Three Classification Results in the Theory of Weighted Hardy Spaces on the Ball

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Abstract

We present a natural family of Hilbert function spaces on the \( d \)-dimensional complex unit ball and classify which of them satisfy that subsets of the ball yield isometrically isomorphic subspaces if and only if there is an analytic automorphism of the ball taking one to the other. We also characterize pairs of weighted Hardy spaces on the unit disk which are isomorphic via a composition operator by a simple criterion on their respective sequences of weights.

Keywords
Hilbert function spaces · Weighted Hardy space · Drury–Arveson space · Multiplier algebras

Mathematics Subject Classification 46E22

1 Introduction

The problem of classification of complete Pick Hilbert function spaces and their multiplier algebras has been considered by several authors in the past, see [2,3,5–7,9,10,15] and also the survey [16]. In this paper we go beyond complete Pick spaces, obtaining some new classification results for weighted Hardy spaces on the disc and on the ball.
We also recover some previous results with simplified proofs, for spaces as well as for multiplier algebras.

Recall that a Hilbert function space on a set $X$ is a Hilbert space $\mathcal{H} \subset \mathbb{C}^X$ for which the evaluation functionals $eval_x : f \mapsto f(x)$ are bounded. These are also known as reproducing kernel Hilbert spaces, or RKHS. By the Riesz representation theorem, if $\mathcal{H}$ is a Hilbert function space on $X$, then for any $x \in X$ there exists $k^\mathcal{H}_x \in \mathcal{H}$ such that for every $f \in \mathcal{H}$:

$$\langle f, k^\mathcal{H}_x \rangle = f(x).$$

The kernel $k^\mathcal{H}$ of $\mathcal{H}$ is the function obtained by un-currying $x \mapsto k^\mathcal{H}_x$, explicitly:

$$k^\mathcal{H}(x, y) = k^\mathcal{H}_y(x) = \langle k^\mathcal{H}_y, k^\mathcal{H}_x \rangle.$$

The first result we will present is a partial classification of weighted Hardy spaces on the unit disk, which we now define. We denote by $\mathbb{C}[[z]]$ the algebra of formal power series in $z$.

**Definition 1.1** Let $w = (w_n)_{n=0}^{\infty}$ be a sequence of positive numbers, we define the weighted Hardy space corresponding to $w$ as the vector space:

$$\mathcal{H}_w = \left\{ \sum a_n z^n \in \mathbb{C}[[z]] : \sum |a_n|^2 w_n < \infty \right\}.$$

With inner product defined by the formula:

$$\langle f, g \rangle = \sum f_n \overline{g_n} w_n.$$

As the reader may easily verify, $\mathcal{H}_w$ is always a Hilbert space. The following proposition gives a sufficient and necessary condition for it to be a Hilbert function space.

**Proposition** (see [4, Exercise 2.1.10]) Let $w = (w_n)_{n=0}^{\infty}$ be a sequence of positive numbers, then $\mathcal{H}_w$ is a Hilbert function space on $\mathbb{D}$ if and only if the power series $\sum w_n^{-1} z^n$ has radius of convergence at least $1$. If $\mathcal{H}_w$ is a Hilbert function space on $\mathbb{D}$ then its reproducing kernel at $x$ is given by:

$$k^\mathcal{H}_w(x) = \sum w_n^{-1} (\overline{x} z)^n.$$

Notice that if the elements of $\mathcal{H}_w$ converge to functions on the disk, then they are automatically holomorphic because they are defined by power series.

We recall the definition of a morphism of Hilbert function spaces.

A bounded operator $T : \mathcal{H} \to \mathcal{E}$ is a morphism of Hilbert function spaces, if $\mathcal{H}, \mathcal{E}$ are Hilbert function spaces on sets $X, Y$ respectively and there exist $\phi : X \to Y$, $f : X \to \mathbb{C}$ such that for any $x \in X$:

$$T(k^\mathcal{H}_x) = f(x) k^\mathcal{E}_\phi(x).$$
A morphism which is an isometry of the underlying Hilbert spaces is called an isometry of Hilbert function spaces. Throughout this paper all morphisms are morphisms of Hilbert function spaces. When we say two Hilbert function spaces are isomorphic or isometric, we mean that they are so as Hilbert function spaces.

These definitions suffice to state the first question we tackled:

### 1.1 When are Two Weighted Hardy Spaces Isomorphic?

We wish to determine when two weighted Hardy spaces are (isometrically) isomorphic via an RKHS isomorphism $T$. If $\mathcal{H}_w \cong \mathcal{H}_u$, then there is some bijective and bounded linear map $T : \mathcal{H}_w \to \mathcal{H}_u$, a bijection $\phi : \mathbb{D} \to \mathbb{D}$ and a non-vanishing function $\lambda : \mathbb{D} \to \mathbb{C}$ such that:

$$\forall s \in \mathbb{D} \colon T (k_s^{H_w}) = \lambda (s) k_{\phi (s)}^{H_u}.$$  

It turns out that such RKHS isomorphisms can be understood more simply through their adjoints, which obtain the simple form of weighted composition operators:

$$T^* h = M_f C_\phi h = f \cdot (h \circ \phi).$$

Here, $\cdot$ can be understood as the pointwise multiplication of functions, and $f (s) = \overline{\lambda (s)}$.

Our main result gives a sufficient condition for when $\mathcal{H}_w \cong \mathcal{H}_u$, and also a necessary condition under the further assumption that $T^*$ is a scalar multiple of a composition operator (i.e., $f = \text{const}$). For the statement of the theorem, we make the following definition; Given two positive sequences, $a_n$ and $b_n$, we say that $a_n \sim b_n$ if there exist $\epsilon > 0$, $M > 0$ such that $0 < \epsilon < \frac{a_n}{b_n} < M$.

**Theorem 1.2** If $w_n \sim u_n$, then $\mathcal{H}_w \cong \mathcal{H}_u$. The isomorphism can be chosen to be isometric if and only if there exists $c > 0$ such that $\frac{w_n}{u_n} = c$. Moreover, if we assume that $\mathcal{H}_w \cong \mathcal{H}_u$ via an isomorphism $T$ such that $T^* = \alpha C_\phi$, then the converse is also true: $w_n \sim u_n$ and the isomorphism can be chosen to be isometric if and only if there exists $c > 0$ such that $\frac{w_n}{u_n} = c$.

