A MULTIFRACTAL FORMALISM FOR HEWITT-STROMBERG MEASURES

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ABSTRACT. In the present work, we give a new multifractal formalism for which the classical multifractal formalism does not hold. We precisely introduce and study a multifractal formalism based on the Hewitt-Stromberg measures and that this formalism is completely parallel to Olsen’s multifractal formalism which based on the Hausdorff and packing measures.

1. INTRODUCTION

In certain circumstances, a measure $\mu$ gives rise to sets of points where $\mu$ has local density of exponent $\alpha$. The dimensions of these sets indicate the distribution of the singularities of the measure. To be more precise, for a finite measure $\mu$ on $\mathbb{R}^n$, the pointwise dimension at $x$ is defined as follows

$$\alpha_\mu(x) = \lim_{{r \to 0}} \frac{\log \mu(B(x, r))}{\log r},$$

wherever this limit exists. For $\alpha \geq 0$, define

$$E(\alpha) = \{ x \in \text{supp } \mu \mid \alpha_\mu(x) = \alpha \}$$

where $B(x, r)$ is the closed ball with center $x$ and radius $r$. The set $E(\alpha)$ may be thought of as the set where the local dimension of $\mu$ equals $\alpha$ or as a multifractal component of $\text{supp } \mu$. The main problem in multifractal analysis is to estimate the size of $E(\alpha)$. This is done by calculating the functions

$$f_\mu(\alpha) = \dim_H(E(\alpha)) \quad \text{and} \quad F_\mu(\alpha) = \dim_P(E(\alpha)) \quad \text{for } \alpha \geq 0.$$

These functions are generally known as the multifractal spectrum of $\mu$ or the singularity spectrum of the measure $\mu$. One of the main problems in multifractal analysis is to understand the multifractal spectrum and the Rényi dimensions and their relationship with each other. During the past 25 years there has been an enormous interest in computing the multifractal spectra of measures in the mathematical literature. Particularly, the multifractal spectra of various classes of measures in Euclidean space $\mathbb{R}^n$ exhibiting some degree of self-similarity have been computed rigorously. The reader can be referred to the paper [42], the textbooks [25, 54] and the references therein. Some heuristic arguments using techniques of Statistical Mechanics (see [32]) show that the singularity spectrum should be finite on a compact interval, noted by $\text{Dom}(\mu)$, and is expected to be the Legendre transform conjugate of the $L^q$-spectrum, given by

$$\tau_\mu(q) = \lim_{{r \to 0}} \frac{\log \left( \sup_{x} \left\{ \sum_{i} \mu(B(x_i, r))^q \right\} \right)}{-\log r},$$

where the supremum is taken over all centered packing $\{B(x_i, r)\}_i$ of $\text{supp } \mu$. That is, for all $\alpha \in \text{Dom}(\mu),

$$f_\mu(\alpha) = \inf_{q \in \mathbb{R}} \left\{ \alpha q + \tau_\mu(q) \right\} = \tau^*_\mu(\alpha). \quad (1.1)$$

The multifractal formalism (1.1) has been proved rigorously for random and non-random self-similar measures [1, 16, 42, 43, 51], for self-conformal measures [26, 27, 28, 29, 37, 37], for self-affine measures [6, 7, 8, 9, 23, 24, 36, 45] and for Moran measures [61, 62, 63, 64]. We note that the proofs of the multifractal formalism (1.1) in the above-mentioned references [1, 10, 12, 13, 14, 36, 37, 42, 43, 45, 52] are all based on the same key idea. The upper bound for $f_\mu(\alpha)$ is obtained by a standard covering argument (involving Besicovitch’s Covering Theorem

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or Vitali’s Covering Theorem). However, its lower bound is usually much harder to prove and is related to the existence of an auxiliary measure (Gibbs measures) which is supported by the set to be analysed. In an attempt to develop a general theoretical framework for studying the multifractal structure of arbitrary measures, Olsen [42], Pesin [53] and Peyrière [55] suggested various ways of defining an auxiliary measure in a very general setting. This formalism was motivated by Olsen’s wish to provide a general mathematical setting for the ideas presented by the physicists Halsey et al. in their seminal paper [32]. In fact, they have been interested in the concept of multifractal spectrum, that is an interesting geometric characteristic for discrete and continuous models of statistical physics. An important thing which should be noted is that there are many measures for which the multifractal formalism does not hold (some examples could be found in [11, 13, 42, 64]). An important question, in which several theorists are interested, is: can we find a necessary and sufficient condition for the multifractal formalism to hold? Another one, asked by Olsen in [42] is: which functions give more information about a multifractal measure, the dimension functions $b_{\mu}$ and $B_{\mu}$ or the spectra functions $f_{\mu}$ and $F_{\mu}$? Olsen gives examples of measures where the dimension functions can be used to split measures which have the same spectrum. In doing this, he implicitly suggests that a return to the physicists’ original idea of calculating the moments of multifractal measures may be the best way to characterize them. It always needs some extra conditions to obtain a minoration for the dimensions of the level sets $E(\alpha)$. Olsen proved the following statement.

**Theorem 1.** [42] Let $\mu$ be a Borel probability measure on $\mathbb{R}^n$. Define $\alpha = \sup_{0 < q} \frac{b_{\mu}(q)}{q}$ and $\bar{\alpha} = \inf_{0 > q} \frac{b_{\mu}(q)}{q}$. Then,

$$\dim_H(E(\alpha)) \leq b_{\mu}^{*}(\alpha) \quad \text{and} \quad \dim_P(E(\alpha)) \leq B_{\mu}^{*}(\alpha) \quad \text{for all} \quad \alpha \in (\underline{\alpha}, \bar{\alpha}).$$

In general, such a minoration is related to the existence of an auxiliary measure which is supported by the set to be analyzed. Olsen also gives a result in such a way and supposes the existence of a Gibbs’ measure (see [42]) at a state $q$ for the measure $\mu$, i.e., the existence of a measure $\nu_q$ on $\text{supp} \mu$ and constants $C > 0$, $\delta > 0$ such that for every $x \in \text{supp} \mu$ and every $0 < r < \delta$,

$$\frac{1}{C} \mu(B(x, r))^q (2r)^{B_{\mu}(q)} \leq \nu_q(B(x, r)) \leq C \mu(B(x, r))^q (2r)^{B_{\mu}(q)}$$

to conclude that

$$\dim_H(E(\alpha)) = \dim_P(E(\alpha)) = b_{\mu}^{*}(\alpha) = B_{\mu}^{*}(\alpha), \quad \text{where} \quad \alpha = -B_{\mu}(q).$$

In general, one needs some degree of similarity to prove the existence of Gibbs measures. For example, in dynamic contexts, the existence of such measures are often natural. For this reason, Ben Nasr et al. in [10, 11, 12, 13] improved Olsen’s result and proposed a new sufficient condition that gives the lower bound. For more details and backgrounds on multifractal analysis as well as their applications the readers may be referred also to the following essential references [4, 5, 15, 17, 18, 19, 39, 41, 44, 47, 56, 57, 58, 59, 60, 61, 62, 63, 64].

In [11, 13, 42, 56, 64], the authors provided some examples for which the classical multifractal formalism does not hold. Indeed, for such examples, the functions $b_{\mu}$ and $B_{\mu}$ differ and $\dim_H(E(\alpha))$ and $\dim_P(E(\alpha))$ are given respectively by the Legendre transform of $b_{\mu}$ and $B_{\mu}$. Motivated by the above papers, the authors in [3] introduced new metric outer measures (multifractal analogues of the Hewitt-Stromberg measure) $\mathcal{H}_{\mu}^{q,t}$ and $\mathcal{P}_{\mu}^{q,t}$ lying between the multifractal Hausdorff measure $\mathcal{H}_{\mu}^{q,t}$ and the multifractal packing measure $\mathcal{P}_{\mu}^{q,t}$, and they used the multifractal density theorems to prove the decomposition theorem for the regularities of these measures. In the present paper, we give a new *multifractal formalism* for which the functions $b_{\mu}$ and $B_{\mu}$ differ. Actually, the main aim of this work is to introduce and study a multifractal formalism based on the Hewitt-Stromberg measures. However, we point out that this formalism is completely parallel to Olsen’s multifractal formalism introduced in [42] which based on the Hausdorff and packing measures. Then, we prove that the lower and upper multifractal Hewitt-Stromberg functions $b_{\mu}$ and $B_{\mu}$ are intimately related to the spectra functions. More precisely, we have

$$f_{\mu}(\alpha) := \underline{\dim}_{MB}(E(\alpha)) \leq b_{\mu}^{*}(\alpha) \quad \text{and} \quad F_{\mu}(\alpha) := \overline{\dim}_{MB}(E(\alpha)) \leq B_{\mu}^{*}(\alpha) \quad \text{for some} \quad \alpha \geq 0.$$
and, if \( P^B_\mu q \cdot (E(-B^\dagger_\mu(q))) > 0 \), then
\[
\overline{\dim} MB (E(-B^\dagger_\mu(q))) = B^\dagger_\mu(-B^\dagger_\mu(q)).
\]

Moreover, we describe a sufficient condition leading to the equalities
\[
f_\mu(\alpha) = F_\mu(\alpha) = F_\mu(\alpha) \quad \text{for some} \quad \alpha \geq 0.
\]

Specifically, if we assume that \( H^q B_\mu(q) \cdot (\text{supp} \mu) > 0 \), then
\[
\overline{\dim} MB (E(-B^\dagger_\mu(q))) = \overline{\dim} MB (E(-B^\dagger_\mu(q))) = B^\dagger_\mu(-B^\dagger_\mu(q)).
\]

We also observe that this sufficient condition is very close to being a necessary and sufficient one, see Theorem 7. In particular, we deal with the case where the lower and upper multifractal Hewitt-Stromberg functions \( b_\mu \) and \( B_\mu \) do not necessarily coincide, see Theorem 8.

We will now give a brief description of the organization of the paper. In the next section we recall the definitions of the various fractal and multifractal dimensions and measures investigated in the paper. The definitions of the Hausdorff and packing measures and the Hausdorff and packing dimensions are recalled in Section 2.1, and the definitions of the Hewitt-Stromberg measures are recalled in Section 2.2, while the definitions of the Hausdorff and packing measures are well-known, we have, nevertheless, decided to include these—there are two main reasons for this: firstly, to make it easier for the reader to compare and contrast the Hausdorff and packing measures with the less well-known Hewitt-Stromberg measures, and secondly, to provide a motivation for the Hewitt-Stromberg measures. Section 2.3 recalls the multifractal formalism introduced in [42]. In Section 2.4 we recall the definitions of the multifractal Hewitt-Stromberg measures and separator functions, and study their properties. Section 2.5 recalls earlier results on the values of the multifractal Hausdorff measure, the multifractal packing measure, the multifractal Hewitt-Stromberg measures and separator functions; this discussion is included in order to motivate our main results presented in Section 3. Section 4 contains concrete examples related to these concepts. The paper is concluded with Section 5 that, lists some open problems.

2. Preliminaries and statements of results
2.1. Hausdorff measure, packing measure and dimensions. While the definitions of the Hausdorff and packing measures and the Hausdorff and packing dimensions are well-known, we have, nevertheless, decided to briefly recall the definitions below. There are several reasons for this: firstly, since we are working in general metric spaces, the different definitions that appear in the literature may not all agree and for this reason it is useful to state precisely the definitions that we are using; secondly, and perhaps more importantly, the less well-known Hewitt-Stromberg measures (see Section 2.2) play an important part in this paper and to make it easier for the reader to compare and contrast the definitions of the Hewitt-Stromberg measures and the definitions of the Hausdorff and packing measures it is useful to recall the definitions of the latter measures; and thirdly, in order to provide a motivation for the Hewitt-Stromberg measures.

