ENDOMORPHISMS AND MODULAR THEORY OF
2-GRAph C*-ALGEBRAS

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ABSTRACT. In this paper, we initiate the study of endomorphisms and modular theory of the graph C*-algebras \( \mathcal{O}_\theta \) of a 2-graph \( \mathbb{F}^+_\theta \) on a single vertex. We prove that there is a semigroup isomorphism between unital endomorphisms of \( \mathcal{O}_\theta \) and its unitary pairs with a twisted property. We characterize when endomorphisms preserve the fixed point algebra \( \mathfrak{F} \) of the gauge automorphisms and its canonical masa \( \mathfrak{D} \). Some other properties of endomorphisms are also investigated.

As far as the modular theory of \( \mathcal{O}_\theta \) is concerned, we show that the algebraic *-algebra generated by the generators of \( \mathcal{O}_\theta \) with the inner product induced from a distinguished state \( \omega \) is a modular Hilbert algebra. Consequently, we obtain that the von Neumann algebra \( \pi(\mathcal{O}_\theta)^\prime\prime \) generated by the GNS representation of \( \omega \) is an AFD factor of type III\(_1\), provided \( \ln \frac{m}{n} \notin \mathbb{Q} \). Here \( m, n \) are the numbers of generators of \( \mathbb{F}^+_\theta \) of degree \((1,0)\) and \((0,1)\), respectively.

This work is a continuation of [11, 12] by Davidson-Power-Yang and [13] by Davidson-Yang.

1. INTRODUCTION

In 2000, Kumjian-Pask generalized higher rank Cuntz-Krieger algebras of Robertson-Steger [32] and introduced the notion of higher rank graphs (or k-graphs) in [21]. Since then, higher rank graphs have been attracting a great deal of attention and extensively studied. See, for example, [16, 20, 21, 23, 24, 27, 28, 29, 31] and the references therein.

Recently, in [11, 12, 13], Davidson, Power and I have systematically studied an interesting and special class of higher rank graphs – 2-graphs with a single vertex, which was initially studied by Power [24]. Roughly speaking, those graphs are given concretely in terms of a finite set of generators and relations of a special type. More precisely, given a permutation \( \theta \) of \( m \times n \), form a unital semigroup \( \mathbb{F}^+_\theta \) with generators...

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e_1, \ldots, e_m \text{ and } f_1, \ldots, f_n \text{ which is free in the } e_i\text{'s and free in the } f_j\text{'s, and has the commutation relations } e_i f_j = f_{j'} e_{i'}, \text{ where } \theta(i, j) = (i', j') \text{ for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n. \mathbb{F}_\theta^+ \text{ is a cancellative semigroup with unique factorization } [24].

It turns out that 2-graph algebras on a single vertex have very nice structures, and provide many very interesting and nontrivial phenomena. We gave a detailed analysis of their representation theory and completely classified their atomic representations in [11]. The dilation theory was studied in [12]. Particularly, it was shown in there that every defect free row contractive representation of \( \mathbb{F}_\theta^+ \) has a unique minimal *-dilation. The characterization of the aperiodicity of \( \mathbb{F}_\theta^+ \) and the structure of the graph C*-algebra \( \mathcal{O}_\theta \) were given in [13]. From those results, one has a very nice and clear picture of those algebras. In [14], some of the results in [11, 12, 13] were further generalized to \( k \)-graphs with a single vertex by Davidson and the author.

The main purpose of this paper is to initiate the study of endomorphisms and modular theory of 2-graph algebras on a single vertex. This was motivated by [9] and [4]. This paper can be regarded as a continuation of [11, 12, 13]. In the seminal paper [9], Cuntz studied the theory of automorphisms of Cuntz algebras \( \mathcal{O}_n \). It is now well-known that there is a one-to-one correspondence between \( \mathcal{U}(\mathcal{O}_n) \), the set of unitaries of \( \mathcal{O}_n \), and \( \text{End}(\mathcal{O}_n) \), the set of unital endomorphisms of \( \mathcal{O}_n \). We should mention that, very recently, the localized automorphisms of Cuntz algebras have been studied in [6, 7, 34].

Since the graph C*-algebra \( \mathcal{O}_\theta \) contains two copies of Cuntz algebras \( \mathcal{O}_m \) and \( \mathcal{O}_n \), which are “connected” by the commutation relations of \( \mathbb{F}_\theta^+ \), one naturally wonders what its unital endomorphisms look like. In this paper, we show that there is a semigroup isomorphism between \( \text{End}(\mathcal{O}_\theta) \), the set of unital endomorphisms of \( \mathcal{O}_\theta \), and \( \mathcal{U}(\mathcal{O}_\theta)^2 \), the set consisting of pairs in \( \mathcal{U}(\mathcal{O}_\theta) \times \mathcal{U}(\mathcal{O}_\theta) \) with a twisted property. It is the twisted property that makes an essential difference between the study of \( \text{End}(\mathcal{O}_n) \) and that of \( \text{End}(\mathcal{O}_\theta) \), and makes \( \text{End}(\mathcal{O}_\theta) \) more involved. However, the twisted property appears here naturally since it decodes the commutation relations determined by \( \theta \). After obtaining the above isomorphism, we study the theory of endomorphisms of \( \mathcal{O}_\theta \) mainly in the vein of [9].

The second part of this paper is devoted to investigating the modular theory of \( \mathcal{O}_\theta \). This was originally motivated by the index theory of endomorphisms. There is a rich literature on this topic. See, for example, [4, 5] and the references therein. Our first main result here is that the algebraic *-algebra generated by \( \mathbb{F}_\theta^+ \) with the inner product, induced
from a distinguished faithful state $\omega$ of $\mathcal{O}_\theta$, is a modular Hilbert algebra. We achieve this by obtaining very explicit expressions of the modular objects associated with $\omega$ in the celebrated Tomita-Takesaki modular theory. Then we give some partial results on the classification of the von Neumann algebra $\pi(\mathcal{O}_\theta)''$ generated from the GNS representation of $\omega$.

This paper is organized as follows. The next section provides some preliminaries on 2-graph algebras. In Section 3, we study the general theory of the endomorphisms of $\mathcal{O}_\theta$. In particular, we prove that there is a semigroup isomorphism between unital endomorphisms of $\mathcal{O}_\theta$ and the unitary pairs of $\mathcal{O}_\theta$ with a twisted property. Some examples are also given there. In Section 4 we characterize when endomorphisms or automorphisms preserve the fixed point algebra $\mathcal{F}$ of the gauge automorphisms and its canonical masa $\mathcal{D}$. Some other properties of endomorphisms are also investigated. In Section 5 the modular theory of 2-graph algebras is given in detail. We prove that the algebraic $*$-algebra generated by the generators of $\mathcal{O}_\theta$ with the inner product induced from a distinguished state $\omega$ is a modular Hilbert algebra. Consequently, we obtain that $\omega$ is a KMS-state with respect to the associated modular automorphism group of the von Neumann algebra $\pi(\mathcal{O}_\theta)''$ generated by $\omega$. As a further consequence, we show that if $\frac{\ln m}{\ln n} \not\in \mathbb{Q}$, then $\pi(\mathcal{O}_\theta)''$ is an AFD factor of type III$_1$. Those results are given in Section 6.

2. Preliminaries

The main source of this section is from [11, 12, 13, 14, 21].

2.1. 2-graphs on a single vertex. A 2-graph on a single vertex is a unital semigroup $\mathbb{F}_\theta^+$, which is generated by $e_1, ..., e_m$ and $f_1, ..., f_n$. The identity is denoted as $\emptyset$. There are no relations among the $e_i$'s, so they generate a copy of the free semigroup on $m$ letters, $\mathbb{F}_m^+$; and there are no relations on the $f_j$'s, so they generate a copy of $\mathbb{F}_n^+$. There are commutation relations between the $e_i$'s and $f_j$'s given by a permutation $\theta$ in $S_{m\times n}$ of $m \times n$:

$$e_if_j = f_je_{\theta(i)}$$

where $\theta(i, j) = (i', j')$.

The semigroup $\mathbb{F}_\theta^+$ has some nice properties. See, for example, [20, 21]. Any word $w \in \mathbb{F}_\theta^+$ has fixed numbers of $e$'s and $f$'s regardless of the factorization. The degree of $w$ is defined as $d(w) = (k, l)$ if there are $k$ $e$'s and $l$ $f$'s, and the length of $w$ is $|w| = k + l$. Moreover, because of the commutation relations, one can write $w$ according to any prescribed pattern of $e$’s and $f$’s as long as the degree is $(k, l)$. For
instance, we can write \( w \) with all \( e \)'s first, all \( f \)'s first, or \( e \)'s and \( f \)'s alternatively if \( d(w) = (k, k) \).

Recall from [21] that the graph \( C^* \)-algebra \( \mathcal{O}_\theta \) of \( \mathbb{F}^+_\theta \) is the universal \( C^* \)-algebra generated by a family of isometries \( \{s_u : u \in \mathbb{F}^+_\theta \} \) satisfying \( s_{\varnothing} = I, s_{uv} = s_us_v \) for all \( u, v \in \mathbb{F}^+_\theta \), and the defect free property:

\[
\sum_{i=1}^m s_{e_i}s^*_{e_i} = I = \sum_{j=1}^n s_{f_j}s^*_{f_j}.
\]

It is well-known that \( \mathcal{O}_\theta \) has standard generators \( s_us^*_v \) and \( \mathcal{O}_\theta = \text{span}\{s_us^*_v : u, v \in \mathbb{F}^+_\theta \} \) ([21] Lemma 3.1). We extend the degree map \( d \) of \( \mathbb{F}^+_\theta \) to the generators of \( \mathcal{O}_\theta \) as follows:

\[
d(s_us^*_v) = d(u) - d(v) \quad \text{for all} \quad u, v \in \mathbb{F}^+_\theta.
\]

To simplify our writing, throughout the paper, we use the following multi-index notation: For all \( x = (x_1, x_2) \in \mathbb{C}^2 \) with \( x_1, x_2 \neq 0 \) and \( k = (k_1, k_2) \in \mathbb{Z}^2 \), let \( x^k := x_1^{k_1}x_2^{k_2} \).

2.2. Gauge automorphisms. It is well-known that the universal property of \( \mathcal{O}_\theta \) yields a family of gauge automorphisms \( \gamma_t \) for \( t = (t_1, t_2) \in \mathbb{T}^2 \) given by

\[
\gamma_t(s_w) = t^{d(w)}s_w \quad \text{for all} \quad w \in \mathbb{F}^+_\theta.
\]

Integrating around \( \mathbb{T}^2 \) yields a faithful expectation

\[
\Phi(X) = \int_{\mathbb{T}^2} \gamma_t(X) \, dt
\]

onto the fixed point algebra \( \mathcal{O}_\theta^\gamma \) of \( \gamma \).

