Decoupling, exponential sums and the Riemann zeta function

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Abstract

We establish a new decoupling inequality for curves in the spirit of [B-D1],[B-D2] which implies a new mean value theorem for certain exponential sums crucial to the Bombieri-Iwaniec method as developed further in [H]. In particular, this leads to an improved bound $|\zeta(1/2+it)| \ll t^{53/342+\epsilon}$ for the zeta function on the critical line.

0 Introduction

The main result of the paper is the essentially sharp bound on the mean-value expression for $r = 6$ (see [H] for details)

$$A_r\left(\frac{1}{N^2}, \frac{1}{N}\right) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \sum_{n \leq N} e(nx_1 + n^2x_2 + N^{\frac{3}{2}}x_3 + N^{\frac{1}{2}}x_4)^{2r} \, dx_1dx_2dx_3dx_4.$$  

(0.1)

It is proven indeed that $A_6 \ll N^{6+\epsilon}$ (see Theorem 2 below). The bound $A_6(\delta, \delta N) \ll \delta N^{7+\epsilon}$, $\frac{1}{N} \leq \delta \leq \frac{1}{N}$, established in [H-K], plays a key role in the refinement of the Bombieri-Iwaniec approach [B-I1] to exponential sums as developed mainly by Huxley (see [H] for an expository presentation). As pointed out in [H], obtaining good bounds on $A_6$ leads to further improvements and this objective was our main motivation.

In [B], we recovered the [H-K] $A_5$-result (in fact in a sharper form) as a consequence of certain general decoupling inequalities related to the harmonic analysis of curves in $\mathbb{R}^d$. Those inequalities were derived from the results in [B-D1] (see also [B-D2]). Theorem 2 will similarly be derived from a decoupling theorem, formulated as Theorem 1. At this point, we do not yet have a full understanding of all the decoupling phenomena for curves and Theorem 1, stated in a more general form than required for the later needs, is a further contribution in this direction.

Let us next briefly recall the structure of the Bombieri-Iwaniec argument. Given an exponential sum

$$\sum_{n \sim M} e(TF(\frac{m}{M}))$$

with $T > M$ and $F$ a smooth function satisfying appropriate derivative conditions, the sum $\sum_{n \sim M}$ is replaced by shorter sums $\sum_{m \in I}$, $I$ ranging over size-$N$ intervals ($N$ a parameter to be chosen). For each $I$, the phase may be replaced by a cubic polynomial and, by Poisson summation, the exponential sum $\sum_{m \in I} e(TF(\frac{m}{M}))$ transformed in a sum of the form

$$\sum_{h \leq H} e(x_1(I)h + x_2(I)h^2 + x_3(I)h^{3/2} + x_4(I)h^{1/2})$$

(0.2)

where the vector $x(I) = (x_j(I))_{1 \leq j \leq 4} \in \mathbb{R}^4$ depends on the interval $I$. 


At this point, one needs to analyze the distributions of
\[ (h, h^2, h^{3/2}, h^{1/2}) \quad (1 \leq h \leq H) \] (0.3)
and
\[ x(I) \text{ with } I \subset \left[ \frac{M}{2}, M \right] \text{ of size } N \] (0.4)
which Huxley refers to as the first and second spacing problems.

Before applying a large sieve estimate, one takes an \( r \)-fold convolution of (0.3) which \( L^2 \)-norm is expressed by mean values of the form (0.1). Roughly speaking, the \( L^2 \)-norm of the distribution (0.4) is bounded by a certain parameter \( B \), which evaluation is highly non-trivial and so far sub-optimal. The only input of this paper is to provide an optimal result for the first spacing problem. It may be applied in various instances discussed in [H] to the effect of providing better bounds. Rather than exploring fully the implications, we limit ourselves here to combining Theorem 2 with the treatment in [H1] and record the corresponding improved bound on \( \zeta \left( \frac{1}{2} + it \right) \).

Recall that the original Bombieri-Iwaniec argument provided the estimate \( |\zeta \left( \frac{1}{2} + it \right)| \ll |t|^{\frac{9}{56} + \varepsilon} \), \( \frac{9}{56} = 0, 16071 \), see [B-I1] [B-I2]. The work of Huxley in [H1] (resp. [H2]) produced the exponents \( \frac{89}{570} = 0, 15614... \) and \( \frac{32}{205} = 0, 15609... \), resp.

while our \( A_6 \)-bound leads to the exponent \( \frac{53}{342} = 0, 15447... \).

Here, we rely on the estimate of the \( B \)-parameter obtained in [H1] and possibly the subsequent work on this matter may lead to a further small improvement.

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1 A decoupling inequality for curves

Let \( \Phi = (\phi_1, \ldots, \phi_d) : [0, 1] \rightarrow \Gamma \subset \mathbb{R}^d \) be a smooth parametrization of a non-degenerate curve in \( \mathbb{R}^d \), more specifically we assume the Wronskian determinant
\[
\det[\phi_j^{(s)}(t)]_{1 \leq j, s \leq d} \neq 0 \text{ for all } t_1, \ldots, t_d \in [0, 1].
\] (1.1)

Let us assume moreover that \( d \) is even. For \( \Omega \subset \mathbb{R}^d \) a bounded set of positive measure, denote \( \|f\|_{L^p_\Omega} = \left( \frac{1}{|\Omega|} \int_{\Omega} |f|^p dx \right)^{\frac{1}{p}} \) the average \( L^p \)-norm and let \( B_\rho \) be the \( \rho \)-cube in \( \mathbb{R}^d \) centered at 0. We prove the following decoupling property in the spirit of results in [B], [B-D2].