Let \( X \) be a metric space, \( E \subseteq X \) and \( t > 0 \). The Hausdorff measure is defined, for \( \varepsilon > 0 \), as follows
\[
H^t_\varepsilon(E) = \inf \left\{ \sum_i \left( \text{diam}(E_i) \right)^t \mid E \subseteq \bigcup_i E_i, \text{diam}(E_i) < \varepsilon \right\}.
\]

This allows to define first the \( t \)-dimensional Hausdorff measure \( H^t(E) \) of \( E \) by
\[
H^t(E) = \sup_{\varepsilon > 0} H^t_\varepsilon(E).
\]

Finally, the Hausdorff dimension \( \dim_H(E) \) is defined by
\[
\dim_H(E) = \sup \left\{ t \geq 0 \mid H^t(E) = +\infty \right\}.
\]

The packing measure is defined, for \( \varepsilon > 0 \), as follows
\[
\overline{\mathcal{P}}^t_\varepsilon(E) = \sup \left\{ \sum_i \left( 2r_i \right)^t \right\}.
\]
where the supremum is taken over all closed balls \( B(x_i, r_i) \) such that \( r_i \leq \varepsilon \) and with \( x_i \in E \) and \( d(x_i, x_j) \geq \frac{r_i + r_j}{2} \) for \( i \neq j \). The \( t \)-dimensional packing pre-measure \( \mathcal{P}^t (E) \) of \( E \) is now defined by

\[
\mathcal{P}^t (E) = \sup_{\varepsilon > 0} \mathcal{P}^t_\varepsilon (E).
\]

This makes us able to define the \( t \)-dimensional packing measure \( \mathcal{P}^t (E) \) of \( E \) as

\[
\mathcal{P}^t (E) = \inf \left\{ \sum_i \mathcal{P}_t (E_i) \ \bigg| \ E \subseteq \bigcup_i E_i \right\},
\]

and the packing dimension \( \dim_P (E) \) is defined by

\[
\dim_P (E) = \sup \left\{ t \geq 0 \ \big| \ \mathcal{P}^t (E) = +\infty \right\}.
\]

2.2. **Hewitt-Stromberg measures and dimensions.** Hewitt-Stromberg measures were introduced in [33, Exercise (10.51)]. Since then, they have been investigated by several authors, highlighting their importance in the study of local properties of fractals and products of fractals. One can cite, for example [30, 31, 35, 50, 65]. In particular, Edgar’s textbook [20, pp. 32-36] provides an excellent and systematic introduction to these measures. Such measures appear also appears explicitly, for example, in Pesin’s monograph [54, 5.3] and implicitly in Mattila’s text [38]. One of the purposes of this paper is to define and study a class of natural multifractal generalizations of the Hewitt-Stromberg measures. While Hausdorff and packing measures are defined using coverings and packings by families of sets with diameters less than a given positive number \( \varepsilon \), say, the Hewitt-Stromberg measures are defined using packings of balls with a fixed diameter \( \varepsilon \). For \( t > 0 \), the Hewitt-Stromberg pre-measures are defined as follows,

\[
\overline{H}^t (E) = \liminf_{r \to 0} N_r (E) \ (2r)^t
\]

and

\[
\underline{H}^t (E) = \limsup_{r \to 0} M_r (E) \ (2r)^t,
\]

where the covering number \( N_r (E) \) of \( E \) and the packing number \( M_r (E) \) of \( E \) are given by

\[
N_r (E) = \inf \left\{ \sharp \{ I \} \ \bigg| \ \left( B(x_i, r) \right)_{i \in I} \text{ is a family of closed balls with } x_i \in E \text{ and } E \subseteq \bigcup_i B(x_i, r) \right\}
\]

and

\[
M_r (E) = \sup \left\{ \sharp \{ I \} \ \bigg| \ \left( B(x_i, r_i) \right)_{i \in I} \text{ is a family of closed balls with } x_i \in E \text{ and } d(x_i, x_j) \geq r \text{ for } i \neq j \right\}.
\]

Now, we define the lower and upper \( t \)-dimensional Hewitt-Stromberg measures, which we denote respectively by \( \mathcal{H}^t (E) \) and \( \mathcal{P}^t (E) \), as follows

\[
\mathcal{H}^t (E) = \inf \left\{ \sum_i \overline{H}_t (E_i) \ \bigg| \ E \subseteq \bigcup_i E_i \right\}
\]

and

\[
\mathcal{P}^t (E) = \inf \left\{ \sum_i \underline{H}_t (E_i) \ \bigg| \ E \subseteq \bigcup_i E_i \right\}.
\]

We recall some basic inequalities satisfied by the Hewitt-Stromberg, the Hausdorff and the packing measure (see [35, 50, Proposition 2.1])

\[
\overline{H}^t (E) \leq \mathcal{P}^t (E) \leq \mathcal{P}^t (E)
\]

and

\[
\mathcal{H}^t (E) \leq \mathcal{H}^t (E) \leq \mathcal{P}^t (E) \leq \mathcal{P}^t (E).
\]

The lower and upper Hewitt-Stromberg dimension \( \dim_{\mathcal{H}B} (E) \) and \( \dim_{\mathcal{P}B} (E) \) are defined by

\[
\dim_{\mathcal{H}B} (E) = \inf \left\{ t \geq 0 \ \big| \ \mathcal{H}^t (E) = 0 \right\} = \sup \left\{ t \geq 0 \ \big| \ \mathcal{H}^t (E) = +\infty \right\}
\]

and

\[
\dim_{\mathcal{P}B} (E) = \inf \left\{ t \geq 0 \ \big| \ \mathcal{P}^t (E) = 0 \right\} = \sup \left\{ t \geq 0 \ \big| \ \mathcal{P}^t (E) = +\infty \right\}.
\]
The lower and upper box dimensions, denoted by $\dim_B(E)$ and $\overline{\dim}_B(E)$, respectively, are now defined by

$$
\dim_B(E) = \lim \inf_{r \to 0} \frac{\log N_r(E)}{-\log r} = \lim \inf_{r \to 0} \frac{\log M_r(E)}{-\log r}
$$

and

$$
\overline{\dim}_B(E) = \lim \sup_{r \to 0} \frac{\log N_r(E)}{-\log r} = \lim \sup_{r \to 0} \frac{\log M_r(E)}{-\log r}.
$$

These dimensions satisfy the following inequalities,

$$
\dim_H(E) \leq \dim_{MB}(E) \leq \overline{\dim}_{MB}(E) \leq \dim_P(E),
$$

$$
\dim_H(E) \leq \dim_P(E) \leq \overline{\dim}_B(E)
$$

and

$$
\dim_H(E) \leq \dim_{MB}(E) \leq \overline{\dim}_B(E).
$$

The reader is referred to [22] for an excellent discussion of the Hausdorff dimension, the packing dimension, lower and upper Hewitt-Stromberg dimension and the box dimensions. In particular, we have (see [22, 40])

$$
\dim_{MB}(E) = \inf \left\{ \sup \dim_B(E_i) \mid E \subseteq \bigcup_i E_i, \ E_i \text{ are bounded in } X \right\}
$$

and

$$
\overline{\dim}_{MB}(E) = \inf \left\{ \sup \overline{\dim}_B(E_i) \mid E \subseteq \bigcup_i E_i, \ E_i \text{ are bounded in } X \right\}.
$$

### 2.3. Multifractal Hausdorff measure and packing measure.

We start by introducing the generalized centered Hausdorff measure $H^{q,t}_\mu$ and the generalized packing measure $P^{q,t}_\mu$. We fix an integer $n \geq 1$ and denote by $\mathcal{P}(\mathbb{R}^n)$ the family of compactly supported Borel probability measures on $\mathbb{R}^n$. Let $\mu \in \mathcal{P}(\mathbb{R}^n)$, $q, t \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\delta > 0$. We define the generalized packing pre-measure by

$$
\overline{P}^{q,t}_\mu(E) = \inf_{\delta > 0} \sup \left\{ \sum_i \mu(B(x_i, r_i)) \left(\frac{2r_i}{\delta}\right)^q \mid \left(B(x_i, r_i)\right)_i \text{ is a centered } \delta\text{-packing of } E \right\}.
$$

In a similar way, we define the generalized Hausdorff pre-measure by

$$
\overline{H}^{q,t}_\mu(E) = \sup_{\delta > 0} \left\{ \sum_i \mu(B(x_i, r_i)) \left(\frac{2r_i}{\delta}\right)^q \mid \left(B(x_i, r_i)\right)_i \text{ is a centered } \delta\text{-covering of } E \right\},
$$

with the conventions $0^q = \infty$ for $q \leq 0$ and $0^q = 0$ for $q > 0$.

The function $\overline{H}^{q,t}_\mu$ is $\sigma$-subadditive but not increasing and the function $\overline{P}^{q,t}_\mu$ is increasing but not $\sigma$-subadditive. That is the reason for which Olsen introduced the following modifications of the generalized Hausdorff and packing measures $H^{q,t}_\mu$ and $P^{q,t}_\mu$:

$$
H^{q,t}_\mu(E) = \sup_{F \subseteq E} \overline{H}^{q,t}_\mu(F) \quad \text{and} \quad P^{q,t}_\mu(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \overline{P}^{q,t}_\mu(E_i).
$$

The functions $H^{q,t}_\mu$ and $P^{q,t}_\mu$ are metric outer measures and thus measures on the Borel family of subsets of $\mathbb{R}^n$. Moreover, there exists an integer $\xi \in \mathbb{N}$, such that $H^{q,t}_\mu \leq \xi P^{q,t}_\mu$. The measure $H^{q,t}_\mu$ is of course a multifractal generalization of the centered Hausdorff measure, whereas $P^{q,t}_\mu$ is a multifractal generalization of the packing measure. In fact, it is easily seen that, for $t \geq 0$, one has

$$
2^{-t} H^{0,t}_\mu \leq H^t \leq H^{0,t}_\mu \quad \text{and} \quad P^{0,t}_\mu = P^t,
$$

where $H^t$ and $P^t$ denote respectively the $t$-dimensional Hausdorff and $t$-dimensional packing measures.

We now define the family of doubling measures. For $\mu \in \mathcal{P}(\mathbb{R}^n)$ and $a > 1$, we write

$$
P_a(\mu) = \lim \sup_{r \to 0} \left( \sup_{x \in \text{supp } \mu} \frac{\mu(B(x, ar))}{\mu(B(x, r))} \right).
$$
We say that the measure $\mu$ satisfies the doubling condition if there exists $a > 1$ such that $P_a(\mu) < \infty$. It is easily seen that the exact value of the parameter $a$ is unimportant:

$$P_a(\mu) < \infty, \text{ for some } a > 1 \text{ if and only if } P_a(\mu) < \infty, \text{ for all } a > 1.$$  

Also, we denote by $\mathcal{P}_D(\mathbb{R}^n)$ the family of Borel probability measures on $\mathbb{R}^n$ which satisfy the doubling condition. We can cite as classical examples of doubling measures, the self-similar measures and the self-conformal ones [42]. In particular, if $\mu \in \mathcal{P}_D(\mathbb{R}^n)$ then $H_{\mu,t}^{q,t} \leq \overline{P}_{\mu,t}^{q,t}$.

The measures $H_{\mu,t}^{q,t}$ and $P_{\mu,t}^{q,t}$ assign in a usual way a multifractal dimension to each subset $E$ of $\mathbb{R}^n$. They are respectively denoted by $\dim_{\mu}^q(E)$, $\dim_{\mu}^{q,t}(E)$ and $\Delta_{\mu,t}^q(E)$ (see [42]) and satisfy

\[
\dim_{\mu}^q(E) = \inf \left\{ t \in \mathbb{R} \mid \dim_{\mu,t}^q(E) = 0 \right\} = \sup \left\{ t \in \mathbb{R} \mid \dim_{\mu,t}^q(E) = +\infty \right\},
\]

\[
\dim_{\mu}^{q,t}(E) = \inf \left\{ t \in \mathbb{R} \mid \dim_{\mu,t}^{q,t}(E) = 0 \right\} = \sup \left\{ t \in \mathbb{R} \mid \dim_{\mu,t}^{q,t}(E) = +\infty \right\},
\]

\[
\Delta_{\mu,t}^q(E) = \inf \left\{ t \in \mathbb{R} \mid \dim_{\mu,t}^{q,t}(E) = 0 \right\} = \sup \left\{ t \in \mathbb{R} \mid \dim_{\mu,t}^{q,t}(E) = +\infty \right\}.
\]

The number $\dim_{\mu}^q(E)$ is an obvious multifractal analogue of the Hausdorff dimension $\dim_{\mu}(E)$ of $E$ whereas $\dim_{\mu}^{q,t}(E)$ and $\Delta_{\mu,t}^q(E)$ are obvious multifractal analogues of the packing dimension $\dim_{\mu}(E)$ and the pre-packing dimension $\Delta(E)$ of $E$ respectively. In fact, it follows immediately from the definitions that

$$\dim_{\mu}(E) = \dim_{\mu,t}^0(E), \quad \dim_{\mu}(E) = \dim_{\mu,t}^0(E) \quad \text{and} \quad \Delta(E) = \Delta_{\mu,t}^0(E).$$

We define the functions

$$b_{\mu}(q) = \dim_{\mu}^q(\supp \mu) \quad \text{and} \quad B_{\mu}(q) = \dim_{\mu,t}^q(\supp \mu).$$

It is well known that the functions $b_{\mu}$ and $B_{\mu}$ are decreasing and $B_{\mu}$ is convex and satisfying $b_{\mu} \leq B_{\mu}$.

2.4. **Multifractal Hewitt-Stromberg measures and separator functions.** In the following, we will set up, for $q, t \in \mathbb{R}$ and $\mu \in \mathcal{P}(\mathbb{R}^n)$, the lower and upper multifractal Hewitt-Stromberg measures $H_{\mu,t}^{q,t}$ and $P_{\mu,t}^{q,t}$.

