Removing Data Heterogeneity Influence Enhances Network Topology Dependence of Decentralized SGD

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Abstract

We consider decentralized stochastic optimization problems where a network of agents each owns a local cost function cooperate to find a minimizer of the global-averaged cost. A widely studied decentralized algorithm for this problem is D-SGD in which each node applies a stochastic gradient descent step, then averages its estimate with its neighbors. D-SGD is attractive due to its efficient single-iteration communication and can achieve linear speedup in convergence (in terms of the network size). However, D-SGD is very sensitive to the network topology. For smooth objective functions, the transient stage (which measures how fast the algorithm can reach the linear speedup stage) of D-SGD is on the order of $O(n/(1-\beta)^2)$ and $O(n^3/(1-\beta)^4)$ for strongly convex and generally convex cost functions, respectively, where $1-\beta \in (0,1)$ is a topology-dependent quantity that approaches 0 for a large and sparse network. Hence, D-SGD suffers from slow convergence for large and sparse networks.

In this work, we study the non-asymptotic convergence property of the D²/Exact-diffusion algorithm. By eliminating the influence of data heterogeneity between nodes, D²/Exact-diffusion is shown to have an enhanced transient stage that are on the order of $O(n/(1-\beta))$ and $O(n^3/(1-\beta)^2)$ for strongly convex and generally convex cost functions, respectively. Moreover, we provide a lower bound of the transient stage of D-SGD under homogeneous data distributions, which coincides with the transient stage of D²/Exact-diffusion in the strongly-convex setting. These results show that removing the influence of data heterogeneity can ameliorate the network topology dependence of D-SGD. Compared with existing decentralized algorithms bounds, D²/Exact-diffusion is least sensitive to network topology.

1 Introduction

Large-scale optimization and learning have become an essential tool in many practical applications. State-of-the-art performances have been reported in various fields such as signal processing, control, reinforcement learning, and deep learning. The amount of data needed to achieve satisfactory results in these tasks is usually very large. Moreover, it has been observed that increasing the size of training data can significantly

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improve the ultimate performance in these tasks. For this reason, the scale of optimization and learning nowadays calls for efficient distributed solutions across multiple computing nodes (e.g., machines), which can handle large-scale datasets effectively.

This work considers a network of $n$ nodes connected through some topology and each node owns a private and local cost function $f_i : \mathbb{R}^d \to \mathbb{R}$. The goal of the network is to find a solution, denoted by $x^*$, of the stochastic optimization problem

$$
\min_{x \in \mathbb{R}^d} f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x), \quad \text{where} \quad f_i(x) = \mathbb{E}_{\xi_i \sim D_i} F(x; \xi_i).
$$

In this problem, $\mathbb{E}_{\xi_i \sim D_i}$ denotes the expectation over the random variable $\xi_i$ with distribution $D_i$. The function $f_i$ is essentially the expectation of some loss function $F(x; \xi_i)$ over $\xi_i$, which represent the local data available at node $i$. Each node $i$ has access to its private cost gradient $\nabla F(x_i; \xi_i)$ (or $\nabla f_i(x)$), but has to communicate to access information from other nodes. In practice, the local data distribution $D_i$ within each node is generally different, and hence, $f_i(x) \neq f_j(x)$ holds for any node $i$ and $j$. For convex costs, the data heterogeneity across the network can be characterized by $b^2 = \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x^*)\|^2$. If all local data samples follow the same distribution $D$, we have $f_i(x) = f_j(x)$ for any $i, j$ and hence $b^2 = 0$.

One of the leading algorithms to solve problem (1) is parallel SGD (P-SGD) [1]. In P-SGD, each node computes its local stochastic gradient, then a global synchronization step is needed to compute the global averaged stochastic gradient and update model parameters (i.e., solution estimates) every iteration. The global synchronization step involves all $n$ nodes and can be implemented via Parameter Server [2, 3] or Ring-Allreduce [4]. In Parameter Server, all nodes need to communicate with a central server, which can cause communication bottlenecks especially when the network bandwidth is limited. On the other hand, Ring-Allreduce organizes all nodes on a ring and divides each local gradient into $n$ chunks for parallel communication. Ring-Allreduce setup can achieve a remarkable saving in bandwidth cost, however, it suffers from a long latency due to its delicate pipelined communication patterns. The significant bandwidth or latency cost in P-SGD is unavoidable for computing the exact global average of all local gradients across $n$ nodes. Therefore, nontrivial communication overhead hampers the scalability of P-SGD.

Decentralized SGD (D-SGD) [5, 6, 7] is a promising alternative to P-SGD due to its ability to reduce the communication overhead [8, 9, 10, 11]. D-SGD is based on local averaging in which each node computes the local average with their direct neighbors as opposed to the global average. Moreover, no global synchronization step is required in D-SGD. On a delicately-designed sparse topology such as one-peer exponential graph [9], each node only needs to communicate with one neighbor per iteration, resulting in a much cheaper communication than P-SGD [12].

Apart from its efficient single-iteration communication, D-SGD can achieve the same linear speedup as P-SGD in convergence asymptotically [8, 10, 9, 13]. Linear speedup refers to a distributed algorithm property where the number of iterations needed to reach an $\epsilon$-accurate solution reduces linearly with the number of nodes. The transient stage [14], which refers to those iterations before an algorithm reaches its linear
speedup stage, is an important metric to measure the convergence performance of decentralized algorithms. The convergence rate and transient stage of D-SGD is very sensitive to the network topology. For strongly convex and smooth objective functions, the transient stage of D-SGD is on the order of $O(n/(1-\beta)^3)$ [14, 13] where $1 - \beta \in (0, 1)$ is a topology-dependent quantity gauging the network connectivity. For generally convex or non-convex objective functions, the transient stage of D-SGD is on the order of $O(n^3/(1-\beta)^4)$ [13]. For a large and sparse network in which $1 - \beta$ will approach to 0, D-SGD will converge very slow, and the solution accuracy achieved by D-SGD may be worse than that obtained by P-SGD. Hence, improving the network topology dependence (i.e., making the convergence rate less sensitive to network topology) in D-SGD is crucial to enhance its convergence rate and solution accuracy.

Many factors in D-SGD contribute to its strong dependence on the network topology. Among them, the data heterogeneity across each node has been shown in [13, 15, 14] to have the most significant influence on topology dependence. This naturally motivates us to examine whether removing the influence of data heterogeneity (i.e., $b^2$) can improve the dependence on the topology (i.e., $1 - \beta$) of D-SGD.

1.1 Main results

This work revisits the $D^2$ algorithm [16], which is also known as Exact-Diffusion [17, 18, 15] or NIDS [19]. $D^2$/Exact-Diffusion is a decentralized optimization algorithm that can remove the influence of data heterogeneity [18, 19], but it is unclear whether $D^2$/Exact-Diffusion has an improved network topology dependence compared to D-SGD in the transient stage. We establish non-asymptotic convergence rates for $D^2$/Exact-Diffusion under both the generally-convex and strongly-convex settings. The established bounds show that $D^2$/Exact-Diffusion has the best network topology dependence compared with existing results. In particular, this paper establishes that $D^2$/Exact-Diffusion converges with rate

$$\frac{1}{T+1} \sum_{k=0}^{T} (E f(\bar{x}^{(k)}) - f(x^*)) = O \left( \frac{\sigma}{\sqrt{nT}} + \frac{\sigma^2}{(1-\beta)^{1/3}T^{2/3}} + \frac{1}{(1-\beta)T} \right)$$

(G-C) (2)

$$\frac{1}{HT} \sum_{k=0}^{T} h_k (E f(\bar{x}^{(k)}) - f(x^*)) = \tilde{O} \left( \frac{\sigma^2}{nT} + \frac{\sigma^2}{(1-\beta)T^2} + \frac{1}{1-\beta} \exp\{-1-\beta\} \right)$$

(S-C) (3)

where $\bar{x}^{(k)} = \frac{1}{n} \sum_{i=1}^{n} x_i^{(k)}$, and $\sigma^2$ denotes the variance of the stochastic gradient $\nabla F(x; \xi_i)$. The weights $h_k \geq 0$ are given in Lemma 8 and $HT = \sum_{k=0}^{T} h_k$. Notation $\tilde{O}(\cdot)$ hides all logarithm factors. Moreover, (G-C) stands for the generally convex scenario while (S-C) stands for strongly convex. Below, we compare this result with D-SGD. To do that, we first interpret the above results using two important metrics.

**Linear speedup.** When $T$ is sufficiently large, the first term $\sigma^2/(nT)$ in (3) (or $\sigma/\sqrt{nT}$ in (2)) will dominate the rate. In this scenario, $D^2$/Exact-Diffusion requires $T = \Omega(1/(nc))$ in (3) (or $T = \Omega(1/(nc^2))$ in (2)) iterations to reach a desired $\epsilon$-accurate solution for strongly convex (or generally convex) problems, which is inversely proportional to the network size $n$. Therefore, we say an algorithm reaches the linear-speedup stage if for some $T$, the term involving $nT$ is dominating the rate. Rates (2) and (3) corroborate
that $D^2$/Exact-Diffusion, similar to P-SGD, can achieve the linear speedup as $T$ increases sufficiently large. A linear speedup result for $D^2$/Exact-Diffusion in the non-convex setting was established in [16].

**Transient stage.** Transient stage is an important metric to measure the scalability of distributed algorithms [14]. It is the amount of iterations needed before an algorithm reaches its linear-speedup stage, *i.e.*, it is the number of iterations $1, \ldots, T$ where $T$ is relatively small so non-$nT$ terms are dominating the rate. For example, let us consider $D^2$/Exact-Diffusion in the strongly convex scenario (3): to reach linear speedup, $T$ has to satisfy $(1 - \beta)T^2 \geq nT$ (*i.e.*, $T \geq n/(1 - \beta)$). Therefore, the transient stage in $D^2$/Exact-Diffusion for strongly-convex scenario requires $\Omega(n/(1 - \beta))$ iterations.

**Comparison with D-SGD.** Table 1 lists the convergence rates for D-SGD and $D^2$/Exact-Diffusion for both generally and strongly convex scenarios. Compared to D-SGD, it is observed in (2) and (3) that $D^2$/Exact-Diffusion has eliminated the data heterogeneity $b^2$ term. Note that the term related to data heterogeneity $b^2$ has the strongest topology dependence on $1 - \beta$ for D-SGD. In Table 2 we list the transient stage of $D^2$/Exact-Diffusion and other existing algorithms. It is observed that $D^2$/Exact-Diffusion has an improved transient stage in terms of $1 - \beta$ compared to D-SGD by removing the influence of data heterogeneity. Gradient tracking methods [20, 21, 22, 23, 24] can also remove the data heterogeneity, but their transient stage established in existing works still suffers from worse network topology dependence than $D^2$/Exact-Diffusion and even D-SGD. In other words, $D^2$/Exact-Diffusion enjoys the state-of-the-art topology dependence in the generally and strongly convex scenarios.

### 1.2 Contributions

This work makes the following contributions:

- We revisit the $D^2$/Exact-Diffusion algorithm [16, 17, 18, 19, 15] and provide its non-asymptotic convergence rate under the *generally-convex* settings. By removing the influence of data heterogeneity, $D^2$/Exact-Diffusion is shown to improve the transient stage of D-SGD from $\Omega(n^3/(1 - \beta)^4)$ to $\Omega(n^3/(1 - \beta)^2)$, which is less sensitive to the network topology.

- We also establish the non-asymptotic convergence rate of $D^2$/Exact-Diffusion under the *strongly-convex* settings. It is established that $D^2$/Exact-Diffusion improves the transient stage of D-SGD from $\Omega(n/(1 - \beta)^2)$ to $\Omega(n/(1 - \beta))$.

| Scenario         | $D^2$/Exact-Diffusion | D-SGD                                      |
|------------------|-----------------------|--------------------------------------------|
| Generally-convex | (2) + $O\left(\frac{b^2}{(1-\beta)^2T^2}\right)$ | (2) + $O\left(\frac{b^2}{(1-\beta)^2T^2}\right)$ |
| Strongly-convex  | (3) + $O\left(\frac{b^2}{(1-\beta)^2T^2}\right)$ | (3) + $O\left(\frac{b^2}{(1-\beta)^2T^2}\right)$ |

Table 1: Convergence rate comparison with D-SGD from [13].
Table 2: Transient stage comparison between D-SGD, gradient tracking, and D\(^2\)/Exact-Diffusion in the strongly convex and generally convex settings. For strongly convex scenario, the transient stage for D-SGD is from \([14, 13]\), and gradient tracking is from \([25]\). For generally-convex scenario, D-SGD is from \([13]\). Note that \(1 - \beta \in (0, 1)\).

We prove that the transient stage of D-SGD is lower bounded by \(\Omega(1/(1 - \beta))\) under homogeneous data (i.e., \(b^2 = 0\)) for the strongly-convex scenario. This bound coincides with the transient stage of D\(^2\)/Exact-Diffusion under heterogeneous data in terms of network topology dependence. This implies that the dependence of D-SGD on the network topology can only match with D\(^2\)/Exact-Diffusion if the data is homogeneous\(^1\), which is an impractical assumption.

### 1.3 Related works

Distributed optimization algorithms can be tracked back to \([26]\). Decentralized gradient descent (DGD) \([6]\) (also referred to as the consensus algorithm in \([27]\)), diffusion \([5, 7, 27]\) (also called DGD or adapt-then-combine DGD in many recent works), and dual averaging \([28]\) are among the earliest decentralized optimization algorithms. However, these algorithms suffer from a bias caused by data heterogeneity even under deterministic settings (i.e., no gradient noise exists) \([29]\) – see more explanation in Sec. 2.4. Numerous algorithms have been proposed to overcome this issue, such as alternating direction method of multipliers (ADMM) methods \([30, 31]\), explicit bias-correction methods (such as EXTRA \([32]\), Exact-Diffusion \([17, 18]\), NIDS \([19]\), and gradient tracking \([20, 21, 22, 23]\) – see \([33]\)), and dual acceleration \([34, 35, 36]\). These algorithms, in the deterministic setting, converge to the exact solution with constant learning rate without any bias.