**Theorem 1.** Let \( \Gamma \) be as above and \( I_1, \ldots, I_d \subset [0, 1] \) subintervals that are \( O(1) \)-separated, let \( N \) be large and
\[ \{I_\tau\} \text{ a partition of } [0, 1] \text{ in } N^{-\frac{1}{d}}\text{-intervals. Then for arbitrary coefficient functions } a_j = a_j(t) \]

\[
\left\| \prod_{j=1}^{d/2} \int_{I_\tau} a_j(t) e(x, \Phi(t)) dt \right\|_{L^2_\#(B_N)}^{2/d} \ll
\]

\[
N^{\frac{1}{2} + \varepsilon} \prod_{j=1}^{d/2} \left[ \sum_{\tau \cap I_\tau < I_\tau} \left\| \int_{I_\tau} a_j(t) e(x, \Phi(t)) dt \right\|_{L^6_\#(B_N)}^6 \right]^{\frac{1}{6d}}
\]

(1.2)

holds, with \(\varepsilon > 0\) arbitrary.

Here \(e(z)\) stands for \(e^{2\pi i z}\) as usual. Strictly speaking, \(L^6_\#(B_N)\) in the r.h.s. of (1.2) should be some weighted space \(L^6_\#(w_N)\) with weight \(1_{B_N}(x) \lesssim w_N(x) \leq (1 + \frac{|x|^2}{N})^{-10d}\), \(\text{supp } \tilde{w}_N \subset B_{\frac{1}{N}}\) (cf. [B-D1] and [B-D2]). For simplicity, this technical point will be ignored here and in the sequel.

Remarks.

(1.3) Obviously (1.2) implies the same inequality for \(B_N\) replaced by any translate.

(1.4) The case \(d = 2\) is an immediate consequence of the \(L^6\)-decoupling inequality for planar curves of non-vanishing curvature

\[
\left\| \int_0^1 a(t) e(x, \Phi(t)) dt \right\|_{L^6(B_N)} \ll N^\varepsilon \left( \sum_{\tau} \left\| \int_{I_\tau} a(t) e(x, \Phi(t)) dt \right\|_{L^6(B_N)}^2 \right)^{\frac{1}{2}}
\]

(1.5)

where \(\Phi : [0,1] \to \Gamma \subset \mathbb{R}^2\) and \(\{I_\tau\}\) as above, established in [B-D1]. In fact, (1.5) will be the main analytical input required for the proof of (1.2).

(1.6) In the language of [B-D1], [B-D2], (1.2) may be reformulated as follows. Let \(\Gamma_1, \ldots, \Gamma_{d/2} \subset \Gamma\) be \(O(1)\)-separated arcs and \(f_1, \ldots, f_\frac{d}{2} \in L^1(\mathbb{R}^d)\) satisfy \(\text{supp } \hat{f}_j \subset \Gamma_j + B_{\frac{1}{N}}\). Denote \(f_\tau = (\hat{f}_{\tau(\cdot)} + B_{\frac{1}{N}})\). The Fourier restriction of \(f\) to the \(\frac{1}{N} \times \cdots \times \frac{1}{N} \times 1_{\mathbb{N}}^{-1} \times 1_{\mathbb{N}}\) tube \(\Phi(I_\tau) + B_{\frac{1}{N}}\).

Then

\[
\left\| \prod_{j=1}^{d/2} \left| f_j \right|^{2/d} \right\|_{L^6_\#(B_N)} \ll N^{\frac{1}{2} + \varepsilon} \prod_{j=1}^{d/2} \left( \sum_{\tau} \left\| f_{\tau,j} \right\|_{L^6_\#(B_N)}^6 \right)^{\frac{1}{6d}}.
\]

(1.7)

(1.8) It may be worthwhile to explain the relation between (1.7) and other known decoupling inequalities for curves in \(\mathbb{R}^d\).

Firstly, with \(\Gamma\) as above and \(\Gamma_1, \ldots, \Gamma_d \subset \Gamma\) \(O(1)\)-separated, one has a \(d\)-linear inequality (the analogue of [B-C-T] for curves)

\[
\left\| \prod_{j=1}^{d} \left| f_j \right|^{\frac{1}{d}} \right\|_{L^2_\#(B_N)} \leq c \prod_{j=1}^{d} \left( \sum_{\tau} \left\| f_{\tau,j} \right\|_{L^2_\#(B_N)}^2 \right)^{\frac{1}{2d}}.
\]

(1.9)

This inequality turns out to be elementary. Using the fact that the map \( I_1 \times \cdots \times I_d \to \mathbb{R}^d : (t_1, \ldots, t_d) \to \Phi(t_1) + \cdots + \Phi(t_d)\) is a diffeomorphism for \(I_1, \ldots, I_d\) \(O(1)\)-separated by assumption (1.1) and Parseval’s theorem,
one sees indeed that
\[
\left\| \prod_{j=1}^{d} \int_{I_j} a_j(t) e(x, \Phi(t)) \, dt \right\|_{L^2(B_N)} \leq c \prod_{j=1}^{d} \| a_j \|_{L^2(I_j)}.
\] (1.10)

On the other hand, one has the \((d - 1)\)-linear inequality (see [B-D2])
\[
\left\| \prod_{j=1}^{d-1} \left| f_j \right| \right\|_{L^{2(d+1)}(B_N)} \lesssim N^{\frac{2}{2(d+1)}} \prod_{j=1}^{d-1} \left[ \sum_{\tau} \left\| f_{j, \tau} \right\|_{L^{2(d+1)}(B_N)} \right]^{\frac{1}{2(d+1)}}
\] (1.11)
and one observes, for \(d\) even, that the pair \((2d + 1, \frac{2(d+1)}{d+1})\) in (1.11) is obtained by interpolation between the pairs \((2d, 2)\) from (1.9) and \((3d, 6)\) from (1.7). The issue of what’s the analogue of Theorem 1 for odd \(d\) will not be considered here. In fact, our main interest is \(d = 4\), which provides the required ingredient for the exponential sum application.

Before passing to the proof of Theorem 1, we make a few preliminary observations.

