A NEW CLASS OF BELL-SHAPED FUNCTIONS

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In memory of Augustyn Kałuża

Abstract. We provide a large class of functions \( f \) that are bell-shaped: the \( n \)-th derivative of \( f \) changes its sign exactly \( n \) times. This class is described by means of Stieltjes-type representation of the logarithm of the Fourier transform of \( f \), and it contains all previously known examples of bell-shaped functions, as well as extended generalised gamma convolutions, including all density functions of stable distributions. The proof involves the representation of \( f \) as the convolution of a Pólya frequency function and a function which is absolutely monotone on \((-\infty, 0)\) and completely monotone on \((0, \infty)\).

In the final part we disprove three plausible generalisations of our result.

1. Introduction and main results

By Rolle’s theorem, for every \( n = 0, 1, 2, \ldots \) the \( n \)-th derivative \( f^{(n)} \) of any smooth function \( f \) which converges to zero at \( \pm \infty \) changes its sign at least \( n \) times. Such \( f \) is said to be bell-shaped if \( f^{(n)} \) changes its sign exactly \( n \) times. Note that in this case \( f^{(n)} \) necessarily converges to zero at \( \pm \infty \) for every \( n = 1, 2, \ldots \)

We say that a locally integrable function (or, more generally, a locally finite measure) \( f \) is weakly bell-shaped if the convolution of \( f \) with the Gauss–Weierstrass kernel \((4\pi t)^{-1/2}e^{-x^2/(4t)}\) is bell-shaped for every \( t > 0 \).

It is elementary to prove that \( e^{-x^2}, (1 + x^2)^{-p} \) and \( x^{-p}e^{-1/x} \mathbf{1}_{(0, \infty)}(x) \) are bell-shaped for \( p > 0 \). I. I. Hirschman proved in [6] that there are no compactly supported bell-shaped functions, thus resolving a conjecture of I. J. Schoenberg. A classical result due to the latter asserts that Pólya frequency functions (which are discussed in Section 4) are bell-shaped; see [16]. T. Simon proved in [15] that positive stable distributions have bell-shaped density functions. Density functions of general stable distributions are known to be bell-shaped when their index of stability is 2 or \( 1/n \) for some \( n = 1, 2, \ldots \): this follows from [5] after correcting an error in the proof, see [15] for a detailed discussion.

W. Jedidi and T. Simon showed in [8] that hitting times of (generalised) diffusions have the same property. The main purpose of this article is to provide a new class of bell-shaped functions which, to the best knowledge of the author, contains all known examples of bell-shaped functions.

Theorem 1.1. Suppose that \( f \) is a locally integrable function which converges to zero at \( \pm \infty \), which decreasing near \( \infty \) and increasing near \(-\infty \). Suppose furthermore that for \( \xi \in \mathbb{R} \setminus \{0\} \) the Fourier transform of \( f \) satisfies

\[
\int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx = \exp \left( -a\xi^2 - ib\xi + c + \int_{-\infty}^{\infty} \left( \frac{1}{i\xi + s} - \frac{1}{s} \right) \mathbf{1}_{\mathbb{R}\setminus(-1,1)}(s) \varphi(s) ds \right)
\]

(with the integral in the left-hand side understood as an improper integral); here \( a \geq 0 \), \( b, c \in \mathbb{R} \) and \( \varphi : \mathbb{R} \to \mathbb{R} \) is a function with the following properties:

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• $\varphi(s) \geq 0$ for $s > 0$ and $\varphi(s) \leq 0$ for $s < 0$;
• for every $k \in \mathbb{Z}$ the function $\varphi(s) - k$ changes its sign at most once;
• we have
\[
\left( \int_{-\infty}^{-1} + \int_{1}^{\infty} \right) \frac{|\varphi(s)|}{|s|^3} ds < \infty; \tag{1.1}
\]
• $|\varphi(s)| \leq 1$ for $s$ in some neighbourhood of 0, and we have
\[
\int_{0}^{1} \frac{1 - \varphi(s)}{s} ds = \infty, \quad \int_{0}^{1} \frac{1 + \varphi(-s)}{s} ds = \infty. \tag{1.2}
\]
Then $f$ is weakly bell-shaped. If in addition $f$ is smooth, then $f$ is bell-shaped. Furthermore, any parameters $a, b, c$ and $\varphi$ with the properties listed above correspond to some weakly bell-shaped function $f$ (which may contain an extra atom at $b$).

The integrability condition \(1.3\) is required in order to assert convergence of the integral in the expression for $Lf$. Condition \(1.2\) is also rather natural: as it is explained in Section 5, it is in a sense equivalent to convergence of the corresponding function $f$ to zero at $\pm \infty$. However, the sign-change condition may seem a bit artificial: we require that there is a non-decreasing sequence of points $s_k \in [-\infty, \infty]$, where $k \in \mathbb{Z}$, such that $\varphi(s) \in [k, k + 1]$ for $s \in (s_k, s_{k+1})$. The function $\varphi$ is therefore uniformly close to a non-decreasing function, but it may have downward jumps of size up to 1, as long as it does not cross any integer. Interestingly, this rather unnatural assumption on $\varphi$ cannot be significantly relaxed, as shown by the examples in Section 6.

The expression for the Fourier transform of $f$ can be written in a Lévy–Khintchine-type fashion (as in Corollary 1.2 below), with Lévy measure $\nu(x)dx$, where for $x > 0$, $\nu(x)$ and $\nu(-x)$ are the Laplace transforms of $\varphi(s) \mathbb{I}_{(0,\infty)}(s)$ and $-\varphi(-s) \mathbb{I}_{(0,\infty)}(s)$, respectively. In particular, $\nu(x)$ and $\nu(-x)$ are completely monotone on $(0, \infty)$.

With the above notation, the function $\varphi$ is non-decreasing if and only if $x\nu(x)$ and $x\nu(-x)$ are completely monotone functions of $x > 0$. This condition characterises a class of functions which is sometimes called extended generalised gamma convolutions, a subclass of the class of (density functions of) infinitely divisible distributions; we refer to Chapter 7 in [2] for a detailed discussion. Theorem 1.1 asserts that all such functions are weakly bell-shaped.

**Corollary 1.2.** Suppose that $f$ is a locally integrable function which converges to zero at $\pm \infty$, is decreasing near $\infty$ and increasing near $-\infty$. Suppose furthermore that for $\xi \in \mathbb{R} \setminus \{0\}$ the Fourier transform of $f$ satisfies
\[
\int_{-\infty}^{\infty} e^{-ix} f(x) dx = \exp \left( -a\xi^2 - ib\xi + c + \int_{-\infty}^{\infty} (e^{-ix} - (1 - i\xi x)e^{-|x|}) \nu(x) dx \right);
\]
here $a \geq 0$, $b, c \in \mathbb{R}$ and $\nu : \mathbb{R} \to \mathbb{R}$ is a function with the following properties:
• $x\nu(x)$ and $x\nu(-x)$ are completely monotone functions of $x > 0$;
• we have
\[
\int_{-1}^{1} x^2 \nu(x) dx < \infty. \tag{1.3}
\]
Then $f$ is weakly bell-shaped. If in addition $f$ is smooth, then $f$ is bell-shaped. Furthermore, any parameters $a, b, c$ and $\nu$ with the properties listed above correspond to some weakly bell-shaped function $f$ (which may contain an extra atom at $b$).

In particular, if $f$ is the density function of an infinitely divisible distribution with Gaussian coefficient $a \geq 0$, drift $b \in \mathbb{R}$ and Lévy measure $\nu(x)dx$, and if $x\nu(x)$ and $x\nu(-x)$ are completely monotone on $(0, \infty)$, then $f$ is weakly bell-shaped.
As a special case, we obtain a result that was given in [5] with an erroneous proof, unless the index of stability is \(2\) or \(1/n\) for some \(n = 1, 2, \ldots\); we refer to [15] for a detailed discussion.

**Corollary 1.3.** All stable distributions on \(\mathbb{R}\) have bell-shaped density functions.

As mentioned above, Corollary 1.3 was proved by T. Simon in [15] in the one-sided case, that is, when the distribution is concentrated on \((0, \infty)\). The argument used in [15] involves factorisation of the density \(f\) of a positive stable distribution into the convolution of a Pólya frequency function and a completely monotone function on \((0, \infty)\). Although this is not stated explicitly in [15], the proof given there can be easily adapted to prove a special case of Theorem 1.1 corresponding to functions \(f\) which are constant zero on \((-\infty, 0)\); this extension is in fact applied in [8]. A remark at the end of [15] explains that a similar proof in the two-sided case is not possible, because one of the convolution factors is no longer completely monotone.

The idea of the proof of Theorem 1.1 is very much inspired by T. Simon’s work: we show that it is enough to represent \(f\) as a convolution of a Pólya frequency function and what we call an **absolutely monotone-then-completely monotone** function. This, however, requires a completely different approach, which turns out to be shorter and more elementary than the method of [15].

Three plausible extensions of Theorem 1.1 are disproved in Section 6. A complete description of all bell-shaped functions remains a widely open problem: there is no good conjecture on their characterisation, and the author would be very surprised if Theorem 1.1 described all of them; this question is closely related to the study of zeroes of holomorphic functions; we refer the reader to [1, 4, 9, 10, 11] for further discussion and references.