For $E \subseteq \supp \mu$, the pre-measure of $E$ is defined by

$$C_{\mu,t}^{q,t}(E) = \limsup_{r \to 0} M_{\mu,t}^q(E)(2r)^t,$$

where

$$M_{\mu,t}^q(E) = \sup \left\{ \sum_i \mu(B(x_i, r))^q \left| (B(x_i, r))_i \text{ is a centered packing of } E \right. \right\}.$$

It’s clear that $C_{\mu,t}^{q,t}$ is increasing and $C_{\mu,t}^{q,t}(\emptyset) = 0$. However it’s not $\sigma$-additive. For this, we introduce the $P_{\mu,t}^{q,t}$-measure defined by

$$P_{\mu,t}^{q,t}(E) = \inf \left\{ \sum_i C_{\mu,t}^{q,t}(E_i) \left| E \subseteq \bigcup_i E_i \text{ and the } E_i's \text{ are bounded} \right. \right\}.$$

In a similar way we define

$$L_{\mu,t}^{q,t}(E) = \liminf_{r \to 0} N_{\mu,t}^q(E)(2r)^t,$$

where

$$N_{\mu,t}^q(E) = \inf \left\{ \sum_i \mu(B(x_i, r))^q \left| (B(x_i, r))_i \text{ is a centered covering of } E \right. \right\}.$$
Theorem 3. by the physicists. The next theorem shows that these functions do indeed have some of these properties. In fact, it is easily seen that, for \( t > 0 \), one has

\[
H^{0,t}_{\mu} = H^t \quad \text{and} \quad P^{0,t}_{\mu} = P^t.
\]

The following result describes some of the basic properties of the multifractal Hewitt-Stromberg measures including the fact that \( H^{q,t}_{\mu} \) and \( P^{q,t}_{\mu} \) are Borel metric outer measures and summarises the basic inequalities satisfied by the multifractal Hewitt-Stromberg measures, the generalized Hausdorff measure and the generalized packing measure.

**Theorem 2.** [3] Let \( q, t \in \mathbb{R} \) and \( \mu \in \mathcal{P}(\mathbb{R}^n) \). Then for every set \( E \subseteq \mathbb{R}^n \) we have

1. the set functions \( H^{q,t}_{\mu} \) and \( P^{q,t}_{\mu} \) are metric outer measures and thus they are measures on the Borel algebra.
2. There exists an integer \( n \in \mathbb{N} \), such that

\[
H^{q,t}_{\mu}(E) \leq H^{n,\xi}_{\mu}(E) \leq \xi P^{q,t}_{\mu}(E) \leq \xi P^{n,\xi}_{\mu}(E).
\]

3. When \( q \leq 0 \) or \( q > 0 \) and \( \mu \in \mathcal{P}_D(\mathbb{R}^n) \), we have

\[
H^{q,t}_{\mu}(E) \leq H^{q,t}_{\mu}(E) \leq P^{q,t}_{\mu}(E) \leq P^{q,t}_{\mu}(E).
\]

The measures \( H^{q,t}_{\mu} \) and \( P^{q,t}_{\mu} \) and the pre-measure \( C^{n,\xi}_{\mu} \) assign in the usual way a multifractal dimension to each subset \( E \) of \( \mathbb{R}^n \). They are respectively denoted by \( b^q_{\mu}(E), B^q_{\mu}(E) \) and \( \Delta^q_{\mu}(E) \).

**Proposition 1.** Let \( q \in \mathbb{R}, \mu \in \mathcal{P}(\mathbb{R}^n) \) and \( E \subseteq \mathbb{R}^n \). Then

1. there exists a unique number \( b^q_{\mu}(E) \in [-\infty, +\infty) \) such that

\[
H^{q,t}_{\mu}(E) = \begin{cases} 
\infty & \text{if} \ t < b^q_{\mu}(E), \\
0 & \text{if} \ b^q_{\mu}(E) < t,
\end{cases}
\]

2. there exists a unique number \( B^q_{\mu}(E) \in [-\infty, +\infty) \) such that

\[
P^{q,t}_{\mu}(E) = \begin{cases} 
\infty & \text{if} \ t < B^q_{\mu}(E), \\
0 & \text{if} \ B^q_{\mu}(E) < t,
\end{cases}
\]

3. there exists a unique number \( \Delta^q_{\mu}(E) \in [-\infty, +\infty) \) such that

\[
C^{q,t}_{\mu}(E) = \begin{cases} 
\infty & \text{if} \ t < \Delta^q_{\mu}(E), \\
0 & \text{if} \ \Delta^q_{\mu}(E) < t.
\end{cases}
\]

In addition, we have

\[
b^q_{\mu}(E) \leq B^q_{\mu}(E) \leq \Delta^q_{\mu}(E).
\]

The number \( b^q_{\mu}(E) \) is an obvious multifractal analogue of the lower Hewitt-Stromberg dimension \( \dim_{MB}(E) \) of \( E \) whereas \( B^q_{\mu}(E) \) is an obvious multifractal analogues of the upper Hewitt-Stromberg dimension \( \overline{\dim}_{MB}(E) \) of \( E \). In fact, it follows immediately from the definitions that

\[
b^0_{\mu}(E) = \dim_{MB}(E) \quad \text{and} \quad B^0_{\mu}(E) = \overline{\dim}_{MB}(E).
\]

**Remark 1.** It follows from Theorem 2 that

\[
\dim_{MB}^q(E) \leq b^q_{\mu}(E) \leq B^q_{\mu}(E) \leq \Delta^q_{\mu}(E) \leq \overline{\dim}_{MB}(E).
\]

The definition of these dimension functions makes it clear that they are counterparts of the \( \tau_{\mu} \)-function which appears in the multifractal formalism. This being the case, it is important that they have the properties described by the physicists. The next theorem shows that these functions do indeed have some of these properties.

**Theorem 3.** Let \( q \in \mathbb{R} \) and \( E \subseteq \mathbb{R}^n \).

1. The functions \( q \mapsto H^{q,t}_{\mu}(E), P^{q,t}_{\mu}(E), C^{q,t}_{\mu}(E) \) are decreasing.
2. The functions \( t \mapsto H^{q,t}_{\mu}(E), P^{q,t}_{\mu}(E), C^{q,t}_{\mu}(E) \) are decreasing.
3. The functions \( q \mapsto b^q_{\mu}(E), B^q_{\mu}(E), \Delta^q_{\mu}(E) \) are decreasing.
The functions \( q \mapsto B^q_\mu(E) \), \( \Delta^q_\mu(E) \) are convex.

Proof. Let \( q \in \mathbb{R} \) and \( E \subseteq \mathbb{R}^n \).

The first and second part of Theorem 3 follows since \( x \mapsto a^x \) is decreasing for all \( a \in [0, 1[ \).

Observe that part (3) of Theorem 3 follows immediately from (1).

We will now prove the part (4). Let \( \alpha \in [0, 1] \) and \( p, s, t \in \mathbb{R} \). Suppose that we have shown that

\[
\sum_i C^{\alpha p + (1-\alpha)q, \alpha t + (1-\alpha)s}_\mu(E) \leq \alpha \left( C^{p,t}_\mu(E) \right)^\alpha \left( C^{q,s}_\mu(E) \right)^{1-\alpha}.
\] (2.1)

Then, for all \( \epsilon > 0 \), we have

\[
\sum_i C^{\alpha p + (1-\alpha)q, \alpha \Delta^p_\mu(E) + (1-\alpha)\Delta^q_\mu(E) + \epsilon}_\mu(E) \leq \alpha \left( C^{p,\Delta^p_\mu(E) + \epsilon}_\mu(E) \right)^\alpha \left( C^{q,\Delta^q_\mu(E) + \epsilon}_\mu(E) \right)^{1-\alpha} = 0.
\]

We therefore conclude that

\[
\Delta^{\alpha p + (1-\alpha)q}_\mu(E) \leq \alpha \Delta^p_\mu(E) + (1-\alpha)\Delta^q_\mu(E) + \epsilon.
\]

Finally, letting \( \epsilon \) tend to 0, then the convexity of \( q \mapsto \Delta^q_\mu(E) \) follows.

We now turn towards the proof of (2.1). Put \( r > 0 \) and \( \left( B(x_i, r) \right)_i \) be a centered packing of \( E \). It follows from Hölder inequality that

\[
\sum_i \mu(B(x_i, r))^{\alpha p + (1-\alpha)q} = \sum_i \left( \mu(B(x_i, r))^p \right)^\alpha \left( \mu(B(x_i, r))^q \right)^{1-\alpha} \leq \left( \sum_i \mu(B(x_i, r))^p \right)^\alpha \left( \sum_i \mu(B(x_i, r))^q \right)^{1-\alpha} \leq \left( M^p_{\mu,r}(E) \right)^\alpha \left( M^q_{\mu,r}(E) \right)^{1-\alpha}.
\]

This shows that

\[
M^{\alpha p + (1-\alpha)q}_{\mu,r}(2r)^{\alpha t + (1-\alpha)s} \leq \left( M^p_{\mu,r}(2r)^t \right)^\alpha \left( M^q_{\mu,r}(2r)^s \right)^{1-\alpha}.
\]

Letting \( r \) tend to 0 we get the result.

We must now show the convexity of \( q \mapsto B^q_\mu(E) \). Let \( \eta > 0 \) and put \( t = B^p_\mu(E) \) and \( s = B^q_\mu(E) \). Since \( P^{q,s+\eta}_\mu(E) = P^{p,t+\eta}_\mu(E) = 0 \), we can choose bounded coverings \( (H_i)_i \) and \( (K_i)_i \) of \( E \) such that

\[
\sum_i C^{q+s+\eta}_{\mu}(H_i) \leq 1 \quad \text{and} \quad \sum_i C^{q+s+\eta}_{\mu}(K_i) \leq 1.
\]
Next, for \(n \in \mathbb{N}^\ast\), let \(E_n = \bigcup_{i,j=1}^{n} (H_i \cap K_j)\), we clearly have

\[
\mathcal{P}_{\mu}^{\alpha p + (1-\alpha)q, \alpha t + (1-\alpha)s + \eta}(E_n) \leq \sum_{i,j=1}^{n} \mathcal{P}_{\mu}^{\alpha p + (1-\alpha)q, \alpha t + (1-\alpha)s + \eta}(H_i \cap K_j)
\]

\[
\leq \sum_{i,j=1}^{n} C_{\mu}^{\alpha p + (1-\alpha)q, \alpha t + (1-\alpha)s + \eta}(H_i \cap K_j)
\]

(2.1)

\[
\leq \sum_{i,j=1}^{n} \left( C_{\mu}^{p,t+\eta}(H_i \cap K_j) \right)^{\alpha} \left( C_{\mu}^{q,s+\eta}(H_i \cap K_j) \right)^{1-\alpha}
\]

Hölder

\[
\leq \left( \sum_{i,j=1}^{n} C_{\mu}^{p,t+\eta}(H_i) \right)^{\alpha} \left( \sum_{i,j=1}^{n} C_{\mu}^{q,s+\eta}(K_j) \right)^{1-\alpha}
\]

\[
\leq \left( \sum_{i=1}^{n} C_{\mu}^{p,t+\eta}(H_i) \right)^{\alpha} \left( \sum_{j=1}^{n} C_{\mu}^{q,s+\eta}(K_j) \right)^{1-\alpha}
\]

\[
\leq n^{\alpha} n^{1-\alpha} = n < \infty.
\]

We now obtain, for all \(n \in \mathbb{N}^\ast\),

\[
\mathcal{B}_{\mu}^{\alpha p + (1-\alpha)q}(E_n) \leq \alpha t + (1-\alpha)s + \eta.
\]

Since clearly \(E \subseteq \bigcup_{n} E_n\), we therefore conclude that

\[
\mathcal{B}_{\mu}^{\alpha p + (1-\alpha)q}(E) \leq \sup_{n} \mathcal{B}_{\mu}^{\alpha p + (1-\alpha)q}(E_n)
\]

\[
\leq \alpha \mathcal{B}_{\mu}^{p}(E) + (1-\alpha) \mathcal{B}_{\mu}^{q}(E) + \eta.
\]

Letting \(\eta\) tend to 0 now yields the desired result. This completes the proof of Theorem 3. \(\square\)

Next we define the multifractal separator functions \(b_{\mu}, B_{\mu}\) and \(\Lambda_{\mu} : \mathbb{R} \to [-\infty, +\infty]\) by

\[
b_{\mu} : q \to b_{\mu}^{q}(\text{supp } \mu), \quad B_{\mu} : q \to B_{\mu}^{q}(\text{supp } \mu) \quad \text{and} \quad \Lambda_{\mu} : q \to \Delta_{\mu}^{q}(\text{supp } \mu).
\]

We also obtain the following corollary providing information about the lower and upper multifractal Hewitt-Stromberg functions.

**Corollary 1.** Let \(q \in \mathbb{R}\). We have

1. For \(q < 1\), \(b_{\mu}(q) \geq 0\).
2. For \(q = 1\), \(b_{\mu}(q) = \Lambda_{\mu}(q) = 0\).
3. For \(q > 1\), \(\Lambda_{\mu}(q) \leq 0\).

