Methods of Equivariant Quantization

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Abstract

This article is a survey of recent work developing a new approach to quantization based on the equivariance with respect to some Lie group of symmetries. Examples are provided by conformal and projective differential geometry: given a smooth manifold $M$ endowed with a flat conformal/projective structure, we establish a canonical isomorphism between the space of symmetric contravariant tensor fields on $M$ and the space of differential operators on $M$. This leads to a notion of conformally/projectively invariant star-product on $T^*M$.

Keywords: Quantization, Lie algebras of vector fields, hypergeometric functions, star-products.

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1 Introduction

This paper is a survey of recent articles \[15, 6, 7, 13\] (see also \[3\]) developing a new viewpoint on quantization.

1.1 Let $\mathcal{D}(M)$ be the space of differential operators on a smooth manifold $M$ and $\mathcal{S}(M) (= \text{Pol}(T^*M))$ the space of symbols, that is, of smooth functions on $T^*M$ polynomial on fibers. Note that $\mathcal{S}(M)$ is naturally isomorphic to the space $\Gamma(STM)$ of symmetric contravariant tensor fields on $M$.

A quantization map (inverse to a symbol map) is a linear map

$$Q : \mathcal{S}(M) \rightarrow \mathcal{D}(M) \quad (1.1)$$

satisfying the following natural condition: for every $P \in \mathcal{S}(M)$ the principal symbol of $Q(P)$ coincides with the homogeneous higher-order part of $P$.

We will be considering a Lie group $G$ acting on $M$ and impose the $G$-equivariance condition; we will then seek a map (1.1) commuting with the $G$-action. We will be, furthermore, interested in the situation where the equivariance condition determines the quantization map in a unique fashion.

1.2 Our main examples of symmetry groups will be:

$$G = \text{SL}(n+1, \mathbb{R}) \quad \text{and} \quad G = \text{SO}(p+1, q+1), \quad p + q = n \quad (= \text{dim } M) \quad (1.2)$$

acting in the standard way on $S^n (\rightarrow \mathbb{R}P^n)$ and on $S^p \times S^q (\rightarrow (S^p \times S^q)/\mathbb{Z}_2)$ (respectively). These two cases are related to projective and conformal differential geometry. Our main purpose is to introduce the projectively/conformally equivariant quantization and prove its uniqueness.

1.3 The crucial property common to these two cases is maximality in the following sense. The corresponding Lie algebras:

$$\mathfrak{g} = \text{sl}(n+1, \mathbb{R}) \quad \text{and} \quad \mathfrak{g} = \text{o}(p+1, q+1), \quad p + q \geq 3 \quad (1.3)$$

can be both viewed as maximal subalgebras of the Lie algebra $\text{Vect}_{\text{Pol}}(\mathbb{R}^n)$ of all polynomial vector fields on $\mathbb{R}^n$ (cf. \[15, 2\]). We believe (although this remains unproven) that the uniqueness of the equivariant quantization map is due to the maximality of the algebras of symmetry.
1.4 Consider the group of all diffeomorphisms \( \text{Diff}(M) \) and the Lie algebra of all smooth vector fields \( \text{Vect}(M) \). An important aspect of our approach is the study of \( \text{Diff}(M) \)- and \( \text{Vect}(M) \)-module structure on the space of differential operators.

Let \( \mathcal{F}_\lambda \) be the space of tensor densities of degree \( \lambda \) on \( M \):

\[
\varphi = f(x^1, \ldots, x^n) |dx^1 \wedge \cdots \wedge dx^n|^\lambda
\] (1.4)

that is, of sections of the line bundle \( \Delta_\lambda(M) = |\Lambda^n T^* M|^{\otimes \lambda} \) over \( M \). Denote \( \mathcal{D}_{\lambda, \mu}(M) \) the space of differential operators

\[
A : \mathcal{F}_\lambda \to \mathcal{F}_\mu.
\] (1.5)

This space is naturally a module over \( \text{Diff}(M) \) and \( \text{Vect}(M) \). Therefore, for a Lie subgroup \( G \subset \text{Diff}(M) \), one obtains a \( G \)-action on \( \mathcal{D}_{\lambda, \mu}(M) \).

