Recovering the characteristic functions of the Sturm-Liouville differential operators with singular potentials on star-type graph with cycle

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Abstract. We consider Sturm-Liouville operators with singular potentials from the class \( W^{-1}_2 \) on star-type graph with cycle, which consist the edges with commensurable lengths. Asymptotic representation for eigenvalues for such operators is obtained. Recovering of the characteristic function the Sturm-Liouville operators with the singular potentials is considered.

Keywords: Sturm-Liouville operators, singular potential, graph with cycle, asymptotic of the spectrum, characteristic function.

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1. Introduction.

This paper is devoted to the inverse spectral problems for differential operators on geometrical graphs. Differential operators on graphs are intensively studied by mathematicians in recent years and have applications in different branches of science and engineering. Inverse problem consists in recovering the potential on a graph from the given spectral characteristics. Recovering of the characteristic function of the Sturm-Liouville operators is the important part of solving such problem. The greatest success in the inverse spectral theory has been achieved for the classical Sturm-Liouville operator and afterwards for higher order differential operators [1-3].

The Sturm-Liouville operator with singular potential has been studied in papers [9-10]. The inverse problem on a finite interval were extensively studied in [11]. There are only few results for Sturm-Liouville operators in different branches of science and engineering. Inverse problem consists in recovering the potential on a graph from the given spectral characteristics. Recovering of the characteristic function of the Sturm-Liouville operators with singular potential is considered for the classical Sturm-Liouville operator and afterwards for higher order differential operators [1-3]. In this paper we consider the recovering of the characteristic function the Sturm-Liouville operators with the singular potentials on star-type graph with cycle, which consist the edges with commensurable lengths. Also we obtain the asymptotic representation for eigenvalues for such operators.

Let \( G \) be a graph with of set of vertices \( \{v_j\}_{j=0}^{p} \) and set of edges \( \{e_j\}_{j=0}^{p} \), \( e_j = [v_0, v_j] \), and edge \( e_0 \) generates the cycle. We suppose that the length of edge \( e_j \) is equal to \( |e_j| \). We consider each edge \( e_j \) as a segment \([0,|e_j]|\) and parameterize it by the parameter \( x \in [0,|e_j]|\). It is convenient for us to choose the orientation such that \( x_j = |e_j| \) corresponds to the vertex \( v_0 \). Lengths of the edges \( e_j, j = 0, p \), we consider as commensurable quantities.

A function \( y \) on graph \( G \) is considered as \( y = [y_j(x_j)]_{j=0}^{p} \), where \( y_j(x_j) \) are correspond to \( e_j, x_j \in [0,|e_j]| \). Let \( q = [q_j(x_j)]_{j=0}^{p} \) be a real-valued function on \( G \) such that \( q_j \in W^{-1}_2[0,|e_j]| \), i.e. \( q_j(x_j) = \sigma_j(x_j) \), where the derivative is considered in the sense of distributions. Function \( \sigma = [\sigma_j(x_j)]_{j=0}^{p} \) we call the potential. The Sturm-Liouville differential operator on the edges \( e_j, j = 0, \overline{p} \) is defined by the following expression:

\[
\ell_jy_j := - (y_j^{[1]})' - \sigma_j(x_j)y_j^{[1]} - \sigma_j^2(x_j)y_j,
\]

where \( y_j^{[1]} := y_j' - \sigma_j(x)y_j \) - is a quasi-derivative, and

\[
dom(\ell_j) = \{ y_j | y_j \in W^1_2[0,|e_j|], \ y_j^{[1]} \in W^1_2[0,|e_j|], \ \ell_jy_j \in L^2[0,|e_j]| \}.
\]

We consider the Sturm-Liouville equation on \( G \):

\[
(\ell_jy_j)(x_j) = \lambda y_j(x_j), \ x_j \in (0,|e_j|), \ y_j \in \dom(\ell_j), \ j = 0, \overline{p} \tag{1.1}
\]

At the internal vertex \( v_0 \) we consider the following matching conditions:

\[
y_j(0) = y_k(|e_k|), \ k = \overline{0,\overline{p}}, \ \sum_{j=0}^{p} y_j^{[1]}(|e_j|) = y_0^{[1]}(0). \tag{1.2}
\]

Let us consider the boundary value problem \( L(G) \) for equation (1.1) with the matching conditions (1.2) and boundary conditions

\[
y_j^{[1]}(0) = 0, \ j = \overline{1,\overline{p}}. \tag{1.3}
\]

Let us define by \( \Lambda \) the eigenvalues of \( L(G) \). We also consider the boundary value problem \( L_k(G), k = \overline{1,\overline{p}} \), for equation (1.1) with the matching conditions (1.2) and boundary conditions

\[
y_j^{[1]}(0) = 0, \ j = \overline{1,\overline{p}}; k, \ y_k(0) = 0. \tag{1.4}
\]

The eigenvalues of \( L_k(G) \) we define as \( \Lambda_k \).
2. Auxiliary propositions.

