A Physicist’s Proof of the Lagrange-Good Multivariable Inversion Formula

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Abstract

We provide yet another proof of the classical Lagrange-Good multivariable inversion formula using the techniques of quantum field theory.

Key words: Lagrange-Good inversion, Reversion, Quantum field theory.

I Introduction

The Lagrange inversion formula [11] is one of the most useful tools in enumerative combinatorics (see [8, 15]). Various efforts have been devoted to finding purely combinatorial proofs and generalizations of this formula. One of the many such generalizations is the extension from the one variable to the multivariable case. Early contributions in this direction can be found in [12, 10, 4, 16, 14], but the credit for the discovery of the general multivariable formula is usually attributed to the mathematical statistician I. J. Good [8]. We recommend [7] for a clear and thorough presentation as well as for more complete references. Quoted from [7] the Lagrange-Good formula says the following.
Theorem 1  Let the formal power series $f_1, \ldots, f_m$ in the variables $x_1, \ldots, x_m$ be defined by

$$f_i = x_i g_i(f_1, \ldots, f_m), \quad 1 \leq i \leq m$$

(1)

for some formal power series $g_i(x_1, \ldots, x_m)$. Then the coefficient of

$$\frac{x_1^{n_1}}{n_1!} \cdots \frac{x_m^{n_m}}{n_m!} \; \text{in} \; \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_m^{k_m}}{k_m!} g_1^{n_1}(x_1, \ldots, x_m) \cdots g_m^{n_m}(x_1, \ldots, x_m)$$

is equal to the coefficient of

$$\frac{x_1^{n_1}}{n_1!} \cdots \frac{x_m^{n_m}}{n_m!} \; \text{in} \; \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_m^{k_m}}{k_m!} \times \frac{1}{\det(\delta_{ij} - x_i g_{ij}(f_1, \ldots, f_m))}$$

where

$$g_{ij}(x_1, \ldots, x_m) \stackrel{\text{def}}{=} \frac{\partial g_i}{\partial x_j}(x_1, \ldots, x_m).$$

(2)

The odd-looking determinant in the denominator was probably one of the reasons this general formula was not discovered until [8]. Remark however that a similar determinantal denominator appeared earlier in the classical MacMahon master theorem [13]. This is no coincidence since the latter is well known to be the linear special case of the Lagrange-Good formula.

We will use the quantum field theory model introduced in [1], which is related to an earlier formula of G. Gallavotti [6] for the Lindstedt series in KAM theory, in order to express the compositional inverse of a power series in the multivariable setting. Our “proof” of the Lagrange-Good formula will follow from this representation of the formal inverse by straightforward and quite natural field theoretical computations which will, in particular, “explain” the determinantal denominator as a normalization factor for a probability measure. We warn the mathematical reader that reckless use will be made of wildly divergent integrals, if understood in the Lebesgue sense, and of quantum field theory terminology. However, when interpreted according to the formalism of our forthcoming article [2], our “proof” becomes a proof. Upon closer inspection, the reader who is familiar with earlier combinatorial proofs of the Lagrange-Good formula, like say in [7] or [5], will undoubtedly have an impression of “déjà vu”. Indeed, the only connected Feynman graphs of our quantum field theory model are either a single tree or a collection of trees branching off a central loop. In fact, we provided the following “proof”,
not so much for its originality, but for its entertainment value, as another instance of the magic of quantum field theory (see [3]).

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II The “proof”

First, we avoid the use of multiindices and write

\[ g_i(x_1, \ldots, x_m) = \sum_{d \geq 0} \frac{1}{d!} \sum_{\alpha_1, \ldots, \alpha_d = 1}^{m} w_{i,\alpha_1,\ldots,\alpha_d}^{[d]} x_{\alpha_1} \cdots x_{\alpha_d} \]  

(3)

where the tensor element \( w_{i,\alpha_1,\ldots,\alpha_d}^{[d]} \) is completely symmetric in \( \alpha_1, \ldots, \alpha_d \). Therefore the \( f_i(x_1, \ldots, x_m) \) are the solutions of

\[ f_i = x_i \sum_{d \geq 0} \frac{1}{d!} \sum_{\alpha_1, \ldots, \alpha_d = 1}^{m} w_{i,\alpha_1,\ldots,\alpha_d}^{[d]} f_{\alpha_1} \cdots f_{\alpha_d} \]  

