E. C. G. Sudarshan and Symmetry in Classical Dynamics, Optics and Quantum Mechanics

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Abstract. We review work initiated and inspired by Sudarshan in relativistic dynamics, beam optics, partial coherence theory, Wigner distribution methods, multimode quantum optical squeezing, and geometric phases. The 1963 No Interaction Theorem using Dirac's instant form and particle World Line Conditions is recalled. Later attempts to overcome this result exploiting constrained Hamiltonian theory, reformulation of the World Line Conditions and extending Dirac's formalism, are reviewed.

Dirac’s front form leads to a formulation of Fourier Optics for the Maxwell field, determining the actions of First Order Systems (corresponding to matrices of Sp(2,R) and Sp(4,R)) on polarization in a consistent manner. These groups also help characterize properties and propagation of partially coherent Gaussian Schell Model beams, leading to invariant quality parameters and the new Twist phase.

The higher dimensional groups Sp(2n,R) appear in the theory of Wigner distributions and in quantum optics. Elegant criteria for a Gaussian phase space function to be a Wigner distribution, expressions for multimode uncertainty principles and squeezing are described.

In geometric phase theory we highlight the use of invariance properties that lead to a kinematical formulation and the important role of Bargmann invariants. Special features of these phases arising from unitary Lie group representations, and a new formulation based on the idea of Null Phase Curves, are presented.

1) The role of symmetry, the analysis of significant group structure and action in physical problems, are recurring themes in E.C.G. Sudarshan’s work. Other speakers have surveyed this in the context of elementary particle physics. In consultation with Professor Thomas Jordan, I will cover mechanics, classical and quantum optics, and some aspects of quantum mechanics.

Some of the work to be described has been done by Sudarshan with collaborators. Others were initiated by him and some of us, carried further by us in later years. As time is limited, we can only afford a fleeting glance at each area.

2) In 1949 Dirac proposed several forms of relativistic dynamical descriptions of physical systems involving ten generators of Poincaré group action—the instant, point and front forms. These are in correspondence with certain subgroups of the Poincaré group—E(3), SO(3,1) and \( G_{2,1} \) respectively, where the last is like the Galilei group in \( 2 + 1 \) dimensions. In each case, the subgroup generators are simple or kinematic, the rest are dynamical or ‘Hamiltonians’. Bakamjian and Thomas are among those who followed up Dirac’s ideas quite early. In 1963, Currie, Jordan and Sudarshan examined the description of a system of relativistic point particles in the instant form, adding the important World Line Condition invented by them. This ensures the existence of objective, observer-independent particle world lines in space time; for each
particle this is the requirement

$$\{K_j, q_k\} = q_j \{H, q_k\},$$

where $K_j$ are the pure Lorentz transformation generators, $H$ the Hamiltonian, and $\{,\}$ are classical Poisson brackets. They then showed that for a two-particle system the Poincaré algebra PB relations plus the World Line Conditions rule out the possibility of any interactions. This was generalised to any number of particles by Leutwyler in 1965. This No Interaction Theorem may be found in many text books, and stood for many years.

In the late 1970’s several authors—Komar, Rohrlich, Todorov, . . . —independently suggested ways to overcome this theorem exploiting another beautiful idea of Dirac—constrained Hamiltonian mechanics. However, none of them seemed to have paid proper attention to the existence and objectivity of particle world lines. In a comprehensive series of papers, Sudarshan and many of us gave a detailed analysis of this problem. The World Line Conditions were developed in a manifestly covariant form, and combined with the Poincaré algebra, it was shown that interactions among relativistic point particles are permitted. But the major problem that remained was separability. An extension of Dirac’s ten generator formalism to a set of eleven generators is involved here, the extra generator being an invariant Hamiltonian giving rise to manifestly covariant evolution equations. Several forms of these theories are available; in some, the translation subgroup of the Poincaré group acts trivially, while in a particular form the homogeneous Lorentz transformations act trivially. In any case, one goes beyond the original Dirac forms of relativistic dynamics in that the evolution parameter is determined dynamically for each state of motion, and not once for all kinematically in each inertial frame. The problem with separability can be traced to this fact.

