HERMITIAN AND SKEW HERMITIAN FORMS OVER LOCAL RINGS

JAMES CRUICKSHANK, RACHEL QUINLAN, AND FERNANDO SZECHTMAN

Abstract. We investigate the structure of possibly degenerate ε-hermitian forms over local rings. We prove classification theorems in the cases where the ring is complete and either the form is nondegenerate or the ring is a discrete valuation ring. In the latter case we describe a complete set of invariants for such forms based on a generalisation of the classical notion of the radical of the form.

1. Introduction

We are concerned with the classification problem of possibly degenerate hermitian or skew hermitian bilinear forms over a local ring in which 2 is a unit. Symmetric and skew symmetric forms are included as a special case, as we allow the underlying involution to be trivial.

In foundational papers of Durfee ([7]) and O’Meara ([14]) the integral theory of quadratic forms over local fields is developed. This is equivalent to the theory of symmetric bilinear forms over a complete local principal ideal domain with finite residue field. Jacobowitz ([8]) extended this integral theory to the case of hermitian forms over local fields equipped with an involution.

More recently, Levchuk and Starikova ([12]) have proved the existence and uniqueness of normal forms for symmetric matrices over local principal ideal domains under certain assumptions on the unit group of the ring.

In a wider setting Bayer-Fluckiger and Fainsilber ([1]) have considered the general problem of equivalence of hermitian or skew hermitian forms over arbitrary rings and are able to prove some quite general reduction theorems for this problem. A broad study of sesquilinear forms and their connection to hermitian forms was carried out by Bayer-Fluckiger, First and Moldovan ([2]) as well as by Bayer-Fluckiger and Moldovan ([3]).

For general information on quadratic and hermitian forms over rings we refer the reader to the textbook of Knus ([9]).

1.1. Outline of the paper. Given the extensive existing literature on this topic and the fact that we aim to provide a unified and self contained exposition, some overlap with previously known results is inevitable. In this section we summarise the results of the paper, emphasising the contributions that we have made.

Throughout, A is a commutative local ring with maximal ideal r and * is an involution of A, that is to say an automorphism of order at most two. Clearly, * induces an involution

2010 Mathematics Subject Classification. 11E08, 11E39, 15A63.
Key words and phrases. local ring, discrete valuation ring, hermitian, skew hermitian.
on the residue field $A/\mathfrak{r}$. If this induced involution is trivial, we say that $\ast$ is ramified, otherwise it is unramified - this distinction will play a decisive role in the sequel. Thus broadly speaking there are four types of forms to consider, ramified hermitian, ramified skew hermitian, unramified hermitian and unramified skew hermitian. However, the last two can be merged into one case since they differ only up to multiplication by a unit.

In Section 2 we present some elementary results on nondegenerate forms of rank one or two in various cases. These essentially form the building blocks for our later classification theorems. In light of this we include all the details so as to make the paper self contained.

In Section 3 we introduce completeness and use it to derive the existence of basis vectors with certain specified properties. A key new result here is Lemma which guarantees the existence of a symplectic pair under appropriate conditions. We also give an example to show that this lemma can fail in the absence of completeness.

In Section 4 we use the results of the previous sections to analyse nondegenerate forms of any rank over arbitrary local rings. In this case there is an associated vector space and form over the residue field and in the case that the ring is also complete we show that equivalence of forms over the local ring reduces to equivalence of these associated forms over the residue field. In particular we demonstrate the existence of a symplectic basis for a nondegenerate skew hermitian form over a complete local ring with ramified involution - see Proposition 1.

The remaining sections of the paper deal with possibly degenerate forms. In this case it is not true without further assumptions that such forms have a nice - in a sense precisely defined in Section 5 - decomposition. However in the case of a discrete valuation ring, that is to say a local principal ideal domain, we are able to prove that any form has such a nice decomposition - this is the content of Theorem 3.

Following this we introduce a generalisation of the classical notion of the radical of a form. In our case we have one generalised radical module for each nonnegative integer. Using these generalised radicals we are able to show that the equivalence problem for a pair of forms over a complete discrete valuation ring can be reduced to the equivalence of a series of pairs of nondegenerate forms over the associated residue field - see Theorem 4. This is a broad generalisation of Theorem D of [4] which dealt only with the special case of symmetric matrices.

Under some additional hypotheses, these spaces essentially turn out to give rise to a complete set of invariants for hermitian and skew hermitian forms over a complete discrete valuation ring. This can be found in Theorem 6.

Also under these hypotheses, we are able to prove, in Theorems 7 and 8 the existence and uniqueness of normal forms in all possible cases. As a consequence of these normal forms we also prove that the congruence class of a hermitian or skew hermitian matrix, over a complete local principal ideal domain satisfying the aforementioned additional hypotheses, depends only on the invariant factors of the matrix - this is stated in Theorem 8. Results analogous to this last theorem have been obtained elsewhere, for polynomial rings over algebraically closed fields see Theorem 4.5 of [6], and for skew symmetric matrices over principal ideal domains see Exercise 4 of Chapter XIV of [11].
1.2. Terminology and notation. We shall (sometimes) use the shorthand \( \overline{a} \) for the residue class \( a + \mathfrak{r} \). Moreover, we extend this shorthand to matrices in the obvious way. We write \( U(A) \) for the multiplicative group of units of \( A \) and recall the fundamental fact that, since \( A \) is local, \( \mathfrak{r} = A - U(A) \).

Let \( R \) be the subring of \( A \) consisting of elements fixed by \( * \). Observe that \( R \) is also a local ring, with maximal ideal \( \mathfrak{m} = R \cap \mathfrak{r} \). Let \( S = \{ a \in A : a^* = -a \} \). Since 2 is assumed to be a unit of \( A \) we have \( A = R \oplus S \). If \( S \subset \mathfrak{r} \) we say that \( * \) is ramified. Otherwise we say that it is unramified. Since \( \mathfrak{r} \) is \( * \)-invariant, \( * \) induces an involution on the residue field \( A/\mathfrak{r} \), which we also denote by \( * \). As indicated in the introduction, \( * \) is ramified if and only if the induced involution on the residue field is trivial.

