TOPICAL REVIEW

The many symmetries of Calabi–Yau compactifications

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Abstract
We review the major mathematical concepts involved in the dimensional reduction of $D = 11$ $\mathcal{N} = 1$ supergravity theory over a Calabi–Yau manifold with non-trivial complex structure moduli resulting in ungauged $D = 5$ $\mathcal{N} = 2$ supergravity theory with hypermultiplets. The latter has a particularly rich structure with many underlying geometries. We reproduce the entire calculation and particularly emphasize its symplectic symmetry and how that arises from the topology of the underlying subspace. The review is intended to fill a specific gap in the literature with the hope that it will be useful to both the beginner and the expert alike.

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1. Introduction

It has long been hoped that the use of Kaluza–Klein techniques to dimensionally reduce string/supergravity (SUGRA) theories will eventually lead to a physically acceptable four-dimensional representation of our universe, i.e. the standard model plus gravity. Unfortunately the number of possible ways of doing so turns out to be (almost) unbelievably high. In fact, the figure $10^{500}$ is often quoted. It is further speculated that a specific choice of vacuum (i.e. the choice of compactification subspace, its topological parameters, etc) would pick the correct four-dimensional structure by some sort of physical ‘natural selection’ mechanism. This problem of the so-called string theory landscape [1–5] is currently the major obstacle in our understanding of string theory as the most promising theory of everything, and is, in fact, the main argument raised by the theory’s critics (and a good argument no doubt) [6, 7]. It then becomes of paramount importance to understand the mathematical techniques of dimensional reduction. These normally involve understanding the geometries and topologies of manifolds with special holonomy, as well as specific types of complex manifolds that arise as a consequence of the dimensional reduction. There is of course a huge literature available on these topics, including discussions written by both physicists and mathematicians. However, there does not seem to be a single source that would act as a tutorial to the beginner, discussing the calculation from the most basic of definitions all the way to completion. This is further complicated by the lack of a unified notation for the various topics. As such it is quite hard for the beginner to follow and reproduce the results in full. This review intends to fill this particular gap. Our choice of specific calculation to reproduce is that of the reduction of 11-dimensional $\mathcal{N}=1$ supergravity over a Calabi–Yau (CY) threefold with non-trivial complex structure moduli. This leads to a five-dimensional $\mathcal{N}=2$ theory with a matter sector comprising an arbitrary number of scalar fields (and their supersymmetric partners), the so-called hypermultiplets. This theory is rarely discussed in the literature, particularly not in the form we review here; another gap we intend to fill.

The study of $\mathcal{N}=2$ supergravity theories in general has gained interest in recent years for a variety of reasons. For example, $\mathcal{N}=2$ branes are particularly relevant to the conjectured equivalence between string theory on anti-de Sitter space and certain superconformal gauge theories living on the boundary of the space (the AdS/CFT duality) [9]. Also interesting is that many results were found to involve the so-called attractor mechanism (e.g. [10–12]), the study of which developed very rapidly with many intriguing outcomes (e.g. [13–15]). From the point of view of dimensional reduction, many $D=4,5$ results were shown to be related to higher dimensional ones via wrapping over specific cycles of manifolds with special holonomy. For example, M-branes wrapping Kähler cycles of a CY threefold [16] dimensionally reduce to black holes and strings coupled to the vector multiplets of five-dimensional $\mathcal{N}=2$ supergravity [17], while M-branes wrapping special Lagrangian cycles reduce to configurations carrying charge under the hypermultiplet scalars [18–22].

In reviewing the literature, one notes that most studies in $\mathcal{N}=2$ SUGRA in any number of dimensions specifically address the vector multiplets sector, setting the hypermultiplets to zero. This is largely due to the fact that the standard representation of the hypermultiplet scalars as coordinates on a quaternionic manifold is somewhat hard to deal with. It has been shown, however, that certain duality maps relate the target space of a given higher dimensional fields’ sector to that of a lower dimensional one [23]. Particularly relevant to this review is the so-called c-map which relates the quaternionic structure of the $D=5$ hypermultiplets

1 Also see [8] for a counter argument.
to the more well-understood special geometric structure of the $D = 4$ vector multiplets. This means that one can recast the $D = 5$ hypermultiplet fields into a form that makes full use of the methods of special geometry. This was done in [24] and applied in the same reference as well as in [20] and others. Using this method, finding solutions representing the five-dimensional hypermultiplet fields often means coming up with ansätze that have special geometric form. This can be, and has been, done by building on the considerable $D = 4$ vector multiplet literature, and in most cases the solutions are remarkably similar. For example, $D = 5$ hypermultiplet couplings to 2-branes and instantons [20, 24] lead to the same type of attractor equations found for the vector multiplets coupled to $D = 4$ black holes (e.g. [25–28]).

Furthermore, it has long been known that quaternionic and special Kähler geometries contain symplectic isometries and that the hypermultiplets action (with or without gravity) is in fact symplectically invariant\textsuperscript{2}. The exploitation of this particular property was recently proposed as a method of constructing solutions to the theory [31]. We include this in our review and emphasize the origin of the symplectic structure of the theory from the topology of the subspace. The discussion is not intended to be exhaustive; rather enough information is presented to achieve an overall, hopefully intuitive, understanding of the process and provide a hands-on first reading. Some part of this review is based on [32]. Further details may be sought out in the given cited texts.

The review is structured in the following way: starting from basic principles, section 2 discusses the various types of complex manifolds needed in the rest of the review. Section 3 focuses on special Kähler geometry (SKG) with particular emphasis on its symplectic structure. Section 4 presents the details of the dimensional reduction of $D = 11$ SUGRA over a CY threefold with non-trivial complex structure moduli. For easy reference we include an appendix on the basics of the language of differential forms on manifolds.

2. Manifolds: from Riemann to Yau

We review the various classes of complex manifolds we will need. Starting with elementary definitions, we write down the different properties with minimal mathematics. The discussion is by no means exhaustive, but enough material is reviewed in preparation for a, hopefully, intuitive understanding of the process of dimensional reduction. Where appropriate, we use the language of differential forms as defined in the appendix.

2.1. Complex and Kähler manifolds

We define the notion of a real $n$-dimensional manifold $M$ as a set of points that behaves locally like $\mathbb{R}^n$ such that $n$ real parameters $(x^1, \ldots, x^n)$ are coordinates on $M$ [33, 34]. Similarly, a complex $k$-dimensional manifold may be defined as a set of points that behaves locally like $\mathbb{C}^k$, where $\{n, k \in \mathbb{Z}\}$. A Riemannian manifold is a manifold on which a smooth symmetric positive-definite metric tensor $g_{\mu\nu}(x^\alpha)$ can be defined, describing a line element on the manifold $d\mathbf{s}^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu$. A manifold is called Lorentzian if its metric has Lorentzian signature\textsuperscript{3}; i.e. it behaves locally like $\mathbb{R}^{1,n-1}$. A Levi-Civita connection (i.e. metric-compatible) may be chosen, leading to the usual expressions for the Christoffel

\textsuperscript{2} This being a straightforward generalization of the ordinary Maxwell dualities first discussed from within the context of supergravity in [29], but generally known for ordinary electrodynamics since 1925 [30].

\textsuperscript{3} Which we take to be $(-+++\cdots)$ throughout.
symbols, the Riemann and Ricci tensors and the Ricci scalar:
\[
\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda k} \left[ (\partial_\mu g_{k\nu}) + (\partial_\nu g_{k\mu}) - (\partial_k g_{\mu\nu}) \right]
\]
\[
R_{\rho\sigma}^{\mu\nu} = \left( \partial_\rho \Gamma_{\nu}^{\sigma\mu} \right) - \left( \partial_\sigma \Gamma_{\nu}^{\rho\mu} \right) + \Gamma_{\rho\sigma}^{\alpha} \Gamma_{\nu}^{\mu\alpha} - \Gamma_{\rho\nu}^{\alpha} \Gamma_{\sigma}^{\mu\alpha}
\]
\[
R_{\mu\nu} = R_{\rho\nu}^{\rho\mu} = \left( \partial_\rho \Gamma_{\nu\rho}^{\mu} \right) - \left( \partial_\mu \Gamma_{\nu\rho}^{\rho} \right) + \Gamma_{\mu\rho}^{\alpha} \Gamma_{\nu\rho}^{\alpha} - \Gamma_{\rho\nu}^{\alpha} \Gamma_{\mu\rho}^{\alpha}
\]
\[
R = g^{\mu\nu} R_{\mu\nu}
\]
(1)

If one now considers manifolds with even dimensions, i.e. \( n = 2k \), then one can, at least locally, 'complexify' \( M \) by pairing \( x^\alpha \) as follows (summation convention not used):
\[
w^\alpha = x^\alpha + \tau^\alpha\bar{\tau}^\alpha + kx^\alpha + k = w^m, \quad m = 1, \ldots, k
\]
\[
\bar{w}^\alpha = x^\alpha + \bar{\tau}^\alpha\tau^\alpha + kx^\alpha + k = \bar{w}^\bar{m}, \quad \bar{m} = \bar{1}, \ldots, \bar{k}
\]
(2)

where the \( \tau \)'s are complex parameters that specify a complex structure on the manifold (more on that later). A general metric on such a manifold is then
\[
d s^2 = g_{mn} dw^m d\bar{w}^\bar{n} + g_{\bar{m}\bar{n}} d\bar{w}^m dw^n + 2g_{mn} dw^m d\bar{w}^\bar{n}.
\]
(3)

Reality of the line element is ensured by the conditions
\[
g_{mn} = g_{\bar{m}\bar{n}}, \quad g_{mn} = g_{\bar{m}\bar{n}}.
\]
(4)

A Hermitian manifold is defined as a complex manifold where there is a preferred class of coordinate systems such that
\[
g_{mn} = g_{\bar{m}\bar{n}} = 0.
\]
(5)

The line element reduces to
\[
d s^2 = 2g_{mn} dw^m d\bar{w}^\bar{n}.
\]
(6)

On any Hermitian manifold, a real 2-form, known as the Kähler form, can be defined as a \((1, 1)\)-form as follows:
\[
K = i g_{mn} dw^m \wedge d\bar{w}^\bar{n}.
\]
(7)

A Kähler manifold is a Hermitian manifold whose Kähler form is closed, i.e.
\[
d K = 0.
\]
(8)

As a closed 2-form, the Kähler form is a member of a cohomology class, namely the second De-Rahm class \([K] \in H^2(M)\). Treating \( K \) as a \((1, 1)\)-form in our complex basis, this corresponds to the \( H^{1,1} \) Dolbeault class. Henceforth we will refer to \( H^{1,1} \) as the Kähler class of the metric. Equation (8) leads to the ‘curl-free’ condition
\[
\partial_m g_{np} - \partial_n g_{mp} = 0,
\]
(9)

which may equivalently be used as the definition of a Kähler manifold\(^4\). This implies that locally the Kähler metric can be determined in terms of a real scalar function, known as the Kähler potential \( K(w, \bar{w}) \). In other words (9) is solved by
\[
g_{mn} = \partial_m \partial_n K \quad \rightarrow \quad K = i(\partial_m \partial_n K) dw^m \wedge d\bar{w}^\bar{n}.
\]
(10)

Obviously, the metric is invariant under changes of the Kähler potential of the form \( K(w, \bar{w}) \rightarrow K(w, \bar{w}) + f_1(w) + f_2(\bar{w}) \), known as the Kähler gauge transformations. It follows then that if two Kähler metrics on \( M \) belong to the same Kähler class, then they can differ only by a Kähler transformation.

\(^4\) Note that not all complex manifolds admit Kähler metrics.
From a computational point of view, condition (8), or equivalently (9), simplifies the properties of the manifold considerably; for example, one finds that

\[ \Gamma^r_{mn} = g^{rp} (\partial_m g_{np}), \quad \Gamma^p_{\bar{m}n} = g^{rp} (\partial_\bar{m} g_{np}) \]  

are the only non-vanishing Christoffel symbols, indicating that parallel transport does not mix the holomorphic with the antiholomorphic components of a vector. Also the non-vanishing components of the Ricci tensor are found to be

\[ R_{m\bar{n}} = \partial_m \partial_{\bar{n}} \ln g, \quad \text{where} \quad g = \det g_{m\bar{n}}. \]  

### 2.2. Issues of global importance

Technically, the assumption that any real 2k-dimensional manifold \( M \) can be made into a complex manifold is only valid locally. Global considerations must be included in order to properly decide if a given manifold is truly complex everywhere. A key element to such considerations is the so-called complex structure of the manifold. Intuitively, it is nothing more than the formalization of multiplication by \( i \) smoothly over the manifold, i.e. an operation on geometrical objects whose square is negative the identity. A tensor \( J \) on \( M \) is called an almost complex structure if it satisfies the condition

\[ J^2 \sim -\mathbb{I} : J^\rho_\mu(x)J^\nu_\rho(x) = -\delta^\nu_\mu, \]  

where the 2k real Greek indices break into \( (m, \bar{n}) \) as before. In components, \( J \) is related to the Kähler form by

\[ K_{\mu\nu} = g_{\nu\rho} J^\rho_\mu, \]  

and is also related to the complex parameters in (2). For example, one common choice is

\[ \tau^\mu_\nu = iJ^\mu_\nu, \quad \bar{\tau}^\mu_\nu = -iJ^\mu_\nu. \]  

Now if a manifold \( M \) has a smooth almost complex structure, it is called an almost complex manifold. An almost complex structure becomes a complex structure when its so-called Nijenhuis tensor

\[ N^\rho_{\mu\nu} = J^\rho_{\mu\nu} \left[ (\partial_\nu J^\sigma_\mu) - (\partial_\mu J^\sigma_\nu) \right] - J^\sigma_{\nu\mu} \left[ (\partial_\nu J^\rho_\sigma) - (\partial_\sigma J^\rho_\nu) \right] \]  

vanishes everywhere. This condition is achieved by demanding that different complex structures on a manifold smoothly patch together. So, any 2k-dimensional real manifold is locally complex (almost complex manifold), but only globally (complex manifold) when it admits a complex structure with vanishing Nijenhuis tensor. This is analogous to the concept that any Riemannian manifold is locally flat, but only globally when the Riemann tensor vanishes everywhere. Consequently one may speak of 'almost Hermitian manifolds', 'almost Kähler manifolds' and so on.