Decentralized stochastic methods (in which the gradient is noisy) have gained a lot of attention recently. Since Decentralized SGD (D-SGD) has the same asymptotic linear speedup as P-SGD \([7, 27, 8, 13, 14]\) but is endowed with a more efficient single-iteration communication, it has been extensively studied in the context of large-scale machine learning (such as deep learning). It is well known that D-SGD is largely affected by the gradient noise, but it was not clear whether the data heterogeneity has a non-trivial influence on the performance of D-SGD. The work \([15]\) clarified that the error caused by the data heterogeneity can be greatly amplified when network topology is sparse, which can be larger than the gradient noise error. The works \([14]\) and \([13]\) also showed that the error term caused by data heterogeneity has the worst dependence on the network topology for D-SGD. It is thus conjectured that removing the influence of data heterogeneity can improve the topology dependence of D-SGD.

\(^1\)The transient stage of D-SGD is lower bounded by \(\Omega(n/(1 - \beta)^2)\) under heterogeneous case \([13, 14]\)
The $D^2$/Exact-Diffusion algorithm and gradient tracking methods have been studied under stochastic settings in [16] and [25, 24, 37, 38], respectively. However, the analysis in [16] and [25, 24, 37, 38] does not reveal whether the removal of data heterogeneity can improve the dependence on network topology. The work [15] studied $D^2$/Exact-Diffusion in the steady-state (asymptotic) regime and for strongly-convex costs. Under this setting, [15] showed that $D^2$/Exact-Diffusion can remove the data heterogeneity term $b^2\gamma^2/(1-\beta)^2$ (where $\gamma$ is the learning rate) from the steady-state mean-square-error performance of D-SGD, and thus has improved steady state error dependence on the network topology. However, it is unclear whether the improved steady-state performance in [15] carries over to D-SGD’s non-asymptotic performance (convergence rate). This paper clarifies the improvement in the non-asymptotic convergence rate in $D^2$/Exact-Diffusion for both generally and strongly convex scenarios. These results demonstrate that removing influence of data heterogeneity improves the dependence on network topology for D-SGD. In addition, our established dependence on network topology for $D^2$/Exact-Diffusion in the strongly-convex scenario coincides with the lower bound of D-SGD with homogeneous dataset.

As to the transient stage of D-SGD, [14] show that it is $\Omega(n/(1-\beta)^2)$ for strongly-convex settings, and [13] show that it is $\Omega(n^3/(1-\beta)^4)$ for both generally-convex and non-convex settings. In comparison, we establish that $D^2$/Exact-Diffusion has an improved $\Omega(n/(1-\beta))$ and $\Omega(n^3/(1-\beta)^2)$ transient stage for strongly and generally convex scenarios, respectively. This work does not study the transient stage of $D^2$/Exact-Diffusion for the non-convex scenario as the current analysis cannot be directly extended to such setting. Note that [16] provides an $\Omega(n^3/(1-\beta)^4)$ transient stage for $D^2$/Exact-Diffusion and [24] establishes an $\Omega(n^3/(1-\beta)^6)$ transient stage for stochastic gradient tracking under the non-convex setting. These transient analysis results, however, are equivalent to or worse than D-SGD in terms of network topology dependence.

There are some recent works [11, 39, 40] that target to alleviate the influence of data heterogeneity on decentralized stochastic momentum SGD. These works can reduce the momentum-incurred data heterogeneity, but cannot entirely remove the influence of data heterogeneity. As a result, the methods in these works do not result in an improved dependence on network topology.

Simultaneously and independently, a recent work [41] has established a similar result to this work. However, [41] studies the transient stage of $D^2$/Exact-Diffusion for the strongly-convex scenario only, but not for the generally-convex scenario. We also prove a lower bound of D-SGD with homogeneous dataset that shows that $D^2$/Exact-Diffusion’s dependence on network topology cannot be worse than D-SGD and always better under the heterogeneous setting. Note that our analysis techniques are also different from [41].

1.4 Notations

Throughout the paper we let $x_i^{(k)} \in \mathbb{R}^d$ denote the the local solution estimate for node $i$ at iteration $k$. Furthermore, we let

$$x^{(k)} = [x_1^{(k)}, \ldots, x_n^{(k)}]^T \in \mathbb{R}^{n \times d}$$  \hspace{1cm} (4)
\[ \nabla F(x^{(k)}; \xi^{(k)}) = [\nabla F_1(x^{(k)}_1; \xi^{(k)}_1), \ldots, \nabla F_n(x^{(k)}_n; \xi^{(k)}_n)]^T \in \mathbb{R}^{n \times d} \]  
\[ \nabla f(x^{(k)}) = [\nabla f_1(x^{(k)}_1), \ldots, \nabla f_n(x^{(k)}_n)]^T \in \mathbb{R}^{n \times d} \]

collect all local variables across the network into a matrix. Note that \( \nabla f(x^{(k)}) = \mathbb{E}[\nabla F(x^{(k)}; \xi^{(k)})] \). We use \( \text{col}\{a_1, \ldots, a_n\} \) and \( \text{diag}\{a_1, \ldots, a_n\} \) to denote a column vector and a diagonal matrix formed from \( a_1, \ldots, a_n \). We let \( \mathds{1}_n = \text{col}\{1, \ldots, 1\} \in \mathbb{R}^n \) denote the one vector and \( I_n \in \mathbb{R}^{n \times n} \) denote the identity matrix. For a symmetric matrix \( A \in \mathbb{R}^{n \times n} \), we let \( \lambda_i(A) \) to be the \( i \)-th largest eigenvalue and \( \rho(A) = \max_i |\lambda_i(A)| \) denote the spectral radius of matrix \( A \). In addition, we let \([n] := \{1, \ldots, n\} \) for any positive integer \( n \). Suppose that \( A \in \mathbb{R}^{n \times n} \) is a positive semidefinite matrix with eigen-decomposition \( A = U \Lambda U^T \) where \( U \in \mathbb{R}^{n \times n} \) is an orthogonal matrix and \( \Lambda \in \mathbb{R}^{n \times n} \) is a non-negative diagonal matrix. Then, we let \( A^{\frac{1}{2}} = U \Lambda^{\frac{1}{2}} U^T \in \mathbb{R}^{n \times n} \) be the square root of the matrix \( A \). Note that \( A^{\frac{1}{2}} \) is also positive semi-definite and \( A^{\frac{2}{2}} \times A^{\frac{1}{2}} = A \).

## 2 Preliminaries and Assumptions

### 2.1 \( D^2/\text{Exact-Diffusion} \) algorithm

To model the decentralized communication, we let \( w_{ij} \geq 0 \) to be the weight used by node \( i \) to scale information flowing from node \( j \). We let \( \mathcal{N}_i \) denote the neighbors of agent \( i \), including node \( i \). If nodes \( j \notin \mathcal{N}_i \) then \( w_{ij} = 0 \), and if \( j \in \mathcal{N}_i \) then \( w_{ij} > 0 \). We let \( W = [w_{ij}] \) denote the \( n \times n \) matrix and assume the following conditions.

**Assumption 1 (Weight matrix)** The network is strongly connected and the weight matrix \( W \) is doubly stochastic and symmetric, i.e., \( W = W^T \) and \( W \mathds{1}_n = \mathds{1}_n \).

**Remark 1** Under Assumption 1, it holds that \( 1 = \lambda_1(W) > \lambda_2(W) \geq \cdots \geq \lambda_n(W) > -1 \). If we let \( \beta = \rho(W - \frac{1}{n} \mathds{1}_n \mathds{1}_n^T) \) be the spectral gap of the weight matrix \( W \), it follows that \( \beta = \max\{|\lambda_2(W)|, |\lambda_n(W)|\} \in (0, 1) \). The spectral gap \( \beta \) reflects the connectivity of the network topology. The scenario \( \beta \to 0 \) implies a well-connected topology. For example, for fully connected topology, we can choose has \( W = \frac{1}{2} \mathds{1}_n \mathds{1}_n^T \) and hence \( \beta = 0 \). In contrast, the scenario \( \beta \to 1 \) implies a badly-connected topology.

**Algorithm 1:** \( D^2/\text{Exact-Diffusion} \)

**Require:** Let \( \bar{W} = (I + W)/2 \) and initialize \( x_i^{(0)} \) arbitrarily; let \( \psi_i^{(0)} = x_i^{(0)} \).

**for** \( k = 0, 1, 2, \ldots \) **do**

\[
\begin{align*}
\text{Sample } \xi_i^{(k)} \text{ and calculate } g_i^{(k)} = \nabla F(x_i^{(k)}; \xi_i^{(k)}); \\
\text{Update } \psi_i^{(k+1)} = x_i^{(k)} - \gamma g_i^{(k)}; \quad \triangleright \text{local gradient descent step} \\
\text{Update } \phi_i^{(k+1)} = \psi_i^{(k+1)} + \bar{w}_{ij} \phi_j^{(k+1)}; \quad \triangleright \text{solution correction step} \\
\text{Update } x_i^{(k+1)} = \sum_{j \in \mathcal{N}_i} \bar{w}_{ij} \phi_j^{(k+1)}; \quad \triangleright \text{communication step}
\end{align*}
\]
Using the notation $x^{(k)}$ and $\nabla F(x^{(k)}; \xi^{(k)})$ introduced in Sec. 1.4, the $D^2$ algorithm [16], also known as Exact-Diffusion in [17, 18, 15] or NIDS in [42], can be written as

$$x^{(k+1)} = \bar{W} \left( 2x^{(k)} - x^{(k-1)} - \gamma \left( \nabla F(x^{(k)}; \xi^{(k)}) - \nabla F(x^{(k-1)}; \xi^{(k-1)}) \right) \right), \quad \forall k = 1, 2, \ldots$$  \hspace{1cm} (7)

where $\bar{W} := (W + I)/2$ and $\gamma$ is a learning rate parameter. The algorithm is initialized with $x^{(1)} = \bar{W} (x^{(0)} - \gamma (\nabla F(x^{(0)}; \xi^{(0)}))$ for any $x^{(0)}$. Different from the vanilla decentralized SGD (D-SGD), $D^2$/Exact-Diffusion exploits the last two consecutive iterates and stochastic gradients to update the current variable. Such algorithm construction is proved to be able to remove the influence of data heterogeneity, see [16, 18, 15, 19].

Recursion (7) can be conducted in a decentralized manner as listed in Algorithm 1 (see [17, 15] for details). A fundamental difference from D-SGD lies in the solution correction step. If $x_i^{(k)} - \psi_i^{(k)}$ is removed, $D^2$/Exact-Diffusion will reduce to D-SGD.

### 2.2 Primal-dual recursion

To ease the analysis, the algorithm recursion (7) can be rewritten into the following primal-dual form [18, 19]:

$$\begin{cases}
x^{(k+1)} = \bar{W} \left( x^{(k)} - \gamma \nabla F(x^{(k)}; \xi^{(k)}) \right) - V y^{(k)}, \\
y^{(k+1)} = y^{(k)} + V x^{(k+1)}, \quad \forall k = 0, 1, 2, \ldots
\end{cases}$$  \hspace{1cm} (8)

where $y = [y_1, \ldots, y_n]^T \in \mathbb{R}^{n \times d}$, $y_i \in \mathbb{R}^d$, and $V = (I - \bar{W})^{-1/2} \in \mathbb{R}^{n \times n}$ is a positive semi-definite matrix. For initialization, we let $y^{(0)} = 0$. To verify the equivalence between (8) and (7), we notice that the first recursion in (8) implies

$$x^{(k+1)} - x^{(k)} = \bar{W} \left( x^{(k)} - x^{(k-1)} - \gamma \left( \nabla F(x^{(k)}; \xi^{(k)}) - \nabla F(x^{(k-1)}; \xi^{(k-1)}) \right) \right)$$

$$- V (y^{(k)} - y^{(k-1)}), \quad \forall k = 1, 2, \ldots$$  \hspace{1cm} (9)

Substituting the second recursion in (8) to the above recursion and using $V^2 = I - \bar{W}$ will result in the $D^2$/Exact-Diffusion algorithm in (7). When $k = 0$, it is straightforward to verify the initialization conditions in (7) and (8) are also equivalent.

### 2.3 Optimality condition

The primal-dual recursion (8) facilitates the following optimality condition of problem (1).

**Lemma 1 (Optimality Condition)** Assume each cost function $f_i(x)$ in problem (1) is convex. Then, there exists some primal-dual pair $(x^*, y^*)$ that satisfies

$$\gamma \bar{W} \nabla f(x^*) + V y^* = 0, \hspace{1cm} (10)$$

$$V x^* = 0, \hspace{1cm} (11)$$

and it holds that $x_1^* = \cdots = x_n^* = x^*$ where $x^*$ is a global solution to problem (1).
The proof of the above lemma is simple and can be referred to [32, Lemma 3.1]. It is worth noting that when there is no gradient noise, i.e., \( \nabla F(x^{(k)}; \xi^{(k)}) = \nabla f(x^{(k)}) \), the fixed point \((x^o, y^o)\) of the primal-dual recursion (8) satisfies the optimality condition (10)–(11). This implies the iterates \( x^{(k)} \) generated by the \( D^2/\text{Exact-Diffusion} \) algorithm will converge to a global solution \( x^* \) of problem (1) in expectation. Such conclusion holds without any assumption whether the data distribution is homogeneous or not.

### 2.4 D-SGD suffers from an intrinsic data-heterogeneity bias

The vanilla D-SGD algorithm (also referred to as diffusion in [5, 7, 27]) [6, 5, 7, 8] iterates as follows:

\[
    x^{(k+1)} = W(x^{(k)} - \gamma \nabla F(x^{(k)}; \xi^{(k)})).
\]

Without any auxiliary variable \( y \) to help correct the gradient direction like D2/Exact-Diffusion recursion (8), D-SGD suffers from a solution deviation caused by data heterogeneity even if there is no gradient noise. Since data heterogeneity exists, we have \( b^2 = \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x^*)\|^2 > 0 \), which implies there exists at least one node \( i \) such that \( \nabla f_i(x^*) \neq 0 \). Suppose there is no gradient noise and \( x^{(k)} \) is initialized as a consensual solution \( x^* = [x^*, \ldots, x^*]^T \) where \( x^* \) is a solution to problem (1), which satisfies \( \mathbb{1}^T_n \nabla f(x^*) = 0 \). Following recursion (12), we have

\[
    x^{k+1} = W(x^* - \gamma \nabla f(x^*)) = x^* - \gamma W \nabla f(x^*) \neq x^*,
\]

in which the last inequality holds because there exist at least one nodes \( i \) such that \( \nabla f_i(x^*) \neq 0 \) and \( w_{ij} > 0 \). Relation (13) implies that even if D-SGD starts from the optimal consensual solution, it will still jump away to a biased solution due to the data heterogeneity. This bias motivates the development of D\(^2\)/Exact-Diffusion.

