Renormalization and Induced Gauge Action on a Noncommutative Space

Harald Grosse** and Michael Wohlgenannt***

Faculty of Physics, University of Vienna
Boltzmanngasse 5, A-1090 Vienna, Austria

Abstract

Field theories on deformed spaces suffer from the IR/UV mixing and renormalization is generically spoiled. In work with R. Wulkenhaar, one of us realized a way to cure this disease by adding one more marginal operator. We review these ideas, show the application to $\phi^3$ models and use heat kernel expansion methods for a scalar field theory coupled to an external gauge field on a $\theta$-deformed space and derive noncommutative gauge actions.

---

**) Talk given by H. Grosse at the 21st Nishinomiya-Yukawa Memorial Symposium Noncommutative geometry and quantum spacetime in physics, Nishinomiya and Kyoto (Japan), 2006.

**) Email:harald.grosse@univie.ac.at

***) Supported by the Austrian Science Fund, project P18657-N16. Email: michael.wohlgenannt@univie.ac.at
§1. Introduction

Four-dimensional quantum field theory suffers from infrared and ultraviolet divergences as well as from the divergence of the renormalized perturbation expansion. Despite the impressive agreement between theory and experiments and despite many attempts, these problems are not settled and remain a big challenge for theoretical physics. Furthermore, attempts to formulate a quantum theory of gravity have not yet been fully successful. It is astonishing that the two pillars of modern physics, quantum field theory and general relativity, seem to be incompatible. This convinced physicists to look for more general descriptions: After the formulation of supersymmetry and supergravity, string theory was developed, and anomaly cancellation forced the introduction of six additional dimensions. On the other hand, loop gravity was formulated, and led to spin networks and space-time foams. Both approaches are not fully satisfactory. A third impulse came from noncommutative geometry developed by Alain Connes, providing a natural interpretation of the Higgs effect at the classical level. This finally led to noncommutative quantum field theory, which is the subject of this contribution. It allows to incorporate fluctuations of space into quantum field theory. There are of course relations among these three developments. In particular, the field theory limit of string theory leads to certain noncommutative field theory models, and some models defined over fuzzy spaces are related to spin networks.

The argument that space-time should be modified at very short distances goes back to Schrödinger and Heisenberg. Noncommutative coordinates appeared already in the work of Peierls for the magnetic field problem, and are obtained after projecting onto a particular Landau level. Pauli communicated this to Oppenheimer, whose student Snyder\textsuperscript{1} wrote down the first deformed space-time algebra preserving Lorentz symmetry. After the development of noncommutative geometry by Connes,\textsuperscript{2} it was first applied in physics to the integer quantum Hall effect. Gauge models on the two-dimensional noncommutative tori were formulated, and the relevant projective modules over this space were classified.

Through interactions with John Madore one of us (H.G.) realized that such Fuzzy geometries allow to obtain natural cutoffs for quantum field theory.\textsuperscript{3} This line of work was further developed together with Peter Prešnajder and Ctirad Klímač.\textsuperscript{4} At almost the same time, Filk\textsuperscript{5} developed his Feynman rules for the canonically deformed four-dimensional field theory, and Doplicher, Fredenhagen and Roberts\textsuperscript{6} published their work on deformed spaces. The subject experienced a major boost after one realized that string theory leads to noncommutative field theory under certain conditions,\textsuperscript{7,8} and the subject developed very rapidly; see e.g.\textsuperscript{9,10}
§2. Noncommutative Quantum Field Theory

The formulation of Noncommutative Quantum Field Theory (NCFT) follows a dictionary worked out by mathematicians. Starting from some manifold $\mathcal{M}$ one obtains the commutative algebra of smooth functions over $\mathcal{M}$, which is then quantized along with additional structure. Space itself then looks locally like a phase space in quantum mechanics. Fields are elements of the algebra respectively a finitely generated projective module, and integration is replaced by a suitable trace operation.

Following these lines, one obtains field theory on quantized (or deformed) spaces, and Feynman rules for a perturbative expansion can be worked out. However some unexpected features such as IR/UV mixing arise upon quantization, which are described below. In 2000 Minwalla, van Raamsdonk and Seiberg realized\(^{11}\) that perturbation theory for field theories defined on the Moyal plane faces a serious problem. The planar contributions show the standard singularities which can be handled by a renormalization procedure. The nonplanar one loop contributions are finite for generic momenta, however they become singular at exceptional momenta. The usual UV divergences are then reflected in new singularities in the infrared, which is called IR/UV mixing. This spoils the usual renormalization procedure: Inserting many such loops to a higher order diagram generates singularities of any inverse power. Without imposing a special structure such as supersymmetry, the renormalizability seems lost; see also.\(^{12,13}\)

However, progress was made recently, when H.G. and R. Wulkenhaar were able to give a solution of this problem for the special case of a scalar four-dimensional theory defined on the Moyal-deformed space $\mathbb{R}^4$.\(^{14}\) The IR/UV mixing contributions were taken into account through a modification of the free Lagrangian by adding an oscillator term with parameter $\Omega$, which modifies the spectrum of the free Hamiltonian. The harmonic oscillator term was obtained as a result of the renormalization proof. The model fulfills then the Langmann-Szabo duality\(^{15}\) relating short distance and long distance behavior. The proof follows ideas of Polchinski. There are indications that a constructive procedure might be possible and give a nontrivial $\phi^4$ model, which is currently under investigation.\(^{16}\) At $\Omega = 1$ the model becomes self-dual, and we are presently studying them in more detail. The noncommutative Euclidean selfdual $\phi^3$ model can be solved using the relationship to the Kontsevich matrix model. This relation holds for any even dimension, but a renormalization still has to be applied. In $D = 2$ and $D = 4$ dimensions the models are super-renormalizable.\(^{17,18}\) In $D = 6$ dimensions, the model is only renormalizable and details are presently worked out.\(^{19}\)

Nonperturbative aspects of NCFT have also been studied in recent years. The most significant and surprising result is that the IR/UV mixing can lead to a new phase denoted
as “striped phase”,
where translational symmetry is spontaneously broken. The existence of such a phase has indeed been confirmed in numerical studies. To understand better the properties of this phase and the phase transitions, further work and better analytical techniques are required, combining results from perturbative renormalization with nonperturbative techniques. Here a particular feature of scalar NCFT is very suggestive: the field can be described as a hermitian matrix, and the quantization is defined nonperturbatively by integrating over all such matrices. This provides a natural starting point for nonperturbative studies. In particular, it suggests and allows to apply ideas and techniques from random matrix theory.

Remarkably, gauge theories on quantized spaces can also be formulated in a similar way. The action can be written as multi-matrix models, where the gauge fields are encoded in terms of matrices which can be interpreted as “covariant coordinates”. The field strength can be written as commutator, which induces the usual kinetic terms in the commutative limit. Again, this allows a natural nonperturbative quantization in terms of matrix integrals.

In the last section, we discuss a formulation of gauge theories related to the approach to NCFT presented here. We start with noncommutative $\phi^4$ theory on canonically deformed Euclidean space with additional oscillator potential. The oscillator potential modifies the free theory and solves the IR/UV mixing problem. We couple an external gauge field to the scalar field via introducing covariant coordinates. As in the classical case, we extract the dynamics of the gauge field from the divergent contributions to the 1-loop effective action. The effective action is calculated using a heat kernel expansion. The technical details are going are presented in.

