Abstract

In order to identify a system (module) embedded in a dynamic network, one has to formulate a multiple-input estimation problem that necessitates certain nodes to be measured and included as predictor inputs. However, some of these nodes may not be measurable in many practical cases due to sensor selection and placement issues. This may result in biased estimates of the target module. Furthermore, the identification problem associated with the multiple-input structure may require determining a large number of parameters that are not of particular interest to the experimenter, with increased computational complexity in large-sized networks. In this paper, we tackle these problems by using a data augmentation strategy that allows us to reconstruct the missing node measurements and increase the accuracy of the estimated target module. To this end, we develop a system identification method using regularized kernel-based methods coupled with approximate inference methods. Keeping a parametric model for the module of interest, we model the other modules as Gaussian Processes (GP) with a kernel given by the so-called stable spline kernel. An Empirical Bayes (EB) approach is used to estimate the parameters of the target module. The related optimization problem is solved using an Expectation-Maximization (EM) method, where we employ a Markov-chain Monte Carlo (MCMC) technique to reconstruct the unknown missing node information and the network dynamics. Numerical simulations on dynamic network examples illustrate the potentials of the developed method.

Key words: System identification; Interconnected systems; Gaussian processes; Estimation algorithms; Sensor placement.

1 Introduction

Modern systems are becoming increasingly complex and largely interconnected, which is evident from the growing interest in the field of data-driven modeling in dynamic networks. Dynamic networks are typically represented as a set of measurable signals (node signals) interconnected through linear dynamic systems (modules), that are possibly driven by external excitation signals. Data-driven model learning in dynamic networks can be classified into two categories - full network identification and local module identification. The former deals with the identification of full network dynamics [12, 17, 35, 39, 42], and the latter deals with learning a single module embedded in a dynamic network given the topology of the network [7, 9, 11, 14, 19, 21, 27–31, 36, 41]. A brief survey on the local module identification has been presented in [37].

In this paper we focus on learning local modules in a dynamic network. In [9, 36], the classical closed loop identification methods [20] like direct method, two-stage method, and joint input-output method have been generalized to a dynamic network setting. A local direct method that can deal with correlated noise has been introduced in [30]. An indirect method to identify local modules has been presented in [14]. Apart from the aforementioned prediction error methods, the direct method and the two-stage method have been extended to a Bayesian setting using regularized kernel-based methods in [27, 28] and [11] respectively. A generalized method that combines direct and indirect approaches to relax the sensing and excitation schemes has been introduced in [31].

The above mentioned direct methods require solving a MISO or MIMO identification problem, resulting in the problem of estimating a large number of parameters that are of no prime interest to the experimenter and the associated optimization problem may become...
computationally infeasible, being nonlinear and large-dimensional. Also, it may be required to perform a preliminary model order selection step for each of the MISO modules, e.g. using complexity criteria such as AIC, BIC, or cross validation [20]. For a relatively large network, one has to test a huge number of combination of candidate model orders, which can become computationally infeasible already in medium-sized networks. For example, for 5 modules in the MISO setup with FIR model structure and orders from 1 to 5, we need to test $5^5$ possible combinations. These problems have been tackled in [25, 27, 28] using regularized kernel-based methods by extending the direct method framework to a Bayesian setting.

When using the direct method for single module identification, the fundamental parallel path/loop condition [9] needs to hold in order to achieve consistent estimates of the target module. This condition states that all parallel paths from the input to the output of the target module and all the loops around the output of the target module must pass through nodes that are measured and included as predictor inputs in the estimation problem. Therefore, it becomes quintessential to measure certain nodes to satisfy the parallel path/loop condition; however, measurement of such nodes might not always be possible. Therefore, to mitigate this issue and achieve reduced bias estimates, it becomes essential to develop identification methods to cope with networks with missing node observations.

In this paper, we introduce a novel identification method that handles the situation of non-measured inputs (i.e., missing node observations) that are required to obtain unbiased target module estimates. To handle the situation of missing node observations, we use a data augmentation strategy [34, 38], where the missing node observations are also estimated along with the parameters of the modules. For reconstructing the missing node information, we use the available information from nodes that lie both upstream and downstream compared to the missing node. To avoid model order selection and reduce the number of nuisance parameters to be estimated, we build on [27, 28] and employ non-parametric regularized kernel-based methods. We keep a parametric model only for the module of interest in order to have an accurate description of its dynamics. The rest of the modules are modeled as zero-mean Gaussian processes, with covariance matrix (kernel) given by a first-order stable spline kernel [6, 24], which encodes stability and smoothness of the processes. By this way of modeling, we reduce the number of estimated parameters and obtain a significant reduction in the variance of the estimates [24].

Using the above approach, we obtain a Gaussian probabilistic description that depends on a vector of parameters that also contains the parameters of interest. We use an Empirical Bayes approach to estimate such a vector; this amounts to maximizing the marginal likelihood of the observed data, obtained by integrating out the dependence on the missing node data and the impulse response of the modules. The solution to the maximization problem is obtained using an iterative scheme based on the Expectation-Maximization (EM) [10] algorithm. The E-step characterizing this scheme involves computing the expected value of a joint log-likelihood. Since in this problem the associated integral does not admit an analytical solution, we employ a Monte Carlo approximation method where samples of the target probability distributions are generated using an instance of the Gibbs sampler [13]. As for the M-step of the EM procedure, we show that it can be split into several small optimization problems that are simple to solve, making the whole optimization routine computationally cheap. Numerical simulation on a dynamic network with missing node observations shows the developed method’s potentials compared to the available classical methods.

This paper is organized as follows. In Section 2, the setup of the dynamic network and the problem statement is defined. Section 3 briefs about the standard direct method. Next, we provide the MIMO model, strategy to reduce the parameters of nuisance modules and solve the missing node observation problem, and solution to the marginal likelihood problem using the MCEM method in Section 4, 5, and 6. Next, numerical simulations and results are provided in Section 7, followed by conclusions. Finally, the technical proofs are provided in the appendix.

2 Problem statement

Following the setup of [36], a dynamic network is built up out of $L$ scalar internal variables or nodes $w_j, j = 1, \ldots, L$, and $K$ external variables $r_k, k = 1, \ldots K$. Each internal variable is described as:

$$w_j(t) = \sum_{i \neq j}^{L} G_{ij}(q)w_l(t) + u_j(t) + v_j(t)$$

where $q^{-1}$ is the delay operator, i.e. $q^{-1}w_j(t) = w_j(t-1)$,

- $G_{ij}$ are strictly proper rational transfer functions, referred to as modules;
- There are no self-loops in the network, i.e. nodes are not directly connected to themselves ($G_{jj} = 0$);
- $v_j(t)$ is the process noise, where the vector process $v = [v_1 \cdots v_L]^T$ is modelled as a stationary stochastic process with rational spectral density $\Phi_v(\omega)$, such that there exists a Gaussian white noise process $e := [e_1 \cdots e_L]^T$, with covariance matrix $\Lambda > 0$ such that $v(t) = \Lambda^{1/2}e(t)$, where $\Lambda(q)$ is monic, square, stable and minimum-phase. The noises are uncorrelated i.e refers to the situation that $\Phi_v(\omega)$ and $\Lambda$ are diagonal.
• \( u_j \) is an input signal, \( u_j(t) = \sum_{k=1}^{K} R_{jk}^0(q)r_k(t) \) with \( r_k \) being the known external signal that can directly be manipulated by the user and \( R_{jk}^0(q) \) is a known stable proper rational transfer function. In some nodes, it may be absent.

When combining the \( L \) node signals we arrive at the full network expression

\[
\begin{bmatrix}
w_1 \\
w_2 \\
\vdots \\
w_L \\
\end{bmatrix} = \begin{bmatrix}
0 & G_{12}^0 & \cdots & G_{1L}^0 \\
G_{21}^0 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & G_{L-1,L}^0 \\
G_{L1}^0 & \cdots & G_{L,L-1}^0 & 0 \\
\end{bmatrix} \begin{bmatrix}
w_1 \\
w_2 \\
\vdots \\
w_L \\
\end{bmatrix} + \begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_L \\
\end{bmatrix} + H^0 \begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_L \\
\end{bmatrix}
\]

which results in the matrix equation:

\[ w = G^0w + u + H^0e. \]  

(2)

It is assumed that the dynamic network is stable, i.e. \((I - G^0)^{-1}\) is stable, and well posed (see [8] for details). The representation (2) is an extension of the dynamic structure function representation [16]. The identification problem to be considered is the problem of identifying one particular module \( G_{ji}^0(q) \) on the basis of a pre-specified subset of nodes \( w_i \), and possibly \( u \), assuming that \( N \) samples of these variables have been observed and the network topology is known. To this end, we choose a parameterization of \( G_{ji}^0(q) \), denoted as \( G_{ji}(q, \theta) \), that describes the dynamics of the module of interest for a certain parameter vector \( \theta = \theta_0 \in \mathbb{R}^{n_{\theta}} \).

3 The direct method and predictor input selection

Let us define \( N_j \) as the set of node indices \( k \) such that \( G_{jk}^0 \neq 0 \), i.e. the node signals in \( N_j \) are the w-in-neighbors of the node signal \( w_j \). Following the definition of a dynamic network in the previous section, each scalar internal variable can be described as:

\[ w_j(t) = \sum_{k \in N_j} G_{jk}^0(q)w_k(t) + u_j(t) + v_j(t) \]  

(3)

The above equation represents a MISO structure and is the starting point of the methodology presented in this paper, which is based on extending the MISO direct method [36]. In the standard MISO direct method for dynamic networks [36], we consider the one-step-ahead predictor [20] of \( u_j(t) \):

\[
\hat{w}_j(t|t-1; \theta) = (1 - H_j^{-1}(q, \theta))w_j(t) + H_j^{-1}(q, \theta)G_{ji}(q, \theta)w_i(t) + H_j^{-1}(q, \theta)\left( \sum_{k \in N_j \setminus \{i\}} G_{jk}(q, \theta)w_k(t) + u_j(t) \right)
\]

Fig. 1. Network example with 4 internal nodes, 2 external excitation signals and a noise sources at each node. Target module is \( G_{31}^0 \) which is a function of the parameter vector \( \theta \).

