MAXIMIZING THE PROBABILITY OF ATTAINING A TARGET PRIOR TO EXTINCTION

DEBASHISH CHATTERJEE, EUGENIO CINQUEMANI, AND JOHN LYGEROS

Abstract. We present a dynamic programming-based solution to the problem of maximizing the probability of attaining a target set before hitting a cemetery set for a discrete-time Markov control process. Under mild hypotheses we establish that there exists a deterministic stationary policy that achieves the maximum value of this probability. We demonstrate how the maximization of this probability can be computed through the maximization of an expected total reward until the first hitting time to either the target or the cemetery set. Martingale characterizations of thrifty, equalizing, and optimal policies in the context of our problem are also established.

§1. Introduction

There are two basic categories of discrete-time controlled Markov processes that deal with random temporal horizons. The first is the well-known optimal stopping problem [Dynkin, 1963], in which the random horizon arises from some dynamic optimization protocol based on the past history of the process. The random ‘stopping time’ thus generated is regarded as a decision variable. This problem arises in, among other areas, stochastic analysis, mathematical statistics, mathematical finance, and financial engineering; see the comprehensive monograph [Peskir and Shiryaev, 2006] for details and further references. The second is relatively less common, and is characterized by the fact that the random horizon arises as a result of an endogenous event of the stochastic process, e.g., the process hitting a particular subset of the state-space, variations in the process paths crossing a certain threshold. This problem arises in, among others, optimization of target-level criteria [Bouakiz and Kebir, 1995; Dubins and Savage, 1976], optimal control of retirement investment funds [Boda et al., 2004], minimization of ruin probabilities in insurance funds [Schmidli, 2008], ‘satisfaction of needs’ problems in economics [Simon, 1957], risk minimizing stopping problems [Ohtsubo, 2003], attainability problems under stochastic perturbations [Diagilova and Kurzhanskii, 2004], and optimal control of Markov control processes up to an exit time [Borkar, 1991].

The problem treated in this article falls under the second category above. In broad strokes, we consider a discrete-time Markov control process with Borel state and action spaces. We assume that there is a certain target set located inside a safe region, the latter being a subset of the state-space. The problem is to maximize the probability of attaining the target set before exiting the safe set (or equivalently, hitting the cemetery set or unsafe region). This ‘reach a good set while avoiding a bad set’ formulation arises in, e.g., air traffic control, where aircraft try to reach their destination while
avoiding collision with other aircraft or the ground despite uncertain weather conditions. It also arises in portfolio optimization, where it is desired to reach a target level of wealth without falling below a certain baseline capital with high probability. Finally, it forms the core of the computation of safe sets for hybrid systems where the 'good' and the 'bad' sets represent states from which a discrete transition into the unsafe set is possible [Gao et al., 2007; Tomlin et al., 2000]. Special cases of this problem have been investigated in, e.g., [Prandini and Hu, 2006; Watkins and Lygeros, 2003] in the context of air traffic applications, [Abate et al., 2008; Prajna et al., 2007] in the context of probabilistic safety, [Boda et al., 2004] in the context of maximizing the probability of attaining a preassigned comfort level of retirement investment funds.

It is clear from the description of our problem in the preceding paragraph that there are two random times involved, namely, the hitting times of the target and the cemetery sets. In this article we formulate our problem as the maximization of an expected total reward accumulated up to the minimum of these two hitting times. As such, this formulation falls under the broad framework of optimal control of Markov control processes up to an exit time, which has a long and rich history. It has mostly been studied as the minimization of an expected total cost until the first time that the state enters a given target set, see e.g., [Borkar, 1991, Chapter II], [Hernández-Lerma and Lasserre, 1999, Chapter 8], and the references therein. In particular, if a unit cost is incurred as long as the state is outside the target set, then the problem of minimizing the cost accumulated until the state enters the target is known variously as the pursuit problem [Eaton and Zadeh, 1962], transient programming [Whittle, 1983], the first passage problem [Derman, 1970; Kushner, 1971], the stochastic shortest path problem [Bertsekas, 2007], and control up to an exit time [Borkar, 1988, 1991; Kesten and Spitzer, 1975]. Here we exploit certain additional structures of our problem in the dynamic programming equations that we derive leading to methods fine-tuned to the particular problem at hand.

Our main results center around the assertion that there exists a deterministic stationary policy that maximizes the probability of hitting the target set before the cemetery set. This maximal probability as a function of the initial state is the optimal value function for our problem. We obtain a Bellman equation for our problem which is solved by the optimal value function. Furthermore, we provide martingale-theoretic conditions characterizing 'thrifty', 'equalizing', and optimal policies via methods derived from [Dubins and Savage, 1976; Karatzas and Sudderth, 2009]; see also [Zhu and Guo, 2006] and the references therein for martingale characterization of average optimality. The principal techniques employed in this article are similar to the ones in [Chatterjee et al., 2008], where the authors studied optimal control of a Markov control process up its first entry time to a safe set. In [Chatterjee et al., 2008] we developed a recovery strategy to enter a given target set from its exterior while minimizing a discounted cost. The problem was posed as one of minimizing the sum of a discounted cost-per-stage function $c$ up to the first entry time $\tau$ to a target set, namely, minimize $E_x^\pi \left[ \sum_{t=0}^{\tau-1} \alpha^t c(x_t, a_t) \right]$ over a class of admissible policies $\pi$, where $\alpha \in ]0, 1[$ is a discount factor. Here we extend this approach to problems with two sets, a target and a cemetery, and the case of $\alpha = 1$.

This article unfolds as follows. The main results are stated in §2. In §2.1 we define the general setting of the problem, namely, Markov control processes on Polish spaces, their transition kernels, and the admissible control strategies. In §2.2 we present our main Theorem (2.10) which guarantees the existence of a deterministic stationary policy that leads to the maximal probability of hitting the target set while avoiding the specified dangerous set, and also provides a Bellman equation that the value function must satisfy. In §2.3 we look at a martingale characterization of the optimal control
problem; thrifty and equalizing policies are defined in the context of our problem, and we establish necessary and sufficient conditions for optimality in terms of thrifty and equalizing policies in Theorem (2.17). We discuss related reward-per-stage functions and their relationships to our problem and treat several examples in §3. Proofs of the main results appear in §4. The article concludes in §5 with a discussion of future work.

§2. MAIN RESULTS

Our main results are stated in this section after some preliminary definitions and conventions.

§2.1. Preliminaries. We employ the following standard notations. Let \( \mathbb{N} \) denote the natural numbers \( \{1, 2, \ldots\} \) and \( \mathbb{N}_0 \) denote the nonnegative integers \( \{0\} \cup \mathbb{N} \). Let \( 1_A(\cdot) \) be the usual indicator function of a set \( A \), i.e., \( 1_A(\xi) = 1 \) if \( \xi \in A \) and 0 otherwise. For real numbers \( a \) and \( b \) let \( a \wedge b := \min\{a, b\} \). A function \( f : X \rightarrow \mathbb{R} \) restricted to \( A \subseteq X \) is depicted as \( f|_A \).

Given a nonempty Borel set \( X \) (i.e., a Borel subset of a Polish space), its Borel \( \sigma \)-algebra is denoted by \( \mathcal{B}(X) \). By convention, when referring to sets or functions, “measurable” means “Borel-measurable.” If \( X \) and \( Y \) are nonempty Borel spaces, a stochastic kernel \( Q \) on \( X \) given \( Y \) is a function \( Q(\cdot|\cdot) \) such that \( Q(\cdot|y) \) is a probability measure on \( X \) for each fixed \( y \in Y \), and \( Q(B|\cdot) \) is a measurable function on \( Y \) for each fixed \( B \in \mathcal{B}(X) \).

We briefly recall some standard definitions below, see, e.g., [Hernández-Lerma and Lasserre, 1996] for further details. A Markov control model is a five-tuple

\[
(X, A, \{A(x) \mid x \in X\}, Q, r)
\]

consisting of a nonempty Borel space \( X \) called the state-space, a nonempty Borel space \( A \) called the control or action set, a family \( \{A(x) \mid x \in X\} \) of nonempty measurable subsets \( A(x) \) of \( A \), where \( A(x) \) denotes the set of feasible controls or actions when the system is in state \( x \in X \) and with the property that the set \( \mathcal{K} := \{(x, a) \mid x \in X, a \in A(x)\} \) of feasible state-action pairs is a measurable subset of \( X \times A \), a stochastic kernel \( Q \) on \( X \) given \( \mathcal{K} \) called the transition law, and a measurable function \( r : \mathcal{K} \rightarrow \mathbb{R} \) called the reward-per-stage function.

(2.2). Assumption. The set \( \mathcal{K} \) of feasible state-action pairs contains the graph of a measurable function from \( X \) to \( A \).

Consider the Markov model (2.1), and for each \( i = 0, 1, \ldots \), define the space \( H_i \) of admissible histories up to time \( i \) as \( H_0 := X \) and \( H_i := \mathcal{K}^i \times X = \mathcal{K} \times H_{i-1}, i \in \mathbb{N} \). A generic element \( h_i \) of \( H_i \), which is called an admissible i-history, or simply i-history, is a vector of the form \( h_i = (x_0, a_0, \ldots, x_{i-1}, a_{i-1}, x_i) \), with \( (x_j, a_j) \in \mathcal{K} \) for \( j = 0, \ldots, i-1 \), and \( x_i \in X \). Hereafter we let the \( \sigma \)-algebra generated by the history \( h_i \) be denoted by \( \mathcal{H}_i \), \( i \in \mathbb{N}_0 \).

Recall that a policy is a sequence \( \pi = (\pi_i)_{i \in \mathbb{N}_0} \) of stochastic kernels \( \pi_i \) on the control set \( A \) given \( H_i \) satisfying the constraint \( \pi_i(A(x_i)|h_i) = 1 \ \forall h_i \in H_i, i \in \mathbb{N}_0 \). The set of all policies is denoted by \( \Pi \). Let \( (\Omega, \mathcal{F}, \mathcal{G}) \) be the measurable space consisting of the (canonical) sample space \( \Omega := \mathbb{F}_\infty = (X \times A)^\infty \) and let \( \mathcal{G} \) be the corresponding product \( \sigma \)-algebra. The elements of \( \Omega \) are sequences of the form \( \omega = (x_0, a_0, x_1, a_1, \ldots) \) with \( x_i \in X \) and \( a_i \in A \) for all \( i \in \mathbb{N}_0 \); the projections \( x_i \) and \( a_i \) from \( \Omega \) to the sets \( X \) and \( A \) are called state and control (or action) variables, respectively.
Let \( \pi = (\pi_i)_{i \in \mathbb{N}_0} \) be an arbitrary control policy, and let \( \nu \) be an arbitrary probability measure on \( X \), referred to as the initial distribution. By a theorem of Ionescu-Tulcea [Rao and Swift, 2006, Chapter 3, §4, Theorem 5], there exists a unique probability measure \( P^\pi_\nu \) on \( (\Omega, \mathcal{F}) \) supported on \( H^\infty \), such that for all \( B \in \mathcal{B}(X) \), \( C \in \mathcal{B}(A) \), \( h_i \in H_j \), \( i \in \mathbb{N}_0 \), we have \( P^\pi_\nu(x_0 \in B) = \nu(B) \).

\[
\begin{align*}
(2.3)a \quad & P^\pi_\nu(a_i \in C \mid h_i) = \pi_i(C \mid h_i) \\
(2.3)b \quad & P^\nu_\pi(x_{i+1} \in B \mid h_i, a_i) = Q(B \mid x_i, a_i).
\end{align*}
\]

(2.4). Definition. The stochastic process \((\Omega, \mathcal{F}, P^\nu_\pi, (x_i)_{i \in \mathbb{N}_0})\) is called a discrete-time Markov control process.