§3. Renormalization of $\phi^4$-theory on the 4D Moyal plane

We briefly sketch the methods used in proving the renormalizability for scalar field theory defined on the 4-dimensional quantum plane $\mathbb{R}^4_\theta$, with commutation relations

$$[x_\mu, x_\nu] = i \theta_{\mu\nu}. \quad (3.1)$$

The IR/UV mixing was taken into account through a modification of the free Lagrangian, by adding an oscillator term which modifies the spectrum of the free Hamiltonian:

$$S = \int d^4x \left( \frac{1}{2} \partial_\mu \phi \ast \partial^\mu \phi + \frac{\Omega^2}{2} (\vec{x}_\mu \phi) \ast (\vec{x}^\mu \phi) + \frac{\mu^2}{2} \phi \ast \phi + \frac{\lambda}{4!} \phi \ast \phi \ast \phi \ast \phi \right) (x). \quad (3.2)$$

Here, $\vec{x}_\mu = 2(\theta^{-1})_{\mu\nu} x^\nu$ and $\ast$ is the Moyal star product

$$(a \ast b)(x) := \int d^4y \frac{d^4k}{(2\pi)^4} a(x+\frac{1}{2}k) b(x+y) e^{iky}, \quad \theta_{\nu\mu} = -\theta_{\mu\nu} \in \mathbb{R}. \quad (3.3)$$
The model is covariant under the Langmann-Szabo duality relating short distance and long distance behavior. At $\Omega = 1$ the model becomes self-dual, and connected to integrable models.

The renormalization proof proceeds by using a matrix base, which leads to a dynamical matrix model of the type:

$$S[\phi] = (2\pi\theta)^2 \sum_{m,n,k,l \in \mathbb{N}^2} \left( \frac{1}{2} \phi_{mn} \Delta_{mn,kl} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \right), \quad (3.4)$$

where

$$\Delta_{m^1 n^1, k^1 l^1} = \left( \mu^2 + \frac{2+2\Omega^2}{\theta} (m^1 + n^1 + m^2 + n^2 + 2) \right) \delta_{n^1, m^1} \delta_{n^2, m^2} \delta_{l^1, k^1} \delta_{l^2, k^2}$$

$$- \frac{2+2\Omega^2}{\theta} \left( \sqrt{k^1 l^1} \delta_{n^1 +1, k^1} \delta_{m^1 +1, l^1} + \sqrt{m^1 n^1} \delta_{n^1 -1, k^1} \delta_{m^1 -1, l^1} \right) \delta_{n^2, l^2} \delta_{l^2, k^2}$$

$$- \frac{2+2\Omega^2}{\theta} \left( \sqrt{k^2 l^2} \delta_{n^2 +1, k^2} \delta_{m^2 +1, l^2} + \sqrt{m^2 n^2} \delta_{n^2 -1, k^2} \delta_{m^2 -1, l^2} \right) \delta_{n^1, k^1} \delta_{m^1, l^1}. \quad (3.5)$$

The interaction part becomes a trace of product of matrices, and no oscillations occur in this basis. The propagator obtained from the free part is quite complicated, in 4 dimensions it is:

$$G_{m^1 n^1, k^1 l^1} = \frac{\theta}{2(1+\Omega)^2} \sum_{v^1 = \frac{|m^1 + n^1|}{2}}^{m^1 + n^1} \sum_{v^2 = \frac{|m^2 + n^2|}{2}}^{m^2 + n^2} B \left( 1 + \frac{\mu^2 \theta}{8\Omega} + \frac{1}{2} (m^1 + k^1 + m^2 + k^2) - v^1 - v^2, 1 + 2v^1 + 2v^2 \right)$$

$$\times \frac{1}{2} \left( 1 + 2v^1 + 2v^2, 1 + 2v^1 + 2v^2 \right) \left( \frac{1 - \Omega^2}{(1 + \Omega)^2} \right) \left( 1 + \Omega, \frac{1}{1 + \Omega} \right)$$

$$\times \prod_{i=1}^{2} \delta_{m^i + k^i, n^i + l^i} \sqrt{\left( \frac{n^i}{v^1 + \frac{n^i - k^i}{2}} \right) \left( \frac{k^i}{v^1 + \frac{k^i - n^i}{2}} \right) \left( \frac{m^i}{v^1 + \frac{m^i - l^i}{2}} \right) \left( \frac{l^i}{v^1 + \frac{l^i - m^i}{2}} \right)}. \quad (3.6)$$

These propagators (in 2 and 4 dimensions) show asymmetric decay properties:

$$\theta^{-1} \Delta_{m^1 n^1, k^1 l^1} \quad (3.7)$$

$$\Omega = 0.1 \quad \mu = 0$$
They decay exponentially on particular directions (in \( l \)-direction in the picture), but have power law decay in others (in \( \alpha \)-direction in the picture). These decay properties are crucial for the perturbative renormalizability of the models.

The proof in\(^{14,31}\) follows the ideas of Polchinski.\(^{32}\) The quantum field theory corresponding to the action (3.4) is defined — as usual — by the partition function

\[
Z[J] = \int \left( \prod_{m,n} d\phi_{mn} \right) \exp \left( -S[\phi] - \sum_{m,n} \phi_{mn} J_{mn} \right). \tag{3.8}
\]

The strategy due to Wilson\(^{33}\) consists in integrating in the first step only those field modes \( \phi_{mn} \) which have a matrix index bigger than some scale \( \theta \Lambda^2 \). The result is an effective action for the remaining field modes which depends on \( \Lambda \). One can now adopt a smooth transition between integrated and not integrated field modes so that the \( \Lambda \)-dependence of the effective action is given by a certain differential equation, the Polchinski equation.

Now, renormalization amounts to prove that the Polchinski equation admits a regular solution for the effective action which depends on only a finite number of initial data. This requirement is hard to satisfy because the space of effective actions is infinite dimensional and as such develops an infinite dimensional space of singularities when starting from generic initial data.

The Polchinski equation can be iteratively solved in perturbation theory where it can be graphically written as

\[
\Lambda \frac{\partial}{\partial \Lambda} = \frac{1}{2} \sum_{m,n,k,l} \sum_{N_1=1}^{N-1} \frac{\partial}{\partial \Lambda} + \frac{1}{4\pi \theta} \sum_{m,n,k,l} \frac{\partial}{\partial \Lambda}. \tag{3.9}
\]

The graphs are graded by the number of vertices and the number of external legs. Then, to the \( \Lambda \)-variation of a graph on the lhs there only contribute graphs with a smaller number of vertices and a bigger number of legs. A general graph is thus obtained by iteratively adding a propagator to smaller building blocks, starting with the initial \( \phi^4 \)-vertex, and integrating over \( \Lambda \). Here, these propagators are differentiated cut-off propagators \( Q_{mn;kl}(\Lambda) \) which vanish (for an appropriate choice of the cut-off function) unless the maximal index is in the interval \([\theta \Lambda^2, 2\theta \Lambda^2]\). As the field carry two matrix indices and the propagator four of them, the graphs are ribbon graphs familiar from matrix models.
It can then be shown that cut-off propagator $Q(\Lambda)$ is bounded by $\frac{C}{\Lambda^\theta}$. This was achieved numerically in\textsuperscript{14} and later confirmed analytically in\textsuperscript{16}. A nonvanishing frequency parameter $\Omega$ is required for such a decay behavior. As the volume of each two-component index $m \in \mathbb{N}^2$ is bounded by $C'\theta^2\Lambda^4$ in graphs of the above type, the power counting degree of divergence is (at first sight) $\omega = 4S - 2I$, where $I$ is the number of propagators and $S$ the number of summation indices.

