INTERMITTENCY AND ALIGNMENT IN STRONG RMHD TURBULENCE

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ABSTRACT

We develop an analytic model of intermittent, three-dimensional, strong, reduced magnetohydrodynamic (RMHD) turbulence with zero cross helicity. We take the fluctuation amplitudes to have a log-Poisson distribution and incorporate into the model a new phenomenology of scale-dependent dynamic alignment between the Elsässer variables $\zeta^{\pm}$. We find that the structure function $\langle |\Delta \zeta^{\pm}|^p \rangle$ scales as $\lambda^{1-p}$, where $\Delta \zeta^{\pm}$ is the variation in $\zeta^{\pm}$ across a distance $\lambda$ perpendicular to the magnetic field. We calculate the value of $\beta$ to be $\approx 0.69$ based on our assumption that the most intense coherent structures are two-dimensional with a volume filling factor $\propto \lambda$. Two consequences of this structure-function scaling are that the total-energy power spectrum is $\propto k^{-1.52}$ and the kurtosis of the fluctuations is $\propto \lambda^{-0.27}$. Our model resolves the problem that the alignment angles defined in different ways exhibit different scalings. Specifically, we find that the energy-weighted average angle between the velocity and magnetic-field fluctuations is $\propto \lambda^{0.21}$, the energy-weighted average angle between $\Delta \zeta^+$ and $\Delta \zeta^-$ is $\propto \lambda^{0.10}$, and the average angle between $\Delta \zeta^+$ and $\Delta \zeta^-$ without energy weighting is $\propto |\ln (L/\lambda)|^{-1/2}$ when $L/\lambda \gg 1$, where $L$ is the outer scale. We also carry out a direct numerical simulation of RMHD turbulence. The scalings in our model are similar to the scalings in this simulation as well as the structure-function scalings observed in the slow solar wind.

Key words: magnetohydrodynamics (MHD) – plasmas – solar wind – Sun: chromosphere – Sun: corona – turbulence

1. INTRODUCTION

Plasma turbulence plays an important role in many astrophysical systems, including accretion flows around black holes, intracluster plasmas in clusters of galaxies, and outflows from stars, including the solar wind. In many of these systems, the energetically dominant component of the turbulence is non-compressive and can be modeled, at least in an approximate way, within the framework of incompressible MHD.

In incompressible RMHD, velocity and magnetic-field fluctuations $(d\mathbf{v}$ and $d\mathbf{B})$ propagate either parallel or anti-parallel to the local background magnetic field $B_{\text{loc}}$, and nonlinear interactions occur only between counter-propagating fluctuations (Iroshnikov 1963; Kraichnan 1965). As a consequence, the energy cascade is anisotropic, producing small-scale structures or “eddies” that satisfy $l \ll l$, where $l(\lambda)$ is the correlation length of an eddy (parallel) to $B_{\text{loc}}$ (Shebalin et al. 1983; Goldreich & Sridhar 1995; Ng & Bhattacharjee 1996; Cho & Vishniac 2000; Galtier et al. 2000; Maron & Goldreich 2001). When $l \ll l$, the components of $d\mathbf{v}$ and $d\mathbf{B}$ perpendicular to $B_{\text{loc}}$ (the “transverse” components) evolve independently of the components parallel to $B_{\text{loc}}$ and are well described by reduced MHD (RMHD; Kadomtsev & Pogutse 1974; Strauss 1976). When $d\mathbf{B} \ll B_{\text{loc}}$ and $\rho_p \ll \lambda \ll l$, where $\rho_p$ is the proton gyroradius, RMHD is a rigorous limit of gyrokinetics and is valid for both strongly and weakly collisional plasmas and for $\beta_{\text{pl}} \ll 1$ and $\beta_{\text{pl}} \gg 1$ (assuming temperature isotropy), where $\beta_{\text{pl}}$ is the ratio of the plasma pressure to the magnetic pressure (Schekochihin et al. 2009). We note that RMHD applies to inertial-range fluctuations satisfying $l \ll l$ and $d\mathbf{B} \ll B_{\text{loc}}$ even when $d\mathbf{B} \sim B_{\text{loc}}$ at the outer scale (Schekochihin et al. 2007, 2009).

In this paper, we propose a phenomenological theory of RMHD turbulence that goes beyond scaling theories for spectra (Iroshnikov 1963; Kraichnan 1965; Goldreich & Sridhar 1995; Boldyrev 2006) and allows us to make predictions concerning the scale dependence of arbitrary-order structure functions and the relative orientation of the turbulent magnetic field and velocity. A new feature of this theory is that it accounts, within one model, for both intermittency and scale-dependent dynamic alignment (SDDA).

The concept of SDDA was introduced by Boldyrev (2005, 2006), who argued that the angle $\phi_\lambda$ between $d\mathbf{v}_\lambda$ and $d\mathbf{B}_\lambda$ decreases with decreasing $\lambda$, where $d\mathbf{v}_\lambda$ and $d\mathbf{B}_\lambda$ are the fluctuations in the velocity and magnetic field at perpendicular scale $\lambda$. As $\phi_\lambda$ decreases, nonlinear interactions in RMHD weaken, causing the power spectrum of the fluctuation energy to flatten relative to models that neglect SDDA.

Intermittency is the phenomenon in which the fluctuation energy is concentrated into an increasingly small fraction of the volume as $\lambda \rightarrow 0$. Intermittency has been measured in hydrodynamic turbulence (e.g., Benzi et al. 1993), solar-wind turbulence (Burlaga 1991; Horbury & Balogh 1997; Sorriso-Valvo et al. 1999; Forman & Burlaga 2003; Bruno et al. 2007; Osman et al. 2012, 2014; Perri et al. 2012; Wan et al. 2012a), numerical simulations of MHD turbulence and RMHD turbulence (Müller & Biskamp 2000; Maron & Goldreich 2001; Müller et al. 2003; Beresnyak & Lazarian 2006; Mininni & Pouquet 2009; Rodriguez Imazio et al. 2013), and hybrid-Vlasov and particle-in-cell simulations of plasma turbulence (Greco et al. 2012; Servidio et al. 2012; Wan 2012b; Karimabadi 2013; Wu 2013). A number of theoretical models have been introduced to describe intermittency, including the log-normal model (Kolmogorov 1962; Gurvich & Yaglom 1967), the “constant-$\beta$” model (Frisch et al. 1978), and multi-fractal models in which the fluctuation amplitudes...
scale differently on different subsets of the volume that have different fractal dimensions (Parisi & Frisch 1985; Paladin & Vulpiani 1987). One such multi-fractal model, based on a log-Poisson probability distribution function (PDF) for the local dissipation rate, was developed by She & Leveque (1994; see also Dubrulle 1994). She & Leveque’s (1994) approach has served as the basis for several studies of intermittency in both compressible and incompressible MHD turbulence (Grauer et al. 1994; Politano & Pouquet 1995; Müller & Biskamp 2000; Boldyrev et al. 2002a, 2002b).

We draw upon ideas from the She–Leveque model to construct an analytic model of strong, intermittent, RMHD turbulence that incorporates a new phenomenology of SDDA. We present this model in Section 2. In Section 3, we present results from a direct numerical simulation of RMHD turbulence and compare these results with our theoretical predictions. In Section 4, we compare our model with observations of solar-wind turbulence, and in Section 5, we discuss our results and the relation between our work and previous turbulence models.

2. ANALYTIC MODEL OF STRONG RMHD TURBULENCE

The equations of RMHD can be written in the form

\[ \frac{\partial \zpm}{\partial t} + vA \cdot \nabla \zpm = -\zpm \cdot \nabla \zpm - \nabla \Pi, \]

\[ \nabla \cdot \zpm = 0, \]

and

\[ B_0 \cdot \zpm = 0, \]

where \( \zpm = \delta v \pm \delta B/\sqrt{4\pi \rho} \) are the Elsässer variables, \( \rho \) is the mass density, \( vA = B_0/\sqrt{4\pi \rho} \) is the Alfvén velocity, \( \Pi = (p + B^2/8\pi)/\rho \), \( p \) is the pressure, \( B = B_0 + \delta B \), and \( B_0 \) is the background magnetic field. Throughout this section, we neglect dissipation and focus on the inertial range.

2.1. Statistical Distribution of Field Increments

We consider the turbulence to be an ensemble of approximately localized \( \zpm \) and \( \zmm \) structures. We define

\[ \Delta \zpm = \zpm(x + 0.5\lambda \delta, t) - \zpm(x - 0.5\lambda \delta, t), \]

where \( \delta \) is a unit vector perpendicular to \( B(x, t) \). We define \( \delta \zpm \) to be \( |\Delta \zpm| \) averaged over the direction of \( \delta \), and we define \( \theta_A \) to be the (positive semi-definite) angle between \( \Delta \zpm \) and \( \Delta \zmm \) averaged over the direction of \( \delta \). We think of \( \delta \zpm(x, t) \) as the characteristic amplitude of the \( \zpm \) structure of scale \( \lambda \) that is located at position \( x \). \( \delta \zpm \) nonlinear interactions cause each structure at scale \( \lambda \) to break up into a number of structures at smaller scales. These smaller structures in turn break up into even smaller structures, and so on.

As can be seen from Equation (1), \( \zpm \) fluctuations propagate with velocity \( \mp vA \). We can thus view \( \zmm \) (\( \zpm \)) structures as wave packets that propagate parallel (anti-parallel) to the background magnetic field while being distorted by nonlinear interactions. The form of the nonlinear term in Equation (1) implies that nonlinear interactions occur only between \( \zpm \) fluctuations and \( \zmm \) fluctuations, and not between fluctuations that propagate in the same direction (Iroshnikov 1963; Kraichnan 1965). The energy cascade in RMHD turbulence can thus be viewed as resulting from “collisions” between counter-propagating wave packets. In the discussion below, we use the terms “structure,” “wave packet,” and “fluctuation” interchangeably.

In the Appendix, we argue that if a \( \delta \zpm \) fluctuation collides with a \( \delta \zmm \) fluctuation that is either much stronger or much weaker than \( \delta \zpm \), then \( \lambda \) changes for both fluctuations (that is, they are sheared by each other), but the fluctuation amplitudes remain approximately the same. We refer to such collisions as “highly balanced.” On the other hand, if \( \delta \zpm \sim \delta \zmm \) (“balanced collisions”), then in general both \( \lambda \) and the fluctuation amplitudes decrease, as in models of MHD turbulence and RMHD turbulence that neglect intermittency (e.g., Goldreich & Sridhar 1995).

