higher dimensional manifolds. It illustrates several points that will be relevant for the non-perturbative quantization of a non-abelian gauge theory:

- whether or not the solution to \( f = 0 \) is ambiguous depends on the topology of the manifold (the circle in our simple example) rather than on the function \( f \).

- one may nevertheless construct a topological characteristic of the manifold that does not depend on continuous deformations of the function \( f \). It is important to note that the topological characteristic generally requires that one retains all solutions to \( f = 0 \).

- certain topological characteristics of the manifold (such as the Euler number of a circle) may vanish.

The latter case also occurs in the conventional quantization of non-abelian gauge theories. Fortunately it can be circumvented by either choosing a different topological characteristic of the gauge-group manifold or by choosing a different manifold.

The first option is difficult to implement in the case of gauge theories. The BRST-quantization generically computes the Morse definition of the (generalized) Euler characteristic of a manifold. To circumvent the problem, we will choose the second option. To see how this works consider again the example above. We could have restricted ourselves to continuous functions \( f \) that vanish at a certain point (call this point the north pole) of the circle and have a non-vanishing positive derivative there. Omitting the contribution from the north pole in the definition Eq. (3) then results in a topological characteristic \( \chi = -1 \) that no longer vanishes. In effect we have replaced the circle by the closed interval with boundary conditions and are computing a (generalized) Euler characteristic of the closed interval. The equivariant BRST-construction below removes certain zero’s in much the same manner. By considering these topological aspects of gauge fixing we found a global topological obstruction (anomaly) to the quantization of an \( SU(n) \) gauge theory on compact manifolds with non-trivial \( \pi_3 \) homotopy group and that the topological characteristic is a constant only in connected sectors of the orbit space. Fortunately the orbit space of a torus with periodic boundary conditions for the gauge fields is connected and the homotopy group \( \pi_3 \) of this manifold is trivial.
2 Gauge fixing, BRST and TQFT

For these reasons, simplicity, and also because this is a physically interesting case, we consider an $SU(n)$ gauge theory on a compact Euclidean hypertorus as space-time. Of course we are eventually interested in the thermodynamic limit where the volume of this torus is taken to be arbitrary large. Why not consider $R^4$ from the outset? Well, topological considerations generally apply to compact manifolds and it would be rather surprising if non-abelian gauge theories could not be defined on them. Compact Euclidean space-time furthermore regulates the infrared behavior of massless theories in a manifestly gauge invariant fashion. The hypertorus also offers several practical advantages compared to other compact manifolds. Most prominently it preserves translations and can be considered in $d$-dimensions: there are $d$ commuting generators, Fourier analysis is relatively simple, and one can still speak of masses, spectra, etc... Best of all, one expects the thermodynamic limit to be rather smooth. The torus is also one of the manifolds that admits physical fermion fields, i.e., quarks, with anti-periodic boundary conditions. Surprisingly one has to choose periodic boundary conditions for the connection, if it couples to quark fields in the fundamental representation. Consequently also the ghost- and antighost- fields are periodic in covariant gauges in this case. I will only discuss covariant gauges in the following, but the topological arguments below apply equally well to any gauge condition that can be continuously deformed to Landau gauge

$$\partial \cdot A = 0 \quad (4)$$

Consider the orbit

$$O_A := \{ AUU^\dagger + UdU^\dagger : U(x) \in SU(n) \} , \quad (5)$$
labeled by a connection $A$ on it. A connection on the orbit Eq.(5) that satisfies Eq.(4) is a local extremum of the bounded Morse functional

$$V_A[U] := \int_T \text{Tr} A^U \cdot A^U \quad (6)$$

\[\text{Twisted boundary conditions for the gauge fields are admissible only in the absence of fundamental fermions. An immediate consequence of the periodic boundary conditions for the gauge fields on a finite torus is that the Pontryagin number vanishes: the space of orbits is connected (the strong CP-problem does not arise in this case, but the $U_A(1)$-problem is not easily solved on a finite torus in covariant gauges either).}\]

\[\text{for the sake of brevity, the notation used here is not entirely consistent. Generally the fields are anti-hermitian forms in the adjoint representation of the Lie-algebra of the group. Euclidean indices and the dependence on the metric of the compact space-time will usually be suppressed.}\]
on the orbit \( O_A \). Generically there are many extrema and thus many solutions to Eq. (4) that form a topological space

\[
E_A := \{ U : \partial \cdot A^U = 0 \}
\]  

