Classical membrane in a time dependent orbifold

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(Dated: May 10, 2009)

Abstract

We analyze classical theory of a membrane propagating in a singular background spacetime. The algebra of the first-class constraints of the system defines the membrane dynamics. A membrane winding uniformly around compact dimension of embedding spacetime is described by two constraints, which are interpreted in terms of world-sheet diffeomorphisms. The system is equivalent to a closed bosonic string propagating in a curved spacetime. Our results may be used for finding a quantum theory of a membrane in the compactified Milne space.

PACS numbers: 46.70.Hg, 11.25.-w, 02.20.Sv
I. INTRODUCTION

In our previous papers we have examined the evolution of a particle \[1, 2\] and a string \[3, 4\] across the singularity of the compactified Milne (CM) space. The case of a membrane is technically more complicated because functions describing membrane dynamics depend on three variables. The Hamilton equations for these functions constitute a system of coupled non-linear equations in higher dimensional phase space. Owing to this complexity, we only try to identify some non-trivial membrane states which propagate through the cosmological singularity.

An action integral of a membrane winding uniformly around compact dimension of CM space (equivalently, a closed string in curved spacetime) is reparametrization invariant. The first-class constraints describing membrane dynamics are generators of gauge transformations in the phase space of the system. We present the relationship between these symmetries. Our results constitute prerequisite for quantization of membrane dynamics in CM space.

The paper is organized as follows: In Sec II we recall a general formalism for propagation of a $p$-brane in a fixed spacetime and we indicate that the Hamiltonian for a membrane winding uniformly around compact dimension of CM space reduces to the Hamiltonian of a string. Sec III concerns the algebra of Hamiltonian constraints of a membrane. The constraint satisfy the Poisson algebra, but may be turned into a Lie algebra by some reinterpretation of constraints. In Sec IV we analyze the algebra of conformal transformations connected with the symmetry of the Polyakov action integral of a string in a fixed gauge and we present a homomorphism between this algebra and the constraints algebra. Some insight into this relationship is given in Sec V. We conclude in Sec VI. Appendix consists of useful details clarifying the content of our paper.

II. GENERAL FORMALISM

The Polyakov action for a test $p$-brane embedded in a background spacetime with metric $g_{\tilde{\mu}\tilde{\nu}}$ has the form

$$ S_p = -\frac{1}{2} \mu_p \int d^{p+1}\sigma \sqrt{-\gamma} \left( \gamma^{ab} \partial_{\tilde{\mu}} X^\tilde{\mu} \partial_{\tilde{\nu}} X^\tilde{\nu} g_{\tilde{\mu}\tilde{\nu}} - (p-1) \right), \quad (1) $$

where $\mu_p$ is a mass per unit $p$-volume, $(\sigma^a) \equiv (\sigma^0, \sigma^1, \ldots, \sigma^p)$ are $p$-brane worldvolume coordinates, $\gamma_{ab}$ is the $p$-brane worldvolume metric, $\gamma := det[\gamma_{ab}]$, $(X^{\tilde{\mu}}) \equiv (X^{\mu}, \Theta) \equiv (T, X^k, \Theta) \equiv (T, X^1, \ldots, X^{d-1}, \Theta)$ are the embedding functions of a $p$-brane, i.e. $X^{\tilde{\mu}} = X^{\tilde{\mu}}(\sigma^0, \ldots, \sigma^p)$, in $d+1$ dimensional background spacetime.

It has been found [5] that the total Hamiltonian, $H_T$, corresponding to the action (1) is the following

$$ H_T = \int d^p \sigma H_T, \quad H_T := AC + A^i C_i, \quad i = 1, \ldots, p \quad (2) $$

where $A = A(\sigma^a)$ and $A^i = A^i(\sigma^a)$ are any functions of $p$-volume coordinates,

$$ C := \Pi_{\tilde{\mu}} \Pi_{\tilde{\nu}} g^{\tilde{\mu}\tilde{\nu}} + \mu_p^2 d\epsilon[\partial_{\tilde{\mu}} X^{\tilde{\mu}} \partial_{\tilde{\nu}} X^{\tilde{\nu}} g_{\tilde{\mu}\tilde{\nu}}] \approx 0, \quad (3) $$

$$ C_i := \partial_i X^{\tilde{\mu}} \Pi_{\tilde{\mu}} \approx 0, \quad (4) $$
and where $\Pi_\mu$ are the canonical momenta corresponding to $X_\mu$. Equations (3) and (4) define the first-class constraints of the system.