To construct an analytic model of RMHD turbulence, we assume that each balanced collision reduces a fluctuation’s amplitude by a constant factor \( \beta \) that satisfies

\[ 0 < \beta < 1, \]

while highly imbalanced collisions reduce \( \lambda \) without reducing a fluctuation’s amplitude. Thus,

\[ \delta \zpm = \overline{\delta \zpm} \beta^q, \]

where \( \overline{\delta \zpm} \) is the amplitude of the fluctuation’s “progenitor” structure at the outer scale (or forcing scale) \( L \), and \( q \) is the number of balanced collisions experienced by the fluctuation during its evolution from scale \( L \) to scale \( \lambda \). For simplicity, we set

\[ \overline{\delta \zpm} = \text{constant}. \]

To determine a plausible functional form for the PDF of \( q \), we consider a hypothetical scenario in which balanced collisions have the property that they reduce a fluctuation’s amplitude without changing its length scale. In this case, balanced collisions are similar to the “modulation defect events” described by She & Waymire (1995), in that a fluctuation’s amplitude can be reduced by a finite factor \( \beta \) during an interval of time in which \( \lambda \) decreases by only an infinitesimal amount. If the length scale of a fluctuation decreases from \( L \) to \( \lambda \), then we can divide the interval \([0, \ln(L/\lambda)]\) into infinitesimal sub-intervals, and within each sub-interval there is an infinitesimal chance that a modulation defect event occurs. Over the entire interval, however, the average number of modulation defect events is finite. If we assume that the probability of a balanced collision is independent of the number of balanced collisions that have already occurred, then \( q \) has a Poisson distribution,

\[ P(q) = \frac{e^{-\mu} \mu^q}{q!}, \]

\( ^5 \) We take the local background field for \( \Delta \zpm \), denoted \( B_{loc} \), to be the sum of \( B_0 \) and the value of \( \delta B(x + 0.5\lambda \delta, t) \) averaged over the direction of \( \delta \). This averaging process filters out magnetic fluctuations at scales \( \lesssim \lambda \), and thus \( B_{loc} \) is approximately the sum of \( B_0 \) and the magnetic fluctuations at scales significantly larger than \( \lambda \). We note that in the RMHD limit, in which \( \delta B \ll B_0 \), the components of \( \delta B \) and \( \delta v \) perpendicular to \( B_{loc} \) are effectively equivalent to the components of \( \delta B \) and \( \delta v \) perpendicular to \( B_0 \).

\( ^6 \) More realistically, \( \overline{\delta \zpm} \) would have its own (scale-independent) distribution reflecting the non-universal details of the outer-scale statistics (e.g., the statistics of the forcing).
where $\mu$ is the as-yet-unknown, scale-dependent, mean value of $q$. In RMHD turbulence, balanced collisions do in fact change $\lambda$, and the probability that a balanced collision occurs may depend upon $q$. Thus, the above arguments do not provide a rigorous justification for Equation (8). We proceed, however, using Equation (8) as a model. We further assume that $\mu$ and $\overline{\delta z}$ are the same for $\delta z^+$ and $\delta z^-$ and thereby restrict our analysis to the case of zero cross helicity.

The median value of $q$ is approximately $\mu$ (Choi 1994), and thus the “typical” value of $\delta z^\lambda$ that best characterizes the bulk of the volume is

$$\delta z^\lambda = \overline{\delta z} \beta^\mu. \quad (9)$$

In contrast, the most intense structures at scale $\lambda$ correspond to $q = 0$ and occur with probability $e^{-\mu}$. Equation (6) implies that the variation in $z^+$ or $z^-$ across such a $q = 0$ structure is $\overline{\delta z}$, independent of $\lambda$. We assume that these structures correspond to sheet-like quasi-discontinuities (current/vorticity sheets) with a volume-filling factor $\propto \lambda$ (cf., Grauer et al. 1994; Politano & Pouquet 1995). Setting $e^{-\mu} \propto \lambda$, we obtain

$$\mu = A + \ln \left( \frac{\lambda}{\lambda} \right), \quad (10)$$

where $A$ is a constant that quantifies the breadth of the distribution at the outer scale. We can thus rewrite Equation (9) in the form

$$\delta z^\lambda = \overline{\delta z} \left( \frac{\lambda}{\lambda} \right)^{-\ln \beta} \quad (11)$$

2.2. Structure Functions

The two-point structure functions $\langle (\delta z^\lambda)^n \rangle$ are the standard measures used to establish the presence of intermittency in turbulence (Kolmogorov 1962; Frisch 1996). From Equations (6) through (10), we obtain

$$\langle (\delta z^\lambda)^n \rangle = \left( \overline{\delta z} \right)^n e^{-\mu} \sum_{q=0}^{\infty} \frac{(\mu \beta^\mu)^q}{q!}. \quad (12)$$

The sum in Equation (12) is simply $e^{\mu \beta^\mu}$. With the use of Equation (10), we thus obtain

$$\langle (\delta z^\lambda)^n \rangle = \left( \overline{\delta z} \right)^n \left( \frac{\lambda}{\lambda} \right)^{\zeta_n}, \quad (13)$$

where

$$\zeta_n = 1 - \beta^\mu. \quad (14)$$

We complete our evaluation of $\zeta_n$, by determining the value of $\beta$, in Section 2.4.

In hydrodynamic turbulence, the rate at which energy is dissipated within a sphere of diameter $\lambda$, denoted $\nu_\lambda$, is

$$\langle \delta v^2 \rangle / \lambda, \quad (6)$$

where $\langle \delta v \rangle$ is the increment in the velocity field across a distance $\lambda$ (Kolmogorov 1962). The fact that $\langle \delta v \rangle$ is independent of $\lambda$ implies that $\langle \delta v^2 \rangle \propto \lambda$ (Kolmogorov 1941a), which is analogous to the condition that $\zeta_3 = 1$. This condition, however, does not arise in RMHD turbulence, because the rate at which $z^\pm$ energy is dissipated within a sphere of diameter $\lambda$ does not scale like $\langle \delta z^\lambda \rangle^2$, but instead depends upon a combination of three different random variables ($\delta z^\lambda$, $\delta z_{\perp}$, and the alignment angle $\theta_\lambda$), as described in Equation (25) below.

Our finding that $\zeta_n \rightarrow 1$ as $n \rightarrow \infty$ differs from previous models of intermittent MHD and hydrodynamic turbulence, in which $\zeta_n \rightarrow \infty$ as $n \rightarrow \infty$, and can be understood on the basis of a simple argument. As $n \rightarrow \infty$, $\langle (\delta z^\lambda)^n \rangle$ is dominated by the most intense fluctuations, which have an amplitude $\overline{\delta z}$ that is independent of scale (as argued in Section 2.1 and the Appendix) and a filling factor $f_\lambda$ that is proportional to $\lambda$. Thus, as $n \rightarrow \infty$, $\langle (\delta z^\lambda)^n \rangle \propto f_\lambda \langle \overline{\delta z} \rangle^n \propto \lambda$.

If, instead of focusing on the limit of large $n$, we focus on the limit of large $\mu$, then the summand in Equation (12) is maximized when $q \approx q_n$, where

$$q_n = \mu \beta^\mu. \quad (15)$$

Terms with $q < q_n$ account for approximately half of the total sum in Equation (12), just as the median of $P(q)$ in Equation (8) is approximately $\mu$ (Choi 1994). The mean value of $q$ is identically $\mu$, and the standard deviation of $q$ is

$$\sigma = \left( \langle (q - \mu)^2 \rangle \right)^{1/2} = \mu^{1/2}. \quad (16)$$

Thus, the fluctuations that make the dominant contribution to $\langle (\delta z^\lambda)^n \rangle$ are $\sim N$ standard deviations out into the tail of the $q$ distribution, where

$$N = \frac{\mu - q_n}{\sigma} = \mu^{1/2} (1 - \beta^\mu). \quad (17)$$

As $\lambda$ decreases, $\mu$ increases, and, as a consequence, $N$ increases. Moreover, $N$ increases to a greater degree when $n$ is larger. This is why $\langle (\delta z^\lambda)^n \rangle$ in Equation (13) decreases more slowly with decreasing $\lambda$ than does $\langle (\delta z^\lambda)^n \rangle$ and why the plot of $\zeta_n$ as a function of $n$ is concave downwards.

2.3. Timescales and Critical Balance

We define the nonlinear timescale

$$\tau_{nl,\lambda} = \frac{\lambda}{\overline{\delta z} \sin \theta_\lambda}, \quad (18)$$

which is the approximate time required for a $z^\pm$ structure at scale $\lambda$ to be altered substantially by the $z^\mp$ structure at scale $\lambda$ at that same location. The factor of $\sin \theta_\lambda$ is included in Equation (18) because, if $z^+ = z^-$ are aligned to within a small angle $\theta$, then $|z^+ \cdot \nabla z^+|$ is reduced by a factor $\sim \theta$ relative to the case in which $\theta \sim 1$ (Boldyrev 2005).

We define the linear timescale

$$\tau_{l,\lambda} = \frac{l^1}{\nu_\lambda}. \quad (19)$$

Here, $l^1$ is the “parallel” correlation length of a $\delta z^\lambda$ structure measured along the local direction of the magnetic field that is obtained by summing $B_0$ with the magnetic fluctuations at scales that exceed $\lambda$ by a factor of at least a few.

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See, e.g., She & Leveque (1994), Grauer et al. (1994), Politano & Pouquet (1995), and Frisch (1996).
In accordance with the critical-balance hypothesis of Goldreich & Sridhar (1995), we assume that
\[ \tau_{nl,\lambda}^+ \sim \tau_{nl,\lambda}^- . \] (20)
We can rewrite Equation (20) as
\[ \chi^\pm = \frac{I_{\lambda}^\pm \delta z_{\lambda}^\pm \sin \theta_\lambda}{\lambda v_A} \sim 1. \] (21)
Equation (20) is also equivalent to the relation
\[ l_{\lambda}^\pm \sim v_A \tau_{nl,\lambda}^\pm , \] (22)
which states that the parallel correlation length of a \( \delta z_{\lambda}^\pm \) fluctuation is roughly the distance it can propagate during its cascade timescale.

When a \( \delta z_{\lambda}^+ \) fluctuation collides with a \( \delta z_{\lambda}^- \) fluctuation and \( \delta z_{\lambda}^+ \sim \delta z_{\lambda}^- \sim \delta z_{\lambda}^* \) (see Equation (9)), nonlinear interactions cause the \( \delta z_{\lambda}^- \) and \( \delta z_{\lambda}^* \) fluctuations to evolve on the same timescale in a strongly coupled and unpredictable way, which impedes the development of alignment. We thus take
\[ \theta_\lambda \sim 1 \quad \tau_{nl,\lambda}^+ \sim \lambda / \delta z_{\lambda}^- . \] (23)
The characteristic parallel correlation length of the median-amplitude fluctuations at scale \( \lambda \) is then
\[ l_{\lambda}^* = \frac{v_A \lambda}{\delta z_{\lambda}^*} . \] (24)

2.4. Nonlinear Interactions and the Refined Similarity Hypothesis

Given our assumption that \( \tau_{nl,\lambda}^+ \sim \tau_{nl,\lambda}^- \), the turbulence is strong, and \( \delta z_{\lambda}^\pm \) energy cascades to smaller scales on the timescale \( \tau_{nl,\lambda} \). We define \( \epsilon_{\lambda}^\pm \) to be the rate at which \( z^\pm \) energy (per unit mass) is dissipated within a sphere of diameter \( \lambda \). In keeping with earlier works on intermittency, we take \( \epsilon_{\lambda}^\pm \) to be “equal in law” to the quantity \( (\delta z_{\lambda}^+)^2 / \tau_{nl,\lambda} \) (Frisch 1996). This means that the nth moment of \( \epsilon_{\lambda}^\pm \) and the nth moment of the quantity \( (\delta z_{\lambda}^+)^2 / \tau_{nl,\lambda} \) scale with \( \lambda \) in the same way for all \( n \). We denote “equality in law” with the symbol \( \sim \) and thus write
\[ \epsilon_{\lambda}^\pm \sim \left( \frac{\delta z_{\lambda}^*}{\lambda} \right)^2 \frac{\delta z_{\lambda}^* \sin \theta_\lambda}{\lambda} . \] (25)
Equation (25) is analogous to Kolmogorov’s 1962 refined similarity hypothesis for hydrodynamic turbulence. We note that the scalings that we derive below do not require full “equality in law,” but just that the averages of the left- and right-hand sides of Equation (25) scale with \( \lambda \) in the same way.

