Dynamics of the non-diagonal Bianchi IX model near the cosmological singularity.

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Abstract

The mathematical structure of the six dimensional physical phase spaces of the non-diagonal Bianchi IX model is analyzed in the neighborhood of the cosmological singularity. Critical points of the Hamiltonian equations appearing at infinities are of the nonhyperbolic type. Specific transformations of the phase space, including projection into finite region, do not change this type of criticality which is difficult for investigation by standard analytical tools. The nonhyperbolicity seems to be a generic feature of considered singular dynamics. The information that can be obtained from the linearized vector field is inconclusive. Making use of the physical Dirac observables as the phase space coordinates lowers substantially the dimensionality of the dynamics arena. Here, using commonly known methods for studying the dynamics may turn out to be quite satisfactory.

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I. INTRODUCTION

The available data of observational cosmology indicate that the Universe emerged from a state with extremely high energy densities of matter fields. Theoretical cosmology, particularly the Belinskii-Khalatnikov-Lifshitz (BKL) scenario (general solution of general relativity), predicts the existence of the initial cosmological singularity with diverging gravitational and matter fields invariants \([1, 2]\). It has been proved that such symmetries of spacetime like isotropy and homogeneity are dynamically unstable in the evolution towards the singularity \([1, 2]\). An existence of the dynamically stable general solution to the Einstein equations with a cosmological singularity means that this classical theory is incomplete\(^1\). It is expected that finding the singularity free quantum cosmology would help in the construction of a theory unifying gravitation and quantum physics. The standard model of cosmology is unable to explain, at the fundamental level, the nature of dark energy and the origin of cosmic inflation. It is expected that quantum BKL theory may bring solution to these problems.

The best prototype for the BKL scenario is the non-diagonal Bianchi IX model \([2, 5]\). We expect that obtaining quantum Bianchi IX model will enable quantization of the BKL theory with the general cosmological singularity. Finding the non-singular quantum BKL theory would mean solving the generic cosmological singularity problem. Such a quantum theory could be used as a realistic model of the very early Universe. Obtaining quantum BKL theory may have fundamental importance for physics and cosmology.

In this paper we examine the mathematical structure of the physical phase spaces of the non-diagonal Bianchi IX model near the cosmological singularity. In our analyses we apply the dynamical systems methods. We try to find some convenient variables in the phase space such that the critical points of the vector field specifying the dynamics are possibly of hyperbolic type. Quantum versions of these variables will be used as building blocks for finding quantum compound observables. The latter are supposed to be measurable observables.

In Sec. II we identify the Hamiltonian of our system, which turns out to be the dynamical constraint, and we show that Hamilton’s equations are equivalent to the Lagrangian equations. In Sec III we analyze the structure of the phase space and identify the subspace of critical points. It turns out to be of nonhyperbolic type. In the rest of the paper we consider two mappings of the phase space to find the one leading to hyperbolic type of critical points. The McGehee type transformation considered in Sec IV leads to another system with nonhyperbolic critical subspace. The commonly used transformation of the Poincaré type, presented in Sec V, leads to the nonhyperbolic critical subspace as well. Finally, in Sec. VI we propose to use the Dirac observables to parametrize the phase space. We conclude in the last section. The Appendix presents the dynamics of the Morse oscillator which motivates our use of the McGehee transformation.

\(^1\) For an excellent introduction to the BKL scenario we recommend a review paper \([3]\).
II. EQUATIONS OF MOTION

A. Lagrangian

An asymptotic form (near the cosmological singularity) of the dynamical equations of the non-diagonal Bianchi IX model reads (see, [4] and equations (2.25)-(2.28) in [5]):

\[
\frac{\partial^2 \ln a}{\partial \tau^2} = \frac{b}{a} - a^2, \quad \frac{\partial^2 \ln b}{\partial \tau^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \quad \frac{\partial^2 \ln c}{\partial \tau^2} = a^2 - \frac{c}{b}.
\] (1)

The solutions to (1) must satisfy the condition:

\[
\frac{\partial \ln a}{\partial \tau} \frac{\partial \ln b}{\partial \tau} + \frac{\partial \ln a}{\partial \tau} \frac{\partial \ln c}{\partial \tau} + \frac{\partial \ln b}{\partial \tau} \frac{\partial \ln c}{\partial \tau} = a^2 + \frac{b}{a} + \frac{c}{b}.
\] (2)

It is easy to verify that (1) can be obtained from the Lagrangian equations of motion

\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}_I} \right) = \frac{\partial L}{\partial x_I}, \quad I = 1, 2, 3,
\] (3)

where \( x_1 := \ln a, \ x_2 := \ln b, \ x_3 := \ln c, \) and \( \dot{x}_I := dx_I/d\tau, \) and where the Lagrangian \( L \) has the form

\[
L := \dot{x}_1 \dot{x}_2 + \dot{x}_1 \dot{x}_3 + \dot{x}_2 \dot{x}_3 + \exp(2x_1) + \exp(x_2 - x_1) + \exp(x_3 - x_2).
\] (4)

B. Hamiltonian

The momenta, \( p_I := \partial L/\partial \dot{x}_I, \) are easily found to be

\[
p_1 = \dot{x}_2 + \dot{x}_3, \quad p_2 = \dot{x}_1 + \dot{x}_3, \quad p_3 = \dot{x}_1 + \dot{x}_2.
\] (5)

The Hamiltonian of the system has the form:

\[
H := p_I \dot{x}_I - L = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{1}{4}(p_1^2 + p_2^2 + p_3^2) - \exp(2x_1) - \exp(x_2 - x_1) - \exp(x_3 - x_2),
\] (6)

which due to the relations (5) and (2) leads to the dynamical constraint:

\[
H = 0.
\] (7)

