ENRICHED ∞-OPERADS

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ABSTRACT. In this paper we initiate the study of enriched ∞-operads. We introduce several models for these objects, including enriched versions of Barwick’s Segal operads and the dendroidal Segal spaces of Cisinski and Moerdijk, and show these are equivalent. Our main results are a version of Rezk’s completion theorem for enriched ∞-operads: localization at the fully faithful and essentially surjective morphisms is given by the full subcategory of complete objects, and a rectification theorem: the homotopy theory of ∞-operads enriched in the ∞-category arising from a nice symmetric monoidal model category is equivalent to the homotopy theory of strictly enriched operads.

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1. Introduction

Operads are a convenient formalism for parametrizing many algebraic structures of interest in mathematics. Roughly speaking, an operad\(^1\) is a structure similar to a category, but instead of morphisms with a single source and target, an operad has multimorphisms with a list of objects as a source; moreover, there is a \(\Sigma_n\)-action on the multimorphisms with \(n\) inputs that permutes these. If \(C\) is a symmetric monoidal category then one can view \(C\) as an operad where a multimorphism \((x_1, \ldots, x_n) \to y\) is given by a morphism \(x_1 \otimes \cdots \otimes x_n \to y\). If \(O\) is an operad we can then define an \(O\)-algebra in \(C\) to be a functor of operads \(O \to C\). Many interesting algebraic structures arise as algebras in this sense, including associative algebras, commutative algebras, and enriched categories with a fixed set of objects.

For many purposes however, it is necessary to generalize operads to enriched operads — here we replace the set of multimorphisms with an object of some symmetric monoidal category. Then we can, for example, describe Lie algebras or Poisson algebras as algebras for operads enriched in abelian groups.

In topology, we often encounter operads enriched in topological spaces or simplicial sets, known as topological and simplicial operads. Indeed, it was this setting that originally motivated the introduction of operads back in the 1970s: \(n\)-fold loop spaces admit natural multiplications where algebraic identities, such as associativity, only hold up to coherent homotopy, and this structure can be codified as the structure of an algebra over a topological operad \(E_n\), defined using spaces of “little discs” in \(\mathbb{R}^n\). These operads were introduced by Boardman–Vogt [4] and May [39], who both proved versions of the recognition principle for \(n\)-fold loop spaces: \(n\)-fold loop spaces are precisely the spaces that admit the structure of a grouplike\(^2\) \(E_n\)-algebra.

For applications in algebraic topology we typically only care about the weak homotopy types of the spaces of multimorphisms in a simplicial or topological operad. We are therefore led to consider the homotopy theory of such operads. This can be done by imposing a model structure, with the weak equivalences being a suitable notion of maps that are “fully faithful and essentially surjective up to homotopy” (often called Dwyer–Kan equivalences). Such a model structure on simplicial operads has been constructed by Cisinski–Moerdijk [12] and by Robertson [46].

Unfortunately, for many purposes this model structure is not as well-behaved as one might have hoped. For example, Boardman and Vogt constructed a tensor product of simplicial operads whose internal Hom gives simplicial operads of algebras, but this is well-known not to be homotopically well-behaved\(^3\) so that these simplicial operads of algebras are typically not homotopically meaningful.

We can improve the situation by replacing simplicial operads by a weakly equivalent, but more flexible, notion in the form of \(\infty\)-operads. Roughly speaking, an \(\infty\)-operad is analogous to a simplicial operad, but composition of multimorphisms is not strictly associative, but rather associative up to coherent homotopy. The first, and by far the best developed, model for \(\infty\)-operads is that of Lurie [38]; other models include the dendroidal sets of Moerdijk–Weiss [40], the dendroidal Segal spaces of Cisinski–Moerdijk [11], and the complete Segal operads of Barwick [1]. (These are all known to be equivalent, due to the results of [1, 27, 10].) There is an analogue of the Boardman–Vogt tensor product for \(\infty\)-operads, and on the \(\infty\)-category of \(\infty\)-operads this is as well-behaved as one might wish.

However, there are other homotopical contexts where we want to consider enriched operads. For example, in algebraic settings we often encounter operads enriched in chain complexes (usually called dg-operads), where we only care about the specific chain complexes up to quasi-isomorphism.

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\(^1\)The term operad is often used only for the single-object version of this notion, with the many-object case we consider known as a coloured operad or multicategory; we have chosen to use the shorter term operad for the more general notion, as seems to be increasingly common in the current literature.

\(^2\)Meaning the induced associative multiplication on the set of connected components makes this a group.

\(^3\)In particular, it does not make simplicial operads a symmetric monoidal model category; see the discussion at https://mathoverflow.net/questions/198205/boardman-vogt-tensor-product.
Similarly, in stable homotopy theory we might want to consider operads enriched in spectra. In both cases model structures with the DK-equivalences as weak equivalences were constructed by Caviglia [5, 6], but in the case of dg-operads only over a field of characteristic zero — indeed, even with a fixed set of objects there do not seem to exist model structures on dg-operads in positive characteristic.

In this paper we extend the \(\infty\)-categorical approach to homotopy-coherent algebraic structures to contexts such as these, by laying the foundations for a theory of enriched \(\infty\)-operads. This allows for enrichment in any symmetric monoidal \(\infty\)-category, thereby giving a well-behaved homotopy theory of \(\infty\)-operads enriched in, for example, spectra, chain complexes (in any characteristic), and modules over a commutative ring spectrum, as well as in more exotic contexts such as quasicoherent sheaves on a derived stack. Although we obtain the same homotopy theory as that presented by model categories of enriched operads (when these exist), our \(\infty\)-categorical approach is better-behaved in several respects — most notably, we obtain the homotopically correct \(\infty\)-categories of algebras for enriched \(\infty\)-operads simply as the right adjoint to a natural tensoring with \(\infty\)-categories.

Although our concerns in the present paper are foundational, we expect the theory of enriched \(\infty\)-operads to have a number of interesting applications, particularly in the context of Koszul duality. Koszul duality for dg-operads was first introduced in [21] and was later studied in, for example, [20, 15, 16, 36, 49]. Though it is currently best understood over a field of characteristic zero (which is the context for the papers just cited), Koszul duality also occurs in spectra [8, 9] where it is closely related to Goodwillie calculus [7]. For spectra, however, it seems likely that a full understanding of Koszul duality requires \(\infty\)-categorical methods, as coalgebraic structures in spectra are difficult to work with using model categories. More generally, Koszul duality should occur for stable symmetric monoidal \(\infty\)-categories — indeed, in this setting Francis–Gaitsgory [14] have used the expected properties of enriched \(\infty\)-operads to obtain Koszul duality equivalences under certain finiteness hypotheses (including in the case of chiral algebras) and also conjectured how this should generalize.

1.1. Main Results. Our first goal is to define \(\infty\)-operads enriched in a symmetric monoidal \(\infty\)-category \(\mathcal{V}\) and set up their “algebraic” homotopy theory (i.e. without inverting the fully faithful and essentially surjective maps). In §2 we do this by considering an enriched analogue of Barwick’s approach to \(\infty\)-operads: we define, given a presentably symmetric monoidal \(\infty\)-category \(\mathcal{V}\), a notion of continuous Segal presheaves on an \(\infty\)-category \(\Delta^{\mathcal{V}}\). We then show that the \(\infty\)-category \(P_{\text{ctsSeg}}(\Delta^{\mathcal{V}})\) of these objects has several pleasant properties:

- it is presentable,
- it is tensored and cotensored over Segal spaces — giving Segal spaces of algebras by adjunction,
- if \(\mathcal{V}\) is the \(\infty\)-category \(\mathcal{S}\) of spaces, with the Cartesian product as symmetric monoidal structure, then it is equivalent to Barwick’s \(\infty\)-category of Segal operads,
- it is functorial for lax symmetric monoidal functors.

To show the last point, we prove that \(P_{\text{ctsSeg}}(\Delta^{\mathcal{V}})\) is equivalent to an alternative model, using algebras in \(\mathcal{V}\) for \(\infty\)-categories \(\Delta^{\mathcal{V}}_{\text{op}}\), for which this functoriality is obvious. (This model also makes sense without assuming presentability, but it is not our main focus as our other results are more easily established using Segal presheaves.)

Just as in the case of Segal spaces, to obtain the “correct” \(\infty\)-category of enriched \(\infty\)-operads we need to invert the fully faithful and essentially surjective maps. We introduce these in §3 and then prove our first main result, an analogue of Rezk’s completion theorem for Segal spaces in this context:

**Theorem 1.1.1.** The \(\infty\)-category \(\text{Opd}_{\infty}^{\mathcal{V}}\) obtained as the localization of \(P_{\text{ctsSeg}}(\Delta^{\mathcal{V}})\) at the fully faithful and essentially surjective morphisms is given by the full subcategory of complete objects, i.e. those whose underlying Segal space is complete in the sense of Rezk.

The proof follows the same outline as the completion theorem for enriched \(\infty\)-categories from [17], which in turn is a variant of Rezk’s original proof [45]; the main new ingredient needed is

\[\text{See also Remark 5.2.4 for a discussion of model structures with a fixed set of objects.}\]
the tensoring of $P_{\text{CtsSeg}}(\Delta_\infty^Y)$ with complete Segal spaces, constructed in §2.8. This moreover gives natural $\infty$-categories of algebras for enriched $\infty$-operads by adjunction.

In §4 we turn to an enriched analogue of the dendroidal Segal spaces of Cisinski and Moerdijk: We again consider a notion of continuous Segal presheaves, now on an $\infty$-category $\Omega^V$ enhancing the dendroidal category $\Omega$. Our main result here extends the comparison result of [10] to the enriched setting:

**Theorem 1.1.2.** There is an equivalence of $\infty$-categories $P_{\text{CtsSeg}}(\Delta_\infty^Y) \simeq P_{\text{CtsSeg}}(\Omega^V)$.

Once the definitions are set up, the proof proceeds essentially as in [10].

Finally, in §5 we prove our third main result, which relates our enriched $\infty$-operads to the existing homotopy theories of operads strictly enriched in model categories. Specifically, we prove the following rectification theorem:

**Theorem 1.1.3.** Suppose $V$ is a nice symmetric monoidal model category, and let $V[W^{-1}]$ denote the symmetric monoidal $\infty$-category obtained by inverting the weak equivalences. Then the $\infty$-category $\text{Opd}_V[W^{-1}]$ of $\infty$-operads enriched in $V[W^{-1}]$ is equivalent to the $\infty$-category $\text{Opd}_V[\text{DK}^{-1}]$, obtained by inverting the Dwyer–Kan equivalences between strictly enriched $V$-operads.

This result applies, for example, with $V$ being simplicial sets (where we recover a result of Cisinski–Moerdijk [12]), symmetric spectra, or chain complexes over a field of characteristic 0. To prove this we first show that $V$-enriched $\infty$-operads with a fixed set of objects are equivalent to algebras in $V$ for the $\infty$-operad obtained from the classical operad for $S$-coloured operads; the rectification result then reduces to a rectification result for operad algebras due to Pavlov and Scholbach [42].

### 1.2. Notation and Terminology.

- We assume the existence of three nested Grothendieck universes; the sets contained in them are called small, large and very large, respectively.
- To the greatest extent possible, we work with $\infty$-categories without mentioning their specific implementation as quasicategories. In particular, we do not distinguish notationally between a category and its nerve and all categorical constructions such as taking (co)limits should be understood in the $\infty$-categorical setting.
- We write $S$ for the $\infty$-category of spaces (or $\infty$-groupoids) and, for an $\infty$-category $C$, we write $P(C)$ for the $\infty$-category $\text{Fun}(C^{op}, S)$ of presheaves of spaces on $C$.
- To ease notation we will often leave the Yoneda embedding implicit, i.e. if $C$ is a small $\infty$-category and $c$ is an object of $C$ we will also use $c$ to denote the presheaf in $P(C)$ represented by $c$.
- We denote the usual simplicial indexing category by $\Delta$.
- We denote the unit of a symmetric monoidal $\infty$-category $V$ by $1_V$, or just $1$ if $V$ is clear from the context.
- We write $\mathbb{F}$ for a skeleton of the category $\text{Fin}$ of finite sets, spanned by $n := \{1, \ldots, n\}$. Similarly, we write $\mathbb{F}_*$ for a skeleton of the category $\text{Fin}_*$ of finite pointed sets, spanned by $\langle n \rangle := n_* := (\{1, \ldots, n\}, *)$.
- For a finite set $K$, we write $K_+$ for the pointed set obtained from $K$ by adjoining a disjoint basepoint. If $|K| = n$, we will often implicitly identify $K_+$ with $\langle n \rangle$ and thus regard $K_+$ as an object of $\mathbb{F}_*$.

For the reader’s convenience we also recall some definitions and notational conventions related to symmetric monoidal $\infty$-categories and $\infty$-operads, as presented in [38].

**Definition 1.2.1.** A morphism $f: \langle m \rangle \to \langle n \rangle$ in $\mathbb{F}_*$ is called inert if the preimage $f^{-1}(i)$ of $i$ has exactly one element for every $i \neq *$. For $1 \leq i, j \leq n$, we write $\rho_i: \langle n \rangle \to \langle 1 \rangle$ for the inert map determined by

$$
\rho_i(j) = \begin{cases} 
1 & \text{if } i = j \\
* & \text{otherwise.}
\end{cases}
$$
A morphism \( f : \langle m \rangle \to \langle n \rangle \) is called active if \( f^{-1}(\ast) = \ast \). The inert and active morphisms form a factorization system on \( \mathbb{F}_* \).

**Definition 1.2.2.** A symmetric monoidal \( \infty \)-category is a coCartesian fibration \( C^\otimes \to \mathbb{F}_* \), such that, for every \( n \geq 0 \), the induced functors \( \rho_{i,1} : C^\otimes_{\langle n \rangle} \to C^\otimes_{\langle 1 \rangle} \) for \( 0 < i \leq n \) exhibit \( C^\otimes_{\langle i \rangle} \) as the product \( C^\otimes_{\langle 1 \rangle} \times C^\otimes_{\langle i \rangle} \). We often denote a symmetric monoidal \( \infty \)-category just by \( C^\otimes \), leaving the coCartesian fibration to \( \mathbb{F}_* \) implicit. Moreover, if \( C^\otimes \) denotes a symmetric monoidal \( \infty \)-category then we write \( C \) for \( C^\otimes_{\langle 1 \rangle} \) and refer to this as the underlying \( \infty \)-category of \( C^\otimes \). In this situation we will also, somewhat informally, refer to \( C^\otimes \) as a symmetric monoidal structure on \( C \).

**Definition 1.2.3.** We say that a symmetric monoidal \( \infty \)-category \( V^\otimes \) is presentably symmetric monoidal if the underlying \( \infty \)-category \( V \) is presentable and the tensor product preserves colimits in each variable.

**Remark 1.2.4.** A symmetric monoidal \( \infty \)-category corresponds (via the straightening equivalence of \([37, \S 3.2]\)) to a functor \( F : \mathbb{F}_* \to \text{Cat}_\infty \) such that the map \( F(\langle n \rangle) \rightarrow F(\langle 1 \rangle)^\times n \) induced by the maps \( \rho_i \) is an equivalence. Thus the definition of symmetric monoidal \( \infty \)-categories is analogous to that of (special) \( \Gamma \)-spaces, introduced as models for \( E_\infty \)-spaces by Segal \([47]\).

**Definition 1.2.5.** An \( \infty \)-operad is a functor \( p : \mathcal{O} \to \mathbb{F}_* \), satisfying the following conditions:

1. For every inert morphism \( f : \langle m \rangle \to \langle n \rangle \) and every object \( x \in \mathcal{O}_{\langle m \rangle} \), there is a \( p \)-coCartesian lift of \( f \) at \( x \).

2. Let \( x \in \mathcal{O}_{\langle m \rangle} \), \( y \in \mathcal{O}_{\langle n \rangle} \) be objects and let \( f : \langle m \rangle \to \langle n \rangle \) be a morphism in \( \mathbb{F}_* \). Let \( \text{Map}_\mathcal{O}(x,y) \) denote the union of those connected components of \( \text{Map}_\mathcal{O}(x,y) \) which lie over \( f \). The coCartesian lifts \( y \to \rho_{i,1}(y) \) of the inert morphisms \( \rho_i : \langle n \rangle \to \langle 1 \rangle, 1 \leq i \leq n \), induce a map

\[
\text{Map}_\mathcal{O}(x,y) \to \prod_{1 \leq i \leq n} \text{Map}_\mathcal{O}(x,\rho_{i,1}(y))
\]

which is an equivalence of spaces.

3. The functors \( \rho_{i,1} : \mathcal{O}_{\langle n \rangle} \to \mathcal{O}_{\langle 1 \rangle} \), \( 1 \leq i \leq n \), given by coCartesian pushforward along \( \rho_i \), give an equivalence

\[
\mathcal{O}_{\langle n \rangle} \to (\mathcal{O}_{\langle 1 \rangle})^\times n.
\]

**Warning 1.2.6.** For \( \infty \)-operads, our notational convention is slightly different from that of \([38]\), where \( \infty \)-operads are generally denoted \( \mathcal{O}^\otimes \), with \( \mathcal{O} \) referring to the \( \infty \)-category \( \mathcal{O}^\otimes_{\langle 1 \rangle} \). For \( \infty \)-operads that are not symmetric monoidal \( \infty \)-categories, however, this \( \infty \)-category is typically not of particular interest and is only rarely referred to. We therefore use the simpler notation \( \mathcal{O} \) for an \( \infty \)-operad, without having a special notation for the \( \infty \)-category \( \mathcal{O}_{\langle 1 \rangle} \).

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2. **Enriched \( \infty \)-Operads as Segal Presheaves**

In this section we introduce and study our first model for \( \mathcal{V} \)-enriched \( \infty \)-operads, as presheaves on a certain \( \infty \)-category \( \Delta^\mathcal{V}_\infty \) satisfying Segal and continuity conditions. In \( \S 2.1 \) we warm up to this by recalling the analogous definition of enriched \( \infty \)-categories as continuous Segal presheaves from \([17]\). In \( \S 2.2 \) we then recall the definition of the category \( \Delta^\mathcal{V}_\infty \) and some of its basic properties.
from [1], before we define $\Delta^Y$ and continuous Segal presheaves on it in §2.3; these form a full subcategory $P_{\text{CtsSeg}}(\Delta^Y)$ of presheaves on $\Delta^Y$. Next, §2.4 introduces an alternative model for enriched $\infty$-operads, using algebras in $\mathcal{V}$ for certain $\infty$-categories $\Delta^X_{\leq}$; we prove this is equivalent to $P_{\text{CtsSeg}}(\Delta^Y)$, but this model has the advantage that certain functorial properties are obvious.

The goal of §2.5 is to show that if we enrich in the $\infty$-category $P(\mathbb{I})$ of presheaves on a small symmetric monoidal $\infty$-category $\mathbb{I}$ (using the Day convolution) then $P_{\text{CtsSeg}}(\Delta^Y)$ is equivalent to Segal presheaves on the much smaller $\infty$-category $\Delta^Y$; as a special case, we see that continuous Segal presheaves for $\mathbb{S}$ are equivalent to Barwick’s Segal operads, i.e. Segal presheaves on $\Delta$. We extend this in §2.6 to get a small presentation of $P_{\text{CtsSeg}}(\Delta^Y)$ when $\mathcal{V}$ is a localization of a presheaf $\infty$-category $P(\mathbb{I})$; this allows us to show that $P_{\text{CtsSeg}}(\Delta^Y)$ is presentable.

In §2.7 we then study inner anodyne maps in $P(\Delta^Y)$, in order to construct the tensoring of $P_{\text{CtsSeg}}(\Delta^Y)$ with Segal spaces in §2.8. Finally, in §2.9 we discuss the underlying enriched $\infty$-category of an enriched $\infty$-operad, which will be needed in §3.

2.1. Enriched $\infty$-Categories as Segal Presheaves. As motivation for our definitions of enriched $\infty$-operads as Segal presheaves, in this subsection we discuss the analogous definition in the simpler setting of enriched $\infty$-categories. This model of enriched $\infty$-categories as Segal presheaves was briefly introduced in [17]. We begin by recalling the definition of Rezk’s Segal spaces, introduced in [45] as a model for $\infty$-categories.

Definition 2.1.1. A presheaf $F: \Delta^{op} \to \mathcal{S}$ is called a Segal space if it satisfies the Segal condition, i.e., for every $[n] \in \Delta^{op}$, the map

$$F([n]) \to F([1]) \times_{F([0])} \cdots \times_{F([0])} F([1]),$$

induced by the maps $[1] \simeq \{i - 1, i\} \to [n]$ and the maps $[0] \to [n]$ in $\Delta$, is an equivalence in $\mathcal{S}$. We write $P_{\text{Seg}}(\Delta)$ for the full subcategory of $P(\Delta)$ spanned by Segal spaces.

Remark 2.1.2. For every $[n] \in \Delta^{op}$, the object $F([1]) \times_{F([0])} \cdots \times_{F([0])} F([1])$ occurring in the previous definition should be thought of as the space of sequences of $n$ composable morphisms. Similarly, $F([n])$ should be interpreted as the space of all these sequences together with sequences of composites of adjacent maps. The Segal condition then says that there exists a homotopy coherent composition for composable morphisms. By the Yoneda Lemma, a presheaf $F: \Delta^{op} \to \mathcal{S}$ is a Segal space if and only if it is local with respect to all spine inclusions $\Delta^1 \amalg \cdots \amalg \Delta^0 \Delta^1 \to \Delta^n$.

Definition 2.1.3. Given a symmetric monoidal $\infty$-category $\mathcal{V}^{\otimes}$, let $\mathcal{V} \to \mathcal{F}_{\mathbb{S}}^{\otimes}$ denote the Cartesian fibration corresponding to the same functor $\mathbb{F}_\ast \to \text{Cat}_{\infty}$ as $\mathcal{V}^{\otimes} \to \mathbb{F}_\ast$. We define the $\infty$-category $\Delta^Y$ by the pullback square

$$\begin{array}{ccc}
\Delta^Y & \longrightarrow & \mathcal{V}^{\otimes} \\
\downarrow & & \downarrow \\
\Delta & \longrightarrow & \mathcal{F}_{\mathbb{S}}^{\otimes},
\end{array}$$

where the functor $V: \Delta^{op} \to \mathbb{F}_\ast$ takes $[n]$ to $\langle n \rangle$ and a morphism $f: [n] \to [m]$ in $\Delta$ to the morphism $V(f): \langle m \rangle \to \langle n \rangle$ in $\mathbb{F}_\ast$ given by

$$V(f)(i) \mapsto \begin{cases} j, & \text{if } f(j - 1) < i \leq f(j), \\ * & \text{otherwise}. \end{cases}$$

Remark 2.1.4. If we regard an object $[n] \in \Delta$ as a linear tree with $n$ vertices and $n + 1$ edges, then the functor $V^{\otimes}$ takes $[n]$ to the disjoint union of the set of vertices of $[n]$ with the base point. The functor $V$ is identical to the composite of the functor $\text{Cut}: \Delta \to \text{Assoc}^{\otimes}$ defined in [38, Construction 4.1.2.9] and the forgetful functor $\text{Assoc}^{\otimes} \to \mathbb{F}_\ast$. In particular, it follows from the construction that an object of $\Delta^Y$ lying over $[n] \in \Delta$ can be identified with a sequence $(v_1, \ldots, v_n)$ of objects of $\mathcal{V}$; we write $[n]_{(v_1)}_{1 \leq i \leq n}$ for this object.
Remark 2.1.5. There is no need to require $\mathcal{V}$ to be symmetric monoidal here — we can equally well work with monoidal $\infty$-categories, which can be described as coCartesian fibrations over $\Delta^{\text{op}}$, satisfying Segal conditions, and this is the definition used in [17]. We have chosen to restrict to the symmetric monoidal case here for consistency with our discussion of $\infty$-operads below.

Definition 2.1.6. Suppose $\mathcal{V}$ is a symmetric monoidal $\infty$-category. A presheaf $F \in \mathcal{P}(\Delta^{\text{op}})$ is a Segal presheaf if for every object $[n]\{v_i\}_{1 \leq i \leq n}$, the map
$$F([n]\{v_i\}_{1 \leq i \leq n}) \to F([1]\{v_1\}) \times_{F([0])} \cdots \times_{F([0])} F([1]\{v_n\}),$$
induced by composition with the Cartesian morphisms over $\rho_i: [1] \to [n]$ and $[0] \to [n]$, is an equivalence.

Definition 2.1.7. Suppose $\mathcal{V}$ is a presentably symmetric monoidal $\infty$-category. Then a presheaf $F \in \mathcal{P}(\Delta^{\text{op}})$ is a continuous Segal presheaf if it is a Segal presheaf and it is continuous in the sense that the functor
$$\mathcal{V}^{\text{op}} \simeq (\Delta^{\text{op}},\mathcal{V})_{[1]} \to \mathcal{S}_{/\mathcal{V}[0]}$$
preserves limits.

Remark 2.1.8. A continuous Segal presheaf $\mathcal{C}$ on $\Delta^{\text{op}}$ encodes a $\mathcal{V}$-enriched $\infty$-category in the following way: $\mathcal{C}([0])$ is the space of objects, and for $x, y \in \mathcal{C}([0])$ the functor
$$\mathcal{C}([1](\cdot,x,y)) : \mathcal{V}^{\text{op}} \to \mathcal{S}$$
given by taking the fibre of $\mathcal{C}([1](\cdot))$ at $x, y$, preserves limits. Since $\mathcal{V}$ is presentable, it is therefore represented by an object $\mathcal{C}(x,y) \in \mathcal{V}$ — this gives the morphisms from $x$ to $y$.

2.2. The Category $\mathcal{D}_\mathcal{F}$. We now wish to introduce a definition of enriched $\infty$-operads analogous to that of enriched $\infty$-categories we just discussed. Our starting point will be Barwick’s definition of $\infty$-operads as Segal presheaves on a category $\mathcal{D}_\mathcal{F}$. In this subsection we recall the definition of this category and its basic structure, before turning to the relevant Segal conditions in the following subsection.

Definition 2.2.1. We let $\mathcal{D}_\mathcal{F} \to \Delta$ be the Grothendieck fibration $\mathcal{D}_\mathcal{F} \to \Delta$ associated to the functor $\text{Fun}(-,\mathcal{F})$, where we view the objects of $\Delta$ as categories in the usual way.

Remark 2.2.2. An object in $\mathcal{D}_\mathcal{F}$ is of the form $([m],f)$, where $[m] \in \Delta$ and $f$ is a functor $[m] \to \mathcal{F}$. We can think of this object as a sequence of morphisms $f(0) \to f(1) \to \ldots \to f(m)$ in $\mathcal{F}$. A morphism $([m],g) \to ([m],f)$ in $\mathcal{D}_\mathcal{F}$ is given by a morphism $\alpha: [n] \to [m]$ in $\Delta$ and a natural transformation $\phi: g \Rightarrow f \circ \alpha$. The morphism $(\alpha,\phi): ([m],g) \to ([m],f)$ is then a Cartesian lift of $\alpha$ at the object $([m], f)$ if and only if the natural transformation $\phi: g \Rightarrow f \circ \alpha$ is a natural isomorphism.

Definition 2.2.3. Let $\mathcal{D}_\mathcal{F}$ be the subcategory of $\mathcal{D}_\mathcal{F}$ containing all the objects, but only the morphisms $(\alpha,\phi): ([n],g) \to ([m],f)$ which satisfy the following conditions:

1. For every $k$, $0 \leq k \leq n$, the morphism $\phi_k : g(k) \to f(\alpha(k))$ is injective.
2. For $k,l$, $0 \leq k \leq l \leq n$, the induced square
$$\begin{array}{ccc}
g(k) & \xrightarrow{\phi_k} & f(\alpha(k)) \\
\downarrow & & \downarrow \\
g(l) & \xrightarrow{\phi_l} & f(\alpha(l)).
\end{array}$$
is a pullback square in $\mathcal{F}$.

We say an object $([m], f) \in \mathcal{D}_\mathcal{F}$ has length $m$.

Notation 2.2.4. For an object $([m], f) \in \mathcal{D}_\mathcal{F}$ and $0 \leq i \leq j \leq m$, let $f^{i,j} : f(i) \to f(j)$ denote the image of the morphism $i \to j$ in $[m]$ under the functor $f: [m] \to \mathcal{F}$. For a subset $S \subseteq f(j)$, let $f(i)_S$ denote the induced fibre product $f(i) \times_{f(j)} S$. We often write $I$ for an object $([m], f) \in \mathcal{D}_\mathcal{F}$, if there
is no need to emphasize the length of the object. For \( I = ([m], f) \) and \( k \in f(m) \), we write \( I_k \) for the object in \( \Delta_{\mathcal{F}} \) given by the sequence
\[
f(0)_{\{k\}} \to \cdots \to f(m-1)_{\{k\}} \to \{k\}.
\]

**Remark 2.2.5.** As mentioned above, an object \( ([m], f) \) in \( \Delta_{\mathcal{F}} \) is given by a sequence of morphisms
\[
f(0) \to f(1) \to \ldots \to f(m)
\]
in \( \mathcal{F} \). Objects in \( \Delta_{\mathcal{F}} \) can be thought of as graphs (with levels), with *vertices* and *edges* given by the sets \( \prod_{i=0}^{m} f(i) \) and \( \prod_{i=0}^{m} f(i) \), respectively. Given a vertex \( v \in f(i) \subseteq \prod_{i=0}^{m} f(i) \), we say that an edge \( e \in \prod_{i=0}^{m} f(i) \) is an incoming edge of \( v \) if and only if \( e \in f(i-1)v \subseteq f(i-1) \) and \( e \) is the unique outgoing edge of \( v \) if and only if \( e \) and \( v \) correspond to the same element in \( f(i) \). Here is an example of a typical object in \( \Delta_{\mathcal{F}} \)

\[
\begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array}
\]

which corresponds to a sequence \( 3 \to 3 \to 4 \) of finite sets. We see that an object \( ([m], f) \) can be regarded as a (non-planar) tree if \( f(m) = 1 \), and more generally as a finite forest containing \( |f(m)| \) trees.

**Definition 2.2.6.** We say a morphism \( \alpha: [n] \to [m] \) in \( \Delta \) is *inert* if it is the inclusion of a subinterval in \([m] \), i.e. if \( \alpha(i) = \alpha(0) + i \) for all \( i \), and *active* if it preserves the boundary, i.e. if \( \alpha(0) = 0 \) and \( \alpha(n) = m \). We say a map \((\alpha, \phi): ([n], g) \to ([m], f) \) in \( \Delta_{\mathcal{F}} \) is
\begin{enumerate}
\item inert if \( \alpha \) is inert in \( \Delta \),
\item active if \( \alpha \) is active in \( \Delta \) and \( \phi_l: g(l) \to f(\alpha(l)) \) is an isomorphism for every \( 0 \leq l \leq n \).
\end{enumerate}

We write \( \Delta_{\mathcal{F}, \text{int}} \) for the subcategory of \( \Delta_{\mathcal{F}} \) containing only the inert morphisms in \( \Delta_{\mathcal{F}} \).

**Remark 2.2.7.** The active and inert maps form a factorization system on \( \Delta \) which can be lifted along the Cartesian fibration \( \Delta_{\mathcal{F}} \to \Delta \), to give an active-inert factorization system on \( \Delta_{\mathcal{F}} \) as follows:

Any morphism \((\alpha, \phi): ([n], g) \to ([m], f) \) in \( \Delta_{\mathcal{F}} \) defines an active-inert factorization \([n] \xrightarrow{\alpha'} [k] \xrightarrow{\alpha''} [m] \) in \( \Delta \) and by Definition 2.2.3 the inert map \( \alpha'' \) can be lifted to a map \((\alpha'', \phi''): ([k], h) \to ([m], g) \) where, for \( 0 \leq l \leq k \), \( h(l) := f(\alpha''(l)) \times f(\alpha(n)) \phi_n(g(n)) \) and \( \phi_l \) is the canonical inclusion \( h(l) \hookrightarrow f(\alpha''(l)) \). It is then clear that \((\alpha, \phi)\) factors as the composite of \((\alpha', \phi')\) and \((\alpha'', \phi'')\) where \( \phi'_l: g(l) \simeq f(\alpha(l)) \times f(\alpha(n)) \phi_n(g(n)) \simeq h(\alpha(l)) \) is an equivalence for every \( 0 \leq l \leq n \).

In Remark 2.2.5 we saw that objects in \( \Delta_{\mathcal{F}} \) can be interpreted as forests. Now we want to introduce and study the simplest examples of forests, which are edges and corollas.

**Definition 2.2.8.** The *edge* is the object \( e := ([0], 1) \in \Delta_{\mathcal{F}} \) — this is the trivial tree with no vertices and a single edge. A *corolla* is a tree with exactly one vertex; more precisely, the \( n \)-corolla \( \mathcal{C}_n \) is \( ([1], n \to 1) \). We write \( \Delta_{\mathcal{F}, \text{el}} \) for the full subcategory of \( \Delta_{\mathcal{F}, \text{int}} \) spanned by the corollas \( \mathcal{C}_n \) and the edge \( e \). For an object \( I \in \Delta_{\mathcal{F}} \), we let \( \Delta_{\mathcal{F}, \text{el}/I} \) denote the category \( \Delta_{\mathcal{F}, \text{el}} \times_{\Delta_{\mathcal{F}, \text{int}}} \Delta_{\mathcal{F}, \text{int}/I} \).

**Remark 2.2.9.** In Remark 2.2.5 we defined the sets of edges and vertices of an object \( ([m], f) \) to be \( \prod_{i=0}^{m} f(i) \) and \( \prod_{i=1}^{m} f(i) \). It clearly follows from the previous definition that the set of edges coincides with the set of morphisms \( e \to ([m], f) \) in \( \Delta_{\mathcal{F}} \), while the set of vertices is given by the set of isomorphism classes of inert morphisms from corollas to \( ([m], f) \). In particular, each vertex \( v \in \prod_{i=1}^{m} f(i) \) corresponds to an isomorphism class of inert morphisms of the form \( \phi: \mathcal{C}_n \to ([m], f) \) such that \( \phi(g(1)) = v \).
Definition 2.2.10. We write $V_{\Delta F}: \Delta F^{op} \to \mathcal{F}_*$ for the functor taking $([m], f) \in \Delta F$ to $(\prod_{i=1}^{m} f(i))_+$ (i.e., the set of vertices of $([m], f)$ when viewed as a forest) and a morphism $(\alpha, \phi): ([n], g) \to ([m], f)$ in $\Delta F$ to the map $(\prod_{i=1}^{m} f(i))_+ \to (\prod_{j=1}^{n} g(j))_+$ given on the component $f(i)$ by the map $f(i) \to (\prod_{j=1}^{n} g(j))_+$ taking $x \in f(i)$ to an object $y \in g(j)$ if $\alpha(j-1) < i \leq \alpha(j)$ and $f^*(\alpha(i))(x) = \phi_j(y)$, and to the base point $*$ otherwise.

Remark 2.2.11. According to Remark 2.2.9 an element $y \in g(j)$ corresponds to an isomorphism class of inert maps of the form $\epsilon_p \to ([n], g)$ where $p = |g(j-1)|$. By composing one of these morphisms with $(\alpha, \phi)$ we obtain a map $\epsilon_p \to ([m], f)$, well-defined up to isomorphism. The image is a subtree of $([m], f)$, given by the sequence

$$p \simeq f(\alpha(j-1))_0 \to \ldots \to f(\alpha(j-1))_{\phi_j(y)} \to 1.$$

Clearly, a corolla $x \in f(i)$ in $([m], f)$ lies in this subtree if and only if $\alpha(j-1) < i \leq \alpha(j)$ and $f^*(\alpha(i))(x) = \phi_j(y)$. Unpacking the previous definition we therefore see that the map $V_{\Delta F}(\alpha, \phi)$ carries a corolla $x \in \prod_{i=1}^{m} f(i)$ to $y \in g(j)$ if $x$ is a corolla of the subtree in $([m], f)$ induced by $y$ and $(\alpha, \phi)$. Since different corollas in $([n], g)$ induce subtrees in $([m], f)$ which have disjoint sets of incoming edges, the object $y$ satisfying the necessary conditions is unique if it exists. Of course, it may happen that there are corollas in $([m], f)$ that are not in the image of any corollas in $([n], g)$; in this case, the map $V_{\Delta F}(\alpha, \phi)$ takes these corollas to the base point.

