Differential topology

The displaced disks problem via symplectic topology

Le problème des disques déplacés via la topologie symplectique

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1. Introduction

Let $\Sigma$ be a closed surface endowed with an area form $\Omega$. We denote by $\text{Homeo}^0(\Sigma)$ the group of area preserving the homeomorphisms of $\Sigma$, by $\text{Homeo}^0_0(\Sigma)$ the path component of identity in $\text{Homeo}^0(\Sigma)$, and by $\text{Ham}(\Sigma)$ the group of Hamiltonian diffeomorphisms of $\Sigma$. We will be studying, $\mathcal{G}$, the $C^0$ closure of $\text{Ham}(\Sigma)$ inside $\text{Homeo}^0_0(\Sigma)$.

The group $\mathcal{G}$ is a well-studied dynamical object: it is precisely the set of elements of $\text{Homeo}^0_0(\Sigma)$ with vanishing mass flow. For a definition of the mass flow homomorphism, which is also known as the mean rotation vector, see Section 5 of [3]. Equivalently, $\mathcal{G}$ can be described as the set of elements of $\text{Homeo}^0_0(\Sigma)$ with zero flux; see Appendix A.5 of [3]. In this note, we only work with the description of $\mathcal{G}$ as the $C^0$ closure of $\text{Ham}(\Sigma)$. It is well known that in the case of $S^2$, $\mathcal{G} = \text{Homeo}^0_0(S^2)$.

Recall that a homeomorphism $\phi$ is said to displace a set $B$ if $\phi(B) \cap B = \emptyset$. For $a > 0$, define $\mathcal{G}_a = \{ \theta \in \mathcal{G} : \theta \text{ displaces a topological disk of area at least } a \}$. The displaced disks problem, posed by F. Béguin, S. Crovisier, and F. Le Roux, asks the following.

Question. (See Béguin, Crovisier, Le Roux [1].) Does the $C^0$ closure of $\mathcal{G}_a$ contain the identity?

The initial motivation of Béguin, Crovisier, and Le Roux for posing this beautiful question is as follows: $\mathcal{G}$ is a normal subgroup of $\text{Homeo}^0(\Sigma)$. Béguin, Crovisier, and Le Roux were interested in knowing whether the conjugacy class of an
The following facts were pointed out to me by Béguin, Crovisier, and Le Roux.

1. **Theorem 1** would not hold without requiring \( \theta \in G \). Indeed, it is not difficult to see that a \( C^0 \)-small translation of the torus displaces a disk with an area nearly equal to half the total area of the torus.

2. **Corollary 2** does not hold for arbitrary elements of \( \text{Homeo}_0(\Sigma) \). See Remark 7.11 of [5] for an example of a homeomorphism of \( S^2 \) whose conjugacy class is \( C^0 \)-dense in \( \text{Homeo}_0(S^2) \). Once it is established that the conjugacy class of a homeomorphism is dense, then it is easy to see that the conjugacy class of that homeomorphism is a \( G_3 \) set. Hence, we conclude that the conjugacy class of a generic homeomorphism of \( S^2 \) is dense.

3. Suppose that \( \Sigma \neq S^2 \). It follows from the work of Gaumbado and Ghys [4] and Entov, Polterovich, and Py [2] that \( G \) carries \( C^0 \)-continuous and homogeneous quasimorphisms; see Theorem 1.2 of [2]. **Corollary 2** follows immediately as homogeneous quasimorphisms are constant on conjugacy classes.

### 2. Proof of Theorem 1

Our proof uses Floer theoretic invariants of Hamiltonian diffeomorphisms. In particular, we use the theory of spectral invariants, or action selectors, introduced by C. Viterbo, M. Schwarz, and Y.-G. Oh [10,8,7]. An important consequence of this theory is that the group of Hamiltonian diffeomorphisms of a closed symplectic manifold \( M \) admits a conjugation invariant norm \( \gamma : \text{Ham}(M) \to [0, \infty) \). Being a conjugation invariant norm, \( \gamma \) satisfies the following axioms:

1. \( \gamma(\phi) \geq 0 \) with equality if and only if \( \phi = \text{Id} \),
2. \( \gamma(\phi) = \gamma(\phi^{-1}) \),
3. \( \gamma(\phi \psi) \leq \gamma(\phi) + \gamma(\psi) \),
4. \( \gamma(\psi \phi \psi^{-1}) = \gamma(\phi) \).

An important feature of \( \gamma \) is the fact that it satisfies the so-called energy-capacity inequality. In the case of a closed surface \( \Sigma \) the energy-capacity inequality states that if \( \phi \in \text{Ham}(\Sigma) \) displaces a disk of area \( a \), then:

\[
a \leq \gamma(\phi).
\]

**Theorem 2** of [9] provides the final step of our solution. According to this theorem, for a closed surface \( \Sigma \), of genus \( g \), there exist constants \( C, \delta > 0 \) such that \( \forall \phi \in \text{Ham}(\Sigma) \) if \( d_{C^0}(\text{Id}, \phi) \leq \delta \), then

\[
\gamma(\phi) \leq C d_{C^0}(\text{Id}, \phi)^{2-2g-1}.
\]  

We now prove **Theorem 1**. For a contradiction, suppose it does not hold and pick a sequence \( \theta_i \in G_\delta \) that converges uniformly to the identity and conclude from Inequality (2) that \( \gamma(\theta_i) \to 0 \). But this is impossible because the energy-capacity inequality (1) implies that \( \gamma|_{G_\delta} \geq a \).

#### 2.1. Extending \( \gamma \) to \( G \)

We will finish this note by showing that the conjugation invariant norm \( \gamma \) extends continuously to \( G \). We need a small modification of Inequality (2). Let \( C, \delta \) denote the constants appearing in this inequality and suppose that \( d_{C^0}(\psi, \phi) \leq \delta \), where \( \psi, \phi \in \text{Ham}(\Sigma) \). Applying Inequality (2) to \( \psi \phi^{-1} \), we obtain that

\[
\gamma(\psi \phi^{-1}) \leq C d_{C^0}(\text{Id}, \psi \phi^{-1})^{2-2g-1}.
\]

\[1 \gamma \] is usually defined on the universal cover of \( \text{Ham} \). For \( \phi \in \text{Ham} \), one can define \( \gamma(\phi) \) by taking infimum over all paths which end at \( \phi \).
\[ C d_{C^0}(\phi, \psi) 2^{-2g-1} \]; the latter inequality follows from the definition of \(d_{C^0}\). Now, using Axiom (iii) of \(\gamma\), we see that \(\gamma'(\psi) - \gamma'(\phi) \leq \gamma(\psi\phi^{-1})\). Hence, \(\gamma'(\psi) - \gamma'(\phi) \leq C d_{C^0}(\psi, \phi)^{2^{-2g-1}}\). Similarly, we obtain the same upper bound for \(\gamma'(\phi) - \gamma'(\psi)\). Therefore, we have proven that \(\forall \psi, \phi \in \text{Ham}(\Sigma)\) if \(d_{C^0}(\psi, \phi) \leq \delta\), then

\[ |\gamma'(\psi) - \gamma'(\phi)| \leq C d_{C^0}(\psi, \phi)^{2^{-2g-1}}. \]

We see that \(\gamma\) is uniformly continuous with respect to \(d_{C^0}\) and so, it extends continuously to \(G\). Clearly, the extension \(\gamma : G \rightarrow \mathbb{R}\) satisfies Inequalities (1) and (2), in addition to the four stated axioms of conjugation invariant norms.

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