DECIDABILITY OF FLOW EQUIVALENCE AND
ISOMORPHISM PROBLEMS FOR GRAPH C*-ALGEBRAS
AND QUIVER REPRESENTATIONS

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Abstract. We note that the deep results of Grunewald and Segal on
algorithmic problems for arithmetic groups imply the decidability of
several matrix equivalence problems involving poset-blocked matrices
over $\mathbb{Z}$. Consequently, results of Eilers, Restorff, Ruiz and Sørensen
imply that isomorphism and stable isomorphism of unital graph C*-algebras
(including the Cuntz-Krieger algebras) are decidable. One can also
decide flow equivalence for shifts of finite type, and isomorphism
of $\mathbb{Z}$-quiver representations (i.e., finite diagrams of homomorphisms of
finitely generated abelian groups).

1. Introduction

This paper concerns algorithmic decidability questions in symbolic dy-
namics and C*-algebras. Recall that, up to conjugacy, a shift of finite type
(SFT) is given by an $n \times n$ 0/1-matrix $A$. The corresponding subshift $\mathcal{X}_A$
consists of all bi-infinite sequences $(x_i)_{i \in \mathbb{Z}}$ over the alphabet $\{1, \ldots, n\}$ such
that $A_{x_i, x_{i+1}} = 1$. We recall that two subshifts are flow equivalent if their
suspicions (or mapping tori) are conjugate, modulo a time change, as flows
over $\mathbb{R}$. One can then ask the natural algorithmic question: given $A$ and
$B$ square 0/1-matrices (of possibly different sizes), decide whether the cor-
responding subshifts $\mathcal{X}_A$ and $\mathcal{X}_B$ are flow equivalent. Parry and Sullivan
[PS75], and then Bowen and Franks [BF77], provided fundamental matrix
invariants for this problem; Franks [Fra84] gave complete invariants for the
irreducible case; and Huang (unpublished) found complete matrix invariants
for the general case. A thorough treatment of complete invariants was given in
[Boy02, BH03], but the question of decidability was left open. This paper
provides the finishing touch on deciding this question. Note that it is still
an open question to decide whether two shifts of finite type are conjugate.
See [LM95] for background on symbolic dynamics.

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a grant from the Simons Foundation (#245268 to Benjamin Steinberg).
Cuntz and Krieger [CK80], motivated in part by the study of flow equivalence of shifts of finite type, introduced a very important class of $C^*$-algebras associated to square nondegenerate $0/1$-matrices. These were generalized to another important class, the class of graph $C^*$-algebras, defined as follows. Let $E = (E^0, E^1)$ be a directed graph, with vertex set $E^0$ and edge set $E^1$, allowed to be finite or countably infinite, and with source and range maps $s, r : E^1 \to E^0$. The graph $C^*$-algebra $C^*(E)$ is the universal $C^*$-algebra generated by a set $\{p_v : v \in E^0\}$ of mutually orthogonal projections and a set $\{s_e : e \in E^1\}$ of partial isometries, satisfying (for all $e, f$ in $E^1$ and $v$ in $E^0$) the relations

\[
s^*_e s_e = p_{r(e)} , \quad s^*_e s_f = 0 \quad \text{if} \ e \neq f , \quad s_e^* s_e \leq p_{s(e)} , \quad p_v = \sum_{s^{-1}(v)} s_e s^*_e \quad \text{if} \ 0 < |s^{-1}(v)| < \infty .
\]

The graph $C^*$-algebras isomorphic to Cuntz-Krieger algebras are those with $E^0$ and $E^1$ finite and with $s^{-1}(v) \neq \emptyset$ for all $v$. A graph $C^*$-algebra is unital (i.e., possesses a unit) if and only if its vertex set $E^0$ (but not necessarily $E^1$) is finite. When $A, B$ are adjacency matrices of finite graphs defining Cuntz-Krieger algebras, flow equivalence of the SFTs defined by $A$ and $B$ implies stable isomorphism of these algebras. (Recall that $C^*$-algebras are stably isomorphic if they become isomorphic upon tensoring with the algebra of compact operators; for unital $C^*$-algebras this is the same as Morita equivalence in the sense of Rieffel.) This connection, made in [CK80], is the heart of a fruitful interaction between symbolic dynamics and the study of Cuntz-Krieger algebras. The interaction in the case of general graph $C^*$-algebras is less direct but still significant. For much more on these algebras and their classification, we refer to the discussion and references of [ERRS16].

