Towards $\mathcal{M}$-Adhesive Categories based on Coalgebras and Comma Categories

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Abstract. In this contribution we investigate several extensions of the powerset that comprise arbitrarily nested subsets, and call them superpower set. This allows the definition of graphs with possibly infinitely nested nodes. Additionally we define edges that are incident to edges. Since we use coalgebraic constructions we refer to these graphs as coalgebraic graphs. The superpower set functors are examined and then used for the definition of $\mathcal{M}$-adhesive categories which are the basic categories for $\mathcal{M}$-adhesive transformation systems. So, we additionally show that coalgebras $\text{Sets}_F$ are $\mathcal{M}$-adhesive categories provided the functor $F: \text{Sets} \to \text{Sets}$ preserves pullbacks along monomorphisms.

Keywords: graph, hierarchy, coalgebra, $\mathcal{M}$-adhesive transformation system
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# 1 Motivation

The main motivation of this paper is the question how to define recursion on a graph’s structure so that we still obtain an $\mathcal{M}$-adhesive transformation systems. Since the recursion construct we use in this contribution is a coalgebraic construction we consequently use the term coalgebraic graphs.

A shorter version can be found in ?. We start with a examples of such graphs to illustrate what we aim at.

*Example 1 (Coalgebraic graphs)*. The coalgebraic graph $G_1 = (N_1, E_1, con_1, nbg_1)$ is given in Fig. 1a and consists of a set of nodes $N$, a set of edges $E$, a contains function $con$ and a neighbour function $nbg$. $con$ yields the set of nodes that for
each node may contain. Those nodes that are mapped to themselves, are considered to be atomic.

Let \( N_1 = \{n_1, n_2, n_3, n_4, n_5, n_6\} \) with:

\[
\text{con}_1(n_i) = \begin{cases} 
  n_i & ; 1 \leq i \leq 3 \\
  \{n_1, n_2\} & ; i = 4 \\
  \{n_3\} & ; i = 5 \\
  \{n_2, \{n_2, n_3\}, n_5\} & ; i = 6 
\end{cases}
\]

The atomic nodes are \( N_1 = \{n_1, n_2, n_3\} \) and \( \text{con}(N_1) = \{n_1, n_2, n_3, \{n_1, n_2\}, \{n_3\}, \{n_2, \{n_1, n_2\}, n_5\}\} \).

We have \( n_1, n_2 \in \text{con}(n_4) \) and \( \{n_1, n_2\} \in \text{con}(n_6) \).

![Diagram of nested nodes and edges](image)

Fig. 1: Examples of graphs with recursive structures

\( E_1 = \{a, b, c\} \) with \( \text{ngb}_1 : a \mapsto \{n_1, n_3\} \)
\( b \mapsto \{n_2, n_5, n_6\} \)
\( c \mapsto \{n_5\} \)

Edge \( b \) is a hyperedge.

In Fig. 1b the coalgebraic graph \( G_2 = (N_2, E_2, \text{ngb}_2) \) with atomic nodes \( N_2 = \{a, b, c, d\} \) and edges \( E_2 = \{x_1, x_2, x_3, x_4\} \) with \( \text{ngb}_2 : x_1 \mapsto \{a, b, c\} \)
\( x_2 \mapsto \{a, b\} \)
\( x_3 \mapsto \{x_2, d\} \)
\( x_4 \mapsto \{a, x_4\} \)
The edge \( x_1 \) is also a hyperedge. The edge \( x_3 \) is attached to edge \( x_2 \) with \( x_2 \in \text{ngb}(x_3) \) and \( x_4 \) is an unary edge that is attached to itself and is denoted by its end \( n_2 \xrightarrow{x_4} \).

\( G_2 \) can be flattened to a hypergraph with the attachment function \( \text{att} := \text{ngb}_2^+ : E \rightarrow \mathcal{P}(V) \) and \( \text{ngb}_2^+ : x_1 \mapsto \{a, b, c\} \)
\begin{align*}
x_2 &\mapsto \{a, b\} \\
x_3 &\mapsto \{a, b, d\} \\
x_4 &\mapsto \{a\}
\end{align*}

\( G_3 \) has an edge, that contains a subgraph. In Fig. 1c the coalgebraic graph \( G_3 = (N_3, E_3, \text{ngb}_2) \) with atomic nodes \( N_2 = \{a, b, c, d, e\} \) and edges \( E_2 = \{x_1, x_2, x_3\} \) with \( \text{ngb}_2 : x_1 \mapsto (abc, \{d, e, x_2, x_3\}) \)
\begin{align*}
x_2 &\mapsto (ed, \emptyset) \\
x_3 &\mapsto (de, \emptyset)
\end{align*}

The concepts investigated in this contribution shall comprise
- usual (un-)directed multi-graphs,
- classic hypergraphs,
- graphs with hyperedges as in DKH97,
- bigraphs Mil06 see Sect. 6.3,
- and various hierarchical graphs see Sect. 6.1.

### 2 Super Power Sets

The superpower set is achieved by recursively inserting subsets of the superpower set into itself. In this contribution we present three possibilities:

1. \( P \) allows atomic nodes.
2. \( \Psi \) only allows sets of nodes.
3. \( P^\omega \) layers the nesting of nodes.

Subsequently, we investigate the properties of each construction and in Subsect. 2.4 we discuss the differences.

#### 2.1 Node Recursion based on \( P \)

**Definition 1 (Superpower set \( P \)).** Given a well-founded set \( M \) and \( \mathcal{P}(M) \) the power set of \( M \) then we define the superpower set \( \mathbb{P}(M) \)
\begin{enumerate}
    \item \( M \subseteq \mathbb{P}(M) \) and \( \mathcal{P}(M) \subseteq \mathbb{P}(M) \)
    \item If \( M' \subseteq \mathbb{P}(M) \) then \( M' \in \mathbb{P}(M) \).
\end{enumerate}

\( \mathbb{P}(M) \) is the smallest set satisfying 1. and 2.

Note that this superpower set construction \( \mathbb{P} \) is well-founded sets with the ordering with respect to the number of parentheses (see Appendix A.2).
Lemma 1 (P is a functor). \( P : \text{Sets} \to \text{Sets} \) is defined for finite sets as in Def. 1 and for functions \( f : M \to N \) by \( f^* : P(M) \to P(N) \) with
\[
f^*(x) = \begin{cases} f(x) & ; x \in M \\ \{ f^*(x') \mid x' \in x \} & ; \text{else} \end{cases}
\]

We use \( f^* \) instead of \( P(f) \) merely for better readability and to stress the similarities of \( P, \mathcal{P} \) and \( \mathcal{P}^\omega \).

Example 2 (Functor \( P \)). Given sets \( M = \{ u, v, w, u', v' \} \) and \( N = \{ n_1, n_2, n_3, n_4, n_5, n_6 \} \) with \( f : M \to N \) and \( f : w \mapsto n_3 ; v \mapsto n_3 ; w \mapsto n_1 \\
u' \mapsto n_5 ; v' \mapsto n_5 ; \)
then we have \( f^* : P(M) \to P(N) \) mapping each subset element by element and for example
\[
f^*(\{ u, v, w, \{ u, v \}, \{ v, \{ w, \emptyset \} \} \}) = \{ n_3, n_1, \{ n_3 \}, \{ n_3, \{ n_1, \emptyset \} \} \}
\]

Lemma 2 (\( P \) preserves injections). Given injective function \( f : M \to N \) then \( f^* : P(M) \to P(N) \) is injective.

Proof. By induction\(^1\) over the depth of the superpower sets, so \( n \) is the number of nested parentheses:

IA (\( n = 0 \)) atomic nodes Given \( x_1, x_2 \in P(M) \) with \( x_1 \neq x_2 \).
Since \( f \) injective, \( f^*(x_1) = f(x_1) \neq f(x_2) = f^*(x_2) \).

IB Let \( f^* : P(M) \to P(N) \) be injective for all sets with at most \( n \) nested parentheses.

IS Given \( M_1, M_2 \in P(M) \) with \( M_1 \neq M_2 \) having both \( n+1 \) nested parentheses.
Let \( x \in M_1 \land x \notin M_2 \).
Hence \( f^*(x) \notin f^*(M_1) \).
\( x \notin M_2 \) implies for all \( m \in M_2 \) that \( x \neq m \).
\( x \) and \( m \) have at most \( n \) nested parentheses.
\( f^*(x) \neq f^*(m) \) for all \( m \in M_2 \) as \( f^* \) is injective for all sets with at most \( n \) nested parentheses.
Thus \( f^*(x) \notin f^*(M_2) \)
So, \( P(M_1) \neq P(M_2) \).

Lemma 3 (\( P \) preserves pullbacks along injective morphisms).

Proof. Given a pullback diagram (\( PB \)) and the diagram (1) in \( \text{Sets} \) with \( g_1 : C \hookrightarrow D \) injective.
Pullbacks and the super powerset functor (see Lemma 2) preserve injections, so \( \pi_B : A \hookrightarrow B, P(\pi_B) : P(A) \hookrightarrow P(B) \) and \( \pi_{P(B)} : P \hookrightarrow P(B) \) are injective.

\(^1\) To be precise this is an Noetherian induction see Appendix A.2
Since \((PB)\) is a pullback diagram we have \(A = \{(b, c) \mid f_1(b) = g_1(c)\}\).

(1) commutes, since \(P\) is a functor.

Let \(P\) be the pullback of \((P(D), f_1^*, g_1^*), \) so \(f_1^* \circ \pi_{P(B)} = \pi_{P(C)} \circ g_1^*\).
Hence, \(P = \{(B', C') \mid f_1^*(B') = g_1^*(C')\} \subseteq P(B) \times P(C)\).
Moreover, there is the unique \(h : P(A) \to P\) s.t. \(h(A') = (\pi_B^*(A'), f\pi_C^*(A'))\)
for all \(A' \subseteq A\) so that the diagrams (2) and (3) commute along \(h\):
\[ \pi_B^* \circ h = \pi_B^* \quad \text{and} \quad \pi_C^* \circ h = \pi_C^*. \]

We define \(\tilde{h} : P \to P(A)\) with
\[
\tilde{h}((X, Y)) = \begin{cases} 
(b, c) ; \text{if } X = b \in B, Y = c \in C \\
(x, y) ; \text{if } x \in X \cap B, y \in Y \cap C, f_1(x) = g_1(y) \\
\quad \quad \cup \bigcup_{(X', Y') \in (X-B) \times (Y-C)} \tilde{h}(X', Y') \quad \text{else}
\end{cases}
\]

and have:

1. \(\tilde{h}\) is well-defined since \(\tilde{h}(X, Y) \in P(A)\).

2. (2) commutes along \(\tilde{h}\), i.e. \(\pi_B^* \circ \tilde{h} = \pi_{P(B)}(X, Y)\) by induction over the number of nested parentheses \(n\):

\textbf{IA} \(n = 0\), i.e. atomic nodes: Given \((b, c) \in P\) with \(b \in B\) and \(c \in C\).
\[ \pi_B^* \circ \tilde{h}(b, c) = \pi_B^* (b, c) = b = \pi_{P(B)} (b, c) \]

\textbf{IB} Let be \(\pi_B^* \circ \tilde{h}(X, Y) = \pi_{P(B)}(X, Y)\) for sets with at most \(n\) nested parentheses.