(7)

The space \( E_A \), by construction, depends only on the orbit \( O_A \). Topological characteristics of \( E_A \) furthermore do not depend on continuous deformations of the orbit (or, for that matter, the gauge condition). We can therefore quantize the gauge theory within a connected sector of its orbit space by constructing a TQFT that computes a non-vanishing topological number of the space \( E_A \). The procedure to construct a TQFT whose partition function is the generalized Euler characteristic of a topological space is well known. This TQFT for the space \( E_A \) indeed corresponds to the conventional (covariant) BRST-quantization of a gauge theory. It is however readily seen that the generalized Euler number of the space \( E_A \) vanishes for a gauge group such as \( SU(n) \), because the gauge condition Eq. (4) is degenerate with respect to global gauge transformations. Thus the manifold corresponding to global (righthanded) gauge-transformations factorizes and \( E \) for \( SU(n) \) is a manifold with the topological structure

\[
E_A \sim B_A \times SU(n)_{\text{global}}
\]  

(8)

If we assume for the moment (this can be verified) that the generalized Euler character of a product space factorizes into the product of the (generalized) Euler characters of the individual spaces,

\[
\chi(M_1 \times M_2) = \chi(M_1) \cdot \chi(M_2)
\]  

(9)

then \( \chi(E_A) = \chi(B_A) \cdot \chi(SU(n)) = \chi(B_A) \cdot 0 \) is at best an indefinite expression, because the Euler character of the compact group manifold \( SU(n) \) vanishes. Since the common \( SU(n) \) factor does not depend on the orbit, we may as well omit it and construct a TQFT that computes the generalized Euler character of the coset space

\[
B_A \simeq E_A / SU(n)
\]  

(10)

The construction of the corresponding TQFT below leads to an equivariant BRST-quantization. For an \( SU(2) \) gauge theory it was shown that \( \chi(B) \) is odd and therefore does not vanish. In certain cases (as for an \( SU(2) \) theory on \( S_4 \) in the trivial sector of the orbit space) one can show that \( \chi(B) \) is in fact finite. For the moment it suffices to observe that there is nothing wrong with computing the generalized Euler character of the space \( B_A \) instead of \( E_A \) and that this topological number is constant on a connected space of orbits.
2.1 The equivariant BRST-algebra

The construction of the TQFT on the gauge group whose partition function is proportional to the generalized Euler character of a coset space proceeds along the general principles outlined in. A similar procedure can also be employed to covariantly gauge fix a lattice gauge theory with nonabelian structure group. Below are the main arguments for an \( SU(n) \) gauge invariance of the continuum theory.

Under infinitesimal righthanded gauge transformations,

\[
\delta U(x) = U(x)\theta(x) ,
\]

where \( \theta(x) \in su(n) \) is an element of the Lie-algebra. We may always decompose

\[
\theta(x) = \tilde{\theta}(x) + \theta ,
\]

into a generator of global gauge transformations and a generator of (pointed) gauge transformations in the coset space \( SU(n)_{\text{gauge}}/SU(n)_{\text{global}} \). To specify \( \tilde{\theta}(x) \) uniquely one can require that it satisfy some (global) condition. Note that this is a linear problem that does not suffer from any ambiguities. However, the condition generally either will not preserve translational invariance or will be nonlocal. One may for instance demand that \( \tilde{\theta}(0) = 0 \) at a preferred space-time point. To manifestly preserve the translational invariance of the model on a compact torus, it is preferable to demand

\[
\int_T dx \tilde{\theta}(x) = 0 .
\]

The first choice would lead to pointed gauge, whereas the second requires the equivariant BRST-algebra below.

The BRST-algebra of a TQFT of Witten type is constructed by “ghostifying” the variation Eq.\((11)\), i.e. one introduces a nilpotent (anticommuting) BRST-variation \( s \), such that

\[
sU(x) = U(x)(c(x) + \omega)
\]

where \( c(x) \) and \( \omega \) are Lie-algebra valued Grassmann ghosts. To uniquely decompose Eq.\((14)\) into global and local ghosts, we eventually will enforce the non-local constraint

\[
\int_T c(x) = 0 .
\]