Now, let $(\mathcal{P}, \preceq)$ be a finite poset. Without loss of generality, we assume that $\mathcal{P} = \{1, \ldots, N\}$ and that $i \preceq j$ implies $i \leq j$. Let $\vec{n} = (n_1, \ldots, n_N)$ be an $N$-tuple of positive integers and put $|\vec{n}| = n_1 + \cdots + n_N$. For any ring $R$ with unit, define $M_{\mathcal{P}, \vec{n}}(R)$ to be the $R$-subalgebra of $M_{|\vec{n}|}(R)$ consisting of all $|\vec{n}| \times |\vec{n}|$-matrices over $R$ with a block form

\[
M = \begin{pmatrix} M_{1,1} & \cdots & M_{1,N} \\ \vdots & \ddots & \vdots \\ M_{N,1} & \cdots & M_{N,N} \end{pmatrix}
\]

with each $M_{i,j}$ an $n_i \times n_j$-matrix over $R$ and such that $M_{i,j} \neq 0$ implies $i \preceq j$; in particular, $M$ is block upper triangular. For example, if each $n_i = 1$, then $M_{\mathcal{P}, \vec{n}}(R)$ is the usual incidence algebra of $\mathcal{P}$ [Sta97].

We denote by $\text{GL}_{\mathcal{P}, \vec{n}}(R)$ the group of units of $M_{\mathcal{P}, \vec{n}}(R)$. If $R$ is commutative, then $\text{SL}_{\mathcal{P}, \vec{n}}(R)$ is the subgroup of matrices $M$ for which each diagonal block $M_{ii}$ has determinant 1. For a subgroup $\Gamma$ of $\text{GL}_{\mathcal{P}, \vec{n}}(R)$, two matrices

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1Following [ERRS16], we use the notation and definition of [FLR00], not [Rae05].
A, B ∈ M_{P,\overline{\pi}}(R) are said to be Γ-equivalent if there are matrices U, V ∈ Γ with UAV = B.

A vast collection of related works followed the introduction of the Cuntz-Krieger C*-algebras in [CK80], in particular, including many papers on graph C*-algebras (consider the citations of [Rae05]). Eventually, Restorff in [Res06] showed that decidability of stable isomorphism for Cuntz-Krieger algebras satisfying Condition II of Cuntz reduces to deciding whether two matrices A, B ∈ M_{P,\overline{\pi}}(Z) are GL_{P,\overline{\pi}}(Z)-equivalent. In related work, Boyle and Huang reduced the question of deciding flow equivalence for shifts of finite type to deciding whether two matrices A, B ∈ M_{P,\overline{\pi}}(Z) are SL_{P,\overline{\pi}}(Z)-equivalent (see [Boy02]; Huang’s alternate development was never published). Eilers, Restorff, Ruiz and Sørensen [ERRS16] reduced the decidability of stable isomorphism of unital graph C*-algebras (a class including the Cuntz-Krieger C*-algebras) to a more general problem of poset blocked equivalence of rectangular matrices, which we describe later. They also reduced decidability of isomorphism of unital graph C*-algebras to yet another matrix equivalence problem.

We shall point out that all these matrix equivalence problems are decidable, by the deep results of Grunewald and Segal [GS80], making the work of Borel and Harish-Chandra [BHC62] effective. We also use Grunewald-Segal [GS80] to prove the decidability of isomorphism of explicitly given commuting diagrams of homomorphisms of finitely many finitely generated abelian groups. (This can be interpreted as decidability of isomorphism of Z-quiver representations.) This applies to diagrams arising as reduced K-webs in the study of flow equivalent SFTs [BH03] and to related diagrams of reduced filtered K-theory arising in the study of certain C*-algebras (e.g. [ABK14, Res06]).

These decidability results, as proved by appeal to [GS80, Algorithm A], do not provide practical decision procedures; the extremely general Algorithm A is not even proved to be primitive recursive.

2. GRUNEWALD AND SEGAL

Following Grunewald and Segal [GS80], by a Q-group we mean a subgroup J of GL_n(\mathbb{C}) (for some n ≥ 1) which is the set of common zeros in GL_n(\mathbb{C}) of finitely many polynomials, with rational coefficients, in the n^2 matrix entries. The Q-group is given explicitly if these polynomials are explicitly given. If R is a subring of \mathbb{C}, then J_R = GL_n(R) \cap J. When there exists an explicitly given linear isomorphism, defined over Q, from a Q-group J of complex matrices onto a Q-group J', which carries J_Z onto (J')_Z, we may avoid mention of the isomorphism and simply refer to J as a Q-group. For example, if J_1 and J_2 are Q-groups, then their direct product J_1 × J_2 is a Q-group, via the embedding (A, B) ↦→ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.

\footnote{However, when such a set of polynomials exists, we usually will not write one out.}
A rational action of a $\mathbb{Q}$-group $J$ is a homomorphism $\rho$ from $J$ into the group of permutations of a subset $W$ of some complex vector space $\mathbb{C}^m$ such that for each $w \in W$, the coordinate entries of the vector $\rho_g(w)$ are rational functions of the $n^2$ entries of the matrix $g$ as $g$ runs through the identity component $J^0$ of $J$; and for $w \in W \cap \mathbb{Q}^m$, these rational functions are ratios of polynomials with rational coefficients. The action is explicitly given if for each $w \in W \cap \mathbb{Q}^m$, (i) there is an effective procedure which produces those coordinate rational functions, and (ii) for each $g \in J_Z$ the vector $\rho_g(w)$ is effectively computable.