The Hamilton equations are

$$X_\mu \equiv \frac{\partial X_\mu}{\partial \tau} = \{X_\mu, H_T\}, \quad \Pi_\mu \equiv \frac{\partial \Pi_\mu}{\partial \tau} = \{\Pi_\mu, H_T\}, \quad \tau \equiv \sigma^0,$$

where the Poisson bracket is defined by

$$\{\cdot, \cdot\} := \int d^p \sigma \left( \frac{\partial}{\partial X_\mu} \frac{\partial}{\partial \Pi_\mu} - \frac{\partial}{\partial \Pi_\mu} \frac{\partial}{\partial X_\mu} \right).$$

(6)

In what follows we restrict our considerations to the compactified Milne, CM, space. The CM space is one of the simplest models of spacetime implied by string/M theory [6]. Its metric is defined by the line element

$$ds^2 = -dt^2 + dx_k dx_k + t^2 d\theta^2 = \eta_{\mu\nu} dx_\mu dx_\nu + t^2 d\theta^2 = g_{\tilde{\mu}\tilde{\nu}} d\tilde{x}_\mu d\tilde{x}_\nu,$$

(7)

where $\eta_{\mu\nu}$ is the Minkowski metric, and $\theta$ parameterizes a circle. Orbifolding $S^1$ to a segment $S^1/\mathbb{Z}_2$ gives the model of spacetime in the form of two planes which collide and re-emerge at $t = 0$. Such model of spacetime has been used in [7, 8]. Our results do not depend on the choice of topology of the compact dimension.

In our previous papers [3, 4] and the present one we analyze the dynamics of a $p$-brane which is winding uniformly around the $\theta$-dimension. The $p$-brane in such a state is defined by the conditions

$$\sigma^p = \theta = \Theta \quad \text{and} \quad \partial_\theta X_\mu = 0 = \partial_\theta \Pi_\mu,$$

(8)

which lead to

$$\frac{\partial}{\partial \theta}(X_\mu) = (0, \ldots, 0, 1) \quad \text{and} \quad \frac{\partial}{\partial \tau}(X_\mu) = (\dot{T}, \dot{X}_k, 0).$$

(9)

The conditions (8) reduce (3)-(6) to the form in which the canonical pair $(\theta, \Pi_\theta)$ does not occur [5]. Thus, a $p$-brane in the winding zero-mode state is described by (3)-(6) with $\tilde{\mu}, \tilde{\nu}$ replaced by $\mu, \nu$. The propagation of a $p$-brane reduces effectively to the evolution of $(p - 1)$-brane in the spacetime with dimension $d$ (while $d + 1$ was the original one).

III. ALGEBRA OF CONSTRAINTS OF A MEMBRANE

In the case of a membrane in the winding zero-mode state the constraints are

$$C = \Pi_\mu(\tau, \sigma) \Pi_\nu(\tau, \sigma) \eta_{\mu\nu} + \kappa^2 T^2(\tau, \sigma) \dot{X}_\mu(\tau, \sigma) \dot{X}_\nu(\tau, \sigma) \eta_{\mu\nu} \approx 0,$$

(10)

$$C_1 = \dot{X}_\mu(\tau, \sigma) \Pi_\mu(\tau, \sigma) \approx 0, \quad C_2 = 0,$$

(11)

where $\dot{X}_\mu := \partial X_\mu/\partial \sigma$, $\sigma := \sigma^1$, $\kappa := \theta_0 \mu_2$, and where $\theta_0 := \int d\theta$.

