On the difference between the eccentric connectivity index and eccentric distance sum of graphs

Yaser Alizadeh \textsuperscript{a}  Sandi Klavžar \textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Hakim Sabzevari University, Sabzevar, Iran  
\text{e-mail: y.alizadeh.s@gmail.com}

\textsuperscript{b} Faculty of Mathematics and Physics, University of Ljubljana, Slovenia  
\text{e-mail: sandi.klavzar@fmf.uni-lj.si}

Abstract

The eccentric connectivity index of a graph $G$ is $\xi^c(G) = \sum_{v \in V(G)} \epsilon(v) \deg(v)$, and the eccentric distance sum is $\xi^d(G) = \sum_{v \in V(G)} \epsilon(v)D(v)$, where $\epsilon(v)$ is the eccentricity of $v$, and $D(v)$ the sum of distances between $v$ and the other vertices. A lower and an upper bound on $\xi^d(G) - \xi^c(G)$ is given for an arbitrary graph $G$. Regular graphs with diameter at most 2 and joins of cocktail-party graphs with complete graphs form the graphs that attain the two equalities, respectively. Sharp lower and upper bounds on $\xi^d(T) - \xi^c(T)$ are given for arbitrary trees. Sharp lower and upper bounds on $\xi^d(G) + \xi^c(G)$ for arbitrary graphs $G$ are also given, and a sharp lower bound on $\xi^d(G)$ for graphs $G$ with a given radius is proved.

Key words: eccentricity; eccentric connectivity index; eccentric distance sum; tree

AMS Subj. Class (2020): 05C12, 05C09, 05C92

1 Introduction

In this paper we consider simple and connected graphs. If $G = (V(G), E(G))$ is a graph and $u, v \in V(G)$, then the distance $d_G(u, v)$ between $u$ and $v$ is the number of edges on a shortest $u, v$-path. The eccentricity of a vertex and its total distance are distance properties of central interest in (chemical) graph theory; they are defined as follows. The eccentricity $\epsilon_G(v)$ of a vertex $v$ is the distance between $v$ and a farthest vertex from $v$, and the total distance $D_G(v)$ of $v$ is the sum of distances between $v$ and the other vertices of $G$. Even more fundamental property of a vertex in (chemical) graph theory is its degree (or valence in chemistry), denoted by $\deg_G(v)$. (We may skip the
index $G$ in the above notations when $G$ is clear.) Multiplicatively combining two out
of these three basic invariants naturally leads to the eccentric connectivity index $\xi^c(G)$, the eccentric distance sum $\xi^d(G)$, and the degree distance $DD(G)$, defined as follows:

$$
\xi^c(G) = \sum_{v \in V(G)} \varepsilon(v) \deg(v).
$$

$$
\xi^d(G) = \sum_{v \in V(G)} \varepsilon(v) D(v).
$$

$$
DD(G) = \sum_{v \in V(G)} \deg(v) D(v).
$$

$\xi^c$ was introduced by Sharma, Goswami, and Madan [17], $\xi^d$ by Gupta, Singh, and Madan [7], and $DD$ by Dobrynin and Kochetova [6] and by Gutman [8]. These three topological indices are well investigated, selected contributions to the eccentric connectivity index are [10, 13, 22], to the eccentric distance sum [11, 14, 21], and to the degree distance [15, 18, 19]. The three invariants were also compared to other invariants, cf. [2, 3, 4, 5, 23]. For information on additional topological indices based on eccentricity see [16].

In [11] the eccentric distance sum and the degree distance are compared, while in [24] the difference between the eccentric connectivity index and the (not defined here) connective eccentricity index is studied. The primary motivation for the present paper, however, are the papers [12, 25] in which $\xi^d(G) - \xi^c(G)$ was investigated. In [25], Zhang, Li, and Xu, besides other results on the two indices, determined sharp upper and lower bounds on $\xi^d(G) - \xi^c(G)$ for graphs $G$ of given order and diameter 2. Parallel results were also derived for sub-classes of diameter 2 graphs with specified one of the minimum degree, the connectivity, the edge-connectivity, and the independence number. Hua, Wang, and Wang [12] extended the last result to general graphs. More precisely, they characterized the graphs that attain the minimum value of $\xi^d(G) - \xi^c(G)$ among all connected graphs $G$ of given independence number. They also proved a related result for connected graphs with given matching number.

