BISHOP’S THEOREM AND DIFFERENTIABILITY OF A
SUBSPACE OF $C_b(K)$

YUN SUNG CHOI, HAN JU LEE, AND HYUN GWI SONG

ABSTRACT. Let $K$ be a Hausdorff space and $C_b(K)$ be the Banach algebra of all complex bounded continuous functions on $K$. We study the Gâteaux and Fréchet differentiability of subspaces of $C_b(K)$. Using this, we show that the set of all strong peak functions in a nontrivial separating separable subspace $H$ of $C_b(K)$ is a dense $G_δ$ subset of $H$, if $K$ is compact. This gives a generalized Bishop’s theorem, which says that the closure of the set of strong peak point for $H$ is the smallest closed norming subset of $H$. The classical Bishop’s theorem was proved for a separating subalgebra $H$ and a metrizable compact space $K$.

In the case that $X$ is a complex Banach space with the Radon-Nikodým property, we show that the set of all strong peak functions in $A_b(B_X) = \{f \in C_b(B_X) : f|_{B_X} \text{ is holomorphic}\}$ is dense. As an application, we show that the smallest closed norming subset of $A_b(B_X)$ is the closure of the set of all strong peak points for $A_b(B_X)$. This implies that the norm of $A_b(B_X)$ is Gâteaux differentiable on a dense subset of $A_b(B_X)$, even though the norm is nowhere Fréchet differentiable when $X$ is nontrivial. We also study the denseness of norm attaining holomorphic functions and polynomials. Finally we investigate the existence of numerical Shilov boundary.

1. Introduction

Let $K$ be a Hausdorff topological space. A function algebra $A$ on $K$ will be understood to be a closed subalgebra of $C_b(K)$ which is the Banach algebra of all bounded complex-valued continuous functions on $K$. The norm $\|f\|$ of a bounded continuous function $f$ on $K$ is defined to be $\sup_{x \in K} |f(x)|$. A function algebra $A$ is called separating if for two distinct points $s, t$ in $K$, there is $f \in A$ such that $f(s) \neq f(t)$.

In this paper, a subspace means a closed linear subspace. For each $t \in K$, let $\delta_t$ be an evaluation functional on $C_b(K)$, that is, $\delta_t(f) = f(t)$ for every $f \in C_b(K)$. A subspace $A$ of $C_b(K)$ is called separating if for distinct points $t, s$ in $K$ we have $\alpha\delta_t \neq \beta\delta_s$ for any complex numbers $\alpha, \beta$ of modulus 1 as a linear functional on $A$. This definition of a separating subspace is a natural extension of the definition of a separating function algebra. In fact, given a separating function algebra $A$

2000 Mathematics Subject Classification. 46B04, 46G20, 46G25, 46B22.

Key words and phrases. Differentiability, Bishop’s theorem, Algebra of holomorphic functions, Boundary for algebra.

This work was supported by grant No. R01-2004-000-10055-0 from the Basic Research Program of the Korea Science & Engineering Foundation.
on $K$ and given two distinct points $t, s$ in $K$, we have $\alpha \delta_t \neq \beta \delta_s$ on $A$ for any nonzero complex numbers $\alpha$ and $\beta$. Otherwise, there are some nonzero complex numbers $\alpha$ and $\beta$ such that $\alpha \delta_t = \beta \delta_s$ on $A$. Let $\gamma = \beta / \alpha$. Choose $f \in A$ so that $f(s) = 1$ and $f(t) \neq 1$. By assumption, $f(t) = \gamma$. Fix a positive number $r$ with $0 < r < 1/\|f\|$. Taking $g = 1 - r \sum_{m=1}^{\infty} r^m f^m = (1 - r) f 1 - rf$, we have $g \in A$ and $g(s) = 1$, which imply $g(t) = \gamma = f(t)$. Hence $\gamma = g(t) = (1 - r) f(t) 1 - rf(t) = (1 - r) \gamma 1 - r \gamma$.

This equation shows that $\gamma = 1$ and $f(t) = 1$, which is a contradiction.

A nonzero element $f \in A$ is called a peak function if there exists only one point $x \in K$ such that $|f(x)| = \|f\|$. In this case, the corresponding point $x$ is said to be a peak point for $A$. A nonzero element $f \in A$ is called a strong peak function if there exists only one point $x \in K$ such that $|f(x)| = \|f\|$ and for any neighborhood $V$ of $x$, there is $\delta > 0$ such that if $y \in K \setminus V$, then $|f(y)| \leq \|f\| - \delta$. In this case, the corresponding point $x$ is called a strong peak point for $A$. We denote by $\rho A$ the set of all strong peak point for $A$. Note also that if $K$ is a compact Hausdorff space, every peak function (resp. peak point for $A$) is a strong peak function (resp. strong peak point for $A$).

A subset $F$ of $K$ is said to be a norming subset for $A$ if for every $f \in A$, we have $\|f\| = \sup_{x \in F} |f(x)|$.

Note that every closed norming subset contains all strong peak points. If $K$ is a compact Hausdorff space, then a closed subset $T$ of $K$ is a norming subset for $A$ if and only if $T$ is a boundary for $A$, that is, for every $f \in A$, we have $\max_{t \in T} |f(t)| = \|f\|$.

A famous theorem of Shilov (see [45, 37]) asserts that if $A$ is a separating function algebra $A$ on a compact Hausdorff space $K$, then there is a smallest closed boundary for $A$, which is called the Shilov boundary for $A$ and denoted by $\partial A$. We shall say that a subspace $A$ of $C_b(K)$ on a Hausdorff space $K$ has the Shilov boundary if there is a smallest closed norming subset for $A$. If $K$ is not compact, a separating function algebra $A$ on $K$ need not have the Shilov boundary (see [6, 14, 31, 34]).

Let $X$ be a real or complex Banach space. We denote by $B_X$ and $S_X$ the closed unit ball and unit sphere of $X$, respectively. The norm $\| \cdot \|$ of $X$ is said to be Gâteaux differentiable (resp. Fréchet differentiable) at $x$ if $\lim_{t \to 0} \frac{\|x + ty\| + \|x - ty\| - 2\|x\|}{t} = 0$.
for every $y \in X$ (resp. uniformly for $y \in S_X$). Notice that if the norm of a nontrivial Banach space is Gâteaux (resp. Fréchet) differentiable at $x$, then $x \neq 0$ and the norm is also Gâteaux (resp. Fréchet) differentiable at $\alpha x$ for any nonzero scalar $\alpha$ (For more details, see [21]).

Let $C$ be a convex subset of a complex Banach space. An element $x \in C$ is said to be an (resp. complex) extreme point of $C$ if for every $y \neq 0$ in $X$, there is a real (resp. complex) number $\alpha$, $|\alpha| \leq 1$ such that $x + \alpha y \notin C$. The set of all (resp. complex) extreme points of $C$ is denoted by $\text{ext}(C)$ (resp. $\text{ext}_C(C)$). A point $x^* \in B_{X^*}$ is said to be a weak-$*$ exposed point of $B_X$, if there exists $x \in S_X$ such that

$$1 = \Re x^*(x) > \Re y^*(x), \quad \forall y^* \in B_{X^*}.$$ 

The corresponding point $x \in S_X$ is said to be a smooth point of $B_X$. It is well-known [21] that the norm of $X$ is Gâteaux differentiable at $x \in S_X$ if and only if $x$ is a smooth point of $B_X$. The set of all weak-$*$ exposed points of $B_X$ is denoted by $\text{wexp}(B_X)$. It is easy to check that $\text{wexp}(B_X) \subset \text{ext}(B_{X^*})$.

We denote by $C_b(K, Y)$ the Banach space of all bounded continuous functions of a Hausdorff space $K$ into a Banach space $Y$ with the sup norm. By replacing the absolute value with the norm of $Y$, the notion of a strong peak function or a norming subset for $C_b(K, Y)$ can be defined in the same way as for $C_b(K)$. Note that $\rho C_b(K) = \rho C_b(K, Y)$ for a nontrivial Banach space $Y$.

Given complex Banach spaces $X, Y$, the following two subspaces of $C_b(B_X, Y)$ are the ones of our main interest:

$$A_b(B_X, Y) = \{ f \in C_b(B_X, Y) : f \text{ is analytic on the interior of } B_X \}$$

$$A_u(B_X, Y) = \{ f \in A_b(B_X, Y) : f \text{ is uniformly continuous on } B_X \}.$$ 

It was shown in [7] that these two function Banach spaces are the same if and only if $X$ is finite dimensional. Hereafter, $A(B_X, Y)$ will represent either $A_b(B_X, Y)$ or $A_u(B_X, Y)$, and we simply write $A(B_X)$ instead of $A(B_X, \mathbb{C})$. For the basic properties of holomorphic functions on a Banach space, see [7, 12, 13, 23]. Note that $\rho A(B_X) = \rho A(B_X, Y)$ for a nontrivial complex Banach space $Y$.

In Section 2, we find a necessary and sufficient condition of $f$ in a subspace $A$ of $C_b(K)$ under which the norm is either Gâteaux or Fréchet differentiable at $f$. The main result of this section is that if $f$ is a strong peak function in $A$, then the norm of $A$ is Gâteaux differentiable at $f$, and the converse is also true for a nontrivial separating subspace $A$ of $C(K)$ on a compact Hausdorff space $K$. Applying them to $A(B_X)$, we show that the norm of $A(B_X)$ is nowhere Fréchet differentiable, if $X$ is nontrivial. The relation between a norm-attaining $m$-homogeneous polynomial and its differentiability was studied in [26].

In Section 3, we give another version of Bishop’s theorem. If $A$ is a nontrivial separating separable subspace of $C(K)$ on a compact Hausdorff space $K$, then the set of all strong peak functions is a dense $G_\delta$-subset of $A$. Using this fact, we obtain Bishop’s theorem which says that if $A$ is a nontrivial separating separable
subspace of $C(K)$, then $\rho A$ is a norming subset for $A$ and its closure is the Shilov boundary for $A$.

Globevnik [31] studied norming subsets for $A(B_X)$, when $X = c_0$. In that paper he showed that neither $A_u(B_X)$ nor $A_b(B_X)$ has the Shilov boundary. In [9], it was shown that $\partial A(B_X) = S_X$ for $X = \ell_p$, $1 \leq p < \infty$. This result was generalized in [15] to show that $\partial A(B_X) = S_X$ for a locally uniformly $c$-convex, order continuous sequence space $X$ (For more details on the $c$-convexity and order continuity of a Banach lattice, see [15, 18, 25, 38, 39, 40]).

In Section 4, it is shown that if $X$ has the Radon-Nikodým property and $Y$ is a nontrivial complex Banach space, then the set of all strong peak functions in $A(B_X, Y)$ is dense in $A(B_X, Y)$. Applying this fact, it is also proved that if $X$ has the Radon-Nikodým property and $Y$ is nontrivial, then $\rho A(B_X)$ is a norming subset for $A(B_X, Y)$. In particular, $\partial A(B_X, Y)$ is the closure of $\rho A(B_X)$, and $\text{ext}_{C}(B_X)$ is also a norming subset for $A(B_X, Y)$.

It is worth-while to remark that Bourgain-Stegall’s perturbed optimization principle [46] is the key method to prove these facts. This method has been used to study the density of the norm-attaining $m$-homogeneous polynomials and holomorphic functions on $X$, when $X$ has the Radon-Nikodým property (see [3, 10, 16]).

In Section 5, we modify the argument of Lindenstrauss [11] with strong peak points and also with uniformly strongly exposed points, and show the density of norm-attaining elements in certain subspaces of $C_b(K, Y)$. We also extend the result of [8] to the vector valued case by changing their proof, which is based on that of Lindenstrauss.

In the last section, we apply Bishop’s theorem to study numerical boundaries for subspaces of $C_b(B_X, X)$. The notion of a numerical boundary was introduced and studied for various Banach spaces $X$ in [5], and it was observed that the smallest closed numerical boundary, called the numerical Shilov boundary, doesn’t exist for some Banach spaces. We show that there exist the numerical Shilov boundaries for most subspaces of $C_b(B_X, X)$, if $X$ is finite dimensional, which is one of the most interesting questions about the existence of the numerical Shilov boundary. In addition, we show the existence of the numerical Shilov boundary for a locally uniformly convex separable Banach space $X$.