Nilpotency of the operation \( s \) implies that

\[
s^2U(x) = 0 \Rightarrow sc(x) + s\omega = -\frac{1}{2}[c(x), c(x)] - [\omega, c(x)] - \frac{1}{2}[\omega, \omega] ,
\]
where $[\cdot,\cdot]$ denotes the commutator graded by the ghost number, i.e., in Eq.(10) it is the anti-commutator since the ghost number of both fields is odd. We recognize that the last term on the RHS of Eq.(14) is global and that the second term is just an infinitesimal rigid gauge transformation of $c(x)$ generated by the ghost $\omega$. Consistency of the constraint Eq.(15) under BRST-variation demands that we also have

\[ \int_T sc(x) = 0 \]  

when Eq.(13) is satisfied. We cannot therefore choose $sc(x) = -\frac{1}{2}[c(x), c(x)] - [\omega, c(x)]$, since $[c(x), c(x)]$ in general does have a constant component even if $c(x)$ doesn’t. (It is at this point that we deviate from pointed gauges, where one demands $c(x=0) = 0$, which is consistent with $sc(x=0) = 0$ even for non-abelian groups but breaks manifest translational invariance.) Consistency of the constraint Eq.(15) under BRST variation forces one to introduce a global ghost field $\phi$ of ghost number two and satisfy Eq.(16) by the algebra

\[ sc(x) = -\frac{1}{2}[c(x), c(x)] - [\omega, c(x)] - \phi \]
\[ s\omega = -\frac{1}{2}[\omega, \omega] + \phi \]
\[ s\phi = -[\omega, \phi] \]  

(18)

where the BRST-variation of $\phi$ ensures the nilpotency of $s$. Evidently the constraint Eq.(17) is an equation for $\phi$. It is worth pausing at this point of the construction to note that the somewhat complicated BRST-structure we have begun to develop is a consequence of the non-abelian nature of the gauge group. In the abelian case all the transformations would have been linear and we obviously could have removed the global invariance by simply imposing the constraint Eq.(17) without further ado. Since the ghosts furthermore decouple in the abelian case, the equivariant construction results in the conventional covariant gauge-fixing for abelian gauge theories. We will see that it gives something new in the non-abelian case and that it is worth pursuing this avenue of attack.

In addition to the usual doublet of covariant gauge constraints

\[ \partial \cdot A^U(x) = 0 \quad \text{and} \quad s\partial \cdot A^U = \partial \cdot D^A^U c(x) - [\omega, \partial \cdot A^U] = 0 \]  

(19)

for $U(x)$ and $c(x)$ we now also have to implement the doublet of global constraints Eq.(15) and Eq.(17). The Lagrange multipliers for the gauge constraints are the local Nakanishi-Lautrup field $b(x)$ and its partner, the anti-ghost $\bar{c}(x)$. They form an equivariant BRST-doublet

\[ sc(x) = -[\omega, \bar{c}(x)] + b(x) \quad \quad \quad sb(x) = -[\omega, b(x)] + [\phi, \bar{c}(x)] \]  

(20)
Note that the anti-ghost $\bar{c}(x)$ transforms under global transformations generated by $\omega$, because the corresponding constraint Eq.(13) does. This is vital and guarantees that the effective action we will construct does not depend on $\omega$. The BRST-transformation of the Nakanishi-Lautrup field $b(x)$ ensures the nilpotency of $s$. Observe that $b(x)$ is not annihilated by $s$ in Eq.(21).

To implement the global constraints Eq.(15) and Eq.(17) we require a BRST-doublet of global fields $\bar{\sigma}, \sigma$. For by now familiar reasons this doublet satisfies
\[ s\sigma = -[\omega, \sigma] + \bar{\sigma} \quad s\bar{\sigma} = -[\omega, \bar{\sigma}] + [\phi, \sigma] . \] (21)

In covariant gauges it is finally important that the constant modes of $\bar{c}(x)$ and $b(x)$ do not couple to the constraints Eq.(15). To avoid uncompensated zero-modes of the anti-ghost and preserve the BRST-invariance of the model on a finite torus, one has to impose an additional BRST-conjugate pair of global constraints:
\[ \int_T \bar{c}(x) = \int_T s\bar{c}(x) = 0 , \] (22)

and introduce associated global fields $\gamma$ and $\bar{\gamma}$ of ghost number 1 and 0 respectively. This second doublet of global Lagrange multipliers satisfies
\[ s\bar{\gamma} = -[\omega, \bar{\gamma}] + \gamma \quad s\gamma = -[\omega, \gamma] + [\phi, \bar{\gamma}] . \] (23)