Not only the target module, but also the modules \( G_{jk}^0(q) \), \( k \in \mathcal{N}_j \setminus \{i\} \), and the noise model \( H_j^0(q) \), are suitably parameterized with additional parameters. The parameter vector of interest \( \theta \) is identified by minimizing the sum of the squared prediction error \( \varepsilon_j(t) = w_j(t) - \hat{w}_j(t|t-1; \theta) \). We note that in this formulation, the prediction error depends also on the additional parameters entering the remaining modules and the noise model, which need to be identified to guarantee consistent estimates of \( \theta \). Therefore, the total number of parameters may grow large if the cardinality of \( \mathcal{N}_j \) is large, with a detrimental effect on the variance of the estimate of \( \theta \) in the case where \( N \) is not very large.

According to [9], it is sufficient to have a set of node signals \( D_j \) to be measured and used as predictor inputs in the direct method, that satisfies an additional parallel path/loop condition and a confounding variable condition.

Condition 1 (parallel path and loop condition [9])

Let \( G_{ji} \) be the target network module to be identified. In the network (2):

• Every path from \( w_i \) to \( w_j \), excluding the path through \( G_{ji} \), passes through a node \( w_k, k \in D_j \), and
• Every loop through \( w_j \) passes through a node in \( w_k, k \in D_j \).

However, if \( Z_j \subseteq D_j \) represent the node signals that are not measured or are inaccessible (i.e. missing node observations) but required to satisfy the above condition, then identification through the direct method using the available signals leads to biased target module estimates [9]. From this, we note that the direct method requires the measurement of the node signals \( w_k \), \( k \in D_j \). For example, consider the network in Figure 1 with diagonal noise spectrum and \( u_2 = r_2, u_4 = r_4 \). In order to identify \( G_{31}^0 \) using the direct method, performing a MISO identification with w_3 as output and \( w_{b_3} = \{w_1, w_4\} \) as inputs when \( w_3 \) is not measured (i.e. missing node observation), leads to estimation of modules in an immersed network.
Let and the setup with classification of different sets of signals is (a network with see, we now estimate from w₁ to w₃ and not the desired target module G₃, which leads to a biased estimate. Confounding variables like e₂ when w₂ is non-measured also create bias in the estimate of the desired estimate. Confounding variables like eᵣ as predicted outputs with graph related to this network, with node signals wᵣ as diagonal, and consider the simultaneous paths from eᵣ to node signals wᵣ, k ∈ X such that serve as predictor inputs with node signals in wᵣ, while these paths do not run through a node in wᵣ.

4 Concepts and Notations

We define the following concepts and notations.

**Definition 1 (confounding variable)** Consider a dynamic network defined in (2) with e a white noise process such that cov(e) being diagonal, and consider the graph related to this network, with node signals w and e. Let wᵣ and wᵦ be two subsets of node signals in w. Then a noise component eᵣ in e is a confounding variable between wᵣ ↦ wᵦ given wᵣ, if in the graph there exist simultaneous paths from eᵣ to node signals wᵣ, k ∈ X and wᵦ, n ∈ X', while these paths do not run through a node in wᵣ.

We will denote wᵣ as the node signals in w that serve as predicted outputs with wᵣ ∈ wᵢ, and wᵦ as the node signals in w that serve as predictor inputs with wᵣ ∈ wᵦ. Next we decompose wᵣ and wᵦ into disjoint sets according to: \( \mathcal{Y} = \mathcal{Q} \cup \mathcal{O} ; \mathcal{D} = \mathcal{Q} \cup \mathcal{U} \) where wᵣ are the node signals that are common in wᵣ and wᵦ, wᵦ are the node signals that are only in wᵦ; wᵣ are the node signals that are only in wᵣ. A pictorial representation of the setup with classification of different sets of signals is presented in figure 3. The remaining node signals in the network are wᵣ with \( \mathcal{Z} = \mathcal{L} \setminus \mathcal{W} \), where \( \mathcal{L} = \{1, 2, \cdots, \mathcal{L}\} \) and \( \mathcal{W} = \mathcal{D} \cup \mathcal{Y} = \mathcal{U} \cup \mathcal{Y} \). We denote \( \mathcal{Y}^{(j)} = \mathcal{Y} \setminus \{j\} \). Let \( wᵣ, \ell ∈ \mathcal{D}^w \subseteq \mathcal{W} \setminus \{j\} \) denote the node signals in w that have unmeasured paths to wⱼ and \( wᵣ, \ell ∈ \mathcal{D}^w \) denote the non-zero excitation signals in u that have unmeasured paths to wⱼ. Also, for all \( k ∈ \mathcal{Y}^{(j)} \), let \( \mathcal{D}^w_k = \mathcal{W} \setminus \{k\} \) and let \( uᵣ, \ell ∈ \mathcal{D}^w_k \subseteq \mathcal{W} \cup \mathcal{Z} \) denote the non-zero excitation signals in u.

5 An empirical Bayes identification technique with missing node observations

5.1 Introduction

We first write a MIMO representation of (part of) our network with inputs \((wᵣ, uᵣ, u)\) and outputs \((wᵦ, wᵦ)\). This is realized by the following steps.

1. Firstly, similar to the reasoning in [30], we write the system’s equations for the measured variables as

\[
\begin{pmatrix} wᵣ \\ wᵦ \\ uᵣ \\ uᵦ \\ wᵦ \\ wᵦ \\ uᵣ \\ uᵦ \end{pmatrix} = \begin{pmatrix} \hat{G} & 0 & \xi_m \\ \hat{G} & 0 & \xi_m \\ 0 & \hat{H} & 0 \\ 0 & \hat{H} & 0 \\ 0 & \xi_m & \xi_m \\ 0 & \xi_m & \xi_m \end{pmatrix} \begin{pmatrix} wᵣ \\ wᵦ \\ uᵣ \\ uᵦ \\ wᵦ \end{pmatrix} + \begin{pmatrix} \hat{R} \\ \hat{R} \end{pmatrix} \begin{pmatrix} u \end{pmatrix} \tag{4}
\]

Figures 2 and 3. Immersed network of network in Figure 1 where the non-measured node w₂ is removed.

\( \xi_m \) a white noise process, while \( \hat{H} \) is monic, stable and stably invertible and the components in \( \hat{G} \) are zero if it concerns a mapping between identical signals. This step is made by removing the non-measured signals wᵦ from the network, while maintaining the second order properties of the remaining signals. This step is referred to as immersion of the nodes in wᵦ [9]. After immersion, we rewrite the system’s equation to structure the noise loops through a node wᵦ.

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model such that there are no confounding variable between \( w_t \leftrightarrow w_s \) given \( w_w \).

(2) As an immediate result of the previous step we can write an expression for the output variables \( w_y \), by considering the upper part of the equation (4), as

\[
\begin{bmatrix}
  w_y \\
  w_y(j)
\end{bmatrix} =
\begin{bmatrix}
  G_{y0} & G_{y1} & \cdots & G_{yj} \\
  G_{y0(j)} & G_{y1(j)} & \cdots & G_{yj(j)}
\end{bmatrix}
\begin{bmatrix}
  w_0 \\
  w_0(j)
\end{bmatrix}
+ \begin{bmatrix}
  H_{y0} & H_{y1} & \cdots & H_{yj} \\
  H_{y0(j)} & H_{y1(j)} & \cdots & H_{yj(j)}
\end{bmatrix}
\begin{bmatrix}
  \xi_0 \\
  \xi_0(j)
\end{bmatrix}
+ Ru
\]

with \( \text{cov}(\xi_0) := \tilde{\Lambda} \).

Now we will show that we can write an expression for the output variables \( w_y \) as in (5).

**Proposition 1** Consider a dynamic network defined by (2), however with \( \text{cov}(\varepsilon) = I \) and \( H^0 \) not necessarily monic. Then,

(1) there exists a representation (4) of the measured node signals \( w_m \), with \( H_m \) monic, stable and stably invertible, and \( \xi_m \), a white noise process, and

(2) for this representation there are no confounding variable between \( w_t \leftrightarrow w_s \) given \( w_w \).

**Proof:** See Proposition 4.2 in [26].

The consequence of Proposition 1 is that the output node signals in \( w_y \) can be explicitly written in the form of (5), in terms of input node signals \( w_p \), excitation signals \( u \) and disturbances, without relying on node signals in \( w_x \). From the result of Proposition 1, it is known that the system’s equations for the output variables \( w_y \) can be written as,

\[
w_y = \tilde{G} w_p + \tilde{H} \xi_y + \tilde{R} u
\]

with \( \xi_y \) a Gaussian white noise process, while \( \tilde{H} \) is full matrix, monic, stable and stably invertible.

For estimation purposes we are going to use a specific form of (6), as formulated next.

**Condition 2** There are no confounding variable between \( w_j \leftrightarrow w_{y \setminus \{j\}} \) given \( w_w \).

**Condition 3** All paths from \( w_h \), \( h \in \mathcal{O} \setminus \{j\} \) to \( w_j \) pass through a node in \( w_w \).

**Proposition 2** Consider the situation of Proposition 1. If Conditions 2 and 3 are satisfied, then there exists a form (6) with \( \tilde{H} \) being (block) lower triangular as follows,

\[
\begin{bmatrix}
  w_j \\
  w_j(j)
\end{bmatrix} =
\begin{bmatrix}
  \tilde{G}_{j0} & \tilde{G}_{j1} & \cdots & \tilde{G}_{j0(j)} \\
  \tilde{G}_{j1(j)} & \tilde{G}_{j1(j)} & \cdots & \tilde{G}_{j1(j)}(j)
\end{bmatrix}
\begin{bmatrix}
  w_0 \\
  w_0(j)
\end{bmatrix}
+ \begin{bmatrix}
  \tilde{H}_{j0} & \tilde{H}_{j1} & \cdots & \tilde{H}_{j0(j)} \\
  \tilde{H}_{j1(j)} & \tilde{H}_{j1(j)} & \cdots & \tilde{H}_{j1(j)}(j)
\end{bmatrix}
\begin{bmatrix}
  \xi_j \\
  \xi_j(j)
\end{bmatrix}
+ \tilde{R} u
\]

where we isolate the target module \( \tilde{G}_{ji}(q) \). \( S_\delta(q), B_\delta(q) \) are strictly proper predictor filters, and \( \xi_y \) is Gaussian white noise with \( \text{cov}(\xi_y) = \tilde{\Sigma} \).

