A Coinductive Approach to Proving Reachability Properties in Logically Constrained Term Rewriting Systems

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Abstract. We introduce a sound and complete coinductive proof system for reachability properties in transition systems generated by logically constrained term rewriting rules over an order-sorted signature modulo builtins. A key feature of the calculus is a circularity proof rule, which allows to obtain finite representations of the infinite coinductive proofs.

1 Introduction

We propose a framework for specifying and proving reachability properties of systems whose behaviour is modelled using transition systems described by logically constrained term rewriting systems (LCTRSs). By reachability properties we mean that a set of target states are reached in all terminating system computations starting from a given set of initial states. We assume transition systems are generated by constrained term rewriting rules of the form

\[ l \rightarrow r \text{ if } \phi, \]

where \( l \) and \( r \) are terms and \( \phi \) is a logical constraint. The terms \( l, r \) may contain both uninterpreted function symbols and function symbols interpreted in a builtin model, e.g., the model of booleans and integers. The constraint \( \phi \) is a first-order formula that limits the application of the rule and which may contain predicate symbols interpreted in the builtin model. The intuitive meaning of a constrained rule \( l \rightarrow r \text{ if } \phi \) is that any instance of \( l \) that satisfies \( \phi \) transitions in one step into a corresponding instance of \( r \).

Example 1. The following set of constrained rewrite rules specifies a procedure for compositeness:

\[
\begin{align*}
\text{init}(n) & \rightarrow \text{loop}(n, 2) \text{ if } \top, \\
\text{loop}(i \times k, i) & \rightarrow \text{comp} \text{ if } k > 1, \\
\text{loop}(n, i) & \rightarrow \text{loop}(n, i + 1) \text{ if } \neg(\exists k. k > 1 \land n = i \times k).
\end{align*}
\]

If \( n \) is not composite, the computation of the procedure is infinite.

Given a LCTRS, which serves as a specification for a transition system, it is natural to define the notion of constrained term \( (t | \phi) \), where \( t \) is an ordinary term (with variables) and \( \phi \) is a logical constraint. The intuitive meaning of such a term is the set of ground instances of \( t \) that satisfy \( \phi \).
Example 2. The constrained term \( \langle \text{init}(n) \mid \exists u.1 < u < n \land n \mod u = 0 \rangle \) defines exactly the instances of \( \text{init}(n) \) where \( n \) is composite.

A reachability formula is a pair of constrained terms \( \langle t \mid \phi \rangle \Rightarrow \langle t' \mid \phi' \rangle \). The intuitive meaning of a reachability formula is that any instance of \( \langle t \mid \phi \rangle \) reaches, along all terminating paths of the transition system, an instance of \( \langle t' \mid \phi' \rangle \) that agrees with \( \langle t \mid \phi \rangle \) on the set of shared variables.

Example 3. The reachability formula
\[
\langle \text{init}(n) \mid \exists u.1 < u < n \land n \mod u = 0 \rangle \Rightarrow \langle \text{comp} \mid \top \rangle
\]
captures a functional specification for the algorithm described in Example 1: each terminating computation starting from a state in which \( n \) is composite reaches the state \( \text{comp} \). Computations that start with a negative number (composite or not) are infinite and therefore vacuously covered by the specification above.

We propose an effective proof system that, given a LCTRS, proves valid reachability formulas such as the one above, assuming an oracle that solves logical constraints. In practice, we use an SMT solver instead of the oracle.

Contributions

1. As computations can be finite or infinite, an inductive approach for reachability is not practically possible. In Section 2, we propose a coinductive approach for specifying transition systems, which is an elegant way to look at reachability, but also essential in handling both finite and infinite executions.
2. We formalize the semantics of LCTRSs as a reduction relation over a particular model that combines order-sorted terms with built-in elements such as integers, booleans, arrays, etc. The new approach, introduced in Section 3, is simpler than the usual semantics for constrained term rewriting systems [18,20,19,13], but it also lifts several technical restrictions that are important for our case studies.
3. We introduce a sound and complete coinductive proof system for deriving valid reachability formulas for transition systems specified by a LCTRS. We present our proof system in two steps: in the first step, we provide a three-rule proof system (Figure 1) for symbolic execution of constrained terms. When interpreting the proof system coinductively, its proof trees can be finite or infinite. The finite proof trees correspond to reachability formulas \( \langle t \mid \phi \rangle \Rightarrow \langle t' \mid \phi' \rangle \) where there is a bounded number of symbolic steps between \( \langle t \mid \phi \rangle \) and \( \langle t' \mid \phi' \rangle \). The infinite proof trees correspond to proofs of reachability formulas \( \langle t \mid \phi \rangle \Rightarrow \langle t' \mid \phi' \rangle \) that hold for an unbounded number of symbolic steps between \( \langle t \mid \phi \rangle \) and \( \langle t' \mid \phi' \rangle \) (obtained, e.g., by unrolling loops). Symbolic execution has similarities to narrowing, but unlike narrowing, where each step computes a possible successor, symbolic execution must consider all successors of a state at the same time.
4. The infinite proof trees above cannot be obtained in finite time in practice. In order to derive reachability formulas that require an unbounded number of symbolic steps in finite time, we introduce a fourth proof rule to the system that we call circularity. The circularity proof rule can be used to compress infinite proof trees into finite proof trees. The intuition is to use as axioms the goals that are to be proven, when they satisfy a guardedness condition. This compression of infinite coinductive trees into finite proof trees via
the guardedness condition nicely complements our coinductive approach. This separation between symbolic execution and circularity answers an open question in [21]. 5. We introduce the RMT tool, an implementation of the proof system that validates our approach on a number of examples. RMT uses an SMT solver to discharge logical constraints. The tool is expressive enough for specifying various transition systems, including operational semantics of programming languages, and proving reachability properties of practical interest and is intended to be the starting point of a library for rewriting modulo builtins, which could have more applications.

Related Work A number of approaches [12132530] to combining rewriting and SMT solving have appeared lately. The rewrite tool Maude [11] has been extended with SMT solving in [25] in order to enable the analysis of open systems. A method for proving invariants based on an encoding into reachability properties is presented in [30]. Both approaches above are restricted to topmost rewrite theories. While almost any theory can be written as a topmost theory [22], the encoding can significantly increase the number of transitions, which raises performance concerns. Our definition for constrained term is a generalization of that of constructor constrained pattern used in [30]. In particular [30] does not allow for quantifiers in constraints, but quantifiers are critical to obtaining a complete proof system, as witnessed by their use in the subsumption rule in our proof system ([subs], Figure 1). The approach without quantifiers is therefore not sufficient to prove reachabilities in a general setting.

A calculus for reachability properties in a formalism similar to LCTRSs is given in [1]. However, the notion of reachability in [1] is different from ours: while we show reachability along all terminating paths of the computation, [1] solves reachability properties of the form \( \exists \bar{x}. t(\bar{x}) \rightarrow^* t'(\bar{x}) \) (i.e. does there exists an instance of \( t \) that reaches, along some path, an instance of \( t' \)).

Work on constrained term rewriting systems appeared in [20191813]. In contrast to this approach to constrained rewriting, our semantics is simpler (it does not require two reduction relations), it does not have restrictions on the terms \( l, r \) in a rule \( l \rightarrow r \models \phi \) and the constraint is an arbitrary first-order formula \( \phi \), possibly with quantifiers, which are crucial to obtain symbolic execution in its full generality. Constrained terms are generalized to guarded terms in [2], in order to reduce the state space.

Reachability in rewriting is explored in depth in [12]. The work by Kirchner and others [17] is the first to propose the use of rewriting with symbolic constraints for deduction. Subsequent work [252013] extends and unifies previous approaches to rewriting with constraints. The related work section in [25] includes a comprehensive account of literature related to rewriting modulo constraints.

Our previous work [921] on proving program correctness was in the context of the K framework [27]. K, developed by Rosu and others, implements semantics-based program verifiers [10] for any language that can be specified by a rewriting-based operational semantics, such as C [15], Java [4] and JavaScript [23]. Our formalism is not more expressive than that of reachability logic [9] for proving
partial correctness of programs in a language-independent manner, but it does have several advantages. Firstly, we make a clear separation between *rewrite rules* (used to define transition systems), for which it makes no sense to have constraints on both the lhs and the rhs, and *reachability formulas* (used to specify reachability properties), for which there can be constraints on both the lhs and the rhs. We provide clear semantics of both syntactic constructs above, which makes it unnecessary to check well-definedness of the underlying rewrite system, as required in [9]. Additionally, this separation, which we see as a contribution, makes it easy to get rid of the top-most restriction in previous approaches. Another advantage is that the proposed proof system is very easy to automate, while being sufficiently expressive to specify real-world applications. Additionally, we work in the more general setting of LCTRSs, not just language semantics, which enlarges the possible set of applications of the technique. We also have several major technical improvements compared to [21], where the proof system is restricted to the cases where unification can be reduced to matching and topmost rewriting. The totality property required for languages specifications, which was quite restrictive, was replaced by a local property in proof rules and all restrictions needed to reduce unification to matching were removed.

In contrast to the work on partial correctness in [10], the approach on reachability discussed here is meant for any LCTRS, not just operational semantics. The algorithm in [10] contains a small source of incompleteness, as when proving a reachability property it is either discharged completely through implication or through circularities/rewrite rules. We allow a reachability rule to be discharged partially by subsumption and partially by other means. Constrained terms are a fragment of Matching Logic (see [26]), where no distinction is made between terms and constraints. Coinduction and circular or cyclic proofs have been proposed in other contexts. For example, circular proof systems have been proposed for first-order logic with inductive predicates in [6] and for separation logic in [5]. In the context of interactive theorem provers, circular coinduction has been proposed as an incremental proof method for bisimulation in process calculi (see [24]). A compositional and incremental approach to coinduction that uses a semantic guardedness check instead of a syntactic check is given in [10].

*Paper Structure* We present coinductive definitions for execution paths and reachability predicates in Section 2. In Section 3, we introduce logically constrained term rewriting with builtins in an order-sorted setting. In Section 4, we propose a sound and complete coinductive calculus for reachability and a circularity rule for compressing infinite proof trees into finite proof trees. Section 6 discusses the implementation before concluding. The proofs and a discussion of coinduction and order-sorted algebras can be found in the Appendix.

2 Reachability Properties: Coinductive Definition

In this section we introduce a class of reachability properties, defined coinductively. A *state predicate* is a subset of states. A *reachability property* is a pair
$P \Rightarrow Q$ of state predicates. Such a reachability property is \textit{demonically valid} iff each execution path starting from a state in $P$ eventually reaches a state in $Q$, or if it is infinite. Since the set of finite and infinite executions is coinductively defined, the set of valid predicates can be defined coinductively as well. Formally, consider a transition system $(M, \rightsquigarrow)$, with $\rightsquigarrow \subseteq M \times M$. We write $\gamma \rightsquigarrow \gamma'$ for $(\gamma, \gamma') \in \rightsquigarrow$. An element $\gamma \in M$ is \textit{irreducible} if $\gamma \not\rightsquigarrow \gamma'$ for any $\gamma' \in M$.

\textbf{Definition 1 (Execution Path).} The set of (complete) execution paths is coinductively defined by the following rules:

\[
\begin{align*}
\gamma \in M, \gamma \text{ irreducible} & \quad \frac{}{\gamma \rightsquigarrow } \\
\gamma_0 \circ \tau & \quad \frac{}{\gamma_0 \rightsquigarrow \text{hd}(\tau)}
\end{align*}
\]

where the function $\text{hd}$ is defined by $\text{hd}(\gamma) = \gamma$ and $\text{hd}(\gamma_0 \circ \tau) = \gamma_0$.

The above definition includes both the finite execution paths ending in an irreducible state and the infinite execution paths, defined as the greatest fixed point of the associated functional (see Appendix A.2).

\textbf{Definition 2 (State and Reachability Predicates).} A state predicate is a subset $P \subseteq M$. A reachability predicate is a pair of state predicates $P \Rightarrow Q$. The predicate $P$ is \textit{runnable} if $P \neq \emptyset$ and for all $\gamma \in P$ there is $\gamma' \in M$ s.t. $\gamma \rightsquigarrow \gamma'$.

A \textit{derivative} measures the sensitivity to change of a quantity. For the case of transition systems, the change of states is determined by the transition relation.

\textbf{Definition 3 (Derivative of a State Predicate).} The derivative of a state predicate $P$ is the state predicate $\partial(P) = \{\gamma' | \gamma \rightsquigarrow \gamma' \text{ for some } \gamma \in P\}$.

As a reachability predicate specifies reachability property of execution paths, we define when a particular execution path satisfies a reachability predicate.

\textbf{Definition 4 (Satisfaction of a Reachability Predicate).} An execution path $\tau$ satisfies a reachability predicate $P \Rightarrow Q$, written $\tau \models P \Rightarrow Q$, iff $\langle \tau, P \Rightarrow Q \rangle \in \nu \text{EPSRP}$, where \text{EPSRP} consists of the following rules:

\[
\begin{align*}
\langle \tau, P \Rightarrow Q \rangle & \quad \frac{}{\text{hd}(\tau) \in P \cap Q} \\
\langle \tau, \partial(P) \Rightarrow Q \rangle & \quad \frac{}{\gamma_0 \in P, \gamma_0 \rightsquigarrow \text{hd}(\tau)}
\end{align*}
\]

The notation $\nu \text{EPSRP}$ stands for the functional of \text{EPSRP} and $\nu \text{EPSRP}$ stands for its greatest fixed point (see Appendix A.2). We coinductively define the set of \textit{demonically valid reachability predicates} over $(M, \rightsquigarrow)$. This allows to use coinductive proof techniques to prove validity of reachability predicates.

