QUANTUM TORSORS

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ABSTRACT. The following text is a short version of a forthcoming preprint about torsors. The adopted viewpoint is an old reformulation of torsors recalled recently by Kontsevich [Kor]. We propose a unification of the definitions of torsors in algebraic geometry and in Poisson manifolds (Example 2 and section 2.2). We introduce the notion of a quantum torsor (Definition 2.1). Any quantum torsor is equipped with two comodule-algebra structures over Hopf algebras and these structures commute with each other (Theorem 3.1). In the finite dimensional case, these two Hopf algebras share the same finite dimension (Proposition 3.1). We show that any Galois extension of a field is a torsor (Example 3.2) and that any torsor is a Hopf-Galois extension (section 3.2). We give also examples of non-commutative torsors without character (Example 3). Torsors can be composed (Theorem 3.2). This leads us to define for any Hopf algebra, a new group-invariant, its torsors invariant (Theorem 3.3). We show how Parmentier’s quantization formalism of “affine Poisson groups” is part of our theory of torsors (Theorem 3.4).

1. Introduction

1.1. General overview. The aim of our work is to give a meaning and to develop a general theory for quantum torsors, starting from the principle that most of the objects of “traditional” commutative geometry should have a counterpart in the framework of non-commutative geometry. In algebraic geometry, torsors are familiar objects. Indeed, we know that they are linked with the problem of inner forms of algebraic groups and classified with the help of Galois cohomology groups: modulo an equivalence relation, torsors over an algebraic group $G$ defined over a field $k$ are in correspondence with a pointed cohomology set $H^1(\text{Gal}(\bar{k}/k), G(\bar{k}))$ where $\bar{k}$ is an algebraic closure of $k$, such that the trivial torsors $X$ on $k$ (the ones with $X(k) \neq \emptyset$) correspond to the trivial cocyle. Therefore, it seems natural to develop a similar theory in the framework of quantum groups or Hopf algebras. On the other hand, in the category of Poisson manifolds, torsors are well-known objects and have been classified by Dazord and Sondaz [DS] under the name of “affine Poisson groups” (for a precursor, see [S]) and quantized by Parmentier [P]. However, the two definitions of torsors in algebraic geometry and in Poisson geometry do not coincide. In algebraic geometry, a torsor is a...
scheme $X$ equipped with a group-action $m : G \times X \to X$ where $G$ is a group-scheme such that the map $(m \times \text{pr}_X) : G \times X \to X \times X$ is an isomorphism of schemes, whereas in the category of Poisson manifolds, a torsor $X$ is (still) the data of a Poisson-Lie group $G$ and a group-action $m : G \times X \to X$ plus certain conditions saying that the stabilizers of the action are all trivial, but in that case, the bijective map $(m \times \text{pr}_X) : G \times X \to X \times X$ is not a Poisson map.

Thus, we are looking for unifying in a same theory the results of algebraic geometry about torsors and Parmentier’s quantization results. Our starting point is an old intrinsic reformulation of affine structures originally suggested by Baer [B] and developed later on by Certaine [C], Vagner [V], Kock [Koc], Weinstein [W] and recalled recently by Kontsevich [Kon]. One of the corollaries of our work is the definition of a non trivial group-invariant $\text{Tor}(H)$ associated with any Hopf algebra $H$ (Theorem 3.3).

1.2. Definition of a torsor. In the category of sets, affine algebraic manifolds over an algebraically closed field or Poisson manifolds, a torsor is a $G$-principal homogeneous space where $G$ is a group in the category. Even though the axioms asserting that $G$ acts transitively can be quantized without any problem, the axiom saying that all stabilizers are trivial is not so easy to quantize, especially when the quantum manifolds have no point (i.e., when the algebras of functions have no character). According to the point of view originally developed by Baer [B], a (classical) torsor is the data of an object $X$ and a composition law $\mu_X : X^3 \to X$ satisfying some associativity relations, called parallelogram relations and analogous to the ones we would get by taking $X = G$ a group and

$$
\mu_G : G^3 \to G \quad (g, h, k) \mapsto gh^{-1}k
$$

By reversing the arrows, Kontsevich noted that it was possible to define in this way all the torsors of the algebraic geometry [Kon]. We will see that it is also possible to extend this idea in non-commutative geometry.

1.3. Classical torsors. In pointed categories such as the category of sets, affine algebraic manifolds over an algebraically closed field or Poisson manifolds, it is easy to see that the map $\mu_X : X^3 \to X$ should verify the following relations:

$$\forall a, b, c, d, e \in X, \quad \mu_X(a, a, b) = b \quad (2)$$

$$\mu_X(a, b, b) = a \quad (3)$$

$$\mu_X(\mu_X(a, b, c), d, e) = \mu_X(a, b, \mu_X(c, d, e)) \quad (4)$$

Conversely, if an object $X$ is equipped with a law $\mu_X$ satisfying these equalities, then we can easily find two groups of the category $G_l(X)$ and $G_r(X)$ acting simply and transitively on $X$, the first one from the left and the second one from the right and $X$ is a principal homogeneous space on $G_l(X)$ or $G_r(X)$. Furthermore, we are
able to classify all the classical torsors. In the category of sets, as well as in the category of affine algebraic manifolds over an algebraically closed field, any torsor is isomorphic to the trivial torsor \((G, \mu_G)\) on a group \(G\). In the category of Poisson manifolds, it can be shown that any torsor is isomorphic to an “affine Poisson group” according to the terminology introduced by Dazord and Sondaz \([DS]\) and thus is identified with a triple \((\mathfrak{g}, \delta, f)\) where \((\mathfrak{g}, \delta)\) is a Lie bialgebra and \(f \in \Lambda^2(\mathfrak{g})\) is a “classical Drinfeld twist” for \(\delta\). The affine Poisson group associated with this triple is defined in the following way: denoting by \((G, P)\) this triple is defined in the following way: denoting by \((G, P)\) the connected, simply connected Poisson-Lie group associated with the bialgebra \((\mathfrak{g}, \delta)\), the affine Poisson group is \(G\) as a manifold equipped with the affine Poisson structure given by the Poisson bivector \(\pi := P_G + f^L\) where \(f^L\) is the left translation of \(f\) on \(G\). On this Poisson manifold, the Poisson-Lie groups acting simply and transitively are \((G, P_G)\) by translation from the left and \((G, P_G + f^L - f^R)\) by translation from the right (\(f^R\) being the right translation of \(f\) on \(G\)).