It is now important to take into account that if three indices of a propagator $Q_{mn;kl}(\Lambda)$ are given, the fourth one is determined by $m + k = n + l$, see (3.6). Then, for simple planar graphs one finds that $\omega = 4 - N$ where $N$ is the number of external legs. But this conclusion is too early, there is a difficulty in presence of completely inner vertices, which require additional index summations. The graph

\begin{equation}
\begin{array}{c}
p_1 + q \\
p_2 + q \\
p_1 + m \\
p_3 + q \\
p_2 + l \\
p_3 + m \\
k \\
m \\
q \\
q' \\
q = n_1 + n_3 - q \\
q' = m_2 + r - q \\
r' = n_2 + r - m_3
\end{array}
\end{equation}

entails four independent summation indices $p_1, p_2, p_3$ and $q$, whereas for the powercounting degree $2 = 4 - N = 4S - 5 \cdot 2$ we should only have $S = 3$ of them. It turns out that due to the quasi-locality of the propagator (the exponential decay in $l$-direction in (3.7)), the sum over $q$ for fixed $m$ can be estimated without the need of the volume factor.

Remarkably, the quasi-locality of the propagator not only ensures the correct powercounting degree for planar graphs, it also renders all nonplanar graphs superficially convergent. For instance, in the nonplanar graphs

\begin{equation}
\begin{array}{c}
m_4 \\
q \\
q' \\
m_3 \\
m_1 \\
n_1 \\
n_4 \\
m_1 \\
n_2 \\
m_3 \\
n_4 \end{array}
\end{equation}

\begin{equation}
\begin{array}{c}
m_2 \\
q \\
r' \\
m_2 \\
n_1 \\
n_2 \\
n_3 \end{array}
\end{equation}

the summation over $q$ and $q, r$, respectively, is of the same type as over $q$ in (3.10) so that the graphs in (3.11) can be estimated without any volume factor.

After all, we have obtained the powercounting degree of divergence

\begin{equation}
\omega = 4 - N - 4(2g + B - 1)
\end{equation}

for a general ribbon graph, where $g$ is the genus of the Riemann surface on which the graph is drawn and $B$ the number of holes in the Riemann surface. Both are directly determined
by the graph. It should be stressed, however, that although the number (3.12) follows from counting the required volume factors, its proof in our scheme is not so obvious: The procedure consists of adding a new cut-off propagator to a given graph, and in doing so the topology \((B, g)\) has many possibilities to arise from the topologies of the smaller parts for which one has estimates by induction. The proof that in every situation of adding a new propagator one obtains (3.12) is given in.\(^{31}\) Moreover, the boundary conditions for the integration have to be correctly chosen to confirm (3.12), see below.

The powercounting behavior (3.12) is good news because it implies that (in contrast to the situation without the oscillator potential) all nonplanar graphs are superficially convergent. However, this does not mean that all problems are solved: The remaining planar two- and four-leg graphs which are divergent carry matrix indices, and (3.12) suggests that these are divergent independent of the matrix indices. An infinite number of adjusted initial data would be necessary in order to remove these divergences.

Fortunately, a more careful analysis shows that the powercounting behavior is improved by the index jump along the trajectories of the graph. For example, the index jump for the graph (3.10) is defined as \(J = \|k - n\|_1 + \|q - l\|_1 + \|m - q\|_1\). Then, the amplitude is suppressed by a factor of order \(\left(\max(m, n \ldots) \frac{1}{\Theta A^2}\right)^J\) compared with the naive estimation. Thus, only planar four-leg graphs with \(J = 0\) and planar two-leg graphs with \(J = 0\) or \(J = 2\) are divergent (the total jumps is even). For these cases, a discrete Taylor expansion about the graphs with vanishing indices is employed. Only the leading terms of the expansion, i.e. the reference graphs with vanishing indices, are divergent whereas the difference between original graph and reference graph is convergent. Accordingly, in this scheme only the reference graphs must be integrated in a way that involves initial conditions. For example, if the contribution to the rhs of the Polchinski equation (3.9) is given by the graph

\[
\Lambda \frac{\partial}{\partial \Lambda} A_{(2)\text{planar,1PI}}^{(2)\text{planar,1PI}} [A] = \sum_{p \in \mathbb{N}^2} \left( \begin{pmatrix} m & l & m & l & m & l & m & l \\ m & l & m & l & m & l & m & l \\ n & n & n & n & n & n & n & n \end{pmatrix} \right) (A),
\]

(3.13)

the \(A\)-integration is performed as follows:

\[
A_{(2)\text{planar,1PI}}^{(2)\text{planar,1PI}} [A] = \int_{\Lambda'} d\Lambda' \sum_{p \in \mathbb{N}^2} \left( \begin{pmatrix} m & l & m & l & m & l & m & l \\ m & l & m & l & m & l & m & l \\ n & n & n & n & n & n & n & n \end{pmatrix} \right) [A']
\]

\[
+ \int_{\Lambda_R} d\Lambda' \sum_{p \in \mathbb{N}^2} \left( \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right) [A'] + A_{(0,0;0,0;0,0;0,0;0,0)}^{(2,1,0)} [A_R].
\]

(3.14)
Only one initial condition, \( A_{00,00,00,00}^{(2,1,0)} \), is required for an infinite number of planar four-leg graphs (distinguished by the matrix indices). We need one further initial condition for the two-leg graphs with \( J = 2 \) and two more initial condition for the two-leg graphs with \( J = 0 \) (for the leading quadratic and the subleading logarithmic divergence). This is one condition more than in a commutative \( \phi^4 \)-theory, and this additional condition justifies a posteriori our starting point of adding one new term to the action (3.2), the oscillator term \( \Omega \).

Knowing the relevant/marginal couplings, we can compute Feynman graphs with sharp matrix cut-off \( N \). The most important question concerns the \( \beta \)-function appearing in the renormalisation group equation which describes the cut-off dependence of the expansion coefficients \( \Gamma_{m_1n_1;\ldots;m_Nn_N} \) of the effective action when imposing normalisation conditions for the relevant and marginal couplings. We have

\[
\lim_{N \to \infty} \left( N \frac{\partial}{\partial N} + N \gamma + \mu_0^2 \beta_{\mu_0} \frac{\partial}{\partial \mu_0^2} + \beta_{\lambda} \frac{\partial}{\partial \lambda} + \beta_{\Omega} \frac{\partial}{\partial \Omega} \right) \Gamma_{m_1n_1;\ldots;m_Nn_N}[\mu_0, \lambda, \Omega, N] = 0, \tag{3.15}
\]

where
\[
\beta_{\lambda} = N \frac{\partial}{\partial N} \left( \lambda_{\text{phys}} \right), \quad \beta_{\Omega} = N \frac{\partial}{\partial N} \left( \Omega_{\text{phys}} \right),
\beta_{\mu_0} = \frac{\mu_0^2}{\partial N} \left( \mu_{\text{phys}} \right), \quad \gamma = N \frac{\partial}{\partial N} \left( \ln Z_{\text{phys}} \right).
\]

