Abstract
In this article we prove that, under certain hypotheses, Morita context algebras that have zero bimodule morphisms have finite \( \phi \)-dimension. We also study the behaviour of the \( \phi \)-dimension for an algebra and its opposite. In particular we show that the \( \phi \)-dimension of an Artin algebra is not symmetric, i.e. there exists a finite dimensional algebra \( A \) such that \( \phi \text{dim}(A) \neq \phi \text{dim}(A^{op}) \).

Keywords: Igusa-Todorov function, finitistic dimension conjecture, Morita context algebras.

1. Introduction
For an Artin algebra \( A \) the finitistic dimension is defined as

\[
\text{findim}(A) = \sup \{ \text{pd}(M) \mid M \in \text{mod}A, \text{pd}M < \infty \}.
\]

The (small) finitistic dimension conjecture states that \( \text{findim}(A) < \infty \). In an attempt to prove this conjecture, K. Igusa and G. Todorov introduced the functions \( \phi \) and \( \psi \) in [11], nowadays called the Igusa-Todorov functions. Later, using the Igusa-Todorov functions, F. Huard and M. Lanzilotta introduced the \( \phi \)-dimension and the \( \psi \)-dimension in [8], and proved that an Artin algebra is selfinjective if and only if its \( \phi \)-dimension (or its \( \psi \)-dimension) is zero. Hence these dimensions show how far an Artin algebra is from being selfinjective, in some sense.

Since M. Auslander proved that the left global dimension is equal to the right global dimension for semi-primary rings, the following question arises naturally due to the relationship of the \( \phi \)-dimension with the global dimension: Is the \( \phi \)-dimension symmetric? i.e. for an Artin algebra \( A \), \( \phi \text{dim}(A) = \phi \text{dim}(A^{op}) \)? In the case of the \( \psi \)-dimension it is not true and an example of radical square zero algebras is given in [13].

The symmetry conjecture of the \( \phi \)-dimension was proved in [13] for radical square zero algebras, in [12] for Gorenstein algebras, and in [11] for truncated path algebras. On the other hand the conjecture is not true for semiperfect coalgebras. In [10] was given an easy example (Example 2.4) of a semiperfect coalgebra \( C \) such that \( \phi \text{dim}(C) \neq \phi \text{dim}(C^{op}) \).

Let \( A \) and \( B \) be two Artin algebras, \( Y \) an \( A-B \)-bimodule, \( X \) a \( B-A \)-bimodule, \( \alpha : X \otimes A Y \to B \) a \( B-B \)-bimodule homomorphism, and \( \beta : Y \otimes_B X \to A \) an \( A-A \)-bimodule homomorphism. Then from the Morita context \( M = (A,Y,X,B,\alpha,\beta) \) we define the Morita context algebra:

\[
\Lambda_{(\alpha,\beta)} = \begin{pmatrix} A & Y \\ X & B \end{pmatrix},
\]

where the multiplication in \( \Lambda_{(\alpha,\beta)} \) is given by

\[
\begin{pmatrix} a & y \\ x & b \end{pmatrix} \cdot \begin{pmatrix} a' & y' \\ x' & b' \end{pmatrix} = \begin{pmatrix} aa' + \beta(y \otimes x') & ay' + yb' \\ xa' + bx' & bb' + \alpha(x \otimes y') \end{pmatrix}.
\]
and the maps $\alpha$ and $\beta$ satisfy $\alpha(x \otimes y)x' = x\beta(y \otimes x')$ and $y\alpha(x \otimes y') = \beta(y \otimes x')$, to make $\Lambda_{(\alpha, \beta)}$ associative. We say that a Morita context algebra has zero bimodule homomorphisms if $\alpha = \beta = 0$.

The family of Morita context algebras with zero bimodule homomorphisms has been extensively studied from the homological point of view. We can cite for example, [3], [5] and [7].

In [8] the authors prove that, under some hypotheses on the bimodule $Y$, a triangular algebra

$\begin{pmatrix} A & Y \\ 0 & B \end{pmatrix}$

has finite $\phi$-dimension if $A$ and $B$ have finite $\phi$-dimensions. A natural extension of triangular matrix algebras is the class of Morita context algebras. For that reason, in this article we focus on Morita context algebras that have the following shape: If $A = \frac{B}{\mathcal{I}}$ and $B = \frac{C}{\mathcal{J}}$ are finite dimensional algebras, $C = \frac{D}{\mathcal{K}}$ has the following hypotheses

H1: $Q_{C0} = Q_{A0} \cup Q_{B0}$.

H2: $Q_{C1} = Q_{A1} \cup Q_{B1} \cup \{\alpha_j : s(\alpha_j) \in Q_{A0}, t(\alpha_j) \in Q_{B0}\}_{j \in J} \cup \{\beta_k : s(\beta_k) \in Q_{B0}, t(\beta_k) \in Q_{A0}\}_{k \in K}$.

H3: $(I_A, I_B, \alpha_\omega, \beta_\omega)$ for $\omega \in Q_{A1}, \beta \in Q_{B1}, \alpha_\omega, \beta_\omega$ where $j \in J, k \in K) \subset I_C$.

H4: $\mathcal{O} = \{\text{addOrb}_A(\Omega_{A}(\Omega_{C}(B_0))) \times \text{addOrb}_B(\Omega_{B}(\Omega_{C}(A_0)))\} \subset K_0(C)$ is syzygy finite.

Note that:

- Hypotheses H1, H2 and H3 imply that for all $M \in \mod C$, $\Omega(M) = N_A \oplus N_B$ where $Y_A \in \mod A$ and $Y_B \in \mod B$. They also imply that the bimodules are left-semisimple.

- Hypotheses H1, H2 and H3 with an equality in Hypothesis H3 imply that the bimodules $X$ and $Y$ are right-projective.

- H4 is straightforward if $A$ and $B$ are syzygy finite or in case of the inclusion in Hypothesis H3 is an equality.

The content of this article can be summarised as follows. Section 2 is devoted to collect some necessary material for the developing of this work. In Section 3, we prove that Morita context algebras with hypotheses H1, H2, H3 and H4 which come from algebras with finite $\phi$-dimension also has finite $\phi$-dimension (Theorem 3.2). The previous result is a generalization of Theorem 5.2 of [8]. We also give an explicit bound when the inclusion of the ideal $I_C$, in Hypothesis 3, is an equality. Similar results are obtained for the finitistic dimension. Finally, in Section 4, we exhibit an example which shows a finite dimensional algebra $A$ such that $\phi\dim(A) \neq \phi\dim(A^{pp})$.