In this paper we continue the investigation along the lines of [12, 25] and proceed as follows. In the rest of this section definitions and some observations needed are listed. In Section 2 we give a lower and an upper bound on $\xi^d(G) - \xi^c(G)$ and in both cases characterize the equality case. The upper bound involves the Wiener index, the first Zagreub index, as well as the degree distance of $G$. In Section 3 we focus on trees and first prove that among all trees $T$ with given order and diameter, $\xi^d(T) - \xi^c(T)$ is minimized on caterpillars. Using this result we give a lower bound on $\xi^d(T) - \xi^c(T)$ for all trees $T$ with given order, the bound being sharp precisely on stars. We also give a sharp upper bound on $\xi^d(T) - \xi^c(T)$ for trees $T$ with given order. In the last section we give a sharp lower bound and a sharp upper bound on $\xi^d(G) + \xi^c(G)$, compare $\xi^d(G)$ with $\xi^c(G)$ for graphs $G$ with not too large maximum degree, and give a sharp lower bound on $\xi^d(G)$ for graphs $G$ with a given radius.
1.1 Preliminaries

The order and the size of a graph $G$ will be denoted by $n(G)$ and $m(G)$, respectively. The star of order $n \geq 2$ is denoted by $S_n$; in other words, $S_n = K_{1,n-1}$. If $n \geq 2$, then the cocktail party graph $CP_{2n}$ is the graph obtained from $K_{2n}$ by removing a perfect matching. The join $G \oplus H$ of graphs $G$ and $H$ is the graph obtained from the disjoint union of $G$ and $H$ by connecting by an edge every vertex of $G$ with every vertex of $H$. The maximum degree of a vertex of $G$ is denoted by $\Delta(G)$. A graph $G$ is regular if all vertices have the same degree. The first Zagreb index $M_1(G)$ of $G$ is the sum of the squares of the degrees of the vertices of $G$. The Wiener index $W(G)$ of $G$ is the sum of distances between all pairs of vertices in $G$.

The diameter $\text{diam}(G)$ and the radius $\text{rad}(G)$ of a graph $G$ are the maximum and the minimum vertex eccentricity in $G$, respectively. A graph $G$ is self-centered if all vertices have the same eccentricity. If this eccentricity is $d$, we further say that $G$ is $d$-self-centered. The eccentricity $\varepsilon(G)$ of $G$ is

$$\varepsilon(G) = \sum_{v \in V(G)} \varepsilon(v).$$

The eccentric connectivity index of $G$ can be equivalently written as

$$\xi^c(G) = \sum_{uv \in E(G)} (\varepsilon(u) + \varepsilon(v)), \quad (1)$$

and the eccentric distance sum as

$$\xi^d(G) = \sum_{\{u,v\} \subseteq V(G)} (\varepsilon(u) + \varepsilon(v))d(u,v). \quad (2)$$

2 The difference on general graphs

In this section we give some sharp upper and lower bounds on $\xi^d(G) - \xi^c(G)$ for an arbitrary graph $G$. The bounds are in terms of the eccentricity, the Wiener index, the first Zagreb index, the degree distance, the maximum degree, the size, and the order of $G$.

**Theorem 2.1** If $G$ is a connected graph, then the following hold.

(i) $\xi^d(G) - \xi^c(G) \geq 2(n(G) - 1 - \Delta(G))\varepsilon(G)$. Moreover, the equality holds if and only if $G$ is a regular graph with $\text{diam}(G) \leq 2$.

(ii) $\xi^d(G) - \xi^c(G) \leq 2n(G)(W(G) - m(G)) + M_1(G) - DD(G)$. Moreover, the equality holds if and only if $G \in \{P_4\} \cup \{CP_{2k} \oplus K_{n(G)-2k} : 0 \leq k \leq n/2\}$. 

3
The equality holds if and only if $D$ that $G$ consequently $D$ simplifies to $v$.

Proof. (i) Let $v$ be a vertex of $G$. If $w$ is not adjacent to $v$, then $d(v, w) \geq 2$ and consequently $D(v) - \deg(v) \geq 2(n(G) - 1 - \Delta(G))$. Thus:

$$\xi^d(G) - \xi^c(G) = \sum_{v \in V(G)} \varepsilon(v)(D(v) - \deg(v))$$

$$\geq \sum_{v \in V(G)} 2\varepsilon(v)(n(G) - 1 - \Delta(G))$$

$$= 2\varepsilon(G)(n(G) - 1 - \Delta(G)).$$

The equality holds if and only if $D(v) - \deg(v) = 2(n(G) - 1 - \Delta(G))$ for every vertex $v$. As the last equality in particular holds for a vertex of maximum degree, we infer that $G$ must be regular. Then the condition $D(v) - \deg(v) = 2(n(G) - 1 - \Delta(G))$ simplifies to

$$D(v) + \Delta(G) = 2n(G) - 2. \quad (3)$$

Suppose that $\text{diam}(G) = d$, and let $x_i, i \in \{2, \ldots, d\}$, be the number of vertices at distance $i$ from $v$. Then $n(G) = 1 + \Delta(G) + x_2 + \cdots + x_d$ and $D(v) = \Delta(G) + 2x_2 + \cdots + dx_d$. Plugging these equalities into (3) yields

$$2\Delta(G) + 2x_2 + \cdots + dx_d = 2 + 2\Delta(G) + 2x_2 + \cdots + 2x_d - 2$$

which implies that $x_3 = \cdots = x_d = 0$, that is, $\text{diam}(G) = 2$. Finally, if $\text{diam}(G) = 2$, then $D(v) = \Delta(G) + 2(n(G) - \Delta(G) - 1)$, so (3) is fulfilled for every regular graph of diameter 2. Clearly, (3) is also fulfilled for graphs of diameter 1, that is, complete graphs.

