CONVERGENCE OF THE GINZBURG-LANDAU APPROXIMATION FOR THE ERICKSEN-LESLIE SYSTEM

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Abstract. We establish the local well-posedness of the general Ericksen-Leslie system in liquid crystals with the initial velocity and director field in $H^3 \times H^2_b$. In particular, we prove that the solutions of the Ginzburg-Landau approximation system converge smoothly to the solution of the Ericksen-Leslie system for any $t \in (0, T^*)$ with a maximal existence time $T^*$ of the Ericksen-Leslie system.

1. Introduction

In the 1960s, Ericksen [8] and Leslie [20] proposed a celebrated hydrodynamic theory to describe the behavior of liquid crystals. The Ericksen-Leslie theory has been widely accepted since then as one of the most successful theories for modeling liquid crystal flows (c.f. [28]). Let $v = (v^1, v^2, v^3)$ be the velocity vector of the fluid and $u = (u^1, u^2, u^3) \in S^2$ the unit direction vector. Then the Ericksen-Leslie system in $\mathbb{R}^3 \times [0, \infty)$ is given by (c.f. [23, 28])

\begin{align*}
\frac{\partial v}{\partial t} + v \cdot \nabla v + \nabla P &= \nabla \cdot (\sigma^E + \sigma^L), \\
\nabla \cdot v &= 0, \\
u \times (\gamma_1 N + \gamma_2 Au - h) &= 0,
\end{align*}

where $P$ represents the pressure and $\sigma^E$ denotes the Ericksen stress tensor given by

\begin{equation}
\sigma^E = -\nabla u^T \frac{\partial W(u, \nabla u)}{\partial (\nabla u)}.
\end{equation}

Here the Oseen-Frank density $W(u, \nabla u)$ takes the form

$W(u, \nabla u) = k_1 (\text{div } u)^2 + k_2 (u \cdot \text{curl } u)^2 + k_3 |u \times \text{curl } u|^2 + (k_2 + k_4) [\text{tr}(\nabla u)^2 - (\text{div } u)^2],
$

where $k_1, k_2, k_3, k_4$ are Frank's elastic constants. The Leslie stress tensor $\sigma^L$ satisfies the constitutive relation

\begin{equation}
\sigma^L = \alpha_1 (u \otimes u : A) u \otimes u + \alpha_2 N \otimes u + \alpha_3 u \otimes N + \alpha_4 A + \alpha_5 (Au) \otimes u + \alpha_6 u \otimes (Au),
\end{equation}

where $\alpha_i, i = 1, 2, \cdot \cdot \cdot, 6$ are the Leslie coefficients. The co-rotational time derivative $N$ of $u$ is defined by

\begin{equation}
N = \partial_t u + v \cdot \nabla u - \Omega u.
\end{equation}

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We denote by $\Omega$ and $A$ the skew-symmetric and symmetric parts of the tensor $\nabla v$ respectively; that is
\[
\Omega = \frac{1}{2}(\nabla v - (\nabla v)^T), \quad A = \frac{1}{2}(\nabla v + (\nabla v)^T).
\]
The molecular field $h$ in (1.3) is given by
\[
h = \nabla \cdot \left( \frac{\partial W(u, \nabla u)}{\partial (\nabla u)} \right) - \frac{\partial W(u, \nabla u)}{\partial u}.
\]

In the sequel, the following assumptions are introduced: Frank’s elastic constants $k_1, k_2, k_3, k_4$ satisfy the strong Ericksen inequalities (c.f. [1])
\[
k_1 > 0, \quad k_2 > |k_4|, \quad k_3 > 0, \quad 2k_1 > k_2 + k_4.
\]
Under which there are positive constants $a, C > 0$ such that the density $W(u, \nabla u)$ is equivalent to a form that satisfies
\[
a|p|^2 \leq W(z, p) \leq C|p|^2, \quad W_{\xi_i \xi_j}(z, p)\xi^k_i \xi^l_j \geq a|\xi|^2
\]
for any $\xi \in M^{3 \times 3}$, any $z \in \mathbb{R}$ and any $p \in M^{3 \times 3}$ (c.f. [9,17]). The Leslie coefficients $\alpha_i, i = 1, 2, \cdots, 6$, are assumed to satisfy the following conditions:
\[
\alpha_1 = \alpha_3 - \alpha_2 > 0, \quad \alpha_2 = \alpha_6 - \alpha_5, \quad \alpha_2 + \alpha_3 = \alpha_6 - \alpha_5,
\]
where the last equation is called the Parodi relation (c.f. [28]). Further, suppose that
\[
\alpha_1 \geq 0, \quad \alpha_4 > 0, \quad \beta := \alpha_5 + \alpha_6 - \frac{\gamma_2}{\gamma_1} \geq 0,
\]
which ensures the energy-dissipation law of the general Ericksen-Leslie system.

The Ericksen-Leslie system (1.1)-(1.3) has attracted much attention in recent years. For the two-dimensional case, Lin-Lin-Wang [22] and Hong [15] independently proved global existence and partial regularity of weak solutions to the simplified system; that is a special case where Frank’s elastic constants in the isotropic case satisfy $k_1 = k_2 = k_3 = 1, k_4 = 0$ and the Leslie tensor is ignored (other than $\alpha_4 \neq 0$). Hong-Xin [18] generalized these results to any positive $k_1, k_2, k_3$, but without the Leslie tensor. Later, Huang-Lin-Wang [19] and Wang-Wang [32] obtained similar results in $\mathbb{R}^2$ for the system (1.1)-(1.3) with the Leslie tensor. Lin-Wang [25], Li-Titi-Xin [21] and Wang-Wang-Zhang [33] established uniqueness of global weak solutions of the the system (1.1)-(1.3).

In three dimensions, the question on global existence of weak solutions to the Ericksen-Leslie system (1.1)-(1.3) remains open. Wen-Ding [36] established the local well-posedness of strong solutions to the simplified system without the Leslie stress tensor in the isotropic case ($k_1 = k_2 = k_3 = 1, k_4 = 0$ and only $\alpha_4 \neq 0$). Later, Fan-Guo [10] and Huang-Wang [14] studied the Serrin and BKM type blow-up criteria for this simplified system, using ideas originating from the celebrated Navier-Stokes equation. For the Ericksen-Leslie system with general Oseen-Frank density and without the Leslie tensor, Hong-Li-Xin [10] proved the local well-posedness and blow-up criterions of strong solutions with initial data $(v_0, u_0) \in H^1(\mathbb{R}^3, \mathbb{R}^3) \times H^2_0(\mathbb{R}^3, S^2)$. For rough initial data, Hineman-Wang [13] established the local well-posedness of solutions to the simplified system with initial velocity $v_0$ and director $u_0$ in uniformly local $L^3$-integrable spaces respectively. See also Wang [31] for the case with initial data in $BMO^{-1} \times BMO$. Recently, Hong-Mei [17] generalized the
result in \[127\] to the case of any positive \(k_1, k_2, k_3\) with initial data in uniformly local \(L^3\) spaces, but without the effect of the Leslie tensor.

Now, we consider the effect of Leslie stress tensor for the general Ericksen-Leslie system in three dimensions. Wang-Zhang-Zhang \[31\] and Wang-Wang \[32\] proved the local well-posedness of solutions to the general Ericksen-Leslie system with initial data \((v_0, u_0) \in H^{2s}(\mathbb{R}^3, \mathbb{R}^3) \times H^{2s}(\mathbb{R}^3, S^2)\) with \(s \geq 2\). In this article, we investigate the local well-posedness of strong solutions to the general Ericksen-Leslie system \(1.1-1.3\) with initial data \((v_0, u_0) \in H^1(\mathbb{R}^3, \mathbb{R}^3) \times H^2(\mathbb{R}^3, S^2)\).

For a given unit vector \(b \in S^2\) and \(m \in \mathbb{N}\), we denote
\[
H^m_b(\mathbb{R}^3; S^2) := \{ u : u - b \in H^m(\mathbb{R}^3, \mathbb{R}^3), |u| = 1 \text{ a.e. in } \mathbb{R}^3 \}.
\]

**Definition 1.** For any \(T > 0\), \((v, u)\) is called a strong solution to the system \(1.1-1.3\) in \(\mathbb{R}^3 \times (0, T)\) if it satisfies the system \(1.1-1.3\) for almost every \((x, t) \in \mathbb{R}^3 \times (0, T)\) and
\[
\begin{align*}
v &\in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)), & \partial_t v &\in L^2(0, T; L^2(\mathbb{R}^3)), & \text{div} v & = 0, \\
u &\in L^\infty(0, T; H^2(\mathbb{R}^3)) \cap L^2(0, T; H^3(\mathbb{R}^3)), & \partial_t u &\in L^2(0, T; H^1(\mathbb{R}^3)), & |u| & = 1.
\end{align*}
\]

Firstly, we prove the local well-posedness of strong solutions to \(1.1-1.3\):

**Theorem 1.** For any \(v_0 \in H^1(\mathbb{R}^3, \mathbb{R}^3)\) and \(u_0 \in H^2(\mathbb{R}^3, S^2)\) with \(\text{div} v_0 = 0\), there is a unique strong solution \((v, u)\) to the system \(1.1-1.3\) in \(\mathbb{R}^3 \times [0, T^*)\) with initial data \((v_0, u_0)\). Moreover, there are two positive constants \(\varepsilon_0\) and \(R_0\) such that at a singular point \(x_i\), the maximal existence time \(T^*\) satisfies
\[
\limsup_{t \to T^*} \int_{B_R(x_i)} |\nabla v(\cdot, t)|^3 + |v(\cdot, t)|^3 \, dx \geq \varepsilon_0,
\]
for any \(R > 0\) with \(R \leq R_0\).

In line with previous efforts, the proof of Theorem 1 utilizes the Ginzburg-Landau approximation. The Ginzburg-Landau functional was introduced in 1950 \(12\) to study the phase transition in superconductivity. For a parameter \(\varepsilon > 0\), the Ginzburg-Landau functional of \(u : \Omega \to \mathbb{R}^3\) is defined by
\[
E_\varepsilon(u; \Omega) := \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 \right) \, dx.
\]

There are many impressive results concerning convergence of the Ginzburg-Landau approximation system as \(\varepsilon \to 0\). In \([7]\), Chen-Struwe proved global existence of weak solutions to the heat flow of harmonic maps using the Ginzburg-Landau approximation. See a further result in \([5]\) on the convergence of the gradient flow of the Ginzburg-Landau approximation. On the other hand, Bethuel, Brezis and Hélein \([33, 34]\) proved asymptotic behavior for minimizers of \(E_\varepsilon\) in two dimensional star-shaped domains as \(\varepsilon \to 0\) (see also \([30]\) for the case of non-star-shaped domains). Motivated by above results, Liu-Liu \([23, 24]\) introduced the Ginzburg-Landau approximation system for the Ericksen-Leslie system
\[
\begin{align*}
\partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon + \nabla P_\varepsilon & = \nabla \cdot (\sigma_\varepsilon^E + \sigma_\varepsilon^L), \\
\nabla \cdot v_\varepsilon & = 0, \\
\gamma_1 N_\varepsilon + \gamma_2 A_\varepsilon u_\varepsilon & = h_\varepsilon
\end{align*}
\]
for $\varepsilon > 0$, where $u_\varepsilon$, $v_\varepsilon$ are the direction and velocity field of the Ginzburg-Landau system and $h_\varepsilon$ is given by

$$h_\varepsilon = \nabla \alpha \left( \frac{\partial W(u_\varepsilon, \nabla u_\varepsilon)}{\partial p_\alpha} - \frac{\partial W(u_\varepsilon, \nabla u_\varepsilon)}{\partial u_\varepsilon} + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \right).$$

Lin-Liu \cite{Lin-Liu} proved global existence of classical solutions in two dimensions and weak solutions in three dimensions to the Ginzburg-Landau system (see also \cite{Lin-Liu} for the $\gamma_2 \neq 0$ case). Lin-Liu \cite{Lin-Liu} also analyzed the limit of solutions ($v_\varepsilon, u_\varepsilon$) of the Ginzburg-Landau system as $\varepsilon \to 0$, but it is not clear that the limiting solution satisfies the original Ericksen-Leslie system with $|u| = 1$. In the study of numerical context, it is a widely used approach to handle the constraint of weak solutions in three dimensions to the Ginzburg-Landau system (see also \cite{Lin-Liu}). Hong \cite{Hong} and Hong-Xin \cite{Hong-Xin} proved the local existence of weak solutions to the simplified system in dimension three with initial velocity and director field in $H^1(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$. We would like to point out that by using the Ginzburg-Landau approximation, Lin-Wang \cite{Lin-Wang} proved global existence of weak solutions to the simplified system in dimension three with initial velocity and director in $L^2 \times H^1$ and hemisphere condition on the director. Building on the ideas of \cite{Lin-Liu} \cite{Hong} \cite{Hong-Xin}, we prove Theorem 1 by establishing the convergence of strong solutions to (1.14)-(1.16), when the Leslie stress tensor is present. One of the key ideas is that when $|u_\varepsilon|$ is close to 1, we handle the singular term $\frac{1 - |u_\varepsilon|^2}{\varepsilon^2}$ using (1.16).

Concerning the Lin-Liu problem on the convergence of the Ginzburg-Landau approximation, Hong-Li-Xin \cite{Hong-Li-Xin} proved the strong convergence of the Ginzburg-Landau approximation to the system (1.1)-(1.3) with the Leslie stress tensor. In this paper, we extend the result in \cite{Hong-Li-Xin} to the general Ericksen-Leslie system (1.1)-(1.4) with the Leslie stress tensor.

**Theorem 2.** For each $\varepsilon > 0$, there is a unique strong solution $(v_\varepsilon, u_\varepsilon)$ to the system (1.14)-(1.16) in $\mathbb{R}^3 \times [0, T^*_\varepsilon)$ with initial data $(v_0, u_0) \in H^1(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ satisfying $\text{div} v_0 = 0$, where $T^*_\varepsilon$ is the maximal existence time. Let $T^*$ be the maximal existence time of the strong solution $(v, u)$ to the system (1.1)-(1.3) with the same initial data $(v_0, u_0)$ in Theorem 1. Then, we have $(\nabla u, v) \in C^\infty(\tau, T; C^\infty_{loc}(\mathbb{R}^3))$ with any $(\tau, T) \subset (0, T^*)$. Moreover, for any $T \in (0, T^*)$, there exists a small positive $\varepsilon_T$ such that $T^*_\varepsilon \geq T$ for any $\varepsilon \leq \varepsilon_T$ and as $\varepsilon \to 0$,

$$(\nabla u_\varepsilon, v_\varepsilon) \to (\nabla u, v), \quad \text{in} \quad L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$$

and

$$(\nabla u_\varepsilon, v_\varepsilon) \to (\nabla u, v), \quad \text{in} \quad C^\infty(\tau, T; C^\infty_{loc}(\mathbb{R}^3))$$

for any $\tau > 0$.

We would like to point out that the smooth convergence in (1.19) is a new result even for the Ericksen-Leslie system without the Leslie stress tensor. One of the
key proofs to Theorem 2 is to establish Proposition 3.1 under the condition that \((v_0, u_0, \varepsilon)\) satisfies
\begin{equation}
\|u_0, v_0\|_{L^2(\mathbb{R}^3)} + \|v_0, \varepsilon\|_{H^1(\mathbb{R}^3)} + \frac{1}{\varepsilon^2}\|1 - |u_0, \varepsilon\|^2\|_{H^1(\mathbb{R}^3)} \leq M
\end{equation}
for a positive constant \(M\) independent of \(\varepsilon\). Note that the condition \((1.20)\) does not involve any condition on \(\|\partial_t u, \varepsilon\|_{L^2(\mathbb{R}^3)}\), which differs from the one in \([17]\). To prove Proposition 3.1 we establish a local estimate on the pressure in Lemma 2.3, and prove Theorem 1. In Section 4, we establish higher order estimates of the Ericksen-Leslie system \((1.1)-(1.3)\). In Section 3, we establish Proposition 3.1 and derive a local \(L^3\)-estimate using an interpolation inequality and a covering argument, which is similar to the argument in \([17]\). By applying Proposition 3.1 we prove that as \(\varepsilon \to 0\), the solutions \((v_\varepsilon, u_\varepsilon)\) of \((1.1)-(1.3)\) converge strongly to the solution \((v, u)\) of the system \((1.1)-(1.3)\) in \(\mathbb{R}^3 \times (0, T_M)\) with a uniform constant \(T_M > 0\) depending only on \(M\).

The second key proof to Theorem 2 is to derive sophisticated higher order estimates of \((v_\varepsilon, \nabla u_\varepsilon)\) with uniform bounds in \(\varepsilon\) in Lemma 4.2, which implies the smooth convergence results of Ginzburg-Landau approximation systems in \(\mathbb{R}^3 \times (0, T_M)\). Let \(T^*\) be the maximal existence time of the solution \((v, u)\) to the Ericksen-Leslie system. For any \(T < T^*\), we choose \(M = 2\sup_{0 \leq \varepsilon \leq T} \|(\nabla u, v)\|_{H^1(\mathbb{R}^3)}\). Then we combine the energy identities in Lemma 4.3 with the higher order estimates to verify that \((v_\varepsilon, u_\varepsilon)\) satisfies \((1.20)\) at \(t = T_M\). Therefore, the solutions \((v_\varepsilon, u_\varepsilon)\) to the Ginzburg-Landau system converge smoothly to the solution \((v, u)\) in \(\mathbb{R}^3 \times (0, 2T_M)\) for sufficiently small \(\varepsilon\). Finally, we establish the smooth convergence of solutions to Ginzburg-Landau approximation systems for any \(T < T^*\).

The paper is organized as follows. In Section 2, we obtain some a priori estimates of the Ericksen-Leslie system \((1.1)-(1.3)\). In Section 3, we establish Proposition 3.1 and prove Theorem 1. In Section 4, we establish higher order estimates of the Ginzburg-Landau approximation system and prove Theorem 2.

2. A PRIORI ESTIMATES

In this section, we derive a priori estimates for strong solutions to the Ginzburg-Landau system \((1.14)-(1.16)\). First, we note that the equation \((1.3)\) is equivalent to
\begin{equation}
\gamma_1 N + \gamma_2 (Au - (u^T Au)) = h - (u \cdot h)u
\end{equation}
by taking the vector cross product to \((1.3)\) with \(u\) and using the fact that \(|u| = 1\).

Then we have the following basic energy identity:

**Lemma 2.1.** Let \((v_\varepsilon, u_\varepsilon)\) be a strong solution to the system \((1.14)-(1.16)\) in \(\mathbb{R}^3 \times (0, T_\varepsilon)\). Then for any \(t \in (0, T_\varepsilon)\) we have
\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^3} \left(\frac{|v_\varepsilon|^2}{2} + W(u_\varepsilon, \nabla u_\varepsilon) + \frac{1}{4\varepsilon^2}(1 - |u_\varepsilon|^2)^2\right) dx + \alpha_4 \int_{\mathbb{R}^3} |A_\varepsilon|^2 dx
\end{equation}
\begin{equation}
+ \alpha_1 \int_{\mathbb{R}^3} |u_\varepsilon^T A_\varepsilon u_\varepsilon|^2 dx + \beta \int_{\mathbb{R}^3} |A_\varepsilon u_\varepsilon|^2 dx + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} |h_\varepsilon|^2 dx = 0.
\end{equation}

**Proof.** Multiplying \((1.14)\) by \(v_\varepsilon\), using \((1.13)\) and integrating by parts yield
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |v_\varepsilon|^2 dx + \int_{\mathbb{R}^3} \sigma^L_\varepsilon : \nabla v_\varepsilon dx = -\int_{\mathbb{R}^3} \sigma^B_\varepsilon : \nabla v_\varepsilon dx.
\end{equation}
Since $A_\varepsilon$ is symmetric and $\Omega_\varepsilon$ is antisymmetric, it follows from (1.16) and (1.10) and (1.17) that

$$
(2.4) \quad \int_{\mathbb{R}^3} \sigma_{\varepsilon}^T : \nabla u_\varepsilon \, dx = \int_{\mathbb{R}^3} \sigma_{\varepsilon}^T : (A_\varepsilon + \Omega_\varepsilon) \, dx
$$

$$
= \int_{\mathbb{R}^3} [\alpha_1 |u_\varepsilon^T A_\varepsilon u_\varepsilon|^2 + \alpha_4 |A_\varepsilon|^2 + (\alpha_5 + \alpha_6)|A_\varepsilon u_\varepsilon|^2 - (\gamma_1 N_\varepsilon + \gamma_2 A_\varepsilon u_\varepsilon) : (\Omega_\varepsilon u_\varepsilon) + \gamma_2 N_\varepsilon \cdot (A_\varepsilon u_\varepsilon)] \, dx
$$

$$
= \alpha_1 \int_{\mathbb{R}^3} |u_\varepsilon^T A_\varepsilon u_\varepsilon|^2 \, dx + \alpha_4 \int_{\mathbb{R}^3} |A_\varepsilon|^2 \, dx + (\alpha_5 + \alpha_6) \frac{\gamma_2^2}{\gamma_1} \int_{\mathbb{R}^3} |A_\varepsilon u_\varepsilon|^2 \, dx - \int_{\mathbb{R}^3} h_\varepsilon^T \Omega_\varepsilon u_\varepsilon \, dx - \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h_\varepsilon^T A_\varepsilon u_\varepsilon \, dx.
$$

Substituting (2.4) into (2.6) and using (1.10), we have

$$
(2.5) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |v_\varepsilon|^2 \, dx + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} |h_\varepsilon|^2 \, dx
$$

$$
= \int_{\mathbb{R}^3} \nabla_i u_\varepsilon^k W_{p_j^k} (u_\varepsilon, \nabla u_\varepsilon) \nabla_j v_\varepsilon \, dx + \int_{\mathbb{R}^3} h_\varepsilon^T \Omega_\varepsilon u_\varepsilon \, dx - \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h_\varepsilon^T A_\varepsilon u_\varepsilon \, dx.
$$

On the other hand, multiplying (1.10) by $\frac{1}{\gamma_1} h_\varepsilon$, integrating over $\mathbb{R}^3$ and using (1.10), we have

$$
(2.6) \quad - \int_{\mathbb{R}^3} \partial_t u_\varepsilon \cdot h_\varepsilon \, dx + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} |h_\varepsilon|^2 \, dx
$$

$$
= \int_{\mathbb{R}^3} (v_\varepsilon \cdot \nabla) u_\varepsilon \cdot h_\varepsilon \, dx - \int_{\mathbb{R}^3} h_\varepsilon^T \Omega_\varepsilon u_\varepsilon \, dx - \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h_\varepsilon^T A_\varepsilon u_\varepsilon \, dx.
$$

It follows from (1.17) and integration by parts that

$$
(2.7) \quad - \int_{\mathbb{R}^3} \partial_t u_\varepsilon \cdot h_\varepsilon \, dx = \int_{\mathbb{R}^3} (\partial_t \nabla_i u_\varepsilon^k W_{p_j^k} (u_\varepsilon, \nabla u_\varepsilon) + \partial_t u_\varepsilon^k W_{p_j^k} (u_\varepsilon, \nabla u_\varepsilon)) \, dx
$$

$$
- \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} \partial_t u_\varepsilon \cdot u_\varepsilon (1 - |u_\varepsilon|^2) \, dx
$$

$$
= \frac{d}{dt} \int_{\mathbb{R}^3} (W(u_\varepsilon, \nabla u_\varepsilon) + \frac{1}{4\varepsilon^2} (1 - |u_\varepsilon|^2)^2) \, dx.
$$

Using (1.13) and integration by parts, we have

$$
(2.8) \quad \int_{\mathbb{R}^3} (v_\varepsilon \cdot \nabla) u_\varepsilon \cdot h_\varepsilon \, dx
$$

$$
= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} (v_\varepsilon \cdot \nabla) u_\varepsilon \cdot u_\varepsilon (1 - |u_\varepsilon|^2) \, dx
$$

$$
+ \int_{\mathbb{R}^3} v_\varepsilon^k \nabla_k u_\varepsilon^j (\nabla_\alpha W_{p_j^k} (u_\varepsilon, \nabla u_\varepsilon) + W_{p_j^k} (u_\varepsilon, \nabla u_\varepsilon)) \, dx
$$

$$
= - \frac{1}{4\varepsilon^2} \int_{\mathbb{R}^3} (v_\varepsilon \cdot \nabla) (1 - |u_\varepsilon|^2)^2 \, dx - \int_{\mathbb{R}^3} \nabla_\alpha v_\varepsilon^k \nabla_k u_\varepsilon^j W_{p_j^k} (u_\varepsilon, \nabla u_\varepsilon) \, dx
$$

$$
- \int_{\mathbb{R}^3} v_\varepsilon^k \nabla_k \nabla_\alpha u_\varepsilon^j W_{p_j^k} (u_\varepsilon, \nabla u_\varepsilon) \, dx - \nabla_k u_\varepsilon^j W_{p_j^k} (u_\varepsilon, \nabla u_\varepsilon) \, dx
$$

$$
= - \int_{\mathbb{R}^3} \nabla_\alpha v_\varepsilon^k \nabla_k u_\varepsilon^j W_{p_j^k} (u_\varepsilon, \nabla u_\varepsilon) \, dx.
$$
Plugging (2.7) into (2.6) gives

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \left( W(u_\varepsilon, \nabla u_\varepsilon) + \frac{1}{4\varepsilon^2} \left(1 - \frac{|u_\varepsilon|^2}{\varepsilon^2}\right)^2 \right) dx + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx \\
= -\int_{\mathbb{R}^3} \nabla_\alpha \nabla^2_{\varepsilon k} \nabla_j \varphi \nabla^3_{\varepsilon} W_{\varepsilon}^j (u_\varepsilon, \nabla u_\varepsilon) dx - \int_{\mathbb{R}^3} h_\varepsilon^T \Omega_\varepsilon u_\varepsilon dx + \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h_\varepsilon^T A_\varepsilon u_\varepsilon dx.
\]

Therefore, summing (2.7) with (2.6) yields (2.8).

