Frequency degeneracy of acoustic waves in
two-dimensional phononic crystals

A N Darinskii\textsuperscript{1}, E Le Clezio\textsuperscript{2} and G Feuillard\textsuperscript{2}

\textsuperscript{1} Institute of Crystallography RAS, Leninskiy pr. 59, Moscow, 119333, Russia
\textsuperscript{2} Université François Rabelais de Tours, ENI Val de Loire, LUSSI, FRE CNRS 2448, rue de la Chocolaterie, BP3410, 41034 Blois, France

E-mail: ADar@ns.crys.ras.ru, emmanuel.leclezio@univ-tours.fr, gfeuillard@univ-tours.fr

Abstract. Degeneracies of acoustic wave spectra in 2D phononic crystals (PC) and PC slabs are studied. A PC structure is constituted of parallel steel rods immersed into water and forming the quadratic lattice. Given the projection $k_z$ of the wave vector on the direction of rods, the bulk wave spectrum of the infinite PC is a set of frequency surfaces $f_i(k_x, k_y)$, $i = 1, 2, \ldots$, where $k_x, y$ are the components of the wave vector in the plane perpendicular to the rods. An investigation is performed of the shape of frequency surfaces in the vicinity of points $(k_{dx}, k_{dy})$, where these surfaces fall into contact. In addition, the evolution of the degeneracy with changing rod radius and cross-section shape is examined. Degeneracy in the spectrum of leaky modes propagating along a single waveguide in a PC slab is also investigated.

1. Introduction
The specific features of the acoustic wave propagation in phononic crystals (PCs) has attracted much attention of researchers during the last decade. Numerous theoretical and experimental works have been essentially aimed at the analysis of the existence conditions of complete stop-bands in the wave spectrum of PCs (e.g., [1]-[4]). The appearance of such stop-bands is of great interest from the engineering application viewpoint.

In the present paper, on the basis of the finite element method, we study the degeneracy in the spectrum of bulk waves and waveguide modes. A PC is constituted of parallel steel rods immersed into water and forming the square lattice. Given the projection $k_z$ of the wave vector $k$ on the direction parallel to the rods, the bulk wave spectrum of the infinite PC represents a set of frequency surfaces $f_i(k_x, k_y)$, $i = 1, 2, \ldots$, where $k_{x, y}$ are the components of $k$ in the plane perpendicular to the rods. We investigate the shape of frequency surfaces in the vicinity of points $(k_{dx}, k_{dy})$ where these surfaces fall into contact. We also study the evolution of the degeneracy with changing rod radius and cross-section shape. In addition, we study the degeneracy in the spectrum of leaky modes propagating along a waveguide in a PC slab.

2. Degeneracy of bulk waves in infinite PC
The frequencies $f_i(k_x, k_y)$, $i = 1, 2, \ldots$, of the perfect PC fall into degeneracy at isolated points $k_{dx,y}$ of the $k_{x,y}$ plane rather than along certain directions like in ”ordinary” crystals [5, 6]. Fig. 1 shows the eight lowest branches of the spectrum (here and below $k_z = 0$). The degeneracy shows up as intersection or touch points. We consider three points marked in Fig. 1.
A branch \( f_i(k_x, k_y) \) near a point where it does not contact another branch is relatively flat. A different situation comes about in the case of degeneracy. In Fig. 2a the shape of frequency surfaces near point 1 is depicted. There are two cones with common apex. Hence, the degeneracy at point 1 can be called conical. The shape of two surfaces at point 2 is plotted in Fig. 2b. In this case the surfaces fall in contact along a line segment directed perpendicular to the edge XM of the Brillouin zone. In a smaller neighborhood of point 2 the surfaces look like two intersecting planes. Therefore such a contact can be called the wedge-type contact. A third type of degeneracy - tangential - takes place at point 3 (Fig. 2c). Surface 3 is a paraboloid. It touches surface 2 which looks like a plane. For further use, the lowest non-degenerate branch 1 separated from surfaces 2 and 3 by a gap is inserted in Fig. 2c.

