Exact solutions of Bianchi I spacetimes which admit Conformal Killing vectors

Michael Tsamparlis · Andronikos Paliathanasis · Leonidas Karpathopoulos.

Received: date / Accepted: date

Abstract We develop a new method in order to classify the Bianchi I spacetimes which admit conformal Killing vectors (CKV). The method is based on two propositions which relate the CKVs of 1+(n-1) decomposable Riemannian spaces with the CKVs of the (n-1) subspace and show that if 1+(n-1) space is conformally flat then the (n-1) spacetime is maximally symmetric. The method is used to study the conformal algebra of the Kasner spacetime and other less known Bianchi type I matter solutions of General Relativity.

1 Introduction

The Bianchi models are spatially homogeneous spacetimes which admit a group of motions $G_3$ acting on spacelike hypersurfaces. These spacetimes include the non-isotropic generalizations of the Friedman-Robertson-Walker (FRW) space-time and have been used in the discussion of anisotropies in a primordial universe and its evolution towards the observed isotropy of the present epoch [34]. The simplest type of these spacetimes are the Bianchi I models for which $G_3$ is the abelian group of translations of the three dimensional Euclidian space $E^3$. In synchronous coordinates the metric of Bianchi I spacetimes is:

$$ ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2 $$  \hfill (1)

where $A(t), B(t), C(t)$ are functions of the time coordinate only and the corresponding KVs are $\{ \partial_t, \partial_x, \partial_y \}$. When two of the metric functions are equal, e.g. $A^2(t) = B^2(t)$, a Bianchi I

M. Tsamparlis · L. Karpathopoulos
Faculty of Physics, Department of Astrophysics - Astronomy - Mechanics University of Athens, Panepistemiopolis, Athens 157 83, Greece

A. Paliathanasis
Dipartimento di Fisica, Universita' di Napoli, "Federico II" Complesso Universitario di Monte S. Angelo, Via Cintia Edificio 6, I-80126 Napoli, Italy
INFN, Sezione di Napoli, Complesso Universitario di Monte S. Angelo, Via Cintia Edificio 6, I-80126 Napoli, Italy
E-mail: paliathanasis@na.infn.it
spacetime \((1)\) reduces to the important class of Locally Rotational Symmetric (LRS) spacetimes \([1]\).

A Conformal Killing Vector (CKV) \(X^a\) is defined by the requirement \(\mathcal{L}_X g_{ab} = 2\psi g_{ab}\) and reduces to a Killing vector (KV) \((\psi = 0)\), to a Homothetic Killing Vector (HV) \((\psi_{,a} = 0)\), and to a Special Conformal Killing Vector (SCKV) \((\psi_{,ab} = 0)\). The effects of these vectors can be seen at all levels of General Relativity, that is, geometry, kinematics and dynamics. At the geometry level the knowledge of a CKV makes possible the choice of coordinates so that the metric is simplified, in the sense that one of the metric components is singled out \([5,6]\). At the level of kinematics the CKVs impose restrictions on the kinematic variables (rotation, expansion and shear) and produce well known results (see for example \([7,8,9,10]\)). Finally at the level of dynamics the CKVs can (and have) been used in various directions, for example to obtain new solutions of the field equations with (hopefully) better physical properties (see for example \([10,11,12,13,14]\)). It becomes evident that it is important that we know the conformal algebra of a given spacetime.

In \([15]\) all LRS spacetimes which admit CKVs have been determined. In the following we determine all Bianchi I spacetimes which are not reducible to LRS spacetimes and admit CKVs.

The general Bianchi I spacetime \((1)\) does not admit CKVs. However, as we will show, there are two families of Bianchi I spacetimes which admit CKVs. One family consists of the conformally flat Bianchi I spacetimes, which admit 15 CKVs and are conformally related to Rebouças and Tiommo (RT) and Rebouças and Teixeira (ART) \([16,17]\) spacetimes. The second family contains the not conformally flat Bianchi I spacetimes, which admit only one proper CKV. In the determination of the CKVs we use the Bilyanov - Defrise - Carter theorem which relates the conformal algebra of conformally related metrics (for details see \([18,19]\)).

