TORSORS ON THE COMPLEMENT OF A SMOOTH DIVISOR

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Abstract. We complete the proof of the Nisnevich conjecture in equal characteristic: for a smooth algebraic variety $X$ over a field $k$, a $k$-smooth divisor $D \subset X$, and a reductive $X$-group $G$ whose base change $G_D$ is totally isotropic, we show that each generically trivial $G$-torsor on $X \setminus D$ trivializes Zariski semilocally on $X$. In mixed characteristic, we show the same when $k$ is a replaced by a discrete valuation ring $\mathcal{O}$, the divisor $D$ is the closed $\mathcal{O}$-fiber of $X$, and either $G$ is quasi-split or $G$ is only defined over $X \setminus D$ but descends to a quasi-split group over $\text{Frac}(\mathcal{O})$ (a Kisin–Pappas type variant). Our arguments combine Gabber–Quillen style presentation lemmas with excision and reembedding dévissages to reduce to analyzing generically trivial torsors over a relative affine line. We base this analysis on the geometry of the affine Grassmannian, and we we give a new proof for the Bass–Quillen conjecture for reductive group torsors over $\mathbb{A}_d$ in equal characteristic.

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1. The Nisnevich conjecture

Inspired by the case of vector bundles that had been conjectured by Quillen in relation to the Bass–Quillen conjecture [Qui76], and also by the Grothendieck–Serre conjecture [Ser58], [Gro58], Nisnevich predicted in [Nis89, Conjecture 1.3] that, for a reductive group scheme $G$ over a smooth variety $X$ over a field $k$ and a $k$-smooth divisor $D \subset X$, every generically trivial $G$-torsor on $X \setminus D$ trivializes Zariski locally on $X$. Recent counterexamples of Fedorov [Fed22b, Proposition 4.1] show that this fails for anisotropic $G$; to bypass them, one considers the following isotropy condition whose relevance for certain problems about torsors has been observed already in [Rag89].

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Definition 1.1 (Čes22a, Definition 8.1). Let $S$ be a scheme and let $G$ be a reductive $S$-group scheme. We say that $G$ is totally isotropic at a point $s \in S$ if each factor in the canonical decomposition

$$G_{\mathcal{O}_{S, s}}^{\text{ad}} \cong \prod_i \text{Res}_{R_i/s}^{\mathcal{O}_{S, s}}(G_i)$$

(1.1.1)

of $\text{[SGA 3\text{III new}, Exposé XXIV, Proposition 5.10 (i)]}$ has a proper parabolic subgroup; here $i$ is a type of connected Dynkin diagrams, $R_i$ is a finite étale $\mathcal{O}_{S, s}$-algebra, and $G_i$ is an adjoint semisimple $R_i$-group with simple geometric $R_i$-fibers of type $i$. If this holds for all $s$, then $G$ is totally isotropic.

Thus, intuitively speaking, $G$ is totally isotropic if each of its simple factors is isotropic. In addition, recall from $\text{[SGA 3\text{III new}, Exposé XXVI, Corollaire 6.12]}$ that, since each $R_i$ is semilocal, it is equivalent to require in Definition 1.1 that each $G_i$ contains $\mathbb{G}_m, R_i$ as an $R_i$-subgroup. For instance, every quasi-split, so also every split, group is totally isotropic, as is any torus.

With the total isotropy in place, the Nisnevich conjecture becomes the following statement.

Conjecture 1.2 (Nisnevich). For a regular semilocal ring $R$, an $r \in R$ that is a regular parameter in the sense that $r \not\in m^2$ for each maximal ideal $m \subset R$, and a reductive $R$-group scheme $G$ such that $G_{R/(r)}$ is totally isotropic, every generically trivial $G$-torsor over $R[\frac{1}{r}]$ is trivial, that is,

$$\text{Ker}(H^1(R[\frac{1}{r}], G) \to H^1(\text{Frac}(R), G)) = \{\ast\}.$$

For instance, in the case when $r$ is a unit, the total isotropicity condition holds for every $G$ and we recover the Grothendieck–Serre conjecture. The condition also holds in the case when $G$ is a torus, and this case follows from the known toral case of the Grothendieck–Serre conjecture, see Čes22b, Section 3.4.2 (1)]. In Fedorov settled the case when $R$ contains an infinite field and $G$ itself is totally isotropic. Other than this, some low dimensional cases are known, see Čes22b, Section 3.4.2—for instance, the case when $R$ is local of dimension $\leq 3$ and $G$ is either $\text{GL}_n$ or $\text{PGL}_n$ is a result of Gabber [Gab81, Chapter I, Theorem 1].

We settle the Nisnevich conjecture in equal characteristic and in some mixed characteristic cases.

Theorem 1.3. Let $R$ be a regular semilocal ring, let $r \in R$ be a regular parameter in the sense that $r \not\in m^2$ for each maximal ideal $m \subset R$, and let $G$ be a reductive $R[\frac{1}{r}]$-group. In the following cases,

$$\text{Ker}(H^1(R[\frac{1}{r}], G) \to H^1(\text{Frac}(R), G)) = \{\ast\},$$

in other words, in the following cases every generically trivial $G$-torsor over $R[\frac{1}{r}]$ is trivial:

1. (§7.1) if $R$ contains a field and $G$ extends to a reductive $R$-group $\mathcal{G}$ with $\mathcal{G}_{R/(r)}$ totally isotropic;

2. (§5.1) if $R$ is geometrically regular over a Dedekind subring $\mathcal{O}$ containing $r$ and $G$ either extends to a quasi-split reductive $R$-group or descends to a quasi-split reductive $\mathcal{O}[\frac{1}{r}]$-group.

The mixed characteristic case (2) is new already for vector bundles, that is, for $G = \text{GL}_n$. In contrast, at least for local $R$, the vector bundle case of the equicharacteristic (1) is due to Bhatwadekar–Rao [BR83, Theorem 2.5], with exceptions when the ground field is finite that have since been removed. When $r \in R^\times$, Theorem 1.3 recovers the equal and mixed characteristic cases of the Grothendieck–Serre conjecture settled in [FP15], [Pan20], [Čes22a], and we reprove them along the way.

The case of (2) in which $G$ descends to an $\mathcal{O}[\frac{1}{r}]$-group but need not extend to a reductive $R$-group was inspired by Kisin–Pappas [KP18, Section 1.4, especially, Lemma 1.4.6], who obtained such a

\footnote{For a ring $A$, recall that an $A$-algebra $B$ is geometrically regular if it is flat and the base change of each of its $A$-fibers to any finite field extension of the corresponding residue field of $A$ is regular, see [SP, Definition 0382]. For instance, $R$ could be a semilocal ring of a smooth algebra over a discrete valuation ring $\mathcal{O}$ with $r$ as a uniformizer.}
statement for some 2-dimensional $R$ under additional assumptions on $G$. The case of (2) when $G$ extends to a quasi-split reductive $R$-group is in the spirit of the mixed characteristic Grothendieck–Serre conjecture, more precisely, assuming further that $r$ is a unit it was settled in [Čes22a].

The geometric version of Theorem 1.3 (1) is the following statement announced in the abstract.

**Theorem 1.4.** For a field $k$, a smooth $k$-scheme $X$, a $k$-smooth divisor $D \subset X$, and a reductive $X$-group scheme $G$ such that $G_D$ is totally isotropic, every generically trivial $G$-torsor $E$ over $X \setminus D$ is trivial Zariski semilocally on $X$, that is, for every $x_1, \ldots, x_m \in X$ that lie in a single affine open, there is an affine open $U \subset X$ containing all the $x_i$ such that $E|_{U \setminus D}$ is trivial.

Theorem 1.4 follows by applying Theorem 1.3 (1) to the semilocal ring of $X$ at $x_1, \ldots, x_m$ (built via prime avoidance, see [SP, Lemma 00DS]) and spreading out. Even when $X$ is affine, the stronger statement that $E$ extends to a $G$-torsor over $X$ is false: for $G = GL_n$, this had been a question of Quillen [Qui76, (3) on page 170] that was answered negatively by Swan in [Swa78, Section 2]. Even for $GL_n$, Theorem 1.4 typically fails if $D$ is singular or if $X$ is singular, see [Lam06, pages 34–35]. We use Theorem 1.3 to reprove the following equal characteristic case of the generalization of the Bass–Quillen conjecture to torsors under reductive group schemes, see [Čes22b, Conjecture 3.6.1].

**Theorem 1.5 (§7.2).** For a regular ring $R$ containing a field and a totally isotropic reductive $R$-group scheme $G$, every generically trivial $G$-torsor over $\AA^d_R$ descends to a $G$-torsor over $R$, equivalently,

$$H^1_{\text{Zar}}(R, G) \xrightarrow{\sim} H^1_{\text{Zar}}(\AA^d_R, G) \quad \text{or, if one prefers,} \quad H^1_{\text{Nis}}(R, G) \xrightarrow{\sim} H^1_{\text{Nis}}(\AA^d_R, G).$$

The equivalence of the three formulations follows from the Grothendieck–Serre conjecture, more precisely, by Theorem 1.3, a $G$-torsor over $\AA^d_R$ is generically trivial, if and only if it is Zariski locally trivial, if and only if it is Nisnevich locally trivial. The generic triviality assumption is needed because, for instance, for every separably closed field $k$ that is not algebraically closed, there are nontrivial PGL$_n$-torsors over $\AA^1_k$, see [CTS21, Theorem 5.6.1 (vi)]. The total isotropy assumption is needed because of [BS17, Proposition 4.9], where Balwe and Sawant show that a Bass–Quillen statement cannot hold beyond totally isotropic $G$. For earlier counterexamples to generalizations of the Bass–Quillen conjecture beyond totally isotropic reductive groups, see [Par78] and [Fed16, Theorem 3 (ii) (whose assumptions can be met thanks to Remark 2.6 (i))].

Theorem 1.5 was established by Stavrova in [Sta22, Corollary 5.5] by a different method, and in the earlier [Sta19, Theorem 4.4] in the case when $R$ contains an infinite field. Prior to that, the case when $R$ is smooth over a field $k$ and $G$ is defined and totally isotropic over $k$ was settled by Asok–Hoyois–Wendt: they used methods of $\AA^1$-homotopy theory of Morel–Voevodsky to verify axioms of Colliot-Thélène–Ojanguren [CTO02] that were known to imply the statement, see [AHW18, Theorem 3.3.7] for infinite $k$ and [AHW20, Theorem 2.4] for finite $k$. As was explained in [Li21], one could also check these axioms directly, without $\AA^1$-homotopy theory. For regular $R$ of mixed characteristic, Theorem 1.5 is only known in sporadic cases, for instance, when $G$ is a torus, see [CTS87, Lemma 2.4], as well as [Čes22b, Section 3.6.4] for an overview.

We obtain Theorem 1.3 by refining the Grothendieck–Serre type strategies used in [Fed22b] and [Čes22a], in particular, our ultimate source of triviality of torsors is the geometry of the affine Grassmannian, even though the latter largely remains behind the curtain of self-contained inputs from the survey [Čes22b] that mildly generalized the corresponding results of Fedorov from [Fed22a]. One key novelty is the following extension result for $G$-torsors over smooth relative curves.

**Theorem 1.6.** Let $R$ be a regular semilocal ring containing a field and let $G$ be a reductive $R$-group.
(a) (Proposition 6.3 and Theorem 5.2). For a smooth affine $R$-scheme $C$ of pure relative dimension $1$ and an $R$-(finite étale) closed $Y \subset C$ such that $G_Y$ is totally isotropic, every $G$-torsor $E$ over $C \setminus Y$ that is trivial away from some $R$-finite closed $Z \subset C$ extends to a $G$-torsor over $C$.

(b) (Theorem 6.4). In (a), if $C = \mathbb{A}^1_R$ and $G$ is totally isotropic, then $E$ is even trivial.

Roughly speaking, extending a $G$-torsor to all of $C$ in Theorem 1.6 corresponds to extending a $G$-torsor in Theorem 1.3 (1) to all of $R$, in effect, to reducing the Nisnevich conjecture to the Grothendieck–Serre conjecture—this is why Theorem 1.6 is crucial for us. Conversely, to reduce Theorem 1.3 (1) to Theorem 1.6 we use a presentation lemma that extends its variants due to Quillen and Gabber: we first use Popescu theorem to pass to the geometric setting of Theorem 1.4 and then show in §3 that, up to replacing $X$ by an affine open neighborhood of $x_1, \ldots, x_m$, we can express $X$ as a smooth relative curve over some affine open of $\mathbb{A}^{d-1}_R$ in such a way that $D$ is relatively finite étale and our generically trivial $G$-torsor over $X$ is trivial away from a relatively finite closed subscheme. With the relative curve setting in hand, we reduce to Theorem 1.6 in Proposition 3.7.

