DEFORMATIONS OF STRONG KÄHLER WITH TORSION METRICS

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Abstract. Existence of strong Kähler with torsion (SKT) metrics on complex manifolds has been shown to be unstable under deformations. We develop a method to compute the complex exterior differentials along a curve of complex manifolds, originating from a base complex manifold, in function of the complex exterior differentials defined on the base complex manifold. We apply such a method in order to find a necessary condition to the existence of SKT metrics on the curve of complex manifolds.

1. Introduction

In this paper, we make use of classical deformation theory and curves of complex structures to prove stability conditions for SKT metrics once the base complex manifold undergoes a deformation of the complex structure. By results of Kodaira and Spencer, we know that the Kähler condition of a manifold, i.e. admitting a metric with closed fundamental form, is stable under infinitesimal deformations of the complex structure. Therefore, it is straightforward to consider those notions that generalize the Kähler condition which naturally arise in the Hermitian setting and study their stability under deformations.

Let \((M, J, g)\) be a Hermitian manifold of complex dimension \(n\). Depending on the closedness of the fundamental form \(\omega\) (or its powers) of \(g\) with respect to certain differential operators, specific structures arise. If \(\partial \bar{\partial} \omega = 0\), the metric is said to be strong Kähler with torsion (SKT). Another notion which generalizes Kählerness is the balanced condition, i.e., \(d\omega^{n-1} = 0\), or equivalently, being \(\omega\) real, \(\partial \bar{\partial} \omega = 0\). It is in fact a particular case of a \(p\)-Kähler metric, i.e., \(d\omega^p = 0\), for \(2 \leq p \leq n - 1\). In respectively [5] and [1], it is proved that the existence of SKT and \(p\)-Kähler metrics is not stable, once the base complex manifold is deformed via a smooth family of complex structures.

Since the existence of SKT metrics on complex manifolds is not stable under deformations, it is worth investigating under which circumstances a SKT metric exists on a deformed complex manifold. In this paper, we show a necessary condition to the existence of SKT metrics on a curve of complex manifolds, see Theorem 5.1. To prove our result, we develop a method to compute the complex differentials \(\partial_t\) and \(\bar{\partial}_t\) acting on functions along a curve of complex manifolds \((M, J_t)\), in function of the complex differentials \(\partial_0\) and \(\bar{\partial}_0\) on the base complex manifold \((M, J_0)\), and in function of the \((0,1)\)-differential form with values in the holomorphic tangent bundle which describes the deformation of the complex structure. In Proposition 4.4 we write explicit formulas for computing \(\partial_t\) and \(\bar{\partial}_t\), and in Theorem 4.6 we approximate such formulas in a way to be applied to our main result. Note that it is not necessary to have any information of the complex coordinates of the deformed complex manifold to apply our method of computing \(\partial_t\) and \(\bar{\partial}_t\), neither it relies on using special algebraic structures, such as structure equations on Lie groups, to compute the complex differentials.

We remark that starting from Theorem 4.6, one could study the existence of other special metrics on complex manifolds, building different necessary conditions to the existence of such metrics.

Finally, we describe two examples of nilmanifolds: the quotient of \(H(3; \mathbb{R}) \times H(3; \mathbb{R})\) by \(H(3; \mathbb{Z}) \times H(3; \mathbb{Z})\) and the Iwasawa manifold, i.e., the quotient of \(H(3; \mathbb{C})\) by \(H(3; \mathbb{Z}[i])\). The general theory that can be applied in these examples (see [4], [5], [2]) describes well the problem of the existence of a SKT metric on deformations of the base complex manifolds. We applied our main result to both the above examples, finding a nice setting in which this kind of computations can be carried on.

The paper is organized in the following way. In section 2, we fix the notation and recall some basic facts of complex geometry. Section 3 is a brief review of the classical deformation theory on complex manifolds, stating the main results of Kodaira and Kuranishi. In section 4, we recall some fundamental properties of curves of complex structures and develop our method of computing the complex differentials along the

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curve of complex manifolds. Section 5 is dedicated to the proof of our main result, i.e., a necessary condition to the existence of a SKT metric on a curve of complex manifolds. Finally, in section 6, we recall some cohomological properties of nilmanifolds and apply our main result to two explicit examples of nilmanifolds.

2. Notations and preliminaries

Let \((M, J, g)\) be an Hermitian manifold, with \(J \in \text{End}(TM)\) the integrable almost-complex structure on \(M\) and \(g\) a Riemannian metric on \(M\) compatible with \(J\). Let \(\omega\) be the \((1,1)\)-fundamental form associated to \(g\) given by \(\omega(\cdot, \cdot) = g(J(\cdot), \cdot)\).

The metric \(g\) is said to be strong Kähler with torsion (or SKT) if

\[
\partial\bar{\partial}\omega = 0,
\]

where \(d = \partial + \partial\bar{\partial}\) is the decomposition induced by the complex structure.

Let \(\pi: E \rightarrow M\) be a complex vector bundle of rank \(r\) over \((M, J, g)\), a \(n\)-dimensional Hermitian manifold. For every \(p, q\), let \(\Lambda^{p,q}(E) = \Lambda^{p,q}(M) \otimes E\) be the bundle of the \((p,q)\)-differential forms on \(M\) with values in \(E\) and let \(\mathcal{A}^{p,q}(E) := \Gamma(M, \Lambda^{p,q}(E))\) be the space of its global \(C^\infty\)-sections.

If \(h\) is an Hermitian metric \(h\) on \(E\), i.e. a smooth Hermitian scalar product on each fibre of \(E\), let us identify \(h\) as a \(\mathbb{C}\)-antilinear isomorphism between \(E\) and its dual \(\mathbb{C}\) and consider the usual \(\mathbb{C}\)-antilinear Hodge \(*\)-operator on \((M, J, g)\) with respect to \(g\) (see [6]). Then

\[
\ast_E: \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{-p,-q}(E^*)
\]

\[
\ast_E(\varphi \otimes s) := \ast(\varphi) \otimes h(s), \quad \text{for } \varphi \otimes s \in \mathcal{A}^{p,q}(E),
\]

is a \(\mathbb{C}\)-antilinear isomorphism depending on the metrics \(g\) and \(h\), such that \(\ast_E \circ \ast_E = (-1)^{p+q}\) on \(\Lambda^{p,q}(M) \otimes E\). In particular, \(h(\alpha, \beta) \ast = 1 = \alpha \ast \ast_E(\beta)\), for \(\alpha, \beta \in \Lambda^{p,q}(E)\).

An element of \(\mathcal{A}^{p,q}(E)\) can be locally written as \(\beta = \sum \beta_i \otimes s_i\), with \(\beta_i \in \mathcal{A}^{p,q}(M)\) and \((s_1, \ldots, s_r)\) a local trivialization of \(E\). Then we can define

\[
\overline{\partial}_E(\beta) := \sum \overline{\partial}(\beta_i) \otimes s_i,
\]

and the Dolbeault cohomology of a holomorphic vector bundle as

\[
H^{p,q}_E(X, E) := \frac{\ker(\overline{\partial}_E: \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E))}{\text{im}(\overline{\partial}_E: \mathcal{A}^{p,q-1}(E) \rightarrow \mathcal{A}^{p,q}(E))}.
\]

The \(*_E\)-operator can be used to define

\[
\overline{\partial}_E := -*_E \ast \overline{\partial}_E \ast_E
\]

and hence, the Laplace operator and its harmonic forms:

\[
\Delta_E := \overline{\partial}_E \ast_E \overline{\partial}_E + \overline{\partial}_E \ast \overline{\partial}_E,
\]

\[
\mathcal{H}^{p,q}(X, E) = \{ \beta \in \mathcal{A}^{p,q}(E) : \Delta_E(\beta) = 0 \}.
\]

Assume that \(M\) is compact. If we define the Hermitian product \(\langle \cdot, \cdot \rangle\) on \(\mathcal{A}^{p,q}(E)\) as

\[
\langle \alpha, \beta \rangle = \int_M h(\alpha, \beta) \ast 1,
\]

the operator \(\overline{\partial}_E \ast\) is the adjoint of \(\overline{\partial}_E\) and the operator \(\Delta_E\) is self-adjoint with respect to \(\langle \cdot, \cdot \rangle\). With these notations, the following Hodge decomposition holds

\[
\mathcal{A}^{p,q}(E) = \overline{\partial}_E(\mathcal{A}^{p,q-1}(E)) \oplus \mathcal{H}^{p,q}(X, E) \oplus \overline{\partial}_E(\mathcal{A}^{p,q+1}(E)),
\]

for any \(p, q\),

and \(\mathcal{H}^{p,q}(X, E)\) is finite-dimensional. Also the map \(\mathcal{H}^{p,q}(X, E)\) projects bijectively onto \(H_{\overline{\partial}_E}^{p,q}(X, E)\) which also is finite-dimensional.

In the following, we will often elide the “\((E)\) symbol from the previous notations.
3. REVIEW OF DEFORMATION THEORY OF COMPLEX STRUCTURES

We will recall the definitions both in the differentiable and complex settings.

Let $B$ be a domain of $\mathbb{R}^m$ (resp. $\mathbb{C}^m$) and $\{M_t\}_{t \in B}$ a family of compact complex manifolds.

**Definition 3.1.** We say that $M_t$ depends differentially (resp. holomorphically) on $t \in B$ and that $\{M_t\}_{t \in B}$ forms a differentiable (resp. holomorphic, or complex analytic) family if there is a differentiable (resp. complex) manifold $\mathcal{M}$ and a differentiable (resp. holomorphic) proper map $\pi$ onto $B$ such that

1. $\pi^{-1}(t) = M_t$ as a complex manifold for every $t \in B$.
2. the rank of the Jacobian of $\pi$ is equal to the dimension (resp. complex dimension) of $B$ at each point of $\mathcal{M}$.