The article is structured as follows. After a formal definition of the class of bell-shaped functions in Section 2, we prove in Section 3 that absolutely monotone-then-completely monotone functions are weakly bell-shaped. The definition of Pólya frequency functions and their variation diminishing property is discussed in Section 4. Section 5 contains the proof of main results. Finally, in Section 6 we discuss some examples and counterexamples.

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### 2. Bell-shaped functions

All functions and measures in this article are Borel, and if \(f\) is a measure, we denote the density function of (the absolutely continuous part of) \(f\) by the same symbol \(f\). The Gauss–Weierstrass kernel is defined by

\[
g_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(4t)}
\]

for \(t > 0\) and \(x \in \mathbb{R}\). We say that a function \(f : \mathbb{R} \to \mathbb{R}\) **changes its sign** exactly \(n\) times if \(n + 1\) is the maximal length \(m\) of a strictly increasing sequence \(x_1, x_2, \ldots, x_m \in \mathbb{R}\) such that \(f(x_j)f(x_{j+1}) < 0\) for \(j = 1, 2, \ldots, m - 1\). If \(f\) is differentiable and \(f'(x) \neq 0\) whenever \(f(x) = 0\), then \(f\) changes its sign \(n\) times if and only if it has \(n\) zeroes.
Definition 2.1. A smooth function function \( f : \mathbb{R} \to \mathbb{R} \) is said to be strictly bell-shaped if for every \( n = 0, 1, 2, \ldots \) the \( n \)-th derivative \( f^{(n)} \) of \( f \) converges to zero at \( \pm \infty \) and \( f^{(n)} \) changes its sign exactly \( n \) times.

A function \( f : \mathbb{R} \to \mathbb{R} \) (or a measure \( f \) on \( \mathbb{R} \)) is said to be weakly bell-shaped if for every \( t > 0 \) the function \( f * g_t \) is well-defined and strictly bell-shaped.

In Section [4] we will see that strictly bell-shaped functions are weakly bell-shaped, and that for a function (or a measure) \( f \) to be weakly bell-shaped it is sufficient to assume that \( f * g_t \) is strictly bell-shaped for some sequence of \( t > 0 \) that converges to zero.

It is easy to see that a pointwise limit of a sequence of functions that change its sign exactly \( n \) times is a function that changes its sign at most \( n \) times. As a consequence, if \( f_k \) is a sequence of strictly bell-shaped functions such that as \( k \to \infty \), the derivatives \( f_k^{(n)}, n = 0, 1, 2, \ldots, \) converge pointwise to the corresponding derivatives \( f^{(n)} \) of some function \( f \), then either \( f \) is constant zero or \( f \) is strictly bell-shaped. This, in turn, implies that if \( f_k \) is a sequence of weakly bell-shaped functions (or measures) that converges vaguely to a function (or a measure) \( f \) as \( k \to \infty \), then either \( f \) is constant zero or \( f \) is weakly bell-shaped. In particular, if \( f \) is a weakly bell-shaped smooth function, then \( f \) is strictly bell-shaped: the derivatives of \( f * g_t \) converge pointwise to the corresponding derivatives of \( f \) as \( t \to 0^+ \).

3. \( \text{AM-\text{CM}} \) Functions

In this section introduce a class of \( \text{AM-\text{CM}} \) functions and we prove that they are weakly bell-shaped. We begin with definitions and notation. We say that a smooth function \( f : \mathbb{R} \to \mathbb{R} \) has a zero of multiplicity \( N \) at \( x \) if \( f^{(n)}(x) = 0 \) for \( n = 0, 1, 2, \ldots, N - 1 \) and \( f^{(N)}(x) \neq 0 \). We denote one-sided limits of a function \( f \) at \( x \) by \( f(x^+) \) and \( f(x^-) \). We will often consider measures that are sums of an absolutely continuous part (which we identify with the corresponding density function) and a Dirac measure. Since we are more tempted to think about these measures as functions with an extra atom, we will call such measures extended functions.

Definition 3.1. A function \( f : \mathbb{R} \to \mathbb{R} \) is completely monotone on an open interval \( I \) if it is smooth in \( I \) and \( (-1)^n f^{(n)}(x) \geq 0 \) for every \( n = 0, 1, 2, \ldots \) and \( x \in I \). When the inequality \( f^{(n)}(x) \geq 0 \) is satisfied instead, then \( f \) is said to be absolutely monotone on \( I \).

We write \( \text{CM}_+ \) for the class of completely monotone functions on \((0, \infty)\) and \( \text{CM} \) for the class of completely monotone functions on \( \mathbb{R} \). Similarly, we write \( \text{AM}_- \) and \( \text{AM} \) for the classes of absolutely monotone functions on \((-\infty, 0)\) and on \( \mathbb{R} \), respectively.

We say that \( f : \mathbb{R} \to \mathbb{R} \) is absolutely monotone-then-completely monotone if it is absolutely monotone on \((-\infty, 0)\) and completely monotone on \((0, \infty)\). More generally, we allow \( f \) to be an extended function, comprising of an absolutely monotone-then-completely monotone function on \( \mathbb{R} \setminus \{0\} \), and possibly a non-negative atom at \( 0 \). We write \( \text{AM-\text{CM}} \) for the class of absolutely monotone-then-completely monotone extended functions.

The Laplace transform of a measure \( \mu \) on \( \mathbb{R} \) or a function \( f : \mathbb{R} \to \mathbb{R} \) is defined by

\[
\mathcal{L}\mu(x) = \int_{\mathbb{R}} e^{-sx} \mu(ds),
\]

\[
\mathcal{L}f(z) = \int_{-\infty}^{\infty} e^{-sz} f(x)dx,
\]

whenever the integrals converge. By Bernstein’s theorem, \( f \in \text{CM}_+ \) if and only if \( f(x) = \mathcal{L}\mu(x) \) for \( x > 0 \), where \( \mu \) is some non-negative measure concentrated on \([0, \infty)\), whose
Laplace transform is convergent on \((0, \infty)\). The measure \(\mu\) is often called the Bernstein measure of the completely monotone function \(f\).

By Fubini’s theorem, a non-negative measure \(\mu\) concentrated on \([0, \infty)\) is the Bernstein measure of a locally integrable completely monotone function \(f\) if and only if
\[
\int_{[0, \infty)} \frac{1}{1 + s} \mu(ds) < \infty. \tag{3.1}
\]

For a detailed treatment of completely monotone functions, we refer the reader to [12].

Denote \(f(x) = f(-x)\). Clearly, \(f \in \mathcal{C}M_+\) if and only if \(\check{f} \in \mathcal{A}M_-,\) and \(f \in \mathcal{C}M\) if and only if \(\check{f} \in \mathcal{A}M\). It follows that \(f \in \mathcal{A}M-\mathcal{C}M\) if and only if \(f\) has a non-negative atom at 0 and there are non-negative measures \(\mu_+\) and \(\mu_-\) concentrated on \([0, \infty)\) such that
\[
f(x) = \mathcal{L}\mu_-(x) \quad \text{for } x < 0,
\]
\[
f(x) = \mathcal{L}\mu_+(x) \quad \text{for } x > 0.
\]

By an analogy with the case of completely monotone functions, we call \(\mu_+\) and \(\mu_-\) the Bernstein measures of the function \(f \in \mathcal{A}M-\mathcal{C}M\).

**Lemma 3.2.** Let \(f(x) = \mathcal{L}\mu(x)\) for \(x > 0\) and \(f(x) = 0\) for \(x \leq 0\), where \(\mu\) is a measure concentrated on \([0, \infty)\) with all moments finite. Then for every \(t > 0\) and \(n = 0, 1, 2, \ldots\) there is a polynomial \(P\) of degree at most \(n - 1\) and a function \(u \in \mathcal{A}M\) such that
\[
(f * g_t)^{(n)}(x) = (P(x) + (-1)^n u(x))g_t(x) \tag{3.2}
\]
for all \(x \in \mathbb{R}\). (For \(n = 0\) we understand that \(P\) is constant zero.)

**Proof.** We proceed by induction. For \(n = 0\) the function
\[
u(x) = \frac{1}{g_t(x)} \int_0^\infty f(y)g_t(x - y)dy
\]
\[
= \int_0^\infty f(y)e^{-(y^2 - 2xy)/(4t)}dy
\]
\[
= \int_0^\infty e^{xy/(2t)} f(y)e^{-y^2/(4t)}dy
\]
\[
= 2t \int_0^\infty e^{sx} f(2ts)e^{-ts^2}ds
\]
is clearly absolutely monotone on \(\mathbb{R}\), as desired.