(ii) If $v \in V(G)$, then clearly $\varepsilon(v) \leq n(G) - \deg(v)$. Then we deduce that

$$\xi^d(G) - \xi^c(G) = \sum_{v \in V(G)} \varepsilon(v)(D(v) - \deg(v))$$

$$\leq \sum_{v \in V(G)} (n(G) - \deg(v))(D(v) - \deg(v))$$

$$= n(G) \sum_{v \in V(G)} (D(v) - \deg(v)) + \sum_{v \in V(G)} \deg(v)^2$$

$$- \sum_{v \in V(G)} \deg(v)D(v)$$

$$= 2n(G)(W(G) - m(G)) + M_1(G) - DD(G).$$

The equality in the above computation holds if and only if $\varepsilon(v) = n(G) - \deg(v)$ holds for all $v \in V(G)$. So suppose that $G$ is a graph for which $\varepsilon(v) = n(G) - \deg(v)$ holds for all $v \in V(G)$ and distinguish the following two cases.

Suppose first that $\text{diam}(G) \geq 3$. Let $P$ be a diametral path in $G$ and let $v$ and $v'$ be its endpoints. Since $\varepsilon(v) = n(G) - \deg(v)$ and $|V(P) \setminus N[v]| = \varepsilon(v) - 1$, it follows
that $n(G) = 1 + \deg(v) + |V(P) \setminus N[v]|$. The latter means that $V(G) = N[v] \cup V(P)$.

Since $\text{diam}(G) = \varepsilon(v) \geq 3$ it follows that $\deg(v') = 1$. Since we have also assumed that $\varepsilon(v') = n(G) - \deg(v')$ holds we see that $\varepsilon(v') = n(G) - 1$ which in turn implies that $G$ is a path. Among the paths $P_n$, $n \geq 4$, the path $P_4$ is the unique one which fulfills the condition $\varepsilon(v) = n - \deg(v)$ for all $v \in V(P_n)$.

Suppose second that $\text{diam}(G) \leq 2$. Then $\varepsilon(v) \in \{1, 2\}$ for every $v \in (G)$. Since $\varepsilon(v) = n(G) - \deg(v)$ it follows that $\deg(v) \in \{n(G) - 1, n(G) - 2\}$. Let $V_1 = \{v: \deg(v) = n(G) - 1\}$ and $V_2 = \{v: \deg(v) = n(G) - 2\}$. Then $V(G) = V_1 \cup V_2$. Clearly, the subgraph of $G$ induced by $V_1$ is complete, and there are all possible edges between $V_1$ and $V_2$. Moreover, the complement of the subgraph of $G$ induced by $V_2$ is a disjoint union of copies of $K_2$, which means that $V_2$ induces a cocktail party graph. In summary, $G$ must be of the form $CP_{2k} \oplus K_{n(G) - 2k}$, where $0 \leq k \leq n/2$. On the other hand, the condition $\varepsilon(v) = n(G) - \deg(v)$ clearly holds for each vertex of $CP_{2k} \oplus K_{n(G) - 2k}$, hence these graphs together with $P_4$ from the previous case are precisely the graphs that attain the equality. 

\[
\text{3 The difference on trees}
\]

In this section we turn our attention to $\xi^d(T) - \xi^c(T)$ for trees $T$, and in particular on extremal trees regarding this difference.

**Theorem 3.1** Among all trees $T$ with given order and diameter, $\min\{\xi^d(T) - \xi^c(T)\}$ is achieved on caterpillars.

**Proof.** Fix the order and diameter of trees to be considered. Let $T$ be an arbitrary tree that is not a caterpillar with this fixed order and diameter. Let $P$ be a diametral path of $T$ connecting $x$ to $y$. Then the eccentricity of each vertex $w$ of $T$ is equal to $\max\{d(w, x), d(w, y)\}$. Let $z \neq x, y$ be a vertex of $P$ and let $T_z$ be a maximal subtree of $T$ which contains $z$ but no other vertex of $P$. We may assume that $z$ can be selected such that $\varepsilon_{T_z}(z) = k \geq 2$, for otherwise $T$ is a caterpillar. Let $u$ be vertex of $T_z$ with $d(u, z) = k - 1$ and let $v$ be the neighbor of $u$ with $d(v, z) = k - 2$. Let $S = N(u) \setminus \{v\}$ and let $s = |S|$. Note that $s > 0$. Let now $T'$ be the tree obtained from $T$ by replacing the edges between $u$ and the vertices of $S$ with the edges between $v$ and the vertices of $S$.

