ON \(p\)-CENTRAL GROUPS

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Abstract. We extend the notion of free \(p\)-central groups for odd primes \(p\) to the case \(p = 2\) by introducing a variant of the lower \(p\)-central series. This enables us to calculate Schur multipliers of free \(p\)-central groups. We also prove that for any \(p\)-central group the exponent of its Schur multiplier divides the exponent of the group, and determine its exponential rank.

1. Introduction

Let \(G\) be a finite \(p\)-group and denote by \(\Omega_i(G)\) the subgroup of \(G\) generated by the elements of order dividing \(p^i\). We call \(G\) a \(p\)-central group if \(p\) is odd and \(\Omega_1(G) \leq Z(G)\), or if \(p = 2\) and \(\Omega_2(G) \leq Z(G)\). These groups are the dual counterpart of powerful \(p\)-groups, which have proved immensely useful in the study of finite \(p\)-groups and pro-\(p\) groups [11, 4].

The power structure of \(p\)-central groups has been explicitly studied by several authors, beginning with Laffey [9, 10], and more recently by Mann [12]. It has been shown that these groups possess certain properties of regular \(p\)-groups, for example, the subgroup \(\Omega_i(G)\) is equal to the subset of all elements of order dividing \(p^i\). However, unlike powerful \(p\)-groups, \(p\)-central groups need not be regular (cf. [2]). A more general approach to the study of these groups has been undertaken by Bubboloni and Corsi Tani in [3], where free \(p\)-central groups are constructed for odd primes \(p\). Their work is based on the properties of the well-known lower \(p\)-central series \(\lambda_i(F_r)\) of the free group \(F_r\) on \(r\)-generators [8, VIII]. More specifically, they proved that the group \(\lambda n_2 G_r = F_r/\lambda n_2+1(F_r)\) is \(p\)-central of exponent \(p^n\), and that any \(r\)-generated \(p\)-group of exponent \(p^n\) is its homomorphic image.

In this note, we extend the results of [3] to the even prime by appropriately adapting the lower \(p\)-central series \(\lambda_i\) to a variant \(\lambda_i\), defined inductively by \(\lambda_1(G) = G\) and \(\lambda_n+1(G) = \lambda_n(G)\{\lambda_n(G), G\}\). This is of course motivated by the definition of \(p\)-central groups. After establishing some basic properties of this series, we put \(\lambda n_2 G_r = F_r/\lambda n_2+1(F_r)\) and prove the following theorem.

Theorem 1.1. The group \(\lambda n_2 G_r\) is an \(r\)-generated 2-central group of exponent \(4^n\). Moreover, any \(r\)-generated 2-central group of exponent \(2^n\) is a homomorphic image of \(\lambda n_2+1 G_r\), and more generally: any finite \(r\)-generated 2-group is a homomorphic image of \(\lambda m G_r\) for some \(m\).

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We obtain several properties of the group $nG_r$ which are known to hold in the odd case. This is done by studying the $\lambda$-series in more detail, the results of which also enable us to provide an explicit formula for the Schur multiplier of free $p$-central groups via Hopf’s formula.

**Theorem 1.2.** The Schur Multiplier $M(nG_r)$ is elementary abelian (resp. free $\mathbb{Z}_4$-module for $p = 2$) of rank $\sum_{i=1}^{n+1} c(r, i)$, where $c(w, g)$ is the number of basic commutators of weight $w$ on $g$ generators in a fixed sequence.

For general $p$-central groups, we prove a classical result concerning the exponent of their Schur multipliers. The theorem is an affirmation of a special case of a conjecture due to Schur, stating that the exponent of $M(G)$ divides $\exp G$ for every finite group $G$. By homological arguments, it suffices to consider this question for finite $p$-groups. The conjecture has turned out to be false in general (cf. [13]), yet it is still not known whether or not it holds true for odd primes $p$. We prove that it is so for the class of $p$-central groups.

**Theorem 1.3.** Let $G$ be a $p$-central group. Then $\exp M(G) \leq \exp G$.

Our final contribution to the study of $p$-central groups is determining their exponential rank, recently introduced by Moravec in [13]. This provides yet another link between $p$-central groups and powerful $p$-groups, as the results are completely the same, and indicates a potentially profounder role of exponential rank. The theorem goes as follows.