To examine the algebra of constraints we ‘smear’ the constraints as follows

$$\tilde{A} := \int_{-\pi}^{\pi} d\sigma \ f(\sigma) A(X_\mu, \Pi_\mu), \quad f \in \{C^\infty[-\pi, \pi] \mid f^{(n)}(-\pi) = f^{(n)}(\pi)\}.$$
The Lie bracket is defined as
\[ \{ \dot{A}, \dot{B} \} := \int_{-\pi}^{\pi} d\sigma \left( \frac{\partial \dot{A}}{\partial X^\mu} \frac{\partial \dot{B}}{\partial \Pi^\mu} - \frac{\partial \dot{A}}{\partial \Pi^\mu} \frac{\partial \dot{B}}{\partial X^\mu} \right). \tag{13} \]

The constraints in an integral form satisfy the algebra
\[ \{ \dot{C}(f_1), \dot{C}(f_2) \} = \dot{C}_1 \left( 4 \kappa^2 T^2 (f_1 \dot{f}_2 - \dot{f}_1 f_2) \right), \tag{14} \]
\[ \{ \dot{C}_1(f_1), \dot{C}_1(f_2) \} = \dot{C}_1 (f_1 \dot{f}_2 - \dot{f}_1 f_2), \tag{15} \]
\[ \{ \dot{C}(f_1), \dot{C}_1(f_2) \} = \dot{C}(f_1 \dot{f}_2 - \dot{f}_1 f_2). \tag{16} \]

Equations (14)-(16) demonstrate that \( C \) and \( C_1 \) are first-class constraints because the Poisson algebra closes. However, it is not a Lie algebra because the factor \( T^2 \) is not a constant, but a function on phase space. Little is known about representations of such type of an algebra. Similar mathematical problem occurs in general relativity (see, e.g. [9]).

The smearing of constraints helps to get the closure of the algebra in an explicit form. A local form of the algebra includes the Dirac delta so the algebra makes sense but in the space of distributions (see Appendix A for more details). It seems that such an arena is inconvenient for finding a representation of the algebra which is required in the quantization process.

The original algebra of constraints may be rewritten in a tractable form by making use of the redefinitions
\[ C_\pm := \frac{C \pm C_1}{2}, \tag{17} \]
where
\[ C := \frac{\text{original } C}{2\kappa T}, \quad C_1 := \text{original } C_1, \tag{18} \]
where ‘original’ means defined by (10) and (11). The new algebra reads
\[ \{ \dot{C}_+(f), \dot{C}_+(g) \} = \dot{C}_+(f \dot{g} - g \dot{f}), \tag{19} \]
\[ \{ \dot{C}_-(f), \dot{C}_-(g) \} = \dot{C}_-(f \dot{g} - g \dot{f}), \tag{20} \]
\[ \{ \dot{C}_+(f), \dot{C}_-(g) \} = 0. \tag{21} \]

The redefined algebra is a Lie algebra.

The redefinition seems to be a technical trick without a physical interpretation. In what follows we show that it corresponds to the specification of the winding zero-mode state of a membrane not at the level of the constraints (10) and (11), but at the level of an action integral.

The Nambu-Goto action for a membrane in the CM space reads
\[ S_{NG} = -\mu_2 \int d^3 \sigma \sqrt{-\det(\partial_a X^\mu \partial_b X^\nu g_{\mu\nu})} \tag{22} \]
\[ = -\mu_2 \int d^3 \sigma \sqrt{-\det(-\partial_a T \partial_b T + T^2 \partial_a \Theta \partial_b \Theta + \partial_a X^k \partial_b X_k)} \tag{23} \]
where \((T, \Theta, X^k)\) are embedding functions of the membrane corresponding to the spacetime coordinates \((t, \theta, x^k)\) respectively.
An action $S_{NG}$ in the lowest energy winding mode, defined by (8), has the form

$$S_{NG} = -\mu_2 \theta_0 \int d^2 \sigma \sqrt{-T^2 \det(-\partial_a T \partial_b T + \partial_a X^k \partial_b X_k)}$$

$$= -\mu_2 \theta_0 \int d^2 \sigma \sqrt{-\det(\partial_a X^\alpha \partial_b X^\beta \tilde{g}_{\alpha \beta})},$$

where $a, b \in \{0, 1\}$ and $\tilde{g}_{\alpha \beta} = T \eta_{\alpha \beta}$. It is clear that the dynamics of a membrane in the state (8) is equivalent to the dynamics of a string with tension $\mu_2 \theta_0$ in the spacetime with the metric $\tilde{g}_{\alpha \beta}$.