2. Preliminaries

Throughout this article $A$ is an Artin algebra and $\mod A$ is the category of finitely generated right $A$-modules, $\ind A$ is the subcategory of $\mod A$ formed by all indecomposable modules, $\mathcal{P}_A \subset \mod A$ is the class of projective $A$-modules. $\mathcal{S}(A)$ is the set of isoclasses of simple $A$-modules and $A_0 = \oplus_{S \in \mathcal{S}(A)} S$. For $M \in \mod A$ we denote by $M^k = \oplus_{i=1}^k M$, by $P(M)$ its projective cover, by $\Omega(M)$ its syzygy and by $l\ell(M)$ its Loewy length. The set $\text{Orb}_{\Omega(M)}$ is the $\Omega$-orbit of the module $M$, i.e. $\text{Orb}_{\Omega(M)} = \{\Omega^n(M)\}_{n \geq 0}$. For a subcategory $\mathcal{C} \subset \mod A$, we denote by $\text{findim}(\mathcal{C})$, $\text{gldim}(\mathcal{C})$ its finitistic dimension and its global dimension respectively and by $\text{add}\mathcal{C}$ the full subcategory of $\mod A$ formed by all the sums of direct summands of every $M \in \mathcal{C}$.

If $Q = (Q_0, Q_1, s, t)$ is a finite connected quiver, $\mathcal{M}_Q$ denotes its adjacency matrix and $kQ$ its associated path algebra. We compose paths in $Q$ from left to right. Given $\rho$ a path in $kQ$, $l(\rho)$, $s(\rho)$ and $t(\rho)$ denote the length, start and target of $\rho$ respectively. For a quiver $Q$, we denote by $J_Q$ the ideal of $kQ$ generated
by all its arrows. If $I$ is an admissible ideal of $kQ$ ($J^2_Q \subset I \subset J^m_Q$ for some $m \geq 2$), we say that $(Q, I)$ is a bounded quiver and the quotient algebra $\frac{kQ}{I}$ is the bound quiver algebra. A relation $\rho$ is an element in $kQ$ such that $\rho = \sum \lambda_i w_i$ where the $\lambda_i$ are scalars (not all zero) and the $w_i$ are paths with $I(w_i) \geq 2$ such that $s(w_i) = s(w_j)$ and $t(w_i) = t(w_j)$ if $i \neq j$. We recall that an admissible ideal $I$ is always generated by a finite set of relations (for a proof see Chapter II.2 Corollary 2.9 of [2]).

For a quiver $Q$, we say that $M = (M^v, T_\alpha)_{v \in Q_0, \alpha \in Q_1}$ is a representation if

- $M^v$ is a $k$-vectorial space for every $v \in Q_0$,
- $T_\alpha : M^{s(\alpha)} \to M^{t(\alpha)}$ for every $\alpha \in Q_1$.

A representation $(M^v, T_\alpha)_{v \in Q_0, \alpha \in Q_1}$ is finite dimensional if $M^v$ is finite dimensional for every $v \in Q_0$. For a path $w = \alpha_1 \ldots \alpha_n$ we define $T_w = T_{\alpha_1} \ldots T_{\alpha_n}$, and for a relation $\rho = \sum \lambda_i \omega_i$ we define $T_\rho = \sum \lambda_i T_{\omega_i}$.

A representation $M = (M^v, T_\alpha)_{v \in Q_0, \alpha \in Q_1}$ of $Q$ is bound by $I$ if we have $T_\rho = 0$ for all relations $\rho \in I$.

Let $M = (M^v, T_\alpha)_{v \in Q_0, \alpha \in Q_1}$ and $M' = (M'^v, T'_\alpha)_{v \in Q_0, \alpha \in Q_1}$ be two representations of the bounded quiver $(Q, I)$, a morphism $f : M \to M'$ is a family $f_v : M^v \to M'^v$ such that for all arrow $\alpha : v \to w$ we have the following commutative diagram:

$$
\begin{array}{c}
M^v \\
\downarrow f_v \\
M'^v
\end{array}
\quad
\begin{array}{c}
T_\alpha \\
\downarrow T'_\alpha \\
T^w
\end{array}
\quad
\begin{array}{c}
M^w \\
\downarrow f_w \\
M'^w
\end{array}
$$

We denote by $\text{Rep}_k(Q, I)$ the category of representations of $(Q, I)$ and by $\text{rep}_k(Q, I)$ the subcategory of $\text{Rep}_k(Q, I)$ formed by finite dimensional representations of $(Q, I)$.

We recall that there is a $k$-linear equivalence of categories

$$
F : \text{Rep}_k(Q, I) \to \text{Mod}_{\frac{kQ}{I}}
$$

that restricts to an equivalence of categories $G : \text{rep}_k(Q, I) \to \text{mod}_{\frac{kQ}{I}}$ (for a proof see Chapter III.1, theorem 1.6 of [2]).

If $A = \frac{kQ}{I}$ and $B = \frac{kQ}{J}$ are finite dimensional algebras, and $C = \frac{M}{I}$ the square algebra has the hypotheses H1 and H2, then the functors $\prod_A : \text{mod} C \to \text{mod} A$ and $\prod_B : \text{mod} C \to \text{mod} B$ are restrictions of the representations, i.e. For a representation $M = (M^v, T_\alpha)_{v \in Q_0, \alpha \in Q_1} \in \text{Rep}_k(Q_C, I_C)$, we have

- $\prod_A (M) = (M^v, T_\alpha)_{v \in Q_0, \alpha \in Q_1}$, and
- $\prod_B (M) = (M'^v, T'_\alpha)_{v \in Q_0, \alpha \in Q_1}$.

For a morphism $f = (f_v)_{v \in Q_0} : M \to M'$, then

- $\prod_A (f) = (f_v)_{v \in Q_0} : \prod_A (M) \to \prod_A (M')$ and
- $\prod_B (f) = (f_v)_{v \in Q_0} : \prod_B (M) \to \prod_B (M)$.

2.1. Igusa-Todorov functions

In this section we exhibit some general facts about the Igusa-Todorov functions for an Artin algebra $A$. The aim is to introduce material which will be used in the following sections.