**Theorem 1.4.** Let $G$ be a $p$-central group.

(a) If $p$ is odd, then $\text{exp} \text{rank}(G) = 0$.

(b) If $p = 2$ and $G$ is not abelian, then $\text{exp} \text{rank}(G) = 1$.

2. The $\lambda$-series

Let the $\lambda$-series of a group $G$ be defined recursively by

$$\lambda_1(G) = G, \quad \lambda_{n+1}(G) = \lambda_n(G)[\lambda_n(G), G].$$

This is a descending series of characteristic subgroups of $G$. Following [8], we also denote

$$N_{n,k}(G) = \gamma_k(G)^{\gamma_{n-k} - 1} \gamma_{n-k-1}(G)^{\gamma_{n-k-2} - 1} \cdots \gamma_n(G)$$

for $1 \leq k \leq n$. Our first aim is to show that the groups $\lambda_i$ and $N_{i,j}$ are tightly correlated. We begin with a lemma.

**Lemma 2.1.** Let $G$ be a group and $m, n, k$ positive integers.

(a) For any $1 \leq i \leq n$, we have $[\gamma_i(G)^{\gamma_{n-i}}, G] \leq N_{n+1,i,1}(G)$.

(b) $[N_{n,k}(G), G] = N_{n+1,k+1}(G)$ and $N_{n,k}(G)^{\gamma_m} \leq N_{n+m,k}(G)$.

**Proof.** Using [8] Lemma 1.2, we have

$$[\gamma_i(G)^{\gamma_{n-i}}, G] \leq \prod_{r=0}^{2(n-i)} \gamma_{1+i,2r}(G)^{2^{(n-i)-r}}.$$
Taking two consecutive terms of this product into account, the inclusion
\[ \gamma_{1+2s+1}(G)^{2^{(n-1)-(2s+1)}} \gamma_{1+2s+2}(G) \leq \gamma_{1+s+2}(G)^{1^{(n-1)-(s+1)}} \]
holds for any \(0 \leq s \leq n-i-1\), since \(1+i2^{s+1} \geq i+s+2\). This proves part (a) of the lemma, from which the inclusion \([N_{n,k}(G), G] \leq N_{n+1,k+1}(G)\) of the first part of (b) follows directly. For the other inclusion, we refer the reader to the proof of Theorem 2.4 of [1], the argument is essentially the same. The second part of (b) follows from the classical Hall-Petrescu formula: \(N_{n,k}(G)^4 \leq N_{n+1,k}(G) \cdot [N_{n,k}(G), N_{n,k}(G)] \leq N_{n+1,k}(G)\). □

An explicit formula for the \(\bar{\lambda}\)-series is thus at hand.

**Proposition 2.2.** Let \(G\) be a group. For any \(n\), we have \(\bar{\lambda}(G)_n = N_{n,1}(G)\).

**Proof.** It suffices to verify that the series \(N_{n,1}(G)\) satisfies the recursion formula for the \(\bar{\lambda}\)-series, and we do this by induction. We plainly have \(\gamma_1(G)^{4^{n-i+1}} \leq (\gamma_1(G)^{4^{n-i}})^4 \leq N_{n,1}(G)^4\) for any \(1 \leq i \leq n\) and \(\gamma_n+1(G) = [\gamma_n(G), G] \leq [N_{n,1}(G), G]\), which implies \(N_{n+1,1}(G) \leq N_{n,1}(G)^4[N_{n,1}(G), G]\). The reverse inclusion follows from Lemma 2.3. □

When the nilpotent quotients of the given group \(G\) are torsion-free (for example when \(G\) is free), another connection between the groups \(\bar{\lambda}_i(G)\) and \(N_{i,j}(G)\) exists. In the proof, we will use a well-known consequence of the Hall-Petrescu formula, which we state beforehand.