The Hamilton equations

\[
\dot{x}_I = \{x_I, H\} = \partial H/\partial p_I, \quad \dot{p}_I = \{p_I, H\} = -\partial H/\partial q_I,
\] (8)

where

\[
\{\cdot, \cdot\} := \sum_{I=1}^{3} \left( \frac{\partial \cdot}{\partial x_I} \frac{\partial}{\partial p_I} - \frac{\partial \cdot}{\partial p_I} \frac{\partial}{\partial x_I} \right).
\] (9)
have the following explicit form:

\[ \dot{x}_1 = \frac{1}{2}(-p_1 + p_2 + p_3), \]  

(10)

\[ \dot{x}_2 = \frac{1}{2}(p_1 - p_2 + p_3), \]  

(11)

\[ \dot{x}_3 = \frac{1}{2}(p_1 + p_2 - p_3), \]  

(12)

\[ \dot{p}_1 = 2 \exp(2x_1) - \exp(x_2 - x_1), \]  

(13)

\[ \dot{p}_2 = \exp(x_2 - x_1) - \exp(x_3 - x_2), \]  

(14)

\[ \dot{p}_3 = \exp(x_3 - x_2). \]  

(15)

Taking derivatives of (10)-(12) and making use of (13)-(15) leads directly to Eqs. (1), which proves that Lagrangian and Hamiltonian formulations are completely equivalent.

The system (10)-(15) presents a set of nonlinear coupled differential equations. The space of solution of the above dynamical system is a 6D region of the phase space \( (x_1, x_2, x_3, p_1, p_2, p_3) \in \mathbb{R}^6 \). This region is bounded by the constraint equation (7). Solving (7) with respect to \( x_3 \) gives

\[ x_3 = x_2 + \log \left[ -e^{2x_1} - e^{-x_1+x_2} - \frac{p_1^2}{4} + \frac{p_1p_2}{2} - \frac{p_2^2}{4} + \frac{p_1p_3}{2} + \frac{p_2p_3}{2} - \frac{p_3^2}{4} \right]. \]  

(16)

Substituting (16) into (10)-(15) we get:

\[ \dot{x}_1 = \frac{1}{2}(-p_1 + p_2 + p_3), \]  

(17)

\[ \dot{x}_2 = \frac{1}{2}(p_1 - p_2 + p_3), \]  

(18)

\[ \dot{p}_1 = 2e^{2x_1} - e^{-x_1+x_2}, \]  

(19)

\[ \dot{p}_2 = e^{2x_1} + 2e^{-x_1+x_2} + \frac{p_1^2}{4} - \frac{p_1p_2}{2} + \frac{p_2^2}{4} - \frac{p_1p_3}{2} - \frac{p_2p_3}{2} + \frac{p_3^2}{4}, \]  

(20)

\[ \dot{p}_3 = -e^{2x_1} - e^{-x_1+x_2} - \frac{p_1^2}{4} + \frac{p_1p_2}{2} - \frac{p_2^2}{4} + \frac{p_1p_3}{2} + \frac{p_2p_3}{2} - \frac{p_3^2}{4}. \]  

(21)

Analytical solution to this system may not exist. For our purposes it is sufficient to understand the local properties of the phase space of the system by using the method of nonlinear dynamical systems [6–10].

### III. DYNAMICAL SYSTEM ANALYSIS OF PHASE SPACE

The local geometry of the phase space is characterized by the nature and position of its critical points\(^2\). These points are locations where the derivatives of all the dynamical variables vanish. These are the points where phase trajectories may start, end, intersect, etc. The trajectories can also begin or end at infinities. Then, after a suitable coordinate transformation projecting an unbounded phase space into a compact region (e.g. the so

\(^2\) In what follows we use the terms critical or fixed points.
called Poincaré projection), these originally infinite critical points become well defined for further analyses. The set of finite and infinite critical points and their characteristic, given by the properties of the Jacobian matrix of the linearized equations at those points, may provide a qualitative description of a given dynamical system.

The above situation is specific to the case when a fixed point is of the hyperbolic type. In the case of the nonhyperbolic fixed point, linearized vector field at the fixed point cannot be used to characterize completely local properties of the phase space.

A. Critical points of the unconstrained system

Inserting $\dot{x}_1 = 0 = \dot{x}_2 = \dot{x}_3 = \dot{p}_1 = \dot{p}_2 = \dot{p}_3$ into l.h.s of (10)-(15) leads to the following set of equations:

$$0 = \frac{1}{2}(-p_1 + p_2 + p_3), \quad (22)$$
$$0 = \frac{1}{2}(p_1 - p_2 + p_3), \quad (23)$$
$$0 = \frac{1}{2}(p_1 + p_2 - p_3), \quad (24)$$
$$0 = 2 \exp(2x_1) - \exp(x_2 - x_1), \quad (25)$$
$$0 = \exp(x_2 - x_1) - \exp(x_3 - x_2), \quad (26)$$
$$0 = \exp(x_3 - x_2), \quad (27)$$

plus the Hamiltonian constraint equation:

$$\frac{1}{2}(p_1p_2 + p_1p_3 + p_2p_3) - \frac{1}{4}(p_1^2 + p_2^2 + p_3^2) - \exp(2x_1) - \exp(x_2 - x_1) - \exp(x_3 - x_2) = 0. \quad (28)$$

Equations (22)-(24):

$$0 = -p_1 + p_2 + p_3, \quad (29)$$
$$0 = p_1 - p_2 + p_3, \quad (30)$$
$$0 = p_1 + p_2 - p_3, \quad (31)$$

yield

$$p_1 = 0 = p_2 = p_3. \quad (33)$$

Equations (25)-(27) lead to the conditions:

$$0 = \exp(2x_1), \quad (34)$$
$$0 = \exp(x_2 - x_1), \quad (35)$$
$$0 = \exp(x_3 - x_2), \quad (36)$$

Conditions (34)-(36) are fulfilled for:

$$x_1 \to -\infty \quad (37)$$
$$x_2 - x_1 \to -\infty \quad (38)$$
$$x_3 - x_2 \to -\infty \quad (39)$$
thus for:

\[ x_1 \to -\infty \]  
\[ x_2 \to -\infty, \ x_2 < x_1 < 0, \]  
\[ x_3 \to -\infty, \ x_3 < x_2 < 0. \]  

Thus, the set of critical points fulfills the following conditions:

\[ p_1 = 0 = p_2 = p_3, \]  
\[ x_1 \to -\infty, \ x_2 \to -\infty, \ x_3 \to -\infty, \]  
\[ x_3 < x_2 < x_1 < 0. \]  

One may easily verify that this set satisfies the Hamiltonian constraint \((28)\).