**Lemma 2.1.** (Fitting Lemma) Let $R$ be a noetherian ring. Consider a left $R$-module $M$ and $f \in \text{End}_R(M)$. Then, for any finitely generated $R$-submodule $X$ of $M$, there is a non-negative integer

$$
\eta_f(X) = \min \{ k : \text{a non-negative integer}; f|_{f^m(X)} : f^m(X) \to f^{m+1}(X), \text{ is injective for } m \geq k \}.
$$

Furthermore, for any $R$-submodule $Y$ of $X$, we have that $\eta_f(Y) \leq \eta_f(X)$. 

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Definition 2.2. \[[11]\] Let $K_0(A)$ be the abelian group generated by all symbols $[M]$, with $M \in \text{mod}A$, modulo the relations

1. $[M] - [M'] = [M'']$ if $M \cong M' \oplus M''$,
2. $[P]$ for each projective module $P$.

Let $\bar{\Omega} : K_0(A) \rightarrow K_0(A)$ be the group endomorphism induced by $\Omega$, i.e. $\bar{\Omega}([M]) = [\Omega(M)]$. We denote by $K_i(A) = \bar{\Omega}(K_{i-1}(A)) = \ldots = \bar{\Omega}^i(K_0(A))$ for $i \geq 1$. We say that a subgroup $G \subset K_0(A)$ is syzygy finite if there is $n \geq 0$ such that $\bar{\Omega}^n(G)$ is finitely generated. For $M \in \text{mod}A$, $(\text{add}M)$ denotes the subgroup of $K_0(A)$ generated by the classes of indecomposable summands of $M$.

For a subcategory $C \subset \text{mod}A$, we denote by $\langle C \rangle \subset K_0(A)$ the free abelian group generated by the classes of direct summands of modules of $C$.

Definition 2.3. \[[11]\] The (right) Igusa-Todorov function $\phi$ of $M \in \text{mod}A$ is defined as

$$\phi_A(M) = \eta_\mathbb{Q}(\text{add}M).$$

In case there is no possible misinterpretation we will use the notation $\phi$ for the Igusa-Todorov $\phi$ function.

Proposition 2.4. \[[9], [11]\] If $M, N \in \text{mod}A$, then we have the following.

1. $\phi(M) = \text{pd}(M)$ if $\text{pd}(M) < \infty$.
2. $\phi(M) = 0$ if $M \in \text{ind}A$ and $\text{pd}(M) = \infty$.
3. $\phi(M) \leq \phi(M \oplus N)$.
4. $\phi(M^k) = \phi(M)$ for $k \geq 1$.
5. $\phi(M) \leq \phi(\Omega(M)) + 1$.

Proposition 2.5. Suppose $G \subset K_0(A)$ is a finitely generated subgroup with $\text{rk}(G) = m$.

1. If $\bar{\Omega}(G) \subset G$, then $\eta_{\mathbb{Q}|G} \leq m$.
2. If $G$ is syzygy finite, then $\eta_{\mathbb{Q}|G} < \infty$

Proof. The proof of item 1 is similar to the proof of Proposition 3.6 (item 3) from \[[13]\]. The proof of item 2 is similar to the proof of Theorem 3.2. from \[[13]\].

The result below follows directly from the fact that the Igusa-Todorov function verifies

$$\phi(M) = \min \{ l : \Omega^l(M) \in \text{add}M \text{ is a monomorphism } \forall s \in \mathbb{N} \}.$$

Proposition 2.6. Given $M \in \text{mod}A$,

$$\phi(A) = \max \{ n \in \mathbb{N} : \bar{\Omega}^n(v) = 0 \text{ and } \bar{\Omega}^{n-1}(v) \neq 0 \text{ for some } v \in \langle \text{add}M \rangle \}.$$

3. Morita context algebras and the Igusa-Todorov $\phi$ function

In this section we compute the $\phi$-dimension of the Morita context algebra $C$ under certain hypotheses on $A$, $B$, the quiver $Q_C$ and the ideal $I_C$.

The proof of the following lemma is similar to Theorem 5.2 of \[[3]\], for this reason it is left to the reader.

Lemma 3.1. Let $A = \frac{kQ_A}{I_A}$ and $B = \frac{kQ_B}{I_B}$ be finite dimensional algebras. Consider $C = \frac{kQ_C}{I_C}$ with the following conditions:

- $Q_{C_0} = Q_{A_0} \cup Q_{B_0}$.
- $Q_{C_1} = Q_{A_1} \cup Q_{B_1} \cup \{ \alpha_j : s(\alpha_j) \in Q_{A_0}, t(\alpha_j) \in Q_{B_0} \}_{j \in J} \cup \{ \beta_k : s(\beta_k) \in Q_{B_0}, t(\beta_k) \in Q_{A_0} \}_{k \in K}$. 

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• \((I_A, I_B, \alpha_j, \beta_k)\) for \(\alpha \in Q_{A1}, \beta \in Q_{B1}, \alpha_j \beta_k, \beta_k \alpha_j\) where \(j \in J, k \in K\) \(\subset I_C\).

Then for every module \(M \in \mod C, \Omega_C(M) = M_1 \oplus M_2\) with \(M_1 \in \mod A\) and \(M_2 \in \mod B\). Suppose \(\Omega_C(Top(M)) = M_1 \oplus M_2\) with \(M_1 \in \mod A\) and \(M_2 \in \mod B\), in case \(M \in \mod A\), then \(M_2 = M_2\), in case \(M \in \mod B\), then \(M_1 = M_1\).

**Theorem 3.2.** Let \(A_i = \frac{kQ_A}{I_i}\) for \(i = 1, \ldots, m\) be finite dimensional algebras. Consider \(C = \frac{kC}{I}\) with the following conditions:

• \(Q_{C0} = \bigcup_{i=1}^m Q_{A0} \cup Q_{B0}\)

• \(Q_{C1} = \bigcup_{i=1}^m Q_{A1} \cup \{\delta_k^i : s(\delta_k^i) \in Q_{A0}, t(\delta_k^i) \in Q_{A0}\}k \in K, i \not= j, j \in \{1, \ldots, m\}\).

• \((I_A, I_B, \alpha_j, \beta_k)\) for \(\alpha \in Q_{A1}, \beta \in Q_{B1}, \alpha_j \beta_k, \beta_k \alpha_j\) where \(j \in J, k \in K\) \(\subset I_C\).