Let \( C_j(x_j, \lambda), S_j(x_j, \lambda) \) be the solutions of equation (1.1) on \( e_j, j = \bar{0, p} \) under initial conditions
\[
C_j(0, \lambda) = S_j(0, \lambda) = 1, \quad C_j(0, \lambda) = S_j(0, \lambda) = 0, \tag{2.1}
\]
From Liouville’s formula we assume that \( (C_j(x_j, \lambda), S_j(x_j, \lambda)) = 1 \), where wronskian \( \langle y, z \rangle = y_z^{[1]} - y^{[1]}_z \). The characteristic functions of the boundary value problem \( L(G) \) we denote as \( \Delta(\lambda, L(G)) \). As in the classical case [16] one can show that the functions \( M_j(\lambda), j = \bar{1, p} \) are meromorphic in \( \lambda \); namely:
\[
M_j(\lambda) = \frac{\Delta(\lambda, L_j(G))}{\Delta(\lambda, L(G))}. \tag{2.2}
\]
The vertex \( v_0 \) separate \( G \) on two parts: \( G = G_0 \cup T \), where \( G_0 \) is the star-type graph with the internal vertex \( v_0 \) and \( T \) is the cycle, generated by the edge \( e_0 \). Taking into account (1.2) and (1.3), one can show, that
\[
\Delta(\lambda, L(T)) = C_0([e_0], \lambda) + S_0^{[1]}([e_0], \lambda) - 2 \tag{2.3}
\]
Denote by \( B_{\epsilon} \), a the class of Paley-Wiener functions of exponential type not greater than \( r \in \mathbb{R} \), belonging to \( L_2(\mathbb{R}) \). It follows from [9-11] that
\[
C_j([e_j], \lambda) = \cos \rho |e_j| + \zeta_j, c(\rho), \quad S_j([e_j], \lambda) = \frac{\sin \rho |e_j|}{\rho} + \frac{1}{\rho} \zeta_j, o(\rho), \tag{2.4}
\]
where \( \zeta_j, c(\rho) \in B_{|e_j|} \) are even functions and \( \zeta_j, o(\rho) \in B_{|e_j|} \) are odd function. Clearly, that
\[
\zeta_j, o(\rho) = \int_0^{[e_j]} K_0(t) \sin \rho t dt, \quad \zeta_j, c(\rho) = \int_0^{[e_j]} K_1(t) \cos \rho t dt, \quad K_0, K_1 \in L_2(0, |e_j|).
\]

Analogously [33], one can prove the following lemma.

**Lemma 2.1.** The characteristic functions of the boundary value problems \( L(G) \) and \( L_k(G) \), \( k = \bar{1, p} \), admits the following representation
\[
\Delta(\lambda, L(G)) = S_0([e_0], \lambda)\Delta(\lambda, L(G_0)) + (-1)^p \Delta(\lambda, L(T)) \prod_{k=1}^{p} C_k([e_k], \lambda)
\]
\[
\Delta(\lambda, L_j(G)) = S_0([e_0], \lambda)\Delta_j(\lambda, L(G_0)) + (-1)^p \Delta(\lambda, L(T))S_j([e_j], \lambda) \prod_{k=1, k \neq j}^{p} C_k([e_k], \lambda) \tag{2.5}
\]
where
\[
\Delta(\lambda, L(G_0)) = (-1)^p \sum_{j=1}^{p} C_j^{[1]}([e_j], \lambda) \prod_{i=1, i \neq j}^{p} C_i([e_i], \lambda)
\]
\[
\Delta(\lambda, L_k(G_0)) = (-1)^{p+1} \left[ S_k([e_k], \lambda) \sum_{j=1, i \neq k}^{p} C_j^{[1]}([e_j], \lambda) \prod_{i=1, i \neq k, j}^{p} C_i([e_i], \lambda) \right] + \sum_{j=1, i \neq k}^{p} C_j^{[1]}([e_j], \lambda) \prod_{i=1}^{p} C_i([e_i], \lambda)
\]

