Cosmological Waveguides for Gravitational Waves

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Abstract

We study the linearized equations describing the propagation of gravitational waves through dust. In the leading order of the WKB approximation, dust behaves as a non-dispersive, non-dissipative medium. Taking advantage of these features, we explore the possibility that a gravitational wave from a distant source gets trapped by the gravitational field of a long filament of galaxies of the kind seen in the large scale structure of the Universe. Such a waveguiding effect may lead to a huge magnification of the radiation flux from distant sources, thus lowering the sensitivity threshold required for a successful detection of gravitational waves by detectors like VIRGO, LIGO and LISA.

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1 Introduction

Detecting gravitational waves is one of the greatest challenges for modern physics both from the theoretical and the experimental points of view [1, 2, 3]. The interest of this enterprise arises from the fact that, in case of success, it will open an entirely new window on the Universe and, as our past experience with radio astronomy teaches, this might have a profound impact on our vision of the Universe. Unfortunately, the detection of a gravitational wave has turned out to be a very difficult task and as of now, almost forty years after the early pioneering work of Weber [4], no one has succeeded yet. The main source of the difficulties is the fact that most of the known astrophysical sources should radiate exceedingly small amounts of gravitational radiation. The strongest sources should be massive, highly non-spherical systems with large internal kinetic energies [5], like colliding and coalescing black holes and neutron stars. Sources of this type last for a short time and should also be very rare, with the consequence that it will be necessary to search large regions of the Universe in order to have a reasonable event rate.

For sources in the high frequency band (1 ÷ 10^4 Hz), accessible to ground based interferometers like VIRGO and LIGO, a rate of several coalescence events per year requires extending the search at distances from the Earth of the order of 10^2 ÷ 10^3 Mpc. At such large distances even the dimensionless amplitudes h of the waves from such strong sources should be in the range of h ~ 10^{-21} ÷ 10^{-22}, which still represents an extremely small signal. There are hopes that the LIGO/VIRGO collaborations [1, 2] will be able to reach or at least closely approach the sensitivity needed for the detection of these signals. The same considerations apply to the sources in the low-frequency band (10^{-4} ÷ 1 Hz) accessible to the forthcoming orbiting LISA detector. Examples of such sources from distant galaxies, at distances of the order of one Gpc from the Earth, are coalescing massive black holes, or compact bodies spiraling into one of them. For a more thorough discussion of these issues and an updated guide to the bibliography we refer the reader to the recent paper by Thorne [6].

This being the state of things, it is our opinion that any mechanism leading to an amplification of the signal from a cosmological source of gravitational waves, thus making its detection easier, should be very welcome. An example of such a mechanism, well established by now in the realm of optical and radio-wave astronomy, is provided by gravitational lensing. In fact, the possibility that the gravitational radiation from distant binary neutron stars may get magnified due to microlensing by compact bodies has been recently considered in the literature [7]. In this paper we propose a different lensing effect: we shall argue that the large scale galactic gravitational fields generated by the filamentary distributions of galaxies seen in the large scale structure of the Universe, primordial cosmic strings evolved in today observable structures, may act as waveguides for gravitational waves. Since a gravitational wave interacts very weakly with practically all kinds of matter, a wave trapped in the interior of such a filament might be able to traverse cosmological distances without being absorbed or scattered in a significant way, and most importantly without undergoing the 1/R attenuation that characterizes propagation in vacuum. If a significant fraction of the radiation from distant
sources is waveguided to us in this way, this might lower in an appreciable way the minimum sensitivity needed for detecting the signal.

The idea of gravitational conductors, waveguides and circuits is not new per se. For example, it was considered in [8] where it was based on the hypothetical existence of a material having a large shear (the "respondium") and capable of reflecting a gravitational wave. Our approach is completely different and not so exotic. Starting from standard General Relativity, we show that a filamentary distribution of dust is capable of trapping in its interior a gravitational wave: this result will be derived below by applying the (lowest-order) WKB approximation to the linearized Einstein’s equations describing the propagation of a weak gravitational wave in a region of space filled with dust. Within these approximations, the trapping of a gravitational wave appears completely analogous to that of a ray of light inside a guide with a specific refraction index profile. Being based on the ray approximation, our approach cannot take into account interference effects, which should be negligible in the high frequency band accessible to VIRGO/LIGO as well as in the low frequency band accessible to LISA. In case they could not be neglected, one could resort to an approximation of the type of the Fock-Leontovich paraxial approximation [9], which, while using the short-wave approximation, allows also to describe interference.

A waveguiding effect by cosmological structures analogous to that described in this paper was shown to be possible in Ref. [10] for the case of electromagnetic waves and in Refs. [11, 12] for massless particles, using the Fock-Leontovich approximation. In Ref. [10], in particular, this mechanism was used to explain in a simple way the huge luminosity of quasars, as compared to their high redshift. They would not be exotic objects hiding in their interior some unknown engine that releases enormous amounts of energy; rather, they might just be protogalaxies whose luminosity is preserved by a waveguide. Furthermore, certain gravitational lensing effects such as “twin” and “brother” quasars [13] could be explained in a similar manner.

The paper is organized as follows. In Sect. 2 we present the linearized theory of gravitational radiation in a region of space-time filled with dust. We show that, in the leading WKB approximation, dust behaves as a non-dispersive and non-dissipative medium. In Sect. 3, we use this approximation to study the possibility that a gravitational wave, coming from a distant source, could be trapped by the gravitational field of a long filamentary structure, thus yielding to a waveguide effect. We estimate the focalization length, the amount of luminosity preserved, and the effective feasibility of the effect. A discussion of the results and the conclusions are drawn in Sect. 4.
2 Linearized Theory

In this Section, we shall derive the linearized equations describing the propagation of a weak gravitational wave through a region of space-time filled with dust. The same problem was discussed in Ref. [16], but the equations derived below are simpler than those of [16]. Before the passage of the wave we thus have an unperturbed background \((g_{ab}, u_a, \rho)\), where \(g_{ab}\) is the space-time metric while \(u_a\) and \(\rho\) denote, respectively, the field of velocities and the density of the dust. The background is assumed to satisfy the exact Einstein’s equations in the presence of dust:

\[
G_{ab} = \kappa \rho u_a u_b ,
\]

where \(\kappa = 8\pi G/c^4\), \(G\) being Newton’s constant, while \(G_{ab}\) is the Einstein’s tensor

\[
G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R .
\]

We also have

\[
\rho > 0 ,
\]

while

\[
u_a u^a = -1 .
\]

We recall that the conservation law for the stress-energy of the dust, combined with the normalization condition (3) implies that the velocities \(u^a\) satisfy the geodesic equation:

\[
u^b \nabla_b u^a = 0 ,
\]

while the density \(\rho\) satisfies the conservation equation:

\[
\nabla_a (\rho u^a) = 0 .
\]