Thus the set of critical points \(S_B\) is given by

\[ S_B : = \{(x_1, x_2, x_3, p_1, p_2, p_3) \in \bar{\mathbb{R}}^6 \mid (x_1 \to -\infty, \ x_2 \to -\infty, \ x_3 \to -\infty) \]  
\[ \land (x_3 < x_2 < x_1 < 0); \ p_1 = 0 = p_2 = p_3 \}, \]  

where \(\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}\).

Stability properties are determined by the eigenvalues of the Jacobian of the system \((10)-(15)\). More precisely, one has to linearize equations \((10)-(15)\) at each point. Inserting \(\vec{x} = \vec{x}_0 + \delta \vec{x}\), where \(\vec{x} = (x_1, x_2, x_3, p_1, p_2, p_3)\), and keeping terms up to 1st order in \(\delta \vec{x}\) leads to an evolution equation of the form \(\delta \dot{\vec{x}} = J \delta \vec{x}\). Eigenvalues of \(J\) describe stability properties at the given point.

The Jacobian \(J\) of the system \((10)-(15)\), evaluated at the set of critical points \(S\), reads:

\[
J = \begin{pmatrix}
0 & 0 & 0 & -1/2 & 1/2 & 1/2 \\
0 & 0 & 0 & 1/2 & -1/2 & 1/2 \\
0 & 0 & 0 & 1/2 & 1/2 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

The characteristic polynomial associated with Jacobian \(J\) is equal to:

\[ P(\lambda) = \lambda^6, \]  

and the eigenvalues are the following:

\( (0, 0, 0, 0, 0, 0) \).

Since the real parts of all eigenvalues of the Jacobian are equal to zero, the set \((46)\) consists of nonhyperbolic fixed points.
B. Critical points of the constrained system

Inserting $\dot{x}_1 = 0 = \dot{x}_2 = \dot{p}_1 = \dot{p}_2 = \dot{p}_3$ into l.h.s. of (17)-(21) leads to the following set of equations:

\begin{align}
0 &= \frac{1}{2}(-p_1 + p_2 + p_3), \\
0 &= \frac{1}{2}(p_1 - p_2 + p_3), \\
0 &= 2e^{2x_1} - e^{-x_1+x_2}, \\
0 &= e^{2x_1} + 2e^{-x_1+x_2} + \frac{p_1^2}{4} - \frac{p_1p_2}{2} + \frac{p_2}{2} - \frac{p_1p_3}{2} - \frac{p_2p_3}{2} + \frac{p_3^2}{4}, \\
0 &= -e^{2x_1} - e^{-x_1+x_2} - \frac{p_1^2}{4} + \frac{p_1p_2}{2} - \frac{p_2}{2} + \frac{p_1p_3}{2} + \frac{p_2p_3}{2} - \frac{p_3^2}{4}.
\end{align}

Equations (49)-(50):

\begin{align}
0 &= \frac{1}{2}(-p_1 + p_2 + p_3), \\
0 &= \frac{1}{2}(p_1 - p_2 + p_3).
\end{align}

yield:

\begin{align}
p_1 &= p_2, \\
p_3 &= 0.
\end{align}

Equations (51)-(53) and the above solutions (56)-(57) lead to the following set of equations:

\begin{align}
0 &= 2e^{2x_1} - e^{-x_1+x_2}, \\
0 &= e^{2x_1} + 2e^{-x_1+x_2}, \\
0 &= -e^{2x_1} - e^{-x_1+x_2}.
\end{align}

Solutions of the above set are the following:

\begin{align}
x_1 &\to -\infty \\
x_2 - x_1 &\to -\infty
\end{align}

or equivalently:

\begin{align}
x_1 &\to -\infty \\
x_2 &\to -\infty, x_2 < x_1.
\end{align}

Thus, the set of critical points of the constrained system (17)-(21) reads:

\begin{align}
p_1 &= p_2, p_3 = 0, \\
x_1 &\to -\infty, x_2 \to -\infty, \\
x_2 &< x_1 < 0.
\end{align}
Taking into account the Hamiltonian constraint \( (6) \) leads to additional condition
\[
x_3 - x_2 \rightarrow -\infty, \tag{70}
\]
or, equivalently, to:
\[
x_3 \rightarrow -\infty, \; x_3 < x_2 < 0. \tag{71}
\]

Requiring that the time derivative of \( x_3 \), Eq \( (16) \), determined from the Hamiltonian constraint \( (6) \) satisfies \( (12) \), leads to Eq \( (24) \) which together with \( (49) \) and \( (50) \) gives finally:
\[
p_1 = 0 = p_2 = p_3, \text{ instead of } (67).
\]

This way we have shown that in both cases of ‘constrained’ and ‘unconstrained’ dynamics the set of critical points \( S \) is defined by Eq. \( (46) \).

