BASIC REMARKS ON LAGRANGIAN SUBMANIFOLDS OF HYPERKÄHLER MANIFOLDS

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Abstract. This note presents basic restrictions on the topology “general” Lagrangian surfaces of hyper-Kähler 4-folds and a remark on the interaction of a Lagrangian subvariety with a Lagrangian fibration of the associated hyper-Kähler variety.

1. Introduction

Lagrangian submanifolds of irreducible symplectic manifolds are known to enjoy interesting properties. To name some of them, they are known to be projective ([2, Proposition 2.1]) even when the symplectic manifold containing them is not and their deformations are unobstructed i.e. the corresponding Hilbert scheme is smooth at any point representing a smooth Lagrangian subvariety (see [3, Section VI.6], see also [9]). In this note, we present some additional properties of Lagrangian submanifolds.

Among the most common examples of Lagrangian subvarieties of hyper-Kähler manifolds, we find curves on K3 surfaces and abelian varieties that appear, for example, as fibers of a Lagrangian fibration.

In the first part of the note, we show that some of the features of these two model Lagrangian subvarieties are common to “most” of the other Lagrangian surfaces.

In the case of curves on a K3, except for $\mathbb{P}^1$, the topological Euler characteristic is non-positive. The first result suggests that, essentially, the topological Euler characteristic of “most” Lagrangian subvarieties is of a given sign (determined by the dimension).

Proposition 1.1. Let $S \subset Y$ be a Lagrangian surface in a hyper-Kähler 4-fold whose deformations cover a dense open subset of $Y$. Then either

- two general surfaces parametrized by the same Hilbert scheme component as $S$, have empty intersection, in which case $\chi_{\text{top}}(S) = 0$;
- or there is a (possibly reducible) curve which is contained in every surface parametrized by the same Hilbert scheme component as $S$;
- or $\chi_{\text{top}}(S) > 0$

Although Lagrangian surfaces whose deformations cover the associated hyper-Kähler 4-fold can have intermediate Kodaira dimension (i.e. 0 or 1), we have the following result about the Albanese dimension.

Theorem 1.2. Let $S \subset Y$ be a Lagrangian surface in a projective hyper-Kähler 4-fold whose deformations cover a dense open subset of $Y$. Then $S$ has maximal Albanese dimension.

The result presented in the second part is concerned with the interaction of a Lagrangian subvariety $X \subset Y$ with a Lagrangian fibration of $Y$.

Proposition 1.3. Let $\pi: Y \to \mathbb{P}^n$ be a Lagrangian fibration of a hyper-Kähler variety $(Y, \omega)$ endowed with a section.

Let $X \subset Y$ be a smooth Lagrangian subvariety which is generically contained in the smooth locus of $\pi$ and is not a fiber. Then either $\pi|_X$ is generically finite or the general fiber of $\pi|_X: X \to \pi(X)$ is a union of abelian varieties.

2. Topology of Lagrangian surfaces

Let $S \subset Y$ be a smooth Lagrangian surface of a hyper-Kähler 4-fold. We have the classical isomorphism $\Omega_S \simeq N_{S/Y}$ which suggests that there is a deep interplay between the
deformation theory of $S$ inside $Y$ and its topology -as already illustrated by the elementary remark (using that the deformations of $S$ are unobstructed) that if $S$ does not deform in $Y$ then $\tau(Y) = 0$.

Let us denote by $H(Y)$ a dense open subset of the Hilbert scheme component containing $[S]$, parametrizing only smooth surfaces. We have the following:

**Lemma 2.1.** Assume that for general $[S], [S'] \in H(Y)$, $S \cap S'$ is a curve. Let us consider, for any $[S] \in H(Y)$, the map $e_S : H(Y) \setminus \{[S]\} \to \text{Div}^{\tau_e}(S)$, $[S'] \mapsto [S' \cap S]$ to the space of effective divisors whose class in $\text{NS}(S)$ is $\gamma_S = [S \cap S']$. Then for any $[S] \in H(Y)$ the image of $e_S$ is a point. The latter is associated to the (possibly reducible) curve contained in every member of $H(Y)$.

**Proof.** Note first that if $e_S$ has finite (0-dimensional) image for a surface $S$ the same is true for any other $[S'] \in H(Y)$.

Indeed, as $H(Y)$ is irreducible, $e_S(H(Y) \setminus \{[S]\})$ consists of a point $[C] \in \text{Div}^{\tau_e}(S)$. Take another $[S'] \in H(Y)$. By definition, $C \subset S'$ and $[C] = \gamma_{S'}$ in $\text{NS}(S')$. For any other $[S''] \in H(Y)$, as $C = S'' \cap S$, $C \subset S'' \cap S'$. Since $[S'' \cap S] = [C]$ in $\text{NS}(S')$, we get $C = S'' \cap S'$ i.e. $\text{Im}(e_{S''}) = \{(C)\}$.

So we just have to prove that there is no surface for which $e_S$ has positive dimensional image.

So assume there is a surface $S$ for which $\text{Im}(e_S)$ is positive dimensional. Then by what we have just seen, $\text{Im}(e_S)$ is positive dimensional for any $[S'] \in H(Y)$. Taking general hyperplanes sections of $H(Y)$, we can find a closed (irreducible) subvariety $M \subset H(Y)$ such that $f : S_M \to Y$, where $S_M$ is the pullback of the universal surface on $M$, is generically finite (dominant) and the restriction $e_{S|M}$ of $e_S$ to $M$ has positive dimensional image for any $[S] \in M$. In particular $\dim(M) = 2$.

If $\dim(\text{Im}(e_{S|M})) = 2$ for a surface $[S] \in M$. Then the general (non-empty) fiber of $e_{S/M}$ is 0-dimensional. For a general $[S'] \in M$, denoting $C = e_{S|M}(S')$, we have $C = S \cap S' \in \text{Im}(e_{S'|M}) \subset \text{Div}^{\tau_e}(S')$. Any other $[S''] \in e_{S'|M}^{-1}([C])$ is also in $e_{S|M}^{-1}([C])$ (as $C \subset S'' \cap S$ and $[C] = [S'' \cap S]$ in $\text{NS}(S)$) so there are finitely of them i.e. $e_{S'|M}^{-1}([C])$ is 0-dimensional. So the general fiber of $e_{S|M}$ is also 0-dimensional, in other words $\dim(\text{Im}(e_{S'})) = 2$ for the general $[S'] \in M$.