The following lemma gives the local energy-dissipation law of the Ginzburg-Landau system (1.14)-(1.16).

**Lemma 2.2.** Let \((v_\varepsilon, u_\varepsilon)\) be a strong solution to the system (1.14)-(1.16) in \(\mathbb{R}^3 \times (0, T_\varepsilon)\). Assume that \(\frac{1}{2} \leq |u_\varepsilon| \leq \frac{3}{2}\) in \(\mathbb{R}^3 \times (0, T_\varepsilon)\). Then for any \(\phi \in C_0^\infty(\mathbb{R}^3)\) and \(s \in (0, T_\varepsilon)\), we obtain

\[
\int_{\mathbb{R}^3} \left( |v_\varepsilon(x,s)|^2 + |\nabla u_\varepsilon(x,s)|^2 + \frac{(1 - |u_\varepsilon(x,s)|^2)^2}{\varepsilon^2} \right) \phi^2 dx \\
+ \int_0^s \int_{\mathbb{R}^3} \left( |\nabla v_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\partial_i u_\varepsilon|^2 + \frac{|\nabla (|u_\varepsilon|^2)|^2}{\varepsilon^2} \right) \phi^2 dx dt \\
\leq C \int_{\mathbb{R}^3} \left( |v_{0,\varepsilon}|^2 + |\nabla u_{0,\varepsilon}|^2 + \frac{(1 - |u_{0,\varepsilon}|^2)^2}{\varepsilon^2} \right) \phi^2 dx \\
+ C \int_0^s \int_{\mathbb{R}^3} (|P_\varepsilon - c_\varepsilon(t)| + |v_\varepsilon|^2) |\nabla \phi| \phi dx dt \\
+ C \int_0^s \int_{\mathbb{R}^3} (|v_\varepsilon|^2 + |\nabla u_\varepsilon|^2) |\nabla u_\varepsilon|^2 \phi^2 dx dt \\
+ C \int_0^s \int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^2 + |v_\varepsilon|^2) |\nabla \phi|^2 dx dt,
\]

where \(C\) is a positive constant independent of \(\varepsilon\) and \(c_\varepsilon(t) \in \mathbb{R}\). In particular, for any \(s \in (0, T_\varepsilon)\), we have

\[
\int_{\mathbb{R}^3} \left( |v_\varepsilon(x,s)|^2 + |\nabla u_\varepsilon(x,s)|^2 + \frac{(1 - |u_\varepsilon(x,s)|^2)^2}{\varepsilon^2} \right) dx \\
+ \int_0^s \int_{\mathbb{R}^3} \left( |\nabla v_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\partial_i u_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\nabla (|u_\varepsilon|^2)|^2 \right) dx dt \\
\leq C \int_{\mathbb{R}^3} |v_{0,\varepsilon}|^2 + |\nabla u_{0,\varepsilon}|^2 + \frac{(1 - |u_{0,\varepsilon}|^2)^2}{\varepsilon^2} dx + C \int_0^s \int_{\mathbb{R}^3} (|v_\varepsilon|^2 + |\nabla u_\varepsilon|^2) |\nabla u_\varepsilon|^2 dx dt.
\]

**Proof.** Multiplying (1.14) by \(v_\varepsilon^3 \phi^2\), integrating over \(\mathbb{R}^3\) and using the similar calculations in (2.3) and (2.4) yield

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \frac{|v_\varepsilon|^2}{2} \phi^2 dx + \int_{\mathbb{R}^3} \alpha_1 |u_\varepsilon|^2 A_\varepsilon u_\varepsilon |\phi|^2 dx + \alpha_4 |A_\varepsilon|^2 |\phi|^2 dx + \beta |A_\varepsilon u_\varepsilon|^2 |\phi|^2 dx \\
= \int_{\mathbb{R}^3} h_\varepsilon^T \Omega_\varepsilon u_\varepsilon \phi^2 dx - \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h_\varepsilon^T A_\varepsilon u_\varepsilon \phi^2 dx + 2 \int_{\mathbb{R}^3} (P_\varepsilon - c_\varepsilon(t)) v_\varepsilon \cdot \nabla \phi \phi dx \\
+ 2 \int_{\mathbb{R}^3} |v_\varepsilon|^2 v_\varepsilon \cdot \nabla \phi \phi dx - 2 \int_{\mathbb{R}^3} \sigma_\varepsilon^T : v_\varepsilon \otimes \nabla \phi \phi dx + \int_{\mathbb{R}^3} \nabla_i u_\varepsilon^3 \nabla_j v_\varepsilon^3 \phi^2 dx
\]
\[ + 2 \int_{\mathbb{R}^3} \nabla_i u^k e W_{fp} v_j \nabla_j \phi \, dx. \]

Multiplying (1.16) by \( \frac{1}{\gamma_1} h \phi^2 \), integrating over \( \mathbb{R}^3 \) and using the similar calculations in (2.7), one has

\[
\begin{align*}
(2.13) \quad & \quad \frac{d}{dt} \int_{\mathbb{R}^3} \left( W(u, \nabla u) + \frac{1}{4\varepsilon^2} (1 - |u|)^2 \right) \phi^2 \, dx + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} |h| \phi^2 \, dx \\
&= - \int_{\mathbb{R}^3} \frac{h^T \Omega e u \phi^2}{\gamma_1} \, dx + \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h^T A e u \phi^2 \, dx + \int_{\mathbb{R}^3} v_e \cdot \nabla u_e \cdot h \phi^2 \, dx \\
&\quad - 2 \int_{\mathbb{R}^3} \partial_t u^i W_{p} \nabla_{ii} \phi \, dx.
\end{align*}
\]

Summing (2.13) with (2.12) and using Young’s inequality yield

\[
(2.14) \quad \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{|v_e|^2}{2} + W(u, \nabla u) + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 \right) \phi^2 \, dx \\
\quad + \frac{\alpha_1}{2} \int_{\mathbb{R}^3} |\nabla v_e|^2 \phi^2 \, dx + \alpha_1 \int_{\mathbb{R}^3} |u^T A e u|^2 \phi^2 \, dx + \beta \int_{\mathbb{R}^3} |A e u|^2 \phi^2 \, dx \\
\leq C \int_{\mathbb{R}^3} (|\nabla v_e|^2 + \gamma_2 |v_e| |\nabla u_e|) \phi^2 \, dx + C \int_{\mathbb{R}^3} |\partial_t u_e| |\nabla v_e| \phi \, dx \\
\quad + C \int_{\mathbb{R}^3} (|\sigma_e^T| + |\nabla v_e| + |P_e - c_e(t)|) |v_e| |\nabla \phi| \phi \, dx \\
\leq \eta \int_{\mathbb{R}^3} |\partial_t u_e|^2 \phi^2 \, dx + \frac{\alpha_4}{4} \int_{\mathbb{R}^3} |\nabla v_e|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} (|\nabla v_e|^2 + |v_e|^2) |\nabla \phi|^2 \, dx \\
\quad + C \int_{\mathbb{R}^3} (|\nabla v_e|^2 + |v_e|^2) |\nabla u_e|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} (|P_e - c_e(t)| + |v_e|^2) |v_e| |\nabla \phi| \phi \, dx,
\]

where \( \eta \) will be chosen later and we have used the facts that

\[
|\sigma_e^T| + |\nabla v_e| \leq C(|A_e| + |\Omega_e| + |N_e|) \leq C(|\nabla v_e| + |\partial_t u_e| + |v_e| |\nabla u_e|)
\]

and

\[
\alpha_4 \int_{\mathbb{R}^3} |A_e|^2 \phi^2 \, dx \geq \frac{\alpha_4}{2} \int_{\mathbb{R}^3} |\nabla v_e|^2 \phi^2 \, dx - C \int_{\mathbb{R}^3} |v_e| |\nabla v_e| |\nabla \phi| \phi \, dx
\]

which follows from integration by parts and using (1.16).

In order to bound the term \( \int_{\mathbb{R}^3} |\partial_t u_e|^2 \phi^2 \, dx \) on the right hand side of (2.14), we multiply (1.16) by \( \partial_t u^i e \phi^2 \) and then integrate it over \( \mathbb{R}^3 \) to obtain

\[
- \int_{\mathbb{R}^3} h \cdot \partial_t u_e \phi^2 \, dx + \gamma_1 \int_{\mathbb{R}^3} |\partial_e u_e|^2 \phi^2 \, dx \\
= - \gamma_1 \int_{\mathbb{R}^3} v_e \cdot \nabla u_e \partial_t u_e \phi^2 \, dx + \int_{\mathbb{R}^3} (\gamma_1 \Omega_e u_e - \gamma_2 A e u_e) \cdot \partial_t u_e \phi^2 \, dx.
\]
It follows from similar calculations in (2.17) that
\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^3} \left( W(u_\varepsilon, \nabla u_\varepsilon) + \frac{1}{4\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \right) \phi^2 \, dx + \gamma_1 \int_{\mathbb{R}^3} |\partial_t u_\varepsilon|^2 \phi^2 \, dx \\
= - \gamma_1 \int_{\mathbb{R}^3} v_\varepsilon \cdot \nabla u_\varepsilon \partial_t u_\varepsilon \phi^2 \, dx - 2 \int_{\mathbb{R}^3} \partial_t W_{\partial u_\varepsilon} \nabla \phi \, dx \\
+ \int_{\mathbb{R}^3} (\gamma_1 \Omega_{\varepsilon} u_\varepsilon - \gamma_2 A_{\varepsilon} u_\varepsilon) \cdot \partial_t u_\varepsilon \phi^2 \, dx \\
\leq \frac{\gamma_1}{4} \int_{\mathbb{R}^3} |\partial_t u_\varepsilon|^2 \phi^2 \, dx + C_1 \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 \phi^2 \, dx \\
+ C \int_{\mathbb{R}^3} |u_\varepsilon|^2 |\nabla u_\varepsilon|^2 \phi^2 + |\nabla u_\varepsilon|^2 |\nabla \phi|^2 \, dx.
\end{equation}

To derive the estimate of \( \int_{\mathbb{R}^3} |\nabla^2 u_\varepsilon|^2 \phi^2 \, dx \), we multiply (1.16) with \( \frac{1}{\gamma_1} \Delta u_\varepsilon \phi^2 \) and integrate the resulting equation over \( \mathbb{R}^3 \). Then, one has
\begin{equation}
- \int_{\mathbb{R}^3} \partial_t u_\varepsilon \cdot \Delta u_\varepsilon \phi^2 \, dx + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} h_\varepsilon \cdot \Delta u_\varepsilon \phi^2 \, dx \\
= \int_{\mathbb{R}^3} v_\varepsilon \cdot \nabla u_\varepsilon \Delta u_\varepsilon \phi^2 \, dx + \int_{\mathbb{R}^3} \left( -\Omega_{\varepsilon} u_\varepsilon + \frac{\gamma_2}{\gamma_1} A_{\varepsilon} u_\varepsilon \right) \cdot \Delta u_\varepsilon \phi^2 \, dx.
\end{equation}

It follows from integration by parts that
\begin{equation}
\int_{\mathbb{R}^3} h_\varepsilon \cdot \Delta u_\varepsilon \phi^2 \, dx = \int_{\mathbb{R}^3} \left( \nabla \alpha W_{\partial u_\varepsilon} - W_{\partial u_\varepsilon} + \frac{1}{\varepsilon} u_\varepsilon (1 - |u_\varepsilon|^2) \right) \Delta u_\varepsilon \phi^2 \, dx \\
= \int_{\mathbb{R}^3} W_{\partial u_\varepsilon} \nabla_{\beta\alpha} u_\varepsilon \nabla_{\gamma\gamma} u_\varepsilon \phi^2 \, dx + \int_{\mathbb{R}^3} W_{\partial u_\varepsilon} \nabla_{\beta\alpha} u_\varepsilon \nabla_{\gamma\gamma} u_\varepsilon \phi^2 \, dx \\
+ 2 \int_{\mathbb{R}^3} W_{\partial u_\varepsilon} (\nabla_{\beta\alpha} u_\varepsilon \nabla_{\gamma\gamma} \phi - \Delta u_\varepsilon \nabla_{\gamma\gamma} \phi) \phi \, dx \\
- \int_{\mathbb{R}^3} W_{\partial u_\varepsilon} \Delta u_\varepsilon \phi^2 \, dx + \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^3} |\nabla (|u_\varepsilon|^2)|^2 \phi^2 \, dx \\
- \int_{\mathbb{R}^3} \frac{1 - |u_\varepsilon|^2}{\varepsilon^2} (|\nabla u_\varepsilon|^2 \phi^2 + \nabla (|u_\varepsilon|^2) \nabla \phi) \phi \, dx.
\end{equation}

and
\begin{equation}
- \int_{\mathbb{R}^3} \partial_t u_\varepsilon \cdot \Delta u_\varepsilon \phi^2 \, dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \phi^2 \, dx + 2 \int_{\mathbb{R}^3} \partial_t u_\varepsilon \cdot \nabla u_\varepsilon \cdot \nabla \phi \, dx.
\end{equation}

Collecting (2.16) – (2.18) and using (1.19) give
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \phi^2 \, dx + \int_{\mathbb{R}^3} \left( \frac{a}{\gamma_1} |\nabla^2 u_\varepsilon|^2 + \frac{1}{2\varepsilon^2} |\nabla u_\varepsilon|^2| \phi^2 \right) \, dx \\
\leq \int_{\mathbb{R}^3} v_\varepsilon \cdot \nabla u_\varepsilon \Delta u_\varepsilon \phi^2 \, dx + \int_{\mathbb{R}^3} \left( -\Omega_{\varepsilon} u_\varepsilon + \frac{\gamma_2}{\gamma_1} A_{\varepsilon} u_\varepsilon \right) \cdot \Delta u_\varepsilon \phi^2 \, dx \\
- 2 \int_{\mathbb{R}^3} \partial_t u_\varepsilon \nabla u_\varepsilon \phi \nabla \phi \, dx - \frac{1}{\gamma_1} \int_{\mathbb{R}^3} W_{\partial u_\varepsilon} (u_\varepsilon, \nabla u_\varepsilon) \nabla_{\beta\alpha} u_\varepsilon \nabla_{\gamma\gamma} \phi \phi \, dx \\
- \frac{2}{\gamma_1} \int_{\mathbb{R}^3} W_{\partial u_\varepsilon} (u_\varepsilon, \nabla u_\varepsilon) \nabla_{\beta\alpha} u_\varepsilon \nabla_{\gamma\gamma} \phi \phi \, dx + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} W_{\partial u_\varepsilon} \Delta u_\varepsilon \phi^2 \, dx \\
+ \frac{1}{\gamma_1} \int_{\mathbb{R}^3} \frac{1 - |u_\varepsilon|^2}{\varepsilon^2} (|\nabla u_\varepsilon|^2 \phi^2 + \nabla (|u_\varepsilon|^2) \nabla \phi \phi) \phi \, dx.
\end{equation}
Lemma 2.3. Let \((u_\varepsilon, v_\varepsilon)\) be a strong solution to the system \((1.14)-(1.16)\) on \(\mathbb{R}^3 \times (0, T_\varepsilon)\). Assume that \(\frac{1}{2} \leq |u_\varepsilon| \leq \frac{3}{2}\) on \(\mathbb{R}^3 \times (0, T_\varepsilon)\). Then for any \(\phi \in C_0^\infty(\mathbb{R}^3)\) and any \(s \in (0, T_\varepsilon)\) we have the estimate

\[
2 \int_{\mathbb{R}^3} \left( |\nabla u_\varepsilon(x, s)|^2 + |\nabla v_\varepsilon(x, s)|^2 + \frac{1}{\varepsilon^2} |\nabla (|u_\varepsilon(x, s)|^2)|^2 \right) \phi^2 dx \\
+ \int_0^s \int_{\mathbb{R}^3} \left( |\nabla^3 u_\varepsilon|^2 + |\nabla^2 v_\varepsilon|^2 + |\nabla \partial_t u_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\nabla^2 |u_\varepsilon|^2|^2 \right) \phi^2 dx dt \\
\leq C \int_{\mathbb{R}^3} \left( |\nabla^2 u_{0, \varepsilon}|^2 + |\nabla v_{0, \varepsilon}|^2 + \frac{1}{\varepsilon^2} |\nabla |u_{0, \varepsilon}|^2|^2 \right) \phi^2 dx \\
+ C \int_0^s \int_{\mathbb{R}^3} \left( |\nabla^3 u_\varepsilon|^2 + |\nabla^2 v_\varepsilon|^2 + |\nabla \partial_t u_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\nabla^2 |u_\varepsilon|^2|^2 \right) \phi^2 dx dt \\
+ C \int_0^s \int_{\mathbb{R}^3} \left( |\nabla u_\varepsilon|^4 + |v_\varepsilon|^4 + |\nabla v_\varepsilon|^4 + |\nabla^2 v_\varepsilon|^2 + |\nabla v_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 \right) (|\nabla \phi|^2 + |\nabla^2 \phi||\phi|) dx dt \\
+ C \int_0^s \int_{\mathbb{R}^3} |P_\varepsilon - c_\varepsilon(t)|^2 (|\nabla \phi|^2 + |\nabla^2 \phi||\phi|) dx dt,
\]

where \(C\) is a positive constant independent of \(\varepsilon\) and \(c_\varepsilon(t) \in \mathbb{R}\). In particular, for any \(s \in (0, T_\varepsilon)\), we have

\[
2 \int_{\mathbb{R}^3} \left( |\nabla u_\varepsilon(x, s)|^2 + |\nabla v_\varepsilon(x, s)|^2 + \frac{1}{\varepsilon^2} |\nabla (|u_\varepsilon(x, s)|^2)|^2 \right) dx \\
+ \int_0^s \int_{\mathbb{R}^3} \left( |\nabla^3 u_\varepsilon|^2 + |\nabla^2 v_\varepsilon|^2 + |\nabla \partial_t u_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\nabla^2 |u_\varepsilon|^2|^2 \right) dx dt \\
\leq C \int_{\mathbb{R}^3} \left( |\nabla^2 u_{0, \varepsilon}|^2 + |\nabla v_{0, \varepsilon}|^2 + \frac{1}{\varepsilon^2} |\nabla |u_{0, \varepsilon}|^2|^2 \right) dx \\
+ C \int_0^s \int_{\mathbb{R}^3} (|v_\varepsilon|^2 + |\nabla u_\varepsilon|^2)(|\nabla v_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2) dx dt.
\]
Proof. For simplicity, denote
\[
g_1 := |v_x|^2 + |A_x u_x|^2, 
g_2 := |v_x|^4 + |A_x u_x|^2.
\]
Multiplying equation (1.14) by \(\Delta v\eta\) (2.24) and integrating over R\(^3\) yield
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla v_x|^2 \phi^2 \, dx = \int_{\mathbb{R}^3} \sigma_x^T \cdot \nabla \Delta v_x \phi^2 \, dx - 2 \int_{\mathbb{R}^3} \partial_t v_x \nabla v_x \cdot \nabla \phi \phi \, dx
\]
(2.23)
\[
- 2 \int_{\mathbb{R}^3} (P_x - c_x(t)) \Delta v_x \nabla \phi \phi \, dx
\]
\[
+ \int_{\mathbb{R}^3} \Delta v_x \cdot (v_x \cdot \nabla v_x \phi^2 + 2\sigma_x^T \cdot \nabla \phi - \nabla \cdot \sigma_x \phi^2) \, dx
\]
\[
= : I_1 + I_2 + I_3 + I_4.
\]
By a similar argument to the one in (2.4), one has
\[
I_1 = \alpha_1 \int_{\mathbb{R}^3} (u_x^T A_x u_x) u_x \cdot u_x : \Delta A_x \phi^2 \, dx + \alpha_4 \int_{\mathbb{R}^3} A_x : \Delta A_x \phi^2 \, dx
\]
\[
+ \beta \int_{\mathbb{R}^3} A_x u_x \otimes u_x : \Delta A_x \phi^2 \, dx + \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h_x^T \Delta A_x u_x \phi^2 \, dx - \int_{\mathbb{R}^3} h_x^T \Delta \Omega_x u_x \phi^2 \, dx
\]
\[
= - \alpha_1 \int_{\mathbb{R}^3} |u_x^T |^2 \phi^2 \, dx - \alpha_4 \int_{\mathbb{R}^3} |A_x|^2 \phi^2 \, dx - \beta \int_{\mathbb{R}^3} |A_{xij} u_x^2 |^2 \phi^2 \, dx
\]
\[
- \alpha_1 \int_{\mathbb{R}^3} A_{x\ell k} \nabla A_{xij} \cdot \nabla (u_x^2 u_x^2 u_x^2 \phi^2) \, dx - 2 \alpha_4 \int_{\mathbb{R}^3} A_{xij} \nabla A_{xij} \cdot \nabla \phi \phi \, dx
\]
\[
- \beta \int_{\mathbb{R}^3} A_{x\ell k} \nabla (u_x^2 u_x^2 \phi^2) \nabla A_{xij} \, dx + \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h_x^T \Delta A_x u_x \phi^2 \, dx - \int_{\mathbb{R}^3} h_x^T \Delta \Omega_x u_x \phi^2 \, dx
\]
\[
\leq - \frac{\alpha_4}{2} \int_{\mathbb{R}^3} |\nabla v_x|^2 \phi^2 \, dx - \alpha_1 \int_{\mathbb{R}^3} |u_x^2 u_x^2 |^2 \phi^2 \, dx - \beta \int_{\mathbb{R}^3} |A_{xij} u_x^2 |^2 \phi^2 \, dx
\]
\[
+ \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h_x^T \Delta A_x u_x \phi^2 \, dx - \int_{\mathbb{R}^3} h_x^T \Delta \Omega_x u_x \phi^2 \, dx + C \int_{\mathbb{R}^3} g_1 + g_2 \, dx.
\]
In the view of (1.14), one has
\[
I_2 = - 2 \int_{\mathbb{R}^3} (\nabla \cdot (\sigma_x^F + \sigma_x^L) - (v_x \cdot \nabla) v_x - \nabla P_x) \nabla v_x \cdot \nabla \phi \phi \, dx
\]
\[
\leq C \int_{\mathbb{R}^3} |\nabla u_x| |\nabla v_x| + |\nabla v_x|^3 |\nabla v_x||\phi| \, dx - 2 \int_{\mathbb{R}^3} \nabla \cdot \sigma_x^L \nabla v_x \cdot \nabla \phi \phi \, dx
\]
\[
+ C \int_{\mathbb{R}^3} |v_x| |\nabla v_x|^2 |\nabla \phi||\phi| \, dx + C \int_{\mathbb{R}^3} |P_x - c_x(t)||\nabla v_x||\nabla (\nabla \phi) ||\phi| \, dx
\]
\[
\leq \frac{\alpha_4}{16} \int_{\mathbb{R}^3} |\nabla v_x|^2 \phi^2 \, dx + \eta_1 \int_{\mathbb{R}^3} |\nabla \partial_x u_x|^2 \phi^2 \, dx
\]
\[
+ C \int_{\mathbb{R}^3} |P_x - c_x(t)|^2 (|\nabla \phi||\phi| + |\nabla \phi||\phi|) \, dx + C \int_{\mathbb{R}^3} g_1 + g_2 \, dx,
\]
where \(\eta_1\) will be chosen later and we utilized the fact that
\[
(2.24) \quad |\nabla \cdot \sigma_x^L| \leq C(|\nabla u_x||\nabla v_x| + |\nabla \partial_x u_x| + |\nabla \partial_x u_x| + |\nabla u_x| |\nabla v_x| + |v_x| |\nabla^2 u_x|).
\]
It follows from Young’s inequality that
\[
I_3 + I_4 \leq \frac{\alpha_4}{16} \int_{\mathbb{R}^3} |\nabla^2 v_\varepsilon|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} \left| P_\varepsilon - c_\varepsilon(t) \right|^2 |\nabla \phi|^2 \, dx + C \int_{\mathbb{R}^3} g_1 + g_2 \, dx.
\]
Substituting \( I_1, I_2, I_3 \) and \( I_4 \) into (2.23) leads us to
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 \phi^2 \, dx + \frac{3\alpha_4}{8} \int_{\mathbb{R}^3} |\nabla^2 v_\varepsilon|^2 \phi^2 \, dx
\leq \eta_1 \int_{\mathbb{R}^3} |\nabla \partial_t u_\varepsilon|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} \left| P_\varepsilon - c_\varepsilon(t) \right|^2 (|\nabla^2 \phi| |\phi| + |\nabla \phi|^2) \, dx
+ \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h_\varepsilon^\gamma \Delta A_\varepsilon u_\varepsilon \phi^2 \, dx - \int_{\mathbb{R}^3} h_\varepsilon^\gamma \Delta \Omega_\varepsilon u_\varepsilon \phi^2 \, dx + C \int_{\mathbb{R}^3} g_1 + g_2 \, dx.
\]
Differentiating (1.10) in \( x_\beta \), multiplying the resulting equation by \( \frac{1}{\gamma_1} \nabla h_\varepsilon \phi^2 \) and integrating over \( \mathbb{R}^3 \), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \nabla \beta u_\varepsilon \cdot \nabla \beta h_\varepsilon \phi^2 \, dx + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} |\nabla h_\varepsilon|^2 \phi^2 \, dx
= -\frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h_\varepsilon^\gamma \Delta A_\varepsilon u_\varepsilon \phi^2 \, dx - \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} \nabla \beta A_\varepsilon \nabla \beta (u_\varepsilon \phi^2) h_\varepsilon \, dx
+ \int_{\mathbb{R}^3} \nabla \beta (v_\varepsilon \cdot \nabla u_\varepsilon) \cdot \nabla \beta h_\varepsilon \phi^2 \, dx =: J_1 + J_2 + J_3.
\]
For \( J_1 \), it follows from integration by parts that
\[
J_1 = -\frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h_\varepsilon^\gamma \Delta A_\varepsilon u_\varepsilon \phi^2 \, dx - \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} \nabla \beta A_\varepsilon \nabla \beta (u_\varepsilon \phi^2) h_\varepsilon \, dx
+ \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} A_\varepsilon \nabla \beta u_\varepsilon \cdot \nabla \beta h_\varepsilon \phi^2 \, dx
\leq -\frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} h_\varepsilon^\gamma \Delta A_\varepsilon u_\varepsilon \phi^2 \, dx + \frac{\alpha_4}{16} \int_{\mathbb{R}^3} |\nabla^2 v_\varepsilon|^2 \phi^2 \, dx
+ \frac{1}{8\gamma_1} \int_{\mathbb{R}^3} |\nabla h_\varepsilon|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} g_1 + g_2 \, dx,
\]
where we have used that \( |h_\varepsilon| \leq C(|\partial_t u_\varepsilon| + |v_\varepsilon||\nabla u_\varepsilon| + |\nabla v_\varepsilon|) \). Similarly, we have
\[
J_2 \leq \int_{\mathbb{R}^3} h_\varepsilon^\gamma \Delta \Omega_\varepsilon u_\varepsilon \phi^2 \, dx + \frac{\alpha_4}{16} \int_{\mathbb{R}^3} |\nabla^2 v_\varepsilon|^2 \phi^2 \, dx
+ \frac{1}{4\gamma_1} \int_{\mathbb{R}^3} |\nabla h_\varepsilon|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} g_1 + g_2 \, dx.
\]
It follows from Young’s inequality that
\[
J_3 \leq \frac{1}{8\gamma_1} \int_{\mathbb{R}^3} |\nabla h_\varepsilon|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} g_1 + g_2 \, dx.
\]
Next we estimate \( J_0 := -\int_{\mathbb{R}^3} \partial_t \nabla \beta u_\varepsilon \cdot \nabla \beta h_\varepsilon \phi^2 \, dx \) on the left hand side of (2.20). By using (1.17), one has
\[
J_0 = -\int_{\mathbb{R}^3} \nabla \beta \partial_t u_\varepsilon \nabla^2 \alpha \frac{W_{p_\varepsilon}}{2} \phi^2 \, dx + \int_{\mathbb{R}^3} \nabla \beta \partial_t u_\varepsilon \nabla \beta W_{u_\varepsilon} \phi^2 \, dx
- \int_{\mathbb{R}^3} \nabla \beta \partial_t u_\varepsilon \nabla \beta \left( \frac{u_\varepsilon (1 - |v_\varepsilon|^2)}{\varepsilon^2} \right) \phi^2 \, dx =: J_{0,1} + J_{0,2} + J_{0,3}.
\]
It follows from integration by parts and (1.3) that