### 2.5 Assumptions

We now introduce some standard assumptions that will be used throughout the paper.

**Assumption 2** (Convexity) Each cost function \( f_i(x) \) is convex.

**Assumption 3** (Smoothness) Each local cost function \( f_i(x) \) is differentiable, and there exists a constant \( L \) such that for each \( x, y \in \mathbb{R}^d \):

\[
    \|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x - y\|, \quad \forall \, i \in [n].
\]

**Assumption 4** (Gradient noise) It is assumed that for any \( k \) and \( i \) that

\[
    \mathbb{E}[\|\nabla F(x^{(k)}_i; \xi^{(k)}_i) - \nabla f_i(x^{(k)}_i)|F^{(k-1)}]\| = 0,
\]

\[
    \mathbb{E}[\|\nabla F(x^{(k)}_i; \xi^{(k)}_i) - \nabla f_i(x^{(k)}_i)|F^{(k-1)}\|^2] \leq \sigma^2
\]

for some constant \( \sigma^2 \geq 0 \). Moreover, we assume \( \xi^{(k)}_i \) is independent of each other for any \( k \) and \( i \). The filtration is defined as \( \mathcal{F}^{(k)} = \{\xi^{(k)}_i\}_{i=1}^n, \{x^{(k)}_i\}_{i=1}^n, \ldots, \{\xi^{(0)}_i\}_{i=1}^n, \{x^{(0)}_i\}_{i=1}^n\}. \)
3 Fundamental Transformation

In this section we will transform the primal-dual update of the D²/Exact-Diffusion algorithm (8) into another equivalent recursion. This transformation, which is inspired from [18, 15], is fundamental to establish the convergence results for D²/Exact-Diffusion.

Let \((x^*, y^*)\) be any pair of variables that satisfies the optimality condition (10) and (11). Subtracting (10) and (11) from the primal-dual recursion (8), we have

\[
\begin{align*}
x^{(k+1)} - x^* &= \bar{W}(x^{(k)} - x^* - \gamma(VF(x^{(k)}; \xi^{(k)}) - \nabla f(x^*))) - V(y^{(k)} - y^*), \\
y^{(k+1)} - y^* &= y^{(k)} - y^* + V(x^{(k+1)} - x^*).
\end{align*}
\]

To simplify the notation, we define \(s^{(k)} := \nabla F(x^{(k)}; \xi^{(k)}) - \nabla f(x^{(k)})\) as the gradient noise at iteration \(k\). By substituting (17) into (18), and recalling that \(I - V^2 = \bar{W}\), we obtain

\[
\begin{bmatrix}
x^{(k+1)} - x^* \\
y^{(k+1)} - y^*
\end{bmatrix} =
\begin{bmatrix}
\bar{W} & -V \\
V\bar{W} & W
\end{bmatrix}
\begin{bmatrix}
x^{(k)} - x^* \\
y^{(k)} - y^*
\end{bmatrix} - \gamma
\begin{bmatrix}
\nabla f(x^{(k)}) - \nabla f(x^*) + s^{(k)} \\
0
\end{bmatrix}.
\]

The main difficulty to analyze the convergence of the above recursion (as well as (17)–(18)) is that terms \(x^{(k)} - x^*\) and \(y^{(k)} - y^*\) are entangled together to update \(x^{(k+1)} - x^*\) or \(y^{(k+1)} - y^*\). For example, the update of \(x^{(k+1)} - x^*\) has to rely on both \(\bar{W}(x^{(k)} - x^*)\) and \(-V(y^{(k)} - y^*)\). In the following, we identify a convenient change of basis and transform (19) into another equivalent form so that the involved iterated variables can be “decoupled”. To this end, we need to introduce a fundamental decomposition lemma. This lemma was first established in [18]. We have improved this lemma by establishing an upper bound of an important term (see (23)) that is critical for our later analysis.

**Lemma 2 (Fundamental Decomposition)** Under Assumption 1, the matrix \(B \in \mathbb{R}^{2n \times 2n}\) in (19) can be diagonalized as

\[
B = \begin{bmatrix}
1 & 0 & 0 & \ell_1^T & \ell_2^T \\
0 & 1 & 0 & X_L/c & X_L/c \\
0 & 0 & X_{-1} & 0 & 0
\end{bmatrix}
\]

for any constant \(c > 0\) where \(D \in \mathbb{R}^{2n \times 2n}\) is a diagonal matrix. Moreover, we have

\[
r_1 = \begin{bmatrix}
I_n \\
0
\end{bmatrix}, \quad r_2 = \begin{bmatrix}
0 & I_n \\
0 & 0
\end{bmatrix}, \quad \ell_1 = \begin{bmatrix}
\frac{1}{n}I_n \\
\frac{1}{n}I_n
\end{bmatrix}, \quad \ell_2 = \begin{bmatrix}
0 \\
\frac{1}{n}I_n
\end{bmatrix}
\]

and \(X_R \in \mathbb{R}^{2n \times 2(n-1)}\), \(X_L \in \mathbb{R}^{2(n-1) \times 2n}\). Also, the matrix \(D_1\) is a diagonal matrix with diagonal entries strictly less than 1 in magnitudes and

\[
\|D_1\| = \bar{\lambda}_2^{1/2}, \quad \text{where} \quad \bar{\lambda}_2 = \frac{1 + \lambda_2(W)}{2}.
\]
Furthermore, it holds that
\[ \|X_L\|X_R\| \leq \hat{\lambda}_n^{-1/2} \]  
where \( \hat{\lambda}_n = (1 + \lambda_n(W))/2 \). (Proof is in Appendix B).

If we left-multiply \( X^{-1} \) to both sides of (19) and use Lemma 2, then can get the transformed recursion given in Lemma 3. The proof of Lemma 3 is shown in [15, Lemma 3] and:

**Lemma 3 (Transformed Recursion)** Under Assumptions 1, the \( D^2/\text{Exact-Diffusion} \) error recursion (19) can be transformed into

\[
\begin{bmatrix}
\bar{z}^{(k+1)} \\
\tilde{z}^{(k+1)}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{n} \tilde{z}^{(k)} - \frac{1}{n} \tilde{z}^{(k)} & D_1 \tilde{z}^{(k)} - \frac{1}{c} D_1 \tilde{g}^{(k)} \\
\end{bmatrix} - \gamma \begin{bmatrix}
\tilde{s}^{(k)} \\
\tilde{s}^{(k)}
\end{bmatrix}
\]

where \( \tilde{g}^{(k)}, \tilde{s}^{(k)} \) and \( \tilde{s}^{(k)} \) are defined as

\[
\begin{align*}
\tilde{g}^{(k)} & \triangleq X_{L,\ell} \left( \nabla f(x^{(k)}) - \nabla f(x^*) \right) \\
\tilde{s}^{(k)} & \triangleq \frac{1}{n} I^T s^{(k)} \\
\tilde{s}^{(k)} & \triangleq X_{L,\ell} s^{(k)}
\end{align*}
\]

and \( X_{L,\ell} \in \mathbb{R}^{2(n-1) \times n} \) is the left part of the matrix \( X_L = [X_{L,\ell} \quad X_{L,r}] \). The relation between the original and the transformed error vectors are

\[
\begin{bmatrix}
\tilde{z}^{(k)} \\
\tilde{z}^{(k)}
\end{bmatrix} =
\begin{bmatrix}
\ell^T \\
X_{L,c}
\end{bmatrix}
\begin{bmatrix}
x^{(k)} - x^* \\
x^{(k)} - y^*
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
x^{(k)} - x^* \\
x^{(k)} - y^*
\end{bmatrix} =
\begin{bmatrix}
-\gamma cX_R \\
\ell^T
\end{bmatrix}
\begin{bmatrix}
\tilde{z}^{(k)} \\
\tilde{z}^{(k)}
\end{bmatrix}.
\]

Note that \( \tilde{z}^{(k)} \in \mathbb{R}^{1 \times d} \) and \( \tilde{z}^{(k)} \in \mathbb{R}^{2(n-1) \times d} \).

**Remark 2 (Error recursion interpretation)** Using the left relation in (28) and the definition of \( \ell_1 \) in (21), it holds that

\[
[\tilde{z}^{(k)}]^T = \frac{1}{n} I^T (x^{(k)} - x^*) = \bar{z}^{(k)} - x^*.
\]

Therefore, \( \tilde{z}^{(k)} \) is the average error, i.e., difference between the averaged variable \( \bar{z}^{(k)} \) and the solution \( x^* \). On the other hand, using the right relation in (28) and the definition of \( r_1 \) in (21), it holds that

\[
x^{(k)} - x^* = \mathbb{1}_n \tilde{z}^{(k)} + cX_{R,u} \tilde{z}^{(k)} = \bar{x}^{(k)} - x^* + cX_{R,u} \tilde{z}^{(k)},
\]

where \( X_{R,u} \in \mathbb{R}^{n \times 2(n-1)} \) is the upper part of matrix \( X_R = [X_{R,u}; X_{R,d}] \). This implies that

\[
cX_{R,u} \tilde{z}^{(k)} = x^{(k)} - \bar{x}^{(k)}.
\]

This implies that \( \tilde{z}^{(k)} \) measures the consensus error between \( x^{(k)} \) and \( \bar{x}^{(k)} \).

One benefit that recursion (24) introduces is the iterates \( \tilde{z}^{(k)} \) and \( \tilde{z}^{(k)} \) are untangled when updating \( \tilde{z}^{(k+1)} \) and \( \tilde{z}^{(k+1)} \). For example, the update of \( \tilde{z}^{(k+1)} \) only relies on \( \tilde{z}^{(k)} \) but not on \( \tilde{z}^{(k)} \). Such decoupling will bring in significant convenience in convergence analysis for \( D^2/\text{Exact-Diffusion} \).
4 Convergence results: Generally-Convex Scenario

With Assumption 4, it is easy to verify that
\[
\mathbb{E}[\|\bar{z}^{(k)}\|^2 | \mathcal{F}^{(k-1)}] \leq \frac{\sigma^2}{n}, \quad k = 1, 2, \ldots
\]  
(31)

We first establish a descent lemma for the D^2/Exact-Diffusion algorithm under the generally-convex setting, which explains how \(\mathbb{E}[\|\bar{z}^{(k)}\|^2]\) evolves with iteration.

**Lemma 4 (Descent Lemma)** Under Assumptions 2-4 and learning rate \(\gamma < \frac{1}{4L}\), it holds that
\[
\mathbb{E}\|\bar{z}^{(k+1)}\|^2 \leq \mathbb{E}\|\bar{z}^{(k)}\|^2 - \gamma (\mathbb{E}f(\bar{z}^{(k)}) - f(x^*)) + \frac{3L\gamma}{2n\lambda_n} \mathbb{E}\|\bar{z}^{(k)}\|^2_F + \frac{\gamma^2\sigma^2}{n}, \quad k = 0, 1, \ldots
\]  
(32)

where \(\bar{\lambda}_n = (1 + \lambda_n(W))/2\). (Proof is in Appendix C.1)

With inequality (32), we have for \(T \geq 0\) that
\[
\frac{1}{T+1} \sum_{k=0}^{T} (\mathbb{E}f(\bar{z}^{(k)}) - f(x^*)) \leq \frac{\mathbb{E}\|\bar{z}^{(0)}\|^2}{\gamma(T+1)} + \frac{3L}{2n\lambda_n(T+1)} \sum_{k=0}^{T} \mathbb{E}\|\bar{z}^{(k)}\|^2_F + \frac{\gamma\sigma^2}{n}.
\]  
(33)

We next bound the ergodic consensus term on the right-hand-side.

**Lemma 5 (Consensus Lemma)** Under Assumptions 1-4 and learning rate \(\gamma \leq \frac{(1-\beta_1)}{2L}\), it holds that
\[
\mathbb{E}\|\bar{z}^{(k+1)}\|^2 \leq \left(1 + \beta_1 \right) \mathbb{E}\|\bar{z}^{(k)}\|^2 + \frac{4n^2\beta_2^2L}{1 - \beta_1} (\mathbb{E}f(\bar{z}^{(k)}) - f(x^*)) + n\gamma^2\beta_1^2\sigma^2, \quad k = 0, 1, \ldots
\]  
(34)

where \(\beta_1 = \bar{\lambda}_2^{1/2}\) and \(\bar{\lambda}_2 = (1 + \lambda_2(W))/2\) (Proof is in Appendix C.2).

**Lemma 6 (Ergodic Consensus Lemma)** Under the same assumptions as Lemma 5, it holds that
\[
\frac{1}{T+1} \sum_{k=0}^{T} \mathbb{E}\|\bar{z}^{(k)}\|^2_F \leq \frac{16n^2\gamma^2\beta_1^2L}{(1 - \beta_1)^2(T+1)} \sum_{k=0}^{T} (\mathbb{E}f(\bar{z}^{(k)}) - f(x^*)) + \frac{2n\gamma^2\beta_1^2\sigma^2}{1 - \beta_1} + \frac{3C}{(1 - \beta_1)(T+1)}
\]  
(35)

where \(C = \mathbb{E}\|\bar{z}^{(0)}\|^2\) is an initialization constant (Proof is in Appendix C.3).

With inequalities (35) and (33), and the fact that \(\beta_1^2 = \bar{\lambda}_2 = (1 + \lambda_2(W))/2 \leq (1 + \beta)/2\) where \(\beta = \rho(W - \frac{1}{n}I I^T)\) is defined in Remark 1, we can show the following convergence result for D^2/Exact-Diffusion in the generally convex scenario.

