A construction of projective bases for irreducible representations of multiplicative groups of division algebras over local fields

David Kazhdan

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Abstract

Let \( F_0 \) be a local non-archimedian field of positive characteristic, \( D_0 \) a skew-field with center \( F_0 \) and \( G_0 := D_0^\times \) the multiplicative group of \( D_0 \). The goal of this paper is to provide a canonical decomposition of any complex irreducible representation \( V \) of \( G_0 \) in a direct sum of one-dimensional subspaces. I will also consider the case when \( F_0 = \mathbb{Q}_p \), \( p = 2, 3, 5, 7 \) or 13.

1 The case of a finite characteristic

Let \( k = \mathbb{F}_q \) be a finite field, \( F := k(t) \) the field of rational functions on the projective line \( \mathbb{P}^1 \) over \( k \).

Let \( S \) be the set of points of \( \mathbb{P}^1 \). For any point \( s \in S \) we denote by \( F_s \) the completion of \( F \) at \( s \), by \( \nu_s : F_s \to \mathbb{Z} \cup +\infty \) the valuation map and by \( \mathcal{O}_s \subset F_s \) the subring of integers. We denote by \( \mathcal{A} \) the ring of adeles for \( F \).

Let \( D \) be a skew-field with the center \( F \) unramified outside \( \{0, \infty\} \), \( D_0 := D \otimes_F F_0 \) and \( D_\infty := D \otimes_F F_\infty \). We have \( \dim_F(D) = n^2 \).

We denote by \( G \) the multiplicative group of \( D \) considered as an algebraic \( F \)-group, write \( G_s := G(F_s) \) and denote by \( \mathcal{C}_c(G_0) \subset \mathcal{C}(G_0) \) the algebra of locally constant compactly supported functions on \( G_0 \).

For any point \( s \in S \) where \( s \neq 0, \infty \), we identify the group \( G(F_s) \) with \( GL_n(F_s) \) and define \( K_s := GL_n(\mathcal{O}_s) \).

Let \( N_\infty : G_\infty \to F_\infty^\times \) be the reduced norm. We define \( K_\infty = \{g \in D_\infty|\nu_\infty(N_\infty(g)) \geq 0\} \) and \( K_\infty^1 = \{g \in D_\infty|\nu_\infty(N_\infty(g - 1)) > 0\} \).

Then \( K_\infty \subset G_\infty \) is an open compact subgroup and \( K_\infty/K_\infty^1 = \mathbb{F}_q^n \). We define \( K^1 := \prod_{s \in S - \{0, \infty\}} K_s \times K_\infty^1 \).

David Kazhdan
kazhdan.david@gmail.com

1 Hebrew University, Jerusalem, Israel
The multiplication defines a map
\[ \kappa : G_0 \times K_1 \times G(F) \to G(\mathbb{A}). \]

This paper is based on the following result

**Proposition 1.1** The map \( \kappa \) is a bijection.

**Proof** The surjectivity follows from Lemma 7.4 in [3]. To show the injectivity it is sufficient to check the equality
\[(D_0^\times \times K_1) \cap G(F) = \{e\},\]
which is obvious. \(\square\)

We denote by \( R \) the space of \( \mathbb{C} \)-valued locally constant functions on \( G(\mathbb{A})/(K_1 \times G(F)) \); let \( \mathcal{H}_s, \ s \neq 0, \infty, \) be the spherical Hecke algebra at \( s \), and \( \mathcal{H} := \prod_{s \in S - \{0, \infty\}} \mathcal{H}_s \).

We have a natural action \( a \mapsto \hat{a} \) of the commutative algebra \( A := \mathcal{H} \otimes \mathbb{C}[K_0^0/K_1^1] \) on \( R \).

**Corollary 1.2**
1. The natural action of the group \( G_0 \) on the space \( X := G(\mathbb{A})/(K_1 \times G(F)) \) is simply transitive. So we can identify \( X \) with \( G_0 \).
2. The restriction to \( G_0 \) defines a \( G_0 \)-equivariant isomorphism \( u : \mathcal{H} \to \mathbb{C}(G_0) \).
3. For any irreducible representation \( V \) of \( G_0 \) the restriction to \( G_0 \) defines an isomorphism \( u_V : \text{Hom}_{G_0}(V^\vee, \mathcal{H}) \to V \), where \( V^\vee \) is the representation dual to \( V \).
4. There exists a map \( \alpha : A \to \mathbb{C}_c(G_0) \) such that
\[ f \star \alpha(a) = \hat{\alpha}(f), \ a \in A, \ f \in \mathbb{C}_c(G_0). \]

Let \( \Xi \) be the set of homomorphisms \( \chi : \mathcal{H} \otimes \mathbb{C}[\mathbb{F}_q^\times] \to \mathbb{C} \).

For any \( \chi \in \Xi \) we define \( V_{\chi} := \{ v \in V | av = \chi(a)v \} \) for all \( a \in \mathcal{H} \otimes \mathbb{C}[\mathbb{F}_q^\times] \). Let \( \Xi_V = \{ \chi \in \Xi | V_{\chi} \neq \{0\} \} \).

**Theorem 1.3**
1. \( \dim(V_{\chi}) = 1 \) for all \( \chi \in \Xi_V \).
2. \( V = \bigoplus_{\chi \in \Xi_V} V_{\chi} \).