As for Theorem 1.6, in §6 we present a series of excision and patching dévissages to reduce to when $C = \mathbb{A}^1_R$ and $C \setminus Y$ descends to a smooth curve defined over a subfield $k \subset R$. In this “constant” case, we show that our $G$-torsor over $C \setminus Y$ is even trivial by the “relative Grothendieck–Serre” theorem of Fedorov from [Fed22a] (with an earlier version due to Panin–Stavrova–Vavilov [PSV15]) that we reprove in Theorem 5.2: for every $k$-algebra $W$, no nontrivial $G$-torsor over $R \otimes_k W$ trivializes over $\text{Frac}(R) \otimes_k W$. As for the excision and patching techniques, we overcome known finite field difficulties with novel versions of the Lindel style embedding in Proposition 2.5 and of Panin’s “finite field tricks” in Lemma 2.7. The wide scope of these techniques makes our overall approach to Theorem 1.3 quite axiomatic, and although we do not pursue this here, it would be interesting to have similar results for other functors, for instance, for the unstable $K_1$-functor studied by Stavrova and her coauthors, compare, for instance, with [Sta22], [Sta19] and earlier articles cited there.

1.7. Notation and conventions. All the rings we consider are commutative and unital. For a point $s$ of a scheme (resp., for a prime ideal $p$ of a ring), we let $k_s$ (resp., $k_p$) denote its residue field. For a global section $s$ of a scheme $S$, we write $S[1_s] \subset S$ for the open locus where $s$ does not vanish. For a semilocal regular ring $R$, we say that an $r \in R$ is a regular parameter if $r \notin m$ for every maximal ideal $m \subset R$. For a ring $A$, we let $\text{Frac}(A)$ denote its total ring of fractions.

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2. Reembeddings into the relative affine space

In problems about torsors, we often have a smooth relative curve $C$ over a semilocal base $S$ and an $S$-finite closed subscheme $Z \subset C$, and we aim to build an étale map $C \to \mathbb{A}^1_S$ that would excisively embed $Z$ into $\mathbb{A}^1_S$. To accommodate for this, we begin by first embedding $Z$ itself as follows.

Proposition 2.1. Let $A$ be a semilocal ring, let $Z$ be a quasi-finite, separated $A$-scheme whose closed points lie above maximal ideals $m \subset A$, let $Y \subset Z$ be an $A$-finite closed subscheme, let $d > 0$, and let $\iota_Y : Y \hookrightarrow \mathbb{A}_A^d$ and $\iota_m : \text{Spec} \ k_m \hookrightarrow \mathbb{A}_A^d$ for every maximal ideal $m \subset A$ be compatible closed immersions. There is a closed immersion $\iota : Z \hookrightarrow \mathbb{A}_A^d$ over $A$ that extends the fixed $\iota_Y$ and $\iota_m$. 
Proof. Zariski Main Theorem [EGA IV, Corollaire 18.12.13] gives an open immersion \( Z \hookrightarrow \tilde{Z} \) into an \( A \)-finite scheme \( \tilde{Z} = \text{Spec}(\tilde{A}) \). Since the closed points of \( Z \) lie above maximal ideals of \( A \), the union of \( Y \) and the \( Z_{k_m} \) is a closed subscheme of \( \tilde{Z} \) that is disjoint from \( \tilde{Z} \setminus Z \). Thus, there is an \( a_{\infty} \in \tilde{A} \) that vanishes on \( \tilde{Z} \setminus Z \) and is a unit on every \( Z_{k_m} \) (so also on \( Z \)), as well as \( a_1, \ldots, a_d \in \tilde{A} \) that are units on \( \tilde{Z} \setminus Z \) and are such that \( a_i/a_{\infty} \) on each \( Z_{k_m} \) (resp., on \( Y \)) is the \( \iota_m \)-pullback (resp., \( \iota_Y \)-pullback) of the \( i \)-th standard coordinate of \( A_{k_m}^d \) (resp., of \( A_A^d \)). Jointly, \( a_1, \ldots, a_d, a_{\infty} \) do not vanish at any point of \( \tilde{Z} \), so they determine a map \( \tilde{\gamma}: \tilde{Z} \to \mathbb{P}_A^d \) such that \( \{a_{\infty} = 0\} \), that is, \( \tilde{Z} \setminus Z \), set theoretically is the \( \tilde{\gamma} \)-pullback of the hyperplane at infinity and on \( Z \) the section \( a_i/a_{\infty} \) is the \( \tilde{\gamma} \)-pullback of the \( i \)-th standard coordinate of \( A_A^d \). By construction, \( \tilde{\gamma} \) restricts to \( \iota_m \) and to \( \iota_Y \).

The schematic image of \( \tilde{\gamma} \) is an \( A \)-finite closed subscheme \( \overline{Z} \subset \mathbb{P}_A^d \); concretely, \( \tilde{\gamma} \) factors through the affine complement of any hypersurfaces in \( \mathbb{P}_A^d \) disjoint from \( \tilde{\gamma}(\tilde{Z}) \) (such a hypersurfaces exists by the avoidance lemma [GLL15, Theorem 5.1]) and the coordinate ring \( \overline{A} \) of \( \overline{Z} \) is the image in \( \tilde{A} \) of the coordinate ring of this hypersurface. Thanks to this description, the image of the finite map \( \tilde{Z} \to \overline{Z} \) contains every minimal prime of \( \overline{A} \), so this map is surjective. Thus, since \( Z \) is the preimage of \( A_A^d \) in \( \tilde{Z} \), it remains to show that the induced map \( Z \to \overline{Z} \) is an open immersion, equivalently, that the finite surjective map \( Z \to \overline{Z} \cap A_A^d \) that is injective on coordinate rings is also surjective on coordinate rings. By the Nakayama lemma [SP, Lemma 00DV], this surjectivity may be checked after passing to the fibers over the closed points of \( \overline{Z} \cap A_A^d \). Since \( Z \to \overline{Z} \cap A_A^d \) is finite surjective, these closed points lift to closed points of \( Z \), and hence lie above maximal ideals \( m \subset A \). The desired surjectivity follows because the \( \iota_m \) are closed immersions and \( \tilde{\gamma} \) restricts to them. \( \square \)

Remark 2.2. The assumption on the closed points of \( Z \) holds in our main case of interest when \( Z \) is \( A \)-finite, and also in some cases when \( A \) is a Dedekind ring. If \( d = 1 \) and the \( Z_{k_m} \) do not meet the zero section of \( A_{k_m}^1 \), then it ensures that \( \iota(Z) \) does not meet the zero section of \( A_A^1 \).

To apply Proposition 2.1 effectively, we need a practical criterion for the existence of the closed immersions \( \iota_m \). For this, we first have to wrestle with the following finite field obstruction.

Definition 2.3. For a ring \( A \), a quasi-finite \( A \)-scheme \( Z \), and an \( A \)-scheme \( X \), there is no finite field obstruction to embedding \( Z \) into \( X \) if for each maximal ideal \( m \subset A \) with \( k_m \) finite,

\[
\{ z \in Z_{k_m} \mid [k_z : k_m] = d \} \leq \{ z \in X_{k_m} \mid [k_z : k_m] = d \} \quad \text{for every} \quad d \geq 1.
\]  

(†)

In practice, \( Z \) occurs as a closed subscheme of a smooth affine \( A \)-scheme, so the following lemma gives an applicable criterion for the existence of the closed immersions \( \iota_m: Z_{k_m} \hookrightarrow A_{k_m}^d \) in Proposition 2.1.

Lemma 2.4. For a finite scheme \( Z \) over a field \( k \) and an open \( U \subset A_k^d \) with \( d > 0 \), there is a closed immersion \( \iota: Z \hookrightarrow U \) if and only if \( Z \) is a closed subscheme of some smooth \( k \)-scheme \( C \) of pure dimension \( d \) and there is no finite field obstruction to embedding \( Z \) into \( U \), in which case we may choose \( \iota \) to extend any fixed embedding \( \iota_Y: Y \hookrightarrow U \) of any closed subscheme \( Y \subset Z \).

Proof. The ‘only if’ is clear, so we fix a closed immersion \( Z \subset C \) as in the statement. By [Čes22b, Proposition 4.1.4] (whose proof uses a presentation theorem similar to Lemma 3.1), every closed point of \( C \) embeds as a closed point of \( A_k^d \). Moreover, if \( k \) is infinite and \( k' \) is the residue field of some closed point \( x \in A_k^d \), then \( k' \) is also the residue field of infinitely many closed points \( y \in U \) (if \( \alpha_1, \ldots, \alpha_d \in k' \) are the images at \( x \) of the standard coordinates of \( A_k^d \), then infinitely many \( y \) for which the corresponding images are \( \alpha_1 + \beta_1, \ldots, \alpha_d + \beta_d \) with \( \beta_i \in k \) will lie in \( U \)). Thus, since there
is no finite field obstruction to embedding $Z$ into $U$, we may embed $Z^{\text{red}}$ into $U$ compatibly with $\iota_Y$.

By the Cohen structure theorem [Mat89, Theorem 29.7], the $n$-th infinitesimal neighborhood in $C$ of a point $z \in Z$ is isomorphic to $\text{Spec}(k[z_1, \ldots, z_d]/(x_1^{n+1}, \ldots, x_d^{n+1}))$. In particular, up to isomorphism this neighborhood does not depend on $C$, so we may extend the fixed embedding of $Z^{\text{red}}$ into $U$ to a similar embedding of the $n$-th infinitesimal neighborhood of $Z$ in $C$ compatibly with $\iota_Y$. This suffices because $Z$ lies in this $n$-th infinitesimal neighborhood for every large enough $n$.  

To transform Proposition 2.1 into a statement that we will use for patching, we now extend [Čes22a, Lemma 6.3] (so also earlier versions due to Panin and Fedorov, see loc. cit.) to arrange that the closed immersion $\iota: Z \hookrightarrow \mathbb{A}^d_A$ built there be excisive as follows.

**Proposition 2.5.** Let $A$ be a semilocal ring, let $U \subset \mathbb{A}^d_A$ with $d > 0$ be an $A$-fiberwise nonempty open, and let $Z$ be a quasi-finite, separated $A$-scheme whose closed points lie above maximal ideals of $A$.

(a) There is a closed immersion $\iota: Z \hookrightarrow U$ iff there is no finite field obstruction to embedding $Z$ into $U$ and $Z$ is a closed subscheme of some $A$-smooth affine scheme $C$ of relative dimension $d$.

(b) If the conditions of (a) hold, then $\iota$ may be chosen to be excisive: then there are an affine open $D \subset C$ containing $Z$ and an étale $A$-morphism $f: D \to U$ that fits into a Cartesian square

$$
\begin{array}{ccc}
Z' & \longrightarrow & D \\
\downarrow & & \downarrow f \\
Z & \longrightarrow & U,
\end{array}
$$

(2.5.1)

in particular, such that $f$ embeds $Z$ as a closed subscheme $Z' \subset U$; for every $A$-finite closed subscheme $Y \subset Z$ and an embedding $\iota_Y: Y \hookrightarrow U$, there are $D$ and $f$ as above with $f|_Y = \iota_Y$.

**Proof.** For the ‘only if’ in (a), it suffices to note that if there is a closed immersion $\iota: Z \hookrightarrow U$, then, by Proposition 2.1, there is also a closed immersion $Z \hookrightarrow \mathbb{A}^d_A$. Thus, we focus on the ‘if’ in its stronger form (b). In particular, we fix an embedding $Z \subset C$ as in (a) and we let $\varepsilon_Z \subset C$ be the first infinitesimal neighborhood of $Z$ in $C$, so that $\varepsilon_Z$ is also quasi-finite and separated over $A$.

By Lemma 2.4 and Proposition 2.1, there is a closed immersion $\tilde{\iota}: \varepsilon_Z \hookrightarrow U$ that extends the fixed $\iota_Y$. By lifting the $\tilde{\iota}$-pullbacks of the standard coordinates of $\mathbb{A}^d_A$, we may extend $\tilde{\iota}$ to an $A$-morphism $\tilde{f}: C \to \mathbb{A}^d_A$. By construction, the $a$ priori open locus of $C$ where $\tilde{f}$ is quasi-finite (see [SP, Lemma 01TI]) contains the points of $Z$. Thus, since $Z$ has finitely many closed points, we may use prime avoidance [SP, Lemma 00DS] to shrink $C$ around $Z$ to arrange that $\tilde{f}$ is quasi-finite. The flatness criteria [EGA IV, Proposition 6.1.5] and [EGA IV, Corollaire 11.3.11] then ensure that $\tilde{f}$ is flat at the points of $Z$, so, by construction, $\tilde{f}$ is even étale at the points of $Z$. Consequently, we may shrink $C$ further around $Z$ to arrange that $\tilde{f}$ is étale and factors through $U$. A section of a separated étale morphism, such as $\tilde{f}^{-1}(\tilde{f}(Z)) \to \tilde{f}(Z)$, is an inclusion of a clopen subset, so, by shrinking $C$ around $Z$ once more, we arrange that $Z = \tilde{f}^{-1}(\tilde{f}(Z))$. This equality means that the square (2.5.1) is Cartesian, so, granted all the shrinking above, it remains to set $D := C$ and $\tilde{f} := f$.  

\[ \Box \]
The following corollary is useful for embedding a finite étale $Z$ into $U$ without an ambient scheme $C$.

**Corollary 2.6.** For a semilocal ring $A$ and an $A$-fiberwise nonempty open $U \subset \mathbb{A}_A^d$ with $d > 0$, an $A$-(finite étale) scheme $Z$ embeds into $U$ if and only if there is no finite field obstruction to it.