Another point of global importance is the question of holonomy groups on a Kähler manifold [35]. Consider a vector \( V^\mu \) on an \( n \)-fold and parallel transport it around a closed loop; generally the vector will not return to itself, but rather rotated by an element of \( GL(n, \mathbb{R}) \). The subset of \( GL(n, \mathbb{R}) \) defined in this way forms the holonomy group of the manifold. The restricted holonomy group would be the subset defined by paths which may be smoothly shrunk to a point (contractable loops). The classification of the restricted holonomy groups of all Riemannian manifolds has been performed by Berger [36], which we list for completeness.

**Berger’s theorem.** Suppose \( M \) is a simply connected manifold of dimension \( n \) and \( g \) is a Riemannian metric on \( M \); then exactly seven restricted, or special, holonomy cases are possible.

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5 A given real manifold can admit many complex structures.
(1) Generic Riemannian manifolds, Hol(g) = SO(n).
(2) Kähler manifolds, where \( n = 2k \) with \( k \geq 2 \) and Hol(g) = U(k) \( \subset SO(2k) \).
(3) CY manifolds, where \( n = 2k \) with \( k \geq 2 \) and Hol(g) = SU(k) \( \subset SO(2k) \).
These are also necessarily Ricci-flat (Yau’s theorem).
(4) Hyper–Kähler manifolds, where \( n = 4k \) with \( k \geq 2 \) and Hol(g) = Sp(k) \( \subset SO(4k) \).
(5) Quaternionic Kähler manifolds, where \( n = 4k \) with \( k \geq 2 \) and Hol(g) = Sp(k) \( \otimes \mathbb{R} \subset SO(4k) \).
(6) Manifolds with \( n = 7 \) and Hol(g) = \( G_2 \subset SO(7) \).
(7) Manifolds with \( n = 8 \) and Hol(g) = Spin(7) \( \subset SO(8) \). The groups \( G_2 \) and Spin(7) are exceptional holonomy groups.

We can categorize the holonomy groups in Berger’s list as follows.

- The Kähler holonomy groups \( U(k), SU(k) \) and \( Sp(k) \). Any Riemannian manifold with one of these is necessarily Kähler.
- The Ricci-flat holonomy groups: \( SU(k), Sp(k), G_2 \) and Spin(7). Any metric with one of these is necessarily Ricci-flat.
- The exceptional holonomy groups: \( G_2 \) and Spin(7). So called because they have properties fundamentally different from the others.

The Berger list may also be understood in terms of the four division algebras in the following way: it is well known that one can define exactly four algebras where, for two quantities \( Z_1 \) and \( Z_2 \), the property \( |Z_1 Z_2| = |Z_1||Z_2| \) is satisfied. These are the real numbers \( \mathbb{R} \), the complex numbers \( \mathbb{C} \), the quaternions \( \mathbb{H} \) and the octonions, or Cayley numbers, \( \mathbb{O} \). The Berger list fits into this classification by noting that

- \( SO(n) \) is a group of automorphisms of \( \mathbb{R}^n \).
- \( U(k) \) and \( SU(k) \) are groups of automorphisms of \( \mathbb{C}^k \).
- \( Sp(k) \) and \( Sp(k) \otimes \mathbb{R} \) are groups of automorphisms of \( \mathbb{H}^k \).
- \( G_2 \) is a group of automorphisms of \( \mathbb{H} \) \( \otimes \mathbb{H} \approx i\mathbb{R}^7 \).
- \( \text{Spin}(7) \) is a group of automorphisms of \( \mathbb{O} \) \( \approx \mathbb{R}^8 \).

It is interesting to note that all of the manifolds on Berger’s list have found applications in theoretical physics. In fact, the \( \mathcal{N} = 2 \) theory we will be discussing makes use of all of them except the exceptional manifolds.

### 2.3. Hodge–Kähler manifolds

We recall that given a Riemannian manifold \( \mathcal{M} \) endowed with a metric \( g_{\mu\nu} \), one can define the vielbeins \( e^\alpha \) and the connection 1-form \( \omega^{\alpha\beta} \) (a.k.a. spin connection) in the following way:

\[
\begin{align*}
\text{d}x^2 &= g_{\mu\nu} \text{d}x^\mu \text{d}x^\nu = \eta_{\mu\nu} e^\mu e^\nu; \\
\omega^\alpha_{\mu} &= e^\beta \Gamma^\alpha_{\mu\beta}; \\
\omega^{\alpha\beta} &= \omega^{\alpha\beta}_{\mu} \text{d}x^\mu,
\end{align*}
\]

(17)

such that the so-called Cartan structure equations define the torsion and curvature 2-forms:

\[
\begin{align*}
\mathfrak{A}^\alpha &= \text{d}e^\alpha + \omega^\beta_\alpha \wedge e^\beta = \frac{1}{2} \omega^\mu_{\alpha\beta} \text{d}x^\mu \wedge \text{d}x^\nu; \\
\mathfrak{R}^{\alpha\beta} &= \text{d}\omega^{\alpha\beta} + \omega^\gamma_{\alpha} \wedge \omega^{\beta\gamma} = \frac{1}{2} \mathfrak{R}^{\mu\alpha\beta} \text{d}x^\mu \wedge \text{d}x^\nu,
\end{align*}
\]

(19)

where the hated indices are raised and lowered by the flat metric \( \eta_{\alpha\beta} \) (which may be either Minkowski or Euclidean depending on the signature of \( g_{\mu\nu} \)), describing a flat space tangent...
to each point on the manifold. They are also sometimes referred to as ‘frame’ indices, as opposed to the manifold’s ‘world’ indices.

Using this language, one can define a topological quantity known as the total Chern form [37], which is a polynomial in the curvature as follows:

\[
C(\mathfrak{g}) = \det \left( 1 + \frac{i}{2\pi} \mathfrak{g} \right) = 1 + c_1(\mathfrak{g}) + c_2(\mathfrak{g}) + \cdots. \tag{20}
\]

The terms \(c_i\) are the so-called Chern classes. They belong to topologically distinct cohomology classes. For example,

\[
c_0 = 1,
\]

\[
c_1 = \frac{i}{2\pi} \text{Tr} \mathfrak{g},
\]

\[
c_2 = \frac{1}{8\pi^2} [\text{Tr} (\mathfrak{g} \wedge \mathfrak{g}) - \text{Tr} \mathfrak{g} \wedge \text{Tr} \mathfrak{g}], \quad \text{etc.} \tag{21}
\]

Furthermore, integrals such as

\[
\int_M c_2(\mathfrak{g}) \quad \text{and} \quad \int_M c_1(\mathfrak{g}) \wedge c_1(\mathfrak{g})
\]

are topologically invariant integers, known as the Chern numbers.

The Chern classes are widely used in classifying invariant quantities in classical field theory [7]. They can also be used to topologically distinguish various types of manifolds. Given the Ricci tensor \(R_{mn}\) of a Kähler manifold, we can define the \((1, 1)\) Ricci form

\[
\tilde{\mathcal{R}} = R_{mn} dw^m \wedge d\bar{w}^n. \tag{22}
\]

Since the Ricci form is necessarily closed \(d\tilde{\mathcal{R}} = 0\), then it defines an equivalence class in \(H^{1,1}\). The first Chern class is simply

\[
c_1 = \frac{i}{2\pi} \tilde{\mathcal{R}}. \tag{23}
\]

Now consider a line bundle \(\mathcal{L}\) over a Kähler manifold \(\mathcal{M}\). By definition, this is a holomorphic vector bundle of rank 1. \(8\) The first Chern class is the only one that exists for such a bundle. In terms of some Hermitian fiber metric \(h\) on \(\mathcal{L}\), and using \((12)\), this is clearly

\[
c_1(\mathcal{L}) = \frac{i}{2\pi} (\bar{\partial} \partial \ln h), \tag{24}
\]

where \(\partial \equiv dw^a \partial_a\) and \(\bar{\partial} \equiv d\bar{w}^\alpha \bar{\partial}_\alpha\). Since \(\mathcal{L}\) is a line bundle, its connection (Christoffel symbol) is a 1-form defined by \(h\) as follows:

\[
\bar{\partial} = \partial \ln h, \quad \bar{\partial} = \bar{\partial} \ln h. \tag{25}
\]

Also, it is known that there exists a correspondence between line bundles and \(U(1)\) bundles. At the level of connections this reduces to

\[
U(1) \text{ connection } \equiv \mathcal{P} = \text{Im} \partial = -\frac{i}{2} (\partial - \bar{\partial}). \tag{26}
\]

Now, if \(c_1(\mathcal{L})\) happens to equal the cohomology class of the manifold’s Kähler form (as may be required by the constraints of supersymmetry for example)

\[
c_1(\mathcal{L}) = [K], \tag{27}
\]

\(6\) Pronounced ‘Chen’.

\(7\) For example, the vector fields of ordinary \(U(1)\) Maxwell and \(SU(2)\) Yang–Mills theories can be treated as fiber bundles on spacetime manifolds where the Chern classes reduce to the special case of the so-called Pontrjagin classes. The first and second such classes represent the ordinary field energy density and Poynting vector. Other such classes are particularly useful in the topological classification of magnetic monopoles.

\(8\) In a more pedestrian physics language, a vector bundle on \(\mathcal{M}\) is a vector field living on a space or spacetime manifold \(\mathcal{M}\).
then we call this a Hodge–Kähler manifold [38]. An equivalent definition is that the exponential of the Kähler potential of the manifold is equal to the metric of the line bundle. So, a Kähler manifold with $L$ is Hodge–Kähler if
\[ h(w, \bar{w}) = e^{K(w, \bar{w})}, \] (28)
which enables us to write
\[ c_1(L) = \frac{i}{2\pi} (\partial \bar{\partial} K), \]
\[ \vartheta = (\partial K), \quad \bar{\vartheta} = (\bar{\partial} K), \]
\[ P = -\frac{i}{2}(\partial K - \bar{\partial} K). \] (29)

A $U(1)$ covariant derivative can then be constructed as follows:
\[ \nabla = d + ipP, \] (30)
or in components
\[ \nabla_n = \partial_n + \frac{p}{2} (\partial_n K), \quad \nabla_{\bar{n}} = \partial_{\bar{n}} - \frac{p}{2} (\partial_{\bar{n}} K), \] (31)
where the so-called Kähler weight $p$ is a constant determined by the choice of basis. For example, a quantity $W$ on $M$ is said to have Kähler weights $(p, \bar{p})$ if
\[ \nabla_n W = \left[ \partial_n + \frac{p}{2} (\partial_n K) \right] W, \quad \nabla_{\bar{n}} W = \left[ \partial_{\bar{n}} - \frac{p}{2} (\partial_{\bar{n}} K) \right] W. \] (32)

Furthermore, if $W$ transforms as a tensor on $M$, then in addition to coupling to $L$ via the $U(1)$ connection it also couples to the metric on $M$ via the ordinary Levi-Civita connection (11). The covariant derivative would then contain both. For example, if $W$ is a vector then
\[ D_n W_m = \nabla_n W_m - \Gamma^r_{nm} W_r, \quad D_{\bar{n}} W_{\bar{m}} = \nabla_{\bar{n}} W_{\bar{m}}, \] (33)
and so on for higher rank tensors.

### 2.4. Special Kähler Manifolds: a first look

Strictly speaking, there are two types of Special Kähler manifolds: dubbed ‘local’ and ‘rigid’. The former describes the fields of a locally supersymmetric theory, i.e. a supergravity theory, while the latter pertains to fields in a flat background. Since our interest is supergravity, we will only discuss the local type. Sometimes, this type of manifolds is referred to simply as ‘special manifolds’ and the geometry that describes it is known as ‘SKG’ or just ‘special geometry’.

