WEIGHTED SOBOLEV REGULARITY AND RATE OF APPROXIMATION OF THE OBSTACLE PROBLEM FOR THE INTEGRAL FRACTIONAL LAPLACIAN

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Abstract. We obtain regularity results in weighted Sobolev spaces for the solution of the obstacle problem for the integral fractional Laplacian. The weight is a power of the distance to the boundary. These bounds then serve us as a guide in the design and analysis of an optimal finite element scheme over graded meshes.

1. Introduction

The purpose of this work is, ultimately, the design of an optimally convergent finite element method for the solution of the obstacle problem for the integral fractional Laplacian which, from now on, we shall simply refer to as the fractional obstacle problem. In addition to the intrinsic interest that the study of unilateral problems with nonlocal operators may give rise to, the fractional obstacle problem appears in the study of optimal stopping times for jump processes; see [33] and [32, Chapter 10]. This, in particular, is used in the modeling of the rational price of a perpetual American option [14]. We also refer the reader to [37] for an account of other applications.

To make matters precise, here we describe the (eventually equivalent) formulations that the fractional obstacle problem may be written as. For \( n \geq 1 \) we let \( \Omega \subset \mathbb{R}^n \) be an open and bounded domain with Lipschitz boundary \( \partial \Omega \) that satisfies the exterior ball condition. For two functions \( f : \Omega \rightarrow \mathbb{R} \) and \( \chi : \Omega \rightarrow \mathbb{R} \), with \( \chi < 0 \) on \( \partial \Omega \), and \( s \in (0, 1) \) we seek a function \( u : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( u = 0 \) in \( \Omega^c \) and it satisfies the complementarity system

\[
\min \{ \lambda, u - \chi \} = 0, \text{ a.e. } \Omega, \quad \lambda := (-\Delta)^s u - f.
\]

This problem can also be written as a constrained minimization problem on the space \( \tilde{H}^s(\Omega) \) (see section 2 for notation). Indeed, if we define the set of admissible functions

\[
\mathcal{K} = \left\{ v \in \tilde{H}^s(\Omega) : v \geq \chi \text{ a.e. } \Omega \right\},
\]

then the solution to the fractional obstacle problem can also be characterized as the (unique) minimizer of the functional

\[
\mathcal{J} : v \mapsto \mathcal{J}(v) = \frac{1}{2} |v|_{\tilde{H}^s(\Omega)}^2 - (f, v),
\]

over the convex set \( \mathcal{K} \). Equivalently, this minimizer \( u \in \mathcal{K} \) solves the variational inequality

\[
(u, u - v)_s \leq (f, u - v), \quad \forall v \in \mathcal{K}.
\]

We refer the reader to section 2.2 and [25] for a more thorough exploration of these formulations and their equivalence. Finally we must mention that among the many, nonequivalent, definitions of the operator \((-\Delta)^s\) that can be provided in a bounded domain, here we choose the so-called integral one; that is, for a sufficiently smooth function \( v : \mathbb{R}^n \rightarrow \mathbb{R} \) we set

\[
(-\Delta)^s v(x) = C(n, s) \text{ p.v.} \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x - y|^{n+2s}} \, dy, \quad C(n, s) = \frac{2^{2s} s \Gamma(s + \frac{n}{2})}{\pi^{n/2} \Gamma(1 - s)}.
\]
The fractional Laplacian of order $s$ is the infinitesimal generator of a $2s$-stable Lévy process. These processes have been widely employed for modeling market fluctuations, both for risk management and option pricing purposes. It is in this context that, as mentioned above, the fractional obstacle problem arises as a pricing model for American options. More precisely, if $u$ represents the rational price of a perpetual American option, modeling the assets prices by a Lévy process $X_t$ and denoting by $\chi$ the payoff function, then $u$ solves \((1.3)\). We refer the reader to [14] for an overview of the use of jump processes in financial modeling.

Taking into account their applications in finance, it is not surprising that numerical schemes for integro-differential inequalities have been proposed and analyzed in the literature; we refer the reader to [22] for a survey on these methods. These applications aim to approximate the price of a number of assets; therefore, the consideration of a logarithmic price leads to problems posed in the whole space $\mathbb{R}^n$. For the numerical solution, it is usual to perform computations on a sufficiently large tensor-product domain. Among the schemes based on Galerkin discretizations, reference [38] utilizes piecewise linear Lagrangian finite elements, while [24] proposes the use of wavelet bases in space. As for approximations of variational inequalities involving integral operators on arbitrary bounded domains, an a posteriori error analysis is performed in [30].

Since the seminal work of Silvestre [37], the fractional obstacle problem started to draw the attention of the mathematical community. Using potential theoretic methods, reference [37] shows that if the obstacle is of class $C^{1,s}$, then the solution to the fractional obstacle problem is of class $C^{1,\alpha}$ for all $\alpha \in (0,s)$; optimal $C^{1,s}$ regularity of solutions was derived assuming convexity of the contact set. The pursuit of the optimal regularity of solutions without a convexity hypothesis, in turn, motivated the celebrated extension by Caffarelli and Silvestre [11] for the fractional Laplacian in $\mathbb{R}^n$. Using this extension technique, Caffarelli, Salsa and Silvestre proved, in [10], the optimal regularity of solutions (cf. Proposition 3.4 below). It is important to notice, however, that this is only a local regularity result. Nothing is said about the boundary behavior of the solution to \((1.3)\). This is a highly nontrivial issue, as it is known that even the solution to a linear problem involving the fractional Laplacian on a very smooth domain possesses limited regularity near the boundary; see [20, 21] and section 2.1 below for details. In addition, regularity results in Hölder spaces are not amenable to the development of an error analysis for a finite element method.

In light of these shortcomings, in this work, following the ideas of [2], we derive global regularity results in weighted Sobolev spaces, which can guide us in the design of an optimally convergent finite element scheme. This is achieved under the assumption of a nondegeneracy condition; namely, the obstacle needs to be sufficiently negative near the boundary and the forcing term must have a sign. If this is the case we have that, essentially, the solution to \((1.3)\) behaves like the solution to the linear problem near the boundary, for which the regularity is known. The regularity results in weighted Sobolev spaces that we obtain allow us to establish convergence rates for the finite element approximations to the fractional obstacle problem.

We must comment that a related analysis for the obstacle problem, corresponding to the spectral fractional Laplacian, was carried out in [25]; we refer the reader to [7] for a comparison between these operators and a survey of numerical methods for fractional diffusion. The recent work [9] also deals with finite element approximations to nonlocal obstacle problems, involving both finite and infinite-horizon kernels. Experiments, carried out for one-dimensional problems with uniform meshes, indicate convergence with order $h^{1/2}$ in the energy norm. However, [9] does not provide an error analysis for the nonlocal obstacle problem. In this paper we show that using suitably graded meshes allows to double the convergence rate in the energy norm. Moreover, a standard argument allows us to extend the results we obtain in this work to nonlocal operators with finite horizon. Finally, we comment that [39] provides regularity results of Lewy–Stampacchia type for the fractional Laplacian. Their use in a numerical setting, however, is not immediate.

The paper is organized as follows. In section 2 we set the notation and assumptions employed in the rest of the work, and review preliminary results about solutions of the linear Dirichlet problem for the fractional Laplacian on bounded domains and the fractional obstacle problem. These results are employed in section 3 to derive weighted Sobolev regularity estimates for solutions of problem \((1.3)\). Then, section 4 applies our regularity estimates to deduce the convergence rate for a finite element approximation of the fractional obstacle problem on bounded domains. This requires the study of a positivity preserving quasi-interpolation operator over fractional order spaces; which is carried out in section 4.1. Finally, the numerical examples presented in section 5 illustrate the sharpness of our theoretical results.
2. Notation and preliminaries

In this section we will introduce some notation and the set of assumptions that we shall operate under. For $n \geq 1$ we let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with Lipschitz boundary $\partial \Omega$ that satisfies the exterior ball condition. The complement of $\Omega$ will be denoted by $\Omega^c$ and the fractional order by $s \in (0,1)$. The ball of radius $R$ and center $x \in \mathbb{R}^n$ will be denoted by $B_R(x)$, and we set $B_R = B_R(0)$. During the course of certain estimates we shall denote by $\omega_{n-1}$ the $(n-1)$-dimensional Hausdorff measure of the unit sphere $\partial B_1$. As usual, we will denote by $C$ a nonessential constant, and its specific value might change from line to line. By $C(A)$ we shall mean a nonessential constant that may depend on $A$. Finally, by $A \approx B$ we mean that $A \leq CB$ and $B \leq CA$.

Unless indicated otherwise, we will follow standard notation regarding function spaces. In particular, the Sobolev space of order $s$ over $\mathbb{R}^n$ is defined as

$$H^s(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n) : \xi \mapsto (1 + |\xi|^2)^{s/2} \mathcal{F}(v)(\xi) \in L^2(\mathbb{R}^n) \right\},$$

with norm

$$\|v\|_{H^s(\mathbb{R}^n)} = \left\| \xi \mapsto (1 + |\xi|^2)^{s/2} \mathcal{F}(v)(\xi) \right\|_{L^2(\mathbb{R}^n)}.$$

In these definitions $\mathcal{F}$ denotes the Fourier transform. The closure of $C_0^\infty(\Omega)$ in $H^s(\mathbb{R}^n)$ will be denoted by $\tilde{H}^s(\Omega)$. This space can also be characterized as follows:

$$\tilde{H}^s(\Omega) := \left\{ v_{|\Omega} : v \in H^s(\mathbb{R}^n), \text{ supp } v \subset \overline{\Omega} \right\}.$$

We comment that, on $\tilde{H}^s(\Omega)$, the natural inner product is equivalent to

$$\langle v, \varphi \rangle_s = \frac{C(n,s)}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, dx \, dy, \quad |v|_{\tilde{H}^s(\Omega)} = (\langle v, v \rangle_s)^{1/2}.$$

The duality pairing between $\tilde{H}^s(\Omega)$ and its dual $\tilde{H}^{-s}(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$. In view of (2.2) we see that, whenever $v \in \tilde{H}^s(\Omega)$ then $(-\Delta)^s v \in H^{-s}(\Omega)$ and that

$$\langle v, \varphi \rangle_s = \langle (-\Delta)^s v, \varphi \rangle, \quad \forall \varphi \in C_0^\infty(\Omega).$$

In section 3 it will become necessary to characterize the behavior of the solution to (1.3) near the boundary. To do so, we must introduce weighted Sobolev spaces, where the weight is a power of the distance to the boundary. We define

$$\delta(x) = \text{dist}(x, \partial \Omega), \quad \delta(x, y) = \min\{\delta(x), \delta(y)\}.$$

Then, for $k \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{R}$, we consider the norm

$$\|v\|_{H^s_k(\Omega)}^2 = \int_\Omega \left( |v(x)|^2 + \sum_{|\beta| \leq k} |\partial^\beta v(x)|^2 \right) \delta(x)^{2\alpha} \, dx,$$

and define $H^s_k(\Omega)$ and $\tilde{H}^s_k(\Omega)$ as the closures of $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$, respectively, with respect to the norm (2.4). We also need to define weighted Sobolev spaces of a non-integer differentiation order, and their zero-trace versions.