We write \( M_m(A) \) for the set of \( m \times m \) matrices over \( A \). The group of invertible \( m \times m \) matrices is denoted by \( GL_m(A) \).

Let \( V \) be a free \( A \)-module of rank \( m > 0 \) and let \( h : V \times V \to A \) be an \( \varepsilon \)-hermitian form, where \( \varepsilon = \pm 1 \). That is to say \( h \) is bi-additive, \( A \)-linear in the second variable and \( h(u, v) = \varepsilon h(u, v)^* \) for all \( u, v \in V \).

The Gram matrix of a list of vectors \( v_1, \ldots, v_k \) with respect to a form \( h \) is the \( k \times k \) matrix whose \((i, j)\)-entry is \( h(v_i, v_j) \). The form \( h \) is said to be nondegenerate if the Gram matrix of any basis of \( V \) belongs to \( GL_m(A) \).

A matrix \( C \in M_m(A) \) is said to be \( \varepsilon \)-hermitian if \( C'^* = \varepsilon C \) where \( C' \) denotes the transpose of \( A \). Of course \( h \) is \( \varepsilon \)-hermitian if and only if \( G \) is \( \varepsilon \)-hermitian, where \( G \) is the Gram matrix of any basis. On the other hand, given an \( \varepsilon \)-hermitian \( C \in M_m(A) \), we can construct an \( \varepsilon \)-hermitian form \( h_C \) on \( V = A^m \), by \( h_C(u, v) = u'^* Cv \).

Matrices \( C, D \in M_m(A) \) are said to be \( * \)-congruent if there exists \( X \in GL_m(A) \) such that \( D = X'^* CX \). Given an \( \varepsilon \)-hermitian form \( h \), then the Gram matrices of any two bases are \( * \)-congruent. On the other hand we say that forms \( h_1 \) on \( V_1 \), respectively \( h_2 \) on \( V_2 \) are equivalent if there is an \( A \)-isomorphism \( \varphi : V_1 \to V_2 \) such that \( h_1(u, v) = h_2(\varphi(u), \varphi(v)) \). Now, \( h_C \) and \( h_D \) as above are equivalent if and only \( C \) and \( D \) are \( * \)-congruent.

2. Small nondegenerate submodules

In contrast to the field case, a linearly independent set is not necessarily a subset of a basis - this is apparent even in the rank one case. In the presence of the form \( h \), however, we have some sufficient conditions.

Given an \( A \)-submodule \( W \) of \( V \), let \( W^\perp = \{ v \in V : h(w, v) = 0 \} \). Then \( W^\perp \) is an \( A \)-submodule of \( V \).

**Lemma 1.** Let \( u \in V \) and suppose that \( h(u, u) \in U(A) \). Then \( V = Au \oplus Au^\perp \). Moreover, both \( Au \) and \( Au^\perp \) are free \( A \)-modules.

**Proof.** Let \( v_1, \ldots, v_m \) be a basis of \( V \). So \( u = \sum c_i v_i \). Now \( h(u, u) = \sum c_i h(u, v_i) \in U(A) \).

Since \( A \) is local this implies that \( c_j \in U(A) \) for some \( j \). Without loss of generality assume that \( c_1 \in U(A) \). Therefore \( u, v_2, \ldots, v_m \) is a basis of \( V \). Now, for \( i = 2, \ldots, m \), let \( w_i = v_i - h(u, v_i)h(u, u)^{-1}u \). Clearly \( u, w_2, \ldots, w_m \) is a basis of \( V \) and \( w_2, \ldots, w_m \) is a basis of \( Au^\perp \). \( \Box \)
In the case that * is ramified and \( h \) is skew hermitian, it is immediate that \( h(u, u) \in \mathfrak{r} \) for all \( u \in V \). So we cannot have any nondegenerate rank one submodules in that case. However we have the following lemma concerning rank two submodules.

**Lemma 2.** Suppose that * is ramified and that \( h \) is skew hermitian. If \( u, v \in V \) satisfy \( h(u, v) \in U(A) \), then \( V = (Au + Av) \oplus (Au + Av)^\perp \). Moreover both summands in this decomposition are free \( A \)-modules.

**Proof.** Let \( v_1, \ldots, v_m \) be a basis of \( V \). So \( u = \sum_i a_i v_i \). Suppose that \( a_i \in \mathfrak{r} \) for all \( i \). Then clearly \( h(u, v) \in \mathfrak{r} \), contradicting our assumption. So without loss of generality we may assume that \( a_1 \in U(A) \). It follows that \( u, v, v_2, \ldots, v_n \) is a basis of \( V \). Write \( v = b_1 u + \sum_{j=2}^m b_j v_j \). If \( b_2, \ldots, b_n \in \mathfrak{r} \), then since \( h(u, u) \in \mathfrak{r} \) it would follow that \( h(u, v) \in \mathfrak{r} \), contradicting our hypothesis. Without loss of generality we may assume that \( b_2 \in U(A) \) and it follows that \( u, v, v_2, \ldots, v_m \) is a basis of \( V \). Since the Gram matrix of \( u \) is invertible in this case it follows that, for \( i = 3, \ldots, m \), there exist (unique) \( a_i, b_i \in A \) such that \( h(u, a_i u + b_i v) = h(u, v_i) \) and \( h(v, a_i u + b_i v) = h(v, v_i) \). Let \( w_i = v_i - (a_i u + b_i v) \). Then \( u, v, w_3, \ldots, w_m \) is a basis of \( V \) and \( u, v, w_3, \ldots, w_3 \) is a basis of \( (Au + Av)^\perp \) as required. \( \square \)

Now we show, excepting the ramified skew hermitian case, that if \( h \) has any unit value, then there is some element \( w \in V \) such that \( h(w, w) \) is a unit.

**Lemma 3.** Let \( u, v \in V \). Suppose that \( h(u, v) \in U(A) \) and that either \( h \) is hermitian or that * is unramified. Then there is some \( w \in V \) such that \( h(w, w) \in U(A) \).