A special Kähler manifold of the local type is defined as a Hodge–Kähler manifold that admits a completely symmetric and covariantly holomorphic tensor $C_{mn\bar{p}}$ and its antiholomorphic conjugate $C_{m\bar{n}p}$ such that the following restriction on the curvature is true:
\[ R_{mn\bar{pq}} = g_{n\bar{p}}g_{mn} + g_{q\bar{p}}g_{mn} - C_{r\bar{q}a}C_{m\bar{n}p}g_{r\bar{a}}. \] (34)

This is generally referred to in the literature as the SKG constraint. The consequences to (34) can be calculated, and a large literature exists on this. However a second, alternative but completely analogous, definition of special Kähler manifolds is more frequently used in the physics literature. It relies heavily on the symplectic symmetry of special manifolds, a topic of particular interest to us, so we will develop this concept in a bit more detail later.
2.5. Calabi–Yau manifolds

In 1954 Calabi proposed the following conjecture: if $\mathcal{M}$ is a complex manifold with a Kähler metric and vanishing first Chern class, then there exists a unique Ricci flat metric for each Kähler class on $\mathcal{M}$. In 1976, Calabi’s conjecture was proven by Yau, also showing that a Ricci flat metric necessarily has $SU(k)$ holonomy, $k$ being the number of complex dimensions of $\mathcal{M}$. We then define CY manifolds as Kähler manifolds with Ricci flat ($c_1 = 0$) metrics.

From its general properties, it turns out that a large number of different CY manifolds exist. It also turns out that defining them explicitly is a difficult task. Indeed, very few explicit CY metrics have ever been written down, and no non-trivial compact ones are known. However, the properties of CY manifolds make it possible to work with them without explicit knowledge of the metric, as far as string/supergravity theory compactifications are concerned. Yau’s theorem in particular guarantees the existence of a metric. On the other hand, this does impose restrictions on how far one can specify solutions in the reduced theory, since generally the solutions will be dependent on the unknown metric of the subspace, as we will see later. We will restrict ourselves to six real-dimensional CY manifolds admitting $SU(3)$ holonomy, since this is the type of interest to string theory in general and to this work in particular.

The importance of this class of manifolds to physics lies in the fact that they admit covariantly constant spinors. As a consequence, it can be shown [39] that string theory compactifications over CY threefolds preserve some supersymmetry (also see [40] and the references therein). Such compactifications have indeed yielded rich and physically interesting theories in lower dimensions. Specifically, the fields in the compactified theory correspond to the parameters that describe possible deformations of the CY threefold. This parameters’ space factorizes, at least locally, into a product manifold $\mathcal{M}_C \otimes \mathcal{M}_K$, with $\mathcal{M}_C$ being the manifold of the complex structure moduli and $\mathcal{M}_K$ being a complexification of the parameters of the Kähler class. These so-called moduli spaces turn out to belong to the category of special Kähler manifolds. In addition, there exists a symmetry in the structures of $\mathcal{M}_C$ and $\mathcal{M}_K$ which lends support to the so-called mirror symmetry hypothesis of CY threefolds [41].

In terms of homology groups, CY threefolds admit a non-trivial $H^3$ that can be Hodge-decomposed as follows:

$$H^3 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}. \quad (35)$$

The full homology structure is summed up by the so-called Hodge diamond

$$
\begin{array}{cccc}
1 & & & \\
& 0 & h_{1,1} & 0 \\
1 & h_{1,2} & h_{2,1} & 1 \\
& 0 & h_{1,1} & 0 \\
& & 0 & \\
\end{array}
\quad (36)
$$

where the Hodge numbers $h$ are the dimensions of the respective homology/cohomology groups the manifold admits\(^9\), so (36) shows that CY threefolds have a single $(3, 0)$ cohomology form, $h_{3,0} = \dim(H^{3,0}) = 1$, which we will call $\Omega$ (the holomorphic volume form) and an arbitrary number of $(1, 1)$- and $(2, 1)$-forms determined by the corresponding $h$’s.\(^{10}\) The Hodge number $h_{2,1}$ determines the dimensions of $\mathcal{M}_C$, while $h_{1,1}$ determines the dimensions of $\mathcal{M}_K$.

---

\(^9\) The equivalent to the Betti numbers for a real manifold.

\(^{10}\) Whose values depend on the particular choice of CY manifold.
The pair \((\mathcal{M}, K)\), where the Kähler form \(K\) of \(\mathcal{M}\) is defined by (7), can be deformed by either deforming the complex structure of \(\mathcal{M}\) or deforming the Kähler form \(K\) (or both). The space of complex structure moduli \(\mathcal{M}_C\), which we will explore in detail in the following section, geometrically corresponds to what is known as a special Lagrangian manifold. In the context of dimensional reduction, such a manifold is defined as a submanifold \(L\) of the CY space, calibrated with respect to \(\text{Re}\Omega\), i.e. the pullback of \(\text{Re}\Omega\) on \(L\) is less than or equal to the volume of \(L\). A more detailed discussion of either the theory of calibrations or the geometry of special Lagrangian manifolds is found in many sources, for example [35]. In string/SUGRA compactifications, each of the two possible deformations yields a different set of fields in the lower dimensional theory. One can interpret this in the following way: the M-branes of compactifications, each of the two possible deformations yields a different set of fields in the \(p\)th component of the CY parameters space of CY manifolds to be the parameter space of Ricci-flat Kähler metrics. Let \(\omega\) be the unique Kähler form that is completely specified by knowledge of the unique \((3, 0)\)-form \(\Omega\) and the arbitrary number of \((2, 1)\)-forms, which we will call \(\chi\). The way the forms \(\chi\) are linked to the complex structure deformations \(\delta g_{mn}\) and \(\delta g_{nh}\) is defined via \(\Omega\) as follows [42]:

\[
\delta g_{\rho\sigma} = -\frac{1}{\|\Omega\|^2} \Omega_m^\rho \Omega_n^\sigma \chi_{mn} \delta z^i, \quad \|\Omega\|^2 \equiv \frac{1}{3!} \Omega_{mpn} \Omega^{mpn},
\]

with the inverse relation

\[
\chi_{i(mnp)} = -\frac{1}{2} \Omega_{mn} \left( \frac{\delta g_{\rho\sigma}}{\delta z^i} \right), \quad \chi_i = \frac{1}{2} \chi_{i(mnp)} dw^m \wedge dw^n \wedge dw^p,
\]

which also defines the parameters, or moduli, of the complex structure \((z^i : i = 1, \ldots, h_{2,1})\). Each \(\chi_i\) defines a \((2, 1)\) cohomology class. The important observation here is that the moduli can be treated as complex coordinates that define a special Kähler metric \(G_{i\bar{j}}\) on \(\mathcal{M}_C\) as follows:

\[
V_{\text{CY}} G_{i\bar{j}}(\delta z^i)(\delta \bar{z}^j) = \frac{1}{4} \int_\mathcal{M} g^{mn} g^{\rho\bar{\rho}} (\delta g_{mn})(\delta g_{\rho\bar{\rho}}).
\]
where \( V_{\text{CY}} \) is the volume of the CY. In differential geometric notation, this gives

\[
G_{ij} = -\int_{\Omega} \chi_i \wedge \bar{\chi}_j = \delta_i \delta_j K = -\partial_i \partial_j \ln \left( i \int_{\Omega} \Omega \wedge \bar{\Omega} \right)
\]  

(42)

which also defines its Kähler potential \( K \). A particularly useful theorem, attributed to Kodaira, states the following relations between \( \Omega \) and \( \chi \):

\[
(\partial_i \Omega) = k_i \Omega + \chi_i, \quad (\partial_i \bar{\Omega}) = k_i \bar{\Omega} + \bar{\chi}_i,
\]

(43)

where the arbitrary coefficients \((k_i, \bar{k}_i)\) may generally depend on the moduli. A reasonable choice for \( k_i \) is in fact

\[
k_i \propto (\partial_i K).
\]

(44)

The following can then be demonstrated:

\[
\int_M \Omega \wedge \bar{\Omega} = -i e^{-K}
\]

\[
\int_M \Omega \wedge \nabla_i \Omega = \int_M \bar{\Omega} \wedge \nabla_i \bar{\Omega} = 0
\]

\[
\int_M \nabla_i \Omega \wedge \nabla_j \bar{\Omega} = i G_{ij} e^{-K},
\]

(45)

where the \( U(1) \) Kähler connection \( \nabla \) is defined by (31). It is well known that the volume of the CY threefold is given by

\[
\text{Vol}(M) = i \int_M \Omega \wedge \bar{\Omega},
\]

(46)

which means that, using the first equation of (45), the Kähler potential of \( M_{\text{C}} \) is related to the volume of \( M \) simply by

\[
\text{Vol}(M) = e^{-K}.
\]

(47)

The space \( M_{\text{C}} \) of complex structure moduli may also be described in terms of the periods of the holomorphic 3-form \( \Omega \). Let \((A^I, B_J)\), where \( I, J, K = 0, \ldots, h_{2,1} \), be a canonical homology basis for \( H^3 \) such that

\[
A^I \cap B_J = \delta^I_J,
\]

\[
B_I \cap A^J = -\delta^I_J,
\]

\[
A^I \cap A^J = B_I \cap B_J = 0,
\]

(48)

and let \((\alpha_I, \beta^J)\) be the dual cohomology basis forms such that

\[
\int_M \alpha_I \wedge \beta^J = \int_{A^I} \alpha_I = \delta^J_I,
\]

\[
\int_M \beta^I \wedge \alpha_J = \int_{B_I} \beta^I = -\delta^I_J,
\]

\[
\int_M \alpha_I \wedge \alpha_J = \int_M \beta^I \wedge \beta^J = 0.
\]

(49)

The periods of \( \Omega \) are then defined by

\[
Z^I = \int_{A^I} \Omega, \quad F_I = \int_{B_I} \Omega.
\]

(50)

---

11 Generally for CY \( k \)-folds where \( k \neq 3 \), expression (46) would have different normalization coefficients.
Now, it can be shown that, locally in the moduli space, the complex structure is entirely
determined by $Z^I$, so one can write $F_I = F_I(Z^J)$. Also, a rescaling $Z^I \rightarrow \lambda Z^I$, where $\lambda$ is
a non-vanishing constant, corresponds to a rescaling of $\Omega$ that does not change the complex
structure, which implies that the $Z$'s are projective coordinates on $\mathcal{M}_C$. In fact, we can choose
a set of independent ‘special coordinates’ $z$ as follows:

$$z^I = \frac{Z^I}{Z^0},$$

which are identified with the complex structure moduli $z_i$. So, given the cohomology basis
defined above, one can invert (50) as follows\(^\text{12}\):

$$\Omega = Z^I \alpha_I - F_I \beta^I,$$

and the Kähler potential of $\mathcal{M}_C$ becomes

$$K = -\text{ln}[i(Z^I F_I - Z^I F_I)].$$

Some of the ingredients of this structure require the knowledge of the Hodge duality
relations (with respect to $\mathcal{M}$) of the forms $(\alpha, \beta)$ \[^{43}\]:

$$\star \alpha_I = (\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) \beta^J - \gamma^{KL} \theta^L \alpha_J,$$

$$\star \beta^I = \gamma^{IK} \theta_{KJ} \beta^J - \gamma^{IJ} \alpha_J,$$

where $\theta_{IJ}$ and $\gamma^{IJ}$ are the real matrices defined by

$$N_{IJ} = \bar{\Omega}^{IJK} \Omega_{JK} Z^{L} Z^{L} Z^{P} Z^{P} Z^{Q} Z^{Q} = \theta_{IJ} - i \gamma_{IJ},$$

where $F_{IJ} = \partial_I F_{J}$ (the derivative is with respect to $Z^I$), $N_{IJ} = \text{Im}(F_{IJ})$, $\gamma^{IJ} \gamma_{JK} = \delta^I_J$ and
$N_{IJ}$ is known as the period matrix. It is also possible to demonstrate the useful relation

$$\bar{Z}^I N_{IJ} Z^J = -\frac{1}{2} e^{-K}.$$\(^{56}\)

One final remark is that it is sometimes possible to further define $F_I$ as the derivative with
respect to $Z^I$ of a scalar function $F$ known as the prepotential, i.e.

$$F_I = \partial_I F \rightarrow F_{IJ} = \partial_I \partial_J F.$$\(^{57}\)

This, however, is avoided by most authors in the more recent literature since it is not
always possible to find such a function. It can also be explicitly shown \[^{44}\] that $F$ is not in
general invariant under symplectic transformations. In addition, some physically interesting
cases arise precisely when a prepotential does not exist. We will then follow convention and
make no further mention of the prepotential.

In conclusion, we note that the crucial observation here is that the curvature of $\mathcal{M}_C$
calculated via the metric $G_{IJ}$ satisfies the special Kähler constraint (34). We will develop this
further using the second definition of special geometry in the following sections.