\begin{equation}
\tag{2.31}
J_{0,1} = \int_{\mathbb{R}^3} \partial_t \nabla^2_{\alpha \beta} u^\epsilon \nabla^2_{\beta \gamma} u^\epsilon \phi^2 dx + \int_{\mathbb{R}^3} \partial_t \nabla^2_{\alpha \beta} u^\epsilon \nabla^2_{\beta \gamma} u^\epsilon \phi^2 dx
+ 2 \int_{\mathbb{R}^3} \partial_t \nabla_{\alpha} u^\epsilon \partial_{\beta} W_{\alpha \beta} \nabla_{\alpha} \phi^2 dx
= \frac{d}{dt} \int_{\mathbb{R}^3} \nabla^2_{\alpha \beta} u^\epsilon \nabla^2_{\beta \gamma} u^\epsilon \phi^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} \nabla_{\alpha} u^\epsilon \nabla_{\beta} \phi^2 dx + 2 \int_{\mathbb{R}^3} \partial_t \nabla_{\alpha} u^\epsilon \partial_{\beta} W_{\alpha \beta} \nabla_{\alpha} \phi^2 dx
\end{equation}

\begin{align*}
&\geq \frac{d}{dt} \int_{\mathbb{R}^3} \nabla^2_{\alpha \beta} u^\epsilon \nabla^2_{\beta \gamma} u^\epsilon \phi^2 dx - \eta_1 \int_{\mathbb{R}^3} |\partial_t u^\epsilon|^2 \phi^2 dx
- \eta_2 \int_{\mathbb{R}^3} |\nabla^3 u^\epsilon|^2 \phi^2 dx - C \int_{\mathbb{R}^3} g_1 + g_2 dx.
\end{align*}

For $J_{0,2}$, Young’s inequality implies

\begin{equation}
\tag{2.32}
J_{0,2} \geq -\eta_1 \int_{\mathbb{R}^3} |\partial_t u^\epsilon|^2 \phi^2 dx - C \int_{\mathbb{R}^3} g_1 + g_2 dx.
\end{equation}

For $J_{0,3}$, noting that (2.24) and the inequality

\begin{equation}
\tag{2.33}
\left| \nabla \left( \frac{|u^\epsilon|^2}{\epsilon^2} \right) \right| \leq C \left( |\nabla \partial_t u^\epsilon| + |\nabla^2 u^\epsilon| + |\nabla^3 u^\epsilon| \right) + C |v^\epsilon| \left( |\nabla u^\epsilon|^2 + |\nabla^2 u^\epsilon| \right)
+ C |\nabla u^\epsilon| \left( |\nabla u^\epsilon|^2 + |\nabla^2 u^\epsilon| + |\partial_t u^\epsilon| + |\nabla u^\epsilon| \right)
\end{equation}

from (1.16) and the assumption $\frac{1}{2} \leq |u^\epsilon| \leq \frac{3}{2}$, one has

\begin{equation}
\tag{2.34}
J_{0,3} = \int_{\mathbb{R}^3} \nabla_{\beta} \left( \frac{|u^\epsilon|^2}{\epsilon^2} \right) \partial_t \nabla_{\alpha} \nabla_{\beta} u^\epsilon \phi^2 dx + \int_{\mathbb{R}^3} \frac{1}{\epsilon^2} \nabla_{\beta} \left( \frac{|u^\epsilon|^2}{\epsilon^2} \right) \partial_t u^\epsilon \nabla_{\alpha} \nabla_{\beta} u^\epsilon \phi^2 dx
- \int_{\mathbb{R}^3} \frac{1}{\epsilon^2} \nabla_{\beta} u^\epsilon \partial_t \nabla_{\alpha} u^\epsilon \phi^2 dx
\geq \frac{2}{\epsilon^2} \int_{\mathbb{R}^3} \nabla \left( |u^\epsilon|^2 \right) |\nabla u^\epsilon|^2 \phi^2 dx - \frac{\alpha_4}{4} \int_{\mathbb{R}^3} |\nabla^2 u^\epsilon|^2 \phi^2 dx
- \eta_1 \int_{\mathbb{R}^3} |\partial_t \nabla u^\epsilon|^2 \phi^2 dx - \eta_2 \int_{\mathbb{R}^3} |\nabla^3 u^\epsilon|^2 \phi^2 dx - C \int_{\mathbb{R}^3} g_1 + g_2 dx.
\end{equation}

Plugging (2.31), (2.32) and (2.34) into (2.30), one has

\begin{equation}
\tag{2.35}
J_{0} \geq \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{1}{2} \nabla_{\alpha \beta} u^\epsilon W_{\alpha \beta} \nabla_{\beta \gamma} u^\epsilon \phi^2 dx + \frac{|\nabla \left( |u^\epsilon|^2 \right)|^2}{2 \epsilon^2} \phi^2 dx
- \int_{\mathbb{R}^3} \left( 3\eta_1 |\partial_t \nabla u^\epsilon|^2 + 2\eta_2 |\nabla^3 u^\epsilon|^2 \right) \phi^2 dx - C \int_{\mathbb{R}^3} g_1 + g_2 dx.
\end{equation}
Substituting (2.27), (2.28), (2.29), (2.35) into (2.26), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{1}{2} \sum a_{ij} W_{\mu,\mu} \nabla u_i \nabla u_j + \frac{1}{4\varepsilon^2} |\nabla u|^2 \right) \phi^2 dx + \frac{1}{2\gamma_1} \int_{\mathbb{R}^3} \nabla h \phi^2 dx \\
\leq 3\alpha_4 \int_{\mathbb{R}^3} \left[ \nabla^2 u \phi^2 dx + 3\eta_1 \int_{\mathbb{R}^3} |\nabla \partial_t u|^2 \phi^2 dx + 2\eta_2 \int_{\mathbb{R}^3} \nabla^3 u \phi^2 dx \\
- \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} \Delta \nabla u \phi^2 dx + \int_{\mathbb{R}^3} h^T \Delta \nabla u \phi^2 dx + C \int_{\mathbb{R}^3} g_1 + g_2 dx.
\]

which, summing with (2.23), integrating over [0, t] and then using (1.9), yields

\[
\int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u(x, s)|^2 + \frac{a}{2} |\nabla^2 u(x, s)|^2 + \frac{1}{4\varepsilon^2} |\nabla (|u|^2)|^2 \right) \phi^2 dx \\
+ \int_{0}^{t} \int_{\mathbb{R}^3} \left( \frac{\alpha_1}{8} |\nabla^2 u|^2 + \alpha_1 |u|^2 \nabla \Delta u \phi^2 + \beta |\nabla \Delta u|^2 \phi^2 \right) dx dt \\
\leq 4\eta_1 \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla \partial_t u|^2 \phi^2 dx dt + 2\eta_2 \int_{0}^{t} \int_{\mathbb{R}^3} \nabla^2 u \phi^2 dx dt + C \int_{0}^{t} \int_{\mathbb{R}^3} g_1 + g_2 dx dt \\
+ C \int_{\mathbb{R}^3} \left( |\nabla v_0|^2 + |\nabla^2 u_0|^2 \phi^2 + \frac{|\nabla (|u_0|^2)|^2}{\varepsilon^2} \right) \phi^2 dx.
\]

In view of (2.36), it remains to estimate $|\nabla \partial_t u|^2$ and $|\nabla^2 u|^2$. Differentiating (1.16) in $x$, multiplying the resulting equation by $\nabla \partial_t u \phi^2$ and integrating over $\mathbb{R}^3$, we have

\[
- \int_{\mathbb{R}^3} \nabla h \cdot \nabla \partial_t u \phi^2 dx + \gamma_1 \int_{\mathbb{R}^3} |\nabla \partial_t u|^2 \phi^2 dx \\
= \int_{\mathbb{R}^3} (\nabla \partial_t (\gamma_1 \nabla u \cdot \nabla u - \gamma_2 \nabla u) - \gamma_1 \nabla \partial_t (v \cdot \nabla u) \cdot \nabla \partial_t u \phi^2 dx \\
\leq \eta_2 \int_{\mathbb{R}^3} |\nabla \partial_t u|^2 \phi^2 dx + C_1 \int_{\mathbb{R}^3} |\nabla^2 u|^2 \phi^2 dx + C \int_{\mathbb{R}^3} g_1 + g_2 dx.
\]

Plugging (2.35) into (2.37) yields

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{1}{2} \sum a_{ij} W_{\mu,\mu} \nabla u_i \nabla u_j + \frac{|\nabla (|u|^2)|^2}{4\varepsilon^2} \right) \phi^2 dx + \gamma_1 \int_{\mathbb{R}^3} |\nabla \partial_t u|^2 \phi^2 dx \\
\leq \int_{\mathbb{R}^3} (3\eta_1 |\nabla \partial_t u|^2 + 3\eta_2 |\nabla^2 u|^2) \phi^2 dx + C_1 \int_{\mathbb{R}^3} |\nabla^2 u|^2 \phi^2 dx + C \int_{\mathbb{R}^3} g_1 + g_2 dx.
\]

On the other hand, taking a derivative $\nabla \partial_t$ of (1.16), multiplying the resulting equation by $\nabla \partial_t \Delta u \phi^2$ and integrating over $\mathbb{R}^3$, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^2 u|^2 \phi^2 dx + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} \nabla h \cdot \nabla \partial_t \Delta u \phi^2 dx \\
= \int_{\mathbb{R}^3} \left( \nabla \partial_t (v \cdot \nabla u - \nabla \Delta u) + \frac{\gamma_2}{\gamma_1} \nabla \partial_t (A u) \right) \cdot \nabla \partial_t \Delta u \phi^2 dx \\
\leq \frac{a}{8\gamma_1} \int_{\mathbb{R}^3} |\nabla^3 u|^2 \phi^2 dx + C_2 \int_{\mathbb{R}^3} |\nabla^2 u|^2 \phi^2 dx + C \int_{\mathbb{R}^3} g_1 + g_2 dx.
\]
For the term \( K_0 := \int_{\mathbb{R}^3} \nabla \beta h_{\varepsilon} \cdot \nabla \beta \Delta u_{\varepsilon} \phi^2 \, dx \), it follows from (1.17) that

\[
(2.40) \quad K_0 = \int_{\mathbb{R}^3} \nabla^2 \beta W_{p_{\alpha}} u_{\varepsilon} \cdot \nabla \beta \Delta u_{\varepsilon} \phi^2 \, dx - \int_{\mathbb{R}^3} \nabla \beta W_{u_{\varepsilon}} \nabla \beta \Delta u_{\varepsilon} \phi^2 \, dx \\
+ \int_{\mathbb{R}^3} \nabla \beta \left( \frac{1 - |u_{\varepsilon}|^2}{\varepsilon^2} \right) \nabla \beta \Delta u_{\varepsilon} \phi^2 \, dx =: K_{0,1} + K_{0,2} + K_{0,3}.
\]

Note that

\[
\nabla^2 \beta W_{p_{\alpha}} (u_{\varepsilon}, \nabla u_{\varepsilon}) = W_{p_{\alpha}} (u_{\varepsilon}, \nabla u_{\varepsilon}) \nabla^3 \beta u_{\varepsilon}^3 + W_{u_{\varepsilon}} (u_{\varepsilon}, \nabla u_{\varepsilon}) \nabla^2 \beta u_{\varepsilon}^2 \\
+ W_{u_{\varepsilon}} (u_{\varepsilon}, \nabla u_{\varepsilon}) \nabla \Delta u_{\varepsilon} u_{\varepsilon} u_{\varepsilon} + W_{u_{\varepsilon} u_{\varepsilon}} (u_{\varepsilon}, \nabla u_{\varepsilon}) \nabla \beta u_{\varepsilon}^4 \\
+ W_{u_{\varepsilon} u_{\varepsilon}} (u_{\varepsilon}, \nabla u_{\varepsilon}) \nabla^2 \beta u_{\varepsilon}^2.
\]

Using integration by parts twice and using (1.3) yield

\[
(2.41) \quad K_{0,1} \geq \int_{\mathbb{R}^3} \nabla^2 \beta W_{p_{\alpha}} \nabla^3 \beta \nabla u_{\varepsilon} \phi^2 \, dx - C \int_{\mathbb{R}^3} \nabla W_{p_{\alpha}} \nabla^3 u_{\varepsilon} \phi \phi \, dx \\
- \frac{a}{4} \int_{\mathbb{R}^3} \nabla^3 u_{\varepsilon} \phi \phi \, dx - C \int_{\mathbb{R}^3} g_1 + g_2 \, dx \\
\geq \frac{3a}{4} \int_{\mathbb{R}^3} \nabla^3 u_{\varepsilon} \phi \phi \, dx - C \int_{\mathbb{R}^3} g_1 + g_2 \, dx.
\]

For \( K_{0,2} \), it follows from Young’s inequality that

\[
(2.42) \quad K_{0,2} \geq -\frac{a}{8} \int_{\mathbb{R}^3} \nabla^3 u_{\varepsilon} \phi \phi \, dx - C \int_{\mathbb{R}^3} \left( |\nabla u_{\varepsilon}|^2 |\nabla^2 u_{\varepsilon}|^2 + |\nabla u_{\varepsilon}|^6 \right) \phi \phi \, dx.
\]

The term \( K_{0,3} \) can be controlled as follows. Since

\[
u_{\varepsilon} \cdot \nabla \beta \Delta u_{\varepsilon} = \frac{1}{2} \nabla \beta \Delta (|u_{\varepsilon}|^2) - \nabla \beta (|\nabla u_{\varepsilon}|^2) - \nabla \beta u_{\varepsilon} \cdot \Delta u_{\varepsilon},
\]

we obtain from integration by parts that

\[
K_{0,3} = \int_{\mathbb{R}^3} \nabla \beta \left( \frac{1 - |u_{\varepsilon}|^2}{\varepsilon^2} \right) u_{\varepsilon} \nabla \beta \Delta u_{\varepsilon} \phi \phi \, dx + \int_{\mathbb{R}^3} \frac{1 - |u_{\varepsilon}|^2}{\varepsilon^2} \nabla \beta u_{\varepsilon} \nabla \beta \Delta u_{\varepsilon} \phi \phi \, dx \\
- \int_{\mathbb{R}^3} \nabla \beta \left( \frac{1 - |u_{\varepsilon}|^2}{\varepsilon^2} \right) \left( \frac{1}{2} \nabla \beta \Delta (|u_{\varepsilon}|^2) - \nabla \beta (|\nabla u_{\varepsilon}|^2) - \nabla \beta u_{\varepsilon} \cdot \Delta u_{\varepsilon} \phi \phi \, dx \\
+ \int_{\mathbb{R}^3} \frac{1 - |u_{\varepsilon}|^2}{\varepsilon^2} \nabla \beta u_{\varepsilon} \nabla \beta \Delta \phi \phi \, dx \\
- \int_{\mathbb{R}^3} \nabla \beta \left( \frac{1 - |u_{\varepsilon}|^2}{\varepsilon^2} \right) \left( \nabla \beta (|\nabla u_{\varepsilon}|^2) + \nabla \beta u_{\varepsilon} \cdot \Delta u_{\varepsilon} \phi \phi \right. \\
+ \int_{\mathbb{R}^3} \frac{1 - |u_{\varepsilon}|^2}{\varepsilon^2} \nabla \beta u_{\varepsilon} \nabla \beta \Delta u_{\varepsilon} \phi \phi \, dx.
\]
Then, by using (2.20), (2.33) and Young’s inequality, it is clear that

\begin{equation}
K_{0,3} \geq \int_{\mathbb{R}^3} \frac{|\nabla^2 (|u_\varepsilon|^2)|^2}{2\varepsilon^2} \phi^2 \, dx - \frac{a}{8} \int_{\mathbb{R}^3} |\nabla^3 u_\varepsilon|^2 \phi^2 \, dx - \eta_1 \int_{\mathbb{R}^3} |\nabla \partial_t u_\varepsilon|^2 \phi^2 \, dx \\
- C_2 \int_{\mathbb{R}^3} |\nabla^2 v_\varepsilon|^2 \phi^2 \, dx - C \int_{\mathbb{R}^3} g_1 + g_2 \, dx.
\end{equation}

Substituting (2.41) - (2.42) and (2.43) into (2.40), we have

\begin{equation}
K_0 \geq \frac{a}{2} \int_{\mathbb{R}^3} |\nabla^3 u_\varepsilon|^2 \phi^2 \, dx + \int_{\mathbb{R}^3} \frac{|\nabla^2 (|u_\varepsilon|^2)|^2}{2\varepsilon^2} \phi^2 \, dx \\
- \eta_1 \int_{\mathbb{R}^3} |\nabla \partial_t u_\varepsilon|^2 \phi^2 \, dx - C_2 \int_{\mathbb{R}^3} |\nabla^2 v_\varepsilon|^2 \phi^2 \, dx - C \int_{\mathbb{R}^3} g_1 + g_2 \, dx.
\end{equation}

Collecting (2.44) with (2.39), one has

\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^2 u_\varepsilon|^2 \phi^2 \, dx + \frac{3a}{8\gamma_1} \int_{\mathbb{R}^3} |\nabla^3 u_\varepsilon|^2 \phi^2 \, dx + \frac{1}{2\gamma_1} \int_{\mathbb{R}^3} \frac{|\nabla^2 (|u_\varepsilon|^2)|^2}{\varepsilon^2} \phi^2 \, dx \\
\leq \eta_1 \int_{\mathbb{R}^3} |\nabla \partial_t u_\varepsilon|^2 \phi^2 \, dx + 2C_2 \int_{\mathbb{R}^3} |\nabla^2 v_\varepsilon|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} g_1 + g_2 \, dx.
\end{equation}

Summing (2.45) with (2.38), integrating over $[0, s]$ and using (1.9) yield

\begin{equation}
\frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{1 + a}{2} |\nabla^2 u_\varepsilon(x, s)|^2 + \frac{|\nabla (|u_\varepsilon(x, s)|^2)|^2}{4\varepsilon^2} \right) \phi^2 \, dx \\
+ \int_0^s \int_{\mathbb{R}^3} \left( \gamma_1 |\nabla \partial_t u_\varepsilon|^2 + \frac{3a}{8\gamma_1} |\nabla^3 u_\varepsilon|^2 + \frac{1}{2\gamma_1} \frac{|\nabla^2 (|u_\varepsilon|^2)|^2}{\varepsilon^2} \right) \phi^2 \, dx dt \\
\leq C \int_{\mathbb{R}^3} \left( |\nabla^2 u_{0,\varepsilon}|^2 + \frac{|\nabla (|u_{0,\varepsilon}|^2)|^2}{\varepsilon^2} \right) \phi^2 \, dx \\
+ 4\eta_1 \int_0^s \int_{\mathbb{R}^3} |\nabla \partial_t u_\varepsilon|^2 \phi^2 \, dx dt + 3\eta_2 \int_0^s \int_{\mathbb{R}^3} |\nabla^3 u_\varepsilon|^2 \phi^2 \, dx dt \\
+ (C_1 + 2C_2) \int_0^s \int_{\mathbb{R}^3} |\nabla^2 v_\varepsilon|^2 \phi^2 \, dx dt + C \int_0^s \int_{\mathbb{R}^3} g_1 + g_2 \, dx dt.
\end{equation}

Multiplying (2.30) by $C_3 = 8\alpha_3^{-1}(C_1 + 2C_2)$, summing with (2.46), and then choosing small constants $\eta_1 = \gamma_1 (8(C_3 + 1))^{-1}$ and $\eta_2 = a(8\gamma_1(2C_3 + 3))^{-1}$, we obtain

\begin{equation}
\int_{\mathbb{R}^3} (|\nabla v_\varepsilon(x, s)|^2 + |\nabla^2 u_\varepsilon(x, s)|^2 + \frac{1}{\varepsilon^2} |\nabla (|u_\varepsilon(x, s)|^2)|^2) \phi^2 \, dx \\
+ \int_0^s \int_{\mathbb{R}^3} \left( |\nabla^2 v_\varepsilon|^2 + |\nabla \partial_t u_\varepsilon|^2 + |\nabla^3 u_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\nabla^2 |u_\varepsilon|^2|^2 \right) \phi^2 \, dx dt \\
\leq C \int_{\mathbb{R}^3} (|\nabla v_{0,\varepsilon}|^2 + |\nabla^2 u_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} |\nabla (|u_{0,\varepsilon}|^2)|^2) \phi^2 \, dx + C \int_0^s \int_{\mathbb{R}^3} g_1 + g_2 \, dx dt.
\end{equation}
Note that integration by parts and Young’s inequality yield
\begin{equation}
(2.48) \int_{\mathbb{R}^3} (|v_e|^2 + |\nabla u_e|^2) |\nabla u_e|^2 \phi^2 \, dx \\
= - \int_{\mathbb{R}^3} (u^2_e - b) \nabla_i (\nabla_i u_e |\nabla u_e|^2 \phi^2 + \nabla_i u_e |v_e|^2 |\nabla u_e|^2 \phi^2) \, dx \\
\leq C \int_{\mathbb{R}^3} (|\nabla^2 u_e|^2 |\nabla u_e|^4 + |v_e|^2 |\nabla v_e|^2 |\nabla u_e|^3 + |v_e|^2 |\nabla u_e|^2 |\nabla^2 u_e| \phi^2) \, dx \\
+ C \int_{\mathbb{R}^3} (|\nabla u_e|^5 + |v_e|^2 |\nabla u_e|^3) |\nabla \phi| \phi \, dx \\
\leq \frac{1}{2} \int_{\mathbb{R}^3} (|v_e|^2 + |\nabla u_e|^2) |\nabla u_e|^4 \phi^2 \, dx + C \int_{\mathbb{R}^3} |\nabla u_e|^2 (|\nabla^2 u_e|^2 + |\nabla v_e|^2) \phi^2 \, dx \\
+ C \int_{\mathbb{R}^3} |v_e|^2 |\nabla^2 u_e|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} (|v_e|^2 + |\nabla u_e|^2) |\nabla u_e|^2 |\nabla \phi| \phi \, dx.
\end{equation}

Therefore, this lemma follows from (2.48) and (2.49).

The following lemma gives a local estimate of pressure under a smallness assumption, see [17] for similar arguments.