By an arithmetic subgroup of $J$, we mean a subgroup $\Gamma \leq J_Z$ of finite index (usually, one allows a subgroup commensurable with $J_Z$, but as pointed out in [GS80] it is enough to consider finite index subgroups by performing a rational change of basis). If $J$ is an explicitly given $\mathbb{Q}$-group, following Grunewald and Segal, we say that the arithmetic subgroup $\Gamma$ is explicitly given if an upper bound on the index of $\Gamma$ in $J_Z$ is given and an effective procedure is given to decide, for each $g \in J_Z$, whether or not $g \in \Gamma$. Most of this paper will only consider $\Gamma = J_Z$, with the exception of Lemma 3.9.

The following stunning result is due to Grunewald and Segal [GS80, Algorithm A].

**Theorem 2.1** (Grunewald/Segal). Let $J$ be an explicitly given $\mathbb{Q}$-group and $\rho$ an explicitly given rational action of $J$ on a subset $W$ of $\mathbb{C}^m$. Let $\Gamma$ be an explicitly given arithmetic subgroup of $J$ (e.g., $\Gamma = J_Z$). There is an algorithm, which given vectors $v, w \in W \cap \mathbb{Q}^m$, decides whether there exists $g \in \Gamma$ such that $\rho_g(v) = w$ (and produces such a $g$, when one exists).

**Remark 2.2.** It is important to note that, implicit in [GS80, Algorithm A], is that $J$, $\rho$ and $\Gamma$ should be considered part of the input (this is the point of $J$ and $\Gamma$ being “explicitly given”), and not just the vectors $v, w \in W \cap \mathbb{Q}^m$, despite the wording of Theorem 2.1 (which mimics that of [GS80, Algorithm A] and seems to imply that they are fixed). For instance, Grunewald and Segal use that the group and the action are part of the input in [GS80, Corollaries 3 and 4]. In our applications to shifts of finite type and graph $C^*$-algebras, the particular $J$, $\rho$ and $\Gamma$ used will be dependent on the input to our decidability questions.

Observe that if $\mathcal{P}$ is a finite poset as above, then $GL_{\mathcal{P},\bar{n}}(\mathbb{C})$ and $SL_{\mathcal{P},\bar{n}}(\mathbb{C})$ are $\mathbb{Q}$-groups. Indeed, $GL_{\mathcal{P},\bar{n}}(\mathbb{C})$ is the subgroup of $GL_{\bar{n}}(\mathbb{C})$ defined by the polynomials over $\mathbb{Z}$ saying that an entry belonging to $M_{i,j}$ with $i \neq j$ is 0. The subgroup $SL_{\mathcal{P},\bar{n}}(\mathbb{C})$ is defined by the additional equations stating that each diagonal block has determinant 1.

We let the $\mathbb{Q}$-group $J = GL_{\mathcal{P},\bar{n}}(\mathbb{C}) \times GL_{\mathcal{P},\bar{n}}(\mathbb{C})$ act on the $\mathbb{C}$ vector space $M_{\mathcal{P},\bar{n}}(\mathbb{C})$ by $(U, V) : A \mapsto UAV^{-1}$; this action is a rational action of $J$. This action restricts to an action of $SL_{\mathcal{P},\bar{n}}(\mathbb{C}) \times SL_{\mathcal{P},\bar{n}}(\mathbb{C})$. Given $A$ in $M_{\mathcal{P},\bar{n}}(\mathbb{Q})$, the polynomials with rational coefficients which compute the entries of $UAV^{-1}$ for $(U, V)$ in $J$ can be effectively computed from $\mathcal{P}$ and $\bar{n}$. We immediately obtain the following corollary of Theorem 2.1.
Corollary 2.3. Given a finite poset \( \mathcal{P} \), a vector \( \vec{n} \) of positive integers and matrices \( A, B \in M_{\mathcal{P}, \vec{n}}(\mathbb{Q}) \), one can decide whether \( A, B \) are \( GL_{\mathcal{P}, \vec{n}}(\mathbb{Z}) \)-equivalent and whether they are \( SL_{\mathcal{P}, \vec{n}}(\mathbb{Z}) \)-equivalent.

As noted in the introduction, Corollary 2.3 combines with the works \[ \text{Boy02, Res06} \] to give the following.

Corollary 2.4. Flow equivalence is decidable for shifts of finite type.

Corollary 2.5. Stable isomorphism is decidable for Cuntz-Krieger algebras satisfying Cuntz’s condition II.

3. Rectangular matrices

We now consider poset blocked matrices with a rectangular structure. This is natural, and necessary for showing that the work of \[ \text{ERRS16} \] implies general decidability results for unital graph C*-algebras. The adjacency matrices for these (directed) graphs have only finitely many vertices, but may have infinitely many edges; the analysis of their adjacency matrices (with “\( \infty \)” an allowed entry) is reduced in \[ \text{ERRS16} \] to the analysis of associated rectangular matrices with integer entries.