We have also computed the pressure distribution in water within the unit cell as a function of the values of \( k_{x,y} = \rho_k \left\{ \frac{\cos \varphi}{\sin \varphi} \right\} + k_{d_{x,y}} \), where \( \rho_k \ll k_{d_{x,y}} \) is the radius of a circle centered on the degeneracy point \( k_{d_{x,y}} \) and \( \varphi \) is the polar angle. Omitting the figures, we note that the distribution changes drastically if we compare, e.g., the patterns at \( \varphi = 0^\circ \) and \( \varphi = 90^\circ \) or \( 180^\circ \). However, at fixed \( \rho_k \), the pattern changes gradually with \( \varphi \) for conical and tangential degeneracies. In the case of the wedge-type degeneracy, the line of contact divides the \( k_{x,y} \) plane into two domains where the distributions are strongly different but quasi-independent of \( \varphi \). The jump occurs near the line of contact. It should be said that the pressure pattern hardly changes from point to point in the vicinity of non-degenerate points.

Let the rod cross-section be an ellipse slightly stretched out in the direction of the coordinate axis \( y \). We find that conical point 1 remains conical, merely shifting along edge GX. On the other hand, the types of contact at points 2 and 3 change. The wedge-type degeneracy at point 2 is transformed into the conical degeneracy lying on the edge XM. The tangential degeneracy at the node M of the Brillouin zone disappears. Instead of it, two conical points come about. They are located on the edge XM and its continuation symmetrically with respect to point M.
Consider the evolution of the tangential degeneracy at point 3 caused by decreasing the radius of the rod circle cross-section; the lattice constant $a$ is fixed. Decreasing the radius decreases the gap between surface 1 and surfaces 2 and 3 (Fig. 2c). The shape of the surfaces remains intact but the curvature of paraboloids 1 and 3 at their apexes increases. Besides, the pressure pattern starts to change around the apex of paraboloid 1, tending to repeat the pressure distribution for branch 3 at the corresponding points. When the radius reaches the critical value $r_c \approx 0.415 \text{ mm}$, all three branches fall into degeneracy. We meet a triple degeneracy. At this moment surfaces 1 and 3 become cones and the pressure distributions for branches 1 and 3 are identical. When the rod radius decreases further the triple degeneracy is lifted. But now branches 1 and 2 are in contact while branch 3 is not degenerate. The paraboloidal shape of surfaces 1 and 3 is getting recovered. As a result, we arrive at the situation representable by Fig. 2c turned upside-down.

It is of interest that the weak-binding approximation allows the described transformations of surfaces 1-3 to be modelled. Let the pressure wave be recovered. As a result, we arrive at the situation representable by Fig. 2c turned upside-down.

In our case, the eigenfunctions are merely plane waves. Their unknown amplitudes have been determined by solving the eigenfunction expansion method.

In view of the symmetry of the structure we put $u(x, y) = V_1 \cos \left(\frac{2\pi}{a} x + \cos \frac{2\pi}{a} y\right) + 2V_2 \cos \frac{2\pi}{a} x \cos \frac{2\pi}{a} y$, where $V_{1,2}$ are small real parameters. The solution is sought in the form $p(x, y) = e^{i(k_x x + k_y y)} \sum_{m,n=\pm 1} a_{mn} e^{i\pi (mx + ny)}$. Here $a_{mn}$ are unknown amplitudes. Assuming $k_{x,y}/k_0 \ll V_{1,2}$, we obtain the dispersion equation:

$$
(\gamma - V_2)[(\gamma - V_2)[(\gamma + V_2)^2 - 4V_2^2] - 4(k_x^2 + k_y^2)(\gamma + V_2)/k_0^2] = 0, \tag{1}
$$

where $\gamma = \omega^2/\omega_0^2 - 1$ and $\omega_0 = ck_0$.

Let $V_1 > 0$ while $V_2 < 0$. If $V_1 \neq |V_2|$, then Eq. (1) has four roots: $\gamma_2 = -|V_2|$ and

$$
\gamma_1 = |V_2| - 2V_1 + \frac{k_x^2 + k_y^2}{k_0^2(|V_2| - V_1)}; \quad \gamma_3 = -|V_2| - \frac{2(k_x^2 + k_y^2)|V_2|}{k_0^2(V_2^2 - V_1^2)}; \quad \gamma_4 = |V_2| + 2V_1 + \frac{k_x^2 + k_y^2}{k_0^2(|V_2| + V_1)}. \tag{2}
$$

The root $\gamma_2$ does not depend on $k_{x,y}$ and represent a plane. Three other roots specify paraboloids. Paraboloid $\gamma_3$ touches plane $\gamma_2$ at $k_{x,y} = 0$. Paraboloid $\gamma_1$ lies below surfaces $\gamma_{2,3}$, when $|V_2| < V_1$, and above them if $|V_2| > V_1$. The root $\gamma_4$ is greater than $\gamma_{1,2,3}$ irrespective of $V_{1,2}$. At last, let $|V_2| = V_1$. The roots $\gamma_{2,4}$ do not change. But now $\gamma_{1,3} = -|V_2| \pm \sqrt{2(k_x^2 + k_y^2)/k_0}$ and we get two cones with common apex. Thus the options $|V_2| < V_1$, $|V_2| = V_1$, and $|V_2| > V_1$ correspond to the three cases we meet in our computations: $r > r_c$, $r = r_c$, and $r < r_c$, respectively.