2. Differentiability of a Subspace of $C_b(K)$

**Definition 2.1.** Let $K$ be a Hausdorff space and $A$ be a subspace of $C_b(K)$. We say that *every norming sequence of $f$ approaches for $A$*, whenever for any two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in $K$ satisfying

\[
\lim_n \alpha f(x_n) = \|f\| = \lim_n \beta f(y_n)
\]

(2.1)
for some complex numbers $\alpha, \beta$ of modulus 1, we have $\lim_n (\alpha g(x_n) - \beta g(y_n)) = 0$ for every $g \in A$. In case that $\lim_n (\alpha g(x_n) - \beta g(y_n)) = 0$ uniformly for $g \in S_A$, we say that every norming sequence of $f$ approaches uniformly for $A$.

It is easy to see that if every norming sequence of $f$ approaches for $A$, and also if $A$ is nontrivial, then $f \neq 0$.

**Theorem 2.2.** Let $A$ be a subspace of $C_b(K)$ and $f \in A$.

(i) The norm $\| \cdot \|$ of $A$ is Gâteaux differentiable at $f$ if and only if every norming sequence of $f$ approaches for $A$.

(ii) The norm $\| \cdot \|$ of $A$ is Fréchet differentiable at $f$ if and only if every norming sequence of $f$ approaches uniformly for $A$.

**Proof.** A slight modification of the proof of (i) gives that of (ii), so we prove only (i). We may assume that $A$ is nontrivial. Assume that $\| \cdot \| : A \to \mathbb{R}$ is Gâteaux differentiable at $f \in S_A$. Take two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in $K$ satisfying

$$\lim_n \alpha f(x_n) = \lim_n \beta f(y_n) = 1 = \|f\|$$

for some complex numbers $\alpha, \beta$ of modulus one. Fix $g \in A$. Since the norm of $A$ is Gâteaux differentiable at $f$, for every $\epsilon > 0$ there is $\delta > 0$ such that

$$\|f + tg\| + \|f - tg\| \leq 2 + \epsilon |t|,$$

for every real number $t$, $|t| < \delta$. For every positive integer $n$ we have

$$|f(x_n) + tg(x_n)| + |f(y_n) - tg(y_n)| \leq 2 + \epsilon |t|,$$

and so

$$\text{Re}(\alpha f(x_n) + t\alpha g(x_n)) + \text{Re}(\beta f(y_n) - t\beta g(y_n)) \leq 2 + \epsilon |t|.$$

Therefore, if $|t| < \delta$,

$$\limsup_n t \text{Re}(\alpha g(x_n) - \beta g(y_n)) \leq \epsilon |t|,$$

which implies that $\lim_n \text{Re}(\alpha g(x_n) - \beta g(y_n)) = 0$. Replacing $g$ by $-ig$, we get $\lim_n \text{Im}(\alpha g(x_n) - \beta g(y_n)) = 0$. Therefore, $\lim_n (\alpha g(x_n) - \beta g(y_n)) = 0$ for every $g \in A$.

For the converse, assume that there is an $f \in S_A$ such that every norming sequence of $f$ approaches for $A$, but $\| \cdot \|$ is not Gâteaux differentiable at $f$. Then there exist $g \in S_A$, a null sequence $\{t_n\}$ of nonzero real numbers and $\epsilon > 0$ such that

$$\|f + t_n g\| + \|f - t_n g\| \geq 2 + \epsilon |t_n|, \quad \forall n \geq 1.$$  \tag{2.2}

Choose sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in $K$ such that for each $n \geq 1$,

$$\|(f + t_n g)(x_n)\| \geq \|(f + t_n g)\| - \frac{1}{n}|t_n|, \quad \|(f - t_n g)(y_n)\| \geq \|(f - t_n g)\| - \frac{1}{n}|t_n|. $$  \tag{2.3}
Then

\[ 1 \geq |f(x_n)| \geq |(f + t_n g)(x_n)| - |t_n g(x_n)| \geq \|f + t_n g\| - \frac{1}{n} |t_n| - |t_n g(x_n)|. \]

So it is clear that \( \lim_{n} |f(x_n)| = 1 \). Similarly, \( \lim_{n} |f(y_n)| = 1 \).

Since every norming sequence of \( f \) approaches \( A \), by passing to a proper subsequence, we may assume that there exist two sequences \( \{x_n\}, \{y_n\} \) and complex numbers \( \alpha, \beta \) of modulus one such that

\[ \lim_{n} \alpha f(x_n) = \lim_{n} \beta f(y_n) = 1 \quad \text{and} \quad \sup_{n \geq 1} |\alpha g(x_n) - \beta g(y_n)| \leq \epsilon/2. \]

Using (2.2), (2.3) and (2.4), we get for any \( n \),

\[ 2 + \epsilon |t_n| - \frac{2}{n} |t_n| \leq \|f + t_n g\| + \|f - t_n g\| - \frac{2}{n} |t_n| \]

\[ \leq |(f + t_n g)(x_n)| + |(f - t_n g)(y_n)| \]

\[ = |(\alpha f + \alpha t_n g)(x_n)| + |(\beta f - \beta t_n g)(y_n)| \]

\[ \leq |(\alpha f + \alpha t_n g)(x_n)| + |\beta f(y_n) - \alpha t_n g(x_n)| + |\alpha t_n g(x_n) - \beta t_n g(y_n)| \]

\[ \leq |\alpha f(x_n) + \alpha t_n g(x_n)| + |\beta f(y_n) - \alpha t_n g(x_n)| + \frac{\epsilon}{2} |t_n| \]

Hence for every \( n \geq 1 \),

\[ 2 + \left( \frac{\epsilon}{2} - \frac{2}{n} \right) |t_n| \leq |\alpha f(x_n) + \alpha t_n g(x_n)| + |\beta f(y_n) - \alpha t_n g(x_n)|. \]

We need the following basic lemma which is proved later.

**Lemma 2.3.** Let \( \varphi : U \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be twice continuously differentiable on a neighborhood \( U \) of \( \xi_0 \in \mathbb{R}^n \). Let \( \epsilon > 0 \). Then there exist \( \delta > 0 \) and a neighborhood \( V \) of \( \xi_0 \) such that for any \( \xi, \zeta \in V \) and \( |\eta| < \delta \),

\[ |\varphi(\xi + \eta) - \varphi(\xi) + \varphi(\zeta - \eta) - \varphi(\zeta)| \leq \epsilon |\eta|. \]

Notice that the function \( \varphi : \mathbb{R}^2 \rightarrow \mathbb{R} \) defined by \( \varphi(\xi) = |\xi| \) is infinitely differentiable on a neighborhood of \((1,0)\), where \( |\cdot| \) is a usual Euclidean norm in \( \mathbb{R}^2 \).

By Lemma 2.3, given \( \epsilon > 0 \), there exist a neighborhood \( V \) of \((1,0)\) and a \( \delta > 0 \) such that for any \( \xi, \zeta \in V \) and \( |\eta| < \delta \),

\[ |\varphi(\xi + \eta) - \varphi(\xi) + \varphi(\zeta - \eta) - \varphi(\zeta)| \leq \epsilon |\eta|/4. \]

We shall identify the complex plane \( \mathbb{C} \) with \( \mathbb{R}^2 \). For each \( n \), set \( \xi_n = \alpha f(x_n) \), \( \zeta_n = \beta f(y_n) \) and \( \eta_n = \alpha t_n g(x_n) \). By (2.4), we may assume that \( \xi_n \) and \( \zeta_n \) are in \( V \) and \( |\eta_n| < \delta \) for any \( n \). By (2.6), for every \( n \),

\[ |\alpha f(x_n) + \alpha t_n g(x_n)| - |f(x_n)| + |\beta f(y_n) - \alpha t_n g(x_n)| - |f(y_n)| \]

\[ \leq |\varphi(\xi_n + \eta_n) - \varphi(\xi_n) + \varphi(\zeta_n - \eta_n) - \varphi(\zeta_n)| \]

\[ \leq \epsilon |\eta_n|/4 = \epsilon |t_n g(x_n)|/4 \leq \epsilon |t_n|/4. \]
By (2.5) and (2.7), we get for every $n$,
\[
2 + \left(\frac{\epsilon}{2} - \frac{2}{n}\right)|t_n| - |f(x_n)| - |f(y_n)| \leq \frac{\epsilon}{4}|t_n|.
\]
This means that
\[
0 \leq 2 - (|f(x_n)| + |f(y_n)|) \leq \left(\frac{\epsilon}{4} + \frac{2}{n}\right)|t_n| < 0
\]
for sufficiently large $n$. This is a contradiction. The proof is done. \hfill \Box

Now we prove Lemma 2.3.

**Proof of Lemma 2.3.** Choose a positive $r > 0$ such that $B(\xi_0, 4r) = \{\xi : |\xi - \xi_0| \leq 4r\}$ is contained in $U$. For any $\xi \in B(\xi_0, r)$ and $|\eta| < r$, by the Taylor formula of $\varphi$ there is $0 \leq t \leq 1$ such that
\[
\varphi(\xi + \eta) - \varphi(\xi) - \nabla \varphi(\xi) \cdot \eta = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j} (\xi + t\eta) \eta_i \eta_j,
\]
where $\nabla \varphi(\xi) \cdot \eta = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial \xi_i} (\xi) \eta_i$. Let $M = \sup_{\xi \in B(\xi_0, 2r)} \frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j}(\xi)$. Then for $\xi \in B(\xi_0, r)$ and $|\eta| < r$,
\[
(2.8) \quad |\varphi(\xi + \eta) - \varphi(\xi) - (\nabla \varphi)(\xi) \cdot \eta| \leq \frac{1}{2} n^2 M|\eta|^2
\]
Notice that the mapping $\xi \to \nabla \varphi(\xi)$ from $B(\xi_0, 4r)$ to $\mathbb{R}^n$ is uniformly continuous. By (2.8), given $\epsilon > 0$ there exists $\delta > 0$ such that for any $\xi, \zeta$ in $B(\xi_0, \delta)$ and for any $|\eta| < \delta$,
\[
|\varphi(\xi + \eta) - \varphi(\xi) - (\nabla \varphi)(\xi) \cdot \eta| \leq \epsilon |\eta|/4
\]
and
\[
|\nabla \varphi(\xi) - \nabla \varphi(\zeta)| \leq \epsilon/2.
\]
Take $V = B(\xi_0, \delta)$. For any $\xi, \zeta$ in $V$ and $|\eta| < \delta$,
\[
|\varphi(\xi + \eta) - \varphi(\xi) + \varphi(\zeta - \eta) - \varphi(\zeta)|
\]
\[
\leq |\varphi(\xi + \eta) - \varphi(\xi) - \nabla \varphi(\xi) \cdot \eta| + |\varphi(\zeta - \eta) - \varphi(\zeta) + \nabla \varphi(\zeta) \cdot \eta|
\]
\[
+ |\nabla \varphi(\xi) \cdot \eta - \nabla \varphi(\xi - \eta) | \cdot |\eta| \leq \epsilon |\eta| / 2 + |\nabla \varphi(\xi) - \nabla \varphi(\zeta) | \cdot |\eta| \leq \epsilon |\eta|.
\]
The proof is done. \hfill \Box

Notice that $X$ and $X^*$ can be regarded as a subspace of $C(B_{X^*})$ and $C_b(B_X)$ respectively, where the weak-* and norm topology is given on $B_{X^*}$ and $B_X$, respectively. By the direct application of Theorem 2.2 we get the following Šmulyan’s theorem.

**Theorem 2.4** (Šmulyan). Let $X$ be a Banach space. Then

(i) The norm of $X$ is Fréchet differentiable at $x \in S_X$ if and only if whenever $x_n^*, y_n^* \in S_{X^*}$, $x_n^*(x) \to 1$ and $y_n^*(x) \to 1$, then $\|x_n^* - y_n^*\| \to 0$. 

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(ii) Then norm of $X^*$ is Fréchet differentiable at $x^* \in S_{X^*}$ if and only if whenever $x_n, y_n \in S_X$, $x^*(x_n) \to 1$ and $x^*(y_n) \to 1$, then $\|x_n - y_n\| \to 0$.

