Spectral decomposition and Gelfand’s theorem

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Abstract
In this paper we are interested in spectral decomposition of an unbounded operator with discrete spectrum. We show that if $A$ generates a polynomially bounded $n$-times integrated group whose spectrum set $\sigma(A) = \{i\lambda_k; k \in \mathbb{Z}^*\}$ is discrete and satisfies $\sum \frac{1}{|\lambda_k - \delta_k|} < \infty$ ($n$ and $\ell$ nonnegative integers), then there exists projectors $(P_k)_{k \in \mathbb{Z}^*}$ such that $\sum P_kx = x$ ($x \in D(A^{n+\ell})$), where $\delta_k = \min \left(\frac{|\lambda_{k+1} - \lambda_k|}{2}, \frac{|\lambda_k - \lambda_{k-1}|}{2}\right)$.

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1 Introduction and notations.
In this paper we deal with the spectral decomposition of a linear operator $A$ over a Banach space $E$ (see for instance [5, Chapter 4] and [6]): For which subspace $D \subset E$ do we have:

$$\sum_{k \in \mathbb{Z}} P_kx = x, \quad (x \in D)$$

(1)

where $\sigma(A) := \{i\lambda_k, k \in \mathbb{Z}\}$, $\lambda_i \neq \lambda_j$, is assumed to be discrete and $P_k$ are the associated eigen-projectors:

$$P_kx := \int_{\gamma_k} (\lambda - A)^{-1} x d\lambda,$$

(2)

whith $\gamma_k := C(i\lambda_k, r_k)$ is the positively oriented circle centered at $i\lambda_k$ and $r_k$ sufficiently small so that $|\lambda_k - \lambda_\ell| > r_k$ for $\ell \neq k$.

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In the case of bounded $C_0$-groups, if $\sup_k \|P_k\| < \infty$ and if $\sum_k |\lambda_k|^{-1} < \infty$ then (1) holds true with $D = D(A)$.

For the generator $A$ of a bounded $C_0$-group on a Banach space $E$, there exists a large literature of spectral theory. The introduction of the spectral theory was strongly motivated by operator algebras (see for instance [24]), and was further developed in the course of applications to groups of automorphisms of operator algebras (see for instance [10]). Always in the framework of the generator of bounded groups, Gelfand’s theorem (1941) [16] gives the equivalence between the spectrum of $A$ is zero, $\sigma(A) = \{0\}$ and $A$ is trivial, $A = 0$.

Recently, related topics to Gelfand’s theorem has been extensively developed in a more general framework than bounded groups. (see [11], [5, Chapter 4], [15], [27] and the references therein).

Unfortunately the framework of bounded groups fails for several applications. A typical example is the Schrödinger operator $i\Delta$ on $L^p(\Omega)$ with Dirichlet or Neumann boundary condition for $p \neq 2$. In fact, for bounded domains $\Omega \subset \mathbb{R}^N$ and using Sobolev embedding, Arendt showed in 1991 [3] that $i\Delta$ (with Dirichlet or Neumann boundary condition) generates a $k$-times integrated group, that is polynomially bounded, for $k > N/2 |1/p - 1/2|$. It should be noticed also that El-Mennaoui and Kyantuo showed that the order $N/2 |1/p - 1/2|$ is optimal, see for instance [13], and hence $i\Delta$ does generate a $C_0$-group in this case.

The notion of integrated semigroups was introduced by Arendt [2] in 1987. An operator $A$ generates a $n$-times integrated semigroup $\{S(t)\}$ ($n \in \mathbb{N}$) if and only if there exists $w \in \mathbb{R}$ such that

$$(\lambda - A)^{-1} = \lambda^n \int_0^\infty e^{-\lambda t} S(t) \, dt$$

for $Re\lambda > w$. The $n$-times integrated group is said to be polynomially bounded if there exists a positive integer $k$ and a positive constant $C$ such that $\|S(t)\| \leq C(1 + |t|^k)$ for all $t$. The $n$-times integrated group is said to be temperate if there exists a positive constant $C$ such that $\|S(t)\| \leq C |t|^n$.

Except spectral distributions, there is no corresponding spectral theory for integrated groups. The notion of spectral distributions have been introduced in 1991. The special case of temperate integrated groups has been extensively studied. See for instance [8, 20, 21, 14, 15].

In this paper, and using elementary calculations, we will show that the decomposition (1) holds true for all $x \in D(A^\ell)$ for some power $\ell$, where $A$ is the generator of polynomially bounded $n$-times integrated group with discrete spectrum.

Throughout this paper we suppose in addition (rescaling $A$ if needed) that $(\lambda_k)_{k \in \mathbb{Z}^+}$ is an increasing sequence, $\lambda_k > 0$ for $k > 0$ and $\lambda_k < 0$ for $k < 0$.
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The paper is organized as follows: in the second section we prove a separation result that permits to find two closed subspaces \( E_- \) and \( E_+ \) invariant by the integrated group such that \( \sigma(A_{|E_-}) \subset i\mathbb{R}^+ \) and \( \sigma(A_{|E_+}) \subset i\mathbb{R}^+ \). In the third section the bounded group case is considered. The fourth section is devoted to the integrated group case (Theorem 3) with some additional conditions on \(|\lambda_k|^{-1}\) and on the distance separating successive elements of the spectrum. In the last section we apply our results to Schrödinger operators on \( L^p \) spaces.