2.6.1. A simple example. To get a more intuitive, as well as visual, understanding of the
subject of moduli spaces (by itself a vast topic), we consider the simplest example of a CY
manifold: the ordinary torus $T^2$ \[^{45}\]. In this case, there are two real periodic degrees of
freedom $x$ and $y$, such that

$$x = x + R_1, \quad y = y + R_2,$$\(^{58}\)

\(^{12}\)The significance of the minus sign in (52) will become apparent when we discuss the symplectic structure behind
these expressions.
corresponding to the $H^1$ homology cycles $A$ and $B$ respectively. The cohomology basis forms would then be

$$\alpha = \frac{dx}{R_1}, \quad \beta = -\frac{dy}{R_2},$$

such that the volume form is the holomorphic $(1, 0)$-form:

$$\Omega_T = dx + i dy = R_1 \alpha - i R_2 \beta = Z \alpha - F \beta$$

$$Z = \int_A \Omega_T = R_1, \quad F = \int_B \Omega_T = i R_2.$$ (59)

The metric, the Kähler form and the ‘volume’$^{13}$ of the torus are then respectively

$$ds^2 = \|\Omega_T\|^2$$

$$\mathcal{K} = \Omega_T \wedge \bar{\Omega}_T$$

$$\text{Vol} = \int_T \Omega_T \wedge \bar{\Omega}_T,$$ (61, 62, 63)

and the Kähler potential of $\mathcal{M}_C$ is

$$\mathcal{K} = -\ln(\text{Vol}).$$ (64)

### 2.7. A note on quaternionic manifolds

The subject of quaternionic manifolds (also known as quaternionic Kähler manifolds) is part of a larger class of geometry referred to as hyper-Kähler geometry, or simply hyper-geometry, since they are manifolds that allow for the existence of more than one Kähler form. This, in fact, is where the hypermultiplet fields derive their name from. Just as there are two types of SKG, there are also two types of hyper-geometry: the rigid and the local. The quaternionic geometry described briefly below is the local case.

Simply put, a manifold is hyper-Kähler if it admits an $SU(2)$ bundle that plays the same role here as the $U(1)$ bundle in special Kähler manifolds. The manifold is called quaternionic if the curvature of this bundle is proportional to the manifold’s Kähler form. The metric of a quaternionic manifold can be written in the form

$$ds^2 = h_{uv}(q) dq^u dq^v,$$ (65)

where $(u, v) = 1, \ldots, 4n$. Such a manifold admits three complex structures $J^x$ that satisfy the quaternionic algebra$^{14}$:

$$J^x J^y = -\delta^{xy} \mathbb{I} + \epsilon^{xyz} J^z, \quad x, y, z = 1, 2, 3.$$ (66)

It follows that we can construct three 2-forms known as the hyper-Kähler forms

$$\mathcal{K}^x = K^x_{uv} dq^u \wedge dq^v,$$ (67)

$$K^x_{uv} = h_{uw}(J^y)^w_v,$$ (68)

generalizing the concept of a Kähler form. The hyper-Kähler forms follow an $SU(2)$ Lie-algebra, in the same way the ordinary Kähler form follows a $U(1)$ Lie algebra.

$^{13}$ In this case ‘volume’ means the surface area of $T^2$.

$^{14}$ As defined in the appendix, a barred Levi-Civita symbol has the usual 0, 1, $-1$ components.
3. Special geometry and symplectic covariance

In this section, we present the second and most common definition of SKG. The language we will use relies heavily on the symplectic structure of special manifolds.

3.1. Principia symplectica

Before delving into special geometry proper, we define the language of symplectic vector spaces and set the notations and conventions that go with it [31]. In group theory, the symplectic group $\text{Sp}(2m, F) \subset \text{GL}(2m, F)$ is the isometry group of a non-degenerate alternating bilinear form on a vector space of rank $2m$ over $F$, where this last is usually either $\mathbb{R}$ or $\mathbb{C}$, although other generalizations are possible. For our purposes, we take $F = \mathbb{R}$ and $m = h_{2,1} + 1$. In other words, $\text{Sp}(2h_{2,1} + 2, \mathbb{R})$ is the group of the real bilinear matrices

$$\Lambda = \begin{bmatrix} \Lambda_{ij} & \Lambda_{ij}^J \\ 21 & 22 \end{bmatrix} \in \text{Sp}(2h_{2,1} + 2, \mathbb{R}), \quad \text{where} \quad I, J = 0, \ldots, h_{2,1} + 1$$

(69)

that leave the totally antisymmetric symplectic matrix

$$S = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \begin{bmatrix} 0 & \delta_I^J \\ -\delta_I^J & 0 \end{bmatrix}$$

(70)

invariant, i.e.

$$\Lambda^T S \Lambda = S, \quad \Lambda^T S^T \Lambda = S^T, \quad \Lambda^{-1} S^{-1} \Lambda^T S = \Lambda^T S^T \Lambda = S^T, \quad \Lambda^{-1} S^{-1} \Lambda^T S = S^{-1} S = \mathbb{1}$$

(71)

or equivalently

$$\begin{aligned}
[11]_{ik} \Lambda_{jk} - [21]_{ik} \Lambda_{jk}^{IK} & = 0, \\
[12]_{ik} \Lambda_{jk}^{IK} - [22]_{ik} \Lambda_{jk}^{IK} & = 0, \\
[11]_{ik} \Lambda_{jk}^{JK} - [21]_{ik} \Lambda_{jk}^{JK} & = \delta_I^J,
\end{aligned}$$

(72)

the last of which implies $|A| = \mathbb{1}$. The inverse of $\Lambda$ is found to be

$$\Lambda^{-1} = S^{-1} \Lambda^T S = \begin{bmatrix} 22 \Lambda_{ij}^J & -12 \Lambda_{ij}^J \\ -21 \Lambda_{ij}^J & 11 \Lambda_{ij}^J \end{bmatrix}$$

(73)

such that, using (71), $\Lambda^{-1} A = S^{-1} \Lambda^T S A = S^{-1} S = \mathbb{1}$ as needed. Also note that $S^{-1} = S^T = -S$. We adopt the language that there exists a vector space $Sp$ such that the symplectic matrix $S$ acts as a metric on that space. Symplectic vectors in $Sp$ can be written in a ‘ket’ notation as follows:

$$|A\rangle = \begin{bmatrix} a_I \\ \tilde{a}_I \end{bmatrix}, \quad |B\rangle = \begin{bmatrix} b_I \\ \tilde{b}_I \end{bmatrix}$$

(74)

On the other hand, ‘bra’ vectors defining a space dual to $Sp$ can be found by contraction with the metric in the usual way, yielding

$$\langle A | = (SA)^T = A^T S^T = (a_I \tilde{a}_I) \begin{bmatrix} 0 & -\delta_I^J \\ \delta_I^J & 0 \end{bmatrix} = (\tilde{a}_I - a_I)$$

(75)

such that the inner product on $Sp$ is the ‘bra(c)ket’:

$$\langle A | B \rangle = A^T S^T B = (\tilde{a}_I - a_I) \begin{bmatrix} b_I \\ \tilde{b}_I \end{bmatrix} = \tilde{a}_I b_I - a_I \tilde{b}_I = -\langle B | A \rangle.$$ 

(76)

In this language, the matrix $\Lambda$ can simply be thought of as a rotation operator in $Sp$. So a rotated vector is

$$|A'\rangle = \pm |\Lambda A\rangle = \pm \Lambda A.$$ 

(77)
This is easily shown to preserve the inner product \((76)\)
\[\langle A'|B' \rangle = (\pm)^2 A^T \Lambda^T S^T \Lambda B = A^T S^T B = \langle A|B \rangle, \tag{78} \]
where \((71)\) was used. In fact, one can define \((71)\) based on the requirement that the inner product is preserved. We also define the symplectic invariant
\[\langle A|\Lambda|B \rangle \equiv \langle A|\Lambda B \rangle = A^T S^T \Lambda B \]
\[= (AA^{-1}|B) = -\langle B\Lambda|A \rangle. \tag{79} \]

The matrix \(\Lambda\) we will be using in the remainder of the review has the property
\[\frac{22}{\Lambda^1 I^J} = -\frac{11}{\Lambda^1 I^J} \rightarrow \Lambda^{-1} = -\Lambda, \tag{80} \]
which, via \((79)\), leads to
\[\langle A|\Lambda|B \rangle = \langle A|\Lambda B \rangle = -\langle A|B \rangle. \tag{81} \]

The choice \((80)\) is not only the natural one. A consequence of it is that \(\Lambda\) is not symmetric, but \(S\Lambda\) is. On the other hand, an equivalent choice would be a symmetric \(\Lambda\), in which case it would be \(S\Lambda\) that satisfies \((80)\).

Now consider the algebraic product of the two symplectic scalars
\[\langle A|B \rangle (C|D) = (A^T S^T B) (C^T S^T D). \tag{82} \]

The ordinary outer product of matrices is defined by
\[\mathbf{B} \otimes \mathbf{C}^T = \left[\begin{array}{ccc} b_1^I c_J & b_1^I c_J & b_1^I c_J \\ \bar{b}_1^I c_J & \bar{b}_1^I c_J & \bar{b}_1^I c_J \end{array}\right], \tag{83} \]
which allows us to rewrite \((82)\):
\[\langle A|B \rangle (C|D) = A^T S^T (\mathbf{B} \otimes \mathbf{C}^T \mathbf{S}) \mathbf{D} = \langle A|\mathbf{B} \otimes \mathbf{C}^T \mathbf{S}^T|D \rangle. \tag{84} \]

Comparing the terms of \((84)\), we see that one way a symplectic outer product can be defined is
\[|B\rangle\langle C| = \mathbf{B} \otimes \mathbf{C}^T \mathbf{S}^T = \left[\begin{array}{cc} b_1^I c_J & -b_1^I c_J \\ \bar{b}_1^I c_J & \bar{b}_1^I c_J \end{array}\right]. \tag{85} \]

Note that the order of vectors in \((85)\) is important, since generally
\[|B\rangle\langle C| = |S|C\rangle \langle B|S|^T. \tag{86} \]

However, if the outer product \(|B\rangle\langle C|\) satisfies the property \((80)\), i.e.
\[|B\rangle\langle C|^{-1} = -|B\rangle\langle C|, \tag{87} \]
then it is invariant under the interchange \(B \leftrightarrow C\):
\[|B\rangle\langle C| = |C\rangle < B \rangle. \tag{88} \]

One can now proceed to develop \(Sp\) vector identities in analogy with ordinary vector spaces. For example, it is useful to note that
\[|A\rangle \langle B|C = \langle B|C\rangle |A\rangle \tag{89} \]
leads to the ‘BAC-CAB’ rule
\[|A\rangle \langle B|C = \langle B|A\rangle |C\rangle - \langle C|A\rangle |B\rangle. \tag{90} \]
3.2. The space of complex structure moduli as a special Kähler manifold

As promised, we discuss the second definition of special Kähler manifolds. Furthermore, since special geometry turns out to be the same geometry that describes the space $M_C$ of complex structure moduli of a CY manifold, we also make the connection and unify the notation.

The definition goes like this: let $L \to M$ denote the complex line bundle whose first Chern class equals the Kähler form $K$ of the Hodge–Kähler manifold $M$. Now consider an additional holomorphic flat vector bundle of rank $(2h^2 + 1, 1 + 2)$ with the structural group $Sp(2h^2 + 1, 1 + 2, \mathbb{R})$ on $M$: $S\mathcal{V} \to M$. Construct a tensor bundle $H = S\mathcal{V} \otimes L$. This then is a special Kähler manifold if for some holomorphic section $|\Psi_1\rangle$ of such a bundle (which is a symplectic vector in the sense of the last section) the Kähler 2-form is given by

$$K = -\frac{i}{2\pi} \partial \bar{\partial} \ln(i\langle \Psi | \bar{\Psi} \rangle), \quad (91)$$

or in terms of the Kähler potential on $M_C$:

$$K = -\ln(i\langle \Psi | \bar{\Psi} \rangle) \to \langle \bar{\Psi} | \Psi \rangle = i e^{-K}. \quad (92)$$

Note that the metric on the bundle is defined via a relation analogous to (28). Now, this exactly describes the space of complex structure moduli $M_C$ if one chooses $|\Psi_1\rangle = (Z_I F_I)$, $I = 0, \ldots, h_{2,1} + 1$, (93)

which, via (92), leads directly to equation (53) defining the Kähler potential of $M_C$. We then identify $M_C$ as a special Kähler manifold with the metric $G_{ij}$. Henceforth, we continue our discussion of $M_C$ using the language of SKG and $Sp(2h^2 + 1, \mathbb{R})$ covariance.