The average dissipation rate within a sphere of diameter \( \lambda \) is independent of \( \lambda \) and thus satisfies the relation
\[ \langle \epsilon_{\lambda}^\pm \rangle = \epsilon, \] (26)
where \( \langle \ldots \rangle \) indicates a spatial average and \( \epsilon \) is the global, average dissipation rate, which is the same for \( z^+ \) and \( z^- \).

Equation (21) is a simplifying assumption. In numerical simulations of RMHD turbulence, \( \chi^\pm \) has a distribution, but this distribution is scale-independent and has a mean of order unity (Mallet et al. 2015).

fluctuations given our assumption that the cross helicity is zero. In forced turbulence that has reached a (statistical) steady state, \( \epsilon \) is also the rate at which energy is injected into the turbulence at the outer scale. Our goal now is to use Equations (25) and (26) to determine the value of \( \beta \).

To average the right-hand side of Equation (25), we consider the spherical trial volume of diameter \( \lambda \) illustrated in Figure 1. We can take the PDF of \( \delta z_{\lambda}^- \) within the trial volume to be determined by Equations (6) and (8). However, once we do so, we cannot also take the value of \( \delta z_{\lambda}^- \) within the trial volume to have the same distribution, because in general \( \delta z_{\lambda}^- \) and \( \delta z_{\lambda}^* \) are correlated.

We assume that the dominant contribution to \( \langle (\delta z_{\lambda}^\pm)^2 \delta z_{\lambda}^* \sin \theta_\lambda / \lambda \rangle \) comes from exceptionally intense \( \delta z_{\lambda}^* \) fluctuations satisfying the inequalities \( \delta z_{\lambda}^- \gg \delta z_{\lambda}^* \) and \( \delta z_{\lambda}^\pm \gg \delta z_{\lambda}^- \). We therefore focus on the case in which these inequalities are satisfied within the trial volume. We assume (and confirm below in Equation (50)) that \( I_{\lambda}^* \gg I_{\lambda}^\pm \) when \( \delta z_{\lambda}^* \gg \delta z_{\lambda}^- \). Because of their comparatively large parallel correlation lengths (and long lifetimes, as we will see in Equation (48)), we refer to structures with \( \delta z_{\lambda}^* \gg \delta z_{\lambda}^- \) as coherent structures.

The \( z^\pm \) fluctuations at scale \( \lambda \) propagate along a local magnetic field obtained by summing \( B_\parallel \) with the magnetic-field fluctuations at length scales exceeding \( \lambda \) by at least some factor of order unity. Because this factor is not uniquely determined, the direction in which \( \delta z_{\lambda}^\pm \) fluctuations propagate is only determined to within an angular uncertainty of order
\[ \Delta \theta_\lambda \sim \frac{\delta z_{\lambda}^*}{v_A} , \] (27)
where we have taken the fluctuations at scales somewhat larger than \( \lambda \) to have amplitudes comparable to \( \delta z_{\lambda}^* \). Here we have assumed that the intense \( \delta z_{\lambda}^* \) fluctuations propagate through a background of median-amplitude \( z^- \) fluctuations, a point that we discuss further in connection with Equation (33) below.
Because of the angular uncertainty $\Delta \theta_\lambda$, a $z^-$ structure of scale $\lambda$ is only able to propagate a distance $\sim \lambda/(\Delta \theta_\lambda) \sim l^*_\lambda$ (see Equation (24)) through a counter-propagating $\delta z^+\lambda$ structure before propagating out of that structure. Because of this, if we follow the magnetic-field lines in the trial volume (Figure 1) back along the magnetic field a distance $l^*_\lambda$, we reach a "source region" in which the $z^-$ fluctuations have not yet interacted with the coherent $\delta z^+\lambda$ structure. Within this source region, the $z^-$ fluctuations are not aligned with the coherent $\delta z^+\lambda$ structure, because they do not yet "know about" the coherent $\delta z^+\lambda$ structure's orientation in space. If we pick two field lines a distance $\lambda$ apart within the trial volume and follow them for a distance $l^*_\lambda$, they will typically separate by a distance of order

$$\xi_\lambda = l^*_\lambda \frac{\delta z^+\lambda}{v_\lambda} = \lambda \frac{\delta z^+\lambda}{\delta z^-\lambda} \gg \lambda. \quad (28)$$

We assume that the coherent $\delta z^+\lambda$ structure remains coherent over a distance of at least $\sim \xi_\lambda$ in the direction of the vector magnetic-field fluctuations associated with the $\delta z^-\lambda$ structure—i.e., throughout the slab depicted in Figure 1. The coherent $\delta z^+\lambda$ structure is thus sheet-like.

We expect (and confirm below in Equation (48)) that the cascade timescale of the coherent $\delta z^+\lambda$ structure is $\gg l^*_\lambda/v_\lambda$, so that the $\delta z^+\lambda$ structure changes very little as a $z^-$ fluctuation propagates from the source region to the trial volume. We make the approximation that during this transit, the $z^-$ fluctuation evolves as if it were acted upon by a linear $z^+$ shear with shearing rate $\delta z^+\lambda/\lambda$ that lasts for a time $l^*_\lambda/v_\lambda$, where the term "linear" refers to the shear's spatial profile (see Equation (75)). In the Appendix, we present an analytic calculation showing that in this approximation the amplitude of the $z^-$ fluctuation is unchanged by the shear and the direction of the $z^-$ fluctuation is rotated into alignment with the $z^+$ structure, so that

$$\sin \theta_\lambda \approx \theta_\lambda \sim \frac{\lambda}{\xi_\lambda} \frac{\delta z^-\lambda}{\delta z^+\lambda} \quad (29)$$

within the trial volume. We also show in the Appendix that, because the $z^-$ fluctuations are sheared at rate $\delta z^+\lambda/\lambda$ for a time $l^*_\lambda/v_\lambda$, their perpendicular scales decrease by a factor of

$$\frac{\delta z^+\lambda}{\lambda} \times \frac{l^*_\lambda}{v_\lambda} = \frac{\xi_\lambda}{\lambda} \gg 1 \quad (30)$$

during their propagation from the source region to the trial volume. The source region in Figure 1 contains $z^-$ fluctuations spanning a range of perpendicular scales. According to the above arguments, the fluctuations at scale $\xi_\lambda$ in the source region make the dominant contribution to the values of $\delta z^+\lambda$ and $\delta z^-\lambda$ sin $\theta_\lambda$ within the trial volume. Thus,

$$\delta z^-\lambda \bigg|_{\text{trial volume}} \simeq \delta z^-\lambda \bigg|_{\text{source region}}. \quad (31)$$

The top half of Figure 2 illustrates the arguments underlying Equation (29) and the scale-reduction factor in Equation (30) for the hypothetical case in which the $z^-$ fluctuations in the source region have square cross sections of scale $\lambda$ in the field-perpendicular plane. The trial volume, the perpendicular scale length of these fluctuations becomes $\sim \lambda^2/\xi_\lambda$ and, because of Equation (29), $\theta_\lambda \ll 1$. The evolution of $\delta z^-\lambda$ can be recovered heuristically by taking the pattern of the $z^-$ fluctuation in the field-perpendicular plane approximately follows the magnetic field lines within the coherent $\delta z^+\lambda$ structure. This causes the perpendicular length scale of the $z^-$ fluctuations to decrease by a factor $\sim \xi_\lambda/\lambda >> 1$ and rotates the $z^-$ fluctuations into alignment with the $\delta z^+\lambda$ structure, decreasing the angle between the $z^+$ and $z^-$ fluctuations to a value $\sim \lambda/\xi_\lambda$.

$$\epsilon_\lambda^+ \approx \left(\frac{\delta z^+\lambda}{\xi_\lambda}\right)^2 \frac{\delta z^-\lambda}{\xi_\lambda} \quad (32)$$

when $\delta z^+\lambda \gg \delta z^-\lambda$, where $\epsilon_\lambda^+$ and $\delta z^+\lambda$ are evaluated within the trial volume and $\delta z^-\lambda$ is evaluated within the source region. We now consider the average of Equation (32). Our assumption that $\delta z^+\lambda \gg \delta z^-\lambda$ in the trial volume decreases the probability that $\delta z^-\lambda$ is much larger than $\delta z^+\lambda$ in the source region, because intense, counter-propagating, $z^+$ structures rapidly annihilate. We thus take the PDF of $\delta z^-\lambda$ within the source region to be negligible at large $\delta z^-\lambda$ and make the approximation that

$$\delta z^-\lambda \bigg|_{\text{source region}} \simeq \delta z^+\lambda. \quad (33)$$

It follows from Equation (11) that $\delta z^+\lambda/\xi_\lambda = (\delta z^+\lambda/\lambda)(\xi_\lambda/\lambda)^{1-\ln \beta}$, and thus, using Equation (28), we can...
rewrite Equation (32) as
\[
\epsilon_\lambda \approx \frac{(\delta_\lambda^z)^{\frac{1}{-\ln \beta}} (\delta_\lambda^z)^{2 + \ln \beta}}{\lambda^\beta}.
\] (34)

We now average Equation (34) over space. For the right-hand side of Equation (34), this is equivalent to averaging over the Poisson distribution of \( g \) in Equation (6), which is given in Equation (8). We thus obtain
\[
\langle \epsilon_\lambda \rangle \sim \frac{(\delta_\lambda^z)^3}{\lambda^{\beta (2 + \ln \beta) - \ln \beta}} \sum_{q=0}^{\infty} \frac{(\mu^{\beta - 1 - \ln \beta}) q}{q!}.
\] (35)

To derive Equation (34), we assumed that \( \delta_\lambda^z > \delta_\lambda^z \), and as a consequence the form of the summand in Equation (35) is incorrect when \( q > \mu \). However, in the inertial range, in which \( \mu \) is formally large, terms with \( q > \mu \) make only a small contribution to the sum in Equation (35), consistent with our assumption that the total \( \zeta^+ \) dissipation rate is dominated by large-\( \delta_\lambda^z \) regions. The sum in Equation (35) equals \( \exp (\mu^{\beta - 1 - \ln \beta}) \), and thus Equation (35) implies that
\[
\langle \epsilon_\lambda \rangle \propto \frac{1}{\lambda^{2 + \ln \beta} \ln \beta}. \tag{36}
\]

Since \( \langle \epsilon_\lambda \rangle \) must be independent of \( \lambda \), we obtain
\[
(2 + \ln \beta) \ln \beta + \beta - 1 - \ln \beta = 0.
\] (37)

There are two solutions to Equation (37): \( \beta \approx 0.136 \) and \( \beta \approx 0.691 \). The solution \( \beta \approx 0.136 \) leads to the scaling \( \delta_\lambda^z \approx \lambda^{0.98} \), which implies that the median variation in \( \zeta^+ \) across a distance \( \lambda \) measured perpendicular to \( B \) is dominated by the outer-scale eddies and not by \( \delta_\lambda^z \), as we have assumed. Thus, the only solution to Equation (37) that is consistent with its derivation is
\[
\beta \approx 0.691.
\] (38)

We note that Equations (26), (35), and (37) imply that
\[
\epsilon \sim \frac{(\delta_\lambda^z)^3}{\epsilon_\lambda L}.
\] (39)

Equation (39) establishes a relationship between the energy input into the turbulence and the two parameters \( \delta_\lambda^z \) and \( A \) that quantify the non-universal features of the outer-scale fluctuations. Given \( \epsilon \) and \( L \), only one of \( \delta_\lambda^z \) and \( A \) is a free parameter in our model.