One can verify that the Hamiltonian corresponding to the string action (25) has the form

$$H_T = \int d\sigma \mathcal{H}_T, \quad \mathcal{H}_T := AC + A^1 C_1,$$

where

$$C := \frac{1}{2\mu_2 \theta_0} \Pi_\alpha \Pi_\beta \eta^{\alpha \beta} + \frac{\mu_2 \theta_0}{2} T \partial_a X^\alpha \partial_b X^\beta \eta_{\alpha \beta} \approx 0, \quad C_1 := \partial_\sigma X^\alpha \Pi_\alpha \approx 0,$$

and $A = A(\tau, \sigma)$ and $A^1 = A^1(\tau, \sigma)$ are any regular functions. Therefore (27) and (18) coincide, which gives an interpretation for the redefinition of the constraints.

**IV. ALGEBRA OF CONFORMAL TRANSFORMATIONS**

The Nambu-Goto action (25) is equivalent to the Polyakov action

$$S_p = -\frac{1}{2} \mu_2 \theta_0 \int d^2 \sigma \sqrt{\gamma} (\gamma^{ab} \partial_a X^\alpha \partial_b X^\beta T \eta_{\alpha \beta})$$

because variation with respect to $\gamma^{ab}$ (and using $\delta \gamma = \gamma^{ab}_\tau \delta \gamma_{ab}$) gives

$$\partial_a X^\alpha \partial_b X^\beta T \eta_{\alpha \beta} - \frac{1}{2} \gamma_{ab} \gamma^{cd} \partial_c X^\alpha \partial_d X^\beta T \eta_{\alpha \beta} = 0.$$

The insertion of (29) into the Polyakov action (28) reproduces the Nambu-Goto action (25).

In the gauge $\sqrt{-\gamma} \gamma^{ab} = 1 - \delta_{ab}$ the action (28) reads

$$S_p = -\mu_2 \theta_0 \int d^2 \sigma (\partial_+ X^\alpha \partial_- X^\beta T \eta_{\alpha \beta})$$

where $\partial_{\pm} = \frac{\partial}{\partial \sigma_{\pm}}$, and where $\sigma_{\pm} := \sigma_0 \pm \sigma_1$.

The least action principle applied to (30) gives the following equations of motion

$$\partial_- (T \partial_+ X^k) + \partial_+ (T \partial_- X^k) = 0$$

$$\partial_- (T \partial_+ T) + \partial_+ (T \partial_- T) + \partial_+ X^\alpha \partial_- X^\beta \eta_{\alpha \beta} = 0,$$

where (29), due to the gauge $\sqrt{-\gamma} \gamma^{ab} = 1 - \delta_{ab}$, reads

$$\partial_+ X^\alpha \partial_+ X^\beta \eta_{\alpha \beta} = 0 = \partial_- X^\alpha \partial_- X^\beta \eta_{\alpha \beta}.$$
On the other hand, the action (30) is invariant under the conformal transformations, i.e. \( \sigma_\pm \longrightarrow \sigma_\pm + \epsilon_\pm(\sigma_\pm) \). It is so because for such transformations we have \( \delta X^\alpha = -\epsilon_- \partial X^\alpha - \epsilon_+ \partial X^\alpha \) and hence

\[
\delta S_p = -\mu_2 \theta_0 \int d^2 \sigma \left( \partial_-(\epsilon_- \partial X^\alpha \partial X^\beta \ T\eta_{\alpha\beta}) + \partial_+(\epsilon_+ \partial X^\alpha \partial X^\beta \ T\eta_{\alpha\beta}) \right),
\]

(34)

which is equal to zero since the fields \( X^\alpha \) either vanish at infinity or are periodic. Now let assume that the fields \( X^\alpha \) satisfy (31) and (32). Then (34) can be rewritten as

\[
\delta S_p = -\mu_2 \theta_0 \int d^2 \sigma \left( \partial_-(\epsilon_- \partial X^\alpha \partial X^\beta \ T\eta_{\alpha\beta}) + \partial_+(\epsilon_+ \partial X^\alpha \partial X^\beta \ T\eta_{\alpha\beta}) \right)
\]

\[
+ \partial_+(\epsilon_+ \partial X^\alpha \partial X^\beta \ T\eta_{\alpha\beta}) + \partial_-(\epsilon_- \partial X^\alpha \partial X^\beta \ T\eta_{\alpha\beta}) \right)
\]

