A QUESTION OF GOL’DBERG AND OSTROVSKII CONCERNING LINEAR DIFFERENTIAL EQUATIONS WITH COEFFICIENTS OF COMPLETELY REGULAR GROWTH

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Dedicated to the memory of Anatoly A. Gol’dberg and Iossif V. Ostrovskii

Abstract. We show that a linear differential equation whose coefficients are entire functions of completely regular growth may have an entire solution of finite order which is not of completely regular growth. This answers a question of Gol’dberg and Ostrovskii.

1. Introduction and results

An entire function $f$ of order $\rho$ is said to be of completely regular growth in the sense of Levin and Pfüger if there exists a $2\pi$-periodic function $h: \mathbb{R} \to \mathbb{R}$ which does not vanish identically such that

$$\log |f(re^{i\theta})| = h(\theta)r^\rho + o(r^\rho)$$

as $r \to \infty$, for $re^{i\theta}$ outside a union of disks $\{z: |z - z_j| < r_j\}$ satisfying

$$\sum_{|z_j| \leq r} r_j = o(r)$$

as $r \to \infty$. For a thorough treatment of entire functions of completely regular growth we refer to [9].

The function $h$ is called the indicator of $f$. The indicator is trigonometrically convex [9, Chapter I, Lemma 6]. Conversely, for every trigonometrically convex function there exists an entire function of completely regular growth which has this function as its indicator [9, Chapter II, Theorem 3].

One may replace the term $r^\rho$ in (1.1) by $r^{\rho(r)}$ with a proximate order $\rho(r)$. We refer to [9, Chapter I, § 12] for the definition of a proximate order.

If $n \in \mathbb{N}$ and $a_0, \ldots, a_{n-1}$ are polynomials, then every solution $f$ of the linear differential equation

$$f^{(n)} + a_{n-1}f^{(n-1)} + \ldots + a_1f' + a_0f = 0$$

is an entire function of completely regular growth. According to [14, Problem 16.2], this is due to Petrenko [10, Chapter IV]. It also follows from the work of Steinmetz ([11]; see also [12, Section 2]).

If the coefficients $a_j$ are transcendental, then the general solution of (1.2) is of infinite order [14, Satz 1]. However, even if (1.2) has a solution of finite order, this solution need
not be of completely regular growth. In fact, Gol’dberg [7, Problem 16.2] showed that every entire function is a solution of some differential equation of the form (1.2) with entire coefficients $a_0, \ldots, a_{n-1}$.

Gol’dberg and Ostrovskii ([7, Problem 16.2], [6, Question 5.5]) asked whether a transcendental entire solution $f$ of (1.2) of finite order must be of completely regular growth if the coefficients $a_0, \ldots, a_{n-1}$ are of completely regular growth. For some further discussion and results concerning this problem we refer to [11] [8] [13].

We will show that the answer to the question of Gol’dberg and Ostrovskii is negative.

**Theorem 1.1.** Let $(z_k)$ be a sequence of distinct points in $C \setminus \{0\}$ which tends to $\infty$. Suppose that the exponent of convergence $\sigma$ of $(z_k)$ is less than 1 so that

$$f(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right)$$

defines an entire function $f$ of order $\sigma$. Suppose also that there exists $C > 0$ such that

$$\left|\frac{f''(z_k)}{f'(z_k)^2}\right| \leq C$$

for all $k \in \mathbb{N}$.

Then there exist entire functions $A_0$ and $B_0$ of order at most $\sigma$ such that

$$f'' + A_0 f' + B_0 f = 0.$$ 

Moreover, given $\rho$ greater than $\sigma$ there exist entire functions $A$ and $B$ of order $\rho$ and of completely regular growth such that

$$f'' + A f' + B f = 0.$$ 

To answer the question by Gol’dberg and Ostrovskii it thus suffices to show that there exists an entire function $f$ which satisfies the hypotheses of Theorem 1.1 but which is not of completely regular growth.