**Proof:** See Proposition 10.3 in [26].

**Condition 4** Let \( G_{ji} \) be the target network module to be identified. In the network (2):

- Every path from \( w_i \) to \( w_j \), excluding the path through \( G_{ji} \), passes through a node \( w_k, k \in \mathcal{W} \), and
- Every loop through \( w_j \) passes through a node in \( w_k, k \in \mathcal{W} \).

In order to guarantee that \( \tilde{G}_{ji}(q) = G_{ji}(q) \), i.e. the target module appearing in equation (7) is the target module of the original network (2) (invariance of target module), we utilize the following result:

**Proposition 3** Consider the situation of Proposition 2. If condition 4 is also satisfied, then we have \( \tilde{G}_{ji}(q) = G_{ji}(q) \).

**Proof:** See Proposition 10.2 in [26].

Based on the above results, we can now re-write (6) in the predictor form as given below.

**Proposition 4** Consider the network as represented by (2) where the set of all nodes \( w_z \) is decomposed into disjoint sets \( w_\varnothing, w_p, w_x \) and \( w_w \) such that Conditions 2, 3 and 4 are satisfied. The network equation for the node \( w_j \) and \( w_k, k \in \mathcal{Y}^{(j)} \) can be written as\(^4\)

\[
\begin{align*}
  w_j(t) &= \hat{w}_j(t|t - 1) + \xi_j(t) \\
  &= S_j(q)(w_j(t) - u_j(t)) + (1 - S_j(q))G_{ji}(q)w_i(t) \\
  &+ \sum_{\ell \in D_{\varnothing}^{n}(i)} S_{\ell}(q)w_{\ell}(t) + u_j(t) + \sum_{\ell \in D_{\varnothing}^{n}(j)} S_{\ell}(q)u_{\ell}(t) + \xi_j(t), \quad (8)
\end{align*}
\]

\[
\begin{align*}
  w_k(t) &= \hat{w}_k(t|t - 1) + \xi_k(t) \\
  &= B_k(q)(w_k(t) - u_k(t)) + \sum_{\ell \in D_{\varnothing}^{n}(k)\setminus\{j\}} B_{\ell}(q)w_{\ell}(t) \\
  &+ \sum_{\ell \in D_{\varnothing}^{n}(k)} B_{\ell}(q)u_{\ell}(t) + u_k(t) + \xi_k(t), \quad (9)
\end{align*}
\]

where \( \delta \) is a confounding variable between \( w_i \leftrightarrow w_j \) given \( w_w \). The lower (block) block triangular structure of \( \tilde{H} \) in (7) allows us to isolate the target module \( G_{ji} \) in (8). Realizing the above representation requires conditions on the selection of node signals in \( w_\varnothing, w_p, w_x \) (i.e. conditions 2, 3 and 4). The conditions are not a restriction. They can always be satisfied by appropriate selection of signals in the sets \( Q, O \) and \( U \).

\(^4\) from now on superscript \( ^0 \) is dropped for convenience.
5.2 Predictor model

The identification method developed in this paper relies on data augmentation strategies [34, 38]. Using this strategy, we treat a non-measured missing node signal as a latent variable which is estimated along with the parameters, while adding measured node signals that are ascendants and descendents of the missing node to our estimation problem. In this way signals that carry information on the missing node signal are incorporated in our estimation procedure. For simplicity, we are going to assume one missing node signal (i.e. the cardinality of $Z_j$ is one). The procedure can simply be generalized to multiple missing node signals.

We are now going to construct a predictor model with inputs $w_y$ and outputs $w_y$ according to the following procedure:

1. Select $w_y$ to be a set of signals, including $w_j$ and $w_m$, that satisfies the parallel path and loop condition 4.
2. Select $w_y$ to include $w_j$, while satisfying Conditions 2 and 3. Include $w_m$ in $w_y$.
3. Select all measured descendant nodes $w_a$ that have an unmeasured path from $w_m$, and include them in the output $w_y$.
4. Select all measured ascendant nodes that have an unmeasured path to $w_m$, and include them in the input $w_y$.
5. Select all measured ascendant nodes that have an unmeasured path to $w_a$, and include them in the input $w_y$.

Note that all signals in $w_y \cup w_y$ are measured node signals, except for the missing node $w_m$. However, for all considerations on confounding variables and unmeasured paths, the missing node signal $w_m$ is considered to be a measured node signal.

Remark 1 Satisfying Conditions 2 and 3 in Step 2 might not be feasible based on measured signals only. In that case the blocking of paths, e.g. the blocking of confounding variables, can be realized by introducing additional missing nodes, for which the same stepwise procedure above is followed.

The result of the above procedure is a MIMO model structure where the outputs are (1) the output of the target module i.e. $w_j$, (2) the missing node output $w_m$, (3) additional nodes $w_a$ that are descendents of the missing node, leading to $Y = \{w_j, w_m, w_a\}$. The inputs of the predictor model are collected in $w_y$ which includes $w_m$. Since $w_m$ and $w_a$ are both in the set $Y^{(l)}$ they can be written according to (9) as,

$$w_m(t) = B_m(q)(w_m(t) - u_m(t)) + \sum_{l \in D_m} B_l(q) w_l(t)$$
$$+ \sum_{l \in D_m} B_{mx}(q) w_d(t) + u_m(t) + \xi_m(t), \quad (10)$$

$$w_a(t) = F_a(q)(w_a(t) - u_a(t)) + \sum_{l \in D_a} F_l(q) w_l(t)$$
$$+ \sum_{l \in D_a} F_{ax}(q) w_d(t) + u_a(t) + \xi_a(t), \quad (11)$$

with \( \text{cov}(\xi_m(t), \xi_a(t)) := \Sigma = \begin{bmatrix} \sigma^2_m & 0 & 0 \\ 0 & \sigma^2_a & \sigma^2_{am} \\ 0 & \sigma^2_{am} & \sigma^2_m \end{bmatrix} \). For notation purposes, we introduce the vector $\mathbf{\xi}(t)$:

$$\mathbf{\xi}(t) := \begin{bmatrix} \xi_j(t) \\ \xi_m(t) \end{bmatrix} = \begin{bmatrix} \sigma^2_j \\ \sigma^2_a \\ \sigma^2_m \end{bmatrix} + \begin{bmatrix} \sigma^2_{am} \\ \sigma^2_{am} \end{bmatrix}, \quad \text{where for} \quad \xi_a(t) = \begin{bmatrix} \xi_j(t) \\ \xi_m(t) \end{bmatrix}.$$

From the above, it is very clear that if we use the additional node in the output, we need to model additional modules. This increase in complexity counterbalances the gain obtained by using more information. In this paper, we develop an identification framework that uses the additional node(s); we also provide the framework that does not use additional nodes as a special case of the former.

5.3 Vector description of the dynamics

In this section, we obtain a vector description of the network dynamics for the available $N$ measurements. For notation purposes, we introduce the $N$-dimensional vector $g_{ji}$ (which will also depend on $\theta$, although we will keep this dependence tacit) as the first $N$ coefficients of the impulse response of $G_{ji}(q, \theta)$. Similarly, we define the vector $s_k$, $k \in \{D^\mu\} \setminus i, s_{jk}, k \in \{D^\nu\} \setminus j$ and $s_j$ as the vectors containing the first $l$ coefficients of the impulse responses of $S_k(q), k \in \{D^\mu\} \setminus i, S_{jk}(q), k \in \{D^\nu\} \setminus j, S_j(q)$, respectively. Similarly, $b_k, b_{mk}, b_m, f_k, f_{ak}, f_a$ are defined as the vectors containing the first $l$ coefficients of the impulse responses of $B_k, B_{mk}, B_m, F_k, F_{ak}, F_a$ respectively. The integer $l$ is chosen large enough to ensure $s_k(l+1), s_{jk}(l+1), s_j(l+1), b_k(l+1), b_{mk}(l+1), b_m(l+1), f_k(l+1), f_{ak}(l+1), f_a(l+1) \sim 0$.

Lemma 1 Let the vector notation for the node $w_k(t)$ be $w_k := [w_k(1) \ldots w_k(N)]^T$ where $k \in \{j, m, a\}$. Considering the parameterization of $G_{ji}(q)$ (i.e. $G_{ji}(q, \theta)$), the network dynamics in (8), (10) and (11) can be rep-
resented in the vector form as:

\[ w_m = \tilde{W}_m b_m + \sum_{k \in D^w_m} W_k b_k + \sum_{k \in D^n_m} R_k b_{mk} + u_m + \xi_m \]

\[ w_j = \tilde{W}_j s_j + W_j g_j + \sum_{k \in D^w_j \setminus \{i\}} W_k s_k + \sum_{k \in D^n_j \setminus \{j\}} R_k s_{jk} + u_j + \xi_j \]

\[ w_a = \tilde{W}_a f_a + \sum_{k \in D^w_a} W_k f_k + \sum_{k \in D^n_a} R_k f_{mk} + u_a + \xi_a \]

where \( \xi_j, \xi_m, \xi_a \) are the vectorized noise and \( r_j, r_m, r_a \) are the vectorized excitation signal. \( \tilde{W}_j, \tilde{W}_m, \tilde{W}_a, W_j, W_k \) and \( R_k \) are Toeplitz matrices constructed from measurements of the respective node and excitation signals.