\textbf{IS} Given \((\hat{X}, \hat{Y}) \in P\) with \(n + 1\) nested parentheses.
Let \(\hat{X} = B \cup X\) with \(B \subseteq B\) and \(X \cap B = \emptyset\).
Let \(\hat{Y} = C \cup Y\) with \(C \subseteq C\) and \(Y \cap C = \emptyset\).
\(X\) and \(Y\) have at most \(n\) nested parentheses.

\[
\pi_B^* \circ \bar{h}(\hat{X}, \hat{Y}) = \pi_B^* \circ \left( \{(x, y) \mid x \in \hat{B}, y \in \hat{C}, f_1(x) = g_1(y)\} \cup \bigcup_{(X', Y') \in (X \times Y)} \bar{h}(X', Y') \right)
\]

\[
= \{x \mid x \in \hat{B} \land \exists y \in \hat{C} : f_1(x) = g_1(y)\} \cup \bigcup_{(X', Y') \in (X \times Y)} \pi_B^* \circ \bar{h}(X', Y')
\]

\[
= \hat{B} \cup \bigcup_{(X', Y') \in (X \times Y)} \pi_B^* \circ \bar{h}(X', Y')
\]

\[
= \hat{B} \cup \bigcup_{(X', Y') \in (X \times Y)} \pi_{P(B)}(X', Y')
\]

\[
= \hat{B} \cup X
\]

\[
= \hat{X}
\]

\[
= \pi_P(\hat{X}, \hat{Y})
\]

Now we show \(P \cong P(A)\):

- \(h \circ \bar{h} = id_P\),
  - since \(\pi_P(B)\) is injective and \(\pi_P(B) \circ h \circ \bar{h} = \pi_B^* \circ \bar{h} = \pi_P(B) \circ id_P\)
- \(\bar{h} \circ h = id_P\),
  - since \(\pi_B^*\) is injective and \(\pi_B^* \circ \bar{h} \circ h = \pi_P(B) \circ h = \pi_B^* = \pi_B^* \circ id_P\).

Based on \(F\)-graphs (see Jä15, Sch99), that is a family of graph categories induced by a comma category construction using a functor \(F\), we can define the category of \(P\)-graphs.

**Example 3 (\(P\)-Graph).** In Fig. 2 the following \(P\)-graph are illustrated and a morphism in between.

\(G_1 = (ngb_1 : E_1 \to P(N_1))\) \quad \(G_2 = (ngb_2 : E_2 \to P(N_2))\)

with \(ngb_1 : x \mapsto \{u\}, \{v\}\) \quad with \(ngb_2 : a \mapsto \{n_1, n_2\}, \{n_1, n_2\}, \{n_3\}, \{n_3\}\)

\(y \mapsto \{u, w\}\) \quad \(b \mapsto \{n_1, n_2\}, \{n_3\}\)

\(z \mapsto \{u, w\}\) \quad \(c \mapsto \{n_3\}\)

Note, that we only have the nesting of nodes, but the nodes that contain others do not have a name themselves. Edges are hyperedges given as a subset of the superpower set, but they may have incident nodes as well as nodes containing nodes.

**Definition 2 (\(P\)-graph and the category of \(P\)-graphs).** The category of \(P\)-graphs \(P \_Graph\) is given by a comma category \(P \_Graph =< Id_{Sets} \downarrow P >\).

\(P\)-graph morphisms are given by mappings of the nodes and edges \(f = (f_N, f_E) : G_1 \to G_2\) with \(f_N : N_1 \to N_2\) and \(f_E : E_1 \to E_2\) so that:
2.2 Node Recursion Based on $\Psi$

**Definition 3 (Superpower set $\Psi$).** Given a well-founded set $M$ and $\mathcal{P}(M)$ the power set of $M$ then we define the superpower set $\Psi(M)$

1. $\mathcal{P}(M) \subset \Psi(M)$
2. If $M' \subset \Psi(M)$ then $M' \in \Psi(M)$.

$\Psi(M)$ is the smallest set satisfying 1. and 2.

Note that this superpower set construction $\Psi$ is well-founded sets with the ordering with respect to the number of parentheses (see Appendix A.2).

The difference between $\mathcal{P}$ and $\Psi$ is that in $\mathcal{P}$ the elements of the underlying set are elements of the superpowerset as well, so $M \subset \mathcal{P}(M)$ but $M \not\subset \Psi$ for an non-empty set $M$.

**Lemma 4 ($\Psi$ is a functor).** $\Psi : \text{Sets} \to \text{Sets}$ is defined for sets as in Def. 3 and for functions $f : M \to N$ by $f^* : \Psi(M) \to \Psi(N)$ with

$$f^*(\{x\}) = \begin{cases} \{f(x)\} & ; x \in M \\ \{f^*(x') | x' \in x\} & ; \text{else} \end{cases}$$

**Example 4 (Functor $\Psi$).** Given sets $M = \{u, v, w, u', v'\}$ and $N = \{n_1, n_2, n_3, n_4, n_5, n_6\}$ with $f : M \to N$ and $f : w \mapsto n_3 ; v \mapsto n_3 ; w \mapsto n_1 ; u' \mapsto n_5 ; v' \mapsto n_5$

then we have $f^* : \Psi(M) \to \Psi(N)$ with for example

$$f^*(\{u, v\}, \{v, \{w, \emptyset\}\}) = \{\{n_3\}, \{n_3, \{n_1, \emptyset\}\}\}$$

**Lemma 5 ($\Psi$ preserves injections).** Given an injective function $f : M \to N$ then $f^* : \Psi(M) \to \Psi(N)$ is injective.
Proof. By induction\(^2\) over the depth of the superpower sets, so \(n\) is the number of nested parentheses:

**IA** (\(n = 1\)) i.e. **sets** Given \(M_1, M_2 \subseteq M\) with \(M_1 \neq M_2\).
Let \(x \in M_1 \land x \notin M_2\). Hence \(f(x) \in f(M_1) \land f(x) \notin f(M_2)\) as \(f\) is injective. So, \(\mathcal{P}(M_1) \neq \mathcal{P}(M_2)\).

**IB** Let \(f^* : \mathcal{P}(M) \to \mathcal{P}(N)\) be injective for all sets with at most \(n\) nested parentheses.

**IS** Given \(M_1, M_2 \in \mathcal{P}(M)\) with \(M_1 \neq M_2\) having both \(n + 1\) nested parentheses.

Given \(x \in M_1 \land x \notin M_2\).
Hence \(f^*(x) \in f^*(M_1)\).
\(x \notin M_2\) implies for all \(m \in M_2\) that \(x \neq m\).
\(x\) and \(m\) have at most \(n\) nested parentheses.
\(f^*(x) \neq f^*(m)\) for all \(m \in M_2\) as \(f^*\) is injective for all sets with at most \(n\) nested parentheses.
Thus \(f^*(x) \neq f^*(M_2)\)
So, \(\mathcal{P}(M_1) \neq \mathcal{P}(M_2)\).

**Lemma 6** \((\mathcal{P}\) preserves pullbacks along injective morphisms).

Proof. Given a pullback diagram \((PB)\) and the diagram (1) in **Sets** with \(g_1 : C \hookrightarrow D\) injective.
Pullbacks and the superpower set functor (see Lemma 5) preserve injections, so \(\pi_B : A \hookrightarrow B, \pi_B(\pi_B) : \mathcal{P}(A) \hookrightarrow \mathcal{P}(B)\) and \(\pi_B(\pi_B) : P \hookrightarrow \mathcal{P}(B)\) are injective.

Since \((PB)\) is a pullback diagram we have \(A = \{(b, c) \mid f_1(b) = g_1(c)\}\).
(1) commutes, since \(\mathcal{P}\) is a functor.

Let \(P\) be the pullback of \((\mathcal{P}(D), f_1^*, g_1^*)\), so \(f_1^* \circ \pi_B = \pi_B^* \circ g_1^*.\)

Hence, \(P = \{(B', C') \mid f_1^*(B') = g_1^*(C')\} \subseteq \mathcal{P}(B) \times \mathcal{P}(C).\)

Moreover, there is the unique \(h : \mathcal{P}(A) \to P\) s.t. \(h(A') = (\pi_B(A'), f_1^*(A'))\) for all \(A' \subseteq A\) so that the diagrams (2) and (3) commute along \(h:\)

\[\pi_B \circ h = \pi_B^* \text{ and } \pi_B(\pi_B) \circ h = \pi_B^*\]

We define \(h : P \to \mathcal{P}(A)\) with
\[
h(X, Y) = \{(x, y) \mid x \in X \cap B, y \in Y \cap C, f_1(x) = g_1(y)\} \cup \bigcup_{(X', Y') \in (X - B) \times (Y - C)} \bar{h}(X', Y')
\]

\(^2\) To be precise this is an Noetherian induction see Appendix A.2
and have:

1. $\bar{h}$ is well-defined since $\bar{h}(X, Y) \in \mathcal{P}(A)$.

2. (2) commutes along $\bar{h}$, i.e. $\pi_B^* \circ \bar{h} = \pi_{\mathcal{P}(B)}(X, Y)$ by induction over the number of nested parentheses $n$:

IA $(n = 1, \text{ i.e. sets}): \pi_B^* \circ \bar{h}(\emptyset, \emptyset) = \pi_B^*(\emptyset) = \emptyset = \pi_{\mathcal{P}(B)}(\emptyset, \emptyset)$

For $X \subseteq B$ and $Y \subseteq C$

\[
\pi_B^* \circ \bar{h}(X, Y) = \pi_{\mathcal{P}(B)}(X, Y)
\]

IB Let be $\pi_B^* \circ \bar{h}(X, Y) = \pi_{\mathcal{P}(B)}(X, Y)$ for sets with at most $n$ nested parentheses.

IS Given $(\hat{X}, \hat{Y}) \in P$ with $n + 1$ nested parentheses.

Let $\hat{X} = \hat{B} \cup X$ with $\hat{B} \subseteq B$ and $X \cap B = \emptyset$.

Let $\hat{Y} = C \cup Y$ with $C \subseteq C$ and $Y \cap C = \emptyset$.

$X$ and $Y$ have at most $n$ nested parentheses.

\[
\pi_B^* \circ \bar{h}(\hat{X}, \hat{Y}) = \pi_{\mathcal{P}(B)}(X, Y)
\]

Now we show $P \cong \mathcal{P}(A)$:

- $h \circ \bar{h} = id_P$, since $\pi_{\mathcal{P}(B)}$ is injective and $\pi_{\mathcal{P}(B)} \circ h \circ \bar{h} = \pi_B^* \circ \bar{h} = \pi_{\mathcal{P}(B)} \circ id_P$

- $h \circ \bar{h} = id_{\mathcal{P}(A)}$, since $\pi_B^*$ is injective and $\pi_B^* \circ h \circ \bar{h} = \pi_{\mathcal{P}(B)} \circ \bar{h} = \pi_B^* = \pi_B^* \circ id_{\mathcal{P}(A)}$. 
Based on $F$-graphs (see Jä15, Sch99), that is a family of graph categories induced by a comma category construction using a functor $F$, we can define the category of coalgebraic $F$-graphs.