Table 1. Canonical dimensions and ghost numbers of the fields.

| field       | $A^U(x)$ | $\psi(x) \& \bar{\psi}(x)$ | $c(x)$ | $\bar{c}(x)$ | $b(x)$ | $\phi$ | $\sigma$ | $\bar{\sigma}$ | $\gamma$ | $\bar{\gamma}$ |
|-------------|---------|-----------------------------|--------|-------------|--------|--------|--------|-------------|--------|-------------|
| dim         | 1       | $3/2$                       | 0      | 2           | 2      | 0      | 4      | 4           | 2      | 2           |
| $\phi\Pi$   | 0       | 0                           | 1      | -1          | 0      | 2      | -2     | -1          | 0      | 1           |

We have completed the construction of the algebra and field content of the TQFT. The canonical dimensions and ghost numbers of the fields is summarized in Table 1. The symmetry generated by $s$ is nilpotent by construction
\[ s^2 = 0 \] (24)
on any functional of the fields in Table 1. The BRST-algebra above is one for the gauge group elements $U(x)$ and the connection $A(x)$ is a background field that does not transform
\[ sA(x) = 0 . \] (25)

Note, however, that
\[ sA^U(x) = -D^A c(x) + [\omega, A^U] , \] (26)

where $D^A$ is the usual covariant derivative of the adjoint representation. Eq.(26) implies a tight relation between the BRST-symmetry of the TQFT and the BRST-symmetry of the corresponding gauge theory.
2.2 The TQFT

The partition function, $Z$, of the TQFT in principle could depend on the background connection $A$ and a set of (gauge) parameters $\{\alpha\}$. Formally the path integral representation of $Z$ is of the form

$$Z[A(x), \{\alpha\}] = \int D[c(x), \bar{c}(x), U(x), \phi, \sigma, \bar{\sigma}, \gamma, \bar{\gamma}] \exp\{S\} ,$$

(27)

where $D[\ldots]$ is a BRST-invariant measure and the effective action $S$

$$S = sW[c(x), \bar{c}(x), U(x), \sigma, \bar{\gamma}; A(x), \{\alpha\}]$$

(28)

is BRST exact, that is $s$ is a nilpotent BRST-symmetry and $S = sW$. Apart from being exact, the effective action, however, has to satisfy several additional criteria in our case. Note that we do not integrate over the constant ghost $\omega$ in Eq.(27) and that the action therefore $S$ should not depend on this field either. Since $\omega$ in Eq.(18), Eq.(21), Eq.(24) and Eq.(23) generates global right-handed gauge transformations of all the fields except itself, $S$ is $\omega$-independent only if $W$ is a globally gauge invariant functional of the fields that does not depend on $\omega$. It immediately follows that $W$ can depend on the background one-form $A(x)$ and $U(x)$ only in the combinations $A^U$ and $U^\dagger U$ that transform in the adjoint. In order to use the TQFT as a gauge-fixing device we consider actions $S$ that depend on $U$ only via $A^U$.

In addition the effective action $S$ should preserve ghost number, be manifestly $SO(4)$ invariant (or at least invariant under the discrete hypercubic subgroup of the hypertorus) and power counting renormalizable in four dimensions. This requires that we construct a manifestly $SO(4)$- and globally gauge invariant local functional $W$ of ghost number $-1$ and canonical dimension 4 from the fields of Table 1. Up to arbitrary normalizations of the fields and constants, the most general power-counting renormalizable action $S$ with the required symmetries thus is the BRST-variation of

$$W = -2 \int_x \text{Tr} \left[ \bar{c}(x) \partial \cdot A^U(x) + \frac{1}{6} \alpha \bar{c}(x) b(x) + \alpha \delta \bar{c}(x) \bar{c}(x)c(x) \\
\alpha \rho \bar{c}(x) A^U(x) A^U(x) - \bar{\gamma} \bar{c}(x) - \sigma c(x) \right] ,$$

(29)

which for $SU(n > 2)$ depends on three gauge parameters: $\alpha, \delta$ and $\rho$. For $SU(2)$, $\rho$ is an irrelevant parameter because it multiplies terms proportional to the symmetric structure constants $d^{abc}$.

