Abstract. We study the existence and orbital stability/instability of periodic standing wave solutions for the Klein-Gordon-Schrödinger system with Yukawa and cubic interactions. We prove the existence of periodic waves depending on the Jacobian elliptic functions. For one hand, the approach used to obtain the stability results is the classical Grillakis, Shatah and Strauss theory in the periodic context. On the other hand, to show the instability results we employ a general criterion introduced by Grillakis, which get orbital instability from linear instability.

1. Introduction.

In this paper we shall investigate the orbital stability of periodic standing wave solutions associated to the Klein-Gordon-Schrödinger system (KG-NLS henceforth),

\begin{align}
\begin{cases}
    iu_t + \frac{1}{2} \Delta u &= -uv \frac{\partial f}{\partial |u|^2}(|u|^2, v) \\
    v_{tt} - \Delta v + m^2 v &= f(|u|^2, v) + v \frac{\partial f}{\partial v}(|u|^2, v),
\end{cases}
\end{align}

when \( f(s, t) = s \) and \( f(s, t) = st \), here \( s, t \in \mathbb{R} \).

If \( f(s, t) = s \) equations in (1) are the so-called Klein-Gordon-Schrödinger system with Yukawa interaction and it describes a system of conserved scalar nucleons interacting with neutral scalar meson. Here \( u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C} \) represents a complex scalar nucleon field and \( v : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) a real scalar meson field. The real constant \( m^2 \) determines the mass of a meson. The full system (1) was motivated by Hayashi’s paper [22].

We restrict ourselves to the case \( d = 1 \). The periodic standing waves we are interested in are of the form

\begin{align}
u(x, t) = e^{i\xi t} \psi_c(x), \quad v(x, t) = \phi_c(x),\end{align}

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where $c \in \mathbb{R}$ and $\psi_c, \phi_c : \mathbb{R} \to \mathbb{R}$ are smooth periodic functions with a fixed period $L > 0$.

Studies related to the stability of stationary waves and well-posedness results for equations (1), have been having a considerable development in recent years. For instance, when $d = 3$, Ohta in [28], obtained a result of stability for stationary states for equations (1) ($f(s,t) = s$) by using the variational approach introduced by Cazenave and Lions in [12]. Later, Kikuchi and Ohta in [25], established a result of orbital instability related to the same equation when the wave-speed $c > 0$ is sufficiently small. On the other hand, Baillon and Chadam in [9] deduced existence of global solutions by using the $L^p - L^q$ estimates for the elementary solutions of the Schrödinger equation. Fukuda and Tsutsumi in [15], discussed the initial boundary value problem of KG-NLS (1) and obtained the global existence of strong solutions in the three-dimensional case, and later the results were improved in [21]. Others contributors can be mentioned as [8], [13], [14], [22], [30], [36].

In the one-dimensional case, Tang and Ding in [34] (see also [35]) studied a result of modulational instability related to the general Klein-Gordon-Schrödinger given by

\begin{align}
\begin{cases}
    iu_t + \alpha u_{xx} + \rho u v + \gamma_1 |u|^2 u = 0 \\
    v_{tt} - \frac{1}{6} v_{xx} + m^2 v + \gamma_2 |v|^3 - \beta |v|^2 = 0,
\end{cases}
\end{align}

where $u$, $v$ and $m^2$ are given as above, $\gamma_i$, $i = 1, 2$ are cubic nonlinear auto-interactions, $\beta$ and $\rho$ are quadratic coupling constants and, $\alpha$ and $c_0$ are constants. Further, it was found that there are a number of possibilities for the modulational instability regions due to the generalized dispersion relation, which relates the frequency and wave-number of the modulating perturbations. When $\gamma_1 = \gamma_2 = 0$, $\alpha = \frac{1}{4}$ and $\beta = \rho = c_0^2 = 1$, equation (3) becomes (1) with $f(s,t) = s$.

In general, the studies about the stability/instability to Klein-Gordon (KG henceforth) and nonlinear Schrödinger (NLS henceforth) equations have attracted a large set of researchers. It is known that both KG and NLS equations, specially with cubic interactions, have wide applications in many physical fields such as nonlinear optics, nonlinear plasmas, condensed matter and so on. Besides, a similar system given by equations in (3) may describe the dynamics of coupled electrostatic upper-hybrid and ion-cyclotron waves in a uniform magnetoplasma (see [34] and [37]).

In a stability/instability approach if one considers the general KG equation

\begin{equation}
    u_{tt} - \Delta u + f(|u|^2)u = 0, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d,
\end{equation}

Grillakis [19] (see also [17], [20]) determined sufficient conditions for the orbital instability of the standing waves $e^{i c t} \varphi(x)$ in the space of radial functions, where $\varphi(x) = \varphi(|x|)$ has a finite number of nodes (with some restrictions on the nonlinearity). Others contributions in this qualitative approach can be mentioned, for example, [31], [32] and [33]. In the periodic context, a recent work due to Natali and Pastor in [27] determined stable/unstable families of periodic standing wave solutions for equation (4) when $f(v) = 1 - v$ and $f(v) = v$ (with $d = 1$) making use of the abstract theory established in [17] and [18].

Next, when the NLS

\begin{equation}
    iu_t + \Delta u + |u|^{p-1}u = 0, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d,
\end{equation}

is considered, a large amount of papers concerning the stability/instability of standing waves can be found in the current literature. In particular, Cazenave and Lions
in [12], determined the existence of stable ones, of the form \( u(x,t) = e^{i\omega t} \phi(x) \), for equation (5) with \( d \geq 1 \) and \( 1 < p < 1 + 4/d \). In [17] and [18] is possible to find a set of sufficient conditions that determines stability/instability of standing waves for that equation. In this case, the theory of stability/instability related to the periodic case (for \( d = 1 \)) have been a terrific development, for instance [1], [5], [6], [7], [16] and [29].

The methods used in the present paper in order to show our stability/instability results will be the ones developed by Grillakis et al. [17], [18] and Grillakis [19], [20]. The main reason for this is because system (1), when \( f(s,t) = s \) or \( f(s,t) = st \), can be seen as an abstract Hamiltonian system

\[
\frac{dU(t)}{dt} = J \mathcal{E}'(U(t)),
\]

where \( U = (u_1, v_1, u_2, v_2) = (\text{Re}u, \text{v}, \text{Im}u, \text{v}) \). \( \mathcal{E} \) represents the energy functional and \( J \) is the skew-symmetric matrix defined by

\[
J = \begin{pmatrix}
0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 1 \\
-1/2 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.
\]

Firstly, we consider the case \( f(s,t) = s \). In order to get explicit solutions we suppose \( \psi_c = \sqrt{2}\phi_c \) in (2). Thus, substituting (2) into (1) (with \( \psi_c = \sqrt{2}\phi_c \)), it follows that \( \phi_c \) must satisfy the ordinary differential equation

\[
-\phi_c'' + 2c\phi_c - 2\phi_c^2 = 0,
\]

where \( 2c = m^2 \). Multiplying equation (8) by \( \phi' \) and integrating once, we obtain

\[
[\phi'_\omega]^2 = \frac{1}{3}[\phi_\omega^3 - 3\omega\phi_\omega + 6B\phi_\omega],
\]

where \( \phi_\omega = 4\phi_c \), \( B\phi_\omega \) is a nonzero integration constant and \( \omega := \omega(c) = 2c \). A positive solution obtained from (9) depending on the \textit{cnoidal} Jacobi elliptic function (see Byrd and Friedman [11]) is given by

\[
\phi_\omega(x) = \beta_2 + (\beta_3 - \beta_2)cn^2\left( \sqrt{\frac{\beta_3 - \beta_1}{12}} x; k \right),
\]

where \( \beta_i, i = 1, 2, 3 \) and the modulus \( k \) depend smoothly on \( \omega \) (and therefore on \( c \)).

