Is the Maxwell–Shafer fish eye lens able to form super-resolved images?

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Abstract
The imaging capabilities of a modification of the Maxwell fish eye lens proposed by Shafer in the early 1990s are studied theoretically. A wave propagator for this system is derived for a two-dimensional version of this system. This propagator is given in simple analytic terms, and its form suggests that the system has the ability to produce super-resolved images. However, a modal analysis of the system shows that this apparent behavior is an artifact resulting from the fact that the system’s input and output are collinear (or coplanar, in the three-dimensional case), and that a realistic implementation of this system would form images that obey the diffraction limit.

1. Introduction

The search for systems that form ‘perfect’ real optical images dates back to early studies of reflectors with conic sections and Cartesian oval refractive interfaces. Such systems have the property of focusing all the rays emerging from a point source onto a second ‘image’ point. A handful of so-called absolute systems have been proposed, which form perfect ray-optical images not only of a single point or a discrete set of points, but of all points within an extended spatial region. Perhaps the best known absolute system is the Maxwell’s fish eye lens [1–3], which is a gradient index distribution that images a volume onto a volume, although several other options have been proposed, such as the Luneburg [2], Eaton [4], and Miñano [5] lenses, amongst others [6, 7]. Two-dimensional versions of the Maxwell fish eye have been implemented on a silicon platform [8] and even on a water tank [9].

There has been much debate recently about whether such absolute systems, particularly the Maxwell fish eye lens, not only forms perfect images ray-optically but also within the wave domain, or at least whether it can achieve images with resolution beyond the diffraction limit [10–20]. If it were possible, such sharp imaging would require specially-designed localized drains to be placed within the intended image position, and the nature of such drains has also sparked discussion [21–28].

Independently of whether or not the diffraction limit can be averted, a problem with the Maxwell fish eye lens is that the image points are embedded within the system itself and hence are not easy to access. For this reason, Shafer [29, 30] proposed a modification of the Maxwell fish eye lens that is not only possible to fabricate in principle, but that functions like a traditional imaging system, forming a perfect ray-optical image of a segment of a plane onto another, both being physically accessible. The Maxwell fish eye is defined by a Lorentzian radial refractive index distribution

\[ n(r) = n_0 \frac{2R^2}{r^2 + R^2}, \]  

where \( r \) is the distance from the system’s center (the origin), \( R \) is a critical radius at which the refractive index takes the value \( n_0 \), which is half of its value at the lens’ center. Shafer proposed two modifications to this system. Firstly, he observed that a spherical mirror can be placed at the critical radius, which perfectly maps the exterior of the fish eye lens onto the interior. This way, the system occupies only a finite region of space and requires refractive indices between \( n_0 \) and \( 2n_0 \). Secondly, he proposed to use only a half of this lens, a hemisphere, as shown in figure 1(a). The flat circular surface of the system can be separated into two halves, one being the
object plane at which the object is placed, and the second being the image plane where the image is formed and potentially detected.

The goal of this article is to study whether the Maxwell–Shafer half fish eye system can form images with sub-wavelength resolution. For simplicity we focus on a two-dimensional version of this system, illustrated in figure 1(b). The refractive index distribution is given in polar coordinates \((r, \phi)\) by equation (1) with \(0 \leq r \leq R\) and \(\phi \in [0, \pi]\), and the edge \(r = R\) is made to be perfectly reflecting, so that the field, assumed to be scalar (or equivalently, transverse-electric), is forced to vanish there. The radial line segment \(\phi = 0\) corresponds to the system’s object line, while the radial segment \(\phi = \pi\) is the system’s image line. In the ray-optics regime, all the rays emanating from a point in the object line for given value of \(r\) will be bent by the refractive index distribution, reflected by the mirror, and then refocused at a point at the image line at the same value of \(r\), as shown in figure 1(b). The ray paths are circular, for reasons mentioned in the next paragraph.

