Product and Moment Formulas for Iterated Stochastic Integrals (associated with Lévy Processes)

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Abstract

In this paper, we obtain explicit product and moment formulas for products of iterated integrals generated by families of square integrable martingales associated with an arbitrary Lévy process. We propose a new approach applying the theory of compensated-covariation stable families of martingales. Our main tool is a representation formula for products of elements of a compensated-covariation stable family, which enables to consider Lévy processes, with both jumps and Gaussian part.

Keywords: Product and moment formulas, iterated integrals, compensated-covariation stable families, chaotic representation property, Lévy processes.

1 Introduction

If \(X\) and \(Y\) are square integrable martingales, the martingale

\[ C(X, Y)_t := [X, Y]_t - \langle X, Y \rangle_t, \quad t \geq 0, \]

is called the compensated-covariation process of \(X\) and \(Y\). Here \([X, Y]\) and \(\langle X, Y \rangle\) denote the quadratic covariation and the predictable quadratic covariation of \(X\) and \(Y\), respectively. A family \(\mathcal{X}\) of square integrable martingales is called compensated-covariation stable if \(C(X, Y) \in \mathcal{X}\) for all \(X, Y \in \mathcal{X}\).

Compensated-covariation stability was introduced by Di Tella and Engelbert in \([3]\) to investigate the predictable representation property (PRP) of families of martingales. Di Tella and Engelbert further exploited this property in \([2]\) to construct families of martingales possessing the chaotic representation property (CRP).

Let \(\mathcal{X} := \{X^\alpha, \alpha \in \Lambda\}\) be a family of square integrable, quasi-left continuous martingales, where \(\Lambda\) is an arbitrary index set. In \([3]\) the following recursive representation formula for products of elements from \(\mathcal{X}\) was shown, provided that \(\mathcal{X}\) is compensated-covariation stable:

\[
\prod_{i=1}^N X^{\alpha_i} = \sum_{1 \leq j_1 < \cdots < j_i \leq N} \left( \prod_{k=1}^N X^{\alpha_k} \right) \cdot X^{\alpha_{j_1}, \ldots, \alpha_{j_i}} \\
+ \sum_{2 \leq 1 \leq j_1 < \cdots < j_i \leq N} \left( \prod_{k=1}^N X^{\alpha_k} \right) \cdot \langle X^{\alpha_{j_1}, \ldots, \alpha_{j_i-1}}, X^{\alpha_{j_i}} \rangle,
\]

(1)

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where \( X^{\alpha_1, \alpha_2} := C(X^{\alpha_1}, X^{\alpha_2}) \) and \( X^{\alpha_1, \ldots, \alpha_i} := C(X^{\alpha_1}, \ldots, X^{\alpha_{i-1}}, X^{\alpha_i}) \).

Relation (1) was the cornerstone to obtain sufficient conditions on \( \mathcal{X} \) for the PRP and the CRP. It will also be the starting point in the present paper to derive a representation formula for products of iterated integrals.

In the first part of this work, we show that the compensated-covariation stability of the family \( \mathcal{X} \), which is required for (1), transfers to the family \( \mathcal{J}_e \) of the elementary iterated integrals generated by \( \mathcal{X} \) (see Definition 3.4 for \( \mathcal{J}_e \)). This straightforwardly leads to a version of the recursive representation formula (1) for products of elements from \( \mathcal{J}_e \) (see Theorem 3.15 below). This recursive representation formula turns out to be crucial for deriving our product and moment formulas. Let \( X \) be a Lévy process such that \( E[|X|^N] < +\infty, N \in \mathbb{N} \). As a by-product of Theorem 3.15, we get in Theorem 3.19 below a recursive representation formula for \( E[X_t^N] \) in terms of lower moments of \( X_t \). This formula seems also to be new and of independent interest.

In the second part of the paper (which is Section 4), using the general theory for compensated-covariation stable families developed in the first part as a tool, we obtain product and moment formulas for the iterated integrals generated by families of martingales associated with a Lévy process. The main idea is to exploit the recursive formula obtained in Theorem 3.15 until we explicitly solve the recursion.

Let \( X \) be a Lévy process with characteristic triplet \((\gamma, \sigma^2, \nu)\) and let \( \mu \) be the measure defined by \( \mu := \sigma^2 \delta_0 + \nu \), where \( \delta_0 \) is the Dirac measure concentrated in zero and \( \nu \) is the Lévy measure of \( X \). With any system of functions \( \Lambda \subseteq L^2(\mu) \) we associate a family \( \mathcal{X}_\Lambda = \{X^{\alpha}, \alpha \in \Lambda\} \) of square integrable martingales setting

\[
X_t^{\alpha} := \alpha(0)\sigma W_t + \int_{[0,t] \times \mathbb{R}} \alpha(x)\tilde{N}(ds, dx),
\]

where \( W \) is the Brownian motion and \( \tilde{N} \) the compensated Poisson random measure appearing in the Lévy-Itô decomposition of \( X \). We recall that, if \( \Lambda \) is a total system (i.e., the linear hull is dense), then the family \( \mathcal{X}_\Lambda \) possesses the CRP (see [2, Theorem 6.6]). As a first step, we show the product formula for the elementary iterated integrals \( \mathcal{J}_e \) generated by \( \mathcal{X}_\Lambda \), provided that \( \mathcal{J}_e \) is compensated-covariation stable (cf. Theorem 4.4 below). From this product formula the moment formula (44) is obtained. In Example 4.6 we illustrate formula (44) in some special cases.

In Theorem 4.10 below, which is the main result of this paper, we extend the product and the moment formula to (non-necessarily elementary) iterated integrals generated by any family \( \mathcal{X}_\Lambda \) satisfying \( \Lambda \subseteq \bigcap_{p \geq 2} L^p(\mu) \). Especially, \( \mathcal{X}_\Lambda \) does not need to be compensated-covariation stable. This extension is important because, to ensure the compensated-covariation stability of \( \mathcal{X}_\Lambda \), it is necessary to require that \( \Lambda \subseteq L^2(\mu) \) is stable under multiplication. However, in some important situations, as in the case of an orthonormal basis of \( L^2(\mu) \), the system \( \Lambda \) may fail to be stable under multiplication.

Thanks to Theorem 4.10 for any Lévy process \( X \), we have at our disposal a rich variety of families of martingales \( \mathcal{X}_\Lambda \) that, according to [2], possess the CRP, and the product and moment formulas hold for the iterated integrals generated by \( \mathcal{X}_\Lambda \).

We now give an overview of results about product and moment formulas which are available in the literature.

Russo and Vallois generalize in [18] the well-known product formula of two iterated integrals generated by a Brownian motion (see [15]) to a version where the iterated integrals are generated by a normal martingale. To define the iterated integrals Russo and Vallois follow the approach of Meyer [14].

In [13], Lee and Shih introduced the multiple integrals associated with a Lévy process following Itô [7] and then they obtained the product formula of two multiple integrals.
We recall that, for Lévy processes, there is a direct relation between iterated integrals, as defined in §3.1 below, and the multiple integrals introduced by Itô in [7] (see [2, Proposition 6.9] and [22, Proposition 7]).

The relation between moments, cumulants, iterated integrals and orthogonal polynomials was studied by Solé and Utzet in [20] and [21] for the case of Teugels martingales and for Lévy processes possessing moments of arbitrary order or an exponential moment.

Peccati and Taqqu gave in [17] product formulas for iterated integrals generated by a Brownian motion. In this case, the basic tool is the hypercontractivity of the iterated integrals. However, hypercontractivity is typical for the Brownian case and it does not hold, for instance, for iterated integrals generated by a compensated Poisson process. This is the reason why for Lévy processes, in general, additional integrability conditions on the integrands are required if product or moment formulas are considered.

In Peccati and Taqqu [17] also moments and cumulants of products for multiple integrals generated by a Poisson random measure have been investigated. This was done on the basis of the Mecke formula (see Mecke [12]) and diagram formulas. For similar results we refer also to Surgailis [23].

In Last et al. [11], again on the basis of the Mecke formula and of diagram formulas, more general product and moment formulas for multiple iterated integrals than those in [17] and [23] have been obtained. We stress that all these results cannot be applied for Lévy processes with a Gaussian part, because in this case the Mecke formula is not applicable.

First results about the product and the moment formulas for Lévy processes with non-vanishing Gaussian part, were obtained by Geiss and Labart in [5, 6], where iterated integrals generated by a simple Lévy process (i.e., the sum of a Brownian motion and of a compensated Poisson process) are considered. The moment formula in Theorem 4.10 below generalizes [5, 6] to arbitrary Lévy processes.

This paper has the following structure: We summarise the necessary background in Section 2. In Section 3 we study the properties of the elementary iterated integrals generated by a compensated-covariation stable family of martingales. In Section 4 we obtain product and moment formulas for the iterated integrals generated by families of martingales associated with a Lévy process. We then conclude giving concrete examples of these families, including Teugels martingales.

2 Preliminary Notions

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space and let \(\mathcal{F}\) be a filtration satisfying the usual conditions. We shall always consider real-valued stochastic processes on a finite time horizon \([0, T]\), \(T > 0\), and assume that \(\mathcal{F} = \mathcal{F}_T\).

Let \(\mathcal{X}\) be a family of processes. By \(\mathcal{F}^{\mathcal{X}}\) we denote the smallest filtration satisfying the usual conditions such that \(\mathcal{X}\) is an adapted family. If \(\mathcal{X} = \{X\}\) we write \(\mathcal{F}^{\mathcal{X}} = \mathcal{F}^X\).

We say that a process \(X\) has a finite moment of order \(N\) if \(E[|X_t|^N] < +\infty, t \in [0, T]\). If this holds for every \(N \in \mathbb{N}\), we say that \(X\) has finite moments of every order.

For a càdlàg process \(X\), we denote by \(\Delta X := X - X_\cdot\) the jump process of \(X\), with \(X_t := \lim_{s \uparrow t} X_s\) for \(t > 0\), and, by convention, \(X_0 := X_0\) so that \(\Delta X_0 = 0\).

In the present paper, \(\mathcal{F}\)-martingales are always assumed to be càdlàg and starting at zero.

A martingale \(X\) belongs to \(\mathcal{H}^p\) if \(\|X\|^p_p := E[|X_T|^p] < +\infty, p \geq 1\), and \((\mathcal{H}^p, \|\cdot\|_p)\) is a Banach space for every \(p \geq 1\) and a Hilbert space for \(p = 2\). We sometimes write \(\mathcal{H}^p(\mathcal{F})\) to specify the filtration.

If \(X, Y \in \mathcal{H}^2\), then there exists a unique predictable process of integrable variation, denoted by
\( \langle X, Y \rangle \) and called the predictable covariation of \( X \) and \( Y \), such that \( XY - \langle X, Y \rangle \in \mathcal{H}^1 \) and \( \langle X, Y \rangle_0 = 0 \).

For \( X \in \mathcal{H}^2 \), the process \( \langle X, X \rangle \) has a continuous version if and only if \( X \) is a quasi-left continuous martingale (cf. [9, Theorem I.4.2]), that is, \( \Delta X_s = 0 \) for every predictable stopping time \( \tau \).

With two semimartingales \( X \) and \( Y \), we associate the process \([X, Y]\), called covariation of \( X \) and \( Y \), defining

\[
[X, Y]_t := \langle X^c, Y^c \rangle_t + \sum_{0 \leq s \leq t} \Delta X_s \Delta Y_s, \quad t \in [0, T],
\]

where \( X^c \) and \( Y^c \) denote the continuous martingale part of \( X \) and \( Y \) respectively. For \( X, Y \in \mathcal{H}^2 \), the process \([X, Y]\) is of integrable variation and \( [X, Y] - \langle X, Y \rangle \in \mathcal{H}^1 \), that is \( \langle X, Y \rangle \) is the predictable compensator of \([X, Y]\) ([9, Proposition I.4.50 b]).

We recall the definition of the stochastic integral with respect to a martingale \( X \in \mathcal{H}^2 \). The space of integrands for \( X \) is given by \( L^2(X) := \{ H \text{ predictable} : \mathbb{E}[H^2 \cdot \langle X, X \rangle_T] < +\infty \} \), where we denote \( H^2 \cdot \langle X, X \rangle_T := \int_0^T H_s^2 d\langle X, X \rangle_s \). For \( X \in \mathcal{H}^2 \) and \( H \in L^2(X) \) we denote by \( H \cdot X \) or \( \int_0^T H_s dX_s \) the stochastic integral of \( H \) with respect to \( X \), characterized as follows: Let \( Z \in \mathcal{H}^2 \). Then \( Z = H \cdot X \) if and only if \( \langle Z, Y \rangle = H \cdot \langle X, Y \rangle \), for every \( Y \in \mathcal{H}^2 \).

Let \( \Lambda \) be an arbitrary parameter set and \( n \leq m \) natural numbers. By \( \alpha_{n,m} \) we denote the ordered \((m-n+1)\)-dimensional tuple \((\alpha_n, \ldots, \alpha_m) \in \Lambda^{m-n+1} \). For \( \alpha_{n_1:m_1} \in \Lambda^{m_1-n_1+1} \) and \( \beta_{n_2:m_2} \in \Lambda^{m_2-n_2+1} \), we denote by \( \alpha_{n_1:n_2}, \beta_{n_2:m_2} \) the \((m_1-n_1+1)+(m_2-n_2+1)\)-dimensional tuple obtained by continuing with \( \beta_{n_2:m_2} \) after \( \alpha_{n_1:m_1} \), that is, \( \alpha_{n_1:n_2}, \beta_{n_2:m_2} := (\alpha_n, \ldots, \alpha_{m_1}, \beta_{n_2}, \ldots, \beta_{m_2}) \).

We use the following notation: For any measure \( \rho \) and any function \( f \in L^1(\rho) \), we denote by \( \rho(f) \) the integral of \( f \) with respect to \( \rho \), that is, \( \rho(f) := \int f(x) \rho(dx) \).

## 3 Iterated integrals and compensated-covariation stability

In this section we introduce compensated-covariation stable families of martingales and iterated integrals generated by such families. We show that if \( \mathcal{X} \subseteq \mathcal{H}^2 \) is a compensated-covariation stable family, then the family of elementary iterated integrals generated by \( \mathcal{X} \) is compensated-covariation stable as well. Using this property, we deduce a formula to represent products and moments of iterated integrals.

### 3.1 Iterated integrals

To begin with, we define elementary iterated integrals generated by a finite family of martingales. For \( X^\alpha \in \mathcal{H}^2 \) with deterministic point brackets \( \langle X^\alpha, X^\alpha \rangle \), we denote

\[
\rho^{\alpha}(dr) := d\langle X^\alpha, X^\alpha \rangle_t, \quad t \in [0, T].
\]

Let now \( m \in \mathbb{N} \) be given and \( X^{\alpha_1}, \ldots, X^{\alpha_m} \in \mathcal{H}^2 \) be such that the predictable covariation \( \langle X^{\alpha_j}, X^{\alpha_k} \rangle \) is a deterministic function for \( j, k = 1, \ldots, m \). Then we denote by \( \rho^{\alpha_1:m} \) the product measure

\[
\rho^{\alpha_1:m} := \rho^{\alpha_1} \otimes \cdots \otimes \rho^{\alpha_m}
\]

on \([0, T]^m, \mathcal{B}([0, T]^m)\).

**Definition 3.1.** (i) The space of bounded measurable functions on \([0, T]\) will be denoted by \( \mathbb{B}_T \). For \( m \geq 1 \) we introduce the tensor product

\[
\mathbb{B}_T^m := \{ F_m = F_1 \otimes \cdots \otimes F_m : F_1, \ldots, F_m \in \mathbb{B}_T \}.
\]
and call the elements $F_{\otimes m}$ elementary functions of order $m$. For $m = 0$ put $F_{\otimes 0} := 1$.

(ii) Let $m \geq 0$ and the martingales $X^{\alpha_1}, \ldots, X^{\alpha_m} \in \mathcal{H}^2$ be fixed. For all $0 \leq n \leq m$, the $n$-fold elementary iterated integral of $F_{\otimes n} \in \mathbb{B}_{\otimes n}^m$ with respect to the martingales $(X^{\alpha_1}, X^{\alpha_2}, \ldots, X^{\alpha_m})$ is defined inductively by letting $J_0 : \mathbb{R} \to \mathbb{R}$ be the identical map and

$$J_{n+1}^\alpha(F_{\otimes n})_t := \int_0^t J_{n}^{\alpha_{n+1}}(F_{\otimes n})_{u-} F_n(u) \, dX^\alpha_u, \quad t \in [0, T]. \quad (4)$$

The following properties of elementary iterated integrals are shown in [8, Lemma 3.2].

**Proposition 3.2.** (i) Let $m \geq 1, X^{\alpha_1}, \ldots, X^{\alpha_m} \in \mathcal{H}^2$ such that $\langle X^{\alpha_j}, X^{\alpha_k} \rangle$ is a deterministic function, $j, k = 1, \ldots, m$. Then we have $J_{m-n}^\alpha(F_{\otimes n}) \in \mathcal{H}^2$ for $m \geq 1$.