3) I turn next to another use of Dirac’s generator formalism, this time the front form. Classical ray optics in the paraxial regime, also called Gaussian optics, is a very well developed subject. Here first order optical systems—FOS’s—act on each incoming ray to yield a single outgoing ray. These systems, such as lenses, magnifiers, rotators and their combinations, are in correspondence with elements of the real symplectic groups $Sp(2, R)$ and $Sp(4, R)$ in the isotropic and anisotropic cases respectively. These groups appear because Fermat’s principle is a variational principle.

Extended to the scalar optical field, we have the subject of Fourier Optics. It was shown by Bacry and Cadilhac in 1981 that when the FOS’s act on scalar waves, the metaplectic groups $Mp(2)$ and $Mp(4)$, double covers of $Sp(2, R)$ and $Sp(4, R)$, enter the picture.

The question arises—is there a natural generalization of scalar Fourier Optics to the full classical Maxwell field, treating polarization consistently and retaining the roles of $Mp(2)$ and $Mp(4)$ in the description of FOS’s? Dirac’s front form showed the way—displayed suitably, the Poincaré algebra contains a 2+1 dimensional Galilei group structure, allowing us to extract two-dimensional canonical pairs of variables from the Poincaré generators. Then by a simple rule paraxial scalar wave optics generalises consistently to paraxial Maxwell wave optics, i.e., to Fourier optics for the Maxwell field. In this regime the actions of FOS’s on polarization are completely determined. The two-dimensional canonical variables are identified starting from the combinations $K_1 - J_2, K_2 + J_1, P_1, P_2$; once these are in hand, the groups $Sp(2, R)$, $Sp(4, R)$ and their metaplectic covers immediately follow. Thus we can say that Fourier Maxwell optics is a unique generalisation of the scalar theory.

Similar ideas have been used by Jagannathan and Khan in the case of the Dirac equation, leading to a satisfactory generalisation of traditional electron optics.

4) Continuing with classical optics, partially coherent (scalar) beams are described using the so-called two-point correlation function. Here again, especially in the context of the widely studied Gaussian Schell Model—GSM—beams, one can exploit the groups $Sp(2, R)$ and $Sp(4, R)$ very effectively. Classification of such beams, their free propagation and passage through general FOS’s, can all be handled in a systematic manner. The use of symmetry and group action yields results going well beyond what was known earlier in the literature. Here are some instances.
In the isotropic case, the IGSM family of beams, we have a three-parameter family, each beam represented by a time like vector in a fictitious $2 + 1$ dimensional Minkowski space $M_{2,1}$; and each FOS acts via a transformation of SO(2,1) in this picture. The general anisotropic case, the AGSM family of beams, is considerably more elaborate. The beams form a ten-parameter family, represented by antisymmetric tensors in a fictitious $3 + 2$ dimensional de Sitter space; and each FOS acts as an SO(3,2) de Sitter transformation on these tensors. Here we recall that Sp(2,R) and Sp(4,R) are two-fold covers of SO(2,1) and SO(3,2) respectively.

This analysis by Sudarshan, Simon and myself presented for the first time the two independent invariant quality parameters for the general AGSM beams, conserved under propagation and FOS action; their identification was possible only thanks to the underlying symmetry structure. Simon and I carried this study further and showed that in the IGSM case a twist or handedness or helicity can be impressed on the two point function. This leads to the family of Twisted Gaussian Schell Model—TGSM—beams, which had not been noticed earlier in the literature. They have subsequently been experimentally produced and theoretically analysed by others.

The most interesting feature here is the upper bound on the twist parameter. The two-point functions in each transverse plane are given by

$$\Gamma_{TGSM}(\rho_1, \rho_2) = e^{-iu(x_1y_2 - x_2y_1)/\lambda}\Gamma_{IGSM}(\rho_1, \rho_2),$$

where $\rho_1 = (x_1, y_1)$, and $\rho_2 = (x_2, y_2)$, are two points in the transverse plane, $\lambda = \lambda/2\pi$ and $u$ is the twist strength. The bound in question is

$$|u| \leq \lambda/\sigma_g^2$$

where $\sigma_g$ is the transverse coherence length. Thus such a phase cannot be impressed on a fully coherent beam for which $\sigma_g \to \infty$!