• \(O = \prod_{i=1}^m \text{addOrb}_A(\prod_{i=1}^m (\Omega_C(A_0))) \subset K_0(C)\) is syzygy finite.

Then \(\phi \dim(C) < \infty\).

The proof of the previous result is analogous to the proof of the following result.

**Theorem 3.3.** Let \(A = \frac{kQ_A}{I}\) and \(B = \frac{kQ_B}{I}\) be finite dimensional algebras. Consider \(C = \frac{kC}{I}\) with the following conditions:

• \(Q_{C0} = Q_{A0} \cup Q_{B0}\)

• \(Q_{C1} = Q_{A1} \cup Q_{B1} \cup \{\alpha_j : s(\alpha_j) \in Q_{A0}, t(\alpha_j) \in Q_{B0}\}j \in J \cup \{\beta_k : s(\beta_k) \in Q_{B0}, t(\beta_k) \in Q_{A0}\}k \in K\).

• \((I_A, I_B, \alpha_j, \beta_k)\) for \(\alpha \in Q_{A1}, \beta \in Q_{B1}, \alpha_j \beta_k, \beta_k \alpha_j\) where \(j \in J, k \in K\) \(\subset I_C\).

• \(O = \text{addOrb}_A(\prod_{i=1}^m (\Omega_C(B0))) \cup \text{addOrb}_B(\Omega_C(A0))) \subset K_0(C)\) is syzygy finite.

Then \(\phi \dim(C) < \infty\) and \(\phi \dim(B) < \infty\).

**Proof.** Consider \(m = \max\{\phi \dim(A), \phi \dim(B)\}\). Let \(A\) and \(B\) be the subgroups of \(K_0(C)\) generated by the indecomposable modules of \(\mod A\) and \(\mod B\) respectively such that are not in \(O\). Note that \(K_1(C) \subset A \times B \times O\), and

\[
\Omega_C([M]) = \begin{cases} 
\Omega_A([M]) + [N] & \text{if } [M] \in A, \\
\Omega_B([M]) + [N] & \text{if } [M] \in B, \\
[N] & \text{if } [M] \in O,
\end{cases}
\]

with \([N] \in O\).

Consider \(v = (v_1, v_2, v_3) \in K_1(C) \subset A \times B \times O\) such that \(\Omega_C^n(v) = (0, 0, 0)\). Since \(\phi \dim(A) < \infty\) and \(\phi \dim(B) < \infty\), then \(\Omega_C^n(v) = (0, 0, v')\) for \(k \geq m\). On the other hand \(\Omega_C^n(0, 0, v') = (0, 0, 0)\), then, by Proposition 2.2 item 2, \(m - n \leq \text{sup}\{\eta_{\Omega_C}^{-1}(D) : D \in O\}\). We conclude that \(\phi \dim(C) \leq \text{sup}\{\eta_{\Omega_C}^{-1}(D) : D \in O\} + m + 1\). Since \(O\) is syzygy finite, then \(\phi \dim(C) < \infty\).

As a particular case of the previous result we have.

**Corollary 3.4.** Let \(A = \frac{kQ_A}{I_A}\) and \(B = \frac{kQ_B}{I_B}\) be finite dimensional algebras. Consider \(C = \frac{kC}{I}\) with the following conditions:

• \(Q_{C0} = Q_{A0} \cup Q_{B0}\).

• \(Q_{C1} = Q_{A1} \cup Q_{B1} \cup \{\alpha_j : s(\alpha_j) \in Q_{A0}, t(\alpha_j) \in Q_{B0}\}j \in J \cup \{\beta_k : s(\beta_k) \in Q_{B0}, t(\beta_k) \in Q_{A0}\}k \in K\).
Proof. Since $\Omega_1(\Omega_2(C_0))$ has finite projective dimension as $A$-module, then $[\text{addOrb}_{\Omega_1}(\Omega_2(\Omega_2(C_0)))]$ is a finitely generated subgroup of $K_2(A)$. Analogously we have that $[\text{addOrb}_{\Omega_2}(\Omega_1(\Omega_2(C_0)))]$ is a finitely generated subgroup of $K_2(B)$. Finally we deduce that $\mathcal{O} = [\text{addOrb}_{\Omega_1}(\Omega_2(\Omega_2(C_0)) \times \text{addOrb}_{\Omega_2}(\Omega_1(\Omega_2(C_0))))]$ is a finitely generated subgroup of $K_2(C)$. \hfill \Box

Proposition 3.5. Let $A = \mathbb{Q}_{A_n}$ and $B = \mathbb{Q}_{B_n}$ be finite dimensional algebras. Consider $C = \frac{\mathbb{Q}}{C}$ with the following conditions:

- $Q_{C_0} = Q_{A_0} \cup Q_{B_0}$.
- $Q_{C_1} = Q_{A_1} \cup Q_{B_1} \cup \{s(\alpha_j) \in Q_{A_0} : t(\alpha_j) \in Q_{B_0}, t(\beta_k) \in Q_{A_0} \}$.
- $I_C = \{I_A, I_B, \alpha_\beta, \beta_k \}$ for $\alpha \in Q_{A_1}, \beta \in Q_{B_1}, \alpha_\beta, \beta_k \alpha_j$ where $j \in J, k \in K \} \subset I_C$.

Then $\phi \dim(C) < \infty$.\hfill \Box

1. We denote by $m = \max \{\phi \dim(A), \phi \dim(B)\}$. \hfill \Box

2. Suppose now that $\phi \dim(B) = \phi \dim(A) + |Q_{B_0}| + 1$. Then we have $\phi \dim(A) = m$, $\phi \dim(B) = 0$ and $\phi \dim(C) = 0$.

Let $v = (v_0, v_1, v_2, v_3, v_4)$ be a vertex of $\mathcal{O}$. Then $v \in K_2(A) \times K_2(B)$.\hfill \Box
2. From now on we denote the arrows of $Q_{C^{op}}$ with the same name as $Q$.

Claim 1: If $M = \Omega(N)$ and $M = (M^v, T_\gamma) \in \text{ind} C$, then $\text{Top}M \in \text{mod} A$ or $\text{Top}M \in \text{mod} B$.