Lemma 2.2.12. The functor $V_{\Delta F}: \Delta F^{op} \to \mathcal{F}_*$ preserves inert-active factorizations.

Proof. Let $(\alpha, \phi): ([n], g) \to ([m], f)$ be an inert map in $\Delta F$ and let $y$ be a corolla in $([n], g)$. We want to show that $V_{\Delta F}(\alpha, \phi)$ is inert, i.e. $V_{\Delta F}(\alpha, \phi)^{-1}(y) \simeq 1$. The previous remark reveals that the set $V_{\Delta F}(\alpha, \phi)^{-1}(y)$ consists of corollas in the subtree induced by $y$ and $(\alpha, \phi)$. Since $\alpha$ is inert, the definition of $\Delta F$ implies that this subtree is a corolla equivalent to $y$.

Now suppose $(\alpha, \phi): ([n], g) \to ([m], f)$ is an active map in $\Delta F$. Then $\phi_k: g(k) \to f(\alpha(k))$ is an isomorphism for all $k$, and we have $\alpha(0) = 0$, $\alpha(n) = m$. This implies that, for every $k$ and $x \in f(k)$, there exists $l$ and $y \in g(l)$ such that $\alpha(l-1) < k \leq \alpha(l)$ and $f^*(\alpha(l))(x) = \phi_l(y)$. Therefore, we see that the functor $V_{\Delta F}$ preserves inert and active morphisms, and in particular inert-active factorizations.

2.3. Segal Presheaves on $\Delta F$. We now want to define an enriched version of Barwick’s Segal operads, which we refer to as Segal presheaves on $\Delta F$. Before we do this, let us first recall Barwick’s definition:

Definition 2.3.1. A presheaf $\mathcal{O}: \Delta F^{op} \to \mathcal{S}$ is called a Segal presheaf if the canonical map

$$\mathcal{O}(I) \to \lim_{J \in (\Delta F_{\alpha(I)})^{op}} \mathcal{O}(J)$$

is an equivalence for every $I \in \Delta F$. In this case we also say that $\mathcal{O}$ satisfies the Segal condition; we write $P_{\text{Seg}}(\Delta F)$ for the full subcategory of $P(\Delta F)$ spanned by the Segal presheaves.

Just as Segal spaces model $\infty$-categories, Segal presheaves on $\Delta F$ model $\infty$-operads. In the latter case we identify $\mathcal{O}(\epsilon_n)$ with the space of multimorphisms with $n$ source objects and one target object. The Segal condition identifies $\mathcal{O}(I)$ for $I \in \Delta F$ as the space of composable trees of such multimorphisms of shape $I$, and the functor $\mathcal{O}$ tells us how to compose this data to a single multimorphism in a homotopy-coherently associative manner.

Remark 2.3.2. In the paper [1] Barwick introduced the theory of operator categories. Speaking somewhat informally, we can think of an operator category as encoding the operations and coherences for a family of algebraic structures; key examples are the category $\mathcal{F}$ of finite sets and the category $\mathcal{O}$ of finite ordered sets. For every operator category $\Phi$, Barwick defines a category $\Delta \Phi$ (see [1, Definition 2.4]) whose objects encode the tree-like structures of composable operations in $\Phi$. The category $\Delta \Phi$ defined above is a special case of this, and Definition 2.3.1 can be extended to the more general notion of Segal presheaves on $\Delta \Phi$ as in [1, Definition 2.6] (where these are
called *Segal* $\Phi$-operads). Our work on enriched Segal presheaves on $\Delta^\Phi$ in this and the next section has an obvious variant for a general operator category $\Phi$, but we have chosen to state our results only for the most important case of symmetric operads in order to present the underlying idea as transparently as possible. Moreover, the lack of a good replacement for the dendroidal category for arbitrary operator categories (except $F$ and $O$) prevents us from generalizing the results of the last two sections.

It is convenient to recall some alternative characterizations of Segal presheaves, for which we need some notation:

**Notation 2.3.3.** For $I = ([n], f)$ in $\Delta^\Phi$, we write $I|_{ij} := ([j - i], f|_{\{i, i+1, \ldots, j\}})$ for $0 \leq i < j \leq n$ and $I|i := ([0], f(i))$.

By rewriting the colimits in Definition 2.3.1 in various ways, we get:

**Proposition 2.3.4.** The following are equivalent for a presheaf $O \in P(\Delta^\Phi)$:

1. $O$ is a Segal presheaf.
2. $O$ is local with respect to the morphisms $I_{\text{Seg}} := \colim_{J \in (\Delta^\Phi \times \text{el})^{\text{op}}} J \to I$ for all $I \in \Delta^\Phi$.
3. $O$ is local with respect to the morphisms
   
   $I|_{12} \times I|_{23} \times \cdots \times I|_{(n-1)n} \to I$

   for all $I \in \Delta^\Phi$, $\prod_{i \in m} ([i], n_i \to 1) \to ([1], n \to m)$, and
   
   $\prod_{i \in m} ([0], 1) \to ([0], m)$,

   for all $m$ (including $m = 0$).
4. $O$ is local with respect to the morphisms
   
   $I_{\text{Seg}} \to I$

   for all $I = ([n], f)$ such that $f(n) = 1$, and
   
   $\prod_{i \in f(n)} I_i \to I$

   for all $I = ([n], f)$, where $I_i = ([n], f_i)$ is obtained by taking the fibres at $i \in f(n)$.
5. $O|_{\Delta^\Phi, \text{el}}$ is the right Kan extension of $O|_{\Delta^\Phi, \text{int}}$. □

We now want to define an enriched version of this model for $\infty$-operads. More precisely, for $V$ a symmetric monoidal $\infty$-category we will define an $\infty$-category $\Delta^V_\Phi$ and then introduce a notion of Segal presheaves on this $\infty$-category.

**Definition 2.3.5.** Given a symmetric monoidal $\infty$-category $V^\otimes$, let $V^\otimes \to F_\ast^{\text{op}}$ denote the Cartesian fibration corresponding to the same functor $F_\ast : \text{Cat}_\infty \to \text{Cat}_\infty$ as $V^\otimes \to F_\ast$. We define the $\infty$-category $\Delta^V_\Phi$ by the pullback square

$$
\begin{array}{ccc}
\Delta^V_\Phi & \to & V^\otimes \\
\downarrow & & \downarrow \\
\Delta^\Phi & \to & F_\ast^{\text{op}}
\end{array}
$$

We also define $\Delta^V_{\Phi, \text{int}}$ and $\Delta^V_{\Phi, \text{el}}$ as the pullbacks $\Delta^V_\Phi \times_{\Delta^\Phi} \Delta^\Phi, \text{int}$ and $\Delta^V_\Phi \times_{\Delta^\Phi} \Delta^\Phi, \text{el}$, respectively.
Remark 2.3.6. Note that the fibre \((\Delta^Y)^I\) is equivalent to \(V\) for a corolla \(c_n \in \Delta^Y\). An object in \(\Delta^Y\) should be thought of as an object \(([m], f) \in \Delta^Y\) together with a labelling of each vertex in \(([m], f)\) by an object of \(V\). Therefore, if \(c_n\) is a corolla in \(\Delta^Y\), we write \(c_n(v)\) for the object in \(\Delta^Y\) lying over \(c_n\) and labeled by \(v \in V\). Given an arbitrary object \(I \in \Delta^Y\) and objects \(v, c \in V^{\Delta^Y}(I)\), we write \(I(v, c)\) for the object in \(\Delta^Y\) corresponding to \((v, c)\) under the equivalence \((\Delta^Y)_I \simeq V^{\Delta^Y}(I)\). We often write \(\pi\) instead of \(I(v, c)\) if we do not want to explicitly introduce notation for the labeling \((v, c)\). In particular, we let \(\pi\) denote the unique object of \((\Delta^Y)_I \simeq \ast\).

Remark 2.3.7. More generally, instead of a symmetric monoidal \(\infty\)-category \(V\) we could consider enriching in a coCartesian fibration \(M \to \Delta^Y\) satisfying Segal conditions in the form

\[
M_I \simeq \prod_{c_n \to I} M_{c_n}
\]

(where the product is over isomorphism classes of inert maps \(c_n \to I\); such an \(M\) is an “internal \(\infty\)-operad” in \(\infty\)-categories. We expect that most of our results should straightforwardly generalize to this setting. However, as we do not wish to develop the theory of “internal \(\infty\)-operads in \(\mathcal{C}\)” (because we are not aware of any interesting examples of these, let alone ones that one might want to enrich \(\infty\)-operads in) we have chosen to consider only enrichment in symmetric monoidal \(\infty\)-categories for simplicity.

Definition 2.3.8. Let \(V\) be a symmetric monoidal \(\infty\)-category. We say a presheaf \(O \in \mathcal{P}(\Delta^Y)\) is a Segal presheaf if for every object \(\pi \in \Delta^Y\) lying over \(I\) in \(\Delta^Y\), the canonical map

\[
\mathcal{O}(\pi) \to \lim_{\psi \in (\Delta^Y, n)} O(\psi^\ast \pi)
\]

is an equivalence, where \(\psi^\ast x \to x\) is the Cartesian lift of the inert map \(\psi\) (corresponding to a coCartesian morphism in \(V^\circ\)). We write \(\mathcal{P}_{\text{Seg}}(\Delta^Y)\) for the full subcategory of \(\mathcal{P}(\Delta^Y)\) spanned by the Segal presheaves.

Definition 2.3.9. Let \(V\) be a presentably symmetric monoidal \(\infty\)-category. We say a presheaf \(O \in \mathcal{P}(\Delta^Y)\) is a continuous Segal presheaf if it is a Segal presheaf and moreover for every \(n\), the functor

\[
O(\epsilon_n(-)) : V^{\text{op}} \simeq (\Delta^Y)^{\text{op}} \to \mathcal{S}/\mathcal{O}(\kappa^{n+1})
\]

induced by the Cartesian lifts of the \(n + 1\) morphisms \(\epsilon \to \epsilon_n\) preserves all small limits. We write \(\mathcal{P}_{\text{ConSeg}}(\Delta^Y)\) for the full subcategory of \(\mathcal{P}(\Delta^Y)\) spanned by the continuous Segal presheaves.

Remark 2.3.10. Continuous Segal presheaves give a model of \(\mathcal{V}\)-enriched \(\infty\)-operads: Given \(O \in \mathcal{P}_{\text{ConSeg}}(\Delta^Y)\) and \(x_1, \ldots, x_n, y \in \mathcal{O}(\epsilon)\), let \(\mathcal{O}(\epsilon_n(v, x_1, \ldots, x_n, y))\) be defined by the pullback square

\[
\begin{tikzcd}
\mathcal{O}(\epsilon_n(v, x_1, \ldots, x_n, y)) \arrow{r} & \mathcal{O}(\epsilon_n(v)) \arrow{d} \\
\{x_1, \ldots, x_n, y\} \arrow{r} & \mathcal{O}(\epsilon)^{\times(n+1)},
\end{tikzcd}
\]

where \(y\) is in the “outgoing” coordinate. Then the presheaf \(\mathcal{O}(\epsilon_n(-, x_1, \ldots, x_n, y)) : V^{\text{op}} \to \mathcal{S}\) is limit-preserving. Since \(V\) is presentable, this means this presheaf is representable by some object

\[
\mathcal{O}(x_1, \ldots, x_n, y) \in V.
\]

This is the object describing the multimorphisms from \((x_1, \ldots, x_n)\) to \(y\) in the enriched \(\infty\)-operad \(\mathcal{O}\).

Remark 2.3.11. To obtain a definition of enriched \(\infty\)-operads in this style when \(V\) is not presentable, we could consider those presheaves \(O\) on \(\Delta^Y\) such that the presheaves \(\mathcal{O}(\epsilon_n(-, x_1, \ldots, x_n, y))\) are all representable. However, in such settings it is more natural to consider an alternative definition of enriched \(\infty\)-operads, such as that of §2.4. Moreover, most of the results we wish to prove
using Segal presheaves only hold in the presentable case, so there is no reason to consider such a generalization.

We end this section by proving some equivalent reformulations of the definitions of Segal presheaves and continuous Segal presheaves. First we give an analogue of Proposition 2.3.4:

**Proposition 2.3.12.** The following are equivalent for a presheaf $\mathcal{O} \in P(\Delta^*_V)$:

1. $\mathcal{O}$ is a Segal presheaf.
2. $\mathcal{O}$ is local with respect to the morphisms
   $$T_{\text{Seg}} := \underbrace{\text{colim}}_{\Phi \in (\Delta^*_e,\alpha,1)} T \to I$$
   for all $T \in \Delta^*_V$.
3. $\mathcal{O}$ is local with respect to the morphisms
   $$T|_{\Delta^*_m} := T|_0 \amalg T|_1 \amalg \cdots \amalg T|_{(n-1)n} \to I$$
   for all $T \in \Delta^*_m$, 
   $$\prod_{i \in m} c_i(v_i) \to ([1], n \to m)(v_1, \ldots, v_n),$$
   for all $n \to m$ (including $m = 0$), and
   $$\prod_{i \in m} c \to ([0], m),$$
   for all $m$ (including $m = 0$).
4. $\mathcal{O}$ is local with respect to the morphisms
   $$T_{\text{Seg}} \to I$$
   for all $T$ over $I = ([n], f)$ such that $f(n) = 1$, and
   $$\prod_{i \in f(n)} T_i \to I$$
   for all $I = ([n], f)$, where $I_i = ([n], f_i)$ is obtained by taking the fibres at $i \in f(n)$.
5. $\mathcal{O}|_{\Delta^*_e,\alpha}$ is the right Kan extension of $\mathcal{O}|_{\Delta^*_e,\alpha}^{\text{op}}$.

To prove this we use the following observation:

**Lemma 2.3.13.** Suppose $p : \mathcal{E} \to \mathcal{B}$ is a Cartesian fibration. If $\mathcal{B}'$ is a full subcategory of $\mathcal{B}$ and $e \in \mathcal{E}$, let $\mathcal{E}' := \mathcal{E} \times_{\mathcal{B}} \mathcal{B}'$, $\mathcal{E}'_p := \mathcal{E}' \times_{\mathcal{E}/e} \mathcal{E}/e$ and $\mathcal{B}'_p(e) := \mathcal{B}' \times_{\mathcal{B}/p(e)} \mathcal{B}/p(e)$. Let $\mathcal{E}'_{\alpha,p}$ denote the full subcategory of $\mathcal{E}_p$ containing only the Cartesian morphisms to $e$. Then $\mathcal{E}'_p \to \mathcal{E}'_p$ is cofinal and $\mathcal{E}'_{\alpha,p} \to \mathcal{B}'_p(e)$ is an equivalence.

**Proof.** By [37, Theorem 4.1.3.1], the map $\mathcal{E}'_p \to \mathcal{E}'_p$ is cofinal if and only if the $\infty$-category $\mathcal{E}'_p := \mathcal{E}'_p \times_{\mathcal{E}_p}(\mathcal{E}'_p)^{\text{op}}$ is weakly contractible for every object $\phi \in \mathcal{E}'_p$. The definition of $\mathcal{E}$ implies that an object in $\mathcal{E}$ is given by a factorization $\phi = e \circ \alpha$ such that $\alpha$ is $p$-Cartesian. Moreover, a morphism from $\phi = e \circ \alpha$ to $\phi = e \circ \alpha_1$ in $\mathcal{E}$ is given by a factorization $\alpha_1 = \alpha' \circ \alpha_0$ such that $\beta_1 \circ \alpha' = \beta_0$. Since $\beta_0$ and $\beta_1$ are Cartesian, [37, Proposition 2.4.1.7] implies that $\alpha'$ is Cartesian as well. We now prove the weak contractibility of $\mathcal{E}$ by showing that it has an initial object given by the factorization $\phi = e \circ \alpha_0$, where $\alpha_0$ is a Cartesian lift of $\alpha$. If an object $\phi$ in $\mathcal{E}$ is given by a factorization $\phi = \beta_0 \circ \alpha_1$, then we have a map from $\phi = \beta_0 \circ \alpha_0$ to $\phi = \beta_1 \circ \alpha_1$ given by factoring $\alpha_1$ into a Cartesian lift $\alpha'$ of $\alpha(1)$ and a map $\alpha'$ lying in the fibre $\mathcal{E}_{\alpha_0}$. Then $\beta_1 \circ \alpha'$ coincides with $\beta_0$ as both are Cartesian lifts of $p(\beta_1) \circ p(\alpha_1) = p(\phi)$. Since every map from $\phi = \beta_0 \circ \alpha_0$ to $\phi = \beta_1 \circ \alpha_1$ is necessarily induced by a Cartesian lift of $p(\alpha_1)$, we see that this map is essentially unique.
To prove that $\mathcal{E}_{/e}' \to \mathcal{B}_{/p(e)}'$ is an equivalence, we first observe that in the commutative diagram

$$
\begin{array}{ccc}
\mathcal{E}_{/e}' & \to & \mathcal{E}_{/e} \\
\downarrow & & \downarrow \\
\mathcal{B}_{/p(e)}' & \to & \mathcal{B}_{/p(e)} \\
\downarrow & & \downarrow \\
\mathcal{E} & \to & \mathcal{E}
\end{array}
$$

the bottom, front, and back squares are Cartesian by definition. Hence, the top square is a pullback and $\mathcal{E}_{/e}' \to \mathcal{B}_{/p(e)}'$ is a Cartesian fibration which restricts to a right fibration $\mathcal{E}_{/e}' \to \mathcal{B}_{/p(e)}'$. By [37, Proposition 2.1.3.4], it is a trivial fibration if all fibres are contractible. But this is clear because the fibre over an object $\psi \in \mathcal{B}_{/p(e)}'$ can be identified with the full subcategory of the fibre $(\mathcal{E}_{/e}')_\psi$ spanned by the Cartesian lifts of $\psi$, which is contractible.

**Proof of Proposition 2.3.12.** The proof that (1) is equivalent to (2), (3), and (4) is the same as in the non-enriched case. By applying Lemma 2.2.6 to the Cartesian fibration $p: \Delta^\text{op}_\mathbb{F} \to \Delta^\text{op}_\mathbb{E}$ and the full subcategory $\Delta^\mathbb{E}_{\text{enr}}$, we obtain that (1) is also equivalent to (5).

**Proposition 2.3.14.** Let $\emptyset$ denote the initial object in $\mathbb{V}$. The following are equivalent for a Segal presheaf $\mathcal{O} \in \mathcal{P}_{\text{Seg}}(\Delta^\text{op}_\mathbb{F})$:

1. $\mathcal{O}$ is continuous.
2. For every $n$, the presheaf
   $$
   \mathcal{O}(\iota_n(-)): \mathbb{V}^\text{op} \simeq (\Delta^\text{op}_\mathbb{F})^\iota_n \to \mathbb{S}
   $$
   preserves limits of diagrams of the form $\phi: J^\text{op} \to \mathbb{V}^\text{op}$ where $\phi(\infty) \simeq \emptyset$, and the natural map $\mathcal{O}(\iota_n(\emptyset)) \to \prod_{n+1} \mathcal{O}(\iota)$ is an equivalence.
3. $\mathcal{O}$ is local with respect to the map $\prod_{n+1} \iota \to c_n(\emptyset)$ and the map $\colim_{\iota} c_n(\phi) \to c_n(\colim_{\iota} \phi)$ for every diagram $\phi$ such that $\phi(\infty) \simeq \emptyset$.
4. For every $n$, the presheaf
   $$
   \mathcal{O}(\iota_n(-)): \mathbb{V}^\text{op} \simeq (\Delta^\text{op}_\mathbb{F})^\iota_n \to \mathbb{S}
   $$
   preserves weakly contractible limits, and the natural map $\mathcal{O}(\iota_n(\emptyset)) \to \prod_{n+1} \mathcal{O}(\iota)$ is an equivalence.
5. $\mathcal{O}$ is local with respect to the map $\prod_{n+1} \iota \to c_n(\emptyset)$ and the map $\colim_{\iota} c_n(\phi) \to c_n(\colim_{\iota} \phi)$ for every weakly contractible diagram in $\mathbb{V}$.

**Proof.** By definition a Segal presheaf $\mathcal{O}$ is continuous if and only if the functor

$$
\mathcal{O}(\iota_n,-): \mathbb{V}^\text{op} \simeq (\Delta^\text{op}_\mathbb{F})^\iota_n \to \mathbb{S}_{/\mathcal{O}(\iota)^{n+1}}
$$

preserves small limits. Since the limit of a functor $\phi: J \to \mathbb{V}^\text{op}$ is the same as the limit of its right Kan extension along $J \hookrightarrow \mathbb{F}$, which takes $\infty$ to $*$, this is equivalent to the preservation of such conical limits and of the terminal object. Moreover, by [18, Lemma 2.2.6] the preservation of conical limits is equivalent to the preservation of weakly contractible limits.

Here the preservation of the terminal object is obviously equivalent to $\mathcal{O}(\iota_n(\emptyset))$ coinciding with the terminal object $\mathcal{O}(\iota)^{n+1} \in \mathbb{S}_{/\mathcal{O}(\iota)^{n+1}}$. Since a functor $\mathbb{V}^\text{op} \to \mathbb{S}_{/\mathcal{O}(\iota)^{n+1}}$ preserves weakly contractible limits if and only its composition with the forgetful functor $\mathbb{S}_{/\mathcal{O}(\iota)^{n+1}} \to \mathbb{S}$ does, we see that conditions (1), (2), and (4) are equivalent. The equivalence between (4) and (5) follows from the fact that a functor $\mathbb{V}^\text{op} \to \mathbb{S}$ preserves all weakly contractible limits if and only if is local with respect
to all maps $\text{colim}_1 c_n(\phi) \to c_n(\text{colim}_3 \phi)$ of presheaves where $\phi$ is a weakly contractible diagram in $\mathcal{V}$; the same argument also shows that (2) and (3) are equivalent.

**Definition 2.3.15.** We call a class $\mathcal{S}$ of morphisms in a cocomplete $\infty$-category $\mathcal{C}$ strongly saturated if

1. it satisfies the 2-of-3 property,
2. it is stable under pushouts along any morphism in $\mathcal{C}$,
3. the full subcategory of $\text{Fun}(\Delta^1, \mathcal{C})$ spanned by $\mathcal{S}$ is stable under small colimits.

For any class of morphisms $\mathcal{S}$, there exists a smallest strongly saturated class $\mathcal{S}'$ that contains $\mathcal{S}$.

**Definition 2.3.16.** If $\mathcal{S}$ is a class of morphisms in $\mathcal{C}$, then we say an object $c \in \mathcal{C}$ is $\mathcal{S}$-local if $\text{Map}_\mathcal{C}(c, \_)$ takes the elements of $\mathcal{S}$ to equivalences in $\mathcal{S}$. A morphism $f : x \to y$ in $\mathcal{C}$ is then an $\mathcal{S}$-equivalence if for every $\mathcal{S}$-local object $c$. Let $\text{Eq}(\mathcal{S})$ denote the class of $\mathcal{S}$-equivalences; if $\mathcal{C}$ is cocomplete, then $\text{Eq}(\mathcal{S})$ is a strongly saturated class by [37, Lemma 5.5.4.11], and $\mathcal{S}' \subseteq \text{Eq}(\mathcal{S})$.

**Remark 2.3.17.** If $\mathcal{S}$ is a small set of morphisms in a presentable $\infty$-category $\mathcal{C}$, then $\text{Eq}(\mathcal{S}) = \mathcal{S}'$ by [37, Proposition 5.5.4.15]. Moreover, if $\mathcal{C}_\mathcal{S}$ denotes the full subcategory of $\mathcal{C}$ spanned by the $\mathcal{S}$-local objects, then $\mathcal{C}_\mathcal{S}$ is again presentable and the inclusion $\mathcal{C}_\mathcal{S} \to \mathcal{C}$ admits a left adjoint, which exhibits $\mathcal{C}_\mathcal{S}$ as the localization of $\mathcal{C}$ that inverts the morphisms in $\mathcal{S}$.

**Definition 2.3.18.** If $\mathcal{S}$ is a class of morphisms in a cocomplete $\infty$-category $\mathcal{C}$, we say that the class $\text{Eq}(\mathcal{S})$ is the strongly saturated class generated by $\mathcal{S}$, and call the elements of $\mathcal{S}$ the generators of $\text{Eq}(\mathcal{S})$.

**Remark 2.3.19.** Our use of the term “generated by $\mathcal{S}$” for the class $\text{Eq}(\mathcal{S})$ (rather than the class $\mathcal{S}'$, which might a priori be smaller than $\text{Eq}(\mathcal{S})$ when $\mathcal{S}$ is not small) is somewhat non-standard, but will be convenient for us: In practice we will be interested in “weak equivalences” of the form $\text{Eq}(\mathcal{S})$ where $\mathcal{S}$ is not small, and we will proceed to find a small set $\mathcal{S}'$ such that $\text{Eq}(\mathcal{S}) = \text{Eq}(\mathcal{S}') = \mathcal{S}'$ (which also implies $\text{Eq}(\mathcal{S}) = \mathcal{S}'$).

**Definition 2.3.20.** We call elements in the strongly saturated class of morphisms in $P(\Delta)$ generated by the set of spine inclusions $\Delta^1 \amalg \Delta^0 \amalg \cdots \amalg \Delta^1 \to \Delta^n$ Segal equivalences. By Remark 2.1.2 and Remark 2.3.17, $P_{\text{Seg}}(\Delta)$ is given by a localization of $P(\Delta)$ with respect to Segal equivalences.

**Definition 2.3.21.** We say that a map $F : G$ in $P(\Delta^V)$ is a Segal equivalence if $\text{Map}(G, \emptyset) \to \text{Map}(F, \emptyset)$ is an equivalence for every Segal presheaf $\emptyset$, and a continuous Segal equivalence if this is an equivalence for every continuous Segal presheaf $\emptyset$. These are both strongly saturated classes of morphisms, since the Segal equivalences are of the form $\text{Eq}(\mathcal{S})$ where $\mathcal{S}$ consists of the maps listed in (2), (3), and (4) of Proposition 2.3.12, while the continuous Segal equivalences are the $\mathcal{S}$-equivalences for these together with the maps listed in (3) or (5) of Proposition 2.3.14. Explicitly, we can view the continuous Segal equivalences as generated by the following morphisms:

1. $T_{\text{Seg}} \to T$ for all $T \in \Delta^V$,
2. $\text{colim}_V c_n(\_ \to c_n(\text{colim}_V \_)$ for every weakly contractible diagram in $\mathcal{V}$,
3. $\prod_n \mathcal{F} \to c_n(\emptyset)$.

Note, however, that this set of morphisms is not small.

Continuous Segal presheaves have an obvious functoriality in colimit-preserving symmetric monoidal functors:

**Lemma 2.3.22.** Suppose $F : \mathcal{V} \to \mathcal{W}$ is a symmetric monoidal colimit-preserving functor between presentably symmetric monoidal $\infty$-categories. Then $F$ induces a functor $\Delta^V \to \Delta^W$, which we also denote $F$, and composition with $F^{op}$ induces a functor $P_{\text{CtsSeg}}(\Delta^V) \to P_{\text{CtsSeg}}(\Delta^W)$. 

2.4. Segal Presheaves vs. Algebras.

Definition 2.4.1. Given a space \( X \), we write \( \Delta^\mathbb{F}_X \rightarrow \Delta^\mathbb{F} \) for the right fibration associated to the functor \( \Delta^\mathbb{F}_X \rightarrow \{x\} \) given by the right Kan extension of the functor \( \{x\} \rightarrow \{x\} \) with value \( X \) along the inclusion \( \{x\} \hookrightarrow \Delta^\mathbb{F}_X \). We write \( I(x_i)_i \) for an object in \( \Delta^\mathbb{F}_X \) and we view \( \Delta^\mathbb{F}_X \) as living over \( \mathbb{F}_x \) via the composite map

\[
\Delta^\mathbb{F}_X \rightarrow \Delta^\mathbb{F}_X \rightarrow \mathbb{F}_x.
\]

For \( X \in \mathcal{S} \) and \( \mathcal{V} \) a symmetric monoidal \( \infty \)-category, we define \( \Delta_{\mathcal{V}}^\mathbb{F} := \Delta^\mathbb{F}_X \times_{\Delta^\mathbb{F}} \Delta_{\mathcal{V}} \) and we write \( I(v, x_i)_i \) or \( T(x_i)_i \) for its objects.

Remark 2.4.2. The right Kan extension \( \Delta^\mathbb{F}_X \rightarrow \mathbb{S} \) takes an object \( I \) to a product of copies of \( X \) indexed by the number of edges of \( I \).

Definition 2.4.3. If \( \mathcal{V} \rightarrow \mathbb{F}_x \) is a symmetric monoidal \( \infty \)-category, then a \( \Delta_{\mathcal{V}}^\mathbb{F}_X \)-algebra in \( \mathcal{V} \) is a functor \( \Delta_{\mathcal{V}}^\mathbb{F}_X \rightarrow \mathcal{V}_\otimes \) over \( \mathbb{F}_x \) that takes the inert morphisms lying over \( \rho \) to coCartesian morphisms. We write \( \text{Alg}_{\Delta_{\mathcal{V}}^\mathbb{F}_X} (\mathcal{V}) \) for the full subcategory of \( \text{Fun}_{\mathcal{V}}(\Delta_{\mathcal{V}}^\mathbb{F}_X, \mathcal{V}_\otimes) \) spanned by the algebras. This is clearly contravariantly functorial in \( \mathcal{X} \), and we write \( \text{Alg}_{\Delta_{\mathcal{V}}^\mathbb{F}_X}/\mathcal{S}(\mathcal{V}) \rightarrow \mathcal{S} \) for the Cartesian fibration associated to the functor \( \mathcal{S}^\otimes \rightarrow \text{Cat}_\infty \) taking \( X \rightarrow \text{Alg}_{\Delta_{\mathcal{V}}^\mathbb{F}_X}/\mathcal{S}(\mathcal{V}) \).

Our goal in this subsection is to construct an equivalence between the \( \infty \)-categories \( \text{P}_{\text{CtsSeg}}(\Delta_{\mathcal{V}}^\mathbb{F}) \) and \( \text{Alg}_{\Delta_{\mathcal{V}}^\mathbb{F}/\mathcal{S}}(\mathcal{V}) \):

Theorem 2.4.4. Let \( \mathcal{V} \) be a presentably symmetric monoidal \( \infty \)-category. There is an equivalence of \( \infty \)-categories over \( \mathcal{S} \),

\[
\text{P}_{\text{CtsSeg}}(\Delta_{\mathcal{V}}^\mathbb{F}) \xrightarrow{\sim} \text{Alg}_{\Delta_{\mathcal{V}}^\mathbb{F}/\mathcal{S}}(\mathcal{V})
\]

To prove this we will show that both sides are equivalent to an intermediate model given by continuous \( \Delta_{\mathcal{V}}^\mathbb{F}_X \)-monoids, in the following sense:

Definition 2.4.5. We say a presheaf \( F \in \text{P}(\Delta_{\mathcal{V}}^\mathbb{F}_X) \) is a \( \Delta_{\mathcal{V}}^\mathbb{F}_X \)-monoid if for every object \( \tilde{T} \) in \( \Delta_{\mathcal{V}}^\mathbb{F}_X \) lying over \( I \) in \( \Delta_{\mathcal{F}} \), the natural map

\[
F(\tilde{T}) \rightarrow \prod_{v \in V_{\mathcal{V}_\otimes}(I)} F(v^* \tilde{T})
\]

is an equivalence, where \( v^* \tilde{T} \rightarrow \tilde{T} \) denotes the Cartesian morphism in \( \Delta_{\mathcal{V}}^\mathbb{F}_X \) over an inert map \( v: \varepsilon_n \rightarrow I \) in \( \Delta_{\mathcal{F}} \) corresponding to \( v \in V_{\mathcal{V}_\otimes}(I) \). We say a \( \Delta_{\mathcal{V}}^\mathbb{F}_X \)-monoid is continuous if for every \( \varepsilon_n \) in \( \Delta_{\mathcal{F}, X} \) over \( \varepsilon_n \) in \( \Delta_{\mathcal{F}} \) the presheaf

\[
\mathcal{V}^\otimes \simeq (\Delta_{\mathcal{V}}^\mathbb{F}_X)_{\varepsilon_n} \rightarrow \mathcal{S}
\]

preserves limits. We write \( \text{P}_{\text{Mon}}(\Delta_{\mathcal{V}}^\mathbb{F}_X) \) for the full subcategory of \( \text{P}(\Delta_{\mathcal{V}}^\mathbb{F}_X) \) spanned by the \( \Delta_{\mathcal{V}}^\mathbb{F}_X \)-monoids and \( \text{P}_{\text{CtsSeg}}(\Delta_{\mathcal{V}}^\mathbb{F}_X) \) for the full subcategory spanned by the continuous \( \Delta_{\mathcal{V}}^\mathbb{F}_X \)-monoids.

Definition 2.4.6. Given a coCartesian fibration \( \varepsilon \rightarrow \mathcal{E} \) corresponding to a functor \( F: \mathcal{E} \rightarrow \text{Cat}_\infty \), let \( \text{P}_{\mathcal{E}}(\varepsilon) \rightarrow \mathcal{E} \) denote the Cartesian fibration associated to the functor

\[
\varepsilon^\otimes \xrightarrow{F^\otimes} \text{Cat}_\infty^\otimes \xrightarrow{P^\otimes} \overline{\text{Cat}}_\infty.
\]

Proposition 2.4.7. Let \( \mathcal{V} \) be a presentably symmetric monoidal \( \infty \)-category. Then there is an equivalence of \( \infty \)-categories \( \text{P}_{\text{CtsSeg}}(\Delta_{\mathcal{V}}^\mathbb{F}_X) \simeq \text{Alg}_{\Delta_{\mathcal{V}}^\mathbb{F}_X}(\mathcal{V}) \), natural in \( X \in \mathcal{S} \).
Proof. Since $\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}} \to \Delta_{\mathcal{F}_X}^{\text{op}}$ is a coCartesian fibration, by [19, Proposition 7.3] we can identify the $\infty$-category $\text{Fun}(\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}}, \mathcal{S})$ with

$$\text{Fun}_{\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}}} (\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}}, \mathcal{P}_{\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}}}) \simeq \text{Fun}_{\mathcal{F}_X} (\Delta_{\mathcal{F}_X}^{\text{op}}, \mathcal{P}_\mathcal{F}_X (\mathcal{V}^{\text{op}, \text{op}})).$$

If $M$ is a continuous $\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}}$-monoid, then the corresponding functor $\Delta_{\mathcal{F}_X}^{\text{op}} \to \mathcal{P}_\mathcal{F}_X (\mathcal{V}^{\text{op}, \text{op}})$ sends $\mathcal{T} \in \Delta_{\mathcal{F}_X}^{\text{op}}$ to an object in the full subcategory

$$\text{Fun}^R((\mathcal{V}^{\text{op}}) \times |V_{\mathcal{V}} (I)|, \mathcal{S}) \to \text{Fun}((\mathcal{V}^{\text{op}}) \times |V_{\mathcal{V}} (I)|, \mathcal{S}) \simeq \mathcal{P}_\mathcal{F}_X (\mathcal{V}^{\text{op}, \text{op}})_{V_{\mathcal{V}} (I)}$$

of functors that preserve limits. Since $\mathcal{V}$ is presentable, this $\infty$-category can be identified with $\mathcal{V}^{\mathcal{V} \times |V_{\mathcal{V}} (I)|}$ under the Yoneda embedding, and the full subcategory of $\mathcal{P}_\mathcal{F}_X (\mathcal{V}^{\text{op}, \text{op}})$ spanned by these objects for all $(n) \in \mathcal{F}_X$ can be identified with $\mathcal{V}^{\mathcal{V}}$. Furthermore, under this equivalence the full subcategory $\mathcal{P} \text{CtsMon}(\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}})$ of $\mathcal{P}(\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}})$ is identified with the full subcategory $\text{Alg}_{\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}}} (\mathcal{V})$ of $\text{Fun}_{\mathcal{F}_X} (\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}}, \mathcal{P}_\mathcal{F}_X (\mathcal{V}^{\text{op}, \text{op}}))$. □

**Proposition 2.4.8.** Let $\mathcal{E} \to \mathcal{S}$ denote the Cartesian fibration for the functor $\mathcal{S}^{\text{op}} \to \text{Cat}_\infty$ taking $X$ to $\mathcal{P}(\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}})$, and let $\mathcal{E} \text{CtsMon}$ denote the full subcategory of $\mathcal{E}$ spanned by the continuous $\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}}$-monoids for all $X \in \mathcal{S}$. There is an equivalence

$$\mathcal{P} \text{CtsSeg}(\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}}) \sim \mathcal{E} \text{CtsMon} \to \mathcal{S}.$$ 

For the proof we need the following two lemmas:

**Lemma 2.4.9.** Suppose $u : \mathcal{D} \to \mathcal{E} : \eta$ is an adjunction with unit transformation $\alpha : \text{id} \to \eta u$, and let the $\infty$-category $\mathcal{E}$ be defined by the pullback square

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{G} & \mathcal{D}^\Delta \\
\downarrow & & \downarrow \text{ev}_1 \\
\mathcal{D} & \xrightarrow{\eta} & \mathcal{D}.
\end{array}$$

(i) The functor $G$ has a left adjoint $F$, which sends $X \to Y$ to $(u(Y), X \to Y \xrightarrow{\alpha_Y} \eta u(Y))$.