It turns out that

\[
\mathfrak{F} := \mathcal{O}_\theta^\gamma = \bigcup_{k \geq 1} \mathfrak{F}_k
\]

is an \( (mn)^\infty \)-UHF algebra, where \( \mathfrak{F}_k = \text{span}\{s_us^*_v : d(u) = d(v) = (k, k) \} \) \( (k \in \mathbb{N}) \) is the full matrix algebra \( M_{(mn)^k} \).

For each \( n = (n_1, n_2) \in \mathbb{Z}^2 \), define a mapping \( \Phi_n \) on \( \mathcal{O}_\theta \) via

\[
\Phi_n(X) = \int_{\mathbb{T}^2} t^{-n}\gamma_t(X)dt \quad \text{for all} \quad X \in \mathcal{O}_\theta.
\]

Then \( \Phi_n \) acts on generators via

\[
\Phi_n(X) = \begin{cases} 
X, & \text{if} \quad d(X) = n, \\
0, & \text{otherwise}.
\end{cases}
\]

So the fixed point algebra \( \mathfrak{F} \) is nothing but \( \text{Ran} \Phi_0 \), which is spanned by the words of degree \((0, 0)\). For every \( X \in \mathcal{O}_\theta \), we also have the formal
series $X \sim \sum_{n \in \mathbb{Z}^2} \Phi_n(X)$ with a Cesaro convergence of the series. Refer to [18] for the details.

Let

$$\mathcal{D} = \text{span}\{s_w s_w^* : d(w) = (k, k), \ k \in \mathbb{N}\},$$

the canonical masa in $\mathfrak{F}$, and

$$N(\mathcal{D}) = \{U \in \mathcal{U}(\mathcal{O}_\theta) : U \mathcal{D} U^* \subseteq \mathcal{D}\},$$

the (unitary) normalizer of $\mathcal{D}$.

Notice that, in the literature, it is usual to interpret $\mathfrak{F}$ (resp. $\mathcal{D}$) as the $\mathrm{C}^*$-algebras generated by $\{s_u s_v^* : d(u) = d(v) = (k, l), k, l \in \mathbb{N}\}$ (resp. $\{s_w s_w^* : d(w) = (k, l), k, l \in \mathbb{N}\}$). But they are the same as those given above because of the defect free property $\sum_{i=1}^m s_{e_i} s_{e_i}^* = I = \sum_{j=1}^n s_{f_j} s_{f_j}^*$. Refer to [13] for more details. We find that it is more convenient to use the above way for our purpose. For example, when we consider the restriction $\lambda_{\{U, V\}}|_\mathfrak{F}$ of an endomorphism $\lambda_{\{U, V\}}$ determined by $(U, V)$, it suffices to take care of the generators $X = s_u s_v^*$ with $d(u) = d(v) = (k, k)$. Using the commutation relations, one can rewrite $X$ as $X = s_{u_1 \cdots u_k} s_{v_1 \cdots v_k}$, where $u_i, v_i \in \mathbb{F}_\theta^+$ with $d(u_i) = d(v_i) = (1, 1)$ ($i = 1, \ldots, k$). Then the action of $\lambda_{\{U, V\}}$ on $X$ is now completely determined by the unitary $W = U \lambda_{e_1}(V) = V \lambda_{e_2}(U)$ (see Section 3 for the notation). We will often use this simple and useful observation later.

2.3. Notation and conventions. We end this section with introducing some notation and conventions. Let $\mathbb{Z}_+$ denote the set of all non-negative integers. If $\mathcal{A}$ is a unital $\mathrm{C}^*$-algebra, by $\mathcal{U}(\mathcal{A})$ we mean the set of all unitaries of $\mathcal{A}$. Let $\mathrm{End}(\mathcal{A})$ denote the semigroup of unital endomorphisms of $\mathcal{A}$, and $\mathrm{Aut}(\mathcal{A})$, $\mathrm{Inn}(\mathcal{A})$ the groups of automorphisms, inner automorphisms of $\mathcal{A}$, respectively. The notation $\mathcal{Z}(\mathcal{A})$ stands for the center of $\mathcal{A}$. In this paper, by endomorphisms we always mean unital endomorphisms.

3. Endomorphisms of 2-graph algebras

In this section, we will study the general theory of unital endomorphisms of $\mathcal{O}_\theta$. Some examples will also be given.

3.1. General Theory. Let $\mathbb{F}_\theta^+$ be a 2-graph on a single vertex generated by $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$. It is easy to see that, for $(p, q) \in \mathbb{Z}_+^2$, the completely positive maps $\lambda_{(p, q)}$ on $\mathcal{O}_\theta$ given by

$$\lambda_{(p, q)}(X) = \sum_{d(w) = (p, q)} s_w X s_w^* \quad \text{for all} \quad X \in \mathcal{O}_\theta$$
are endomorphisms of $\mathcal{O}_\theta$. Those endomorphisms are said to be canonical. Let $\epsilon_1$ and $\epsilon_2$ be the standard generators of $\mathbb{Z}_2^2$. One can easily check that $\lambda_{(p,q)} = \lambda_{\epsilon_1}^p \lambda_{\epsilon_2}^q = \lambda_{\epsilon_1}^q \lambda_{\epsilon_2}^p$, and that $\lambda_{(p,q)}(X) s_w = s_w X$ for all $X \in \mathcal{O}_\theta$ and $w$ with $d(w) = (p,q)$.

Let $\mathcal{U}(\mathcal{O}_\theta)^2_{\text{twi}} \subseteq \mathcal{U}(\mathcal{O}_\theta) \times \mathcal{U}(\mathcal{O}_\theta)$ be the family of unitary pairs satisfying a twisted property. More precisely,

$$\mathcal{U}(\mathcal{O}_\theta)^2_{\text{twi}} = \{(U, V) \in \mathcal{U}(\mathcal{O}_\theta) \times \mathcal{U}(\mathcal{O}_\theta) : U \lambda_{\epsilon_1}(V) = V \lambda_{\epsilon_2}(U)\}.$$ 

Then there is a bijective correspondence between $\mathcal{U}(\mathcal{O}_\theta)^2_{\text{twi}}$ and $\text{End}(\mathcal{O}_\theta)$, as shown below.

**Theorem 3.2.** The mapping $\Psi : \mathcal{U}(\mathcal{O}_\theta)^2_{\text{twi}} \rightarrow \text{End}(\mathcal{O}_\theta)$,

$$(U, V) \mapsto \lambda_{(U,V)} : s_{e_i} \mapsto U s_{e_i}, \quad s_{f_j} \mapsto V s_{f_j} \quad (i = 1, \ldots, m, \ j = 1, \ldots, n)$$

is a bijection. Its inverse is given by

$$\lambda \mapsto \left( \sum_{i=1}^m \lambda(s_{e_i}) s_{e_i}^*, \sum_{j=1}^n \lambda(s_{f_j}) s_{f_j}^* \right).$$

**Proof.** Let $(U, V) \in \mathcal{U}(\mathcal{O}_\theta)^2_{\text{twi}}$. Define

$$\lambda_{(U,V)}(s_{e_i}) = U s_{e_i}, \quad \lambda_{(U,V)}(s_{f_j}) = V s_{f_j}, \quad i = 1, \ldots, m, \ j = 1, \ldots, n.$$ 

In what follows, we prove that $\lambda_{(U,V)}$ preserves the commutation relations given by $\theta$. That is, if $e_{i_j} f_{j_1} = f_{j_2} e_{i_2}$ where $\theta(i,j) = (i', j')$, then $\lambda_{(U,V)}(s_{e_i}) \lambda_{(U,V)}(s_{f_j}) = \lambda_{(U,V)}(s_{f_j}) \lambda_{(U,V)}(s_{e_i})$. Indeed, let $W := U \lambda_{\epsilon_1}(V) = V \lambda_{\epsilon_2}(U)$. Since $e_{i_j} f_{j_1} = f_{j_2} e_{i_2}$, we have

$$U \sum_{i=1}^m s_{e_i} V s_{e_i}^* = W \implies U s_{e_i} V = W s_{e_i} \implies U s_{e_i} V s_{f_j} = W s_{e_i} f_j;$$

$$V \sum_{j'=1}^n s_{f_{j'}} U s_{f_{j'}}^* = W \implies V s_{f_{j'}} U = W s_{f_{j'}} \implies V s_{f_{j'}} U s_{e_{i'}} = W s_{f_{j'}} e_{i'}.$$ 

Thus $U s_{e_i} V s_{f_j} = V s_{f_{j'}} U s_{e_{i'}}$, which says that $\lambda_{(U,V)}$ preserves the commutation relations.

By the universal property of $\mathcal{O}_\theta$, the mapping $\lambda_{(U,V)}$ defines an endomorphism of $\mathcal{O}_\theta$.

Conversely, suppose $\lambda \in \text{End}(\mathcal{O}_\theta)$. Set

$$U = \sum_{i=1}^m \lambda(s_{e_i}) s_{e_i}^*, \quad V = \sum_{j=1}^n \lambda(s_{f_j}) s_{f_j}^*, \quad W = \sum_{i=1}^m \sum_{j=1}^n \lambda(s_{e_i} f_j) s_{e_i} f_j^*.$$ 

Then it is straightforward to verify that $U, V, W$ all are unitaries, and that

$$\lambda(s_{e_i}) = U s_{e_i}, \quad \lambda(s_{f_j}) = V s_{f_j}, \quad \lambda(s_{e_i} f_j) = W s_{e_i} f_j.$$
Furthermore, $U, V, W$ have the following relations:

\[ U^* W = \sum_{k=1}^{m} s_{e_k} \lambda(s_{e_k})^* \sum_{k,l} \lambda(s_{e_k f_l}) s_{e_k f_l}^* \]

\[ = \sum_{k=1}^{m} s_{e_k} \left( \sum_{l=1}^{n} \lambda(s_{f_l}) s_{f_l}^* \right) s_{e_k}^* \]

\[ = \sum_{k=1}^{m} s_{e_k} V s_{e_k}^* \]

\[ = \lambda_{e_1}(V). \]

Similarly,

\[ V^* W = \lambda_{e_2}(U). \]

Therefore, $(U, V)$ satisfies the twisted property required in the proposition: $U \lambda_{e_1}(V) = V \lambda_{e_2}(U) (= W)$. We proved $(U, V) \in U(O^2)_{\text{twi}}$.

It is easy to check that the above two processes are inverses of each other. This ends the proof.

Some remarks are in order.