**Remark 1.1.** Assume that \( M \) is orientable and choose a volume form on \( M \), one then naturally identifies \( \mathcal{D}_{\lambda, \mu}(M) \) with \( \mathcal{D}(M) \) (i.e. with \( \mathcal{D}_{0,0}(M) \)) as a vector space. One can, therefore, consider \( \mathcal{D}_{\lambda, \mu}(M) \) as a two-parameter family of \( \text{Diff}(M) \)- and \( \text{Vect}(M) \)-actions on the same space, \( \mathcal{D}(M) \).

It worth noticing that many important examples of differential operators acting on tensor densities have already been studied by the classics. The most important case \( \mathcal{D}_{\frac{1}{2}, \frac{1}{2}} \) (of operators on half-densities) naturally arises in geometric quantization (see e.g. [12]). We refer to recent articles [3, 14, 10, 16] (and references therein) for a systematic study and classification of the modules \( \mathcal{D}_{\lambda, \mu}(M) \).

1.5 In many cases we have to assume that the \( G \)-action cannot be defined *globally* on \( M \). An example is provided when \( M \) is not compact (e.g., \( M = \mathbb{R}^n \)) and one can only deal with a \( g \)-action. Moreover, there are many cases when even the \( g \)-action is only locally-defined. The most general framework is provided by the notion of a flat \( G \)-structure (defined by a local action of \( G \) on \( M \), compatible with a local identification of \( M \) with some homogeneous space \( G/H \)). Existence of such structures on a given smooth manifold is a difficult (and widely open) problem of topology (see e.g. [17]).

We will assume that \( M \) is endowed either with a flat projective or a flat conformal structure.

1.6 In the particular case \( n = 1 \), both conformal and projective structures coincide. We refer to [3] (see also [18]) for a thorough study of \( \text{sl}(2, \mathbb{R}) \)-equivariant quantization and of the corresponding invariant star-product.
2 Existence results

In this section we formulate our most general existence theorems. We will treat the basic examples and give some explicit formulæ later.

2.1 Symbols with values in tensor densities

The space $S(M)$ of contravariant symmetric tensor fields on $M$ is naturally a $\text{Diff}(M)$- and $\text{Vect}(M)$-module. We will, however, need to consider a one-parameter family of $\text{Diff}(M)$- and $\text{Vect}(M)$-actions on this space. Put

$$S_\delta = S(M) \otimes_{C^\infty(M)} F_\delta.$$  \hspace{1cm} (2.1)

The space $S_\delta$ is also, naturally, a $\text{Diff}(M)$- and $\text{Vect}(M)$-module.

2.2 Action of $\text{Vect}(M)$ on the space of symbols

Space $S(M)$ is a Poisson algebra isomorphic as a $\text{Vect}(M)$-module to the space of smooth functions on $T^*M$ polynomial on the fibers. The action of $X \in \text{Vect}(M)$ on $S$ is given by the Hamiltonian vector field

$$L_X = \frac{\partial X}{\partial \xi_i} \frac{\partial}{\partial x^i} - \frac{\partial X}{\partial x^i} \frac{\partial}{\partial \xi_i},$$  \hspace{1cm} (2.2)

where $(x^i, \xi_i)$ are local coordinates on $T^*M$ (we identified vector fields on $M$ with the first-order polynomials on $T^*M$, that is $X = X^i \xi_i$ in (2.2)). The Poisson bracket on $S(M)$ is usually called the (symmetric) Schouten bracket (see e.g. [8]).

The $\text{Vect}(M)$-action on $S_\delta$ is of the form:

$$L_X^\delta = L_X + \delta D(X)$$  \hspace{1cm} (2.3)

where

$$D(X) = \partial_i X^i, \quad \partial_i = \partial / \partial x^i.$$  \hspace{1cm} (2.4)

Note that the formula (2.3) does not depend on the choice of local coordinates. The space $S_\delta$ is naturally the space of symbols corresponding to the space of differential operators $D_{\lambda, \mu}$ such that $\mu - \lambda = \delta$.