### 2.5. Consistency of the Strong-Turbulence Assumption

As described in Section 2.4, the type of nonlinear interaction that is most effective at shearing large-amplitude \( \delta_\lambda^z \) structures involves the typical \( \delta_\lambda^z \approx \delta_\lambda^z \) structures (Equation (33)) at scale \( \xi_1 \), which exceeds \( \epsilon_\lambda \) to a degree that depends on the amplitude \( \alpha \) (see Equation (28)). These are the \( \zeta^+ \) structures in the source region that, upon shearing by an intense \( \delta_\lambda^z \) structure, become the \( \delta_\lambda^z \) structures in the trial volume in Figure 1 (see Equation (31)) and enter into the computation of the average cascade power \( \langle (\delta_\lambda^z)^2 \delta_\lambda^z \sin \theta / \lambda \rangle \) within the trial volume. The parallel correlation length of the typical \( \delta_\lambda^z \) fluctuations in the source region in Figure 1 is \( \sim l_{\epsilon_\lambda}^z \), and thus the correlation time of the \( \delta_\lambda^z \) fluctuation in the trial volume is \( \sim \tau_\lambda \). However, in the inertial range, in which \( \epsilon_\lambda^z \) is given in Figure 1, we have just summarized in Section 2.5, an intense \( \delta_\lambda^z \) fluctuation is cascaded primarily by collisions with \( \zeta^+ \) fluctuations whose perpendicular scale prior to colliding was \( \xi_1 \), which significantly exceeds \( \lambda \). Thus, the cascade is local in \( \lambda \) if the scales of the interacting fluctuations are evaluated at the same point in space (e.g., the trial volume in Figure 1), but nonlocal if the perpendicular scale of \( \delta_\lambda^z \) is evaluated in the trial volume in Figure 1 while the perpendicular scale of the \( \zeta^+ \) fluctuation is evaluated in the source region depicted in this figure. We note that Equations (22) and (40) imply that, when \( \delta_\lambda^z \approx \delta_\lambda^z \),
\[
l_{\epsilon_\lambda}^z \approx l_{\epsilon_\lambda}^z.
\] (41)

Thus, just before the nonlinear interaction begins, the \( \zeta^+ \) fluctuations that dominate the shearing of a large-amplitude, coherent \( \delta_\lambda^z \) structure have the same parallel correlation length as that of \( \delta_\lambda^z \) structure. In this sense, the cascade could be described as “local in parallel length scale” (cf. Beresnyak & Lazarian 2008).

### 2.7. Inertial-Range Scalings

From Equations (13), (14), and (38), the second-order structure function satisfies the relation
\[
\left\langle \left( \delta_\lambda^z \right)^2 \right\rangle \propto \lambda^{1 - \beta} \approx \lambda^{0.52},
\]
which corresponds to an inertial-range \( \zeta^\pm \) power spectrum
\[
E(k_\perp) \propto k_\perp^{-1.52},
\] (43)
where $k_\perp$ is the wave-vector component perpendicular to $B_0$. Equation (13) implies that the flatness obeys the scaling

$$\frac{\langle (\delta z^+_\perp)^4 \rangle}{\langle (\delta z^+_\perp)^2 \rangle^2} = \left( \frac{\lambda}{e^4 L} \right)^{1-(1-\beta)^2} \propto \lambda^{-0.27},$$  

(44)

which exemplifies how intermittency increases with decreasing $\lambda$. The kurtosis is the flatness minus 3. Since both quantities become $\gg 1$ as $\lambda \to 0$, the kurtosis also scales like $\lambda^{-0.27}$ in the small-$\lambda$ limit. We emphasize that the parameter $A$ does not affect the exponents in any of the power-law scalings in our model (nor the fact that $\theta^\perp_\lambda$ in Equation (64) below decreases logarithmically as $\lambda/L$ decreases to very small values).

For reference, Equation (11) implies that the amplitude of a “typical” structure is

$$\delta z^+_\perp = \delta z \left( \frac{\lambda}{e^4 L} \right)^{-\ln \beta} \propto \lambda^{0.37},$$  

(45)

and hence

$$\frac{\langle (\delta z^+_\perp)^2 \rangle^{1/2}}{\delta z^+_\perp} = \left( \frac{\lambda}{e^4 L} \right)^{\ln \beta/(1-\beta)^2/2} \propto \lambda^{-0.11}.$$  

(46)

This shows that at small $\lambda/L$ the rms fluctuation amplitude is much larger than the median fluctuation amplitude. Equation (45) implies via Equation (24) that

$$I^+_\perp \propto \lambda^{1+\ln \beta} \propto \lambda^{0.63}.$$  

(47)

Equations (29), (31), and (33) imply that, when $\delta z^+_\perp \gg \delta z^+_\parallel$,

$$\tau_{\text{mix}} \sim \frac{\lambda}{\delta z^+_\perp} \left( \frac{\delta z^+_\perp}{\delta z^+_\parallel} \right)^{1+\ln \beta}.$$  

(48)

The energy cascade timescale of the most intense fluctuations $\tau_{\text{mix}}$ follows from setting $\delta z^+_\perp = \delta z^+_\parallel$ in Equation (48), which, together with Equation (45), yields

$$\tau_{\text{mix}} \propto \lambda^{1+\ln \beta} \propto \lambda^{0.40}.$$  

(49)

Finally, Equations (22) and (48) yield

$$I^+_\parallel \sim I^+_\perp \left( \frac{\delta z^+_\perp}{\delta z^+_\parallel} \right)^{1+\ln \beta} \gg I^+_\perp$$  

(50)

(which confirms an assumption to this effect in Section 2.4).

2.8. Alignment

We define the average alignment angles

$$\theta^\perp_\lambda = \frac{\langle |\Delta z^+_\perp \times \Delta z^-_{\parallel} |^2 \rangle}{\langle |\Delta z^+_\perp |^2 \rangle^2}$$  

(51)

and

$$\theta^{(vb)}_\lambda = \frac{\langle |\Delta v_{\perp} \times \Delta b_{\parallel} | \rangle}{\langle |\Delta v_{\perp} |^2 \rangle^2},$$  

(52)

where $\Delta v_{\perp} = (\Delta z^+_\perp + \Delta z^-_{\perp})/2$, $\Delta b_{\parallel} = (\Delta z^+_\parallel - \Delta z^-_{\parallel})/2$, and $\langle . . . \rangle$ now denotes averages over volume as well as the direction of the unit vector $\delta$ defined following Equation (4). To evaluate $\theta^\perp_\lambda$, we set

$$\langle |\Delta z^+_\perp \times \Delta z^-_{\parallel} | \rangle \approx \langle \delta z^+_\perp \delta z^-_{\parallel} \sin \theta_{\parallel} \rangle.$$  

(53)

We assume (and verify below) that in the inertial range the dominant contribution to $\langle \delta z^+_\perp \delta z^-_{\parallel} \sin \theta_{\parallel} \rangle$ comes from regions in which $\delta z^+_\perp \gg \delta z^+_\parallel$ or $\delta z^-_{\parallel} \gg \delta z^+_\parallel$. Since $\langle \delta z^+_\perp \delta z^-_{\parallel} \sin \theta_{\parallel} \rangle$ is symmetric with respect to the interchange of $z^+$ and $z^-$, we can estimate $\langle \delta z^+_\perp \delta z^-_{\parallel} \sin \theta_{\parallel} \rangle$ by keeping only the contribution from regions in which $\delta z^+_\perp \gg \delta z^-_{\parallel}$. We then evaluate this contribution by considering a spherical trial volume of diameter $\lambda$ as in Figure 1 and approximating $\delta z^-_{\parallel}$ and $\sin \theta_{\parallel}$ within the trial volume using Equations (29), (31), and (33). We then average over the log-Poisson PDF of $\delta z^+_\perp$ and make use of Equation (37) to obtain

$$\langle |\Delta z^+_\perp \times \Delta z^-_{\parallel} | \rangle \approx \delta z^+_\perp \left( \frac{\lambda}{e^4 L} \right)^{1+(\beta-1)\beta^{-\ln \beta}} \propto \lambda^{0.73},$$  

(54)

Using the same approach and setting

$$\langle |\Delta z^+_\perp \times \Delta z^-_{\parallel} | \rangle \approx \delta z^+_\perp \left( \frac{\lambda}{e^4 L} \right)^{1+\ln \beta} \propto \lambda^{0.63},$$  

(55)

we obtain

$$\langle |\Delta z^+_\perp \times \Delta z^-_{\parallel} | \rangle \approx \delta z^+_\perp \left( \frac{\lambda}{e^4 L} \right)^{(\beta-1)\beta^{-\ln \beta}} \propto \lambda^{0.10}.$$  

(56)

Combining Equations (54) and (56), we find that

$$\theta^\perp_\lambda \sim \left( \frac{\lambda}{e^4 L} \right)^{(\beta-1)\beta^{-\ln \beta}} \propto \lambda^{0.21}.$$  

(58)

The above scalings reflect the contributions to $\langle \delta z^+_\perp \delta z^-_{\parallel} \sin \theta_{\parallel} \rangle$ and $\langle \delta z^+_\parallel \delta z^-_{\perp} \rangle$ from regions in which $\delta z^+_\perp \gg \delta z^+_\parallel$ or $\delta z^-_{\perp} \gg \delta z^+_\perp$. An upper bound on the contribution from the remaining regions in which $\delta z^+_\perp \lesssim \delta z^+_\parallel$ can be obtained by setting $\delta z^+_\parallel = \delta z^+_\perp$ and $\sin \theta_{\parallel} \sim 1$ in $\langle \delta z^+_\perp \delta z^-_{\perp} \sin \theta_{\parallel} \rangle$ and $\langle \delta z^+_\parallel \delta z^-_{\perp} \rangle$. The resulting upper bounds become negligibly small compared to the values in Equations (54) and (56) as $\lambda/L \to 0$, consistent with our assumption that the large-$\delta z^\perp$ regions make the dominant contributions to $\langle \delta z^+_\perp \delta z^-_{\parallel} \sin \theta_{\parallel} \rangle$ and $\langle \delta z^+_\perp \delta z^-_{\perp} \rangle$.