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## 2 Some general separation results

Given a linear operator \( T \) on a Banach space \( E \) with spectrum \( \sigma(T) = \sigma_1 \cup \sigma_2 \), union of two closed disjoint sets with \( \sigma_2 \) compact, it is well known that \( P_2 x := \frac{1}{2\pi} \int_\Gamma (\lambda - T)^{-1} x \, d\lambda \ (x \in E) \), where \( \Gamma \) is any bounded path surrounding \( \sigma_2 \) and not containing in its interior any point of \( \sigma_1 \), defines a bounded projector and then we have the following decomposition: \( E = E_1 \oplus E_2 \), where \( E_2 := P_2 E = Im P_2 \) and \( E_1 := ker P_2 \), and if we denote by \( T_1 := T|_{E_1} \) and \( T_2 := T|_{E_2} \), the part of \( T \) on \( E_1 \) and \( E_2 \) respectively, then \( \sigma(T_2) = \sigma_2 \) and \( \sigma(T_1) = \sigma_1 \). But if \( \sigma_1 \) and \( \sigma_2 \) are not bounded, the associated projectors are not necessarily bounded. Consider, for example, the operator \( T = \frac{d}{dx} \) on the space \( E = L^1([0, 2\pi]) \). It is well known that the spectrum is \( \sigma(T) = i\mathbb{Z} \) and the projector defined by

\[
P \left[ \sum_{k=-n}^{n} a_k e^{ikx} \right] := \sum_{k=-n}^{n} a_k e^{ikx}
\]

is not bounded on \( E \) (see for example [23, Part II, Chapter 3 e.]).

The aim of this section is to show the following “separation” theorem:

**Theorem 1** Let \( A \) be the generator of an \( n \)-times integrated group \((S(t))_{t \in \mathbb{R}}\) satisfying \( \|S(t)\| \leq M(1+|t|^m) \ (m \in \mathbb{R}^+) \). Assume that \( \sigma(A) = \{i\lambda_n; \ n \in \mathbb{Z}^* \} = \sigma^+ \cup \sigma^- \) where \( \sigma^+ := \{i\lambda_n; \ n \geq 1 \} \subset i\mathbb{R}^+ \) and \( \sigma^- := \{i\lambda_n; \ n \leq -1 \} \subset i\mathbb{R}^*_- \). Then there exists two closed subspaces \( E_- \) and \( E_+ \) of \( E \) invariant by \((S(t))_{t \in \mathbb{R}}\) such that \( D(A) \subset E_- \oplus E_+ \) and \( A_- := A_{|E_-} \) (resp. \( A_+ := A_{|E_+} \)) generates \( n \)-times integrated group satisfying the same boundedness property and whose spectrum \( \sigma(A_-) = \sigma^- \) (resp. \( \sigma(A_+) = \sigma^+ \)).

For all \( k \in \mathbb{Z}^* \), define the spectral projection \( P_k \) by

\[
P_k := \frac{1}{2i\pi} \int_{\gamma_k} R(\lambda, A) \, d\lambda,
\]
where \( \gamma_k := \{ z \in \mathbb{C}; \ |z - i\lambda_k| = \varepsilon_k \} \) with \( \varepsilon_k > 0 \) such that \( \mathbb{D}(i\lambda_k, \varepsilon_k) \cap \sigma(A) = \{ i\lambda_k \} \). It is clear that \( P_k \) is a bounded projection, \( P_k x \in D(A) \) for all \( x \in E \), and \( P_k P_{k'} = 0 \) for all \( k \neq k' \).

We will need the following generalization of Gelfand’s theorem, for the proof we refer to [12, Proposition 8.1].

**Proposition 1** [12, Proposition 8.1]
Let \( A \) be the generator of a \( n \)-times integrated group \((S(t))_{t \in \mathbb{R}}\) satisfying \( \|S(t)\| \leq M(1 + |t|^m) \), with \( m \geq n \) then \( \sigma(A) = \emptyset \) if and only if \( E = \{ 0 \} \).

**Remark 1** This implies the following weaker version (see also [12, Proposition 8.1]): \( \sigma(A) = \{ 0 \} \) if and only if \( A^{m-n+1} = 0 \).

In the following we are concerned with the convergence of the series \( \sum P_k x \). A first result is:

**Proposition 2** \( F := \bigcup_{k \in \mathbb{Z}^*} P_k(E) \) is dense in \( E \).

**Proof**: Setting \( G := E/F \), it suffices to see that this Banach space is trivial. Denote by \( B \) the part of \( A \) in \( G \). \( B \) generates an \( n \)-times integrated group that is polynomially bounded as \((S(t))\) and the spectrum \( \sigma(B) = \emptyset \). Using proposition 1 \( G = \{ 0 \} \). \( \square \)

Given \( x \in F \), there exists \( y \in E \), \( k \in \mathbb{Z}^* \) such that \( x = P_k y \), then \( P_\ell x = 0 \) for all \( \ell \neq k \), thus we obtain the convergence of the series \( \sum P_k x \) to \( x \) for all \( x \in F \).

In order to give more precise information on the convergence of the series, we start by showing that the convergence of the series \( \sum_{k>0} P_k x \) and \( \sum_{k<0} P_k x \) for some regular \( x \).