No matter how complicated the general covariant effective action finally is, the most important point at this stage is that $S$ is BRST-exact. One can prove that the equivariant BRST-algebra we derived is free of anomalies for
a hypertorus as spacetime. Once this is established, it is not difficult to show that $Z[A(x), \alpha, \delta, \rho]$ in fact does not depend on variations of its parameters,
\[
\frac{\delta Z}{\delta A(x)} = \frac{\partial Z}{\partial \alpha} = \frac{\partial Z}{\partial \delta} = \frac{\partial Z}{\partial \rho} = 0 ,
\]
(30)
since these variations are proportional to expectation values of $s$-exact functionals. This establishes that Eq. (27) is a gauge-parameter independent constant on a connected sector of the parameter and orbit space.

The interesting question is whether this constant is normalizable or whether $Z$ vanishes identically. One only has to prove that $Z$ is normalizable (or not) for at least one orbit $A(x)$ and some (admissible) value of the gauge parameters. For simplicity we will only consider gauges with $\delta = 1, \rho = 0$ in the following. For this choice of the gauge parameters one obtains the simplest effective action
\[
S|_{\delta=1, \rho=0} = -2 \int \text{Tr} \left[ b(x) \partial \cdot A^U(x) - \bar{c}(x) \partial \cdot D^A c(x) \right. \\
\left. + \alpha \left\{ 4b^2(x) + b(x)[\bar{c}(x), c(x)] - \bar{c}^2(x)c^2(x) \right\} \\
- \bar{\gamma}b(x) - \gamma \bar{c}(x) - \sigma c(x) + \sigma c^2(x) + \sigma \phi \right] .
\]
(31)
This action of the TQFT can be further simplified by using the equations of motion of some of the fields. The equation of motion of the topological ghost $\phi$ sets $\sigma = 0$ and those of $\bar{\sigma}$ and $\gamma$ eliminate the constant modes of the ghost and anti-ghost fields on the torus. Finally, the equation of motion of the Nakanishi-Lautrup field $b(x)$ is
\[
b(x) = \bar{\gamma} - \alpha^{-1} \partial \cdot A^U(x) - [\bar{c}(x), c(x)] .
\]
(32)
We can thus rewrite $\tilde{Z}[A] := Z[A(x), \alpha, \delta = 1, \rho = 0]$ as the relatively transparent functional integral
\[
\tilde{Z}[A] = N(\alpha) \int d\gamma \int D'[c(x), \bar{c}(x), U(x)] \exp S_{GF}[c(x), \bar{c}(x), A^U(x), \bar{\gamma}; \alpha] ,
\]
(33)
with an $\alpha$-dependent normalization $N$ and a measure $D'$ of the dynamical fields that does not include constant modes of the ghost and anti-ghost. The effective action in Eq. (33) is
\[
S_{GF}[c(x), \bar{c}(x), A(x), \bar{\gamma}; \alpha] = 2 \int \text{Tr} \left[ (\partial \cdot A(x))^2/(2\alpha) + \bar{c}(x)D^A \cdot \partial c(x) \right. \\
\left. - \bar{\gamma}[\bar{c}(x), c(x)] + \bar{\gamma}^2/(2\alpha) \right]
\]
(34)
after dropping a surface-term $\int \text{Tr} \partial \cdot A(x)$ that vanishes on a torus with periodic boundary conditions for the connection.
Note that the effective action Eq.(34) is unbounded below for $\alpha < 0$. This does not contradict Eq.(30) and $\bar{Z}$ does not depend on the value of $\alpha$ in the open interval $\alpha \in (0, \infty)$. Landau-gauge is, however, can only be defined as the limit $\alpha \to 0_+$ (which generally is not equivalent to simply setting $\alpha = 0$ in Eq.(31)).

3 Why deal with an equivariant BRST-algebra?

Apart from an interchange of the ghost and anti-ghost, the action Eq.(34) we have constructed is that of conventional covariant gauges when we set $\bar{\gamma} = 0$. There is one other important difference: the measure $D\gamma$ of the functional integral Eq.(33) no longer includes an integration over constant ghost and anti-ghost modes. The importance of these differences lies in the fact that one can show that the TQFT for conventional covariant gauge fixing, described by a partition function

$$Z_{\text{conv.}}[A, \alpha] := \int D[c(x), \bar{c}(x), U(x)] \exp S_{GF}[c(x), \bar{c}(x), A^U(x), \bar{\gamma} = 0; \alpha] ,$$

vanishes for $\alpha > 0$ on any finite hypertorus whereas Eq.(33) is normalizable. The argument that Eq.(33) vanishes relies on the fact that the effective action of Eq.(34) is BRST-exact with respect to the on-shell nilpotent BRST-symmetry $\tilde{s}$ defined by

$$\tilde{s}U(x) = U(x)c(x), \quad \tilde{s}c(x) = -c^2(x) \tilde{s}\bar{c}(x) = -\alpha^{-1} \partial \cdot A^U(x) - [\bar{c}(x), c(x)] .$$