More precisely, for each $p_0 \in \mathcal{M}$ there is a local diffeomorphism (resp. biholomorphism) from $U_j \ni p_0$

$$p \mapsto (z^1, \ldots, z^n, t^1, \ldots, t^m)$$

to a domain in $\mathbb{C}^n \times \mathbb{R}^m$ (resp. $\mathbb{C}^n \times \mathbb{C}^m$), where $\pi(p) = (t^1, \ldots, t^m)$ are differentiable (resp. complex) coordinates around $p$, and such that for a fixed $t \in \mathbb{R}^m$ (resp. $\mathbb{C}^m$), $(z^1, \ldots, z^n)$ are complex coordinates on $M_t$. If $U_t \cap U_j \neq \emptyset$, the coordinates $z_i, z_j$ are related via the transition functions $f^t_{ij}$,

$$z_i^t = f^t_{ij}(z_j, t),$$

where $f^t_{ij}(z_j, t)$ is differentiable (resp. holomorphic) in $(z_j, t)$ and holomorphic in $z_j$ for a fixed $t$.

It follows from the definition that every $M_t$, for $t \in B$, is a submanifold (resp. complex submanifold) of $\mathcal{M}$.

**Definition 3.2.** If $M, N$ are compact complex manifolds, we say that $M$ is a differentiable (resp. holomorphic) deformation of $N$ if there exists a differentiable (resp. holomorphic) family $\{M_t\}_{t \in B}$ over a domain $B$ of $\mathbb{R}^m$ (resp. $\mathbb{C}^m$), with $M_{t_0} = M, M_{t_1} = N$ for some $t_0, t_1 \in B$.

Note that $M$ and $N$ are diffeomorphic as differentiable manifolds, see [10, pag. 147].

In the holomorphic setting, deformation theory proceeds then with the description of the tools which allow us to determine the existence and study in detail deformations of the complex structure of a complex manifold. Let $\mathcal{M} = \{M_t\}_{t \in B}$, and $\pi: \mathcal{M} \to B$ with $B := B_r(0) \subset \mathbb{C}^m$, be a holomorphic family over $B$. Let $f^t_{ij}$ be the holomorphic transition functions on $\mathcal{M}$.

If we denote by $\Theta_t$ the sheaf of holomorphic vector fields on $M_t$, we can define $\frac{\partial M_t}{\partial t^r} = \theta_{ijr}(t)$, with

$$\theta_{ijr}(t) = \sum_{\alpha=1}^{n} \frac{\partial f^t_{ij}(z_j, t)}{\partial t^r} \frac{\partial}{\partial z^\alpha},$$

which is an element of the cohomology with values in $\Theta_t$, i.e. $H^1(M_t, \Theta_t)$. Let us set $\frac{\partial}{\partial t^r} := \sum_{j=1}^{m} \frac{\partial}{\partial t^r}$.

**Definition 3.3.** We define the infinitesimal deformation of $M$ as

$$\frac{\partial M_t}{\partial t^r} := \sum_{\nu=1}^{m} \frac{\partial M_t}{\partial t^\nu} \in H^1(M_t, \Theta_t)$$

Under assumptions on the cohomology space $H^2(M, \Theta)$, Kodaira, Niremberger, and Spencer proved a theorem of existence of complex analytic deformations. Indeed, let $B$ be the ball centered in $0 \in \mathbb{C}$ of radius $r > 0$, i.e. $B = B_r(0) \subset \mathbb{C}$.

**Theorem 3.4.** Let $M$ be a compact complex manifold and assume $H^2(M, \Theta) = 0$. Then for any $\theta \in H^1(M, \Theta)$ there is a complex analytic family $\{M_t : t \in B\}$, and

$$\frac{\partial M_t}{\partial t^r} |_{t=0} = \theta$$

A $C^\infty$ vector $(p, q)$–form on a complex manifold $M$ with $\dim_{\mathbb{C}} M = n$ is a differentiable section of the bundle $\Lambda^{p.q}(M) \otimes T^1 \Omega(M)$, which can be locally described as

$$\psi = \sum_{i=1}^{n} \psi^i \otimes \frac{\partial}{\partial z^i},$$

with $(z_1, \ldots, z_n)$ holomorphic coordinates on $M$ and $\psi^i$ a $(p, q)$–differential form on $M$, for $i = 1, \ldots, n$. The operator $\mathcal{D}$ acts on such forms as in (2.1).
Results of deformation theory assure that to each infinitesimal deformation \( \frac{\partial M}{\partial t} |_{t=0} \in H^1(M, \Theta) \) as in Definition 3.3, corresponds a unique \((0,1)\)-vector form on \( M \) which is \( \overline{\partial} \)-closed. Indeed, let \( X \) be the sheaf of \( C^\infty \) vector fields on \( M \), which can be thought as the \((0,0)\)-vector forms on \( M \); the Dolbeault isomorphism implies:

\[
H^1(M, \Theta) \cong \Gamma(M, \overline{\partial} \mathcal{X}) / \Gamma(M, \overline{\partial}^* \mathcal{X}) \cong H^{0,1}_\partial(M, T^{1,0}M).
\]

We can actually describe the complex structure on each \( M_t \), \( t \in B \) via a \( C^\infty(0,1) \)-vector form \( \Psi(t) \), defined starting from the local transition functions \( f_{ij}^t \) (see [10, pag. 150]). If such \( \Psi(t) \) is locally written as

\[
\Psi(t) = \sum_{i,\lambda=1}^n \psi_{i\lambda}(z, t) d\bar{z}^\lambda \otimes \frac{\partial}{\partial z^i},
\]

the (local) holomorphic functions on \( M_t \) are defined as the differentiable functions defined on open sets of \( M \) such that

\[
\left( \overline{\partial} + \sum_{i,\lambda=1}^n \psi_{i\lambda}(z, t) d\bar{z}^\lambda \otimes \frac{\partial}{\partial z^i} \right) f(z) = 0.
\]

On the space of \( C^\infty \) vector forms a bracket can be defined in the following way. Let \( \Psi = \sum \psi^\alpha \partial_\alpha \) and \( \Xi = \sum \xi^\alpha \partial_\alpha \) be respectively \((0,p)\)- and \((a,q)\)-vector forms, where \( \partial_\alpha = \frac{\partial}{\partial z^\alpha} \). Then

\[
[\Psi, \Xi] := \sum_{\alpha,\beta=1}^n \left( \partial_\alpha \psi^\alpha \wedge \xi^\beta - (-1)^p \psi^\alpha \wedge \partial_\alpha \xi^\beta \right) \partial_\beta.
\]

In particular \([ , ]\) is bilinear and satisfies the following:

1. \( [\Psi, \Xi] = -(-1)^p [\Xi, \Psi] \)
2. \( \overline{\partial} [\Psi, \Xi] = [\overline{\partial} \Psi, \Xi] + (-1)^p [\Psi, \overline{\partial} \Xi] \)
3. \( (-1)^p [\Xi, \Psi] + (-1)^q [\Xi, \Phi] + (-1)^{p+q} [\Phi, [\Psi, \Xi]] = 0 \)

if \( \Psi \) is a \((0,p)\)-form, \( \Xi \) a \((0,q)\)-form and \( \Phi \) a \((0,r)\)-form.

A classical result shows that the deformations of the complex structure on a compact complex manifold which give rise to integrable complex structures can be characterized according to the following theorem.

**Theorem 3.5.** If \( \pi: \mathcal{M} \to B = B_r(0) \) is a complex analytic family of compact complex manifolds, then the complex structure on each \( M_t \) is \( \pi^{-1}(t) \) is represented by a vector \((0,1)\)-form \( \Psi(t) \) on \( M_t \) such that \( \Psi(0) = 0 \) and

\[
\overline{\partial} \Psi(t) + \frac{1}{2} [\Psi(t), \Psi(t)] = 0 \quad \text{(Maurer-Cartan equation)}.
\]

As by Theorem 3.4, existence of complex deformations of a compact complex manifold is assured if \( H^2(M, \Theta) = 0 \). However, if this property does not hold, a more general theory, known as Kuranishi theory, can be applied.

Let \( M \) be a compact complex manifold and define \( \mathcal{A}_q := \Gamma(M, \Lambda^0,q(M) \otimes T^{1,0}M) \), for \( q \in \mathbb{Z}_+ \). Fix an Hermitian metric \( h \) on \( M \), extend it to \( \mathcal{A}_q \) and denote it by the same symbol \( h \). Define and inner product on \( \mathcal{A}_q \) by

\[
\langle \Psi, \Xi \rangle = \int_M h(\Psi, \Xi) \ast 1,
\]

where \( \Psi, \Xi \in \mathcal{A}_q \), \( \ast \) is the \( \mathbb{C} \)-antilinear Hodge operator. We also define the Laplacian on \( \mathcal{A}_q \) by

\[
\Box = \overline{\partial} \overline{\partial}^* + \overline{\partial}^* \overline{\partial},
\]

where \( \overline{\partial}^* \) is the adjoint operator of \( \overline{\partial} \) with respect to the Hermitian metric \( h \). The space of harmonic forms is

\[
\mathcal{H}^q = \{ \Psi \in \mathcal{A}_q | \Box \Psi = 0 \} \cong H^q(M, \Theta).
\]

The Hodge theory induces a decomposition on the space \( \mathcal{A}_q \) as a direct sum of orthogonal subspaces:

\[
\mathcal{A}_q = \mathcal{H}^q \oplus \Delta \mathcal{A}_q.
\]

The operator \( \Box : \mathcal{A}_q \to \Delta \mathcal{A}_q \) is well defined and acts on \( \mathcal{A}_q \) as the projection onto \( \Delta \mathcal{A}_q \), whereas the operator \( \Box \) is the (well-defined) projection operator onto \( \mathcal{H}^q \).
Theorem 3.6 (Kuranishi). Let $M$ be a compact complex manifold, $\{\eta_\nu\}$ a base for $\mathcal{H}^1$. Let $\Psi(t)$ be the solution of the equation

$$\Psi(t) = \eta(t) - \frac{1}{2} t^* G[\Psi(t), \Psi(t)],$$

where $\eta(t) = \sum_{\nu=1}^{m} t_\nu \eta_\nu$, $|t| < r$, $r > 0$, and let $S = \{ t \in B_r(0) : H[\Psi(t), \Psi(t)] = 0 \}$. Then For each $t \in S$, $\Psi(t)$ determines a complex structure $M_t$ on $M$.