Note that in the setting of Theorem 1, (1.9) also implies the inequality
\[
\left\| \prod_{j=1}^{d/2} \left[ \sum_{\tau \subset I_j} \int_{I_{\tau}} a_j(t) e(x, \Phi(t)) \, dt \right] \right\|_{L^2(B_N)} \leq c \prod_{j=1}^{d/2} \left[ \sum_{\tau \subset I_j} \left\| \int_{I_{\tau}} a_j(t) e(x, \Phi(t)) \, dt \right\|_{L^2(B_N)} \right]^{\frac{1}{2}},
\] (1.12)
To see this, take \(f_j(x) = \frac{1}{N} \sum_{0 \leq k < N} \varepsilon_k e(x, \Phi(k/n))\) for \(j = \frac{d}{2} + 1, \ldots, d\) with \(\varepsilon_k = \pm 1\) independent random variables and average over \(\{\varepsilon_k\}\), noting that \(\mathbb{E}_\varepsilon [|f_j|^2] \asymp 1\) and \(\mathbb{E}_\varepsilon [|f_{\tau}|^2] \asymp N^{-\frac{1}{2}}\).

There is also the trivial bound
\[
\left\| \prod_{j=1}^{d/2} \int_{I_j} a_j(t) e(x, \Phi(t)) \, dt \right\|_{L^\infty(B_N)} \leq N^{\frac{d}{2}} \prod_{j=1}^{d/2} \left\| \int_{I_j} a_j(t) e(x, \Phi(t)) \, dt \right\|_{L^\infty(B_N)}^{1/2},
\] (1.13)

Interpolation between (1.12) and (1.13) using appropriate wave packet decomposition as explained in [B-D1] (note that it is essential here that the \(I_r\) are \(N^{-\frac{1}{2}}\)-intervals) gives
\[
\left\| \prod_{j=1}^{d/2} \int_{I_j} a_j(t) e(x, \Phi(t)) \, dt \right\|_{L^{3d}(B_N)} \leq CN^{d/2} \prod_{j=1}^{d/2} \left[ \sum_{\tau \subset I_j} \left\| \int_{I_{\tau}} a_j(t) e(x, \Phi(t)) \, dt \right\|_{L^6(B_N)}^6 \right]^{1/6},
\] (1.14)
with \(\{I_r\}\) a partition in \(N^{-\frac{1}{2}}\)-intervals.
More generally, if $\Delta = \Delta_K \subset \mathbb{R}^d$ is a $K$-cube, we have (by translation)

\[
\left\| \prod_{j=1}^{d/2} \int_{I_j} a_j(t)e(x.\Phi(t))dt \right\|_{L^d_\infty(\Delta)}^{2/d} \leq CK^{1/3} \prod_{j=1}^{d/2} \left[ \sum_{I_{\tau} \subset I_j} \left\| \int_{I_{\tau}} a_j(t)e(x.\Phi(t))dt \right\|_{L^6_\infty(\Delta)}^6 \right]^{1/3} \tag{1.15}
\]

where $\{I_\tau\}$ is now a partition in $K^{-\frac{d}{3}}$-intervals.

The main point of (1.15) is to provide a preliminary $L^6 - L^3$ inequality; the prefactor $K^{1/3}$ is not important for what follows as it will be improved to $K^{\varepsilon}$ using a bootstrap argument.

Returning to (1.1), it follows from the mean value theorem that

\[
| \det[\phi_i'(t_j)]_{1 \leq i,j \leq d} | \sim \prod_{i \neq j} |t_i - t_j| \tag{1.16}
\]

By (1.16) and since $\phi''(t) = \lim_{s \to 0} \frac{1}{s} (\phi'(t + s) - \phi'(t))$, it follows that for $t_1 < \cdots < t_{d/2} \in [0,1]$ $O(1)$-separated,

\[
| \phi'(t_1) \wedge \phi''(t_1) \wedge \phi'(t_2) \wedge \phi''(t_2) \wedge \cdots \wedge \phi'(t_{d/2}) \wedge \phi''(t_{d/2}) | > c \tag{1.17}
\]

holds.

**Proof of Theorem 1.**

Introduce numbers $b(N) > 0$ for which the inequality, with arbitrary $\{a_j\}$,

\[
\left\| \prod_{j=1}^{d/2} \int_{I_j} a_j(t)e(x.\Phi(t))dt \right\|_{L^d_\infty(B_N)}^{2/d} \leq b(N)N^{\frac{d}{6}} \prod_{j=1}^{d/2} \left[ \sum_{I_{\tau} \subset I_j} \left\| \int_{I_{\tau}} a_j(t)e(x.\Phi(t))dt \right\|_{L^6_\infty(B_N)}^6 \right]^{1/3} \tag{1.18}
\]

holds. Our aim is to establish a bootstrap inequality. By (1.14), $b(N) \leq N^{1/6}$. With $K < N$ to specify, partition $B_N$ in $K$-cubes $\Delta = \Delta_K$. We may bound for each $\Delta$ (since the inequalities for $B_K$ and $\Delta_K$ are equivalent)

\[
\int_\Delta \left\| \prod_{j=1}^{d/2} \int_{I_j} a_j(t)e(x.\Phi(t))dt \right\|_{L^6_\infty(\Delta)}^6 \ leq \tag{1.19}
\]

\[
b(K)^{3d} K^{d/2} \prod_{j=1}^{d/2} \left[ \sum_{I_{\tau} \subset I_j} \left\| \int_{I_{\tau}} a_j(t)e(x.\Phi(t))dt \right\|_{L^6_\infty(\Delta)}^6 \right]
\]
with \( \{ I_\sigma \} \) a partition in \( K^{-\frac{1}{2}} \)-intervals. Summation over \( \Delta \subset B_N \) implies then

\[
\int_{B_N} \prod_{j=1}^{d/2} \left| \int_{I_j} a_j(t) c(x, \Phi(t)) dt \right|^6 dx \leq b(K)^{3d} K^{\frac{d}{2}} \sum_{I_{\sigma_1} \subset I_1, \ldots, I_{\sigma_d/2} \subset I_{d/2}} \left\{ \int_{B_N} \prod_{j=1}^{d/2} \left| \int_{I_{\sigma_j}} a_j(t) c((x + z_j), \Phi(t)) dt \right|^6 dx \right\} \prod_j dz_j.
\]