**Proof.** Replacing \( v \) by \( h(u, v)^{-1} v \) if necessary, we can assume without loss of generality that \( h(u, v) = 1 \). Now suppose that \( h \) is hermitian. Then

\[
2 = h(u, v) + h(v, u) = h(u + v, u + v) - h(u, u) - h(v, v).
\]

Now since \( 2 \in U(A) \) and \( A \) is local, we conclude that at least one of \( h(u, u) \), \( h(v, v) \) or \( h(u + v, u + v) \) belongs to \( U(A) \). Finally in the case where \( h \) is skew hermitian and * is unramified, choose \( b \in U(A) \cap S \). Now observe that \( bh \) is hermitian and that \( bh(u, v) \in U(A) \) if and only if \( h(u, v) \in U(A) \). \( \square \)

In summary the results of this section show that if \( h \) has any unit value then it is possible to break off a nondegenerate submodule of rank one except in the ramified skew hermitian case. In the latter case, no nondegenerate rank one submodules can possibly exist, but it is possible to break off a nondegenerate submodule of rank two.

### 3. Complete local rings

Having given conditions sufficient to ensure the existence of nondegenerate rank one and two submodules, in this section we introduce some natural conditions on the ring that will guarantee the existence of unit length basis vectors, or in the ramified skew hermitian case, the existence of a symplectic pair. Recall that a symplectic pair is an ordered pair of vectors \( (u, v) \in V^2 \) satisfying \( h(u, u) = h(v, v) = 0 \) and \( h(u, v) = -h(v, u) = 1 \). We also recall that a symplectic basis is a basis \( v_1, w_1, v_2, w_2, \ldots, v_l, w_l \) such that each pair \( (v_i, w_i) \)
is a symplectic pair and such that \( h(v_i, v_j) = h(w_i, w_j) = h(v_i, w_j) = h(w_j, v_i) = 0 \) for \( i \neq j \).

Observe that \( h(au, au) = a^*ah(u, u) \), so it is natural to investigate the image of the so-called norm map, \( N : a \mapsto a^*a \). To this end, we follow Cohen (3) and say that \( A \) is \textit{complete} if \( \bigcap_{n=1}^{\infty} r^n = 0 \) and \( A \) is metrically complete with respect to its \( r \)-adic metric. We observe that in the present context, where we seek solutions to certain quadratic equations, it is natural to restrict our attention to complete rings.

**Lemma 4.** Suppose that \( A \) is complete. Suppose that \( b \in U(R) \) and that \( a^*a \equiv b \mod r \) for some \( a \in A \). Then there is some \( c \in A \) such that \( c^*c = b \).

**Proof.** First we observe that since \( A \) is complete, \( R \) is also complete. Since \( 2 \in R \), it follows from Hensel’s Lemma (see [4], Theorem 4) that the squaring map from \( 1 + m \) to \( 1 + m \) is a surjection. Now \( a \in U(A) \) since \( b \in U(A) \), therefore \( b(a^*a)^{-1} \in 1 + m \). Hence there is some \( \delta \in m \) such that \( b(a^*a)^{-1} = (1 + \delta)^2 \). So \( b = (a(1 + \delta))^*a(1 + \delta) \).

**Corollary 1.** Suppose that \( A \) is complete and that \( u \in V \), \( b \in U(A) \) and \( b^* = \varepsilon b \). If \( h(u, u) \equiv b \mod r \) then there is some \( w \in Au \) such that \( h(w, w) = b \).

**Proof.** Note that \( h(u, u)^{-1}b \in 1 + m \). By Lemma 4 there is some \( c \in A \) such that \( c^*c = h(u, u)^{-1}b \). Let \( w = cu \). □

Now we turn to the nondegenerate two dimensional submodules in the case where \( * \) is ramified and \( h \) is skew hermitian.

**Lemma 5.** Suppose that \( A \) is complete and that \( * \) is ramified. Suppose also that \( h \) is skew hermitian and that \( h(u, v) \in U(A) \). Then there is some symplectic pair \( (u', v') \in V^2 \) such that \( Au' + Av' = Au + Av \).

**Proof.** By replacing \( v \) by \( h(u, v)^{-1}v \) we can assume that \( h(u, v) = 1 \). Now observe that for \( b \in A \), \( h(u + bv, u + bv) = 0 \) if and only if

\[
(1) \\
b^*bh(v, v) + (b - b^*) + h(u, u) = 0
\]

Since both \( h(v, v) \) and \( h(u, u) \) belong to \( S \) we can apply Lemma 4 (below) to conclude that Equation (1) has a solution in \( r \). So assume that \( b \in r \) satisfies (1) and let \( u' = u + bv \). Clearly, \( Au + Av = Au' + Av \). Moreover, \( h(u', v) = 1 + b^*h(v, v) \in U(A) \). Now, let \( v' = h(u', v)^{-1}(v + \frac{1}{2}h(v, v)(h(u', v)^*)^{-1}u') \). A straightforward calculation, using \( h(u', u') = 0 \), demonstrates that \( (u', v') \) is a symplectic pair. Moreover, since \( h(u', v) \) is a unit, it is clear that \( Au' + Av' = Au' + Av = Au + Av \). □

To complete the proof of Lemma 5 we need the following.

**Lemma 6.** Suppose that \( A \) is complete and that \( * \) is ramified. Given \( \alpha, \beta, \gamma \in A \) satisfying \( \alpha, \gamma \in S \) and \( \beta \in U(A) \), there is some \( t \in A\gamma \) such that

\[
\alpha t^*t + \beta t - t^*\beta^* + \gamma = 0.
\]
Proof. We define sequences \((\beta_i)\) and \((\gamma_i)\) as follows. Let \(\gamma_1 = \gamma\) and \(\beta_1 = \beta\). Given \(\beta_i\) and \(\gamma_i\), let

\[
\gamma_{i+1} = -\frac{\alpha \gamma_i^2}{4\beta_i^* \beta_i}, \quad \beta_{i+1} = \beta_i + \frac{\alpha \gamma_i}{2\beta_i^*}.
\]

