SOME QUANTUM DYNAMICAL SEMI-GROUPS WITH QUANTUM STOCHASTIC DILATION

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Abstract. We consider the GNS Hilbert space $\mathcal{H}$ of a uniformly hyper-finite $C^*$-algebra and study a class of unbounded Lindbladian arises from commutators. Exploring the local structure of UHF algebra, we have shown that the associated Hudson-Parthasarathy type quantum stochastic differential equation admits a unitary solution. The vacuum expectation of homomorphic cocycle, implemented by the Hudson-Parthasarathy flow, is conservative and gives the minimal semi-group associated with the formal Lindbladian. We also associate conservative minimal semi-groups to another class of Lindbladian by solving the corresponding Evan-Hudson equation.

1. Introduction

Quantum dynamical semi-groups (QDS) appear naturally when one studies the evolution of irreversible open quantum systems. QDS are non-commutative analogue to Markov semi-groups in classical probability. For a uniformly continuous semi-groups, the generator is a bounded, conditionally completely positive (CCP) map. In [6], Lindblad proved that for hyper-finite von Neumann algebras, which includes the case of $\mathcal{B}(\mathcal{H})$, the generator $\mathcal{L}$ of uniformly continuous QDS can be written as $\mathcal{L}(X) = \phi(X) + G^* X + XG$, where $\phi$ is a completely positive map and $G \in \mathcal{B}(\mathcal{H})$. In [1] Christensen and Evans proved that for general $C^*$-algebras, the generator of a uniformly continuous QDS exhibits the similar structure.

For the case of a strongly continuous QDS, structure of the generator is not well understood, Kato [5] and Davies [2] studied some unbounded operators or forms similar to above on $\mathcal{B}(\mathcal{H})$ and gave a construction of one-parameter semi-groups, so-called minimal semi-group. Under certain assumptions, Davies in [3] showed that the unbounded generator have a similar form as for the bounded case, thus extends the Lindblad’s result to strongly continuous QDS. However, these semi-groups need not preserve the identity, i.e., need not be Markov. Generally such unbounded operator or form referred as Lindbladian. Starting with a Lindbladian, a similar construction of a minimal semi-group was done for any von Neumann algebra in [10].

In this article, we have considered Hudson-Parthasarathy (HP) quantum stochastic differential equation associated with a model of unbounded Lindbladian and construct the QDS by taking vacuum expectation. There are various attempts

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to study HP quantum stochastic differential equation with unbounded coefficients, for example see [4, 10] and references therein.

In section 2, we discuss briefly QDS and some results of quantum stochastic calculus and quantum stochastic differential equations (QSDE) with bounded operator coefficients. In section 3, a class of unbounded Lindblad form are defined on the GNS space of UHF $C^*$-algebra and properties of structure maps are studied. Finally, exploring the local structure of UHF algebra, it is shown that the associated HP equation admits a unitary solution. This implies that the expectation semi-group of the homomorphic co-cycle implemented by this unitary is conservative and therefore the unique (also minimal) $C_0$-contraction semi-group associated with the given form. The model is very special and hence simple enough to allow the construction of the minimal semi-group, without any of the machineries of the abstract theories, mentioned earlier in the introduction.

2. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the von Neumann algebra of bounded linear operators on $\mathcal{H}$.

**Definition 2.1.** A quantum dynamical semi-group on a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a semi-group $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$ of completely positive maps on $\mathcal{A}$ with the following properties:

(i) $\mathcal{T}_t(I) \leq I$, for all $t \geq 0$.

(ii) $\mathcal{T}_t$ is a ultra-weakly continuous operator i.e. normal for all $t \geq 0$.

(iii) for each $a \in \mathcal{A}$, the map $t \rightarrow \mathcal{T}_t(a)$ is continuous with respect to the ultra-weak topology on $\mathcal{A}$.

A QDS is called Markov or Conservative if $\mathcal{T}_t(I) = I$ for every $t$.