**Theorem 1 (Convergence Property)** Under Assumptions 1-4 and learning rate
\[
\gamma = \min \left\{ \left(1 - \beta \right) \frac{\bar{\lambda}_n}{28L}, \left(\frac{r_0}{r_1(T+1)}\right)^\frac{1}{2}, \left(\frac{r_0}{r_2(T+1)}\right)^\frac{1}{2} \right\}
\]  
(36)

where \(r_0, r_1\) and \(r_2\) are constants defined in (95), and \(\bar{\lambda}_n\) is bounded away from 0, then it holds that
\[
\frac{1}{T+1} \sum_{k=0}^{T} (\mathbb{E}f(\bar{z}^{(k)}) - f(x^*)) = O \left( \frac{\sigma^2}{\sqrt{nT}} + \frac{\sigma^2}{(1 - \beta)^2T^2} + \frac{1}{(1 - \beta)T} \right).
\]  
(37)

(Proof is in Appendix C.4)
Remark 3 The bound in (37) is a result of the bound (102) where we ignored the term $\tilde{\lambda}_2/\tilde{\lambda}_n$ from (37). It is because $\tilde{\lambda}_2/\tilde{\lambda}_n$ can be regarded as a constant when $\tilde{\lambda}_n$ is bounded away from zero. For example, if $W = (3I + W)/4$, we have $\tilde{\lambda}_n \geq 1/2$ so that $\tilde{\lambda}_2/\tilde{\lambda}_n \leq 1/\tilde{\lambda}_n \leq 2$.

Corollary 1 (Transient stage) Under the same assumptions as Theorem 1, the transient stage for $D^2$/Exact-Diffusion is on the order of $\Omega((n^3)^2)$. 

Proof To achieve the linear speedup stage, $T$ has to be large enough so that 

$$\frac{\sigma^2}{(1-\beta)^2T^2} \leq \frac{\sigma}{\sqrt{nT}}, \quad \frac{1}{(1-\beta)T} \leq \frac{\sigma}{\sqrt{nT}},$$

which is equivalent to $T \geq \max\{\frac{n^3}{(1-\beta)^2}, \frac{n^3}{(1-\beta)^2}\} = \Omega((n^3)^2)$. 

Remark 4 It is established in [13] that the transient stage of $D$-SGD for the generally convex scenario is on the order of $\Omega((n^3)^2)$. By removing the influence of the data heterogeneity, $D^2$/Exact-Diffusion improves the transient stage to $\Omega((n^3)^2)$, which has a better network topology dependence on $1-\beta$.

5 Convergence results: Strongly-Convex Scenario

5.1 Convergence analysis of $D^2$/Exact-Diffusion

In this subsection we examine the convergence property of $D^2$/Exact-Diffusion algorithm in the strongly convex scenario and clarifies its transient stage.

Assumption 5 (Strongly convex) Each $f_i(x)$ is strongly convex, i.e., there exists a constant $\mu > 0$ such that for any $x, y \in \mathbb{R}^d$ we have:

$$f_i(y) \geq f_i(x) + \langle \nabla f_i(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2, \quad \forall i \in [n]. \quad (39)$$

Lemma 7 (Descent Lemma) When Assumptions 3-5 hold and learning rate $\gamma < \frac{1}{4L}$, it holds for $k = 1, 2, \ldots$ that

$$E\|\tilde{z}^{(k+1)}\|^2 \leq \left(1 - \frac{\gamma \mu}{2}\right)E\|\tilde{z}^{(k)}\|^2 - \gamma(Ef(\tilde{x}^{(k)}) - f(x^*)) + \frac{5L \gamma}{2n \lambda_n} E\|\tilde{z}^{(k)}\|^2_F + \frac{\gamma^2 \sigma^2}{n} \quad (40)$$

(Proof is in Appendix D.1)

With inequality (40), we have

$$Ef(\tilde{x}^{(k)}) - f(x^*) \leq \left(1 - \frac{\gamma \mu}{2}\right) \frac{E\|\tilde{z}^{(k)}\|^2}{\gamma} - \frac{E\|\tilde{z}^{(k+1)}\|^2}{\gamma} + \frac{5L \gamma}{2n \lambda_n} E\|\tilde{z}^{(k)}\|^2_F + \frac{\gamma^2 \sigma^2}{n} \quad (41)$$
If we take the uniform average for both sides over \( k = 0, \ldots, T \), the term \( (1 - \frac{\mu}{k}) \frac{\|z^{(k)}\|^2}{r} \) from the \( k \)-th iteration cannot cancel the term \( -\frac{\|Ez^{(k)}\|^2}{\gamma} \) from the \((k+1)\)-th iteration. We next take the weighted average for both sides over \( k = 0, \ldots, T \) so that

\[
\frac{1}{HT} \sum_{k=0}^{T} h_k (E f(\bar{x}^{(k)}) - f(x^*)) \leq \frac{1}{HT} \sum_{k=0}^{T} h_k \left( \frac{(1 - \frac{\mu}{k})E\|z^{(k)}\|^2}{\gamma} - \frac{E\|z^{(k+1)}\|^2}{\gamma} \right) + \frac{5L}{2nHT\lambda_n} \sum_{k=0}^{T} h_k E\|\hat{z}^{(k)}\|_F^2 + \frac{\gamma\sigma^2}{n},
\]

(42)

where \( h_k \geq 0 \) is some weight to be determined, and \( HT = \sum_{k=0}^{T} h_k \). If we let \( h_k = (1 - \frac{\mu}{k})h_{k+1} \) for \( k = 0, 1, \ldots \), the above inequality becomes

\[
\frac{1}{HT} \sum_{k=0}^{T} h_k (E f(\bar{x}^{(k)}) - f(x^*)) \leq h_0 E\|\hat{z}^{(0)}\|_F^2 \leq \frac{5L}{2nHT\lambda_n} \sum_{k=0}^{T} h_k E\|\hat{z}^{(k)}\|_F^2 + \frac{\gamma\sigma^2}{n}.
\]

(43)

We next bound the ergodic consensus term in the right-hand-side.

**Lemma 8 (Ergodic Consensus Lemma)** Under Assumptions 1, 3, 4, and 5 and if learning rate satisfies

\[
\gamma \leq \frac{(1-\beta_1)^{1/2}}{2L\lambda_2},
\]

then it holds that

\[
\frac{1}{HT} \sum_{k=0}^{T} h_k E\|\hat{z}^{(k)}\|_F^2 \leq \frac{4C\beta_0}{HT(1-\beta_1)} + \frac{2n\gamma^2\beta_0^2\sigma^2}{1-\beta_1} + \frac{64n\gamma^2\beta_0^2L}{(1-\beta_1)^2HT} \sum_{k=0}^{T} h_k (E f(\bar{x}^{(k)}) - f(x^*))
\]

(44)

where \( C = E\|\hat{z}^{(0)}\|_F^2 \), the positive weights \( \{h_k\}_{k=0}^{\infty} \) satisfy

\[
h_k \leq h_\ell \left( 1 + \frac{1-\beta_1}{4} \right)^{k-\ell} \quad \text{for any } k \geq 0 \text{ and } 0 \leq \ell \leq k,
\]

(45)

and \( HT = \sum_{k=0}^{T} h_k \). (Proof is in Appendix D.2)

With inequalities (43) and (44), and the fact that \( \beta_0^2 = \bar{\lambda}_2 = (1 + \lambda_2(W))/2 \leq (1 + \beta)/2 \) where \( \beta = \rho(W - \frac{1}{n} \mathbf{1}\mathbf{1}^T) \), we can establish the convergence property of \( D^2/\text{Exact-Diffusion} \) as follows.

**Theorem 2 (Convergence Property)** Under Assumptions 1, 3, 4, and 5, if

\[
\gamma = \min \left\{ \frac{2\ln(2n\mu\sigma_0^2CT^2)/\sigma^2}{\mu T}, \frac{1-\beta_1}{2L} \left( \frac{\lambda_2^{1/2}}{\lambda_1^{1/2}} \right) \right\},
\]

(46)

and \( \bar{\lambda}_n \) is bounded away from zero, it holds that

\[
\frac{1}{HT} \sum_{k=0}^{T} h_k (E f(\bar{x}^{(k)}) - f(x^*)) = \tilde{O} \left( \frac{\sigma^2}{nT} + \frac{\sigma^2}{(1-\beta)T^2} + \frac{1}{1-\beta} \exp\{-(1-\beta)T\} \right).
\]

(47)

where \( h_k \) and \( HT \) are defined in Lemma 8. Notation \( \tilde{O}(\cdot) \) hides all logarithm factors. (Proof is in Appendix D.3)

**Remark 5** Comparing the bound in (47) and a more accurate bound in (128), it is observed we ignored the term \( \bar{\lambda}_2/\bar{\lambda}_n \) in (47). It is because \( \bar{\lambda}_2/\bar{\lambda}_n \) can be regarded as a constant when \( \bar{\lambda}_n \) is bounded away from zero.
Corollary 2 (Transient stage) Under the same assumptions as Theorem 2, the transient stage for D^2/Exact-Diffusion in the strongly convex scenario is on the order of $\tilde{\Omega}(\frac{n}{1-\beta})$.

Proof The third term (47) decays exponentially fast and hence can be ignored compared to the first two terms. To reach the linear speedup, it is enough to set

$$\frac{\sigma^2}{(1-\beta)T^2} \leq \frac{\sigma^2}{nT},$$

which amounts to $T \geq \frac{n}{1-\beta}$.

We use $\tilde{\Omega}(\cdot)$ rather than $\Omega(\cdot)$ because some logarithm factors are hidden inside.

Remark 6 It is established in [14, 13] that the transient stage of D-SGD for the strongly-convex scenario is on the order of $\Omega(\frac{n}{(1-\beta)^2})$. By removing the influence of the data heterogeneity, D^2/Exact-Diffusion improves the transient stage to $\Omega(\frac{n}{1-\beta})$ which has an improved dependence on $1-\beta$.

5.2 Transient stage lower bound of the homogeneous D-SGD

In Sec. 5.1, we have shown that D^2/Exact-Diffusion, by removing the influence of data heterogeneity, can improve the transient stage of D-SGD from $\Omega(\frac{n}{(1-\beta)^2})$ to $\Omega(\frac{n}{1-\beta})$. In this section, we ask what is the optimal transient stage of D-SGD if the data distributions are homogeneous (i.e., there is no influence of data heterogeneity)? Can D-SGD have a better network topology dependence than D^2/Exact-Diffusion in certain scenarios? The answer reveals that D-SGD dependence on the network topology can match D^2/Exact-Diffusion only under the homogeneous setting and always worse in heterogeneous setting. In any cases, D-SGD cannot be more robust to network topology than D^2/Exact-Diffusion.

Theorem 3 (Lower Bound) There exists strongly convex and smooth functions and certain type of gradient noise such that when learning rate $\gamma$ is set as in (46), the D-SGD algorithm (12) with homogeneous data distributions will require at least $\tilde{\Omega}(1/(1-\beta))$ iterations to reach the linear speedup stage for any weight matrix $W$ satisfying Assumption 1 (Proof is in Appendix E).

From Corollary 2 and Theorem 3, we observe that the transient stage of D^2/Exact-Diffusion in the strongly-convex scenario coincides with the lower bound of homogeneous D-SGD in terms of the dependence on network topology (i.e., the influence of $\beta$). This imply that D-SGD has the same dependence on $1-\beta$ as D^2/Exact-Diffusion under the impractical homogeneous case and worse dependence in the heterogeneous case [13, 14]. Hence, the dependence of D^2/Exact-Diffusion on network topology is no worse than D-SGD and always better under the practical heterogeneous case.

Remark 7 Note that there is still a mismatch in dependence on the network size $n$. It is possible that the established lower bound in Theorem 3 is loose in $n$. It is also possible that the transient stage of D^2/Exact-Diffusion has a potential to be independent of $n$. We will leave this line of research as a future work.
6 Conclusion and Discussion

In this work, we revisited the D\textsuperscript{2}/Exact-Diffusion algorithm [16, 17, 18, 19, 15] and studied its non-asymptotic convergence rate under both the generally-convex and strongly-convex settings. By removing the influence of data heterogeneity, D\textsuperscript{2}/Exact-Diffusion is shown to improve the transient stage of D-SGD from $\Omega(n^3/(1-\beta)^4)$ to $\Omega(n^3/(1-\beta)^2)$ and from $\Omega(n/(1-\beta)^2)$ to $\Omega(n/(1-\beta))$ for the generally convex and strongly-convex settings, respectively. This result shows that D\textsuperscript{2}/Exact-Diffusion [16, 17, 18, 19, 15] is less sensitive to the network topology. For the strongly-convex scenario, we also proved that our transient stage bound coincides with the lower bound of homogeneous D-SGD in terms of network topology dependence, which implies that D\textsuperscript{2}/Exact-Diffusion cannot have worse network dependence than D-SGD and has a better dependence under the heterogeneous setting.

There are still several open questions to answer for the family of data-heterogeneity-corrected methods such as EXTRA, D\textsuperscript{2}/Exact-Diffusion, and gradient-tracking. First, it is unclear whether an improved transient stage dependence on $1-\beta$ can be established for non-convex settings. Note that the analysis used in this paper utilizes the convexity of $f(x)$ and cannot be directly extended to the non-convex settings. Second, it is also unclear whether the transient stage of gradient-tracking methods can be enhanced. Although gradient-tracking methods remove the data heterogeneity effect, their existing transient stage results are even worse than D-SGD, see [25, 24]. Third, while data-heterogeneity-corrected methods are endowed with superior convergence properties in terms of robustness to heterogeneous data or network topology dependence, D-SGD can still empirically outperforms them in deep learning applications, see [40, 11]. Great efforts may still be needed to fill in the gap between theory and real implementations.

A Notations and Preliminaries

We first review some notations and facts.

