COMPOSITION + HOMOTOPY = CUBES

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ABSTRACT. The goal of this article is to emphasize the role of cubical sets in enriched categories theory and infinity-categories theory. We show in particular that categories enriched in cubical sets provide a convenient way to describe many infinity-categories appearing in the context of homological algebra.

CONTENTS

Introduction 1
1. Monoidal categories and enriched categories 4
2. Reduced cubical sets and cubical sets with connections 12
3. Homotopy theory of cubical sets 16
4. Nerve functors 20
5. Applications 23
References 27

INTRODUCTION

The goal of this article is to emphasize the role of cubes and cubical sets when dealing with compositions of homotopies.

Indeed, let \((E, \otimes, 1)\) be a monoidal model category, together with the choice of an interval \(1 \sqcup 1 \hookrightarrow H \twoheadrightarrow 1\). One can think of the category of simplicial sets with the interval \(\Delta[1]\), of the category of chain complexes with the cellular model of the interval, or of the category of differential graded coalgebras with the cellular model of the interval. Then let \((A, \gamma, \eta)\) be a monoid in \(E\) (for instance a simplicial monoid or a differential graded algebra depending on our choice of category \(E\)). A point of \(A\) is a morphism \(a: 1 \rightarrow A\) in the category \(E\). Then, a path between two points in \(A\) is the data of a morphism \(H \rightarrow A\). Using the product on \(A\), one can define the product of two paths \(f\) and \(g\) as follows

\[
H \otimes H \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\gamma} A.
\]

The product of \(f\) with \(g\) is thus a morphism from \(H \otimes H\) to \(A\); that is a square of \(A\). Similarly, the product of a \(n\)-cube \(f: H^\otimes n \rightarrow A\) with a \(m\)-cube \(g: H^\otimes m \rightarrow A\) is a \(n + m\)-cube \(f \cdot g: H^\otimes (n+m) \rightarrow A\). The same phenomenon appears when dealing with a “monoid with many objects”, that is a category enriched over \(E\). This enlightens the fact that cubes appear naturally when mixing homotopy with composition. In the case where the monoidal structure is Cartesian, that is \(\otimes\) is the categorical product \(\times\), for instance for simplicial sets, then the interval \(H\) has a diagonal map \(H \rightarrow H \otimes H\). Then, the product of two paths which is a square induces another path, the diagonal of this square

\[
H \rightarrow H \otimes H \rightarrow A \otimes A \xrightarrow{\eta} A.
\]

There exists a category of cubes \(\Box_r\) (the \(r\) stands for reduced), described in details in the book [Cis06], which roughly consists of cubes of various dimensions \(\Box_r^k\), together with face inclusions

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\( \delta_i : \square^n \to \square^{n+1} \) and contraction along a direction \( \sigma_i : \square^n \to \square^{n-1} \). This category has a monoidal product given by
\[ \square^n \otimes \square^m = \square^{n+m}. \]

In a similar way as simplicial sets \( \text{sSet} = \text{Fun}(\Delta^{op}, \text{Set}) \) are gluings of points, lines, triangles, tetrahedrons, \ldots, reduced cubical sets \( \square_{r-\text{Set}} = \text{Fun}(\square_{c}^{op}, \text{Set}) \) are gluings of points, lines, squares, cubes, \ldots Moreover, reduced cubical sets represent all the possible homotopy types as well as simplicial sets; indeed, the category \( \square_{r-\text{Set}} \) has a model structure Quillen equivalent to the Kan–Quillen model category of simplicial sets. Besides, cubical sets inherit a monoidal structure from that of cubes.

Then all the discussion above about compositions and homotopies is encompassed in the following proposition.

**Proposition** ([Cis06]). Let \((E, \otimes, 1)\) be a monoidal model category. Then, the data of an interval \( H \) of \( E \), is essentially the data of a monoidal Quillen adjunction
\[ \square_{r-\text{Set}} \xrightarrow{L_H} E. \]

Moreover, this induces another adjunction
\[ \text{Cat}_{\square_{r-\text{Set}}} \xrightarrow{L_H} \text{Cat}_E \]

between categories enriched in reduced cubical sets and categories enriched in \( E \), which is a Quillen adjunction, when the categorical model structure on \( \text{Cat}_E \) exists.

Besides, let us consider the simplicial set \( \Delta[3] \).

\[ \begin{array}{ccc}
(0) & \rightarrow & (1) \\
\downarrow & & \downarrow \\
(2) & \rightarrow & (3)
\end{array} \]

Seen as an infinity-category, it has 4 objects, that is 0, 1, 2, 3. Its morphisms are generated by the edges \((ij)\) for any integers \(0 \leq i < j \leq 3\). Then there are exactly 4 morphisms from 0 to 3 that is \((03), (01)(13), (02)(23)\) and \((01)(12)(23)\). They are organized into a square as follows.

\[ \begin{array}{ccc}
(01)(13) & \rightarrow & (01)(12)(23) \\
\downarrow & & \downarrow \\
(03) & \rightarrow & (02)(23) \\
\downarrow & & \downarrow \\
(013) & \rightarrow & (012)(23) \\
\end{array} \]
More generally, the morphisms of $\Delta[n]$ from $i$ to $j$ are organized into a $j - i - 1$-cube for $i < j$. In particular the morphisms from $0$ to $n$ are organized into a $n - 1$-cube. The face maps $\Delta[n - 1] \to \Delta[n]$ induce face maps between cubes. This seems to be the beginning of a functor $W_r$ from $\Delta$ to the category $\text{Cat}_{\square_r}$ of categories enriched over reduced cubical sets which has the shape that we hoped earlier. Indeed, see for instance [Mal09]. Unfortunately, such a morphism from $\square_r[2]$ to $\square_r[1]$ does not exist. Therefore, one needs to enhance reduced cubes and reduced cubical sets by adding this map $\gamma$ to obtain respectively the category $\square_r$ of cubes with connections and the category $\square_r-\text{Set}$ of cubical sets with connections; see for instance [Mal09].

Actually, there exists a functor $W_r$ from the category $\Delta$ to the category $\text{Cat}_{\square_r}$ of categories enriched over cubical sets such that $W_r(n)$ would be the cubical category with $n + 1$ objects, $0, \ldots, n$ and

$$W_{n,0}(0,n) = \square_r[n - 1],$$

and more generally,

$$W_{n,i}(i,j) = \square_r[j - i - 1],$$

for any integers $0 < i < j \leq n$. However, the degeneracy map $\sigma_1 : \Delta[3] \to \Delta[2]$ would give a functor $W_{r,2} \to W_{r,1}$ corresponding at the level of mapping spaces to a map $\gamma : \square_r[2] \to \square_r[1]$ mimicking the behavior of the function

$$[0,1] \times [0,1] \to [0,1]
(x,y) \mapsto \max(x,y).$$

Unfortunately, such a morphism from $\square_r[2]$ to $\square_r[1]$ does not exist. Therefore, one needs to enhance reduced cubes and reduced cubical sets by adding this map $\gamma$ to obtain respectively the category $\square_r$ of cubes with connections and the category $\square_r-\text{Set}$ of cubical sets with connections; see for instance [Mal09].

Actually, there exists a functor $W_r$ from the category $\Delta$ to the category $\text{Cat}_{\square_r}$ of categories enriched over cubical sets such that $W_r(n)$ has the shape that we hoped earlier. Indeed, $W_n := W_r(n)$ has $n + 1$ objects, $0, \ldots, n$ and

$$W_{n,0}(0,n) = \square_r[n - 1].$$

It induces an adjunction

$$\text{sSet} \xrightarrow{W_r} \text{Cat}_{\square_r}.$$

The usual adjunctions relating simplicial sets to categories enriched over a monoidal model category $E$ factorizes through this one. Moreover, this is a Quillen adjunction if the category of simplicial sets is endowed with the Joyal model structure. This two facts coupled with some Reedy theory lead us to the following theorem.

**Theorem.** Let $E$ be a monoidal model category and suppose that the category $\text{Cat}_E$ of categories enriched over $E$ has a categorical model structure. Then, for any Reedy cofibrant replacement $F$ of the cosimplicial $E$-enriched category $n \mapsto [n]$, the induced adjunction

$$\text{sSet} \xrightarrow{F} \text{Cat}_E,$$

is a Quillen adjunction.

The use of cubical sets is particularly efficient when dealing with enriched model structures. Simplicial model categories are model categories $M$ enriched, tensored and cotensored over the category of simplicial sets satisfying an additional axiom which implies that the simplicial category of fibrant-cofibrant objects is a model of the infinity-category that is presented by the model category $M$. Replacing the category of simplicial sets by another monoidal model category $E$, one obtains the notion of an $E$-model category; see [Hov99]. By the following proposition, any $E$-model category $M$ has the structure of a cubical model category.

**Proposition.** Let $(E, \otimes, 1)$ be a monoidal model category and let $M$ be an $E$-model category. Then, any choice of an interval (resp. monoidal interval) $H$ in $E$ induces the structure of a $\square_r-\text{Set}$-model category (resp. $\square_r-\text{Set}$-model category) on $M$.

In particular, if $M$ is a simplicial model category, then it has an induced structure of a cubical model category. We are also interested in the cases where $M$ is an $E$-model category for some $E$ but it is not a simplicial model category; for instance the Joyal model category and the model category of algebras over a nonsymmetric differential graded operad where $E$ is respectively the Joyal model category and the model category of differential graded coalgebras.
Layout. This article is organized as follows. In the first section, we describe in details the theory of enriched categories and recall some results about their homotopy theory. The two next sections deal with the categories of reduced cubical sets and cubical sets with connections and their model structures. Many of the results given there are due to Cisinski. The fourth part makes a link between simplicial sets and enriched categories. The final section applies the material developed to concrete examples.

What is new in this article. Some results given here were already known. Indeed, many of the results about cubical homotopy are consequences of the work of Cisinski ([Cis06]). The idea that the homotopy coherent nerve functors for dg categories and simplicial categories factor through categories enriched over cubical sets was already in [RZ16]. Moreover, similar ideas to those of Section 4, already appeared independently in [KV18]. However, to the best of my knowledge, this is the first time that cubical sets are used systematically to study enriched categories. Therefore, sections 4 and 5 develop new mathematics.

Conventions.

- The category of simplicial sets is denoted $sSet$. It is usually endowed with the Kan-Quillen model structure. If we endow it with the Joyal model structure, we write $sSet_J$.
- We denote by $[n]$ the category with $n+1$ objects $0, \ldots, n$ such that
  \[
  \text{hom}_{[n]}(i, j) = \begin{cases} 
  \ast & \text{if } i \leq j, \\
  \emptyset & \text{otherwise}.
  \end{cases}
  \]
  Furthermore, $\delta^A_i : [n] \to [n+1]$ is the only injective functor which omits the objects $i$ in $[n+1]$ and $\sigma^A_i : [n] \to [n-1]$ is the only surjective functor which sends the objects $i$ and $i+1$ to $i$. All these categories $[n]$ and these functors generate the category $\Delta$.
- Let $\mathcal{U} < \mathcal{U}'$ be two universes. Usually, we work with categories whose sets of morphisms are $\mathcal{U}$-small and whose set of objects is $\mathcal{U}$-large (that is a subset of $\mathcal{U}$). In particular, these categories are $\mathcal{U}'$-small, in the sense that their sets of objects and morphisms are $\mathcal{U}'$-small. Then, when performing constructions on a category considered as an object, or when working with «the category of categories», we assume working with $\mathcal{U}'$-small categories.

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1. MONOIDAL CATEGORIES AND ENRICHE D CATEGORIES

1.1. Bilinear monoidal categories. In this section, we recall the notions of a monoidal adjunction, of a monoidal natural transformation and of a bilinear monoidal category.

Definition 1. Let $(E, \otimes, 1)$ and $(F, \otimes, 1)$ be two monoidal categories. A monoidal adjunction between them is an adjunction

\[
E \xrightarrow{L} F,
\]

together with a natural isomorphism $L(X) \otimes L(Y) \simeq L(X \otimes Y)$ which makes the left adjoint $L$ a strong monoidal functor. Subsequently, $R$ obtains the structure of a lax monoidal functor.

The structure of a monoidal functor on $R$ is given by the composition

\[
RA \otimes RB \xrightarrow{\eta} R(LA \otimes LB) \simeq R(LRA \otimes LRB) \xrightarrow{R(\epsilon \otimes \epsilon)} R(A \otimes B),
\]

where $\eta$ and $\epsilon$ are respectively the unit and the counit of the adjunction.

Lemma 1. The following diagram commutes

\[
\begin{array}{ccc}
L(RA \otimes RB) & \xrightarrow{\sim} & LR(A \otimes B) \\
\downarrow & & \downarrow \epsilon \\
LRA \otimes LRB & \xrightarrow{\epsilon \otimes \epsilon} & A \otimes B
\end{array}
\]
for any objects $A$ and $B$ of $F$, where the upper horizontal map and the left vertical map correspond respectively to the monoidal structure of the functor $R$ and to the strong monoidal structure of the functor $L$.

**Proof.** The two morphisms from $L(RA \otimes RB)$ to $A \otimes B$ correspond to the same morphism from $RA \otimes RB$ to $R(A \otimes B)$, that is the one defining the monoidal structure of $R$. \hfill $\square$

**Definition 2.** Let $(E, \otimes, 1)$ and $(F, \otimes, 1)$ be two monoidal categories, and let $F, G : C \to D$ be two lax monoidal functors between them. A natural transformation $\phi : F \to G$ is monoidal if the following diagrams commute

$$
\begin{align*}
F(X) \otimes F(Y) & \xrightarrow{\phi(X) \otimes \phi(Y)} F(X \otimes Y) \\
G(X) \otimes G(Y) & \xrightarrow{\phi(X \otimes Y)} G(X \otimes Y)
\end{align*}
$$

for any objects $X, Y$ of the category $C$.