We note that the Markov control process in Definition (2.4) is not necessarily Markovian in the usual sense due to the dependence on the entire history \( h_i \) in \((2.3)a\); however, it is well-known [Hernández-Lerma and Lasserre, 1996, Proposition 2.3.5] that if \((\pi_i)_{i \in \mathbb{N}_0}\) is restricted to a suitable subclass of policies, then \((x_i)_{i \in \mathbb{N}_0}\) is a Markov process.

Let \( \Phi \) denote the set of stochastic kernels \( \varphi \) on \( A \) given \( X \) such that \( \varphi(A(x)|x) = 1 \) for all \( x \in X \), and let \( \mathcal{F} \) denote the set of all measurable functions \( f : X \rightarrow A \) satisfying \( f(x) \in A(x) \) for all \( x \in X \). The functions in \( \mathcal{F} \) are called measurable selectors of the set-valued mapping \( X \ni x \rightarrow A(x) \subseteq A \). Recall that a policy \( \pi = (\pi_i)_{i \in \mathbb{N}_0} \in \Pi \) is said to be randomized Markov if there exists a sequence \((\varphi_i)_{i \in \mathbb{N}_0}\) of stochastic kernels \( \varphi_i \in \Phi \) such that \( \pi_i(h_i) = \varphi_i(h_i|x) \forall h_i \in H_j \), \( i \in \mathbb{N}_0 \); deterministic Markov if there exists a sequence \((f_i)_{i \in \mathbb{N}_0}\) of functions \( f_i \in \mathcal{F} \) such that \( \pi_i(h_i) = \delta(f_i(x_i))(h_i) \); deterministic stationary if there exists a function \( f \in \mathcal{F} \) such that \( \pi_i(h_i) = \delta(f(x_i))(h_i) \). As usual let \( \Pi, \Pi_{RM}, \Pi_{DS} \), and \( \Pi_{DS} \) denote the set of all randomized history-dependent, randomized Markov, deterministic Markov, and deterministic stationary policies, respectively. The transition kernel \( Q \) in \((2.3)b\) under a policy \( \pi = (\varphi_i)_{i \in \mathbb{N}_0} \in \Pi_{RM} \) is given by \( (Q(\cdot|x, \varphi_i))_{i \in \mathbb{N}_0} \), which is defined as the transition kernel \( \mathcal{B}(X) \times X \ni (B, x) \rightarrow Q(B|x, \varphi_i(x)) := \int_{A(x)} \varphi_i(da|x)Q(B|x, a) \). Occasionally we suppress the dependence of \( \varphi_i \) on \( x \) and write \( Q(B|x, \varphi_i) \) in place of \( Q(B|x, \varphi_i(x)) \), and \( r(x, \varphi_i) := \int_{A(x)} \varphi_i(da|x)r(x, a) \).

We simply write \( f^\infty \) for a policy \((f, f, \ldots) \in \Pi_{DS} \).

§2.2. Maximizing the Probability of Hitting a Target before a Cemetery Set. Let \( O \) and \( K \) be two nonempty measurable subsets of \( X \) with \( O \subseteq K \). Let

\[
\tau := \inf\{t \in \mathbb{N}_0 \mid x_t \in O\} \quad \text{and} \quad \tau' := \inf\{t \in \mathbb{N}_0 \mid x_t \in X \setminus K\}
\]

be the first hitting times of the above sets.¹ These random times are stopping times with respect to the filtration \((\mathcal{F}_n)_{n \in \mathbb{N}_0} \). Suppose that the objective is to maximize the probability that the state hits the set \( O \) before exiting the set \( K \); in symbols the objective is to attain

\[
V^*(x) := \sup_{\pi \in \Pi} V(\pi, x) := \sup_{\pi \in \Pi} P^\pi_x(\tau < \tau', \tau < \infty),
\]

where the sup is taken over a class \( \Pi \) of admissible policies.

(2.7). Admissible policies. It is clear at once that the class of admissible policies for the problem \((2.6)\) is different from the classes considered in §2.1. Indeed, since the process is killed at the stopping time \( \tau \land \tau' \), it follows that the class of admissible policies should also be truncated at the stage \( \tau \land \tau' - 1 \). For a given stage \( t \in \mathbb{N}_0 \), we define the \( t \)-th policy element \( \pi_t \) only on the set \( \{t < \tau \land \tau'\} \). Note that with this definition \( \pi_t \) becomes a \( \mathcal{F}_t \land \tau \land \tau' \)-measurable randomized control. It is also immediate

¹As usual we set the infimum over an empty set to be \( \infty \).
from the definitions of $\tau$ and $\tau'$ that if the initial condition $x \in O \cup (X \setminus K)$, then the set of admissible policies is empty in the sense that there is nothing to do by definition. Indeed, in this case $\tau \land \tau' = 0$ and no control is needed. We are thus interested only in $x \in K \setminus O$, for otherwise the problem is trivial. In other words, the domain of $\pi_t$ is contained in the ‘spatial’ region $\{(x, a) \in \mathbb{R} | x \in K \setminus O, a \in A(x)\}$. Equivalently, in view of the definitions of the ‘temporal’ elements $\tau$ and $\tau'$, $\pi_t$ is well-defined on the set $\{t < \tau \land \tau'\}$. We re-define $\mathbb{K} := \{(x, a) \in \mathbb{R} | x \in K \setminus O, a \in A(x)\}$, and also let $\mathbb{F}$ to be the set of measurable selectors of the set-valued map $K \setminus O \ni x \mapsto A(x) \subseteq A$.

Throughout this subsection we shall denote by $\Pi_M$ the class of Markov policies such that if $(\pi_t)_{t \in \mathbb{N}_0} \in \Pi_M$, then $\pi_t$ is defined on $\mathbb{K}$ for each $t$.

(2.8). Recall that a transition kernel $Q$ on a measurable space $X$ given another measurable space $Y$ is said to be strongly Feller if the mapping $y \mapsto \int_X g(x)Q(dx|y)$ is continuous and bounded for every measurable and bounded function $g : X \to \mathbb{R}$. A function $g : \mathbb{K} \to \mathbb{R}$ is upper semicontinuous (u.s.c.) if for every sequence $(x_j, a_j)_{j \in \mathbb{N}} \subseteq \mathbb{K}$ converging to $(x, a) \in \mathbb{K}$, we have $\limsup_{j \to \infty} g(x_j, a_j) \leq g(x, a)$; or, equivalently, if for every $r \in \mathbb{R}$, the set $\{(x, a) \in \mathbb{K} | g(x, a) \geq r\}$ is closed in $\mathbb{K}$. A set-valued map $\Psi : X \to Y$ between topological spaces is upper hemi-continuous at a point $x$ if for every neighborhood $U$ of $\Psi(x)$ there exists a neighborhood $V$ of $x$ such that $z \in V$ implies that $\Psi(z) \subseteq U$; $\Psi$ is upper hemi-continuous if it is upper hemi-continuous at every $x$ in its domain. If $X$ is equipped with a $\sigma$-algebra $\Sigma$, then the set-valued map $\Psi$ is called weakly measurable if $\Psi^t(G) \in \Sigma$ for every open $G \subseteq Y$, where $\Psi^t$ is the lower inverse of $\Psi$, defined by $\Psi^t(A) := \{x \in X | \Psi(x) \cap A \neq \emptyset\}$. See, e.g., [Aliprantis and Border, 2006, Chapters 17-18] for further details on set-valued maps. Whenever $B \subseteq X$ is a nonempty measurable set and we are concerned with any set-valued map $B \ni x \mapsto A(x) \subseteq A$, we let $B$ be equipped with the trace of $\mathcal{B}(X)$ on $B$. Let $b \mathcal{B}(X)^{+}$ denote the convex cone of nonnegative, bounded, and measurable real-valued functions on $X$, and we define $\bar{b} := \{g \in L_\infty(X) | g \upharpoonright_{K \setminus O} = 0, \|g\|_{L_\infty(X)} \leq 1\}$.

(2.9). Assumption. In addition to Assumption (2.2), we stipulate that

(i) the set-valued map $K \setminus O \ni x \mapsto A(x) \subseteq A$ is compact-valued, upper hemi-continuous, and weakly measurable;

(ii) the transition kernel $Q$ on $X$ given $\mathbb{K}$ is strongly Feller, i.e., the mapping $\mathbb{K} \ni (x, a) \mapsto \int_X Q(dy|x, a)g(y)$ is continuous and bounded for all bounded and measurable functions $g : X \to \mathbb{R}$.

The following theorem gives basic existence results for the problem (2.6); a proof is presented in §4.1.

(2.10). Theorem. Suppose that Assumption (2.9) holds, and that $\tau \land \tau'$ is finite for every policy in $\Pi_M$. Then:

(i) The value function $V^*$ is a pointwise bounded and measurable solution to the Bellman equation in $\psi$:

\[
\psi(x) = 1_{X \setminus O}(x) + 1_{K \setminus O}(x) \max_{a \in A(x)} \int_X Q(dy|x, a)1_k(y)\psi(y) \quad \forall x \in X.
\]

Moreover, $V^*$ is minimal in $\bar{b} \cap b \mathcal{B}(X)^{+}$.

\[\text{2What we call “set-valued maps” are “correspondences” in [Aliprantis and Border, 2006].}\]
(ii) There exists a measurable selector \( f, \in \mathbb{F} \) such that \( f_\pi(x) \in A(x) \) attains the maximum in (2.11) for each \( x \in K \setminus O \), which satisfies

\[
V^*(x) = \begin{cases} 
1 & \text{if } x \in O, \\
\int_K Q(dy|x, f_\pi)V^*(y) & \text{if } x \in K \setminus O, \\
0 & \text{otherwise},
\end{cases}
\]

(2.12)

where \( V^* \) is as defined in (3.1). Moreover, the deterministic stationary policy \( f_\pi^\infty \) is optimal. Conversely, if \( f_\pi^\infty \) is optimal, then it satisfies (2.12).

(2.13). As a matter of notation we shall henceforth represent the functional equation (2.12) with the less formal version:

\[
V^*(x) = 1_\mathcal{O}(x) + 1_{K \setminus O}(x)\int_K Q(dy|x, f_\pi)V^*(y) \quad \forall x \in X.
\]

(2.14)

Note that the measure \( Q(\cdot|x, f_\pi) \) is not well-defined for \( x \in O \cup (X \setminus K) \) for \( f \in \mathbb{F} \) in view of the definition in paragraph (2.7). As such the integral \( \int_K Q(dy|x, f_\pi)V^*(y) \) is undefined for \( x \in O \cup (X \setminus K) \). However, to preserve the form of (2.11) and simplify notation, we shall stick to the representation (2.14) by defining any object that is written as an integral of a bounded measurable function with respect to the measure \( Q(\cdot|x, f_\pi) \) to be 0 whenever \( x \in O \cup (X \setminus K) \) and \( f \in \mathbb{F} \).

§ 2.3. A Martingale Characterization. We now return to the more general class of all possible policies (not just Markovian), denoted by \( \Pi \).

Fix an initial state \( x \in X \) and a policy \( \pi \in \Pi \). For each \( n \in \mathbb{N} \) we define the random variable \( W_n(\pi, x) := \sum_{i=0}^{n-1} a_{\pi + i}(x_i) \). Let us consider the process \( (\zeta_n)_{n \in \mathbb{N}} \) defined by

\[
\zeta_0 := V^*(x),
\]

\[
\zeta_n := W_n(\pi, x) + 1_{K \setminus O}(x(n-1)\wedge \tau')(1_{K \setminus O}(x(n-1)\wedge \tau')(x(n-1)\wedge \tau')), 
\]

(2.15)

We follow the basic framework of [Karatzas and Sudderth, 2009].