2. Quantum torsors

2.1. Non-commutative torsors. Let us consider now a commutative field \(k\) and a commutative \(k\)-algebra \(A\) without any zero divisor. For example, \(A = k\) or \(A = k[[h]]\). We will work in the category of \(A\)-unitary associative algebras.

Definition 2.1. An \(A\)-torsor is a quintuple \((T, m_T, 1_T, \mu_T, \theta_T)\) where \((T, m_T, 1_T)\) is an \(A\)-algebra, \(\mu_T : T \rightarrow T \otimes_A T^{\text{op}} \otimes_A T\) is an \(A\)-algebra morphism and \(\theta_T : T \rightarrow T\) is an \(A\)-algebra automorphism satisfying the following axioms:

\[
\forall x \in T, \quad (Id_T \otimes m_T) \circ \mu_T(x) = x \otimes 1_T \tag{6}
\]
\[
(Id_T \otimes Id_{T^{\text{op}}} \otimes \mu_T) \circ \mu_T = (\mu_T \otimes Id_{T^{\text{op}}} \otimes Id_T) \circ \mu_T \tag{8}
\]
\[
\theta_T \circ (\mu_T \otimes Id_{T^{\text{op}}} \otimes Id_T) \circ \mu_T = (Id_T \otimes \mu_T^{\text{op}} \otimes Id_T) \circ \mu_T \tag{9}
\]
\[
(\theta_T \otimes \theta_T \otimes \theta_T) \circ \mu_T = \mu_T \circ \theta_T \tag{10}
\]

with \(\mu_T^{\text{op}} := \tau_{(13)} \circ \mu_T\) and \(\theta_T^{(3)} := (Id_T \otimes Id_{T^{\text{op}}} \otimes \theta_T \otimes Id_{T^{\text{op}}} \otimes Id_T)\). If \(m_T = \mu_T^{\text{op}}\), the torsor is said to be commutative. If \(\mu_T = \mu_T^{\text{op}}\), the torsor is said to be endowed with a commutative law.

Note 1. If \((T, m_T, 1_T, \mu_T, \theta_T)\) is an \(A\)-torsor, then \(\theta_T\) is fully determined by \(m_T\) and \(\mu_T\). For instance, if the torsor is either commutative or endowed with a commutative law, then \(\theta_T = Id_T\).

Remark 1. Of course, a given algebra needs not carry a torsor structure. For example, if \(\text{char}(k) \neq 2\), then there is no \(k\)-torsor structure on the \(k\)-algebra \(k[X]/(X^2)\).

Remark 2. If \((T, m_T, 1_T, \mu_T, \theta_T)\) is an \(A\)-torsor, then \((T^{\text{op}}, m_T^{\text{op}}, 1_T, \mu_T^{\text{op}}, \theta_T)\) is also an \(A\)-torsor, called its opposite torsor.
We will use generalized Sweedler notations. If \((T, m_T, 1_T, \mu_T, \theta_T)\) is an \(A\)-torsor, then for all \(x \in T\), forgetting the symbol \(\sum\), we denote \(\mu_T(x) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)}\) and \(\mu_T^{(n)}(x) = x^{(1)} \otimes \ldots \otimes x^{(2n+1)}, n \in \mathbb{N}\), where \(\mu_T^{(n)}\) satisfies the induction \(\mu_T^{(0)} := \text{Id}_T\) and \(\mu_T^{(n)} = (\mu_T^{(n-1)} \otimes \text{Id}_{T^\text{op}} \otimes \text{Id}_T) \circ \mu_T\).

Then, the torsor axioms show that for any odd integer \(i\), we have:
\[
x^{(1)} \otimes \ldots \otimes x^{(i-1)} \otimes x^{(i)(1)} \otimes x^{(i)(2)} \otimes x^{(i)(3)} \otimes x^{(i+1)} \otimes \ldots \otimes x^{(2n-1)} = \mu_T^{(n)}(x)
\]
and, for any even integer \(i\),
\[
x^{(1)} \otimes \ldots \otimes x^{(i-1)} \otimes x^{(i)(1)} \otimes x^{(i)(2)} \otimes x^{(i)(3)} \otimes x^{(i+1)} \otimes \ldots \otimes x^{(2n-1)} = x^{(1)} \otimes \ldots \otimes x^{(i-1)} \otimes x^{(i+2)} \otimes \theta_T(x^{(i+1)}) \otimes x^{(i)} \otimes x^{(i)(3)} \otimes \ldots \otimes x^{(2n+1)}
\]
In particular, we see that for all \(x \in T\), \(\theta_T(x) = x^{(1)}x^{(2)(3)}x^{(2)(2)}x^{(2)(1)}x^{(3)}\).

As apparent with the study of Example 1 below which shows that \(\theta_T\) is an analogue of the square of the antipode in a Hopf algebra, it seems necessary to introduce \(\theta_T\) in the definition of a non-commutative torsor.