3. Asymptotic of the spectrums. Recovering the characteristic function

The eigenvalues \( \Lambda \) can be numbered as \( \{\lambda_{nk}\}_{k=0, \mu_0 - 1, n \in \mathbb{N}} \cup \{\lambda_{nk}\}_{k=\mu_0 m, n \in \mathbb{Z}} \), where \( m \in \mathbb{N} \), \( \mu_0 \) is the multiplicity of the zero eigenvalue of the boundary value problem \( L(G) \) with the zero potential.

**Lemma 3.1.** Following asymptotic behavior are valid for the eigenvalues of the boundary value problem \( L(G) \):
\[
\sqrt{\lambda_{nk}} =: \rho_{nk} = \begin{cases} \tau n + \varepsilon_{nk}, & k = 0, \mu_0 - 1, \quad n \in \mathbb{N}, \\ |\tau n + \alpha_k| + \varepsilon_{nk}, & k = \mu_0 m, \quad n \in \mathbb{Z}, \end{cases} \tag{3.1}
\]
where \( \varepsilon_{nk} \in l_{2\mu_k}, \tau \in \mathbb{R}, \mu_k \in \mathbb{N} \).
Its sufficient to investigate zeros of the function $d$.

By virtue of the fact, that function $d$ is analytic at a point $\alpha_k$, $k = 0, m$, where $m \in \mathbb{N}$. We define the multiplicity of the $\alpha_k$ as $\mu_k$, $k = 0, m$. Clearly, that $\alpha_0 = \ldots = \alpha_{\mu_0} = 0$. For simplicity we assume, that $\tau/2$ are not the zero of $d_0(\rho)$.

Then we conclude:

$$\rho_0^k := \left\{ \begin{array}{ll}
n\pi n, & k = 0, \mu_0 - 1, \quad n \in \mathbb{N}, \\
|\pi n + \alpha_k|, & k = \mu_0, m, \quad n \in \mathbb{Z}
\end{array} \right.$$ \hspace{1cm} (3.3)

Let us consider $\delta := \min_{\alpha_k \neq \alpha_j, k = 0, m} |\alpha_k - \alpha_j|$. By the sequence $\rho_0^k$, $n \in \mathbb{N}$ we assume the sequence $\rho_0^{nk}$, $(n, k) \in \{0, \mu_0 - 1\} \times \mathbb{N} \cup \{\mu_0, m\} = \mathbb{Z}$ put in order of increasing, such that $\rho_0^k \neq \rho_0^n$ for $n \neq k$. Define $\Gamma_n := \{\lambda, |\lambda| = (\rho_0^k + \delta)^2, n \in \mathbb{N}\}$. Clearly, that

$$|\zeta_{G, e}(\rho)| \leq C_1 e^{G[|\rho_0^k + \delta|]}, \quad |\Delta_0(\lambda, L(G))| \geq C_2 e^{G[|\rho_0^k + \delta|]}, \quad \rho \in \partial \Gamma_n$$

Analogously [7], by the Rouche’s theorem, one can show:

$$\rho_{nk} := \left\{ \begin{array}{ll}
n\pi n + \varepsilon_{nk}, & k = 0, \mu_0 - 1, \quad n \in \mathbb{N}, \\
|\pi n + \alpha_k| + \varepsilon_{nk}, & k = \mu_0, m, \quad n \in \mathbb{Z},
\end{array} \right.$$ \hspace{1cm} (3.4)

Without restricting the generality we consider the case $n \geq 0$ with $k = \mu_0, m$. Substituting (3.3) into (3.2), we obtain

$$d_0(\alpha_k + \varepsilon_{nk}) + \zeta_{G, e}(\rho_{nk}) = 0 \hspace{1cm} (3.5)$$

Clearly, that

$$\zeta_{G, e}(\rho_{nk}) = \int_0^{|G|} K(t) \cos(n\pi + \alpha_k + \varepsilon_{nk})t dt, \quad K(t) \in L_2(0, |G|)$$ \hspace{1cm} (3.6)