(4)

which can be rewritten as the direct reversion problem, with unknowns \( f_1, \ldots, f_m \),

\[ y_i = \Gamma_i(f) \]  

(5)

with \( y_i \stackrel{\text{def}}{=} x_i w_i^{[0]} \) and

\[ \Gamma_i(f) \stackrel{\text{def}}{=} f_i - \sum_{d \geq 1} \frac{1}{d!} \sum_{\alpha_1, \ldots, \alpha_d = 1}^{m} x_i w_{i,\alpha_1,\ldots,\alpha_d}^{[d]} f_{\alpha_1} \cdots f_{\alpha_d} \]  

(6)

that is

\[ \Gamma_i(f) = \sum_{d \geq 1} \frac{1}{d!} \sum_{\alpha_1, \ldots, \alpha_d = 1}^{m} \eta_{i,\alpha_1,\ldots,\alpha_d}^{[d]} f_{\alpha_1} \cdots f_{\alpha_d} \]  

(7)

with

\[ \eta_{i,\alpha}^{[1]} \stackrel{\text{def}}{=} \delta_{i\alpha} - x_i w_{i,\alpha}^{[1]} \text{ for } i, \alpha \in \{1, \ldots, m\} \]  

(8)
and
\[ \eta_{i, \alpha_1 \ldots \alpha_d} \equiv -x_{i} w_{i, \alpha_1 \ldots \alpha_d} \]
for \( d \geq 2 \), and \( i, \alpha_1, \ldots, \alpha_d \in \{1, \ldots, m\} \).

It was shown in [1] (see [2] for more detail) that the solution of such a reversion problem is given by the perturbation expansion of the following quantum field theory one-point function
\[ f_{i} = \int d\phi d\phi \phi_{i} e^{-\phi \bar{\Gamma}(\phi) + \bar{\phi} y} \]
(10)

Here \( \bar{\phi}_1, \ldots, \bar{\phi}_m, \phi_1, \ldots, \phi_m \) are the components of a complex Bosonic field. The integration is over \( \mathbb{C}^m \) with the measure
\[ d\phi d\phi \equiv \prod_{i=1}^{m} \left( \frac{d(\text{Re} \phi_i) d(\text{Im} \phi_i)}{\pi} \right) \]
(11)
we used the notation \( \bar{\phi} \bar{\Gamma}(\phi) \equiv \sum_{i=1}^{m} \bar{\phi}_i \Gamma_i(\phi_1, \ldots, \phi_m) \), and \( \bar{\phi} y \equiv \sum_{i=1}^{m} \bar{\phi}_i y_i \).

If \( \Omega(\phi, \bar{\phi}) \) is a function of the fields, we use the notation
\[ < \Omega(\phi, \bar{\phi}) >_{U} \equiv \int d\phi d\phi \Omega(\phi, \bar{\phi}) e^{-\phi \bar{\Gamma}(\phi) + \bar{\phi} y} \]
(12)
for the corresponding unnormalized correlation function, and
\[ < \Omega(\phi, \bar{\phi}) >_{N} \equiv \frac{1}{Z} < \Omega(\phi, \bar{\phi}) >_{U} \]
(13)
for the corresponding normalized correlation function, where the normalization factor is
\[ Z \equiv \int d\phi d\phi e^{-\phi \bar{\Gamma}(\phi) + \bar{\phi} y} \]
(14)
Finally we denote by \( < \ldots >_{C} \) the connected correlation functions, also known as cumulants or semi-invariants in mathematical statistics and probability theory.