2. Wave propagator

To study this system, one can exploit the well-known fact that both ray and wave solutions within the two-dimensional fish eye correspond to the stereographic mapping of the corresponding solutions for propagation constrained to the surface of a sphere [2, 3]. In the ray model, rays are great circles (geodesics) of the sphere, and these great circles map onto circles on the plane. The mirror circle of radius \(R\) for the fish eye corresponds to the mapping of the equator, and each hemisphere is mapped stereographically from the opposite pole so that both are mapped onto the interior of the mirror. We can then use the well-known wave solutions for a sphere and map them onto the plane. Consider for simplicity that the sphere’s radius is also \(R\), and that position is uniquely specified by the polar angle \(\theta\) and the azimuth \(\phi\). The field generated by a monochromatic point source at \(\theta = 0\) (the sphere’s north pole) and fully absorbed by a perfect sink at \(\theta = \pi\) (the south pole) is given by [22]:

\[
U_{\text{Sph}}(c) = \exp(\text{i} \alpha c) \left[ L_\alpha(c) - \frac{2\text{i}}{\pi} Q_\alpha(c) \right],
\]

where \(\alpha = \sqrt{(kr)^2 + 1/4} - 1/2\), \(L_\alpha\) and \(Q_\alpha\) are Legendre functions of the first and second kinds, respectively, and \(c = \cos \theta\). (Appropriate Green’s functions for three and higher dimensions have also been studied [31].) The location of the sources and sinks can be rotated by an angle \(\phi'\) within the \(\phi = 0\) plane by replacing \(c\) with the expression for the projection (divided by \(R\)) of a generic point \((\theta, \phi')\) onto the line joining the new source-sink pair, given by
\[ C'(\theta', \theta; \phi) = \sin \theta \sin \theta' \cos \phi + \cos \theta \cos \theta'. \]

The stereographical mapping is given by the simple relation \( r = R \tan(\theta/2) \), which implies \( \sin \theta = 2rR/(R^2 + r^2) \) and \( \cos \theta = (R^2 - r^2)/(R^2 + r^2) \). The point source solution over the sphere is then mapped stereographically onto the fish eye coordinates \((r, \phi)\) by simply replacing the function \( C \) by either of the expressions

\[ C_{\pm}(r', r; \phi) = \frac{4\pi' r^2 \cos \phi \pm \left( R^2 - r^2 \right) \left( R^2 - r'^2 \right)}{\left( R^2 + r^2 \right) \left( R^2 + r'^2 \right)}, \]

where the choice of sign depends on whether the projection is from the north or the south pole. Since we are considering a fish eye with a mirror at \( R \), we must combine the two solutions, one projected from the north pole (representing the field emanating from the source) and one from the south pole (representing the field reflected at the mirror), such that they cancel at \( r = R \):

\[ U_{\text{Fish}}(r, \phi; r') = U_{\text{sph}}[C_+(r', r; \phi)] - U_{\text{sph}}[C_-(r', r; \phi)]. \]

This solution does vanish at the mirror since \( C_-(r', R; \phi) = C_+(r', R; \phi) \). \( U_{\text{Fish}} \) is then the solution corresponding to a point source at \( r = r' \), \( \phi = 0 \) and a perfect sink at \( r = r' \), \( \phi = \pi \).

The next goal is to construct a wave propagator that gives the field at any point \( 0 \leq r \leq R, 0 \leq \phi \leq \pi \) given its knowledge at the object radial line \( \phi = 0 \). Such propagator must not only satisfy the wave equation but it should also reduce to a Dirac delta distribution at the object line and propagate away from it towards increasing \( \phi \). Since the wave equation is of second order, the propagation integral in general requires the prescription of both the field and its normal derivative at the initial line. However, under the assumption of a definite sense of propagation across the initial line, the initial conditions can be relaxed and a propagator can be constructed that involves only the prescription of the field and not of its normal derivative. This is the standard strategy that leads, for example, to the Rayleigh–Sommerfeld formula for propagation in free space. The key for constructing the Rayleigh–Sommerfeld propagator (which is exact) is to use the directional derivative of the point source solution in the direction normal to the initial plane. Let us try such an approach in this problem, where the normal derivative corresponds to a derivative with respect to \( \phi \) scaled by the radial position of the point source:

\[ K(r', r; \phi) = \frac{2}{r'} \frac{\partial U_{\text{Fish}}(r, \phi; r')}{\partial \phi} = \frac{-8\pi' \sin \phi}{\left( R^2 + r^2 \right) \left( R^2 + r'^2 \right)} U_{\text{Fish}}(r, \phi; r'), \]

where a factor of two was also included for normalization purposes. This propagator indeed reduces to a Dirac delta of \( r' - r \) at the \( \phi = 0 \) radial line, hence reproducing the prescribed initial field. Further, for small \( \phi \) it behaves like a dipolar wave propagating away from \((r', 0)\) towards larger values of \( \phi \). Therefore, it is expected that the field at any point \((r, \phi)\) can be calculated from the knowledge of the incoming field at the object radial line \( \phi = 0 \) according to the propagation integral

\[ U(r, \phi) = \hat{K}_\phi U(r, 0) = \int_0^R U(r', 0) K(r', r; \phi) dr'. \]