Suppose now that property (3.2) holds for some fixed \(n\), and apply it to the function \(g\) defined by \(g(x) = -f'(x)\) for \(x > 0\), \(g(x) = 0\) for \(x \leq 0\). Note that on \((0, \infty)\), \(g\) is the Laplace transform of the measure \(s\mu(ds)\), and \(s\mu(ds)\) also has all moments finite, so \(g\) satisfies the assumptions for (3.2). It follows that
\[
(g * g_t)^{(n)} = (P + (-1)^n u)g_t
\]
for some polynomial \(P\) of degree \(n - 1\) and some \(u \in \mathcal{A}M\). Recall that \(\mu\) is a finite measure, so that \(f(0^+) = \mu(\mathbb{R})\) is finite (this is where the assumption on the moments of \(\mu\) is needed). It follows that
\[
(f * g_t)'(x) = f(0^+)g_t(x) - g * g_t(x)
\]
for all \(x \in \mathbb{R}\). Therefore,
\[
(f * g_t)^{(n+1)} = f(0^+)g_t^{(n)}(x) - (g * g_t)^{(n)}(x)
\]
\[
= \left(\frac{f(0^+)g_t^{(n)}(x)}{g_t(x)} - P(x) - (-1)^n u(x)\right)g_t(x)
\]
Corollary 3.3. Let \( g_t^{(n)} / g_t \) be a polynomial of degree \( n \) (namely, it is the \( n \)-th Hermite polynomial evaluated at \( x / \sqrt{2t} \)), up to multiplication by a constant, the desired result for the derivative of degree \( n + 1 \) follows. This completes the proof by induction. \( \square \)

By replacing \( f \) with \( \tilde{f} \), we immediately obtain the following corollary.

**Corollary 3.3.** Let \( f(x) = L_\mu(-x) \) for \( x < 0 \) and \( f(x) = 0 \) for \( x \geq 0 \), where \( \mu \) is a measure concentrated on \([0, \infty)\) with all moments finite. Then for every \( t > 0 \) and \( n = 0, 1, 2, \ldots \) there is a polynomial \( P \) of degree at most \( n - 1 \) and a function \( v \in \mathcal{CM} \) such that

\[
(f * g_t)^{(n)}(x) = (P(x) + v(x))g_t(x)
\]

for all \( x \in \mathbb{R} \). (For \( n = 0 \) we understand that \( P \) is constant zero.) \( \square \)

In order to prove the main theorem of this section, we need one more technical result.

**Lemma 3.4.** Suppose that \( u \in \mathcal{AM}, v \in \mathcal{CM} \), and \( P \) is a polynomial of degree at most \( n \), with coefficient at \( x^n \) non-negative. Then the equation

\[
P(x) + u(x) + (-1)^n v(x) = 0,
\]

if not satisfied for all \( x \in \mathbb{R} \), has at most \( n \) real solutions (counting multiplicities).

**Proof.** Again, we proceed by induction. For \( n = 0 \), the left-hand side of (3.3) is either constant zero or everywhere strictly positive, as desired. Suppose now that the assertion of the lemma is true for some \( n \). Consider a function \( f = P + u + (-1)^{n+1} v \), where \( u \in \mathcal{AM}, v \in \mathcal{CM} \), and \( P \) is a polynomial of degree at most \( n + 1 \), with coefficient at \( x^{n+1} \) non-negative. Suppose, that \( f \) has more than \( n + 1 \) real zeroes (counting multiplicities). By Rolle’s theorem and the definition of the multiplicity of a zero, the derivative of \( f \) has more than \( n \) zeroes (counting multiplicities). However,

\[
f' = P' + u' + (-1)^n (-v'),
\]

and in the right-hand side \( u' \in \mathcal{AM}, -v' \in \mathcal{CM} \) and \( P' \) is a polynomial of degree at most \( n \), with coefficient at \( x^n \) non-negative. By the induction hypothesis, \( f' \) either has at most \( n \) zeroes (counting multiplicities) or it is constant zero. Since we already know that \( f' \) has at least \( n + 1 \) zeroes, it follows that \( f' \) is identically zero, which means that \( f \) is constant. Since \( f \) has at least one zero, \( f \) is constant zero, as desired. This completes the proof by induction. \( \square \)

We note that an (extended) function \( f \in \mathcal{AM} \) is locally integrable if and only if the corresponding Bernstein measures \( \mu_+ \) and \( \mu_- \) satisfy condition (3.1), and any such measures are Bernstein measures of some \( f \in \mathcal{AM} \).

**Theorem 3.5.** If \( f \in \mathcal{AM} \), \( f \) is locally integrable, \( f \) is not identically zero, but \( f \) converges to zero at \( \pm \infty \), then \( f \) is weakly bell-shaped.

**Proof.** Recall that \( f \) may have a non-negative atom at zero, and denote \( m = f(\{0\}) \geq 0 \). On \( \mathbb{R} \setminus \{0\} \), \( f \) is a pure function, completely monotone on \((0, \infty)\) and absolutely monotone on \((-\infty, 0)\). We first assume that the Bernstein measures \( \mu_+, \mu_- \) corresponding to \( f \) have all moments finite. In this case, by Lemma 3.2 and Corollary 3.3 for every \( t > 0 \) and \( n = 0, 1, 2, \ldots \) there are functions \( u \in \mathcal{AM} \) and \( v \in \mathcal{CM} \), as well as a polynomial \( P \) of degree at most \( n - 1 \), such that

\[
(f * g_t)^{(n)}(x) = (-1)^n \left( m \frac{(-1)^n g_t^{(n)}(x)}{g_t(x)} + P(x) + u(x) + (-1)^n v(x) \right) g_t(x)
\]
for all $x \in \mathbb{R}$. (For $n = 0$ we understand that $P$ is constant zero). However, $(-1)^n \frac{g_k^{(n)}}{g_k}$ is a polynomial of degree $n$, with coefficient at $x^n$ positive. Therefore, by Lemma 3.3, the equation $(f * g_k)^{(n)}(x) = 0$ has at most $n$ solutions (counting multiplicities). As a consequence, $f * g_k$ is strictly bell-shaped. Since $t > 0$ is arbitrary, $f$ is weakly bell-shaped, as desired.

For general Bernstein measures $\mu_+, \mu_-$ we proceed by approximation. For $k = 1, 2, \ldots$ we define the extended function $f_k \in \mathcal{AM}-\mathcal{CM}$ so that $f_k(\{0\}) = f(\{0\})$ and on $\mathbb{R} \setminus \{0\}$,

$$f_k(x) = \begin{cases} e^{sx} \mu_+(dx) & \text{for } x < 0, \\ e^{-sx} \mu_+(dx) & \text{for } x > 0. \end{cases}$$

By the first part of the proof, for every $k = 1, 2, \ldots$ the extended function $f_k$ is weakly bell-shaped, unless it is identically zero. Furthermore, the density functions of $f_k$ converge monotonically to the density function of $f$ as $k \to \infty$, and so $f$ is either identically zero or weakly bell-shaped. \hfill \Box

**Example 3.6.** For any $p \in (0, 1)$ the functions $f(x) = |x|^{-p}$ and $g(x) = x^{-p} \mathbb{1}_{(0,\infty)}(x)$ are weakly bell-shaped.

**Remark 3.7.** Let us say that an extended function $f$ is *strictly bell-shaped in the broad sense* if we only require that $f^{(n)}$ converges to zero at $\pm \infty$ and changes its sign exactly $n$ times for $n = 1, 2, \ldots$, but not necessarily for $n = 0$. Similarly, we introduce the notion of a *weakly bell-shaped function in the broad sense*.

By repeating the proof of Theorem 3.5 it is easy to see that a locally integrable extended function $f$ is weakly bell-shaped in the broad sense if we assume that $f(\{0\}) \geq 0$, $f$ is locally integrable, $-f'$ is completely monotone on $(0, \infty)$ and $f'$ is absolutely monotone on $(-\infty, 0)$. In this case we have

$$f(x) = c_- + \int_{(0,\infty)} (e^{sx} - e^{-s}) \mu_-(ds) \quad \text{for } x < 0,$$

$$f(x) = c_+ + \int_{(0,\infty)} (e^{-sx} - e^{-s}) \mu_+(ds) \quad \text{for } x > 0,$$

where $c_-$ and $c_+$ are arbitrary real constants, and $\mu_-$ and $\mu_+$ are arbitrary non-negative measures concentrated on $(0, \infty)$ such that

$$\int_{(0,\infty)} \frac{s}{(1+s)^2} \mu_+(ds) < \infty, \quad \int_{(0,\infty)} \frac{s}{(1+s)^2} \mu_-(ds) < \infty.$$

Such a function $f$ need not be positive, and it may converge to $-\infty$ at $\infty$ or at $-\infty$. The derivative $f'$ of $f$ is, however, completely monotone on $(0, \infty)$, and $-f'$ is absolutely monotone on $(-\infty, 0)$.

**Example 3.8.** For any $p \in (0, 1)$ the functions $f(x) = -|x|^p$ and $g(x) = -x^p \mathbb{1}_{(0,\infty)}(x)$ are weakly bell-shaped in the broad sense. As can be directly checked, for any $p \in (0, 1/2)$ the function $h(x) = -(1 + x^2)^{-p}$ is strictly bell-shaped in the broad sense.

4. **Pólya frequency functions**

The class of Pólya frequency functions is the closure (with respect to vague convergence of measures) of the class of convolutions of exponential distributions. The best way do describe this class involves the Fourier transform (or the characteristic function), which we identify with the restriction of the Laplace transform to the imaginary axis $i\mathbb{R}$. For
this reason we re-use the notation and denote the Fourier transform of \( f \) by \( \mathcal{L}f(z) \), where we understand that \( z \in i\mathbb{R} \).