**Remark 3.3.** Because of Theorem 3.2, we can and do use $\lambda(U, V)$ to denote the endomorphism of $O^2$ uniquely determined by the unitary pair $(U, V) \in U(O^2)_{\text{twi}}$. Using this notation, the canonical endomorphism $\lambda(p, q)$ is equal to $\lambda(U, V)$, where

\[ U = \sum_{i=1}^{m} \sum_{d(w)=(p,q)} s_{we_i} s_{e_i w}^*, \quad V = \sum_{j=1}^{n} \sum_{d(w)=(p,q)} s_{w f_j} s_{f_j w}^*. \]

**Remark 3.4.** From the proof of Theorem 3.2, we actually have $W = U \lambda_{e_1}(V) = V \lambda_{e_2}(U)$. So any two of the three unitaries $U, V, W$ completely determine the endomorphism $\lambda(U, V)$.

**Remark 3.5.** As we shall see later, the twisted property in Theorem 3.2 makes the study of $\text{End}(O^2)$ more interesting, and gives the essential difference between the studies of $\text{End}(O^2)$ and $\text{End}(O_n)$.

We should also admit that, in general, it is not easy to check if the twisted property in Theorem 3.2 holds for a given unitary pair. However, we do have a lot of examples. Before giving our examples, let us first consider the composition of two endomorphisms. We will see that $U(O^2)_{\text{twi}}$ is a unital semigroup with the multiplication induced from the composition. The following lemma is adapted from [9, Proposition 1.1] and the discussion immediately following.
Lemma 3.6. Endomorphisms of $O_\theta$ have the following properties.

(i) If $\lambda_{(U_i,V_i)} \in \text{End}(O_\theta)$ ($i = 1, 2$), then $\lambda_{(U_2,V_2)} \lambda_{(U_1,V_1)} = \lambda_{(U,V)}$, where

$$U = \lambda_{(U_2,V_2)}(U_1)U_2, \quad V = \lambda_{(U_2,V_2)}(V_1)V_2.$$ 

(ii) Suppose $\mathbb{F}_9^+$ is aperiodic and $\lambda_{(U,V)} \in \text{End}(O_\theta)$. Then

$\lambda_{(U,V)} \in \text{Aut}(O_\theta) \iff \lambda_{(U,V)}(U_0) = U^*, \quad \lambda_{(U,V)}(V_0) = V^*$

for some $U_0, V_0 \in \mathcal{U}(O_\theta)$.

(iii) The mapping

$$\varphi: \mathcal{U}(O_\theta) \to \text{Inn}(O_\theta), \quad W \mapsto \lambda_{(U,V)},$$

is surjective, where $(U,V) = (W\lambda_{e_1}(W)^*, W\lambda_{e_2}(W)^*)$. Moreover, $\ker \varphi \subseteq Z(O_\theta)$.

Proof. (i) For simplicity, put $\lambda_i := \lambda_{(U_i,V_i)}$ ($i = 1, 2$). Then

$$\lambda_2(\lambda_1(s_{e_i})) = \lambda_2(U_1s_{e_i}) = \lambda_2(U_1)\lambda_2(s_{e_i}) = \lambda_2(U_1)U_2s_{e_i} = U s_{e_i},$$

$$\lambda_2(\lambda_1(s_{f_j})) = \lambda_2(V_1s_{f_j}) = \lambda_2(V_1)\lambda_2(s_{f_j}) = \lambda_2(V_1)V_2s_{f_j} = V s_{f_j},$$

where $U = \lambda_2(U_1)U_2$, $V = \lambda_2(V_1)V_2$. Hence the composition endomorphism $\lambda_2\lambda_1$ is determined by $(U,V)$, i.e., $\lambda_2\lambda_1 = \lambda_{(U,V)}$.

(ii) It suffices to check the sufficiency. Since $\mathbb{F}_9^+$ is aperiodic, $O_\theta$ is simple. See, e.g., [13] (or Lemma 13). So $\lambda_{(U,V)}$ is injective. But if $\lambda_{(U,V)}(U_0) = U^*$ and $\lambda_{(U,V)}(V_0) = V^*$ for some $U_0, V_0 \in \mathcal{U}(O_\theta)$, then $\lambda_{(U,V)}(U_0s_{e_i}) = \lambda_{(U,V)}(U_0)\lambda_{(U,V)}(s_{e_i}) = U^*U s_{e_i} = s_{e_i}$, and, similarly, $\lambda_{(U,V)}(V_0s_{f_j}) = s_{f_j}$. This gives the surjectivity of $\lambda_{(U,V)}$. Hence $\lambda_{(U,V)}$ is an automorphism.

(iii) It is sufficient to show that $\lambda_{(U,V)} \in \text{Inn}(O_\theta)$ iff $U = W\lambda_{e_1}(W)^*$ and $V = W\lambda_{e_2}(W)^*$ for some $W \in \mathcal{U}(O_\theta)$. Clearly, $\lambda_{(U,V)} \in \text{Inn}(O_\theta)$ iff $\lambda_{(U,V)}(s_{e_i}) = W s_{e_i} W^*$ and $\lambda_{(U,V)}(s_{f_j}) = W s_{f_j} W^*$ ($i = 1, ..., m, j = 1, ..., n$) for some $W \in \mathcal{U}(O_\theta)$. But we also have $\lambda_{(U,V)}(s_{e_i}) = U s_{e_i}$ and $\lambda_{(U,V)}(s_{f_j}) = V s_{f_j}$. Thus

$$\lambda_{(U,V)} \in \text{Inn}(O_\theta) \iff \begin{cases} Us_{e_i} = W s_{e_i} W^*, & V s_{f_j} = W s_{f_j} W^* \text{ for some } W \in \mathcal{U}(O_\theta) \\ Us_{e_i}s_{e_i} = W s_{e_i} W^* s_{e_i}, & V s_{f_j} s_{f_j} = W s_{f_j} W^* s_{f_j} \\ \sum_{i=1}^m Us_{e_i}s_{e_i} = \sum_{i=1}^m W s_{e_i} W^* s_{e_i}, & \sum_{j=1}^n V s_{f_j} s_{f_j} = \sum_{j=1}^n W s_{f_j} W^* s_{f_j} \end{cases} \iff U = W\lambda_{e_1}(W^*), \quad V = W\lambda_{e_2}(W^*),$$

where $i = 1, ..., m, j = 1, ..., n$. (The property of $(U,V) \in \mathcal{U}(O_\theta)^2$ is automatic.) This ends the proof. □
By Theorem 3.2 and Lemma 3.6 (i), we immediately have

**Corollary 3.7.** Let \((U_1, V_1), (U_2, V_2) \in \mathcal{U}(\mathcal{O}_\theta)^2\). Then, under the multiplication
\[
(U_2, V_2) \cdot (U_1, V_1) := (\lambda_{(U_2, V_2)}(U_1)U_2, \lambda_{(U_2, V_2)}(V_1)V_2),
\]
the set \(\mathcal{U}(\mathcal{O}_\theta)^2\) is a unital semigroup, and the bijection \(\Psi\) in Theorem 3.2 is a unital semigroup isomorphism.

### 3.8. Examples
We now give some examples of non-canonical endomorphisms of \(\mathcal{O}_\theta\).

**Example 3.9.** Consider \(\mathbb{F}_\theta^+\) with \(m = n\) and \(\theta\) the flip relation: \(e_i f_j = f_i e_j\) for all \(i, j = 1, \ldots, m\). Let \(U\) be any unitary of \(\mathcal{O}_\theta\). Then \((U, U) \in \mathcal{U}(\mathcal{O}_\theta)^2\) and so it gives an endomorphism of \(\mathcal{O}_\theta\) by Theorem 3.2. Indeed,
\[
(U, U) \in \mathcal{U}(\mathcal{O}_\theta)^2 \iff U \lambda_{e_1}(U) = U \lambda_{e_2}(U) \iff \lambda_{e_1}(U) = \lambda_{e_2}(U)
\]
\[
\iff (\sum_{k=1}^m s_{ek} U s_{ek}^*) s_{ei} = (\sum_{l=1}^m s_{fl} U s_{fl}^*) s_{ei} (i = 1, \ldots, m)
\]
\[
\iff s_{f j}^* s_{ei} U = s_{f j}^* (\sum_{l=1}^m s_{fl} U s_{fl}^*) s_{ei} (i, j = 1, \ldots, m)
\]
\[
\iff s_{f j}^* s_{ei} U = U s_{f j}^* s_{ei} (i, j = 1, \ldots, m).
\]
Therefore, \((U, U) \in \mathcal{U}(\mathcal{O}_\theta)^2\) iff \(U\) commutes with \(s_{f j}^* s_{ei}\) (\(i, j = 1, \ldots, m\)).

But for flip algebras, we have \(s_{f j}^* s_{ei} = \delta_{ij} s_{f j}^* s_{ei}\), and \(s_{f j}^* s_{ei}\) is a unitary in \(\mathcal{Z}(\mathcal{O}_\theta)\) (cf. [13]). Here, as usual, \(\delta_{ij}\) denotes the Kronecker delta function. Hence we have \((U, U) \in \mathcal{U}(\mathcal{O}_\theta)^2\) for any \(U \in \mathcal{U}(\mathcal{O}_\theta)\).

Furthermore, one can check that if \(U = W \lambda_{e_1}(W)^*\) for some \(W \in \mathcal{U}(\mathcal{O}_\theta)\) then \(\lambda_{(U, U)}\) is an inner automorphism, as \(W \lambda_{e_1}(W)^* = W \lambda_{e_2}(W)^*\) in this case.

**Example 3.10.** Let \(\mathbb{F}_\theta^+\) be a 2-graph. Let \(U, V \in \mathcal{U}(\mathcal{O}_\theta)\). If
\[
UV = VU, \ U s_{f j} = s_{f j} U, \ V s_{e i} = s_{e i} V (i = 1, \ldots, m, \ j = 1, \ldots, n),
\]
then one can easily check that \((U, V) \in \mathcal{U}(\mathcal{O}_\theta)^2\) and so \((U, V)\) determines an endomorphism. In particular, \((U, V)\) with \(U, V \in \mathcal{Z}(\mathcal{O}_\theta)\) gives an endomorphism of \(\mathcal{O}_\theta\).

**Example 3.11.** Consider \(\mathbb{F}_\theta^+\) with the identity relation \(\theta\): \(e_i f_j = f_j e_i\) for all \(i = 1, \ldots, m, \ j = 1, \ldots, n\). Then we have \(\mathcal{O}_\theta \cong \mathcal{O}_m \otimes \mathcal{O}_n\) by [21 Corollary 3.5 (iv)]. Let \(U\) (resp. \(V\)) be unitaries in the Cuntz algebras \(\mathcal{C}^*(s_{e_1}, \ldots, s_{e_m}) \cong \mathcal{O}_m\) (resp. \(\mathcal{C}^*(s_{f_1}, \ldots, s_{f_n}) \cong \mathcal{O}_n\)). Then, from Example 3.10, \((U, V)\) gives an endomorphism of \(\mathcal{O}_\theta\). Furthermore,
if \( U \) (resp. \( V \)) determines an automorphism of \( \mathcal{O}_m \) (resp. \( \mathcal{O}_n \)), then \( \lambda_{(U,V)} \in \text{Aut}(\mathcal{O}_\theta) \) by Lemma 3.6 (ii). Moreover, it is easy to see that \( \lambda_{(U,V)} = \lambda_U \otimes \lambda_V \) by considering the actions on generators, where \( \lambda_U \) (resp. \( \lambda_V \)) are endomorphisms of \( \mathcal{O}_m \) (resp. \( \mathcal{O}_n \)) as defined in [9].