(iii) The norm of $X$ is Gâteaux differentiable at $x \in S_X$ if and only if whenever $x_n^*, y_n^* \in S_{X^*}$, $x_n^*(x) \to 1$ and $y_n^*(x) \to 1$, then $x_n^* - y_n^* \overset{w^*}{\to} 0$.

(iv) The norm of $X^*$ is Gâteaux differentiable at $x^* \in S_{X^*}$ if and only if whenever $x_n, y_n \in S_X$, $x^*(x_n) \to 1$ and $x^*(y_n) \to 1$, then $x_n - y_n \overset{w}{\to} 0$.

**Proposition 2.5.** Let $K$ be a Hausdorff space and $A$ be a subspace of $C_b(K)$.

(i) If $f$ is a strong peak function in $A$, then every norming sequence of $f$ approaches for $A$. Hence the norm $\| \cdot \|$ is Gâteaux differentiable at every strong peak function.

(ii) Assume in addition that $A$ is a separating subspace of $C(K)$ on a compact Hausdorff space $K$ and that $f$ is a nonzero element of $A$. Then the norm $\| \cdot \|$ of $A$ is Gâteaux differentiable at $f$ if and only if $f$ is a strong peak function.

In this case, the set of all weak-$*$ exposed points of $B_{A^*}$ is

$$w^*_{ex} B_{A^*} = \{ \alpha \delta_t : t \in \rho A, |\alpha| = 1 \}.$$  

**Proof.** (i) Suppose that $f$ is a strong peak function at $x_0$ and that there exist two sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in $K$ satisfying

$$\lim_n \alpha f(x_n) = \lim_n \beta f(y_n) = \|f\|$$

for some complex numbers $\alpha, \beta$ of modulus one. Then, two sequences converge to $x_0$ in $K$ and $\alpha = \beta$. It is clear that $\lim_n (\alpha g(x_n) - \beta g(y_n)) = 0$ for every $g \in A$. This completes the proof of (i).

(ii) It is enough to prove the necessity. We may assume $\|f\| = 1$. Since the norm $\| \cdot \|$ is Gâteaux differentiable at $f$, $f$ is a smooth point of $B_A$. Choose $t \in K$ and $\alpha, |\alpha| = 1$ such that $\alpha f(t) = 1$. Then the evaluation functional $\alpha \delta_t \in S_{A^*}$ is a weak-$*$ exposed point of $B_{A^*}$. Since $A$ is separating, $\alpha \delta_t \neq \beta \delta_s$ on $A$ if $t \neq s$ in $K$ and $\alpha, \beta \in S_C$. If $s \neq t$,

$$\|f\| = |f(t)| = 1 > \max \{ \text{Re} \beta f(s) : \beta \in S_C \} = |f(s)|.$$  

Hence $f$ is a peak function in $A$.  

Consider the product space $K \times B_{Y^*}$, where $B_{Y^*}$ is equipped with the weak-$*$ topology. Given a subspace $A$ of $C_b(K, Y)$, consider the map $\varphi: f \in A \mapsto \tilde{f} \in C_b(K \times B_{Y^*})$ defined by

$$\tilde{f}(x, y^*) = y^* f(x), \quad \forall (x, y^*) \in K \times B_{Y^*}.$$  

Then $\varphi$ is a linear isometry, and its image $\tilde{A}$ of $A$ is also a subspace of $C_b(K \times B_{Y^*})$. In particular, we shall say that the subspace $A$ of $C_b(K, Y)$ is *separating* if the following conditions hold:

(i) If $x \neq y$ in $K$, then $\delta_{(x,x^*)} \neq \delta_{(y,y^*)}$ on $\tilde{A}$ for $x^*, y^* \in S_{Y^*}$.

(ii) Given $x \in K$ with $\delta_x \neq 0$ on $A$, we have $\delta_{(x,x^*)} \neq \delta_{(x,y^*)}$ on $\tilde{A}$ for $x^* \neq y^*$ in $\text{ext}(B_{Y^*})$.  

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By applying Theorem 2.2 and Proposition 2.5 to the subspace $\tilde{A}$ of $C_b(K \times B_{Y^*})$, we get the following.

**Corollary 2.6.** Let $K$ be a Hausdorff space, $Y$ a Banach space and $A$ a subspace of $C_b(K,Y)$. Then the following hold:

(i) The norm $\| \cdot \|$ of $A$ is Gâteaux differentiable (resp. Fréchet differentiable) at $f$ if and only if whenever there exist sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ in $K$ and $\{\alpha f_n\}_{n=1}^\infty$, $\{\beta f_n\}_{n=1}^\infty$ in $S_Y$ such that

$$\lim_{n \to \infty} x_n^* f(x_n) = \|f\| = \lim_{n \to \infty} y_n^* f(y_n),$$

we get

$$\lim_{n \to \infty} (x_n^* g(x_n) - y_n^* g(y_n)) = 0, \quad \forall g \in A.$$  

( resp. $\lim_{n \to \infty} (x_n^* g(x_n) - y_n^* g(y_n)) = 0$ uniformly for $g \in S_A$).

(ii) If $f$ is a strong peak function at $x_0 \in K$ and $f(x_0)/\|f(x_0)\|_Y$ is a smooth point of $B_Y$, then the norm $\| \cdot \|$ of $A$ is Gâteaux differentiable at $f$.

(iii) Assume in addition that $A$ is a separating subspace of $C_b(K,Y)$ on a compact Hausdorff space $K$ and that $f$ is a nonzero element of $A$. Then the norm of $A$ is Gâteaux differentiable at $f$ if and only if $f$ is a strong peak function at some $x_0$ and $f(x_0)/\|f(x_0)\|_Y$ is a smooth point of $B_Y$. In this case, the set of all weak-$*$ exposed points of $B_{A^*}$ is

$$w^*\text{exp}B_{A^*} = \{\alpha \delta_{(x,y^*)} : \exists \text{ a strong peak function } f \text{ such that } y^* f(x) = \|f\| \text{ and } y^* \in w^*\text{exp}(B_{Y^*}) \},$$

where $\delta_{(x,y^*)}(f) = y^* f(x)$ for all $f \in C_b(K,Y)$.

**Proposition 2.7.** If we denote by $G'(f)$ the Gâteaux differential of the norm at $f$, then

$$G'(f)(g) = \lim_{n} \Re(\alpha g(x_n)),$$

for a sequence $\{x_n\}_n$ in $K$ and a complex number $\alpha$ of modulus 1 satisfying $\lim_n \alpha f(x_n) = \|f\|.$

**Proof.** Since $\Re(\alpha f(x_n) + \alpha t g(x_n)) \leq \|f + t g\|$, we have $t \Re(\alpha g(x_n)) \leq \|f + t g\| - \Re(\alpha f(x_n))$ for all real $t$. Hence for $t > 0$,

$$\limsup_{n} \Re(\alpha g(x_n)) \leq \lim_{n} \frac{\|f + t g\| - \Re(\alpha f(x_n))}{t} = \frac{\|f + t g\| - \|f\|}{t},$$

and for $t < 0$,

$$\liminf_{n} \Re(\alpha g(x_n)) \geq \lim_{n} \frac{\|f + t g\| - \Re(\alpha f(x_n))}{t} = \frac{\|f + t g\| - \|f\|}{t}.$$ 

Therefore, it is easy to see that

$$\lim_{n} \Re(\alpha g(x_n)) = \lim_{t \to 0} \frac{\|f + t g\| - \|f\|}{t} = G'(f)(g).$$
This completes the proof. □

We apply Theorem 2.2 to show that the norm of $A(B_X)$ is nowhere Fréchet differentiable, if $X$ is nontrivial.

**Proposition 2.8.** Suppose that $X$ is a nontrivial complex Banach space and that $f$ is a strong peak function in $A(B_X)$. Then every norming sequence of $f$ doesn’t approach uniformly for $A(B_X)$.

**Proof.** Let $f \in A(B_X)$ be a strong peak function at some $x_0 \in S_X$. After a proper rotation, we may assume that $f(x_0) = \|f\|$. Let $x_n = e^{i/n}x_0$ for every positive integer $n$. It is easy to see that $\{x_n\}$ is a norming sequence and each $x_n$ is a strong peak point for $A(B_X)$, so there is a strong peak function $g_n \in A(B_X)$ such that $g_n(x_n) = 1 = \|g_n\|$ and $|g_n(x_0)| < 1/2$. Hence we get for every $n$,

$$ |g_n(x_n) - g_n(x_0)| \geq |g_n(x_n)| - |g_n(x_0)| \geq 1/2. $$

Since $\lim_n f(x_n) = \|f\| = f(x_0)$, (2.9) implies that every norming sequence of $f$ doesn’t approach uniformly for $A(B_X)$. □

**Theorem 2.9.** Suppose that $X$ is a nontrivial complex Banach space. The norm $\|\cdot\|$ of $A(B_X)$ is nowhere Fréchet differentiable.

**Proof.** Suppose that the norm of $A(B_X)$ is Fréchet differentiable at some $f$. By Theorem 2.2, every norming sequence of $f$ approaches uniformly for $A(B_X)$. This implies that $f$ is a strong peak function which contradicts Proposition 2.8. In fact, suppose that there is a sequence $\{x_n\}$ in $S_X$ such that

$$ \lim |f(x_n)| = \|f\|. $$

By passing to a proper subsequence, we may assume that there is a complex number $\alpha$, $|\alpha| = 1$ such that $\lim \alpha f(x_n) = \|f\|$. We claim that the sequence $\{x_n\}$ is Cauchy. Otherwise, there exist subsequence $\{x_{n_k}\}$ and $\delta > 0$ such that for $\|x_{n_{k+1}} - x_{n_k}\| \geq \delta$ for every $k$. Since

$$ \lim \alpha f(x_{n_k}) = \|f\| = \lim \alpha f(x_{n_{k+1}}), $$

and since every norming sequence of $f$ approaches uniformly for $A(B_X)$, we have $\lim_n |g(x_{n_{k+1}}) - g(x_{n_k})| = 0$ uniformly in $g \in S_{A(B_X)}$. Since $S_X \subset S_{A(B_X)}$, we have that $\lim_k \|x_{n_{k+1}} - x_{n_k}\| = 0$, which is a contradiction. Let $x_0$ be a limit of $\{x_n\}$. Suppose that there is another sequence $\{y_n\}$ in $S_X$ such that $\lim_n |f(y_n)| = \|f\|$. By choosing an appropriate subsequence, we may assume that there is a complex number $\beta$, $|\beta| = 1$ such that $\lim_n \beta f(y_n) = \|f\|$. Then $\lim_n \beta f(y_n) = \|f\| = \lim_n \alpha f(x_n)$. Since every norming sequence of $f$ approaches uniformly for $A(B_X)$, $\alpha = \beta$ and $\lim_n \|x_n - y_n\| = 0$. Therefore, $\lim_n y_n = \lim_n x_n = x_0$. This shows that $f$ is a strong peak function at $x_0$. □

**Remark 2.10.** When $X = \{0\}$, it is easy to see that $A(B_X)$ is isometrically isomorphic to $\mathbb{C}$. Thus the norm is Fréchet differentiable everywhere except zero.
3. Bishop’s theorem

Bishop showed in [9] that if $K$ is a compact metrizable and if $\rho A$ is the set of all (strong) peak points for a separating function algebra $A$, then

$$\max_{t \in \rho A} |f(t)| = \|f\| \quad \text{for every } f \in A.$$  

We now give another version of Bishop’s theorem from the results in the previous section.

**Theorem 3.1** (Bishop’s theorem). Let $A$ be a nontrivial separating separable subspace of $C(K)$ on a compact Hausdorff space $K$. Then the set of all peak functions in $A$ is a dense $G_δ$-subset of $A$. In particular, $\rho A$ is a norming subset for $A$ and $\partial A = \bar{\rho A}$.

