Finite groups with $\mathbb{P}$-subnormal primary cyclic subgroups

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Abstract

A subgroup $H$ of a group $G$ is called $\mathbb{P}$-subnormal in $G$ whenever either $H = G$ or there is a chain of subgroups $H = H_0 \subset H_1 \subset \ldots \subset H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for all $i$. In this paper, we study the groups in which all primary cyclic subgroups are $\mathbb{P}$-subnormal.

Keywords: finite group, supersolvable group, primary subgroup, cyclic subgroup

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Introduction

We consider finite groups only. A. F. Vasilyev, T. I. Vasilyeva and V. N. Tyutyunnov in [1] introduced the following definition. Let $\mathbb{P}$ be the set of all prime numbers. A subgroup $H$ of a group $G$ is called $\mathbb{P}$-subnormal in $G$ whenever either $H = G$ or there is a chain

$$H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

of subgroups such that $|H_i : H_{i-1}|$ is prime for all $i$.

Let $|G| = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, where $p_1 > p_2 > \ldots > p_k$, $a_i \in \mathbb{N}$. We say that $G$ has an ordered Sylow tower of supersolvable type if there exist normal subgroups $G_i$ with

$$1 = G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_{k-1} \subset G_k = G,$$

and where each factor $G_i/G_{i-1}$ is isomorphic to a Sylow $p_i$-subgroup of $G$ for all $i$. We denote by $\mathfrak{D}$ the class of all groups which have an ordered Sylow tower of supersolvable type. It is well known that $\mathfrak{D}$ is a hereditary saturated formation.

In [1] finite groups with $\mathbb{P}$-subnormal Sylow subgroups were studied. A group $G$ is called w-supersolvable if every Sylow subgroup of $G$ is $\mathbb{P}$-subnormal.
in $G$. Denote by $w\mathfrak{U}$ the class of all $w$-supersolvable groups. Observe that the class $\mathfrak{U}$ of all supersolvable groups is included into $w\mathfrak{U}$. In [1], the authors proved that the class $w\mathfrak{U}$ is a saturated hereditary formation; every group in $w\mathfrak{U}$ possesses an ordered Sylow tower of supersolvable type; all metanilpotent and all biprimary subgroups in $w\mathfrak{U}$ are supersolvable.

In [2] the following problem was proposed.

To describe the groups in which all primary cyclic subgroups are $\mathbb{P}$-subnormal.

In this note we solve this problem. Denote by $\mathfrak{X}$ the class of groups whose primary cyclic subgroups are all $\mathbb{P}$-subnormal. It is easy to verify that $\mathfrak{U} \subset w\mathfrak{U} \subset \mathfrak{X}$.

**Theorem.**
1. A group $G \in w\mathfrak{U}$ if and only if $G$ possesses an ordered Sylow tower of supersolvable type and every biprimary subgroup of $G$ is supersolvable.
2. The class $\mathfrak{X}$ is a hereditary saturated formation.
3. A group $G \in \mathfrak{X}$ if and only if $G$ possesses an ordered Sylow tower of supersolvable type and every biprimary subgroup of $G$ with cyclic Sylow subgroup is supersolvable.
4. Every minimal non-$\mathfrak{X}$-group is a biprimary minimal non-supersolvable group whose non-normal Sylow subgroup is cyclic.

1 Preliminary results

We use the standard notation of [3]. The set of all prime divisors of $|G|$ is denoted $\pi(G)$. We write $[A]B$ for a semidirect product with a normal subgroup $A$. If $H$ is a subgroup of $G$, then $\text{Core}_G H = \bigcap_{x \in G} x^{-1}Hx$ is called the core of $H$ in $G$. If a group $G$ contains a maximal subgroup $M$ with trivial core, then $G$ is said to be primitive and $M$ is its primitivator. We will use the following notation: $S_n$ and $A_n$ are the symmetric and the alternating groups of degree $n$, $E_{p^t}$ is the elementary abelian group of order $p^t$, $Z_m$ is the cyclic group of order $m$, $D_8$ is the dihedral group of order 8, $Z(G)$, $\Phi(G)$, $F(G)$, $G'$ is the center, the Frattini subgroup, the Fitting subgroup and the derived subgroup of $G$ respectively.

**Lemma 1.** Let $H$ be a subgroup of a solvable group $G$, and assume that $|G : H|$ is a prime number. Then $G/\text{Core}_G H$ is supersolvable.

**Proof.** By hypothesis, $|G : H| = p$, where $p$ is a prime number. If $H = \text{Core}_G H$, then $G/H$ is cyclic of order $p$ and $G/\text{Core}_G H$ is supersolvable, as required. Assume now that $H \neq \text{Core}_G H$, i.e., $H$ is not normal in $G$. It
follows that $G/\text{Core}_G H$ contains a maximal subgroup $H/\text{Core}_G H$ with trivial core. Hence, $G/\text{Core}_G H$ is primitive and the Fitting subgroup $F/\text{Core}_G H$ of $G/\text{Core}_G H$ has prime order $p$. Since

$$F/\text{Core}_G H = C_{G/\text{Core}_G H}(F/\text{Core}_G H),$$

it follows that

$$(G/\text{Core}_G H)/(F/\text{Core}_G H) \cong H/\text{Core}_G H$$

is isomorphic to a cyclic group of order dividing $p - 1$. Thus, $G/\text{Core}_G H$ is supersolvable.

**Lemma 2.** 1. A group $G$ is supersolvable if and only if all of its maximal subgroups have prime indices.

2. Every subgroup of a supersolvable group is $P$-subnormal.

**Proof.** 1. This is Huppert’s classic theorem, see [3, Theorem VI.9.5].

2. The statement follows from 1 of the lemma.

Immediately, using the definition of $P$-subnormality, we deduce the following properties.