**Example 1.** The trivial torsor of a Hopf algebra

Let \((H, m_H, \Delta_H, \eta_H, \varepsilon_H, S_H)\) be an \(A\)-Hopf algebra. Then, \((H, m_H, 1_H, \mu_H, \theta_H)\) is an \(A\)-torsor with \(1_H = \eta_H(1), \mu_H := (\text{Id}_H \otimes S_H \otimes \text{Id}_H) \circ \Delta_H^{(2)}\) and \(\theta_H := S_H^2\).

**Example 2.** Affine torsors in algebraic geometry

As said in introduction, in algebraic geometry, a torsor is the data of a scheme \(X\), a group-scheme \(G\) and an action \(m : G \times X \rightarrow X\) such that the map:
\[
(m \times \text{pr}_X) : G \times X \rightarrow X \times X
\]
is an *isomorphism of schemes*. As noted by Kontsevich \[Ko\], in the language of *affine* scheme, \(X\) correspond to a comodule-algebra \(A\) over an Hopf algebra \(H\) and the map \(\mu_X : X^3 \rightarrow X\) obtained by composition:
\[
X^3 = X^2 \times X \rightarrow G \times X \times X \rightarrow G \times X \rightarrow X
\]
where the second map is obtained by forgetting the second factor in \(G \times X \times X\) gives rise to a torsor-structure on \(A\) (with of course \(\theta_A = \text{Id}_A\)). In other terms, the torsors of algebraic geometry are the *commutative* torsors for the definition \[2.1\]. A good example of torsor without any point that we should keep in mind is \(X = \text{Isom}(M(n), A)\) where \(A\) is a *simple central* algebra. In this case, the left group (i.e., the group acting simply transitively from the left on the torsor) is \(\text{PGL}(n)\) and the right group is \(A^\times/k^\times\) if \(k\) denotes the ground field.

**Example 3.** Quadratic extensions of a field

The following examples of torsor structures on algebras (without character if \(d \not\in (k^\times)^2\) are nothing but *subcases* of the previous example. If \(k\) is a commutative field and if \(d \in k^\times\), then the \(k\)-algebra \(A = k[X]/(X^2 - d)\) can be equipped with a
\(k\)-torsor structure by \(\mu_A(x) = x \otimes x^{-1} \otimes x, x = \text{cl}(X)\) and \(\theta_A = \text{Id}_A\). If \(\text{char}(k) = 2\) and \(A = k[X]/(X^2 - d)\) or \(A = k[X]/(X^2 - X - d)\) with \(d \in k\), the law \(\mu_A\) defined on \(A\) by \(\mu_A(x) = 1 \otimes 1 \otimes x + 1 \otimes x \otimes 1 + x \otimes 1 \otimes 1\) and \(x = \text{cl}(X)\) gives to \(A\) a \(k\)-torsor structure with \(\theta_A = \text{Id}_A\).

**Example 4.** Generalization: Galois extension of a field

Let \(K = k[T]/(P)\) be a Galois extension of a field \(k\), \(t := \text{cl}(T)\) a primitive element in \(K\), \(G := \text{Gal}(K/k)\) the Galois group of the extension and \(H := k^G := (kG)^*\) the natural Hopf algebra of functions on \(G\) with values in \(k\). First, note that there is a natural structure of \(H^\text{op}\)-left comodule algebra on \(K\) given by a morphism \(\Delta_K: K \to H \otimes_k K\). Second, note that we can identify \(K \otimes_k K\) with \(K[T]/(P)\) such that \(t_1 := t \otimes 1 \in K \otimes_k K\) is identified to \(\text{cl}(T) \in K[T]/(P)\) and \(t_2 := 1 \otimes t\) is identified to \(t \in K \subset K[T]/(P)\). Third, note that there is a natural isomorphism between \(K^G := H \otimes_k K\) (the algebra of functions on \(G\) with values in \(K\)) and \(K \otimes_k K \cong K[T]/(P)\). Under this isomorphism, \(x \otimes y \in K \otimes_k K\) is associated to \(\sum_{\sigma \in G} 1_{\sigma} \otimes \sigma(x)y \in H \otimes K\) where \(1_{\sigma}\) denotes the natural idempotent in \(H\) associated to \(\sigma\) by \(<1_{\sigma}, \tau> = \delta_{\sigma, \tau}\) for all \(\tau \in G\) and \(1_{\sigma} \otimes x \in K^G\) is associated to \(P_{\sigma} \times (1 \otimes x)\) with \(P_\sigma = \prod_{\tau \neq \sigma} \frac{\tau(t_1) - \tau(t_2)}{\tau(t_{2}) - \tau(t_{1})}\). Therefore, by composing \(\Delta_K\) with the embedding of \(H\) in \(K^G = H \otimes_k K \cong K \otimes_k K\), we get an algebra morphism:

\[
\mu_K: K \longrightarrow K \otimes_k K \otimes_k K \\
x \longmapsto \sum_{\sigma \in G} P_{\sigma} \otimes \sigma(x)
\]

We have \(\mu_K^G(x) := \tau(1_3) \circ \mu_K(x) = \sum_{\sigma} \sigma(x) \otimes P_{\sigma^{-1}}\) for all \(x \in K\) and \((g \otimes \text{Id})(P_h) = P_{gh^{-1}}\) and \((\text{Id} \otimes g)(P_h) = P_{gh}\) for any elements \(g, h \in G\). From this, it can be shown that \((K, m_K, 1_K, \mu_K, \text{Id}_K)\) is a \(k\)-torsor and that \(G\) is a group of automorphisms for the torsor \((K, m_K, 1_K, \mu_K, \text{Id}_K)\). In other terms, Galois doesn’t classify only torsors of the algebraic geometry; Galois is also a torsor!