We will use (and slightly augment) notations from \[ \text{ERRS16} \]. Take \( \mathcal{P} \) and \( N \) as above. Let \( \vec{m} \) and \( \vec{n} \) be nonnegative elements of \( \mathbb{Z}^N \). Set \( \mathcal{I} = \{ i: m_i > 0 \}, \mathcal{J} = \{ j: n_j > 0 \}, m = \sum m_i, n = \sum n_j \). We impose the nontriviality requirement that \( \mathcal{I} \) and \( \mathcal{J} \) are nonempty, with \( \mathcal{I} \cup \mathcal{J} = \mathcal{P} \). For \( R \) a subring of \( \mathbb{C} \), define \( M_{\mathcal{P}, \vec{m}, \vec{n}}(R) \) to be the set of \( m \times n \) matrices with \( \mathcal{I} \times \mathcal{J} \) block form, with \( M_{ij} \) an \( m_i \times n_j \) matrix over \( R \) such that \( M_{ij} \neq 0 \) implies \( i \preceq j \).

As in \[ \text{ERRS16} \], \( m_i = 0 \) can be viewed as producing an empty block row indexed by \( i \), and similarly \( n_j = 0 \) corresponds to an empty block column. \( \mathcal{I} \) and \( \mathcal{J} \) are posets, with the order inherited from \( \mathcal{P} \).

Given a tuple \( \vec{n} \) over \( \mathbb{Z}_+ \), with \( \mathcal{J} \) indexing the indices \( j \) at which \( n_j > 0 \), and with \( \mathcal{J} \) nonempty, we let \( M_{\mathcal{P}, \vec{n}}(R) \) denote \( M_{\mathcal{P}, \vec{m}, \vec{n}}(R) \). If all entries of \( \vec{n} \) are positive, then this agrees with \( M_{\mathcal{P}, \vec{n}}(R) \) as defined earlier; in general, \( M_{\mathcal{P}, \vec{n}}(R) \) is the set of matrices over \( R \) with \( \mathcal{J} \times \mathcal{J} \) block structure corresponding to the poset \( \mathcal{J} \) and the associated positive entries of \( \vec{n} \).

Let \( GL_{\mathcal{P}, \vec{n}}(R) \) be the group of units of \( M_{\mathcal{P}, \vec{n}}(R) \), with \( SL_{\mathcal{P}, \vec{n}}(R) \) its subgroup of matrices \( M \) such that for \( n_j > 0 \), we have \( \det M_{jj} = 1 \).

Example 3.1. Let \( \mathcal{P} = \{1, 2, 3, 4, 5\} \) be the poset such that \( i \preceq j \) iff \( i = j, i = 1 \) or \((i, j) \in \{(2, 5), (3, 4)\} \). Let \( \vec{m} = (1, 0, 1, 0, 1) \) and \( \vec{n} = (1, 1, 0, 1, 1) \). Then \( \mathcal{I} = \{1, 3, 5\} \) and \( \mathcal{J} = \{1, 2, 4, 5\} \). We display some general matrix forms:

\[
\begin{array}{c}
\text{SL}_{\mathcal{P}, \vec{m}}(\mathbb{Z}) & \text{M}_{\mathcal{P}, \vec{m}, \vec{n}}(\mathbb{Z}) & \text{GL}_{\mathcal{P}, \vec{n}}(\mathbb{Z}) \\
\begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \pm 1 & * & * & * \\ 0 & \pm 1 & 0 & * \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}
\end{array}
\]
in which each $\ast$ is an arbitrary entry from $\mathbb{Z}$. \hfill \Box

If $U \in M_{P,\tilde{m}}(R)$, $M \in M_{P,\tilde{m},\tilde{n}}(R)$ and $V \in M_{P,\tilde{n}}(R)$, then $UMV \in M_{P,\tilde{m},\tilde{n}}(R)$. The rule $\rho_{(U,V)}: M \mapsto UMV^{-1}$ defines an action of the $\mathbb{Q}$-group $\text{GL}_{P,\tilde{m}}(\mathbb{C}) \times \text{GL}_{P,\tilde{n}}(\mathbb{C})$ on $M_{P,\tilde{m},\tilde{n}}(\mathbb{C})$. This is an explicitly given rational action. The next result follows immediately from Theorem 2.1.

**Corollary 3.2.** Suppose $H$ is an explicitly given $\mathbb{Q}$-group and $H$ is a subgroup of $\text{GL}_{P,\tilde{m}}(\mathbb{C}) \times \text{GL}_{P,\tilde{n}}(\mathbb{C})$ (given by an explicit embedding defined over $\mathbb{Q}$). Then given matrices $A, B$ in $M_{P,\tilde{m},\tilde{n}}(\mathbb{Q})$, there is an algorithm which decides whether there exists $(U, V)$ in $H$ such that $UAV^{-1} = B$ (and produces such a $(U, V)$, when one exists).