Note that some similar results for the multiplier algebras of a specific family of weighted Hardy spaces are obtained in [5, Section 7].

We now present a few more definitions and the rest of the questions we tackled. Recall that the multiplier algebra of a Hilbert function space $\mathcal{H}$ on $X$ is the set

$$M(\mathcal{H}) = \left\{ f \in \mathbb{C}^X : \forall h \in \mathcal{H}, \, fh \in \mathcal{H} \right\}.$$  

The elements of $M(\mathcal{H})$ are called multipliers. Every multiplier defines a bounded multiplication operator on $\mathcal{H}$. Under the assumption that for every $x \in X$, there exists $h \in \mathcal{H}$ such that $h(x) \neq 0$ (which will always hold in our case), $M(\mathcal{H})$ is a Banach algebra with respect to the operator norm [17, Section 6.3.2].
Definition 1.3  Let \( \mathcal{H} \) be a Hilbert function space \( \mathcal{H} \) on \( X \). For every \( A \subset X \) we define a subspace:

\[
\mathcal{H}_A = \overline{\text{span}} \{ k_a : a \in A \}.
\]

This is clearly a Hilbert function space with respect to the restriction of the inner product on \( \mathcal{H} \). We denote the corresponding multiplier algebra \( \mathcal{M}_A = M(\mathcal{H}_A) \).

The following example is the subject of our second classification result. We denote the unit ball in \( \mathbb{C}^d \) by \( B_d \).

Example 1.4 Let \( d \) be a positive integer and \( t \in (0, \infty) \), then by [1, p. 100, Remark 8.10] and [14, p. 54] there exists a Hilbert function space \( \mathcal{H}^t_d \) on \( B_d \) with kernel given by:

\[
k_{\mathcal{H}^t_d}(x, y) = \frac{1}{(1 - \langle x, y \rangle_{\mathbb{C}^d})^t}.
\]

The function \( z \mapsto z^t \) is defined using the principal branch of the logarithm on the half plane \( P = \{ z \in \mathbb{C} : \Re(z) > 0 \} \). Note that when \( d \) is equal to 1, \( \mathcal{H}^t_1 \) is a weighted Hardy space.

We can now present the rest of the problems that will be addressed in this paper:

1.2 Classification of \( \mathcal{H}_A \) up to Isometric Isomorphism

We define two subsets of the disk to be congruent if there exists a bi-holomorphic automorphism of \( B_d \) taking one to the other.

We note that for \( t = 1 \) the theorem follows from the results of [7, Section 4] (see also the survey paper [16]). Our proof, relying mostly on linear algebra, is simpler and more direct. The partial classification that will be presented in this paper is as follows:

Theorem 1.5  Let \( \mathcal{H} = \mathcal{H}^t_d \) be as in Example 1.4. If \( t \in (0, 2] \) then for any two subsets \( A, B \subset B_d \), \( \mathcal{H}_A \) is isometric to \( \mathcal{H}_B \) if and only if \( A \) and \( B \) are congruent. If \( t > 2 \) there exist non-congruent subsets that yield isometric subspaces of \( \mathcal{H} \).

1.3 Classification of \( \mathcal{M}_A \) up to Isometric Isomorphism

We present a solution in the case where \( \mathcal{H} \) is the classical Hardy space on the disk. That is, the weighted Hardy space \( \mathcal{H}_w \) with \( w_n = 1 \) for all \( n \). In the notation of Example 1.4, \( \mathcal{H} = \mathcal{H}^1 \). We define for a subset \( A \subset \mathbb{D} \):

\[
S(A) := \{ x \in \mathbb{D} : k_x \in \mathcal{H}_A \}.
\]
We will assume that $S(A) = A$, $S(B) = B$. This is quite a natural assumption because clearly for any two subsets $A, A' \subset \mathbb{D}$:

$$\mathcal{H}_A = \mathcal{H}_{A'} \iff S(A) = S(A').$$

Moreover, the kernel functions $\{k^H_x\}_{x \in \mathbb{D}}$ are easily seen to be linearly independent. Therefore if $A \subset \mathbb{D}$ is finite, then automatically $S(A) = A$.

**Theorem 1.6** Let $\mathcal{H}$ be the classical Hardy space on the disk, then for any two subsets $A, B \subset \mathbb{D}$ such that $S(A) = A$ and $S(B) = B$, $M_A$ and $M_B$ are isometrically isomorphic as Banach algebras if and only if $A$ and $B$ are congruent.

Note that this result follows as a special case of [7, Theorem 5.10]. Again, the method presented here is new and more direct. Also note (although we will not use this fact) that if $A$ is an infinite proper subset such that $S(A) = A$, then by applying [7, Proposition 2.2] we find that $A$ is the joint zero set of some bounded analytic functions on the disk. By the main result of [8, Theorem 2.1] this implies that $A$ is a Blaschke sequence. In general $S(A) = A$ if and only if $A$ is a Blaschke sequence.

2 Proof of Theorem 1.2

We remind the reader the statement of the theorem:

**Theorem 1.2**

If $w_n \sim u_n$, then $\mathcal{H}_w \cong \mathcal{H}_u$. The isomorphism can be chosen to be isometric if and only if there exists $c > 0$ such that $\frac{w_n}{u_n} = c$. Moreover, if we assume that $\mathcal{H}_w \cong \mathcal{H}_u$ via an isomorphism $T$ such that $T^* = \alpha C_\phi$ (where $\phi : \mathbb{D} \to \mathbb{D}$ is a bijection), then the converse is also true: $w_n \sim u_n$, and the isomorphism can be chosen to be isometric if and only if there exists $c > 0$ such that $\frac{w_n}{u_n} = c$.