Observe that \(\beta_{i+1}\) is a unit since \(\beta_{i+1} \equiv \beta_i \mod r\). Also \(\gamma_{i+1}\) is skew hermitian, since \(\alpha\) and \(\gamma_i\) (inductively) are both skew hermitian. Now we check that \(-\frac{1}{2} \sum_{i=1}^{\infty} \gamma_i/\beta_i\) is the required solution. Define \(f_i(t) = \alpha t^* t + \beta_i t - t^* \beta_i^* + \gamma_i\). Using the fact that \(\gamma_i\) is skew hermitian, an easy calculation shows that \(f_i(t - \gamma_i/2\beta_i) = f_{i+1}(t)\). Therefore \(f_{k+1}(0) = f_k(-\gamma_k/2\beta_k) = f_{k-1}(-\gamma_k/2\beta_k - \gamma_{k-1}/2\beta_{k-1}) = \cdots = f_1(-\frac{1}{2} \sum_{i=1}^{k} \gamma_i/\beta_i)\). Now \(f_{k+1}(0) = \gamma_k\). Since \(\gamma_1 \in r\), it is clear that \(\gamma_k \in r^k\) for all \(k\). Therefore \(f_1(-\frac{1}{2} \sum_{i=1}^{k} \gamma_i/\beta_i) \in r^k\) as required.

The following example shows that if \(A\) is not complete it may be possible to find a nondegenerate rank two skew hermitian submodule that does not have a symplectic basis.

**Example 1.** Fix an odd prime \(p\) such that \(p + 1\) is not the sum of two squares (e.g. \(p = 5\)) and let \(A\) be the extension of \(R = \mathbb{Z}(\sqrt{-p})\) obtained by adjoining a square root of \(p\). So \(A = R \oplus R\sqrt{p}\) is a local principal ideal domain with maximal ideal \(A\sqrt{p}\). Let \(*\) be the involution of \(A\) that fixes elements of \(R\) and maps \(\sqrt{p}\) to \(-\sqrt{p}\) and consider the nondegenerate skew hermitian form \(h_M\) where \(M = \begin{pmatrix} \sqrt{p} & 1 \\ -1 & \sqrt{p} \end{pmatrix}\). We claim that there is no isotropic basis vector of \(A^2\). Suppose that \(\begin{pmatrix} 1 \\ a + b\sqrt{p} \end{pmatrix}\) was such an vector (a similar argument applies to the case \(\begin{pmatrix} a + b\sqrt{p} \\ 1 \end{pmatrix}\)). A straightforward calculation shows that we must have \(pb^2 - 2b - 1 = a^2\). Writing \(b = \frac{c}{d}\) for integers \(c\) and \(d\), we see that \(pc^2 - 2cd - d^2\) must be the square of an integer. But \(pc^2 - 2cd - d^2 = (p + 1)c^2 - (d + c)^2\), so \((p + 1)c^2\) is the sum of two squares. By assumption, \(p + 1\) is not the sum of two squares, so neither is \((p + 1)c^2\) for any integer \(c\).

4. Nondegenerate forms

We set \(V(-1) = 0\) and

\[
V(i) = \{v \in V : h(V, v) \subset r^i\}, \quad i \geq 0.
\]

Observe that \(V(i)\) is an \(A\)-submodule of \(V\), \(V(i + 1) \subset V(i)\) and \(rV(i - 1) \subset V(i)\) for all \(i \geq 0\). We note in passing that, if \(\bigcap_{i \geq 0} r^i = 0\) then \(\bigcap_{i \geq 0} V(i) = \{v \in V : h(v, V) = 0\}\) is the radical of \(h\). Now, since \(rV(i) \subset V(i + 1)\), we see that

\[
W(i) = V(i)/(rV(i - 1) + V(i + 1))
\]

inherits a \(A/r\)-vector space structure. The form \(h\) induces a map

\[
h_i : W(i) \times W(i) \to r^i/r^{i+1}.
\]
In the special case \(i = 0\), we see that \(h_0\) is an \(\varepsilon\)-hermitian form \(V/V(1) \times V/V(1) \to A/\mathfrak{r}\). Observe that \(h\) is nondegenerate if and only if \(h_0\) is nondegenerate.

**Lemma 7.** Suppose that \(h\) is nondegenerate. Then \(V(i) = \mathfrak{r}^iV\) for all \(i \geq 0\).

**Proof.** The inclusion \(\mathfrak{r}^iV \subset V(i)\) is clear. For the reverse inclusion, let \(v \in V(i)\) and let \(\{v_1, \ldots, v_m\}\) be a basis of \(V\), so that \(v = a_1v_1 + \cdots + a_mv_m\) for some \(a_i \in A\). Since \(h\) is nondegenerate, given any \(1 \leq j \leq m\), there is \(u \in V\) such that \(h(u, v_j) = 1\) and \(h(u, v_k) = 0\) for all \(k \neq j\). It follows that \(h(u, v) = a_j\), whence \(a_j \in \mathfrak{r}^i\) and a fortiori \(v \in \mathfrak{r}^iV\). \(\square\)

**Theorem 1.** Suppose that \(h\) is nondegenerate.

1. If \(*\) is unramified or if \(h\) is hermitian then \(V\) has a basis whose Gram matrix is diagonal.
2. If \(*\) is ramified and \(h\) is skew hermitian then \(V\) has a basis whose Gram matrix is the direct sum of \(m/2\) matrices, each of which is an invertible \(2 \times 2\) skew hermitian matrix.

**Proof.** Since \(h\) is nondegenerate the Gram matrix of a basis is invertible. Since \(A\) is local some entry of this Gram matrix must be a unit. Therefore there exist vectors \(u, v\) such that \(h(u, v) \in U(A)\). The results of Section 2 show that in the ramified skew hermitian case there is a nondegenerate rank two submodule of \(V\) and that in all other cases there is a nondegenerate rank one submodule. Moreover, if \(U\) is this rank one or two submodule then by Lemma 4 or Lemma 5 \(V = U \oplus U^\perp\) and \(U^\perp\) is a free submodule of \(V\). Now, it is clear that \(h|_{U^\perp}\) is also nondegenerate. The required conclusions follow by induction on the rank of \(V\). \(\square\)

**Theorem 2.** Suppose that \(A\) is complete and let \(h\) and \(h'\) be nondegenerate \(\varepsilon\)-hermitian forms on \(V\). Then \(h\) and \(h'\) are equivalent if and only if \(h_0\) and \(h'_0\) are equivalent. In particular, if \(*\) is ramified and \(h\) and \(h'\) are skew hermitian, then \(h\) and \(h'\) are equivalent.