**Lemma 2.3.** Let \(G\) be a group and \(x, y \in G\). Suppose \(p\) is a prime and \(k\) a positive integer. Then
\[ (xy)^p \equiv x^p y^p \mod \gamma_2((x, y))^p \prod_{i=1}^k \gamma_{p^i}((x, y))^{p^{k-i}}. \]
If furthermore \(x\) and \([x, y]\) belong to \(H \leq G\), then
\[ [x^p, y] \equiv [x, y]^p \mod \gamma_2(H)^p \prod_{i=1}^k \gamma_{p^i}(H)^{p^{k-i}}. \]

**Proposition 2.4.** Let \(G\) be a group and \(n, k\) positive integers.
(a) For any \(x, y \in \gamma_k(G)\), we have
\[ (xy)^{4^{n-k+1}} \equiv x^{4^{n-k+1}} y^{4^{n-k+1}} \mod N_{n,k}(G) \cap N_{n+1,k-1}(G). \]
(b) Any element of \(N_{n,k}(G)\) can be written in the form \(a_k^{4^{n-k}} a_{k+1}^{4^{n-k-1}} \cdots a_n\) for suitable \(a_j \in \gamma_j(G)\).
(c) Suppose that the groups \(G/\gamma_j(G)\) are torsion-free for all \(1 \leq j \leq n\). Then \(\bar{\lambda}_n(G) \cap \gamma_k(G) = N_{n,k}(G)\).

**Proof.** Note first that \(\gamma_a(G)^4 \leq N_{n,k}(G)\) whenever \(a \geq k\) and \(a+b \geq n\).
(a) We use Lemma 2.3. Note that \(\gamma_2((x, y))^4 \leq \gamma_2(k-1)(G)\). Taking two consecutive terms of the product \(\prod_{i=1}^{2^{(n-k+1)}} \gamma_2((x, y))^2^{2^{(n-k+1)}-1}\) into account, we see
that for any $0 \leq s \leq n - k$, the group
\[
\gamma_{2^{s+1}+1}(k-1)(G)\cdot \gamma_{2^{s+1}+2}(k-1)(G)\cdot 2^{(n-k+1)-(2s+1)}
\]
is contained in $\gamma_{2^{s+1}+1}(k-1)(G)\cdot 2^{(n-k+1)-(2s+2)}$, which is itself a subgroup of the intersection $N_{n,k}(G) \cap N_{n+1,k-1}(G)$, since $2^{2s+1}(k-1) + (n-k+1) - (s+1) \geq 2^{2s+1} + n - s \geq n + 1$.

(b) This is proved by reverse induction on $k$, the case $k = n$ being trivial. Suppose $x \in N_{n,k-1}(G) = \gamma_{k-1}(G)\cdot 2^{n-k+1}\cdot \gamma_k(G)\cdot 2^{n-k} \cdots \gamma_n(G)$. Then there exist elements $x_1, \ldots, x_l \in \gamma_{k-1}(G)$ such that $x = x_1^{2^{n-k-1}} \cdots x_l^{2^{n-k-1}} c$, where $c \in N_{n,k}(G)$. We finish off using (a) followed by the induction hypothesis. 

(c) We prove the claim by induction on $k$. First note that the inclusion $N_{n,k}(G) \subset \bar{\lambda}_n(G)\cap \gamma_k(G)$ holds without the additional assumptions. Now take any $x \in \bar{\lambda}_n(G)\cap \gamma_k(G)$. By the induction hypothesis and (a), there exist elements $a \in \gamma_{k-1}(G)$ and $y \in N_{n,k}(G)$ such that $x = a^{2^{n-k+1}} y$. This implies $a^{2^{n-k+1}} \in \gamma_k(G)$, and as the group $\gamma_{k-1}(G)/\gamma_k(G)$ is torsion-free, we have $a \in \gamma_k(G)$ and thus $x \in N_{n+1,k}(G)$. 

We now prove further properties of the $\bar{\lambda}$-series which are analogues of the well-known properties of the lower $p$-central series.