Then the curves $S' \cap S$ cover a dense open subset of $S$ and as the deformations of $S$ parametrized by $M$ cover a dense open subset of $Y$, those curves cover a dense open subset of $Y$. Denoting $C$ the universal curve over $\text{Div}^{\tau_e}(S)$, the generic fiber of $g_{S} : e_{S'|M}^{-1}C \to S$ has dimension 1. Now take a general point $y \in Y$ and a general surface $[S] \in M$ passing through $y$. Then there is a 1-dimensional family of divisors of $S$ of the form $S' \cap S$, with $[S'] \in M$ passing through $y$. So $f$ is not generically finite.

We are left with the case when $\dim(\text{Im}(e_{S|M})) = 1$ for the general surface $[S] \in M$. In this case, the curves $S' \cap S$ cover again $S$. As the deformations of $S$ parametrized by $M$ cover $Y$, those curves cover $Y$. Let $y \in Y$ be a general point and $C_y$ a curve of the form $S_1 \cap S_2$ containing $y$. As the general fiber of $e_{S_1|M}$ is 1-dimensional, there is a 1-dimensional family $[S] \in M$ such that $S_1 \cap S_2 = C_y$. So $f$ is not generically finite. □

**Proof of Proposition 1.1.** For two surfaces $[S_1], [S_2]$ in $H(Y)$, we have

$$\int_{[S_1]}([S_2]) = \int_{[S_2]}([S]) = \int_{S} i^*\omega(\Omega_{S/Y}) = \int_{S} e_{\text{top}}(\text{NS}_{S/Y}) = \chi_{\text{top}}(S),$$

using $\Omega_{S/Y} \simeq \Omega_{S}$ and where $i : S \hookrightarrow Y$.

According to Lemma 2.1, if there is no common curve to the surfaces in $H(Y)$, the intersection of two general surfaces is either empty, in which case 0 $\leq [S_1] \cdot [S_2] = \chi_{\text{top}}(S)$ or 0-dimensional $0 < [S_1] \cdot [S_2] = \chi_{\text{top}}(S)$.
Remark 2.2. (1) As the following example shows, it is necessary that the deformations of $S$ cover $Y$: Let $C \subset \Sigma$ be a smooth curve of genus $> 1$ on a $K3$ surface. Then, as explained in [3] Section 8, $\mathbb{P}(T_{\Sigma|C}) \subset \Sigma^{[2]}$ is Lagrangian and
\[
\chi_{\text{top}}(\mathbb{P}(T_{\Sigma|C})) = \int c_2(T_{\mathbb{P}(T_{\Sigma|C})}) = \int (pr^*c_1(T_{\Sigma|C}) + 2c_1(\mathcal{O}_{T_{\mathbb{P}(T_{\Sigma|C})}}(1))) \cdot pr^*c_1(T_{\Sigma|C}) = 2(2 - 2g(C)) < 0.
\]
As $h^0(\Omega_{\mathbb{P}(T_{\Sigma|C})}) = h^0(\omega_C)$ and deformations of the Lagrangian subvarieties $C \subset \Sigma$ and $\mathbb{P}(T_{\Sigma|C}) \subset \Sigma^{[2]}$ are unobstructed, the deformations of $\mathbb{P}(T_{\Sigma|C})$ are induced by deformations of $C$; so the deformations of $\mathbb{P}(T_{\Sigma|C})$ are contained in the branched locus $E \simeq \mathbb{P}(S)$ of $q : Bl_\Delta_{\Sigma}(\Sigma^2) \rightarrow \Sigma^{[2]}$.

Moreover $\mathbb{P}(T_{\Sigma|C}) \cap \mathbb{P}(T_{\Sigma|C}) = \cup_{i=1}^{2g(C)-2} \mathbb{P}(\Sigma_{P_i})$ where $\{p_1, \ldots, p_{2g(C)-2}\} = C_1 \cap C_2$ and as $\mathcal{O}_\Sigma(C)$ is base point free ([4] Lemma 2.3), the surfaces $\mathbb{P}(T_{\Sigma|C})$ have no curve in common.

Actually, in this case, any Lagrangian surface $S \subset E$ is of the form $\mathbb{P}(T_{\Sigma|C})$ for a smooth curve $C \subset \Sigma$.

Indeed, denoting by $\tau : Bl_\Delta_{\Sigma}(\Sigma^2) \rightarrow \Sigma^2$ the blow-up, by $\omega$ the symplectic form on $\Sigma^{[2]}$ and by $\omega_C$ the one on $\Sigma$, we have $q^*\omega = \tau^*(pr_1^*\omega_{\Sigma} + pr_2^*\omega_{\Sigma})$. So, denoting $j_E : E \hookrightarrow Bl_\Delta_{\Sigma}(\Sigma^2)$, we get
\[
j_E^*q^*\omega = j_E^*\tau^*(pr_1^*\omega_{\Sigma} + pr_2^*\omega_{\Sigma}) = \tau^*|_{\Delta_{\Sigma}}(pr_1^*\omega_{\Sigma} + pr_2^*\omega_{\Sigma}) = 2\tau^*|_{\Delta_{\Sigma}}\omega_{\Sigma}.
\]

Now, let $S \subset E$ be a Lagrangian surface. Assume $\tau_{\Sigma} : S \rightarrow \Sigma$ is generically finite. Let $U \subset \Sigma$ be open subset over which $\tau_{\Sigma}$ is étale. By the above description of the restriction of the symplectic form, $\omega_{\Sigma}$ is non-degenerate (symplectic) on $\tau_{\Sigma}^{-1}(U)$, in particular non-zero, thus $S$ is not Lagrangian. So $\tau_{\Sigma}$ cannot be surjective and by semi-continuity of the dimension of the fibers of $\tau_{\Sigma} : S \rightarrow \text{Im}(\tau_{\Sigma})$, all the fibers are 1-dimensional i.e. are fibers of $\tau_E : E \rightarrow \Sigma$. Denoting $C \subset \Sigma$ its image, we get $S \simeq \mathbb{P}(T_{\Sigma|C})$. So $C$ is smooth.

In particular, any deformation of a Lagrangian surface $S$ contained in the rigid uniruled divisor $E$ stays in the latter.

(2) Examples of families covering $Y$ whose members have curves in common can be constructed. Let $C \subset \Sigma$ be a smooth curve of genus $\geq 2$ on a $K3$ surface and $V_p \subset |\mathcal{O}_\Sigma(C)|$ be the linear system of curves passing through $p \in \Sigma$ (generically with multiplicity one). The curves parametrized by $V_p$ cover $\Sigma$.