**Lemma 2.4.** Let \((u_e, v_e)\) be a strong solution to (1.1a)-(1.1b) \(R^3 \times (0, T_e)\) and \(\phi\) be a cut-off function satisfying \(0 \leq \phi \leq 1, \text{supp } \phi \subset B_{2R}(x_0)\) for some \(x_0 \in \mathbb{R}^3\) and \(|\nabla \phi| \leq \frac{C}{R}\). For any \(s \in (0, T_e)\), assume that \(\frac{1}{2} \leq |u_e| \leq \frac{2}{3}\) and
\begin{equation}
(2.49) \sup_{0 \leq t \leq s} \int_{B_R(x_0)} |\nabla u_e| (x, t)|^3 + |v_e(x, t)|^3 \, dx \leq \delta^3.
\end{equation}

Then for any \(t \in (0, T_e)\), there exists a \(c_e(t) \in \mathbb{R}\) such that the pressure \(P_e\) satisfies the following estimate
\begin{equation}
(2.50) \int_0^s \int_{\mathbb{R}^3} |P_e - c_e(t)|^2 \phi^2 \, dx dt \\
\leq C \sup_{x_0 \in \mathbb{R}^3} \int_0^s \int_{B_R(x_0)} \frac{\delta^2}{R^2} (|\nabla u_e|^2 + |v_e|^2) + \delta^2 (|\nabla^2 u_e|^2 + |\nabla v_e|^2) \, dx dt \\
+ C \sup_{x_0 \in \mathbb{R}^3} \int_0^s \int_{B_R(x_0)} (|\partial_t u_e|^2 + |\nabla v_e|^2) \, dx dt.
\end{equation}

**Proof.** Adapting the proof from Lemma 2.3 [17], we take divergence on both sides of (1.1a), then the pressure \(P_e\) satisfies the elliptic equation
\begin{equation}
(2.51) -\Delta P_e = \nabla^2_{ij} [\nabla_i u_e W_{i,j}^e(u_e, \nabla u_e) + v_i^j v_j^e - \sigma^e] \quad \text{on } \mathbb{R}^3 \times [0, T_e],
\end{equation}
which implies
\[
P_e = R_i R_j (F^{ij}) = \nabla_i u_e W_{i,j}(u_e, \nabla u_e) + v_i^j v_j^e - \sigma^e,
\]
where \(R_i\) is the \(i\)-th Riesz transform on \(\mathbb{R}^3\). Then we have
\begin{equation}
(2.52) (P_e - c_e(t)) \phi = R_i R_j (F^{ij}) \phi + [\phi, R_i R_j] (F^{ij}) - c_e \phi
\end{equation}
for a cut-off function \(\phi\), where the commutator \([\phi, R_i R_j]\) is defined by
\[
[\phi, R_i R_j](\cdot) = \phi R_i R_j(\cdot) - R_i R_j(\cdot) \phi.
\]
Since
\[
|F^{ij}| \leq C(|\nabla u_e|^2 + |v_e|^2 + |\partial_t u_e| + |\nabla v_e|)
\]
and the Riesz operator maps $L^q$ into $L^q$ spaces for any $1 < q < +\infty$, we have

\begin{align*}
(2.53) \quad &\int_0^s \int_{\mathbb{R}^3} |\mathcal{R}_i \mathcal{R}_j(F^{ij})|^2 \ dx \ dt \\
&\leq C \int_0^s \int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^4 + |v_\varepsilon|^4 + |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) \phi^2 \ dx \ dt \\
&\leq C \delta^2 \int_0^s \int_{B_{2R}(x_0)} |\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 \ dx \ dt + \frac{C \delta^2}{R^2} \int_0^s \int_{B_{2R}(x_0)} |\nabla u_\varepsilon|^2 + |v_\varepsilon|^2 \ dx \ dt \\
&\quad + C \int_0^s \int_{B_{2R}(x_0)} |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 \ dx \ dt,
\end{align*}

where we have utilized the following estimate that

\begin{align*}
(2.54) \quad &\int_0^s \int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^4 + |v_\varepsilon|^4) \phi^2 \ dx \ dt \\
&\leq \int_0^s \left( \sup_{0 \leq t \leq s, x_0 \in \mathbb{R}^3} \int_{B_{R}(x_0)} |\nabla u_\varepsilon|^3 + |v_\varepsilon|^3 \ dx \right)^{\frac{2}{3}} \left( \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^6 + |v_\varepsilon|^6 \ dx \right)^{\frac{1}{3}} \ dt \\
&\leq C \delta^2 \int_0^s \int_{B_{2R}(x_0)} |\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 \ dx \ dt + \frac{C \delta^2}{R^2} \int_0^s \int_{B_{2R}(x_0)} |\nabla u_\varepsilon|^2 + |v_\varepsilon|^2 \ dx \ dt.
\end{align*}

Since $\text{supp} \ \phi \subset B_{2R}(x_0)$, the commutator can be expressed as

\begin{align*}
(2.55) \quad &|[\phi, \mathcal{R}_i \mathcal{R}_j](F^{ij})(x, t) - c_\varepsilon(t)\phi(x)| \\
&= \int_{\mathbb{R}^3} \frac{(\phi(x) - \phi(y))(x_i - y_i)(x_j - y_j)}{|x - y|^5} F^{ij}(y, t) \ dy - c_\varepsilon(t) \phi(x) \\
&= \int_{B_{4R}(x_0)} \frac{(\phi(x) - \phi(y))(x_i - y_i)(x_j - y_j)}{|x - y|^5} F^{ij}(y, t) \ dy \\
&\quad + \phi(x) \int_{\mathbb{R}^3 \setminus B_{4R}(x_0)} \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^5} F^{ij}(y, t) \ dy - c_\varepsilon(t) \\
&= : f_1(x, t) + f_2(x, t).
\end{align*}

Note that

\begin{align*}
|f_1(x, t)| \leq \frac{C}{R} \int_{\mathbb{R}^3} \frac{(|\nabla u_\varepsilon|^2 + |v_\varepsilon|^2 + |\partial_t u_\varepsilon| + |\nabla v_\varepsilon|) \chi_{B_{4R}(x_0)}}{|x - y|^2} \ dx \ dy
\end{align*}

and the Hardy-Littlewood-Sobolev inequality holds by (c.f. [24])

\begin{align*}
\|I_\alpha(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^r(\mathbb{R}^n)}, \quad \frac{1}{q} = 1 - \frac{\alpha}{n},
\end{align*}
where $I_\infty(f) = \int_{\mathbb{R}^n} \frac{f(x)}{|x-y|^s} \, dy$. Then it follows from Hölder’s inequality and standard covering arguments that

$$
(2.56) \quad \int_0^s \int_{\mathbb{R}^3} |f_2(x, s)|^2 \, dx \, dt \leq CR^{-2} \int_0^s \|(F^{ij}) \chi_{B_R(x_0)}\|^2_{L^2(\mathbb{R}^3)} \, dt
$$

$$
\leq \frac{C}{R^2} \int_0^s \||\nabla u_\varepsilon| + |v_\varepsilon|\chi_{B_R(x_0)}\|^2_{L^2(\mathbb{R}^3)} \||\nabla u_\varepsilon| + |v_\varepsilon|\chi_{B_R(x_0)}\|^2_{L^2(\mathbb{R}^3)} \, dt
$$

$$
+ \frac{C}{R^2} \int_0^s \|\chi_{B_R(x_0)}\|^2_{L^2(\mathbb{R}^3)} \|(|\partial_t u_\varepsilon| + |\nabla v_\varepsilon|)\chi_{B_R(x_0)}\|^2_{L^2(\mathbb{R}^3)} \, dt
$$

$$
\leq \frac{C\delta^2}{R^2} \int_0^s \int_{B_R(x_0)} |\nabla u_\varepsilon|^2 + |v_\varepsilon|^2 \, dx \, dt + C \int_0^s \int_{B_R(x_0)} |\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 \, dx \, dt,
$$

where $\chi_{B_R(x_0)}(x) = 1$ for $x \in B_R(x_0)$ and 0 otherwise. As in Lemma 3.2 of [13], to estimate the term involving $f_2(x, t)$, we choose

$$
c_\varepsilon(t) = \int_{\mathbb{R}^3 \setminus B_R(x_0)} \frac{(x_{0k} - y_1)(x_{0j} - y_j)}{|x_0 - y|^{5}} F^{ij}(y, t) \, dy,
$$

which is finite for any approximation data $(v, u, \partial_t u) \in \dot{H}^1(\mathbb{R}^3) \times \dot{H}^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Then, one has

$$
|f_2(x, t)| \leq C R \phi(x) \int_{\mathbb{R}^3 \setminus B_R(x_0)} \frac{(|\nabla u_\varepsilon|^2 + |v_\varepsilon|^2 + |\partial_t u_\varepsilon| + |\nabla v_\varepsilon|)(y)}{|x_0 - y|^{4}} \, dy,
$$

due to the fact (c.f. [28]) that

$$
\left| \frac{(x_i - y_1)(x_j - y_j)}{|x-y|^5} - \frac{(x_{0i} - y_1)(x_{0j} - y_j)}{|x_0 - y|^{5}} \right| \leq C \frac{|x_0 - x|}{|x_0 - y|^{4}}.
$$

Upon relabeling and using Hölder’s inequality we observe

$$
(2.57) \quad \int_0^s \int_{\mathbb{R}^3} |f_2(z, s)|^2 \, dz \, dt
$$

$$
\leq CR^5 \int_0^s \sum_{k=4}^{\infty} \frac{C}{(kR)^4} \int_{B_{k+1}R(x_0) \setminus B_kR(x_0)} |F^{ij}(x, t)|^2 \, dx \, dt
$$

$$
\leq CR^5 \int_0^s \sum_{k=4}^{\infty} \frac{C}{(kR)^4} \int_{B_{k+1}R(x_0) \setminus B_kR(x_0)} |F^{ij}|^2 \, dx \cdot |B_{k+1}R \setminus B_kR| \, dt
$$

$$
\leq C \sup_{x_0 \in \mathbb{R}^3} \int_0^s \sum_{k=4}^{\infty} \frac{C\delta^2}{R^2} \int_{B_kR(x_0)} (|\nabla u_\varepsilon|^2 + |v_\varepsilon|^2) + C\delta^2 (|\nabla^2 u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) \, dx \, dt
$$

$$
+ C \sup_{x_0 \in \mathbb{R}^3} \int_0^s \int_{B_kR(x_0)} (|\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) \, dx \, dt,
$$

where the last step follows from (2.53). Now combine (2.53), (2.56) and (2.57), then apply standard covering arguments to complete the proof. \qed
3. Local existence

In this section, we prove the local well-posedness of the general Ericksen-Leslie system (1.1)-(1.3) with initial data \((v_0, u_0) \in H^1(\mathbb{R}^3, \mathbb{R}^3) \times H^2_b(\mathbb{R}^3, S^2)\) by using the Ginzburg-Landau approximation approach. The following lemma states the local well-posedness of the Ginzburg-Landau approximation system (1.14)-(1.16).

**Lemma 3.1.** Let \((v_{0, \varepsilon}, u_{0, \varepsilon})\) be the initial data satisfying

\[
\begin{align*}
(3.1) & \quad u_{0, \varepsilon} - b \in H^2(\mathbb{R}^3), \quad v_{0, \varepsilon} \in H^1(\mathbb{R}^3), \quad \text{div } v_{0, \varepsilon} = 0,
\end{align*}
\]

where \(b\) is a constant unit vector. Then there exists a constant \(T_\varepsilon > 0\) such that the system (1.14)-(1.16) with initial data \((v_{0, \varepsilon}, u_{0, \varepsilon})\) has a unique strong solution \((v_\varepsilon, u_\varepsilon)\) in \(\mathbb{R}^3 \times (0, T_\varepsilon)\) satisfying

\[
\begin{align*}
& u_\varepsilon \in L^\infty(0, T_\varepsilon, H^1(\mathbb{R}^3)) \cap L^2(0, T_\varepsilon; H^2(\mathbb{R}^3)), \quad \partial_t u_\varepsilon \in L^2(\mathbb{R}^3 \times (0, T_\varepsilon)),
& u_\varepsilon \in L^\infty(0, T_\varepsilon, H^2_b(\mathbb{R}^3)) \cap L^2(0, T_\varepsilon; H^3_b(\mathbb{R}^3)), \quad \partial_t u_\varepsilon \in L^2(0, T_\varepsilon; H^1(\mathbb{R}^3)).
\end{align*}
\]

**Proof.** The local well-posedness of the system (1.14)-(1.16) follows from the standard contraction mapping principle. We omit the proof and refer to the similar argument in Lin-Liu [24]. \(\square\)

The following proposition gives the uniform estimates of solutions \((u_\varepsilon, v_\varepsilon)\) in Lemma 3.1.

**Proposition 3.1.** Let \((v_{0, \varepsilon}, u_{0, \varepsilon})\) be the initial data in Lemma 3.1 satisfying

\[
\begin{align*}
(3.2) & \quad \frac{3}{4} \leq |u_{0, \varepsilon}| \leq \frac{5}{4}, \quad \|u_{0, \varepsilon} - b\|_{H^2}^2 + \|v_{0, \varepsilon}\|_{H^1}^2 + \varepsilon^{-2} \|(1 - |u_{0, \varepsilon}|^2)\|_{H^1}^2 \leq M,
\end{align*}
\]

for some constant \(M\) independent of \(\varepsilon\). Then there is a uniform constant \(T_M\) in \(\varepsilon\) such that the system (1.14)-(1.16) with initial data \((v_{0, \varepsilon}, u_{0, \varepsilon})\) has a unique strong solution \((u_\varepsilon, v_\varepsilon)\) in \(\mathbb{R}^3 \times [0, T_M]\) satisfying

\[
\begin{align*}
(3.3) & \quad \frac{3}{4} \leq |u_\varepsilon| \leq \frac{5}{4} \quad \text{in } \mathbb{R}^3 \times [0, T_M]
\end{align*}
\]

and

\[
\begin{align*}
(3.4) & \quad \sup_{0 \leq t \leq T_M} \left( \|v_\varepsilon\|_{H^1}^2 + \|
abla u_\varepsilon\|_{H^1}^2 + \varepsilon^{-2} \|(1 - |u_\varepsilon|^2)\|_{H^1}^2 \right) + \|
abla v_\varepsilon\|_{L^2(0, T_M; H^1)}^2 + \|
abla^2 u_\varepsilon\|_{L^2(0, T_M; H^1)}^2 + \|
abla^2 v_\varepsilon\|_{L^2(0, T_M; H^1)}^2 + \|
abla (|u_\varepsilon|^2)\|_{L^2(0, T_M; H^1)}^2 \leq C_M,
\end{align*}
\]

provided \(\varepsilon \leq \varepsilon_M\), where \(T_M, \varepsilon_M, C_M\) are positive constants only depend on \(M\).

**Proof.** For the initial data \((v_{0, \varepsilon}, u_{0, \varepsilon})\) satisfying (3.1) and (3.2), it follows from the Sobolev embedding \(H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)\) with the constant \(C_1\) that for any \(0 < \delta < 1,\)
there exists a $R_0 = \frac{\delta^2}{c_1 P M}$ such that

\begin{equation}
(3.5) \quad \sup_{x_0 \in \mathbb{R}^3} \int_{B_{R_0}(x_0)} |\nabla u_{0,\varepsilon}|^3 + |v_{0,\varepsilon}|^3 + \frac{|1 - |u_{0,\varepsilon}|^2|^3}{\varepsilon^3} \, dx \\
\leq CR_0^3 \left( \sup_{x_0 \in \mathbb{R}^3} \int_{B_{R_0}(x_0)} |\nabla u_{0,\varepsilon}|^6 + |v_{0,\varepsilon}|^6 + \frac{|1 - |u_{0,\varepsilon}|^2|^6}{\varepsilon^6} \, dx \right)^\frac{1}{2} \\
\leq R_0^3 C_1 \left( \|\nabla u_{0,\varepsilon}\|_{H^1(\mathbb{R}^3)} + \|v_{0,\varepsilon}\|_{H^1(\mathbb{R}^3)} + \varepsilon^{-1} \|(|1 - |u_{0,\varepsilon}|^2)\|_{H^1(\mathbb{R}^3)} \right)^3 \\
\leq C_1^3 (R_0 M)^2 = \frac{\delta^3}{\varepsilon^3},
\end{equation}

where $L > 1$ is an absolute constant independent of $\varepsilon$ and $M$ to be chosen. By Lemma 3.1 there exists a unique strong solution to the system (1.14)-(1.16) in $\mathbb{R}^3 \times [0, T_\varepsilon]$ with initial data $(u_{0,\varepsilon}, v_{0,\varepsilon})$. Since the solution $(u_\varepsilon, v_\varepsilon)$ is continuous, which follows from the Sobolev inequality, there is a time $T_\varepsilon \in (0, T_\varepsilon)$ such that

\begin{equation}
(3.6) \quad \frac{1}{2} \leq |u_\varepsilon| \leq \frac{3}{2} \quad \text{in } \mathbb{R}^3 \times [0, T_\varepsilon)
\end{equation}

and

\begin{equation}
(3.7) \quad \sup_{0 \leq t \leq T_\varepsilon} \int_{B_{R_0}(x)} (|v_\varepsilon|^3 + |\nabla u_\varepsilon|^3) \, dx \leq \delta^3.
\end{equation}

Now we shall show that (3.6) and (3.7) hold true for some uniform time $T$ by using the local energy estimate (2.10), (2.21) and (2.50). Let $\phi \in C_0^\infty(B_{2R_0}(x_0))$ be a cut-off function with $\phi \equiv 1$ on $B_{R_0}(x_0)$ and $|\nabla \phi| \leq \frac{C}{R_0}$ and $|\nabla^2 \phi| \leq \frac{C}{R_0^2}$. It follows from (2.10) that

\begin{equation}
(3.8) \quad \sup_{0 \leq t \leq T_\varepsilon} \frac{1}{R_0} \int_{B_{2R_0}(x_0)} |v_\varepsilon|^2 + |\nabla u_\varepsilon|^2 + \varepsilon^{-2}(|1 - |u_\varepsilon|^2|^2) \, dx \\
+ \frac{1}{R_0} \int_0^{T_\varepsilon} \int_{B_{2R_0}(x_0)} |\nabla^2 u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + \varepsilon^{-2}|\nabla |u_\varepsilon|^2|^2 \, dxdt \\
\leq C \frac{1}{R_0} \int_{B_{2R_0}(x_0)} |v_{0,\varepsilon}|^2 + |\nabla u_{0,\varepsilon}|^2 + \frac{|1 - |u_{0,\varepsilon}|^2|^2}{\varepsilon^2} \, dx \\
+ \frac{C}{R_0} \int_0^{T_\varepsilon} \int_{B_{2R_0}(x_0)} |\nabla u_\varepsilon|^2 + |v_\varepsilon|^2 \, dxdt \\
+ \frac{C}{R_0} \int_0^{T_\varepsilon} \int_{B_{2R_0}(x_0)} |\nabla u_\varepsilon|^4 + |v_\varepsilon|^4 \, dxdt \\
+ \frac{C}{R_0} \int_0^{T_\varepsilon} \int_{B_{2R_0}(x_0)} |P_\varepsilon - c_\varepsilon(t)| |v_\varepsilon| \, dxdt \\
=: I_1 + I_2 + I_3 + I_4.
\end{equation}

For $I_1$, since the ball $B_{2R_0}(x_0)$ can be covered by finitely many number, which is independent of $R_0$, balls $B_{R_0}(y)$ with $y \in \mathbb{R}^3$, we obtain from Hölder’s inequality,
standard covering arguments and (3.5) that
\[ I_1 \leq C \left| B_{R_0} \right|^{1/3} \left( \int_{B_{2R_0}(x_0)} \left| u_{0,\varepsilon} \right|^3 + \left| v_{0,\varepsilon} \right|^3 + \frac{\left| 1 - \left| u_{0,\varepsilon} \right| \right|^3}{\varepsilon^3} \right)^{2/3} \]
\[ \leq C \left( \sup_{y \in \mathbb{R}^3} \int_{B_{R_0}(y)} \left| u_{0,\varepsilon} \right|^3 + \left| v_{0,\varepsilon} \right|^3 + \frac{\left| 1 - \left| u_{0,\varepsilon} \right| \right|^3}{\varepsilon^3} \right)^{2/3} = \frac{C \delta^2}{L^2}. \]

Similarly, using (3.7), one has
\[ I_2 \leq C \left| B_{R_0} \right|^{1/3} \int_0^{T^1} \left( \int_{B_{2R_0}(x_0)} \left| v_{\varepsilon} \right|^3 + \left| \nabla u_{\varepsilon} \right|^3 \right)^{2/3} \ dt \]
\[ \leq \frac{C}{R_0} \int_0^{T^1} \left( \sup_{y \in \mathbb{R}^3} \int_{B_{R_0}(y)} \left| v_{\varepsilon} \right|^3 + \left| \nabla u_{\varepsilon} \right|^3 \right)^{2/3} \ dt \leq \frac{C \delta^2 T^1}{R_0}. \]

Similar to (2.54), we employ the Sobolev inequality for \( I_3 \) and estimate of \( I_2 \) to compute
\[ I_3 \leq \frac{C}{R_0} \int_0^{T^1} \left( \int_{B_{2R_0}(x_0)} \left| \nabla u_{\varepsilon} \right|^3 + \left| v_{\varepsilon} \right|^3 \right)^{2/3} \left( \int_{B_{2R_0}(x_0)} \left| \nabla u_{\varepsilon} \right|^6 + \left| v_{\varepsilon} \right|^6 \right) \ dt \]
\[ \leq \frac{C \delta^2}{R_0} \sup_{y \in \mathbb{R}^3} \int_0^{T^1} \int_{B_{R_0}(y)} \left| \nabla^2 u_{\varepsilon} \right|^2 + \left| \nabla v_{\varepsilon} \right|^2 + R_0^2 \left| \nabla u_{\varepsilon} \right|^2 + R_0^{-2} \left| \nabla v_{\varepsilon} \right|^2 \ dx dt \]
\[ \leq \frac{C \delta^2}{R_0} \sup_{y \in \mathbb{R}^3} \int_0^{T^1} \int_{B_{R_0}(y)} \left| \nabla^2 u_{\varepsilon} \right|^2 + \left| \nabla v_{\varepsilon} \right|^2 \ dx dt + \frac{C \delta^2 T^1}{R_0}. \]

For \( I_4 \), it follows from Young’s inequality, (2.60) and the estimate of \( I_2 \) that
\[ I_4 \leq \frac{\delta^2}{R_0} \int_0^{T^1} \int_{B_{2R_0}(x_0)} \left| P_\varepsilon - c_\varepsilon(s) \right|^2 \ dx dt + \frac{C \delta}{R_0} \int_0^{T^1} \int_{B_{2R_0}(x_0)} \left| v_{\varepsilon} \right|^2 \ dx dt \]
\[ \leq \frac{C \delta^2}{R_0} \sup_{y \in \mathbb{R}^3} \int_0^{T^1} \int_{B_{R_0}(y)} \left( \left| \nabla u_{\varepsilon} \right|^2 + \left| v_{\varepsilon} \right|^2 \right) + \delta^2 \left( \left| \nabla^2 u_{\varepsilon} \right|^2 + \left| \nabla v_{\varepsilon} \right|^2 \right) \ dx dt \]
\[ + \frac{C \delta^2}{R_0} \sup_{y \in \mathbb{R}^3} \int_0^{T^1} \int_{B_{R_0}(y)} \left( \left| \partial_t u_{\varepsilon} \right|^2 + \left| \nabla v_{\varepsilon} \right|^2 \right) \ dx dt + I_2 \]
\[ \leq \frac{C \delta^2}{R_0} \sup_{y \in \mathbb{R}^3} \int_0^{T^1} \int_{B_{R_0}(y)} \left( \left| \partial_t u_{\varepsilon} \right|^2 + \left| \nabla v_{\varepsilon} \right|^2 + \left| \nabla^2 u_{\varepsilon} \right|^2 \right) \ dx dt + \frac{C \delta^2 T^1}{R_0}. \]

Substituting estimates of \( I_i \), \( i = 1, 2, 3, 4 \), into (3.8) and taking supremum in \( x_0 \), we obtain from choosing \( \delta < 1 \) such that \( C \delta^2 < \frac{1}{4} \)
\[ \sup_{0 \leq t \leq T^1} \int_{B_{R_0}(x_0)} \left| \nabla u_{\varepsilon} \right|^2 + \left| v_{\varepsilon} \right|^2 + \varepsilon^{-2} \left( \left| u_{\varepsilon} \right|^2 \right) \ dx \]
\[ + \frac{1}{2R_0} \sup_{x_0 \in \mathbb{R}^3} \int_0^{T^1} \int_{B_{R_0}(x_0)} \left| \nabla^2 u_{\varepsilon} \right|^2 + \left| \nabla v_{\varepsilon} \right|^2 + \left| \partial_t u_{\varepsilon} \right|^2 + \frac{\left| \nabla u_{\varepsilon} \right|^2}{\varepsilon^2} \ dx dt \]
\[ \leq \frac{C \delta^2}{L^2} + \frac{C \delta^2 T^1}{R_0}. \]
On the other hand, it follows from (2.21) that

\begin{equation}
(3.10) \quad R_0 \sup_{0 \leq t \leq T_1} \int_{B_{2R_0}(x_0)} \left( |\nabla v_\varepsilon(x, t)|^2 + |\nabla^2 u_\varepsilon(x, t)|^2 + \frac{1}{\varepsilon^2} |\nabla (|u_\varepsilon|^2)(x, t)|^2 \right) dx \\
+ R_0 \int_0^{T_1} \int_{B_{2R_0}(x_0)} |\nabla^2 v_\varepsilon|^2 + |\nabla \partial_t u_\varepsilon|^2 + |\nabla^3 u_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\nabla^2 |u_\varepsilon||^2|^2 dx dt \\
\leq C R_0 \int_{B_{2R_0}(x_0)} \left( |\nabla v_{0,\varepsilon}|^2 + |\nabla^2 u_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} |\nabla (|u_{0,\varepsilon}|^2)(x, t)|^2 \right) dx \\
+ C R_0 \int_0^{T_1} \int_{B_{2R_0}(x_0)} \left( |v_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right) (|\nabla v_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2) dx dt \\
+ \frac{C}{R_0} \int_0^{T_1} \int_{B_{2R_0}(x_0)} |\nabla u_\varepsilon|^4 + |v_\varepsilon|^4 + |\partial_t u_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 dx dt \\
+ \frac{C}{R_0} \int_0^{T_1} \int_{B_{2R_0}(x_0)} |P_\varepsilon - c_\varepsilon(t)|^2 dx dt =: I_5 + I_6 + I_7 + I_8.
\end{equation}