To prove the theorem, we need the following facts:

1. If $\phi$ fixes the origin and $T^* = \alpha C_\phi$ then $T$ is diagonalized over the monomials.
2. If $T^* = \alpha C_\phi$ with $\phi$ which fixes the origin, then $\mathcal{H}_w$ and $\mathcal{H}_u$ are isomorphic via $T$ if and only if $w_n \sim u_n$. The isomorphism is isometric if and only if there exists $c > 0$ such that $\frac{w_n}{u_n} = c$.
3. If $\mathcal{H}_w, \mathcal{H}_u$ are isomorphic via $T^* = \alpha C_\phi$, then they are also isomorphic via $C_\phi$.
4. If $\mathcal{H}_w, \mathcal{H}_u$ are isomorphic via $T^* = C_\phi$ then they are also isomorphic via some $\widetilde{T}^* = C_{\phi'}$ with $\phi'$ which fixes the origin.

Using these results, we can prove our main theorem.

**Proof** If $w_n \sim u_n$, then we can construct such an isomorphism via the construction described in the proof of 2 (and the same construction works for the isometric case). If $\mathcal{H}_w \cong \mathcal{H}_u$ via $T$ with $T^* = \alpha C_\phi$, then from 3, we can without loss of generality assume that $T^* = C_\phi$. Moreover, from 4, we might as well assume that $\phi$ fixes the origin, and so by 2, the result follows. $\square$
Note that the lemmas we use rely heavily on $T^*$ being a scalar multiple of a composition operator - the general case of a weighted composition operator is more subtle, and the proofs we gave do not hold for this case. Thus, we could only prove our classification result under the given assumption.

We now proceed to prove the four facts stated above.

**Lemma 2.1** If $\phi$ fixes the origin and $T^* = \alpha C_\phi$ then $T$ is diagonalized over the monomials.

**Proof** First, note that if $C_\phi : H_w \to H_u$ is a composition operator, then $\phi$ is analytic. This is since $C_\phi (id) = \phi$, which means that $\phi \in \text{Im} (C_\phi) \subset H_u$. But all functions in $H_u$ are analytic, and so the result follows.

Now, if $\phi$ fixes the origin, then since $\phi$ is an analytic bijection of the disk, then $\phi$ is a rotation - $\phi(z) = e^{i \theta} z$. We thus get:

$$T^* z^n = \alpha C_\phi z^n = \alpha e^{i n \theta} z^n.$$  

Which means that $T^*$ is diagonalized over the monomials, and so $T$ is as well. $\square$

**Lemma 2.2** $H_w$ and $H_u$ are isomorphic via $T$ such that $T^* = \alpha C_\phi$ with $\phi$ which fixes the origin if and only if $w_n \sim u_n$. The isomorphism is isometric if and only if there exists $c > 0$ such that $\frac{w_n}{u_n} = c$.

**Proof** $\Rightarrow$: Suppose that $H_w \cong H_u$ via $T$ such that $T^* = \alpha C_\phi$ with $\phi$ which fixes the origin. Then by the previous lemma, we know that there is some $\alpha \in \mathbb{C}, \theta \in \mathbb{R}$ such that $T^* z^n = \alpha e^{i n \theta} z^n$ for all $n \in \mathbb{N}$. Note that $T^*$ is by definition bounded and invertible, and so there exist $\epsilon > 0, M > 0$ such that:

$$\epsilon \| z^n \|_{H_u} \leq \| T^* z^n \|_{H_w} \leq M \| z^n \|_{H_u}$$

$$\Rightarrow \epsilon \sqrt{u_n} \leq \| \alpha e^{i n \theta} z^n \|_{H_u} \leq M \sqrt{u_n}$$

$$\Rightarrow \epsilon \sqrt{u_n} \leq |\alpha| \sqrt{w_n} \leq M \sqrt{u_n}$$

$$\Rightarrow \frac{\epsilon^2}{|\alpha|^2} \leq \frac{w_n}{u_n} \leq \frac{M^2}{|\alpha|^2}.$$  

And so $w_n \sim u_n$. Moreover, if $T$ is isometric, then the two sequences are proportional, since:

$$\sqrt{u_n} = \| z^n \|_{H_u} = \| T^* z^n \|_{H_w} = |\alpha| \| z^n \|_{H_w} = |\alpha| \sqrt{w_n}$$

$$\Rightarrow w_n = c.$$  

$\Leftarrow$: Given sequences $w_n, u_n$ such that $w_n \sim u_n$, we can choose $\alpha_n = \frac{w_n}{u_n}$ and construct the linear map:

$$T : H_w \to H_u, \ T z^n = \alpha_n z^n.$$
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$T$ is clearly invertible, and is in fact an RKHS isomorphism, since:

\[
T (k_s^w) = T \left( \sum_{n=0}^{\infty} \frac{\bar{w}_n}{w_n} z^n \right) = \sum_{n=0}^{\infty} \frac{\bar{w}_n}{w_n} T (z^n) = \sum_{n=0}^{\infty} \bar{w}_n \cdot \frac{z^n}{u_n} = \sum_{n=0}^{\infty} \frac{\bar{w}_n}{u_n} \cdot z^n = k_s^u.
\]

From here we also see that $T^* = C_{\phi}$ with $\phi = id$. Furthermore, if $\frac{w_n}{u_n} = c > 0$, then choose $\alpha_n = \sqrt{c}$. $T$ is now an isometry:

\[
\|T z_n\|_{\cal H_u} = \|\sqrt{c} z^n\|_{\cal H_u} = \|c u_n\| = \|u_n\| = \|z^n\|_{\cal H_w}.
\]

By a similar computation, one can conclude that $T$ is again an isomorphism, such that $T^* = \frac{1}{\sqrt{c}} C_{\phi}$ with $\phi = id$.

Lemma 2.3 If $\cal H_w$, $\cal H_u$ are isomorphic via $T^* = \alpha C_{\phi}$, then they are also isomorphic via $C_{\phi}$

Proof Naturally, $C_{\phi}$ induces an RKHS isomorphism onto its image. But its image is $\frac{1}{\alpha} \cal H_u = \cal H_u$.

The proof of the next lemma is an adaptation of the “disk trick” which was first put forward by Orr Shalit and Baruch Solel, and has been applied to various operator algebraic settings in which there is an action of the circle (see [18] for more details).