(ii) The composite functor $\mathcal{E} \xrightarrow{G} \mathcal{D}^\Delta \xrightarrow{\text{ev}_0} \mathcal{D}$ has a left adjoint, given by $X \mapsto (u(X), X \xrightarrow{\alpha_X} \eta u(X))$.

Moreover, this is fully faithful.

Proof. Since $\text{ev}_1$ is a coCartesian fibration, (i) is a special case of (the dual of) [26, Lemma 4.14]. The functor $\text{ev}_0$ is given by composition with the inclusion $i_0 : \{0\} \hookrightarrow \Delta^1$; it is right adjoint to the constant diagram functor $c : \mathcal{D} \to \Delta^1$, since this can be described as left Kan extension along $i_0$. The composite $\text{ev}_0 G$ is therefore right adjoint to $Fc$, which indeed takes $X \in \mathcal{D}$ to $(u(X), X \xrightarrow{\alpha_X} \eta u(X))$. To see that $Fc$ is fully faithful, we must show that the unit map $X \to \text{ev}_0 GFcX$ is an equivalence, which is clear.

**Lemma 2.4.10.** Let $X$ be a space, and let $\pi$ denote the projection $\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}} \to \Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}}$. Then a presheaf $F : \Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}} \to \mathcal{S}$ such that $F(\pi(x)) \simeq *$ for all $x \in X$ is a continuous $\Delta_{\mathcal{F}_X}^{\mathcal{V}, \text{op}}$-monoid if and only if the left Kan extension $\pi F$ is a continuous Segal presheaf.
Proof. We first show that $\pi_!$ gives an identification between $\Delta_{\mathcal{E},X}^{V,op}$-monoids and Segal presheaves. At the end we will see that this restricts to an equivalence of continuous objects.

Suppose $F: \Delta_{\mathcal{E},X}^{V,op} \to S$ is a functor such that $F(e(x)) \simeq *$ for all $x \in X$. For $I \in \Delta_\ast$, let $E_{\Delta_\ast}(I)$ denote the set of edges of $I$. Since $\pi$ is a cocartesian fibration, we can compute the left Kan extension along $\pi$ fibrewise; for $\overline{T}$ in $\Delta_{\mathcal{E}}^{V,op}$ this gives

$$\pi_!F(\overline{T}) \simeq \colim_{(x_i)_I \in (\Delta_{\mathcal{E},X}^{V,op})_I} F(\overline{T}(x_i)_I) \simeq \colim_{(x_i)_I \in X \times |E_{\Delta_\ast}(I)|} F(\overline{T}(x_i)_I).$$

In particular, we have that

$$\pi_!F(e) \simeq \colim_{x \in X} * \simeq X.$$

Moreover, the natural transformation from $F$ to the constant functor with value $*$ induces natural morphisms $\pi_!F(\overline{T}) \to X \times |E_{\Delta_\ast}(I)|$. Hence, there is a commutative square

$\begin{tikzcd}
\pi_!F(\overline{T}) & \lim_{\phi \in (\Delta_{\mathcal{E},X}^{V,op})_I} \pi_!F(\phi \overline{T}) \\
X \times |E_{\Delta_\ast}(I)| & \lim_{J \to I \in (\Delta_{\mathcal{E},X}^{V,op})_I} X \times |E_{\Delta_\ast}(J)|,
\end{tikzcd}$

where the bottom horizontal map is an equivalence, since the set $E_{\Delta_\ast}(I)$ is isomorphic to $\colim_{J \to I \in (\Delta_{\mathcal{E},X}^{V,op})_I} E_{\Delta_\ast}(J)$.

It follows that the top horizontal morphism in the square is an equivalence if and only if it gives an equivalence on all fibres over $X \times |E_{\Delta_\ast}(I)|$. The functor $\text{colim}: \text{Fun}(T,S) \to S_T$ is an equivalence for any space $T$, with inverse given by taking fibres, so the fibre of $\pi_!F(\overline{T}) \simeq \colim_{(x_i)_I \in X \times |E_{\Delta_\ast}(I)|} F(\overline{T}(x_i)_I)$ at $(x_i)_I$ is naturally equivalent to $F(\overline{T}(x_i)_I)$. Since limits commute, it follows that the map on fibres at $(x_i)_I$ in the square is the natural map

$$F(\overline{T}(x_i)_I) \to \lim_{\phi \in (\Delta_{\mathcal{E},X}^{V,op})_I} F(\phi \overline{T}(x_i)_I) \simeq \prod_{v \in V_{\Delta_\ast}(I)} F(v \ast \overline{T}(x_i)_I),$$

where the equivalence follows from the assumption that $F(e(x))$ is the terminal object for all $x \in X$.

It follows that $\pi_!F$ is a Segal presheaf if and only if the above squares are all pullbacks, and this in turn holds if and only if $F$ is a monoid.

By definition, $\pi_!F$ is continuous if and only if for all $n$, the functor

$$\pi_!F(e_n(-)): V^n \simeq (\Delta_\ast^{V,op})_{e_n} \to S_{X \times (n+1)} \simeq \text{Fun}(X \times (n+1), S)$$

preserves limits. Limits in functor $\infty$-categories are computed objectwise, so this holds if and only if for all $(x_i)_{i=1,\ldots,n+1} \in X \times (n+1)$ the composite of this functor with evaluation at $(x_i)_I$ preserves limits. But this composite can be identified with $F(e_n(-, (x_i)_I)): V^n \to S$; this preserves limits for all $n$ and $(x_i)_I \in X \times (n+1)$ precisely if $F$ is a continuous $\Delta_{\mathcal{E},X}^{V,op}$-monoid.

**Proof of Proposition 2.4.8.** The functor $\text{ev}_\gamma: P(\Delta_\gamma^V) \to S$ has a right adjoint, which takes $X \in S$ to the presheaf $\tau_X: \Delta_\gamma^{V,op} \to \Delta_\gamma^V \xrightarrow{\Delta_X} S$ corresponding to the right fibration $\Delta_X \to \Delta_\gamma^V$. Let $\mathcal{E}' \to S$ be the Cartesian fibration for the functor $X \mapsto P(\Delta_\gamma^V)_{/\tau_X}$. Then we can apply Lemma 2.4.9 to the pullback diagram

$\begin{tikzcd}
\mathcal{E}' & P(\Delta_\gamma^V)_{/\tau_X} \\
S & P(\Delta_\gamma^V)
\end{tikzcd}$
to conclude that the forgetful functor \( \mathcal{E}' \to P(\Delta^V) \) has a fully faithful left adjoint, which takes a presheaf \( F \) to the adjunction unit \( F \to \tau_F(\mathcal{E}) \). The image of \( P(\Delta^V) \) consists of precisely those maps \( G \to \tau_X \) such that the restriction \( G(e) \to X \) is an equivalence.

By [19, Corollary 9.8], left Kan extension along \( \Delta^{V,op}_{\mathcal{E},X} \to \Delta^V_{\mathcal{E},X} \) gives an equivalence between \( \mathcal{E} \) and \( \Delta^V_{\mathcal{E},X} \). Since this equivalence is natural in \( X \), it gives an equivalence over \( S \) between \( \mathcal{E} \) and \( \mathcal{E}' \). Moreover, the image of \( P(\Delta^V) \) in \( \mathcal{E}' \) corresponds under this equivalence to the full subcategory of functors \( F : \Delta^{V,op}_{\mathcal{E},X} \to S \) such that \( F(e(x)) \simeq * \) for all \( x \in X \).

It thus remains to show that this equivalence restricts further to \( P_{\text{CtsSeg}}(\Delta^V) \simeq \mathcal{E}_{\text{CtsMon}} \); this is exactly the statement of Lemma 2.4.10.

\[ \square \]

**Proof of Theorem 2.4.4.** Combine Propositions 2.4.8 and 2.4.7. \[ \square \]

We record the functoriality of \( \text{Alg}_{\Delta^V/S}(\mathcal{V}) \) in the symmetric monoidal \( \infty \)-category \( \mathcal{V} \):

**Proposition 2.4.11.**

(i) If \( F : \mathcal{V} \to \mathcal{W} \) is a lax monoidal functor, then \( F \) induces a functor 
\[ F_* : \text{Alg}_{\Delta^V/S}(\mathcal{V}) \to \text{Alg}_{\Delta^W/S}(\mathcal{W}). \]

(ii) If \( F : \mathcal{V} \to \mathcal{W} \) is a symmetric monoidal left adjoint, with (lax monoidal) right adjoint \( G \), then there is an adjunction
\[ F_* : \text{Alg}_{\Delta^V/S}(\mathcal{V}) \rightleftarrows \text{Alg}_{\Delta^W/S}(\mathcal{W}) : G_*. \]

(iii) If \( L : \mathcal{V} \to \mathcal{W} \) is a symmetric monoidal localization with (lax monoidal) fully faithful right adjoint \( i \), then the right adjoint \( i_* : \text{Alg}_{\Delta^V/S}(\mathcal{W}) \to \text{Alg}_{\Delta^V/S}(\mathcal{V}) \) is fully faithful with image those algebras \( A : \Delta^{op}_{\mathcal{E},X} \to \mathcal{V}^O \) such that for every corolla \( \tilde{e}_n \) in \( \Delta^{op}_{\mathcal{E},X} \) over \( e_n \in \Delta_{\mathcal{E}} \), the image \( A(\tilde{e}_n) \) lies in \( i(\mathcal{W}) \).

\[ \text{Proof.} \text{ (i) is obvious from the definition of } \text{Alg}_{\Delta^V/S}(\mathcal{V}), \text{ (ii) follows from } [38, \text{ Corollary 7.3.2.7}], \text{ and (iii) is immediate from (ii)}. \text{ } \[ \square \]

We can restate this in terms of continuous Segal presheaves:

**Corollary 2.4.12.**

(i) If \( F : \mathcal{V} \to \mathcal{W} \) is a lax monoidal functor, then \( F \) induces a functor \( F_* : P_{\text{CtsSeg}}(\Delta^V) \to P_{\text{CtsSeg}}(\Delta^W) \).

(ii) If \( F : \mathcal{V} \to \mathcal{W} \) is a symmetric monoidal left adjoint, with (lax monoidal) right adjoint \( G \), then there is an adjunction
\[ F_* : P_{\text{CtsSeg}}(\Delta^V) \rightleftarrows P_{\text{CtsSeg}}(\Delta^W) : G_* \]

Moreover, the functor \( G_* \) can be identified with \( F^* \).

(iii) If \( L : \mathcal{V} \to \mathcal{W} \) is a symmetric monoidal localization with (lax monoidal) fully faithful right adjoint \( i \), then the right adjoint \( i_* : P_{\text{CtsSeg}}(\Delta^W) \to P_{\text{CtsSeg}}(\Delta^V) \) is fully faithful with image those \( O \in P_{\text{CtsSeg}}(\Delta^V) \) where the presheaves \( O(e_n(-,x_1,\ldots,x_n,y)) \) are representable by objects in \( \mathcal{W} \).

\[ \text{Proof.} \text{ The only claim that requires proof is the identification of } G_* \text{ with } F^* \text{ in (ii). For } O \in P_{\text{CtsSeg}}(\Delta^W), \text{ let } A \text{ be the corresponding object of } \text{Alg}_{\Delta^V/S}(\mathcal{W}). \text{ We have a natural equivalence} \]
\[ \text{Map}_{\mathcal{V}^{\Delta^V}}((w,c),A(I(x_i)),I(w,c,x_i)) \simeq O(I(w,c,x_i)). \]

Then we get
\[ F^* O(I(v,c,x_i)) \simeq O(I(Fv,c,x_i)) \simeq \text{Map}_{\mathcal{V}^{\Delta^V}}(F(v,c),A(I(x_i)),I(v,c,AI(x_i))). \]

\[ \text{i.e. } F^*O \text{ corresponds to } G_*A \text{ under the identification of } P_{\text{CtsSeg}}(\Delta^V) \text{ with } \text{Alg}_{\Delta^V/S}(\mathcal{V}). \text{ } \[ \square \]
2.5. Enrichment in Presheaves. If \( \mathcal{U} \) is a small symmetric monoidal \( \infty \)-category, then by [38, Proposition 4.8.1.12] the \( \infty \)-category \( \mathbf{P}(\mathcal{U}) \) of presheaves on \( \mathcal{U} \) has a unique symmetric monoidal structure such that the tensor product preserves colimits in each variable and the Yoneda embedding \( y: \mathcal{U} \to \mathbf{P}(\mathcal{U}) \) is symmetric monoidal; this is the Day convolution [22]. This induces a fully faithful functor \( \Delta_\mathcal{U}^{\infty} \to \Delta_\mathcal{U}^{\infty} \), which we also denote \( y \), and our goal in this section is to show that composition with \( y \) induces an equivalence of \( \infty \)-categories between \( \mathbf{P}_{\text{CtsSeg}}(\Delta_\mathcal{U}^{\infty}) \) and the \( \infty \)-category \( \mathbf{P}_{\text{Seg}}(\Delta_\mathcal{U}^{\infty}) \) of Segal presheaves on \( \Delta_\mathcal{U}^{\infty} \). In particular, taking \( \mathcal{U} \) to be a point, this implies that continuous Segal presheaves on \( \Delta_\mathcal{U}^{\infty} \) are equivalent to Segal presheaves on \( \Delta_\mathcal{U}^{\infty} \).

We first prove that the functor \( y^*: \mathbf{P}(\Delta_\mathcal{U}^{\infty}) \to \mathbf{P}(\Delta_\mathcal{U}^{\infty}) \) given by composition with \( y \) has a right adjoint given by right Kan extension, despite \( \mathbf{P}(\Delta_\mathcal{U}^{\infty}) \), and our goal in this section is to show that composition with \( y \) induces an equivalence of \( \infty \)-categories between \( \mathbf{P}_{\text{CtsSeg}}(\Delta_\mathcal{U}^{\infty}) \) and the \( \infty \)-category \( \mathbf{P}_{\text{Seg}}(\Delta_\mathcal{U}^{\infty}) \) of Segal presheaves on \( \Delta_\mathcal{U}^{\infty} \). In particular, taking \( \mathcal{U} \) to be a point, this implies that continuous Segal presheaves on \( \Delta_\mathcal{U}^{\infty} \) are equivalent to Segal presheaves on \( \Delta_\mathcal{U}^{\infty} \).

**Proposition 2.5.1.** The restriction
\[
\mathbf{P}(\Delta_\mathcal{U}^{\infty}) \xrightarrow{y^*} \mathbf{P}(\Delta_\mathcal{U}^{\infty})
\]
has a fully faithful right adjoint \( y_* \), given by right Kan extension.

**Proof.** For a presheaf \( F \in \mathbf{P}(\Delta_\mathcal{U}^{\infty}) \), the right Kan extension \( y_* F \) along \( y \) exists if for every \( F \in \mathbf{P}(\Delta_\mathcal{U}^{\infty}) \) and \( \mathcal{T} = (I, (\phi_c)_{c \in V_{\omega}(I)}) \in \Delta_\mathcal{U}^{\infty} \), the diagram
\[
(\Delta_\mathcal{U}^{\infty})_{\mathcal{T}} := (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}} \times (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}} \to (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}} \to (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}}
\]
has a limit in \( \mathbf{S} \). Since \( \mathbf{S} \) has all small limits, it is enough to show the \( \infty \)-category \( (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}} \) is essentially small. As a pullback of the left fibration \( (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}} \to (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}} \), the map \( (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}} \to (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}} \) is a left fibration whose target is essentially small since \( \mathcal{U} \) is essentially small. Hence, it suffices to show that the fibre over any object \( \mathcal{J} = (J, (\psi_d)_{d \in V_{\omega}(J)}) \in (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}} \) is small. It follows from the definition of \( (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}} \) that this fibre can be identified with
\[
\text{Map}_{\Delta_\mathcal{U}^{\infty}}(J, \mathcal{J}) \simeq \text{Map}_{\mathcal{U} \omega}(J, I) \times \text{Map}_{\mathcal{U} \omega}(V_{\omega}(J), V_{\omega}(I)) \text{Map}_{\mathcal{U} \omega}(J, \mathcal{J})
\]
It suffices to show that \( \text{Map}_{\mathcal{U} \omega}(J, \mathcal{J}) \) is small as the other two mapping spaces in the pullback are obviously small. The Cartesian fibration \( \mathbf{P}(\mathcal{U}) \to \mathbf{F}^{\text{op}} \) induces a map
\[
\text{Map}_{\mathcal{U} \omega}(J, \mathcal{J}) \to \text{Map}_{\mathcal{U} \omega}(V_{\omega}(J), V_{\omega}(I))
\]
whose fibre over \( \mathcal{J} \) is given by
\[
\prod_{d \in V_{\omega}(J)} \text{Map}_{\mathcal{U} \omega}(\psi_d, \phi_{c})_{c \subseteq d}
\]
The \( \infty \)-category \( \mathbf{P}(\mathcal{U}) \) is locally small by [37, Example 5.4.1.8], so these mapping spaces are small. \( \square \)

**Theorem 2.5.2.** Let \( \mathcal{U}^{\infty} \) be a small symmetric monoidal \( \infty \)-category. The fully faithful functor \( y_*: \mathbf{P}(\Delta_\mathcal{U}^{\infty}) \to \mathbf{P}(\Delta_\mathcal{U}^{\infty}) \) restricts to an equivalence \( \mathbf{P}_{\text{Seg}}(\Delta_\mathcal{U}^{\infty}) \simeq \mathbf{P}_{\text{CtsSeg}}(\Delta_\mathcal{U}^{\infty}) \).

Before we turn to the proof of this theorem, we prove the preliminary result that the functors \( y^* \) and \( y_* \) restrict to functors between the subcategories of Segal objects. We first consider the easy case of \( y^* \):

**Lemma 2.5.3.** The left adjoint functor \( y^*: \mathbf{P}(\Delta_\mathcal{U}^{\infty}) \to \mathbf{P}(\Delta_\mathcal{U}^{\infty}) \) takes continuous Segal presheaves to Segal presheaves.

**Proof.** Consider an object \( \mathcal{T} \in \Delta_\mathcal{U}^{\infty} \) and an inert map \( \psi: c_n \to I \) in \( \Delta_\mathcal{U} \). Since the Yoneda embedding is symmetric monoidal, we have \( \psi^* y(\mathcal{T}) \simeq c_n(y_{v_i}) \simeq y(\psi^* \mathcal{T}) \) for some \( v_i \in \mathcal{V} \). Hence, for every continuous Segal presheaf \( \mathcal{O} \in \mathbf{P}_{\text{CtsSeg}}(\Delta_\mathcal{U}^{\infty}) \) we have
\[
y^*(\mathcal{T}) = O(g(\mathcal{T})) \simeq \lim_{\psi \in (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}}} O(\psi^* y(\mathcal{T})) \simeq \lim_{\psi \in (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}}} O(\psi(\mathcal{T})) = \lim_{\psi \in (\Delta_\mathcal{U}^{\infty})_{\mathcal{T}}} y^*(\psi(\mathcal{T})),
\]
where the first equivalence follows from the assumption that $\mathcal{O}$ is a continuous Segal presheaf (and so in particular a Segal presheaf).

Now we consider the opposite direction:

**Proposition 2.5.4.** The functor $g_*: \mathcal{P}(\Delta^1_{\mathcal{F}}) \hookrightarrow \mathcal{P}(\Delta^1_{\mathcal{F}})$ takes objects in $\mathcal{P}_{\text{Seg}}(\Delta^1_{\mathcal{F}})$ to objects in $\mathcal{P}_{\text{CtsSeg}}(\Delta^1_{\mathcal{F}})$.

The proof of this requires some preliminary lemmas:

**Lemma 2.5.5.** For $I \in \Delta_2$, the functor

$$g^*I(-, \ldots, -): \mathcal{P}(\Delta^1_I) \to \mathcal{P}(\Delta^1_{\mathcal{F}})$$

preserves weakly contractible colimits in each variable.

**Proof.** Pick a vertex $c' \in V_{\mathcal{A}}(I)$. We fix the labels $X_c \in \mathcal{P}(\Delta^1_I)$ for all vertices $c \neq c'$, and write $T_{c'}(-): \mathcal{P}(\Delta^1_I) \to \mathcal{P}(\Delta^1_{\mathcal{F}})$ for the functor $X_c \mapsto I(X_c)$. Then we wish to show that the functor $g^*T_{c'}$ preserves weakly contractible colimits, which is equivalent to the natural map

$$\text{Map}_{\mathcal{A}}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1) \to \text{Map}_{\mathcal{A}}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1)$$

being an equivalence of spaces for all $\mathcal{A} \in \mathcal{P}(\Delta^1_{\mathcal{F}})$ and all diagrams $\phi: I \to \mathcal{P}(\Delta^1_{\mathcal{F}})$ with $\mathcal{A}$ weakly contractible. It suffices to check this map is an equivalence when $\mathcal{F}$ is (the Yoneda image of) an object $\mathcal{A} = J(v_x)_{x \in V_{\mathcal{A}}(J)}$ in $\Delta^1_{\mathcal{F}}$. In this case we have $\text{Map}_{\mathcal{A}}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1) \simeq \text{colim Map}_{\mathcal{A}^1}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1)$ and therefore it suffices to prove that the natural map

$$\text{colim Map}_{\mathcal{A}_I^1}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1) \to \text{Map}_{\mathcal{A}_I^1}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1)$$

is an equivalence. This map can be identified with the horizontal map in the commutative triangle

$$\text{colim Map}_{\mathcal{A}_I^1}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1) \to \text{Map}_{\mathcal{A}_I^1}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1)$$

It therefore suffices to show that this map gives an equivalence on the fibres over every map $\phi: I \to J$ in $\Delta^1_{\mathcal{F}}$. Since $\mathcal{A}_I^1 \to \mathcal{A}_J^1$ is a Cartesian fibration and colimits of spaces are preserves by pullbacks, the map on fibres at $\phi$ is

$$\text{colim Map}_{\mathcal{A}_I^1}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1) \to \text{Map}_{\mathcal{A}_I^1}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1)$$

Under the equivalence $(\mathcal{A}_I^1, \mathcal{A}_J^1) \simeq \mathcal{P}(\Delta^1_{\mathcal{F}})$, the object $g(\mathcal{A})$ corresponds to $g(v_x)_{x \in V_{\mathcal{A}}(J)}$ and the object $\mathcal{F}(X_c)$ corresponds to $(\bigotimes_{c \in V_{\mathcal{A}}(J)} X_c)_{x \in V_{\mathcal{A}}(J)}$, so we have an equivalence

$$\text{Map}_{\mathcal{A}_I^1}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1) \simeq \prod_{x \in V_{\mathcal{A}}(J)} \text{Map}_{\mathcal{A}_I^1}(g(v_x), \bigotimes_{c \in V_{\mathcal{A}}(J)} X_c)$$

$$\simeq \prod_{x \in V_{\mathcal{A}}(J)} \left( \bigotimes_{c \in V_{\mathcal{A}}(J)} X_c \right) (v_x).$$

There are two cases to consider: If $V_{\mathcal{A}}(\phi)$ takes $c'$ to the base point, then $X_c$ does not appear in this expression, and so the functor $\text{Map}_{\mathcal{A}_I^1}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1)$ is constant. The map

$$\text{colim Map}_{\mathcal{A}_I^1}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1) \to \text{Map}_{\mathcal{A}_I^1}(\mathcal{A}_I^1, \text{colim Map}\, \mathcal{A}_I^1)$$

is therefore an equivalence because $I$ is weakly contractible (which implies that the colimit cocone of a constant diagram is constant). (Note that in this case we do not have an equivalence if $I$ fails...
to be weakly contractible.) On the other hand, if \( V_{\Delta^0}(\phi) \) takes \( c' \) to a corolla \( x' \in V_{\Delta^0}(I) \) then \( \text{Map}_{(\Delta^0)^n(I)}(y(\mathcal{T}), \phi^* T_{c'}(-)) \) can be identified with

\[
\left( \neg \otimes \bigotimes_{c \in V_{\Delta^0}(\phi)^{-1}(x')} X_c \right)(v_x) \times \prod_{x \in V_{\Delta^0}(I)} \left( \bigotimes_{c \in V_{\Delta^0}(\phi)^{-1}(x)} X_c \right)(v_x).
\]

This functor preserves colimits, since the tensor product on \( P(\mathbb{U}) \) preserves colimits in each variable, colimits in presheaves are computed objectwise, and the Cartesian product of spaces preserves colimits in each variable.

\[\text{Definition 2.5.6.}\]

For \( I \in \Delta_\mathbb{F} \) and \( c' \in V_{\Delta^0}(I) \), let \( \text{Sub}_{\mathcal{C}}(I) \) denote the partially ordered set of inert maps \( I' \to I \) such that \( c' \) is not in the image of \( I' \), and inert maps between them. We write \( I \setminus c' \) for the colimit in \( P(\Delta_\mathbb{F}) \) of \( I' \to I \) in \( \text{Sub}_{\mathcal{C}}(I) \).

\[\text{Lemma 2.5.7.}\]

Suppose \( \mathcal{T} \) is an object of \( \Delta^0_P(\mathbb{U}) \), lying over \( I \in \Delta_\mathbb{F} \) such that the corolla \( c' \) is labelled by \( \emptyset \). Let \( \mathcal{T} \setminus c' \) denote the colimit in \( P(\Delta^0_P(\mathbb{U})) \) of \( \mathcal{T} \), over all \( I' \to I \) in \( \text{Sub}_{\mathcal{C}}(I) \), where \( \mathcal{T} \to \mathcal{T} \) is the Cartesian morphism over this map with target \( \mathcal{T} \). Then \( y^*(\mathcal{T} \setminus c') \to y^*\mathcal{T} \) is an equivalence.

\[\text{Proof.}\]

It suffices to show that the induced map

\[
\text{Map}_{P(\Delta^0_P(\mathbb{U}))}(\mathcal{T}, y^*(\mathcal{T} \setminus c')) \to \text{Map}_{P(\Delta^0_P(\mathbb{U}))}(\mathcal{T}, y^*(\mathcal{T}))
\]

is an equivalence for all \( \mathcal{T} \in \Delta^0_P(\mathbb{U}) \). This map can be identified with the top horizontal map in the commutative triangle

\[
\text{colim}_{I' \to I} \text{Map}_{\Delta^0_P(\mathbb{U})}(y(\mathcal{T}), \mathcal{T}) \to \text{Map}_{\Delta^0_P(\mathbb{U})}(y(\mathcal{T}), \mathcal{T}) \to \text{Map}_{\Delta^0_P(\mathbb{U})}(y(\mathcal{T}), \mathcal{T})
\]

It suffices to show that this map is an equivalence on the fibre over every map \( \phi : J \to I \) in \( \Delta_\mathbb{F} \).

There are two cases to consider, according to whether the map \( V_{\Delta^0}(\phi) \) takes the corolla \( c' \) to a corolla in \( V_{\Delta^0}(J) \) or to the base point. If this map sends \( c' \) to a corolla \( x' \in V_{\Delta^0}(J) \), then the fibre of the left-hand map is empty (since the map to \( \text{Map}_{\Delta^0}(J, I) \) factors through the subset \( \text{colim}_{I' \to I} \text{Map}_{\Delta^0}(J, I')) \)). On the other hand, if \( J := J(v_x \in V_{\Delta^0}(J)) \) and \( \mathcal{T} := I(X_c \in V_{\Delta^0}(I)) \) (with \( X_c = \emptyset \)), then we can identify the fibre of the right-hand map with

\[
\left( \emptyset \otimes \bigotimes_{c \in V_{\Delta^0}(\phi)^{-1}(x')} X_c \right)(v_x) \times \prod_{x \in V_{\Delta^0}(J)} \left( \bigotimes_{c \in V_{\Delta^0}(\phi)^{-1}(x)} X_c \right)(v_x),
\]

as in the proof of Lemma 2.5.5. Since the tensor product on \( P(\mathbb{U}) \) preserves colimits, colimits in presheaves are computed objectwise, and the Cartesian product of spaces preserves colimits, this is also empty.

It remains to consider the case where \( V_{\Delta^0}(\phi) \) takes \( c' \) to the base point in \( V_{\Delta^0}(J) \). Observe that, since the maps \( \psi : I' \to I \) are inert, there is at most one map \( \phi' : J \to I' \) such that \( \phi = \psi \circ \phi' \). If the active-inert factorization of \( \phi \) is \( J \xrightarrow{\delta} K \xrightarrow{\beta} I \), then such a map \( \phi' \) exists if and only if \( \beta \) factors through \( I' \) (with such a factorization being unique if it exists). We can therefore identify the functor

\[
(I' \to I) \mapsto \text{Map}_{\Delta^0_P(\mathbb{U})}(y(\mathcal{T}), \mathcal{T}) \phi
\]

with the left Kan extension of its restriction to \( \text{Sub}_{\mathcal{C}}(I) \). Moreover, we can identify this restriction with the constant functor with value \( \text{Map}_{(\Delta^0_P)^n(I)}(y(\mathcal{T}), \alpha^* \mathcal{K}) \). This space is also equivalent to...
Map_{\Delta^n}((y(\mathcal{V}), y(\mathcal{T})), \phi), \) which is the fibre of Map_{\Delta^n}(y(\mathcal{V}), y(\mathcal{T})) at \phi. Since the category Sub_{\mathcal{X}}(I)_{K/} is weakly contractible, and fibre products of spaces commute with colimits, this means that the map
\[
\colim_{n \to \infty} \text{Map}_{\Delta^n}(y(\mathcal{V}), y(\mathcal{T})) \to \text{Map}_{\Delta^n}(y(\mathcal{V}), y(\mathcal{T}))
\]
is an equivalence on the fibre over \phi. \hfill \Box

\textbf{Proof of Proposition 2.5.4.} To prove that \textit{y}, takes Segal presheaves to continuous Segal presheaves it suffices to show that the left adjoint \textit{y}*, takes a collection of generating continuous Segal equivalences in \textit{P}(\Delta^n_{F(U)}) to Segal equivalences in \textit{P}(\Delta^n_{F(U)}). We consider the generators
\[
\prod_{n+1} \mathcal{V} \to (\mathcal{V}, \emptyset), \quad (n \geq 0),
\]
\[
\colim_{j} \mathcal{V} \to \mathcal{V} (\mathcal{V} \in \Delta^n_{F(U)}).
\]
As special cases of Lemmas 2.5.7 and 2.5.5, respectively, we have that \textit{y}*, takes the first two types of maps to equivalences. It then remains to show that \textit{y} Seg \to \textit{y} Seg is a Segal equivalence for all \textit{y} in \Delta^n_{F(U)}. Writing \textit{y} := I(X_e)_{e \in V_{\Delta^n_{X}}} (I), we will prove this in several steps for increasing numbers of elements in \textit{y}.

\textit{Step 1:} First suppose all the labels \textit{X} lie in the essential image of the Yoneda embedding. Then \textit{y} Seg \to \textit{y} Seg is the image under \textit{y} of a generating Segal equivalence in \textit{P}(\Delta^n_{F(U)}) and so is obviously taken to a Segal equivalence by \textit{y}.*

\textit{Step 2:} Next consider the case where the labels \textit{X} are either the initial object \emptyset or in the essential image of the Yoneda embedding. We induct on the number of corollas labelled by \emptyset (the case where this is zero being Step 1). Suppose the corolla \textit{c} is labelled by \emptyset and consider the commutative square
\[
\begin{array}{ccc}
\mathcal{V} \setminus \mathcal{V} & \to & \mathcal{V} \setminus \mathcal{V} \\
\mathcal{V} & \to & \mathcal{V}
\end{array}
\]
where \(\mathcal{V} \setminus \mathcal{V} := \text{colim}_{t \to \text{Sub}_{\mathcal{X}}(I)} \mathcal{V} \setminus \mathcal{V} \). This colimit can be identified with the colimit of the same diagram as \textit{y} Seg, except that the corolla \textit{c} corresponding to \textit{c} is replaced by \(\text{colim}_{\text{c} \to \mathcal{V}} \mathcal{V} \setminus \mathcal{V} \). Thus by Lemma 2.5.7, the functor \textit{y}* takes the vertical maps in the square to equivalences. By the inductive hypothesis, \textit{y} Seg \setminus \mathcal{V} \setminus \mathcal{V} \to \textit{y} Seg \setminus \mathcal{V} \setminus \mathcal{V} \text{ Seg} is a Segal equivalence, since it is a colimit of \textit{y} Seg \setminus \mathcal{V} \setminus \mathcal{V} \text{ Seg} where the number of corollas in \textit{y} Seg labelled by \emptyset is at least one less than in \textit{y} Seg. By the 2-of-3 property, we conclude that \textit{y} Seg \setminus \mathcal{V} \setminus \mathcal{V} \text{ Seg} is also a Segal equivalence.

\textit{Step 3:} Next consider the case where the labels \textit{X} are either the initial object \emptyset or in the essential image of the Yoneda embedding. We induct on the number of these corollas (with the case where they are all of size 0 or 1 covered by Step 2). Suppose the corolla \textit{c} is labelled by \textit{v} \prod \mathcal{V} with \textit{v} \in \textit{U}. Then by Lemma 2.5.5 the map \textit{y} Seg \setminus \mathcal{V} \setminus \mathcal{V} \text{ Seg} can be identified with the pushout
\[
y Seg \setminus \mathcal{V} \setminus \mathcal{V} \text{ Seg} \to \textit{y} Seg \setminus \mathcal{V} \setminus \mathcal{V} \text{ Seg} \to \textit{y} Seg \setminus \mathcal{V} \setminus \mathcal{V} \text{ Seg}.
\]
This is a pushout of maps that are Segal equivalences by the inductive hypothesis, and hence this is also a Segal equivalence.

\textit{Step 4:} Any presheaf on \textit{U} can be written as a sifted (hence weakly contractible) colimit of finite coproducts of elements in \textit{U}. By Lemma 2.5.5 the map \textit{y} Seg \setminus \mathcal{V} \setminus \mathcal{V} \text{ Seg} for general labels \textit{X} is therefore a colimit of the maps we proved were Segal equivalences in Step 3. This map is therefore also a Segal equivalence. \hfill \Box

It is now easy to complete the proof of Theorem 2.5.2:
Proof of Theorem 2.5.2. Since $y_*$ is fully faithful, Proposition 2.5.4 implies that it restricts to a fully faithful functor $P_{\text{Seg}}(\Delta^I_g) \to P_{\text{CtsSeg}}(\Delta^I_g)$. It therefore only remains to prove that this restricted functor is essentially surjective as well. Suppose $\mathcal{O} \in P_{\text{CtsSeg}}(\Delta^I_g)$; we will show that the unit map $u: \mathcal{O} \to y_*y^*(\mathcal{O})$ is an equivalence. As $y^*\mathcal{O}$ lies in $P_{\text{Seg}}(\Delta^I_g)$ by Lemma 2.5.3, this is a map of continuous Segal presheaves by Proposition 2.5.4. To show that $u$ is an equivalence it is therefore enough to show it gives an equivalence when evaluated at $\mathcal{T}$ and $c_n(\phi)$ with $\phi \in P(\mathcal{U})$. Since $y_*$ is fully faithful, we have $y^* \circ y_* \simeq \text{id}$ and so for $X \in \Delta^I_g$ we get
\[
y_*y^*(\mathcal{O})(y(X)) \simeq \text{Map}(y(X), y_*y^*(\mathcal{O})) \simeq \text{Map}(X, y_*y^*(\mathcal{O})) \simeq \text{Map}(X, y^*\mathcal{O}) \simeq \mathcal{O}(y(X)).
\]
In particular, this applies to $c$ and $(c_n, g(v))$ with $v \in \mathcal{U}$. But for a general $\phi \in P(\mathcal{U})$ there exists a small diagram $f: c \to \mathcal{U}$ with colimit $\phi$, and so a Segal equivalence $\text{colim}_\mathcal{C}(c_n, f^s) \to c_n(\phi)$, which means $u$ is an equivalence also when evaluated at $c_n(\phi)$.