(ii) Let $n \geq 1, X^{\beta_1}, \ldots, X^{\beta_n} \in \mathcal{H}^2$ such that $\langle X^{\beta_j}, X^{\beta_k} \rangle$ is a deterministic function, $j, k = 1, \ldots, n$. Then for any $F_{\otimes m} \in \mathbb{B}_{\otimes m}^m$ and $G_{\otimes n} \in \mathbb{B}_{\otimes n}^n$ it holds $\mathbb{E}[J_{m-n}^\alpha(F_{\otimes n}), J_{m-n}^\beta(G_{\otimes n})] = 0$, if $m \neq n$, while, if $m = n$,

$$\mathbb{E}[J_{m}^\alpha(F_{\otimes m}), J_{m}^\beta(G_{\otimes m})] = \int_0^T \cdots \int_0^{t_{n-1}} \int_0^{t_{n-2}} \cdots \int_0^{t_1} F_1(t_1) \cdots F_m(t_m) \, d\langle X^{\alpha_1}, X^{\beta_1} \rangle_1 \cdots d\langle X^{\alpha_n}, X^{\beta_n} \rangle_{n-1}. \quad (5)$$

The next lemma concerns some properties of elementary iterated integrals, which will be useful in this paper.

**Lemma 3.3.** Let $X^{\alpha_j} \in \mathcal{H}^2$ have moments of every order for $j = 1, \ldots, m$.

(i) For every $F_{\otimes m} \in \mathbb{B}_{\otimes m}^m$ and every $p > 0$ we have the estimate

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \left| J_{m}^{\alpha}(F_{\otimes m})_t \right|^p \right] < \infty. \quad (6)$$

(ii) Assume that $p \geq 1$. Let $X^{\alpha_1}, \ldots, X^{\alpha_m} \in \mathcal{H}^2$ be such that

1. $\langle X^{\alpha_j}, X^{\alpha_k} \rangle$ is a deterministic function for every $j, k = 1, \ldots, m$ and $n \geq 1$;
2. each $X^{\alpha_j}$ has finite moments of every order;
3. $X^{\alpha_j} \to X^\alpha$ in $\mathcal{H}^{2p}$, as $n \to +\infty$.

Then, for every $F_{\otimes m} \in \mathbb{B}_{\otimes m}^m$, we have

$$\lim_{n \to +\infty} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| J_{m}^{\alpha}(F_{\otimes m})_t - J_{m}^{\alpha}(F_{\otimes m})_t \right|^p \right] = 0.$$

**Proof.** We prove both the statements by induction on $m$. We start proving (i). We only consider the case $p \geq 1$, since the case $0 < p < 1$ immediately follows by the case $p = 1$ and Jensen’s inequality. If $m = 0$ there is nothing to prove. We now assume that (6) holds for an arbitrary $p \geq 1$ and every $j \leq m$, and we show it for $m + 1$. By Burkholder–Davis–Gundy’s inequality, from now on BDG’s inequality, (see [8], Theorem 2.34)

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \left| J_{m+1}^{\alpha}(F_{\otimes m+1})_t \right|^p \right] \leq C_p \mathbb{E} \left[ \left( J_{m+1}^{\alpha}(F_{\otimes m+1})_t \cdot J_{m+1}^{\alpha}(F_{\otimes m+1})_t \right)^{p/2} \right]$$

$$= C_p \mathbb{E} \left[ \left( J_{m+1}^{\alpha}(F_{\otimes m})_t^2 \cdot J_{m+1}^{\alpha}(F_{\otimes m})_t \right)^{p/2} \right]$$

$$\leq C_p \mathbb{E} \left[ \sup_{t \in [0, T]} \left| J_{m+1}^{\alpha}(F_{\otimes m})_t \right|^{2p} \right]^{1/2} \mathbb{E} \left[ \left| J_{m+1}^{\alpha}(F_{\otimes m})_t \right|^{p} \right]^{1/2}.$$
where \( c_{m+1} = \sup_{t \in [0,T]} |F_{m+1}(t)| < +\infty \). Notice that

\[
\mathbb{E}\left[ |X_{T}^{\alpha_{m+1}}, X_{T}^{\alpha_{m+1}}|^{p} \right] \leq c_{(2p)} \mathbb{E}[|X_{T}^{\alpha_{m+1}}|^{2p}] < \infty,
\]

where the constant \( c_{(2p)} \) arises from the use of BDG’s inequality and Doob’s martingale inequality. Hence, the right hand side of the above estimate is finite by the induction hypothesis. The proof of (i) is complete. Concerning (ii) we observe that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |J_{m}^{\alpha_{m}}(F_{\otimes m})_{t} - J_{m}^{\alpha_{m}}(F_{\otimes m})_{t}|^{p} \right] \leq 2^{p-1} \left\{ \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_{0}^{t} \left( J_{m-1}^{\alpha_{m-1}}(F_{\otimes m-1})_{u} - J_{m-1}^{\alpha_{m-1}}(F_{\otimes m-1})_{u} \right) F_{m}(u) dX_{u}^{\alpha_{m}} \right|^{p} \right] + \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_{0}^{t} J_{m-1}^{\alpha_{m-1}}(F_{\otimes m-1})_{u} F_{m}(u) d\left( X_{u}^{\alpha_{m}} - X_{u}^{\alpha_{m}} \right) \right|^{p} \right] \right\} \leq 2^{p-1} C_{p} c_{m} \left\{ \mathbb{E} \left[ \sup_{t \in [0,T]} \left( J_{m-1}^{\alpha_{m-1}}(F_{\otimes m-1})_{t} - J_{m-1}^{\alpha_{m-1}}(F_{\otimes m-1})_{t} \right)^{2p} \right]^{1/2} \right\}^{1/2} \mathbb{E} \left[ \left( X_{T}^{\alpha_{m}} - X_{T}^{\alpha_{m}} \right)^{2p} \right]^{1/2},
\]

where we similarly as above used BDG’s inequality and H"older’s inequality. Because of the convergence assumptions, we see that \( \mathbb{E} \left[ |X_{T}^{\alpha_{m}} - X_{T}^{\alpha_{m}}|^{p} \right] \) is bounded in \( n \) and \( \mathbb{E} \left[ \left( X_{T}^{\alpha_{m}} - X_{T}^{\alpha_{m}} \right)^{2p} \right] \) converges to zero, as \( n \to +\infty \). Furthermore, because for \( m = 1 \) we have

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| J_{1}^{\alpha_{1}}(F_{1})_{t} - J_{1}^{\alpha_{1}}(F_{1})_{t} \right|^{2p} \right] \to 0, \quad n \to +\infty,
\]

we can assume, for every \( k \leq m - 1 \),

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| J_{k}^{\alpha_{k}}(F_{\otimes k})_{t} - J_{k}^{\alpha_{k}}(F_{\otimes k})_{t} \right|^{2p} \right] \to 0, \quad n \to +\infty.
\]

Hence, (ii) follows by induction and the proof of the lemma is complete. \( \square \)

As a next step we introduce some linear spaces of elementary iterated integrals and the definition of the chaotic representation property.

**Definition 3.4.** Let \( \mathcal{H} := \{X^{\alpha}, \alpha \in \Lambda\} \subseteq \mathcal{H}^{2} \), where \( \Lambda \) is an arbitrary parameter set. We assume that \( \langle X^{\alpha}, X^{\beta} \rangle \) is a deterministic function, for each \( \alpha, \beta \in \Lambda \).

(i) Let \( \mathcal{J}_{0} := \mathbb{R} \) be the space of 0-fold elementary iterated integrals.

(ii) Let \( m \geq 1 \) and \( \alpha_{1}, \ldots, \alpha_{m} \in \Lambda \). By \( \mathcal{J}_{m}^{\alpha_{m}} \) we denote the linear hull of all \( J_{m}^{\alpha_{m}}(F_{\otimes m}) \) with respect to \( (X_{m}^{\alpha_{m}}, X_{m}^{\alpha_{m}}, \ldots, X_{m}^{\alpha_{m}}) \) from \( \mathcal{H}^{2} \) and \( F_{\otimes m} \in \mathbb{F}^{m}_{T} \).

(iii) For all \( m \geq 1 \), we introduce

\[
\mathcal{J}_{m} := \text{Span} \left( \bigcup_{\alpha_{m} \in \Lambda^{m}} \mathcal{J}_{m}^{\alpha_{m}} \right), \quad \mathcal{J}_{c} := \text{Span} \left( \bigcup_{m \geq 0} \mathcal{J}_{m} \right),
\]

We call \( \mathcal{J}_{c} \) the space of elementary iterated integrals generated by \( \mathcal{H} \).

(iv) If \( \mathcal{J}_{c} \) is dense in \( (\mathcal{H}^{2}(\mathbb{F}), \| \cdot \|_{2}) \), we say that \( \mathcal{H} \) possesses the chaotic representation property (CRP) with respect to \( \mathbb{F} \).
Remark 3.5. We now briefly recall an alternative formulation of the CRP. For details we refer to [2, Proposition 3.7 and Theorem 3.11].

(i) From Proposition 3.2 (ii), we deduce the equivalent representation \( \mathcal{J}_e = \bigoplus_{m \geq 0} \mathcal{J}_m \). Denoting now by \( \mathcal{J} \) and \( \mathcal{J}_m \) respectively the closure of \( \mathcal{J}_e \) and \( \mathcal{J}_m \) in \( (\mathcal{H}^2, \| \cdot \|_2) \), we see that \( \mathcal{J} = \bigoplus_{m \geq 0} \mathcal{J}_m \), and the \( \mathcal{J} \) has the CRP if and only if \( \mathcal{H}^2(\mathbb{F}) = \bigoplus_{m \geq 0} \mathcal{J}_m \).

(ii) If furthermore \( \mathcal{J} \) consists of countably many martingales (that is, \( \Lambda \) is a countable index set) which are pairwise orthogonal, we can write

\[
\mathcal{J} = \bigoplus_{m \geq 0} \bigoplus_{\alpha \in \Lambda_m} \mathcal{J}^{\alpha \, m}_m,
\]

\( \mathcal{J}^{\alpha \, m}_m \) denoting the closure of \( \mathcal{J}^{\alpha \, m}_m \) in \( (\mathcal{H}^2, \| \cdot \|_2) \).

We conclude this section introducing the iterated integrals as an isometric extension of the elementary iterated integrals. For \( t \in [0, T] \) and \( m \geq 1 \), we introduce the set

\[
M^m_t := \{(t_1, \ldots, t_m) : 0 \leq t_1 < \ldots < t_m < t\}.
\]

We recall the definition of \( \rho^\alpha \) and \( \rho^{\alpha \, m} \) given in (2) and (3), respectively. For \( m \geq 1 \), we denote by \( \mathcal{E}^{\rho \, m} \) the linear subspace of \( L^2(M^m_t, \rho^{\alpha \, m}) \) generated by \( F_{\bigotimes_m} \in \mathbb{B}^{\bigotimes_m}_T \) restricted to \( M^m_t \). Notice that \( \mathcal{E}^{\rho \, m} \) is dense in \( L^2(M^m_t, \rho^{\alpha \, m}) \). From (5) we have the isometry relation

\[
\| J^{\alpha \, m}_m(F) \|_{L^2(\mathbb{P})} = \| F \|_{L^2(M^m_t, \rho^{\alpha \, m})}
\]

between \( \{ F := F_{\bigotimes_m} 1_{M^m_t} : F_{\bigotimes_m} \in \mathbb{B}^{\bigotimes_m}_T \} \) and \( L^2(\mathbb{P}) \). We linearly extend \( J^{\alpha \, m}_m(\cdot) \) to \( \mathcal{E}^{\rho \, m} \), and by continuity, to \( L^2(M^m_t, \rho^{\alpha \, m}) \), denoting this extension again by \( J^{\alpha \, m}_m(\cdot) \). Recall that \( J_0 \) denotes the identity map on \( \mathbb{R} \).

Definition 3.6. We call the mapping \( J^{\alpha \, m}_m(F), F \in L^2(M^m_T, \rho^{\alpha \, m}) \), defined above an iterated integral with respect to \( X^{\alpha_1}, \ldots, X^{\alpha_m} \).

In the next proposition we summarize the properties of iterated integrals.

Proposition 3.7. Let \( m \geq 1 \), \( \alpha_1, \ldots, \alpha_m \in \Lambda \), \( F \in L^2(M^m_T, \rho^{\alpha \, m}) \) and let \( X^{\alpha_1}, \ldots, X^{\alpha_m} \in \mathcal{H}^2 \) be such that \( \langle X^{\alpha_j}, X^{\alpha_k} \rangle \) is a deterministic function for \( j, k = 1, \ldots, m \).

(i) \( J^{\alpha \, m}_m(F_{\otimes m}) \) belongs to \( \mathcal{H}^2 \) and it is quasi-left continuous.

(ii) Let moreover \( n \geq 1 \), \( \beta_1, \ldots, \beta_n \in \Lambda \), \( G \in L^2(M^m_T, \rho^{\beta \, n}) \) and let \( X^{\beta_1}, \ldots, X^{\beta_n} \in \mathcal{H}^2 \) be such that \( \langle X^{\beta_j}, X^{\beta_k} \rangle \) is a deterministic function, \( j, k = 1, \ldots, n \). Then, for every \( t \in [0, T] \), we have: If \( m \neq n \), then \( E \left[ J^{\alpha \, m}_m(F) J^{\beta \, n}_n(G) \right] = 0 \), while, if \( m = n \),

\[
E \left[ J^{\alpha \, m}_m(F), J^{\beta \, n}_n(G) \right] = \int_0^t \int_0^{t_m} \cdots \int_0^{t_m} F(t_1, \ldots, t_m) G(t_1, \ldots, t_m) \, d\langle X^{\alpha_1}, X^{\beta_1} \rangle_{t_1} \cdots d\langle X^{\alpha_n}, X^{\beta_n} \rangle_{t_m}.
\]

3.2 Compensated-covariation stable families of martingales

Definition 3.8. Let \( \Lambda \) be an arbitrary parameter set and \( \mathcal{X} := \{ X^\alpha, \alpha \in \Lambda \} \subseteq \mathcal{H}^2 \).

(i) For every \( \alpha_1, \alpha_2 \in \Lambda \) we define the process

\[
X^{\alpha_1 \, 2} := [X^{\alpha_1}, X^{\alpha_2}] - \langle X^{\alpha_1}, X^{\alpha_2} \rangle
\]
which we call the compensated-covariation process of $X^{\alpha_1}$ and $X^{\alpha_2}$.

(ii) The family $\mathcal{K}$ is called compensated-covariation stable if for every $\alpha_1, \alpha_2 \in \Lambda$ it holds that $X^{\alpha_1 \omega} \in \mathcal{K}$.

(iii) Let $\mathcal{K}$ be a compensated-covariation stable family and let $\alpha_1, \ldots, \alpha_m \in \Lambda$ with $m \geq 2$. The process $X^{\alpha_{1:m}}$ is defined recursively by

$$
X^{\alpha_{1:m}} := [X^{\alpha_{1:m-1}}, X^{\alpha_m}] - \langle X^{\alpha_{1:m-1}}, X^{\alpha_m} \rangle
$$

and called $m$-fold compensated-covariation of the ordered $m$-tuple of martingales $(X^{\alpha_1}, \ldots, X^{\alpha_m})$. We sometimes use also the notation $X^{\alpha_1, \ldots, \alpha_m}$ instead of $X^{\alpha_{1:m}}$.

Notice that if the family $\mathcal{K}$ is compensated-covariation stable, then $X^{\alpha_{1:m}} \in \mathcal{K}$ holds, for every $\alpha_1, \ldots, \alpha_m$ in $\Lambda$ and $m \geq 2$.

As a toy-example of a compensated-covariation stable family consider the family $\mathcal{K} := \{X\}$, where $X_t = N_t - \lambda t$, $t \geq 0$, and $N$ is a homogeneous Poisson process with parameter $\lambda$. In this case we have $\langle X, X \rangle_t = \lambda t$ and

$$
[X, X]_t - \langle X, X \rangle_t = \sum_{s \leq t} (\Delta N_s)^2 - \lambda t = \sum_{s \leq t} \Delta N_s - \lambda t = X_t.
$$

More generally, let $N$ be a simple point process, i.e., $N$ is a càdlàg adapted increasing process taking value in the set $\mathbb{N}$ of natural numbers such that $N_0 = 0$ and $\Delta N \in \{0, 1\}$. Let $NP$ be the predictable compensator of $N$ and assume that $NP$ is continuous (i.e., that $N$ is quasi-left continuous). Then the family $\mathcal{K} := \{X\} \subseteq \mathcal{H}^2_{\text{loc}}$, where $X := N - NP$, is compensated-covariation stable.

Next we state a representation formula for products of martingales from $\mathcal{H}^2$. Thanks to the quasi-left continuity which we assume here, the representation formula is a simpler version of the one shown in [3, Proposition 3.3].

**Proposition 3.9.** Let $\mathcal{K} := \{X^{\alpha}, \alpha \in \Lambda\}$ be a compensated-covariation stable family of quasi-left continuous martingales in $\mathcal{H}^2$. For every $N \geq 1$ and $\alpha_1, \ldots, \alpha_N \in \Lambda$, we have

$$
\prod_{i=1}^{N} X^{\alpha_i} = \sum_{i=1}^{N} \sum_{1 \leq j_1 < \cdots < j_i \leq N} \left( \prod_{k=1}^{N} X^{\alpha_j}_{t_k} \right) \cdot X^{\alpha_{j_1} \ldots \alpha_{j_i}} + \sum_{i=2}^{N} \sum_{1 \leq j_1 < \cdots < j_i \leq N} \left( \prod_{k=1}^{N} X^{\alpha_j}_{t_k} \right) \cdot \langle X^{\alpha_{j_1} \ldots \alpha_{j_i}} \rangle,
$$

where $X^{\alpha_{j_1} \ldots \alpha_{j_i}}$ denotes the left-hand side of the above expression.