Certain constraints on the $Sp$ vector $|\Psi\rangle$ are imposed as part of the definition, or, from the point of view of $M_C$, can also follow as consequences of equations (45); these are

$$\langle \Psi | \partial_i \Psi \rangle = 0 \quad \langle \nabla_i \Psi | \nabla_j \Psi \rangle = 0. \quad (94)$$

Now, it can be easily demonstrated that the matrix

$$\Lambda = \begin{bmatrix} \gamma^{IK} \theta_{KJ} & \gamma^{IJ} \\ (\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) & -\gamma^{IK} \theta_{KI} \end{bmatrix} \quad (95)$$

satisfies the symplectic condition (71), where $\gamma$ and $\theta$ are defined by (55). Its inverse is then

$$\Lambda^{-1} = -\Lambda = \begin{bmatrix} -\gamma^{IK} \theta_{KI} & \gamma^{IJ} \\ (\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) & -\gamma^{IK} \theta_{KI} \end{bmatrix}. \quad (96)$$

The symplectic structure manifest here is a consequence of the topology of the CY manifold $M$, the origins of which can be traced to the completeness relations (49); clearly

$$\int_M \begin{bmatrix} \alpha_I \wedge \alpha_J & \alpha_I \wedge \beta^J \\ \beta^I \wedge \alpha_J & \beta^I \wedge \beta^J \end{bmatrix} = \begin{bmatrix} 0 & \delta^J_I \\ -\delta^J_I & 0 \end{bmatrix} = S. \quad (97)$$

In fact, if one defines the symplectic vector

$$|\Theta \rangle = \begin{bmatrix} \beta^I \cr \alpha_I \end{bmatrix}, \quad (98)$$

then it is easy to check that

$$\int_M \Theta \wedge \Theta^T = S^T \to \int_M |\Theta \rangle \wedge |\Theta \rangle = -I. \quad (99)$$
and that the Hodge duality (54) is equivalent to a rotation in symplectic space:

\[ * \Theta = | \Lambda \Theta | . \]  

(100)

Also note that (52) can similarly be rewritten as

\[ \Omega = 0 | \Theta | \Psi | . \]  

(101)

Next, we construct a basis in \( Sp \). Properly normalized, the periods vector (93) provides such a basis

\[ | V \rangle = e^{\frac{i}{2} | \Psi |} = \left( \frac{L^I}{M_I} \right) , \]  

(102)

such that, using (92)

\[ \langle \bar{V} | V \rangle = (L^I \bar{M}_I - \bar{L}^I M_I) = i. \]  

(103)

From the point of view of physics, equation (103) is the condition required to obtain an \( \mathcal{N} = 2 \) SUGRA action in the Einstein frame. If it were not true, then the Einstein–Hilbert term would have the form

\[ i \langle V | \bar{V} \rangle \sqrt{|g|} R. \]  

(104)

Since \( | V \rangle \) is a scalar in the \((i, j, k)\) indices, it couples only to the \( U(1) \) bundle via the Kähler covariant derivative (31) as follows:

\[ | \nabla_i V \rangle = \left[ \partial_i + \frac{1}{2} (\partial_i K) \right] V , \quad | \nabla_i \bar{V} \rangle = \left[ \partial_i - \frac{1}{2} (\partial_i K) \right] \bar{V} \]  

\[ | \nabla_i \bar{V} \rangle = \left[ \partial_i + \frac{1}{2} (\partial_i K) \right] \bar{V} , \quad | \nabla_i V \rangle = \left[ \partial_i - \frac{1}{2} (\partial_i K) \right] V . \]  

(105)

In other words, the Kähler weights of \( | V \rangle \) are \((1, -1)\). Using this, one can construct the orthogonal \( Sp \) vectors

\[ | U_i \rangle = | \nabla_i V \rangle = \left( \frac{\nabla_i L^I}{\nabla_i M_I} \right) = \left( \frac{f_i^I}{h_{ii}} \right) . \]  

(106)

\[ | U_j \rangle = | \nabla_j \bar{V} \rangle = \left( \frac{\nabla_j \bar{L}^I}{\nabla_j \bar{M}_I} \right) = \left( \frac{\bar{f}_j^I}{\bar{h}_{jj}} \right) , \]  

(107)

with the same Kähler weights as \( | V \rangle \), i.e.

\[ | \nabla_i U_j \rangle = \left[ \partial_i + \frac{1}{2} (\partial_i K) \right] U_j , \quad | \nabla_i U_j \rangle = \left[ \partial_i - \frac{1}{2} (\partial_i K) \right] U_j \]  

\[ | \nabla_j U_j \rangle = \left[ \partial_j + \frac{1}{2} (\partial_j K) \right] U_j , \quad | \nabla_j U_j \rangle = \left[ \partial_j - \frac{1}{2} (\partial_j K) \right] \bar{U}_j . \]  

(108)

Note that \( | U_i \rangle \) also couples to the metric \( G_{ij} \) via the Levi-Civita connection. So its full covariant derivative is defined by (33)

\[ | \nabla_i U_j \rangle = \nabla_i U_j - \Gamma_{ij}^k | U_k \rangle \]  

\[ | \nabla_i U_j \rangle = | \nabla_i U_j \rangle = \Gamma_{ij}^k | U_k \rangle \]  

\[ | \nabla_i U_j \rangle = | \nabla_i U_j \rangle = | \nabla_i U_j \rangle - \Gamma_{ij}^k | U_k \rangle . \]  

(109)

It can be demonstrated that these quantities satisfy the properties

\[ | \nabla_i \bar{V} \rangle = | \nabla_i V \rangle = 0 , \]  

(110)

\[ \langle U_i | U_j \rangle = \langle U_i | U_j \rangle = 0 , \]  

(111)

\[ \langle \bar{V} | U_i \rangle = \langle V | U_i \rangle = \langle V | U_i \rangle = \langle \bar{V} | U_i \rangle = 0 , \]  

(112)

\[ | \nabla_j U_i \rangle = G_{ij} | V \rangle , \quad | \nabla U_j \rangle = G_{ij} | \bar{V} \rangle , \]  

\[ G_{ij} = (\partial_i \partial_j K) = -i \langle U_i | U_j \rangle . \]  

(113)

(114)
Note that (110) implies
\[ |\psi\rangle = |\psi\rangle = 0. \] (115)

This definition of special Kähler manifolds is directly related to the first definition in section 2.4 via the identification
\[ |D_j U_j\rangle = G^{jk} C_{ijk} |U_i\rangle, \] (116)
which leads to
\[ C_{ijk} = -i |D_j U_j\rangle |U_k\rangle. \] (117)

The following identities may now be derived:
\[
\begin{align*}
\mathcal{N}_{ij} L^i &= M_i, & \tilde{\mathcal{N}}_{ij} f^i_j &= h_{ij}, \\
\tilde{\mathcal{N}}_{ij} \tilde{L}^i &= \tilde{M}_i, & \mathcal{N}_{ij} f^i_j &= h_{ij}, \\
\gamma_{ij} L^i L^j &= \frac{1}{2}, & G_{ij} &= 2\gamma_{ij} f^i_j f^j_i, \\
\end{align*}
\] (118)
as well as the very useful
\[
\gamma^{ij} = 2(L^i L^j + G^{ij} f^i_j f^j_i).
\]
\[
\begin{align*}
(\gamma_{ij} + \gamma^{kl} \theta_{lkl}) &= 2(M_i \tilde{M}_j + G^{ij} h_{ij} h_{ij}), \\
\gamma^{ij} \theta_{KJ} &= 2(\tilde{L}^i M_j + G^{ij} f^i_j h_{ij}) + i\delta^i_j \\
&= 2(L^i \tilde{M}_j + G^{ij} h_{ij} f^i_j) - i\delta^i_j \\
&= (L^i \tilde{M}_j + \tilde{L}^i M_j) + G^{ij} (f^i_j h_{ij} + h_{ij} f^i_j). \quad (120)
\end{align*}
\]

Note that the last formula in (119) in particular implies that the imaginary part of the period matrix \( \text{Im} \mathcal{N}_{ij} = -\gamma_{ij} \) acts as a metric in the \( (I, J, K) \) indices, and that \( f^i_j \) are the vielbeins relating it to the special Kähler metric \( G_{ij} \) similar to (17). In other words, these relations provide a connection between the SKG structure and the Sp space. We will now exploit this. Equations (120) lead to a second form for the symplectic matrix (95)
\[
\Lambda = \begin{bmatrix}
L^i M_j + L^i M_j & -2(L^i L^j + G^{ij} f^i_j f^j_i) \\
+ G^{ij} (f^i_j h_{ij} + h_{ij} f^i_j) & -2(L^i \tilde{M}_j + \tilde{L}^i M_j) \\
2(M_i \tilde{M}_j + G^{ij} h_{ij} h_{ij}) & -G^{ij} (f^i_j h_{ij} + h_{ij} f^i_j)
\end{bmatrix} \] (121)

with inverse
\[
\Lambda^{-1} = -\Lambda = \begin{bmatrix}
-(L^i \tilde{M}_j + \tilde{L}^i M_j) & 2(L^i \tilde{L}^j + G^{ij} f^i_j f^j_i) \\
-G^{ij} (f^i_j h_{ij} + h_{ij} f^i_j) & (L^i \tilde{M}_j + \tilde{L}^i M_j) \\
-2(M_i \tilde{M}_j + G^{ij} h_{ij} h_{ij}) & +G^{ij} (f^i_j h_{ij} + h_{ij} f^i_j)
\end{bmatrix}. \] (122)

By inspection, one can write down the following important result:
\[
\Lambda = |V\rangle \langle \tilde{V}| + |\tilde{V}\rangle \langle V| + G^{ij} |U_i\rangle \langle U_j| + G^{ij} |U_j\rangle \langle U_i| \\
\Lambda^{-1} = -|V\rangle \langle \tilde{V}| - |\tilde{V}\rangle \langle V| - G^{ij} |U_i\rangle \langle U_j| - G^{ij} |U_j\rangle \langle U_i|. \] (123)

In other words, the rotation matrix in Sp is expressible as the outer product of the basis vectors. Note that since \( \Lambda \) satisfies the property (80), it is invariant under the interchange
$V \leftrightarrow \bar{V}$ and/or $U_i \leftrightarrow U_j$. This makes manifest the fact that $\Lambda$ is a real matrix, $\Lambda = \bar{\Lambda}$. Now, applying $\Lambda^{-1} \Lambda = 1$, we end up with the condition

$$|\bar{V}\rangle\langle V| + G^{ij} U_i \langle U_j| = |V\rangle\langle \bar{V}| + G^{ij} \langle U_i| U_j| - i,$$

which can be checked explicitly using (120). This can be used to write $\Lambda$ in an even simpler form:

$$\Lambda = 2|V\rangle\langle \bar{V}| + 2G^{ij}|U_i\rangle \langle U_j|,$$

$$\Lambda^{-1} = -2|\bar{V}\rangle\langle V| + 2G^{ij}\langle U_i| U_j| - i.$$  

It can further be shown that

$$D_i \Lambda = \nabla_i \Lambda = \partial_i \Lambda = 2|U_i\rangle \langle \bar{V}| + 2|\bar{V}\rangle \langle U_i| + 2G^{ij} G^{kp} C_{ijk} \langle U_p| (U_k|.$$  

Finally, we note that our discussion here is based on a definition of special manifolds that is not the only one in existence. See, for instance, [44] for details. Explicit examples of special manifolds in various dimensions are given in, for example, [46].

4. $D = 5 \mathcal{N} = 2$ supergravity with hypermultiplets

In this section, we review the derivation of ungauged $D = 5 \mathcal{N} = 2$ SUGRA via the dimensional reduction of $D = 11$ SUGRA over a CY manifold $M$. Specifically, we look at the case where only the complex structure of $M$ is deformed. For the sake of compactness and clarity, we emphasize the use of differential forms on the spacetime manifold, following the definitions in the appendix.

4.1. Dimensional reduction

The unique supersymmetric gravity theory in 11 dimensions has the following bosonic action:

$$S_{11} = \int_{11} \left( \mathcal{R} \ast 1 - \frac{1}{2} \mathcal{F} \wedge \ast \mathcal{F} - \frac{1}{6} \mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F} \right),$$

where $\mathcal{R}$ is the $D = 11$ Ricci scalar, $\mathcal{A}$ is the 3-form gauge potential and $\mathcal{F} = d\mathcal{A}$. The dimensional reduction is traditionally done using the metric
d

$$dx^2 = G_{MN} dx^M dx^N = e^{\tau g_{\mu\nu}} dx^\mu e^{-\tau} k_{ab} dx^a dx^b$$

$$M, N = 0, \ldots, 10 \quad \mu, \nu = 0, \ldots, 4 \quad a, b = 1, \ldots, 6,$$  

where $g_{\mu\nu}$ is the target five-dimensional metric, $k_{ab}$ is a metric on the six-dimensional compact subspace $M$, the dilaton $\tau$ is a function in $x^\mu$ only and the warp factors are chosen to give the conventional coefficients in five dimensions, guaranteeing that the gravitational term in the action will have the standard Einstein–Hilbert form. We choose a complex structure on $M$ such that

$$k_{ab} dx^a \wedge dx^b = k_{mn} du^m \wedge du^n + k_{nh} du^n \wedge du^h + 2k_{m\bar{n}} \wedge du^m \wedge du^\bar{n},$$

where the holomorphic and antiholomorphic indices $(m, n; m, \bar{n})$ are three dimensional on $M$. The Hermiticity condition (5) demands that $k_{mn} = k_{\bar{m}\bar{n}} = 0$, while the Ricci tensor for $M$ is set to zero as dictated by Yau’s theorem. Furthermore, we consider the case where only the complex structure is deformed, which requires $\delta k_{mn} = 0$ and $(\delta k_{mn}, \delta k_{\bar{m}\bar{n}}) \neq 0$, as discussed earlier.
Now, the flux compactification of the gauge field is done by expanding $A$ into two forms: one is the five-dimensional gauge field $A$ while the other contains the components of $A$ on $\mathcal{M}$ written in terms of the cohomology forms $(\alpha_I, \beta_I)$ as follows:

$$A = A + \sqrt{2}(\xi^I \alpha_I + \bar{\xi}^I \beta_I) \quad (I = 0, \ldots, h_{2,1}),$$

$$F = dA = F + F_\mu \, dx^\mu,$$

$$F_\mu \, dx^\mu = \sqrt{2}[\alpha_I + (\partial_\mu \xi^I) \beta_I] \wedge dx^\mu.$$

(130)

Because of the 11-dimensional Chern–Simons term, the coefficients $\xi^I$ and $\bar{\xi}^I$ appear as pseudo-scalar axion fields in the lower dimensional theory. We also note that $A$ in five dimensions is dual to a scalar field which we will call $a$ (known as the universal axion). The set $(a, \sigma, \xi^0, \bar{\xi}^0)$ is known as the universal hypermultiplet\(^1\). The rest of the hypermultiplets are $(z_i, \bar{z}_i, \xi_i, \bar{\xi}_i; i = 1, \ldots, h_{2,1})$, where we will recognize the $z_i$'s as the CY's complex structure moduli. Note that the total number of scalar fields in the hypermultiplets sector is $4(h_{2,1} + 1)$ (each hypermultiplet has four real scalar fields) which comprises a quaternionic manifold as noted earlier. Also included in the hypermultiplets are the fermionic partners of the hypermultiplet scalars known as the hyperini (singular: hyperino). However, in what follows, we will only discuss the bosonic part of the action. The hyperini, as well as the gravitini, will make their appearance in the SUSY variation equations later. The bits and pieces one needs for the dimensional reduction are as follows.