*Proof.* By Proposition 2.5 and Mazur’s theorem, the set of all peak functions in $A$ is a dense $G_δ$-subset of $A$. It is clear that every closed boundary for $A$ contains $\rho A$. Hence we have only to show that $\rho A$ is a norming subset for $A$. For each $f \in A$, there is a sequence $\{f_n\}$ of peak functions such that $\|f_n - f\| \to 0$ as $n \to \infty$. Then

$$-\|f - f_n\| + \|f_n\| \leq -|f(x_n) - f_n(x_n)| + |f_n(x_n)| \leq |f(x_n)| \leq |f(x_n) - f_n(x_n)| + |f_n(x_n)| \leq \|f - f_n\| + \|f_n\|,$$

where $x_n$ is a peak point for $f_n$ for each $n$. Hence $\lim_n |f(x_n)| = \lim_n \|f_n\| = \|f\|$. Notice that $x_n \in \rho A$. Therefore, $\rho A$ is a norming subset for $A$. The proof is done. \(\square\)

The following example given in [9] shows that the separability assumption in Theorem 3.1 is necessary. Let $J$ be an uncountable set and let $I_\alpha = [0,1]$ for each $\alpha \in J$. Then the product space $K = \prod_{\alpha \in I_\alpha}$ is a compact non-metrizable space, and $C(K)$ is not separable. Using the Stone-Weierstrass theorem, it is not difficult to check that for every $f \in C(K)$, there is a countable subset $\Delta \subset J$ such that whenever $x$ and $y$ in $K$ satisfy $x_\alpha = y_\alpha$ for every $\alpha \in \Delta$, $f(x) = f(y)$ holds. It is easy to see that there is no peak function in $C(K)$. In particular, the norm of $C(K)$ is nowhere Gâteaux differentiable by Proposition 2.5.

**Example 3.2.** Let $X = \ell_2^2$ be the 2-dimensional complex Euclidean space, and let $A$ be the set of restrictions to $B_X$ of the elements of $X^*$, which is a closed subspace of $C(B_X)$. Given two distinct points $x, y \in B_X$, there is $f \in A$ such that $f(x) \neq f(y)$, but it is easy to see that the subspace $A$ is not separating. The set $T_1 = \{(x_1, x_2) : (x_1, x_2) \in S_X, \ x_2 \geq 0\}$ and $T_2 = \{(x_1, -x_2) : (x_1, x_2) \in S_X, \ x_2 \geq 0\}$ are two closed norming subsets for $A$. However, $T_1 \cap T_2 = \{(x_1, 0) : |x_1| = 1\}$ is not a norming subset for $A$, so $A$ doesn’t have the Shilov boundary. Therefore we cannot omit the separation assumption in Theorem 3.1.

The following is a consequence of Corollary 2.6 and Theorem 3.1.
Corollary 3.3. Let $Y$ be a Banach space and let $A$ be a nontrivial separating separable subspace of $C(K,Y)$ on a compact Hausdorff space $K$. Then the set 
\[ \{ f \in A : f \text{ is a peak function at some } t \in K, f(t)/\|f\| \text{ is a smooth point of } B_Y \} \]
is a dense $G_\delta$-subset of $A$. In particular, $\rho A$ is a norming subset for $A$ and $\partial A = \overline{\rho A}$.

Notice that if $K$ is a compact metric space and $Y$ is separable, then every subspace $A$ of $C_b(K,Y)$ is separable. Indeed, we can regard $A$ as a subspace of $C(K \times B_Y^*)$ and $K \times B_Y^*$ is a compact metrizable space.

From the proof of Theorem 3.1 we have the following proposition.

Proposition 3.4. Given a Banach space $Y$, let $A$ be a nontrivial subspace of $C_b(K,Y)$ on a Hausdorff space $K$. Suppose that the set of all strong peak functions in $A$ is dense. Then $\rho A$ is a norming subset for $A$ and $\partial A = \overline{\rho A}$.

Assume that $A$ is a subspace of $C(K)$ on compact Hausdorff space $K$ and for any two distinct points $s, t$ in $K$, there is $f \in A$ such that $f(s) \neq f(t)$. Then the mapping $x \mapsto \delta_x$ from $K$ into the weak-$*$ compact subset $B_{A^*}$ is an injective homeomorphism and we shall identify $K$ with its image in $B_{A^*}$.

Proposition 3.5. Suppose that $A$ is a subspace of $C(K)$ on a compact Hausdorff space $K$ and that for two distinct points $t, s \in K$, there is $f \in A$ such that $f(t) \neq f(s)$. Then $A$ is separable if and only if $K$ is metrizable.

Proof. Recall that if $A$ is separable, then the weak-$*$ compact set $B_{A^*}$ is metrizable. Since $K$ is embedded in $B_{A^*}$, it is metrizable. For the converse, notice that if $K$ is metrizable, the Stone-Weierstrass theorem shows that $C(K)$ is separable, and so is its subspace $A$. \hfill \Box

4. Density of strong peak functions in $A(B_X,Y)$.

Let $C$ be a closed convex and bounded set in a Banach space $X$. The set $C$ is said to have the Radon-Nikodým property if for every probability space $(\Omega, \mathcal{B}, \mu)$ and every $X$-valued countably additive measure $\tau$ on $\mathcal{B}$ such that $\tau(A)/\mu(A) \in C$ for every $A \in \mathcal{B}$ with $\mu(A) > 0$, there is a Bochner measurable $f : \Omega \to X$ such that
\[ \tau(A) = \int_A f(\omega) \, d\mu(\omega), \quad A \in \mathcal{B}. \]

A Banach space $X$ is said to have the Radon-Nikodým property if its unit ball $B_X$ has the Radon-Nikodým property. For the basic properties and useful information on the Radon-Nikodým property, see [24, 27, 36].

Let $D$ be a metric space. We say that a function $\varphi : D \to \mathbb{R}$ strongly exposes $D$ if there is $x \in D$ such that
\[ \varphi(x) = \sup\{ \varphi(y) : y \in D \} \]
and whenever there is a sequence \( \{x_n\} \) in \( D \) satisfying \( \lim_n \varphi(x_n) = \varphi(x) \), the sequence \( \{x_n\} \) converges to \( x \).

The important Bourgain-Stegall’s perturbed optimization theorem \cite{46} says that if a closed bounded convex subset \( D \) of \( X \) has the Radon-Nikodým property and if \( \varphi : D \to \mathbb{R} \) is a bounded above upper semi-continuous function, then the set

\[
\{ x^* : \varphi + x^* \text{ strongly exposes } D \}
\]

is a dense \( G_\delta \)-subset of \( X^* \).

Let \( X \) and \( Y \) be complex Banach spaces. Notice that \( f \in A(B_X, Y) \) is a strong peak function if and only if \( \|f(\cdot)\| \) strongly exposes \( B_X \).

**Definition 4.1.** A function \( f \in A(B_X, Y) \) is said to attain its norm strongly on \( B_X \) if there is \( x_0 \in S_X \) such that whenever \( \lim_n \|f(x_n)\| = \|f\| \) for a sequence \( \{x_n\} \) in \( B_X \), it has a subsequence \( \{x_{n_k}\} \) converging to \( \alpha x_0 \) for some \( |\alpha| = 1 \).

Acosta, Alaminos, García and Maestre \cite{3} showed that if \( X \) has the Radon-Nikodým property, then for every \( f \in A(B_X, Y) \), every natural number \( N \) and every \( \epsilon > 0 \), there are \( x_1^*, \ldots, x_N^* \in X^* \) and \( y_0 \in Y \) such that the \( N \)-homogeneous polynomial \( Q \) on \( X \), given by \( Q(x) = x_1^*(x) \cdots x_N^*(x)y_0 \) satisfies that \( \|Q\| < \epsilon \) and \( f + Q \) attains its norm. For our application we prove the following stronger version.

**Theorem 4.2.** Let \( X \) be a complex Banach space with the Radon-Nikodým property. Suppose that \( f \in A(B_X, Y), N \geq 1 \) and \( \epsilon > 0 \). Then there are \( x_1^*, x_2^* \in X^* \) and \( y_0 \in Y \) such that the \( N \)-homogeneous polynomial \( Q \) on \( X \), given by \( Q(x) = [x_1^*(x)]^{N-1}x_2^*(x)y_0 \), satisfies that \( \|Q\| < \epsilon \) and \( f + Q \) strongly attains its norm. In particular, the set of all strongly norm-attaining functions is dense in \( A(B_X, Y) \).

**Proof.** We may assume that \( X \neq 0 \). Fix \( f \in A(B_X, Y) \) and define a function \( g : B_X \to \mathbb{C} \) as the following:

\[
g(x) = \max\{ \|f(\lambda x)\| : \lambda \in \mathbb{C}, |\lambda| \leq 1 \}. \tag{4.1}
\]

It is clearly bounded, because \( f \) is an element of \( A(B_X, Y) \).

For the proof of the upper semi-continuity of \( g \), suppose that a sequence \( \{x_n\}_{n=1}^\infty \) in \( B_X \) converges to \( x \). Then for each \( n \), there is a complex number \( \lambda_n \) such that \( |\lambda_n| = 1 \) and \( g(x_n) = \|f(\lambda_n x_n)\| \). For any convergent subsequence \( \{\lambda_{n_k}\} \) of \( \{\lambda_n\} \) with the limit \( \lambda \), we get

\[
\lim_{k \to \infty} \|f(\lambda_{n_k} x_{n_k})\| = \|f(\lambda x)\| \leq g(x).
\]

Hence \( \limsup_n g(x_n) \leq g(x) \). This means that \( g \) is upper semi-continuous.

By Bourgain-Stegall’s perturbed optimization theorem, there is \( x^* \in X^* \) such that \( \|x^*\| < \epsilon \) and \( g + \text{Re } x^* \) strongly exposes \( B_X \) at \( x_0 \).

We claim that \( \text{Re } x^*(x_0) \neq 0 \). Assume that \( \text{Re } x^*(x_0) = 0 \). Then \( g(x_0) + \text{Re } x^*(x_0) = g(-x_0) + \text{Re } x^*(-x_0) \). So \( x_0 = 0 \). Notice that for each \( x \in B_X \),
\( g(0) = \|f(0)\| \leq \|f(x)\| \leq g(x) \) by the maximum modulus theorem. Since \( g + \text{Re} \, x^* \) strongly exposes \( B_X \) at 0,

\[
g(0) = \sup \{g(x) + \text{Re} \, x^*(x) : x \in B_X\}
= \sup \{g(x) + |x^*(x)| : x \in B_X\}.
\]

Hence \( g(0) \leq g(x) \leq g(x) + |x^*(x)| \leq g(0) \) for any \( x \in B_X \). This means that \( x^* = 0 \) and \( g \) is constant on \( B_X \). This is a contradiction to that \( g \) strongly exposes \( B_X \) at 0. Therefore \( \text{Re} \, x^*(x_0) \neq 0 \).

Then \( \|x_0\| = 1 \). Indeed, it is clear that \( x_0 \neq 0 \), because \( x^* \neq 0 \) and \( g \) is nonnegative. If \( 0 < \|x_0\| < 1 \), then

\[
g(x_0) + \text{Re} \, x^*(x_0) = \sup \{g(x) + \text{Re} \, x^*(x) : x \in B_X\}
= \sup \{g(x) + |x^*(x)| : x \in B_X\}
\]

shows that \( \text{Re} \, x^*(x_0) = |x^*(x_0)| \) and

\[
g(x_0) + |x^*(x_0)| < g\left(\frac{x_0}{\|x_0\|}\right) + |x^*\left(\frac{x_0}{\|x_0\|}\right)| = g\left(\frac{x_0}{\|x_0\|}\right) + \text{Re} \, x^*\left(\frac{x_0}{\|x_0\|}\right).
\]

This is a contradiction to the fact that \( g + \text{Re} \, x^* \) strongly exposes \( B_X \) at \( x_0 \).

