Quantum Resonance near Optimal Eavesdropping in Quantum Cryptography

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Abstract

We find a resonance behavior in the disturbance when an eavesdropper chooses a near-optimal strategy intentionally or unintentionally when the usual Bennett-Brassard cryptographic scheme is performed between two trusted parties. This phenomenon tends to disappear when eavesdropping strategy moves far from the optimal one. Therefore, we conjecture that this resonant effect is a characteristic for the eavesdropping strategy near to optimal one. We argue that this effect makes the quantum cryptography more secure against the eavesdropper’s attack.
Recently, there are a lot of activities in the various applications of the quantum information theories\[1\]. Among them the most important ones are application to quantum computer and quantum cryptography. While the physical realization of the quantum computer seems to be far from the embodiment in a few years, developing quantum cryptography based on the BB84\[2\] and Ekert91\[3\] protocols is at the stage of the industrial era\[4\].

The main issue of the quantum cryptography is to determine how secure the quantum cryptographic scheme compared to the classical scheme. This issue can be turned into the following question: how much information the eavesdropper (Eve) can gain when a secret key is established between two trusted parties(Alice and Bob)?

Of course, the answer of the question is dependent on the eavesdropping strategies. The authors in Ref.\[5, 6\] computed the optimal mutual information between Alice and Eve in the usual BB84 protocol and the final results are

\[
I_{xy} = \frac{1}{2} \phi \left[ 2 \sqrt{D_{uv}(1 - D_{uv})} \right] \quad I_{uv} = \frac{1}{2} \phi \left[ 2 \sqrt{D_{xy}(1 - D_{xy})} \right],
\]

where \(\phi(z) = (1 + z) \log_2(1 + z) + (1 - z) \log_2(1 - z)\) and the subscripts denote the conjugate basis Alice and Bob use during BB84 process. The disturbance \(D\) is Bob’s observable error rate.

Subsequently, the BB84 protocol has been extended to the case that Alice and Bob use the three conjugate bases\[7\]. It has been shown that this extended scheme is more secure against the optimal eavesdropping. In order to find more secure quantum cryptographic protocols, recently, much attention is paid to the qutrit\[8, 9\], qudit\[9, 10, 11\] and continuous-variable systems\[12\]. The optimal eavesdropping on noisy states is also fully discussed very recently in Ref.\[13\].

Instead of the optimal eavesdropping strategy we would like to discuss, in this letter, on the near-optimal eavesdropping in usual BB84 scenario. We will show that an interesting quantum resonance occurs in the disturbance between Alice and Bob when Eve’s eavesdropping strategy is near to optimal.

First, we consider a simple case that Eve uses one-qubit probe. Eve makes contact her probe with the qubit between Alice and Bob and gets her probe to be entangled. We restrict ourselves into the case when Alice chooses \(x - y\) basis with notation \(|x\rangle \equiv |0\rangle\) and \(|y\rangle \equiv |1\rangle\).
We choose the entangled states as following

\[ |x\rangle \rightarrow |X\rangle = a|00\rangle + b|11\rangle \]
\[ |y\rangle \rightarrow |Y\rangle = \delta (-b|00\rangle + a|11\rangle) + \sqrt{1-\delta^2} (c|10\rangle + d|01\rangle) \]  

with \(a^2 + b^2 = c^2 + d^2 = 1\).

According to BB84 scenario, Alice will announce bases which she used to establish a secret key through public channel. After the announcement Eve performs an appropriate quantum-mechanical measurement on her probe to gain information on the Alice’s qubit. In order to maximize the information gain one can show that Eve takes a POVM measurement with complete set of positive operators \(\{E_0 = |E_0\rangle\langle E_0|, E_1 = |E_1\rangle\langle E_1|\}\), where

\[ |E_0\rangle = \frac{-1}{\sqrt{2}} \epsilon(ac - bd) \sqrt{1 + \cos \varphi} |0\rangle + \frac{1}{\sqrt{2}} \epsilon(ac - bd) \sqrt{1 - \cos \varphi} |1\rangle \]  
\[ |E_1\rangle = \frac{1}{\sqrt{2}} \sqrt{1 - \cos \varphi} |0\rangle + \frac{1}{\sqrt{2}} \epsilon(ac - bd) \sqrt{1 + \cos \varphi} |1\rangle \]  