**Proof.** This follow immediately from the above theorem and definitions. \(\square\)

2.5. **Some characterizations of** \(b_{\mu}(q)\) **and** \(B_{\mu}(q)\). In this section, we investigate the relation between the lower and upper multifractal Hewitt-Stromberg functions \(b_{\mu}\) and \(B_{\mu}\) and the multifractal box dimension, the multifractal packing dimension and the multifractal pre-packing dimension. We first note that there exists a unique number \(\Theta_{\mu}^{q}(E) \in [-\infty, +\infty]\) such that

\[
\mathcal{L}_{\mu}^{q,t}(E) = \begin{cases} 
\infty & \text{if } t < \Theta_{\mu}^{q}(E), \\
0 & \text{if } \Theta_{\mu}^{q}(E) < t.
\end{cases}
\]
Proposition 2. Let $q \in \mathbb{R}$ and $\mu$ be a compact supported Borel probability measure on $\mathbb{R}^n$. Then for every $E \subseteq \text{supp} \mu$ we have

$$\Theta^q_\mu(E) = \liminf_{r \to 0} \frac{\log N^q_{\mu,r}(E)}{-\log r} \quad \text{and} \quad \Delta^q_\mu(E) = \limsup_{r \to 0} \frac{\log M^q_{\mu,r}(E)}{-\log r}.$$ 

Proof. We will prove the first equality, the second one is similar. Suppose that

$$\liminf_{r \to 0} \frac{\log N^q_{\mu,r}(E)}{-\log r} > \Theta^q_\mu(E) + \epsilon$$

for some $\epsilon > 0$. Then we can find $\delta > 0$ such that for any $r \leq \delta$, $N^q_{\mu,r}(E) r^{\Theta^q_\mu(E) + \epsilon} > 1$

and then

$$\log N^q_{\mu,r}(E) - \log r \geq 2^{\Theta^q_\mu(E) + \epsilon}$$

which is a contradiction. We therefore infer

$$\liminf_{r \to 0} \frac{\log N^q_{\mu,r}(E)}{-\log r} \leq \Theta^q_\mu(E) + \epsilon \quad \text{for any} \quad \epsilon > 0.$$ 

The proof of the following statement

$$\liminf_{r \to 0} \frac{\log N^q_{\mu,r}(E)}{-\log r} \geq \Theta^q_\mu(E) - \epsilon \quad \text{for any} \quad \epsilon > 0$$

is identical to the proof of the above statement and is therefore omitted. \qed

Remark 2. Here we follow the approach of Olsen in [42, 45, 48, 49].

1. The multifractal dimensions $\Theta^q_\mu(E)$ and $\Delta^q_\mu(E)$ of $E$ represent the upper and lower multifractal box-dimension. In particular, we have

$$\Theta^q_\mu(E) = \liminf_{r \to 0} \frac{\log N^q_{\mu,r}(E)}{-\log r} \quad \text{and} \quad \Delta^q_\mu(E) = \limsup_{r \to 0} \frac{\log M^q_{\mu,r}(E)}{-\log r}.$$ 

2. Let us introduce the multifractal generalization of the $q$-dimensions called also relative Rényi $q$-dimensions

based on integral representations. Let $\mu$ be a probability measure on $\mathbb{R}^n$. For $q \in \mathbb{R} \setminus \{0\}$, we write

$$D^q_\mu = \liminf_{r \to 0} \frac{1}{q \log r} \log \int \mu(B(x,r))^q \, d\mu(x),$$

and

$$\overline{D}^q_\mu = \limsup_{r \to 0} \frac{1}{q \log r} \log \int \mu(B(x,r))^q \, d\mu(x).$$

Now we define the generalized entropies due to Rényi by,

$$h^q_\mu(\mu) = \frac{1}{q - 1} \log M^q_{\mu,r}(\text{supp} \mu) \quad \text{for} \quad q \neq 1$$

and

$$h^1_\mu(\mu) = \inf \left\{ -\sum \mu(E_i) \log \mu(E_i) \mid \{E_i\}, \text{is a partition of} \quad \text{supp} \mu \right\}.$$ 

We define the upper and lower Rényi $q$-dimensions $\Theta^q_\mu$ and $\Delta^q_\mu$ of $\mu$ by

$$\Theta^q_\mu = \limsup_{r \to 0} \frac{\log \overline{h}^q_\mu(\mu)}{-\log r} \quad \text{and} \quad \Delta^q_\mu = \liminf_{r \to 0} \frac{\log \underline{h}^q_\mu(\mu)}{-\log r}.$$ 

If $D^q_\mu = \overline{D}^q_\mu$ (respectively $\underline{D}^q_\mu = \overline{D}^q_\mu$) we refer to the common value as the relative Rényi $q$-dimension of $\mu$ and denote it $D^q_\mu$ (respectively $\overline{D}^q_\mu$). Finally define $\underline{D}^q_\mu(q), \overline{D}^q_\mu(q), \underline{D}^q_\mu(q)$ and $\overline{D}^q_\mu(q)$ : $\mathbb{R} \to [-\infty, +\infty]$ by

$$\underline{D}^q_\mu(q) = (1 - q)\overline{D}^{q-1}_\mu, \quad \overline{D}^q_\mu(q) = (1 - q)\underline{D}^{q-1}_\mu$$

and

$$\underline{D}^q_\mu(q) = (1 - q)\underline{D}^{q-1}_\mu, \quad \overline{D}^q_\mu(q) = (1 - q)\overline{D}^{q-1}_\mu.$$ 

Let $q \in \mathbb{R}$ and $\mu \in \mathcal{P}_D(\mathbb{R}^n)$. Then the following holds

$$\Delta^q_\mu(q) = D^q_\mu(q) \lor \overline{D}^q_\mu(q) = \underline{D}^q_\mu(q) \lor \overline{D}^q_\mu(q).$$
(3) We define the multifractal Minkowski volume as follows. Let $E$ be a subset of $\mathbb{R}^n$ and $r > 0$. We denote by $B(E, r)$ the open $r$ neighbourhood of $E$, i.e.,

$$B(E, r) = \left\{ x \in \mathbb{R}^n \mid \text{dist}(x, E) < r \right\}.$$ \hspace{1cm}

For a real number $q$ and a Borel measure $\mu$ on $\mathbb{R}^n$, we define the multifractal Minkowski volume $V_{\mu, r}^q(E)$ of $E$ with respect to the measure $\mu$ by

$$V_{\mu, r}^q(E) = \frac{1}{r^n} \int_{B(E, r)} \mu(B(x, r))^q d\mathcal{L}^n(x).$$

Here $\mathcal{L}^n$ denotes the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$. The importance of the Rényi dimensions in multifractal analysis together with the formal resemblance between the multifractal Minkowski volume $V_{\mu, r}^q(E)$ and the moments $\int_E \mu(B(x, r))^q d\mu(x)$ used in the definition the Rényi dimensions may be seen as a justification for calling the quantity $V_{\mu, r}^q(E)$ for the multifractal Minkowski volume. Using the multifractal Minkowski volume we can define multifractal Minkowski dimensions. For a real number $q$ and a Borel measure $\mu$ on $\mathbb{R}^n$, we define the lower and upper multifractal Minkowski dimension of $E$, by

$$\dim_{M, \mu}^q(E) = \liminf_{r \to 0} \frac{\log V_{\mu, r}^q(E)}{-\log r} \quad \text{and} \quad \dim_{M, \mu}^q(E) = \limsup_{r \to 0} \frac{\log V_{\mu, r}^q(E)}{-\log r}.$$

We note the close similarity between the multifractal Minkowski dimensions and $\Delta_{\mu}^q$. Indeed, the equality (2.2) shows that this similarity is not merely a formal resemblance. In fact, for $q \geq 1$, the multifractal Minkowski dimensions and $\Delta_{\mu}^q$ coincide, i.e. for $q \geq 1$ and $\mu \in \mathcal{P}_D(\mathbb{R}^n)$, we have

$$\Delta_{\mu}(q) = \dim_{M, \mu}^{q-1}(\text{supp } \mu) \vee \dim_{M, \mu}^{q-1}(\text{supp } \mu). \quad (2.2)$$

**Proposition 3.** Let $q \in \mathbb{R}$ and $\mu$ be a compact supported Borel probability measure on $\mathbb{R}^n$. Then for every $E \subseteq \text{supp } \mu$ we have

$$b_{\mu}^q(E) = \sup_{F \subseteq E} \left\{ \inf \left\{ \sup_{i} \Theta_{\mu}^q(F_i) \mid F \subseteq \bigcup_i F_i, \ F_i \text{ are bounded in } \mathbb{R}^n \right\} \right\}$$

and

$$B_{\mu}^q(E) = \inf \left\{ \sup_{i} \Delta_{\mu}^q(E_i) \mid E \subseteq \bigcup_i E_i, \ E_i \text{ are bounded in } \mathbb{R}^n \right\}.$$

**Proof.** Denote

$$\beta = \sup_{F \subseteq E} \left\{ \inf \left\{ \sup_{i} \Theta_{\mu}^q(F_i) \mid F \subseteq \bigcup_i F_i, \ F_i \text{ are bounded in } \mathbb{R}^n \right\} \right\}.$$ \hspace{1cm}

Assume that $\beta < b_{\mu}^q(E)$ and take $\alpha \in (\beta, b_{\mu}^q(E))$. Then, for all $F \subseteq E$, there exists $\{ F_i \}$ of bounded subset of $F$ such that $F \subseteq \bigcup_i F_i$, and $\sup_i \Theta_{\mu}^q(F_i) < \alpha$. Now observe that $L_{\mu}^q(\alpha)(F_i) = 0$ which implies that $\overline{H}_{\mu}^{q, \alpha}(F) = 0$. This implies that $H_{\mu}^{q, \alpha}(E) = 0$. It is a contradiction. Now suppose that $b_{\mu}^q(E) < \beta$, then, for $\alpha \in (b_{\mu}^q(E), \beta)$, we have $H_{\mu}^{q, \alpha}(E) = 0$. It follows from this that $\overline{H}_{\mu}^{q, \alpha}(F) = 0$ for all $F \subseteq E$. Thus, there exists $\{ F_i \}$ of bounded subset of $F$ such that $F \subseteq \bigcup_i F_i$, and $\sup_i L_{\mu}^{q, \alpha}(F_i) < \infty$. We conclude that, $\sup_i \Theta_{\mu}^q(F_i) \leq \alpha$. It is also a contradiction.

The proof of the second statement is identical to the proof of the statement in the first part and is therefore omitted. \hfill \Box

**Proposition 4.** If $q \in \mathbb{R}$ and $\mu \in \mathcal{P}_D(\mathbb{R}^n)$, then for any subset $E$ of $\text{supp } \mu$, we have

$$B_{\mu}^q(E) = B_{\mu}^q(E).$$

**Proof.** This follows easily from Propositions 2 and 3, Propositions 2.19 and 2.22 in [42] and Lemma 4.1 in [46]. \hfill \Box

**Remark 3.** The results developed by Falconer in [22] are obtained as a special case of the multifractal results by setting $q = 0$. \hfill \Box
3. A multifractal formalism for Hewitt-Stromberg measures

Multifractal analysis was proved to be a very useful technique in the analysis of measures, both in theory and applications. The upper and lower local dimensions of a measure $\mu$ on $\mathbb{R}^n$ at a point $x$ are respectively given by:
\[
\overline{\alpha}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad \underline{\alpha}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},
\]
where $B(x, r)$ denote the closed ball of center $x$ and radius $r$. We refer to the common value as the local dimension of $\mu$ at $x$, and denote it by $\alpha_\mu(x)$.

The level set of the local dimension of $\mu$ contains a crucial information on the geometrical properties of $\mu$. The aim of the multifractal analysis of a measure is to relate the Hausdorff and packing dimensions of these levels sets to the Legendre transform of some concave (convex) function (see for example [2, 10, 11, 12, 13, 17, 41, 42]). For $\alpha \geq 0$, we define the fractal sets,
\[
E^\alpha = \{ x \in \text{supp} \mu \mid \overline{\alpha}_\mu(x) \leq \alpha \}; \quad \overline{E}_\alpha = \{ x \in \text{supp} \mu \mid \overline{\alpha}_\mu(x) \geq \alpha \}
\]
and
\[
E^\alpha = \{ x \in \text{supp} \mu \mid \underline{\alpha}_\mu(x) \leq \alpha \}; \quad E_\alpha = \{ x \in \text{supp} \mu \mid \underline{\alpha}_\mu(x) \geq \alpha \}.
\]
Also, let
\[
E(\alpha) = \overline{E}^\alpha \cap \underline{E}_\alpha = \{ x \in \text{supp} \mu \mid \alpha_\mu(x) = \alpha \}.
\]

Theorem 4 allows us to consider the relationship between the lower and upper multifractal Hewitt-Stromberg functions $b_\mu$ and $B_\mu$ and the multifractal spectra. We start by giving an upper bound theorem. For $\mu \in \mathcal{P}(\mathbb{R}^n)$, set
\[
\alpha_{\min} = \sup_{0 < q} \frac{b_\mu(q)}{q}, \quad \alpha_{\max} = \inf_{0 > q} \frac{b_\mu(q)}{q}, \quad \beta_{\min} = \sup_{0 < q} \frac{B_\mu(q)}{q}, \quad \beta_{\max} = \inf_{0 > q} \frac{B_\mu(q)}{q}.
\]
Before stating this formally, we remind the reader that if $\varphi : \mathbb{R} \to \mathbb{R}$ is a real valued function, then the Legendre transform $\varphi^* : \mathbb{R} \to [-\infty, +\infty]$ of $\varphi$ is defined by
\[
\varphi^*(x) = \inf_y \left( xy + \varphi(y) \right).
\]
Now, we can state our multifractal formalism.