In [1], it was proved that the “flip-flop” unitary \( U = s_{e_2}s_{e_1}^* + s_{e_1}s_{e_2}^* \) gives an outer automorphism of \( C^* (s_{e_1}, s_{e_2}) \cong \mathcal{O}_2 \). In the following example, we give an outer automorphism of \( \mathcal{O}_\theta \) by using two “mixing” flip-flop unitaries.

**Example 3.12.** Suppose \( m = n \) and \( \theta \) is the flip relation. Let

\[
U = \sum_{j=1}^{m} s_{sf_j}s_{e_j}^* = s_{e_i}s_{f_i}.
\]

Notice that \( U \in \mathcal{Z}(\mathcal{O}_\theta) \) (cf. [13]). From Example 3.10 \((U, U^*)\) gives an endomorphism \( \lambda_{(U,U^*)} \). A simple calculation shows that \( \lambda_{(U,U^*)} \) is an involution: \( \lambda_{(U,U^*)}^2 = \text{id} \). In particular, \( \lambda_{(U,U^*)} \) is an automorphism.

In what follows, we show that \( \lambda_{(U,U^*)} \) is actually outer. To the contrary, suppose that \( \lambda_{(U,U^*)} = \text{Ad} W \) for some \( W \in \mathcal{U}(\mathcal{O}_\theta) \). Then

\[
\lambda(s_{e_i}) = Us_{e_i} = Ws_{e_i}W^* = s_{f_i},
\]

\[
\lambda(s_{f_i}) = U^*s_{f_i}W = Ws_{f_i}W^* = s_{e_i}
\]

for all \( i = 1, \ldots, m \). Therefore

\[
W s_{e_i} W^* = s_{f_i} W = W^* s_{e_i} W, \quad W s_{f_i} W^* = s_{e_i} W = W^* s_{f_i} W.
\]

We have \( W^2 s_{e_i} = s_{e_i} W^2 \) and \( W^2 s_{f_i} = s_{f_i} W^2 \) for \( i = 1, \ldots, m \). So \( W^2 \in \mathcal{Z}(\mathcal{O}_\theta) \). Also, from above we have \( U = W \lambda_e(W^*) \). This implies \( U^* W = \lambda_{e_i}(W) \) and \( W U = W^2 \lambda_e(W^*) = \lambda_{e_i}(W) \) as \( W^2 \in \mathcal{Z}(\mathcal{O}_\theta) \).

So \( U^* W = W U \). Since \( U \in \mathcal{Z}(\mathcal{O}_\theta) \), we have \( W U = U^* W \). Namely, \( U = U^* \). This is ridiculous. (Indeed, \( U = U^* \Rightarrow s_{e_i}^* s_{f_i} s_{f_j} = s_{f_j}^* s_{e_i} s_{f_i} \Rightarrow s_{e_i}^* s_{f_i} f_j = s_{e_j} \Rightarrow s_{e_i}^* s_{f_i} s_{f_j} = I \Rightarrow s_{e_i e_j} = s_{f_i f_j} \), a contradiction.) Therefore \( \lambda_{(U,V)} \) is outer.

By [12, 13], we have \( \mathcal{O}_\theta \cong C(\mathbb{T}) \otimes \mathcal{O}_m \). From this point of view, one can see that the above \( \lambda_{(U,V)} \cong \varphi \otimes \text{id} \), where \( \varphi \in \text{Aut}(C(\mathbb{T})) \) given by \( \varphi(z) = \bar{z} \) (the conjugate of \( z \)), and \( \text{id} \) is the trivial automorphism of \( \mathcal{O}_m \).

Curiously, if we replace \((U, V)\) in Example 3.12 to \((U, V) = (s_{e_2}s_{e_1}^* + s_{e_1}s_{e_2}^* + s_{f_2}s_{f_1}^* + s_{f_1}s_{f_2}^*)\), it is rather easy to see that \( \lambda_{(U,V)} \) is also an automorphism. But we do not know if it is outer in this case. The essential difference is that \( U \) is not in \( \mathcal{Z}(\mathcal{O}_\theta) \) any more.

It follows from Example 3.10 that each pair \((U^k, U^{*k})\) with \( k \geq 1 \) indeed gives an endomorphism \( \lambda_k := \lambda_{(U^k, U^{*k})} \). It is not hard to see
that $\lambda_k$ with $k \geq 2$ is not an automorphism, and that $\lambda_k|_{\mathfrak{F}} = \text{id}$. Since we will not use this fact later, we omit the details here.

**Example 3.13.** Let $m = n$ and $\theta$ be the identity relation. Let $U = \sum_{j=1}^m s_j s_j^*$. One can easily check that $(U, U^*) \in \mathcal{U}(\mathcal{O}_\theta)^2_{tw}$, and so it determines an automorphism $\lambda_{(U, U^*)}$. But, unlike Example 3.12, we do not know if $\lambda_{(U, U^*)}$ is outer in this case. Using the same argument as in Example 3.12, we still have $W^2 \in \mathcal{Z}(\mathcal{O}_\theta)$, and so $W^2$ is a scalar as $\mathcal{O}_\theta$ is now simple [13]. However, the issue here is that $U$ is not in $\mathcal{Z}(\mathcal{O}_\theta)$ any more.

4. **Endomorphisms preserving some Subalgebras**

In this section, we study endomorphisms of $\mathcal{O}_\theta$ which preserve the fixed point algebra $\mathfrak{F}$ of the gauge automorphisms $\gamma_t$ ($t \in \mathbb{T}^2$) and its diagonal subalgebra $\mathfrak{D}$. Recall from Section 2 that $\mathfrak{D}$ is the canonical masa of $\mathfrak{F}$. When $\mathbb{F}_t^+$ is aperiodic, we actually have more: $\mathfrak{D}$ is also a masa in $\mathcal{O}_\theta$, and $\mathfrak{F}$ has the trivial relative commutant; moreover, the converse is also true. We should mention that it is well-known that these hold true for Cuntz algebras $\mathcal{O}_n$ (see, e.g., [9, Section 1] and [15, Corollary 3.3]).

**Lemma 4.1.** The following statements are equivalent.

(i) $\mathbb{F}_t^+$ is aperiodic.

(ii) $\mathcal{O}_\theta$ is simple.

(iii) $\mathfrak{D}' \cap \mathcal{O}_\theta = \mathfrak{D}$.

(iv) $\mathfrak{F}' \cap \mathcal{O}_\theta = C I$.

**Proof.** By [13] or [31], (i) and (ii) are equivalent.

That (i) is equivalent to (iii) is the main result of [17]. (Notice that $\mathfrak{D}$ here is the same as $\mathfrak{D}$ in [17] in terms of groupoid terminology.)

(iii)⇒(iv): This is directly from the simple fact that the relative commutant of an algebra is contained in that of a subalgebra.

(iv)⇒(i): If $\mathbb{F}_t^+$ is periodic, by a result in [13], we have $C(\mathbb{T}) \subseteq \mathcal{Z}(\mathcal{O}_\theta) \subseteq \mathfrak{F}' \cap \mathcal{O}_\theta$ and so $\mathfrak{F}' \cap \mathcal{O}_\theta \neq CI$.

Because the proofs in the rest of this subsection need the property that $\mathfrak{D}$ is a masa in $\mathcal{O}_\theta$, we assume that $\mathbb{F}_t^+$ is aperiodic by Lemma 4.1.

The proofs of Propositions 4.2 and 4.3 below are adapted from those of [9, Proposition 1.2] and [9, Propositions 1.3–1.5], respectively. We sketch the proofs here, and only give the full details which are different from there.

**Proposition 4.2.** Suppose that $\mathbb{F}_t^+$ is aperiodic. Let $\lambda_{(U,V)} \in \text{End}(\mathcal{O}_\theta)$ and $W := U\lambda_{e_1}(V)(= V\lambda_{e_2}(U))$. Then we have the following.
This ends the proof of our claim.

(i) \( W \in \mathfrak{F} \Rightarrow \lambda_{(U,V)}(\mathfrak{F}) \subseteq \mathfrak{F} \). Moreover, \( \lambda_{(U,V)}(\mathfrak{F}) \subseteq \mathfrak{F} \Rightarrow W \in \mathfrak{F} \) provided that \( \lambda_{(U,V)}(\mathfrak{F})' \cap O_\theta = \mathbb{C}I \).

(ii) If \( U, V \in \mathfrak{F} \), then \( \lambda_{(U,V)} \in \text{Aut}(O_\theta) \iff \lambda_{(U,V)}(\mathfrak{F}) = \mathfrak{F} \).

**Proof.** (i) We first claim that

\[
\lambda_{(U,V)}(X) = \lim_{k \to \infty} \text{Ad}(W \lambda_{(1,1)}(W) \cdots \lambda_{(1,1)}^k(W))(X) \quad \text{for all} \quad X \in \mathfrak{F}. \quad (1)
\]

It suffices to check (1) for the generators \( X \) of \( \mathfrak{F} \). But if \( X_1 \in \mathfrak{F}_1 \), then \( X_1 \) can be written as \( X_1 = s_{i_1} s_{i_1}^* \) with \( d(u_1) = d(v_1) = (1, 1) \). Recall that \( \lambda_{(U,V)}(s_w) = W s_w \) for all \( w \in \mathbb{F}_\theta^+ \) with \( d(w) = (1, 1) \). So we have \( \lambda_{(U,V)}(X_1) = \text{Ad}(W)(X_1) \). Now one can prove inductively

\[
\lambda_{(U,V)}(X_k) = \text{Ad}(W \lambda_{(1,1)}(W) \cdots \lambda_{(1,1)}^{k-1}(W))(X_k) \quad \text{for all} \quad X_k \in \mathfrak{F}_k.
\]

To this end, first notice that simple calculations yield the following relations: For all \( \lambda \in \mathbb{F}_\theta^+ \) with \( d(\lambda) = (1, 1) \),

\[
s_w \lambda_{(1,1)}^{k-1}(W) = \lambda_{(1,1)}^{k}(W)s_w \quad (i = 1, \ldots, m, j = 1, \ldots, n, k \geq 1).
\]