**Example 5.** Non-commutative torsors without character

Let \(k\) be a field and \(n\) be a non-negative integer. Suppose that \(k\) contains an element \(q\) which is a \(n\)-th primitive root of 1. For any \(\alpha\) and \(\beta\) in \(k^\times\), we denote by \(A_{\alpha, \beta}\) the non-commutative algebra with unit and without character given by generators: \(x, y\) and relations: \(x^n = \alpha, y^n = \beta\) and \(xy = qyx\). The algebra \(A_{\alpha, \beta}\) is a non-trivial cyclic algebra and \(\dim_k A_{\alpha, \beta} = n^2\). If \(n = 2\), then \(A_{\alpha, \beta}\) is an algebra of quaternions. There is a natural structure of non-commutative torsor endowed with a commutative law on \(A_{\alpha, \beta}\) given by \(\mu(x) = x \otimes x^{-1} \otimes x, \mu(y) = y \otimes y^{-1} \otimes y\) and \(\theta = \text{Id}\).

2.2. Poisson torsors.

**Definition 2.2.** A \(k\)-Poisson torsor is a sextuple \((T, m_T, 1_T, \mu_T, \theta_T, \{, \})_T\) such that 1) \((T, m_T, 1_T, \mu_T, \theta_T, \{, \})_T\) is a commutative \(k\)-torsor, 2) \((T, m_T, 1_T, \{, \})_T\) is a \(k\)-Poisson algebra and 3) the maps \(\mu_T\) and \(\theta_T\) are Poisson maps where the Poisson structure on \(T \otimes_k T \otimes_k T\) is given by the bracket: \(\{x \otimes y \otimes z, x' \otimes y' \otimes z'\} =\)
\{x,x'\} \otimes yy' \otimes zz' - xx' \otimes \{y,y'\} \otimes zz' + xx' \otimes yy' \otimes \{z,z'\} \text{ for any elements } x,x',y,y',z,z' \text{ belonging to } T.

**Proposition 2.1.** If \( G \) is a Poisson affine group, then \((G,\mu_G)\) is a classical Poisson torsor and \((\mathcal{O}_G,\mathcal{O}_G,\infty_{\mathcal{O}_G})\) can be naturally endowed with a Poisson torsor structure. Conversely, if \( X \) is a smooth manifold and if \((\mathcal{O}_X,\mathcal{O}_X,\infty_{\mathcal{O}_X})\) can be endowed with a Poisson torsor structure then \( X \) is isomorphic to a Poisson affine group.

However, Poisson torsor structures on commutative algebras without character should exist (\(?\)), even if we are unable to give a single example. A way to get such a torsor should be to consider an affine torsor \( X \) without any point on a field \( k \) (cf Example 2) and two \( r \)-matrices \( r_I \) and \( r_r \) of \( g_I := \text{Lie}\{\mathcal{G}(\mathcal{X})\} \) and \( g_r := \text{Lie}\{\mathcal{G}(\mathcal{X})\} \) such that the Schouten algebraic brackets \([r_I,r_I]\) and \([r_r,r_r]\) are equal as elements of \( g_I \otimes \ell \cong g_r \otimes \ell \) where \( k \) is an algebraic closure of \( k \). Indeed, in this case, the bivector field \( r_I - r_r \) is a Poisson bivector on \( X \), where \( r_I \) (resp. \( r_r \)) is the left action of \( r_I \) (resp. \( r_r \)) on \( X \). Therefore, this problem is intimately linked with the problem of classification of \( r \)-matrices in Lie algebras over a non-algebraically closed field.

**Definition 2.3.** A quantization of a \( k \)-Poisson torsor \((T_{cl},m_{T_{cl}},1_{T_{cl}},\mu_{T_{cl}},\theta_{T_{cl}},{\{,\}_{cl}})\) is the data of a \( k[[\hbar]] \)-torsor \((T,m_T,1_T,\mu_T,\theta_T)\) such that the \( k[[\hbar]] \)-algebra \((T,m_T,1_T)\) is a quantization of the \( k \)-Poisson algebra \((T_{cl},m_{T_{cl}},1_{T_{cl}},{\{,\}_{cl}})\) and such that \( \mu_T = \mu_{T_{cl}} \) (mod \( \hbar \)) and \( \theta_T = \theta_{T_{cl}} \) (mod \( \hbar \)).

In the future, our goal is to classify all torsors corresponding to certain classes of Hopf algebras. The problem seems to be difficult. Indeed, when the ground field \( k \) is not (necessarily) algebraically closed, affine torsors of the algebraic geometry are classified by means of non-abelian cohomology. On the other hand, when \( k \) is algebraically closed, the study of classical torsors in the category of Poisson manifolds \([\text{DS}]\) shows that the Drinfeld twists should play a role in the classification.

### 3. The results

We present below our main results about torsors. As before, in all this section, \( k \) stands for a commutative field and \( A \) for a commutative \( k \)-algebra without any zero divisor.

#### 3.1. The Hopf algebras \( H_i(T) \) and \( H_r(T) \)

In the classical case, on any torsor, there are two groups acting simply and transitively. In the non-commutative setting, we can also find two Hopf algebras co-acting on a given torsor.