As it is well-known, the main ingredient on the theory developed in [17]–[20] is the spectral properties of the linear operator arising in the linearized equation around the traveling wave. Here, by making use of the Floquet theory we can determine that matrix operators

\[
\mathcal{L}_{R, cn} = \begin{pmatrix}
\frac{d^2}{dx^2} + 2c - 2\phi & -2\sqrt{2}\phi \\
-2\sqrt{2}\phi & -\frac{d^2}{dx^2} + 2c
\end{pmatrix}.
\]
and

\[
\mathcal{L}_{I,\text{cn}} = \begin{pmatrix}
-\frac{d^2}{dx^2} + 2c - 2\phi & 0 \\
0 & 1
\end{pmatrix}
\]

have the spectral properties required in \([18]\) and \([20]\). This allows us to prove our stability/instability results concerning the cnoidal solution in \((10)\).

Secondly, we consider the case \(f(s,t) = st\). In this case, if we substitute \((2)\) into \((1)\) with \(\psi_c = \phi_c\), we obtain the differential equation (after integrating once)

\[
[\phi_c']^2 = -\phi_c^4 + 2c\phi_c^2 + 2B_{\phi_c}.
\]

Thus, a dnoidal wave solution can be found:

\[
\phi_c(\xi) = \eta \text{dn}(\eta \xi; k),
\]

where \(\eta > 0\) depends smoothly of the wave-speed \(c > 0\).

The linear operators arising in this case are:

\[
\mathcal{L}_{R,\text{dn}} = \begin{pmatrix}
-\frac{d^2}{dx^2} + 2c - 2\phi^2 & -4\phi^2 \\
-4\phi^2 & -\frac{d^2}{dx^2} + 2c - 2\phi^2
\end{pmatrix}
\]

and

\[
\mathcal{L}_{I,\text{dn}} = \begin{pmatrix}
-\frac{d^2}{dx^2} + 2c - 2\phi^2 & 0 \\
0 & 1
\end{pmatrix},
\]

which also possess the spectral properties needed in \([18]\) and \([20]\) that guarantees our stability/instability results.

We point out that our orbital stability results will be with respect to periodic perturbations having the same fundamental period as the corresponding traveling wave, whereas the instability results will be with respect to periodic perturbations having twice the fundamental period as the corresponding traveling wave.

The question about global well-posedness in the energy space \(H_{\text{per}}^1([0,L]) \times H_{\text{per}}^1([0,L]) \times L_{\text{per}}^2([0,L]),\) associated to system \((1)\) can be established by a direct application of Kato’s classical theory (see \([30]\)). Many other results of local and global well-posedness for these two equations can be found in the current literature, as for example \([8]\), \([10]\), \([13]\), \([36]\).

In order to show the current findings, the paper is organized as follows. In Section 2 we present an explicit family of periodic solutions related to equation \((1)\) when \(f(s,t) = s\) and study their orbital stability/instability. In Section 3 we consider the case \(f(s,t) = st\) and establish the existence and stability/instability of another family of periodic solutions.

**Notation.** For \(s \in \mathbb{R}\), the Sobolev space \(H_{\text{per}}^s([0,L])\) is the set of all periodic distributions such that

\[
\|f\|_{H_{\text{per}}^s([0,L])}^2 := \|f[s]\|^2 \equiv L \sum_{k=-\infty}^{+\infty} (1 + |k|)^s |\hat{f}(k)|^2 < \infty,
\]
where \( \hat{f} \) is the Fourier transform of \( f \). The symbols \( sn(\cdot; k), \ dn(\cdot; k) \) and \( cn(\cdot; k) \) will denote the Jacobian elliptic functions of \( snoidal \), \( dnoidal \) and \( cnoidal \) type, respectively. Quantities \( \text{Re}(z) \) and \( \text{Im}(z) \) denote, respectively, the real and imaginary parts of the complex number \( z \).

Note that system (1) can be written in three distinct form: besides (1), we can write it as a first-order system (in \( t \)) and, finally, separate the real and imaginary parts of \( u \). We use any one of these forms without further comments.

2. Orbital stability of cnoidal wave solutions for system (1)

This section is concerned with the existence and orbital stability/instability of periodic solutions to the KG-NLS system

\[
\begin{cases}
  iu_t + \frac{1}{2}u_{xx} = -vu \\
v_{tt} - v_{xx} + m^2 v = |u|^2
\end{cases}
\]

of the form

\[
(u(x, t), v(x, t)) = (e^{ict} \sqrt{2}\phi_c(x), \phi_c(x))
\]

where \( \phi_c : \mathbb{R} \to \mathbb{R} \) is a smooth positive periodic function with a fixed period \( L > 0 \), \( t \in \mathbb{R} \) and \( c > 0 \). In fact, for \( 2c = m^2 \), equation (17) becomes

\[
-\phi''_c + 2c\phi_c - 2\phi^2_c = 0.
\]

2.1. Existence of standing waves. Here our goal consists in showing that equation (19) has a smooth branch, \( c \in I \mapsto \phi_c \), of positive periodic solutions with a fixed period \( L > 0 \) for some parameter interval \( I \). First we define

\[
\varphi_c := 4\phi_c.
\]

Letting \( 2c = \omega \), we obtain from (19) and (20) that

\[
-\varphi'' + \omega \varphi - \frac{1}{2} \varphi^2 = 0.
\]

The next step is to deduce a \( L \)-periodic solution \( \varphi = \varphi_\omega \) for (21). Indeed, multiplying equation (21) by \( \varphi' \) and integrating once, we get

\[
[\varphi']^2 = \frac{1}{3}[-\varphi^3 + 3\omega\varphi^2 + 6B_\varphi]
\]

where \( B_\varphi \) is a nonzero integration constant and \( \beta_1, \beta_2, \beta_3 \) are the zeros of the polynomial \( F(t) = -t^3 + 3\omega t^2 + 6B_\varphi \). We can suppose, without loss of generality, that \( \beta_3 > \beta_2 > \beta_1 \) and \( \beta_3 > 0 \). Therefore, we should have the relations

\[
\begin{cases}
  \beta_1 + \beta_2 + \beta_3 = 3\omega, \\
  \beta_2\beta_1 + \beta_3\beta_1 + \beta_3\beta_2 = 0, \\
  \beta_1\beta_2\beta_3 = 6B_\varphi,
\end{cases}
\]
A periodic solution for the differential equation (21) is obtained from the standard direct integration method (Byrd and Friedman [11], see also [2–5]), namely,

\[ \varphi_\omega(x) = \beta_2 + (\beta_3 - \beta_2)cn^2 \left( \sqrt{\frac{\beta_3 - \beta_1}{12}} x; k \right), \quad k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}. \]