Let \( X_k = s_{u_1} \cdots s_{v_k} \in \mathfrak{F}_k, \) where \( d(u_i) = d(v_i) = (1, 1) \) \((i = 1, \ldots, k)\). Put \( X_{k-1} = s_{u_2} \cdots s_{v_k} \). Obviously, \( X_{k-1} \in \mathfrak{F}_{k-1} \). Then use the inductive assumption and the relations given above to obtain

\[
\lambda_{(U,V)}(X_k) = \lambda_{(U,V)}(s_{u_1} X_{k-1} s_{v_1}^*)
\]

\[
= Ws_{u_1} \text{Ad}(W \lambda_{(1,1)}(W) \cdots \lambda_{(1,1)}^{k-2}(W))(X_{k-1}) s_{v_1}^* W^* 
\]

\[
= W \cdot s_{u_1} W \cdot \lambda_{(1,1)}(W) \cdots \lambda_{(1,1)}^{k-2}(W) X_{k-1} 
\]

\[
\lambda_{(1,1)}^{k-2}(W)^* \cdots \lambda_{(1,1)}(W)^* \cdot W^* s_{v_1}^* \cdot W^* 
\]

\[
= W \cdot \lambda_{(1,1)}(W) s_{u_1} \cdot \lambda_{(1,1)}(W) \cdots \lambda_{(1,1)}^{k-2}(W) X_{k-1} 
\]

\[
\lambda_{(1,1)}^{k-2}(W)^* \cdots \lambda_{(1,1)}(W)^* \cdot s_{v_1}^* \lambda_{(1,1)}(W)^* \cdot W^* 
\]

\[
= \cdots 
\]

\[
= W \lambda_{(1,1)}(W) \cdots \lambda_{(1,1)}^{k-1}(W) s_{u_1} X_{k-1} s_{v_1}^* \lambda_{(1,1)}^{k-1}(W)^* \cdots \lambda_{(1,1)}(W)^* W^* 
\]

\[
= \text{Ad}(W \lambda_{(1,1)}(W) \cdots \lambda_{(1,1)}^{k-1}(W))(X_k).
\]

This ends the proof of our claim.

That \( W \in \mathfrak{F} \Rightarrow \lambda_{(U,V)}(\mathfrak{F}) \) is directly derived from (1).

We now assume that \( \lambda_{(U,V)}(\mathfrak{F})' \cap O_\theta = \mathbb{C}I \). It is easy to check

\[
\gamma_t \lambda_{(U,V)} \gamma_t^{-1} = \lambda_{(\gamma_t(\gamma_t(U)), \gamma_t(V))}. \quad (2)
\]

Since \( \lambda_{(U,V)}(\mathfrak{F}) \subseteq \mathfrak{F} \) and \( \gamma_t(X) = X \) for all \( X \in \mathfrak{F} \), then (2) yields

\[
\lambda_{(U,V)}(X) = \lambda_{(\gamma_t(U), \gamma_t(V))}(X) \quad \text{for all} \quad X \in \mathfrak{F}. \quad (3)
\]
Direct calculations give
\[
\gamma_t(U)\lambda_{v_1}(\gamma_t(V)) = \gamma_t(U\lambda_{v_1}(V)) = \gamma_t(W).
\] (4)

We now claim \(W^*\gamma_t(W) \in \lambda_{(U,V)}(\mathfrak{F})' \cap \mathcal{O}_\theta\). Indeed, if \(X_1 = s_{u_1}s_{v_1}^*\) with \(d(u_1) = d(v_1) = (1,1)\), from (3) and (4) we derive \(WX_1W^* = \gamma_t(W)X_1\gamma_t(W)^*\). Thus
\[
W^*\gamma_t(W)\tilde{X}_1 = \tilde{X}_1W^*\gamma_t(W), \quad \text{where} \quad \tilde{X}_1 := X_1.
\]
Then putting \(X_2 = s_{u_1u_2}s_{v_1v_2}^*\) to (3) and using (4), we have
\[
W^*\gamma_t(W)\tilde{X}_2 = \tilde{X}_2W^*\gamma_t(W),
\]
where \(\tilde{X}_2 := s_{u_1}W_{u_2}s_{v_2}^*W_{v_1}^*\). Set
\[
\tilde{X}_n := s_{u_1}W_{u_2}\cdots W_{u_n}s_{v_n}^*W_{v_2}\cdots W_{v_1}^*s_{v_1}^*.
\]
Here the degrees of all above \(u_i\)'s and \(v_i\)'s are \((1,1)\). By induction, we can obtain
\[
W^*\gamma_t(W)\tilde{X}_n = \tilde{X}_nW^*\gamma_t(W).
\]
Furthermore, from the above process one also has \(\tilde{X}_n \in \mathfrak{F}\) as \(\lambda_{(U,V)}(\mathfrak{F})' \subseteq \mathfrak{F}\). Therefore, we have \(W^*\gamma_t(W)\) commutes with every element in
\[
\tilde{\mathfrak{F}} = \bigcup_{n \geq 1} \text{span}\{\tilde{X}_n\} = \text{span}\{s_u\lambda_{(U,V)}(\mathfrak{F})s_u^* : d(u) = d(v) = (1,1)\}.
\]

Now one can check that \(\gamma_t(W)W^* \in \lambda_{(U,V)}(\mathfrak{F})' \cap \mathcal{O}_\theta\). This ends the proof of our claim.

Therefore, from our assumption, we obtain \(\lambda_t(W) = \alpha W\) for some \(\alpha \in \mathbb{T}\). This forces \(W \in \mathfrak{F}\). We are done.

(ii) Let \(U, V \in \mathfrak{F}\). Suppose \(\lambda_{(U,V)} \in \text{Aut}(\mathcal{O}_\theta)\). Obviously, \(W \in \mathfrak{F}\). This implies \(\lambda_{(U,V)}(\mathfrak{F}) \subseteq \mathfrak{F}\) from (1). Also, as \(\gamma_t(U) = U\) and \(\gamma_t(V) = V\), the identity (2) yields \(\lambda_{(U,V)}^{-1}\gamma_t = \gamma_t\lambda_{(U,V)}^{-1}\). Thus, \(\lambda_{(U,V)}(\mathfrak{F}) \subseteq \mathfrak{F}\). Therefore, \(\lambda_{(U,V)}(\mathfrak{F}) = \mathfrak{F}\).

For the other direction, since \(\mathbb{F}_\theta^+\) is aperiodic, \(\mathcal{O}_\theta\) is simple by Lemma 4.1. So \(\lambda_{(U,V)}\) is injective. Since \(U^*, V^* \in \mathfrak{F}\) and \(\lambda_{(U,V)}(\mathfrak{F}) = \mathfrak{F}\), there are \(U_0, V_0 \in \mathfrak{F}\) such that \(\lambda_{(U,V)}(U_0) = U^*, \lambda_{(U,V)}(V_0) = V^*\). This implies that \(\lambda_{(U,V)}\) is surjective as in the proof of Lemma 3.6 (ii). So \(\lambda_{(U,V)}\) is an automorphism.

Clearly, by Remark 3.3 all canonical endomorphisms \(\lambda_{(p,q)}\) satisfy the corresponding unitary pairs \((U, V) \in \mathfrak{F} \times \mathfrak{F}\). So \(\lambda_{(p,q)}(\mathfrak{F}) \subseteq \mathfrak{F}\), and \(\lambda_{(p,q)} \in \text{Aut}(\mathcal{O}_\theta)\) if and only if \(\lambda_{(p,q)}(\mathfrak{F}) = \mathfrak{F}\).

Before stating the following result, we recall that \(N(\mathfrak{D})\) is the unitary normalizer of \(\mathfrak{D}\).
Proposition 4.3. Suppose that $\mathbb{F}_\theta^+$ is aperiodic and $\lambda_{(U,V)} \in \text{End}(\mathcal{O}_\theta)$. Let $W := U\lambda_1(V)(= V\lambda_2(U))$. Then the following hold true.

(i) The fixed point algebra for $\{\lambda_{(U,V)} : U, V \in \mathcal{U}(\mathcal{D})\}$ is $\mathcal{D}$.

(ii) $\{\lambda_{(U,V)} : U, V \in \mathcal{U}(\mathcal{D})\} = \{\lambda \in \text{Aut}(\mathcal{O}_\theta) : \lambda|_{\mathcal{D}} = \text{id}\}$.

(iii) $\lambda_{(U,V)}(\mathcal{D}) \subseteq \mathcal{D} \iff W \in N(\mathcal{D})$.

If, in addition, $\lambda_{(U,V)} \in \text{Aut}(\mathcal{O}_\theta)$, then

$$\lambda_{(U,V)}(\mathcal{D}) = \mathcal{D} \iff W \in N(\mathcal{D}).$$

(iv) If $U, V \in \mathcal{D}$, then $\lambda_{(U,V)} \in \text{Aut}(\mathcal{O}_\theta) \iff \lambda_{(U,V)}(\mathcal{D}) = \mathcal{D}$.

**Proof.** (i) To simply our writing, we use $\text{Fix}$ to denote the fixed point algebra of $\{\lambda_{(U,V)} : U, V \in \mathcal{D}\}$.

In order to check $\mathcal{D} \subseteq \text{Fix}$, it is sufficient to check that all generators of $\mathcal{D}$ are in $\text{Fix}$. Represent a generator $X$ of $\mathcal{D}$ as

$$X = s_{u_1} \cdots s_{u_k} s_{u_1}^* \cdots s_{u_k}^*,$$

where $u_i \in \mathbb{F}_\theta^+$ with $d(u_i) = (1, 1)$ ($i = 1, \ldots, k$). Then, since both $W$ and $s_{u_1} \cdots s_{u_k} s_{u_1}^* \cdots s_{u_k}^*$ ($1 \leq \ell \leq k$) are in $\mathcal{D}$, they commute. Hence we have

$$\lambda_{(U,V)}(X) = \lambda(s_{u_1}) \cdots \lambda(s_{u_k}) \lambda(s_{u_k})^* \cdots \lambda(s_{u_1})^*$$

$$= W s_{u_1} \cdots W s_{u_k} s_{u_1}^* \cdots s_{u_k}^* W^* s_{u_1}^* \cdots s_{u_k}^* W^*$$

$$= W s_{u_1} \cdots s_{u_k} s_{u_1}^* \cdots s_{u_k}^* W^*$$

$$= \cdots$$

$$= X.$$

Thus $\mathcal{D} \subseteq \text{Fix}$.

We now prove $\text{Fix} \subseteq \mathcal{D}$. For any $D \in \mathcal{U}(\mathcal{D})$, by Lemma 3.6 we have $\text{Ad}(D) \in \{\lambda_{(U,V)} : U, V \in \mathcal{U}(\mathcal{D})\}$. So if $X \in \text{Fix}$, then $\text{Ad}(D)(X) = X$. That is, $D X D^* = X$ for all $D \in \mathcal{U}(\mathcal{D})$. So we have $X \in \mathcal{D}$ as $\mathcal{D}$ is a masa of $\mathcal{O}_\theta$ by Lemma 4.1. This takes care of (i).