**Proof.** The ‘only if’ is clear. For the ‘if,’ by Proposition 2.5 (a), it is enough to embed $Z$ into $\mathbb{A}_A^d$, so we may assume that $U = \mathbb{A}_A^d$. It then suffices to show that $Z \cong \text{Spec}(A')$ with an $A'$ that may be generated by $d$ elements as an $A$-algebra. Thus, since $A'$ is a finite $A$-module and $A$ is semilocal, the Nakayama lemma [SP, Lemma 00DV] allows us to replace $A$ by the product of the residue fields of its maximal ideals, so we may assume that $A$ is a field $k$. In this case, $Z$ is a disjoint union of spectra of finite separable field extensions $k$ and, since there is no finite field obstruction to embedding $Z$ into $\mathbb{A}_k^d$, such an embedding exists by the primitive element theorem. □

To bypass the finite field obstruction in practice, we will modify $Z$ via the following lemma. It extends [Čes22a, Lemma 6.1] (so also earlier versions by Panin) and is built on Panin’s “finite field tricks.”

**Lemma 2.7.** Let $A$ be a semilocal ring, let $Z$ be a quasi-finite, separated $A$-scheme, let $Y \subset Z$ be an $A$-finite closed subscheme, and let $X$ be an $A$-scheme such that for every maximal ideal $m \subset A$ some closed subscheme of $X_{k_m}$ is of finite type over $k_m$, positive dimensional, and geometrically irreducible.

(a) There is a finite étale surjection $\tilde{Z} \rightarrow Z$ such that there is no finite field obstruction to embedding $\tilde{Z}$ into $X$, moreover, for every large $N > 0$ we may find such a $\tilde{Z}$ of the form $\tilde{Z}_0 \cup \tilde{Z}_1$ with $\tilde{Z}_i \cong \text{Spec}(\mathcal{O}_Z[t]/(f_i(t))) \rightarrow Z$ surjective and $f_i$ monic of constant degree $N + i$.

(b) Fix any sufficiently divisible $n \geq 0$ and suppose that $Y = Y_0 \cup Y_1$ such that there is no finite field obstruction to embedding $Y_0$ into $X$. Then (a) holds with the requirement that

$\tilde{Y} := Y \times_Z \tilde{Z}$ is a disjoint union $\tilde{Y} = \tilde{Y}_0 \cup \tilde{Y}_1$

such that $\tilde{Y}_0 \rightarrow Y_0$ and each connected component of $\tilde{Y}_1$ is a scheme over a finite $\mathbb{Z}$-algebra $B$ each of whose residue fields $k$ of characteristic $p$ satisfies

$\#k > n \cdot \deg(\tilde{Z}/Z)$. 

Part (b) is a critical statement that we will use in §6 to bypass finite field difficulties of [Fed22b], and a typical case is when $Y = Y_0 = \text{Spec}(A)$ is an $A$-rational point of $Z$. To be clear, in (b) the $\mathbb{Z}$-algebra $B$ depends on the connected component of $\tilde{Y}_1$ in question.

**Proof.** We may replace $Z$ by any $A$-finite scheme containing $Z$ as an open, so we use the Zariski Main Theorem [EGA IV, Corollaire 18.12.13] to assume that $Z = \text{Spec}(A')$ for an $A$-finite $A'$. To explain the role of the assumption on $X$, recall that by the Weil conjectures [Poo17, Theorem 7.7.1 (ii)], it implies that for every $m > 0$, every maximal ideal $m \subset A$ with $k_m$ finite, and every large $d > 0,$

$\{ z \in X_{k_m} \mid [k_z : k_m] = d \} \geq m. \hspace{1cm} (2.7.1)$

In (a), we let $N > 2$ be sufficiently large and choose the following monic polynomials: for each closed point $z \in Z$ with $k_z$ finite (resp., infinite), a monic $f_z(t) \in k_z[t]$ that is irreducible of degree $N$ (resp., that is the product of $N$ distinct monic linear polynomials). We let $f_0(t) \in A'[t]$ be a monic
polynomial that simultaneously lifts all the \( f_z(t) \), and we define a monic \( f_1(t) \in \mathcal{A}'[t] \) analogously with \( N \) replaced by \( N + 1 \). Granted that \( N \) is large enough, by (2.7.1), the resulting \( \tilde{Z}_i \) settle (a).

In (b), to satisfy the “sufficiently divisible” requirement it suffices to make sure that \( n \) is divisible by all the positive residue characteristic of \( A \). Granted this, for each \( N > 2 \) we choose

- an \( f_{Y_0}(t) \in \mathbb{Z}[t] \) that is the product of \( t \) and a monic polynomial of degree \( N - 1 \) whose reduction modulo every prime \( p \mid n \) is irreducible;
- a monic \( f_{Y_1}(t) \in \mathbb{Z}[t] \) of degree \( N \) whose reduction modulo every prime \( p \mid n \) is irreducible;
- for each closed point \( z \in Z \) not in \( Y \) with \( k_z \) finite (resp., infinite), an \( f_z(t) \in k_z[t] \) that is irreducible of degree \( N \) (resp., that is the product of \( N \) distinct monic linear polynomials).

We write \( Y_i = \text{Spec}(A'_i) \) and view \( f_{Y_i}(t) \) as an element of \( A'_i[t] \). Since \( Y \) and the closed points of \( Z \) not in \( Y \) form a closed subscheme of \( Z \), there is a monic polynomial \( f_0(t) \in \mathcal{A}'[t] \) whose image in \( A'_i[t] \) (resp., in \( k_z[t] \) for each closed point \( z \in Z \) not in \( Y \)) is \( f_{Y_i}(t) \) (resp., \( f_z(t) \)). With the resulting \( \tilde{Z}_0 \) defined by this \( f_0(t) \) as in (a), let \( \tilde{Y}_0 \) be component of \( Y_0 \times_Z \tilde{Z}_0 \) cut out by the factor \( t \) of \( f_{Y_0}(t) \) to arrange that \( \tilde{Y}_0 \rightarrow Y_0 \). By the choice of the \( f_{Y_1}(t) \), each connected component of the complement of \( \tilde{Y}_0 \) in \( Y \times_Z \tilde{Z}_0 \) is an algebra over some finite \( \mathbb{Z} \)-algebra \( B \) each of whose residue fields \( k \) of characteristic \( p > 0 \) with \( p \mid n \) has degree either \( N - 1 \) or \( N \) over \( \mathbb{F}_p \).

We repeat the construction with \( N \) replaced by \( N + 1 \), except that now we let \( f_{Y_0}(t) \in \mathbb{Z}[t] \) be a monic polynomial of degree \( N + 1 \) whose reduction modulo every prime \( p \mid n \) is irreducible, to build a monic \( f_1(t) \in \mathcal{A}'[t] \) of degree \( N + 1 \). For the resulting \( \tilde{Z}_1 \), by construction, each connected component of \( Y \times_Z \tilde{Z}_1 \) is an algebra over some finite \( \mathbb{Z} \)-algebra \( B \) each of whose residue fields \( k \) of characteristic \( p > 0 \) with \( p \mid n \) has degree \( N + 1 \) over \( \mathbb{F}_p \). Overall, with the resulting \( \tilde{Y}_1 \) complementary to \( \tilde{Y}_0 \), every connected component of \( \tilde{Y}_1 \) is an algebra over some finite \( \mathbb{Z} \)-algebra \( B \) each of whose residue fields \( k \) of characteristic \( p \mid n \) has degree \( N - 1, N \), or \( N + 1 \) over \( \mathbb{F}_p \). For large \( N \), such a \( k \) satisfies

\[
\#k > nN(N + 1) = n \cdot \deg(\tilde{Z}/Z).
\]

By construction, the number of closed points of \( \tilde{Z} \) not in \( \tilde{Y}_0 \) with a finite residue field is bounded as \( N \) grows and the degree of the residue field of every such closed point over the corresponding \( \mathbb{F}_p \) is \( \geq \varepsilon N \) for some \( \varepsilon > 0 \) (that depends on the degrees of \( k_z \) over the \( \mathbb{F}_p \), but not on \( N \)). In particular, for large \( N \), by (2.7.1), there is no finite field obstruction to embedding \( \tilde{Z} \) into \( X \). \( \square \)

**Remark 2.8.** The \( A \)-finite \( Z \) that is to be modified as in Lemma 2.7 to avoid the finite field obstruction to embedding it into \( X \) often occurs as a closed subscheme of a smooth affine \( A \)-scheme \( C \), and it is useful to lift the resulting \( \tilde{Z} \rightarrow Z \) to a finite étale cover \( \tilde{C} \rightarrow D \) of an affine open neighborhood \( D \subset C \) of \( Z \). Since the \( \tilde{Z}_i \) are explicit, this is possible to arrange: it suffices to lift each \( f_i(T) \) to a monic polynomial with coefficients in the coordinate ring of the semilocalization of \( C \) at the closed points of \( Z \) (built via prime avoidance [SP, Lemma 00DS]) and to spread out.

Throughout this article, we will analyze torsors that are trivial away from a closed subscheme \( Z \). For this, the following basic glueing technique of Moret-Bailly [MB96] (with a more restrictive version implicit already in [FR70, Proposition 4.2]) will let us take advantage of excisive squares like (2.5.1).
Lemma 2.9 ([Čes22b, Proposition 4.2.1]). For a scheme $S$, a closed subscheme $Z \subset S$ that is locally cut out by a finitely generated ideal, an affine, flat map $f$ that fits into a Cartesian square

$$
\begin{array}{ccc}
Z' & \xrightarrow{f} & S \\
\downarrow & & \downarrow f \\
Z'' & \xrightarrow{} & S'
\end{array}
$$

and embeds $Z$ as a closed subscheme $Z' \subset S'$ (so $Z \cong Z' \times_{S'} S$ by the Cartesianness requirement), and a quasi-affine, flat, finitely presented $S'$-group $G$, base change induces an equivalence of categories

$$\{G\text{-torsors over } S'\} \xrightarrow{\sim} \{G\text{-torsors over } S\} \times_{\{G\text{-torsors over } S'\}} \{G\text{-torsors over } S'\backslash Z'\},$$

in particular, a $G$-torsor over $S$ descends to $S'$ if and only if it does so away from $Z'$.

\[ \square \]

Remark 2.10. If the flat map $f$ is locally of finite presentation, then the excisive condition on $Z$ and $Z'$ implies that $f$ is étale at the points of $Z$. This means that it then induces an isomorphism between the formal completion of $S$ along $Z$ and that of $S'$ along $Z'$.

3. A REDUCTION TO STUDYING $G$-TORSORS OVER A RELATIVE CURVE

For reducing the Grothendieck–Serre or Nisnevich type questions to studying torsors on the relative affine spaces via reembeddings like in Proposition 2.5, regularity or smoothness assumptions enter in a crucial way. This is thanks to geometric presentation lemmas in the spirit of Gabber’s refinement [Gab94, Lemma 3.1] of the Quillen presentation lemma [Qui73, Section 7, Lemma 5.12], which itself is a variant of the Noether normalization theorem. Reductive groups or their torsors play no role in this geometric step that first expresses a smooth affine variety as a smooth relative curve as follows.

Lemma 3.1. For a smooth, affine, irreducible scheme $X$ of dimension $d > 0$ over a field $k$ that is either finite or of characteristic $0$,$^2$ points $x_1, \ldots, x_m \in X$, a proper closed subscheme $Z \subset X$, and a $k$-smooth divisor $D \subset X$, there are an affine open $X' \subset X$ containing $x_1, \ldots, x_m$, an affine open $S \subset \mathbb{A}^{d-1}_k$, and a smooth morphism

$$f : X' \to S$$

of relative dimension 1 such that

$$X' \cap Z = f^{-1}(S) \cap Z \text{ is } S\text{-finite and } X' \cap D = f^{-1}(S) \cap D \text{ is } S\text{-finite étale}.$$

Proof. In the case $d = 1$, we may choose $X' = X$ and $S = \text{Spec}(k)$, so we assume that $d > 1$. We also replace each $x_i$ by a specialization to reduce to $x_i$ being a closed point (see [SP, Lemma 02J6]), and in this case we will force each $f(x_i)$ to be the origin of $\mathbb{A}^{d-1}_k$. We embed $X$ into some projective space $\mathbb{P}^N_k$ and then form closures to arrange that $X$ is an open of a projective $\overline{X} \subset \mathbb{P}^N_k$ of dimension $d$ with $\overline{X} \setminus X$ of dimension $\leq d - 1$ and that there are

- a projective $\overline{D} \subset \overline{X}$ of dimension $d - 1$ with $\overline{D} \setminus D$ of dimension $\leq d - 2$, and
- a projective $\overline{Z} \subset \overline{X}$ of dimension $\leq d - 1$ with $\overline{Z} \setminus Z$ of dimension $\leq d - 2$.

We use the avoidance lemma [GLL15, Theorem 5.1] and postcompose with a Veronese embedding to build a hyperplane $H_0$ not containing any $x_i$ such that $(\overline{X} \setminus X) \cap H_0$ is of dimension $\leq d - 2$ (to force the dimension drop, choose appropriate auxiliary closed points and require $H_0$ to not contain them). By the Bertini theorem [Poo04, Theorem 1.3] of Poonen if $k$ is finite and by the Bertini

---

$^2$The assumption on $k$ is likely not optimal but it will suffice and we do not wish to further complicate the proof.
theorem of [Čes22a, second paragraph of the proof of Lemma 3.2] applied both to $X$ and to $D$ in place of $X$ if $k$ is of characteristic 0, there is a hypersurface $H_1 \subset \mathbb{P}_k^N$ such that

- $H_1$ contains $x_1, \ldots, x_m$;
- $X \cap H_1$ (resp., $D \cap H_1$) is $k$-smooth of dimension $d - 1$ (resp., $d - 2$);
- $Z \cap H_1$ is (resp., $(D \setminus D) \cap H_1$ and $(Z \setminus Z) \cap H_1$ are) of dimension $\leq d - 2$ (resp., $\leq d - 3$);
- $(\mathcal{X} \setminus \mathcal{X}) \cap H_0 \cap H_1$ is of dimension $\leq d - 2$.