- $W = [w_{ij}] \in \mathbb{R}^{n \times n}$ is a symmetric and doubly stochastic combination matrix
- $\bar{W} = (I + W)/2 \in \mathbb{R}^{n \times n}$
- $V = (I - \bar{W})^{1/2} = (I - \bar{W})^{1/2}$ and hence $I - \bar{W} = V^2$
- $\lambda_i(W)$ is the $i$-th largest eigenvalue of matrix $W$, and $\tilde{\lambda}_i(W) = (1 + \lambda_i(W))/2$ is the $i$-th largest eigenvalue of matrix $\bar{W}$. Note that $\lambda_i(W) \in (-1, 1)$ and $\tilde{\lambda}_i(W) \in (0, 1)$ for $i = 2, \ldots, n$.
- Let $\Lambda = \text{diag}\{\lambda_1(W), \ldots, \lambda_n(W)\} \in \mathbb{R}^{n \times n}$. It holds that $W = QAQ^T$ where $Q = [q_1, q_2, \ldots, q_n] \in \mathbb{R}^{n \times n}$ is an orthogonal matrix and $q_1 = \frac{1}{\sqrt{n}} \mathbb{1}_n$.
- $\bar{W} = QAQ^T$ where $\bar{\Lambda} = (I + \Lambda)/2$.
- $V = Q(I - \bar{\Lambda})^{1/2}Q^T$
If a matrix $A \in \mathbb{R}^{n \times n}$ is normal, i.e., $AA^T = A^TA$, it holds that $A = UDU^*$ where $D$ is a diagonal matrix and $U$ is a unitary matrix.

If a matrix $\Pi \in \mathbb{R}^{n \times n}$ is a permutation matrix, it holds that $\Pi^{-1} = \Pi^T$.

**Smoothness.** Since each $f_i(x)$ is assumed to be $L$-smooth in Assumption 3, it holds that $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$ is also $L$-smooth. As a result, the following inequality holds for any $x, y \in \mathbb{R}^d$:

$$f_i(x) - f_i(y) - \frac{L}{2} \|x - y\|^2 \leq \langle \nabla f_i(y), x - y \rangle$$

(49)

**Smoothness and convexity.** If each $f_i(x)$ is further assumed to be convex (see Assumption 2), it holds that $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$ is also convex. For this scenario, it holds for any $x, y \in \mathbb{R}^d$ that:

$$\|\nabla f(x) - \nabla f(x^*)\|^2 \leq 2L(f(x) - f(x^*))$$

(50)

$$f_i(x) - f_i(y) \leq \langle \nabla f_i(x), x - y \rangle$$

(51)

**Submultiplicativity of the Frobenius norm.** Given matrices $W \in \mathbb{R}^{n \times n}$ and $y \in \mathbb{R}^{n \times d}$, it holds that

$$\|Wy\|_F \leq \|W\|_2 \|y\|_F. \quad (52)$$

To verify it, by letting $y_j$ be the $j$-th column of $y$, we have $\|Wy\|_F^2 = \sum_{j=1}^{d} \|Wy_j\|_2^2 \leq \sum_{j=1}^{d} \|W\|_2^2 \|y_j\|_2^2 = \|W\|_2^2 \|y\|_F^2$. 

### B The Fundamental Decomposition

We now analyze the eigen-decomposition of matrix $B$:

$$B = \begin{bmatrix} \tilde{W} & -V \\ V\tilde{W} & \tilde{W} \end{bmatrix}. \quad (53)$$

**Proof.** Using $\tilde{W} = Q\tilde{\Lambda}Q^T$, it holds that

$$B = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \tilde{\Lambda} & -(I - \tilde{\Lambda})^{1/2} \\ \tilde{\Lambda}(I - \tilde{\Lambda})^{1/2} & \tilde{\Lambda} \end{bmatrix} \begin{bmatrix} Q^T & 0 \\ 0 & Q^T \end{bmatrix}. \quad (54)$$

Note that $(I - \tilde{\Lambda})^{1/2} = (I - \tilde{\Lambda})^{1/2}\tilde{\Lambda}$ because both $\tilde{\Lambda}$ and $I - \tilde{\Lambda}$ are diagonal matrices. We next introduce

$$E_{(i)} = \begin{bmatrix} \tilde{\lambda}_i & -(1 - \tilde{\lambda}_i)^{1/2} \\ \tilde{\lambda}_i(1 - \tilde{\lambda}_i)^{1/2} & \tilde{\lambda}_i \end{bmatrix} \in \mathbb{R}^{2 \times 2} \quad (55)$$

and

$$E = \text{BlockDiag}\{E_{(1)}, \ldots, E_{(n)}\} \in \mathbb{R}^{2n \times 2n} \quad (56)$$

where $\tilde{\lambda}_i = \lambda_i(\tilde{W})$, and $E$ is a block diagonal matrix with each $i$-th block diagonal matrix as $E_{(i)}$. It is easy to verify that there exists some permutation matrix $\Pi$ such that

$$B = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \Pi E \Pi^T \begin{bmatrix} Q^T & 0 \\ 0 & Q^T \end{bmatrix}. \quad (57)$$
Next we focus on the matrix $E_{(i)}$ defined in (55). Note that $E_{(1)} = I$. For $i \geq 2$, it holds that

$$E_{(i)} = \begin{bmatrix}
1 & 0 \\
0 & \bar{\lambda}^{1/2}
\end{bmatrix} \begin{bmatrix}
\lambda_i & -[\lambda_i(1 - \bar{\lambda}_i)]^{1/2} \\
[\lambda_i(1 - \bar{\lambda}_i)]^{1/2} & \lambda_i
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & \bar{\lambda}_i^{-1/2}
\end{bmatrix}$$

(58)

Since $G_{(i)}$ is skew symmetric and hence normal, it holds that (see Appendix A)

$$G_{(i)} = U_{(i)} D_{(i)} U_{(i)}^*$$

where $D_{(i)} = \text{diag}\{\sigma_1(G_{(i)}), \sigma_2(G_{(i)})\}$,

(59)

In the above expression, $\sigma_1(G_{(i)})$ and $\sigma_2(G_{(i)})$ are complex eigenvalues of $G_{(i)}$. Moreover, it holds that $|\sigma_1(G_{(i)})| = |\sigma_2(G_{(i)})| = \bar{\lambda}_i^{1/2} < 1$. The quantity $U_{(i)} \in \mathbb{R}^{2 \times 2}$ is a unitary matrix. Next we define $C = \text{BlockDiag}\{C_{(1)}, \ldots, C_{(n)}\}$, $U = \text{BlockDiag}\{U_{(1)}, \ldots, U_{(n)}\}$, and $D = \text{BlockDiag}\{D_{(1)}, \ldots, D_{(n)}\}$. By substituting (58) and (59) into (57), we have

$$B = d \begin{bmatrix}
Q & 0 \\
0 & Q
\end{bmatrix} \PiCU D_{(i)} U_{(i)}^* C_{(i)}^{-1} \Pi^T \begin{bmatrix}
Q^T & 0 \\
0 & Q^T
\end{bmatrix} d^{-1}$$

(60)

where $d$ is any positive constant. Next we define

$$X = d \begin{bmatrix}
Q & 0 \\
0 & Q
\end{bmatrix} \PiCU, \quad X^{-1} = U_{(i)}^* C_{(i)}^{-1} \Pi^T \begin{bmatrix}
Q^T & 0 \\
0 & Q^T
\end{bmatrix} d^{-1}.$$  

(61)

By letting $d = \sqrt{n}$ and considering the structure of $Q$, $\Pi$, $C$, and $U$, it is easy to verify that

$$X = [r_1 \ r_2 \ X_R] \text{ where } r_1 = \begin{bmatrix}
1_n \\
0
\end{bmatrix}, \quad r_2 = \begin{bmatrix}
0 \\
1_n
\end{bmatrix}$$

(62)

$$X^{-1} = [\ell_1 \ \ell_2 \ X_L^T]^T \text{ where } \ell_1 = \begin{bmatrix}
\frac{1}{n}1_n \\
0
\end{bmatrix}, \quad \ell_2 = \begin{bmatrix}
0 \\
\frac{1}{n}1_n
\end{bmatrix}$$

(63)

With (60)–(63), it holds that

$$B = XDX^{-1}$$

(64)

where $X$ and $X^{-1}$ take the form of (62) and (63), and

$$D = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & D_1
\end{bmatrix}$$

(65)

and $D_1$ is a diagonal matrix with complex entries. The magnitudes of the diagonal entries in $D_1$ are all strictly less than 1. Next we evaluate the quantity $\|X\||X^{-1}|$:

$$\|X\||X^{-1}| \leq \left\| \begin{bmatrix}
Q & 0 \\
0 & Q
\end{bmatrix} \right\| \Pi \|C\| \|C^{-1}\| \|U\| \|U^*\| \Pi^T \left\| \begin{bmatrix}
Q^T & 0 \\
0 & Q^T
\end{bmatrix} \right\|$$
\[
\begin{align*}
&\leq \max_i \{\tilde{\lambda}_i^{-1/2}\} = \tilde{\lambda}_n^{-1/2}
\end{align*}
\]  
where (a) holds because \( Q \) is orthogonal, \( U \) is unitary, and \( \Pi^T \Pi = I \). Note that

\[
X_R = XS, \quad \text{and} \quad X_L = S^TX^{-1}
\]

where \( S = [e_3, \ldots, e_{2n}] \in \mathbb{R}^{2n \times (2n-1)} \) and \( e_j \) is the \( j \)-th column of the identity matrix \( I_{2n} \). It then holds that

\[
\|X_R\|\|X_L\| \leq \|X\|\|S\|\|S^T\|\|X^{-1}\| = \|X\|\|X^{-1}\| \leq \tilde{\lambda}_n^{-1/2}
\]

\[\text{(66)}\]

\section{Convergence Analysis for Generally-Convex Scenario}

\subsection{Proof of Lemma 4}

\textbf{Proof} From (28) we have

\[
[x^{(k)}]^T = \left[\frac{1}{n}1^T(x^{(k)} - x^*)\right]^T = \bar{x}^{(k)} - x^* \in \mathbb{R}^d,
\]

where \( \bar{x}^{(k)} = \frac{1}{n}1^Tx^{(k)} \) and \( x^* \) is the global solution to problem (1). With this relation, the first line of (24) becomes

\[
\bar{x}^{(k+1)} - x^* = \bar{x}^{(k)} - x^* - \frac{\gamma}{n}1^T(\nabla f(x^{(k)}) - \nabla f(x^*)) - \gamma \tilde{s}^{(k)}.
\]

\[\text{(69)}\]

The above equality implies that

\[
\mathbb{E}[\|\bar{x}^{(k+1)} - x^*\|^2 | \mathcal{F}^{(k)}] \leq \|\bar{x}^{(k)} - x^* - \frac{\gamma}{n}1^T(\nabla f(x^{(k)}) - \nabla f(x^*))\|^2 + \frac{\gamma^2\sigma^2}{n}
\]

\[\text{(70)}\]

Note that the first term can be expanded as follows.

\[
\|\bar{x}^{(k)} - x^* - \frac{\gamma}{n}\sum_{i=1}^n[\nabla f_i(x_i^{(k)}) - \nabla f_i(x^*)]\|^2
\]

\[\text{(A)}\]

\[
= \|\bar{x}^{(k)} - x^*\|^2 - \frac{2\gamma}{n}\sum_{i=1}^n \langle \bar{x}^{(k)} - x^*, \nabla f_i(x_i^{(k)}) - \nabla f_i(x^*) \rangle + \gamma^2 \|\frac{1}{n}\sum_{i=1}^n[\nabla f_i(x_i^{(k)}) - \nabla f_i(x^*)]\|^2
\]

\[\text{(B)}\]

We now bound the term (A):

\[
= \frac{2\gamma}{n}\sum_{i=1}^n \langle f_i(\bar{x}^{(k)}) - f_i(x_i^{(k)}), \nabla f_i(x_i^{(k)}) - \nabla f_i(x^*) \rangle
\]

\[
= \frac{2\gamma}{n}\sum_{i=1}^n \langle \bar{x}^{(k)} - x_i^{(k)}, \nabla f_i(x_i^{(k)}) \rangle
\]

\[
\geq \frac{2\gamma}{n}\sum_{i=1}^n \langle f_i(\bar{x}^{(k)}) - f_i(x_i^{(k)}), -L\|\bar{x}^{(k)} - x_i^{(k)}\|^2 \rangle + \frac{2\gamma}{n}\sum_{i=1}^n \langle f_i(x_i^{(k)}) - f_i(x^*), \nabla f_i(x_i^{(k)}) \rangle
\]

\[\text{(a)}\]
\[
\begin{align*}
&= \frac{2\gamma}{n} \sum_{i=1}^{n} \left( f_i(\bar{x}^{(k)}) - f_i(x^*) \right) - \frac{\gamma L}{n} \|\bar{x}^{(k)} - x^{(k)}\|_F^2 \\
&= 2\gamma \left( f(\bar{x}^{(k)}) - f(x^*) \right) - \frac{\gamma L}{n} \|\bar{x}^{(k)} - x^{(k)}\|_F^2,
\end{align*}
\]  

(72)

where (a) holds because of each \( f_i(x) \) is convex and \( L \)-smooth. We next bound term (B):

\[
\begin{align*}
&\gamma^2 \frac{1}{n} \sum_{i=1}^{n} \| \nabla f_i(x_i^{(k)}) - \nabla f_i(x^*) \|^2 \\
&= \gamma^2 \frac{1}{n} \sum_{i=1}^{n} \| \nabla f_i(x_i^{(k)}) - \nabla f_i(\bar{x}^{(k)}) + \nabla f_i(\bar{x}^{(k)}) - \nabla f_i(x^*) \|^2 \\
&\leq \frac{2\gamma^2 L^2}{n} \|\bar{x}^{(k)} - x^{(k)}\|_F^2 + 2\gamma^2 \| \nabla f(\bar{x}^{(k)}) - \nabla f(x^*) \|^2 \\
&\leq \frac{2\gamma^2 L^2}{n} \|\bar{x}^{(k)} - x^{(k)}\|_F^2 + 4L\gamma^2 \left( f(\bar{x}^{(k)}) - f(x^*) \right),
\end{align*}
\]

(73)

where the last inequality holds because \( f(x) \) is \( L \)-smooth. Substituting (73) and (72) into (71), we have

\[
\begin{align*}
&\| \bar{x}^{(k)} - x^* - \frac{\gamma}{n} \sum_{i=1}^{n} \nabla f_i(x_i^{(k)}) \|^2 \\
&\leq \| \bar{x}^{(k)} - x^* \|^2 - 2\gamma(1 - 2L\gamma) \left( f(\bar{x}^{(k)}) - f(x^*) \right) + \left( \frac{2L}{n} + \frac{2\gamma^2 L^2}{n} \right) \|\bar{x}^{(k)} - x^{(k)}\|_F^2 \\
&\leq \| \bar{x}^{(k)} - x^* \|^2 - 2\gamma \| \nabla f(x^*) \|^2 + \frac{3\gamma L}{2n} \|\bar{x}^{(k)} - x^{(k)}\|_F^2,
\end{align*}
\]