(ii) Assume that $\lambda_{(U,V)}|_{\mathcal{D}} = \text{id}$. For any $D \in \mathcal{D}$, we have $s_{e_i} D s_{e_i}^* = \lambda_{(U,V)}(s_{e_i}) D s_{e_i}^* = U s_{e_i} D s_{e_i}^* U^*$ ($i = 1, \ldots, m$). So $U \in \mathcal{D}$. Similarly, we have $V \in \mathcal{D}$. Hence

$$\{\lambda_{(U,V)} : U, V \in \mathcal{U}(\mathcal{D})\} \supseteq \{\lambda \in \text{End}(\mathcal{O}_\theta) : \lambda|_{\mathcal{D}} = \text{id}\}.$$

The inclusion $\subseteq$ is directly from (i). This proves (ii). Clearly, by Lemma 3.6 (ii) every element in the above set is an automorphism.

(iii) Set $\lambda = \lambda_{(U,V)}$. Assume first $W \in N(\mathcal{D})$. From (1) we have $\lambda(\mathcal{D}) \subseteq \mathcal{D}$. Now suppose $\lambda(\mathcal{D}) \subseteq \mathcal{D}$. For $1 \leq i \leq m, 1 \leq j \leq n$, we have $\mathcal{D} \supseteq \lambda(s_{e_i f_j} \mathcal{D} s_{e_i f_j}^*) = W s_{e_i f_j} \mathcal{D} s_{e_i f_j}^* W^*$. Thus $W \in N(\mathcal{D})$.

To finish the proof of (iii), it suffices to show that if $\lambda \in \text{Aut}(\mathcal{O}_\theta)$, then $\lambda(\mathcal{D}) \subseteq \mathcal{D}$ actually implies $\lambda(\mathcal{D}) = \mathcal{D}$. In the sequel, we show
\[ \lambda^{-1}(\mathcal{D}) \subseteq \mathcal{D}. \] Arbitrarily choose \( D_1, D_2 \in \mathcal{D} \). As \( \lambda(\mathcal{D}) \subseteq \mathcal{D} \), we have
\[ \lambda(\lambda^{-1}(D_1)D_2) = D_1\lambda(D_2) = \lambda(D_2)D_1 = \lambda(D_2\lambda^{-1}(D_1)). \]

As \( \lambda \) is an automorphism, we have \( \lambda^{-1}(D_1)D_2 = D_2\lambda^{-1}(D_1) \). This implies \( \lambda^{-1}(D_1) \in \mathcal{D}' \). As \( \mathcal{D} \) is a masa in \( \mathcal{O}_\theta \) by Lemma 4.1, it follows that \( \lambda^{-1}(D_1) \in \mathcal{D} \). Therefore, \( \lambda^{-1}(\mathcal{D}) \subseteq \mathcal{D} \) from the arbitrariness of \( D_1 \).

(iv) Clearly \( W \in N(\mathcal{D}) \) as \( U, V \in \mathcal{D} \). From (iii), \( \lambda(U,V)(\mathcal{D}) \subseteq \mathcal{D} \).

As a consequence of Proposition 4.3 (i) and Lemma 3.6, we get

**Corollary 4.4.** Suppose \( \mathbb{T}_\theta^+ \) is aperiodic. Then \( \{\lambda(U,V) \in \text{End}(\mathcal{O}_\theta) : U, V \in \mathcal{U}(\mathcal{D})\} \) is a maximal abelian subgroup of \( \text{Aut}(\mathcal{O}_\theta) \).

**Proof.** By Proposition 4.3 (ii), every element in \( \{\lambda(U,V) \in \text{End}(\mathcal{O}_\theta) : U, V \in \mathcal{U}(\mathcal{D})\} \) is an automorphism. It now suffices to notice that, for \( D \in \mathcal{U}(\mathcal{D}) \), the identity \( \lambda(U,V) \text{Ad}(D) = \text{Ad}(D)\lambda(U,V) \) implies that \( D^*\lambda(U,V)(D) \in Z(\mathcal{O}_\theta) = \mathbb{C}I \).}

**4.5. Unitarily implemented automorphisms.** In [36], Voiculescu constructed a family of unitarily implemented automorphisms of the Cuntz algebras \( \mathcal{O}_n \) from a subgroup \( \mathcal{U}(n,1) \) of the general linear group \( GL_n(\mathbb{C}) \). This result plays a very important role in many places. See, e.g., [10, 24]. Our original main purpose was to “naturally” generalize this result to 2-graph algebras. But, as we have mentioned in Section 3, for a given pair of unitaries \( (U, V) \) of \( \mathcal{O}_\theta \), in practice, it is hard to check if \( (U, V) \) determines an endomorphism because of the twisted property. So to know if it gives an automorphism becomes a much more challenging task. Thus, in this direction, so far we are only able to generalize the above result in the case of \( \theta = \text{id} \).

In order to state our results, we first need some notation. Following [3], let \( J = \begin{bmatrix} -1 & 0 \\ 0 & I_n \end{bmatrix} \) and
\[ \mathcal{U}(n,1) = \left\{ A = \begin{bmatrix} a_0 & h_1^* \\ h_2 & A_1 \end{bmatrix} \in GL_{n+1}(\mathbb{C}) : A^*JA = J \right\}. \]
Here $a_0 \in \mathbb{C}$, $A_1$ is an $n \times n$ matrix, and $h_1, h_2$ are column vectors in $\mathbb{C}^n$. It is well-known that for each $A \in \mathcal{U}(n, 1)$, there is a unitary $U_A \in \mathcal{O}_n$ determined by $A$, whose formula can be found in [3, 36].

We are now ready to give a family of unitarily implemented automorphisms of $\mathcal{O}_{id}$ constructed from $\mathcal{U}(m, 1) \times \mathcal{U}(n, 1)$.

**Proposition 4.6.** Every pair $(A, B) \in \mathcal{U}(m, 1) \times \mathcal{U}(n, 1)$ determines a unitarily implemented automorphism of $\mathcal{O}_{id}$. Furthermore, there is an action $\alpha$ of $\mathcal{U}(m, 1) \times \mathcal{U}(n, 1)$ on $\mathcal{O}_{id}$ given by

$$\alpha(A, B)(s_{e_i}) = U_A s_{e_i}, \quad \alpha(A, B)(s_{f_j}) = V_B s_{f_j}$$

for all $1 \leq i \leq m, 1 \leq j \leq n$. Here, $U_A$ (resp. $V_B$) are unitaries determined by $A$ (resp. $B$) in the Cuntz algebras $\mathcal{O}_m = \mathcal{C}^*(s_{e_1}, \ldots, s_{e_m})$ (resp. $\mathcal{O}_n = \mathcal{C}^*(s_{f_1}, \ldots, s_{f_n})$).

**Proof.** Let $(A, B) \in \mathcal{U}(m, 1) \times \mathcal{U}(n, 1)$. As in [9], let $\lambda_{U_A} \in \text{End}(\mathcal{O}_m)$ and $\lambda_{V_B} \in \text{End}(\mathcal{O}_n)$ denote the endomorphisms determined by $U_A$ and $V_B$, respectively. Then, from [36, 2.9], $\lambda_{U_A}$ and $\lambda_{V_B}$ are actually unitarily implemented automorphisms of $\mathcal{O}_m$ and $\mathcal{O}_n$, respectively. Since $\theta = \text{id}$, from Example 3.11, we have $\lambda_{(U_A, V_B)} \in \text{Aut}(\mathcal{O}_{id})$ and $\lambda_{(U_A, V_B)} = \lambda_{U_A} \otimes \lambda_{V_B}$. Therefore $\lambda_{(U_A, V_B)}$ is also unitarily implemented.

The mapping $\alpha$ given in the proposition is an action because simple calculations yield

$$U_{A_2 A_1} = \alpha_{(A_2, B_2)}(U_{A_1})U_{A_2}, \quad V_{B_2 B_1} = \alpha_{(A_2, B_2)}(V_{B_1})V_{B_2}$$

for all $A_1, A_2 \in \mathcal{U}(m, 1)$ and $B_1, B_2 \in \mathcal{U}(n, 1)$. \hfill \Box

5. MODULAR THEORY OF 2-GRAF ALGEBRAS

Recall that $\Phi = \int_{\mathbb{T}^2} \gamma_t dt$ is the faithful conditional expectation of $\mathcal{O}_\theta$ onto the $(mn)^\infty$-UHF algebra $\mathfrak{F}$. Let $\tau$ be the unique faithful normalized trace on $\mathfrak{F}$. Define $\omega = \tau \Phi$. Then $\omega$ is a faithful state on $\mathcal{O}_\theta$. Also notice that $\omega \gamma_t = \omega$ ($t \in \mathbb{T}^2$), i.e., $\omega$ is invariant under the gauge automorphisms $\gamma_t$ of $\mathcal{O}_\theta$.

Let $L^2(\mathcal{O}_\theta)$ be the GNS Hilbert space determined by the state $\omega$. So the inner product on $\mathcal{O}_\theta$ is defined by $\langle A|B \rangle = \omega(A^*B)$ for all $A, B \in \mathcal{O}_\theta$. (Notice that the inner product here is linear in the second variable.) Let $A \in \mathcal{O}_\theta$ and denote the left action of $A$ by $\pi(A)$, that is, $\pi(A)B = AB$ for all $B \in \mathcal{O}_\theta$. Let $\mathcal{O}_{\theta_{\text{gen}}}$ denote the algebra as the finite linear span of the generators of $\mathcal{O}_\theta$: $\mathcal{O}_{\theta_{\text{gen}}} = \text{span}\{s_u s_v^* : u, v \in \mathbb{F}_\theta^+\}$.

For brevity, in what follows, let $\mathbf{n} := (m, n)$, where $m, n$ are the numbers of generators of $\mathbb{F}_\theta^+$ of degree $(1, 0)$ and $(0, 1)$, respectively.

\footnote{This proof is due to the referee. It is easier and shorter than the original one.}
The first lemma below gives some identities on the tracial state $\tau$ on $\mathfrak{F}$ and generalizes \cite[Lemma 3.1]{4}.

**Lemma 5.1.** Suppose $u, v \in \mathbb{F}_\theta^+$ with $d(u) = d(v)$. Then

$$\tau(s_u X s_u^*) = \delta_{u,v} n^{-d(u)} \tau(X) \quad \text{for all} \quad X \in \mathfrak{F}.$$  

Here, as usual, $\delta_{u,v} = 1$ if $u = v$; $0$, otherwise.