In Eq.(3) \(\epsilon(x) = x/|x|\) is usual alternating function and

\[ \cos \varphi = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \]  

with \(\alpha = (a^2 - c^2) - \delta^2(b^2 - c^2)\) and \(\beta = \delta \sqrt{1 - \delta^2}(ac - bd)\). Then, following Ref. [5], it is straightforward to compute the Eve’s average information gain \(G\):

\[ G = q_0G_0 + q_1G_1 \]  

where

\[ q_0 = \frac{1}{2} + \frac{1}{4}(1 - \delta^2)(a^2 - b^2 + c^2 - d^2) \cos \varphi - \frac{1}{2} \delta \sqrt{1 - \delta^2} |ac - bd| \sin \varphi \]  
\[ q_1 = \frac{1}{2} - \frac{1}{4}(1 - \delta^2)(a^2 - b^2 + c^2 - d^2) \cos \varphi + \frac{1}{2} \delta \sqrt{1 - \delta^2} |ac - bd| \sin \varphi \]  
\[ G_0 = \frac{1}{4q_0} \left\{(a^2 - b^2 - c^2 + d^2) + \delta^2(a^2 - b^2 + c^2 - d^2)\right\} \cos \varphi + 2\delta \sqrt{1 - \delta^2} |ac - bd| \sin \varphi \]  
\[ G_1 = \frac{1}{4q_1} \left\{(a^2 - b^2 - c^2 + d^2) + \delta^2(a^2 - b^2 + c^2 - d^2)\right\} \cos \varphi + 2\delta \sqrt{1 - \delta^2} |ac - bd| \sin \varphi \]  

Using Eq.(6) one can compute the mutual information between Alice and Eve, which is

\[ I_{AE} = \frac{1}{2} [q_0\phi(G_0) + q_1\phi(G_1)] \]  

where \(\phi(z) = (1 + z) \log_2(1 + z) + (1 - z) \log_2(1 - z)\).
Now, let us discuss on the Bob’s error rate $D_B$, which is usually called disturbance. Firstly, let us consider $d_{\lambda u}$ (or $d_{\lambda v}$), which is the probability that Bob gets a wrong result conditioned upon Alice sending $|u\rangle$ (or $|v\rangle$) and Eve measuring $\lambda$, where $|u\rangle = (|x\rangle + |y\rangle)/\sqrt{2}$ and $|v\rangle = (|x\rangle - |y\rangle)/\sqrt{2}$. Then, it is easy to show

$$d_{\lambda u} = 1 - \frac{\langle U| (|u\rangle\langle u|) \otimes E_{\lambda} |U\rangle}{\langle U| \mathbb{I} \otimes E_{\lambda} |U\rangle}$$

$$d_{\lambda v} = 1 - \frac{\langle V| (|v\rangle\langle v|) \otimes E_{\lambda} |V\rangle}{\langle V| \mathbb{I} \otimes E_{\lambda} |V\rangle}$$  \hspace{1cm} (8)

where $|U\rangle = (|X\rangle + |Y\rangle)/\sqrt{2}$ and $|V\rangle = (|X\rangle - |Y\rangle)/\sqrt{2}$. If Eve has chosen the optimal strategy ab initio, $d_{\lambda u}$ and $d_{\lambda v}$ should coincide with each other. Since, however, we are considering the non-optimal case, we cannot expect $d_{\lambda u} = d_{\lambda v}$ in general. Although it is straightforward to compute $d_{\lambda u}$ and $d_{\lambda v}$, we will not present the explicit expressions in this letter due to their lengthy expressions. As expected, $d_{\lambda u}$ is different from $d_{\lambda v}$ except $\delta = 0$ case.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The plot of $D$-dependence of the mutual information between Alice and Eve when $\delta = 0$ and $a = 0.01$. Because of the fact that $d_{\lambda}$ is dependent on $\lambda$, the mutual information for the $\delta = 0$ case is slightly smaller than that for the optimal case.}
\end{figure}

