Numerical Estimation of Heat Flux and Convective Heat Transfer Coefficient in a One Dimensional Rectangular Plate by Levenberg-Marquardt Method

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Abstract

Background/Objectives: The objective of this paper is to simultaneously obtain heat flux and convective heat transfer coefficient for a one dimensional heat transfer problem through an inverse method. Methods/Statistical Analysis: The problem deals with a rectangular aluminium plate subjected to constant heat flux to a face, while convection and radiation occur on the opposite face. The temperature distributions with respect to space and time are calculated using full implicit “Finite Difference Method”. The geometry is simplified to a lumped system and fourth order “Runge-Kutta” is used to solve the governing equation which is then represented as a forward model. Sensitivity analysis has also been carried out in the present study. Finally, the unknown heat flux and convective heat transfer coefficient are estimated using “Levenberg-Marquardt Method” for the known temperature distribution. Findings: The two parameters discussed in the work cannot be measured and they can only be inferred by some means. As an inverse problem, Levenberg Marquardt algorithm, which is quite popular because of its gradient information, is used to estimate the unknown quantities for noisy data and also it has been proven that the method estimates with reasonable accuracy for different noise levels. Application/Improvements: Instead of using noise added surrogated data, one can perform experiments and obtain the temperature distribution which can be used as input to the inverse method. The inverse method can also be combined with probabilistic method in order to avoid getting trapped in the local minima/maxima.

Keywords: Conjugate, Finite Difference, Levenberg-Marquardt, Runge-Kutta, Sensitivity

1. Introduction

Many real life applications involve a situation where a body is subjected to heat flux and it has to dissipate this heat through convection and radiation to its surroundings. Many studies have been performed on the steady state and transient conditions which vary with the properties of the body and also those of the surroundings.

A similar situation can be emulated using a rectangular plate which is subjected to a heat flux. A nonlinear heat transfer problem usually involves different types of boundary conditions which makes the governing differential equations complicated. These complicated equations cannot always be directly integrated which results in the need of approximation methods. One such method is fourth order Range-Kutta method which approximates a function when its derivative and initial value are available.

Sensitivity analysis provides a measure of the sensitivity of the estimated temperature with respect to changes in the parameter. The quantity of heat flux that the plate which is being subjected to and the convective heat transfer coefficient can be simultaneously estimated using inverse heat transfer techniques which require basic temperature distributions.

Inverse heat transfer techniques are majorly classified into deterministic and stochastic techniques. Levenberg-Marquardt method is one of the deterministic approaches to determine the unknown parameters. This method

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2. Problem Description

Consider a rectangular plate which is subjected to a constant heat flux on one of its faces and convection and radiation occur on the opposite face as shown in the Fig 1, all other faces are assumed to be insulated. The conduction is assumed to be one dimensional along the direction OX as shown in the Figure 1.

The governing differential equation for the transient case of the above model is:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

(1)

The boundary conditions are:

$$q''' = -k \frac{\partial T}{\partial x} \text{ at } x=0$$

(2)

$$-k \frac{\partial T}{\partial x} = h(T - T_\infty) + \sigma \varepsilon (T^4 - T_\infty^4) \text{ at } x=L$$

(3)

$$T = T_\infty \text{ at } t=0$$

(4)

where,

x – Spatial position (m)

\(t\) – Time instant (s)

T – Temperature \([T(x,t)] (K)\)

\(\alpha\) – Thermal diffusivity \((m^2/s)\)

k – Thermal conductivity of the plate \((W/m\,K)\)

\(q''\) – Heat flux \((W/m^2)\)

h – Convection heat transfer coefficient of surroundings \((W/m^2\,K)\)

\(T_\infty\) – Temperature of the surroundings (K)

\(\sigma\) – Stefan-Boltzmann’s constant \((W/m^2\,K^4)\)

\(\varepsilon\) – Emissivity of the plate

\(q''\), \(\alpha\), k, h, \(T_\infty\) and \(\varepsilon\) have been considered as constants in the further evaluation of the equations.