(2.16). Definition. A policy \( \pi \in \Pi \) is called thrifty at \( x \in X \) if \( V^*(x) = \Lambda^\pi(x) \), and equalizing at \( x \in X \) if \( \Lambda^\pi(x) = V^*(\pi, x) \). The action \( a_n \), defined on \( \{\tau \wedge \tau' > n\} \), is said to conserve \( V^* \) at \( x_n \) if \( 1_\mathcal{O}(x_n) + 1_{K \setminus O}(x_n)\int_K Q(dy|x_n, a_n)V^*(y) = V^*(x_n) \).

Connections between equalizing, thrifty, and optimal policies for our problem (2.6) are established by the following

(2.17). Theorem. A policy \( \pi \in \Pi \) is

- equalizing at \( x \in X \) if and only if
  \[
  \lim_{n \to \infty} \mathbb{E}_x^\pi 1_{K \setminus O}(x(n-1)\wedge \tau')(1_{K \setminus O}(x(n-1)\wedge \tau')(x(n-1)\wedge \tau')) = 0;
  \]
- optimal at \( x \in X \) if and only if \( \pi \) is both thrifty and equalizing.

A connection between thrifty policies, the process \( (\zeta_n)_{n \in \mathbb{N}} \) defined in (2.15), and actions conserving the optimal value function \( V^* \) is established by the following

(2.18). Theorem. For a given policy \( \pi \in \Pi \) and an initial state \( x \in X \) the following are equivalent:

(i) \( \pi \) is thrifty at \( x \);
(ii) \((\zeta_n)_{n \in \mathbb{N}}\) is a \(\left(\mathfrak{F}_n\right)_{n \in \mathbb{N}}\)-martingale under \(P^\pi_x\);

(iii) \(P^\pi_x\)-almost everywhere on \(\{\tau \wedge \tau' > n\}\) the action \(a_n\) conserves \(V^\pi\).

It is possible to make a connection, relying purely on martingale-theoretic arguments, between the process \((\zeta_n)_{n \in \mathbb{N}}\) and the value function corresponding to an optimal policy. This is the content of the following theorem, which may be viewed as a partial converse to Theorem (2.18).

(2.19) Theorem. Suppose that either one of the stopping times \(\tau\) and \(\tau'\) defined in (2.5) is finite for every policy in \(\Pi\). Let \(V'\) be a nonnegative measurable function such that \(V'|_O = 1\), \(V'|_{X \setminus K} = 0\), and bounded above by 1 elsewhere. For a policy \(\pi \in \Pi\) define the process \((\zeta_n')_{n \in \mathbb{N}}\) as

\[
\begin{align*}
\zeta'_0 & := V'(x), \\
\zeta'_n & := W_n(\pi, x) + 1_{K \setminus O}(x_{(n-1)\wedge \tau})(1_K \cdot V')(x_{n \wedge \tau' \wedge \tau}), \quad n \in \mathbb{N},
\end{align*}
\]

where \(W_n(\pi, x)\) is as in (2.15). If for some policy \(\pi^* \in \Pi\) the process \((\zeta_n')_{n \in \mathbb{N}}\) is a \(\left(\mathfrak{F}_n\right)_{n \in \mathbb{N}}\)-martingale under \(P^\pi_x\), then \(V'(x) = P^\pi_x(\tau < \tau', \tau < \infty)\).

Proofs of the above results are presented in §4.2.

§ 3. Discussion and Examples

Let us look at the stopped process \((x_{1\wedge(t-1)\wedge \tau})_{t \in \mathbb{N}}\). It is clear that in this case \(V_n(\pi, x) = 1\) whenever \(x \in O\) and \(V_n(\pi, x) = 0\) whenever \(x \in X \setminus K\) for all policies in \(\Pi_M\); otherwise for \(x \in K \setminus O\) we have

\[
V_n(\pi, x) := P^\pi_x(\tau < \tau', \tau < n) = P^\pi_x(x_{1\wedge \tau' \wedge \tau} \in O) + P^\pi_x(x_{1\wedge \tau' \wedge \tau} \in K \setminus O, x_{2\wedge \tau' \wedge \tau} \in O) + \ldots + P^\pi_x(x_{1\wedge \tau' \wedge \tau}, \ldots, x_{(n-2)\wedge \tau' \wedge \tau} \in K \setminus O, x_{(n-1)\wedge \tau' \wedge \tau} \in O).
\]

Since the \(k\)-th term on the right-hand side is \(E^\pi_x\left[\prod_{i=1}^{k-1} 1_{K \setminus O}(x_{i\wedge \tau' \wedge \tau})1_{O}(x_{k\wedge \tau' \wedge \tau})\right]\), it follows that

\[
V_n(\pi, x) = E^\pi_x\left[\sum_{i=1}^{(n-1)\wedge \tau' \wedge \tau} \left(\prod_{i=0}^{t-1} 1_{K \setminus O}(x_i)\right)1_{O}(x_i)\right] = E^\pi_x\left[\sum_{i=1}^{(n-1)\wedge \tau' \wedge \tau} 1_{O}(x_i)\right] = E^\pi_x\left[\sum_{i=1}^{(n-1)\wedge \tau' \wedge \tau} 1_{O}(x_i)\right].
\]

We note that \(V_n(\pi, x) = 0\) whenever \(x \in O \cup (X \setminus K)\). A policy that maximizes \(V_n(\pi, x)\) is defined only on the set \(K \setminus O\), and it is left undefined elsewhere. Once the process exits \(K \setminus O\) or the stage reaches \(n - 1\), the task of our control policy is over. Such a deterministic stationary policy (which exists, as demonstrated below) with a measurable selector \(f \in F\) should be represented as \(f^{\tau \wedge \tau'} := (f, f, \ldots, f)\) since it is applied only for the first \(\tau \wedge \tau'\) stages; however, for notational brevity we simply write \(f^\infty\) hereafter.

Quite clearly, letting \(n \to \infty\), the monotone convergence theorem gives

\[
V(\pi, x) = \lim_{n \to \infty} V_n(\pi, x) = P^\pi_x(\tau < \tau', \tau < \infty) = E^\pi_x\left[\sum_{t=1}^{\tau \wedge \tau'} 1_{O}(x_t)\right] = E^\pi_x\left[\sum_{t=1}^{\tau \wedge \tau'} 1_{O}(x_{t\wedge \tau' \wedge \tau})\right].
\]
We note that by definition, the random sum inside the expectation on the right-hand side of the last equality above is the limit of partial (finite) sums, and this ensures that the term inside the expectation is defined on the event \( \{ \tau \land \tau' < \infty \} \). By definition note that

\[
 (3.1) \quad V^\star(x) = \sup_{\pi \in \Pi_n} V(\pi, x) = \sup_{\pi \in \Pi_n} E_n^\pi \left[ \sum_{t=1}^{\tau \lor \tau'} 1_{O}(x_t) \right].
\]

Consider again the value-iteration functions defined by

\[
 (3.2) \quad \begin{cases} 
 v_0(x) := 1_{O}(x) \\ v_n(x) := 1_{O}(x) + 1_{K \smallsetminus O}(x)\max_{a \in X(x)} \int_X Q(dy|x,a)1_k(y) v_{n-1}(y) 
 \end{cases}
\]

for \( x \in X \) and \( n \in \mathbb{N} \). The function \( v_n \) is clearly identifiable with the optimal value function for the problem of maximizing \( P^{\tau_n} (\tau < \tau', \tau < n) \) of the process stopped at the \((n-1)\)-th stage, \( n \in \mathbb{N} \). To get a intuitive idea, fix a deterministic Markov policy \( \pi' = (f_t)_{t \in \mathbb{N}} \) and take the first iterate \( v_1 \). (Of course the assumption underlying the notation \( (f_t)_{t \in \mathbb{N}} \) is that \( f_t \) is defined on \( \{ t < \tau \land \tau' \} \).) It is immediately clear that the reward at the first step is 1 if and only if \( x \in X \) and 0 otherwise, and that is precisely \( v_0 \) irrespective of the policy. For the second iterate note the reward under the policy \( \pi' \) is \( v_1(x) = 1_{O}(x) + 1_{K \smallsetminus O}(x)Q(O,x,f_0(x)) \). This is because the reward is 1 if \( x \in O \) and the process terminates at the first stage, and \( x \in K \smallsetminus O \) and the reward at the second stage is the probability of hitting \( O \) at the second stage. Of course there is no reward if \( x \in X \setminus K \). Similarly, for the third iterate the reward is \( v_2(x) = 1_{O}(x) + 1_{K \smallsetminus O}(x)\int_X Q(dx|x,f_0(x))\{ 1_{O}(x) + 1_{K \smallsetminus O}(x)Q(O,x,f_0(x)) \} \). Note that only those sample paths that stay in \( K \smallsetminus O \) at the first step contribute to the reward at the second stage, only those sample paths that stay in \( K \smallsetminus O \) for the first and the second stages contribute to the reward at the third stage, and so on.

\S 3.1. A general setting and various special cases. Our problem (2.6) can be viewed as a special case of a more general setting. To wit, consider a nonnegative upper semicontinuous reward-per-stage function \( r : X \rightarrow \mathbb{R}_{\geq 0} \) and the problem of maximizing the total reward up to (and including) the hitting time \( \tau \land \tau' \), i.e., maximize \( E_{\xi} [ \sum_{i=0}^{\tau \lor \tau'} r(x_i,a_i) ] \) over a class of policies. This corresponds to maximization of the reward until exit from the set \( K \smallsetminus O \). The value-iteration functions \( v_n(x) \) corresponding to this problem can be written down readily: for \( x \in X \) and \( n \in \mathbb{N} \) let

\[
 v'_0(x) := \sup_{a \in X(x)} r(x,a)1_{O \cup (X \setminus K)}(x),
\]

\[
 v'_n(x) := \sup_{a \in X(x)} \left[ r(x,a)1_{O \cup (X \setminus K)}(x) + 1_{K \smallsetminus O}(x) \int_X Q(dy|x,a)v'_{n-1}(y) \right].
\]

Our problem (2.6) corresponds to the case of \( r(x,a) = 1_{O}(x) \). Modulo the additional technical complications involving integrability of the value-iteration functions at each stage and the total reward corresponding to initial conditions being well-defined real numbers, the analysis of this more general problem can be carried out in exactly the same way as we do below for the problem (2.6). While the above more general problem treats both the target set \( O \) and the cemetery state \( X \setminus K \) equally, the bias towards the target set \( O \) is provided in our problem (2.6) by the special structure of the reward \( r(x,a) = 1_{O}(x) \).

From the general framework it is not difficult to arrive at reward-per-stage functions that are meaningful in the context of reachability, avoidance, and safety. For the sake
of simplicity, till the end of this subsubsection we suppose that for all initial conditions and admissible policies \( \pi \in \Pi \) the stopping times \( \tau \) and \( \tau' \) are finite \( P_\pi \)-almost surely. With this assumption in place, let us look at some examples:

- Consider a discounted version of our problem (2.6), namely, let

\[
V^{(1)}(\pi, x) := E_\pi^x \left[ \sum_{t=0}^{\tau'} \alpha^t 1_0(x_t) \right],
\]

where \( \alpha \in [0, 1[ \) is a constant discount factor. From the definitions of \( \tau \) and \( \tau' \) it follows that \( \sum_{t=0}^{\tau_\pi} \alpha^t 1_0(x_t) = \alpha^\tau 1_{[\tau < \tau']} \), and in view of the range of \( \alpha \) it follows that maximization of \( V^{(1)} \) over admissible policies leads to small values of \( \tau \) on the set \( \{ \tau < \tau' \} \) on an average, but it is silent about the values of \( \tau \) on \( \{ \tau > \tau' \} \).