**Definition 3.** A bilinear monoidal category $(E, \otimes, 1)$ is a monoidal category such that $E$ is cocomplete and such that the bifunctor $- \otimes -$ commutes with colimits separately on both sides.

For any such bilinear monoidal category, the element $1 \in C$ induces a cocontinuous functor $i : \text{Set} \to E$ such that $i(*) = 1$. This functor has a right adjoint $S$ such that $S(X) = \text{hom}_E(1, X)$.

$$
\begin{array}{c}
\text{Set} \xrightarrow{i} E \\
S \xleftarrow{\text{adjunction}}
\end{array}
$$

**Proposition 1.** The functor $i$ is strong monoidal. Therefore, the adjunction $i \dashv S$ is monoidal.

**Proof.** Since the monoidal structure is bilinear, then for any sets $X$ and $Y$ we have

$$
i(X) \otimes i(Y) \simeq (\bigsqcup_{a \in X} 1) \otimes (\bigsqcup_{b \in Y} 1) \simeq \bigsqcup_{(a, b) \in X \times Y} 1 \otimes 1 \simeq \bigsqcup_{(a, b) \in X \times Y} 1 \simeq i(X \times Y).
$$

\hfill $\square$

**1.2. Day monoidal product.** Let $(A, \otimes, 1)$ be a small category endowed with a monoidal structure. Then, the opposite category $A^{\text{op}}$ inherits a monoidal structure. We denote by $A - \text{Set}$ the category of presheaves over $A$, that is functors from $A^{\text{op}}$ to $\text{Set}$.

**Definition 4.** For any presheaves $X, Y$ over $A$, the Day product $X \otimes Y$ is the following left Kan extension

$$
\begin{array}{c}
\begin{array}{c}
A^{\text{op}} \times A^{\text{op}} \xrightarrow{X \times Y} \text{Set} \times \text{Set} \\
\otimes \xrightarrow{X \otimes Y}
\end{array} \\
A^{\text{op}}
\end{array}
$$

The Day product may also be defined in the following way. Both $X$ and $Y$ are colimits of representables

$$
\begin{cases}
X \simeq \text{colim}_{a \in A/X} a, \\
Y \simeq \text{colim}_{a \in A/Y} a.
\end{cases}
$$

Then,

$$
X \otimes Y \simeq \text{colim}_{(a, a') \in A/X \times A/Y} a \otimes a'.
$$

**Proposition 2.** ([Day70]) The Day product defines a bilinear monoidal structure on the category $A - \text{Set}$. Moreover, the Yoneda embedding functor $A \to A - \text{Set}$ is strong monoidal.
1.3. Enriched categories.

Definition 5. Let \((E, \otimes, 1)\) be a monoidal category. A category enriched over \(E\) (or \(E\)-category) \((\mathcal{C}, m, n)\) is the data of

\[\begin{align*}
\triangleright & \text{ a set of objects } \text{Ob}(\mathcal{C}), \\
\triangleright & \text{ for any objects } x, y \in \text{Ob}(\mathcal{C}), \text{ an element } \mathcal{C}(x, y) \text{ of the category } E, \\
\triangleright & \text{ an associative composition } m_{x,y,z} : \mathcal{C}(x, y) \otimes \mathcal{C}(y, z) \to \mathcal{C}(x, z), \\
\triangleright & \text{ a unit for this composition } u_x : 1 \to \mathcal{C}(x, x) \text{ for any object } x.
\end{align*}\]

A functor \(F\) between two such \(E\)-categories \((\mathcal{C}, m, n)\) and \((\mathcal{C}', m', n')\) is the data of

\[\begin{align*}
\triangleright & \text{ a function from } \text{Ob}(\mathcal{C}) \text{ to } \text{Ob}(\mathcal{C}') \text{ also denoted } F, \\
\triangleright & \text{ for any objects } x, y \in \text{Ob}(\mathcal{C}), \text{ a morphism } F_{x,y} : \mathcal{C}(x, y) \to \mathcal{C}'(F(x), F(y)), \\
\triangleright & \text{ which commutes with the composition and the unit in the sense that}
\end{align*}\]

\[
\begin{align*}
\begin{cases}
m'_{F(x), F(y), F(z)} (F_{x,y} \otimes F_{y,z}) = F_{x,z}m_{x,y,z}, \\
u_{F(x)} = F_{x,x}u_x,
\end{cases}
\end{align*}
\]

for any objects \(x, y, z\) of \(\mathcal{C}'\).

This defines the category \(\text{Cat}_E\) of \(E\)-categories.

Forgetting the composition and the unit in \(E\)-categories, one gets the notion of an \(E\)-quiver.

Definition 6. An \(E\)-quiver \(\mathcal{D}\) is the data of a set of objects \(\text{Ob}(\mathcal{D})\) together with an element \(\mathcal{D}(x, y)\) of \(E\) for any objects \(x, y \in \text{Ob}(\mathcal{D})\). A morphism \(F\) of quivers from \(\mathcal{D}\) to \(\mathcal{D}'\) is the data of a function from \(\text{Ob}(\mathcal{D})\) to \(\text{Ob}(\mathcal{D}')\) also denoted \(F\) and, for any objects \(x, y \in \text{Ob}(\mathcal{D})\), a morphism \(F_{x,y} : \mathcal{D}(x, y) \to \mathcal{D}'(F(x), F(y))\). We denote by \(\text{Quiv}_E\) the category of \(E\)-quivers.

Lemma 2. Suppose that the monoidal category \(E\) is bilinear. Then, the forgetful functor \(O : \text{Cat}_E \to \text{Quiv}_E\) has a left adjoint \(T\) such that for any quiver \(\mathcal{D}\)

\[\begin{align*}
\triangleright & \text{ the set } \text{Ob}(T \mathcal{D}) \text{ is exactly the set } \text{Ob}(\mathcal{D}), \\
\triangleright & \text{ for any objects } x, y \in \text{Ob}(\mathcal{D})
\end{align*}\]

\[
\begin{align*}
\begin{cases}
\mathcal{T}(x, y) = \cup_{n \geq 1} \cup_{x_0=x, x_1,..., x_n=y} \mathcal{D}(x_0, x_1) \otimes \cdots \otimes \mathcal{D}(x_{n-1}, x_n) & \text{if } x \neq y, \\
\mathcal{T}(x, x) = 1 \cup \cup_{n \geq 1} \cup_{x_0=x, x_1,..., x_n=x} \mathcal{D}(x_0, x_1) \otimes \cdots \otimes \mathcal{D}(x_{n-1}, x_n) & ,
\end{cases}
\end{align*}\]

\[\begin{align*}
\triangleright & \text{ the composition is given by the concatenation of tensors.}
\end{align*}\]

Moreover, the adjunction \(T \dashv O\) is monadic.

Proof. Straightforward. \qed

Lemma 3. If the category \(E\) is cocomplete, then the category \(\text{Quiv}_E\) is cocomplete. Moreover, for any regular cardinal \(\lambda\), if \(E\) is a \(\lambda\)-presentable category, then, the category \(\text{Quiv}_E\) is \(\lambda\)-presentable.

Proof. It is straightforward to prove that the category \(\text{Quiv}_E\) is stable under small coproducts. Let us prove that it has all cokernels. Consider the following diagram of \(E\)-quivers.

\[
\begin{align*}
\mathcal{D} & \xrightarrow{F} \mathcal{D}' \\
\mathcal{D} & \xrightarrow{G} \mathcal{D}'
\end{align*}\]

The cokernel \(\mathcal{D}''\) of \(F\) and \(G\) is the following \(E\)-quiver:

\[\begin{align*}
\triangleright & \text{ its set of objects is the cokernel of the underlying functions of } F \text{ and } G \text{ from } \text{Ob}(\mathcal{D}) \text{ to } \text{Ob}(\mathcal{D}'). \text{ Therefore, it is the quotient of the set } \text{Ob}(\mathcal{D}'') \text{ by the relation } F(x) \sim G(x) \text{ for any object } x \text{ of } \mathcal{D}. \text{ Let us denote by } K \text{ the surjection from } \text{Ob}(\mathcal{D}') \text{ to } \text{Ob}(\mathcal{D}''). \text{ It is clear that at the level of objects of } \mathcal{D}, KF = KG. \\
\triangleright & \text{ For any objects } x, y \in \text{Ob}(\mathcal{D}''), \mathcal{D}'(x, y) \text{ is the cokernel in } E \text{ of the following diagram.}
\end{align*}\]
Besides, if $E$ is $\lambda$-presentable, then the category $\text{Quiv}_E$ of $E$-quivers is generated under $\lambda$-filtered colimits by $E$-quivers $\mathcal{D}$ whose sets of objects are $\lambda$-small and such that for any objects $x, y, \mathcal{D}(x, y)$ is $\lambda$-small. The (possibly large) set of isomorphisms classes of such $E$-quivers is actually a small set.

**Lemma 4.** Suppose that the monoidal category $(E, \otimes, 1)$ is bilinear. Then the category $\text{Cat}_E$ has all filtered colimits and the forgetful functor $O$ preserves filtered colimits.

**Proof.** Let $D : I \to \text{Cat}_E$ be a filtered diagram and let

$$\mathcal{D} = \text{colim}_i O \circ D.$$  

We denote by $F(i)$ the morphism of $E$-quivers $D(i) \to \mathcal{D}$ for any object $i \in I$. The set of objects $\text{Ob}(\mathcal{D})$ is the colimit of the diagram $i \in I \to \text{Ob}(D(i))$. Moreover, for any two of its element $x, y$,

$$\mathcal{D}(x, y) = \text{colim}_{i}(i,F(i)(x')=x,F(i)(y')=y)D(i)(x', y') \simeq \text{colim}_{i}(i,F(i)(x')=x,F(i)(y')=y)D(i)(x', y') .$$

Then, for any three objects $x, y, z$ of $\mathcal{D}$,

$$\mathcal{D}(x, y) \otimes \mathcal{D}(y, z) = \left(\text{colim}_{i}(i,F(i)(x')=x,F(i)(y')=y)D(i)(x', y')\right) \otimes \left(\text{colim}_{i}(i,F(i)(y')=y,F(i)(z')=z)D(i)(y', z')\right) \simeq \text{colim}_{i}(i,F(i)(x')=x,F(i)(y')=y,F(i)(z')=z)D(i)(x', y') \otimes D(i)(y', z') .$$

The inclusion functor $(i,F(i)(x') = x, F(i)(y') = y, F(i)(z') = z) \mapsto (i,F(i)(x') = x, F(i)(y') = y, i, F(i)(y') = y, F(i)(z') = z)$ is final. Thus

$$\mathcal{D}(x, y) \otimes \mathcal{D}(y, z) \simeq \text{colim}_{i}(i,F(i)(x')=x,F(i)(y')=y,F(i)(z')=z)D(i)(x', y') \otimes D(i)(y', z') .$$

Besides, the following cocone

$$D(i)(x', y') \otimes D(i)(y', z') \xrightarrow{m_{x'y'z'}} D(i)(x', z') \xrightarrow{\eta_{x'y'z'}} \mathcal{D}(x, y),$$

induces a composition morphism $m_{x'y'z'}^{\mathcal{D}} : \mathcal{D}(x, y) \otimes \mathcal{D}(y, z) \to \mathcal{D}(x, z)$. Moreover, the composite map

$$1 \xrightarrow{\eta_{x'y'}} D(i)(x', y') \xrightarrow{F(i)_{x'y'}} \mathcal{D}(x, x)$$

does not depend on the choice of $i$ and $x' \in \text{Ob}(D(i))$ such that $F(i)(x') = x$. Then, it defines a morphism $\eta_x : 1 \to \mathcal{D}(x, x)$. Similar arguments about final diagrams as those used above show that $\eta_x$ is a unit for the composition and that the composition is associative. We thus have defined the structure of an $E$-category on $\mathcal{D}$. Finally, it is straightforward to prove that $\mathcal{D}$ equipped with this structure is the colimit of the diagram $D$. 

**Theorem 1.** Let $\lambda$ be regular cardinal. Suppose that the category $E$ is a $\lambda$-presentable bilinear monoidal category. Then the category $\text{Cat}_E$ is $\lambda$-presentable.

**Proof.** It follows from the fact that the monad $O \circ T$ preserves filtered colimits. 

**Definition 7.** We denote by $*_1$ the $E$-enriched category with one object $0$ such that

$$*_1(0, 0) = 1_E.$$  

**Definition 8.** For any object $X$ of $E$, let $[1]_X$ be the $E$-category with two objects $0$ and $1$ such that

$$[1]_X(0, 0) = [1]_X(1, 1) = 1, \quad [1]_X(0, 1) = X, \quad [1]_X(1, 0) = 0.$$  

This defines a functor $[1] : E \to \text{Cat}_E$. 