Then using the definition of $R_0$ from (3.5) and the initial condition (3.2) we have

\[ I_5 \leq C M R_0 \leq \frac{C \delta^2}{C_1^2 L^2}. \]

For $I_6$, we utilize the similar argument as the estimate of $I_3$ to obtain

\[ I_6 \leq C R_0 \delta^2 \int_0^{T_1} \int_{B_{2R_0}(x_0)} \left( |\nabla^2 v_\varepsilon|^2 + |\nabla \partial_t u_\varepsilon|^2 + |\nabla^3 u_\varepsilon|^2 \right) dx dt \\
+ \frac{C \delta^2}{R_0} \int_0^{T_1} \int_{B_{2R_0}(x_0)} \left( |v_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 \right) dx dt \\
\leq C \delta^2 R_0 \sup_{x_0 \in \mathbb{R}^3} \int_0^{T_1} \int_{B_{R_1}(x_0)} \left( |\nabla^2 v_\varepsilon|^2 + |\nabla \partial_t u_\varepsilon|^2 + |\nabla^3 u_\varepsilon|^2 \right) dx dt + \frac{C \delta^2}{L^2} + \frac{C \delta^2 T_1}{R_0^2},
\]

where (3.9) is used in the last step. By the estimate of $I_3$ and (3.9), it is clear that

\[ I_7 \leq \frac{C \delta^2}{L^2} + \frac{C \delta^2 T_1}{R_0^2}. \]

For the pressure term $I_8$, it follows from (2.50), (3.9) and the estimate of $I_2$ that

\[ I_8 \leq C \sup_{x_0, x_\varepsilon \in \mathbb{R}^3} \int_0^{T_1} \int_{B_{R_0}(x_0)} \frac{\delta^2}{R_0^2} (|\nabla u_\varepsilon|^2 + |v_\varepsilon|^2) + \delta^2 (|\nabla^2 u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx dt \\
+ \frac{C}{R_0} \sup_{x_0, x_\varepsilon \in \mathbb{R}^3} \int_0^{T_1} \int_{B_{R_0}(x_0)} (|\partial_t u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx dt \leq \frac{C \delta^2}{L^2} + \frac{C \delta^2 T_1}{R_0^2}. \]
Substituting estimates of $I_i$, $i = 5, 6, 7, 8$ into (3.10) and taking supremum in $x_0$, we have

\begin{equation}
R_0 \sup_{0 \leq t \leq T_\varepsilon} \int_{B_{R_0}(x_0)} (|\nabla u_\varepsilon(x, t)|^2 + |\nabla^2 u_\varepsilon(x, t)|^2 + \frac{1}{\varepsilon^2} |\nabla (|u_\varepsilon|^2)(x, t)|^2) \, dx \\
+ R_0 \sup_{x_0 \in \mathbb{R}^3} \int_{0}^{T_\varepsilon} \int_{B_{R_0}(x_0)} |\nabla v_\varepsilon|^2 + |\nabla \partial_t u_\varepsilon|^2 + |\nabla^3 u_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\nabla^2 |u_\varepsilon|^2|^2 \, dx \, dt
\leq C \frac{C \delta^2}{L^2} + C \frac{\delta^2 T_1^2}{R_0^2}.
\end{equation}

Therefore, using the Gagliardo-Nirenberg interpolation inequality, (3.10) and (3.11), we obtain that

\[
\sup_{0 \leq t \leq T_\varepsilon} \int_{B_{R_0}(x_0)} |\nabla u_\varepsilon|^3 + |v_\varepsilon|^3 \, dx \\
\leq C \sup_{0 \leq t \leq T_\varepsilon, x \in \mathbb{R}^3} \left( \frac{1}{R_0} \int_{B_{R_0}(x_0)} |\nabla u_\varepsilon|^2 + |v_\varepsilon|^2 \right)^{3/2} \\
+ C \sup_{0 \leq t \leq T_\varepsilon} \left( \frac{R_0 \int_{B_{R_0}(x_0)} |\nabla^2 u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 \, dx}{3/2} \right) \\
\leq \left( \frac{C \delta^2}{L^2} + \frac{C \delta^2 T_1^2}{R_0^2} \right)^{3/2} \leq \frac{\delta^3}{2}
\]

from choosing $L = 2\sqrt{C_2 + 1}$ and $T_\varepsilon^1 \leq \sigma R_0^2$ with $\sigma \leq 2 C_3^{-1}$. Hence, (3.7) is verified up to the uniform time $T_M = \sigma R_0^2 =: CM^{-2}$. It remains to verify (4.10) on $[0, T_M)$ for sufficiently small $\varepsilon$. First, It follows from Lemma 2.2 that, for any $s \in (0, T_M)$,

\[
\int_{\mathbb{R}^3} |v_\varepsilon(x, s)|^2 + |\nabla u_\varepsilon(x, s)|^2 + \frac{|1 - |u_\varepsilon(x, s)|^2|^2}{\varepsilon^2} \, dx \\
+ \int_{0}^{s} \int_{\mathbb{R}^3} \left( |\nabla v_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + \frac{|\nabla |u_\varepsilon|^2|^2}{\varepsilon^2} \right) \, dx \, dt
\leq C \int_{\mathbb{R}^3} |\nabla u_0\varepsilon|^2 + |v_0\varepsilon|^2 + \frac{|1 - |u_0\varepsilon|^2|^2}{\varepsilon^2} \, dx + C \int_{0}^{s} \int_{\mathbb{R}^3} (|v^3_\varepsilon|^2 + |\nabla u_\varepsilon|^2) \, dx \, dt \, dx \\
\leq CM + C \sum_{i=1}^{\infty} \int_{0}^{s} \left( \int_{B_{R_0}(x_i)} (|v^3_\varepsilon|^3 + |\nabla u_\varepsilon|^3) \, dx \right)^{2/3} \left( \int_{B_{R_0}(x_i)} |\nabla u_\varepsilon|^6 \, dx \right)^{1/3} \, dt \\
\leq CM + C \delta^2 \sum_{i=1}^{\infty} \int_{0}^{s} \int_{B_{R_0}(x_i)} |\nabla^2 u_\varepsilon|^2 + R_0^{-2} |\nabla u_\varepsilon|^2 \, dx \, dt \\
\leq CM + C \delta^2 \int_{0}^{s} \int_{\mathbb{R}^3} |\nabla^2 u_\varepsilon|^2 \, dx \, dt + \frac{C \delta^2 T_1^1}{R_0^2} \sup_{0 \leq t \leq s} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \, dx,
\]
Choosing \( \delta \) and \( T_1 \) sufficiently small such that \( C\delta^2 \leq 1/2 \) and noting \( s \leq T_M = \sigma R_0^2 \) for \( \sigma \) sufficiently small, we have

\[
(3.12) \quad \sup_{0 \leq t \leq s} \int_{\mathbb{R}^3} \left( |v_\varepsilon|^2 + |\nabla u_\varepsilon|^2 + \varepsilon^{-2}(1 - |u_\varepsilon|^2)^2 \right) \left( \cdot, t \right) dx
\]
\[
+ \int_0^s \int_{\mathbb{R}^3} \left( |\nabla v_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + \varepsilon^{-2}|\nabla (|u_\varepsilon|^2)|^2 \right) dxdt \leq CM.
\]

Then, from Lemma 2.3 we derive

\[
\int_{\mathbb{R}^3} |\nabla^2 u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 + C_\varepsilon^2 \int_{\mathbb{R}^3} \left( |\nabla v_\varepsilon|^2 \right) dxdt
\]
\[
+ \int_0^s \int_{\mathbb{R}^3} \left( |\nabla^2 v_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\nabla^3 u_\varepsilon|^2 | \right) dxdt
\]
\[
\leq CM + C \int_0^s \int_{\mathbb{R}^3} \left( |\nabla v_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right) dxdt
\]
\[
\leq CM + C\delta^2 \int_0^s \int_{\mathbb{R}^3} \left( |\nabla^2 v_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\nabla^3 u_\varepsilon|^2 \right) dxdt
\]
\[
+ \frac{C\delta^2}{R_0^2} \int_0^s \int_{\mathbb{R}^3} \left( |\nabla v_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 \right) dxdt.
\]

Noting that \( C\delta^2 < \frac{1}{2} \), and using (3.12) we find

\[
(3.13) \quad \int_{\mathbb{R}^3} |\nabla^2 u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 + \frac{|\nabla(|u_\varepsilon|^2)|^2}{\varepsilon^2} \right) dxdt
\]
\[
+ \int_0^s \int_{\mathbb{R}^3} \left( |\nabla^2 v_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + |\nabla^3 u_\varepsilon|^2 + \frac{1}{\varepsilon^2}|\nabla^2 |u_\varepsilon|^2| \right.) dxdt
\]
\[
\leq CM \left( 1 + \frac{\delta^2}{R_0^2} \right).
\]

Therefore, we obtain from the Gagliardo–Nirenberg interpolation that

\[
\| 1 - |u_\varepsilon|^2 \|_{L^\infty(\mathbb{R}^3)} \leq C(\varepsilon^2 M)^\frac{2}{3} \left( \| \nabla^2 u_\varepsilon \|_{L^2(\mathbb{R}^3)}^\frac{2}{3} + \| \nabla u_\varepsilon \|_{L^6(\mathbb{R}^3)}^\frac{2}{6} \right)
\]
\[
\leq \varepsilon^{\frac{2}{3}} C^4 (M^5 + \sqrt{(M^2 + M^6)^2}) \leq \frac{9}{16},
\]

for all \( \varepsilon < \varepsilon_M := \frac{\sqrt{2^5}}{C^4 + \sqrt{M^2 + M^6}} \), which gives (3.6). In the view of (3.12) and (3.13), we have proved the assertion (3.4). \( \square \)

Now we can give the proof of local existence of strong solutions to (1.11)–(1.13) stated in Theorem 1.

**Proof of Theorem 1** By Proposition 5.1 there exist two positive constants \( T_0 \) and \( \varepsilon_* \) independent of \( \varepsilon \) such that for any \( \varepsilon \leq \varepsilon_* \), the strong solutions \( (u_\varepsilon, v_\varepsilon) \) to (1.11)–(1.13) satisfy

\[
(u_\varepsilon, v_\varepsilon) \in L^\infty(0, T_0; H^1(\mathbb{R}^3) \times H^2_0(\mathbb{R}^3)) \cap L^2(0, T_0; H^2(\mathbb{R}^3) \times H^3_0(\mathbb{R}^3)),
\]
\[
(\partial_t v_\varepsilon, \partial_t u_\varepsilon) \in L^2(0, T_0; L^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3))
\]
and (5.4) holds. It is clear that multiplying \( (u_\varepsilon - b) \) with (4.16) and using estimates in Proposition 5.1, we find \( \|(u_\varepsilon - b)\|_{L^\infty(0, T_0; L^2(\mathbb{R}^3))} < C \).
Since the pressure $P_\varepsilon$ satisfies \([25,22]\), it follows from using the elliptic estimate and the Sobolev inequality that
\[
\int_0^{T_0} \int_{\mathbb{R}^3} |P_\varepsilon|^2 \, dx \, dt \leq \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^4 + |v_\varepsilon|^4 + |\partial_\varepsilon u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 \, dx \, dt \leq C
\]
and
\[
\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla P_\varepsilon|^2 \, dx \, dt \leq C \int_0^{T_0} \int_{\mathbb{R}^3} (|\nabla \sigma_\varepsilon^E|^2 + |\nabla \sigma_\varepsilon^L|^2 + |\nabla (v_\varepsilon \otimes v_\varepsilon)|^2) \, dx \, dt
\]
\[
\leq C \int_0^{T_0} \|\nabla u_\varepsilon\|_{L^2} \|\nabla u_\varepsilon\|_{L^6} (\|\nabla v_\varepsilon\|^2_{L^6} + \|\nabla u_\varepsilon\|^2_{L^6} + \|\partial_\varepsilon u_\varepsilon\|^2_{L^6}) \, dt
\]
\[
+ C \int_0^{T_0} (\|v_\varepsilon\|_{L^2} \|v_\varepsilon\|_{L^6} \|\nabla v_\varepsilon\|^2_{L^6} + \|\nabla^2 v_\varepsilon\|^2_{L^2} + \|\nabla \partial_\varepsilon u_\varepsilon\|^2_{L^6} + \|\nabla u_\varepsilon\|^2_{L^6}) \, dt \leq C.
\]

Then, by the Aubin-Lions Lemma, there is a subsequence, still denoted by $(v_\varepsilon, u_\varepsilon, P_\varepsilon)$ and a solution $(v, u, p)$ such that for any $R \in (0, \infty)$
\[
v_\varepsilon \to v \text{ in } L^2(0, T_0; H^1(B_R)) \cap C([0, T_0]; L^2(B_R))
\]
\[
v_\varepsilon \to v \text{ in } L^2(0, T_0; H^2(\mathbb{R}^3)), \quad \partial_0 v_\varepsilon \to \partial_0 v \text{ in } L^2(\mathbb{R}^3 \times (0, T_0)),
\]
\[
u_\varepsilon \to u \text{ in } L^2(0, T_0; H^2(B_R)) \cap C([0, T_0]; H^1(B_R)),
\]
\[
u_\varepsilon \to u \text{ in } L^2(0, T_0; H^3(\mathbb{R}^3)), \quad \partial_0 u_\varepsilon \to \partial_0 u \text{ in } L^2(0, T_0; H^1(\mathbb{R}^3)),
\]
\[
P_\varepsilon \to P \text{ in } L^2(0, T_0; H^1(\mathbb{R}^3)),
\]
where $|u| = 1$ due to $\sup_{0 \leq t \leq T_0} \int_{\mathbb{R}^3} \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \leq C$. It can be checked that $(v, u)$ satisfies \([1.3] - [1.6]\) based on the above compactness, see \([10]\) for more details. Indeed, \([1.3]\) follows from taking cross product with $u_\varepsilon$ twice in \([1.10]\) and standard weak convergence argument. The uniqueness of strong solutions to \([1.1] - [1.3]\) follows from the $L^2$ estimates for the difference between two solutions, we refer to \([13]\) for more details.

Next, we check the characterization of the maximal existence $T^*$. Let $(u, v)$ be the solution to the Ericksen-Leslie system \((1.1) - (1.3)\) in $\mathbb{R}^3 \times [0, T)$ with $T < T^*$. Then we have
\[
(3.14) \quad \frac{d}{dt} \int_{\mathbb{R}^3} (|\nabla v|^2 + |\nabla^2 u|^2 + \nabla_{\alpha\beta} u_\varepsilon^T W_{\varepsilon_{\alpha\beta}} \nabla_{\gamma\delta} u_\varepsilon^2) \, dx
\]
\[
+ \int_{\mathbb{R}^3} (|\nabla^2 v|^2 + |\partial_0 \nabla u|^2 + |\nabla^3 u|^2) \, dx
\]
\[
\leq C \int_{\mathbb{R}^3} (|v|^2 + |\nabla u|^2)(|\nabla v|^2 + |\nabla^2 u|^2 + |\partial_0 u|^2) \, dx
\]
provided that
\[
\sup_{0 \leq t \leq T, x \in \mathbb{R}^3} \int_{B_R(x)} |\nabla u|^3 + |v|^3 \, dx \leq \varepsilon_0
\]
for some $\varepsilon_0 > 0$ and some $R > 0$. 

By a standard open cover \( \{ B_R(x_i) \}_{i=1}^{\infty} \) of \( \mathbb{R}^3 \) (at each \( x \in \mathbb{R}^3 \), there is at most a fixed number of intersection of open balls), we obtain

\[
(3.15) \quad \int_{\mathbb{R}^3} (|\nabla v(T)|^2 + |\nabla^2 u(T)|^2) \, dx + \int_0^T \int_{\mathbb{R}^3} (|\nabla^2 v|^2 + |\partial_t \nabla u|^2 + |\nabla^3 u|^2) \, dx dt \leq C + C \varepsilon \int_0^T \int_{B_R(x_i)} \left( \sum_{\gamma} |\nabla^\gamma f| + |\nabla^\gamma g| \right) \left( \sum_{\gamma} |\nabla^\gamma f| + |\nabla^\gamma g| \right) \, dt,
\]

Choosing \( \varepsilon_0 \) sufficiently small, \( (u(T^*), v(T^*)) \in H^2_0(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \) and by the local existence result, the solution can be extended passing \( T \), so \( T^* \) is the maximal existence time.

4. Smooth convergence of the Ginzburg-Landau system

In this section, we prove that the Ginzburg-Landau system smoothly converge to the Ericksen-Leslie system once away from the initial time and until the maximal existence time. First, we derive higher order uniform estimates of solutions to the Ginzburg-Landau system. To do that, the following lemma, which are essentially from the Gagliardo-Nirenberg interpolation inequality (c.f. \([2], [34]\)), will be frequently used.

**Lemma 4.1.** For any index \( \alpha, \beta, \gamma \in \mathbb{N}^3 \), it holds that

\[
\|\nabla^\alpha (fg)\|_{L^2} \leq C \sum_{|\gamma|=|\alpha|} (\|f\|_{L^\infty} \|\nabla^\gamma g\|_{L^2} + \|g\|_{L^\infty} \|\nabla^\gamma f\|_{L^2}),
\]

\[
\|\nabla^\alpha f \nabla^\beta g\|_{L^2} \leq C \left( \sum_{|\gamma|=|\alpha|+|\beta|} \|g\|_{L^\infty} \|\nabla^\gamma f\|_{L^2} + \sum_{|\gamma|=|\alpha|+|\beta|-1} \|\nabla^\alpha f\|_{L^\infty} \|\nabla^\gamma g\|_{L^2} \right),
\]

where the commutator \( [\nabla^\alpha, f] \nabla^\beta g \) is defined by

\[
[\nabla^\alpha, f] \nabla^\beta g = \nabla^\alpha (f \nabla^\beta g) - f \nabla^\alpha (\nabla^\beta g).
\]

The following lemma shows that the solution to the Ginzburg-Landau system obtained in Proposition 3.1 is uniformly smooth once away from the initial time.

**Lemma 4.2.** Let \( (\nu, u) \) be the strong solution, obtained in Proposition 3.1, to the system \((1.1), (1.10)\) in \( \mathbb{R}^3 \times [0, T_M] \). Then it holds for any \( \tau > 0 \), \( s \in (\tau, T_M) \) and any integer \( l \geq 0 \) that

\[
(4.1) \quad \int_{\mathbb{R}^3} \left( |\nabla^l \nu(s)|^2 + |\nabla^{l+1} u(s)|^2 + \frac{|\nabla^l (|u(s)|^2)|^2}{\varepsilon^2} \right) \, dx + \int_\tau^s \int_{\mathbb{R}^3} \left( |\nabla^{l+1} \nu|^2 + |\nabla^{l+2} u|^2 + \frac{|\nabla^{l+1} (|u|^2)|^2}{\varepsilon^2} \right) \, dx dt \leq C(\tau, l, M, s),
\]

where \( C \) is a positive constant independent of \( \varepsilon \). For simplicity, \( \nabla^l \) is denoted as multi-derivatives with index \( \alpha \) of order \( l \).
Proof. We prove this lemma by induction. In the view of Proposition 5.1 \[1.1\] holds for \(l = 0, 1\). Assume that \(1.1\) holds for \(l = 0, 1, \ldots, k\) with \(k \geq 1\). Next, we show that \(1.1\) holds for \(l = k + 1\).

Firstly, we define the following energy and dissipation terms

\[
\mathcal{E}_m := \|\nabla^m u_z\|_{L^2}^2 + \|\nabla^{m+1} u_z\|_{L^2}^2, \quad \mathcal{D}_m := \mathcal{E}_{m+1} + \|\nabla^m \partial_t u_z\|_{L^2}^2
\]

for any integer \(m\), and

\[
\mathcal{A}_\infty := \|v_z\|_{L^\infty}^2 + \|\nabla u_z\|_{L^\infty}^2
\]
to simplify notations in the sequel.

Now we prove \(1.1\) for \(l = k + 1\). Applying \(\nabla^\nu\) with index \(\nu\) of order \(k + 1\) to \(1.14\), multiplying the resulting equation by \(\nabla^\nu v_z\), integrating over \(\mathbb{R}^3\) and using \(1.15\), we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^\nu v_z|^2 \, dx = -\int_{\mathbb{R}^3} \nabla^\nu \sigma_{ijz} \nabla^\nu \nabla_j v_z^i \, dx + \int_{\mathbb{R}^3} \nabla^\nu (v_z^i v_z^j) \nabla^\nu \nabla_j v_z^i \, dx
\]

\[
+ \int_{\mathbb{R}^3} \nabla^\nu (\nabla_i u_z^\nu \nabla^\nu u_z^i) \nabla^\nu \nabla_j v_z^i \, dx =: I_1 + I_2 + I_3.
\]

To estimate \(I_1\), we write

\[
I_1 = -\int_{\mathbb{R}^3} \mathcal{T}_{i,j,\nu}^L \nabla^\nu \nabla_j v_z^i \, dx - \int_{\mathbb{R}^3} \mathcal{R}_{i,j,\nu}^L \nabla^\nu \nabla_j v_z^i \, dx =: I_{1,1} + I_{1,2},
\]

where \(\mathcal{T}^L_{i,j,\nu}\) is the highest order derivatives defined by

\[
\mathcal{T}_{i,j,\nu}^L := \alpha_1 u_z^i \nabla^\nu A_{eim} u_z^m u_z^j + \alpha_2 \nabla^\nu N_z^i u_z^j + \alpha_3 u_z^i \nabla^\nu N_z^j
\]

\[
+ \alpha_4 \nabla^\nu A_{eij} + \alpha_5 \nabla^\nu A_{eij} u_z^j + \alpha_6 \nabla^\nu A_{eij} u_z^i
\]

and the remainder \(\mathcal{R}_{i,j,\nu}^L\) is given by

\[
\mathcal{R}_{i,j,\nu}^L := \alpha_1 [\nabla^\nu, u_z^i] \nabla^\nu A_{eim} u_z^m u_z^j + \alpha_2 [\nabla^\nu, u_z^i] N_z^j + \alpha_3 [\nabla^\nu, u_z^i] N_z^j
\]

\[
+ \alpha_5 [\nabla^\nu, u_z^i] A_{eij} + \alpha_6 [\nabla^\nu, u_z^j] A_{eij}.
\]

Note that \(u_z^i \nabla^\nu \Omega_{eij} u_z^j = \nabla^\nu A_{eij} \Omega_{eij} = 0\). By a similar argument to one in \(2.4\), we obtain

\[
I_{1,1} = -\int_{\mathbb{R}^3} \left( \alpha_1 |u_z^i \nabla^\nu A_{eij} u_z^j|^2 + \alpha_4 |\nabla^\nu A_{eij}|^2 + \beta |\nabla^\nu A_{eij} u_z^j|^2 \right) \, dx
\]

\[
+ \int_{\mathbb{R}^3} \nabla^\nu h_z^i \left( \nabla^\nu \Omega_{eij} u_z^j - \frac{\gamma_2}{\gamma_1} \nabla^\nu A_{eij} u_z^j \right) \, dx
\]

\[
+ \int_{\mathbb{R}^3} |\nabla^\nu, u_z^j| A_{eij} \left( \frac{\gamma_2^2}{\gamma_1} \nabla^\nu A_{eij} u_z^j - \gamma_2 \nabla^\nu \Omega_{eij} u_z^j \right) \, dx
\]

\[
\leq -\int_{\mathbb{R}^3} \left( \alpha_1 |u_z^i \nabla^\nu A_{eij} u_z^j|^2 + \alpha_4 |\nabla^\nu A_{eij}|^2 + \beta |\nabla^\nu A_{eij} u_z^j|^2 \right) \, dx
\]

\[
+ \int_{\mathbb{R}^3} \nabla^\nu h_z^i \left( \nabla^\nu \Omega_{eij} u_z^j - \frac{\gamma_2}{\gamma_1} \nabla^\nu A_{eij} u_z^j \right) \, dx
\]

\[
+ \delta_1 \|\nabla^{k+2} v_z\|_{L^2}^2 + C \|u_z\|_{L^\infty}^2 \|\nabla^\nu, u_z\|_{L^2} \|\nabla^\nu v_z\|_{L^2}^2
\]

\[
\leq -\int_{\mathbb{R}^3} \left( \alpha_1 |u_z^i \nabla^\nu A_{eij} u_z^j|^2 + \alpha_4 |\nabla^\nu A_{eij}|^2 + \beta |\nabla^\nu A_{eij} u_z^j|^2 \right) \, dx
\]
\[ + \int_{\mathbb{R}^3} \nabla^\nu h_\nu^I \left( \nabla^\nu \Omega_{\nu ij} u_\nu^I - \frac{\gamma_2}{\gamma_1} \nabla^\nu A_{\nu ij} u_\nu^I \right) \, dx \]

where we have used the estimate from Lemma 4.1 that

\[ (4.7) \quad \| \nabla^\nu, u_\nu \|_{L^2} \leq C(\| v_\nu \|_{L^\infty} \| \nabla^{k+2} u_\nu \|_{L^2} + \| v_\nu \|_{L^\infty} \| \nabla^{k+1} v_\nu \|_{L^2}), \]