To get a more quantitative idea of the role that the discount factor \( \alpha \) plays, let \( \bar{\tau} \) be a random variable independent of the Markov control process defined in Definition (2.4), \( \bar{\tau} \) with distribution function \( P(\bar{\tau} = n) = (1 - \alpha)\alpha^n \) for all \( n \in \mathbb{N}_0 \). In a standard way we construct the product probability measure \( P^\infty \otimes P \) and denote the expectation with respect to this measure as \( E_x^{\pi, \bar{\tau}}[\cdot] \). We can write

\[
V^{(1)}(\pi, x) = E_x^{\pi, \bar{\tau}} \left[ \sum_{t=0}^{\bar{\tau} \wedge \tau'} \alpha^t 1_0(x_t) \right] = (1 - \alpha)^{-1} E_x^{\pi, \bar{\tau}} \left[ 1_0(x_\bar{\tau}) 1_{[\bar{\tau} < \tau']} \right].
\]

In view of the definitions of \( \tau \) and \( \tau' \) we get

\[
V^{(1)}(\pi, x) = (1 - \alpha)^{-1} E_x^{\pi, \bar{\tau}} \left[ 1_{\{\bar{\tau} = \tau, \tau < \tau'\}} \right].
\]

This alternative characterization shows that maximization of \( V^{(1)} \) over admissible policies leads to smaller values of \( \bar{\tau} \) compared to \( \tau' \); moreover, the random variable \( \bar{\tau} \) gives a quantitative idea of how the choice of \( \alpha \) determines the outcome since \( \bar{\tau} \) is a geometric random variable with parameter \( (1 - \alpha) \). Choosing a small \( \alpha \) implies smaller \( \bar{\tau} \) with higher probability and may appear to be profitable; however, in certain problems it is possible that the set \( O \) may be reachable at \( \bar{\tau} \) with small probability and the corresponding event of interest \( \{ \bar{\tau} = \tau, \tau < \tau' \} \) may be relatively small for a given initial condition \( x \). Moreover, the factor \( (1 - \alpha)^{-1} \) is small for small values of \( \alpha \), and contributes to this phenomenon.

A second quantitative view of the role of \( \alpha \) is offered by the fact that \( V^{(1)}(\pi, x) = E_x^{\pi, \bar{\tau}} \left[ \sum_{t=0}^{\tau_\pi \wedge \tau'} 1_0(x_t) \right] \). Indeed, we have

\[
E_x^{\pi, \bar{\tau}} \left[ \sum_{t=0}^{\tau_\pi \wedge \tau'} 1_0(x_t) \right] = E_x^{\pi, \bar{\tau}} \left[ \sum_{t=0}^{\bar{\tau}} 1_0(x_t) \right] = E_x^{\pi, \bar{\tau}} \left[ \sum_{t=0}^{\infty} \alpha^t 1_0(x_t) \right] = \sum_{t=0}^{\infty} \alpha^t 1_0(x_t).
\]

\( \bar{\tau} \) is defined in a standard way by enlarging the probability space.
In this setting we do not have the $(1 - \alpha)^{-1}$ factor outside the expectation as in the second version of $V^{(1)}$ above, and it demonstrates that maximizing $V^{(1)}(\pi, x)$ over admissible policies leads to maximizing the probability of the event $\{\tau < \tilde{\tau} \wedge \tau'\}$, where $\alpha$ controls the values of $\tilde{\tau}$ as before.

- Consider the reward-per-stage function $r(x, a) = 1_{O}(x) - 1_{X\setminus O}(x)$. Under integrability assumption on $\tau \wedge \tau'$ under all admissible policies, we have
  \[
  V^{(2)}(\pi, x) := E_{x}^{\tau} \left[ \sum_{t=0}^{\tau \wedge \tau'} (1_{O}(x_{t}) - 1_{X\setminus O}(x_{t})) \right]
  = E_{x}^{\tau} \left[ \sum_{t=0}^{\tau \wedge \tau'} (1_{O}(x_{t}) - 1_{K\setminus O}(x_{t}) - 1_{X\setminus K}(x_{t})) \right]
  = P_{x}^{\tau}(\tau < \tau') - P_{x}^{\tau}(\tau' < \tau) - E_{x}^{\tau}[\tau \wedge \tau'].
  \]
  Clearly, maximization of $V^{(2)}$ over admissible policies leads to both the maximal enlargement of the set $\{\tau < \tau'\}$ and minimization of the hitting time $\tau$ on this set.

- Consider $r(x, a) = 1_{O}(x) - 1_{X\setminus K}(x)$. This leads to the expected total reward until escape from $K \setminus O$ as
  \[
  V^{(3)}(\pi, x) := E_{x}^{\tau} \left[ \sum_{t=0}^{\tau \wedge \tau'} (1_{O}(x_{t}) - 1_{X\setminus K}(x_{t})) \right] = P_{x}^{\tau}(\tau < \tau') - P_{x}^{\tau}(\tau' < \tau).
  \]
  Since $P_{x}^{\tau}(\tau < \tau') + P_{x}^{\tau}(\tau' < \tau) = 1$, maximization of $V^{(3)}$ over admissible policies maximizes the probability of the event $\{\tau < \tau'\}$. Thus, maximizing $V^{(3)}(\pi, x)$ over $\pi \in \Pi$ is a different formulation of the objective of our problem (2.6). The above analysis also shows that the same objective results if we take the reward-per-stage function to be $1_{O}(x) - \gamma 1_{X\setminus K}(x)$ for any $\gamma \geq 0$.

- Suppose that $\tau \wedge \tau'$ is integrable for all admissible policies and consider the reward-per-stage $r(x, a) = 1_{K\setminus O}(x)$. Let
  \[
  V^{(4)}(\pi, x) := E_{x}^{\tau} \left[ \sum_{t=0}^{\tau \wedge \tau'} 1_{K\setminus O}(x_{t}) \right].
  \]
  Maximization of $V^{(4)}$ over admissible policies leads to large values of $\tau \wedge \tau'$ on an average. This is a form of safety problem, the state stays inside $K \setminus O$ for as long as possible on an average.

- Suppose that $\tau \wedge \tau'$ is integrable for all admissible policies and consider $r(x, a) = \gamma 1_{O}(x) - 1_{K\setminus O}(x)$ for $\gamma \geq 1$. Consider
  \[
  V^{(5)}(\pi, x) := E_{x}^{\tau} \left[ \sum_{t=0}^{\tau \wedge \tau'} (\gamma 1_{O}(x_{t}) - 1_{K\setminus O}(x_{t})) \right],
  \]
  we see that $V^{(5)}(\pi, x) = \gamma P_{x}^{\tau}(\tau < \tau') - E_{x}^{\tau}[\tau \wedge \tau']$. We see that maximization of $V^{(5)}$ over admissible policies leads to a balance between maximizing the probability that the state hits the set $O$ before getting out of $K$ and exiting $K$ quickly. This is because it is more profitable to exit from $K$ and get a zero reward than incur negative reward by prolonging the duration of stay in $K \setminus O$. The factor $\gamma$ decides the priorities of the two alternatives. It is trivially clear that $\gamma = 1$ leads to rapid exit from $K$ if the initial condition is in $K \setminus O$.

Not all the reward-per-stage functions mentioned above can be handled in our present framework. In particular, we make the crucial assumption that the reward-per-stage function is nonnegative, which does not hold in some of the cases above. However, under appropriate growth-rate conditions on the reward-per-stage function, the nonnegativity assumption can be dispensed with.
In classical finite or infinite-horizon optimal control problems a translation of the (fixed) reward-per-stage function would not change the solution to the problem. However, translations of the reward-per-stage function in random-horizon problems may lead to drastically different policies. We give two examples:

- Consider the reward-per-stage functions $r'(x,a) = 1_{O}(x) - 1_{X \prec K}(x)$ and $r''(x,a) = 2 \cdot 1_{O}(x) + 1_{K \prec O}(x)$; in this case we translate $r'$ on $X$ by 1, i.e., $r'' = r' + 1$. On the one hand, maximizing $E_{x}^{π} \left[ \sum_{t=0}^{\tau} r'(x_{t},a_{t}) \right]$ yields a policy that $P^{π}(\tau < \tau')$ as we have seen before (this is $V^{(3)}$ above). On the other hand, maximizing $E_{x}^{π} \left[ \sum_{t=0}^{\tau \land \tau'} r''(x_{t},a_{t}) \right]$ yields a policy that tries to keep the state in $K \setminus O$ for as long as possible, and at each stage accrue a reward of 1, which is certainly better than jumping to $O$ and accruing a reward of 2 at most.

- Consider $r'(x,a) = 1_{O}(x) - 1_{X \prec K}(x)$ and $r''(x,a) = -1_{O}(x) - 3 \cdot 1_{X \prec K}(x)$; in this case we translate $r'$ by −2 only on its support $O \cup (X \setminus K)$. We have noted above that maximizing $E_{x}^{π} \left[ \sum_{t=0}^{\tau \land \tau'} r'(x_{t},a_{t}) \right]$ yields a policy that maximizes $P^{π}(\tau < \tau')$. However, maximizing $E_{x}^{π} \left[ \sum_{t=0}^{\tau \land \tau'} r''(x_{t},a_{t}) \right]$ yields a policy that tries to keep the state in $K \setminus O$ for the longest possible duration to avoid incurring negative reward.

§ 3.2. Further examples. For one-dimensional stochastic processes initialized somewhere between two different levels $a$ and $b$, problems such as calculating the probability of hitting the level $a$ before the level $b$ are fairly common, e.g., in random walks, Brownian motion, and diffusions, see, e.g., [Levin et al., 2009, Chapters 2-3], [Revuz and Yor, 1999]. It is possible to obtain explicit expressions of these probabilities in a handful of cases.

Let us consider a controlled Markov chain $(x_{t})_{t \in \mathbb{N}_{0}}$ with a finite state-space $X = \{1, 2, \ldots, m\}$ and a transition probability matrix $Q = [q_{ij}(a)]_{m \times m}$, where $a$ is the action or control variable. Let $O \subsetneq X$, $K \subsetneq X$ be subsets of $X$ with $O \subsetneq K$. Since $X$ is finite, Assumption (2.9) is satisfied. Consider the problem (2.6) in the context of this Markov chain $(x_{t})_{t \in \mathbb{N}_{0}}$ initialized at some $i_{0} \in K \setminus O$. By Theorem (2.10) the optimal value function $V^{*}$ must satisfy the equation

$$V^{*}(i) = 1_{O}(i) + 1_{K \setminus O}(i) \max_{a \in \mathcal{A}(i)} \sum_{j \in K} q_{ij}(a)V^{*}(j)$$

$$= 1_{O}(i) + 1_{K \setminus O}(i) \max_{a \in \mathcal{A}(i)} \sum_{j \in O} q_{ij}(a) + \sum_{j \in K \setminus O} q_{ij}(a)V^{*}(j)$$

for all $i \in X$. If the control actions are finite in number, searching for a maximizer over an enumerated list all control actions corresponding to each of the states may be possible if the state and action spaces are not too large. However, the memory requirement for storing such enumerated lists clearly increases exponentially with the dimension of the state and action spaces if the Markov chain is extracted by a discretization procedure based on a grid on the state-space of a discrete-time Markov process evolving, for example, on a subset of Euclidean space. As an alternative, it is possible to search for a maximizer from a parametrized family of functions (vectors) by applying well-known suboptimal control strategies [Bertsekas, 2007, Chapter 6], [Bertsekas and Tsitsiklis, 1996; Powell, 2007]. Note that in the case of an uncontrolled Markov chain the equation above reduces to $V^{*}(i) = 1_{O}(i) + 1_{K \setminus O}(i) \left( \sum_{j \in O} q_{ij} + \sum_{j \in K \setminus O} q_{ij}V^{*}(j) \right)$, and can be solved as a linear equation on $K \setminus O$ for the vector $V^{*}|_{K \setminus O}$. Thus, solving for $V^{*}$ yields a method of calculating the probability of hitting $O$ before hitting $X \setminus K$ in uncontrolled Markov chains, and can serve as a verification tool [Kwiatkowska et al., 2007].
In certain cases of uncountable state-space Markov chains the policies and value functions corresponding to maximization of \(P^x_\tau(\tau < \tau', \tau < n)\) can be explicitly calculated for small values of \(n\). As an illustration, consider a scalar linear controlled system

\[
x_{t+1} = x_t + a_t + w_t, \quad x_0 = x, \quad t \in \mathbb{N}_0.
\]

Here \(x_t \in \mathbb{R}\) is the state of the system at time \(t\), \(a_t\) is the action or control at time \(t\) taking values in \([-1, 1]\), and \((w_t)_{t \in \mathbb{N}_0}\) is a sequence of independent and identically distributed (i.i.d) standard normal random variables treated as noise inputs to the system. Let us suppose that our target set is \(O = [-1, 1]\), safe set is \(K = [-3, 3]\), and let us find a greedy policy for our problem, i.e., a policy that maximizes \(P^x_\tau(\tau < \tau', \tau < 2)\).