1 Throughout this paper the summation over repeated indices is understood.
2.3 Isomorphism of $\mathfrak{g}$-modules

Let now $M$ be a endowed with a flat projective or conformal structure (see e.g. [17] and Section 3.1 for precise definition). The most general results of [15, 13, 6, 7] can be formulated as follows.

**Theorem 2.1.** For generic values of $\delta$ there exists an isomorphism of $\mathfrak{sl}(n+1, \mathbb{R})$- or $\mathfrak{o}(p+1, q+1)$-modules (respectively):

$$\tilde{Q}_{\lambda, \mu} : \mathcal{S}_\delta \xrightarrow{\cong} \mathcal{D}_{\lambda, \mu}, \quad \delta = \mu - \lambda$$ (2.5)

which is unique provided the principal symbol be preserved at each order.

There exist values of $\delta$ such that the modules $\mathcal{S}_\delta$ and $\mathcal{D}_{\lambda, \mu}$ with $\mu - \lambda = \delta$ are not isomorphic for generic $\lambda$. These particular values are called resonant.

**Remark 2.2.**

(a) In the projective case the resonance values of $\delta$ are $\delta = k/(n+1)$, where $k$ is integer $\geq n+1$ (see [13]).

(b) In the conformal case the structure of resonances is much more complicated (see [7]).

(c) In the one-dimensional case, $M = S^1$ or $\mathbb{R}$, the above theorem still holds true but the resonances are simply $\delta = 1, 3/2, 2, 5/2, \ldots$ and appear in [3, 9]. (The projective or conformal structure is then replaced by the natural $\mathfrak{sl}(2, \mathbb{R})$ action.)

2.4 Definition of quantization map

Let us introduce a new operator on symbols that will eventually insure the symmetry of the corresponding differential operators. Define $\mathcal{I}_\hbar : \mathcal{S}_\delta \rightarrow \mathcal{S}_\delta[[\hbar i]]$ by

$$\mathcal{I}_\hbar(P)(\xi) = P(i\hbar \xi).$$ (2.6)

Note that we will understand $\hbar$ either as a formal parameter or as a fixed real number, depending upon the context. We will introduce the quantization map $\mathcal{Q}_{\lambda, \mu; \hbar} : \mathcal{S}_\delta \rightarrow \mathcal{D}_\lambda[[\hbar i]]$ depending on $\hbar$:

$$\mathcal{Q}_{\lambda, \mu; \hbar} = \tilde{Q}_{\lambda, \mu} \circ \mathcal{I}_\hbar$$ (2.7)

where $\tilde{Q}_{\lambda, \mu}$ is the isomorphism from Theorem 2.1.
2.5 Symmetric quantized operators

Let us recall that if $\lambda + \mu = 1$, there exists, for compactly-supported densities, a $\text{Vect}(M)$-invariant pairing $\mathcal{F}_\lambda^c \otimes \mathcal{F}_\mu^c \to \mathbb{C}$ defined by

$$\varphi \otimes \psi \mapsto \int_M \varphi \psi \,.$$ (2.8)

We can now formulate the important

**Corollary 2.3.** For generic $\delta$ and $\lambda + \mu = 1$ the quantization

$$\hat{P} = Q_{1 - \lambda, \delta}^\lambda(P)$$ (2.9)

of any symbol $P \in S_\delta$ is a symmetric (formally self-adjoint) operator.

**Proof.** Let us denote by $A^*$ the adjoint of $A \in D_{1 - \lambda}$ with respect to the pairing (2.8). Consider the symmetric operator

$$\text{Symm}(Q_{\lambda, \mu; \hbar}(P)) = \frac{1}{2} (Q_{\lambda, \mu; \hbar}(P) + (Q_{\lambda, \mu; \hbar}(P))^*)$$

which exists whenever $\lambda + \mu = 1$. Notice that it has the same principal symbol as $Q_{\lambda, \mu; \hbar}(P)$. Now, the map $\text{Symm}(Q_{\lambda, \mu; \hbar})$ is obviously $\mathfrak{g}$-equivariant. The result follows then from the uniqueness of $\mathfrak{g}$-module isomorphism — see Theorem 2.1. \( \square \)

2.6 Invariant star-products

The special and most important value $\delta = 0$ is non-resonant. The existence and uniqueness of the isomorphism (2.5) in this case has been proven in [15, 7].