Corollary 2.5.8. There is an equivalence of $\infty$-categories
\[
P_{\text{Seg}}(\Delta^I_g) \simeq P_{\text{CtsSeg}}(\Delta^I_g),
\]
where the $\infty$-category $\mathcal{S}$ of spaces is equipped with the Cartesian monoidal structure.

Proof. Applying Theorem 2.5.2 to the trivial $\infty$-category $* \to \mathcal{U}$ we get an equivalence $P_{\text{Seg}}(\Delta^I_g) \simeq P_{\text{CtsSeg}}(\Delta^I_g)$, where $P_{\text{Seg}}(\Delta^I_g)$ coincides with $P_{\text{CtsSeg}}(\Delta^I_g)$ by definition.\qed

2.6. Enrichment in Localizations of Presheaves and Presentability. In this subsection we will show that if $\mathcal{V}$ is the symmetric monoidal localization of $P(\mathcal{U})$ at a set of morphisms, then the $\infty$-category of continuous Segal presheaves on $\Delta^I_g$ is equivalent to a certain localization of $P_{\text{Seg}}(\Delta^I_g)$. This will also allow us to show that $P_{\text{CtsSeg}}(\Delta^I_g)$ is presentable when $\mathcal{V}$ is presentably symmetric monoidal.

Consider a small symmetric monoidal $\infty$-category $\mathcal{U}$ and a set $\mathcal{S}$ of morphisms in $P(\mathcal{U})$ which is compatible with the symmetric monoidal structure on $P(\mathcal{U})$, in the sense that the strongly saturated class generated by $\mathcal{S}$ is closed under tensor products. We write $P_{\mathcal{S}}(\mathcal{U})$ for the full subcategory of $P(\mathcal{U})$ spanned by the $\mathcal{S}$-local objects. Then by [38, Proposition 2.2.1.9] the $\infty$-category $P_{\mathcal{S}}(\mathcal{U})$ inherits a symmetric monoidal structure such that the localization $P(\mathcal{U}) \to P_{\mathcal{S}}(\mathcal{U})$ is a symmetric monoidal functor.

Definition 2.6.1. If $\mathcal{U}$ and $\mathcal{S}$ are as above, then a presheaf in $P(\Delta^I_g)$ is a continuous $\mathcal{S}$-Segal presheaf if it is a continuous Segal presheaf and it is local with respect to the maps $c_n(s)$ where $s$ is in $\mathcal{S}$. Similarly, a presheaf in $P(\Delta^I_g)$ is an $\mathcal{S}$-Segal presheaf if it is a Segal presheaf and is local with respect to the maps $y^*c_n(s)$ where $s$ is in $\mathcal{S}$. We write $P_{\text{CtsSeg}}(\Delta^I_g)$ and $P_{\text{Seg}}(\Delta^I_g)$ for the full subcategories of $P(\Delta^I_g)$ and $P(\Delta^I_g)$ spanned by the continuous $\mathcal{S}$-Segal presheaves and the $\mathcal{S}$-Segal presheaves, respectively. These are by definition the localizations of $P(\Delta^I_g)$ and $P(\Delta^I_g)$ at the continuous $\mathcal{S}$-Segal equivalences and the $\mathcal{S}$-Segal equivalences, these being the morphisms in the strongly saturated classes generated (in the sense of Definition 2.3.18) by the continuous Segal equivalences together with the maps $c_n(s)$ and the Segal equivalences together with the maps $y^*c_n(s)$, respectively.

Remark 2.6.2. A Segal presheaf $\mathcal{O} \in P_{\text{Seg}}(\Delta^I_g)$ is an $\mathcal{S}$-Segal presheaf if and only if the presheaf $\mathcal{O}(c_n(\cdot)): \mathcal{U}^{op} \to \mathcal{S}$ is $\mathcal{S}$-local for all $n$. Similarly, a continuous Segal presheaf $\mathcal{O} \in P_{\text{CtsSeg}}(\Delta^I_g)$ is a continuous $\mathcal{S}$-Segal presheaf if and only if the presheaf $\mathcal{O}(c_n(\cdot, x_1, \ldots , x_n, y)): P(\mathcal{U})^{op} \to \mathcal{S}$ is representable, and the representing object in $P(\mathcal{U})$ is $\mathcal{S}$-local: for $s : v \to v'$ in $\mathcal{V}$ we have a commutative diagram
\[
\begin{array}{ccc}
\text{Map}(c_n(v'), \mathcal{O}) & \xrightarrow{c_n(s)^*} & \text{Map}(c_n(v), \mathcal{O}) \\
\downarrow & & \downarrow \\
\text{Map}(\prod_{n+1} e, \mathcal{O}), & & \text{Map}(\prod_{n+1} e, \mathcal{O})
\end{array}
\]
and $c_\alpha(s)^*$ is an equivalence if and only if it gives an equivalence on fibres over each point of $\text{Map}(\prod_{n+1} c, \mathcal{O}) \simeq \mathcal{O}(c)^{n+1}$, and over $(x_1, \ldots, x_n, y)$ we get the map

$$\text{Map}_P(v', \mathcal{O}(x_1, \ldots, x_n; y)) \rightarrow \text{Map}_P(v, \mathcal{O}(x_1, \ldots, x_n; y)),$$

where $\mathcal{O}(x_1, \ldots, x_n; y)$ is the object representing $\text{Map}(c_\alpha(-), \mathcal{O}(x_1, \ldots, x_n, y))$.

**Corollary 2.6.3.** Let $\mathcal{U}$ and $\mathcal{S}$ be as above.

(i) The equivalence $y^* : \text{P}Cts\text{-Seg}(\Delta^U_P) \sim \text{P}\text{-Seg}(\Delta^U_S)$ restricts to an equivalence

$$\text{P}Cts\text{-Seg}(\Delta^U_P) \sim \text{P}\text{-Seg}(\Delta^U_S).$$

(ii) Let $L_\mathcal{S}$ denote the localization $P(\mathcal{U}) \to \mathcal{P}(\mathcal{S}(\mathcal{U})$. This is symmetric monoidal, and so induces a functor $\Delta^U_P \to \Delta^U_S$, which we also denote $L_\mathcal{S}$. Then the functor $L_\mathcal{S}^P : \text{P}(\Delta^U_P) \to \text{P}(\Delta^U_S)$ given by composition with $L_\mathcal{S}^P$ restricts to an equivalence

$$\text{P}Cts\text{-Seg}(\Delta^U_P) \sim \text{P}Cts\text{-Seg}(\Delta^U_S),$$

In particular, we have an equivalence $\text{P}Cts\text{-Seg}(\Delta^U_P) \sim \text{P}\text{-Seg}(\Delta^U_S)$, which describes the continuous Segal presheaves on $\Delta^U_P$ as a localization of $\text{P}(\Delta^U_S)$.

**Proof.** Part (i) is obvious, since $\text{P}Cts\text{-Seg}(\Delta^U_P)$ and $\text{P}\text{-Seg}(\Delta^U_S)$ are, respectively, the localization of $\text{P}Cts\text{-Seg}(\Delta^U_P)$ at a certain set of maps and the localization of $\text{P}\text{-Seg}(\Delta^U_S)$ at the image of this set under the equivalence $y^*$ of Theorem 2.5.2.

To prove (ii), first recall that the functor $L_\mathcal{S}^P$ restricts to a functor

$$\text{P}Cts\text{-Seg}(\Delta^U_P) \to \text{P}Cts\text{-Seg}(\Delta^U_S),$$

by Lemma 2.3.22. Now observe that by Corollary 2.4.12, this functor is fully faithful, and by Remark 2.6.2 its image is precisely $\text{P}Cts\text{-Seg}(\Delta^U_S)$.

We now consider a key special case of this corollary:

**Definition 2.6.4.** Let $\mathcal{U}$ be a small symmetric monoidal $\infty$-category, and suppose $\kappa$ is a regular cardinal such that $\mathcal{U}$ has all $\kappa$-small colimits and the tensor product preserves these in each variable separately. A Segal presheaf $\mathcal{O} \in \text{P}\text{-Seg}(\Delta^U_S)$ is $\kappa$-continuous if for every $n$,

1. the functor $\mathcal{O}(c_\alpha(-)) : \mathcal{U}^{op} \to \mathcal{S}$ preserves $\kappa$-small weakly contractible limits,
2. $\mathcal{O}(c_\alpha(\emptyset)) \to \prod_{n+1} \mathcal{O}(e)$ is an equivalence.

We write $\text{P}_\kappa\text{-Seg}(\Delta^U_S)$ for the full subcategory of $\text{P}(\Delta^U_S)$ spanned by the $\kappa$-continuous Segal presheaves.

**Proposition 2.6.5.** Let $\mathcal{U}$ be as in Definition 2.6.4. Take $S_\kappa$ to be the class of morphisms of the form colim $y(\phi) \to y(\text{colim } \phi)$ where $\phi$ is a $\kappa$-small diagram in $\mathcal{U}$ and $y$ is the Yoneda embedding; we take $S_\kappa^U \subseteq S_\kappa$ to be a subset corresponding to a set of representatives for equivalence classes of pushouts and $\kappa$-small coproducts in $\mathcal{U}$. Then the following are equivalent for $\mathcal{O} \in \text{P}\text{-Seg}(\Delta^U_S)$:

(i) $\mathcal{O}$ is a $\kappa$-continuous Segal presheaf,
(ii) $\mathcal{O}$ is an $S_\kappa$-Segal presheaf,
(iii) $\mathcal{O}$ is an $S_\kappa^U$-Segal presheaf.

**Proof.** By definition, $\mathcal{O}$ is an $S_\kappa$-Segal presheaf if for every $\kappa$-small diagram $\phi : \mathcal{I} \to \mathcal{U}$, the morphism

$$\text{Map}(y^*c_\alpha(y(\text{colim } \phi)), \mathcal{O}) \to \text{Map}(y^*c_\alpha(y(\text{colim } \phi)), \mathcal{O})$$

is an equivalence. This holds if and only if it holds for all $\kappa$-small weakly contractible diagrams and the initial object, since these generate the $\kappa$-small colimits. For $\kappa$-small weakly contractible colimits, we have $y^*c_\alpha(y(\text{colim } \phi)) \simeq \text{colim } y^*c_\alpha(y(\phi))$ by Lemma 2.5.5, so we can identify the morphism with $\mathcal{O}(c_\alpha(\text{colim } \phi)) \to \lim \mathcal{O}(c_\alpha(\emptyset))$, corresponding to condition (1) in Definition 2.6.4. For the initial object, we instead have $y^*c_\alpha(\emptyset) \simeq \prod_{n+1} e$ by Lemma 2.5.7, corresponding to condition (2). This proves that (i) and (ii) are equivalent. The equivalence of (ii) and (iii) is immediate, since $\kappa$-small colimits are generated by pushouts and $\kappa$-small coproducts. \qed
Corollary 2.6.6. Let $\mathcal{U}$ be a small symmetric monoidal $\infty$-category, and suppose $\kappa$ is a regular cardinal such that $\mathcal{U}$ has all $\kappa$-small colimits and the tensor product preserves these. Then the Yoneda embedding induces an equivalence

$$p_{\text{CtsSeg}}(\Delta_\mathcal{U}^{\text{Ind}_{\mathcal{U}}\mathcal{U}}) \simeq p_{\kappa\text{-Seg}}(\Delta_\mathcal{V}^\mathcal{Y}).$$

Proof. Apply Corollary 2.6.3 with $S$ being the set $S_{\kappa}'$ from Proposition 2.6.5.

Remark 2.6.7. Since $S_{\kappa}'$ is a small set, the $\infty$-category $p_{\kappa\text{-Seg}}(\Delta_\mathcal{V}^\mathcal{Y})$ is the full subcategory of $p(\Delta_\mathcal{V}^\mathcal{Y})$ of presheaves that are local with respect to a small set of maps. It follows that $p_{\kappa\text{-Seg}}(\Delta_\mathcal{V}^\mathcal{Y})$ is an accessible localization of $p(\Delta_\mathcal{V}^\mathcal{Y})$. In particular, the $\infty$-category $p_{\text{CtsSeg}}(\Delta_\mathcal{U}^{\text{Ind}_{\mathcal{U}}\mathcal{U}}) \simeq p_{\kappa\text{-Seg}}(\Delta_\mathcal{V}^\mathcal{Y})$ is presentable. We can use this result to see that $p_{\text{CtsSeg}}(\Delta_\mathcal{U}^{\mathcal{Y}})$ is presentable. If we write $\kappa$-accessibly symmetric monoidal for some cardinal $\kappa$.

Corollary 2.6.8. Suppose $\mathcal{V}$ is a presentably symmetric monoidal $\infty$-category. Then $p_{\text{CtsSeg}}(\Delta_\mathcal{U}^{\mathcal{V}})$ is a presentable $\infty$-category.

This follows from Remark 2.6.7 and the following technical observation:

Proposition 2.6.9. Suppose $\mathcal{V}_\omega$ is a presentably symmetric monoidal $\infty$-category. Then there exists a regular cardinal $\kappa$ such that $\mathcal{V}$ is $\kappa$-presentable and the full subcategory $\mathcal{V}_\kappa$ of $\kappa$-compact objects is closed under the tensor product.

To prove Proposition 2.6.9 it is convenient to prove a slightly more general result, for accessibly symmetric monoidal $\infty$-categories in the following sense:

Definition 2.6.10. Given a regular cardinal $\kappa$, we say a symmetric monoidal $\infty$-category $\mathcal{V}$ is $\kappa$-accessibly symmetric monoidal if the underlying $\infty$-category $\mathcal{V}$ is $\kappa$-accessible and the tensor product preserves $\kappa$-filtered colimits separately in each variable. We say $\mathcal{V}$ is accessibly symmetric monoidal if it is $\kappa$-accessibly symmetric monoidal for some cardinal $\kappa$.

Lemma 2.6.11. Let $\kappa$ and $\kappa'$ be two regular cardinals such that $\kappa \ll \kappa'$ (in the sense of [37, Definition 5.4.2.8]). If $\mathcal{C}$ is a $\kappa$-accessible $\infty$-category, then an object in $\mathcal{C}$ is $\kappa'$-compact if and only if it is a $\kappa'$-small $\kappa$-filtered colimit of $\kappa$-compact objects.

Proof. Let $\mathcal{C}_{\kappa}$ and $\mathcal{C}_{\kappa'}$ denote the full subcategories of $\mathcal{C}$ spanned by the $\kappa$-compact and $\kappa'$-compact objects, respectively. If we write $\mathcal{C}_{\kappa'}$ for the full subcategory of $\mathcal{C}$ spanned by the colimits of all $\kappa'$-small $\kappa$-filtered diagrams in $\mathcal{C}_{\kappa}$, then $\mathcal{C}_{\kappa'}$ is essentially small. This is because $\mathcal{C}$ is locally small and the collection of all equivalence classes of $\kappa'$-small $\kappa$-filtered diagrams is bounded. Since $\mathcal{C}_{\kappa'}$ is closed under $\kappa'$-small colimits by [37, Corollary 5.3.4.15], the $\infty$-category $\mathcal{C}_{\kappa'}$ is a full subcategory of $\mathcal{C}_{\kappa'}$.

The proof of [37, Proposition 5.4.2.11] shows that $\mathcal{C}_{\kappa'}$ generates $\mathcal{C}$ under small $\kappa'$-filtered colimits. According to [37, Lemma 5.4.2.4], the $\infty$-category $\mathcal{C}_{\kappa'} \subseteq \mathcal{C} = \text{Ind}_{\kappa'}(\mathcal{C}_{\kappa})$ is given by the idempotent completion of $\mathcal{C}_{\kappa'}$. Since $\mathcal{C}_{\kappa'}$ is already idempotent complete by [37, Proposition 4.4.5.15], the $\infty$-categories $\mathcal{C}_{\kappa'}$ and $\mathcal{C}_{\kappa'}$ coincide.

Definition 2.6.12. Let $\mathcal{V}_\omega \to \mathbb{F}_*$ be a symmetric monoidal $\infty$-category. For a cardinal $\kappa$, let $\mathcal{V}_\kappa$ denote the full subcategory of $\mathcal{V}$ spanned by $\kappa$-compact objects in $\mathcal{V}$. We write $\mathcal{V}_{\kappa,\omega}$ for the full subcategory of $\mathcal{V}_\omega$ spanned by objects lying in $(\mathcal{V}_\kappa)^n \subseteq (\mathcal{V})^n \simeq \mathcal{V}_n$ for all $n \geq 1$.

Remark 2.6.13. We emphasize that $\mathcal{V}_{\kappa,\omega}$ is in general not the full subcategory of the $\infty$-category $\mathcal{V}_\omega$ spanned by $\kappa$-compact objects. Moreover, it is in general also not a symmetric monoidal $\infty$-category, though it is an $\omega$-operad.

Proposition 2.6.14. Suppose $\mathcal{V}$ is a $\kappa$-accessibly symmetric monoidal $\infty$-category. Then there exists a regular cardinal $\kappa'$ such that the full subcategory $\mathcal{V}_{\kappa'}$ is a symmetric monoidal $\infty$-category.
Proof. Since \( \mathcal{V} \) is \( \kappa \)-accessibly symmetric monoidal, the \( \infty \)-category \( \mathcal{V}^{\kappa, \otimes} \) is essentially small and we can choose a cardinal \( \kappa' \) such that \( \kappa' \gg \kappa, 1 \in \mathcal{V}^{\kappa'} \) and \( v \otimes w \in \mathcal{V}^{\kappa'} \) for any pair of objects \( v, w \in \mathcal{V}^{\kappa} \).

If \( w \in \mathcal{V}^{\kappa'} \), then, by Lemma 2.6.11, there exists a \( \kappa' \)-small \( \kappa \)-filtered colimit diagram \( f : \mathcal{P} \to \mathcal{V} \) such that \( f(i) \in \mathcal{V}^{\kappa} \) and \( w = \text{colim}_{i \in I} f(i) \). Since \( \mathcal{V} \) is \( \kappa \)-accessibly symmetric monoidal, the tensor product preserves \( \kappa \)-filtered colimits in each variable. Therefore, for every \( v \in \mathcal{V}^{\kappa} \), we have that \( v \otimes (\text{colim}_{i \in I} f(i)) \simeq \text{colim}_{i \in I} (v \otimes f(i)) \) is a \( \kappa' \)-small colimit of \( \kappa' \)-compact objects, and is thus \( \kappa' \)-compact by [37, Corollary 5.3.4.15]. Thus we see that \( v \otimes w \in \mathcal{V}^{\kappa'} \), if \( v \in \mathcal{V}^{\kappa} \) and \( w \in \mathcal{V}^{\kappa} \). Applying the same argument to the other variable, we deduce that \( \mathcal{V}^{\kappa} \) is closed under tensor products, and so it is a symmetric monoidal subcategory of \( \mathcal{V} \). \( \square \)

Since every presentably symmetric monoidal \( \infty \)-category is by definition \( \kappa \)-accessibly symmetric monoidal for some \( \kappa \), Proposition 2.6.9 follows as a special case.

2.7. Inner Anodyne Maps. Recall that a morphism in \( \mathcal{P}(\Delta) \) is inner anodyne if it is in the weakly saturated class generated by the inner horn inclusions \( \Lambda^k_n \to \Delta^k_n \) (viewed as discrete simplicial spaces). In this subsection we will consider an analogous class of inner anodyne maps in \( \mathcal{P}(\Delta^\gamma_\mathcal{V}) \) and show that these are Segal equivalences.

Definition 2.7.1. Let \( \mathcal{C} \) be a cocomplete \( \infty \)-category.

1. We say a class of morphisms in \( \mathcal{C} \) is weakly saturated if it is closed under pushouts, transfinite compositions and retracts.

2. Given a class \( \mathcal{S} \) of morphisms in \( \mathcal{C} \), we write \( \langle \mathcal{S} \rangle \) for the smallest weakly saturated class of morphisms in \( \mathcal{C} \) containing \( \mathcal{S} \) and we say that \( \langle \mathcal{S} \rangle \) is generated by \( \mathcal{S} \).

Remark 2.7.2. In the model category literature, the 1-categorical analogue of a weakly saturated class is usually just called a saturated class. Following [37], we use the term weakly saturated to avoid confusion with the strongly saturated classes relevant to accessible localizations of presentable \( \infty \)-category (cf. Definition 2.3.15). It is then clear that from the definition that strongly saturated classes are also weakly saturated.

Definition 2.7.3. Let \( p \) denote the Cartesian fibration \( \Delta_\mathcal{V} \to \Delta \) and let \( I \) be an object of \( \Delta_\mathcal{V} \) with \( p(I) = [n] \). For \( 1 \leq k \leq n - 1 \), we define \( \Lambda^k_n I \) by the pullback

\[
\begin{array}{ccc}
\Lambda^k_n I & \longrightarrow & I \\
\downarrow & & \downarrow \\
p^*\Lambda^k_n & \longrightarrow & p^*(\Delta^k_n)
\end{array}
\]

If \( \mathcal{V} \) is a symmetric monoidal \( \infty \)-category and \( \mathcal{T} \) is an object of \( \Delta^\gamma_\mathcal{V} \) over \( I \), we similarly define \( \Lambda^k_n \mathcal{T} \) by the pullback

\[
\begin{array}{ccc}
\Lambda^k_n \mathcal{T} & \longrightarrow & \mathcal{T} \\
\downarrow & & \downarrow \\
(pq)^*\Lambda^k_n & \longrightarrow & (pq)^*(\Delta^k_n)
\end{array}
\]

where \( q \) denotes the Cartesian fibration \( \Delta^\gamma_\mathcal{V} \to \Delta_\mathcal{V} \). We refer to the inclusions \( \Lambda^k_n \mathcal{T} \hookrightarrow \mathcal{T} \) as inner horn inclusions and we say a map in \( \mathcal{P}(\Delta^\gamma_\mathcal{V}) \) is inner anodyne if it lies in the weakly saturated class generated by the inner horn inclusions (for all \( \mathcal{T} \) in \( \Delta^\gamma_\mathcal{V} \)).

Remark 2.7.4. In \( \mathcal{P}(\Delta) \), a presheaf is a Segal space if and only if it has the right lifting property with respect to the inner anodyne maps. The analogous statement for Segal presheaves on \( \Delta^\gamma_\mathcal{V} \) is
not true for the inner anodyne maps as defined here, since they clearly do not generate the maps
\[
\prod_{i \in m} c_{n_i}(v_i) \to ([1], n \to m)(v_1, \ldots, v_m),
\]
\[
\prod_{i \in m} e \to ([0], m).
\]
It should be possible to enlarge the class of inner anodyne maps to include “inner horns” related also to these maps, but we will not consider this issue here.

Our goal is now to prove the following:

**Proposition 2.7.5.** The inner anodyne maps in \(P(\Delta^V_\infty)\) are Segal equivalences.

We will prove this at the end of this subsection after making some formal observations. The starting point for the proof is the following reinterpretation of the proof of [32, Lemma 3.5]:

**Proposition 2.7.6 (Joyal–Tierney).** Every inner horn inclusion \(\Lambda^n_k \hookrightarrow \Delta^n\) can be constructed from the spine inclusions \(\Delta^n_{\text{Seg}} \to \Delta^i\) \((i \leq n)\) by a finite sequence of compositions, pushouts, and right cancellations.

**Proposition 2.7.7.** In \(P(\Delta)\) inner horn inclusions generate the strongly saturated class of Segal equivalences.

**Proof.** Inner horn inclusions are Segal equivalences as they are generated by the spine inclusions according to the previous proposition. Hence, it suffices to show that the spine inclusions, which generate the Segal equivalences, lie in the strongly saturated class generated by the inner horn inclusions. This follows from [31, Proposition 2.13] which says that spine inclusions are inner anodyne, and so even lie in the weakly saturated class generated by the inner horn inclusions. \(\square\)

Since pullbacks in an \(\infty\)-topos preserve colimits, we get the following:

**Proposition 2.7.8.** Suppose \(\mathcal{S}\) is a set of morphisms in an \(\infty\)-topos \(X\). If

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\xi} & & \downarrow{\eta} \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

is a pullback square, let \(\eta^*\mathcal{S}\) denotes the class of morphisms \(\eta^*A \to \eta^*B\) given by pullbacks

\[
\begin{array}{ccc}
\eta^*A & \xrightarrow{\eta^*s} & \eta^*B \\
\downarrow{\eta^*s} & & \downarrow{\eta^*s} \\
A & \xrightarrow{s} & B
\end{array}
\]

for all \(s \in \mathcal{S}\) and all maps \(B \to Y'\).

(i) If \(f'\) is a (possibly transfinite) composite of morphisms in \(\mathcal{S}\), then \(f\) is a (transfinite) composite of morphisms in \(\eta^*\mathcal{S}\).

(ii) If \(f'\) is a cobase change of a morphism in \(\mathcal{S}\), then \(f\) is a cobase change of a morphism in \(\eta^*\mathcal{S}\).

(iii) If \(f'\) is a retract of a morphism in \(\mathcal{S}\), then \(f\) is a retract of a morphism in \(\eta^*\mathcal{S}\).

(iv) If \(f'\) is obtained from \(\mathcal{S}\) by right cancellation (i.e. there exists \(g'\) in \(\mathcal{S}\) such that \(f'g'\) is in \(\mathcal{S}\)) then \(f\) is obtained from \(\eta^*\mathcal{S}\) by right cancellation.

Combining this observation with the small object argument, we have:
Corollary 2.7.9. Suppose $\mathcal{S}$ is a set of morphisms in an $\infty$-topos $X$. If

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\xi} & & \downarrow{g} \\
X' & \xrightarrow{f'} & Y'
\end{array}
$$

is a pullback square and $f'$ lies in the weakly saturated class $\langle \mathcal{S} \rangle$ generated by $\mathcal{S}$, then $f$ lies in the weakly saturated class $\langle \eta^*\mathcal{S} \rangle$. □

Lemma 2.7.10. Let $p: \mathcal{E} \to \mathcal{C}$ be a Cartesian fibration and let $p^*: \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{E})$ denote the functor given by composition with $p$. If $\sigma \to \tau$ is a $p$-Cartesian morphism lying over $s \to t$ in $\mathcal{C}$, then there is a pullback square in $\mathcal{P}(\mathcal{E})$

$$
\begin{array}{ccc}
\sigma & \xrightarrow{p^*s} & \tau \\
\downarrow{p^*} & & \downarrow{p^*}
\end{array}
$$

where the vertical maps are the adjunction units $\sigma \to p^*p_!\sigma \simeq p^*s$ and $\tau \to p^*p_!\tau \simeq p^*t$, respectively.

Proof. We have to show that for every presheaf $F \in \mathcal{P}(\mathcal{E})$, the commutative square

$$
\begin{array}{ccc}
\text{Map}_{\mathcal{P}(\mathcal{E})}(F,\sigma) & \xrightarrow{\phi} & \text{Map}_{\mathcal{P}(\mathcal{E})}(F,\tau) \\
\downarrow & & \downarrow \\
\text{Map}_{\mathcal{P}(\mathcal{C})}(p^*s,F) & \xrightarrow{\phi} & \text{Map}_{\mathcal{P}(\mathcal{C})}(p^*t,F)
\end{array}
$$

is a pullback square of spaces. Since every object in the presheaf category $\mathcal{P}(\mathcal{E})$ is given by a colimit of representable objects we can assume without loss of generality that $F$ is represented by an object $\sigma \in \mathcal{E}$ lying over $x \in \mathcal{C}$. In this case, the adjunction $(p_!,p^*)$ implies that the above square is of the form

$$
\begin{array}{ccc}
\text{Map}_E(\sigma,\sigma) & \xrightarrow{\phi} & \text{Map}_E(\sigma,\tau) \\
\downarrow & & \downarrow \\
\text{Map}_C(x,s) & \xrightarrow{\phi} & \text{Map}_C(x,t),
\end{array}
$$

which is a pullback of spaces because the morphism $\sigma \to \tau$ was $p$-Cartesian. □

Definition 2.7.11. Let $p: \mathcal{E} \to \mathcal{C}$ be a Cartesian fibration between small $\infty$-categories and let $p^*: \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{E})$ denote the functor on presheaves given by composition with $p$. For $F \in \mathcal{P}(\mathcal{E})$ and $G \in \mathcal{P}(\mathcal{C})$, we say a morphism $F \to p^*(G)$ is simple if for every map $\sigma: c \to G$ where $c$ is representable, in the pullback

$$
\begin{array}{ccc}
X & \xrightarrow{p^*(c)} & F \\
\downarrow{p^*(c)} & & \downarrow{p^*(G)}
\end{array}
$$

the presheaf $X$ is representable, and the adjoint map $p_!X \to c$ is an equivalence (i.e. $X$ is representable by an object whose image under $p$ is $c$). In this case we often write $F_\sigma$ for the presheaf $X$.

Remark 2.7.12. The previous definition and Lemma 2.7.10 immediately imply that for any $\tau \in \mathcal{E}$ the counit map $\tau \to p^*\tau$ is simple and a pullback of a simple map is simple.
Notation 2.7.13. Let $\mathcal{S}$ be a class of morphisms $f : K \to L$ in $P(\mathcal{C})$ such that $L$ is representable. We define $p^* \mathcal{S} := \{ p^* s, s \in \mathcal{S} \}$ and we write $\mathcal{S}_E$ for the class of morphisms $Y \to X$ in $P(\mathcal{E})$ which fit in a pullback square

$$
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
p^* K & \longrightarrow & p^* L,
\end{array}
$$

where $X$ is representable, the adjoint map $p(X) \simeq p_X \to L$ is an equivalence and the bottom horizontal map lies in $p^* \mathcal{S}$.

Lemma 2.7.14. Let $p^* : P(\mathcal{C}) \to P(\mathcal{E})$ denote the functor induced by composition with a Cartesian fibration $p : \mathcal{E} \to \mathcal{C}$ between small $\infty$-categories and let $\mathcal{S}$ be a class of morphisms in $P(\mathcal{C})$ whose codomains are representable. Suppose there is a simple map $f : X \to p^* L$ and a map $K \to L \in \langle \mathcal{S} \rangle$, then the natural map $Y := p^* K \times_{p^* L} X \to X$ lies in $\langle p^* \mathcal{S} \rangle$.

Proof. Since the functor $p^*$ is left adjoint to the functor $p_*$ given by right Kan extension, it preserves colimits. Therefore, by the small object argument it takes every element in $\langle \mathcal{S} \rangle$ to $\langle p^* \mathcal{S} \rangle$. In particular, the bottom horizontal morphism of the pullback square

$$
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
p^* K & \longrightarrow & p^* L
\end{array}
$$

lies in $\langle p^* \mathcal{S} \rangle$. Then Proposition 2.7.9 implies that $Y \to X$ lies in $\langle f^* p^* \mathcal{S} \rangle$ and each generator $f^* p^* s$, $s \in \mathcal{S}$, is given by two pullback squares

$$
\begin{array}{ccc}
f^* p^* A & \longrightarrow & f^* p^* B \\
\downarrow & & \downarrow \\
p^* A & \longrightarrow & p^* B
\end{array}
$$

Therefore, to prove the lemma, it suffices to show that each of these maps $f^* p^* s$ lie in $\mathcal{S}_E$. Since $s \in \mathcal{S}$ and its codomain $B$ is representable, the assumption that $f : X \to p^* L$ is simple implies that $f^* p^* B$ is representable and $p(f^* p^* B) \simeq p^* B$. Hence, the map $f^* p^* s$ lies in $\mathcal{S}_E$ by definition. □

Proof of Proposition 2.7.5. It’s enough to show that every inner horn inclusion $\Lambda^n_k \T \to \T$ in $\Delta^n_m$ is a Segal equivalence: this implies that the class of inner anodyne maps, which is the weakly saturated class generated by these maps, must be contained in the strongly saturated class of Segal equivalences. Given an inner horn inclusion $j : \Lambda^n_k \T \to \T$, by definition it comes with a pullback diagram

$$
\begin{array}{ccc}
\Lambda^n_k \T & \longrightarrow & \T \\
\downarrow & & \downarrow \\
(pq)^* \Lambda^n_k & \longrightarrow & (pq)^* \Delta^n
\end{array}
$$

Let $\mathcal{S}$ denote the set of spine inclusions $\Delta^m_k \to \Delta^m_k$, then Propositions 2.7.6 and 2.7.8 imply that $\jmath$ is obtained from $\eta^* \mathcal{S}$ by a finite sequence of pushouts, compositions, and right cancellations. It follows that $\jmath$ is contained in the strongly saturated class generated by $\eta^* \mathcal{S}$. 


Since the right vertical map \( \eta : \mathcal{T} \to (pq)^* \Delta^n \) is simple by Remark 2.7.12, as in the proof of Lemma 2.7.14 we see that the maps in \( \eta^* \mathcal{S} \) are among the generating Segal equivalences of Proposition 2.3.12(3). Hence \( \mathcal{J} \) must be contained in the strongly saturated class of Segal equivalences, as required.

By applying Lemma 2.7.14 to the set \( \mathcal{S} = \{ \Delta^n_k \to \Delta^n, n \geq 0, 0 < k < n \} \) of inner horn inclusions in \( P(\Delta) \) we also obtain the following:

**Corollary 2.7.15.** Let \( p : \Delta^n_F \to \Delta \) and \( q : \Delta^n_G \to \Delta \) denote the Cartesian fibrations given by the obvious projections. If \( f : X \to (pq)^* L \) is a simple map in \( P(\Delta^n_F) \) and \( K \to L \) is an inner anodyne map in \( P(\Delta) \), then the natural map \( Y := (pq)^* K \times_{(pq)^* L} X \to X \) in \( P(\Delta^n_R) \) is inner anodyne (and so in particular a Segal equivalence, by Proposition 2.7.5).

2.8. Tensoring with Segal Spaces. Our goal in this section is to prove that for \( V \) a presentably symmetric monoidal \( \infty \)-category, the \( \infty \)-category \( P_{\text{CtsSeg}}(\Delta) \) is a module over the symmetric monoidal \( \infty \)-category \( P_{\text{Seg}}(\Delta) \) of Segal spaces. This will be useful in the next section as it allows us to reduce a number of proofs to the case of Segal spaces.

**Definition 2.8.1.** Let \( \mathcal{U} \) be a small symmetric monoidal \( \infty \)-category, and let \( p : \Delta^{\mathcal{U}}_F \to \Delta \) denote the composite of the two Cartesian fibrations \( \Delta^{\mathcal{U}}_F \to \Delta_F \) and \( \Delta_F \to \Delta \). Precomposition with \( p^* \) gives a functor \( p^* : P(\Delta) \to P(\Delta^{\mathcal{U}}_F) \). This functor preserves products (since it is a right adjoint), and so can be viewed as a symmetric monoidal functor, with both \( \infty \)-categories equipped with their Cartesian symmetric monoidal structures. In other words, \( p^* \) is a morphism of commutative algebra objects in \( \text{Cat}_{\infty} \) (or even in \( \text{Pr}^{\text{cof}} \)) and so induces on \( P(\Delta^{\mathcal{U}}_F) \) the structure of a module over \( P(\Delta) \) (e.g. by [38, Corollary 3.4.1.7]), given by the functor \( P(\Delta^{\mathcal{U}}_F) \times P(\Delta) \to P(\Delta^{\mathcal{U}}_F) \) that takes \( (I,K) \) to \( I \times p^*(K) \). Since the Cartesian product in \( \mathcal{S} \) preserves colimits in each variable, as does \( p^* \), the functor \( - \times p^*(-) \) preserves colimits in each variable.

Our main goal in this section is to prove the following result:

**Theorem 2.8.2.** For \( \mathcal{U} \) a small symmetric monoidal \( \infty \)-category, the \( P(\Delta) \)-module structure on \( P(\Delta^{\mathcal{U}}_F) \) induces a \( P_{\text{Seg}}(\Delta) \)-module structure on \( P_{\text{Seg}}(\Delta^{\mathcal{U}}_F) \) where the tensoring

\[
\otimes : P_{\text{Seg}}(\Delta^{\mathcal{U}}_F) \times P_{\text{Seg}}(\Delta) \to P_{\text{Seg}}(\Delta^{\mathcal{U}}_F)
\]

is given by \( 0 \otimes X = L(0 \times p^* X) \), where \( L \) denotes the localization \( P(\Delta^{\mathcal{U}}_F) \to P_{\text{Seg}}(\Delta^{\mathcal{U}}_F) \). In particular, the tensoring preserves colimits in each variable.