\textbf{Definition 5 (Valid Reachability Predicates, Coinductively).} We say that $P \Rightarrow Q$ is demonically valid, and we write $(M, \rightsquigarrow) \models P \Rightarrow Q$, iff $P \Rightarrow Q \in \nu \text{DVP}$, where \text{DVP} consists of the following rules:

\[
\begin{align*}
\text{[Subsumption]} & \quad P \Rightarrow Q \subseteq Q \\
\text{[Step]} & \quad \frac{\partial(P \setminus Q) \Rightarrow Q}{P \Rightarrow Q} P \setminus Q \text{ runnable}.
\end{align*}
\]
The condition $P \setminus Q$ runnable in the second rule is essential to avoid the cases where execution is stuck. These blocking states have no successor in $\partial(P \setminus Q)$ and, in the absence of the condition, we would wrongly conclude that they satisfy $P \Rightarrow Q$. The terminating executions are captured by [Subsumption].

The following proposition justifies our definition of demonically valid reachability predicates.

**Proposition 1.** Let $P \Rightarrow Q$ be a reachability predicate. We have $(M, \cdot \cdot \cdot) \models^\forall P \Rightarrow Q$ iff any execution path $\tau$ starting from $P$ (hd($\tau$) $\in$ $P$) satisfies $P \Rightarrow Q$.

### 3 Logically Constrained Term Rewriting Systems

In this section we introduce our formalism for LCTRSs. We interpret LCTRSs in a model combining order-sorted terms with builtins such as integers, booleans, etc. Logical constraints are first-order formulas interpreted over the fixed model.

We assume a built-in model $M^b$ for a many-sorted built-in signature $\Sigma^b = (S^b, F^b)$, where $S^b$ is a set of built-in sorts that includes at least the sort $\text{Bool}$ and $F^b$ is the $S^b$-sorted set of built-in function symbols. We assume that the set interpreting the sort $\text{Bool}$ in the model $M^b$ is $M^b_{\text{bool}} = \{\top, \bot\}$. We use the standard notation $M_\sigma$ for the interpretation of the sort/symbol $\sigma$ in the model $M$. The set $\text{CF}^b$, defined as the set of (many-sorted) first-order formulas with equality over the signature $\Sigma^b$, is the set of built-in constraint formulas. Functions returning $\text{Bool}$ play the role of predicates and terms of sort $\text{Bool}$ are atomic formulas. We will assume that the built-in constraint formulas can be decided by an oracle (implemented as an SMT solver).

A signature modulo builtins is an order-sorted signature $\Sigma = (S, \leq, F)$ that includes $\Sigma^b$ as a subsignature and such that the only builtin constants in $\Sigma$ are elements of the builtin model $\{c \mid c \in F_{\leq,s}, s \in S^b\} = M^b_s$ — therefore the signature might be infinite. By $F_{w,s}$ we denoted the set of function symbols of arity $w$ and result sort $s$. $\Sigma^b$ is called the builtin subsignature of $\Sigma$ and $\Sigma^c = (S, \leq, (F \setminus F^b) \cup \bigcup_{s \in S^b} F_{\leq,s})$ the constructor subsignature of $\Sigma$. We let $X$ be an $S$-sorted set of variables.

We extend the builtin model $M^b$ to an $(S, \leq, F)$-model $M^c$ defined as follows:

- $M^c_s = T_{\Sigma^c,s}$, for each $s \in S \setminus S^b$ ($M^c_s$ is the set of ground constructor terms of sort $s$, i.e. terms built from constructors applied to builtin elements);
- $M^c_f = M^b_f$ for each builtin function symbol $f \in F^b$;
- $M^c_f$ is the term constructor $M^c_f(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$, for each non-builtin function symbol $f \in F \setminus F^b$.

By fixing the interpretation of the non-builtin function symbols, we can reduce constraint formulas to built-in constraint formulas by relying on an unification algorithm described in detail in [4]. We also make the standard assumption that $M_s \neq \emptyset$ for any $s \in S$.

**Example 4.** Let $\Sigma^b = (S^b, F^b)$, where $S^b = \{\text{Int}, \text{Bool}\}$ and $F^b$ include the usual operators over $\text{Booleans}$ ($\lor, \land, \ldots$) and over the $\text{Integers}$ ($+, -, \times, \ldots$). The builtin model $M^b$ interprets the above sorts and operations as expected.
We consider the signature modulo builtins $\Sigma = (S, \leq, F)$, where the set of sorts $S = \{ \text{Cfg}, \text{Int}, \text{Bool} \}$ consists of the builtin sorts and an additional sort $\text{Cfg}$, where the subsorting relation $\leq \subseteq S \times S = \emptyset$ is empty, and where the set of function symbols $F$ includes, in addition to the builtin symbols in $F^b$, the following function symbols: $\text{init} : \text{Int} \to \text{Cfg}$, $\text{loop} : \text{Int} \times \text{Int} \to \text{Cfg}$, $\text{comp} : \text{Cfg}$.

We have that $M_{\text{Cfg}}^\Sigma = \{ \text{init}(i) \mid i \in \mathbb{Z} \} \cup \{ \text{loop}(i, j) \mid i, j \in \mathbb{Z} \} \cup \{ \text{comp} \}$.

The set $\text{CF}$ of constraint formulas is the set of first-order formulas with equality over the signature $\Sigma$. The subset of the builtin constraint formulas is denoted by $\text{CF}^b$. Let $\text{var}(\phi)$ denote the set of variables freely occurring in $\phi$. We write $M, \alpha \models \phi$ when the formula $\phi$ is satisfied by the model $M^\Sigma$ with a valuation $\alpha : X \to M^\Sigma$.

**Example 5.** The constraint formula $\phi \overset{\text{def}}{=} \exists u.1 < u < n \land n \ mod \ u = 0$ is satisfied by the model $M^\Sigma$ defined in Example 4 and any valuation $\alpha$ such that $\alpha(n)$ is a composite number.

**Definition 6 (Constrained Terms).** A constrained term $\varphi$ of sort $s \in S$ is a pair $(t | \varphi)$, where $t \in T_{\Sigma,s}(X)$ and $\varphi \in \text{CF}$.

**Example 6.** Continuing the previous example, the following is a constrained term: $(\text{init}(n) \mid \exists u.1 < u < n \land n \ mod \ u = 0)$.

We consistently use $\varphi$ for constrained terms and $\phi$ for constraint formulas.

**Definition 7 (Valuation Semantics of Constraints).** The valuation semantics of a constraint $\varphi$ is the set $[\varphi] \overset{\text{def}}{=} \{ \alpha : X \to M^\Sigma \mid M^\Sigma, \alpha \models \varphi \}$.

**Example 7.** Continuing the previous example, we have that $[\exists u.1 < u < n \land n \ mod \ u = 0] = \{ \alpha : X \to M^\Sigma \mid \alpha(n) \text{ is composite} \}$.

**Definition 8 (State Predicate Semantics of Constrained Terms).** The state predicate semantics of a constrained term $(t | \varphi)$ is the set $[[t | \varphi]] \overset{\text{def}}{=} \{ \alpha(t) \mid \alpha \in [\varphi] \}$.

**Example 8.** Continuing the previous example, we have that $[[\text{init}(n) \mid \exists u.1 < u < n \land n \ mod \ u = 0]] = \{ \text{init}(n) \mid n \text{ is composite} \}$.

We now introduce our formalism for logically constrained term rewriting systems. Syntactically, a rewrite rule consists of two terms (the left hand side and respectively the right hand side), together with a constraint formula. As the two terms could `share some variables`, these shared variables should be instantiated consistently in the semantics:

**Definition 9 (LCTRS).** A logically constrained rewrite rule is a tuple $(l, r, \varphi)$, often written as $l \rightarrow_r \varphi$, where $l, r$ are terms in $T_{\Sigma}(X)$ having the same sort, and $\varphi \in \text{CF}$. A logically constrained term rewriting system $\mathcal{R}$ is a set of logically constrained rewrite rules. $\mathcal{R}$ defines an order-sorted transition relation $\rightarrow_{\mathcal{R}}$ on $M^\Sigma$ as follows: $t \rightarrow_{\mathcal{R}} t'$ iff there exist a rule $l \rightarrow_r \varphi$ in $\mathcal{R}$, a context $c[l]$, and a valuation $\alpha : X \to M^\Sigma$ such that $t = \alpha(c[l])$, $t' = \alpha(c[r])$ and $M^\Sigma, \alpha \models \varphi$. 


Example 9. We recall the LCTRS given in the introduction:
\[ R = \begin{cases} \text{init}(n) \rightarrow \text{loop}(n, 2) \text{ if } \top, \\
\text{loop}(i \times k, i) \rightarrow \text{comp} \text{ if } k > 1, \\
\text{loop}(n, i) \rightarrow \text{loop}(n, i + 1) \text{ if } \neg(\exists k. k > 1 \land n = i \times k) \end{cases}. \]

A LCTRS \( R \) defines a sort-indexed transition system \((M^\Sigma, \leadsto_R)\). As each constrained term \( \varphi \) defines a state predicate \([\varphi]\), it is natural to specify reachability predicates as pairs of constrained terms sharing a subset of variables. The shared variables must be instantiated in the same way by the execution paths connecting states specified by the two constrained terms.

Definition 10 (Reachability Properties of LCTRSs). A reachability formula \( \varphi \Rightarrow \varphi' \) is a pair of constrained terms, which may share variables. We say that a LCTRS \( R \) demonically satisfies \( \varphi \Rightarrow \varphi' \), written \( R \models^\forall \varphi \Rightarrow \varphi' \), iff \((M^\Sigma, \leadsto_R) \models^\forall [\sigma(\varphi)] \Rightarrow [\sigma(\varphi')] \) for each \( \sigma : \text{var}(\varphi) \cap \text{var}(\varphi') \rightarrow M^\Sigma \).

Example 10. Continuing the previous example, we have that the reachability formula \( \langle \text{init}(n) \mid \exists u. 1 < u < n \land n \mod u = 0 \rangle \Rightarrow \langle \text{comp} \mid \top \rangle \) is demonically satisfied by the constrained rule system \( R \) defined in Example 9:
\[ R \models^\forall (\text{init}(n) \mid \exists u. 1 < u < n \land n \mod u = 0) \Rightarrow (\text{comp} \mid \top). \]
We have checked the above reachability formula against \( R \) mechanically, using an implementation of the approach described in this paper.

4 Proving Reachability Properties of LCTRSs

We introduce two proof systems for proving reachability properties in transition systems specified by LCTRSs. The first proof system formalizes symbolic execution in a LCTRS, in the following sense: a reachability formula \( \varphi \Rightarrow \varphi' \) can be proven if either the left-hand side \( \varphi \) can be derived infinitely many times (and therefore all execution paths starting with \( \varphi \) are infinite), or if some derivative is an instance of the right-hand side \( \varphi' \), i.e., all the execution paths starting with \( \varphi \) reach a state that is an instance of \( \varphi' \). Note that this intuition holds when the proof system is interpreted coinductively, where infinite proof trees are allowed. Unfortunately, these infinite proof trees have a limited practical use because they cannot be obtained in finite time.

In order to solve this limitation, we introduce a second proof system, which contains an additional inference rule, called circularity. The circularity rule allows to use the reachability formula to be proved as an axiom. This allows to fold infinite proof trees into finite proof trees, which can be obtained in finite time. Adding the reachability formulas that are to be proved as axioms seems at first to be unsound, but it corresponds to a natural intuition: when reaching a proof obligation that we have handled before, there is no need to prove
it again, because the previous reasoning can be reused (possibly leading to an infinite proof branch). However, the circularity rule must be used in a guarded fashion in order to preserve soundness. We introduce a simple criterion to select the sound proof trees.

4.1 Derivatives of Constrained Terms

Our proof system relies on the notion of derivative at the syntactic level:

**Definition 11 (Derivatives of Constrained Terms).** The set of derivatives of a constrained term \( \varphi \triangleq (t \mid \phi) \) w.r.t. a rule \( l \Rightarrow r \) is

\[
\Delta_{l,r,\phi_r}(\varphi) \triangleq \{ [c[r] \mid \phi'] \mid \phi' \triangleq \phi \land t = c[l] \land \phi_r, c[l] \text{ an appropriate context and } \phi' \text{ is satisfiable} \},
\]

where the variables in \( l \Rightarrow r \) are renamed such that \( \text{var}(l, r, \phi_r) \) and \( \text{var}(\varphi) \) are disjoint. If \( \mathcal{R} \) is a set of rules, then \( \Delta_{\mathcal{R}}(\varphi) = \bigcup_{(l,r,\phi_r) \in \mathcal{R}} \Delta_{l,r,\phi_r}(\varphi) \).

A constrained term \( \varphi \) is \( \mathcal{R} \)-derivable if \( \Delta_{\mathcal{R}}(\varphi) \neq \emptyset \).

**Example 11.** Continuing the previous examples, we have that

\[
\Delta_{\mathcal{R}}(\langle \text{init}(n) \mid \exists u. 1 < u < n \land n \mod u = 0 \rangle) = \{ \langle \text{loop}(n, 2) \mid \exists u. 1 < u < n \land n \mod u = 0 \rangle \}.
\]

In the above case, \( \Delta_{\mathcal{R}} \) includes only the derivative computed w.r.t. the first rule in \( \mathcal{R} \), because the constraints of the ones computed w.r.t. the other rules are unsatisfiable. Intuitively, the derivatives of a constrained term denote all its possible successor configurations in the transition system generated by \( \mathcal{R} \).

The symbolic derivatives and the concrete ones are related as expected:

**Theorem 1.** Let \( \varphi \triangleq (t \mid \phi) \) be a constrained term, \( \mathcal{R} \) a constrained rule system, and \( (M^\Sigma, \rightarrow_{\mathcal{R}}) \) the transition system defined by \( \mathcal{R} \). Then \( \Delta_{\mathcal{R}}(\varphi) = \partial([\varphi]) \).