Suppose $\text{Top}M \notin \text{mod} A$ and $\text{Top}M \notin \text{mod} B$. This implies that there exist vertices $v_0 \in Q_{C_0}$, $v_1 \in Q_{A_0}$, $v_2 \in Q_{B_0}$ and paths $\rho_1 : v_1 \rightarrow v_0$ and $\rho_2 : v_2 \rightarrow v_0$ such that $T_{\rho_i}(x_i) = x_0 \neq 0$ for $i = 1, 2$ and $x_i \in M^{v_i}$ for $i = 0, 1, 2$. Consider an indecomposable projective module $P = (P^v, T_\gamma)$ and a non trivial morphism $f : M \rightarrow P$ such that $f_{\rho_0}(x_0) = y_0$ with $y_0 \in P^{v_0}$. We conclude that there are non null vectors $y_1 \in P^{v_1}$ and $y_2 \in P^{v_2}$ such $T_{\rho_i}(y_i) = y_0$ for $i = 1, 2$, which is a contradiction since $\text{Top}P$ is a simple module.

Claim 2: Let $P = (P^v, T_\gamma) \in \mathcal{P}_C$ be indecomposable with $\text{Top}P \in \text{mod} A$. If $v_0 \in Q_{A_0}$ is such that $P^{v_0} \neq 0$, then $T_{\beta_j}$ is injective for all $\beta_j : v_0 \rightarrow Q_{B_0}$.

It follows since for every nontrivial path $\rho : v_0 \rightarrow w$ in $Q_A$, $\rho \beta_j \neq 0$ whenever $t(\rho) = s(\beta_j)$.

Now consider $M$ non simple indecomposable module such that $M = \Omega(N)$ with $\text{Top}(M) \in \text{mod} A$. Then $M$ is a submodule of a projective module $P$ with $\text{Top}P \subset \text{mod} A$ and we conclude that claim 2 is also true for the module $M$.

We denote by $\mathcal{T} = \{ S_{v_0} \in \mathcal{S}(C) \}$ and there is $\beta_k$ or $\alpha_j$ with $s(\beta_k) = v_0$ or $s(\alpha_j) = v_0$. Let $f : \Omega(\text{mod} C^{op}) \rightarrow \text{mod} A^{op} \times \text{mod} B^{op} \times \langle C_0 \rangle$ be the function defined in the indecomposable modules as follows

\[
f(M) = \begin{cases} 
(0, 0, S_{v_0}) & \text{if } M = S_{v_0} \in \mathcal{T} \\
(\prod A(M), 0, 0) & \text{if we are not in the first case and } \text{Top}(M) \in \text{mod} A \\
(0, \prod B(M), 0) & \text{if we are not in the first case and } \text{Top}(M) \in \text{mod} B
\end{cases}
\]

Claim 3: $f : \Omega(\text{mod} C^{op}) \rightarrow \text{mod} A^{op} \times \text{mod} B^{op} \times \langle C_0 \rangle$ is a monomorphism of groups and $f(\Omega_{C^{op}}(M)) = (\Omega_{A^{op}}(M_1), \Omega_{B^{op}}(M_2), 0)$ if $f(M) = (M_1, M_2, 0)$.

If $M = M_1 \oplus M_2 \oplus S$ where

- $S$ is a direct sum of modules of $\mathcal{T}$.
- $M_1$ has no direct summand that are simple modules of $\mathcal{T}$ and $\text{Top}(M_1) \in \text{mod} A^{op}$.
- $M_2$ has no direct summand that are simple modules of $\mathcal{T}$ and $\text{Top}(M_1) \in \text{mod} B^{op}$.

Then we have

- Let $N_1 = \Omega_{C^{op}}(M_1)$, then it has not direct summand that are simple modules of $\mathcal{T}$ and $\text{Top}(N_1) \subset \text{mod} A^{op}$ and the following diagram is commutative with exact rows.

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \Omega_{A^{op}}(\prod A(N_1)) & \longrightarrow & \prod A^{op}(P) & \longrightarrow & \prod A^{op}(M_1) & \longrightarrow & 0 \\
\Pi_{A^{op}} & \downarrow & \Pi_{A^{op}} & \downarrow & \Pi_{A^{op}} & \downarrow & \Pi_{A^{op}} & \\
0 & \longrightarrow & N_1 & \longrightarrow & P & \longrightarrow & M_1 & \longrightarrow & 0
\end{array}
\]

- Let $N_2 = \Omega_{C^{op}}(M_2)$, then it has not direct summand that are simple modules of $\mathcal{T}$ and $\text{Top}(N_2) \subset \text{mod} B^{op}$, and the following diagram is commutative with exact rows.

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \Omega_{B^{op}}(\prod B^{op}(N_2)) & \longrightarrow & \prod B^{op}(P) & \longrightarrow & \prod B^{op}(M_2) & \longrightarrow & 0 \\
\Pi_{B^{op}} & \downarrow & \Pi_{B^{op}} & \downarrow & \Pi_{B^{op}} & \downarrow & \Pi_{B^{op}} & \\
0 & \longrightarrow & N_2 & \longrightarrow & P & \longrightarrow & M_2 & \longrightarrow & 0
\end{array}
\]
If $S = 0$ then $f(\Omega_{C^{op}}(M)) = (\Omega_{A^{op}}(M_1), \Omega_{B^{op}}(M_2), 0)$.

On the other hand if $f(M_1 \oplus M_2 \oplus S) = f(M'_1 \oplus M'_2 \oplus S')$, then $\prod_{A^{op}}(M_1) = \prod_{A^{op}}(M'_1)$, $\prod_{B^{op}}(M_2) = \prod_{B^{op}}(M'_2)$ and $S = S'$. Since $\prod_{A^{op}}(M_1) = \prod_{A^{op}}(M'_1)$ and $T_{\alpha}$ are injective for every representation in $\Omega(\text{mod}C^{op})$, then $M_1 = M'_1$. The same follows from $\prod_{B^{op}}(M_2) = \prod_{B^{op}}(M'_2)$.

Finally we deduce, with similar computations to part 1, that

$$\phi\dim(C^{op}) \leq \max\{\phi\dim(A^{op}), \phi\dim(B^{op})\} + |Q_{A^{op}}| + |Q_{B^{op}}| + 1$$

\[\Box\]

**Remark 3.6.** In the Hypothesis of Proposition 3.5, we have

1. $\phi\dim(C) < \infty \iff \phi\dim(A) < \infty$ and $\phi\dim(B) < \infty$.
2. From the proof of Proposition 3.5 we deduce that if $A$ and $B$ ($A^{op}$ and $B^{op}$) are syzygy finite then $C$ ($C^{op}$) is also syzygy finite.