**Theorem 2 (Matrix version).** Suppose that \(A\) is complete and let \(C, D \in \text{GL}_m(A)\). Then \(C\) and \(D\) are \(*\)-congruent if and only if \(\overline{C}\) and \(\overline{D}\) are \(*\)-congruent over \(A/\mathfrak{r}\). In particular, if \(*\) is ramified and \(C\) and \(D\) are skew hermitian, then \(C\) and \(D\) are \(*\)-congruent.

**Proof.** The only if direction is obvious. For the other direction suppose that \(\overline{C}\) and \(\overline{D}\) are \(*\)-congruent. So there is some matrix \(X \in M_m(A)\) such that \(\overline{X} \in \text{GL}_m(A/\mathfrak{r})\) and \(X^*CX \equiv D \mod \mathfrak{r}\). Since \(A\) is local, \(X \in \text{GL}_m(A)\) and \(C\) is \(*\)-congruent to \(X^*CX\). So replacing \(C\) by \(X^*CX\) we can, without loss of generality, reduce to the case that \(C \equiv D \mod \mathfrak{r}\). Equivalently, we may assume without loss of generality that \(h(u, v) \equiv h'(u, v) \mod \mathfrak{r}\) for all \(u, v \in V\).

Now we deal with the case where \(*\) is ramified and \(h\) is skew hermitian. In this case, Lemma 3 implies that \(V\) has symplectic bases relative to \(h\) and \(h'\), whence they are equivalent.

In all other cases, Lemmas 4 and 5 ensure the existence of a basis \(v_1, \ldots, v_m\) of \(V\) whose corresponding Gram matrix relative to \(h\), say \(M\), is diagonal with only units on the diagonal. By Corollary 4 there is some \(w_1 \in Av_1\) such that \(h'(w_1, w_1) = h(v_1, v_1)\).
Moreover, it is clear that, for $j \geq 2$, $h'(w_1, v_j) \equiv h'(v_1, v_j) \equiv h(v_1, v_j) \equiv 0 \mod r$. For $j \geq 2$, let $w_j = v_j - h'(w_1, v_j)h'(w_1, w_1)^{-1}w_1$. Now $w_1, \ldots, w_m$ is a basis of $V$ whose Gram matrix with respect to $h'$ is of the form

$$
\begin{pmatrix}
M_{11} & 0 \\
0 & N
\end{pmatrix}.
$$

Moreover $N \equiv \text{diag}(M_{22}, \ldots, M_{mm}) \mod r$. Inductively, we may assume that there is some $X \in GL_{m-1}(A)$ such that $X^*NX = \text{diag}(M_{22}, \ldots, M_{mm})$. It follows that there is a basis of $V$ whose Gram matrix with respect to $h'$ is equal to $M$. Note that Corollary 1 provides the base case of the induction.

We list some noteworthy corollaries and special cases of Theorem 2.

**Proposition 1.** If $A$ is complete, $*$ is ramified and $h$ is skew hermitian and nondegenerate, then $V$ has a symplectic basis.

We have a canonical imbedding $R/m \to A/r$ and we will view $R/m$ as a subfield of $A/r$ by means of this imbedding. If $*$ is ramified then $R/m = A/r$. Suppose $*$ is unramified. Then $R/m$ is the fixed field of an automorphism of $A/r$ of order 2. If $A/r$ is quadratically closed (in the sense that it has no extensions of degree 2), then by the Diller-Dress theorem (see [10], p. 235), $R/m$ is a Euclidean field (this is an ordered field wherein every nonnegative element is a square).

**Proposition 2.** Suppose that $A$ is complete and that $A/r$ is quadratically closed. If $*$ is unramified and $h$ is a nondegenerate hermitian form then $V$ has a basis whose Gram matrix is diagonal and such that all the diagonal entries are $\pm 1$. Moreover, given any two such bases, the signatures of the corresponding Gram matrices are the same.

*Proof.* By Theorem 2 it suffices to prove this result in the case that $A$ is a field (i.e. $r = 0$). By our above remarks, $A$ is the quadratic extension of a Euclidean field. The result in this case goes exactly as in a classical case when $A = \mathbb{C}$.

Similarly we have

**Proposition 3.** Suppose that $A$ is complete and that $A/r$ is quadratically closed. If $*$ is unramified and $h$ is a nondegenerate skew hermitian form then, given $b \in U(A) \cap S$, $V$ has a basis whose Gram matrix is diagonal and such that all the diagonal entries are $\pm b$. Moreover, given any two such bases, the number of occurrences of $b$ are the same in each of the corresponding Gram matrices.

*Proof.* Apply the previous proposition to the hermitian form $b^{-1}h$.

**Proposition 4.** Suppose that $A$ is complete and that $A/r$ is quadratically closed. If $*$ is ramified and $h$ is a nondegenerate hermitian form then $V$ has an orthonormal basis.

*Proof.* This follows by combining Theorem 2 with the classical result that every nondegenerate symmetric form over a quadratically closed field of characteristic not 2 has an orthonormal basis.
5. Discrete Valuation Rings

For the remainder of the paper $A$ will be a discrete valuation ring, that is, a local principal ideal domain. A uniformiser of $A$ is a generator of the maximal ideal.

**Lemma 8.** If $*$ is a nontrivial involution of $A$ then $A$ has a uniformiser $y$ such that $y^* = -y$.

**Proof.** Suppose that $r = Az$. Now $z = \frac{1}{2}(z + z^*) + \frac{1}{2}(z - z^*)$ and since $A$ is local, at least one of $\frac{1}{2}(z + z^*)$ or $\frac{1}{2}(z - z^*)$ must lie in $r - r^2$. So we can certainly choose some $w$ such that $w^* = \pm w$ and $r = Aw$. Suppose that $w^* = w$. Since $*$ is nontrivial there is some nonzero $u \in A$ such that $u^* = -u$. Now $u = bw^k$ for some $b \in U(A)$ and since $w^* = w$, we conclude that $b^* = -b$. Now $y = bw$ is the required generator of $r$. \hfill $\square$

Following the previous lemma we fix $y$ so that $Ay = r$ and so that $y^* = -y$ in the case that $*$ is a nontrivial involution. We agree that $y^\infty = 0$.