There is a \( \lambda_0 \) such that \( |\lambda_0| = 1 \) and \( g(x_0) = \|f(\lambda_0 x_0)\| \). Let \( x_1 = \lambda_0 x_0 \) and choose \( x_1^* \in X^* \) with \( x_1^*(x_1) = 1 = \|x_1^*\| \). Define \( h : B_X \to Y \) by

\[
h(x) = f(x) + \lambda_1 x^*_1(x) N^{-1} x^*(x) \frac{f(x_1)}{\|f(x_1)\|},
\]

where the complex number \( \lambda_1 \) is properly chosen so that

\[
\|f(x_1)\| + \lambda_1 x^*(x_1) = \|f(x_1)\| + |x^*(x_1)|.
\]

It is clear that \( h \in A(B_X, Y) \) and notice that we get for every \( x \in B_X \),

\[
(4.2) \quad \|h(x)\| \leq \|f(x)\| + |x^*(x)| \leq g(x) + |x^*(x)| \leq \sup \{g(x) + |x^*(x)| : x \in B_X\}
= \sup \{g(x) + \text{Re} \, x^*(x) : x \in B_X\} = g(x_0) + \text{Re} \, x^*(x_0).
\]

Hence \( \|h\| = g(x_0) + \text{Re} \, x^*(x_0) \) because \( \text{Re} \, x^*(x_0) = |x^*(x_0)| \) and

\[
\|h(x_1)\| = \|f(x_1)\| + \lambda_1 x^*(x_1) = \|f(x_1)\| + |x^*(x_1)|
= g(x_0) + |x^*(x_0)| = g(x_0) + \text{Re} \, x^*(x_0).
\]

We shall show that \( h \) strongly attains its norm at \( x_0 \). Suppose that \( \lim_n \|h(x_n)\| = \|h\| = g(x_0) + \text{Re} \, x^*(x_0) \). Choose a sequence \( \{\alpha_n\} \) of complex numbers so that \( |\alpha_n| = 1 \) and

\[
g(x_n) + |x^*(x_n)| = g(\alpha_n x_n) + \text{Re} \, x^*(\alpha_n x_n), \quad \forall n \geq 1.
\]

Then (4.2) shows that

\[
\lim_n g(\alpha_n x_n) + \text{Re} \, x^*(\alpha_n x_n) = g(x_0) + \text{Re} \, x^*(x_0).
\]
Since $g + \text{Re } x^*$ strongly exposes $B_X$ at $x_0$, $\{\alpha_n x_n\}$ converges to $x_0$. Hence there is a subsequence of $\{x_n\}$ which converges to $\alpha x_0$ for some $|\alpha| = 1$. This implies that $h$ strongly attains its norm at $x_0$ and $\|f - h\| \leq \epsilon$. The proof is done. \hfill \Box

Remark 4.3. In (4.1) $g$ is continuous, because it is the supremum of a family of continuous functions, that is, it is lower semi-continuous.

**Theorem 4.4.** Suppose that a complex Banach space $X$ has the Radon-Nikodým property and $Y$ is a nontrivial complex Banach space. Then the following hold:

(i) The set of all strong peak functions in $A(B_X, Y)$ is dense in $A(B_X, Y)$. 
In particular, the set of all smooth points of $B_{A(B_X, Y)}$ is dense in $S_{A(B_X, Y)}$ if the Banach space $Y$ is smooth.

(ii) $\rho A(B_X)$ is a norming subset for $A(B_X, Y)$, and $\partial A(B_X, Y) = \overline{\rho A(B_X)}$. 
In particular, $\text{ext}_C(B_X)$ is a norming subset for $A(B_X, Y)$.

**Proof.** (i) Suppose that $f \in A(B_X, Y)$ strongly attains its norm at $x_0$. We shall show that given $\epsilon > 0$ there is $\tilde{g} \in A_u(B_X, Y)$ such that $\|\tilde{g}\| \leq \epsilon$ and $f + \tilde{g}$ is a strong peak function in $A(B_X, Y)$.

Since $f$ strongly attains its norm at $x_0$, there is a complex number $\alpha$ of modulus 1 such that $\|f(\alpha x_0)\| = \|f\|$. Choose $x^* \in S_X$ so that $x^*(x_0) = 1$ and take a peak function $g \in A_u(\mathbb{D})$ such that $g(\alpha) = 1$ and $|g(\gamma)| < 1$ for any $\gamma \neq \alpha$, where $\mathbb{D}$ is the closed unit disc in $\mathbb{C}$. Define $h : B_X \to Y$ by

$$h(x) = f(x) + \epsilon g(x^*(x)) \frac{f(\alpha x_0)}{\|f(\alpha x_0)\|}.$$  

(4.3) 

It is easy to see that $h \in A(B_X, Y)$ and $\|h(x)\| \leq \|f\| + \epsilon = \|h(\alpha x_0)\|$ for all $x \in B_X$. We claim that $h$ is a strong peak function at $\alpha x_0$. Suppose that $\lim_n \|h(x_n)\| = \|h\|$. For each $n$, we have

$$\|h(x_n)\| \leq \|f(x_n)\| + \epsilon |g(x^*(x_n))| \leq \|f\| + \epsilon = \|h\|.$$

Hence $\lim_n \|f(x_n)\| = \|f\|$ and $\lim_n |g(x^*(x_n))| = 1$. Since $g$ is a peak function at $\alpha$, $\{x^*(x_n)\}$ converges to $\alpha$. Now for any subsequence of $\{x_n\}$, there is a further subsequence $\{y_k\}$ which converges to $\eta x_0$ for some unit complex number $\eta$, because $\lim_n \|f(x_n)\| = \|f\|$ and $f$ strongly attains its norm. Thus $\alpha = \lim_k x^*(y_k) = \eta$. This means that every subsequence of $\{x_n\}$ has a further subsequence converging to $\alpha x_0$, so $\lim_n x_n = \alpha x_0$. Take $\tilde{g}(x) = \epsilon g(x^*(x)) \frac{f(\alpha x_0)}{\|f(\alpha x_0)\|}$. Then $\|\tilde{g}\| \leq \epsilon$ and $f + \tilde{g}$ is a strong peak function. Hence we can conclude from Theorem 4.2 that the set of all strong peak functions in $A(B_X, Y)$ is dense in $A(B_X, Y)$. The rest of proof follows from Corollary 2.7 (ii).

(ii) The proof follows from (i), Proposition 3.4 and the fact that every peak point for $A(B_X, Y)$ is a complex extreme point of $B_X$ (see [33]). \hfill \Box

**Remark 4.5.** Notice that for any natural number $m$ the peak function $g$ at $\alpha$ in (4.3) can be chosen to be a polynomial $g(\gamma) = (\bar{\alpha} \gamma + 1)^m / 2^m$ of degree $m$. In
particular, the function $\tilde{g}(x) = \epsilon^{\frac{\bar{\alpha}(x^*(x))}{2}} \frac{m f(ax_0)}{\|f(ax_0)\|}$ is a polynomial of degree $m$ and of rank 1.

Recall that a Banach space $X$ is said to be \textit{locally uniformly convex} if $x \in S_X$ and there is a sequence $\{x_n\}$ in $B_X$ satisfying $\lim_n \|x_n + x\| = 2$, then $\lim_n \|x_n - x\| = 0$.

Let $X$ be a complex Banach space. A point $x \in S_X$ is called a \textit{strong complex extreme point} of $B_X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{0 \leq \theta \leq 2\pi} \|x + e^{i\theta}y\| \geq 1 + \delta$$

for all $\|y\| \geq \epsilon$. A complex Banach space $X$ is said to be \textit{locally uniformly c-convex} if every $x \in S_X$ is a strong complex extreme point of $B_X$. Notice that if a complex Banach space $X$ is locally uniformly convex, then $X$ is locally uniformly c-convex. For more details on the local uniform c-convexity, see [25, 40].

A complex Banach space $X$ is said to be \textit{locally uniformly c-convex} if every $x \in S_X$ is a strong complex extreme point of $B_X$. Notice that if a complex Banach space $X$ is locally uniformly convex, then $X$ is locally uniformly c-convex. For more details on the local uniform c-convexity, see [25, 40].

**Remark 4.6.** Let $X$ be a complex Banach space and let

$$A_{wu}(B_X) = \{f \in A_u(B_X) : f \text{ is weakly uniformly continuous on } B_X\}$$

$$A_{wb}(B_X) = \{f \in A_b(B_X) : f \text{ is weakly continuous on } B_X\}.$$ 

We shall denote by $A_w(B_X)$ one of $A_{wu}(B_X)$ and $A_{wb}(B_X)$. The proof of Theorem 4.4 and Remark 4.5 show that the set of all strong peak functions for $A_w(B_X)$ is dense in $A_w(B_X)$ if $X$ has the Radon-Nikodým property.

It is a natural question that the set of all strong peak functions in either $A(B_X)$ or $A_w(B_X)$ is dense, if $X$ has the \textit{analytic Radon-Nikodým property}. The answer is negative in $A_w(B_X)$ as observed in [30]. Recall that a complex Banach space $X$ is said to have the \textit{analytic Radon-Nikodým property} if for every bounded analytic function $f$ from the open unit disc of $\mathbb{C}$ into $X$, it has the a.e. radial limits

$$f(e^{i\theta}) = \lim_{r \uparrow 1} f(re^{i\theta}) \quad a.e. \theta.$$ 

For more details on the analytic Radon-Nikodým property, see [11, 30].

Notice that $L_1[0, 1]$ is uniformly c-convex and has the analytic Radon-Nikodým property (cf. [38]). Let $X = L_1[0, 1]$. We shall show that $A_w(B_X)$ does not contain any strong peak function. Indeed, suppose that $f \in A_w(B_X)$ is a strong peak function at $x$. For each $n \geq 1$, let

$$U_n = \{y \in B_X : |f(y)| > \|f\| - 1/n\}.$$
Then $U_n$ is a relative weak neighborhood of $x$ for every $n$. Since $L_1[0,1]$ has the Daugavet property, we can choose a sequence $\{x_n\}$ (see [17]) such that

$$x_n \in U_n, \quad \|x_n - x\| \geq 1, \quad \forall n \geq 1.$$  

This is a contradiction to that $f$ is a strong peak function at $x$.

5. **Density of norm-attaining elements in a subspace of $C_b(K,Y)$**

Let $X$ be a complex Banach space. An element $x \in B_X$ is said to be a *strongly exposed point* for $B_X$ if there is a linear functional $f \in B_X^*$ such that $f(x) = 1$ and whenever there is a sequence $\{x_n\}$ in $B_X$ satisfying $\lim_n \Re f(x_n) = 1$, we get $\lim_n \|x_n - x\| = 0$. A set $\{x_\alpha\}$ of points on $S_X$ is called *uniformly strongly exposed* (u.s.e.), if there are a function $\delta(\epsilon)$ with $\delta(\epsilon) > 0$ for every $\epsilon > 0$, and a set $\{f_\alpha\}$ of elements of norm 1 in $X^*$ such that for every $\alpha$, $f_\alpha(x_\alpha) = 1$, and for any $x$,

$$\|x\| \leq 1 \text{ and } \Re f_\alpha(x) \geq 1 - \delta(\epsilon) \text{ imply } \|x - x_\alpha\| \leq \epsilon.$$  

In this case we say that $\{f_\alpha\}$ uniformly strongly exposes $\{x_\alpha\}$. Lindenstrauss [11] Proposition 1 showed that if $S_X$ is the closed convex hull of a set of u.s.e. points, then $X$ has property $A$, that is, for every Banach space $Y$, the set of norm-attaining elements is dense in $L(X,Y)$, the Banach space of all bounded operators of $X$ into $Y$. Modifying his argument and also applying strong peak points instead of u.s.e. points, we study the density of norm-attaining elements in a subspace of $C_b(K,Y)$. Notice that if $S_X$ is the closed convex hull of a set $E$ of u.s.e. points, then $E$ is a norming set for $L(X,Y)$.

**Theorem 5.1.** Let $(K,d)$ be a complete metric space, $Y$ a Banach space and $A$ a subspace of $C_b(K,Y)$. Assume that there exist a norming subset $\{x_\alpha\}_\alpha \subset K$ for $A$ and a family $\{\varphi_\alpha\}_\alpha$ of functions in $C_b(K)$ such that each $\varphi_\alpha$ is a strong peak function at $x_\alpha$. Assume also that $A$ contains $\varphi_\alpha^* \otimes y$ for each $y \in Y$ and $n \geq 1$. Then the set of norm-attaining elements of $A$ is dense in $A$.