**Definition 2.5** (weighted fractional Sobolev spaces). Let $0 < \ell \in \mathbb{R} \setminus \mathbb{Z}$ and $\alpha \in \mathbb{R}$. Assume that $k \in \mathbb{N} \cup \{0\}$ and $\sigma \in (0,1)$ are the unique numbers such that $\ell = k + \sigma$. The weighted fractional Sobolev space is

$$H^s_\ell(\Omega) = \left\{ v \in H^s_k(\Omega) : |\partial^\beta v|_{H^s_\ell(\Omega)} < \infty \forall \beta \in \mathbb{N}^n, \ |\beta| = k \right\},$$

where

$$|v|_{H^s_\ell(\Omega)}^2 = \int_\Omega \sum_{|\beta| \leq k} |\partial^\beta v|^2 \delta(x)^{2\alpha} \, dx.$$

We endow this space with the norm

$$\|v\|_{H^s_\ell(\Omega)}^2 = \|v\|^2_{H^s_k(\Omega)} + \sum_{|\beta| = k} |\partial^\beta v|_{H^s_\ell(\Omega)}^2.$$

Similarly, the zero-trace weighted Sobolev space is

$$\tilde{H}^s_\ell(\Omega) = \left\{ v \in \tilde{H}^s_k(\Omega) : |\partial^\beta v|_{\tilde{H}^s_\ell(\Omega)} < \infty \forall \beta \in \mathbb{N}^n, \ |\beta| = k \right\},$$

and define $\tilde{H}^s_\ell(\Omega)$ as the closure of $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$, respectively, with respect to the norm (2.4). We also need to define weighted Sobolev spaces of a non-integer differentiation order, and their zero-trace versions.
with the norm
\[ \|v\|_{H^s_0(\Omega)}^2 = \|v\|_{H^s_0(\Omega)}^2 + \sum_{|\beta|=k} |\partial^\beta v|_{L^2_0(\mathbb{R}^n)}^2. \]

Spaces like the ones defined above have been considered, for example, in [2] in connection with the study of the regularity properties of the solution to the linear fractional Poisson problem. However, unlike [2], the spaces \( H^s_a(\Omega) \) and \( H^s_0(\Omega) \) require functions to belong respectively to \( H^k_a(\Omega) \) and \( H^k_0(\Omega) \), instead of \( H^k(\Omega) \). This is a weaker condition and that shall become important below.

We remark also that, during our discussion, we will make use of the norms and seminorms of \( H^s_a(\omega) \) and \( \tilde{H}^s_a(\omega) \), where \( \omega \) is a Lipschitz subdomain of \( \Omega \). If that is the case, the weight \( \delta \) will always refer to the distance to \( \partial \Omega \).

As a final preparatory step, we recall an interior regularity result for \( s \)-harmonic functions over balls.

**Lemma 2.6** (balayage). \( \text{Let } w \in L^\infty(\mathbb{R}^n) \text{ be such that } (-\Delta)^s w = 0 \text{ in } B_R. \text{ Then, } w \in C^\infty(B_{R/2}). \)

**Proof.** According to [24] formula (1.6.11'), in the ball \( B_R \), any \( s \)-harmonic function \( w \) can be represented using a Poisson kernel:
\[ w(x) = \int_{B_R} w(y) P(x, y) \, dy, \]
where
\[ P(x, y) = C \left( \frac{R^2 - |x|^2}{|y|^2 - R^2} \right)^s \frac{1}{|x-y|^n}. \]
Consequently, whenever \( x \in B_{R/2} \), it is legitimate to differentiate to any order the representation above.

### 2.1. The linear problem.
Here we consider the linear version of (1.3); that is, we formally set \( \chi = -\infty \) to arrive at the problem: given \( g \in H^{-s}(\Omega) \) we seek for \( w_g \in H^s(\Omega) \) such that
\[ (-\Delta)^s w_g = g \text{ in } \Omega, \quad w_g = 0 \text{ in } \Omega^c. \]
Identity (2.3) yields the existence and uniqueness of a solution to this problem. In addition, since the kernel is positive, we have a nonlocal maximum principle.

**Proposition 2.8** (nonlocal maximum principle). \( \text{Let } g \in H^{-s}(\Omega) \text{ be such that } g \geq 0 \text{ in } \Omega, \text{ then we have that } w_g \geq 0 \text{ in } \Omega. \)

**Proof.** See [34] Proposition 4.1. \( \square \)

The investigation of the regularity of the solution to (2.7) has been an active area of research in recent years. Solutions to this problem are known to possess limited boundary regularity. Namely, the behavior
\[ w_g(x) \approx \text{dist}(x, \partial \Omega)^s, \]
is expected independently of the smoothness of the domain \( \Omega \) and right hand side \( g \). Assuming \( \Omega \) is smooth, this behavior can be precisely quantified in terms of Hörmander regularity [21]; for Lipschitz domains satisfying the exterior ball condition it can also be expressed in terms of the reduced Hölder regularity of solutions [35],
\[ \|w_g\|_{C^s(\mathbb{R}^n)} \leq C\|g\|_{L^\infty(\Omega)}. \]
For one-dimensional or radial domains, these regularity estimates can be further sharpened by deriving explicit expressions for the map \( w \mapsto (-\Delta)^s [\text{dist}(\cdot, \partial \Omega)^s w] \) in terms of expansions in bases consisting of special functions, see [3] [17]. Of importance in the design of optimally convergent finite element schemes is [2], where regularity in spaces similar to those introduced in Definition 2.5 was derived. Below we extend and modify these results to fit the framework that we are adopting here.

**Theorem 2.9** (weighted regularity of \( w_g \)). \( \text{Let } \Omega \text{ be a bounded Lipschitz domain satisfying the exterior ball condition. Let } g \in C^{1-s}(\overline{\Omega}) \text{ and } w_g \text{ be the unique solution of (2.7). Then, for every } \varepsilon > 0, \text{ we have that } w_g \in H^{1-s-2\varepsilon}(\Omega), \text{ with the estimate} \)
\[ \|w_g\|_{H^{1-s-2\varepsilon}(\Omega)} \leq \frac{C(\Omega, s)}{\varepsilon} \|g\|_{C^{1-s}(\overline{\Omega})}. \]
Proof. We must first notice that, as mentioned before, the spaces of Definition 2.5 do not require integrability of the derivatives of functions with respect to Lebesgue measure but with respect to $\delta^{2\alpha}(x)\,dx$. Since, in this case, $\alpha = 1/2 - \varepsilon > 0$, this is a weaker condition, as it allows certain blow up of the derivatives near the boundary. Consequently, for $s \in (1/2, 1)$, the result follows from [2] Proposition 3.12.

In the case $s \in (0, 1/2)$ one can set $\beta = 1 - s$ in [2] Theorem 3.2 to obtain that $w_g \in C^{1-s}(\Omega)$ with the estimate
\[
\|w_g\|_{C^s(\Omega)} + \sup_{x \in \Omega} \delta(x)^{1-s}|\nabla w_g(x)| + \sup_{x,y \in \Omega} \delta(x,y)\frac{\|\nabla w_g(x) - \nabla w_g(y)\|}{|x-y|^s} \leq C(\Omega, s)\|g\|_{C^{1-s}(\Omega)}.
\]
Notice that the middle term in this estimate yields that $w_g \in \tilde{H}^{1}_{1/2-\varepsilon}(\Omega)$, while the last one can be used to conclude the thesis. 

2.2. The fractional obstacle problem: known results. Let us now review the known results about the solution to the fractional obstacle problem \((1.3)\). First we remark that existence and uniqueness of a solution immediately follows from standard arguments, and that this solution is also the minimizer of the functional $J$ over the set $K$. Since this will be useful when dealing with approximation, it is now our intention to explore the equivalence of \((1.3)\) with the complementarity system \((1.1)\). To do so, we first define to coincidence and non-coincidence sets, respectively, by
\[
\Lambda = \{x \in \Omega : u(x) = \chi(x)\}, \quad N = \Omega \setminus \Lambda.
\]

Proposition 2.10 \((1.3) \implies (1.1)\). Let $\Omega$ be a bounded and Lipschitz domain that satisfies the exterior ball condition. Let $\chi \in C(\Omega)$ satisfy $\chi \leq 0$ on $\partial\Omega$ and $f \in L^p(\Omega)$ for some $p > n/2s$. In this setting, the function $u \in H^s(\Omega)$ that solves \((1.3)\) satisfies $u \in C(\Omega)$ as well as the complementarity conditions \((1.1)\).

Proof. Since $u \in K$, then we have that $u - \chi \geq 0$ a.e. $\Omega$. Let now $0 \leq \varphi \in C_0^\infty(\Omega)$ and observe that the function $v = u + \varphi \in K$. This particular choice of test function in \((1.3)\) implies that
\[
\langle u, \varphi \rangle_s \geq \langle f, \varphi \rangle,
\]
and, using \((2.3)\) we conclude that
\[
\langle (-\Delta)^s u - f, \varphi \rangle \geq 0, \quad \forall \varphi \in C_0^\infty(\Omega), \, \varphi \geq 0.
\]
In other words, $\lambda \geq 0$ in the sense of distributions.

On the other hand, according to [24] Theorem 1.2, the assumptions imply that $u \in C(\Omega)$ and, consequently, $N$ is an open set. Let $\varphi \in C_0^\infty(N)$ and notice that, for a sufficiently small $\varepsilon$ we have that $v = u \pm \varepsilon \varphi \in K$. Using these test functions in \((1.3)\) then implies that
\[
\langle \lambda, \varphi \rangle = 0, \quad \forall \varphi \in C_0^\infty(N),
\]
as we intended to show. 

We will also make use of the following continuous dependence result.

Lemma 2.11 (continuous dependence). Let $\chi \in L^\infty(\Omega)$, $f = 0$, and $u \in H^s(\Omega)$ solve \((1.3)\). Then, we have that $u \in L^\infty(\Omega)$ with
\[
\max\{\chi, 0\} \leq u \leq \|\max\{\chi, 0\}\|_{L^\infty(\Omega)} \quad \text{a.e. } \Omega.
\]

Proof. See [24] Corollary 4.2].

Below we will introduce further assumptions on the data $f$ and $\chi$ that will allow us to apply the previous results.

3. Regularity

Having established the existence of solutions and its equivalent characterization as the solution of \((1.1)\), we now begin with the study of its regularity. To do so, we must introduce some notation. First, we will always assume that the obstacle $\chi$ is at least continuous and strictly negative near the boundary. For a positive number $\kappa > 0$ we introduce a kernel $K_\kappa \in C^\infty(\mathbb{R}^n)$ that is such that
\[
K_\kappa(z) = \frac{C(n, s)}{|z|^{n+2s}}, \quad |z| \geq \kappa,
\]
and is extended smoothly for $|z| < \kappa$.

Finally, to concisely quantify the smoothness assumption on the right hand side $f$ we introduce

$$\mathcal{F}_s(\Omega) = \begin{cases} 
C^{2,1-2s}(\Omega), & s \in \left(0, \frac{1}{2}\right), \\
C^{1,2-2s}(\Omega), & s \in \left[\frac{1}{2}, 1\right].
\end{cases}$$

3.1. **Interior regularity.** The interior regularity of the solution to (1.3) will follow from the regularity for the case $\Omega = \mathbb{R}^n$ as detailed in [10]. Let us first slightly extend the main result in that work.

**Lemma 3.2** (regularity in $\mathbb{R}^n$). Let $u \in \tilde{H}^s(\mathbb{R}^n)$ solve (1.3) with $\Omega = \mathbb{R}^n$. If $\chi \in C^{2,1}(\mathbb{R}^n)$, $f \in \mathcal{F}_s(\mathbb{R}^n)$, and $f$ is such that $|f(x)| \leq C|x|^{-\sigma}$ for some $\sigma > 2s$ as $|x| \to \infty$, then we have $u \in C^{1,s}(\mathbb{R}^n)$.