**Definition 4.1.** An integrable function \( f : \mathbb{R} \to \mathbb{R} \) is a \textit{Pólya frequency function} if its Fourier transform satisfies

\[
\mathcal{L}f(z) = e^{az^2-bz} \prod_{n=1}^{N} \frac{e^{z/z_n}}{1 + z/z_n}
\]

for all \( z \in i\mathbb{R} \); here \( a \geq 0, \ b \in \mathbb{R}, \ N \in \{0, 1, 2, \ldots, \infty\}, \ z_n \in \mathbb{R} \setminus \{0\} \), and

\[
\sum_{n=1}^{N} \frac{1}{|z_n|^2} < \infty.
\]

For simplicity, we abuse the notation and agree that Dirac measures \( \delta_b \) (which correspond to \( a = 0 \) and \( N = 0 \)) are also Pólya frequency functions, so formally Pólya frequency function is an extended function.

The definition can be equivalently phrased as follows: \( f \) is a Pólya frequency function if and only if \( f \) is the convolution of the Gauss–Weierstrass kernel \( g_a \) (if \( a > 0 \)), the Dirac measure \( \delta_b \) and the (finite or infinite) family of shifted exponential measures

\[
|z_n|e^{-z_n(x-1)} \mathbb{I}_{(0,\infty)}(z_n(x-1))dx
\]

with mean 0 and variance \( |z_n|^{-2} \).

**Definition 4.2.** An integrable extended function \( f \) is said to be a \textit{variation diminishing convolution kernel} if for all bounded functions \( g \) the function \( f \ast g \) changes its sign at most as many times as the function \( g \) does.

We recall the following fundamental result of Schoenberg, proved originally in [13, 14].

**Theorem 4.3** (Schoenberg; see Chapter IV in [16]). An integrable function is a variation diminishing convolution kernel if and only if it is a Pólya frequency function, up to multiplication by a constant. \( \square \)

We remark that we will in fact only need the direct part of the above theorem, that is, the fact that Pólya frequency functions are variation diminishing convolution kernels. The proof of this statement is relatively simple: it is not very difficult to show that \( e^{-x} \mathbb{I}_{(0,\infty)}(x) \) is a variation diminishing convolution kernel, and then the desired result follows by approximation of any Pólya frequency function by convolutions of exponentials \( |z|e^{-z^2} \mathbb{I}_{(0,\infty)}(zx) \) (up to translation). On the other hand, the converse part of Theorem 4.3 is a deep result that involves the concept of total positivity.

Since the Gauss–Weierstrass kernel is strictly bell-shaped, Schoenberg’s theorem asserts that Pólya frequency functions are weakly bell-shaped. It also implies the following properties of bell-shaped functions, which were already mentioned in the previous section.

**Corollary 4.4.** If \( f \) is a strictly bell-shaped function, then it is also a weakly bell-shaped function. A non-negative extended function \( f \) is weakly bell-shaped if and only if \( f \ast g_t \) is well-defined for every \( t > 0 \) and strictly bell-shaped for some sequence of \( t > 0 \) that converges to 0.

**Proof.** For every \( t > 0 \), the Gauss–Weierstrass kernel \( g_t \) is a Pólya frequency function, and therefore it is a variation diminishing convolution kernel. Therefore, if \( f \) is strictly bell-shaped, so is \( f \ast g_t \) for every \( t > 0 \). Similarly, if \( f \) is a non-negative extended function such that \( f \ast g_t \) is strictly bell-shaped for some \( t > 0 \) and \( f \ast g_{t+s} \) is well-defined for some \( s > 0 \), then \( f \ast g_{t+s} = (f \ast g_t) \ast g_s \) is strictly bell-shaped. \( \square \)
5. Synthesis

Clearly, the convolution of a bell-shaped function with a variation diminishing convolution kernel is again a bell-shaped function. In this section we describe the class of convolutions of AM-CM functions (which we already know to be bell-shaped) and Pólya frequency functions (which are precisely variation diminishing convolution kernels) in terms of Fourier transform. Recall that we identify the Fourier transform of \( f \) with the restriction of the Laplace transform \( \mathcal{L}f \) to the imaginary axis \( i\mathbb{R} \).

For integrable (extended) functions \( f \) the Fourier transform is defined as a convergent integral. We will consider a more general situation, described in the following definition.

**Definition 5.1.** We say that an (extended) function \( f \) is regular if it is a sum of an integrable (extended) function \( g \) and a locally integrable function \( h \) with the following properties: \( h \) converges to zero at \( \pm \infty \), \( h \) is continuously differentiable and \( h' \) is an integrable function.

If \( f \) is a regular (extended) function and \( z \in i\mathbb{R} \setminus \{0\} \), then the Fourier transform \( \mathcal{L}f(z) \) is well-defined as an improper integral: if \( f = g + h \) as in the above definition, then

\[
\mathcal{L}f(z) = \lim_{\alpha \to -\infty} \lim_{\beta \to \infty} \int_{\alpha}^{\beta} e^{-zx} f(x) dx
\]

\[= \int_{-\infty}^{\infty} e^{-zx} g(x) dx + \lim_{\beta \to \infty} \lim_{\alpha \to -\infty} \frac{1}{z} \left( -e^{-z\beta} h(\beta) - e^{-z\alpha} h(\alpha) - \int_{\alpha}^{\beta} e^{-zx} h'(x) dx \right) \]

\[= \int_{-\infty}^{\infty} e^{-zx} g(x) dx + \frac{1}{z} \int_{-\infty}^{\infty} e^{-zx} h'(x) dx. \]

It is easy to see that if we know in advance that \( f \) is regular, then the Fourier transform \( \mathcal{L}f(z) \) (with \( z \in i\mathbb{R} \setminus \{0\} \)) describes \( f \) completely. Furthermore, all AM-CM extended functions which converge to zero at \( \pm \infty \) are regular.

The description of Fourier transforms of AM-CM functions requires basic theory of Stieltjes functions.

**Definition 5.2.** A function \( F \) defined on \( \mathbb{C} \setminus (-\infty, 0] \) is a Stieltjes function if it is given by the formula

\[
F(z) = m + \int_{[0,\infty)} \frac{1}{z + s} \mu(ds) \tag{5.1}
\]

for some \( m \geq 0 \) and a non-negative measure \( \mu \) concentrated on \([0, \infty), \) such that the integral in the right-hand side is finite for some (or, equivalently, for all) \( z \in \mathbb{C} \setminus (-\infty, 0] \).

For a detailed discussion of the properties of Stieltjes functions we refer to Chapters 2 and 6–7 in [12].

Stieltjes functions are precisely Laplace transforms of locally integrable completely monotone function on \((0, \infty), \) possibly with an atom at 0. More precisely, let \( \mu \) be as in (5.1), and define \( f \) to be an extended function such that \( f(x) = \mathcal{L}\mu(x) \) for \( x > 0, \) \( f(x) = 0 \) for \( x < 0, \) and such that \( f(\{0\}) = m. \) Then the Laplace transform of \( f \) agrees with \( F \) defined in (5.1) for all \( z \) such that \( \text{Re} z > 0. \) If in addition \( \mu(\{0\}) = 0, \) or, equivalently, if \( f \) converges to zero at \( \infty, \) then \( f \) is regular and \( \mathcal{L}f(z) = F(z) \) also for all \( z \in i\mathbb{R} \setminus \{0\}. \)

Any Stieltjes function \( F(z) \) satisfies \( F(z) \geq 0 \) when \( z > 0 \) and \( \text{Im} F(z) \leq 0 \) when \( \text{Im} z > 0. \) Furthermore, these two properties completely characterise the class of Stieltjes functions. If \( F \) is not constant zero, then \( \text{arg} F \) is a harmonic function in the upper
complex half-plane, with values in \((-\pi, 0]\). By Herglotz’s theorem, \(\arg F\) is the Poisson integral of its boundary values. Writing

\[
\varphi(s) = -\frac{1}{\pi} \lim_{t \to 0^+} \arg F(-s + it),
\]

we obtain a function \(\varphi : \mathbb{R} \to [0, 1]\) such that \(\varphi(s) = 0\) for \(s < 0\) and

\[
\arg F(z) = \int_0^\infty \text{Im} \left( \frac{1}{z + s} \varphi(s) \right) ds
\]

when \(\text{Im} z > 0\). It follows that for some \(c \in \mathbb{R}\),

\[
F(z) = \exp \left( c + \int_0^\infty \left( \frac{1}{z + s} - \frac{1}{s} \mathbf{1}_{[1,\infty)}(s) \right) \varphi(s) ds \right)
\]

(5.3)

(5.2)

and dominated convergence, \(\mu(\{0\}) = \lim_{t \to 0^+} \int_{[0,\infty)} \frac{t^2}{t^2 + s^2} \mu(ds) = \lim_{t \to 0^+} (-t \text{Im} F(it))\).