COMPOSITION + HOMOTOPY = CUBES 7
1.4. Adjunction between categories of enriched categories. Let \( G \) be a lax monoidal functor from \((E, \otimes, 1)\) to \((F, \otimes, 1)\). Since \( G \) is monoidal one can define a functor from \( \text{Cat}_E \) to \( \text{Cat}_F \) also denoted \( G \) such that for any \( E \)-category \( (\mathcal{C}, m, u) \):

- \( \text{Ob}(G(\mathcal{C})) = \text{Ob}(\mathcal{C}) \),
- \( G(\mathcal{C})(x, y) = G(\mathcal{C}(x, y)) \),
- the composition is defined as follows

\[
G(\mathcal{C}(x, y)) \otimes G(\mathcal{C}(y, z)) \to G(\mathcal{C}(x, y) \otimes \mathcal{C}(y, z)) = G(\mathcal{C}(x, z)).
\]

**Proposition 3.** Let \( L \dashv R \) be a monoidal adjunction between two monoidal categories \((E, \otimes, 1)\) and \((F, \otimes, 1)\). Then, the extended functor \( L : \text{Cat}_E \to \text{Cat}_F \) is left adjoint to the extended functor \( R : \text{Cat}_F \to \text{Cat}_E \).

**Lemma 5.** The unit and the counit of the adjunction are monoidal natural transformations. In other words, for any objects of \( E \) \( X \) and \( Y \) and any objects of \( F \) \( A \) and \( B \), the following diagrams commute.

\[
\begin{array}{ccc}
X \otimes Y & \xrightarrow{\eta_{X \otimes Y}} & RL(X \otimes Y) \\
\eta_{X \otimes Y} & & \downarrow \cong \\
RL(X) \otimes RL(Y) & \xrightarrow{\eta_L \otimes \eta_Y} & LRL(X) \otimes LRL(Y) \\
\end{array}
\]

\[
\begin{array}{ccc}
LR(A \otimes LR(B) & \xrightarrow{\epsilon_{A \otimes B}} & A \otimes B \\
\epsilon_{A \otimes B} & & \downarrow \cong \\
LR(A) \otimes B & \xrightarrow{\epsilon_L \otimes \eta_Y} & L(X) \otimes L(Y)
\end{array}
\]

**Proof.** Let us prove that the first diagram commutes. Proving that the second diagram commutes may be done using the same arguments. The composite morphism \( X \otimes Y \to RL(X) \otimes RL(Y) \to RL(X \otimes Y) \) corresponds to a morphism \( f : L(X \otimes Y) \to L(X \otimes Y) \) through the adjunction \( L \dashv R \).

This morphism \( f \) is the identity of \( L(X \otimes Y) \) because the following diagram commutes.

\[
\begin{array}{ccc}
L(X \otimes Y) & \xrightarrow{L(\eta_{X \otimes Y})} & L(RL(X \otimes RL(Y)) \\
\downarrow \cong & & \downarrow \cong \\
L(X) \otimes L(Y) & \xrightarrow{L(\eta_X) \otimes L(\eta_Y)} & LRL(X) \otimes LRL(Y) \\
\end{array}
\]

\[
\begin{array}{ccc}
LRL(X) \otimes LRL(Y) & \xrightarrow{\epsilon_L(X) \otimes \epsilon_L(Y)} & L(X) \otimes L(Y) \\
\end{array}
\]

The fact that the right square commutes is a consequence of Lemma 1. Finally, since \( f \) is the identity of \( L(X \otimes Y) \), then the composite morphism \( X \otimes Y \to RL(X) \otimes RL(Y) \to RL(X \otimes Y) \) is the unit \( \eta_{X \otimes Y} \). □

**Proof of Proposition 3.** By Lemma 5, for any \( E \)-category \( (\mathcal{C}, m, u) \), the map \( \eta_\mathcal{C} : \mathcal{C} \to RL(\mathcal{C}) \) is indeed a functor of \( E \)-categories. Similarly, for any \( F \)-category \( (\mathcal{C}', m', u') \), the map \( \epsilon_{\mathcal{C}'} : LR(\mathcal{C}') \to \mathcal{C}' \) is indeed a functor of \( F \)-categories. It is then straightforward to prove that the composite functors

\[
L(\mathcal{C}) \xrightarrow{L(\eta_\mathcal{C})} LRL(\mathcal{C}) \xrightarrow{\epsilon_L(\mathcal{C})} L(\mathcal{C}'),
\]

\[
R(\mathcal{C}') \xrightarrow{\eta_R(\mathcal{C}')} RLR(\mathcal{C}') \xrightarrow{\epsilon_R(\mathcal{C}')} R(\mathcal{C}'),
\]

are respectively the identity of \( L(\mathcal{C}) \) and the identity of \( R(\mathcal{C}') \). □

Suppose that the monoidal category \( E \) is bilinear. We know that the adjunction \( i \dashv S \) relating \( E \) to sets is monoidal. Thus, it extends to an adjunction also denoted \( i \dashv S \) which relates \( E \)-categories to small categories.

\[
\text{Cat} \xrightarrow{i} \text{Cat}_E \xrightarrow{S} \text{Cat}
\]
**Definition 9.** Let \((\mathcal{C}, m, u)\) be an E-category. We call \(S(\mathcal{C})\) the underlying category of \(\mathcal{C}\).

**1.5. Monoidal model categories.**

**Definition 10 (Monoidal model category).** Let \((E, \otimes, \mathds{1})\) be a monoidal category equipped with a model structure. It is said to be a monoidal model category if
- the monoidal structure is bilinear,
- the monoidal unit \(\mathds{1}\) is cofibrant,
- for any cofibrations \(f : X \to X'\) and \(g : Y \to Y'\), the morphism \(X' \otimes Y \sqcup X \otimes Y' \to X' \otimes Y'\) is a cofibration; moreover, this is a weak equivalence whenever either \(f\) or \(g\) is.

**Proposition 4.** Let \((E, \otimes, \mathds{1})\) be a monoidal model category. For any cofibrant object \(X\) and for any weak equivalence \(Y \to Y'\) between cofibrant objects, the morphism \(X \otimes Y \to X \otimes Y'\) is a weak equivalence between cofibrant objects as well as the morphism \(Y \otimes X \to Y' \otimes X\).

**Proof.** It is a consequence of the definition of a monoidal model category and of K. Brown’s lemma.

**Lemma 6 (K. Brown’s lemma).** Let \(F : C \to D\) be a functor between model categories. Suppose that \(F\) sends acyclic cofibrations between cofibrant objects to weak equivalences. Then it preserves weak equivalences between cofibrant objects.

**Proof.** Let \(f : X \to Y\) be a weak equivalence between two cofibrant objects of \(C\). Consider the following factorisation of the map \((f, \text{Id}_Y)\)

\[X \sqcup Y \xrightarrow{} Z \xrightarrow{\sim} Y,\]

by a cofibration followed by a weak equivalence. Since both \(X\) and \(Y\) are cofibrant, then the morphisms \(X \to Z\) and \(Y \to Z\) are both acyclic cofibrations. Then, the morphisms \(F(X) \to F(Z)\) and \(F(Y) \to F(Z)\) are weak equivalences. So, by the 2-out-of-3 rule, the morphism \(F(Z) \to F(Y)\) is a weak equivalence. So the composite morphism \(F(X) \to F(Z) \to F(Y)\) is also a weak equivalence. □

**Corollary 1.** Let \((E, \otimes, \mathds{1})\) be a monoidal model category. Let \(X \to Y\) be a weak equivalence between cofibrant objects of \(E\). Then, the morphism \(X^\otimes n \to Y^\otimes n\) is a weak equivalence between cofibrant objects for any integer \(n \in \mathbb{N}\).

**Proof.** It follows from an induction: if \(X^\otimes n \to Y^\otimes n\) is a weak equivalence between cofibrant objects, then

\[X^\otimes n \otimes X \to X^\otimes n \otimes Y \to Y^\otimes n \otimes Y\]

is a sequence of weak equivalences between cofibrant objects by Proposition 4. □

If \((E, \otimes, \mathds{1})\) is a monoidal model category, then we can define a tensor product on the homotopy category \(Ho(E)\) as follows:

\[X \otimes_{Ho(E)} Y := \pi(QX \otimes QY),\]

where \(QX\) and \(QY\) are cofibrant replacement of \(X\) and \(Y\) and \(\pi\) is the localisation functor \(\pi : E \to Ho(E)\).

**Proposition 5.** [Hov99, Theorem 4.3.2] This tensor products is part of a bilinear monoidal structure on the category \(Ho(E)\). Moreover, the localisation functor \(\pi : E \to Ho(E)\) is lax monoidal.

**Definition 11.** Let \((E, \otimes, \mathds{1})\) and \((F, \otimes, \mathds{1})\) be two monoidal model categories. A Quillen monoidal adjunction relating \(E\) to \(F\) is an adjunction

\[\begin{array}{ccc}
E & \xrightarrow{L} & F \\
\xleftarrow{R} & & \\
\end{array}\]

which is both a Quillen adjunction and a monoidal adjunction.

**Lemma 7.** Let us use the notations of the above definition. Then the left derived functor \(LL : Ho(E) \to Ho(F)\) is strong monoidal. Moreover, the canonical natural transformation from \(LL \circ \pi_{Ho(E)}\) to \(\pi_{Ho(E)} \circ L\) is a monoidal natural transformation.
1.6. Enriched, tensored an cotensored category. In this section, we recall the definition of tensored-cotensored-enriched category over a not necessarily symmetric monoidal category.

Definition 12 ((Co)tensorisation). Let \((E, \otimes, 1)\) be a monoidal category and let \(C\) be a category.

\(\triangleright\) A tensorisation of \(E\) on \(C\) is a functor
\[-\triangleleft: C \times E \rightarrow C\]

together with functorial isomorphisms
\[
\begin{align*}
X \triangleleft (A \otimes B) &\simeq (X \triangleleft A) \triangleleft B, \\
X \triangleleft 1 &\simeq X,
\end{align*}
\]
for any \(X \in C\), any \(A, B \in E\); these functors are compatible with the monoidal structure of \(E\) in the sense that the following diagrams are commutative:

\[
\begin{align*}
\begin{array}{ccc}
(X \triangleleft A) \triangleleft B & \rightarrow & X \triangleleft (A \otimes (B \otimes C)) \\
\downarrow & & \downarrow \\
(X \triangleleft A) \triangleleft (B \otimes C) & \rightarrow & X \triangleleft (A \otimes (B \otimes C)),
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ccc}
(X \triangleleft 1) \triangleleft A & \rightarrow & X \triangleleft (1 \otimes A) \\
\downarrow & & \downarrow \\
X \triangleleft A & \rightarrow & X \triangleleft A.
\end{array}
\end{align*}
\]

\(\triangleright\) A cotensorisation of \(E\) on \(C\) is a functor:
\[
\langle-,-\rangle: E^{op} \times C \rightarrow C
\]

together with functorial isomorphisms
\[
\begin{align*}
\langle A \otimes B, X \rangle &\simeq \langle A(B, X) \rangle, \\
\langle 1, X \rangle &\simeq X,
\end{align*}
\]
such that the duals of the above diagrams are commutative.

Definition 13 (Category tensored-cotensored-enriched over a monoidal category). Let \((E, \otimes, 1)\) be a monoidal category and let \(C\) be a category. We say that \(C\) is tensored-cotensored-enriched over \(E\) if there exists three functors:
\[
\begin{align*}
\{ -,- \}: E^{op} \times C &\rightarrow E \\
-\triangleleft -: C \times E &\rightarrow C \\
\langle -,- \rangle: E^{op} \times C &\rightarrow C
\end{align*}
\]

together with functorial isomorphisms
\[
\text{hom}_C(X \triangleleft A, Y) \simeq \text{hom}_E(A, \{X, Y\}) \simeq \text{hom}_C(X, \langle A, Y \rangle),
\]
for any \(X, Y \in C\) and any \(A \in E\), such that \(-\triangleleft-\) is a tensorisation of \(E\) on \(C\) or, equivalently, such that \(\langle-,-\rangle\) is a cotensorisation.

Definition 14. Let \(C\) be a category tensored-cotensored-enriched over \(E\). We denote by \(C^{[E]}\) the \(E\)-category whose objects are the objects of \(C\) and such that for any \(x, y \in C\),
\[
C^{[E]}(x, y) = \{x, y\}.
\]
The composition is given by the adjoint of the following composite map
\[
x \triangleleft (\{x, y\} \otimes \{y, z\}) \simeq (x \triangleleft \{x, y\}) \triangleleft \{y, z\} \rightarrow y \triangleleft \{y, z\} \rightarrow z,
\]
and the unit \(1 \rightarrow \{x, x\}\) by the adjoint of the map
\[
x \triangleleft 1 \simeq x.
\]

Proposition 6. Suppose that \((E, \otimes, 1)\) is a bilinear monoidal category. Then the category \(S(C^{[E]}))\) is canonically isomorphic to \(C\).
Proof. The category $S(C^{[E]})$ has the same object as $C$. Moreover, for any two such objects $x, y$

$$S((x, y)) \simeq \text{hom}_E(1_E, \{x, y\}) \simeq \text{hom}_C(x \triangleleft 1_E, y) \simeq \text{hom}_C(x, y).$$

\[ \square \]

Now, consider two monoidal categories $(E, \otimes, 1)$ and $(F, \otimes, 1)$ and a monoidal adjunction

$$E \xleftarrow{L} \xrightarrow{R} F.$$  

Proposition 7. Let $C$ be a category tensored-cotensored-enriched over $F$. Then $C$ is also tensored-cotensored-enriched over $E$.

Proof. The enrichment is given by $R(\{−, −\})$, the tensoring by $− \odot L(−)$ and the cotensoring by $[L(−), −].$ \[ \square \]

1.7. Model category enriched over a monoidal model category. In this subsection $(E, \otimes, 1)$ is a monoidal model category.