In particular, by passing to intersections with $H_1$, we are left with an analogous situation with $d$ replaced by $d - 1$. Therefore, by iteratively applying the Bertini theorem in this way, we build hypersurfaces $H_1, \ldots, H_{d-1}$ such that

1. the $x_1, \ldots, x_m$ lie in $H_1 \cap \ldots \cap H_{d-1}$ but not in $H_0$;
2. $X \cap H_1 \cap \ldots \cap H_{d-1}$ (resp., $D \cap H_1 \cap \ldots \cap H_{d-1}$) is $k$-smooth of dimension 1 (resp., $k$-étale);
3. $(\mathcal{D} \setminus \mathcal{D}) \cap H_1 \cap \ldots \cap H_{d-1} = (\mathcal{Z} \setminus \mathcal{Z}) \cap H_1 \cap \ldots \cap H_{d-1} = \emptyset$.
4. $(\mathcal{X} \setminus \mathcal{X}) \cap H_0 \cap H_1 \cap \ldots \cap H_{d-1} = \emptyset$.

By letting $1, w_1, \ldots, w_{d-1}$ be the degrees of the hypersurfaces $H_0, H_1, \ldots, H_{d-1}$ and choosing defining equations $h_i$ of the $H_i$, we determine a projective morphism $\tilde{f}: \tilde{X} \to \mathbb{P}_k(1, w_1, \ldots, w_{d-1})$ from the weighted blowup $\tilde{X} := \text{Bl}(h_0, \ldots, h_{d-1})$ to the weighted projective space such that the diagram

$$
\begin{array}{ccc}
\mathcal{X} \setminus \mathcal{X} \setminus H_0 & \subset & \mathcal{X} \setminus (H_0 \cap \ldots \cap H_{d-1}) \subset \tilde{X} \\
\downarrow f & & \downarrow \tilde{f} \\
\mathbb{A}_k^{d-1} \subset \mathbb{P}_k(1, w_1, \ldots, w_{d-1}) & \longrightarrow & \mathbb{P}_k(1, w_1, \ldots, w_{d-1})
\end{array}
$$

commutes, where the bottom left arrow is the inclusion of the open locus where the first standard coordinate of $\mathbb{P}_k(1, w_1, \ldots, w_{d-1})$ does not vanish, see [Čes22a, Sections 3.4 and 3.5]. By (i), each $f(x_i)$ is the origin of $\mathbb{A}_k^{d-1}$. By (ii) and the dimensional flatness criterion [EGA IV2, Proposition 6.1.5], at every point of the fiber above the origin of $\mathbb{A}_k^{d-1}$, the map $f$ is smooth of relative dimension 1 and its restriction to $D$ is étale. Since $\tilde{f}$ is projective, (iii)-(iv) and the openness of the quasi-finite locus [SP, Lemma 01TI] ensure that for some affine open neighborhood of the origin $S \subset \mathbb{A}_k^{d-1}$ both $f^{-1}(S) \cap Z$ and $f^{-1}(S) \cap D$ are $S$-finite (see also [SP, Lemma 02OG]). In conclusion, any affine open of $f^{-1}(S)$ that contains all the $x_i$ and all the points of $Z$ and $D$ that lie above the origin of $\mathbb{A}_k^{d-1}$ becomes a sought $X'$ after possibly shrinking $S$ further. \hfill \Box

In the semilocal Dedekind setting of Theorem 1.3 (2), we need the following version of the geometric presentation lemma that does not have the aspect about the smooth divisor $D$.

**Lemma 3.2** ([Čes22a, Proposition 4.1]). For a smooth, affine scheme $X$ of relative dimension $d > 0$ over a semilocal Dedekind ring $\mathcal{O}$, points $x_1, \ldots, x_m \in X$, and a closed subscheme $Z \subset X$ of codimension $\geq 2$, there are an affine open $X' \subset X$ containing $x_1, \ldots, x_m$, an affine open $S \subset \mathbb{A}_\mathcal{O}^{d-1}$, and a smooth morphism $f: X' \to S$ of relative dimension 1 such that $X' \cap Z$ is $S$-finite. \hfill \Box

To use the geometric presentation lemmas effectively to study torsors, we have to cope with the issue that a general reductive $X'$-group need not descend to $S$ along the map $f$. We overcome this with Lemma 3.5 about equating reductive groups, which is a variant of [PSV15, Theorem 3.6] of Panin–Stavrova–Vavilov and combines ideas from [Čes22a, Lemma 5.1] with the survey [Čes22b, Chapter 6].
Definition 3.3: For a ring $A$ and an ideal $I \subset A$, we consider the following property of a set-valued functor $\mathcal{F}$ defined on the category of $A$-algebras:

for every $x \in \mathcal{F}(A/I)$, there are a faithfully flat, finite, étale $A$-algebra $\tilde{A}$, an $A/I$-point $a: \tilde{A} \to A/I$, and an $\tilde{x} \in \mathcal{F}(\tilde{A})$ whose $a$-pullback is $x$. ($\star$)

Remark 3.4. Let $f: \mathcal{F} \to \mathcal{F}'$ be a map of functors on the category of $A$-algebras and, for a $y \in \mathcal{F}'(A)$, let $\mathcal{F}_y \subset \mathcal{F}$ denote the $f$-fiber of $y$. If $\mathcal{F}'$ has property ($\star$) with respect to $I \subset A$ and, for every faithfully flat, finite, étale $A$-algebra $\tilde{A}$ and every $y \in \mathcal{F}'(\tilde{A})$, the fiber $(\mathcal{F}_x|_{\tilde{A}})_y$ has property ($\star$) with respect to any ideal $\tilde{I} \subset \tilde{A}$ with $\tilde{A}/\tilde{I} \cong A/I$, then $\mathcal{F}$ itself has property ($\star$) with respect to $I \subset A$. This straightforward dévissage is useful in practice for dealing with short exact sequences.

Lemma 3.5. For a Noetherian semilocal ring $A$ whose local rings are geometrically unibranch, an ideal $I \subset A$, a reductive $A$-groups $G$ and $G'$ that on geometric $A$-fibers have the same type, maximal $A$-tori $T \subset G$ and $T' \subset G'$, and an $A/I$-group isomorphism

$$\iota: G_{A/I} \overset{\sim}{\to} G'_{A/I}$$

such that $\iota(T_{A/I}) = T'_{A/I}$, there are a faithfully flat, finite, étale $A$-algebra $\tilde{A}$ equipped with an $A/I$-point $a: \tilde{A} \to A/I$ and an $\tilde{A}$-group isomorphism $\tilde{\iota}: G_{\tilde{A}} \overset{\sim}{\to} G'_{\tilde{A}}$ whose $a$-pullback is $\iota$ and such that $\tilde{\iota}(T_{\tilde{A}}) = T'_{\tilde{A}}$.

Proof. The claim is that the functor

$$X := \text{Isom}_g((G,T),(G',T'))$$

that parametrizes those group scheme isomorphisms between base changes of $G$ and $G'$ that bring $T$ to $T'$ has property ($\star$) with respect to $I \subset A$. By [SGA 3, Exposé XXIV, Corollaires 1.10 et 2.2 (i)], the normalizer $N_{G_{\text{ad}}}(T_{\text{ad}})$ of the $A$-torus $T_{\text{ad}} \subset G_{\text{ad}}$ induced by $T$ acts freely on $X$ and, thanks to the assumption about the geometric fibers of $G$ and $G'$, the quotient $\overline{X} := X/N_{G_{\text{ad}}}(T_{\text{ad}})$ is a faithfully flat $A$-scheme that becomes constant étale locally on $A$. The geometric unibranchness assumption then allows us to apply [Čes22b, Example 6.2.1] to conclude that $\overline{X}$ has property ($\star$) with respect to $I \subset A$. By Remark 3.4, we may therefore replace $A$ by a finite étale cover to reduce to showing that every $N_{G_{\text{ad}}}(T_{\text{ad}})$-torsor has property ($\star$). However, $N_{G_{\text{ad}}}(T_{\text{ad}})$ is an extension of a finite étale $A$-group scheme by $T_{\text{ad}}$ (see, for instance, [Čes22b, Section 1.3.2]), so we may repeat the same reduction based on Remark 3.4 and be left with showing that every $T_{\text{ad}}$-torsor has property ($\star$) with respect to $I \subset A$. This, however, is a special case of [Čes22b, Corollary 6.3.2] (that is based on building an equivariant projective compactification of the $A$-torus $T_{\text{ad}}$ using toric geometry). □

Remark 3.6. Lemma 3.5 continues to hold if instead of the maximal $A$-tori $T \subset G$ and $T' \subset G'$, the groups $G$ and $G'$ come equipped with fixed quasi-pinnings extending Borel $A$-subgroups $B \subset G$ and $B' \subset G'$, and if $\iota$ and $\tilde{\iota}$ are required to respect these quasi-pinnings, see [Čes22a, Lemma 5.1].

We are ready to show that a torsor over a regular local ring of equal characteristic can be lifted to a smooth relative curve equipped with a section. We directly state this lifting construction in its “relative version” that includes an auxiliary $W$, but the main case of interest is $W = \text{Spec}(k)$.

Proposition 3.7. For a regular semilocal ring $R$ containing a field $k$, a reductive $R$-group $G$, a nonempty open $U \subset \text{Spec}(R)$, a quasi-compact and quasi-separated $k$-scheme $W$, and a $G$-torsor $E$ over $W \times_k U$ that is trivial over $W \otimes_k \text{Frac}(R)$, there are

(i) an open $\mathcal{U} \subset \mathcal{A}^1_R$ that is affine if so is $U$ such that $\mathcal{A}^1_R \setminus \mathcal{U}$ is finite over $R$;
(ii) an $R$-finite closed subscheme $Z \subset A^1_R$ that contains $A^1_R \setminus \mathcal{U}$ and satisfies $(A^1_R \setminus \mathcal{U})(R) \neq \emptyset$;

(iii) a section $s \in Z(R)$ given by the origin $\{ t = 0 \}$ of $A^1_R$ such that $s|_U$ factors through $\mathcal{U}$; and

(iv) a $G$-torsor $\mathcal{E}$ over $W \times_k \mathcal{U}$ that is trivial over $W \times_k (\mathcal{U}\setminus Z)$ such that

$$(\text{id}_W \times_k s|_U)^*(\mathcal{E}) \cong E$$

as $G$-torsors over $W \times_k U$.

If $U = \text{Spec}(R[\frac{1}{r}])$ for a regular parameter $r \in R$ (resp., if $U = \text{Spec}(R)$), then we may arrange that $A^1_R \setminus \mathcal{U}$ be $R$-(finite étale) (resp., that $\mathcal{U} = A^1_R$) and that $G_{A^1_R \setminus \mathcal{U}}$ is totally isotropic if so is $G_{R/(r)}$.

Proof. Let $\mathbb{F} \subset k$ be the prime subfield and consider the $k$-algebra $k \otimes_{\mathbb{F}} R$. The composition $R \xrightarrow{a} k \otimes_{\mathbb{F}} R \xrightarrow{b} R$, in which the second map uses the $k$-algebra structure of $R$, is the identity. The base change of $E$ along $\text{id}_W \times_k a|_U$ is a $G$-torsor $E'$ over $W \times_{\mathbb{F}} U$. Thus, if we settle the claim with $\mathbb{F}$ in place of $k$, in particular, build a $G$-torsor $\mathcal{E}'$ over $W \times_{\mathbb{F}} \mathcal{U}$ as in (iv), then base change along $b|_U$ will settle the claim over the original $k$. This reduces us to the case when $k = \mathbb{F}$.

Since $k$ is now perfect, Popescu theorem [SP, Theorem 07GC] expresses $R$ as a filtered direct limit of smooth $k$-algebras. Thus, by passing to connected components of $\text{Spec}(R)$ and doing a limit argument, we may assume that $R$ is a semilocal ring of a smooth, affine, irreducible $k$-scheme $X$ of dimension $d \geq 0$, that $U$ is the base change of an open $\mathcal{V} \subset X$ that equals $X$ if $U = \text{Spec}(R)$, that $G$ (resp., $E$) is defined over all of $X$ (resp., all of $W \times_k \mathcal{V}$), and, if $r$ is in play, that $r$ is a global section of $X$ that cuts out a complementary to $\mathcal{V}$ divisor $D \subset X$ such that $G_{D}$ is totally isotropic if so is $G_{R/(r)}$. Since $k$ is perfect and $D$ is regular at its points in $\text{Spec}(R)$, we may shrink $X$ (using [SP, Lemma 00DS]) to make sure that $D$ is $k$-smooth. Since $E$ is trivial over $W \otimes_k \text{Frac}(X)$, there is a proper closed subscheme $\mathcal{Z} \subset X$ containing $X \setminus \mathcal{V}$ such that $E$ is trivial over $W \times_k (\mathcal{V} \setminus \mathcal{Z})$. If $d = 0$, then $E$ is trivial and we may choose $C = A^1_R$, so we assume that $X$ is of dimension $d > 0$. Finally, we use [SGA 3II, Exposé XIV, Corollaire 3.20] to shrink $X$ further to make $G$ have a maximal torus $T$ defined over all of $X$.