**Lemma 1** For all \( x \in E \) and all \( k \in \mathbb{Z}^* \), the mapping \( \lambda \mapsto R(\lambda, A)P_k x \) is analytic on \( \mathbb{C} \setminus \{ i\lambda_k \} \) and there exists \( c_k > 0 \) such that for all \( \lambda \notin \mathbb{D}(i\lambda_k, \varepsilon_k) \)

\[
\|R(\lambda, A)P_k\| \leq \frac{C_k}{\text{dist}(\lambda, \gamma_k)}.
\]

**Proof**: Let \( k \in \mathbb{Z}^* \) and \( x \in E \). Since \( R(\lambda, A)P_k x = \frac{1}{2i\pi} \int_{\gamma_k} R(\lambda, A) R(\mu, A) x d\mu \), the resolvent equation implies

\[
R(\lambda, A)P_k x = \frac{1}{2i\pi} \int_{\gamma_k} \frac{R(\lambda, A) x - R(\mu, A) x}{\mu - \lambda} d\mu = \frac{1}{2i\pi} \int_{\gamma_k} \frac{R(\lambda, A) x}{\mu - \lambda} d\mu - \frac{1}{2i\pi} \int_{\gamma_k} \frac{R(\mu, A) x}{\mu - \lambda} d\mu.
\]
For $\lambda \notin \overline{D}(i\lambda_k, \varepsilon_k)$, \( \frac{1}{2i\pi} \int_{\gamma_k} \frac{R(\lambda, A)x}{\mu - \lambda} d\mu = 0 \), then
\[
R(\lambda, A)P_kx = -\frac{1}{2i\pi} \int_{\gamma_k} \frac{R(\mu, A)x}{\mu - \lambda} d\mu,
\]
thus \( \|R(\lambda, A)P_kx\| \leq \|x\| \varepsilon_k \sup_{\mu \in \gamma_k} \|R(\mu, A)\| \). It suffices then to take
\[
C_k := \varepsilon_k \sup_{\mu \in \gamma_k} \|R(\mu, A)\|.
\]

**Proposition 3** There exists \( \mathbb{P} \in \mathcal{L}(D(A), E) \) such that
\[
\mathbb{P} \left[ \sum_{k=-N}^{N} A^{-m-1}P_kx \right] = \sum_{k=1}^{k=N} A^{-m-1}P_kx \quad (N \in \mathbb{N}^*, x \in E), \quad (3)
\]
\[
(\mathbb{P}A^{-m-1})^2 = \mathbb{P}A^{-2m-2}.
\]

**Proof:** Let \( \delta > 0 \) be such that \( \delta < \min(\lambda_1, |\lambda_1|) \) and consider \( (R_n)_{n \in \mathbb{N}^*} \) an increasing sequence of strictly positive numbers satisfying for all \( N \geq 1 \)
\[
\overline{D}(0, R_N) \cap \sigma(A) = \{i\lambda_k; \quad -N \leq k \leq N, \quad k \neq 0 \}.
\]
Let \( \Gamma_N \) be the positively oriented path \( \Gamma_N := [-R_N, -\delta] \cup C^+(0, \delta) \cup [\delta, R_N] \) (see the figure below).

Since \( 0 \notin \sigma(A) \), consider, for \( N \in \mathbb{N}^* \) and \( x \in E \), the bounded operator \( Q_NA^{-m-1}x := \frac{1}{2i\pi} \int_{\Gamma_N} R(\lambda, A)A^{-m-1}x d\lambda \). We will first show that \( Q_NA^{-m-1} \) converges strongly in \( E \). The resolvent equation:
\[
R(\lambda, A)A^{-k}x = \frac{A^{-k}x}{\lambda} + \cdots + \frac{A^{-1}x}{\lambda^k} + \frac{R(\lambda, A)x}{\lambda^k}
\]
implies
\[ Q_N A^{-m-1} x = \frac{A^{-m-1} x}{2} + \sum_{j=2}^{m+1} \frac{1}{2i\pi} \int_{\Gamma_N} (-\lambda)^{-j} A^{-(m+1-j)} x \, d\lambda \]
\[ + \frac{1}{2i\pi} \int_{\Gamma_N} (-\lambda)^{-m-1} R(\lambda, A) x \, d\lambda. \]

Since \( \int_{\Gamma_N} (-\lambda)^{-j} A^{-(m+1-j)} x \, d\lambda = -\int_{C^+(0,R_N)} (-\lambda)^{-j} A^{-(m+1-j)} x \, d\lambda \), whose modulus tends to zero as \( N \to \infty \) for \( j > 1 \), it suffices to prove the convergence of the last term. Using the bound on the integrated group \( (S(t)) \) we have \( (Re\lambda \neq 0) \)
\[ \| R(\lambda, A) \| \leq C|\lambda|^{n-m+1} \left( \frac{1}{|Re\lambda|} + \frac{1}{|Re\lambda|^{m+1}} \right) \]
so
\[ \lim_{N \to \infty} \frac{1}{2i\pi} \int_{\Gamma_N} R(\lambda, A)(-\lambda)^{-m-1} d\lambda \]
exists. Denote by \( QA^{-m-1} x \) its limit.

Set \( \mathbb{P} := Q - \frac{1}{2} Id \) and let’s show the equality (3).