2.1 Forward Model

By the application of full implicit Finite Difference method to the equations 1–4, we obtain

$$\frac{T_i^{p+1} - 2T_i^{p} + T_i^{p-1}}{(\Delta x)^2} = \frac{1}{\alpha} \frac{T_i^{p+1} - T_i^p}{\Delta t}$$

(5)

$$q''' = -k \frac{T_i^{p+1} - T_i^{p-1}}{\Delta x} \text{ at } x=0$$

(6)

$$-k \frac{T_i^{p+1} - T_i^{p-1}}{\Delta x} = h(T_i^{p+1} - T_\infty) + \sigma \varepsilon (T_i^{p+1} - T_\infty^4) \text{ at } x=L,i=i_{max}$$

(7)

$$T_i^0 = T_\infty \text{ for any } i \text{ at } t=0$$

(8)

where,

i – spatial node along OX \((0 \leq i \leq i_{max})\) (here \(i_{max}\) corresponds to spatial node at \(x=L)\)

p – time node (where \(p=0\) corresponds to \(t=0)\)

\(\Delta x\) – length of each spatial element along OX

\(\Delta t\) – time interval

The RHS of equation (7) is approximated by taking \(T_i^{p}\) instead of \(T_i^{p+1}\) for easier evaluation of the temperature distribution. By considering an aluminum plate with properties, \(\alpha = 8.546399 \times 10^{-5} m^2/s\), \(k = 202.4 W/m\,K\) and \(\varepsilon = 0.07\) having dimensions of \((250 \times 150 \times 6) mm\) (where \(L = 6 mm)\), which is subjected to a constant heat flux \(q'' = 201.7333 W/m^2\) \((250 \times 150) mm^2\) cross-section] and exposed to \(T_\infty = 303.9303 K\) and \(h = 4.5 W/m^2\,K\), the temperature distributions at nodes along OX at various time instants are obtained.

From Figure 2, it is evident that the maximum temperature difference along OX is of the order \(10^{-2}\) and hence the plate can be considered as a lumped system. This is
parameter yields a small change in temperature. Hence the estimation of the parameter becomes extremely difficult in such a case as there is a wide range of the values of the parameter for the same temperature.

Sensitivity analysis was carried out for the parameters $q''$, $h$ and $\varepsilon$ in order to determine which parameter can be easily estimated based on the curves obtained.

It has been assumed that, $d q'' = 10^{-5} q''$, $d h = 10^{-5} h$, and $d \varepsilon = 10^{-5} \varepsilon$.

The plot obtained is:

From Figure 5, it can be noticed that $q''$ (heat flux) and $h$ (convective heat transfer coefficient) can be easily estimated as their saturation values are far away from zero.

**2.2 Lumped System**

The plate has now been considered as a lumped system, the governing differential equation changes into

$$pL \frac{dT}{dt} = q' - \kappa(T - T_\infty) - \sigma \varepsilon (T^4 - T_\infty^4)$$  \hspace{1cm} (9)

Equation (9) is solved using 4th order Range-Kutta method to obtain the temperature distributions with respect to time.

**2.3 Time Independence**

The optimum time interval for carrying out Range-Kutta method was considered to be $\Delta t = 60 \text{s}$ in order to have a maximum error of the order $10^{-7}$. The error is defined as the difference between corresponding temperatures with time intervals $2\Delta t$ and $\Delta t$. Eqn. 9 represents heating of an aluminium plate. When the LHS of Eqn.9 is zero, then the plate is assumed to have been reached steady state. Figure 3 shows the temperature distribution for the heating portion of the plate.

For any inverse approach temperature distribution must be known. The temperature distribution can be obtained either by experimental or numerical. The temperature obtained numerically is called as surrogated data and shown in Figure 4.

**2.4 Sensitivity Analysis**

Sensitivity analysis provides a measure of the sensitivity of the estimated temperature with respect to changes in the parameter. A small value of the magnitude of a parameter’s sensitivity indicates that large change in the parameter yields a small change in temperature. Hence the estimation of the parameter becomes extremely difficult in such a case as there is a wide range of the values of the parameter for the same temperature.

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2.5 Inverse Method

2.5.1 Levenberg-Marquardt Method

The parameters $q''$ and $h$ are estimated using two parameter Levenberg-Marquardt method. The algorithm which has been used by this method is as follows:

$$\Omega^k = diag \left( \left[ J^k \right]^T J^k \right)$$  \hspace{1cm} (10)

where,
- $\Omega^k$ – diagonal matrix for $k$th iteration
- $J^k$ – sensitivity matrix for $k$th iteration
- $\left[ J^k \right]^T$ – transpose of $J^k$

Here, $J^k$ is a two dimensional matrix, $J^k = \left( \frac{\partial T_p}{\partial q''} \right)$ and $J^k = \left( \frac{\partial T_p}{\partial h} \right)$, where, $p$ corresponds to time instant.