The space $S$ is called the space of Kuranishi. Its defining property holds for a $\Psi(t)$ satisfying (3.3) if and only $\Psi(t)$ satisfies Maurer-Cartan equation (3.2). The proof of Theorem 3.6 shows that a $(0,1)$-vector form $\Psi(t)$ satisfying equation (3.3) can be constructed as a converging power series

$$\Psi(t) = \sum_{\mu=1}^{\infty} \psi_\mu(t)$$

in which the forms

$$\psi_\mu(t) = \sum \psi_{\nu_1 \cdots \nu_{\mu}} t^{\nu_1} \cdots t^{\nu_{\mu}}, \quad \psi_{\nu_1 \cdots \nu_{\mu}} \in \mathcal{A}_1$$

are determined via a recursive formula. In fact, if $\{\eta_\nu\}_{\nu=1}^{n}$ is a base for $\mathcal{H}^1 \cong H^1(M, \Theta)$ and we set $\psi_1(t) = \sum_{\nu=1}^{m} \eta_\nu t_\nu$, equation (3.3) assures that each term $\psi_\mu$ can be computed as

$$\psi_\mu(t) = -\frac{1}{2} t^* G \left( \sum_{\kappa=1}^{\mu-1} [\psi_\kappa(t), \psi_{\mu-\kappa}(t)] \right).$$

In general $S$ can have singularities and hence may not have a structure of smooth manifold. Nonetheless, $\{M_t\}_{t \in S}$ can be still be interpreted as a complex analytic family, see [7].

4. Curves of Complex Structures

Let $(M, J)$ be a complex manifold of complex dimension $n$. On the space $\Gamma(M, End(TM))$ of $C^\infty$ sections of the vector bundle $End(TM) = T^*M \otimes TM$ we will consider the usual topology. Then every almost complex structure $J$ in a neighborhood of $J$ can be uniquely represented as

$$\hat{J} = (I + L)J(I + L)^{-1},$$

with $L \in End(TM)$ such that $LJ + JL = 0$ and det$(I + L) \neq 0$ (see, e.g., [3]). It turns out that $\hat{J}$ is integrable if and only if

$$\Psi := \frac{1}{2} (I + L)J(I + L)^{-1} \in \mathcal{A}_1$$

satisfies the Maurer-Cartan equation (3.2), i.e.,

$$\bar{\partial} \Psi + \frac{1}{2} [\Psi, \Psi] = 0,$$

(see, [3, Theorem 5.16]). Let $t \mapsto J_t$ be a smooth curve of (integrable) complex structures on $M$, with $J_0 = J$. Note that the curve $t \mapsto J_t$ is a differentiable deformation of the complex structure $J$ of $M$, according to Definition 3.1. Then, as recalled above, for $-\epsilon < t < \epsilon$, and some $\epsilon > 0$,

$$J_t = (I + L_t)J(I + L_t)^{-1},$$

where

$$L_tJ + JL_t = 0, \quad L_t = tL + o(t).$$

Finally, we set

$$\Psi_t = \frac{1}{2} (L_t - iJL_t) \in \mathcal{A}_1.$$

We need the following known lemma. We give the proof for completeness.

Lemma 4.1. Let $z$ be a complex vector field of type $(1, 0)$ and $\alpha$ be a complex 1-form of type $(1, 0)$ with respect to the complex structure $J$. Then $(I + L_t)z$ and $(I + L_t)\alpha$ (resp. $(I + L_t)\overline{z}$ and $(I + L_t)\overline{\alpha}$) are of type $(1, 0)$ (resp. $(0, 1)$) with respect to the complex structure $J_t$. Moreover, $L_tz$ and $L_t\alpha$ are of type $(0, 1)$ and $L_t\overline{z}$ and $L_t\overline{\alpha}$ are of type $(1, 0)$ with respect to the complex structure $J$. 
Proof. Let us prove the lemma for \( z \in \Gamma(M, T^{1,0}M) \). The other cases are analogous. We get
\[
J_t((I + L_t)z) = (I + L_t)J_z = i(I + L_t)z,
\]
i.e., \((I + L_t)z\) is of type \((1, 0)\) for the complex structure \(J_t\). Moreover,
\[
JL_tz = -L_tJz = -iL_tz,
\]
i.e., \(L_tz\) is of type \((0, 1)\) for the complex structure \(J\).

Since \( \det(I + L_t) \neq 0 \) for \(-\epsilon < t < \epsilon\), the map \( I + L_t \) is an isomorphism between the spaces of complex vector fields of type \((1, 0)\) (resp. \((0, 1)\)) with respect to the complex structure \(J\) and complex vector fields of type \((1, 0)\) (resp. \((0, 1)\)) with respect to the complex structure \(J_t\). The same holds for the spaces of \((1, 0)\) and \((0, 1)\) forms. As another consequence of lemma 4.1, note that we have an explicit verification of \(\Psi_t \in \mathcal{A}_1\) since by definition \(\Psi_t(\overline{z}) = L_t\overline{z} \in \Gamma(M, T^{1,0}M)\), for \(z \in \Gamma(M, T^{1,0}M)\).

Our goal is to understand how the differential operators \(\partial_t\) and \(\overline{\partial}_t\), namely the operators \(\partial\) and \(\overline{\partial}\) with respect to the complex structure \(J_t\), act on functions.

**Lemma 4.2.** Let \(w = (I + L_t)z\) be a complex vector field of type \((1, 0)\) with respect to the complex structure \(J_t\). Let \(f : M \to \mathbb{C}\) be a \(C^\infty\) function. Then
\[
\partial_t f(w) = \partial f(z) + \overline{\partial}f(\overline{\Psi_t(z)}).
\]

**Proof.** We compute
\[
2\partial_t f((I + L_t)z) = (df - iJ_t df)((I + L_t)z) = df(z) + df(L_t z) - idf((I + L_t)J_z) = \partial f(z) + \overline{\partial}f(L_t z) + df((I + L_t)z) = 2\partial f(z) + 2\overline{\partial}f(L_t z).
\]

Finally, since \(L_t \in \text{End}(TM)\), then
\[
L_t z = L_t \overline{z} = \overline{\Psi_t(z)}.
\]

Analogously, we get the following.

**Lemma 4.3.** Let \(\overline{w} = (I + L_t)\overline{z}\) be a complex vector field of type \((0, 1)\) with respect to the complex structure \(J_t\). Let \(f : M \to \mathbb{C}\) be a \(C^\infty\) function. Then
\[
\overline{\partial}_t f(\overline{w}) = \overline{\partial} f(\overline{z}) + \partial f(\overline{\Psi_t(z)}).
\]

**Proof.** By making use of Lemma 4.1 and the characterization of \(J_t\) (4.1), we compute
\[
2\overline{\partial}_t f(\overline{w}) = (df + iJ_t df)((I + L_t)\overline{z}) = df(\overline{z}) + df(L_t(\overline{z})) + idf((I + L_t)J\overline{z}) = \overline{\partial} f(\overline{z}) + \partial f(L_t \overline{z}) + df((I + L_t)\overline{z}) = 2\overline{\partial} f(\overline{z}) + 2\partial f(L_t \overline{z}).
\]

Since \(L_t \overline{z} = \Psi_t(\overline{z})\), we conclude.

Now, fix a local frame \(\{v_1, \ldots, v_n\}\) of complex vector fields of type \((1, 0)\) and the corresponding dual local coframe \(\{\xi^1, \ldots, \xi^n\}\) of \((1, 0)\)-forms with respect to the complex structure \(J\). Then, locally we have
\[
\Psi_t = (\psi_t)_{i}^{j} \otimes v_j.
\]

By the very definition of \(\Psi_t\) and Lemma 4.1, we get that \(\{v_i + \overline{(\psi_t)i} \overline{v_j}\}\) is a local frame of complex vector fields of type \((1, 0)\) and \(\{\xi^j + \overline{\xi}^i \psi_t^i\}_{j}\) is a local coframe of \((1, 0)\)-forms with respect to the complex structure \(J_t\). Let us introduce the following matrix notation:
\[
v = (v_1, \ldots, v_n)^T, \quad \xi = (\xi^1, \ldots, \xi^n), \quad \psi_t = ((\psi_t)^i_j)_{ij}.
\]

The following products between matrices, vectors and covectors satisfy the rules of matrix products. Then, if \(f : M \to \mathbb{C}\) is a \(C^\infty\) function, since \(\partial_t f\) is a \((1,0)\)-form with respect to \(J_t\), we have
\[
\partial_t f = (\xi + \overline{\xi} \psi_t) a_t,
\]
Let \( M \) be a smooth curve of complex structures on \( (M,J) \), such that \( J_0 = J \). We have:
\[
\partial_t f = (\xi + \zeta \psi_i)(v(f) + \overline{\psi_i}(\overline{v}(f))) + o(t),
\]
\[
\overline{\partial}_t f = (\xi + \zeta \psi_i)(\overline{v}(f) + \psi_i v(f)) + o(t).
\]

The only thing to note is that \( (I + \psi_i v)^{-1} = I + o(t) \).