**Theorem 2.2.** [6, 1, 9, 10] A bounded map $\mathcal{L}$ on the von Neumann algebra $\mathcal{B}(\mathcal{H})$ is the infinitesimal generator of a uniformly continuous QDS $(\mathcal{T}_t)_{t \geq 0}$ if and only if it can be written as

$$\mathcal{L}(X) = \sum_{n=1}^{\infty} L_n^* XL_n + G^* X + XG,$$

for all $X \in \mathcal{B}(\mathcal{H})$, where $L_n$'s and $G$ are in $\mathcal{B}(\mathcal{H})$ and the series on the right side converges strongly, with $G$ generator of a contraction semi-group in $\mathcal{H}$. The QDS is Markov if and only if

$$\text{Re}(G) = -\frac{1}{2} \sum_{n=1}^{\infty} L_n^* L_n.$$

For more general QDS, the generator can be understood as one coming from a similarly defined quadratic form on $\mathcal{H}$, e.g., for $X \in \mathcal{B}(\mathcal{H})$,

$$\langle u, \mathcal{L}(X)v \rangle \equiv \langle u, XGv \rangle + \langle Gu, Xv \rangle + \sum_{n=1}^{\infty} \langle L_n u, XL_n v \rangle$$

(2.1)
where these \(L_n\) and \(G\) are unbounded operators, \(G\) is the generator of a \(C_0\)-contraction semi-group in \(\mathcal{H}\) such that \(\text{Dom}(G) \subseteq \text{Dom}(L_n)\), for each \(n\) and

\[
\langle u, \mathcal{L}(I)v \rangle \equiv \langle u, Gv \rangle + \langle Gu, v \rangle + \sum_{n=1}^{\infty} \langle L_n u, L_n v \rangle = 0, \quad (2.2)
\]

for all \(u, v \in \text{Dom}(G)\).

Conversely, let \(G\) be the generator (not necessarily bounded) of a \(C_0\)-contraction semi-group in \(\mathcal{H}\) and \(L_n\) be a family of closed densely defined linear operators in \(\mathcal{H}\) with \(\text{Dom}(G) \subseteq \text{Dom}(L_n)\) and let \(\mathcal{L}\) define formally by (2.1) satisfies (2.2).

Then the aim is to construct a canonical (minimal) semigroup associated with the formal Lindbladian \(\mathcal{L}\), for some results in this direction see [2, 5, 10].

We conclude this section with a brief discussion of Quantum stochastic calculus developed by Hudson and Parthasarathy. We state a result of existence and uniqueness of unitary solution for QSDE. For detail see [9, 10].

For a separable Hilbert space \(\mathcal{H}\), let \(\Gamma_{\text{sym}}(\mathcal{H})\) denotes the symmetric Fock space over \(\mathcal{H}\). For any \(u \in \mathcal{H}\), we denote by \(e(u)\), the exponential vector in \(\Gamma_{\text{sym}}(\mathcal{H})\) associated with \(u\):

\[
e(u) = \bigoplus_{n \geq 0} \frac{1}{\sqrt{n!}} u \otimes^n.
\]

Given a contraction \(T\) on \(\mathcal{H}\), the second quantization \(\Gamma(T)\) on \(\Gamma_{\text{sym}}(\mathcal{H})\) is defined by \(\Gamma(T)e(u) = e(Tu)\) and extends to a contraction on \(\Gamma_{\text{sym}}(\mathcal{H})\). Moreover, if \(T\) is an isometry (respectively unitary), then so is \(\Gamma(T)\).

Let us write \(\Gamma_{\text{sym}}\) for the symmetric Fock space \(\Gamma_{\text{sym}}(L^2(\mathbb{R}^+, k))\), where \(k\) is a Hilbert space with an orthonormal basis \(\{e_l : 1 \leq l \leq m\}\). Following result provides a nice criterion for the existence of a unitary solution for an HP type QSDE with bounded coefficients.

**Theorem 2.3.** [9] Let \(\mathcal{H}, \{L_i : 1 \leq i \leq m\}, \{S_j^i : 1 \leq i, j \leq m\}\) are bounded operators in \(\mathcal{H}\) satisfying the following conditions:

(i) \(\mathcal{H}\) is self-adjoint.

(ii) \(\sum_{1 \leq i, j \leq m} S_j^i \otimes |e_i\rangle\langle e_j|\) is a unitary operator in \(\mathcal{H} \otimes k\).