(1.20)

Fix \( I_{\sigma_j} = [t_j, t_j + K^{-\frac{1}{2}}] \subset I_j \) and write for \( t = t_j + s \in I_{\sigma_j} \)

\[
(x + z_j)_j t \Phi(t) = (x + z_j)_j t \Phi(t_j) + (x + z_j)_j t \Phi'(t_j) s + \frac{1}{2}(x + z_j)_j t \Phi''(t_j) s^2 + o(1)
\]

(1.21)

provided

\[
N = o(K^{3/2}).
\]

(1.22)

The inner integral in (1.20) may then be replaced by

\[
\int_{B_N} \prod_{j=1}^{d/2} \left| \int_0^{K^{-\frac{1}{2}}} a_j(t_j + s) c((x + z_j)_j t \Phi'(t_j) s + \frac{1}{2}(x + z_j)_j t \Phi''(t_j) s^2) ds \right|^6 dx
\]

(1.23)

the \( o(1) \)-term in (1.21) producing a harmless smooth Fourier multiplier that may be ignored.

Next, since \( t_1 < t_2 < \cdots < t_{d/2} \) are \( O(1) \)-separated, (1.17) applies and therefore the map \( \mathbb{R}^d \to \mathbb{R}^d : x \mapsto (x, \Phi'(t_1), \frac{1}{2} x, \Phi''(t_1), \ldots, x, \Phi'(t_{d/2}), \frac{1}{2} x, \Phi''(t_{d/2})) \) is a linear homeomorphism. The image measure of the normalized measure on \( B_N \) may be bounded by the normalized measure on \( B_{CN} \), up to a factor and

\[
(1.23) \lesssim \prod_{j=1}^{d/2} \int_{|u||v|<CN} \left| \int_0^{K^{-\frac{1}{2}}} a_j(t_j + s) e(us + vs^2) ds \right|^6 du dv.
\]

(1.24)

This factorization is the main point in the argument.

We may now apply (after rescaling \( s = k^{-\frac{1}{2}} s_1 \)) to each factor in (1.24) the \( 2D \)-decoupling inequality (1.5) with \( \Gamma \) the parabola \((s_1, s_1^2)\) and perform a decoupling at scale \( (CN)\frac{1}{2} \). Thus, by another change of variables,

\[
(1.24) \ll N^\frac{e}{K} \prod_{j=1}^{d/2} \left[ \sum_{I_\tau \subset I_{\sigma_j}} \left\| \int_{I_\tau} a_j(t) c(x, \Phi(t)) dt \right\|_{L^6_p(\mathbb{R}^d)} \right]^3
\]

with \( \{ I_\tau \} \) a partition in \( N^{-\frac{1}{2}} \)-intervals

\[
\ll N^\frac{e}{K} \left( \frac{N}{K} \right)^\frac{d}{2} \prod_{j=1}^{d/2} \left[ \sum_{I_\tau \subset I_{\sigma_j}} \left\| \int_{I_\tau} a_j(t) c(x, \Phi(t)) dt \right\|_{L^6_p(B_N)} \right]^3.
\]

(1.25)

Substituting (1.25) in (1.20) leads to the estimate

\[
b(K)^{3d} N^\frac{e}{K} \prod_{j=1}^{d/2} \left[ \sum_{I_\tau \subset I_j} \left\| \int_{I_\tau} a_j(t) c(x, \Phi(t)) dt \right\|_{L^6_p(B_N)} \right]^3.
\]

(1.26)
Recalling (1.22), one may conclude that
\[ b(N) \ll b(N^{2/3})N^\varepsilon \]
and Theorem 1 follows.

\section{A mean value theorem}

From now on, we focus on \( d = 4 \) (in view of the application to exponential sums) and consider \( \Phi : [0, 1] \to \Gamma \subset \mathbb{R}^4 \) satisfying (1.1). If \( I_1, I_2 \subset \{1, \ldots, N\} \) are \( \sim N \) separated, we get from Theorem 1

\[ \left\| \prod_{j=1}^2 \sum_{n \in I_j} a_n e \left( \Phi \left( \frac{n}{N} \right) \cdot x \right) \right\|_{L^2_\mu(B_N)} \ll \]

\[ N^{\frac{1}{2} + \varepsilon} \prod_{j=1}^2 \left( \sum_{J \subset I_j} \sum_{n \in J} a_n e \left( \Phi \left( \frac{n}{N} \right) \cdot x \right) \right)^6 \] \( L^6_\mu(B_N) \) \( \alpha \)

with \( \{J\} \) a partition of \( \{1, \ldots, N\} \) in \( N^{\frac{1}{2}} \)-intervals.

Again in view of the application, specify

\[ \phi_1(t) = t, \phi_2(t) = t^2 \] \( (2.2) \)

and assume

\[ |\phi'''_3| > c. \] \( (2.3) \)

In order to perform a further decoupling in (2.1), we enlarge the domain \( B_N \), considering first

\[ \Omega = [0, N] \times [0, N^{3/2}] \times [0, N^{3/2}] \times [0, N] \]

which we partition in \( N \)-cubes \( \Delta_N \).