(74)

where the last inequality holds when \( \gamma \leq \frac{1}{4L} \). Substituting (74) into (70) and taking expectation on the filtration \( \mathcal{F}^{(k)} \), we achieve

\[
\mathbb{E}[\|\bar{x}^{(k+1)} - x^*\|^2] \leq \mathbb{E}[\|\bar{x}^{(k)} - x^*\|^2 - 2\gamma \mathbb{E}[\nabla f(\bar{x}^{(k)}) - f(x^*)] + \frac{3\gamma L}{2n} \mathbb{E}[\|\bar{x}^{(k)} - x^{(k)}\|_F^2 + \frac{\gamma^2 \sigma^2}{n}]
\]

(75)

From relation (28), it holds that

\[
\bar{x}^{(k)} - x^* = I_n \bar{z}^{(k)} + cX_{R,u} \tilde{z}^{(k)} = \bar{z}^{(k)} - x^* + cX_{R,u} \tilde{z}^{(k)},
\]

(76)

where \( X_{R,u} \in \mathbb{R}^{n \times 2(n-1)} \) is the upper part of matrix \( X_R = [X_{R,u}; X_{R,d}] \). The above relation implies that

\[
\bar{x}^{(k)} - x^{(k)} = cX_{R,u} \tilde{z}^{(k)}.
\]

(77)

Substituting the above fact and \( \tilde{z}^{(k)}^T = \bar{z}^{(k)} - x^* \) into (75), we achieve

\[
\begin{align*}
&\mathbb{E}[\|\tilde{z}^{(k+1)}\|^2] \leq \mathbb{E}[\|\tilde{z}^{(k)}\|^2 - 2\gamma(\mathbb{E}[\nabla f(\bar{x}^{(k)}) - f(x^*)] + \frac{3\gamma L}{2n} \mathbb{E}[\|\bar{x}^{(k)} - x^{(k)}\|_F^2 + \frac{\gamma^2 \sigma^2}{n}]
\end{align*}
\]

(78)

where the last inequality holds because

\[
\|X_{R,u}\| = \| \begin{bmatrix} I_n & 0 \end{bmatrix} X_R \| \leq \| \begin{bmatrix} I_n & 0 \end{bmatrix} \| \cdot \|X_R\| = \|X_R\|.
\]

(79)

By setting \( c^2 = \|X_L\|^2 \) and recalling that \( \|X_L\|^2 \|X_R\| \leq \tilde{\lambda}_n^{1/2} \) from Lemma 2, we achieve (32).
C.2 Proof of Lemma 5

Proof From the second line in (24), it holds that

\[ z^{(k+1)} = D_1 z^{(k)} - \frac{\gamma}{c} D_1 \dot{g}^{(k)} - \frac{\gamma}{c} D_1 \dot{s}^{(k)} \]  

(80)

We next introduce \( \beta_1 = \|D_1\| \). With (22), we know that \( \beta_1 = \lambda_2^{1/2} \). By taking mean-square for both sides of the above recursion, we achieve

\[ \mathbb{E}[\|z^{(k+1)}\|^2_F | \mathcal{F}^{(k)}] \]

\[ = \|D_1 z^{(k)} - \frac{\gamma}{c} D_1 \dot{g}^{(k)}\|^2_F + \frac{\gamma^2}{c^2} \mathbb{E}\|D_1 \dot{s}^{(k)}\|^2_F \]

\[ \overset{(a)}{\leq} \frac{1}{t} \|D_1\|^2 \|\dot{z}^{(k)}\|^2_F + \frac{\gamma^2}{(1-t)c^2} \|D_1\|^2 \|X_L, t\|^2 \|\nabla f(x^{(k)}) - \nabla f(x^*)\|^2_F + \frac{\gamma^2}{c^2} \|D_1\|^2 \|X_L, t\|^2 n \sigma^2 \]

\[ \overset{(b)}{\leq} \beta_1 \|\dot{z}^{(k)}\|^2_F + \frac{\gamma^2 \beta^2 L^2}{(1 - \beta_1)c^2} \|X_L\|^2 \|\nabla f(x^{(k)}) - \nabla f(x^*)\|^2_F + \frac{n \gamma^2 \beta^2}{c^2} \|X_L\|^2 \sigma^2 \]  

(81)

where inequality (a) holds because of relation (25), (27), and the Jensen’s inequality for any \( t \in (0, 1) \), and inequality (b) holds by letting \( t = \beta_1 = \|D_1\| \) and the fact that

\[ \|X_L, t\| = \|X_L \left[ \begin{array}{c} I_n \\ 0 \end{array} \right] \| \overset{\leq}{=} \|X_L \left[ \begin{array}{c} I_n \\ 0 \end{array} \right] \| = \|X_L\|. \]

(82)

Note that

\[ \|\nabla f(x^{(k)}) - \nabla f(x^*)\|^2_F = \|\nabla f(x^{(k)}) - \nabla f(x^{(k)}) + \nabla f(x^{(k)}) - \nabla f(x^*)\|^2_F \]

\[ \leq 2 \|\nabla f(x^{(k)}) - \nabla f(x^{(k)})\|^2_F + 2 \|\nabla f(x^{(k)}) - \nabla f(x^*)\|^2_F \]

\[ \overset{(a)}{\leq} 2L^2 \|x^{(k)} - \bar{x}^{(k)}\|^2_F + 4nL(f(\bar{x}^{(k)}) - f(x^*)) \]

\[ \overset{(77)}{\leq} 2c^2 L^2 \|X_R\|^2 \|\dot{z}^{(k)}\|^2_F + 4nL(f(\bar{x}^{(k)}) - f(x^*)) \]  

(83)

where inequality (a) holds because \( f(x) \) is convex and \( L \)-smooth. Substituting (83) into (81), and taking expectations over the filtration \( \mathcal{F}^{(k)} \), we achieve the result

\[ \mathbb{E}[\|\dot{z}^{(k+1)}\|^2_F \leq \beta_1 \mathbb{E}[\|\dot{z}^{(k)}\|^2_F + \frac{2 \gamma^2 \beta^2 L^2}{1 - \beta_1} \|X_L\|^2 \|X_R\|^2 \mathbb{E}[\|\dot{z}^{(k)}\|^2_F \]

\[ + \frac{4n \gamma^2 \beta^2 L}{(1 - \beta_1)c^2} \|X_L\|^2 \|\mathbb{E}[f(\bar{x}^{(k)}) - f(x^*)] + \frac{n \gamma^2 \beta^2}{c^2} \|X_L\|^2 \sigma^2 \]

(84)

By setting \( \gamma \) sufficiently small such that

\[ \beta_1 + \frac{2 \gamma^2 \beta^2 L^2}{1 - \beta_1} \|X_L\|^2 \|X_R\|^2 \leq \frac{1 + \beta_1}{2}, \]

(85)

we achieve

\[ \mathbb{E}[\|\dot{z}^{(k+1)}\|^2_F \leq \left( \frac{1 + \beta_1}{2} \right) \mathbb{E}[\|\dot{z}^{(k)}\|^2_F + \frac{4n \gamma^2 \beta^2 L}{(1 - \beta_1)c^2} \|X_L\|^2 \|\mathbb{E}[f(\bar{x}^{(k)}) - f(x^*)] + \frac{n \gamma^2 \beta^2}{c^2} \|X_L\|^2 \sigma^2 \]

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where the last equality holds by setting \( c^2 = \|X_L\|^2 \). To satisfy (85), it is enough to set (recall (23))

\[
\gamma \leq \frac{(1 - \beta_1)\lambda_1^{1/2}}{2\beta_1L} = \frac{(1 - \beta_1)\lambda_1^{1/2}}{2\bar{\lambda}_2^{1/2}L}.
\]

(87)

C.3 Proof of Lemma 6

Proof Keep iterating (34) we achieve for \( k = 1, 2, \ldots \) that

\[
E\|\hat{z}^{(k)}\|_F^2 \leq \left(\frac{1 + \beta_1}{2}\right)^k E\|\hat{z}^{(0)}\|_F^2 + \frac{2n\gamma^2\beta_1^2\sigma^2}{1 - \beta_1} + \frac{8n\gamma^2\beta_1^2L}{1 - \beta_1} \sum_{\ell=0}^{k-1} \left(\frac{1 + \beta_1}{2}\right)^{k-1-\ell} (E(f(\hat{x}^{(\ell)})) - f(x^*))
\]

(88)

Recall from (77) that \( x^{(k)} - \hat{x}^{(k)} = cX_{R,u}\hat{z}^{(k)} \) for any \( k = 0, 1, \ldots \) where \( X_{R,u} \in \mathbb{R}^{n \times 2(n-1)} \) is not a full column-rank matrix. Even if we initialize \( x^{(0)} = \hat{x}^{(0)} \), the variable \( \hat{z}^{(0)} \) may not be zero. We let \( C = E\|\hat{z}^{(0)}\|_F^2 \)

be a positive constant. By taking average over \( k = 1, 2, \ldots, T \), we achieve

\[
\frac{1}{T} \sum_{k=1}^{T} E\|\hat{z}^{(k)}\|_F^2 \leq \frac{C}{T} \sum_{k=1}^{T} \left(\frac{1 + \beta_1}{2}\right)^k + \frac{2n\gamma^2\beta_1^2\sigma^2}{1 - \beta_1} + \frac{8n\gamma^2\beta_1^2L}{(1 - \beta_1)T} \sum_{k=1}^{T} \sum_{\ell=0}^{k-1} \left(\frac{1 + \beta_1}{2}\right)^{k-1-\ell} (E(f(\hat{x}^{(\ell)})) - f(x^*))
\]

\[
\leq \frac{C}{T} \sum_{k=1}^{T} \left(\frac{1 + \beta_1}{2}\right)^k + \frac{2n\gamma^2\beta_1^2\sigma^2}{1 - \beta_1} + \frac{8n\gamma^2\beta_1^2L}{(1 - \beta_1)T} \sum_{k=1}^{T} \left[ \sum_{\ell=0}^{T-1} \left(\frac{1 + \beta_1}{2}\right)^{k-1-\ell} \right] (E(f(\hat{x}^{(k)})) - f(x^*))
\]

\[
\leq \frac{2C}{T(1 - \beta_1)} + \frac{16n\gamma^2\beta_1^2L}{(1 - \beta_1)^2T} \sum_{k=0}^{T-1} (E(f(\hat{x}^{(k)})) - f(x^*)) + \frac{2n\gamma^2\beta_1^2\sigma^2}{1 - \beta_1}
\]

(89)

Since \( \frac{1}{T+1} \sum_{k=0}^{T} E\|\hat{z}^{(k)}\|_F^2 = \left( \sum_{k=1}^{T} E\|\hat{z}^{(k)}\|_F^2 + E\|\hat{z}^{(0)}\|_F^2 \right)/(T+1) \) and \( 1 < \frac{1}{1 - \beta_1} \), we achieve the result (35). ■

C.4 Proof of Theorem 1

Proof From (32), we have

\[
E(f(\hat{x}^{(k)})) - f(x^*) \leq \frac{E\|\hat{z}^{(k)}\|_F^2 - E\|\hat{z}^{(k+1)}\|_F^2}{\gamma} + \frac{3L}{2n\lambda_n} E\|\hat{z}^{(k)}\|_F^2 + \frac{\gamma\sigma^2}{n}
\]

(90)
By taking average over $k = 0, 1, \ldots, T$, we have

$$
\frac{1}{T+1} \sum_{k=0}^{T} (\mathbb{E}f(\bar{x}^{(k)}) - f(x^*)) \leq \frac{\mathbb{E}\|\bar{z}^{(0)}\|^2}{\gamma(T+1)} + \frac{3L}{2n\lambda_n(T+1)} \sum_{k=0}^{T} \mathbb{E}\|\bar{z}^{(k)}\|_p^2 + \frac{\gamma\sigma^2}{n} \leq \frac{24L^2\gamma^2\beta_1^2}{(T+1)(1-\beta_1)^2\lambda_n} \sum_{k=0}^{T} (\mathbb{E}f(\bar{x}^{(k)}) - f(x^*)) + \frac{3L\gamma^2\beta_1^2\sigma^2}{\lambda_n(1-\beta_1)} + \frac{\gamma\sigma^2}{n} + \frac{C}{\gamma(T+1)} + \frac{9LC}{2n(T+1)(1-\beta_1)\lambda_n}
$$

(35)

If $\gamma$ is sufficiently small such that

$$
\frac{24L^2\gamma^2\beta_1^2}{(1-\beta_1)^2\lambda_n} \leq \frac{1}{2},
$$

inequality (91) becomes

$$
\frac{1}{T+1} \sum_{k=0}^{T} (\mathbb{E}f(\bar{x}^{(k)}) - f(x^*)) \leq \frac{6L\gamma^2\beta_1^2\sigma^2}{(1-\beta_1)\lambda_n} + \frac{2\gamma\sigma^2}{n} + \frac{2C}{\gamma(T+1)} + \frac{9LC}{n(T+1)(1-\beta_1)\lambda_n}
$$

(93)

To satisfy (92), it is enough to let

$$
\gamma \leq \frac{(1-\beta_1)\lambda_n^{1/2}}{7L\lambda_n^{1/2}}
$$

(94)

The way to choose step-size $\gamma$ is adapted from Lemma 15 in [13]. For simplicity, we let

$$
B^{(k)} = \mathbb{E}f(\bar{x}^{(k)}) - f(x^*), \quad r_0 = 2C, \quad r_1 = \frac{2\gamma^2}{n}, \quad r_2 = \frac{6L\gamma^2\beta_1^2\sigma^2}{(1-\beta_1)\lambda_n}, \quad r_3 = \frac{9LC}{n(1-\beta_1)\lambda_n}
$$

(95)

and inequality (93) becomes

$$
\frac{1}{T+1} \sum_{k=0}^{T} B^{(k)} \leq \frac{r_0}{(T+1)\gamma} + r_1\gamma + r_2\gamma^2 + \frac{r_3}{T+1}
$$