**Proof** As follows from [2] and [6] we have a direct sum decomposition
\[ V = \bigoplus_{\chi \in \Xi_V} V_{\chi} \]

where the subspaces \( V_{\chi} \subset V \) are \( \mathcal{H} \times G_\infty \)-invariant and the representation \( \tilde{\rho}_{\chi} \) of \( \mathcal{H} \times G_\infty \) on \( V_{\chi} \) is irreducible. Since \( \mathcal{H} \) is commutative this implies the irreducibility of the restriction \( \rho_{\chi} \) of \( \tilde{\rho}_{\chi} \) to \( G_\infty \). By definition we can consider \( \rho_{\chi} \) as a representation of the quotient group \( G_\infty/K_1^1 = \mathbb{Z} \times \mathbb{F}_q^\times \) where \( 1 \in \mathbb{Z} \) acts by the Frobenius automorphism on \( \mathbb{F}_q^\times \). It is easy to see that the restriction of any irreducible representation of the group \( \mathbb{Z} \times \mathbb{F}_q^\times \) to \( \mathbb{F}_q^\times \) is the direct sum of distinct one-dimensional representations. \(\square\)
2 The case of special p-adic fields

In this section we consider the case when $F_0 = \mathbb{Q}_p$, $p = 2, 3, 5, 7$ or 13. Let $\mathbb{A}^f \subset \mathbb{A}$ be the subring of finite adeles of $\mathbb{Q}$, $D$ be a quaternion algebra over $\mathbb{Q}$. Fix a maximal order $\mathcal{O} \subset D$ and denote by $\text{Cl}(D)$ the class number of $D$, that is the number of isomorphism classes of left $\mathcal{O}$-modules $M$ that admit an embedding $M \hookrightarrow D$. For any prime $\ell$ we denote by $\mathcal{O}_\ell \subset D(\mathbb{Q}_\ell)$ the completion of $\mathcal{O}$ at $\ell$ and write $\widehat{\mathcal{O}} := \prod_\ell \mathcal{O}_\ell \subset D(\mathbb{A}^f)$.

The following statement is contained in [8] (see especially Theorem 5.7).

Claim 2.1.1. The class number $\text{Cl}(D)$ does not depend on a choice of a maximal order $\mathcal{O}$.

Claim 2.1.2. The class number $\text{Cl}(D)$ is equal to the size of the two-sided quotient $D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}^f)/\widehat{\mathcal{O}}^\times$.

Let $D = D^p$ be the quaternion algebra over $\mathbb{Q}$ ramified at $p$ and $\infty$. The following result is in [1].

Claim 2.2 $\text{Cl}(D^p) = 1$ if $p = 2, 3, 5, 7$ or 13.

Let $\kappa : D^\times(\mathbb{Q}) \times D^\times(\mathbb{R}) \times \widehat{\mathcal{O}}^\times \rightarrow D^\times(\mathbb{A})$ be the product map.

Corollary 2.3 The map $\kappa$ is onto.

We denote by $N : D(\mathbb{A}) \rightarrow \mathbb{A}$ the reduced norm and by $||| : \mathbb{A}^\times \rightarrow \mathbb{R}_{>0}$ the product of local norms. So $||a|| = 1$ for $a \in \mathbb{Q}^\times$.

Let $H \subset D^\times$ be the subgroup formed by the elements of reduced norm 1. Let $\mathcal{H}_\ell$ be the spherical Hecke algebra at $\ell$, and $\mathcal{H} := \bigotimes_{\ell \neq p} \mathcal{H}_\ell$. For $\ell \neq p$ we write $H_\ell$ for $SL(2, \mathbb{Z}_\ell) \subset SL(2, \mathbb{Q}_\ell) = H(\mathbb{Q}_\ell)$. We write $H^1 := H_\mathbb{R} \times \prod_{\ell \neq p} H_\ell$ and denote by $R$ the space of $\mathbb{C}$-valued locally constant functions on $H(\mathbb{A})/(H(\mathbb{Q}) \times H^1)$. The algebra $\mathcal{H}$ acts on $R$.

Let $\kappa_1 : H(\mathbb{Q}) \times H_\mathbb{R} \times H_{\mathbb{R}} \times \prod_{\ell \neq p} H_\ell \rightarrow H(\mathbb{A})$ be the product map.

Claim 2.4 $\kappa_1$ is onto and $\text{Ker}(\kappa_1) = \pm 1$.

Proof The surjectivity follows from Corollary 2.3 and the triviality of the divisor class group for $\mathbb{Q}$. The equality $\text{Ker}(\kappa_1) = \pm 1$ is clear.

Corollary 2.5 For any irreducible representation $V$ of $H_p$ trivial on $-\text{Id}$ we may identify the space $\text{Hom}_{H_p}(V, R)$ with $V^\vee$.

So we have an action of the algebra $\mathcal{H}$ on $V^\vee$. For any morphism $\chi : \mathcal{H} \rightarrow \mathbb{C}$ we define $V_\chi^\vee := \{ \lambda \in V^\vee \mid h\lambda = \chi(h)\lambda \ \forall h \in \mathcal{H} \}$. Let $\Sigma_V = \{ \chi : \mathcal{H} \rightarrow \mathbb{C} \mid V_\chi^\vee \neq 0 \}$. Now the multiplicity one theorem for $SL_2$ (see [7]) implies the validity of the following statement.
Theorem 2.6 For any irreducible representation $V$ of $H_p$ trivial on $-\text{Id}$ we have

1. $\dim V^{\vee}_X = 1 \forall X \in \mathcal{E}_V$.
2. $V^{\vee} = \bigoplus_{X \in \mathcal{E}_V} V^{\vee}_X$.

Question 2.7 Is it possible to extend this construction to the quaternion algebra over $\mathbb{Q}_p$ for all primes $p$?

Remark 2.8 The paper [5] was influenced by [4] and is concerned with the understanding of the local Langlands conjecture. This short paper is a streamlined version of [5].

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