**Proof:** We denote by \( W_k \in \mathbb{R}^{N \times I} \) the Toeplitz matrix of the vector \( \vec{w}_k := \begin{bmatrix} w_k(1) \cdots w_k(N-1) \end{bmatrix}^T \), \( k \in D^w_j \cup D^w_m \cup D^w_a \cup Y \) and \( W_j \) is \( \mathbb{R}^{N \times N} \) the Toeplitz matrix of the vector \( \vec{w}_j := \begin{bmatrix} w_i(1) \cdots w_i(N-1) \end{bmatrix}^T \). Let \( R_\ell \in \mathbb{R}^{N \times I} \) be the Toeplitz matrix of the vector \( \vec{u}_\ell := \begin{bmatrix} u_\ell(1) \cdots u_\ell(N-1) \end{bmatrix}^T \) where \( \ell \in Y \cup D^m_n \cup D^w_m \cup D^w_j \).

Similarly, we denote by \( \tilde{W}_k \in \mathbb{R}^{N \times I} \) the Toeplitz matrix of the vector \( \vec{w}_k := \begin{bmatrix} 0 & w_k(1) & \cdots & -w_k(N-2) \end{bmatrix}^T \), \( k \in \{i,j\} \), and by \( G_\theta \) the Toeplitz of \( g_{ij} \). Considering the parameterization of \( G^0_{ij} \) and the above established notations, we can rewrite the network dynamics in (8) as (13), (11) as (14), and (10) as (12) where \( \tilde{W}_j := W_j - R_j + G_\theta \tilde{W}_i, W_m := W_m - R_m, \tilde{W}_a := W_a - R_a \).

### 5.4 Strategy to reduce the number of parameters for nuisance modules

Before we move to an estimation scheme that can deal with the missing node \( w_m \), we explain how we can avoid having to estimate a huge number of parameters in the nuisance modules, i.e., the modules that we need to identify but that are not the target module of interest. For this we follow the work in [28], where no missing nodes are assumed and extend it to the situation where there are missing node(s). Our goal is to limit the number of parameters necessary to describe \( w_j, w_a \) and \( w_m \), in order to increase the accuracy of the estimated parameter vector of interest \( \theta \). Therefore, while we keep a parametric model for \( G_{ji} \), for the remaining impulse responses in (12), (13) and (14), we use nonparametric models induced by Gaussian processes [32]. The choice of Gaussian processes is motivated by the fact that, with a suitable choice of the prior covariance matrix, we can get a significant reduction in the variance of the estimated impulse responses [24]. Therefore, we model \( s_k, k \in j \cup D^w_j - \{i\}, s_{jk}, k \in D^w_j \cup D^w_m \cup b_{mk}, k \in D^w_m, f_k, k \in a \cup D^w_a, f_{ak}, k \in D^w_a \) as independent Gaussian processes (vectors in this case) with zero-mean. The covariance matrix of these vectors, usually referred to as a kernel in this context, is chosen to be corresponding to the so-called *First-order Stable Spline kernel*. The general structure of this kernel is given by \( \lambda(K_{\beta x}, y) = \lambda_{\beta_{\max}(x, y)} \), where \( \beta \in [0, 1) \) is a hyperparameter that regulates the decay of the realizations of the corresponding Gaussian vector, while \( \lambda \geq 0 \) tunes their amplitude. The choice of this kernel is motivated by the fact that it enforces favorable properties such as stability and smoothness in the estimated impulse responses [22], [23]. Therefore, we have that

\[ s_k \sim \mathcal{N}(0, \lambda_k^0 K_{\beta_k^0}) \quad k \in j \cup D^w_j - \{i\}, \]

\[ s_{jk} \sim \mathcal{N}(0, \lambda_{jk}^0 K_{\beta_{jk}^0}) \quad k \in D^w_j \cup D^w_m \]

\[ b_{mk} \sim \mathcal{N}(0, \lambda_{mk}^0 K_{\beta_{mk}^0}) \quad k \in D^w_m, \]

\[ f_k \sim \mathcal{N}(0, \lambda_k^0 K_{\beta_k^0}) \quad k \in a \cup D^w_a, \]

\[ f_{ak} \sim \mathcal{N}(0, \lambda_{ak}^0 K_{\beta_{ak}^0}) \quad k \in D^w_a, \]

where we have assigned different hyperparameters to the impulse response priors to guarantee flexible enough models. In (12) - (14) we have terms that are multiplication of the Toeplitz matrix related to missing node \( w_m \) and the impulse response models, where both the missing node data and the prior hyperparameters of the impulse models are unknown and need to be estimated. In order to tackle the identifiability issues, we set the hyperparameters \( \lambda_m^0, \lambda_m^b, \lambda_m^s \) (i.e. \( \lambda \)‘s corresponding to the modules having missing node as inputs) to be 1.

### 5.5 Incorporating Empirical Bayes approach

We now explain how the parameters of the priors and the target module are estimated using the Empirical Bayes approach. For this, we define

\[ s := \begin{bmatrix} s_j^T & s_c^T & \cdots & s_c^T & s_{jk}^T & s_{jk}^1 & \cdots & s_{jk}^p \end{bmatrix}^T, \]

where \( c_1, \ldots, c_p \) and \( k_1, \ldots, k_p \) are the elements of the set \( D^w_j \setminus \{i\} \) and \( D^w_j \setminus \{j\} \), and

\[ W := \begin{bmatrix} \tilde{W}_j & W_{c_1} & W_{c_2} & \cdots & R_{k_{p-1}} & R_{k_p} \end{bmatrix}, \]

\[ K_1 := \text{diag}\{\lambda_j^0 K_{\beta_j^0}, \lambda_{c_1}^0 K_{\beta_{c_1}^0}, \ldots, \lambda_{k_{p+1}}^0 K_{\beta_{k_{p+1}}^0}\}. \]

Analogously we define

\[ b := \begin{bmatrix} b_m^T & b_{c_1}^T & \cdots & b_{c_p}^T & b_{mk_1}^T & b_{mk_2}^T & \cdots & b_{mk_p}^T \end{bmatrix}^T, \]

\[ R := \begin{bmatrix} \tilde{W}_m & W_{c_1} & W_{c_2} & \cdots & R_{k_{p-1}} & R_{k_p} \end{bmatrix}, \]
\[ \mathbf{K}_2 := \text{diag}\{K_{\beta_m^a}, K_{\beta_{c_1}}, \ldots, K_{\beta_{c_p}} \} \]

where \( c_1, \ldots, c_p \) and \( k_1, \ldots, k_p \) are the elements of the set \( D_m^w \) and \( D_m^\alpha \), respectively, and

\[ f := \begin{bmatrix} f_a^T & f_{c_1}^T & \ldots & f_{c_p}^T & f_{a_k_1}^T & \ldots & f_{a_k_p}^T \end{bmatrix}^T, \]

\[ \mathbf{Q} := \begin{bmatrix} \mathbf{W}_a & \mathbf{W}_{c_1} & \mathbf{W}_{c_2} & \ldots & \mathbf{R}_{k_{p-1}} & \mathbf{R}_{k_p} \end{bmatrix}, \]

\[ \mathbf{K}_3 := \text{diag}\{\lambda_{a_1}^f, \lambda_{c_1}^f, \lambda_{c_2}^f, \ldots, \lambda_{a_k}^f, \lambda_{a_k}^f \} \]

where \( c_1, \ldots, c_p \) and \( k_1, \ldots, k_p \) are the elements of the set \( D_m^a \) and \( D_m^\alpha \) respectively. Using the above, we can rewrite (12), (13) and (14) in compact form and we obtain the following model:

\[
\begin{pmatrix} w_j \\ w_a \\ w_m \\ w_y \\ w_i \\ s \end{pmatrix} = \begin{pmatrix} \mathbf{W} & 0 & 0 & s & [W_{j1}] \\ 0 & Q & 0 & f & 0 \\ 0 & 0 & R & b & 0 \\ g & 0 & u & u_m & \xi_j \\ u_a & u & 0 & \xi_a & \xi_m \\ \xi_y & \xi_i & \xi_f & \xi_y & \xi_y \end{pmatrix},
\]

\[ s \sim \mathcal{N}(0, \mathbf{K}_1), \]

\[ b \sim \mathcal{N}(0, \mathbf{K}_2), \]

\[ f \sim \mathcal{N}(0, \mathbf{K}_3), \]

\[ \xi_y \sim \mathcal{N}(0, \Sigma), \]

where \( s, b, f \) and \( \xi_y \) are mutually independent and with \( \Sigma = \Sigma \otimes I_N \). We note that the above model depends upon the vector of parameters

\[ \eta := \begin{bmatrix} \theta \lambda_j^s & \ldots & \lambda_{j,k_p}^s & \lambda_{c_1}^b & \ldots & \lambda_{c_p}^b & \lambda_{a_1}^f & \ldots & \lambda_{a_k}^f & \beta_{c_1}^s & \ldots & \beta_{c_p}^s & \beta_{c_1}^b & \ldots & \beta_{c_p}^b & \beta_{a_1}^f & \ldots & \beta_{a_k}^f & \sigma_j^2 & \sigma_m^2 & \sigma_a^2 & \sigma_{am} \end{bmatrix}, \]

which contains the parameter vector of the target module, the hyperparameters of the kernels of the impulse response models of the other modules, and the parameters related to the covariance of the noise corrupting \( w_j(t), w_a(t) \) and \( w_m(t) \). Note that \( \theta \) appears in \( g_{ji} \) while the other parameters appear in \( g \) and in covariance of \( \xi_y \). Therefore, we focus on the estimation of \( \eta \), since it contains the parameter of interest \( \theta \). For this, we apply an Empirical Bayes (EB) approach, where the estimate of \( \eta \) is obtained by maximizing the marginal likelihood of the observed data \( w_y = [w_j^T, w_a^T]^T \), obtained by integrating out the dependence on the missing node data and the impulse response of the modules,

\[ \hat{\eta} = \arg \max_\eta p(w_y; \eta). \]

**Remark 2** If we do not consider additional node \( w_a \), we can remove the extra layer of equation in \( w_j \). In the above model (30), it will be the second (block) row of equation in \( w_j \) and therefore we need not model \( f \). Now the model will depend upon the vector of parameters,

\[ \eta := \begin{bmatrix} \theta \lambda_j^s & \ldots & \lambda_{j,k_p}^s & \lambda_{c_1}^b & \ldots & \lambda_{c_p}^b & \beta_{c_1}^s & \ldots & \beta_{c_p}^s & \beta_{c_1}^b & \ldots & \beta_{c_p}^b & \beta_{a_1}^f & \ldots & \beta_{a_k}^f & \sigma_j^2 & \sigma_m^2 & \sigma_a^2 & \sigma_{am} \end{bmatrix}. \]

We do not need to estimate extra parameters \( \lambda_{a_1}^f, \lambda_{a_2}^f, \ldots, \lambda_{a_k}^f \), \( \beta_{a_1}^f, \ldots, \beta_{a_k}^f \), \( \sigma_j^2, \sigma_m^2, \sigma_a^2, \sigma_{am} \).