For clarity, we next address a minor point directly.

**Corollary 3.3.** Suppose $H$ is an explicitly given $\mathbb{Q}$-group and $H$ is a subgroup of $\text{GL}_{P,\tilde{m}}(\mathbb{C}) \times \text{GL}_{P,\tilde{n}}(\mathbb{C})$ (given by an explicit embedding defined over $\mathbb{Q}$). Then given matrices $A, B$ in $M_{P,\tilde{m},\tilde{n}}(\mathbb{Q})$, there is an algorithm which decides whether there exists $(U, V)$ in $H$ such that $UAV = B$ (and produces such a $(U, V)$, when one exists).

**Proof.** Let $H^\circ$ be the image of $H$ under the map $(U, V) \mapsto (U, V^{-1})$. The following are equivalent: (i) there exists $(U, V)$ in $H_{\mathbb{Z}}$ with $UAV = B$; (ii) there exists $(U, V)$ in $(H^\circ)_{\mathbb{Z}}$ with $UAV^{-1} = B$. Even if $H \neq H^\circ$, the group $H^\circ$ is an explicitly given $\mathbb{Q}$-group in $\text{GL}_{P,\tilde{m}}(\mathbb{C}) \times \text{GL}_{P,\tilde{n}}(\mathbb{C})$. Corollary 3.2 applies with $H^\circ$ in place of $H$, and this decides (ii). \hfill \Box

Eilers, Restorff, Ruiz and Sørensen reduced the problem of deciding stable isomorphism of two unital graph $C^*$-algebras to the problem of deciding, given $A, B$ in $M_{P,\tilde{m},\tilde{n}}(\mathbb{Z})$, whether there exists $(U, V)$ in $\text{SL}_{P,\tilde{m}}(\mathbb{Z}) \times \text{SL}_{P,\tilde{n}}(\mathbb{Z})$ such that $UAV = B$ (see [ERRST15], Theorems 3.1 and 14.2). As $\text{SL}_{P,\tilde{m}}(\mathbb{Z}) \times \text{SL}_{P,\tilde{n}}(\mathbb{Z})$ is an explicitly given $\mathbb{Q}$-group in $\text{GL}_{P,\tilde{m}}(\mathbb{C}) \times \text{GL}_{P,\tilde{n}}(\mathbb{C})$, their work implies the following result.

**Theorem 3.4.** Stable isomorphism of unital graph $C^*$-algebras is decidable.

Below, for $R$ a subring of $\mathbb{C}$ and $M$ an $n \times m$ matrix over $\mathbb{C}$, $\text{im}_R(M)$ denotes $\{Mz \in \mathbb{C}^n : z \in R^m\}$.

**Theorem 3.5.** Suppose that $x, y$ are column vectors in $\mathbb{Z}^n$ and $H$ is an explicitly given $\mathbb{Q}$-group which is a subgroup of $\text{GL}_{P,\tilde{m}}(\mathbb{C}) \times \text{GL}_{P,\tilde{n}}(\mathbb{C})$ (via an explicitly given embedding defined over $\mathbb{Q}$). Then given matrices $A, B$ in $M_{P,\tilde{m},\tilde{n}}(\mathbb{Q})$, there is an algorithm which decides whether there exists $(U, V) \in H_{\mathbb{Z}}$ such that the following hold:

\begin{align}
(3.6) \quad & UAV^{-1} = B, \quad \text{and} \\
(3.7) \quad & (V^{-1})^T x - y \in \text{im}_\mathbb{Z}(B^T).
\end{align}

The algorithm produces such a $(U, V)$, when one exists.
Proof. Corollary 3.2 decides whether there exists \((U, V) \in H_Z\) such that (3.6) holds, and if so produces such a \((U, V)\). If \((U, V)\) doesn’t exist, the problem is decided; given such a \((U, V)\), after replacing \((A, B, x, y)\) with \((UAV^{-1}, B, (V^{-1})^T x, y)\), it remains to produce a deciding algorithm in the case \(A = B\). We leave this step to Lemma 3.9 below. \(\square\)

Remark 3.8. If \(C\) is an \(m \times n\) integer matrix, then the set of \(m \times m\) matrices \(D\) with \(D \cdot \text{im}_C(C) \subseteq \text{im}_C(C)\) is the vanishing set of an explicitly given set of polynomials over \(\mathbb{Q}\) (assuming \(C\) is given explicitly). Indeed, using standard linear algebra over \(\mathbb{Q}\), we can find a \(k \times m\) matrix \(M\) over \(\mathbb{Q}\) so that \(Mx = 0\) if and only if \(x \in \text{im}_C(C)\). Then we are looking for the matrices \(D\) such that \(MDC = 0\), which is an explicitly given set of polynomial equations over \(\mathbb{Q}\) in the entries of \(D\).