The greedy policy tries to maximize \(P_x(x_{\tau} \in [-1, 1]) = P_x(x + a + w \in [-1, 1]) = \mathcal{N}(1 - x - a) - \mathcal{N}(-1 - x - a) =: G(x, a)\), where \(\mathcal{N}\) is the cumulative distribution function of the standard normal random variable. The function \(G\) can be expressed in terms of the complementary error function \(^4\) as \(G(x, a) = \frac{1}{2} \left( \text{erfc}(\frac{1}{\sqrt{2}}(1 - x - a)) - \text{erfc}(\frac{1}{\sqrt{2}}(-1 - x - a)) \right)\), and \(\arg \max_{a \in [-1, 1]} G(x, a)\) can be solved in closed form. Indeed, \(\frac{\partial G}{\partial a}(x, a) = \frac{1}{\sqrt{\pi}} \left( e^{-\frac{1}{2}(x+a+1)^2} - e^{-\frac{1}{2}(x+a-1)^2} \right) = 0\) gives \(a^* = f_\tau(x) = -x\) as the unconstrained optimizer. Since \(a \in [-1, 1]\), we have the constrained maximizer as \(f_\tau(x) = -\text{sat}(x)\), where sat\((\cdot)\) is the standard saturation function. \(^5\) In other words, we get a bang-bang controller since \(x - \text{sat}(x) \neq 0\) on the interior of \(K \setminus O\). It is easy to discern the maximizer from the accompanying figure. The corresponding maximal probability is found by substituting the above optimizer back into the dynamic programming equation, and this yields \(V^*_\tau(x) = 1_{[O]}(x) + \frac{1}{2} 1_{K \setminus O}(x) \left( \text{erf}(\frac{1}{\sqrt{2}}(x - \text{sat}(x) + 1)) - \text{erf}(\frac{1}{\sqrt{2}}(x - \text{sat}(x) - 1)) \right)\). For \(n = 3\) it turns out that we can no longer compute the optimizer corresponding to the first stage in closed form; the optimizer for the second stage is, of course, \(f_\tau(x) = -\text{sat}(x)\) calculated above. It is also evident from the accompanying figure that even in this simple example there will arise nontrivial issues with nonconvexity for \(n \geq 3\).

### §3.3. Uniqueness of optimal policies.

So far in our discussion we have not addressed the issue of uniqueness of the optimal policy in our problem (2.6). (Theorem (2.10) shows that an optimal policy exists, so the uniqueness question is meaningful.) It becomes clear from considerations of the geometry of the sets \(O\) and \(K\) in simple examples that the optimal controller \(f_\tau\) in Theorem (2.10)(ii) is nonunique in general. For instance, consider the linear system considered in (3.3) above with initial condition \(x_0 = 0\), and let \(O = [-2, -1]\) and \(K = [-3, 3]\). Since the noise is symmetric about the origin, from symmetry considerations it immediately follows that the optimal controller \(f_\tau\) is nonunique at the origin. Note that \(f_\tau\) is, of course, defined on \(K \setminus O\).

\(^4\)Recall that the complementary error function is defined as \(\text{erfc}(r) := \frac{2}{\sqrt{\pi}} \int_{r}^{\infty} e^{-t^2} dt = 1 - \text{erf}(r)\), where \(\text{erf}(\cdot)\) is the standard error function.

\(^5\)Recall that the standard saturation function is defined as \(\text{sat}(r) = r\) if \(|r| < 1\), 1 if \(r \geq 1\) and \(-1\) otherwise.
§ 3.4. Relation to a probabilistic safety problem. Let us digress a little and consider the following probabilistic safety problem: maximize the probability that the state remains inside a safe set $C \subseteq X$ for $n$ stages, beginning from an initial condition $x \in C$. This, as mentioned earlier, is the probabilistic safety problem addressed in [Abate et al., 2008]. Of course the probability of staying inside $C$ for the first $n$ stages is given by

$$P_x^n(\bigcap_{t=0}^{n-1} \{x_t \in C\}) = E_x^n[\prod_{t=0}^{n-1} 1_{\{x_t \in C\}}].$$

If $\sigma$ is the first exit time from $C$, then $P_x^n(\bigcap_{t=0}^{n-1} \{x_t \in C\}) = E_x^n[\prod_{t=0}^{\sigma-1\wedge(n-1)} 1_{\{x_t \in C\}}]$. Therefore, in this particular problem there is no difference between the maximal values of $E_x^n[\prod_{t=0}^{n-1} 1_{\{x_t \in C\}}]$ or $E_x^n[\prod_{t=0}^{\sigma-1\wedge(n-1)} 1_{\{x_t \in C\}}]$. However, the policies arising from the two different maximizations are quite unlike each other. Indeed, whereas the former yields a deterministic Markov policy [Abate et al., 2008] whose every element is defined on all of $X$, the stopping time version yields a deterministic Markov policy whose $t$-th element $\pi_t$ is defined on the set $\{t < \sigma \land n\}$, just as discussed in paragraph (2.7). On the one hand note that the reward in the former case is not affected by further application of the control actions once the state has exited the safe set $C$; the policy resulting from this formulation, however, dictates that the control actions are carried out until (and including) the $(n-2)$-th stage nonetheless. On the other hand, the reward in the latter stopping time version saturates at the stage the state leaves $C$ and future control actions are not defined.

It is interesting to note that the Bellman equation developed for probabilistic safety and reachability in [Abate et al., 2008] may be obtained as a special case of (2.11) in Theorem (2.10) above. This comes as no surprise. The problem of maximizing the probability of staying inside a (measurable) safe set $C \subseteq X$ for $N$ steps is given by the maximization of $E_x^N[\prod_{t=0}^{\sigma(N-1)} \xi_t(x_t)]$, where $\sigma$ is the first time to exit $C$ and this clearly translates to minimizing $P_x^N(\tau < N)$. In our setting, if we let $K$ be the entire state-space $X$, $C = X \setminus O$, and $\tau$ the first time to hit the set $O$, then our problem (2.6) is precisely that in [Abate et al., 2008] with the exception of maximization in place of minimization. It must be mentioned however, that the analysis carried out in [Abate et al., 2008] relies on the approach in [Bertsekas and Shreve, 1978] and is purely analytical; the strong Feller assumption on the transition kernel in our formulation plays no role there.

§ 4. Proofs

This section collects the proofs of the various results in §2.

§ 4.1. Proof of Theorem (2.10). We recall a few standard results about set-valued maps first, followed by sequence of lemmas before getting to the proof of Theorem (2.10). The various definitions in paragraphs (2.7), (2.8), and (2.13) will be employed without further reference. Just as in §2.2, for the purposes of this subsection, we let $\Pi_M$ denote the set of admissible Markov policies such that $\pi_t$ is defined on $K$ whenever $(\pi_t)_{t \in \mathbb{N}} \in \Pi_M$.

(4.1) Proposition ([Aliprantis and Border, 2006, Lemma 17.30]). Let $\Psi : X \rightarrow Y$ be an upper hemicontinuous set-valued map between topological spaces with nonempty compact values, and let $f : \text{Graph}(\Psi) \rightarrow \mathbb{R}$ be upper semicontinuous. Define the function $m : X \rightarrow \mathbb{R}$ by $m(x) := \max_{y \in \Psi(x)} f(x, y)$. Then the function $m$ is upper semicontinuous.

\footnote{Recall that $\text{Graph}(\Psi)$ is the set $\{(x, \Psi(x)) \mid x \in X\} \subseteq X \times Y$, the graph of the set-valued map $\Psi$.}
(4.2). Proposition ([Aliprantis and Border, 2006, Theorem 18.19]). Let \( X \) be a separable metrizable space and \((S, \Sigma)\) a measurable space. Let \( \Psi : S \rightarrow X \) be a weakly measurable correspondence with nonempty compact values, and suppose \( f : S \times X \rightarrow \mathbb{R} \) is a Carathéodory function. Let us also define the function \( m : S \rightarrow \mathbb{R} \) by \( m(s) := \max_{x \in \Psi(s)} f(s, x) \), and the correspondence \( \mu : S \rightarrow X \) of maximizers by \( \mu(s) := \{ x \in \Psi(s) \mid f(s, x) = m(s) \} \). Then the argmax correspondence \( \mu \) is measurable and admits a measurable selector.

(4.3). Definition. For \( u \in b\mathcal{B}(X)^+ \cap \bar{B} \) we define the mapping \( Tu \)

\[
X \ni x \rightarrow Tu(x) := 1_A(x) + 1_{K \setminus O}(x) \sup_{a \in A(x)} \int_K Q(dy|x, a)u(y) \in \mathbb{R}_{\geq 0}.
\]

The operator \( T \) is called the dynamic programming operator corresponding to the problem (2.6).

(4.5). Lemma. Suppose that Assumption (2.9) holds. Then the dynamic programming operator \( T \) defined in (4.4) takes \( b\mathcal{B}(X)^+ \cap \bar{B} \) into itself. Moreover, there exists a measurable selector \( f \in \mathbb{F} \) such that

\[
Tu(x) = 1_A(x) + 1_{K \setminus O}(x) \int_K Q(dy|x, f(y))u(y) \quad \forall \ x \in X.
\]

Proof. Fix \( u \in b\mathcal{B}(X)^+ \cap \bar{B} \). Since the transition kernel \( Q \) is strongly Feller on \( K \), the mapping

\[
K \ni (x, a) \rightarrow S(x, a) := \int_X Q(dy|x, a)1_K(y)u(y) \in \mathbb{R}_{\geq 0}
\]

is continuous on \( K \). Also, \( S(x, a) \) is bounded whenever \( u \) is, a bound of \( S \) being the essential supremum norm of \( u \). Therefore, since \( A(x) \) is compact for each \( x \in X \), the function \( S^*(x) := \sup_{a \in A(x)} S(x, a) \) is well-defined on \( K \setminus O \), i.e., the sup is attained on \( A(x) \) for \( x \in K \setminus O \). We also note that since \( K \setminus O \) is a measurable set, by Assumption (2.9)

\[
\circ \quad \text{the correspondence } K \setminus O \ni x \rightarrow A(x) \subseteq A \text{ is upper hemicontinuous, and since } S \text{ is continuous on } K, \text{ the map } K \setminus O \ni x \rightarrow S^*(x) := \max_{a \in A(x)} S(x, a) \in \mathbb{R}_{\geq 0} \text{ is an u.s.c. function by Proposition (4.1);}
\]

\[
\circ \quad \text{the correspondence } K \setminus O \ni x \rightarrow A(x) \subseteq A \text{ is weakly measurable, and since } S \text{ is continuous on } K \text{ (and therefore is a Carathéodory function), there exists a measurable selector } f \in \mathbb{F} \text{ such that } S^*(x) = S(x, f(x)) \text{ for all } x \in K \setminus O \text{ by Proposition (4.2).}
\]

It follows at once that \( X \ni x \rightarrow Tu(x) = 1_A(x) + 1_{K \setminus O}(x) \int_K Q(dy|x, f(x))u(y) \in \mathbb{R}_{\geq 0} \) is a member of the set \( b\mathcal{B}(X)^+ \), and the assertion follows.