**Theorem 3.1.** Let \((T,m_T,1_T,\mu_T,\theta_T)\) be an \( A \)-torsor. We denote by \( H_i(T,m_T,1_T,\mu_T,\theta_T) \) or shortly \( H_i(T) \) or \( H_i(\mu_T) \) if there is no confusion, the subset of \( T \otimes_A T^{op} \) defined as \( \{ x \in T \otimes_A T^{op}/(1_T \otimes \text{Id}_{T^{op}} \otimes \theta_T \otimes \text{Id}_{T^{op}}) \circ (\mu_T \otimes \text{Id}_{T^{op}})(x) = (\text{Id}_T \otimes \mu_T^{op})(x) \} \). Then,
1. if \( x \in H_1(T) \), \( m_T(x) \) is a scalar denoted by \( \varepsilon_{H_1(T)}(x)1_T \);
2. if \( x \in H_1(T) \), \( \Delta_{H_1(T)}(x) := (\mu_T \otimes \text{Id}_{T^{op}})(x) \in H_1(T) \otimes_A H_1(T) \);
3. if \( x \in H_1(T) \), \( S_{H_1(T)}(x) := \pi_{(12)} \circ (\text{Id}_{T} \otimes \theta_T)(x) \in H_1(T) \);
4. By defining \( m_{H_1(T)} \) as the restriction of \( m_T \otimes m_T^{op} \) to \( H_1(T) \) and \( \eta_{H_1(T)} : A \rightarrow H_1(T) \) as given by \( \eta_{H_1(T)}(1) = 1_T \otimes 1_T \), then the following sextuple \((H_1(T), m_{H_1(T)}, \Delta_{H_1(T)}, \varepsilon_{H_1(T)}, S_{H_1(T)})\) is an \( A \)-Hopf algebra.
5. \( \text{Im}\mu_T \subset H_1(T) \otimes_A T \) and \( \mu_T : T \rightarrow H_1(T) \otimes_A T \) gives to \( T \) a left \( H_1(T) \)-comodule-algebra structure.

Likewise, we denote by \( H_r(T, m_T, 1_T, \mu_T, \theta_T) \) or shortly \( H_r(T) \) or \( H_r(\mu_T) \) if there is no confusion, the subset of \( T^{op} \otimes_A T \) defined as

\[
H_r(T) := \{ x \in T^{op} \otimes_A T / (\text{Id}_{T^{op}} \otimes \theta_T \otimes \text{Id}_{T^{op}} \otimes \text{Id}_T) \circ (\text{Id}_{T^{op}} \otimes \mu_T)(x) = (\mu_T^{op} \otimes \text{Id}_T)(x) \}.
\]

Then, \( H_r(T) \) can be equipped with a natural structure of \( A \)-Hopf algebra, \( \text{Im}\mu_T \subset T \otimes_A H_r(T) \) and the map \( \mu_T : T \rightarrow T \otimes_A H_r(T) \) defines a right \( H_r(T) \)-comodule-algebra structure on \( T \). Moreover, the two co-actions of \( H_1(T) \) and \( H_r(T) \) on \( T \) commute.

**Example 6.** Let \((H, m_H, \Delta_H, \eta_H, \varepsilon_H, S_H)\) be a Hopf algebra equipped with its trivial torsor structure \((H, m_H, 1_H, \mu_H, \theta_H)\). Then, \( H_1(H) = (\text{Id}_H \otimes S_H) \circ \Delta(H), H_r(H) = (S_H \otimes \text{Id}_H) \circ \Delta(H) \) and \( i_{t,H} := (\text{Id}_H \otimes \varepsilon_H) \) (resp. \( i_{r,H} := (\varepsilon_H \otimes \text{Id}_H) \)) establishes a Hopf algebra isomorphism between \( H_1(H) \) and \( H \) (resp. \( H_r(H) \) and \( H \)).

**Example 7.** If \( K/k \) is a Galois extension of a field \( k \) equipped with its torsor structure seen in 4, then \( H_1(T) \cong (kG^{op})^\times \) and \( H_r(T) \cong (kG)^\times \) as Hopf algebras.

**Example 8.** Let \( A_{\alpha,\beta} \) be the non-commutative algebra-torsor considered in Example 3. Then, \( H_1(A_{\alpha,\beta}) = H_r(A_{\alpha,\beta}) \) is generated by the elements \( x \otimes x^{-1} \) and \( y \otimes y^{-1} \) and thus is isomorphic to the algebra of functions on \( \mathbb{Z}/\times \mathbb{Z} \times \mathbb{Z}/\times \mathbb{Z} \).

**Note 2.** If \( (T, m_T, 1_T, \mu_T, \theta_T) \) is an \( A \)-torsor, then the map \((\theta_T \otimes \text{Id}_T)\) (resp. \( (\text{Id}_T \otimes \theta_T)\)) is a Hopf algebra isomorphism from \( H_1(\mu_T) \) to \( H_r(\mu_T^{op}) \) (resp. \( H_r(\mu_T) \) to \( H_1(\mu_T^{op}) \)).

**Note 3.** If \( (T, m_T, 1_T, \mu_T, \theta_T) \) is an \( A \)-torsor endowed with a commutative law, then \( H_1(T) = H_r(T) \) (in \( T \otimes_A T \)).

**Note 4.** Let \( (T, m_T, 1_T, \mu_T, \theta_T) \) be an \( A \)-torsor and \( \varepsilon : T \rightarrow A \) a character. If \( \dim_k T < \infty \) or \( \varepsilon \circ \theta_T = \varepsilon \), then \( T \) is isomorphic as an algebra to its left or right Hopf algebra with an identification between \( \varepsilon \) and the co-unity of the Hopf algebra.
3.2. Torsors and Hopf-Galois extensions. The notion of Hopf-Galois extension was introduced by Chase and Sweedler in 1969 in the commutative case \[CS\] and later by Kreimer and Takeuchi in the non-commutative case \[KT\] as a generalization of the Galois theory where the Galois groups are replaced by Hopf algebras. Let \(k\) be a (commutative) field and \(H\) an Hopf algebra. The axioms for a left \(H\)-Galois extension \(A\) of \(k\) are the ones we would get by taking formula (11) as a reference for defining torsors over Hopf algebras \[Schn\]. By definition, a left \(H\)-torsor \(T\) is a group \(T\) acted on by \(H\) and \(k\)-algebras. Let \(H\) be a \(k\)-Hopf algebra. The generalization of the Galois theory where the Galois groups are replaced by Hopf algebras \[CS\] and later by Kreimer and Takeuchi in the non-commutative case \[KT\] as a torsor was introduced by Chase and Sweedler in 1969 in the commutative case.