**Claim A:** $\xi^d(T) - \xi^c(T) > \xi^d(T') - \xi^c(T')$.

Set $X_d = \xi^d(T) - \xi^d(T')$ and $X_c = \xi^c(T) - \xi^c(T')$. To prove the claim it is equivalent to show that $X_d - X_c > 0$.

For a vertex $w \in V(G) \setminus (S \cup \{u\})$ we have $D_{T'}(w) = D_{T}(w) - s$ and $\varepsilon_{T'}(w) \leq \varepsilon_T(w)$. Moreover if $w \in S$, then $\varepsilon_{T'}(w) = \varepsilon_T(w) - 1$ and $D_{T'}(w) = D_{T}(w) + n - s - 2$. With
these facts in hand we can compute as follows.

\[
X_d = \sum_{w \in V(T)} \varepsilon_T(w)D_T(w) - \sum_{w \in V(T')} \varepsilon_{T'}(w)D_{T'}(w) \\
= \varepsilon_T(u)D_T(u) - \varepsilon_{T'}(u)D_{T'}(u) + \varepsilon_T(v)D_T(v) - \varepsilon_{T'}(v)D_{T'}(v) \\
+ \sum_{w \in S} \varepsilon_T(w)D_T(w) - \varepsilon_{T'}(w)D_{T'}(w) \\
+ \sum_{w \in V(T)-(S \cup \{u,v\})} \varepsilon_T(w)D_T(w) - \varepsilon_{T'}(w)D_{T'}(w) \\
\geq s(\varepsilon_T(v) - \varepsilon_T(u)) + \sum_{w \in V(T)-(S \cup \{u,v\})} \varepsilon_T(w)s \\
+ \sum_{w \in S} ((D_T(w) - n + 2 + s) - \varepsilon_T(u)(-n + 2 + s)) \\
= -s + \sum_{w \in V(T)-(S \cup \{u,v\})} \varepsilon_T(w)s \\
+ \sum_{w \in S} ((D_T(w) - n + 2 + s) - \varepsilon_T(u)(-n + 2 + s)) \\
= -s + \sum_{w \in V(T)-(S \cup \{u,v\})} \varepsilon_T(w)s \\
+ s(n - s - 2)\varepsilon_T(u) + s(D_T(u) + n - 2) \\
= [\varepsilon(T) - \varepsilon_T(u)(s + 2) - s + 1 + (n - s - 2)\varepsilon_T(u) + D_T(u) + n - 3].
\]

Similarly, but shorter, we get that $X_c = 2s$. Thus

\[
X_d - X_c \geq s[\varepsilon(T) - \varepsilon_T(u)(s + 2) \\
+ (n - s - 2)\varepsilon_T(u) + D_T(u) + n - 4] > 0.
\]

This proves Claim A. If $T'$ is not a caterpillar, we can repeat the construction as many times as required to arrive at a caterpillar. Since at each step the value of $\xi^d - \xi^c$ is decreased, the minimum of this difference is indeed achieved on caterpillars. \qed

**Theorem 3.2** If $T$ is a tree of order $n \geq 3$, then

\[
\xi^d(T) - \xi^c(T) \geq 4n^2 - 12n + 8.
\]

Moreover, equality holds if and only if $T = S_n$.  

6
Proof. Let $n \geq 3$ be a fixed integer. By Theorem 3.1 it suffices to consider caterpillars. More precisely, let $T$ be a caterpillar of order $n$ and with $\text{diam}(T) = d \geq 3$. Then we wish to prove that $\xi^d(T) - \xi^e(T) > \xi^d(S_n) - \xi^e(S_n) = 4n^2 - 12n + 8$. The latter equality is straightforward to check, for the strict inequality we proceed as follows.

Let $w, z \in V(T)$ be two adjacent vertices of eccentricities $d-1$ and $d-2$, respectively. Let $S = N(w) \setminus \{z\}$ and set $s = |S|$. As $\varepsilon(w) = d - 1$, we have $s \geq 1$. Let further $S_1 = V(G) \setminus (S \cup \{w, z\})$. Construct now a tree $T'$ from $T$ by replacing the edges between $w$ and the vertices of $S$ with the edges between $z$ and the vertices of $S$. Note that $\text{deg}_T(w) = \text{deg}_{T'}(w) + s = 1 + s$ and $\text{deg}_T(z) = \text{deg}_{T'}(z) - s$, while the other vertices have the same degree in $T$ and $T'$. Further, it is straightforward to verify the following relations:

$$D_T(w) = D_{T'}(w) - s, \quad \varepsilon_T(w) = \varepsilon_{T'}(w);$$
$$D_T(z) = D_{T'}(z) + s, \quad \varepsilon_T(z) \leq \varepsilon_{T'}(z) \leq \varepsilon_{T'}(z) + 1;$$
$$D_T(x) = D_{T'}(x) + n - s - 2, \quad \varepsilon_T(x) = \varepsilon_{T'}(x) + 1 \; (x \in S);$$
$$D_T(y) = D_{T'}(y) + s, \quad \varepsilon_T(y) \leq \varepsilon_{T'}(y) \leq \varepsilon_{T'}(y) + 1 \; (y \in S_1).$$