(4.7). Lemma. Suppose that hypotheses of Theorem (2.10) hold. If \( u \in b\mathcal{B}(X)^+ \cap \bar{B} \) satisfies the inequality \( u \leq Tu \) pointwise on \( X \), then also \( u \leq V^+ \) pointwise on \( X \), where \( T \) is the dynamic programming operator in (4.4).

Proof. By definition of \( T \) it is clear that we only need to examine the validity of the assertion on \( K \setminus O \). Suppose that \( u \in b\mathcal{B}(X)^+ \cap \bar{B} \) satisfies the inequality \( u \leq Tu \)

\[
\footnote{Recall that a Carathéodory function \( f : S \times X \rightarrow Y \) is a mapping that is measurable in the first variable and continuous in the second, where \((S, \Sigma)\) is a measurable space and \(X, Y\) are topological spaces. In particular, if \( X \) is a separable and metrizable space, and \( Y \) is a metrizable space, every Carathéodory function \( f : S \times X \rightarrow Y \) is jointly measurable [Aliprantis and Border, 2006, Lemma 4.51]; this is clearly true in the Carathéodory functions we consider.}
\]
pointwise on $X$. By Lemma (4.5) we know that there exists $f \in \mathcal{F}$ satisfying

$$Tu(x) = 1_0(x) + 1_{K \setminus O}(x) \int_{K} Q(dy|x,f)u(y) \quad \forall x \in K \setminus O.$$  

A straightforward calculation shows that if $u \leq Tu$ then $Tu \leq T \circ Tu$ on $K \setminus O$. Fix $x \in K \setminus O$. Applying the inequality $u \leq Tu$ repeatedly we have

$$u(x) \leq 1_0(x) + 1_{K \setminus O}(x) \int_{K} Q(d\xi_1|x,f)u(\xi_1)$$

$$\leq 1_0(x) + 1_{K \setminus O}(x) \int_{K} Q(d\xi_1|x,f)\left[1_0(\xi_1) + 1_{K \setminus O}(\xi_1) \int_{K} Q(d\xi_2|\xi_1,f)u(\xi_2)\right]$$

$$\ldots$$

and after $n$ steps

$$u(x) \leq 1_0(x) + 1_{K \setminus O}(x) \int_{K} Q(d\xi_1|x,f)\left[1_0(\xi_1) + \ldots \right.$$  

$$\left.\ldots + 1_{K \setminus O}(\xi_{n-2}) \int_{K} Q(d\xi_{n-1}|\xi_{n-2},f)\left[1_0(\xi_{n-1}) + 1_{K \setminus O}(\xi_{n-1}) \int_{K} Q(d\xi_n|\xi_{n-1},f)u(\xi_n)\right]\ldots\right]$$

$$= \left(1_0(x) + 1_{K \setminus O}(x) \int_{K} Q(d\xi_1|x,f)\left[1_0(\xi_1) + \ldots \right.$$  

$$\left.\ldots + 1_{K \setminus O}(\xi_{n-2}) \int_{K} Q(d\xi_{n-1}|\xi_{n-2},f)\right]\right)$$

$$+ \left(1_{K \setminus O}(x) \int_{K \setminus O} Q(d\xi_1|x,f) \int_{K \setminus O} Q(d\xi_2|\xi_1,f)\ldots \int_{K} Q(d\xi_n|\xi_{n-1},f)u(\xi_n)\right).$$

We claim that the right-hand side of the last equality above is

$$E_x^{f^\infty} \left[\sum_{i=0}^{(n-1)\wedge \tau'} 1_0(x_i)\right] + E_x^{f^\infty} \left[1_{K \setminus O}(x_{(n-1)\wedge \tau'})(1_k \cdot u)(x_{(n-1)\wedge \tau'})1_{[\tau \wedge \tau' < \infty]}\right],$$

where $1_k \cdot u(\xi) := 1_k(\xi)u(\xi)$ for $\xi \in X$. To see this note that the first term is clear by definition. The second term above is due to the fact that only those trajectories that stay in $K \setminus O$ for $n$ steps (i.e., from stage 0 through stage $n - 1$) contribute to the integrand that features $u$, and this accounts for the factor $1_{K \setminus O}(x_{(n-1)\wedge \tau'})$. Since $\{\tau \wedge \tau' < \infty\}$ is a full measure set, the factor $1_{[\tau \wedge \tau' < \infty]}$ does not change the value of the integral. Taking the limit of the first term above as $n \to \infty$, the monotone convergence theorem gives

$$\lim_{n \to \infty} E_x^{f^\infty} \left[\sum_{i=0}^{(n-1)\wedge \tau'} 1_0(x_i)\right] = E_x^{f^\infty} \left[\sum_{i=0}^{\tau \wedge \tau'} 1_0(x_i)\right] = V(f^\infty, x) \leq V^*(x),$$

where the last inequality follows from the definition of $V^*$. Since $u$ is bounded and nonnegative, taking the limit of the second term above as $n \to \infty$, the dominated convergence theorem gives

$$\lim_{n \to \infty} E_x^{f^\infty} \left[1_{K \setminus O}(x_{(n-1)\wedge \tau'})(1_k \cdot u)(x_{(n-1)\wedge \tau'})1_{[\tau \wedge \tau' < \infty]}\right]$$

$$= E_x^{f^\infty} \left[1_{K \setminus O}(x_{\tau \wedge \tau'})(1_k \cdot u)(x_{\tau \wedge \tau'})1_{[\tau \wedge \tau' < \infty]}\right]$$

$$= 0.$$
since \(1_{K \setminus O}(x_{\tau \land \tau'}) = 0\) on the set \(\{\tau \land \tau' < \infty\}\) by definition of the stopping times \(\tau\) and \(\tau'\). Substituting back we see that \(u(x) \leq V^\ast(x)\), and the assertion follows since \(x \in K \setminus O\) is arbitrary.

(4.8). **Lemma.** Suppose that Assumption (2.9) holds. Then the value iteration functions \((v_n)_{n \in \mathbb{N}_0}\) defined in (3.2) satisfy \(v_n \uparrow V^\ast\), and the function \(V^\ast\) satisfies the Bellman equation (2.11).

**Proof.** From the definition of the value-iteration functions \((v_n)_{n \in \mathbb{N}_0}\) in (3.2) we see that \((v_n)_{n \in \mathbb{N}_0}\) is a monotone increasing sequence bounded above by \(1_x\). Therefore there exists a measurable function \(v^\ast : X \to [0,1]\) such that \(v_n \uparrow v^\ast\) pointwise on \(X\). By definition of \(v_n\) we have

\[
E_x^\tau \left[ \sum_{t=0}^{(n-1)\wedge \tau \land \tau'} 1_{0}(x_t) \right] \leq \sup_{\pi \in \Pi_M} E_x^\tau \left[ \sum_{t=0}^{(n-1)\wedge \tau \land \tau'} 1_{0}(x_t) \right] = v_n(x),
\]

and the monotone convergence theorem shows that

\[
v^\ast(x) = \lim_{n \to \infty} v_n(x) \geq \lim_{n \to \infty} E_x^\tau \left[ \sum_{t=0}^{(n-1)\wedge \tau \land \tau'} 1_{0}(x_t) \right] = E_x^\tau \left[ \sum_{t=0}^{\tau} 1_{0}(x_t) \right] = V(\pi, x).
\]

Taking the supremum over \(\pi \in \Pi_M\) on the right-hand side shows that \(v^\ast \geq V^\ast\) pointwise on \(X\). Note that \(v_n|_{\mathcal{L}^\ast} = 1\) and \(v_n|_{\mathcal{L} \setminus \mathcal{K}} = 0\) for all \(n\); therefore \(v^\ast|_{\mathcal{L}^\ast} = 1\) and \(v^\ast|_{\mathcal{L} \setminus \mathcal{K}} = 0\).

Let us define the maps

\[
\mathbb{K} \ni (x, a) \longrightarrow T' v_n(x, a) := \int_K Q(dy|x, a)v_n(y) \in [0,1],
\]

\[
\mathbb{K} \ni (x, a) \longrightarrow T' v^\ast(x, a) := \int_K Q(dy|x, a)v^\ast(y) \in [0,1].
\]

We note that the transition kernel \(Q\) is strongly Feller by Assumption (2.9), and therefore \(T' v_n, n \in \mathbb{N}_0\) and \(T' v^\ast\) are continuous functions on \(\mathbb{K}\). Moreover, for all \(n \in \mathbb{N}_0\) we define

\[
T' v_n(x, a) = T' v^\ast(x, a) = 1 \quad \text{for} \quad x \in O \quad \text{and} \quad a \in A(x),
\]

\[
(4.9) \quad T' v_n(x, a) = T' v^\ast(x, a) = 0 \quad \text{for} \quad x \in X \setminus K \quad \text{and} \quad a \in A(x),
\]

Since \(v_n \uparrow v^\ast\) pointwise on \(X\), it follows from the definitions above and the monotone convergence theorem that for all \(x \in X\) and \(a \in A(x)\)

\[
T' v_n(x, a)1_{K \setminus O}(x) \uparrow T' v^\ast(x, a)1_{K \setminus O}(x).
\]

Fix \(x \in K \setminus O\). Since \(T' v_n\) and \(T' v^\ast\) are continuous functions on \(\mathbb{K}\), for each \(n \in \mathbb{N}_0\) both \(\sup_{a \in A(x)} T' v_n(x, a)\) and \(\sup_{a \in A(x)} T' v^\ast(x, a)\) are attained on \(A(x)\). From the definition of \((v_n)_{n \in \mathbb{N}_0}\) in (3.2) we have \(\max_{a \in A(x)} T' v_n(x, a) \leq \sup_{a \in A(x)} T' v^\ast(x, a)\) for all \(n \in \mathbb{N}_0\). Also, \(\max_{a \in A(x)} T' v_n(x, a)\) is a nondecreasing sequence of numbers bounded above by 1, and therefore it attains a limit. If this limit is strictly less than \(\max_{a \in A(x)} T' v^\ast(x, a)\), standard easy arguments may be invoked to show that the sequence of continuous functions \((T' v_n(x, \cdot))_{n \in \mathbb{N}_0}\) cannot converge pointwise to \(T' v^\ast(x, \cdot)\) on \(A(x)\), which contradicts (4.10). It follows that whenever \(x \in K \setminus O\),

\[
v^\ast(x) = \lim_{n \to \infty} v_n(x) = \lim_{n \to \infty} T v_{n-1}(x)
\]

\[
= \lim_{n \to \infty} \max_{a \in A(x)} T' v_{n-1}(x, a) = \max_{a \in A(x)} T' v^\ast(x, a)
\]

\[
= T v^\ast(x).
\]
Together with (4.9) this shows that \( \nu^* \) satisfies the Bellman equation (2.11) pointwise on \( X \), i.e., \( \nu^* = T \nu^* \). We have already seen above that \( \nu^* \geq V^* \) pointwise on \( X \). Since \( \nu^* = T \nu^* \), the reverse inequality follows from Lemma (4.7). Therefore, we conclude that \( \nu^* = V^* \) identically on \( X \).