Define an associative bilinear operation (depending on $\lambda$)

$$*_{\lambda; \hbar} : \mathcal{S} \otimes \mathcal{S} \to \mathcal{S}[[\hbar]]$$ (2.10)

such that

$$Q_{\lambda; \hbar}(P *_{\lambda; \hbar} Q) = Q_{\lambda; \hbar}(P) \circ Q_{\lambda; \hbar}(Q).$$ (2.11)

Recall that an associative operation $*_{\hbar} : \mathcal{S} \otimes \mathcal{S} \to \mathcal{S}[[\hbar]]$ is called a **star-product** if it is of the form

$$P *_{\hbar} Q = PQ + \frac{i\hbar}{2} \{P, Q\} + O(\hbar^2)$$ (2.12)

where $\{\cdot, \cdot\}$ stands for the Poisson bracket on $T^*M$, and is given by bi-differential operators at each order in $\hbar$. 

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Theorem 2.4. The associative, conformally invariant, operation $\ast_{\lambda;\hbar}$ defined by \( (2.11) \) is a star-product if and only if $\lambda = \frac{1}{2}$.

Let us emphasize that this theorem provides us precisely with the value of $\lambda$ used in geometric quantization and, in some sense, links the latter to deformation quantization.

2.7 Quantum Hamiltonians in (pseudo-)Riemannian case

Assume now that a (pseudo-)Riemannian metric $g$ on $M$ is fixed.

To recover the traditional Schrödinger picture of quantum mechanics, one needs to associate to the operator $\hat{P}$ resulting from our quantization map \( (2.9) \) an operator

$$\hat{P} : \mathcal{F}_0^C \to \mathcal{F}_0^C$$

(2.13)
on the space of complex-valued functions on a conformally flat manifold $(M, g)$.

Definition 2.5. Using the natural identification $\mathcal{F}_0^C \to \mathcal{F}_\lambda^C$ between tensor densities and smooth functions given by

$$f \mapsto f |\text{Vol}_g|^{\lambda},$$

(2.14)
one can define the differential operator $\hat{P}$ by the commutative diagram

$$\begin{array}{ccc}
\mathcal{F}_0^C & \xrightarrow{\hat{P}} & \mathcal{F}_0^C \\
|\text{Vol}_g|^{\lambda} \downarrow & & |\text{Vol}_g|^{\mu} \downarrow \\
\mathcal{F}_\lambda^C & \xrightarrow{\hat{P}} & \mathcal{F}_\mu^C \\
\end{array}$$

(2.15)
where $\hat{P}$ is given by \( (2.9) \) in the case $\lambda + \mu = 1$.

We call the operator $\hat{P}$ corresponding to a polynomial $P$ the quantum Hamiltonian.

Remark 2.6. An important feature of equivariant quantization in the conformal case is, of course, that the metric $g$ can be chosen conformally flat (as a representative of the conformal structure on $M$).
3 Examples: quantizing quadratic Hamiltonians

Let us consider a homogeneous second order polynomial $H$ on $T^*M$. In arbitrary local coordinates it is of the form:

$$H = g^{ij}(x) \xi_i \xi_j. \quad (3.1)$$

To illustrate the general Theorem 2.1 we will give explicit formulae for the quantization in this case.

We will need special coordinate systems adopted to projective and conformal structures.

3.1 Local coordinates

The following description can be considered as a definition of flat projective and conformal structures.

**Definition 3.1.** A smooth manifold $M$ is endowed with a projective (or conformal) structure if there exists an atlas $U_i$ on $M$ with local coordinates $(x^1, \ldots, x^n)$ in each chart $U_i$ that are accorded on $U_i \cap U_j$ in the following way.

1. **Projective case:** The linear span of the following $n(n+1)/2$ vector fields

$$\langle \partial_i, \ x^i \partial_j, \ x^i \mathcal{E} \rangle, \quad \text{where} \quad \mathcal{E} = x^i \partial_j. \quad (3.2)$$

is independent on the choice of a chart (i.e., coincide on each $U_i \cap U_j$). The vector fields (3.2) obviously generate the Lie algebra isomorphic to $\text{sl}(n+1,\mathbb{R})$ and, therefore, one obtains a (locally defined) $\text{sl}(n+1,\mathbb{R})$-action on $M$.

