Dense coding capacity of a quantum channel

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We consider the fundamental protocol of dense coding of classical information assuming that noise affects both the forward and backward communication lines between Alice and Bob. Assuming that this noise is described by the same quantum channel, we define its dense coding capacity by optimizing over all adaptive strategies that Alice can implement, while Bob encodes the information by means of Pauli operators. Exploiting techniques of channel simulation and protocol stretching, we are able to establish the dense coding capacity of Pauli channels in arbitrary finite dimension, with simple formulas for depolarizing and dephasing qubit channels.

Introduction. – One of the most essential resources for quantum communication and information processing is entanglement. Quantum entanglement describes correlations outside the classical realm and it is at the core of the realization of many quantum tasks, including quantum teleportation, quantum cryptography, and dense coding. The dense coding protocol allows two parties to transmit classical information encoded on quantum systems with the aid of shared entanglement. By employing a bipartite entangled state, it is possible to encode 2 log₂ d bits of classical information in a d-dimensional system, thus overcoming the upper bound log₂ d on the unassisted classical capacity.

In ideal conditions, a dense coding scheme exploits a noiseless quantum channel between Alice and Bob. Through this quantum channel, Alice sends to Bob part B of a bipartite entangled state σAB. Once received by Bob, system B is subject to a Pauli operator UX with probability PX. The encoded system is sent back to Alice through the second use of the noiseless quantum channel. At the output, Alice implements a joint quantum measurement on A and B to retrieve the classical information. In this case, the capacity C(σAB) is

\[ C(σ_{AB}) = \max \{ \log_2 d, \log_2 d + S(σ_B) − S(σ_{AB}) \}, \]

where σB = TrA σAB and S(σ) := −Tr(σ log₂ σ) is the Von Neumann entropy. For a maximally-entangled resource state σAB one has \( C = 2 \log_2 d \).

In a realistic scenario, noise must be explicitly included in the protocol. For instance, noise can affect the transmission of quantum systems from the sender (Bob) to the receiver (Alice), after the entangled resource state has been perfectly distributed. This is the typical scenario in the definition of entanglement-assisted protocols whose capacity is known. More realistically, noise may also affect the distribution itself of the resource state from Alice to Bob. This scenario has been previously studied, where it has been called “two-sided” noisy dense coding but not capacity has been established.

This is the aim of this manuscript where the two-sided protocol is formulated in a general feedback-assisted fashion. Here the round-trip transmission of the quantum systems between Alice and Bob is interleaved by two adaptive quantum operations (QOs) performed by Alice, which are optimized and updated on the basis of the previous rounds. At the same time, Bob may also optimize his classical encoding strategy, i.e., the probability distribution of his Pauli encoders. Optimizing over these protocols we define the dense coding capacity of a quantum channel between Alice and Bob. We then use simulation techniques that allow us to simplify the structure of the protocol and derive a single-letter upper bound for this capacity. This quantity is explicitly computed for a Pauli channel in arbitrary d dimension, with remarkably simple formulas for qubit channels, such as the depolarizing and the dephasing channel.

FIG. 1: Two-sided noisy dense coding over a quantum channel. Alice prepares locally the bipartite state σ′A′ and sends system A′ to Bob who performs the Pauli unitary encoding Ux and sends the system back through the quantum channel. At the output Alice performs a joint positive-valued operator measure (POVM) in order to retrieve x. In an adaptive version of the protocol, Alice performs quantum operations (QOs) on her input and output systems which are generally updated and optimized round-by-round. These QOs may also be conditioned by an extra assisting variable which is communicated back by Bob.
Dense coding protocol.– Let us recall the expressions of Pauli operators in a \(d\)-dimensional Hilbert space. On a computational basis \(\{|j\}\) we may define the two shift operators

\[
X|j\rangle = |j + 1\rangle, \quad Z|j\rangle = e^{i\psi}|j\rangle,
\]

where \(\oplus\) is modulo \(d\) addition and \(\psi = \exp(2i\pi/d)\). We may then consider the \(d^2\) Pauli operators \(X^iZ^m\) that, for simplicity, we denote by \(U_x\) with collapsed index \(x = i,m\). For \(d = 2\), these operators provide the standard qubit Pauli operators \(X, Y, Z\) plus the identity \(I\). In the following we use the compact notation \(U_x = U_x^0\).