The first important problem with the above approach of parameter \( \eta \) inference in (31) is that we need to deal with the unknown missing node observation \( w_m \). Secondly, due to the incomplete model, the marginal pdf of \( w_y \) (i.e. \( p(w_y; \eta) \)) does not admit an analytical expression and cannot be computed under a closed-form solution. Adding to it, the maximization problem does not admit an explicit solution. In the next section, we study how to solve the marginal likelihood problem through a dedicated iterative scheme.

### 6 Parameter Inference

In this section, we provide the approach to deal with the missing node \( w_m \) and the above discussed problems and solve the marginal likelihood problem in (31). We use the strategy of data augmentation [34, 38] to deal with the unknown missing node observations. In this data augmentation strategy, we treat the unknown node signal as auxiliary variables which are estimated along with the parameters in \( \eta \). This data augmentation strategy has been used for state inference in identification of state-space models [33]. There are various methods that use the data augmentation strategy like the EM algorithm [10] for a Frequentist formulation of the identification problem and the Gibbs sampler [13] for a Bayesian formulation.

For the problem in (31), which is a Frequentist formulation, we solve it by deriving an iterative solution scheme through the EM algorithm. For this, we need to first define the latent variables whose estimation simplifies the computation of the marginal likelihood. The first natural choice is \( w_m \), which is the missing node observation. Also, \( s, b, f \) are latent variables. Then, the solution to (31) using the EM algorithm is obtained by iterating among the following two steps:

- **E-Step**: Given an estimate \( \hat{\eta}^{(n)} \) computed at the \( n^{th} \) iteration, compute

\[ Q^{(n)}(\eta) = E[\log p(w_y, w_m, s, f, b; \eta)], \]

where the expectation of the joint log-likelihood of \( w_y, w_m, s, f, b \) is taken with respect to the posterior \( p(w_m, s, f|w_y; \hat{\eta}^{(n)}) \);
• **M-step**: Update $\eta$ by solving

$$
\hat{\eta}^{(n+1)} = \arg \max_{\eta} Q^{(n)}(\eta). \tag{33}
$$

When the two steps are iterated, convergence to a stationary point of the marginal likelihood (which can be a local minima or global minima) is ensured [5]. In the next section, we show the clear advantage of using the EM algorithm. We have transformed the original marginal likelihood problem (31) to a sequence of problems that require solving (33) using the EM algorithm. We show that, when we use the EM method, the nonlinear optimization problem becomes a problem of iteratively constructing analytical solutions and solving scalar optimization problems, which significantly simplifies solving (31).

Also, the E-step in the algorithm involves computing expectation with respect to the posterior distribution $p(w_m, s, p, f|w_y)$, which is non-Gaussian and does not have an analytical form. Thus the integral in (32) is not tractable. In the next section, we present a solution to this problem by using a Markov Chain Monte Carlo (MCMC) method, Gibbs sampler.

### 6.1 Computation of E-step

In order to perform the E-step we resort to the Monte Carlo approximation of (32). This method has been introduced in [40] and is known as Monte Carlo Expectation Maximization (MCEM). In this, we approximate (32) as,

$$
Q^{(n)}(\eta) \approx \frac{1}{M} \sum_{i=1}^{M} \log p(w_y, \bar{w}_m^{(i,n)}, \bar{s}^{(i,n)}, \bar{p}^{(i,n)}, \bar{f}^{(i,n)}; \eta), \tag{34}
$$

where $\bar{s}^{(i,n)}, \bar{p}^{(i,n)}, \bar{f}^{(i,n)}$, and $\bar{w}_m^{(i,n)}$ are samples drawn at the $n^{th}$ iteration from the posterior $p(w_m, s, p, f|w_y; \eta^{(n)})$. In order to draw samples from the posterior, we use the Gibbs sampler. The idea behind the Gibbs sampler, which is a MCMC method, is to generate samples from a desired target distribution by simulating a Markov chain, with the target distribution as its stationary distribution. The Gibbs sampler produces samples from the posterior distribution by iteratively sampling each random variable conditioned on all other random variables [13]. Therefore to create samples from the joint posterior distribution, starting from an initialization $\bar{s}^{(0,n)}, \bar{p}^{(0,n)}, \bar{f}^{(0,n)}, \bar{w}_m^{(0,n)}$, we iteratively perform Algorithm 1 for a large number of iterations keeping the hyperparameters value fixed. Normally, we discard first few samples since the Markov chain will be poorly mixed and the obtained samples will be far away from the stationary distribution, which is the target distribution for the Gibbs sampler. Therefore, we discard the first $B$ samples, and this is known as burn-in period. If the burn-in period is large enough, then we produce samples that come from the stationary distribution. Similar to the other MCMC techniques, the generated samples can be correlated, but we need independent samples. Instead of sampling individual variables, a group of variables can be sampled to tackle this. This is called blocking Gibbs sampling algorithm or blocked Gibbs sampler [18]. This is done by choosing blocks of variables, not necessarily disjoint, and then sampling jointly from the variables in each block in turn, conditioned on the remaining variables. We adopt this approach of Gibbs sampling in this paper. Another approach called thinning can be used to reduce the correlation in generated samples, where after the burn-in period each sample can be collected after $\kappa$ iterations.

It is important to note that in order to use the Gibbs sampler, the above conditional distributions should be known and we should be able to generate samples from them. This is a requirement when using Gibbs sampling. Gibbs sampling is a simple and effective sampling method, provided we know the conditional distributions. Next we show that, in our situation, the conditional distributions that we require have a convenient form.

**Proposition 5** Consider the model in (30). The conditional distributions of $s, p, f, w_m$ are Gaussian and given by,

$$
p(w_m|w_y, s, b, f) \sim \mathcal{N}(\mu_w, P_w), \tag{35}
$$

$$
p(s|w_y, w_m, b, f) \sim \mathcal{N}(\mu_s, P_s), \tag{36}
$$

$$
p(b|w_y, w_m, s, f) \sim \mathcal{N}(\mu_b, P_b), \tag{37}
$$

$$
p(f|w_y, w_m, s, b) \sim \mathcal{N}(\mu_f, P_f), \tag{38}
$$

---

5 There are other joint posterior approximation techniques like Variational Bayes approximations [3] and other MCMC methods [15], which can also be applied. In this paper we resort to Gibbs sampler. Gibbs sampler does not require any tuning of proposal distribution and does not include any rejection step.

6 The choice of burn-in period is a non-trivial problem which is outside the scope of this paper and methods to address this problem have been provided in [15].
where,

$$P_w = (\tilde{\mu_2}^\top \Sigma^{-1} \tilde{\mu_2} + \Lambda_{22})^{-1} \bigg[ \Lambda_{21} \tilde{\mu_2} - \tilde{\mu_2} \bigg( \Lambda_{12} \tilde{\mu_2}^\top \bigg)^{-1} \bigg], \quad (39)$$

$$\mu_w = P_w (\tilde{\mu_2}^\top \Lambda_{11} \mu_1 \Lambda_{12}) \tilde{w_y}, \quad (40)$$

$$P_s = (K^{-1} + \tilde{\Sigma}^{-1} W)^{-1},$$

$$\mu_s = P_s \tilde{w}^\top \Sigma^{-1} (w_y - \tilde{\mu_5}), \quad (44)$$

$$P_f = (K^{-1} + \tilde{\Lambda} \Sigma^{-1} \tilde{\mu})^{-1}, \quad (45)$$

$$\mu_f = P_f \tilde{Q}^\top \Sigma^{-1} (w_y - \tilde{\mu_6}). \quad (46)$$

Proof: Collected in the appendix. The expressions for \( \bar{W}, \bar{R}, \bar{Q}, \bar{\mu_1}, \bar{\mu_2}, \bar{\mu_4}, \bar{\mu_5} \) and \( \bar{\mu_6} \) are provided in the appendix.

Therefore, it is easy to setup the Gibbs sampler and sample from the joint posterior distribution, thereby approximating (32) using (34).

Remark 3 When we do not consider additional node \( w_a \), we do not consider the extra layer of equation in \( w_y \) (i.e. \( w_y \) becomes \( (w_j, w_m) \)) and we need not model \( f \). Therefore, we discard the use of \( f \) and expressions related to it. The conditional distributions follow the same equations as above.

6.2 Computation of M-step

Next we move to the M-step where we update the vector of parameters according to (33). We need to maximize (34) with respect to the vector of parameters in \( \eta \). We will now show that the optimization problem can be split into several independent optimization problems that depend on different components of the vector of parameters \( \eta \). We can split the optimization problem as,

$$Q^{(n)}(\eta) = \arg\max_{\eta} \frac{1}{M} \sum_{i=1}^{M} \log p(w_y, s^{(i,n)}, b^{(i,n)}, f^{(i,n)}; \mu^{(i,n)}; \eta)$$

$$= A + B + C + D,$$  \quad (47)
\[
\lambda_k^{(n+1)} = \frac{1}{T} \left( \frac{1}{k} \right) (K_{\beta_k^{(n+1)}})^{-1} s_k^{(n)} + \text{tr} \left( (K_{\beta_k^{(n+1)}})^{-1} S_k^{(n)} \right).
\]

The updates for \( \beta \) and \( \lambda \) for impulse responses \( b_k, k \in m \cup D_m, b_m, k \in D_m \), and \( f_k, k \in a \cup D_a, f_a, k \in D_a \) are updated analogously by using \( b_m^{(n)}, \tilde{B}_m^{(n)} \) and \( f_m^{(n)}, \tilde{F}_m^{(n)} \) respectively.