1. The metric components

$$G_{\mu\nu} = e^{\tilde{\sigma}} g_{\mu\nu}, \quad G^{\mu\nu} = e^{-\tilde{\sigma}} g^{\mu\nu},$$
$$G_{ab} = e^{-\tilde{\sigma}} k_{ab}, \quad G^{ab} = e^{\tilde{\sigma}} k^{ab},$$
$$G = \det G_{MN} = e^{\tilde{\sigma}} g,$$
$$g = \det g_{\mu\nu}, \quad k = \det k_{ab}.\quad (131)$$

2. The Christoffel symbols

$$\Gamma^\mu_{\nu\rho} = \tilde{\Gamma}^\mu_{\nu\rho}[g] + \frac{1}{3} \left[ \delta^\mu_{\nu} (\partial_\rho \sigma) + \delta^\mu_{\rho} (\partial_\nu \sigma) - \delta^\mu_{\nu} \delta_{\rho\sigma} (\partial_\kappa \sigma) \right],$$
$$\Gamma^a_{\mu b} = \frac{1}{6} e^{-\sigma} k_{ab} (\partial^a \sigma) - \frac{1}{3} e^{-\sigma} (\partial^a k_{ab}),$$
$$\Gamma^a_{b c} = \frac{1}{6} k^{ac} (\partial_b k_{cb}) - \frac{1}{3} \delta^a_b (\partial_c \sigma),$$
$$\Gamma^a_{b c} = \tilde{\Gamma}^a_{b c}[k],\quad (132)$$

where the (`') and the (``') refer to the purely five- and six-dimensional components respectively.

Now, calculating the 11-dimensional Ricci scalar based on this gives

$$\sqrt{|G|} R = \sqrt{|g|} \left[ R[g] - \frac{1}{2} (\partial_\mu \sigma)(\partial^\mu \sigma) - \frac{1}{2} k^{mn} k^{rp} (\partial_m k_{nr})(\partial_p k_{nb}) \right],$$

(133)

where we have used $\tilde{R}_{ab} = 0$ since $\mathcal{M}$ is Ricci-flat, as well as dropped all total derivatives and terms containing $k_{mn}, k_{ab}$ and $\delta_{mn}$. Using (41) one gets

$$\int_1^{11} R \star 1 = \int d^5 x \sqrt{|g|} \left[ R - \frac{1}{2} (\partial_\mu \sigma)(\partial^\mu \sigma) - G_{ij}(\partial_\mu z^i)(\partial^\mu z^j) \right].\quad (134)$$

\(^1\) So called because it appears in all CY compactifications, irrespective of the detailed structure of the CY manifold. The dilaton $\sigma$ is proportional to the natural logarithm of the volume of $\mathcal{M}$.
where we have normalized the volume of the compact space to $V_C = 1$. Next, the Maxwell term is

$$-\frac{1}{2} F \wedge \star F \rightarrow -\frac{1}{2} \frac{1}{4!} F_{\mu \nu \rho \sigma} F^{\mu \nu \rho \sigma} = -\frac{1}{48} \left( e^{-2\sigma} F_{\mu \nu \rho \sigma} F^{\mu \nu \rho \sigma} + e^{2\sigma} F_{\mu} F^{\mu} \right). \quad (135)$$

Substituting, we get

$$-\frac{1}{2} \int \mathcal{F} \wedge \star \mathcal{F} = -\frac{1}{48} \int d^5 x \sqrt{|g|} e^{-2\sigma} F_{\mu \nu \rho \sigma} F^{\mu \nu \rho \sigma}$$

$$- \int d^5 x \sqrt{|g|} \left[ (\partial_\mu \xi^I)(\partial^\mu \xi^J) \int_M \alpha_I \wedge \star \alpha_J 
+ (\partial_\mu \xi^I)(\partial^\mu \bar{\xi}^J) \int_M \beta^I \wedge \star \beta^J \right] \right], \quad (137)$$

where the Hodge star on the right-hand side is with respect to $M$. Now, using (49) and (54) we end up with

$$-\frac{1}{2} \int \mathcal{F} \wedge \star \mathcal{F} = -\int d^5 x \sqrt{|g|} \left\{ \frac{1}{48} e^{-2\sigma} F_{\mu \nu \rho \sigma} F^{\mu \nu \rho \sigma} 
+ e^{2\sigma} \left[ (\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) (\partial_\mu \xi^I)(\partial^\mu \xi^J) + (\partial_\mu \xi^I)(\partial^\mu \bar{\xi}^J) \right] \right\}. \quad (138)$$

Finally, the Chern–Simons term gives

$$-\frac{1}{6} \int A \wedge \mathcal{F} \wedge \mathcal{F} = -\frac{1}{6} \int d^5 x \left[ \xi^I F \wedge d\bar{\xi}^I \int_M \alpha_I \wedge \beta^I - \bar{\xi}_I F \wedge d\xi^I \int_M \alpha_J \wedge \beta^I \right]$$

$$= -\frac{1}{2} \int d^5 x \left[ (\xi^I d\bar{\xi}_I - \bar{\xi}_I d\xi^I) \right]. \quad (139)$$

To sum up, the ungauged five-dimensional $N = 2$ supergravity bosonic action with vanishing vector multiplets is

$$S_5 = \int \left\{ \mathcal{R} \wedge \mathcal{F} - \frac{1}{2} d\sigma \wedge d\sigma - G_{ij} dz^i \wedge \star dz^j - F \wedge (\xi^I d\bar{\xi}_I - \bar{\xi}_I d\xi^I) 
- \frac{1}{2} e^{-2\sigma} F \wedge \star F - e^{2\sigma} \left[ (\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) d\xi^I \wedge \star d\xi^J \right] \right\}.$$
$$d^4[e^\sigma \gamma^I K \theta_J K \, d\xi^J + e^\sigma \gamma^{IJ} \, d\tilde{\xi}_J + \zeta^I \star F] = 0$$
$$d^4[e^\sigma (\gamma_{IJ} + \gamma^{KL} \theta_I K \theta_J L) \, d\xi^J + e^\sigma \gamma^I K \theta_K \, d\xi^J - \tilde{\xi}_I \star F] = 0,$$

where for compactness we have defined

$$X = (\gamma_{IJ} + \gamma^{KL} \theta_I K \theta_J L) \, d\xi_J \wedge \star d\xi_I + \gamma^{IJ} \, d\tilde{\xi}_J \wedge \star d\tilde{\xi}_I + 2\gamma^{IK} \theta_K \, d\xi_J \wedge \star d\xi_I,$$

as well as used the Bianchi identity $dF = 0$ to get the given form of (144). From a five-dimensional perspective, the moduli $(z^i, z^{\bar{i}})$ behave as scalar fields. We recall, however, that the behavior of the other fields is dependent on the moduli; i.e. they are functions in them. Hence it is possible to treat (142) as constraints that can be used to reduce the degrees of freedom of the other field equations.

Equations (143) and (144) are clearly the statements that the forms

$$J_2 = e^{-2\sigma} F + \star(\xi^l \, d\tilde{\xi}_l - \tilde{\xi}_l \, d\xi^l)$$
$$J_5 = e^\sigma \gamma^K K \theta_J K \, d\xi_J + e^\sigma \gamma^{J I} \, d\tilde{\xi}_J + \zeta^I \star F$$
$$\tilde{J}_{5/1} = e^\sigma (\gamma_{IJ} + \gamma^{KL} \theta_K \theta_J L) \, d\xi_J + e^\sigma \gamma^I K \theta_K \, d\xi_J - \tilde{\xi}_I \star F$$

are conserved. These are, in fact, Noether currents corresponding to certain isometries of the quaternionic manifold defined by the hypermultiplets as discussed in various sources [23, 47]. From a five-dimensional perspective, they can be thought of as the result of the invariance of the action under particular infinitesimal shifts of $A$ and $(\zeta, \tilde{\zeta})$ [24, 48]. The charge densities corresponding to them can then be found in the usual way by

$$Q_2 = \int J_2, \quad Q_5 = \int J_5, \quad \tilde{Q}_{5/1} = \int \tilde{J}_{5/1}.$$

The geometric way of understanding these charges is noting that they descend from the 11-dimensional electric and magnetic M-brane charges; hence the $(2, 5)$ labels. M2-branes wrapping special Lagrangian cycles of $\mathcal{M}$ generate $Q_2$ while the wrapping of M5-branes excite $(Q_5, \tilde{Q}_{5/1})$.

Finally, for completeness purpose we also give $da$, where $a$ is the universal axion dual to $A$. Since (143) is equivalent to $d^2 a = 0$, we conclude that

$$da = e^{-2\sigma} \star F - (\xi^l \, d\tilde{\xi}_l - \tilde{\xi}_l \, d\xi^l),$$

where $a$ is governed by the field equation

$$d^4[e^{2\sigma} \, da + e^{2\sigma} (\xi^l \, d\tilde{\xi}_l - \tilde{\xi}_l \, d\xi^l)] = 0$$

as a consequence of $dF = 0$. Both terms involving $F$ in (140) could then be replaced by the single expression

$$S_a = \frac{1}{2} \int e^{2\sigma} [(da + (\xi^l \, d\tilde{\xi}_l - \tilde{\xi}_l \, d\xi^l)] \wedge \star [da + (\xi^l \, d\tilde{\xi}_l - \tilde{\xi}_l \, d\xi^l)].$$

### 4.2. Supersymmetry

In this section we briefly outline the derivation of the five-dimensional SUSY variation equations from the 11-dimensional one. As before, enough material is reviewed for an overall understanding rather than a detailed description. On a CY threefold, there are two

---

16 This is the reverse situation to that of [24], where the (dual) Euclidean theory was studied.
17 Alternatively, one may dualize the action by introducing $a$ as a Lagrange multiplier and modifying the action accordingly [48].
supercovariantly constant Killing spinors [49] that may be defined, as usual, in terms of the Dirac matrices acting as ‘creation’ and ‘annihilation’ operators on the spinors as follows:

$$\Gamma_\mu \eta_+ = 0, \quad \Gamma_\mu \eta_- = 0,$$

(151)

where once again the hatted indices are flat indices on a tangent space. It follows then that

$$\Gamma_{\hat{m}\hat{n}\hat{p}} \eta_+ = \pm \delta_{\hat{m}\hat{n}\hat{p}} \eta_-,$$

(152)

where $\Gamma_{\hat{m}\hat{n}\hat{p}}$ is the antisymmetrized product of $\Gamma_\hat{m}$. We use (152) to define the spinors in terms of the CY (3, 0)-form as follows:

$$\Gamma_{\mu\nu\rho} \eta_+ = \Omega_{\mu\nu\rho} \eta_-,$$

(153)

Now, the 11-dimensional $\mathcal{N} = 1$ spinor $\Pi$ may be expanded in terms of the five-dimensional $\mathcal{N} = 2$ spinors $\epsilon_1$ and $\epsilon_2$ as follows:

$$\Pi = \epsilon_1 \otimes \eta_+ + \epsilon_2 \otimes \eta_-.$$

(154)

The strategy is to write the 11-dimensional gravitino equation, expand in terms of the five-dimensional spinors similarly to (154) and then identify the terms that are dependent on the (2, 1)- and (1, 2)-forms $\chi_i$ and $\chi^i$, or, via Kodaira’s formula (43), $(\nabla, \Omega)$ and $(\nabla, \bar{\Omega})$. These are taken to represent the hyperin, and their sum is identified as the hyperino variation equations. The rest of the terms, dependent on the (3, 0)- and (0, 3)-forms, become the $\mathcal{N} = 2$ gravitino equations.