Let \( I_1, I_2 \) be as above. Application of (2.1) on \( \Delta_N \) gives

\[ \left\| \prod_{j=1}^2 \sum_{n \in I_j} a_n e \left( \Phi \left( \frac{n}{N} \right) \cdot x \right) \right\|_{L^2_\mu(\Delta_N)} \ll \]

\[ N^{\frac{1}{2} + \varepsilon} \left[ \prod_{j=1}^2 \left( \sum_{J \subset I_j} \sum_{n \in J} a_n e \left( \Phi \left( \frac{n}{N} \right) \cdot x \right) \right)^6 \right] \] \( L^6_\mu(\Delta_N) \) \( \alpha \)

and summing over \( \Delta_N \)

\[ \left\| \prod_{j=1}^2 \sum_{n \in I_j} \cdots \right\|_{L^2_\mu(\Omega)} \ll \]

\[ N^{\frac{1}{2} + \varepsilon} \left[ \sum_{J_1 \subset I_1} \int_{B_N \times B_N} dz dz' \int_{\Omega} dx \sum_{n \in J_1} a_n e \left( \Phi \left( \frac{n}{N} \right) \cdot (x + z) \right)^6 \right] \sum_{n \in J_2} a_n e \left( \Phi \left( \frac{n}{N} \right) \cdot (x + z') \right)^6. \] \( (2.4) \)
Let \( J_1 = [h_1, h_1 + N^{\frac{1}{2}}] \), \( J_2 = [h_2, h_2 + N^{\frac{1}{2}}] \) with \( h_1 - h_2 \asymp N \). Write for \( n \in J_1 \), \( n = h_1 + m \), recalling (2.2)

\[
\Phi \left( \frac{n}{N} \right) (x + z) = \Phi \left( \frac{h_1}{N} \right) (x + z) + \frac{m}{N} \left( x_1 + z_1 + \frac{2h_1}{N} (x_2 + z_2) + \phi'(\frac{h_1}{N}) (x_3 + z_3) + \phi''(\frac{h_1}{N}) (x_4 + z_4) \right) + \frac{m^2}{N^2} \left( x_2 + \frac{1}{2} \phi''(\frac{h_1}{N}) x_3 \right) + O(1) \tag{2.5}
\]

recalling that \( |x| < N \) and \( |y_2|, |y_3| < N^{3/2}, |y_4| < N \) while \( |m| < N^{\frac{1}{4}} \). Proceed similarly for \( \Phi \left( \frac{n}{N} \right) (x + z'), n \in J_2 \).

Observe that \( z_1, z_1' \) have range \( [0, N] \), so that periodicity considerations and a change of variables in \( z_1, z_1' \) permit to replace the phase (2.5) by

\[
\frac{m}{N} z_1 + \frac{m^2}{N^2} \left( x_2 + \frac{1}{2} \phi''(\frac{h_1}{N}) x_3 \right)
\]

and

\[
\frac{m}{N} z_1' + \frac{m^2}{N^2} \left( x_2 + \frac{1}{2} \phi''(\frac{h_2}{N}) x_3 \right).
\]

Since \( h_1 - h_2 \asymp N \) and (2.3), one more change of variables in \( x_2, x_3 \) gives the phases

\[
\begin{cases}
m u_1 + \frac{m^2}{N^{1/2}} w_1 \\
m u_2 + \frac{m^2}{N^{1/2}} w_2
\end{cases}
\tag{2.6}
\]

with \( u_1, u_2, w_1, w_2 \) ranging in \([0, 1]\). Hence we obtain again a factorization of the integrand in (2.4), i.e.

\[
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} du_1 du_2 dw_1 dw_2 \left| \sum_{m_1 < \sqrt{N}} a_{h_1 + m_1} e \left( m_1 u_1 + \frac{m_1^2}{N^{1/2}} w_1 \right) \right|^6 \left| \sum_{m_2 < \sqrt{N}} a_{h_2 + m_2} e \left( m_2 u_2 + \frac{m_2^2}{N^{1/2}} w_2 \right) \right|^6 \tag{2.7}
\]

and the 2D-decoupling result applied to each factor enables to make a further decoupling at scale \( N^{1/4} \). This clearly permits to bound (2.4) by

\[
N^{\frac{1}{4} + \varepsilon} N^{-\frac{1}{2} + \varepsilon} \left[ \sum_{j'_1 \subset J_1} \int_{j'_1} \int_{0}^{1} \left| \sum_{n \in j'_1'} a_n e(nu_1) \right|^6 \left| \sum_{n \in j'_2} a_n e(nu_2) \right|^6 d u_1 d u_2 \right]^{\frac{1}{12}} \tag{2.8}
\]

with \( \{j'_i\} \) a partition in \( N^{\frac{1}{4}} \)-intervals.

If instead we consider a translate \( \Omega + y \) of \( \Omega \), the expression (2.8) needs to be modified replacing \( a_n \) by \( a_n e \left( \Phi \left( \frac{n}{N} \right) y \right) \).

Finally, consider the domain (according to Huxley’s \( A_6 \)-problem)

\[
\tilde{\Omega} = [0, N] \times [0, N^2] \times [0, N^2] \times [0, N]
\]
which we partition in domains $\Omega_\alpha = \Omega + y_\alpha$ with $\Omega$ as above. Thus for each $\alpha$ (2.8) implies

$$\left\| \prod_{j=1}^2 \left[ \sum_{n \in I_j} \prod_{k=1}^4 a_n \ e\left( \frac{n}{N} \cdot \nu_0 \right) e\left( n u_1 \right) \right] \right\|_{L^1(\Omega_\alpha)} \ll N^{\frac{1}{2} + \varepsilon} \left[ \sum_{J_1 \subset I_1} \int_0^1 \int_0^1 \left| \sum_{n \in J_1} a_n \ e\left( \frac{n}{N} \cdot \nu_0 \right) \right| \left| \sum_{n \in J_1} a_n \ e\left( n u_1 \right) \right| \right]^6 \left| \int_0^1 \cdots \int_0^1 \int_0^1 \int_0^1 du_1 du_2 \right|^\frac{1}{18}.$$ 

and

$$\left\| \prod_{j=1}^2 \left[ \sum_{n \in I_j} \prod_{k=1}^4 a_n \ e\left( \frac{n}{N} \cdot \nu_0 \right) e\left( n u_1 \right) \right] \right\|_{L^1(\Omega)} \ll N^{\frac{1}{2} + \varepsilon} \left[ \sum_{J_1 \subset I_1} \int_0^1 \int_0^1 \left| \sum_{n \in J_1} a_n \ e\left( \frac{n}{N} \cdot \nu_0 \right) \right| \left| \sum_{n \in J_1} a_n \ e\left( n u_1 \right) \right| \right]^6 \left| \int_0^1 \cdots \int_0^1 \int_0^1 \int_0^1 dydu_1 du_2 \right|^\frac{1}{18}. \quad (2.9)$$