For \( I_{1,2} \) involving \( R_{ij,\nu}^L \), we first estimate \( \| R_{ij,\nu}^L \|_{L^2} \). By using Lemma 4.1 several times, we obtain

\[ \| \alpha_1 [\nabla^\nu, u_\nu^I] A_{\epsilon_{it}} \|_{L^2} \leq C(\| v_\nu \|_{L^\infty} \| \nabla^{k+2} (u_\nu \# u_\nu) \|_{L^2} + \| v_\nu \|_{L^\infty} \| \nabla^{k+1} v_\nu \|_{L^2}) \]

where the notation \# denotes the multi-linear map with constant coefficients in the sequel. Similarly,

\[ \| \alpha_2 [\nabla^\nu, u_\nu^I] A_{\epsilon_{it}} + \alpha_3 [\nabla^\nu, u_\nu^I] A_{\epsilon_{it}} \|_{L^2} \leq C(\| v_\nu \|_{L^\infty} \| \nabla^{k+2} u_\nu \|_{L^2} + \| v_\nu \|_{L^\infty} \| \nabla^{k+1} v_\nu \|_{L^2}). \]

For the commutator involving \( N_\nu \) in \( R_{ij,\nu}^L \), we first write

\[ [\nabla^\nu, u_\nu^I] N_\nu^I = [\nabla^\nu, u_\nu^I] \partial_t u_\nu^I + [\nabla^\nu, u_\nu^I] (u_\nu^I \nabla \nu u_\nu^I) - [\nabla^\nu, u_\nu^I] (\Omega_{\epsilon_{ti}} u_\nu^I) \]

\[ = \sum_{|\mu| = k} \mu_{i}^\nu \nabla^\nu - u_\nu^I \partial_t u_\nu^I + \sum_{|\mu| = 0} \mu_{i}^\nu \nabla^\nu - u_\nu^I \partial_t u_\nu^I \]

\[ + [\nabla^\nu, u_\nu^I] \nabla (v_\nu^I) - [\nabla^\nu, u_\nu^I] \Omega_{\epsilon_{ti}} + u_\nu^I [\nabla^\nu, u_\nu^I] \Omega_{\epsilon_{ti}}, \]

where we have used (4.8) and the fact that

\[ (4.9) \quad [\nabla^\nu, f_1] (f_2 f_3) = [\nabla^\nu, f_1 f_2] f_3 - f_1 [\nabla^\nu, f_2] f_3 \]

for any functions \( f_1, f_2 \) and \( f_3 \). Then, we apply Lemma 4.1 to yield

\[ (4.10) \quad \| [\nabla^\nu, u_\nu^I] \nabla (v_\nu^I) \|_{L^2} \leq C(\| v_\nu \|_{L^\infty} \| \nabla^{k+2} u_\nu \|_{L^2} + \| v_\nu \|_{L^\infty} \| \nabla^{k+1} u_\nu \|_{L^2} + \| v_\nu \|_{L^\infty} \| \nabla^{k+1} v_\nu \|_{L^2}) \]

and

\[ (4.11) \quad \| - [\nabla^\nu, u_\nu^I] \Omega_{\epsilon_{ti}} + u_\nu^I [\nabla^\nu, u_\nu^I] \Omega_{\epsilon_{ti}} \|_{L^2} \leq C(\| v_\nu \|_{L^\infty} \| \nabla^{k+2} u_\nu \|_{L^2} + \| v_\nu \|_{L^\infty} \| \nabla^{k+1} v_\nu \|_{L^2}). \]
It follows from the Hölder and Sobolev inequalities that

\begin{equation}
\left| \sum_{|\mu| = k} \left( \sum_{|\mu| = 0}^{k-1} \left| \nabla^\mu u_\xi \varphi \partial_{\xi} u_\epsilon \right| \right) \right|_{L^2} \leq C \left\| \nabla u_\epsilon \right\|_{L^2} + C \sum_{|\mu| = 0}^{k-1} \left| \nabla^\mu \varphi \partial_{\xi} u_\epsilon \right|_{L^2}
\end{equation}

\begin{equation}
\leq C \left\| \nabla u_\epsilon \right\|_{L^2} + C \sum_{|\mu| = 0}^{k-1} \left| \nabla^\mu \varphi \partial_{\xi} u_\epsilon \right|_{L^2}.
\end{equation}

Thus, we obtain

\begin{equation}
\left\| \mathcal{R}_{\epsilon} \right\|_{L^2} \leq C \left\| v_\epsilon \right\|_{L^\infty} \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2} + C \left\| \nabla u_\epsilon \right\|_{L^\infty} \left\| \nabla^{k+1} v_\epsilon \right\|_{L^2} + C \left\| \nabla v_\epsilon \right\|_{L^\infty} \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2} + C \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2}.
\end{equation}

Therefore, it follows from Young’s inequality that

\begin{equation}
I_{1,2} \leq \delta_1 \left\| \nabla^{k+1} v_\epsilon \right\|_{L^2}^2 + C \left\| \nabla u_\epsilon \right\|_{L^\infty} \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2}^2 + C \left( \Lambda_{\infty} + \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2} \right) \mathcal{E}_{k+1} + C \mathcal{E}_k \Lambda_{\infty} \left\| \nabla u_\epsilon \right\|_{L^2}^2 + C \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2} ^2.
\end{equation}

Hence, we obtain from (4.13) and (4.14) that

\begin{equation}
I_1 \leq - \int_{\mathbb{R}^3} \left( \alpha_1 |u_\xi^2| \nabla^\nu A_{\xi j} u_{\xi}^2 + \alpha_1 |\nabla^\nu A_{\xi j} u_{\xi}^2 + \beta |\nabla^\nu A_{\xi j} u_{\xi}^2 | \right) \, dx
\end{equation}

\begin{equation}
+ \int_{\mathbb{R}^3} \nabla^\nu h_{\xi} \left( \nabla^\nu A_{\xi j} u_{\xi}^2 - \frac{2\epsilon}{\gamma} \nabla^\nu A_{\xi j} u_{\xi}^2 \right) \, dx
\end{equation}

\begin{equation}
+ 2\delta_1 \left\| \nabla^{k+2} v_\epsilon \right\|_{L^2}^2 + C \left\| \nabla u_\epsilon \right\|_{L^\infty} \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2}^2 + C \left( \Lambda_{\infty} + \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2} \right) \mathcal{E}_{k+1} + C \mathcal{E}_k \Lambda_{\infty} \left\| \nabla u_\epsilon \right\|_{L^2}^2 + C \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2}^2.
\end{equation}

To estimate $I_2, I_3$, we note that

\begin{equation}
\nabla_i u_\epsilon^2 W_{\rho_\xi} = u_\epsilon \varphi u_\epsilon \varphi \nabla v_\epsilon + \nabla u_\epsilon \varphi \nabla v_\epsilon
\end{equation}

and apply Lemma 1.4 to yield

\begin{equation}
\left\| \nabla^\nu (u_\xi^2 v_\epsilon^2) \right\|_{L^2} \leq C \left\| v_\epsilon \right\|_{L^\infty} \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2},
\end{equation}

\begin{equation}
\left\| \nabla^\nu (\nabla_i u_\epsilon^2 W_{\rho_\xi}) \right\|_{L^2} \leq C \left( \left\| \nabla u_\epsilon \right\|_{L^\infty} \left\| \nabla^{k+2} u_\epsilon \right\|_{L^2} + \left\| \nabla u_\epsilon \right\|_{L^\infty} \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2} \right).
\end{equation}

Thus, we obtain from Young’s inequality that

\begin{equation}
I_2 + I_3 \leq \delta_1 \int_{\mathbb{R}^3} \left| \nabla^{k+2} v_\epsilon \right|^2 \, dx + C \left\| v_\epsilon \right\|_{L^\infty} \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2}^2
\end{equation}

\begin{equation}
+ C \left\| \nabla u_\epsilon \right\|_{L^\infty} \left\| \nabla^{k+2} u_\epsilon \right\|_{L^2} + C \left\| \nabla^{k+1} u_\epsilon \right\|_{L^2} \left\| \nabla u_\epsilon \right\|_{L^\infty}^2.
\end{equation}
Substituting (4.15) and (4.18) into (4.4), and using (1.11), one has

\[
\begin{aligned}
(4.19) \quad & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^\nu v_\varepsilon|^2 \, dx + \alpha_4 \int_{\mathbb{R}^3} |\nabla^\nu A_\varepsilon|^2 \, dx \\
& \leq \int_{\mathbb{R}^3} \nabla^{k+1} h^i \left( \nabla^{k+1} \Omega_{eij} u_\varepsilon^j - \frac{\gamma_2}{\gamma_1} \nabla^{k+1} A_{eij} u_\varepsilon^j \right) \, dx \\
& + 3\delta_1 ||\nabla^{k+2} v_\varepsilon||^2_{L^2} + C ||\nabla u_\varepsilon|| \|\nabla^k \partial_t u_\varepsilon\|_{L^2}^2 + C(\Lambda_\infty + ||\nabla^k u_\varepsilon\|_{L^2}^2) E_{k+1} \\
& + C E_k \Lambda_\infty ||\nabla^k u_\varepsilon||^2_{L^2} + C ||\nabla^k u_\varepsilon\|_{L^2} ||\nabla^k u_\varepsilon||_{L^2}.
\end{aligned}
\]

Applying $\nabla^\nu$, with index $\nu$ of order $k+1$, to (4.14), multiplying the resulting equation by $\frac{1}{\gamma_1} \nabla^\nu h_\varepsilon$ and integrating over $\mathbb{R}^3$ give

\[
(4.20) \quad - \int_{\mathbb{R}^3} \partial_t \nabla^\nu u_\varepsilon^i \nabla^\nu h_\varepsilon^i \, dx + \frac{1}{\gamma_1} ||\nabla^\nu h_\varepsilon||^2_{L^2} \\
= \int_{\mathbb{R}^3} \nabla^\nu h_\varepsilon^i \left( \frac{\gamma_2}{\gamma_1} \nabla^\nu A_{eij} u_\varepsilon^j - \nabla^\nu \Omega_{eij} u_\varepsilon^j \right) \, dx \\
- \int_{\mathbb{R}^3} \left( ||\nabla^\nu, u_\varepsilon^j|| \Omega_{eij} \nabla^\nu u_\varepsilon^j - \frac{\gamma_2}{\gamma_1} \nabla^\nu (v_\varepsilon \cdot \nabla u_\varepsilon^j) \right) \, dx \\
\leq \frac{1}{\gamma_1} ||\nabla^\nu h_\varepsilon||^2_{L^2} + \int_{\mathbb{R}^3} \nabla^\nu h_\varepsilon^i \left( \frac{\gamma_2}{\gamma_1} \nabla^\nu A_{eij} u_\varepsilon^j - \nabla^\nu \Omega_{eij} u_\varepsilon^j \right) \, dx \\
+ C(||v_\varepsilon||_{L^\infty} ||\nabla^{k+2} u_\varepsilon||_{L^2}^2 + ||\nabla u_\varepsilon||_{L^\infty} ||\nabla^{k+1} v_\varepsilon||_{L^2}^2),
\]

where, in the last step, we have used (4.7) and (4.11)

\[
(4.21) \quad ||\nabla^\nu (v_\varepsilon \cdot \nabla u_\varepsilon)||_{L^2} \leq C(||v_\varepsilon||_{L^\infty} ||\nabla^{k+2} u_\varepsilon||_{L^2} + ||\nabla u_\varepsilon||_{L^\infty} ||\nabla^{k+1} v_\varepsilon||_{L^2}).
\]

For the term

\[
J_0 := - \int_{\mathbb{R}^3} \partial_t \nabla^\nu u_\varepsilon^i \nabla^\nu h_\varepsilon^i \, dx
\]

on the left hand side of (4.20), integration by parts yields

\[
\begin{aligned}
&= - \int_{\mathbb{R}^3} \partial_t \nabla^\nu u_\varepsilon^i \nabla^\nu v_\varepsilon \nabla_\alpha W_{\rho_\alpha}^i \, dx \\
&= \int_{\mathbb{R}^3} \partial_t \nabla^\nu u_\varepsilon^i \nabla_\alpha W_{\rho_\alpha}^i \, dx \\
&= \int_{\mathbb{R}^3} \partial_t \nabla^\nu a_\varepsilon^i W_{\rho_\alpha}^i \nabla^\nu \gamma u_\varepsilon^j \, dx \\
&= \int_{\mathbb{R}^3} \partial_t \nabla^\nu a_\varepsilon^i W_{\rho_\alpha}^i \nabla^\nu \gamma u_\varepsilon^j \, dx \\
&= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \nabla^\nu \alpha u_\varepsilon^i W_{\rho_\alpha}^i \nabla^\nu \gamma u_\varepsilon^j \, dx \\
&= \int_{\mathbb{R}^3} \partial_t \nabla^\nu a_\varepsilon^i \nabla_\alpha \left( W_{\rho_\alpha}^i \nabla^\nu \gamma u_\varepsilon^j \right) \, dx \\
&= \int_{\mathbb{R}^3} \partial_t \nabla^\nu a_\varepsilon^i \nabla_\alpha \left( W_{\rho_\alpha}^i \nabla^\nu \gamma u_\varepsilon^j \right) \, dx.
\end{aligned}
\]
where $e_{ij}$ denotes the index of taking one derivative with respective to $x_{ij}$. Therefore, in view of (1.17), we have

\begin{equation}
(4.22) \quad J_0 = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \nabla^\nu \nabla^\alpha u_\varepsilon^i W_{p_\alpha} u^i_\varepsilon \nabla^\nu \nabla^\gamma u_\varepsilon^j dx - \int_{\mathbb{R}^3} \partial_t \nabla^\nu u_\varepsilon^i \nabla^\mu \left( \frac{1}{\varepsilon^2} u^j_\varepsilon \right) dx
\end{equation}

\begin{align*}
&= - \frac{1}{2} \int_{\mathbb{R}^3} \nabla^\nu \nabla^\alpha u_\varepsilon^i \partial_t W_{p_\alpha} u_\varepsilon^i \nabla^\nu \nabla^\gamma u_\varepsilon^j dx - \int_{\mathbb{R}^3} \partial_t \nabla^\nu u_\varepsilon^i \nabla^\mu \left( \left[ \nabla^\nu \varepsilon, W_{p_\alpha} u_\varepsilon^i \right] \nabla^\gamma \nabla^\mu u_\varepsilon^j \right) dx \\
&\quad + \int_{\mathbb{R}^3} \partial_t \nabla^\nu u_\varepsilon^i \left( \nabla^\nu W_{u_\varepsilon^j} - \nabla^\nu \varepsilon \nabla^\alpha \left( W_{p_\alpha} u_\varepsilon^i \nabla^\beta u_\varepsilon^j \right) \right) dx \\
&= : J_{0,1} + J_{0,2} + J_{0,3} + J_{0,4} + J_{0,5}.
\end{align*}

Using the fact

\begin{equation}
\nabla^\nu (|u_\varepsilon|^2) = 2 \nabla^\nu u_\varepsilon^i u_\varepsilon^j + \sum_{|\mu| = 1} \binom{\nu}{\mu} \nabla^{\nu - \mu} u_\varepsilon^i \nabla^\mu u_\varepsilon^j,
\end{equation}

we can rewrite $J_{0,2}$ as

\begin{equation}
(4.23) \quad J_{0,2} = - \int_{\mathbb{R}^3} \left\{ \partial_t (\nabla^\nu u_\varepsilon^i u_\varepsilon^j) - \nabla^\nu u_\varepsilon \partial_t u_\varepsilon \right\} \nabla^\nu \left( \frac{1}{\varepsilon^2} |u_\varepsilon|^2 \right) dx
\end{equation}

\begin{align*}
&= - \int_{\mathbb{R}^3} \partial_t \nabla^\nu u_\varepsilon^i \nabla^\nu \left( \frac{1}{\varepsilon^2} |u_\varepsilon|^2 \right) dx \\
&= \frac{1}{4} \frac{d}{dt} \int_{\mathbb{R}^3} \left| \nabla^\nu (|u_\varepsilon|^2) \right|^2 dx - \int_{\mathbb{R}^3} \partial_t \nabla^\nu u_\varepsilon^i \nabla^\nu \left( \frac{1}{\varepsilon^2} |u_\varepsilon|^2 \right) dx \\
&\quad + \int_{\mathbb{R}^3} \left( \nabla^\nu u_\varepsilon \partial_t u_\varepsilon - \sum_{|\mu| = 1} \binom{\nu}{\mu} \nabla^{\nu - \mu} u_\varepsilon^i \nabla^\mu \partial_t u_\varepsilon^j \right) \nabla^\nu \left( \frac{1}{\varepsilon^2} |u_\varepsilon|^2 \right) dx \\
&= : \frac{1}{4} \frac{d}{dt} \int_{\mathbb{R}^3} \left| \nabla^\nu (|u_\varepsilon|^2) \right|^2 dx + B_1 + B_2.
\end{align*}

To estimate $J_{0,2}$, we have to control

\begin{align*}
\left\| \varepsilon^{-2} \nabla^\nu, u_\varepsilon \right\|_{L^2} (1 - |u_\varepsilon|^2) \quad \text{and} \quad \left\| \varepsilon^{-2} \nabla^\nu (1 - |u_\varepsilon|^2) \right\|_{L^2}.
\end{align*}

Due to the fact that $\frac{3}{4} \leq |u_\varepsilon| \leq \frac{5}{4}$, the equation (1.10) gives

\begin{equation}
\varepsilon^{-2} (1 - |u_\varepsilon|^2) = |u_\varepsilon|^{-2} \left( \gamma_1 N_\varepsilon + \gamma_2 A_\varepsilon u_\varepsilon - u_\varepsilon^t \nabla A_\varepsilon W_{p_\alpha} + u_\varepsilon^t W_{u_\varepsilon} \right).
\end{equation}

Note that

\begin{equation}
\nabla A_\varepsilon W_{p_\alpha} = u_\varepsilon \nabla u_\varepsilon \nabla^2 u_\varepsilon + u_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon + \nabla^2 u_\varepsilon
\end{equation}

and

\begin{equation}
W_{u_\varepsilon} = u_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon.
\end{equation}
Then we use (4.9) to write
\[
\varepsilon^{-2}[\nabla^\nu, u_\varepsilon] (1 - |u_\varepsilon|^2) = \gamma_1[\nabla^\nu, u_\varepsilon|u_\varepsilon|^{-2}]N_\varepsilon - \gamma_1 u_\varepsilon[\nabla^\nu, |u_\varepsilon|^{-2}]N_\varepsilon \\
+ \gamma_2[\nabla^\nu, u_\varepsilon \cdot u_\varepsilon]A_\varepsilon - \gamma_2 u_\varepsilon[\nabla^\nu, u_\varepsilon]A_\varepsilon \\
+ [\nabla^\nu, |u_\varepsilon|^{-2} u_\varepsilon \cdot u_\varepsilon]u_\varepsilon \cdot u_\varepsilon + |u_\varepsilon|^{-2} u_\varepsilon \cdot u_\varepsilon \nabla^2 u_\varepsilon \\
- u_\varepsilon \cdot \nabla^\nu, |u_\varepsilon|^{-2} u_\varepsilon \cdot u_\varepsilon + |u_\varepsilon|^{-2} u_\varepsilon \nabla^2 u_\varepsilon \\
+ [\nabla^\nu, |u_\varepsilon|^{-2} u_\varepsilon \cdot u_\varepsilon] \nabla u_\varepsilon \cdot \nabla u_\varepsilon \\
- u_\varepsilon \nabla^\nu, |u_\varepsilon|^{-2} u_\varepsilon \cdot u_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \\
= : \sum_{i=1}^{8} G_i.
\]

We apply Lemma 4.3 to estimate $G_i$, $i = 1, \ldots, 8$. Firstly, we claim that for any $u_\varepsilon$ satisfying $\frac{3}{4} \leq |u_\varepsilon| \leq \frac{5}{4}$ and any $k \geq 0$

\[(4.24) \quad \|\nabla^{k+1}(|u_\varepsilon|^{-2})\|_{L^2} \leq C\|\nabla^{k+1}u_\varepsilon\|_{L^2}.
\]

Indeed, by direct calculations, we have

\[\|\nabla^{k+1}(|u_\varepsilon|^{-2})\|_{L^2} \leq C \sum_{|\mu_1| + \cdots + |\mu_m| = k+1} \|\nabla^{\mu_1}u_\varepsilon \cdots \nabla^{\mu_m}u_\varepsilon\|_{L^2} \]

Then, it follows from the Hölder and Gagliardo-Nirenberg interpolation inequalities that

\[\|\nabla^{k+1}(|u_\varepsilon|^{-2})\|_{L^2} \leq C \sum_{|\mu_1| + \cdots + |\mu_m| = k+1} \|\nabla^{\mu_1}u_\varepsilon \cdots \nabla^{\mu_m}u_\varepsilon\|_{L^2} \]

\[\leq C \sum_{|\mu_1| + \cdots + |\mu_m| = k+1} \|\nabla^{\mu_1}u_\varepsilon\|_{L^{r_1}} \cdots \|\nabla^{\mu_m}u_\varepsilon\|_{L^{r_m}} \]

\[\leq C\|\nabla^{k+1}u_\varepsilon\|_{L^{1/2}} \|u_\varepsilon\|_{L^{1/2}}^{1-\alpha_1} \cdots \|\nabla^{k+1}u_\varepsilon\|_{L^{1/2}} \|u_\varepsilon\|_{L^{1/2}}^{1-\alpha_m} \leq C\|\nabla^{k+1}u_\varepsilon\|_{L^2},
\]

where $r_i$ and $\alpha_i$, $i = 1 \cdots m$ satisfies

\[\sum_{i=1}^{n} \frac{1}{r_i} = \frac{1}{2}, \quad \frac{1}{r_i} = \frac{|\mu_i|}{3} + \left(\frac{1}{2} - \frac{k+1}{3}\right)\alpha_i + \frac{1 - \alpha_i}{\infty}, \quad \text{so that} \quad \sum_{i=1}^{n} \alpha_i = 1.
\]

Then, it follows from an expression like (4.8) and arguments like (4.8)- (4.12) that

\[\|G_1\|_{L^2} \leq C\|\nabla(|u_\varepsilon|^{-2})\|_{L^2} + C\|\nabla u_\varepsilon\|_{L^2} \|\nabla^{k+2}(|u_\varepsilon|^{-2})\|_{L^2} \]

\[+ C\|\nabla u_\varepsilon\|_{H^{-k+1}} \|\nabla u_\varepsilon\|_{H^k} + C\|\nabla u_\varepsilon\|_{L^\infty} \|\nabla^{k+2}(|u_\varepsilon|^{-2})\|_{L^2} \]

\[+ C\|\nabla u_\varepsilon\|_{L^{1/2}} \|\nabla^{k+1}u_\varepsilon\|_{L^2} + C\|\nabla u_\varepsilon\|_{L^{1/2}} \|\nabla^{k+1}u_\varepsilon\|_{L^2} \]

\[\leq C\|\nabla u_\varepsilon\|_{L^{1/2}} \|\nabla^{k+2}u_\varepsilon\|_{L^2} + C\|\nabla u_\varepsilon\|_{L^{1/2}} \|\nabla^{k+2}u_\varepsilon\|_{L^2} \]

\[+ C\|\nabla u_\varepsilon\|_{L^1} \|\nabla u_\varepsilon\|_{L^\infty} + C\|\nabla u_\varepsilon\|_{L^{1/2}} \|\nabla^{k+1}u_\varepsilon\|_{L^2} \]

\[+ C\|\nabla u_\varepsilon\|_{L^1} \|\nabla^{k+2}u_\varepsilon\|_{L^2} + C\|\nabla u_\varepsilon\|_{L^{1/2}} \|\nabla^{k+2}u_\varepsilon\|_{L^2}.
\]

Similarly, we have the same estimate for $\|G_2\|_{L^2}$ as $\|G_1\|_{L^2}$. Here we have used

\[\|\nabla^{k+2}(|u_\varepsilon|^{-2})\|_{L^2} + \|\nabla^{k+2}(|u_\varepsilon|^{-2})\|_{L^2} \leq C\|\nabla^{k+2}(|u_\varepsilon|^{-2})\|_{L^2} + C\|\nabla^{k+2}(|u_\varepsilon|^{-2})\|_{L^2} \leq C\|\nabla^{k+2}(|u_\varepsilon|^{-2})\|_{L^2}.
\]
from (1.24). By using Lemma 1.1 again, it is easy to derive
\[\|G_3 + G_4\|_{L^2} \leq C\|v_\varepsilon\|_{L^\infty} \|\nabla^{k+2} u_\varepsilon\|_{L^2} + C\|\nabla u_\varepsilon\|_{L^\infty} \|\nabla^{k+1} v_\varepsilon\|_{L^2},\]
\[\|G_5 + G_6\|_{L^2} \leq C\|\nabla u_\varepsilon\|_{L^\infty} \|\nabla^{k+2} u_\varepsilon\|_{L^2},\]
\[\|G_7 + G_8\|_{L^2} \leq C\|\nabla u_\varepsilon\|_{L^\infty} \|\nabla^{k+1} u_\varepsilon\|_{L^2}.\]

Therefore, we obtain
\[
\begin{align*}
&\|\varepsilon^{-2}[\nabla^\nu, w_\varepsilon^2](1 - |u_\varepsilon|^2)\|_{L^2} \\
&\leq C\||\nabla u_\varepsilon||\nabla^k \partial_t u_\varepsilon||_{L^2}^2 + C(\Lambda_\infty + \|\nabla \partial_t u_\varepsilon\|_{L^2}^2)\mathcal{E}_{k+1} \\
&\quad + C\mathcal{E}_k \Lambda_\infty \|\nabla u_\varepsilon\|_{L^\infty} + C\|\nabla \partial_t u_\varepsilon\|_{H^{k-1}}^2 \|\nabla u_\varepsilon\|_{H^k}^2.
\end{align*}
\]