**Proof:** See the appendix.

To tackle the identifiability issues, we have fixed the hyperparameters \( \lambda_m^\ast, \lambda_b^\ast \) and \( \lambda_f^\ast \) (i.e. \( \lambda \)'s corresponding to the modules having missing node as inputs) to be 1. We now provide means to update the respective \( \beta \) hyperparameters of the kernel of the corresponding modules using the following proposition.

**Proposition 7** The updates of kernel's hyperparameters related to the impulse response of modules with missing node as inputs are obtained by solving the scalar optimization problem in the domain \([0, 1] \),

\[
\hat{\beta}_m^{(n+1)} = \arg \min_{\beta_m^\ast} \text{log det}(K_{\beta_m^\ast}) + s_k^{(n)} (K_{\beta_m^\ast})^{-1} s_k^{(n)} + \text{tr} \left( (K_{\beta_m^\ast})^{-1} S_k^{(n)} \right).
\]

The updates for \( \beta_b^\ast \) and \( \beta_f^\ast \) for impulse responses \( b_m \) and \( f_m \) are updated analogously by using \( b_m^{(n)}, \tilde{B}_m^{(n)} \) and \( f_m^{(n)}, \tilde{F}_m^{(n)} \) respectively.

**Proof:** See the appendix.

**Remark 4** The optimization problem in (49) and (51) can be difficult to perform in practice when the determinant of the kernel has a very low value or when the inversion of the kernel becomes difficult. To tackle this, we exploit the factorization of the first order stable spline kernel as in [4] by writing \( K_{\beta} = LD(\beta)L^T \), where \( L \) is lower-triangular with known entries (essentially, an “integrator”) and \( D(\beta) \) is diagonal with entries essentially being an exponential functions of \( \beta \). Using the above technique also increases the computation speed of the algorithm.

**Remark 5** When we do not consider an additional node \( w_a \), we need to update only the hyperparameter \( \lambda \)'s and \( \beta \)'s related to impulse responses in \( s, b \).

### 6.2.2 Update of \( \theta \) and noise covariance

Following (47), the updates of \( \theta \) and the noise covariance parameters in \( \eta \) are independent of the kernel hyperparameters. Following a reasoning similar to [2], \( \theta \) and \( \Sigma \) are updated as per the **Proposition 8**.

**Proposition 8** Let \( \tilde{\epsilon}_j^{(i,n)}(\theta) = w_j - W_j g_j(\theta) - u_j - \)

\[
W(i,n) \tilde{g}^{(i,n)}, \tilde{\Sigma} = \left[ \sigma_a^2 \sigma_m^2 \right], \text{ and} \]

\[
\tilde{\epsilon}^{(i,n)}(t) = \begin{bmatrix} u(t) \\ w_m(t)(t) \end{bmatrix} - \begin{bmatrix} u(t) \\ u_m(t) \end{bmatrix} - \begin{bmatrix} \tilde{Q}^{(i,n)}(t, \ast) & 0 \\ 0 & R^{(i,n)}(t, \ast) \end{bmatrix} \begin{bmatrix} \tilde{f}(i,n) \\ \tilde{g}(i,n) \end{bmatrix},
\]

where \( \tilde{Q}^{(i,n)}(t, \ast), R^{(i,n)}(t, \ast) \) corresponds to the \( t \)th row of the matrix \( Q, R \) respectively, with \( w_m \) in the matrices substituted with \( w_m^{(i,n)} \). Define

\[
\tilde{\epsilon}^{(i,n)}(t) = \frac{1}{M} \sum_{i=1}^{M} \tilde{\epsilon}^{(i,n)}(t),
\]

\[
\tilde{E}^{(i,n)}(t) = \frac{1}{M} \sum_{i=1}^{M} (\tilde{\epsilon}^{(i,n)}(t) - \tilde{\epsilon}^{(i,n)}(t))(\tilde{\epsilon}^{(i,n)}(t) - \tilde{\epsilon}^{(i,n)}(t))^T.
\]

Then

\[
\tilde{\theta}^{(n+1)} = \arg \min_{\theta} \left[ \tilde{g}_{ji}^T \tilde{A}(n) \tilde{g}_{ji} - 2\tilde{b}(n)^T \tilde{g}_{ji} \right],
\]

\[
\tilde{\Sigma}^{(n+1)} = \frac{1}{N} \left( \sum_{i=1}^{N} \tilde{\epsilon}^{(i,n)}(t, \tilde{\theta}^{(n+1)}) \tilde{\epsilon}^{(i,n)}(t, \tilde{\theta}^{(n+1)})^T + \tilde{E}^{(n)}(t) \right).
\]

**Proof:** See the appendix.

**Remark 6** If \( g_{ji} \) is linearly parameterized in \( \theta \) (e.g. in case of FIR models), the above problem related to the update of \( \theta \) becomes quadratic and a closed-form solution is achieved. That is, if \( g_{ji} = M \theta \) where \( M \in \mathbb{R}^{N \times n_a} \), then

\[
\tilde{\theta}^{(n+1)} = (M^T \tilde{A}(n) M)^{-1} M^T \tilde{b}(n).
\]

Note that, as shown in [28], we can update the parameter of the target module (i.e. \( \theta \)) using similar analytical expression as in (53) for any other rational model structures as well (e.g. BJ models). This can be done by following the similar approach of [28]. From (52), we can observe that the update of \( \Sigma \) in each iteration of the MCEM algorithm is a closed form analytical solution and the update of \( \theta \) is a nonlinear least-squares problem with decision variables being the parameters of the target module which are fewer than a direct PEM that includes the nuisance modules parameters as well in the
problem as decision variables. Also the result of Propositions 6 and 7 show that the update of kernel hyperparameter \( \beta \)'s are scalar optimization problems and \( \lambda \)'s have closed form solutions. Therefore, we have obtained a fast iterative procedure that follows simple rules to update the parameters, and provides a local solution to the marginal likelihood problem (31) under the presence of missing node observations. Algorithm 2 summarizes the steps to follow to obtain \( \hat{\eta} \) and therefore \( \hat{\theta} \).

Algorithm 2 Algorithm for local module identification in dynamic networks under missing node observations

(1) Set \( n = 0 \), Initialize \( \hat{\eta}^{(0)} \).
(2) Run Gibbs sampler according to Algorithm 1 and collect \( M \) samples after discarding first \( B \) samples for burn-in period using the result of Proposition 5.
(3) Update kernel hyperparameters using the result of Proposition 6 and 7.
(4) Update \( \hat{\theta}^{(n+1)} \) and \( \hat{\Sigma}^{(n+1)} \) using result of proposition 8.
(5) Update \( \hat{\eta}^{(n+1)} \) using the above updated values of the parameters.
(6) Set \( n = n + 1 \).
(7) Repeat from steps (2) to (8) until convergence.

The initialization can be done by randomly choosing \( \eta \) considering the constraints of the hyperparameters. The convergence criterion for the algorithm depend on the value of \( \| \hat{\eta}^{(n+1)} - \hat{\eta}^{(n)} \| \). This value should be small for convergence so that the algorithm can be terminated. A value of \( 10^{-2} \) is considered for the numerical simulations in Section 7. The other convergence criterion is the maximum number of iterations. It is taken as 50. For the numerical simulations, the initialization of the latent variables \( (s, p, f, w_m) \) for the Gibbs sampler are taken as a zero vector. The number of samples \( M \) for the Gibbs sampler is taken as 100 and the burn-in period \( B \) is equal to 2000.

Remark 7 The above developed method can be applied when we have the input of the target module (i.e. \( w_i \)) as the missing node observation with slight modifications in the above results. However, in this case, we might face identifiability issues of the target module. This is because of the fact that we can estimate the missing node signal and the target module up to a scaling factor. This has been a common issue in blind system identification [1].

7 Numerical simulations

Numerical simulations are performed to evaluate the performance of the developed method. The simulations are performed on the dynamic network depicted in Figure 1. The goal is to identify \( G_3 \), which is the target module. The modules of the network in Figure 1 are given by,

\[
G_0^{31} = \frac{q^{-1} + 0.05q^{-2}}{1 + q^{-1} + 0.6q^{-2}} = \frac{b_0^{q^{-1}} + b_2^{q^{-2}}}{1 + a_1^{q^{-1}} + a_2^{q^{-2}}}
\]

\[
G_0^{32} = \frac{0.225q^{-1}}{1 + 0.5q^{-1}};
\]

\[
G_0^{33} = \frac{1.184q^{-1} - 0.647q^{-2} + 0.151q^{-3} - 0.082q^{-4}}{1 - 0.8q^{-1} + 0.279q^{-2} - 0.048q^{-3} + 0.01q^{-4}};
\]

\[
G_0^{14} = G_2^{21} = \frac{0.4q^{-1} - 0.5q^{-2}}{1 + 0.3q^{-1}}; H_1^{0} = \frac{1}{1 + 0.2q^{-1}};
\]

\[
G_0^{12} = G_2^{24} = \frac{0.4q^{-1} + 0.5q^{-2}}{1 + 0.3q^{-1}}; H_2^{0} = \frac{1}{1 + 0.3q^{-1}};
\]

\[
H_4^{0} = \frac{1}{1 - 0.505q^{-1} + 0.165q^{-2} - 0.01q^{-3}};
\]

\[
\frac{1}{1 - 0.729q^{-1} + 0.236q^{-2} - 0.0109q^{-3}}.
\]

For estimation \( G_0^{31} \) using the direct method, we need to measure \( w_1, w_2, w_3 \) and \( w_4 \) and solve a 3-input/1-output MISO identification problem with \( w_i(t) \), \( w_2(t) \) and \( w_4(t) \) as inputs. \( w_2 \) needs to be included as predictor input in order to satisfy the parallel path/loop condition (i.e. Condition 1) and \( w_4 \) needs to be included as predictor input to satisfy the confounding variable condition. Now, we consider the case where we cannot measure \( w_2 \), which leads to lack of consistency in the direct method since we cannot satisfy the parallel path/loop condition. In this case, we resort to the approach developed in this paper and resort to the following options:

(1) consider \( w_2 \) as missing node, i.e. \( w_m = w_2 \), and consider the predictor model with \( w_p = \{w_3, w_2\} \).