Now consider the scheme depicted in Fig. 1 where the communication line between Alice and Bob is affected by a completely positive trace-preserving (CPTP) map \(E\). Alice’s resource state \(\sigma_{AA'}\) is defined on a \(d\times d\)-dimensional Hilbert space. Part \(A'\) is sent to Bob who encodes classical variable \(X = \{x, \pi_x\}\) by means of \(d^2\) Pauli operators \(U_x^i\) which are chosen with probability \(\pi_x\). In this way, Bob generates the state

\[
\sigma_{AB'}(x) := [I_A \otimes (U_x^i \circ E)|A']\sigma_{AA'},
\]

where \(I_x := I\rho^x\) is the identity map. Once system \(B'\) is sent back through the channel, Alice receives the output system \(A''\) in the state \(\rho_{AA''}(x) := (I_x \otimes E_x)\sigma_{AA'}\) where we have defined the encoding channel

\[
E_x := E \circ U_x \circ E.
\]

In order to retrieve the value of \(x\), Alice performs a joint quantum measurement on \(A\) and \(A''\). Asymptotically (i.e., for many repetitions of the protocol), the accessible information of Alice’s output ensemble \(\{\pi_x, \rho_{AA''}(x)\}\) is given by the Holevo bound

\[
\chi(\{\pi_x, \rho_{AA''}(x)\}) = \sup_{\pi,\rho} \left\{ S(\sum_x \pi_x \rho_{AA''}(x)) - \sum_x \pi_x S(\rho_{AA''}(x)) \right\}
\]

where the output state \(\rho_{AA''}(x)\) depends on the encoded classical information and the sequence of QOs \(Q := \{Q_1, Q_2, \ldots, Q_n\}\). Alice’s deferred measurement will be done on the final state. For large \(n\), and optimizing the Holevo information of the ensemble \(P\) over all the possible sequences \(Q\), we define the dense coding capacity (DCC) of the quantum channel \(E\) as

\[
C_D(E) := \sup_{Q} \max \lim_{n \to \infty} n^{-1} \chi(\{\pi_{x_n}, \rho_{A_n}^n(x_n)\}).
\]

Note that this definition is more general than a regularized version \(C_D^\infty(E)\) of Eq. (6), where Alice prepares a large multipartite input state, sends part of this state through \(n\) uses of the round-trip and then performal global measurement of the total output. In fact, Eq. (7) assumes that Alice’s input can also be updated round-by-round on the basis of feedback from Bob.

Single-letter upper bound.– We now exploit a number of ingredients from recent literature to derive a computable upper bound for the DCC. Recall that, for any finite-dimensional quantum channel \(E\), we may write the simulation \(E(\rho) = \mathcal{T}(\rho \otimes \sigma)\), where \(\mathcal{T}\) is a trace-preserving LOCC and \(\sigma\) a resource state [17]. Furthermore, suppose that the channel is covariant with respect to Pauli operators so that, for any Pauli \(U\), we may write \(E \circ U = U' \circ E\) for some generally different Pauli \(U'\). In this case the channel is Pauli-covariant and we may write \(E(\rho) = T_{\text{tele}}(\rho \otimes \sigma_E)\), where \(T_{\text{tele}}\) is a teleportation LOCC and \(\sigma_E\) is the channel’s Choi matrix, i.e., \(\sigma_E := I_A \otimes E(\Phi_{AB})\) with \(\Phi_{AB}\) being a maximally-entangled state. Note that the Pauli unitaries \(U_x\) are jointly Pauli-covariant, i.e., we may cer-
tainly write $\mathcal{U}_e \circ \mathcal{U} = \mathcal{U}' \circ \mathcal{U}_e$ where $\mathcal{U}'$ is the same for any $x$ (since a Pauli operator either commutes or anti-commutes with another Pauli operator). Therefore, if $\mathcal{E}$ is Pauli-covariant, we also have that the encoding channel $\mathcal{E}_e$ is jointly Pauli-covariant. We may therefore write the channel simulation $\mathcal{E}_e(\rho) = T_{\text{tele}}(\rho \otimes \sigma_{\mathcal{E}_e})$ in terms of its Choi matrix.