Note that the “action” \( S(\phi, \bar{\phi}) \equiv \phi \bar{\Gamma}(\phi) - \bar{\phi} y \) in the exponential can be separated into quadratic and nonquadratic parts by writing
\[ \Gamma(\phi) = C^{-1} \phi - H(\phi) \]
(15)
with

\[ [C^{-1}]_{ij} \stackrel{\text{def}}{=} \eta^{[1]}_{i,j} = \delta_{ij} - x_i w^{[1]}_{i,j} \]  \hspace{1cm} (16)

and

\[ H(\phi) \stackrel{\text{def}}{=} \sum_{d \geq 2} \sum_{\alpha_1, \ldots, \alpha_d = 1} \left( -\frac{1}{d!} \right) \eta^{[d]}_{i,\alpha_1 \ldots \alpha_d} \phi_{\alpha_1} \cdots \phi_{\alpha_d} . \]  \hspace{1cm} (17)

\( C \) is the free propagator of our theory, \( \overline{\phi} H(\phi) \) is the interaction potential and \( \overline{\phi} y \) contains the sources which can be treated as particular vertices of the interaction. Therefore

\[ e^{-S(\overline{\phi}, \phi)} = e^{-\overline{\phi} C^{-1} \phi + \overline{\phi} H(\phi) + \overline{\phi} y} \]  \hspace{1cm} (18)

and we let

\[ d\mu_C(\overline{\phi}, \phi) \stackrel{\text{def}}{=} \frac{d\overline{\phi} d\phi}{\text{det} C} e^{-\overline{\phi} C^{-1} \phi} \]  \hspace{1cm} (19)

be the normalized complex Gaussian measure with covariance \( C \). As a result

\[ Z = \int d\overline{\phi} d\phi \ e^{-S(\overline{\phi}, \phi)} \]  \hspace{1cm} (20)

\[ = (\text{det} C) \int d\mu_C(\overline{\phi}, \phi) \ e^{\overline{\phi} H(\phi) + \overline{\phi} y} . \]  \hspace{1cm} (21)

Now, by the standard rules of perturbative quantum field theory,

\[ \log \left( \int d\mu_C(\overline{\phi}, \phi) \ e^{\overline{\phi} H(\phi) + \overline{\phi} y} \right) \]

is the sum over connected vacuum Feynman diagrams built using the propagators

\[ \begin{array}{c}
\text{i} \\
\text{---} \\
\text{j}
\end{array} = C_{ij} \]  \hspace{1cm} (22)

the \( H \)-vertices

\[ \begin{array}{c}
\text{i} \\
\text{\bullet} \\
\text{\alpha}_1 \swarrow \text{\alpha}_2 \\
\text{\alpha}_d \searrow \\
\text{j}
\end{array} = -\eta^{[d]}_{i,\alpha_1 \ldots \alpha_d} \]  \hspace{1cm} (23)
with \( d \geq 2 \), and the \( y \)-vertices

\[
\bullet = y_i .
\]  

(24)

These diagrams are made of a single oriented loop of \( H \)-vertices linked by free propagators \( C \), on which tree diagrams terminating with \( y \)-vertices are hooked. Since the sum over such tree diagrams builds the one-point function \( < \phi_i >_N = f_i = \Gamma^{-1}(y) \), it is easy to see that

\[
\log \left( \int d\mu_C(\phi, \phi) \ e^{\phi H(\phi) + \bar{\phi}y} \right) = \sum_{k \geq 1} \frac{1}{k} \text{tr} \left[ C \partial H(\Gamma^{-1}(y)) \right]^k
\]

(25)

where \( \partial H(z) \) is the matrix with entries \( \frac{\partial H}{\partial z_j}(z) \). Therefore

\[
Z = (\det C) e^{-\text{tr} \log(I - C \partial H(\Gamma^{-1}(y)))}
\]

(26)

or

\[
Z^{-1} = \det \left( C^{-1}(I - C \partial H(\Gamma^{-1}(y))) \right)
\]

(27)

\[
= \det \left( C^{-1} - \partial H(\Gamma^{-1}(y)) \right)
\]

(28)

Now note that

\[
\partial \Gamma(\phi) = C^{-1} - \partial H(\phi)
\]

(29)

so

\[
Z^{-1} = \det \left( \partial \Gamma(\Gamma^{-1}(y)) \right) = \det(\partial \Gamma(f)) .
\]

(30)

Now we also have by (8)

\[
\Gamma_i(f) = f_i - (x_i g_i(f) - x_i w_i^{[0]})
\]

(31)

and thus

\[
[\partial \Gamma(f)]_{ij} = \frac{\partial}{\partial f_j} \left( f_i - x_i g_i(f) + x_i w_i^{[0]} \right)
\]

\[
= \delta_{ij} - x_i g_{ij}(f)
\]

(32)

(33)
that is

\[ Z = \frac{1}{\det(\delta_{ij} - x_i g_{ij}(f))} \] (34)

which is our interpretation of the determinantal denominator in the Lagrange-Good formula as a normalization factor for a probability measure.