Let $\mu^k$ be damping parameter.

Suppose the temperature measurement matrix is
$$Y = (Y_1, Y_2, \ldots, Y_p)^T$$, where, $p = 1, 2, \ldots, P$.

Here, $P$ – total number of measurements. Let $q''$ be the initial guess of $q''$ and $h^0$ be the initial guess of $h$.

Assuming, $\mu^0 = 0.001$, the following steps are followed.

Step 1: Using the available estimates $q''^k$, $h^k$ the temperature matrix $T(q''^k, h^k) = (T_1, T_2, \ldots, T_p)$ is obtained.

Step 2: $S(q''^k, h^k) = [Y - T(q''^k, h^k)]^T \times [Y - T(q''^k, h^k)]$ is computed.

Step 3: Sensitivity matrix $J^k$ is computed and $\Omega^k$ is evaluated using current values of $q''^k$ and $h^k$.

Step 4: $\Delta q''^k = q''^{k+1} - q''^k$ and $\Delta h^k = h^{k+1} - h^k$ are computed using the equation,

$$\left[ J^k \right]^T J^k + \mu^k \Omega^k \Delta \hat{X}^k = (2J^k)^T \left[ Y - T(q''^k, h^k) \right]$$  \hspace{1cm} (12)

where, $\Delta \hat{X}^k (1,1) = \Delta q''^k$ and $\Delta \hat{X}^k (2,1) = \Delta h^k$.

Step 5: New estimates $q''^{k+1}$ and $h^{k+1}$ are computed as

$$q''^{k+1} = q''^k + \Delta q''^k \hspace{1cm} (13(i))$$

$$h^{k+1} = h^k + \Delta h^k \hspace{1cm} (13(ii))$$

Step 6: Using the new estimates $q''^{k+1}$ and $h^{k+1}$, $T(q''^{k+1}, h^{k+1})$ is computed from equation (9) and $S(q''^{k+1}, h^{k+1})$ is defined by equation (11).

Step 7: If $S(q''^{k+1}, h^{k+1}) = S(q''^k, h^k)$, replace $\mu^k$ by $10 \mu^k$ and return to step 4.

Step 8: If $S(q''^{k+1}, h^{k+1}) < S(q''^k, h^k)$, accept the new estimates $q''^{k+1}$, $h^{k+1}$ and replace $\mu^k$ by $0.1 \mu^k$.

Step 9: The iterative procedure is stopped when the condition $\sum \left| q''^{k+1} - q''^k \right| < \delta 1$ and $\left| h^{k+1} - h^k \right| < \delta 2$, where $\delta 1$ and $\delta 2$ are the prescribed tolerances and $\|\cdot\|$ is the vector Euclidean.

LIMITATION: $|c|^T J^k C^T J^k | \neq 0$ where, $\|\cdot\|$ is determinant.

In Figure 6 X denotes the unknown parameters, in our scenario $q''$ and $h$, which are estimated using the Levenberg-Marquardt method.

3. Results and Discussion

3.1 Estimation of Heat Flux and Convective Heat Transfer Coefficient for the Surrogated Data with Different Noise Levels

In this section the estimation of heat flux and convective heat transfer coefficient for the noise added surrogated data is attempted. Experiments always contain inherent noise and in order to mimic the same scenario, the surrogated data is now added with noise at different levels and finally, the unknown heat flux and convective heat transfer coefficient are estimated using the retrieval methodology explained in the previous section. Noise level represents a set of random values within a certain range.

Figure 7 is the flowchart indicating the estimation of unknown parameters for noise added surrogated data. The noise added surrogated data or perturbed data is fed into the inverse method i.e. Levenberg Marquardt algorithm to determine the unknown heat flux and convective heat transfer coefficient.

Figure 8 represents the addition of noise to all the points of the surrogated data. In the following Tables 1-3, the retrieved values (obtained from Levenberg-Marquardt method for heat flux and convective heat transfer coefficient are tabulated along with their initial guesses.