**Proof.** Assume first that $f = 0$. In this case the assertion is the content of [10, Corollary 6.10].

We now reduce the inhomogeneous case to the previous one by invoking the function $w_f$ defined, for $\Omega = \mathbb{R}^n$, in (2.7). Indeed, the function $U = u - w_f$ solves (1.3) with right hand side $f = 0$ and obstacle $\chi - w_f$. Thus, to be able to invoke the reasoning for the homogeneous case, we must ensure that $\chi - w_f \in C^{2,1}(\mathbb{R}^n)$. Since $\chi \in C^{2,1}(\mathbb{R}^n)$ a sufficient condition for this, according to [37, Propositions 2.8 and 2.9], is that $\chi - w_f \in C^{2,1}(\mathbb{R}^n)$. To show the boundedness of $w_f$ we use its explicit representation; see [23, Formula (1.1.12)] and [37, Formula (2.3)]

$$w_f(x) = C(n, -s) \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-2s}} \, dy.$$

Indeed, using the decay of $f$ we can estimate

$$|w_f(x)| \leq \|f\|_{L^\infty(B_R)} \int_{B_R} \frac{1}{|y|^{n-2s}} \, dy + C \int_{B_R} \frac{|x + y|^{-\sigma}}{|y|^{n-2s}} \, dy \leq M.$$ 

Since $w_f \in C^{2,1}(\mathbb{R}^n)$, we deduce $u = U + w_f \in C^{1,s}(\mathbb{R}^n)$, and conclude the proof. \hfill $\Box$

With this result at hand we can establish the local regularity of the solution to (1.3). The idea is to use a localization argument. However, we stress that if $0 \leq \eta \leq 1$ is a smooth cut-off function such that $\eta = 1$ in $\{\chi > 0\}$, then

$$(-\Delta)^s(\eta u) \neq \eta(-\Delta)^s u \quad \text{in } \{\eta = 1\}$$

because of the nonlocal structure of $(-\Delta)^s$. Consequently, we cannot deduce regularity of $\eta u$ directly from that of $(-\Delta)^s u$. This is the difficulty we confront now.

**Remark 3.3** (Cauchy principal values). At this point we must warn the reader about a technical aspect of our discussion. Namely, in what follows we will proceed formally and “evaluate” expressions of the form

$$\int_{\mathbb{R}^n} \frac{w(y)}{|x - y|^{n+2s}} \, dy, \quad \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x - y|^{n+2s}} \, dy,$$

for some function $w : \mathbb{R}^n \to \mathbb{R}$. Evidently, these integrals do not necessarily converge. We are doing this to avoid unnecessary technicalities, and what we mean in these cases is to compute the principal value of these integrals which, in the sense of distributions, is always meaningful. In other words, substitutions of the form

$$\int_{\mathbb{R}^n} \frac{w(y)}{|x - y|^{n+2s}} \, dy \leftarrow \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{w(y)}{|x - y|^{n+2s}} \, dy$$

need to be made below.

**Proposition 3.4** (local Hölder regularity). Let $\Omega$ be a bounded Lipschitz domain, $\chi \in C^{2,1}(\Omega)$ with $\chi < 0$ on $\partial\Omega$, and $f \in \mathcal{F}_s(\Omega)$. Then the solution $u \in \tilde{H}^s(\Omega)$ of (1.3) verifies $u \in C^{1,s}(\Omega)$.

**Proof.** Let us assume, for the time being, that $f = 0$. Let $D \Subset \Omega$ be open. Let $\eta \in C_0^\infty(\Omega)$ be a smooth cutoff function such that

$$D \cup \{\chi > 0\} \Subset \{\eta \equiv 1\}, \quad \text{supp}(\eta) \Subset \Omega, \quad 0 \leq \eta \leq 1.$$
Define $U = \eta u$. We claim that $U \geq \chi$ in $\mathbb{R}^n$. Indeed, if $\chi > 0$ then $\eta = 1$ and $U = u \geq \chi$. On the other hand, since $0 \leq \eta \leq 1$ we can multiply the inequality $u \geq \chi$ by $\eta$ to conclude that

$$U = \eta u \geq \eta \chi,$$

which, if $\chi \leq 0$, implies that $U \geq \chi$.

The objective is now to show that $(-\Delta)^s U$ lies above some smooth function, for if that is the case we can appeal to Lemma 3.2 to conclude that $U \in C^{1,s}(\mathbb{R}^n)$, but since $U = u$ on $D$ the local Hölder regularity of $u$ will follow. To accomplish this we choose $\tau > 0$ sufficiently small so that

$$\text{dist}(\text{supp}(\eta), \partial \Omega) > 2\tau, \quad \text{dist}(D \cup \{\chi > 0\}, \partial\{\eta = 1\}) > 2\tau.$$

We also introduce the disjoint partition $\mathbb{R}^n = A_1 \cup A_2 \cup A_3$, where

$$A_1 = \{x \in \Omega : \text{dist}(x, D \cup \{\chi > 0\}) \leq \tau\},$$

$$A_2 = \{x \in \Omega : \text{dist}(x, \partial \Omega) > \tau, \text{dist}(x, D \cup \{\chi > 0\}) > \tau\},$$

$$A_3 = \{x \in \Omega : \text{dist}(x, \partial \Omega) \leq \tau\} \cup \Omega^c.$$

We examine the behavior of $U$ in each one of these sets:

- $x \in A_1$: In this case $\eta(x) = 1$ and we can write

$$(-\Delta)^s U(x) = (-\Delta)^s u(x) + C(n, s) \int_{\mathbb{R}^n} \frac{(1 - \eta(y))u(y)}{|x - y|^{n+2s}} \, dy.$$

Using that $(-\Delta)^s u(x) = \lambda(x) \geq 0$ and that $\eta = 1$ in $B_\tau(x)$ we see that

$$(-\Delta)^s U(x) = (-\Delta)^s u(x) + C(n, s) \int_{\mathbb{R}^n} \frac{(1 - \eta(y))u(y)}{|x - y|^{n+2s}} \, dy \geq (K_\tau \ast (1 - \eta)u)(x).$$

- $x \in A_3$: In this case the strategy is essentially the same. In this case we have that $\eta \equiv 0$ in $B_\tau(x)$, so that

$$(-\Delta)^s U(x) = C(n, s) \int_{\mathbb{R}^n} -\eta(y)u(y) \frac{y}{|x - y|^{n+2s}} \, dy = -(K_\tau \ast \eta u)(x).$$

- $x \in A_2$: Notice that here we have that dist$(x, \partial \Omega) > \tau$ and dist$(x, \{\chi > 0\}) > \tau$. Consequently $\chi < 0$ in $B_\tau(x)$ and, using Lemma 2.11 we conclude that $u \in L^\infty(B_\tau(x))$ and that

$$\chi < 0 \leq u \text{ a.e. } B_\tau(x).$$

From the complementarity conditions (1.1) it then follows that $u$ is $s$-harmonic in $B_\tau(x)$. According to Lemma 2.6 this implies that $u \in C^\infty(B_{\tau/2}(x))$. In conclusion, we have that $U = \eta u \in C^\infty(B_{\tau/2}(x))$. Let $\mathcal{E} U \in C^\infty_0(\mathbb{R}^n)$ be a smooth extension of $U$ outside of $B_{\tau/2}(x)$. These observations then allow us to obtain that

$$(-\Delta)^s U(x) = C(n, s) \int_{\mathbb{R}^n} \frac{\mathcal{E} U(x) - U(y)}{|x - y|^{n+2s}} \, dy = (-\Delta)^s \mathcal{E} U(x) + C(n, s) \int_{B_{\tau/2}(x)^c} \frac{\mathcal{E} U(y) - U(y)}{|x - y|^{n+2s}} \, dy.$$

Finally, since $\mathcal{E} U = U$ in $B_{\tau/2}(x)$ we conclude that

$$(-\Delta)^s U(x) = (-\Delta)^s \mathcal{E} U(x) + (K_{\tau/2} \ast (\mathcal{E} U - U))(x).$$

Collecting the previous three cases we arrive at

$$(-\Delta)^s U(x) \geq \begin{cases} (K_\tau \ast (1 - \eta)u)(x), & x \in A_1, \\ (K_\tau \ast \eta u)(x) + (K_{\tau/2} \ast (\mathcal{E} \eta u - \eta u))(x), & x \in A_2, \\ (K_\tau \ast \eta u)(x), & x \in A_3. \end{cases}$$

(3.5)

Because the kernels $K_\tau$ are smooth, the right hand side of the expression above is smooth as well. In addition since, in view of Lemma 2.11 $u$ is bounded, we can estimate the right hand side of (3.5) for $x \in A_3$ by

$$|(K_\tau \ast \eta u)(x)| \leq C \int_{\Omega} \frac{1}{|x - y|^{n+2s}} \, dy \leq C \text{dist}(x, \Omega)^{-n-2s} \quad \text{as } |x| \to \infty,$$

which gives the decay required in Lemma 3.2. As a consequence, we have that $(-\Delta)^s U$ is above some smooth function with sufficiently rapid decay and $U \geq \chi$ in $\mathbb{R}^n$. We can then invoke Lemma 3.2 to conclude that $U \in C^{1,s}(\mathbb{R}^n)$. This in turn implies $u \in C^{1,s}(D)$ as desired.

In the general case $f \neq 0$ we apply the same reduction of Lemma 3.2. This completes the proof. \qed
Remark 3.6 (local regularity estimate). Notice that, from (3.5), one can establish an estimate of \(|u|_{C^{1,s}(\overline{\Omega})}\) in terms of \(f\), \(\chi\) and, more importantly, \(\tau\) which, essentially, measures how close the set \(\{\chi > 0\}\) is to the boundary \(\partial \Omega\).

An immediate consequence of the local Hölder regularity is a local Sobolev regularity estimate.

Corollary 3.7 (local Sobolev regularity). In the setting of Proposition 3.4 we have that, for every \(\varepsilon > 0\), the solution to (1.3) satisfies \(u \in H^{1+s-\varepsilon}_{loc}(\Omega)\) with the estimate

\[ |u|_{H^{1+s-\varepsilon}(\Omega)} \leq C(n)|D|^{1/2} \text{diam}(D)^{\varepsilon/4}|u|_{C^{1,s}(\overline{\Omega})}, \]

where \(D \subseteq \Omega\) is any open set and \(\text{diam}(D)\) denotes the diameter of \(D\).