Setting $X_d = \xi^d(T) - \xi^d(T')$ we have:

$$X_d = \sum_{v \in \{w, z\}} D_T(v)\varepsilon_T(v) - D_{T'}(v)\varepsilon_{T'}(v) + \sum_{v \in S} D_T(v)\varepsilon_T(v) - D_{T'}(v)\varepsilon_{T'}(v)$$
$$+ \sum_{v \in S_1} D_T(v)\varepsilon_T(v) - D_{T'}(v)\varepsilon_{T'}(v)$$
$$\geq s(\varepsilon_{T'}(z) - \varepsilon_T(w)) + \sum_{v \in S} D_T(v)\varepsilon_T(v) - (D_T(v) - (n - s - 2))(\varepsilon_T(v) - 1)$$
$$+ \sum_{v \in S_1} D_T(v)\varepsilon_T(v) - (D_{T'}(v) - s)\varepsilon_T(v)$$
$$\geq -s + (n - s - 2)\varepsilon_T(v) + \sum_{v \in S} D_T(v) - s(n - s - 2) + s \sum_{v \in S} \varepsilon_T(v)$$
$$\geq -s + 3s(n - s - 2) + s(2(n - s - 2) + 2s + 1) - s(n - s - 2)$$
$$+ 3s(n - s - 2)$$
$$= 5s(n - s - 3) + 2s(n - 1).$$
Similarly, setting $X_c = \xi^c(T) - \xi^c(T')$, we have

$$X_c = \sum_{v \in \{w, z\}} (\deg_T(v) \varepsilon_T(v) - \deg_{T'}(v) \varepsilon_{T'}(v))$$

\[ + \sum_{v \in S} (\deg_T(v) \varepsilon_T(v) - \deg_{T'}(v) \varepsilon_{T'}(v)) \]

\[ + \sum_{v \in S_1} (\deg_T(v) \varepsilon_T(v) - \deg_{T'}(v) \varepsilon_{T'}(v)) \]

\[ \leq s \varepsilon_T(w) + \deg_T(z) \varepsilon_T(z) - (\deg_T(z) + s)(\varepsilon_T(z) - 1) \]

\[ + s + \sum_{v \in S_1} \deg_T(v) \varepsilon_T(v) - \deg_{T'}(v) \varepsilon_{T'}(v) \]

\[ = 2s + \deg_T(z) + \sum_{v \in S_1} \deg(v) \]

Therefore,

$$X_d - X_c \geq (5s(n - s - 3) + 2s(n - 1)) - (2n - 3) > 0,$$

that is, $\xi^d(T) - \xi^c(T) > \xi^d(T') - \xi^c(T')$. Repeating the above transformation until $S_n$ is constructed finishes the argument. 

To bound the difference $\xi^d(T) - \xi^c(T)$ for an arbitrary tree $T$ from above, we first recall the following result.

**Lemma 3.3** [14, Theorem 2.1] Let $w$ be a vertex of graph $G$. For non-negative integers $p$ and $q$, let $G(p, q)$ denotes the graph obtained from $G$ by attaching to vertex $w$ pendant paths $P = wv_1 \cdots v_p$ and $Q = wu_1 \cdots u_q$ of lengths $p$ and $q$, respectively. Let $G(p + q, 0) = G(p, q) - wu_1 + v_pv_1$. If $r = \varepsilon_G(w)$ and $r \geq p \geq q \geq 1$, then

$$\xi^d(G(p + q, 0)) - \xi^d(G(p, q)) \geq \frac{pq}{6} \left[ 6D_G(w) + p(2p - 3) + q(2q - 3) + 3pq - 12r + 6n(G)(p + q + r + 1) + 6 \sum_{v \in V(G)} \varepsilon(v) \right].$$