Passing in local complex coordinates \( \{z^i\}_i \), for \( v_i = \frac{\partial}{\partial z^i} \) and \( \xi^i = dz^i \), we get the following general formulas:
\[
\partial_t f = (dz^i + tdz^i \psi_i^j) \left( \frac{\partial f}{\partial z^i} + t \psi_i v^k \frac{\partial f}{\partial z^k} \right) + o(t),
\]
\[
\overline{\partial}_t f = (dz^i + tdz^i \psi_i^j) \left( \frac{\partial f}{\partial z^i} + t \psi_i v^k \frac{\partial f}{\partial z^k} \right) + o(t).
\]

5. Main Result

Let \( (M,J,g,\omega) \) be a Hermitian manifold and assume that the metric is SKT, i.e., \( \partial \overline{\partial} \omega = 0 \). Let \( t \rightarrow J_t \) be a smooth curve of complex structures on \( M \) such that \( J_0 = J \). We are interested to find a necessary condition for the existence of a SKT metric \( g_t \) on \( (M,J_t) \), which converges to \( g \) as \( t \) approaches 0. If \( \omega_t \) is a SKT metric on \( (M,J_t) \), i.e., \( \partial_t \overline{\partial}_t \omega_t = 0 \), for \( t \in (-\epsilon, \epsilon) \), then we obtain \( \frac{\partial}{\partial t} (\partial_t \overline{\partial}_t \omega_t) |_{t=0} = 0 \). Applying Theorem 4.6 to explicitly calculate in coordinates this necessary condition, we obtain the following theorem.

Theorem 5.1. Let \( (M,J,g,\omega) \) be a Hermitian manifold of complex dimension \( n \). Let \( t \rightarrow J_t \) be a smooth curve of complex structures on \( M \) such that \( J_0 = J \). Let \( \Psi_t \in A_1 \), locally written as \( \Psi_t = (\psi_i^j)_{i,j} dz^i \otimes \frac{\partial}{\partial z^j} \), where \( \{z^i\}_i \) are local complex coordinates on \( (M,J) \), be associated to \( J_t \) by equation (4.3). Then:
\[
\omega_t = (\omega_t)_{ij} (dz^i + (\psi_t^j_k) dz^k) \wedge (dz^j + (\overline{\psi_t}^j_k) \overline{dz}^k).
\]
(5.2) \[
\frac{\partial^2 \omega_{ij}}{\partial z^k \partial z^l} \psi_k^j dz^k \partial \overline{\omega}^m_i + \partial \overline{\omega}^m_i \psi_k^j dz^k \partial \omega_{ij} + \partial \overline{\omega}^m_i \left( \frac{\partial (\omega_{ij})}{\partial t} dz^j \right) = 0, \]
(5.3) \[
\frac{\partial}{\partial t} \left( \frac{\partial \omega_{ij}}{\partial z^k} \psi_k^j dz^k \partial \overline{\omega}^m_i \right) = \partial \overline{\omega}^m_i \psi_k^j dz^k \partial \omega_{ij}, \]

Proof. If the metric \(\partial \overline{\omega}_{t=0} = 0\), then we get \(\frac{\partial}{\partial t} \left( \partial \overline{\omega}_{t=0} \right) = 0\). Let us compute explicitly \(\frac{\partial}{\partial t} \left( \partial \overline{\omega}_{t=0} \right) = 0\), using the local complex coordinates \(\{ z^i \} \). First of all, let us set the notation
\[
\theta^i = dz^i + (\psi_i)^j dz^j, \quad \lambda^i = dz^i + t \psi_i^j dz^j, \quad \eta_{ij} = (\omega_{ij}) \]
so that
\[
\omega_t = \eta_{ij} \theta^i \wedge \theta^j = \eta_{ij} \lambda^i \wedge \lambda^j + o(t). \]

Now, let us compute \(\overline{\omega}_{t=0} \), using equations (4.5). We begin calculating \(\partial \omega_{t=0} \).
\[
\partial \omega_{t=0} = \partial \eta_{ij} \wedge \lambda^i \wedge \lambda^j + \partial \eta_{ij} \lambda^i \wedge \lambda^j - \eta_{ij} \lambda^i \wedge \partial \lambda^j + o(t)
= \left( \frac{\partial \eta_{ij}}{\partial z^k} + t \frac{\partial \eta_{ij}}{\partial \overline{\omega}^m_i} \right) \lambda^i \wedge \lambda^j
- \eta_{ij} \left( \frac{\partial \psi_i^j}{\partial z^k} + t \frac{\partial \psi_i^j}{\partial \overline{\omega}^m_i} \right) \lambda^i \wedge d\overline{\psi}^j \wedge \lambda^j
= A_{ij} \lambda^i \wedge \lambda^j + B_{ij} \lambda^i \wedge d\overline{\psi}^j \wedge \lambda^j + C_{ij} \lambda^i \wedge \lambda^j + o(t).
\]

Then, we compute \(\overline{\omega}_{t=0} \).
\[
\overline{\omega}_{t=0} = \overline{\omega}_{t=0} A_{ij} \lambda^i \wedge \lambda^j + \overline{\omega}_{t=0} B_{ij} \lambda^i \wedge d\overline{\psi}^j \wedge \lambda^j
+ \overline{\omega}_{t=0} C_{ij} \lambda^i \wedge \lambda^j + o(t),
\]

where
\[
\overline{\omega}_{t=0} A_{ij} = \left( \frac{\partial^2 \eta_{ij}}{\partial z^k \partial z^l} \right) \lambda^k \wedge \lambda^l + \lambda^k \wedge \lambda^l + o(t),
\overline{\omega}_{t=0} B_{ij} = \left( \frac{\partial^2 \eta_{ij}}{\partial z^k \partial \overline{z}^l} \right) \lambda^k \wedge d\overline{\psi}^l + o(t),
\overline{\omega}_{t=0} C_{ij} = \left( \frac{\partial^2 \eta_{ij}}{\partial \overline{z}^k \partial \overline{z}^l} \right) \lambda^k \wedge \lambda^l + o(t).
\]

If the metric \(\omega_t \) is SKT, then \(\frac{\partial}{\partial t} \left( \partial \overline{\omega}_{t=0} \right) = 0\), and
\[
\frac{\partial}{\partial t} \left( \partial \overline{\omega}_{t=0} \right) = \left( \frac{\partial}{\partial z^k \partial z^l} \right) \left( \frac{\partial (\omega_{ij})}{\partial t} dz^j \right) d\overline{\omega}^m_i + \frac{\partial (\omega_{ij})}{\partial z^k \partial \overline{z}^l} \partial \overline{\omega}^m_i \psi_k^j dz^k + \frac{\partial (\omega_{ij})}{\partial \overline{z}^k \partial \overline{z}^l} \partial \overline{\omega}^m_i \psi_k^j dz^k,
\]
\[
\frac{\partial}{\partial t} \left( \overline{\omega}_{t=0} A_{ij} \right) = \left( \frac{\partial^2 \omega_{ij}}{\partial z^k \partial z^l} \right) \lambda^k \wedge \lambda^l + \omega_{ij} \lambda^k \wedge \lambda^l,
\frac{\partial}{\partial t} \left( \overline{\omega}_{t=0} B_{ij} \right) = \left( \frac{\partial^2 \omega_{ij}}{\partial z^k \partial \overline{z}^l} \right) \lambda^k \wedge d\overline{\psi}^l + \omega_{ij} \lambda^k \wedge d\overline{\psi}^l,
\frac{\partial}{\partial t} \left( \overline{\omega}_{t=0} C_{ij} \right) = \left( \frac{\partial^2 \omega_{ij}}{\partial \overline{z}^k \partial \overline{z}^l} \right) \lambda^k \wedge \lambda^l + \omega_{ij} \lambda^k \wedge \lambda^l,
\]
\[
\frac{\partial}{\partial t} \lambda^k_{t=0} = \psi_k^j dz^j, \quad \frac{\partial}{\partial t} \overline{\lambda}^j_{t=0} = \overline{\psi}_i^j dz^j \wedge d\overline{\psi}^i.
\]
Thus, $\frac{\partial}{\partial t}(\partial_t \omega_t)_{|t=0} = 0$ if and only if the following three conditions holds:

$$
\begin{align*}
&\left(\frac{\partial^2 \omega_{ij}}{\partial z^j \partial \overline{z}^i} \psi^h_r - \frac{\partial \omega_{jk} \partial \overline{\psi}^i_r}{\partial \overline{z}^i} - \frac{\partial \omega_{ih} \partial \overline{\psi}^k_r}{\partial \overline{z}^k} + \frac{\partial^2 \omega_{ij}}{\partial \overline{z}^i \partial \overline{z}^j} \psi^h_r - \frac{\partial^2 \omega_{ik} \partial \overline{\psi}^j_r}{\partial \overline{z}^j} - \frac{\partial^2 \omega_{jk} \partial \overline{\psi}^i_r}{\partial \overline{z}^i} \right) dz^i d\overline{z}^j = 0, \\
&\left(\frac{\partial^2 \omega_{ij}}{\partial \overline{z}^i \partial \overline{z}^j} \partial_t \omega_{kl} \partial \overline{\psi}^m_r \partial \overline{\psi}^r_k + \frac{\partial^2 \omega_{ij}}{\partial \overline{z}^i \partial \overline{z}^j} \partial_t \omega_{kl} \partial \overline{\psi}^r_k + \partial_t \omega_{ij} \partial \overline{\psi}^j_r \partial \overline{\psi}^r_k \right) dz^i d\overline{z}^j = 0, \\
&\left(\frac{\partial \omega_{ij} \partial \overline{\psi}^i_r}{\partial \overline{z}^i} \partial \overline{\psi}^j_r + \partial_t \omega_{ij} \partial \overline{\psi}^i_r \partial \overline{\psi}^j_r + \partial_t \omega_{ij} \partial \overline{\psi}^j_r \partial \overline{\psi}^i_r \right) dz^i d\overline{z}^j = 0;
\end{align*}
$$

and the last equations are a rewriting of equations (5.1), (5.2) and (5.3).