Moreover, we conclude from identities (23) and (24) that the roots \( \beta_1, \beta_2, \beta_3 \) must satisfy \( \beta_1 < 0 < \beta_2 < \beta_3 < 3\omega \) and function in (24) has fundamental period given by

\[ T_\varphi = \frac{4\sqrt{3}}{\sqrt{\beta_3 - \beta_1}} K(k_\varphi), \]

where \( k = k_\varphi \) is the so-called elliptic modulus and \( K \) represents the complete elliptic integral of the first kind defined by

\[ K(k) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}. \]

From relations in (23) we obtain, after some calculations, that

\[ \beta_2^2 + (\beta_3 - 3\omega)\beta_2 + (\beta_3^2 - 3\omega) = 0, \]

whose value of \( \beta_2 \) is given by

\[ \beta_2 = \frac{1}{2} \left( 3\omega - \beta_3 + \sqrt{9\omega^2 + 6\omega\beta_3 - 3\beta_3^2} \right). \]

Therefore, from (23) and (27), we deduce

\[ \beta_3 - \beta_1 = \frac{1}{2} \left( 3\beta_3 - 3\omega + \sqrt{9\omega^2 + 6\omega\beta_3 - 3\beta_3^2} \right) \]

and

\[ \beta_3 - \beta_2 = \frac{1}{2} \left( 3\beta_3 - 3\omega - \sqrt{9\omega^2 + 6\omega\beta_3 - 3\beta_3^2} \right). \]

Identities (28) and (29) allow us to conclude that the modulus \( k \) must satisfy

\[ k^2 = \frac{3\beta_3 - 3\omega - \sqrt{9\omega^2 + 6\omega\beta_3 - 3\beta_3^2}}{3\beta_3 - 3\omega + \sqrt{9\omega^2 + 6\omega\beta_3 - 3\beta_3^2}}. \]

Moreover, thanks to (29) we obtain the inequality \( 0 < \beta_2 < 2\omega < \beta_3 < 3\omega \).

Let \( \omega > 0 \) be fixed, from (30), asymptotic properties of \( K \) and the fact that \( \beta_3 \in (2\omega, 3\omega) \rightarrow T_\varphi(\beta_3) \) is a strictly increasing function (see Theorem 2.1 below) it follows that \( T_\varphi > 2\pi/\sqrt{\omega} \). This means that for any \( L > 0 \) fixed, choosing \( \omega > 0 \) such that \( \omega > 4\pi^2/L^2 \) there is a unique \( \beta_3 = \beta_3(\omega) \in (2\omega, 3\omega) \) such that the corresponding cnoidal wave given by (24) has fundamental period \( T_\varphi = L \).

**Remark 1.** The solitary wave solution for equation (21) can be determined from the asymptotic properties of Jacobi elliptic function \( cn \) given in (24). In fact, for \( \omega > 0 \) fixed, if \( \beta_1, \beta_2 \rightarrow 0 \) (then \( \beta_3 \rightarrow 3\omega \)), we get, \( k \rightarrow 1^- \). On the other hand, since \( cn(\cdot, 1^-) \approx sech(\cdot) \), we obtain the single-humped function

\[ \varphi_\omega(x) = 3\omega sech^2 \left( \frac{\sqrt{\omega}}{2} x \right), \]

which is the solitary wave solution for equation (21).
Next, we shall construct a smooth curve, \( \omega \mapsto \varphi_\omega \), of cnoidal wave solutions for equation (21).

**Theorem 2.1.** Let \( L > 0 \) be arbitrary but fixed. Consider \( \omega_0 > 4\pi^2/L^2 \) and the unique \( \beta_{3,0} = \beta_3(\omega_0) \in (2\omega_0, 3\omega_0) \) such that \( T_{\varphi_{\omega_0}} = L \), then

1. there exists an interval \( I(\omega_0) \) around \( \omega_0 \), an interval \( B(\beta_{3,0}) \) around \( \beta_{3,0} \) and a unique smooth function \( \Gamma : I(\omega_0) \to B(\beta_{3,0}) \) such that \( \Gamma(\omega_0) = \beta_{3,0} \) and

\[
\frac{4\sqrt{6}}{\sqrt{3\beta_3 - 3\omega + \sqrt{9\omega^2 + 6\omega\beta_3 - 3\beta_3^2}}} K(k) = L,
\]

where \( \omega \in I(\omega_0) \), \( \beta_3 = \Gamma(\omega) \in B(\beta_{3,0}) \) and \( k^2 = k^2(\omega) \in (0,1) \) is given by (30).

2. The cnoidal wave solution in (24), \( \varphi_\omega(\cdot; \beta_1, \beta_2, \beta_3) \), determined by \( \beta_i = \beta_i(\omega) \), \( i = 1, 2, 3 \), has fundamental period \( L \) and satisfies (21). Moreover, the mapping

\[
\omega \in I(\omega_0) \mapsto \varphi_\omega \in H_{\text{per}}^n([0, L]), \quad n = 0, 1, \ldots
\]

is a smooth function.

3. \( I(\omega_0) \) can be chosen as \((4\pi^2/L^2, +\infty)\).

**Proof.** The proof is an application of the Implicit Function Theorem and it follows closely the arguments in Angulo and Linares [5, Theorem 3.1] (see also [3] and [4]). Thus, we omit it here. \( \square \)

**Remark 2.** From the Implicit Function Theorem, Theorem 2.1 and (25) we can conclude that function \( \Gamma : I(\omega_0) \to B(\beta_{3,0}) \) given in Theorem 2.1 is a strictly increasing function. Moreover, \( dk/d\omega > 0 \) (this fact can also be found in [1] or [7]).

**Remark 3.** From Theorem 2.1 and identity (26) we are in a position to conclude that

\[
\omega = \frac{(16K^2\sqrt{1-k^2+k^4})/L^2}{\beta_3} = \frac{16K^2}{L^2} \left[ \sqrt{1-k^2+k^4+k^2} \right], \quad \beta_3 = 16K^2 \left[ \sqrt{1-k^2+k^4+k^2} \right]/L^2, \quad \beta_3 - \beta_1 = 48K^2/L^2 \text{ and } \beta_3 - \beta_2 = 48k^2K^2/L^2.
\]

As a consequence of Theorem 2.1, we immediately have:

**Corollary 1.** Let \( c \in (2\pi^2/L^2, +\infty) \) and \( \omega(c) = 2c \). Then the cnoidal wave \( \phi_c = \varphi_{\omega(c)}/4, \) where \( \varphi_{\omega(c)} \) is give in Theorem 2.1, has fundamental period \( L \) and satisfies (19). Moreover, the mapping

\[
c \in \left(\frac{2\pi^2}{L^2}, +\infty\right) \mapsto \phi_c \in H_{\text{per}}^n([0, L]), \quad n = 0, 1, \ldots
\]

is a smooth function. In addition, \( dk/dc > 0 \).