**Proof.** We may assume that $\varphi_\alpha(x_\alpha) = 1$ for each $\alpha$. Let $f \in A$ with $\|f\| = 1$ and $\epsilon$ with $0 < \epsilon < 1/3$ be given. We choose a monotonically decreasing sequence $\{\epsilon_k\}$ of positive numbers so that

$$2 \sum_{i=1}^\infty \epsilon_i < \epsilon, \quad 2 \sum_{i=k+1}^\infty \epsilon_i < \epsilon_k^2, \quad \epsilon_k < \frac{1}{10k}, \quad k = 1, 2, \ldots$$

We next choose inductively sequences $\{f_k\}_{k=1}^\infty, \{x_{\alpha_k}\}_{k=1}^\infty$ satisfying

$$(5.1) \quad f_1 = f$$

$$(5.2) \quad \|f_k(x_{\alpha_k})\| \geq \|f\| - \epsilon_k^2$$

$$(5.3) \quad f_{k+1}(x) = f_k(x) + \epsilon_k \varphi_{\alpha_k}(x) \cdot f_k(x_{\alpha_k})$$

$$(5.4) \quad |\varphi_{\alpha_k}(x)| > 1 - 1/k \quad \text{implies} \quad d(x, x_{\alpha_k}) < 1/k,$$
where \( \tilde{\varphi}_{\alpha_j} \) is \( \varphi_{\alpha_j}^{n_j} \) for some positive integer \( n_j \). Having chosen these sequences, we verify the following hold:

\begin{align*}
(5.6) \quad & \|f_j - f_k\| \leq 2 \sum_{i=j}^{k-1} \epsilon_i, \quad \|f_k\| \leq 4/3, \quad j < k, \quad k = 2, 3, \ldots \\
(5.7) \quad & \|f_{k+1}\| \geq \|f_k\| + \epsilon_k \|f_k\| - 2\epsilon_k^2, \quad \quad k = 1, 2, \ldots \\
(5.8) \quad & \|f_k\| \geq \|f_j\| \geq 1, \quad j < k, \quad k = 2, 3, \ldots \\
(5.9) \quad & |\tilde{\varphi}_{\alpha_j}(x_{a_k})| > 1 - 1/j, \quad j < k, \quad k = 2, 3, \ldots .
\end{align*}

Assertion (5.6) is easy by using induction on \( k \). By (5.3) and (5.4),

\[
\|f_{k+1}\| \geq \|f_k(x_{a_k})\| = \|f_k(x_{a_k})(1 + \epsilon_k \tilde{\varphi}_{\alpha_k}(x_{a_k}))\| \\
= \|f_k(x_{a_k})(1 + \epsilon_k\|f_k\| - \epsilon_k^2)(1 + \epsilon_k)\| \\
\geq \|f_k\| + \epsilon_k \|f_k\| - 2\epsilon_k^2,
\]

so the relation (5.7) is proved. Therefore (5.8) is an immediate consequence of (5.2) and (5.7). For \( j < k \), by the triangle inequality, (5.3) and (5.6), we have

\[
\|f_{j+1}(x_{a_k})\| \geq \|f_k(x_{a_k})\| - \|f_k - f_{j+1}\| \\
\geq \|f_k\| - \epsilon_k^2 - 2 \sum_{i=j+1}^{k-1} \epsilon_i \geq \|f_{j+1}\| - 2\epsilon_j^2.
\]

Hence by (5.4) and (5.7),

\[
\epsilon_j |\tilde{\varphi}_{\alpha_j}(x_{a_k})| : \|f_j\| + \|f_j\| \geq \|f_{j+1}(x_{a_k})\| \geq \|f_{j+1}\| - 2\epsilon_j^2 \\
\geq \|f_j\| + \epsilon_j \|f_j\| - 4\epsilon_j^2,
\]

so that

\[
|\tilde{\varphi}_{\alpha_j}(x_{a_k})| \geq 1 - 4\epsilon_j > 1 - 1/j
\]

and this proves (5.9). Let \( \hat{f} \in A \) be the limit of \( \{f_k\} \) in the norm topology. By (5.1) and (5.6), \( \|\hat{f} - f\| = \lim_n \|f_n - f\| \leq 2 \sum_{i=1}^{\infty} \epsilon_i \leq \epsilon \) holds. The relations (5.5) and (5.9) mean that the sequence \( \{x_{a_k}\} \) converges to a point \( \tilde{x} \), say and by (5.3), we have \( \|\hat{f}\| = \lim_n \|f_n\| = \lim_n \|f_n(x_{a_k})\| = \|\hat{f}(\tilde{x})\| \). Hence \( \hat{f} \) attains its norm. This concludes the proof. \( \square \)

Let \( A \) be the closed linear span of the constant 1 and \( X^* \) as a subspace of \( C_0(B_X) \). Notice that if \( X \) is locally uniformly convex, then every element of \( S_X \) is a strong peak point for \( A \). Therefore, every element of \( S_X \) is a strong peak point for \( A(B_X, Y) \) for every complex Banach space \( Y \), and \( \rho_A(B_X) \) is a norming subset for \( A(B_X, Y) \). Indeed, if \( x \in S_X \), choose \( x^* \in S_{X^*} \) so that \( x^*(x) = 1 \). Set \( f(y) = x^*(y)+1 \) for \( y \in B_X \). Then \( f \in A \) and \( f(x) = 1 \). If \( \lim_n |f(x_n)| = 1 \) for some sequence \( \{x_n\} \) in \( B_X \), then \( \lim_n x^*(x_n) = 1 \). Since \( |x^*(x_n)+x^*(x)| \leq |x_n+x| \leq 2 \) for every \( n \), \( \|x_n+x\| \rightarrow 2 \) and \( \|x_n-x\| \rightarrow 0 \) as \( n \rightarrow \infty \). Similarly it is easy to see that every strongly exposed point for \( B_X \) is a strong peak point for \( A \).
It was shown in [15] that if a Banach sequence space $X$ is locally uniformly $c$-convex and order continuous, then the set of all strong peak points for $A(B_X)$ is dense in $S_X$. Therefore, the set of all strong peak points for $A(B_X, Y)$ is dense in $S_X$ for every complex Banach space $Y$, and $\rho A(B_X)$ is a norming subset for $A(B_X, Y)$. For the definition of a Banach sequence space and order continuity, see [15, 28, 42]. By the remarks above, we get the following.

**Corollary 5.2.** Suppose that $X$ and $Y$ are complex Banach spaces and $\rho A(B_X)$ is a norming subset for $A(B_X, Y)$. Then the set of norm-attaining elements is dense in $A(B_X, Y)$. In particular, if $X$ is locally uniformly convex, or if it is a locally uniformly $c$-convex, order continuous Banach sequence space, then the set of norm-attaining elements is dense in $A(B_X, Y)$.

The complex Banach space $c_0$ renormed by Day’s norm is locally uniformly convex [19, 20], but it doesn’t have the Radon-Nikod´ ym property [24]. In addition, it is a locally uniformly $c$-convex and order continuous Banach sequence space.

**Example 5.3.** A function $\varphi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if $\varphi$ is even, convex continuous and vanishing only at zero. Let $w = \{ w(n) \}$ be a weight sequence, that is, a non-increasing sequence of positive real numbers satisfying $\sum_{n=1}^{\infty} w(n) = \infty$. Given a sequence $x$, $x^*$ is the decreasing rearrangement of $|x|$.

An Orlicz-Lorentz sequence space $\lambda_{\varphi, w}$ consists of all sequences $x = \{ x(n) \}$ such that for some $\lambda > 0$,

$$\varphi(\lambda x) = \sum_{n=1}^{\infty} \varphi(\lambda x^*(n))w(n) < \infty,$$

and equipped with the norm $\| x \| = \inf\{ \lambda > 0 : \varphi(x/\lambda) \leq 1 \}$, which is a Banach sequence space. We say an Orlicz function $\varphi$ satisfies condition $\delta_2$ ($\varphi \in \delta_2$) if there exist $K > 0$, $u_0 > 0$ such that $\varphi(u_0) > 0$ and the inequality

$$\varphi(2u) \leq K \varphi(u)$$

holds for $u \in [0, u_0]$.

If $\varphi \in \delta_2$, then $\lambda_{\varphi, w}$ is locally uniformly $c$-convex [15] and order continuous [28]. Notice that if $\varphi(t) = |t|^p$ for $p \geq 1$ and $w = 1$, then $\lambda_{\varphi, w} = \ell_p$. The characterization of the local uniform convexity of an Orlicz-Lorentz function space is given in [28, 35] and the characterization of the local uniform $c$-convexity of a complex function space is given in [40].

Extending the result of Lindenstrauss mentioned in the beginning of this section, Payá and Saleh [44] showed that if $B_X$ is the closed absolutely convex hull of a set of u.s.e. points, then the set of norm-attaining elements is dense in $L^n(X)$, the Banach space of all bounded $n$-linear forms on $X$. We study a similar question for the space of polynomials from $X$ into $Y$. In particular, if a set of u.s.e.
points on $S_X$ is a norming set for the Banach space $P(^nX,Y)$ of all bounded $n$-homogeneous polynomials from $X$ into $Y$, then the set of norm-attaining elements is dense in $P(^nX,Y)$.

**Theorem 5.4.** Let $X$ and $Y$ be Banach spaces and $n \in \mathbb{N}$. Suppose that a set $E$ of u.s.e. points on $S_X$ is a norming set of $P(^nX,Y)$. Then the set of all norm-attaining elements is dense in $P(^nX,Y)$. Especially, if $E$ is dense in $S_X$, then the set of norm-attaining elements is dense in $P(^nX,Y)$.

Moreover, if the set of strongly exposed points of $B_X$ is dense in $S_X$, then the set of norm-attaining elements is dense in $A(B_X,Y)$ for complex Banach spaces $X$ and $Y$.

**Proof.** Suppose that a set $E$ of u.s.e. points on $S_X$ is a norming set of $P(^nX,Y)$. Let $P \in P(^nX,Y)$, $\|P\| = 1$, and $0 < \epsilon < 1/3$ be given. We first choose a monotonically decreasing sequence $\{\epsilon_k\}$ of positive numbers so that

\[
4 \sum_{i=1}^{\infty} \epsilon_i < \epsilon < \frac{1}{3}, \quad 4 \sum_{i=k+1}^{\infty} \epsilon_i \leq \epsilon_k, \quad \epsilon_k < \frac{1}{10k}, \quad k = 1, 2, \ldots.
\]

Using induction, we next choose sequences $\{P_k\}_{k=1}^{\infty}$ in $P(^nX,Y)$, $\{x_k\}_{k=1}^{\infty}$ in $E$ and $\{x_k^*\}_{k=1}^{\infty}$ in $S_X$, so that

\[
\|P_1\| = P
\]
\[
\|P_k(x_k)\| \geq \|P_k\| - \epsilon_k^2 \quad \text{and} \quad \|x_k\| = 1, \quad \|x_k^*(x_k)\| = 1,
\]

where $\{x_k^*\}$ uniformly strongly exposes $\{x_k\}$,

\[
P_{k+1}(x) = P_k(x) + \epsilon_k (x_k^*(x))^{n} P_k(x_k).
\]

Having chosen these sequences, we see that the followings hold:

\[
\|P_j - P_k\| \leq \frac{4}{3} \sum_{i=1}^{k-1} \epsilon_i, \quad \|P_k\| \leq \frac{4}{3}, \quad j < k
\]
\[
\|P_{k+1}\| \geq \|P_k\| + \epsilon_k \|P_k\| - \epsilon_k^2 - \epsilon_k^3
\]
\[
\|P_{k+1}\| \leq \|P_k\| + \epsilon_k \|x_k^*(x_1)\|^n \|P_k\| + \epsilon_k^2 + 2 \cdot \frac{4}{3} \sum_{i=k+1}^{l-1} \epsilon_i, \quad k + 1 < l.
\]
The assertion (5.14) can easily be proved by induction and (5.15) follows directly from (5.13). To see (5.16), for $k + 1 < l$ we have

$$\|P_{k+1}\| \leq \|P_l\| + \|P_{k+1} - P_l\|$$

$$\leq \|P_l(x_i)\| + \|x^*_k(x_l)|^n\|P_k\| + \|x^*_k(x_l)|^n\|P_l\| + 2 \cdot \frac{4}{3} \sum_{i=k+1}^{l-1} \epsilon_i,$$

By (5.14), the sequence $\{P_k\}$ converges in the norm topology to $Q \in P^{(n)X,Y}$ satisfying $\|P - Q\| < \epsilon$.