From one side, for \( 1 \leq k \leq N \), by Cauchy formula and Lemma 1 we have:
\[ Q_N A^{-m-1} P_k x = \frac{1}{2i\pi} \int_{\Gamma_N} R(\lambda, A) A^{-m-1} P_k x \, d\lambda \]
\[ = \int_{C^+(0,R_N)} R(\lambda, A) A^{-m-1} P_k x \, d\lambda + P_k A^{-m-1} P_k x \]
\[ = -\frac{1}{2i\pi} \int_{C^+(0,R_N)} \sum_{j=1}^{m+1} (-\lambda)^{-j} A^{-(m+1-j)} P_k x \, d\lambda \]
\[ -\frac{1}{2i\pi} \int_{C^+(0,R_N)} (-\lambda)^{-m-1} R(\lambda, A) P_k x \, d\lambda + A^{-m-1} P_k x \]
\[ = A^{-m-1} P_k x + \frac{1}{2i\pi} \int_{C^+(0,R_N)} R(\lambda, A) P_k x + A^{-1} P_k x \, d\lambda \]
\[ \to A^{-m-1} P_k x + \frac{A^{-m-1} P_k x}{2} \quad \text{as} \ N \to \infty \]
\[ = \frac{3}{2} A^{-m-1} P_k x = QA^{-m-1} P_k x. \]

Similarly, for \( -N \leq k \leq -1 \), we get
\[ \lim_{N \to +\infty} Q_N A^{-m-1} P_k x = QA^{-m-1} P_k x = \frac{1}{2} A^{-m-1} P_k x. \]

Thus we have proved the equality (3). Now since \( F \) is dense in \( E \) and \( \mathbb{P} A^{-1} \in \mathcal{L}(E) \), we see that \( (\mathbb{P} A^{-m-1})^2 = \mathbb{P} A^{-2m-2} \). \( \square \)
Remarks 2

1- Notice that \( \mathbb{P}x = -\frac{1}{2i\pi} \int_{\Gamma} \frac{R(\lambda, A)A^{m+1}x}{(-\lambda)^{m+1}} \, d\lambda \) for all \( x \in D(A^{m+1}) \), where 
\[ \Gamma := \{ -\infty, -\delta \} \cup C^+(0, \delta) \cup \delta, +\infty \].

2- In general the operator \( \mathbb{P} \) is not bounded.

For example consider the rotation group \( (T(t))_{t \in \mathbb{R}} \) on \( L^p_{2\pi}(\mathbb{R}) \) with \( 1 \leq p < +\infty \), where \( L^p_{2\pi}(\mathbb{R}) \) the Banach space of measurable \( 2\pi \)-periodic functions on \( \mathbb{R} \) satisfying \( \int_{0}^{2\pi} |f(x)|^p \, dx < +\infty \). The rotation group is defined by \( T(t)f(x) := f(x+t) \) for \( f \in L^p_{2\pi}(\mathbb{R}) \). The generator \( A \) of \( (T(t))_{t \in \mathbb{R}} \) is defined by \( Af = f' \) and \( D(A) = \{ f \in L^p_{2\pi}(\mathbb{R}), f' \in L^p_{2\pi}(\mathbb{R}) \} \). \( f' \) is the derivative in the distribution sense. It is easy to verify that the spectrum of \( A \), \( \sigma(A) = i\mathbb{Z} \) and the eigenvectors are exactly the Fourier basis \( (e^{int})_{n \in \mathbb{Z}} \). Therefore \( \mathbb{P}(f) \) is the Fourier series associated to \( f \). It is well known that for \( p \neq 2 \), \( (e^{int})_{n \in \mathbb{Z}} \) is not a basis of \( L^p_{2\pi}(\mathbb{R}) \) so \( \mathbb{P}(f) \) does not converge in \( L^p_{2\pi}(\mathbb{R}) \) except for \( p = 2 \). Consequently \( \mathbb{P} \) is defined only on the domain of \( A \).

Proof of Theorem 1

Set \( E_- := \{ x \in E; \mathbb{P}A^{-m-1}x = 0 \} = \ker \mathbb{P}A^{-m-1} \) and \( E_+ := \{ x \in E; \mathbb{P}A^{-m-1}x = A^{-1}x \} \). \( E_- \) and \( E_+ \) are two closed subspaces of \( E \). It is clear that \( D(A^{m+1}) \subseteq E_0 + E_1 \) and that \( E_0 \) and \( E_1 \) are invariant by \( (S(t))_{t \in \mathbb{R}} \). If \( x \in E_0 \cap E_1 \), then \( A^{-m-1}x = \mathbb{P}A^{-m-1}x = 0 \) and so \( x = 0 \).

Let \( A_- \) be the part of \( A \) in \( E_- \). It’s clear that \( \sigma(A_-) \subseteq \sigma(A) \). To show that \( \sigma(A_-) = \sigma_- \) it suffices to verify that \( \sigma(A_-) \cap \sigma^+ = \emptyset \). Suppose that there exists \( k \in \mathbb{N}^* \) such that \( i\lambda_k \in \sigma(A_-) \). There exists then a sequence \( (x_n)_{n \in \mathbb{N}^*} \subseteq D(A_-) \) such that for all \( j \in \mathbb{N}^* \) we have \( \|x_j\| = 1 \) and \( \|Ax_j - i\lambda_kx_j\| \leq \frac{1}{j} \).

From one side for \( \lambda \in \Gamma \) we have 
\[ \left\| \frac{-x_j}{\lambda - i\lambda_k} + R(\lambda, A)x_j \right\| = \left\| \frac{R(\lambda, A)(Ax_j - i\lambda_kx_j)}{\lambda - i\lambda_k} \right\| \leq \frac{C}{j \|\lambda - \lambda_k\|} \]

where \( c := \sup_{\lambda \in \Gamma} \|R(\lambda, A)\| \).