Define

\[
L_j^i = \begin{cases} 
S_j^i - \delta_j^i & \text{if } 1 \leq i, j \leq m; \\
L_i & \text{if } 1 \leq i \leq m, j = 0; \\
-\sum_{1 \leq k \leq m} L_k^* S_j^k & \text{if } 1 \leq j \leq m, i = 0; \\
-(i\mathcal{H} + \frac{1}{2} \sum_{1 \leq k \leq m} L_k^* L_k) & \text{if } i = j = 0;
\end{cases} \quad (2.3)
\]

where \(\delta_j^i\) is the Kronecker’s delta function. Then there exists a unique unitary process \(U_t\) satisfying the QSDE on \(\mathcal{H} \otimes \Gamma_{\text{sym}}\)

\[
U_t = I + \sum_{i,j=0}^{m} \int_0^t U_s L_j^i N_j^i(ds), \quad (2.4)
\]
where $\Lambda^0_0$ is time, for $i, j \geq 1, \Lambda^j_i$ is conservation, $\Lambda^0_i$ is creation and $\Lambda^j_0$ is annihilation processes.

Let $(U_t)_{t \geq 0}$ be a unitary process satisfying (2.4). Then the family of homomorphisms $\{J_t : t \geq 0\}$ defined by

$$J_t(X) = U_t^* (X \otimes I) U_t, \ X \in \mathcal{B}(\mathcal{H}).$$

satisfies the QSDE

$$J_t(X) = X \otimes I + \sum_{i,j=0}^m \int_0^t J_s \theta^i_j(X) \Lambda^j_i(ds) \quad (2.5)$$

where

$$\theta^i_j(X) = XL^i_j + (L^i_j)^* X + \sum_{k=1}^m (L^k_i)^* XL^k_j, \ \forall \ i, j \geq 0.$$ 

In particular $\theta^0_0$ is given by,

$$\theta^0_0(X) = \sum_{k=0}^m L^*_k XL_k + XL^0_0 + L^0_0 X, \quad (2.6)$$

is the generator of a QDS $(\mathcal{T}_t)_{t \geq 0}$ and the homomorphic co-cycle $J_t$ dilates $\mathcal{T}_t$ in the sense that

$$\langle u e(0), U_t^* (X \otimes I) U_t v e(0) \rangle = \langle u, \mathcal{T}_t(X) v \rangle, \forall u, v \in \mathcal{H} \text{ and } X \in \mathcal{B}(\mathcal{H}). \quad (2.7)$$

The QDS $\mathcal{T}_t$ is called the vacuum expectation of $J_t$. This homomorphic co-cycle $J_t$, implemented by the HP flow $U_t$, is known as an HP dilation of the QDS $\mathcal{T}_t$.

Consider the time reversal operator $R_t$ on $L^2(\mathbb{R}_+, \mathcal{B}_k)$ defined by

$$R_t(f)(s) := \begin{cases} f(t-s) & \text{if } s \leq t; \\ f(s) & \text{if } s > t. \end{cases} \quad (2.8)$$

Observe that $R_t$ is a self-adjoint unitary. Thus the second quantization $\Gamma(R_t)$ is so. For a bounded process $U_t$, define the dual process $\tilde{U}_t$ by

$$\tilde{U}_t := (1 \otimes \Gamma(R_t)) U_t^* (1 \otimes \Gamma(R_t)).$$

**Proposition 2.4.** [10] Let $U_t$ be a bounded process satisfying the QSDE (2.4). Then the dual process $\tilde{U}_t$ will satisfy the QSDE of the similar form given by,

$$\tilde{U}_t = I + \sum_{i,j=0}^m \int_0^t \tilde{U}_s L^*_i \Lambda^j_i(ds).$$
3. Examples of Quantum Dynamical Semi-groups

In this section, we have constructed a class of formal Lindbladian on the GNS space of a UHF \( C^* \)-algebra \( \mathcal{A} \). In Theorem 3.5 we will show that the associated HP equation admits a unitary solution.

Let us consider the UHF \( C^* \)-algebra \( \mathcal{A} \) as the \( C^* \)-inductive limit of the infinite tensor product of the matrix algebra \( M_N(\mathbb{C}) \),

\[
\mathcal{A} = \bigotimes_{j \in \mathbb{Z}^d} M_N(\mathbb{C})
\]

The algebra \( \mathcal{A} \) can be interpreted as inductive limit of full matrix algebras. For \( x \in M_N(\mathbb{C}) \) and \( j \in \mathbb{Z}^d \), \( x^{(j)} \) denotes an element of \( \mathcal{A} \) with \( x \) in the \( j^{th} \) component and identity everywhere else. We shall call the elements of the form \( \prod_{i \geq 1} x_i^{(j)} \) to be simple tensor elements in \( \mathcal{A} \). For a simple tensor element \( x \in \mathcal{A} \), let \( x_{(j)} \) be the \( j^{th} \) component of \( x \). Support ‘\( \text{supp}(x) \)’ of \( x \) is defined to be the subset \( \{ j \in \mathbb{Z}^d; x_{(j)} \neq I \} \).