It worth noticing that coordinate changes on $U_i \cap U_j$ in this case are nothing but linear-fractional transformations.

2. **Conformal case:** The linear span of the following $(n+2)(n+1)/2$ vector fields

$$\langle \partial_i, \ x_i \partial_j - x_j \partial_i, \ \mathcal{E}, \ x_j x^i \partial_i - 2x_i \mathcal{E} \rangle \quad (3.3)$$

coincide on each $U_i \cap U_j$ (we have used the notation $x_i = g_{ij} x^j$ where the flat metric $g = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$ is of signature $(p, q)$). Note that the vector fields (3.3) generate the Lie algebra isomorphic to $\text{o}(p+1, q+1)$.

The coordinate changes are then given by the conformal transformations.
3.2 The quantization map in the second-order case

We are now ready to give explicit expressions of our quantization maps. For the sake of simplicity we restrict ourself to the most interesting case \( \lambda = \mu = 1/2 \) (see [15, 6, 7] for the general formulæ). We will use coordinates adopted to a projective or conformal structure on \( M \).

(1) Projective case: The quantization map (2.7) associates to the polynomial (3.1) the following second-order differential operator

\[
Q_{\frac{1}{2}\hbar}(H) = -\hbar^2 \left( g^{ij} \partial_i \partial_j + \partial_j(g^{ij}) \partial_i + \frac{(n+1)}{4(n+2)} \partial_i \partial_j(g^{ij}) \right)
\]  

(3.4)
on the space \( \mathcal{F}_{1/2}^C \) of half-densities.

(2) Conformal case: One has

\[
Q_{\frac{1}{2}\hbar}(H) = -\hbar^2 \left( g^{ij} \partial_i \partial_j + \partial_j(g^{ij}) \partial_i + \frac{n}{4(n+1)} \partial_i \partial_j(g^{ij}) + \frac{n}{4(n+1)(n+2)} \partial_j \partial_i(g^{ij}) \right)
\]  

(3.5)

Let us also recall here for comparison the explicit expression of the most famous and standard quantization procedure, called the \textit{Weyl} or \textit{symmetric} quantization.

(3) Weyl quantization: In the case of second-order polynomials

\[
Q_{\text{Weyl}}(H) = -\hbar^2 \left( g^{ij} \partial_i \partial_j + \partial_j(g^{ij}) \partial_i + \frac{1}{4} \partial_i \partial_j(g^{ij}) \right)
\]  

(3.6)

This formula makes sense globally only in the case \( M = \mathbb{R}^n \) and it is then defined uniquely (up to normalization) by equivariance with respect to the standard \( \text{sp}(2n, \mathbb{R}) \)-action on \( T^*M = \mathbb{R}^{2n} \).

Remark 3.2. It can be checked that:

(a) the explicit formulæ (3.4,3.5) are independent of the choice of the adopted coordinate system;

(b) in both cases the differential operator \( Q_{\frac{1}{2}\hbar}(H) \) is symmetric.
3.3 Quantizing the geodesic flow

Assume now that there exists (a non-degenerate) metric

\[ g = g_{ij} dx^i dx^j \]  

(3.7)
corresponding to the polynomial (3.1), i.e. \( H = g^{-1} \).

Let us apply the procedure of Section 2.7 to obtain the quantum Hamiltonian. It is easy to see that the fact that the operators (3.4–3.6) are symmetric implies in each case (1), (2), (3) that the corresponding quantum Hamiltonian has the general form:

\[ \hat{H} = -\hbar^2 (\Delta_g + U) \]  

(3.8)
where \( \Delta_g \) is the Laplace-Beltrami operator and the function \( U \) (the potential) is different in each of the three cases.

The potential can be easily computed in each case (1), (2), (3). In particular, in the conformal case one can choose \( g \) conformally flat. It has been shown in [6] that in such a situation

\[ \hat{H} = -\hbar^2 \left( \Delta_g - \frac{n^2}{4(n-1)(n+2)} R_g \right) \]  

(3.9)
where \( R_g \) stands for the scalar curvature of \((M, g)\). It worth noticing that this result differs from other known versions of the quantization of the geodesic flow (cf. [4] and references therein).