With these preparations, Lemma 3.1 allows us to shrink $X$ around $\text{Spec}(R)$ to arrange that there exist an affine open $S \subset A^{d-1}_k$ and a smooth morphism $f : X \to S$ of relative dimension 1 such that $\mathcal{Z}$ is $S$-finite and $D$ is $S$-(finite étale). We base change $f$ along the map $\text{Spec}(R) \to S$ to obtain

- a smooth affine $R$-scheme $C$ of pure relative dimension 1 (base change of $X$);
- an open $\mathcal{U} \subset C$ (base change of $\mathcal{V}$) with $C \setminus \mathcal{U}$ finite over $R$ such that $\mathcal{U} = C$ if $U = \text{Spec}(R)$;
- an $R$-finite closed subscheme $Z \subset C$ (base change of $\mathcal{Z}$) that contains $C \setminus \mathcal{U}$;
- a section $s \in C(R)$ (induced by the “diagonal” section) such that $s|_U$ factors through $\mathcal{U}$;
- a reductive $C$-group scheme $\mathcal{G}$ with $s^*(\mathcal{G}) \cong G$ (base change of $G$) such that $\mathcal{G}_{C \setminus \mathcal{U}}$ is totally isotropic if so is $G_{R/(r)}$;
- a maximal $C$-torus $\mathcal{T} \subset \mathcal{G}$ (base change of $T$) with $s^*(\mathcal{T}) \cong T$; and
- a $\mathcal{G}$-torsor $\mathcal{E}$ over $W \times_k \mathcal{U}$ (base change of $E$) that is trivial over $W \times_k (\mathcal{U} \setminus Z)$ such that

$$(\text{id}_W \times_k s|_U)^*(\mathcal{E}) \cong E$$

as $G$-torsors over $W \otimes_k U$.

By construction, if $r$ is in play, then $C \setminus \mathcal{U}$ is $R$-(finite étale) and is contained in $Z$. Moreover, we replace $Z$ by $Z \cup s$ if needed to arrange that $s \in Z(R)$. By Lemma 3.5 and spreading out, there is a finite étale cover $\tilde{C}$ of some affine open neighborhood of $Z$ in $C$ such that $s$ lifts to some $\tilde{s} \in \tilde{C}(R)$ and $\mathcal{G}_{\tilde{C}} \cong G_{\tilde{C}}$, compatibly with an already fixed such isomorphism after pullback along $\tilde{s}$. Thus, we
may replace $C$ and $s$ by $\tilde{C}$ and $\tilde{s}$ and replace $Z$, $\mathcal{U}$, $\mathcal{G}$, $\mathcal{E}$ by their corresponding base changes to reduce to the case when $\mathcal{G}$ is $G_C$. An analogous reduction based on Lemma 2.7 (b) and Remark 2.8 instead allows us to assume that there is no finite field obstruction to embedding $Z$ into $\mathbb{A}^1_R$ or even into $\mathbb{G}_{m,R}$. Thus, Proposition 2.5 gives an affine open $D \subset C$ containing $Z$ and a Cartesian square

\[
\begin{array}{ccc}
Z' & \longrightarrow & D \\
\downarrow & & \downarrow \pi \\
Z'' & \longrightarrow & \mathbb{G}_{m,R}
\end{array}
\]

in which the map $\pi$ is étale and embeds $Z$ as a closed subscheme $Z' \subset \mathbb{G}_{m,R}$. These properties of the square persist after restricting to the open complements of the common closed subscheme $C \setminus \mathcal{U}$ of $Z$ and $Z'$ (so also of $D$ and $\mathbb{G}_{m,R}$) and then by also forming a fiber product with $W$ over $k$. Thus, by Lemma 2.9, we may descend $\mathcal{E}|_{W \times_k (D \setminus \mathcal{U})}$ to a $G$-torsor $\mathcal{E}'$ over $W \times_k (\mathbb{A}^1_R \setminus (C \setminus \mathcal{U}))$ that is trivial over $W \times_k (\mathbb{A}^1_R \setminus Z')$. It then remains to replace $Z \subset C$ and $\mathcal{E}$ by $Z' \subset \mathbb{A}^1_R$ and $\mathcal{E}'$, to let $\mathbb{A}^1_R \setminus (C \setminus \mathcal{U})$ be the new $\mathcal{U}$, and to change the coordinate of $\mathbb{A}^1_R$ to make $s$ be the section $\{t = 0\}$. □

**Remark 3.8.** To complement the statement of Proposition 3.7, we recall from [Čes22a, Remark 6.4] that $\mathcal{U}$ is a principal affine open of $\mathbb{A}^1_R$ whenever its complement underlies an $R$-(finite locally free) closed subscheme of $\mathbb{A}^1_R$, for instance, whenever $\mathbb{A}^1_R \setminus \mathcal{U}$ is $R$-(finite étale).

In mixed characteristic, the following variant of Proposition 3.7 improves [Čes22a, Proposition 4.2].

**Proposition 3.9.** For a semilocal ring $R$ that is flat, geometrically regular over a Dedekind subring $\mathcal{O}$, an $r \in \mathcal{O}$, a reductive $R[\frac{1}{r}]$-group $G$ that either extends to a quasi-split reductive $R$-group or descends to a quasi-split $\mathcal{O}[\frac{1}{r}]$-group, and a generically trivial $G$-torsor $E$ over $R[\frac{1}{r}]$, there are

- (a) an $R$-finite closed subscheme $Z \subset \mathbb{A}^1_R$ that satisfies $(\mathbb{A}^1_R \setminus Z)(R) \neq \emptyset$; and
- (b) a $G$-torsor $\mathcal{E}$ over $\mathbb{A}^1_{R[\frac{1}{r}]}$ that is trivial over $\mathbb{A}^1_{R[\frac{1}{r}]} \setminus Z$ whose restriction to the zero section is $E$.

**Proof.** We begin by recalling that to every reductive group $H$ over a scheme $S$ one associates an $S$-torus $T_H$, the abstract maximal torus of $H$ defined by étale descent on $S$ as follows. Étale locally on $S$, the group $H$ has a Borel $B \subset H$, and, letting $\mathcal{B}_u(B) \subset B$ denote the unipotent radical, one sets

$T_H := B/\mathcal{B}_u(B)$.

Up to a canonical isomorphism, this $T_H$ does not depend on the choice of $B$, and so it descends to the original $S$: indeed, any two Borels are Zariski locally conjugate and, up to multiplying by a section of $B$, the conjugating section is unique [SGA 3[III new, exposé XXVI, proposition 1.2, corollaire 5.2], so it suffices to note the conjugation action of $B$ on $T_H$ is trivial because the latter is abelian.

With this in mind, in the setting of Proposition 3.9 we have the following triviality of $T_G$-torsors.

**Claim 3.9.1.** The abstract maximal torus of $G$ has no nontrivial generically trivial torsors over $R[\frac{1}{r}]$: $H^1(R[\frac{1}{r}], T_G) \hookrightarrow H^1(\text{Frac}(R[\frac{1}{r}]), T_G)$.

**Proof.** Thanks to our assumption on $G$, the torus $T_G$ is the base change of a torus $\mathcal{T}$ defined over a ring $A$ that is either $R$ or $\mathcal{O}[\frac{1}{r}]$. By [CTS87, Proposition 1.3], this $\mathcal{T}$ has a flasque resolution

$0 \to \mathcal{F} \to \text{Res}_{A'/A}(\mathbb{G}_m) \to \mathcal{T} \to 0$, 
where $A'$ is a finite étale $A$-algebra and $\mathcal{F}$ is a flasque $A$-torus. For now, all we need to know about flasque tori is that, by the regularity of $R[\frac{1}{t}]$ and [CTS87, Proposition 1.4, Theorem 2.2 (ii)],

$$H^2(R[\frac{1}{t}], \mathcal{F}) \hookrightarrow H^2(\text{Frac}(R[\frac{1}{t}]), \mathcal{F}).$$

This reduces our desired claim to the vanishing $\text{Pic}(R[\frac{1}{t}]) \otimes_A A' \cong 0$, which is argued as follows. In the case $A = R$, the ring $A'$ is again regular semilocal, so every line bundle on $A'[\frac{1}{t}]$ extends to a line bundle on $A'$, and hence is trivial, to the effect that $\text{Pic}(A'[\frac{1}{t}]) = 0$, as desired. In the case $A = \mathcal{O}[\frac{1}{t}]$, by [Ser79, Chapter I, Section 4, Proposition 8], the normalization of $\mathcal{O}$ in $A'$ is a finite $\mathcal{O}$-algebra $\mathcal{O}'$, in particular, $\mathcal{O}'$ is again a Dedekind ring. Thus, $R \otimes_{\mathcal{O}} \mathcal{O}'$ is a finite $R$-algebra, and hence is semilocal, but is also flat and geometrically regular over $\mathcal{O}'$, so it is regular by [SP, Lemma 033A]. Since $R[\frac{1}{t}] \otimes_A A'$ is a localization of $R \otimes_{\mathcal{O}} \mathcal{O}'$, it again follows that $\text{Pic}(R[\frac{1}{t}]) \otimes_A A' \cong 0$, as desired. □

Getting back to the proof of Proposition 3.9, we now use Popescu theorem [SP, Theorem 07GC] and a limit argument to reduce to the case when $R$ is a semilocal ring of a smooth affine $\mathcal{O}$-scheme $X$. By passing to connected components if needed, we may assume that $X$ is connected, of constant relative dimension $d$ over $\mathcal{O}$. If $d = 0$, then $R$, and so also $R[\frac{1}{t}]$, is a semilocal Dedekind ring, and $E$ is trivial by [Guo20, Theorem 1]; therefore, we lose no generality by assuming that $d > 0$. By shrinking $X$ if needed, we may assume that $G$ (resp., $E$) begins life over $X$ (resp., over $X[\frac{1}{t}]$).

In the case when our original $G$ extended to a reductive $R$-group, we shrink $X$ further and use [SGA 3II, exposé XIV, sections 3.8–3.9] to make $G$ extend to a quasi-split reductive $X$-group $G'$ and to fix a quasi-pinning of $G'$ over $X$, in particular, to fix a Borel $X$-subgroup $B \subset G'$. In the case when our original $G$ descends to an $\mathcal{O}[\frac{1}{t}]$-group, we shrink $X$ further to make sure that our new $G$ over $X[\frac{1}{t}]$ still descends to a quasi-split reductive group over $\mathcal{O}[\frac{1}{t}]$ and we fix a quasi-pinning of this descended group, in particular, we fix its Borel $\mathcal{O}[\frac{1}{t}]$-subgroup $B$.

By applying the valuative criterion of properness to $E/B_{X[\frac{1}{t}]}$, we may choose an open $U \subset X[\frac{1}{t}]$ with complement of codimension $\geq 2$ such that $E_U$ to reduces to a generically trivial $B$-torsor $E^B$ over $U$. By purity for torsors under tori [CTS79, corollaire 6.9], the $T_G$-torsor $E^B/B_u(B)$ over $U$ extends to a generically trivial $T_G$-torsor over $X[\frac{1}{t}]$. By shrinking $X$ and applying Claim 3.9.1, we trivialize this $T_G$-torsor, to the effect that $E_U$ then reduces to a $B_u(B)$-torsor.

Since the complement $X[\frac{1}{t}] \setminus U$ is of codimension $\geq 2$, its closure $\mathcal{Z}$ in $X$ is also of codimension $\geq 2$. Thus, by Lemma 3.2, we may shrink $X$ around $\text{Spec}(R)$ to arrange that there exist an affine open $S \subset \mathbb{A}^{d-1}_R$ and a smooth morphism $f : X \to S$ of relative dimension 1 such that $\mathcal{Z}$ is $S$-finite. Similarly to the proof of Proposition 3.7, we base change $f$ along the map $\text{Spec}(R) \to S$ to obtain

- a smooth affine $R$-scheme $C$ of pure relative dimension 1;
- an $R$-finite closed subscheme $Z \subset C$;
- a section $s \in C(R)$;
- a reductive $C[\frac{1}{t}]$-group $\mathcal{G}$ with $(s|_{R[\frac{1}{t}]})^*(\mathcal{G}) \cong G$;
- a quasi-pinning of $\mathcal{G}$, in particular, a Borel $C[\frac{1}{t}]$-subgroup $\mathcal{B} \subset \mathcal{G}$, whose pullback by $s|_{R[\frac{1}{t}]}$ is the chosen quasi-pinning of $G$; and
- a $\mathcal{G}$-torsor $\mathcal{E}$ over $C[\frac{1}{t}]$ whose restriction to $(C[Z])[\frac{1}{t}]$ reduces to a $B_u(\mathcal{B})$-torsor such that $(s|_{R[\frac{1}{t}]})^*(\mathcal{E}) \cong E$ as $G$-torsors over $R[\frac{1}{t}]$. 