For a general element \( x \in \mathcal{A} \) such that \( x = \sum_{n=1}^{\infty} c_n x_n \) with simple tensor elements \( x_n \) and complex coefficients \( c_n \), define \( \text{supp}(x) = \bigcup_{n \geq 1} \text{supp}(x_n) \). For any \( \Delta \subset \mathbb{Z}^d \), let \( \mathcal{A}_\Delta \) denotes the \( * \)-sub algebra generated by the elements of \( \mathcal{A} \) with support in \( \Delta \). For \( j = (j_1, j_2, \cdots, j_d) \in \mathbb{Z}^d \), define \( \|j\| = \max\{|j_i| : 1 \leq i \leq d\} \) and set \( \Delta_n = \{ j \in \mathbb{Z}^d; \|j\| \leq n \} \), \( \partial \Delta_n = \{ j \in \mathbb{Z}^d; \|j\| = n \} \). We say an element \( x \in \mathcal{A} \) is local if \( x \in \mathcal{A}_\Delta \) for some \( p \geq 1 \). The unique normalized trace \( \text{tr} \) on \( \mathcal{A} \) is given by \( \text{tr}(x) = \frac{1}{N^n} \text{Tr}(x) \), for \( x \in M_N(\mathbb{C}) \), where \( \text{Tr} \) denotes the matrix trace.

The trace \( \text{tr} \) is a faithful normal state on \( \mathcal{A} \). The algebra \( \mathcal{A} \) can be represented as vectors in the Hilbert space \( \mathcal{H} = L^2(\mathcal{A}, \text{tr}) \), the GNS Hilbert space for \( (\mathcal{A}, \text{tr}) \), and as an element of \( \mathcal{B}(\mathcal{H}) \) by left multiplication. We write \( \mathcal{A}_{\text{loc}} \) for the dense \( * \)-algebra generated by local elements.

Consider a formal element of the type

\[
r := \sum_{n=1}^{\infty} W_n \text{ such that } \sum_{n=1}^{\infty} \|W_n\| = \infty,
\]

where each \( W_n \) belongs to \( \mathcal{A}_{\Delta_n} \). Let us denote formally

\[
\sum_{n=1}^{\infty} W_n^* \text{ by } r^*.
\]

Now, if we set \( \mathcal{C}_r(x) = [r, x] = \sum_{n=1}^{\infty} [W_n, x] \) for \( x \in \mathcal{A}_{\text{loc}} \), clearly it is well defined since \( [W_n, x] = 0 \) for all \( n > m \) when \( x \) is in finite dimensional algebra \( \mathcal{A}_{\Delta_m} \subseteq \mathcal{A}_{\text{loc}} \).

Thus we have a densely defined linear operator \( (\mathcal{C}_r, \mathcal{A}_{\text{loc}}) \) in \( \mathcal{H} \).

**Lemma 3.1.** Let \( r \) be as above and \( n \geq 1 \). Consider the element \( r_n = \sum_{k=1}^{n} W_k \) in \( \mathcal{A} \) and define a bounded operator \( \mathcal{C}_r^{(n)} \) on \( \mathcal{H} \) by setting \( \mathcal{C}_r^{(n)}(x) = [r_n, x] = \sum_{k=1}^{n} [W_k, x] \) for \( x \in \mathcal{A}_{\text{loc}} \). Then for each \( n \geq 1 \), \( \mathcal{A}_{\Delta_n} \) is an invariant subspace for \( \mathcal{C}_r \) and \( \mathcal{C}_r^{(n)} \).
Also for \( m \geq p \),

\[
C_r|_{A_{\Delta_p}} = C_r^{(m)}|_{A_{\Delta_p}} = C_r^{(p)}|_{A_{\Delta_p}}.  \tag{3.1}
\]

**Proof.** For \( x \) is in \( A_{\Delta_n} \), \([W_k, x] = 0\) for \( k > n \). Thus \([r, x] = [r_n, x] \in A_{\Delta_n} \) and \( A_{\Delta_n} \) is an invariant subspace under \( C_r \) and \( C_r^{(n)} \). Now for \( x \in A_{\Delta_p} \) and \( m \geq p \), it is easy to see that \( C_r(x) = C_r^{(m)}(x) = C_r^{(p)}(x) \) in \( A_{\Delta_p} \), thereby showing that the operator \( C_r \) is a positive self-adjoint operator in \( H \). Then by standard theorem of von Neumann, \( C_r^{(*)} \) is an invariant under \( C_r \). Furthermore, the operator \( G := \frac{1}{2} C_r^{(*)} \) generates a \( C_0 \)-contraction semi-group \( S_t \) in \( H \).