Simon and I have also completely analysed this twist phase in the general AGSM context.

5) A related very fruitful idea of Sudarshan from 1979 is the adaptation of Wigner distribution methods to paraxial partially coherent optics. In this way the complete range of wave optical phenomena can be cast into the ray language, provided that one allows for what he called ‘bright’ and ‘dark rays’. The actions of FOS’s via the integral Huyghens kernel on the field amplitude reduce to much simpler point transformations by matrices on the generalised rays, the matrices belonging to Sp(2,R) or Sp(4,R) as the case may be. This approach has been exploited in the beam problems described earlier; it also leads to the famous ABCD Law, a matrix version of the fractional Mobius transformation, accompanying FOS action.

6) The higher dimensional real symplectic groups Sp(2n,R) play an important role in the non-relativistic quantum mechanics of multimode systems, especially in connection with Wigner distributions. For example, the Wigner quality of a real phase space function $W(q, p)$—whether it is the Wigner distribution associated with some bonafide quantum state—is very subtle and difficult to capture in finite terms in general. What is true is that under Sp(2n,R) action, or more precisely action by its metaplectic double cover Mp(2n), the transformation rule is very simple as mentioned earlier:

$$S \in Sp(2n, R) \rightarrow \bar{U}(S) \text{ on Hilbert space, determined upto sign;}$$

Density matrix $\hat{\rho} \rightarrow \hat{\rho}' = \bar{U}(S)\hat{\rho}\bar{U}(S)^{-1} \iff$

Wigner distribution $W'(q, p) = W(S^{-1}(q, p))$, where we view $(q, p)$ as a 2n-component real column vector. In the particular case of Gaussian phase space distributions we showed that the Wigner quality is easily expressible using Williamson’s Theorem.

$$W(q, p) = \pi^{-n}(\det G)^{1/2} \exp \left\{ - (q \ p)G \left( \begin{array}{c} q \\ p \end{array} \right) \right\} ,$$
$G$ real symmetric $2n$ dimensional positive definite;

Williamson: $S^TGS = \text{diag}(\kappa_1, \ldots, \kappa_n, \kappa_1, \ldots, \kappa_n)$, suitable $S \in Sp(2n, R),$

$$\kappa_j > 0, j = 1, 2, \ldots, n;$$

$W(q, p)$ is a Wigner distribution $\iff 0 < \kappa_j \leq 1, j = 1, \ldots, n.$

These conditions can also be expressed explicitly in terms of the polynomial $Sp(2n, R)$ invariants formed from the matrix $G$.

These groups are also relevant in multimode quantum optics problems, since they preserve the fundamental Heisenberg commutation relations whether written in real or complex forms. Here I may overlap a little bit with tomorrow’s session on quantum optics. For a general state $\hat{\rho}$ of an $n$-mode quantum optical system, the variance matrix or second order moments matrix is defined in terms of hermitian quadrature components as

$$V = \begin{pmatrix} V_1 & V_2 \\ V_2^T & V_3 \end{pmatrix},$$

$$(V_1)_{jk} = Tr(\hat{\rho} \hat{q}_j \hat{q}_k) \quad (V_2)_{jk} = \frac{1}{2} Tr(\hat{\rho}\{\hat{q}_j, \hat{p}_k\}),$$

$$(V_3)_{jk} = Tr(\hat{\rho} \hat{p}_j \hat{p}_k) \quad j, k = 1, 2, \ldots, n.$$  

(For simplicity we have assumed that the mean values of $\hat{q}_j$ and $\hat{p}_j$ vanish). The uncertainty principles can be stated in an explicitly $Sp(2n, R)$ invariant manner:\(\text{V} + \frac{i}{2} \beta \geq 0, \beta = \begin{pmatrix} 0 & 1_{n \times n} \\ -1_{n \times n} & 0 \end{pmatrix} = \text{Symplectic metric matrix}\)