Following O’Meara [14] we say that an $\varepsilon$-hermitian matrix $M \in M_m(A)$ has an O’Meara decomposition if

$$M = \bigoplus_{i=0}^s a_i M_i,$$

where each $a_i \in A$ and every $M_i$ is an invertible hermitian or skew hermitian matrix.

Our next result is an extension of Theorem 1 to arbitrary forms, possibly degenerate, and establishes the existence of an O’Meara decomposition for the Gram matrices of such forms.

**Theorem 3.**

1. If $*$ is unramified then $V$ has a basis whose Gram matrix is diagonal.
2. If $*$ is ramified then $V$ has a basis whose Gram matrix is the direct sum of a diagonal matrix (possibly of size zero) and a number (possibly zero) of $2 \times 2$ blocks each of the form $y^d B$ where $B$ is an invertible skew hermitian $2 \times 2$ matrix.

**Proof.** We prove this by induction on the rank of $V$. If $h$ is identically zero then the theorem is true. Now suppose that $h$ is not identically zero. Let $d = \min\{j : h(V, V) \subset r^j\}$. Since $A$ is a domain there is a unique form $h'$ on $V$ such that $h = y^d h'$. Clearly $h'$ is $\varepsilon$-hermitian if $y^* = y$ and is $(-1)^d \varepsilon$-hermitian if $y^* = -y$. Moreover, by construction, there exist $u, v \in V$ such that $h'(u, v) \in U(A)$.

If $h'$ is skew hermitian and $*$ is ramified then, by Lemma 2 there is some nondegenerate rank two submodule $U$ such that $V = U \perp U^\perp$. If $h'$ is hermitian or if $*$ is unramified then, by Lemma 1 and Lemma 3 there is some nondegenerate rank one submodule $U$ such that $V = U \perp U^\perp$.

In either case $U$ has an O’Meara decomposition of the required form with respect to $h'$ and by induction $U^\perp$ has an O’Meara decomposition of the required form with respect to $h'$. Together these yield the required decomposition for $h$. \hfill $\square$

Next we consider the structure of the space $W(i)$ in more detail. By Theorem 3 we have

$$V = U_0 \perp U_1 \perp U_2 \perp \cdots \perp U_\infty$$
where $U_i = 0$ for all but finitely many $i$ and, for $U_i \neq 0$, the Gram matrix of a basis, say $B_i$, of $U_i$ is equal to $y^i M_i$ with $M_i$ invertible. Thus, if $B$ is the union of all $B_i$ then $B$ is a basis of $V$ with Gram matrix

$$M = M_0 \oplus y^1 M_1 \oplus y^2 M_2 \oplus \cdots \oplus y^n M_\infty,$$

where almost all summands have size 0.

**Lemma 9.** For every nonnegative integer $i$, we have

$$V(i) = r^i U_0 \perp r^{i-1} U_1 \perp r^{i-2} U_2 \perp \cdots \perp U_i \perp U_{i+1} \perp U_{i+2} \perp \cdots \perp U_\infty.$$

**Proof.** The operation of passing from $V$ to $V(i)$ is compatible with orthogonal decompositions, so the result follows immediately from Lemma 7 and the decomposition (2). □

**Corollary 2.** For every nonnegative integer $i$, we have $\dim_{A/\tau} W(i) = \rank_{A/\tau} U_i$.

Making use of the uniformiser $y$ for $A$, we may alter the above map $W(i) \times W(i) \rightarrow r^i/\tau^{i+1}$ and define an $A/\tau$-valued form $h_i : W(i) \times W(i) \rightarrow A/\tau$ by

$$h_i(u + rV(i-1) + V(i+1), v + rV(i-1) + V(i+1)) = y^{-i} h(u, v) + r, \quad u, v \in V(i).$$

Note that $h_i$ is $\epsilon$-hermitian if $*$ is trivial and is $\epsilon(-1)^i$-hermitian if $*$ is nontrivial.

**Corollary 3.** For every nonnegative integer $i$, the form $h_i$ is nondegenerate. In fact, the Gram matrix of the basis $\overline{B}_i$ of $W(i)$ relative to $h_i$ is $\overline{M}_i$, which is invertible or has size 0.

It is clear that if $h$ and $h'$ are equivalent forms on $V$, then $h_i$ and $h'_i$ are equivalent over $A/\tau$ for $i \geq 0$. The converse is, of course, not necessarily true (see Example 1).

**Theorem 4.** Suppose that $A$ is complete and that $h$ and $h'$ are $\epsilon$-hermitian forms on $V$. If, for each nonnegative integer $i$, the forms $h_i$ and $h'_i$ are equivalent over $A/\tau$ then $h$ and $h'$ are equivalent.

**Proof.** This is an immediate consequence of Theorem 2 (in its matrix version), Corollary 3 and the decomposition (3). □

We next obtain the following classification theorem for $*$-congruence classes of $\epsilon$-hermitian matrices

**Theorem 5.** Assume that $A$ is complete.

1. Suppose that $M \in M_m(A)$ is $\epsilon$-hermitian. Then $M$ is $*$-congruent to a matrix of the form $\bigoplus_{i=0}^\infty y^i M_i$, where for each $M_i$ is either of size zero or is an invertible matrix. Moreover, for $0 \leq i < \infty$,
   (a) if $*$ is unramified then $M_i$ is a diagonal matrix.
   (b) if $*$ is ramified and $(y^i)^* = \epsilon y^i$ then $M_i$ is diagonal. Whereas if $*$ is ramified and $(y^i)^* = -\epsilon y^i$ then $M_i$ is a direct sum of copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

2. Given $\epsilon$-hermitian matrices $M = \bigoplus_{i=0}^\infty y^i M_i$ and $N = \bigoplus_{i=0}^\infty y^i N_i$ in $M_m(A)$ such that each $M_i$ (resp. $N_i$) is either of size zero or is an invertible matrix, then
(a) if * is unramified, $M$ and $N$ are *-congruent over $A$ if and only if for each $0 \leq i < \infty$, $M_i$ is *-congruent to $N_i$ over $A/\mathfrak{m}$.