In the inertial range, the dominant contribution to $\langle \delta z^+_\perp \rangle^2$ comes from regions in which $\delta z^+_\perp$ is unusually large. In most of these regions, $\delta z^+_\perp \gg \delta z^-_{\perp}$ and $|\Delta v_{\perp}| \approx |\Delta b_{\parallel}| \approx \delta z^+_\perp/2$. Keeping only the contribution to $\langle |\Delta v_{\perp}| \rangle |\Delta b_{\parallel}|$ from the regions that make the dominant contributions to $\langle \delta z^+_\perp \delta z^-_{\perp} \rangle$ and $\langle \delta z^+_\perp \delta z^-_{\parallel} \rangle$, we obtain the estimate $\langle |\Delta v_{\perp}| \rangle |\Delta b_{\parallel}| \approx (\langle \delta z^+_\perp \rangle^2)/2$. Since $\Delta v_{\perp} \times \Delta b_{\parallel} = \Delta z^-_{\perp} \times \Delta z^+_\parallel$, Equations (13) and (54) imply that

$$\theta^{(vb)}_\lambda \sim \left( \frac{\lambda}{e^4 L} \right)^{\beta+1-(\beta-1)\beta^{-\ln \beta}} \propto \lambda^{0.04}.$$  

(59)

Finally, we define a third average alignment angle

$$\theta^\perp_\lambda = \left( \frac{\langle |\Delta z^+_\perp \times \Delta z^-_{\perp} | \rangle}{\langle |\Delta z^+_\perp |^2 \rangle^2} \right).$$  

(59)
The angle $\theta_\lambda^*$ is the volume average of the (sine of the) angle between $\Delta z^+_{\lambda}$ and $\Delta z^-_{\lambda}$, whereas $\theta_\lambda^+$ is a weighted average of (the sine of) this angle that is dominated by regions in which the fluctuation amplitudes are large. If initially unaligned $\delta z^+_{\lambda}$ and $\delta z^-_{\lambda}$ fluctuations collide and $\delta z^+_{\lambda} \sim \delta z^-_{\lambda}$, then both fluctuations evolve nonlinearly on the same timescale in an unpredictable and disordered manner, which prevents the development of strong alignment. Building upon this idea, we estimate $\theta_\lambda^*$ as follows. We consider a new trial volume that is halfway between two source regions, one for $\delta z^+_{\lambda}$ and one for $\delta z^-_{\lambda}$, which are separated by a distance $2L_0^\lambda$. Because this distance is twice as large as the typical parallel correlation length that characterizes the bulk of the volume, we take $\delta z^-_{\lambda}$ in the two source regions to be statistically independent. This assumption of statistical independence breaks down for exceptionally strong fluctuations with large values of $L_0^\lambda$, but such fluctuations account for only a small fraction of the volume and thus introduce only a small amount of error into our estimate of $\theta_\lambda^*$. We then set $\theta_\lambda^* = 2/\pi$ in the trial volume if $\delta z^+_{\lambda}$ and $\delta z^-_{\lambda}$ (where $\lambda = e\lambda$) in the two different source regions are equal to within a factor of order unity, and otherwise we set $\theta_\lambda^* = 0$. We choose $\theta_\lambda^* = 2/\pi$ for unaligned fluctuations because this is the average value of $|\sin \phi|$ when $\phi$ is uniformly distributed between 0 and $2\pi$. We compare $\delta z^+_{\lambda}$ and $\delta z^-_{\lambda}$ in the two source regions at scale $\lambda$ because we assume that the fluctuations cascade from scale $\lambda$ to scale $\lambda$ as they propagate from the source regions to the trial volume. This leads to the estimate

$$\theta_\lambda^* = \frac{2e^{-2\mu_1}}{\pi} \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} \frac{\mu_1^{q_1+q_2}}{q_1! q_2!},$$

where $m_1$ is some integer of order unity and

$$\mu_1 = \mu - 1.$$  

We rewrite Equation (60) in the form

$$\theta_\lambda^* = \frac{2e^{-2\mu_1}}{\pi} \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} \frac{\mu_1^{2q_1+q_2}}{q_1! (q_1 + q_2)!} + \sum_{q_2=0}^{\infty} \sum_{q_2=0}^{\infty} \frac{\mu_1^{2q_2+q_1}}{q_2! (q_2 + q_1)!},$$

(62)

which is equivalent to

$$\theta_\lambda^* = \frac{2e^{-2\mu_1}}{\pi} \sum_{n=-m_1}^{m_1} I_n(2\mu_1),$$

where $I_n(x)$ is the $n$th-order modified Bessel function of the first kind. As $\lambda \to 0$,

$$\theta_\lambda^* \propto \left[ \ln \left( \frac{L}{\lambda} \right) \right]^{-1/2},$$

(64)

which decreases more slowly than any positive power of $\lambda$.

2.9. Cross Correlation

Equations (13), (55), and (66) yield the relation

$$\frac{\langle \delta z^+_{\lambda} \delta z^-_{\lambda} \rangle}{\langle \delta z^+_{\lambda} \rangle \langle \delta z^-_{\lambda} \rangle} \sim \left( \frac{\lambda}{e^{\ln 2^{1/2} - 1}} \right) \propto \lambda^{0.12},$$

(65)

which implies that $\delta z^+_{\lambda}$ and $\delta z^-_{\lambda}$ become anti-correlated at sufficiently small scales. However, this anti-correlation grows extremely slowly as $\lambda$ decreases. This very slow growth of anti-correlation results from the near cancellation of two competing effects. First, we argued in Equation (33) that when $\delta z^+_{\lambda} \gg \delta z^-_{\lambda}$ in the trial volume in Figure 1, the likelihood that $\delta z^+_{\lambda} \gg \delta z^-_{\lambda}$ in the source region is decreased because intense, counter-propagating fluctuations rapidly annihilate. This effect acts to make $\delta z^+_{\lambda}$ and $\delta z^-_{\lambda}$ anti-correlated to an increasing degree as $\lambda$ decreases, because moments of the $\delta z^\pm_{\lambda}$ distribution are increasingly dominated by exceptionally intense structures at smaller scales (see, e.g., the discussion following Equation (17)). On the other hand, a large-amplitude, coherent $\delta z^+_{\lambda}$ structure amplifies $\delta z^-_{\lambda}$ to a value $\sim \delta z^+_{\lambda}$, which exceeds $\delta z^-_{\lambda}$. Thus, a sheet-like coherent $\delta z^+_{\lambda}$ structure produces a weaker, sheet-like, $\delta z^-_{\lambda}$ structure at the same location. On its own, this effect would act to make $\delta z^+_{\lambda}$ and $\delta z^-_{\lambda}$ positively correlated, to a degree that would increase at smaller scales, again because the moments of the $\delta z^\pm_{\lambda}$ distribution become increasingly dominated by exceptionally intense fluctuations as $\lambda$ decreases.

3. COMPARISON WITH NUMERICAL SIMULATIONS

In this section, we compare our model with numerical simulations of RMHD turbulence. We focus on results from a new numerical simulation and briefly discuss previously published simulations.

3.1. Numerical Method

Using the code described by Chen et al. (2011), we solve the RMHD equations numerically within a box of dimensions $L_x$, $L_y$, and $L_z$ in the $x$, $y$, and $z$ directions, respectively, where the $z$ axis is aligned with $B_0$. We discretize each axis with 1024 evenly spaced grid points and impose triply periodic boundary conditions on $z^\pm$. The code computes spatial derivatives in the $x$ and $y$ directions using a pseudo-spectral algorithm, but uses a finite-difference technique (upwind differencing) to compute spatial derivatives in the $z$ direction.

Prior to solving Equation (1) numerically, we add to the right-hand side of this equation a forcing term $F^\pm$ and a dissipation term $-\nu |z|^{-1/2} |\nabla \psi|^2 + \nu_1 (\partial/\partial z) \nabla \psi$. We set $n = 8$. We include the $\nu_1$ term to stabilize the upwind-differencing scheme, but we choose the constant $\nu_1$ to be sufficiently small that the $\nu_1$ term accounts for only a small fraction (6.8%) of the total dissipation power. We set $F^+ = -F^-$, so that we force the velocity field but not the magnetic field. We take the Fourier transform of the forcing term $F_k^\pm$ to be nonzero only at wavenumbers for which $1 \leq k_L L_0 / 2\pi \leq 2$ and $|k_L| L_0 / 2\pi = 1$. At each of these wavenumbers, we randomize $F_k^\pm$ at each time step, keeping the rms amplitude of $F_k^\pm$ constant in time and independent of $k$.  

\[ \theta_\lambda^* \]
We set $\mathcal{z}^{\pm} = 0$ at the beginning of the simulation and run the code for two large-eddy turnover times until the energy reaches an approximately steady value. (We define the large-eddy turnover time to be $L_\perp/\delta \xi_{\mathrm{rms}}$.) We then continue the simulation and save ten snapshots of the turbulent fields, with consecutive snapshots separated in time by one large-eddy turnover time. We use these snapshots to evaluate the power spectrum, structure functions, and average alignment angles in this simulation, as described further in Sections 3.2 through 3.4.

To obtain a dimensionless measure of the hyperdissipation coefficient $\nu_\eta$, we first define a characteristic dissipation scale $\eta$ as follows. We define an effective energy cascade timescale $\tau_\lambda$ by setting $\epsilon \sim (\delta \xi_{\lambda, \mathrm{rms}})^2/\tau_\lambda$ within the inertial range, where $\delta \xi_{\lambda, \mathrm{rms}}$ is the rms amplitude of $\delta \xi^\pm$. If the power spectrum scales as $E(k) \propto k^{2 \lambda}$, then $(\delta \xi_{\lambda, \mathrm{rms}})^2 \propto \lambda^{1-\epsilon}$. Since $\epsilon$ is independent of $\lambda$, $\tau_\lambda \propto \lambda^{1-\epsilon}$ in the inertial range. We assume that the fluctuations are not strongly aligned at the outer scale, which we take to be comparable to the perpendicular box size $L_\perp$. Thus, $\tau_\lambda \sim L_\perp/\delta \xi_0$, $\delta \xi_0 \sim (L_\perp)^{1/3}$, and $\tau_\lambda \sim L_\perp^{2/3} \epsilon^{-1/3} (\lambda/L_\perp)^{\epsilon-1}$ in the inertial range, where $\delta \xi_0$ is the rms value of $\delta \xi^\pm$ at $\lambda = L_\perp$. We define $\eta$ to be that scale at which this inertial-range estimate of $\tau_\lambda$ equals the dissipation timescale $\lambda^2/\nu_\eta$:

$$\eta = \left( \frac{n_\lambda}{\epsilon} \right)^{1/3(n+i+1)} L_\perp^m, \quad (66)$$

where

$$m = \frac{3i + 5}{3(n + i + 1)}. \quad (67)$$

Equation (66) is the same as Equation (1) of Beresnyak (2014b) except for the factor of $L_\perp^m$, which is unity when $i$ is taken to be $-5/3$. The maximum resolvable perpendicular wavenumber in our dealiased simulation is $k_{\max} = 2\pi/(3\Delta)$, where $\Delta$ is the grid spacing in the $x$ and $y$ directions. For the values of $L_\perp$, $\nu_\eta$, and $\epsilon$ in our simulation,

$$k_{\max} \eta = \begin{cases} 2.17 & \text{if } i \text{ is set equal to } -3/2 \\ 1.86 & \text{if } i \text{ is set equal to } -5/3. \end{cases} \quad (68)$$

### 3.2. Power Spectra

The average of the $z^+$ and $z^-$ power spectra in our RMHD simulation, denoted $E(k_{\perp})$, is plotted in the left panel of Figure 3. This power spectrum exhibits a scaling similar to the $k_{\perp}^{1.52}$ scaling predicted in Equation (43). This $k_{\perp}^{1.52}$ scaling is also in good agreement with the power spectra found by Perez et al. (2012, 2014b) in numerical simulations of RMHD turbulence. We note, however, that Beresnyak (2012, 2014b) carried out a series of numerical simulations of RMHD turbulence and argued that the rms amplitude of $z^\pm$ fluctuations at the dissipation scale varies with Reynolds number in his simulations in the manner that would be expected if $E(k_{\perp})$ were proportional to $k_{\perp}^{-5/3}$ (cf. Beresnyak 2014a; Perez et al. 2014a, 2014b).