Lemma 2.4 If $T^* = C_{\phi}$ is an isomorphism, then there is an isomorphism $\tilde{T}$ such that $\tilde{T}^* = \phi'$ and $\phi'$ fixes the origin

Proof Throughout this proof, in an effort to minimize verbiage, we will call such composition operators isomorphisms as well.

Since $\cal H_w \cong \cal H_u$, we can define the following non-empty set:

\[
O_1 = \{ \lambda \in \mathbb{D} : \exists \psi \in Aut (\mathbb{D}) , \psi (\lambda) = 0 , C_{\psi} : \cal H_w \to \cal H_u \text{ is an isomorphism} \}.
\]

If we show that $0 \in O_1$, then we can conclude that there is an isomorphism $C_{\phi} : \cal H_w \to \cal H_u$ with $\phi$ which fixes the origin, and so we are done.

Let $\lambda_* \in O_1$, which corresponds to some $C_{\phi}$, where $\phi \in Aut (\mathbb{D})$. If $\lambda_* = 0$, we are done. Otherwise, we define the circle going through $\lambda_*$:

\[
C_{\lambda_*} = \{ e^{-i0} \lambda_* : \theta \in \mathbb{R} \}.
\]

Denote $A = \phi (C_{\lambda_*}) \subset \mathbb{D}$ (see Fig. 1). Note that $A$ is a circle going through the origin. It is a circle as the conformal image of a circle (since $\phi \in Aut (\mathbb{D})$), and $0 \in A$ since $\phi (\lambda_*) = 0$ by definition. We also define:

\[
O_2 = \{ \lambda \in \mathbb{D} : \exists \tau \in Aut (\mathbb{D}) , \tau (\lambda) = 0 , C_{\tau} \in Aut (\cal H_w) \}.
\]
where \( \text{Aut} (\mathcal{H}_w) \) is the set of all isomorphism from \( \mathcal{H}_w \) to itself.

Claim \( O_1 \) and \( O_2 \) are invariant under rotations. In particular, \( C_{\lambda_s} \subset O_1 \).

Indeed, let \( \lambda \in O_1 \). We wish to show that \( e^{i\theta} \lambda \in O_1 \), for all \( \theta \in \mathbb{R} \). Clearly, \( C_{e^{-i\theta} \lambda} \in \text{Aut} (\mathcal{H}_w) \). Thus, if \( \lambda \) corresponds to some \( C_f \), then \( e^{i\theta} \lambda \) will correspond to an isomorphism \( C_g \) where \( g (z) = f (e^{-i\theta} \lambda) \) (so \( C_g = C_{e^{-i\theta} C_f} \)), and so indeed \( e^{i\theta} \lambda \in O_1 \). Similarly, one can show that \( O_2 \) is invariant under rotations.

Claim \( A \subset O_2 \).

Let \( \phi (\lambda') \in A \), where \( \lambda' \in C_{\lambda_s} \). Since \( \lambda' \in O_1 \), we have some \( \psi \in \text{Aut} (\mathbb{D}) \) such that \( \psi (\lambda') = 0 \). Moreover, if we define \( \tau = \psi \circ \phi^{-1} \), we have that \( C_{\tau} = C_{\phi^{-1} C_{\psi} \phi^{-1}} \in \text{Aut} (\mathcal{H}_w) \) and:

\[
\tau (\phi (\lambda')) = \psi (\phi^{-1} (\phi (\lambda'))) = \psi (\lambda') = 0.
\]

And so \( \phi (\lambda') \in O_2 \). We thus conclude that \( A \subset O_2 \).

Define \( [A] \) to be the open disk enclosed by \( A \). Since \( O_2 \) is invariant under rotations, we can rotate \( A \) about the origin and obtain a large disk around the origin (see Fig. 2), which contains \( [A] \), which means that \( [A] \subset O_2 \). Note that \( \phi^{-1} ([A]) \) is exactly the open disk enclosed by \( C_{\lambda_s} \). This is since \( \phi^{-1} \) must send \( [A] \) to the interior or the exterior of \( C_{\lambda_s} \) (which is a Jordan curve) in \( \mathbb{C} \), but \( \phi^{-1} ([A]) \) must also be bounded in the disk and so it must be the interior of the disk enclosed by \( C_{\lambda_s} \).

Since \( \phi^{-1} ([A]) \) is a disk centered around the origin, we conclude that \( 0 \in \phi^{-1} ([A]) \). But this means that \( 0 \in O_1 \) as well. Indeed, Since \( 0 \in C_{\lambda_s} = \phi^{-1} ([A]) \), we can write \( 0 = \phi^{-1} (a) \), where \( a \in [A] \subset O_2 \). By definition of \( O_2 \), there is some \( \tau \in \text{Aut} (\mathbb{D}) \) such that \( \tau (a) = 0 \) and \( C_{\tau} \in \text{Aut} (\mathcal{H}_w) \). Then if we look at \( \psi \in \text{Aut} (\mathbb{D}), \psi = \tau \circ \phi \), then \( C_{\psi} = C_{\phi} C_{\tau} : \mathcal{H}_w \to \mathcal{H}_u \) is an isomorphism, and moreover:

\[
\psi (0) = \tau (\phi (\phi^{-1} (a))) = \tau (a) = 0.
\]
We can thus conclude that $0 \in O_1$, and so by the discussion above this completes the proof. \hfill \Box

Note that Lemma 2.4 can be proven using a different approach. By Corollary 9.10 of [10], one can show that if $T^* = C_\phi$ with $\phi$ which does not fix the origin, then $H_2$ is invariant under $\text{Aut}(\mathbb{D})$. This gives that $C_z$ is an isomorphism between $H_1$ and $H_2$ as well. Since Corollary 9.10 of [10] holds for spaces on the ball as well, this approach has the benefit of giving a possible generalization of our result to the ball as well, although we do not pursue this idea here.

3 Proofs of Theorems 1.5 and 1.6

We first present a useful characterization of Hilbert function space isometries. Let $\mathcal{H}$ and $\mathcal{E}$ be Hilbert function spaces on sets $X$ and $Y$ respectively. We say that a function $\phi : X \to Y$ induces a morphism of Hilbert function spaces $T : \mathcal{H} \to \mathcal{E}$ if $T(k^\mathcal{H}_z)$ is proportional to $k^\mathcal{E}_{\phi(z)}$ for all $z \in X$.