By [38, Proposition 2.2.1.9] to prove Theorem 2.8.2 it suffices to show that the module structure on \( P(\Delta^{\mathcal{U}}_F) \) is compatible with the Segal equivalences in the following sense:

**Proposition 2.8.3.** Suppose \( f : F \to G \) is a Segal equivalence in \( P(\Delta^{\mathcal{U}}_F) \) and \( g : K \to L \) is a Segal equivalence in \( P(\Delta) \). Then \( f \times p^*(g) : F \times p^*(K) \to G \times p^*(L) \) is a Segal equivalence in \( P(\Delta^{\mathcal{U}}_F) \).

As already mentioned in Definition 2.8.1 the tensor functor \( - \times p^*(-)(-) \) preserves colimits in each variable. This allows us to verify the claim of the proposition by reducing it to a few key special cases of Segal equivalences.

**Proposition 2.8.4.** Given an object \( \mathcal{T} \in \Delta^{\mathcal{U}}_F \) lying over \( [n] \in \Delta \) and a Segal equivalence \( U \to \Delta^n \), we write \( f : \mathcal{T}\{U \to \mathcal{T} \) for the map \( \mathcal{T} \times_{p^*(\Delta^n)} p^*(U) \to \mathcal{T} \). For every Segal equivalence \( g : K \to L \) in \( P(\Delta) \), the map

\[
f \times p^*(g) : \mathcal{T}\{U \to \mathcal{T} \times p^*(K) \to \mathcal{T} \times p^*(L)
\]

is a Segal equivalence.

**Proof.** We first observe that we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{T}\{U \to \mathcal{T} \times p^*(K) & \xrightarrow{f \times p^*(g)} & \mathcal{T} \times p^*(L) \\
\downarrow & & \downarrow \\
p^*(U) \times p^*(K) & \longrightarrow & p^*(\Delta^n) \times p^*(L) \longrightarrow p^*(\Delta^n)
\end{array}
\]
consisting of two pullback squares where the middle vertical map is simple in the sense of Definition 2.7.11 by Remark 2.7.12. We first assume that the maps \( U \to \Delta^n \) and \( K \to L \) are inner horn inclusions. Then [37, Corollary 2.3.2.4] says that the product \( U \times K \to \Delta^n \times L \) of inner horn inclusions is inner anodyne and Corollary 2.7.15 shows that the left upper horizontal map \( f \times p^*(g) \) is inner anodyne and hence a Segal equivalence by Proposition 2.7.5. If \( U \to \Delta^n \) is an arbitrary Segal equivalence in \( P(\Delta) \), then according to Proposition 2.7.7 it is strongly generated by inner horn inclusions. Since the class of morphisms \( f \) such that \( f \times p^*(\text{id}_K) \) is a Segal equivalence is clearly strongly saturated, we see that \( f \times p^*(\text{id}_K) \) is a Segal equivalence by the assumption that \( f \) lies over a Segal equivalence \( U \to \Delta^n \). By fixing the object \( \overline{T} \in \Delta^U_{\mathcal{P}} \), the same argument shows that \( \text{id}_\overline{T} \times p^*(g) \) is a Segal equivalence for every Segal equivalence \( g \). Hence, the map \( f \times p^*(g) \) which is the composite of \( f \times p^*(\text{id}_K) \) and \( \text{id}_\overline{T} \times p^*(g) \) is a Segal equivalence. 

Before we complete the proof of Proposition 2.8.3 we need to understand more explicitly what the result of tensoring a corolla with \( \Delta^1 \) looks like, for which it is convenient to introduce some notation:

Definition 2.8.5. For an object \( \overline{T} \in \Delta^U_{\mathcal{P}} \) lying over \([1] \in \Delta\), let \( \overline{T}^+, \overline{T}^- \to \overline{T} \) denote the Cartesian morphisms lying over \( s_0, s_1 : [2] \to [1] \), respectively. If \( \overline{T} = \epsilon_n(v) \) then we also write \( \overline{T}^\pm \) as \( \epsilon_n^\pm(v) \) — these objects are clearly functorial in \( v \in \mathcal{U} \). The objects \( \epsilon_n^\pm(v) \) thus lie over \(([2], n^\pm) : n \to 1\) and \(([2], n^\pm) : n \to 1\), respectively, with the unary vertices labelled by \( 1 \) and the \( n \)-ary vertex by \( v \) in both cases.

Lemma 2.8.6. For an object \( \overline{T} \in \Delta^U_{\mathcal{P}} \) lying over \([1] \in \Delta\), there is an equivalence

\[
\overline{T}^+ \amalg_{\overline{U}^+} \overline{T}^- \to \overline{T} \times p^*(\Delta^1).
\]

In particular, for \( \overline{T} := \epsilon_n(v) \), we have an equivalence \( \epsilon_n^+ (v) \amalg \epsilon_n^-(v) \to \epsilon_n(v) \times p^*(\Delta^1) \), natural in \( v \).

Proof. Let \( \sigma^\pm : \Delta^2 \to \Delta^1 \times \Delta^1 \) denote the two non-degenerate 2-simplices of \( \Delta^1 \times \Delta^1 \), taking \((0, 1, 2)\) to \((0, 0), (0, 1), (1, 1)\) and \((0, 0), (1, 0), (1, 1)\), respectively, and let \( \delta : \Delta^1 \to \Delta^1 \times \Delta^1 \) denote the diagonal map. Then the map \( \Delta^2 \amalg \Delta^2 \to \Delta^1 \times \Delta^1 \) induced by \( \sigma^\pm \) and \( \delta \) is an equivalence. Since pullbacks in \( P(\Delta^U_{\mathcal{P}}) \) preserve colimits, we have a natural pullback square

\[
\begin{array}{ccc}
(\overline{T} \times p^*(\Delta^1))_{\sigma^+} & \amalg & (\overline{T} \times p^*(\Delta^1))_{\sigma^-} \\
\downarrow & & \downarrow \\
p^*(\Delta^2) & \amalg & p^*(\Delta^2) \\
\downarrow & \sim & \downarrow \\
p^*(\Delta^1) \times p^*(\Delta^1) & \sim & p^*(\Delta^1) \times p^*(\Delta^1).
\end{array}
\]

The right vertical map is clearly the pullback of \( \overline{T} \to p^*(\Delta^1) \) along the projection \( p^*(\Delta^1) \times p^*(\Delta^1) \to p^*(\Delta^1) \) and therefore simple by Remark 2.7.12. This implies that \( (\overline{T} \times p^*(\Delta^1))_{\sigma^+} \simeq \overline{T}^\pm \) and \( (\overline{T} \times p^*(\Delta^1))_{\sigma^-} \simeq \overline{T} \), which completes the proof. 

Proof of Proposition 2.8.3. It remains to prove that the map \( F \times p^*(K) \to G \times p^*(K) \) is a Segal equivalence in \( P(\Delta^U_{\mathcal{P}}) \) for every \( K \in P(\Delta) \) and every Segal equivalence \( f : F \to G \). Since Segal equivalences are closed under colimits and \( \times p^*(\cdot) \) preserves colimits in each variable, we may assume that \( K \) is a representable presheaf \( \Delta^n \). Then, by Proposition 2.8.4, the two vertical morphisms in the commutative diagram

\[
\begin{array}{ccc}
F \times p^*\Delta_{\text{Seg}}^n & \to & G \times p^*\Delta_{\text{Seg}}^n \\
\downarrow & & \downarrow \\
F \times p^*(\Delta^n) & \to & G \times p^*(\Delta^n)
\end{array}
\]
are Segal equivalences. It therefore suffices to prove that the top horizontal morphism is a Segal equivalence, and using the definition of $\Delta^n_{\text{Seg}}$ as a colimit we see that for this it is enough to show that $f \times p^* (\Delta^1)$ is a Segal equivalence.

It suffices to check this for $f$ in a class of generating Segal equivalences; by Proposition 2.3.12 we can consider the morphisms

- $\bar{T}|_{\Delta^n_{\text{Seg}}} := \bar{T}|_{01} \sqcup \bar{T}|_{12} \sqcup \bar{T}|_{p2} \cdots \bar{T}|_{(n-1)n} \to \bar{T}$ for all $\bar{T} \in \Delta^n_{\text{Seg}}$,
- $\coprod_{i \in m} ([1], n \to v_i) \to ([1], n \to m, v_1, \ldots, v_m)$, for all $n \to m$ (including $m = 0$),
- $\coprod_{i \in m} ([0], 1) \to ([0], m)$, for all $m$ (including $m = 0$).

For the first class of maps the result was proved in Proposition 2.8.4. For $f : \coprod \bar{T}_i \to \bar{T}$ in the second class of maps, by Lemma 2.8.6 we can identify $f \times p^* (\Delta^1)$ with

$$\coprod_i (\bar{T}_i \sqcup \bar{T}_i) \to \bar{T} \sqcup \bar{T}.$$

Since colimits commute, it suffices to show that the maps $\coprod_i \bar{T}_i \to \bar{T}$ are Segal equivalences. We consider the case of $\bar{T}$; the proof of the other case is the same. We have a commutative square

$$\begin{array}{ccc}
\coprod_i \bar{T}_i|_{\Delta^n_{\text{Seg}}} & \to & \bar{T}|_{\Delta^n_{\text{Seg}}} \\
\downarrow & & \downarrow \\
\coprod_i \bar{T}_i & \to & \bar{T},
\end{array}$$

where the vertical maps are Segal equivalences. By the 2-of-3 property, we only need to see that the upper horizontal map given by $\coprod_i ((\bar{T}_i)|_{01} \sqcup (\bar{T}_i)|_{12}) \to \bar{T}|_{01} \sqcup \bar{T}|_{12}$ is a Segal equivalence. It follows from Definition 2.8.5 that $(\bar{T}_i)|_{k,k+1} \simeq (\bar{T}_i)|_{k+1}$ for $0 \leq k \leq 1$ and that there exists $m_1$ such that $m_1 \sqcup \ldots \sqcup m_1 \simeq m$ and $(\bar{T}_i)|_1 \simeq \coprod_{m_1} \bar{\tau}$. Therefore, the upper horizontal map of the square is a pushout of maps $\coprod m \bar{T}_i \to \bar{T}_i \simeq ([0], n)$ and $\coprod (\bar{T}_i)|_{k,k+1} \to \bar{T}_i|_{k,k+1}$ which are Segal equivalences.

It remains to consider the third class of maps. The canonical equivalence $\sigma : \Delta^1 \hookrightarrow \Delta^0 \times \Delta^1$ induces a pullback square

$$\begin{array}{ccc}
([0], n)_\sigma & \to & ([0], n) \times p^* (\Delta^1) \\
\downarrow & & \downarrow \\
p^* (\Delta^1) & \simeq & p^* (\Delta^0) \times p^* (\Delta^1)
\end{array}$$

where the horizontal maps are equivalences. By Remark 2.7.12, the right vertical map, which is a pullback of $([0], n) \to p^* (\Delta^0)$ along the projection $p^* (\Delta^0) \times p^* (\Delta^1) \to p^* (\Delta^0)$, is simple. Therefore, the presheaf $([0], n)_\sigma$ is represented by $\tau^* ([0], n)$ given by the Cartesian lift $\tau^* ([0], n) \to ([0], n)$ of the projection $\tau : \Delta^1 \to \Delta^0$. Hence, the the presheaf $([0], n) \times p^* (\Delta^1)$ is represented by $I(1_c)_{c \in V_{\Delta^0} (I)}$, where $I = ([1], n \xrightarrow{\text{id}} n)$ and $\text{id}$ denotes the unit in $\mathcal{U}$. In particular, we have $\bar{\tau} \times p^* (\Delta^1) \simeq c_1 (1)$. Therefore, it is clear that the map

$$\coprod_{c \in \mathcal{U}} \bar{\tau} \times p^* (\Delta^1) \simeq \coprod_{c \in \mathcal{U}} c_1 (1) \to I(1_c)_{c \in V_{\Delta^0} (I)} \simeq ([0], n) \times p^* (\Delta^1)$$

is a Segal equivalence. \hfill \Box

This completes the proof of Theorem 2.8.2. As a consequence, we get:

**Corollary 2.8.7.** Let $\mathcal{U}$ be a small symmetric monoidal $\infty$-category and $S$ a set of morphisms in $P(\mathcal{U})$ compatible with the symmetric monoidal structure. Then the $P_{\text{Seg}} (\Delta^3_\text{Seg})$-module structure on $P_{\text{Seg}} (\Delta^3_\text{Seg})$ induces a $P_{\text{Seg}} (\Delta^3_\text{Seg})$-module structure on $P_{\text{Seg}} (\Delta^3_\text{Seg})$. 

Proof. We must show that for \( s : X \to Y \) in \( \mathcal{S} \), the map
\[
y^*c_n(s) \times p^*K : y^*c_n(X) \times p^*K \to y^*c_n(Y) \times p^*K
\]
is an \( \mathcal{S} \)-Segal equivalence for all \( K \) in \( P(\Delta) \). As in the proof of Proposition 2.8.3, we can use Proposition 2.8.4 to reduce to the case where \( K = \Delta^1 \).

Using Lemma 2.8.6 we can (since \( y^*c_n(X) \) is a colimit of the objects considered there) write the map \( y^*c_n(s) \times p^*\Delta^1 \) as
\[
y^*c_n(X) \amalg y^*c_n(Y) \to y^*c_n(Y).
\]
It then suffices to show that the morphisms \( y^*c_n(X) \to y^*c_n(Y) \) are both \( \mathcal{S} \)-Segal equivalences. We consider the case of \( y^*c_n^+ \); the proof for \( c_n^- \) is the same. We then have a commutative diagram
\[
\begin{array}{ccc}
(\amalg_n \mathcal{S}_n(1)) \amalg \amalg_n \mathcal{S}_n \mathcal{S}^n(X) & \longrightarrow & (\amalg_n \mathcal{S}_n(1)) \amalg \amalg_n \mathcal{S}_n \mathcal{S}^n(\mathcal{S}^n(Y)) \\
\downarrow & & \downarrow \\
y^*c_n^+(X) & \longrightarrow & y^*c_n^+(Y).
\end{array}
\]
Here the top horizontal map is an \( \mathcal{S} \)-Segal equivalence, as it is a pushout of \( y^*c_n(X) \to y^*c_n(Y) \). Since the Yoneda embedding \( y : \mathcal{U} \to P(\mathcal{U}) \) is symmetric monoidal, the vertical maps are of the form \( y^*c_n^+(Z) \to y^*c_n^+(Z) \). In particular, they are Segal equivalences and by the 2-of-3 property, the bottom horizontal map is also an \( \mathcal{S} \)-Segal equivalence. \( \square \)

As a special case, we get:

**Corollary 2.8.8.** Let \( \mathcal{V} \) be a presentably symmetric monoidal \( \infty \)-category. Then \( \mathcal{P} \mathcal{C} t \mathcal{S} e g(\Delta^V_\mathcal{P}) \) has a \( \mathcal{P} \mathcal{S} e g(\Delta) \)-module structure such that the tensoring preserves colimits in each variable.

**Proof.** Since \( \mathcal{V} \) is presentably symmetric monoidal, by Proposition 2.6.9 we can choose a regular cardinal \( \kappa \) such that \( \mathcal{V}^\kappa \) is a symmetric monoidal subcategory of \( \mathcal{V} \), and \( \mathcal{V} \simeq \text{Ind}_\kappa \mathcal{V}^\kappa \) as a symmetric monoidal \( \infty \)-category. The result is then a special case of Corollary 2.8.7 applied to \( \mathcal{V}^\kappa \) with \( \mathcal{S} \) as in the proof of Corollary 2.6.6. \( \square \)

Applying the adjoint functor theorem, we get:

**Corollary 2.8.9.** Let \( \mathcal{V} \) be a presentably symmetric monoidal \( \infty \)-category. The tensor product \( \otimes : \mathcal{P} \mathcal{C} t \mathcal{S} e g(\Delta^V_\mathcal{P}) \times \mathcal{P} \mathcal{S} e g(\Delta) \to \mathcal{P} \mathcal{C} t \mathcal{S} e g(\Delta^V_\mathcal{P}) \) induces
\[
\text{Alg}_{\mathcal{S}eg}(\mathcal{V})(-): (\mathcal{P} \mathcal{C} t \mathcal{S} e g(\Delta^V_\mathcal{P}))^{\text{op}} \times \mathcal{P} \mathcal{C} t \mathcal{S} e g(\Delta^V_\mathcal{P}) \to \mathcal{P} \mathcal{S} e g(\Delta)
\]
such that
\[
\text{Map}_{\mathcal{P} \mathcal{S} e g(\Delta)}(\mathcal{X}, \text{Alg}_{\mathcal{S}eg}(\mathcal{V})(\mathcal{P})) \simeq \text{Map}_{\mathcal{P} \mathcal{C} t \mathcal{S} e g(\Delta^V_\mathcal{P})}(\mathcal{O} \otimes \mathcal{X}, \mathcal{P})
\]
and a cotensor product
\[
(-)^{\mathcal{P}} : \mathcal{P} \mathcal{C} t \mathcal{S} e g(\Delta^V_\mathcal{P}) \times (\mathcal{P} \mathcal{S} e g(\Delta))^{\text{op}} \to \mathcal{P} \mathcal{C} t \mathcal{S} e g(\Delta^V_\mathcal{P})
\]
such that
\[
\text{Map}_{\mathcal{P} \mathcal{C} t \mathcal{S} e g(\Delta^V_\mathcal{P})}(\mathcal{O}, \mathcal{P}^\mathcal{V}) \simeq \text{Map}_{\mathcal{P} \mathcal{C} t \mathcal{S} e g(\Delta^V_\mathcal{P})}(\mathcal{O} \times \mathcal{X}, \mathcal{P}).
\]
Moreover, both of these functors preserve limits in each variable.

**Remark 2.8.10.** Since \( \mathcal{P} \mathcal{C} t \mathcal{S} e g(\Delta^V_\mathcal{P}) \) is a module over \( \mathcal{P} \mathcal{S} e g(\Delta) \), for \( \mathcal{O} \in \mathcal{P} \mathcal{C} t \mathcal{S} e g(\Delta^V_\mathcal{P}) \) and \( \mathcal{X}, \mathcal{Y} \in \mathcal{P} \mathcal{S} e g(\Delta) \) we have a natural equivalence
\[
(\mathcal{O} \otimes \mathcal{X}) \otimes \mathcal{Y} \simeq \mathcal{O} \otimes (\mathcal{X} \times \mathcal{Y}).
\]
This induces a natural equivalence
\[
\mathcal{O}^{\mathcal{X} \times \mathcal{Y}} \simeq (\mathcal{O}^\mathcal{X})^\mathcal{Y}.
\]
2.9. The Underlying Enriched $\infty$-Category. In this subsection we define the underlying enriched $\infty$-category of an enriched $\infty$-operad, by extracting from a (continuous) Segal presheaf on $\Delta^V_\mathbb{R}$ a (continuous) Segal presheaf on $\Delta^V$. Here we assume without further comment that the analogues of the results of §2.5–2.8 also hold for Segal presheaves on $\Delta^V$, by simpler versions of the arguments given above.

Definition 2.9.1. Let $u: \Delta \to \Delta^\mathbb{R}$ denote the functor taking $[n]$ to $(\{n\}, 1 \to 1 \to \cdots \to 1)$. We write $\mathfrak{U}: \Delta^V \hookrightarrow \Delta^\mathbb{R}$ for the map given by the pullback

$$
\begin{array}{ccc}
\Delta^V & \rightarrow & \Delta^\mathbb{R} \\
\downarrow & & \downarrow \\
\Delta & \hookrightarrow & \Delta^\mathbb{R}.
\end{array}
$$

Proposition 2.9.2. Let $\mathfrak{U}$ be a small symmetric monoidal $\infty$-category. The functors $\mathfrak{U}$ and $\mathfrak{U}^*$ satisfy:

(i) $\mathfrak{U}: P(\Delta^U) \to P(\Delta^\mathbb{R}_\mathbb{R})$ preserves Segal presheaves.

(ii) $\mathfrak{U}: P(\Delta^\mathbb{R}_\mathbb{R}) \to P(\Delta^\mathbb{R}_\mathbb{R})$ preserves Segal equivalences.

(iii) $\mathfrak{U}^*: P(\Delta^\mathbb{R}_\mathbb{R}) \to P(\Delta^\mathbb{R}_\mathbb{R})$ preserves Segal presheaves.

(iv) The adjunction $\mathfrak{U} \dashv \mathfrak{U}^*$ restricts to an adjunction between the $\infty$-categories of Segal presheaves.

Proof. For a Segal presheaf $F: \Delta^\mathbb{R}_\mathbb{R}, op \to \mathcal{S}$, we must show that $\mathfrak{U}F$ is a Segal presheaf on $\Delta^\mathbb{R}_\mathbb{R}$. This is clear, since $\mathfrak{U}F(\overline{1}) \simeq \emptyset$ for any $1$ not in the image of $u$. Moreover, the images of the generating Segal equivalences in $P(\Delta^\mathbb{R}_\mathbb{R})$ under $\mathfrak{U}$ are clearly among the generating Segal equivalences in $P(\Delta^\mathbb{R}_\mathbb{R})$. This proves (ii), and (iii) is an immediate consequence of (ii). Then (iv) follows since both functors preserve the full subcategories of Segal presheaves.

The $\infty$-category $P_{Seg}(\Delta^U)$ is tensored and cotensored over $P_{Seg}(\Delta)$, with $\mathcal{C} \otimes X$ for $\mathcal{C} \in P_{Seg}(\Delta^U)$ and $X \in P_{Seg}(\Delta)$ given by the localization of $\mathcal{C} \times q^*X$ for $q$ the projection $\Delta^U \to \Delta$. This is compatible with the tensoring of $P_{Seg}(\Delta^\mathbb{R}_\mathbb{R})$.

Proposition 2.9.3. For $\mathcal{C} \in P_{Seg}(\Delta^U)$ and $X \in P_{Seg}(\Delta)$ there is a natural equivalence

$$
\mathfrak{U}(\mathcal{C} \otimes X) \simeq (\mathfrak{U}\mathcal{C}) \otimes X.
$$

Proof. For $\mathcal{O}$ a Segal presheaf on $\Delta^\mathbb{R}_\mathbb{R}$ and $X \in P_{Seg}(\Delta)$, the object $\mathcal{O} \otimes X$ is defined as the localization of $\mathcal{O} \times p^*X$, where $p$ is the composite $\Delta^\mathbb{R}_\mathbb{R} \to \Delta^U \hookrightarrow \Delta$ of the obvious projections. We clearly have $\mathfrak{U}^*(\mathcal{O} \otimes p^*X) \simeq \mathfrak{U}^*\mathcal{O} \otimes q^*X$. Moreover, $\mathfrak{U}^*(\mathcal{O} \otimes X)$ is a Segal presheaf since $\mathfrak{U}^*$ preserves these. Hence, the map $\mathfrak{U}^*(\mathcal{O} \otimes p^*X) \to \mathfrak{U}^*(\mathcal{O} \otimes X)$ factors through the localization $(\mathfrak{U}^*\mathcal{O}) \otimes X$ to give a natural map $(\mathfrak{U}^*\mathcal{O}) \otimes X \to \mathfrak{U}^*(\mathcal{O} \otimes X)$. Applying this with $\mathcal{O} = \mathfrak{U}\mathcal{C}$ and combining it with the counit for the adjunction we get a natural map

$$
\mathcal{C} \otimes X \to \mathfrak{U}^*\mathfrak{U}\mathcal{C} \otimes X \to \mathfrak{U}^*(\mathfrak{U}\mathcal{C} \otimes X),
$$

which induces the required natural map

$$
\mathfrak{U}(\mathcal{C} \otimes X) \to \mathfrak{U}\mathcal{C} \otimes X
$$

by adjunction. Since $\mathfrak{U}$ preserves Segal equivalences and colimits, as do the two tensor products in each variable, it suffices to check that the map

$$
\mathfrak{U}([1](v) \otimes \Delta^1) \to \mathfrak{U}([1](v) \otimes \Delta^1) \simeq c_1(v) \otimes \Delta^1
$$

is an equivalence, which is clear from Lemma 2.8.6 and its analogue for $\Delta^U$.

Corollary 2.9.4. There is a natural equivalence $\mathfrak{U}^*(\mathcal{O}^X) \simeq (\mathfrak{U}^*\mathcal{O})^X$.

Proof. We have natural equivalences

$$
\text{Map}(\mathcal{C}, \mathfrak{U}^*\mathcal{O}^X) \simeq \text{Map}(\mathfrak{U}\mathcal{C} \otimes X, \mathcal{O}) \simeq \text{Map}(\mathfrak{U}\mathcal{C} \otimes X, \mathcal{O}) \simeq \text{Map}(\mathfrak{U}\mathcal{C}, (\mathfrak{U}^*\mathcal{O})^X).
$$
The preceding results are compatible with localization, in the following sense:

**Corollary 2.9.5.** Suppose $\mathcal{U}$ is a small symmetric monoidal $\infty$-category and $\mathcal{S}$ is a set of morphisms in $P(\mathcal{U})$ compatible with the symmetric monoidal structure. Then:

(i) The adjunction $\overline{\pi} \dashv \pi^*$ restricts to an adjunction

$$
\overline{\pi} : P_{\mathcal{S}}(\Delta^\mathcal{U}) \rightleftarrows P_{\mathcal{S}}(\Delta^\mathcal{V}) : \pi^*.
$$

(ii) The functor $\overline{\pi}$ is compatible with the tensoring with $P_{\mathcal{S}}(\Delta)$ in the sense that there is a natural equivalence $\overline{\pi}(\xi \otimes X) \simeq \overline{\pi}(\xi) \otimes X$ for $\xi \in P_{\mathcal{S}}(\Delta^\mathcal{U})$ and $X \in P_{\mathcal{S}}(\Delta)$.

(iii) There is a natural equivalence $\overline{\pi}^* (\Omega^n) \simeq (\pi^* \Omega)^n$ for $\Omega \in P_{\mathcal{S}}(\Delta^\mathcal{V})$ and $X \in P_{\mathcal{S}}(\Delta)$.

As usual, this extends to the presentable case by choosing a suitable subcategory of $\kappa$-compact objects, giving:

**Corollary 2.9.6.** Suppose $\mathcal{V}$ is a presentably symmetric monoidal $\infty$-category. Then

(i) The adjunction $\overline{\pi} \dashv \pi^*$ restricts to an adjunction

$$
\overline{\pi} : P_{\text{CtsSeg}}(\Delta^\mathcal{V}) \rightleftarrows P_{\text{CtsSeg}}(\Delta^\mathcal{V}) : \pi^*.
$$

(ii) The functor $\overline{\pi}$ is compatible with the tensoring with $P_{\text{CtsSeg}}(\Delta)$ in the sense that there is a natural equivalence $\overline{\pi}(\xi \otimes X) \simeq \overline{\pi}(\xi) \otimes X$ for $\xi \in P_{\text{CtsSeg}}(\Delta^\mathcal{V})$ and $X \in P_{\text{CtsSeg}}(\Delta)$.

(iii) There is a natural equivalence $\overline{\pi}^* (\Omega^n) \simeq (\pi^* \Omega)^n$ for $\Omega \in P_{\text{CtsSeg}}(\Delta^\mathcal{V})$ and $X \in P_{\text{CtsSeg}}(\Delta)$.

**Warning 2.9.7.** Note that here the functors $\overline{\pi}$ and $\pi^*$ for continuous Segal presheaves for $\mathcal{V}$ refer to those obtained from the analogous functors for Segal presheaves on $\mathcal{V}^\kappa$ for a suitable cardinal $\kappa$. In particular, $\overline{\pi}$ is not just given by a left Kan extension along $\Delta^\mathcal{V}_{\text{op}} \to \Delta^\mathcal{V}_{\text{op}}$.

**Remark 2.9.8.** Since the $\infty$-category $P_{\text{CtsSeg}}(\Delta^\mathcal{V})$ models $\mathcal{V}$-enriched $\infty$-categories by [17, Theorem 4.5.3], we call $\pi^* (\Omega)$ the underlying $\mathcal{V}$-enriched $\infty$-category of $\Omega$ for every continuous Segal presheaf $\mathcal{O} \in P_{\text{CtsSeg}}(\Delta^\mathcal{V})$.

3. The Completion Theorem

So far we have considered the “algebraic” theory of enriched $\infty$-operads. However, just as in the case of Segal spaces, to produce the correct homotopy theory of enriched $\infty$-operads we need to invert the class of fully faithful and essentially surjective morphisms. In [45] Rezk showed that the localization of Segal spaces at the fully faithful and essentially surjective morphisms is given by the full subcategory of complete objects, which are those Segal spaces whose space of objects is equivalent to their “classifying space of equivalences”. Our main goal in this section is to prove the analogous result for enriched $\infty$-operads, in the form of continuous Segal presheaves on $\Delta^\mathcal{V}$.

In §3.1 we review the definition of complete objects in Segal spaces and in enriched $\infty$-categories, and in §3.2 we extend this to define complete continuous Segal presheaves on $\Delta^\mathcal{V}$. Then we introduce fully faithful and essentially surjective functors in §3.3, and a notion of “pseudo-equivalences” in §3.4; these are morphisms with a “pseudo-inverse”, i.e. an inverse up to natural equivalence. Lastly, in §3.5 we prove our analogue of Rezk’s completion theorem.

3.1. Completeness for Segal Spaces and Enriched $\infty$-Categories. To fix notation, we recall the definition of complete Segal spaces from [45] and the analogous definition of complete enriched $\infty$-categories from [17].

**Definition 3.1.1.** Let $E^n$ denote the (contractible) category with $n + 1$ objects $0, 1, \ldots, n$ and a unique morphism between any pair of objects. We also denote the Segal space corresponding to this category by $E^n$.

**Remark 3.1.2.** The category $E^1$ is the “generic isomorphism”, so giving a morphism of Segal spaces $E^1 \to X$ corresponds to giving two objects of $X$ and an equivalence between them. Similarly, giving a map $E^n \to X$ amounts to specifying $n + 1$ equivalent objects in $X$. 

Definition 3.1.3. A Segal space $F$ is complete if the map

$$F_0 \to \text{Map}(E^1, F)$$

induced by the map $E^1 \to E^0 \simeq \ast$, is an equivalence in $\mathcal{S}$. We write $\text{P}_{\text{Seg}}(\Delta)$ for the full subcategory of $\text{P}_{\text{Seg}}(\Delta)$ spanned by the complete Segal spaces.

Definition 3.1.4. We can view the category $E^n$ as enriched in the initial monoidal $\infty$-category $\ast$ and so by transferring the enrichment we get for any symmetric monoidal $\infty$-category $\mathcal{V}$ an enriched $\infty$-category $E^n_{\mathcal{V}}$ with morphisms $E^n_{\mathcal{V}}(i, j) \simeq \mathcal{V}$ for all $i, j$. We will usually denote this simply as $E^n$, leaving $\mathcal{V}$ implicit. A $\mathcal{V}$-enriched $\infty$-category is complete if it is local with respect to $E^1 \to E^0$. Viewing enriched $\infty$-categories as continuous Segal presheaves on $\Delta^\vee$ (for $\mathcal{V}$ presentably symmetric monoidal) we write $\text{P}_{\text{CCS}}(\Delta^\vee)$ for the full subcategory of $\text{P}(\Delta^\vee)$ spanned by the complete continuous Segal presheaves.

Definition 3.1.5. For every $\mathcal{C} \in \text{P}_{\text{CtsSeg}}(\Delta^\vee)$, let $t_n(\mathcal{C})$ denote the space $\text{Map}_{\text{P}_{\text{CtsSeg}}(\Delta^\vee)}(E^n, \mathcal{C})$. We call a map $E^1 \to \mathcal{C}$ in $\text{P}_{\text{CtsSeg}}(\Delta^\vee)$ an equivalence in $\mathcal{C}$ and the space $t_1(\mathcal{C})$ the space of equivalences of $\mathcal{C}$. We write $\mathcal{C}^\ast$ for the colimit of the simplicial object $\iota_\ast(\mathcal{C}) = \text{Map}_{\text{P}_{\text{CtsSeg}}(\Delta^\vee)}(E^\ast, \mathcal{C})$ and call this the classifying space of equivalences in $\mathcal{C}$.

Remark 3.1.6. We refer the reader to [17, §§5.1–5.2] for a much more substantial treatment of equivalences in enriched $\infty$-categories.

3.2. Complete Segal Presheaves.

Definition 3.2.1. If $\mathcal{O}$ is a Segal presheaf on $\Delta^\mathcal{F}$, then the underlying $\infty$-category $u^\ast \mathcal{O}$ (see §2.9) is a Segal space. We say $\mathcal{O}$ is complete if its underlying Segal space $u^\ast \mathcal{O}$ is a complete Segal space. Similarly, $\mathcal{O} \in \text{P}_{\text{CtsSeg}}(\Delta^\mathcal{F})$ is complete if $\mathcal{O}^\mathcal{F}$ is a complete object in $\text{P}_{\text{CtsSeg}}(\Delta^\mathcal{V})$; we write $\text{P}_{\text{CCS}}(\Delta^\mathcal{F})$ for the full subcategory of $\text{P}_{\text{CtsSeg}}(\Delta^\mathcal{F})$ spanned by the complete objects.

Notation 3.2.2. We will also write $\text{Opd}^\mathcal{V}_{\text{CCS}}$ for the $\infty$-category $\text{P}_{\text{CCS}}(\Delta^\mathcal{V})$ when we do not wish to emphasize the specific implementation of enriched $\infty$-operads as continuous Segal presheaves.

Notation 3.2.3. For $\mathcal{O} \in \text{P}_{\text{CtsSeg}}(\Delta^\mathcal{F})$ we abbreviate $t_n(\mathcal{O}) := t_n(\mathcal{O})$. Then $\mathcal{O}$ is complete if and only if the canonical map $t_1(\mathcal{O}) \to t_0(\mathcal{O})$ is an equivalence in $\mathcal{S}$.

Lemma 3.2.4. For every continuous Segal presheaf $\mathcal{O} \in \text{P}_{\text{CtsSeg}}(\Delta^\mathcal{V})$, the objects $t_0(\mathcal{O})$ and $\mathcal{O}(\overline{\mathcal{T}})$ are equivalent in $\mathcal{S}$.

Proof. For every object $\mathcal{O} \in \text{P}_{\text{CtsSeg}}(\Delta^\mathcal{V})$, we have the following equivalences:

$$t_0(\mathcal{O}) \simeq \text{Map}(E^0, \mathcal{O}) \simeq \text{Map}(\mathcal{O}E^0, \mathcal{O}) \simeq \mathcal{O}(\mathcal{T}) \simeq \mathcal{O}(\overline{\mathcal{T}}).$$

Remark 3.2.5. This implies that the functor $t_0$ preserves colimits.

Remark 3.2.6. For a presentably symmetric monoidal $\infty$-category $\mathcal{V}^\otimes$, there exists a unique colimit-preserving functor $F_0: \mathcal{S} \to \mathcal{V}$ extending $\ast \mapsto 1$. This is symmetric monoidal, and it has a lax monoidal right adjoint $G: \mathcal{V} \to \mathcal{S}$ given by $\text{Map}_\mathcal{V}(1, -)$. By [17, Theorem 4.4.7, Theorem 4.5.3], the right adjoint induces a map $G_\ast: \text{P}_{\text{CtsSeg}}(\Delta^\mathcal{V}) \to \text{P}_{\text{Seg}}(\Delta)$ which carries a Segal presheaf to its underlying Segal space. Moreover, [17, Proposition 5.1.11] implies that this functor detects complete objects, meaning $\mathcal{C} \in \text{P}_{\text{CtsSeg}}(\Delta^\mathcal{V})$ is complete if and only if $G_\ast(\mathcal{C})$ is a complete Segal space.