(4.11) Lemma. Let \( f^\infty \) be a deterministic stationary policy. Then we have

\[
V(f^\infty, x) = \begin{cases} 
1 & \text{if } x \in O, \\
\int_K Q(dy|x, f)V(f^\infty, y) & \text{if } x \in K \cap O, \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. For \( x \in O \cup (X \setminus K) \) the assertions are trivial. Fix \( x \in K \cap O \). From the definition of \( V(f^\infty, x) \) we have

\[
V(f^\infty, x) = E^{f^\infty} \left[ \sum_{t=0}^{\tau' \land t} 1_o(x_t) \bigg| x_0 = x \right]
\]

\[
= E^{f^\infty} \left[ 1_o(x_0) 1_{\tau' \land t = 0} + \sum_{t=1}^{\tau' \land t > 0} 1_o(x_t) \bigg| x_0 = x \right]
\]

\[
= 1_o(x) + E^{f^\infty} \left[ \sum_{t=1}^{\tau' \land t > 0} 1_o(x_t) \bigg| x_0 = x \right].
\]

Since \( \{\tau \land \tau' > 0\} = \{x_0 \in K \cap O\} \) and this event is \( \mathcal{F}_0 \)-measurable,

\[
E^{f^\infty} \left[ \sum_{t=1}^{\tau' \land t > 0} 1_o(x_t) \bigg| x_0 = x \right] = 1_{K \cap O}(x) E^{f^\infty} \left[ \sum_{t=1}^{\tau' \land t > 0} 1_o(x_t) \bigg| x_0 = x \right].
\]

Therefore,

\[
V(f^\infty, x) = 1_o(x) + 1_{K \cap O}(x) E^{f^\infty} \left[ \sum_{t=1}^{\tau' \land t > 0} 1_o(x_t) \bigg| x_0 = x \right]
\]

\[
= 1_o(x) + 1_{K \cap O}(x) E^{f^\infty} \left[ \sum_{t=1}^{\tau' \land t > 0} 1_o(x_t) \bigg| x_0 = x \right].
\]

Considering the fact that \( V(f^\infty, x) = 0 \) for \( x \in X \setminus K \) by definition, the Markov property shows that the second term on the right-hand side above equals

\[
1_{K \setminus O}(x) E^{f^\infty} \left[ E^{f^\infty} \left[ \sum_{t=1}^{\tau' \land t > 0} 1_o(x_t) \bigg| x_0 = x \right] \bigg| x_0 = x \right]
\]

\[
= 1_{K \setminus O}(x) \int_K Q(dy|x, f) E^{f^\infty} \left[ \sum_{t=1}^{\tau' \land t > 0} 1_o(x_t) \bigg| x_1 \land t = y \right]
\]

\[
= 1_{K \setminus O}(x) \int_K Q(dy|x, f) V(f^\infty, y).
\]

Collecting the above equations we obtain (4.12), and this completes the proof.

Proof of Theorem (2.10). (i) Note that by definition \( V^* \) is nonnegative. The fact that \( V^* \) satisfies the Bellman equation follows from Lemma (4.8). In view of the definition of \( \bar{B} \) in Theorem (2.10) and Lemma (4.8) we conclude that \( V^* \) is minimal in \( b\mathbb{B}(X)^\cap \bar{B} \).
because \(u = T u\) pointwise on \(K \setminus O\) implies that \(u \leq V^*\) pointwise on \(K \setminus O\) for any \(u \in \mathcal{B}(X)^+ \cap \mathcal{B}\).

(ii) Lemma (4.5) guarantees the existence of a selector \(f_s \in F\) such that (2.12) holds. Iterating the inequality (2.12) (or (2.14)) it follows as in the proof of Lemma (4.7) that for \(x \in X,

\[
V'(x) = E_x^\tau \left[ \sum_{t=0}^{(n-1)\wedge \tau^*} 1_o(x_t) + E_x^\tau \left[ 1_{K \setminus O}(x_{(n-1)\wedge \tau^*}) (1_K V^*)(x_{n\wedge \tau^*}) \right] \right].
\]

Taking limits as \(n \to \infty\) on the right, the monotone and dominated convergence theorems give \(V'(x) = V(f^\infty, x)\). Since \(x\) is arbitrary, \(V'(\cdot) = V(f^\infty, \cdot)\) on \(K \setminus O\) and that \(f^\infty\) is an optimal policy. Conversely, by Lemma (4.11) it follows that under the stationary deterministic strategy \(f^\infty\) we have (4.12) with \(f_s\) in place of \(f\), which is identical to (2.12).}

### § 4.2. Proofs of the results in § 2.3.

For the purposes of this subsection we let \(\Pi\) denote the set of admissible policies such that \(\pi_i\) is defined on \(K\) whenever \((\pi_i)_{i \in \mathbb{N}_0} \in \Pi\).

\[(4.13). \text{Lemma.} \text{ For every policy } \pi \in \Pi \text{ and initial state } x \in X \text{ the processes } (\zeta_n)_{n \in \mathbb{N}_0} \text{ and } (1_{K \setminus O}(x_{(n-1)\wedge \tau^*})(1_K V^*)(x_{n\wedge \tau^*}))_{n \in \mathbb{N}_0} \text{ are both nonnegative } (\mathfrak{F}_n)_{n \in \mathbb{N}_0} \text{ supermartingales under } \mathbb{P}_x^\pi.\]

**Proof.** It is clear that both processes are nonnegative and \((\mathfrak{F}_n)_{n \in \mathbb{N}_0}\)-adapted. Fix \(n \in \mathbb{N}\), an initial state \(x \in X\), a policy \(\pi \in \Pi\), and on the event \(\{\tau \wedge \tau' > n\}\) fix a history \(h_n = (x, a_0, x_1, a_1, \ldots, x_{n-1}, a_{n-1}, x_n)\). Let \(a_n := \pi_n(h_n)\) on \(\{\tau \wedge \tau' > n\}\). Then

\[
\zeta_{n+1} = W_n(\pi, x) + 1_{K \setminus O}(x_{n\wedge \tau^*})(1_K V^*)(x_{(n+1)\wedge \tau^*})
\]

\[
= W_n(\pi, x) + 1_0(x_{n\wedge \tau^*}) 1_{\{\tau^* = n\}} + 1_{K \setminus O}(x_{n\wedge \tau^*})(1_K V^*)(x_{(n+1)\wedge \tau^*})
\]

\[
= W_n(\pi, x) + 1_{\{\tau^* = n\}} 1_0(x_{n\wedge \tau^*}) + 1_{K \setminus O}(x_{n\wedge \tau^*})(1_K V^*)(x_{(n+1)\wedge \tau^*}).
\]

Since \(x_{n\wedge \tau^*} \in O\) \(\subseteq \{\tau \wedge \tau' = n\}\), we have

\[
1_{\{\tau^* = n\}} 1_0(x_{n\wedge \tau^*}) + 1_{K \setminus O}(x_{n\wedge \tau^*})(1_K V^*)(x_{(n+1)\wedge \tau^*})
\]

\[
= 1_{\{\tau^* = n\}} 1_0(x_{n\wedge \tau^*}) + 1_{K \setminus O}(x_{n\wedge \tau^*})(1_K V^*)(x_{(n+1)\wedge \tau^*}).
\]

Since \(\{\tau \wedge \tau' > n - 1\} = \{x_{(n-1)\wedge \tau^*} \in K \setminus O\}\), it follows that

\[
\zeta_{n+1} = W_n(\pi, x) + 1_{K \setminus O}(x_{(n-1)\wedge \tau^*})(1_K V^*)(x_{(n+1)\wedge \tau^*}).
\]

Therefore, keeping in mind the definition of \(a_n\) above,

\[
E_x^{\pi}[\zeta_{n+1} | \mathfrak{F}_{n\wedge \tau^*}] = W_n(\pi, x) + 1_{K \setminus O}(x_{(n-1)\wedge \tau^*})(1_K V^*)(x_{n\wedge \tau^*}, a_n)
\]

\[
\leq W_n(\pi, x) + 1_{K \setminus O}(x_{(n-1)\wedge \tau^*}) V^*(x_{n\wedge \tau^*})
\]

\[
= \zeta_n,
\]

where the inequality holds \(\mathbb{P}_x^\pi\)-almost surely. Therefore, the process \((\zeta_n)_{n \in \mathbb{N}_0}\) is a nonnegative \((\mathfrak{F}_n)_{n \in \mathbb{N}_0}\)-supermartingale, and hence also \((\mathfrak{F}_n)_{n \in \mathbb{N}_0}\)-supermartingale. Considering that the sequence \((W_n(\pi, x))_{n \in \mathbb{N}_0}\) is nondecreasing, from the definitions in (2.15) and the fact that the process \((\zeta_n)_{n \in \mathbb{N}_0}\) is a \((\mathfrak{F}_n)_{n \in \mathbb{N}_0}\)-supermartingale we see that the process \((1_{K \setminus O}(x_{(n-1)\wedge \tau^*})(1_K V^*)(x_{n\wedge \tau^*}))_{n \in \mathbb{N}_0}\) is also a \((\mathfrak{F}_n)_{n \in \mathbb{N}_0}\)-supermartingale under \(\mathbb{P}_x^\pi\).
Proof of Theorem (2.17). Lemma (4.13) confirms that both of the two adapted processes \((\zeta_n)_{n \in \mathbb{N}}\) and \((1_{K \times O}(x_{(n-1)\wedge \tau^*})(1_K V^*)(x_{n\wedge \tau^*}))_{n \in \mathbb{N}}\) converge almost surely and are nonincreasing in expectation, both under \(P^\pi_x\). Let \(\Lambda^\pi(x) := \lim_{n \to \infty} E^\pi_x[\zeta_n]\). We then have
\[
V^*(x) = E^\pi_x[\zeta_0] \geq \lim_{n \to \infty} E^\pi_x[\zeta_n] \\
(4.15) \quad = \lim_{n \to \infty} \left( E^\pi_x[W_n(\pi, x)] + E^\pi_x[1_{K \times O}(x_{(n-1)\wedge \tau^*})(1_K V^*)(x_{n\wedge \tau^*})] \right) \\
\geq V(\pi, x).
\]
The assertion is now an immediate consequence of (4.15).

Proof of Theorem (2.18). Suppose that (i) holds. Since \(E^\pi_x[\zeta_n]\) is nonincreasing with \(n\) it follows that \(E^\pi_x[\zeta_{n+1}] = E^\pi_x[\zeta_n] = \ldots = E^\pi_x[\zeta_0] = V^*(x)\) for every \(n \in \mathbb{N}\). Therefore, equality must hold \(P^\pi_x\)-almost surely in (4.14), and (ii) follows.

Suppose that (ii) holds. Then equality holds in (4.14) almost surely under \(P^\pi_x\), and therefore \(P^\pi_x\)-almost everywhere on the set \([x_{n\wedge \tau^*} \in K \setminus O] = \{\tau \wedge \tau^* > n\}\) we have \(T^\tau V^*(x_n, a_n) = V^*(x_n)\), and (iii) follows.

Suppose that (iii) holds. Then taking expectations in (4.14) we arrive at \(E^\pi_x[\zeta_{n+1}] = E^\pi_x[\zeta_j] = \ldots = E^\pi_x[\zeta_0] = V^*(x)\). As a result we have \(\Lambda^\pi(x) = V^*(x)\), and (i) follows.