### 3.3. Structure Functions

The scale $\lambda$ in Equation (13) is a distance measured perpendicular to the magnetic field $B$. To compare Equation (13) with our numerical results, we wish to evaluate structure functions in our numerical simulation that are functions of distance perpendicular to $B$. Because the RMHD Equations are invariant with respect to the simultaneous rescalings $z \rightarrow az$ and $v \rightarrow av$, where $a$ is an arbitrary factor, our numerical solution represents an entire family of solutions. Since RMHD is asymptotically exact in the limit that $L_\parallel/L_\perp \rightarrow \infty$ and $\delta B/B_0 \rightarrow 0$, we focus on values of $a$ for which $\delta B \ll B_0$ and $L_\perp \ll L_\parallel$. In this limit, the $xy$ plane is effectively perpendicular to the full magnetic field $B$. We therefore compute structure functions from Elsässer increments obtained by evaluating $z^\pm$ at pairs of points that have the same Cartesian coordinate $z$. 
We consider 32 logarithmically spaced separation lengths \( \Delta_j \), ranging from \( L_\perp/256 \) to \( L_\perp/4 \),
\[
\Delta_j = 2^{-8+6j/3}L_\perp,
\]
where \( j = 0, 1, 2, \ldots, 31 \). For each value of \( \Delta_j \) in each of the ten snapshots, we pick \( 5 \times 10^5 \) pairs of points \( r_{ij}^k = (x_i \pm 0.5\Delta_j\cos \phi_i, y_i \pm 0.5\Delta_j\sin \phi_i, z_i) \), whose centroids \( (x_i, y_i, z_i) \) are chosen randomly from a uniform distribution covering the entire numerical domain, where the angle \( \phi_i \) is a uniformly distributed random number of radians between \( 0 \) and \( 2\pi \). We then evaluate the Elsässer increments \( (\delta z^{+}_j)^p \) at \( \lambda = \Delta_j \) by averaging \( |\Delta z^{+}_j|^p \) over the \( 5 \times 10^5 \) values of \( i \) in each of the 10 snapshots. We plot 10 such structure functions in the middle panel of Figure 3. More precisely, we plot the dimensionless quantities \( S_{n_j} = S_n(\Delta_j) \) with \( 1 \leq n \leq 10 \), where
\[
S_n(\lambda) = \left( \frac{\langle (\delta z^{+}_j)^p \rangle^2 + \langle (\delta z^{-}_j)^p \rangle^2}{0.33\langle \delta z_{\text{rms}}^+ \rangle^p} \right)^{1/2},
\]
and \( \delta z_{\text{rms}} \) is the rms amplitude of \( z^{\pm} \). We include the factor of 0.33 in Equation (70) to separate the curves in the middle panel of Figure 3.

To identify the scale range in our simulation that best approximates an inertial range, we search for the \( \lambda \) interval within which the structure functions most closely resemble power laws. Specifically, we consider intervals of the form \( \Delta l \leq \lambda \leq \Delta l+1 \), where \( 1 \leq l \leq 29 \) and \( 1 \leq j \leq 30 - l \). For each such interval, we evaluate the quantity
\[
V_{l,j} = \left[ \frac{1}{10} \sum_{n=1}^{10} \left( \Gamma_{n,j+l} - \Gamma_{n,j} \right)^2 \right]^{1/2},
\]
where
\[
\Gamma_{n,j} = \frac{\ln(S_{n,j+1}/S_{n,j-1})}{\ln(\Delta_{j+1}/\Delta_{j-1})}.
\]
In words, \( \Gamma_{n,j} \) is the power-law index of \( S_n \) at \( \lambda = \Delta_j \), and \( V_{l,j} \) is the rms variation of the power-law indices of the first ten integer-order structure functions across the \( \lambda \) interval (\( \Delta_j, \Delta_{j+1} \)). For each \( l \), we define \( j_l \) to be the value of \( j \) that minimizes \( V_{l,j_l} \), and we define \( V_{\text{min}}(l) = V_{l,j_l} \). We then consider the \( \lambda \) intervals (\( \Delta_{j_l}, \Delta_{j_l+1} \)) to be candidate inertial ranges. We find that \( V_{\text{min}}(l) \) decreases steadily as \( l \) decreases from 29 to 7, and that \( V_{\text{min}} \) starts to fluctuate as \( l \) decreases further. We take the optimal value of \( l \) to be 7 for three reasons. First, \( l = 7 \) corresponds to a sufficient range of scales (a factor of \( \approx 2.5 \) in \( \lambda \)) that a power-law scaling can be clearly seen in the plots of \( S_n \). Second, \( V_{\text{min}}(7) = 0.052 \), so that the quality of the power-law fits is high. Third, \( V_{\text{min}}(l) \) exceeds \( V_{\text{min}}(7) \) for any larger value of \( l \). The value of \( j_l \) is 19, and thus we take the inertial range in our simulation to be the interval \( \Delta_{19} \leq \lambda \leq \Delta_{26} \), or equivalently
\[
0.050L_\perp < \lambda < 0.13L_\perp.
\]
The range of scales in Equation (73) is indicated by the vertical dashed lines in the middle panel of Figure 3.

Within the scale range defined in Equation (73), we fit each \( S_n \) from \( n = 1 \) to \( n = 10 \) to a power law of the form \( \lambda^\nu \). We plot the resulting values of \( \zeta_n \) in the right panel of Figure 3. We estimate error bars on each value of \( \zeta_n \) by considering each of the 10 snapshots separately, computing the structure functions in each snapshot, fitting these structure functions to power laws of the form \( \lambda^\nu \) over the scale range in Equation (73), and computing the standard deviation \( \sigma_n \) of these ten values of \( \zeta_n \). The error bars in the right panel of Figure 3 then connect the points \( (\nu, \zeta_n) \) to the points \( (\nu, \zeta_n \pm \sigma_n/\sqrt{10}) \). For comparison, we also plot Equation (14) with \( \beta = 0.691 \) and the result from Grauer et al. (1994) (cf. Politano & Pouquet 1995) for the Elsässer-structure-function scaling exponents in incompressible MHD.

\[
\zeta_n^{(\text{Grauer})} = 1 + n - \left( \frac{1}{2} \right)^{n/4}.
\]
As can be seen in this figure, our numerical results agree well with our analytic model, but not with Equation (74). Values of \( \zeta_n \) similar to (but slightly larger than) ours were obtained by Rodriguez Imazio et al. (2013) for velocity and magnetic-field structure functions in a \( 512^2 \times 32 \) simulation of compressible RMHD turbulence. Significantly larger values of \( \zeta_n \) were obtained by Müller & Biskamp (2000) and Müller et al. (2003) in simulations of incompressible MHD turbulence (without the RMHD approximation).

3.4. Average Alignment Angles

We compute \( \theta^{\pm}_\alpha \), \( \theta^{(\text{vb})}_\alpha \), and \( \theta^{(\text{iv})}_\alpha \) in our numerical simulation by averaging the right-hand sides of Equations (51), (52), and (59), respectively, over the same set of Elsässer increments used to evaluate the structure functions in Section 3.3. The resulting angles are plotted in Figure 4. The values of \( \theta^{(\text{vb})}_\alpha \) and \( \theta^{(\text{iv})}_\alpha \) in our simulation have scalings that are in reasonable agreement with Equations (58) and (63). The scaling of \( \theta^{(\text{vb})}_\alpha \) in our simulation is close to the \( \lambda^{0.05} \) scaling in Equation (57) at \( \lambda < 0.05L_\perp \), but is steeper at larger \( \lambda \). Although scales \( \lambda \leq 0.05L_\perp \) lie outside the range of scales in Equation (73), \( \theta^{(\text{vb})}_\alpha \) is in some ways like the ratio of two structure functions, as is \( \theta^{(\text{iv})}_\alpha \) (Mason et al. 2008; Perez et al. 2012). In hydrodynamic turbulence, the power-law scalings of structure-function ratios that are present in the inertial range are observed to extend to smaller \( \lambda \) values than the power-law scalings of individual structure functions, a phenomenon referred to as extended self similarity (Benzi et al. 1993). This analogy suggests that the scaling of \( \theta^{(\text{vb})}_\alpha \) at \( \lambda < 0.05L_\perp \) could reflect inertial-range physics. A similar argument was made by Mason et al. (2008) to explain the \( \theta^{(\text{vb})}_\alpha \) scalings in their simulations of incompressible MHD turbulence. On the other hand, the steeper scaling of \( \theta^{(\text{iv})}_\alpha \) at \( \lambda > 0.05L_\perp \) is not predicted by our model. Whether the \( \sim \lambda^{0.10} \) scaling seen at \( \lambda < 0.05L_\perp \) represents an asymptotic inertial-range scaling that would persist in numerical simulations with higher resolution and broader inertial ranges is an open question.

Perez et al. (2012) carried out a series of RMHD simulations on numerical grids as large as \( 2048^3 \) and found that \( \theta^{(\text{vb})}_\alpha \propto \lambda^{0.22} \), in close agreement with our prediction that
which are influenced by couplings with compressive modes that are not included in RMHD (Viñas and Goldstein 1991; Ghosh et al. 1993; Del Zanna et al. 2001; Nariyuki et al. 2007; Chandran 2008b). The extent to which these properties of solar-wind turbulence affect the observables that we discuss is not yet clear.