(b) if * is ramified, $M$ and $N$ are *-congruent over $A$ if and only if for each $0 \leq i < \infty$ such that $(y)^* = \varepsilon y^i$, $M_i$ is *-congruent to $N_i$ over $A/\mathfrak{m}$, and for each $0 \leq i < \infty$ such that $(y)^* = -\varepsilon y^i$, the size of $M_i$ is equal to the size of $N_i$.

Proof. This follows immediately from Theorem 2, Theorem 3 and Lemma 5.

Thus, for matrices over a complete discrete valuation ring with residue field characteristic not 2, the *-congruence problem essentially reduces to the *-congruence problem over its residue field.

Example 2. (cf. Example 1.1 of [12]) Given any field $F$, consider the complete, local, nonprincipal domain $A = F[[X,Y]]$ and the *-hermitian (with trivial *) matrix

$$M = \begin{pmatrix} 0 & X \\ X & Y \end{pmatrix}.$$ 

We claim that $M$ has no O’Meara decomposition. Suppose, to the contrary, that it does. Since $M$ is not invertible, then $M$ must be *-congruent to a diagonal matrix $D = \text{diag}(a,b)$. Thus $D = T'MT$ for some $T \in GL_2(A)$. Taking determinants yields $ab = -t^2X^2$, where $t \in U(A)$. Since $(X)$ is a prime ideal, it follows that $a \in (X)$ or $b \in (X)$. Suppose, without loss of generality, that $a = cX$ for some $c \in A$. Cancelling, we obtain $cb = -t^2X$. If $b \notin (X)$ then a repetition of the preceding argument yields that $b \in U(A)$, against the fact that the hermitian form $h = h_M$ takes no unit values. We are thus forced to conclude that $b = dX$ for some $d \in A$, which implies that $c, d \in U(A)$. Now, since $M$ is congruent to $D$, there is $u \in V = A^2$ such that $h(u,u) = cX$. Let $e_1, e_2$ be the canonical basis of $V$. Then $u = fe_1 + ge_2$ for some $f, g \in A$, whence $2fgX + g^2Y = cX$, that is, $g^2Y = (c - 2fg)X$. We infer that $g \in (X)$ and a fortiori $c - 2fg \in U(A)$. Thus $(c - 2fg)X \in (Y)$ but neither $(c - 2fg)$ nor $X$ are in $(Y)$, which contradicts the fact that $(Y)$ is a prime ideal.

The following proposition is not needed for the sequel. However, we include it to shed light on the structure of the rings under consideration.

Proposition 5. Suppose * is a nontrivial involution of $A$. Then $R$ is a discrete valuation ring and $A = R[z] = R \oplus Rz$, where $z^* = -z$ and $z^2 = x \in R$. Moreover, $x \in U(R)$ if * is unramified and $x \in \mathfrak{m}$ otherwise. In either case, $x$ is not a square in $R$.

Proof. By Lemma 8 we can choose $y \in S \cap (\mathfrak{r} - \mathfrak{r}^2)$.

Suppose first that $S \subset \mathfrak{r}$. Since $\mathfrak{m} = \mathfrak{r} \cap R$, we know that if $b \in \mathfrak{m}$ then $b = cy$ for some $c \in A$. Clearly, since $b \in R$ and $y \in S$, we must have $c \in S \subset \mathfrak{r}$. Therefore $c \in Ay$ and we conclude that $b \in Ry^2$. So in this case $\mathfrak{m} = Ry^2$. Since $2 \in U(A)$, we have $A = R \oplus S$. If $s \in S$ then $sy \in \mathfrak{r} \cap R = \mathfrak{m}$, whence $sy = ry^2$ for some $r \in R$, so $s = ry$ and a fortiori $S = Ry$. Moreover, in this case, let $x = y^2$ and observe that $x$ cannot be a square of any element of $R$. 
On the other hand, if \( S \not\subset r \) then we may choose \( w \in U(A) \) such that \( w^* = -w \). Then \( z = wy \in R \cap (r - r^2) \) and it is clear that \( m = Rz \) in this case. If \( t \in S \) then \( tw^{-1} \in R \), so \( t \in Rw \), which gives \( S = Rw \). Now let \( x = w^2 \in R \) and once again, one readily checks that \( x \) is not a square in \( R \).

In both cases \( R \) is a local domain with principal maximal ideal, so every ideal is principal.

\[ \square \]

6. Invariants and normal forms

We are finally in a position to show that, after imposing additional hypotheses, the sequence \( d_i = \dim_{A/r} W(i), i \geq 0 \), is a complete set of invariants for equivalence classes of \( \varepsilon \)-hermitian forms.

Let \( B \) stand for the fixed field of the involution that \( * \) induces on \( A/r \). Then \( B = A/r \) if \( * \) is ramified and \( B = R/m \) (viewed as a subfield of \( A/r \)) if \( * \) is unramified. In either case, we have a norm map \( N : A/r \to B \) given by \( a + r \to aa^* + r \). One of the aforementioned hypotheses is that \( N \) be surjective.

**Theorem 6.** Suppose that \( A \) is complete and \( N \) is surjective. Let \( V \) and \( V' \) be free \( A \)-modules, both of rank \( m \), and let \( h : V \times V \to A \) and \( h' : V' \times V' \to A \) be \( \varepsilon \)-hermitian forms. Then \( h \) and \( h' \) are equivalent if and only if \( d_i = d'_i \) for all \( i \geq 0 \).

**Proof.** This follows from Corollary 3 and Theorem 4 using the fact that \( N \) is surjective. \( \square \)

We may use Theorem 6 to obtain normal forms for \( \varepsilon \)-hermitian matrices.