(96)

Now we let

$$
\gamma = \min \left\{ \frac{(1-\beta_1)\lambda_n^{1/2}}{7L\lambda_n^{1/2}}, \left( \frac{r_0}{r_1(T+1)} \right)^{\frac{1}{2}}, \left( \frac{r_0}{r_2(T+1)} \right)^{\frac{1}{2}} \right\}
$$

(97)

- If $\left( \frac{r_0}{r_1(T+1)} \right)^{\frac{1}{2}}$ is the smallest, we let $\gamma = \left( \frac{r_0}{r_2(T+1)} \right)^{\frac{1}{2}}$. With $\left( \frac{r_0}{r_1(T+1)} \right)^{\frac{1}{2}} \leq \left( \frac{r_0}{r_2(T+1)} \right)^{\frac{1}{2}}$, (96) becomes

$$
\frac{1}{T+1} \sum_{k=0}^{T} B^{(k)} \leq 2r_2^{\frac{1}{2}} \left( \frac{r_0}{T+1} \right) + r_1 \left( \frac{r_0}{r_1(T+1)} \right)^{\frac{1}{2}} + \frac{r_3}{T+1}
$$

$$
\leq 2r_2^{\frac{1}{2}} \left( \frac{r_0}{T+1} \right)^{\frac{1}{2}} + \left( \frac{r_0r_1}{r_1(T+1)} \right)^{\frac{1}{2}} + \frac{r_3}{T+1}.
$$

(98)

- If $\left( \frac{r_0}{r_1(T+1)} \right)^{\frac{1}{2}}$ is the smallest, we let $\gamma = \left( \frac{r_0}{r_1(T+1)} \right)^{\frac{1}{2}}$. With $\left( \frac{r_0}{r_1(T+1)} \right)^{\frac{1}{2}} \leq \left( \frac{r_0}{r_2(T+1)} \right)^{\frac{1}{2}}$, (96) becomes

$$
\frac{1}{T+1} \sum_{k=0}^{T} B^{(k)} \leq 2\left( \frac{r_0r_1}{T+1} \right)^{\frac{1}{2}} + \frac{r_0r_2}{r_1(T+1)} + \frac{r_3}{T+1}
$$

$$
\leq 2\left( \frac{r_0r_1}{T+1} \right)^{\frac{1}{2}} + \frac{r_0}{T+1} + \frac{r_0}{T+1} + \frac{r_3}{T+1}.
$$

(99)
• If \( \frac{(1-\beta_1)\lambda_1^{1/2}}{7L\lambda_n^{1/2}} \leq \left( \frac{r_0}{r_2(T+1)} \right)^{1/2} \) and \( \frac{(1-\beta_1)\lambda_1^{1/2}}{7L\lambda_n^{1/2}} \leq \left( \frac{r_0}{r_2(T+1)} \right)^{1/2} \), we let \( \gamma = \frac{(1-\beta_1)\lambda_1^{1/2}}{7L\lambda_n^{1/2}} \) and (96) becomes

\[
\frac{1}{T+1} \sum_{k=0}^{T} B^{(k)} \leq \frac{7L\lambda_1^{1/2}r_0}{(1-\beta_1)(T+1)\lambda_n^{1/2}} + \left( \frac{r_0r_1}{T+1} \right)^{1/2} + r_2^2 \left( \frac{r_0}{T+1} \right)^{1/2} + \frac{r_3}{T+1}. \tag{100}
\]

Combining (98), (99) and (100), we have

\[
\frac{1}{T+1} \sum_{k=0}^{T} B^{(k)} \leq \frac{7L\lambda_2^{1/2}r_0}{(1-\beta_1)(T+1)\lambda_n^{1/2}} + 2\left( \frac{r_0r_1}{T+1} \right)^{1/2} + 2r_2^2 \left( \frac{r_0}{T+1} \right)^{1/2} + \frac{r_3}{T+1}. \tag{101}
\]

Substituting constants \( r_0, r_1, \) and \( r_2, \) we have the final result:

\[
\frac{1}{T+1} \sum_{k=0}^{T} B^{(k)} = O\left( \frac{\sigma}{\sqrt{nT}} + \frac{\sigma\bar{\lambda}_2^{1/2}}{(1-\beta_1)^{1/2}T^{1/2}} \left( \frac{\tilde{\lambda}_2}{\lambda_n^{1/2}} \right)^{1/2} + \frac{1}{(1-\beta_1)T} \left( \frac{\tilde{\lambda}_2}{\lambda_n^{1/2}} \right)^{1/2} \right). \tag{102}
\]

With \( \tilde{\lambda}_2 = \bar{\lambda}_2 \leq (1+\beta)/2, \) we have

\[
1 - \beta_1 = \frac{1-\beta_2}{1+\beta_1} \geq \frac{1-\beta}{2(1+\beta_1)} \geq \frac{1-\beta}{4}. \tag{103}
\]

Substituting (103) to (97) and (102), we have the result in Theorem 1.

\[\square\]

### D Convergence Analysis for Strongly-Convex Scenario

#### D.1 Proof of Lemma 7

**Proof** Since each \( f_i(x) \) is strongly convex, it holds that

\[
f_i(x) - f_i(y) + \frac{\mu}{2} \|x - y\|^2 \leq \langle \nabla f_i(x), x - y \rangle, \quad \forall \ x, y \in \mathbb{R}^d \tag{104}
\]

Let \( x = x_i^{(k)} \) and \( y = x^* \), we have

\[
f_i(x_i^{(k)}) - f_i(x^*) + \frac{\mu}{2} \|x_i^{(k)} - x^*\|^2 \leq \langle \nabla f_i(x_i^{(k)}), x_i^{(k)} - x^* \rangle. \tag{105}
\]

Following arguments from (71) to (73), and replacing the bound in (72) with

\[
\begin{align*}
&\frac{2\gamma}{n} \sum_{i=1}^{n} \langle \tilde{x}^{(k)} - x^*, \nabla f_i(x_i^{(k)}) - \nabla f_i(x^*) \rangle \\
&= \frac{2\gamma}{n} \sum_{i=1}^{n} \langle \tilde{x}^{(k)} - x_i^{(k)}, \nabla f_i(x_i^{(k)}) \rangle + \frac{2\gamma}{n} \sum_{i=1}^{n} \langle x_i^{(k)} - x^*, \nabla f_i(x^*) \rangle \\
&\geq \frac{2\gamma}{n} \sum_{i=1}^{n} \left( f_i(\tilde{x}^{(k)}) - f_i(x_i^{(k)}) - \frac{L}{2} \|\tilde{x}^{(k)} - x_i^{(k)}\|^2 \right) + \frac{2\gamma}{n} \sum_{i=1}^{n} \left( f_i(x_i^{(k)}) - f_i(x^*) + \frac{\mu}{2} \|x_i^{(k)} - x^*\|^2 \right) \\
&= \frac{2\gamma}{n} \sum_{i=1}^{n} \left( f_i(\tilde{x}^{(k)}) - f_i(x^*) \right) - \frac{\gamma}{n} \|\tilde{x}^{(k)} - x^{(k)}\|_F^2 + \frac{\gamma}{n} \|x^{(k)} - x^*\|_F^2 \\
&\geq 2\gamma (f(\tilde{x}^{(k)}) - f(x^*)) - \frac{\gamma(L + \mu)}{n} \|\tilde{x}^{(k)} - x^{(k)}\|_F^2 + \frac{\gamma\mu}{2} \|\tilde{x}^{(k)} - x^*\|_F^2. \tag{106}
\end{align*}
\]
we achieve a slightly different bound from (74):

\[
\|\bar{x}^{(k)} - x^* - \frac{\gamma}{n} \sum_{i=1}^n \nabla f_i(x_i^{(k)})\|^2 \leq (1 - \frac{\gamma \mu}{2})\|\bar{x}^{(k)} - x^*\|^2 - 2\gamma(1 - 2L\gamma)(f(\bar{x}^{(k)}) - f(x^*)) + \left(\frac{\gamma(L + \mu)}{n} + \frac{2\gamma^2 L^2}{n}\right)\|\bar{x}^{(k)} - x^*\|^2 + \frac{5\gamma L}{n} \|\bar{x}^{(k)} - x^*\|^2 \leq (1 - \frac{\gamma \mu}{2})\|\bar{x}^{(k)} - x^*\|^2 - \gamma(f(\bar{x}^{(k)}) - f(x^*)) + \frac{5\gamma L}{n} \|\bar{x}^{(k)} - x^*\|^2
\]

(107)

where the last inequality holds when \(\gamma \leq \frac{1}{4L}\). With (107), we can follow arguments (75)-(79) to achieve the result in (40).

\[\]
Furthermore, with condition (45), we have \( h_k \leq h_0(1 + \frac{1 - \beta_1}{2})^k \) for any \( k = 0, 1, \ldots \). This implies
\[
\sum_{k=0}^{T} h_k \left(\frac{1 + \beta_1}{2}\right)^k \leq h_0 \sum_{k=0}^{T} \left(1 + \frac{1 - \beta_1}{4}\right)^k \left(\frac{1 + \beta_1}{2}\right)^k \leq h_0 \sum_{k=0}^{T} \left(\frac{3 + \beta_1}{4}\right)^k \leq \frac{4h_0}{1 - \beta_1} \tag{113}
\]
Substituting (113) into (112) and dividing both sides by \( H_T = \sum_{k=0}^{T} h_k \), we achieve the final result in (44). ■

D.3 Proof of Theorem 2

The following proof is inspired by [43].

Proof With descent inequality (40), we have
\[
\mathbb{E}f(\bar{x}^{(k)}) - f(x^*) \leq (1 - \frac{\gamma}{2}) \mathbb{E}\|\bar{z}^{(k)}\|^2 - \frac{\mathbb{E}\|\bar{z}^{(k+1)}\|^2}{\gamma} + \frac{5L}{2n\lambda_n} \mathbb{E}\|\bar{z}^{(k)}\|^2 + \gamma\sigma^2
\tag{114}
\]
Taking the weighted average over \( k \), it holds that (we let \( B^{(k)} = \mathbb{E}f(\bar{x}^{(k)}) - f(x^*) \))
\[
\frac{1}{H_T} \sum_{k=0}^{T} h_k B^{(k)} \leq \frac{1}{H_T} \sum_{k=0}^{T} h_k \left(\frac{1 - \frac{2\mu}{\gamma}}{\gamma}\|\bar{z}^{(k)}\|^2 - \frac{\mathbb{E}\|\bar{z}^{(k+1)}\|^2}{\gamma}\right) + \frac{5L}{2nH_T\lambda_n} \sum_{k=0}^{T} h_k \mathbb{E}\|\bar{z}^{(k)}\|^2 + \gamma\sigma^2
\tag{115}
\]
If we let \( h_k = (1 - \frac{2\mu}{\gamma})h_{k+1} \) for \( k = 0, 1, \ldots \), the above inequality becomes
\[
\frac{1}{H_T} \sum_{k=0}^{T} h_k B^{(k)} \leq \frac{h_0 C}{H_T\gamma} + \frac{5L}{2nH_T\lambda_n} \sum_{k=0}^{T} h_k \mathbb{E}\|\bar{z}^{(k)}\|^2 + \gamma\sigma^2
\tag{116}
\]
Since \( h_k = (1 - \frac{2\mu}{\gamma})h_{k+1} \), we have
\[
h_k = h_\ell \left(\frac{1}{1 - \frac{2\mu}{\gamma}}\right)^{k-\ell}, \text{ for any } k \geq 0 \text{ and } 0 \leq \ell \leq k.
\tag{117}
\]
If \( \gamma \) is sufficiently small such that
\[
1 - \frac{2\mu}{\gamma} \leq 1 + \frac{1 - \beta_1}{4}, \text{ (it is enough to set } \gamma \leq \frac{1 - \beta_1}{2\mu}\)
\tag{118}
\]
then \( \{h_k\}_{k=0}^{\infty} \) satisfy condition (45). As a result, we can substitute inequality (44) into (116) to achieve
\[
\frac{1}{H_T} \sum_{k=0}^{T} h_k B^{(k)} \leq \frac{h_0 C}{H_T\gamma} + \frac{\gamma\sigma^2}{n} + \frac{10LC\lambda_0 h_0}{nH_T(1 - \beta_1)\lambda_n} + \frac{5L\gamma^2\beta_1^2\sigma^2}{\lambda_n(1 - \beta_1)} + \frac{160\gamma^2\beta_1^2 L^2}{(1 - \beta_1)^2\lambda_n H_T} \sum_{k=0}^{T} h_k B^{(k)}.
\tag{119}
\]
If \( \gamma \) is sufficiently small such that
\[
\frac{160\gamma^2\beta_1^2 L^2}{(1 - \beta_1)^2\lambda_n} \leq \frac{1}{2}, \text{ (it is enough to set } \gamma \leq \frac{1 - \beta_1}{18L} \left(\frac{\delta^{1/2}}{1/2}\right)\)
\tag{120}
\]
it holds that
\[
\frac{1}{H_T} \sum_{k=0}^{T} h_k B^{(k)} \leq \frac{2h_0 C}{H_T\gamma} + \frac{20LC\lambda_0}{nH_T(1 - \beta_1)\lambda_n} + \frac{2\gamma\sigma^2}{n} + \frac{20L\gamma^2\beta_1^2\sigma^2}{\lambda_n(1 - \beta_1)}.
\tag{121}
\]
Since $H_T \geq h_T = (1 - \frac{4n}{T})^{-T}$, we have

\[
\frac{1}{H_T} \sum_{k=0}^{T} h_k B^{(k)} \leq \frac{2h_0 C}{\gamma} \left(1 - \frac{\gamma \mu}{2}\right)^T + \frac{20LCh_0}{n(1 - \beta_1)\lambda_n} \left(1 - \frac{\gamma \mu}{2}\right)^T + \frac{2\gamma \sigma^2}{n} + \frac{20L\gamma^2 \beta_1^2 \sigma^2}{\lambda_n(1 - \beta_1)}
\]

\[
\leq \left(\frac{2h_0 C}{\gamma} + \frac{20LCh_0}{n(1 - \beta_1)\lambda_n}\right) \exp\left(-\frac{\gamma \mu T}{2}\right) + \frac{2\gamma \sigma^2}{n} + \frac{20L\gamma^2 \beta_1^2 \sigma^2}{\lambda_n(1 - \beta_1)}
\]

and $\gamma$ needs to satisfy condition (120). When $T$ is large enough such that the constant

\[
\frac{2 \ln(2n\mu h_0 CT^2/\sigma^2)}{\mu T} \leq \frac{1 - \beta_1}{18L} \left(\frac{\tilde{\lambda}_n^{1/2}}{\tilde{\lambda}_2^{1/2}}\right),
\]

we set

\[
\gamma = \frac{2 \ln(2n\mu h_0 CT^2/\sigma^2)}{\mu T} \text{ so that } \exp\left(-\frac{\gamma \mu T}{2}\right) = \frac{\sigma^2}{2n\mu h_0 CT^2}
\]