Furthermore, again by dominated convergence,

\[
\lim_{t \to 0^+} t \text{Re} F(it) = \lim_{t \to 0^+} \left( m t + \int_{[0,\infty)} \frac{t s}{t^2 + s^2} \mu(ds) \right) = 0.
\]

Therefore, \(\mu(\{0\})\) is the limit of \(|tF(it)|\) as \(t \to 0^+\). It follows that if \(F\) is also given by (5.3), then \(\mu(\{0\}) = 0\) if and only if

\[
\lim_{t \to 0^+} t \exp \left( \int_0^\infty \text{Re} \left( \frac{1}{it + s} - \frac{1}{s} \mathbf{1}_{[1,\infty)}(s) \right) \varphi(s) ds \right) = 0.
\]

(5.5)

The Stieltjes function \(1/z\) corresponds to the function \(\varphi(s) = \mathbf{1}_{(0,\infty)}(s)\) in representation (5.3). Therefore, evaluating \(|1/z|\) at \(z = it\) leads to

\[
\frac{1}{t} = \exp \left( \int_0^\infty \text{Re} \left( \frac{1}{it + s} - \frac{1}{s} \mathbf{1}_{[1,\infty)}(s) \right) ds \right) = 0.
\]

This allows us to re-write condition (5.5) as

\[
\lim_{t \to 0^+} \exp \left( \int_0^\infty \text{Re} \left( \frac{1}{it + s} - \frac{1}{s} \mathbf{1}_{[1,\infty)}(s) \right) (\varphi(s) - 1) ds \right) = 0,
\]

or, equivalently,

\[
\lim_{t \to 0^+} \int_0^\infty \left( \frac{s}{t^2 + s^2} - \frac{1}{s} \mathbf{1}_{[1,\infty)}(s) \right) (1 - \varphi(s)) ds = \infty.
\]
By dominated convergence, the integral over $[1, \infty)$ converges to zero as $t \to 0^+$. On the other hand, by monotone convergence, the integral over $(0, 1)$ converges as $t \to 0^+$ to the integral of $s^{-1}(1 - \varphi(s))$. This completes the proof of (5.4).

Replacing $f$ by $\tilde{f}$ leads to a similar description of Laplace transforms of absolutely monotone functions on $(-\infty, 0)$. By combining these two results, we obtain the following statement.

**Proposition 5.3.** (a) Suppose that $f \in \mathcal{AMCM}$, $f$ is locally integrable and $f$ converges to zero at $\pm\infty$. Then for some $m \geq 0$ and some measure $\mu$ concentrated on $\mathbb{R} \setminus \{0\}$, non-negative on $(0, \infty)$ and non-positive on $(-\infty, 0)$, we have

$$\mathcal{L}f(z) = m + \int_{\mathbb{R} \setminus \{0\}} \frac{1}{z + s} \mu(ds)$$

(5.6)

for all $z \in i\mathbb{R} \setminus \{0\}$. Furthermore, if $f$ is not constant zero, then for some $c \in \mathbb{R}$ and some function $\varphi : \mathbb{R} \to [-1, 1]$, non-negative on $(0, \infty)$ and non-positive on $(-\infty, 0)$, we have

$$\mathcal{L}f(z) = \exp \left(c + \int_{-\infty}^{\infty} \left( \frac{1}{z + s} - \frac{1}{s} \mathbf{1}_{\mathbb{R} \setminus (-1, 1)}(s) \right) \varphi(s) ds \right)$$

(5.7)

for all $z \in i\mathbb{R} \setminus \{0\}$. Additionally, $\varphi$ satisfies

$$\int_0^1 \frac{1 - \varphi(s)}{s} ds = \infty, \quad \int_0^1 \frac{1 + \varphi(-s)}{s} ds = \infty.$$  

(5.8)

(b) If $c \geq 0$ and $\mu$ is a measure concentrated on $\mathbb{R} \setminus \{0\}$, non-negative on $(0, \infty)$ and non-positive on $(-\infty, 0)$, and the integral in the right-hand side of (5.6) is finite for some (or, equivalently, for all) $z \in i\mathbb{R} \setminus \{0\}$, then there is a corresponding locally integrable $f \in \mathcal{AMCM}$ such that $f$ converges to zero at $\pm\infty$ and (5.6) holds for $z \in i\mathbb{R} \setminus \{0\}$.

(c) If $c > 0$, $\varphi : \mathbb{R} \to [-1, 1]$, $\varphi$ is non-negative on $(0, \infty)$ and non-positive on $(-\infty, 0)$, and condition (5.8) is satisfied, then there is a corresponding locally integrable $f \in \mathcal{AMCM}$ such that $f$ converges to zero at $\pm\infty$ and (5.6) holds for $z \in i\mathbb{R} \setminus \{0\}$. \hfill \Box

If $\mu_+$ and $\mu_-$ are the Bernstein measures corresponding to $f \in \mathcal{AMCM}$, then the measure $\mu$ in the above result can be written as $\mu = \mu_+ - \mu_-$, that is, $\mu(A) = \mu_+(A) - \mu_-(A)$ for all sets $A$. Similarly, if $\varphi_+$ and $\varphi_-$ correspond to the representation (5.3) of $\mu_+$ and $\mu_-$, then the function $\varphi$ in the theorem satisfies $\varphi(s) = \varphi_+(s) - \varphi_-(s)$. In particular, formula (5.2) with $F = \mathcal{L}f$ remains valid.

We stress that condition (5.8) in part (c) is required to assert that the measures $\mu_+$ and $\mu_-$ have no atoms at 0, so that there is a corresponding $f \in \mathcal{AMCM}$ which converges to zero at $\pm\infty$.

The next result is merely a reformulation of the definition of a Pólya frequency function.

**Proposition 5.4.** For an integrable extended function $f$, the following conditions are equivalent:

(a) $f$ is a Pólya frequency function, that is, the Fourier transform of $f$ has the representation

$$\mathcal{L}f(z) = e^{a z^2 - b z} \prod_{n=1}^N \frac{e^{z/n}}{1 + z/n}$$
for all \( z \in i\mathbb{R} \), where, as in Definition 4.1, \( a > 0 \), \( b \in \mathbb{R} \) and
\[
\sum_{n=1}^{N} \frac{1}{|z_n|^2} < \infty;
\]
(b) for some non-decreasing function \( \varphi : \mathbb{R} \to \mathbb{Z} \) which satisfies the condition
\[
\int_{-\infty}^{\infty} \frac{|\varphi(s)|}{|s|^3} \, ds < \infty,
\]
the Fourier transform of \( f \) satisfies
\[
\mathcal{L} f(z) = \exp \left( az^2 - bz + \int_{-\infty}^{\infty} \left( \frac{1}{z+s} - \frac{1}{s} + \frac{z}{s^2} \right) \varphi(s) \, ds \right)
\]
for all \( z \in i\mathbb{R} \).

Proof. Suppose that \( z \in i\mathbb{R} \). Observe that if \( z_n > 0 \), then
\[
z/z_n - \log(1 + z/z_n) = \int_{z_n}^{\infty} \left( \frac{1}{s + z} - \frac{1}{s} + \frac{z}{s^2} \right) \, ds,
\]
while if \( z_n < 0 \), then
\[
z/z_n - \log(1 + z/z_n) = - \int_{-\infty}^{z_n} \left( \frac{1}{s + z} - \frac{1}{s} + \frac{z}{s^2} \right) \, ds.
\]
Therefore, if \( f \) is a Pólya frequency function such that \( \mathcal{L} f \) has the representation given in the statement of the theorem, and if
\[
\varphi(s) = \sum_{n=1}^{N} \left( \mathbb{I}_{(0,\infty)}(z_n) \mathbb{I}_{[z_n,\infty)}(s) - \mathbb{I}_{(-\infty,0)}(z_n) \mathbb{I}_{(-\infty,z_n)}(s) \right),
\]
then \( \varphi : \mathbb{R} \to \mathbb{Z} \) is non-decreasing, \( \varphi(0) = 0 \),
\[
\int_{-\infty}^{\infty} \frac{\varphi(s)}{s^3} \, ds = \frac{1}{2} \sum_{n=1}^{N} \frac{1}{|z_n|^2};
\]
and
\[
\prod_{n=1}^{N} \frac{e^{z/z_n}}{1 + z/z_n} = \exp \left( \int_{-\infty}^{\infty} \left( \frac{1}{z+s} - \frac{1}{s} + \frac{z}{s^2} \right) \varphi(s) \, ds \right),
\]
as desired. Conversely, if \( \varphi : \mathbb{R} \to \mathbb{Z} \) is a right-continuous non-decreasing function and \( |s^{-3}\phi(s)| \) is integrable, then necessarily \( \varphi(s) = 0 \) for \( s \) in some neighbourhood of 0. It follows that \( \varphi \) can be written in the form given in (5.10), and (5.11) remains valid. Therefore, there is a corresponding Pólya frequency function \( f \) such that (5.9) is satisfied.

We are now in position to describe the class of convolutions of \( \mathcal{A} \mathcal{M} - \mathcal{C} \mathcal{M} \) functions and Pólya frequency functions.