(3.16)

Here, \( Z \) is the wavefunction renormalisation. To one-loop order one finds

\[
\beta_{\lambda} = \frac{\lambda_{\text{phys}}^2 (1 - \Omega_{\text{phys}}^2)}{48\pi^2 \left( 1 + \Omega_{\text{phys}}^2 \right)^3}, \quad \beta_{\Omega} = \frac{\lambda_{\text{phys}} \Omega_{\text{phys}} (1 - \Omega_{\text{phys}}^2)}{96\pi^2 \left( 1 + \Omega_{\text{phys}}^2 \right)^3}, \quad \beta_{\mu_0} = -\frac{\lambda_{\text{phys}} \mu_0^2 (4N \ln(2) + \frac{(8 + 4\mu_0^2 \mu_{\text{phys}}^2) \Omega_{\text{phys}}^2}{(1 + \Omega_{\text{phys}}^2)^2})}{48\pi^2 \theta \mu_{\text{phys}}^2 (1 + \Omega_{\text{phys}}^2)^3}, \quad \gamma = \frac{\lambda_{\text{phys}} \Omega_{\text{phys}}^2}{96\pi^2 \left( 1 + \Omega_{\text{phys}}^2 \right)^3}. \tag{3.17}
\]

Eq. (3.17) shows that the ratio of the coupling constants \( \lambda / \Omega \) remains bounded along the renormalization group flow up to first order. Starting from given small values for \( \Omega_R, \lambda_R \) at \( N_R \), the frequency grows in a small region around \( \ln N_R = 48\pi^2 / \lambda_R \) to \( \Omega \approx 1 \). The coupling constant approaches \( \lambda_{\infty} = \frac{\lambda_{\Omega}}{\Omega_R} \), which can be made small for sufficiently small \( \lambda_R \). This leaves the chance of a nonperturbative construction of the model.

In particular, the \( \beta \)-function vanishes at the self-dual point \( \Omega = 1 \), indicating special properties of the model.

§4. Nontrivial solvable \( \phi^3 \) model

In the 4-dimensional scalar noncommutative \( \phi^3 \) model is considered, with additional oscillator-type potential in order to avoid the problem of IR/UV mixing. The model is
defined by the action\(^{17,18}\)

\[
\tilde{S} = \int_{\mathbb{R}_4^2} \frac{1}{2} \partial_i \phi \partial_i \phi + \frac{\mu^2}{2} \phi^2 + \Omega^2 (\tilde{x}_i \phi)(\tilde{x}_i \phi) + \frac{i\tilde{\lambda}}{3!} \phi^3
\]  

(4.1)
on the 4-dimensional quantum plane. The dynamical object is the scalar field \(\phi = \phi^\dagger\), which is a self-adjoint operator acting on the representation space \(\mathcal{H}\) of the algebra \(\mathcal{A}\). The action is chosen to be written with an imaginary coupling \(i\tilde{\lambda}\), assuming \(\tilde{\lambda}\) to be real. The reason is that for real coupling \(\tilde{\lambda}' = i\tilde{\lambda}\), the potential would be unbounded from above and below, and the quantization would seem ill-defined. The quantization is completely well-defined for imaginary \(i\tilde{\lambda}\), and allows analytic continuation to real \(\tilde{\lambda}' = i\tilde{\lambda}\) in a certain sense which will be made precise below. Therefore we accept for now that the action \(\tilde{S}\) is not necessarily real. Using the commutation relations \(\mathcal{A}\), the derivatives \(\partial_i\) can be written as inner derivatives \(\partial_i f = -i[x_i, f]\). Therefore the action can be written as

\[
\tilde{S} = \int - (\tilde{x}_i \phi \tilde{x}_i \phi - \tilde{x}_i \tilde{x}_i \phi \phi) + \Omega^2 \tilde{x}_i \phi \tilde{x}_i \phi + \frac{\mu^2}{2} \phi^2 + \frac{i\tilde{\lambda}}{3!} \phi^3
\]  

(4.2)
using the cyclic property of the integral. For the “self-dual” point \(\Omega = 1\), this action simplifies further to

\[
\tilde{S} = \int (\tilde{x}_i \tilde{x}_i + \frac{\mu^2}{2})\phi^2 + \frac{i\tilde{\lambda}}{3!} \phi^3 = \text{Tr} \left( \frac{1}{2} J \phi^2 + \frac{i\lambda}{3!} \phi^3 \right).
\]  

(4.3)
Here we replaced the integral by \(\int = (2\pi\theta)^2 \text{Tr}\), and introduce

\[
J = 2(2\pi\theta)^2 \left( \sum_i \tilde{x}_i \tilde{x}_i + \frac{\mu^2}{2} \right), \quad \lambda = (2\pi\theta)^2 \tilde{\lambda}.
\]  

(4.4)
In\(^{17,18}\) it has been shown that noncommutative Euclidean selfdual \(\phi^3\) model can be solved using matrix model techniques, and is related to the KdV hierarchy. This is achieved by rewriting the field theory as Kontsevich matrix model, for a suitable choice of the eigenvalues in the latter. The relation holds for any even dimension, and allows to apply some of the known, remarkable results for the Kontsevich model to the quantization of the \(\phi^3\) model.\(^{36,37}\)

In order to quantize the theory, we need to include a linear counterterm \(-\text{Tr}(i\lambda)\phi\) to the action (the explicit factor \(i\lambda\) is inserted to keep most quantities real), and – as opposed to the 2-dimensional case\(^{17}\) – we must also allow for a divergent shift

\[
\phi \to \phi + i\lambda c
\]  

(4.5)
of the field \(\phi\). These counterterms are necessary to ensure that the local minimum of the cubic potential remains at the origin after quantization. The latter shift implies in particular
that the linear counterterm picks up a contribution $-Tr(i\lambda)(a + cJ)\phi$ from the quadratic term. Therefore the linear term should be replaced by $-Tr(i\lambda)A\phi$ where

$$A = a + cJ,$$

(4.6)

while the other effects of this shift $\phi \to \phi + i\lambda c$ can be absorbed by a redefinition of the coupling constants (which we do not keep track of). We are thus led to consider the action

$$S = Tr\left(\frac{1}{2}J\phi^2 + \frac{i\lambda}{3!}\phi^3 - (i\lambda)A\phi - \frac{1}{3(\lambda i)^2}J^3 - JA\right),$$

(4.7)

involving the constants $i\lambda$, $a$, $c$ and $\mu^2$. The additional constant terms in (4.7) are introduced for later convenience. By suitable shifts in the field $\phi$, one can now either eliminate the linear term or the quadratic term in the action,

$$S = Tr\left(-\frac{1}{2i\lambda}M^2\tilde{\phi} + \frac{i\lambda}{3!}\tilde{\phi}^3\right) = Tr\left(\frac{1}{2}MX^2 + \frac{i\lambda}{3!}X^3 - \frac{1}{3(\lambda i)^2}M^3\right)$$

(4.8)

where

$$\tilde{\phi} = \phi + \frac{1}{i\lambda}J = X + \frac{1}{i\lambda}M$$

(4.9)

and

$$M = \sqrt{J^2 + 2(\lambda i)^2A} = \sqrt{\tilde{J}^2 + 2(\lambda i)^2a - (\lambda i)^4c^2}$$

$$\tilde{J} = J + (\lambda i)^2c.$$  

(4.10)