Proof. For \(x, y \in \Omega\) Proposition 3.4 implies the bound

\[ |\nabla u(x) - \nabla u(y)| \leq |u|_{C^{1,s}(\overline{\Omega})}|x - y|^s. \]

This bound, together with integration in polar coordinates, allow us to estimate directly the requisite seminorm as follows:

\[ |u|_{H^{1+s-\varepsilon}(D)}^2 = \int_{D \times D} \frac{|\nabla u(x) - \nabla u(y)|^2}{|x - y|^{n+2s-2\varepsilon}} \, dy \, dx \leq |u|_{C^{1,s}(\overline{\Omega})}^2 \int_{D \times D} \frac{1}{|x - y|^{n-2\varepsilon}} \, dy \, dx \]

\[ \leq |u|_{C^{1,s}(\overline{\Omega})}^2 \text{diam}(D)^{\varepsilon/2} \int_0^{\text{diam}(D)} \zeta^{-1+2\varepsilon} \, d\zeta = |u|_{C^{1,s}(\overline{\Omega})}^2 \text{diam}(D)^{\varepsilon} \cdot \frac{1}{\varepsilon}. \]

This is the asserted estimate. \(\square\)

3.2. Boundary regularity. Let us now study the behavior of the solution to (1.3) near the boundary of the domain \(\partial \Omega\). It is here that the weighted Sobolev spaces introduced in Definition 2.5 shall become important. We begin by recalling that we are assuming the obstacle \(\chi\) is a smooth function that is negative in a neighborhood of the boundary \(\partial \Omega\). In other words, we have

\[ \varrho = \text{dist}(\{\chi > 0\}, \partial \Omega) > 0. \]

In a similar spirit to Remark 3.6, the regularity estimates near the boundary will depend on \(\varrho\). We now choose \(\tau \in (0, \varrho/4)\) and define a layer around the boundary of width \(\tau\), i.e.

\[ \mathcal{B}_\tau = \{x \in \overline{\Omega} : \text{dist}(x, \partial \Omega) < \tau\}. \]

Let \(\eta \in C^\infty(\mathbb{R}^n)\) be a smooth cutoff function such that

\[ 0 \leq \eta \leq 1, \quad \eta(x) = 1 \forall x \in \mathcal{B}_{2\tau}, \quad \text{dist}(\text{supp}(\eta), \{\chi > 0\}) > \tau. \]

We finally set \(N_\eta = \text{supp}(\eta)\).

Having introduced all the necessary notation, we proceed to establish the boundary regularity of \(u\).

Proposition 3.10 (boundary Hölder regularity). Let \(\chi \in C^{2,1}(\Omega) \cap L^\infty(\Omega)\) and \(f = 0\). With the notation introduced above, the function \((-\Delta)^s(\eta u)\) is smooth in \(N_\eta\). In particular, it holds that

\[ \|(-\Delta)^s(\eta u)\|_{C^{1-s}(\mathbb{R}^n)} \leq C\|\eta\|_{C^{1,s}(\mathbb{R}^n)}, \chi, s, n, \Omega, \varrho). \]

Proof. As in the proof of Proposition 3.4 we define \(U = \eta u\) and consider separately two cases:

- \(x \in \mathcal{B}_{3\tau/2}\): In this case \(\eta(x) = 1\) and, consequently, we write

\[ (-\Delta)^s U(x) = (-\Delta)^s u(x) + C(n, s) \int_{\mathbb{R}^n} \frac{(1 - \eta(y))u(y)}{|x - y|^{n+2s}} \, dy. \]

Notice that using, once again, Lemma 2.11, we have that \(u(x) \geq 0 > \chi(x)\) and so the complementarity conditions (1.1) imply that \(\lambda(x) = (-\Delta)^s u(x) = 0\). In addition, since \(1 - \eta(y)\) vanishes in \(\mathcal{B}_{\tau/2}(x)\), we deduce that

\[ (-\Delta)^s U(x) = (K_{\tau/2} \ast (1 - \eta)u)(x), \]

which gives

\[ \|(-\Delta)^s U\|_{C^{1-s}(\mathcal{B}_{3\tau/2})} \leq \|K_{\tau/2}\|_{C^{1-s}(\mathbb{R}^n)}\|(1 - \eta)u\|_{L^1(\mathbb{R}^n)} \leq C(\varrho, \chi). \]
\[ x \in N_\eta \setminus \overline{B_{3\eta/3}^c} \] In this case we still have that \( \lambda = (-\Delta)^s u = 0 \) in \( B_\tau(x) \). Consequently, we can proceed as in the case \( x \in A_2 \) in the proof of Proposition 3.11 to conclude that \((-\Delta)^s U\) is smooth in this region. In this case, constructing a smooth extension \( U \) outside \( B_{\tau/6}(x) \), and vanishing on \( B_{\tau/3}(x)^c \) we obtain

\[ \|(-\Delta)^s U\|_{C^{1,-s}(N_\eta \setminus \overline{B_{3\eta/3}^c})} \leq \|(-\Delta)^s \mathcal{E} U\|_{C^{1,-s}(N_\eta \setminus \overline{B_{3\eta/3}^c})} + \|K_{\tau/2} * (\mathcal{E} U - U)\|_{C^{1,-s}(N_\eta \setminus \overline{B_{3\eta/3}^c})}. \]

Notice that, since this extension satisfies \( \mathcal{E} U \equiv 0 \) in \( B_\tau \), we have that

\[ \|(-\Delta)^s \mathcal{E} U\|_{C^{1,-s}(N_\eta \setminus \overline{B_{3\eta/3}^c})} \leq C(s) \|\mathcal{E} U\|_{C^{1,-s}(\overline{B_{3\eta/3}}^c)} \leq C(s) \|u\|_{C^{1,-s}(\overline{B_{3\eta/3}}^c)}. \]

In addition, since

\[ \|K_{\tau/2} * (\mathcal{E} U - U)\|_{C^{1,-s}(N_\eta \setminus \overline{B_{3\eta/3}^c})} \leq C(\tau, \chi), \]

we deduce

\[ \|(-\Delta)^s U\|_{C^{1,-s}(N_\eta \setminus \overline{B_{3\eta/3}^c})} \leq C(\|u\|_{C^{1,-s}(\overline{B_{3\eta/3}^c})}, \theta, \chi). \]

Estimate (3.14) follows immediately. \( \square \)

The boundary Hölder regularity of Proposition 3.10 can be translated to a weighted Sobolev regularity.

**Theorem 3.12** (boundary regularity). Let \( \Omega \) be a bounded Lipschitz domain satisfying the exterior ball condition, and let \( \chi \in C^{2,1}(\Omega) \), \( f = 0 \) and \( u \in H^s(\Omega) \) solve (3.8). Let \( g \) be defined in (3.8) and \( B_\tau \) in (3.9). Then, for every \( \varepsilon > 0 \) we have that \( u \in H^{1+s-2\varepsilon}(B_\tau) \) with the estimate

\[ \|u\|_{H^{1+s-2\varepsilon}(B_\tau)} \leq \frac{C(\|u\|_{C^{1,-s}(\overline{B_{3\eta/3}^c})}, \chi, s, n, \Omega, \theta)}{\varepsilon}, \]

where the weight \( \delta \) refers to \( \text{dist}(\cdot, \partial \Omega) \). Moreover, we have the estimate

\[ \iint_{B_\tau \times \Omega} \frac{|\nabla u(x)|^2}{|x - y|^{n+2s-4\varepsilon}} \delta(x, y)^{1-2\varepsilon} \, dy \, dx \leq \frac{C(\|u\|_{C^{1,-s}(\overline{B_{3\eta/3}^c})}, \chi, s, n, \Omega, \theta)}{\varepsilon^2}. \]

**Proof.** Define \( \tilde{f} = (-\Delta)^s(\eta u) \) and notice that, on \( N_\eta \), the function \( \eta u \) coincides with \( w_f \), that is, it solves

\[ (-\Delta)^s w = \tilde{f}, \quad \text{in } N_\eta, \quad w = 0, \quad \text{in } N_\eta^c. \]

Since \( \tilde{f} \in C^{1,-s}(\overline{N_\eta}) \) according to Proposition 3.11 we can apply Theorem 2.10 to conclude that \( \eta u = w_f \in H^{1+s-2\varepsilon}(N_\eta) \) with

\[ \|\eta u\|_{H^{1+s-2\varepsilon}(N_\eta)} \leq \frac{C(\|u\|_{C^{1,-s}(\overline{B_{3\eta/3}^c})}, \chi, s, n, \Omega, \theta)}{\varepsilon}. \]

Notice that, in this estimate, the weight used to define the norm is the distance to \( \partial N_\eta \). However, owing to the definition of \( B_\tau \), we have that for all \( x \in B_\tau \) this coincides with \( \text{dist}(x, \partial \Omega) \). In addition, since \( \eta \equiv 1 \) on \( B_\tau \) we can conclude that \( u \in H^{1+s-2\varepsilon}(B_\tau) \), with the corresponding estimate (3.13). Finally, recalling the definition of \( H^{1+s-2\varepsilon}(N_\eta) \) and restricting the integration to \( B_\tau \times \Omega^c \) instead of \( \mathbb{R}^n \times \mathbb{R}^n \), the previous inequality yields (3.14). \( \square \)

### 3.3. Global regularity

We now proceed to blend the interior and boundary regularity results. To do so, we must first obtain a weighted localization result à la Faermann [18], which accounts for the non-additive structure of fractional Sobolev norms with respect to disjoint domain partitions.

**Proposition 3.15** (localization). Let \( s \in (0,1), \alpha \in [0,1) \) and \( \Omega \) be a bounded Lipschitz domain. Assume there is a decomposition \( \overline{\Omega} = \bigcup_1^s \Omega_i \), where the subdomains \( \Omega_i \) are open and pairwise disjoint. Then, for any \( v \in H^s(\Omega) \) it holds that

\[ \|v\|_{H^s(\Omega)}^2 \leq \sum_i \left[ \iint_{\Omega_i \times S_i} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \delta(x, y)^{2\alpha} \, dy \, dx + \frac{2\nu_{n-1}}{s\delta_{\alpha}^s} \|v\|_{L^2(\Omega_i)}^2 \right], \]

where

\[ S_i := \bigcup_{j: \Omega_j \cap \Omega_i \neq \emptyset} \Omega_j, \]

and \( \delta_i = \text{dist}(\Omega_i, \Omega \setminus S_i) \).
Proof. Given an element $\Omega_i$ of the partition, we define $D_i = \Omega \setminus S_i$. Then,

$$|v|_{H^s(\Omega)}^2 = \sum_i \left[ \iint_{\Omega_i \times S_i} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \delta(x,y)^{2s} \, dy \, dx + \iint_{\Omega_i \times D_i} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \delta(x,y)^{2s} \, dy \, dx \right].$$

We may bound the integrals on the right hand side as follows

$$\frac{1}{2} \iint_{\Omega_i \times D_i} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \delta(x,y)^{2s} \, dy \, dx \leq \int_{\Omega_i} \frac{|v(x)|^2}{|x - y|^{n+2s}} \, dy \, dx + \int_{D_i} \frac{|v(y)|^2}{|x - y|^{n+2s}} \, dy \, dx =: J_{i,1} + J_{i,2}.$$

Let us show that $\sum_i J_{i,1} = \sum_i J_{i,2}$. Indeed, we write

$$\sum_i J_{i,2} = \sum_i \int_{\Omega} \chi_{D_i}(y) |v(y)|^2 \int_{\Omega_i} \frac{\delta(x,y)^{2s}}{|x - y|^{n+2s}} \, dy \, dx = \sum_i \int_{\Omega} |v(y)|^2 F(y) \, dy,$$

where

$$F(y) = \sum_i \chi_{D_i}(y) \int_{\Omega_i} \frac{\delta(x,y)^{2s}}{|x - y|^{n+2s}} \, dx.$$

Next, we write the integral over $\Omega$ as a sum of integrals over subdomains $\Omega_j$. Observe that, if $y \in \Omega_j$, then

$$\chi_{D_i}(y) = \begin{cases} 1 & \Omega_i \cap \Omega_j = \emptyset \implies \Omega_j \subset D_i, \\ 0 & \Omega_i \cap \Omega_j \neq \emptyset, \end{cases}$$

Thus, for $y \in \Omega_j$, $\Omega_i \cap \Omega_j = \emptyset$ implies $\Omega_j \subset D_i$ and

$$F(y) = \sum_i \chi_{D_i}(y) \int_{\Omega_i} \frac{\delta(x,y)^{2s}}{|x - y|^{n+2s}} \, dx = \int_{\Omega_j} \frac{\delta(x,y)^{2s}}{|x - y|^{n+2s}} \, dx,$$

whence

$$\sum_i J_{i,2} = \sum_j \int_{\Omega_j} |v(y)|^2 \int_{D_j} \frac{\delta(x,y)^{2s}}{|x - y|^{n+2s}} \, dx \, dy = \sum_j J_{j,1}.$$

Therefore, we have shown that

$$|v|_{H^s(\Omega)}^2 \leq \sum_i \left[ \iint_{\Omega_i \times S_i} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \delta(x,y)^{2s} \, dy \, dx + 4J_{i,1} \right].$$

Finally, $J_{i,1}$ is easily bounded by noticing that $\delta(x,y) \leq \delta(x)$ and that $D_i \subset \Omega \setminus B_\delta(x)$ for all $x \in \Omega_i$, $y \in D_i$. Integrating in polar coordinates gives

$$J_{i,1} \leq \omega_{n-1} \int_{\Omega_i} |v(x)|^2 \delta(x)^{2s} \int_{\delta_i} \rho^{-1-2s} \, d\rho \, dx = \frac{\omega_{n-1}}{2s\delta^s_i} \|v\|^2_{L^2(\Omega_i)};$$

and concludes the proof. \qed

We are now in position to prove the global regularity of solutions of the fractional obstacle problem.