We would like to discuss the $\delta = 0$ case briefly. In this case the most optimal conditions derived in Ref. \cite{5} are satisfied. The only one this case does not satisfy is the fact that $d_{\lambda} \equiv d_{\lambda u} = d_{\lambda v}$ is dependent on $\lambda$. This fact makes the mutual information $I_{AE}(\delta = 0)$
to be slightly smaller than the optimal value Eq. (11) as shown in Fig. 1. In Fig. 1 the disturbance $D$ is defined as an average Bob’s error rate

$$D = \sum_{\lambda} q_{\lambda} d_{\lambda} = \frac{1}{2}(1 - ac - bd).$$

(9)

Since the $\delta = 0$ case does satisfy the almost optimal conditions, we guess that the mutual information $I_{AE}$ for this case is maximum on condition that Eve uses the single-qubit probe.

Now let us consider the $\delta \neq 0$ case. Since, in this case, $d_{\lambda u}$ is different from $d_{\lambda v}$, we should define the disturbance as

$$D = p_u D_u + p_v D_v$$

(10)

where $p_i$’s are the prior probabilities that Alice sends signal $i$, and

$$D_u \equiv \sum_{\lambda} q_{\lambda} d_{\lambda u} \quad D_v \equiv \sum_{\lambda} q_{\lambda} d_{\lambda v}.$$  

(11)

In this letter we take a reasonable assumption that the two signals are equiprobable, i.e. $p_u = p_v = 1/2$.

FIG. 2: The $c$-dependence of $D$ when $\delta = 0.05$ and $a = 0.5$. Fig. 2 implies that there exists a resonance-like phenomenon in the Bob’s error rate when Eve chooses the near-optimal eavesdropping strategy. The two peaks in the figure are originated from $D_u$ and $D_v$ respectively.

Fig. 2 is a $c$-dependence of $D$ when $\delta = 0.05$ and $a = 0.5$. Fig. 2 shows that there exist two sharp peaks, which looks like a resonance phenomenon. The left and right peaks are
originated from $D_v$ and $D_u$ respectively. The reason why the peaks appear in the disturbance can be explained as follows. Under some circumstances the numerator $\langle U|(|u\rangle\langle u|\otimes E_\lambda)\rangle U\rangle$ is slightly smaller than the denominator $\langle U|\mathbb{I} \otimes E_\lambda|U\rangle$ in $d_{u\lambda}$ in the wide range of parameter space. Thus, $d_{u\lambda}$ becomes very small in this region. If however, there are some points where the numerator $\langle U|(|u\rangle\langle u|\otimes E_\lambda)\rangle U\rangle$ approaches zero, this makes a sharp increase at these points even if the denominator $\langle U|\mathbb{I} \otimes E_\lambda|U\rangle$ is slightly larger than the numerator. Similar phenomenon can occurs for $d_{u\lambda}$, which gives different peak.

Numerical calculation shows that these sharp peaks disappear when $\delta$ increases. This fact makes us to conjecture that this resonance-like phenomenon happens in the near-optimal strategy because the $\delta = 0$ case can play a role as an optimal strategy on condition that Eve uses a single-qubit probe.