**Lemma 3.** Suppose that $H$ is a subgroup of $G$, and let $N$ be a normal subgroup of $G$. Then the following hold:

1) if $H$ is $P$-subnormal in $G$, then $(H \cap N)$ is $P$-subnormal in $N$, and $HN/N$ is $P$-subnormal in $G/N$;

2) if $N \subseteq H$ and $H/N$ is $P$-subnormal in $G/N$, then $H$ is $P$-subnormal in $G$;

3) if $H$ is $P$-subnormal in $K$, and $K$ is $P$-subnormal in $G$, then $H$ is $P$-subnormal in $G$;

4) if $H$ is $P$-subnormal in $G$, then $H^g$ is $P$-subnormal in $G$ for each element $g \in G$.

**Example 1.** The subgroup $H = A_4$ of the alternating group $G = A_5$ is $P$-subnormal. If $x \in G \setminus H$, then $H^x$ is $P$-subnormal in $G$. The subgroup $D = H \cap H^x$ is a Sylow 3-subgroup of $G$ and $D$ is not $P$-subnormal in $H$. Therefore, an intersection of two $P$-subnormal subgroups is not $P$-subnormal. Moreover, if $H$ is $P$-subnormal in $G$ and $K$ is an arbitrary subgroup of $G$, in general, their intersection $H \cap K$ is not $P$-subnormal in $K$.

However, this situation is impossible if $G$ is a solvable group.

**Lemma 4.** Let $G$ be a solvable group. Then the following hold:
1) if \( H \) is \( \mathbb{P} \)-subnormal in \( G \), and \( K \) is a subgroup of \( G \), then \( (H \cap K) \) is \( \mathbb{P} \)-subnormal in \( K \);

2) if \( H_i \) is \( \mathbb{P} \)-subnormal in \( G \), \( i = 1, 2 \), then \( (H_1 \cap H_2) \) is \( \mathbb{P} \)-subnormal in \( G \).

**Proof.** 1. It is clear that in the case \( H = G \) the statement is true. Let \( H \neq G \). According to the definition of \( \mathbb{P} \)-subnormality, there exists a chain of subgroups

\[ H = H_0 \subset H_1 \subset \ldots \subset H_n = G \]

such that \(|H_i : H_{i-1}|\) is a prime number for any \( i \). We will use induction by \( n \).

Consider the case when \( n = 1 \). In this situation, \( H = H_{n-1} \) is a maximal subgroup of prime index in \( G \). By Lemma 1, \( G/N \) is supersolvable, \( N = \text{Core}_G H \). Since, by Lemma 2 (2), every subgroup of a supersolvable group is \( \mathbb{P} \)-subnormal, we have

\[ H/N \cap KN/N = N(H \cap K)/N \]

is \( \mathbb{P} \)-subnormal in \( KN/N \). Lemma 3 (2) implies that \( N(H \cap K) \) is \( \mathbb{P} \)-subnormal in \( KN \). It means that there exists a chain of subgroups

\[ N(H \cap K) = A_0 \subset A_1 \subset \ldots \subset A_{m-1} \subset A_m = NK \]

such that \(|A_i : A_{i-1}| \in \mathbb{P}\) for all \( i \). Since

\[ N(H \cap K) \subseteq A_i \subseteq NK, \]

we have \( A_i = N(A_i \cap K) \) and \( H \cap K \subseteq A_i \cap K \) for all \( i \). We introduce the notation \( B_i = A_i \cap K \). It is clear that

\[ B_{i-1} \subseteq B_i, \quad A_i = N(A_i \cap K) = NB_i, \quad N \cap B_i = N \cap A_i \cap K = N \cap K \]

for all \( i \). Since \( N \subseteq H \), we have

\[ B_0 = A_0 \cap K = N(H \cap K) \cap K = (N \cap K)(H \cap K) = H \cap K, \]

\[ B_m = A_m \cap K = KN \cap K = K. \]

Moreover,

\[
\frac{|A_i : A_{i-1}|}{|NB_i|} = \frac{|N||B_i||N \cap B_{i-1}|}{|N \cap B_i||N||B_{i-1}|} = \frac{|B_i : B_{i-1}|}{|N \cap B_i : N \cap B_{i-1}|} = \frac{|B_i : B_{i-1}|}{|N \cap K : N \cap K|} = |B_i : B_{i-1}|.
\]
Now we have a chain of subgroups

\[ H \cap K = B_0 \subset B_1 \subset \ldots \subset B_{m-1} \subset B_m = K, \quad |B_i : B_{i-1}| \in \mathbb{P}, \quad 1 \leq i \leq m, \]

which proves that the subgroup \( H \cap K \) is \( \mathbb{P} \)-subnormal in \( K \).

Let \( n > 1 \). Since \( H_{n-1} \) is a maximal subgroup of prime index in \( G \), and \( G \) is solvable, thus, as it was proved, \( H_{n-1} \cap K \) is \( \mathbb{P} \)-subnormal in \( K \). The subgroup \( H \) is \( \mathbb{P} \)-subnormal in the solvable group \( H_{n-1} \) and the induction is applicable to them. By induction,

\[ H \cap (H_{n-1} \cap K) = H \cap K \]

is \( \mathbb{P} \)-subnormal in \( H_{n-1} \cap K \). By Lemma 3 (3), \( H \cap K \) is \( \mathbb{P} \)-subnormal in \( K \).

2. Let \( H_i \) is \( \mathbb{P} \)-subnormal in \( G \), \( i = 1, 2 \). It follows from 1 of the lemma, that \( (H_1 \cap H_2) \) is \( \mathbb{P} \)-subnormal in \( H_2 \). Now by Lemma 3 (3), we obtain that \( (H_1 \cap H_2) \) is \( \mathbb{P} \)-subnormal in \( G \).

**Lemma 5.** Let \( H \) be a subnormal subgroup of a solvable group \( G \). Then \( H \) is \( \mathbb{P} \)-subnormal in \( G \).