4 Universal formula in the projective case

In the projective case there exists an explicit formula for the quantization map that is valid for polynomial functions on \( T^* M \) of an arbitrary order \( k \) (see [15, 13]). In this section we will give a new (but still equivalent) expression of the projectively equivariant quantization map. As above, we will consider only the case \( \lambda = \mu = 1/2 \).

4.1 The Euler and divergence operators

Fix an arbitrary system of local coordinates \((x^1, \ldots, x^n)\) on \( M \) and introduce the following differential operators acting locally on the space of symbols (i.e., on the space of polynomials \( \mathbb{C}[x^1, \ldots, x^n, \xi_1, \ldots, \xi_n] \)):

\[ E = \xi_i \frac{\partial}{\partial \xi_i} + \frac{n}{2}, \quad D = \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial x^i} \]  

(4.1)
4.2 The confluent hypergeometric function

Recall that the confluent hypergeometric function is a series in $z$ depending on two parameters $a, b \in \mathbb{R}$ (or $\mathbb{C}$):

$$F \left( \begin{array}{c} a \\ b \end{array} \bigg| z \right) = \sum_{m=0}^{\infty} \frac{(a)_m z^m}{(b)_m m!}$$

where

$$(a)_m := a(a+1) \cdots (a+m-1)$$

(see, e.g., [11]).

4.3 The explicit formula

Identify locally polynomials with differential operators by the normal ordering map

$$\sigma : A^{i_1 \cdots i_k}_{i_1 \cdots i_k} \partial_{i_1} \cdots \partial_{i_k} \mapsto A^{i_1 \cdots i_k}_{i_1 \cdots i_k} \xi_{i_1} \cdots \xi_{i_k}$$

(4.2)

Using this identification we will work with differential operators as with polynomials.

The following theorem provides an explicit expression for the projectively equivariant quantization map (2.7) in terms of a “non-commutative” confluent function.

**Theorem 4.1.** The projectively equivariant quantization is of the form

$$Q_{\frac{1}{h}} = F \left( \begin{array}{c} 2E \\ i\hbar D \end{array} \bigg| \frac{4}{i\hbar} \right)$$

(4.3)

for every system of local coordinates adopted to the projective structure.

In other words, one has

$$Q_{\frac{1}{2}; h} = \sum_{m=0}^{\infty} C_m(E) (i\hbar D)^m$$

(4.4)

where the expression

$$C_m(E) = \frac{1}{m!} \frac{(E + 1/2)(E + 3/2) \cdots (E + m - 1/2)}{(2E + m)(2E + m + 1) \cdots (2E + 2m - 1)}$$

(4.5)

is also understood as an infinite (Taylor) series in $E$. 

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Remark 4.2.

(a) If the manifold $M$ is endowed with a projective structure, then choosing an adapted coordinate system, the formula (4.3) is well-defined globally on $T^*M$.

(b) One easily checks that for the second-order polynomials the formula (4.3) coincides with (3.4).

The expression (4.3) is a particular case (for $\lambda = 1/2$) of the quantization map given in [15] after restriction to the space of $k$-th order polynomials. Unfortunately, we do not know the analog of the formula (4.3) in the conformal case.

4.4 Comparison with the Weyl quantization

The Weyl quantization map, $Q_{\text{Weyl}}$, retains the very elegant form:

\[
Q_{\text{Weyl}} = \exp \left( \frac{i\hbar}{2} D \right) = \text{Id} + \frac{i\hbar}{2} D - \frac{\hbar^2}{8} D^2 + O(\hbar^3)
\]

(4.6)

(see [1]). Comparing (4.6) with (4.3) one notes that the exponential is, indeed, the most elementary hypergeometric function.

4.5 An open problem

It is well-known that the Weyl quantization can be extended for a much larger class of functions on $\mathbb{R}^{2n}$ than polynomials, namely to the space of pseudo-differential symbols.

We formulate here the following problem. It it true that the formula (4.3) is also valid for pseudo-differential symbols? If the answer is positive, it would be interesting to compute its integral expression.
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