**Proposition 3.2.** The operator \((C_r, A_{loc})\) is closable.

**Proof.** We shall show that \( A_{loc} \subseteq Dom(C_r^{(*)}) \) and for \( x \in A_{loc} \), \( C_r^{(*)}(x) = C_{r^*}(x) = [r^*, x] \), thereby showing that the operator \( C_r^{(*)} \) is densely defined and therefore \((C_r, A_{loc})\) is closable. Indeed for \( x \in A_{loc} \), there exists \( p \geq 1 \) such that \( x \in A_{\Delta_p} \).

We denote by \( \tilde{C}_r \), the closure of a densely defined, closable operator \( C_r \). Note here that for a operator \( T \) on \( H \), \( T^* = \tilde{T}^* \), if \( T \) is closable. Then by standard theorem of von Neumann, \( C_r^{(*)} \tilde{C}_r \) is a positive self-adjoint operator in \( H \) and \( Dom(C_r^{(*)} \tilde{C}_r) \) is a core for \( \tilde{C}_r \). Furthermore, the operator \( G := -\frac{1}{2} C_r^{(*)} \tilde{C}_r \) generates a \( C_0 \)-contraction semi-group \( S_t \) in \( H \).

**Proposition 3.3.** For \( n \geq 1 \), define the bounded operator \( G^{(n)} \) on \( H \) by

\[
G^{(n)} := -\frac{1}{2} C_r^{(*)} C_r^{(n)}. \tag{3.2}
\]

Then each \( A_{\Delta_n} \) is an invariant under \( G^{(n)} \). Furthermore,

\[
G^{(m)}|_{A_{\Delta_p}} = G^{(p)}|_{A_{\Delta_p}} = G|_{A_{\Delta_p}} \text{ if } m \geq p. \tag{3.3}
\]

**Proof.** By Lemma 3.1, we have \( A_{\Delta_n} \) invariant under \( C_r \) and \( C_r^{(n)} \) and for \( m \geq p \), the identity \( C_r|_{A_{\Delta_n}} = C_r^{(m)}|_{A_{\Delta_p}} \) holds. As \( C_r^{(*)}(x) = C_{r^*}(x), \forall x \in A_{loc} \), we have \( C_r^{(*)}|_{A_{\Delta_p}} = C_r^{(p)}|_{A_{\Delta_p}} \) and hence result follows.

**Proposition 3.4.** The subspace \( A_{loc} \) is a core for the operator \( G \).
Since these operators $C_i$ is for $A_{\Delta_n}$, Now by Lemma 3.1, for any $k \geq 0$, $G^k(x) = G^{(n)k}(x) \in A_{\Delta_n}$ and it follows that the series $\sum_{k \geq 0} e^{tG^{(n)k}}$ converges strongly in $A_{\Delta_n}$. Therefore, we have, $S_i x = S_i^{(n)} x = e^{tG^{(n)}} x$ for $x \in A_{\Delta_n}$. Thus, $S_i$ leaves $A_{\text{loc}}$ invariant and by Nelson’s theorem [8], the core property follows. 

Now consider the sesquilinear form, Lindbladian, $\mathcal{L}(X)$ with the domain $A_{\text{loc}} \times A_{\text{loc}} \subseteq \text{Dom}(G) \times \text{Dom}(G)$ given by

$$
(u, \mathcal{L}(X)v) \equiv \langle u, XGv \rangle + \langle Gu, Xv \rangle + \langle \tilde{C}_r u, X\tilde{C}_r v \rangle.
$$

(3.4)

By definition of $G$, it is clear that $\langle u, \mathcal{L}(I)v \rangle = \langle u, Gv \rangle + \langle Gu, v \rangle + \langle \tilde{C}_r u, \tilde{C}_r v \rangle = 0$.