We begin with the $D = 11$ gravitino variation:

$$\delta_\Pi \psi_M = (\partial_M \Pi) + \frac{1}{8} \omega_M^{\hat{a}\hat{b}} \Gamma_{\hat{L}\hat{S}} \Pi - \frac{1}{288} F_{LNPQ} (\Gamma_M^{LNPQ} - 8 \delta_M^{\hat{L}} \Gamma^{N\hat{P}\hat{Q}}) \Pi.$$

(155)

Based on the metric (128), we collect the relevant beins

$$e^a_v = e^\hat{a}_v N^{\hat{a}}_v, \quad e^\hat{a}_b = e^{-\hat{a}} W_b \hat{a},$$

(156)

The non-vanishing components of the spin connections are then

$$\omega_\mu^{\hat{a}\hat{b}} = \hat{\partial}_\mu \hat{a}^{\hat{b}} [g] - \frac{1}{2} \left( N^{\hat{a}\hat{b} \mu} \hat{N}^{\hat{b} \mu} - N^{\hat{a} \hat{b} \mu} \hat{N}^{\hat{a} \mu} \right) (\partial_\mu \sigma),$$

$$\omega_\mu^{\hat{a} \hat{b}} = W^{\hat{a} \hat{b}} (\partial_\mu W^{\hat{a} \hat{b}}) - \frac{1}{4} W^{\hat{a} \hat{b} \hat{c}} W^{\hat{a} \hat{c}} (\partial_\mu k_{\hat{d}}),$$

$$\omega_\mu^{\hat{a} \hat{b}} = e^{-\hat{a}} N^{\hat{a} \hat{b} \mu} \left[ \hat{\partial}_\mu (\partial_\mu \sigma) - \frac{1}{2} W^{\hat{a} \hat{b}} (\partial_\mu k_{\hat{d}}) \right]$$

(157)

The spin connections carrying $\hat{a}$ and $\hat{b}$ indices break down into $(\hat{m}\hat{n})$, $(\hat{n}\hat{m})$, $(\hat{m}\hat{n})$ and $(\hat{n}\hat{m})$ pieces. Based on the relations between the deformations of the metric and the cohomology forms, such as (39), the non-vanishing ones can be written in terms of variations of the moduli $z_i$. For example, one can straightforwardly show that

$$\Gamma^a_{\mu\nu} = \frac{1}{2 \|\Omega\|^2} \Omega^{\rho\sigma\rho} \chi_{\rho \mu \nu} (\partial_\mu z^c).$$

(158)

To deal with the components $F_\mu$ of the field strength, we note that, up to an exact form, one can always expand any three forms in terms of the (3, 0)- and (2, 1)-forms dual to the homology decomposition (35) as follows [24, 50]:

$$F_\mu = i e^{-\hat{c}/2} B_\mu \Omega - i e^{\hat{c}/2} G^{\hat{i}} j (\nabla j B_\mu) (\nabla, \Omega) + c.c.$$

$$B_\mu = \int F_\mu \wedge \bar{\Omega},$$

(159)
where the quantities $B_\mu$ and $\bar{B}_\mu$ are the coefficients of the expansion, found in the usual way by making use of (45), and c.c. represents the complex conjugate of previous terms. The 3-form $F_\mu$ then becomes

$$F_\mu = i \sqrt{2} \left[ M_I \left( \partial_\mu \xi^I \right) + L^I \left( \partial_\mu \bar{\xi}^I \right) \right] \bar{\Omega} - i \sqrt{2} G^j_I \left[ h_{ij} \left( \partial_\mu \xi^j \right) + f^j_I \left( \partial_\mu \bar{\xi}^j \right) \right] (\nabla_j \bar{\Omega}) + \text{c.c.} \quad (160)$$

Putting everything together, we find that we can write the resulting $D = 5$ equations as follows: the gravitini variations

$$\delta_\epsilon \psi^A = \tilde{\nabla}_\epsilon A + [G] \epsilon \epsilon^B$$

$$[G] = \begin{bmatrix} \frac{1}{4} (v - \bar{v} - Y) & -\bar{u} \\ u & -\frac{1}{4} (v - \bar{v} - Y) \end{bmatrix}, \quad (161)$$

where the indices $A$ and $B$ run over $(1, 2)$, $\tilde{\nabla}$ is given by

$$\tilde{\nabla} = dx^\mu \left( \partial_\mu + \frac{1}{4} \omega^\mu_\phi \Gamma^{\phi_\bar{\mu}} \right)$$

as usual and

$$u = e^{\frac{\sigma}{2}} (M_I \ d\xi^I + L^I \ d\bar{\xi}^I)$$

$$\bar{u} = e^{\frac{\sigma}{2}} (\bar{M}_I \ d\bar{\xi}^I + \bar{L}^I \ d\xi^I)$$

$$v = \frac{1}{2} d\sigma + \frac{1}{2} e^{-\sigma} \ast F$$

$$\bar{v} = \frac{1}{2} d\bar{\sigma} - \frac{1}{2} e^{-\bar{\sigma}} \ast \bar{F}. \quad (163)$$

The quantity $Y$ is proportional to the $U(1)$ connection $P$ defined by (29); explicitly

$$Y = \frac{\tilde{Z}^I N_{IJ} dZ^J - Z^I N_{IJ} d\tilde{Z}^J}{Z^I N_{IJ} \tilde{Z}^J}, \quad (164)$$

where, as before, $N_{IJ} = \text{Im}(F_{IJ})$ encoding the dependence of $F_I$ on $Z^I$. The matrix $G$ is the $Sp(1)$ connection of the quaternionic manifold described by the action$^{18}$. One can derive $G$ based on this alone with no reference to the higher dimensional theory, as was done in [23] for the four-dimensional case. The hyperini equations are

$$\delta_\epsilon \xi^I_1 = e^{1I}_\mu \Gamma^\mu_\epsilon_1 e_2 - e^{2I}_\mu \Gamma^\mu_\epsilon_2$$

$$\delta_\epsilon \xi^I_2 = e^{2I}_\mu \Gamma^\mu_\epsilon_1 e_2 + e^{1I}_\mu \Gamma^\mu_\epsilon_2, \quad (165)$$

written in terms of the quantities

$$e^{1I} = e^{1I}_\mu \ dx^\mu = \left( \begin{array}{c} \mu \\ E^i \end{array} \right)$$

$$e^{2I} = e^{2I}_\mu \ dx^\mu = \left( \begin{array}{c} \nu \\ \bar{E}^i \end{array} \right)$$

$$E^i = e^\bar{\sigma} e^j \left( h_{ji} \ d\xi^j + f^j_I \ d\bar{\xi}^j \right)$$

$$\bar{E}^i = e^\sigma e^j \left( h_{ji} \ d\xi^j + f^j_I \ d\bar{\xi}^j \right). \quad (167)$$

$^{18}$ Recall from Berger’s list that a quaternionic manifold has $Sp(h_2,1) \otimes Sp(1)$ holonomy.
and the beins of the special Kähler metric
\[ e^i = e^i_j dz^j, \quad \bar{e}^i = e^i_j \bar{dz}^j \]
\[ G_{ij} = e^k_i e^j_k \delta_{ij}. \]  

These quantities may also be used to make the connection between the special Kähler language we are using here and the quaternionic language used more abundantly in the literature. Quaternionic vielbeins may be defined as follows:
\[ V^{\Gamma A} = \left( \begin{array}{c} e^{1\Gamma} \\ \bar{e}^{2\Gamma} \\ -e^{2\Gamma} \end{array} \right), \quad \Gamma = 1, \ldots, 2(h_{2,1} + 1), \quad A = 1, 2, \]

(169)

such that
\[ \int h_{uv} dq^u \wedge \star dq^v = 2 \int (u \wedge \star \bar{u} + v \wedge \star \bar{v} + \delta_{ij} e^i \wedge \star \bar{e}^j + \delta_{ij} E^i \wedge \star \bar{E}^j), \]

(170)

where \( h_{uv} \) is the quaternionic metric with coordinates \( q^u \), the hypermultiplet scalars. This is tantamount to demonstrating the c-map, which relates the quaternionic form of the hypermultiplets in \( D = 5 \) to the SKG form of the vector multiplets in \( D = 4 \). The proof that this is, in fact, a quaternionic structure as defined in section 2.7 is somewhat tedious. The interested reader may consult [23].

4.3. The theory in manifestly symplectic form

For the sake of completeness, we also give a recently proposed form of the \( N = 2 \) theory [31], clearly highlighting its symplectic structure. Since the action is invariant under rotations in \( Sp \), it is clear that \( R, \sigma, dz \) and \( F \) are themselves symplectic invariants. The axion fields \((\zeta, \tilde{\zeta})\), however, can be thought of as components of an \( Sp \) ‘axions vector’. If we define
\[ |\Xi\rangle = \left( \begin{array}{c} \zeta^I \\ -\tilde{\zeta}^I \end{array} \right), \quad |d\Xi\rangle = \left( \begin{array}{c} d\zeta^I \\ -d\tilde{\zeta}^I \end{array} \right), \]

(171)

then
\[ \langle \Xi | d\Xi\rangle = \zeta^I d\tilde{\zeta}^I - \tilde{\zeta}^I d\zeta^I, \]

(172)
as well as
\[ \langle \partial_{\mu} \Xi | \partial^\mu \Xi \rangle = -(\gamma_L^J + \gamma_{LJ} \theta_{IK} \theta_{JL}) (\partial_{\mu} \zeta^I) (\partial^\mu \zeta^J) - \gamma_J^I (\partial_{\mu} \tilde{\zeta}^J) (\partial^\mu \tilde{\zeta}^I), \]

(173)
such that (145) becomes
\[ X = (\gamma_L^J + \gamma_{LJ} \theta_{IK} \theta_{JL}) d\zeta^I \wedge \star d\zeta^J + \gamma^J_I d\tilde{\zeta}^J \wedge \star d\tilde{\zeta}^I + 2\gamma^J_K \theta_{JK} d\zeta^I \wedge \star d\tilde{\zeta}^J - \langle \partial_{\mu} \Xi | \partial^\mu \Xi \rangle \star 1. \]

(174)

As a consequence of this language, the field expansion (130) could be rewritten as
\[ A = \mathcal{A} + \sqrt{2} (\Theta | \Xi), \]
\[ F = d\mathcal{A} = F + \sqrt{2} (\Theta | d\Xi). \]

(175)
The bosonic action in manifest symplectic covariance is hence

\[ S_2 = \int \left[ R \mathbf{1} - \frac{1}{2} d\sigma \wedge \star d\sigma - G_{ij} \, dz^i \wedge \star dz^j - F \wedge (\mathbb{Z} \mid d\mathbb{Z}) - \frac{1}{2} \, e^{-2\sigma} F \wedge \star F + e^\sigma (\delta\mu_\mathbb{Z}|A|\partial^\mu_\mathbb{Z}) \star 1 \right]. \]  

(176)

The equations of motion are now

\[ (\Delta \sigma) \star 1 + e^\sigma (\delta\mu_\mathbb{Z}|A|\partial^\mu_\mathbb{Z}) \star 1 + e^{-2\sigma} F \wedge \star F = 0 \]  

(177)

\[ (\Delta z^i) \star 1 + \Gamma^i_{jk} \, dz^j \wedge \star dz^k + \frac{1}{2} e^\sigma G_{ij} \partial^j (\delta\mu_\mathbb{Z}|A|\partial^\mu_\mathbb{Z}) \star 1 = 0 \]  

(178)

\[ d^i (e^{-2\sigma} F + \star (\mathbb{Z} | d\mathbb{Z})) = 0 \]  

(179)

\[ d^i (e^\sigma (A d\mathbb{Z}) + \star F | \mathbb{Z}) = 0. \]  

(180)

Note that, as is usual for Chern--Simons actions, the explicit appearance of the gauge potential \(|\mathbb{Z}\)| in (179) and (180) does not have an effect on the physics since

\[ d^i \star (\mathbb{Z} | d\mathbb{Z}) \longrightarrow d (\mathbb{Z} | d\mathbb{Z}) = (d\mathbb{Z} | d\mathbb{Z}) \]  

(181)

\[ d^i \star F \mid \mathbb{Z} \longrightarrow d (F \mid \mathbb{Z}) = F \wedge |d\mathbb{Z}|, \]

where the Bianchi identities on \(A\) and \(|\mathbb{Z}|\) were used. The Noether currents and charges become

\[ J_2 = e^{-2\sigma} F + \star (\mathbb{Z} | d\mathbb{Z}) \]  

\[ |J_2| = e^\sigma (A d\mathbb{Z}) + \star F | \mathbb{Z} \]  

(182)

\[ Q_2 = \int J_2, \quad |Q_2| = \int |J_2|. \]

The equations of the universal axion (148), (149) and (150) are now

\[ da = e^{-2\sigma} \star F - (\mathbb{Z} | d\mathbb{Z}), \]  

(183)

\[ d^i [e^{2\sigma} \, da + e^\sigma (\mathbb{Z} | d\mathbb{Z})] = 0 \quad \text{and} \]

(184)

\[ S_a = \frac{1}{2} \int e^{2\sigma} [da + (\mathbb{Z} | d\mathbb{Z})] \wedge \star [da + (\mathbb{Z} | d\mathbb{Z})]. \]  

(185)

The gravitini equations can be explicitly written as follows:

\[ \delta \psi^1 = \tilde{\psi} \epsilon_1 + \frac{i}{2} (e^{-\sigma} \star F - Y) \epsilon_1 - e^\frac{\epsilon}{2} (\tilde{\psi} | d\mathbb{Z}) \epsilon_2 \]  

(186)

\[ \delta \psi^2 = \tilde{\psi} \epsilon_2 - \frac{i}{2} (e^{-\sigma} \star F - Y) \epsilon_2 + e^\frac{\epsilon}{2} (\tilde{\psi} | d\mathbb{Z}) \epsilon_1, \]  

(187)

while the hyperini variations are

\[ \delta \xi^0 = e^\frac{\epsilon}{2} (\tilde{\psi} | d\mathbb{Z}) \Gamma^\mu \epsilon_1 - \left[ \frac{1}{2} (\partial_\mu \sigma) - \frac{i}{2} e^{-\sigma} (\star F)_\mu \right] \Gamma^\mu \epsilon_2 \]  