Note that equations (5.1) and (5.3) depend only on $\omega$ and $\psi$ and their derivatives. Moreover, if a priori $(\omega_t)_{ij}$ are known to be independent of the time $t$, then also equation (5.2) depends only on $\omega$ and $\psi$ and their derivatives.

6. Applications

The conditions in Theorem 5.1 hold in general for deformations on any SKT manifold. However, if we consider the class of nilmanifolds, exploiting the theory developed in [4] and [8], we can find a nice field of application for our results.

6.1. Deformations of Abelian complex structures on nilmanifolds. We recall some definitions.

Definition 6.1. A nilmanifold $M$ is a compact quotient $M = \Gamma \backslash G$ of a simply-connected nilpotent Lie group $G$ by a uniform discrete subgroup $\Gamma$.

More precisely, a nilmanifold $M$ is said to be $k$-step if the descending series of its Lie algebra $\mathfrak{m}$,

$$
\mathfrak{m}^{(0)} = \mathfrak{m}, \quad \mathfrak{m}^{(1)} = [\mathfrak{m}^{(0)}, \mathfrak{m}], \quad \ldots \quad \mathfrak{m}^{(j)} = [\mathfrak{m}^{(j-1)}, \mathfrak{m}]
$$

vanishes for $j \geq k$.

Let $M$ be endowed with an invariant almost complex structure $J$. We denote by $\mathfrak{m}_C = \mathfrak{m}^{1,0} \oplus \mathfrak{m}^{0,1}$ the decomposition on complex invariant vector field on $M$ induced by $J$ and, extending $J$ to complex forms, by $\mathfrak{m}^{(p,q)}$ the invariant $(p,q)$-form on $M$. If $J$ induces a decomposition on the $\mathfrak{m}^{(p,q)} = \oplus_{p+q=k} \mathfrak{m}^{(p,q)}$ such that

$$
d(\mathfrak{m}^{(1,0)}) \subset \mathfrak{m}^{(1,1)},
$$

then $J$ is said to be Abelian. Such almost complex structures are integrable by definition.

Let $(M, J)$ is a nilmanifold with $J$ Abelian complex structure and let $\mathfrak{m}$ be the Lie algebra associated to $M$. As in [8] and [4], let us consider the sequence $\mathfrak{m}^{(0,k)} \otimes \mathfrak{m}^{1,0}$ on which the linear operator $\overline{J}$ acts in the following way. If $V \in \mathfrak{m}^{1,0}$, $\overline{U} \in \mathfrak{m}^{0,1}$, we set

$$
\overline{J} V = \frac{1}{2}(\overline{[\overline{U}, V]} - iJ[\overline{U}, V])
$$

so that $\overline{J} : \mathfrak{m}^{1,0} \rightarrow \mathfrak{m}^{(0,1)} \otimes \mathfrak{m}^{1,0}$ can be extended to a linear map on every space $\mathfrak{m}^{(0,k)} \otimes \mathfrak{m}^{1,0}$ by

$$
\overline{J}_k (\omega \otimes V) := (-1)^k \omega \wedge \overline{J} V, \quad \omega \in \mathfrak{m}^{(0,k)}, V \in \mathfrak{m}^{1,0}.
$$

It turns out that $\overline{J}$ so defined coincides with the same operator defined in Section 1, in (2.1), on left invariants differentiable vector forms. Moreover, $\overline{J}$ it is a differential, $i.e.$ $\overline{J}_k \circ \overline{J}_{k-1} \equiv 0$. This allows one to define the $j$th cohomology of the complex $\mathfrak{m}^{(0,*)} \otimes \mathfrak{m}^{1,0}$ by

$$
H^j_{\overline{J}}(\mathfrak{m}^{1,0}) := \ker(\overline{J}_j : \mathfrak{m}^{(0,j)} \otimes \mathfrak{m}^{1,0} \rightarrow \mathfrak{m}^{(0,j+1)} \otimes \mathfrak{m}^{1,0}) / \overline{J}_{j-1}(\mathfrak{m}^{(0,j-1)} \otimes \mathfrak{m}^{1,0}).
$$

The following result, due to [4], links (6.1) and the Dolbeault cohomology with values in the holomorphic tangent bundle.

Theorem 6.2. Let $M$ be a $k$-step nilmanifold with an Abelian complex structure. Then there are natural isomorphisms

$$
H^j(M, T^{1,0}M) \cong H^j_{\overline{J}}(\mathfrak{m}^{1,0}), \quad \text{for } j \in \{0,1,\ldots, k-1\}.
$$
It is possible to find harmonic representatives for the Dolbeault cohomology with values in $T^{1,0}M$ by looking at the invariant cohomology. In fact, let us fix an appropriate invariant Hermitian metric $g$ on $M$ as in $[4, \S 2]$. Extending $g$ to every $\mathfrak{m}^{(0,1)} \otimes \mathfrak{m}^{1,0}$ we denote by $\overline{\partial}$ the formal adjoint of $\overline{\partial}$ in the invariant setting with respect to $g$ and by $\Delta := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ the Laplacian operator with respect to $\overline{\partial}$. Let us set $\text{im}^i \overline{\partial}_{j-1}$ as the orthogonal complement of $\text{im} \overline{\partial}_{j-1}$ with respect to that fixed metric. Then, the following holds (see [8]).

**Theorem 6.3.** The space $\text{im}^i \overline{\partial}_{j-1} \subset \ker \overline{\partial}_j$ is a space of harmonic representatives for the Dolbeault cohomology $H^j(M, T^{1,0}M)$. Also, if $\mu \in \mathfrak{m}^{(0,1)} \otimes \mathfrak{m}^{1,0}$, then $\overline{\partial} \mu$ with respect to the $L^2$-norm on the compact manifold $M$ is equal to $\overline{\partial} \mu$ with respect to the Hermitian inner product on the finite-dimensional vector space $\mathfrak{m}^{(0,1)} \otimes \mathfrak{m}^{1,0}$.

We note that due to Hodge theory in the invariant setting, each injection map $\iota_j : \text{im}^j \overline{\partial}_{j-1} \subset \ker \overline{\partial}_j \to H^j$ is an isomorphism, for $j \in \{0, 1, \ldots, k-1\}$.

The bracket of vector forms (3.1) can be defined also in the invariant setting on cohomology classes in $H^j(M, T^{1,0}M)$. If $\mathfrak{d} \otimes V, \mathfrak{d} \otimes V'$ is representative for an element in $H^j(M, T^{1,0}M)$, we define

$$\{ \mathfrak{d} \otimes V, \mathfrak{d} \otimes V' \} := \mathfrak{d} \wedge \iota_\mathfrak{d} V \otimes V + \mathfrak{d} \wedge \iota_\mathfrak{d} V \mathfrak{d} \otimes V',$$

where by $i_\mathfrak{d} V \mathfrak{d}$ we mean the contraction of the form $V \mathfrak{d}$ with the vector field $V'$. We observe that this definition coincides with (3.1) on $H^j(M, T^{1,0}M)$.

We are now set to describe the Kuranishi method in this setting (as in [8]), which will yield a power series construction of the $(0, 1)$-vector form representing the deformations of the complex structure.

Let $\{ \beta_1, \ldots, \beta_N \}$ be an orthonormal base of representatives for $H^1(M, T^{1,0}M)$. For any $\mathbf{t} := (t_1, \ldots, t_N) \in \mathbb{C}^N$, let $\mu(\mathbf{t}) = t_1 \beta_1 + \cdots + t_N \beta_N$ and set $\Psi_1 = \mu$. If $\overline{\partial}^*$ is the formal adjoint of $\overline{\partial}$ with respect to $g$ and $\Delta = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, we define by recursion the terms $\Psi_{r}(\mathbf{t})$, with $r \geq 2$, as in equation (3.4):

$$\Psi_{r}(\mathbf{t}) = -\frac{1}{2} \sum_{s=1}^{r-1} \overline{\partial}^* \mathcal{G} \{ \Psi_s(\mathbf{t}), \Psi_{r-s}(\mathbf{t}) \} = -\frac{1}{2} \sum_{s=1}^{r-1} \overline{\partial}^* \mathcal{G} \{ \Psi_s(\mathbf{t}), \Psi_{r-s}(\mathbf{t}) \},$$

where $\mathcal{G}$ is the Green’s operator, which inverts $\Delta$ on the orthogonal complement of the space of harmonic forms.

Let $\{ \gamma_1, \ldots, \gamma_P \}$ be an orthonormal base for the space of $(0, 2)$-vector forms. If $f_{\mathbf{t}}(\mathbf{t})$ is the $L^2$-inner product of $\langle \{ \Psi(\mathbf{t}), \Psi(\mathbf{t}) \}, \gamma_\mathbf{k} \rangle$, i.e. the projection of $\{ \Psi(\mathbf{t}), \Psi(\mathbf{t}) \}$ onto $H^2$. Kuranishi theorem (3.6) guarantees the existence of $\epsilon > 0$ such that

$$S = \{ s \in \mathbb{C}^N : |s| < \epsilon, \ f_{\mathbf{t}}(\mathbf{t}) = 0, j = 1, \ldots, P \}$$

forms a family of deformations, thus proving the following theorem.

**Theorem 6.4.** Let $M$ be a nilmanifold with Abelian invariant complex structure $J$. Then the deformations arising from $J$, parametrized as in (6.5) are all invariant complex structures.

Furthermore, in [4] a characterization Abelian deformation is given.

**Theorem 6.5.** A parameter $\mu \in H^1(M, T^{1,0}M)$ defines an integrable infinitesimal Abelian deformation if and only if $\overline{\partial} \mu = 0$ and

$$\{ \mu, \mathfrak{d} \} = 0, \text{ for } \mathfrak{d} \in \mathfrak{m}^{(0,1)}.$$

### 6.2 Applications on examples.

In the following example, we will apply the theory on Abelian nilmanifolds so far introduced together with Theorem 5.1 to study the SKT condition on products of the real Heisenberg group.