**Proof.** Let $X \in \mathfrak{F}$. Since $d(u) = d(v)$, we have

$$\tau(s_u X s_v^*) = \tau(s_u X s_u^* s_v s_v^*) = \tau(s_v^* s_u X s_u^*) = \delta_{u,v} \tau(s_u X s_u^*). \quad (5)$$

In particular, we have

$$\tau(s_u s_v^*) = \delta_{u,v} \tau(s_u s_v^*).$$

Making use of the defect free property, we have

$$\sum_{d(u) = d(v)} \tau(s_u s_u^*) = \tau \left( \sum_{d(u) = d(v)} s_u s_u^* \right) = \tau(I) = 1.$$  

Since there are only $n^{d(v)}$ such $u$’s with $d(u) = d(v)$, we obtain

$$\tau(s_u s_u^*) = n^{-d(u)}.$$

Therefore

$$\tau(s_u s_v^*) = \delta_{u,v} n^{-d(u)} . \quad (6)$$

Now we claim that for any $(p, q) \in \mathbb{Z}_+^2$, we have

$$\tau(X) = \sum_{d(v) = (p,q)} \tau(s_v X s_v^*) \quad \text{for all} \quad X \in \mathfrak{F}.$$  

To this end, it suffices to check it for the generators $X = s_{w_1} s_{w_2}^*$ of $\mathfrak{F}$. But then from (6) it follows that

$$\sum_{d(v) = (p,q)} \tau(s_v X s_v^*) = \sum_{v} \tau(s_v s_{w_1} s_{w_2}^* s_v^*)$$

$$= \sum_{v} \delta_{w_1,w_2} n^{-d(w_1)-d(v)} \quad \text{(by (6))}$$

$$= \delta_{w_1,w_2} n^{-d(w_1)}$$

$$= \tau(X) \quad \text{(by (6))}.$$
On the other hand, for any $X \in \mathcal{F}$, similar to the proof of (5) we have
\[
\tau(s_uXs_u^*) = \tau(s_uXss_v^*s_us_u^*) \\
= \tau(ss_v^*s_us_u^*Xs_v^*) \\
= \tau(s_us_v^*).
\]
Thus
\[
\tau(s_uXs_u^*) = n^{-d(u)}\tau(X) \quad \text{for all } X \in \mathcal{F}.
\]
Combining this identity with (5) completes the proof of the lemma.

We now begin to give the modular objects in the celebrated Tomita-Takesaki modular theory. Define an operator $S$ on $O_{\theta c} \subset L^2(O_\theta)$ by
\[
S(A) = A^* \quad \text{for all } A \in O_{\theta c}.
\]
Clearly, $S$ is anti-linear. Define another anti-linear operator $F$ on $O_{\theta c}$, which acts on generators by
\[
F(s_us_v^*) = n^{d(u)-d(v)}sv^*s_{u},
\]
and then extend it anti-linearly.

We shall show that $F$ is indeed the adjoint of $S$. The key step in its proof is to make full use of the close relations between $\omega$ and $S$, $F$. The basic idea behind here is to convert the computations involved with $\langle S(A)|B \rangle$ and $\langle F(B)|A \rangle$ to those related to $\omega$. But then $\omega|\mathcal{F} = \tau$ is a trace, and so we can invoke the commutativity of $\tau$ and apply Lemma 5.1. If we use this approach to Cuntz algebras, it seems that the proof here is more unified than that in [4]. More importantly, if one applies the approach in [4] directly, it seems that he/she could only deal with a very special class of 2-graph algebras, i.e., those with the identity relation ($\theta = \text{id}$).

**Lemma 5.2.** Let $S, F$ be defined as above. Then $F$ is the adjoint of $S$: $\langle S(A)|B \rangle = \langle F(B)|A \rangle$ for all $A, B \in O_{\theta c}$.

**Proof.** It suffices to show that $\langle S(A)|B \rangle = \langle F(B)|A \rangle$ holds for all generators $A, B$ of $O_{\theta c}$. So we let $A = s_{u_1}s_{v_1}^*$ and $B = s_{u_2}s_{v_2}^*$. From the definition of the degree map $d$ for the generators of $O_\theta$, we have $d(A) = d(u_1) - d(v_1)$ and $d(B) = d(u_2) - d(v_2)$.

First observe from the definitions of $S, F$ that
\[
\langle S(A)|B \rangle = \langle A^*|B \rangle = \omega(AB)
\]
and
\[
\langle F(B)|A \rangle = n^{d(B)}\langle B^*|A \rangle = n^{d(B)}\omega(BA).
\]
If \( d(A) + d(B) \neq 0 \), then \( AB, BA \) are either 0, or not in \( \mathcal{F} \). So (7) and (8) implies \( \langle S(A)|B \rangle = 0 = \langle F(B)|A \rangle \). We are done. Thus we now suppose that \( d(A) + d(B) = 0 \) in what follows.

**Case 1.** \( d(A) = (-s, -t) \) with \( s, t \geq 0 \). Then \( d(B) = (s, t) \). So, making full use of commutation relations of \( \mathbb{F}_q^+ \), we can rewrite \( A, B \) as

\[
A = A's_u^*, \quad B = s_v B',
\]

for some \( u, v \in \mathbb{F}_q^+ \) with \( d(u) = d(v) = (s, t) \) and some generators \( A', B' \) in \( \mathcal{F} \).

Clearly, we now have

\[
\omega(AB) = \omega(A's_u^*s_v B') = \delta_{u,v} \omega(A'B') = \delta_{u,v} \tau(A'B').
\]  

(9)

On the other hand, it follows from Lemma 5.1 that

\[
\omega(BA) = \omega(s_v B' A's_u^*) = \delta_{u,v} n^{-d(u)} \omega(B'A') = \delta_{u,v} m^{-s} n^{-t} \tau(B'A').
\]

(10)

As \( \tau \) is a trace on \( \mathcal{F} \), from (7), (8), (9) and (10) we proved \( \langle S(A)|B \rangle = \langle F(B)|A \rangle \).

**Case 2.** \( d(A) = (s, -t) \) with \( s, t \geq 0 \). Then \( d(B) = (-s, t) \). As above, we rewrite \( A, B \) as

\[
A = s_{u_1} A's_{v_1}^*, \quad B = s_{v_2} B's_{u_2}^*,
\]

for some \( u_1, u_2, v_1, v_2 \in \mathbb{F}_q^+ \) and some generators \( A', B' \in \mathcal{F} \) with

\[
d(u_1) = d(u_2) = (s, 0), \quad d(v_1) = d(v_2) = (0, t).
\]

We have from Lemma 5.1 that

\[
\omega(AB) = \omega(s_{u_1} A's_{v_1}^* s_{v_2} B's_{u_2}^*) = \delta_{v_1,v_2} \omega(s_{u_1} A'B's_{u_2}^*) = \delta_{v_1,v_2} \delta_{u_1,u_2} n^{-d(u_1)} \omega(A'B') = \delta_{u_1,u_2} \delta_{v_1,v_2} m^{-s} \tau(A'B'),
\]

(9')

and

\[
\omega(BA) = \omega(s_{v_2} B's_{u_2}^* s_{u_1} A's_{v_1}^*) = \delta_{u_1,u_2} \omega(s_{v_2} B'A's_{v_1}^*) = \delta_{u_1,u_2} \delta_{v_1,v_2} n^{-d(v_1)} \omega(B'A') = \delta_{u_1,u_2} \delta_{v_1,v_2} n^{-t} \tau(B'A').
\]

(10')

It follows from (7), (8), (9') and (10') that \( \langle S(A)|B \rangle = \langle F(B)|A \rangle \).
Case 3. \( d(A) = (-s, -t) \) with \( s, t \geq 0 \). The proof is completely similar to Case 1.

Case 4. \( d(A) = (s, -t) \) with \( s, t \geq 0 \). The proof is completely similar to Case 2.

Therefore, \( F \) is the adjoint of \( S \).

From Lemma 5.2, we particularly obtained that both \( F \) and \( S \) are closable ([30, Theorem VIII.1]). By abusing notation, we still use \( F, S \) to denote their corresponding closures. The following is well-known: \( S \) and \( F \) have polar decompositions

\[
S = J \Delta^\frac{1}{2} = \Delta^{-\frac{1}{2}} J, \quad F = J \Delta^{-\frac{1}{2}} = \Delta^\frac{1}{2} J,
\]

where \( \Delta = FS \), and \( J \) is an anti-unitary operator with \( J = J^* \) and \( J^2 = I \). So far we have obtained all modular objects.

Moreover, by Lemma 5.2 we have

\[
J(s_u s^*_v) = n \frac{d(u) - d(v)}{2} s_u s^*_v,
\]

and

\[
\Delta^z(s_u s^*_v) = n^{z(d(v) - d(u))} s_u s^*_v \quad (z \in \mathbb{C}).
\]

Here \( m^z = \exp(z \ln m) \).

We are now in a position to prove the first main result in this section.

**Theorem 5.3.** The algebra \( \mathcal{O}_{\theta_c} \) with the inner product \( \langle \cdot | \cdot \rangle: \langle A | B \rangle = \omega(A^* B) \), is a modular Hilbert algebra.

**Proof.** The proof can now be easily adapted from [4, Lemma 3.2]. In order to check all axioms of a modular Hilbert algebra (which is called a Tomita algebra in [4]), the only thing that is not very obvious here is the fact that every \( \Delta^z \) is multiplicative on \( \mathcal{O}_{\theta_c} \). We will prove this below.

Arbitrarily choose two generators of \( \mathcal{O}_{\theta_c} \): \( X = s_{u_1} s^*_{v_1} \) and \( Y = s_{u_2} s^*_{v_2} \). Let \( d(u_2) \lor d(v_1) = (p, q) \). Then

\[
s^*_{v_1} s_{u_2} = \sum s^*_{w_1} s_{w_2},
\]

where the sum is over all \( w_1, w_2 \in \mathbb{F}^+_{\theta} \) such that

\[
v_1 w_1 = u_2 w_2 \quad \text{and} \quad d(w_1) + d(v_1) = (p, q)
\]
(cf. [21, 27]). We now have

\[ \Delta^z(XY) = \Delta^z \left( \sum_{w_1, w_2} s_{w_1} s_{w_2}^* s_{v_1}^* s_{v_2}^* \right) \]

\[ = \sum_{w_1, w_2} n^{z(k_1, \ell_1)} s_{w_1} s_{v_2}^* \]

\[ = \sum_{w_1, w_2} n^{z(k_2, \ell_2)} s_{w_1} s_{v_2}^* \] (by (11))

\[ = n^{z(k_2, \ell_2)} XY \]

\[ = \Delta^z(X) \Delta^z(Y), \]

where

\[ (k_1, \ell_1) := d(v_2) + d(w_2) - d(u_1) - d(w_1), \]

\[ (k_2, \ell_2) := d(v_1) + d(v_2) - d(u_1) - d(u_2). \]

Therefore \( \Delta^z \) is multiplicative on \( \mathcal{O}_{\theta_c}. \)

The above operators \( S \) (resp. \( J \)) are called the Tomita operator (resp. modular conjugation, modular operator) of \( \mathcal{O}_{\theta_c}. \) Let \( \pi(\mathcal{O}_\theta)^\prime \) be the von Neumann algebra generated by the GNS representation of the state \( \omega. \) Then \( \pi(\mathcal{O}_\theta)^\prime \) is nothing but the left von Neumann algebra of \( \mathcal{O}_{\theta_c} (35). \) The celebrated Tomita-Takesaki modular theory says that

\[ \Delta^{\prime \prime \dagger} \pi(\mathcal{O}_\theta)^\prime \Delta^{-\prime \prime \dagger} = \pi(\mathcal{O}_\theta)^\prime \quad (t \in \mathbb{R}), \quad J \pi(\mathcal{O}_\theta)^\prime J = \pi(\mathcal{O}_\theta)^\prime. \]

Refer to [33, 35] for more information on the Tomita-Takesaki modular theory.