(35)

which leads to

\[
\partial_- T_{++} = 0, \quad \partial_+ T_{--} = 0
\]

(36)

where

\[
T_{++} = \epsilon_+ \partial_+ X^\alpha \partial_+ X^\beta \ T\eta_{\alpha\beta}, \quad T_{--} = \epsilon_- \partial_- X^\alpha \partial_- X^\beta \ T\eta_{\alpha\beta}.
\]

(37)

One can verify that the vector fields \( \epsilon_- \partial_- \) and \( \epsilon_+ \partial_+ \) satisfy the following Lie algebra

\[
[f_+ \partial_+, g_+ \partial_+] = (f_+ \dot{g}_+ - g_+ \dot{f}_+) \partial_+, \quad
\]

(38)

\[
[f_- \partial_-, g_- \partial_-] = (f_- \dot{g}_- - g_- \dot{f}_-) \partial_-, \quad
\]

(39)

\[
[f_+ \partial_+, g_- \partial_-] = 0.
\]

(40)

The constraints algebra (19)-(21) defined on the phase space is the representation of the algebra of the conformal transformations (38)-(40) defined on the constraints surface (33). The Lie algebra homomorphism is defined by

\[
\tilde{\mathcal{C}}_+(f(\sigma)) \longrightarrow f(\sigma_+) \partial_+, \quad \tilde{\mathcal{C}}_-(f(\sigma)) \longrightarrow f(\sigma_-) \partial_-, \quad
\]

(41)

where \( \sigma_\pm \in \mathbb{R} \) and \( \sigma \in \mathbb{S} \).

\vspace{1cm}

V. TRANSFORMATIONS GENERATED BY CONSTRAINTS

An action integral of a string is invariant with respect to smooth and invertible maps of worldsheet coordinates

\[
(\tau, \sigma) \rightarrow (\tau', \sigma').
\]

(42)

These diffeomorphisms considered infinitesimally form an algebra of local fields \( -\epsilon(\tau, \sigma) \partial \tau \) and \( -\eta(\tau, \sigma) \partial \sigma \) (we refer to their actions on the fields as \( \delta_\epsilon \) and \( \delta_\eta \), respectively). Mapping (42) leads to the infinitesimal changes of the fields \( X^\mu(\tau, \sigma) \) and \( \Pi_\mu(\tau, \sigma) = \partial L/\partial \dot{X}^\mu = \mu(\frac{1}{2}g_{\mu\nu}\dot{X}^\nu - \frac{A_1}{A}g_{\mu\nu}\dot{X}^\nu) \) as follows

\[
\delta X^\mu = \delta_\epsilon X^\mu + \delta_\eta X^\mu = \epsilon \dot{X}^\mu + \eta \dot{X}^\mu, \quad \delta \Pi_\mu = \epsilon \dot{\Pi}_\mu + \epsilon(A_1 \Pi_\mu + \mu A_{\mu\nu}\dot{X}^\nu) + (\eta \Pi_\mu)'.
\]

(43)
The transformations \([43]\) are defined along curves in the phase space with coordinates \((X^\mu, \Pi_\mu)\) and are expected to be generated by the first-class constraints \(\hat{C}\) and \(\hat{C}_1\) according to the theory of gauge systems [10, 11]. One verifies that

\[
\{X^\mu, \hat{C}(\varphi)\} = \frac{\partial \varphi}{\partial \mu} \Pi_\mu, \quad \{\Pi_\mu, \hat{C}(\varphi)\} = -\frac{\partial \varphi}{2\mu}(\Pi_\alpha \Pi_\beta g_{\alpha \beta, X^\nu} + \dot{X}^\alpha g_{\alpha \beta, X^\nu}) + \mu(\varphi g_{\mu \nu} \dot{X}^\nu)',
\]

(44)

\[
\{X^\mu, \hat{C}_1(\phi)\} = \phi \dot{X}^\mu, \quad \{\Pi_\mu, \hat{C}_1(\phi)\} = (\phi \Pi_\mu)',
\]

(45)

where \(\phi(\sigma, \tau)\) and \(\varphi(\sigma, \tau)\) are smearing functions depending on two variables, and the integration defining the smearing of the constraints \(C\) and \(C_1\) does not include the integration with respect to \(\tau\) variable (see (12)).