This algebra is nothing but the equivariant BRST-algebra generated by $s$ on-shell (i.e., when Eq.(32) holds) and all the global fields are formally set to zero. The slightly unusual transformation of the anti-ghost in Eq.(36) just interchanges the role of ghost and anti-ghost in the effective action. The argument below can also be repeated with the more conventional action and corresponding BRST-symmetry. The main point is that a nilpotent BRST-symmetry also exists for the TQFT corresponding to conventional covariant gauges. It guarantees that $Z_{\text{conv.}}$ also does not depend on on the connection. It thus suffices to show that $Z_{\text{conv.}}$ vanishes for a particular orbit. We choose the trivial orbit represented by the connection $A(x) = 0$. This is an example of a degenerate orbit, since

$$D^A \eta(x) = 0$$

has a non-trivial solution $\eta(x) \neq 0$. Other examples of degenerate orbits are those corresponding to non-abelian monopole configurations, for which the connection is invariant under a $U(1)$ subgroup of $SU(n)$ that defines the magnetic charge of the monopole.
To appreciate the significance of degenerate orbits note that Eq.(37) implies that the connection \( A \) is invariant under a particular (infinitesimal) gauge transformation generated by \( \eta(x) \) (hence is “degenerate”). Since the definition Eq.(5) together with Eq.(37) shows that

\[
D^A U(x) \eta(x) U^\dagger(x) = U(x) [D^A \eta(x)] U^\dagger(x) = 0,
\]

for any gauge transformation \( U(x) \), it furthermore does not matter which connection we pick to represent the degenerate orbit. We can evidently find a gauge transformation \( U_0(x) \) and a corresponding connection \( A_0(x) = A U_0 \) on the degenerate orbit for which \( \eta_0(x) = U_0(x) \eta(x) U_0^\dagger(x) \) is diagonal. For \( \eta_0 \) in the Cartan subalgebra Eq.(38) requires that

\[
d\eta_0(x) = 0 \quad \text{and} \quad [A_0(x), \eta_0(x)] = 0,
\]

separately\(^{11}\), since the commutator is not diagonal. The zero-form \( c \eta_0 \) thus does not depend on space-time and the connection \( A_0(x) \) is in a sub-algebra of \( su(n) \). The mode \( \eta_0 \) is normalizable on any compact Euclidean manifold such as a hypertorus. Since \( \eta(x) \) and \( \eta_0 \) are related by a unitary transformation, we furthermore have that

\[
\text{Tr} \eta^\dagger(x) \eta(x) = \text{Tr} \eta_0^\dagger \eta_0 = \text{const.}
\]

is independent of the space-time point \( x \). Eq.(40) not only implies that the modes satisfying Eq.(37) are normalizable on a compact space-time manifold for any connection of the degenerate orbit, but also that there is no solution to Eq.(37) in the space of functions \( \eta(x) \) satisfying the condition \( \eta(0) = 0 \). Recalling that \( \eta(x) \) is the generator of an infinitesimal gauge transformation, the restriction to \( \eta(0) = 0 \) is evidently equivalent to considering only gauge transformations of the pointed gauge group \( SU(n)_{\text{gauge}}/SU(n)_{\text{global}} \) rather than those of \( SU(n)_{\text{gauge}} \). Thus there are no degenerate orbits with respect to the pointed gauge group for which we constructed the equivariant BRST-algebra.