Let $A_{\text{loc}} \oslash \mathcal{E}$ be the linear span of $\{ x \otimes e(f) : x \in A_{\text{loc}}, f \in L^2(\mathbb{R}_+, \mathbb{C}) \}$. Then the set $A_{\text{loc}} \oslash \mathcal{E}$ is a dense subspace of $\mathcal{H} \otimes \Gamma_{\text{sym}}$.

**Theorem 3.5.** Consider the HP type QSDE in $A_{\text{loc}} \otimes \mathcal{E}$

$$
U_t = I + \int_{0}^{t} U_s G ds + \int_{0}^{t} U_s \tilde{C}_r a^{\dagger}(ds) - \int_{0}^{t} U_s C_r a (ds),
$$

(3.5)

where $a^\dagger$, $a$ are creation and annihilation processes respectively. The QSDE (3.5) admits a unitary solution $U_t$. Moreover, the expectation semi-group $(T_t)_{t \geq 0}$ of the homomorphically co-cycle $J_t(X) = U_t^\ast (X \otimes I) U_t$ is the unique (minimal) semi-group associated with the formal Lindbladian $\mathcal{L}$ in (3.4) and is conservative.

**Proof.** Recall that the UHF algebra $\mathcal{A}$ can be approximated by finite dimensional algebras, namely $A_{\Delta_n} = \prod_{\|j\| \leq n} M_N(\mathbb{C})$ and $A_{\text{loc}} = \bigcup_{n=0}^{\infty} A_{\Delta_n}$. For $n \geq 0$, consider the following QSDE in $A_{\text{loc}} \otimes \mathcal{E}$,

$$
U_t^{(n)} = I + \int_{0}^{t} U_s^{(n)} G^{(n)} ds + \int_{0}^{t} U_s^{(n)} C_r^{(n)} a^{\dagger}(ds) - \int_{0}^{t} U_s^{(n)} C_r^{(n)*} a (ds).
$$

(3.6)

By Theorem 2.3, the QSDE 3.6 admits a unitary solution $U_t^{(n)}$ on $\mathcal{H} \otimes \Gamma_{\text{sym}}$.

We now show that the operators $U_t^{(n)}$ satisfy some compatibility condition, that is for $n \geq m$,

$$
U_t^{(n)} |_{A_{\Delta_m}} = U_t^{(m)} |_{A_{\Delta_m}}.
$$

(3.7)

Here the symbol $T|_{A_{\Delta_m}}$ means the restriction of $T$ to the subspace $A_{\Delta_m} \otimes \Gamma_{\text{sym}}$.

Since these operators $C_r^{(m)}, C_r^{(m)*}$ and $G^{(m)}$ leave $A_{\Delta_m}$ invariant, the restriction $U_t^{(m)} |_{A_{\Delta_m}}$ satisfies the following QSDE in $A_{\Delta_m} \otimes \mathcal{E}$,

$$
U_t^{(m)} |_{A_{\Delta_m}} = I |_{A_{\Delta_m}} + \int_{0}^{t} U_s^{(m)} |_{A_{\Delta_m}} G^{(m)} |_{A_{\Delta_m}} ds
$$

(3.8)

$$
+ \int_{0}^{t} U_s^{(m)} |_{A_{\Delta_m}} C_r^{(m)} |_{A_{\Delta_m}} a^{\dagger}(ds) - \int_{0}^{t} U_s^{(m)} |_{A_{\Delta_m}} C_r^{(m)*} |_{A_{\Delta_m}} a (ds).
$$
For \( n \geq m \), consider the QSDE in \( A_{\Delta_m} \otimes \mathcal{E} \),

\[
U_t^{(n)}|_{A_{\Delta_m}} = I|_{A_{\Delta_m}} + \int_0^t U_s^{(n)}|_{A_{\Delta_m}} G^{(n)}|_{A_{\Delta_m}} ds + \int_0^t U_s^{(n)}|_{A_{\Delta_m}} \mathcal{C}_{r}^{(n)} a^\dagger(ds) - \int_0^t U_s^{(n)}|_{A_{\Delta_m}} \mathcal{C}_{r}^{(n)*} a(ds). \tag{3.9}
\]

With reference to Lemma 3.1, equation (3.3) and Theorem 2.3, the unitary processes \( U_t^{(n)}|_{A_{\Delta_m}} \) and \( U_t^{(m)}|_{A_{\Delta_m}} \) satisfy the same QSDE in \( A_{\Delta_m} \otimes \mathcal{E} \). Therefore, by uniqueness of solution in Theorem 2.3, (3.7) follows.