**Proposition 3.1.** Let \((T, m_T, 1_T, \mu_T, \theta_T)\) be a \(k\)-torsor. Then, \(T\) is a left \(H_1(T)\)-Galois extension of \(k\) and a right \(H_1(T)\)-Galois-extension of \(k\). In particular, in the finite dimensional case, we have \(\dim_k T = \dim_k H_1(T) = \dim_k H_1(T)\).

3.3. Composition of torsors. The \(A\)-torsors form a category. We can naturally define the notion of torsors morphism, sub-torsors, quotient torsors and tensor product of two torsors. If \((T_i, m_{T_i}, 1_{T_i}, \mu_{T_i}, \theta_{T_i}), i = 1, 2\) are two torsors with a Hopf algebra isomorphism between \(H_1(T_1)\) and \(H_1(T_2)\), then the following theorem shows that we can compose the two torsors to get a third torsor whose Hopf algebra co-acting from the left (resp. from the right) is isomorphic to \(H_1(T_1)\) (resp. \(H_1(T_2)\)).

**Theorem 3.2.** Let \((T_i, m_{T_i}, 1_{T_i}, \mu_{T_i}, \theta_{T_i}), i = 1, 2\) be two \(A\)-torsors where \(A\) is as above. Let us assume that there is an isomorphism \(\Phi : H_1(T_1) \rightarrow H_1(T_2)\) of \(A\)-Hopf algebra and set

\[T_\Phi := T_1 \otimes_A T_2 := \{ x \in T_1 \otimes_A T_2 / (Id_{T_1} \otimes \Phi \otimes Id_{T_2}) \circ (\mu_{T_1} \otimes Id_{T_2})(x) = (Id_{T_1} \otimes \mu_{T_2})(x) \} \cdot\]

Let \(m_\Phi\) and \(\theta_\Phi\) be the restrictions of \(m_{T_1} \otimes m_{T_2}\) and \(\theta_{T_1} \otimes \theta_{T_2}\) to \(T_\Phi\) and let \(\mu_\Phi\) be the map defined on \(T_\Phi \subset T_1 \otimes_A T_2\) with the help of the generalized Sweedler notations by:

\[\mu_\Phi(x_i \otimes y_i) = \tau_{(34)}(x_{i}^{(1)} \otimes \Phi(x_{i}^{(2)} \otimes x_{i}^{(3)}) \otimes x_{i}^{(4)} \otimes x_{i}^{(5)} \otimes y_i)\]

(14)

where \(\tau_{(34)} : T_1 \otimes_A T_2 \otimes_A T_2^{op} \otimes_A T_1 \otimes_A T_1 \otimes_A T_2 \rightarrow T_1 \otimes_A T_2 \otimes_A T_1 \otimes_A T_2^{op} \otimes_A T_1 \otimes_A T_2\) denotes the permutation morphism of the third and fourth factors, \(H_1(T_2)\) being imbedded in \(T_2 \otimes_A T_2^{op}\). Then,

1. \((T_\Phi, m_\Phi, 1_{T_1} \otimes 1_{T_2})\) is an \(A\)-algebra, \(\text{Im}_\Phi \subset T_\Phi \otimes T_\Phi^{op} \otimes T_\Phi\), \(\text{Im}_\theta_\Phi \subset T_\Phi\) and \((T_\Phi, m_\Phi, 1_{T_1} \otimes 1_{T_2}, \mu_\Phi, \theta_\Phi)\) is an \(A\)-torsor.
2. The restriction of the map $\tau(34) \circ (\text{Id}_{T_1} \otimes \Phi \otimes \text{Id}_{T_1^{op}}) \circ (\mu_{T_1} \otimes \text{Id}_{T_1^{op}})$ to $H_1(T_1) \subset T_1 \otimes_A T_1^{op}$ gives rise to an $A$-Hopf algebra isomorphism between $H_1(T_1)$ and $H_1(T_0)$, whose inverse map is $(\text{Id}_{T_1} \otimes \varepsilon_{H_1(T_2)} \otimes \text{Id}_{T_1^{op}}) \circ \tau(34)$.

3. The restriction of the map $\tau(12) \circ (\text{Id}_{T_2^{op}} \otimes \Phi^{-1} \otimes \text{Id}_{T_2}) \circ (\text{Id}_{T_2^{op}} \otimes \mu_{T_2})$ to $H_r(T_2) \subset T_2^{op} \otimes_A T_2$ gives rise to an $A$-Hopf algebra isomorphism between $H_r(T_2)$ and $H_r(T_0)$, whose inverse map is $(\text{Id}_{T_2^{op}} \otimes \varepsilon_{H_r(T_1)} \otimes \text{Id}_{T_2}) \circ \tau(12)$.

