A NOTE ON THE JACOBI FIELDS ON MANIFOLDS

HWAJEONG KIM*

Abstract. We consider Jacobi fields as the first derivatives for $\mathcal{E}$, the energy of harmonic extensions, in a given manifold. In this paper we see that the Jacobi field is bounded by the given boundary map. Here we give no restriction concerned with the curvature for the given manifold.

1. Introduction

A minimal surface with a given boundary is known to be a solution of the Plateau Problem. The calculus of variations for minimal surfaces with given boundary was a very difficult topic for mathematicians for a long time. In 1980’s M. Struwe developed the critical point theory for the solutions of the Plateau Problem in Euclidean space. Furthermore, he discussed conditions for the existence of the unstable minimal surface and constructed the Morse theory for the solutions of the Plateau Problem developing morse theory on a locally convex closed set ([10], [11], [12]).

The critical point theory for the solutions of generalised Plateau problem, where the underlying Euclidean space is replaced by some Riemannian manifold, was discussed by J. Hohrein ([4]). Here the curvatures of the manifolds were nonpositive. Hohrein also gave the conditions for the existence of unstable minimal surfaces, the boundary of which is of one-circle type. In 2009 and 2011, the unstable minimal surfaces of annulus type with boundary of two-circles type in a manifold was discussed by author ([5],[6], [7],[8]), here some manifolds of positive curvature were also considered. Furthermore the morse theory for the minimal surfaces of annulus type in a manifold is developed recently in [9].

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*Corresponding author.
In the above works the minimal surfaces are characterized as the critical points of $\mathcal{E}$ which is the energy of the harmonic extension $\mathcal{F}$ in a Riemannian manifold with metric $h$ for a boundary parametrization $x$ as follows:

$$\mathcal{E} : \mathcal{M} \rightarrow \mathbb{R}$$

$$x \mapsto E(\mathcal{F}(x)) := \frac{1}{2} \int |d\mathcal{F}(x)|_h^2 d\omega.$$  

In Euclidean spaces the derivative of $\mathcal{E}$ is easy to check because of the linear property of the harmonic extension operator defined on the space of boundary parametrizations. In manifolds, we can’t expect the linear property of the harmonic extensions, so the discussion on the derivatives of $\mathcal{E}$ is more complicated than in Euclidean spaces.

The first derivative of $\mathcal{E}$ is known to be the Jacobi field, a vector field along a harmonic mapping $f$ as a weak solution of the following ([4], [5]):

$$\int \langle \nabla J, \nabla X \rangle + \langle \text{tr} R(J, df) df, X \rangle d\omega = 0, \quad X \in H^{1,2}_0 \cap L^\infty(\cdot, f^*TN).$$

We ask about the boundedness and the uniqueness of the Jacobi fields which will follow from the maximum principle. The maximum principle for Jacobi field is discussed in [4], but the nonpositive curvature condition was demanded for the manifolds.

In Theorem 3.1 we will give another proof for the maximum principle for the Jacobi field as the solution of the above system (1). Here we demand no condition for the manifolds. In fact, in [7], [9] the minimal surfaces in some manifolds with positive sectional curvature have been considered.

2. Set-up

2.1. Some definitions

Let $(N, h)$ be a connected, oriented, complete Riemannian manifold of dimension $n \geq 2$ with coordinate chart $(y^1, \cdots, y^n)$, embedded isometrically and properly into some $\mathbb{R}^k$ as a closed submanifold by means of the map $\eta$ ([2]).

Indicating

$$B := \{w = (z^1, z^2) \in \mathbb{R}^2 \mid |w| < 1\}$$
we define
\[ H^{1,2} \cap C^0(B, N) := \{ f \in H^{1,2} \cap C^0(B, \mathbb{R}^k) | f(B) \subset N \} \]
with norm \( \| f \|_{1,2,0} := \| df \|_{L^2} + \| f \|_{C^0} \), where \( df \) is a section of \( T^*B \otimes f^*TN \),
\[ df = f^\alpha dz^\alpha \otimes \frac{\partial}{\partial y^\alpha} \circ f. \]

Now set
\[ T_f H^{1,2} \cap C^0(B, N) \cong \{ V \in H^{1,2} \cap C^0(B, \mathbb{R}^k) | V(\cdot) \in T_f(\cdot)N \} \]
= \( H^{1,2} \cap C^0(B, f^*TN) \),
with norm
\[ \| V \| := \left( \int_B |\nabla f V|^2 h \, dw \right)^{\frac{1}{2}} + \| V \|_{C^0} \cong \left( \int_B |dV|^2_{\mathbb{R}^k} \, d\omega \right)^{\frac{1}{2}} + \| V \|_{C^0}. \]

Let \( \Gamma \) be a Jordan curve in \( N \) diffeomorphic to \( S^1 := \partial B \)
\[ H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma) := \{ u \in H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k) | u(\partial B) = \Gamma \}, \]
with \( \| u \|_{\frac{1}{2},2,0} := ||d\mathcal{H}(u)||_{L^2} + \| u \|_{C^0} \), where \( \mathcal{H}(u) \) is the harmonic extension in \( \mathbb{R}^k \) with \( \mathcal{H}(u)|_{\partial B}(\cdot) = u(\cdot) \).