**Proposition 2.5.** Let $G$ be a group and $m, n$ positive integers. Then
\[
\bar{\lambda}_n(G)\cdot 2^{m} \leq \bar{\lambda}_{n+m}(G) \quad \text{and} \quad [\bar{\lambda}_n(G), \bar{\lambda}_m(G)] \leq \bar{\lambda}_{n+m}(G).
\]

*Proof.* The first inclusion is clear. We prove the second one by induction on $m$. By the Three Subgroups Lemma, $[\bar{\lambda}_n(G), [\bar{\lambda}_m(G), G]] \leq \bar{\lambda}_{n+m+1}(G)$. It remains to prove that $[\bar{\lambda}_n(G), \bar{\lambda}_m(G)]\cdot 2^{4}$ is contained in $\bar{\lambda}_{n+m+1}(G)$. Picking any $x \in \bar{\lambda}_n(G)$ and $y \in \bar{\lambda}_m(G)$, we have $[x, y] \in \bar{\lambda}_{n+m}(G)$ and $[x, y, x] \in \bar{\lambda}_{n+m+2}(G)$ by induction. Putting $H = \langle x, [x, y]\rangle$ and using Lemma 2.3 we get
\[
[x, y, x]^4 = [x, y]^4 \mod \gamma_2(H)\cdot 2^{4}\cdot \gamma_4(H).
\]
Now $[x, y]^4 \in \bar{\lambda}_{n+m}(G)$, $\leq \bar{\lambda}_{n+m+1}(G)$, and $x_2(H) \leq \bar{\lambda}_{n+2}(G)$, which implies $\gamma_2(H)\cdot 2^{4}\cdot \gamma_4(H) \leq \gamma_2(H) \leq \bar{\lambda}_{n+m+1}(G)$. This concludes our proof. 

Continuing in the manner of [3], we provide a version of [8] VIII, Theorem 1.8).

**Lemma 2.6.** Let $G$ be a group and $n$ a positive integer. If $x \in \gamma_j(G)$ and $y \in \gamma_{j+1}(G)$ for some $1 \leq j \leq n$, then $(xy)^{2^{n-j}} \equiv x^{2^{n-j}} \mod N_{n+1,j}(G)$.

*Proof.* Both $x$ and $y$ are elements of $\gamma_j(G)$. By Proposition 2.4 with $k = i + 1$, we have $(xy)^{2^{n-i}} \equiv x^{2^{n-i}} y^{2^{n-i}} \mod N_{n+1,i}(G)$. Since $y \in N_{n+1,i}(G)$, we also have $y^{2^{n-i}} \in N_{n+1,i}(G)$ by Proposition 2.1, hence the lemma.

**Theorem 2.7.** Let $G$ be a group and $n, k$ positive integers.

(a) There is an epimorphism $\beta$ from the direct product
\[
\frac{\gamma_k(G)}{\gamma_k(G)\cdot \gamma_{k+1}(G)} \times \frac{\gamma_{k+1}(G)}{\gamma_{k+1}(G)\cdot \gamma_{k+2}(G)} \times \cdots \times \frac{\gamma_n(G)}{\gamma_n(G)\cdot \gamma_{n+1}(G)}
\]
onto the quotient \( N_{n,k}(G)/N_{n+1,k}(G) \), given by
\[ \beta(\bar{a}_k, \bar{a}_{k+1}, \ldots, \bar{a}_n) = a_k^{n-k} a_{k+1}^{n-k-1} \cdots a_n N_{n+1,k}(G), \]
where \( a_j \in \gamma_j(G) \) and \( \bar{a}_j = a_j \gamma_j(G)^{\gamma_{j+1}(G)} \).

(b) Suppose that the groups \( G/\gamma_j(G) \) are torsion-free for all \( 1 \leq j \leq n \). Then the map \( \beta \) of (a) is an isomorphism.

Proof. (a) It follows from the previous Lemma that the mapping \( \beta \) is well-defined. It is a homomorphism by Proposition 2.4 (a), and it is surjective by (b) of the same proposition.