We now imagine perturbing the background solution by a small perturbation \((\hat{g}_{ab}, \hat{u}_a, \hat{\rho})\). Plugging the perturbed solution into Einstein’s equations and retaining all the terms up to first order in the perturbation, we get the following linearized equations:

\[
\hat{G}_{ab} = \kappa [\hat{\rho} u_a u_b + \rho (\hat{u}_a u_b + u_a \hat{u}_b)] ,
\]

where

\[
\hat{G}_{ab} = - \frac{1}{2} \nabla^c \nabla_c \hat{\gamma}_{ab} + \nabla_{(a} \nabla^{c} \hat{\gamma}_{b)c} - \frac{1}{2} g_{ab} \nabla^c \nabla^d \hat{\gamma}_{cd} + R^c_{c d} \hat{\gamma}_{ab} + \frac{1}{2} R_{ab} R^{cd} \hat{\gamma}_{cd} .
\]

In this equation \(\nabla_a\) and \(R_{abc} \, ^d\) represent, respectively, the covariant derivative and the Riemann tensor relative to the background metric \(g_{ab}\). Indices are raised and lowered using \(g_{ab}\) and its inverse, while \(\hat{\gamma}_{ab}\) is defined as

\[
\hat{g}_{ab} = \hat{\gamma}_{ab} - \frac{1}{2} g_{ab} \hat{\gamma} ,
\]

\*We shall follow in this paper the notations and conventions of [15]. In particular, the signature of the metric is chosen to be \((- +++)\). Moreover \(A_{(bc)} := \frac{1}{2}(A_{bc} + A_{cb})\).
where
\[ \hat{\gamma} = g^{ab} \hat{\gamma}_{ab} . \] (10)

We have to add to Eq. (7) the equation obtained linearizing the analogue of the normalization condition Eq. (4) for the perturbed velocity field \( u_a + \hat{u}_a \):
\[ 2u^a \hat{u}_a = \hat{g}_{ab} u^a u^b . \] (11)

In deriving this equation, we used:
\[ \hat{g}^{ab} = -g^{ac} g^{bd} \hat{g}_{cd} \] (12)
and
\[ \hat{u}^a = g^{ab} \hat{u}_b - g^{ab} u^c \hat{g}_{bc} . \] (13)

Thus, Eqs. (7) and (11) are the equations that describe the propagation of a weak gravitational wave in a medium made of dust. We refer the reader to Ref. [16] for a proof of their linear stability. Let now \( h^a_b \) be the projector onto the hyperplane orthogonal to the 4-velocity \( u^a \):
\[ h^a_b = \delta^a_b + u^a u^b . \] (14)

Upon multiplying Eq. (7) by \( h^a_e h^b_f \), we get:
\[ h^a_e h^b_f \hat{G}_{ab} = 0 . \] (15)

Upon multiplying now Eq. (7) by \( u^a u^b \) and using Eq. (11) to eliminate \( u^a \hat{u}_a \), we also get:
\[ \kappa \hat{\rho} = \hat{G}_{ab} u^a u^b + \kappa \rho \hat{g}_{ab} u^a u^b . \] (16)

Finally, by multiplying Eq. (7) by \( u^b \) and using again Eq. (11) together with Eq. (14), we get:
\[ \kappa \rho \hat{u}_a = -\left( \hat{G}_{cd} u^c u^d u_a + \hat{G}_{ab} u^b \right) - \frac{\kappa}{2} \rho \hat{g}_{cd} u^c u^d u_a . \] (17)

Equations (13), (14) and (17) are completely equivalent to Eqs. (7) and (11). This can be seen by plugging Eq. (14) in Eq. (15) and then taking the terms involving the velocities \( u_a \) on the right hand side: upon using Eqs. (14) and (17), one then recovers the right hand side of Eq. (7). Finally, Eq. (11) can be obtained back from Eq. (17) by simply multiplying it by \( u^a \).

Now, Eq. (15) is the only evolution equation governing the perturbation, while Eqs. (16) and (17) just represent constraints. It is quite remarkable that the evolution equations depend only on the perturbation of the metric, while the constraints are explicitly solved with respect to the perturbations of the velocities and density. Thus, we focus on Eqs. (15): they can be simplified a bit by a suitable choice of gauge. Under an infinitesimal diffeomorphism
\[ x'^a = x^a - \xi^a , \] (18)
\( \hat{\gamma}_{ab} \) transforms as:
\[ \hat{\gamma}'_{ab} = \hat{\gamma}_{ab} + 2 \nabla_{(a} \xi_{b)} - g_{ab} \nabla^c \xi_c . \] (19)
Using the method discussed in [16], it is easy to prove that by means of a suitable choice of \( \xi_a \) it is always possible to achieve the so-called transverse gauge:

\[
\nabla^b \hat{\gamma}_{ab} = 0 . 
\]

(20)

We notice that this transversality condition does not fix completely the gauge, as it is preserved by any infinitesimal diffeomorphism such that:

\[
\nabla^a \nabla_a \xi_b + R^a_b \xi_a = 0 .
\]

(21)

In the transverse gauge, the second and third terms in the expression of \( \hat{\mathcal{G}}_{ab} \) in Eq. (8) drop out and thus we are left with the following equations:

\[
\frac{1}{2} \nabla^c \nabla_c \hat{\gamma}_{ab} - R^c_{cd} \hat{\gamma}_{cd} - R_{(a} \hat{\gamma}_{b)c} + \frac{1}{2} \hat{\gamma}_{ab} - \frac{1}{2} \nabla^c R^d_{cd} \hat{\gamma}_{cd} = 0 ,
\]

(22)

that are to be solved together with Eq. (20).