**Proof.** Since \( H \) is subnormal in \( G \), and \( G \) is solvable, then there exists a series

\[ H = H_0 \subset H_1 \subset \ldots \subset H_{n-1} \subset H_n = G, \]

such that \( H_i \) is normal in \( H_{i+1} \) for all \( i \). Working by induction on \( |G| \), we can assume that \( H \) is \( \mathbb{P} \)-subnormal in \( H_{n-1} \). Since \( G/H_{n-1} \) is solvable, the composition factors of \( G/H_{n-1} \) have prime orders. Thus, there is a chain of subgroups

\[ H_{n-1} = G_0 \subset G_1 \subset \ldots \subset G_{m-1} \subset G_m = G \]

such that \( G_j \) is normal in \( G_{j+1} \) and \( |G_{j+1}/G_j| \in \mathbb{P} \) for all \( j \). This means that \( H_{n-1} = G_0 \) is \( \mathbb{P} \)-subnormal in \( G \). Using Lemma 3 (3), we deduce that \( H \) is a \( \mathbb{P} \)-subnormal in \( G \).

**Example 2.** The subgroup \( Z(SL(2, 13)) \) of the non-solvable group \( SL(2, 13) \) is normal, but is not \( \mathbb{P} \)-subnormal. This follows from the fact that the identity subgroup is not \( \mathbb{P} \)-subnormal in \( PSL(2, 13) = SL(2, 13)/Z(SL(2, 13)) \).

**Lemma 6.** Let \( A \) be a \( p \)-subgroup of a group \( G \). Then \( A \) is subnormal in \( G \) if and only if \( A \subseteq O_p(G) \).

**Proof.** The statement follows from Theorem 2.2 [4].

**Lemma 7.** Let \( A \) be a \( p \)-subgroup of a group \( G \). If \( |G : N_G(A)| = p^\alpha \), \( \alpha \in \mathbb{N} \), then \( A \) is subnormal in \( G \).
Proof. Let \( P \) be a Sylow \( p \)-subgroup of \( G \) with the property that \( P \) contains \( A \). Then

\[
G = N_G(A)P, \quad A^G = A^{N_G(A)}P = A^P \subseteq P,
\]

so \( A^G \subseteq O_p(G) \). It is clear that \( A \) is subnormal in \( G \).

**Lemma 8.** Let \( p \) be the largest prime divisor of \( |G| \), and let \( A \) be a \( p \)-subgroup of \( G \). If \( A \) is \( P \)-subnormal in \( G \), then \( A \) is subnormal in \( G \).

Proof. Let \( |A| = p^\alpha \). Since \( A \) is \( P \)-subnormal in \( G \), then there exists a series

\[
A = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_{t-1} \subseteq A_t = G, \quad |A_i : A_{i-1}| \in P, \quad 1 \leq i \leq t.
\]

Since \( |A_1 : A_0| \in P \), we have

\[
|A_1| = p^{1+\alpha} \text{ or } |A_1| = p^\alpha q, \quad p \neq q.
\]

If \( |A_1| = p^{1+\alpha} \), then \( A \) is a normal subgroup of \( A_1 \). If \( |A_1| = p^\alpha q \), then \( p > q \) and again \( A \) is normal in \( A_1 \). Suppose we already know that \( A \) is subnormal in \( A_j \). Using Lemma 6 we have, \( A \subseteq O_p(A_j) \). Since \( |A_{j+1} : A_j| \in P \), we obtain

\[
|A_{j+1}| = p|A_j| \text{ or } |A_{j+1}| = q|A_j|, \quad p \neq q.
\]

If \( |A_{j+1}| = p|A_j| \), then, by Lemma 7, \( O_p(A_j) \subseteq O_p(A_{j+1}) \), and \( A \) is subnormal in \( A_{j+1} \). If \( |A_{j+1}| = q|A_j| \), \( p \neq q \), then \( p > q \). Consider the set of left cosets of \( A_j \) in \( A_{j+1} \). We know that \( A_{j+1}/\text{Core}_{A_{j+1}}A_j \) is isomorphic to a subgroup of the symmetric group \( S_q \) and any Sylow \( p \)-subgroup of \( A_{j+1} \) is contained in \( \text{Core}_{A_{j+1}}A_j \). Since \( A \) is subnormal in \( A_j \), so \( A \) is subnormal in \( \text{Core}_{A_{j+1}}A_j \). Since \( \text{Core}_{A_{j+1}}A_j \) is normal in \( A_{j+1} \), it follows that \( A \) is subnormal in \( A_{j+1} \). Therefore, \( A \) is subnormal in \( A_i \) for all \( i \). This implies that \( A \) is subnormal in \( G \).

**Corollary.** (\cite{I, Proposition 2.8}) Every \( w \)-supersolvable group possesses an ordered Sylow tower of supersolvable type.

Proof. Use induction on \( |G| \). Let \( G \) be a \( w \)-supersolvable group, and assume that \( p \) is the largest prime divisor of \( |G| \). Let \( P \) be a Sylow \( p \)-subgroup of \( G \). By Lemma 8, \( P \) is normal in \( G \). It follows by Lemma 3 (1), that any quotient group of a \( w \)-supersolvable group is \( w \)-supersolvable. Working by induction on \( |G| \), we deduce that \( G/P \) possesses an ordered Sylow tower of supersolvable type, so \( G \) possesses an ordered Sylow tower of supersolvable type. The corollary is proved.
Recall that a Schmidt group is a finite non-nilpotent group all of whose proper subgroups are nilpotent. Given a class \( \mathfrak{F} \) of groups. By \( \mathcal{M}(\mathfrak{F}) \) we denote the class of all minimal non-\( \mathfrak{F} \)-groups. A group \( G \) is a minimal non-\( \mathfrak{F} \)-group if \( G \notin \mathfrak{F} \) but all proper subgroups of \( G \) belong to \( \mathfrak{F} \). Clearly, the class \( \mathcal{M}(\mathfrak{N}) \) consists of Schmidt groups. Here \( \mathfrak{N} \) denotes the class of all nilpotent groups. We will need the properties of groups from \( \mathcal{M}(\mathfrak{N}) \) and \( \mathcal{M}(\mathfrak{U}) \).