For a general pair $([C_1],[C_2]) \in V_p^2/i$, $i$ being the natural involution, $C_1$ and $C_2$ intersect transversally (in particular at $p$). As explained in [3] Section 8, $Bl_{C_1 \cap C_2}(C_1 \times C_2) \subset \Sigma^{[2]}$ is a Lagrangian submanifold.

All the smooth Lagrangian submanifolds parametrized by an open subset of $V_p^2/i$ contain the rational curve $\mathbb{P}(T_{\Sigma,p})$. Moreover the universal surface over this open subset dominates $\Sigma^{[2]}$ since one can find a member of $V_p$ through any point of $\Sigma$.

As $\mathcal{O}_\Sigma(C)$ is base point free ([4] Lemma 2.3]), for the full Hilbert scheme of $Bl_{C_1 \cap C_2}(C_1 \times C_2) \subset \Sigma^{[2]}$, there is no curve common to every member of it. Moreover, the example does not contradict $\chi_{\text{top}}(S) \geq 1$.

The following strenghtening of Lemma 2.1 is natural:

Question 2.3. (General position for Hilbert scheme): Given a Hilbert scheme component $H$ parametrizing (generically) smooth (irreducible) subvarieties of a smooth projective variety $Y$ that cover it, is the intersection of two general members of $H$ dimensionally transverse?

Now, let $S$ be a projective surface of Albanese dimension 1. Then the Albanese morphism factors through a fibration (flat with connected fibers) $\text{alb}_S : S \rightarrow B$ over a smooth curve of genus $g(S) = h^{1,0}(S)$ ([1]) such that $J(B) \simeq \text{Alb}(S)$. 
We recall that having Albanese dimension 1 is a topological property ([7, Section 2.2]).

Now assume $S \subset Y$ is a Lagrangian surface of a projective hyper-Kähler that has Albanese dimension 1 and such that the deformations of $S$ in $Y$ cover the latter. Then we have $2 \leq \dim(\mathcal{H}(Y)) = h^{1,0}(S) = g(B)$ (using that the deformations of $S$ are unobstructed and $\Omega_S \simeq N_{S/Y}$) and any surface parametrized by $\mathcal{H}(Y)$ has then Albanese dimension 1.

Up to taking an étale cover of (an open subset of) $\mathcal{H}(Y)$ (which does not change the tangent spaces) we have the relative Albanese fibration

$$S \xrightarrow{\overline{\alpha S}} B \xrightarrow{\mathcal{H}(Y)}.$$ 

We have $H^0(N_{S/Y}) \simeq H^0(\Omega_S) \xrightarrow{\overline{\alpha S}} H^0(\omega_B)$ so that the general section $\sigma_S = \overline{\alpha S} \ast \sigma_B$ of $\Omega_S$ vanishes along a disjoint union $\sigma_S = \cup_{i=1}^{2g(B)-2} F_i = \overline{[\alpha S^{-1}(\text{div}(\sigma_B))]}$ of smooth fibers of $\overline{\alpha S}$.

One can think of the zero locus of a section of $N_{S/Y}$ as an “infinitesimal intersection” of $S$ with one of its first order deformation. So here this infinitesimal intersection is not dimensionally transverse. Theorem 1.2 will follow from Lemma 2.1 after we have proven that this infinitesimal picture can be integrated to a dimensionally non-transverse actual intersection.

More precisely, we will prove that the deformations of $S$ fixing $Z = Z_{\sigma_S}$ are unobstructed.

Let us first prove that the fact that such $\mathcal{O}_S(Z)$ comes from the base of the fibration is an open property.

**Lemma 2.4.** Let $S' \subset Y$ be a deformation of $S$ fixing $Z$. Then $\mathcal{O}_{S'}(Z)$ can be written $\overline{\alpha S'} \ast \mathcal{L}$ for a degree $2g(B') - 2$ line bundle.

**Proof.** As $\overline{\alpha S'} \omega_B \simeq \mathcal{O}_S(Z)$ and having trivial Chern class is topological, $[\overline{\alpha S'} \ast \kappa_B'] = [Z]$ in $NS(S')$. So we can write $\overline{\alpha S'} \ast \kappa_B' = Z + \ell$ in $\text{Pic}(S')$ for some $\ell \in \text{Pic}^0(S')$. Intersecting with $\overline{\alpha S'} \ast \kappa_B'$ we get $0 = \int_{S'} \overline{\alpha S'} \ast \kappa_B' = \int_{S'} \overline{\alpha S'} \ast \kappa_B' + \int_{S'} \ell \overline{\alpha S'} \ast \kappa_B'$. As $\ell \in \text{Pic}^0(S')$, we get $\int_{S'} Z \cdot \overline{\alpha S'} \ast \kappa_B' = 0$. But as $g(B') \geq 2$, $\omega_{B'}$ is ample, so $Z$ is contracted by $\overline{\alpha S'}$. Now let $H$ be a ample divisor on $Y$, the invariance of $H$-degree of the fibers of the Albanese fibrations gives that the components of $Z$ are actual fibers of $\overline{\alpha S'}$ (not just components of fibers). Hence the result ($\overline{\alpha S'}$ is flat). \qed

As any degree $2g - 2$ line bundle $L$ of a genus $g$ curve $B$ is either $\omega_B$ or satisfies $h^0(L) = g - 1$, the difference between $\mathcal{O}_{S'}(Z)$ and $\overline{\alpha S'} \omega_B$ can be detected by their respective number of linearly independent sections. So we will prove that all the sections of $\mathcal{O}_S(Z) \simeq \overline{\alpha S} \omega_B$ deform.

Let us recall some results of deformation theory. They can be found in [6] and [8] for example.

Set the deformation functor

$$H^Z_y: \mathcal{A} \rightarrow \text{Sets}$$

$$\mathcal{A} \mapsto \{ Z \times \text{Spec}(A) \subset S_A \subset Y \times \text{Spec}(A), S_A \text{ flat over } A \text{ and } S_A \otimes k \simeq S \}$$

where $\mathcal{A}$ is the category of local artinian $\mathbb{C}$-algebra with residue field $\mathbb{C}$.

By cocycle computation, we get the following proposition (which is essentially [6, Lemma 1.4.3]).

**Proposition 2.5.** The functor $H^Z_y$ is pro representable (subfunctor of the local Hilbert functor) with tangent space $H^0(N_{S/Y}(-Z))$ and obstruction space $H^1(N_{S/Y}(-Z))$. 