2.2. **Spectral analysis.** We start establishing the basic framework for the stability study introduced in [18] and [20]. As we have already mentioned, we can write system (17) as a Hamiltonian system

\[
\frac{dU(t)}{dt} = JE'(U(t)),
\]

where \( U = (u_1, v_1, u_2, v_2) = (\text{Re}(u), v, \text{Im}(u), v_1) \), \( J \) is the matrix defined in (7) and \( E \) is the energy functional

\[
E(U) = \frac{1}{2} \int_0^L \left[ u_{1,x}^2 + u_{2,x}^2 + v_{1,x}^2 + v_{2,x}^2 + m^2v_1^2 - 2v_1(u_{1,x}^2 + u_{2,x}^2) \right] dx.
\]
We remind the reader that system (17) also preserves the $L^2$-norm of $u$, that is, the quantity
\[
\mathcal{F}(U) = \int_0^L (u_1^2 + u_2^2)dx
\]
is a conserved quantity of system (17).

In what follow in this section we will denote $\Phi = (\sqrt{2}\phi, \phi, 0, 0)$, where $\phi$ is the cnoidal wave given in Corollary 1.

It is well-known that to apply the abstract Grillakis et al. theory we need to study the spectral properties of operator
\[
\mathcal{L}_{cn} := \mathcal{E}''(\Phi) + c\mathcal{F}''(\Phi) = \begin{pmatrix} \mathcal{L}_{\mathcal{R},cn} & 0 \\ 0 & \mathcal{L}_{\mathcal{I},cn} \end{pmatrix},
\]
where
\[
\mathcal{L}_{\mathcal{R},cn} = \begin{pmatrix} \frac{d^2}{dx^2} + 2c - 2\phi & -2\sqrt{2}\phi \\ -2\sqrt{2}\phi & -\frac{d^2}{dx^2} + 2c \end{pmatrix}
\]
and
\[
\mathcal{L}_{\mathcal{I},cn} = \begin{pmatrix} -\frac{d^2}{dx^2} + 2c - 2\phi & 0 \\ 0 & 1 \end{pmatrix}.
\]

We begin by studying the spectra of operators $\mathcal{L}_{\mathcal{R},cn}$ and $\mathcal{L}_{\mathcal{I},cn}$. More precisely, we have:

**Theorem 2.2.** Let $\phi = \phi_c$ be the cnoidal wave solution given by Corollary 1. Then

(i) operator $\mathcal{L}_{\mathcal{R},cn}$ in (35) defined in $L^2_{\text{per}}([0,L]) \times H^2_{\text{per}}([0,L])$ whose domain is $H^2_{\text{per}}([0,L]) \times H^1_{\text{per}}([0,L])$ has exactly one negative eigenvalue which is simple; zero is a simple eigenvalue whose eigenfunction is $(2\phi/3, \sqrt{2}\phi/3)$. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.

(ii) Operator $\mathcal{L}_{\mathcal{I},cn}$ in (36) defined in $L^2_{\text{per}}([0,L]) \times L^2_{\text{per}}([0,L])$ whose domain is $H^2_{\text{per}}([0,L]) \times L^2_{\text{per}}([0,L])$ has only non-negative eigenvalues being zero the first one which is simple with eigenfunction $(\phi, 0)$. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.

We point out that from Weyl’s essential spectrum theorem, all operators we study here have only point spectrum.

Before proving Theorem 2.2, we note that operator $\mathcal{L}_{\mathcal{R},cn}$ can be diagonalized under a similarity transformation. In fact, we consider
\[
A_R = \begin{pmatrix} 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1 \end{pmatrix}.
\]
Then operator $L_{DR} := A_R L_{R,cn} A_R^{-1}$, is a diagonal operator given by

$$L_{DR} = \begin{pmatrix} L_{1,cn} & 0 \\ 0 & L_{3,cn} \end{pmatrix},$$

with

$$L_{1,cn} = -\frac{d^2}{dx^2} + 2c - 4\phi$$

and

$$L_{3,cn} = -\frac{d^2}{dx^2} + 2c + 2\phi.$$

The next lemma give us the spectral properties of operators $L_{1,cn}$ and $L_{2,cn}$, where $L_{1,cn}$ is defined in (38) and

$$L_{2,cn} = -\frac{d^2}{dx^2} + 2c - 2\phi.$$

**Lemma 2.3.** Let $\phi = \phi_c$ be the cnoidal wave given by Corollary 1. Then the following spectral properties hold:

(i) operator $L_{1,cn}$ in (38) defined in $L^2_{per}([0, L])$ with domain $H^2_{per}([0, L])$ has exactly one negative eigenvalue which is simple; zero is an eigenvalue which is simple with eigenfunction $\phi'$. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.

(ii) Operator $L_{2,cn}$ in (40) defined in $L^2_{per}([0, L])$ with domain $H^2_{per}([0, L])$ has no negative eigenvalues; zero is an eigenvalue, simple with eigenfunction $\phi$. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.

**Proof.** (i) The main point is that the periodic eigenvalue problem associated to operator $L_{1,cn}$,

$$\begin{cases} L_{1,cn} f = \lambda f \\ f(0) = f(L), \quad f'(0) = f'(L), \end{cases}$$

is equivalent (under the transformation $\Psi(x) = f(\theta x), \theta^2 = 12/(\beta_3 - \beta_1)$) to the periodic eigenvalue problem

$$\begin{cases} \frac{d^2}{dx^2}\Psi + [\delta - 12k^2 sn^2(x;k)]\Psi = 0 \\ \Psi(0) = \Psi(2K(k)), \quad \Psi'(0) = \Psi'(2K(k)), \end{cases}$$

associated to the Lamé equation with

$$\delta = \frac{12}{\beta_3 - \beta_1}(\lambda + \beta_3 - 2c) = \frac{12}{\beta_3 - \beta_1}(\lambda + \beta_3 - \omega).$$

The proof of this item follows from Floquet Theory and the basic ideas can be found in [5, Theorem 4.1] (see also [2]). However, for the sake of clearness we will list the basic facts. Indeed, since $L_{1,cn}\phi' = 0$ and $\phi'$ has two zeros in $[0, L)$, then zero is the second or the third eigenvalue of $L_{1,cn}$. Next, since $\delta_1 = 4 + 4k^2$ is an eigenvalue to (42) with eigenfunction $\Psi_1(x) = cn(x) sn(x) dn(x)$, we have that $\lambda_1 = 0$ is a
eigenvalue to (41) whose eigenfunction is $\phi'$. On the other hand, from Ince [23] we have that the Lamé polynomials,

$$\Psi_0(x) = dn(x)[1 - (1 + 2k^2 - \sqrt{1 - k^2 + 4k^4})sn^2(x)],$$

$$\Psi_2(x) = dn(x)[1 - (1 + 2k^2 + \sqrt{1 - k^2 + 4k^4})sn^2(x)],$$

are the eigenfunctions related to other two eigenvalue $\delta_0, \delta_2$ given by,

$$\delta_0 = 2 + 5k^2 - 2\sqrt{1 - k^2 + 4k^4}, \quad \delta_2 = 2 + 5k^2 + 2\sqrt{1 - k^2 + 4k^4}.$$  

Since $\Psi_0$ has no zeros in $[0, 2K]$ it follows that $\delta_0$ will be the first eigenvalue to (42). Since $\delta_0 < \delta_1$ for all $k \in (0, 1)$, we obtain from (43) that $\lambda_0 < 0$. Therefore $\lambda_0$ is the first eigenvalue to $L_{1, cn}$ which is simple. Moreover, since $\delta_1 < \delta_2$ for every $k \in (0, 1)$, we obtain from (43) that $\lambda_2 > 0$. This fact implies that $\lambda_2$ is the third eigenvalue to $L_{1, cn}$ and therefore $\lambda_1 = 0$ results to be simple.