Proceeding as before, let $J_1' = [h_1, h_1 + N^{\frac{1}{2}}], J_2' = [h_2, h_2 + N^{\frac{1}{2}}], h_1 - h_2 \approx N$.

Write for $n \in J_1', n = h_1 + m$

$$\Phi\left( \frac{n}{N} \right) \cdot \nu + n u_1 =$$

$$\Phi\left( \frac{h_1}{N} \right) \cdot \nu + h_1 u_1 + m\left( u_1 + \frac{y_1}{N} + 2\left( \frac{h_1}{N} \right) y_2 + \frac{1}{N} \phi'_3\left( \frac{h_1}{N} \right) y_3 + \frac{1}{N} \phi'_4\left( \frac{h_1}{N} \right) y_4 \right)$$

$$+ \frac{m^2}{N^2} \left( y_2 + \frac{1}{N} \phi''_3\left( \frac{h_1}{N} \right) y_3 \right) + O(1).$$

Since $|y_2|, |y_3| < N^2, |y_4| < N$ and $|m| < N^{\frac{1}{2}}$.

Again by periodicity, (2.3) and change of variables, we obtain the phases

$$m u_1 + m^2 w_1$$

and

$$m u_2 + m^2 w_2$$

with $u_1, u_2, w_1, w_2 \in [0, 1]$ and the $L^6$-norms are bounded by the $\ell^2$-norms of the coefficients. In conclusion, we proved that

$$\left\| \prod_{j=1}^2 \left[ \sum_{n \in I_j} a_n e\left( \frac{n}{N} \cdot x \right) \right] \right\|_{L^1(\Omega)} \ll N^{\frac{1}{2} + \varepsilon} ||a||_{\infty} \quad (2.10)$$

with $\Phi$ satisfying (1.1), (2.2), (2.3), i.e.

$$\phi_1(t) = t, \phi_2(t) = t^2, |\phi_3''| > c \text{ and } \left| \begin{array}{cc} \phi''_3(s) & \phi''_4(s) \\ \phi''_3(t) & \phi''_4(t) \end{array} \right| > c \text{ for } s, t \in [0, 1]. \quad (2.11)$$

The following statement is the mean value estimate for $A_6$ in $[H]$. 

9
Theorem 2.

\[ \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left| \sum_{n \leq N} e(nx_1 + n^2 x_2 + N^{\frac{3}{2}} n^{3/2} x_3 + N^{\frac{5}{2}} n^2 x_4) \right|^{12} dx_1 dx_2 dx_3 dx_4 \leq N^{6 + \varepsilon}. \]  

(2.12)

Proof. Let \( I \subset [1, N] \) be an interval of the form \([N_0, N_0 + M]\), \( 100M < N_0 \leq N \), and assume \( I_1, I_2 \subset I \) subintervals of size \( \sim M \) that are \( \sim M \)-separated.

We first estimate

\[ \int \left\{ \prod_{j=1}^2 \left| \sum_{n \in I_j} e(nx_1 + n^2 x_2 + N^{1/2} n^{3/2} x_3 + N^{1/2} n^{1/2} x_4) \right|^6 \right\} dx. \]  

(2.13)

Clearly (2.13) amounts to the number of solutions of the system

\[
\begin{align*}
& m_1 + m_2 + m_3 - m_4 - m_5 - m_6 = m_7 + m_8 + m_9 - m_{10} - m_{11} - m_{12} \\
& m_1^2 + m_2^2 + m_3^2 - m_4^2 - m_5^2 - m_6^2 = m_7^2 + m_8^2 + m_9^2 - m_{10}^2 - m_{11}^2 - m_{12}^2 \\
& (N_0 + m_1)^{3/2} + (N_0 + m_2)^{3/2} + (N_0 + m_3)^{3/2} - (N_0 + m_4)^{3/2} - (N_0 + m_5)^{3/2} - (N_0 + m_6)^{3/2} = \\
& (N_0 + m_7)^{3/2} + (N_0 + m_8)^{3/2} + (N_0 + m_9)^{3/2} - (N_0 + m_{10})^{3/2} - (N_0 + m_{11})^{3/2} - (N_0 + m_{12})^{3/2} + O(N^{-\frac{1}{2}}) \\
& (N_0 + m_1)^{\frac{1}{2}} + \cdots + (N_0 + m_6)^{1/2} = \\
& (N_0 + m_7)^{\frac{1}{2}} + \cdots + (N_0 + m_{12})^{1/2} + O(N^{-\frac{1}{2}}). \\
\end{align*}
\]  

(2.14)

(2.15)

(2.16)

(2.17)

with \( m_1, \ldots, m_6 \in I_1' = I_1 - N_0; m_7, \ldots, m_{12} \in I_2' = I_2 - N_0 \).

Write \( (N_0 + m)^{3/2}, (N_0 + m)^{1/2} \) in the form

\[
(1 + O\left(\frac{M}{N_0}\right)t + \cdots), \phi_4(t) \sim t^4.
\]

Hence \( \Phi(t) = (t, t^2, \phi_3(t), \phi_4(t)) \) satisfies (2.11).