**Lemma 9.** ([5], [6]) Let \( S \in \mathcal{M}(\mathfrak{N}) \). Then the following statements hold:
1) \( S = [P]\langle y \rangle \), where \( P \) is a normal Sylow \( p \)-subgroup, and \( \langle y \rangle \) is a non-normal cyclic Sylow \( q \)-subgroup, \( p \) and \( q \) are distinct primes, \( y^q \in Z(S) \);
2) \( |P/P'| = p^m \), where \( m \) is the order of \( p \) modulo \( q \);
3) if \( P \) is abelian, then \( P \) is an elementary abelian \( p \)-group of order \( p^m \) and \( P \) is a minimal normal subgroup of \( S \);
4) if \( P \) is non-abelian, then \( Z(P) = P' = \Phi(P) \) and \( |P/Z(P)| = p^m \);
5) \( Z(S) = \Phi(S) = \Phi(P) \times \langle y^q \rangle \); \( S' = P, P' = (S')' = \Phi(P) \);
6) if \( N \) is a proper normal subgroup of \( S \), then \( N \) does not contain \( \langle y \rangle \) and either \( P \subseteq N \) or \( N \subseteq \Phi(S) \).

**Lemma 10.** ([7]) Let \( G \in \mathcal{M}(\mathfrak{U}) \). Then the following statements hold:
1) \( G \) is solvable and \( |\pi(G)| \leq 3 \);
2) if \( G \) is not a Schmidt group, then \( G \) possesses an ordered Sylow tower of supersolvable type;
3) \( G \) has a unique normal Sylow subgroup \( P \) and \( P = G^\mathfrak{U} \);
4) \( |P/\Phi(P)| > p \) and \( P/\Phi(P) \) is a minimal normal subgroup of \( G/\Phi(G) \);
5) the Frattini subgroup \( \Phi(P) \) of \( P \) is supersolvable embedded in \( G \), i.e., there exists a series
   \[
   1 \subset N_0 \subset N_1 \ldots \subset N_m = \Phi(P)
   \]
such that \( N_i \) is a normal subgroup of \( G \) and \( |N_i/N_{i-1}| \in \mathbb{P} \) for all \( i \);
6) let \( Q \) be a complement to \( P \) in \( G \), then \( Q/Q \cap \Phi(G) \) is a minimal non-abelian group or a cyclic group of prime power order;
7) all maximal subgroups of non-prime index are conjugate in \( G \), and moreover, they are conjugate to \( \Phi(P)Q \).

We now present new properties of w-supersolvable groups.

**Lemma 11.**
1. If \( G \in \mathcal{M}(\mathfrak{U}) \) and \( |\pi(G)| = 3 \), then \( G \) is w-supersolvable.
2. \( \mathcal{M}(\mathfrak{U}) \setminus w\mathfrak{U} = \{ G \in \mathcal{M}(\mathfrak{U}) \mid |\pi(G)| = 2 \} \).
3. If \( G \in w\mathfrak{U} \), then the derived length of \( G/\Phi(G) \) is at most \( |\pi(G)| \).
Proof. 1. Let $G \in M(U)$ and $|\pi(G)| = 3$. By Lemma 10, $G = [P]([Q]|R)$, where $P$, $Q$ and $R$ are Sylow subgroups of $G$. The subgroup $P$ is normal in $G$, and using Lemma 5, we see that $P$ is $P$-subnormal in $G$. The subgroup $PQ$ is normal in $G$, and by Lemma 5, $PQ$ is $P$-subnormal in $G$. Since $PQ$ is supersolvable, it follows by Lemma 2 (2), that $Q$ is $P$-subnormal in $PQ$. By Lemma 3 (3), $Q$ is $P$-subnormal in $G$. Since $G/P \simeq QR$, so $G/P$ is supersolvable and $PR/P$ is $P$-subnormal in $G/P$ by Lemma 2 (2). Hence $PR$ is $P$-subnormal in $G$ by Lemma 3 (2). Since $PR$ is supersolvable, we see that $R$ is $P$-subnormal in $PR$ by Lemma 2 (2). Hence $R$ is $P$-subnormal in $G$ by Lemma 3 (3). We conclude that all Sylow subgroups of $G$ are $P$-subnormal in $G$. Therefore, $G$ is $w$-supersolvable.

2. If $G \in M(U) \setminus wU$, then $|\pi(G)| = 2$ by assertion 1 of the lemma. Conversely, let $G \in \{M(U) \mid |\pi(G)| = 2\}$. Suppose that $G \in wU$. Then by Theorem 2.13 (2) [1], the group $G$ is supersolvable, this is a contradiction.

3. By theorem 2.13 (3) [1], $G/F(G)$ has only abelian Sylow subgroups. By theorem VI.14.16 [3], the derived length of $G/F(G)$ is at most $|\pi(G/F(G))|$. Since $G$ has an ordered Sylow tower of supersolvable type, so $|\pi(G/F(G))| \leq |\pi(G)| - 1$. But if $G$ is solvable, then the quotient group $F(G)/\Phi(G)$ is abelian, and we conclude that the derived length of $G/\Phi(G)$ does not exceed $|\pi(G)|$.

2 Finite groups with $P$-subnormal primary cyclic subgroups

Example 3. There are three non-isomorphic minimal non-supersolvable groups of order 400:

$$[E_{5^2}(<a><b>), |a| = |b| = 4.$$ Numbers of these groups in the library of SmallGroups [8] are [400,129], [400,130], [400,134]. Sylow 2-subgroups of these groups are non-abelian and have the form: $[Z_4 \times Z_2]Z_2$ and $[Z_4]Z_4$. Let $G$ be one of these groups. All subgroups of $G$ are $P$-subnormal except the maximal subgroup $<a><b>$. Therefore, these groups belong to the class $\mathcal{X}$.