We finish this section with two results on Morita context algebras for the finitistic dimension. The first one, the theorem below, has its hypotheses similar to Theorem 3.3.

**Theorem 3.7.** Let $A = \frac{kQ_A}{I_A}$ and $B = \frac{kQ_B}{I_B}$ be finite dimensional algebras. Consider $C = \frac{kr}{I_C}$ with the following conditions:

- $Q_{C0} = Q_{A0} \cup Q_{B0}$.
- $Q_{C1} = Q_{A1} \cup Q_{B1} \cup \{\alpha_j : s(\alpha_j) \in Q_{A0}, t(\alpha_j) \in Q_{B0}\}_{j \in J} \cup \{\beta_k : s(\beta_k) \in Q_{B0}, t(\beta_k) \in Q_{A0}\}_{k \in K}$.
- $\{I_A, I_B, \alpha_j, \beta_k : \alpha \in Q_{A1}, \beta \in Q_{B1}, \alpha_j, \beta_k \in Q_{A0} \cup Q_{B0}\}_{j \in J, k \in K} \subset I_C$.
- $\mathcal{O} = \text{addOrb}_{\Omega_A}(\Omega_C(B_0)) \oplus \text{addOrb}_{\Omega_B}(\Omega_C(A_0)) \subset K_0(C)$ is syzygy finite.
- $\text{findim}(A) < \infty$ and $\text{findim}(B) < \infty$.

Then $\text{findim}(C) < \infty$.

**Proof.** We begin the proof with a straightforward claim, it follows from Lemma 3.1.

**Claim:** If $M \in \text{mod}A$ (mod$B$), then $\Omega_C(M) = \Omega_A(M) \oplus N (\Omega_B(M) \oplus N)$ where $[N] \in \mathcal{O}$.

Consider $M \in \text{mod}A$ (mod$B$) such that $\text{pd}_C(M) < \infty$. Since $\text{pd}_A(M) \leq \text{pd}_C(M) + \text{findim}[N : [N] \in \mathcal{O}]$ and $\mathcal{O}$ is syzygy finite, then $\text{pd}_A(M) < \infty$ (\text{pd}_B(M) < \infty).

Let $M \in \text{mod}C$ such that $\text{pd}_C(M) < \infty$. By Lemma 3.1 $\Omega_C(M) = N_A \oplus N_B$ where $N_A \in \text{mod}A$, $N_B \in \text{mod}B$ and $\Omega^k_A(N_A) = \Omega^k_B(N_B) = 0$ if $k = \max\{\text{findim}(A), \text{findim}(B)\}$.

Finally, by the claim above $[\Omega^{k+1}(M)] \in \mathcal{O}$. Hence $\text{pd}(\Omega^{k+1}(M)) < \infty$ and in particular

$$\text{pd}(M) \leq k + 1 + \text{findim}[N : [N] \in \mathcal{O}]$$

\[\Box\]

The following proposition has its hypotheses similar to Theorem 3.5.

**Proposition 3.8.** Let $A = \frac{kQ_A}{I_A}$ and $B = \frac{kQ_B}{I_B}$ be finite dimensional algebras. Consider $C = \frac{kr}{I_C}$ with the following conditions:

- $Q_{C0} = Q_{A0} \cup Q_{B0}$.
- $Q_{C1} = Q_{A1} \cup Q_{B1} \cup \{\alpha_j : s(\alpha_j) \in Q_{A0}, t(\alpha_j) \in Q_{B0}\}_{j \in J} \cup \{\beta_k : s(\beta_k) \in Q_{B0}, t(\beta_k) \in Q_{A0}\}_{k \in K}$.
• \((I_A, I_B, \alpha \beta, \beta \beta)\) for \(\alpha \in Q_{A1}, \beta \in Q_{B1}, \alpha \beta, \beta \beta \in K\) = \(I_C\).

• \(\mathcal{O} = [\text{addOrb}_{Q_A}(\Pi_M(\Omega_C(B_0))) \times \text{addOrb}_{Q_B}(\Pi_M(\Omega_C(A_0)))] \subset K_C\) is syzygy finite.

• findim(A) < \infty and findim(B) < \infty.

Then findim(C) < \infty.

Proof. Consider \(M \in \Omega(\text{mod}C)\), such that pd\(_{C_{op}}(M) < \infty\). Note, by Claim 3 of Proposition 5.3, that for \(M \in \Omega(\text{mod}C_{op})\) we have the following:

• \(M = M_1 \oplus M_2 \oplus S\) where \(M_1 \in \text{mod}A_{op}, M_2 \in \text{mod}B_{op},\) and \(S\) is a semisimple module, and

• \(\Omega(k_{C_{op}}(f(M_1))) = \Omega(k_{A_{op}}(M_2)), \Omega(k_{C_{op}}(f(M_2))) = \Omega(k_{B_{op}}(M_2))\).

Since pd\(_{C_{op}}(M) < \infty\), then pd\(_{C_{op}}(M_1) < \infty\) and pd\(_{C_{op}}(M_2) < \infty\), hence pd\(_{C_{op}}(M_1) < \text{pd}_{A_{op}}(M_1) < \text{findim}(A)\) and pd\(_{C_{op}}(M_2) < \text{pd}_{B_{op}}(M_2) < \text{findim}(B)\). Since \(S \in \text{addC}_{op}\) and it has finite nonisomorphic indecomposable modules, then pd\(_{C_{op}}(M) < \text{max}\{\text{findim}(A), \text{findim}(B), \text{findim}(C)\}\) and we conclude that findim(C) < max{findim(A), findim(B), findim(C)} + 1.

\[\square\]

4. (Counter)examples

In this section we present two examples. The first one (Example 5.4 of 3) shows that Hypotheses H1, H2 and H4 are not enough to obtain the results of the previous section for the Igusa-Todorov \(\phi\) function. The second is an example of an algebra where its \(\phi\)-dimension is not symmetric, i.e. \(\phi\text{dim}(A)\) and \(\phi\text{dim}(A_{op})\) do not agree.