We will prove that $H^{Z,Y}_S$ is unobstructed i.e. that any infinitesimal deformation can be extended to an actual deformation. Letting $A_n = \mathbb{C}[t]/(t^{n+1})$, we have the following criterion:

**Proposition 2.6.** ([8] Corollary 1.1.7) The functor $H^{Z,Y}_S$ is unobstructed if and only if for any $n \geq 0$ the natural map $H^{Z,Y}_S(A_{n+1}) \to H^{Z,Y}_S(A_n)$ (induced by $A_{n+1} \to A_n$) is surjective.

We will use the $T^1$-lifting principle (see [8] section VI.3.6) to prove that the maps $H^{Z,Y}_S(A_{n+1}) \to H^{Z,Y}_S(A_n)$ are surjective.

Let us introduce $D_n = A_n[\epsilon]/(\epsilon^2)$, $C_n = D_n/(t^n)$. There are projections $C_{n+1} \to D_n$, $D_n \to C_n \to A_n$. Let us also introduce the homomorphism of $\mathbb{C}$-algebras $\delta : A_{n+1} \to D_n$, $t \mapsto t + \epsilon$. It is injective. Likewise, let us define $\delta' : A_n \to C_n$, $t \mapsto t + \epsilon$.

Given a $[S_n] \in H^{Z,Y}_S(A_n)$, we denote $H^{Z,Y}_S(D_n)_{S_n}$ the fiber of $H^{Z,Y}_S(A_n) \to H^{Z,Y}_S(A_n)$ over $[S_n]$.

The $T^1$-lifting principle consists of the following

**Proposition 2.7.** ([8] Lemmas VI.3.7, VI.3.8) For a given $n \geq 0$, if for any $[S_{n+1}] \in H^{Z,Y}_S(A_{n+1})$, the map $H^{Z,Y}_S(D_{n+1})_{S_{n+1}} \to H^{Z,Y}_S(D_n)_{S_n}$, where $S_n = S_{n+1}/A_n$, is surjective then $H^{Z,Y}_S(D_{n+1}) \to H^{Z,Y}_S(C_{n+1})$ and $H^{Z,Y}_S(A_{n+2}) \to H^{Z,Y}_S(A_{n+1})$ are surjective.

Actually the surjectivity of $H^{Z,Y}_S(A_{n+2}) \to H^{Z,Y}_S(A_{n+1})$ is derived from the surjectivity of $H^{Z,Y}_S(D_{n+1}) \to H^{Z,Y}_S(C_{n+1})$ by applying $H^{Z,Y}$ to the commutative diagram ([8] Section VI.3.6)

\[ \begin{array}{ccc}
0 & \longrightarrow & \mathbb{C}[t^{n+2}] \\
& & \downarrow{(n+2)} \\
0 & \longrightarrow & A_{n+2} \\
& & \delta \\
0 & \longrightarrow & A_{n+1} \\
& & \delta' \\
& & 0
\end{array} \]

We recall the following results of deformation theory that can be found in [8] for first order deformation. Lifting objects from $A_n$ to $D_n$ works a lot like first order deformations of the objects.

The following Lemma is essentially [8] Lemma I.4.3.

**Lemma 2.8.** For $[S_n] \in H^{Z,Y}_S(A_n)$, there is a natural 1 to 1 correspondence between $H^{Z,Y}_S(D_n)_{S_n}$ and $H^0(N_{S_n/A_n}(−(Z \times \text{Spec}(A_n))))$.

Following [8], we introduce tools to deal with deformation of lines bundles. The proof are almost identical to those found in loc. cit. one has just to be careful of some additional automorphisms that appear.

Since $S$ is smooth, any infinitesimal deformation of $S$ is locally trivial ([8] Theorem 1.2.4]).

Let $\mathcal{V} = \{V_i\}$ be an affine open cover $S$ and $[S_n] \in H^{Z,Y}_S(A_n)$. We have $A_n$-isomorphisms $\theta_i : V_i \times \text{Spec}(A_n) \to S_n|V_i$ which gives for each (ordered) pair $(i,j)$ the gluing automorphisms $\theta_{ij} = \theta_i^{-1} \theta_j : V_{ij} \times \text{Spec}(A_n) \to V_{ij} \times \text{Spec}(A_n)$. The proof of the following proposition follows the one of [8] Proposition 1.2.9].

**Proposition 2.9.** For $[S_n] \in H^{Z,Y}_S(A_n)$, there is a 1 to 1 correspondence between

\[ \{S_{D_n} \to \text{Spec}(D_n), \text{ flat and restricting to } S_n\}/\text{isom} \]

and $H^1(T_{S_n/A_n})$.

**Proof.** We just show one direction. Let $S_{D_n} \to \text{Spec}(D_n)$ be an extension of $S_n \to \text{Spec}(A_n)$. We can choose the affine cover $\mathcal{V}$ of $S$ so that $S_{D_n|V_i}$ and $S_n|V_i$ are trivial. So we have isomorphisms of deformations

\[ \tilde{\theta}_i : V_i \times \text{Spec}(A_n) \to S_n|V_i \text{ and } \tilde{\theta}_i : V_i \times \text{Spec}(D_n) \to S_{D_n|V_i} \]

with $\tilde{\theta}_i$ restricting to $\theta_i$ and automorphisms

\[ \theta_{ij} : V_{ij} \times \text{Spec}(A_n) \to V_{ij} \times \text{Spec}(A_n) \text{ and } \tilde{\theta}_{ij} : V_{ij} \times \text{Spec}(D_n) \to V_{ij} \times \text{Spec}(D_n). \]
Set $V_{ij} \simeq \text{Spec}(B_{ij})$. Denoting $i_n : B_{ij} \otimes A_n \rightarrow B_{ij} \otimes D_n$ the natural inclusion, $q_n : B_{ij} \otimes D_n \rightarrow B_{ij} \otimes A_n$ the surjective $B_{ij} \otimes A_n$-algebra homomorphism and $p_n : B_{ij} \otimes D_n \rightarrow B_{ij} \otimes A_n$, the projection on the $e$ component (which is a $B_{ij} \otimes A_n$-homomorphism), we can define $\eta_{ij} = i_n \circ \theta_{ij} \circ q_n + e_i \circ \theta_{ij} \circ p_n$. A direct calculation shows that it is an endomorphism of $D_n$-algebras.