Example 4. The general linear group $GL(2,7)$ contains the symmetric group $S_3$ which acts irreducibly on the elementary abelian group $E_{7^2}$ of order 49. The semidirect product $[E_{7^2}]S_3$ is a minimal non-supersolvable group, it has subgroups of orders 14 and 21. Every primary cyclic subgroup of the group $[E_{7^2}]S_3$ is $P$-subnormal. Therefore, these groups belong to the class $\mathcal{X}$. 
Example 5. Non-supersolvable Schmidt groups do not belong to the class \( \mathcal{X} \). We verify this fact. Let \( S = [P]Q \) be a non-supersolvable Schmidt group. Suppose that \( S \in \mathcal{X} \). It follows that \( Q \) is \( \mathbb{P} \)-subnormal in \( S \) and \( Q \) is contained in some subgroup \( M \) of prime index. Therefore, \( M = P_1 \times Q \), where \( P_1 \) is a normal subgroup of \( S \) with the property \( |P/P_1| = p \). By the properties of Schmidt groups, see Lemma 9, we have \( |P/\Phi(P)| > p \) and \( P/\Phi(P) \) is a chief factor of \( S \). We have a contradiction.

Lemma 12. Suppose that all cyclic \( p \)-subgroups of a group \( G \) are \( \mathbb{P} \)-subnormal and let \( N \) be a normal subgroup of \( G \). Then all cyclic subgroups of \( N \) and \( G/N \) are \( \mathbb{P} \)-subnormal.

Proof. Lemma 3 (1) implies that all cyclic \( p \)-subgroups of the normal group \( N \) are \( \mathbb{P} \)-subnormal in \( N \). Let \( A/N \) be a cyclic \( p \)-subgroup of \( G/N \) and assume that \( a \in A \setminus N \). Let \( P \) be a Sylow \( p \)-subgroup of \( \langle a \rangle \). By hypothesis, \( P \) is \( \mathbb{P} \)-subnormal in \( G \). Since \( PN/N = AN/N \), it follows by Lemma 3 (1), that \( A/N \) is \( \mathbb{P} \)-subnormal in \( G/N \).

Lemma 13. 1. If every primary cyclic subgroup of a group \( G \) is \( \mathbb{P} \)-subnormal, then \( G \) possesses an ordered Sylow tower of supersolvable type.

2. \( \mathcal{U} \subset w\mathcal{U} \subset \mathcal{X} \subset \mathcal{D} \).

Proof. 1. Let \( P \) be a Sylow \( p \)-subgroup of \( G \), where \( p \) is the largest prime divisor of \( |G| \). If \( a \in P \), then by hypothesis, the subgroup \( \langle a \rangle \) is \( \mathbb{P} \)-subnormal in \( G \). By Lemma 8, the subgroup \( \langle a \rangle \) is subnormal in \( G \), and by Lemma 6, \( \langle a \rangle \subseteq O_p(G) \). Since \( a \) is an arbitrary element of \( P \), we see that \( P \subseteq O_p(G) \), and hence \( G \) is \( p \)-closed. By Lemma 12, the conditions of the lemma are inherited by all quotient groups of \( G \). Applying induction on \( |G| \), we see that \( G/P \) possesses an ordered Sylow tower of supersolvable type, and thus \( G \) possesses an ordered Sylow tower of supersolvable type.

2. By Lemma 2 (2), we have the inclusion \( \mathcal{U} \subseteq w\mathcal{U} \). It follows from Example 4, that \( [E_{72}]_3 \) is non-supersolvable and \( [E_{72}]_3 \in w\mathcal{U} \setminus \mathcal{U} \). Therefore, \( \mathcal{U} \subseteq w\mathcal{U} \).

We verify the inclusion \( w\mathcal{U} \subseteq \mathcal{X} \). Suppose that \( G \in w\mathcal{U} \), and let \( A \) be an arbitrary primary cyclic subgroup of \( G \). Then \( A \) is a \( p \)-subgroup for some \( p \in \pi(G) \). By Sylow’s theorem, \( A \) is contained in some Sylow \( p \)-subgroup \( P \) of the group \( G \). Since \( G \in w\mathcal{U} \), it follows that \( P \) is \( \mathbb{P} \)-subnormal in \( G \). By Lemma 2 (2), \( A \) is \( \mathbb{P} \)-subnormal in \( P \), and by Lemma 3 (3), \( A \) is \( \mathbb{P} \)-subnormal in \( G \). Therefore, \( G \in \mathcal{X} \). The group \( [E_{52}]_Q \) from Example 3 is a biprimary minimal non-supersolvable group, \( Q \) is non-cyclic. The group \( [E_{52}]_Q \in \mathcal{X} \setminus w\mathcal{U} \), therefore, \( w\mathcal{U} \subset \mathcal{X} \).
By the above assertion of the lemma, $\mathfrak{X} \subseteq \mathfrak{D}$. Since there exist non-supersolvable Schmidt groups which have an ordered Sylow tower of supersolvable type (for example, $[E_5^2]Z_3$), and they do not belong to the class $\mathfrak{X}$, it follows that $[E_5^2]Z_3 \in \mathfrak{D} \setminus \mathfrak{X}$. Therefore, $\mathfrak{X} \subset \mathfrak{D}$.

**Lemma 14.** Let $G$ be a minimal non-supersolvable group. The group $G \not\in \mathfrak{X}$ if and only if $G$ is a biprimary group whose non-normal Sylow subgroup is cyclic.