The next step is the stretching of the protocol as explained in Fig. 3. Thanks to this procedure the output state can be decomposed in a tensor product of Choi matrices up to a global $QO \Lambda$, i.e., we may write

$$\rho_n^A(x_n) = \Lambda(\sum_{\pi_n} \sigma_{\mathcal{E}_e} \otimes \sigma_{\mathcal{E}_e} \otimes \ldots \otimes \sigma_{\mathcal{E}_e}^{n_{x_n}}),$$

where $n_{x_n}$ is the number of $x_i$ occurrences in the message $x_n$. This given by $n_{x_n} = n_{\pi_x}$ where $\pi_x = \sum_{j \neq i} \pi_{x_n}$ is the marginal probability. Thanks to Eq. (7) we may simplify the Holevo quantity in Eq. (4). In fact, by using $(\ast)$ the contractivity under CPTP maps of the Holevo quantity, and $(\otimes)$ the subadditivity of the von Neumann entropy $S$ under tensor products, we may write

$$\chi(\{\pi_{x_n}, \rho_n^A(x_n)\}) \leq \chi(\{\pi_{x_n}, \otimes_{i=1}^n \sigma_{\mathcal{E}_e}^{n_{x_n}}\})$$

$$\leq \sum_{x_n} S(\sum_{\pi_{x_n}} \pi_{x_n} \otimes \sigma_{\mathcal{E}_e}^{n_{x_n}}) - \sum_{x_n} \pi_{x_n} S(\otimes_{i=1}^n \sigma_{\mathcal{E}_e}^{n_{x_n}})$$

$$\leq \sum_{x_n} n_{x_n} S(\sum_{\pi_{x_n}} \pi_{x_n} \sigma_{\mathcal{E}_e}^{n_{x_n}}) + \ldots + n_{x_n} S(\sum_{\pi_{x_n}} \pi_{x_n} \sigma_{\mathcal{E}_e}^{n_{x_n}}) - \sum_{x_n} \pi_{x_n} S(\sigma_{\mathcal{E}_e}) - \ldots - \sum_{x_n} \pi_{x_n} S(\sigma_{\mathcal{E}_e})$$

$$\leq n S(\sum_{x_n} \pi_{x_n} \sigma_{\mathcal{E}_e}) - n \sum_{x_n} \pi_{x_n} S(\sigma_{\mathcal{E}_e})$$

$$= n \chi(\{\pi_x, \sigma_{\mathcal{E}_e}\}),$$

where $\pi_x$ is the marginal probability of a generic letter $x$ and the Choi matrix $\sigma_{\mathcal{E}_e}$ is defined in Eq. (8). Note that, in the last inequality of Eq. (10), we also use the fact that a random code $[1, 2, 3]$, i.e., a code where the codewords are randomly chosen with an iid distribution equal to the marginal probability $\pi_x$, is known to achieve the Holevo bound for discrete memoryless quantum channels [24, 25].

By using Eq. (10) in the definition of Eq. (7), we may then get rid of the supremum over $Q$ and the asymptotic limit in $n$. We may therefore write a single-letter upper bound for the DCC of a Pauli-covariant channel $\mathcal{E}$ as

$$C_D(\mathcal{E}) \leq \max_{\pi_x} \chi(\{\pi_x, \sigma_{\mathcal{E}_e}\})$$

(11)

where $\pi_x$ is the marginal probability distribution of Bob’s encoding variable, and $\sigma_{\mathcal{E}_e}$ is the Choi matrix of the encoding channel $\mathcal{E}_e$ in Eq. (3). Note that the upper bound in Eq. (11) may be reached asymptotically by a non-adaptive protocol where Alice prepares maximally-entangled states $\Phi_{A'A'}$ and sends $A'$ through the channel, while Bob applies independent Pauli operators $\mathcal{U}_e$ with optimized probability $\pi_x$. Therefore, for a Pauli-covariant channel we conclude that

$$C_D(\mathcal{E}) = C_D^{(1)}(\mathcal{E}) = \max_{\pi_x} \chi(\{\pi_x, \sigma_{\mathcal{E}_e}\})$$

(12)

Remarkably, no adaptiveness or regularization is needed to achieve the best possible dense coding performance with a Pauli-covariant channel.