Define \( U_t \) on \( \mathcal{A}_{loc} \otimes \mathcal{E} \) by setting

\[
U_t(x \otimes e(f)) = U_t^{(n)}(x \otimes e(f)) \text{ if } x \in A_{\Delta_n}.
\]

and extending linearly. Since the family \( U_t^{(n)} \) satisfies the compatibility condition (3.7), \( U_t \) is well defined on \( \mathcal{A}_{loc} \otimes \mathcal{E} \), and for \( x \in A_{\Delta_n} \) we have

\[
U_t(x \otimes e(f)) = U_t^{(m)}(x \otimes e(f)) = U_t^{(n)}(x \otimes e(f)), \forall n \geq m. \tag{3.10}
\]

Hence \( U_t^{(n)} \) converges strongly to \( U_t \) on \( \mathcal{A}_{loc} \otimes \mathcal{E} \) and \( U_t \) extends to a contraction operator on \( \mathcal{H} \otimes \Gamma_{sym} \). As \( \mathcal{A}_{loc} \otimes \mathcal{E} \) is dense in \( \mathcal{H} \otimes \Gamma_{sym} \), (3.10) gives that \( U_t^{(n)} \) converges strongly to \( U_t \) on \( \mathcal{H} \otimes \Gamma_{sym} \) as well and the limit \( U_t \) is an isometry.

For \( U_t^{(n)} \), consider the dual process \( \tilde{U}_t^{(n)} = (1 \otimes \Gamma(R_t))U_t^{(n)*} (1 \otimes \Gamma(R_t)) \). Then by Proposition 2.4, \( \{\tilde{U}_t^{(n)}\} \) satisfies the following QSDE in \( \mathcal{A}_{loc} \otimes \mathcal{E} \),

\[
\tilde{U}_t^{(n)} = I + \int_0^t \tilde{U}_s^{(n)} G^{(n)*} ds + \int_0^t \tilde{U}_s^{(n)} \mathcal{C}_{r}^{(n)*} a(ds) - \int_0^t \tilde{U}_s^{(n)} \mathcal{C}_{r}^{(n)*} a^\dagger(ds). \tag{3.11}
\]

The equation (3.11) is identical to (3.6) except that \( \mathcal{C}_{r}^{(n)} \) is replaced by \( \mathcal{C}_{r}^{(n)*} \). So similar arguments yield that the operators \( \tilde{U}_t^{(n)} \) also satisfy the compatibility condition and converge strongly to an isometry and because \( \tilde{U}_t^{(n)} \) and \( \Gamma(R_t) \) are unitaries, the sequence \( U_t^{(n)*} \) of unitaries converges strongly and thus it must converge to \( U_t^* \). Hence \( U_t^* \) is an isometry, so \( U_t \) is a unitary process.

It remains to prove that \( U_t \) satisfies the QSDE (3.5). As \( U_t \) is a unitary process, the quantum stochastic integral on the right-hand side of (3.5) makes sense. Thus, it is enough to establish that integrals on the right-hand side of (3.6) converge to integrals in (3.5). For \( xe(f) \in \mathcal{A}_{loc} \otimes \mathcal{E} \), we have

\[
\| \int_0^t (U_s^{(n)}G^{(n)} - U_s G) ds(xe(f)) \| \leq \int_0^t \| (U_s^{(n)}G^{(n)} - U_s G)(xe(f)) \| ds,
\]

hence by (3.3) and (3.10), it converges to 0. By estimates of quantum stochastic integrals [9], we have
\[
\| \int_0^t (U_s^{(n)} C_r^{(n)} - U_s C_r) a^\dagger (ds)(xe(f)) \|^2 \\
\leq 2 e^{\int_0^t (1 + f(s)^2) ds} \int_0^t \| (U_s^{(n)} C_r^{(n)} - U_s C_r) xe(f) \|^2 (1 + |f(s)|^2) ds.
\]

Therefore, by (3.1) and (3.10),
\[
\lim_{n \to \infty} \int_0^t (U_s^{(n)} C_r^{(n)} - U_s C_r) a^\dagger (ds)(xe(f)) \|^2 = 0.
\]

Convergence of annihilation term follows from a simpler estimate and using (3.2), (3.1) and (3.10). Thus \(U_t\) is a unitary solution to the QSDE (3.5).