(b) We are proving that \( \beta \) is injective. So suppose \( a_j, b_j \) are elements of \( \gamma_j(G) \), \( k \leq j \leq n \), for which
\[ a_k^{n-k} a_{k+1}^{n-k-1} \cdots a_n \equiv b_k^{n-k} b_{k+1}^{n-k-1} \cdots b_n \mod N_{n+1,k}(G). \]
We prove that this implies \( a_j \equiv b_j \mod \gamma_j(G)^{\gamma_{j+1}(G)^4} \) for all \( k \leq j \leq n \) by induction on \( j \). By hypothesis, \( a_l \equiv b_l \mod \gamma_l(G)^{\gamma_{l+1}(G)^4} \) for all \( k \leq l \leq j - 1 \), so \( a_l^{n-l-1} \equiv b_l^{n-l-1} \mod N_{n,k}(G) \) by Lemma 2.6. This now implies
\[ a_j^{n-j} a_{j+1}^{n-j-1} \cdots a_n \equiv b_j^{n-j} b_{j+1}^{n-j-1} \cdots b_n \mod N_{n+1,k}(G) \cap \gamma_j(G). \]
By Proposition 2.4 (c), the intersection \( N_{n+1,k}(G) \cap \gamma_j(G) \) equals \( N_{n+1,j}(G) \). Note that the obtained congruence also holds when \( j = k \). Hence
\[ a_j^{n-j} \equiv b_j^{n-j} \mod \gamma_j(G)^{\gamma_{j+1}(G)}. \]
The group \( \gamma_j(G)/\gamma_{j+1}(G) \) is abelian, so there exists an element \( c \in \gamma_j(G) \) such that \( a_j^{n-j+1} \equiv b_j^{n-j+1} \mod \gamma_{j+1}(G) \). The assumption that the factor group \( \gamma_j(G)/\gamma_{j+1}(G) \) is also torsion-free now gives us the desired congruence \( a_j \equiv b_j \mod \gamma_j(G)^{\gamma_{j+1}(G)}. \)

We will primarily be interested in the case when \( G \) is a free group of finite rank. In this case, the stated theorem is fully applicable, and so the factor groups of the \( \lambda \)-series are particularly easy to describe.

Theorem 2.8. Let \( F_r \) be the free group of rank \( r \) and \( n \) a positive integer.

(a) The group \( \tilde{\lambda}_n(F_r)/\tilde{\lambda}_{n+1}(F_r) \) can be embedded in \( \tilde{\lambda}_{n+1}(F_r)/\tilde{\lambda}_{n+2}(F_r) \). Moreover, there exists a base \( A_n \mod \tilde{\lambda}_{n+1} \) of \( \tilde{\lambda}_n/\tilde{\lambda}_{n+1} \) such that \( A_n \) is independent in \( \tilde{\lambda}_{n+1}/\tilde{\lambda}_{n+2} \).

(b) If \( C_n \) is the set of all basic commutators of weight \( n \) in a fixed sequence, then the image of \( B_n := C_1^{n-1} \cup C_2^{n-2} \cup \cdots \cup C_n \) in \( \bar{\lambda}_n(F_r)/\bar{\lambda}_{n+1}(F_r) \) is a base.

(c) The map \( \varphi_n : \tilde{\lambda}_n(F_r) \rightarrow \tilde{\lambda}_{n+1}(F_r)/\tilde{\lambda}_{n+2}(F_r) \) given by \( \varphi_n(x) = x^4 \tilde{\lambda}_{n+2}(F_r) \) is a homomorphism and \( \ker \varphi_n = \tilde{\lambda}_{n+1} \).

Proof. For (a) and (b) we invoke Theorem 2.7 and refer the reader to [3, Theorem 2.5], the proofs there are directly applicable in the case \( p = 2 \). To prove (c), we first show that the maps \( \varphi_n \) are indeed homomorphisms. When \( n \geq 2 \), this is true because for any \( x, y \in \tilde{\lambda}_n(F_r) \), we have \( (xy)^4 = x^4 y^4 c \) for some \( c \in \gamma_2(\tilde{\lambda}_n(F_r)) \leq \tilde{\lambda}_{2n}(F_r) \leq \tilde{\lambda}_{n+2}(F_r) \). And when \( n = 1 \), this is true by Proposition 2.4 (a). We
now show that ker $\varphi_n(F_r) = \tilde{\lambda}_n+1(F_r)$. Clearly $\tilde{\lambda}_n+1(F_r) \leq \ker \varphi_n$, so pick any $x \in \lambda_n(F_r)$ with the property $x^4 \in \lambda_{n+2}(F_r)$. By (a), we can choose a base $A_n \mod \tilde{\lambda}_{n+1}(F_r)$ of $\lambda_n(F_r)/\lambda_{n+1}(F_r)$ such that $A_n \mod \tilde{\lambda}_{n+2}(F_r)$ is independent in $\tilde{\lambda}_{n+1}(F_r)/\tilde{\lambda}_{n+2}(F_r)$. Expanding $x$ with respect to this base and using the fact that $\varphi_n$ is a homomorphism, we conclude $x \in \lambda_{n+1}(F_r)$. \hfill \Box