Definition 15 (Homotopical enrichment). Let $M$ be a model category. We say that $M$ is homotopically enriched over $E$ if it enriched over $E$ and if for any cofibration $f : X \to X'$ in $M$ and any fibration $g : Y \to Y'$ in $M$, the morphism in $E$:

$$\{X', Y\} \to \{X', Y\} \times_{\{X, Y\}} \{X, Y\}$$

is a fibration. Moreover, we require this morphism to be a weak equivalence whenever $f$ or $g$ is a weak equivalence.

Definition 16. A $E$-model category is a model category $M$ tensored-cotensored-enriched over $E$ such that the enrichment of $M$ over $E$ is homotopical.

One can show that the enrichment is homotopical if for any cofibration $f : X \to Y$ in $M$ and any cofibration $g : A \to B$ in $E$, the morphism in $M$:

$$X \triangleleft B \sqcup_{X \triangleleft A} Y \triangleleft A \to Y \triangleleft B$$

is a cofibration, and it is a weak equivalence whenever $f$ or $g$ is a weak equivalence.

Proposition 8. Consider a monoidal Quillen adjunction $E \dashv F$. If $M$ is an $F$-model category, then it has a canonical structure of a $E$-model category.

Proof. We know from Proposition 7 that $M$ is tensored-cotensored-enriched over $E$. The enrichment is homotopical because the functor $E \to F$ is a left Quillen functor. \[ \square \]

1.8. Categorical model structure. This section recalls model structures on categories enriched over a monoidal model category $E$ in the vein of [Lur09, A.3.1]. There are other results related to this subject. See for instance [BM13], [Cav14].

Let $(E, \otimes, 1)$ be a monoidal model category.

Definition 17. Let $\pi_0$ be the following composite functor

$$E \xrightarrow{\pi} \text{Ho}(E) \xrightarrow{S} \text{Set}.$$  

where $\text{Ho}(E)$ is the homotopy category of $E$ and $S$ is the Yoneda functor $\text{hom}_{\text{Ho}(E)}(1, −)$. Since the localization functor $E \to \text{Ho}(E)$ and $S : \text{Ho}(E) \to \text{Set}$ are both lax monoidal, then $\pi_0$ is lax monoidal.

Definition 18. If it exists, the categorical model structure on the category $\text{Cat}_E$ is the model structure such that

1. the weak equivalences are the functors $F$ from $(\mathcal{C}, \mu, u)$ to $(\mathcal{C}', \mu', u')$ such that $F_{x,y} : \mathcal{C}(x, y) \to \mathcal{C}'(F(x), F(y))$ is a weak equivalence of $E$ for any $x, y \in \text{Ob}(\mathcal{C})$, and such that $\pi_0(F) : \pi_0(\mathcal{C}) \to \pi_0(\mathcal{C}')$ is an essentially surjective functor.

2. the (large) set of cofibrations is the smallest subset of the set of functors which is stable under pushouts, transfinite compositions and retracts and which contains the functor $\emptyset \to \ast_1$ and, for any cofibration $f : X \to Y$ of $E$, the functor $[1]_f : [1]_X \to [1]_Y$. 

7
Theorem 2. ([Lur09, A.3.2.4]) Suppose that \((E, \otimes, 1)\) is a combinatorial monoidal model category such that every object is cofibrant and such that weak equivalences are stable under filtered colimits. Then, the category \(\text{Cat}_E\) admits the categorical model structure which is moreover left proper and combinatorial. A set of generating cofibration is
\[
\{\emptyset \rightarrow *\} \sqcup \{[1]/f \text{ is a generating cofibration of } E\}.
\]

Remark 1. Lurie assumed \(E\) to be symmetric monoidal in his result. The proof does not rely on this assumption. Actually, this result is a consequence (after some work) of the theory of combinatorial model categories; see for instance [Ros09].

Consider a monoidal Quillen adjunction between monoidal model categories.
\[
E \xleftarrow{L} \xrightarrow{R} F.
\]

Proposition 9. Suppose that the category \(\text{Cat}_E\) and \(\text{Cat}_F\) have categorical model structures. Suppose moreover that the functor \(L : E \rightarrow F\) preserves weak equivalences (for instance if any object of \(E\) is cofibrant). Then the adjunction \(\text{Cat}_E \xleftarrow{L} \xrightarrow{R} \text{Cat}_F\) is a Quillen adjunction.

Proof. It is straightforward to prove that \(R : \text{Cat}_F \rightarrow \text{Cat}_E\) preserves acyclic fibrations. So \(L : \text{Cat}_E \rightarrow \text{Cat}_F\) preserves cofibrations. Let us prove that it preserves weak equivalences. Let \(F : (\mathcal{C}, \gamma) \rightarrow (\mathcal{D}, \gamma)\) be weak equivalence of \(\text{Cat}_E\). Since the functor \(L : E \rightarrow F\) preserves weak equivalences, then the map \(L(F) : (L\mathcal{C})(x, y) \rightarrow (L\mathcal{D})(F(x), F(y))\) is a weak equivalence for any objects \(x, y \in \text{Ob}(\mathcal{C})\). Besides, the functors \(\pi_0 : E \rightarrow \text{Set}\) and \(\pi_0 \circ L : E \rightarrow \text{Set}\) are lax monoidal and there exists a monoidal natural transformation between them (Lemma 7). We thus obtain a natural transformation between the functor \(\pi_0 : \text{Cat}_E \rightarrow \text{Cat}\) and the functor \(\pi_0 \circ L : \text{Cat}_E \rightarrow \text{Cat}\). So, we have the following commuting square diagram of categories
\[
\begin{array}{ccc}
\pi_0(\mathcal{C}) & \longrightarrow & \pi_0L\mathcal{C} \\
\downarrow \pi_0(F) & & \downarrow \pi_0(F) \\
\pi_0\mathcal{D} & \longrightarrow & \pi_0L\mathcal{D},
\end{array}
\]
whose horizontal arrows are isomorphisms on objects. Since \(\pi_0F\) is essentially surjective, then \(\pi_0LF\) is also essentially surjective. 

2. Reduced Cubical Sets and Cubical Sets with Connections

This section deals with cubical sets. We describe two categories: the category of reduced cubical sets and the category of cubical sets with connections. The first one is described in [Cis06, §8.3], in [Jar02] and in [Jar06] and the second one in [Mal09] and [Cis14, Example 1.6].

Remark 2. There are many different categories called cubical sets. See for instance [GM03], and [Isa09].

2.1. Segments. We give here the notions of a segment and of an interval inspired from [BM06]. Note that a segment (resp. an interval) in the sense of [BM06] is a monoidal segment (resp. monoidal interval) for us.

Definition 19. Let \((E, \otimes, 1)\) be a monoidal category. A segment of \(E\) is an object \(H\) of \(E\) together with maps
\[
1 \sqcup 1 \xrightarrow{(\delta^0_H, \delta^1_H)} H \xrightarrow{\sigma_H} 1
\]
which factorizes the morphism \(1 \sqcup 1 \rightarrow 1\). Moreover, in a monoidal model category, an interval is a segment such that the map \((\delta^0_H, \delta^1_H)\) is a cofibration and the map \(\sigma_H\) is a weak equivalence.

Notice that one can also define a segment as a functor \(F : \Delta_{\leq 1} \rightarrow E\) such that \(F(\Delta_0) = 1\).
Definition 20. Let \((E, \otimes, 1)\) be a monoidal category. A monoidal segment \((H, \delta^0_H, \delta^1_H, \sigma_H, \gamma_H)\) is a segment

\[
1 \sqcup 1 \xrightarrow{(\delta^0_H, \delta^1_H)} H \xrightarrow{\sigma_H} 1
\]

together with a map \(\gamma_H : H \otimes H \to H\) such that

\begin{itemize}
  \item the product \(\gamma_H\) is associative, that is \(\gamma_H(Id_H \otimes \gamma_H) = \gamma_H(\gamma_H \otimes Id_H)\),
  \item the product has a unit given by \(\delta^0_H : 1 \to H\),
  \item the morphism \(\sigma_H\) is a morphism of monoids,
  \item the morphism \(\delta^1_H : 1 \to H\) is absorbant, that is the following diagram commutes
\end{itemize}

\[
\begin{array}{ccc}
H & \xrightarrow{\sigma_H} & H \\
\downarrow{\gamma_H} & & \downarrow{\sigma_H} \\
1 & \xrightarrow{\delta^1_H} & H
\end{array}
\]

A morphism of monoidal segments from \((H, \delta^0_H, \delta^1_H, \sigma_H, \gamma_H)\) to \((H', \delta^0_{H'}, \delta^1_{H'}, \sigma_{H'}, \gamma_{H'})\) is a morphism of segment \(f : H \to H'\) such that \(f \gamma_H = \gamma_{H'}(f \otimes f)\). In a monoidal model category, a monoidal segment which is also an interval is called a monoidal interval.

2.2. **Cubes.** This subsection deals with the categories of reduced cubes and cubes with connections.

For any integers \(n \in \mathbb{N}\), we denote by \(\square^n\) the \(n\)-times product of the set \(\{0, 1\}\)

\[
\square^n := \{0, 1\}^n.
\]

Notice that the full subcategory of the category of sets spanned by the objects \(\square^n\) has a symmetric monoidal structure given by the cartesian product. Consider the following functions

\[
\begin{align*}
\delta^0 &: \square^0 \to \square^1, \\
\delta^1 &: \square^0 \to \square^1, \\
\sigma &: \square^1 \to \square^0, \\
\gamma &: \square^2 \to \square^1,
\end{align*}
\]

defined by

\[
\begin{align*}
\delta^0(*) &= 0, \\
\delta^1(*) &= 1, \\
\sigma(i) &= * \forall i \in \square^1, \\
\gamma(i, j) &= \max(i, j) \forall i, j \in \square^1.
\end{align*}
\]

Tensoring \(\delta^0, \delta^1, \sigma\) and \(\gamma\) with identities, one obtains the following maps

\[
\begin{align*}
\delta^0_i &= Id^i \times \delta^0 \times Id^{n-i} : \square^n \simeq \square^i \times \square^0 \times \square^{n-i} \to \square^{n+1}, \\
\delta^1_i &= Id^i \times \delta^1 \times Id^{n-i} : \square^n \simeq \square^i \times \square^0 \times \square^{n-i} \to \square^{n+1}, \\
\sigma_i &= Id^i \times \sigma \times Id^{n-i-1} : \square^n \simeq \square^i \times \square^1 \times \square^{n-i-1} \to \square^{n-1}, \\
\gamma_i &= Id^i \times \gamma \times Id^{n-i-1} : \square^n \simeq \square^i \times \square^2 \times \square^{n-i-2} \to \square^{n-1},
\end{align*}
\]
for any $n$. The maps $\delta^0_i$ and $\delta^1_i$ are called cofaces, the maps $\sigma_i$ are called codegeneracies and the maps $\gamma_j$ are called connections. These functions satisfy the following relations

\[
\begin{align*}
\delta^c_i \delta^c_j &= \delta^c_{i+1} \delta^c_j, & \text{if } i \leq j , \\
\sigma_i \sigma_j &= \sigma_j \sigma_{i+1}, & \text{if } i \geq j , \\
\sigma_i \delta^0_j &= \delta^c_{i-1} \sigma_j, & \text{if } i < j , \\
\sigma_i \delta^1_j &= \text{Id}, \\
\sigma_i \gamma_j &= \delta^c_{i-1} \gamma_j, & \text{if } i < j , \\
\gamma_i \gamma_j &= \gamma_{i+1} \gamma_j, & \text{if } i > j , \\
\gamma_i \delta^0_j &= \gamma_i \delta^c_{j+1} = \text{Id}, \\
\gamma_i \delta^1_j &= \gamma_i \delta^c_{j+1} = \delta^c_j \sigma_i, \\
\gamma_i \delta^0_j &= \delta^c_{i-1} \gamma_j, & \text{if } i+1 < j , \\
\gamma_i \delta^1_j &= \delta^c_{i-1} \gamma_{i-1}, & \text{if } i > j . 
\end{align*}
\]

**Definition 21.** The category of reduced cubes $\square_r$ is the subcategory of sets whose objects are the sets $\square^n_n$ for $n \in \mathbb{N}$ and whose morphisms are generated by the cofaces $\delta^0_i$ and $\delta^1_i$ and the codegeneracies $\sigma_i$. The category of cubes with connections $\square_c$ is the subcategory of sets whose objects are the sets $\square^n_n$ for $n \in \mathbb{N}$ and whose morphisms are generated by the cofaces $\delta^0_i$ and $\delta^1_i$, the codegeneracies $\sigma_i$ and the connections $\gamma_j$.

**Remark 3.** Beware! The category of cubes with connections that we consider is not the same as the category considered by [GM03], but it is the category considered in [Mal09].

The rewriting rules given above give us the following proposition.

**Proposition 10.** [GM03] Any morphism in the category $\square_r$ may be uniquely written as a sequence

\[\delta^c_{i_1} \cdots \delta^c_{i_n} \sigma_{j_1} \cdots \sigma_{j_m},\]

where $i_1 > \cdots > i_n$ and $j_1 < \cdots < j_m$. Similarly, any morphism in the category $\square_c$ may be uniquely written as a sequence

\[\delta^c_{i_1} \cdots \delta^c_{i_n} \gamma_{j_1} \cdots \gamma_{j_m} \sigma_{k_1} \cdots \sigma_{k_l},\]

where $i_1 > \cdots > i_n$, $j_1 < \cdots < j_m$ and $k_1 < \cdots < k_l$.