**Example 6.6** ($H(3; \mathbb{R}) \times H(3; \mathbb{R})$). Let $H(3; \mathbb{R})$ be the real Heisenberg group, i.e. the nilpotent group

$$\begin{pmatrix} 1 & x & w \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : (x, y, w \in \mathbb{R})$$

with the Lie bracket

$$[A, B] = \begin{pmatrix} 0 & x & y \\ -x & 0 & -w \\ -y & w & 0 \end{pmatrix}.$$
If we define $G := H(3; \mathbb{R}) \times H(3; \mathbb{R})$, i.e. the group of matrices of the form

$$\begin{pmatrix}
1 & x & w & 0 & 0 & 0 \\
0 & 1 & y & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & s & v \\
0 & 0 & 0 & 0 & 1 & u \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

for $x, y, w, u, v \in \mathbb{R}$

and we define analogously $\Gamma := H(3; \mathbb{Z}) \times H(3; \mathbb{Z})$ we set $M := \Gamma \setminus G$. It is easy to check $M$ is a 2-step nilmanifold.

A global invariant coframe of 1-forms on $G$, hence on $M$, is given by

$$\begin{align*}
ed^1 &= dx, \quad e^2 = dy \\
ed^3 &= ds, \quad e^4 = du \\
ed^5 &= -dw + xdy, \quad e^6 = -dv + sdu.
\end{align*}$$

We define an almost complex structure $J$ on $M$ by setting

$$Je^1 = -e^2, \quad Je^3 = -e^4, \quad Je^5 = -e^6$$

so that

$$\varphi^1 := e^1 + ie^2, \quad \varphi^2 := e^3 + ie^4, \quad \varphi^3 := e^5 + ie^6$$

is a global coframe of $(1,0)$-forms on $M$. The structure equations with respect to (6.8) are given by

$$\begin{align*}
d\varphi^1 &= 0 \\
d\varphi^2 &= 0 \\
d\varphi^3 &= \frac{1}{2}\varphi^1 \circ T - \frac{1}{2}\varphi^2 \circ T.
\end{align*}$$

We notice that $J$ is an Abelian complex structure (henceforth, integrable).

We will consider the following metric as the initial metric

$$\omega = \frac{i}{2} (\varphi^1 \circ T + \varphi^2 \circ T + \varphi^3 \circ T)$$

which locally can be written as

$$\omega = \frac{i}{2} (1 + \frac{1}{4}|z|^2) dz \circ T + \frac{1}{8} z^2 \circ T dz \circ T - \frac{1}{4} z \circ T dz^3
- \frac{1}{2} \frac{1}{z^2} dz^2 \circ T + \frac{i}{2} (1 + \frac{1}{4}|z|^2) dz \circ T + \frac{i}{4} \frac{1}{z^2} dz^3
+ \frac{1}{4} \frac{1}{z} dz^3 \circ T - \frac{1}{4} \frac{1}{z} dz^2 \circ T + \frac{i}{2} \frac{1}{z} dz^2
$$

From (6.9), it is clear that $\partial \partial \omega = 0$, i.e. $\omega$ is a SKT metric on $M$.

By simple computations, we can find that the holomorphic coordinates which induce $J$ on $M$ are given by

$$\begin{align*}
z_1 &= x + iy \\
z_2 &= s + iu \\
z_3 &= \left( w + \frac{1}{4}(s^2 + w^2) - \frac{1}{2} xy \right) + i \left( v - \frac{1}{4}(x^2 + y^2) - \frac{1}{2} su \right)
\end{align*}$$

We can then rewrite locally (6.7) and the dual invariant vector fields $\{\varphi_1, \varphi_2, \varphi_3\}$:

$$\begin{align*}
\varphi_1 &= dz^1 \\
\varphi_2 &= dz^2 \\
\varphi_3 &= -dz^3 - \frac{1}{2} z^1 dz^1 + \frac{1}{2} z^2 dz^2
\end{align*}$$

$$\begin{align*}
\varphi_1 &= \frac{\partial}{\partial z^1} - \frac{i}{2} \frac{z^1}{z^2} \frac{\partial}{\partial z^2} \\
\varphi_2 &= \frac{\partial}{\partial z^2} + \frac{i}{2} \frac{z^2}{z^3} \frac{\partial}{\partial z^3} \\
\varphi_3 &= -\frac{\partial}{\partial z^3}
\end{align*}$$

We now proceed with making use of the tools we introduced in the beginning of this section. Since we can restrict ourselves to the invariant setting, we start by finding an invariant basis for $\mathcal{H}^1$. 
Let us fix a generic element of $\alpha \in \mathfrak{m}^{*(0,1)} \otimes \mathfrak{m}^{1,0}$,

$$\alpha := \sum_{i,\lambda} a_{i\lambda} \varphi^i \otimes \varphi_\lambda, \quad (a_{i\lambda}) \in \mathbb{C}^0.$$  

The element $\alpha$ is harmonic if and only if

$$\bar{\partial} \alpha = \overline{\partial^*} \alpha = 0.$$  

By imposing (6.14) and using (2.1) and (2.2), we get the following conditions on the coefficients:

$$
\begin{align*}
  a_{13} &= 0 \\
  a_{23} &= 0 \\
  a_{12} &= i a_{21}.
\end{align*}
$$

Therefore, an invariant basis for $\mathcal{H}^1$ is given by

$$\{ \varphi^1 \otimes \varphi_1, \varphi^2 \otimes \varphi_2, i \varphi^2 \otimes \varphi_1 + \varphi^1 \otimes \varphi_2, \varphi^1 \otimes \varphi_3, \varphi^2 \otimes \varphi_3, \varphi^3 \otimes \varphi_3 \},$$

so that we can fix $\Psi_1(t) \in \mathfrak{m}^{*(0,1)} \otimes \mathfrak{m}^{1,0}$ as

$$\Psi_1(t) := t_{11} \varphi^1 \otimes \varphi_1 + t_{22} \varphi^2 \otimes \varphi_2 + it_{21} \varphi^2 \otimes \varphi_1 + t_{12} \varphi^1 \otimes \varphi_2$$

$$+ t_{31} \varphi^3 \otimes \varphi_3 + t_{32} \varphi^2 \otimes \varphi_3 + t_{33} \varphi^3 \otimes \varphi_3,$$

with $t = (t_{11}, t_{22}, t_{31}, t_{32}, t_{33}) \in \mathbb{C}^5$. Now we can apply the recursive formula (6.4) to obtain the terms $\Psi_j(t)$, for $j \geq 2$. With easy computations, it can be checked that

$$\mathcal{G} \overline{\partial^*} \{ \Psi_1(t), \Psi_1(t) \} = 0,$$

so that $\Psi_2(t) = 0$. By induction on the recursive formula for the following terms, we can see that $\Psi_j(t) = 0$ for $j \geq 2$. Therefore, the deformation $(0,1)$-form $\Psi(t)$ coincides with $\Psi_1(t)$. This allows us to make use of Theorem 6.5 to describe deformations which preserve the Abelian condition. In particular, we must impose that $\{ \Psi(t), \overline{\varphi^j} \} = 0$ for any element $\overline{\varphi^j}$ of the basis of $\mathfrak{m}^{*(0,1)}$. This gives us the following condition:

$$t_{21} = 0.$$

Therefore, $\Psi(t) \in \mathfrak{m}^{*(0,1)} \otimes \mathfrak{m}^{1,0}$ describing deformations of the complex structure (6.8) which preserve the Abelian conditions can be written as

$$\Psi(t) := t_{11} \overline{\varphi^1} \otimes \varphi_1 + t_{22} \overline{\varphi^2} \otimes \varphi_2 + t_{31} \overline{\varphi^3} \otimes \varphi_3 + t_{32} \overline{\varphi^2} \otimes \varphi_3 + t_{33} \overline{\varphi^3} \otimes \varphi_3,$$

for $(t_{11}, t_{22}, t_{31}, t_{32}, t_{33}) \in \mathbb{C}^5$ in a neighborhood $B$ of $0 \in \mathbb{C}^5$.

We will now construct a curve of complex structures arising from this family of deformations. If $(a_{11}, a_{22}, a_{31}, a_{32}, a_{33}) \in B$ is a fixed point, we define the segment $\gamma: [0, 1] \to \mathbb{C}^5$ such that $\gamma(t) := (ta_{11}, ta_{22}, ta_{31}, ta_{32}, ta_{33})$. The form $\Psi(t)$ is then

$$\Psi_t := t(a_{11} \overline{\varphi^1} \otimes \varphi_1 + a_{22} \overline{\varphi^2} \otimes \varphi_2 + a_{31} \overline{\varphi^1} \otimes \varphi_3 + a_{32} \overline{\varphi^2} \otimes \varphi_3 + a_{33} \overline{\varphi^3} \otimes \varphi_3), \quad t \in [0, 1]$$

which has local expression

$$\Psi_t \equiv t \cdot \Psi$$

with

$$\Psi = a_{11}(d\varphi^1 \otimes \frac{\partial}{\partial z^1}) + a_{22}(d\varphi^2 \otimes \frac{\partial}{\partial z^2}) + a_{33}(d\varphi^3 \otimes \frac{\partial}{\partial z^3})$$

$$+ (\frac{i}{2}a_{11} \delta^1 - a_{31} - \frac{i}{2}a_{33} \delta^1)(d\varphi^1 \otimes \frac{\partial}{\partial z^3}) + (\frac{1}{2}a_{22} \delta^2 - a_{32} - \frac{1}{2}a_{33} \delta^2)(d\varphi^2 \otimes \frac{\partial}{\partial z^3}).$$