**Theorem 3.17** (global regularity for $f = 0$). Let $\Omega$ be a bounded Lipschitz domain satisfying the exterior ball condition, $\chi \in C^{2,1}(\Omega)$ with $\phi > 0$, where $\phi$ is defined in \[\text{[3.3]}\], and $f = 0$. In this setting, the solution $u \in \bar{H}^s(\Omega)$ of \[\text{[3.3]}\] satisfies $u \in \bar{H}^{1+\varepsilon-2\tau}(\Omega)$ for all $\varepsilon > 0$, with the estimate

$$|u|_{\bar{H}^{1+\varepsilon-2\tau}(\Omega)} \leq \frac{C(\chi,s,n,\Omega)}{\varepsilon}.$$

**Proof.** Splitting

$$|u|_{\bar{H}^{1+\varepsilon-2\tau}(\Omega)} = |u|_{\bar{H}^{1+\varepsilon-2\tau}(\Omega)} + \frac{2}{|y|^{n+2s-4\varepsilon}} \delta(x,y)^{1-2\varepsilon} \, dy \, dx;$$

we treat the two terms on the right hand side separately, starting from the second one. In order to bound the integral over $B_\tau \times \Omega^c$, we use estimate \[\text{[3.14]}\], whereas if $x \in \Omega \setminus B_\tau$ and $y \in \Omega^c$, then $\delta(x,y) \leq \delta(x) \leq \text{diam}(\Omega)$,

$$\int_{\Omega^c} \frac{1}{|x - y|^{n+2s-4\varepsilon}} \, dy \leq \frac{\omega_{n-1}}{2s - 2\varepsilon},$$

and concludes the proof.
Corollary 3.18 (global regularity for results. To shorten notation, for any it follows that
ball condition, Proposition 3.15 then implies that
\[ A \equiv \lambda \subseteq \chi \text{ (regularity of) Theorem 3.19} \]
We thus deduce
\[ \int_{\Omega \times \mathbb{R}^n} \frac{|\nabla u(x)|^2}{|x-y|^{n+2-4\varepsilon}} \delta(x,y)^{1-2\varepsilon} \, dy \, dx \leq C \int_{\Omega} |\nabla u(x)|^2 \, dx \leq C\|u\|_{C^{1,\varepsilon}(\Omega \setminus B_\varepsilon)}^2. \]
We thus deduce
\[ \int_{\Omega \times \mathbb{R}^n} \frac{|\nabla u(x)|^2}{|x-y|^{n+2-4\varepsilon}} \delta(x,y)^{1-2\varepsilon} \, dy \, dx \leq C\left(\|u\|_{C^{1,\varepsilon}(\Omega \setminus B_\varepsilon)}^2, \chi, s, n, \Omega, \varrho\right). \]

We now tackle the first term of the splitting. Let \( K \subseteq \Omega \) be a compact subset of \( \Omega \) such that \( \{ \chi > 0 \} \subseteq K \), \( A_r := K \cap B_r \neq \emptyset \) and \( K \cup B_r = \Omega \). We consider now the disjoint splitting into open sets
\[ K \setminus A_r = K_r \cup K_{r}^B, \quad B_r \setminus A_r = B_r^B \cup B_{r}^\varepsilon, \]
where \( K_B \) and \( B_K \) are layers that separate the interior set \( K \), the boundary set \( B_\varepsilon \) and \( A_r \).

Owing to Corollary 3.7 we know that \( u \in H^{1+s-2\varepsilon}(K) \subseteq H^{1/2}_\varepsilon(K) \). In addition, from Theorem 3.12 it follows that \( u \in H^{1/2}_\varepsilon(B_r) \). The main idea now is to use Proposition 3.15 to glue together these two results. To shorten notation, for any \( D_1, D_2 \subseteq \Omega \) we define
\[ \mathcal{I}(D_1, D_2) = \int_{D_1 \times D_2} \frac{|\nabla u(x) - \nabla u(y)|^2}{|x-y|^{n+2-4\varepsilon}} \delta(x,y)^{1-2\varepsilon} \, dy \, dx. \]

Proposition 3.15 then implies that
\[ \|u\|_{H^{1+s-2\varepsilon}(\Omega)}^2 \leq \mathcal{I}(K_1, K_1 \cup K_B) + \mathcal{I}(K_B, K_1 \cup K_B \cup A_r) + \mathcal{I}(A_r, K_B \cup A_r \cup B_K) + \mathcal{I}(B_K, A_r \cup B_K \cup B_\varepsilon) + \mathcal{I}(B_\varepsilon, B_K \cup B_\varepsilon) + C(n, s, K)\|u\|_{L^2(\Omega)}^2. \]
Since \( \Omega \) is bounded and \( 1/2 - \varepsilon > 0 \) we have that \( \|u\|_{H^{1+s-2\varepsilon}(\Omega)}^2 \leq C\|u\|_{L^2(\Omega)}^2 \leq C \). On the other hand, all the integrals are computed over subsets of either \( K \times K \) or \( B_\varepsilon \times B_\varepsilon \). As announced, we apply Corollary 3.7 and Theorem 3.12——upon recalling that \( \|u\|_{C^{1,\varepsilon}(\Omega \setminus B_\varepsilon)}^2 \) is bounded whenever \( \chi \in C^{1,\varepsilon}(\Omega) \)——to finish the proof.

We conclude the discussion about the regularity of \( u \) by treating the nonhomogeneous case \( f \neq 0 \).

Corollary 3.18 (global regularity for \( f \neq 0 \)). Let \( \Omega \) be a bounded Lipschitz domain satisfying the exterior ball condition, \( \chi \in C^{1,\varepsilon}(\Omega) \) with \( \varepsilon > 0 \), where \( \varepsilon \) is defined in (3.8). Assume, in addition, that \( 0 \leq f \in F_s(\Omega) \) and let \( u \in H^s(\Omega) \) be the solution to (1.3). In this setting, for every \( \varepsilon > 0 \) we have that \( u \in H^{1+s-2\varepsilon}(\Omega) \) with the estimate
\[ \|u\|_{H^{1+s-2\varepsilon}(\Omega)}^2 \leq C\varepsilon, \]
where the constant depends on \( \chi, s, n, \Omega, \varrho \) and \( \|f\|_{F_s(\Omega)}^2 \).

Proof. We will, as before, reduce this to the case \( f = 0 \). As before, let \( w_f \) solve (2.7) and define \( U = u - w_f \). We observe then that \( U \) solves the obstacle problem (1.3) with right hand side \( f = 0 \) and obstacle \( \chi - w_f \).
To conclude then that \( U \) has the regularity properties of Theorem 3.17 we must be able to assert that \( \chi - w_f \in C^{1,\varepsilon}(\Omega) \) and that \( \chi - w_f < 0 \) on a layer near \( \partial \Omega \).
A sufficient condition to obtain that \( \chi - w_f \in C^{1,\varepsilon}(\Omega) \) is to require \( \chi \in C^{2,1}(\Omega) \) and \( f \in F_s(\Omega) \). See [21 formula (7.11)].

On the other hand, since \( f \geq 0 \), the nonlocal maximum principle given in Proposition 2.8 implies that \( w_f \geq 0 \) and, consequently, since \( \varrho > 0 \) we have that \( \chi - w_f < 0 \) on a layer around \( \partial \Omega \) of size at least \( \varrho \).
Finally, since \( u = U + w_f \) we can combine the estimates of Theorems 3.17 and 2.9 to conclude.

Finally, since it will be useful in the sequel, we present a regularity result for \( \lambda \).

Theorem 3.19 (regularity of \( \lambda \)). Let \( \lambda \) be defined in (1.1). In the setting of Corollary 3.18 we have that \( \lambda \in C^{1-\varepsilon}(\Omega) \).

Proof. We begin by observing that \( \varrho > 0 \) according to (3.3) and the coincidence set \( \Lambda \subseteq \Omega \). Consequently \( \lambda \equiv 0 \) in \( N \) and the arguments then mimic the ideas used to prove interior regularity in Proposition 3.4.
We introduce a smooth cutoff function \( \eta \) such that \( \eta \equiv 1 \) on \( \Lambda \) and, for some \( \varepsilon > 0 \),
\[ \text{dist}(\text{supp}(\eta), \partial \Omega) > 2\varepsilon, \quad \text{dist}(\Lambda, \partial \{\eta = 1\}) > 2\varepsilon. \]
Define now
\[ \Lambda_\varepsilon = \{ x \in \Omega : \text{dist}(x, \Lambda) \leq \varepsilon \} \]
and let \( x \in \Lambda_\varepsilon \). Since \( \eta \equiv 1 \) on \( B_\varepsilon(x) \), we are now in a similar situation to the case \( x \in A_1 \) in the proof of Proposition 3.4. Then we have
\[
(-\Delta)^s u(x) = (-\Delta)^s(\eta u)(x) - C(n, s) \int_{\mathbb{R}^n} \frac{(1 - \eta(y))u(y) - (\eta u)(x) - (K_\varepsilon \ast (1 - \eta)u)(x)}{|x - y|^{n+2s}} \, dy.
\]
where the last identity holds since \( 1 - \eta \) vanishes on \( B_\varepsilon(x) \). Now, since \( u \in C^{1,s}(\Omega) \) and \( \eta \) is smooth, we deduce that \( \eta u \in C^{1,s}(\mathbb{R}^n) \) and \( (-\Delta)^s(\eta u) \in C^{1,s}(\mathbb{R}^n) \). This implies \( (-\Delta)^s u \in C^{1,s}(\Lambda_\varepsilon) \).