By replacing $Z$ by $Z \cup s$ if needed, we arrange that $s \in Z(R)$. In the case when $G$ descends to an $O[\frac{1}{r}]$-group, since $\mathcal{G}$ is a base change of $G$ along the $O$-morphism $C[\frac{1}{r}] \to X[\frac{1}{r}]$, we have $\mathcal{G} \cong G_{C[\frac{1}{r}]}$ and $\mathcal{B} \cong B_{C[\frac{1}{r}]}$. In the case when $G$ extends to a reductive $X$-group $G'$, our $\mathcal{G}$ extends to its base change, namely, to a reductive $C$-group $\mathcal{G}'$ with $s^*(\mathcal{G}') \cong G'$ such that $\mathcal{G}'$ has a quasi-pinning extending that of $\mathcal{G}$ whose $s$-pullback is the chosen quasi-pinning of $G'$, in particular, that it has a Borel $C$-subgroup $\mathcal{B}' \subset \mathcal{G}'$ extending $\mathcal{B}$ with $s^*(\mathcal{B}') \cong B_R$. Thus, in this case Remark 3.6 and spreading out allow us to replace $C$ by a suitable finite étale cover of a neighborhood of $Z$ to arrange that $\mathcal{G}' \cong G'_C$, and $\mathcal{B}' \cong B_C$ compatibly with pullbacks under $s$, so that also $\mathcal{G} \cong G_{C[\frac{1}{r}]}$ and $\mathcal{B} \cong B_{C[\frac{1}{r}]}$ compatibly with pullbacks under $s|_{R[\frac{1}{r}]}$. In conclusion, in all cases we may assume that

$$\mathcal{G} = G_{C[\frac{1}{r}]} \quad \text{and} \quad \mathcal{B} = B_{C[\frac{1}{r}]}.$$  

At this point, we use Lemma 2.7 (b) and Remark 2.8 to change $C$ again to arrange that, in addition, there is no finite field obstruction to embedding $Z$ into $G_{m, R}$. Proposition 2.5 then gives us an affine open $D \subset C$ containing $Z$ and a Cartesian square

$$\begin{array}{ccc}
z & \to & D \\
\downarrow & & \downarrow \pi \\
z' & \to & G_{m, R}
\end{array}$$

in which the map $\pi$ is étale and embeds $Z$ as a closed subscheme $Z' \subset \mathbb{A}^1_\mathbb{R}$. By excision for $\mathcal{R}_u(B)$-torsors, more precisely, by [Čes22a, Lemma 7.2 (b) and Example 7.3], the restriction of $\mathcal{E}$ to $D \setminus Z$ descends to a $G$-torsor over $\mathbb{A}^1_{R[\frac{1}{r}]} \setminus Z'$ such that this descent reduces to a $\mathcal{R}_u(B)$-torsor. Thus, by glueing of Lemma 2.9, we may descend $\mathcal{E}$ to a $G$-torsor over $\mathbb{A}^1_{R[\frac{1}{r}]}$ whose restriction to $\mathbb{A}^1_{R[\frac{1}{r}]} \setminus Z'$ reduces to a $\mathcal{R}_u(B)$-torsor. By renaming $Z'$ to $Z$ we therefore reduce to the case when $C = \mathbb{A}^1_\mathbb{R}$, so by a change of coordinates on $\mathbb{A}^1_\mathbb{R}$, we may also assume that $s$ is the zero section of $\mathbb{A}^1_\mathbb{R}$. It now remains to apply the avoidance lemma, for instance, [Čes22a, Lemma 3.1], to enlarge $Z$ to be the vanishing locus of a suitable hypersurface in $\mathbb{P}^1_\mathbb{R}$ to arrange that $\mathbb{A}^1_\mathbb{R} \setminus Z$ is affine: indeed, then $\mathbb{A}^1_{R[\frac{1}{r}]} \setminus Z$ will also be affine, and so over it the torsor $\mathcal{E}$ will not merely reduce to a $\mathcal{R}_u(B)$-torsor but, by [SGA 3 III new, exposé XXVI, corollaire 2.2], will even be trivial, as desired. \hfill $\square$

4. TORSORS OVER $\mathbb{A}^d_A$ VIA THE AFFINE GRASSMANNIAN

Propositions 3.7 and 3.9 reduce us to analyzing $G$-torsors over relative affine line $\mathbb{A}^1_A$. We turn to general aspects of this in the present section, more precisely, we use techniques built on the geometry of the affine Grassmannian to present partial results on the following general conjecture.

Conjecture 4.1 ([Čes22b, Conjecture 3.5.1]). For a ring $A$ and a totally isotropic reductive $A$-group scheme $G$, every $G$-torsor over $\mathbb{A}^d_A$ that is trivial away from an $A$-finite closed subscheme is trivial.

We recall from [Čes22b, Section 3.5.2] that Conjecture 4.1 holds for any $A$ when $G$ is either semisimple simply connected, or split, or a torus. For a general $G$, we now show that the torsor in question trivializes after pulling back along any map $\mathbb{A}^1_A \to \mathbb{A}^1_A$ given by $t \mapsto t^d$ for a sufficiently divisible $d$ (Lemma 4.3 (b)) and also after pulling back by an $A$-point of $\mathbb{A}^1_A$ (Theorem 4.4 (ii)). In Proposition 5.5, we will establish Conjecture 4.1 in the case when $A$ is regular and contains a field.

Under additional assumptions, the triviality after pulling back by an $A$-point is [Fed22a, Theorem 5], and we will follow Fedorov’s strategy that is based on the geometry of the affine Grassmannian. The
latter will enter through (self-contained) citations to the survey article [Čes22b] that mildly generalized some of Fedorov’s key steps. We will also use the following general form of Quillen patching.

**Lemma 4.2.** (Gabber, see [Čes22b, Corollary 5.1.5 (b)]). For a ring $A$ and a locally finitely presented $A$-group algebraic space $G$, a $G$-torsor (for the fpf topology) over $\mathbb{A}^d_A$ descends to a $G$-torsor over $A$ if and only if it does so Zariski locally on $\text{Spec}(A)$. \hfill $\square$

**Lemma 4.3.** Let $A$ be a ring, let $G$ be a totally isotropic reductive $A$-group scheme, and let $E$ be a $G$-torsor over $\mathbb{A}^1_A$ that is trivial away from some $A$-finite closed subscheme $Z \subset \mathbb{A}^1_A$.

(a) If, for some extension of $E$ to a $G$-torsor $\tilde{E}$ over $\mathbb{P}^1_A$ obtained by gluing $E$ with the trivial torsor over $\mathbb{P}^1_A \setminus Z$ and for every prime ideal $p \subset A$, the $G^{\text{ad}}$-torsor over $\mathbb{P}^1_{kp}$ induced by $\tilde{E}$ lifts to a generically trivial $(G^{\text{ad}})^{\text{sc}}$-torsor over $\mathbb{P}^1_{kp}$, then $E$ is trivial.

(b) For any $m > 0$ divisible by the $A$-fibral degrees of the isogeny $(G^{\text{ad}})^{\text{sc}} \to G^{\text{ad}}$, the pullback of $E$ along any finite flat map $\mathbb{A}^1_A \to \mathbb{A}^1_A$ of degree $m$ that extends to a map $\mathbb{P}^1_A \to \mathbb{P}^1_A$ is trivial.

**Proof.** In (b), we extend $E$ to a $G$-torsor $\tilde{E}$ over $\mathbb{P}^1_A$ as in (a) and consider the pullback of this extension under our map $\mathbb{P}^1_A \to \mathbb{P}^1_A$. By [Čes22b, Lemma 5.3.5] (or [Fed22a, Proposition 2.3]), the choice of $m$ ensures that the fibral condition of (a) holds for this pullback, so (b) follows from (a).

In (a), it suffices to show that both $E$ and the restriction of $\tilde{E}$ to the complementary affine line $\mathbb{P}^1_A \setminus \{t = 0\}$ descend to $G$-torsors over $A$: both of these descents will agree with the restriction of $\tilde{E}$ to $t = 1$, which will agree with the restriction of $\tilde{E}$ to $t = \infty$ and hence be trivial, and then $E$ will also be trivial. By Quillen patching of Lemma 4.2, for the descent claim we may replace $A$ by its localization at a maximal ideal to reduce to the case of a local $A$.

Once $A$ is local, we will directly show that both $E$ and the restriction of $\tilde{E}$ to $\mathbb{P}^1_A \setminus \{t = 0\}$ are trivial. For this, we first show that we may modify $Z$ so that it does not meet $t = 0$. Namely, if the residue field $k$ of $A$ is infinite, then there is some $s \in (\mathbb{A}^1_A \setminus (Z \cup \{t = 0\}))\langle A \rangle$ and, by [Čes22b, Proposition 5.3.6] (which uses the total isotropity assumption, the fibral assumption on $\tilde{E}$, and is based on geometric input about the affine Grassmannian in the style of [Fed22a, Theorem 6]), the restriction of $\tilde{E}$ to $\mathbb{P}^1_A \setminus s$ is a trivial $G$-torsor, so that we may replace $Z$ by $s$ to arrange the desired $Z \cap \{t = 0\} = \emptyset$. In contrast, if the residue field $k$ of $A$ is finite, then there is some large $n$ such that $\mathbb{A}^1_k \setminus (Z_k \cup \{t = 0\})$ contains a finite étale subscheme $y$ that is the union of a point valued in the field extension of $k$ of degree $n$ and a point valued in the field extension of $k$ of degree $n + 1$. Both of these components of $y$ are cut out by monic polynomials with coefficients in $k$, so $y$ lifts to an $A$-(finite étale) closed subscheme $Y \subset \mathbb{A}^1_A \setminus (Z \cup \{t = 0\})$ that is a disjoint union of an $A$-(finite étale) closed subscheme of degree $n$ and an $A$-(finite étale) closed subscheme of degree $n + 1$. In particular, both $\mathcal{O}(n)$ and $\mathcal{O}(n + 1)$ restrict to trivial line bundles on $\mathbb{P}^1_A \setminus Y$, and hence so does $\mathcal{O}(1)$. Thus, by [Čes22b, Proposition 5.3.6] once more, $\tilde{E}$ is trivial on $\mathbb{P}^1_A \setminus Y$, to the effect that in the case when $k$ is finite we may replace $Z$ by $Y$ to again arrange that $Z \cap \{t = 0\} = \emptyset$.

Once our $Z \subset \mathbb{A}^1_A$ does not meet $\{t = 0\}$, it suffices to apply [Čes22b, Proposition 5.3.6] twice to conclude that $\tilde{E}$ restricts to the trivial torsor both on $\mathbb{P}^1_A \setminus \{t = \infty\}$ and on $\mathbb{P}^1_A \setminus \{t = 0\}$, as desired. \hfill $\square$

The following result is our eventual source of triviality of torsors under reductive groups.
Theorem 4.4. For a smooth scheme $C$ of relative dimension $d > 0$ over a ring $A$, a reductive $A$-group $G$, a $G$-torsor $E$ over $C$ that is trivial away from some $A$-finite closed subscheme, and an affine $C$-scheme $W$ (for instance, $W$ could be an $A$-point $s \in C(A)$), if either

(i) $W$ is semilocal; or

(ii) $G_W$ is totally isotropic and $C = \mathbb{A}^d_A$;

then the $G$-torsor $E_W$ over $W$ is trivial.

Parts (i) and (ii) are generalizations of [Fed22a, Theorem 4] and [Fed22a, Theorem 5], respectively.

Proof. Base change to $W$ reduces us to the case when $W$ is an $A$-point $s$, in which we need to show that $s^*(E)$ is trivial. Any $A$-point of $\mathbb{A}^d_A$ factors through some $\mathbb{A}^{d-1}_A$-point, so in (ii) we may replace $A$ by $A[t_1, \ldots, t_{d-1}]$ to reduce to $d = 1$. We then change the variable of $\mathbb{A}^1_A$ to transform $s$ into the section $t = 0$. This ensures that $s$ lifts to an $A$-point along any map $\mathbb{A}^1_A \to \mathbb{A}^1_A$ given by $t \mapsto t^d$. Thus, in (ii) we may pull back along such a map and conclude from Lemma 4.3 (b).

In the remaining case (i), we let $E$ be a $G$-torsor over $C$ that is trivial away from an $A$-finite closed subscheme $Z \subset C$ containing $s$. By working locally around $s$, we may first arrange that $C$ be affine and then apply Lemma 2.7 (b) and Remark 2.8 to arrange that $Z$ have no finite field obstruction to embedding it into $\mathbb{A}^d_A$. By then combining Proposition 2.5 and Lemma 2.9, we reduce the case when $C = \mathbb{A}^d_A$. Arguing as in (ii), we may then even assume that $d = 1$ and that $s$ is the section $t = 0$, and we extend $E$ to a $G$-torsor $\tilde{E}$ over $\mathbb{P}^1_A$ by glueing $E$ with the trivial torsor over $\mathbb{P}^1_A \setminus Z$. We let $m$ be the least common multiple of the $A$-fibral degrees of the isogeny $(G^{ad})^\text{sc} \to G^{ad}$, and, as in the proof of Lemma 4.3 (b), replace $\tilde{E}$ by its pullback along the map $\mathbb{P}^1_A \to \mathbb{P}^1_A$ given by $[x : y] \mapsto [x^m : y^m]$ to arrange that, for every maximal ideal $m \subset A$, the $G^{ad}$-torsor over $\mathbb{P}^1_{k_m}$ induced by $\tilde{E}$ lifts to a generically trivial $(G^{ad})^\text{sc}$-torsor over $\mathbb{P}^1_{k_m}$ (see [Čes22b, Lemma 5.3.5]).