**Proof.** Let $G \in \mathcal{M}(\mathfrak{U}) \setminus \mathfrak{X}$. If $|\pi(G)| = 3$, then by Lemma 11 (1), $G \in w\mathfrak{U}$. Since $w\mathfrak{U} \subset \mathfrak{X}$, we have $G \in \mathfrak{X}$, which contradicts the choice of $G$. So, if $G \in \mathcal{M}(\mathfrak{U}) \setminus \mathfrak{X}$, then $|\pi(G)| = 2$ and $G = [P]Q$, where $P$ is a Sylow $p$-subgroup of $G$, $Q$ is a Sylow $q$-subgroup of $G$. Suppose that $Q$ is non-cyclic, and let $a \in Q$. Since $P \langle a \rangle$ is a proper subgroup of $G$, we deduce that $P \langle a \rangle$ is supersolvable. Lemma 2 (2) implies that $\langle a \rangle$ is a $\mathbb{P}$-subnormal subgroup of $P \langle a \rangle$. Since $P \langle a \rangle$ is subnormal in $G$, it follows by Lemma 5, that $P \langle a \rangle$ is a $\mathbb{P}$-subnormal subgroup of $G$. Now by Lemma 3 (3), we deduce that $\langle a \rangle$ is $\mathbb{P}$-subnormal in $G$. Applying Lemma 3 (4), we can conclude that all cyclic $q$-subgroups of $G$ are $\mathbb{P}$-subnormal in $G$. Lemma 5 implies that all cyclic $p$-subgroups of $G$ are $\mathbb{P}$-subnormal in $G$. Thus $G \in \mathfrak{X}$. We have a contradiction. Therefore, the assumption is false and $Q$ is cyclic.

Conversely, let $G \in \mathcal{M}(\mathfrak{U})$, $|\pi(G)| = 2$ and a non-normal Sylow subgroup $Q$ of $G$ is cyclic. Assume that $G \in \mathfrak{X}$. This implies that $Q$ is $\mathbb{P}$-subnormal in $G$, and so both Sylow subgroups of the group $G$ are $\mathbb{P}$-subnormal. Now, by Theorem 2.13 (2) [1], $G$ is supersolvable, which is a contradiction.

**Lemma 15.** ([9]) If $P$ is a normal Sylow subgroup of a group $G$, then $\Phi(P) = \Phi(G) \cap P$.

**Theorem.** 1. A group $G \in w\mathfrak{U}$ if and only if $G$ possesses an ordered Sylow tower of supersolvable type and every biprimary subgroup of $G$ is supersolvable.

2. The class $\mathfrak{X}$ is a hereditary saturated formation.

3. A group $G \in \mathfrak{X}$ if and only if $G$ possesses an ordered Sylow tower of supersolvable type and every biprimary subgroup of $G$ with cyclic Sylow subgroup is supersolvable.

4. Every minimal non-$\mathfrak{X}$-group is a biprimary minimal non-supersolvable group whose non-normal Sylow subgroup is cyclic.

**Proof.** 1. If a group $G \in w\mathfrak{U}$, then $G$ possesses an ordered Sylow tower of supersolvable type by the corollary of Lemma 8, and every biprimary subgroup
of $G$ is $\omega$-supersolvable by Lemma 4 (1). We conclude by Lemma 10 (2), that every biprimary subgroup of $G$ is supersolvable.

Conversely, suppose that a group $G$ possesses an ordered Sylow tower of supersolvable type and every biprimary subgroup of $G$ is supersolvable. Assume that $G$ is not $\omega$-supersolvable. Let us choose among all such groups a group with the smallest possible $|\pi(G)|$. Then $|\pi(G)| \geq 3$ and $G$ contains a Sylow $r$-subgroup $R$ such that $R$ is not $P$-subnormal in $G$. Let $p \in \pi(G)$, where $p$ is the largest prime divisor of $|G|$ and let $P$ be a Sylow $p$-subgroup of $G$. Since $G$ possesses an ordered Sylow tower of supersolvable type, we deduce that $P$ is normal in $G$. By hypothesis, the subgroup $PR$ is supersolvable, and we deduce by Lemma 2 (2) that $R$ is $P$-subnormal in $PR$. It is clear that $G/P$ possesses an ordered Sylow tower of supersolvable type and all of its biprimary subgroups are supersolvable. Since $|\pi(G/P)| = |\pi(G)| - 1$, it follows by the inductive hypothesis, that $G/P$ is $\omega$-supersolvable. Therefore, the Sylow $r$-subgroup $PR/P$ is $P$-subnormal in $G/P$. Lemma 3 (2) implies that the subgroup $PR$ is $P$-subnormal in $G$, and hence by Lemma 3 (3), the subgroup $R$ is $P$-subnormal in $G$. This is a contradiction.

2. By Lemma 13 (1), the class $\mathfrak{X}$ consists of finite groups which have an ordered Sylow tower of supersolvable type, so we can apply Lemma 4. Let $G \in \mathfrak{X}$ and suppose that $H$ is an arbitrary subgroup of $G$. If $A$ is a cyclic primary subgroup of $H$, then $A$ is $P$-subnormal in $G$. By Lemma 4 (1), the subgroup $A$ is $P$-subnormal in $H$, and hence $\mathfrak{X}$ is a hereditary class.