Lemma 3.1 Let $\mathcal{H}$ and $\mathcal{E}$ be Hilbert function spaces on $X$ and $Y$ respectively, and assume that $z \mapsto k^\mathcal{H}_z(z)$ is nowhere vanishing. Then for any function $\phi : X \to Y$ the following are equivalent:

1. The function $\phi$ induces an isometric isomorphism between $\mathcal{H}$ and $\mathcal{E}$.
2. There exists a function $f : X \to \mathbb{C}$ such that for all $z, w \in X$:

$$k^\mathcal{H}_z(w) = f(z)\overline{f(w)}k^\mathcal{E}_{\phi(z)}(\phi(w)).$$
Proof  (1) $\implies$ (2) If $k^H_x \mapsto f(x)k^E_{\phi(x)}$ extends to an isometry for some $f : X \to \mathbb{C}$, then for any $z, w \in X$:

$$k^H_z(w) = \langle k^H_z, k^H_w \rangle = \langle f(w)k^E_{\phi(z)}, f(w)k^E_{\phi(w)} \rangle = f(z)f(w)k^E_{\phi(z)}(\phi(w))$$

as required.

(2) $\implies$ (1) Assume that $k^H_z(w) = f(z)f(w)k^E_{\phi(z)}(\phi(w))$ holds for all $z, w \in X$. Then for all $z, w \in X$:

$$\langle k^H_z, k^H_w \rangle = k^H_z(w) = f(z)f(w)k^E_{\phi(z)}(\phi(w)) = \langle f(w)k^E_{\phi(z)}, f(w)k^E_{\phi(w)} \rangle$$

Notice that because $z \mapsto k^H_z(z)$ is nowhere vanishing, so is $f$. Because any function which is orthogonal to all kernel functions is everywhere zero, $\{k^H_x\}_{x \in X}$ and $\{k^E_y\}_{y \in Y}$ are dense in $\mathcal{H}$ and $\mathcal{E}$ respectively. Therefore $k^H_x \mapsto f(x)k^E_{\phi(x)}$ extends to an isometric isomorphism between $\mathcal{H}$ and $\mathcal{E}$ (note that we need $f$ to be nowhere vanishing to guarantee surjectivity). 

\[\Box\]

Remark 3.2 If $X = \{x_1, \ldots, x_n\}$ then it will be convenient to state the above condition in matrix form as:

\[(a) : \begin{bmatrix} k^H(x_i, x_j) \end{bmatrix} = \begin{bmatrix} f(x_i)f(x_j)k^E(\phi(x_i), \phi(x_j)) \end{bmatrix}\]

Or when $k^E(\phi(x_i), \phi(x_j))$ does not vanish as:

\[(b) : \begin{bmatrix} k^H(x_i, x_j) \\ k^E(\phi(x_i), \phi(x_j)) \end{bmatrix} = \begin{bmatrix} f(x_i)f(x_j) \end{bmatrix}\]

3.1 Moving Towards a Proof of Theorem 1.5

Throughout this section fix $d \in \mathbb{N}$, $t \in (0, \infty)$, and let $\mathcal{H} = \mathcal{H}_d'$ be the Hilbert function spaces on $\mathbb{B}_d$ defined as in Example 1.4. We denote the kernel function of $\mathcal{H}$ by $k$.

We denote the bi-holomorphic automorphisms of the ball by Aut($\mathbb{B}_d$). From now on we omit the adjective “bi-holomorphic”; by automorphism we shall mean bi-holomorphic automorphism. The following Proposition is well known. We include a proof for completeness.

\textbf{Proposition 3.3} For any $\phi \in \text{Aut}(\mathbb{B}_d)$ there exists $f : \mathbb{B}_d \to \mathbb{C}$ such that $k_x \mapsto f(x)k_{\phi(x)}$ extends to an isometric automorphism of $\mathcal{H}$. 

Proof Let \( \phi \in \text{Aut}(\mathbb{B}_d) \) and assume that \( \phi(a) = 0 \) then by [19, Theorem 2.2.2], we have for all \( x, y \in \mathbb{B}_d \):

\[
1 - \langle \phi(x), \phi(y) \rangle = (1 - |a|^2) \frac{1 - \langle x, y \rangle}{(1 - \langle x, a \rangle)(1 - \langle a, y \rangle)}.
\]

This implies that for \( f : x \mapsto \frac{\sqrt{1 - |a|^2}}{1 - \langle x, a \rangle} \) the following equality holds for all \( x, y \in \mathbb{B}_d \):

\[(c) : \ k(\phi(x), \phi(y)) f^t(x) f^t(y) = k(x, y).\]

Indeed, using the principal branch \( z^t = e^{t \log(z)} \), the identity \( z^t w^t = (zw)^t \) holds for any two complex numbers in the half plane \( P = \{ z \in \mathbb{C} : \Re(z) > 0 \} \). This implies (c) for all \( x, y \in \mathbb{B}_d \) sufficiently close to the origin. Then (c) follows for all \( x, y \in \mathbb{B}_d \) by analyticity of both sides of the equation. By Lemma 3.1 this shows that \( k_x \mapsto f(x)^t k_{\phi(x)} f(y) \) extends to an isometric automorphism of \( \mathcal{H} \). \( \square \)

Corollary 3.4 If \( A, B \subset \mathbb{B}_d \) are congruent, then \( \mathcal{H}_A \) is isometrically isomorphic to \( \mathcal{H}_B \).

Proof This follows from Proposition 3.1 and Proposition 3.3 because the restriction of an isometric isomorphism is an isometric isomorphism. \( \square \)

We will call \( \mathcal{H} \) faithful if for any two subsets \( A, B \subset \mathbb{B}_d \), every \( \phi : A \to B \) which induces an isometric isomorphism \( \mathcal{H}_A \to \mathcal{H}_B \) may be extended to an automorphism of \( \mathbb{B}_d \).

The last ingredient for the proof of Theorem 1.5 is:

Proposition 3.5 \( \mathcal{H} \) is faithful if and only if \( t \in (0, 2] \).