From the other side we have \( \mathbb{P}x = \frac{1}{2i\pi} \int_{\Gamma} \frac{R(\lambda, A)A^{m+1}x}{(-\lambda)^{m+1}} \, d\lambda \) for \( x \in D(A^{m+1}) \) and since \( \frac{1}{2i\pi} \int_{\Gamma} \frac{x_j}{(-\lambda)^{m+1}(\lambda - i\lambda_k)} \, d\lambda \to \frac{1}{2i\pi} \int_{\Gamma} \frac{C}{j \lambda(\lambda - i\lambda_k)} \, d\lambda \) for \( j \to \infty \). Hence \( \|\mathbb{P}A_-^{m-1}x_j\| \leq \frac{1}{2i\pi} \int_{\Gamma} \frac{C}{j \lambda(\lambda - i\lambda_k)} \, d\lambda \). Thus \( \lim_{j \to \infty} \|\mathbb{P}A_-^{m-1}x_j\| = 0 \), but \( x_j \in E_- \), so \( \mathbb{P}A_-^{m-1}x_j = 0 \) hence \( \lim_{n \to \infty} \|x_j\| = 0 \). This is absurd since \( \|x_j\| = 1 \). Therefore \( \sigma(A_-) \cap \sigma^+ = \emptyset \) and the spectrum of \( A_- \) is \( \sigma_- \). \( \square \)
3 Bounded $C_0$-groups case

The case of bounded groups is more classical and spectral theory must lead to simple proofs. Indeed, the spectral projections can be written in the form $P_k = \int_\mathbb{R} \mathcal{F}(f_k)(t)T(t) \, dt$ where $f_k(\lambda_j) = \delta_{kj}$. Assume for the moment that $c := \sup_k \|P_k\| < \infty$. For $x \in D(A)$, and using Gelfand’s theorem, one has $P_k Ax = \lambda_k P_k x$ and hence $\|P_k x\| \leq c|\lambda_k|^{-1}\|Ax\|$. Under the hypothesis that $\sum_k |\lambda_k|^{-1} < \infty$ it follows that $\sum_k P_k x$ is convergent for all $x \in D(A)$.

The problem now is to choose suitable functions $f_k$ so that $\sup_k \|\mathcal{F}f_k\|_{L^1}$ is finite. For this take

$$f_k(t) := f\left(\frac{t - \lambda_k}{\delta_k}\right)$$

where $f$ is any $C^\infty$ function of support in $(-1,1)$ that is equal to 1 on a neighborhood of zero, and $\delta_k := \frac{1}{2}\min(\lambda_k - \lambda_{k-1}, \lambda_{k+1} - \lambda_k)$. Then we get

$$\|P_k\| \leq \|\mathcal{F}f_k\|_{L^1} = \|\mathcal{F}f\|_{L^1}.$$ 

We have thus proved the following theorem. However, aiming to introduce our method, we give another proof based on the contour method.

**Theorem 2** Let $A$ be the generator of a bounded $C_0$-group $(T(t))_{t \in \mathbb{R}}$ with discrete spectrum $\sigma(A) = \{i\lambda_k, k \in \mathbb{Z}^*\}$. Assume that $\sum_{k \in \mathbb{Z}^*} |\lambda_k| < \infty$, then $\sum_{k \in \mathbb{Z}^*} P_k x = x$ ($x \in D(A)$) and $\sum_{k \in \mathbb{Z}^*} i\lambda_k P_k x = Ax$ ($x \in D(A^2)$).

**Proof:** Let’s show first that $\sup_k \|P_k\| < \infty$, where $P_k$ is defined by (2).

Remark that we can write

$$P_k x = \frac{1}{2\pi i} \int_{\gamma_k} (\lambda - A)^{-1} \left[1 + \frac{(\lambda - i\lambda_k)^2}{r_k^2} \right] \, d\lambda$$

since $\lambda_k$ is a simple pole for $(\lambda - A)^{-1}$. Now since $A$ generates a bounded group we have $\|\text{Re}(\lambda)(\lambda - A)^{-1}\| \leq c$. Thus

$$\|P_k\| \leq \frac{1}{2\pi i} \int_0^{2\pi} \|e^{i\theta} (\lambda - A)^{-1}\| |1 + e^{2i\theta} r_k| \, d\theta$$

$$\leq \frac{1}{\pi} \int_0^{2\pi} \|r_k \cos \theta (e^{i\theta} - A)^{-1}\| \, d\theta \leq 2c.$$ 

Now, for $x \in D(A)$ and using Gelfand’s theorem, we have

$$P_k x = \frac{1}{\lambda_k} AP_k x = \frac{1}{\lambda_k} P_k Ax$$

hence

$$\|P_k x\| = \frac{1}{|\lambda_k|} \|P_k Ax\| \leq \frac{2c}{|\lambda_k|} \|Ax\|.$$ 

Therefore, $\sum_k P_k x$ is normally convergent for all $x \in D(A)$. Since $\sum_k P_k x = x$ for all $x \in F$ we get the result using the density of $F$ in $D(A)$.
4 Integrated groups case

As we saw in the preceding section, the spectral theory could give easily the spectral decomposition in the case of bounded groups. But this method does not work for the more general class, while contour method still applicable.

In this section we will give more appropriate conditions on the behavior of the spectrum to get convergence.

The following is an "integrated" version of theorem 2. Here $n$ and $\ell$ are fixed integers.