The Jacobian \( J \) of the system \( (17)-(21) \) evaluated at the set of critical points \( S \) reads:
\[
J = \begin{pmatrix}
0 & 0 & -1/2 & 1/2 & 1/2 \\
0 & 0 & 1/2 & -1/2 & 1/2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The characteristic polynomial associated with Jacobian \( J \) is equal to:
\[
P(\lambda) = \lambda^5, \tag{72}
\]
so the eigenvalues are the following:
\[
(0, 0, 0, 0, 0) \tag{73}
\]
which is consistent with the result obtained for the case of the unconstrained dynamics.

In both cases of constraint and unconstraint dynamics we are dealing with the nonhyperbolic type of critical points. Thus, getting insight into the structure of the space of orbits near such points requires an examination of the exact form of the vector field defining the phase space of our dynamical system. The information obtained from linearization is inconclusive.

The results we have obtained so far have the following properties:

1. The phase space is higher dimensional.

2. The critical points we are dealing with are of nonhyperbolic type.

3. The set of critical points \( S_B \), defined by \( (46) \), is not a set of isolated points, but a 3-dimensional continuous subspace of \( \mathbb{R}^6 \).

4. The critical subspace \( S_B \) is situated in an asymptotic region of phase space with infinite values of its variables.

Lower dimensional phase spaces can be easily analyzed by making mapping to known solved cases available in textbooks. Our six dimensional phase space needs sophisticated tools to be used. The nonhyperbolicity means that one cannot avoid direct examination of the original nonlinear set of equations defining the dynamics. The corresponding linearized set of equations is unable to reveal the nature of dynamics in the neighborhoods of fixed
points. The vector field may bifurcate there. However, suitable methods seem to be available (see, e.g. [7, 8]) to deal with such type of criticality. The third property is challenging. It needs discovering new methods within dynamical systems theory. Nonisolated critical points are atypical ones. The standard methods presented in textbooks [6, 8] cannot cope with such a case. Making perturbations around critical points at infinities is not a well defined procedure. However, this problem may be avoided by mapping unbounded subspaces onto finite ones so the fourth property should not lead to serious problems.

IV. THE MCGEHEE TYPE VARIABLES

In this section, we apply the McGehee type transformations [11] which has been successfully applied to analyses of dynamical systems of general relativity (see, e.g. [12, 13]). In particular, we try to extend McGehee’s transformations applied to the analysis of the Morse oscillator. The functional form of the vector field specific to Morse’s oscillator looks similarly to our vector field (10)-(15). This is why we hope to get profits using the McGehee mapping, which in the case of the Morse oscillator turns nonhyperbolic fixed points into hyperbolic ones. For completeness, we present in Appendix A the case of the Morse oscillator

We define the transformation

$$\mathbb{R}^6 \ni (x_1, x_2, x_3, p_1, p_2, p_3) \longrightarrow (u_1, u_2, u_3, v_1, v_2, v_3) \in \mathbb{R}_+^3 \times \mathbb{R}^3$$

(74)
as follows:

$$(u_1, u_2, u_3, v_1, v_2, v_3) := (\exp x_1, \exp x_2, \exp x_3, p_1, p_2, p_3).$$

(75)

We would like to interpret the dynamics in the new coordinates in terms of our original coordinates. This is possible if the mapping is canonical [14, 15]:

$$\{u_k, u_l\}_{x,p} = 0 = \{v_k, v_l\}_{x,p}, \quad \{u_k, v_l\}_{x,p} = \delta_{kl},$$

(76)

where

$$\{\cdot, \cdot\}_{x,p} := \sum_{k=1}^{3} \left( \frac{\partial \cdot}{\partial x_k} \frac{\partial \cdot}{\partial p_k} - \frac{\partial \cdot}{\partial p_k} \frac{\partial \cdot}{\partial x_k} \right)$$

(77)

One may easily verify that

$$\{u_k, u_l\}_{x,p} = 0 = \{v_k, v_l\}_{x,p}, \quad \{u_k, v_l\}_{x,p} = u_k \delta_{kl}.$$ 

(78)

Therefore, the transformation (74) is not canonical. Thus, the phase space in the new variables corresponds to a different dynamical system, i.e. the transformation maps the phase space of our original system to the phase space of a new system. As long as we stay at the classical level, i.e. we do not plan making quantization of our classical system, such procedure may be reasonable. It is so if the new system turns out to have simpler dynamics than the original one. In the rest of the present section we verify this expectation.

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3 For an introduction to this problem we recommend item 11 on page 722 of [8]
A. Vector field

Under this transformation, the vector field (10)-(15) turns into

\[ 2 \dot{u}_1 = u_1(-v_1 + v_2 + v_3), \]  
\[ 2 \dot{u}_2 = u_2(v_1 - v_2 + v_3), \]  
\[ 2 \dot{u}_3 = u_3(v_1 + v_2 - v_3), \]  
\[ \dot{v}_1 = 2u_1^2 - u_2/u_1, \]  
\[ \dot{v}_2 = u_2 - u_3/u_2, \]  
\[ \dot{v}_3 = u_3/u_2. \]  

(79) \hspace{1cm} (80) \hspace{1cm} (81) \hspace{1cm} (82) \hspace{1cm} (83) \hspace{1cm} (84)

To have well defined vector field, possibly of class \( C^r \), we rescale the time variable as follows:

\[ \dot{f} := \frac{df}{d\tau} = \frac{df}{ds}dsd\tau = \frac{1}{u_1u_2}df = \frac{1}{u_1u_2}f'. \]  

(85)

Making use of (85) turns (79)-(84) into:

\[ 2u'_1 = u_1^2u_2(-v_1 + v_2 + v_3), \]  
\[ 2u'_2 = u_1u_2^2(v_1 - v_2 + v_3), \]  
\[ 2u'_3 = u_1u_2u_3(v_1 + v_2 - v_3), \]  
\[ v'_1 = 2u_1^2u_2 - u_2^2, \]  
\[ v'_2 = u_2 - u_1u_3, \]  
\[ v'_3 = u_1u_3. \]  