### 3. Free 2-central groups

We now use the results of the previous section to extend the theory of [3] to the case $p = 2$. For any positive integers $n$ and $r \geq 2$, we set

$$nG_r = \frac{F_r}{\lambda_{n+1}(F_r)}.$$

**Theorem 3.1.** The group $nG_r$ is a 2-central group of exponent $4^n$ and of nilpotency class $n$. Its $\lambda$-series is equal to $\tilde{\lambda}_n(nG_r) = \tilde{\lambda}_n(F_r)/\tilde{\lambda}_{n+1}(F_r)$, and we have $\Omega_2(nG_r) = \tilde{\lambda}_{n+1-1}(nG_r)$. The order of $nG_r$ equals $4^{b_1+\cdots+b_n}$, where $b_i$ is the number of basic commutators of weight at most $i$ on $r$ generators. In particular, $|\Omega_2(nG_r)| = 4^{b_n}$.

**Proof.** The claim on the shape of the $\lambda$-series is easily deduced by induction on $i$. Then the claims on the order, nilpotency class and exponent follow from the previous two theorems. The fact that $nG_r$ is a 2-central group follows from Theorem 2.8 (c), because the $\lambda$-series of a group is a central series. The final claim on the $\Omega$-subgroups of $nG_r$ is proved identically as in [3] Theorem 3.5]. \hfill \Box

Finally, we establish the importance of the group $nG_r$ in the context of $p$-central groups.

**Theorem 3.2.** Let $G$ be a 2-central group with $r$ generators and of exponent $2^n$. Then $G$ is a homomorphic image of $[\frac{2}{r}]^{+1}G_r$. Moreover, any finite 2-group with $r$ generators is a homomorphic image of $mG_r$ for some $m$.

**Proof.** In any 2-central group $G$ of exponent $4^n$, the $\Omega$-series $\cdots \leq \Omega_{2n-2} \leq \Omega_n = G$ is central (see [2]), thus $\tilde{\lambda}_i(G) \leq \Omega_{n-2i+2}(G)$ for all $i \geq 1$. In particular, $\tilde{\lambda}_{[\frac{n}{2}]+1}(G) = 1$. The presentation homomorphism $F_r \to G$ thus induces the required epimorphism. For the second part we only need to notice that for a finite 2-group $G$ of exponent $2^n$ an of nilpotency class $c$, we have $\tilde{\lambda}_{n+c-1}(G) = 1$ by Proposition 2.5. \hfill \Box

### 4. Schur multipliers

In this section, we present two results concerning the Schur multipliers of $p$-central groups. The first one is a complete description of the Schur multiplier of the free $p$-central groups $nG_r$ for any prime $p$.

**Theorem 4.1.** The Schur Multiplier $M(nG_r)$ is elementary abelian (resp. free $\mathbb{Z}_4$-module for $p = 2$) of rank $\sum_{i=2}^{n+1} c(r,i)$, where $c(w,g)$ is the number of basic commutators of weight $w$ on $g$ generators (calculable by Witt’s formula).
Proof. For odd $p$, Hopf’s formula and [8, VIII, 1.8] give us an isomorphism

$$M(^nG_r) \cong \frac{\gamma_2(F_r) \cap \lambda_{n+1}(F_r)}{[F_r, \lambda_{n+1}(F_r)]} = \frac{\gamma_2(F_r)^{p^r-1} \cdots \gamma_{n+1}(F_r)}{\gamma_2(F_r)^{p^n} \cdots \gamma_{n+2}(F_r)}.$$ 