**Lemma 3.4** Let $G$, $p$, $q$, $G(p, q)$, and $G(p + q, 0)$ be as in Lemma 3.3. Then

$$\xi^c(G(p + q, 0)) - \xi^c(G(p, q)) \leq q(3p + 2m(G) - 1).$$
Proof. Let \( \text{deg}'(v) \) and \( \varepsilon'(v) \) (resp. \( \text{deg}(v) \) and \( \varepsilon(v) \)) denote the degree and the eccentricity of \( v \) in \( G(p + q, 0) \) (resp. \( G(p, q) \)). Then we have:

\[
\begin{align*}
\text{deg}'(w) &= \text{deg}(w) - 1, \quad \varepsilon'(w) \leq \varepsilon(w) + q; \\
\text{deg}'(v_i) &= \text{deg}(v_i), \quad i \in [p - 1], \quad \text{deg}'(v_p) = \text{deg}(v_p) + 1; \\
\varepsilon'(v_i) &\leq \varepsilon(v_i) + q, \quad i \in [p]; \\
\text{deg}'(u_j) &= \text{deg}(u_j), \quad \varepsilon'(u_j) = \varepsilon(u_j) + p; \\
\varepsilon'(x) &\leq \varepsilon(x) + q, \quad x \in V(G).
\end{align*}
\]

Moreover, the degrees of vertices in \( G(p + q, 0) \) do not decrease. Calculating the difference of contributions of vertices in \( \xi'_c \) for \( G(p + q, 0) \) and \( G(p, q) \), we can estimate the difference \( X_c = \xi'_c(G(p + q, 0)) - \xi'_c(G(p, q)) \) as follows:

\[
X_c \leq \sum_{w \neq x \in V(G)} \text{deg}(x)q + \sum_{i=1}^{q} \text{deg}(u_i)p + \sum_{i=1}^{\lfloor \frac{p}{2} \rfloor} \text{deg}(v_i)q + \varepsilon(v_p) + (\text{deg}(w) - 1)(r + q) - \text{deg}(w)r \\
= (2m(G) - \text{deg}(w))q + (2q - 1)p + pq + p + q(\text{deg}(w) - 1) \\
= 2qm(G) + 3pq - q.
\]

\(\square\)

Theorem 3.5 If \( T \) is a tree of order \( n \), then

\[
\xi^d(T) - \xi^c(T) \leq \begin{cases} 
\frac{25n^4}{96} - \frac{n^3}{6} - \frac{89n^2}{48} + \frac{19n}{6} - \frac{45}{32}; & n \text{ odd}, \\
\frac{25n^4}{96} - \frac{n^3}{6} - \frac{43n^2}{24} + \frac{19n}{6} - 2; & n \text{ even}.
\end{cases}
\]

Moreover, equality holds if and only if \( T = P_n \).

Proof. The right side of the above inequality is equal to \( \xi^d(P_n) - \xi^c(P_n) \). (The value of \( \xi^d(P_n) \) has been determined in [14], while it is straightforward to deduce \( \xi^c(P_n) \). Combining the two formulas, the polynomials from the right hand side of the inequality are obtained.) Suppose now that \( T \neq P_n \). Then there is always a vertex \( w \) of degree at least 3 such that we can apply Lemmas 3.3 and 3.4. Setting

\[
X_{dc} = (\xi^d(T(p + q, 0)) - \xi^c(p + q, 0)) - (\xi^d(T(p, q)) - \xi^c(p, q))
\]

Page 9
we have:

\[
X_{dc} \geq pqD_{T}(w) + \frac{pq}{6} \left( p(2p - 3) + q(2q - 3) \right) + \left( \frac{pq}{2} - 2pqr \right) + pqn(T)(p + q + r + 1) + pq \sum_{v \in V(T)} \epsilon(v) - (2qm(T) + 3pq - q) \\
= pq(D_{T}(w) + \sum_{v \in V(T)} \epsilon(v) - 3) + \frac{pq}{6} \left( 2p^2 - 3p + 2q^2 - 3q + 3pq \right) + pqr(n(G) - 2) + q(pn(T)(p + q + r) - 2m(T) + 1) > 0
\]

and the result follows.

\[\square\]

4 Further comparison

In this concluding section we give sharp lower and upper bounds on \(\xi^d(G) + \xi^c(G)\), compare \(\xi^d(G)\) with \(\xi^c(G)\) for graphs \(G\) with \(\Delta(G) \leq \frac{2}{3}(n - 1)\), and give a sharp lower bound on \(\xi^d(G)\) for graphs \(G\) with a given radius.

**Theorem 4.1** If \(G\) is a connected graph, then the following hold.

(i) \(\xi^d(G) + \xi^c(G) \leq 2(n(G) - 1)\epsilon(G) + 2\text{diam}(G)(W(G) + m(G) - 2^{(n(G))})\).

(ii) \(\xi^d(G) + \xi^c(G) \geq 2(n(G) - 1)\epsilon(G) + 2\text{rad}(G)(W(G) + m(G) - 2^{(n(G))})\).

Moreover, each of the equalities holds if and only if \(G\) is a self-centered graph.