(188)

\[ \delta \xi^0 = e^\frac{\epsilon}{2} (\tilde{\psi} | d\mathbb{Z}) \Gamma^\mu \epsilon_2 + \left[ \frac{1}{2} (\partial_\mu \sigma) + \frac{i}{2} e^{-\sigma} (\star F)_\mu \right] \Gamma^\mu \epsilon_1. \]
\[ \delta_\epsilon \xi^i_j = \epsilon^z \left( \epsilon^j (U_j | \partial \mu) \xi^i_\mu \Gamma^\mu_\nu \epsilon_1 - \epsilon^i_j (\partial_\mu \xi^j_\nu) \Gamma^\mu_\nu \epsilon_2 \right) \]

Finally a useful set of identities was derived in [31] which we reproduce here for easy reference:

\[ \begin{align*}
\delta G_{ij} & = G_{ij} \Gamma^k_{ri} \partial \nu \Gamma^\nu_k + G_{ik} \Gamma^j_{ri} \partial \nu \Gamma^\nu_k \\
\delta G^{ij} & = -G^{pj} \Gamma^i_{rk} \partial \nu \Gamma^\nu_k + G^{ik} \Gamma^j_{rk} \partial \nu \Gamma^\nu_k \\
|dV| & = dz^i |U_i| - i\mathcal{P}|V| \\
|dV| & = dz^i |U_i| + i\mathcal{P}|V| \\
|dU_i| & = G_{ij} |dz^j| |V| + \Gamma^i_{jk} |dz^k| |U_j| + G^{ij} |dz^j| |U_i| - i\mathcal{P}|U_i| \\
|dU_i| & = G_{ij} |dz^j| |V| + \Gamma^i_{jk} |dz^k| |U_j| + G^{ij} |dz^j| |U_i| + i\mathcal{P}|U_i| \\
d\Lambda & = (\partial_i \Lambda) dz^i + (\partial_i \Lambda) dz^i,
\end{align*} \]

where \( \mathcal{P} \) is the \( U(1) \) connection defined by (29)

\[ \mathcal{P} = -\frac{i}{2} [(\partial_i \mathcal{K}) dz^i - (\partial_i \mathcal{K}) dz^i], \]

and \((\partial_i \Lambda, \partial_i \Lambda)\) are given by (126).

**Appendix. Differential forms on manifolds**

In this appendix we review the language of differential forms used in various locations in the text. Clearly, reading this review requires more knowledge of differential forms, Hodge theory and topology than is reviewed here. The purpose of this appendix is then to simply set the notation and collect in one place all the equations necessary to reproduce the various details in the review.

Consider a \( D \)-dimensional Riemannian/Lorentzian manifold \( \mathcal{M} \). A differential form \( \omega \) or \( \omega_p \) of order \( p \) on \( \mathcal{M} \), also known as a \( p \)-form, is a totally antisymmetric tensor of type \((0, p)\). It may be defined in terms of the differentials \( dx^\mu \), themselves 1-forms, acting as basis in this case, in the standard way

\[ \omega = \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \quad (A.1) \]

where the so-called wedge product \( \wedge \) is defined such that the following properties are satisfied:

1. \( dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = 0 \) if some index \( \mu_i \) appears at least twice.
2. \( dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = -dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \) on the exchange of two adjacent indices.
3. \( dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \) is linear in each \( dx^{\mu_i} \).

Clearly the components \( \omega_{\mu_1 \cdots \mu_p} \) are themselves antisymmetric such that \( \omega \) is non-vanishing. It can be shown that the wedge product of forms \( \omega_p, \eta \) and \( \xi \) satisfies the following.

1. \( \omega_p \wedge \omega_p = 0 \) if \( p \) is odd.
2. \( \omega_p \wedge \eta = -(-1)^p \eta \wedge \omega_p \).
3. \( \xi \wedge \omega \wedge \eta = \xi \wedge (\omega \wedge \eta) \).

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The so-called exterior derivative $d = dx^\nu \partial_\nu$ is defined as an operator that maps $p$-forms into $(p + 1)$-forms as follows:

$$\lambda_{p+1} = d\omega_p = \frac{1}{p!} \left( \partial_\nu \omega_{\mu_1 \cdots \mu_p} \right) dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p},$$  \(A.2\)

satisfying the product rule

$$d(\omega_p \wedge \eta_r) = (d\omega_p) \wedge \eta_r + (-1)^p \omega_p \wedge (d\eta_r),$$  \(A.3\)
as well as the very important

$$d^2 = 0.$$  \(A.4\)

An operator satisfying (A.4) is called nilpotent. Differential forms that can be written as exterior derivatives of other forms, such as $\lambda = d\omega$, are called exact, while forms whose exterior derivative vanishes, e.g. $d\lambda = 0$, are called closed. Because of (A.4), exact forms are always closed, while the converse is not necessarily true.

One of the most beautiful theorems in mathematics involves the integration of $p$-forms and is known as Stokes’ theorem, which is a generalization of the familiar theorem by the same name in $\mathbb{R}^3$, as well as the divergence theorem and the fundamental theorem of calculus. It states

$$\int_M d\omega = \int_{\partial M} \omega,$$  \(A.5\)

where $\partial M$ denotes the $(D - 1)$-dimensional boundary of $M$, unless of course $M$ is closed, in which case $\omega$ is closed as well and the right-hand side vanishes. Equation (A.5) is sometimes referred to as the fundamental theorem of calculus on manifolds.

Note that all our definitions so far are metric independent. We now choose a metric $g_{\mu\nu}$ on $M$ with either Riemannian or Lorentzian signatures. We also define the Levi-Civita totally antisymmetric symbol in the following way:

$$\bar{\epsilon}_{\mu_1 \cdots \mu_D} = \begin{cases} 
+1 & \text{for even permutations of the indices,} \\
-1 & \text{for odd permutations of the indices,}
\end{cases}$$

\(A.6\)

where $\bar{\epsilon}_0 \cdots D-1$ or $\bar{\epsilon}_1 \cdots D = +1$. Defined this way, $\bar{\epsilon}_{\mu_1 \cdots \mu_p}$ does not transform as a tensor, hence the name ‘symbol’. One way of defining a Levi-Civita tensor is described below. The volume form over $D$ dimensions is defined by

$$\varepsilon_D = \sqrt{|g|} \, dx^1 \wedge \cdots \wedge dx^D = \frac{1}{D!} \varepsilon_{\mu_1 \cdots \mu_D} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D},$$  \(A.7\)

where the unbarred $\varepsilon_{\mu_1 \cdots \mu_D}$ (the components of $\varepsilon_D$, which does transform as a tensor) are defined by

$$\varepsilon_{\mu_1 \cdots \mu_D} = \sqrt{|g|} \varepsilon_{\mu_1 \cdots \mu_D},$$

$$\varepsilon^{\mu_1 \cdots \mu_D} = \frac{1}{\sqrt{|g|}} \bar{\epsilon}^{\mu_1 \cdots \mu_D}.$$  \(A.8\)

The indices of $\varepsilon_{\mu_1 \cdots \mu_D}$ are raised and lowered by $g_{\mu\nu}$ while those of $\bar{\epsilon}_{\mu_1 \cdots \mu_D}$ are raised and lowered by the flat metric (either Minkowski or Euclidean depending on the signature of $g_{\mu\nu}$). Clearly

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D} = \varepsilon^{\mu_1 \cdots \mu_D} dx^1 \wedge \cdots \wedge dx^D.$$  \(A.9\)

Note that in most of the literature the nomenclature $\sqrt{|g|} \, d^Dx$ is used as a substitute for $\sqrt{|g|} \, dx^1 \wedge \cdots \wedge dx^D$ which is technically the correct volume element.
Based on all this, we define the Hodge-duality operator $\star$, mapping $p$-forms into $(D-p)$-forms, as follows:

$$\star \left( dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \right) = \frac{1}{(D-p)!} \varepsilon_{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_D} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_D}$$

$$k_{D-p} = \star \omega_p = \frac{1}{p! (D-p)!} \varepsilon_{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_D} \omega^{\mu_1 \cdots \mu_p} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_D}$$

$$= \frac{1}{(D-p)!} k_{\mu_{p+1} \cdots \mu_D} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_D}. \quad (A.10)$$

Note that, in this language, the volume form is the Hodge dual of the identity, i.e.

$$\star 1 = \sqrt{|g|} \, dx^1 \wedge \cdots \wedge dx^D. \quad (A.11)$$

Furthermore, it can be straightforwardly shown that

$$\star \star \omega_p = (-1)^p (D-p) q \omega_p$$

$$\star^{-1} = (-1)^p (D-p) q \star,$$  

where $\star^{-1}$ is the inverse Hodge dual and $q$ is the number of eigenvalues of the metric with a minus sign, i.e. if $\mathcal{M}$ is Riemannian then $q = 0$, while if it is Lorentzian then $q = 1$.

We can now define an inner product of forms. This is

$$(\omega_p, \eta_p) = \int_\mathcal{M} \omega_p \wedge \star \eta_p,$$  

where

$$\omega_p \wedge \star \eta_p = \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} \eta^{\mu_1 \cdots \mu_p} \sqrt{|g|} \, dx^1 \wedge \cdots \wedge dx^D. \quad (A.14)$$

The inner product is clearly symmetrical:

$$\omega_p \wedge \star \eta_p = \eta_p \wedge \star \omega_p$$

$$(\omega_p, \eta_p) = (\eta_p, \omega_p). \quad (A.15)$$

We can also use the Hodge dual to define the so-called adjoint exterior derivative operator

$$d^* \omega_p = (-1)^{p(D+p+1)} q \star d \star \omega_p,$$  

which maps a $p$-form down to a $(p-1)$-form:

$$\star d \star \omega_p = \star d \kappa_{D-p} = \star \tau_{D-p+1} = \phi_{D-(p+1)} = \phi_{p-1}. \quad (A.17)$$

Also note that since $d$ is nilpotent, then so is $d^*$:

$$d^{2*} \propto \star d \star \star d \star \propto d^2 \star = 0. \quad (A.18)$$

For calculational convenience, we explicitly give the action of the adjoint exterior derivative on a $p$-form $\omega$ in $D$ dimensions

$$d^* \omega = (-1)^{p(D+1)} q \star \left( \nabla^{\mu_1} \omega_{\mu_1 \mu_2 \cdots \mu_p} \right) dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}, \quad (A.19)$$

where $\nabla_\mu$ is the usual Levi-Civita connection with respect to the metric on $\mathcal{M}$. Certain useful theorems involving $d^*$ can be proven. For example, one can show that

$$(d \omega, \eta) = (\omega, d^* \eta). \quad (A.20)$$

In analogy with the exterior derivative, one says that a form $\lambda$ that can be written as $\lambda = d^! \omega$ is co-exact, while one that satisfies $d^! \lambda = 0$ is co-closed. Clearly, co-exact forms are always co-closed, while the converse is not necessarily true.
Finally, we define the Laplacian operator on p-forms by

$$\Delta = (d + d^\dagger)^2 = dd^\dagger + d^\dagger d.$$  \hspace{1cm} (A.21)

A p-form that satisfies

$$\Delta \omega = 0$$ \hspace{1cm} (A.22)

is called harmonic, and (A.22) is known as the harmonic condition. A form is harmonic if and only if it is both closed and co-closed. Harmonic forms clearly play a fundamental role in physics. To demonstrate this, consider the following: using (A.19), the Laplacian of a 0-form scalar field $f$, i.e. an ordinary function, in $D = 5$ spacetime, leads to the familiar expression:

$$\Delta f = d^\dagger d f = \nabla^\mu \nabla_\mu f = \nabla^2 f,$$ \hspace{1cm} (A.23)

where we have used $d^\dagger f = 0$. Also consider the Laplacian for a general Abelian gauge potential $A$ in $D$ dimensions. We define $F = dA$ as usual and write

$$\Delta A = dd^\dagger A + d^\dagger dA = dd^\dagger A + d^\dagger F.$$ \hspace{1cm} (A.24)

Because of the gauge freedom of $A$, we normally choose $d^\dagger A = 0$, which is the generalized Lorenz gauge condition\(^{20}\) leading to the more familiar $\nabla_\mu A^{\mu} = 0$. Hence

$$\Delta A = d^\dagger dA = d^\dagger F,$$ \hspace{1cm} (A.25)

which, in physical theory, may or may not vanish depending on the presence or absence of sources. For example, in ordinary Maxwell theory in $D = 4$ flat spacetime, the expression $d^\dagger F = J$ leads to the ordinary Gauss and Ampère laws, provided that $J$ is the current 1-form. The Bianchi identity $d^\dagger F = 0$, resulting from the fact that the $U(1)$-form $F$ is exact, leads to the Faraday and no-monopoles laws. It is clearly straightforward to extend the formalism of differential forms to complex manifolds. We will not do so here but rather refer the interested reader to more detailed discussions of this vast topic, such as [34].

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19 Also sometimes known as the Laplace–de Rahm operator, to differentiate between it and the ordinary Laplacian acting on scalar functions, which is a special case as we will see.

20 Note that this refers to the Danish physicist L. Lorenz and not the Dutch H. Lorentz of Lorentz transformations fame. Confusing the two names is a recurring error in the literature.
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