**Proposition 11.** Both the category of reduced cubes and the category of cubes with connections inherit a strict monoidal structure from the cartesian product of sets. In both cases, the unit is $\square^0$ and

\[\square^n \otimes \square^m = \square^{n+m},\]

for any integers $n, m$.

**Proof.** The proposition follows from long but straightforward checkings. \qed

**Remark 4.** The monoidal category $(\square_r, \otimes, \square^0)$ is not symmetric monoidal. Indeed, the following diagram does not commute

\[
\begin{array}{ccc}
\square^1 & \rightarrow & \square^0 \otimes \square^1 \\
\downarrow & & \downarrow \delta^0 \otimes \text{Id} \\
\square^1 \otimes \square^0 & \rightarrow & \square^1 \otimes \square^1 = \square^2
\end{array}
\]

For the same reason, the monoidal category $(\square_c, \otimes, \square^0)$ is not symmetric monoidal.
Proposition 12 (Cisinski). Let \((\mathcal{C}, \otimes, 1)\) be a monoidal category. The category of strong monoidal functors from \(\square_r\) to \(\mathcal{C}\) and monoidal natural transformations is equivalent to the category of segments of \(\mathcal{C}\). Similarly, the category of strong monoidal functors from \(\square_c\) to \(\mathcal{C}\) and monoidal natural transformations is equivalent to the category of monoidal segments of \(\mathcal{C}\).

Proof. The first statement may be found in [Cis06, Proposition 8.4.6]. The proof of the second statement relies on the same arguments that we give here. Consider a strong monoidal functor \(F\) from \(\square_c\) to \(\mathcal{C}\). Then, \(F(\square^1)\) has the structure of a monoidal segment. Conversely, any monoidal segment \(H\) of \(\mathcal{C}\) induces a strong monoidal functor \(L^H_c\) from \(\square_c\) to \(\mathcal{C}\) defined by

\[
L^H_c(\square^n) = H \otimes (H \otimes (\cdots \otimes H)),
\]

where \(H\) appears \(n\) times in this formula. \(\square\)

2.3. Cubical sets.

Definition 22. The category of reduced cubical sets \(\square_r-\text{Set}\) is the category of presheaves on \(\square_r\), that is functors from \(\square_r^{pp}\) to \(\text{Set}\). The category of cubical sets with connections \(\square_c-\text{Set}\) is the category of presheaves on \(\square_c\), that is functors from \(\square_c^{pp}\) to \(\text{Set}\).

Notations.

- We will denote by \(\square_r[n]\) (resp. \(\square_c[n]\)) the Yoneda embedding of \(\square^n\) in the category \(\square_r-\text{Set}\) (resp. \(\square_c-\text{Set}\)).
- For any reduced cubical set or any cubical set with connections \(X\) and any integer \(n \in \mathbb{N}\), \(X(n)\) will denote the set \(X(\square^n)\).

Example 1. We will often deal with the following cubical sets.

- We denote by \(\partial\square_r[n]\) the union of all the faces of \(\square_r[n]\), that is

\[
\partial\square_r[n](\square^m) = \{ f \in \text{hom}_{\square_c}(\square^m, \square^n) \mid \text{there is a factorisation } f = \delta_{n-1}^i g \}.
\]

- For any \((\epsilon, i) \in \{0, 1\} \times \{1, \ldots, n\}\), let \(\partial^\epsilon,i[n]\) be the \(\epsilon\)-cap of \(\partial\square_r[n]\), that is

\[
\partial^\epsilon,i[n](\square^m) = \{ f \in \text{hom}_{\square_c}(\square^m, \square^n) \mid \text{there is a factorisation } f = \delta_{n-1}^{\epsilon,i'} g \text{ with } (\epsilon', i') \neq (\epsilon, i) \}.
\]

- One can define in a similar way the cubical sets with connections \(\partial\square_c[n]\) and \(\partial^\epsilon,i[n]\).

Remark 5. Since the categories \(\square_r-\text{Set}\) and \(\square_c-\text{Set}\) are presentable and since the bifunctor \(- \otimes -\) commutes with colimits on both sides, then one can show that the monoidal categories \(\square_r-\text{Set}\) and \(\square_c-\text{Set}\) are biclosed, that is the functor \(- \otimes X\) and the functor \(X \otimes -\) have both right adjoints.

Proposition 13. [Cis06, 8.4.23] Let \((\mathcal{C}, \otimes, 1)\) be a monoidal complete bilinear category. The category of cocontinuous strong monoidal functors from \(\square_r-\text{Set}\) to \(\mathcal{C}\) with monoidal natural transformations is equivalent to the category of segments of \(\mathcal{C}\). Similarly, the category of cocontinuous strong monoidal functors from \(\square_c-\text{Set}\) to \(\mathcal{C}\) with monoidal natural transformations is equivalent to the category of monoidal segments of \(\mathcal{C}\).

Proof. It is a straightforward consequence of Proposition 12. \(\square\)

Moreover, any functor \(L_r : \square_r-\text{Set} \to \mathcal{C}\) which is cocontinuous has a right adjoint. Then, any adjunction

\[
\square_r-\text{Set} \xrightarrow{L_r} \mathcal{C},
\]

is essentially determined by the image under the functor \(L_r\) of \(\square_r[1]\). The same result holds if we replace the category \(\square_c-\text{Set}\) by the category \(\square_c-\text{Set}\).

Notations. Let \((\mathcal{C}, \otimes, 1)\) be a bilinear monoidal category. Consider a segment \(H\) in \(\mathcal{C}\). The adjunction relating \(\mathcal{C}\) to reduced cubical sets induced by this segment will be denoted \(L^H_c - R^H_c\). If \(H\) is a monoidal segment, the induced adjunction relating \(\mathcal{C}\) to cubical sets with connections will be denoted \(L^H_c - R^H_c\).

Remark 6. Notice that for any monoidal segment \(H\),

\[
L^H_c \circ L_r[1] \simeq L^H_r.
\]
Lemma 8. [Cis06, Lemme 8.4.36] For any integer $n \in \mathbb{N}$ and for any integers $i, j \in \mathbb{N}$ such that $i + j = n$, we have
\[
\partial \square_r[n] \simeq \partial \square_r[i] \otimes \square_r[j] \cup \partial \square_r[j] \otimes \partial \square_r[i] \square_r[i] \otimes \partial \square_r[j].
\]
Moreover,
\[
\begin{align*}
\Pi_+^n[n] & \simeq \partial \square_r[n - 1] \otimes \square_r[1] \cup \partial \square_r[n - 1] \otimes \partial \square_r[0] \square_r[n - 1] \otimes \partial \square_r[0], \\
\Pi_0^1[n] & \simeq \square_r[1] \otimes \partial \square_r[n - 1] \cup \partial \square_r[0] \otimes \square_r[n - 1] \square_r[1] \otimes \partial \square_r[0] \cup \partial \square_r[n - 1] \otimes \partial \square_r[0], \\
\Pi_{\delta^i}^n[n] & \simeq \Pi_+^i[j] \otimes \square_r[j] \cup \Pi_{\delta^i}^n[j] \otimes \partial \square_r[j] \square_r[i] \otimes \partial \square_r[j].
\end{align*}
\]

The same results hold in the category of cubical sets with connections.

3. Homotopy theory of cubical sets

This section deals with the homotopy theories of reduced cubical sets and of cubical sets with connections. We first recall their model structures from the work of Cisinski and Jardine and give additional results on the model structure on cubical sets with connections. We then study the Reedy model structure on cocubicals and its link to functors from cubical sets (reduced or with connections) to any other monoidal model category.

3.1. Homotopy theory of reduced cubical sets. We know that the unit of the cartesian monoidal structure on simplicial sets is the final object $\Delta[0]$. Moreover, the map $\Delta[0] \sqcup \Delta[0] \to \Delta[0]$ may be factorised as follows
\[
\begin{array}{ccc}
\Delta[0] \sqcup \Delta[0] & \overset{\delta^0 \sqcup \delta^1}{\longrightarrow} & \Delta[1] \\
\downarrow & & \downarrow \\
\Delta[0] & \to & \Delta[0].
\end{array}
\]

By Proposition 13, this induces a strong monoidal cocontinuous functor $L_r^{\Delta[1]} : \Box_r \to \operatorname{sSet}$ and hence a monoidal adjunction between reduced cubical sets and simplicial sets
\[
\begin{array}{ccc}
\Box_r & \overset{L_r^{\Delta[1]}}{\longrightarrow} & \operatorname{sSet} \\
\downarrow & & \downarrow \\
\Box_r & \rightleftarrows & \operatorname{sSet}.
\end{array}
\]

One can transfer the usual model structure on simplicial sets to reduced cubical sets.

Theorem 3. [Cis06][Jar02] There exists a combinatorial proper model structure on reduced cubical sets such that
- the cofibrations are the monomorphisms,
- the weak equivalences are the morphisms $f$ such that $L_r^{\Delta[1]}(f)$ is a weak equivalence,
- the generating cofibrations are the injections $\partial(\square)_r[n] \hookrightarrow \square_r[n]$,
- the generating acyclic cofibrations are the injections $\Pi_{\delta^i}^n[n] \hookrightarrow \square_r[n]$, 
- the adjunction $L_r^{\Delta[1]} : \Box_r \to \operatorname{sSet}$ is a Quillen equivalence,
- $(\Box_r, \Box_r, \Box_r)$ is a monoidal model category,
- the functor $K_r^{\Delta[1]}$ preserves and reflects weak equivalences.

Remark 7. Jardine and Cisinski described independently two model structures on the category of cubical sets; see [Jar02] and [Cis06]. They are actually the same (see [Jar06]). Besides, the fact that this model structure is right proper is a direct consequence of Cisinski’s theory. However, this property seems hard to prove in Jardine’s framework which is more topological.

3.2. Homotopy theory of cubical sets with connections. Consider the following sequence of adjunctions
\[
\begin{array}{ccc}
\Box_r & \overset{L_{\Box_r}^{\Delta[1]}}{\longrightarrow} & \Box_r \\
\downarrow & & \downarrow \\
\Box_c & \overset{L_{\Box_c}^{\Delta[1]}}{\longrightarrow} & \operatorname{sSet}.
\end{array}
\]

One can also transfer the usual model structure on simplicial sets to cubical sets with connections.

Theorem 4 (After Cisinski). There exists a combinatorial proper model structure on the category $\Box_c \to \operatorname{sSet}$ such that
- cofibrations are monomorphisms,
\[\text{weak equivalences are maps } f \text{ such that } L^\Delta(1)[f] \text{ is a weak equivalence,}\]
\[\text{a set of generating cofibrations is given by the maps}\]
\[\{\partial \square[n] \rightarrow \square[n] \mid n \in \mathbb{N}\},\]
\[\text{sets of generating acyclic cofibrations is given by the maps}\]
\[\{\epsilon^n \mid n \in \mathbb{N}\}.\]

Moreover, this is a monoidal model category. Finally, the adjunction \(L_c^{\Delta[1]} \dashv R_c^{\Delta[1]}\) is a Quillen equivalence as well as the adjunction \(L_c^{\square[1]} \dashv R_c^{\square[1]}\).

**Proof.** We know from [Cis14] that there exists a monoidal cofibrantly generated model structure on \(\square_c\)-Set whose weak equivalences are maps \(f : X \rightarrow Y\) such that the induced morphism of simplicial sets
\[N(\square_c/X) \rightarrow N(\square_c/Y)\]
is a weak equivalence and which admits the generating cofibrations and the generating acyclic cofibrations that we gave in the theorem. Besides, in the category \(\square_c\) the category of subobjects of \(\square^n\) is equivalent to the \(n\)-times product of the poset \([1] = \{0 \rightarrow 1\}\) and its simplicial nerve is \(\Delta[1]^n\). This is actually \(L_c^{\Delta[1]}(\square[n])\). Since the category \(\square\) is a regular skeletal category in the sense of [Cis06, §8.2], by [Cis06, Proposition 8.2.28], we have a natural transformation \(N(\square_c/X) \rightarrow L_r^{\Delta[1]}(X)\) which is objectwise a weak equivalence. This shows that the weak equivalences are the maps \(f : X \rightarrow Y\) such that \(L_r^{\Delta[1]}(f)\) is a weak equivalence. Finally, we know that the category \(\square_c\) is a test category. Hence, the functor
\[X \in \square_c - \text{Set} \mapsto \square_c/X \in \text{Cat} \mapsto N(\square_c/X) \in \text{sSet}\]
induces an equivalence of categories between the homotopy category of cubical sets with connections and the homotopy category of simplicial sets. Since the morphism \(N(\square_c/X) \rightarrow L_r^{\Delta[1]}(X)\) is an equivalence for any object \(X\), then \(L_c^{\Delta[1]}\) is an equivalence of categories at the level of homotopy categories. \(\square\)

### 3.3. The Reedy model structure on cocubical objects

In this subsection and until the end of this section, what we describe holds in the category of reduced cubical sets and in the category of cubical sets with connections. Therefore, we will not use the indices \(c\) or \(r\) but use the notation \(\square\) and talk about cubical sets.

**Proposition 14.** The category \(\square\) has a Reedy structure such that
\[\text{the degree of } \square^n \text{ is } n,\]
\[\text{the degree raising morphisms are the composites of cofaces,}\]
\[\text{the degree lowering morphisms are the composites of codegeneracies (or the composites of codegeneracies and connections).}\]

Let \(E\) be a model category. We know that the category \(\text{Fun}(\square, E)\) of cocubical objects in \(E\) is equivalent to the category \(\text{Fun}_{\square}(\square-\text{Set}, E)\) of functors from cubical sets to \(E\) which preserve colimits. Therefore, we will often assimilate a cubical object to such a functor. We know that a cubical object \(F\) also induces a functor \(F^!\) from \(E\) to cubical sets defined by
\[F^!(X) := \text{hom}_E(F(-), X),\]
and which is right adjoint to \(F^!\). Besides a map \(F \rightarrow G\) induces a natural transformation \(G^! \rightarrow F^!\).