A further simplification arises if the wavelength \( \lambda \) of the perturbation is much smaller than all the typical lengths that characterize the background. We thus assume that the distances over which the velocities \( u_a \) vary in an appreciable way and the average radius of curvature of the background \( |R_{abcd}|^{-1/2} \) are both much larger than \( \lambda \). Under these conditions, it is legitimate to make the WKB ansatz:

\[
\hat{\gamma}_{ab}(x, \epsilon) = \text{Re}\{f_{ab}(x, \epsilon) \exp(i/\epsilon S(x))\} ,
\]

(23)

where \( f_{ab}(x, \epsilon) \) is a slowly varying amplitude having the asymptotic expansion (for \( \epsilon \to 0 \)):

\[
f_{ab}(x, \epsilon) = \sum_{n=0}^{\infty} \left( \frac{\epsilon}{i} \right)^n f_{ab}^{(n)}(x) .
\]

(24)

Plugging this expansion in Eq. (22) and equating to zero the coefficients of each power of \( \epsilon \), one gets a system of recursive equations for the phase \( S(x) \) and for the amplitudes \( f_{ab}^{(n)}(x) \). To order \( \epsilon^{-2} \) we get:

\[
(g^{ab} l_a l_b) (h_c^e h_d^f f_{cd}^{(0)}) = 0 ,
\]

(25)

where \( l_a \) is the wave vector:

\[
l_a := \partial_a S .
\]

(26)

As for the transversality condition Eq. (21), it gives to order \( \epsilon^{-1} \):

\[
l_b^{(0)} f_{ab}^{(0)} = 0 .
\]

(27)

Equation (25) admits two types of solutions, depending on which of the two factors between the brackets vanishes. Consider first:

\[
h_a^e h_b^d f_{cd}^{(0)} = 0 .
\]

(28)
This equation implies that $f^{(0)}_{ab}$ is of the form:

$$f^{(0)}_{ab} = v_{(a}u_{b)} := \frac{1}{2}(v_{a}u_{b} + u_{a}v_{b}) ,$$  \hspace{1cm} (29)$$
where $v_{a}$ in an arbitrary vector field. Upon multiplying both sides of the above equation by $2l^{b}$, and imposing Eq. (27), one finds that $v_{a}$ must be proportional to $u_{a}$, $v_{a} = v_{u_{a}}$. But then, multiplying Eq. (28) by $l^{a}l^{b}$, one gets that $f^{(0)}_{ab}$ can be different from zero only if $\omega := -l^{a}u_{a} = 0$. This means that the wavefronts are time-like surfaces, and then this type of solutions cannot describe a gravitational wave. For this reason, we shall not consider anymore this case in the rest of the paper.

The second class of solutions of Eq. (25) is characterized by phase-functions $S(x)$ satisfying the eikonal equation:

$$g^{ab}(\partial_{a}S)(\partial_{b}S) = 0 .$$  \hspace{1cm} (30)$$
The wavefronts are thus null hypersurfaces (we exclude the possibility of singular wavefronts and thus assume that $l_{a}$ does not vanish at any point) describing waves that propagate through the dust with the speed of light. We recall that the integral curves $x^{a}(v)$, called rays, of the null vector field $l_{a}$:

$$\frac{dx^{a}}{dv} = (g^{ab}\partial_{b}S)(x(v)) ,$$  \hspace{1cm} (31)$$
with $S(x)$ a solution of the eikonal equation, are null geodesics of the background metric $g_{ab}$ providing bicharacteristics of Eq. (22).

By looking at the next order in the formal expansion of Eq. (22), at order $\epsilon^{-1}$ one gets a continuity equation for the amplitude $f^{(0)}_{ab}(x)$:

$$h^{c}_{a}h^{d}_{b}(\nabla_{l} + \theta)f^{(0)}_{cd} = 0 ,$$  \hspace{1cm} (32)$$
where we have set

$$\nabla_{l} := l^{a}\nabla_{a} , \quad \theta := \frac{1}{2}\nabla_{a}l^{a} .$$  \hspace{1cm} (33)$$
Now, Eq. (32) is equivalent to

$$(\nabla_{l} + \theta)f^{(0)}_{ab} = v_{(a}u_{b)} ,$$  \hspace{1cm} (34)$$
where $v_{a}(x)$ is again an arbitrary vector field. Eq. (27) together with the geodesic condition satisfied by $l_{a}$, $\nabla_{l}l_{a} = 0$, imply that $v_{a}$ must be identically zero. To see this, multiply both sides of Eq. (34) by $2l^{a}$. The left-hand side is then seen to vanish:

$$2l^{b}(\nabla_{l} + \theta)f^{(0)}_{ab} = 2\nabla_{l}(l^{b}f^{(0)}_{ab}) = 0 ,$$
and thus we have for the right-hand side:

$$0 = 2l^{b}v_{(a}u_{b)} = (l^{b}v_{b})u_{a} - \omega v_{a} .$$  \hspace{1cm} (35)$$
This equation implies that $v_{a}$ must be proportional to $u_{a}$: $v_{a} = v_{u_{a}}$; but then it easily follows that it must vanish because after multiplying Eq. (34) by $l^{q}l^{b}$ we get

$$0 = v_{a}\omega^{2} ,$$
which implies that $v$ vanishes, because $\omega$, being the time-component of a non-vanishing null vector, is different from zero. We conclude that the amplitude $f^{(0)}_{ab}$ satisfies the transport equation:

\[
(\nabla_l + \theta) f^{(0)}_{ab} = 0 .
\]  

(36)

It follows from this equation that the expansion of $G_{ab}$ vanishes up to order $\epsilon^{-1}$ included and thus, according to Eqs. (16) and (17), the dust suffers no density or velocity perturbations to this order.

As it is usual in the WKB approximation, Eq. (36) converts into an ordinary first-order differential equation for the restriction $f_{ab}^{(0)}(x(v))$ of the amplitude $f_{ab}^{(0)}(x)$ to the rays (31) and thus its solutions are uniquely determined by the knowledge of $f_{ab}^{(0)}$ on a surface $\Sigma$ cutting each ray at one point. We can further simplify Eq. (36) by a suitable choice of a tetrad field, which we now describe. We take $u^a$ and $k^a := h^a_{b\ell} l^b$ as the first two elements of the tetrad, while the remaining two, that we call $e^1_a$, $e^2_a$ are defined as follows. We pick up an arbitrary point on each ray and choose for $e^1_a$, $e^2_a$ there two arbitrary space-like unit vectors such that:

\[
e^i_a u^a = e^i_a k^a = 0 , \quad e^a_1 e^a_2 = 0 .
\]

(37)

The vectors $e^i_a$ are then transported along the ray according to the “quasiparallel” transport rule [17] given by the equation

\[
\nabla_l e^i_a = - e^{ib} \nabla_l q^a_b ,
\]

(38)

where $q^a_b$ is the orthogonal projector onto the plane spanned by $u^a$ and $k^a$:

\[
q^a_b = - u^a u_b + \frac{1}{\omega^2} k^a k_b .
\]

(39)

In [17] it is shown that this rule of transport preserves inner products and that $e^i_a$ remain orthogonal to $u^a$ and $k^a$ all along the ray. Using the tetrad so defined, one can see that the most general amplitude $f_{ab}^{(0)}$ satisfying the gauge condition Eq. (27) can be decomposed as

\[
f_{ab}^{(0)} = A_+ e^1_{ab} + A_\times e^\times_{ab} + A_{tr} e^{tr}_{ab} + B_t e^t_{(a} b) + B_l e^l_{a b} ,
\]

(40)

where

\[
e^1_{ab} = e^1_a e^1_b - e^2_a e^2_b , \quad e^\times_{ab} = e^1_a e^2_b + e^2_a e^1_b , \quad e^{tr}_{ab} = e^1_a e^1_b + e^2_a e^2_b .
\]