Definition 3.2.7. A morphism in $\text{P}_{\text{CCS}}(\Delta^\mathcal{V})$ is called a local equivalence if it lies in the strongly saturated class of maps generated by $u_s(s^0)$, where $s^0$ denotes the canonical map $E^1 \to E^0$ in $\text{P}_{\text{CtsSeg}}(\Delta^\mathcal{F})$. Equivalently, a morphism $f: \mathcal{O} \to \mathcal{O}'$ is a local equivalence if and only if for every complete object $\mathcal{P}$ the induced map $\text{Map}(\mathcal{O}', \mathcal{P}) \to \text{Map}(\mathcal{O}, \mathcal{P})$ is an equivalence.
3.3. Fully Faithful and Essentially Surjective Functors.

Definition 3.3.1. Given two objects \( \mathcal{O}, \mathcal{P} \in P_{\text{CtsSeg}}(\Delta^V) \), we say that a morphism \( f: \mathcal{O} \to \mathcal{P} \) is

- **fully faithful** if the commutative square
  \[
  \begin{array}{ccc}
  \mathcal{O}(\tau_n) & \xrightarrow{f(\tau_n)} & \mathcal{P}(\tau_n) \\
  \downarrow & & \downarrow \\
  \mathcal{O}(\tau)^{n+1} & \xrightarrow{f(\tau)^{n+1}} & \mathcal{P}(\tau)^{n+1}
  \end{array}
  \]

  is Cartesian for every \( \tau_n \in \Delta^V \) lying over a corolla \( \tau_n \) in \( \Delta^V \).

- **essentially surjective** if the induced functor \( \mathfrak{P}^*(f): \mathfrak{P}(\mathcal{O}) \to \mathfrak{P}(\mathcal{P}) \) of the underlying \( \mathcal{V} \)-enriched \( \infty \)-categories is essentially surjective, i.e. the map \( \pi_0(\tau f): \pi_0(\tau) \to \pi_0(\tau') \) is a surjection of sets.

Remark 3.3.2. A morphism \( f: \mathcal{O} \to \mathcal{P} \) of Segal presheaves on \( \Delta^V \) is fully faithful if and only if for all \( \tau_n \in \Delta^V \) and \( x_1, \ldots, x_n, y \in \mathcal{O}(\tau) \) the natural map \( \mathcal{O}(\tau_n(x_1, \ldots, x_n, y)) \to \mathcal{P}(\tau_n(f x_1, \ldots, f x_n, f y)) \) is an equivalence. Since these are both representable presheaves in \( \mathcal{V} \), this is equivalent to the representing morphism \( \mathcal{O}(x_1, \ldots, x_n; y) \to \mathcal{P}(f(x_1, \ldots, f x_n; y)) \) being an equivalence in \( \mathcal{V} \), which is what we would expect the notion of fully faithful to mean.

Remark 3.3.3. By [17, Lemma 5.3.4], a functor of Segal presheaves is essentially surjective in our sense if and only for every object \( x \in \iota_0 \mathcal{P} \) there exists an equivalence (i.e. a functor of \( \mathcal{V} \)-enriched \( \infty \)-categories \( E^1 \to \mathfrak{P}(\mathcal{P}) \)) connecting \( x \) to an object lying in the image of \( \iota_0 f \) in \( \iota_0 \mathcal{P} \).

Lemma 3.3.4. Evaluation at \( \tau \in \Delta^V \) gives a map \( \mathfrak{ev}_\tau: P_{\text{CtsSeg}}(\Delta^V) \to S \), which is a Cartesian fibration by Proposition 2.4.8. A morphism \( f: \mathcal{O} \to \mathcal{P} \) in \( P_{\text{CtsSeg}}(\Delta^V) \) is \( \mathfrak{ev}_\tau \)-Cartesian if and only if \( f \) is fully faithful.

Proof. Suppose \( f \) lies over \( \mathfrak{ev}_\tau(f): X \to Y \in S \). We factor the morphism \( f \) in \( P_{\text{CtsSeg}}(\Delta^V) \) into a morphism \( g: \mathcal{O} \to \mathfrak{ev}_\tau(f)^*(\mathcal{P}) \) lying over \( \iota_0 f \) followed by an \( \mathfrak{ev}_\tau \)-Cartesian morphism \( h: \mathfrak{ev}_\tau(f)^*(\mathcal{P}) \to \mathcal{P} \) lying over \( \mathfrak{ev}_\tau(f) \). If \( f \) is fully faithful, then we wish to show that \( g \) is an equivalence in \( P_{\text{CtsSeg}}(\Delta^V) \), which is equivalent to requiring \( g \) to be an equivalence in the fibre \( \mathcal{P}_{\text{CtsSeg}}(\Delta^V)_X \). Hence, we need to show that \( g(\tau): \mathcal{O}(\tau) \to \mathfrak{ev}_\tau(f)^*(\mathcal{P})(\tau) \) is an equivalence for every object \( \tau \in \Delta^V \). For every object \( \mathcal{P}(\tau_n) \in \Delta^V \) lying over a corolla, the \( \infty \)-groupoid \( \mathfrak{ev}_\tau(f)^*(\mathcal{P})(\tau_n) \) is given by the pullback \( X^{n+1} \times_{Y^{n+1}} \mathcal{P}(\tau_n) \) which is equivalent to \( \mathcal{O}(\tau_n) \) since \( f \) is fully faithful. The Segal condition in the definition of \( \mathcal{P}_{\text{CtsSeg}}(\Delta^V)_X \) then implies that \( \mathcal{O}(\tau) \) is equivalent to \( \mathfrak{ev}_\tau(f)^*(\mathcal{P})(\tau) \) for every \( \tau \in \Delta^V \).

Conversely, if \( f \) is an \( \mathfrak{ev}_\tau \)-Cartesian morphism over \( \mathfrak{ev}_\tau(f): X \to Y \), then \( \mathcal{O}(\tau_n) \) is equivalent to \( X^{n+1} \times_{Y^{n+1}} \mathcal{P}(\tau_n) \), and thus \( f \) is fully faithful.

Proposition 3.3.5. Fully faithful and essentially surjective morphisms in \( P_{\text{CtsSeg}}(\Delta^V) \) satisfy the 2-of-3 property.

Proof. As [17, Proposition 5.3.9].

Proposition 3.3.6. A fully faithful and essentially surjective morphism in \( P_{\text{CtsSeg}}(\Delta^V) \) between complete objects is an equivalence.

Proof. Let \( f: \mathcal{O} \to \mathcal{P} \) be a fully faithful and essentially surjective morphism in \( P_{\text{CtsSeg}}(\Delta^V) \). It follows from Definition 3.3.1 that the map \( \mathfrak{P}^*(f) \) between the underlying \( \mathcal{V} \)-enriched \( \infty \)-categories is fully faithful and essentially surjective, and it is then an equivalence by [17, Corollary 5.3.8]. This implies that the Cartesian fibration \( \mathfrak{ev}_\tau: P_{\text{CtsSeg}}(\Delta^V) \to S \) induced by the evaluation at \( \tau \) carries \( f \) to an equivalence in \( S \). Since the fully faithful map \( f \) is then an \( \mathfrak{ev}_\tau \)-Cartesian lift of the equivalence \( \mathfrak{ev}_\tau(f) \) by Lemma 3.3.4, it has to be an equivalence as well.
3.4. Pseudo-Equivalences. In this subsection we consider morphisms that admit an inverse up to natural equivalence, which we call pseudo-equivalences. We will show that these are both local equivalences and fully faithful and essentially surjective, which will be used to prove the completion theorem in the next subsection.

Definition 3.4.1. Let \( d^0, d^1 : E^0 \to E^1 \) be the maps induced by the two inclusions \( 1 \leftrightarrow 2 \) of the sets of objects. A natural equivalence of morphisms from 0 to \( \mathcal{P} \) is a morphism \( h : \mathcal{O} \otimes E^1 \to \mathcal{P} \) \((\mathcal{O} \times E^1 \to \mathcal{P} \) in case of \( \mathcal{P}_{n}(\Delta) \)). We say that \( f \) and \( g \) are naturally equivalent if there exists a natural equivalence \( h \) such that \( h \circ (id \otimes d^0) \simeq f \) and \( h \circ (id \otimes d^1) \simeq g \).

Definition 3.4.2. For a morphism \( f : \mathcal{O} \to \mathcal{P} \in \mathcal{P}_{n}(\Delta) \), a pseudo-inverse of \( f \) is a morphism \( g : \mathcal{P} \to \mathcal{O} \) such that there exist natural equivalences \( \phi : id_{\mathcal{O}} \to g \circ f \) and \( \psi : f \circ g \to id_{\mathcal{P}} \). We call a morphism \( f \) a pseudo-equivalence if it has a pseudo-inverse \( g \) and we call the quadruple \( (f,g,\phi,\psi) \) a pseudo-equivalence datum.

The following proposition is an operadic variant of [17, Proposition 5.5.3].

Proposition 3.4.3. If \( f : \mathcal{O} \to \mathcal{P} \) is a pseudo-equivalence, then it is fully faithful and essentially surjective.

Proof. Let \((f,g,\phi,\psi)\) be a pseudo-equivalence datum associated to the pseudo-equivalence \( f \). By Remark 3.3.3, to prove the essential surjectivity it suffices to find for every object \( y \in \mathcal{O} \) an object \( x \) and an equivalence \( f(x) \simeq y \). But by defining \( x := \psi(g(y)) \), we get an equivalence \( f \circ g(y) \simeq y \) induced by the natural equivalence \( \psi \).

For the full faithfulness of \( f \), we have to show that for every corolla \( \epsilon_n \in \Delta_n^\mathcal{P} \) and every object \( x = (x_1, \ldots, x_n, x) \in \mathcal{O}(e)^{n+1} \) the map \( g(\tau_n) : \mathcal{O}(\tau_n) \to \mathcal{P}(\tau_n) \) restricts to an equivalence of fibres \( f_x : \mathcal{O}(\tau_n(x)) \to \mathcal{P}(\tau_n(f(x))) \). Using the natural equivalence \( \phi : id_{\mathcal{O}} \to g \circ f \), the map \( g(\tau_n) \) restricts to a map \( g_\epsilon(\tau_n,f) \) of fibres \( \mathcal{P}(\tau_n(f(x))) \to \mathcal{O}(\tau_n(g f(x))) \) and we have \( id \simeq g_\epsilon \circ f \). It remains to see that \( f_\epsilon \circ g_\epsilon f \simeq id \). The natural equivalence \( \psi : f \circ g \to id_{\mathcal{P}} \) induces an equivalence \( f_\epsilon \circ g_\epsilon f \simeq id \) and \( \psi : id_{\mathcal{O}} \to g \circ f \) gives an equivalence \( f_\epsilon \circ g_\epsilon f \simeq f \). Hence, we have \( f_\epsilon \circ g_\epsilon f \simeq f_\epsilon \circ g_\epsilon f \simeq id \).

Corollary 3.4.4. Pseudo-equivalences between complete objects are equivalences in \( \mathcal{P}_{n}(\Delta) \).

Proof. By the previous proposition, pseudo-equivalences in \( \mathcal{P}_{n}(\Delta) \) are fully faithful and essentially surjective, and Proposition 3.3.6 implies that such maps between complete objects are in \( \mathcal{P}_{n}(\Delta) \).

Similarly to [17, Lemma 5.5.7], we have the following result:

Lemma 3.4.5. For every object \( \mathcal{C} \in \mathcal{P}_{n}(\Delta) \) and every pseudo-equivalence \( f : \mathcal{C} \to \mathcal{D} \) in \( \mathcal{P}_{n}(\Delta) \), the cotensor product \( \mathcal{O} : \mathcal{D} \to \mathcal{C} \) is a pseudo-equivalence in \( \mathcal{P}_{n}(\Delta) \).

Proof. If \((f,g,\phi,\psi)\) is a pseudo-equivalence datum associated to \( f \), then the natural equivalences \( \phi : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \) and \( \psi : \mathcal{C} \times \mathcal{D} \to \mathcal{D} \) in \( \mathcal{P}(\Delta) \) induce maps \( \mathcal{O}^{\mathcal{C}} \simeq (\mathcal{O}^{\mathcal{C}} \times \mathcal{D})^{\mathcal{C}} \) and \( \mathcal{O}^{\mathcal{D}} \simeq (\mathcal{O}^{\mathcal{D}} \times \mathcal{C})^{\mathcal{D}} \) in \( \mathcal{P}_{n}(\Delta) \), respectively. If \( \phi^{f} : \mathcal{O}^{\mathcal{C}} \times \mathcal{D} \to \mathcal{C}^{\mathcal{D}} \) and \( \psi^{f} : \mathcal{O}^{\mathcal{D}} \times \mathcal{C} \to \mathcal{D}^{\mathcal{C}} \) denote the corresponding adjoint maps, then \( \phi^{f} \) and \( \psi^{f} \) are natural equivalences and one readily checks that \( \mathcal{O}^{\mathcal{D}} \simeq \mathcal{O}^{f} \).

Lemma 3.4.6. For every complete object \( \mathcal{O} \in \mathcal{P}_{n}(\Delta) \), the map \( \mathcal{O}^{\mathcal{O}} : \mathcal{O} \simeq \mathcal{O}^{\mathcal{O}} \to \mathcal{O}^{\mathcal{E}^{1}} \) induced by the unique map \( s^{0} : \mathcal{E}^{1} \to \mathcal{E}^{0} \) in \( \mathcal{P}_{n}(\Delta) \) is an equivalence.

Proof. By [17, Definition 5.5.6], the map \( s_{0} \) is a pseudo-equivalence in \( \mathcal{P}_{n}(\Delta) \) and Lemma 3.4.5 therefore implies that \( \mathcal{O}^{\mathcal{O}} \) is a pseudo-equivalence in \( \mathcal{P}_{n}(\Delta) \). Since \( \mathcal{O}^{\mathcal{E}^{1}} \simeq \mathcal{O} \) is complete by assumption, it suffices to show that \( \mathcal{O}^{\mathcal{E}^{1}} \) is also complete by Corollary 3.4.4.

The adjunctions \( \frac{\mathcal{O}_{1}}{\mathcal{O}} \to \mathcal{O} \) and \(- \otimes \mathcal{E}^{1} \vdash (-)^{\mathcal{E}^{1}} \) provide a chain of equivalences

\[
\begin{align*}
\phi_{0}(\mathcal{O}^{\mathcal{E}^{1}}) = \text{Map}(\mathcal{E}^{0}, \mathcal{O}_{1}^{\mathcal{E}^{1}}) & \simeq \text{Map}(\mathcal{P}(\mathcal{E}^{0} \otimes \mathcal{E}^{1}), 0) \\ & \simeq \text{Map}(\mathcal{P}(\mathcal{E}^{1}), 0) \simeq \text{Map}(\mathcal{E}^{1}, \mathcal{O}_{1}^{\mathcal{O}}).
\end{align*}
\]
Similarly, we have the equivalence \( \iota_1(O^{E^I}) \simeq \text{Map}(E^1 \otimes E^1, \pi^*(O)) \). It follows that the map \( \iota_0(O^{E^I}) \to \iota_1(O^{E^I}) \) can be identified with \( \text{Map}(d_{E^I} \otimes s^0, \pi^*(O)) : \text{Map}(E^1, \pi^*(O)) \to \text{Map}(E^1 \otimes E^1, \pi^*(O)) \). Since \( O \) is complete and \( d_{E^I} \otimes s^0 \) is a local equivalence by [17, Lemma 5.4.7], the map \( \iota_0(O^{E^I}) \to \iota_1(O^{E^I}) \) is an equivalence, and hence \( O^{E^I} \) is complete. \( \square \)

**Lemma 3.4.7.** For every object \( O \in \text{P}_{\text{CtsSeg}}(\Delta^Y) \), the map \( \text{id}_O \otimes s^0 : O \otimes E^1 \to O \otimes E^0 \simeq O \) induced by \( s^0 : E^1 \to E^0 \) is a local equivalence.

**Proof.** The map \( \text{id}_O \otimes s^0 \) is a local equivalence if and only if the induced map

\[
(id_O \otimes s^0)^* : \text{Map}(O, \mathcal{P}) \to \text{Map}(O \otimes E^1, \mathcal{P})
\]

is an equivalence for every complete object \( \mathcal{P} \). By adjunction, this is equivalent to requiring \( \text{Map}(O, \mathcal{P}) \to \text{Map}(O, \mathcal{P} E^I) \) to be an equivalence, which is true for every complete object \( \mathcal{P} \) by the previous lemma. \( \square \)

**Proposition 3.4.8.** Every pseudo-equivalence is a local equivalence.

**Proof.** Let \( f : O \to \mathcal{P} \) be a pseudo-equivalence in \( \text{P}_{\text{CtsSeg}}(\Delta^Y) \) and let \( (f, g, \phi, \psi) \) be a corresponding pseudo-equivalence datum. We want to show that the map \( f^* : \text{Map}(\mathcal{P}, O) \to \text{Map}(O, O) \) is an equivalence for every complete object \( O \). It follows from Definition 3.4.2 that

\[
f^* g^* \simeq (id \otimes d^0)^* \phi^* \text{ and } id^* \simeq (id \otimes d^1)^* \phi^*.
\]

Similarly, we have

\[
g^* f^* \simeq (id \otimes d^1)^* \psi^* \text{ and } id^* \simeq (id \otimes d^0)^* \psi^*.
\]

Hence, we only need to show that the morphisms \( (id \otimes d^0)^* \) and \( (id \otimes d^1)^* \) are equivalent in \( S \).

For every \( O' \in \text{P}_{\text{CtsSeg}}(\Delta^Y) \), the map

\[
(id \otimes s^0)^* \circ (id \otimes d^1)^* : \text{Map}(O', O) \to \text{Map}(O', O)
\]

is equivalent to the identity. Since \( (id \otimes s^0)^* \) is a local equivalence by Lemma 3.4.7, the map \( (id \otimes s^0)^* \) is an equivalence for every complete object \( O \). Therefore, the maps \( (id \otimes d^0)^* \) and \( (id \otimes d^1)^* \) are equivalent, because both are right inverses of the equivalence \( (id \otimes s^0)^* \). \( \square \)

**Proposition 3.4.9.** The tensor product \( \otimes : \text{P}_{\text{CtsSeg}}(\Delta^Y) \times \text{P}_{\text{Seg}}(\Delta) \to \text{P}_{\text{CtsSeg}}(\Delta^Y) \) of Corollary 2.8.8 induces a tensor product on the complete objects

\[
\otimes : \text{Opd}^Y_{\infty} \times \text{Cat}_{\infty} \to \text{Opd}^Y_{\infty}
\]

(i.e. \( \text{P}_{\text{Ccs}}(\Delta^Y) \times \text{P}_{\text{Cstag}}(\Delta) \to \text{P}_{\text{Ccs}}(\Delta^Y) \)), which preserves colimits in each variable.

**Proof.** It suffices to show that for \( O \in \text{P}_{\text{CtsSeg}}(\Delta^Y) \), the map \( O \otimes E^1 \to O \otimes E^0 \) is a local equivalence, and for \( C \in \text{P}_{\text{Seg}}(\Delta) \) the map \( \pi(E^1) \otimes \mathcal{C} \to \pi(E^0) \otimes \mathcal{C} \) is a local equivalence.

By adjunction, the first claim is equivalent to \( \mathcal{P} \simeq \mathcal{P} E^0 \to \mathcal{P} E^1 \) being an equivalence for every complete object \( \mathcal{P} \), which follows from Lemma 3.4.6.

By Corollary 2.9.6(ii) we have equivalences \( \pi(E^1) \otimes \mathcal{C} \simeq \pi(E^0) \otimes \mathcal{C} \) and \( \pi(E^0) \otimes \mathcal{C} \simeq \pi(E^1) \otimes \mathcal{C} \). The map \( E^1 \otimes \mathcal{C} \to E^0 \otimes \mathcal{C} \) is then a local equivalence in \( \text{P}_{\text{Seg}}(\Delta) \) by [17, Proposition 5.5.9] and the claim follows from the fact that \( \pi \) obviously preserves local equivalences. \( \square \)

Applying the adjoint functor theorem, we get:

**Corollary 3.4.10.** The tensor product \( \otimes : \text{Opd}^Y_{\infty} \times \text{Cat}_{\infty} \to \text{Opd}^Y_{\infty} \) induces

\[
\text{Alg}_{\text{Seg}}(\cdot) : (\text{Opd}^Y_{\infty})^{\text{op}} \times \text{Opd}^Y_{\infty} \to \text{Cat}_{\infty}
\]

such that

\[
\text{Map}_{\text{Cat}_{\infty}}(X, \text{Alg}_{\text{Seg}}(\cdot)) \simeq \text{Map}_{\text{Opd}^Y_{\infty}}(O \otimes X, \mathcal{P})
\]

and a cotensor product

\[
(\cdot)^{-1} : \text{Opd}^Y_{\infty} \times \text{Cat}^{\text{op}}_{\infty} \to \text{Opd}^Y_{\infty}
\]

and a cotensor product

\[
(\cdot)^{-1} : \text{Opd}^Y_{\infty} \times \text{Cat}^{\text{op}}_{\infty} \to \text{Opd}^Y_{\infty}
\]
such that
\[ \text{Map}_{\text{Opo}}(\mathcal{O}, P^X) \simeq \text{Map}_{\text{Opo}}(\mathcal{O} \otimes \mathcal{X}, P). \]
Moreover, both of these functors preserve limits in each variable.

3.5. Completion. Our goal in this subsection is to prove that \( P_{\text{CCS}}(\Delta^\mathbb{Y}_P) \) is the localization of \( P_{\text{CtsSeg}}(\Delta^\mathbb{Y}_P) \) at the fully faithful and essentially surjective morphisms. We do this by the strategy introduced by Rezk [45] and applied to enriched \( \infty \)-categories in [17, 5.5]: We define a functor that takes every continuous Segal presheaf \( \mathcal{O} \) to a complete object \( \hat{\mathcal{O}} \) with a natural map \( \mathcal{O} \to \hat{\mathcal{O}} \), and check that this map is both a local equivalence and fully faithful and essentially surjective. This functor is defined as follows:

**Definition 3.5.1.** Given an object \( \mathcal{O} \in P_{\text{CtsSeg}}(\Delta^\mathbb{Y}_P) \), we write \( \hat{\mathcal{O}} \) for the colimit in \( P_{\text{CtsSeg}}(\Delta^\mathbb{Y}_P) \) of the simplicial object \( \mathcal{O}^{E*} \), and \( l_0 : \mathcal{O} \to \hat{\mathcal{O}} \) for the natural map from \( \mathcal{O} \simeq \mathcal{O}^{E^0} \) to this colimit.

For the rest of this section we choose a regular cardinal \( \kappa \) such that \( P_{\text{CtsSeg}}(\Delta^\mathbb{Y}_P) \simeq P_{\kappa-\text{Seg}}(\Delta^\mathbb{Y}_P) \). The key observation that makes the proof work is that the colimit \( \hat{\mathcal{O}} \) can be computed in presheaves on \( \Delta^\mathbb{Y}_P \):

**Proposition 3.5.2.** For \( \mathcal{O} \in P_{\kappa-\text{Seg}}(\Delta^\mathbb{Y}_P) \), the geometric realization \( \hat{\mathcal{O}} := |\mathcal{O}^{E*}| \), computed in \( P(\Delta^\mathbb{Y}_P) \), is a \( \kappa \)-continuous Segal presheaf.

**Proof.** For every object \( (\tau_n) \in \Delta^\mathbb{Y}_n \) lying over a corolla and every morphism \([m] \to [n]\) in \( \Delta^{op} \), there is a commutative diagram
\[
\begin{array}{ccc}
\mathcal{O}^{E^n}(\tau_n) & \longrightarrow & \mathcal{O}^{E^n}(\tau_n) \\
\downarrow & & \downarrow \\
\mathcal{O}^{E^n}(\tau)_{k+1} & \longrightarrow & \mathcal{O}^{E^n}(\tau)_{k+1}.
\end{array}
\]

Since \( E^n \to E^m \) is a pseudo-equivalence by [17, Corollary 5.5.6], Lemma 3.4.5 and Proposition 3.4.3 imply that the functor \( \mathcal{O}^{E^n} \to \mathcal{O}^{E^m} \) is fully faithful, which by definition means this commutative square is Cartesian. In other words, the natural transformation \( \tau : \mathcal{O}^{E*}(\tau_n) \to \mathcal{O}^{E*}(\tau)_{k+1} \) between the two simplicial diagrams is Cartesian in the sense of [37, Definition 6.1.3.1]. Since \( \mathcal{S} \) is an \( \infty \)-topos, by [37, Theorem 6.1.3.9] the commutative square
\[
\begin{array}{ccc}
\mathcal{O}^{E^n}(\tau_n) & \longrightarrow & |\mathcal{O}^{E*}(\tau_n)| \\
\downarrow & & \downarrow \\
\mathcal{O}^{E^n}(\tau)_{k+1} & \longrightarrow & |\mathcal{O}^{E*}(\tau)_{k+1}|
\end{array}
\]
is also Cartesian. The surjectivity of the bottom horizontal map on connected components and the pullback condition then imply that each fibre of \( |\mathcal{O}^{E*}(\tau_n)| \to |\mathcal{O}^{E*}(\tau)_{k+1}| \) is equivalent to one of \( \mathcal{O}^{E^n}(\tau_n) \to \mathcal{O}^{E^n}(\tau)_{k+1} \).

Using this we first check that \( \hat{\mathcal{O}} \) is \( \kappa \)-continuous (in the sense of Definition 2.6.4), starting with condition (2). To see that the map \( |\mathcal{O}^{E*}(\tau_n)(\emptyset)| \to |\mathcal{O}^{E*}(\tau)(\emptyset)| \) is equivalent to one of \( \mathcal{O}^{E^n}(\tau_n)(\emptyset) \to \mathcal{O}^{E^n}(\tau)(\emptyset) \), we observe that this factors as
\[
|\mathcal{O}^{E*}(\tau_n)(\emptyset)| \to |\mathcal{O}^{E*}(\tau)(\emptyset)| \to |\mathcal{O}^{E*}(\tau)(\emptyset)| \times (n+1),
\]
where the first map is an equivalence since it is a colimit of equivalences (as \( \mathcal{O}^{E^n} \) is a \( \kappa \)-continuous Segal presheaf for each \( n \)) and the second map is an equivalence since simplicial colimits commute with products.

We now check condition (1) from Definition 2.6.4. Fix a \( \kappa \)-small weakly contractible diagram \( q : K \to \mathcal{V}^\kappa \). We want to show that the canonical map \( \hat{\mathcal{O}}(\xi_k, \text{colim}_K q) \to \lim_K \hat{\mathcal{O}}(\xi_k, q) \) is an
equivalence in $S / |\mathcal{E}^n_\kappa|$. To see this, it suffices to verify that the front square in the following commutative diagram in $S$

$$
\begin{array}{c}
\mathcal{O}^{E^0}(\kappa, \lim_K q) \\
\downarrow \\
\lim_K |\mathcal{O}^{E^0}(\kappa, q)|
\end{array}
\quad
\begin{array}{c}
\mathcal{O}^{E^0}(\kappa, \lim_K q) \\
\downarrow \\
\lim_K |\mathcal{O}^{E^0}(\kappa, q)|
\end{array}
\quad
\begin{array}{c}
\mathcal{O}^{E^0}(\epsilon)^{k+1} \\
\downarrow \\
\mathcal{O}^{E^0}(\epsilon)^{k+1}
\end{array}
$$

is a pullback square. The horizontal maps of the back square are equivalences by the assumption that $\mathcal{O}^{E^0}(\epsilon) \simeq \mathcal{O}$ is a continuous Segal presheaf. We also saw above that the square on the left side is Cartesian. Moreover, since $K$ is weakly contractible (so the limit cone on a constant diagram is constant) and limits commute with pullbacks, the square on the right side is also Cartesian. This implies that the front square is a pullback too, and so $|\mathcal{O}^{E^0}|$ satisfies the required condition. The Segal condition for $|\mathcal{O}^{E^0}|$ holds by a completely analogous argument, so $|\mathcal{O}^{E^0}|$ is indeed a $\kappa$-continuous Segal presheaf.

\[\square\]

**Corollary 3.5.3.** For every $\mathcal{O} \in \text{P}_{\text{CtsSeg}}(\Delta^\vee_V)$, the canonical map

$$
\overline{\mathcal{O}^*} \to \overline{\mathcal{O}^*}
$$

is an equivalence, where by $\overline{\mathcal{O}^*}$ we mean the colimit $|\overline{\mathcal{O}^*}|$ in $\text{P}_{\text{CtsSeg}}(\Delta^\vee)$.

**Proof.** Since there is a natural equivalence $\overline{\mathcal{O}^*}(\kappa)^{E^*} \simeq (\overline{\mathcal{O}^*})^{E^*}$ by Corollary 2.9.6, there is a natural map from the colimit $\overline{\lim_K \mathcal{O}^{E^0}(\kappa, q)}$ to $\overline{\mathcal{O}^*}$.

Let $F$ denote the colimit of $\mathcal{O}^{E^0}$ in $\text{P}(\Delta^\vee_V)$. Then $\overline{\mathcal{O}}$ is the localization of $F$ at the $\kappa$-continuous Segal equivalences, but by Proposition 3.5.2 the presheaf $F$ is already a $\kappa$-continuous Segal presheaf, and so $\overline{\mathcal{O}} \simeq F$. The functor $\overline{\mathcal{O}^*} : \text{P}(\Delta^\vee_V) \to \text{P}(\Delta^\vee)$ preserves colimits, so $\overline{\mathcal{O}^*} F$ is the colimit of $\overline{\mathcal{O}^*}(\kappa)^{E^*} \simeq (\overline{\mathcal{O}^*})^{E^*}$. Moreover, $\overline{\mathcal{O}^*}$ preserves $\kappa$-continuous Segal presheaves by Corollary 2.9.5, and so $\overline{\mathcal{O}^*} F$ is a $\kappa$-continuous Segal presheaf. Thus $\overline{\mathcal{O}^*} F$ is the colimit $\overline{\lim_K \mathcal{O}^{E^0}(\kappa, q)}$ in $\text{P}_{\text{CtsSeg}}(\Delta^\vee)$, as required.

\[\square\]

**Corollary 3.5.4.** For any $\mathcal{O} \in \text{P}_{\text{CtsSeg}}(\Delta^\vee_V)$, the object $\overline{\mathcal{O}}$ is complete and the map $l_\mathcal{O} : \mathcal{O} \to \overline{\mathcal{O}}$ is a local equivalence.

**Proof.** By Corollary 3.5.3 the underlying enriched $\infty$-category $\overline{\mathcal{O}}$ is equivalent to $\overline{\mathcal{O}^*}$, which is complete by [17, Theorem 5.6.2]; by definition, this means $\overline{\mathcal{O}}$ is also complete.

As the class of local equivalences is strongly saturated, it is closed under colimits of morphisms. Therefore, to see that $l_\mathcal{O}$ is a local equivalence it suffices to show that for every map $[m] \to [n]$ in $\Delta$ the map $\mathcal{O}^{E^m} \to \mathcal{O}^{E^n}$ is a local equivalence. The map $E^m \to E^n$ is a pseudo-equivalence in $\text{P}_{\text{Seg}}(\Delta)$ by [17, Corollary 5.5.6]. Then Lemma 3.4.5 implies that $\mathcal{O}^{E^m} \to \mathcal{O}^{E^n}$ is a pseudo-equivalence and so a local equivalence by Proposition 3.4.8.

\[\square\]

**Corollary 3.5.5.** The functor $\widehat{(-)} : \text{P}_{\text{CtsSeg}}(\Delta^\vee_V) \to \text{P}_{\text{CCS}}(\Delta^\vee)$ is left adjoint to the inclusion $\text{P}_{\text{CCS}}(\Delta^\vee) \hookrightarrow \text{P}_{\text{CtsSeg}}(\Delta^\vee)$.
Proof. By [37, Proposition 5.2.7.8], the functor \((\tilde{-})\) is left adjoint to the inclusion \(\mathcal{P}_{\text{CtsSeg}}(\Delta^Y_p) \subseteq \mathcal{P}_{\text{CtsSeg}}(\Delta^Y_p)\) if and only if for every \(O \in \mathcal{P}_{\text{CtsSeg}}(\Delta^Y_p)\) and every \(\Phi \in \mathcal{P}_{\text{CtsSeg}}(\Delta^Y_p)\) the canonical map \(l_O : O \to \tilde{O}\) induces an equivalence
\[
\text{Map}_{\mathcal{P}_{\text{CtsSeg}}(\Delta^Y_p)}(\tilde{O}, \Phi) \to \text{Map}_{\mathcal{P}_{\text{CtsSeg}}(\Delta^Y_p)}(O, \Phi).
\]
Since \(\Phi\) is complete, this follows from \(l_O\) being a local equivalence, which we just saw in Corollary 3.5.4. □

**Proposition 3.5.6.** The map \(l_O : O \to \tilde{O}\) is fully faithful and essentially surjective for all \(O \in \mathcal{P}_{\text{CtsSeg}}(\Delta^Y_p)\).

Proof. For the essential surjectivity we only need to check that the map \(u^*(l_O) : u^*(O) \to u^*(\tilde{O})\) of the underlying enriched \(\infty\)-categories is essentially surjective. This easily follows from Corollary 3.5.3 and [17, Theorem 5.6.2]. To see that the map is fully faithful, recall from the proof of Proposition 3.5.2 that we have a Cartesian square
\[
\begin{array}{ccc}
O^n((\tau_n)) & \xrightarrow{f} & \hat{O}^n((\tau_n)) \\
\downarrow & & \downarrow \\
O^n((\varepsilon)^{n+1}) & \xrightarrow{\hat{f}} & \hat{O}^n((\varepsilon)^{n+1})
\end{array}
\]
By Proposition 3.5.2 (and the fact that geometric realization commutes with finite products in \(S\)) this says that the commutative square
\[
\begin{array}{ccc}
O(\tau_n) & \xrightarrow{f} & \hat{O}(\tau_n) \\
\downarrow & & \downarrow \\
O(\varepsilon)^{n+1} & \xrightarrow{\hat{f}} & \hat{O}(\varepsilon)^{n+1}
\end{array}
\]
is Cartesian. In other words, \(O \to \tilde{O}\) is fully faithful. □

Putting together our results so far, we can now prove the main result of this section:

**Theorem 3.5.7.** A map in \(\mathcal{P}_{\text{CtsSeg}}(\Delta^Y_p)\) is a local equivalence if and only if it is fully faithful and essentially surjective.

Proof. Every map \(f : O \to \Phi\) in \(\mathcal{P}_{\text{CtsSeg}}(\Delta^Y_p)\) gives a commutative diagram
\[
\begin{array}{ccc}
O & \xrightarrow{f} & \Phi \\
\downarrow_{l_O} & & \downarrow_{l_\Phi} \\
\tilde{O} & \xrightarrow{\tilde{f}} & \tilde{\Phi}
\end{array}
\]
where the vertical maps are local equivalences by Corollary 3.5.4 as well as fully faithful and essentially surjective by Proposition 3.5.6. Since fully faithful and essentially surjective maps satisfy the 2-of-3 property by Postisiton 3.3.5, the map \(\tilde{f}\) is fully faithful and essentially surjective if and only if \(\tilde{f}\) is so. Corollary 3.5.4 implies that the map \(\tilde{f}\) is a map between complete objects and so by Proposition 3.3.6 it is fully faithful if and only if it is an equivalence. Similarly, using the 2-of-3 property for local equivalences we see that \(f\) is a local equivalence if and only if \(\tilde{f}\) is an equivalence. Thus the map \(f\) is a local equivalence if and only if it is fully faithful and essentially surjective. □

From Corollary 3.5.5 and Theorem 3.5.7 we immediately get:
Corollary 3.5.8. The adjunction

\[
(\bigvee) : P_{CtsSeg}(\Delta^V_\infty) \rightleftharpoons P_{CCS}(\Delta^V_\infty)
\]

(where the right adjoint is the inclusion) exhibits \(P_{CCS}(\Delta^V_\infty)\) as the localization of \(P_{CtsSeg}(\Delta^V_\infty)\) with respect to the class of fully faithful and essentially surjective morphisms.

Remark 3.5.9. Suppose \(\mathcal{V}\) is a large symmetric monoidal \(\infty\)-category, not necessarily presentable, and let \(Opd^\infty_\mathcal{V}\) denote the full subcategory of \(\text{Alg}_{\mathcal{V}^{op}/S}(\mathcal{V})\) spanned by the complete objects. By embedding \(\mathcal{V}\) in a presentably symmetric monoidal \(\infty\)-category in a larger universe, it follows by exactly the same argument as in the proof of [17, Theorem 5.6.6] that the inclusion \(Opd^\infty_\mathcal{V} \hookrightarrow \text{Alg}_{\mathcal{V}^{op}/S}(\mathcal{V})\) has a left adjoint that exhibits \(Opd^\infty_\mathcal{V}\) as the localization at the fully faithful and essentially surjective morphisms.