From (6.17), Lemma (4.1), and the identification between $\Psi(t)$ and $L_t$, we can compute the global invariant coframe $\{ \varphi^j \}_{j=1}^3$ of the deformation in terms of $\{ \varphi^j \}_{j=1}^3$ and $\{ \theta^j = dz^j + \Psi_t(dz^j) \}_{j=1}^3$

$$\begin{align*}
   \varphi^1_t &= \varphi^1 + (a_{11} + a_{31}) \varphi^1 + ta_{11} \varphi^1 \\
   \varphi^2_t &= \varphi^2 + (a_{22} + a_{32}) \varphi^2 + t a_{22} \varphi^2 \\
   \varphi^3_t &= \varphi^3 + (a_{31} + a_{33} + a_{33}) \varphi^3 + t a_{31} \varphi^3 + t a_{33} \varphi^3.
\end{align*}$$

If we consider the metric having

$$\omega_t = \frac{i}{2}(\varphi^1_T + \varphi^2_T + \varphi^3_T),$$

$$\omega_{T1} = \frac{i}{2}(\varphi^1_T + \varphi^2_T + \varphi^3_T),$$

$$\omega_{T2} = \frac{i}{2}(\varphi^1_T + \varphi^2_T + \varphi^3_T),$$

$$\omega_{T3} = \frac{i}{2}(\varphi^1_T + \varphi^2_T + \varphi^3_T).$$

$$\omega_{T1} = \frac{i}{2}(\varphi^1_T + \varphi^2_T + \varphi^3_T),$$

$$\omega_{T2} = \frac{i}{2}(\varphi^1_T + \varphi^2_T + \varphi^3_T),$$

$$\omega_{T3} = \frac{i}{2}(\varphi^1_T + \varphi^2_T + \varphi^3_T).$$

$$\omega_{T1} = \frac{i}{2}(\varphi^1_T + \varphi^2_T + \varphi^3_T),$$

$$\omega_{T2} = \frac{i}{2}(\varphi^1_T + \varphi^2_T + \varphi^3_T),$$

$$\omega_{T3} = \frac{i}{2}(\varphi^1_T + \varphi^2_T + \varphi^3_T).$$
as fundamental form, clearly, for $t \to 0$ we have that $\omega_t$ tends to the metric $\omega$ previously defined in (6.10).

We can check the SKT condition for $\omega_t$ by rewriting the structure equations for the complex structure having (6.21) as a basis for the invariant (1,0)-forms

$$
\begin{align*}
&\left\{ 
\begin{aligned}
d\varphi^1_t &= 0 \\
d\varphi^2_t &= 0 \\
d\varphi^3_t &= \frac{1-t_{33}}{1-t_{22}} [\varphi^1_t \wedge \varphi^2_t - \frac{1-t_{33}}{1-t_{22}} \varphi^2_t \wedge \varphi^1_t].
\end{aligned}
\right.
\end{align*}
$$

Imposing $\partial_t \bar{\partial} t \omega = 0$, we obtain

$$
\partial_t \bar{\partial} (t \omega_t) = 0 \iff \left( \frac{i}{(1-t_{11}^2)(1-t_{22}^2)} \right) (t_{33} - t_{33}) \varphi^1_t \wedge \varphi^2_t = 0 \iff t_{33} \in \mathbb{R}.
$$

Actually, we can say more. Let

$$
\Omega_t = \frac{i}{2} \sum_{\alpha, \beta = 1}^3 g_{\alpha \beta} (z, t) \varphi_\alpha^t \wedge \varphi_\beta^t
$$

be a SKT Hermitian metric on $M$ with respect to $J_t$, with $g_{\alpha \beta} (z, t)$ local functions. From the positiveness of $\Omega_t$, we must have that $g_{33} (z, t) \neq 0$. If we compute the $L^2$ product of $* \omega_t$ and $\partial_t \bar{\partial} \Omega_t$, integrating by parts, we get

$$
0 = \langle \partial_t \bar{\partial} \Omega_t, * \omega_t \rangle = \langle \Omega_t, * \partial_t \bar{\partial} \omega_t \rangle = \frac{i}{2} \int_M \varphi^1_t \wedge \varphi^2_t \wedge \sum g_{\alpha \beta} \varphi_{\alpha}^t
$$

$$
= \frac{i}{2} \int_M g_{33} (z, t) \varphi^1_t \wedge \varphi^2_t \wedge \varphi^3_t.
$$

Therefore, if $t_{33} \in \mathbb{C} \setminus \mathbb{R}$, there exists no SKT metrics on the deformed Abelian complex structure.

Let us apply Theorem 5.1 to $\omega_t$ to check other possible necessary conditions to the property of being SKT. Under a change of basis from $\{ \varphi^j_t \}_{j=1}^3$ to $\{ \theta^j_t \}_{j=1}^3$, the metric $\omega_t$ can be written as

$$
\omega_t = \frac{i}{2} \left( 1 + \frac{1}{4} |z|^2 \right) \theta^1 \wedge \theta^2 + \frac{1}{8} z^2 \theta^3 - \frac{1}{4} z \theta^1 \wedge \theta^3
$$

$$
- \frac{1}{4} z^2 \theta^2 \wedge \theta^3 + \frac{i}{2} \left( 1 + \frac{1}{4} |z|^2 \right) \theta^2 \wedge \theta^3
$$

$$
+ \frac{1}{4} z \theta^3 \wedge \theta^2 - \frac{i}{2} z^2 \theta^1 \wedge \theta^3 + \frac{i}{2} \theta^1 \wedge \theta^2.
$$

The matrix $\psi = \{ \psi_{ij}^j \}_{i,j=1}^3$ representing the form $\psi$ is

$$
\psi = \begin{pmatrix}
a_{11} & 0 & -\frac{1}{8} a_{11} z^1 - a_{31} - \frac{1}{2} a_{33} z^1 \\
0 & a_{22} & \frac{1}{2} a_{22} z^2 - a_{32} - \frac{1}{2} a_{33} z^2 \\
0 & 0 & a_{33}
\end{pmatrix}
$$

Recalling the local expression (6.11) and using (6.25) and (6.26), we now have all the ingredients to apply Theorem 5.1.

As a first remark, by looking at (6.25) we note that for any $i, j \in \{1, 2, 3\}$,

$$
\frac{\partial (\omega_t)^i}{\partial \theta^j} = 0,
$$

since the coefficients $(\omega_t)_{ij}$ do not depend on $t$.

Also, the terms $\omega_{ij}$ and $\omega_{ji}$ for $i, j \in \{1, 2, 3\}$ are linear in either $z^1, z^2, z^3$ or their conjugates, therefore

$$
\frac{\partial^2 \omega_{ij}}{\partial \theta^k \partial \theta^l} = \frac{\partial^2 \omega_{ji}}{\partial \theta^k \partial \theta^l} = 0, \text{ for every } i, j, k, l \in \{1, 2, 3\}.
$$

With the aid of these remarks, by elementary computations it turns out that all the necessary conditions are trivially verified, therefore our theorem does not give any non-trivial conditions.

In the following example we apply Theorem 5.1 on consecutive deformations on the Iwasawa manifold.

**Example 6.7** (Iwasawa manifold). Let us consider the complex Heisemberg group

$$
H(3; \mathbb{C}) = \left\{ \begin{pmatrix}
1 & z_1 & z_3 \\
0 & 1 & z_2 \\
0 & 0 & 1
\end{pmatrix} : z_1, z_2, z_3 \in \mathbb{C} \right\}
$$
and its cocompact discrete subgroup

\[ H(3; \mathbb{Z}[i]) = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_1, z_2, z_3 \in \mathbb{Z}[i] \right\}. \]

Define the Iwasawa manifold as \( X = H(3; \mathbb{Z}[i]) \backslash H(3; \mathbb{C}) \). \( X \) is a compact complex 3-dimensional nilmanifold which do not admit Kähler metrics. Let \( z^1, z^2, z^3 \) be the standard complex coordinate system on \( H(3; \mathbb{C}) \).

The following \((1,0)\)-forms on \( H(3; \mathbb{C}) \) are invariant for the left action of \( H(3; \mathbb{Z}[i]) \), so they give rise to a global coframe for \( (T^{1,0}X)^* \):

\[
\begin{align*}
\varphi^1 &= dz^1 \\
\varphi^2 &= dz^2 \\
\varphi^3 &= dz^3 - z^1 dz^2.
\end{align*}
\]

Therefore the structure equations are

\[
\begin{align*}
d\varphi^1 &= 0 \\
d\varphi^2 &= 0 \\
d\varphi^3 &= -\varphi^1 \wedge \varphi^2.
\end{align*}
\]

In [2, Section 3], the authors, following Nakamura in [9, pag. 95], compute and classify small deformations of the complex structures of the complex manifold \( X \). With the same notation of [2], consider the local system of complex coordinates for the complex structure \( J_t \) at \( t = (0, t_{12}, t_{21}, 0, 0, 0) \in \mathbb{C}^6 \) given by

\[
\begin{align*}
\zeta^1 &= z^1 + t_{12} z^2 \\
\zeta^2 &= z^2 + t_{21} z^3 \\
\zeta^3 &= z^3 + t_{21} |z|^2.
\end{align*}
\]