4.1. Structure Functions in the Solar Wind

In Figure 5 we plot the scaling exponents of the inertial-range structure functions of the magnetic field obtained by several authors in analyses of measurements from the Helios, Wind, STEREO, and Ulysses spacecraft. All of these measurements were taken during years of low solar activity (i.e., near solar minimum). The legend in each panel indicates whether the results apply to slow wind, fast wind, or both. The results in the lower-left panel correspond to intervals of time in which $80^\circ \leq \theta_{VB} \leq 90^\circ$, where $\theta_{VB}$ is the angle between the background magnetic field $B$ and the quasi-radial solar-wind outflow velocity $V$. According to Taylor’s hypothesis (Taylor 1938), the structure functions computed from such time intervals correspond to structure functions $\langle (\delta B_i r^\lambda) \rangle$ in which $\lambda$ is measured nearly perpendicular to $B$. In contrast, during the time intervals used to obtain the results in the lower-middle panel, $0 \leq \theta_{VB} \leq 10^\circ$. The structure functions computed from these time intervals correspond to structure functions $\langle (\delta B_i r^\lambda) \rangle$ in which $\lambda$ is measured nearly parallel to $B$.

One difference between our model and the data points in Figure 5 is that we compute scaling exponents for Elsässer structure functions $\langle (\delta \xi_x \delta B_z + \delta \xi_z \delta B_x) \rangle$, while the data points shown correspond to structure functions of the magnetic field. As $\lambda / L_\perp \to 0$, we expect the two types of exponents to approach each other for the following reasons. The regions that make the dominant contributions to both types of structure functions correspond to the small fraction of the volume in which $\delta \xi_x$ or $\delta \xi_z$ is exceptionally large. In most of these regions, one Elsässer fluctuation, say $\delta \xi_x$, dominates over the other, and $\Delta B_\lambda \simeq (1/2) \Delta \xi_x$. The top-left panel of Figure 5 shows that the measured values of $\zeta_{\xi_x}$ in the slow solar wind are close to the values in our model. In contrast, the lower-right panel shows that the measured values of $\zeta_{\xi_x}$ in high-helioographic-latitude fast solar wind are significantly higher than the values in our model. A possible explanation for this difference is Dasso et al.’s (2005) observation that slow solar wind and fast solar wind exhibit different types of anisotropy at comparatively large scales within the inertial range, with fast-wind fluctuations varying most rapidly parallel to $B$ and slow-wind fluctuations varying most rapidly perpendicular to $B$. Another factor that could help explain this difference is that $\theta_{VB}$ tends to be smaller in high-latitude fast wind than in low-latitude slow wind.

The dependence of $\zeta_{\xi_x}$ on $\theta_{VB}$ is illustrated in the lower-left and lower-middle panels of Figure 5. When $\theta_{VB} < 10^\circ$, $\zeta_{\xi_x}$ is larger than in either Equations (14) or (74). In contrast, when $\theta_{VB} > 80^\circ$, the $\zeta_{\xi_x}$ measurements are consistent with our model to within the observational uncertainties. Thus, it is possible that our model applies to fast-wind structure functions when $\theta_{VB}$ is sufficiently large. On the other hand, the error bars are large enough that fast-wind structure functions could be closer to Equation (74) than to our model even when $\theta_{VB} > 80^\circ$.

The scaling exponents in the top-middle and top-right panels of Figure 5 are close to the values in our model, particularly if
the $\zeta_q$ values for $B_r$ (the radial component of $B$) are excluded from the comparison. A reason to exclude $B_r$ is that, on average, $B_r$ contains a larger contribution than $B_{\phi}$ or $B_z$ from the component of the fluctuating magnetic field that is parallel to the background magnetic field, whereas $B_{\phi}$ and $B_z$ are to a larger degree dominated by the transverse, RMHD-like components of the fluctuating field. The top-right panel resembles the pure-slow-wind results (top-left panel) more than it resembles the all-$\theta_{B\parallel}$ fast-wind results (lower-right panel), consistent with the fact that the data set used by Salem et al. (2009) contained primarily slow-wind measurements. For example, the fraction of their data in which $V < 500 \text{ km s}^{-1}$ was 70.5% and the fraction in which $V < 600 \text{ km s}^{-1}$ was 83.3% (C. Salem 2015, private communication). The scaling exponents in the top-middle panel are also much closer to the pure-slow-wind results than they are to the all-$\theta_{B\parallel}$ fast-wind results. For this panel, the fraction of the data that comes from slow-wind measurements is not available to us.

4.2. Average Alignment Angles in the Solar Wind

Podesta et al. (2009) measured the angle $\theta_{\lambda \parallel}^{(vb)}$ in four solar-wind streams with average outflow velocities ranging from 398 to 457 km s$^{-1}$. These authors obtained power-law fits of the form $\theta_{\lambda \parallel}^{(vb)} \propto \lambda^m$ over a restricted range of $\lambda$ corresponding to timescales $\tau$ in the spacecraft frame ranging from $2 \times 10^3$ to $2 \times 10^4$ s, with $m$ ranging from 0.27 to 0.36. These timescales correspond to length scales comparable to the outer scale $L_\perp$, and thus are not directly comparable to the results of our RMHD model, which applies when $\lambda \ll L_\perp$. Podesta et al. (2009) found that as $\tau$ decreases below $10^3$ s, $\theta_{\lambda \parallel}^{(vb)}$ reaches a minimum at a value of $\tau$ between $\sim 10^2$ s and a few times $10^2$ s, and that $\theta_{\lambda \parallel}^{(vb)}$ then increases as $\tau$ decreases further. They argued that the behavior of $\theta_{\lambda \parallel}^{(vb)}$ at $\tau < 10^3$ s could be strongly affected by uncertainty in the velocity measurements and concluded that they could not rule out a power-law scaling of $\theta_{\lambda \parallel}^{(vb)}$ in the inertial range. The comparison between our model and this study is thus inconclusive.

5. DISCUSSION

Intermittency has qualitatively different effects upon the energy cascades rates in hydrodynamic turbulence and RMHD turbulence. In hydrodynamic turbulence, an intense vorticity structure interacts with itself. The concentration of fluctuation energy into a decreasing fraction of the volume as $\lambda$ decreases thus reduces the energy cascade timescale in the energetically dominant regions, to an increasing degree as $\lambda \to 0$. Intermittency in hydrodynamic turbulence thus acts to steepen the inertial-range power spectrum. For example, $E(k) \propto k^{-1.71}$
in the She–Leveque model, whereas $E(k) \propto k^{-5/3}$ in Kolmogorov’s (1941) theory. In RMHD, since only counterpropagating fluctuations interact, the concentration of $\delta \xi_+^x$ energy into a tiny fraction of the volume makes it difficult for a $\delta \xi_-^x$ fluctuation to “find” and interact with the dominant $\delta \xi_+^x$ fluctuations. This in turn increases the energy cascade timescale, to an increasing degree as $\lambda \rightarrow 0$, causing the inertial-range power spectrum to flatten relative to models of RMHD turbulence that neglect intermittency, a point first made by Maron & Goldreich (2001).

Like She & Leveque (1994) and She & Waymire (1995), we assume that the fluctuation amplitudes have a log-Poisson PDF and make an assumption about the dimension of the most intense structures. On the other hand, the PDF of the fluctuation amplitudes (and the PDF of the dissipation rate) in the She–Leveque model has three parameters, whereas the PDF in our model has just two: $\beta$ and $\mu$ (Equations (6) and (8)). (We do not count the overall normalization of the fluctuation amplitudes—e.g., $\xi_0$—as a parameter of the PDF in either model, because this normalization does not affect the inertial-range scalings.) Our PDF has one fewer parameter because of our argument that highly imbalanced collisions reduce a fluctuation’s length scale without affecting its amplitude, which implies that the amplitude of the most intense ($q = 0$) fluctuations is independent of $\lambda$. In order to determine the extra free parameter in their model, She & Leveque (1994) introduced an extra assumption concerning the scaling of the energy dissipation rate of the most intense fluctuations.

In this paper, we draw heavily upon Boldyrev’s (2005, 2006) argument that alignment within the field-perpendicular plane plays an important role in the energy cascade. However, our treatment of SDDA differs from Boldyrev’s. In his theory, there is a single characteristic alignment angle at each scale. In our model, at each scale $\theta_\lambda$ varies systematically with the fluctuation amplitudes (Equation (29)). Boldyrev (2006) argued that a larger fluctuation amplitude reduces alignment. In our model, given a scale $\lambda$, larger fluctuation amplitudes are associated with enhanced alignment, a phenomenon observed by Beresnyak & Lazarian (2006) in numerical simulations of incompressible MHD turbulence. Also, in our model, there are two distinct mechanisms for aligning $\Delta v_\perp$ and $\Delta b_\perp$ fluctuations in regions where the fluctuation amplitudes are large. First, intense $\delta \xi_+^x$ fluctuations rotate weaker $\delta \xi_-^x$ fluctuations into alignment, as illustrated in Figure 2, which reduces $\theta_\lambda^{(vb)}$ because $\Delta v_\perp \times \Delta b_\perp = \Delta \xi_+^x \times \Delta \xi_-^x/2$. Second, when the fluctuations are intermittent, the turbulence becomes locally imbalanced at small scales (cf. Perez & Boldyrev 2009), with either $\delta \xi_+^x \gg \delta \xi_-^x$ or $\delta \xi_-^x \gg \delta \xi_+^x$ in the regions containing most of the fluctuation energy. In such locally imbalanced regions, the velocity and magnetic-field fluctuations are nearly parallel or anti-parallel, regardless of whether $z^+$ and $z^-$ are aligned (Grappin et al. 2013; Wicks et al. 2013a, 2013b). This second effect is why $\theta_\lambda^{(vb)}$ decreases more quickly than $\theta_\lambda$ as $\lambda/L$ decreases to small values.

Grauer et al. (1994), Politano & Pouquet (1995), and Müller & Biskamp (2000) developed models of intermittent, incompressible, MHD turbulence based on the approach of She & Leveque (1994) and the assumption that $\epsilon_\chi^+ \sim (\delta \xi_+^x)^2/\lambda \sigma_\lambda$. Müller & Biskamp (2000) also developed a She–Leveque-like model of incompressible MHD turbulence under the assumption that $\epsilon_\chi^\pm \sim (\delta \xi_\chi^\pm)^3/\lambda$. A major difference between our approach and these previous studies is that we set $\epsilon_\chi^\pm \sim (\delta \xi_\chi^\pm)^2/\lambda \sigma_\lambda$, accounting for alignment and treating $\delta \xi_+^x$ and $\delta \xi_-^x$ as separate but correlated random variables.

6. CONCLUSION

We have constructed an analytic model of intermittent, threedimensional, strong RMHD turbulence that incorporates a new phenomenology of scale-dependent dynamic alignment. We restrict our analysis to the case of “globally balanced” turbulence, in which the cross helicity is zero. There are three main assumptions in our model. First, we take the fluctuation amplitudes to have a scale-dependent, log-Poisson PDF. In Section 2.1, we describe how this assumption can be motivated by treating a fluctuation’s evolution as a random, quantized, multiplicative process, as in the work of She & Waymire (1995). Second, we assume that the most intense $\delta \xi_+^x$ fluctuations are two-dimensional current/vorticity sheets with a volume filling factor $\propto \lambda$. Third, we assume that the turbulence obeys a refined similarity hypothesis (Equation (25)) that includes the effect of dynamic alignment.