**Remark 3.10.** For compensated-covariation stable families with deterministic point brackets, sufficient conditions for the CRP (see Definition [3.4](iv)) are known. We briefly recall these results, for an extensive study of which we refer to [2].

The set $\mathcal{K}$ of polynomials generated by $\mathcal{K}$ is defined as the linear hull of products of elements of $\mathcal{K}$ taken at different deterministic times. Clearly, the completed $\sigma$-algebra generated by $\mathcal{K}$ coincides with $\mathcal{K}^\mathcal{F}$. In [2, Theorem 5.8], the following result has been established:

The family $\mathcal{K} \subseteq \mathcal{H}^2(\mathcal{F}^\mathcal{F})$ possesses the CRP with respect to $\mathcal{F}^\mathcal{F}$ provided that

(i) $\mathcal{K}$ is compensated-covariation stable;

(ii) $(X^{\alpha_1}, X^{\alpha_2})$ is deterministic, for every $\alpha_1, \alpha_2 \in \Lambda$;

(iii) the family $\mathcal{K}$ of polynomials generated by $\mathcal{K}$ is dense in $L^2(\Omega, \mathcal{F}^\mathcal{F}, \mathbb{P})$.

We recall that to ensure (iii), it is sufficient that for every $X \in \mathcal{K}$, the random variable $|X_t|$ possesses an (arbitrarily small) exponential moment for every $t \in [0, T]$: That is, there exist $c_t > 0$ such that $\mathbb{E}[\exp(\frac{c_t}{X_t})] < +\infty$, for every $t \in [0, T]$. For an elementary proof of this well-known result see, for example, [3, Appendix A].
3.3 Representing powers of processes

As an application of Proposition 3.9 we show how to represent powers of a Brownian motion, of a homogeneous Poisson process and then, more generally, of a Lévy process with finite moments of every order.

**Brownian Motion.** Let $X$ be a standard Brownian motion. Then $\mathcal{F} = \{X\} \cup \{0\}$ is a compensated-covariation stable family. Furthermore $\langle X, X \rangle_t = t$ and $X^{\alpha_n} = 0$, for $m \geq 2$. Therefore (11) becomes

$$X_t^N = N \int_0^t X_s^{N-1} dX_s + \frac{N(N - 1)}{2} \int_0^t X_s^{N-2} ds,$$

which is in fact Itô’s formula applied to $X^N$.

**Homogeneous Poisson Process.** We now denote by $N$ be a homogeneous Poisson process with parameter $\lambda > 0$. Then, setting $X_t := N_t - \lambda t$, $\mathcal{F} = \{X\}$ is a compensated-covariation stable family. Furthermore, $X^{\alpha_n} = X$, for every $m \geq 1$. From (11) we deduce

$$X_t^N = \sum_{i=1}^N \binom{N}{i} \int_0^t X_s^{N-i} dX_s + \lambda \sum_{i=2}^N \binom{N}{i} \int_0^t X_s^{N-i} ds.$$

**Lévy Processes.** Let $\mathbb{F}$ be a filtration satisfying the usual conditions and let $X$ be a Lévy process relative to $\mathbb{F}$, that is: $X$ is càdlàg, stochastically continuous, $\mathbb{F}$-adapted, $X_{t+s} - X_t$ is independent from $\mathcal{F}_s$ and $X_{t+s} - X_t \sim X_t$, for all $s, t \geq 0$, and $X_0 = 0$. The Lévy-Itô decomposition of $X$ is given by

$$X_t = \gamma t + W^\sigma_t + \int_{[0,t] \times \{|x| > 1\}} xN(dx, dx) + \int_{[0,t] \times \{|x| \leq 1\}} x\tilde{N}(dx, dx),$$

where $\gamma \in \mathbb{R}$, $W^\sigma$ denotes a Wiener process relative to $\mathbb{F}$ with variance function $\mathbb{E}[W^\sigma_t^2] = \sigma^2 t$, and $N$ is the jump measure of $X$, which is a Poisson random measure relative to $\mathbb{F}$. By $\tilde{N} := N - \nu \otimes \lambda_+$ we denote the compensated Poisson random measure associated with $N$, where $\lambda_+$ denotes the Lebesgue measure on $[0, T]$ and $\nu$ the Lévy measure of $X$ (i.e., $\nu$ is a $\sigma$-finite measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $x \mapsto x^2 \land 1 \in L^1(\nu)$). We call $(\gamma, \sigma^2, \nu)$ the characteristic triplet of $X$.

We now assume that $X$ has finite moments of every order. This implies that, for $p_i(x) := x^i$, we have $p_i \in L^2(\nu)$ for all $i \geq 1$ and $p_i \in L^1(\nu)$ for all $i \geq 2$.

As in [16] or [20] we now define the power-jump processes of $X$ and the family of Teugels martingales, setting $L^{(i)} := X$ and

$$L^{(i)}_t = \sum_{s \leq t} (\Delta X_s)^i = \int_{[0,t] \times \mathbb{R}} p_i(x)N(dx, dx), \quad n \geq 2.$$

The process $L^{(i)}$ is called $i$-th power jump asset of $X$. It can be seen that $L^{(i)}$ is a Lévy process with Lévy measure given by the image measure $\nu^{(i)} = \nu \circ p_i^{-1}$, $i \geq 1$. Since $\nu^{(i)}(p_j^2) = \nu(p_{2j}) < +\infty$, for $j \geq 1$, $L^{(i)}$ has finite moments of every order, for every $i \geq 1$ and

$$\mathbb{E}[L^{(i)}_1] = \gamma + \nu(1_{\{|p_1| > 1\}}), \quad \mathbb{E}[L^{(i)}_1] = \nu(p_1), \quad i \geq 2.$$
Hence, $X_t^{(i)} := L_t^{(i)} - E[L_t^{(i)} | t]$ is a square integrable martingale for $i \geq 1$ and it is of finite variation for $i \geq 2$. Furthermore, the identities $\langle X^{(i)}, X^{(j)} \rangle_t = (\nu(p_{i+j}) + \sigma^2 1_{i=j=1}) t$ and

$$[X^{(i)}, X^{(j)}] - \langle X^{(i)}, X^{(j)} \rangle = X^{(i+j)}, \quad i, j \geq 1,$$

hold. This shows that the family $\mathcal{X} := \{X^{(i)}, i \geq 1\}$ is compensated-covariation stable. It is easy to recognize that the $i$-fold compensated-covariation process of the $i$-tuple of martingales $(X^{(1)}, \ldots, X^{(1)})$ equals $X^{(i)}$, for every $i \geq 2$. The family $\mathcal{X}$ is the family of Tegels martingales. Notice that the family $\mathcal{X}$ can be seen as the compensated-covariation stable hull of the process $X^{(1)}$.

As a consequence of Proposition 3.9 applied to the family $\mathcal{X}$, with $\alpha_1 = \ldots = \alpha_N = 1$, we can express the $N$-th power of $X^{(1)}$ as follows:

$$\left( X_t^{(1)} \right)^N = \sum_{i=1}^{N} \binom{N}{i} \int_0^t (X_{s-}^{(1)})^{N-i} dX_t^{(i)} + \sum_{i=2}^{N} \binom{N}{i} \int_0^t (X_{s-}^{(1)})^{N-i} d\langle X^{(i-1)}, X^{(1)} \rangle_s \tag{17}$$

In the same way one can obtain formulas for the power of each Tegels martingale $X^{(n)}, n \geq 2$:

$$\left( X_t^{(n)} \right)^N = \sum_{i=1}^{N} \binom{N}{i} \int_0^t (X_{s-}^{(n)})^{N-i} dX_s^{(n)} + \sum_{i=2}^{N} \binom{N}{i} \int_0^t (X_{s-}^{(n)})^{N-i} \nu(p_i) ds.$$  

### 3.4 Compensated-covariation stability of $\mathcal{J}_e$

In this subsection we show that, if $\mathcal{X} = \{X^\alpha, \alpha \in \Lambda \} \subseteq \mathcal{H}^2$ is compensated-covariation stable and $\langle X^\alpha, X^\beta \rangle$ is deterministic for every $\alpha, \beta \in \Lambda$, then $\mathcal{J}_e \subseteq \mathcal{H}^2$ and $\mathcal{J}_e$ is compensated-covariation stable. In this way we can apply (11) to compute products of elementary iterated integrals. Notice that, for $Y, X \in \mathcal{J}_e$, the process $\langle Y, X \rangle$ is not deterministic in general.

We shall often need the following assumption.

**Assumption 3.11.** (i) The family $\mathcal{X} := \{X^\alpha, \alpha \in \Lambda \} \subseteq \mathcal{H}^2$ is compensated-covariation stable.

(ii) $\mathcal{X}$ consists of quasi-left continuous martingales.

(iii) $\langle X^\alpha, X^\beta \rangle$ denotes the continuous version of the predictable covariation between $X^\alpha$ and $X^\beta$ and it is a deterministic function of time for every $\alpha, \beta \in \Lambda$.

The following lemma will be used to show that $\mathcal{J}_e$ is compensated-covariation stable. Recall that the notation $\mathbb{B}_T$ and $\mathbb{B}^{\otimes m}_T$ was introduced in Definition 3.11 (i).

**Lemma 3.12.** Let $\mathcal{X}$ satisfy Assumption 3.11. For $J_m^{\alpha_m}(F_{\otimes_m}), \mathcal{J}_n^{\beta_n}(G_{\otimes_n}) \in \mathcal{J}_e$ the processes $M$ and $N$ defined by

$$M_t := \int_0^t R_{m,n}^u H(u) dX^\gamma_u, \quad N_t := \int_0^t \int_0^t \sum_{s=0}^{\nu} R_{m,n}^{u,s} K(u) d\langle X^\delta, X^\eta \rangle_u H(s) dX^\gamma_s, \quad t \in [0, T],$$

with $R_{m,n} := J_m^{\alpha_m}(F_{\otimes_m}) \mathcal{J}_n^{\beta_n}(G_{\otimes_n})$, belong to $\mathbb{B}^{\otimes m+n+1}_T$, for all $\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_n; \gamma, \delta, \eta \in \Lambda$; for all $F_{\otimes_m} \in \mathbb{B}^{\otimes m}_T$ and $G_{\otimes_n} \in \mathbb{B}^{\otimes n}_T$; for all $H, K \in \mathbb{B}_T$ and every $m, n \in \mathbb{N}$.  

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Proof. For an arbitrary $m \in \mathbb{N}$, we prove the lemma for every $n \in \mathbb{N}$ by induction on $n$. If $n = 0$, then $R^{m,0} = J^{m}_{m-1}(F \otimes m)$ and so $M = J^{m-1}_{m-1}(F \otimes m \otimes H)$, which clearly belongs to the linear space $\bigoplus_{k=0}^{m} \mathcal{F}_k$. We show that $N$ belongs to $\bigoplus_{k=0}^{m+1} \mathcal{F}_k$ if $n = 0$. The function $\tilde{F}(t) := \int_{0}^{t} K(u)d\langle X^\delta, X^\eta \rangle_u$ is deterministic, bounded and, by continuity of $\langle X^\delta, X^\eta \rangle$, also continuous. By integration by parts and continuity of $\tilde{F}$, we get

$$\int_{0}^{t} R^{m,0}_{u-} K(u)d\langle X^\delta, X^\eta \rangle_u = R^{m,0}_{t-} F(t) - \int_{0}^{t} \tilde{F}(u)dR^{m,0}_{u}$$

$$= R^{m,0}_{t-} F(t) - \int_{0}^{t} J^{m-1}_{m-1}(F \otimes m)(u)\tilde{F}(u)dX^\alpha_m$$

$$= R^{m,0}_{t-} F(t) - J^{m-1}_{m-1}(F \otimes m \tilde{F} \otimes H)_{t}.$$  \hspace{1cm} (18)

Therefore

$$\int_{0}^{t} \int_{0}^{s-} R^{m,0}_{u-} K(u)d\langle X^\delta, X^\eta \rangle_u H(s)dX^\gamma_s$$

$$= \int_{0}^{t} R^{m,0}_{u-} \tilde{F}(s)H(s)dX^\gamma_s - J^{m+1}_{m+1}(F \otimes m \tilde{F} \otimes H)_{t}$$

$$= J^{m+1}_{m+1}(F \otimes m \tilde{F} \otimes H)_{t} - J^{m+1}_{m+1}(F \otimes m \tilde{F} \otimes H)_{t},$$  \hspace{1cm} (19)

which again belongs to $\bigoplus_{k=0}^{m+1} \mathcal{F}_k$ and this proves the basis of the induction. Now we assume that $M_t := \int_{0}^{t} R^{m,j}_{u-} H(u)dX^\gamma_u$, $N_t := \int_{0}^{t} \int_{0}^{s-} R^{m,j}_{u-} K(u)d\langle X^\delta, X^\eta \rangle_u H(s)dX^\gamma_s$

belong to $\bigoplus_{k=0}^{m+j+1} \mathcal{F}_k$, for every $j \leq n$, every $m$, every $H,F \in \mathbb{B}_T$ and every $\gamma, \delta, \eta \in \Lambda$ and prove the statement for $n+1$. By integration by parts we have

$$R^{m,n+1}_{t-} = \int_{0}^{t} R^{m-1,n+1}_{u-} F_m(u)dX^\alpha_u + \int_{0}^{t} R^{m,n}_{u-} G_{n+1}(u)dX^\beta_{n+1}$$

$$+ \int_{0}^{t} R^{m-1,n}_{u-} F_m(u)G_{n+1}(u)d\langle X^\alpha_u, X^\beta_{n+1} \rangle_u,$$  \hspace{1cm} (20)

$$= \int_{0}^{t} R^{m-1,n+1}_{u-} F_m(u)dX^\alpha_u + \int_{0}^{t} R^{m,n}_{u-} G_{n+1}(u)dX^\beta_{n+1}$$

$$+ \int_{0}^{t} R^{m-1,n}_{u-} F_m(u)G_{n+1}(u)dX^\alpha_u,\beta_{n+1},$$  \hspace{1cm} (21)

$$= \int_{0}^{t} R^{m-1,n+1}_{u-} F_m(u)dX^\alpha_u$$

$$+ \int_{0}^{t} R^{m,n}_{u-} G_{n+1}(u)dX^\beta_{n+1}$$

$$+ \int_{0}^{t} R^{m-1,n}_{u-} F_m(u)G_{n+1}(u)d\langle X^\alpha_u, X^\beta_{n+1} \rangle_u,$$  \hspace{1cm} (22)

where in (22) the compensated-covariation of $X^\alpha_u$ and $X^\beta_{n+1}$ appears. Because of the induction hypothesis, (21) belongs to $\bigoplus_{k=0}^{m+n+1} \mathcal{F}_k$. Therefore, $\int_{0}^{t} H(s)dX^\gamma_s$ belongs to $\bigoplus_{k=0}^{m+n+2} \mathcal{F}_k$. We now discuss (22). The family $\mathcal{X}$ is compensated covariation stable. Therefore, the process $X^\alpha_u,\beta_{n+1}$ belongs to $\mathcal{X}$. Obviously $F_mG_{n+1} \in \mathbb{B}_T$. Hence, by the induction hypothesis, (22) belongs to $\bigoplus_{k=0}^{m+n+1} \mathcal{F}_k$ and therefore $\int_{0}^{t} H(s)dX^\gamma_s$ belongs to $\bigoplus_{k=0}^{m+n+1} \mathcal{F}_k \subseteq \bigoplus_{k=0}^{m+n+2} \mathcal{F}_k$. From the induction hypothesis, it also immediately follows that $\int_{0}^{t} H(s)dX^\gamma_s$ belongs to $\bigoplus_{k=0}^{m+n+2} \mathcal{F}_k$. We now show that $\int_{0}^{t} H(s)dX^\gamma_s$ belongs to $\bigoplus_{k=0}^{m+n+2} \mathcal{F}_k$. For this it suffices to verify that (20) belongs to $\bigoplus_{k=0}^{m+n+1} \mathcal{F}_k$. By integration by parts applied to $R^{m-1,n+1}_{m-1}(F \otimes m)_{n+1}(G \otimes m)$ we
get
\[
\int_0^t R_{u-n+1}^{m-1} F_m(u) dX_u^{\alpha_m} = \int_0^t \int_0^s R_{u-n+1}^{m-1} F_m(u) dX_u^{\alpha_m} F_m(s) dX_s^{\alpha_m}
\]
\[
+ \int_0^t \int_0^s R_{u-n+1}^{m-1} G_n(u) dX_u^{\alpha_m} F_m(s) dX_s^{\alpha_m}
\]
\[
+ \int_0^t \int_0^s R_{u-n+1}^{m-1} F_m(u) dX_u^{\alpha_m} - F_m(s) dX_s^{\alpha_m}
\]
\[
+ \int_0^t \int_0^s R_{u-n+1}^{m-1} G_n(u) dX_u^{\alpha_m} - F_m(s) dX_s^{\alpha_m}.
\]

In other words, because of the induction hypothesis, we can rewrite this expression as
\[
\int_0^t R_{u-n+1}^{m-1} F_m(u) dX_u^{\alpha_m} = \int_0^t \int_0^s R_{u-n+1}^{m-1} F_m(u) dX_u^{\alpha_m} F_m(s) dX_s^{\alpha_m} + V_t
\]
where \( V \in \bigoplus_{k=0}^{m+n+1} J_k \). Iterating this procedure \( m-1 \) times, we get
\[
\int_0^t R_{u-n+1}^{m-1} F_m(u) dX_u^{\alpha_m} = U_t + \int_0^t \int_0^s \cdots \int_0^s \beta_{n+1}^{\alpha_1} (G_{\otimes_{n+1}})_{t_1} - F_1(t_1) dX_{t_1}^{\alpha_1} \cdots F_{m-1}(t_{m-1}) dX_{t_{m-1}}^{\alpha_m} - F_m(t_m) dX_{t_{m}}^{\alpha_m},
\]
where \( U \in \bigoplus_{k=0}^{m+n+1} J_k \). Using the definition of the elementary iterated integral, we get
\[
\int_0^t R_{u-n+1}^{m-1} F_m(u) dX_u^{\alpha_m} = \beta_{n+m+1}^{\alpha_1} (G_{\otimes_{n+1}} \otimes F_{\otimes_{m}}) + U_t
\]
which belongs to \( \bigoplus_{k=0}^{m+n+2} J_k \) and this completes the proof for \( M \). To prove the statement for \( N \) we have to verify that
\[
N_t := \int_0^t \int_0^s R_{u-n+1}^{m,n+1} K(u) d\langle X^{\delta}, X^{\eta} \rangle H(s) dX_s^{\delta}
\]
belongs to \( \bigoplus_{k=0}^{m+n+2} J_k \), for every \( m \), every \( H, K \in \mathcal{B}_T \) and every \( \gamma, \delta, \eta \in \Lambda \). From the previous step, \( \int_0^t R_{u-n+1}^{m,n+1} H(u) dX_u^{\delta} \in \bigoplus_{k=0}^{m+n+2} J_k \), for every \( m \), every \( H \in \mathcal{B}_T \) and every \( \gamma \in \Lambda \). Hence, with similar computations as in (13) and (19), using the representation of \( R_{m,n+1} \) obtained in (20) – (23), we deduce the claim from the induction hypothesis. The proof of the lemma is complete.