Our proof systems allows to replace any reachability formula by an equivalent one. Two reachability formulas, \( \varphi_1 \Rightarrow \varphi'_1 \) and \( \varphi_2 \Rightarrow \varphi'_2 \), are equivalent, written \( \varphi_1 \Rightarrow \varphi'_1 \equiv \varphi_2 \Rightarrow \varphi'_2 \), if, for all LCTRSs \( \mathcal{R} \),

\[
\mathcal{R} \vdash^\forall \varphi_1 \Rightarrow \varphi'_1 \text{ if } \mathcal{R} \vdash^\forall \varphi_2 \Rightarrow \varphi'_2.
\]

We write \([\varphi] \subseteq_{\text{shared}} [\varphi']\) iff for each \( \sigma : \text{var}(\varphi) \cap \text{var}(\varphi') \rightarrow M^\Sigma \), we have \([\sigma(\varphi)] \subseteq [\sigma(\varphi')]\). The next result, used in our proof system, shows that inclusion of the state predicate semantics of two constrained terms can be expressed as a constraint formula, when the shared variables are instantiated consistently.

**Proposition 2.** The inclusion \([\langle t \mid \phi \rangle] \subseteq_{\text{shared}} [\langle t' \mid \phi' \rangle]\) holds if and only if

\[
M^\Sigma \models \phi \rightarrow (\exists \bar{x})(t = t' \land \phi'),
\]

where \( \bar{x} \models \text{var}(t', \phi') \setminus \text{var}(t, \phi) \).

4.2 Proof System for Symbolic Execution

The first proof system, DSTEP, derives sequents of the form \( \langle t_1 \mid \phi_1 \rangle \Rightarrow \langle t_r \mid \phi_r \rangle \). The proof system consists of three proof rules presented in Figure 1 and an implicit structural rule that allows to replace reachability formulas by equivalent
Theorem 2. Let $R$ be a LCTRS. For any reachability formula $\phi \Rightarrow \phi'$, we have $R \models^\exists \phi \Rightarrow \phi'$ iff $\phi \Rightarrow \phi' \in \nu \mathcal{DSTEP}(R)$.

Example 12. Consider the LCTRS $R$ defined in Example 9. The proof tree for the reachability formula $\langle \text{init}(n) \mid \psi \rangle \Rightarrow \varphi_r$, where $\psi \trianglerighteq \exists u.1 < u < n \land u \mod n = 0$ denotes the fact that $n$ is composite and $\varphi_r \trianglerighteq \langle \text{comp} \mid \top \rangle$, is infinite.
with ability formulas. Then the set of rules DCC

\[\text{Definition 12 (Demonic circular coinduction).} \]

goals as axioms to fold infinite of the infinite proof trees. The next inference rule is intended to use the initial

As we said at the beginning of the section, the use of

4.3 Extending the Proof System with a Circularity Rule

The right branch of the above proof tree is infinite, and:

\[\phi_2 \triangleq \neg \exists k. k > 1 \land n = 2 \times k \quad \phi_3 \triangleq \neg \exists k. k > 1 \land n = 3 \times k \]

Note that in the presentation of the tree above, we used the structural rule to replace reachability formulas by equivalent reachability formulas as follows:

\[\langle \text{comp} \mid \psi \land \phi_a \land \neg (\text{comp} = \text{comp} \land T) \rangle \Rightarrow \varphi_r \quad \equiv \quad \langle \text{comp} \mid \bot \rangle \Rightarrow \varphi_r,\]

\[\langle \text{comp} \mid \psi \land \phi_2 \land \phi_b \land \neg (\text{comp} = \text{comp} \land T) \rangle \Rightarrow \varphi_r \quad \equiv \quad \langle \text{comp} \mid \bot \rangle \Rightarrow \varphi_r,\]

\[\langle \text{loop}(n', 2) \mid T \land \text{init}(n') = \text{init}(n) \land \psi \rangle \Rightarrow \varphi_r \quad \equiv \quad \langle \text{loop}(n, 2) \mid \psi \rangle \Rightarrow \varphi_r,\]

\[\langle \text{loop}(n', i' + 1) \mid \psi \land \phi_2' \rangle \Rightarrow \varphi_r \quad \equiv \quad \langle \text{loop}(n, 3) \mid \psi \land \phi_2 \rangle \Rightarrow \varphi_r,\]

where \(\phi_2' \triangleq \text{loop}(n, 2) = \text{loop}(n', i') \land \neg \exists k. k > 1 \land n' = i' \times k.\) The ticks appear in the formulas above because, to compute derivatives, we used the following fresh instance of \(\mathcal{R}:\)

\[\mathcal{R} = \begin{cases} 
\text{init}(n') \Rightarrow \text{loop}(n', 2) \text{ if } T, \\
\text{loop}(i' \times k', i') \Rightarrow \text{comp} \text{ if } k' > 1, \\
\text{loop}(n', i') \Rightarrow \text{loop}(n', i' + 1) \text{ if } \neg (\exists k. k > 1 \land n' = i' \times k) 
\end{cases} \]

4.3 Extending the Proof System with a Circularity Rule

As we said at the beginning of the section, the use of DSTEP is limited because of the infinite proof trees. The next inference rule is intended to use the initial goals as axioms to fold infinite DSTEP-proof trees into sound finite proof trees.

**Definition 12 (Demonic circular coinduction).** Let \(G\) be a finite set reachability formulas. Then the set of rules \(\text{DCC}(\mathcal{R}, G)\) consists of \(\text{DSTEP}(\mathcal{R})\), together with

\[\langle \text{circ} \mid t_1 \phi_t \land \phi \land \phi_c \rangle \Rightarrow \varphi_r,\]

\[\langle \text{circ} \mid t_1 \phi_t \land \neg \phi \rangle \Rightarrow \varphi_r,\]

\[\langle \text{circ} \mid t_1 \phi \rangle \Rightarrow \varphi_r,\]

\[\phi \equiv \exists \text{var}(t_1, \phi_c). t_1 = t_1 \phi_t \land \phi_c,\]

\[\langle t_1 \phi_t \rangle \Rightarrow \langle t_1 \phi_c \rangle \in G\]

where \(\langle t_1 \phi_t \rangle \Rightarrow \langle t_1 \phi_c \rangle\) is a rule in \(G\) whose variables have been renamed with fresh names.

The idea is that \(G\) should be chosen conveniently so that \(\text{DCC}(\mathcal{R}, G)\) proves \(G\) itself. We call such goals \(G\) (that are used to prove themselves) *circulairies*.

The intuition behind the rule is that the formula \(\phi\) defined in the rule holds when a circularity can be applied. In that case, it is sufficient to continue the current proof obligation from the rhs of the circularity \(\langle t_1 \phi_t \land \phi \land \phi_c \rangle\). The cases when \(\phi\) does not hold (the circularity cannot be applied) are captured by the proof obligation \(\langle t_1 \phi_t \land \neg \phi \rangle \Rightarrow \varphi_r\).
Of course, not all proof trees under DCC(\(\mathcal{R}, G\)) are sound. The next two definitions identify a class of sound proof trees (cf. Theorem 3).

**Definition 13.** Let PT be a proof tree of \(\varphi \Rightarrow \varphi'\) under DCC(\(\mathcal{R}, G\)). A \([\text{circ}]\) node in PT is guarded iff it has as ancestor a \([\text{der}]\) node. PT is guarded iff all its \([\text{circ}]\) nodes are guarded.

**Definition 14.** We write \((\mathcal{R}, G) \vdash^\forall \varphi \Rightarrow \varphi'\) iff there is a proof tree of \(\varphi \Rightarrow \varphi'\) under DCC(\(\mathcal{R}, G\)) that is guarded. If \(F\) is a set of reachability formulas, we write \((\mathcal{R}, G) \vdash^\forall F\) iff \((\mathcal{R}, G) \vdash^\forall \varphi \Rightarrow \varphi'\) for all \(\varphi \Rightarrow \varphi' \in F\).

The criterion stated by Definition 13 can be easily checked in practice. The following theorem states that the guarded proof trees under DCC are sound.

**Theorem 3 (Circularity Principle).** Let \(\mathcal{R}\) be a constrained rule system and \(G\) a set of goals. If \((\mathcal{R}, G) \vdash^\forall G\) then \(\mathcal{R} \models^\forall G\).

Theorem 3 can be used by finding a set of circularities and using them in a guarded fashion to prove themselves. Then the circularity principle states that such circularities hold.

**Example 13.** In order to prove \(\langle \text{init}(n) | \psi_1 \rangle \Rightarrow \langle \text{comp} | \top \rangle\), we choose the following set of circularities

\[ G = \{ \langle \text{init}(n) | \psi_1 \rangle \Rightarrow \langle \text{comp} | \top \rangle, \langle \text{loop}(n, i) | 2 \leq i \land \exists u. i \leq u < n \land n \mod u = 0 \rangle \Rightarrow \langle \text{comp} | \top \rangle \} \]

The second circularity is inspired by the infinite branch of the proof tree under DSTEP. We will show that \((\mathcal{R}, G) \vdash^\forall G\), and by Theorem 3, it follows that all reachability formulas in \(G\) hold in \(\mathcal{R}\).

**First circularity.** To obtain a proof of the first circularity, \(\langle \text{init}(n) | \psi_1 \rangle \Rightarrow \langle \text{comp} | \top \rangle\), we replace the infinite subtree rooted at \(\langle \text{loop}(n, 2) | \psi \rangle \Rightarrow \varphi_r\) in Example 12 by the following finite proof tree (that uses \([\text{circ}]\)):

\[
\frac{\langle \text{comp} | \bot \rangle \Rightarrow \varphi_r, \langle \text{comp} | \psi \land \phi \land \top \rangle \Rightarrow \varphi_r}{\langle \text{comp} | \psi \land \phi \land \top \rangle \Rightarrow \varphi_r} \quad [\text{subs}]
\]

\[
\frac{\langle \text{loop}(n, 2) | \psi \land \neg \phi \rangle \Rightarrow \varphi_r, \langle \text{loop}(n, 2) | \psi \land \neg \phi \rangle \Rightarrow \varphi_r}{\langle \text{loop}(n, 2) | \psi \rangle \Rightarrow \varphi_r} \quad [\text{circ}]
\]

where \(\phi \triangleq \exists u'. n'. \text{loop}(n, 2) = \text{loop}(n', i') \land 2 \leq i' \land \exists u. i' \leq u < n' \land n' \mod u = 0\).

**Second circularity.** To complete the proof of \(G\), we have to find a finite proof tree for \(\langle \text{loop}(n, i) | 2 \leq i \land \exists u. i \leq u < n \land n \mod u = 0 \rangle \Rightarrow \langle \text{comp} | \top \rangle\) as well. This is also obtained using \([\text{circ}]\) as follows:

\[
\frac{\langle \text{comp} | \bot \rangle \Rightarrow \varphi_r}{\langle \text{comp} | \psi_1 \land \psi_n \rangle \Rightarrow \varphi_r} \quad [\text{subs}]
\]

\[
\frac{T_1 \quad T_2}{\langle \text{loop}(n, i + 1) | \psi_1 \land \psi_n \rangle \Rightarrow \varphi_r} \quad [\text{circ}]
\]

\[
\frac{T_1 \quad T_2}{\langle \text{loop}(n, i) | \psi_1 \rangle \Rightarrow \langle \text{comp} | \top \rangle} \quad [\text{der}]^\forall
\]
where
\[
\psi_a \triangleq k' > 1 \land \text{loop}(n, i) = \text{loop}(i' \times k', i'),
\]
\[
\psi_b \triangleq \neg \exists k. k > 1 \land n = i \times k,
\]
\[
\psi_1 \triangleq 2 \leq i \land \exists u. u \leq u < n \land n \mod u = 0.
\]

The subtree

\[
\begin{array}{c}
T_1 \\
\hline
T_2
\end{array}
\]

is:

\[
\begin{array}{c}
\langle \text{comp} | \bot \rangle \Rightarrow \varphi_r \quad \text{[axiom]} \\
\langle \text{comp} | \psi_1 \land \psi_b \land \psi_c \rangle \Rightarrow \varphi_r \quad \text{[subs]} \\
\langle \text{loop}(n, i + 1) | \psi_1 \land \psi_b \land \neg \psi_c \rangle \Rightarrow \varphi_r \quad \text{[axiom]} \\
\langle \text{loop}(n, i + 1) | \psi_1 \land \psi_b \rangle \Rightarrow \varphi_r \quad \text{[circ]}
\end{array}
\]

where
\[
\psi_c \triangleq \exists n', i'. \text{loop}(n, i + 1) = \text{loop}(n', i') \land 2 \leq i' \land \exists u. u \leq u < n' \land n' \mod u = 0.
\]

The constraint \( \psi_c \) holds when the circularity can be applied and therefore this branch is discharged immediately by \text{subs} and \text{axiom}. The other branch, when the circularity cannot be applied, is discharged directly by \text{axiom}, as \( \psi_1 \land \psi_b \land \neg \psi_c \) is unsatisfiable (\( \psi_1 \) says that \( n \) has a divisor between \( i \) and \( n \), \( \psi_b \) says that \( i \) is not a divisor of \( n \), and \( \psi_c \) that \( n \) has a divisor between \( i + 1 \) and \( n \)).