(4.11)

This has precisely the form of the Kontsevich model.\textsuperscript{37)}

The quantization of the model (4.7) resp. (4.8) is defined by an integral over all Hermitian $N^2 \times N^2$ matrices $\phi$, where $N$ serves as a UV cutoff. The partition function is defined as

$$Z(M) = \int D\tilde{\phi} \exp(-Tr\left(-\frac{1}{2i\lambda}M^2\tilde{\phi} + \frac{i\lambda}{3!}\tilde{\phi}^3\right)) = e^{F(M)},$$

(4.12)

which is a function of the eigenvalues of $M$ resp. $\tilde{J}$. Since $N$ is finite, we can freely switch between the various parametrizations (4.7), (4.8) involving $M$, $J$, $\phi$, or $\tilde{\phi}$. Correlators or “$n$-point functions” are defined through

$$\langle \phi_{i_1j_1}...\phi_{i_nj_n}\rangle = \frac{1}{Z} \int D\phi \exp(-S)\phi_{i_1j_1}...\phi_{i_nj_n},$$

(4.13)

keeping in mind that each $i_n$ denotes a double-index.\textsuperscript{18)}

\textsuperscript{18)} for the quantization, the integral for the diagonal elements is then defined via analytical continuation, and the off-diagonal elements remain hermitian since $J$ is diagonal.
This allows to write down closed expressions for the genus expansion of the free energy, and also for some $n$-point functions by taking derivatives and using the equations of motion. It turns out that the required renormalization is determined by the genus 0 sector only, and can be computed explicitly. As for the renormalization procedure, see.\textsuperscript{17)–19) All contributions in a genus expansion of any $n$-point function correlation function are finite and well-defined for finite coupling. This implies but is stronger than perturbative renormalization. One thus obtains fully renormalized models with nontrivial interaction which are free of IR/UV diseases. All this shows that even though the $\phi^3$ may appear ill-defined at first, it is in fact much better under control than other models.

§5. Induced gauge theory

Since elementary particles are most successfully described by gauge theories it is a big challenge to formulate consistent gauge theories on non-commutative spaces. Let $u$ be a unitary element of the algebra such that the scalar fields $\phi$ transform covariantly:

$$\phi \mapsto u^* \phi u, \quad u \in G.$$ \hspace{1cm} (5.1)

For a purpose which will become clear in the sequel, we rewrite the action (3.2) using $\partial_{\mu} f = -i[\tilde{x}_{\mu}, f]$, and obtain

$$S_0 = \int d^4 x \left( \frac{1}{2} \phi * [\tilde{x}_{\nu}, [\tilde{x}_{\nu}, \phi]]_s + \frac{\Omega^2}{2} \phi * \{\tilde{x}_{\nu}, \{\tilde{x}_{\nu}, \phi\}_s\}_s + \frac{\mu^2}{2} \phi * \phi + \frac{\lambda}{4!} \phi * \phi * \phi \phi \right) (x).$$ \hspace{1cm} (5.2)

The approach employed here makes use of two basic ideas. First, it is well known that the $*$-multiplication of a coordinate - and also of a function, of course - with a field is not a covariant process. The product $x^\mu * \phi$ will not transform covariantly,

$$x^\mu * \phi \nrightarrow u^* x^\mu * \phi u.$$

Functions of the coordinates are not effected by the gauge group. The matter field $\phi$ is taken to be an element of a left module.\textsuperscript{38)} The introduction of covariant coordinates

$$\tilde{X}_\nu = \tilde{x}_\nu + A_\nu$$ \hspace{1cm} (5.3)

finds a remedy to this situation.\textsuperscript{39)} The gauge field $A_\mu$ and hence the covariant coordinates transform in the following way:

$$A_\mu \mapsto i u^* \partial_\mu u + u^* A_\mu * u,$$

$$\tilde{X}_\mu \mapsto u^* \tilde{X}_\mu * u.$$ \hspace{1cm} (5.4)
Using covariant coordinates we can construct an action invariant under gauge transformations. This action defines the model for which we shall study the heat kernel expansion:

\[
S = \int d^4x \left( \frac{1}{2} \phi \star [\tilde{X}_\nu, [\tilde{X}^\nu, \phi]]_\star + \frac{\Omega^2}{2} \phi \star \{X^\nu, \{X_\nu, \phi\}_\star\}_\star 
+ \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right)(x) .
\] (5.5)

Secondly, we apply the heat kernel formalism. The gauge field \( A_\mu \) is an external, classical gauge field coupled to \( \phi \). In the following sections, we will explicitly calculate the divergent terms of the one-loop effective action. In the classical case, the divergent terms determine the dynamics of the gauge field.\(^{(28), (40), (41)}\) There have already been attempts to generalise this approach to the non-commutative realm; for non-commutative \( \phi^4 \) theory see.\(^{(42), (43)}\) First steps towards gauge kinetic models have been done in.\(^{(44)–(46)}\) However, the results there are not completely comparable, since we have modified the free action and expand around \( -\nabla^2 + \Omega^2 \tilde{x}^2 \) rather than \( -\nabla^2 \).

Recently, A. de Goursac, J.-Chr. Wallet and R. Wulkenhaar\(^{(47)}\) published a paper, where they also computed the effective action for a similar model in coordinate space. They have evaluated relevant Feynman diagrams and obtained the same results as presented here.

5.1. The model

The expansion of the action (6.5) yields

\[
S = S_0 + \int d^4x \frac{1}{2} \phi \star \left( 2iA^\nu \star \partial_\nu \phi - 2i\partial_\nu \phi \star A^\nu \\
+ 2(1 + \Omega^2)A_\nu \star A^\nu \star \phi - 2(1 - \Omega^2)A_\nu \star \phi \star A^\nu \\
+ 2\Omega^2 \{\tilde{x}_\nu, (A^\nu \star \phi + \phi \star A^\nu)\}_\star \right),
\] (5.6)

where \( S_0 \) denotes the free part of the action \((3.2)\) independent of \( A \). Now we compute the second derivative:

\[
\frac{\delta^2 S}{\delta \phi^2}(\psi) = \frac{2}{\theta} H^0 \psi + \frac{\lambda}{3!} \left( \phi \star \phi \star \psi + \psi \star \phi \star \phi + \phi \star \phi \star \psi \right) \\
+ i\partial_\nu A^\nu \star \psi - i\psi \star \partial_\nu A^\nu + 2iA^\nu \star \partial_\nu \psi - 2i\partial_\nu \psi \star A^\nu \\
+ (1 + \Omega^2)A_\nu \star A^\nu \star \psi - 2(1 - \Omega^2)A_\nu \star \psi \star A^\nu + (1 + \Omega^2)\psi \star A_\nu \star A^\nu \\
+ 2\Omega^2 \left( \tilde{x}_\nu \cdot (A^\nu \star \psi + \psi \star A^\nu) + \tilde{x}_\nu \cdot \psi \star A^\nu + \psi \star (\tilde{x}_\nu \cdot \psi) \right),
\] (5.7)

where

\[
H^0 = \frac{\theta}{2} \left( -\frac{\partial^2}{\partial x_\nu \partial x^{\nu}} + 4\Omega^2 \tilde{x}_\nu \tilde{x}^{\nu} + \mu^2 \right).
\] (5.8)
The oscillator term is considered as a modification of the free theory. We use the the following parametrisation of $\theta_{\mu\nu}$:

$$
(\theta_{\mu\nu}) = \begin{pmatrix}
0 & \theta \\
-\theta & 0 \\
0 & \theta \\
-\theta & 0
\end{pmatrix},
(\theta_{\mu\nu}^{-1}) = \begin{pmatrix}
0 & -1/\theta \\
1/\theta & 0 \\
0 & -1/\theta \\
1/\theta & 0
\end{pmatrix}.
$$

We expand the fields in the matrix base of the Moyal plane,

$$
A^\nu(x) = \sum_{p,q \in \mathbb{N}^2} A^\nu_{pq} f_{pq}(x), \phi(x) = \sum_{p,q \in \mathbb{N}^2} \phi_{pq} f_{pq}(x), \psi(x) = \sum_{p,q \in \mathbb{N}^2} \psi_{pq} f_{pq}(x). 
$$

This choice of basis simplifies the calculations. In the end, we will again represent the results in the $x$-basis. Usefull properties of this basis are reviewed in the Appendix of.  