To estimate \(\|\varepsilon^{-2}\nabla^\nu(1 - |u_\varepsilon|^2)\|_{L^2}\), we extract terms of higher order derivatives and write
\[
\nabla^\nu \left( \frac{1 - |u_\varepsilon|^2}{\varepsilon^2} \right) = \gamma_1|u_\varepsilon|^{-2}\nabla^\nu N_\varepsilon + \gamma_1[\nabla^\nu, |u_\varepsilon|^{-2}]N_\varepsilon + \gamma_2|u_\varepsilon|^{-2} u_\varepsilon^2 \nabla^\nu A_{\varepsilon ij} \\
+ \gamma_2[\nabla^\nu, |u_\varepsilon|^{-2} u_\varepsilon^2]A_{\varepsilon ij} + u_\varepsilon \nabla u_\varepsilon \nabla^\nu u_\varepsilon^2 \\
+ \nabla^\nu |u_\varepsilon|^2 u_\varepsilon + u_\varepsilon \nabla^\nu u_\varepsilon^2 + \nabla^\nu |u_\varepsilon|^2 u_\varepsilon + |\nabla^\nu, u_\varepsilon^2|^{1/2} u_\varepsilon^2 \\
+ u_\varepsilon \nabla u_\varepsilon \nabla^\nu u_\varepsilon^2 + |\nabla^\nu, u_\varepsilon|^1 u_\varepsilon^2 \nabla u_\varepsilon \nabla^\nu u_\varepsilon.
\]

Then, by using similar arguments to derive (4.25), it is clear that
\[
\begin{align*}
\|\nabla^\nu \left( \frac{1 - |u_\varepsilon|^2}{\varepsilon^2} \right)\|_{L^2}^2 &\leq C\|\nabla^{k+1} \partial_t u_\varepsilon\|_{L^2}^2 + C\|\nabla^{k+2} u_\varepsilon\|_{L^2}^2 + C\|\nabla^{k+3} u_\varepsilon\|_{L^2}^2 \\
&\quad + C\||\nabla u_\varepsilon||\nabla^k \partial_t u_\varepsilon||_{L^2}^2 + C(\Lambda_\infty + \|\nabla \partial_t u_\varepsilon\|_{L^2}^2)\mathcal{E}_{k+1} \\
&\quad + C\mathcal{E}_k \Lambda_\infty \|\nabla u_\varepsilon\|_{L^\infty} + C\|\nabla \partial_t u_\varepsilon\|_{H^{k-1}}^2 \|\nabla u_\varepsilon\|_{H^k}^2.
\end{align*}
\]

Therefore, using (1.12), (1.29) and Young’s inequality, we have
\[
|B_1 + B_2| \leq \delta_1\|\nabla^{k+2} v_\varepsilon\|_{L^2}^2 + \delta_2\|\nabla^{k+1} \partial_t u_\varepsilon\|_{L^2}^2 + \delta_3\|\nabla^{k+3} u_\varepsilon\|_{L^2}^2 \\
+ C\||\nabla u_\varepsilon||\nabla^k \partial_t u_\varepsilon||_{L^2}^2 + C(\Lambda_\infty + \|\nabla \partial_t u_\varepsilon\|_{L^2}^2)\mathcal{E}_{k+1} \\
+ C\mathcal{E}_k \Lambda_\infty \|\nabla u_\varepsilon\|_{L^\infty}^2 + C\|\nabla \partial_t u_\varepsilon\|_{H^{k-1}}^2 \|\nabla u_\varepsilon\|_{H^k}^2.
\]

Substituting (4.27) into (4.26), one has
\[
\begin{align*}
J_{0.2} \geq & \frac{1}{4} \frac{d}{dt} \int_{\mathbb{R}^3} \left|\nabla^\nu(|u_\varepsilon|^2)\right|^2 dx - \delta_1\|\nabla^{k+2} v_\varepsilon\|_{L^2}^2 - \delta_2\|\nabla^{k+1} \partial_t u_\varepsilon\|_{L^2}^2 \\
&\quad - \delta_3\|\nabla^{k+3} u_\varepsilon\|_{L^2}^2 - C\||\nabla u_\varepsilon||\nabla^k \partial_t u_\varepsilon||_{L^2}^2 - C\mathcal{E}_k \Lambda_\infty \|\nabla u_\varepsilon\|_{L^\infty}^2 \\
&\quad - C(\Lambda_\infty + \|\nabla \partial_t u_\varepsilon\|_{L^2}^2)\mathcal{E}_{k+1} + C\|\nabla \partial_t u_\varepsilon\|_{H^{k-1}}^2 \|\nabla u_\varepsilon\|_{H^k}^2.
\end{align*}
\]

For \(J_{0.3}\), it follows from the Hölder, Sobolev and Young inequalities that
\[
\begin{align*}
J_{0.3} \leq & C \int_{\mathbb{R}^3} |\partial_t u_\varepsilon||\nabla^{k+2} u_\varepsilon| dx \leq C\|\nabla^{k+2} u_\varepsilon\|_{L^6} \|\partial_t u_\varepsilon\|_{L^3} \|\nabla^{k+2} u_\varepsilon\|_{L^2} \\
&\leq \delta_3\|\nabla^{k+3} u_\varepsilon\|_{L^2}^2 + C\|\partial_t u_\varepsilon\|_{H^1}^2 \|\nabla^{k+2} u_\varepsilon\|_{L^2}^2.
\end{align*}
\]

For \(J_{0.4}\), we first rewrite the commutator in the integral as
\[
\nabla^\alpha \left( [\nabla^\nu, \gamma_\beta] |\nabla^\beta u_\varepsilon|_{L^2}^2 \right) + [\nabla^\nu, \gamma_\beta] |\nabla^\beta u_\varepsilon|_{L^2}^2 - \nabla^\alpha W_{\gamma_\beta} \gamma_\beta u_\varepsilon^2.
\]
Since $W_{p_{t1}p_{t2}} = u_{\varepsilon} \# u_{\varepsilon} + 1$, then, utilizing Lemma 4.1 gives
\[
\| \nabla_{\alpha} \left( \nabla^\nu u_{\varepsilon}, W_{p_{t1}p_{t2}} \right) \|_{L^2} \leq C \| \nabla u_{\varepsilon} \|_{L^\infty} \| \nabla^{k+2} u_{\varepsilon} \|_{L^2}.
\]
Thus, it follows from Young’s inequality that
\[
|J_{0,4}| \leq \delta_2 \| \nabla^{k+1} \partial_t u_{\varepsilon} \|_{L^2}^2 + C \| \nabla u_{\varepsilon} \|_{L^\infty} \| \nabla^{k+2} u_{\varepsilon} \|_{L^2}^2.
\]
For $J_{0,5}$, note that $W_{u_{\varepsilon} \# u_{\varepsilon}} \nabla_{\beta} u_{\varepsilon} = u_{\varepsilon} \# u_{\varepsilon} \# u_{\varepsilon}$, then Lemma 4.1 gives
\[
\| \nabla^\nu(u_{\varepsilon} \# \nabla u_{\varepsilon}) \|_{L^2} \leq C(\| \nabla u_{\varepsilon} \|_{L^\infty} \| \nabla^{k+2} u_{\varepsilon} \|_{L^2} + \| \nabla u_{\varepsilon} \|_{L^\infty} \| \nabla^{k+1} u_{\varepsilon} \|_{L^2}).
\]
Hence, we can obtain that
\[
|J_{0,5}| \leq \delta_2 \| \nabla^{k+1} \partial_t u_{\varepsilon} \|_{L^2}^2 + C(\| \nabla u_{\varepsilon} \|_{L^\infty}^2 \| \nabla^{k+3} u_{\varepsilon} \|_{L^2}^2 + \| \nabla u_{\varepsilon} \|_{L^\infty} \| \nabla^{k+2} u_{\varepsilon} \|_{L^2}^2).
\]
Substituting (4.28) - (4.32) into (4.22), we compute
\[
|J_0| \geq \frac{d}{dt} \int_{\mathbb{R}^3} \frac{1}{2} \nabla^\nu \nabla_{\alpha} u_{\varepsilon}^j W_{p_{t1}p_{t2}} \nabla^\nu \nabla_{\gamma} u_{\varepsilon}^j + \frac{1}{4\varepsilon^2} \| \nabla^\nu(\nu_{\varepsilon}^2) \|_{L^2}^2 dx
\]
\[
- \delta_1 \| \nabla^{k+2} v_{\varepsilon} \|_{L^2}^2 - 3\delta_2 \| \nabla^{k+1} \partial_t u_{\varepsilon} \|_{L^2}^2 - 2\delta_3 \| \nabla^{k+3} u_{\varepsilon} \|_{L^2}^2,
\]
\[
+ C(\| \nabla u_{\varepsilon} \|_{L^\infty}^2 \| \nabla^{k+2} u_{\varepsilon} \|_{L^2}^2 + \| \nabla u_{\varepsilon} \|_{L^\infty} \| \nabla^{k+2} u_{\varepsilon} \|_{L^2}^2) E_k + C E_k \Lambda \| \nabla u_{\varepsilon} \|_{L^\infty}^2 + C \| \nabla \partial_t u_{\varepsilon} \|_{H_{k-1}}^2 \| \nabla u_{\varepsilon} \|_{H_k}^2.
\]
Summing (4.14) with (4.33) yields
\[
|J_0| \geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \| \nabla^\nu v_{\varepsilon} \|_{L^2}^2 dx + \alpha_4 \int_{\mathbb{R}^3} \| \nabla^\nu A_{\varepsilon} \|_{L^2}^2 dx + \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^3} \| \nabla^\nu h_{\varepsilon} \|_{L^2}^2 dx
\]
\[
\leq 3\delta_1 \| \nabla^{k+2} v_{\varepsilon} \|_{L^2}^2 + 3\delta_2 \| \nabla^{k+1} \partial_t u_{\varepsilon} \|_{L^2}^2 + 2\delta_3 \| \nabla^{k+3} u_{\varepsilon} \|_{L^2}^2,
\]
\[
+ C(\| \nabla u_{\varepsilon} \|_{L^\infty}^2 \| \nabla^{k+2} u_{\varepsilon} \|_{L^2}^2 + \| \nabla u_{\varepsilon} \|_{L^\infty} \| \nabla^{k+2} u_{\varepsilon} \|_{L^2}^2) E_k + C E_k \Lambda \| \nabla u_{\varepsilon} \|_{L^\infty}^2 + C \| \nabla \partial_t u_{\varepsilon} \|_{H_{k-1}}^2 \| \nabla u_{\varepsilon} \|_{H_k}^2.
\]
Using integration by parts and (1.16), we note that
\[
\alpha_4 \int_{\mathbb{R}^3} \| \nabla^\nu A_{\varepsilon} \|_{L^2}^2 dx = \frac{\alpha_4}{2} \int_{\mathbb{R}^3} \| \nabla^\nu v_{\varepsilon} \|_{L^2}^2 dx.
\]
Then, it remains to estimate terms involving $\nabla^{k+3} u_{\varepsilon}$ and $\nabla^{k+1} \partial_t u_{\varepsilon}$. Applying $\nabla^\nu$, with index $\nu$ of order $k+1$, to (1.16) and multiplying the resulting equation by
\[ \nabla^\nu \partial_t u_\varepsilon, \] we have

\[ (4.37) \quad - \int_{\mathbb{R}^3} \nabla^\nu h_\varepsilon \cdot \nabla^\nu \partial_t u_\varepsilon \, dx + \gamma_1 \int_{\mathbb{R}^3} |\nabla^\nu \partial_t u_\varepsilon|^2 \, dx \]

\[ = \int_{\mathbb{R}^3} (\nabla^\nu (\gamma_1 \Omega_\varepsilon u_\varepsilon - \gamma_2 A_\varepsilon u_\varepsilon) - \gamma_1 \nabla^\nu (v_\varepsilon \cdot \nabla u_\varepsilon)) \cdot \nabla^\nu \partial_t u_\varepsilon \, dx \]

\[ \leq \frac{\gamma_1}{4} \int_{\mathbb{R}^3} |\nabla^\nu \partial_t u_\varepsilon|^2 \, dx + C \| (\nabla^\nu u_\varepsilon) \|_{L^2}^2 \]

\[ + C \| (\nabla^\nu, u_\varepsilon) \|_{L^2}^2 + C \| (\nabla^\nu, u_\varepsilon) A_\varepsilon \|_{L^2}^2 + C \| \nabla^\nu \nabla (v_\varepsilon \cdot u_\varepsilon) \|_{L^2}^2 \]

where we have used (4.7) and Lemma 4.1 in the last step. Plugging (4.33) into (4.37) yields

\[ (4.38) \quad \frac{d}{dt} \int_{\mathbb{R}^3} \frac{1}{2} \nabla^\nu \nabla_{\alpha u_\varepsilon} W_{p_\varepsilon p_\varepsilon'} \nabla^\nu \nabla_{\gamma u_\varepsilon} + \frac{1}{4 \varepsilon^2} (\nabla^\nu (|u_\varepsilon|^2))^2 \, dx + \frac{3 \gamma_1}{4} \| \nabla^\nu \partial_t u_\varepsilon \|_{L^2}^2 \]

\[ \leq (C_1 + \delta_1) \| \nabla^{k+2} u_\varepsilon \|_{L^2}^2 + 3 \delta_2 \| \nabla^{k+1} \partial_t u_\varepsilon \|_{L^2}^2 + 2 \delta_3 \| \nabla^{k+3} u_\varepsilon \|_{L^2}^2 \]

\[ + C \| \nabla u_\varepsilon \| \| \nabla \partial_t u_\varepsilon \|_{L^2}^2 + C (\Lambda_{\infty} + \| \partial_t u_\varepsilon \|_{H^1}) E_{k+1} \]

\[ + C E_k \Lambda_{\infty} \| \nabla u_\varepsilon \|_{L^\infty} + C \| \nabla \partial_t u_\varepsilon \|_{H^{k-1}} \| \nabla u_\varepsilon \|_{L^2}^2. \]

Applying \( \nabla^\nu \nabla_{\beta} \), with index \( \nu \) of order \( k+1 \), to (1.19), multiplying by \( \nabla^\nu \nabla_{\beta} u_\varepsilon \) and integrating by parts, it follows from a similar argument as the one in (4.37) that

\[ (4.39) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^\nu u_\varepsilon|^2 \, dx - \frac{1}{\gamma_1} \int_{\mathbb{R}^3} \nabla^\nu \nabla_{\beta} h_\varepsilon \cdot \nabla^\nu \nabla_{\beta} u_\varepsilon \, dx \]

\[ = - \int_{\mathbb{R}^3} \nabla^\nu \nabla_{\beta} ((v_\varepsilon \cdot \nabla u_\varepsilon) - \Omega_\varepsilon u_\varepsilon + \frac{\gamma_2}{\gamma_1} A_\varepsilon u_\varepsilon) \cdot \nabla^\nu \nabla_{\beta} u_\varepsilon \, dx \]

\[ = \int_{\mathbb{R}^3} \nabla^\nu \nabla_{\beta} ((v_\varepsilon \cdot \nabla u_\varepsilon) - \Omega_\varepsilon u_\varepsilon + \frac{\gamma_2}{\gamma_1} A_\varepsilon u_\varepsilon) \cdot \nabla^\nu \Delta u_\varepsilon \, dx \]

\[ \leq \delta_4 \| \nabla^{k+3} u_\varepsilon \|_{L^2}^2 + C_2 \| \nabla^{k+2} u_\varepsilon \|_{L^2}^2 + C \| v_\varepsilon \|_{L^\infty} \| \nabla^{k+2} u_\varepsilon \|_{L^2}^2 + C \| \nabla u_\varepsilon \|_{L^\infty} \| \nabla^{k+1} u_\varepsilon \|_{L^2}^2. \]

To estimate the term \( K_0 := - \frac{1}{\gamma_1} \int_{\mathbb{R}^3} \nabla^\nu \nabla_{\beta} h_\varepsilon \cdot \nabla^\nu \nabla_{\beta} u_\varepsilon \, dx, \)

we use (1.17) and integration by parts to get

\[ (4.40) \quad K_0 = \frac{1}{\gamma_1} \int_{\mathbb{R}^3} \nabla^\nu \nabla_{\beta} u_\varepsilon W_{p_\varepsilon p_\varepsilon'} \nabla^\nu \nabla_{\alpha \beta} u_\varepsilon \, dx \]

\[ - \frac{1}{\gamma_1} \int_{\mathbb{R}^3} \nabla^\nu \nabla_{\beta} \left( \frac{(1 - |u_\varepsilon|^2) u_\varepsilon}{\varepsilon^2} \right) \nabla^\nu \nabla_{\beta} u_\varepsilon \, dx \]

\[ + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} (\nabla^\nu, W_{p_\varepsilon p_\varepsilon'}) \nabla_{\beta} u_\varepsilon \nabla^\nu \nabla_{\alpha \beta} u_\varepsilon \, dx \]

\[ - \frac{1}{\gamma_1} \int_{\mathbb{R}^3} (\nabla^\nu W_{p_\varepsilon u_\varepsilon} \nabla^\nu \Delta u_\varepsilon - \nabla^\nu (W_{p_\varepsilon u_\varepsilon} \nabla_{\beta} u_\varepsilon) \nabla^\nu \nabla_{\alpha \beta} u_\varepsilon) \, dx \]

\[ =: K_{0,1} + K_{0,2} + K_{0,3} + K_{0,4}. \]
Hence, we have
\[ K_{0,1} \geq \frac{a}{\gamma_1} \| \nabla^\nu \nabla^2 u_\varepsilon \|_{L^2}^2. \]

For \( K_{0,2} \), it can be rewritten as follows
\[
K_{0,2} = \frac{1}{\gamma_1 \varepsilon^2} \int_{\mathbb{R}^3} (\nabla^\nu \nabla (|u_\varepsilon|^2)) \int_{\mathbb{R}^3} \nabla^\nu \nabla^2 u_\varepsilon^i \, dx
\]
\[
= \frac{1}{2\gamma_1 \varepsilon^2} \int_{\mathbb{R}^3} |\nabla^\nu \nabla^2 u_\varepsilon^i| \, dx - \frac{1}{\gamma_1 \varepsilon^2} \int_{\mathbb{R}^3} \nabla^\nu \nabla (|u_\varepsilon|^2) |\nabla^\nu u_\varepsilon^i| \, dx
\]
\[
= \frac{1}{2\gamma_1 \varepsilon^2} \int_{\mathbb{R}^3} |\nabla^\nu \nabla^2 u_\varepsilon^i| \, dx + \frac{1}{\gamma_1 \varepsilon^2} \int_{\mathbb{R}^3} \nabla^\nu (|u_\varepsilon|^2) |\nabla^\nu \nabla u_\varepsilon^i| \, dx
\]
\[
= \frac{1}{\gamma_1 \varepsilon^2} \int_{\mathbb{R}^3} (|\nabla^\nu, u_\varepsilon^i| (1 - |u_\varepsilon|^2)) \nabla^\nu \Delta u_\varepsilon^i \, dx,
\]
where we have used the fact
\[
\nabla^\nu [\nabla^\nu, f] g = [\nabla^\nu \nabla^\nu, f] g - \nabla^\nu f \nabla^\nu g
\]
for two functions \( f \) and \( g \). Then, it follows from \( 4.42 \), \( 4.42a \) and Lemma 4.1 that
\[
\frac{1}{\gamma_1 \varepsilon^2} \int_{\mathbb{R}^3} \nabla^\nu (|u_\varepsilon|^2) \nabla^\nu \nabla u_\varepsilon^i \, dx
\]
\[
\leq \varepsilon^{-2} \nabla^\nu (|u_\varepsilon|^2) \| L^2 \| \| \nabla^\nu \nabla u_\varepsilon^i \|_{L^2}
\]
\[
\leq \varepsilon^{-2} \nabla^\nu (|u_\varepsilon|^2) \| L^2 \| \| \nabla^\nu \nabla u_\varepsilon^i \|_{L^2}
\]
\[
\leq \delta_1 \| \nabla^k+2 u_\varepsilon \|_{L^2}^2 + \delta_2 \| \nabla^k+1 \partial_t u_\varepsilon \|_{L^2}^2 + \delta_3 \| \nabla^k+3 u_\varepsilon \|_{L^2}^2 + C \| \nabla u_\varepsilon \| \| \nabla^k \partial_t u_\varepsilon \|_{L^2}^2
\]
\[
+ C(\Lambda_\infty + \| \nabla \partial_t u_\varepsilon \|_{L^2}) E_{k+1} + C E_k \Lambda_\infty \| \nabla u_\varepsilon \|_{L^\infty}^2 + C \| \nabla \partial_t u_\varepsilon \|_{H^{k-1}} \| \nabla u_\varepsilon \|_{H^k}^2.
\]
and
\[
\frac{1}{\gamma_1 \varepsilon^2} \int_{\mathbb{R}^3} (|\nabla^\nu, u_\varepsilon^i| (1 - |u_\varepsilon|^2)) \nabla^\nu \Delta u_\varepsilon^i \, dx
\]
\[
\leq C \| \nabla^\nu u_\varepsilon \|_{L^2} (\varepsilon^{-2} (1 - |u_\varepsilon|^2)) \| L^2 \| \| \nabla^k+3 u_\varepsilon \|_{L^2}
\]
\[
\leq \delta_3 \| \nabla^k+3 u_\varepsilon \|_{L^2}^2 + C \| \nabla u_\varepsilon \| \| \nabla^k \partial_t u_\varepsilon \|_{L^2}^2 + C(\Lambda_\infty + \| \nabla \partial_t u_\varepsilon \|_{L^2}) E_{k+1}
\]
\[
+ C E_k \Lambda_\infty \| \nabla u_\varepsilon \|_{L^\infty}^2 + C \| \nabla \partial_t u_\varepsilon \|_{H^{k-1}} \| \nabla u_\varepsilon \|_{H^k}^2.
\]

Hence, we have
\[
K_{0,2} \geq \frac{1}{2\gamma_1 \varepsilon^2} \int_{\mathbb{R}^3} \nabla^\nu \nabla (|u_\varepsilon|^2) \| L^2 \| \| \nabla^k+2 \partial_t u_\varepsilon \|_{L^2}^2 - \delta_1 \| \nabla^k+3 u_\varepsilon \|_{L^2}^2 - \delta_2 \| \nabla^k+1 \partial_t u_\varepsilon \|_{L^2}^2
\]
\[
- \delta_3 \| \nabla^k+3 u_\varepsilon \|_{L^2}^2 - C \| \nabla u_\varepsilon \| \| \nabla^k \partial_t u_\varepsilon \|_{L^2}^2 - C E_k \Lambda_\infty \| \nabla u_\varepsilon \|_{L^\infty}^2
\]
\[
- C(\Lambda_\infty + \| \nabla \partial_t u_\varepsilon \|_{L^2}) E_{k+1} + C \| \nabla \partial_t u_\varepsilon \|_{H^{k-1}} \| \nabla u_\varepsilon \|_{H^k}^2.
\]

By using \( 4.31 \) and Young’s inequality that
\[
|K_{0,3} + K_{0,4}| \leq \delta_3 \| \nabla^k+3 u_\varepsilon \|_{L^2}^2 + C \| \nabla u_\varepsilon \|_{L^\infty}^2 \| \nabla^k+2 u_\varepsilon \|_{L^2}^2
\]
\[
+ C \| \nabla u_\varepsilon \|_{L^\infty} \| \nabla^k+1 u_\varepsilon \|_{L^2}^2.
\]
Substituting (4.41)–(4.43) into (4.40), the inequality (4.39) reads as
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left| \nabla^\alpha \varphi \nabla \varphi \right|^2 dx + \frac{\alpha}{\gamma_1} \left| \nabla^\alpha \varphi \nabla \varphi \right|^2 L^2 + \frac{1}{2} \frac{d}{dt} \nabla^\alpha \nabla \varphi \left( |\varphi|^2 \right) L^2 \\
\leq (C_2 + \delta_1) \left| \nabla^k v \right|^2 L^2 + \delta_2 \left| \nabla^{k+1} \partial_t \varphi \right|^2 L^2 + 3 \delta_3 \left| \nabla^{k+3} \varphi \right|^2 L^2 \\
+ C \left( \nabla \varphi \left| \nabla \partial_t \varphi \right| \right)^2 L^2 + C(\Lambda_\infty + \left| \nabla \partial_t \varphi \right|^2 L^2) \mathcal{E}_{k+1} \\
+ C \mathcal{E}_{k+1} \nabla \varphi \left| \nabla \varphi \right|^2 L^2 + C \left( \nabla \partial_t \varphi \right)^2 H_{k+1} \left| \nabla \varphi \right|^2 L^2.
\end{equation}