Note that in both proof trees of the two circularities in \( G \), in order to apply the \[ \text{circ} \] rule, we used the following fresh instance of the second circularity:

\[
\langle \text{loop}(n', i') | 2 \leq i' \land \exists u. u \leq u < n' \land n' \mod u = 0 \rangle \Rightarrow \langle \text{comp} | \top \rangle.
\]

The proof trees for both goals (circularities) in \( G \) are guarded. We have shown therefore that \( (R, G) \vdash^Y G \). By the Circularity Principle (Theorem 3), we obtain that \( R \vdash^Y \forall \), and therefore

\[
R \vdash^Y \{ \langle \text{init} | n \mid \exists u. 1 < u < n \land n \mod u = 0 \rangle \Rightarrow \langle \text{comp} | \top \rangle, \langle \text{loop} | n, i \mid 2 \leq i \land \exists u. u \leq u < n \land n \mod u = 0 \rangle \Rightarrow \langle \text{comp} | \top \rangle \}
\]

which includes what we wanted to show of our transition system defined \( R \) in the running example.

5 Implementation

We have implemented the proof system for reachability in a tool called RMT (for rewriting modulo theories). RMT is open source and can be obtained from

[http://github.com/ciobaca/rmt/](http://github.com/ciobaca/rmt/)

To prove a reachability property, the RMT tool performs a bounded search in the proof system given above. The bounds can be set by the user. We have also tested the tool on reachability problems where we do not use strong enough circularities. In these cases, the tool will not find proofs. A difficulty that appears when a proof fails, difficulty shared by all deductive approaches to correctness, is that it is not known is the specification is wrong or if the circularities are not strong enough. Often, analysing the failing proof tree, the user may have
the chance to find a hint for the missing circularities, if any. In addition, proofs might also fail because of the incompleteness of the SMT solver. In addition to the running example, we have used RMT on a number of examples, summarized in the table below:

| LCTRS Reachability Property |  
|----------------------------|---|
| Computation of $1 + \ldots + n$ | Result is $n \times (n + 1)/2$ |
| Comp. of $gcd(u, v)$ by rptd. subtractions | Result matches builtin $gcd$ function |
| Comp. of $gcd(u, v)$ by rptd. divisions | Result matches builtin $gcd$ function |
| Mult. of two naturals by rptd. additions | Result matches builtin $\times$ function |
| Comp. of $1^2 + \ldots + n^2$ | Result is $n(n+1)(2n+1)/6$ |
| Comp. of $1^2 + \ldots + n^2$ w/out multiplications | Result is $n(n+1)(2n+1)/6$ |
| Semantics of an IMPerative language | Program computing $1 + \ldots + n$ is correct |
| Semantics of a FUNctional language | Program computing $1 + \ldots + n$ is correct |

Implementation details. RMT contains roughly 5000 lines of code, including comments and blank lines. RMT depends only on the standard C++ libraries and it can be compiled by any relatively modern C++ compiler out of the box. At the heart of RMT is a hierarchy of classes for representing variables, function symbols and terms. Terms are stored in DAG format, with maximum structure sharing. The RMT tool relies on an external SMT solver to check satisfiability of constraints. By default, the only dependency is the Z3 SMT solver, which should be installed and its binary should be in the system path. A compile time switch allows to use any other SMT solver that supports the SMTLIB interface, such as CVC4 [3]. In order to reduce constraints over the full signature to constraints over the builtin signature, RMT uses a unification modulo builtins algorithm (see [7]), which transforms any predicate $t_1 = t_2$ (where the terms $t_1, t_2$ can possibly contain constructor symbols) into a set of builtin constraints.

6 Conclusion and Future Work

We introduced a coinduction based method for proving reachability properties of logically constrained term rewriting systems. We use a coinductive definition of transition systems that unifies the handling of finite and infinite executions. We propose two proof systems for the problem above. The first one formalizes symbolic execution in LCTRSs coinductively, with possibly infinite proof trees. This proof system is complete, but its infinite proof trees cannot be used in practice as proofs. In the second proof system we add to symbolic execution a circularity proof rule, which allows to transform infinite proof trees into finite trees. It is not always possible to find finite proof trees, and we conjecture that establishing a given reachability property is higher up in the arithmetic hierarchy.

We also proposed a semantics for logically constrained term rewriting systems as transition systems over a model combining order-sorted terms with builtin elements such as booleans, integers, etc. The proposed semantics has the advantage of being simpler than the usual semantics of LCTRSs defined in [20], which requires two reduction relations (one for rewriting and one for computing). The
approach proposed here also removes some technical constraints such as variable inclusion of the rhs in the lhs, which is important in modelling open systems, where the result of a transition is non-deterministically chosen by the environment. In addition, working in an order-sorted setting is indispensable in order to model easily the semantics of programming languages.

In fact, proving program properties, like correctness and equivalence, is one application of our method. A tool such as C2LCTRS (http://www.trs.cm.is.nagoya-u.ac.jp/c2lctrs/) can be used to convert the semantics of a C program into a LCTRS and then RMT can prove reachability properties of the C program. Additionally, the operational semantics of any language can be encoded as a LCTRS and then program correctness is reducible to a particular reachability formula. But our approach is not limited to programs, as any system that can be modelled as a LCTRS is also amenable to our approach. We define reachability in the sense of partial correctness (infinite execution paths are not considered). Therefore termination should be established in some other way, as it is an orthogonal concern. Our approach to reachability and LCTRSs extends to working modulo AC (or more generally, modulo any set of equations E), but we have not formally presented this to preserve brevity and simplicity. For future work, we would like to test our approach on other interesting problems that arise in various domains. In particular, it would be interesting to extend our approach to reachability in the context of program equivalence. An interesting challenge is to add defined operations to the algebra underlying the constrained term rewriting systems, which would allow a user to define their own functions, which are not necessarily built-in.

Acknowledgements. We thank the anonymous reviewers for their valuable suggestions. This work was supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS/CCCDI - UEFISICDI, project number PN-III-P2-2.1-BG-2016-0394, within PNCDI III.

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A Preliminaries

A.1 Order-Sorted Algebra.

In this subsection we recall the main definitions and notations from order-sorted algebra we use in this paper. More details can be found, e.g., in [14].

An order-sorted signature $\Sigma \triangleq (S, \leq, F)$ consists of:

1. A set $S$ of sorts,
2. An $S^* \times S$-indexed family $F = \{F_{w,s} \mid w \in S^*, s \in S\}$ of sets whose elements are called operation symbols, and
3. A partial order $\leq \subseteq S \times S$,

such that the following monotonicity condition is satisfied:

$$f \in F_{w_1,s_1} \cap F_{w_2,s_2} \text{ and } w_1 \leq w_2 \text{ imply } s_1 \leq s_2.$$ 

We often write $f : s_1 \times \cdots \times s_n \to s$ for $f \in F_{w,s}$ with $w = s_1 \ldots s_n$. We write $\varepsilon$ for the empty sequence of sorts. A connected component of $(S, \leq)$ is an $\sim$-equivalence class, where $\sim$ is the smallest equivalence relation containing $\leq$.

Given an order-sorted signature $\Sigma \triangleq (S, \leq, F)$, a $\Sigma$-model ($\Sigma$-algebra) $M$ consists of:

- An $S$-indexed family $\{M_s \mid s \in S\}$ of carrier sets such that $s \leq s'$ implies $M_s \subseteq M_{s'}$;
- A function $M_f : M_w \to M_s$ for each operation symbol $f \in F_{w,s}$ such that if $f \in F_{w_1,s_1} \cap F_{w_2,s_2}$ and $w_1 \leq w_2$ then the corresponding functions $M_f : M_{w_1} \to M_{s_1}$ and $M_f : M_{w_2} \to M_{s_2}$ agree on $M_{w_1}$.

Let $\Sigma \triangleq (S, \leq, F)$ be an order-sorted signature and let $X \triangleq \{X_s \mid s \in S\}$ be an $S$-indexed family of variables such that $s \neq s'$ implies $X_s \cap X_{s'} = \emptyset$. The $S$-indexed family $T_{\Sigma}(X) = \{T_{\Sigma,s}(X) \mid s \in S\}$ of $\Sigma$-terms with variables $X$ is inductively defined as follows:

- $X_s \subseteq T_{\Sigma,s}(X)$;
- if $f \in F_{w,s}$, $w = s_1 \ldots s_n$, and $t_i \in T_{\Sigma,s_i}(X)$ for $i = 1, \ldots, n$, then the expression $f(t_1, \ldots, t_n)$ belongs to $T_{\Sigma,s}(X)$;
- if $s \leq s'$ then $T_{\Sigma,s}(X) \subseteq T_{\Sigma,s'}(X)$.

We also make the standard assumption that constant symbols are not followed by parentheses.

$T_{\Sigma}(X)$ can be organised as a $\Sigma$-model by considering $T_{\Sigma}(X)_f : T_{\Sigma}(X)_w \to T_{\Sigma}(X)_s$ that maps the terms $t_1, \ldots, t_n$ into $f(t_1, \ldots, t_n)$, where $f \in F_{w,s}$. The set of ground terms is $T_{\Sigma} = T_{\Sigma}(\emptyset)$. In this paper we consider order-sorted signatures $(S, \leq, F)$ that are preregular [14], i.e., each term $t$ in $T_{\Sigma}(X)$ has a least sort $l(t)$. For this case, $T_{\Sigma}$ is a initial order-sorted $\Sigma$-algebra and $T_{\Sigma}(X)$ is a free order-sorted $\Sigma$-algebra. A context $c[\cdot]$ is a term including exactly one

\footnote{If $w = s_1 \ldots s_n$ then $M_w = M_{s_1} \times \cdots \times M_{s_n}$.}
A.2 Coinduction.

We briefly recall from [28] (Chapter 2) the coinductive definitions and the coinduction proof technique defined using inference rules.

Given a set \( U \), a ground inference rule over \( U \) is a tuple \((a_1, \ldots, a_n, a)\), often written

\[
\frac{a_1 \ldots a_n}{a}
\]

where \( a, a_1, \ldots, a_n \in U \).

Given a set \( R \) of inference rules over \( U \) and a subset \( X \subseteq U \), the one-step closure of \( X \) with respect to \( R \) is the set \( \hat{R}(X) \subseteq U \) defined as follows:

\[
\hat{R}(X) \triangleq \left\{ a \left| \frac{a_1 \ldots a_n}{a} \in R, a_1, \ldots, a_n \in X \right. \right\}.
\]

\( \hat{R} \) is a monotone endofunction defined over \( \mathcal{P}(U) \), which is a complete lattice. Hence \( \hat{R} \) has a least fixed point and a greatest fixed point, by the Fixed Point Theorem (see [28]). By \( \nu \hat{R} \) we denote the greatest fixed point of \( \hat{R} \), that is the largest set \( X \subseteq U \) such that \( X \supseteq \hat{R}(X) \) (equivalently, \( X \) is the largest set such that \( X = \hat{R}(X) \)). We say that a set is coinductively defined if it is the greatest fixed point \( \nu \hat{R} \) of some ground rule system \( R \).

\[ \text{Here we consider only case when the set of the premises is finite.} \]
Example 14. The system LIST, given below, coinductively defines the possibly infinite lists over integers:

\[
\begin{align*}
[A] & \quad \text{nil} \\
[B] & \quad \ell, z \quad z \in \mathbb{Z}
\end{align*}
\]

Let \(\mathcal{U}\) be any set including all (finite and infinite) strings over \(\mathbb{Z} \cup \{\text{nil}, ,\}\). Note that \([B]\) is a rule scheme; the ground rules are obtained by instantiating \(z\) and \(\ell\) with concrete integers and elements in \(\mathcal{U}\), respectively. We have

\[
\hat{\text{LIST}}(X) = \{z, \ell \mid z \in \mathbb{Z}, \ell \in X\} \cup \{\text{nil}\}.
\]

The set of possibly infinite (i.e., finite and infinite) lists is the greatest fixed point, \(\mathbb{Z}^\infty = \nu \hat{\text{LIST}}\). We obtain

\[
\mathbb{Z}^\infty = \mathcal{U} \cap \text{LIST}(\mathcal{U}) \cap \text{LIST}^2(\mathcal{U}) \cap \ldots
\]

\[
= \mathcal{U} \cap \{(z_1, u \mid z_1 \in \mathbb{Z}, u \in \mathcal{U}\} \cup \{\text{nil}\})
\]

\[
\cap \{(z_1, z_2, u \mid z_1, z_2 \in \mathbb{Z}, u \in \mathcal{U}\} \cup \{z_1, \text{nil} \mid z_1 \in \mathbb{Z}\} \cup \{\text{nil}\})
\]

\[
\cap \ldots
\]

by Kleene’s Theorem. The set of infinite lists over \(\mathbb{Z}\) is the greatest fixed point of the system consisting only of rule \([B]\), i.e. \(\mathbb{Z}^\omega = \nu [B]\).

To prove that some element \(x\) is in \(\nu \hat{R}\), we often use the well-known coinduction principle:

**Proposition 3 (The Coinduction Principle).** Let \(X \subseteq U\) be a set such that \(X \subseteq \hat{R}(X)\). If \(x \in X\), then \(x \in \nu \hat{R}\).

The coinduction principle can be represented in a more compact way by the following inference rule:

\[
\frac{X \subseteq \hat{R}(X)}{X \subseteq \nu \hat{R}}
\]

**Definition 15.** A rule \(r\) is called coadmissible for \(R\) if \(\nu \hat{R} = \nu \hat{R} \cup \{r\}\).

**Remark 1.** Adding a coadmissible rule \(R\) does not change the greatest fixed point. Usually, a coadmissible rule can be used only finitely many times in a proof tree. Otherwise we may have unsound proofs, like that consisting only of coadmissible rules. The main idea behind of a coadmissible rule \(r\) is that if its premises can be derived using rules from \(R\) (i.e. there are proof trees under \(R\) for its premises), then we may find a proof tree under \(R\) of its conclusion. Then, we can show by induction that any proof tree under \(R \cup \{r\}\) where \(r\) is applied only finitely many times, can be transformed into a proof tree under \(R\) for the same conclusion.