3. Leaky modes in PC slabs

Consider the spectrum of a PC slab containing six rod rows and immersed into water. The waveguide in the slab is created between rows 3 and 4 by changing the interlayer distance along the $y$-direction. Due to symmetry the computations can be carried out within the cell that includes only one-half of the slab putting on the lower boundary $\partial p/\partial y = 0$ or $p = 0$. To determine the wave field in water above the cell we use the eigenfunction expansion method. In our case, the eigenfunctions are merely plane waves. Their unknown amplitudes have been introduced into the finite element problem as additional degrees of freedom defined only on the upper boundary.

In view of periodicity along the axis $x$ the pressure produced in water is represented in the form $p(x, y) = e^{i(qx - \omega t)} \sum_{n} a_n(y) e^{i2\pi n/a} x$, where $a_n(y)$ is the $y$-dependent amplitude of the mode with $k_y = q + 2\pi n/a$. We put $q = 0$ and compute the mode frequencies as a function of the parameter $w$ determining the waveguide width. The modes are attenuated because of the radiation of the energy into the interior of water surrounding the slab. Therefore the corresponding frequencies $f_{lk} = f_{lk}' - if_{lk}''$ are complex (the index $k = 1, 2, \ldots$ labels modes).
Fig. 3 shows the \( \omega \)-dependence of \( f'_{lk} \) and \( f''_{lk} \), when \( f'_{lk} \) falls into the third stop-band (1046 kHz < \( f < 1232 \) kHz in Fig. 1). In this frequency range three radiative waves (\( k_x = 0, 2\pi/a \)) are generated in homogeneous water above and below the slab.

The real parts \( f'_{lk}(\omega) \) of symmetric and antisymmetric branches intersect at a number of points. The imaginary parts \( f''_{lk}(\omega) \) and \( f''_{lj}(\omega) \) at the \( \omega \)-values corresponding to the intersection points of curves \( f'_{lk}(\omega) \) and \( f'_{lj}(\omega) \) are relatively close or even nearly coincide. However, the imaginary part is far smaller than the real part so that the latter can be viewed as the main characteristics of modes and, hence, the merging of the real parts of frequencies can be treated as a degeneracy of the leaky mode spectrum in the slab. Note that the total merging \( f'_{lk}(\omega) = f'_{lj}(\omega) \) and \( f''_{lk}(\omega) = f''_{lj}(\omega) \) is unlikely to be met, since there is only one real variable \( \omega \), whereas the real and imaginary parts of frequency are independent functions of \( \omega \).

Figure 3. Real and imaginary parts of mode frequencies versus the parameter \( \omega \) determining the waveguide width \( L_w = (1 + \omega)a \).

4. Conclusion

The bulk wave spectrum of an infinite PC possesses the degeneracy points of the same three types as the types of acoustic axes in "ordinary" crystals. The conical degeneracy is stable, whereas the tangential and wedge-type degeneracies are generally unstable with respect of perturbation and transformed into the conical degeneracy. It is also found that three frequency surfaces can fall into contact. In the example given, the occurrence of the triple degeneracy is accompanied by the change of the type of contact. A degeneracy also arises in the mode spectrum of PC slabs. The frequency curves of leaky modes of different types, i.e., symmetric and antisymmetric, can cross at a number of points. However, the degeneracy occurs not in any stop-band. For instance, it is absent in the first stop-band, at least when \( q = 0 \).

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References

[1] Psarobas I E and Sigalas M 2002 Phys. Rev. B 66 052302
[2] Khelif A, Choujaa A, Benchabane S, Djafari-Rouhani B and Laude V 2004 Appl. Phys. Lett. 84 4400
[3] Zhang W Y, Liu Z X and Wang Z L 2005 Phys. Rev. B 71 195114
[4] Chen J J, Zhang K W, Gao J and Cheng J C 2006 Phys. Rev. B 73 094307
[5] Alshits V I, Sarychev A V and Shuvalov A L 1985 Sov. Phys. JETP 62 531
[6] Shuvalov A L 1998 Proc. R. Soc. Lond. A 454 2911