Summing (4.41) with (4.38) yields
\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^3} \left| \nabla^\alpha \varphi \nabla \varphi \right|^2 dx + \frac{3 \gamma_1}{4} \left| \nabla^\alpha \partial_t \varphi \right|^2 L^2 + \frac{\alpha}{\gamma_1} \left| \nabla^\alpha \varphi \nabla \varphi \right|^2 L^2 + \frac{1}{2} \frac{d}{dt} \left( \left| \nabla^\alpha \varphi \nabla \varphi \right|^2 L^2 \right) \\
\leq \hat{C}_1 \left( \left| \nabla^k v \right|^2 L^2 + 4 \delta_2 \left| \nabla^{k+1} \partial_t \varphi \right|^2 L^2 + 5 \delta_3 \left| \nabla^{k+3} \varphi \right|^2 L^2 \right) \\
+ C \left( \nabla \varphi \left| \nabla \partial_t \varphi \right| \right)^2 L^2 + C \left( \Lambda_\infty + \left| \partial_t \varphi \right|^2 H_{k+1} \right) \mathcal{E}_{k+1} \\
+ C \mathcal{E}_{k+1} \nabla \varphi \left| \nabla \varphi \right|^2 L^2 + C \left( \nabla \partial_t \varphi \right)^2 H_{k+1} \left| \nabla \varphi \right|^2 L^2,
\end{equation}
where \( \hat{C}_1 = C_1 + C_2 + 2 \delta_1 \). Multiplying (4.38) by \( \hat{C}_2 := \max\{1, 4 \alpha_4^{-1}(\hat{C}_2 + 1)\} \), adding with (4.45) and taking a summation over all the index \( \nu \) of order \( k + 1 \), we can obtain
\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^3} \nabla^{k+1} \nabla^\alpha \varphi \nabla \varphi |\nabla \varphi|^2 dx + \frac{1}{2} \frac{d}{dt} \nabla^{k+1} \left( \nabla^\alpha \varphi \nabla \varphi \right) \left( \nabla^\alpha \varphi \nabla \varphi \right)^2 dx \\
+ \left( 4 \gamma_1 \hat{C}_1 \right) - \left| \nabla^{k+1} \left( \nabla^\alpha \varphi \nabla \varphi \right) \right|^2 L^2 \\
\leq C \left| \nabla \varphi \right|^2 \left( \nabla \partial_t \varphi \right)^2 L^2 + C \left( \partial_t \varphi \right)^2 H_{k+1} \mathcal{E}_{k+1} + C \mathcal{E}_{k+1} \nabla \varphi \left| \nabla \varphi \right|^2 L^2 \\
+ C \left( \nabla \partial_t \varphi \right)^2 H_{k+1} \left| \nabla \varphi \right|^2 L^2,
\end{equation}
where we have chosen \( \delta_1 = \alpha_4/12, \delta_2 = \gamma_1(16(\hat{C}_2 + 1))^{-1}, \delta_3 = a(2\gamma_1(2\hat{C}_2 + 5))^{-1} \) and \( \hat{a} = \min\{1, \frac{2}{\gamma_1}, \frac{\sqrt{2}}{\gamma_1}\} \). Furthermore, it follows from using uniform estimates of the strong solution in Proposition 6.1 that for any \( \delta > 0 \) there exists a \( R_0 \) depending only on \( M \) such that
\begin{equation}
\sup_{0 \leq T \leq T_M} \int_{B_{R_0} (x_0)} |v| \left| \nabla \varphi \right|^3 dx \leq \delta^3.
\end{equation}
By standard covering argument, we have
\begin{equation}
C \left| \nabla \varphi \right|^2 \left( \nabla \partial_t \varphi \right)^2 L^2 \leq C \sum_i \left( \int_{B_{R_0} (x_i)} \left| \nabla \partial_t \varphi \right|^6 dx \right)^{1/3} \delta^2 \leq C \delta^2 \left| \nabla^{k+1} \partial_t \varphi \right|^2 L^2
\end{equation}
so that we can choose \( \delta \) small enough satisfying \( C \delta^2 = \frac{3}{2} \) to conclude
\begin{equation}
\frac{d}{dt} \mathcal{E}_{k+1} (t) + \frac{d}{dt} \int_{\mathbb{R}^3} \nabla^{k+1} \nabla^\alpha \varphi \nabla \varphi |\nabla \varphi|^2 dx + \frac{1}{2} \frac{d}{dt} \nabla^{k+1} \left( \nabla^\alpha \varphi \nabla \varphi \right) \left( \nabla^\alpha \varphi \nabla \varphi \right)^2 dx \\
+ \frac{\hat{a}}{2} \mathcal{D}_{k+1} (t) + \left( 4 \gamma_1 \hat{C}_1 \right) - \left| \nabla^{k+1} \left( \nabla^\alpha \varphi \nabla \varphi \right) \right|^2 L^2 \\
\leq C(\Lambda_\infty + \left| \partial_t \varphi \right|^2 H_{k+1}) \mathcal{E}_{k+1} + C \mathcal{E}_{k+1} \nabla \varphi \left| \nabla \varphi \right|^2 L^2 + C \left( \sum_{m=0}^{k} \mathcal{E}_m \right) \left( \sum_{m=1}^{k} \mathcal{D}_m \right).
\end{equation}
By inductive assumptions, it holds for any \(l = 0, 1, \cdots, k\) with \(k \geq 1\), any \(\tau > 0\) and any \(s \in (\tau, T_M]\) that
\[
\mathcal{E}_l(s) + \int_\tau^s \mathcal{D}_l(t) + \varepsilon^{-2} \|\nabla^{l+1}(|u_\varepsilon(t)|^2)\|_{L^2}^2 \, dt \leq C(\tau, l).
\]

Applying the mean value theorem in (4.49) for \(l = k\), there exists a \(\tau_\varepsilon \in (\tau, 2\tau)\) such that
\[
\mathcal{E}_{k+1}(\tau_\varepsilon) + \varepsilon^{-2} \|\nabla^{k+1}(|u_\varepsilon(\tau_\varepsilon)|^2)\|_{L^2}^2 \leq C(\tau, k).
\]

On the other hand, by the Sobolev embedding and Proposition 3.1, one has
\[
(\Lambda_\infty + \|\partial_t u_\varepsilon\|_{H^1}^2) \leq C(\|v_\varepsilon\|_{H^2}^2 + \|\nabla u_\varepsilon\|_{H^2}^2 + \|\partial_t u_\varepsilon\|_{H^1}^2) \leq C(D_0 + D_1).
\]

Moreover, the Sobolev embedding and (4.49) imply
\[
\|\nabla u_\varepsilon\|_{L^\infty} \leq C \|\nabla u_\varepsilon\|_{H^2} \leq C(\|\nabla^{k+2} u_\varepsilon\|_{L^2} + 1)
\]
when \(k \geq 1\). Therefore, we apply (4.49)-(4.52) into (4.48)
\[
\frac{1}{2} \frac{d}{dt} \mathcal{E}_{k+1}(t) + \frac{d}{dt} \int_{\mathbb{R}^3} \nabla^{k+1} \nabla \alpha u_\varepsilon(W_{s, \psi} - p^\varepsilon) + \nabla^{k+1} \nabla A u_\varepsilon + \frac{1}{2} \nabla^{k+1} (|u_\varepsilon|^2) \, dx
\]
\[
+ \frac{\hat{a}}{2} D_{k+1}(t) + (4\gamma_1 \varepsilon^2)^{-1} \nabla^{k+1} (|u_\varepsilon|^{-2}) \|_{L^2}^2 \leq C(D_0 + D_1) \mathcal{E}_{k+1} + C \sum_{m=1}^k D_m.
\]

We apply the Gronwall inequality in (4.53) for \(t \in (\tau, s)\) and conclude that (4.1) holds for \(l = k + 1\) on the \((2\tau, s)\). Since \(\tau\) is an arbitrary positive constant, we prove (4.1) for any \(s \in (\tau, T_M]\) and \(l = k + 1\) which completes a proof of this lemma.

Next, we have the following strong convergence lemma

**Lemma 4.3.** Let \((v_\varepsilon, u_\varepsilon)\) be the strong solution, obtained in Proposition 3.1,\(^7\) to the system (1.14)-(1.16) in \(\mathbb{R}^3 \times [0, T_M]\). Then, for any \(t \in [0, T_M]\), we have
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} |v_\varepsilon(t)|^2 \, dx = \int_{\mathbb{R}^3} |v(t)|^2 \, dx,
\]
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} |\nabla u_\varepsilon(t)|^2 \, dx = \int_{\mathbb{R}^3} |\nabla u(t)|^2 \, dx,
\]
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} \varepsilon^{-2} (1 - |u_\varepsilon(t)|^2) \, dx = 0.
\]

**Proof.** It follows from Lemma 2.1 that
\[
\int_{\mathbb{R}^3} \left(\frac{|v_\varepsilon(t)|^2}{2} + W(u_\varepsilon, \nabla u_\varepsilon)(t) + \frac{1}{4\varepsilon^2}(1 - |u_\varepsilon(t)|^2)^2\right) \, dx
\]
\[
+ \alpha_4 \int_0^t \int_{\mathbb{R}^3} |A_\varepsilon|^2 \, dxdt + \alpha_1 \int_0^t \int_{\mathbb{R}^3} |u_\varepsilon A_\varepsilon|^2 \, dxdt
\]
\[
+ \beta_1 \int_0^t \int_{\mathbb{R}^3} |A_\varepsilon u_\varepsilon|^2 \, dxdt + \frac{1}{\gamma_1} \int_0^t \int_{\mathbb{R}^3} |\gamma_1 N_\varepsilon + \gamma_2 A_\varepsilon u_\varepsilon|^2 \, dxdt
\]
\[
= \int_{\mathbb{R}^3} \left(\frac{|v_0|^2}{2} + W(u_0, \nabla u_0)\right) \, dx.
\]
By the lower semi-continuity, we have
\begin{align}
(4.58) \quad \int_{\mathbb{R}^3} |v(t)|^2 \, dx & \leq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^3} |v_\varepsilon(t)|^2 \, dx, \\
\int_{\mathbb{R}^3} W(u, \nabla u)(t) \, dx & \leq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^3} W(u_\varepsilon, \nabla u_\varepsilon)(t) \, dx, \\
\int_0^t \int_{\mathbb{R}^3} |A|^2 \, dxdt & \leq \liminf_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}^3} |A_\varepsilon|^2 \, dxdt, \\
\int_0^t \int_{\mathbb{R}^3} |u^T A u|^2 \, dxdt & \leq \liminf_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}^3} |u_\varepsilon^T A_\varepsilon u_\varepsilon|^2 \, dxdt, \\
\int_0^t \int_{\mathbb{R}^3} |A|^2 \, dxdt & \leq \liminf_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}^3} |A_\varepsilon|^2 \, dxdt, \\
\int_0^t \int_{\mathbb{R}^3} |\gamma_1 N + \gamma_2 Au|^2 \, dxdt & \leq \liminf_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}^3} |\gamma_1 N_\varepsilon + \gamma_2 A_\varepsilon u_\varepsilon|^2 \, dxdt.
\end{align}

On the other hand, using a similar argument in Lemma 2.4 (c.f. [34]), one has
\begin{align*}
\int_{\mathbb{R}^3} \left( \frac{|v(t)|^2}{2} + W(u, \nabla u)(t) \right) \, dx & + (\alpha_1 + \frac{\gamma_2^2}{\gamma_1}) \int_0^t \int_{\mathbb{R}^3} |u^T A u|^2 \, dxdt \\
& + \alpha_4 \int_0^t \int_{\mathbb{R}^3} |A|^2 \, dxdt + \beta \int_0^t \int_{\mathbb{R}^3} |A|^2 \, dxdt + \frac{1}{\gamma_1} \int_0^t \int_{\mathbb{R}^3} |h - (u \cdot h)| u|^2 \, dxdt \\
& = \int_{\mathbb{R}^3} \left( \frac{|v_0|^2}{2} + W(u_0, \nabla u_0) \right) \, dx.
\end{align*}

It follows from (1.3) that
\begin{align*}
\int_0^t \int_{\mathbb{R}^3} |h - (u \cdot h)| u|^2 \, dxdt & = \int_0^t \int_{\mathbb{R}^3} |\gamma_1 N + \gamma_2 (Au - (u^T A) u^-)| u|^2 \, dxdt \\
& = \int_0^t \int_{\mathbb{R}^3} |\gamma_1 N + \gamma_2 Au|^2 \, dxdt - \frac{\gamma_2^2}{\gamma_1} \int_0^t \int_{\mathbb{R}^3} |u^T A u|^2 \, dxdt,
\end{align*}
where we have used \(u \cdot N = 0\) due to the fact that \(|u| = 1\). Hence, we have the energy identity for the Ericksen-Leslie system (1.1)-(1.3) that
\begin{align}
(4.59) \quad \int_{\mathbb{R}^3} \left( \frac{|v(t)|^2}{2} + W(u, \nabla u)(t) \right) \, dx & + (\alpha_1 + \frac{\gamma_2^2}{\gamma_1}) \int_0^t \int_{\mathbb{R}^3} |u^T A u|^2 \, dxdt + \alpha_4 \int_0^t \int_{\mathbb{R}^3} |A|^2 \, dxdt \\
& + \beta \int_0^t \int_{\mathbb{R}^3} |A|^2 \, dxdt + \frac{\gamma_2^2}{\gamma_1} \int_0^t \int_{\mathbb{R}^3} |\gamma_1 N + \gamma_2 Au|^2 \, dxdt \\
& = \int_{\mathbb{R}^3} \left( \frac{|v_0|^2}{2} + W(u_0, \nabla u_0) \right) \, dx.
\end{align}

Comparing (4.57) with (4.59) and using (4.58), we first obtain (4.56). Repeating the comparison of (4.57) and (4.59), we have (4.54) and
\begin{align*}
\int_{\mathbb{R}^3} W(u, \nabla u) \, dx & = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} W(u_\varepsilon, \nabla u_\varepsilon) \, dx
\end{align*}
which implies (4.55), since \(W(u, \nabla u)\) satisfies (1.9).

Now we give a proof of Theorem 2.
Proof of Theorem 2 Let \((u, v)\) be the strong solution to the Ericksen-Leslie system \((1.1)-(1.3)\) in \(\mathbb{R}^3 \times [0, T^*)\) with initial data \((u_0, v_0) \in H_0^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\), where \(T^*\) is its maximal existence time. Given any \(T \in (0, T^*)\), set

\[
M = 2 \sup_{0 \leq t \leq T} \| (\nabla u, v) \|_{H^1(\mathbb{R}^3)}^2.
\]

By Proposition 3.1, the system \((1.1)-(1.3)\) with initial data \((u_0, v_0)\) has a unique strong solution \((u_\varepsilon, v_\varepsilon)\) in \(\mathbb{R}^3 \times [0, T_M]\) satisfying

\[
\frac{3}{4} \leq |u_\varepsilon| \leq \frac{5}{4}
\]

and

\[
\frac{1}{2} \sup_{0 \leq t \leq T_M} \left( \| u_\varepsilon \|_{H^1}^2 + \| \nabla u_\varepsilon \|_{H^1}^2 + \varepsilon^{-2} \| (1 - |u_\varepsilon|^2) \|_{H^1}^2 \right)
\]

\[
+ \| \nabla^2 u_\varepsilon \|_{L^2(0, T_M; H^1)}^2 + \| \partial_t u_\varepsilon \|_{L^2(0, T_M; H^1)}^2 + \varepsilon^{-2} \| \nabla (|u_\varepsilon|^2) \|_{L^2(0, T_M; H^1)}^2 \leq CM
\]

for any \(\varepsilon \leq \varepsilon_M\). Next, applying Lemma 3.2, one has the following higher estimates

\[
(v_\varepsilon, \nabla v_\varepsilon) \in L^\infty(\tau, T_M; H^k(\mathbb{R}^3)) \cap L^2(\tau, T_M; H^{k+1}(\mathbb{R}^3)),
\]

\[
(\partial_t v_\varepsilon, \partial_t \nabla u_\varepsilon) \in L^2(\tau, T_M; H^{k-1}(\mathbb{R}^3)),
\]

which have uniform bounds in \(\varepsilon\), for any \(k \geq 2\). It follows from the Aubin-Lions Lemma that there exists a subsequence such that

\[
v_\varepsilon \to v \quad \text{in} \quad C[\tau, T_M; H^{k-1}(B_R(0))]
\]

\[
u_\varepsilon \to u \quad \text{in} \quad C[\tau, T_M; H^k(B_R(0))]
\]

for any \(k \geq 2\) and \(R \in (0, \infty)\). This together with (4.61) implies that

\[
(u_\varepsilon, v_\varepsilon) \to (u, v) \quad \text{in} \quad C[\tau, T_M; C_\text{loc}^\infty(\mathbb{R}^3)]
\]

with \(|u| = 1\). By Theorem 1 (\(u, v\)) must be the unique solution to the Ericksen-Leslie system \((1.1)-(1.3)\). Since \((u, v)\) is unique and any sequence \((u_\varepsilon, v_\varepsilon)\) has a convergent subsequence \((u_\varepsilon, v_\varepsilon)\), then the sequence \((u_\varepsilon, v_\varepsilon)\) converges to \((u, v)\) in \(C[\tau, T_M; C_\text{loc}^\infty(\mathbb{R}^3)]\). Then, using the equations \((1.1)-(1.3)\), it is not difficult to prove the smooth convergence in \(t\); that is

\[
(u_\varepsilon, v_\varepsilon) \to (u, v) \quad \text{in} \quad C^\infty([\tau, T_M]; C_\text{loc}^\infty(\mathbb{R}^3)).
\]

Now, we prove that \(T_M\) can be extended to \(T\).

Suppose that \(T_M < T\). Then it follows from Lemma 4.2 and integration by parts that

\[
\lim_{\varepsilon \to 0} \| (\nabla v_\varepsilon - \nabla v)(T_M) \|_{L^2}^2 \leq C \lim_{\varepsilon \to 0} \| (v_\varepsilon - v)(T_M) \|_{L^2}^2 \| \nabla^2 v_\varepsilon - \nabla^2 v(T_M) \|_{L^2}^2 = 0.
\]

Similarly,

\[
\lim_{\varepsilon \to 0} \| (\nabla^2 u_\varepsilon - \nabla^2 u)(T_M) \|_{L^2}^2 = 0, \quad \lim_{\varepsilon \to 0} \varepsilon^{-2} \| \nabla (1 - |u_\varepsilon|^2)(T_M) \|_{L^2}^2 = 0
\]

Therefore, we obtain

\[
\lim_{\varepsilon \to 0} \left( \| v_\varepsilon(T_M) \|_{H^1(\mathbb{R}^3)}^2 + \| \nabla u_\varepsilon(T_M) \|_{H^1(\mathbb{R}^3)}^2 + \varepsilon^{-2} \| (1 - |u_\varepsilon|^2)^2(T_M) \|_{H^1(\mathbb{R}^3)}^2 \right)
\]

\[
= \| v(T_M) \|_{H^1(\mathbb{R}^3)}^2 + \| \nabla u(T_M) \|_{H^1(\mathbb{R}^3)}^2 \leq \frac{M}{2}.
\]

Hence, for sufficiently small \(\varepsilon\), one has

\[
\| v_\varepsilon(T_M) \|_{H^1(\mathbb{R}^3)}^2 + \| \nabla u_\varepsilon(T_M) \|_{H^1(\mathbb{R}^3)}^2 + \varepsilon^{-2} \| (1 - |u_\varepsilon|^2)^2(T_M) \|_{H^1(\mathbb{R}^3)}^2 \leq M.
\]
Moreover, (4.56) and Lemma 4.2 imply \( \frac{3}{4} \leq |u_\varepsilon(T_M)| \leq \frac{5}{4} \) for sufficiently small \( \varepsilon \). Therefore, using \( (u_\varepsilon(T_M), u_\varepsilon(T_M)) \) as a new initial data at \( t = T_M \) and applying Proposition 3.1 again, we can extend the strong solution \( (u_\varepsilon, v_\varepsilon) \) to the time \( T_1 = \min\{T, 2T_M\} \). By the same argument above, it is obvious that
\[
(u_\varepsilon, v_\varepsilon) \to (u, v) \quad \text{in} \quad C^\infty(\tau, T_1]; C^\infty_{loc}(\mathbb{R}^3)).
\]

We repeat the above two steps and establish the convergence up to \( T \) for any \( T < T^* \). This completes the proof of Theorem 2. \( \square \)

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References

[1] J. M. Ball, Mathematics and liquid crystals, Molecular Crystals and Liquid Crystals 647 (2017) 1-27.
[2] T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3D Euler equation, Commun. Math. Phys. 94 (1984) 61–66.
[3] F. Bethuel, H. Brezis, F. Hélein, Asymptotics for the minimization of a Ginzburg-Landau functional, Calc. Var. & PDEs 1 (1993) 123-148.
[4] F. Bethuel, H. Brezis, G. Orlandi, Asymptotics for the Ginzburg-Landau Equation in Arbitrary Dimensions, Journal of Functional Analysis 186 (2001) 432–520.
[5] F. Bethuel, G. Orlandi, D. Smets, Convergence of the parabolic Ginzburg-Landau equation to motion by mean curvature, Annals of Mathematics, 163 (2006) 37–163.
[6] C. Cavaterra, E. Rocca, H. Wu, Global weak solution and blow-up criterion of the general Ericksen-Leslie system for nematic liquid crystal flows, J. Differential Equations 255 (2013) 24–57.
[7] Y. Chen, M. Struwe, Existence and partial regular results for the heat flow for harmonic maps, Math. Z. 201 (1989) 83-103.
[8] J. L. Ericksen, Conservation laws for liquid crystals, Trans. Soc. Rheol. 5 (1961) 23-34.
[9] J. L. Ericksen, Inequalities in liquid crystals theory, Phys. Fluids 9, (1966) 12051207.
[10] J. Fan, B. Guo, Regularity criterion to some liquid crystal models and the Landau-Lifshitz equations in \( \mathbb{R}^3 \), Sci. China Ser. A 51 (2008) no. 10, 1787-1797.
[11] F. C. Frank, On the theory of liquid crystals, Discuss. Faraday Soc. 25 (1958) 19-28.
[12] V. Ginzburg, L. Landau, On the theory of superconductivity, Zh. Eksp. Teor. Fiz. 20 (1950) 1064-1082.
[13] J. Hineman, C. Wang, Well-posedness of Nematic Liquid Crystal flow in \( L^3_{uloc}(\mathbb{R}^3) \), Arch. Rational Mech. Anal. 210 (2013) 177-218.
[14] T. Huang, C. Wang, Blow up criterion for nematic liquid crystal flows, Comm. Part. Diff. Eqns. 37 (2012) 875-884.
[15] M.-C. Hong, Global existence of solutions of the simplified Ericksen-Leslie system in dimension two, Calc. Var. & PDEs 40 (2011) 15-36.
[16] M.-C. Hong, J. Li, Z. Xin, Blow-up Criteria of Strong Solutions to the Ericksen-Leslie System in \( \mathbb{R}^3 \), Commun. Partial Differ. Equ. 39 (2014) 1284-1328.
[17] M.-C. Hong, Y. Mei, Well-posedness of the Ericksen-Leslie system with the Oseen-Frank energy in \( L^3_{uloc}(\mathbb{R}^3) \), Calc. Var. & PDEs 58 (2019) no. 1, Art. 3, 38 pp.
[18] M.-C. Hong, Z. Xin, Global existence of solutions of the Liquid Crystal flow for the Oseen-Frank model in \( \mathbb{R}^2 \), Adv. Math. 231 (2012) 1364-1400.
[19] J. Huang, F.-L. Lin, C. Wang, Regularity and Existence of Global Solutions to the Ericksen-Leslie System in \( \mathbb{R}^3 \), Comm. Math. Phys. 331 (2014) 805-850.
[20] F. M. Leslie, Some constitutive equations for liquid crystals, Arch. Rational Mech. Anal. 28 (1968), 265-283.
[21] J. Li, E. S. Titi, Z. Xin, On the uniqueness of weak solutions to the Ericksen-Leslie liquid crystal model in \( \mathbb{R}^2 \), Math. Models Methods Appl. Sci. 26 (2016), no. 4, 803-822.
[22] F.-H. Lin, J. Lin, C. Wang, Liquid crystal flow in two dimension, Arch. Rational Mech. Anal. 197 (2010) 297-336.
[23] F.-H. Lin, C. Liu, Nonparabolic dissipative systems modelling the flow of liquid crystals, Comm. Pure Appl. Math. 48 (1995) 501-537.
[24] F.-H. Lin, C. Liu, Existence of solutions for the Ericksen-Leslie System, Arch. Rational Mech. Anal. 154 (2000) 135-156.
[25] F.-H. Lin, C. Wang, Global Existence of Weak Solutions of the Nematic Liquid Crystal Flow in Dimension Three, Comm. Pure Appl. Math. vol. LXIX (2016) 1532-1571.
[26] C. Liu, N. J. Walkington, Approximation of Liquid Crystal Flows, SIAM J. Numer. Anal. 37 (2000) 725741.
[27] C. W. Oseen, The theory of liquid crystals, Trans. Faraday Soc. 29 (1933) 833-899.
[28] I. W. Stewart, The Static and Dynamic Continuum Theory of Liquid Crystals, Taylor and Francis (2004).
[29] M. Struwe, On the evolution of harmonic maps of Riemannian surfaces, Commun. Math. Helv. 60 (1985) 558-581.
[30] M. Struwe, On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2 dimensions, Diff. Int. Eqs. 7 (1994) 1613-1624.
[31] C. Wang, Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data, Arch. Rational Mech. Anal. 200 (2011) 1-19.
[32] M. Wang and W. Wang, Global existence of weak solution for the 2-D Ericksen-Leslie system Calc. Var. & PDEs 51 (2014) 915-962.
[33] M. Wang, W. Wang and Z. Zhang, On the uniqueness of weak solution for the 2-D Ericksen-Leslie system, Discrete Contin. Dyn. Syst. Ser. B 21 (2016), no. 3, 919-941.
[34] W. Wang, P. Zhang and Z. Zhang, Well-Posedness of the Ericksen-Leslie System, Arch. Rational Mech. Anal. 210 (2013) 837-855.
[35] N. J. Walkington, Numerical approximation of nematic liquid crystal flows governed by the Ericksen-Leslie equations, ESAIM: Mathematical Modelling and Numerical Analysis 45 (2011) 523-540.
[36] H. Wen, S. Ding, Solutions of incompressible hydrodynamic flow of liquid crystals Nonlinear Anal. 12 (2011) 1510-1531.
[37] Stein, E.: Singular integrals and differentiability properties of functions. Princeton Univ Press, Princeton, N.J. (1970)

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