By assumption $\overline{\theta}_{ij} - \eta_{ij}$ is trivial mod $e$. Writing $\epsilon \phi_{ij} = \overline{\theta}_{ij} - \eta_{ij}$, for $x, y \in B_{ij} \otimes D_n$, we find $\phi_{ij}(xy) = \theta_{ij}(q_n(x)) \phi_{ij}(y) + \theta_{ij}(q_n(y)) \phi_{ij}(x)$, which can be written

$$(\theta_{ij}^{-1} \phi_{ij})(xy) = q_n(x)(\theta_{ij}^{-1} \phi_{ij})(y) + q_n(y)(\theta_{ij}^{-1} \phi_{ij})(x) = x \cdot (\theta_{ij}^{-1} \phi_{ij})(y) + y \cdot (\theta_{ij}^{-1} \phi_{ij})(x)$$

i.e. $(\theta_{ij}^{-1} \phi_{ij}) \in \text{Der}_{B_{ij} \otimes D_n}(B_{ij} \otimes D_n, B_{ij} \otimes A_n) \simeq \text{Hom}_{B_{ij} \otimes D_n}(\Omega_{B_{ij} \otimes D_n/D_n}, B_{ij} \otimes A_n) \simeq \text{Der}_{B_{ij} \otimes D_n}(B_{ij} \otimes A_n, B_{ij} \otimes A_n) = \Gamma(V_{ij} \times \text{Spec}(A_n), T_{V_{ij} \times \text{Spec}(A_n)/\text{Spec}(A_n)})$.

Since on $V_{ijk} \times \text{Spec}(D_n)$, $\theta_{ij} \theta_{jk} = \overline{\theta}_{ik}$, looking at $\phi_{ij}$ as a map from $B_{ij} \otimes A_n$ to itself, we find

$$\theta_{ij} \phi_{jk} + \phi_{ij} \theta_{jk} = \phi_{ik}$$

which can be re-written

$$\theta_{ij} \phi_{jk}(\theta_{ij}^{-1} \phi_{jk}) + \theta_{ij}((\theta_{ij}^{-1} \phi_{ij}) \theta_{jk}) = \theta_{ik}(\theta_{ik}^{-1} \phi_{ik})$$

and pre-composing with $\theta_{ij}^{-1}$ and post-composing with $\theta_i$, we get

$$\theta_j(\theta_{ij}^{-1} \phi_{ij}) \theta_{ij}^{-1} + \theta_k(\theta_{jk}^{-1} \phi_{jk}) \theta_{ik}^{-1} = \theta_k(\theta_{ik}^{-1} \phi_{ik}) \theta_{ik}^{-1}$$

and means that $\{\theta_j(\theta_{ij}^{-1} \phi_{ij}) \theta_{ij}^{-1} = \theta_i \phi_{ij} \theta_{ij}^{-1}\}$ defines a Čech 1-cocycle i.e. an element of $H^1(T_{S_n/A_n})$. □

Let $d : \mathcal{O}_{S_n} \rightarrow \Omega_{S_n/A_n}$ be the natural $A_n$-derivation. We can define a homomorphism of sheaves of abelian groups $\mathcal{O}_{S_n}^c \rightarrow \Omega_{S_n/A_n}$, $a \mapsto \frac{da}{a}$. It gives rise to a group homomorphism $c : H^1(S_n, \mathcal{O}_{S_n}^c) \rightarrow H^1(S_n, \Omega_{S_n/A_n})$.

As $\Omega_{S_n/A_n}$ is locally free, $H^1(S_n, \Omega_{S_n/A_n}) \simeq \text{Ext}^1(T_{S_n/A_n}, \mathcal{O}_{S_n})$ so that to any line bundle $L_n \in \text{Pic}(S_n)$ we can associate an extension

$$0 \rightarrow \mathcal{O}_{S_n} \rightarrow \mathcal{E}_{L_n} \rightarrow T_{S_n/A_n} \rightarrow 0$$

defined by $c(L_n)$.

For a line bundle $L_n$ represented by the cocycle $\{(V_{ij} \times \text{Spec}(A_n), f_{ij})\}$ with $f_{ij} \in \Gamma(V_{ij} \times \text{Spec}(A_n), \mathcal{O}_{V_{ij} \times \text{Spec}(A_n)})$, the sheaf $\mathcal{E}_{L_n}[\theta_i(V_{ij} \times \text{Spec}(A_n))]/\theta_i(V_{ij} \times \text{Spec}(A_n))$ is isomorphic to $(\mathcal{O}_{S_n} \oplus T_{S_n/A_n})[\theta_i(V_{ij} \times \text{Spec}(A_n))]$ and sections $(a_i, d_i)$ of $(\mathcal{O}_{S_n} \oplus T_{S_n/A_n})[\theta_i(V_{ij} \times \text{Spec}(A_n))]$ are identified on $\theta_i(V_{ij} \times \text{Spec}(A_n))$ if and only if $d_i = d_j$ and $a_i - a_j \equiv \frac{d_i(\theta_i(f_{ij}))}{d_j(\theta_i(f_{ij}))}$.

(we recall that as the cocycle relation translates into $\theta_i(f_{ij}) \theta_j(f_{jk}) = \theta_i(f_{ik})$ for any triple, so that, for example, $f_{ji} \theta_{ji}(f_{jj}) = f_{jj}$.)

The proof of the following theorem follows the one of [S Theorem 3.3.11].

**Theorem 2.10.** Let $(S_n, L_n)$ be a (projective) deformation of a pair $(S, L)$ (with $S$ smooth projective) over $A_n$. There is a 1 to 1 correspondence between

$$\{(S_{D_n}, L_{D_n})\}, \text{ lifting } (S_n, L_n) \text{ on } D_n$/isom

and $H^1(S_n, \mathcal{E}_{L_n})$.

If $L_{D_n}$ has cocycle representation $\{(V_{ij,n} \times \text{Spec}(A_1), f_{ij} + e_i g_{ij})\}$, (using $D_n \simeq A_n \otimes A_1$) where $V_{ij,n} \subset S_n$ is the affine open subset isomorphic to $V_{ij} \times \text{Spec}(A_n)$, restricting to $V_{ij} \subset S$, with $g_{ij} \in \Gamma(V_{ij,n} \times \text{Spec}(A_1), \mathcal{O}_{V_{ij,n} \times \text{Spec}(A_1)})$, and $\{(V_{ij,n}, d_{ij})\} \in H^1(T_{S_n/A_n})$ is the class of the extension $S_{D_n}$ of $S_n$, then the associated class in $H^1(\mathcal{E}_{L_n})$ is represented by the cocycle $\{(V_{ij,n}, \frac{d_{ij}}{\theta_{ij}(f_{ij})})\}$. 
Given a deformation $(S_n, L_n)$ of a pair $(S, L)$ over $A_n$, we can define a homomorphism of sheaves