Proposition 3.5.10. The \(\infty\)-category \(Opd^\infty_\mathcal{V}\) is functorial in \(\mathcal{V}\) with respect to lax symmetric monoidal functors. Moreover, if \(F : \mathcal{V} \to \mathcal{W}\) is a colimit-preserving symmetric monoidal functor then \(F_* : Opd^\infty_\mathcal{V} \to Opd^\infty_\mathcal{W}\) preserves colimits; thus \(Opd^\infty_\mathcal{V}\) gives a functor \(\text{CAlg}(\text{Pr}^L) \to \text{Pr}^L\).

Proof. The previous remark and [17, Proposition 5.7.4] imply that the functor \(F_* : \text{Alg}_{\mathcal{V}^{op}/S}(\mathcal{V}) \to \text{Alg}_{\mathcal{W}^{op}/S}(\mathcal{W})\) induced by a lax monoidal functor \(F : \mathcal{V} \to \mathcal{W}\) gives a functor \(Opd^\infty_\mathcal{V} \to Opd^\infty_\mathcal{W}\) if \(F_*\) preserves fully faithful and essentially surjective functors. This can be proven analogously to [17, Lemma 5.7.5]. The second claim follows from an argument similar to that used in the proof of [17, Lemma 5.7.7]. \(\square\)

4. Enriched \(\infty\)-Operads as Dendroidal Segal Presheaves

In this section we consider an enriched version of the dendroidal Segal spaces of Cisinski–Moerdijk; our main result is that this approach is equivalent to that using \(\infty\)-operads. We begin in \S 4.1 by briefly reviewing the definition and basic properties of the dendroidal category \(\Omega\). In \S 4.2 we then introduce \(\infty\)-categories \(\Omega^{\mathcal{V}}\) for \(\mathcal{V}\) a symmetric monoidal \(\infty\)-category and define (continuous) Segal presheaves on \(\Omega^{\mathcal{V}}\); we also discuss the dendroidal analogues of many of the results from \S 2. As a preliminary to the comparison result, in \S 4.3 we observe that we can replace the \(\infty\)-category \(\Delta^V_\infty\) by a full subcategory \(\Delta^V_{\text{loc}}\), before proving the comparison in \S 4.4.

4.1. The Dendroidal Category. The dendroidal category \(\Omega\) was first introduced by Moerdijk and Weiss in [40] as a category of trees whose morphisms are given by maps of free operads. Here we recall a more combinatorial reformulation of this definition due to Kock [33].

Definition 4.1.1. A polynomial endofunctor is a diagram of sets

\[
T_0 \overset{\sigma}{\leftarrow} T_2 \overset{p}{\rightarrow} T_1 \overset{t}{\rightarrow} T_0.
\]

We call a polynomial endofunctor as above a tree if the following conditions are satisfied:

- The sets \(T_i\) are all finite.
- The function \(t\) is injective.
- The function \(s\) is injective and there is a unique element \(r\) called the root in the complement of its image.
- Define a successor function \(\sigma : T_0 \to T_0\) by \(\sigma(r) = r\) and \(\sigma(e) = t(p(e))\) for \(e \in s(T_2)\). Then for every \(e\) there exists some \(k \geq 0\) such that \(\sigma^k(e) = r\).

Definition 4.1.2. Let \(T\) be a tree and let \(e, e' \in T_0\). We say \(e\) and \(e'\) are comparable, if there is some \(k \geq 0\) such that either \(\sigma^k(e) = e'\) or \(\sigma^k(e') = e\), and incomparable otherwise.

Remark 4.1.3. The intuition behind this notion of a “tree” is as follows: We interpret \(T_0\) as the set of edges of the tree, \(T_1\) as the set of vertices, and \(T_2\) as the set of pairs \((v, e)\) where \(e\) is an incoming edge of \(v\). The function \(s\) is the projection \(s(v, e) = e\), the function \(p\) is the projection \(p(v, e) = v\), and the function \(t\) assigns to each vertex its unique outgoing edge.
**Definition 4.1.4.** For a tree $T$ given by $T_0 \xleftarrow{s} T_2 \xrightarrow{p} T_1 \xrightarrow{t} T_0$, we call an edge $e \in T_0$ a leaf if it does not lie in the image of $t$, and an inner edge if it lies in the image of $t$ and $e \neq r$.

**Remark 4.1.5.** The name “polynomial endofunctor” comes from the fact that such a diagram induces an endofunctor of $\text{Set}_{/X_0}$ given by $t_! p_* s^*$. We refer the reader to [33] for a more thorough discussion of this.

**Definition 4.1.6.** A morphism of polynomial endofunctors $f : X \rightarrow Y$ is a commutative diagram

$$
\begin{array}{cccc}
X_0 & \xleftarrow{f_0} & X_2 & \xrightarrow{f_2} \xrightarrow{f_1} X_1 \xrightarrow{f_1} \xrightarrow{f_0} X_0 \\
Y_0 & \xleftarrow{f_0} & Y_2 & \xrightarrow{f_1} Y_1 \xrightarrow{f_1} \xrightarrow{f_0} Y_0 
\end{array}
$$

such that the middle square is Cartesian. We write $\Omega_{\text{int}}$ for the category of trees and morphisms of polynomial endofunctors between them; we will refer to these morphisms as the inert morphisms between trees, or as embeddings of subtrees.

**Remark 4.1.7.** By [33, Proposition 1.1.3] every morphism of polynomial endofunctors between trees is injective, which justifies calling these morphisms embeddings.

**Definition 4.1.8.** A tree $T$ is called a corolla if it has only one vertex, i.e. $T_1$ is a one-element set. For $n \leq 0$, we write $C_n$ for the corolla given by

$$
n + 1 \xleftarrow{\eta} n \rightarrow \{n + 1\} \xleftarrow{\eta} n + 1.
$$

More generally, for a finite set $A$ we let $C_A$ denote the corolla $A_+ \xleftarrow{\eta} A \rightarrow \ast \xleftarrow{\eta} A_+$ (which is of course isomorphic to $C_{|A|}$). We write $\eta$ for the edge, namely the trivial tree $1 \xleftarrow{\eta} 0 \rightarrow 0 \xleftarrow{\eta} 1$.

**Definition 4.1.9.** We define $\Omega_{\text{el}}$ to be the full subcategory of $\Omega_{\text{int}}$ spanned by the corollas $C_n$, $n \geq 0$, and the edge $\eta$. For a tree $T$ we write $\Omega_{\text{el}/T}$ for the pullback $\Omega_{\text{el}} \times_{\Omega_{\text{int}}} \Omega_{\text{int}/T}$.

**Definition 4.1.10.** For a tree $T$, we write $\text{sub}(T)$ for the set of subtrees of $T$, meaning the set of morphisms $T' \rightarrow T$ in $\Omega_{\text{int}}$, and we write $\text{sub}'(T)$ for the set of subtrees of $T$ with a marked leaf, meaning the set of pairs of morphisms $(\eta \rightarrow T', T' \rightarrow T)$. We then write $\mathcal{T}$ for the polynomial endofunctor

$$
T_0 \xleftarrow{\text{sub}'(T)} \xrightarrow{\text{sub}(T)} \rightarrow T_0,
$$

where the first map sends a marked subtree to its marked edge, the second is the obvious projection, and the third sends a subtree to its root.

**Definition 4.1.11.** The dendroidal category $\Omega$ has trees as objects and the morphisms of polynomial endofunctors $\mathcal{T} \rightarrow \mathcal{T}'$ as morphisms from $T$ to $T'$.

**Remark 4.1.12.** By [33, Corollary 1.2.10], the polynomial endofunctor $\mathcal{T}$ is in fact the free polynomial monad generated by $T$, and the category $\Omega$ is a full subcategory of the Kleisli category of the monad for free polynomial monads. This means that a morphism $T \rightarrow S$ in $\Omega$ can be identified with a map of polynomial endofunctors $T \rightarrow \mathcal{T}$. It follows that $\Omega_{\text{int}}$ is a subcategory of $\Omega$.

**Definition 4.1.13.** A map $T \rightarrow T'$ in $\Omega$ is active if it takes the leaves of $T$ to the leaves of $T'$ (bijectively) and the root of $T$ to the root of $T'$.

**Remark 4.1.14.** By [33, Proposition 1.3.13] the inert and active morphisms form a factorization system on $\Omega$.}
Remark 4.1.15. There is a fully faithful functor $u_\Omega: \Delta \to \Omega$ which can viewed as the embedding of the full subcategory of linear trees into $\Omega$. More precisely, the functor $u$ takes an object $[n] \in \Delta$ to the tree
\[ n + 1 \overset{s}{\leftarrow} \{2, \ldots, n + 1\} \overset{p}{\to} \{1, \ldots, n\} \overset{t}{\to} n + 1, \]
where $s, t$ are canonical inclusions and $p(i) = i - 1$ for $1 \leq i \leq n$.

Definition 4.1.16. The functor $V_\Omega: \Omega^{\text{op}} \to \text{Fin}_*$ takes an object $T$ corresponding to a diagram
\[ T_0 \overset{f_0}{\leftarrow} T_2 \overset{p}{\to} T_1 \overset{f_1}{\to} T_0 \]
to $T_{1,+} \in \text{Fin}_*$ and a morphism $f: T \to T'$ in $\Omega$ corresponding to a diagram
\[ T'_0 \overset{f_0}{\leftarrow} T'_2 \overset{p}{\to} T'_1 \overset{f_1}{\to} T'_0 \]
to the morphism $V_\Omega(f): T_{1,+} \to T_{1,+}$ defined by
\[ V_\Omega(f)(x) = \begin{cases} y, & x \text{ is a vertex of the subtree } f_1(y), \\ * & x \text{ is not in the image of } f. \end{cases} \]

The following observation shows that the functor $V_\Omega$ is well-defined:

Lemma 4.1.17. If $\phi: T \to S$ is a map in $\Omega$ and $t \neq t' \in T_1$, then $\phi(t)$ and $\phi(t')$ are two subtrees of $S$ with disjoint sets of vertices.

Proof. The map $\phi$ is given by a morphism of polynomial endofunctors of the form
\[ T_0 \overset{\phi}{\leftarrow} \text{sub}'(T) \overset{r}{\to} \text{sub}(T) \to T_0 \]
\[ S_0 \overset{\phi}{\leftarrow} \text{sub}'(S) \overset{r}{\to} \text{sub}(S) \to S_0. \]

By identifying $T_1$ with the subset of $\text{sub}(T)$ consisting of corollas whose roots are given by elements in $T_1$, we regard $\phi(t)$ and $\phi(t')$ as two subtrees in $S$. Let us assume that there exists a corolla $C_n$ lying in the intersection of $\phi(t)$ and $\phi(t')$. Without loss of generality we can assume that there exists a path from a leaf $t'$ of the subtree $\phi(t')$ to the root $r'$ of $\phi(t')$ which passes through the root of $C_n$ and the root $t$ of $\phi(t)$. In particular, this path induces a linear ordering $t'' \leq r \leq r'$. Since the middle square in the diagram is Cartesian, $\phi$ carries the leaves of $t''$ to leaves of $\phi(t'')$. Therefore, the path in $T$ going from $t$ to $t''$ necessarily passes through a unique leaf $t_{r''}$ of the corolla corresponding to $t''$. We then obtain another linear ordering $t \leq t_{r''} \leq t'$. As this contradicts [33, Proposition 1.3.7], the subtrees $\phi(t)$ and $\phi(t')$ necessarily have disjoint sets of corollas. □

Lemma 4.1.18. The functor $V_\Omega$ preserves active-inert factorizations.

Proof. If $f: T \to T'$ in $\Omega$ is inert, then the subtree $f_1(x)$ is a corolla for every $x \in T_1$, so that $V_\Omega(f)$ is inert away from the base point, i.e. $V_\Omega(f)$ is inert. Similarly, if $f$ is active, then it is easy to see that every corolla of $T'$ must lie in the subtree $f_1(x)$ for some $x \in T_1$; thus nothing is sent to the base point, i.e. $V_\Omega(f)$ is active. □

4.2. Segal Presheaves on $\Omega^\Lambda$. We now introduce the dendroidal analogue of the Segal presheaves of §2.3. We will state the analogues of many of the results of §2 in this setting, but without giving the proofs, seeing as they are entirely analogous.

Definition 4.2.1. A presheaf $\mathcal{O}: \Omega^{\text{op}} \to S$ is called a Segal presheaf (or a dendroidal Segal space) if the canonical map
\[ \mathcal{O}(T) \to \lim_{S \in (\Omega_{1+(T)})^{\text{op}}} \mathcal{O}(S) \]
is an equivalence for every $T \in \Omega$. We write $P_{\text{Seg}}(\Omega)$ for the full subcategory of $P(\Omega)$ spanned by the Segal presheaves.

**Definition 4.2.2.** Let $\mathcal{V}$ be a symmetric monoidal $\infty$-category. By identifying $\text{Fin}_*$ with its skeleton $\mathbb{F}_*$, we define the $\infty$-category $\Omega^\mathcal{V}$ by the pullback

$$
\begin{array}{ccc}
\Omega^\mathcal{V} & \longrightarrow & \mathcal{V}_{\emptyset} \\
\downarrow & & \downarrow \\
\Omega & \longrightarrow & \mathcal{V}^\text{op}_{\emptyset}.
\end{array}
$$

We also write $\Omega^\mathcal{V}_{\text{int}}$ and $\Omega^\mathcal{V}_{\text{cl}}$ for the pullbacks $\Omega_{\text{int}} \times \Omega^\mathcal{V}$ and $\Omega_{\text{cl}} \times \Omega^\mathcal{V}$, respectively. We let $\mathbf{T}$ or $T(\psi_{c})_{c \in T_{1}}$ denote an object in $\Omega^\mathcal{V}$ lying over an object $T \in \Omega$.

**Remark 4.2.3.** An object in $\Omega^\mathcal{V}$ should be thought of as a tree in $\Omega$ whose vertices are labeled by objects of $\mathcal{V}$.

**Definition 4.2.4.** Given a symmetric monoidal $\infty$-category $\mathcal{V}_{\infty}$, a presheaf $\mathcal{O} : \Omega^{\mathcal{V}, \text{op}} \to \mathcal{S}$ is called a Segal presheaf on $\mathcal{V}$ if for every object $\mathbf{T}$ of $\Omega^\mathcal{V}$ lying over $T \in \Omega$, the canonical map

$$
\mathcal{O}(\mathbf{T}) \to \lim_{\psi \in (\Omega_{\text{cl}}/T)^{\text{op}}} \mathcal{O}(\psi^{*}(\mathbf{T}))
$$

is an equivalence, where $\psi^{*}(\mathbf{T}) \to \mathbf{T}$ is the Cartesian lift of the inert map $\psi$ (corresponding to a coCartesian morphism in $\mathcal{V}_{\infty}$). We write $P_{\text{Seg}}(\mathcal{V})$ for the full subcategory of $P(\mathcal{V})$ spanned by the Segal presheaves. Similarly, we define the full subcategory $P_{\text{Seg}}(\mathcal{V}^\mathcal{V}_{\text{int}})$ of Segal presheaves on $\mathcal{V}^\mathcal{V}_{\text{int}}$.

**Definition 4.2.5.** Let $\mathcal{V}$ be a presentably symmetric monoidal $\infty$-category. We say a presheaf $\mathcal{O} \in P(\Omega^\mathcal{V})$ is a continuous Segal presheaf if it is a Segal presheaf and moreover the functor

$$
\mathcal{V}^{\text{op}} \simeq (\Omega^{\mathcal{V}, \text{op}})_{C_{n}} \to \mathcal{S}/(\mathcal{O}(\mathcal{F}_{n}))_{n+1},
$$

induced by the Cartesian lifts of the $n+1$ morphisms $\eta \to C_{n}$, preserves all small limits in $\mathcal{V}$ for every $n$. We write $P_{\text{CtsSeg}}(\Omega^\mathcal{V})$ for the full subcategory of $P(\mathcal{V})$ spanned by the continuous Segal presheaves. Similarly, we define the full subcategories $P_{\text{CtsSeg}}(\mathcal{V}^\mathcal{V}_{\text{int}})$ and $P_{\text{CtsSeg}}(\mathcal{V}^\mathcal{V}_{\text{cl}})$ of continuous Segal presheaves on $\mathcal{V}^\mathcal{V}_{\text{int}}$ and continuous presheaves on $\mathcal{V}^\mathcal{V}_{\text{cl}}$.

**Proposition 4.2.6.** The following are equivalent for a presheaf $\mathcal{O} \in P(\mathcal{V})$:

1. $\mathcal{O}$ is a Segal presheaf.
2. $\mathcal{O}$ is local with respect to the morphisms $\mathbf{T}_{\text{Seg}} := \text{colim}_{C \in (\Omega_{\text{cl}}/T)^{\text{op}}} C \to \mathbf{T}$.
3. $\mathcal{O}|_{\mathcal{V}^{\text{op}}_{\text{cl}}}$ is the right Kan extension of $\mathcal{O}|_{\mathcal{V}^{\text{op}}_{\text{cl}}}$.

**Proof.** As Proposition 2.3.12. \qed

**Proposition 4.2.7.** The following are equivalent for a Segal presheaf $\mathcal{O} \in P_{\text{Seg}}(\mathcal{V})$:

1. $\mathcal{O}$ is continuous.
2. For every $n$, the presheaf

$$
\mathcal{O}(C_{n}(\cdot)) : \mathcal{V}^{\text{op}} \simeq (\Omega^{\mathcal{V}, \text{op}})_{C_{n}} \to \mathcal{S}
$$

preserves weakly contractible limits, and the natural map $\mathcal{O}(C_{n}(\emptyset)) \to \prod_{n+1} \mathcal{O}(\overline{\mathcal{F}})$ is an equivalence.
3. $\mathcal{O}$ is local with respect to the map $\prod_{n+1} \overline{\mathcal{F}} \to C_{n}(\emptyset)$ and the map $\text{colim}_{3} C_{n}(\phi) \to C_{n}(\text{colim}_{3} \phi)$ for every weakly contractible diagram $\phi$ in $\mathcal{V}$.
4. $\mathcal{O}$ is local with respect to the map $\prod_{n+1} \overline{\mathcal{F}} \to C_{n}(\emptyset)$ and the map $\text{colim}_{3} C_{n}(\phi) \to C_{n}(\text{colim}_{3} \phi)$ for every diagram $\phi$ such that $\phi(-\infty) \simeq \emptyset$. 

2.3.14 and 2.6.6

2.4.4

2.5.2

2.6.8

Inclusion

\{ \eta \} \to \Omega^{op}

We write \( T \) for an object in \( \Omega_X \) lying over \( T \in \Omega \), and we view \( \Omega^{op}_X \) as living over \( F_* \) via the composite map

\[
\Omega^{op}_X \to \Omega^{op} \xrightarrow{V_\eta} F_*.
\]

Remark 4.2.9. This right Kan extension \( \Omega^{op} \to S \) takes an object \( T \) to a product of copies of \( X \) indexed by the number of edges of \( T \).

Definition 4.2.10. If \( \mathcal{V} \) is a symmetric monoidal \( \infty \)-category, then an \( \Omega^{op}_X \)-algebra in \( \mathcal{V} \) is a functor \( \Omega^{op}_X \to \mathcal{V} \) over \( F_* \) that takes the inert morphisms to coCartesian morphisms. We write \( \text{Alg}_{\Omega^{op}_X}(\mathcal{V}) \) for the full subcategory of \( \text{Fun}_{\mathcal{V}}(\Omega^{op}_X, \mathcal{V}) \) spanned by the algebras. This is clearly functorial in \( X \), and we write \( \text{Alg}_{\Omega^{op}_X}(\mathcal{V}) \) for the Cartesian fibration associated to the functor \( \mathcal{S}^{op} \to \text{Cat}_\infty \) taking \( X \) to \( \text{Alg}_{\Omega^{op}_X}(\mathcal{V}) \).

Theorem 4.2.11. Let \( \mathcal{V} \) be a presentably symmetric monoidal \( \infty \)-category. There is an equivalence of \( \infty \)-categories over \( S \),

\[
P_{\text{CtsSeg}}(\Omega^V) \xrightarrow{\sim} \text{Alg}_{\Omega^{op}/S}(\mathcal{V})
\]

Proof. As Theorem 2.4.4.

Theorem 4.2.12. Let \( \mathcal{U} \) be a small symmetric monoidal \( \infty \)-category. Let \( y : \Omega^{\text{tr}} \to \Omega^{P(\mathcal{U})} \) denote the functor induced by the Yoneda embedding \( \mathcal{U} \to \text{P}(\mathcal{U}) \), viewed as a symmetric monoidal functor. Then right Kan extension along \( y \) gives a fully faithful functor \( y_* : \text{P}(\Omega^{\text{tr}}) \hookrightarrow \text{P}(\Omega^{P(\mathcal{U})}) \) which restricts to an equivalence \( \text{P}_{\text{Seg}}(\Omega^{\text{tr}}) \xrightarrow{\sim} \text{P}_{\text{CtsSeg}}(\Omega^{P(\mathcal{U})}) \).

Proof. As Theorem 2.5.2.

Building on this, there are also obvious dendroidal variants of the results of §2.6. We will not spell these out explicitly, except for the following analogue of Corollaries 2.6.6 and 2.6.8, which we will need below:

Corollary 4.2.13. Suppose \( \mathcal{V} \) is a presentably symmetric monoidal \( \infty \)-category. Then \( \mathcal{V} \) is \( \kappa \)-presentably symmetric monoidal for some regular cardinal \( \kappa \), and we have an equivalence

\[
P_{\text{CtsSeg}}(\Omega^V) \xrightarrow{\sim} P_{\kappa-\text{Seg}}(\Omega^{V_{\kappa}}).
\]

In particular, \( P_{\text{CtsSeg}}(\Omega^V) \) is a presentable \( \infty \)-category.

4.3. Reduction to \( \Delta^1_\infty \). Just as in [10], in order to connect \( \Delta_\infty \) to \( \Omega \) it is convenient to introduce an intermediate category. While \( \Omega \) consists of trees, the objects of \( \Delta_\infty \) can be thought of as “forests” of trees with levels; the category \( \Delta^1_\infty \) is then the full subcategory of trees:

Definition 4.3.1. As in [10] we write \( \Delta^1_\infty \) for the full subcategory of \( \Delta_\infty \) spanned by the objects \( ([n], f) \) such that \( f([n]) = 1 \). Since \( \Delta_{\infty, \text{el}} \) is contained in \( \Delta^1_\infty \) we can define Segal presheaves on \( \Delta^1_\infty \) as presheaves \( \mathcal{O} \in \text{P}(\Delta^1_\infty) \) such that for every \( I \in \Delta^1_\infty \) the canonical map

\[
\mathcal{O}(I) \to \lim_{J \in (\Delta_{\infty, \text{el}}/I)^{op}} \mathcal{O}(J)
\]
is an equivalence. Let \( i \) denote the inclusion \( \Delta_{\mathbb{I}} \hookrightarrow \Delta_{\mathbb{F}} \). If \( \mathcal{V} \) is a symmetric monoidal \( \infty \)-category, we define \( \Delta_{\mathbb{F}}^{1,\mathcal{V}} \) by the pullback square

\[
\begin{array}{ccc}
\Delta_{\mathbb{F}}^{1,\mathcal{V}} & \xrightarrow{\tau} & \Delta_{\mathbb{F}}^{\mathcal{V}} \\
\downarrow & & \downarrow \\
\Delta_{\mathbb{I}} & \xrightarrow{i} & \Delta_{\mathbb{F}}
\end{array}
\]

we also similarly define \( \Delta_{\mathbb{F},\text{int}}^{1,\mathcal{V}} \). We also define Segal presheaves on \( \Delta_{\mathbb{F}}^{1,\mathcal{V}} \) and, when \( \mathcal{V} \) is presentably symmetric monoidal, continuous Segal presheaves on \( \Delta_{\mathbb{F}}^{1,\mathcal{V}} \), by the obvious variants of Definitions 2.3.8 and 2.3.9; we write \( \text{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}) \) and \( \text{P}_{\text{CtsSeg}}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}) \) for the full subcategories of \( \text{P}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}) \) spanned by the Segal presheaves and the continuous Segal presheaves, respectively.

In [10, Lemma 2.11] we proved that the functor \( i^* : \text{P}(\Delta_{\mathbb{F}}) \to \text{P}(\Delta_{\mathbb{I}}) \) given by composition with \( i \) restricts to an equivalence \( \text{P}_{\text{Seg}}(\Delta_{\mathbb{F}}) \to \text{P}_{\text{Seg}}(\Delta_{\mathbb{I}}) \). We now want to prove an enriched analogue of this statement:

**Proposition 4.3.2.** Let \( \mathcal{V} \) be a symmetric monoidal \( \infty \)-category.

(i) The functor \( \tau : \text{P}(\Delta_{\mathbb{F}}) \to \text{P}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}) \) restricts to an equivalence

\[
\text{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}) \xrightarrow{\sim} \text{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}).
\]

(ii) If \( \mathcal{V} \) is presentably symmetric monoidal, then this restricts further to an equivalence

\[
\text{P}_{\text{CtsSeg}}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}) \xrightarrow{\sim} \text{P}_{\text{CtsSeg}}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}).
\]

**Lemma 4.3.3.** Let \( \mathcal{V} \) be a symmetric monoidal \( \infty \)-category.

(i) The functor \( \tau \) preserves Segal presheaves.

(ii) The functor \( \tau \) has a right adjoint \( \tau_* \), given by right Kan extension along \( \tau^\text{op} \).

(iii) The functor \( \tau_* \) preserves Segal presheaves.

**Proof.** A presheaf on \( \Delta_{\mathbb{F}}^{1,\mathcal{V}} \) is a Segal presheaf if and only if it is local with respect to the maps \( \mathcal{T}_{\text{Seg}} \to \mathcal{T} \) for \( \mathcal{T} \) in \( \Delta_{\mathbb{F}}^{1,\mathcal{V}} \). For \( \mathcal{O} \in \text{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}) \) we have natural equivalences

\[
\text{Map}_{\text{P}(\Delta_{\mathbb{F}}^{1,\mathcal{V}})}(\mathcal{T}, \tau^* \mathcal{O}) \simeq \mathcal{O}(\mathcal{T}) \simeq \text{Map}_{\text{P}(\Delta_{\mathbb{F}}^{1,\mathcal{V}})}(\mathcal{T}, \mathcal{O}),
\]

\[
\text{Map}_{\text{P}(\Delta_{\mathbb{F}}^{1,\mathcal{V}})}(\mathcal{T}_{\text{Seg}}, \tau^* \mathcal{O}) \simeq \text{Map}_{\text{P}(\Delta_{\mathbb{F}}^{1,\mathcal{V}})}(\mathcal{T}, \mathcal{O}),
\]

where the second arises by moving the colimit in \( \mathcal{T}_{\text{Seg}} \) outside and then using equivalences of the first kind. Since \( \tau(\mathcal{T})_{\text{Seg}} \to \tau(\mathcal{T}) \) is a generating Segal equivalence in \( \text{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}) \), it follows that \( \tau_* \mathcal{O} \) is a Segal presheaf. Thus we have proved (i).

To prove (ii), it suffices to show that for \( F \in \text{P}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}) \) and \( \mathcal{T} \) in \( \Delta_{\mathbb{F}} \), the limit \( \text{lim}_{\mathcal{T} \in (\Delta_{\mathbb{F}}^{1,\mathcal{V}}, \tau^* \mathcal{O})} F(\mathcal{T}) \) exists in \( \mathcal{S} \). By Lemma 2.3.13 there is a coinitial map to \( (\Delta_{\mathbb{F}}^{1,\mathcal{V}}, \tau^* \mathcal{O}) \) from an \( \infty \)-category equivalent to \( (\Delta_{\mathbb{F}}^{1,\mathcal{V}}, \tau^* \mathcal{O}) \); this is small, and \( \mathcal{S} \) has all small colimits, so the right Kan extension along \( \tau^\text{op} \) exists.

To prove (iii), it suffices to show that \( \tau \) takes a collection of generating Segal equivalences in \( \text{P}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}) \) to Segal equivalences in \( \text{P}(\Delta_{\mathbb{F}}^{1,\mathcal{V}}) \). By Proposition 2.3.12 it is enough to check this for the images of the maps

\[
\mathcal{T}_{\text{Seg}} \to \mathcal{T}
\]

for all \( \mathcal{T} \) in \( \Delta_{\mathbb{F}}^{1,\mathcal{V}} \subseteq \Delta_{\mathbb{F}}^{\mathcal{V}} \), and

\[
\prod_{i \in I(f(n))} \mathcal{T}_i \to \mathcal{T}
\]

for all \( I = ([n], f) \), where \( \mathcal{T}_i = ([n], f_i) \) is obtained by taking the fibres at \( i \in f(n) \).
Since $\tau$ is fully faithful, the images of the first class of maps are clearly just the generating Segal equivalences in $P(\Delta^1_\mathbf{F})$. Moreover, from the definition of $\Delta^1_\mathbf{F}$ it is easy to see that for $\mathcal{T} \in \Delta^1_\mathbf{F}$ we have

$$\prod_{i \in f(n)} \text{Map}_{\Delta^1_\mathbf{F}}(\mathcal{T}, \mathcal{T}_i) \cong \text{Map}_{\Delta^1_\mathbf{F}}(\mathcal{T}, \mathcal{T})_i,$$

so that $\tau \left( \prod_{i \in f(n)} \mathcal{T}_i \right) \rightarrow \tau \mathcal{T}$ is an equivalence. This gives (iii).

Proof of Proposition 4.3.2. By Lemma 4.3.3, the adjunction $\tau^* \dashv \tau_*$ restricts to an adjunction

$$\tau^*: P(\Delta^1_\mathbf{F}) \rightleftarrows P(\Delta^1_\mathbf{F}) : \tau_*.$$

Since $\tau$ is fully faithful, the right Kan extension $\tau_*$ is also fully faithful, and so we automatically have that the counit map $\tau \tau_* \rightarrow \text{id}$ is an equivalence. If $\mathcal{O}$ is a Segal presheaf on $\Delta^1_\mathbf{F}$, then so is $\tau_\mathcal{T} \mathcal{O}$. To show that the unit map $\mathcal{O} \rightarrow \tau_\mathcal{T} \mathcal{O}$ is an equivalence, it therefore suffices to see that this map of presheaves is an equivalence when evaluated at $\tau$ and $\mathcal{C}_n(v)$ for $v \in \mathcal{V}$. Since these objects all lie in $\Delta^1_\mathbf{F}$, this follows from $\tau$ being fully faithful. This proves (i), and (ii) is clear since this only depends on objects in the image of $\tau$. \qed

4.4. Comparison. In [10] we defined a functor $\tau: \Delta^1_\mathbf{F} \rightarrow \mathcal{O}$ and proved that composition with $\tau$ induces an equivalence $\tau^*: P_{\text{Seg}}(\mathcal{O}) \rightleftarrows P_{\text{Seg}}(\Delta^1_\mathbf{F})$. In this subsection we will prove the enriched analogue of this result. To state this, we first introduce some notation:

Definition 4.4.1. Let $\tau: \Delta^1_\mathbf{F} \rightarrow \mathcal{O}$ be the functor constructed in [10, §4]; by [10, Lemmas 4.4 and 4.5] this is compatible with the inert–active factorization systems and restricts to a functor $\tau_{\text{int}}: \Delta^1_{\mathbf{F}, \text{int}} \rightarrow \mathcal{O}_{\text{int}}$ and an equivalence $\tau_{\text{el}}: \Delta_{\mathbf{F}, \text{el}} \rightleftarrows \mathcal{O}_{\text{el}}$. Given a symmetric monoidal $\infty$-category $\mathcal{V}$, let $\tau: \Delta^1_{\mathbf{F}, \mathcal{V}} \rightarrow \mathcal{V}$ be the functor defined by the pullback square

$$\begin{array}{ccc}
\Delta^1_{\mathbf{F}, \mathcal{V}} & \xrightarrow{\tau} & \mathcal{V} \\
\downarrow & & \downarrow \\
\Delta^1_{\mathbf{F}} & \xrightarrow{\tau} & \mathcal{V}.
\end{array}$$

We write $\tau_{\text{int}}: \Delta^1_{\mathbf{F}, \text{int}} \rightarrow \mathcal{O}_{\text{int}}^\mathcal{V}$ and $\tau_{\text{el}}: \Delta^1_{\mathbf{F}, \text{el}} \rightarrow \mathcal{O}_{\text{el}}^\mathcal{V}$ for the appropriate restrictions of $\tau$.

Remark 4.4.2. By [10, Lemma 4.4], the functor $\tau$ restricts to an equivalence $\tau_{\text{el}}: \Delta_{\mathbf{F}, \text{el}} \rightleftarrows \mathcal{O}_{\text{el}}$. The pullback $\tau_{\text{el}}: \Delta^1_{\mathbf{F}, \text{el}} \rightarrow \mathcal{O}^\mathcal{V}_{\text{el}}$ is therefore also an equivalence.

Our goal is then to prove:

Theorem 4.4.3. Let $\mathcal{U}$ be a small symmetric monoidal $\infty$-category. The functor $\tau^*: P(\mathcal{U}) \rightarrow P(\Delta^1_\mathbf{F})$ restricts to an equivalence

$$P_{\text{Seg}}(\mathcal{U}) \rightarrow P_{\text{Seg}}(\Delta^1_\mathbf{F}).$$

Before we turn to the proof, we note that as an immediate consequence our models for enriched $\infty$-operads as continuous Segal presheaves are equivalent. This is a special case of the following observation:

Corollary 4.4.4. Let $\mathcal{U}$ be a small symmetric monoidal $\infty$-category and let $S$ be a small set of morphisms in $P(\mathcal{U})$, compatible with the symmetric monoidal structure. Then:

(i) Composition with $\tau: \Delta^1_{\mathbf{F}, S(\mathcal{U})} \rightarrow \mathcal{U}$ restricts to an equivalence

$$P_{S, \text{Seg}}(\mathcal{U}) \rightleftarrows P_{S, \text{Seg}}(\Delta^1_{\mathbf{F}, S(\mathcal{U})}).$$

(ii) Composition with $\tau: \Delta^1_{\mathbf{F}, S(\mathcal{U})} \rightarrow \mathcal{O}^\mathcal{P}(\mathcal{U})$ restricts to an equivalence

$$P_{\text{CtsSeg}}(\mathcal{U}^\mathcal{P}(\mathcal{U})) \rightleftarrows P_{\text{CtsSeg}}(\Delta^1_{\mathbf{F}, S(\mathcal{U})}).$$
Proof. By Theorem 4.4.3 composition with $\mathfrak{r}$ restricts to an equivalence $\mathfrak{r}^* : \mathcal{P}_{\text{Seg}}(\Omega^U) \xrightarrow{\sim} \mathcal{P}_{\text{Seg}}(\Delta^1_{\mathcal{F}})^U)$. The full subcategories $\mathcal{P}_{\text{Seg}}(\Omega^U)$ and $\mathcal{P}_{\text{Seg}}(\Delta^1_{\mathcal{F}})^U)$ consist of Segal presheaves whose restrictions to $\Omega^U$ and $\Delta^1_{\mathcal{F}, \text{el}}$ satisfy a locality condition, as in Definition 2.6.1. Since $\mathcal{T}_{\text{el}} : \Delta^1_{\mathcal{F}, \text{el}} \rightarrow \Omega^U$ is an equivalence by Remark 4.4.2, these subcategories correspond under the equivalence $\mathfrak{r}$. This gives (i).