One way to check that condition (1.3) is satisfied for a given function $f$ is to establish a lower bound for $|f'|$ in some disk around $z_k$. Since $f''/(f')^2 = -(1/f')'$ one can then obtain an upper bound for $|f''(z_k)/(f'(z_k)^2)|$ from Cauchy’s integral formula.

**Theorem 1.2.** Let $0 < \rho < 1$, let $(r_k)$ be an increasing sequence of positive numbers tending to infinity and let $(n_k)$ be a sequence of natural numbers satisfying $n_k \sim r_k^{\rho}$ as $k \to \infty$.

If $(r_k)$ tends to $\infty$ sufficiently fast, then

$$f(z) = \prod_{k=1}^{\infty} \left(1 - \left(\frac{z}{r_k}\right)^{n_k}\right)$$

defines an entire function $f$ of order $\rho$ and lower order $0$ which satisfies the hypotheses of Theorem 1.1.

Since the order and lower order of the function $f$ given in Theorem 1.2 are different, this function cannot be of completely regular growth. As noted above, combined with Theorem 1.1 this solves the problem posed by Gol’dberg and Ostrovskii.

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2. Proofs of Theorems 1.1 and 1.2

A standard interpolation result says that given sequences \((z_k)\) and \((w_k)\) in \(\mathbb{C}\), with \(z_k \to \infty\), there exists an entire function \(h\) satisfying \(h(z_k) = w_k\). This result was also used by Gol’dberg [7] in his proof that every entire function \(f\) satisfies an equation of the form (1.2) with entire coefficients \(a_0, \ldots, a_{n-1}\).

The standard proof of this interpolation result is as follows. Let \(f\) be an entire function with simple zeros at the points \(z_k\) and let \(g\) be a meromorphic function having simple poles at the \(z_k\) with principal part \(u_k/(z - z_k)\), where \(u_k = w_k/f'(z_k)\). Then \(h := fg\) has the required property.

We will control the growth of \(g\) and hence \(h\) using the following lemma [5, Chapter 5, Theorem 6.1]. Here and in the following we use the standard terminology of Nevanlinna theory as given in [5].

**Lemma 2.1.** Let \((z_k)\) and \((u_k)\) be sequences in \(\mathbb{C} \setminus \{0\}\), with \(z_k \to \infty\) and

\[
\sum_{k=1}^{\infty} \left| \frac{u_k}{z_k} \right| < \infty.
\]

Then

\[
g(z) = \sum_{k=1}^{\infty} \frac{u_k}{z - z_k}
\]

defines a meromorphic function \(g\) which satisfies \(m(r, g) = o(1)\) as \(r \to \infty\).

**Proof of Theorem 1.1.** Put \(u_k = -f''(z_k)/f'(z_k)^2\). Since the exponent of convergence of \((z_k)\) is less than 1 we deduce from (1.3) that (2.1) holds. Let \(g\) be defined by (2.2) and put \(A_0 = fg\). Then

\[
A_0(z_k) = u_k f'(z_k) = -\frac{f''(z_k)}{f'(z_k)}
\]

for all \(k \in \mathbb{N}\). It follows that \(f''(z_k) + A_0(z_k)f'(z_k) = 0\) for all \(k \in \mathbb{N}\) and hence that the function \(B_0\) defined by

\[
B_0 = \frac{-f'' + A_0 f'}{f}
\]

is entire. Thus (1.4) is satisfied.

Moreover, it follows from Lemma 2.1 that

\[
T(r, g) = N(r, g) + o(1) = N\left(r, \frac{1}{r}\right) + o(1) \leq T(r, f) + O(1).
\]

Thus the order of \(g\) does not exceed that of \(f\), which is equal to \(\sigma\). It now follows from the definition of \(A_0\) and \(B_0\) that their orders are also not larger than \(\sigma\).