**Lemma 5.5.** A regular extended function \( f \) is the convolution of a locally integrable \( g \in \mathcal{A} \mathcal{M} - \mathcal{C} \mathcal{M} \) and a Pólya frequency function \( h \) if and only if
\[
\mathcal{L} f(z) = \exp \left( az^2 - bz + c + \int_{-\infty}^{\infty} \left( \frac{1}{z+s} - \frac{1}{s} + \frac{z}{s^2} \right) \mathbb{I}_{\mathbb{R}\setminus(-1,1)}(s) \varphi(s) \, ds \right)
\]
for all \( z \in i\mathbb{R} \setminus \{0\} \); here \( a \geq 0 \), \( b, c \in \mathbb{R} \) and \( \varphi : \mathbb{R} \to \mathbb{R} \) has the following properties: \( \varphi(s) \geq 0 \) for \( s > 0 \), \( \varphi(s) \leq 0 \) for \( s < 0 \);
\[
\text{for all } k \in \mathbb{Z} \text{ the function } \varphi(s) - k \text{ changes its sign at most once;}
\]
we have

\[
\left( \int_{-\infty}^{-1} + \int_{1}^{\infty} \right) \frac{|\varphi(s)|}{|s|^3} \, ds < \infty; \tag{5.14}
\]

and finally \( |\varphi(s)| \leq 1 \) for \( s \) in some neighbourhood of \( 0 \), and

\[
\int_{0}^{1} \frac{1 - \varphi(s)}{s} \, ds = \infty, \quad \int_{0}^{1} \frac{1 + \varphi(-s)}{s} \, ds = \infty. \tag{5.15}
\]

In this case \( g \) converges to zero at \( \pm \infty \). Furthermore, any parameters \( a, b, c \) and \( \varphi \) with the above properties correspond to some regular extended function \( f \).

**Proof.** Suppose that \( g \in \mathcal{AM}\mathcal{CM} \), \( g \) converges to zero at \( \pm \infty \) and \( h \) is a Pólya frequency function. Then the convolution \( f = g * h \) is regular (that is, it converges to zero at \( \pm \infty \)), and we have \( \mathcal{L}f(z) = \mathcal{L}g(z) \mathcal{L}h(z) \) for all \( z \in i\mathbb{R} \setminus \{0\} \). By Propositions 5.3 and 5.4 there are constants \( \tilde{a} \geq 0, \tilde{b}, \tilde{c} \in \mathbb{R} \) and functions \( \varphi_1 : \mathbb{R} \to [−1, 1] \) and \( \varphi_2 : \mathbb{R} \to \mathbb{Z} \), satisfying appropriate conditions, such that for all \( z \in i\mathbb{R} \setminus \{0\} \) we have

\[
\mathcal{L}f(z) = \exp \left( \tilde{a} z^2 - \tilde{b} z + \tilde{c} + \int_{-\infty}^{\infty} \left( \frac{1}{z + s} - \frac{1}{s} \mathbb{1}_{\mathbb{R} \setminus (-1,1)}(s) \right) \varphi_1(s) \, ds \right) \left( 1 - \frac{1}{s} + \frac{z}{s^2} \right) \varphi_2(s) \, ds.
\]

This is equivalent to (5.12) with \( \varphi(s) = \varphi_1(s) + \varphi_2(s) \) and \( a = \tilde{a} \),

\[
b = \tilde{b} + \left( \int_{-\infty}^{-1} + \int_{1}^{\infty} \right) \frac{1}{s^2} \varphi_1(s) \, ds - \int_{-1}^{1} \frac{1}{s^2} \varphi_2(s) \, ds,
\]

\[
c = \tilde{c} - \int_{-1}^{1} \frac{1}{s} \varphi_2(s) \, ds. \tag{5.16}
\]

Indeed, \( \varphi_1 \) is bounded, and \( \varphi_2(s) = 0 \) for \( s \) in some neighbourhood of \( 0 \), so that all integrals in (5.16) are convergent. It remains to verify that \( \varphi \) has all the desired properties.

Since both \( \varphi_1 \) and \( \varphi_2 \) are non-negative on \( (0, \infty) \) and non-positive on \( (-\infty, 0) \), \( \varphi \) has the same property. Integrability of \( |\varphi(s)|^3/|s|^3 \) over \( \mathbb{R} \setminus (-1, 1) \) follows from integrability of \( |\varphi_2(s)|^3/|s|^3 \) and from \( \varphi_1 \) being bounded. Furthermore, \( \varphi(s) \in [\varphi_2(s), \varphi_2(s) + 1] \) for \( s > 0 \), while \( \varphi(s) \in [\varphi_2(s) - 1, \varphi_2(s)] \) for \( s < 0 \), which easily implies that for \( k \in \mathbb{Z} \setminus \{0\} \) the function \( \varphi(s) - k \) changes its sign at most once, at a point of discontinuity of \( \varphi_2 \); this change takes place at the argument \( s \) such that \( \varphi_2(s^-) < k \leq \varphi_2(s^+) \) when \( k > 0 \), and at the argument \( s \) such that \( \varphi_2(s^-) \leq k < \varphi_2(s^+) \) when \( k < 0 \) (if such \( s \) does not exists, then \( \varphi(s) - k \) does not change its sign). When \( k = 0 \), \( \varphi(s) - k \) changes its sign at \( s = 0 \). Finally, property (5.15) is satisfied, because \( \varphi_1 \) satisfies condition (5.8) in Proposition 5.3 and \( \varphi_2(s) = 0 \) for \( s \) in some neighbourhood of \( 0 \).

We proceed with the proof the converse implication. Suppose that \( f \) is a regular function that satisfies (5.12), with all conditions listed in the statement of the lemma. In particular, for \( k \in \mathbb{Z} \setminus \{0\} \) the function \( \varphi(s) - k \) changes its sign at most once for all \( k \in \mathbb{Z} \setminus \{0\} \); let us denote the argument at which this change occurs by \( s_k \), namely,

\[
s_k = \sup \{ s \in \mathbb{R} : \varphi(s) \leq k \} \quad \text{if } k > 0,
\]

\[
s_k = \inf \{ s \in \mathbb{R} : \varphi(s) \geq k \} \quad \text{if } k < 0
\]

(possibly \( s_k = \infty \) for some \( k > 0 \) and \( s_k = -\infty \) for some \( k < 0 \)). We also set \( s_0 = 0 \), and we let \( \varphi_2(s) = k \) for \( s \in [s_k, s_{k+1}] \) when \( k \geq 0 \) and \( \varphi_2(s) = k \) for \( s \in [s_{k-1}, s_k] \) when \( k \leq 0 \). We also define \( \varphi_1(s) = \varphi(s) - \varphi_2(s) \).
It is easy to see that \( \varphi_1 : \mathbb{R} \to [-1, 1] \) and that \( \varphi_1 \) is non-negative on \((0, \infty)\) and non-negative on \((-\infty, 0)\). From the definition of \( \varphi_2 \) it also follows that \( \varphi_2 \) maps \( \mathbb{R} \) into \( \mathbb{Z} \) and \( \varphi_2 \) is non-decreasing. By the assumption, \(|\varphi(s)| \leq 1\) for \( s \) in some neighbourhood of \( 0 \), and so \( s_{-1} < 0 \) and \( s_1 > 0 \). Therefore, \( \varphi_2(s) = 0 \) for \( s \in (-s_{-1}, s_1) \), and hence \(|\varphi_2(s)|/s^3\) is integrable on \((-1, 1)\). Since \(|\varphi_2(s)| \leq |\varphi(s)|\), the function \(|\varphi_2(s)|/s^3\) is also integrable over \( \mathbb{R} \setminus (-1, 1) \). Finally, since \( \varphi_2(s) = 0 \) for \( s \) in some neighbourhood of \( 0 \) and \( \varphi \) satisfies condition (5.15), the function \( \varphi_1 \) satisfies condition (5.8).

With the above information, we let \( \tilde{a} = a \) and we define \( \tilde{b} \) and \( \tilde{c} \) so that relations (5.16) are satisfied. We conclude that there is an \( AM-\text{CM} \) extended function \( g \) which converges to zero at \( \pm \infty \) and which has representation (5.7) with parameters \( \tilde{c} \) and \( \varphi_1 \), as well as a Polya frequency function \( h \) which corresponds to parameters \( \tilde{a} \), \( \tilde{b} \) and \( \varphi_2 \) in representation (5.9). By the first part of the proof, the function \( g \ast h \) is regular and has the same Fourier transform as the function \( f : \mathcal{L}f(z) = \mathcal{L}g(z) \mathcal{L}h(z) \) for all \( z \in i\mathbb{R} \setminus \{0 \} \). Thus, \( f = g \ast h \), as desired.

Finally, note that in the above construction we only used the properties of the parameters \( a, b, c \) and \( \varphi \), so for any parameters that satisfy the conditions listed in the statement of the lemma there are corresponding \( g \) and \( h \), which define a regular extended function \( f = g \ast h \).

\[\square\]

**Proof of Theorem 1.1.** The desired result follows from Lemma 5.3 combined with Theorem 5.5 and the fact that the convolution of a weakly bell-shaped function with a variation diminishing convolution kernel is again a weakly bell-shaped function.

We remark that the function \( \varphi \) in Theorem 1.1 can be identified by studying the holomorphic extension of \( \mathcal{L}f \). More precisely, \( \varphi \) is given by formula (5.2), where we take \( F \) to be the holomorphic extension of \( \mathcal{L}f \) to the upper complex half-plane (this extension is given again by (5.12), and where \( \arg F(z) \) denotes the continuous version of the complex argument of \( F(z) \), determined uniquely by an additional condition \(|\arg F(z)| < \pi\) for \( z \) in some neighbourhood of \( 0 \).