Proof of Theorem (2.19). It follows readily from the definition of the stopping times \(\tau\) and \(\tau^*\) that the process \((\zeta'_n)_{n \in \mathbb{N}}\) defined in (2.20) is a bounded process, and by assumption it is a \((\zeta'_n)_{n \in \mathbb{N}}\)-martingale under \(P^\pi_x\). Doob’s Optional Sampling Theorem [Rao and Swift, 2006, Theorem 2, p. 422] applied to \((\zeta'_n)_{n \in \mathbb{N}}\) at the stopping time \(\tau \wedge \tau^*\) gives us
\[
E^\pi_x[\zeta'_{\tau \wedge \tau^*}] = E^\pi_x[\zeta'_0] = V^*(x),
\]
where the last equality follows from the definition of \(\zeta'_0\). From (2.15) we get
\[
E^\pi_x[\zeta'_{\tau \wedge \tau^*}] = E^\pi_x\left[ W_{\tau \wedge \tau^* - 1}(\pi, x) + 1_{K \times O}(x_{\tau \wedge \tau^* - 1})(1_K \cdot V^*)(x_{\tau \wedge \tau^*}) \right] \\
= E^\pi_x\left[ \sum_{t=0}^{\tau - 1} 1_{O}(x_t) + 1_{K \times O}(x_{\tau - 1})(1_K \cdot V^*)(x_{\tau^*}) \right] \\
= E^\pi_x\left[ 1_{K \times O}(x_{\tau - 1})(1_K \cdot V^*)(x_{\tau^*}) \right].
\]
By definition of \(\tau\) and \(\tau^*\), \(1_{K \times O}(x_{\tau - 1})\) equals 1 on \(\{\tau \wedge \tau^* < \infty\}\), and by our hypotheses the set \(\{\tau \wedge \tau^* < \infty\}\) is a \(P^\pi_x\)-full-measure set. Continuing from the last equality above we arrive at
\[
E^\pi_x[\zeta'_{\tau \wedge \tau^*}] = E^\pi_x\left[ 1_{\{\tau \wedge \tau^* < \infty\}}(1_K \cdot V^*)(x_{\tau \wedge \tau^*}) \right] \\
= E^\pi_x\left[ 1_{\{\tau \wedge \tau^* < \infty\}}(1_{t < \tau} 1_{K}(x_t)V^*(x_t) + 1_{t > \tau} 1_{K}(x_t)V^*(x_t)) \right] \\
(4.16) \quad = E^\pi_x\left[ 1_{\{\tau \wedge \tau^* < \infty\}} 1_{\{t < \tau\}} \right] \\
= P^\pi_x(\tau < \tau^*, \tau < \infty),
\]
where the equality in (4.16) follows from the assumptions on \(V^*\) and the definitions of \(\tau\) and \(\tau^*\). Collecting the equations above we get \(V^*(x) = P^\pi_x(\tau < \tau^*, \tau < \infty)\) as asserted.

It is of interest to note that the hypotheses of Theorem (2.19) requires at least one of the stopping times \(\tau\) or \(\tau^*\) to be finite. Let us examine the case of \(\tau \wedge \tau^*\) being \(\infty\) on a set of positive probability. Following the proof of Theorem (2.19), we see that in this case
we have to agree on the value of $V'(x_{\tau \land \tau'})$ on $\{\tau \land \tau' = \infty\}$. If $\lim_{n \to \infty} V'_{n}(\pi^{*}, x)$ exists, then we can always let $V'(x_{\tau \land \tau'})$ take this value on the set $\{\tau \land \tau' = \infty\}$. However, the context of the problem offers another alternative, namely, to set $V'(x_{\tau \land \tau'}) = 0$ on $\{\tau \land \tau' = \infty\}$. This is because if $x_t \in K \setminus O$ for all $t \in \mathbb{N}_0$, then the value of $x_{\tau \land \tau'}$ is of no consequence at all.

§5. Conclusions and Future Work

The purpose of this article was to present a dynamic programming based solution to the problem of maximizing the probability of attaining a target set before hitting a cemetery set, and furnish an alternative martingale characterization of optimality in terms of thrifty and equalizing policies. Several related problems of interest were sketched in §3.1. Some of these problems do not admit an immediate solution in the dynamic programming framework we established here because of our central assumption that the cost-per-stage function is nonnegative. This issue deserves further investigation.

The results in this article also provide clear indications to the possibility of developing verification tools for probabilistic computation tree logic [Kwiatkowska et al., 2007] in terms of dynamic programming operators. This matter is under investigation and will be reported in [Ramponi et al., 2009]. Implementation of the dynamic-programming algorithm in this article is challenging due to integration over subsets of the state-space, and suboptimal policies are needed. In this context development of a possible connection with ‘greedy-time-optimal’ policies [Meyn, 2008, Chapters 4, 7], originally proposed as a tractable alternative to optimal policies in demand-driven large-scale production systems, is being sought.

Acknowledgement

The authors thank Onésimo Hernández-Lerma for helpful suggestions and pointers to references, and Sean Summers for posing the problem.

References

Abate, A., Prandini, M., Lygeros, J., Sastry, S., 2008. Probabilistic reachability and safety for controlled discrete time stochastic hybrid systems. Automatica 44 (11), 2724–2734.
Aliprantis, C., Border, K. C., 2006. Infinite Dimensional Analysis: a Hitchhiker’s Guide, 3rd Edition. Springer-Verlag, Berlin.
Bertsekas, D., Tsitsiklis, J., 1996. Neuro-Dynamic Programming. Athena Scientific.
Bertsekas, D. P., 2007. Dynamic Programming and Optimal Control, 3rd Edition. Vol. 2. Athena Scientific.
Bertsekas, D. P., Shreve, S. E., 1978. Stochastic Optimal Control: the Discrete-Time Case. Vol. 139 of Mathematics in Science and Engineering. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York.
Boda, K., Filar, J. A., Lin, Y., Spanjers, L., 2004. Stochastic target hitting time and the problem of early retirement. IEEE Transactions on Automatic Control 49 (3), 409–419.
Borkar, V. S., 1988. A convex analytic approach to Markov decision processes. Probability Theory and Related Fields 78 (4), 583–602.
Borkar, V. S., 1991. Topics in Controlled Markov Chains. Vol. 240 of Pitman Research Notes in Mathematics Series. Longman Scientific & Technical, Harlow.
Bouakiz, M., Kebir, Y., 1995. Target-level criterion in markov decision processes. Journal of Optimization Theory and Applications 86 (1), 1–15.

Chatterjee, D., Cinquemani, E., Chaloulos, G., Lygeros, J., 2008. Stochastic control up to a hitting time: optimality and rolling-horizon implementation. http://arxiv.org/abs/0806.3008.

Derman, C., 1970. Finite State Markovian Decision Processes. Vol. 67 of Mathematics in Science and Engineering. Academic Press, New York.

Digal’lova, I. A., Kurzhanski˘ı, A. B., 2004. The attainability problem under stochastic perturbations. Differential’nye Uravneniya 40 (11), 1494–1499, 1582.

Dubins, L. E., Savage, L. J., 1976. Inequalities for Stochastic Processes (How to Gamble if You Must). Dover Publications Inc., New York, corrected republication of the 1965 edition.

Dynkin, E. B., 1963. Optimum choice of the stopping moment of a Markov process. Doklady Akademii Nauk SSR 150, 238–240.

Eaton, J. H., Zadeh, L. A., 1962. Optimal pursuit strategies in discrete-state probabilistic systems. Transactions of the ASME Ser. D. J. Basic Engineering 84, 23–29.

Gao, Y., Lygeros, J., Quincampoix, M., 2007. On the reachability problem for uncertain hybrid systems. IEEE Transactions on Automatic Control 52 (9), 1572–1586.

Hernández-Lerma, O., Lasserre, J. B., 1996. Discrete-Time Markov Control Processes: Basic Optimality Criteria. Vol. 30 of Applications of Mathematics. Springer-Verlag, New York.

Hernández-Lerma, O., Lasserre, J. B., 1999. Further Topics on Discrete-Time Markov Control Processes. Vol. 42 of Applications of Mathematics. Springer-Verlag, New York.

Karatzas, I., Sudderth, W., 2009. Two characterizations of optimality in dynamic programming. Applied Mathematics & Optimization; http://www.springerlink.com/content/340m82862p817446/?p=f54a99eb8ef3432cb45724c3d5e86ba&pi=0.

Kesten, H., Spitzer, F., 1975. Controlled Markov chains. Annals of Probability 3, 32–40.

Kushner, H., 1971. Introduction to Stochastic Control. Holt, Rinehart and Winston, Inc., New York.

Kwiatkowska, M., Norman, G., Parker, D., 2007. Stochastic model checking. In: Lecture Notes in Computer Science. Vol. 4486. Springer-Verlag.

Levin, D. A., Peres, Y., Wilmer, E. L., 2009. Markov Chains and Mixing Times. American Mathematical Society, USA, with an Appendix written by James G. Propp and David B. Wilson.

Meyn, S. P., 2008. Control Techniques for Complex Networks. Cambridge University Press, Cambridge.

Ohtsubo, Y., 2003. Value iteration methods in risk minimizing stopping problems. Journal of Computational and Applied Mathematics 152 (1-2), 427–439.

Peskir, G., Shiryaev, A. N., 2006. Optimal Stopping and Free-Boundary Problems. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel.

Powell, W. B., 2007. Approximate Dynamic Programming. Wiley Series in Probability and Statistics. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ.

Prajna, S., Jadbabaie, A., Pappas, G. J., 2007. A framework for worst-case and stochastic safety verification using barrier certificates. IEEE Transactions on Automatic Control 52 (8), 1415–1428.

Prandini, M., Hu, J., 2006. A stochastic approximation method for reachability computations. In: Stochastic Hybrid Systems. Vol. 337 of Lecture Notes in Control and Information Sciences. Springer, Berlin, pp. 107–139.

Ramponi, F., Chatterjee, D., Summers, S., Lygeros, J., 2009. On the connections between PCTL and Dynamic Programming. http://arxiv.org/abs/0910.4738.
Rao, M. M., Swift, R. J., 2006. Probability Theory with Applications, 2nd Edition. Vol. 582 of Mathematics and Its Applications. Springer-Verlag.

Revuz, D., Yor, M., 1999. Continuous Martingales and Brownian Motion, 3rd Edition. Vol. 293 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin.

Schmidli, H., 2008. Stochastic Control in Insurance. Probability and its Applications. Springer-Verlag London Ltd., London.

Simon, H. A., 1957. Models of man, social and rational. Mathematical essays on rational human behavior in a social setting. John Wiley & Sons Inc., New York.

Tomlin, C. J., Lygeros, J., Sastry, S., 2000. A game theoretic approach to controller design for hybrid systems. Proceedings of IEEE 88, 949–969.

Watkins, O., Lygeros, J., 2003. Stochastic reachability for discrete-time systems: an application to aircraft collision avoidance. In: 42nd IEEE Conference on Decision and Control. Vol. 5. pp. 5314–5319.

Whittle, P., 1983. Optimization Over Time. Vol. II of Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons Ltd., Chichester.

Zhu, Q., Guo, X., 2006. A semimartingale characterization of average optimal stationary policies for Markov decision processes. Journal of Applied Mathematics and Stochastic Analysis, Art. ID 81593. http://www.hindawi.com/GetArticle.aspx?doi=10.1155/JAMSA/2006/81593.

AUTOMATIC CONTROL LABORATORY, ETL I19, ETH ZURICH, PHYSIKSTRASSE 3, 8092 ZURICH, SWITZERLAND
E-mail address: chatterjee@control.ee.ethz.ch
URL: http://control.ee.ethz.ch/~chatterd

INRIA-GRENoble - RHôNE-ALPES, 655 AVENUE DE L’EUROPE, MONTRONNOT, 38334 SAINT ISMIER CEDEX, FRANCE
E-mail address: Eugenio.Cinquemani@inria.fr
URL: http://ibis.inrialpes.fr/article941.html

AUTOMATIC CONTROL LABORATORY, ETL I22, ETH ZURICH, PHYSIKSTRASSE 3, 8092 ZURICH, SWITZERLAND
E-mail address: lygeros@control.ee.ethz.ch
URL: http://control.ee.ethz.ch/~lygeros