(86) \hspace{1cm} (87) \hspace{1cm} (88) \hspace{1cm} (89) \hspace{1cm} (90) \hspace{1cm} (91)

We also redefine the constraint (7) by multiplying it by the factor \( u_1u_2 \). The new form of the constraint reads:

\[ 0 = u_1u_2H =: K, \]  

(92)

where

\[ K = \frac{1}{2}u_1u_2(v_1v_2 + v_1v_3 + v_2v_3) - \frac{1}{4}u_1u_2(v_1^2 + v_2^2 + v_3^2) - u_1^2u_2 - u_2^2 - u_1u_3. \]  

(93)

B. Critical points

All considerations carried out so far have been done with the assumption that \( u_1 > 0, u_2 > 0 \) and \( u_3 > 0 \). From now we lift this assumption down, i.e. we may have \( u_1 \geq 0, u_2 \geq 0 \) and \( u_3 \geq 0 \). One can say that we extend the phase space of our system to a bigger one.

Let us identify the critical points of the vector field (86)-(91). We find that the set of the critical points is defined to be:

\[ S_M := \{(u_1, u_2, u_3, v_1, v_2, v_3) \in \mathbb{R}^6 \mid u_1 = 0 = u_2, u_3 \geq 0, (v_1, v_2, v_3) \in \mathbb{R}^3\}. \]  

(94)

It is clear that any element of \( S_M \) satisfies the constraint (93). One may verify that the transformation (74) does not map the critical subspace \( S_B \) into \( S_M \).
Let us examine the type of criticality specific to the set of critical points (94). It is not difficult to find that the Jacobian of (86)-(91) at any point of the critical set $S_M$ reads:

$$
J = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\nu_3 & 0 & 0 & 0 & 0 & 0 \\
\nu_3 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

The characteristic polynomial associated with $J$ reads: $P(\lambda) = \lambda^6$. Thus, $\lambda = 0$ is an eigenvalue of $J$ of multiplicity 6. Therefore, the set $S_M$ consists of nonisolated nonhyperbolic fixed points.

C. Hamiltonian structure

Suppose that (93) is a Hamiltonian of some new system. Let us derive the corresponding Hamilton’s equations. One gets:

$$
2u_1u'_1 = u_1^2u_2(-v_1 + v_2 + v_3), \quad (95)
$$
$$
2u_2u'_2 = u_1u_2^2(v_1 - v_2 + v_3), \quad (96)
$$
$$
2u_3u'_3 = u_1u_2u_3(v_1 + v_2 - v_3), \quad (97)
$$
$$
u_1v'_1 = 2\nu_1^2u_2 - u_2^2, \quad (98)
$$
$$u_2v'_2 = u_2^2 - u_1u_3, \quad (99)
$$
$$u_3v'_3 = u_1u_3, \quad (100)
$$

which looks similarly to the system (86)-(91), but is still different. Thus, (93) cannot be the Hamiltonian of the original vector field (86)-(91).

V. THE POINCARÉ TYPE VARIABLES

Since examination of phase space at ‘infinite region’, (10), is difficult mathematically, we change coordinates of the phase space to map the set of critical points (10) onto a finite region. We map the infinite space $\mathbb{R}^6$ into a finite Poincaré sphere, parameterized by Cartesian coordinates $(X_1, X_2, X_3, P_1, P_2, P_3)$, as follows:

$$
x_1 =: \frac{X_1}{1 - r}, \quad (101)
$$
$$
x_2 =: \frac{X_2}{1 - r}, \quad (102)
$$
$$
x_3 =: \frac{X_3}{1 - r}, \quad (103)
$$
$$
p_1 =: \frac{P_1}{1 - r}, \quad (104)
$$
$$
p_2 =: \frac{P_2}{1 - r}, \quad (105)
$$
$$
p_3 =: \frac{P_3}{1 - r}. \quad (106)
$$
where \( r^2 = X_1^2 + X_2^2 + X_3^2 + P_1^2 + P_2^2 + P_3^2 \). We also rescale the time parameter \( \tau \) by defining the new time parameter \( T \) such that \( dT := dr/(1 - r) \). In these coordinates our phase space is contained within a sphere of radius one – ‘infinities’ correspond to \( r = 1 \).

If the mapping is canonical, we should have:

\[
\{X_t, X_k\}_{x,p} = 0 = \{P_t, P_k\}_{x,p}, \quad \{X_t, P_k\}_{x,p} = \delta_{tk}. \tag{107}
\]

The map (101)-(106) is not canonical, because we have:

\[
\{X_k, X_t\}_{x,p} = (1 - r)g(a)(x_kp_t - x_t p_k), \tag{108}
\]

\[
\{P_k, P_t\}_{x,p} = (1 - r)f(a)(x_kp_t - x_t p_k), \tag{109}
\]

\[
\{X_k, P_t\}_{x,p} = (1 - r)^2\delta_{kt} - (1 - r)(f(a)x_kx_t + g(a)p_kp_t), \tag{110}
\]

where \( a := r^2/(1 - r)^2 \), \( f(a) \neq 0 \), \( g(a) \neq 0 \). It is clear that there is no chance to satisfy (107) for any \( r \) including the limit \( r \to 1 \).