We can now prove that \( J_c \) is compensated-covariation stable.

**Theorem 3.13.** Let \( \mathcal{X} \) satisfy Assumption [3.11] Then the family \( J_c \) of the elementary iterated integrals generated by \( \mathcal{X} \) is compensated-covariation stable and any \( M \in J_c \) is quasi-left-continuous.

**Proof.** Let \( M, N \in J_c \). By linearity we can assume without loss of generality that \( M = J_m^\alpha_m(F_{\otimes m}) \) and \( N = J_n^\beta_n(G_{\otimes n}) \). Because of (3.4) we have
\[
[M, N]_t = \langle M, N \rangle_t = \int_0^t \int_0^s R_{u-n+1}^{m,n+1} K(u) d\langle X^{\delta}, X^{\eta} \rangle H(s) dX_s^{\delta}.
\]

(24)
Since by assumption $\mathcal{X}$ is compensated-covariation stable, we have $X_{\alpha \beta} \in \mathcal{X}$. Hence, $F_n G_m$ being bounded, the compensated-covariation stability of $\mathcal{J}_e$ immediately follows from Lemma 3.12. To see the quasi-left continuity of $M$ it is enough to observe that

$$\langle M, M \rangle = (J^a_{m-1} (F_{\otimes m-1}) - F_m)^2 \cdot \langle X_{\alpha \alpha}, X_{\alpha \alpha} \rangle$$

(25)

and hence $\langle M, M \rangle$ is continuous by the continuity of $\langle X_{\alpha \alpha}, X_{\alpha \alpha} \rangle$. The quasi-left continuity of $M$ is a consequence of the last statement of [3.13, Theorem I.4.2]. The proof of the theorem is now complete.

As a next step we study the $m$-fold compensated covariation process built from elements of $\mathcal{J}_e$. For $J_{m}^{\alpha n}(F_{\otimes m}), J_{n}^{\beta n}(G_{\otimes n}) \in \mathcal{J}_e$ we denote by

$$J_{m,n}^{\alpha \beta}(F_{\otimes m} \otimes G_{\otimes n}) := [J_{m}^{\alpha n}(F_{\otimes m}), J_{n}^{\beta n}(G_{\otimes n})] - \langle J_{m}^{\alpha n}(F_{\otimes m}), J_{n}^{\beta n}(G_{\otimes n}) \rangle$$

their compensated-covariation process. For any $j \in \mathbb{N}$ we introduce the notation

$$F_j := F_1 \otimes \cdots \otimes F_m.$$

Because of Theorem 3.13 for $j = 1, \ldots, N$, $\alpha^j_{1:m_j} \in \Lambda^{m_j}$ and $J_{m_j}^{\alpha_j}(F_{\otimes m_j}) \in \mathcal{J}_e$, we can inductively define

$$J_{m_1, \ldots, m_n}^{\alpha_1 \cdots \alpha_N}(F_{\otimes m_1} \otimes \cdots \otimes F_{\otimes m_N}) :=
\left\{ J_{m_1, \ldots, m_{N-1}}^{\alpha_1 \cdots \alpha_{N-1}}(F_{\otimes m_1} \otimes \cdots \otimes F_{\otimes m_{N-1}}), J_{m_N}^{\alpha_N}(F_{\otimes m_N}) \right\} - \langle J_{m_1, \ldots, m_{N-1}}^{\alpha_1 \cdots \alpha_{N-1}}(F_{\otimes m_1} \otimes \cdots \otimes F_{\otimes m_{N-1}}), J_{m_N}^{\alpha_N}(F_{\otimes m_N}) \rangle.$$ (26)

Proposition 3.14. Let $\mathcal{X}$ satisfy Assumption 3.11. Then

$$J_{m_1, \ldots, m_N}^{\alpha_1 \cdots \alpha_N}(F_{\otimes m_1} \otimes \cdots \otimes F_{\otimes m_N}) = \left( \prod_{j=1}^{N} J_{m_j}^{\alpha_j}(F_{\otimes m_j}) \right) \cdot X_{\alpha_1 \cdots \alpha_N}.$$ (27)

Proof. For $N = 2$ the statement coincides with (24). Then, the formula immediately follows by induction from (26) and Theorem 3.13.

Notice that the integrator appearing on the right-hand side of (27) is the $m$-fold compensated-covariation process of the ordered $m$-tuple of martingales $(X_{\alpha_1}, \ldots, X_{\alpha_N})$.

3.5 A product formula for elementary iterated integrals

We now exploit (11) for elementary iterated integrals generated by a compensated-covariation stable family of martingales. Observe that to prove Proposition 3.9 for a family of martingales $\mathcal{X} = \{X_\alpha, \alpha \in \Lambda\}$, it is not needed that the predictable covariation $\langle X_{\alpha}, X_{\alpha} \rangle$ is deterministic but it is sufficient that the family $\mathcal{X}$ is compensated-covariation stable and consists of quasi-left continuous martingales. This is important because for $X, Y \in \mathcal{J}_e$, the process $\langle X, Y \rangle$ is continuous but not deterministic, in general (cf. (25)).
Theorem 3.15. Let $\mathcal{X}$ satisfy Assumption 3.11. Then, for $J_{m_j}^j(F_{m_j}^j) \in \mathcal{J}_c$, $j = 1, \ldots, N$, $N \geq 2$, we have

$$
\prod_{j=1}^N J_{m_j}^j(F_{m_j}^j)
= \sum_{i=1}^N \sum_{1 \leq j_1 < \ldots < j_i \leq N} \left( \prod_{k=1}^N J_{m_k}^k(F_{m_k}^k) - \prod_{\ell=1}^i J_{m_j_{\ell-1}}^j(F_{m_j_{\ell-1}}^j) \right) \cdot X_{m_j_{\ell-1}}^{a_{m_j_{\ell-1}}} \cdots X_{m_j}^{a_{m_j}}
+ \sum_{i=2}^N \sum_{1 \leq j_1 < \ldots < j_i \leq N} \left( \prod_{k=1}^N J_{m_k}^k(F_{m_k}^k) - \prod_{\ell=1}^i J_{m_j_{\ell-1}}^j(F_{m_j_{\ell-1}}^j) \right) \cdot \left( X_{m_j_{\ell-1}}^{a_{m_j_{\ell-1}}} \cdots X_{m_j}^{a_{m_j}} \right).
$$

(28)

Proof. We have $\mathcal{J}_c \subseteq \mathcal{H}^2$. Moreover, from Theorem 3.13, $M \in \mathcal{J}_c$ is quasi-left continuous and $\mathcal{J}_c$ is compensated-covariation stable. So we can introduce the compensated-covariation processes and by Proposition 3.9

$$
\prod_{j=1}^N J_{m_j}^j(F_{m_j}^j) =
\sum_{i=1}^N \sum_{1 \leq j_1 < \ldots < j_i \leq N} \left( \prod_{k=1}^N J_{m_k}^k(F_{m_k}^k) - \prod_{\ell=1}^i J_{m_j_{\ell-1}}^j(F_{m_j_{\ell-1}}^j) \right) \cdot X_{m_j_{\ell-1}}^{a_{m_j_{\ell-1}}} \cdots X_{m_j}^{a_{m_j}}
+ \sum_{i=2}^N \sum_{1 \leq j_1 < \ldots < j_i \leq N} \left( \prod_{k=1}^N J_{m_k}^k(F_{m_k}^k) - \prod_{\ell=1}^i J_{m_j_{\ell-1}}^j(F_{m_j_{\ell-1}}^j) \right) \cdot \left( X_{m_j_{\ell-1}}^{a_{m_j_{\ell-1}}} \cdots X_{m_j}^{a_{m_j}} \right).
$$

The statement follows from Proposition 3.14.

In the next step we obtain a recursive formula for computing moments of products from $\mathcal{J}_c$. To ensure their existence we make the following assumption:

Assumption 3.16. Each martingale in $\mathcal{X}$ has finite moments of every order.

Let $\mathcal{X}$ satisfy Assumption 3.11. We observe that then, according to [2, Corollary 4.5], $\mathcal{X}$ fulfills Assumption 3.16 if there exists $\beta \in \Lambda$ such that $\langle X_\beta^t, X_\beta \rangle_t < \langle X_\beta^T, X_\beta \rangle_T, t < T$.

Before we come to the moment formula we need the following technical result. For the proof see [2, Lemma 5.5].

Lemma 3.17. Let $\mathcal{X} := \{ X^\alpha, \alpha \in \Lambda \} \subseteq \mathcal{H}^2$ satisfy Assumption 3.16 and let $A$ be a deterministic process of finite variation. We define the process $K$ by

$$
K := \prod_{i=1}^N X_{-i}^\alpha, \quad \alpha_i \in \Lambda, \quad i = 1, \ldots, N.
$$

Then the process $K \cdot A$ is of integrable variation.
Let $A$ and $K$ be as in Lemma 3.17. Then by $\mathbb{E}[K] \cdot A$ we denote the integral with respect to $A$ of the $A$-integrable function $t \mapsto \mathbb{E}[K_t]$.

Now we are ready to state and prove a recursive moment formula for products of elementary iterated integrals.

**Corollary 3.18.** Let $\mathcal{X}$ satisfy Assumptions 3.11 and 3.16. Then, for $J_{m_j}^{\alpha_{m_j}^j}(F_{\otimes m_j}^j) \in \mathcal{J}_e$, $j = 1, \ldots, N$, $N \geq 2$, the following formula holds:

$$
\mathbb{E} \left[ \prod_{j=1}^N J_{m_j}^{\alpha_{m_j}^j}(F_{\otimes m_j}^j) \right] = \sum_{i=2}^N \sum_{1 \leq j_1 < \ldots < j_i \leq N} \mathbb{E} \left[ \prod_{k=1}^N J_{m_k}^{\alpha_{m_k}^k}(F_{\otimes m_k}^k) - \prod_{\ell=1}^i J_{m_{j_\ell}}^{\alpha_{m_{j_\ell}}^\ell}(F_{\otimes m_{j_\ell}}^\ell) - F_{\otimes m_{j_\ell}}^\ell \right] \cdot \langle X^{\alpha_{j_1}^{j_1}}, \ldots, X^{\alpha_{j_{i-1}}^{j_{i-1}}}, X^{\alpha_{j_i}^{j_i}} \rangle.
$$

(29)

**Proof.** Because of Lemma 3.3, if $\mathcal{X}$ satisfies Assumption 3.16, the family $\mathcal{J}_e$ of elementary iterated integrals also does. Therefore the left-hand side of (28) is integrable. Furthermore, because $(X^\alpha, X^\alpha)$ is deterministic, for every $\alpha \in \Lambda$, we conclude by Lemma 3.17 that the integrands of the stochastic integrals in the first term on the right-hand side of (28) belong to $L^2(X^\alpha)$, for every $\alpha \in \Lambda$. Therefore each summand in the first term on the right-hand side of (28) belongs to $\mathcal{M}$. Analogously, we have by Lemma 3.17 and Assumption 3.11 that the second term on the right-hand side of (28) is integrable. We can therefore consider the expectation and apply the theorem of Fubini to conclude the proof. 

Notice that by taking $m_j = 1$ and $F_j^1 = 1$, $j = 1, \ldots, N$ in (29) one can recursively compute expressions like $\mathbb{E}[\prod_{j=1}^N X_{t_j}^\alpha]$. Indeed, in this special case, the second product on the right-hand side of (29) is identically equal to one, while the first product consists of a number of factors which is strictly smaller than $N$. For example, if $X$ is a Brownian motion, (29) corresponds to taking the expectation in (12) and we get $\mathbb{E}[X_{t_i}^{2N+1}] = 0$, $N \geq 0$, and

$$
\mathbb{E}[X_{t_i}^{2N}] = (2N-1)!!(\sqrt{t})^{2N}, \quad N \geq 2,
$$

where $N!!$ denotes the double factorial of $N$.

If $X$ is a compensated Poisson process with parameter $\lambda$, then by taking the expectation in (13) (or by simplifying (29)) we obtain

$$
\mathbb{E}[X_{t_i}^N] = \lambda \sum_{i=2}^N \binom{N}{i} \int_0^t \mathbb{E}[(X_s)^{N-i}] ds
$$

which is the formula in [17, Proposition 3.3.4 and Example 3.3.5]. The resulting polynomials

$$
\mathbb{E}[X_t^2] = \lambda t, \quad \mathbb{E}[X_t^4] = \lambda t, \quad \mathbb{E}[X_t^6] = 3\lambda^2 t^2 + \lambda t, \quad \mathbb{E}[X_t^8] = 10\lambda^2 t^2 + \lambda t \ldots,
$$

also called centred Touchard polynomials.

If, more generally, $X$ is a Lévy process (see Subsection 3.5) with finite moments of every order, we derive from Corollary 3.18 the recursive formula (3.19) below for the central moments of $X$. This formula allows to compute the moments of a Lévy process recursively without taking derivatives of its characteristic function.
**Theorem 3.19.** Let $X$ be a Lévy process with characteristic triplet $(\gamma, \sigma^2, \nu)$, such that $X$ has finite moments of every order. Let us define the centred process $X^{(1)}$ by $X^{(1)}_t := X_t - \mathbb{E}[X_t]$. Then, for every $N \geq 1$, we have
\[
\mathbb{E}[(X^{(1)}_t)^N] = \sum_{i=2}^{N} \binom{N}{i} (\nu + \sigma^2 1_{\{i=2\}}) \int_0^t \mathbb{E}[(X^{(1)}_s)^{N-i}] ds.
\] (30)

Consequently, for $N \geq 1$, the non-central moments of $X$ are given by
\[
\mathbb{E}[X^N_t] = \sum_{i=1}^{N} \binom{N}{i} (\nu + \sigma^2 1_{\{i=2\}}) \int_0^t \mathbb{E}[(X^{(1)}_s)^{i-k}] ds.
\] (31)

**Proof.** The family $\mathcal{X} := \{X^{(i)}, i \geq 1\}$ defined by (16) is compensated-covariation stable, consists of quasi-left continuous martingales and is such that $(X^{(i)}, X^{(j)})$ is deterministic. Hence, $\mathcal{X}$ satisfies Assumptions 3.11. Furthermore, because of the last paragraph in Subsection 3.3, $\mathcal{X}$ satisfies also Assumption 3.16. Hence, from the identity $(X^{(1)}_t)^N = \prod_{i=1}^{N} J_1^{(1)}(t)$, it follows that (30) is a special case of Corollary (3.18). To see (31), we observe that the identity $X_t = X^{(1)}_t + \mathbb{E}[X_t]$ holds. Thus, $\mathbb{E}[X^N_t] = \sum_{i=1}^{N} \binom{N}{i} \mathbb{E}[(X^{(1)}_s)^i] \mathbb{E}[X_s]^{N-i}$ and (31) immediately follows from (30) because of (15). The proof is complete. 

### 4 Iterated integrals with respect to Lévy processes

In this section $X$ is a Lévy process with characteristic triplet $(\gamma, \sigma^2, \nu)$, and we denote by $N$ the jump measure of $X$ and by $\tilde{N} := N - \nu \otimes \delta_0$ the compensated Poisson random measure. For $\alpha \in L^2(\nu)$ and $t \geq 0$ we use the notation
\[
1_{[0,t]}\alpha \ast \tilde{N} := \int_{[0,t] \times \mathbb{R}} \alpha(x)\tilde{N}(ds, dx).
\]

We set $\mu := \sigma^2 \delta_0 + \nu$, where $\delta_0$ is the Dirac measure concentrated in zero. For $\alpha \in L^2(\mu)$, we define
\[
X^\alpha := \alpha(0)W^\sigma + 1_{[0,\cdot) \times (\mathbb{R} \setminus \{0\})} \alpha \ast \tilde{N}.
\] (32)

We recall that, if $\alpha \in L^2(\mu)$, then the process $X^\alpha$ has the following properties:

(i) $(X^\alpha, \mathbb{F})$ is a Lévy process with characteristic triplet $(-\int_{|x|>1} \alpha(x)\nu(dx), \alpha(0)^2 \sigma^2, \nu \circ \alpha^{-1})$.