(ii) It follows immediately from the Floquet theory since $L_{2, cn} \phi = 0$ and $\phi$ has no zeros in the interval $[0, L]$.  \hfill \Box

Proof of Theorem 2.2. (i) Since $\phi$ is strictly positive, it follows that operator $L_{3, cn}$ is strictly positive and $\sigma(L_{3, cn}) \geq 2c$, where $\sigma(L_{3, cn})$ denotes the spectrum of $L_{3, cn}$. Next, let $\tilde{f} = (g, h)^t \neq 0$ be such that $LD_R \tilde{f} = f \neq 0$, then $L_{1, cn}g = 0$ and $L_{1, cn}h = 0$. Thus, from Lemma 2.3 we have $h \equiv 0$ and $g = \alpha \phi'$, for some nonzero real constant $\alpha$. Hence, the kernel of $LD_R$ is generated by $(\alpha \phi', 0)^t$. This implies that the kernel of $LR_{cn}$ is 1-dimensional and generated by $(2\phi'/3, \sqrt{2} \phi'/3)$.

Next we consider $\lambda < 0$ and $\tilde{f} = (g, h)^t \neq 0$ such that $LD_R \tilde{f} = \lambda \tilde{f}$, then $h \equiv 0$ and $L_{1, cn}g = \lambda g$. Therefore, from Lemma 2.3 we must have $\lambda = \lambda_0$ and $g = \beta \chi_0$, where $\lambda_0$ is the unique negative eigenvalue of $L_{1, cn}$ and $\chi_0$ is the corresponding eigenfunction. This implies part (i) of the lemma.

(ii) It follows immediately from Lemma 2.3 since operator $L_{2, cn}$ has no negative eigenvalues and zero is a simple eigenvalue. \hfill \Box

Theorem 2.2 give us the spectral properties needed to prove our stability results when we consider periodic perturbations having the same fundamental period as the corresponding wave itself. However, we are also interested in perturbations having twice the fundamental period as the corresponding wave. In this regard, the following lemma is useful.

Lemma 2.4. Let $\phi = \phi_c$ be the cnoidal wave solution given by Corollary 1. Then, the linear operator $L_{1, cn}$ in (38) defined in $L^2_{per}([0, 2L])$ with domain $H^2_{per}([0, 2L])$ has its first four eigenvalues simple, being the eigenvalue zero the fourth one with eigenfunction $\phi'$. Moreover, if $\chi_1$ and $\chi_2$ denote the eigenfunctions associated to the second and third eigenvalues then $\chi_i \perp \phi, i = 1, 2$.

Proof. The proof follows the same steps as in Lemma 2.3 and a more detailed proof can be found in [5, Theorem 4.2]. It is easy to see that

$$\tilde{\delta}_0 = 5 + 2k^2 - 2\sqrt{4 - k^2 + k^4}, \quad \tilde{\delta}_1 = 5 + 5k^2 - 2\sqrt{4 - 7k^2 + 4k^4}$$

are the first two eigenvalues for the semi-periodic eigenvalue problem associated with the Lamé equation in (42). The eigenfunctions are given, respectively, by

$$\Psi_0(x) = cn(x)[1 - (2 + k^2 - \sqrt{4 - k^2 + k^4})sn^2(x)]$$

$$\Psi_1(x) = sn(x)[3 - (2 + k^2 - \sqrt{4 - 7k^2 + 4k^4})sn^2(x)]$$
Hence, if $\rho_0$ and $\rho_1$ are the first two eigenvalues for the semi-periodic eigenvalue problem associated with operator $L_{1,cn}$, we obtain from (43) (with $\delta$ and $\lambda$ replaced, respectively, with $\delta$ and $\rho$) that (see e.g. [26, Theorem 2.1])

$$
\lambda_0 < \rho_0 < \rho_1 < \lambda_1 = 0,
$$

where $\lambda_0$ and $\lambda_1$ are given in Lemma 2.3. The orthogonality condition follows from the explicit forms of $\chi_1$ and $\chi_2$. This completes the proof of the lemma. \(\square\)

2.3. **Orbital stability.** First of all, let us make clear our notion of orbital stability. Since system (17) has phase and translation symmetries, that is, if $(u(x,t), v(x,t))$ is a solution of (17), so are

$$
(\epsilon^x u(x,t), v(x,t)) =: P_s(u,v)(x,t),
$$

and

$$
(\epsilon^t u(x,t), v(x,t)) =: \Phi_s(u,v)(x,t),
$$

for any $x_0, s \in \mathbb{R}$, our definition of orbital stability in this subsection is as follows:

**Definition 2.5.** A standing wave solutions for (17), $(e^{i\epsilon \psi_c(x)}, \phi_c(x))$, is said to be orbitally stable in $X = H^1_{\text{per}}([0,L]) \times H^1_{\text{per}}([0,L]) \times L^2_{\text{per}}([0,L])$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $(u_0, v_0, v_1)$ is a solution of (17) with $\|\psi(t) = (u_0, v_0, v_1)\| \leq \epsilon$. Then the solution $u(t) = (u, v, v_1)$ of (17) with $\psi(0) = (u_0, v_0, v_1)$ exists globally and satisfies

$$
\sup_{\epsilon > 0} \inf_{\delta > 0} \|\psi(t) - (e^i\epsilon \psi_c(x), T_y \phi_c(x), 0)\| < \epsilon.
$$

Otherwise, $(e^{i\epsilon \psi_c(x)}, \phi_c(x))$ is said to be orbitally unstable in $X$.

Here, $T_y g(x) = g(x+y)$. Our stability result reads as follows:

**Theorem 2.6.** Let $\phi_c$ be the corresponding cnoidal wave obtained in Corollary 1. Then the periodic wave solution $(\sqrt{2} e^{i\epsilon \psi_c(x)}, \phi_c)$ is orbitally stable in $X$ by the periodic flow of system (17).

**Proof.** We apply the abstract Stability Theorem in [18]. From Theorem 2.2 we see that operator $L_{cn}$ has the spectral properties required in Grillakis et al. [18] to apply the abstract theorem. Indeed, Theorem 2.2 implies that operator $L_{cn}$ has only one negative eigenvalue which is simple and its kernel is 2-dimensional. Moreover, the remainder of the its spectrum consists on a discrete and positive set of eigenvalues.