We obtain for (5.7):

$$
\frac{\theta}{2} \left( \frac{\delta^2 S}{\delta \phi^2} (f_{mn}) \right)_{lk} = H^0_{kl;mn} + \frac{\theta}{2} V_{kl;mn} \equiv H_{kl;mn},
$$

where

$$
H^0_{mn;kl} = \left( \frac{\mu^2 \theta}{2} + (1 + \Omega^2)(n^1 + m^1 + 1) + (1 + \Omega^2)(n^2 + m^2 + 1) \right) \delta_{n^1, k^1} \delta_{m^1, l^1} \delta_{n^2, k^2} \delta_{m^2, l^2}
$$

$$
- (1 - \Omega^2) \left( \sqrt{m^1 n^1} \delta_{n^1, 1, k^1} \delta_{m^1, 1, l^1} + \sqrt{m^1 n^1} \delta_{n^1, -1, k^1} \delta_{m^1, 1, l^1} \right) \delta_{n^2, k^2} \delta_{m^2, l^2}
$$

$$
- (1 - \Omega^2) \left( \sqrt{m^2 n^2} \delta_{n^2, 1, k^2} \delta_{m^2, 1, l^2} + \sqrt{m^2 n^2} \delta_{n^2, -1, k^2} \delta_{m^2, 1, l^2} \right) \delta_{n^1, k^1} \delta_{m^1, l^1} 
$$

is the field-independent part and

$$
V_{kl;mn} = \left( \frac{\lambda}{3!} \phi \star \phi + (1 + \Omega^2)(\bar{X}^\nu \star \overline{\bar{X}^\nu} - \bar{x}^2) \right)_{lmnk} \\
+ \left( \frac{\lambda}{3!} \phi \star \phi + (1 + \Omega^2)(\bar{X}^\nu \star \overline{\bar{X}^\nu} - \bar{x}^2) \right)_{nmlk} \\
+ \left( \frac{1}{3!} \delta_{lm} \delta_{nk} - 2 (1 - \Omega^2) A_{vl;lm} A^\nu_{nk} \right) \\
+ (1 - \Omega^2)i \sqrt{2} \left( \sqrt{n^1 m^1} A_{l^1 m^1}^{(1+)} \delta_{n^1, 1, l^1} - \sqrt{n^1 m^1} A_{l^1 m^1}^{(1-)} \delta_{n^1, -1, l^1} \right) k^2 n^2 \\
+ \sqrt{n^2 m^2} A_{l^2 m^2}^{(2+)} \delta_{n^2, 1, l^2} - \sqrt{n^2 m^2} A_{l^2 m^2}^{(2-)} \delta_{n^2, -1, l^2} \\
+ (1 - \Omega^2)i \sqrt{2} \left( - \sqrt{n^1 m^1} A_{l^1 m^1}^{(1+)} \delta_{n^1, 1, l^1} + \sqrt{n^1 m^1} A_{l^1 m^1}^{(1-)} \delta_{n^1, -1, l^1} \right) k^2 n^2 \\
- \sqrt{m^2 m^2} A_{n^2 k^2}^{(2+)} \delta_{m^2, 1, l^2} + \sqrt{m^2 m^2} A_{n^2 k^2}^{(2-)} \delta_{m^2, -1, l^2} \right) k^2 n^2. 
$$
We have used the definitions
\[ A^{(1\pm)} = A^1 \pm iA^2, \quad A^{(2\pm)} = A^3 \pm iA^4. \]  (5.13)

The heat kernel \( e^{-tH^0} \) of the Schrödinger operator (5.8) can be calculated from the propagator given in. \(^{14}\) In the matrix base of the Moyal plane, it has the following representation:

\[
\begin{align*}
(e^{-tH^0})_{mn;kl} &= e^{-2\sigma^2} \delta_{m+k,n+l} \prod_{i=1}^{2} K_{m^i,n^i;k^i,l^i}(t), \\
K_{m,m+a;l+a,l}(t) &= \left[ \sum_{u=0}^{\min(m,l)} \sqrt{\binom{m}{u} \binom{l}{u} \binom{\alpha + m}{m-u} \binom{\alpha + l}{l-u}} \right] \\
&\times e^{-4\Omega t} \left( \frac{1}{1 - \Omega^2} (1 - e^{-4\Omega t})^{m+l-2a} \right) \frac{4\Omega}{(1 + \Omega)^2} \frac{\alpha + 2a + 1}{(1 + \Omega)} \\
&\quad \times \left( 1 - e^{-2\Omega t} \text{sinh}(2\Omega t) \right)^{m+l-2a} X_\Omega(t)^n, \quad \text{for } \Omega = 1, \quad \text{simplifies a lot,} \\
&\quad \text{and for the heat kernel we obtain the following simple expression:} \\
&\quad (e^{-tH^0})_{mn;kl} = \delta_{ml} \delta_{kn} e^{-2\sigma^2} \prod_{i=1}^{2} e^{-2t(m^i + n^i)}, \\
&\quad K_{mn;kl}(t) = \delta_{ml} \prod_{i=1}^{2} e^{-2t(m^i + k^i)},
\end{align*}
\]  (5.14)-(5.20)

where \( 2\sigma^2 = (\mu^2\theta/2 + 4\Omega) \), and we have defined
\[
X_\Omega(t) = \frac{4\Omega}{(1 + \Omega)^2 e^{2\Omega t} - (1 - \Omega)^2 e^{-2\Omega t}}. \quad \text{(5.17)}
\]

For \( \Omega = 1 \), the interaction part of the action simplifies a lot,
\[
\begin{align*}
V_{kl;mn} &= \left( \frac{\lambda}{3!} \phi \ast \phi + 2 \left( \bar{X}^\mu \ast \bar{X}^\mu - \bar{x}^2 \right) \right)_{lm} \delta_{nk} \\
&\quad + \left( \frac{\lambda}{3!} \phi \ast \phi + 2 \left( \bar{X}^\mu \ast \bar{X}^\mu - \bar{x}^2 \right) \right)_{nk} \delta_{ml} + \frac{\lambda}{3!} \phi_{lm} \phi_{nk}, \quad \text{(5.18)}
\end{align*}
\]

and for the heat kernel we obtain the following simple expression:
\[
\begin{align*}
(e^{-tH^0})_{mn;kl} &= \delta_{ml} \delta_{kn} e^{-2\sigma^2} \prod_{i=1}^{2} e^{-2t(m^i + n^i)}, \quad \text{(5.19)} \\
K_{mn;kl}(t) &= \delta_{ml} \prod_{i=1}^{2} e^{-2t(m^i + k^i)}, \quad \text{(5.20)}
\end{align*}
\]

where \( \sigma^2 = \frac{\mu^2\theta}{4} + 2 \).