In (i), we will eventually obtain the conclusion from [Čes22b, Proposition 5.3.6]. To prepare for applying it, consider the canonical decomposition as in (1.1.1):

$$G^{ad} \cong \prod_i H_i \text{ with } H_i \cong \text{Res}_{A_i/A}(G_i),$$

where $A_i$ is a finite étale $A$-algebra and $G_i$ is an adjoint $A_i$-group scheme with simple geometric fibers. For each $i$, consider the projective, smooth $A$-scheme $X_i$ parametrizing parabolic subgroups of $H_i$ (see [SGA 3 new, Exposé XXVI, Corollaire 3.5]). For each $i$, consider the closed subscheme $\text{Spec}(A/I_i) \subset \text{Spec}(A)$ that is the disjoint union of those maximal ideals $m \subset A$ such that $(H_i)_m$ is isotropic, in other words, such that $(H_i)_m$ has a proper parabolic subgroup (see [SGA 3 new, Exposé XXVI, Corollaire 6.12]), and fix such parabolic subgroups to obtain an $x_i \in X_i(A/I_i)$. By [Čes22b, Lemma 6.2.2] (which is based on Bertini theorem), there are a faithfully flat, finite, étale $A$-scheme $Y_i$ equipped with an $A/I_i$-point $y_i \in Y_i(A/I_i)$ and an $A$-morphism $Y_i \to X_i$ that maps $y_i$ to $x_i$, so that, in particular, $(H_i)_{Y_i}$ is totally isotropic for every $i$.

By Lemma 2.7 (a), there is a finite étale cover $\pi: \tilde{Y} \to \bigsqcup Y_i$ such that there is no finite field obstruction to embedding $\tilde{Y}$ into $\mathbb{A}^1_A \setminus (Z \cup s)$ and $\tilde{Y} = \tilde{Y}' \cup \tilde{Y}''$ with $\tilde{Y}'$ (resp., $\tilde{Y}'$) surjective over $\bigsqcup Y_i$ of constant degree $N$ (resp., $N + 1$) for some $N > 0$. By Corollary 2.6, we may therefore find an embedding $\tilde{Y} \to \mathbb{A}^1_A$ whose image does not meet $Z$ or $s$. By construction, for each $i$ and each maximal ideal $m \subset A$ such that $(H_i)_m$ is isotropic, $Y_i$ has a $k_m$-point, and so the $k_m$-fiber of the preimage $\tilde{Y}_i := \pi^{-1}(Y_i)$ has two disjoint clopens that have degrees $N$ and $N + 1$ over $k_m$. Consequently, for each such $i$ and $m$, the line bundle $\mathcal{O}(1)$ is trivial over $(\mathbb{P}^1_{k_m})_{\tilde{Y}_i}$. Thus, since
(\tilde{Y} \cup \{ t = \infty \}) \cap Z = \emptyset$, we may apply [Čes22b, Proposition 5.3.6] to conclude that $E$ is trivial over $\mathbb{A}_A^1 \setminus \tilde{Y}$. In particular, since $\tilde{Y}$ is disjoint from $s$, the pullback $s^*(E)$ is also trivial, as desired. □

5. Conclusions about the Grothendieck–Serre conjecture

We have everything in place to settle our mixed characteristic cases of the Nisnevich conjecture.

5.1. Proof of Theorem 1.3 (2). We have a semilocal ring $R$ that is flat and geometrically regular over a Dedekind subring $\mathcal{O}$, an $r \in \mathcal{O}$, a reductive $R[1/r]$-group $G$ that either extends to a quasi-split reductive $R$-group or descends to a quasi-split reductive $\mathcal{O}[1/r]$-group, and a generically trivial $G$-torsor $E$ over $R[1/r]$. We need to show that $E$ is trivial.

Proposition 3.9 supplies a $G$-torsor $\mathcal{E}$ over $\mathbb{A}_R^{1}[1/r]$ that is trivial away from some $R[1/r]$-finite $Z \subset \mathbb{A}_R^{1}[1/r]$ and whose pullback by the zero section is $E$. Our $R[1/r]$-group $G$ is quasi-split, so it is totally isotropic. Thus, we may apply Theorem 4.4 (ii) to $\mathcal{E}$ to conclude that $E$ is trivial, as desired. □

In equal characteristic, the approach to the Nisnevich conjecture is based on the following relative version of the Grothendieck–Serre conjecture that is a mild improvement to [Fed22a, Theorem 1] (with an earlier more restrictive case due to Panin–Stavrova–Vavilov [PSV15, Theorem 1.1]).

**Theorem 5.2.** For a regular semilocal ring $R$ containing a field $k$, a reductive $R$-group $G$, and an affine $k$-scheme $W$, no nontrivial $G$-torsor over $W \otimes_k R$ trivializes over $W \otimes_k \text{Frac}(R)$ if either

(i) $G$ is totally isotropic;

(ii) $W \otimes_k R$ is semilocal, for instance, if $W = \text{Spec}(k)$;

in particular, if $U \subset \mathbb{A}_k^1$ is a nonempty open and $G$ is totally isotropic, then every generically trivial $G$-torsor on $U \otimes_k R$ is trivial.

**Proof.** The claim about $U$ follows from the rest and the following lemma applied to $K = \text{Frac}(R)$.

**Lemma 5.3** ([Gil02, Corollaire 3.10]). For a reductive group $G$ over a field $k$ and an open $U \subset \mathbb{P}_k^1$, each generically trivial $G$-torsor $E$ over $U$ reduces to a torsor under a maximal $K$-split subtorus of $G$; in particular, if $U \subset \mathbb{A}_k^1$, then, since $U$ has no nontrivial line bundles, $E$ is a trivial $G$-torsor. □

The claim about $U$ implies the parenthetical claim about the functor $H^1(-, G)$: indeed, every finite étale $R$-scheme $\mathcal{Y}$ is reduced and is also a $k$-scheme, so any map $\mathcal{Y} \to \text{Spec} B$ with $B$ a finite $Z$-algebra factors through a union of finitely many reduced points, to the effect that the claim about $U$ implies that every $G$-torsor over $\mathbb{A}_\mathcal{Y}^1 \setminus \mathcal{Y}$ as in (6.3.1) is trivial as soon as it is trivial away from $Z$.

As for the main claim of Theorem 5.2, we let $E$ be a $G$-torsor over $W \otimes_k R$ that trivializes over $W \otimes_k \text{Frac}(R)$. By Proposition 3.7, there are an $R$-finite closed subscheme $Z \subset \mathbb{A}_R^1$, an $s \in Z(R)$, and a $G$-torsor $\mathcal{E}$ over $W \times_k \mathbb{A}_R^1$ that is trivial over $W \times_k (\mathbb{A}_R^1 \setminus Z)$ such that

$$(\text{id}_W \times_k s)^*(\mathcal{E}) \cong E$$

as $G$-torsors over $W \otimes_k R$.

Let $A$ be the coordinate ring of $W \otimes_k R$ and view $\mathcal{E}$ as a $G$-torsor over $\mathbb{A}_A^1$ that is trivial away from an $A$-finite closed subscheme $\mathcal{Z} := W \times_k Z \subset \mathbb{A}_A^1$. We need to show that the pullback of $\mathcal{E}$ along a given $A$-point $z := \text{id}_W \times_k s$ of $\mathcal{Z}$ is trivial. This, however, is a special case of Theorem 4.4. □
We conclude the section with the promised case of Conjecture 4.1 when $A$ is regular and contains a field. This uses the following variant of Quillen patching that reduces Conjecture 4.1 to local $A$.

Lemma 5.4 ([Čes22b, Corollary 5.1.9]). For a ring $A$ and a locally finitely presented $A$-group scheme $G$, every $G$-torsor over $\mathbb{A}^1_A$ that is trivial away from an $A$-finite closed subscheme is trivial as soon as the same holds with $A$ replaced by its localization $A_m$ for every maximal ideal $m \subset A$. □

Proposition 5.5. Conjecture 4.1 holds in the case when the ring $A$ is regular and contains a field, in other words, for a totally isotropic reductive group scheme $G$ over a regular ring $A$ that contains a field, every $G$-torsor over $\mathbb{A}^d_A$ that is trivial away from an $A$-finite closed subscheme is trivial.

Proof. We may replace $A$ by $A[t_1, \ldots, t_{d-1}]$ to reduce to the case when $d = 1$. We may then apply Lemma 5.4 to reduce further to the case when our regular ring $A$ is local. At this point we may apply the relative Grothendieck–Serre conjecture, more precisely, Theorem 5.2 to replace $A$ by its fraction field. Once $A$ is a field with $d = 1$, the claim becomes a special case of Lemma 5.3. □

6. Extending $G$-torsors over a finite étale subscheme of a relative curve

The final preparation to the equicharacteristic case of the Nisnevich conjecture is a result about extending $G$-torsors over a finite étale closed subscheme of a smooth relative curve that we deduce in Proposition 6.3 from the reembedding techniques of §2. For wider applicability, we present this extension result axiomatically—it loosely amounts to a reduction of the Nisnevich conjecture to the Grothendieck–Serre conjecture. The equicharacteristic relative Grothendieck–Serre conjecture settled in Theorem 5.2 supplies the required axiomatic assumptions in our main case of interest.

Definition 6.1. For a ring $A$, a contravariant, set-valued functor $F$ on the category of $A$-schemes of the form $S\setminus Z$ with $S$ a smooth affine $A$-scheme of pure relative dimension $d$ and $Z \subset S$ an $A$-quasi-finite closed subscheme, is excisive if for all $Z \subset S$ and $Z' \subset S'$ as above and all Cartesian squares

$$
\begin{array}{ccc}
Z' & \to & S \\
\downarrow & & \downarrow f \\
Z' & \to & S'
\end{array}
$$

with $f$ étale that induces an indicated isomorphism $Z \xrightarrow{\sim} Z'$, we have

$$
F(S') \to F(S) \times_{F(S\setminus Z)} F(S'\setminus Z').
$$

For instance, by Lemma 2.9, for a quasi-affine, flat, finitely presented $A$-group $G$, the functor $H^1(-, G)$ is excisive. The following lemma is instrumental for the aforementioned ‘excision tricks.’

Lemma 6.2. Let $A$ be a ring, let $S$ be an $A$-scheme, let $Y \subset S$ be an $A$-(separated étale) closed subscheme that is locally cut out by a finitely generated ideal, and consider the decomposition

$$
Y \times_A Y = \Delta \sqcup Y'
$$

in which $\Delta \subset Y \times_A Y$ is the diagonal copy of $Y$. The following square is Cartesian:

$$
\begin{array}{ccc}
\Delta & \to & S_Y \setminus Y' \\
\downarrow & & \downarrow \\
Y & \to & S
\end{array}
$$
in particular, if $S$ is as in Definition 6.1 and $F$ is an excisive functor, then an element of $F(S \setminus Y)$ extends to $F(S)$ if and only if its pullback to $F((S \setminus Y)_Y)$ extends to $F(S_Y \setminus Y')$; for instance, for a quasi-affine, flat, finitely presented $S$-group scheme $G$, a $G$-torsor over $S \setminus Y$ extends to a $G$-torsor over $S$ if and only if its base change to $(S \setminus Y)_Y$ extends to a $G$-torsor over $S_Y \setminus Y'$.

Proof. The claimed decomposition $Y \times_A Y = \Delta \sqcup Y'$ exists because any section of a separated étale morphism, such as the projection $Y \times_A Y \to Y$, is both a closed immersion and an open immersion. Thus, the square in question is Cartesian because the étale map $S_Y \setminus Y' \to S$ induces an isomorphism $\Delta \xrightarrow{\sim} Y$. The claim about $F$ is then immediate from Definition 6.1. 

We are ready for our key axiomatic extension result, which extends Fedorov’s [Fed22b, Proposition 2.6].

Proposition 6.3. Let

- $A$ be a semilocal ring,
- $C$ be a smooth affine $A$-scheme of pure relative dimension $d > 0$,
- $Y \subset C$ be an $A$-(finite étale) closed subscheme, and
- $F$ be an excisive, pointed set valued functor as in Definition 6.1.

Suppose that for all finite étale $Y$-schemes $\mathcal{Y}$ and integers $m \leq \deg(Y/A)$ such that $\mathcal{Y}$ is a scheme over a finite $\mathbb{Z}$-algebra $B$ for which $\mathbb{A}^d_B$ contains $m$ disjoint copies of $\text{Spec}(B)$, some $\mathcal{Y}' \subset \mathbb{A}^d_B$ that is a union of $m$ disjoint copies of $\mathcal{Y}$ and every $\mathcal{Y}$-finite closed subscheme $Z \subset \mathbb{A}^d_B$ containing $\mathcal{Y}'$,

$$\text{Ker}(F(\mathbb{A}^d_B) \to F(\mathbb{A}^d_B/Z)) \to \text{Ker}(F(\mathbb{A}^d_B/\mathcal{Y}') \to F(\mathbb{A}^d_B/Z)), \quad (6.3.1)$$

that is, every element of $F(\mathbb{A}^d_B/\mathcal{Y}')$ that trivializes away from some $\mathcal{Y}$-finite $Z \subset \mathbb{A}^d_B$ extends to $F(\mathbb{A}^d_B)$. Then, for every $A$-finite closed subscheme $Z \subset C$ containing $Y$,

$$\text{Ker}(F(C) \to F(C/Z)) \to \text{Ker}(F(C\setminus Y) \to F(C\setminus Z)), \quad (6.3.2)$$

that is, every element of $F(C\setminus Y)$ that trivializes away from some $A$-finite $Z \subset C$ extends to $F(C)$. 