The comparison of \([43]\) with \([44]-[45]\) gives specific relations between these two transformations. For the action of the constraints along curves in the phase space, which are solutions to the equations of motion, we get

\[
\{X^\mu, \hat{C}(\varphi)\} = \delta_\varphi X^\mu - \delta_{\varphi_1} X^\mu, \quad \{\Pi_\mu, \hat{C}(\varphi)\} = \delta_\varphi \Pi_\mu - \delta_{\varphi_1} \Pi_\mu,
\]

(46)

\[
\{X^\mu, \hat{C}_1(\phi)\} = \delta_\phi X^\mu, \quad \{\Pi_\mu, \hat{C}_1(\phi)\} = \delta_\phi \Pi_\mu.
\]

(47)

Since \(A\) and \(A^1\) (see (C2)) are invariant with respect to conformal isometries with respect to the worldsheet metric, the solutions to the equations of motion with fixed \(A\) and \(A^1\) have still some gauge freedom. The reduction of transformations \([16]-[17]\) to the conformal transformations \(\sigma_\pm \rightarrow \sigma_\pm + \rho_\pm(\sigma_\pm)\) for the curves in the orthonormal gauge \(A = 1\) and \(A^1 = 0\), leads to

\[
\frac{1}{2}(\delta_{\rho_\pm} \pm \delta_{\rho_\pm}')F(X^\mu(\sigma, \tau), \Pi_\mu(\sigma, \tau)) = \{F(X^\mu(\sigma, \tau), \Pi_\mu(\sigma, \tau)), \hat{C}_\pm(\rho_\pm)\},
\]

(48)

where \(F\) is a smooth function on phase space. One may show that (48) corresponds to the transformations defined by the algebra \([33]-[40]\) but limited to the solutions of the equations of motion. On the other hand, the transformations \([18]\) and \([16]-[17]\) coincide with the algebra \([19]-[21]\), for fixed \(\tau\).

Now, we can see that the homomorphism \([11]\) represents the reduction of the algebra of general conformal transformations (for fields not necessarily satisfying the equations of motion) to the algebra of generators of conformal transformations acting on curves \((X^\mu, \Pi_\mu)\) for fixed \(\tau\). The latter algebra is equivalent to the algebra of generators \(\hat{C}\) and \(\hat{C}_1\) acting on the phase space \((X^\mu, \Pi_\mu)\).

VI. CONCLUSIONS

In this paper we have considered states of membrane winding uniformly around compact dimension of the background space. Dynamics of a membrane in such special states is equivalent to the dynamics of a closed string in curved target space. However, the problem of quantization of a string in curved spacetime has not been solved yet (see, e.g. [14]). The construction of satisfactory quantum theory of membrane presents a challenge.

The first-class constraints specifying the dynamics of a membrane propagating in the compactified Milne space satisfy the algebra which is a Poisson algebra. Methods for finding a self-adjoint representation of such type of an algebra are very complicated [3, 14]. We
overcome this problem by the reduction and redefinition of the constraints algebra. Resulting algebra is a Lie algebra which simplifies the problem of quantization of the membrane dynamics.

We have found a homomorphism between the algebra of conformal transformations and the algebra of transformations generated by the first-class constraints of the system. This may enable the construction of quantum dynamics of a membrane by making use of representations of conformal algebra. Details concerning quantization procedure will be presented elsewhere [15].