5.2. Method

The regularised one-loop effective action is given by
\[
\Gamma_\Omega^\epsilon[\phi] = -\frac{1}{2} \int^\infty_\epsilon dt \frac{dt}{t} \text{Tr} \left( e^{-tH} - e^{-tH^0} \right). \quad \text{(5.21)}
\]
In order to proceed, we use the Duhamel formula. We have to iterate the identity

\[ e^{-tH} - e^{-tH^0} = \int_0^t d\sigma \frac{d}{d\sigma} \left( e^{-\sigma H} e^{-(t-\sigma)H^0} \right) \]

\[ = - \int_0^t d\sigma e^{-\sigma H} \frac{\theta}{2} V e^{-(t-\sigma)H^0}, \quad (5.22) \]

giving

\[ e^{-tH} = e^{-tH^0} - \frac{\theta}{2} \int_0^t dt_1 e^{-t_1H^0} V e^{-(t-t_1)H^0} \]

\[ + \left( \frac{\theta}{2} \right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-t_2H^0} V e^{-(t_1-t_2)H^0} V e^{-(t-t_1)H^0} + \ldots \quad (5.23) \]

We thus obtain

\[ \Gamma_{11}^\epsilon = \frac{\theta}{4} \int_\epsilon^\infty dt \text{ Tr } V e^{-tH^0} - \frac{\theta^2}{8} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \text{ Tr } V e^{-t' H^0} V e^{-(t-t')H^0} \]

\[ + \frac{\theta^3}{16} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \int_0^{t'} dt'' \text{ Tr } V e^{-t'' H^0} V e^{-(t'-t'')H^0} V e^{-(t-t'')H^0} \]

\[ - \frac{\theta^4}{32} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \int_0^{t'} \int_0^{t''} dt''' \text{ Tr } V e^{-t''' H^0} V e^{-(t''-t''')H^0} V e^{-(t'-t''')H^0} \]

\[ + O(\theta^5). \quad (5.24) \]

Divergences occur up to fourth order only, higher order contributions are finite.

Operators $H^0$ and $V$ entering the heat kernel obey obvious scaling relations. Defining

\[ v = \frac{V}{1 + \Omega^2}, \]

\[ h^0 = \frac{H^0}{1 + \Omega^2}, \]

and the auxiliary parameter $\tau$

\[ \tau = t \left( 1 + \Omega^2 \right). \]

This leads to operators depending beside on $\theta$ only on the following three parameters:

\[ \rho = \frac{1 - \Omega^2}{1 + \Omega^2}, \]

\[ \tilde{\epsilon} = \epsilon \left( 1 + \Omega^2 \right), \quad (5.25) \]

\[ \tilde{\mu}^2 = \frac{\mu^2 \theta}{1 + \Omega^2}. \]

The task of this paper is to extract the divergent contributions of the expansion (5.24). In order to do so, we expand the integrands for small auxiliary parameters. The divergencies
are due to infinite sums over indices occurring in the heat kernel but not in the gauge field \( A \). After integrating over the auxiliary parameters, we obtain the divergent contributions provided in the next section. In the end, we convert the results to \( x \)-space using
\[
\sum_m B_{mm} = \frac{1}{4\pi^2 \theta^2} \int d^4x B(x),
\]
where \( B(x) = \sum_{m,n} B_{mn} f_{mn}(x) \).

5.3. Resulting gauge action

The explicit calculation is very tedious and is given in detail in.\(^{30}\) We have to insert the expressions \((5.12)\) and \((5.14)\) into the expansion \((5.24)\) of the effective action, order by order. Although the method is not manifestly gauge invariant, various terms from different orders add up to a gauge invariant final expression. Collecting all the terms together, we get for the divergent contributions of the effective action
\[
\Gamma_{II}^{\epsilon} = \frac{1}{192\pi^2} \int d^4x \left\{ \frac{24}{\epsilon \theta} (1 - \rho^2)(\tilde{X}_\nu \star \tilde{X}_\nu - \tilde{x}^2) \right. \\
+ \ln \epsilon \left( \frac{12}{\theta} (1 - \rho^2)(\tilde{\mu}^2 - \rho^2)(\tilde{X}_\nu \star \tilde{X}_\nu - \tilde{x}^2) \right. \\
\left. + 6(1 - \rho^2)^2((\tilde{X}_\mu \star \tilde{X}_\mu)^2 - (\tilde{x}^2)^2) - \rho^4 F_{\mu\nu} F^{\mu\nu} \right) \right\},
\]
where the field strength is given by
\[
F_{\mu\nu} = -i[\bar{x}_\mu, A_\nu]_* + i[\bar{x}_\nu, A_\mu]_* - i[A_\mu, A_\nu]_*.
\]

5.4. Conclusions

Our main result is summarised in Eqn. \((5.26)\): Both, the linear in \( \epsilon \) as well as the logarithmic in \( \epsilon \) divergent term, turn out to be gauge invariant. The logarithmically divergent part is an interesting candidate for a renormalisable gauge interaction. As far as we know, this action did not appear before in string theory. The sign of the term quadratic in the covariant coordinates may change depending on whether \( \tilde{\mu}^2 \leq \rho^2 \). This reflects a phase transition. In a forthcoming work (H.G. and H. Steinacker, in preparation), we were able to analyse in detail an action like \((5.26)\) in two dimensions. The case \( \Omega = 1 \) (\( \rho = 0 \)) is of course of particular interest. One obtains a matrix model. In the limit \( \Omega \to 0 \), we obtain just the standard deformed Yang-Mills action. Furthermore, the action \((5.26)\) allows to study the limit \( \theta \to \infty \).

In addition, we will attempt to study the perturbative quantisation. One of the problems of quantising action \((5.26)\) is connected to the tadpole contribution, which is non-vanishing and hard to eliminate. The Paris group arrived at similar conclusions.
References