Although a general $d > 0$ requires no extra work, the main case is $d = 1$, in which Theorem 5.2 supplies the assumption (6.3.1) when $A$ is regular of equicharacteristic and $F(-) = H^1(-, G)$ for a reductive $A$-group $G$ such that $G_Y$ totally isotropic. Roughly, Proposition 6.3 reduces the extendability property (6.3.2) to the case when $C = \mathbb{A}^d_A$ and $Y$ is “constant.”

Proof. For the proof, it is convenient to generalize our setup as follows. We assume that $C \subset C'$ is an open immersion of smooth affine $A$-schemes of pure relative dimension $d$ such that $Y' := C' \setminus C$ is $A$-(finite étale), that our assumption (6.3.1) holds with $m \leq \deg((Y \cup Y')/A)$, and that we seek to show (6.3.2) for every $A$-finite $Z \subset C'$ containing $Y$ and $Y'$. Of course, the case $C' = C$ recovers the original claim, and the formulation with an arbitrary $C'$ is equivalent because after extending to an element of $F(C)$ we may extend further to an element of $F(C')$. For intermediate reductions, however, it is convenient to require that our $A$-finite $Z$ lives in $C'$ instead of the possibly smaller $C$.

In the setup with a $C'$, we fix an $F$ satisfying the assumptions and an $\alpha \in \text{Ker}(F(C\setminus Y) \to F(C\setminus Z))$ that we wish to extend over $Y$. We then use Lemma 6.2 to base change along $Y \to \text{Spec}(A)$ and shrink the base changed $C$ by removing the off-diagonal part of $Y \times_A Y$ to reduce to the case when $Y \cong \text{Spec}(A)$. Moreover, we decompose $A$ to reduce to the case when $\text{Spec}(A)$ is connected, so that $\deg(Y/A)$ is a well-defined integer. We then let $n$ be the product of $\deg((Y \cup Y')/A)$ and all the
primes \( p \) with either \( p \leq \deg((Y \cup Y')/A) \) or \( p \not\in A^x \). We combine Lemma 2.7 (b) with Remark 2.8 to find an affine open \( D \subset C' \) containing \( Z \) as well as a finite étale cover \( \tilde{C}' \to D \) such that

\[
\tilde{Y} := Y \times_{C'} \tilde{C}' \text{ decomposes as } \tilde{Y} = \tilde{Y}_0 \sqcup \tilde{Y}_1 \text{ where } \tilde{Y}_0 \to \Spec(A),
\]

each component of \( \tilde{Y}_1 \) or of \( \tilde{Y}' := \tilde{C}' \setminus \tilde{C} \) with \( \tilde{C} := (D)/Y' \times_{C'} \tilde{C}' \) is an algebra over some finite \( \mathbb{Z} \)-algebra \( B \) each of whose residue fields \( k' \) of characteristic \( p \mid n \) satisfies

\[
\# k' > \deg((\tilde{Y} \cup \tilde{Y}')/A),
\]

and there is no finite field obstruction to embedding \( \tilde{Z} := \tilde{Z} \times_{C'} \tilde{C}' \to \mathbb{A}_A^d \). By construction,

\[
\begin{array}{ccc}
\tilde{Y}_0 & \to & \tilde{C} \setminus \tilde{Y}_1 \\
\downarrow & & \downarrow \\
Y' & \to & C \cap D
\end{array}
\]

is a Cartesian square. Thus, since \( F \) is excisive, to extend \( \alpha \) over \( Y \) we may first restrict to \( C \cap D \) and then pass to \( \tilde{C} \setminus \tilde{Y}_1 \). That is, we may replace \( Y \subset C \subset C' \) by \( \tilde{Y}_0 \subset \tilde{C} \setminus \tilde{Y}_1 \subset \tilde{C}' \) and \( \alpha \) by its pullback to \( \tilde{C} \setminus \tilde{Y} \) to reduce to the case when each connected component of \( Y' \) is an algebra over some finite \( \mathbb{Z} \)-algebra \( B \) each of whose residue fields \( k' \) of characteristic \( p \mid n \) satisfies \( \# k' > \deg((Y \cup Y')/A) \) and there is no finite field obstruction to embedding \( Z \) into \( \mathbb{A}_A^d \). By Proposition 2.5, such an embedding exists, more precisely, there are an affine open \( D \subset C' \) containing \( Z \) and a Cartesian square

\[
\begin{array}{ccc}
Z' & \to & D \\
\downarrow & & f \\
Z'' & \to & \mathbb{A}_A^d
\end{array}
\]

in which the map \( f \) is étale and embeds \( Z \) as a closed subscheme \( Z' \subset \mathbb{A}_A^d \). The square remains Cartesian after passing to the complements of the \( A \)-finite étale \( Y \cup Y' \) viewed inside \( Z \) (so also inside \( Z' \)). Thus, for the purpose of extending \( \alpha \) over \( Y \), we may use the excisive property of \( F \) to patch the restriction of \( \alpha|_{D \setminus (Y \cup Y')} \) with the origin in \( F(\mathbb{A}_A^d \setminus Z') \) to reduce to the case when \( C' = \mathbb{A}_A^d \).

In conclusion, we reduced to the case when \( C' = C = \mathbb{A}_A^d \) and \( Y \cong \Spec(A) \sqcup y \) such that each connected component of \( y \) is an algebra over some finite \( \mathbb{Z} \)-algebra \( B \) each of whose residue fields \( k' \) of characteristic \( p \mid n \) satisfies \( \# k' > \deg(Y/A) \). Moreover, we may assume that \( A \) itself is an algebra over some such finite \( \mathbb{Z} \)-algebra \( B \); indeed, once we settle this case, we may combine it with Lemma 6.2 to iteratively extend \( \alpha \) over each connected component of \( y \), and hence to reduce to the case when \( y = \emptyset \), in which we may simply choose \( B = \mathbb{Z} \).

Granted the reductions above, we now induct on the number of disjoint copies of \( \Spec(A) \) contained in \( Y \) to reduce to when \( Y \cong \bigsqcup \Spec(A) \). Indeed, suppose that \( Y \) has a connected component \( W \) that does not map isomorphically to \( \Spec(A) \), so that \( W \) is of degree \( \geq 2 \) over \( A \). Since \( W \times_A W \) contains the diagonal copy of \( W \) as a clopen (compare with Lemma 6.2), the \( W \)-finite étale closed subscheme \( Y \times_A W \subset \mathbb{A}_W^d \) contains strictly more disjoint copies of \( W \) than \( Y \) contained disjoint copies of \( \Spec(A) \). Thus, by the inductive hypothesis, the pullback of \( \alpha \) to \( \mathbb{A}_W^d \setminus (Y \times_A W) \) extends over \( Y \times_A W \). By Lemma 6.2, this implies that \( \alpha \) extends over \( W \). By repeating this for each possible \( W \), we effectively shrink \( Y \) until we reduce to the desired base case when \( Y \cong \bigsqcup \Spec(A) \).

To treat this last case, we set \( m := \deg(Y/A) \) and note that, by Proposition 2.5, for any closed subscheme \( \mathcal{Y}' \subset \mathbb{A}_A^d \) that is a disjoint union of \( m \) copies of \( \Spec(A) \), there are an affine open \( D \subset \mathbb{A}_A^d \) containing \( Z \) and a Cartesian square as in (6.3.3) such that \( f \) maps \( Y \) isomorphically onto \( \mathcal{Y}' \). Since \( F \) is excisive, Lemma 2.9 then reduces us to the case when \( Y = \mathcal{Y}' \) inside \( \mathbb{A}_A^d \). At this
point, we will finally use the assumption (6.3.1) on $F$. Namely, by the arranged condition on the residue fields of $B$, there is a $B$-(finite étale) closed subscheme of $\mathbb{A}^d_B$ that is a union of $m$ disjoint copies of $\text{Spec}(B)$, and its base change is then a closed subscheme of $\mathbb{A}^d_A$ that is a union of $m$ disjoint copies of $\text{Spec}(A)$. This means that (6.3.3) applies to some closed subscheme $\mathcal{Y}' \subset \mathbb{A}^d_A$ that is a disjoint union of $m$ copies of $\text{Spec}(A)$ and, as we have already argued, this implies our claim about extending over $Y$.

\begin{corollary}
For a regular semilocal ring $R$ containing a field, a totally isotropic reductive $R$-group scheme $G$, and an $R$-(finite étale) closed subscheme $Y \subset \mathbb{A}^1_R$, no nontrivial $G$-torsor over $\mathbb{A}^1_R \setminus Y$ becomes trivial over $\mathbb{A}^1_R \setminus Z$ for some $R$-finite closed subscheme $Z \subset \mathbb{A}^1_R$ containing $Y$, that is,

$$\text{Ker}(H^1(\mathbb{A}^1_R \setminus Y, G) \to H^1(\mathbb{A}^1_R \setminus Z, G)) = \{\ast\}.$$\end{corollary}

\textbf{Proof.} By Proposition 6.3 (with Theorem 5.2), every $G$-torsor over $\mathbb{A}^1_R \setminus Y$ that is trivial over $\mathbb{A}^1_R \setminus Z$ extends to a $G$-torsor over $\mathbb{A}^1_R$. This $Y = \emptyset$ case, however, is covered by Theorem 5.2. \hfill \square

7. The Nisnevich and the generalized Bass–Quillen conjectures over a field

\subsection{Proof of Theorem 1.3 (1)}
We have a regular semilocal ring $R$ containing a field, a regular parameter $r \in R$, a reductive $R$-group $\mathcal{G}$ with $\mathcal{G}_{R/(r)}$ totally isotropic, and a generically trivial $G$-torsor $E$ over $R[\frac{1}{r}]$. We need to show that $E$ is trivial. Equivalently, by the Grothendieck–Serre conjecture, more precisely, by Theorem 5.2 (ii), we need to extend $E$ to a $\mathcal{G}$-torsor $\mathcal{E}$ over $R$. For this, by patching supplied by Lemma 2.9 and a limit argument, we may semilocalize $R$ along the union of those maximal ideals $m \subset R$ that contain $r$ and reduce ourselves to the case when $r$ lies in every maximal ideal $m \subset R$. By Proposition 3.7, there are an $R$-finite closed subscheme $Z \subset \mathbb{A}^1_R$, an $R$-(finite étale) closed subscheme $Y \subset Z$, a section $s \in \mathbb{A}^1_R(R)$ such that $s|_{R[\frac{1}{r}]}$ factors through $\mathbb{A}^1_R \setminus Y$ and $\mathcal{G}_Y$ is totally isotropic, and a $\mathcal{G}$-torsor $\mathcal{E}$ on $\mathbb{A}^1_R \setminus Y$ that is trivial over $\mathbb{A}^1_R \setminus Z$ such that

$$(s|_{R[\frac{1}{r}]})^*(\mathcal{E}) \cong E \quad \text{as $\mathcal{G}$-torsors over $R[\frac{1}{r}]$.}$$

By Proposition 6.3 (with Theorem 5.2), this $\mathcal{G}$-torsor $\mathcal{E}$ extends to a $\mathcal{G}$-torsor over $\mathbb{A}^1_R$. Thus, by pulling back along $s$, our $\mathcal{G}$-torsor $E$ extends to a desired $\mathcal{G}$-torsor $\mathcal{E}$ over $R$. \hfill \square

\subsection{Proof of Theorem 1.5}
We have a regular ring $R$ containing a field, a totally isotropic reductive $R$-group $G$, and a generically trivial $G$-torsor $E$ over $\mathbb{A}^d_R$. We need to show that $E$ descends to a $G$-torsor over $R$. For this, by induction on $d$, we may assume that $d = 1$. By Quillen patching of Lemma 4.2, we may assume that $R$ is local. In this key local case, we will show that $E$ is trivial.

For this, by Proposition 5.5, it suffices to show that $E$ is trivial on $\mathbb{A}^1_R \setminus Z$ for some $R$-finite closed subscheme $Z \subset \mathbb{A}^1_R$. By a limit argument, it therefore suffices to show that $E$ becomes trivial over the localization of $R[t]$ obtained by inverting all the monic polynomials. By the change of variables $x := t^{-1}$, this localization is the localization of $\mathbb{P}^1_R$ along the section $\infty$, and hence is isomorphic to

$$(R[x]_1 + xR[x])[\frac{1}{x}].$$

The ring $R' := R[x]_1 + xR[x]$ is regular, local, and shares its fraction field with $\mathbb{A}^1_R$. In particular, the base change of $E$ to $R'$ is generically trivial. Thus, since $x$ is a regular parameter of $R'$, Theorem 1.3 (1) implies that this base change of $E$ is trivial, as desired. \hfill \square