Let $N^*$ be such a number, that $\forall n > N^* \varepsilon_{nk} < 1/4$. Then from the kadee-1/4 theorem in the complex case (see [24]) we conclude, that $\{\cos(n + \varepsilon_{nk})x\}_n$ and $\{\sin(n + \varepsilon_{nk})x\}_n$, where

$$\varepsilon_{nk}^* = \left\{ \begin{array}{ll}
0, & n \leq N^*, \\
\varepsilon_{nk}/\pi, & n > N^*,
\end{array} \right.$$  

are the Riesz basis. Then change the variables $s = \tau t$ and taking into account

$$\cos(n\pi + \alpha_k + \varepsilon_{nk})t = \cos \alpha_k t \cos(n\pi + \varepsilon_{nk})t - \sin \alpha_k t \sin(n\pi + \varepsilon_{nk})t,$$

from (3.6) we assume, that

$$\zeta_{G, e}(\rho_{nk}) = \int_0^\pi A_k(t) \cos(n + \varepsilon_{nk})t dt + \int_0^\pi B_k(t) \sin(n + \varepsilon_{nk})t dt \in l_2, \quad A_k, B_k \in L_2(0, \pi)$$ \hspace{1cm} (3.7)

By virtue of the fact, that function $d_0(\rho)$ are analytic at a point $\alpha_k$, we obtain, that

$$d_0(\alpha_k + \varepsilon_{nk}) = \sum_{j=0}^{\mu_k} d_0^{(j)}(\alpha_k) \frac{\varepsilon_{nk}^j}{j!} + O(\varepsilon_{nk}^{\mu_k+1}), \quad n \to \infty$$
From (5.5) it follows, that
\[
\frac{1}{\mu_k} d^{(\mu_k)}(\alpha_k) e^{\mu_k^n} + O(e^{\mu_k^n + 1}) + \zeta |G|, c(\rho_{nk}) = 0, \quad n \to \infty
\]
Consequently,
\[
e^{\mu_k^n} = \zeta |G|, c(\rho_{nk}) \frac{1}{1 + O(e^n)} \in l_2, \quad \varepsilon_{nk} \in l_{2\mu_k}.
\]
Thus, we obtain (3.1).

Define
\[
\lambda_{nk} := \left\{ \begin{array}{ll}
\lambda_{nk}, & \lambda_{nk} \neq 0, \\
1, & \lambda_{nk} = 0,
\end{array} \right.
\]
\[
\lambda_{nk}^0 := \left\{ \begin{array}{ll}
\lambda_{nk}^0, & \lambda_{nk}^0 \neq 0, \\
1, & \lambda_{nk}^0 = 0,
\end{array} \right.
\]
\[
\lambda_{jk} := \left\{ \begin{array}{ll}
\lambda_{jk}, & \lambda_{jk} \neq 0, \\
1, & \lambda_{jk} = 0,
\end{array} \right.
\]
\[
\lambda_{jk}^0 := \left\{ \begin{array}{ll}
\lambda_{jk}^0, & \lambda_{jk}^0 \neq 0, \\
1, & \lambda_{jk}^0 = 0,
\end{array} \right.
\]
The following theorem give us the formulas for the recovering the characteristic function of the boundary value problem \(L(G)\) and \(L_j(G), \ j \in \mathbb{R}_p\), by the spectra \(\Lambda \ni \Lambda_j, \ j = 1, p\) respectively.

**Theorem 3.1.** The specification of the spectrums \(\Lambda\) and \(\Lambda_j\) uniquely determines the characteristic functions respectively by the formula
\[
\Delta(\lambda, L(G)) = (-1)^{\mu_0} \frac{\partial}{\partial \lambda^{\mu_0}} \Delta_0(\lambda, L(G)) = \sum_{n=0}^{\mu_0 - 1} \prod_{k=0}^{\nu_0 - 1} \lambda_{nk} - \lambda
\]
\[
\prod_{n=-\infty}^{\mu_0} \prod_{k=0}^{\nu_0 - 1} \lambda_{nk}^{01} \left(1 + \lambda_{nk}^{0} - \lambda_{nk}^{01} \right)
\]
\[
\Delta(\lambda, L_j(G)) = (-1)^{\mu_0} \frac{\partial}{\partial \lambda^{\mu_0}} \Delta_0(\lambda, L_j(G)) = \sum_{n=0}^{\mu_0 - 1} \prod_{k=0}^{\nu_0 - 1} \lambda_{jk} - \lambda
\]
\[
\prod_{n=-\infty}^{\mu_0} \prod_{k=0}^{\nu_0 - 1} \lambda_{jk}^{01} \left(1 + \lambda_{jk}^{0} - \lambda_{jk}^{01}\right).
\]