In the literature one finds very few cases of Bianchi I spacetimes which admit proper CKVs. For example even the CKV found by Maartens and Mellin \([20]\) is really a CKV in an LRS spacetime and not in a Bianchi I spacetime \([15]\). The difficulty lies in the fact that the direct solution of the conformal equations in Bianchi I spacetimes is a major task. Thus an alternative simpler method is needed to solve this problem and this is what it is developed in the following sections. It is to be noted that using the Petrov classification and the Bilyanov - Defrise - Carter theorem McIntosh and Steele \([21]\) have determined all vacuum Bianchi I spacetimes which admit a homothety.

One extra advantage of the proposed method is that one can use it to prove/test if a given Bianchi I spacetime admits a CKV or not. For example as it will be shown the two well known anisotropic Bianchi I solutions that is, the Kasner solution \((1)\) and the anisotropic dust solution \([22]\), which have formed the basis of many studies of anisotropic universes, do not admit a proper CKV; in particular the Kasner spacetime admits a HV.

The structure of the paper is as follows. In section 2 we present two propositions required for the computation of the CKVs in Bianchi I spacetimes. In sections 3 and 4 we apply the results of section 2 and we determine all Bianchi I spacetimes which admit CKVs. In section 5 we consider the application of these results in various Bianchi I metrics found in the literature. Finally in section 6 we discuss our results.

---

1 These spacetimes are 1+3 spacetimes in which the 3d hypersurface is a maximally symmetric space with positive and negative curvature scalar respectively
2 Preliminaries

As it has been remarked in the last section the computation of CKVs of Bianchi I spacetimes by direct solution of the conformal equations is a difficult task. Thus we have developed an indirect method which is based on the two Propositions discussed below. The first is proposition 1 which has been given in [23] for spacetimes \((n = 4)\) and below is generalized\(^2\) to \(n\)-dimensional Riemannian spaces as follows.

Proposition 1 A decomposable \(1 + (n - 1)\) \((n \geq 3)\) Riemannian space \(g_{ab}\) with line element (Greek indices take the values \(1, \ldots, n\) and Latin indices the values \(0, \ldots, n\))

\[
ds^2 = \varepsilon dt^2 + h_{\mu\nu}(x^\sigma) dx^\mu dx^\nu
\]  

admits a proper CKV \(X^a\) if and only if the \((n - 1)\) space \(h_{\mu\nu}(x^\sigma)\) admits a gradient proper CKV \(\xi^\mu\). In particular the two vector fields are related as follows

\[
X^a = -\frac{\varepsilon}{p} \dot{\lambda}(t) \Psi(x^\sigma) \partial_t + \frac{1}{p} \lambda(t) \xi^\mu(x^\sigma) + H^\mu(x^\sigma)
\]  

where:
- \(p\) is a non vanishing constant,
- \(\Psi(x^\sigma)\) is the conformal factor of the CKV \(\xi^\mu\) and satisfies the condition

\[
\Psi_{\mu\nu} = p \Psi h_{\mu\nu}
\]  

that is, \(\Psi_{\mu\nu}\) is a gradient CKV of the \(n - 1\) space
- \(\lambda(t)\) satisfies the linear second order equation

\[
\ddot{\lambda}(t) + \varepsilon p \dot{\lambda}(t) = 0
\]  

where \(\dot{\lambda} = \frac{d\lambda}{dt}\).
- \(H^\mu\) is a KV or a HV of the \(n - 1\) metric \(h_{\mu\nu}(x^\sigma)\).

From proposition 1 follows that a proper CKV of the \(n - 1\) metric \(h_{\mu\nu}\) generates two proper CKVs for the \(n\) metric \(g_{ab}\).