$$M : \mathcal{E}_{L_n} \rightarrow H^0(S_n, L^n)^* \otimes_{A_n} L_n$$

in the following way: let $\{(V_{ij} \times \text{Spec}(A_n), f_{ij})\}$ be a cocycle representation of $L_n$. Let $V \subset S_n$ be an open set and $\eta \in \Gamma(V, \mathcal{E}_{L_n})$; it is given by a system $\{(a_i, d_i) \in \Gamma(V \times \text{Spec}(A_n)), \mathcal{O}_{S_n}(V) \times \Gamma(V \times \text{Spec}(A_n), T_{S_n/A_n})\}$ such that $d_i = d_j$ and $a_j - a_i = \frac{d_i(\theta(f_{ij}))}{\theta(d_{ij})}$ on $V \cap \theta(V_i \times \text{Spec}(A_n))$. For every section $s = \{(s_i \in \Gamma(\theta(V_i \times \text{Spec}(A_n)), \mathcal{O}_{S_n}))\} \in H^0(S_n, L_n)$ set

$$M(\eta)(s_i) = a_i s_i + d_i(s_i).$$

As done in [S. 3.3.4], a direct calculation, that on $V \cap \theta(V_j \times \text{Spec}(A_n)), f_{ij} \theta^{-1}(M(\eta)(s_j)) = \theta^{-1}(M(\eta)(s_i))$ so that $M(\eta)(s) \in \Gamma(V, L_n)$. Let $\eta \in H^1(\mathcal{E}_{L_n})$ be given by the system $\{\theta(V_{ij,n}), (a_{ij}, d_{ij})\}$; $M$ induces a $A_n$-linear map

$$M_1(\eta) : H^0(L_n) \rightarrow H^1(L_n) \quad (s_i) \mapsto (a_{ij} s_i + d_{ij}(s_i)).$$

The proof of the following proposition follows the one of [S. Proposition 3.3.14].

**Proposition 2.11.** Let $(S_n, L_n)$ be a deformation of the pair $(S, L)$ over $A_n$ and $(S_{D_n}, L_{D_n})$ a lifting of $(S_n, L_n)$ to $D_n$ defined by a class $\eta \in H^1(\mathcal{E}_{L_n})$. Then a section $s \in H^0(L_n)$ extends to a section of $L_{D_n}$ if and only if $s \in \ker(M_1(\eta))$.

Now, we can prove Theorem 1.2.

**Proof of Theorem 1.2.** Let us prove that $H^2_S$ is unobstructed by induction. Let $U = \{U_i \simeq \text{Spec}(P_i)\}$ be an affine open cover of $Y$ and for each $i$, $(x_{1,i}, x_{2,i}, x_{3,i})$ be a regular sequence such that $S \cap U_i \simeq \text{Spec}(P_i/(x_{1,i}, x_{2,i}))$ and $Z \cap U_i \simeq \text{Spec}(P_i/(x_{1,i}, x_{2,i}, x_{3,i}))$.

The line bundle $\mathcal{O}_S(Z)$ has the following cocyclic representation $\{\{U_{ij} \cap S, \frac{d_{ij}(U_{ij})}{\theta(d_{ij})}\}\}$. We have $h^0(\Omega_S(-Z)) = h^0(\Omega_S(\text{alb}^{-1}(Z))) = 1$, so let $\sigma \in H^0(\Omega_S(-Z)) \simeq H^0(N_{S/Y}(-Z)) \simeq \text{Hom}_{\mathcal{O}_S}(\mathcal{I}_{S/Y}/\mathcal{I}_S^2, \mathcal{I}_{S/Y}/\mathcal{I}_S)$. It gives rise to a first order deformation $S_1 \subset Y \times \text{Spec}(A_1)$ of $S$ fixing $Z$. Looking at $\sigma_{U_i \cap S} \in \text{Hom}_{\mathcal{O}_S(U_i)}(P_i/(x_{1,i}, x_{2,i})[x_{1,i}] \otimes P_i/(x_{1,i}, x_{2,i}, x_{3,i}), (x_{3,i}))$ as a couple acting by scalar product we have $\sigma_{U_i \cap S} = (x_{3,i} a_{i1}, x_{3,i} b_{i1})$ for some $a_{i1}, b_{i1} \in P_i/(x_{1,i}, x_{2,i})$ and $S_1 \cap (U_i \times \text{Spec}(A_1) \simeq \text{Spec}(P_i \otimes A_1/(x_{1,i} + x_{3,i} a_{i1}, x_{2,i} + x_{3,i} b_{i1}))$, with $\theta^2 = 0$.

As $(x_{1,i}, x_{2,i}, x_{3,i})/(x_{1,i} + x_{3,i} a_{i1}, x_{2,i} + x_{3,i} b_{i1}) \simeq (\frac{x_{3,i}}{x_{1,i}})$, the Cartier divisor $Z \times \text{Spec}(A_1)$ on $S_1$ admits the representation $\{(S_1 \cap (U_i \times \text{Spec}(A_1)), \frac{x_{3,i}}{x_{1,i}})\}$. We also have the associated line bundle $\mathcal{O}_{S_1}(Z \times \text{Spec}(A_1))$ on $S_1$.

Looking at the Čech resolution of the exact sequence

$$0 \rightarrow T_S \rightarrow T_{Y|S} \rightarrow N_{S/Y} \rightarrow 0$$

by the snake lemma, we see that the class $\kappa_0(\sigma) \in H^1(T_S)$ associated to this first order deformation is represented by a cocycle $\{\{U_{ij} \cap S, d_{ij}(U_{ij}) - d_{ij}(U_{ij})\}\}$, where $d_{ij} \in \text{Der}_{\mathcal{C}}(P_i, P_i/(x_{1,i}, x_{2,i}))$ satisfies $d_{ij}(x_{1,i}) = x_{3,i} a_{i1}, d_{ij}(x_{2,i}) = x_{3,i} b_{i1}$ and $d_{ij}$ is the associated abstract (before pre and post composition by coordinate chart). Given those derivations, we can write an explicit trivialization of the open subset $S_1 \cap (U_i \times \text{Spec}(A_1))$ suited to Theorem 2.11. Consider the commutative diagram with exact rows

$$\begin{array}{c}
0 \rightarrow (x_{1,i}, x_{2,i}) \otimes A_1 \rightarrow P_i \otimes A_1 \rightarrow (P_i/(x_{1,i}, x_{2,i})) \otimes A_1 \rightarrow 0 \\
\Bigg| \psi_i \Bigg| \downarrow \psi_{i+1+t d_i} \downarrow \psi_i \Bigg| \downarrow \\
0 \rightarrow (\psi_{i+1}(x_{1,i}), \psi_{i+1}(x_{2,i})) \rightarrow P_i \otimes A_1 \rightarrow P_i \otimes A_1/(x_{1,i} + x_{3,i} a_{i1}, x_{2,i} + x_{3,i} b_{i1}) \rightarrow 0
\end{array}$$