Example 4.1. Let \(C = \frac{kQ}{I}\) be an algebra where \(Q\) is

\[\text{\includegraphics{example4.png}}\]

and \(I = (\alpha_i \alpha_{i+1} - \bar{\beta}_i \beta_{i+1}, \beta_i \beta_{i+1} - \bar{\beta}_i \bar{\beta}_{i+1}, \alpha_i \bar{\beta}_{i+1}, \bar{\alpha}_i \alpha_{i+1}, \beta_i \bar{\beta}_{i+1}, \bar{\beta}_i \bar{\beta}_{i+1}\), for \(i \in \mathbb{Z}_4, J^3\) Consider \(A = kQ_A\) and \(B = kQ_B\) where

• \(Q_A\) is the full quiver formed by the vertices 1 and 2.

• \(Q_B\) is the full quiver formed by the vertices 3 and 4.

It is easy to show that hypotheses H1, H2 and H4 are verified. However it was showed in 3 that \(\phi\text{dim}(C) = \infty\).
Example 4.2. Consider the finite dimensional algebra $C_{p,q} = \frac{kQ}{I_{p,q}}$ where $Q$ is the following quiver

and $I_{p,q}$ is generated by

- $\gamma_i \in \mathbb{Z}$ for all $i$, $i = 1, \ldots, m$,
- $\alpha_i, \beta_i, \alpha_i', \beta_i' \in \mathbb{Z}$ for all $i$,
- $\alpha_i' - \alpha_i - \beta_i - \beta_i' \in \mathbb{Z}$ for all $i$,
- $\alpha_3, \beta_3, \alpha_3', \beta_3' \in \mathbb{Z}$ and
- $\alpha_3, \beta_3, \beta_3' \in \mathbb{Z}$ for all $i$.

Since $\text{rad}^3(C_{p,q}) = 0$, then $\text{ll}(M) \leq 3$, and $\text{ll}(\Omega(M)) \leq 2$ for all $M \in \text{mod}C_{p,q}$.

Let $Q \subset Q$ be the full subquiver with vertices $\{a_1, a_2, a_3, b_1, b_2, b_3, c_0\}$. Consider $B_{p,q} = \frac{kQ}{I_{p,q} \cap Q}$. If $M$ is in $\text{mod}B$, then the Top of the indecomposable modules of $\Omega(M)$ is in

- $c_0$ (level 0),
- $a_1$ or $b_1$ (level 1),
- $a_2$ or $b_2$ (level 2) and
- $a_3$ or $b_3$ (level 3).

Consider the following modules

- $M_{i,\lambda,\nu}$ such that $\text{supp}(M_{i,\lambda,\nu}) = a_i \frac{\alpha_i}{\alpha_i'} a_{i+1}$ and $M_{i,\lambda,\nu} = \cdots k^n \frac{J_{\lambda}}{1_k n} k^n \cdots$
- $M'_{i,\lambda,\nu}$ such that $\text{supp}(M'_{i,\lambda,\nu}) = a_i \frac{\alpha_i}{\alpha_i'} a_{i+1}$ and $M'_{i,\lambda,\nu} = \cdots k^n \frac{J_{\lambda}'}{1_k n} k^n \cdots$
- $M_{i,\nu}$ such that $\text{supp}(M_{i,\nu}) = a_i \frac{\alpha_i}{\alpha_i'} a_{i+1}$ and $M_{i,\nu} = \cdots k^n \frac{\frac{1_k n}{(1,1_k n)}}{0} k^{n+1} \cdots$
- $\tilde{M}_{i,\nu}$ such that $\text{supp}(\tilde{M}_{i,\nu}) = a_i \frac{\alpha_i}{\alpha_i'} a_{i+1}$ and $\tilde{M}_{i,\nu} = \cdots k^{n+1} \frac{(1,1_k n)}{0} k^n \cdots$
\[ N_{i,\lambda,n} \text{ such that } \text{supp}(N_{1,\lambda,n}) = b_i \frac{\beta_i}{\beta_i'} b_i+1 \text{ and } N_{i,\lambda,n} = \cdots \frac{k^n}{J_{\lambda}} \frac{J_n}{k^n} \cdots \]

\[ N'_{i,\lambda,n} \text{ such that } \text{supp}(N'_{1,\lambda,n}) = b_i \frac{\beta_i}{\beta_i'} b_i+1 \text{ and } N'_{i,\lambda,n} = \cdots \frac{k^n}{J_{\lambda}} \frac{J_n}{k^n} \cdots \]

\[ N_{i,n} \text{ such that } \text{supp}(N_{1,n}) = b_i \frac{\beta_i}{\beta_i'} b_i+1 \text{ and } N_{i,n} = \cdots \frac{(1_{k^n,0})}{(0,1_{k^n})} \frac{k^n}{(0,1_{k^n})} \cdots \]

\[ \bar{N}_{i,n} \text{ such that } \text{supp}(\bar{N}_{1,n}) = b_i \frac{\beta_i}{\beta_i'} b_i+1 \text{ and } \bar{N}_{i,n} = \cdots \frac{(1_{k^n,0})}{(0,1_{k^n})} \frac{k^n}{(0,1_{k^n})} \cdots \]

\[ M_{0,\lambda,\mu,n} \text{ such that } \text{supp}(M_{0,\lambda,\mu,n}) = b_i \frac{\beta_i}{\beta_i'} c_0 \frac{a_1}{a_0} a_1 \text{ and } M_{0,\lambda,\mu,n} = \cdots \frac{k^n}{J_{\lambda}} \frac{J_n}{k^n} \frac{1_{k^n}}{1_{k^n}} \frac{k^n}{k^n} \cdots \]

\[ M'_{0,\lambda,\mu,n} \text{ such that } \text{supp}(M'_{0,\lambda,\mu,n}) = b_i \frac{\beta_i}{\beta_i'} c_0 \frac{a_1}{a_0} a_1 \text{ and } M'_{0,\lambda,\mu,n} = \cdots \frac{k^n}{J_{\lambda}} \frac{J_n}{k^n} \frac{1_{k^n}}{1_{k^n}} \frac{k^n}{k^n} \cdots \]

\[ M_{0,n} \text{ such that } \text{supp}(M_{0,n}) = b_i \frac{\beta_i}{\beta_i'} c_0 \frac{a_1}{a_0} a_1 \text{ and } M_{0,n} = \cdots \frac{(1_{k^n,0})}{(0,1_{k^n})} \frac{k^n}{(0,1_{k^n})} \frac{(1_{k^n,0})}{(0,1_{k^n})} \cdots \]

\[ \bar{M}_{0,n} \text{ such that } \text{supp}(\bar{M}_{0,n}) = b_i \frac{\beta_i}{\beta_i'} c_0 \frac{a_1}{a_0} a_1 \text{ and } \bar{M}_{0,n} = \cdots \frac{(1_{k^n,0})}{(0,1_{k^n})} \frac{k^n}{(0,1_{k^n})} \frac{(1_{k^n,0})}{(0,1_{k^n})} \cdots \]

for \( i = 1, 2, 3, \lambda \in k, \ n \in \mathbb{N} \) and \( a_4 = b_4 = c_0 \)