For the proof we will need the following two lemmas:

Lemma 3.6 Let \( A, B \subset \mathbb{C}^d \), and assume that \( f : A \to B \) preserves the inner product, meaning that for any \( a, a' \in A \):

\[
\langle a, a' \rangle = \langle f(a), f(a') \rangle.
\]

Then there exists a unitary operator \( U : \mathbb{C}^d \to \mathbb{C}^d \) extending \( f \).

Proof By basic linear algebra, there exists a unitary operator \( U' : \text{span } A \to \text{span } B \). Choose orthonormal bases \( \{ e_i \}_{i=1}^r \), \( \{ e'_i \}_{i=1}^r \) for \( (\text{span } A)^\perp \) and \( (\text{span } B)^\perp \) respectively, and let \( T \) be the partial isometry defined by linearly extending \( e_i \mapsto e'_i \); it is easy to check that

\[
U = U' \oplus T : \text{span } A \oplus (\text{span } A)^\perp \to \text{span } B \oplus (\text{span } B)^\perp
\]

is the required isometry. \( \square \)
Lemma 3.7 Let \( t \in (0, \infty) \) and \( U = \mathbb{D} \setminus \{0\} \). The function \( g_t : U \to \mathbb{C}, z \mapsto (1-z)^t \) defined by the principal branch of the logarithm is injective if \( t \leq 2 \). If \( t > 2 \) there exists \( z, w \in U \) such that \( z \neq w \), \( g_t(z) = g_t(w) \) and \( |z| = |w| \).

**Proof** Assume first that \( t \in (0, 2] \) and \( g_t(z) = g_t(w) \) for \( z, w \in U \). Let \( z' = 1 - z \) and \( w' = 1 - w \). Since \( e^{t \text{Log} z'} = e^{t \text{Log} w'} \) there exists an integer \( m \) such that \( t (\text{Arg}(z') - \text{Arg}(w')) = 2\pi m \). Since \( \Re(z'), \Re(w') > 0 \) we have \( \text{Arg}(z'), \text{Arg}(w') \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and therefore:

\[
\pi |m| \leq \frac{2\pi |m|}{t} = \text{Arg}(z') - \text{Arg}(w') \in (-\pi, \pi).
\]

Therefore we must have \( m = 0 \) which implies \( z = w \). If \( t > 2 \), then we can find \( z \in U \) such that \( t \text{Arg}(z') = \pi \) (in particular \( z \neq \overline{z} \)). For \( w = \overline{z} \) we get \( t (\text{Arg}(z') - \text{Arg}(w')) = 2\pi \) and therefore \( g_t(z) = g_t(w) \). As we wanted to show. \( \square \)

**Proof of Proposition 3.5**

**Proof** We first assume that \( t \in (0, 2] \) and show that \( \mathcal{H} \) is faithful.

Let \( A, B \) be subsets of \( \mathbb{H}_d \) and assume that \( T : \mathcal{H}_A \to \mathcal{H}_B \) is an isometry induced by \( \phi : A \to B \). We deal first with the case where \( 0 \in A \cap B \) and \( \phi(0) = 0 \).

Let \( a_2, a_3 \in A \setminus \{0\} \), and denote \( a_1 = 0 \). By Lemma 3.1 there exists \( \delta : \{a_1, a_2, a_3\} \to \mathbb{C} \) such that:

\[
\begin{bmatrix}
\delta(a_i) \delta(a_j)
\end{bmatrix} = \begin{bmatrix}
k(a_i, a_j)
k(\phi(a_i), \phi(a_j))
\end{bmatrix} = \begin{bmatrix}
(1 - \langle \phi(a_i), \phi(a_j) \rangle)^t
\end{bmatrix}
\]

Notice the identity \((z^{-1})^t = (z^t)^{-1}\) holds for \( \Re(z) > 0 \) because \( \text{Log}(z^{-1}) = -\text{Log}(z) \). This gives us:

\[
\text{Rank} \begin{bmatrix}
(1 - \langle \phi(a_i), \phi(a_j) \rangle)^t
\end{bmatrix} = 1.
\]

Which implies that the \( 2 \times 2 \) minors of the matrix vanish.

Because \( a_1 = \phi(a_1) = 0 \), if \( i = 1 \) or \( j = 1 \) we have:

\[
k(a_i, a_j) = k(\phi(a_i), \phi(a_j)) = 1.
\]

Therefore the following \( 2 \times 2 \) minor vanishes:

\[
0 = \det \begin{bmatrix}
1 & \frac{1}{1 - \langle a_i, a_j \rangle^t}
\end{bmatrix} = \begin{bmatrix}
(1 - \langle \phi(a_i), \phi(a_j) \rangle)^t
\end{bmatrix} - 1.
\]

Therefore \((1 - \langle \phi(a_2), \phi(a_3) \rangle)^t = (1 - \langle a_2, a_3 \rangle)^t \). By Lemma 3.7 this implies \( \langle \phi(a_2), \phi(a_3) \rangle = \langle a_2, a_3 \rangle \). Since \( a_2, a_3 \in A \setminus \{0\} \) were arbitrary this shows that
\( \phi \) preserves the inner product and therefore it may be extended to a linear isometry by Lemma 3.5.

For the general case, we use the transitivity of \( \text{Aut} (\mathbb{B}_d) \) (see [19, Theorem 2.2.3]). Choose some \( a \in A \), by transitivity there exist automorphisms \( \psi, \theta \in \text{Aut} (\mathbb{B}_d) \) such that:

\[
\psi(0) = a, \quad \theta(\phi(a)) = 0.
\]

By Proposition 3.3, \( \psi \) and \( \theta \) induce isometric isomorphisms and therefore

\[
g := \theta \circ \phi \circ \psi : \psi^{-1}(A) \to \theta(B)
\]

induces an isometric isomorphism between \( \mathcal{H}_{\psi^{-1}(A)} \) and \( \mathcal{H}_{\theta(B)} \) and fixes the origin. By the first case it extends to an automorphism \( \tilde{g} \in \text{Aut} (\mathbb{B}_d) \). Then \( \theta^{-1} \circ \tilde{g} \circ \psi^{-1} \) is an automorphism of the ball which extends \( \phi \). This shows that if \( t \in (0, 2] \) then \( \mathcal{H} \) is faithful.