We make extensive use of sets coinductively defined by rules. The underlying set \(U\) will be understood each time from the shape of rules: e.g., if the hypotheses and the conclusion are pairs of execution paths and formulas, then the set \(U\) is the set of all these pairs.
B Proofs of Helper Results

The following result is a direct consequence of Definition 5:

Corollary 1. If \( (M, \leadsto) \models^\forall P \Rightarrow Q \) then \( (M, \leadsto) \models^\forall \partial(P \setminus Q) \Rightarrow Q \).

The disjunction of valid predicates with the same target is a valid predicate as well:

Proposition 4. If \( (M, \leadsto) \models^\forall P_i \Rightarrow Q \) for \( i = 1, 2 \), then \( (M, \leadsto) \models^\forall P_1 \cup P_2 \Rightarrow Q \).

Proof. We show that the set \( X = \{P_1 \cup P_2 \Rightarrow Q \mid (M, \leadsto) \models^\forall P_i \Rightarrow Q, i = 1, 2\} \) is backward closed w.r.t. \( \overline{\Delta \text{DVP}} \), i.e. \( X \subseteq \overline{\Delta \text{DVP}}(X) \). Note first that if \( (M, \leadsto) \models^\forall P \Rightarrow Q \) then obviously \( P \Rightarrow Q \in X \) since \( P = P \cup \emptyset \) and \( (M, \leadsto) \models^\forall \emptyset \Rightarrow Q \).

Let \( P_1 \cup P_2 \Rightarrow Q \in X \). We have \( \partial((P_1 \cup P_2) \setminus Q) = \partial(P_1 \setminus Q) \cup \partial(P_2 \setminus Q) \), which implies \( \partial((P_1 \cup P_2) \setminus Q) \Rightarrow Q \in X \) (since \( (M, \leadsto) \models^\forall \partial(P_i \setminus Q) \Rightarrow Q, i = 1, 2 \)).

It follows that \( P_1 \cup P_2 \Rightarrow Q \in \overline{\Delta \text{DVP}}(X) \) by the rule [Step]. \( \square \)

Corollary 2. \( X = \nu \overline{\Delta \text{DVP}} \), where \( X \) is the set from the proof of Proposition 4.

Proposition 4 allows to extend DVP with the following coadmissible inference rule:

\[
\begin{array}{c}
P_1 \Rightarrow Q \quad P_2 \Rightarrow Q \quad P_1 \neq \emptyset \neq P_2
\end{array}
\]

The following result shows that the set of demonic valid reachability predicates is closed under the subset relation:

Proposition 5. If \( (M, \leadsto) \models^\forall P \Rightarrow Q \) and \( P' \subseteq P \) then \( (M, \leadsto) \models^\forall P' \Rightarrow Q \).

Proof. We show that the set \( X = \{P' \Rightarrow Q \mid P' \subseteq P, (M, \leadsto) \models^\forall P \Rightarrow Q\} \) is backward closed w.r.t. \( \overline{\Delta \text{DVP}} \), i.e. \( X \subseteq \overline{\Delta \text{DVP}}(X) \).

Let \( P' \Rightarrow Q \in X \). It follows that there is \( P \) such that \( (M, \leadsto) \models^\forall P \Rightarrow Q \) and \( P' \subseteq P \). Let \( PT \) be a proof tree of \( P \Rightarrow Q \) under \( \overline{\Delta \text{DVP}} \). We distinguish the following two cases:

1. \( P \subseteq Q \). It follows that \( P' \subseteq Q \) and hence \( P' \Rightarrow Q \in \overline{\Delta \text{DVP}}(X) \).
2. The unique child of the root is \( \partial(P \setminus Q) \Rightarrow Q \). We have \( \partial(P' \setminus Q) \subseteq \partial(P \setminus Q) \) and hence \( \partial(P' \setminus Q) \Rightarrow Q \in X \), which implies \( P' \Rightarrow Q \in \overline{\Delta \text{DVP}}(X) \). \( \square \)

To show the demonic validity of a reachability predicate, we have to find a proof tree only for the state predicate of not already reached target states:

Proposition 6. If \( (M, \leadsto) \models^\forall P \Rightarrow Q \) if \( (M, \leadsto) \models^\forall P \setminus Q \Rightarrow Q \).

Proof. Reverse implication (\( \Leftarrow \)). We have \( P = (P \setminus Q) \cup (P \cap Q) \). Since \( P \cap Q \subseteq Q \), we obviously have \( (M, \leadsto) \models^\forall P \cap Q \Rightarrow Q \). The conclusion follows for by applying Proposition 5.

Direct implication (\( \Rightarrow \)). Since \( P \setminus Q \subseteq P \), the conclusion follows by Proposition 6. \( \square \)
A starting state of a demonically valid reachability predicate that is not in the target state predicate must be runnable:

**Proposition 7.** If \((M, \leadsto) \models^V P \Rightarrow Q\) then \(P \setminus Q\) is runnable.

**Proof.** It follows directly from the definition of DVP. \(\square\)

**Corollary 3.** If \(P \cap Q = \emptyset\) and \((M, \leadsto) \models^V P \Rightarrow Q\) then \(P\) is runnable.

**Proposition 8.** If \(\text{var}(\varphi_1) \cap \text{var}(\varphi') = \text{var}(\varphi_2) \cap \text{var}(\varphi')\) and \([\sigma(\varphi_1)] = [\sigma(\varphi_2)]\) for all \(\sigma : \text{var}(\varphi_1) \cap \text{var}(\varphi') \rightarrow M^\Sigma\), then \(\varphi_1 \Rightarrow \varphi' \equiv_R \varphi_2 \Rightarrow \varphi'\).

**Proof.** \(\varphi_1 \Rightarrow \varphi'\) and \(\varphi_2 \Rightarrow \varphi'\) define the same reachability predicate for each \(\sigma : \text{var}(\varphi_1) \cap \text{var}(\varphi') \rightarrow M^\Sigma\). \(\square\)

**Proposition 9.** \(\langle t \mid \phi \rangle \Rightarrow \varphi' \equiv \langle z \mid z = t \land \phi \rangle \Rightarrow \varphi',\) where \(z\) is a fresh variable (it does not appear in \(\langle t \mid \phi \rangle \Rightarrow \varphi'\)).

**Proof.** We obviously have \(\text{var}(\langle t \mid \phi \rangle) \cap \text{var}(\varphi') = \text{var}(\langle z \mid z = t \land \phi \rangle) \cap \text{var}(\varphi')\) and \([\sigma(\langle t \mid \phi \rangle)] = [\sigma(\langle z \mid z = t \land \phi \rangle)]\) for all \(\sigma : \text{var}(\varphi_1) \cap \text{var}(\varphi') \rightarrow M^\Sigma\) (we used here the fact that \(\sigma(z) = z\)). Then we apply Proposition 8. \(\square\)

**Proposition 10.** If \(\langle t \mid \phi_1 \rangle \Rightarrow \varphi' \equiv \langle t_i' \mid \phi_i' \rangle \Rightarrow \varphi'\) and \(\text{var}(\langle t \mid \phi_1 \rangle) \cap \text{var}(\varphi') = \text{var}(\langle t_i' \mid \phi_i' \rangle) \cap \text{var}(\varphi')\) for \(i = 1, 2\), then

\[
\langle t \mid \phi_1 \lor \phi_2 \rangle \Rightarrow \varphi'
\]

is equivalent to

\[
\langle z \mid (z = t_i' \land \phi_i') \lor (z = t_i'' \land \phi_i'') \rangle \Rightarrow \varphi',
\]

where \(z\) is a fresh variable.

**Proof.** We have \(\langle t_i' \mid \phi_i' \rangle \Rightarrow \varphi'\) equivalent to \(\langle z \mid z = t_i' \land \phi_i' \rangle \Rightarrow \varphi'\) by Proposition 9. Let \(\mathcal{R}\) be a constrained rule system, \(\sigma : \text{var}(\langle t \mid \phi_1 \lor \phi_2 \rangle) \cap \text{var}(\varphi') \rightarrow M^\Sigma\) and assume that

\[
(M^\Sigma, \leadsto_{\mathcal{R}}) \models^V [\sigma(\langle t \mid \phi_1 \lor \phi_2 \rangle)] \Rightarrow [\sigma(\varphi')].
\]

Note that \(\text{var}(\langle t \mid \phi_1 \lor \phi_2 \rangle) \cap \text{var}(\varphi') = \text{var}(\langle z \mid (z = t_i' \land \phi_i') \lor (z = t_i'' \land \phi_i'') \rangle)\) by the hypotheses. Since \([\sigma(\langle t \mid \phi_1 \lor \phi_2 \rangle)] \leq [\sigma(\langle z \mid z = t_i' \lor \phi_i' \rangle)]\), it follows that

\[
(M^\Sigma, \leadsto_{\mathcal{R}}) \models^V [\sigma(\langle t \mid \phi_1 \lor \phi_2 \rangle)] \Rightarrow [\sigma(\varphi')]
\]

by Proposition 8, \(i = 1, 2\). By Proposition 10 we obtain

\[
(M^\Sigma, \leadsto_{\mathcal{R}}) \models^V [\sigma(\langle z \mid z = t_i' \lor \phi_i' \rangle)] \Rightarrow [\sigma(\varphi')]
\]

for \(i = 1, 2\), which implies

\[
(M^\Sigma, \leadsto_{\mathcal{R}}) \models^V [\sigma(\langle z \mid z = t_i' \lor \phi_i' \rangle)] \cup [\sigma(\langle z \mid z = t_i'' \lor \phi_i'' \rangle)] \Rightarrow [\sigma(\varphi')]
\]

by Proposition 4. Since

\[
[\sigma(\langle z \mid z = t_i' \lor \phi_i' \rangle)] \cup [\sigma(\langle z \mid z = t_i'' \lor \phi_i'' \rangle)] = [\sigma(\langle z \mid z = t_i' \land \phi_i' \rangle \lor (z = t_i'' \land \phi_i''))]
\]

it follows that

\[
(M^\Sigma, \leadsto_{\mathcal{R}}) \models^V [\sigma(\langle z \mid z = t_i' \land \phi_i' \rangle \lor (z = t_i'' \land \phi_i''))] \Rightarrow [\sigma(\varphi')].
\]

Since \(\sigma\) defined over \(\text{var}(\langle t \mid \phi_1 \lor \phi_2 \rangle) \cap \text{var}(\varphi')\) is arbitrary, we have proved that \(\mathcal{R} \models^V (t \mid \phi_1 \lor \phi_2 \Rightarrow \varphi')\ implies \mathcal{R} \models^V (z \mid z = t_i' \land \phi_i' \lor (z = t_i'' \land \phi_i'')) \Rightarrow \varphi'.\)

The converse implication is proven in a similar way. \(\square\)
Remark 2. We assume that \( \var(\varphi_1) \cap \var(\varphi') = \var(\varphi_2) \cap \var(\varphi') \) and \( \llbracket \varphi_2 \rrbracket = \llbracket \varphi_2 \rrbracket \) whenever \( \varphi_1 \Rightarrow \varphi' \equiv_R \varphi_2 \Rightarrow \varphi' \). The first equality says that the shared variables by the lhs and rhs are preserved by the equivalence and the second one is needed to be sure that the two constrained terms have the same syntactic variables (see below).

The following result is useful for case analysis:

**Proposition 11.** If \( M^\Sigma \models \phi \iff (\phi_1 \lor \phi_2) \), \( \langle t | \phi_1 \rangle \Rightarrow \varphi' \) and \( \langle t | \phi_2 \rangle \Rightarrow \varphi' \) are in \( \nu \text{DSTEP}(R) \), and \( \var(\langle t | \phi_1 \rangle) \cap \var(\varphi') = \var(\langle t | \phi_2 \rangle) \cap \var(\varphi') \), then \( \langle t | \phi \rangle \Rightarrow \varphi' \) is in \( \nu \text{DSTEP}(R) \).

and it allows to extend \( \text{DSTEP}(R) \) with the following inference rule:

**Definition 16 (Coadmissible rule for reachability formulae).**

\[
\frac{\text{[disj]} \quad \langle t | \phi_1 \rangle \Rightarrow \varphi', \langle t | \phi_2 \rangle \Rightarrow \varphi' \quad M^\Sigma \models \phi \iff \phi_1 \lor \phi_2}{\langle t | \phi \rangle \Rightarrow \varphi'}
\]

**Proof (of Proposition 11).** Let \( A \) be the set
\[
\{ \langle t | \phi \rangle \Rightarrow \varphi' | \langle t | \phi_1 \rangle \Rightarrow \varphi', \langle t | \phi_2 \rangle \Rightarrow \varphi' \in \nu \text{DSTEP}(R), M^\Sigma \models \phi \iff (\phi_1 \lor \phi_2) \}.
\]

Note that \( \nu \text{DSTEP}(R) \subseteq A \) since \( \phi \) is equivalent to \( \phi \lor \phi \), which implies \( \nu \text{DSTEP}(R) \subseteq \nu \text{DSTEP}(R)(A) \) (\( A \)). We show that \( A \) is backward-closed w.r.t. \( \text{DSTEP}(R) \), i.e. \( A \subseteq \nu \text{DSTEP}(R)(A) \). Let \( \langle t | \phi \rangle \Rightarrow \varphi' \in A \), where \( M^\Sigma \models \phi \iff (\phi_1 \lor \phi_2) \). Let \( PT_i \) a proof tree for \( \langle t | \phi_i \rangle \Rightarrow \varphi' \) under \( \text{DSTEP}(R) \), \( i = 1, 2 \). We distinguish the following cases, according to the definition of \( PT_i \), \( i = 1, 2 \):

1. \( M^\Sigma \models \phi_i \iff \bot \), \( i \in \{1, 2\} \) (\( PT_i \) consists of [axiom]). Then \( \langle t | \phi_1 \lor \phi_2 \rangle \Rightarrow \varphi' \) is equivalent to \( \langle t | \phi_{3 - i} \rangle \Rightarrow \varphi' \), which is in \( \nu \text{DSTEP}(R) \) and hence in \( \nu \text{DSTEP}(R)(A) \) by (\( A \)).