(41)

$e^1_{ab}$ and $e^\times_{ab}$ represent the usual traceless transverse (T-T) modes of a gravitational wave, while $e^{tr}_{ab}$ is a transverse mode with non-vanishing trace; for the last two terms in Eq. (40), they represent longitudinal modes. The coefficients $A_+$, $A_\times$, $A_{tr}$, $B_t$ and $B_l$ satisfy a simple transport equation. Upon inserting Eq. (40) in Eq. (36) and taking its trace, we find:

\[
(\nabla_l + \theta) A_{tr} = 0 .
\]

(42)
Contracting now Eq. (36) with any of the traceless modes appearing in Eq. (40), in view of the fact that they are orthogonal to each other, and observing that $e_a \nabla_l e^a = 0$, which is a consequence of the transport rule Eq. (38), one obtains:

$$
(\nabla_l + \theta)A_+ = (\nabla_l + \theta)A_\times = (\nabla_l + \theta)B_i = (\nabla_l + \theta)B = 0 .
$$

(43)

We now show that by means of a gauge transformation of the type (21) it is always possible to achieve that $A_{tr}$, $B_i$, and $B$ vanish on some initial surface $\Sigma$ and, due to the above transport equations, this implies that they vanish all along the rays crossing $\Sigma$. For this purpose, we consider a family of vector fields $\xi(x, \epsilon)$ of the form

$$
\xi_a(x, \epsilon) = e^{i/\epsilon} S(x) \sum_{n=1}^{\infty} \left( \frac{\epsilon}{l} \right)^n \xi^{(n)}_a(x) .
$$

(44)

This ensures that gauge transforming a $\hat{\gamma}_{ab}$ of the form (23) according to Eq. (19) the new $\hat{\gamma}_{ab}$ is still of the form (23). In particular, we find for $f^{(0)}_{ab}$

$$
f^{(0)}_{ab} = f^{(0)}_{ab} + 2l_a \epsilon^{(1)} b - (l_a \epsilon^{(1)} c) g_{ab} .
$$

(45)

Upon imposing Eq. (21) on the field $\xi(x, \epsilon)$, one can see that the coefficient $\xi^{(1)}_a(x)$ must satisfy the transport equation

$$
(\nabla_l + \theta)\xi^{(1)}_a = 0 .
$$

(46)

The important thing is that the value of $\xi^{(1)}_a(x)$ on $\Sigma$ is completely arbitrary and it is easy to verify that it always possible to choose it in such a way that

$$
A_{tr}' = B_i' = B' = 0 \text{ on } \Sigma .
$$

(47)

Consider now the effective energy-momentum tensor $\hat{T}^{ab}$ [18] and the effective “graviton number” $N^a$, respectively, defined as

$$
\hat{T}^{ab} = \frac{1}{4\kappa} (|A_+|^2 + |A_\times|^2) l^a l^b ,
$$

$$
N^a = \frac{1}{4\kappa h} (|A_+|^2 + |A_\times|^2) l^a .
$$

(48)

The transport equations for $A_+$ and $A_\times$, together with the geodesic equation for $l_a$, imply that their covariant divergences vanish:

$$
\nabla_a \hat{T}^{ab} = 0 , \quad \nabla_a N^a = 0 .
$$

(49)

Using the identity $\Gamma^a_{ab} = \partial_a g/(2g)$, where $g = \text{det} ||g_{ab}||$, we can rewrite the equation $\nabla_a N^a = 0$ in the form of a conservation law:

$$
4\kappa h \nabla_a N^a = \partial_a (|A|^2 l^a) + |A|^2 l^a \frac{1}{2g} \partial_a g = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} |A|^2 l^a) = 0 .
$$

(50)
where \( A^2 = (|A_+|^2 + |A_\times|^2) \). Upon integrating this equation on the 4-volume spanned by a ray bundle, and using Gauss’ theorem, one proves that the number of gravitons in the bundle is conserved and observer-independent.

The conclusion of this analysis is that, in the lowest WKB approximation, the physical degrees of freedom of a gravitational wave propagating through dust are represented, as in vacuum, by two traceless-transverse modes, \( e^+_{ab} \) and \( e^\times_{ab} \). The transport equations for \( A_+ \) and \( A_\times \), Eqs. (43), show that these modes propagate along null geodesics of the background gravitational field, again as it happens in vacuum. The dust behaves as a non-dispersive, non-dissipative medium that preserves the polarization of the wave and the total graviton number in a ray bundle.

3 The Waveguiding Effect

It is well known that, when the WKB approximation is valid, a weak gravitational wave propagates in vacuum much like an electromagnetic wave. One is thus led to the expectation that a gravitational wave passing by some massive body may undergo a lensing effect, much in the same way as it happens to light. The results of the previous Section show that under many respects a gravitational wave propagating through dust does not behave differently than in vacuum. In particular, we saw that dust behaves as a non-dissipative medium which implies that a gravitational wave can traverse large amounts of dust without suffering a significant absorption by the medium. This suggested to us the idea that the long, filamentary sequences of galaxies seen in the large scale structure of the Universe may act as “optical” guides for the gravitational radiation that propagates in their interior: the gravitational wave from a source placed at one end of such a filament (or inside it) may get trapped by the large scale galactic gravitational field and consequently be able to traverse huge distances without undergoing the \( 1/R \) attenuation that characterizes propagation in vacuum, resulting in a possibly strong magnification of the source for an observer placed at the other end of the guide. The possibility of an analogous phenomenon for the electromagnetic radiation was studied in ref. [10].

In this Section we shall study this phenomenon in an highly idealized situation. We shall consider a point source \( S \) of monochromatic gravitational waves, placed at a large distance from an observer \( O \). Between them, there is a long cylinder \( C \) filled with dust. We assume that \( S, O \) and the cylinder are at rest and that \( S \) and \( O \) are roughly aligned with the axis of the cylinder. For simplicity, we imagine that the distribution of dust inside the cylinder is stationary, while the density of matter outside the cylinder is negligible. Moreover, we assume that the background gravitational field \( g_{ab} \), generated by the dust, is weak (but much stronger than the perturbation) and that the relative motion of the dust is much slower than the speed of light, which is generally the case in astrophysical situations. Under these conditions the background can be described using the Newtonian approximation and so there will be a quasi-Minkowskian coordinate system \( \{ x^\mu \} \equiv \{ ct, x, y, z \} \) where the dust is practically at rest and
the gravitational field $g_{\mu\nu}$ of the background can be expressed in terms of the Newtonian potential $\Phi$ generated by the dust:

$$
\begin{align*}
g_{00} &= -\left(1 + \frac{2\Phi}{c^2}\right), \\
g_{ij} &= \left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij},
\end{align*}
$$

with $i, j = 1, 2, 3$ \hspace{1cm} (51)

\[ |\hat{g}_{\mu\nu}| \ll \frac{2|\Phi|}{c^2} \ll 1. \hspace{1cm} (52)\]

We assume that the time variation of the potential $\Phi$, due to the motion of the dust, is so slow that it can be neglected and thus we take $\Phi$ to be time-independent. We choose the origin and orientation of the Minkowskian coordinate system such that the axis of the cylinder $C$ coincides with the $z$-axis. Its top and bottom faces have radius $D$ and have $z$ coordinates respectively equal to $z = 0$ and $z = L$. The source $S$ is placed outside the cylinder at the point $P_S$ of coordinates $(x = a, y = 0, z = -l)$, with $l > 0$ and $|a| \ll D$.