Besides, (31) can be rewritten as

\[ \Gamma_i(f) - y_i = f_i - x_i g_i(f) \] (35)

that is (10) becomes

\[ f_i = \frac{1}{Z} \int d\phi d\bar{\phi} \phi_i e^{-\bar{\phi}\phi + \bar{\phi}x g(\phi)} \] (36)

with \( \bar{\phi}x g(\phi) \overset{\text{def}}{=} \sum_{i=1}^{m} \bar{\phi}_i x_i g_i(\phi) \) and

\[ Z = \int d\phi d\bar{\phi} e^{-\bar{\phi}\phi + \bar{\phi}x g(\phi)} . \] (37)

Now

\[ \frac{f_1^{k_1}}{k_1!} \cdots f_m^{k_m}}{k_m!} \times \frac{1}{\det(\delta_{ij} - x_i g_{ij}(f))} = \frac{Z < \phi_1 >^{k_1} \cdots < \phi_m >^{k_m}}{k_1! \cdots k_m!} \] (38)

but

\[ < \phi_1^{k_1} \cdots \phi_m^{k_m} >_N = < \phi_1 >^{k_1}_C \cdots < \phi_m >^{k_m}_C \] (39)

because a connected graph can hook to at most one of the sources \( \phi_i \). As a result

\[ \frac{f_1^{k_1}}{k_1!} \cdots f_m^{k_m} \times \frac{1}{\det(\delta_{ij} - x_i g_{ij}(f))} = \int d\phi d\bar{\phi} \frac{\phi_1^{k_1}}{k_1!} \cdots \frac{\phi_m^{k_m}}{k_m!} e^{-\bar{\phi}\phi + \bar{\phi}x g(\phi)} . \] (40)

Now it all becomes very simple since, on expanding \( e^{\bar{\phi}x g(\phi)} \), one gets

\[ \frac{f_1^{k_1}}{k_1!} \cdots f_m^{k_m} \times \frac{1}{\det(\delta_{ij} - x_i g_{ij}(f))} = \\
\sum_{n_1, \ldots, n_m=0}^{+\infty} \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_m^{n_m}}{n_m!} \int d\mu_1(\bar{\phi}, \phi) \prod_{a=1}^{m} \left( \frac{\phi_k^{k_a}}{k_a!} \right) \prod_{b=1}^{m} (\phi_b g_b(\phi))^{n_b} \] (41)
where \( d\mu_I(\overline{\phi}, \phi) \defeq d\phi d\phi e^{-\phi \phi} \), the Gaussian measure with covariance equal to the identity matrix. Now, by integration of the \( \phi \)'s by parts,

\[
\int d\mu_I(\overline{\phi}, \phi) \prod_{a=1}^m \left( \frac{\phi_a^k}{k_a!} \right) \prod_{b=1}^m (\overline{\phi_b} g_b(\phi))^n_b = \int d\mu_I(\overline{\phi}, \phi) \Omega(\phi)
\]

with

\[
\Omega(\phi) \defeq \frac{\partial^n_{\phi} \ldots \partial^n_{\phi}}{\partial_{\phi_1^{n_1}} \ldots \partial_{\phi_m^{n_m}}} \left( \frac{\phi_1^{k_1}}{k_1!} \ldots \frac{\phi_m^{k_m}}{k_m!} g_1(\phi)^{n_1} \ldots g_m(\phi)^{n_m} \right).
\]

Since \( \Omega(\phi) \) only depends on \( \phi \), \( \int d\mu_I(\overline{\phi}, \phi) \Omega(\phi) \) is equal to the constant term of \( \Omega(\phi) \) which is easily seen to be the coefficient of

\[
\frac{\phi_1^{n_1}}{n_1!} \ldots \frac{\phi_m^{n_m}}{n_m!} \quad \text{in} \quad \frac{\phi_1^{k_1}}{k_1!} \ldots \frac{\phi_m^{k_m}}{k_m!} g_1(\phi)^{n_1} \ldots g_m(\phi)^{n_m}.
\]

which concludes our “proof”.

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