1) H. S. Snyder, “Quantized Space-Time,” Phys. Rev. 71 (1947) 38–41.
2) A. Connes, “Noncommutative differential geometry,” Inst. Hautes Etudes Sci. Publ. Math. 62 (1986) 257.
3) H. Grosse and J. Madore, “A noncommutative version of the Schwinger model,” Phys. Lett. B283 (1992) 218–222.
4) H. Grosse, C. Klimcik, and P. Presnajder, “Towards finite quantum field theory in noncommutative geometry,” Int. J. Theor. Phys. 35 (1996) 231–244, hep-th/9505175.
5) T. Filk, “Divergencies in a field theory on quantum space,” Phys. Lett. B376 (1996) 53–58.
6) S. Doplicher, K. Fredenhagen, and J. E. Roberts, “The quantum structure of space-time at the Planck scale and quantum fields,” Commun. Math. Phys. 172 (1995) 187–220, hep-th/0303037.
7) V. Schomerus, “D-branes and deformation quantization,” JHEP 06 (1999) 030, hep-th/9903205.
8) N. Seiberg and E. Witten, “String theory and noncommutative geometry,” JHEP 09 (1999) 032, hep-th/9908142.
9) R. J. Szabo, “Quantum field theory on noncommutative spaces,” Phys. Rept. 378 (2003) 207–299, hep-th/0109162.
10) M. R. Douglas and N. A. Nekrasov, “Noncommutative field theory,” Rev. Mod. Phys. 73 (2002) 977–1029, hep-th/0106048.
11) S. Minwalla, M. Van Raamsdonk, and N. Seiberg, “Noncommutative perturbative dynamics,” JHEP 02 (2000) 020, hep-th/9912072.
12) I. Chepelev and R. Roiban, “Renormalization of quantum field theories on noncommutative $\mathbb{R}^d$. I: Scalars,” JHEP 05 (2000) 037, hep-th/9911098.
13) I. Chepelev and R. Roiban, “Convergence theorem for non-commutative Feynman graphs and renormalization,” JHEP 03 (2001) 001, hep-th/0008090.
14) H. Grosse and R. Wulkenhaar, “Renormalisation of $\phi^4$ theory on noncommutative $\mathbb{R}^4$ in the matrix base,” Commun. Math. Phys. 256 (2005) 305–374, hep-th/0401128.
15) E. Langmann and R. J. Szabo, “Duality in scalar field theory on noncommutative phase spaces,” Phys. Lett. B533 (2002) 168–177, hep-th/0202039.
16) V. Rivasseau, F. Vignes-Tourneret, and R. Wulkenhaar, “Renormalization of noncommutative $\phi^4$-theory by multi-scale analysis,” Commun. Math. Phys. 262 (2006) 565–594, hep-th/0501036.
17) H. Grosse and H. Steinacker, “Renormalization of the noncommutative phi**3 model
through the Kontsevich model,” *Nucl. Phys.* **B746** (2006) 202–226, hep-th/0512203

18) H. Grosse and H. Steinacker, “A nontrivial solvable noncommutative $\phi^3$ model in 4 dimensions,” *JHEP* **08** (2006) 008, hep-th/0603052.

19) H. Grosse and H. Steinacker, “Exact renormalization of a noncommutative $\phi^3$ model in 6 dimensions,” hep-th/0607235.

20) S. S. Gubser and S. L. Sondhi, “Phase structure of non-commutative scalar field theories,” *Nucl. Phys.* **B605** (2001) 395–424, hep-th/0006119.

21) W. Bietenholz, F. Hofheinz, and J. Nishimura, “Phase diagram and dispersion relation of the non-commutative lambda phi**4** model in $d = 3$,” *JHEP* **06** (2004) 042, hep-th/0404020.

22) X. Martin, “A matrix phase for the $\phi^4$ scalar field on the fuzzy sphere,” *JHEP* **04** (2004) 077, hep-th/0402230.

23) J. Ambjorn, Y. M. Makeenko, J. Nishimura, and R. J. Szabo, “Nonperturbative dynamics of noncommutative gauge theory,” *Phys. Lett.* **B480** (2000) 399–408, hep-th/0002158.

24) U. Carow-Watamura and S. Watamura, “Noncommutative geometry and gauge theory on fuzzy sphere,” *Commun. Math. Phys.* **212** (2000) 395–413, hep-th/9801195.

25) H. Steinacker, “Quantized gauge theory on the fuzzy sphere as random matrix model,” *Nucl. Phys.* **B679** (2004) 66–98, hep-th/0307075.

26) W. Behr, F. Meyer, and H. Steinacker, “Gauge theory on fuzzy $S^2 \times S^2$ and regularization on noncommutative $\mathbb{R}^4$,” *JHEP* **07** (2005) 040, hep-th/0503041.

27) P. B. Gilkey, “Invariance theory, the heat equation and the Atiyah-Singer index theorem.”

28) D. V. Vassilevich, “Heat kernel expansion: User’s manual,” *Phys. Rept.* **388** (2003) 279–360, hep-th/0306138.

29) H. Grosse and M. Wohlgenannt, “Noncommutative QFT and renormalization,” *J. Phys. Conf. Ser.* **53** (2006) 764–792, hep-th/0607208.

30) H. Grosse and M. Wohlgenannt, “Induced gauge theory on a noncommutative space,” hep-th/0703169.

31) H. Grosse and R. Wulkenhaar, “Power-counting theorem for non-local matrix models and renormalisation,” *Commun. Math. Phys.* **254** (2005) 91–127, hep-th/0305066.

32) J. Polchinski, “Renormalization and effective Lagrangians,” *Nucl. Phys.* **B231** (1984) 269–295.

33) K. G. Wilson and J. B. Kogut, “The renormalization group and the epsilon expansion,” *Phys. Rept.* **12** (1974) 75–200.

34) H. Grosse and R. Wulkenhaar, “The beta-function in duality-covariant
noncommutative $\phi^4$ theory,” *Eur. Phys. J.* **C35** (2004) 277–282, hep-th/0402093.
35) V. Rivasseau, “From perturbative to constructive renormalization,”. Princeton, USA: Univ. Pr. (1991) 336 p. (Princeton series in physics).
36) C. Itzykson and J. B. Zuber, “Combinatorics of the modular group. 2. the Kontsevich integrals,” *Int. J. Mod. Phys.* **A7** (1992) 5661–5705, hep-th/9201001.
37) M. Kontsevich, “Intersection theory on the moduli space of curves and the matrix Airy function,” *Commun. Math. Phys.* **147** (1992) 1–23.
38) B. Jurčo, P. Schupp, and J. Wess, “Noncommutative gauge theory for Poisson manifolds,” *Nucl. Phys.* **B584** (2000) 784–794, hep-th/0005005.
39) J. Madore, S. Schraml, P. Schupp, and J. Wess, “Gauge theory on noncommutative spaces,” *Eur. Phys. J.* **C16** (2000) 161–167, hep-th/0001203.
40) A. H. Chamseddine and A. Connes, “The spectral action principle,” *Commun. Math. Phys.* **186** (1997) 731–750, hep-th/9606001.
41) E. Langmann, “Generalized Yang-Mills actions from Dirac operator determinants,” *J. Math. Phys.* **42** (2001) 5238–5256, math-ph/0104011.
42) V. Gayral, “Heat-kernel approach to UV/IR mixing on isospectral deformation manifolds,” *Annales Henri Poincare* **6** (2005) 991–1023, hep-th/0412233.
43) V. Gayral, J. M. Gracia-Bondia, and F. R. Ruiz, “Trouble with space-like noncommutative field theory,” *Phys. Lett.* **B610** (2005) 141–146, hep-th/0412235.
44) D. V. Vassilevich, “Non-commutative heat kernel,” *Lett. Math. Phys.* **67** (2004) 185–194, hep-th/0310144.
45) V. Gayral and B. Iochum, “The spectral action for Moyal planes,” *J. Math. Phys.* **46** (2005) 043503, hep-th/0402147.
46) D. V. Vassilevich, “Heat kernel, effective action and anomalies in noncommutative theories,” *JHEP* **08** (2005) 085, hep-th/0507123.
47) A. de Goursac, J.-C. Wallet, and R. Wulkenhaar, “Noncommutative induced gauge theory,” hep-th/0703075.
48) H. Grosse and R. Wulkenhaar, “Renormalisation of $\phi^4$ theory on noncommutative $\mathbb{R}^2$ in the matrix base,” *JHEP* **12** (2003) 019, hep-th/0307017.