**Example 5 ($\mathcal{P}$-Graph).** In Fig. 3 the following $\mathcal{P}$-graphs are illustrated and a morphism in between.

\[
G_3 = \langle \text{ngb}_3 : E_3 \rightarrow \mathcal{P}(N_3) \rangle \quad \text{with } c_3 : x \mapsto \{\{u\}, \{v\}\}
\]
\[y \mapsto \{\{u\}, \{w\}\} \quad z \mapsto \{\{u\}, \{w\}\}\]

\[
G_4 = \langle \text{ngb}_4 : E_4 \rightarrow \mathcal{P}(N_4) \rangle \quad \text{with } \text{ngb}_4 : a \mapsto \{\{n_1\}, \{n_4\}\}
\]
\[b \mapsto \{\{n_1, n_4\}, \{n_3\}, \{n_3\}\} \quad c \mapsto \{\{n_3\}\}\]

Note, that we only have the recursion of nodes, but the nodes that contain others do not have a name themselves. Edges are hyperedges given as a subset of the superpower set, but they only have neighbours that are nodes containing nodes. They cannot have incident vertices.

**Definition 4 ($\mathcal{P}$-graphs and the category $\mathcal{P}$-Graph).** The category of coalgebraic graphs $\mathcal{P}$-Graph is given by a comma category $\mathcal{P}$-Graph $= \langle \text{Id}_{\text{Sets}} \downarrow \mathcal{P} \rangle$.

Morphisms are given by mappings of the nodes and edges $f = (f_N, f_E) : G_1 \rightarrow G_2$ with $f_N : N_1 \rightarrow N_2$ and $f_E : E_1 \rightarrow E_2$ so that:
\[
- f_N^* \circ \text{con}_1 = \text{con}_2 \circ f_N \\
- f_E^* \circ \text{ngb}_1 = \text{ngb}_2 \circ f_E
\]

**2.3 Node recursion based on $\mathcal{P}^\omega$**

**Definition 5 (Superpower set $\mathcal{P}^\omega$).** Given a set well-founded $M$ we define $\mathcal{P}^0(M) = M$ and $\mathcal{P}^1(M) = \mathcal{P}(M)$ the power set of $M$. Then $\mathcal{P}^{i+1}(M) = \mathcal{P}(\mathcal{P}^i(M))$.
\[ \mathcal{P}^\omega(M) = \bigcup_{i \in \mathbb{N}_0} \mathcal{P}^i(M) \]

Note that this superpower set construction \( \mathcal{P}^\omega \) is well-founded sets with the ordering with respect to the number of parentheses (see Appendix A.2).

This construction differs from the other notions, as in each subset there are only subsets that the same depth in terms of nesting. So, for some non-empty set \( M \) with \( m \in M \), we have \( \{m, M\} \notin \mathcal{P}^\omega(M) \) but \( \{m, M\} \in \mathcal{P}(M) \) and \( \{m, M\} \in \mathcal{P}(M) \).

**Lemma 7 (\( \mathcal{P}^\omega \) is a functor).** \( \mathcal{P}^\omega : \text{Sets} \to \text{Sets} \) is defined for sets as in Def. 5 and for functions \( f : M \to N \) by \( f^\ast : \mathcal{P}^\omega(M) \to \mathcal{P}^\omega(N) \) with

\[
\begin{align*}
\{ f(x) \} ; x \in M \\
\{ f^\ast(x') \mid x' \in x \} ; \text{ else}
\end{align*}
\]

**Example 6 (Functor \( \mathcal{P}^\ast \)).** Given sets \( M = \{u, v, w, u', v'\} \) and \( N = \{n_1, n_2, n_3, n_4, n_5, n_6\} \) with \( f : M \to N \) and \( f : w \mapsto n_3 ; v \mapsto n_3 ; w \mapsto n_1 \)
\( w' \mapsto n_5 ; v' \mapsto n_5 \);

then we have \( f^\ast : \mathcal{P}^\omega(M) \to \mathcal{P}^\omega(N) \) with for example

\[ f^\ast(\{\{u, v\}, \{\emptyset, \{w, \emptyset\}\}\}) = \{\{n_3\}, \{\emptyset, \{n_1\}\}\} \]

**Lemma 8 (\( \mathcal{P}^\omega \) preserves injections).** Given injective function \( f : M \to N \) then \( f^\ast : \mathcal{P}^\omega(M) \to \mathcal{P}^\omega(N) \) is injective.

**Proof.** By induction\(^3\) over the depth of the superpower sets, so \( n \) is the number of nested parentheses:

**IA (\( n = 0 \)) atomic nodes** Given \( x_1, x_2 \in \mathcal{P}^\omega(M) \) with \( x_1 \neq x_2 \).

Since \( f \) injective, \( f^\ast(x_1) = f(x_1) \neq f(x_2) = f^\ast(x_2) \).

**IB** Let \( f^\ast : \mathcal{P}^\omega(M) \to \mathcal{P}^\omega(N) \) be injective for all sets with at most \( n \) nested parentheses.

**IS** Given \( M_1, M_2 \in \mathcal{P}^\omega(M) \) with \( M_1 \neq M_2 \) having both \( n+1 \) nested parentheses.

Let \( x \in M_1 \cap x \notin M_2 \).

Hence \( f^\ast(x) \in f^\ast(M_1) \).

\( x \notin M_2 \) implies for all \( m \in M_2 \) that \( x \neq m \).

\( x \) and \( m \) have at most \( n \) nested parentheses.

\( f^\ast(x) \neq f^\ast(m) \) for all \( m \in M_2 \) as \( f^\ast \) is injective for all sets with at most \( n \) nested parentheses.

Thus \( f^\ast(x) \neq f^\ast(M_2) \)

So, \( \mathcal{P}^\omega(M_1) \neq \mathcal{P}^\omega(M_2) \).

**Lemma 9 (\( \mathcal{P}^\omega \) preserves pullbacks along injective morphisms).**

**Proof.** Given a pullback diagram \((PB)\) and the diagram \((1)\) in \textbf{Sets} with \( g_1 : C \to D \) injective.

\(^3\) To be precise this is an Noetherian induction see Appendix A.2
Pullbacks and the superpower set functor (see Lemma 8) preserve injections, so \( \pi_B : A \to B, P^\omega(\pi_B) : P^\omega(A) \to P^\omega(B) \) and \( \pi_{P^\omega(B)} : P \to P^\omega(B) \) are injective.

Since \( (PB) \) is a pullback diagram we have \( A = \{(b, c) \mid f_1(b) = g_1(c)\} \).

Let \( P \) be the pullback of \( (P^\omega(D), f_1^*, g_1^*) \), so \( f_1^* \circ \pi_{P^\omega(D)} = \pi_{P^\omega(C)} \circ g_1^* \).

Hence, \( P = \{(B', C') \mid f_1^*(B') = g_1^*(C')\} \subseteq P^\omega(B) \times P^\omega(C) \).

Moreover, there is the unique \( \tilde{h} : P^\omega(A) \to P \) s.t. \( h(A') = (\pi_B^*(A'), f_{\pi_C^*(A')} \rangle \) for all \( A' \subseteq A \) so that the diagrams (2) and (3) commute along \( h \): \( \pi_{P^\omega(B)} \circ \tilde{h} = \pi_B^* \) and \( \pi_{P^\omega(C)} \circ \tilde{h} = \pi_C^* \).

We define \( \tilde{h} : P \to P^\omega(A) \) with

\[
\tilde{h}(X, Y) = \begin{cases} 
(b, c) & \text{if } X = b \in B, Y = c \in C \\
(x, y) & \text{if } x \in X \cap B, y \in Y \cap C, f_1(x) = g_1(y) \\
\cup \bigcup_{(x', y') \in (X-B) \times (Y-C)} \tilde{h}(X', Y') & \text{else}
\end{cases}
\]

and have:

1. \( \tilde{h} \) is well-defined since \( \tilde{h}(X, Y) \in P^\omega(A) \).

2. (2) commutes along \( \tilde{h} \), i.e. \( \pi_B^* \circ \tilde{h} = \pi_{P^\omega(B)}(X, Y) \) by induction over the number of nested parentheses \( n \):

**IA** \( (n = 0, \text{ i.e. atomic nodes}) \): Given \( (b, c) \in P \) with \( b \in B \) and \( c \in C \), \( \pi_B^* \circ \tilde{h}(b, c) = \pi_B^*(b, c) = b = \pi_{P^\omega(B)}(b, c) \)

**IB** Let be \( \pi_B^* \circ \tilde{h}(X, Y) = \pi_{P^\omega(B)}(X, Y) \) for sets with at most \( n \) nested parentheses.

**IS** Given \( (\hat{X}, \hat{Y}) \in P \) with \( n + 1 \) nested parentheses.

Let \( \hat{X} = \hat{B} \cup X \) with \( \hat{B} \subseteq B \) and \( X \cap B = \emptyset \).
Let \( \hat{Y} = \hat{C} \cup Y \) with \( \hat{C} \subseteq C \) and \( Y \cap C = \emptyset \).
$X$ and $Y$ have at most $n$ nested parentheses.

$$\pi_B^* \circ \bar{h}(\hat{X}, \hat{Y})$$

$$= \pi_B^* \circ \left( \{(x, y) \mid x \in \hat{B}, y \in \hat{C} : f_1(x) = g_1(y)\} \cup \bigcup_{(X', Y') \in (X \times Y)} \bar{h}(X', Y') \right)$$

$$= \{x \mid x \in \hat{B} \land \exists y \in \hat{C} : f_1(x) = g_1(y)\} \cup \bigcup_{(X', Y') \in (X \times Y)} \pi_B^* \circ \bar{h}(X', Y')$$

$$= \hat{B} \cup \bigcup_{(X', Y') \in (X \times Y)} \pi_B^* \circ \bar{h}(X', Y')$$

$$= \hat{B} \cup \bigcup_{(X', Y') \in (X \times Y)} \pi_{\mathcal{P}^\omega(B)}(X', Y')$$

$$= \hat{B} \cup X$$

$$= \hat{X}$$

$$= \pi_{\mathcal{P}^\omega(B)}(\hat{X}, \hat{Y})$$

Now we show $\mathcal{P} \cong \mathcal{P}^\omega(\mathcal{A})$:

- $h \circ \bar{h} = \text{id}_\mathcal{P}$,
  - since $\pi_{\mathcal{P}^\omega(B)}$ is injective and $\pi_{\mathcal{P}^\omega(B)} \circ h \circ \bar{h} = \pi_B^* \circ \bar{h} = \pi_{\mathcal{P}^\omega(B)} \circ \text{id}_\mathcal{P}$
- $\bar{h} \circ h = \text{id}_{\mathcal{P}^\omega(\mathcal{A})}$,
  - since $\pi_B^*$ is injective and $\pi_B^* \circ \bar{h} \circ h = \pi_{\mathcal{P}^\omega(B)} \circ \bar{h} = \pi_B^* = \pi_B^* \circ \text{id}_{\mathcal{P}^\omega(\mathcal{A})}$.

Based on $F$-graphs (see Jä15, Sch99), that is a family of graph categories induced by a comma category construction using a functor $F$, we can define the category of coalgebraic $F$-graphs.

**Example 7 (Coalgebraic $F$-Graph based on $\mathcal{P}^\omega$).** In Fig. 4 the following coalgebraic $F$-graphs are illustrated and a morphism in between.