Also, it is easy to see that our smooth curve of periodic solutions given in Corollary 1 is a family of critical points for the functional $\mathcal{H} = \mathcal{E} + c\mathcal{F}$, that is,

$$
\mathcal{H}'(\sqrt{2} \phi_c, \phi_c, 0, 0) = 0.
$$

Finally, we need to study the convexity of the real function

$$
d(c) = \mathcal{E}(\sqrt{2} \phi_c, \phi_c, 0, 0) + c\mathcal{F}(\sqrt{2} \phi_c, \phi_c, 0, 0).
$$

From (48) we obtain that $d'(c) = \mathcal{F}(\sqrt{2} \phi_c, \phi_c, 0, 0)$. Hence, from Theorem 2.1,

$$
d''(c) = \frac{d}{dc} \left( \int_0^L \phi_{c}^2(x)dx \right)
$$

$$
= \frac{1}{8} \frac{d}{d\omega} \left( \int_0^L \phi_{c}^2(x)dx \right).
$$
In order to finish the proof, we just need to show that $d''(c) > 0$. To show this, we integrate equation (21) over $[0, L]$ to conclude that

$$
\int_0^L \varphi''_c(x)dx = 2\omega \int_0^L \varphi'_c(x)dx.
$$

However, to verify that $d''(c) > 0$ it is sufficient to prove that $\Upsilon(\omega) = \int_0^L \varphi'_c(x)dx$ is a strictly increasing function. In fact, from Byrd and Friedman’s book [11] we deduce from the fact $\beta_1 - \beta_2 = 48k^2K^2/L^2$, that

$$
\Upsilon(\omega) = \frac{16}{L} \omega \left( K(k)^2 \sqrt{1 - k^2 + k^4 + 1 - 2k^2} + 3K(k)[E(k) - k^2K(k)] \right).
$$

Since

$$
f(k) = K(k)^2 \sqrt{1 - k^2 + k^4 + 1 - 2k^2} + 3K(k)[E(k) - k^2K(k)]
$$

(52)

function $f : (0, 1) \to \mathbb{R}$ defined in (52) is a strictly increasing function with respect to the modulus. Moreover, from Remark 3 we have $dk/d\omega > 0$. Thus, we conclude that $\Upsilon(\omega)$ is a strictly increasing function and therefore $d''(c) > 0$. \hfill \Box

**Remark 4.** By using the same framework as presented in this section (with the necessary modifications), one shows the stability of the solitary standing wave solution $(\sqrt{2}e^{i\omega t}\phi_c(x), \phi_c(x))$, $x \in \mathbb{R}$, $c > 0$, for equation (1) with $f(s, t) = s$, where

$$
\phi_c(x) = 6c \text{sech}^2 \left( \frac{c}{\sqrt{2}} x \right),
$$

is the solitary wave given in Remark 1.

### 2.4. Orbital instability

Let $\phi = \phi_c$ be given in Corollary 1. This section is devoted to prove that the corresponding standing wave is unstable in the space $H^1_{per}([0, 2L]) \times H^1_{per}([0, 2L]) \times L^2_{per}([0, 2L])$, that is, the standing waves are unstable with respect to periodic perturbations having twice the fundamental period as the corresponding wave. The ideas to obtain such result is to get orbital instability from the linear instability of the zero solution for the linearization of (17) around the orbit generated modulus phase, $\{P_{\omega}(\sqrt{2}\phi, \phi, 0, 0); t \in \mathbb{R}\}$, where $P_s$ is the transformation defined in (47). Note that $P_s$ (acting on real-valued functions) can be described as

$$
P_s \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos s & 0 & -\sin s & 0 \\ 0 & 1 & 0 & 0 \\ \sin s & 0 & \cos s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}.
$$

(53)

The clever reader has already noted that our notion of stability here is slightly different from that one in Subsection 2.3. In fact, our definition is only modulo phase.

**Definition 2.7.** Let $Y = H^1_{per}([0, 2L]) \times H^1_{per}([0, 2L]) \times L^2_{per}([0, 2L])$. The orbit generated modulus phase, $\{P_{sc}(\psi_c, \phi_c, 0); s \in \mathbb{R}\}$, is said to be orbitally stable in $Y$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $z_0 \in Y$ and $\|z_0 - (\psi_c, \phi_c, 0)\|_Y < \delta$, then the solution $z(t)$ of (17) with $z(0) = z_0$ exists for all $t$ and

$$
\sup_{t \geq 0} \inf_{s \in \mathbb{R}} \|z(t) - P_s(\psi_c, \phi_c, 0)\|_Y < \varepsilon.
$$
we see that
\[ \lambda = \{ P_{cs}(\sqrt{2}\phi, \phi, 0, 0), \ s \in \mathbb{R} \}, \]
we get the equation
\[ \frac{dV}{dt} = J\mathcal{L}_{cn}V + O(\|V\|^2), \]
where \( V(t) = P_{(-cs)}U(t) - (\sqrt{2}\phi, \phi, 0, 0) \), \( J \) and \( \mathcal{L}_{cn} \) are the operators defined in (7) and (34), respectively.

In order to prove that (54) has the zero solution as an unstable solution we know that it is sufficient to prove that \( J\mathcal{L}_{cn} \) has finitely many eigenvalues with strictly positive real part. Moreover, this implies that orbit \( \Omega \) is orbitally unstable (see [18], [19]). Actually, we prove:

**Theorem 2.8.** Let \( \phi = \phi_c \) be the cnoidal wave given by Corollary 1. Then the orbit
\[ \Omega = \{ P_{cs}(\sqrt{2}\phi, \phi, 0, 0), \ s \in \mathbb{R} \} \]
is orbitally unstable in the space \( Y = H^1_{per}([0, 2L]) \times H^1_{per}([0, 2L]) \times L^2_{per}([0, 2L]). \)

**Proof.** From Lemma 5.6 and Theorem 5.8 in [18], we know that \( J\mathcal{L}_{cn} \) has finitely many eigenvalues with strictly positive real part. Hence we only have to prove that \( J\mathcal{L}_{cn} \) has at least one eigenvalue with strictly positive real part. To prove this, we use the approach introduced by Grillakis [20]. We start by defining
\[ Z = [\text{Ker}(\mathcal{L}_{R, cn}) \cup \text{Ker}(\mathcal{L}_{T, cn})]^+, \]
\[ \mathcal{L}_{R, cn} = \text{restriction of } \mathcal{L}_{R, cn} \text{ on } Z \cap H^2_{per}([0, 2L]), \]
\[ \mathcal{L}_{T, cn}^{-1} = \text{restriction of } \mathcal{L}_{T, cn}^{-1} \text{ on } Z \cap H^2_{per}([0, 2L]). \]
With this definitions, Theorem 2.6 in [20] states that \( J\mathcal{L}_{cn} \) has exactly
\[ \max\{ n(\mathcal{L}_{R, cn}), n(\mathcal{L}_{T, cn}^{-1}) \} - d(C(\mathcal{L}_{R, cn}) \cap C(\mathcal{L}_{T, cn}^{-1})) \]
\[ \pm \text{ cone of real eigenvalues, where } C(\mathcal{L}) = \{ z \in Z; \langle Lz, z \rangle_{L^2_{per}} < 0 \} \text{ denotes the negative cone of operator } \mathcal{L} \text{ and } d(C(\mathcal{L})) \text{ denotes the dimension of a maximal linear subspace that is contained in } C(\mathcal{L}). \]

Hence we just need to prove that the number in (55) is strictly positive. We first observe that since zero is the first eigenvalue of \( \mathcal{L}_{T, cn} \) (see Theorem 2.2) it follows that such operator is a positive operator on \( Z \) and so \( C(\mathcal{L}_{T, cn}^{-1}) = 0 \) and \( n(\mathcal{L}_{T, cn}^{-1}) = 0 \). Thus, the number in (55) reduces to \( n(\mathcal{L}_{R, cn}) \).