According to the WKB approximation developed in the previous Section, the propagation of the wave emitted by $S$ is described by the eikonal equation, Eq. (30). Since the background is time-independent, we can make for the eikonal $S(x)$ the ansatz:

$$
S(t, x, y, z) = k \left\{ \bar{S}(x, y, z) - ct \right\},
$$

where $k = \frac{2\pi \nu_0}{c}$, $\nu_0$ being the frequency of the wave. Plugging this expression in the eikonal equation (30) and using for the background the expression in Eq. (51) we find

$$
\frac{1}{2} \left\{ \left(\partial_x \bar{S}\right)^2 + \left(\partial_y \bar{S}\right)^2 + \left(\partial_z \bar{S}\right)^2 \right\} + \frac{2\Phi}{c^2}(x, y, z) = \frac{1}{2}.
$$

While deriving Eq. (54), we have divided the eikonal equation by $2g^{ii}$, expanding inverses and products of $g_{\mu\nu}$ to first order in $\Phi/c^2$. Equation (54) has the same form of the eikonal equation describing a light-wave propagating in a medium with a refractive index $n = 1 - 2\Phi/c^2$.

In order to study the propagation of the wave inside the cylinder, we shall not search for the solution of Eq. (54) that describes a spherical wave originating from $P_S$, but we shall rather focus our attention on the corresponding rays $\vec{x}(v)$. We observe that Eq. (54) has the same form as the Hamilton-Jacobi equation for a unit mass particle, moving in $\mathbb{R}^3$ with Hamiltonian:

$$
\bar{H}(x^i, l_i) = \frac{1}{2}(l_x^2 + l_y^2 + l_z^2) + U(x, y, z),
$$

where

$$
U(x, y, z) = \frac{2\Phi}{c^2}.
$$

Thus, the rays $\vec{x}(v)$ are solutions of the Hamilton’s equations:

$$
\frac{dx^i}{dv} = \frac{\partial \bar{H}}{\partial l_i}, \quad \frac{dl_i}{dv} = -\frac{\partial \bar{H}}{\partial x^i}, \quad i = 1, 2, 3
$$

subject to the constraint $\bar{H} = 1/2$. 

11
We imagine that, before reaching the bottom face $\Sigma_0$ of the cylinder, the wave traverses a region of space where the gravitational field is negligible and so the incoming wave entering the cylinder will not differ appreciably from a spherical wave with center in $P_S$. If $Q$ is a point of $\Sigma_0$ of coordinates $(\zeta, \xi, 0)$, the eikonal $\bar{S}(\zeta, \xi, 0)$ will be roughly equal to the distance $R = [(\zeta - a)^2 + \xi^2 + l^2]^{1/2}$ of $Q$ from the source. We shall focus our attentions on the rays that cross $\Sigma_0$ in directions approximately parallel to the axis of the cylinder and this requires
\[
\frac{(\zeta - a)^2 + \xi^2}{l^2} \ll 1. \tag{58}
\]
The weak-field condition, $\Phi/c^2(x, y, z) \ll 1$, ensures that the direction of these rays will continue to be practically parallel to the $z$-axis all along the cylinder and this allows us to use the paraxial approximation to study their propagation. We start by taking the following Taylor expansion of the eikonal of the incoming wave:
\[
\bar{S}|_{\Sigma_0}(\zeta, \xi) = R \approx \left( l + \frac{(\zeta - a)^2 + \xi^2}{2l} \right). \tag{59}
\]
From it we compute the components $l_x(\zeta, \xi)$ and $l_y(\zeta, \xi)$ of the gradient of $\bar{S}(x)$ on the points of $\Sigma_0$. The third component $l_z(\zeta, \xi)$ should be computed from the constraint $H = 1/2$ (of course, we have to pick up the positive solution for $l_z$, as our wave propagates in the positive $z$-direction) but due to the paraxial condition $l_x(\zeta, \xi) \ll 1$, $l_y(\zeta, \xi) \ll 1$, and to the weak-field condition $\Phi/c^2 \ll 1$, we can make the approximation $l_z(\zeta, \xi) = 1$. In conclusion, we can say with a good approximation that on $\Sigma_0$ the rays $\vec{x}(v)$ associated with the spherical wave from $S$ are such, that
\[
x(0; \zeta, \xi) = \zeta, \quad y(0; \zeta, \xi) = \xi, \quad z(0; \zeta, \xi) = 0,
\]
\[
l_x(0; \zeta, \xi) = \frac{\zeta - a}{l}, \quad l_y(0; \zeta, \xi) = \frac{\xi}{l}, \quad l_z(0; \zeta, \xi) = 1, \tag{60}
\]
where we have labelled each ray by the coordinates $(\zeta, \xi)$ of the point, where it crosses $\Sigma_0$, and we have arbitrarily chosen 0 for the value of the parameter $v$ at that point. Thus, in order to see how the wave propagates inside the cylinder, in a small solid angle parallel to the axis, we just have to find the solutions of the Hamilton’s equations (57), using Eqs. (60) as initial conditions. Consider now the case that the cylinder is much longer than thicker, $L \gg D$ and that the distribution of dust inside it is independent of $z$ so that, away from the caps, the potential $\Phi$ will be only a function of $(x, y)$. The $z$-components of Eqs. (57) are then trivial to integrate and give $z(v) = v$ and $l_z(v) = 1$. We can thus use $z$ as a parameter in the place of $v$. As for the projections of Eqs. (57) in the $(x, y)$ plane, they are formally identical to the equations of motion for a point particle on a plane, having a mass equal to one and moving in the potential $U(x, y)$. It is obvious that if the potential $U(x, y)$ has the shape of a two-dimensional well, there may exist a subset $\Sigma^* \subset \Sigma_0$, such that for $(\zeta, \xi) \in \Sigma^*$, the solutions $(x(z; \zeta, \xi), y(z; \zeta, \xi))$ represent bounded motions and, when this happens, it means that the rays that cross $\Sigma^*$ will get trapped inside the cylinder. We thus have that the gravitational field of the cylinder may effectively act as a sort of waveguide for the gravitational radiation.
In order to examine this phenomenon in greater detail, let us make the simplifying assumption that the density $\rho_0$ of the dust is uniform. Then $U(x, y)$ inside the cylinder is equal to the potential of a harmonic oscillator:

$$U(x, y) = \frac{1}{2} \omega^2 (x^2 + y^2) \quad \text{for} \quad (x^2 + y^2) \leq D^2,$$

where

$$\omega^2 = \frac{4\pi G \rho_0}{c^2}. \quad (62)$$

If $D$ is infinite, the motions $(x(z; \zeta, \xi), y(z; \zeta, \xi))$ are all bounded and thus all the rays will be trapped by the cylinder. Upon solving the Hamilton’s equations, we get

$$x(z; \zeta, \xi) = \zeta \left[ \cos(\omega z) + \frac{1}{\omega l} \sin(\omega z) \right] - \frac{a}{\omega l} \sin(\omega z),$$

$$y(z; \zeta, \xi) = \xi \left[ \cos(\omega z) + \frac{1}{\omega l} \sin(\omega z) \right]. \quad (63)$$

Let now $L_{\text{foc}}$ be the solution of the transcendental equation:

$$\tan(\omega L_{\text{foc}}) = -l \omega \quad \pi/2 < \omega L_{\text{foc}} < \pi. \quad (64)$$

Assume that $L > L_{\text{foc}}$ and let $N$ be the integer such that $NL_{\text{foc}} < L < (N + 1)L_{\text{foc}}$. We see from Eqs. (64) that the rays are focussed at the focal points $F_n$ of coordinates

$$F_n \equiv \{-(-1)^n a \cos(\omega L_{\text{foc}}), 0, L_{\text{foc}} + (n - 1)\frac{\pi}{\omega}\},$$

$$n = 1, 2, \cdots N \quad (65)$$

This means that inside the cylinder there will form a series of alternatively inverted and right images of the source. We shall now compute the specific luminosity $L_n$ of the image at $F_n$, assuming that the source $S$ has a specific luminosity $L_S$. According to the definition of $L_S$, and assuming that $S$ radiates isotropically, the number $dN_S$ of gravitons with energy in the range $\hbar d\omega_s$ emitted by $S$ during the proper time interval $d\tau_S$ in the solid angle $d\Omega_S$ around the $z$-axis is equal to

$$dN_S = d\tau_S d\Omega_S d\omega_s \frac{L_S}{4\pi \omega_S}. \quad (66)$$

These gravitons form a narrow bundle. According to what we said above, they will be focussed at $F_n$, and we let $d\Omega_n$ be the solid angle they span at $F_n$. By the graviton number conservation, Eq. (64), $dN_S$ must be equal to the number of gravitons $dN_n$ emitted by the image during the corresponding time interval $d\tau_n$, in the interval of frequency $d\omega_n$ in the solid angle $d\Omega_n$:

$$dN_S = dN_n = d\tau_n d\Omega_n d\omega_n \frac{L_n}{4\pi \omega_n}. \quad (67)$$

Since the source and the image are at rest, $d\tau_S = d\tau_n$, $\omega_S = \omega_n$, and $d\omega_S = d\omega_n$. Thus, we have

$$L_n = L_S \left| \frac{d\Omega_S}{d\Omega_n} \right|. \quad (68)$$
In view Eqs. (63), it is easy to verify that
\[
\frac{d\Omega_n}{d\Omega_S} = \cos^2(\omega L_{foc})(1 + \omega^2l^2)^2.
\] (69)

If \(\omega l \ll 1\) we see from Eq. (64) that \(L_{foc} \approx \pi/\omega - l\). Equation (69) then implies \(L_n \approx L_S(1 - \omega^2l^2) \approx L_S\), which shows that the images have practically the same specific luminosity of the source. From the point of view of the observer \(O\), it will be as if the “waveguide” had driven the source from \(P_S\) to \(F_N\). Being much closer to him than the real source, the image at \(F_N\) will obviously appear to him much brighter.

So far, we have proceeded as if the radius of the cylinder had been infinite. The effect of \(D\) being finite is that in general not all the rays entering the cylinder will be focussed at \(F_N\). Outside the cylinder, the strength of the gravitational field decreases rapidly as one goes farther from its axis and so in general a ray that escapes out of it will not be focussed at \(F_N\). This suggests that a simple criterion for a ray to be focussed is that it should always remain inside the cylinder while traversing it: \(x(z)^2 + y(z)^2 < D^2\) for \(0 < z < L\). Assuming again that \(\omega l \ll 1\), we see from Eqs. (63) that this condition is satisfied only by the gravitons emitted by \(S\) in a solid angle \(\Delta\Omega_S\) around the direction of the \(z\)-axis of order:
\[
\Delta\Omega_S \approx \frac{D}{L_{foc}}\pi.
\] (70)

Since, for \(\omega l \ll 1\), this is also roughly equal to the angle \(\Delta\Omega_N\) spanned by these rays at the focal point \(F_N\), \(\Delta\Omega_S\) provides a measure of the degree of alignment of \(O\) with the axis of the cylinder needed for \(O\) to observe the image at \(F_N\).

In order to have an estimate of the above numbers in realistic situations, let us imagine that our waveguide is constituted by a sequence of galaxies forming a filamentary structure of the kind observed in the large scale structure of the Universe. Taking for \(D\) the typical size of a galaxy, \(D \approx 10^4\) pc, and for \(\rho_0\) the typical density of matter inside a galaxy, \(\rho_0 \approx 5 \cdot 10^{-24}\) g/cm\(^3\), we find that the weak-field approximation for the background, the second inequality in Eq. (72), is satisfied, because near the edge of the cylinder, where the galactic gravitational field is strongest, \(2\Phi/c^2 \approx \omega^2D^2/2 \approx 10^{-6}\). Plugging the above value for \(\rho_0\) in Eqs. (61) and (64), we get that \(L_{foc} \approx 10^2\) Mpc, which is the typical scale of the large structures of the Universe and represents also the expected distance of coalescence events, as was discussed in the Introduction. The corresponding angular opening \(\Delta\Omega_S\) of the bundle of rays that are efficiently trapped is of the order of 1′.