Despite it satisfying the above properties, the behavior of this propagator is surprising in two respects: (i) it reduces to a delta function not only at the object line but also at the image line:

\[ K(r', r; \pi) = i \exp \left[ i \sqrt{(kR)^2 + 1/4} \pi \right] \delta(r' - r); \]

(ii) while \( \phi \) seems to play the role of the propagation parameter, the propagator is not additive in this variable under consecutive propagations:

\[ \hat{K} \phi \hat{K} \phi \neq \hat{K} \phi + \phi. \]

The first of these two surprising observations strongly suggests that the Maxwell–Shafer half fish eye lens does indeed form perfect, superresolved images, and that this perfect image is accessible from the exterior of the system. The second observation alerts us, however, to a possible problem with this interpretation: for example, if one propagates only over an angle \( \phi = \pi/2 \) and then performs a consecutive propagation integral over the same angle, one obtains an image that is diffraction limited rather than perfect.

**3. Modal analysis**

A closer analysis of this system in terms of its modes helps resolve this issue. This is easiest to appreciate in the cases where \( k = \sqrt{l(l+1)} / R \), where \( l \) is a positive even integer. (An analogous description exists for odd \( l \).) The
modes of angular propagation for this system are then simply given by the projections of the corresponding modes for the sphere:

\[ U_{lm}^{\text{prop}}(r, \phi) = \sqrt{\frac{2(l - 2m + 1)(2m - 1)!}{2l - 2m + 1}} \exp \left[ i(l - 2m + 1)\phi \right] L_{l-2m+1}^{(l-2m+1)} \left( \frac{R^2 - r^2}{R^2 + r^2} \right), \quad 1 \leq m \leq l/2, \]

which are orthonormal, with weight factor $1/r$, over any radial line. The real part of these propagating (and hence complex) modes for $l = 10$ is shown in the left column of Figure 2. The fact that there is a limited number $l/2$ of modes of angular propagation suggests that the system only has $l/2$ independent communication channels, and therefore limited resolution consistent with the diffraction limit. However, the fact that the object and image lines are collinear allows the inclusion of other solutions with higher spatial frequency oscillations that reproduce the same field profile at both radial lines. These fields are not modes in $\phi$; instead they can be thought of as standing evanescent waves. We can build a set of these solutions analytically in the following way:

\[ U_{lm}^{\text{ev}}(r, \phi) = \exp \left[ 2mi \arctan \left( \frac{R^2 - r^2}{2Rr \cos \phi} \right) \right] L_{l-2m+1}^{(l-2m+1)} \left( \frac{2Rr \sin \phi}{R^2 + r^2} \right), \quad m > l/2, \]

where $L_l^{(\mu)}(c)$ are solutions to the associated Legendre equation given by

\[ L_l^{(\mu)}(c) = \left( 1 - c^2 \right)^{\mu/2} \frac{h_{l,\mu}(c)}{h_{l,\mu}(0)}, \quad h_{l,\mu}(c) = \sum_{\mu=0}^{4l} a_{l,\mu} (c - 1)^\mu, \]
with coefficients $a_{i,l,\mu}$ satisfying the recurrence relation

$$a_{i,l,\mu} = a_{i-1,l,\mu} \frac{l(l+1) - \mu(\mu+1) - (i-1)(i+2\mu)}{2i(i+\mu)}, \quad (13)$$

with $a_{0,l,\mu} = 1$. These evanescent modes are orthonormal over the object line ($\phi = 0$) with weight factor $R/(R^2 + r^2)$, and they are also orthogonal to all the propagating modes under the same weight. The first few of these modes are shown in the central column of figure 2 for $l = 10$. Together, the sets of modes in equations (10) and (11) can be used to expand any field over the object radial line, and because both the propagating and evanescent modes satisfy $U_{lm}(r, \pi) = (-1)^{l+1}U_{lm}(r, 0)$, they form a perfect image (up to a global phase) of the object field. The equivalence of this combination of propagating and evanescent modes with the propagator in equation (6) was verified by calculating, both through the propagator and the modal expansion, the propagation of an initial localized field given by $|r(R - r)/R^2| \exp(-200r^2/R^2)$, which requires both sets of modes in comparable amounts. Figure 3(a) shows the intensity of this solution resulting from using the modes, for a system with $l = 10$. The corresponding solution using the propagator is not shown, as it is indistinguishable.