To prove (ii), observe that we have a commutative square

$$
\begin{array}{ccc}
P_{\text{CtsSeg}}(\Omega^P(U)) & \xrightarrow{\mathfrak{r}} & P_{\text{CtsSeg}}(\Delta^1_{\mathcal{F}})^P(U)) \\
\downarrow & & \downarrow \\
P_{\text{Seg}}(\Omega^U) & \xrightarrow{\mathfrak{r}_*} & P_{\text{Seg}}(\Delta^1_{\mathcal{F}})^U),
\end{array}
$$

where the vertical maps, given by composition with the maps induced by the Yoneda embedding $U \rightarrow \mathcal{P}(U)$, are equivalences by analogues of Corollary 2.6.6 for $\Omega$ and $\Delta^1_{\mathcal{F}, \text{el}}$. Since the bottom horizontal map is an equivalence by (i), so is the upper horizontal map. □

**Corollary 4.4.5.** If $\mathcal{V}$ is a presentably symmetric monoidal $\infty$-category, then the functor $\mathcal{V} : \mathcal{P}(\Omega^V) \rightarrow \mathcal{P}(\Delta^1_{\mathcal{F}})^V)$ restricts to an equivalence

$$P_{\text{CtsSeg}}(\Omega^V) \rightarrow P_{\text{CtsSeg}}(\Delta^1_{\mathcal{F}})^V).$$

Proof. Since $\mathcal{V}$ is presentably symmetric monoidal, by Proposition 2.6.9 we can choose a regular cardinal $\kappa$ such that $\mathcal{V}^\kappa$ is a symmetric monoidal subcategory of $\mathcal{V}$, and $\mathcal{V} \simeq \text{Ind}_\kappa\mathcal{V}^\kappa$ as a symmetric monoidal $\infty$-category. The result is then a special case of Corollary 4.4.4 applied to $\mathcal{V}^\kappa$ with $S$ as in the proof of Corollary 2.6.6. □

We now turn to the proof of Theorem 4.4.3. Our approach closely follows that of [10, Theorem 5.1], and where the proofs are obtained from the unenriched version simply by adding superscript $\mathcal{V}$s and overlines, we will not repeat them here.

**Lemma 4.4.6.** The functor $\mathfrak{r}^* : \mathcal{P}(\Omega^V) \rightarrow \mathcal{P}(\Delta^1_{\mathcal{F}})^V)$ preserves Segal presheaves.

Proof. As [10, Lemma 4.5]. □

**Lemma 4.4.7.** Composition with the functor $\mathfrak{r}_{\text{int}} : \Delta^1_{\mathcal{F}, \text{int}} \rightarrow \Omega^V_{\text{int}}$ induces an equivalence

$$\mathcal{P}_{\text{Seg}}(\Omega^V_{\text{int}}) \xrightarrow{\sim} \mathcal{P}_{\text{Seg}}(\Delta^1_{\mathcal{F}, \text{int}}).$$

Proof. As [10, Lemma 5.2]. □

We let $\mathcal{T}_\Omega$ and $\mathcal{T}_{\Delta^1_{\mathcal{F}, \text{el}}}$ denote the inclusions $\Omega^V_{\text{int}} \rightarrow \Omega^V$ and $\Delta^1_{\mathcal{F}, \text{int}} \rightarrow \Delta^1_{\mathcal{F}}$, respectively. Moreover, we write $\mathcal{T}_{\Omega}$ and $\mathcal{T}_{\Delta^1_{\mathcal{F}, \text{el}}}$ for the localizations $\mathcal{P}(\Omega^V) \rightarrow \mathcal{P}_{\text{Seg}}(\Omega^V)$ and $\mathcal{P}(\Delta^1_{\mathcal{F}})^V) \rightarrow \mathcal{P}_{\text{Seg}}(\Delta^1_{\mathcal{F}})^V)$, respectively.

**Lemma 4.4.8.**

(i) The functor $\mathcal{T}_\Omega : \mathcal{P}_{\text{Seg}}(\Omega^V) \rightarrow \mathcal{P}_{\text{Seg}}(\Omega^V_{\text{int}})$ has a left adjoint $\mathcal{F}_{\Omega} := \mathcal{T}_{\Omega} \mathcal{T}_\Omega^!$, and the adjunction $\mathcal{F}_{\Omega} \dashv \mathcal{T}_\Omega$ is monadic.

(ii) The functor $\mathcal{T}_{\Delta^1_{\mathcal{F}, \text{el}}} : \mathcal{P}_{\text{Seg}}(\Delta^1_{\mathcal{F}})^V) \rightarrow \mathcal{P}_{\text{Seg}}(\Delta^1_{\mathcal{F}, \text{int}})$ has a left adjoint $\mathcal{F}_{\Delta^1_{\mathcal{F}, \text{el}}} := \mathcal{T}_{\Delta^1_{\mathcal{F}, \text{el}}} \mathcal{T}_{\Delta^1_{\mathcal{F}, \text{el}}}^!$, and the adjunction $\mathcal{F}_{\Delta^1_{\mathcal{F}, \text{el}}} \dashv \mathcal{T}_{\Delta^1_{\mathcal{F}, \text{el}}}$ is monadic.

Proof. As [10, Lemma 5.3]. □

**Proposition 4.4.9.** The functor $\mathfrak{r}^* : \mathcal{P}(\Omega^V) \rightarrow \mathcal{P}(\Delta^1_{\mathcal{F}})^V)$ preserves Segal equivalences.
Proof. We need to prove that for $T$ in $\Omega^V$, the map $\tau^* T_{\text{Seg}} \to \tau^* T$ is a Segal equivalence in $P(\Delta^1_{\text{Seg}})$. We prove this by inducting on the number of vertices of $T$, noting that the statement is vacuous if $T$ has zero or one vertices.

For $T$ in $\Omega^V$, let $\partial T$ and $(\partial T)_{\text{Seg}}$ be as in [10, Definition 5.7]. Writing $\pi$ for the projection $\Omega^V \to \Omega$, we then define $\partial T$ for $T$ in $\Omega^V$ by the pullback square

$$
\partial T \longrightarrow T \\
\pi^* \partial T \longrightarrow \pi^* T.
$$

We also define $(\partial T)_{\text{Seg}}$ similarly. By [10, Lemma 5.8] the natural map $(\partial T)_{\text{Seg}} \to T_{\text{Seg}}$ is an equivalence; since pullbacks in $P(\Omega^V)$ preserve colimits, we see that the natural map $(\partial T)_{\text{Seg}} \to T_{\text{Seg}}$ is a pullback of this map, and hence this is also an equivalence. We have a commutative square

$$
(\partial T)_{\text{Seg}} \longrightarrow \partial T \\
T_{\text{Seg}} \longrightarrow T.
$$

Here the upper horizontal morphism is a colimit of generating Segal equivalences for trees with fewer vertices than $T$, and is therefore mapped to a Segal equivalence in $P(\Delta^1)$. By [10, Proposition 5.6] it follows that $\tau^* \partial T \to \tau^* T$ is an inner anodyne map in $P(\Delta^1)$, and hence a Segal equivalence by Proposition 2.7.5.

Corollary 4.4.10. The functor $\tau_*$ given by right Kan extension along $\tau$ restricts to a functor $\tau_* : P_{\text{Seg}}(\Delta^1) \to P_{\text{Seg}}(\Omega^V)$, right adjoint to $\tau^*$. 

Proof. As [10, Lemma 5.5].

Lemma 4.4.11. The canonical map $\tau_* \tau^* F \sim \tau^* \tau_* F \to F$ is an equivalence for $F \in P_{\text{Seg}}(\Delta^1)$. 

Proof. As [10, Lemma 5.5].

Proof of Theorem 4.4.3. We have a commutative square

$$
P_{\text{Seg}}(\Omega^V) \longrightarrow P_{\text{Seg}}(\Delta^1) \\
\tau^* \downarrow \quad \tau_* \downarrow \\
P_{\text{Seg}}(\Omega^V_{\text{int}}) \longrightarrow P_{\text{Seg}}(\Delta^1_{\text{int}}).
$$
where the lower horizontal morphism is an equivalence by Lemma 4.4.7 and the vertical morphisms are monadic right adjoints by Lemma 4.4.8. It follows from [38, Corollary 4.7.4.16] that to show \( \mathcal{T}_*^{\mathcal{T}_+} \) is an equivalence it is enough to prove that the canonical natural transformation \( \mathcal{T}_{\Delta^1}^{\mathcal{T}_+} \to \mathcal{T}_*^{\mathcal{T}_+} \) is an equivalence. But by Corollary 4.4.10 both functors are left adjoints, so this transformation is an equivalence if and only if the corresponding transformation of right adjoints \( (\mathcal{T}_*)^{-1} \mathcal{T}_{\Delta^1}^{\mathcal{T}_+} \) is an equivalence, which it is by Lemma 4.4.11.

**Definition 4.4.12.** Let \( u_\Omega^*: P(\Omega) \to P(\Delta) \) be the functor induced by the inclusion \( u_\Omega: \Delta \to \Omega \) of Remark 4.1.15. If \( \Omega \) is a Segal presheaf on \( \Omega \), then \( u_\Omega^* \) is a Segal space, and we say that \( \Omega \) is complete if \( u_\Omega^* \) is a complete Segal space. Similarly, we say that a continuous Segal presheaf \( \Omega \in P_{\operatorname{CtsSeg}}(\Omega^V) \) is complete if \( \pi_\Omega \) is complete in \( P_{\operatorname{CtsSeg}}(\Delta^V) \) is complete, where \( \pi_\Omega: \Delta^V \to \Omega^V \) denotes the pullback of \( u_\Omega \) along the projection \( \Omega^V \to \Omega \).

Since \( u_\Omega = \pi \circ u \), the complete objects correspond under the equivalence \( \mathcal{T}_* \), giving:

**Corollary 4.4.13.** Let \( V \) be a presentably symmetric monoidal \( \infty \)-category. Composition with \( \mathcal{T}_* \) induces an equivalence
\[
P_{\operatorname{CCS}}(\Omega^V) \xrightarrow{\sim} P_{\operatorname{CCS}}(\Delta^V_{\mathcal{T}_*}).
\]

5. **Rectification of Enriched \( \infty \)-Operads**

In this section we relate our homotopy theory of enriched \( \infty \)-operads to the existing literature on model categories of enriched operads. In §5.1 we relate our dendroidal model to algebras for the \( (\infty-)\) operads for coloured operads, and in §5.2 we prove a rectification result for \( \infty \)-operads enriched in a symmetric monoidal \( \infty \)-category coming from a nice symmetric monoidal model category.

**5.1. Operads for Operads.** In this subsection we will prove that for a set \( S \), algebras for \( \Omega_S^{\operatorname{op}} \) in a symmetric monoidal \( \infty \)-category are equivalent to algebras for the operad for \( S \)-coloured operads. We will do this by showing that \( \Omega_S^{\operatorname{op}} \) is an “approximation” to this operad, in the sense of [38, §2.3.3]. Our first task is therefore to give a convenient definition of these operads, which requires some observations about certain pushouts in \( \Omega \) and \( \Omega_S \):

**Lemma 5.1.1.** Given an active morphism \( C_1 \to T \) and an inert morphism \( C_1 \to S \) in \( \Omega \), the pushout
\[
\begin{array}{ccc}
C_1 & \to & T \\
\downarrow & & \downarrow \\
S & \to & S \amalg_{C_1} T
\end{array}
\]
exists in \( \Omega \).

**Proof.** It follows from [33, Proposition 1.1.19] that for any tree \( S \) the “Segal diagram” \( \Omega_{el/S}^{\operatorname{op}} \to \Omega_{\operatorname{int}} \) is a colimit diagram. The composite diagram \( \Omega_{el/S}^{\operatorname{op}} \to \Omega \) is therefore also a colimit, since this is given by composition with the free polynomial monad functor, which is a left adjoint. Thus the pushout \( S \amalg_{C_1} T \), if it exists, is equivalent to the colimit of the diagram \( \Omega_{el/S}^{\operatorname{op}} \to \Omega \) obtained from the Segal diagram of \( S \) by replacing the corolla \( C_1 \) by the tree \( T \). This colimit is still given by grafting of trees, and so exists by [33, Proposition 1.1.19].

**Remark 5.1.2.** The tree \( S \amalg_{C_1} T \) is obtained by substituting the corolla \( C_1 \) in \( S \) by the tree \( T \). We refer to this as **substituting** the tree \( T \) into \( S \). If we are given also an active morphism \( C_j \to U \) and an inert morphism \( C_j \to T \), then this procedure is associative in the sense that \( (S \amalg_{C_1} T) \amalg_{C_j} U \) is
canonically isomorphic to $S \amalg_{C_1} (T \amalg_{C_1} U)$, since we have the following commutative diagram where all squares are pushouts:

$$
\begin{array}{ccc}
C_j & \longrightarrow & U \\
\downarrow & & \downarrow \\
C_i & \longrightarrow & T \\
\downarrow & & \downarrow \\
S & \longrightarrow & S \amalg_{C_1} T \\
\end{array}
$$

Here $X$ can be identified with both $(S \amalg_{C_1} T) \amalg_{C_1} U$ and $S \amalg_{C_1} (T \amalg_{C_1} U)$.

The analogous result also holds in $\Omega X$ for any space $X$. To see this, we use:

**Lemma 5.1.3.** Let $\mathcal{E} \rightarrow \mathcal{B}$ be a right fibration corresponding to a functor $F: \mathcal{B}^{op} \rightarrow \mathcal{S}$. Suppose

$$
\begin{array}{ccc}
a & \xrightarrow{f} & b \\
f' \downarrow & & \downarrow g \\
b' & \xrightarrow{g'} & c
\end{array}
$$

is a pushout diagram in $\mathcal{B}$ and that $F$ takes this to a pullback square in $\mathcal{S}$. Then for any morphisms $\tilde{b} \xleftarrow{\tilde{f}} \tilde{a} \xrightarrow{\tilde{f}'} \tilde{b}'$ lying over $b \leftarrow a \rightarrow b'$ there is a pushout square in $\mathcal{E}$ lying over the given pushout square in $\mathcal{B}$.

**Proof.** We have $\mathcal{E}_c \simeq \mathcal{E}_b \times_{\mathcal{E}_a} \mathcal{E}_{b'}$, and the morphisms $\tilde{f}, \tilde{f}'$ determine a point $\tilde{c} \in \mathcal{E}_c$ and a commutative square

$$
\begin{array}{ccc}
\tilde{a} & \xrightarrow{\tilde{f}} & \tilde{b} \\
\tilde{f}' \downarrow & & \downarrow \tilde{g} \\
\tilde{b}' & \xrightarrow{\tilde{g}'} & \tilde{c}
\end{array}
$$

For $\tilde{x}$ in $\mathcal{E}$ lying over $x \in \mathcal{B}$ we then have a commutative square

$$
\begin{array}{ccc}
\text{Map}_\mathcal{E}(\tilde{c}, \tilde{x}) & \longrightarrow & \text{Map}_\mathcal{E}(\tilde{b}, \tilde{x}) \times_{\text{Map}_\mathcal{E}(a, x)} \text{Map}_\mathcal{E}(\tilde{b}', \tilde{x}) \\
\downarrow & & \downarrow \\
\text{Map}_\mathcal{B}(c, x) & \longrightarrow & \text{Map}_\mathcal{B}(b, x) \times_{\text{Map}_\mathcal{B}(a, x)} \text{Map}_\mathcal{B}(b', x).
\end{array}
$$

Since the bottom horizontal morphism is an equivalence, to show the top horizontal morphism is an equivalence it suffices to prove this square is Cartesian. This is equivalent to the map between the fibres at any $\phi: c \rightarrow x$ being an equivalence. This map can be identified with

$$
\text{Map}_\mathcal{E}_c(\tilde{c}, \phi^* \tilde{x}) \rightarrow \text{Map}_\mathcal{E}_b(\tilde{b}, g^* \phi^* \tilde{x}) \times_{\text{Map}_\mathcal{E}_a(a, f^* g^* \phi^* \tilde{x})} \text{Map}_\mathcal{E}_b(\tilde{b}', g'^* \phi^* \tilde{x}),
$$

which is an equivalence since $\mathcal{E}_c$ is a pullback. 

$\square$
Lemma 5.1.4. Let $X$ be a space. Given an active morphism $C_i \to T$ and an inert morphism $C_i \to S$ in $O$, and morphisms $\tilde{C}_i \to \tilde{T}$ and $\tilde{S} \to \tilde{C}_i \tilde{T}$

$$
\begin{array}{c}
\tilde{C}_i \\
\downarrow \\
\tilde{T} \\
\downarrow \\
\tilde{S} \\
\downarrow \\
\tilde{C}_i \tilde{T}
\end{array}
$$

exists in $O_X$ and forgets to a pushout in $O$.

Proof. Using Lemma 5.1.3 and Lemma 5.1.1 this follows from the observation that in this situation the sets of edges give a pushout. □

Definition 5.1.5. Let $S$ be a set. The operad $\textbf{Op}_S$ is defined as follows: The objects of $\textbf{Op}_S$ are pairs $(A, \alpha)$ where $A$ is a finite set and $\alpha: A \to S$ is a function; equivalently, the objects are objects $\tilde{C}_A$ of $O_S$ lying over the corolla $C_A$ in $O$. A multimorphism $((A_1, \alpha_1), \ldots, (A_n, \alpha_n)) \to (B, \beta)$ is given by an object $\tilde{T}$ of $O_S$ together with inert maps $\tilde{C}_{A_i} \to \tilde{T}$ such that each hits a distinct vertex of $\tilde{T}$ and all vertices are hit, and an active map $\tilde{C}_B \to \tilde{T}$. (In other words, the tree $\tilde{T}$ is assembled from the corollas $\tilde{C}_{A_i}$ in such a way that the labels match up, the tree $\tilde{T}$ has $|B|$ leaves, and the labels of the leaves and root match those of $\tilde{C}_B$.) More precisely, a multimorphism is an isomorphism class of this data. It is convenient to represent this as a cospan $\prod\tilde{C}_{A_i} \to \tilde{T} \leftarrow \tilde{C}_B$ (which can be thought of as living in the category obtained by freely adjoining coproducts to $O_S$). Composition is given by substitution of trees: Given a multimorphism $\phi: (A_1, \ldots, A_n) \to B$ as above and another multimorphism $\psi: (C_1, \ldots, C_m) \to A_i$ corresponding to $\prod_j \tilde{C}_{C_j} \to \tilde{S} \leftarrow \tilde{C}_{A_i}$, then the composite $\psi \circ \phi$ is given by the cospan $\prod \tilde{C}_{A_i} \coprod \prod \tilde{C}_{C_j} \to \tilde{T} \coprod \tilde{C}_{A_i} \tilde{S} \leftarrow \tilde{C}_B$.

The composition is associative because of the associativity of tree substitution discussed in Remark 5.1.2, and it is easy to see that the other requirements for an operad are satisfied.

Recall that if $O$ is an operad, we can define its category of operators $O^\otimes \to \textbf{Fin}_*$. This has objects lists $(x_1, \ldots, x_n)$ of objects of $O$, with a morphism $(x_1, \ldots, x_n) \to (x'_1, \ldots, x'_m)$ given by a morphism $\phi: (n) \to (m)$ in $\textbf{Fin}_*$ and for all $i = 1, \ldots, m$ a morphism $(x_i)_{j \in \phi^{-1}(j)} \to x'_i$. Equivalently, we can replace the skeleton $\textbf{Fin}_*$ by $\textbf{Fin}_*$ and take the objects to be $(A \in \textbf{Fin}_, (x_i)_{i \in A})$. Applying this construction to the operad $\textbf{Op}_S$ gives a category $\textbf{Op}_S \to \textbf{Fin}_*$: this is an $\infty$-operad. We now wish to define a functor from $O_S^\otimes$ to $\textbf{Op}_S$.

Definition 5.1.6. We first define the functor $\Theta: O^{\text{op}} \to \textbf{Op}_*$ over $\text{Fin}_*$ as follows: If $T \in O^{\text{op}}$ corresponds to a diagram $T_0 \leftarrow T_2 \to T_1 \to T_0$

then $\Theta(T)$ is the sequence $(T_{2,t})_{t \in T_1}$. For a morphism $f: T \to T'$ in $O^{\text{op}}$, given by a diagram

$$
\begin{array}{ccccccc}
T_0' & \leftarrow & T_2' & \rightarrow & T_1' & \rightarrow & T_0' \\
\downarrow f_0 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\
T_0 & \leftarrow & \text{sub}(T) & \rightarrow & \text{sub}(T) & \rightarrow & T_0 
\end{array}
$$

the morphism $\Theta(f)$ is given by the morphism $V_\Theta(f): T_{1,+} \to T_{1,+}'$ in $\text{Fin}_*$ as in Definition 4.1.16 together with the subtrees $f_1(x) \in \text{sub}(T)$ for $x \in T_1'$ and the isomorphisms between $T_{2,t}'$ and the
leaves of this tree given by the pullback square

\[
\begin{array}{ccc}
T_2' \times \{x\} & \rightarrow & \{x\} \\
\downarrow & & \downarrow \\
\text{sub}'(T)_{f_1(x)} & \rightarrow & \{f_1(x)\}
\end{array}
\]

It is easy to see that this is compatible with composition, because composition in \(\Omega\) can be described in terms of substitution of trees.

Now for a set \(S\), we define \(\Theta_S: \Omega^\text{op}_S \rightarrow \mathcal{O}p_S\) in the same way, just carrying the labelling of edges along.

We wish to prove that the functor \(\Theta_S\) is an approximation in the sense of [38, Definition 2.3.3.6], which we first recall for the reader’s convenience:

**Definition 5.1.7.** Suppose \(p: \emptyset \rightarrow \mathbb{F}_n\) is an \(\infty\)-operad and \(\mathcal{C}\) an \(\infty\)-category. We call a functor \(f: \mathcal{C} \rightarrow \emptyset\) an approximation to \(\emptyset\), if it satisfies the following conditions:

1. Let \(p' = p \circ f\), let \(c \in \mathcal{C}\) be an object and let \(p'(c) = \langle n \rangle\). For every \(1 \leq i \leq n\), there is a locally \(p'\)-coCartesian morphism \(\alpha_i: c \rightarrow c_i \in \mathcal{C}\) lying over the inert map \(\rho^i: \langle n \rangle \rightarrow \langle 1 \rangle\) given by the projection at the \(i\)th element, and the morphism \(f(\alpha_i)\) in \(\emptyset\) is inert.
2. Let \(c \in \mathcal{C}\) and let \(\alpha: u \rightarrow f(c)\) be an active morphism in \(\emptyset\). There exists an \(f\)-Cartesian morphism \(\pi: \pi \rightarrow c\) lifting \(\alpha\).

**Proposition 5.1.8.** The functor \(\Theta_S: \Omega^\text{op}_S \rightarrow \mathcal{O}p_S\) is an approximation.

*Proof.* We need to verify the two conditions mentioned in the previous definition.

Let \(p\) denote the projection \(\mathcal{O}p_S \rightarrow \mathbb{F}_n\) and let \(q: \Omega^\text{op}_S \rightarrow \Omega^\text{op}\) be the natural left fibration, then by construction we have \(p \circ q = V_G\). For an object \(T \in \Omega^\text{op}_S\) lying over some \(T \in \Omega^\text{op}\) and an element \(i \in V_G(T)\), there is an inert map \(T \rightarrow C_n\) in \(\Omega^\text{op}\) which lifts \(\rho^i\) and it is easy to check that it is locally \(V_G\)-coCartesian. Since \(q\) is a left fibration, we can lift \(T \rightarrow C_n\) to a locally \(V_G\)-coCartesian fibration.

To simplify the notation we give the remainder of the proof in the case where \(S\) is a singleton; the general case is proved by the same argument. Consider an active morphism \(\phi: (A_b)_{b \in B} \rightarrow \Theta(T)\) in \(\mathcal{O}p\). This corresponds to giving, for every vertex \(v\) of \(T\), a tree \(S_v\) whose leaves are identified with the incoming edges of \(v\). We can view this as specifying active maps \(S_v \rightarrow C_n\) in \(\Omega^\text{op}\) where \(T \rightarrow C_n\) is the inert map corresponding to the vertex \(v\), and then define a new tree \(T'\) with an active map \(T' \rightarrow T\) in \(\Omega^\text{op}\) by iterating the construction of Remark 5.1.2. In other words, the tree \(T'\) is (slightly informally) given by \(T \sqcup \bigsqcup v S_v\). It is easily verified using the definition of morphisms and composition in \(\mathcal{O}p\) that the map \(T' \rightarrow T\) is \(\Theta\)-Cartesian over \(\phi\). \(\Box\)

Applying [38, Theorem 2.3.3.23], this implies:

**Corollary 5.1.9.** If \(V\) is a symmetric monoidal \(\infty\)-category, then the functor \(\Theta_S^*: \text{Alg}_{\mathcal{O}p_S}(V) \rightarrow \text{Alg}_{\Omega^\text{op}_S}(V)\), induced by composition with \(\Theta_S\), is an equivalence.

**Definition 5.1.10.** Let \(\text{Alg}_{\mathcal{O}p/\text{Set}}(V) \rightarrow \text{Set}\) denote the Cartesian fibration corresponding to the functor \(\text{Set}^\text{op} \rightarrow \text{Cat}_\infty\) taking \(S\) to \(\text{Alg}_{\mathcal{O}p_S}(V)\), and let \(\text{Alg}_{\Omega^\text{op}/\text{Set}}(V) \rightarrow \text{Set}\) denote the pullback of the Cartesian fibration \(\text{Alg}_{\Omega^\text{op}/S}(V) \rightarrow S\) along the inclusion \(\text{Set} \hookrightarrow S\). Since the functors \(\Theta_S\) are natural in \(S\), they induce a functor

\[
\Theta^*: \text{Alg}_{\mathcal{O}p/\text{Set}}(V) \rightarrow \text{Alg}_{\Omega^\text{op}/\text{Set}}(V)
\]

over \(\text{Set}\) that preserves Cartesian morphisms.

**Corollary 5.1.11.** The functor

\[
\Theta^*: \text{Alg}_{\mathcal{O}p/\text{Set}}(V) \rightarrow \text{Alg}_{\Omega^\text{op}/\text{Set}}(V)
\]
is an equivalence.

Proof. Since this is a functor between Cartesian fibrations that preserves Cartesian morphisms, it is an equivalence as it is an equivalence on fibres over every \( S \in \text{Set} \) by Corollary 5.1.9.

**Proposition 5.1.12.** If \( \mathcal{V} \) is a presentably symmetric monoidal \( \infty \)-category, then the inclusion \( \text{Alg}_{\mathcal{G}^w/\text{Set}(\mathcal{V})} \hookrightarrow \text{Alg}_{\mathcal{G}^w/\mathcal{S}(\mathcal{V})} \) induces an equivalence

\[
\text{Alg}_{\mathcal{G}^w/\text{Set}(\mathcal{V})}[\text{FFES}^{-1}] \sim \to \text{Alg}_{\mathcal{G}^w/\mathcal{S}(\mathcal{V})}[\text{FFES}^{-1}],
\]

where FFES denotes the class of fully faithful and essentially surjective morphisms on both sides.

**Proof.** As [17, Theorem 5.3.17].

**Corollary 5.1.13.** There is an equivalence of \( \infty \)-categories

\[
\text{Alg}_{\mathcal{G}_{\text{op}}/\text{Set}(\mathcal{V})}[\text{FFES}^{-1}] \simeq \text{Opd}_{\mathcal{V}}^\Sigma.
\]

**Proof.** Combine the equivalences of Corollary 5.1.11, Proposition 5.1.12, Theorem 4.2.11, Theorem 4.4.3, Proposition 4.3.2, and Theorem 3.5.7.

5.2. **Rectification.** In this subsection we will prove that for a suitable monoidal model category \( \mathcal{V} \), the \( \infty \)-category \( \text{Opd}_{\mathcal{V}}[W^{-1}] \) of \( \infty \)-operads enriched in the symmetric monoidal \( \infty \)-category \( \mathcal{V}[W^{-1}] \), obtained by inverting the weak equivalences \( W \) in \( \mathcal{V} \), is equivalent to the \( \infty \)-category obtained from operads strictly enriched in \( \mathcal{V} \).

**Definition 5.2.1.** Let \( \mathcal{V} \) be a symmetric monoidal model category. An operad \( \mathcal{O} \) is called **admissible** for \( \mathcal{V} \) if there is a model structure on \( \text{Alg}_\mathcal{O}(\mathcal{V}) \) such that the weak equivalences and fibrations are those maps whose underlying maps in \( \mathcal{V} \) are weak equivalences and fibrations, respectively.

**Proposition 5.2.2** (Pavlov–Scholbach). Suppose the operad \( \mathcal{O}_S \) is admissible for \( \mathcal{V} \) for a set \( S \), and let \( W_S \) denote the class of weak equivalences in \( \text{Alg}_{\mathcal{O}_S}(\mathcal{V}) \). Then the natural functor of \( \infty \)-categories

\[
\text{Alg}_{\mathcal{O}_S}(\mathcal{V})[W_S^{-1}] \rightarrow \text{Alg}_{\mathcal{O}_S}(\mathcal{V}[W^{-1}])
\]

is an equivalence, where \( \mathcal{V}[W^{-1}] \) denotes the symmetric monoidal \( \infty \)-category induced by \( \mathcal{V} \), with \( W \) being the class of weak equivalences in \( \mathcal{V} \).

**Proof.** The operad \( \mathcal{O}_S \) is \( \Sigma \)-cofibrant (which here just means that the \( \Sigma_n \)-actions are all free), so this is a special case of [42, Theorem 7.10]. (As stated, this result requires a simplicial model category, but since our operad \( \mathcal{O}_S \) is merely an operad in sets the same proof goes through without this assumption.)

**Examples 5.2.3.** Proposition 5.2.2 applies to the following model categories:

(i) the category \( \text{Set}_\Delta \) of simplicial sets, equipped with the Kan–Quillen model structure,

(ii) the category \( \text{Top} \) of compactly generated weak Hausdorff spaces, equipped with the usual model structure,

(iii) the category \( \text{Ch}_k \) of chain complexes of \( k \)-vector, spaces, where \( k \) is a field of characteristic 0 (or more generally a ring containing \( \mathbb{Q} \)), equipped with the projective model structure,

(iv) the category \( \text{Sp}^{\Sigma} \) of symmetric spectra, equipped with the positive stable model structure.

These are the standard examples of model categories for which all operads (and so in particular the operads \( \mathcal{O}_S \) for all \( S \)) are admissible, as discussed in [43, §7]. Further examples of such model categories include orthogonal spectra [34], orthogonal \( G \)-spectra, stable module categories, and the “folk” model structure on categories [50]. Unfortunately, we are not aware of any examples of symmetric monoidal model categories for which the operads \( \mathcal{O}_S \) are admissible other than those for which all operads are admissible.

**Remark 5.2.4.** There is a substantial literature on model structures for operads enriched in a model category (monochromatic or with a fixed set of objects), and more generally for algebras over a fixed operad in a model category. We list some key results of this kind:
• Hinich [28, 29] constructed a model structure for monochromatic operads in chain complexes over a ring containing $\mathbb{Q}$.
• Berger–Moerdijk [2] constructed a model structure on reduced monochromatic operads (i.e. ones with no nullary operations) in suitable model categories with a commutative Hopf interval, including simplicial sets, topological spaces, and chain complexes over a ring. They later extended this result to algebras for coloured operads in $[3]$, giving model structures for operads with any fixed set of colours as a special case.
• Kro [35] extended the work of Berger–Moerdijk to get a model structure for monochromatic reduced operads in orthogonal spectra.
• Elmendorf–Mandell [13] constructed a model structure on algebras over any simplicial operad in symmetric spectra (in Top); this was used by Gutiérrez–Vogt [23] to obtain model structures on operads in symmetric spectra with a fixed set of colours.
• Most recently, Pavlov–Scholbach [42] have studied general assumptions on a model category under which all operads are admissible, and applied this to symmetric spectra in general model categories in [44].

Remark 5.2.5. It follows from the results of [41] that for any presentably symmetric monoidal $\infty$-category $V$ there exists a symmetric monoidal simplicial combinatorial model category modelling $V$ for which all (simplicial) operads are admissible.

Remark 5.2.6. Since the operads $\text{Op}_S$ are $\Sigma$-cofibrant, work of Spitzweck [48] shows that for any cofibrantly generated symmetric monoidal model category $V$, the category $\text{Alg}_{\text{Op}_S}(V)$ has a semi-model structure. We strongly suspect that this is sufficient for Proposition 5.2.2 to hold. However, for the proof to go through for semi-model categories one would have to extend most of Lurie’s results relating model categories to $\infty$-categories, such as the connection between homotopy colimits in model categories and colimits in the associated $\infty$-category, to the semi-model case.

Corollary 5.2.7. Suppose $V$ is a symmetric monoidal model category for which the operads $\text{Op}_S$ are admissible for all sets $S$. Then there is an equivalence

$$\text{Opd}^V[W^{-1}] \xrightarrow{\sim} \text{Alg}_{\text{Op/}S}(V[W^{-1}])$$

over $\text{Set}$, where $W$ denotes the morphisms that are bijective on objects and given by weak equivalences on all multimorphism objects.

Proof. The forgetful functor $\text{Opd}^V \to \text{Set}$ taking an operad in $V$ to its set of colours is a Grothendieck fibration (and opfibration) with fibre $\text{Alg}_{\text{Op}_S}(V)$. Applying [25, Corollary 4.22] or [30, Proposition 2.1.4] it follows that $\text{Opd}^V[W^{-1}] \to \text{Set}$ is the Cartesian (and coCartesian) fibration corresponding to the functor taking $S$ to $\text{Alg}_{\text{Op}_S}(V)[W^{-1}]$. Thus the functor

$$\text{Opd}^V[W^{-1}] \to \text{Alg}_{\text{Op/}S}(V[W^{-1}])$$

over $\text{Set}$ is a functor between Cartesian fibrations that preserves Cartesian morphisms. It is therefore an equivalence as it is an equivalence on fibres at each $S$ by Proposition 5.2.2. □

Remark 5.2.8. In the situation above there is a model structure on $\text{Opd}^V$ where the morphisms in $W$ are the weak equivalences by [25, Proposition 4.25] or [24, Theorem 3.0.12].

Definition 5.2.9. If $V$ is a symmetric monoidal model category, we say a morphism $F: O \to O'$ of $V$-enriched operads is a Dwyer–Kan equivalence if:

1. The map $O(x_1, \ldots, x_n, y) \to O'(F(x_1), \ldots, F(x_n); F(y))$ is a weak equivalence in $V$ for all $x_1, \ldots, x_n, y$ in $O$.
2. The functor $V \to hV$ to the homotopy category is symmetric monoidal, so to a $V$-enriched operad $O$ we can associate an $hV$-enriched operad $hO$. The induced functor of $hV$-enriched operads $hF: hO \to hO'$ is essentially surjective (i.e. its underlying functor of enriched categories is essentially surjective).
Theorem 5.2.10. Suppose $V$ is a symmetric monoidal model category for which the operads $Op_S$ are admissible for all sets $S$. Then there is an equivalence

$$\text{Opd}_V^{[DK^{-1}]} \simeq \text{Opd}_\infty^{[W^{-1}]}$$

where $DK$ denotes the class of Dwyer–Kan equivalences.

Proof. The class of Dwyer–Kan equivalences clearly corresponds under the equivalence of Corollary 5.2.7 to the class of fully faithful and essentially surjective morphisms in $\text{Alg}_{Op/\text{Set}}(V^{[W^{-1}]})$, so we get an equivalence

$$\text{Opd}_V^{[DK^{-1}]} \simeq \text{Alg}_{Op/\text{Set}}(V^{[W^{-1}]})^{[FFES^{-1}]}.$$

The result now follows by combining this with the equivalence of Corollary 5.1.13. □

Remark 5.2.11. By Example 5.2.3, we can apply Theorem 5.2.10 to get the following comparisons:
- The homotopy theory of simplicial operads is equivalent to that of $\infty$-operads; this was already shown by Cisinski–Moerdijk [12].
- The homotopy theory of spectral operads, or more precisely operads enriched in symmetric spectra, is equivalent to that of spectral $\infty$-operads, i.e. $\infty$-operads enriched in the $\infty$-category of spectra.
- If $k$ is a ring containing $\mathbb{Q}$, then the homotopy theory of dg-operads over $k$, i.e. operads enriched in chain complexes of $k$-modules, is equivalent to that of $\infty$-operads enriched in the derived $\infty$-category of $k$.

Remark 5.2.12. In good cases, there is a model structure on $\text{Opd}_V$ with the Dwyer–Kan equivalences as weak equivalences. Such model structures were constructed by Cisinski–Moerdijk [12] and Robertson [46] for simplicial operads, and by Caviglia [5] for a general class of model categories that also includes topological spaces and chain complexes over a field of characteristic 0. In unpublished work [6], Caviglia has furthermore extended this result so that it also applies to symmetric spectra.

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