Lemma 3.9. Suppose \(m, n\) are integers; \(A\) is an \(m \times n\) matrix with integer entries; and \(J\) is an explicitly given \(\mathbb{Q}\)-group in \(GL_m(\mathbb{C}) \times GL_n(\mathbb{C})\). Then there is an algorithm which decides, given \(x, y\) in \(\mathbb{Z}^n\), whether there exists \((U, V)\) in \(J_Z\) such that

\[
(3.10) \quad UAV^{-1} = A, \quad \text{and}
\]

\[
(3.11) \quad (V^{-1})^T x - y \in \text{im}_{\mathbb{Z}}(A^T). 
\]

Proof. Let \(J_A = \{(U, V) \in J : UAV^{-1} = A\}\), an explicitly given \(\mathbb{Q}\)-group. Set \(H_A = \{V : (U, V) \in J_A\}\). Let \(E = \text{End}(\text{im}_{\mathbb{C}}(A^T))\) be the set of \(n \times n\) matrices \(M\) over \(\mathbb{C}\) such that \(\{Mw : w \in \text{im}_{\mathbb{C}}(A^T)\} \subseteq \text{im}_{\mathbb{C}}(A^T)\). Define \(K_A\) to be the set of matrices \(K\) in \(GL_{m+1}(\mathbb{C})\) with block form \((K_{ij})_{0 \leq i, j \leq m}\), with each \(K_{ij} \in \mathbb{C}^{n \times n}\); \(K_{00} \in H_A\) (or equivalently, \((K_{00}^T)^{-1} \in H_A\)); \(K_{ii} = I\) for \(1 \leq i \leq m\); \(K_{ij} = 0\) if \(1 \leq i \neq j\); and \(K_{0j} \in E\) if \(j \geq 1\). Visually, we have

\[
(3.12) \quad K = \begin{pmatrix}
K_{00} & K_{01} & K_{02} & \cdots & K_{0m} \\
0 & I & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I
\end{pmatrix}.
\]

We claim that \(K_A\) is a group. To show this, it suffices to show, given \(V\) in \(H_A\), that \(V^T \in E\) and \((V^{-1})^T \in E\). (This follows from considering, given \(K, L\) in \(K_A\) with block forms from (3.12), the block forms of \(K^{-1}\) and \(KL\); for \(1 \leq j \leq m\), \((K^{-1})_{0j} = -K_{00}^{-1}K_{0j}\) and \((KL)_{0j} = K_{00}L_{0j} + K_{0j}\).) Because \(H_A\) is a group, it suffices to show \(V^T \in E\). For this, pick \(U\) such that \((U, V) \in J_A\), and note

\[
(3.13) \quad UAV^{-1} = A \implies UA = AV \implies A^TU^T = V^TA^T.
\]

Let \(A^T w, w \in \mathbb{C}^m\), be an arbitrary element of \(\text{im}_{\mathbb{C}}(A^T)\). Then

\[
V^T(A^T w) = A^T(U^T w) \in \text{im}_{\mathbb{C}}(A^T).
\]
The group $K_A$ is an explicitly given $\mathbb{Q}$-group by Remark 3.8. Set $W = (\mathbb{C}^n)^{m+1}$, writing $w$ in $W$ as $w = (w^{(0)}, \ldots, w^{(m)})$. There is an explicitly given rational action $\kappa$ of $K_A$ on $W$, given for $K$ in $K_A$ by the rule

$$
(\kappa_K w)^{(0)} = \sum_{j=0}^{m} K_{0j} w^{(j)},
$$

$$
(\kappa_K w)^{(j)} = w^{(j)}, \quad 1 \leq j \leq m.
$$

Define the $\mathbb{Q}$-group $\tilde{K}_A = \{(M, K) \in \text{GL}_m(\mathbb{C}) \times K_A : (M, (K_0^{-1})^T) \in J_A\}$; it is explicitly given. There is an explicitly given rational action $\rho$ of $\tilde{K}_A$ on $W$ by $\rho_{(M,K)} w = \kappa_K w$.

Let $\Gamma$ be the subgroup of $(\tilde{K}_A)_{\mathbb{Z}}$ consisting of those $(M, K)$ such that $K_{0j} \cdot \text{im}_{\mathbb{Z}}(A^T) \subseteq \text{im}_{\mathbb{Z}}(A^T)$ for $j \geq 1$. We claim that $\Gamma$ is an explicitly given arithmetic subgroup of $\tilde{K}_A$. To see that it is a subgroup, we again use (3.13), but this time in the case that $(U, V) \in (J_A)_{\mathbb{Z}}$. Let $b_1, \ldots, b_k$ be a basis for the free abelian group $A = \text{im}\mathbb{Q}(A^T) \cap \mathbb{Z}^n \supseteq \text{im}_{\mathbb{Z}}(A^T)$. Let $b_i = A^T c_i$ with $c_i \in \mathbb{Q}^m$. Let $\ell$ be a common denominator for the entries of the $c_i$. Then $\ell \cdot A \subseteq \text{im}_{\mathbb{Z}}(A^T)$. It now follows easily that if $(M, K) \in (\tilde{K}_A)_{\mathbb{Z}}$, then $\ell K_{0j} \cdot \text{im}_{\mathbb{Z}}(A^T) \subseteq \ell \cdot A \subseteq \text{im}_{\mathbb{Z}}(A^T)$ for $j \geq 1$. We conclude that $\Gamma$ has finite index in $(\tilde{K}_A)_{\mathbb{Z}}$ and it is clearly explicitly given.