From Table 1, for the noise level ±0.5K, the surrogated data is added with various random values ranging between ±0.5K.

From Tables 1–3, it is observed that the deviation of the retrieved values from the actual values for a given heat flux and convective heat transfer coefficient increases with the increase in the noise level. The decrease in the deviation of retrieved $h$ value from the actual value with the increasing heat flux for a certain noise level is an implication of the higher saturation value obtained in the sensitivity analysis which leads to a smaller range of $h$ that are attributed to a certain temperature distribution.

From these observations it can be inferred that even at a reasonable sensitivity of the thermocouple or errors in the measurement due to various sources can be alleviated.
Figure 6. Flowchart indicating the iterative procedure for Levenberg-Marquardt method.

Figure 7. Representation of noise addition.

Figure 8. Temperature vs time (with and without noise addition).
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Table 1. \( q''=500 \, \text{W/m}^2 \)

| Noise Level | Initial guess of \( q'' \) | Retrieved value of \( q'' \) | Initial guess of \( h \) | Retrieved value of \( h \) |
|-------------|-----------------------------|-------------------------------|--------------------------|---------------------------|
| ±0.5K       | 250                         | 498.3647                      | 3.5                      | 4.4832                    |
| ±1.0K       | 300                         | 496.7267                      | 3.25                     | 4.4664                    |
| ±1.5K       | 400                         | 495.0858                      | 4                        | 4.4496                    |

Table 2. \( q''=750 \, \text{W/m}^2 \)

| Noise Level | Initial guess of \( q'' \) | Retrieved value of \( q'' \) | Initial guess of \( h \) | Retrieved value of \( h \) |
|-------------|-----------------------------|-------------------------------|--------------------------|---------------------------|
| ±0.5K       | 550                         | 748.3185                      | 3.25                     | 4.4882                    |
| ±1.0K       | 650                         | 746.6349                      | 3.5                      | 4.4764                    |
| ±1.5K       | 700                         | 744.9491                      | 4                        | 4.4646                    |

Table 3. \( q''=1000 \, \text{W/m}^2 \)

| Noise Level | Initial guess of \( q'' \) | Retrieved value of \( q'' \) | Initial guess of \( h \) | Retrieved value of \( h \) |
|-------------|-----------------------------|-------------------------------|--------------------------|---------------------------|
| ±0.5K       | 800                         | 998.2698                      | 3.5                      | 4.4907                    |
| ±1.0K       | 850                         | 996.5378                      | 3.75                     | 4.4813                    |
| ±1.5K       | 900                         | 994.8040                      | 4.1                      | 4.4720                    |

Figure 9. Parity Plot of Table I for noise level ±0.5K.

by using LM algorithm and the unknown parameters can be estimated to a reasonable accuracy.

The results obtained can be validated using parity plots, which have been presented below for some of the heat flux and noise level combinations that are presented in the Tables I-III.

Parity plot indicating the agreement between the simulated temperatures against the experimental or noise added surrogated temperatures are shown in Figures 9-11. The simulated temperature in the parity plot is obtained by solving the forward model for the estimated values of heat flux and heat transfer coefficient. For example, in Figure 9 the data comprises of temperatures obtained for \( q''=500 \, \text{W/m}^2 \) and \( h=5 \, \text{W/m}^2 \) and their corresponding temperatures for \( q''=498.3647 \, \text{W/m}^2 \) and \( h=4.4832 \, \text{W/m}^2 \) (estimated values from Table I). The close agreement between the simulated temperatures and noise added temperatures strongly corroborates that the inverse methodology incorporated for the simultaneous estimation of heat flux and heat transfer coefficient is a highly potential tool in the field of inverse estimation.

4. Conclusions

A one dimensional heat transfer problem has been considered and adjudged as a lumped system (material having high thermal conductivity has been used so that it can be considered as a lumped system). Sensitivity analysis was carried out to determine whether the heat flux and convective heat transfer coefficient can be estimated or not. Sensitivity analysis curves differed for different values of
heat flux (especially their saturation values). Longer time interval resulted in lower accuracy in temperature measurement for surrogated data. Based on the amount of the accuracy required, length of the time interval can be chosen. Heat flux and convective heat transfer coefficient for surrogated data with noise levels were estimated using Levenberg-Marquardt method.

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