**Proof.** Let us consider the case of the boundary value problem \(L(G)\). Using Hadamard’s factorization theorem, one can show
\[
\Delta(\lambda, L(G)) = \frac{C}{\lambda_0} \prod_{n=0}^{\mu_0 - 1} \prod_{k=0}^{\nu_0 - 1} \lambda_{nk}^{01} \left(1 + \lambda_{nk}^{0} - \lambda_{nk}^{01} \right)
\]
\[
\prod_{n=-\infty}^{\mu_0} \prod_{k=0}^{\nu_0 - 1} \lambda_{nk}^{01} \left(1 + \lambda_{nk}^{0} - \lambda_{nk}^{01}\right).
\]
Clearly, that \(\forall \lambda \in [-\infty, 0]\)
\[
\left| \sum_{n=0}^{\mu_0 - 1} \lambda_{nk}^{01} \lambda_{nk}^{0} - \lambda \right| \leq \sum_{n=0}^{\mu_0 - 1} \sum_{k=0}^{\nu_0 - 1} \left| \lambda_{nk}^{01} \lambda_{nk}^{0} - \lambda \right| \leq \sum_{n=0}^{\mu_0 - 1} \frac{2 \varepsilon_{nk}}{\rho_{nk}^{01}} + \sum_{n=0}^{\mu_0 - 1} \left| \varepsilon_{nk} \right|^2
\]
Using Hölder’s inequality and \(\varepsilon_{nk} \in l_{2\mu_k}\), one can show
\[
\sum_{n=0}^{\mu_0 - 1} \sum_{k=0}^{\nu_0 - 1} \frac{2 \varepsilon_{nk}}{\rho_{nk}^{01}} + \sum_{n=0}^{\mu_0 - 1} \left| \varepsilon_{nk} \right|^2 < \infty,
\]
Then using Weierstrass M-test, we obtain the uniform convergence of the series
\[
\sum_{n=0}^{\mu_0 - 1} \sum_{k=0}^{\nu_0 - 1} \lambda_{nk}^{01} \lambda_{nk}^{0} - \lambda
\]
where \(\lambda \in [-\infty, 0]\). Then we obtain the uniform convergence of the infinite product
\[
\prod_{n=0}^{\mu_0 - 1} \prod_{k=0}^{\nu_0 - 1} \left(1 + \lambda_{nk}^{01} \lambda_{nk}^{0} - \lambda \right)
\]
Clearly, that
\[
\lim_{\lambda \to -\infty} \prod_{n=0}^{\mu_0 - 1} \prod_{k=0}^{\nu_0 - 1} \left(1 + \lambda_{nk}^{01} \lambda_{nk}^{0} - \lambda \right) = 1, \quad \forall N \in \mathbb{N}.
\]
It follows from the Moore-Osgood theorem, that
\[
\lim_{\lambda \to -\infty} \lim_{N \to \infty} \prod_{n=0}^{\mu_0 - 1} \prod_{k=0}^{\nu_0 - 1} \left(1 + \lambda_{nk}^{01} \lambda_{nk}^{0} - \lambda \right) = 1
\]
Using $\varepsilon_{nk} \in l_2\mu_k$, one can show the convergence of the infinite product

$$
\prod_{n=0}^{\infty} \prod_{k=0}^{\mu_0-1} \frac{\lambda_{nk}}{\mu_{nk}} = \prod_{n=0}^{\infty} \prod_{k=0}^{\mu_0-1} \left( 1 + 2\frac{\varepsilon_{nk}}{\rho_{nk}} + \left( \frac{\varepsilon_{nk}}{\rho_{nk}} \right)^2 \right)
$$

Clearly, that

$$
C_0 = (-1)^{\mu_0} \frac{\partial^{\mu_0}}{\partial \lambda^{\mu_0}} \Delta_0(\lambda, L(G)) \bigg|_{\lambda=0} .
$$

Using lemma 2.1 and (3.2), we assume, that $\lim_{\lambda \to -\infty} \frac{\Delta(\lambda, L(G))}{\Delta_0(\lambda, L(G))} = 1$. Consequently, using (3.9) we obtain (3.8). Analogously, one can prove (3.8) for the boundary value problem $L_j(G)$. □

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