2. The rule corresponding to the root of \( PT_i \) is [subb] for \( i = 1, 2 \). By Proposition 9 we may assume that the child of the \( PT_i \)’ root is of the form
\[
\langle z | \varphi \rangle, \ i = 1, 2 \text{, where } \varphi'' \text{ is of the form } (\exists \bar{x}) z = t'' | \varphi', \text{ and } \bar{x} = \var(\varphi') \setminus \var(t''_i | \varphi''_i) = \var(\varphi') \setminus \var(t | \phi_i), \ i = 1, 2 \text{ (by the hypotheses of the proposition and Remark 2).}
\]
Since \( \langle t | \phi_i \rangle \Rightarrow \varphi' \) is equivalent to \( \langle z | z = t''_i \land \varphi''_i \rangle \Rightarrow \varphi' \), it follows that \( \langle t | \phi_1 \lor \phi_2 \rangle \Rightarrow \varphi' \) is equivalent to \( \langle z | (z = t''_1 \land \varphi''_1) \lor (z = t''_2 \land \varphi''_2) \rangle \Rightarrow \varphi' \) by Proposition 11. It follows that
\[
\langle z | (z = t''_1 \land \varphi''_1) \lor (z = t''_2 \land \varphi''_2) \rangle \Rightarrow \varphi'
\]
is equivalent to
\[
\langle z | (z = t''_1 \land \varphi''_1) \lor (z = t''_2 \land \varphi''_2) \rangle \Rightarrow \varphi'
\]
which is in \( A \) and hence \( \langle t | \phi_1 \lor \phi_2 \rangle \Rightarrow \varphi' \in \nu \text{DSTEP}(R)(A) \) by [subb].

3. The rule corresponding to the root of \( PT_i \) is [der] for \( i = 1, 2 \). We assume that the children of the root of \( PT_i \) are of the form \( \langle t'' | \varphi''_i \rangle \Rightarrow \varphi' \) with
\[
\langle t'' | \varphi''_i \rangle \in \Delta_R(t''_i | \varphi''_i) \]
and
\( \langle t \mid \phi_j \rangle \Rightarrow \varphi' \equiv (t_i' \mid \phi''_j) \Rightarrow \varphi' \)
where \( j \in J''_i \) and \( i = 1, 2 \). We also assume that
\[ \Delta_R(\langle t \mid \phi_j \rangle) = \{ \langle t_j \mid \phi''_j \rangle \mid j \in J_i \} \]
where \( i = 1, 2 \). For \( j \in J_i \) there is \( j'' \in J''_i \) such that
\[ \langle t_j \mid \phi''_j \rangle \Rightarrow \varphi' \equiv (t''_j \mid \phi''''_j) \Rightarrow \varphi' \]
by Remark \( 3 \). Since \( \langle t''_j \mid \phi''''_j \rangle \Rightarrow \varphi' \in \nu DSTEP(\mathcal{R}) \), it follows that \( \langle t_j \mid \phi''_j \rangle \Rightarrow \varphi' \in \nu DSTEP(\mathcal{R}) \) by the implicit equivalence rule, \( i = 1, 2 \). Note that \( \phi''_j \) is of the form (or equivalent to) \( \phi_1 \land \phi', \) where \( \phi' \) depends only on \( t \) and the applied rule, which implies
\[ \Delta_R(\langle t \mid \phi_1 \lor \phi_2 \rangle) = \{ \langle t_j \mid (\phi_1 \lor \phi_2) \land \phi' \rangle \mid j \in J_1 \lor J_2 \} \]
Using the equivalence between \( (\phi_1 \lor \phi_2) \land \phi' \) and \( (\phi_1 \lor \phi') \land (\phi_2 \lor \phi') \), we obviously obtain \( \langle t \mid (\phi_1 \lor \phi_2) \land \phi' \rangle \Rightarrow \varphi' \in \mathcal{R} \), and hence
\[ \langle t \mid \phi_1 \lor \phi_2 \rangle \Rightarrow \varphi' \in \nu DSTEP(\mathcal{R})(A) \]
by \( \text{[der]} \)

4. The rule corresponding to the root of \( PT \) is \( \text{[subs]} \) and the rule corresponding to the root of \( PT_{3-i} \) is \( \text{[der]} \), \( i \in \{1, 2\} \). Note that the rule \( \text{[subs]} \) cannot be applied twice consecutively, so the child of the root of \( PT \) corresponds to either \( \text{[axiom]} \) or \( \text{[der]} \). The rest of the proof for this case is similar to the case 1 or to the case 3.

\[ \square \]

The next result shows that it is fine to relax the constraints of some goals (a kind of generalization).

**Proposition 12.** If \( (\mathcal{R}, G) \models \forall \langle t \mid \phi \rangle \Rightarrow \varphi' \) then \( (\mathcal{R}, G) \models \forall \langle t \mid \phi \land \phi'' \rangle \Rightarrow \varphi' \), where \( \phi'' \) is a constraint formula.

**Proof.** Let \( PT \) a guarded proof tree for \( \langle t \mid \phi \rangle \Rightarrow \varphi' \) under \( DCC(\mathcal{R}, G) \). We transform \( PT \) into a guarded proof tree \( PT' \) for \( \langle t \mid \phi \land \phi'' \rangle \Rightarrow \varphi' \) under \( DCC(\mathcal{R}, G) \) as follows: The root \( \langle t \mid \phi \rangle \Rightarrow \varphi' \) is transformed into \( \langle t \mid \phi \land \phi'' \rangle \Rightarrow \varphi' \). Assuming that the current node \( \langle t_1 \mid \phi_1 \rangle \Rightarrow \varphi' \) is transformed into \( \langle t'_1 \mid \phi'_1 \rangle \Rightarrow \varphi' \) with \( M \Sigma \models \phi'_1 \leftrightarrow \phi_1 \land \phi'' \), its children are transformed according to the inference rule used to obtain the current node (if the rule involves an equivalence of the conclusion, then \( \langle t_1 \mid \phi_1 \rangle \Rightarrow \varphi' \) is the used equivalent formula):

1. \( \text{[axiom]} \). \( M \Sigma \models \phi_1 \leftrightarrow \bot \) implies \( M \Sigma \models \phi'_1 \leftrightarrow \bot \) and there are no children in this case.
2. \( \text{[subs]} \). The unique child of \( \langle t_1 \mid \phi_1 \rangle \Rightarrow \varphi' \) is of the form \( \langle t_1 \mid \phi''_1 \rangle \Rightarrow \varphi' \) with \( M \Sigma \models \phi''_1 \leftrightarrow \phi_1 \land \phi_2 \) and it is transformed into \( \langle t_1 \mid \phi''_1 \land \phi'' \rangle \Rightarrow \varphi' \). Note that \( \phi_2 \) does not depend on \( \phi_1 \), so it is the same with that for \( \langle t_1 \mid \phi_1 \land \phi'' \rangle \Rightarrow \varphi' \). Obviously, \( \langle t'_1 \mid \phi'_1 \rangle \Rightarrow \varphi' \) and \( \langle t_1 \mid \phi''_1 \land \phi'' \rangle \Rightarrow \varphi' \) form an instance of \( \text{[subs]} \).
3. \( \text{[der]} \). The children of the current node in \( PT \) are of the form \( \langle t_j' \mid \phi'_j \rangle \Rightarrow \varphi' \), \( j \in J \). We have \( \Delta_R(\langle t_1 \mid \phi_1 \rangle) = \{ \langle t'_j \mid \phi'_j \rangle \mid j \in J' \} \) with \( M \Sigma \models \phi'_1 \leftrightarrow \phi_1 \land \phi'' \) and \( J' \subseteq J \) by the definition of \( \Delta_R \). We may have \( J' \subset J \) because
some of $\phi_1^{j} \land \phi''$ could become unsatisfiable. The children of $(t_1' \parallel \phi_1') \Rightarrow \phi'$ are $(t_1' \parallel \phi_2') \Rightarrow \phi'$, $j \in J'$. Obviously, the new node is an instance of $[\text{der}']$.

4. [circ]. The current node in $PT$ has two children of the form

$(t_1' \parallel \phi_1 \land \phi_1') \Rightarrow \phi'$ and $(t_1 \parallel \phi_1 \land \neg \phi_1') \Rightarrow \phi'$.

The children are transformed into

$(t_1' \parallel \phi_1 \land \phi'' \land \phi_1') \Rightarrow \phi'$ and $(t_1 \parallel \phi_1 \land \phi'' \land \neg \phi_1') \Rightarrow \phi'$, respectively.

Since $PT$ is guarded it follows that $PT'$ is guarded as well. 

\[\square\]

C  Proofs of Results from the Paper

Proposition 1. Let $P \Rightarrow Q$ be a reachability predicate. We have $(M, \sim) \models^\gamma P \Rightarrow Q$ if and only if any execution path $\tau$ starting from $P$ ($\text{hd}(\tau) \in P$) satisfies $P \Rightarrow Q$.

Proof. Reverse implication ($\Rightarrow$). Let $X$ denote the set $\{(\tau, P \Rightarrow Q) \mid \text{hd}(\tau) \in P, (M, \sim) \models^\gamma P \Rightarrow Q\}$. We show that $X$ is backward closed w.r.t. $\text{EPSRP}$, i.e. $X \subseteq \text{EPSRP}(X)$. Let $(\tau, P \Rightarrow Q)$ be in $X$. If $\text{hd}(\tau) \in Q$ then $(\tau, P \Rightarrow Q)$ is in $\text{EPSRP}(X)$ by the first rule of $\text{EPSRP}$. If $\text{hd}(\tau) \in P \setminus Q$ then $(M, \sim) \models^\gamma \partial(P \setminus Q) \Rightarrow Q$ and $P \setminus Q$ is runnable (the root of the proof tree for $P \Rightarrow Q$ corresponds to the second rule in $\text{DVP}$). It follows that $\tau$ is of the form $\gamma_0 \leadsto \tau'$. We have $\text{hd}(\tau') \in \partial(P \setminus Q)$ by the definition of $\partial(\cdot)$. It follows that $(\tau', \partial(P \setminus Q) \Rightarrow Q) \in X$, which implies $(\tau, P \Rightarrow Q) \in \text{EPSRP}(X)$ by the second rule of $\text{EPSRP}$. This finishes the proof of "if" direction.

Direct implication ($\Leftarrow$). Let $Y$ be the set $\{P' \Rightarrow Q \mid (\forall \gamma)\text{hd}(\tau) \in P' \Rightarrow \tau \models^\gamma P' \Rightarrow Q\}$. We show that $Y$ is backward closed w.r.t. $\text{DVP}$, i.e. $Y \subseteq \text{DVP}(Y)$, and we then apply the coinduction rule. Let $P' \Rightarrow Q \in Y$. For any $\gamma \in P' \setminus Q$ there is an execution path $\tau$ starting from $\gamma$. Since $\gamma \models^\gamma P' \Rightarrow Q$ and $\gamma \not\in Q$ it follows that $\tau = \gamma \circ \tau'$ for certain $\tau'$ with $\tau' \models^\gamma \partial(P') \Rightarrow Q$ (by the definition of $\models^\gamma$), which implies $P' \setminus Q$ runnable. Moreover, we observe that $\gamma \models^\gamma P' \Rightarrow Q$ implies $\tau \models^\gamma \tau' \Rightarrow Q \Rightarrow Q$ for any $\tau$ starting from $P' \setminus Q$, i.e. $(P' \setminus Q) \Rightarrow Q \in Y$. We show now that $\partial(P' \setminus Q) \Rightarrow Q \in Y$. Let $\tau'$ be an execution path with $\text{hd}(\tau') \models^\gamma \tau' \in \partial(P' \setminus Q)$. There is $\gamma \in P' \setminus Q$ such that $\gamma \sim \gamma'$ by the definition of $\partial$. Then $\tau \models^\gamma P' \setminus Q \Rightarrow Q$ by the definition of $Y$, where $\tau = \gamma \circ \tau'$, which implies $\tau' \models^\gamma \partial(P' \setminus Q) \Rightarrow Q$ by the definition of $\models^\gamma$. Since $\tau'$ is arbitrary, it follows that $\partial(P' \setminus Q) \Rightarrow Q \in Y$, and hence $P' \Rightarrow Q \in \text{DVP}(Y)$ by the rule [Step]. Since $P' \Rightarrow Q \in Y$ is arbitrary, it follows that $Y \subseteq \text{DVP}(Y)$. This finishes the proof of the "only if" direction. 

\[\square\]

Proposition 2. The inclusion $\langle t \parallel \phi \rangle \subseteq_{\text{shared}} \langle t' \parallel \phi' \rangle$ holds if and only if $M^E \models \phi \rightarrow (\exists \bar{x})(t = t' \land \phi')$, where $\bar{x} \triangleq \text{var}(t', \phi') \setminus \text{var}(t, \phi)$.

Proof. Reverse implication ($\Rightarrow$). Assume that $M^E \models \phi \rightarrow (\exists \bar{x})(t = t' \land \phi')$ and consider $\sigma : \text{var}(t, \phi) \cap \text{var}(t', \phi') \rightarrow M^E$. We have to prove that $[\sigma((t \parallel \phi))] \subseteq [\sigma((t' \parallel \phi'))]$. Let $\alpha$ be a valuation such that $M^E, \alpha \models \sigma(\phi)$. Then the valuation
In this paper we assume more than Corollary 4 claims, namely that

Then we apply Theorem 1.