**Theorem 4.** Let $\alpha \geq 0$, then the following hold

(1) \[ \alpha_{\min} \leq \inf \overline{\alpha}_\mu(x) \leq \sup \overline{\alpha}_\mu(x) \leq \beta_{\max} \]
and
\[ \beta_{\min} \leq \inf \underline{\alpha}_\mu(x) \leq \sup \underline{\alpha}_\mu(x) \leq \alpha_{\max}. \]

(2) \[
\dim_{MB} E(\alpha) = \begin{cases} b_\mu(\alpha) & \text{if } \alpha \in (\alpha_{\min}, \alpha_{\max}) \\ 0 & \text{if } \alpha \notin (\alpha_{\min}, \alpha_{\max}). \end{cases}
\]

(3) \[
\dim_{MB} E(\alpha) = \begin{cases} B_\mu(\alpha) & \text{if } \alpha \in (\alpha_{\min}, \alpha_{\max}) \\ 0 & \text{if } \alpha \notin (\alpha_{\min}, \alpha_{\max}). \end{cases}
\]

**Proof.** This theorem follows immediately from the following lemmas. \(\square\)

**Lemma 1.** If $\mu \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha \geq 0$, then

(1) \[ E^\alpha = \emptyset \text{ for } \alpha < \beta_{\min} \quad \text{and} \quad \overline{E}_\alpha = \emptyset \text{ for } \alpha > \beta_{\max}. \]
Lemma 2. Let $\mu \in \mathcal{P}(\mathbb{R}^n)$, $\alpha \geq 0$, $q, t \in \mathbb{R}$ and $\delta > 0$ such that $\delta \leq \alpha q + t$. Then the following hold.

(1) (a) $H^{\alpha q + t + \delta}(E^t) \leq 2^{\alpha q + t} H_{\mu}^{q, \ell}(E^{q, \ell})$ for $0 \leq q$.
(b) $H^{\alpha q + t + \delta}(E_n^t) \leq 2^{\alpha q + t} H_{\mu}^{q, \ell}(E_n^{q, \ell})$ for $0 \geq q$.
(c) If $0 \leq \alpha q + b_\mu(q)$ then
\[ \dim_{MB}(E^t) \leq \inf_{q \geq 0} \alpha q + b_\mu(q) \quad \text{and} \quad \dim_{MB}(E_n^t) \leq \inf_{q \leq 0} \alpha q + b_\mu(q). \]
In particular $\dim_{MB}(E_{\alpha q + t + \delta}) \leq \alpha$.

(2) (a) $P^{\alpha q + t + \delta}(E^t) \leq 2^{\alpha q + t} P_{\mu}^{q, \ell}(E^{q, \ell})$ for $0 \leq q$.
(b) $P^{\alpha q + t + \delta}(E_n^t) \leq 2^{\alpha q + t} P_{\mu}^{q, \ell}(E_n^{q, \ell})$ for $0 \geq q$.
(c) If $0 \leq \alpha q + b_\mu(q)$ then
\[ \overline{\dim}_{MB}(E^t) \leq \inf_{q \geq 0} \alpha q + B_\mu(q) \quad \text{and} \quad \overline{\dim}_{MB}(E_n^t) \leq \inf_{q \leq 0} \alpha q + B_\mu(q). \]
In particular $\overline{\dim}_{MB}(E_{\alpha q + t + \delta}) \leq \alpha$.

Proof. An exhaustive proof of this lemma would require considerable repetition. To avoid this we prove (1)-(a) and (2)-(a).
(1) Clearly, the statement is true for $q = 0$. For $m \in \mathbb{N}$, write

$$E_m = \left\{ x \in \overline{E}^\alpha \mid \frac{\log \mu(B(x, r))}{\log r} \leq \alpha + \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{m} \right\}.$$ 

Fix $m \in \mathbb{N}$ and $r > 0$ such that $0 < r < \frac{1}{m}$. Let $\left( B(x_i, r) \right)_i$ be a centered covering of $E_m$. Next, we observe that

$$\frac{\log \mu(B(x, r))}{\log r} \leq \alpha + \frac{\delta}{q} \implies \mu(B(x, r))^q \geq r^{\alpha q + \delta}$$

$$\implies \sum_i \mu(B(x_i, r))^q \geq N_r(E_m)^q r^{\alpha q + \delta}$$

$$\implies N_{\mu, r}^q(E_m) \geq N_r(E_m)^q r^{\alpha q + \delta}$$

$$\implies L_{\mu, t}^q(E_m) \geq 2^{-\alpha q - \delta} \mathbb{H}^{\alpha q + \delta + t}(E_m)$$

$$\implies H_{\mu, t}^q(E_m) \geq \mathbb{H}_{\mu, t}^q(E_m) \geq 2^{-\alpha q - \delta} \mathbb{H}_{\mu, t}^{\alpha q + \delta + t}(E_m).$$

Now from this and since $E_m \not\subset \overline{E}^\alpha$, we can deduce that

$$H_{\mu, t}^q(\overline{E}^\alpha) \geq 2^{-\alpha q - \delta} \mathbb{H}_{\alpha q + \delta + t}(E_m).$$

(2) Once again for $q = 0$ the statement is well known. For $m \in \mathbb{N}$, put

$$E_m = \left\{ x \in \overline{E}^\alpha \mid \frac{\log \mu(B(x, r))}{\log r} \leq \alpha + \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{m} \right\}.$$ 

We therefore fix $m \in \mathbb{N}$ and $r > 0$ such that $0 < r < \frac{1}{m}$. Let $\left( B(x_i, r) \right)_{i \in \{1, \ldots, M_r(E_m)\}}$ be a packing of $E_m$. Then we have

$$\frac{\log \mu(B(x, r))}{\log r} \leq \alpha + \frac{\delta}{q} \implies \mu(B(x, r))^q \geq r^{\alpha q + \delta}$$

$$\implies \sum_i \mu(B(x_i, r))^q \geq M_r(E_m)^q r^{\alpha q + \delta}$$

$$\implies M_{\mu, r}^q(E_m) \geq M_r(E_m)^q r^{\alpha q + \delta}$$

$$\implies C_{\mu, t}^q(E_m) \geq 2^{-\alpha q - \delta} \mathbb{H}^{\alpha q + \delta + t}(E_m)$$

$$\implies P_{\mu, t}^q(E_m) \geq 2^{-\alpha q - \delta} \mathbb{H}_{\mu, t}^{\alpha q + \delta + t}(E_m).$$

However, since $E_m \not\subset \overline{E}^\alpha$, we conclude that

$$P_{\mu, t}^q(\overline{E}^\alpha) \geq 2^{-\alpha q - \delta} \mathbb{P}^{\alpha q + \delta + t}(\overline{E}^\alpha)$$

which yields the desired result. 

Our purpose of the following theorems is to propose a sufficient condition that gives the lower bound.

**Theorem 5.** Let $q, t \in \mathbb{R}$, and $\alpha > 0$ such that $\alpha q + t \geq 0$. Let $A \subseteq E(\alpha)$ is a Borel set.

(1) If $H_{\mu, t}^q(A) > 0$, then

$$\dim_{MB} E(\alpha) \geq \alpha q + t.$$ 

In particular, if the multifractal function $b_\mu$ is differentiable at $q$, then, provided that $b_\mu'(-b_\mu'(q)) \geq 0$ and $H_{\mu, b_\mu'(q)}(E(-b_\mu'(q))) > 0$, we have

$$\dim_{MB} E(-b_\mu'(q)) = b_\mu^*(b_\mu'(q)).$$
(2) If $\mathcal{P}^{q,t}_\mu(A) > 0$, then
\[ \overline{\dim}_{MB} E(\alpha) \geq \alpha q + t. \]

In particular, if the multifractal function $\mathcal{B}_\mu$ is differentiable at $q$, then, provided that $\mathcal{B}_\mu^* \left( -\mathcal{B}_\mu'(q) \right) \geq 0$ and $\mathcal{P}^{q,t}_\mu(\mu(B(-\mathcal{B}_\mu'(q))) > 0$, we have
\[ \overline{\dim}_{MB} E \left( -\mathcal{B}_\mu'(q) \right) = \mathcal{B}_\mu^* \left( -\mathcal{B}_\mu'(q) \right). \]

Proof. This follows easily from Theorem 4 and the following lemma. \hfill \Box

Lemma 3. Let $\mu \in \mathcal{P}(\mathbb{R}^n)$, $\alpha \geq 0$, $q, t \in \mathbb{R}$ and $\delta > 0$ such that $\delta \leq \alpha q + t$. Then we have the following

1. (a) If $A \subset E^\alpha$, is Borel then $\mathcal{P}^{\alpha q + t - \delta}_\mu(A) \geq 2^{\alpha q - \delta} \mathcal{P}^{q,t}_\mu(A)$ for $0 \geq q$.
   
   (b) If $A \subset E^\alpha$, is Borel then $\mathcal{P}^{\alpha q + t - \delta}_\mu(A) \geq 2^{\alpha q - \delta} \mathcal{P}^{q,t}_\mu(A)$ for $0 \leq q$.
   
   In particular, if $\mu(A) > 0$ then $\overline{\dim}_{MB}(A) \geq \alpha$.

2. (a) If $A \subset E^\alpha$, is Borel then $\mathcal{H}^{\alpha q + t - \delta}_\mu(A) \geq 2^{\alpha q - \delta} \mathcal{H}^{q,t}_\mu(A)$ for $0 \geq q$.
   
   (b) If $A \subset E^\alpha$, is Borel then $\mathcal{H}^{\alpha q + t - \delta}_\mu(A) \geq 2^{\alpha q - \delta} \mathcal{H}^{q,t}_\mu(A)$ for $0 \leq q$.
   
   In particular, if $\mu(A) > 0$ then $\overline{\dim}_{MB}(A) \geq \alpha$.

Proof. An exhaustive proof of this theorem would require considerable repetition. For this we only prove (1)-(a) and (2)-(a), the other assertions are similar.

(1) (a) Clearly the statement is true for $q = 0$. For $m \in \mathbb{N}$, write
\[
E_m = \left\{ x \in A \mid \frac{\log \mu(B(x, r))}{\log r} \leq \alpha - \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{m} \right\}.
\]

Let $m \in \mathbb{N}$ and $r > 0$ such that $0 < r < \frac{1}{m}$. Let $\left( B(x_i, r) \right)_i$ be a centred packing of $E_m$. We have
\[
\frac{\log \mu(B(x_i, r))}{\log r} \leq \alpha - \frac{\delta}{q} \implies \mu(B(x_i, r))^q \leq r^{\alpha q - \delta} \\implies \sum_i \mu(B(x_i, r))^q \leq M_r(E_m) r^{\alpha q - \delta} \\implies M^{\alpha q}_\mu r(E_m) \leq M_r(E_m) r^{\alpha q - \delta} \\implies C^{q,t}_\mu(E_m) \leq 2^{-\alpha q + \delta} \mathcal{P}^{\alpha q - \delta + t}_\mu(E_m) \\implies \mathcal{P}^{q,t}_\mu(E_m) \leq 2^{-\alpha q + \delta} \mathcal{P}^{\alpha q - \delta + t}_\mu(E_m).
\]

Finally, since $E_m \nsubseteq A$ we conclude that
\[
\mathcal{P}^{q,t}_\mu(A) \leq 2^{-\alpha q + \delta} \mathcal{P}^{\alpha q - \delta + t}_\mu(A).
\]

(2) (a) It is well known that the statement is true for $q = 0$. For $m \in \mathbb{N}$, we define the set $E_m$ by
\[
E_m = \left\{ x \in A \mid \frac{\log \mu(B(x, r))}{\log r} \leq \alpha - \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{m} \right\}.
\]
Next, fix $m \in \mathbb{N}$ and $r > 0$ such that $0 < r < \frac{1}{m}$. Let $\left( B(x_i, r) \right)_{i \in \{1, \ldots, N_r(F)\}}$ be a centred covering of $F \subset E_m$. We get
\[
\frac{\log \mu(B(x_i, r))}{\log r} \leq \alpha - \frac{\delta}{q} \quad \Rightarrow \quad \mu(B(x_i, r))^q \leq r^{q\alpha - \delta}
\]
\[
\Rightarrow \quad \sum_i \mu(B(x_i, r))^q \leq N_r(F) r^{q\alpha - \delta}
\]
\[
\Rightarrow \quad N^q_{\mu, r}(F) \leq N_r(F) r^{q\alpha - \delta}
\]
\[
\Rightarrow \quad L^{q,t}_{\mu}(F) \leq 2^{\alpha + \delta} H^{q - \delta + t}(F)
\]
Putting these together we have that
\[
H^{q, t}_{\mu}(A) \leq 2^{-\alpha + \delta} H^{q - \delta + t}(A).
\]
This proves the lemma.