Finally, since \( f \in \mathcal{F}_s(\Omega) \subset C^{1,s}(\Omega) \) we conclude that \( \lambda = (-\Delta)^s u - f \in C^{1,s}(\Lambda_\varepsilon) \).

\[ \square \]

4. Finite element approximation

Having obtained, in section 3, regularity estimates in weighted Sobolev spaces here we will apply them to the derivation of near optimal rates of convergence for a finite element approximation of (1.3) over graded meshes. Let us then begin by describing the discrete framework that we will adopt. First, to avoid technicalities we shall assume, from now on, that \( \Omega \) is a polytope. Next, we introduce a family \( \{ \mathcal{T}_h \}_{h>0} \) of conforming and simplicial triangulations of \( \Omega \) which we assume shape regular, i.e. we have that
\[
\sigma = \sup_{h>0} \sup_{T \in \mathcal{T}_h} \frac{h_T}{\rho_T} < \infty,
\]
where \( h_T = \text{diam}(T) \) and \( \rho_T \) is the diameter of the largest ball contained in \( T \). The vertices of \( \mathcal{T}_h \) will be denoted by \( \{ x_i \} \). We comment that we assume that the elements \( T \in \mathcal{T}_h \) are closed. In this case, the star or patch of \( T \in \mathcal{T}_h \) is defined as
\[
S^1_T = \bigcup \{ T' \in \mathcal{T}_h : T \cap T' \neq \emptyset \}.
\]
We also introduce the star of \( S^1_T \),
\[
S^2_T = \bigcup \{ T' \in \mathcal{T}_h : S^1_T \cap T' \neq \emptyset \}.
\]
Below, when discussing positivity preserving interpolation over fractional order smoothness spaces we partition \( \mathcal{T}_h \) into two classes, interior and boundary elements, as follows:
\[
\mathcal{N}^\circ_h = \{ T \in \mathcal{T}_h : S^1_T \cap \partial \Omega = \emptyset \}, \quad \mathcal{N}^\partial_h = \{ T \in \mathcal{T}_h : S^1_T \cap \partial \Omega \neq \emptyset \}.
\]

On the basis of the triangulation \( \mathcal{T}_h \) we define \( V_h \) as the space of continuous, piecewise affine functions on \( \mathcal{T}_h \) that vanish on \( \partial \Omega \). The Lagrange nodal basis of \( V_h \) will be denoted by \( \{ \varphi_i \} \) and
\[
S_i = \text{supp}(\varphi_i).
\]
We will denote by \( B_i \) the maximal ball, centered at \( x_i \), and contained in \( S_i \). If \( r_i \) is the radius of \( B_i \), and \( h_i = \text{diam}(S_i) \) by shape regularity of the mesh we have the equivalences \( r_i \approx h_i \approx h_T \), for all \( T \subset S_i \).

4.1. Positivity preserving interpolation over fractional order spaces. Below it will become necessary to introduce a discrete version of the admissible set \( \mathcal{K} \) defined in [12]. In addition, when performing the analysis of the method it will become necessary that an interpolator of the exact solution belongs to this discrete admissible set. Since we assume that \( \chi \in C^{2,1}(\Omega) \) and \( f \in \mathcal{F}_s(\Omega) \), we have that \( u \in C^2(\Omega) \) as a consequence of Proposition 2.10. Therefore one could, in principle, use the Lagrange interpolation operator. It turns out, however, that this operator does not possess suitable stability and approximation properties with respect to fractional order Sobolev spaces. For this reason, we will use instead the operator \( I_h \) introduced in [12] which we now describe.

Definition 4.2 (positivity preserving interpolation operator). Let \( I_h : L^1(\Omega) \rightarrow V_h \) be defined by
\[
I_h v = \sum_{i : x_i \in \Omega} \left( \frac{1}{|B_i|} \int_{B_i} v(x) \, dx \right) \varphi_i.
\]
Notice that, since the sum is only over interior vertices of $T_h$, we indeed have that $I_h v$ vanishes on $\partial\Omega$, whence $I_h v \in V_h$. In addition, by construction, this operator is positivity preserving: we have that $I_h v \geq 0$ whenever $v \geq 0$. Moreover, since for every $x_i \in \Omega$ the ball $B_i$ is symmetric with respect to $x_i$ we have the following exactness property for $I_h$

\begin{equation}
I_h v(x_i) = v(x_i), \quad \forall v \in P_1(B_i),
\end{equation}

where by $P_1(E)$ we denote the space of polynomials of degree one over the set $E$. Notice however, that this operator is not a projection. In general, if $v_h \in V_h$ then $I_h v_h \neq v_h$; see \cite{31} for details. The following result summarizes the local stability and approximation properties of $I_h$.

**Proposition 4.4** (properties of $I_h$). Let $p \in [1, \infty]$, $I_h$ be the operator introduced in Definition 4.2 and $T \in T_h$. Then, there are constants independent of $T$ and $h$ such that

$$
\|I_h v\|_{L^p(T)} \leq C\|v\|_{L^p(S^1)}, \quad \forall v \in L^p(\Omega),
$$

and

$$
\|\nabla I_h v\|_{L^p(T)} \leq C\|\nabla v\|_{L^p(S^1)}, \quad \forall v \in W^{1,p}(\Omega).
$$

As a consequence, for $k \in \{1, 2\}$, we also have the estimate

$$
\|v - I_h v\|_{L^p(T)} \leq C h_T^k \|v\|_{W^{k,p}(S^1)}, \quad \forall v \in W^{k,p}(\Omega) \cap W^{1,p}(\Omega).
$$

**Proof.** See \cite{12} Lemmas 3.1 and 3.2. \hfill $\square$

We need to obtain similar properties in fractional order Sobolev spaces, and for that we will follow the ideas of \cite{13}. We begin with a local stability result.

**Proposition 4.5** (stability of $I_h$). Let $s \in (0,1)$ and $T \in T_h$. There is a constant $C(n, \sigma)$ dependent only on the dimension $n$ and the shape regularity parameter $\sigma$ of the mesh, such that the estimate

$$
\left\| \frac{I_h v(x) - I_h v(y)}{|x - y|^{n+2s}} \right\|_{L^1(T \times S^1_T)} \leq \frac{C(n, \sigma)}{1 - s} \int_{S^1_T} \left( \frac{1}{|B_i|} \int_{B_i} \rho \right)^2 dx
$$

holds for all $v \in L^1(\Omega)$.

**Proof.** From Definition 4.2 it follows that, if $x \in T$ and $y \in S^1_T$, then

$$
I_h v(x) - I_h v(y) = \sum_{i:y_i \in S^1_T} \left( \frac{1}{|B_i|} \int_{B_i} \rho \right) (\varphi_i(x) - \varphi_i(y)).
$$

In addition we observe that, by shape regularity the number of terms in this sum is uniformly bounded by a constant that depends only on $\sigma$. Thus, by Hölder’s inequality we have that

$$
\left\| \frac{I_h v(x) - I_h v(y)}{|x - y|^{n+2s}} \right\|_{L^1(T \times S^1_T)} \leq C(\sigma) \int_{S^1_T} \left( \frac{1}{|B_i|} \int_{B_i} \rho \right)^2 dx.
$$

From mesh regularity it follows that \(|\varphi_i|_{C^{0,1}(\Omega)} \leq C(\sigma) h_T^{-1}\) uniformly in $i$ and that

$$
\alpha(x) = \max_{z \in S^1_T} |x - z| \leq C(\sigma) h_T.
$$

These two observations then imply that

$$
\left\| \frac{I_h v(x) - I_h v(y)}{|x - y|^{n+2s}} \right\|_{L^1(T \times S^1_T)} \leq C(\sigma) \int_{S^1_T} \left( \frac{1}{|B_i|} \int_{B_i} \rho \right)^2 dx \leq C(n, \sigma) \int_j^{\alpha(x)} \rho^{1-2s} \rho dx.
$$

From this the asserted estimate immediately follows. \hfill $\square$

Let now $S \subset \mathbb{R}^n$. It is well-known that for every $v \in W^{k,1}(S)$ there is a unique polynomial $P_k v$ of degree $k$ that satisfies

\begin{equation}
\int_S \partial^\alpha (v - P_k v) dx = 0, \quad \forall \alpha \in \mathbb{N}^n, \; |\alpha| \leq k.
\end{equation}

We shall also need the following fractional Poincaré inequality.
Proposition 4.7 (fractional Poincaré). Let $s \in (0, 1)$, $\alpha \in [0, s)$ and $S$ be a domain which is a finite union of overlapping star-shaped domains $S_i$ with respect to balls $B_i$, $i = 1, \ldots, I$. Then, there exists a constant $C > 0$, depending on the chunkiness of $S_i$ and the amount of overlap between the subdomains $S_i$, such that, for any $i \in \{1, \ldots, I\}$, we have
\begin{equation}
\|v - \nabla x_i\|_{L^2(S)} \leq C \text{diam}(S)^{s-\alpha} |v|_{H^s(S)}, \quad \forall v \in H^s(S),
\end{equation}
where $x_i = \frac{1}{|S_i|} \int_{S} v(x) \, dx$.

Proof. We must first observe that when $S$ is itself star-shaped, the result is proved in [3, Proposition 4.8].

In the general case, the result is an easy modification of the arguments used to show [13, Theorem 7.1]; see also [29, Corollary 3.2] and [27, Corollary 4.4]. For brevity we skip the details. \hfill \Box

Notice that, as a consequence of the fractional Poincaré inequality (4.8), we have that, whenever $\ell \in (1, 2)$ and $\alpha \in [0, \ell - 1)$, there are constants that depend only on $\sigma$ such that, for every $v \in H^\alpha(S^2_\ell)$, the polynomial $P_\ell v$, defined by (4.6), satisfies
\begin{equation}
\|v - P_\ell v\|_{L^2(S^2_\ell)} \leq C h^{-\alpha}_{T} |v|_{H^\alpha(S^2_\ell)}, \quad \forall v \in H^\alpha(S^2_\ell),
\end{equation}
where
\begin{equation}
\|\nabla (v - P_\ell v)\|_{L^2(S^2_\ell)} \leq C h^{-\alpha-1}_{T} |v|_{H^\alpha(S^2_\ell)}.
\end{equation}

Interpolating these two inequalities we can obtain that, whenever $s \in [0, 1]$, $\ell \in (1, 2)$. $\alpha \in [0, \ell - 1)$ there is a constant that depends only on $\sigma$ for which
\begin{equation}
|v - P_\ell v|_{H^s(S^2_\ell)} \leq C h^{s-\alpha}_{T} |v|_{H^\alpha(S^2_\ell)}.
\end{equation}

With these estimates at hand, we now proceed to obtain local interpolation error estimates for $I_\ell$ of Definition 4.2. We must do this separately for interior and boundary elements, as defined in (4.1). We first give the interior estimate and next the boundary estimate.

Proposition 4.10 (interior interpolation estimate). Let $N^\ell_h$ be defined in (4.1) and $T \in N^\ell_h$. Assume, in addition, that $s \in (0, 1)$, $\ell \in (1, 2)$, and that $I_\ell$ is the positivity preserving interpolator of Definition 4.2. Then, there is a constant $C(n, \sigma, \ell)$ that depends only on the dimension, the shape regularity parameter $\sigma$, and $\ell$ such that
\begin{equation}
\iint_{T \times S^2_\ell} \frac{|(v - I_\ell v)(x) - (v - I_\ell v)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \leq \frac{C(n, \sigma, \ell)}{1-s} h^{2s-2\alpha}_{T} |v|_{H^\alpha(S^2_\ell)}^2.
\end{equation}

Proof. We begin by writing $v - I_\ell v = (v - P_\ell v) + (P_\ell v - I_\ell v)$, where $P_\ell v \in P_\ell$ is the polynomial defined by (4.6) over $S^2_\ell$.