$G_5 = (\text{ngb}_5 : E_5 \rightarrow \mathcal{P}^\omega(N_5))$ \hspace{1cm} $G_6 = (\text{ngb}_6 : E_6 \rightarrow \mathcal{P}^\omega(N_6))$

with $\text{ngb}_5 : x \mapsto \{\{u\}, \{v\}\}$ \hspace{1cm} with $\text{ngb}_6 : a \mapsto \{\{n_1\}, \{n_2\}\}$

$y \mapsto \{\{u\}, \{w\}\}$ \hspace{1cm} $b \mapsto \{\{n_1, n_2\}, \{n_3\}\}$

$z \mapsto \{\{u\}, \{w\}\}$ \hspace{1cm} $c \mapsto \{\{n_3\}\}$

Note, that we only have the recursion of nodes, but the nodes that contain others do not have a name themselves. Edges are hyperedges given as a subset of the superpower set, but they cannot have incident vertices.

**Definition 6 ($\mathcal{P}^\omega$-Graph and the category of coalgebraic graphs).** The category of $\mathcal{P}^\omega$-graphs $\mathcal{P}^\omega_{-}\text{Graph}$ is given by a comma category $\mathcal{P}^\omega_{-}\text{Graph} =< \text{Id}_{\text{Sets}} \downarrow \mathcal{P}^\omega >$.

$\mathcal{P}^\omega$-graph morphisms are given by mappings of the nodes and edges $f = (f_N, f_E) : G_1 \rightarrow G_2$ with $f_N : N_1 \rightarrow N_2$ and $f_E : E_1 \rightarrow E_2$ so that:
Fig. 4: $\mathcal{P}^\omega$-graph morphism

- $f_N^* \circ \text{con}_1 = \text{con}_2 \circ f_N$
- $f_E^* \circ \text{ngb}_1 = \text{ngb}_2 \circ f_E$

2.4 Differences between $\mathcal{P}$, $\mathfrak{P}$ and $\mathcal{P}^\omega$

We have for any set $S$ that $\mathfrak{P}(M) \subseteq \mathcal{P}(M)$ and $\mathcal{P}^\omega(M) \subseteq \mathcal{P}(M)$. The graphs in examples Ex. ??, Ex. 5, and Ex. 7 and the superpower set functors are listed in the table below stating which graph can be constructed using which functor.

|   | $\mathfrak{P}$ | $\mathcal{P}$ | $\mathcal{P}^\omega$ |
|---|---------------|---------------|-------------------|
| G1 | yes | yes | yes       |
| G2 | yes | yes | no see 1 |
| G3 | no see 2 | yes | yes |
| G4 | no see 3 | yes | no see 4 |
| G5 | no see 2 | yes | yes |
| G6 | no see 5 | yes | yes |

1. because $\{\{n_1\}, \{n_2\}\} \notin \mathcal{P}^\omega(\{n_1, n_2, n_3\})$
2. because $u, w \notin \mathfrak{P}(\{u, v, w\})$
3. because $\{n_1, \{n_2\}\} \notin \mathcal{P}(\{n_1, n_2, n_3\})$
4. because $\{n_1, \{n_2\}\} \notin \mathcal{P}^\omega(\{n_1, n_2, n_3\})$
5. because $n_1 \notin \mathfrak{P}(\{n_1, n_2, n_3\})$

3 $\mathcal{M}$-adhesive Categories of $F$-Coalgebras

A endofunctor $F : \mathbf{Sets} \rightarrow \mathbf{Sets}$ gives rise the category of coalgebras $\mathbf{Sets}_F$ with $M \overset{\alpha_M}{\rightarrow} F(M)$ – also denoted by $(M, \alpha_M)$ – being the objects and morphisms $f : (M, \alpha_M) \rightarrow (N, \alpha_N)$ – called $F$-homomorphism – so that (1) commutes in $\mathbf{Sets}$ (see Rut00):

$$
\begin{array}{ccc}
M & \overset{\alpha_M}{\rightarrow} & F(M) \\
\downarrow & & \downarrow \ \ (1) \\
N & \overset{\alpha_N}{\rightarrow} & F(N)
\end{array}
$$

Lemma 10 (Pullbacks along injections in $\mathbf{Sets}_F$). Given a functor $F : \mathbf{Sets} \rightarrow \mathbf{Sets}$ that preserves pullbacks along an injective morphism, then $\mathbf{Sets}_F$ has pullbacks along an injective $F$-homomorphism.
Proof. Given \((B, \alpha_B) \xrightarrow{\ell} (D, \alpha_D) \xrightarrow{\ell'} (C, \alpha_C)\). Then we have (PB1) in \textbf{Sets} below. This results in (PB2) since \(F\) preserves pullbacks along an injective morphism. \(\alpha_A\) is the unique induced morphism for \(F(g) \circ (\alpha_C \circ \pi_C) = F(f) \circ (\alpha_B \circ \pi_B)\). So, (1) commutes.

\((A, \alpha_A)\) is pullback in \(\textbf{Sets}_F\) because we have the diagrams below in \(\textbf{Sets}\):

The comparison object \((X, \alpha_X)\) with \(f \circ g' = g \circ f'\) in \(\textbf{Sets}_F\) leads in \(\textbf{Sets}\) to the induced morphisms \(h : X \to A\) and \(F(h) : F(X) \to F(A)\) commuting the corresponding triangles. As \(F(A)\) is pullback in \(\textbf{Sets}\), we have \(F(h) \circ \alpha_X = \alpha_A \circ h\). Hence, \(h\) is the induced morphism in \(\textbf{Sets}_F\) as well.

Corollary 1 (Pullbacks along injections in \(\textbf{Sets}_F\)). Given a functor \(F : \textbf{Sets} \to \textbf{Sets}\) that preserves pullbacks along an injective morphism, then \(\textbf{Sets}_F\) has pullbacks along an injective \(F\)-homomorphism.

Concerning the Vertical Weak VK Square

Definition 7 (Class of monomorphisms \(\mathcal{M}\)). Let \(\mathcal{M}\) be a class of monomorphisms in \(\textbf{Sets}\) that is PO-PB-compatible, that is:

1. Pushouts along \(\mathcal{M}\)-morphisms exist and \(\mathcal{M}\) is stable under pushouts.
2. Pullbacks along \(\mathcal{M}\)-morphisms exist and \(\mathcal{M}\) is stable under pullbacks.
3. \(\mathcal{M}\) contains all identities and is closed under composition.

According to Prop. 4.7 in \textit{Rut00} if \(f : M \to N\) is injective in \(\textbf{Sets}\) then \(f\) is an \(F\)-monomorphism in \(\textbf{Sets}_F\). Obviously the class of all injective functions \(\mathcal{M}_F = \{(A, \alpha_A) \xrightarrow{f} (B, \alpha_B) \mid f \text{ is injective in } \textbf{Sets}\}\) is PO-PB-compatible.

Theorem 1 ((\(\textbf{Sets}_F, \mathcal{M}_F\) is an \(\mathcal{M}\)-Adhesive Category). If \(F\) preserves pullbacks along injective morphisms, then \((\textbf{Sets}_F, \mathcal{M}_F)\) is an \(\mathcal{M}\)-adhesive category.
Proof. \((\mathbf{Sets}_F, \mathcal{M}_F)\) is an \(\mathcal{M}\)-adhesive category:

1. Pushouts in \(\mathbf{Sets}_F\) along \(m \in \mathcal{M}_F\) exist, since \(\mathbf{Sets}_F\) is finitely cocomplete (Thm 4.2 Rut00) for arbitrary \(F : \mathbf{Sets} \to \mathbf{Sets}\).

2. and they are vertical weak VK squares, i.e. for all commutative cubes (2) where all vertical morphisms \(a, b, c, d\) are in \(\mathcal{M}\) with the given pushout (1) in the bottom and the back squares being pullbacks.

Then following holds:
The top square is a pushout if and only if the front squares are pullbacks.

![Diagram](image)

Since (finite) colimits and pullbacks along \(\mathcal{M}\)-morphisms are constructed on the underlying set, square (1) and the VK-cube are given for the underlying sets as well.

\[
\begin{array}{c}
(A, \alpha_A) \xrightarrow{m \in \mathcal{M}} (B, \alpha_B) \\
\downarrow \\
(C, \alpha_C) \xrightarrow{n} (D, \alpha_D) \\
\end{array}
\]

\[
\begin{array}{c}
\text{(1)} \\
\downarrow \\
\text{(2)} \\
\end{array}
\]

⇒ If in \(\mathbf{Sets}_F\) \((B', \alpha_{B}') \to (D', \alpha_{D}') \leftarrow (C', \alpha_{C}')\) is pushout over \((B', \alpha_{B}') \leftarrow (A', \alpha_{A}') \to (C', \alpha_{C}')\) then \(B' \to D' \leftarrow C'\) is pushout over \(B' \leftarrow A' \to C'\) in \(\mathbf{Sets}\) as pushout are constructed on the underlying sets. As \(\mathbf{Sets}\) together with the class of injective morphisms is an adhesive category (see Thm 4.6 in EEPT06), we have the front squares are pullbacks in \(\mathbf{Sets}\). As the vertical morphisms are injective the front squares are pullbacks in \(\mathbf{Sets}_F\) provided that \(F\) preserves pullbacks along injections.

⇐ Let the front squares be pullbacks in \(\mathbf{Sets}_F\), then the squares of the underlying sets are pullbacks in \(\mathbf{Sets}\). So the top square in pushout in \(\mathbf{Sets}\) and hence in \(\mathbf{Sets}_F\).

The same holds for many-sorted coalgebras over sets. Graphs with undirected edges can be considered as many sorted coalgebras using the functor \(F : \mathbf{Sets} \times \mathbf{Sets} \to \mathbf{Sets} \times \mathbf{Sets}\) with \(F(N, E) = (N, E)\) \((\langle 1, <s,t> \rangle) (1, N \times N)\) where \(1\) is the final object and ! the corresponding final morphism, see e.g. Rut00.

**Corollary 2.** If \(F : \mathbf{Sets} \times \mathbf{Sets} \to \mathbf{Sets} \times \mathbf{Sets}\) preserves pullbacks along injective morphisms, then \((\mathbf{Sets} \to \mathbf{Sets})_F, \mathcal{M}_F)\) is an \(\mathcal{M}\)-adhesive category for a class of monomorphisms \(\mathcal{M}\) in \(\mathbf{Sets}\) that is PO-PB-compatible.

The following corollary allows \(\mathcal{M}\) transformation systems for various dynamic systems based on \(F\)-coalgebras of functors the preserve pullbacks along injective morphisms.
Corollary 3 (M transformation systems for F-coalgebras). We obtain

- M-transformation systems for finitely branching non-deterministic transition systems Sets_{P_{fin}}, where \( (Q, \alpha_Q : Q \to P_{fin}(Q) \) as finite power set functor \( P_{fin} \) preserves pullbacks along injective morphisms.
- M-transformation systems for infinite binary trees Sets_{A \times \Sigma} over an alphabet \( A \) with since the product functor preserves limits.
- M-transformation systems for labelled transition systems over a signature \( \Sigma \) with Sets_{P(\Sigma \times \Sigma)}, since the composition preserves pullback-preservation.

Example 8 (Transformation of a finitely branching non-deterministic transition system). Given the transition system \( (K, \alpha_K) \) with \( K = \mathbb{N} \) and \( \alpha_K(n) = \{2n + 1, 2n + 2\} \) that is a full infinite binary tree. The rule is \( L \leftarrow K \rightarrow R \) and the morphisms are all set inclusions. The application of the rule to \( Q \) leads to the transformation step given in as illustrated in Fig. 5.

\[ \text{Fig. 5: Transformation step in } Coalg_{P_{fin}} \]

Although Kah14 has already approached the coalgebraic representation of DPO-transformations this approach is far more general as it considers arbitrary coalgebras based on functors preserving pullbacks.