(ii) $X^\alpha \in \mathcal{H}^2(\mathbb{F})$ and $(X^\alpha, X^\alpha_t) = t \mu(\alpha^2)$.

(iii) $\Delta X^\alpha = \alpha(\Delta X) 1_{\Delta X \neq 0}$ and hence $\Delta X^\alpha$ is bounded, if $\alpha$ is bounded.

(iv) Let $\beta \in L^2(\mu)$. Then $(X^\alpha, X^\beta) = 0$ if and only if $\mu(\alpha \beta) = 0$.

For a system $\Lambda \subseteq L^2(\mu)$, we put
\[
\mathcal{X}_\Lambda := \{X^\alpha, \alpha \in \Lambda\}.
\] (33)

**Assumption 4.1.** Let $\Lambda$ be a set of real-valued functions with the following properties:

(i) $\Lambda \subseteq L^1(\mu) \cap L^2(\mu)$;

(ii) $\Lambda$ is total in $L^2(\mu)$;

(iii) $\Lambda$ is stable under multiplication, and $1_{\mathbb{R} \setminus \{0\}} \alpha \in \Lambda$ whenever $\alpha \in \Lambda$;

(iv) $\Lambda$ is a system of bounded functions.
We observe that a system $\Lambda$ satisfying Assumption 4.1 always exists: Obviously, we can choose $\Lambda := \{\alpha = c1_{[0]} + 1_{(a,b]}, \ a, b \in \mathbb{R} : a < b, \ 0 \notin [a,b]; \ c \in \mathbb{R} \} \cup \{0\}$ as an example.

**Proposition 4.2.** Let $\Lambda$ satisfy Assumption 4.1 Then $\mathcal{X}_\Lambda \subseteq \mathcal{H}^2(\mathbb{F})$. For $\alpha_1, \ldots, \alpha_m \in \Lambda$ the compensated-covariation process $X^{\alpha_1,m}$, $m \geq 2$, has the following form:

$$X^{\alpha_1,m} = \left(1_{[0,\cdot] \times (\mathbb{R}\setminus\{0\})} \prod_{k=1}^m \alpha_k\right) \ast \bar{N}. \quad (34)$$

Moreover,

(i) $\mathbb{F}_{\mathcal{X}_\Lambda} = \mathbb{F}^X$;

(ii) $\mathbb{E}[\exp(\lambda |X_t|)] < +\infty$ for every $X \in \mathcal{X}_\Lambda$, $\lambda > 0$, $t \in [0,T]$,

(iii) $(X,Y)$ is deterministic for every $X,Y \in \mathcal{X}_\Lambda$,

(iv) $\mathcal{X}_\Lambda$ possesses the CRP with respect to $\mathbb{F}^X$.

**Proof.** For the “Moreover” part we refer to [2, Proposition 6.4 and Proposition 6.5]. To show (34), we notice that

$$[X^{\alpha_1},X^{\alpha_2}]_t = \alpha_1(0)\alpha_2(0)\sigma^2 t + \int_{(0,t] \times \mathbb{R}\setminus\{0\}} \alpha_1(x)\alpha_2(x) N(ds,dx)$$

and

$$(X^{\alpha_1},X^{\alpha_2})_t = \alpha_1(0)\alpha_2(0)\sigma^2 t + v(\alpha_1\alpha_2) t.$$ 

Then, since $\alpha_1\alpha_2 \in L^1(\mu) \cap L^2(\mu)$, we get $X^{\alpha_1,2} = \left(1_{[0,\cdot] \times (\mathbb{R}\setminus\{0\})} \alpha_1 \alpha_2\right) \ast \bar{N}$, and (34) follows from (10) by induction. \qed

### 4.1 Products of elementary iterated integrals

The aim of this subsection is to deduce a product formula for elements from $\mathcal{J}_e$, where we assume that $\mathcal{J}_e$ is generated by $\mathcal{X}_\Lambda$ and $\Lambda \subseteq L^2(\mu)$ satisfies Assumption 4.1.

By Definition 3.1 an elementary iterated integral generated by the martingales $X^{\alpha_1}, \ldots, X^{\alpha_m}$, $\alpha_1, \ldots, \alpha_m \in \Lambda$, is given by

$$J_{m_1,\ldots,m_n}(F_{m-1})_t := \int_0^t J_{m_1,\ldots,m_n-1}(F_{m-1})_u F_m(u) \ dX_u, \quad t \in [0,T], \ m \geq 1.$$

Our aim is to determine a product formula for

$$\prod_{j=1}^N J_{m_{j_1},\ldots,m_{j_N}}(F_{j_1})_t, \quad N \in \mathbb{N},$$

(given in equation 4.2 below) by iterating formula (28) until the inner elementary iterated integrals appearing as integrands will reduce to deterministic functions. In this way it will be also possible to determine a formula for the moment of products of elementary iterated integrals which generalizes the isometry relation 5. For this goal we need to introduce some combinatoric rules and notations.

First of all we fix $N$, that is, the number of factors of the product, and then we fix $m_1, \ldots, m_N$, that is, the order of each factor. We then order all the functions $F_{j_k} \in \mathbb{B}_T, k = 1, \ldots, m_j, j = 1, \ldots, N$, consecutively: Set

$$\overline{m}_0 := 0,$$

$$\overline{m}_j := m_1 + \ldots + m_j, \quad j = 1, \ldots, N.$$
We then define
\[
(F_{m_{j-1}+1}, \ldots, F_{m_j}) := (F^j_1, \ldots, F^j_{m_j}), \quad j = 1, \ldots, N,
\]
and set
\[
F_{\otimes m_{j-1}+1 m_j} := F_{m_{j-1}+1} \otimes \cdots \otimes F_{m_j}, \quad j = 1, \ldots, N.
\]
Analogously, we define
\[
(\alpha_{m_{j-1}+1}, \ldots, \alpha_{m_j}) := (\alpha^j_1, \ldots, \alpha^j_{m_j}), \quad j = 1, \ldots, N.
\]
Like in (3), we put
\[
\rho \alpha^j_{m_{j-1}+1} := \rho \alpha^j_{m_{j-1}+1} \otimes \cdots \otimes \rho \alpha^j_{m_j},
\]
where
\[
d\rho \alpha^j_{m_{j-1}+1} (t_1, \ldots, t_{m_j}) = \left( \prod_{i=m_{j-1}+1}^{m_j} R_{\alpha^j_i} \right) d\lambda (t_1, \ldots, t_{m_j}).
\]
With this notation we get:
\[
\prod_{j=1}^{N} J_{m_j} (F_{\otimes m_j}) = \prod_{j=1}^{N} J_{m_j} \alpha^j_{m_{j-1}+1} (F_{\otimes m_{j-1}+1 m_j}). \quad (35)
\]
Notice that a stochastic integral with respect to $X^\alpha$ is in fact the sum of a stochastic integral with respect to $\alpha(0)W^\sigma$ and one with respect to $(1_{[0,\cdot]} \times \langle \cdot \rangle_0(\alpha)) * \tilde{N}$. For convenience we shall write these two integrals instead of the integral with respect to $X^\alpha$ to recognize whether we are integrating with respect to the Brownian part or with respect to the jump part. To this aim, let $B \subseteq \{1, 2, \ldots, m_N\}$ denote the set of indices for which we integrate with respect to the Brownian part of $X^{\alpha_k}$, $k = 1, \ldots, m_N$. We define
\[
\alpha^B_k := \begin{cases} 
\alpha^1_k 1_{\{0\}} & \text{if } k \in B, \\
\alpha^1_k 1_{\mathbb{R}\setminus\{0\}} & \text{if } k \notin B.
\end{cases}
\]
Then
\[
\prod_{j=1}^{N} J_{m_j} \alpha^j_{m_{j-1}+1} (F_{\otimes m_{j-1}+1 m_j}) = \sum_{B \subseteq \{1, 2, \ldots, m_N\}} \prod_{j=1}^{N} J_{m_j} \alpha^B_k (F_{\otimes m_{j-1}+1 m_j}), \quad (36)
\]
which implies that, to obtain the multiplication formula, it suffices to transform the product on the right-hand side of (36) into a sum of iterated integrals, for any $B \subseteq \{1, 2, \ldots, m_N\}$. We observe that the extreme cases $B = \{1, 2, \ldots, m_N\}$ and $B = \emptyset$ correspond to the case of integration with respect to the Brownian part only and with respect to the jump part only, respectively. Let us consider, as an illustrating example, the case $N = 2, m_1 = m_2 = 1$. Here we have $B \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and
\[
\begin{align*}
\alpha^{(1)}_1 &= \alpha^1 1_{\{0\}}, & \alpha^{(1)}_2 &= \alpha^1 1_{\mathbb{R}\setminus\{0\}}, & \alpha^{(2)}_1 &= \alpha^1 1_{\mathbb{R}\setminus\{0\}}, & \alpha^{(2)}_2 &= \alpha^1 1_{\{0\}}, \\
\alpha^{(1,2)}_1 &= \alpha^1 1_{\{0\}}, & \alpha^{(1,2)}_2 &= \alpha^1 1_{\mathbb{R}\setminus\{0\}}, & \alpha^{(1)}_1 &= \alpha^1 1_{\mathbb{R}\setminus\{0\}}, & \alpha^{(1)}_2 &= \alpha^1 1_{\{0\}},
\end{align*}
\]
Notice that (28) holds also for $\prod_{j=1}^{N} \alpha_{m_{j-1}+1}^{m_{j}}(F_{\otimes m_{j-1}+1,m_{j}})$ and reads as

$$
\prod_{j=1}^{N} \alpha_{m_{j-1}+1}^{m_{j}}(F_{\otimes m_{j-1}+1,m_{j}}) = \sum_{r=1}^{N} \sum_{1 \leq j_{1} < \ldots < j_{r} \leq N} \left\{ \left( \prod_{k=1}^{r} \alpha_{m_{k-1}+1}^{m_{k}}(F_{\otimes m_{k-1}+1,m_{k}}) \right) \times \left( \prod_{q=1}^{r} F_{m_{j_{q}}} \right) \right\} \cdot X^{\alpha_{m_{1}}^{m_{1}} \ldots \alpha_{m_{r}}^{m_{r}}} (37)
$$

We observe that the integrands on the right-hand of (37) consist of products of elementary iterated integrals. These elementary iterated integrals are either of the same order $B_{\alpha_{1}},...,$ $B_{\alpha_{N}}$, or of the diminished order $m_{j_{i}}$. We will repeatedly apply (37) to the integrands on the right-hand side of (37) until we get integrands that consist of iterated integrals of order zero, that is, they are equal to one.

**The integrators in (37).** In order to determine the exact structure of the outcome of the procedure explained above, we want to specify which integrators $dX^{\alpha_{m_{1}}^{m_{1}} \ldots \alpha_{m_{r}}^{m_{r}}}$ can occur in (37). If $r = 1$, we have

$$
X^{\alpha_{m_{j_{1}}}^{m_{j_{1}}}} = \alpha_{m_{j_{1}}}^{m_{j_{1}}}(0)W^\sigma, \quad \text{if} \quad m_{j_{1}} \in B,
$$

$$
X^{\alpha_{m_{j_{1}}}^{m_{j_{1}}}} = (1_{\mathbb{R}(0)} \mathbb{1}_{(0)} \alpha_{m_{j_{1}}}) \hat{N}, \quad \text{if} \quad m_{j_{1}} \in B^c = \{1,2,\ldots,m_{N} \} \setminus B.
$$

From (34) we conclude that for $r \geq 2$

$$
X^{\alpha_{m_{j_{1}}}^{m_{j_{1}}} \ldots \alpha_{m_{j_{r}}}^{m_{j_{r}}}} = X^{1_{\mathbb{R}(0)} \prod_{k=1}^{r} \alpha_{m_{j_{k}}}} = \begin{cases} 
1_{(0] \times (\mathbb{R}(0))} \prod_{k=1}^{r} \alpha_{m_{j_{k}}} \hat{N}, & \text{if} \quad \{m_{j_{1}}, \ldots, m_{j_{r}}\} \subseteq B^c, \\
0, & \text{otherwise}.
\end{cases}
$$

Furthermore, it holds

$$
\langle X^{\alpha_{m_{j_{1}}}^{m_{j_{1}}} \ldots \alpha_{m_{j_{r}}}^{m_{j_{r}}}} \rangle_{t} = t \mathcal{V}(\alpha_{m_{j_{1}}}^{m_{j_{1}}} \alpha_{m_{j_{2}}}^{m_{j_{2}}}), \quad \text{if} \quad \{m_{j_{1}}, m_{j_{2}}\} \subseteq B^c,
$$

$$
\langle X^{\alpha_{m_{j_{1}}}^{m_{j_{1}}} \ldots \alpha_{m_{j_{r}}}^{m_{j_{r}}}} \rangle_{t} = t \mathcal{V}(\alpha_{m_{j_{1}}}^{m_{j_{1}}} \alpha_{m_{j_{2}}}^{m_{j_{2}}} \ldots \alpha_{m_{j_{r}}}^{m_{j_{r}}}), \quad \text{if} \quad \{m_{j_{1}}, m_{j_{2}}, \ldots, m_{j_{r}}\} \subseteq B^c,
$$

and for $r \geq 3$,

$$
\langle X^{\alpha_{m_{j_{1}}}^{m_{j_{1}}} \ldots \alpha_{m_{j_{r-1}}}^{m_{j_{r-1}}} \ldots \alpha_{m_{j_{r}}}^{m_{j_{r}}}} \rangle_{t} = \begin{cases} 
t \mathcal{V}(\prod_{k=1}^{r} \alpha_{m_{j_{k}}}), & \text{if} \quad \{m_{j_{1}}, \ldots, m_{j_{r}}\} \subseteq B^c, \\
0, & \text{otherwise}.
\end{cases}
$$
The integrators in the iteration steps. On the right-hand side of (37), integrals with respect to martingales and with respect to deterministic point-bracket processes appear. If we apply (37) to the integrands, we will get “mixed” iterated integrals where the integrators are both martingales and point brackets. We will use the superscript $i$ where $i = 1$ stands for “martingale” and $i = 0$ for point bracket. If $S \subseteq \{1, \ldots, m_N\}$ we define
\[
\alpha_{(S)} := \prod_{t \in S} \alpha_t, \quad \bar{\alpha}_{(S)} := 1_{\mathbb{R}\setminus\{0\}} \alpha_{(S)}.
\]
For $i \in \{0, 1\}$, we will write, summarizing the analysis of the integrators above,
\[
d Y_t^{\alpha_{(S,i)}} := \begin{cases} 
  \nu(\alpha_{(S)})dt & \text{if } i = 0, \ S \subseteq B^c, \ |S| \geq 2, \\
  \sigma^2 \alpha_{(S)}(0)dt & \text{if } i = 0, \ S \subseteq B, \ |S| = 2, \\
  \alpha_{(S)}(0)dW_t^\sigma & \text{if } i = 1, \ S \subseteq B, \ |S| = 1, \\
  dX_t^{\bar{\alpha}_{(S)}} & \text{if } i = 1, \ S \subseteq B^c, \ |S| \geq 1, \\
  0, & \text{otherwise}. 
\end{cases}
\]

Algorithm to build identification rules for the integrands. In (37), the function $t \mapsto \prod_{q=1}^{L_q=1} F_{m_q}(t)$ appears on the right hand side. Since we want to use the tensor product $\bigotimes_{i=1}^N F_{m_i} \otimes \cdots \otimes m_i$ as integrand in the final formula, we need to identify $t := m_1 = \ldots = m_N$ to get this ordinary product from the tensor product. These ordinary products arise from each application of (37), hence we want to derive an identification rule that describes which of the variables $t_1, \ldots, t_{m_q}$ need to be identified in each step. Set
\[
[\bar{m}_N] := \{1, 2, \ldots, m_N\}.
\]
$\Pi(m_1, \ldots, m_N)$ will denote the set of all those partitions $s = (S_1, \ldots, S_k)$ of the set $[\bar{m}_N]$ which can be built using the following backward induction from $k$ till 1:

First Step: $k$ Choose a non-empty subset
\[
S_k \subseteq \{1, \ldots, m_N\}.
\]

Step: $k-1+1 \rightarrow k-1$ Assume that $S_k, S_{k-1}, \ldots, S_{k-l+1}$ (for some $l = 1, \ldots, k-1$) have been chosen. By $I_1, \ldots, I_N$ we denote the sets which contain exactly those indices of the $F_j$’s which are used in the same elementary iterated integral on the left-hand side of (35), that is,
\[
I_1 := \{1, \ldots, m_1\} \quad \text{and} \quad I_r := \{m_{r-1}+1, \ldots, m_r\}, \quad r = 2, \ldots, N.
\]

Then $S_{k-1}$ denotes a non-empty subset of the set consisting of the largest elements of $I_1, \ldots, I_N$ which have not yet been chosen for $S_k, S_{k-1}, \ldots, S_{k-l+1}$ already: If
\[
L_r^{k-1} = \sup \{I_r \setminus (S_k \cup S_{k-1} \cup \ldots \cup S_{k-l+1})\}, \quad r = 1, \ldots, N,
\]
where we use the convention $\sup \emptyset = -\infty$, then
\[
S_{k-1} \subseteq \{L_r^{k-1} : L_r^{k-1} \neq -\infty, \ r = 1, \ldots, N\}.
\]

Notice that we require that $\bigcup_{r=1}^k S_r = [\bar{m}_N]$. Any such $s = (S_1, \ldots, S_k) \in \Pi(m_1, \ldots, m_N)$ will be called an identification rule. We summarize this procedure in the following definition.
Definition 4.3. Let $\Pi(m_1,\ldots,m_N)$ be the set of all partitions $s = (S_1,\ldots,S_{|s|})$ of $\{1,2,\ldots,N\}$ (here $|s|$ stands for the number of sets which belong to $s$), where $N \leq |s| \leq m_N$, such that

(i) each $S_\ell$ contains at most one element of $\{m_\ell-1+1,\ldots,m_\ell\}$, that is,

$$|S_\ell \cap \{m_\ell-1+1,\ldots,m_\ell\}| \leq 1 \quad \text{for all} \quad \ell = 1,\ldots,|s|, \quad r = 1,\ldots,N,$$

(ii) the elements of each $\{m_\ell-1+1,\ldots,m_\ell\}$ appear ordered within $s$, that is, if for $1 \leq \ell < k \leq |s|$ it holds $a \in S_\ell \cap \{m_\ell-1+1,\ldots,m_\ell\}$ and $b \in S_k \cap \{m_\ell-1+1,\ldots,m_\ell\}$, then $a < b$.