In addition
\[ T_u H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma) \]
:= \( \{ \xi \in H^{\frac{1}{2},2} \cap C^0(\partial B, u^*TN) | \xi(\cdot) \in T_u(\cdot)\Gamma, \text{ for all } \xi \in \partial B \} \)
= \( H^{\frac{1}{2},2} \cap C^0(\partial B, u^*TT) \).

Finally, the energy of \( f \in H^{1,2}(B, N) \) is defined by
\[ E(f) := \frac{1}{2} \int_B |df|^2 \, dw = \frac{1}{2} \int_B h_{\alpha\beta} \circ f f^\alpha_{\beta} f^\beta_{\alpha} \, dw. \]

2.2. The functional \( \mathcal{E} \)

Let
\[ \mathcal{S}(\Gamma) = \{ X \in H^{1,2}(\overline{B}, N) | X|_{\partial B} \text{ is weakly monotone onto } \Gamma \}. \]
and
\[ M = \{ X|_{\partial B} | X \in \mathcal{S}(\Gamma) \}. \]

We then have all the necessary properties for the further discussion (see [7]). In order that the paper is self-contained, we give a summary of the necessary properties of the above setting:

1. \( M \) is complete with respect to \( \| \cdot \|_{\frac{1}{2},2,0} \).
2. for $x \in M$, there exist a unique harmonic extension of disc type on $B$.

Given $x \in M$ we take the following variations:

observing $\partial B \cong \mathbb{R}/2\pi$, for a given oriented $x \in M$ there exists a weakly monotone map $w_x \in C^0(\mathbb{R}, \mathbb{R})$ with $w_x(\theta + 2\pi) = w_x(\theta) + 2\pi$ such that $x(\theta) = \gamma(\cos(w_x(\theta)), \sin(w_x(\theta))) =: \gamma \circ w(\theta)$. In addition $w_x = \tilde{w}_x + Id$ for some $\tilde{w}_x \in C^0(\partial B, \mathbb{R})$.

Denoting the Dirichlet integral by $D$ and the $\mathbb{R}^k$-harmonic extension by $H$, let

$$W_{\mathbb{R}^k} := \{w \in C^0(\mathbb{R}, \mathbb{R}) | w \text{ is weakly monotone, } w(\theta + 2\pi) = w(\theta) + 2\pi; D(\mathcal{H}(\gamma \circ w)) < \infty \}.$$  

Clearly, $W_{\mathbb{R}^k}$ is convex (for further details, refer to [10]).

For $w \in W_{\mathbb{R}^k}$, $w - w_x$ is actually a map on $\partial B$ and can be considered as a tangent vector along $\tilde{w}_x$. So we can define

$$\mathcal{T}_x = \{d\gamma((w - w_x) \frac{d}{d\theta} \circ \tilde{w}_x) | w \in W_{\mathbb{R}^k} \text{ and } \gamma \circ w_x = x\}.$$  

We have then (see [7]):

1. $\mathcal{T}_x$ is convex in $T_xH^{1,2} \cap C^0(\partial B; \Gamma)$, since $W_{\mathbb{R}^k}$ is convex,
2. For $\xi = d\gamma((w - w_x) \frac{d}{d\theta} \circ \tilde{w}) \in \mathcal{T}_x$ we have that $\exp_x \xi = \gamma \circ w$,
3. there exists $l > 0$, depending on $\gamma$, such that if $\|\xi\|_{\mathcal{T}_x} < l$, then $\exp_x \xi \in M$, for any $x \in M$.

Now we introduce the following set-up.

**Definition 2.1.** (i) We take $M$ and for $x \in M$ the variations from $\mathcal{T}_x$.

(ii) We define a harmonic extension operator $\mathcal{F} : M \to H^{1,2}(B, N)$ such that for $x \in M$ $\mathcal{F}(x)$ is the harmonic extension with $x$ on the boundary $C$.