Substituting the above $\gamma$ into (122), we achieve

\[
\frac{1}{H_T} \sum_{k=0}^{T} h_k B^{(k)} = \tilde{O} \left(\frac{\sigma^2}{nT} + \frac{\sigma^2}{(1 - \beta_1)T^2} \frac{\tilde{\lambda}_2}{\lambda_n^{1/2}}\right)
\]

Otherwise, if $T$ is small so that

\[
\frac{2 \ln(2n\mu h_0 CT^2/\sigma^2)}{\mu T} \geq \frac{1 - \beta_1}{18L} \left(\frac{\tilde{\lambda}_n^{1/2}}{\tilde{\lambda}_2^{1/2}}\right),
\]

we let

\[
\gamma = \frac{1 - \beta_1}{18L} \left(\frac{\tilde{\lambda}_n^{1/2}}{\tilde{\lambda}_2^{1/2}}\right) \leq \frac{2 \ln(2n\mu h_0 CT^2/\sigma^2)}{\mu T}
\]

Substituting the above $\gamma$ as well as the above inequality into (122), we achieve

\[
\frac{1}{H_T} \sum_{k=0}^{T} h_k B^{(k)} = \tilde{O} \left(\frac{1}{1 - \beta_1} \left(\frac{\tilde{\lambda}_2}{\tilde{\lambda}_n}\right)^{1/2} \exp\left(-(1 - \beta_1)\frac{\tilde{\lambda}_n}{\tilde{\lambda}_2} \frac{1}{2}T\right) + \frac{\sigma^2}{nT} + \frac{\sigma^2}{(1 - \beta_1)T^2} \left(\frac{\tilde{\lambda}_2}{\lambda_n^{1/2}}\right)^{1/2}\right).
\]

Combining (125) and (128), substituting relation (103) to bound $1 - \beta_1$, and ignoring the constant $\tilde{\lambda}_2/\tilde{\lambda}_n$, we achieve the result in (47). We can also achieve the learning rate expression (46) with (127) and (103).

### E Proof of Theorem 3

**Proof** We consider the minimization problem of the form (1) with $f_i(x) = \frac{1}{2}\|x\|^2$ where $x \in \mathbb{R}$. Under such setting, it holds that $f_i(x) = f_j(x)$ for any $i, j \in [n]$ and there is no heterogeneity, i.e., $b^2 = 0$. The D-SGD algorithm in this setting will iterate as follows:

\[
x^{(k+1)} = \bar{W}(x^{(k)} - \gamma x^{(k)} - \gamma s^{(k)}) = (1 - \gamma)\bar{W}x^{(k)} - \gamma \bar{W}s^{(k)}
\]
where \( \mathbf{x} \in \mathbb{R}^n \) is a vector, and \( \mathbf{s} \in \mathbb{R}^n \) is the gradient noise. Moreover, we assume each element of the noise follows standard Gaussian distribution, i.e., \( s_{i(k)} \sim \mathcal{N}(0, \sigma^2) \), and \( s_{i(k)} \) is independent of each other for any \( k \) and \( i \). We also assume the gradient noise \( \mathbf{s}(k) \) is independent of \( \mathbf{x}(\ell) \) for any \( \ell \leq k \). With these assumptions, it holds that \( \mathbb{E}[\mathbf{s}(k)(\mathbf{s}(k)^T)] = \sigma^2 \mathbf{I} \in \mathbb{R}^{n \times n} \). With (129), we have

\[
\mathbf{x}^{(k+1)} = (1 - \gamma) \frac{1}{n} \mathbf{1} \mathbf{1}^T \mathbf{x}^{(k)} - \frac{1}{n} \mathbf{1} \mathbf{1}^T \mathbf{s}^{(k)}.
\]

Subtracting the above recursion from (129), we have

\[
\mathbf{x}^{(k+1)} - \bar{\mathbf{x}}^{(k+1)} = (1 - \gamma)(\bar{\mathbf{W}} - \frac{1}{n} \mathbf{1} \mathbf{1}^T)(\mathbf{x}^{(k)} - \bar{\mathbf{x}}^{(k)}) - \gamma(\bar{\mathbf{W}} - \frac{1}{n} \mathbf{1} \mathbf{1}^T)\mathbf{s}^{(k)}.
\]

We next define matrix \( \mathbf{R} = \bar{\mathbf{W}} - \frac{1}{n} \mathbf{1} \mathbf{1}^T \). Note that \( \mathbf{s}(k) \) is independent of \( \mathbf{x}(k) \). By taking the mean-square-expectation over both sides of the above equality, we have

\[
\mathbb{E}\|\mathbf{x}^{(k+1)} - \bar{\mathbf{x}}^{(k+1)}\|^2 = \mathbb{E}\|(1 - \gamma)\mathbf{R}(\mathbf{x}^{(k)} - \bar{\mathbf{x}}^{(k)})\|^2 + \gamma^2 \mathbb{E}\|\mathbf{R}\mathbf{s}^{(k)}\|^2
\]

\[
\overset{(131)}{=} \mathbb{E}\|(1 - \gamma)^2\mathbf{R}^2(\mathbf{x}^{(k-1)} - \bar{\mathbf{x}}^{(k-1)}) - \gamma(1 - \gamma)\mathbf{R}^2\mathbf{s}^{(k-1)}\|^2 + \gamma^2 \mathbb{E}\|\mathbf{R}\mathbf{s}^{(k)}\|^2
\]

\[
= \mathbb{E}\|(1 - \gamma)^2\mathbf{R}^2(\mathbf{x}^{(k-1)} - \bar{\mathbf{x}}^{(k-1)})\|^2 + \gamma^2(1 - \gamma)^2 \mathbb{E}\|\mathbf{R}\mathbf{s}^{(k-1)}\|^2 + \gamma^2 \mathbb{E}\|\mathbf{R}\mathbf{s}^{(k)}\|^2
\]

\[
= \ldots
\]

\[
= \|(1 - \gamma)^{k+1}\mathbf{R}^{k+1}(\mathbf{x}^{(0)} - \bar{\mathbf{x}}^{(0)})\|^2 + \gamma^2 \sum_{\ell=0}^{k} \mathbb{E}\|(1 - \gamma)^{\ell}\mathbf{R}^{\ell+1}\mathbf{s}^{(k-\ell)}\|^2
\]

In the above derivations, we used the fact that \( \mathbf{s}(k) \) is independent of \( \mathbf{x}(k) \) for any \( k \). Next we examine

\[
\mathbb{E}\|(1 - \gamma)^{\ell}\mathbf{R}^{\ell+1}\mathbf{s}^{(k-\ell)}\|^2
\]

\[
\overset{(132)}{=} (1 - \gamma)^{2\ell}\mathbb{E}\{\text{tr}([\mathbf{s}^{(k-\ell)}]^T \mathbf{R}^{\ell+1} \mathbf{R}^{\ell+1} \mathbf{s}^{(k-\ell)})\}
\]

\[
\overset{(a)}{=} (1 - \gamma)^{2\ell}\mathbb{E}\{\text{tr}([\mathbf{R}^{2(\ell+1)} \mathbf{s}^{(k-\ell)}]^T \mathbf{s}^{(k-\ell)})\}
\]

\[
\overset{(b)}{=} (1 - \gamma)^{2\ell}\mathbb{E}\{\text{tr}([\mathbf{R}^{2(\ell+1)}]^T \mathbf{s}^{(k-\ell)} \mathbf{s}^{(k-\ell)})\}
\]

\[
\overset{(c)}{=} (1 - \gamma)^{2\ell}\sum_{i=2}^{n} \lambda_i^{2(\ell+1)}
\]

\[
\overset{(d)}{=} \sigma^2 (1 - \gamma)^{2\ell} \lambda_2^{2(\ell+1)}
\]

where (a) holds because \( \mathbb{E}[\mathbf{s}(k)(\mathbf{s}(k)^T)] = \sigma^2 \mathbf{I} \in \mathbb{R}^{n \times n} \) for any \( k \), and (b) holds because \( \mathbf{R} = \bar{\mathbf{W}} - \frac{1}{n} \mathbf{1} \mathbf{1}^T \) is symmetric and can be eigen-decomposed as \( \mathbf{R} = \mathbf{U} \Lambda \mathbf{U}^T \) in which \( \mathbf{U} \) is an orthogonal matrix. The eigenvalue
\( \lambda_i \) is defined as \( \lambda_i(W) \). With (133), we have
\[
\gamma^2 \sum_{\ell=0}^{k} \mathbb{E} \| (1 - \gamma)^{\ell} R^{\ell+1} s^{(k-\ell)} \|^2 \geq \gamma^2 \sigma^2 \lambda_2^2 \sum_{\ell=0}^{k} (1 - \gamma)^{2\ell} \lambda_2^{2\ell} = \gamma^2 \sigma^2 \lambda_2^2 \frac{1 - (1 - \gamma)^{2(k+1)} \lambda_2^{2(k+1)}}{1 - (1 - \gamma)^2 \lambda_2^2}.
\] (134)
Substituting (134) into (132), we achieve
\[
\mathbb{E} \| x^{(k)} - \bar{x}^{(k)} \|^2 \geq \gamma^2 \sigma^2 \lambda_2^2 \frac{1 - (1 - \gamma)^{2\kappa} \lambda_2^{2\kappa}}{1 - (1 - \gamma)^2 \lambda_2^2}
\] (135)
Since \( \frac{1}{n} \mathbb{E} \| x^{(k)} - x^* \|^2 = \mathbb{E} \| \bar{x}^{(k)} - x^* \|^2 + \frac{1}{n} \mathbb{E} \| x^{(k)} - \bar{x}^{(k)} \|^2 \), we have
\[
\frac{1}{n} \mathbb{E} \| x^{(k)} - x^* \|^2 \geq \frac{\sigma^2}{n} \gamma^2 \lambda_2^2 \frac{1 - (1 - \gamma)^{2\kappa} \lambda_2^{2\kappa}}{1 - (1 - \gamma)^2 \lambda_2^2}
\] (136)
To guarantee D-SGD to achieve the linear speedup, we require a sufficiently large \( k \) so that \( \frac{1}{n} \mathbb{E} \| x^{(k)} - x^* \|^2 \leq \frac{\sigma^2}{nk} \) (note that P-SGD will achieve the linear speedup \( \frac{\sigma^2}{nk} \) for the strongly-convex scenario). Thus, it is necessary to have that
\[
\frac{\sigma^2}{n} \gamma^2 \lambda_2^2 \frac{1 - (1 - \gamma)^{2\kappa} \lambda_2^{2\kappa}}{1 - (1 - \gamma)^2 \lambda_2^2} \leq \frac{\sigma^2}{2nk}
\] (137)
In other words, \( k \) has to be sufficiently large such that the above inequality holds. When \( f_i(x) = \frac{1}{2} \| x \|^2 \), it holds that \( \mu = L = 1 \). Now we set the same step-size as in Theorem 2, i.e.,
\[
\gamma = \frac{2 \ln(2n\mu h_0 C T^2/\sigma^2)}{\mu k} = \frac{2 \ln(2n\mu h_0 C k^2/\sigma^2)}{k}
\] (138)
Note that
\[
(1 - \gamma)^{2\kappa} \lambda_2^{2\kappa} \leq (1 - \gamma)^{2\kappa} \leq \exp(-2k\gamma) \quad (139)
\]
\[
\bar{C} \sigma^2 \quad \text{when} \quad k \geq \sqrt{\frac{2C \sigma^2}{n}}
\]
\[
1 - (1 - \gamma) \lambda_2 \leq 2(1 - \lambda_2) \quad (140)
\]
where \( \bar{C} \) is some constant independent of \( n, k, \) and \( \sigma^2 \). Substituting the above relations into (137), we achieve
\[
\frac{\sigma^2}{2nk} \geq \frac{\sigma^2}{n} \gamma^2 \lambda_2^2 \frac{1 - (1 - \gamma)^{2\kappa} \lambda_2^{2\kappa}}{1 - (1 - \gamma)^2 \lambda_2^2} \geq \frac{\sigma^2}{n} \gamma^2 \lambda_2^2 \frac{1 - (1 - \gamma)^{2\kappa} \lambda_2^{2\kappa}}{2(1 - (1 - \gamma) \lambda_2)} \quad (139)
\]
\[
\geq \frac{\sigma^2}{n} \gamma^2 \lambda_2^2 \frac{1}{4(1 - (1 - \gamma) \lambda_2)} \quad (140)
\]
\[
\geq \frac{\gamma^2 \sigma^2 \lambda_2^2}{8n(1 - \lambda_2)} = \frac{\ln^2(2n\mu h_0 C k^2/\sigma^2) \sigma^2 \lambda_2^2}{2n(1 - \lambda_2) k^2}
\] (141)
from which we achieve
\[
k = \tilde{\Omega}\left( \frac{\lambda_2^2}{1 - \lambda_2} \right)
\] (142)
Combining (139), (140) and (142), we finally achieve the lower bound of the transient stage of D-SGD:
\[
k = \tilde{O}\left( \max\left\{ \sqrt{\frac{2C \sigma^2}{n}} \cdot \frac{\lambda_2}{1 - \lambda_2}, \frac{\lambda_2^2}{1 - \lambda_2} \right\} \right) = \tilde{\Omega}\left( \frac{\lambda_2}{1 - \lambda_2} \right)
\] (143)
when $\lambda_2 \to 1$.

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