Now let us consider the expectation semi-group \((T_t)_{t \geq 0}\) of the homomorphic co-cycle \(J_t(\cdot) = U_t^* \cdot \otimes f U_t\). As \(U_t\) is a unitary process, the QDS \((T_t)_{t \geq 0}\) is conservative minimal semi-group associated with the form (3.4).

We conclude by a small remark on the Lindbladian
\[
\mathcal{L}(X) = \frac{1}{2} \sum_{j=1}^{\infty} \{ W_j^* \delta_j(X) + \delta_j^\dagger(X) W_j \}, \forall X \in \mathcal{A}_{loc}
\]
where \(W_j \in \mathcal{A}_{\delta_{\Delta_j}}, \delta_j(X) = [X, W_j], \delta_j^\dagger(X) = (\delta_j(X^*))^* = [W_j^*, X].\) Though each component \(W_j^* \delta_j(\cdot) + \delta_j^\dagger(\cdot) W_j\) are bounded maps, \(\mathcal{L}\) is unbounded due to presence of infinitely many components (like in [7]). For \(n \geq 1\), define a bounded map \(\mathcal{L}^{(n)}(X) = \frac{1}{2} \sum_{j=1}^{n} \{ W_j^* \delta_j(X) + \delta_j^\dagger(X) W_j \}, \forall X \in \mathcal{A}.\) Note that for \(X \in \mathcal{A}_{\Delta_n}, \delta_k(X) = \delta_k^\dagger(X) = 0\) and \(\mathcal{L}^{(k)}(X) = \mathcal{L}^{(n)}(X)\) for every \(k > n.\)

**Remark 3.6.** The HP equation on \(\mathcal{H} \otimes \Gamma_{sym}(L^2(\mathbb{R}_+, k))\), where \(k\) is a separable Hilbert space with an orthonormal basis \(\{ e_j : j \geq 1 \},\)
\[
U_t = I + \int_0^t U_s \left( -\frac{1}{2} \sum_{j=1}^{\infty} W_j^* W_j ds + \sum_{j=1}^{\infty} \int_0^t U_s W_j a_j^\dagger (ds) - \sum_{j=1}^{\infty} \int_0^t U_s W_j a_j (ds) \right)
\]
may not make sense as \(-\frac{1}{2} \sum_{j=1}^{\infty} W_j^* W_j\) may have a trivial domain or not a generator of a \(C_0\) semigroup on \(\mathcal{H}.\) However, there exist a homomorphic co-cycle \(J_t : \mathcal{A} \to \mathcal{A}_{\Delta}^\ast \otimes \mathcal{B}(\Gamma_{sym})\) satisfying the Evan-Hudson equation, for \(X \in \mathcal{A}_{loc},\)
\[
J_t(X) = X \otimes I + \int_0^t J_s(\mathcal{L}(X) ds + \sum_{j=1}^{\infty} \int_0^t J_s(\delta_j(X) a_j^\dagger (ds) + \sum_{j=1}^{\infty} \int_0^t J_s(\delta_j^\dagger(X) a_j (ds).
\]
The expectation semi-group \((T_t)_{t \geq 0}\) of the homomorphic co-cycle \(J_t\) is conservative minimal semi-group associated with the Lindbladian (3.12).

This can be seen similarly as for HP equation in theorem 3.5 by constructing \(J_t\) as a strong limit of homomorphic co-cycles \(\{ J_t^{(n)} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\Gamma_{sym}) \}\).
where $J_t^{(n)}$ satisfies the Evan-Hudson equation, for $X \in \mathcal{B}(\mathcal{H})$,

$$J_t^{(n)}(X) = X \otimes I + \int_0^t J_s^{(n)}(L^{(n)}(X))ds + \sum_{j=1}^n \int_0^t J_s^{(n)}(\delta_j(X))a_j^\dagger(ds)$$

$$+ \sum_{j=1}^n \int_0^t J_s^{(n)}(\delta_j^\dagger(X))a_j(ds)$$

with bounded structure maps and finite degree of freedom (see [9]). In fact, $J_t^{(n)}$ takes $\mathcal{A}_{\Delta_n}$ to $\mathcal{A}_{\Delta_n} \otimes \mathcal{B}(\Gamma_{sym})$, and for $X \in \mathcal{A}_{\Delta_n}$,

$$J_t(X) = J_t^{(m)}(X) = J_t^{(n)}(X), \forall m \geq n.$$

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