**Theorem 6.** Let $q \in \mathbb{R}$ and suppose that $H^{q, A_{\mu}}_{\mu}(\text{supp } \mu) > 0$. Then,
\[
\dim_B \left( E - N_{\mu, +}^{q}(q) \cap E - N_{\mu, -}^{q}(q) \right) \geq \begin{cases} -N_{\mu, -}^{q}(q)q + A_{\mu}(q), & \text{for } q \leq 0, \\ -N_{\mu, +}^{q}(q)q + A_{\mu}(q), & \text{for } q \geq 0. \end{cases}
\]

**Proof.** It is well known from Lemma 3 that for all $\delta > 0$ and $t \in \mathbb{R},$
\[
\begin{cases}
H^{-N_{\mu, +}^{q}(q)q + t - \delta}(E(q)) \geq 2^{-N_{\mu, +}^{q}(q)q - \delta} H^{q, t}_{\mu}(E(q)), & \text{for } q \geq 0, \\
H^{-N_{\mu, -}^{q}(q)q + t - \delta}(E(q)) \geq 2^{-N_{\mu, -}^{q}(q)q - \delta} H^{q, t}_{\mu}(E(q)), & \text{for } q \leq 0
\end{cases}
\]
where the set $E(q)$ is defined by
\[
E(q) = E - N_{\mu, +}^{q}(q) \cap E - N_{\mu, -}^{q}(q).
\]

Theorem 6 is then an easy consequence of the following lemma.

**Lemma 4.** One has $H^{q, A_{\mu}}_{\mu}(\text{supp } \mu \setminus E(q)) = 0.$

**Proof.** Let us introduce, for $\alpha$ and $\beta$ in $\mathbb{R}$
\[X_\alpha = \text{supp } \mu \setminus E_\alpha, \quad \text{and} \quad Y_\beta = \text{supp } \mu \setminus E_\beta.\]
It clearly suffices to prove that
\[H^{q, A_{\mu}}_{\mu}(X_\alpha) = 0, \quad \text{for all } \alpha < -N_{\mu, +}^{q}(q) \quad (3.1)\]
and
\[H^{q, A_{\mu}}_{\mu}(Y_\beta) = 0, \quad \text{for all } \beta > -N_{\mu, -}^{q}(q). \quad (3.2)\]
Indeed, it is clear that
\[
0 \leq H^{q, A_{\mu}}_{\mu}(\text{supp } \mu \setminus (E_{-N_{\mu, +}^{q}(q)} \cap E_{-N_{\mu, -}^{q}(q)})) \leq H^{q, A_{\mu}}_{\mu}(\text{supp } \mu \setminus E_{-N_{\mu, +}^{q}(q)}) + H^{q, A_{\mu}}_{\mu}(\text{supp } \mu \setminus E_{-N_{\mu, -}^{q}(q)}) \leq H^{q, A_{\mu}}_{\mu}(X_\alpha) + \sum_{\beta > -N_{\mu, -}^{q}(q)} H^{q, A_{\mu}}_{\mu}(Y_\beta) = 0.
\]
We only have to prove that (3.1). The proof of (3.2) is identical to the proof of (3.1) and is therefore omitted.

Let \( \alpha < -\Lambda_{\mu+}'(q) \) and \( t > 0 \), such that \( \Lambda_{\mu}(q + t) < \Lambda_{\mu}(q) - \alpha t \), we have

\[
C_{\mu}^{q+t,\Lambda_{\mu}(q)-\alpha t}(\text{supp } \mu) = 0.
\]

For \( x \in X_\alpha \) and \( \delta > 0 \), we can find \( \lambda_x \geq 2 \) and \( \frac{\delta}{\lambda_x} < r_x \), such that

\[
\mu(B(x, r_x)) > r_x^\alpha.
\]

The family \( \{B(x, r_x)\}_{x \in X_\alpha} \) is a centered \( \delta \)-covering of \( X_\alpha \). Then, we can choose a finite subset \( J \) of \( \mathbb{N} \) such that the family \( \{B(x_i, r_{x_i})\}_{i \in J} \) is a centered \( \delta \)-covering of \( X_\alpha \). Take \( \lambda = \sup_{i \in J} \lambda_{x_i} \), then for all \( i \), we have

\[
\mu(B(x_i, \delta)) \geq \mu(B(x_i, r_{x_i})) > r_{x_i}^\alpha \geq \left(\frac{\delta}{\lambda}\right)^\alpha.
\]

Since \( \{B(x_i, \delta)\}_{i \in J} \) is a centered covering of \( X_\alpha \). Then, using Besicovitch’s covering theorem, we can construct \( \xi \) finite sub-families \( \{B(x_{ij}, \delta)\}_{j \in I} \) \( \xi \) such that each \( X_\alpha \subseteq \bigcup_{i=1}^\xi \bigcup_{j} B(x_{ij}, \delta) \) and \( \{B(x_{ij}, \delta)\}_{j} \) is a packing of \( X_\alpha \). We clearly have

\[
\mu(B(x_{ij}, \delta)) \leq \lambda^{\alpha t} \mu(B(x_{ij}, \delta)^t) \delta^{\Lambda_{\mu}(q) - \alpha t}.
\]

It therefore follows that

\[
N_{\mu}^t(X_\alpha) \delta^{\Lambda_{\mu}(q)} \leq \lambda^{\alpha t} \xi M_{\mu}^{\alpha t+t}(X_\alpha) \delta^{\Lambda_{\mu}(q) - \alpha t}.
\]

Letting \( \delta \to 0 \) now yields

\[
H_{\mu}^{\alpha}(X_\alpha)(\text{supp } \mu) \leq L_{\mu}^{\alpha}(X_\alpha)(\text{supp } \mu) \leq 2^{\alpha t} \lambda^{\alpha t} \xi C_{\mu}^{q+t,\Lambda_{\mu}(q)-\alpha t}(X_\alpha) \leq 2^{\alpha t} \lambda^{\alpha t} \xi C_{\mu}^{q+t,\Lambda_{\mu}(q)-\alpha t}(\text{supp } \mu) = 0.
\]

Remark that, in the last inequality, we can replace \( X_\alpha \) by any arbitrary subset of \( X_\alpha \). Then, we can finally conclude that

\[
H_{\mu}^{\alpha}(X_\alpha)(\text{supp } \mu) \leq 2^{\alpha t} \lambda^{\alpha t} \xi C_{\mu}^{q+t,\Lambda_{\mu}(q)-\alpha t}(\text{supp } \mu) = 0.
\]

This completes the proof of (3.1).

The following result proves that the condition \( H_{\mu}^{\alpha}(X_\alpha)(\text{supp } \mu) > 0 \) is very close to being a necessary and sufficient condition for the validity of our multifractal formalism.

**Theorem 7.** Let \( q \in \mathbb{R} \) and \( \mu \) be a compact supported Borel probability measure on \( \mathbb{R}^n \). Suppose that one of the following hypotheses is satisfied,

1. \( \dim_{MB}(E_{-\mathcal{N}_{\mu+}'(q)} \cap \overline{E}^{-\mathcal{N}_{\mu-}'(q)}) \geq -\mathcal{N}_{\mu+}'(q)q + \Lambda_{\mu}(q), \quad \text{for } q \leq 0. \)

2. \( \dim_{MB}(E_{-\mathcal{N}_{\mu+}'(q)} \cap \overline{E}^{-\mathcal{N}_{\mu-}'(q)}) \geq -\mathcal{N}_{\mu-}'(q)q + \Lambda_{\mu}(q), \quad \text{for } q \geq 0. \)

Then,

\[
\mathcal{B}_{\mu}(q) = \mathcal{B}_{\mu}(q) = \Lambda_{\mu}(q).
\]

In other words,

\[
H_{\mu}^{\alpha}(\text{supp } \mu) > 0 \quad \text{for all } \quad t < \Lambda_{\mu}(q).
\]

**Proof.** We have, for \( q \geq 0 \)

\[
E_{-\mathcal{N}_{\mu+}'(q)} \cap \overline{E}^{-\mathcal{N}_{\mu-}'(q)} \subseteq E^{-\mathcal{N}_{\mu-}'(q)},
\]

it follows immediately that

\[
-\mathcal{N}_{\mu-}'(q)q + \Lambda_{\mu}(q) \leq \dim_{MB}(E_{-\mathcal{N}_{\mu+}'(q)} \cap \overline{E}^{-\mathcal{N}_{\mu-}'(q)}) \leq \dim_{MB}(E^{-\mathcal{N}_{\mu-}'(q)}).
\]

Now, suppose that \( \alpha = -\mathcal{N}_{\mu-}'(q) \). We only prove the case where \( q \geq 0 \). The other one is very similar and is therefore omitted. We have

\[
\dim_{MB}(E^{\alpha}) \geq \alpha q + \Lambda_{\mu}(q).
\]
Since $\mu_b(q) \leq \mu_B(q) \leq \Lambda_\mu(q)$, we only have to prove that $\mu_b(q) \geq \Lambda_\mu(q)$. Let $t < \Lambda_\mu(q)$ and choose $\beta$, such that $\beta < \alpha$. Then, $\beta q + t < \alpha q + \Lambda_\mu(q)$. For $p \in \mathbb{N}$, we consider the set

$$F_p = \left\{ x \in \mathbb{E}^\alpha \mid \mu(B(x, r)) \geq r^\beta, \ 0 < r < \frac{1}{p} \right\}. $$

It is clear that $F_p \not\supset \mathbb{E}^\alpha$ as $p \to \infty$. It follows that, there exists $p > 0$, such that

$$\dim_{MB}(F_p) = \beta q + t \Rightarrow H^{\beta q + t}(F_p) > 0.$$ 

Let $0 < r < \frac{1}{p}$ and\( \left( B(x_i, r) \right) \) be a centered covering of $F_p$. Then,

$$\sum_i \mu(B(x_i, r))^q r^t \geq \sum_i r^{\beta q + t} \geq N_r(F_p) r^{\beta q + t}.$$

We conclude that

$$N^q_{p, r}(F_p) (2r)^t \geq 2^t N_r(F_p) r^{\beta q + t}$$

and then

$$L^{q, t}_{\mu}(F_p) \geq 2^{-\beta q} H^{\beta q + t}(F_p).$$

This implies that

$$H^{q, t}_{\mu}(\text{supp } \mu) \geq H^{q, t}_{\mu}(\mathbb{E}^\alpha) \geq H^{q, t}_{\mu}(F_p) \geq 2^{-\beta q} H^{\beta q + t}(F_p) > 0.$$

It therefore follows that $t \leq \mu_b(q)$. Finally, we get

$$\mu_b(q) = \mu_B(q) = \Lambda_\mu(q).$$

\[ \square \]

**Corollary 2.** Assume that $H^{q, \Lambda_\mu(q)}_{\mu}(\text{supp } \mu) > 0$ hold for all $q \in \mathbb{R}$ and that $\Lambda_\mu$ is differentiable at $q$. Let $\alpha = -\Lambda_\mu(q)$, there holds

$$\dim_{MB}(E(\alpha)) = \dim_{MB}(E(\alpha)) = \mu_b^*(\alpha) = \mu_B^*(\alpha) = \Lambda_\mu^*(\alpha).$$

**Remark 4.** The results of Theorems 6, 7 and Corollary 2 hold if we replace the multifractal function $\Lambda_\mu$ by the function $\mu_b$.

Now, we deal with the case where the lower and upper multifractal Hewitt-Stromberg functions $\mu_b$ and $\mu_B$ do not necessarily coincide.