In the above discussion, we have treated our waveguide as if the distribution of matter inside it was homogeneous. Obviously this is not what happens inside the galaxies, where a significant fraction of matter is concentrated in compact bodies like the stars.† Considering the recent results of microlensing collaborations like MACHO, OGLE, DUO, and EROS, it is very likely that a large amount of the matter inside galaxies is built up of baryonic compact objects weakly interacting with electromagnetic radiation like planets, brown dwarfs or black holes so that all the following considerations apply also to these galactic components [19], [20], [21], [22].
galactic field is distorted on small scales by the local field of nearby stars and we have to analyze
the effect of these local disturbances on the focusing properties of the waveguide. A simple
order-of-magnitude estimate shows that in fact it should be negligible. We start by computing
the distance \( d(x) \) such that a ray, while traversing the cylinder, passes with probability \( x \) at a
distance smaller than \( d(x) \) from a star. For this purpose, consider the tube \( T \) of radius \( d(x) \)
around the ray. Assuming that the ray is roughly parallel to the axis of the cylinder and that
the stars are distributed in the cylinder with uniform density \( n_* \), the probability of finding a
star inside \( T \) is

\[
x = 1 - (1 - d^2(x)/D^2)^{n_* \pi D^2 L} \approx n_* \pi d^2(x) L .
\]

This formula is easy to understand, since \( (1 - d^2(x)/D^2)^{n_* \pi D^2 L} \) is the probability that throwing
at random \( n_* \pi D^2 L \) stars inside the cylinder no one of them falls inside \( T \). Now, since the
deflection \( \alpha(x) \) suffered by a ray passing at a distance \( d(x) \) from a star with mass \( M \) is
approximately \( \alpha(x) \approx 4GM/c^2 d(x) \), we have

\[
\alpha(x) \approx \frac{4Gc}{\sqrt{\pi M^2 n_* L x}} .
\]

Taking \( L \approx 10^8 \) Mpc, \( M n_* \approx 0.12 M_\odot pc^{-3} \) and assuming that the stars have a mass of
the order of the mass of the sun \( M_\odot \), we get

\[
\alpha(x) \approx 2 \times 10^{-4} \frac{1}{\sqrt{x}} \text{arcsec}.\quad (73)
\]

For \( x = 0.01 \), the deflection angle is of order of \( 10^{-3} \) arcsec. This means that, while traversing
the cylinder from one end to the other, a ray has a probability of 1% of being deflected by a
star by an angle of approximately one thousandth of arcsecond. Now consider the bundle of
rays emitted by \( S \) in a solid angle of opening \( \Delta \Omega_S \). While traversing the cylinder, the cross
section of this bundle will expand up to a point where it becomes comparable to the cross
section of the cylinder and then, after traversing a distance of order \( L_{\text{foc}} \), it will shrink to a
point. The rays in the bundle will thus suffer an average deflection of order \( D/L_{\text{foc}} \approx 1' \), which
is much greater than the deflection due to the occasional passages near a star as estimated
above. We thus conclude that the stars have a negligible effect on the propagation of the rays
inside the cylinder.

4 Discussion and conclusions

In this paper, we have shown that a filamentary distribution of dust may act as a waveguide
for gravitational radiation. We derived this result studying the short wave-length limit of
the linearized Einstein’s equations that describe the propagation of a weak gravitational wave
through dust. In this approximation, a gravitational wave behaves in essentially the same
way as an electromagnetic wave propagating in vacuum: in particular, the gravitational wave
propagates along geodesics of the background gravitational field generated by the dust. The
dust itself behaves as a non-dispersive and non-dissipative medium, preserving the polarization state of the gravitational wave. In this geometrical optic limit, the waveguiding effect has a very simple explanation: the gravitational field generated by a filament of dust traps in its interior a paraxial bundle of rays, with the result that, while traversing the filament, the amplitude $h$ of the gravitational wave does not undergo the $1/R$ attenuation characterizing the propagation in vacuum of spherical waves. We showed also that for a roughly uniform density of dust, the guide produces real images of the source, having approximately the same absolute “luminosity” as the real source. In some sense, the waveguide ”draws” the source closer to the observer: if the true distance to the source is $R$, its image brightness will correspond to that of a similar source at the closer distance

$$R_{\text{eff}} = R - l - L,$$

where $L$ is the length of the filament and $l$ is the distance of the source from the top of the filament (see also [10]).

Even though the effect described in this paper is basically a gravitational lensing effect, it is worth pointing out that it differs in some important points from the standard gravitational lensing situations. When discussing the gravitational lensing of light, it is normally sufficient to treat the deflector as a thin lens; moreover it is usually assumed that the gravitational field of the deflector can be treated using the newtonian limit, which requires the impact parameter $r_0$ to be much larger than the gravitational $r_g$ radius of the lens $r_0 \gg r_g$. This is not a very stringent constraint, because of the smallness of $r_g$; a much more stringent condition is that the light ray should pass close enough to the lens, for the deflection to be sensible, but not so close to cross it, because this would easily lead to the absorption or scattering of the light ray by the matter constituting the lens. In order to obtain the waveguiding effect described in this paper, we need to reverse two of the three above conditions: while we still assume the weak field condition, which is well respected in realistic astrophysical situations, an efficient trapping of the gravitational wave requires the lens to be very thick (in fact our filamentary lenses are much thicker than wider), and that the wave passes well inside the distribution of matter, in the region where the newtonian potential well is deeper and thus more confining. These conditions do not represent a problem since gravitational waves, differently from electromagnetic waves [24, 25], travel nearly unscathed through all forms and amounts of intervening matter and can thus traverse our long filaments without being significantly absorbed or scattered. Finally, we should mention that even though in this paper we have limited the discussion of the waveguiding effect to a dust distribution with the shape of a filament, more general geometries, planar for example, analogous to those used for optical waveguides, could be considered.

If the waveguiding effect described in this paper could be really observed, it might significantly lower the sensitivity threshold required to successfully detect a gravitational wave. In order to address this issue several questions have to be answered. The first one is: what is the probability that waveguides for gravitational radiation exist? Furthermore, what is the probability that a significant fraction of the flux from sources of gravitational radiation will get trapped?
The first point seems, from our point of view, very likely. The observations in the last twenty years tell us that filamentary or slab-like structures are quite common at large scales (see, e.g. [27],[28],[29],[30]). Furthermore, their sizes (∼10^2 ÷ 10^3 Mpc) are comparable with the focalization length of waveguides, as we have shown above. The hypotheses of their origin, starting from some fundamental theory and in connection with primordial structures as cosmic strings, domain walls and textures are among the most investigated topics of theoretical cosmology [29],[31],[32],[33].