Assume that \( t > 2 \). Then by Lemma 3.7 there exists \( z, w \in \mathbb{D} \setminus \{0\} \) such that \( (1 - z^t) = (1 - w)^t \), \( |z| = |w| \) and \( z \neq w \). It is easy to see that we can choose \( a_1, a_2, b_1, b_2 \) in \( \mathbb{B}_d \) such that \( ||a_i|| = ||b_i|| \) and

\[
\langle a_1, a_2 \rangle = z, \quad \langle b_1, b_2 \rangle = w.
\]

Indeed, one could simply take:

\[
a_1 = \frac{z}{\sqrt{|z|}} e_1, \quad a_2 = \sqrt{|z|} e_1, \quad b_1 = \frac{w}{\sqrt{|w|}} e_1, \quad b_2 = \sqrt{|w|} e_1
\]

where \( e_1 \) is the vector \( (1, 0, \ldots, 0) \in \mathbb{C}^d \). Put \( A = \{0, a_1, a_2\} \), \( B = \{0, b_1, b_2\} \) and let \( \phi : A \to B \) be given by \( \phi(0) = 0, \phi(a_i) = b_i \). Then we get the following equality of matrices:

\[
\begin{bmatrix}
k(a_i, a_j) \\
k(\phi(a_i), \phi(a_j))
\end{bmatrix}_{i,j=1,2,3} =
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]

Therefore, by Lemma 3.1, \( \phi \) induces an isometry between \( \mathcal{H}_A \) and \( \mathcal{H}_B \). On the other hand, any automorphism of the ball fixing the origin must be a unitary (see [19, Theorem 2.2.5]), so \( \phi \) cannot be extended to an automorphism. This shows that if \( t > 2 \), then \( \mathcal{H} \) is not faithful.

We can now restate the theorem and prove it:

**Theorem 1.5** Let \( \mathcal{H} = \mathcal{H}^{t,d} \) be as in Example 1.4. If \( t \in (0, 2] \) then for any two subsets \( A, B \subset \mathbb{B}_d \), \( \mathcal{H}_A \) is isometric to \( \mathcal{H}_B \) if and only if \( A \) and \( B \) are congruent. If \( t > 2 \) there exist non-congruent subsets that yield isometric subspaces of \( \mathcal{H} \).
Proof We have already constructed examples of non-congruent subsets yielding isometric subsets of $\mathcal{H}$ for $d \in \mathbb{N}$, $t > 2$ in the proof of Proposition 3.5.

Assume that $t \in (0, 2]$, $d \in \mathbb{N}$ and let $A, B \subset \mathbb{D}_d$. Then $\mathcal{H}$ is faithful by Proposition 3.5. Therefore if $T : \mathcal{H}_A \rightarrow \mathcal{H}_B$ is an isometry induced by a bijection $\phi : A \rightarrow B$, then $\phi$ extends to an automorphism of the ball which shows that $A$ and $B$ are congruent. Conversely if $A$ is congruent to $B$ then $\mathcal{H}_A$ and $\mathcal{H}_B$ are isometric by Corollary 3.4. $\square$

3.2 Proof of Theorem 1.6

Throughout this section let $\mathcal{H} = \mathcal{H}_1$ be the Hilbert function space on $\mathbb{D}$ defined as in Example 1.4 (this space is known as the classical Hardy space and is often denoted $H^2$).

Our goal is then to prove that for any 2 subsets $A, B \subset \mathbb{D}$, with $S(A) = A$, $S(B) = B$, $\mathcal{M}_A$ and $\mathcal{M}_B$ are isometric if and only if $A$ and $B$ are congruent.

If $\phi$ is the restriction of an automorphism which takes $B$ to $A$, then by Corollary 3.5 it induces an isometry between $\mathcal{H}_B$ and $\mathcal{H}_A$. It is well known that this implies that the pull back:

$$\phi^* : \mathcal{M}_A \rightarrow \mathcal{M}_B, \ f \mapsto f \circ \phi.$$ 

Is a well-defined isometric isomorphism, see for example [1, Section 2.6, p. 25].

Theorem 3.8 For any $A \subset \mathbb{D}$ and $f \in \mathcal{M}_A$, there exists an extension of $f$ to the unit disk $\tilde{f} \in \mathcal{M}_D$ such that:

$$||\tilde{f}||_{\mathcal{M}_D} = ||f||_{\mathcal{M}_A}.$$ 

Proof The proof is an adaptation of the well know proof of Pick’s theorem via commutant lifting (see [1, p. 163]) and is included for completeness.

Let $0 \neq f \in \mathcal{M}_A$. We wish to extend $f$ to a multiplier on the disk of norm $||f||_{\mathcal{M}_A}$, by normalizing we may assume that $||f||_{\mathcal{M}_A} = 1$. Let $S \in B(\mathcal{H})$ be the operator corresponding to multiplication by $z$. Notice that for all $\lambda \in \mathbb{D}$, $k_\lambda$ is an eigenvector of $S^*$ (see, e.g., [17, Proposition 6.3.5]). This implies that $\mathcal{H}_A$ is $S^*$-invariant. Let $T \in B(\mathcal{H}_A)$ be the operator corresponding to multiplication by $f$. Then $T^*$ commutes with the restriction of $S^*$ to $\mathcal{H}_A$, because their adjoints commute. By a corollary to the commutant lifting theorem (see [1, Corollary 10.30]), this implies that $f$ extends to a multiplier $\tilde{f} \in \mathcal{M}_D$ of norm at most one. The inequality $||\tilde{f}||_{\mathcal{M}_D} \geq ||T^*|| = 1$ holds because $T^*$ is the restriction of the adjoint of multiplication by $f$ to $\mathcal{H}_A$. $\square$

Definition The pseudo-hyperbolic metric on the disk $\rho$ is a metric on $\mathbb{D}$ given by the formula:

$$\rho(x, y) = \left| \frac{x - y}{1 - \overline{x}y} \right|.$$
**Theorem** (Schwarz–Pick Lemma, see [12, Theorem 4, p. 15]) Let \( f : \mathbb{D} \to \mathbb{D} \) be holomorphic. Then for any \( x, y \in \mathbb{D} \):

\[
\rho(f(x), f(y)) \leq \rho(x, y).
\]

If equality is achieved for some pair of points, then \( f \) is a conformal automorphism of the disk.