There is now a well-defined mapping

$$\beta: \frac{\gamma_2(F_r)}{\gamma_2(F_r)^{p^n} \gamma_3(F_r)} \times \cdots \times \frac{\gamma_{n+1}(F_r)}{\gamma_{n+1}(F_r)^{p^n} \gamma_{n+2}(F_r)} \rightarrow \frac{\gamma_2(F_r)^{p^r-1} \cdots \gamma_{n+1}(F_r)}{\gamma_2(F_r)^{p^n} \cdots \gamma_{n+2}(F_r)},$$

given by $\beta(\bar{a}_2, \bar{a}_3, \ldots, \bar{a}_{n+1}) = a_2^{p^r-1} a_3^{p^n-2} \cdots a_{n+1}^{p^{n+1}} \gamma_2(F_r)^{p^n} \cdots \gamma_{n+2}(F_r)$, where $a_i \in \gamma_i(F_r)$ and $\bar{a}_i = a_i \gamma_0(F_r)^{p} \gamma_{i+1}(F_r)$. Moreover, $\beta$ is a bijective homomorphism (cf. [8, VIII, 1.8 and 1.9]).

For $p = 2$, the proof is essentially the same, referring now to Theorem 2.7 and Proposition 2.8 (b) in providing the isomorphism between the product

$$\frac{\gamma_2(F_r) \gamma_3(F_r) \cdots \gamma_{n+1}(F_r)}{\gamma_2(F_r)^{p^n} \gamma_3(F_r)^{p^{n-1}} \cdots \gamma_{n+1}(F_r)^{p^n}}$$

and the multiplier $M(^nG_r)$. □

Combining the above with Theorem 3.2 and [3, Theorem 4.1], we obtain the following corollary, dualising a result of [11] on powerful $p$-groups.

**Corollary 4.2.** Every finite $p$-group is a quotient of a $p$-group whose Schur multiplier is of exponent $p$ for odd $p$, 4 for $p = 2$.

Our second result is a positive solution to a special case of the conjecture stating that the exponent of the multiplier $M(G)$ divides the exponent of $G$ for any finite $p$-group $G$. We prove this by referring to Schur’s theory of covering groups (cf. [7, VI]).

**Proposition 4.3.** Let $G$ be a finite $p$-central group. Then $\exp M(G) \leq \exp G$.

**Proof.** Let $H$ be a covering group of $G$, so that $G \cong H/Z$ with $M(G) \cong Z \leq Z(H) \cap H'$, and let us denote $\exp G = p^e$. It suffices to prove that $H' \leq \Omega_e(H)$. Since $Z$ is central in $H$, we have $[\Omega_e(H), H] = 1$. Using [5, Theorem 2.5 (ii)], we get

$$(H')^{p^e} \leq \prod_{i=1}^{e} [H^{p^{e-i}}, i(p-1)+1] H.$$ 

Since the group $G = H/Z$ is $p$-central, we have $(H/Z)^{p^{e-i}} \leq \Omega_i(H/Z) \leq Z_i(H/Z)$, from which we conclude $[H^{p^{e-i}}, H] \leq Z$. Since $i(p-1) \geq i$ for all $1 \leq i \leq e$ and $Z$ is central in $H$, the right hand side of the above product is trivial. This completes our proof. □

5. **Exponential rank**

Following [13], we determine the exponential rank of $p$-central groups. Recall that the exponent semigroup of a finite $p$-group $G$ is

$$\mathcal{E}(G) = \{ n \in \mathbb{Z} | (xy)^n = x^n y^n \text{ for all } x, y \in G \}.$$
Let $\exp(G/Z(G)) = p^r$. By [13] Proposition 3.2, there exists a unique nonnegative integer $r$ such that $E(G) = p^{r+1}Z \cup (p^{r+1}Z + 1)$. The exponential rank of the group $G$ is defined to be $\exp(G) = r$. When $G$ is regular, its exponential rank is equal to 0, and when $G$ is powerful, we similarly have $\exp(G) = 0$ for odd $p$, $\exp(G) = 1$ for $p = 2$ if $G$ is not abelian.

In the course of turning our attention to the exponential rank of $p$-central groups, let us first recall an old result of Laffey [10] Theorem 3.