The second point depends on the efficiency of production of gravitational radiation. As we said in the Introduction, for rates of the order of a few events per year, the expected distance of high frequency sources like coalescing neutron-stars, should be in the range of 10^2 ÷ 10^3 Mpc, which is also the order of magnitude of the focalization length of our waveguides. If the probability that a wave from such a source gets trapped by a waveguide is large enough, this effect might be seen already by the VIRGO and LIGO interferometers in the next few years. Since the filamentary structures that constitute our waveguides extend over cosmological distances, the waveguiding effect may have an important impact on the detection of gravitational waves in the low-frequency band that will be the target of the LISA interferometer and the candidates for such a kind of sources are quite numerous. First of all we have to stress the fact that, obviously, all the sources we are considering have extragalactic origin and can be sited in distant galaxies. This fact implies that our sources may have high redshifts so that in the frequency and in all the quantities characterizing a gravitational wave we have to include a factor (1+z). A large amount of candidates sources could be formed by binary evolved stars as double systems of white dwarfs, black holes and white dwarfs, or black holes and black holes. Such structures are very common in the Universe, they are present in all galaxies and, if they are tight bound, the gravitational radiation emission fluxes are conspicuous. Another class of objects could be quasars [34] if in their core a very massive black hole is present and massive objects fall into it. Finally, waves from primordial phase transitions and from cosmic strings [32] could be good candidate sources.

REFERENCES

[1] A. Abramovici et al., Science 256, 325 (1992).
[2] C. Bradaschia et al., Nucl. Instrum & Methods A 289, 518 (1990).
[3] P. Bender, A. Brillet, I. Ciufolini, K. Danzmann, R. Hellings, J. Hough, A. LObo, M. Sandford, and P. Touboul, LISA, Laser interferometer space antenna for gravitational wave measurements: ESA Assessment Study Report, R. Reinhard, ESTEC, 1994.
[4] J. Weber, Phys. Rev. 117, 306 (1960).
[5] K.S. Thorne, Three Hundreds Years of Gravitation, p. 330, S.W. Hawking and W. Israel eds. (Cambridge Univ. Press, 1987).
[6] K.S. Thorne, *Gravitational Radiation: a new window onto the Universe*, GRP-468 (1997), gr-qc/9704042.

[7] Y. Wang, A. Stebbins, and E.L. Turner, *Phys. Rev. Lett.* **77**, 2875 (1996).

[8] W. H. Press, *Gen. Relativ. Grav.* **11**, 105 (1979).

[9] A. M. Leontovich, and V. A. Fock, Zh. Éksp. Teor. Fiz., **16**, 557 (1946).

[10] S. Capozziello, R. de Ritis, V. Man’ko, A.A. Marino, and G. Marmo, *Phys. Scripta* **56**, 315 (1997).

[11] V.V. Dodonov, and V.I. Man’ko, *Gravitational waveguide*, Preprint of the P.N.Lebedev Physical Institute, No. 255 (Moscow, 1988); *J. Soviet Laser Research* (Plenum Press), **10**, 240 (1989); *Invariants and Evolution of Nonstationary Quantum Systems* Proceedings of the P.N. Lebedev Physical Institute, **183**, Nova Science, N.Y. (1989).

[12] V.V. Dodonov, O.V. Man’ko, and V.I. Man’ko, in: *Squeezed and Correlated States of Quantum Systems* Proceedings of the Lebedev Physical Institute, **205** p. 217, Nova Science, N.Y. (1993).

[13] V.V. Dodonov, in: *Proceedings of the First International Sakharov Conference* Moscow May 21–25 (1992), p. 241, Eds. L. V. Keldysh, and V. Ya. Fainberg, Nova Science, N.Y. (1993).

[14] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (W.H. Freeman and Company, San Francisco, 1973).

[15] R. M. Wald, *General Relativity* (Chicago University Press, Chicago, 1984).

[16] J. Ehlers, A.R. Prasanna, and R.A. Breuer, *Class. Quantum Grav.* **4**, 253 (1987).

[17] R.A. Breuer, and J. Ehlers, *Proc. Roy. Soc. Lond. A* **374**, 65 (1981).

[18] R.A. Isaacson, *Phys. Rev.* **166**, 1263 (1968).

[19] B. Paczynski, *Gravitational Lenses*, Lecture Notes in Physics **406**, p. 163, Springer–Verlag, Berlin (1992);
R. Kaiser, *Gravitational Lenses*, Lecture Notes in Physics **404**, p. 143, Springer–Verlag, Berlin (1992).

[20] M. Bartelmann, and P. Schneider, *Astron. Astrophys.* **268**, 1 (1993);
R. Kormann, P. Schneider, and M. Bartelmann, *Astron. Astrophys.* **284**, 285 (1994).

[21] P.V. Blioh, and A.A. Minakov, *Gravitational Lenses*, Naukova Dumka, Kiev (1989) [in Russian];
P. Schneider, *Cosmological Applications of Gravitational Lensing*, Lecture Notes in Physics, eds. E. Martinez–Gonzales, J.L. Sanz, Springer Verlag, Berlin (1996), astro-ph/9512047 (1995).
[22] P. Schneider, J. Ehlers, and E.E. Falco, *Gravitational Lenses*, Springer–Verlag, Berlin (1992).

[23] C.W. Allen, *Astrophysical quantities* (Athlone Press, London, 1973).

[24] F.A.E. Pirani, and H. Bondi, *Lectures on General Relativity* (Prentice–Hall, Inc. Englewood Cliffs, New Jersey, 1964).

[25] N. Straumann, *General Relativity and Relativistic Astrophysics* (Springer–Verlag, Berlin, 1984)

[26] V. I. Man’ko, in *Lee Methods in Optics*, Lecture Notes in Physics, Eds. S. Mondragon and K.-B. Wolf, 250, 193 (1986), Springer, Berlin.

[27] P.J.E. Peebles, *Principles of Physical Cosmology* (Princeton Univ. Press., Princeton, 1993)

[28] U. Borgeest, *Astron. Astrophys.* 128, 162 (1983);
B. Fort, and Y. Mellier, *Astron. Astrophys. Rev.* 5, 239 (1994);
J.A. Frieman, D.D. Harari, and G.C. Surpi, *Phys. Rev.* D 50, 4895 (1994).

[29] E.W. Kolb, and M.S. Turner, *The Early Universe* (Addison–Wesley Pub. Co., Menlo Park 1990).

[30] V. de Lapparent, M.J. Geller, and J.P. Huchra, *Ap. J.* 302, L1 (1986).
M.J. Geller, and J.P. Huchra, *Science* 246, 897 (1989).

[31] E. L. Turner, D. P. Schneider, B. F. Burke, J. N. Hewitt, G. L. Langston, J. E. Gunn, C. R. Lawrence, and M. Schmidt, *Nature*, 321, 142 (1986).

[32] A. Vilenkin, *Ap. J.* 282, L51 (1984); *Phys. Rep.* 121, 263 (1985);

[33] J.R. Gott III, *Ap. J.* 288, 422 (1985).

[34] H.C. Arp, *Quasars, Redshifts and Controversies*, Interstellar Media, Berkeley, §7 (1987).