We add to \( w_2 \) the measured node signals that have unmeasured paths to \( w_2 \) (i.e. \( w_1, w_3 \)) and the missing node signal \( w_2 \). Therefore \( w_p = \{w_1, w_4, w_2\} \), \( w_m = \{w_1, w_2, w_3, w_4\} \), \( w_p = \{w_3\} \) and \( w_m = \{w_2\} \).

By this signal selection , we can verify that the conditions 2, 3 and 4 are satisfied. This identification strategy is mentioned below as MC-EBDM.

(2) consider \( w_2 \) as missing node and add the descendant \( w_1 \) of \( w_2 \) as additional output, i.e. \( w_m = w_2, w_4 = w_1 \) and consider the model with \( w_p = \{w_3, w_2, w_1\} \). We add to \( w_2 \) the measured node signals that have unmeasured paths to \( w_2 \) (i.e. \( w_1, w_4 \)) and the missing node signal \( w_2 \). Therefore \( w_p = \{w_1, w_4, w_2\} \), \( w_m = \{w_1, w_2, w_3, w_4\} \), \( w_p = \{w_3\} \) and \( w_m = \{w_1, w_2\} \).

By this signal selection , we can verify that the conditions 2, 3 and 4 are satisfied. This identification strategy is mentioned below as MC-EBDMA.

We compare the following identification strategies:

MC-EBDM This is the method developed in this paper, namely Empirical Bayes Direct method with Monte Carlo sampling to deal with missing nodes; in particular, this estimator does not use additional node(s) and considers the predictor model
\{w_1, w_2, w_4\} \rightarrow \{w_3, w_2\};

**MC-EBDMA** This is a variant of MC-EBDM that uses \(w_1\) as additional node; and considers the predictor model \(\{w_1, w_2, w_4\} \rightarrow \{w_1, w_2, w_3\}\)

**EBDM+M** This is the EBDM method developed in [28]; this estimator does not encompass missing nodes and considers the predictor model \(\{w_1, w_4\} \rightarrow \{w_3\}\); in other words it discards the non-measured node signal \(w_2\);

**EBDM** This is the same estimator as the previous one, with the assumption that the missing node \(w_2\) is measurable (oracle assumption). We use this estimator as an upper bound of the performance of our developed method to reconstruct the missing node observation and identify the target module. Therefore, it considers a predictor model \(\{w_1, w_2, w_4\} \rightarrow \{w_3\}\) where \(w_2\) is known;

**DM+TO** This is the standard MISO direct method first proposed in [36], with the assumption that the missing node \(w_2\) is measurable (oracle assumption). Therefore, it considers a predictor model \(\{w_1, w_2, w_4\} \rightarrow \{w_3\}\) where \(w_2\) is known and assumes a fully parametric model structure. Note that in order to avoid biased target module estimates in the direct method framework, we need \(w_2\) to be measured and included as one of the predictor inputs [9];

**DM+TO+M** This is the same estimator as the previous one; it assumes a fully parametric model structure and has no specific way to deal with missing nodes and considers a predictor model \(\{w_1, w_4\} \rightarrow \{w_3\}\).

We run 50 independent Monte Carlo experiments where the data are generated using known reference signals \(r_2(t)\) and \(r_4(t)\) that are realizations of white noise with unit variance. The number of data samples is \(N = 150\). The noise sources \(e_1(t), e_2(t), e_3(t)\), and \(e_4(t)\) have variance 0.05, 0.08, 0.5, 0.1, respectively. We assume that we know the model order of \(G_{ji}^0(q)\). For the method DM+TO+M, we solve a 2-input/1-output MISO identification problem with \(w_1(t)\) and \(w_4(t)\) as inputs, which should lead to a biased target module estimate [9]. As for DM+TO, with the assumption that the missing node \(w_2\) is measurable (oracle assumption), we solve a 3-input/1-output MISO identification problem with \(w_1(t), w_2(t)\), and \(w_4(t)\) as inputs in order to compare the results of our developed method. For both these cases we consider that the model orders of all the modules in the MISO structure are known. Analogously, EBDM considers a 3-input/1-output MISO identification problem, while EBDM+M solves a 2-input/1-output MISO identification problem. For MC-EBDM, MC-EBDMA, EBDM, EBDM+M we choose \(l = 15\).

To evaluate the performance of the methods, we use the standard goodness-of-fit metric,

\[
\text{Fit}_{imp} = 1 - \frac{\|\hat{g}_{ji} - \hat{g}_{ji}\|_2}{\|g_{ji}^0 - \hat{g}_{ji}\|_2}; \quad \text{Fit}_{\theta} = 1 - \frac{\|\theta^0 - \hat{\theta}\|_2}{\|\theta^0 - \theta\|_2},
\]

where \(\text{Fit}_{imp}\) and \(\text{Fit}_{\theta}\) are the fit of the estimated impulse response and estimated parameters of the target module respectively. \(\hat{g}_{ji}^0\) is the true value of the impulse response of \(G_{ji}^0\), \(\hat{g}_{ji}\) is the impulse response of the estimated target module, \(\hat{g}_{ji}\) is the sample mean of \(g_{ji}^0\), \(\theta^0\) is the true parameter of the target module, \(\hat{\theta}\) is the estimated value of the parameter and \(\theta\) is the sample mean of \(\theta^0\). The box plots of the fit of the impulse response and box plots of the fit of the parameters of \(G_{31}^0(q)\) are shown in Figure 4 and 5 respectively for the above mentioned methods.

---

**Fig. 4.** Box plot of the fit of the impulse response of \(\hat{G}_{31}\) obtained using different methods. EBDM and DM+TO assumes that the missing node \(w_2\) is measurable (oracle assumption) and use it for the estimation.

**Fig. 5.** Box plot of the fit of the parameters of \(\hat{G}_{31}\) obtained using different methods.

It can be noted that both MC-EBDMA, despite considering \(w_2\) to be non-measured, achieve significantly better fit than the other methods that do not consider \(w_2\) to be known (i.e. DM+TO+M and EBDM+M). Comparing with methods that consider \(w_2\) to be known (i.e. DM+TO and EBDM), the novel estimator MC-EBDMA performs better than the direct method; furthermore, MC-EBDMA achieves a fit comparable to the fit obtained by the oracle EBDM. Also, the performance of MC-EBDM is poor compared to other methods, and thus shows the importance of including additional node(s) (i.e. MC-EBDMA). In Figure 6, we show the
Identification of a local module in a dynamic network typically leads to a MISO or MIMO identification problem where all node signals that are part of the predictor model are assumed to be available from measurements. In this paper, we have introduced an effective method for dealing with the situation that one or more of the required available node signals are not measured, but reconstructed as part of a Bayesian parameter estimation problem, by using a data-augmentation strategy. By using a regularized kernel-based approach, the method also circumvents an extensive model order selection step for all the modules in the predictor model, and offers a substantial reduction in the number of parameters to be estimated. For the optimization, an EM algorithm is followed where the E-step is solved by the particle approximation MCEM approach, by drawing samples from the posterior using Gibbs sampler. The introduced framework is not dependent on this particular choice, but allows alternative steps as well, like the Variational Bayes EM (VBEM) [3]. Numerical simulations performed with a dynamic network example shows promising results, and illustrate the potentials of the developed method to reconstruct the missing node observations and provide reduced variance estimates.

8 Conclusions

Identification of a local module in a dynamic network typically leads to a MISO or MIMO identification problem where all node signals that are part of the predictor model are assumed to be available from measurements. In this paper, we have introduced an effective method for dealing with the situation that one or more of the required available node signals are not measured, but reconstructed as part of a Bayesian parameter estimation problem, by using a data-augmentation strategy. By using a regularized kernel-based approach, the method also circumvents an extensive model order selection step for all the modules in the predictor model, and offers a substantial reduction in the number of parameters to be estimated. For the optimization, an EM algorithm is followed where the E-step is solved by the particle approximation MCEM approach, by drawing samples from the posterior using Gibbs sampler. The introduced framework is not dependent on this particular choice, but allows alternative steps as well, like the Variational Bayes EM (VBEM) [3]. Numerical simulations performed with a dynamic network example shows promising results, and illustrate the potentials of the developed method to reconstruct the missing node observations and provide reduced variance estimates.

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A Proof of Proposition 5

Let us first consider the conditional distribution \( p(w_m|u_y, s, p, f) \). We first write,

\[
\begin{align*}
  w_m &= R_{m}b + W_m b_m + u_m + \xi_m, \quad (A.1) \\
  u_y &= W_{g,m} g_m + W_m g_m + W_j g_{ji} + u_y + \xi_y, \quad (A.2)
\end{align*}
\]

where \( W_{g,m} \) is constructed after excluding \( W_m \) in the matrix \( W_D \) and \( g_m \) is the vector constructed after excluding \( s_m \) and \( f_m \) in \( g \). Here, \( g_m = [s_m f_m^\top] \) and \( W_m = blkdiag(W_m, W_m) \). Grouping terms in (A.1) and (A.2) we get,

\[
\begin{align*}
  w_m &= \mu_3 + \bar{B}_m w_m + \xi_m, \quad (A.3) \\
  u_y &= \mu_1 + \mu_2 w_m + \xi_y, \quad (A.4)
\end{align*}
\]

where \( \mu_1 = W_{g,m} g_m + W_j g_{ji} + u_y, \mu_2 = [\bar{S}_m \bar{F}_m], \mu_3 = R_{m}b + u_m \) and \( \Sigma_y = blkdiag(\sigma_y^2 I_N, \sigma_y^2 I_N) \). By the law of conditional expectation and ignoring terms independent of \( w_m \), we write

\[
\log p(w_m|u_y, b, s, f) \equiv \log p(w_m|s, b, f) + \log p(u_y|s, b, f)
\]

\[
\approx -\frac{1}{2}||\begin{bmatrix} u_y & w_m \end{bmatrix} - \begin{bmatrix} \mu_1 & \mu_2 \\ \bar{B}_m \end{bmatrix} w_m ||^2_{\Sigma_y^{-1}}
\]

\[
\approx -\frac{1}{2}||w_m||^2_{P_w^{-1}} + w_m^\top P_w^{-1} \mu_w, \quad (A.5)
\]

where \( \mu_w = P_w(\mu_2^\top \Sigma_w^{-1} \bar{u}_y + \Lambda_{A_2} \Lambda_{A_2} \bar{u}_y - \Lambda_1 \Sigma_1 \Sigma_2^{-1} \bar{u}_y) \). The above log density is quadratic and represents a Gaussian distribution with covariance \( P_w \) and mean \( \mu_w \).