From (2.14), (2.15), (2.18), (2.19), inequalities (2.16), (2.17) may be replaced by

\[
\phi_3\left(\frac{m_1}{M}\right) + \cdots + \phi_3\left(\frac{m_{12}}{M}\right) < O(N^{-\frac{1}{2}} N_0^{3/2} M^{-3})
\]  

(2.20)

\[
\phi_4\left(\frac{m_1}{M}\right) + \cdots + \phi_4\left(\frac{m_{12}}{M}\right) < O(N^{-\frac{1}{2}} N_0^{7/2} M^{-4}).
\]  

(2.21)

The number of solutions of (2.14), (2.15), (2.20), (2.21) may be evaluated by

\[ \int \left\{ \prod_{j=1}^2 \left| \sum_{m \in I_j'} e(mx_1 + m^2 x_2 + \frac{N^\frac{3}{2} M^3}{N_0^{3/2}} \phi_3\left(\frac{m}{M}\right) x_3 + \frac{N^\frac{5}{2} M^4}{N_0^{7/2}} \phi_4\left(\frac{m}{M}\right) x_4 \right|^6 \right\} dx. \]  

(2.22)
According to (2.10), (2.22) and hence (2.13) are bounded by

\[
M^{6+\varepsilon} \left\{ 1 + \frac{N_0^{3/2}}{N^{2} M} \right\} \left\{ 1 + \frac{N_0^{7/2}}{N^{1/2} M^3} \right\} \ll N^{4+\varepsilon} M^2. \tag{2.23}
\]

Returning to (2.12), let \( B(N)N^6 \) be a bound on the l.h.s. We use the same reduction procedure to multi-linear (here bi-linear) inequalities as in \([\text{B, B-D2}](\text{and originating from [B-G]}). \) Denote \( K \) a large constant and partition \([1, N] \) in intervals \( I_0, I_1, \ldots, I_K, \) where \( |I_0| = \frac{100N}{K} \) and \( |I_s| = \left( 1 - \frac{40}{K} \right) \frac{N}{K} = M_0 \) for \( 1 \leq s \leq K. \)

Bound

\[
\int \left| \sum_{n \leq N} \right|_{12}^2 \leq 2^{12} \int \left| \sum_{n \in I_0} \right|_{12}^2 + (2K)^{12} \sum_{1 \leq s \leq K} \int \left| \sum_{n \in I_s} \right|_{12}^2. \tag{2.24}
\]

The first term of (2.24) is bounded by \( 2^{12}100^6K^{-6}b \left( \frac{100N}{K} \right) N^6. \)

For the remaining terms, write \( I_s = [N_s, N_s + M_0], \) \( N_s > 100M_0, \) and make a further partition of \( I_s \) in consecutive intervals \( I_{s,1}, \ldots, I_{s,K} \) of size \( M_1 = M_0 \). The key point (going back to \([\text{B-G}]) is an estimate of the form

\[
\int \left| \sum_{n \in I_s} \right|_{12}^2 \leq 4^{12} \sum_{s_1 \leq K} \int \left| \sum_{n \in I_{s_1}} \right|_{12}^2 + K^{18} \sum_{s_1, s_2 \leq K} \int \left\{ \left| \sum_{n \in I_{s_1}} \right|_{6} \left| \sum_{n \in I_{s_2}} \right|_{6} \right\}. \tag{2.25}
\]

Recall that (2.25) follows from considering the (pointwise in \( x \)) decreasing rearrangement \( \eta_1 \geq \eta_2 \geq \cdots \geq \eta_K \) of the sequence \( \left| \sum_{n \in I_{s_1}} \right|_{1 \leq s_1 \leq K} \) and distinguishing the cases \( \eta_4 < \frac{1}{K} \eta_1 \) and \( \eta_4 \geq \frac{1}{K} \eta_1. \)

Application of (2.23) gives for \( |s_1 - s_2| \geq 2 \)

\[
\int \left\{ \left| \sum_{n \in I_{s_1}} \right|_{6} \left| \sum_{n \in I_{s_2}} \right|_{6} \right\} \ll N^{6+\varepsilon}
\]

and the second sum in (2.25) contributes at most for \( C(K)N^{6+\varepsilon}. \) Replace the second term in the r.h.s. of (2.24) by

\[
(2K)^{12}4^{12} \sum_{s \leq s_1 \leq K} \int \left| \sum_{n \in I_{s_1}} \right|_{12}^2.
\]

Repeating the procedure, partition each \( I_{s,s_1} \) in intervals \( I_{s,s_1,s_2} \) of size \( M_2 = \frac{M_1}{K} \) and apply the decomposition (2.25) for each \( \sum_{n \in I_{s,s_1}} \) etc.

In general, one gets bilinear contributions of the form

\[
(2K)^{12}4^{12\alpha} K^{18} \sum_{J,J'} \int \left\{ \left| \sum_{n \in J} \right|_{6} \left| \sum_{n \in J'} \right|_{6} \right\}. \tag{2.26}
\]

where the sum extends over pairs \( J, J' \) of intervals of size \( M_\alpha = \frac{N}{K^{\alpha+1}}, \alpha \geq 1 \) that are at least \( M_\alpha \)-separated and contained in an interval of the form \([N_0, N_0 + KM_\alpha], KM_\alpha < \frac{1}{100}N_0. \) Again by (2.23)

\[
\int \left\{ \left| \sum_{n \in J} \right|_{6} \left| \sum_{n \in J'} \right|_{6} \right\} \ll N^{4+\varepsilon} M_\alpha^2
\]

implying that

\[
(2.26) \ll C(K)^{4^{12\alpha}} \frac{N}{M_\alpha} N^{4+\varepsilon} M_\alpha^2 \ll N^{6+\varepsilon} \left( \frac{4^{12}}{K} \right)^{\alpha}.
\]
Summing over $\alpha$ eventually leads to the bound
\[ 2^{12}100^6 K^{-6} b\left(\frac{100N}{K}\right) N^6 + N^{6+\varepsilon}. \] (2.27)

On the l.h.s. of (2.24). Therefore
\[ b(N) \leq 2^{12}100^6 K^{-6} b\left(\frac{100N}{K}\right) + C_\varepsilon N^\varepsilon \]
implying $b(N) \ll N^\varepsilon$ and Theorem 2.