The insertion of (101)-(106) into (10)-(15) gives:

\[
\begin{align*}
\left(\frac{X_1}{1 - r}\right)' &= \frac{1}{2}(-P_1 + P_2 + P_3), \tag{111} \\
\left(\frac{X_2}{1 - r}\right)' &= \frac{1}{2}(P_1 - P_2 + P_3), \tag{112} \\
\left(\frac{X_3}{1 - r}\right)' &= \frac{1}{2}(P_1 + P_2 - P_3), \tag{113} \\
\left(\frac{P_1}{1 - r}\right)' &= (1 - r)(2\exp \frac{2X_1}{1 - r} - \exp \frac{X_2 - X_1}{1 - r}), \tag{114} \\
\left(\frac{P_2}{1 - r}\right)' &= (1 - r)(\exp \frac{X_2 - X_1}{1 - r} - \exp \frac{X_3 - X_2}{1 - r}), \tag{115} \\
\left(\frac{P_3}{1 - r}\right)' &= (1 - r)(\exp \frac{X_3 - X_2}{1 - r}), \tag{116}
\end{align*}
\]

where prime denotes derivative with respect to the new time parameter \( T \).

To find the fixed points we insert \( X_1' = 0 = X_2' = X_3' = P_1' = P_2' = P_3' \) into (111)-(116) by using the elementary formulas:

\[
r' = \frac{d}{dT}r = (X_1X_1' + X_2X_2' + X_3X_3' + P_1P_1' + P_2P_2' + P_3P_3')/r \tag{117}
\]

and, e.g.

\[
\frac{d}{dT}\left(\frac{X_1}{1 - r}\right) = \frac{X_1'(1 - r) + X_1r'}{(1 - r)^2}. \tag{118}
\]

After rearrangement of terms we finally get:

\[
- P_1 + P_2 + P_3 = 0, \tag{119}
\]

\[
P_1 - P_2 + P_3 = 0, \tag{120}
\]

\[
P_1 + P_2 - P_3 = 0, \tag{121}
\]

\[
2\exp \frac{2X_1}{1 - r} - \exp \frac{X_2 - X_1}{1 - r} = 0, \tag{122}
\]

\[
\exp \frac{X_2 - X_1}{1 - r} - \exp \frac{X_3 - X_2}{1 - r} = 0, \tag{123}
\]

\[
\exp \frac{X_3 - X_2}{1 - r} = 0. \tag{124}
\]
The solution to (119)-(121) reads: \( P_1 = 0 = P_2 = P_3 \). The equations (122)-(124) can be satisfied in the limit \( r \to 1 \) if

\[
\lim_{r \to 1^-} \exp \frac{2X_1}{1-r} = 0 = \lim_{r \to 1^-} \exp \frac{X_2 - X_1}{1-r} = \lim_{r \to 1^-} \exp \frac{X_3 - X_2}{1-r},
\]

which leads to the condition: \( X_3 < X_2 < X_1 < 0 \). Therefore, the critical subspace is defined to be:

\[
S_P := \{(X_1, X_2, X_3, P_1, P_2, P_3) \mid (X_3 < X_2 < X_1 < 0) \land (P_1 = 0 = P_2 = P_3)\}. \tag{126}
\]

It is not difficult to verify that the transformation (101)-(106) does not map \( S_B \) into \( S_P \).

It is clear that any point of \( S_P \), in the limit \( r \to 1^- \), satisfies the constraint (7), which in the variables (101)-(106) has the form:

\[
\frac{1}{2(1-r)^2} (P_1 P_2 + P_1 P_3 + P_2 P_3) - \frac{1}{4(1-r)^2} (P_1^2 + P_2^2 + P_3^2)
\]

\[
- \exp \frac{2X_1}{1-r} - \exp \frac{X_2 - X_1}{1-r} - \exp \frac{X_3 - X_2}{1-r} = 0. \tag{127}
\]

One can resolve (either manually or by symbolic computations) the nonlinear vector field (111)-(116) with respect to the derivatives \( X_1', X_2', \ldots, P_3' \), and find the corresponding Jacobian. Its value at any point of the subspace \( S_P \) (in the limit \( r \to 1 \)) turns out to be a six dimensional zero matrix. It means that linearization of the exact vector field, at the set of critical points \( S_P \), cannot help in the understanding of the mathematical structure of the space of orbits of considered vector field. An examination of the nonlinearity cannot be avoided. One may say, formally, that the set \( S_P \) consists of the nonhyperbolic type of fixed points.

### VI. THE DIRAC TYPE VARIABLES

Solutions to Eqs. (10)-(15) define the kinematical phase space \( \mathcal{F}_k \). The physical phase space \( \mathcal{F}_p \) is defined by those solutions to (10)-(15), which additionally satisfy Eq. (7).

The physical Dirac observable \( \mathcal{O} \) is defined to be a function on the phase space that weakly Poisson commutes with first-class constraints of the dynamical system [18–20]. In our case there is only one constraint (7) so the equation defining the physical Dirac observables reads

\[
\{\mathcal{O}, H\} \approx 0, \quad \text{where} \quad \{·, ·\} := \sum_{k=1}^{3} \left( \frac{\partial}{\partial x_k} \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_k} \frac{\partial}{\partial x_k} \right), \tag{129}
\]

and where the sign \( \approx \) means that (129) has to be treated as a weak equation in Dirac’s sense [18]. Functions which satisfy (129) strongly define the kinematical observables. So they are defined by the solutions to the following equation

\[
(-p_1 + p_2 + p_3) \frac{\partial \mathcal{O}}{\partial x_1} + (p_1 - p_2 + p_3) \frac{\partial \mathcal{O}}{\partial x_2} + (p_1 + p_2 - p_3) \frac{\partial \mathcal{O}}{\partial x_3}
\]

\[
+ 2(e^{2x_1} - e^{x_2-x_1}) \frac{\partial \mathcal{O}}{\partial p_1} + 2(e^{x_2-x_1} - e^{x_3-x_2}) \frac{\partial \mathcal{O}}{\partial p_2} + 2e^{x_3-x_2} \frac{\partial \mathcal{O}}{\partial p_3}
\]

\[
= 0. \tag{130}
\]
The physical observable can be obtained from the solutions to Eqs. \((130)\) satisfying the constraint \((28)\).