3.4. The group Tor($H$). If $f : (T_1, m_{T_1}, 1_{T_1}, \mu_{T_1}, \theta_{T_1}) \rightarrow (T_2, m_{T_2}, 1_{T_2}, \mu_{T_2}, \theta_{T_2})$ is an $A$-torsor morphism, then it can be shown that $(f \otimes f^{op})(H_1(T_1)) \subset H_1(T_2)$ and that $f_t := (f \otimes f^{op})(H_1(T_1)) : H_1(T_1) \rightarrow H_1(T_2)$ is an $A$-Hopf algebra morphism. Likewise, we define a Hopf morphism $f_r : H_r(T_1) \rightarrow H_r(T_2)$. Consequently, if $H$ and $H'$ are two $A$-Hopf algebras, on the set $\widehat{\text{Tor}}(H, H')$ of the septuples $(T, m_T, 1_T, \mu_T, \theta_T, i_{l,T}, i_{r,T})$ where $(T, m_T, 1_T, \mu_T, \theta_T)$ is an $A$-torsor and where $i_{l,T} : H_l(A) \rightarrow H$ and $i_{r,T} : H_r(A) \rightarrow H'$ are two $A$-Hopf algebra isomorphisms, we can define a relation $\sim_{H,H'}$ by $(T_1, m_{T_1}, 1_{T_1}, \mu_{T_1}, \theta_{T_1}, i_{l,T_1}, i_{r,T_1}) \sim_{H,H'} (T_2, m_{T_2}, 1_{T_2}, \mu_{T_2}, \theta_{T_2}, i_{l,T_2}, i_{r,T_2})$ if and only if there exists an $A$-torsor isomorphism $f : (T_1, m_{T_1}, 1_{T_1}, \mu_{T_1}, \theta_{T_1}) \rightarrow (T_2, m_{T_2}, 1_{T_2}, \mu_{T_2}, \theta_{T_2})$ such that $i_{l,T_1} = i_{l,T_2} \circ f_l$ and $i_{r,T_1} = i_{r,T_2} \circ f_r$. This relation is an equivalence relation. The quotient set is denoted by Tor($H, H'$). Moreover, if $H$, $H'$, $H''$ are three $A$-Hopf algebras, we have a natural map:

$$\text{Tor}(H, H') \times \text{Tor}(H', H'') \rightarrow \text{Tor}(H, H'')$$

where $T := T_1 \otimes A T_2$ is by Theorem 3.2 an $A$-torsor and $\Phi := i_{l,T_1}^{-1} \circ i_{l,T_2}$, $i_{l,T} := i_{l,T_1} \circ (\text{Id}_{T_1} \otimes \varepsilon_{H_l(T_2)} \otimes \text{Id}_{T_1^{op}}) \circ \tau(34)$ and $i_{r,T} := i_{r,T_2} \circ (\text{Id}_{T_2^{op}} \otimes \varepsilon_{H_r(T_1)} \otimes \text{Id}_{T_2}) \circ \tau(12)$.

It can be shown that the map defined in (15) is associative and compatible with the equivalence relation $\sim_{H,H'}$. Thus, we define a composition law on Tor($H$) := Tor($H, H$) which is denoted by $\ast$.

Theorem 3.3. Let $(H, m_H, \Delta_H, \eta_H, \varepsilon_H, S_H)$ be a Hopf algebra over a commutative field $k$. The set Tor($H$) equipped with the law $\ast$ is a group whose unit element is the class of the trivial torsor $(H, m_H, 1_H, \mu_H, \theta_H, i_{l,H}, i_{r,H})$ (the notations are the same as those of Examples 4 and 5). The inverse of the class of $(T, m_T, 1_T, \mu_T, \theta_T, i_{l,T}, i_{r,T})$ is equal to the class of the opposite torsor of $T : (T^{op}, m_{T^{op}}, 1_T, \mu_{T^{op}}, \theta_T, i_{l,T^{op}}, i_{r,T^{op}})$ with $i_{l,T^{op}} := i_{r,T} \circ (\text{Id}_T \otimes \theta_T)^{-1}$ and $i_{r,T^{op}} := i_{l,T} \circ (\theta_T \otimes \text{Id}_T)^{-1}$ (see Remark 4 and Note 5).

The group Tor($H$) is called the torsor invariant of the $A$-Hopf algebra $H$. As we saw with the study of Example 5 together with Example 8, this group is far from trivial. Given the link with the Hopf-Galois extensions theory, it should be seen as a subgroup of Bigal($H$) introduced earlier by Schauenburg [Schau] and also as a generalization of the Harisson group in Galois theory.
3.5. Cotorors “la Parmentier”. We show that Parmentier’s formalism which allows to quantize affine Poisson groups can be subsumed in our theory of torsors or cotorors whose definition is given below.

**Definition 3.1.** An $A$-cotoror is a quintuple $(C, \Delta_C, \varepsilon_C, \nu_C, \theta_C)$ where $(C, \Delta_C, \varepsilon_C)$ is an $A$-coalgebra, $\nu_C : C \otimes_A C^{\text{cop}} \otimes_A C \to C$ an $A$-coalgebra morphism and $\theta_C : C \to C$ a coalgebra automorphism satisfying the following axioms:

\[
\begin{align*}
\nu_C \circ (\Delta_C \otimes \text{Id}_C) &= \varepsilon_C \otimes \text{Id}_C \quad (16) \\
\nu_C \circ (\text{Id}_C \otimes \Delta_C) &= \text{Id}_C \otimes \varepsilon_C \quad (17) \\
\nu_C \circ (\nu_C \otimes \text{Id}_C) &= \nu_C \circ (\text{Id}_C \otimes \nu_C) \quad (18) \\
\nu_C \circ (\nu_C \otimes \text{Id}_C) \circ \theta_C^{(3)} &= \nu_C \circ (\text{Id}_C \otimes \nu_C^{\text{op}} \otimes \text{Id}_C) \quad (19) \\
\nu_C \circ (\theta_C \otimes \text{Id}_C) &= \nu_C \circ \theta_C \quad (20)
\end{align*}
\]

with $\nu_C^{\text{op}} := \nu_C \circ \tau_{(13)}$ and $\theta_C^{(3)} := (\text{Id}_C \otimes \nu_C^{\text{cop}} \otimes \theta_C \otimes \text{Id}_C^{\text{cop}} \otimes \text{Id}_C)$.