(The content here is the same as in the conditions for a Gaussian phase space function to be a Wigner distribution.) Thus $V$ can arise from some quantum state if and only if this matrix condition is satisfied. Now we can ask: when does $V$ describe a squeezed state? The delicate point here is that squeezing is a property invariant under $U(n)$ alone and not under the full group $Sp(2n, R)$, where $U(n)$ is the total photon number conserving maximal compact subgroup of $Sp(2n, R)$. A squeezed state is of course non classical in the sense specified by Sudarshan’s diagonal coherent state representation of 1963.\(^{20}\) This signature of non-classicality can be expressed in terms of the variance matrix $V$ as:\(^{19}\)

$$\hat{\rho} \text{ is a squeezed state } \iff \text{ Some diagonal element of } S(U)VS(U)^T \text{ for some } U \in U(n) < \frac{1}{2}$$

$$\iff \ell(V) = \text{least eigen value of } V < \frac{1}{2}.$$  

Here $S(U)$ is the $2n$-dimensional real form of the unitary matrix $U$ of $n$ dimensions, as it occurs in the defining representation of $Sp(2n, R)$. It is worth emphasizing that while the uncertainty principles are $Sp(2n, R)$ invariant, squeezing is only $U(n)$ invariant; and even though in general $V$ cannot be brought to diagonal form using only elements of $U(n)$, the squeezing condition is expressible in terms of the least eigen value of $V$.

All these results show the close connections between group structures in canonical mechanics, classical optics, and certain quantum optical signatures of non-classicality.

7) My last item in this retrospective concerns contributions to the theory of the geometric phase in quantum mechanics. I think both Professor Simon and I were introduced to this subject
by a talk given by Sudarshan in Kottayam, a town in his home state of Kerala, in 1986. We three did some initial work together, then Simon and Professor Marmo and some others including me took it somewhat further. I will only present a brief sketch of our ideas.

Let me first mention briefly what Sudarshan, Simon and I did in 1986. We found a way of exploiting Hamilton’s representation of elements of SU(2) by turns—great circle arcs on $S^2$—for polarization optics problems and then connected up with geometric phases for two-level systems. We also generalized SU(2) turns to screws for SU(1,1); later some of us treated SL(2,C) in the same spirit. Within polarization optics per se Simon and I were able to give a minimal 3-component scheme to synthesize the most general SU(2) polarization gadget.

At this point I turn to geometric phases, and come back to turns later.

Michael Berry’s original 1983 discovery of the geometric phase was in the context of cyclic evolution in the framework of the quantum adiabatic theorem. Later this was generalised to non-adiabatic and even non-cyclic evolution by Aharonov and Anandan, and by Samuel and Bhandari. Then Simon and I were able to give a purely kinematical treatment, not involving quantum dynamics at all. It turned out that the geometric phase is the simplest kinematic expression associated with any Hilbert space curve $C$ which possesses invariance under two independent transformations:

(i) Local phase changes of $C$,

(ii) Reparametrisations of $C$.

This approach led to the geometric phase for the non-cyclic case in an immediate and extremely elementary manner:

$$\varphi[C] = \varphi_{\text{total}}[C] - \varphi_{\text{dyn}}[C],$$

$$\varphi_{\text{total}}[C] = \arg(\psi(s_1),\psi(s_2)),$$

$$\varphi_{\text{dyn}}[C] = -i \int_{s_1}^{s_2} ds \left( \psi(s), \frac{d\psi(s)}{ds} \right).$$