To prove the existence of entire functions \(A\) and \(B\) with the properties stated, let \(H\) be any entire function which has completely regular growth of order \(\rho\). Since \(\rho > \sigma\), it follows from [9, Chapter III, § 4] that \(Hf\) and \(Hf'\) are of completely regular growth, with the same indicator as \(H\). We may choose \(H\) such that the indicator of \(H\) is positive.

(For example, we may choose \(H\) such that the indicator \(h\) of \(H\) satisfies \(h(\theta) \equiv c\) for some \(c > 0\), noting that the function \(h\) defined this way is trigonometrically convex.)
It then follows that \( A := A_0 + Hf \) and \( B := B_0 - Hf' \) are also of completely regular growth, again with the same indicator as \( H \). Finally, (1.4) yields that \( f, A \) and \( B \) satisfy (1.5).

**Remark.** If the indicator of \( H \) is not positive, then the functions \( A \) and \( B \) defined in the above proof need not be of completely regular growth. However, the results in [1, 2, 3] imply in particular that there exists \( c \in \mathbb{C} \setminus \{0\} \) such that \( A := A_0 + cHf \) and \( B := B_0 - cHf' \) have completely regular growth. And (1.5) is also satisfied for these functions.

**Proof of Theorem 1.2.** It is not difficult to see that if \((r_k)\) tends to \(\infty\) sufficiently fast, then (1.6) defines an entire function \( f \) of order \(\rho\) and lower order \(0\). Moreover, letting \((r_k)\) tend to \(\infty\) sufficiently fast we can achieve that if \(z \in \mathbb{C}\) with \(|z| \sim r_k\), then

\[
(2.3) \quad f(z) \sim \prod_{j=1}^{k} \left(1 - \left(\frac{z}{r_j}\right)^{n_j}\right) \sim \left(1 - \left(\frac{z}{r_k}\right)^{n_k}\right) (-1)^{k-1} \prod_{j=1}^{k-1} \left(\frac{z}{r_j}\right)^{n_j}
\]

and

\[
(2.4) \quad \frac{zf'(z)}{f(z)} = \sum_{j=1}^{\infty} n_j \left(\frac{z}{r_j}\right)^{n_j} - 1 = \sum_{j=1}^{k-1} n_j + n_k \left(\frac{z}{r_k}\right)^{n_k} - 1 + o(1)
\]

as \(k \to \infty\).

A zero \(\xi\) of \(f\) has the form \(\xi = \omega_k r_k\) where \(\omega_k^{n_k} = 1\). For \(z = \omega_k r_k (1 + \xi/n_k)\) with \(|\xi| \leq 1\) we have \((z/r_k)^{n_k} \to e^\xi\) as \(k \to \infty\). The disk \(\{\zeta: |\zeta| \leq 1\}\) corresponds to the disk \(\{z: |z - \xi| \leq r_k/n_k\}\). Since we may assume that \(\sum_{j=1}^{k-1} n_j = o(n_k)\) it follows from (2.4) that for large \(k\) the disk \(\partial D_\xi\) contains no zero of \(f'\).

We deduce from (2.3) and (2.4) that if \(z \in \partial D_\xi\), then

\[
|f'(z)| = \frac{1}{|z|} \cdot |f(z)| \cdot \left|\frac{zf'(z)}{f(z)}\right| \sim \frac{1}{r_k} \cdot \prod_{j=1}^{k-1} \left(\frac{r_j}{r_k}\right)^{n_j} \cdot n_k |e^\xi|.
\]

Hence

\[
\left|\frac{f''(\xi)}{f'(\xi)^2}\right| = \frac{1}{2\pi i} \int_{\partial D_\xi} \frac{dz}{f'(z)(z - \xi)^2} \leq \frac{n_k}{r_k} \max_{z \in \partial D_\xi} \frac{1}{|f'(z)|} \leq (1 + o(1)) e \sum_{j=1}^{k-1} \left(\frac{r_j}{r_k}\right)^{n_j} = o(1)
\]

as \(k \to \infty\). Thus \(f\) satisfies the hypothesis of Theorem 1.1. 

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