We can endow the category \(\text{Fun}(\square, E)\) of cocubical objects in \(E\) with the Reedy model structure (see for instance [Hov99, Theorem 5.2.5]), that is,
\[\text{the weak equivalences are the morphisms } F \rightarrow G \text{ such that } F(\square[n]) \rightarrow G(\square[n]) \text{ is a weak equivalence in } E \text{ for any } n \in \mathbb{N},\]
\[\text{the cofibrations (resp. acyclic cofibrations) are the morphisms } F \rightarrow G \text{ such that}\]
\[F(\square[n]) \cup_{\partial(\square[n])} G(\partial(\square[n])) \rightarrow G(\square[n])\]
is a cofibration (resp. an acyclic cofibration) in \(E\) for any \(n \in \mathbb{N}\.\]
**Proposition 15.** Let $F \to G$ be a Reedy cofibration of cocubical objects of $E$ and let $p : X \to Y$ be a cofibration of $E$. Then if one of these two maps is also a weak equivalence, then the morphism

$$G^i(X) \to G^i(Y) \times_{F^i(Y)} F^i(X)$$

is an acyclic fibration.

Equivalently, for any Reedy cofibration $F \to G$ and for any cofibration of cubical sets $A \to B$, the morphism in $E$

$$F(B) \cup_{F(A)} G(A) \to G(B)$$

is a cofibration and it is an acyclic cofibration if $F \to G$ is acyclic. In particular, a Reedy cofibrant functor $F$ preserves cofibrations.

**Proof of Proposition 15.** Consider a square diagram as follows

$$\begin{array}{ccc}
\partial(\square[n]) & \to & G^i(X) \\
\downarrow & & \downarrow \\
\square[n] & \to & G^i(Y) \times_{F^i(Y)} F^i(X).
\end{array}$$

It induces another square diagram in $E$

$$\begin{array}{ccc}
F(n) \cup_{\partial F(n)} \partial G(n) & \to & X \\
\downarrow & & \downarrow \\
G(n) & \to & Y.
\end{array}$$

This square diagram has a lifting because one of the vertical maps is a weak equivalence. So the first square diagram has also a lifting. \qed

**Corollary 2.** Let $F$ and $G$ be two Reedy cofibrant cocubical objects of $E$ and consider a weak equivalence $F \to G$. Then, for any cubical set $A$, the morphism $F(A) \to G(A)$ is a weak equivalence and for any fibrant object $X$ of $E$, the morphism $G^i(X) \to F^i(X)$ is a weak equivalence.

**Proof of Corollary 2.** It is a straightforward consequence of Lemma 6 and Proposition 15. \qed

### 3.4. Intervals and Quillen adjunctions.

Let $(E, \otimes, 1)$ be a monoidal model category. Let us choose a segment $1 \sqcup 1 \to H \to 1$ (or a monoidal segment in the case of cubical sets with connections). We know that such a segment induces a monoidal adjunction $L^H \dashv R^H$ relating cubical sets to $E$.

**Proposition 16.** The adjunction $L^H \dashv R^H$ is a Quillen adjunction if and only if $H$ is an interval, that is the morphism $1 \sqcup 1 \to H$ is a cofibration and the morphism $H \to 1$ is a weak equivalence.

**Lemma 9.** The functor $L^H$ preserves cofibrations if and only if the map $1 \sqcup 1 \to H$ is a cofibration.

**Proof.** If $L^H$ preserves cofibrations, then the map

$$1 \sqcup 1 \simeq L^H(\partial \square[1]) \to L^H(\square[1]) \simeq H$$

is a cofibration. Conversely, suppose that the map $1 \sqcup 1 \to H$ is a cofibration. Let us prove by induction that, for any integer $n \in \mathbb{N}$, the map

$$L^H(\partial \square[n]) \to L^H(\square[n])$$

is a cofibration. Since $E$ is a monoidal model category, the map $\emptyset \to 1$ is a cofibration. So the result holds for $n = 0$. Suppose that it holds for some integer $n$. Then the morphism

$$L^H(\partial \square[n + 1]) \simeq (1 \sqcup 1) \otimes L^H(\square[n]) \cup_{(1 \sqcup 1) \otimes L^H(\partial \square[n])} H \otimes L^H(\partial \square[n]) \to H \otimes L^H(\square[n]) \simeq L^H(\square[n + 1])$$

is a cofibration. So the result holds at the stage $n + 1$. \qed
Proof of Proposition 16. Suppose that \( H \) is an interval and let us prove that \( L^H \) is left Quillen. We already know from Lemma 9 that it preserves cofibrations. So it suffices to show that it preserves acyclic cofibrations. Let \( n \) be an integer, let \( 0 \leq i \leq n \) and let \( \epsilon \in \{0, 1\} \). We will denote the opposite sign of \( \epsilon \) by \( \overline{\epsilon} \). Then \( L^H(\sqcup^i,\epsilon[i]) \) is the colimit of the following diagram

\[
\begin{array}{ccc}
L^H(\partial(\Box[i-1])) \otimes 1 & \rightarrow & L^H(\partial(\Box[i-1])) \otimes H \\
\downarrow & & \downarrow \\
L^H(\Box[i-1]) \otimes 1 & & L^H(\Box[i-1]) \otimes H
\end{array}
\]

Since \( L^H(\partial(\Box[i-1])) \) is cofibrant by Lemma 9, and since the map \( 1 \rightarrow H \) is an acyclic cofibration, then the morphism \( L^H(\partial(\Box[i-1])) \otimes 1 \rightarrow L^H(\partial(\Box[i-1])) \otimes H \) is an acyclic cofibration. So, the map \( L^H(\Box[i-1]) \rightarrow L^H(\sqcup^i,\epsilon[i]) \) is also an acyclic cofibration. Besides, since \( H^{\otimes i-1} \) is cofibrant and since the map \( 1 \rightarrow H \) is an acyclic cofibration, then the map

\[
\delta_i^\Box : L^H(\Box[i-1]) \simeq H^{\otimes i-1} \otimes 1 \rightarrow H^{\otimes \overline{i}} \simeq L^H(\Box[i])
\]

is a weak equivalence. So the map \( L^H(\sqcup^i,\epsilon[i]) \rightarrow L^H(\Box[i]) \) is a weak equivalence. It is even an acyclic cofibration since the map \( \sqcup^i,\epsilon[i] \rightarrow \Box[i] \) is a cofibration and since \( L^H \) preserves cofibrations. Besides, \( L^H(\sqcup^i,\epsilon[n]) \) is the colimit of the following diagram

\[
\begin{array}{ccc}
L^H(\sqcup^i,\epsilon[i]) \otimes L^H(\partial(\Box[n-i])) & \leftarrow & L^H(\Box[i]) \otimes L^H(\partial(\Box[n-i])) \\
\downarrow & & \\
L^H(\sqcup^i,\epsilon[i]) \otimes L^H(\Box[n-i]) & & \end{array}
\]

Using the same arguments as in the paragraph just above, we can prove that the map \( L^H(\sqcup^i,\epsilon[n]) \rightarrow L^H(\Box[n]) \) is an acyclic cofibration. So \( L^H \) preserves acyclic cofibrations. Therefore, it is a left Quillen functor. The converse implication is straightforward. \( \square \)

Consider a morphism if intervals \( H \rightarrow H' \). By Proposition 13, it induces a unique natural transformation \( L^H \rightarrow L^{H'} \). Thus, we obtain a natural transformation \( R^{H'} \rightarrow R^H \).

**Proposition 17.** Consider a morphism of interval \( H \rightarrow H' \). This is in particular a weak equivalence. Then, for any fibrant object \( X \) on \( E \), the morphism \( R^{H'}(X) \rightarrow R^H(X) \) is a weak equivalence. Moreover, for any cubical set \( X \), the morphism \( L^H(X) \rightarrow L^{H'}(X) \) is a weak equivalence.

**Proof of Proposition 17.** By Proposition 16, we know that the functor \( L^H \) and \( L^{H'} \) are both Reedy cofibrant. Moreover, we know by Corollary 1 that the map \( L^H \rightarrow L^{H'} \) is an equivalence. We conclude by Corollary 2. \( \square \)

**Corollary 3.** The two following propositions are equivalent.

1. There exists an interval \( H \) of \( E \) such that the adjunction \( L^H \dashv R^H \) is a Quillen equivalence.
2. For any interval \( H \) of \( E \), the adjunction \( L^H \dashv R^H \) is a Quillen equivalence.

**Proof.** The second statement implies the first one since there exists intervals. Besides, suppose that the first statement is true, and let \( H' \) be an other interval. There exists a sequence of weak equivalences of intervals as follows

\[
H \sim H'' \sim H'.
\]

This induces a sequence of natural transformations of functors from cubical sets to \( \text{Ho}(E) \)

\[
L^H \rightarrow L^{H''} \leftarrow L^{H'}.\]

By Proposition 17, these natural transformations are isomorphisms. So \( L^{H'} \) is an equivalence of categories as well as \( L^H \). \( \square \)
3.5. Cubical model categories.

Proposition 18. Let \((E, \otimes, 1)\) be a monoidal model category and let \(M\) be an \(E\)-model category. Then, any choice of interval (resp. monoidal interval) \(H\) in \(E\) induces a structure of a \(\square_{\mathbb{C}}\-\text{Set-model category (resp. } \square_{\mathbb{C}}\-\text{Set-model category)} on \(M\).

Proof. This is a consequence of Proposition 8.

4. Nerve functors

In this section, we study the link between quasi-categories and cubical categories. This allows us to give precise conditions making an adjunction relating the Joyal category of simplicial sets to the category \(\text{Cat}_{\mathbb{E}}\) to be a Quillen adjunction.

Notations. The category of reduced cubical categories (that is \(\square_{\mathbb{C}}\-\text{Set-enriched categories}) and the category of cubical categories with connections (that is \(\square_{\mathbb{C}}\-\text{Set-enriched categories}) are denoted respectively \(\text{Cat}_{\mathbb{E}}\) and \(\text{Cat}_{\square_{\mathbb{C}}}\).

4.1. From cubical categories to quasi-categories. For any integer \(n \in \mathbb{N}\), let \((W_n, \mu, u)\) be the following category enriched in cubical sets with connections

\(\triangleright\) its set of objects is \(\{0, \ldots, n\}\),
\(\triangleright\) for any \(i, j \in \{0, \ldots, n\}\), \(W_n(i, j) = \square_{\mathbb{C}}[j - i - 1]\), with the convention \(\square_{\mathbb{C}}[-1] = \square_{\mathbb{C}}[0]\) and \(\square_{\mathbb{C}}[m] = \emptyset\) for any \(m < -1\),
\(\triangleright\) the composition is defined for any \(i < j < k\) by

\[W_n(i, j) \otimes W_n(j, k) \simeq W_n(i, j) \otimes \square_{\mathbb{C}}[0] \otimes W_n(j, k) \xrightarrow{\text{Id} \otimes \delta^k_1 \otimes \text{Id}} W_n(i, j) \otimes \square_{\mathbb{C}}[1] \otimes W_n(j, k) \simeq W_n(i, k).\]

In other words, it is given by the map \(\delta^1_{j-i-1} : \square_{\mathbb{C}}[k - i - 2] \to \square_{\mathbb{C}}[k - i - 1]\).

Remark 8. The cubical category \(W_n\) is obtained by applying the Boardman–Vogt construction to the category \([n]\). See \([BM06]\).

Proposition 19. The construction \((W_n, \mu, u)\) does is a cubical category. Moreover, the assignment \(n \mapsto W_n\) defines a functor from the category \(\Delta\) to the category \(\text{Cat}_{\square_{\mathbb{C}}}\) of cubical categories.

Proof. The composition that we defined in \(W_n\) is associative since for any \(0 \leq i < j < k < l \leq n\)

\[\delta^1_{l-i-1} \circ \delta^1_{j-i-1} = \delta^1_{l-i-2} \circ \delta^1_{j-i-2}.\]

Moreover, any coface morphism \(\delta^i_A : [n] \to [n + 1]\) in the category \(\Delta\) induces a cubical functor \(W_n \to W_{n+1}\) which is the function \(\delta^i_A\) on objects and such that for any \(j < i < l\) the morphism \(W_n(j, k) \to W_{n+1}(j, k + 1)\) is given by

\[\square_{\mathbb{C}}[k - j - 1] \xrightarrow{\delta^i_A} \square_{\mathbb{C}}[k - j].\]

Similarly, any codegeneracy morphism \(\sigma^i_A : [n] \to [n - 1]\) in the category \(\Delta\) induces a cubical functor \(W_n \to W_{n-1}\) which is the function \(\sigma^i_A\) on objects and such that for any \(j < i < k\) the morphism \(W_n(j, k) \to W_{n-1}(j, k - 1)\) is given by

\[\square_{\mathbb{C}}[k - j - 1] \xrightarrow{\gamma_{j-i-1}} \square_{\mathbb{C}}[k - j - 2].\]

Moreover, the morphism \(W_n(i, k) \to W_{n-1}(i, k - 1)\) is \(\sigma_0 : \square_{\mathbb{C}}[k - i - 1] \to \square_{\mathbb{C}}[k - j - 2]\) and the morphism \(W_n(j, i + 1) \to W_{n-1}(j, i)\) is \(\sigma_{j-i-1} : \square_{\mathbb{C}}[j - i] \to \square_{\mathbb{C}}[j - i - 1]\).