For a function $(u_1,\ldots,u_{m_N}) \mapsto H(u_1,\ldots,u_{m_N})$ and $s = (S_1,\ldots,S_k)$, we define the following identification of variables:

$$H_s(t_1,\ldots,t_k) \text{ is derived from } H(u_1,\ldots,u_{m_N}) \text{ by replacing all } u_j \text{ with } j \in S_r \text{ by the same } t_r \text{ for } r = 1,\ldots,k. \quad (39)$$

As already indicated above the identification rules $s = (S_1,\ldots,S_k)$ will be used to describe the integrands which result from the iteration of (37). In the first step, which is (37) itself, the factor $\prod_{q=1}^J F_{m_q}$ appears. Notice that

$$\prod_{q=1}^J F_{m_q} = \prod_{t \in S_k} F_t \quad \text{provided that} \quad S_k = \{m_{j_1},\ldots,m_{j_k}\}.$$

At the same time, the iterated integrals with order $m_{j_1},\ldots,m_{j_k}$, which appear on the left-hand side of (37), have its order diminished by 1 on the right-hand side. If we apply (37) to the integrands of the right-hand side, then the product $\prod_{t \in S_k} F_t$ will appear, where

$$S_{k-1} \subseteq (\{m_1,\ldots,m_N\} \setminus S_k) \cup \{m_{j_1}-1,\ldots,m_{j_k}-1\}.$$

We repeatedly apply (37) and, finally, we have $\prod_{r=1}^k \prod_{j \in S_r} F_j(t_r)$. This product we get from the tensor product $\bigotimes_{j=1}^N F_{m_{j-1}+1,m_{j+1}}(u_1,\ldots,u_{m_N}) = F_1(u_1) \times \ldots \times F_{m_N}(u_{m_N})$ applying the identification rule $s = (S_1,\ldots,S_k)$:

$$\left( \bigotimes_{j=1}^N F_{m_{j-1}+1,m_{j+1}} \right)_s (t_1,\ldots,t_k) = \prod_{r=1}^k \prod_{j \in S_r} F_j(t_r).$$

Identification rules similar to the above ones appear also in Peccati and Taqqu [17, Chapter 7] and Last et al. [11] (see also the references therein).

The set $I_s$. We still need to pay attention to the fact that each application of (37) produces integrals with respect to both compensated-covariation processes and point brackets. For any identification rule $s = (S_1,\ldots,S_{|s|}) \in \Pi(m_1,\ldots,m_N)$ we define the sets

$$I_s = \{i := (i_1,\ldots,i_{|s|}) : i_\ell \in i(S_\ell), \ell = 1,\ldots,|s|\},$$

where

$$i(S_\ell) := \begin{cases} \{0\}, & \text{if } |S_\ell| = 2, S_\ell \subseteq B, \\ \{1\}, & \text{if } |S_\ell| = 1, \\ \{0,1\}, & \text{if } |S_\ell| \geq 2, S_\ell \subseteq B^c, \\ \emptyset, & \text{otherwise.} \end{cases}$$
So, in view of (38), if for example \(|S_\ell| = 2\) and \(S_\ell \subseteq B\), then we have just one integral, hence \(i(S_\ell)\) contains one element. This integral is with respect to \(dr\), which we indicate here by \(i(S_\ell) = \{0\}\). If \(|S_\ell| = 1\), then we also have just one integral but the integrator is a martingale, so we use \(i(S_\ell) = \{1\}\), and so on.

We denote by \(M^s_{\ell}\) the simplex \(M^m_{\ell}\) (cf. (7)) with \(m := |s|\). Then we get

\[
\prod_{j=1}^N J_{m_j}^{\rho_{m_j-1, m_j}} (F \otimes \mathcal{F}) s
= \sum_{B \subseteq [m_N]} \sum_{s \in \Pi(m_1, \ldots, m_N)} \sum_{i \in k \subseteq M^s_{\ell}} \left( \bigotimes_{j=1}^N F \otimes \mathcal{F} \right) \left( t_1, \ldots, t_{|s|} dY_{t_1}^{\alpha_{|s|}} \cdots dY_{t_{|s|}}^{\alpha_{|s|}} \right) s. \tag{40}
\]

**Avoiding zeros in the summation.** The right-hand side of (40) contains many terms which are zero: From (38) we see that this happens, for example, if there is a set \(S_\ell\) containing elements from both, \(B\) and \(B^c\). Another case where the integral is zero is if there is a set \(S_\ell \subseteq B\) with \(|S_\ell| \geq 3\). We will exclude these cases by defining

\[
\Pi_{\leq 2, \geq 1} (B, B^c; m_1, \ldots, m_N) := \{ s = (S_1, \ldots, S_{|s|}) \in \Pi(m_1, \ldots, m_N) : \forall \ell = 1, \ldots, |s| \text{ it holds} \\
(S_\ell \cap B = S_\ell \text{ and } |S_\ell| = 2) \quad \text{or} \quad S_\ell \cap B^c = S_\ell \text{ and } |S_\ell| \geq 2 \}. \tag{41}
\]

Hence we have derived the following theorem:

**Theorem 4.4.** Let \(X\) be a Lévy process and suppose that \(\Lambda \subseteq L^2(\mu)\) satisfies Assumption 4.7. For \(J_{m_j}^{\rho_{m_j-1, m_j}} (F^j) \in \mathcal{F}_t, \quad j = 1, \ldots, N, \) with \(F^j := F \otimes \mathcal{F}_{m_j} \in \mathbb{B}_{m_j}\), the following product formula holds:

\[
\prod_{j=1}^N J_{m_j}^{\rho_{m_j-1, m_j}} (F^j) t
= \sum_{B \subseteq [m_N]} \sum_{s \in \Pi_{\leq 2, \geq 1} (B, B^c; m_1, \ldots, m_N)} \sum_{i \in k \subseteq M^s_{\ell}} \left( \bigotimes_{j=1}^N F \otimes \mathcal{F} \right) \left( t_1, \ldots, t_{|s|} dY_{t_1}^{\alpha_{|s|}} \cdots dY_{t_{|s|}}^{\alpha_{|s|}} \right) s. \tag{42}
\]

Let us now denote by \(\lambda^{|s|}\) the \(|s|\)-dimensional Lebesgue measure restricted to the simplex \(M^{|s|}_{\ell}\) (cf. (7)). Our aim is to compute the expectation of (42). We shall call the resulting relation **moment formula**.

Before we define

\[
\Pi_{\geq 2, \geq 2} (B, B^c; m_1, \ldots, m_N) := \{ s = (S_1, \ldots, S_{|s|}) \in \Pi(m_1, \ldots, m_N) : \forall \ell = 1, \ldots, |s| \text{ it holds} \\
(S_\ell \cap B = S_\ell \text{ and } |S_\ell| = 2) \quad \text{or} \quad S_\ell \cap B^c = S_\ell \text{ and } |S_\ell| \geq 2 \}. \tag{43}
\]

**Corollary 4.5.** Let \(X\) be a Lévy process and suppose that \(\Lambda \subseteq L^2(\mu)\) satisfies Assumption 4.7. Then for \(J_{m_j}^{\rho_{m_j-1, m_j}} (F^j) \in \mathcal{F}_t, \quad j = 1, \ldots, N, \) where \(F^j = F \otimes \mathcal{F}_{m_j} \in \mathbb{B}_{m_j}\), the moment formula

\[
\mathbb{E} \left[ \prod_{j=1}^N J_{m_j}^{\rho_{m_j-1, m_j}} (F^j) t \right] = \sum_{B \subseteq [m_N]} \sum_{s \in \Pi_{\geq 2, \geq 2} (B, B^c; m_1, \ldots, m_N)} \left( \prod_{S_\ell \subseteq B} \nu(\alpha_{|S_\ell|}) \right) \left( \prod_{S_\ell \subseteq B} (\alpha_{|S_\ell|}(0) \sigma^2) \right) \\
\times \int_{M^s_{\ell}} \left( \bigotimes_{i=1}^N F^j \right) \left( t_1, \ldots, t_{|s|} \right) d\lambda^{|s|}(t_1, \ldots, t_{|s|}) \tag{44}
\]

holds.
Proof. First we observe that, according to (38), the process $Y_{\beta,j}^{\alpha}$ can be deterministic only if $|S_i| \geq 2$ and $i_\ell = 0$, and in this case, its explicit expression is

$$Y_{\beta,j}^{\alpha} = \begin{cases} v(\alpha_{(S_1)}), & \text{if } S_\ell \subseteq B^c, \\ \sigma^2 \alpha_{(S_1)}(0), & \text{if } S_\ell \subseteq B \text{ and } |S_\ell| = 2. \end{cases}$$

If $Y_{\beta,j}^{\alpha}$ is random, then $i_\ell = 1$ and $Y_{\beta,j}^{\alpha} \in \mathcal{H}^2$. We now take the expectation in (42) and notice that, if at least one of the $Y_{\beta,j}^{\alpha}$ is random, we have

$$\mathbb{E} \left[ \int_{M^n} \left( \bigotimes_{j=1}^{N} F_j \right) (t_1, \ldots, t_{|S|}) dY_{t_1}^{\alpha} \ldots dY_{t_{|S|}}^{\alpha} \right] = 0$$

because the integrand is bounded. This implies that only integrals with $s \in \Pi_{2,\geq 2}(B, B^c; m_1, \ldots, m_N)$ will appear, and since $i_\ell = 0$ for all $\ell$ there is no sum over $i \in I_s$, so that we get (44). The proof is complete. \qed

Example 4.6. We illustrate the moment formula (44) with some examples. Choose $\alpha_1, \ldots, \alpha_5$ and let $X^{\alpha_1}, \ldots, X^{\alpha_5}$ be given by (32).

1. For $N = 3$ we assume that $m_1 = 1, m_2 = 1$ and $m_3 = 2$. We want to apply (44) to compute

$$\mathbb{E} \left[ J^\alpha_1 (F^1), J^\alpha_2 (F^2), J^\alpha_3 (F^3) \right].$$

By Definition 4.3 we get the identification rules

$$s_1 = (\{1,3\}, \{2,4\}) \quad \text{and} \quad s_2 = (\{1,4\}, \{2,3\}).$$

In this case $m_3 = 4$. The subsets of $[m_3] = \{1,2,3,4\}$ which are relevant for the construction of $B$ are the elements of the family $\mathcal{B} := \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{1,2,3,4\}, \emptyset\}$, since $\Pi_{2,\geq 2}(B, B^c; 1,1,2) = \emptyset$ for $B \notin \mathcal{B}$. We now compute $\Pi_{2,\geq 2}(B, B^c; 1,1,2)$ for $B \in \mathcal{B}$. For shortness, we write $\Pi_{2,\geq 2}(B, B^c)$. We have

$$\Pi_{2,\geq 2}(\{1,3\}, \{2,4\}) = \Pi_{2,\geq 2}(\{2,4\}, \{1,3\}) = s_1,$$

$$\Pi_{2,\geq 2}(\{1,4\}, \{2,3\}) = \Pi_{2,\geq 2}(\{2,3\}, \{1,4\}) = s_2,$$

and finally we have

$$\Pi_{2,\geq 2}(m_3, \emptyset) = \Pi_{2,\geq 2}(\emptyset, m_3) = \{s_1, s_2\}.$$

We stress that $\Pi_{2,\geq 2}(m_3, \emptyset)$ corresponds to the case where only integration with respect to the Brownian part is considered, while $\Pi_{2,\geq 2}(\emptyset, m_3)$ corresponds to the opposite case where only integration with respect to the jump part is considered. Hence,

$$\mathbb{E} \left[ J^\alpha_1 (F^1), J^\alpha_2 (F^2), J^\alpha_3 (F^3) \right] = C_1 \int_0^t \int_0^r F^1(u) F^2(r) F^3(u, r) \, du \, dr + C_2 \int_0^t \int_0^r F^1(r) F^2(u) F^3(u, r) \, du \, dr,$$

where

$C_1 = v(\alpha_1 \alpha_3) v(\alpha_2 \alpha_4) + v(\alpha_1 \alpha_3)(\alpha_2 \alpha_4)(0) \sigma^2 + v(\alpha_2 \alpha_4)(\alpha_1 \alpha_3)(0) \sigma^2 + (\alpha_1 \alpha_2 \alpha_3 \alpha_4)(0) \sigma^4$

and

$C_2 = v(\alpha_2 \alpha_3) v(\alpha_1 \alpha_4) + v(\alpha_2 \alpha_3)(\alpha_1 \alpha_4)(0) \sigma^2 + v(\alpha_1 \alpha_4)(\alpha_2 \alpha_3)(0) \sigma^2 + (\alpha_1 \alpha_2 \alpha_3 \alpha_4)(0) \sigma^4.$
2. For $N = 3$ with $m_1 = 1$, $m_2 = 1$ and $m_3 = 3$ we notice that $\Pi_{j=2,3}(B,B; m_1,m_2,m_3) = \emptyset$, for every $B \subseteq \{1,2,3,4,5\}$. This can be seen because by Definition 4.3 (i) each element of \{\overline{m}_{2,1}, \overline{m}_{2,2}, \overline{m}_{2,3}\} has to be in a different partition set. So we should have at least 3 partition sets, each with (at least) 2 elements. This is impossible as $m_1 + m_2 + m_3 = 5$. Hence,

$$E\left[J_1^{\alpha_1}(F^1), J_1^{\alpha_2}(F^2), J_2^{\alpha_3,\alpha_3}(F^3)\right] = 0.$$ 

It is easy to see that this observation can be generalized as follows: If there exists a $m_j$ such that $m_j > m_1 + \ldots + m_{j-1} + m_{j+1} + \ldots + m_N$ then

$$E\left[\prod_{j=1}^N J_{m_{j-1}+1,m_j}(F^j)\right] = 0.$$ 

Moment formulas similar to (44) have also been obtained by Peccati and Taqqu in [17, Corollary 7.4.1] and by Last et al. in [11, Theorem 1] for multiple integrals (similar to the multiple integrals introduced by Itô in [7]) generated by a compensated Poisson random measure. The proofs in [17] and [11] rely on Mecke’s Formula (see [12]), which is not applicable if the Lévy process has a Gaussian part. Here we use (37) instead of Mecke’s Formula. We stress that for the case without Gaussian part the moment formula in [11, Theorem 1] only requires an $L^1$-condition. This is due to the fact that for $\sigma = 0$ the multiple integrals can be considered pathwise.

**Remark 4.7** (Product and moment formula for linear combinations of elementary functions). Let $\mathcal{E}^m_T$ denote the linear subspace of $L^2(M^m_T, \rho^{\alpha_{m-1}+1})$ generated by $F_{\alpha_{m_j}} \in \mathcal{E}_T^{\rho_{m_j}}$ restricted to $M^m_T$. The elementary iterated integrals can be uniquely linearly extended to elements of $\mathcal{E}^m_T$. Hence, if $F^j \in \mathcal{E}^m_T$ has the representation

$$F^j = \sum_{k=1}^{M_j} F^j_{\alpha_{m_{j-1}+1}^{\alpha_{m_j}}}, \quad j = 1, \ldots, N,$$

then

$$\prod_{j=1}^N J_{m_{j-1}+1,m_j}(F^j) = \prod_{j=1}^N \left( \sum_{k=1}^{M_j} J_{m_{j-1}+1,m_j}(F^j_{\alpha_{m_{j-1}+1}^{\alpha_{m_j}}}) \right).$$

Using the formula

$$\prod_{j=1}^N \left( \sum_{k=1}^{M_j} a^{j,k} \right) = \prod_{j_1=1}^{M_1} \ldots \prod_{j_N=1}^{M_N} a^{1,j_1} a^{2,j_2} \ldots a^{N,j_N},$$

valid for real numbers $a^{j,k}$, $k = 1, \ldots, N_j$; $j = 1, \ldots, N$, the multi-linearity of the tensor product in Theorem 4.4 and the linearity of the identification rule, it is clear that (42) extends to this more general case. From Corollary 4.5 it is now clear that the moment formula (44) holds also for $F^j \in \mathcal{E}^m_T$.