**Proof of Corollary 1.2.** Suppose that \( a \geq 0, b, c \in \mathbb{R}, \nu \) is a function such that \( x\nu(x) \) and \( x\nu(-x) \) are completely monotone functions of \( x > 0 \), and in addition

\[
\int_{-1}^{1} x^2 \nu(x) dx < \infty.
\]

(5.17)

Let \( \pi_+ \) and \( \pi_- \) denote the Bernstein measures corresponding to the completely monotone functions \( x\nu(x) \) and \( x\nu(-x) \) (with \( x > 0 \)), respectively. Finally, let \( \varphi_+(s) = \mu_+((0, s)) \) and \( \varphi_-(s) = \mu_-((0, s)) \) for \( s > 0 \). Integrating by parts, we obtain

\[
\nu(x) = \frac{1}{x} \int_{[0, \infty)} e^{-sx} \pi_+(ds) = \int_0^\infty e^{-sx} \varphi_+(s) ds = \mathcal{L} \varphi_+(x),
\]

for \( x > 0 \), and similarly \( \nu(x) = \mathcal{L} \varphi_-(-x) \) for \( x < 0 \). Condition (5.17) translates easily into

\[
\int_1^\infty \frac{\varphi_+(s)}{s^3} ds < \infty, \quad \int_1^\infty \frac{\varphi_-(s)}{s^3} ds < \infty.
\]

Let \( \varphi(s) = \varphi_+(s) \) for \( s > 0 \), \( \varphi(s) = -\varphi_-(s) \) for \( s < 0 \) and \( \varphi(0) = 0 \). Then \( \varphi \) is non-decreasing and it satisfies

\[
\left( \int_{-\infty}^{-1} + \int_1^\infty \right) \frac{|\varphi(s)|}{|s|^3} ds < \infty.
\]
Furthermore, \( \varphi(0) = 0 \) and \( \varphi \) is continuous at 0, which easily implies that
\[
\int_0^1 \frac{1 - \varphi(s)}{s} \, ds = \infty, \quad \int_0^1 \frac{1 + \varphi(-s)}{s} \, ds = \infty.
\]

We find that for \( z \in i\mathbb{R} \setminus \{0\} \), with the integrals with respect to \( x \) understood as improper integrals,
\[
\int_{-\infty}^{\infty} (e^{-xz} - (1 - xz)e^{-|x|}) \nu(x) \, dx
= \int_{-\infty}^{0} (e^{-xz} - (1 - xz)e^{-x}) \nu(x) \, dx + \int_{0}^{\infty} (e^{-xz} - (1 - xz)e^{-x}) \nu(x) \, dx
= \int_{-\infty}^{0} \left( -\frac{1}{z + s} + \frac{1}{|1 + s|} - \frac{z}{(1 + s)^2} \right) \varphi_{-}(s) \, ds
+ \int_{0}^{\infty} \left( \frac{1}{z + s} - \frac{1}{|1 + s|} + \frac{z}{(1 + s)^2} \right) \varphi_{+}(s) \, ds
= \int_{-\infty}^{\infty} \left( \frac{1}{z + s} - \text{sign} s \frac{z}{1 + |s|} + \frac{z}{(1 + |s|)^2} \right) \varphi(s) \, ds.
\]
The above calculation involves simply the use of Fubini’s theorem when \( \nu(x) \) is integrable near \( \infty \), or, equivalently, if \(|s^{-1} \varphi(s)|\) is integrable near zero. In the general case the same argument works when \( \text{Re } z > 0 \) in the integral over \( x > 0 \) and when \( \text{Re } z < 0 \) in the integral over \( x < 0 \), and the desired identity follows by continuity of both integrals as \( z \) approaches \( i\mathbb{R} \setminus \{0\} \); we omit the details.

It follows that
\[
\mathcal{L} f(z) = \exp \left( a z^2 - bz + c + \int_{-\infty}^{\infty} \left( \frac{1}{z + s} - \left( \frac{\text{sign } s}{1 + |s|} - \frac{z}{(1 + |s|)^2} \right) \right) \varphi(s) \, ds \right)
= \exp \left( \tilde{a} z^2 - \tilde{b} z + \tilde{c} + \int_{-\infty}^{\infty} \left( \frac{1}{z + s} - \left( \frac{\text{sign } s}{s} - \frac{z}{s^2} \right) \right) \mathbb{1}_{\mathbb{R}\setminus(-1,1)}(s) \varphi(s) \, ds \right),
\]
where \( \tilde{a} = a \),
\[
\tilde{b} = b + \int_{-\infty}^{\infty} \left( \frac{1}{s^2} \mathbb{1}_{\mathbb{R}\setminus(-1,1)}(s) - \frac{1}{(1 + |s|)^2} \right) \, ds,
\]
\[
\tilde{c} = c + \int_{-\infty}^{\infty} \left( \frac{1}{s} \mathbb{1}_{\mathbb{R}\setminus(-1,1)}(s) - \frac{\text{sign } s}{1 + |s|} \right) \, ds.
\]
We have thus proved that all assumptions of Theorem 1.1 are satisfied, and so its assertion applies to \( f \).

Finally, the above calculations, combined with the final assertion of Theorem 1.1, also show that any parameters \( a, b, c \) and \( \nu \) satisfying the assumptions listed in the statement of the corollary correspond to some function \( f \) with the desired properties. \( \square \)

Proof of Corollary 1.3. The density function of the Lévy measure of a stable distribution is of the form
\[
\nu(x) = c_+ x^{-\alpha} \quad \text{for } x > 0,
\]
\[
\nu(x) = c_- (-x)^{-\alpha} \quad \text{for } x < 0,
\]
where \( c_+, c_- \geq 0 \) and \( \alpha \in (0, 2) \). Therefore, \( x \nu(x) = c_+ x^{-\alpha} \) and \( x \nu(-x) = c_- x^{-\alpha} \) are completely monotone functions of \( x > 0 \), and the desired result follows from Corollary 1.2. \( \square \)
Remark 5.6. Suppose that \( g \) is a locally integrable extended function, possibly with a non-negative atom at 0, and \( g' \) is completely monotone on \((0, \infty)\) and \(-g'\) is absolutely monotone on \((-\infty, 0)\). In Remark 4.7 we observed that this class of functions is weakly bell-shaped in the broad sense. It is easy to see that any such \( g \) can be convolved with any Pólya frequency function \( h \) (because \(|g|\) grows at most at a linear rate at \( \pm \infty \), while \( h \) has exponential decay at \( \pm \infty \)), and the convolution \( f = g \ast h \) is again weakly bell-shaped in the broad sense.

Apparently, the Fourier transform of \( g \) (defined, for example, in the sense of distributions) admits a description similar to that of Proposition 5.3 in terms of extended complete Bernstein functions, discussed in Section 6.2 in [12]. It would be interesting to study this further and derive an analogue of Lemma 5.5 for the class of convolutions \( f = g \ast h \) discussed above.

6. Discussion

The sign-change condition in Theorem 1.1 requires \( \varphi(s) - k \) to change its sign at most once for \( k \in \mathbb{Z} \). As discussed in the introduction, this assumption seems rather artificial, and it is natural to ask to what extent it can be relaxed. In this section we disprove three natural conjectures and discuss several examples. We used Wolfram Mathematica 10 computer algebra system for the (otherwise tiresome, but elementary) evaluation of some explicit expressions.

6.1. Infinitely divisible distributions with AM-ÇM Lévy measure. We might hope that the sign-change condition is completely superfluous. With the notation of Corollary 1.2 this would amount to relaxing the condition on \( \nu \) to the following one: \( \nu(x) \) and \( \nu(-x) \) are completely monotone functions of \( x > 0 \), that is, \( \nu \in \text{AM-ÇM} \). Indeed, for \( x > 0 \), \( \nu(x) \) and \( \nu(-x) \) are the Laplace transforms of \( \varphi(s) \mathbb{I}_{(0, \infty)}(s) \) and \( -\varphi(-s) \mathbb{I}_{(0, \infty)}(s) \), respectively.

It turns out, however, that this is not the case: we provide an example of a function \( f \) which is not bell-shaped, which is constant zero in \((\infty, 0)\), and which satisfies all assumptions of Theorem 1.1 except the sign-change condition. In other words, there is an infinitely divisible distribution on \((0, \infty)\) with Lévy measure \( \nu(x)dx \) for a completely monotone \( \nu \), which is not bell-shaped.

Consider a function \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
\mathcal{L} f(z) = \frac{(1 + z/4)^3}{(1 + z/2)^3(1 + z/17)^4}
\]

for \( z \in i\mathbb{R} \). This corresponds to \( a = c = 0 \), \( b = \frac{67}{68} \) and \( \varphi(s) = 3 \cdot \mathbb{I}_{(2, 4)}(s) + 4 \cdot \mathbb{I}_{(17, \infty)}(s) \) in Theorem 1.1 or \( a = 0 \) and \( \nu(x) = x^{-1}(3e^{-2x} - 3e^{-3x} + 4e^{-6x}) \mathbb{I}_{(0, \infty)}(x) \) in Corollary 1.2. Clearly, the only assumption of Theorem 1.1 which is violated by \( f \) is the sign-change condition: when \( k = 1 \) or \( k = 2 \), \( \varphi(s) - k \) changes its sign three times.