**Proof.** (i) Partition the pairs of vertices of \(G\) into neighbors and non-neighbors, and using \([\Pi]\), we can compute as follows:

\[
\xi^d(G) = \sum_{\{u,v\} \subseteq V(G)} (\epsilon(u) + \epsilon(v))d(u, v) \\
= \sum_{uv \in E(G)} (\epsilon(u) + \epsilon(v)) + 2 \sum_{\{u,v\} \subseteq V(G)} (\epsilon(u) + \epsilon(v)) \\
+ \sum_{\{u,v\} \subseteq V(G)} (\epsilon(u) + \epsilon(v))(d(u, v) - 2) \\
= \xi^c(G) + \sum_{\{u,v\} \subseteq V(G)} (\epsilon(u) + \epsilon(v)) - 2\xi^c(G) \\
+ \sum_{\{u,v\} \subseteq V(G)} (\epsilon(u) + \epsilon(v))(d(u, v) - 2) \\
\leq -\xi^c(G) + 2(n(G) - 1)\epsilon(G) + 2\text{diam}(G)(W(G) + m(G) - 2^{(n(G))}).
\]
The inequality above becomes equality if and only if \( \varepsilon(v) = \text{diam}(G) \) for every \( v \in V(G) \). That is, the equality holds if and only if \( G \) is a self-centered graph.

(ii) This inequality as well as its equality case are proved along the same lines as (i). The only difference is that the inequality \( \varepsilon(u) + \varepsilon(v) \leq 2\text{diam}(G) \) is replaced by \( \varepsilon(u) + \varepsilon(v) \geq 22\text{rad}(G) \). \( \square \)

In our next result we give a relation between \( \xi^d(G) \) and \( \xi^c(G) \) for graph \( G \) with maximum degree at most \( \frac{2}{3}(n(G) - 1) \).

**Theorem 4.2** If \( G \) is a graph with \( \Delta(G) \leq \frac{2}{3}(n - 1) \), then \( \xi^d(G) \geq 2\xi^c(G) \). Moreover, the equality holds if and only if \( G \) is 2-self-centered, \( \frac{2}{3}(n(G) - 1) \)-regular graph.

**Proof.** Set \( n = n(G) \) and let \( v \) be a vertex of \( G \). Since \( \text{deg}(v) < n - 1 \) we have \( \varepsilon(v) \geq 2 \). Therefore \( D(v) \geq 2(n - 1) - \text{deg}(v) \) with equality holding if and only if \( \varepsilon(v) = 2 \). Using the assumption that \( \text{deg}(v) \leq \frac{2}{3}(n - 1) \), equivalently, \( 2n - 2 \geq 3 \text{deg}(v) \), we infer that \( \varepsilon(v)D(v) \geq 2\varepsilon(v)\text{deg}(v) \). Summing over all vertices of \( G \) the inequality is proved. Its derivation also reveals that the equality holds if and only if \( \text{deg}(v) = \frac{2}{3}(n - 1) \) and \( \varepsilon(v) = 2 \) for each vertex \( v \in V(G) \). \( \square \)

To conclude the paper we give a lower bound on the eccentric distance sum in terms of the radius of a given graph. Interestingly, the cocktail-party graphs are again among the extreme graphs.

**Theorem 4.3** If \( G \) is a graph with \( \text{rad}(G) = r \), then

\[
\xi^d(G) \geq (n(G) - 1 + \binom{r}{2})\varepsilon(G).
\]

Equality holds if and only if \( G \) is a complete graph or a cocktail-party graph.

**Proof.** Set \( n = n(G) \) and let \( v \in V(G) \). Let \( P \) be a longest path starting in \( v \). Separately considering the neighbors of \( v \), the last \( \varepsilon(v) - 2 \) vertices of \( P \), and all the other vertices, we can estimate that

\[
D(v) \geq \text{deg}(v) + (3 + \cdots + \varepsilon(v)) + 2(n - 1 - \text{deg}(v) - (\varepsilon(v) - 2))
\]

\[
= 2n - \text{deg}(v) + \frac{\varepsilon(v)^2 - 3\varepsilon(v)}{2} - 1.
\]

Since \( n - \text{deg}(v) \geq \varepsilon(v) \) for every vertex \( v \in V(G) \), we have \( D(v) \geq n + \varepsilon(v) + \frac{\varepsilon(v)^2 - 3\varepsilon(v)}{2} - 1 \). Consequently, having the fact \( \varepsilon(v) \geq r \) in mind, we get \( D(v) \geq n - 1 + \binom{r}{2} \). Multiplying this inequality by \( \varepsilon(v) \) and summing over all vertices of \( G \) the claimed inequality is proved.

From the above derivation we see that the equality can hold only if \( \varepsilon(v) = r = n - \text{deg}(v) \) holds for every \( v \in V(G) \). From the equality part of the proof of Theorem 2.1(ii) we know that this implies \( \text{diam}(G) \leq 2 \). For the equality we must also have \( D(v) = n - 1 + \binom{r}{2} \) for every \( v \). If \( r = 2 \) this means that \( D(v) = n \) and hence \( \text{deg}(v) = n - 2 \). It follows that \( G \) is a cocktail-party graph. And if \( r = 2 \), then we get a complete graph. \( \square \)
Acknowledgements

Sandi Klavžar acknowledges the financial support from the Slovenian Research Agency (research core funding P1-0297 and projects J1-9109, J1-1693, N1-0095).