Using (4.9) with $\alpha = 0$ the first difference can be estimated as follows:
\begin{equation}
\iint_{T \times S^2_\ell} \frac{|(v - P_\ell v)(x) - (v - P_\ell v)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \leq |v - P_\ell v|_{H^s(S^2_\ell)}^2 \leq C h^{2s-2\alpha}_{T} |v|_{H^\alpha(S^2_\ell)}^2.
\end{equation}

On the other hand, since $P_\ell v \in P_\ell(S^2_\ell)$ it follows from (4.3), that $I_\ell, P_\ell v \in S^2_\ell$ and to control the second term we only need to invoke the stability estimate of Proposition 4.5 to arrive at
\begin{equation}
\iint_{T \times S^2_\ell} \frac{|(P_\ell v - I_\ell v)(x) - (P_\ell v - I_\ell v)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \leq \frac{C(n, \sigma)}{1-s} h^{-2s}_{T} \sum_{i \in S^1_\ell} \frac{1}{|B_i|} \|v - P_\ell v\|_{L^2(B_i)}^2 \leq \frac{C(n, \sigma)}{1-s} h^{-2s}_{T} |v - P_\ell v|_{L^2(S^2_\ell)}^2.
\end{equation}

Setting $s = \alpha = 0$ in (4.9) yields the desired estimate. \hfill \Box

As a final preparatory step we obtain local interpolation error estimates for elements in $N^\ell_h$.

Proposition 4.11 (boundary interpolation estimate). Let $N^\ell_h$ be defined in (4.1) and $T \in N^\ell_h$. Assume, in addition, that $s \in (0, 1)$, $\ell \in (1, 2)$, $\alpha \in [0, 1/2)$, and that $I_\ell$ is the positivity preserving interpolation operator of Definition 4.2. Then, there is a constant $C(n, \sigma, \ell)$ that depends only on the dimension, the shape regularity parameter $\sigma$, and $\ell$ such that, for all $v \in H^\alpha(S^2_\ell)$, we have
\begin{equation}
\iint_{T \times S^2_\ell} \frac{|(v - I_\ell v)(x) - (v - I_\ell v)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \leq \frac{C(n, \sigma, \ell)}{1-s} h^{2s-2\alpha}_{T} |v|_{H^\alpha(S^2_\ell)}^2.
\end{equation}
Proof. As in the proof of Proposition 4.10 we decompose \( v - I_h v = (v - P_1 v) + (P_1 v - I_h v) \) and estimate the first term using (4.9) to obtain
\[
\int_{T \times S_T^1} \frac{|(v - P_1 v)(x) - (v - P_1 v)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \leq C h_T^{2(\ell - s - \alpha)} |v|_{H^s(T)}^2.
\]

The estimate of the term \( P_1 v - I_h v \) is now more delicate, as we cannot exploit the symmetries that \( T \in \mathcal{N}_h^0 \) afforded us in Proposition 4.10. Instead, we will follow the ideas used to obtain [12, Lemma 3.2], where a similar difficulty is handled by further decomposing this term into
\[
P_1 v - I_h v = I_h (P_1 v - v) + (P_1 v - I_h P_1 v).
\]
The local stability of \( I_h \), shown in Proposition 4.5 and estimate (4.3) for \( s = 0 \) allow us to bound the first term:
\[
\int_{T \times S_T^1} \frac{|I_h (P_1 v - v)(x) - I_h (P_1 v - v)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \leq \frac{C(n, \sigma, \ell)}{1 - s} h_T^{2(\ell - s - \alpha)} |v|_{H^s(T)}^2.
\]

Next, we notice that the difference \( P_1 v - I_h P_1 v \) can be written, for \( x \in S_T^1 \), as
\[
(P_1 v - I_h P_1 v)(x) = \sum_{j: x_j \in S_T^1} (P_1 v(x_j) - I_h P_1 v(x_j)) \varphi_j(x);
\]
where now the summation must include the vertices \( x_j \in S_T^1 \cap \partial \Omega \), where \( I_h P_1 v(x_j) = 0 \) but \( P_1 v(x_j) \neq 0 \) in general. Since, by shape regularity, the number of indices in this sum is uniformly bounded and \( 0 \leq \varphi_j \leq 1 \), we can proceed as in Proposition 4.5 to obtain
\[
\int_{T \times S_T^1} \frac{|(P_1 v - I_h P_1 v)(x) - (P_1 v - I_h P_1 v)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \leq \frac{C(n, \sigma, \ell)}{1 - s} h_T^{2(\ell - s - \alpha)} \sum_{j: x_j \in S_T^1} ((P_1 v - I_h P_1 v)(x_j))^2.
\]

The objective is now to show that, for all indices in the indicated range,
\[
((P_1 v - I_h P_1 v)(x_j))^2 \leq C h_T^{h_T^{n+2(\ell - s - \alpha)}} |v|_{H^s(T)}^2,
\]
as this will imply the result. If \( x_j \in \Omega \) then we get
\[
(P_1 v - I_h P_1 v)(x_j) = 0,
\]
in view of (4.3). On the other hand if \( x_j \in \partial \Omega \), then \( I_h P_1 v(x_j) = 0 \). Let \( x_j \in e_j \subset \partial \Omega \cap S_T^1 \) be a face and recall the scaled trace inequality
\[
\|v\|_{L^2(\e_j)} \leq C \left( h^{-1/2} \|v\|_{L^2(T)} + h^{1/2} \|\nabla v\|_{L^2(T)} \right) \quad \forall v \in H^1(T).
\]
This, for \( w = v - P_1 v \), together with an inverse inequality and the fact that \( v|_{e_j} = 0 \), yields
\[
\|P_1 v(x_j)\| \leq C h^{-1/2}_T \|P_1 v\|_{L^2(e_j)} = C h^{-1/2}_T \|P_1 v - v\|_{L^2(e_j)} \leq C h^{-1/2}_T \left( h^{1/2}_T \|v - P_1 v\|_{L^2(T)} + h^{1/2}_T \|\nabla (v - P_1 v)\|_{L^2(T)} \right).
\]

Property \( v|_{e_j} = 0 \) is a consequence of [26, Theorem 2.3], because \( v \in \tilde{H}^s_0(\Omega) \subset \tilde{H}^s_0(\Omega) \). An application of (4.3) for \( s = 0 \) and \( s = 1 \) allows us to conclude the proof.

Remark 4.12 (\( s = 0 \)). We briefly comment that the conclusion of Proposition 4.11 can be extended to \( s = 0 \) to obtain that if \( T \in \mathcal{N}_h^0 \), \( s, \ell \) and \( \alpha \) are as in Proposition 4.11 then we have
\[
\|v - I_h v\|_{L^2(T)} \leq C h^{\ell - s} |v|_{H^s_T(S^2_T)},
\]
for every \( v \in H^s_T(S^2_T) \). The proof is a slight modification of the arguments needed for \( s > 0 \) and, for brevity, we skip the details.

We are now finally in position to prove global interpolation error estimates. While the results of Propositions 4.10 and 4.11 may allow us to obtain error estimates over quasi-uniform meshes for functions in \( H^\ell(\Omega) \), \( \ell \in (1, 2) \), the regularity results of Section 3 show that these may be of little use for the approximation of problem (1.3). We will, instead, exploit the regularity estimates in weighted Sobolev spaces that we have obtained. For that, a grading of the mesh near the boundary must be introduced, which will compensate the singular behavior of the solution.
For this reason, from now on we will assume that \( n = 2 \), and that given a mesh parameter \( h > 0 \) and \( \mu \in [1, 2] \) every element \( T \in \mathcal{T}_h \) satisfies

\[
\begin{align*}
&\left\{ h_T \approx C(\sigma) h^\mu, \quad T \in \mathcal{N}_h^0 \\
&h_T \approx C(\sigma) h \operatorname{dist}(T, \partial \Omega)^{(\mu-1)/\mu}, \quad T \in \mathcal{N}_h^s.
\end{align*}
\]

\( (G) \)

**Remark 4.13** (dimension of \( V_h \)). Following [5] Lemma 4.1 it is not difficult to see that the space \( V_h \) constructed over the mesh \( \mathcal{T}_h \) that satisfies \( (G) \) will satisfy

\[
\dim V_h \approx h^{-2}\log h.
\]

Indeed, since we are in two dimensions and the mesh is assumed shape regular, we have that

\[
\dim V_h \leq 3 \sum_{T \in \mathcal{T}_h} 1 \leq 3 \left( \sum_{T \in \mathcal{N}_h^0} 1 + \sum_{T \in \mathcal{N}_h^s} 1 \right) \leq C(\sigma) \left( \sum_{T \in \mathcal{N}_h^0} h_T^{-2} \int_T dx + \sum_{T \in \mathcal{N}_h^s} h_T^{-2} \int_T dx \right).
\]

Over \( \mathcal{N}_h^0 \), because \( \cup_{T \in \mathcal{N}_h^0} T \) defines a layer around the boundary of thickness about \( h^{\mu} \), we have

\[
\sum_{T \in \mathcal{N}_h^0} h_T^{-2} \int_T dx \leq Ch^{-2\mu} \sum_{T \in \mathcal{N}_h^0} \int_T dx \leq Ch^{-\mu}.
\]

On the other hand, for \( \mathcal{N}_h^s \) we have

\[
\sum_{T \in \mathcal{N}_h^s} h_T^{-2} \int_T dx \leq C h^{2} \int_{h^\mu}^{\text{diam}(\Omega)} \rho^{-2(\mu-1)/\mu} \, d\rho.
\]

In conclusion, since \( \mu \in [1, 2] \) implies \(-2(\mu-1)/\mu \in [-1, 0]\), we get

\[
\dim V_h \approx h^{-2} + h^{-2} \int_{h^\mu}^{\text{diam}(\Omega)} \rho^{-2(\mu-1)/\mu} \, d\rho \leq C h^{-2}\log h.
\]

**Remark 4.14** (two vs. three dimensions). It is well known that, in three dimensions, it is not possible to construct shape regular graded meshes that are able to capture boundary singularities like the ones we proved in Theorem 3.17 see [1] Section 4.2.3. Anisotropic meshes must be introduced to optimally handle singularities near edges, and this will violate the shape regularity assumption.

Let us now show a global interpolation estimate for functions in \( \widetilde{H}^{1+s-2\varepsilon}(\Omega) \), in two dimensions, over graded meshes that satisfy \( (G) \).

**Theorem 4.15** (global interpolation estimate). Let \( n = 2 \) and \( \mathcal{T}_h \) be shape regular and satisfy \( (G) \) with \( \mu = 2 \). Then, there is a constant \( C \) that depends only on \( s, \Omega \) and \( \sigma \) such that

\[
|v - I_h v|_{\widetilde{H}^{s}(\Omega)} \leq C h^{1-2\varepsilon} |v|_{\widetilde{H}^{1+s-2\varepsilon}(\Omega)}
\]

(4.16)

for all \( v \in \widetilde{H}^{1+s-2\varepsilon}(\Omega) \).