By (5.15) and (5.16) we have, for every $l > k + 1$,

$$\epsilon_k \|P_l\| - \epsilon_l \leq \epsilon_k \|x^*_k(x_l)|^n\|P_k\| + 2 \epsilon_l^2,$$

and hence $1 - 4\epsilon_k < \|x^*_k(x_l)|^n\|.

Since $A$ is uniformly strongly exposed, $\{x_n\}$ has norm convergent subsequence by Lemma 6 in [1]. Let $x_0$ be a limit of that subsequence. Then we have $\|Q(x_0)\| = \|Q\|$. The rest of the proof follows from Corollary 5.2 because every strongly exposed point for $B_X$ is a strong peak point for $A(B_X)$.

Lindenstrauss [11, Theorem 1] proved that the set of all bounded linear operators of $X$ to $Y$ with norm-attaining second adjoint is dense in $L(X,Y)$. In 1996 Acosta [2] extended this result to bilinear forms, and in 2002 Aron, Garcia and Maestre [8] showed that this is also true for scalar-valued 2-homogeneous bounded polynomials. Recently, Acosta, Garcia and Maestre [1] extended it to $n$-linear mappings.

We extend the result of [8] to the vector valued case by modifying their proof, which is originally based on that of Lindenstrauss. A bounded $n$-homogeneous polynomial $P \in P^{(n)X,Y}$ has an extension $\overline{P} \in P^{(n)X^{**},Y^{**}}$ to the bidual $X^{**}$ of $X$, which is called the Aron-Berner extension of $P$. In fact, $\overline{P}$ is defined in the following way.

Let $X_1, \ldots, X_n$ be an arbitrary collection of Banach spaces and let $\mathcal{L}^{(n)}(X_1 \times \cdots \times X_n)$ denote the space of bounded $n$-linear forms. Given $z_i \in X_i^{**}, 1 \leq i \leq n$, define $\overline{z}_i$ from $\mathcal{L}^{(n)}(X_1 \times \cdots \times X_i \times X_{i+1}^{**} \times \cdots \times X_n^{**})$ to $\mathcal{L}^{(n-1)}(X_1 \times \cdots \times X_{i-1} \times X_{i+1}^{**} \times \cdots \times X_n^{**})$ by

$$\overline{z}_i(T)(x_1, \ldots, x_{i-1}, x_{i+1}^{**}, \ldots, x_n^{**}) = \langle z_i, T(x_1, \cdots, x_{i-1}, \bullet, x_{i+1}^{**}, \ldots, x_n^{**}) \rangle,$$

where $T(x_1, \cdots, x_{i-1}, \bullet, x_{i+1}^{**}, \ldots, x_n^{**})$ is a linear functional on $X_i$ defined by $\bullet \mapsto T(x_1, \cdots, x_{i-1}, \bullet, x_{i+1}^{**}, \ldots, x_n^{**})$ and $\langle z, x^* \rangle$ is the duality between $X_i^{**}$ and $X_i^*$. 

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The map $\mathcal{z}_i$ is a bounded operator with norm $\|z_i\|$. Now, given $T \in \mathcal{L}(^n(X_1 \times \cdots \times X_n))$, define the extended $n$-linear form $\mathcal{T} \in \mathcal{L}(^n(X_1^{**} \times \cdots \times X_n^{**}))$ by

$$\mathcal{T}(z_1, \ldots, z_n) := \mathcal{z}_1 \circ \cdots \circ \mathcal{z}_n(T).$$

For a vector-valued $n$-linear mapping $L \in \mathcal{L}(^n(X_1 \times \cdots \times X_n), Y)$, define

$$\mathcal{T}(x_1^{**}, \ldots, x_n^{**})(y^*) = y^* \circ L(x_1^{**}, \ldots, x_n^{**}),$$

where $x_i^{**} \in X_i^{**}, 1 \leq i \leq n$ and $y^* \in Y^*$. Then $\mathcal{T} \in \mathcal{L}(^n(X_1^{**} \times \cdots \times X_n^{**}), Y^{**})$ has the same norm as $L$. Let $S \in \mathcal{L}_s(^nX, Y)$ be the symmetric $n$-linear mapping corresponding to $P$, then $S$ can be extended to an $n$-linear mapping $\mathcal{S} \in \mathcal{L}(^nX^{**}, Y^{**})$ as described above. Then the restriction

$$\mathcal{P}(z) = \mathcal{S}(z, \ldots, z)$$

is called the Aron-Berner extension of $P$. Given $z \in X^{**}$ and $w \in Y^*$, we have

$$\mathcal{P}(z)(w) = w \circ \mathcal{P}(z).$$

Actually this equality is often used as definition of the vector-valued Aron-Berner extension based upon the scalar-valued Aron-Berner extension. Davie and Gamelin [17, Theorem 8] proved that $\|P\| = \|\mathcal{P}\|$. It is also worth to remark that $\mathcal{S}$ is not symmetric in general.

**Theorem 5.5.** Let $X$ and $Y$ be Banach spaces. The subset of $\mathcal{P}(^2X,Y)$ each of whose elements has the norm-attaining Aron-Berner extension is dense in $\mathcal{P}(^2X,Y)$.

**Proof.** Let $P \in \mathcal{P}(^2X,Y), \|P\| = 1$, and let $S$ be the symmetric bilinear mapping corresponding to $P$. Let $\epsilon$ with $0 < \epsilon < 1/4$ be given. We first choose a monotonically decreasing sequence $\{\epsilon_k\}$ of positive numbers which satisfies the following conditions:

$$(5.17) \quad 8 \sum_{i=1}^\infty \epsilon_i < \epsilon < \frac{1}{4}, \quad 8 \sum_{i=k+1}^\infty \epsilon_i < \epsilon_k^2 \quad \text{and} \quad \epsilon_k < \frac{1}{10k}, \quad k = 1, 2, \ldots.$$  

Using induction, we next choose sequences $\{P_k\}_{k=1}^\infty$ in $\mathcal{P}(^2X,Y), \{x_k\}_{k=1}^\infty$ in $S_X$ and $\{f_k\}_{k=1}^\infty$ in $S_Y$, so that

$$(5.18) \quad P_1 = P, \quad \|P\| = 1$$

$$(5.19) \quad f_k(P_k(x_k)) = \|P_k(x_k)\| \geq \|P_k\| - \epsilon_k^2$$

$$(5.20) \quad P_{k+1}(x) = P_k(x) + \epsilon_k (f_k(S_k(x_k, x)))^2 P_k(x_k),$$
where each $S_k$ is the symmetric bilinear mapping corresponding to $P_k$. Having chosen these sequences, we see that the following hold:

\begin{align}
&&\|P_j - P_k\| &\leq 4 \left(\frac{5}{4}\right)^k \sum_{i=j}^{k-1} \epsilon_i, & j < k \tag{5.21}
&&\|P_{k+1}\| &\geq \|P_k\| + \epsilon_k \|P_k\|^3 - 4\epsilon_k^2 \tag{5.22}
&&\|P_{j+1}(x_k)\| &> \|P_{j+1}\| - 2\epsilon_j, & j < k \tag{5.23}
&&\|f_j(S_j(x_j, x_k))\|^2 &\geq \|P_j\|^2 - 6\epsilon_j, & j < k \tag{5.24}
\end{align}

By (5.21) and the polarization formula \cite{23}, the sequences $\{P_k\}$ and $\{S_k\}$ converge in the norm topology to $Q$ and $T$, say, respectively. Clearly $T$ is the symmetric bilinear mapping corresponding to $Q$, and $\|P - Q\| < \epsilon$.

Let $\eta > 0$ be given. Then there exists $j_0 \in \mathbb{N}$ such that

$$\|Q - P_j\| \leq \|T - S_j\| < \eta$$

for all $j \geq j_0$,

hence $\|P_j\| \geq \|Q\| - \eta$ for all $j \geq j_0$.

By

$$\|T - S_j\| \geq |f_j(T(x_j, x_k)) - f_j(S_j(x_j, x_k))|$$

and (5.24), we have

$$|f_j(T(x_j, x_k))| \geq |f_j(S_j(x_j, x_k))| - \|T - S_j\| \geq \sqrt{\|P_j\|^2 - 6\epsilon_j - \eta} \geq \sqrt{(\|Q\| - \eta)^2 - 6\epsilon_j - \eta}$$

for all $k > j \geq j_0$. Let $z \in X^{**}$ is a weak-$\star$ limit point of the sequence $\{x_k\}$. Then for all $j \geq j_0$

$$\|T(x_j, z)\| \geq \sqrt{(\|Q\| - \eta)^2 - 6\epsilon_j - \eta}.$$

Hence $\|T(z, z)\| \geq \|Q\| - 2\eta$. Since $\eta > 0$ is arbitrary, we have

$$\|Q(z)\| = \|T(z, z)\| \geq \|Q\| = \|\overline{Q}\|.$$

We finally investigate a version of Theorem 2 in \cite{41} relating with the complex convexity. Recall that a complex Banach space $X$ is said to be strictly $c$-convex if $S_X = \text{ext}_C(B_X)$.

**Theorem 5.6.** Let $X$ be a Banach space with property $A$. Then

1. If $X$ is isomorphic to a strictly $c$-convex space, then $B_X$ is the closed convex hull of its complex extreme points.
2. If $X$ is isomorphic to a locally uniformly $c$-convex space, then $B_X$ is the closed convex hull of its strong complex extreme points.
Proof. We prove only (2). We shall use the fact ([22, 25]) that \( x \in S_X \) is a strong complex extreme point of \( B_X \) if and only if for each \( \epsilon > 0 \), there is \( \delta > 0 \) such that
\[
\inf \left\{ \int_0^{2\pi} \| x + e^{i\theta} y \|^2 \frac{d\theta}{2\pi} : y \in X, \| y \| \geq \epsilon \right\} \geq 1 + \delta.
\]
For the proof of (1), use the fact ([22, 25]) that \( x \in S_X \) is a complex extreme point of \( B_X \) if and only if for any nonzero \( y \in X, \int_0^{2\pi} \| x + e^{i\theta} y \|^2 \frac{d\theta}{2\pi} > 1 \).

Let \( C \) be the closed convex hull of the strong complex extreme points of \( B_X \). Suppose that \( C \neq B_X \). Then there are \( f \in X^* \) with \( \| f \| = 1 \) and \( \delta, \ 0 < \delta < 1 \) such that \( |f(x)| < 1 - \delta \) for \( x \in C \). Let \( \| \cdot \| \) be a locally uniformly \( c \)-convex norm on \( X \), which is equivalent to the given norm \( \| \cdot \| \), such that \( \| x \| \leq \| x \| \) for \( x \in X \). Let \( Y \) be the space \( X \oplus_2 \mathbb{C} \) with the norm \( \|(x, c)\| = (\|x\|^2 + |c|^2)^{1/2} \).

Then \( Y \) is locally uniformly \( c \)-convex. Otherwise, there exist \( (x, c) \in S_{X \oplus_2 \mathbb{C}}, \epsilon > 0 \) and a sequence \( \{(x_n, c_n)\} \) such that for every \( n \geq 1 \), \( \|(x_n, c_n)\| \geq \epsilon \) and
\[
\lim_n \int_0^{2\pi} \|(x, c) + e^{i\theta}(x_n, c_n)\|^2 \frac{d\theta}{2\pi} = 1.
\]
Since the norm is plurisubharmonic,
\[
1 = \|x\|^2 + |c|^2 \leq \int_0^{2\pi} \|(x, c) + e^{i\theta}(x_n, c_n)\|^2 \frac{d\theta}{2\pi} = \int_0^{2\pi} \|x + e^{i\theta} x_n\|^2 \frac{d\theta}{2\pi} + \int_0^{2\pi} |c + e^{i\theta} c_n|^2 \frac{d\theta}{2\pi} \to 1.
\]
So
\[
\lim_{n \to \infty} \int_0^{2\pi} \|x + e^{i\theta} x_n\|^2 \frac{d\theta}{2\pi} = \|x\|^2 \quad \text{and} \quad \lim_{n \to \infty} \int_0^{2\pi} |c + e^{i\theta} c_n|^2 \frac{d\theta}{2\pi} = |c|^2.
\]
Since both \( (X, \| \cdot \|) \) and \( \mathbb{C} \) are locally uniformly \( c \)-convex, we get \( \lim_n \|(x_n, c_n)\| = \lim |c_n| = 0 \), which is a contradiction to \( \inf_n \|(x_n, c_n)\| \geq \epsilon \).