**Theorem 5.1.** Let $G$ be a $p$-central group. Then $\exp(G') = \exp(G/Z(G))$.

Next we prove a lemma based on Hall’s collection formula.

**Lemma 5.2.** Let $G$ be a $p$-central group. For any $x, y \in G$ such that $[x, y] \in \Omega_{i-\epsilon}(G)$ for some $i \geq 0$ and $\epsilon = 0$ for odd $p$, $\epsilon = 1$ for $p = 2$, we have $x^{p^i} y^{p^i} = (xy)^{p^i}$.

**Proof.** By Hall’s collection formula, we have

$$x^{p^i} y^{p^i} = (xy)^{p^i} c_2^{(p^i)} c_3^{(p^i)} \cdots c_p^{(p^i)}$$

for some $c_j \in \gamma_j([x, y])$. It thus suffices to prove that $\exp(\gamma_j([x, y]))$ divides the binomial $\binom{p^i}{j}$ for all $2 \leq j \leq p^i$. Let first $p$ be odd. Since the $\Omega$-series of $G$ is central, we have $\gamma_j([x, y]) \leq \Omega_{i+2-j}(G)$ for all $j \geq 2$. Our claim is thus reduced to the fact that $p^{i+2-j}$ divides $\binom{p^i}{j}$. Now let $[k]_p$ denote the largest integer such that $p^{[k]_p}$ divides $k$. A straightforward induction shows $\binom{p^i}{j} = [j]_p$. The inequality in question is therefore equivalent to $[j]_p \leq j - 2$ for all $j \geq 2$, which evidently holds since $[j]_p \leq \lceil \log_p j \rceil$. When $p = 2$, the series $1 = \Omega_0(G) \leq \Omega_2(G) \leq \cdots \leq G$ is central and thus $\gamma_j([x, y]) \leq \Omega_{i+3-2j}(G)$ for all $j \geq 2$. It now analogously suffices to prove the inequality $i + 3 - 2j \leq i - [j]_2$, which is again trivial. \(\square\)

Finally, we state and prove our theorem.

**Theorem 5.3.** Let $G$ be a $p$-central group.

(a) If $p$ is odd, then $\exp(G) = 0$.

(b) If $p = 2$ and $G$ is not abelian, then $\exp(G) = 1$.

**Proof.** Suppose $\exp G/Z(G) = p^r$. By Theorem [9,2] we have $\gamma_2(G) \leq \Omega_e(G)$. For $p$ odd, Lemma [5,2] shows that the mapping $x \mapsto x^{p^i}$ is an endomorphism of $G$, which is exactly our claim. Now let $p = 2$. The same reasoning shows that the mapping $x \mapsto x^{2^{i+1}}$ is an endomorphism of $G$, so we have $\exp(G) \leq 1$. If $G$ is abelian, then clearly $\exp(G) = 0$. Suppose now that there exists a nonabelian 2-central group $G$ with $\exp(G) = 0$. Since $\gamma_2(G) \leq \Omega_e(G)$, we obtain $\gamma_2(G) \leq \Omega_{i+2-i}(G)$ for all $2 \leq i \leq e$, and consequently $\gamma_2^i(G) \leq \Omega_{i-1}(G)$. Lemma [2,2] thus implies

$$(xy)^{2^i} \equiv x^{2^i} y^{2^i} \mod \gamma_2^i([x, y])^{2^{i-1}}.$$  

The corresponding terms in $\gamma_2^i([x, y])^{2^{i-1}}$ can be computed using the commutator collection process [6, pp. 165–166]. We obtain

$$(xy)^{p^i} = x^{p^i} y^{p^i} [y, x]^{(p^i)} [y, x, x]^{(p^i)} [y, x, y]^{(p^i)2}.$$
Since $\exp \gamma_2(G) = 2^e$, the fact that the $\Omega$-series is central in $G$ gives $\exp \gamma_3(G) \leq 2^{e-1}$. We are also assuming $\exp \text{rank}(G) = 0$, so the above equality implies $[y, x] \in \Omega_{e-1}(G)$ for all $x, y \in G$, but this is clearly a contradiction. □

Note that the obtained result coincides with the one for powerful $p$-groups.

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