On the other hand, there are degenerate connections with respect to the full gauge group. Since \( \eta(x) \) satisfying Eq.(37) is also a zero-mode of the Faddeev-Popov (FP) operator \( \partial \cdot D^A \), degenerate orbits are on the Gribov horizon, that is \( \det \partial D^A = 0 \) for any connection on a degenerate orbit and in particular for all those connections of the degenerate orbit that satisfy a covariant gauge condition. Thus there are normalizable ghost and anti-ghost zero-modes on any degenerate orbit, leading to a vanishing FP-determinant on the whole orbit. One can also evaluate \( \hat{Z}_{\text{conv.}}[A(x) = 0, \alpha] \) semiclassically by choosing the

\(^{11}\)I am indebted to D. Zwanziger for this argument.
gauge parameter $\alpha$ sufficiently small and show more rigorously that $Z_{\text{conv.}}$ is proportional to $\chi(SU(n))$ and indeed vanishes.

On the other hand $\tilde{Z}$ defined by Eq.(33) does not suffer from Grassmannian zero-modes. The modified FP-determinant $\det[D^A \cdot \partial + \hat{\gamma}]$ generically does not vanish on a degenerate orbit. Constant ghost modes have been eliminated and the determinant depends on the global ghost $\bar{\gamma}$. By evaluating $\tilde{Z}$ semiclassically, one can prove that the partition function of the TQFT is normalizable.

Because $\tilde{Z}$ does not depend on the orbit and does not vanish, one can insert $\tilde{Z}$ in the functional integral of the gauge theory and perform the change of variables

$$A^U \to A$$

(41)

to decouple the integration over gauge transformations in each connected sector of the orbit space. The change of variables Eq.(41) implies that the gauge-fixed theory inherits the equivariant BRST-symmetry of the TQFT. Physical observables of this theory are expectation values of globally gauge- and BRST-invariant functionals with vanishing ghost number.

### 3.1 The role of the global ghost $\bar{\gamma}$

In addition to the usual dynamical fields, the equivariantly quantized non-abelian gauge theory also depends on the global ghost $\bar{\gamma}$. As far as the dynamical fields are concerned, this global field is a somewhat unconventional “mass”-parameter for the ghosts. One can compute the effective measure for this moduli-space order-by-order in the loop expansion for the dynamical fields. For $n_f = n$ quark flavors it was shown that there is a nontrivial ultraviolet fixed point

$$<\bar{\gamma}> |_{g \to 0, \mu \to \infty, \Lambda_{\text{MS}}(\mu, g)\text{fixed}} \neq 0$$

(42)

in the thermodynamic limit of the hypertorus. For $n_f < n$ the existence of such a fixed point in the more complicated moduli-space with $\delta \neq 1$ was also established. Although Eq.(42) implies that the global gauge symmetry of the equivariantly gauge-fixed theory is broken, the corresponding Goldstone poles do not appear in observables of the equivariant cohomology of the model, since these are globally gauge invariant (just as the Goldstone poles of spontaneously broken chiral symmetry are not found in chirally invariant correlation functions). This is not to say that the corresponding Goldstone singularities in unphysical correlators have no effect—they could even be the cause for confinement in covariantly quantized gauge theories. Eq.(42) furthermore does not
imply that the equivariant BRST-symmetry is broken, if the gap equation

$$ <\bar{c}(x)> = 0 = <\bar{\gamma}> - <[\bar{c}(x), c(x)]> ,$$

(43)
is satisfied. Eq.(43) is a consequence of Eq.(20), Eq.(32) and translational invariance. Since a renormalization group invariant minimum of the effective potential on the moduli-space is just another definition of the asymptotic scale parameter of the theory, the expectation value Eq.(42) is related to $\Lambda_{\text{MS}}$ by a one-loop calculation. $<\text{Tr}\bar{\gamma}^2>$ furthermore is the expectation value of the scale anomaly when all dynamical fields have been integrated out. The resulting relation between the expectation value of the energy-momentum tensor of $SU(n)$ gauge theory with $n_f = n = 3$ quark flavors and the asymptotic scale parameter compares favorably with phenomenological estimates from QCD sumrules.\(^1\) Power corrections to the asymptotic behavior of physical correlation functions proportional to $\text{Tr}\bar{\gamma}^2, \text{Tr}\bar{\gamma}^3, \ldots$ etc. arise order by order in the loop expansion for the dynamical fields when Eq.(42) holds, even in the chiral limit of a massless theory. Note the absence of dimension 2 contributions, a consequence of the fact that physical correlators are globally gauge invariant and $\bar{\gamma}$ is a field of canonical dimension 2 in the adjoint. The power corrections therefore have the structure required by Wilson’s Operator Product Expansion for a gauge theory without scalars. It will be interesting to compare the two asymptotic expansions on a more quantitative level.

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