Let us prove that \( n(\mathcal{L}_{R, cn}) = 2 \). Indeed, from the definition of \( \mathcal{L}_{R, cn} \) and Lemma 2.4 we see that \( n(\mathcal{L}_{R, cn}) \leq n(\mathcal{L}_{T, cn}) \leq 3 \). Now let \( \lambda_0, \lambda_1, \) and \( \lambda_2 \) be the three negative eigenvalues of operator \( \mathcal{L}_{1, cn} \) given by Lemma 2.4 with respective eigenfunctions \( \chi_0, \chi_1 \) and \( \chi_2 \). Thus,
\[ \chi_0 = (2\chi_0/3, \sqrt{2}\chi_0/3), \quad \chi_1 = (2\chi_1/3, \sqrt{2}\chi_1/3) \quad \text{and} \quad \chi_2 = (2\chi_2/3, \sqrt{2}\chi_2/3) \]
are eigenfunctions of \( \mathcal{L}_{R, cn} \) associated to eigenvalues \( \lambda_0, \lambda_1, \) and \( \lambda_2, \) respectively. Since \( \text{Ker}(\mathcal{L}_{T, cn}) \) is generated by \( \hat{\Phi} = (\phi, 0) \) (see Theorem 2.2), we obtain from Lemma 2.4 that
\[ \langle \chi_j, \hat{\Phi} \rangle_{L^2_{per} \times L^2_{per}} = \frac{2}{3} \langle \chi_j, \phi \rangle_{L^2_{per}} = 0, \quad j = 1, 2. \]
Moreover, \( \langle \chi_0, \Phi \rangle_{L^2_{per} \times L^2_{per}} = 2/3 \langle \chi_0, \phi \rangle_{L^2_{per}} > 0 \) since \( \chi_0 \) and \( \phi \) are strictly positive functions. This completes the proof of the Theorem. \( \square \)

3. ORBITAL STABILITY OF DINOIDAL WAVE SOLUTIONS FOR SYSTEM (1).

The purpose of this section is to establish the stability/instability of dinoidal wave solutions for system (1) when \( f(s, t) = st \), namely,

\[
\begin{align*}
  iu_t + \frac{1}{2} u_{xx} &= -v^2 u \\
  v_t - v_{xx} + m^2 v &= 2|u|^2 v.
\end{align*}
\]  

(56)

Here we look for solutions of the form

\[ u(x, t) = e^{ict} \phi_c(x), \quad v(x, t) = \phi_c(x) \]

where \( \phi_c \) is a smooth \( L \)-periodic real-valued function and \( c > 0 \) is the wave-speed. Substituting this form in (56) we see, after integration, that \( \phi_c \) must satisfy

\[
[\phi'_c]^2 = -\phi^4_c + 2c\phi^2_c + 2B_{\phi_c},
\]

for some real constant \( B_{\phi_c} \). A smooth positive periodic solution for (57) is given by (see e.g., [1] or [11])

\[
\phi_c(x) = \eta \text{dn}(\eta x; k), \quad k \in (0, 1),
\]

(58)

where

\[
\eta = \frac{2K}{L} \quad \text{and} \quad c = \frac{2K^2}{L^2}(2 - k^2).
\]

(59)

**Remark 5.** The solitary standing wave solution related to equation (57) with \( B_{\phi_c} \equiv 0 \) can be obtained from equation (58), namely,

\[
\phi_c(x) = \sqrt{c} \text{sech}(\sqrt{c}x), \quad x \in \mathbb{R}.
\]

(60)

If we consider the same steps as in those for proving Corollary 1, for each \( L > 0 \) we can construct a smooth branch (depending on \( c \)) of dinoidal waves having fundamental period \( L \). The next proposition summarizes these results.

**Proposition 1.** Let \( L > 0 \) fixed and \( c \in (\pi^2/L^2, +\infty) \). Then the dinoidal wave \( \phi_c \) given by (58) has fundamental period \( L \) and satisfies (57). Moreover, the mapping

\[
c \in (\pi^2/L^2, +\infty) \rightarrow \phi_c \in H^2_{per}(0, L)
\]

is a smooth function and the modulus \( k = k(c) \) satisfies \( dk/dc > 0 \).

**Remark 6.** The fact that \( dk/dc > 0 \) follows immediately from (59) and the Inverse Function Theorem.

3.1. Spectral analysis. Let \( \phi = \phi_c \) be the dinoidal wave given by Proposition 1. As in Subsection 2.2, we first note that system (56) can be write as an infinite-dimensional Hamiltonian system, namely,

\[
\frac{dU}{dt} = JE'_1(U(t)),
\]

where \( U = (u_1, v_1, u_2, v_2) = (\text{Re}(u), v, \text{Im}(u), v_t) \), \( J \) is the skew-symmetric matrix defined in (7) and \( E'_1 \) is the energy functional

\[
E_1(U) = \frac{1}{2} \int_0^L \left[ u_{1,x}^2 + u_{2,x}^2 + v_2^2 + v_1^2 + m^2 v_1^2 - 2u_1^2(u_1^2 + u_2^2) \right] dx.
\]

(61)
We observe that functional $\mathcal{F}$ given in (33) is also a conserved quantity of system (56). Hence, the linearized operator is this case is given by

$$L_{\text{dn}} := \mathcal{L}'_1(\phi, \phi, 0, 0) + c\mathcal{F}'(\phi, \phi, 0, 0) = \begin{pmatrix} \mathcal{L}_{\text{R},\text{dn}} & 0 \\ 0 & \mathcal{L}_{\text{I},\text{dn}} \end{pmatrix},$$

where

$$\mathcal{L}_{\text{R},\text{dn}} = \begin{pmatrix} -\frac{d^2}{dx^2} + 2c - 2\phi^2 & -4\phi^2 \\ -4\phi^2 & -\frac{d^2}{dx^2} + 2c - 2\phi^2 \end{pmatrix}$$

and

$$\mathcal{L}_{\text{I},\text{dn}} = \begin{pmatrix} -\frac{d^2}{dx^2} + 2c - 2\phi^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