**Theorem 3** Let $A$ be the generator of an $n$-times integrated group $(S(t))_{t \in \mathbb{R}}$ satisfying for all $t$

$$\|S(t)\| \leq M(1 + |t|^n),$$  \hspace{1cm} (4)

with discrete spectrum $\sigma(A) = \{i\lambda_k; k \in \mathbb{Z}^*\} \subset i\mathbb{R}^*$. Assume that

$$\sum_{k \in \mathbb{Z}^*} \frac{1}{|\lambda_k|^{\ell} \delta_k^n} < \infty,$$  \hspace{1cm} (5)

where

$$\delta_k := \min \left( \frac{|\lambda_{k+1} - \lambda_k|}{2}, \frac{|\lambda_{k-1} - \lambda_k|}{2} \right),$$

then

$$\sum_{k \in \mathbb{Z}^*} P_k x = x \hspace{1cm} \text{for all } x \in D(A^{n+\ell}).$$  \hspace{1cm} (6)

**Proof:** Using theorem 1 we can suppose that $\sigma(A) = \{i\lambda_n; n \in \mathbb{N}^*\} \subset i\mathbb{R}_+^*$. Let $k \in \mathbb{Z}^*$ and set $A_k := (A - i\lambda_k Id) \mid P_k(E)$ the part of $A - i\lambda_k Id$ on $P_k(E)$. $A_k$ is a bounded operator on $P_k(E)$, hence it generates, in particular, an $n$-times integrated group $(S_k(t))_{t \in \mathbb{R}}$ that is temperate at infinity (i.e. $\|S_k(t)\| = O(|t|^{2n})$) and whose spectrum $\sigma(A_k) = \{0\}$. Hence, using Remark 1 we have $A_k^{n+1} = 0$. Thus we have

$$R(i\lambda_k + \lambda, A)P_k x = \sum_{j=1}^{n} \frac{A_k^j}{(\lambda - i(\lambda_\ell - \lambda_k))^{j+1}} P_\ell x$$

for all $|\lambda| = \delta_k$.

Observing that the mapping $\lambda \mapsto \lambda^{2(n+1)} R(i\lambda + \lambda) x$ is analytic from the disc $\overline{D}(0, \delta_k)$ into $P_k(E)$, denote by

$$P_N x := \frac{1}{2i\pi} \int_{|\lambda| = \delta_k} \left[ 1 + \frac{\lambda^{2(n+1)}}{\delta_k^{2(n+1)}} \right]^{n+1} \sum_{k=1}^{N} R(i\lambda_k + \lambda, A) x d\lambda,$$  \hspace{1cm} (with $\delta_k := \min \left( \frac{|\lambda_{k+1} - \lambda_k|}{2}, \frac{|\lambda_{k-1} - \lambda_k|}{2} \right)$)

\footnote{Since the uniform boundedness of the projectors is not true in general in this case.}
then
\[
\begin{aligned}
\{ \mathbb{P}_N P_\ell(x) &= P_\ell(x) \quad \text{for } |\ell| \leq N \\
\mathbb{P}_N P_\ell(x) &= 0 \quad \text{for } |\ell| > N
\end{aligned}
\]

We deduce that for all \( x \in F := \bigcup_{k>1} P_k(E) \) the sequence \( (\mathbb{P}_N x)_{N \in \mathbb{N}^*} \) converges to \( x \).

Using the following resolvent equation:
\[
R(\lambda, A)A^{-n}x = \frac{A^{-n}x}{\lambda} + \cdots + \frac{A^{-1}x}{\lambda^{n-1}} + \frac{R(\lambda, A)x}{\lambda^n}
\]
we get
\[
\mathbb{P}_N A^{-n}x = \frac{1}{2\pi i} \int_{|\lambda| = \delta} \left[ 1 + \frac{\lambda^{2(2n+1)}}{\delta^{2(2n+1)}} \right]^{n+1} \sum_{k=1}^N \frac{R(i\lambda_k + \lambda, A)}{(\lambda + i\lambda_k)^n} x \, d\lambda
\]

Now for \( \text{Re}(\lambda) > 0 \), we have \( (\lambda + i\lambda_k)^{-n} R(i\lambda_k + \lambda, A)x = \int_0^{+\infty} e^{-tRe(\lambda)} e^{-\lambda t} S(t)x \, dt \).

Thus, for \( x \in D(A^{n+\ell}) \), setting \( U_N(\lambda)x := \sum_{k=1}^N (\lambda + i\lambda_k)^{-n} R(i\lambda_k + \lambda, A)x \), we have
\[
U_N(\lambda)x = \sum_{k=1}^N \int_0^{+\infty} e^{-i\lambda_k t} e^{-\lambda t} S(t)x \, dt
\]
\[
= \int_0^{+\infty} \left[ \sum_{k=1}^N e^{-i\lambda_k t} \right] e^{-\lambda t} S(t)x \, dt
\]