Corollary 4. If \( \alpha_1(y) = \text{if } y \in \text{var}(t, \phi) \cap \text{var}(t', \phi') \) then \( \sigma(y) \) else \( \alpha(y) \), satisfies \( M^\Sigma, \alpha_1 \models \phi \). There is a valuation \( \alpha'_1 \) such that \( \alpha'_1(y) = \alpha_1(y) \), for each \( y \not\in \bar{x} \), and \( M^\Sigma, \alpha'_1 \models (t \equiv t' \land \phi') \). Since \( \text{var}(t, \phi) \cap \text{var}(t', \phi') \) and \( \bar{x} \) are disjoint, it follows that \( \alpha'_1(y) = \sigma(y) \) for each \( y \in \text{var}(t, \phi) \cap \text{var}(t', \phi') \).

Let \( \alpha' \) be a valuation with \( \alpha'(y) = \alpha'_1(y) \) for each \( y \not\in \text{var}(t, \phi) \cap \text{var}(t', \phi') \). We obviously have \( M^\Sigma, \alpha' \models \phi \) and \( \alpha'(\phi') = \alpha'_1(t') = \alpha_1(t) = \alpha(t) \). Since \( \alpha \) is arbitrary, we obtain \( [\sigma((t \mid \phi))] \subseteq [\sigma((t' \mid \phi'))] \).

Direct implication (\( \Rightarrow \)). Assume that \( [[t \mid \phi]] \subseteq \text{shared} [[t' \mid \phi'] \). We have to prove that \( M^\Sigma, \alpha \models \phi \Rightarrow (\exists x)(t = t' \land \phi') \). Let \( \alpha \) be a valuation such that \( M^\Sigma, \alpha \models \phi \). Consider \( \sigma: \text{var}(t, \phi) \cap \text{var}(t', \phi') \rightarrow M^\Sigma \) given by \( \sigma(y) = \alpha(y) \). Since \( [\sigma((t \mid \phi))] \subseteq [\sigma((t' \mid \phi'))] \) it follows that there is \( \alpha' \) such that \( \alpha(\sigma(t)) = \alpha'(\sigma(t')) \) and \( M^\Sigma, \alpha' \models \phi' \). We may assume w.l.o.g. that \( \alpha'(y) = \alpha(y) \) for \( y \not\in \text{var}(t', \phi') \).

Let \( \alpha'_1 \) be the valuation such that \( \alpha'_1(y) = \text{if } y \in \text{var}(t, \phi) \cap \text{var}(t', \phi') \) then \( \sigma(y) \) else \( \alpha(y) \). We obviously have \( \alpha'_1(y) = \sigma(y) \) for each \( y \not\in \bar{x} \), \( M^\Sigma, \alpha'_1 \models \phi' \) (since \( M^\Sigma, \alpha' \models \phi' \)).

Hence \( M^\Sigma, \alpha'_1 \models (t \equiv t' \land \phi') \), which implies \( M^\Sigma, \alpha \models (\exists x)(t = t' \land \phi') \). Since \( \alpha \) is arbitrary, it follows that \( M^\Sigma, \alpha \models \phi \Rightarrow (\exists x)(t = t' \land \phi') \). 

Theorem 1. Let \( \varphi \equiv (t \mid \phi) \) be a constrained term, \( R \) a constrained rule system, and \( (M^\Sigma, \rightarrow_R) \) the transition system defined by \( R \). Then \([\Delta_R(\varphi)] = \partial([\varphi]).\)

Proof. \( \subseteq. \) Let \( \gamma_2 \in [\Delta_R(\varphi)] \). There is a rule \( l \rightarrow r \text{ if } \phi_l \) in \( R \), a context \( c[\cdot] \) and a valuation \( \alpha \) such that \( \gamma_2 = \alpha(c[r]) \) and \( M^\Sigma, \alpha \models \phi' \). We may use \( \Delta_1, \phi_0 \).

Recall that the variables in \( l \rightarrow r \text{ if } \phi_l \) are possibly renamed in order to be disjoint from those in \( \phi \). \( M^\Sigma, \alpha \models \phi' \) implies \( M^\Sigma, \alpha \models \phi \) and \( \alpha(t) = \alpha(c[l]) \).

It follows that \( \gamma_1 = \alpha(c[l]) = \alpha(t) \) is in \( [\varphi] \) and \( \gamma_1 \rightarrow_R \gamma_2 \). Hence \( \gamma_2 \in [\partial([\varphi])] \).

\( \supseteq. \) Let \( \gamma_2 \in [\partial([\varphi])] \). It follows that there is \( \gamma_1 \in [\varphi] \) s.t. \( \gamma_1 \cal{R} \gamma_2 \) by the definition of \( \cal{R}. \) The transition step \( \gamma_1 \cal{R} \gamma_2 \) implies that there exists a rule \( l \rightarrow r \text{ if } \phi_l \hspace{1cm} \) in \( R \), a context \( c[\cdot] \), and a valuation \( \alpha_1 : X \rightarrow M^\Sigma \) such that \( \gamma_1 = \alpha_1(c[l]) \), \( \gamma_2 = \alpha_1(c[r]) \) and \( M^\Sigma, \alpha_1 \models \phi \). Since \( \gamma_1 \in [\varphi] \), there exists \( \alpha_2 : X \rightarrow M^\Sigma \) such that \( \gamma_1 = \alpha_2(t) \) and \( M^\Sigma, \alpha_2 \models \phi \). We consider \( \alpha \) such that \( \alpha(x) = \alpha_1(x) \) for \( x \in \text{var}(l \rightarrow r \text{ if } \phi_l) \) and \( \alpha(y) = \alpha_2(y) \) for \( y \in \text{var}(\varphi) \); this can be achieved by renaming the variables occurring in the rule. The equality \( \alpha(t) = \alpha(c[l]) \) follows from \( \alpha(t) = \gamma_1 = \alpha(c[l]) \). We obtain \( M^\Sigma, \alpha \models c[l] \equiv t \land \phi_l \land \phi \), which implies \( \gamma_2 \in [\Delta_R(\varphi)] \).

Corollary 4. If \( \varphi_1 \Rightarrow \varphi' \equiv_R \varphi_2 \Rightarrow \varphi \) then \([\Delta_R(\varphi_1)] = [\Delta_R(\varphi_2)]\).

Proof. We have \([\varphi_1] = [\varphi_2] \) by Remark 2 which implies \( \partial([\varphi_1]) = \partial([\varphi_2]) \). Then we apply Theorem 1.

Remark 3. In this paper we assume more than Corollary claims, namely that \( \varphi_1 \Rightarrow \varphi' \equiv_R \varphi_2 \Rightarrow \varphi' \) implies

for each \( \varphi''_i \in \Delta_R(\varphi_1) \) there is \( \varphi''_{3-i} \in \Delta_R(\varphi_{3-i}) \) such that \( \varphi''_i \Rightarrow \varphi' \equiv_R \varphi_{3-i} \Rightarrow \varphi' \), \( i = 1, 2. \)
In this way the symbolic execution given by the system $\text{DSTEP}(\mathcal{R})$ (see below) is preserved by the equivalence.

**Theorem 2.** Let $\mathcal{R}$ be a LCTRS. For any reachability formula $\varphi \Rightarrow \varphi'$, we have $\mathcal{R} \models^\forall \varphi \Rightarrow \varphi'$ iff $\varphi \Rightarrow \varphi' \in \nu \text{DSTEP}(\mathcal{R})$.

**Proof.** Direct implication (soundness). Let $A$ be the set

$$\{[\sigma(\varphi)] \Rightarrow [\sigma(\varphi')] | \sigma : \text{var}(\varphi) \cap \text{var}(\varphi') \rightarrow M^\Sigma, \varphi \Rightarrow \varphi' \in \nu \text{DSTEP}(\mathcal{R})\}.$$ 

The conclusion of the theorem follows by showing that the set $A$ is backward closed w.r.t. $\text{DVP}$, i.e. $A \subseteq \text{DVP}(A)$. Let $\varphi \Rightarrow \varphi' \in \nu \text{DSTEP}(\mathcal{R})$ and $\sigma : \text{var}(\varphi) \cap \text{var}(\varphi') \rightarrow M^\Sigma$. There is a proof tree $PT$ of $\varphi \Rightarrow \varphi'$ under $\text{DSTEP}(\mathcal{R})$, where $\varphi \Rightarrow \varphi'$ is the root. We proceed by case analysis on the DSTEP rule applied to the root $\varphi \Rightarrow \varphi' \triangleq (t | \phi) \Rightarrow (t' | \phi')$.

1. [axiom]. Then $M^\Sigma \vdash \phi \leftrightarrow \perp$ and $[\sigma(\varphi)] = [\sigma(\varphi')] \in \text{DVP}(A)$ by the first rule of DVP.

2. [subs]. The root has one child $(t'' | \phi'' \land \neg \phi''') \Rightarrow (t' | \phi')$, where $\varphi \Rightarrow \varphi' \equiv (t'' | \phi'') \Rightarrow (t' | \phi')$ (recall that the equivalence rule is implicitly applied) and $M^\Sigma \vdash \phi'' \leftrightarrow (\exists X)(t'' = t' \land \phi'), X = \text{var}(t', \phi') \setminus \text{var}(t'', \phi''')$. The side condition of the rule ensures that $M^\Sigma \vdash \neg(\phi'' \equiv \perp)$. We have $[\sigma((t'' | \phi''))] = [\sigma((t' | \phi'))]$ by Proposition 2 which implies the equality $[\sigma((t'' | \phi'' \land \neg \phi'''))] = [\sigma((t' | \phi''))] \cap [\sigma((t' | \phi'))]$. Moreover, from $[\sigma((t'' | \phi''))] = [\sigma((t'' | \phi'' \land \neg \phi'''))] \cup [\sigma((t'' | \phi'' \land \neg \phi'''))]$ we obtain $[\sigma((t'' | \phi''))] = [\sigma((t'' | \phi''))] \setminus [\sigma((t'' | \phi'' \land \neg \phi'''))]$. We obviously have $(t'' | \phi'' \land \neg \phi''') \Rightarrow (t' | \phi') \in \nu \text{DSTEP}(\mathcal{R})$ (as a child of the proof tree root) and hence we get $[\sigma((t'' | \phi'' \land \neg \phi'''))] = [\sigma((t' | \phi'))] \in A$. Moreover, $[\sigma((t | \phi))] = [\sigma((t'' | \phi''))] \setminus [\sigma((t' | \phi'))]$ by Remark 2. We distinguish two subcases:

2.1. $M^\Sigma \vdash \phi'' \equiv \top$. Then $[\sigma((t' | \phi'))] = [\sigma((t'' | \phi''))]$, which implies $[\sigma((t | \phi))] \subseteq [\sigma((t' | \phi'))]$. We obtain $[\sigma(\varphi)] \Rightarrow [\sigma(\varphi')] \in \text{DVP}(A)$ by the first rule of DVP.

2.2. $M^\Sigma \vdash (\neg \phi'' \equiv \top)$. The children of $(t'' | \phi'' \land \neg \phi''') \Rightarrow (t' | \phi')$ in $PT$ are given by the rule [der*]. We proceed as in the case 3, and we get that $[\sigma((t'' | \phi'' \land \neg \phi'''))]$ is runnable. It follows $[\sigma((t | \phi))] \Rightarrow [\sigma((t' | \phi'))] \in \text{DVP}(A)$ by the rule [Step] of DVP.

3. [der*]. The root has a set of children $\{(t^j | \phi^j) \Rightarrow \varphi' | j \in J\}$, where $J = \{1, \ldots, n\}$. $M^\Sigma \vdash \phi^j \leftrightarrow (\phi'' \land \phi_j')$ with $\phi_j'$ depending only on $t''$ and the applied rule, and $\varphi \Rightarrow \varphi' \equiv (t'' | \phi') \Rightarrow (t' | \phi')$. We have:

$$\langle t^j | \phi^j \rangle \Rightarrow \varphi' \in \nu \text{DSTEP}(\mathcal{R})$$ implies by Prop. 9
$$\langle z | z = t^j \land \phi^j \rangle \Rightarrow \varphi' \in \nu \text{DSTEP}(\mathcal{R})$$ implies by Prop. 11
$$\langle z \bigvee_{j \in J} (z = t^j \land \phi^j) \rangle \Rightarrow \varphi' \in \nu \text{DSTEP}(\mathcal{R})$$ implies by def. of $A$
We distinguish two cases: 

\[ \bigvee_{j \in J} (z = t^j \land \phi^j) \Rightarrow [\sigma(\phi')] \in A \]

Since \( \bigcup_{j \in J} [\sigma((t^j | \phi^j))] = [\sigma\left( z \bigvee_{j \in J} (z = t^j \land \phi^j) \right)] \), by Remark 2 it follows that \( \bigcup_{j \in J} [\sigma((t^j | \phi^j))] \Rightarrow [\sigma(\phi')] \in A \).

We have \( [\sigma((t^j | \phi^j))] \Rightarrow [\sigma(\phi')] \in A \) since \( (t^j | \phi^j) \Rightarrow \phi' \in \nu \text{DSTEP}(\mathcal{R}) \), for each \( j \in J \), and hence \( \bigcup_{j \in J} [\sigma((t^j | \phi^j))] \Rightarrow [\sigma(\phi')] \in A \) by the definition of \( A \).

We have \( \bigcup_{j \in J} [\sigma((t^j | \phi^j))] = [\Delta(\bigcup_{j \in J} (t^j | \phi^j))] = [\Delta(t^j | \phi^j)] \) by Theorem 1 and Remark 2, and hence the equality \( \bigcup_{j \in J} [\sigma((t^j | \phi^j))] = [\sigma(t | \phi)] \).