In order to check the validity of our conjecture let us consider the case that Eve chooses the near-optimal strategy with her two-qubit probe. We assume that entanglement between Alice’s and Eve’s qubits is given by

\begin{equation}
|X\rangle = \sqrt{s}|x\rangle|\xi_x\rangle + \sqrt{1-s}|y\rangle|\zeta_x\rangle \tag{12}
\end{equation}

\begin{equation}
|Y\rangle = \sqrt{s}|y\rangle \left[ \sqrt{1-\delta^2}|\xi_y\rangle + \delta \left( -\sqrt{1-\beta}|\Psi_{xy}^+\rangle + \sqrt{\beta}|\Psi_{xy}^-\rangle \right) \right] + \sqrt{1-s}|x\rangle \left[ \sqrt{1-\delta^2}|\zeta_y\rangle + \delta \left( -\sqrt{1-\alpha}|\Phi_{xy}^+\rangle + \sqrt{\alpha}|\Phi_{xy}^-\rangle \right) \right] \nonumber
\end{equation}

where

\begin{align}
|\xi_x\rangle &= \sqrt{\alpha}|\Phi_{xy}^+\rangle + \sqrt{1-\alpha}|\Phi_{xy}^-\rangle \\
|\xi_y\rangle &= \sqrt{\alpha}|\Phi_{xy}^+\rangle - \sqrt{1-\alpha}|\Phi_{xy}^-\rangle \\
|\zeta_x\rangle &= \sqrt{\beta}|\Psi_{xy}^+\rangle - \sqrt{1-\beta}|\Psi_{xy}^-\rangle \\
|\zeta_y\rangle &= \sqrt{\beta}|\Psi_{xy}^+\rangle + \sqrt{1-\beta}|\Psi_{xy}^-\rangle.
\end{align}

(13)

The states $|\Phi_{xy}^\pm\rangle$ and $|\Psi_{xy}^\pm\rangle$ denote the maximally entangled Bell basis as follows:

\begin{align}
|\Phi_{xy}^\pm\rangle &= \frac{1}{\sqrt{2}} (|x\rangle|y\rangle \pm |y\rangle|x\rangle) \\
|\Psi_{xy}^\pm\rangle &= \frac{1}{\sqrt{2}} (|x\rangle|y\rangle \pm |y\rangle|x\rangle).
\end{align}

(14)

The reason why we choose Eq.(12) is that the entangled states $|X\rangle$ and $|Y\rangle$ with $\delta = 0$ provides an optimal mutual information to Eve as shown in Ref.5. Thus, we want to find a resonance phenomenon when $\delta$ is small to check the validity of our guess.

The disturbance $D$ can be computed numerically by making use of the symbolic calculation. Fig. 3 is a plot of $\alpha$-dependence of $D$ when $\beta = 1.8 - \alpha$, $s = 0.5$ and $\delta = 0.05$. As expected the disturbance $D$ exhibits a resonance behavior with varying $\alpha$. This phenomenon tends to disappear with increasing $\delta$. Thus, this resonant behavior seems to be a
FIG. 3: The $\alpha$-dependence of the disturbance $D$ when Eve uses the entanglement Eq.(12). The other constants are fixed by $\beta = 1.8 - \alpha$, $s = 0.5$ and $\delta = 0.05$. As expected the disturbance $D$ exhibits a sharp resonance. This effect disappears with increasing $\delta$, which means that Eve’s eavesdropping strategy is far from optimal one. Thus, this resonance seems to occur in the near-optimal strategy.

characteristic for the near-optimal eavesdropping strategy. Unlike Fig. 2, Fig. 3 shows one peak. This is due to the fact that $D_u$ and $D_v$ have peaks at the same point.

In this letter we report on the resonance phenomenon in the disturbance when Eve chooses the near-optimal eavesdropping strategy. In reality eavesdropper cannot perform the exact optimal strategy due to the various nature’s non-linear and/or decoherence effects. If eavesdropper takes an near-optimal strategy, this resonance effect increases a possibility for the two trusted parties to realize the eavesdropping attack. As a result, the resonance discussed in this letter makes the quantum cryptography more and more secure. It is of highly important, in this reason, to verify this resonance phenomenon in the quantum cryptographic experiment.

Acknowledgement: This work was supported by the Kyungnam University Foundation
Grant, 2008.

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