Note that

- \( M_{1,0} = M_{2,0} = S_{a_2}, M_{2,0} = M_{3,0} = S_{a_3}, M_{3,0} = M_{0,0} = S_{c_0} \)
- \( N_{1,0} = N_{2,0} = S_{b_2}, N_{2,0} = \bar{N}_{3,0} = S_{b_3}, N_{3,0} = M_{0,0} = S_{c_3}, M_{0,0} = M_{1,0} \oplus \bar{N}_{1,0} = S_{a_1} \oplus S_{b_1} \)
- \( M'_{i,\lambda,n} = M_{i,\lambda,n}, N'_{i,\lambda,n} = N_{i,\lambda,n}, M'_{0,\lambda,\mu,n} = M_{0,\lambda,\mu,n} \) if \( \mu \neq 0, \lambda \neq 0 \).
- \( \Omega(M_{1,\lambda,n}) = M_{2,\lambda,n}, \Omega(M_{2,\lambda,n}) = M_{3,\lambda,n}, \Omega(M_{3,\lambda,n}) = M_{0,\lambda,\lambda,n} \) and \( \Omega(M_{0,\lambda,\mu,n}) = M_{1,\lambda,n} \oplus N_{1,\mu,n} \).
- \( \Omega(M'_{1,0,n}) = M'_{2,0,n}, \Omega(M'_{2,0,n}) = M'_{3,0,n}, \Omega(M'_{3,0,n}) = M'_{0,0,0,n} \) and \( \Omega(M'_{0,0,0,n}) = M'_{1,0,n} \oplus N'_{1,0,n} \).
- \( \Omega(M_{1,n}) = M_{2,n-1}, \Omega(M_{2,n}) = M_{3,n-1}, \Omega(M_{3,n}) = M_{0,n-1} \) and \( \Omega(M_{0,n}) = M_{1,n-1} \oplus N_{1,n-1} \) for \( n \geq 1 \).
- \( \Omega(M'_{1,n}) = M'_{2,n+1}, \Omega(M'_{2,n}) = M'_{3,n+1}, \Omega(M'_{3,n}) = M'_{0,n+1} \) and \( \Omega(M'_{0,n}) = M'_{1,n+1} \oplus \bar{N}_{1,n+1} \).
- \( \Omega(N_{1,\lambda,n}) = N_{2,\lambda,n}, \Omega(N_{2,\lambda,n}) = N_{3,\lambda,n} \) and \( \Omega(N_{3,\lambda,n}) = M_{0,\lambda,\lambda,n} \).
- \( \Omega(N'_{1,0,n}) = N'_{2,0,n}, \Omega(N'_{2,0,n}) = N'_{3,0,n} \) and \( \Omega(N'_{3,0,n}) = M'_{0,0,0,n} \)
\[\Omega(N_{1,n}) = N_{2,n-1}, \Omega(N_{2,n}) = N_{3,n-1}, \Omega(N_{3,n}) = N_{0,n-1} \text{ and } \Omega(M_{0,n}) = M_{1,n-1} \oplus N_{1,n-1} \text{ for } n \geq 1.\]

\[\Omega(N_{1,n}) = \tilde{N}_{2,n+1}, \Omega(N_{2,n}) = \tilde{N}_{3,n+1}, \Omega(N_{3,n}) = \tilde{N}_{0,n+1} \text{ and } \Omega(\tilde{N}_{0,n}) = \tilde{M}_{1,n+1} \oplus \tilde{N}_{1,n+1}.\]

Consider \(\mathcal{V}\) the full subcategory generated by \(d\) direct sum of the following family of modules

\[\{M_{i,\lambda,n}, M'_{i,0,0,n}, M_{i,n}, N_{j,\lambda,n}, N'_{i,0,n}, N_{i,n}, N'_{j,n}, M_{0,\lambda,\mu,n}, M'_{0,0,0,n} \text{ for } i = 0, 1, 2, 3, j = 1, 2, 3 \text{ and } \lambda \in k\}\]

Claim 1: \(\phi \text{dim}(\mathcal{V}) = 4.\)

It follows from the fact that \(\bar{\Omega}(\mathcal{V})\) is an monomorphism.

Claim 2: \(\Omega^2(\text{mod}B_{p,q}) \subset \mathcal{V} \subset \Omega(\text{mod}C_{p,q}) \text{ and } K_2(C_{p,q}) \subset \langle [\mathcal{V}], [S_{c_{m-1}}], [S_{c_{m-2}}], \ldots, [S_{c_1}] \rangle\)

Note that \(\Omega(S_{c_1}) = M_{0,q,1,1},\)

- \([\Omega(S_{c_1})] \notin \bar{\Omega}(\mathcal{V})\),
- \([\Omega^3(S_{c_1})] \notin \langle \Omega^2([\mathcal{V}]), [\Omega(S_{c_1})] \rangle\),
- \([\Omega^3(S_{c_1})] \notin \langle \Omega^2([\mathcal{V}]), [\Omega(S_{c_1})], [\Omega^2(S_{c_1})] \rangle\) and
- \([\Omega^4(S_{c_1})] \notin \langle \Omega^4([\mathcal{V}]), [\Omega(S_{c_1})], [\Omega^2(S_{c_1})], [\Omega^3(S_{c_1})] \rangle\).

Claim 3: \(\phi \text{dim}(C_{p,q}) = 5.\)

It follows from the fact that \(\bar{\Omega}(\mathcal{V})\) is a monomorphism.

On the other hand \(\phi \text{dim}(C_{p,q}^{op}) \geq m - 1\) because \(\text{id}_{C_{p,q}}(S_{c_1}) = m - 1.\)

Note that \(C_{p,q}\) also does not verify Proposition \(3.5.2\), just take \(A = \frac{1}{Q}Q^{e}\) where \(Q^{e}\) is the subquiver of \(Q\) generated by \(\{c_0, c_1, \ldots, c_{m+1}\}\) and \(B_{p,q}\) as before.

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