**Proof.** From the localization estimate (3.16) of Proposition 3.15 we obtain

\[
|v - I_h v|_{\widetilde{H}^{s}(\Omega)} \leq \sum_{T \in \mathcal{T}_h} \left[ \int_{T \times S_T^1} \frac{|(v - I_h v)(x) - (v - I_h v)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx + \frac{2\omega_{n-1}}{sh^2} \|v - I_h v\|_{L^2(T)}^2 \right].
\]

To shorten notation, for \( T \in \mathcal{T}_h \), we set

\[
\mathcal{I}_T = \int_{T \times S_T^1} \frac{|(v - I_h v)(x) - (v - I_h v)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx.
\]

To control the term \( \mathcal{I}_T \), we follow the notation (4.1) and consider two cases:

- \( T \in \mathcal{N}_h^0 \): In this case we apply Proposition 4.10 with \( \ell = 1 + s - 2\varepsilon \) and use the mesh grading condition \( (G) \) with \( \mu = 2 \) to obtain that

\[
\mathcal{I}_T \leq \frac{C(\sigma)}{1-s} h^{2(1-2\varepsilon)} \operatorname{dist}(T, \partial \Omega)^{1-2\varepsilon} |v|_{\widetilde{H}^{1+s-2\varepsilon}(S_T^1)}^2.
\]
In addition since, for all \( x, y \in S^2_k \), we have that \( \text{dist}(T, \partial \Omega) \approx \delta(x, y) \), the right hand side of the previous expression can be modified so that the final estimate reads
\[
I_T \leq \frac{C(\sigma)}{1 - s} h^{2(1-2\varepsilon)}|v|^{2}_{H^{1+1-2\varepsilon}(S^2_k)}.
\]

\( \bullet \) \( T \in N^\partial_k \): We now use Proposition \ref{prop:interp1} with \( \alpha = 1/2 - \varepsilon \) and \( \ell = 1 + s - 2\varepsilon \) and obtain
\[
I_T \leq \frac{C(\sigma)}{1 - s} h^{2(1-2\varepsilon)}|v|^{2}_{H^{1+1-2\varepsilon}(S^2_k)} \leq \frac{C(\sigma)}{1 - s} h^{2(1-2\varepsilon)}|v|^{2}_{H^{1+1-2\varepsilon}(S^2_k)}
\]
as a consequence of the grading condition \( \mathcal{G} \) for \( \mu = 2 \).

Gathering the estimates obtained in the previous two steps we conclude that
\[(4.17) \sum_{T \in T_h} I_T \leq C h^{2(1-2\varepsilon)}|v|^{2}_{H^{1+1-2\varepsilon}(\Omega)}.
\]

Next we need to control the local \( L^2 \)-interpolation errors. We set
\[
L_T = \frac{1}{h_T^2} \|v - I_h v\|_{L^2(T)}^2
\]
and, again, consider two cases:

\( \bullet \) \( T \in N^\partial_k \): Interpolating the estimates of Proposition \ref{prop:interp1} for \( k = 1 \) and \( k = 2 \) we have
\[
L_T \leq C h^{2(1-2\varepsilon)}|v|^{2}_{H^{1+1-2\varepsilon}(S^2_k)}.
\]

Then, as in the first case for \( I_T \), we can use the mesh grading condition \( \mathcal{G} \) and the fact that, for all \( x, y \in S^2_k \), \( \delta(x, y) \approx \text{dist}(T, \partial \Omega) \) to obtain
\[
L_T \leq C(\sigma, s) h^{2(1-2\varepsilon)}|v|^{2}_{H^{1+1-2\varepsilon}(S^2_k)}.
\]

\( \bullet \) \( T \in N^\partial_k \): Owing to Remark \ref{rem:interp2} we have
\[
L_T \leq C(\sigma) h^{1-2\varepsilon}|v|^{2}_{H^{1+1-2\varepsilon}(S^2_k)}
\]
and use the mesh grading condition \( \mathcal{G} \) for \( \mu = 2 \).

In conclusion we have shown that
\[(4.18) \sum_{T \in T_h} L_T \leq C h^{2(1-2\varepsilon)}|v|^{2}_{H^{1+1-2\varepsilon}(\Omega)}.
\]

Adding \( (4.17) \) and \( (4.18) \) allows us to conclude that
\[
|v - I_h v|_{H^s(\Omega)} \leq C h^{1-2\varepsilon}|v|_{H^{1+1-2\varepsilon}(\Omega)}.
\]

Finally, to bound the full \( H^s(\Omega) \)-seminorm we need to provide a bound for the term
\[
T_\mathcal{O} = \int_\Omega \|v - I_h v\|^2 \int_{\Omega^2} \frac{1}{|x - y|^{n+2\varepsilon}} \, dy \, dx.
\]

To do so, if \( s \neq 1/2 \) we employ the inequality
\[
T_\mathcal{O} \leq C(s) \begin{cases} 
\|v\|_{H^s(\Omega)}, & s \in \left( 0, \frac{1}{2} \right), \\
\|v\|^2_{H^s(\Omega)}, & s \in \left[ \frac{1}{2}, 1 \right),
\end{cases}
\]
whose proof is implicit in the proof of \cite[Corollary 2.6]{2} and uses the fractional Hardy-type inequality of \cite[Theorem 1.1 (T1)]{16} in the case \( s > 1/2 \) and is the content of \cite[Theorem 1.4.4.4]{19} for \( s < 1/2 \). On the other hand, if \( s = 1/2 \), it suffices to realize that for any \( \varepsilon > 0 \) we have
\[
T_\mathcal{O} \leq C \text{diam}(\Omega)^{2\varepsilon} \int_\Omega \frac{|v - I_h v|^2}{\delta(x)^{1+2\varepsilon}} \, dx
\]
and apply, once again, the fractional Hardy-type inequality of \cite[Theorem 1.1 (T1)]{16}. \( \square \)
4.2. The numerical scheme and its analysis. Having studied the interpolation operator \( I_h \), introduced in Definition 4.2, we can finally proceed to present and analyze the numerical scheme we use to approximate the solution of (4.20). In essence, this is a direct discretization inspired by the approximation of classical obstacle-type problems and their analyses; see [8, 28].

We begin by introducing a discrete version of the admissible set as follows:

\[
K_h = \{ v_h \in V_h : v_h \geq I_h \chi \}.
\]

Then the discrete problem reads: find \( u_h \in K_h \) such that

\[
(u_h, u_h - v_h)_s \leq (f, u_h - v_h), \quad \forall v_h \in K_h.
\]

The existence and uniqueness of a solution to (4.20) is standard. The approximation properties of this scheme are presented below.

**Theorem 4.21** (convergence). Let \( u \) be the solution to (4.23) and \( u_h \) be the solution to (4.20), respectively. Assume that \( \chi \in C^{2,1}(\Omega) \) with \( \varrho \), defined in (3.8), satisfying \( \varrho > 0 \), and that \( f \in F_s(\Omega) \). If \( n = 2 \), \( \Omega \) is a convex polygon, and the mesh \( T_h \) satisfies the grading hypothesis (4) with \( \mu = 2 \), then we have that

\[
|u - u_h|_{H^s(\Omega)} \leq C h^{1 - 2\varepsilon},
\]

where \( C > 0 \) depends on \( \chi, s, n, \Omega, \varrho \) and \( \|f\|_{F_s(\Omega)} \). In particular, setting \( \varepsilon = |\log h|^{-1} \) we obtain

\[
|u - u_h|_{H^s(\Omega)} \leq C h |\log h|.
\]

**Proof.** After all the discussion about regularity of section 3 and preparatory steps, the proof of this result follows more or less standard arguments; see [8, Theorem 4.1]. However, it requires a combination of Sobolev and Hölder regularity results on the solution as it was first exploited in [28, Theorems 3.1 and 4.4].

We begin by writing

\[
|u - u_h|^2_{H^s(\Omega)} = (u - u_h, u - I_h u)_s + (u - u_h, I_h u - u_h)_s
\]

so that

\[
|u - u_h|^2_{H^s(\Omega)} \leq |u - I_h u|^2_{H^s(\Omega)} + 2(u - u_h, I_h u - u_h)_s \leq C \left( h^{2(1 - 2\varepsilon)} + (u - u_h, I_h u - u_h)_s \right),
\]

where in the last inequality we applied Theorem 4.15. In conclusion, it remains to bound the second term. To do this we use (2.8) to obtain

\[
(u, I_h u - u_h)_s = \langle \{(-\Delta)^s u, I_h u - u_h \rangle.
\]

In addition, since \( I_h \) is positivity preserving, we have that \( I_h u \in K_h \) and so it is a legitimate test function for (4.20). Subtracting these two inequalities then yields

\[
(u - u_h, I_h u - u_h)_s \leq \langle \lambda, I_h u - u_h \rangle = \int_\Omega \lambda (I_h u - u_h) \, dx
\]

where we have used the regularity \( \lambda \in C^{1,\gamma}(\Omega) \) of Theorem 3.19 to transform the pairing into an integral. Next, we apply the complementarity conditions (1.11) to conclude that \( \lambda(u - \chi) = 0 \). Finally, we use, once again, the complementarity conditions to see that \( \lambda \geq 0 \) and, since \( u_h \in K_h \), then the middle term can be dropped. Consequently,

\[
(u - u_h, I_h u - u_h)_s \leq \int_\Omega \lambda [I_h(u - \chi) - (u - \chi)] \, dx = \sum_{T \in T_h} \int_T \lambda [I_h(u - \chi) - (u - \chi)] \, dx = \sum_{T \in T_h} \mathcal{J}_T.
\]

We continue by partitioning the terms in the previous sum into three cases:

- **Case 1:** \( T \subseteq N \): The complementarity condition (1.11) then implies that \( \lambda = 0 \), whence \( \mathcal{J}_T = 0 \).
- **Case 2:** \( T \) is such that \( S_T \subseteq \Lambda \): In this case \( u = \chi \) and, again, \( \mathcal{J}_T = 0 \).
Theorem 4.21 in terms of degrees of freedom as follows described in [6, Section 5.3]. A brief explanation on how to construct graded meshes satisfying (G) can be performed in two-dimensional domains, and we illustrate the qualitative differences between fractional Laplacian in terms of Jacobi polynomials and an s

Then, it holds that

\[ (-Δ)^s u(x) = ˜f(x), \quad x ∈ B_1. \]

We now consider a smooth obstacle χ that coincides with u in \( \overline{B_{1/5}} \) and modify ˜f in \( B_{1/5} \) so that within this contact set the strict inequality \( (-Δ)^s u > ˜f \) holds. More precisely, we extend χ to \( B_1 \setminus B_{1/5} \) by using the Taylor polynomial of order two of u on \( ∂B_{1/5} \) and set

\[ f(x) = ˜f(x) - 100 \left( \frac{1}{5} - |x| \right)_+. \]

We carried out computations for \( s \in \{0.1, 0.9\} \) using graded meshes satisfying (G) with \( µ = 2 \) and different mesh size parameters h. Figure 1 shows that the observed convergence rates are in good agreement with either Theorem 4.21 or Remark 4.22.
Figure 1. Computational rate of convergence for the discrete solutions to the fractional obstacle problems described in section 5.1 using graded meshes. The left panel shows the errors for $s = 0.1$ and the right one for $s = 0.9$. The rate observed in both cases is $\approx \dim(V_h)^{-1/2}$, in agreement with the theory.

5.2. Qualitative behavior. Finally, we consider problem (1.3), posed in the unit ball $B_1 \subset \mathbb{R}^2$, with $f = 0$ and the obstacle

$$\chi(x) = \frac{1}{2} - |x - x_0|, \quad \text{with } x_0 = (1/4, 1/4).$$

Figure 2 shows computed solutions for $s \in \{0.1, 0.5, 0.9\}$ over meshes graded according to (G) with $\mu = 2$ and 24353 degrees of freedom (this corresponds to $h \approx 0.025$).

Figure 2. Discrete solutions to the fractional obstacle problem for $s = 0.1$ (left), $s = 0.5$ (center) and $s = 0.9$ (right), computed over meshes with 24353 degrees of freedom, and graded accorded to (G) with $\mu = 2$. Top: lateral view. Bottom: top view, with the discrete contact set highlighted.

Qualitative differences between solutions for different choices of $s$ are apparent. While for $s = 0.9$ the discrete solution resembles what is expected for the classical obstacle problem, the solution for $s = 0.1$ is much flatter in the non-contact set $N$. Moreover, taking into account that the solution of the fractional
obstacle problem is non-negative in $\Omega$ and that $u = \chi_+$ in the formal limit of $s = 0$, it is apparent that the area of the contact set $\Lambda$ decreases with $s$. This fact is verified by the experiments presented in Figure 2.

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