Definition 23. Let \(W_{\mathbb{C}} \dashv N^\mathbb{C}\) be the adjunction relating simplicial sets to \(\square_{\mathbb{C}}\-\text{Set-enriched-categories}\) such that \(W_{\mathbb{C}}\) is the left Kan extension of the cosimplicial object \(n \mapsto W_n\) and

\[N^\mathbb{C}(C)(n) = \text{hom}_{\text{Cat}_{\square_{\mathbb{C}}}}(W_n, C).\]

Remark 9. The adjunction \(L^\mathbb{C}_{\Delta^1} \circ W_{\mathbb{C}} \dashv N^\mathbb{C} \circ R^\mathbb{C}_{\Delta^1}\) is the adjunction \(\mathbb{C} \dashv N\) of the book Higher Topos Theory [Lur09, §1.1.5].

Remark 10. Note that the construction above already appeared in [RZ16]. They showed that the adjunction \(W_{\mathbb{C}} \dashv N^\mathbb{C}\) factors the adjunction \(\mathbb{C} \dashv N\) of the book Higher Topos Theory [Lur09, §1.1.5] as well as the dg nerve of [Lur12, §1.3.1].
Definition 24 (Joyal, Lurie). [Lur09] The Joyal model category 
\( sSet \) is the category of simplicial 
sets \( sSet \) equipped with the model structure whose cofibrations are monomorphisms and weak 
equivalences are maps \( f \) such that \( W_c(f) \) is a weak equivalence for the categorical model structure 
on \( \text{Cat}_{\Delta} \). The fibrant objects are the quasi-categories. Moreover, this is a cartesian closed 
monoidal model category.

Actually, Lurie shows (and uses this fact as a definition) in [Lur09] that weak equivalences for 
the Joyal model structure are maps \( f \) such that \( C(f) = (L^\Delta_c \circ W_c)(f) \) is a weak equivalence. Since 
the functor \( L^\Delta_c \) preserves and reflects weak equivalences, these are exactly the maps \( f \) such that 
\( W_c(f) \) is a weak equivalence.

Lemma 10. The functor \( W_c \) preserves cofibrations. Hence it is a left Quillen functor.

Proof. It suffices to show that for any integer \( n \), the map \( W_c(\partial \Delta[n]) \to W_n \) is a cofibration. This 
follows from the fact that the following square of cubical categories is a pushout

\[
\begin{array}{ccc}
[1]_{\partial \Delta[n]} & \longrightarrow & W_c(\partial \Delta[n]) \\
\downarrow & & \downarrow \\
[1]_{\Delta[n]} & \longrightarrow & W_n .
\end{array}
\]

\[\square\]

4.2. Nerve functors.

Definition 25. Let \( C \) be a cocomplete category. A functor \( F : \Delta \to C \) induces an adjunction

\[
sSet \xrightarrow{\leftarrow} C,
\]

where the left adjoint \( F_l \) is the left Kan extension of \( F \) and where

\[
F_l(X)_n = \text{hom}_C(F(n), X).
\]

This right adjoint functor is called a nerve functor. If \( C \) is a model category, the functor \( F \) (or 
equivalently the functor \( F^l \)) is said to be homotopy coherent if the adjunction \( F_l \dashv F^l \) is a Quillen 
adjunction with respect to the Joyal model structure.

We are interested by the case where \( C \) is the category \( \text{Cat}_E \) equipped with the categorical model 
structure; where \( E \) is a monoidal model category. We know that there exists an adjunction

\[
\text{Cat} \xrightarrow{i} \text{Cat}_E .
\]

Since the category \( \Delta \) is a full subcategory of the category \( \text{Cat} \), this provides us with a cosimplicial 
object in \( \text{Cat}_E \), that is

\[
i([-]) : n \mapsto i([n]) ,
\]

that we refer to using the notation \( n \mapsto [n] \).

Theorem 5. Let \( E \) be a monoidal model category and suppose that the category \( \text{Cat}_E \) has a categorical model 
structure. Then, for any Reedy cofibrant replacement \( F \) of the cosimplicial \( E \)-enriched category \( n \mapsto [n] \), 
the nerve \( F^l \) is homotopy coherent.

Note first that the fact that \( F \) is Reedy cofibrant implies that the functor \( F_l : sSet \to \text{Cat}_E \) 
preserves cofibrations. So, it suffices to check that it preserves weak equivalences. One way to 
prove it is to show that the functor \( F_l \) sends the maps

\[
\Lambda^k[n] \to \Delta[n], \quad 0 < k < n ,
\]

and the map \( * \to N(* \leftrightarrow *) \) to equivalences. We actually take a shortcut by using the fact that \( W_c \) 
is a left Quillen functor.
Recall that the category $\text{Fun}(\Delta, \text{Cat}_E)$ of cosimplicial $E$-categories carries a Reedy model structure where a map $F \rightarrow G$ is a cofibration (resp. an acyclic cofibration) if the map

$$F_{\ast}(\Delta[n]) \cup F_{\ast}(\partial \Delta[n]) \rightarrow G_{\ast}(\partial \Delta[n])$$

is a cofibration (resp. an acyclic cofibration) for any integer $n$.

**Lemma 12.** For any monomorphism of simplicial sets $X \rightarrow Y$ and any Reedy cofibration $F \rightarrow G$ in the category $\text{Fun}(\Delta, \text{Cat}_E)$, then the map

$$F_{\ast}(Y) \cup F_{\ast}(X) \rightarrow G_{\ast}(X)$$

is a cofibration. Moreover, this is an acyclic cofibration if $F \rightarrow G$ is an acyclic cofibration.

In particular, for any simplicial set $X$ and for any Reedy acyclic cofibration $F \rightarrow G$, the map $F_{\ast}(X) \rightarrow G_{\ast}(X)$ is an acyclic cofibration.

**Proof.** The proof is the same as Proposition 15 using the standard Reedy structure on the category $\Delta$. $\square$

**Lemma 11.** If a Reedy cofibrant replacement $F$ of the cosimplicial $E$-category $n \mapsto [n]$ is homotopy coherent, then all its Reedy cofibrant replacements are homotopy coherent.

**Proof.** Let $G$ be a cofibrant replacement of $[-]$. Then, let us consider the following factorisation in the Reedy model category of cosimplicial $E$-categories

$$F \sqcup G \rightarrow G' \rightarrow [-].$$

Since $F$ and $G$ are cofibrant, then, the morphisms $F \rightarrow G'$ and $G \rightarrow G'$ are both acyclic cofibrations for the Reedy model structure. Let $f : X \rightarrow Y$ be a morphism of simplicial sets which is a weak equivalence for the Joyal model structure. Consider the following diagram

$$
\begin{array}{ccc}
F_{\ast}(X) & \longrightarrow & G'_{\ast}(X) & \longrightarrow & G_{\ast}(X) \\
\downarrow & & \downarrow & & \downarrow \\
F_{\ast}(Y) & \longrightarrow & G'_{\ast}(Y) & \longrightarrow & G_{\ast}(Y)
\end{array}
$$

Since $F$ is homotopy coherent, then the left vertical arrow is a weak equivalence of $E$-enriched categories. By Lemma 11, all the horizontal arrows are weak equivalences. So, by the 2-out-of-3 rule, the middle vertical arrow and then the right vertical arrow are weak equivalences. So $G$ is homotopy coherent. $\square$

**Proof of Theorem 5.** First, let $G$ be a Reedy cofibrant replacement of the functor

$$\Delta \rightarrow \text{Cat}_{\square},$$

$$n \rightarrow [n].$$

Then the functor $L_r^\square_{[1]} \circ G$ is also Reedy cofibrant. Besides, by Lemma 10, the functor

$$\Delta \rightarrow \text{Cat}_{\square},$$

$$n \rightarrow W_n.$$ is an homotopy coherent Reedy cofibrant replacement of the functor $n \mapsto [n]$. So by Lemma 12, $L_r^\square_{[1]} \circ G$ is homotopy coherent. Since the functor $L_r^\square_{[1]}$ reflects weak equivalences, then $G$ is also homotopy coherent. Then, for any interval $H$ of $E$, the cosimplicial $E$-enriched category $L_r^H \circ G$ is homotopy coherent. So again by Lemma 12, $F$ is homotopy coherent. $\square$

**Proposition 20.** We use the same notation as in Theorem 5. The following propositions are equivalent.

1. There exists an interval $H$ of $E$ such that the adjunction $L_r^H \dashv R_r^H$ relating reduced cubical sets to $E$ is a Quillen equivalence.
2. For any Reedy cofibrant replacement $F$ of the cosimplicial object $n \mapsto [n]$ of $\text{Cat}_E$ the adjunction $F_{\ast} \dashv F^\ast$ is a Quillen equivalence.
3. There exists a Reedy cofibrant replacement $F$ of the cosimplicial object $n \mapsto [n]$ of $\text{Cat}_E$ such that the adjunction $F_{\ast} \dashv F^\ast$ is a Quillen equivalence.
**Lemma 13.** The adjunction $W_c ⊣ N^c$ is a Quillen equivalence for the Joyal model structure.

**Proof.** It is a consequence of the fact that the adjunction $L_{c}^{A[1]} ⊣ W_c ⊣ N^c ⊣ R_{c}^{A[1]}$ relating simplicial sets to simplicial categories is a Quillen equivalence (see for instance [Lur09, Proposition 2.2.4.1]) and the adjunction $L_{c}^{A[1]} ⊣ R_{c}^{A[1]}$ is a Quillen equivalence. \[ \Box \]

**Proof of Proposition 20.** The equivalence between (2) and (3) follows from the same arguments as those used to prove corollary 3. Then, let $F : Δ \to \text{Cat}_{\square}$ be a Reedy cofibrant replacement of $n \to [n]$. By the equivalence between (2) and (3) (for $E = \square_c\text{-Set}$) and by Lemma 13, the adjunction $L_{\square}^{c[1]} ⊣ F_1 \circ F^3 ⊣ R_{\square}^{c[1]}$ is a Quillen equivalence. Besides, the adjunction

$$\text{Cat}_{\square} \xrightarrow{L_{\square}^{c[1]}} \text{Cat}_{\square} \xleftarrow{R_{\square}^{c[1]}} \text{Cat}_{E}$$

is a Quillen equivalence. So, by the 2-out-of-3 rule, the adjunction $F_1 \circ F^3$ is also a Quillen equivalence. Now, suppose (1). Then, the adjunction $L_{\square}^{H} ⊣ F_1 \circ F^3 ⊣ R_{\square}^{H}$ is a Quillen equivalence, which implies (2). Conversely, suppose (3), then for any interval $H$ of $E$, the adjunction $L_{\square}^{H} ⊣ F_1 \circ F^3 ⊣ R_{\square}^{H}$ is a Quillen equivalence and since $F_1 \circ F^3$ is a Quillen equivalence, then the adjunction

$$\text{Cat}_{\square} \xrightarrow{L_{\square}^{H}} \text{Cat}_{E} \xleftarrow{R_{\square}^{H}} \text{Cat}_{E}$$

is also a Quillen equivalence. In particular, for any reduced cubical set $X$ and any fibrant object $Y$ of $E$, a morphism $L_{\square}^{H}([1]_X) \to [1]_Y$ is an equivalence if and only if the adjoint morphism $[1]_X \to R_{\square}^{H}[1]_Y$ is an equivalence. This rewrites as: $L_{\square}^{H}(X) \to Y$ is an equivalence if and only if the adjoint morphism $X \to R_{\square}^{H}Y$ is an equivalence; that is, the adjunction

$$\square_c\text{-Set} \xrightarrow{L_{\square}^{H}} E \xleftarrow{R_{\square}^{H}} E$$

is a Quillen equivalence. \[ \Box \]

5. **APPLICATIONS**

The goal of this final section is to describe various contexts where cubical categories appear.

Let $(E, \otimes, 1)$ be a monoidal model category and let $H$ be a monoidal interval. We know that it induces a Quillen monoidal adjunction $L_{c}^{H} ⊣ R_{c}^{H}$ relating cubical sets with connections to $E$ which extends to the level of enriched categories.

$$\text{Cat}_{\square} \xrightarrow{L_{c}^{H}} \text{Cat}_{E} \xleftarrow{R_{c}^{H}} \text{Cat}_{E}$$

Moreover, any $E$-model category $M$ has an induced structure of a $\square_c\text{-Set}$-model category. In this section, we describe three examples of such a monoidal model category $E$: the simplicial sets with the Joyal model structure, the chain complexes and the differential graded coalgebras.

5.1. **A remark about the Boardman–Vogt construction.** A theory of homotopy coherent nerve is developed in [MW07, §6]. Roughly, for any monoidal model category $E$ (satisfying some conditions) equipped with a monoidal interval $H$, there exists a endofunctor $W_H : \text{Cat}_E \to \text{Cat}_E$ called the Boardman–Vogt construction together with a natural transformation $W_H \to \text{Id}$ such that the functor

$$W_H C \to C$$

is a cofibrant replacement of $C$ provided the unit maps $1 \to C(x, x)$ are cofibrations. More generally, any functor $F : C \to D$ which is injective on objects and such that the maps $F_{x,y} : C(x, y) \to D(F(x), F(y))$ are cofibrations in $E$ induces a cofibration of $E$-enriched categories

$$W_H(F) : W_H C \to W_H D.$$
Then, the functor \( n \in \Delta \mapsto W_H[n] \in \text{Cat}_E \) induces an adjunction

\[
\text{sSet} \xrightarrow{W_H} \text{Cat}_E 
\]

where \( N^H \) is an homotopy coherent nerve.