(iii) We then define

$$\mathcal{E} : M \to \mathbb{R}$$

$$x \mapsto E(\mathcal{F}(x)) := \frac{1}{2} \int_B |d\mathcal{F}(x)|_h^2 d\omega.$$
3. Jacobi fields

In this section we discuss on the first derivative of the harmonic operator

\[ \mathcal{F} : M \longrightarrow H^{1,2} \cap C^0(\overline{B}, N) \]

\[ x \mapsto \mathcal{F}(x). \]

Consider a 2-parameter variation \( f_{st} \) such that \( f_{st}(0, t) \) are harmonic and \( f_{st}(s, 0)|_{\partial M} = f|_{\partial M} \) with

\[ \frac{\partial f_{st}}{\partial s} \bigg|_{s,t=0} = v, \quad \frac{\partial f_{st}}{\partial t} \bigg|_{s,t=0} = w, \]

then

\[ \frac{\partial^2 E(f_{st})}{\partial s \partial t} \bigg|_{s,t=0} = \int \langle \nabla f v, \nabla f w \rangle - \langle \text{tr} R(v, df)df, w \rangle d\omega = 0. \]

Hence, a Jacobi field which is a vector field along a harmonic mapping \( f \) as a weak solution of

\[ \int \langle \nabla J, \nabla X \rangle + \langle \text{tr} R(J, df)df, X \rangle d\omega = 0, \quad \text{for all } X \in H^{1,2} \cap L^\infty(\cdot, f^*TN), \]

is a natural candidate for the derivative of harmonic operators \( \mathcal{F} \).

For \( \xi \in H^{1,2} \cap C^0(\partial B, x^*TT) \), a weak Jacobi field along \( \mathcal{F} \) with boundary \( \xi \), denoted by \( J_\mathcal{F} \), has the following minimal property:

\[ I(J_\mathcal{F}) := \int_B |\nabla^3 J_\mathcal{F}|^2 - \langle \text{tr} R(J_\mathcal{F}, df)df, J_\mathcal{F} \rangle d\omega \]

\[ \leq \int_B |\nabla^3 X|^2 - \langle \text{tr} R(X, df)df, X \rangle d\omega, \]

for all \( X \in H^{1,2}(B, \mathcal{F}^*TN) \) with \( X|_{\partial B} = \xi \). The analogous property holds for Jacobi fields along the harmonic extension on \( B \).

**Theorem 3.1.** Let \( \mathcal{F} \in H^{1,2} \cap C^1 \) be harmonic in \( N \), \( \mathcal{F}|_{\partial B} = \eta \). For \( Y \in H^{1,2} \cap C^0(B; \mathcal{F}^*TN) \), \( I_Y(V) = I(Y + V) \), where \( I \) is defined as above.

Assume \( V \) is the minimizer of \( I_Y \) on \( H^{1,2}_0 \cap C_0 \) then \( A = Y + V \) is essentially bounded and

\[ \|A\|_\infty \leq \|\eta\|_{C_0(\partial\Omega; x^*TT)} \]
Proof. Let $k := \|\eta\|_{C^0} = \max\{|A(z)| : z \in \partial \Omega\}$. We define

$$[A]_k^l := \begin{cases} l & |A(x)| > l \\ |A(x)| & l \geq |A(x)| \geq k \\ k & |A(x)| \leq k \end{cases}$$

and

$$W := C([A]_k^l - k)A \in H^{1,2}_0(\Omega; \mathcal{F}^*TN)$$

where $C$ is a constant to be determined later. By computation

$$\nabla W = \begin{cases} C\langle A, \nabla A \rangle \frac{A}{|A|} + ([A]_k^l - k)\nabla A, & k < |A| < l \\ 0, & \text{otherwise} \end{cases}$$

Taking $W$ as a test vector field

$$0 = \int_B \left( \langle \nabla A, \nabla W \rangle - \langle \text{tr}R(A, d\mathcal{F})d\mathcal{F}, W \rangle \right)d\omega$$

$$= C\int_{B \cap \{z: k < |A(z)| \leq l\}} |\nabla A|^2 ([A]_k^l - k)d\omega$$

$$+ C\int_{B \cap \{z: k < |A(z)| \leq l\}} \frac{1}{|A|} \langle A, \nabla A \rangle^2 d\omega$$

$$- \int_{B \cap \{z: k < |A(z)| \leq l\}} \langle \text{tr}R(A, d\mathcal{F})d\mathcal{F}, A \rangle ([A]_k^l - k)d\omega.$$

If $\nabla A \equiv 0$ or $[A]_k^l - k \equiv 0$ a.e., we have done.

Otherwise, $|\nabla A|^2 ([A]_k^l - k) \geq d > 0$ on some subset with positive measure, so there exists $C > 0$ such that the above $I + II \geq 0$, here we note that $|II| < \infty$. This means

$$\langle A, \nabla A \rangle \equiv 0 \text{ and } |\nabla A|^2 \equiv 0 \text{ a.e. on } \{|z : k < |A(z)| \leq l\}.$$

Therefore by definition of $[A]_k^l$

$$\nabla [A]_k^l \equiv 0 \text{ a.e. for } k < |A(z)| \leq l$$

and

$$|A(z)| \leq k \text{ a.e. for } z \in B.$$

$\Box$
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Hwajeong Kim
Department of Mathematics, Hannam University,
Daejeon 306-791, Korea
hwajkim@hnu.kr