**Dense coding capacity of Pauli channels.—** The main result in Eq. (12) can be applied to any Pauli channel at any finite dimension $d$. For any $d \geq 2$, a Pauli channel takes the form

$$\mathcal{E}^d(\rho) = \sum_{k \neq 0} p_{kk} \left( X^k Z^r \right) \rho \left( X^k Z^r \right) \rho,$$

where $p_{kk}$ is a probability distribution, and $X$ and $Z$ are the $d$-dimensional shift operators in Eq. (2). For this channel, we may easily write an explicit formula for its DCC capacity. In evaluating the Holevo bound, we notice that von Neumann entropy $S(\sum_{\pi_x} \pi_x \sigma_{\mathcal{E}_e})$ is maximized by the uniform probability $\pi_x = 1/d^2$ and we can write $S(\sum_{\pi_x} \pi_x \sigma_{\mathcal{E}_e}) = \log_2 d^2$. Then, using the invariance of the entropy under unitary transformations, one has $\sum_{\pi_x} \pi_x S(\sigma_{\mathcal{E}_e}) = S[\mathcal{I}_A \otimes \mathcal{E}^d(\sigma_{\mathcal{E}_e})]$.

Therefore, for the Holevo quantity in Eq. (12) we may write

$$C_D(\mathcal{E}^d) = \log d^2 - S[\mathcal{I}_A \otimes \mathcal{E}^d(\sigma_{\mathcal{E}_e})] \leq S[\mathcal{I}_A \otimes \mathcal{E}^d(\sigma_{\mathcal{E}_e})].$$

(14)

As expected this is strictly less than the entanglement-assisted classical capacity of the channel, given by [12, 13]

$$C_E(\mathcal{E}) = \log d^2 - S(\sigma_{\mathcal{E}_e}).$$

(15)

Consider a qubit depolarizing channel, which is a Pauli channel of the form

$$\mathcal{E}_{\text{depol}}(\rho) = \frac{1 - 3}{4} \rho + \frac{1}{4} (X \rho X + Y \rho Y + Z \rho Z),$$

(16)
for some probability $p$. Then, it is straightforward to see that
\[ C_D(\mathcal{E}_{\text{dep}}^2) = 2 - h_2(\alpha) - \alpha \log 3, \]
where $h_2(x) = -x \log x - (1-x) \log(1-x)$ is the binary entropy function
and $\alpha := 3/4p(2-p)$. Then, consider a qubit dephasing channel, which takes the form
\[
\mathcal{E}_{\text{dep}}(\rho) = (1-p)\rho + pZ\rho Z. \tag{17}
\]
Its DCC is equal to the following expression
\[
C_D(\mathcal{E}_{\text{dep}}^2) = 2[1 - h_2(p)], \tag{18}
\]
for $p \leq 1/2$ and zero otherwise.

Conclusion.– In this work we have considered the most general adaptive protocol for the dense coding of classical
information in a realistic scenario where noise affects both the communication lines between Alice and Bob.
Assuming that this noise is modelled by the same quantum
channel, we define its dense coding capacity as the maximum amount of classical information (per round-
trip use) that Bob can transmit to Alice. We assume
that Bob is implementing Pauli encoders with an optimized
probability distribution and Alice is using quantum registers that are adaptively updated and optimized
in the process. For Pauli-covariant channel, we find that
this capacity reduces to a single-letter version based on a
protocol which is non-adaptive and one-shot (i.e., using iid input states). In particular, we can establish exact formulas for the dense coding capacity of Pauli channels.

Note that our approach departs from the definition
of entanglement-assisted classical capacity of a quantum
channel [12,13], where it is implicitly required that the
parties either have a noiseless side quantum channel for
distributing entangled sources or they have previously met and stored quantum entanglement in ideal long-
life quantum memories. Our treatment and definition of dense coding capacity removes these assumptions
assuming that the entanglement source is itself distributed
through the noisy channel and, therefore, it is realistically
degraded by the environment. Because of this feature,
our capacity can also be seen as an upper bound for the key rates of two-way quantum key distribution protocols
that are related to the dense coding idea [26–30].

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ing protocol, where Bob uses arbitrary encoding unitaries
instead of Pauli operators. Correspondingly, we may in-
roduce a generalized dense coding capacity $C_D(\mathcal{E})$ as in
Eq. (17) but replacing the maximization as $\max_{x_n} \rightarrow
\max_{U_n, x_n} \rightarrow \max_{U_n, x_n}$, where $U_n$ are sequences of arbitrary unitary
encoders. By definition, we have $C_D(\mathcal{E}) \leq \tilde{C}_D(\mathcal{E})$.
However we conjecture that an equality should hold, due
to the fact that the basis of Pauli operators represents
the optimal choice in the noise-less scenario.
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