**Lemma 14.** The functor \( W_H \) is isomorphic to the composite functor \( L_H \circ c \).

**Proof.** It follows from the fact that the functor \( n \mapsto W_H[n] \) is isomorphic to the functor \( n \mapsto L_H n \). \( \square \)

Therefore, the functor \( N^H \) is isomorphic to the functor \( N \circ L^H \).

#### 5.2. The underlying \((\infty, 1)\)-category of an \((\infty, 2)\)-category

Let us endow the category of simplicial sets with the Joyal model structure which is a monoid al model structure.

**Definition 26.** Let \( I \) be the groupoid with two objects 0 and 1 such that

\[
I(i, j) = \ast, \quad \forall i, j \in \{0, 1\}.
\]

In this context, the simplicial nerve of the groupoid \( I \)

\[
H = N(I)
\]

is a monoidal interval. Subsequently, there exists a monoidal Quillen adjunction relating cubical sets to simplicial sets with the Joyal model structure.

\[
\square \xrightarrow{c} \text{Set} \xrightarrow{L^H} \text{sSet} 
\]

**Proposition 21.** Let \( X \) be a quasi-category. Then \( R^H_X(X) \) is canonically equivalent to \( R^H_{\square^H}(\text{Core}(X)) \) where \( \text{Core}(X) \) is the maximal Kan complex contained in \( X \).

**Proof.** On the one hand, for any integer \( n \), we have

\[
\text{hom}_{\square \setminus \text{Set}}(\square c[n], R^H_{\square^H}(X)) \simeq \text{hom}_{\text{sSet}}(L^H_{\square c}[n], X) \\
\simeq \text{hom}_{\text{sSet}}(H^{\square^n}, X) \\
\simeq \text{hom}_{\text{sSet}}(N(1^n), X) \\
\simeq \text{hom}_{\text{quasi-categories}}(N(1^n), \text{Core}(X)) \text{ since } N(1^n) \text{ is a Kan complex} \\
\simeq \text{hom}_{\text{sSet}}(N(1^n), \text{Core}(X)) \\
\simeq \text{hom}_{\square \setminus \text{Set}}(\square c[n], R^H_{\square^H}(\text{Core}(X))).
\]

Therefore, the canonical map \( R^H_{\square^H}(\text{Core}(X)) \rightarrow R^H_{\square}(X) \) is an isomorphism. On the other hand, in the Kan–Quillen model structure on simplicial sets whose fibrant objects are Kan complexes, the inclusion \( \Delta [1] \rightarrow N(1) \) is an equivalence. So, by Proposition 17, we obtain an equivalence \( R^H_{\Delta [1]}(\text{Core}(X)) \rightarrow R^H_{\square^H}(\text{Core}(X)) \). \( \square \)

The adjunction \( L^H \dashv R^H \) extends to the level of enriched categories.

\[
\text{Cat}_\square \xrightarrow{L^H} \text{Cat} \xrightarrow{R^H} \text{Cat}_\square
\]

where the category \( \text{Cat}_\Delta \) of simplicial categories is equipped with the categorical model structure induced by the Joyal model structure on simplicial sets. Any simplicial category \( C \) whose mapping objects are quasi-categories (which is the case for any fibrant object of \( \text{Cat}_\Delta \)), represents an \((\infty, 2)\)-category. The underlying \((\infty, 1)\)-category is the simplicial category with the same objects but whose mapping space between any two objects \( x \) to \( y \) is

\[
\text{Core}(C(x, y)).
\]
Let us denote it by Core(\mathcal{C}). Then, the \((\infty,1)\)-category represented by Core(\mathcal{C}) is equivalent to the \((\infty,1)\)-category represented by \(R^\Delta_1(\text{Core}(\mathcal{C}))\) (indeed, they yield the same quasi-category from the usual homotopy coherent nerve functors). The proposition just above implies that
\[
R^\Delta_1(\text{Core}(\mathcal{C})) \simeq R^H c\mathcal{C}.
\]
Hence, \(R^H c\mathcal{C}\) represents the underlying \((\infty,1)\)-category of \(\mathcal{C}\).

Besides, by Proposition 18, any \(\text{sSet}_I\)-model category inherits a structure of a cubical model category. For instance, we have the following proposition.

**Proposition 22.** The data of
\[
\delta^0, \delta^1 : \mathbb{K} \to C[1], \sigma : C[1] \to \mathbb{K} \text{ and } \gamma : C[1] \otimes C[1] \to C[1]
\]
defined by the formulas
\[
\begin{align*}
\delta^0(1) &= (0) \\
\delta^1(1) &= (1) \\
\sigma(0) &= \sigma(1) = 1 \\
\sigma(01) &= 0 \\
\gamma((0) \otimes (x)) &= \gamma(x \otimes (0)) = x \\
\gamma((1) \otimes (1)) &= (1) \\
\gamma((01) \otimes (1)) &= \gamma((1) \otimes (01)) = 0.
\end{align*}
\]

This gives an adjunction \(W_{C[1]} \dashv N^C[1]\) relating differential graded categories to simplicial set. The right adjoint functor \(N^C[1]\) is the dg-nerve of dg-categories described in [Lur12, §1.3.1]. Besides, any dgMod-model category has an induced structure of a \(\Box_c\)–Set-model category.

**Proposition 23.** The category of chain complexes of \(\mathbb{K}\)-modules is a \(\Box_c\)–Set-model category.

5.4. The coalgebraic nerve. Here \(\mathbb{K}\) is a field.

5.4.1. A coalgebraic model of the interval.

**Definition 27.** A counital coassociative coalgebra \((V, \omega, \tau)\) is a comonoid in the category of chain complexes. We denote by uCog the category of such coalgebras.

Since \(\mathbb{K}\) is a field, the category of counital coassociative coalgebras admits a monoidal model structure whose cofibrations and weak equivalences are respectively degreewise injections and quasi-isomorphisms; see [GG99]. The chain complex monoidal interval \(C[1]\) has the structure of a coalgebra as follows
\[
\begin{align*}
\tau(0) &= \tau(1) = 1 \\
\omega(i) &= (i) \otimes (i) \text{ for } i \in \{0, 1\} \\
\omega(01) &= (0) \otimes (01) + (01) \otimes (1).
\end{align*}
\]
Moreover, a straightforward checking leads to well known following result.

**Lemma 15.** The data of \((C[1], \omega, \tau, \delta^0, \delta^1, \sigma, \gamma)\) defines a monoidal interval in the category of counital coassociative coalgebras.
Therefore, any uCog-model category has an induced structure of a □c−Set-model category. We will study the example of dg associative algebras.

5.4.2. The uCog-model category of $A_\infty$-algebras. The remaining of this article is devoted to the description through our cubical approach of the higher structures appearing in the study in of associative algebras in chain complexes over a field. What is done there can easily be extended to the case of algebras over a nonsymmetric operad using the theory developed in [Gri16].

Definition 28. A differential graded (dg) associative algebra (or dg algebra for short) $(A, m)$ is the data of a chain complex $A$ together with an associative product $m : A \otimes A \to A$.

Definition 29. A dg conilpotent coassociative coalgebra (or dg conilpotent coalgebra for short) $(C, w)$ is the data of a chain complex $C$ together with a coproduct $w : C \to C \otimes C$ which is coassociative that is $(w \otimes \text{Id}) \circ w = (\text{Id} \otimes w) \circ w$ and conilpotent, that is, for any element $x \in C_m$, there exists an integer $n$ such that

\[ w^{(n)}(x) := (w \otimes \text{Id} \otimes \cdots \otimes w) \circ (w(x)) = 0. \]

There exists an adjunction

\[ \text{NilCog} \xleftrightarrow{\Omega} \text{Alg} \]

relating dg algebras to dg conilpotent coalgebras. The right adjoint $B$ called the bar functor is defined as follows.

- The underlying graded coalgebra of $B A$ is the cofree conilpotent coalgebra on the suspension of $A$,
  \[ T_s A := \bigoplus_{n \geq 1} s A \otimes^n \]
  whose coproduct is given by
  \[ w(s x_1 \otimes \cdots \otimes s x_n) = \sum_{i=1}^{n-1} (s x_1 \otimes \cdots s x_i) \otimes (s x_{i+1} \otimes \cdots \otimes s x_n). \]

- The differential $d_{B A}$ on $B A$ is the only coderivation on $T_s A$ whose projection on the co-generators $s A$ is the following composite map
  \[ T_s A \to s A \oplus s A \otimes s A \to s A \]
  \[ s x \mapsto -s d_{A} x \]
  \[ s x \otimes s y \mapsto (-1)^{|x|} s y_A (x \otimes y) \]

The fact that $d_{B A}^2 = 0$ follows from the fact that $d_A^2 = 0$, that $\gamma_A$ is associative and that $\gamma_A$ and $d_A$ satisfy the Leibniz equation.

Proposition 24. There exists a model structure on the category of dg algebras whose fibrations are degree-wise surjections and whose weak equivalences are quasi-isomorphisms.

Theorem 6. [LH03] There exists a model structure on the category dg conilpotent coalgebras transferred through the adjunction $\Omega \dashv B$, that is

- a cofibration is a morphism $f$ such that $\Omega(f)$ is a cofibration of algebras,
- a weak equivalence is a morphism $f$ such that $\Omega(f)$ is a quasi-isomorphism.
![](image)

Moreover, for any dg algebra $A$, the morphism

\[ \Omega B A \to A \]

is a cofibrant replacement of $A$, which implies that the adjunction $\Omega \dashv B$ is a Quillen equivalence.

Remark 25. [AJ13][Gri16] The category NilCog and the category Alg are both uCog-model categories. Moreover, there exists a natural isomorphism of counital coassociative coalgebras

\[ \{C, B A\} \simeq \{\Omega C, A\}, \]

for any dg conilpotent coalgebra $C$ and for any dg algebra $A$. 

Let us give a hint on what are these enrichments. On the one hand, dg algebras are canonically cotensored over counital coassociative coalgebras. Indeed, for any dg algebra $A$ and any counital coassociative coalgebra $V$, the chain complex $[V, A]$ has the canonical structure of a dg algebra called the convolution algebra. Then, the tensorisation and the enrichment may be obtained by the adjoint functor theorem. On the other hand, the category of dg conilpotent coalgebras is canonically tensored over counital coassociative coalgebras. Indeed, for any dg conilpotent coalgebra $C$ and any counital coassociative coalgebra $V$, the chain complex $C \otimes V$ has the canonical structure of a dg conilpotent coalgebra. Then, the cotensorisation and the enrichment may be obtained with the adjoint functor theorem.

5.4.3. The infinity category of dg algebras. Restricting the $u\text{Cog}$-enriched category of dg conilpotent coalgebras to bar constructions of dg algebras (which are in particular fibrant-cofibrant dg conilpotent coalgebras) we obtain an $u\text{Cog}$-enriched category which is essentially the same as the $u\text{Cog}$-enriched category $\mathcal{A}l_{\text{g}}$ whose objects are dg algebras and such that

$$\mathcal{A}l_{\text{g}}(A, A') := \{ B.A, B.A' \}.$$ 

Then, using the interval $C[1]$, one obtains a $\square_\ast$-Set-enriched category $\mathcal{A}l_{\text{g}}^{\square}$ with the same objects as $\mathcal{A}l_{\text{g}}$ and such that

$$\mathcal{A}l_{\text{g}}^{\square}(A, A')(n) = \text{hom}_{u\text{Cog}}(B.A \otimes C[1]^\otimes n, B.A') \approx \text{hom}_{u\text{Cog}}(C[1]^\otimes n, \{ B.A, B.A' \}) \approx \text{hom}_{u\text{Cog}}(C[1]^\otimes n, \{ \Omega B.A, A' \}) \approx \text{hom}_{\mathcal{A}l_{\text{g}}}(\Omega B.A, [C[1]^\otimes n, A']) .$$

Finally, the nerve $N^c \mathcal{A}l_{\text{g}}^{\square}$ of this cubical category is the quasi-category whose $n$ vertices are the data of

1. $n+1$ dg algebras $A_0, \ldots, A_n$,
2. for any integers $0 \leq i < j \leq n$, a morphism of dg conilpotent coalgebras

$$f_{i,j} : (BA_i) \otimes C[1]^\otimes j-i-1 \to BA_j ,$$

which is equivalent to the data of a morphism of dg algebras

$$\Omega BA_i \to [C[1]^\otimes j-i-1, A_j] ,$$

3. such that the following diagram commutes

$$\begin{array}{ccc}
BA_i \otimes C[1]^\otimes j-i-1 \otimes C[1]^\otimes k-j-1 & \xrightarrow{f_{i,j} \otimes \text{Id}} & BA_j \otimes C[1]^\otimes k-j-1 \\
\downarrow \quad \quad \quad \downarrow f_{i,k} \quad \quad \quad \quad \quad \downarrow \\
BA_i \otimes C[1]^\otimes j-i-1 \otimes 1 \otimes C[1]^\otimes k-j-1 & \xrightarrow{\text{Id} \otimes \delta^i \otimes \text{Id}} & BA_k \\
\downarrow f_{i,k} \quad \quad \quad \downarrow & & \\
BA_i \otimes C[1]^\otimes k-i-1 & \xrightarrow{f_{i,k}} & BA_k
\end{array}$$

for any integers $0 \leq i < j < k \leq n$. 

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