Denote by \( X_t \) the deformed complex manifold with complex structure \( J_t \). Equivalently \( X_t \) could be viewed as \( \Gamma_t \backslash H(3; \mathbb{C}) \), where \( \Gamma_t \) is the group generated by the transformations

\[
\begin{align*}
\zeta^1 &= \zeta^1 + \eta_1 + t_{12} \eta_2 \\
\zeta^2 &= \zeta^2 + \eta_2 + t_{21} \eta_1 \\
\zeta^3 &= \zeta^3 + \eta_3 + \zeta^1 \eta_2 + t_{21} \eta_1 (\zeta^1 + \eta_1)
\end{align*}
\]

varying \( \eta = (\eta_1, \eta_2, \eta_3) \in (\mathbb{Z}[i])^3 \), and \( H(3; \mathbb{C}) \) is considered with coordinates \( \zeta^1, \zeta^2, \zeta^3 \). Consider the \((1,0)\)-forms invariant for the action of \( \Gamma_t \) on \( H(3; \mathbb{C}) \) given by

\[
\begin{align*}
\varphi^1_t &= d\zeta_t \\
\varphi^2_t &= d\zeta^2_t \\
\varphi^3_t &= -d\zeta^3_t - z^1 dz^1 t_{21} \zeta^1_t,
\end{align*}
\]

where

\[
z^1 = \frac{1}{1 - t_{21} i_{12}} \left( \zeta^1_t + \frac{t_{12}}{t_{21} i_{12} - 1} \zeta^2_t \right).
\]

Their structure equations are

\[
\begin{align*}
d\varphi^1_t &= 0 \\
d\varphi^2_t &= 0 \\
d\varphi^3_t &= \sigma_{12} \varphi^1_t \wedge \varphi^1_t + \sigma_{1T} \varphi^1_t \wedge \varphi^2_t + \sigma_{2T} \varphi^1_t \wedge \varphi^2_t,
\end{align*}
\]

where

\[
\begin{align*}
\sigma_{12} &= -\frac{1}{|t_{12} i_{21} - 1|^2} \\
\sigma_{1T} &= \frac{t_{21}}{|t_{12} i_{21} - 1|^2} \\
\sigma_{2T} &= -\frac{t_{12}}{|t_{12} i_{21} - 1|^2}.
\end{align*}
\]

The vector form in \( \mathcal{A}_J \) describing the deformation can be explicitly written as

\[
\Psi(t) = t_{21} dz^1 \otimes \frac{\partial}{\partial z^2} + t_{21} z^1 dz^2 \otimes \frac{\partial}{\partial z^1} + t_{12} d\bar{z}^2 \otimes \frac{\partial}{\partial \bar{z}^1} - t_{12} t_{21} \bar{z}^1 d\bar{z}^2 \otimes \frac{\partial}{\partial \bar{z}^3}.
\]

We are interested to find parameters \( t_{12}, t_{21} \) such that the manifold \( X_t \) admits a SKT metric. Note that, according to structure equations (6.28), the complex structure \( J_t \) is invariant. By [11, Proposition 21] if
there exists a SKT metric on \(X_t\), then there is also a left invariant one. By direct computations using structure equations (6.28), one gets that if \(g\) is a left invariant Hermitian metric on \(X_t\), then \(g\) is SKT if and only if
\[
\bar{\partial}_t \bar{\partial}_t (\varphi_t^3 \wedge \varphi_t^4) = 0,
\]
and
\[
\bar{\partial}_t \bar{\partial}_t (\varphi_t^3 \wedge \varphi_t^4) = (\bar{\sigma}_{17} \bar{\sigma}_{28} + \bar{\sigma}_{17} \bar{\sigma}_{28} - \bar{\sigma}_{12} \sigma_{12}) \varphi_t^{12\overline{12}}.
\]
where
\[
\sigma_{17} \sigma_{28} + \bar{\sigma}_{17} \sigma_{28} - \sigma_{12} \bar{\sigma}_{12} = -\frac{2 \Re (t_{12} \bar{t}_{21}) + 1}{|t_{12} \bar{t}_{21} - 1|^2},
\]
and by \(\bar{\partial}_t, \bar{\partial}_t\) we indicate the complex differentials on \(X_t\). Therefore, taking \(t_{12} = -i\) and \(t_{21} = \frac{i}{2}\), the manifold \(X_t\) admits a SKT metric. Note that, for these values of \(t_{12}, t_{21}\), we retrieve the values of the parameters \(t = s = 1\) in [5, Section 2]. At this point, we want to apply another deformation to the complex manifold \(X_t\).

To simplify the notation, for \(t = (0, -i, \frac{i}{2}, 0, 0, 0)\), set \(M = X_t\), \(\xi^1 = \xi_t^1\), \(\xi^2 = \xi_t^2\), \(2\pi = -z^1\). Thus we recall
\[
\begin{cases}
\xi^1 = d\xi^1 \\
\xi^2 = d\xi^2 \\
\xi^3 = d\xi^3 + i\alpha d\xi^1 + 2\pi d\xi^2,
\end{cases}
\]
and
\[
\begin{cases}
d\xi^1 = 0 \\
d\xi^2 = 0 \\
d\xi^3 = -\frac{4}{9} \xi^1 \wedge \xi^2 + \frac{2}{9} \xi^1 \wedge \zeta^1 + \frac{4}{9} \xi^2 \wedge \zeta^2.
\end{cases}
\]
Let us deform the complex structure in the following way: for \(-\epsilon < t < \epsilon\), set
\[
\begin{cases}
\xi_t^1 = \xi^1 \\
\xi_t^2 = \xi^2 + t \xi^3 \\
\xi_t^3 = \xi^3
\end{cases}
\]
their structure equations are
\[
(6.29)
\begin{cases}
d\xi_t^1 = 0 \\
d\xi_t^2 = 0 \\
d\xi_t^3 = -\frac{4}{9} \xi_t^1 \wedge \xi_t^2 - \frac{4}{9} \xi_t^2 \wedge \xi_t^1 + \frac{2}{9} \xi_t^1 \wedge \zeta_t^1 + \frac{4}{9} \xi_t^2 \wedge \zeta_t^2.
\end{cases}
\]
therefore the almost complex structure defined by declaring \(\xi_t^1, \xi_t^2, \xi_t^3\) as \((1, 0)\)-forms is integrable and defines a complex manifold \(M_t\). Note that \(\xi_t^1, \xi_t^2, \xi_t^3\) are invariant forms on \(M_t\) and that the complex structure of \(M_t\) is invariant. The vector form in \(A_t\) describing the deformation can be explicitly written as
\[
\Psi'(t) = t d\xi^2 \otimes \frac{\partial}{\partial \bar{\xi}^2} - 2 t m d\xi^2 \otimes \frac{\partial}{\partial \bar{\xi}^3}.
\]
To apply Theorem 5.1, we need a basis of \((1, 0)\)-forms of the type \(\theta^i = d\xi^i + (\psi_t)^i_s d\xi^s\), where \(\Psi'(t) = (\psi_t)^i_s d\xi^s \otimes \frac{\partial}{\partial \bar{\xi}^j}\). Then, write
\[
\begin{cases}
\theta^1 = d\xi^1 \\
\theta^2 = d\xi^2 + t d\bar{\xi}^2 \\
\theta^3 = d\xi^3 - 2 t m d\bar{\xi}^3
\end{cases}
\]
Writing the \(\xi_t^i\) in function of the \(\theta^i\), one obtains
\[
(6.30)
\begin{cases}
\xi_t^1 = \theta^1 \\
\xi_t^2 = \theta^2 \\
\xi_t^3 = \theta^3 + i \alpha \theta^1 + 2 \pi \theta^2
\end{cases}
\]
Let \(\omega_t\) be the fundamental form of any left invariant Hermitian metric on \(M_t\). Then, \(\omega_t\) can be expressed in the following way,
\[
(6.31)
2 \omega_t = i (r^2 \xi_t^{11} + s^2 \xi_t^{22} + t^2 \xi_t^{33}) + u \xi_t^{12} - \bar{u} \xi_t^{21} + v \xi_t^{13} - \bar{v} \xi_t^{31} + w \xi_t^{23} - \bar{w} \xi_t^{32},
\]
where \( r, s, l \) are smooth real valued functions in \( t \in (-\epsilon, \epsilon) \) and \( u, v, w \) are smooth complex valued functions in \( t \in (-\epsilon, \epsilon) \), satisfying the following conditions
\[
 r^2 > 0, \quad r^2 s^2 - |u|^2 > 0, \quad r^2 s^2 l^2 - 2\text{Re}(iu\bar{w}_t) > r^2 |u|^2 + s^2 |v|^2 + l^2 |w|^2.
\]
It is a long but straightforward computation to substitute equations (6.30) in equation (6.31) and to apply Theorem 5.1 on \( \omega_t \). One obtains that if \( \omega_t \) is SKT for \( t \in (-\epsilon, \epsilon) \setminus \{0\} \), then \( \frac{\partial \omega}{\partial t} |_{t=0} = 0 \). Therefore, if \( \frac{\partial \omega}{\partial t} |_{t=0} \neq 0 \), then \( \omega_t \) is not SKT for \( t \in (-\epsilon, \epsilon) \setminus \{0\} \). In fact, following the same method used in [5, Section 2] to prove the non-stability under small deformations of the SKT condition, one could proceed as follows. By direct computations using structure equations (6.29), one can show that \( M_t \) does not admit a SKT invariant metric for \( t \neq 0 \), since \( \partial_t \bar{\omega}(\xi_1^2 \cap \xi_2^2) = 0 \) iff \( t = 0 \). Again by [11, Proposition 21], if there exists a SKT metric on \( M_t \), then there is also a left invariant one. Thus, \( M_t \) does not admit any SKT metric for \( t \neq 0 \). Therefore, our method is weaker than the one in [5] in this case.

**Remark 6.8.** As a conclusion, on a nilmanifold \( M \) with an invariant complex structure \( J_0 \), an invariant coframe of \((1,0)\)-forms for the deformed complex structure \( J_t \) yields affordable computations to verify if the manifold admits (invariant) SKT metrics, through the use of structure equations. In these cases, our necessary condition on the existence of SKT metrics gives us less information (due to the approximation of the first term of the Taylor expansion) and requires more calculations than the direct computation of the SKT condition. On the other hand, note that Theorem 5.1 has a wide range of applications, since it requires no hypothesis on the base complex manifold.

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