While both the propagator and modal approaches just described seem to suggest that perfect imaging is indeed possible, we discuss why in practice it is not. The key observation is that the evanescent modes in equation (11) for integer $m$ are not the only valid solutions of this type. The solutions corresponding to half-integer $m > 1/2$ are also valid solutions of the wave equation that meet the required boundary conditions. Let us define $U_{lm}^{ev}(r, \phi) = U_{lm-1/2}^{ev}(r, \phi)$ for $m > 1/2$. These evanescent modes satisfy $U_{lm}^{ev}(r, \pi) = (-1)^lU_{lm}^{ev}(r, 0)$. Therefore, they have the opposite parity as $U_{lm}^{ev}$, and hence they are not included in the propagator described earlier, as their presence would destroy the property of perfect imaging. The first few of these modes for $l = 10$ are shown in the right column of figure 2.

Perfect imaging would then require the system to support (and be able to couple in and out of the system) the evanescent modes $U_{lm}^{ev}$ and not the evanescent modes $U_{lm}^{ev}$. However, this is not what would happen in a normal situation, as we now show. For example, consider an incident field coupled into the object line of the system via a waveguide of width slightly smaller than $R$ surrounded by perfect reflectors with refractive index $n$, significantly larger than $2n_0$. Let this incident field correspond to a high-order mode of this waveguide, such that it couples essentially only to the evanescent modes of the fish eye. The question is whether this field can be collected by an identical waveguide attached to the image line of the lens, as shown in figure 4. The boundary conditions at the object and image lines require that the field and its normal derivative are continuous across these interfaces. Additionally, we require that there is no incoming wave from the waveguide coupled to the image line. Note that each of the two sets of evanescent modes has the ability to independently reproduce the initial field profile. However, to meet all the required boundary conditions (in value and normal derivative), both sets of evanescent modes are required in comparable amounts. Given the opposite parities $U_{lm}^{ev}(r, 0) = -U_{lm}^{ev}(r, \pi) = 2\sqrt{2/\pi} \sin [2m \arctan ((1 - r^2)/(2r))]$ and $U_{lm}^{ev}(r, 0) = U_{lm}^{ev}(r, \pi) = 2\sqrt{2/\pi} \sin [(2m - 1)\arctan ((1 - r^2)/(2r))]$, and the fact that the modes are real (so that their normal derivatives have the same phase as the modes themselves), the combination of both sets of modes largely cancels at the image line, so that the system essentially reflects the incident field, as shown in figure 4. Of course, this is simply a case of total internal reflection.

A similar calculation is shown in figure 3(b) for the localized Gaussian field in figure 3(a); if an appropriate mixture of both types of evanescent modes is used, their contribution is localized within the vicinity of the input line, so that the image is diffraction-limited rather than perfect. It is interesting to note that essentially the same diffraction-limited result is obtained if the propagator is used in two stages of $\phi = \pi/2$. The reason for this is that all information about the evanescent modes contained in the propagator is lost at the intermediate stage $\phi = \pi/2$, since this is a nodal line of these modes.
4. Concluding remarks

We have shown that the images formed by a Maxwell–Shafer fish eye lens are constrained by the standard diffraction limit, at least when the field is coupled into and out of the system in a standard way. This is an absolute system only from the ray-optical point of view, as it was intended. Note that these conclusions also hold for the three-dimensional version of this system. In that case, the evanescent modes are constrained to the vicinity of the disk containing the object and image, at which they have a profile similar to the propagating modes shown here for different values of both $l$ and $m$. Such modes can also be separated into two subsets with opposite parities, and the inclusion of both destroys the possibility of sub-wavelength resolution.

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Figure 4. To waveguides with refractive index significantly larger than that of the lens at its center are attached to the input and output lines. If a highly oscillating mode is injected into the first, the solution inside the lens is a mixture of both types of evanescent modes that is localized near the vicinity of the input line. Therefore, essentially all the light is reflected into the input waveguide and none is transmitted out of the output one.
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