For the purpose of quantization, which we plan to carry out in the future, we rewrite the classical dynamics in terms of the Dirac observables. More precisely, we give the physical meaning to the following factorization, consistent with the Darboux theorem [15, 16],

\[
\sum_{k=1}^{3} (dx_k \wedge dp_k) = \sum_{\alpha=1}^{2} (dq_\alpha \wedge d\pi_\alpha) - dt \wedge dH,
\]  \hspace{1cm} (131)

where \((p_m, x_n) \in \mathcal{F}_k\), which can be made in the neighborhood of the constraint surface \(H = 0\). It is clear that the transformation \((x_1, x_2, x_3, p_1, p_2, p_3) \rightarrow (q_1, q_2, t, \pi_1, \pi_2, H)\) underlying \((131)\) is canonical, but not unique.

Now, suppose that \((q_\alpha, \pi_\beta)\) are the Dirac observables. On the constraint surface, due to \((131)\) and considerations on time-dependent vector fields and differential forms in \([15–17]\), we propose the following:

\[
0 = \{q_\alpha, H\}_{|H=0} =: \{q_\alpha, H_T\}_{q,\pi} - \frac{d}{dT} q_\alpha \quad \Rightarrow \quad \frac{d}{dT} q_\alpha = \{q_\alpha, H_T\}_{q,\pi} = \frac{\partial H_T}{\partial \pi_\alpha},
\]  \hspace{1cm} (132)

and

\[
0 = \{\pi_\alpha, H\}_{|H=0} =: \{\pi_\alpha, H_T\}_{q,\pi} - \frac{d}{dT} \pi_\alpha \quad \Rightarrow \quad \frac{d}{dT} \pi_\alpha = \{\pi_\alpha, H_T\}_{q,\pi} = -\frac{\partial H_T}{\partial q_\alpha},
\]  \hspace{1cm} (133)

where

\[
\{\cdot, \cdot\}_{q,\pi} := \sum_{\alpha=1}^{2} \left( \frac{\partial}{\partial q_\alpha} \frac{\partial}{\partial \pi_\alpha} - \frac{\partial}{\partial \pi_\alpha} \frac{\partial}{\partial q_\alpha} \right).
\]  \hspace{1cm} (134)

Therefore, the dynamics is generated by the Hamiltonian \(H_T\) (to be found) and is parameterized by an evolution parameter \(T\). In this new setting the system has no dynamical constraints. One may examine the above dynamics using the dynamical systems methods [7, 8]. The phase space is now only four dimensional, which simplifies analysis of the original dynamics, \((10)-(15)\), defined in six dimensional phase space.

The factorization procedure described above make sense, but not in the neighborhood of the subspace of the critical points \(S_B\) defined by \((16)\), and it is not defined uniquely. However, we expect that taking into account the possible structure of the set of critical points of the vector field \((132)-(133)\) may emerge a natural choice for \(H_T\) and \(T\).

**VII. CONCLUSIONS**

The novelty of our approach is atypical formulation of the Hamiltonian description of the non-diagonal Bianchi IX dynamics. Our point of departure is taking the Lagrange equations in the asymptotic region near the cosmological singularity. Next, we make an educated guess what the Lagrangian and Hamiltonian could be which would lead to these equations by using the principle of least action. Usually one starts with an exact form of the Lagrangian (or Hamiltonian) and analyzing dynamics one gets an asymptotic form of the dynamics. Our approach guarantees that we are really dealing with the prototype of the BKL scenario as the BKL theory was derived by generalization of the Bianchi IX model dynamics considered in this paper [2, 3].
Taking into account the Hamiltonian constraint in two ways, before or after identifying the critical points, leads to the same results concerning the set of critical points of the vector field of the considered dynamical system.

The circumstance that critical points occur in asymptotic regions seems to be not an obstacle because one may project an unbounded subspace onto a finite region.

The original phase space presented in Sec. III is higher dimensional. Since all eigenvalues of the Jacobian (corresponding to the nonlinear vector field) are purely imaginary, no reduction to lower dimensional phase space is possible, because the center manifold theory \cite{7,8} cannot be applied. Since all the critical points are nonhyperbolic, the information obtained from linearization is inconclusive. One has to apply, e.g. the theory of normal forms \cite{7,8} to get insight into the structure of the space of orbits in the neighborhoods of the critical points.

Additional difficulty results from the fact that the set of critical points is not a set of isolated points, but a 3-dimensional continuous subspace of $\mathbb{R}^6$. Perturbation of the vector field in the neighborhood of such a space, seems to require making analyses in transverse space to the space of critical points.

The mapping of the original system by making use of sophisticated transformations into other systems (helpful in analyses of other gravitational systems) does not simplify the problem. The critical points are still nonhyperbolic. This is the case of both the McGehee and the Poincaré transformations. In the former case, the ‘miracle’ of getting away from the nonhyperbolicity observed in the case of Morse’s oscillator does not occur. Apart from this, these mappings are obtained by non-canonical transformations. Such type of transformations of phase space lead to different dynamical systems so quantization of the dynamics before and after such transformation would give different results.

We believe that parameterizing the phase space by the Dirac observables, proposed in Sec. VI, is the best choice. Since the physical phase space defined there is only four dimensional, we expect that the subspace of critical points has even lower dimensionality so it can be examined by the phase portrait methods \cite{9,10}. Finding the Hamiltonian that generates the dynamics which is no longer the dynamical constraint, outlined in Sec. VI, may enable the construction of the quantum evolution of the Bianchi IX universe. Preliminary results obtained in the papers \cite{17,21,23} for the FRW and the Bianchi I models which make use, to some extent, the above ideas are encouraging.

The realization of the factorization described in Sec. VI and quantization of the resulting theory may be done, but it is beyond the scope of the present paper. This intriguing issue will be the subject of our next paper.