By Lemma 12, the class $\mathfrak{X}$ is closed under homomorphic image. By induction on the order of $G$, we verify that the class $\mathfrak{X}$ is closed under subdirect products. Let $G$ be a group of least order with the following properties:

$$G/N_i \in \mathfrak{X}, \ i = 1, 2, \ N_1 \cap N_2 = 1, \ G \not\in \mathfrak{X}.$$ 

In this case, $G$ has a primary cyclic subgroup $A$ which is not $P$-subnormal in $G$. Since $G/N_i \in \mathfrak{X}, \ i = 1, 2$, it follows that $AN_i/N_i$ is $P$-subnormal in $G/N_i$, and thus by Lemma 4 (2), $AN_1 \cap AN_2$ is $P$-subnormal in $G$. If $K = AN_1N_2$ is a proper subgroup of $G$, then $K/N_i \in \mathfrak{X}$ because $\mathfrak{X}$ is a hereditary class. By the induction hypothesis, $K \in \mathfrak{X}$. It follows that $A$ is $P$-subnormal in $AN_1 \cap AN_2$, and by Lemma 3 (3), $A$ is $P$-subnormal in $G$, which is a contradiction. Therefore, $G = AN_1N_2$. Assume that $G = AN_1$. Then

$$N_2 \cong N_1N_2/N_1 \subseteq G/N_1 \cong A/A \cap N_1.$$ 

Thus $N_2$ is cyclic and $AN_2$ is supersolvable by Theorem VI.10.1 [3]. It follows that $A$ is $P$-subnormal in $AN_2$, $AN_2$ is $P$-subnormal in $G$, and by Lemma 3 (3),
A is $\mathbb{P}$-subnormal in $G$, which is a contradiction. Thus our assumption is false and $AN_1 \neq G \neq AN_2$.

The subgroup $D = N_1 \cap AN_2$ is normal in $AN_2 = H \neq G$. Hence the group $H$ contains two normal subgroups $D$ and $N_2$ such that

$$D \cap N_2 \subseteq N_1 \cap N_2 = 1,$$

$$H/N_2 \subset G/N_2 \in \mathcal{X}, \ H/N_2 \in \mathcal{X},$$

$$G/N_1 = (AN_2)N_1/N_1 \simeq AN_2/N_1 \cap AN_2 = H/D \in \mathcal{X}.$$  

By the inductive assumption, $H \in \mathcal{X}$. It follows that $A$ is $\mathbb{P}$-subnormal in $H$, $H$ is $\mathbb{P}$-subnormal in $G$, and hence $A$ is $\mathbb{P}$-subnormal in $G$. This contradicts the fact that $G$ has a primary cyclic subgroup which is not $\mathbb{P}$-subnormal in $G$. Thus $\mathcal{X}$ is formation.

We prove that $\mathcal{X}$ is a saturated formation by induction on $|G|$. Suppose that $\Phi(G) \neq 1$ and $G/\Phi(G) \in \mathcal{X}$. Since by Lemma 13, the quotient group $G/\Phi(G)$ possesses an ordered Sylow tower of supersolvable type, it follows that $G$ possesses an ordered Sylow tower of supersolvable type.

Let $N$ be a minimal normal subgroup of $G$. It is clear that

$$\Phi(G)N/N \subseteq \Phi(G)/N, \ G/\Phi(G)N \simeq (G/\Phi(G))/(\Phi(G)N/\Phi(G)) \in \mathcal{X},$$

$$(G/N)/(\Phi(G)/N) \simeq ((G/N)/(\Phi(G)N/N))/(\Phi(G)/N/\Phi(G)) \in \mathcal{X},$$

because

$$(G/N)/(\Phi(G)/N) \simeq G/\Phi(G)N \in \mathcal{X}.$$  

By the inductive hypothesis, we have $G/N \in \mathcal{X}$. Since $\mathcal{X}$ is a formation, this implies that $N$ is a unique minimal normal subgroup of $G$, $N \subseteq \Phi(G)$, $N$ is a $p$-subgroup for the largest $p \in \pi(G)$, and $O_{p'}(G) = 1$. Let $P$ be a Sylow $p$-subgroup of $G$, $P$ is normal in $G$.

Suppose that $G$ has a primary cyclic subgroup $A$ which is not $\mathbb{P}$-subnormal in $G$. Since $G/N \in \mathcal{X}$, it follows that the quotient group $AN/N$ is $\mathbb{P}$-subnormal in $G/N$, and by Lemma 3 (2), $AN$ is $\mathbb{P}$-subnormal in $G$. By Lemma 3 (3), we see that $A$ is not $\mathbb{P}$-subnormal in $AN$, and Lemma 5 implies that the orders of $A$ and $N$ are coprime. Therefore, $AP/N$ is a biprimary subgroup in which the Sylow subgroups $AN/N$ and $P/N$ are both $\mathbb{P}$-subnormal. Theorem 2.13 (2) \cite{1} implies that $AP/N$ is supersolvable. By Lemma 15, $\Phi(P) = P \cap \Phi(G)$, thus $N \subseteq \Phi(P) \subseteq \Phi(AP)$, and by Theorem VI.8.6 \cite{3}, we deduce that $AN$ is supersolvable. Lemma 2 (2) implies that $A$ is $\mathbb{P}$-subnormal in $AN$, which is a contradiction.
3. Let $G \in \mathfrak{X}$ and $B$ is a biprimary group with cyclic Sylow subgroup $R$. By Lemma 16 (1), $G$ possesses an ordered Sylow tower of supersolvable type. If $R$ is normal in $B$, then $B/R$ is primary, it follows that $B$ is supersolvable. If $R$ is not normal in $B$, then $B = PR$, where $P$ is a normal Sylow subgroup of $B$. By hypothesis, we conclude that $R$ is $\mathbb{P}$-subnormal in $G$, and by Lemma 4 (1), $R$ is $\mathbb{P}$-subnormal in $B$. Hence, by Lemma 10 (2), $B$ is supersolvable.