We argue that the largest contribution to the average $z^+$ cascade power at any inertial-range scale $\lambda$ comes from regions in which $\xi_\chi^+ \gg \xi_\chi^- \gg \xi_\chi^0$, where $\xi_\chi^0$ is the typical (median) fluctuation amplitude at scale $\lambda$. We then develop an approximate theory describing how a large-amplitude, coherent $\xi_\chi^+$ structure interacts with a much weaker $z^-$ fluctuation. We show that during an interaction, the $z^-$ fluctuation cascades rapidly to smaller scales without a reduction in amplitude and rotates into alignment with the coherent $\xi_\chi^+$ structure. By accounting for these effects, we compute the average $z^+$ cascade power using the assumed log-Poisson PDF of $\xi_\chi^\pm$.

This log-Poisson PDF has two free parameters, $\mu$ and $\beta$ (see Equations (6) and (10)). Our assumption that the most intense fluctuations form two-dimensional structures with a filling factor $\propto \lambda$ determines $\mu$ up to an additive constant $A$, which affects neither the power-law scalings in our model nor the fact that the typical alignment angle $\theta_\lambda$ (Equation (59)) decreases logarithmically as $\lambda \rightarrow 0$. The condition that the average cascade power is independent of $\lambda$ then determines $\beta$. Once we have determined $\mu$ and $\beta$, we compute the scalings of the $z^+$ power spectrum, higher-order structure functions, and three different average alignment angles. Given the assumptions stated above, the power-law scalings in our model and the logarithmic scaling of $\theta_\lambda$ do not depend upon any free parameters.

To test our theoretical results, we carry out a direct numerical simulation of RMHD turbulence on a numerical grid consisting of 1024$^3$ grid points. We compute the first 10 integer-order structure functions, the $z^+$ power spectrum, and 3 different average alignment angles. The scalings of these quantities are similar to the scalings in our phenomenological theory. Our model is also consistent with results from several previously published RMHD simulations.

Although the assumptions underlying our RMHD model do not apply to the full range of turbulent fluctuations contained in the solar wind, we compare our model with measured scaling exponents of solar-wind magnetic-field structure functions. We find that the Elsässer-field scaling exponents in our model are close to the magnetic-field scaling exponents observed in the slow solar wind, but significantly smaller than the magnetic-
field scaling exponents measured in high-latitude fast-solar-wind streams. The Elsässer-field scaling exponents in our model are also consistent (to within comparatively large error bars) with the magnetic-field scaling exponents observed in low-latitude fast-solar-wind streams when \( \theta_{BW} > 80^\circ \), where \( \theta_{BW} \) is the angle between the solar-wind outflow velocity and the magnetic field.

There are a number of ways in which our model could be improved. As presented, our model can approximate a broad distribution of outer-scale fluctuation amplitudes through the parameter \( A \), but the outer-scale distribution is then forced to be log-Poisson. A more realistic approach might be to allow the parameter distribution of outer-scale fluctuations to be random with a distribution that could be adjusted so as to model different forcing mechanisms in forced turbulence or different initial conditions in decaying turbulence. Our finding that unusually intense fluctuations cascade more slowly than median-amplitude fluctuations (Equation 48) suggests that exceptionally intense fluctuations have larger dissipation scales than median-amplitude fluctuations, at least for some dissipation mechanisms such as viscosity and resistivity. A useful direction for future research would be to develop this idea further by exploring the consequences of intermittency for the transition between the inertial and dissipation ranges within the framework of our analytic model. It would also be useful to extend our model to allow for nonzero cross helicity in order to investigate how intermittency affects strong “imbalanced” RMHD turbulence. Finally, inhomogeneity of the background plasma can fundamentally alter RMHD turbulence by causing the non-WKB refection of Alfvén waves (Heinemann & Gilbert 1980). This linear coupling between counter-propagating Alfvén waves occurs in the solar atmosphere and solar wind (Dmitruk et al. 2002; Cranmer & van Ballegooijen 2005; Verdini & Velli 2007; Chandran & Hollweg 2009) and can modify the power spectrum and energy-cascade timescales in solar-wind turbulence (Velli et al. 1989; Verdini et al. 2012; Perez & Chandran 2013). Extending our model to account for background inhomogeneity and non-WKB wave reflection would be helpful for understanding intermittent turbulence in the inner heliosphere.

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### APPENDIX

**HIGHLY IMBALANCED COLLISIONS**

In this appendix, we consider “highly imbalanced collisions” between a large-amplitude, sheet-like, coherent, \( \delta z^+ \) structure and smaller-amplitude \( z^- \) fluctuations. We begin by considering the effects of such collisions on the weaker, \( z^- \) fluctuations.

For this part of our analysis, we make the simplifying approximation that the \( z^+ \) field has the form of a linear shear within the volume of the \( \delta z^+ \) structure.\(^{11}\) We further assume that the evolution of \( z^- \) within the volume of the coherent \( \delta z^+ \) structure does not depend strongly on the properties of the \( z^+ \) field outside of the structure. This assumption allows us to choose a convenient form for \( z^+ \) throughout all of space,

\[
    z^+ = S(z, t) x \hat{y},
\]

where \( S(z, t) \) is the shearing rate and \((x, y, z)\) are Cartesian coordinates chosen so that \( B_0 \) is in the \( z \) direction. The RMHD equations can be rewritten in the form (Shokohihiin et al. 2007)

\[
    \frac{\partial}{\partial t} \nabla^2 \psi^+ \pm v_\Lambda \frac{\partial}{\partial z} \nabla^2 \psi^+ = -\frac{1}{2} \left( \{ \psi^+, \nabla^2 \psi^- \} + \{ \psi^-, \nabla^2 \psi^+ \} \right),
\]

(76)

where \( \nabla^2 = \hat{x} \partial/\partial x + \hat{y} \partial/\partial y, \{ g, h \} = \hat{z} \cdot (\nabla g \times \nabla h) \) for any functions \( g \) and \( h \), and \( \psi^+ \) and \( \psi^- \) are the Elsässer stream functions, which satisfy \( z^+ = \hat{z} \times \nabla \psi^\pm \). Equation (75) then implies that \( \psi^+ = S x^2/2 \) to within an arbitrary additive function of the \( z \) coordinate and time. Upon substituting this value of \( \psi^+ \) into Equation (76), we obtain

\[
    \left( \frac{\partial}{\partial t} + S_{x^2} \frac{\partial}{\partial y} + v_\Lambda \frac{\partial}{\partial z} \right) \nabla^2 \psi^- = -S \frac{\partial^2 \psi^-}{\partial z^2 \partial y}.
\]

(77)

Although \( \psi^+ = S x^2/2 \) is not localized, we take \( \psi^- \) to vanish sufficiently rapidly as \( x^2 + y^2 \rightarrow \infty \) that

\[
    f = \frac{1}{(2\pi)^3} \int dx dy \psi^- e^{-ik_x x - ik_y y}
\]

(78)

is defined and the Fourier transforms in \( x \) and \( y \) of each term in Equation (77) are defined. The Fourier transform of Equation (77) yields

\[
    \left( \frac{\partial}{\partial t} - S k_y \frac{\partial}{\partial k_x} + v_\Lambda \frac{\partial}{\partial k_z} \right) (k_z^2 f) = -S k_x k_y f.
\]

(79)

To solve Equation (79), we define a family of trajectories in \( k_z - z \) space through the equations \( dk_z/dt = -S k_y, dz/dt = v_\Lambda \). The total time derivative of any function \( G(k_x, k_y, z(t), t) \) along one of these trajectories is then \( (d/dt)G = (d/dt) - S k_y \partial/\partial k_x + v_\Lambda \partial/\partial k_z \). Since \( (d/dt)k_z^2 = -2S k_x k_y \), we can rewrite Equation (79) as \( (d/dt)(k_z^2 f) = 0 \). The solution to Equation (79) is thus

\[
    f(k_x, k_y, z, t) = \frac{k_z^2}{k_z^2} f_0(k_{x0}, k_y, z_0),
\]

(80)

where

\[
    z_0 = z - v_\Lambda t, \quad k_{x0} = k_x + k_y H,
\]

\[
    H = \int_0^t S(z_0 + v_\Lambda t', t') dt',
\]

(81)

\(^{11}\) We use the term “linear” to refer to the functional form of \( z^+ \) in Equation (75), and not to imply that the amplitude of the \( \delta z^+ \) structure is small or that the turbulence is weak.
We now use these results to obtain an approximate description of the evolution of $\zeta$ fluctuations as they propagate a distance $l_{\phi}$ from the source region to the trial volume depicted in Figure 1 in the limit that $\zeta_{\phi,0} \gg \zeta_{\phi,u}$. Because the $\zeta$ fluctuations in the source region have not yet interacted with the coherent $\zeta_{\phi,u}$ structure depicted in Figure 1, they do not yet “know about” the orientation of this structure. The typical case is thus that $k_{\perp} \sim k_{\parallel}$, so that the $\zeta$ fluctuations are not initially aligned with the $\zeta_{\phi,u}$ structure. We set $S = \zeta_{\phi,u}/\chi$, which is the shearing rate associated with the $\zeta_{\phi,u}$ structure, and we set $t = l_{\phi}/v_{\phi}$, which is the time it takes the $\zeta$ fluctuations to propagate from the source region to the trial volume. This leads to $H = \zeta_{\phi,u}/S \gg 1$, so that Equation (84) applies. The condition $k_{\parallel} v_{\phi} = \text{constant}$ and Equation (84) then lead to Equations (29) and (31). Equation (84) also implies that the perpendicular length scale of the $\zeta$ fluctuations decreases by a factor of $\zeta_{\phi,u}/\zeta_{\phi,0}$ as the $\zeta$ fluctuations propagate from the source region to the trial volume.

Finally, we consider how highly imbalanced collisions affect the larger-amplitude, coherent $\zeta_{\phi,u}$ structure. As can be seen in the bottom half of Figure 2, the $\zeta_{\phi,u}$ fluctuations within a sheet-like coherent $\zeta_{\phi,u}$ structure have been sheared in such a way that they resemble a smaller-amplitude, counter-propagating, current/vorticity sheet that is nearly aligned with the coherent $\zeta_{\phi,u}$ structure. The nature of the effect of this $\zeta_{\phi,u}$ current/vorticity sheet on the original $\zeta_{\phi,u}$ structure is also effectively linear shearing. Because of this, we can repeat the arguments leading from Equations (75) to (83), interchanging the roles of $\zeta^+$ and $\zeta^-$. We thus conclude that highly imbalanced collisions between a sheet-like coherent $\zeta_{\phi,u}$ structure and much weaker $\zeta$ fluctuations cause the scale but not the amplitude of the $\zeta_{\phi,u}$ structure, as argued in Section 2.1. We also note that the scale of the $\zeta_{\phi,u}$ structure can either increase or decrease, depending on the directions of the vector fluctuations in the “colliding” fluctuations (i.e., depending on the relative signs of the two terms on the right-hand side of Equation (83), when we have interchanged the roles of $\zeta^+$ and $\zeta^-$ in Equations (75) through (83) in order to describe the evolution of $\zeta_{\phi,u}^+$).
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