where $\psi_i$ is the isomorphism induced by the automorphism (as $d_i$ is a derivation) $\psi_i$. Then under this trivialization the local equation of $Z \times \text{Spec}(A_1)$ is $x_{3,i} - t d_i(x_{3,i})$ so that
the interesting cocycle representation (in view of Theorem 2.10) of \(\mathcal{O}_{S_i}(Z \times \text{Spec}(A_1))\) is 
\[
\{(S_i \cap U_{ij}) \times \text{Spec}(A_1), \frac{d_1(x_{3,i})}{x_{3,j}}(1 + t(\text{d}(\frac{d_1(x_{3,i})}{x_{3,j}}) - \text{d}(\frac{d_1(x_{3,i})}{x_{3,j}})))\}.
\]

So the class \(\eta_0(\sigma) \in H^1(\mathcal{E}\mathcal{O}_{S_i}(Z))\) associated to this first order deformation of the pair 
\((S, \mathcal{O}_S(Z))\) induced by \(\sigma \in H^0(\mathcal{N}_{S/Y})\) has, according to Theorem 2.10, cocycle representation 
\[
\{(U_{ij} \cap S), \frac{d_1(x_{3,i})}{x_{3,j}}(1 + t(\text{d}(\frac{d_1(x_{3,i})}{x_{3,j}}) - \text{d}(\frac{d_1(x_{3,i})}{x_{3,j}})))\}.
\]

Now, let \(s \in H^0(\mathcal{O}_S(Z)) \simeq H^0(\text{alb}_S \omega_B)\) represented by \((U_i \cap S, s_i)\). We have
\[
M_1(\eta_0(\sigma))(s) = \{(U_{ij} \cap S), \frac{d_1(x_{3,i})}{x_{3,j}}(1 + t(\text{d}(\frac{d_1(x_{3,i})}{x_{3,j}}) - \text{d}(\frac{d_1(x_{3,i})}{x_{3,j}})))\}
\]
\[
M_1(\eta_0(\sigma))(s) = \{(U_{ij} \cap S, d_i(s_i) - \frac{d_1(x_{3,i})}{x_{3,j}}(1 + t(\text{d}(\frac{d_1(x_{3,i})}{x_{3,j}}) - \text{d}(\frac{d_1(x_{3,i})}{x_{3,j}})))\}
\]
where \(\delta\) is the Čech differential i.e. 
\[
M_1(\eta_0(\sigma))(s) = \{(U_{ij} \cap S, d_i(s_i) - \frac{d_1(x_{3,i})}{x_{3,j}}(1 + t(\text{d}(\frac{d_1(x_{3,i})}{x_{3,j}}) - \text{d}(\frac{d_1(x_{3,i})}{x_{3,j}}))))\}
\]

Now we go through the \(T^1\)-lifting principle. We have \(h^0(\mathcal{N}_{S_i/Y} \times \text{Spec}(A_1))\) 
\[
h^0(\mathcal{N}_{S_1/A_1}(\text{alb}_{S_1/A_1}(T_{B_1/A_1})) = 1. \text{ Choose a generator } \sigma_1 \in H^0(\mathcal{N}_{S_1/Y} \times \text{Spec}(A_1))(-Z \times \text{Spec}(A_1)).\]

By Lemma 2.9 it gives rise to an extension \(S_0\) of \(S_1\) fixing \(Z\). We have \(\sigma(U \cap S \times \text{Spec}(A_1)) = 0\) for some \(a_i, b_i \in P_i \otimes A_1/(x_{1,i} + x_{3,i})\) and then \(S_0\) is an extension after Lemma 2.4 implies 
\[
\mathcal{O}_{S_0}(Z \times \text{Spec}(A_1)) \simeq \text{alb}_{S_0} \omega_{B_0/A_1}.
\]

As \((x_{1,i}, x_{2,i}, x_{3,i})/(x_{1,i} + x_{3,i})\) has the following cocycle representation 
\[
((S_i \cap U_{ij} \times \text{Spec}(A_1)), d_i(\frac{d_1(x_{3,i})}{x_{3,j}}(1 + t(\text{d}(\frac{d_1(x_{3,i})}{x_{3,j}}) - \text{d}(\frac{d_1(x_{3,i})}{x_{3,j}}))))\}
\]
and the class \(\eta_1(\sigma_1) \in H^1(\mathcal{E}\mathcal{O}_{S_i}(Z \times \text{Spec}(A_1))\) associated to the 
\[
((S_i \cap U_{ij} \times \text{Spec}(A_1)), d_i(\frac{d_1(x_{3,i})}{x_{3,j}}(1 + t(\text{d}(\frac{d_1(x_{3,i})}{x_{3,j}}) - \text{d}(\frac{d_1(x_{3,i})}{x_{3,j}})))))
\]

A similar computation as above shows that 
\[
M_1(\eta_1(\sigma_1))(s) = 0 \in H^1(\mathcal{O}_{S_i}(Z \times \text{Spec}(A_1))\) for any section \(s \in H^0(\mathcal{O}_{S_i}(Z \times \text{Spec}(A_1))\). So by Proposition 2.11 
\[
h^0(\mathcal{O}_{S_0}(Z \times \text{Spec}(A_1)) \simeq \text{alb}_{S_0} \omega_{B_0/A_1}.
\]
The map $H^{2,Y}_{S}(D_1)_{S_1} \simeq H^0(\Omega_{S_1/A_1}(-(Z \times \text{Spec}(A_1)))) \to H^{2,Y}_{S}(D_0)_{S} \simeq H^0(\Omega_{S}(-Z))$ is obviously surjective. So by Proposition 2.7, $H^{2,Y}_{S}(A_2) \to H^{2,Y}_{S}(A_1)$ is also surjective i.e. there is an extension $S_2$ of $S_1$ fixing $Z$.