The side-condition of the inference rule implies \( [\{t | \phi\}] = [\{t' | \phi'\}] \) runnably, and hence \( [\sigma(t | \phi)] \) runnably. Since the side-condition of the inference rule implies \( [\{t | \phi\}] = [\{t' | \phi'\}] \), it follows that \( [\sigma(t | \phi)] \Rightarrow [\sigma(t' | \phi')] \in \text{DVP}(A) \) by the rule [Step] of DVP.

**Reverse implication (completeness).** Assume that \( \mathcal{R} \models^\nu \phi \Rightarrow \phi' \), i.e., \( (M^\varphi, \sim_\mathcal{R}) \models^\nu [\sigma(\phi)] \Rightarrow [\sigma(\phi')] \) for all \( \sigma \in \text{SH} \), where \( \text{SH} = \{ \sigma \mid \sigma : \text{var}(\phi) \cap \text{var}(\phi') \rightarrow M^\phi \} \). We prove by coinduction that there is a proof of \( \phi \Rightarrow \phi' \) under DSTEP(\mathcal{R}).

We distinguish two cases:

1. if there exists \( \sigma \in \text{SH} \) such that \( [\sigma(\phi)] \cap [\sigma(\phi')] \neq \emptyset \), then we start our proof tree by a [subs] node:

\[
\begin{align*}
\text{[subs]} & \quad \varphi \land \neg \phi \Rightarrow \varphi' \\
& \quad \varphi \Rightarrow \varphi',
\end{align*}
\]

where \( \phi = \exists x.t_i = t_r \land \phi_r \) is the constraint in Rule [subs], Figure 1 under the assumption that \( \varphi = (t_i | \phi_i) \) and \( \varphi' = (t_r | \phi_r) \). By \( \varphi \land \neg \phi \) we denoted the constrained term \( (t_i | \phi_i \land \neg \phi) \). As \( [\sigma(\phi)] \cap [\sigma(\phi')] \neq \emptyset \) for some \( \sigma \), it follows that \( \phi \) is satisfiable and therefore [subs] can be applied. We have that \( R \models^\nu \varphi \land \neg \phi \Rightarrow \varphi' \) and also that, for any \( \sigma \in \text{SH} \), \( [\sigma(\phi \land \neg \phi)] \cap [\sigma(\phi')] = \emptyset \) (by the definition of \( \phi \)). Therefore, we continue to build the proof tree of \( \varphi \land \neg \phi \Rightarrow \varphi' \) coinductively (directly going into the second case, with \( \varphi \land \neg \phi \) playing the role of \( \varphi \), and \( \varphi' \) the role of \( \varphi' \)).

2. if for all \( \sigma \in \text{SH} \), \( [\sigma(\phi)] \cap [\sigma(\phi')] = \emptyset \), we distinguish two more cases:

(a) if for all \( \sigma \in \text{SH} \), \( [\sigma(\phi)] = \emptyset \), then we construct a proof tree of \( \phi \Rightarrow \varphi' \) as follows:

\[
\text{[axiom]} \quad \varphi \Rightarrow \varphi'.
\]

(b) if there exists \( \sigma \in \text{SH} \) such that \( [\sigma(\phi)] \neq \emptyset \), let \( \text{SH}_1 = \{ \sigma \in \text{SH} \mid [\sigma(\phi)] \neq \emptyset \} \) and \( \text{SH}_2 = \{ \sigma \in \text{SH} \mid [\sigma(\phi)] = \emptyset \} \). We have that \( \text{SH}_1 \neq \emptyset \).

We have that for all \( \sigma \in \text{SH}_1 \):

(A) \( [\sigma(\phi)] \cap [\sigma(\phi')] = \emptyset \), (B) \( [\sigma(\phi)] \neq \emptyset \), and (C) \( R \models^\nu \sigma(\phi) \Rightarrow \sigma(\phi') \).

Therefore, rule [Step] must have been applied with \( P = [\sigma(\phi)] \) and \( Q = [\sigma(\phi')] \) to justify \( R \models^\nu \sigma(\phi) \Rightarrow \sigma(\phi') \) (for all \( \sigma \in \text{SH}_1 \)). But \( P \cap Q = \emptyset \) and therefore \( P \setminus Q = P \). Which means
We first introduce some notations. Let \( G \) be a set of goals. If \( \varphi \vdash G \), then \( \varphi \) is derivable in \( \mathcal{R} \).

Indeed, as for all \( \sigma \in \mathcal{SH} \), \( [\sigma(\varphi)] \) is runnably for all \( \rho \in [\varphi] \), there is a rewrite rule \( l \rightarrow r \mathcal{R} \varphi_{tr} \), a ground context \( c[l] \) and \( \rho' : \text{var}(l, r, \varphi_{tr}) \) such that \( \rho(c[l]) = \rho'(c[l]) \) and \( \rho(\varphi_{tr}) = \top \).

This means that \( \varphi \vdash \bigvee_{j \in \{1, \ldots, n\}} 3\varphi_j \) if \( \varphi_j \) is valid in rule \( \text{der}\) in Figure 11 and therefore it can be applied:

\[
\text{[der]} \quad \langle \upsilon \mid \phi \rangle \Rightarrow \phi', j \in \{1, \ldots, n\} \\
\varphi \Rightarrow \phi'.
\]

Next we show that the coinduction hypothesis can be applied on all hypotheses \( \langle \upsilon \mid \phi \rangle \Rightarrow \phi' \). Indeed, it is sufficient to show that \( \varphi \vdash \langle \upsilon \mid \phi \rangle \Rightarrow \phi' \). First, notice that \( \text{var}(\varphi) \cap \text{var}(\varphi') = \text{var}(\Delta_R(\varphi)) \cap \text{var}(\varphi') \) (taking the derivative of \( \varphi \) preserves the common variables with \( \varphi' \)). By Proposition 10 it is sufficient to show that for all \( \sigma \in \mathcal{SH} \), \( \varphi \vdash [\sigma(\Delta_R(\varphi))] \Rightarrow [\sigma(\varphi')] \). But \( [\sigma(\Delta_R(\varphi))] = [\Delta_R(\varphi)] = \delta([\sigma(\varphi)]) \), and \( \varphi \vdash \delta([\sigma(\varphi)]) \Rightarrow [\sigma(\varphi')] \) is already known to hold.

We have shown that in whenever \( \varphi \vdash \varphi' \), we can build a proof tree of \( \varphi \Rightarrow \varphi' \) under \( \text{DS}
\]

\[\square\]

**Theorem 3 (Circularity Principle).** Let \( \mathcal{R} \) be a constrained rule system and \( G \) a set of goals. If \( (\mathcal{R}, G) \vdash \vdash G \) then \( \varphi \vdash \mathcal{R} \).

**Proof.** We first introduce some notations. Let \( \square \) be the partial order over proof trees of DCC defined as follows: \( PT_1 \square PT_2 \) iff \( \text{circOut}(PT_1) \) is a subtree of \( \text{circOut}(PT_2) \), where \( \text{circOut}(PT) \) is the tree obtained from \( PT \) by removing all subtrees having a [circ] root. If \( \text{circHeight}(PT) \) denote the length of the shortest path from the root to a circ-node in the proof tree \( PT \) under DCC, then \( PT_1 \square PT_2 \) implies \( \text{circHeight}(PT_1) \leq \text{circHeight}(PT_2) \). The main idea of the proof is to transform a guarded proof tree \( PT \) under DCC(\( \mathcal{R}, G \)) for a \( \varphi \Rightarrow \varphi' \) in \( \mathcal{R} \) into a proof tree \( PT' \) under \( \text{DS} \) for the same formula, where \( PT' \) is the lub \( \bigcup_{i \geq 0} PT_i \) of a chain \( PT = PT_0 \square PT_1 \square PT_2 \square \cdots \) with the property that \( \text{circHeight}(PT_i) < \text{circHeight}(PT_{i+1}) \) (this ensures that the limit \( \bigcup_{i \geq 0} PT_i \) has no [circ]-nodes). We show how \( PT_{i+1} \) is obtained from \( PT_i \). Let \( \langle t_i \mid \phi_i \rangle \Rightarrow \varphi' \) a circ node that gives \( \text{circHeight}(PT_i) \), i.e. its children are \( (t_c, \phi_c \wedge \phi_1 \wedge \phi_2) \Rightarrow \varphi' \) and \( (t_c, \phi_1 \wedge \neg \phi_2) \Rightarrow \varphi' \), where \( M \models \phi'' \leftrightarrow \exists \varphi(t_c, \phi_c)(t_c = t_c \wedge \phi_2) \) and \( \langle t_c, \phi_c \rangle \Rightarrow (t_c', \phi_c) \in G \).

\( (\mathcal{R}, G) \vdash \vdash G \) implies \( (\mathcal{R}, G) \vdash \vdash \langle t_c \mid \phi_c \rangle \Rightarrow \langle t_c' \mid \phi_c' \rangle \) and hence we obtain

\( (\mathcal{R}, G) \vdash \vdash (t_c \mid \phi_c \wedge \phi_1 \wedge \phi_2) \Rightarrow (t_c' \mid \phi_c') \) by Proposition 12.
Let $PT_c$ be a guarded proof tree for $\langle t_c \mid \phi_c \land \phi_1 \land \phi_2'' \rangle \Rightarrow \langle t'_c \mid \phi'_c \rangle$ under $\text{DCC}(R, G)$ and we want to transform it into a proof tree $PT'_c$ under the same proof system for $\langle t_c \mid \phi_c \land \phi_1 \land \phi_2'' \rangle \Rightarrow \varphi'$. We may replace all the rules $\langle t'_c \mid \phi'_c \rangle$ by $\varphi'$ in $PT_c$ (in the sense that they are valid instances of the $\text{DCC}(R, G)$ rules) excepting the nodes that are instances of the inference rule $\text{subs}$, because this rule involves the right-hand side of the reachability formula. Let $\langle t_j \mid \phi_j \rangle \Rightarrow \langle t'_c \mid \phi'_c \rangle$ be a $\text{subs}$-node in $PT_c$, i.e. its child is (or equivalent to) $\langle t_j \mid \phi_j \land \neg \phi_j'' \rangle \Rightarrow \langle t'_c \mid \phi'_c \rangle$, where $M^S = \phi_j'' \leftrightarrow (\exists X)(t_j = t'_c \land \phi'_c)$ and $X = \text{var}(t'_c, \phi'_c) \setminus \text{var}(t_j, \phi_j)$. Our intention is to transform this node into a $\text{disj}$-node $\langle t_j \mid \phi_j \rangle \Rightarrow \varphi'$ with the children $\langle t_j \mid \phi_j \land \phi_j'' \rangle \Rightarrow \varphi'$ and $\langle t_j \mid \phi_j \land \neg \phi_j'' \rangle \Rightarrow \varphi'$. In order to obtain $PT'_c$, a valid proof tree, we have to add to it, as a subtree of the new node, a proof tree for $\langle t_j \mid \phi_j \land \phi_j'' \rangle \Rightarrow \varphi'$ or for a formula equivalent to it. We know that $\langle t'_c \mid \phi'_c \land \phi_1 \land \phi_2'' \rangle \Rightarrow \varphi'$ is a node in $PT_j$ and hence there is a proof tree for it. We show that $[\{t_j \mid \phi_j \land \phi_j'' \}] \subseteq [\{t'_c \mid \phi'_c \land \phi_1 \land \phi_2'' \}]$. Let $\rho$ be in $[\phi_j \land (\exists X)(t_j = t'_c \land \phi'_c)]$. Note that $X = \text{var}(\varphi'_c) \setminus \text{var}(\phi_j)$. There is $\rho'$ such that $\rho'(y) = \rho(y)$ for all $y \notin X$ and $M^S, \rho' \models t_j = t'_c \land \phi'_c$. Since $\rho$ and $\rho'$ coincide on $\text{var}(\phi_j)$ we obtain $M^S, \rho' \models \phi_j \land t_j = t'_c \land \phi'_c$ and hence $M^S, \rho' \models \phi'_c \land (\exists X)(t_j = t'_c \land \phi'_c)$. Now the proof of the inclusion is finished.

Since $\text{var}(\{t'_c \mid \phi'_c \land \phi_1 \land \phi_2'' \}) = \text{var}(\phi_c \land \phi_j'' \land \varphi')$ and $\langle t_j \mid \phi_j \rangle \Rightarrow \langle t'_c \mid \phi'_c \rangle$ is a node in the proof tree of $\langle t_c \mid \phi_c \land \phi_1 \land \phi_2'' \rangle \Rightarrow \langle t'_c \mid \phi'_c \rangle$ under $\text{DCC}(R, G)$, it follows that $\text{var}(\{t_j \mid \phi_j \}) \cap \text{var}(\varphi') = \text{var}(\phi_c \land \phi_j'' \land \varphi')$. It follows that $\langle t_j \mid \phi_j \land \phi_j'' \rangle \Rightarrow \varphi'$ is equivalent to $\langle t'_c \mid t_j = t'_c \land \phi_j \land \phi_j'' \land \varphi' \rangle \Rightarrow \varphi'$ and the later one has a proof tree under $\text{DCC}(R, G)$ by Proposition 12. This proof tree is added as the subtree of $\langle t_j \mid \phi_j \land \phi_j'' \rangle \Rightarrow \varphi'$. Now the transformation of $PT_c$ into $PT'_c$ is completely described.

The proof tree $PT_{i+1}$ is the result of processing all $\text{circ}$ nodes that give $\text{circHeight}(PT_i)$. The relation $PT_i \sqsubseteq PT_{i+1}$ is given by the fact that $PT_i$ is guarded. Moreover, we have $\text{circHeight}(PT_i) < \text{circHeight}(PT_{i+1})$. □