**APPENDIX A: LOCAL FORM OF THE CONSTRAINTS ALGEBRA**

One can verify that the constraints (10) and (11) satisfy the algebra

\[ \{C(\sigma), C'(\sigma')\} = 8\kappa^2 T^2(\sigma) C_1(\sigma) \frac{\partial}{\partial \sigma} \delta(\sigma' - \sigma) + 4\kappa^2 \delta(\sigma' - \sigma) \frac{\partial}{\partial \sigma} (T^2(\sigma) C_1(\sigma)), \]

\[ \{C(\sigma), C_1(\sigma')\} = 2 C(\sigma) \frac{\partial}{\partial \sigma} \delta(\sigma' - \sigma) + \delta(\sigma' - \sigma) \frac{\partial}{\partial \sigma} C(\sigma), \]

\[ \{C_1(\sigma), C_1(\sigma')\} = 2 C_1(\sigma) \frac{\partial}{\partial \sigma} \delta(\sigma' - \sigma) + \delta(\sigma' - \sigma) \frac{\partial}{\partial \sigma} C_1(\sigma), \]

where \( \partial X^\mu(\sigma')/\partial X^\nu(\sigma) = \delta^\mu_\nu \delta(\sigma' - \sigma) = \partial \Pi_\nu(\sigma')/\partial \Pi_\mu(\sigma) \) (with other partial derivatives being zero), and where the Poisson bracket is defined to be

\[ \{\cdot, \cdot\} := \int_{-\pi}^{\pi} d\sigma \left( \frac{\partial}{\partial X^\mu} \frac{\partial}{\partial \Pi_\mu} - \frac{\partial}{\partial \Pi_\mu} \frac{\partial}{\partial X^\mu} \right). \] 

**APPENDIX B: RELATION BETWEEN GAUGES**

The least action principle applied to the Nambu-Goto action, \( \delta S_{NG} = 0 \), gives

\[ \partial_a \left( \frac{\partial_b X^\alpha \partial_b X^\beta g_{\alpha\beta}}{-\det(\partial_a X^\alpha \partial_b X^\beta g_{\alpha\beta})} \partial_a X_\mu - \frac{\partial_a X^\alpha \partial_b X^\beta g_{\alpha\beta}}{-\det(\partial_a X^\alpha \partial_b X^\beta g_{\alpha\beta})} \partial_b X_\mu \right) \]

\[ - \frac{(\partial_a X^\alpha \partial_b X^\beta g_{\alpha\beta}) \partial_b X^\alpha \partial_b X^\beta - (\partial_a X^\alpha \partial_b X^\beta g_{\alpha\beta}) \partial_a X^\alpha \partial_b X^\beta}{2\sqrt{-\det(\partial_a X^\alpha \partial_b X^\beta g_{\alpha\beta})}} g_{\alpha\beta,\mu} = 0. \] 

(B1)

In the case of the Polyakov action the least action principle, \( \delta S_p = 0 \), gives

\[ \partial_a (\sqrt{-\gamma} \gamma^{ab} \partial_b X_\mu) = -\frac{1}{2} \sqrt{-\gamma} \gamma^{ab} \partial_a X^\alpha \partial_b X^\beta g_{\alpha\beta,\mu}, \]

\[ \partial_a X^\alpha \partial_b X^\beta g_{\alpha\beta} - \frac{1}{2} \gamma^{ab} \gamma^{cd} \partial_c X^\alpha \partial_d X^\beta g_{\alpha\beta} = 0. \] 

(B3)

On the other hand, the Hamilton equations read

\[ \dot{X}^\mu = \{X^\mu, H_T\} \approx A^1 \Pi_\mu g^{\mu\nu} + A^1 \partial_\sigma X^\mu, \]

\[ \dot{\Pi}_\mu = \{\Pi_\mu, H_T\} \approx -A^1 \frac{1}{2\mu} (\Pi_\alpha \Pi_\beta \partial g^{\alpha\beta}/\partial X^\nu + \mu^2 \partial_\sigma X^\alpha \partial_\sigma X^\beta \partial g_{\alpha\beta}/\partial X^\mu) + \mu \partial_\sigma (A g_{\nu\mu} \partial_\sigma X^\nu) \]

\[ + \partial_\sigma (A^1 \Pi_\mu), \]