Conversely, suppose that $G$ possesses an ordered Sylow tower of supersolvable type and every biprimary subgroup of $G$ with cyclic Sylow subgroup is supersolvable. Assume that $G \notin \mathfrak{X}$. Let us choose among all such groups a group $G$ with the smallest possible order. Then $G$ contains a cyclic non-$\mathbb{P}$-subnormal $r$-subgroup $R$. Since $G$ possesses an ordered Sylow tower of supersolvable type, we deduce that a Sylow $p$-subgroup $P$ for the largest prime $p \in \pi(G)$ is normal. If $p = r$, then $R \subseteq P$, it follows that $R$ is $\mathbb{P}$-subnormal in $G$, which is a contradiction. Thus $p \neq r$ and $PR$ is biprimary with cyclic Sylow subgroup $R$. By hypothesis, $PR$ is supersolvable, and by Lemma 2 (2), $R$ is $\mathbb{P}$-subnormal in $PR$. The quotient group $G/P$ possesses an ordered Sylow tower of supersolvable type and every its biprimary subgroup with cyclic Sylow subgroup is supersolvable. Thus $G/P \in \mathfrak{X}$. Since $PR/P$ is a cyclic $r$-subgroup, we see that $PR/P$ is $\mathbb{P}$-subnormal in $G/P$. It follows by Lemma 3 (2) that $PR$ is $\mathbb{P}$-subnormal in $G$. Now by Lemma 3 (3), we obtain that $R$ is $\mathbb{P}$-subnormal in $G$. This is a contradiction. The assertion is proved.

4. Let $G \in \mathcal{M}(\mathfrak{X})$, and let $q$ be the smallest prime divisor of $|G|$. Consider an arbitrary proper subgroup $H$ of $G$. Since $H \in \mathfrak{X}$, so by Lemma 13 (1), the subgroup $H$ has an ordered Sylow tower of supersolvable type, in particular, $H$ is $q$-nilpotent. By Theorem IV.5.4 [3], the group $G$ is either $q$-nilpotent or a $q$-closed Schmidt group. If $G$ is a $q$-closed Schmidt group, then $G$ is a biprimary minimal non-supersolvable group whose non-normal Sylow subgroup is cyclic. In this case, the statement is true.

Suppose that $G$ is a $q$-nilpotent group. Then $G = [G_{q'}]G_q$. Since $G_{q'} \in \mathfrak{X}$, it follows by Lemma 13 (1), that $G_{q'}$ possesses an ordered Sylow tower of supersolvable type, and thus $G$ possesses an ordered Sylow tower of supersolvable type. Let $N$ be a minimal normal subgroup of $G$.

First, assume that $\Phi(G) = 1$. In this case, $G = [N]M$, where $M$ is some maximal subgroup of $G$. Since $G \notin \mathfrak{X}$, then $G$ contains a primary cyclic non-$\mathbb{P}$-subnormal subgroup. Let $A$ be a subgroup of least order among these subgroups. Since

$$AN/N \subseteq G/N \simeq M \in \mathfrak{X}, \ AN/N \simeq A/A \cap N,$$
it follows that $AN/N$ is $\mathbb{P}$-subnormal in $G/N$, and by Lemma 3 (2), the subgroup $AN$ is $\mathbb{P}$-subnormal in $G$. If $AN \neq G$, then $AN \in \mathfrak{X}$, it follows that $A$ is $\mathbb{P}$-subnormal in $AN$, and by Lemma 3 (3), the subgroup $A$ is $\mathbb{P}$-subnormal in $G$, which is a contradiction. Therefore $AN = G$, in particular, $G$ is biprimary. Let $H$ be an arbitrary maximal subgroup of $G$. Then either $A^x \subseteq H$, $x \in G$ or $N \subseteq H$. If $A^x \subseteq H$, then $A^x = H$ because $N$ is a minimal normal subgroup of $AN = G$ and $H$ is cyclic. If $N \subseteq H$, then by the Dedekind identity, $H = (A \cap H)N$. By the choice of $A$ we can conclude that $A \cap H$ is $\mathbb{P}$-subnormal in $G$. Now by Theorem 2.13 (2) \[\square\], $H$ is supersolvable. So in the case of $\Phi(G) = 1$, we proved that $G$ is a biprimary minimal non-supersolvable group.

Let $\Phi(G) \neq 1$. According to statement 2 of the theorem, $G/\Phi(G) \not\in \mathfrak{X}$, and so by the inductive hypothesis, $G/\Phi(G)$ is a biprimary minimal non-supersolvable group. It follows from the structure of such groups that $G/\Phi(G)$ possesses an ordered Sylow tower. Since $\pi(G) = \pi(G/\Phi(G))$, we deduce that $G$ is a biprimary group which possesses an ordered Sylow tower: $G = [P]Q$, where $P$ and $Q$ are Sylow subgroups of $G$. Since $G \not\in \mathfrak{X}$, then there exists a primary cyclic non-$\mathbb{P}$-subnormal subgroup. Let $A$ be a subgroup of least order among these subgroups. If $PA \neq G$, then $PA \in \mathfrak{X}$, and thus $A$ is $\mathbb{P}$-subnormal in $PA$. Since $PA$ is $\mathbb{P}$-subnormal in $G$, it follows by Lemma 3 (3) that $A$ is $\mathbb{P}$-subnormal in $G$, this is a contradiction. Therefore, $PA = G$. Let $H$ be an arbitrary maximal subgroup of $G$. Then either $P \subseteq H$ or $A^x \subseteq H$, $x \in G$. If $P \subseteq H$, then $H = [P](A \cap H)$ by the Dedekind identity. By the choice of $A$, we can conclude that $A \cap H$ is $\mathbb{P}$-subnormal in $G$. Now by Theorem 2.13 (2) \[\square\], the subgroup $H$ is supersolvable. If $A^x \subseteq H$, then $H = [P \cap H]A^x$. Since $H \in \mathfrak{X}$, we deduce that $A^x$ is $\mathbb{P}$-subnormal in $H$, and thus $H$ is supersolvable by Theorem 2.13 (2) \[\square\]. Hence, in the case of $\Phi(G) \neq 1$, we proved that $G$ is a biprimary minimal non-supersolvable group.

Therefore, in any case every minimal non-$\mathfrak{X}$-group is a biprimary minimal non-supersolvable group. We conclude by Lemma 14, that every non-normal Sylow subgroup of $G$ is cyclic.

The theorem is proved.

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