### 4.2 Extensions of the product and moment formula

For practical applications Assumption 4.1 might not be desirable because it requires that $\Lambda$ is stable under multiplication. For example, if $\Lambda$ is an orthogonal basis of $L^2(\mu)$, in general, it fails to be stable under multiplication. However, $\Lambda$ being an orthogonal basis is especially interesting because then $\mathcal{X}_\Lambda$, defined in (33), consists of countably many orthogonal martingales, and it possesses the CRP with respect to $\mathbb{P}^X$ in the simpler form given in Remark 3.3. Therefore, in the present subsection, we extend (42) and (44) to products of iterated integrals generated by $\mathcal{X}_\Lambda$, where $\Lambda$ is an arbitrary system in $\Lambda \subseteq \bigcap_{p \geq 1} L^p(\mu)$. 

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To begin with, we prove the following technical lemma. We denote by $\mathbb{B}_\mathbb{R}$ the space of bounded measurable functions on $\mathbb{R}$.

**Lemma 4.8.** Let $\alpha \in L^2(\mu) \cap L^p(\mu)$, $p > 2$, and let $\Lambda := L^1(\mu) \cap \mathbb{B}_\mathbb{R}$. Then there exists $(\alpha_k)_{k \geq 1} \subseteq \Lambda$ such that $\alpha_k \to \alpha$ in $L^q(\mu)$ as $k \to +\infty$ for every $2 \leq q \leq p$.

**Proof.** For $\alpha \in L^2(\mu) \cap L^p(\mu)$ we define $\eta := \alpha1_{\{|\alpha| \leq 1\}}$ and $\gamma := \alpha1_{\{|\alpha| > 1\}}$. Then, $\alpha = \eta + \gamma$, where $\eta \in L^2(\mu)$ and $\gamma \in L^p(\mu)$. Because $\Lambda$ is dense in $L^q(\mu)$, for every $q > 1$, there exist $(\eta_k)_{k \geq 1}$ and $(\gamma^k)_{k \geq 1}$ in $\Lambda$ such that $\eta_k \to \eta$ in $L^2(\mu)$ and $\gamma^k \to \gamma$ in $L^p(\mu)$ as $k \to +\infty$, respectively. Since $\mu(\{|\alpha| > 1\}) < +\infty$, we have $\gamma^k \to \gamma$ also in $L^2(\mu)$. Notice that, because $\eta$ is bounded by one, we can also assume that $(\eta^k)_{k \geq 1}$ is uniformly bounded by one. But then $\eta^k \to \eta$ in $L^p(\mu)$. A sequence $(\alpha^k)_{k \geq 1} \subseteq \Lambda$ converging to $\alpha$ in $L^2(\mu)$ and in $L^p(\mu)$ is given, for example, by $\alpha^k := \eta^k1_{\{|\alpha| \leq 1\}} + \gamma^k1_{\{|\alpha| > 1\}} \in \Lambda$, $k \geq 1$. Finally, if $(\alpha^k)_{k \geq 1}$ converges to $\alpha$ in $L^2(\mu)$ and $L^p(\mu)$, then it also converges to $\alpha$ in $L^q(\mu)$ for every $2 \leq q \leq p$. \qed

We now come to the following proposition, which, in particular, extends (42) to $\mathcal{J}_e^{\mathcal{F}_X}$, provided that $\Lambda \subseteq \bigcap_{p \geq 2} L^p(\mu)$.

**Proposition 4.9.** Assume $X$ is a Lévy process with characteristics $(\gamma, \sigma^2, \nu)$. Let $\Lambda \subseteq \bigcap_{p \geq 2} L^p(\mu)$ and let $\mathcal{F}_X, \mathcal{F}_{\Lambda}$ denote the associated family of martingales. Then

(i) $X^\alpha$ possesses finite moments of every order, for every $\alpha \in \Lambda$.

(ii) $\Lambda$ is a total system in $L^2(\mu)$ if and only if $\mathcal{F}_X$ possesses the chaotic representation property with respect to $\Sigma^X$.

(iii) The product formula (42) and the moment formula (44) hold for the family $\mathcal{J}_e^{\mathcal{F}_X}$ of elementary iterated integrals generated by $\mathcal{F}_X$.

(iv) Let $F^j \in \mathcal{C}_T^{\mathcal{M}}$, $j = 1, \ldots, N$. Then the product formula (42) and the moment formula (44) hold for the product $\prod_{j=1}^N f_{\alpha_{m_{j-1}+1:m_j}}(F^j)$.

**Proof.** First we verify (i). By [10, Corollary 2.12], there exists a constant $C_{\alpha} > 0$ such that

$$||X_\alpha^\alpha||_{L^p(\mathbb{F})}^p \leq C_{\alpha}T \left(||\alpha||_{L^2(\mu)}^p + ||\alpha||_{L^p(\mu)}^p\right). \quad (45)$$

Hence, $\mathcal{F}_X$ is a family of martingales with finite moments of every order if $\Lambda \subseteq \bigcap_{p \geq 2} L^p(\mu)$. To see (ii) we refer to [2, Theorem 6.6]. To show (iii), let $\Lambda := L^1(\mu) \cap \mathbb{B}_\mathbb{R}$. Then $\Lambda$ satisfies Assumption 4.1 and, according to Theorem 4.4, the multiplication formula (42) holds for $\mathcal{J}_e^{\mathcal{F}_X}$. We now show by approximation that (42) holds also for $\mathcal{J}_e^{\mathcal{F}_{\Lambda}}$ if $\Lambda \subseteq \bigcap_{p \geq 2} L^p(\mu)$. We set $q_0 := 2m_NN$. By Lemma 4.8 for every $\alpha \in \Lambda$ there exists a sequence $(\alpha^k)_{k \geq 1} \subseteq \Lambda$ such that $\alpha^k \to \alpha$ in $L^2(\mu)$ and in $L^{q_0}(\mu)$ as $k \to +\infty$. Since (42) implies $X_T^\alpha - X_T^\alpha = X_T^{\alpha - \alpha}$ a.s., by (45) we get

$$||X_T^{\alpha} - X_T^\alpha||_{L^{q_0}(\mathbb{F})} = ||X_T^{\alpha - \alpha}||_{L^{q_0}(\mathbb{F})} \to 0 \quad as \quad k \to +\infty.$$

Let now $(\alpha_k^k)_{k \in \mathbb{N}} \subseteq \Lambda$ and $\alpha_k^k \to \alpha_k$ in $L^2(\mu)$ and in $L^{q_0}(\mu)$, for every $i = 1, \ldots, m_N$. Then, setting $F^j := F^{\alpha_{m_{j-1}+1:m_j}}$ for every $j = 1, \ldots, N$, Lemma 3.2 (ii) with $p = N$ implies

$$\mathbb{E} \left[ \sup_{t \in [0,T]} \left| J_{m_{j-1}+1:m_j}^\alpha(F^j)_t - J_{m_{j-1}+1:m_j}^{\alpha_{m_{j-1}+1:m_j}}(F^j)_t \right|^{N} \right] \to 0 \quad as \quad k \to +\infty.$$

This yields, as a consequence of Hölder’s inequality,

$$\prod_{j=1}^N J_{m_{j-1}+1:m_j}^\alpha(F^j)_t \to \prod_{j=1}^N J_{m_{j-1}+1:m_j}^{\alpha_{m_{j-1}+1:m_j}}(F^j)_t \quad in \quad L^1(\mathbb{F}) \quad as \quad k \to +\infty.$$
for every $t \in [0, T]$. For the right-hand side of (42), we will show that

$$\int_{M^k} \left( \bigotimes_{j=1}^N F^j \right) (t_1, \ldots, t_{|\gamma|}) dY^\alpha_{\gamma(1)} \ldots dY^\alpha_{\gamma(|\gamma|)}$$

converges in $L^2(\mathbb{P})$. For this, we first consider the convergence of the processes $Y^\alpha_{\gamma(1)} \ldots$.

We define $Y^\alpha_{\gamma(1)} \ldots$ according to (38). We use the representation (38) and discuss the convergence cases separately. Notice that, since we have chosen the sequence $\alpha_j^k \to \alpha_j$ in $L^q(\mu)$ for every $2 \leq q \leq q_0 = 2^{m_n}N$, $j = 1, \ldots, N$, and $\mu = \sigma^2 \delta_0 + \nu$, it follows that

$$v(\alpha^k_{\gamma(1)}) = v\left( \prod_{\ell \in \mathcal{S}_\gamma} \alpha^k_{\ell} \right) \to v\left( \prod_{\ell \in \mathcal{S}_\gamma} \alpha_{\ell} \right) = v(\alpha_{\gamma(1)}) \quad \text{and} \quad \alpha^k_{\gamma(1)}(0) \to \alpha_{\gamma(1)}(0) \quad \text{as} \quad k \to +\infty.$$
We notice that the constant \( c^k \) remains bounded as \( k \to +\infty \). Indeed, \( c^k \) can be explicitly computed and consists of products of terms of the form \( v((\alpha_{(S_j)i_j})^2) \) or \( v(\alpha_{(S_j)i_j}) \), as well as \( T \) and \( \sigma^2 \) for \( j = 1, \ldots, r - 1 \), and similarly, but not dependent on \( k \), for \( j = r + 1, \ldots, |s| \). The constant \( C^k := C(T, \alpha_{(S_j)i_j}^k, \alpha_{(S_j)i_j}^k) \) which arises from Itô’s isometry or Cauchy-Schwarz inequality concerning the integrator \( (Y_{\alpha_{(S_j)i_j}} - Y_{\alpha_{(S_j)i_j}}) \), can assume the values \( C^k = \sigma^2(\alpha_{(S_j)}(0) - \alpha_{(S_j)}(0))^2 \) or \( C^k = v((\alpha_{(S_j)} - \alpha_{(S_j)})^2) \) when the integrator is in \( \mathcal{H}^2 \), while we would have \( C^k = T(v(\alpha_{(S_j)} - \alpha_{(S_j)}))^2 \) or \( C^k = T \sigma^2(\alpha_{(S_j)}(0) - \alpha_{(S_j)}(0))^2 \) for deterministic integrators. But in any case,

\[
C^k = C(T, \alpha_{(S_j)i_j}^k, \alpha_{(S_j)i_j}^k) \to 0 \quad \text{as} \quad k \to +\infty.
\]

The proof of the product formula for this more general case is complete. For the moment formula \((44)\) we observe that this is a direct consequence of the product formula \((42)\) and the proof can be given as in Corollary \((4.5)\). Clearly \((iv)\) is a direct consequence of \((iii)\) because of the linearity of the iterated integrals and the multi-linearity of the tensor product. The proof of the proposition is complete.

We now generalize \((42)\) and \((44)\) to the case in which the functions \( F^j \) need not to be bounded but, rather, satisfy some integrability condition. This is the main result of the present paper. For the definition of \( s \) recall Definition \((4.3)\) and the relations \((41)\) and \((39)\).

**Theorem 4.10.** Let \( X \) be a Lévy process with characteristic triplet \((\gamma, \sigma^2, \nu)\). Let \( \alpha_1, \ldots, \alpha_{|s|} \) belong to \( \bigcap_{p \geq 2} L^p(\mu) \) and assume that \( F^j \in L^2(M_{T^p}, \rho \alpha_{\gamma-j+1+m}) \), \( j = 1, \ldots, N \), are such that

\[
\int_{M_{T^p}} \left( \bigotimes_{j=1}^N F^j \right)_s (t_1, \ldots, t_{|s|}) d\lambda(t_1, \ldots, t_{|s|}) < \infty, \quad \forall B \subseteq \mathbb{R}_N, \forall s \in \Pi_{\geq 2, \geq 1} (B, B^c; m_1, \ldots, m_N). \tag{48}
\]

Then, the product formula \((42)\) and the moment formula \((44)\) extend to \( \prod_{j=1}^N \left( \int_{M_{T^p}} \alpha_{\gamma-j+1+m} \mathbb{E} F^j \right)_t \) for every \( t \in [0, T] \).

Condition \((48)\) is an \( L^2 \)-bound on \( \left( \bigotimes_{j=1}^N F^j \right)_s \) and ensures that all the stochastic integrals appearing on the right-hand side of \((42)\) are square integrable martingales.

Notice that \((48)\) is fulfilled, for example, whenever \( F^j \in L^{2N}(M_{T^p}, \rho \alpha_{\gamma-j+1+m}) \) for any \( j = 1, \ldots, N \), (see Remark \((4.11)\) below).

**Proof of Theorem 4.10.** We choose \( \Lambda \subseteq \bigcap_{p \geq 2} L^p(\mu) \) such that \( \alpha_1, \ldots, \alpha_{|s|} \in \Lambda \) and recall that, because of Proposition \((4.9)\) \((iv)\), the elements of \( \mathcal{F}_\Lambda \) possess moments of every order. From Proposition \((4.9)\) \((iv)\), we know that the product rule \((42)\) extends to the case \( \Lambda \subseteq \bigcap_{p \geq 2} L^p(\mu) \) for \( F^j \in \mathcal{C}_{T^p} \), \( j = 1, \ldots, N \). For later use, we derive an estimate for the second moment of the iterated integrals on the right hand side of this extension of \((42)\). Let \( F^j \in \mathcal{C}_{T^p}, \quad j = 1, \ldots, N \), and \( \Lambda \subseteq \bigcap_{p \geq 2} L^p(\mu) \).

Applying Itô’s isometry every time the process \( Y_{\alpha_{(S_j)i_j}} \) is random, and Hölder’s inequality whenever it is deterministic, we find a constant \( c(T, \alpha_{(S_j)i_j}, \alpha_{(S_j)i_j}) \alpha, \nu) > 0 \), such that

\[
\mathbb{E} \left[ \left( \int_{M_{T^p}} \left( \bigotimes_{j=1}^N F^j \right)_s (t_1, \ldots, t_{|s|}) d\lambda_{t_1}^{\alpha_{(S_j)i_j}} \ldots d\lambda_{t_{|s|}}^{\alpha_{(S_j)i_j}} \right)^2 \right] \leq c(T, \alpha_{(S_j)i_j}, \alpha_{(S_j)i_j}) \alpha, \nu) \int_{M_{T^p}} \left( \bigotimes_{j=1}^N F^j \right)_s (t_1, \ldots, t_{|s|}) d\lambda|s|(t_1, \ldots, t_{|s|}). \tag{49}
\]
The estimate (49) implies that we may extend by linearity and continuity the iterated integral appearing on the left-hand side of (49) to those functions \( F_j \in L^2(M_T^{(j)}, \rho_{\alpha_j-1+1/\pi_j}), \) \( j = 1, \ldots, N, \) which furthermore satisfy the integrability condition (48). We now divide the proof into three steps.

Step 1. In this first step we are going to show that the product formula (42) extends to indicator functions \( F_j = 1_{A_j} \in L^2(M_T^{(j)}, \rho_{\alpha_j-1+1/\pi_j}). \) By [4, Theorem 2.40], given an \( \varepsilon > 0 \) and a Borel set \( A_j \in \mathcal{B}([0, T]^{m_j}), \) there exists an \( M \geq 1 \) and disjoint rectangles \( R_{j,1}, \ldots, R_{j,M} \subseteq M_T^{(j)} \) whose sides are intervals such that

\[
\int_{M_T^{(j)}} \left| 1_{A_j} - \sum_{i=1}^M 1_{R_{j,i}} \right| d\lambda_{m_j} = \lambda_{m_j} \left( A_j \Delta \bigcup_{i=1}^M R_{j,i} \right) < \varepsilon,
\]

where \( A \Delta B := (A \setminus B) \cup (B \setminus A) \) denotes the symmetric difference of \( A \) and \( B, \) and we used the identity \( 1_{A \Delta B} = |1_A - 1_B|. \) We notice that \( \sum_{i=1}^M 1_{R_{j,i}} \in \rho_{\alpha_j} \) and \( 0 \leq \sum_{i=1}^M 1_{R_{j,i}} \leq 1. \) Hence, since \( \rho_{\alpha_j} \) is absolutely continuous with respect to \( \lambda_{m_j}, \) there are sequences \( (F_k^j)_{k=1}^\infty \subseteq \rho_{\alpha_j} \) with \( |F_k^j| \leq 1 \) such that

\[
F_k^j \to 1_{A_j} \quad \text{in} \quad L^2(M_T^{(j)}, \rho_{\alpha_j-1+1/\pi_j}), \quad \text{as} \quad k \to +\infty, \quad j = 1, \ldots, N.
\]

To estimate the difference below we first use a telescopic sum, apply the Cauchy-Schwartz inequality and then use the product formula (42) from Proposition 4.9(iv) so that

\[
E \left[ \left( \sum_{j=1}^N J_{m_j}^{\alpha_j-1+1/\pi_j} (F_k^j)_{t} - \prod_{j=1}^N J_{m_j}^{\alpha_j-1+1/\pi_j} (F^j)_{t} \right)^2 \right] \leq c \sum_{r=1}^N E \left[ \left( \sum_{j=1}^r J_{m_j}^{\alpha_j-1+1/\pi_j} (F_k^j)_{t} - \prod_{j=1}^r J_{m_j}^{\alpha_j-1+1/\pi_j} (F^j)_{t} \right)^2 \right]
\]

\[
= c \sum_{r=1}^N E \left[ \left( \sum_{B \subseteq [m_N]} \sum_{s \subseteq \{i \in \mathbb{N} : i < r\} \sum_{j=1}^r \int_{M_t^{(j)}} \left( \bigotimes_{j=1}^r F_k^j \otimes (F_r^{r+1} - F_{r+1}^r) \otimes \bigotimes_{i=r+2}^N F_i^j \right) (t_1, \ldots, t_{|s|}) \right. \right.
\]

\[
\left. \left. \times dY_{(s)}^{(j_1)} \ldots dY_{(s)}^{(j_{|s|})} \right) \right)^2 \right],
\]

where \( c > 0 \) is a constant. By (49) we conclude that the right-hand side of (51) converges to zero. Indeed, since the integrand is bounded, by dominated convergence and (50) we get

\[
\int_{M_t^{(j)}} \left( \bigotimes_{j=1}^r F_k^j \otimes (F_r^{r+1} - F_{r+1}^r) \otimes \bigotimes_{i=r+2}^N F_i^j \right)^2 d\lambda_{m_j}(t_1, \ldots, t_{|s|}) \to 0 \quad \text{as} \quad k \to +\infty.
\]