Let \( V \) be the operator from \( X \) into \( Y \) defined by \( Vx = (x, Mf(x)) \), where \( M > 2/\delta \). Then \( V \) is an isomorphism (into) and the same is true for every operator sufficiently close to \( V \). We have
\[
\|V\| \geq M, \quad \|Vx\| \leq (1 + (M - 2)^2)^{1/2} \quad \text{for} \ x \in C.
\]
It follows that operators sufficiently close to \( V \) cannot attain their norm at a point belonging to \( C \). To conclude its proof we have only to show that if \( T \) is an isomorphism (into) which attains its norm at a point \( x \) and if the range of \( T \) is locally uniformly \( c \)-convex, then \( x \) is a strong complex extreme point of \( B_X \).

We may assume that \( \|Tx\| = \|T\| = 1 \). If \( x \) is not a strong complex extreme point, then there are \( \epsilon > 0 \) and a sequence \( \{y_n\} \subset X \) such that \( \|y_n\| \geq \epsilon \) for every \( n \) and
\[
\lim_n \int_0^{2\pi} \|x + e^{i\theta} y_n\|^2 \frac{d\theta}{2\pi} = 1.
\]
Then
\[ 1 \leq \int_0^{2\pi} \|Tx + e^{i\theta}Ty_n\|^2 \frac{d\theta}{2\pi} \leq \int_0^{2\pi} \|x + e^{i\theta}y_n\|^2 \frac{d\theta}{2\pi} \]
shows that \( \{Ty_n\} \) converges to 0, because the range of \( T \) is locally uniformly \( c\)-convex. Therefore, \( \{y_n\} \) converges to 0, which is a contradiction. \( \square \)

6. APPLICATIONS TO A NUMERICAL BOUNDARY

Let \( \Pi_X = \{(x, x^*) : \|x\| = \|x^*\| = 1 = x^*(x)\} \subset S_X \times S_{X^*} \). We denote by \( \tau \) the product topology of the space \( B_X \times B_{X^*} \), where the topologies on \( B_X \) and \( B_{X^*} \) are the norm topology of \( X \) and the weak-* topology of \( X^* \), respectively. It is easy to see that \( \Pi_X \) is a \( \tau \)-closed subset of \( B_X \times B_{X^*} \). Let \( \pi_1 \) be the projection from \( \Pi_X \) onto \( S_X \) defined by \( \pi_1(x, x^*) = x \) for every \( (x, x^*) \in \Pi_X \). It is not difficult to see that \( \pi_1 \) is a closed map.

The spatial numerical range of \( f \in C_b(B_X, X) \) is defined by
\[ W(f) = \{x^*f(x) : (x^*, x) \in \Pi_X\}, \]
and the numerical radius of \( f \in C_b(B_X, X) \) is defined by \( \nu(f) = \sup\{\lambda : \lambda \in W(f)\} \). For a subspace \( A \subset C_b(B_X, X) \) we say that \( B \subset \Pi_X \) is a numerical boundary for \( A \) if
\[ \nu(f) = \sup_{(x, x^*) \in B} |x^*(f(x))|, \forall f \in A, \]
and that \( A \) has the numerical Shilov boundary if there is a smallest closed numerical boundary for \( A \). The numerical boundary was introduced and studied in [3] for various Banach spaces, and it was observed that the numerical Shilov boundary doesn’t exist for some Banach spaces. We first show that there exist the numerical Shilov boundaries for most subspaces of \( C_b(B_X, X) \) if \( X \) is finite dimensional. Notice that as a topological subspace of \( B_X \times B_{X^*} \), \( \Pi_X \) is a compact metrizable space if \( X \) is finite dimensional.

**Theorem 6.1.** Let \( X \) be a finite dimensional Banach space. Suppose that a subspace \( H \) of \( C_b(B_X, X) \) contains the functions of the form
\[(6.1) \quad 1 \otimes x, \quad y^* \otimes z, \quad \forall x \in X, \quad \forall z \in X, \quad \forall y^* \in X^*. \]
Then \( H \) has the numerical Shilov boundary.

**Proof.** Consider the linear map \( f \mapsto \tilde{f} \) from \( H \) into \( C(\Pi_X) \) defined by
\[ \tilde{f}(x, x^*) = x^*f(x). \]
Notice that \( \nu(f) = \|\tilde{f}\| \) for every \( f \in H \). Let \( \overline{H} \) be the closure of the image \( \tilde{H} \) in \( C(\Pi_X) \). Then \( \overline{H} \) is a separable subspace of \( C(\Pi_X) \).

We claim that \( \overline{H} \) is separating. Let \( (s, s^*) \neq (t, t^*) \in \Pi_X \) and let \( \alpha, \beta \in S_{\mathbb{C}} \). If \( \alpha t^* \neq \beta s^* \), then choose \( x \in S_X \) such that \( \alpha t^*(x) \neq \beta s^*(x) \). Set \( f = 1 \otimes x \in H \). Then
\[ \alpha \delta_{(t, t^*)}(\tilde{f}) = \alpha \tilde{f}(t, t^*) = \alpha t^*(x) \neq \beta s^*(x) = \beta \tilde{f}(s, s^*) = \beta \delta_{(s, s^*)}(\tilde{f}). \]
If $\alpha t^* = \beta s^*$, then $t \neq s$, and choose $z^* \in S_X^*$ such that $z^*(t) \neq z^*(s)$. Set $f = z^* \otimes t \in H$. Then $\beta s^*(t) = \alpha \neq 0$ and

$$\alpha f(t, t^*) = \alpha z^*(t)t^*(t) = \beta z^*(t)s^*(t) \neq \beta z^*(s)s^*(t) = \beta f(s, s^*),$$

hence $\alpha \delta(t, t^*) \neq \beta \delta(s, s^*) (f)$. Therefore $\overline{H}$ is a separating separable subspace of $C(\Pi_X)$. By Theorem 3.1 there is the Shilov boundary $\partial \overline{H} \subset \Pi_X$ for $\overline{H}$. It is clear that for every $f \in H$,

$$v(f) = \|\tilde{f}\| = \max_{(t, t^*) \in \partial \overline{H}} |t^* f(t)|.$$  

We shall show that if $T \subset \Pi_X$ is a closed numerical boundary for $H$, then $T$ is a closed boundary for $\overline{H}$. Fix $g \in \overline{H}$ and choose a sequence $\{f_n\}_{n=1}^\infty$ in $H$ such that $\lim_n \|g - \tilde{f}_n\| = 0$. For each $n$, there exists $(t_n, t^*_n) \in T$ such that $|t_n f_n(t_n)| = v(f_n) = \|\tilde{f}_n\|$. So $\|g\| = \lim_n \|\tilde{f}_n\| = \lim_n |t_n f_n(t_n)|$ and

$$|\tilde{f}_n(t_n, t^*_n) - g(t_n, t^*_n)| \leq \|\tilde{f}_n - g\| \to 0.$$  

This shows that $\|g\| = \lim_n |g(t_n, t^*_n)|$ and

$$\|g\| = \sup_{(t, t^*) \in T} |g(t, t^*)| = \max_{(t, t^*) \in T} |g(t, t^*)|.$$  

Therefore, $T$ is a closed boundary for $\overline{H}$ and so $\partial \overline{H}$ is contained in $T$, which means that $\partial \overline{H}$ is the smallest closed subset satisfying

$$v(f) = \max_{(t, t^*) \in \partial \overline{H}} |t^* f(t)|, \quad \forall f \in H.$$  

The proof is done. \qed

**Example 6.2.** Let $X = \ell^2_\infty$ be the 2-dimensional space $\mathbb{C}^2$ with the sup norm. Let $H$ be the subspace of $C_b(B_X, X)$ spanned by all $f \otimes x$, $f \in X^*$ and $x \in X$. In fact, $H$ is isometrically isomorphic to the Banach space $L(X)$ of bounded linear operators from $X$ into $X$. It is easy to see that $v(T) = \|T\|$ for $T \in H = L(X)$. Take

- $S_1 = \{(x, x^*) : x = (x_1, 1) \in S_X, |x_1| = 1, x^* = (0, 1) \text{ or } (\bar{x}_1, 0)\}$,
- $S_2 = \{(x, x^*) : x = (x_1, -1) \in S_X, |x_1| = 1, x^* = (0, -1) \text{ or } (\bar{x}_1, 0)\}.$

It is easy to see that for each $T \in H$,

$$v(T) = \|T\| = \sup_{(x, x^*) \in S_1} |x^* T x| = \sup_{(x, x^*) \in S_2} |x^* T x|.$$  

However, $S_1$ and $S_2$ are disjoint closed subsets of $\Pi_X$, so $H$ doesn’t have the numerical Shilov boundary. In particular, we cannot weaken the assumption of Theorem 6.1.

Applying the Mazur theorem, we next prove the existence of the numerical Shilov boundary for some subspaces of $C_b(B_X, X)$, when $X$ is separable.
Theorem 6.3. Let $X$ be a separable Banach space. Suppose that $A$ is a subspace of $C_b(B_X)$ such that every element in $A$ is uniformly continuous on $S_X$ and the set of all strong peak points for $A$ is dense in $S_X$. If a subspace $H$ of $C_b(B_X, X)$ contains the functions of the form:

$$ (6.2) \quad f \otimes y, \quad \forall f \in A, \quad \forall y \in X, $$

then $H$ has the numerical Shilov boundary. In particular, it is the set

$$ \{ (x, x^*) : x \text{ is a smooth point of } B_X \}^\tau. $$

Proof. Let $\Gamma = \{ (x, x^*) : x \text{ is a smooth point of } B_X \}$. We shall show that $\Gamma^\tau$ is the numerical Shilov boundary for $H$. Notice that by Mazur’s theorem, the set of smooth points of $B_X$ is dense in $S_X$. Therefore, $\pi_1(\Gamma)$ is dense in $S_X$. By [13, Theorem 2.5], $\Gamma^\tau$ is a closed numerical boundary for $H$, that is,

$$ v(f) = \max_{(t, t^*) \in \Gamma^\tau} |t^* f (t)|, \quad \forall f \in H. $$

Suppose that $C$ is a closed numerical boundary for $H$. Then it is easy to see that $\pi_1(C)$ is a closed subset of $S_X$, and $\pi_1(C)$ contains all strong peak points for $A$. Since the set of all strong peak point for $A$ is dense in $S_X$, $\pi_1(C) = S_X$. Therefore $\Gamma \subset C$, and hence $\Gamma^\tau \subset C$. This completes the proof. □

If $X$ is a smooth Banach space in Theorem 6.3, then it is easily seen that the numerical Shilov boundary for $H$ is $\Pi_X$, which is proved in [5].

Corollary 6.4. Let a separable Banach space $X$ be locally uniformly convex and $A$ be a closed linear span of the constant 1 and $X^*$ as a subspace of $C_b(B_X)$. Suppose that $H$ is a Banach space of uniformly continuous functions from $B_X$ into $X$, which contains the functions of the form (6.2). Then the numerical Shilov boundary for $H$ exists.

Corollary 6.5. Suppose that a Banach sequence space $X$ is locally uniformly $c$-convex and also order continuous. If $H$ is a Banach space of uniformly continuous functions from $B_X$ into $X$ which contains the functions of the form:

$$ f \otimes y, \quad \forall f \in A_u(B_X), \quad \forall y \in X, $$

then $H$ has the numerical Shilov boundary.

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Department of Mathematics, POSTECH, San 31, Hyoja-dong, Nam-gu, Pohang-shi, Kyungbuk, Republic of Korea, +82-054-279-2712

E-mail address: mathchoi@postech.ac.kr

E-mail address: hahnju@postech.ac.kr

E-mail address: hyuns@postech.ac.kr

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