3.1 Endofunctors and Pullbacks

A functor \( F : C \to D \) is called (weak) pullback preserving if it maps (weak) pullback squares to (weak) pullback squares: that is, if \( F \) is weakly pullback preserving in the category \( C \) then \( F \) is weakly pullback preserving in the category \( D \). (Def. 4.2.1(ii)
This obviously implies that $F$ preserves pullbacks weakly, i.e. it maps pullback squares to (weak) pullback squares (Rut00, Ada05). Many coalegebraic results require that $F$ preserves pullbacks weakly. This comprises the power-set functors, and arbitrary product, coproduct, power or composite of functors weakly preserving pullbacks (Ada05). Jac16 states that a weak pullback preserving functor preserves (ordinary) pullbacks of monos (exercise 4.2.5 in Jac16). So, we have

- $F$ is (weak) pullback preserving $\implies$ $F$ preserves pullbacks weakly.
- $F$ is (weak) pullback preserving $\implies$ $F$ preserves pullback along monomorphims.

Seemingly, all three notions hold for the power-set functors, and arbitrary product, coproduct, power or composite of functors. But it remains an open question if these notions are equivalent at least in $\text{Sets}$ or in arbitrary categories.

### 4 Edge Recursion

In this section we investigate the recursion of edges, yielding nested edges where neighbours of edges can again be edges, as in Fig. 1a. In Rut00 it is shown that graphs with undirected edges can be considered as many sorted coalgebras using the functor $F : \text{Sets} \times \text{Sets} \to \text{Sets} \times \text{Sets}$ with $F(V,E) = (V,E) \mapsto (1, V \times V)$ where $1$ is the final object and $!$ the corresponding final morphism. In Jac15 the notion of $F$-graphs based on comma-categories is investigated and in Jac16 extended to coalgebras. The functors investigated in Jac16 are the product, the coproduct and several powerset functors. This can be extended to various types of nested edges.

**Definition 8 (Nested hyperedges).** Given a set of nodes $N$ and a set of edges $E$ and a function yielding the neighbours $\text{ngb} : E \to \mathcal{P}(V \sqcup E)$. Then the category of coalgebras $\text{Coalg}_{F_1}$ over $F_1 : \text{Sets} \times \text{Sets} \to \text{Sets} \times \text{Sets}$ with $F_1(N,E) = (1, \mathcal{P}(V \sqcup E))$ yields the category of graphs with nested hyperedges. The class $\mathcal{M}$ is given by the class of pairs of injective morphisms $< f_N, f_E >$.

**Lemma 11 (\((\text{Coalg}, \mathcal{M})\) is an $\mathcal{M}$-adhesive category).**

Proof. $F$ preserves pullbacks along monomorphisms, as the first component is a pullback of the final object in $\text{Sets}$ and the powerset functor preserves pullback of monos (see Lemma 14) and the coproduct functor as well (see Lemma 12).

**Corollary 4 (Nested undirected edges).** Given a set of nodes $N$ and a set of edge names $E$ and a function yielding the neighbours $\text{ngb} : E \to \mathcal{P}^{(1,2)}(V \sqcup E)$. Then the category of coalgebras $\text{Coalg}_{F_2}$ over $F_2 : \text{Sets} \times \text{Sets} \to \text{Sets} \times \text{Sets}$ with $F_2(N,E) = (1, \mathcal{P}^{(1,2)}(N \sqcup E))$ yields the category of graphs with nested undirected edges. The class $\mathcal{M}$ is given by the class of pairs of injective morphisms $< f_N, f_E >$, and $(\text{Coalg}_{F_2}, \mathcal{M})$ is an $\mathcal{M}$-adhesive category.
Corollary 5 (Nested directed edges). Given a set of vertices \( N \) and a set of edge names \( E \) and a function yielding the neighbours \( \text{ngb} : E \to (N \uplus E) \times (V \uplus E) \). Then the category of coalgebras \( \text{Coalg}_{F_3} \) over \( F_3 : \text{Sets} \times \text{Sets} \to \text{Sets} \times \text{Sets} \) with \( F_3(N, E) = (1, N \uplus E) \times (N \uplus E) \) yields the category of graphs with nested directed edges.

The class \( M \) is given by the class of pairs of injective morphisms \( < f_N, f_E > \) and \( (\text{Coalg}_{F_3}, M) \) is an \( M \)-adhesive category.

And again, these edge concepts can be mixed as well see Jä15.

5 Coalgebraic Graphs

Based on the above reached results we can now define graph where nodes and edges are nested.

Example 9 (Nested nodes). Nested nodes can be constructed using the coalgebra \( \text{Coalg}_P \) based on the superpower set functor \( P \). Given a set \( N \) the function \( \text{con} : N \to P(N) \) gives the nodes contained in a given node. This function yields an \( M \)-adhesive category; the category of coalgebras \( \text{Coalg}_P \) over \( P : \text{Sets} \to \text{Sets} \) with the class \( M \) of injective morphisms.

The nesting of nodes can also be defined allowing the different kinds of nesting using some functor \( F : \text{Sets} \to \text{Sets} \), so we have the contains function \( \text{con} : N \to F(N) \). This yields an \( M \)-adhesive category where \( G \) may be one of the (super-)power functors, e.g. \( P, P^{(1,2)}, P^2 \) or \( P^\omega \) or any other functor preserving pullbacks of injections.

To obtain coalgebraic graphs as given in Sect. 1 we construct coalgebraic graphs as \( G = (N, E, \text{con} : N \to F(N, E), \text{ngb} : E \to G(N, E)) \). These graphs can be considered to be an coalgebra over \( F : \text{Sets} \times \text{Sets} \to \text{Sets} \times \text{Sets} \) with \( F(N, E) = (F(N, E), G(N, E)) \).

Definition 9 (Coalgebraic graph). Let the coalgebraic graph functor \( F : \text{Sets} \times \text{Sets} \to \text{Sets} \times \text{Sets} \) be given with \( F(N, E) = (F(N, E), G(N, E)) \) where \( F \) preserves pullbacks along monomorphisms provided \( F : \text{Sets} \times \text{Sets} \to \text{Sets} \) and \( G : \text{Sets} \times \text{Sets} \to \text{Sets} \) preserve injections and pullbacks along injective morphisms, then \( \text{Coalg}_F \) is the category of the corresponding coalgebraic graphs.

According to Lemma 15 in Appendix A.1 the functor \( F \) preserves pullbacks along monomorphisms., so the have:

Corollary 6 (Coalgebraic graphs yield an \( M \)-adhesive category.). The class \( M \) is given by the class of pairs of injective morphisms \( < f_N, f_E > \) and \( (\text{Coalg}_F, M) \) is an \( M \)-adhesive category.

The definition of coalgebraic graphs is chosen to be quite open and comprises the usual graph types, as (un-) directed and (hyper-) graphs as well as various hierarchical graphs (see Subsect. 5.1).

Obviously, in non-hierarchical graph types the contains function is superfluous, so below it is given by the final morphism \(!\).
Undirected graphs are given by $F = 1$ and $G = \prod \circ (P^{(1,2)} \times 1)$, so the objects in $\text{Coalg}_F$ are given by $(N, E) \xrightarrow{\text{con}} (1, P^{(1,2)}(N))$.

Since $G(N, E) = \prod \circ (P^{(1,2)} \times 1)(N, E) = \prod ((P^{(1,2)}(N), 1) \cong P^{(1,2)}(N)$, an undirected graph is given by $G = (N, E, \text{con}, \text{ngb})$ with $\text{con} = !$ and $\text{ngb} : E \to P^{(1,2)}(N)$.

Directed graphs can be given by $F = 1$ and $G = \prod \circ X^2 \times 1$, so we have $(N, E) \xrightarrow{\text{con}} (1, N \times N)$ since $G(N, E) = \prod \circ X^2 \times 1(N, E) = N \times N$.

So, a directed graph is an object in the category $\text{Coalg}_F$ given by $G = (N, E, \text{con}, \text{ngb})$ with $\text{con} = !$ and $\text{ngb} : E \to N \times N$.

It would be interesting to know whether the evolving categories correspond to the usual ones, e.g. is the category $\text{Coalg}_F$ isomorphic to the usual category of graphs (as in EEPT06) given as a functor category $[S, \text{Sets}]$ for the schema category $S = \bullet \rightarrow \rightarrow \rightarrow \rightarrow \bullet$.

Classical hypergraphs, where edges are attached to a set of nodes, are given by $F = 1$ and $G = \mathcal{P} \times 1$.

Hypergraphs as in hyperedge replacement DKH97, where edges are attached to a string of nodes, is hence given by $F = 1$ and $G = (\_^* \times 1)$ with $\_^*$ the free monoid functor.

The definition in EEPT06 uses the indexed comma categories, hence the relation between $\text{ComCat} (\text{Id}_{\text{Sets}}, (\_^* \times \{1, 2\})$ and the evolving coalgebra $\text{Coalg}_F$ needs to be investigated.

Place-Transition nets can be considered to be objects in $\text{Coalg}_F$, with $F = 1$ and $G = ((\_^* \times (\_^*)) \circ X^2) \times 1$. The extension of the hierarchy concepts in this contribution to Petri nets is probably worth exploring.

Since all the involved functors preserve pullbacks of injective morphisms (see Subsect. A.1), we immediately obtain $\mathcal{M}$-adhesive categories.

Subsequently, we omit the final functor $1$ for better readability.

5.1 Examples of Hierarchical Graphs

In the following we relate the notion developed above to concepts of hierarchical graphs in the literature using both concepts, comma category and colaggebra.

1. The comma category $< \text{Id}_{\text{Sets}} \downarrow \mathcal{P} >$ as used in Ex. 3 with is an $\mathcal{M}$-adhesive category because of the comma-category construction (see Theorem 4.15 (construction of (weak) adhesive HLR categories in EEPT06) and $\mathcal{P}$ preserving pullbacks of injections. It yields hierarchical graphs with hyperedges between nodes and containers of nodes, but containers do not have an explicit name.

2. Combining the nested nodes based on the superpower set functor $\mathcal{P}$ as in Ex. 9 with usual edges concepts leads to various types of hierarchical graphs and is closely related to hierarchical graphs in the sense of BKK05. In this
case hierarchical graphs are given by \( G = (N, E, con : N \to \mathcal{P}(N),\ ngb : E \to H(N)) \). \( H \) determines edge type. Typical choices for \( H \) are \( \mathcal{P} \) or \((\_)^*\) for hyperedges, \( \mathcal{P}^{(1,2)} \) for undirected edges or directed edges with \( X^2 : \text{Sets} \to \text{Sets} \times \text{Sets} \) with \( X^2(N) = N \times N \). For an example see Sect. 6.1.1.

We use a coalgebra over \( \mathcal{F}_1 : \text{Sets} \times \text{Sets} \to \text{Sets} \times \text{Sets} \) with \( \mathcal{F}(N, E) = \mathcal{F}(N) \times H(N) \), then \((\text{Coalg}_{\mathcal{F}_1}, \mathcal{M})\) is an \( \mathcal{M}\)-adhesive category.