Let \( \sigma_z(\pi(X)) = \Delta^{iz} \pi(X) \Delta^{-iz} \) for all \( z \in \mathbb{C} \) and \( X \in \mathcal{O}_{\theta_c}. \) We now give the formula of the modular automorphisms \( \sigma_t \) (\( t \in \mathbb{R} \)) of \( \pi(\mathcal{O}_\theta)^\prime \) on generators, and show that \( \omega \) is a \( \sigma \)-KMS state. Refer to [2, Chapter 5] and [35, Section 13] for KMS-states.

**Proposition 5.4.** Let \( \omega \) be the state given at the beginning of this section, \( \Delta \) the modular operator of \( \mathcal{O}_{\theta_c}, \) and \( \sigma_z \) (\( z \in \mathbb{C} \)) the operator defined as above. Then

(i) the group of modular automorphisms \( \sigma_t \) (\( t \in \mathbb{R} \)) of the von Neumann algebra \( \pi(\mathcal{O}_\theta)^\prime \) on \( L^2(\mathcal{O}_\theta) \) is given on the generators by

\[ \sigma_t(\pi(s_u s_v^*)) := \Delta^{it} \pi(s_u s_v^*) \Delta^{-it} = n^{it(d(v) - d(u))} \pi(s_u s_v^*); \]
(ii) $\omega$ is a $\sigma$-KMS state over $\pi(O_\theta)'$:

$$\omega(AB) = \omega(\sigma_t(B)A) \quad \text{for all} \quad A, B \in \pi(O_{\theta c});$$

(iii) $\omega$ is the unique $\sigma$-KMS state over $\pi(O_\theta)'$, provided $\frac{\ln m}{\ln n} \not\in \mathbb{Q}$.

**Proof.** The proofs of (i) and (ii) below are borrowed from [4, Lemma 3.3 and its remarks].

(i) From the proof of Theorem 5.3, we know that $\Delta^z : O_{\theta c} \rightarrow O_{\theta c}$ is an algebra homomorphism. This implies that

$$\pi(\Delta^z(A)) = \Delta^z\pi(A)\Delta^{-z} \quad \text{for all} \quad A \in O_{\theta c}, \ z \in \mathbb{C}.$$  

The rest of the proof of (i) is done by direct computation.

(ii) It is proved by the following calculations:

$$\omega(AB) = \langle A^*|B \rangle = \langle S(A)|B \rangle = \langle F(B)|A \rangle = \omega(SF(B)A)$$

$$= \omega(\Delta^{-1}(B)A) = \omega(\sigma_t(B)A).$$

(iii) In the sequel, we naturally identify the C*-algebra $\pi(O_\theta)$ with the C*-algebra $O_\theta$ (not as a subspace of $L^2(O_\theta)$) as $\pi$ is faithful. A straightforward calculation yields the following relation:

$$\sigma_t = \gamma_{(m^{-it}, n^{-it})} \quad \text{for all} \quad t \in \mathbb{R}.$$  

As $\frac{\ln m}{\ln n} \not\in \mathbb{Q}$, by Kronecker’s Theorem, the set $\{(m^{-it}, n^{-it}) : t \in \mathbb{R}\}$ is dense in $\mathbb{T}^2$. Thus the modular automorphisms $\{\sigma_t : t \in \mathbb{R}\}$ determine the gauge automorphisms $\{\gamma_t : t \in \mathbb{T}^2\}$.

Now we can use a similar argument of [22, Theorem 2] (also cf. [2, Chapter 5]) to prove the uniqueness of $\omega$. We only sketch it here. Suppose $\omega'$ is also a KMS state for $\sigma$ at value $\beta$.

We first show that $\beta$ has to be finite. To the contrary, suppose that $\beta = \infty$ (or $-\infty$). However, from (i) one can see that the functions

$$t \mapsto \omega'(\pi(s_{e_i})^*\sigma_t(\pi(s_{e_i}))) = m^{-it}\omega'(\pi(s_{e_i})^*\pi(s_{e_i})) = m^{-it}\omega'(I) = m^{-it},$$

$$\text{(or} \quad t \mapsto \omega'(\pi(s_{e_i})\sigma_t(\pi(s_{e_i})^*)) = m^{it}\omega'(\pi(s_{e_i}s_{e_i}^*)))$$

do not have bounded analytic extensions to the upper (or lower) half-planes. Here we used the simple fact that $\omega'(\pi(s_{e_i}s_{e_i}^*)) \neq 0$ because of the defect free property. Now from [2 Proposition 5.3.19] and its immediately preceding remark, we get a contradiction. Therefore, $\beta$ is finite.

Since $\omega'$ is a $(\sigma, \beta)$-KMS state, $\omega'$ is invariant under $\sigma_t$, namely, $\omega'\sigma_t = \omega'$ ($t \in \mathbb{R}$). Hence from the relations between $\sigma$ and $\gamma$ given above, we obtain $\omega'\gamma_t = \omega'$ for all $t \in \mathbb{T}^2$. Hence, $\omega'\Phi = \omega'$. On the other hand, from the KMS condition (i.e., $\omega'(AB) = \omega'(\sigma_{i\beta}(B)A)$), we have that $\omega'|_\mathfrak{F}$ is a normalized trace on $\mathfrak{F}$. The uniqueness of the
normalized trace on $\mathfrak{F}$ concludes $\omega' \equiv \omega$. Now from (ii), we also have $\beta = 1$. Therefore, $\omega$ is the unique $\sigma$-KMS state over $\pi(O_0)'$. \hfill \blacksquare

6. Some remarks on the classification of $\pi(O_0)'$

In this short section, we begin with the following observation.

**Lemma 6.1.** $\pi(\mathfrak{F})'$ is a $\mathrm{II}_1$ factor.

**Proof.** It is known that $\pi$ and the GNS representation of $\omega|_{\mathfrak{F}}$ (the restriction $\omega|_{\mathfrak{F}}$ of $\omega$ to $\mathfrak{F}$, and so $\omega|_{\mathfrak{F}} = \tau$) are quasi-equivalent. The lemma now follows from [26, Theorem 2.5]. \hfill \blacksquare

Recall that if $\mathcal{M}$ is a von Neumann algebra, the *Connes invariant* $S(\mathcal{M})$ is the intersection over all faithful normal states of the spectra of their corresponding modular operators [19]. Connes classified type III factors as follows. A factor $\mathcal{M}$ is said to be of type $\mathrm{III}_0$ if $S(\mathcal{M}) = \{0, 1\}$; type $\mathrm{III}_\lambda$ if $S(\mathcal{M}) = \{0, \lambda^n : n \in \mathbb{Z}\}$ ($0 < \lambda < 1$); type $\mathrm{III}_1$ if $S(\mathcal{M}) = \{0\} \cup \mathbb{R}^+$. We are now able to obtain the following result on partially classifying the von Neumann algebra $\pi(O_0)'$.

**Corollary 6.2.** If $\frac{\ln m}{\ln n} \notin \mathbb{Q}$, then $\pi(O_0)'$ is an AFD factor of type $\mathrm{III}_1$.

**Proof.** Since $O_0$ is amenable ([21]), it is known that $\pi(O_0)'$ is AFD. Furthermore, by Proposition 5.4 (iii) and [2, Theorem 5.3.30], we obtain that $\pi(O_0)'$ is a factor.

Also, from Proposition 5.4 one can show that the fixed point algebra of $\sigma$ is $\pi(\mathfrak{F})'$ as $\frac{\ln m}{\ln n} \notin \mathbb{Q}$. It follows from Lemma 6.1 and [33, Section 28.3] that the Connes spectrum coincides with the spectrum of the modular operator. That is, $S(\pi(O_0)') = \text{Sp}(\Delta)$. But from the definition of $\Delta$, it is easy to see that

$$\text{Sp}(\Delta) = \{m^an^b : a, b \in \mathbb{Z}\} = [0, \infty)$$

as $\frac{\ln m}{\ln n} \notin \mathbb{Q}$. So $\pi(O_0)'$ is of type $\mathrm{III}_1$. \hfill \blacksquare

**Remark 6.3.** Notice that if $\frac{\ln m}{\ln n} \in \mathbb{Q}$, i.e., $m^a = n^b$ for some $a, b \in \mathbb{N}$ with $\text{gcd}(a, b) = 1$, then $\pi(\mathfrak{F})'$ is a proper subalgebra of $\pi(O_0)'$, the fixed point algebra of the modular automorphisms $\sigma_t$ ($t \in \mathbb{R}$). Indeed, one can check that

$$\pi(O_0)' = \{\pi(\mathfrak{F}), \pi(s_{ev_1}s_{ev_1}^*), \pi(s_{f_{ev_2}e_{u_2}v_2}^*): (|u_i|, |v_i|) = k_i(a, b), k_i \in \mathbb{N}\}'$$

$\supseteq \pi(\mathfrak{F})'$.\end{proof}
Remark 6.4. Since $\frac{\ln m}{\ln n} \notin \mathbb{Q}$, the 2-graph $F^+_\theta$ in Corollary 6.2 is aperiodic. Clearly, if $F^+_\theta$ is periodic, then $\pi(O_\theta)^{\text{tr}}$ is not a factor (cf. [13]). A natural question is if the converse is true: If $F^+_\theta$ is aperiodic, is $\pi(O_\theta)^{\text{tr}}$ a factor? Moreover, by Corollary 6.2 we obtain the type of $\pi(O_\theta)^{\text{tr}}$ when $\frac{\ln m}{\ln n} \notin \mathbb{Q}$. If the converse is true, a further question arises: What is the type of $\pi(O_\theta)^{\text{tr}}$ if $\frac{\ln m}{\ln n} \in \mathbb{Q}$ and $F^+_\theta$ is aperiodic? Also, it would be interesting to study the index theory of endomorphisms of $\pi(O_\theta)^{\text{tr}}$. We will leave those as future research.

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