Integrating \( \ell \) times by parts, we get
\[
U_N(\lambda)x = \sum_{k=1}^N \int_0^{+\infty} e^{-i\lambda_k t} e^{-\lambda t} S(t)x \, dt
\]
\[
= \int_0^{+\infty} \left[ \sum_{k=1}^N e^{-i\lambda_k t} \right] e^{-\lambda t} S(t)x \, dt
\]
\[
= \int_0^{+\infty} \sum_{k=1}^N e^{-i\lambda_k t} e^{-\lambda t} \left[ S(t)(A - \lambda)^\ell + f(\lambda, A, t) \right] x \, dt,
\]

where \( f(\lambda, A, t) := \sum_{0 \leq \alpha, \beta \leq \ell} \sum_{0 \leq \gamma \leq n} a_{\alpha, \beta, \gamma} \lambda^{\alpha} t^\beta A^\gamma \) is a polynomial on \( t \), \( a_{\alpha, \beta, \gamma} \) are constants. Using \( (4) \) we get
\[
\|U_N(\lambda)x\| \leq c_1 \|x\|_{A^\ell} \sum_{k=1}^N \frac{1}{|\lambda_k|^{\ell}} \int_0^{+\infty} \left( \sum_{0 \leq j \leq n} t^j \right) e^{-tRe(\lambda)} \, dt
\]
\[
\leq c_2 \|x\|_{A^\ell} \sum_{k=1}^N \frac{1}{|\lambda_k|^{\ell}} \sum_{1 \leq j \leq n+1} \frac{1}{|Re(\lambda)|^j},
\]
where \(c_1\) and \(c_2\) are two strictly positive constants. Then, denoting by \(C_k^+ = \{ \text{Re} z > 0, |z| = \delta_k \}\) and by \(P_N^+\)

\[
P_N^+ x := \frac{1}{2i\pi} \int_{C_k^+} \left[ 1 + \frac{\lambda^{2(2n+1)}}{\delta_k^{2(2n+1)}} \right]^{n+1} \sum_{k=1}^{N} \frac{R(i\lambda_k + \lambda, A)}{\lambda + i\lambda_k}^n x d\lambda,
\]

we get

\[
\|P_N A^{-n-1} x\| \leq c_3 \delta_k \|x\|_{A^\ell} \sum_{k=1}^{N} \frac{1}{|\lambda_k|^\ell} \int_{-\pi/2}^{\pi/2} \left| 1 + e^{2i\theta(2n+1)} \right|^{n+1} \times \\
\sum_{1 \leq j \leq n+1} \left| \delta_k \cos \theta \right|^j d\theta \leq c_5 \|x\|_{A^\ell} \sum_{0 \leq k \leq N} \sum_{0 \leq j \leq n} \frac{1}{|\lambda_k|^\ell \delta_k^j} \int_{-\pi/2}^{\pi/2} \frac{\cos(\theta(2n+1))^{n+1}}{\cos \theta |\cos \theta|^j} d\theta.
\]

Thus (we may assume that \(\delta_k \leq 1\))

\[
\|P_N A^{-n-1} x\| \leq c_5 \|x\|_{A^\ell} \sum_{0 \leq k \leq N} \sum_{0 \leq j \leq n} \frac{1}{|\lambda_k|^\ell \delta_k^j} \int_{-\pi/2}^{\pi/2} \frac{\cos(\theta(2n+1))^{n+1}}{\cos \theta |\cos \theta|^j} d\theta.
\]

Since the integrals

\[
\int_0^{2\pi} \frac{|\cos \theta(2n+1)|^{n+1}}{|\cos \theta|^j} d\theta = 2 \int_{-\pi/2}^{\pi/2} \frac{|\sin \theta(2n+1)|^{n+1}}{|\sin \theta|^j} d\theta < \infty
\]

are finite for \(j\), and using \(\Box\) there exists \(c > 0\) such that for all \(x \in E\) we have

\[
\left\| P_N A^{-(n+1)} x \right\| \leq c \|x\|.
\]

But for \(x \in F\), \(\lim_{N \to +\infty} P_N x = x\) and \(F\) is dense in \(E\) then for \(x \in D(A^{n+1})\)

\[
\lim_{N \to +\infty} P_N x = x.
\]

\[\Box\]

5 Applications: Schrödinger operator on a compact manifold

Consider the Schrödinger operator \(H_p = -\Delta + V\), where \(\Delta\) is the Laplace-Beltrami operator on \(L^p(\mathcal{M})\), \(\mathcal{M}\) is a compact manifold with Dirichlet or Neumann manifold in the nonempty boundary case and \(V\) a regular potential. The operator \(H_2\) generates on \(L^2(\mathcal{M})\) a self-adjoint operator. Following the same technique as in \([3, \text{Theorem 4.3, p.37}]\) and by Sobolev imbeddings, the operator \(iH_p\) generates on \(L^p(\mathcal{M})\) a \(n\)-times integrated group \(\{S(t)\}\), for \(n > \frac{N}{2} \left| \frac{1}{p} - \frac{1}{2} \right|\), with \(1 < p < \infty\). Now by a similar calculation as in
[22] Proposition 3.1, p. 61] the integrated group is polynomially bounded: \( \|S(t)\| \leq C(1 + |t|^n) \) for some positive constant \( C \) and all \( t \in \mathbb{R} \).

The basic fact in this section is the \( p \)-spectral independence of the Schrödinger operator on \( L^p \) for \( 1 \leq p < \infty \) (see [18], [4] and the references therein). Moreover, it’s well known that the spectrum of \(-\Delta + V\) is a strictly increasing sequence \((\lambda_k)\) and if \((\mu_k)\) are the eigenvalues counted with their multiplicities then \( \mu_k \sim Ck^{2/N} \) as \( k \to \infty \) where \( C \) is a positive constant independent of \( k \). Indeed, for any \( k \) we have

\[
\mu_k(-\Delta) - \|V\|_\infty \leq \mu_k(-\Delta + V) \leq \mu_k(-\Delta) + \|V\|_\infty.
\]

To apply Theorem 3 we need an estimate for \( \lambda_{k+1} - \lambda_k \).