Using the notation from [H], Theorem 2 implies

**Corollary 3.** Let $\frac{1}{N^2} \leq \delta \leq 1, \frac{1}{N} \leq \Delta \leq 1$. Then
\[ A_6(\delta, \Delta) = \int \left| \sum_{n \leq N} e\left( n x_1 + n^2 x_2 + \frac{1}{\delta} \left( \frac{n}{N} \right)^{3/2} x_3 \right) + \frac{1}{\Delta} \left( \frac{n}{N} \right)^{1/2} x_4 \right|^{12} dx \ll \delta \Delta N^{9+\varepsilon}. \] (2.28)

Considering the major arc contribution, (2.28) is clearly seen to be essentially best possible.

### 3 Applications to exponential sums

Let $F$ be a smooth function on $[\frac{1}{T}, 1]$ satisfying the condition
\[ |F'''(x)| > c > 0 \] (3.1)
and define
\[ S = \sum_{m=M} e\left( FT\left( \frac{m}{M} \right) \right). \] (3.2)

In what follows, we assume $M < \sqrt{T}$, in view of the application to $|\zeta(\frac{1}{2} + it)|$. We use notation and background from [H] and also rely on [H-W] and §8 in [H1]. For simplicity we ignore logarithmic and $T^\varepsilon$ factors.

Once the parameter $1 \ll N < M$ is chosen, $R$ is defined by the relation
\[ M^3 \sim R^2 NT. \] (3.3)

From the large sieve bound (cf. [H-W], (3.14)), we obtain
\[ |S|^r \ll \frac{M^r}{N^{r/2}} + \sum_{Q} \left( \frac{MR}{NQ} \right)^{r-1} \frac{R^r}{Q^{r/2}} \left[ A_r BH^5 N R^2 \left( 1 + \frac{Q}{N} \right) \right]^{r/2} \] (3.4)
with $Q$ running over dyadic values in the range $R < Q < R^2$, $H = \frac{NQ}{R^2}$, and where with the notation of Corollary 3
\[ A_r = A_r(\delta, H\delta), \delta = \frac{R^2}{N^2} \] (3.5)
and $B = B(Q)$ is a quantity related to the so-called ‘second spacing problem’ mentioned in the Introduction. At
this point, we invoke the treatment and estimate (8.8) from [H1]

\[ B \ll \left( \frac{Q}{R} \right)^{2/3} \frac{M^2 R^{14}}{N^{20}}. \]  

(3.6)

Huxley sets \( r = 5 \) and relies on the bound \( A_5(\delta, \delta H) \ll \delta H^7 \) obtained in [H-K].

Using our estimate (2.28), it follows that

\[ A_6 \ll \delta^2 H^9. \]  

(3.7)

Applying (3.4) with \( r = 6 \) and (3.7) leads to a bound

\[ |S|^6 \ll \frac{M^6}{N^3} + \sum Q \frac{M^6}{N^7/3} \frac{R}{Q^{5/3}} \left( 1 + \frac{Q}{N} \right)^{1/2} \ll \frac{M^6}{N^3} + \frac{M^6}{N^7/3 R^{2/3}} \]  

(3.8)

\[ |S| \ll \frac{M}{N^{7/18} R^{1/9}} \]  

(3.9)

(recalling that \( N \geq R \)).

We choose \( N \) and \( R \) according to [H1] in order to obtain the bound (3.6) on \( B \). Note that this discussion relates to the second spacing problem and is independent of the choice of \( r \).

Thus for \( T^{1/2} \gg M \gg T^{114} \), set

\[ N \asymp MT^{-1/4}, \quad R \asymp MT^{-5/12} \]  

(3.10)

which gives

\[ |S| \ll M^{1/2} T^{-5/12} + \varepsilon. \]  

(3.11)

For \( T^{1/2} \ll M \ll T^{114} \), set

\[ N \asymp M^{1/2} T^{-1/4}, \quad R \asymp M^{1/2} T^{-5/12} \]  

(3.12)

and (3.9) implies

\[ |S| \ll M^{1/2} T^{-1/4} + \varepsilon. \]  

(3.13)

For \( T^{1/3} \ll M \ll T^{2/7} \), set

\[ N \asymp R^2 \asymp \left( \frac{M^3}{T} \right)^{1/2} \]  

(3.14)

which gives

\[ |S| \ll M^{1/3} T^{-4/7} + \varepsilon. \]  

(3.15)

It follows from (3.11), (3.13), (3.15) that

\[ \frac{1}{\sqrt{M}} |S| \ll T^{123/19} + \varepsilon \]  

(3.16)

as long as \( M > T^{114}, \quad \frac{23}{31} = 0.4035 \ldots \)

Finally, we invoke the estimate with \( M = T^{\alpha} \) (see [H1], Theorem 3)

\[ |S| \ll T^{(4+103\alpha)+\varepsilon} \text{ for } 0, 3870 \ldots = \frac{12}{31} < \alpha < 1 \]  

(3.17)

to conclude (3.16) if \( \alpha \leq 0, 406 \ldots \) while the exponent pair \( (\frac{4299}{43860}, \frac{29507}{43860}) \) from Table 17.3 in [H] gives (3.16) for \( \alpha < 0, 3896 \ldots \).
In conclusion, we proved

**Theorem 4.**

\[ \left| \zeta\left(\frac{1}{2} + it\right) \right| \ll T^{\frac{53}{342}} + \epsilon. \]

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