Thanks to its invariances, $\varphi[C]$ is both geometric and a ray-space quantity, so its argument is actually the ray space image of the Hilbert space curve $C$. A direct connection to the Bargmann invariants then became apparent. These invariants are expressions of the form

$$\Delta^{(3)}(\psi_1,\psi_2,\psi_3) = (\psi_1,\psi_2)(\psi_2,\psi_3)(\psi_3,\psi_1), \psi's \in H,$$

with generalisations to four or more vectors. They are cyclically symmetric, and also invariant under independent phase changes in each vector, so they are actually ray space quantities. The key result is a connection between the phase of a Bargmann invariant defined by its vertices, and the (kinematic) geometric phase associated with a geodesic triangle with the same vertices:

$$\arg \Delta^{(3)}(\psi_1,\psi_2,\psi_3) = -\varphi[\text{geodesic triangle with vertices } \psi_1,\psi_2,\psi_3].$$

The geodesics here are determined by the Fubini–Study ray space metric.

In later work two significant results emerged and are worth mentioning:

a) Unitary irreducible representations of Lie groups lead naturally to associated geometric phases which can be systematically analysed. Reminiscent of the Wigner–Eckart theorem, the ‘dynamical’ parts of these phases split into (sums of) two factors: an algebraic piece determined by the initial fiducial vector and the generators of the representation; and a geometric piece involving the integral of a Maurer–Cartan form along a curve in the group manifold. This result has been exploited to survey all possible geometric phases that can arise from representations of SU(3), phases for three-level systems, etc.
b) The role of ray space geodesics in relating Bargmann invariants and geometric phases was mentioned earlier. It turns out that the most natural Hilbert space curves relevant for geometric phases are not geodesics but something we have called Null Phase Curves. They are much more general and numerous than geodesics which form only a very special example. In general, while there is a unique geodesic connecting any two points, there are infinitely many null phase curves connecting them. These curves are characterised entirely in terms of Bargmann invariants, and they help us express the connection between these invariants and geometric phases in the most general form:

\[ \mathcal{C} = \{ \psi(s) | s_1 \leq s \leq s_2 \} \text{ is a Null Phase Curve } \iff \]

\[ \arg \Delta^{(3)}(\psi(s), \psi(s'), \psi(s'')) = 0 \text{ for all } s, s', s''; \]

\[ \arg \Delta^{(3)}(\psi_1, \psi_2, \psi_3) = -\varphi[\mathcal{C}_{12} \cup \mathcal{C}_{23} \cup \mathcal{C}_{31}], \]

\[ \mathcal{C}_{12} = \text{any Null Phase Curve from } \psi_1 \text{ to } \psi_2, \ldots \]

Unlike geodesics, these curves are not defined as solutions to any (finite order) ordinary differential equations, but have in a sense only a non-local characterisation. But it is they that are really relevant for geometric phases, and so deserve further study. An approach to geometric phases based entirely on Bargmann invariants and null phase curves has been developed. I conclude this account of our work on the geometric phase with two points. Simon and I gave a complete analysis of the connection between turns and two-level geometric phases; and we also showed that the classical optical Guoy phase, expressed as the phase of a Bargmann invariant, is probably the earliest instance of the geometric phase!

Conclusion

It is a privilege to have been invited to be here and pay tribute to Professor E. C. G. Sudarshan. He taught me many things—but sadly no Sanskrit—and we worked closely together for close to thirty years. He strengthened an already existing profound and deep admiration and respect I had for everything Dirac said and did. I might mention some other work we did together with Jacob Kuriyan in the area of unitary representations of non-compact simple Lie groups. We called it the Master Analytic Representation—a combination of Weyl’s unitary trick and Dirac’s analytic continuation of the dimension of a group representation.

For all this and so much else—thanks, George, and wish you good health and happiness.