In what follows we give applications for which one can use our result. Unfortunately, in many cases the condition (5) is not satisfied (which shows the relevance of this condition).

5.1 Schrödinger operator on \( L^p(S^N) \)

Consider the Laplace-Beltrami operator \(-\Delta\) on the sphere \( S^N \). It is well known (see for instance [26, Corollary 4.3]) that the spectrum of the Laplace operator \(-\Delta\) on the space \( L^2(S^N) \) (and hence on \( L^p \)) is the set \( \{k(k + N - 1), k \in \mathbb{Z}\} \). In this case one can apply Theorem 3 with \( \ell > 1/2 \) and we have the decomposition (6).

It is well known that the dimension of the eigen-space associated to the eigenvalue \( k(k + N - 1) \) is \( m(k) := (n + 2k - 1)\frac{(N+k-2)!}{(n-1)!} \). Each eigenvalue \( k(k + N - 1) \) of \(-\Delta\) is split into \( m(k) \) eigenvalues of the operator \( H_p := -\Delta + V \), with some regular potential \( V \) and the distance between two successive eigenvalues is now the distance between the two successive \( \mu_{k,\ell} \) and \( \mu_{k,\ell+1} \) (supposed ordered). Several authors studied the asymptotic of the \((\mu_{k,\ell})\) as \( k \to \infty \), and in particular, Grigis gave in the two dimensional case (see [17]) a very simple form of potential \( V(x) = 4x_1x_2 \), with the notation \( S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\} \), for which the difference \( \mu_{k,\ell+1} - \mu_{k,\ell} = O(k^{-\infty}) \) is exponentially small. See also [11, Theorem 7.9] for another example. This shows the relevance of the hypothesis (5).

5.2 Harmonic oscillator on \( L^p(\mathbb{R}^N) \)

Consider the operator \( H_p := -\Delta + |x|^2 \) on \( L^p(\mathbb{R}^N) \). For \( p = 2 \), \( H_2 \) is self-adjoint and its spectrum is \( \{2k + N; k \in \mathbb{N}\} \) (see for instance [26] Chapter 8, section 6). In this case the distances between successive eigenvalues are bounded below. Using the same technics as in the last subsection Theorem 3 is applicable and we have the decomposition

\[
\sum_{k \geq 1} P_k x = x \quad (x \in D(H_n^\ell)),
\]

where \( n > \frac{N}{2} \left| \frac{1}{p} - \frac{1}{2} \right| \) and \( \ell > 1 \).
5.3 Schrödinger operator on a 2-D flat torus

Consider the Laplace-Beltrami operator $H = -\Delta_T$ on the torus $T := \mathbb{R}^2/\Gamma$ where $\Gamma := \mathbb{Z}e_1 + \mathbb{Z}e_2$ and $e_1 = (a, 0)$, $e_2 = (0, b)$, $a$ and $b$ are positive real numbers. Using again Sobolev imbeddings, the operator $iH$ generates on $L^p(T)$ a $n$-times integrated group $\{S(t)\}$, for $n \geq \frac{N}{2} \left| \frac{1}{p} - \frac{1}{2} \right|$, with $1 < p < \infty$. On $L^p(T)$, $1 \leq p < \infty$, the spectrum $\sigma(H)$ is formed of eigenvalues and is given by $\{a'm^2 + b'n^2; m, n \in \mathbb{Z}\}$ (see for instance [9, Chapitre 3]), where $a' := (2\pi/a)^2$ and $b' := (2\pi/b)^2$. In order to apply Theorem 3 we need to estimate the decay of $\delta_k$, the distance between successive eigenvalues. For this we need the following algebraic proposition:

Proposition 4 [19, Proposition D.1.2 and Remark D.1.2.2] Let $\alpha$ be an algebraic number of degree $d \geq 2$. There exists a constant $C(\alpha) > 0$ such that

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{C(\alpha)}{q^d}$$

for all $\frac{p}{q} \in \mathbb{Q}$.

Corollary 1 Let $\alpha := b^2/a^2$ and $d$ be as in the last proposition and denote by $\{\lambda_n, n \in \mathbb{N}\}$ the spectrum of $i\Delta_T$ the Schrödinger operator on $L^p(T)$, $1 \leq p < \infty$. Then there exists projectors $P_k$ such that

$$\sum_{k \geq 1} P_k x = x \quad (x \in D((-\Delta)^{n(d-1)+2})),
$$

where $n > \frac{N}{2} \left| \frac{1}{p} - \frac{1}{2} \right|$.

Proof. Denote by $0 < \bar{\lambda} = a'm^2 + b'n^2 < \lambda = a'm^2 + b'n^2$ be two successive large eigenvalues. Then

$$|\lambda - \bar{\lambda}|/b' = |n^2 - \bar{n}^2 + \alpha(m^2 - \bar{m}^2)| = |m^2 - \bar{m}^2| \left| \frac{n^2 - \bar{n}^2}{m^2 - \bar{m}^2} + \alpha \right|
$$

and by the last proposition

$$\frac{\lambda - \bar{\lambda}}{b'} \geq \frac{C(\alpha)}{|m^2 - \bar{m}^2|^d} \geq \frac{C}{\lambda^{d-1}}.$$

Therefore $\lambda_k^{d-1}\delta_k \geq \text{const}$ (with the notations of Theorem 3). Applying Theorem 3 and since the series $\sum_{n \geq 1} \sum_{m \geq 1} \frac{1}{(am^2 + bm^2)^d}$ is convergent, one gets the desired result. $\square$

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