The cotorors theory can be developed in the same way as the torsors one. In particular, any cotoror can be equipped with two coalgebra-module structures over Hopf algebras and the two actions commute. If $(T, m_T, 1_T, \mu_T, \theta_T)$ is an $A$-torsor, then $(T^*, m_T^*, \eta_T^*, \mu_T^*, \theta_T^*)$ is an $A$-cotoror, where $\eta_T : A \to T$ is defined by $\eta_T(1) = 1_T$. The converse is true in the finite dimensional case.

Now, let $(H, m_H, \Delta_H, \eta_H, \varepsilon_H, S_H)$ be an $A$-Hopf algebra and let $F \in H \otimes_A H$ be a Drinfeld twist [D], i.e., an element $F$ satisfying the equations:

\[
(F \otimes 1)(\Delta \otimes \text{Id}_H)(F) = (1 \otimes F)(\text{Id}_H \otimes \Delta_H)(F) \quad (21)
\]

\[
(\varepsilon_H \otimes \text{Id}_H)(F) = (\text{Id}_H \otimes \varepsilon_H)(F) = 1 \quad (22)
\]

Then, it is known that $u_F := m_H \circ (\text{Id}_H \otimes S_H)(F)$ is an invertible element of $H$ whose inverse is $u_F^{-1} = m_H \circ (S_H \otimes \text{Id}_H)(F^{-1})$ and that if we set $\Delta_F := F \Delta_H F^{-1}$ and $S_F := u_F S_H u_F^{-1}$, then the sextuple $(H, m_H, \Delta_F, \eta_H, \varepsilon_H, S_F)$ denotes shortly by $H_F$ is an $A$-Hopf algebra. Moreover, if $H = U_h(g)$ is a QUE algebra, $\delta := \lim_{h \to 0} h^{-1}(\Delta_H - \Delta_H^{op})$ and $f := \lim_{h \to 0} h^{-1}(F - 1 \otimes 1)$, then the triple $(H, \Delta_H F^{-1}, \varepsilon_H)$ is a $k$-coalgebra which is a quantization of the affine Poisson group given by the triple $(g, \delta, f)$ [F].

**Theorem 3.4.** Let $(H, m_H, \Delta_H, \eta_H, \varepsilon_H, S_H)$ be an $A$-Hopf algebra and let $F \in H \otimes_A H$ be a Drinfeld twist. Set $(C, \Delta_C, \varepsilon_C) := (H, \Delta_H F^{-1}, \varepsilon_H)$ and let $\theta_C$ be the $A$-linear map defined on $C$ by $\forall x \in C, \theta_C(x) = S_H^2(x)S_H(u_F)u_F^{-1}$ with $u_F := m_H \circ (\text{Id}_H \otimes S_H)(F)$ and

\[
\nu_C : C \otimes_A C^{\text{cop}} \otimes_A C \to C \quad \begin{array}{ccc}x \otimes y \otimes z & \mapsto & x u_F S_H(y)z \end{array}
\]

Then $\theta_C$ is an $A$-coalgebra automorphism for the $A$-coalgebra $(C, \Delta_C, \varepsilon_C), \nu_C$ is a $A$-coalgebra morphism, and the quintuple $(C, \Delta_C, \varepsilon_C, \nu_C, \theta_C)$ is an $A$-cotoror.
whose Hopf algebra acting from the left (resp. from the right) on $C$ is isomorphic to $H$ (resp. $H_F$).

In particular, if we denote by $i_{l,C}$ and $i_{r,C}$ two isomorphisms from $H$ and $H_F$ to $H_l(C)$ and $H_r(C)$, we see that, in the case where $A = k$ and $C$ is a finite dimensional $k$-vector space, then the septuple $(C^*, \Delta^*_C, \varepsilon^*_C(1), \nu^*_C, \theta^*_C, i_{l,C}^*, i_{r,C}^*) \in \hat{\text{Tor}}(H, H_F)$. Moreover, if $F$ commutes with the comultiplication $\Delta_H$, then $H_F = H$. Therefore, we see that $\hat{\text{Tor}}(H)$ contains a subgroup isomorphic to $\text{Aut}(H)^2 \times \{ F \in (H \otimes H)^\times / F \text{ is a Drinfeld twist and } F \text{ commutes with } \text{Im}(\Delta_H) \}$. On the other hand, according to Note 3, to get a torsor which is not of the Parmentier type, it is enough to consider torsors endowed with a commutative law whose underlying algebra does not possess any character. This is the case of torsors of Example 3 or Example 5. This last example gives also torsors which are neither “Parmentier type” torsors nor torsors arising from algebraic geometry.

4. Open problems

1. Find a non-trivial Poisson torsor structure on a commutative algebra without any character and try to quantize it. Classify all Poisson torsors.
2. Find $\text{Tor}(H)$ in simple cases.
3. Study $\text{Tor}(H)$ when the Hopf algebra $H$ is a quantized enveloping algebra.
4. In particular, if $(g, \delta)$ a $k$-Lie bialgebra, if $f$ is a classical Drinfeld twist and if $(U^e_k(g), \Delta)$ is a quantization of $(g, \delta)$, is it true or not that every quantization of the Poisson torsor given by the triple $(g, \delta, f)$ is a “Parmentier” type torsor, that is to say given by a quantization $F$ of $f$?
5. Study the case where $H = CT$ is equipped with its natural Hopf algebra structure, with $\Gamma$ a finite group, by using some results due to Movshev [M] or to Etingof-Gelaki [EG] about twists in this Hopf algebra.
6. Find and classify all “low dimensional” torsors.
7. Generally speaking, classify all torsors and cotorsors for a large class of Hopf algebras.
8. A classical torsor together with its opposite torsors can be seen as a groupoid on a basis with two elements. Are our axioms compatible with the notion of a quantum groupoid [Ma]?
9. Is it possible to find a Hopf-Galois extension which is not a torsor in our sense?
10. Establish a link with the Galois cohomology which classifies torsors in algebraic geometry.

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