As mentioned after Proposition 2.7, we have a commutative diagram

$$
\begin{array}{ccc}
S_2 \in H^{2,Y}_{S}(A_2) & \longrightarrow & H^{2,Y}_{S}(A_1) \ni S_1 \\
\downarrow h^{2,Y}_{x}(\delta) & & \downarrow h^{2,Y}_{x}(\delta') \\
S_{D_1} \in H^{2,Y}_{S}(D_1) & \longrightarrow & H^{2,Y}_{S}(C_1).
\end{array}
$$

Now, as $\delta : A_2 \to D_1$ is injective, the image of $\text{Spec}(D_1) \to \text{Spec}(A_2)$ is dense. Since the component of the (relative) Picard scheme of $S$ containing $\mathcal{O}_S(Z)$ is proper and $\mathcal{O}_{S_{D_1}}(Z \times \text{Spec}(D_1)) \simeq \widetilde{alb}_{S_{D_1}} \omega_{B_{1}}$, we get $\mathcal{O}_{S_{D_1}}(Z \times \text{Spec}(A_2)) \simeq \widetilde{alb}_{S_2} \omega_{B_2}$. So again $h^0(N_{S_2/Y \times \text{Spec}(A_2)}(-(Z \times \text{Spec}(A_2)))) = 1$ and we can continue the induction.

So a first order deformation of $S$ fixing $Z$ extends to an actual deformation of $S$ fixing $Z$. So for a general $S'$ in the Hilbert scheme component of $S$, $S \cap S'$ meet along a divisor of the linear system $|\widetilde{alb}_B \omega_B|$. As $\widetilde{alb}_B$ is base point free, Lemma 2.1 gives a contradiction. □

3. Interaction with Lagrangian fibrations

We recall the following Proposition found in [3, Proposition 3.5]:

**Proposition 3.1.** Let $\pi : M \to B$ be a smooth Lagrangian fibration with a Lagrangian section. There is a unramified surjective holomorphic map $f : T^*B \to M$, $T^*B$ being the total space of the cotangent bundle of $B$, which commutes with the projection to $B$ and the map $\pi$. Moreover the pull-back of the symplectic form $\omega$ on $M$ by $f$ coincides with the standard symplectic structure of $T^*B$.

**Proof of Proposition 3.1.** Let $U \subset \mathbb{P}^n$ be the maximal Zariski open subset over which $\pi$ is smooth. According to the above Proposition 3.4, there is a unramified surjective holomorphic morphism $f : T^*U \to \pi^{-1}(U)$, which commutes with the respective projections on $U$ and such that $f^*\omega$ is the canonical symplectic form $\omega_{\text{can}}$ on $T^*U$.

Assume $X \cap \pi^{-1}(U) \neq \emptyset$. It is then a Lagrangian submanifold of $(\pi^{-1}(U), \omega|_{\pi^{-1}(U)})$.

As $f$ is a local biholomorphism, $Z^0 : = f^{-1}(X \cap \pi^{-1}(U))$ is a manifold. The same reason implies that $Z^0$ is Lagrangian submanifold of $(T^*U, \omega_{\text{can}})$.

Assume $\pi(Z^0)$ is a proper subvariety of $U$. Choose a coordinates chart $(U', (z_1, \ldots, z_n))$ of $U$ centered at a point $u \in \text{pr}(Z^0) = \pi(X) \cap U$ which is a smooth point of $\text{pr}(Z^0)$ and over which $\pi|_X$ (thus $\text{pr}$) is smooth, so that $\text{pr}(Z^0) \cap U' = \{z_{k+1} = \cdots = z_n = 0\}$.

We recall that in this chart, $\omega_{\text{can}} = \sum_i d\xi_i \wedge dz_i$ where $(z_1, \ldots, z_n, \xi_1, \ldots, \xi_n)$ are the cotangent coordinates associated to $(z_1, \ldots, z_n)$. Let us analyze the affine manifold $Z^0_u \subset \mathbb{P}^{-1}(u) = T^*_u U \simeq \mathbb{C}^n$.

For any $z \in Z^0_u$, as $\text{pr}$ (because $\pi|_X$ is) smooth above $u$, we have the exact sequence

$$0 \to T_{(u,z),pr} \to T_z Z^0 \to T_u \text{pr}(Z^0) \to 0$$

and an isomorphism $T_z Z^0_u \simeq T_{(u,z),pr}$.

The space $T_u \text{pr}(Z^0)$ is generated by $\frac{\partial}{\partial z_i}, \ldots, \frac{\partial}{\partial z_n}$ and any $v \in T_z Z^0_u$ can be written $v = \sum_i a_i \frac{\partial}{\partial z_i}$. As $Z^0$ is Lagrangian, we have

$$0 = \omega_{\text{can}}(v, \frac{\partial}{\partial z_i}) = a_i, \ i = 1, \ldots, r.$$ 

So, we have $T_z Z^0_u \subset \text{Span}(\frac{\partial}{\partial z_i}, k + 1 \leq i \leq n)$, which gives an equality as they have the same dimension.

The tangent space of the complex manifold $Z^0_u \subset \mathbb{C}^n$ at each point is thus equal (as an affine space) to a fixed subspace, so $Z^0_u$ is a finite union of linear spaces.

So the generic fiber of $\text{pr}|_{Z^0}$ is linear. Now looking at the projection $f : Z^0 \to X \cap \pi^{-1}(U)$
we see that the general fibers of \( \pi|_X \) are (union of) compact complex manifolds that admit a surjective (unramified) holomorphic morphism from an affine space, so they are complex tori.

\[ \square \]

**Example 3.2.** (1) Let \( \Sigma \) be an Enriques surface and \( q : S \rightarrow \Sigma \) its universal cover. Take \( L \in \text{Pic}(\Sigma) \) a line bundle giving rise to an elliptic fibration on \( \Sigma \rightarrow |L| \). Then \( q^*L \) also gives rise to an elliptic fibration on \( S \).

Now, \( q^*L \) induces a Lagrangian fibration on \( \pi : S^{[2]} \rightarrow \mathbb{P}^2 \). On the other hand, we have an embedding \( \Sigma \hookrightarrow S^{[2]} \), \( x \mapsto q^{-1}(x) \) and as \( h^{2,0}(\Sigma) = 0 \), \( \Sigma \) is a Lagrangian subvariety of \( S^{[2]} \). Then \( \pi|_{\Sigma} \) is the elliptic fibration of \( \Sigma \) given by \( L \) and \( \pi(\Sigma) \) is a conic.

(2) Let \( C \subset S \) a smooth curve which is a multisection of a K3 surface admitting an elliptic fibration \( S \rightarrow \mathbb{P}^1 \). Then \( C^{[2]} \subset S^{[2]} \) is a Lagrangian subvariety for which \( f|_{C^{[2]}} \) is a finite morphism.

**Remark 3.3.** In these examples, \( \bar{\pi}|_X : X \rightarrow \pi(X) \) is flat.

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