3. A hierarchy where the edges are refined by subnets is obtained by the neighbouring function \( \text{ngb} : E \to (N)^* \times \mathcal{P}^\omega(N) \) that maps edges to a pair where the first component defines the incident nodes and the second component defines the nodes contained by the edges. This nesting is layered as it is defined by the functor \( \mathcal{P}^\omega \), see Def. 5. The resulting graphs are given by \( G = (N, E, \text{ngb} : E \to N^* \times \mathcal{P}^\omega(N)) \). The category of such graphs is given by the comma category \( <\text{Id}_{\text{Sets}} \downarrow \mathcal{G}> \) with the functor \( \mathcal{G} = ((\_)^* \times \mathcal{P}^\omega) \circ X^2 \).

Note \( \mathcal{G}(N) = ((\_)^* \times \mathcal{P}^\omega) \circ X^2(N) = ((\_)^* \times \mathcal{P}^\omega)(N, N) = (N)^* \times \mathcal{P}^\omega(N) \).

4. For hierarchies, where the edges between nodes may have other parents than the nodes and where the edges may contain subgraphs (as in Pal04) the graphs can be given by the functions \( \text{con} : N \to \mathcal{P}(N \uplus E) \) and \( \text{ngb} : E \to \mathcal{P}(N) \times \mathcal{P}(N \uplus E) \). We use then a coalgebra with \( \mathcal{F}_2(N, E) = (\mathcal{P}(N \uplus E), \mathcal{P}(N) \times \mathcal{P}(N \uplus E)) \). \( \mathcal{F}(N \uplus E) \) yields nested sets of nodes and edges and \( \mathcal{P}(N) \) yields the incident nodes of an hyperedge. To obtain an \( \mathcal{M}\)-adhesive category we construct \( \mathcal{F} \) from other functors that yield the \( \mathcal{M}\)-adhesive category \( \text{Coalg}_{\mathcal{F}_2} \).

5. Multiple hierarchies can be constructed as \( \mathcal{M}\)-adhesive categories using a copying functor \( X^i : \text{Sets} \to \text{Sets} \times \text{Sets} \times \ldots \times \text{Sets} \). The the containment function \( \text{con} : N \to \prod_i \circ X^i \circ \mathcal{P}(N) \) yields for each node \( i \) different nestings. For edges we may use hyperedges \( \text{ngb} : E \to \mathcal{P}(N) \). The corresponding \( \mathcal{M}\)-adhesive category \( \text{Coalg}_{\mathcal{F}_4} \) is given by \( \mathcal{F}(N, E) = (\prod_i \circ X^i \circ \mathcal{P}(N), \mathcal{P}(N)) \) and corresponds to the multi-hierarchical graphs in Sect. 6.2.

6. For bigraphs, see Sect. 6.3 we use the following functions \( \text{con} : N \to \mathcal{P}(N) \) and \( \text{ngb} : E \to \mathcal{P}(N \uplus E) \times \mathcal{P}(N \uplus E) \). Again we obtain an \( \mathcal{M}\)-adhesive category \( \text{Coalg}_{\mathcal{F}_4} \) with \( \mathcal{F}_4(N, E) = (\mathcal{P}(N), \mathcal{P}(N \uplus E) \times \mathcal{P}(N \uplus E)) \) constructed from other functors.

7. The functions \( \text{con} : N \to \mathcal{P}(N) \) and \( \text{ngb} : E \to N \times N \times \mathcal{P}(E) \) allow the description of graph grouping and give rise to the category of coalgebraic graphs \( \text{Coalg}_{\mathcal{F}_5} \) with \( \mathcal{F}_5(N, E) = (\mathcal{P}(N), N \times N \times \mathcal{P}(E)) \) that corresponds roughly to the the graph grouping in Sect. 6.4.
Summarizing, we have:

|   | Definition | Categorical construction | Description |
|---|------------|--------------------------|-------------|
| 1 | \( \text{ngb} : E \to \mathcal{P}(N) \) | comma category \(< \text{Id}_{\mathcal{Sets}} \downarrow \mathcal{P} >\) | hyperedges between nodes and containers of nodes, but container have no explicit name, see Ex. 3 |
| 2 | \( \text{con} : N \to \mathcal{P}(N) \) \( \text{ngb} : E \to \mathcal{H}(N) \) | coalgebra \( \text{Coalg}_{\mathcal{F}_1} \) \( \mathcal{F}_1 = \mathcal{P} \times \mathcal{H} \) | hierarchical graphs, \( \mathcal{H} \) determines edge type, see Sect. 6.1.1 |
| 3 | \( \text{ngb} : E \to N^* \times \mathcal{P}^{\omega}(N) \) | comma category \(< \text{Id}_{\mathcal{Sets}} \downarrow \mathcal{G} >\) with \( \mathcal{G} = (\mathcal{X}^* \times \mathcal{P}^{\omega}) \circ \mathcal{X}^2 \) | hierarchical graphs, see Sect. 6.1.2 |
| 4 | \( \text{con} : N \to \mathcal{P}(N \uplus E) \) \( \text{ngb} : E \to \mathcal{P}(N) \times \mathcal{P}(N \uplus E) \) | \( \text{Coalg}_{\mathcal{F}_2} \) with \( \mathcal{F}_2 = (\mathcal{P} \circ \mathcal{X}) \times (\mathcal{P} \times (\mathcal{P} \circ \mathcal{X})) \circ (\mathcal{X}^2 \times \text{Id}_{\mathcal{Sets}}) \) | hierarchical graphs see Sect. 6.1.3 |
| 5 | \( \{\text{con}_i\}_{i<n} : N \to \mathcal{P}(N \uplus E) \) \( \text{ngb} : E \to \mathcal{P}(N) \) | \( \text{Coalg}_{\mathcal{F}_3} \) with \( \mathcal{F}_3 = (\mathcal{P} \circ \mathcal{X}) \circ \mathcal{X} \times \mathcal{P} \) | multi-hierarchical graphs, see Sect. 6.2 |
| 6 | \( \text{con} : N \to \mathcal{P}(N) \) \( \text{ngb} : E \to \mathcal{P}(N \uplus E) \times \mathcal{P}(N \uplus E) \) | \( \text{Coalg}_{\mathcal{F}_4} \) with \( \mathcal{F}_4 = (\mathcal{P}(N), \mathcal{P}(N \uplus E) \times \mathcal{P}(N \uplus E)) \) | bigraphs, see Sect. 6.3 |
| 7 | \( \text{con} : N \to \mathcal{P}(N) \) \( \text{ngb} : E \to N \times N \times \mathcal{P}(E) \) | \( \text{Coalg}_{\mathcal{F}_5} \) with \( \mathcal{F}_5(N, E) = (\mathcal{P}(N), N \times N \times \mathcal{P}(E)) \) | graph grouping, see Sect. 6.4 |

Table 1: Involved functors

5.2 Properties of Nested Nodes and Edges

Definition 10 (Properties of nested nodes).

1. Nodes are unique if \( c \) is injective.
2. Vertices are the atomic nodes that refer to themselves: \( V = \{ n \mid c(n) = n \} \)
3. Nodes are containers if \( c(n) \in \mathcal{P}(N) - N \)
4. The set of nodes is well-founded if and only if
   - \( X \in N \land Y \in \text{con}(X) \) implies, that \( Y \in \text{con}(N) \)
   - \( X \in \text{con}(N) \land Y \in (X - N) \) implies, that \( Y \in \text{con}(N) \)
5. The set of nodes is hierarchical if and only if \( \text{con}(n) \cap \text{con}(n') \neq \emptyset \) implies \( n = n' \).

**Definition 11 (Properties of nested edges).**

1. The set of atomic hyperedges \( E := \{ e \in E \mid \text{ngb}(e) \in \mathcal{P}(N) \} \).
2. Edges are node-based if the function \( \text{ngb}^+ : E \rightarrow \mathcal{P}(N) \) defined by \( \text{ngb}^+(e) = \{ n \in N \mid n \in \text{ngb}(e) \} \cup \bigcup_{x \in \text{ngb}(e)} \text{ngb}^+(x) \) is well-defined.
3. Edges are atomic if they are node-based and if the function \( \text{ngb}^+(E) \subseteq V \) only yields vertices.

Analogously the properties for (un-)directed edges.

### 6 Transformations of Hierarchical Graphs

Here we argue to what extent known concepts can be considered as \( \mathcal{M} \)-adhesive categories of hierarchical graphs. The detailed, mathematical investigation of each of these examples is beyond the scope of this paper. Labels and attributes are not considered in this paper, but labelled or attributed graphs yield \( \mathcal{M} \)-adhesive categories (see EEPT06, EGH10) and at least labels can be introduced into coalgebraic constructions (see Rut00, Ada05).

#### 6.1 Hierarchical Graphs

Many possibilities to define hierarchical graphs have already been investigated, e.g. BH01, DHP02, Bus02, Pal04, BKK05, BCM10. In ES95 the possibility of infinitely recursive hierarchies has already been introduced as an infinite number of type layers. Here we sketch how three of them, namely DHP02, BKK05 and Pal04, can be considered in this framework.

1. **Hierarchical Graphs as in BKK05**

   In this approach graphs are grouped into packages via a coupling graph. A hierarchical graph is a system \( H = (G, D, B) \), where \( G \) is a graph some graph type, \( P \) is a rooted directed acyclic graph, and \( B \) is a bipartite coupling graph whose partition contains the nodes of \( N_G \) and of \( N_P \). All edges are oriented from the first \( N_G \) to the second set of nodes \( N_P \) and every node in \( N_G \) is connected to at least one node in \( N_P \). For this approach we can consider coalgebraic graphs in the coalgebra category \( \text{Coalg}_{\mathcal{F}_1} \) (see Table 1) with \( \text{con} : N \rightarrow \mathcal{P}(N) \) being well-founded. Additionally a completeness condition, stating that each atomic node is within some package, has to hold:

   \[ \forall n \in N : \text{con}(n) = n \Rightarrow \exists p \in N : n \in \text{con}(p) \]

   The packages are the nodes that are not atomic. The edge function is given by \( \text{ngb} : E \rightarrow \mathcal{H}(N) \) where \( \mathcal{H}(N) \) determines the type of the underlying

Fig. 6: Hierarchical graph as in BKK05
graphs. In Fig. 6 we have an example with two packages, that uses directed edges. So based on $H = X^2$ we can give this example as a coalgebraic graph.

We have $N = \{n, m, x, y, z, p_1, p_2, p_3\}$

with $con(v) = \begin{cases} v ; & \text{if } v \in \{n, m, x, y, z\} \\ \{x, y, z\} ; & \text{if } v = p_1 \\ \{n, m\} ; & \text{if } v = p_2 \\ \{p_1, p_2\} ; & \text{if } v = p_3 \end{cases}$

and $ngb : \begin{cases} a \mapsto (y, x) \\ b \mapsto (y, z) \\ c \mapsto (m, n) \\ e \mapsto (z, n) \end{cases}$

2. Hierarchical Hypergraphs as in DHP02

Hypergraphs $H = (V, E, att, lab)$ in DHP02 consist of two finite sets $V$ and $E$ of vertices and hyperedges. These are equipped with an order, so the attachment function is defined by $att : E \to V^*$. The hierarchy is given in layers, in the sense that subsets in the same layer have the same nesting depth. So, edges are within one layer. Hierarchical graphs $< G, F, cts : F \to \mathcal{H} > \in \mathcal{H}$ are given with special edges $F$ that contain potentially hierarchical subgraphs. Fig. 7a depicts a hierarchical graph that can be considered to be a graph in the comma category $< \text{Id}_{\text{Sets}} \downarrow G >$ (see Table 1). The graph $G = (N, E, ngb)$ with $ngb : E \to N^* \times P^\omega (N)$ is defined so that edges are node-based.