**Theorem 8.** Let $q \in \mathbb{R}$ and $\mu$ be a compact supported Borel probability measure on $\mathbb{R}^n$.

1. If the multifractal function $\mu_b$ is differentiable at $q$, then, provided that $\mu_b^*\left( -b_\mu(q) \right) \geq 0$ and $H^{q, \mu_b(q)}(E\left( -b_\mu(q) \right)) > 0$, we have

$$\dim_H E\left( -b_\mu(q) \right) = \dim_{MB} E\left( -b_\mu(q) \right) = \mu_b^*\left( -b_\mu(q) \right).$$

2. If the multifractal function $\mu_B$ is differentiable at $q$, then, provided that $\mu_B^*\left( -B_\mu(q) \right) \geq 0$ and $P^{q, \mu_B(q)}(E\left( -B_\mu(q) \right)) > 0$, we have

$$\dim_P E\left( -B_\mu(q) \right) = \dim_{MB} E\left( -B_\mu(q) \right) = \mu_B^*\left( -B_\mu(q) \right).$$

**Proof.** The proof is similar to the one of Theorem 5. \[ \square \]

4. **Examples**

In this section, more motivations and examples related to these concepts, will be discussed.
4.1. Example 1. The classical multifractal formalism has been proved rigorously for random and non-random self-similar measures [42, 43], for self-affine measures [10, 45], for quasi self-similar measures [41], for quasi-Bernoulli measures [10], for graph directed self-conformal measures [42] and for some Moran measures [61, 62]. Specifically, we have

\[ b_\mu(q) = b_\mu(q) = B_\mu(q) = B_\mu(q) \]

and for some \( \alpha \geq 0 \), we get

\[ f_\mu(\alpha) = f_\mu(\alpha) = F_\mu(\alpha) = F_\mu(\alpha) = b_\mu^*(\alpha) = b_\mu^*(\alpha) = B_\mu^*(\alpha) = B_\mu^*(\alpha) \]

4.2. Example 2: Multifractal formalism of homogeneous Moran measures. We will start by defining the homogeneous Moran sets. Let \( \{n_k\}_k \) and \( \{\Phi_k\}_{k \geq 1} \) be respectively two sequences of positive integers and positive vectors such that

\[ \Phi_k = \left( c_{k1}, c_{k2}, \ldots, c_{kn_k} \right), \quad \sum_{j=1}^{n_k} c_{kj} \leq 1, \ k \in \mathbb{N}. \]

For any \( m, k \in \mathbb{N} \), such that \( m \leq k \), let

\[ D_{m,k} = \left\{ (i_m, i_{m+1}, \ldots, i_k) \mid 1 \leq i_j \leq n_j, m \leq j \leq k \right\} \]

and

\[ D_k = D_{1,k} = \left\{ (i_1, i_2, \ldots, i_k) \mid 1 \leq i_j \leq n_j, 1 \leq j \leq k \right\}. \]

We also set

\[ D_0 = \emptyset \quad \text{and} \quad D = \cup_{k \geq 0} D_k, \]

Considering \( \sigma = (i_1, i_2, \ldots, i_k) \in D_k \), \( \tau = (j_1, j_2, \ldots, j_m) \in D_{k+1,m} \), we set

\[ \sigma * \tau = (i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_m). \]

**Definition 1.** Let \( X \) be a complete metric space and \( I \subseteq X \) a compact set with no empty interior (for convenience, we assume that the diameter of \( I \) is 1). The collection \( \mathcal{F} = \{I_\sigma, \sigma \in D\} \) of subsets of \( I \) is said to have a homogeneous Moran structure, if it satisfies the following conditions (MSC):

- **a:** \( I_0 = I \).
- **b:** For all \( k \geq 1 \), \( (i_1, i_2, \ldots, i_{k-1}) \in D_{k-1} \), \( I_{i_1i_2 \ldots i_{k-1}} \) is a subset of \( I_{i_1i_2 \ldots i_{k-1}} \) and
  \[ I^*_{i_1i_2 \ldots i_{k-1}, i_k} \cap I^*_{i_1i_2 \ldots i_{k-1}, i'_k} = \emptyset, \quad 1 \leq i_k < i'_k \leq n_k, \]
  where \( I^* \) denotes the interior of \( I \).
- **c:** For all \( k \geq 1 \) and \( 1 \leq j \leq n_k \), taking \( (i_1, i_2, \ldots, i_{k-1}, j) \in D_k \), we have
  \[ 0 < c_{kj} = c_{i_1i_2 \ldots i_{k-1}j} = \frac{|I_{i_1i_2 \ldots i_{k-1}j}|}{|I_{i_1i_2 \ldots i_{k-1}}|} < 1, \quad k \geq 2, \]
  where \( |I| \) denotes the diameter of \( I \).

Suppose that \( \mathcal{F} \) is a collection of subsets of \( I \) having a homogeneous Moran structure. We call \( E = \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} I_\sigma \), a homogeneous Moran set determined by \( \mathcal{F} \), and call \( \mathcal{F}_k = \{\sigma \mid \sigma \in D_k\} \) the \( k \)-order fundamental sets of \( E \). \( I_\sigma \) is called the original set of \( E \). We assume \( \lim_{k \to \infty} \max_{\sigma \in D_k} |I_\sigma| = 0 \). Then, for all \( i \in D \), the set \( \bigcap_{n \geq 1} I_{i_{1i_2 \ldots i_n}} \) is a single point. We use the abbreviation \( w\|_k \) for the first \( k \) elements of the sequence

\[ w = (i_1, i_2, \ldots, i_k, \ldots) \in D, \quad I_k(w) = I_{w\|_k} = I_{i_1i_2 \ldots i_k}. \]

Here, we consider a class of homogeneous Moran sets \( E \) which satisfy a special property called the strong separation condition (SSC), i.e., take \( I_\sigma \in \mathcal{F} \). Let \( I_{(i_1, i_2, \ldots, i_{k+1})} \) be the \( n_{k+1} \) basic intervals of order \( k+1 \) contained in \( I_\sigma \) arranged from the left to the right. Then we assume that for all \( 1 \leq i \leq n_{k+1} - 1 \),

\[ \text{dist}(I_{\sigma i_1}, I_{\sigma(i+1)}) \geq \delta_k |I_\sigma|, \quad \text{for all} \ i \neq j, \]
where \((\delta_k)_k\) is a sequence of positive real numbers, such that

\[0 < \delta = \inf_k \delta_k.\]

We now define a Moran measure. Let \(\left\{ p_{i,j} \right\}_{j=1}^{n_i}\) be the probability vectors, i.e. \(p_{i,j} > 0\) and \(\sum_{j=1}^{n_i} p_{i,j} = 1\) \((i = 1, 2, 3, \ldots)\), suppose that \(p_0 = \inf \{p_{i,j}\} > 0\). Let \(\mu\) be a mass distribution on \(E\), such that for any \(I_{\sigma}\) \((\sigma \in D_k)\)

\[\mu(I_{\sigma}) = p_{1,\sigma_1}p_{1,\sigma_2} \cdots p_{1,\sigma_k}\quad \text{and} \quad \mu \left( \sum_{\sigma \in D_k} I_{\sigma} \right) = 1,\]

we call \(\mu\) be Moran measure.

Finally we define an auxiliary function \(\beta_k(q)\) as follows: for all \(k \geq 1\) and \(q \in \mathbb{R}\), there is a unique number \(\beta_k(q)\) satisfying

\[
\sum_{\sigma \in D_k} p_0^k |I_{\sigma}|^{\beta_k(q)} = 1.
\]

Set

\[
\underline{\beta}(q) = \liminf_{k \to +\infty} \beta_k(q) \quad \text{and} \quad \overline{\beta}(q) = \limsup_{k \to +\infty} \beta_k(q).
\]

**Theorem 9.** Suppose that \(E\) is a homogeneous Moran set satisfying (SSC) and \(\mu\) is the Moran measure on \(E\),

1. then for all \(q \in \mathbb{R}\),

\[
b_\mu(q) = b_\mu(q) = \Theta_\mu^q(\text{supp } \mu) = \underline{\beta}(q)
\]

and

\[
B_\mu(q) = B_\mu(q) = N_\mu^q(\text{supp } \mu) = \overline{\beta}(q).
\]

2. Suppose that \(\beta'(q)\) exists and for this real number \(q\)

(a) there is \(k_0 \in \mathbb{N}\) such that \(\underline{\beta}(q) \leq \beta_k(q)\) for all \(k \geq k_0\), or

(b) there is some \(c > 0\) and \(n_0 \in \mathbb{N}\) such that \(\beta_{k_i}(q) - \underline{\beta}(q) \leq \frac{c}{k_i}\) for all \(k_i \geq n_0\) with \(\beta_{k_i}(q) < \underline{\beta}(q)\).

(c) \(\underline{\beta}(q)\) is smooth. Then there exist numbers \(0 \leq \underline{\alpha} \leq \overline{\alpha}\) such that

\[
f_\mu(\alpha) = f_\mu(\alpha) = \begin{cases} 
b_{\mu}^*(\alpha) = b_{\mu}^*(\alpha) = \overline{\beta}(\alpha), & \text{if } \alpha \in (\underline{\alpha}, \overline{\alpha}) \\
0, & \text{if } \alpha \notin (\underline{\alpha}, \overline{\alpha}). \end{cases}
\]

3. Suppose that \(\overline{\beta}'(q)\) exists and for this real number \(q\)

(a) there is \(k_0 \in \mathbb{N}\) such that \(\overline{\beta}(q) \geq \beta_k(q)\) for all \(k \geq k_0\), or

(b) there is some \(c > 0\) and \(n_0 \in \mathbb{N}\) such that \(\beta_{k_i}(q) - \overline{\beta}(q) \leq \frac{c}{k_i}\) for all \(k_i \geq n_0\) with \(\beta_{k_i}(q) < \overline{\beta}(q)\).

(c) \(\overline{\beta}(q)\) is smooth. Then there exist numbers \(0 \leq \underline{\gamma} \leq \overline{\gamma}\) such that

\[
F_\mu(\alpha) = F_\mu(\alpha) = \begin{cases} 
B_{\mu}^*(\alpha) = B_{\mu}^*(\alpha) = \overline{\beta}(\alpha), & \text{if } \alpha \in (\underline{\gamma}, \overline{\gamma}) \\
0, & \text{if } \alpha \notin (\underline{\gamma}, \overline{\gamma}). \end{cases}
\]

4. If the limit \(\liminf_{k \to +\infty} \beta_k(q) = \beta(q)\) exists, and for all \(k \geq 1\), \(k(\beta(q) - \beta_k(q)) < +\infty\), suppose that \(\alpha = -\beta'(q)\) exists, then

\[
f_\mu(\alpha) = f_\mu(\alpha) = F_\mu(\alpha) = b_{\mu}(\alpha) = \overline{\beta}(\alpha) = B_{\mu}(\alpha) = B_\mu^*(\alpha).
\]

**Proof.** All of the ideas needed to prove Theorem 9 can be found in [63, 64], Propositions 2, 3 and 4 and Theorem 8. \(\square\)
4.2.1. Moran measures for which the classical multifractal formalism is valid. Let

\[ n_k = \begin{cases} 
2, & \text{k is odd number}, \\
3, & \text{k is even number}.
\end{cases} \]

\[ c_k = \begin{cases} 
1, & \text{k is odd number}, \\
2, & \text{k is even number}.
\end{cases} \]

where \(0 < r_1 < \frac{1}{2}\) and \(0 < r_2 < \frac{1}{3}\). Put

\[ p_{k,j} = \begin{cases} 
p_{1,j}, & \text{k is odd number, } 1 \leq j \leq 2, \\
p_{2,j}, & \text{k is even number, } 1 \leq j \leq 3,
\end{cases} \]

where

\[ \sum_{j=1}^{2} p_{1,j} = 1 \quad \text{and} \quad \sum_{j=1}^{3} p_{2,j} = 1. \]

We therefore conclude that

\[ \beta_k(q) = \begin{cases} 
-\log \sum_{j=1}^{2} p_{1,j}^q - \frac{k-1}{k+1} \log \sum_{j=1}^{3} p_{2,j}^q, & \text{k is odd number}, \\
-\log \sum_{j=1}^{2} p_{1,j}^q - \log \sum_{j=1}^{3} p_{2,j}^q, & \text{k is even number},
\end{cases} \]

and

\[ \beta(q) = \lim_{k \to +\infty} \beta_k(q) = \frac{-\log \sum_{j=1}^{2} p_{1,j}^q - \log \sum_{j=1}^{3} p_{2,j}^q}{\log r_1 + \log r_2}. \]

This clearly implies that \(k(\beta(q) - \beta_k(q)) < +\infty\) and \(\beta'(q)\) exists. Now, it follows immediately from Theorem 9 that

\[ f_{\mu}(\alpha) = f_\mu(\alpha) = F_{\mu}(\alpha) = f_{\mu}(\alpha) = B_{\mu}(\alpha) = B_{\mu}(\alpha) = B_{\mu}(\alpha). \]

4.2.2. Moran measures for which the classical multifractal formalism does not hold. Let \(\{T_k\}_{k \geq 1}\) be a sequence of integers such that

\[ T_1 = 1, \quad T_k < T_{k+1} \quad \text{and} \quad \lim_{k \to +\infty} \frac{T_{k+1}}{T_k} = +\infty. \]