By inspecting the Laplace transform of \( f \) only, it is easy to see that \( f \) is positive on \((0, \infty)\), constant zero on \((\infty, 0)\), it is integrable with integral 1, and it is twice continuously differentiable on \( \mathbb{R} \). It follows that \( f''(x) > 0 \) for \( x > 0 \) arbitrarily close to 0, as well as for \( x > 0 \) arbitrarily large. Using Mathematica, we find that

\[
f(x) = \frac{83521}{36450000}((284 + 888x + 360x^2)e^{-2x} - (284 + 5148x + 45630x^2 - 494325x^3)e^{-17x}),
\]
which leads to
\[
\begin{align*}
f''(\frac{1}{4}) &= \frac{83521}{23528000000}(-38168123e^{-17/4} - 92032e^{-1/2}) < 0, \\
f''(\frac{1}{2}) &= \frac{83521}{23528000000}(271849e^{-17/2} - 64e^{-1}) > 0, \\
f''(\frac{3}{4}) &= \frac{83521}{23528000000}(1787319463e^{-51/4} - 24448e^{-3/2}) < 0.
\end{align*}
\]
Therefore, \( f'' \) changes its sign at least 4 times (in fact — exactly 4 times).

The above function \( f \) is not infinitely smooth. This can be circumvented by convolving \( f \) with a smooth, sufficiently localised Pólya frequency function (or the Gauss–Weierstrass kernel if we do not require the resulting function to be zero in \((-\infty, 0)\)).

6.2. Self-decomposable distributions. With the notation of Corollary 1.2 \( f \) is the density function of a positive self-decomposable distribution if \( f \) is positive on \((0, \infty)\), constant zero on \((-\infty, 0)\), it is integrable with integral 1 and \( x\nu(x) \) is a non-increasing function of \((0, \infty)\). T. Simon conjectured in [15] that density functions of positive self-decomposable distributions are (weakly) bell-shaped. This also turns out to be false, the counter-example being the same as in the previous section. Indeed, we have
\[
x\nu(x) = 3e^{-2x} - 3e^{-4x} + 4e^{-17x}
\]
for \( x > 0 \), which is easily shown to be decreasing: the function
\[
2e^{-2x}(x\nu(x))' = -3 + 6e^{-2x} - 34e^{-15x}
\]
attains its global maximum at \( x = \frac{1}{13} \log \frac{85}{2} \), and the maximal value is
\[
-3 + \frac{26}{9} \left( \frac{3}{85} \right)^{2/13} < 0.
\]
As before, convolution of \( f \) with a sufficiently localised Pólya frequency function concentrated on \((0, \infty)\) (which is always the density function of a positive self-decomposable distribution) turns this example into a smooth density function of a positive self-decomposable distribution which is not bell-shaped.

6.3. Functions corresponding to \( \varphi \) uniformly close to non-decreasing functions.

The function \( \varphi \) in Theorem 1.1 is allowed to decrease by at most 1 compared to its previous maximal value. For this reason it is natural to conjecture that Theorem 1.1 remains valid at least when we relax the the sign-change condition to the following one: \( \varphi(s_2) - \varphi(s_1) \geq -1 \) when \( s_1 \leq s_2 \). This is, however, again false, as can be seen by considering the following simple example. Note that the argument used below can be easily adapted to show that an even stronger condition: \( \varphi(s_2) - \varphi(s_1) \geq -\varepsilon \) when \( s_1 \leq s_2 \), with arbitrarily small \( \varepsilon > 0 \), is not sufficient.

We consider \( f : \mathbb{R} \to \mathbb{R} \) such that
\[
\mathcal{L} f(z) = \frac{1 + z/2}{(1 + z)^{3/2}}
\]
for \( z \in i\mathbb{R} \). This corresponds to \( a = c = 0, b = 1 \) and \( \varphi(s) = \frac{s}{2} \mathbb{1}_{[1,2)}(s) + \frac{1}{2} \mathbb{1}_{[2,\infty)}(s) \) in Theorem 1.1 and \( f \) is easily found to be equal to
\[
f(x) = \frac{1}{2\sqrt{\pi}} \frac{1 + 2x}{\sqrt{x}} e^{-x} \mathbb{1}_{(0,\infty)}(x).
\]
We claim that for sufficiently small \( t > 0 \), the convolution \( f * g_t \) of \( f \) and the Gauss–Weierstrass kernel \( g_t \) is not bell-shaped: \( f * g_t \) changes its sign more than 8 times.

From characterisation of the integrals of Stieltjes functions and the expression for \( \mathcal{L} f \) it follows that \( f \) is not completely monotone on \((0, \infty)\), and indeed using Mathematica we easily find that \( f^{(8)}(4) = \frac{11598575}{67108864} e^{-4\pi - 1/2} < 0 \). Therefore, \( (f * g_t)^{(8)}(4) = f^{(8)} * g_t(4) < 0 \) for sufficiently small \( t > 0 \).
Let \( g(x) = (2\sqrt{\pi})^{-1} x^{-1/2} \mathbb{1}_{(0,\infty)}(x) \) and \( h(x) = (g * g_1)(x) \). Since \( g \) is completely monotone on \((0, \infty)\) and locally integrable, \( g \) is weakly bell-shaped, and therefore \( h \) changes its sign 8 times. Furthermore, \( (g) \) is positive on \((0, \infty)\), so \( h \) is positive near \( \infty \). If follows that there is an increasing sequence \( x_1, x_2, \ldots, x_9 \) such that \((-1)^{j-1} h(x_j) > 0\) for \( j = 1, 2, \ldots, 9 \).

Note that \( f - g \) is a bounded function. Therefore, with the usual notation \( \|f\|_1 = \int_{-\infty}^{\infty} |f(x)| \, dx \) and \( \|f\|_\infty = \sup \{|f(x)| : x \in \mathbb{R}\} \),

\[
\|(f * g_t)(x) - (g * g_t)(x)\|_1 = |(f - g) * g_t(\cdot)|_1 \leq \|f - g\|_\infty \|g_t(\cdot)\|_1
\]

for all \( x \in \mathbb{R} \). Furthermore, \( g * g_t(x) = t^{-1/4} g(t^{-1/2} x) \), so \( g * g_t(\cdot) = t^{-17/4} h(t^{-1/2} \cdot) \).

It follows that

\[
\|t^{17/4} (f * g_t)(\cdot)(t^{1/2} \cdot) - h(\cdot)\|_1 \leq t^{1/4} \|f - g\|_\infty \|g_1(\cdot)\|_1
\]

for all \( x \in \mathbb{R} \). By considering \( x = x_j \), we conclude that if \( t > 0 \) is sufficiently small, we have \((-1)^{j-1} (f * g_t)(t^{1/2} x_j) > 0\) for \( j = 1, 2, \ldots, 9 \). Making \( t > 0 \) even smaller if necessary, we may assume that \( t^{1/2} x_9 < 4 \). Since \((f * g_t)(4) < 0\), \((f * g_t)(\cdot)\) changes its sign at least 9 times.

6.4. **Previously known classes of bell-shaped functions.** The following classes of functions that have been previously shown to be bell-shaped are included in Theorem 1.1:

- The Gauss–Weierstrass kernel \( g_t \), which can be obtained by setting \( a = t, b = c = 0 \) and \( \varphi(s) = 0 \) in Theorem 1.1.

- The functions \( f_p(x) = (1 + x^2)^{-p} \), where \( p > 0 \). Note that \( f_p \) is bell-shaped, because \((1 + x^2)^p f_p^{(n)}(x)\) is a polynomial of degree \( n \) for \( n = 0, 1, 2, \ldots \). The Fourier transform of \( f_p \) is given by

\[
\mathcal{L} f_p(z) = c_p |z|^{p - 1/2} K_{p-1/2}(|z|)
\]

for \( z \in \mathbb{R} \), and it corresponds to \( a = 0 \) and

\[
\varphi(s) = \frac{1}{\pi} \arg(i J_{p-1/2}(|s|) - Y_{p-1/2}(|s|)) \text{ sign } s
\]

in Theorem 1.1. Here \( c_p > 0 \) is a constant, \( K_p, J_p \) and \( Y_p \) denote appropriate Bessel functions and \( \arg \) stands for the continuous version of the complex argument of a zero-free function, determined uniquely by the condition \(|\varphi(s)| < \pi\) for \( s \) in a neighbourhood of \( 0 \). The above expression for \( \varphi \) can be derived using formula (5.2) (with \( F = \mathcal{L} f_p \)) and well-known properties of Bessel functions; we refer to Section 5 and to the proof of Theorem 1 in [4] for additional details.

- The functions \( f_p(x) = x^{-p} e^{-1/x} \mathbb{1}_{(0,\infty)}(x) \) for \( p > 0 \). Since \( e^{1/x} x^{-p-2n} f_p^{(n)}(x) \) is equal on \((0, \infty)\) to a polynomial of degree \( n \), \( f_p \) is indeed bell-shaped. Since \( f_p \) is a density function of a hitting times of a diffusion (discussed below), it is indeed included in Theorem 1.1; we refer to Section 5 for further discussion.

- Pólya frequency functions, which correspond to non-decreasing functions \( \varphi \) in Theorem 1.1 that only take integer values.

- Density functions of all stable distributions, including those concentrated on \((0, \infty)\), considered in [15].

- Density functions of hitting times of generalised diffusions, studied in [8]. These are indeed included in Theorem 1.1 because they can be represented as convolutions of Pólya frequency functions and completely monotone functions on \((0, \infty)\), as discussed in [8].
A NEW CLASS OF BELL-SHAPED FUNCTIONS

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