As in Section 2, we see that operator $\mathcal{L}_{\text{R},\text{dn}}$ also can be diagonalized by a similarity transformation. Indeed, let

$$B_{\text{R}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

Then the operator $B_{\text{R}}\mathcal{L}_{\text{R},\text{dn}}B_{\text{R}}^{-1} =: \mathcal{L}_{\text{DR}}$ is a diagonal operator given by

$$\mathcal{L}_{\text{DR}} = \begin{pmatrix} \mathcal{L}_{1,\text{dn}} & 0 \\ 0 & \mathcal{L}_{3,\text{dn}} \end{pmatrix},$$

where

$$\mathcal{L}_{1,\text{dn}} = -\frac{d^2}{dx^2} + 2c - 6\phi^2$$

and

$$\mathcal{L}_{3,\text{dn}} = -\frac{d^2}{dx^2} + 2c + 2\phi^2$$

Now we can prove:

**Theorem 3.1.** Let $\phi = \phi_c$ be the dnoidal wave solution given by Proposition 1. Then,

(i) operator $\mathcal{L}_{\text{R},\text{dn}}$ in (63) defined in $L^2_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])$ whose domain is $H^2_{\text{per}}([0, L]) \times H^2_{\text{per}}([0, L])$ has exactly one negative eigenvalue which is simple; zero is a simple eigenvalue. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.

(ii) Operator $\mathcal{L}_{\text{I},\text{dn}}$ in (64) defined in $L^2_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])$ whose domain is $H^2_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])$ has only non-negative eigenvalues being zero the first one which is simple. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.
Proof. The procedure here follows from the same arguments in Theorem 2.2 and therefore we will give only the main steps.

To prove (i), we observe that operator $L_{1,\text{dn}}$ in (66), defined in $L^2_{\text{per}}([0, L])$ with domain $H^1_{\text{per}}([0, L])$ has exactly one negative eigenvalue and zero is a simple eigenvalue with eigenfunction $\phi'$. Moreover, operator $L_{3,\text{dn}}$ in (67) (defined in $L^2_{\text{per}}([0, L])$ with domain $H^2_{\text{per}}([0, L])$) is such that $\sigma(L_{3,\text{dn}}) \geq 2c$. Hence from (65) and arguments similar as the ones in Theorem 2.2 we prove part (i).

To prove part (ii), we just note that zero is the first eigenvalue of operator $L_{2,\text{dn}} = -\frac{d^2}{dx^2} + 2c - 2\phi^2$ with eigenfunction $\phi$. So the proof is completed. □

Remark 7. To show the spectral properties of operator $L_{1,\text{dn}}$ used in the proof of Theorem 3.1 one needs to see that the periodic eigenvalue problem associated to $L_{1,\text{dn}}$, posed on the interval $[0, L]$, is equivalent to the periodic eigenvalue problem associated to the Lamé operator

$$L_{\text{Lame}} = -\frac{d^2}{dx^2} + 6k^2 \text{sn}^2(x; k),$$

posed on the interval $[0, 2K]$ (see [1], [7]).

Next lemma is in the same spirit of Lemma 2.4.

Lemma 3.2. Let $\phi = \phi_c$ be the cnoidal wave solution given by Proposition 1. Then, linear operator $L_{1,\text{dn}}$ in (66) defined in $L^2_{\text{per}}([0, 2L])$ with domain $H^1_{\text{per}}([0, 2L])$ has its first four eigenvalues simple, being the eigenvalue zero the fourth one with eigenfunction $\phi'$. Moreover, if $\xi_1$ and $\xi_2$ denote the eigenfunctions associated to the second and third eigenvalues then $\xi_i \perp \phi$, $i = 1, 2$.

Proof. The proof follows combining the periodic and semi-periodic eigenvalues problems associated to operator $L_{1,\text{dn}}$ with the equivalent problem associated to the Lamé operator in (68) (see e.g., [7]). □

3.2. Stability results. The procedure here follows the same steps as in Subsection 2.3. Our stability theorem reads as follows:

Theorem 3.3. Let $\phi_c$ be the dnoidal wave solution given by Proposition 1. Then the periodic wave solution $(e^{iEt}\phi_c, \phi_c)$ is orbitally stable in $X$ by the periodic flow of system (56) in the sense of Definition 2.5.

Proof. We apply the abstract Stability Theorem in [18] again. It follows from Theorem 3.1 that operator $L_{\text{dn}}$ has a unique negative eigenvalue, its kernel is two-dimensional and the remainder of the spectrum is bounded away from zero. Moreover, by our construction

$$E_1'(\phi_c, \phi_c, 0, 0) + cF'(\phi_c, \phi_c, 0, 0) = 0.$$

Hence, it remains only to show that $d''(c) > 0$ where

$$d(c) = E(\phi_c, \phi_c, 0, 0) + cF(\phi_c, \phi_c, 0, 0).$$

From (69) we have $d'(c) = F(\phi_c, \phi_c, 0, 0)$. Therefore,

$$d''(c) = \frac{d}{dc} \left( \int_0^L \phi_c^2(x)dx \right)$$

(70)

$$= \frac{4}{L} \frac{d}{dc} \left( K(k) \int_0^K \text{dn}^2(x)dx \right) = \frac{4}{L} \frac{d}{dk} (K(k)E(k)) \frac{dk}{dc}.$$
Since \( k \in (0, 1) \mapsto K(k)E(k) \) is a strictly increasing function and \( dk/dc > 0 \) (see Proposition 1) we see from (70) that \( d''(c) > 0 \). This completes the proof of the theorem. \(\square\)

**Remark 8.** As mentioned in Remark 4, by similar arguments presented in this section, one can obtain the stability of the solitary standing wave solution of the form \((e^{ict}\phi_c(x), \phi_c(x))\), \(x \in \mathbb{R}, c > 0\), for equation (1) with \(f(s, t) = st\), where \(\phi_c\) is given by (60).

### 3.3. Instability results

The idea here is to prove a similar result as the one in Subsection 2.4, in which we prove an orbital instability result taking the advantage that the linearized system has the zero solution as an unstable solution.

We first observe that linearizing system (56) around the orbit,

\[ \Gamma = \{ P_s(\phi_c, \phi_c, 0, 0) \mid s \in \mathbb{R} \}, \]

where \(P_s\) is defined in (47), we face the equation

\[ \frac{dW}{dt} = JLdnW + O(\|W\|^2), \]

where \(W(t) = P(-ct)U(t) - (\phi_c, \phi_c, 0, 0)\), \(J\) and \(Ldn\) are defined in (7) and (62), respectively.

Thus we can prove:

**Theorem 3.4.** Let \(\phi_c\) be the dnoidal wave given by Proposition 1. Then the orbit

\[ \Gamma = \{ P_s(\phi_c, \phi_c, 0, 0) \mid s \in \mathbb{R} \} \]

is orbitally unstable in \(Y\) in the sense of Definition 2.7.

**Proof.** The proof follows the same analytic-functional approach introduced in Theorem 2.8 (with obvious modifications), taking into account Theorem 3.1 and Lemma 3.2 instead of Theorem 2.2 and Lemma 2.4. So, we will omit the details. \(\square\)

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