We define the family

\[ n_i = \begin{cases} 
2, & \text{if } T_{2k-1} \leq i < T_{2k}, \\
3, & \text{if } T_{2k} \leq i < T_{2k+1}.
\end{cases} \]

\[ c_i = \begin{cases} 
1, & \text{if } T_{2k-1} \leq i < T_{2k}, \\
2, & \text{if } T_{2k} \leq i < T_{2k+1}.
\end{cases} \]

where \(0 < r_1 < \frac{1}{2}\) and \(0 < r_2 < \frac{1}{3}\). Put

\[ p_{i,j} = \begin{cases} 
p_{1,j}, & \text{if } T_{2k-1} \leq i < T_{2k}, 1 \leq j \leq 2, \\
p_{2,j}, & \text{if } T_{2k} \leq i < T_{2k+1}, 1 \leq j \leq 3,
\end{cases} \]

where

\[ \sum_{j=1}^{2} p_{1,j} = 1 \quad \text{and} \quad \sum_{j=1}^{3} p_{2,j} = 1. \]
We therefore conclude from this
\[ \beta_k(q) = \frac{\log \sum_{\sigma \in D_k} \mu(I_\sigma)^q}{-\log c_1 \ldots c_k}. \]

Finally, if \( N_k \) is the number of integers \( i \leq k \) such that \( p_{i,j} = p_{1,j} \), we have
\[ \beta_k(q) = -\frac{N_k}{k} \log (p_{1,1}^q + p_{1,2}^q) + (1 - \frac{N_k}{k}) \log (p_{2,1}^q + p_{2,2}^q + p_{2,3}^q). \]

Observing that
\[ \lim k \to +\infty \frac{N_k}{k} = 0 \quad \text{and} \quad \lim k \to +\infty \frac{N_k}{k} = 1. \]

We can then conclude that
\[ \lim k \to +\infty \beta_k(q) = \inf \left\{ \frac{\log (p_{1,1}^q + p_{1,2}^q)}{-\log r_1}, \frac{\log (p_{2,1}^q + p_{2,2}^q + p_{2,3}^q)}{-\log r_2} \right\}, \]
and
\[ \lim k \to +\infty \beta_k(q) = \sup \left\{ \frac{\log (p_{1,1}^q + p_{1,2}^q)}{-\log r_1}, \frac{\log (p_{2,1}^q + p_{2,2}^q + p_{2,3}^q)}{-\log r_2} \right\}. \]

It results that for \( 0 < q < 1 \), we have
\[ b_\mu(q) = B_\mu(q) = \beta(q) = \lim k \to +\infty \beta_k(q) = \frac{\log (p_{1,1}^q + p_{1,2}^q)}{-\log r_1} < \frac{\log (p_{2,1}^q + p_{2,2}^q + p_{2,3}^q)}{-\log r_2}. \]

and, for \( q < 0 \) or \( q > 1 \),
\[ b_\mu(q) = B_\mu(q) = \beta(q) = \lim k \to +\infty \beta_k(q) = \frac{\log (p_{1,1}^q + p_{1,2}^q)}{-\log r_1} < \frac{\log (p_{2,1}^q + p_{2,2}^q + p_{2,3}^q)}{-\log r_2}. \]

4.3. Example 3. In the following, we give an example of a measure for which the lower and upper multifractal Hewitt-Stromberg functions are different and the Hausdorff and packing dimensions of the level sets of the local Hölder exponent \( E(\alpha) \) are given by the Legendre transform respectively of lower and upper multifractal Hewitt-Stromberg functions. Take \( 0 < p < \hat{p} \leq 1/2 \) and a sequence of integers
\[ 1 = t_0 < t_1 < \ldots < t_n < \ldots, \quad \text{such that} \quad \lim n \to +\infty \frac{t_{n+1}}{t_n} = +\infty. \]

The measure \( \mu \) assigned to the diadic interval of the n-th generation \( I_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n} \) is
\[ \mu(I_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n}) = \prod_{j=1}^n \varepsilon_j, \]
where
\[ \begin{cases} 
\text{if } t_{2k-1} \leq j < t_{2k} \quad \text{for some } k, \quad \varepsilon_j = p \text{ if } \varepsilon_j = 0, \quad \varepsilon_j = 1 - p \text{ otherwise,} \\
\text{if } t_{2k} \leq j < t_{2k+1} \quad \text{for some } k, \quad \varepsilon_j = \hat{p} \text{ if } \varepsilon_j = 0, \quad \varepsilon_j = 1 - \hat{p} \text{ otherwise.}
\end{cases} \]

Now, for \( q \in \mathbb{R} \) we define,
\[ \tau(q) = \log_2 \left( p^q + (1 - p)^q \right) \quad \text{and} \quad \hat{\tau}(q) = \log_2 \left( \hat{p}^q + (1 - \hat{p})^q \right). \]

It results from [11, 13] that
\[ \begin{cases} 
b_\mu(q) = B_\mu(q) = \tau(q) < \hat{\tau}(q) = B_\mu(q) = B_\mu(q), \quad \text{for } 0 < q < 1, \\
b_\mu(q) = B_\mu(q) = \hat{\tau}(q) < \tau(q) = B_\mu(q) = B_\mu(q), \quad \text{for } q < 0 \text{ or } q > 1.
\end{cases} \]

Then we have the following result,

**Theorem 10.** Let \( \alpha \geq 0 \).
(1) For \( \alpha \in \left( -\log_2(1 - \hat{p}), -\log_2(\hat{p}) \right) \), then we have

\[
f_\mu(\alpha) = f_\mu(\alpha) = b_\mu^*(\alpha) = b_\mu^*(\alpha).
\]

(2) For \( \alpha \in \left( -\log_2(1 - \hat{p}), -\log_2(\hat{p}) \right) \) \& \( \left( -B_{\mu,+}^0(0), -B_{\mu,0}^0(0) \right) \cup \left( -B_{\mu,+}^0(1), -B_{\mu,-}^0(1) \right) \), we have

\[
F_\mu(\alpha) = F_\mu(\alpha) = B_{\mu,+}^*(\alpha) = B_{\mu,+}^*(\alpha).
\]

**Proof.** All of the ideas needed to prove this theorem can be found in [11, Proposition 9], Propositions 2, 3 and 4 and Theorem 8. \( \Box \)

4.4. **Example 4.** Given a class of exact dimensional measures (inhomogeneous multinomial measures) whose support is the whole interval \([0, 1]\), the multifractal functions \(b_\mu, b_{\mu}, B_\mu\) and \(B_{\mu}\) are analytic and agree at two points only 0 and 1 (for more details, see [56]). These measures satisfy our multifractal formalism in the sense that, for \( \alpha \) in some interval, the Hausdorff dimension of the level sets \(E(\alpha)\) is given by the Legendre transform of lower multifractal Hewitt-Stromberg function and their packing dimension by the Legendre transform of the upper multifractal Hewitt-Stromberg function. More specifically,

\[
b_\mu(q) = b_\mu(q) < B_\mu(q) = B_\mu(q) \quad \text{for all } q \notin \{0, 1\},
\]

\[
f_\mu(\alpha) = f_\mu(\alpha) = b_\mu^*(\alpha) = b_\mu^*(\alpha)
\]

and

\[
F_\mu(\alpha) = F_\mu(\alpha) = B_{\mu,+}^*(\alpha) = B_{\mu,+}^*(\alpha)
\]

for some \( \alpha \).

5. **Open problems**

Motivated by some results and examples developed in [42, 43, 44, 45, 47, 65], we therefore ask the following questions.

(1) Let \( \mu \in \mathcal{P}_D(\mathbb{R}^n), E \subseteq \text{supp } \mu, p, q \in \mathbb{R} \) and \( \alpha \in [0, 1] \). Then, the following problem remains open:

\[
b_{\mu,(1-\alpha)q}^\alpha(E) \leq \alpha B_{\mu}^p(E) + (1 - \alpha) b_{\mu}^q(E) ?
\]

(2) Let \( q \in \mathbb{R} \) and assume that \( B_{\mu}(q) = b_{\mu}(q) \). Are the measures \( H_{\mu}^{q,t} \) and \( P_{\mu}^{q,t} \) proportional, i.e. does there exists a constant \( c_q > 0 \) such that

\[
P_{\mu}^{q,t} \mid_{\text{supp } \mu} = c_q H_{\mu}^{q,t} \mid_{\text{supp } \mu} ?
\]

Even though it seems rather unlikely that the lower and upper multifractal Hewitt-Stromberg measures are proportional in general, the ratio of the measures \( H_{\mu}^{q,t} \mid_{\text{supp } \mu} \) and \( P_{\mu}^{q,t} \mid_{\text{supp } \mu} \) might still be bounded. We therefore ask the following question: Does there exists a number \( 0 < c_q < +\infty \) such that

\[
H_{\mu}^{q,t} \mid_{\text{supp } \mu} \leq P_{\mu}^{q,t} \mid_{\text{supp } \mu} \leq C_q H_{\mu}^{q,t} \mid_{\text{supp } \mu} ?
\]

(3) Let \( q \in \mathbb{R} \) and assume that \( b_{\mu}(q) = b_{\mu}(q) \). Are the measures \( H_{\mu}^{q,t} \) and \( H_{\mu}^{q,t} \) proportional, i.e. does there exists a constant \( C_q > 0 \) such that

\[
H_{\mu}^{q,t} \mid_{\text{supp } \mu} = C_q H_{\mu}^{q,t} \mid_{\text{supp } \mu} ?
\]

Even though it seems rather unlikely that the multifractal Hausdorff measure and the lower multifractal Hewitt-Stromberg measure are proportional in general, the ratio of the measures \( H_{\mu}^{q,t} \mid_{\text{supp } \mu} \) and \( H_{\mu}^{q,t} \mid_{\text{supp } \mu} \) might still be bounded. We therefore ask the following question: Does there exists a number \( 0 < C_q < +\infty \) such that

\[
H_{\mu}^{q,t} \mid_{\text{supp } \mu} \leq H_{\mu}^{q,t} \mid_{\text{supp } \mu} \leq C_q H_{\mu}^{q,t} \mid_{\text{supp } \mu} ?
\]

(4) Let \( p, q \in \mathbb{R} \) and assume that \( b_{\mu}(q) \) is differentiable at \( p \) and \( q \) with \( b_{\mu}'(p) \neq b_{\mu}'(q) \). Then, the following problem remains open:

\[
H_{\mu}^{p,b_{\mu}(p)} \perp H_{\mu}^{q,b_{\mu}(q)} ?
\]
(5) Let \( p, q \in \mathbb{R} \) and assume that \( B_{\mu}(q) \) is differentiable at \( p \) and \( q \) with \( B_{\mu}^p(p) \neq B_{\mu}^q(q) \). Then, the following problem remains open:

\[
P_{\mu} B_{\mu}(p) \quad \text{and} \quad P_{\mu} B_{\mu}(q)
\]

(6) Is it true that the weaker condition \( b_{\mu}(q) = b_{\mu}(p) \) is sufficient to obtain the conclusion of Theorem 8?

(7) Let \( \mu \in \mathcal{P}_D(\mathbb{R}^n), \nu \in \mathcal{P}_D(\mathbb{R}^m) \) and \( q, s, t \in \mathbb{R} \). Assume that \( c > 0, E \subseteq \mathbb{R}^n, F \subseteq \mathbb{R}^m, H \subseteq \mathbb{R}^{n+m} \) and \( H(y) = \{ x \colon (x, y) \in H \} \). Then, the following problem remains open:

\[
\int H_{\mu}^{q,s}(H(y)) \, dH_{\nu}^{q,t}(y) \leq c \, H_{\mu \times \nu}^{q,s+t}(H)
\]

\[
H_{\mu \times \nu}^{q,s+t}(E \times F) \leq c \, H_{\mu}^{q,s}(E) \, P_{\nu}^{q,t}(F)
\]

\[
\int H_{\mu}^{q,s}(H(y)) \, dP_{\nu}^{q,t}(y) \leq c \, P_{\mu \times \nu}^{q,s+t}(H)
\]

\[
P_{\mu \times \nu}^{q,s+t}(E \times F) \leq c \, P_{\mu}^{q,s}(E) \, P_{\nu}^{q,t}(F)
\]

and

\[
b_{\mu}^q(E) + b_{\mu}^t(F) \leq b_{\mu \times \nu}^{q,s+t}(E \times F) \leq b_{\mu}^q(E) + b_{\mu}^t(F) \leq B_{\mu \times \nu}^q(E \times F) \leq B_{\mu}^q(E) + B_{\nu}^t(F)\]

(8) The multifractal Hausdorff dimension function \( b_{\mu} \) and the lower multifractal Hewitt-Stromberg function \( b_{\mu} \) do not necessarily coincide. Motivated by the results developed in [40], we conjecture that there exist Borel probability measures \( \mu \) on \( \mathbb{R}^n \) such that

\[
b_{\mu} \neq B_{\mu} = B_{\mu} \quad \text{or} \quad b_{\mu} < B_{\mu} = B_{\mu} < \Lambda_{\mu} \quad \text{for all} \quad q \neq 1.
\]

In particular, this will imply that

\[
f_{\mu}(\alpha) = b_{\mu}^*(\alpha) < f_{\mu}(\alpha) = F_{\mu}(\alpha) = \Lambda_{\mu}(\alpha) = b_{\mu}^*(\alpha) = B_{\mu}^*(\alpha) \quad \text{for some} \quad \alpha \geq 0.
\]

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