It is well known that \( \mathcal{M}_\mathbb{D} \) is equal to the algebra of bounded analytic functions on the disk equipped with the supremum norm (see, e.g., [17, Example 6.3.9]). This algebra is usually denoted by \( H^\infty \). In this paper we keep to the notation \( \mathcal{M}_\mathbb{D} \) as it emphasizes the role \( H^\infty \) plays in the proof. The following lemma is well known. We include the proof for completeness.

**Lemma 3.9** Let \( \psi : \mathcal{M}_\mathbb{D} \to \mathbb{C} \) be a unital morphism of algebras such that \( \psi(z) = a \in \mathbb{D} \). Then \( \psi \) is given by evaluation at \( a \).

**Proof** Let \( f \in \mathcal{M}_\mathbb{D} \). By Taylor’s Theorem we can find an analytic function \( g : \mathbb{D} \to \mathbb{C} \) such that:

\[
\frac{f(z) - f(a)}{z - a} = g(z).
\]

It follows that \( g \) is bounded on the disk, which implies \( g \in \mathcal{M}_\mathbb{D} \). We calculate:

\[
\psi(f) = \psi(f(a) + (z - a)g(z)) = f(a) + (\psi(z) - a)\psi(g(z)) = f(a)
\]

As we wanted to show. \( \square \)

We say that a morphism of algebras \( \Phi : \mathcal{M}_A \to \mathcal{M}_B \) is a pull-back by \( \phi : B \to A \) if \( \Phi(f) = f \circ \phi \) for all \( f \in \mathcal{M}_A \). We denote such a pull-back by \( \Phi = \phi^* \).

**Proposition 3.10** If \( \Phi : \mathcal{M}_A \to \mathcal{M}_B \) is an isometric isomorphism of Banach algebras, where \( A = S(A) \) and \( B = S(B) \), then it is a pull-back by \( \phi : B \to A \) where \( \phi \) is the restriction of an automorphism of the disk.

We will prove this in two stages:

**Claim** There exists \( \phi : B \to A \) for which \( \Phi = \phi^* \).

**Proof** Let \( \phi = \Phi(z) \), i.e. \( \phi \) is the image of the identity function under \( \Phi \). We will show that \( \Phi = \phi^* \).

It is easy to see that \( ||z||_{\mathcal{M}_A} \leq 1 \) (by [17, Proposition 6.3.15], for example) and therefore we get \( ||\phi||_{\mathcal{M}_B} \leq 1 \). By Theorem 3.8 we get an extension \( \tilde{\phi} \in \mathcal{M}_\mathbb{D} \) of \( \phi \), such that \( ||\tilde{\phi}||_{\infty} \leq 1 \). Notice that \( \phi \) cannot be constant because \( z \) is not constant and \( \Phi^{-1} \) fixes the constant functions (it is an isomorphism of algebras). By the maximum modulus principle we get that \( \tilde{\phi}(\mathbb{D}) \subset \mathbb{D} \).
Let $b \in B$ and define the following homomorphism of algebras:

$$\psi : \mathcal{M}_D \to \mathbb{C}, \ f \mapsto \Phi(f|_A)(b).$$

Here $f|_A$ is the restriction of $f$ to $A$ (this is a multiplier by [17, Proposition 6.3.5.]). Because $\psi(z) = \phi(b) \in \mathbb{D}$, by Lemma 3.9 $\psi$ is given by evaluation at $\phi(b)$. We show that $\phi(b) \in A$ by adapting [1, Theorem 9.27].

We assume that $\phi(b) \in \mathbb{D} \setminus A$ and reach a contradiction. This implies $k_{\phi(b)} \notin \mathcal{H}_A$ because $S(A) = A$. Therefore we can define a bounded operator $T$ on $\mathcal{H}_A \oplus \text{span}\{k_{\phi(b)}\}$ sending $\mathcal{H}_A$ to zero and fixing $k_{\phi(b)}$. The adjoint of $T$ is a multiplier by [17, Proposition 6.3.5], which extends to a multiplier $g \in \mathcal{M}_D$ by Theorem 3.8. Now we notice that $g(a) = 0$ for all $a \in A$ by definition. This implies:

$$1 = g(\phi(b)) = \psi(g) = \Phi(g|_A)(b) = \Phi(0)(b) = 0.$$

We reached a contradiction, so it cannot be that $\phi(b) \notin A$. This shows that the image of $\phi$ is contained in $A$. By Theorem 3.8 for any $f \in \mathcal{M}_A$ there exists an extension $\tilde{f}$ of $f$ to $\mathcal{M}_D$. We calculate:

$$\Phi(f)(b) = \Phi(\tilde{f}|_A)(b) = \psi(\tilde{f}) = f(\phi(b))$$

Therefore $\Phi = \phi^*$, as we wanted to show. \qed

Claim The extension $\tilde{\phi}$ is a conformal automorphism of the disk.

Proof Because $\tilde{\phi} : \mathbb{D} \to \mathbb{D}$ is analytic, the Schwarz–Pick Lemma implies that for any $x, y \in B$:

$$\rho(\phi(x), \phi(y)) = \rho(\tilde{\phi}(x), \tilde{\phi}(y)) \leq \rho(x, y).$$

But $(\phi^{-1})^* = \Phi^{-1}$ is also an isometric isomorphism, so by reversing the argument we get that for any $x, y \in B$:

$$\rho(x, y) = \rho(\phi^{-1}(\phi(x)), \phi^{-1}(\phi(y))) \leq \rho(\phi(x), \phi(y)).$$

We deduce that $\tilde{\phi}$ must be an automorphism by the Schwarz–Pick lemma because it preserves the pseudo-hyperbolic metric on $B$. This finishes the proof of the proposition, and therefore proves Theorem 1.6. \qed

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