Let $a_1^T, \ldots, a_m^T$ denote the columns of $A^T$. Given $x, y$ in $\mathbb{Z}^n$, we claim that the following are equivalent.

1. $\exists (U, V)$ in $J_{\mathbb{Z}}$ such that $(V^{-1})^T x - y \in \text{im}_{\mathbb{Z}}(A^T)$ and $UAV^{-1} = A$.
2. $\exists (M, K)$ in $\Gamma$ such that $\rho_{(M,K)} : (x, a_1^T, \ldots, a_m^T) \mapsto (y, a_1^T, \ldots, a_m^T)$.

Let us check the claim. Given $(U, V)$ in $J_{\mathbb{Z}}$ from (1), we have $V \in H_A$, and there are integers $r_1, \ldots, r_m$ such that $(V^{-1})^T x - y = r_1 a_1^T + \cdots + r_m a_m^T$. Define $(M, K)$ in $\Gamma$ by setting $K_{00} = (V^{-1})^T$, $K_{0j} = -r_j I$ for $1 \leq j \leq m$, and $M = U$. Then $\kappa_K : (x, a_1^T, \ldots, a_m^T) \mapsto (y, a_1^T, \ldots, a_m^T)$.

Conversely, suppose $(M, K)$ in $\Gamma$, satisfies (2). Set $U = M$ and $V = (K_0^{-1})^T$. Then $(U, V) \in J_{\mathbb{Z}}$, and

$$
(V^{-1})^T x - y = -K_{01} a_1^T - \cdots - K_{0m} a_m^T \in \text{im}_{\mathbb{Z}}(A^T).
$$

Because $(M, K) \in \Gamma$, we have $(U, V) \in J_{\mathbb{Z}}$. This finishes the proof of the claim.

By Theorem 2.1, there is an algorithm deciding whether (2) holds, because $\kappa$ is an explicitly given rational action of $K_A$ on $W$. Therefore there is an algorithm deciding (1). \hfill \Box

Eilers, Restorff, Ruiz and Sørensen reduced the problem of deciding isomorphism of two unital graph $C^*$-algebras to the case of Theorem 3.5 (after $V^{-1}$ in (3.10) and (3.11) is replaced with $V$) in which $J = \text{SL}_{p,\tilde{p}}(\mathbb{Z}) \times \text{SL}_{p,\tilde{p}}(\mathbb{Z})$ and the vectors $x, y$ are the vectors with every entry equal to 1 (see [ERRS16 Theorem 14.3]). It follows that their work implies the following result.
Theorem 3.14. Isomorphism of unital graph $C^*$-algebras is decidable.

4. ISOMORPHISM OF DIAGRAMS AND QUIVER REPRESENTATIONS

The purpose of this section is to show that the isomorphism problem is decidable for finite diagrams of homomorphisms of finitely generated abelian groups. These include diagrams of the sort that appear in full and reduced K-webs as invariants of operator algebras or flow equivalence.

Let $Q = (Q_0, Q_1)$ be a finite directed graph, hereafter called a quiver, with vertex set $Q_0$ and edge set $Q_1$. We shall write $s(e)$ and $r(e)$ for the source and target of an edge $e$, respectively. By a $\mathbb{Z}$-representation $\Phi = (A, \phi)$ of $Q$ we mean an assignment of a finitely generated abelian group $A_v$ to each vertex $v \in Q_0$ and a homomorphism $\phi_e : A_{s(e)} \to A_{r(e)}$. A morphism $f : (A, \phi) \to (B, \rho)$ is a collection of homomorphisms $f_v : A_v \to B_v$, one for each $v \in Q_0$, such that the diagram

$$
\begin{array}{ccc}
A_{s(e)} & \xrightarrow{\phi_e} & A_{r(e)} \\
\downarrow f_{s(e)} & & \downarrow f_{r(e)} \\
B_{s(e)} & \xrightarrow{\phi_e} & B_{r(e)}
\end{array}
$$

commutes for all $e \in Q_1$. The category of $\mathbb{Z}$-representations of $Q$ will be denoted $\text{rep}_{\mathbb{Z}}(Q)$.