Hence, we have shown that \( Z^k := \prod_{j=1}^N J_{m_j}^{\alpha_j-1+1/\pi_j} (F_k^j)_{t} \) is a Cauchy sequence in \( L^2(\mathbb{P}). \) So, there exists \( Z \in L^2(\mathbb{P}) \) such that \( Z^k \to Z \) in \( L^2(\mathbb{P}) \) as \( k \to \infty. \) On the other side, since \( J_{m_j}^{\alpha_j-1+1/\pi_j} (F_k^j)_{t} \) converges to \( J_{m_j}^{\alpha_j-1+1/\pi_j} (F^j)_{t} \) in \( L^2(\mathbb{P}) \) as \( k \to \infty, \) we also have that \( Z^k \to \prod_{j=1}^N J_{m_j}^{\alpha_j-1+1/\pi_j} (F^j)_{t} \) in probability as \( k \to +\infty. \) This implies

\[
Z = \prod_{j=1}^N J_{m_j}^{\alpha_j-1+1/\pi_j} (F^j)_{t} \quad \text{a.s.},
\]

because of the uniqueness of the limit in probability. Thus, \( \prod_{j=1}^N J_{m_j}^{\alpha_j-1+1/\pi_j} (F^j)_{t} \in L^2(\mathbb{P}) \) for every \( t \in [0, T]. \) From Proposition 4.9(iv), we know that \( Z^k \) satisfies the product formula (42) for every \( k. \)
We now discuss the convergence of the corresponding right hand side in (42) for $Z^k$. Similarly as for (51) but now with $F^j$ instead of and $F^j$, we see that

$$
E \left[ \left( \int_{M^N_T} ^{N} \left( \bigotimes_{j=1} ^N F^j_k - \bigotimes_{j=1} ^N F^j \right) \right) \mathcal{d} \lambda(t_1, \ldots, t_{|\mathcal{S}|}) \right] \rightarrow 0 \quad \text{as} \quad k \rightarrow +\infty.
$$

This means that also $Z$ satisfies (42). Thus, (42) holds for $F^j = 1_{A_j} \in L^2(M_T^{m_j}, \rho^{\alpha m_j-1+1 \pi_j})$. The proof of the first step is complete.

**Step 2.** We observe that, as a consequence of Step 1, by linearity, the product formula (42) also holds for simple functions from $L^2(M_T^{m_j}, \rho^{\alpha m_j-1+1 \pi_j})$.

**Step 3.** In this step we show the product formula (42) for arbitrary functions $F^j \in L^2(M_T^{m_j}, \rho^{\alpha m_j-1+1 \pi_j})$, $j = 1, \ldots, N$, which furthermore satisfy the integrability condition (48). Any $F^j \in L^2(M_T^{m_j}, \rho^{\alpha m_j-1+1 \pi_j})$ can be pointwise approximated by a sequence $(F^j_k)_{k \geq 1}$ of simple functions such that $|F^j_k| \uparrow |F^j|$ as $k \rightarrow +\infty$ (see, for example, [4, Theorem 2.10]). Because of this and (48), we get

$$
\int_{M^N_T} ^{N} \left( \bigotimes_{j=1} ^N F^j_k \right) (t_1, \ldots, t_{|\mathcal{S}|}) \mathcal{d} \lambda(t_1, \ldots, t_{|\mathcal{S}|}) \leq \int_{M^N_T} ^{N} \left( \bigotimes_{j=1} ^N F^j \right) (t_1, \ldots, t_{|\mathcal{S}|}) \mathcal{d} \lambda(t_1, \ldots, t_{|\mathcal{S}|}) < +\infty.
$$

This implies that we may extend by linearity and continuity the iterated integrals on the right hand side of (42) for $Z^k$ to those $F^j \in L^2(M_T^{m_j}, \rho^{\alpha m_j-1+1 \pi_j})$, $j = 1, \ldots, N$, which satisfy (48). Using the second step, we can conclude like in the first step that (51) holds also for simple functions. The difference is that now the integrand

$$
\left( \bigotimes_{j=1} ^r F^j_k \otimes (F^j_{r+1} - F^j_{r+1}) \otimes \bigotimes_{i=r+2} ^N F^j_i \right)
$$

is not bounded. However, because of (55), we can apply dominated convergence to get (52). Clearly, we can identify the $L^2(\mathbb{P})$-limit $Z$ of the corresponding Cauchy sequence $Z^k$ as in (53). Because of the second step, $Z^k$ satisfies (42). To complete the proof we verify that also for the general case we have the correct limit expression on the right hand side of (42) and, hence, that the multiplication formula holds also for the limit $Z$. An estimate like (49) ensures that the addends on the right hand side of (42) belong all to $L^2(\mathbb{P})$ for any $F^j \in L^2(M_T^{m_j}, \rho^{\alpha m_j-1+1 \pi_j})$, $j = 1, \ldots, N$, such that (48) holds. Hence, (54) follows from (48) and (55), because of $|F^j_k| \uparrow |F^j|$ as $k \rightarrow +\infty$, $j = 1, \ldots, N$. This proves (42) for the general case.

To see (44), because of (42) and since each summand on the right-hand side of (42) belongs to $L^2(\mathbb{P})$, we can proceed as in the proof of Corollary 4.5. The proof of the theorem is now complete. 

**Remark 4.11.** A sufficient condition for (48) is

$$
F^j \in L^{2N}(M_T^{m_j}, \lambda^{m_j}), \quad j = 1, \ldots, N.
$$

To see this, we recall that, from (39), to apply the identification rule $s: H(u_1, \ldots, u_{|\mathcal{S}|}) \mapsto H_s(t_1, \ldots, t_k)$, we need $H$ to depend on $u_1, \ldots, u_{|\mathcal{S}|}$. Therefore, we extend each $F^j$ which depends on the variables $u_{|\mathcal{S}|-1+1}, \ldots, u_{|\mathcal{S}|}$ constantly to all variables $u_1, \ldots, u_{|\mathcal{S}|}$ and denote the extension by $\hat{F}^j(u_1, \ldots, u_{|\mathcal{S}|})$. Since

$$
|S_t \cap \{u_{|\mathcal{S}|-1+1}, \ldots, u_{|\mathcal{S}|}\}| \leq 1,
$$

the identification rule

$$
s: \hat{F}^j(u_1, \ldots, u_{|\mathcal{S}|}) \mapsto \hat{F}^j_k(t_1, \ldots, t_{|\mathcal{S}|})
$$
causes only a one-one renaming but no identification among the variables $u_{m_{i-1}+1}, \ldots, u_{m_i}$. We have

$$\left( \bigotimes_{j=1}^{N} F_j \right) \left( t_1, \ldots, t_{|s|} \right) = (F_1)^s(t_1, \ldots, t_{|s|}) \times \ldots \times (F_N)^s(t_1, \ldots, t_{|s|}).$$

Consequently, by Hölder’s inequality,

$$\int_{M^n} \left( \bigotimes_{j=1}^{N} F_j \right)^2 \left( t_1, \ldots, t_{|s|} \right) d\lambda(t_1, \ldots, t_{|s|}) \leq \prod_{j=1}^{N} \left( \int_{M^n} (F_j)^{2N} \left( t_1, \ldots, t_{|s|} \right) d\lambda(t_1, \ldots, t_{|s|}) \right)^{\frac{1}{N}} \leq \prod_{j=1}^{N} T^{|s|} \left( \int_{M^n} (F_j)^{2N} \left( t_1, \ldots, t_{|s|} \right) d\lambda(t_1, \ldots, t_{|s|}) \right)^{\frac{1}{N}}.$$

4.3 Examples for $\mathcal{D}_{\Lambda}$

We conclude this section giving examples of $\mathcal{D}_{\Lambda}$ satisfying the assumptions of Theorem 4.10. We observe that, if $\alpha \in L^2(\mu)$, then the Lévy measure $\nu^{\alpha}$ of the square integrable martingale and $\mathbb{F}$-Lévy process $X^{\alpha}$ is the image measure of $\nu$ under the mapping $\alpha$ (see [1], Definition 7.6), that is,

$$\nu^{\alpha}(dx) = (\nu \circ \alpha^{-1})(dx). \quad (56)$$

The main point of this part is to construct families of martingales $\mathcal{D}_{\Lambda}$ possessing moments of every order. Because of the equivalence between the totality of a system $\Lambda$ in $L^2(\mu)$ and the CRP of the family $\mathcal{D}_{\Lambda}$, we will consider systems $\Lambda$ which are total in $L^2(\mu)$ or, more specifically, orthogonal bases of $L^2(\mu)$.

Dyadic Intervals. This example is very simple but it is interesting because it holds without further assumptions on the Lévy measure. Let $X$ be a Lévy process relative to $\mathbb{F}$ and let $(\gamma, \sigma^2, \nu)$ denote its characteristic triplet. Let $\mathcal{D}$ denote the set of dyadic numbers and define

$$\hat{\Lambda} := \{ 1_{(a,b]} 1_{\{0\}} + 1_{(a,b]} 1_{\mathbb{R}\setminus\{0\}}, \ a, b \in \mathcal{D} : 0 \notin [a,b] \} \cup \{0\}.$$  

Then $\hat{\Lambda}$ is total in $L^2(\mu)$, consists of countably many functions and satisfies Assumption 4.1. Clearly, we can orthonormalize $\hat{\Lambda}$ in $L^2(\mu)$ and obtain an orthonormal basis $\Lambda$ of $L^2(\mu)$ consisting of bounded functions. The associated family $\mathcal{D}_{\Lambda}$ consists of countably many orthogonal martingales and it possesses the CRP with respect to $\mathbb{F}^X$ (cf. Proposition 4.9) in the simpler version of Remark 3.5. Furthermore, from Theorem 4.10 we get (42) and (44) for the iterated integrals generated by $\mathcal{D}_{\Lambda}$.

Teugels martingales. Teugels martingales were already discussed in [3,3]. We notice that, to introduce Teugels martingales as square integrable martingales, it is enough that $X$ possesses moments of arbitrary order, that is, all monomials $p_n(x) := x^n$ belong to $L^2(\nu)$, for every $n \geq 1$. We assume that there exist constants $\hat{\lambda}, \varepsilon > 0$ such that $x \mapsto 1_{(-\varepsilon, \varepsilon)} e^{\lambda|x|/2}$ belongs to $L^2(\nu)$ and define $h_1(x) := 1_{\{0\}} + 1_{\mathbb{R}\setminus\{0\}}(x)p_1(x)$ and $h_n(x) := 1_{\mathbb{R}\setminus\{0\}}(x)p_n(x), \ n \geq 2$. The system $\hat{\Lambda} := \{ h_n, \ n \geq 1 \}$ belongs to $L^2(\mu)$ and it is total. Furthermore, the identity $X^{(i)} = X^{h_i}$ holds, where $X^{(i)}$ denotes the $i$th Teugels martingale as introduced in [3,3]. The associated family $\mathcal{D}_{\Lambda}$ of Teugels martingales is compensated-covariation stable and, according to Proposition 4.9(ii), it possesses the CRP with respect to $\mathbb{F}^X$. However, the system $\hat{\Lambda}$ does not satisfy Assumption 4.1 because it does not consist
of bounded functions and it is not stable under multiplication. Let \( v^h \) be the Lévy measure of the martingale \( X^h \). Then, from (56), we get
\[
\int_{\mathbb{R}} |x|^m v^h(dx) = \int_{\mathbb{R}} |x|^{m+n} v(dx) < +\infty.
\]
Hence \( \tilde{\Lambda} \subseteq \cap_{p \geq 2} L^p(\mu) \), and the family \( \mathcal{H}_{\tilde{\Lambda}} \) is contained in \( \cap_{p \geq 2} \mathcal{H}^p \). From Theorem 4.10 we get (42) and (44) for the iterated integrals generated by \( \mathcal{H}_{\tilde{\Lambda}} \). The family \( \mathcal{H}_{\tilde{\Lambda}} \) is the family of Hermite polynomials. Therefore, the associated family of martingales \( \mathcal{H}_{\tilde{\Lambda}} \) consists of countably many orthogonal martingales and satisfies the CRP with respect to \( \mathbb{P}^X \) in the simpler form of Remark 3.5. Moreover, the product and the moment formula hold for the iterated integrals generated by \( \mathcal{H}_{\tilde{\Lambda}} \).

**Hermite polynomials.** Assume that \( X \) is a Lévy process with Lévy measure \( v \) which is equivalent to the Lebesgue measure on \( \mathbb{R} \), that is,
\[
v(dx) = h(x)dx, \quad h(x) > 0.
\]
This is, for example, the case if \( X \) is an \( \alpha \)-stable Lévy process (see [19, Chapter 3]). In this case, if \( (H_n)_{n \geq 1} \) is the family of Hermite polynomials, setting \( g(x) := (h(x))^{-1/2} e^{-x^2/2} \), the family \( \Lambda := \{P_n, n \geq 1\} \), where \( P_n := 1_{\{0\}} H_n(0) + 1_{\mathbb{R}\setminus\{0\}} gH_n \) is an orthogonal basis of \( L^2(\mu) \). However, in general, we cannot expect that the family \( \mathcal{H}_{\tilde{\Lambda}} \) has moments of every order, that is, that the inclusion \( \Lambda \subseteq \cap_{p \geq 2} L^p(\mu) \) holds. From (56), using the definition of \( g \) and of Hermite polynomials (see [15, §1.1]) we see that
\[
\int_{\mathbb{R}\setminus\{0\}} |x|^{2m} v^h(dx) = \frac{1}{n!} \int_{\mathbb{R}\setminus\{0\}} (h(x))^{1-m} e^{-mx^2} |q_{2m+n}(x)| dx
\]
where \( q_n \) denotes a polynomial function of order \( n \), \( n \geq 1 \). Therefore, to ensure that \( \mathcal{H}_{\tilde{\Lambda}} \) has finite moments of every order, we derive the condition on \( h \)
\[
\int_{\mathbb{R}\setminus\{0\}} (h(x))^{1-m} e^{-mx^2} |x|^{2m+n} dx < +\infty, \quad \text{for every } m, n \geq 1.
\]
Hence, if \( h \) satisfies (58), the moment formula (44) holds for the iterated integrals generated by the orthogonal family \( \mathcal{H}_{\tilde{\Lambda}} \).

Observe that \( \alpha \)-stable processes are an example of a family of Lévy processes satisfying (58). We remark that, for an \( \alpha \)-stable Lévy process \( X \), the family \( \mathcal{H}_{\tilde{\Lambda}} \) has finite moments of every order but the Lévy process \( X \) itself may possess infinite moments. This is the case, for example, if \( X \) is a Cauchy process.

**Haar basis.** We consider a Lévy process \( X \) with Lévy measure \( v \) as in (57). Let \( \lambda \) be the Lebesgue measure on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) and \( \{\psi_{jk} := 2^j \psi(2^j x - k), x \in \mathbb{R}, j, k \in \mathbb{Z}\} \subseteq L^2(\lambda) \) be the Haar basis. The generic element of the Haar basis can be written as
\[
\psi_{jk}(x) := 2^j \left[ \left( \frac{1}{2^j} \frac{x+k+1}{2^j} \right) - \left( \frac{1}{2^j} \frac{x+k+1}{2^j} \right) \right]
\]
and its support is \( \{\psi_{jk} \neq 0\} = \left( \frac{k}{2^j}, \frac{k+1}{2^j} \right), j, k \in \mathbb{Z} \). Note that the boundary points of these intervals are dyadic rational numbers. Defining \( \Lambda := \{\psi_{jk}(0) 1_{\{0\}} + 1_{\mathbb{R}\setminus\{0\}} h^{-1/2} \psi_{jk}, j, k \in \mathbb{Z}\} \) we obtain an orthogonal basis of \( L^2(\mu) \). The system \( \mathcal{H}_{\tilde{\Lambda}} \) is a family of orthogonal martingales. In general, the
martingales in $\mathcal{X}_\Lambda$ do not have finite moments of every order. Let us denote be $\nu^{\psi, k}$ the Lévy measure of the martingales $X^{\psi, k}$. Then, from (56), we get
\[
\int_{\mathbb{R}\setminus\{0\}} |x|^{2m} \nu^{\psi_k}(dx) = \int_{\mathbb{R}\setminus\{0\}} (h(x))^{1-m} |\psi_{jk}(x)|^{2m} \nu(dx).
\]
That is, we obtain the following condition on $h$
\[
\int_a^b (h(x))^{1-m} \nu(dx) < +\infty, \quad \text{for all dyadic rational numbers} \ a \text{ and } b: \ 0 \notin [a,b]. \tag{59}
\]
Hence, if $h$ satisfies (59), we get (44) for the iterated integrals generated by $\mathcal{X}_\Lambda$. Notice that (59) is satisfied, for example, if $h > 0$ is bounded over all intervals whose extremes are dyadic rational numbers. We again find that if $X$ is an $\alpha$-stable Lévy process, condition (59) is satisfied.

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