**Theorem 7.** Suppose that \( A \) is complete and \( N \) is surjective.

1. If \( * \) is nontrivial, let \( M \in M_m(A) \) be skew hermitian (resp. hermitian). Then \( M \) is \( * \)-congruent to one and only one matrix of the form \( \bigoplus_{i=0}^{\infty} y^i M_i \), where every \( M_i \) of size \( > 0 \) is equal to the direct sum of copies of \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) if \( i \) is even (resp. odd) and equal to the identity matrix if \( i \) is odd (resp. even).
2. If \( * \) is trivial, let \( M \in M_m(A) \) be skew symmetric (resp. symmetric). Then \( M \) is \( * \)-congruent to one and only one matrix of the form \( \bigoplus_{i=0}^{\infty} y^i M_i \), where every \( M_i \) of size \( > 0 \) is equal to the direct sum of copies of \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) (resp. equal to the identity matrix).

Recall that two matrices \( M, N \in M_m(A) \) are said to be equivalent if \( PMQ = N \) for some \( P, Q \in \text{GL}_m(A) \). Assuming that \( A \) is complete and \( N \) is surjective, we may use Corollary 3 and Theorem 4 to see that the problem of \( * \)-congruence of matrices reduces to the problem of equivalence of matrices, whose answer is well-known. Since every invertible matrix is equivalent to the identity matrix, we have

**Theorem 8.** Suppose \( A \) is complete and \( N \) is surjective. Then two \( \varepsilon \)-hermitian matrices \( M, N \in M_m(A) \) are \( * \)-congruent if and only if they have the same invariant factors.
What are the possible invariant factors of an \( \varepsilon \)-hermitian matrix? By above this amounts to asking what are the sequences that arise as \((d_i)_{i \geq 0}\) for some \( \varepsilon \)-hermitian form. Let us call such sequences \( \varepsilon \)-realisable. The answer is an immediate consequence of Theorem 7.

**Proposition 6.** Suppose that \( A \) is complete and \( N \) is surjective. Let \((d_i)\) be a sequence of nonnegative integers.

1. If \( * \) is trivial and \( \varepsilon = 1 \) then \((d_i)\) is \( \varepsilon \)-realisable.
2. If \( * \) is trivial and \( \varepsilon = -1 \) then \((d_i)\) is \( \varepsilon \)-realisable if and only if \( d_i \) is even for all \( i \).
3. If \( * \) nontrivial, then \((d_i)\) is \( \varepsilon \)-realisable if and only if \( (-1)^i = -\varepsilon \).

The analogue of Theorem 7 in the case when \( * \) is unramified and \( A/\mathfrak{r} \) is assumed to be quadratically closed is an immediate consequence of Proposition 2 and Theorem 5.

**Theorem 9.** Suppose that \( A \) is complete with quadratically closed residue field. Assume that \( * \) is unramified and fix \( b \in S \cap U(A) \). Let \( M \in M_m(A) \) be hermitian (resp. skew hermitian). Then \( M \) is \( * \)-congruent to one and only one matrix of the form \( \bigoplus_{i=0}^{\infty} y^i M_i \), where every \( M_i \) of size \( > 0 \) is equal to a diagonal matrix \( \text{diag}(1, \ldots, 1, -1, \ldots, -1) \) (resp. \( \text{diag}(b, \ldots, b, -b, \ldots, -b) \)).

**References**

1. Eva Bayer-Fluckiger and Laura Fainsilber, *Non-unimodular Hermitian forms*, Invent. Math. 123 (1996), no. 2, 233–240, DOI 10.1007/s002220050024. MR1374198
2. Eva Bayer-Fluckiger, Uriya A. First, and Daniel A. Moldovan, *Hermitian categories, extension of scalars and systems of sesquilinear forms*, Pacific J. Math. 270 (2014), no. 1, 1–26, DOI 10.2140/pjm.2014.270.1. MR3245846
3. Eva Bayer-Fluckiger and Daniel Arnold Moldovan, *Sesquilinear forms over rings with involution*, J. Pure Appl. Algebra 218 (2014), no. 3, 417–423, DOI 10.1016/j.jpaa.2013.06.012. MR3124208
4. Yonglin Cao and Fernando Szechtman, *Congruence of symmetric matrices over local rings*, Linear Algebra Appl. 431 (2009), no. 9, 1687–1690, DOI 10.1016/j.laa.2009.06.003. MR2555069
5. I.S. Cohen, *On the structure and ideal theory of complete local rings*, Transactions of the American Mathematical Society 59 (1946), no. 1, 54-106.
6. Dragomir Z. Dokovic and Fernando Szechtman, *Solution of the congruence problem for arbitrary Hermitian and skew-Hermitian matrices over polynomial rings*, Math. Res. Lett. 10 (2003), no. 1, 1–10, DOI 10.4310/MRL.2003.v10.n1.a1. MR1960118
7. William H. Durfee, *Congruence of quadratic forms over valuation rings*, Duke Math. J. 11 (1944), 687–697. MR0011073
8. Ronald Jacobowitz, *Hermitian forms over local fields*, Amer. J. Math. 84 (1962), 441–465, DOI 10.2307/2372982. MR0150128
9. Max-Albert Knus, *Quadratic and Hermitian forms over rings*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 294, Springer-Verlag, Berlin, 1991. With a foreword by I. Bertuccioni. MR1096299
10. T.Y. Lam, *Introduction to Quadratic Forms over Fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005.
11. Serge Lang, *Algebra*, 2nd ed., Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1984. MR783636
12. V. Levchuk and O. Starikova, *Quadratic forms of projective spaces over rings*, Sb. Math. 197 (2006), 887-899.
[13] V. Levchuk and Starikova O., *A normal form and schemes of quadratic forms*, J. Math. Sci. 152 (2008), 558-570.

[14] O. T. O’Meara, *Quadratic forms over local fields*, Amer. J. Math. 77 (1955), 87–116, DOI 10.2307/2372423. MR0067163

**School of Mathematics, Statistics and Applied Mathematics, National University of Ireland Galway, Ireland**

*E-mail address: james.cruickshank@nuigalway.ie*

**School of Mathematics, Statistics and Applied Mathematics, National University of Ireland Galway, Ireland**

*E-mail address: rachel.quinlan@nuigalway.ie*

**Department of Mathematics and Statistics, University of Regina, Canada**

*E-mail address: fernando.szechtman@gmail.com*