Let us now consider the conditional distribution \( p(b|u_y, w_m, s, f) \). By the law of conditional expectation and ignoring terms independent of \( s \), we write

\[
\log p(b|u_y, w_m, s, f) \equiv \log p(u_y, w_m|s, b, f) + p(b)
\]

\[
\approx -\frac{1}{2}||u_y - \bar{u}_y - \bar{W}w_m||^2_{\Sigma_y^{-1}} - \frac{1}{2}||b||^2_{K_b^{-1}},
\]

\[
\approx -\frac{1}{2}||b||^2_{P_b^{-1}} + b^\top P_b^{-1} \mu_b, \quad (A.8)
\]

where \( \mu_b = P_b(\mu_2^\top \Sigma_y^{-1} \bar{u}_y + \Lambda_{A_2} \Lambda_{A_2} \bar{u}_y - \Lambda_1 \Sigma_1 \Sigma_2^{-1} \bar{u}_y) \). The above log density is quadratic and represents a Gaussian distribution with covariance \( P_b \) and mean \( \mu_b \).

B Proof of Proposition 1

From (47) we have,

\[
C = \arg \max \left[ \sum_k Q_{s_k}^{(n)}(\lambda_k^s, \beta_k^s) + \frac{1}{M} \sum_{i=1}^M \log p(\bar{e}^{(i)_m}; \beta_m^s) \right]
\]
with \( k = \{j, c_1, \ldots, c_p, jk_1, \ldots, jk_p\} \) where \( c_1, \ldots, c_p \) and \( k_1, \ldots, k_p \) are the elements of the set \( D_j \setminus \{i, m\} \) and \( D_j \setminus \{j\} \) respectively, and

\[
Q_{\lambda_k^*(n)}(\lambda_k^*, \beta_k^*) = \frac{1}{M} \sum_{i=1}^{M} \log p(s_{i}^{(i,n)}; \lambda_k^*, \beta_k^*)
\]

\[
\approx \frac{1}{M} \sum_{i=1}^{M} \log |\det(\lambda_k^* K_{\beta_k^*})| - \text{tr}\left( (\lambda_k^* K_{\beta_k^*})^{-1} s_k^{(n)} s_k^{(n)\top} \right)
\]

\[
\approx - \log |\det(\lambda_k^* K_{\beta_k^*})| - \text{tr}\left( (\lambda_k^* K_{\beta_k^*})^{-1} \tilde{s}_k^{(n)} \right) - s_k^{(n)\top} (\lambda_k^* K_{\beta_k^*})^{-1} s_k^{(n)}. \quad (B.1)
\]

Next, the proof follows the procedure used in [4]. We partially differentiate (B.1) with respect to \( \lambda_k^* \) and equate to zero to get the \( \lambda_k^* \) expression. Substituting this \( \lambda_k^* \) in (B.1) we get the expression for (48) using which we obtain \( \beta_k^{(n+1)} \). Equation (50) is the expression of \( \lambda_k^* \) after substituting \( \beta_k^{(n+1)} \).

### C Proof of Proposition 7

Considering \( \arg \max \frac{1}{M} \sum_{i=1}^{M} \log p(s_{i}^{(i,n)}; \beta_m^*) \) in (B.1) and expanding it as in (B.1) where \( \lambda_m^* = 1 \), we get the result of the proposition.

### D Proof of Proposition 8

From (47) we have,

\[
A = \arg \min_{\theta, \sigma_j^2} \frac{1}{M} \sum_{i=1}^{M} \left[ N \log \sigma_j^2 + \frac{1}{\sigma_j^2} \varepsilon_j^{(i,n)\top} \varepsilon_j^{(i,n)} \right]
\]

\[
+ \arg \min_{\Sigma} \frac{1}{M} \sum_{i=1}^{M} \left[ \sum_{t=1}^{N} \log |\det(\Sigma)| \right.
\]

\[
+ \sum_{t=1}^{N} \text{tr}(\Sigma^{-1} \varepsilon_j^{(i,n)}(t) \varepsilon_j^{(i,n)}(t)^\top) \right]
\]

\[
= A_1 + A_2.
\]

We now write,

\[
A_2 = \arg \min_{\Sigma} \left[ \sum_{t=1}^{N} \log |\det(\Sigma)| \right.
\]

\[
+ \sum_{t=1}^{N} \text{tr}(\Sigma^{-1} \left[ \tilde{e}_j^{(i,n)}(t) \tilde{e}_j^{(i,n)}(t)^\top + \tilde{e}_j^{(i,n)}(t) \right]) \right].
\]

For the optimization problem \( A_2 \), we can follow the similar reasoning as the maximum likelihood proof in [2], which yields the result of the proposition for estimating parameters in \( \tilde{e}_j \).

Now considering \( A_1 \), we can write

\[
A_1 = \arg \min_{\theta, \sigma_j^2} \left[ N \log \sigma_j^2 + \frac{1}{\sigma_j^2} \sum_{i=1}^{M} \varepsilon_j^{(i,n)\top} \varepsilon_j^{(i,n)} \right]. \quad (D.1)
\]

We notice that the optimum with respect to \( \theta \) does not depend on the optimal value of \( \sigma_j^2 \). Then, we can first update \( \theta \) and then use its optimal value to update \( \sigma_j^2 \). In order to find \( \hat{\theta}^{(n)} \), \( \sigma_j^2 \) is fixed to \( \hat{\sigma}_j^{(2(n))} \) and substituted in Eq. (D.1). After substitution, the terms that are independent of \( \theta \) can be removed from the objective function since it becomes a constant. Then we get,

\[
\hat{\theta}^{(n+1)} = \arg \min_{\theta} \sum_{i=1}^{M} \varepsilon_j^{(i,n)\top} \varepsilon_j^{(i,n)} \quad (D.2)
\]

Let us define \( \tilde{g}^{(i,n)} \in \mathbb{R}^N \) be a vector such that, if \( N \leq l \), \( \tilde{g}^{(i,n)} \) is the vector of first \( N \) elements of \( \tilde{g}^{(i,n)} \) and if \( N > l \), \( \tilde{g}^{(i,n)} \) is a vector with the first \( l \) elements equal to \( \tilde{g}^{(i,n)} \) and the remaining ones equal to 0. Let \( \tilde{S}^{(i,n)} \), \( W_i^{(i,n)} \in \mathbb{R}^N \times N \) be the Toeplitz matrix of \( \tilde{g}^{(i,n)} \) and \( \tilde{w}_i \) respectively. Then

\[
\tilde{X}^{(i,n)} = \begin{bmatrix} W_j - R_j \ W_{c_1} \cdots R_{k_p} \end{bmatrix}, \quad \tilde{Y}^{(i,n)} = \tilde{S}^{(i,n)} W_i.
\]

We now re-write, \( \tilde{W}^{(i,n)} \tilde{g}^{(i,n)} = \tilde{X}^{(i,n)} \tilde{g}^{(i,n)} + G_{\theta} W_i \tilde{g}^{(i,n)} + \tilde{Y}^{(i,n)} g_j ji \). Therefore,

\[
\hat{\theta}^{(n+1)} = \arg \min_{\theta} \sum_{i=1}^{M} \left[ -2 \tilde{w}_j^{\top} W_{ji} g_{ji} - 2 \tilde{w}_j^{\top} \tilde{Y}^{(i,n)} g_j ji \right]
\]

\[
+ g_{ji}^{\top} W_{ji} g_{ji} + 2 g_{ji}^{\top} W_{ji} r_j^{\top} + 2 \tilde{g}^{(i,n)\top} \tilde{X}^{(i,n)} \tilde{S}^{(i,n)} \tilde{g}^{(i,n)}
\]

\[
+ 2 g_{ji}^{\top} \tilde{X}^{(i,n)\top} \tilde{Y}^{(i,n)} g_{ji} + 2 \tilde{S}^{(i,n)\top} \tilde{X}^{(i,n)\top} \tilde{S}^{(i,n)} g_{ji} \bigg].
\]

Defining \( \hat{A}^{(n)} = \sum_{i=1}^{M} [W_{ji}^{\top} W_{ji} + 2W_{ji}^{\top} \tilde{Y}^{(i,n)} + \tilde{S}^{(i,n)\top} \tilde{Y}^{(i,n)}] \),

and

\[
\hat{b}^{(n)} = \sum_{i=1}^{M} \left[ w_j^{\top} W_{ji} + w_j^{\top} \tilde{Y}^{(i,n)} - r_j^{\top} W_{ji} - r_j^{\top} \tilde{Y}^{(i,n)} \right]
\]

\[
- \tilde{S}^{(i,n)\top} \tilde{X}^{(i,n)\top} W_{ji} - \tilde{S}^{(i,n)\top} \tilde{X}^{(i,n)\top} \tilde{Y}^{(i,n)} \bigg],
\]

we get the statement of the proposition for updating \( \hat{\theta}^{(n+1)} \).

In order to find \( \hat{\sigma}_j^{(2(n))} \), \( \theta \) is fixed to \( \hat{\theta}^{(n+1)} \) and substituted in Eq. (D.1). After substitution, \( A_1(\sigma_j^{2}, \hat{\theta}^{(n+1)}) \) is differentiated w.r.t. \( \sigma_j^{2} \) and equated to zero which leads to the result of the proposition.