(B5)
which in the case $g_{\alpha\beta} = T\eta_{\alpha\beta}$ give

$$\dot{X}^\mu = \{X^\mu, H_T\} \approx \frac{1}{\mu T} \Pi^\mu \eta^{\mu\mu} + A^1 \partial_\sigma X^\mu, \quad (B6)$$

$$\dot{\Pi}_\mu = \{\Pi_\mu, H_T\} \approx -A \frac{\delta_\mu}{2\mu} \Pi^\alpha \eta^{\alpha\beta} T^2 + \mu^2 \partial_\sigma X^\alpha \partial_\beta X^\beta \eta_{\alpha\beta} + \mu \partial_\sigma (AT\eta_{\mu\nu} \partial_\sigma X^\nu) + \partial_\sigma (A^1 \Pi_\mu). \quad (B7)$$

Now, we are ready to find the relations among $\gamma_{\alpha\beta}, A, A^1$ and the induced metric. It is not difficult to see that

$$\frac{1}{\sqrt{-\det(\partial_a X^\mu \partial_b X^\nu g_{\mu\nu})}} \left(\begin{array}{cc}
-\partial_\sigma X^\mu \partial_\sigma X^\nu g_{\mu\nu} & \partial_\gamma X^\mu \partial_\gamma X^\nu g_{\mu\nu} \\
\partial_\gamma X^\mu \partial_\gamma X^\nu g_{\mu\nu} & -\partial_\sigma X^\mu \partial_\sigma X^\nu g_{\mu\nu}
\end{array}\right) = -\sqrt{-\gamma} \gamma^{ab}, \quad (B8)$$

$$\left(\begin{array}{cc}
\frac{1}{A} & -\frac{A^1}{A} \\
-A^1 & -A + \frac{(A^1)^2}{A}
\end{array}\right) = -\sqrt{-\gamma} \gamma^{ab}. \quad (B9)$$

For instance, $\sqrt{-\gamma} \gamma^{ab} = (-1)^a \delta_{ab}$ translates into $A = 1$ and $A^1 = 0$.

There exists an interesting discussion of the ADM like gauges in the context of a constrained Hamiltonian approach to the bosonic p-branes in the Minkowski space $[16, 17]$. We postpone finding the relation between our choice of gauges and the ADM type and its usefulness in the context of the singularity problem to our next papers.

**APPENDIX C: POSITION-VELOCITY AND PHASE SPACES**

The position-velocity space is a space of pairs of fields $(X^\mu(\sigma), \dot{X}^\mu(\sigma))$, whereas the space of pairs $(X^\mu(\sigma), \Pi_\mu(\sigma))$ defines a phase space. The transformation

$$\{X^\mu, \dot{X}^\mu\} \rightarrow \{X^\mu, \Pi_\mu = \frac{\mu}{\sqrt{-g}} \left(-\dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + (\dot{X} \dot{X}) g_{\mu\nu} \dot{X}^\nu\right)\} \quad (C1)$$

is a surjection onto the surface $C = 0 = C^1$. It becomes a bijection for fixed

$$A := -\frac{\sqrt{-g}}{\dot{X}}, \quad A^1 := \frac{(\dot{X} \dot{X})}{(\dot{X})^2}, \quad (C2)$$

where $\dot{X}^\mu \dot{X}^\nu g_{\mu\nu} > 0$ and $\dot{X}^\mu \dot{X}^\nu g_{\mu\nu} < 0$, and $g < 0$. We say that such choice of $A, A^1$ defines the $(A, A^1)$-sector. Thus, the mapping

$$\{X^\mu(\sigma), \Pi_\mu(\sigma)\} \rightarrow \{X^\mu(\sigma), \dot{X}^\mu(\sigma) = \frac{A}{\mu} \Pi_\mu + A^1 \dot{X}^\mu\} \quad (C3)$$

defines the one-to-one correspondence between the phase space surface $C = 0 = C^1$ and the $(A, A^1)$-sector. If $A$ and $A^1$ depend on $\tau$, then the $(A, A^1)$-sector and the correspondence depend on $\tau$ as well. All $(A, A^1)$-sectors are equivalent ($A \neq 0$) in the sense that all solutions to dynamics are mapped from one sector to another by a diffeomorphism $[12]$. 

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ACKNOWLEDGMENTS

This work has been supported by the Polish Ministry of Science and Higher Education Grant NN 202 0542 33.

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