References

[1] Dirac P A M 1949 Rev. Mod. Phys., 21, 392
[2] Currie D G, Jordan T F, and Sudarshan E C G 1963 Rev. Mod. Phys., 35, 350
[3] Leutwyler H 1965 Nuovo Cimento, 37, 556
[4] Dirac P A M 1950 Can. J. Math., 2, 147 (1950)
[5] Komar A 1978 Phys. Rev., D18, 1881, 1887, 3617; Rohrlich F 1979 Ann. Phys. (NY), 117, 292; Todorov I T 1979 Istanbul Lectures
[6] Mukunda N and Sudarshan E C G 1981 Phys. Rev., D23, 2210; Sudarshan E C G, Mukunda N, and Goldberg J N 1981 Phys. Rev., D23, 2218; Goldberg J N, Sudarshan E C G, and Mukunda N 1981 Phys. Rev., D23, 2231; Balachandran A P, Marmo G, Mukunda N, Nilsson J S, Simoni A, Sudarshan E C G, and Zaccaria F 1984 J. Math. Phys., 25, 167
[7] Balachandran A P, Marmo G, Mukunda N, Nilsson J S, Simoni A, Sudarshan E C G, and Zaccaria F 1982 Nuovo Cimento, 67A, 121
[8] Bacry H and Cadilhac M 1981 Phys. Rev., A23, 2533
[9] Sudarshan E C G, Simon R, and Mukunda N 1983 Phys. Rev., A28, 2921; Mukunda N, Simon R, and Sudarshan E C G 1983 Phys. Rev., A28, 2942
[10] Mukunda N, Simon R, and Sudarshan E C G 1985 J. Opt. Soc. Am., A2, 416
[11] Jagannathan R, Simon R, Sudarshan E C G and Mukunda N 1989 Phys. Lett., A134, 457; Jagannathan R and Khan S A 1996 Quantum theory of the optics of charged particles, in Advances in Imaging and Electron Physics vol. 97, ed. Hawkes P W, Academic Press, San Diego, pp. 257–358
[12] Simon R, Sudarshan E C G, and Mukunda N 1984 Phys. Rev., A29, 3273
[13] Simon R, Sudarshan E C G, and Mukunda N 1985 Phys. Rev., A31, 2419
[14] Simon R and Mukunda N 1993 J. Opt. Soc. Am., A10, 95
[15] Simon R and Mukunda N 1998 J. Opt. Soc. Am., A15, 2373
[16] Sudarshan E C G 1979 Phys. Lett., 73A, 269; Physica,96A, 315
[17] Simon R, Mukunda N, and Sudarshan E C G 1988 Optics Commun.,65, 322
[18] Simon R, Sudarshan E C G, and Mukunda N 1988 Phys. Lett., A124, 223; Phys. Rev., A37, 3028
[19] Simon R, Mukunda N, and Biswadeb Dutta 1994 Phys. Rev., A49, 1567
[20] Sudarshan E C G 1963 Phys. Rev. Lett., 10, 277
[21] Simon R, Mukunda N, and Sudarshan E C G 1989 Pramana, 32, 769
[22] Simon R, Mukunda N, and Sudarshan E C G 1989 Phys. Rev. Lett., 62, 1331; J. Math. Phys., 30, 1000
[23] Simon R, Chaturvedi S, Srinivasan V, and Mukunda N 2006 Int. J. Theoret. Phys., 45, 2075
[24] Simon R and Mukunda N 1990 Phys. Lett., 138, 474; 143A,165
[25] Berry M V 1984 Proc. R. Soc. London, A392, 45
[26] Aharonov Y and Anandan J 1987 Phys. Rev. Lett., 58, 1593; Samuel J and Bhandari R 1988 Phys. Rev. Lett., 60, 2339
[27] Mukunda N and Simon R 1993 Ann. Phys. (NY), 228, 205
[28] Mukunda N and Simon R 1993 Ann. Phys. (NY), 228, 269
[29] Khanna G, Mukhopadhyay S, Simon R, and Mukunda N 1997 Ann. Phys. (NY), 253, 55; Arvind, Mallesh K S and Mukunda N 1997 J. Phys., A30, 2417
[30] Rabei E M, Arvind, Mukunda N, and Simon R 1999 Phys. Rev., A60, 3397
[31] Mukunda N, Arvind, Ercolessi E, Marmo G, Morandi G, and Simon R 2003 Phys. Rev., A67, 042114
[32] Simon R and Mukunda N 1992 J. Phys., A25, 6135
[33] Simon R and Mukunda N 1993 Phys. Rev. Lett., 70, 880