References

[1] S. Chen, S. Li, Y. Wu, L. Sun, Connectivity, diameter, minimal degree, independence number and the eccentric distance sum of graphs, *Discrete Appl. Math.* **247** (2018) 135–146.

[2] P. Dankelmann, M.J. Morgan, S. Mukwembi, H.C. Swart, On the eccentric connectivity index and Wiener index of a graph, *Quaest. Math.* **37** (2014) 39–47.

[3] K.Ch. Das, N. Trinajstić, Relationship between the eccentric connectivity index and Zagreb indices, *Comput. Math. Appl.* **62** (2011) 1758–1764.

[4] K. Ch. Das, M.J. Nadjafi-Arani, Comparison between the Szeged index and the eccentric connectivity index, *Discrete Appl. Math.* **186** (2015) 74–86.

[5] K.Ch. Das, G. Su, L. Xiong, Relation between degree distance and Gutman index of graphs, *MATCH Commun. Math. Comput. Chem.* **76** (2016) 221–232.

[6] A.A. Dobrynin, A.A. Kochetova, Degree distance of a graph: a degree analogue of the Wiener index, *J. Chem. Inf. Comput. Sci.* **34** (1994) 1082–1086.

[7] S. Gupta, M. Singh, A.K. Madan, Eccentric distance sum: A novel graph invariant for predicting biological and physical properties, *J. Math. Anal. Appl.* **275** (2002) 386–401.

[8] I. Gutman, Selected properties of the Schultz molecular topological index, *J. Chem. Inf. Comput. Sci.* **34** (1994) 1087–1089.

[9] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.

[10] P. Hauweele, A. Hertz, H. Mélot, B. Ries, G. Devillez, Maximum eccentric connectivity index for graphs with given diameter, *Discrete Appl. Math.* **268** (2019) 102–111.

[11] H. Hua, H. Wang, X. Hu, On eccentric distance sum and degree distance of graphs, *Discrete Appl. Math.* **250** (2018) 262–275.

[12] H. Hua, H. Wang, M. Wang, The difference between the eccentric distance sum and eccentric connectivity index, *Ars Combin.* **144** (2019) 3–12.
[13] A. Ilić, I. Gutman, Eccentric connectivity index of chemical trees, *MATCH Commun. Math. Comput. Chem.* 65 (2011) 731–744.

[14] A. Ilić, G. Yu, L. Feng, On the eccentric distance sum of graph, *J. Math. Anal. Appl.* 381 (2011) 590–600.

[15] S. Li, Y. Song, H. Zhang, On the degree distance of unicyclic graphs with given matching number, *Graphs Combin.* 31 (2015) 2261–2274.

[16] A.K. Madan, H. Dureja, Eccentricity based descriptors for QSAR/QSPR. In: Novel Molecular Structure Descriptors - Theory and Applications II, I. Gutman, B. Furtula (Eds.), *Univ. Kragujevac, Kragujevac* (2010) 91–138.

[17] V. Sharma, R. Goswami, A.K. Madan, Eccentric connectivity index: A novel highly discriminating topological descriptor for structure - property and structure - activity studies, *J. Chem. Inf. Comput. Sci.* 37 (1997) 273–282.

[18] A.I. Tomescu, Unicyclic and bicyclic graphs having minimum degree distance, *Discrete Appl. Math.* 156 (2008) 125–130.

[19] H. Wang, L. Kang, Further properties on the degree distance of graphs, *J. Comb. Optim.* 31 (2016) 427–446.

[20] H. Wiener, Structural determination of paraffin boiling points, *J. Amer. Chem. Soc.* 69 (1947) 17–20.

[21] Y.-T. Xie, S.-J. Xu, On the maximum value of the eccentric distance sums of cubic transitive graphs, *Appl. Math. Comput.* 359 (2019) 194–201.

[22] K. Xu, K.C. Das, H. Liu, Some extremal results on the connective eccentricity index of graphs, *J. Math. Anal. Appl.* 433 (2016) 803–817.

[23] K. Xu, X. Li, Comparison between two eccentricity-based topological indices of graphs, *Croat. Chem. Acta* 89 (2016) 499–504.

[24] K. Xu, Y. Alizadeh, K.Ch. Das, On two eccentricity-based topological indices of graphs, *Discrete Appl. Math.* 233 (2017) 240–251.

[25] H. Zhang, S. Li, B. Xu, Extremal graphs of given parameters with respect to the eccentricity distance sum and the eccentric connectivity index, *Discrete Appl. Math.* 254 (2019) 204–221.