3. Hierarchical Graphs as in Pal04

are obtained from hypergraphs by adding a parent assigning function to them. Nodes and edges can be assigned as a child of any other node or edge. These correspond to coalgebraic graphs in the category $\text{Coalg}_{F_2}$ (see Table 1). The parent function coincides with $con : N \to P(N \cup E)$ being well-founded and hierarchical and $ngb : E \to P(N) \times P(N \cup E)$ since edges can have children as well.
In Fig. 8 the nodes \( N = \{1, 2, 3, 6, 8, 9, 11\} \) and the contains function \( con : N \rightarrow \mathcal{P}(N \cup E) \), yield the nodes and their children. The hyperedges \( E = \{4, 5, 7, 10\} \) with \( ngb : E \rightarrow \mathcal{P}(N \times \mathcal{P}(N \cup E)) \) yield the edges. Note in this example the edges are not nested.

Contains and neighbour function are given by

\[
\begin{align*}
con &: 1 \mapsto \{1\}, 8 \mapsto \{1, 8\}, 2 \mapsto \{2\}, 9 \mapsto \{2, 9\}, \\
& 3 \mapsto \{1, 2, 3\}, 11 \mapsto \{8, 9, 11\}, 6 \mapsto \{6\}, \\
\end{align*}
\]

\[
\begin{align*}
ngb &: 4 \mapsto (\{1, 2\}, \emptyset), 5 \mapsto (\{2, 6\}, \emptyset), \\
& 7 \mapsto (\{3, 6, 11\}, \emptyset), 10 \mapsto (\{8, 9\}, \emptyset).
\end{align*}
\]

### 6.2 Multi-Hierarchical Graphs

In ŠLP\textsuperscript{+}17 multiple hierarchies have been suggested, first ideas can be found in Pal04. A finite set of child nesting functions is specified that relate nodes to set of nodes and edges. This corresponds to a finite family \( \{con_1 : N \mapsto \mathcal{P}(N \cup E)\}_{i<n} \) that are well-founded and hierarchical. For transformations of multi-hierarchical graphs there is the \( M \)-adhesive category \( \mathbf{Coalg}_{\mathbb{F}_3} \) of coalgebraic graphs (see Table 1).

### 6.3 Bigraphs as an hierarchy

Bigraphs Mil06 originate in process calculi for concurrent systems and provide a graphical model of computation. A bigraph is composed of two graphs: a place graph and a link graph. They emphasize interplay between physical locality and
virtual connectivity. Reaction rules allow the reconfiguration of bigraphs. A bigraphical reactive system consists of a set of bigraphs and a set of reaction rules, which can be used to reconfigure the set of bigraphs. Bigraphs may be composed and have a bisimulation that is a congruence wrt. composition. Categorically, bigraphs are given as morphisms in a symmetric partial monoidal category where the objects are interfaces. This construction corresponds to ranked graphs as given in GH97 where morphisms are given by a isomorphism class of concrete directed graphs with interfaces. Ehr02 discusses extensively the relation of bigraphs to graph transformations. In GRJD16 a functor that flattens bigraphs into ranked graphs is provided that encodes the topological structure of the place graph into the node names. In BMPT14 bigraphs are shown to be essentially the same as gs-graphs that present the place and the link graph within one graph. We also represent bigraphs within one graph, where the hierarchical structure is given by a superpower set of nodes and the link structure is given by nested hyperedges. Here we abstract from the categorical foundations and give bigraphs as a special cases of hierarchical graphs. Hence, we ignore their categorical structure, but we obtain a transformation system. Nevertheless, often only the graphical representation of bigraphs is used WW12, Wor13, BCRS16. A bigraph is a 5-tuple: \( (V, E, \text{ctrl}, \text{prnt}, \text{link}) : \langle k, X \rangle \rightarrow \langle m, Y \rangle \), where \( V \) is a set of nodes, \( E \) is a set of edges, \( \text{ctrl} \) is the control map that assigns controls to nodes, \( \text{prnt} \) is the parent map that defines the nesting of nodes, and \( \text{link} \) is the link map that defines the link structure. The notation \( \langle k, X \rangle \rightarrow \langle m, Y \rangle \) indicates that the bigraph has \( k \) holes (sites) and a set of inner names \( X \) and \( m \) regions, with a set of outer names \( Y \). These are respectively known as the inner and outer interfaces of the bigraph.

Below we illustrate the relation of bigraphs to coalgebraic graphs in \( \text{Coalg}_{\mathcal{F}_4} \) (see Table 1) in an example. In Fig. 9a we have an introductory example from \( \text{Mil06} \) that we represent as a coalgebraic graph. The coalgebraic graphs in the \( \mathcal{M} \)-adhesive category \( \text{Coalg}_{\mathcal{F}_4} \) need to have well-founded and hierarchical nodes, where the contains function represents the parent function, so \( \text{con} = \text{prnt} \). The
function `cntrl` yields basically the in- and out-degree of each node. The `link` function yields hyperedges, which we represent as directed hyperedges. Hyperedges connecting outer names are represented as directed hyperedges with the arc itself as the target, those connecting inner names as directed hyperedges with the arc itself as the source. The regions correspond to the roots 0, 1 of the forests given by `con` and the site are the distinguished atomic nodes 0, 1, 2. The nodes \( N = \{0, 1, v_0, v_1, v_2, v_3, 0, 1, 2\} \) and the contains function `con : N → P(N)`, yield the place graph. The directed nested hyperedges \( E = \{e_1, e_2, e_3, e_4, e_5\} \) with \( ngb : E → P(N ⊎ E) × P(N ⊎ E) \) yield the link graph. We have:

\[
\begin{align*}
\text{con} : & 0 → \{v_0, v_2\} & \quad & \text{and} \quad \text{ngb} : e_1 → (\{v_1, v_2, v_3\}, \{v_1, v_2, v_3\}) \\
1 & → \{v_3, 1\} & y_0 & → (\{v_2\}, \{v_2\}) \\
v_0 & → \{v_1\} & y_1 & → (\{v_2, v_3\}, \{v_2, v_3\}) \\
v_1 & → \{0\} & x_0 & → (\{x_0\}, \{y_1\}) \\
v_2 & → v_2 & x_1 & → (\{x_1\}, \{v_3\}) \\
v_3 & → \{2\}
\end{align*}
\]

\( \bar{i} \rightarrow i; \text{for } 0 ≤ i ≤ 2 \)

Assuming `con` to be just well-founded we obtain bigraphs with sharing as in SC15.

### 6.4 Graph Grouping

JPR17 aims at a fundamentally different application area, namely graph grouping to support data analysts making decisions based on very large graphs. Here, a graph hierarchy is established to cope with large amounts of data and to aggregate them. Graph grouping operators produce a so-called summary graph containing super vertices and super edges. A super vertex stores the properties representing the group of nodes, and a super edge stores the properties representing the group of edges. Basically this leads to a contains function `con : N → P(N)` that are well-founded but not necessarily hierarchical and a neighbour function `ngb : E → N × N × P(E)`. These can be given as coalgebraic graphs in the category of coalgebras \( \text{Coalg}_{\mathcal{F}_5} \) (see Table 1) that is \( \mathcal{M} \)-adhesive.

But clearly this graph grouping is only sensible for attributed graphs since these used to abstract the data.

### 7 Related Work

#### 7.1 Recursive Sets

A set \( M \) of integers is said to be recursive (see e.g. Rog87) if there is a total recursive function \( f(x) \) such that \( f(x) = 1 \) for \( x \in M \) and \( f(x) = 0 \) for \( x \notin M \).
Any recursive set is also recursively enumerable.
Finite sets, sets with finite complements, the odd numbers, and the prime numbers are all examples of recursive sets. The union and intersection of two recursive sets are themselves recursive, as is the complement of a recursive set.

7.2 Recursive Graphs

Bea76 is concerned with recursive function theory that are analogous to certain problems in chromatic graph theory and introduces the following definition of recursive graphs according to Rem86. A recursive graph $G = (V, E)$ is recursive, if $V$, the set of vertices is a recursive subset of the natural numbers $\mathbb{N}$ and $E$, the set of edges is a recursive subset of $\mathbb{N}^{(2)}$, the set of unordered pairs from $\mathbb{N}$. These recursive graphs have an infinite amount of nodes that need to be computed by a recursive function.

8 Conclusion

We have presented a novel approach to hierarchies in graphs and graph transformations. This approach supports the use of the mature and extensive theory of algebraic graph transformations for graphs with many different and also uncommon hierarchy concepts. The aim of our approach is not a generalisation of hierarchy concepts in graph transformation but a possibility to access algebraic graph transformation for graphs with a wide spectrum of hierarchy concepts.

We have presented an approach to graphs that allows arbitrarily nested nodes and edges being attached to nodes, sets of nodes and edges. This gives rise to an abstract notion of graphs based on different functors.

The vision is a clear and simple access that provides a potential user with the hierarchical technique that is most adequate for the purpose. This requires a much deeper treatment of the hierarchical concepts at the abstract categorical level as well as an intuitive representation of these concepts.

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A Appendix

A.1 Functors Preserving Pullbacks (along Monomorphisms)

The functors $\prod$ and $1$ are essentially limit constructions and hence compatible with a pullback construction.

The coproduct in $\text{Sets}$ is the disjoint union and can be considered a functor from the functor category $[I, \text{Sets}]$ to $\text{Sets}$ where $I$ a small, discrete category.
Lemma 12 ([I : I \to \text{Sets} preserves pullbacks]).

Proof. Given pullbacks (PB) in \text{Sets} for the index category I and diagram (1).

\begin{align*}
A_i & \rightarrow^{\pi_i} B_i \quad (PB) \quad \downarrow \quad \downarrow f_i \\
C_i & \rightarrow^{g_i} D_i \quad (PB)
\end{align*}

(1) commutes since \(\prod\) is a functor.

For each \(X\) with \(g_i \circ \hat{f}_i = f_i \circ \hat{g}_i\) there is the unique \(h : X \to \prod(A_i)\) with \(h(x) = (c_i, b_i)\) with \(\hat{g}_i(x) = b_i\) and \(\hat{f}_i(x) = c_i\), so that (2) and (3) commute.

The copy functor \(X^n\) takes one set \(S\) and copies the set \(n\)-times yielding an object \((S, S, \ldots, S)\) in the category \(\text{Sets}^n\).

Definition 12 (Copy functor \(X\)).

\(X^1 = \text{Id}_{\text{Sets}} : \text{Sets} \to \text{Sets}\) and
\(X^{n+1} : \text{Sets} \to \text{Sets}^{n+1}\) with \(X^{n+1}(S) = X^n(S) \times S\) for sets \(S\) and
\(X^{n+1}(f) = X^n(f) \times f\) for functions \(f : S \to S'\).

If \(n\) is not in the focus we may omit it.

Obviously, \(X\) is well-defined and preserves injections.

Note, \(X\) differs from a discrete schema \(S\) since \(S\) chooses sets that may be different.