The path ring $\mathbb{Z}Q$ is the ring defined as follows. As an abelian group, it has basis the set of directed paths in $Q$, including an empty path $P_v$ for each vertex $v \in Q_0$. The product of two basis elements $q$ and $r$, is their concatenation, if defined, and otherwise is 0. We follow the convention here of concatenating edges from right to left, as if we were composing functions. For example, the path

$$u \xrightarrow{e} v \xrightarrow{f} w$$

is denoted $fe$. Note that $\mathbb{Z}Q$ is finitely generated as a ring by the $P_v$ with $v \in Q_0$ and the edges $e \in Q_1$. Also note that $\mathbb{Z}Q$ is unital with $1 = \sum_{v \in Q_0} P_v$ a decomposition into orthogonal idempotents. Let $\mathbb{Z}Q$-mod denote the category of (unital) left $\mathbb{Z}Q$-modules which are finitely generated over $\mathbb{Z}$. Then, analogously to the well studied case of representations of quivers over fields [ASS06], there is an equivalence of categories between $\text{rep}_{\mathbb{Z}}(Q)$ and $\mathbb{Z}Q$-mod. We state here how the equivalence behaves on objects because we want to show that it can be done algorithmically. The fact that this is an equivalence of categories follows from a more general result of Mitchell [Mit72, Theorem 7.1] applied to the free category generated by $Q$.

If $(A, \phi)$ is a $\mathbb{Z}$-representation of $Q$, then we obtain a left $\mathbb{Z}Q$-module, finitely generated over $\mathbb{Z}$, by taking as the underlying abelian group $A = \bigoplus_{v \in Q_0} A_v$. The empty path $P_v$ acts as the projection to the summand $A_v$ (so it is the identity on $A_v$ and annihilates $A_w$ with $w \neq v$). A non-empty path
\[ p = e_n \cdots e_2 e_1 \text{ from } v \text{ to } w \text{ acts on } A_v \text{ by the composition } \phi_{e_n} \cdots \phi_{e_2} \phi_{e_1} \]
and is zero on all summands \( A_u \) with \( u \neq v \). Conversely, if \( A \) is a left \( \mathbb{Z}Q \)-module, then we define \( A_v = P_v A \) for \( v \in Q_0 \). From the orthogonal decomposition \( 1 = \sum_{v \in Q_0} P_v \) it follows easily that \( A = \bigoplus_{v \in Q_0} A_v \) and that if \( e \in Q_1 \), then \( eA_v = 0 \) if \( v \neq s(e) \) and \( eA_{s(e)} \subseteq A_{r(e)} \). Thus we can define \( \phi_e : A_{s(e)} \to A_{r(e)} \) by \( \phi_e(a) = ea \).

For algorithmic problems, we assume that \( \mathbb{Z} \)-representations \((A, \phi)\) of \( Q \) are given by providing a finite presentation for each group \( A_v \) and giving the image under \( \phi_e \) of each generator of \( A_{s(e)} \). We assume that \( \mathbb{Z}Q \)-modules, finitely generated over \( \mathbb{Z} \), are given via finite presentations as abelian groups and with the action of each edge and each empty path \( P_v \) on the generators specified. Clearly, there is a Turing machine which can turn such a presentation of a \( \mathbb{Z} \)-representation into such a presentation of a \( \mathbb{Z}Q \)-module, finitely generated over \( \mathbb{Z} \) (and vice versa). Therefore, to algorithmically decide isomorphism of \( \mathbb{Z} \)-representations of \( Q \) is equivalent to deciding the isomorphism problem for \( \mathbb{Z}Q \)-modules which are finitely generated over \( \mathbb{Z} \). But Grunewald and Segal \([GS80, Corollary 4]\) solved the isomorphism problem for \( R \)-modules finitely generated over \( \mathbb{Z} \) when \( R \) is a finitely generated ring. Consequently, we have the following.

**Theorem 4.1.** There is an algorithm that given as input a finite quiver \( Q \) and two \( \mathbb{Z} \)-representations, decides whether the representations are isomorphic.

The reader is referred to \([BH03]\) for the definitions of full and reduced \( K \)-webs in the following corollary.

**Corollary 4.2.** Let \( \mathcal{P} \) be a finite poset. There is an algorithm which, given matrices in \( M_{\mathcal{P},\bar{\pi}}(\mathbb{Z}) \), decides whether their full \( K \)-webs are isomorphic, and there is an algorithm which decides whether their reduced \( K \)-webs are isomorphic.

In \([BH03]\), for matrices \( A, B \) within a subclass of \( M_{\mathcal{P},\bar{\pi}}(\mathbb{Z}) \) sufficient to address problems of stable isomorphism, it was shown that two matrices are \( GL_{\mathcal{P},\bar{\pi}}(\mathbb{Z}) \) equivalent if and only if the reduced \( K \)-webs of \( I - A \) and \( I - B \) are isomorphic. Thus Corollary 4.2 gives an alternate route to proving Corollary 2.5.

In \([BH03]\), there is also a characterization of flow equivalence of shifts of finite type in terms of more refined isomorphism relations of reduced \( K \)-webs. We believe that the work of Grunewald and Segal can also be applied to show decidability of isomorphisms of quiver representations satisfying such constraints. We will not attempt this here.

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