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Appendix A: Morse oscillator

In this section we consider the Morse oscillator\(^4\). It is defined by the Hamiltonian:

\[
H(x, y) = \frac{y^2}{2} - \beta \left( e^{-x} - \frac{1}{2} e^{-2x} \right),
\]

(A1)

where \((x, y) \in \mathbb{R}^2\) and \(0 < \beta \in \mathbb{R}\) is a constant. The Hamilton equations read:

\[
\dot{x} = \frac{\partial H}{\partial y} = y, \quad (A2)
\]

\[
\dot{y} = -\frac{\partial H}{\partial x} = -\beta \left( e^{-x} - e^{-2x} \right), \quad (A3)
\]

where dot denotes derivative with respect to the time parameter \(t\). The critical points of the vector field \((A2)-(A3)\), obtained by solving \(\dot{x} = 0 = \dot{y}\), are easily found to be \((x_1, y_1) = (0, 0) =: c_1\) and \((x_2, y_2) = (\infty, 0) =: c_2\).

The characteristic equation of the Jacobian at \(c_1\), corresponding to \((A2)-(A3)\), reads:

\[
0 = \det \left( J(c_1) - \lambda I \right) = \det \left( \begin{array}{cc} -\lambda & 1 \\ \beta & -\lambda \end{array} \right) = (\lambda^2 - \beta),
\]

(A4)

and has the solution

\[
(-\sqrt{\beta}, \sqrt{\beta}), \quad (A5)
\]

which means that the critical point at \(c_1\) is of a hyperbolic type since the real parts of both eigenvalues are different from zero.

The characteristic polynomial corresponding to the critical point at \(c_2\) has the form:

\[
0 = \det \left( J(c_2) - \lambda I \right) = \det \left( \begin{array}{cc} -\lambda & 1 \\ 0 & -\lambda \end{array} \right) = \lambda^2,
\]

(A6)

with the solution

\[
(0, 0). \quad (A7)
\]

Since both eigenvalues are purely imaginary, the critical point at \(c_2\) is of a nonhyperbolic type.

Now, let us apply the McGehee type transformation of the variables [8]:

\[
u := e^{-x/2}, \quad (A8)
v := y. \quad (A9)
\]

and rescale the time as follows:

\[
dt/ds = -2/u. \quad (A10)
\]

The equations \((A2)-(A3)\) are mapped into the system:

\[
\dot{u} = v, \quad (A11)
\]

\[
\dot{v} = 2\beta(u - u^3), \quad (A12)
\]

where prime denotes derivative with respect to the new time parameter \(s\).

---

\(^4\) It is a special case of the so called driven Morse oscillator [8].
It is clear that the vector field (A11)-(A12) has three fixed points: $(u_1, v_1) = (0, 0) =: f_1$, $(u_2, v_2) = (0, 1) =: f_2$ and $(u_3, v_3) = (0, -1) =: f_3$. The Jacobian corresponding to (A11)-(A12) reads:

$$J(u, v) = \begin{pmatrix} 0 & 1 \\ 2\beta(1 - u^2) & 0 \end{pmatrix}.$$  \hspace{1cm} (A13)

It is easy to verify that $J(f_1) = J(f_2) = J(f_3)$. The characteristic polynomial equation, e.g. for $J(f_1)$, reads: $\lambda^2 - 2\beta = 0$, and has the solution $(-\sqrt{2\beta}, \sqrt{2\beta})$. \hspace{1cm} (A14)

Thus, all three critical points are of hyperbolic type.

It is interesting to examine the role of the rescaling of the time (A10). Without the rescaling, the mapping (A8)-(A9) turns the system (A2)-(A3) into the vector field:

$$\begin{align*}
\dot{u} &= -uv/2, \\
\dot{v} &= \beta(u^4 - u^2),
\end{align*}$$

(A15)

which has three critical points at: $(0, v) =: e_1$, $(-1, 0) =: e_2$, $(1, 0) =: e_3$. The first one is in fact a manifold as $v \in \mathbb{R}$. The Jacobian corresponding to the vector field (A15)-(A16) reads:

$$\tilde{J}(u, v) = \begin{pmatrix} -v/2 & -u/2 \\ 2\beta(2u^2 - u) & 0 \end{pmatrix}.$$  \hspace{1cm} (A17)

The characteristic polynomial $\det(\tilde{J}(e_1) - \lambda I) = 0$ has two solutions $(0, -v/2)$, so $e_1$ is a nonhyperbolic fixed point. Since $\tilde{J}(e_2) = -\tilde{J}(e_3)$, the characteristic polynomials in both cases have the same roots, which read: $(-i\sqrt{\beta}, i\sqrt{\beta})$. Thus, both $e_2$ and $e_3$ are centers. We conclude that the time rescaling is crucial.

One may easily verify that we have:

$$\{u, u\}_{x,y} = 0 = \{v, v\}_{x,y}, \quad \{u, v\}_{x,y} = -u/2,$$

(A18)

which means that the transformation (A8)-(A9) is not canonical.

Let us verify if the new vector field (A11)-(A12) can be given the structure of the Hamiltonian vector field. It is not difficult to find that (A11)-(A12) can be obtained via Hamilton’s principle from the Hamiltonian $K(u, v)$ defined to be:

$$K(u, v) := v^2/2 - \beta(u^2 - u^4/2) = y^2/2 - \beta(e^x - e^{-2x}/2) = H(x, y).$$  \hspace{1cm} (A19)

Thus, the transformation (A8)-(A10) leads from one Hamiltonian system to another one, and the relation between both systems reads:

$$y\dot{x} - H(x, y) = v\dot{u} - K(u, v).$$  \hspace{1cm} (A20)

The relation (A20) looks like a standard relation between two Hamiltonian systems related by canonical transformations with the generating function equal to zero [14], but now the time derivatives on both sides of (A20) are determined with respect to different variables related by (A10).

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