The symbol \(\times^n\) is used here to construct tuples of sets, whereas the product functor \(\prod\) yields tuples of elements, so in these terms \(\prod \circ X^2(S) = S \times S\) with \(\times\) the set-theoretic cartesian product.

Lemma 13 (The copy functor \(X\) preserves pullbacks.).

Proof. Given pullback (PB) in \(\text{Sets}\):
leading to the following pullback in the category $\text{Sets}^n$:

$$
\begin{array}{c}
X \\
\downarrow h \\
(A, A, \ldots, A) \xrightarrow{(1)} (B, B, \ldots, B) \\
\downarrow f \\
(C, C, \ldots, C) \xrightarrow{(3)} (D, D, \ldots, D)
\end{array}
$$

(1) commutes since $X$ is a functor.

For each $X = (X_1, X_2, \ldots, X_n)$ with $(g, g, \ldots, g) \circ \hat{f} = (f, f, \ldots, f) \circ \hat{g}$ there is the induced pullback morphism $h : X \to (A, A, \ldots, A)$ with $h = (h_1, h_2, \ldots, h_n)$ and $h_i : X_i \to A$ are the induced pullback morphism of $(PB)$, so that (2) and (3) commute.

Next we investigate some power set functors.

**Definition 13** ($\mathcal{P}, \mathcal{P}^{(i,j)}, \mathcal{P}_{\text{fin}} : \text{Sets} \to \text{Sets}$ with

- $\mathcal{P}(M) = \{M' \subseteq M\}$,
- $\mathcal{P}^{(i,j)}(M) = \{M' \subseteq M \mid i \leq |M| \leq j\}$ and
- $\mathcal{P}_{\text{fin}}(M) = \{M' \subseteq M \mid |M| \in \mathbb{N}\}$

so that $f : M \to N$ with $\mathcal{P}(f)(M') = f(M')$, the same for $\mathcal{P}^{(i,j)}(f)$ and $\mathcal{P}_{\text{fin}}(f)$

Contrary to Lemma A.39 in EEPT06 the covariant powerset functor $\mathcal{P}$ does not preserve pullbacks.

Given a pullback diagram $(PB)$ in $\text{Sets}$ with $D = \{d\}$, $B = \{1, 2\}$ and $C = \{c, c'\}$.

Then $A = \{(1, c), (2, c), (1, c'), (2, c')\}$ is PB with the corresponding projections.

$(\mathcal{P}, \pi_{\mathcal{P}(B)}, \pi_{\mathcal{P}(C)})$ is the pullback in $\text{Sets}$ over $(\mathcal{P}(D), \mathcal{P}(f_1), \mathcal{P}(g_1))$.

$P = \{(B, C) \mid f_1(B) = g_1(C)\} = \{\langle \emptyset, \emptyset \rangle\} \cup \{\{1,\}, \{2,\}, \{1, 2\}\} \times \{\{c,\}, \{c'\}, \{c, c'\}\}$.

Unfortunately $|P| = 10 \neq 16 = |\mathcal{P}(A)|$.

**Lemma 14** ($\mathcal{P} : \text{Sets} \to \text{Sets}$ preserves pullbacks along injective functions).

**Proof.** Given a pullback (PB) in $\text{Sets}$ and diagram (1) as above with $g_1 : C \hookrightarrow A$ injective.

Pullbacks and the powerset functor preserve injections, so
\(\pi_B : A \hookrightarrow B, \mathcal{P}(\pi_B) : \mathcal{P}(A) \hookrightarrow \mathcal{P}(B)\) and \(\pi_{\mathcal{P}(b)} : P \hookrightarrow \mathcal{P}(B)\) are injective.

Since \((P)\) is a pullback diagram we have \(A = \{(b, c) \mid f_1(b) = g_1(c)\}\).

(1) commutes, since \(\mathcal{P}\) is a functor.

\(h : \mathcal{P}(A) \to P\) is given by \(h(A^{'}) = (\mathcal{P}(\pi_B)(A^{'})\), \(\mathcal{P}(\pi_C)(A^{'})\)) the induced PB morphism so that, \(\mathcal{P}(\pi_B) \circ h = \pi_{\mathcal{P}(B)}\) and \(\mathcal{P}(\pi_C) \circ h = \pi_{\mathcal{P}(C)}\).

We define \(\tilde{h} : P \to \mathcal{P}(A)\) with

\(\tilde{h}(X, Y) = \{(x, y) \mid x \in X, y \in Y, f_1(x) = g_1(y)\} = (X \times Y) \cap A\)

and have:

1. \(\tilde{h}\) is well-defined since \(\tilde{h}(X, Y) \in \mathcal{P}(A)\).
2. (2) commutes along \(\tilde{h}\), i.e. \(\mathcal{P}(\pi_B) \circ \tilde{h} = \pi_{\mathcal{P}(B)}(X, Y)\)

For \((X, Y) \in P\) we have

\[
\mathcal{P}(\pi_B) \circ \tilde{h}(X, Y) = \mathcal{P}(\pi_B)(\{(x, y) \mid x \in X, y \in Y, f_1(x) = g_1(y)\})
= \{x \mid x \in X, \exists y \in Y : f_1(x) = g_1(y)\}
= X
= \pi_{\mathcal{P}(B)}(X, Y)
\]

We show \(P \cong \mathcal{P}(A)\):

- \(h \circ \tilde{h}(X, Y) = id_P\):
  For each \((X, Y) \in P\) we have
  \[
h \circ \tilde{h}(X, Y) = h(\{(x, y) \mid x \in X, y \in Y, f_1(x) = g_1(y)\})
= h(\{(x, y) \mid x \in X, y \in Y, f_1(x) = g_1(y)\})
= (\{x \mid x \in X \land \exists y \in Y : f_1(x) = g_1(y)\}, \{y \mid y \in Y \land \exists x \in X : f_1(x) = g_1(y)\})
= (X, Y)
\]

- \(\tilde{h} \circ h = id_{\mathcal{P}(A)}\):
  Since \(\mathcal{P}(\pi_B)\) is injective \(\mathcal{P}(\pi_B) \circ \tilde{h} \circ h = \pi_{\mathcal{P}(B)} \circ h = \mathcal{P}(\pi_B) = \mathcal{P}(\pi_B) \circ id_{\mathcal{P}(A)}\)
  we have \(\tilde{h} \circ h = id_{\mathcal{P}(A)}\)

Corollary 7 \((\mathcal{P}^{i,j}, \mathcal{P}_{fin} : \text{Sets} \to \text{Sets} \text{ preserve pullbacks along injective functions})\).
Lemma 15 (F preserves pullbacks along monos). Let the coalgebraic graph functor $F : \text{Sets} \times \text{Sets} \to \text{Sets} \times \text{Sets}$ be given with $F(N,E) = (F(N),G(E))$. $F$ preserves pullbacks along monomorphisms provided $F : \text{Sets} \times \text{Sets} \to \text{Sets}$ and $G : \text{Sets} \times \text{Sets} \to \text{Sets}$ preserve injections and pullbacks along injective morphisms.

Proof. Given pullback $(PB1)$ and $(PB2)$ in $\text{Sets}$ with the monomorphisms $f_N$ and $f_E$.

$$A_N \xrightarrow{\pi_N} B_N \xleftarrow{\pi_N (PB1)} \xrightarrow{f_N} C_N \xrightarrow{g_N} D_N$$

leading to the following pullbacks, due to the assumption for $F$

$$X \xrightarrow{h} F(A_N, A_E) \xrightarrow{F(\pi_B, \pi_E)} F(B_N, B_E) \xleftarrow{f} \xrightarrow{g} F(C_N, C_E) \xrightarrow{g} F(D_N, D_E)$$

and due to the assumption for $G$

$$G(A_N, A_E) \xrightarrow{G(\pi_B, \pi_E)} G(B_N, B_E) \xleftarrow{f} \xrightarrow{g} G(C_N, C_E) \xrightarrow{g} G(D_N, D_E)$$

Then we have in the category $\text{Sets} \times \text{Sets}$:

$$X \xrightarrow{h} F(A_N, A_E) \xrightarrow{(2)} F(B_N, B_E) \xleftarrow{g} \xrightarrow{f} \xrightarrow{g} F(C_N, C_E) \xrightarrow{g} F(D_N, D_E)$$

(1) commutes since $F$ is a functor.

For each $X$ with $F(g_N, g_E) \circ \hat{f} = F(f_N, f_E) \circ \hat{g}$ there is the induced pullback morphism $h : X \to F(A_N, A_E)$ with $(h_1, h_2)$, so that (2) and (3) commute. $h_1 : X \to F(A_N, A_E)$ is the induced pullback morphism of $(PB3)$ for the projection of the first component and $h_2 : X \to G(A_N, A_E)$ is the induced pullback morphism of $(PB4)$ for the projection of the second component.
A.2 Noetherian Induction

A set $S$ together with a binary relation $R$ over $S$ is well-founded if and only if every non-empty subset $S' \subseteq S$ has a minimal element; that is,

$$\forall S' \subseteq S \ (S' \neq \emptyset \Rightarrow \exists x \in S' \ \forall s \in S' \ (s, x) \notin R)$$

Examples of relations that are not well-founded include the negative integers $\mathbb{Z}^-$ with the usual order, since any unbounded subset has no least element. The powerset of a set together with the inclusion of sets is a well-founded set if and only if the set is finite. Note, that the superpower sets are well-founded even for infinite sets, since we employ a different order.

Well-founded sets allow Noetherian induction:

Given a property $P$ for an order relation $R$ of a wellfounded set $S$, then we have:

- $P(x)$ holds for all minimal elements of $S$.
- $x \in S$ and $P(x)$ holds for all $yR x$ then $P(x)$ holds.

Then $P(x)$ holds for all $x \in S$.

Lemma 16 ($\mathbb{P}(S)$ is well-founded). Let $R \subseteq \mathbb{P}(S) \times \mathbb{P}(S)$ be given by $A R B$ iff $np(A) \leq np(B)$ with $np : \mathbb{P}(S) \rightarrow \mathbb{N}$

- $s \in S \Rightarrow np(s) = 0$
- $np(\emptyset) = 1$
- $(X_i)_{i \in I} \in S \Rightarrow np(\{X_i | i \in I\}) = \max\{np(X_i) | i \in I\} + 1$ with $I \subseteq \mathbb{N}$

Proof. Due to the inductive definition of $\mathbb{P}$.

Lemma 17 ($\mathbb{P}(S)$ is well-founded). Let $R \subseteq \mathbb{P}(S) \times \mathbb{P}(S)$ be given by $A R B$ iff $np(A) \leq np(B)$ with $np : \mathbb{P}(S) \rightarrow \mathbb{N}$

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Proof. Due to the inductive definition of $\mathbb{P}$.

Lemma 18 ($\mathbb{P}^\omega(S)$ is wellfounded). Let $R \subseteq \mathbb{P}^\omega(S) \times \mathbb{P}^\omega(S)$ be given by $A R B$ iff $np(A) \leq np(B)$ with $np : \mathbb{P}^\omega(S) \rightarrow \mathbb{N}$

- $s \in S \Rightarrow np(s) = 0$
- $np(\emptyset) = 1$
- $(X_i)_{i \in I} \in S \Rightarrow np(\{X_i | i \in I\}) = \max\{np(X_i) | i \in I\} + 1$ with $I \subseteq \mathbb{N}$

Proof. Due to the inductive definition of $\mathbb{P}^\omega$. 