The crucial result of proposition 1 is that the gradient CKVs of the \((n - 1)\) space are of the specific form \(^4\). Furthermore if the \((n - 1)\) space has constant non vanishing Ricciscal
capital \(R\), then the constant \(p\) is given by the expression

\[
p = \frac{R}{(n - 1)(n - 2)} \quad (n \geq 3).
\]  

The second Proposition 2 concerns the \(1 + (n - 1)\) decomposable spacetimes which admit CKV\(^3\).

Proposition 2 The metric \(^2\) is conformally flat if and only if the \((n - 1)\) metric \(h_{\mu\nu}(x^\sigma)\) is the metric of a space of constant curvature \((n \geq 3)\).

\(^2\) The generalization of the proof of \(^3\) to \(n\) dimensions is similar to the one for \(n = 4\) and we omit it.
\(^3\) The proof of Proposition \(^2\) is given in appendix \(^5\).
In addition to these propositions we recall the following result (see [23]).

The metric of a space of constant non-vanishing curvature of dimension \( n \) admits \( n + 1 \) gradient CKVs.

From proposition 2 it follows that as far as the admittance of CKVs is concerned, the connected \( 1 + (n - 1) \) decomposable spaces are classified in two major classes.

i) Class A: The \( 1 + (n - 1) \) space is conformally flat. Then the \( (n - 1) \) space is not conformally flat and the \( 1 + (n - 1) \) space admits \( \frac{(n+1)(n+2)}{2} \) CKVs.

ii) Class B: The \( 1 + (n - 1) \) space is not conformally flat. Then the \( (n - 1) \) space is not a space of constant curvature.

For a space conformally related to a \( 1 + (n - 1) \) decomposable space this classification of CKVs remains the same since all conformally related spaces admit the same conformal algebra.

We conclude that the parameter in the classification of the connected \( 1 + (n - 1) \) spacetimes which admit CKVs is the constancy or not of the curvature scalar of the \( (n - 1) \) space. Using this observation and propositions 1 and 2 we are able to determine all Bianchi I spacetimes which admit CKVs.

2.1 CKVs of Bianchi I spacetimes

In the generic line element (1) of Bianchi I spacetime, we consider the coordinate transformation \( dt = C(\tau) d\tau \) and get

\[
 ds^2 = C^2(\tau) \left( d\tau^2 + ds^2_{(3)} \right)
\]  

(7)

where \( ds^2_{(3)} \) is the three dimensional metric

\[
 ds^2_{(3)} = -d\tau^2 + B^2_1(\tau) dy^2 + A^2_1(\tau) dx^2
\]

(8)

with \( A^2_1(\tau) = \frac{A^2_1(\tau)}{C^2(\tau)}, \ B^2_1(\tau) = \frac{B^2_1(\tau)}{C^2(\tau)} \). Applying a second transformation \( d\tau = B^2_1(\tau) d\tau \) and \( \Gamma^2(\tilde{\tau}) = \frac{B^2_1(\tau)}{B^2_1(\tau)} \) the three dimensional metric (8) becomes

\[
 ds^2_{(3)} = B^2_1(\tilde{\tau}) ds^2_{1+2}
\]

(9)

where

\[
 ds^2_{1+2} = dy^2 + ds^2_{(2)}
\]

(10)

and

\[
 ds^2_{(2)} = -d\tilde{\tau}^2 + \Gamma^2(\tilde{\tau}) ds^2.
\]

(11)

The two dimensional metric (11) is conformally flat. Indeed if we introduce the new variable \( d\tilde{\tau} = \Gamma(\tilde{\tau}) d\tilde{\tau} \) the metric \( ds^2_{(2)} \) becomes

\[
 ds^2_{(2)} = \Gamma^2(\tilde{\tau})(-d\tilde{\tau}^2 + ds^2).
\]

(12)

* All 2d metrics are conformally flat.
The 2d metric $\eta_{AB} = \text{diag}(-1, 1)$ admits the three KVs

$$P_\tau = \partial_\tau, \quad P_i = \partial_i, \quad r = x \partial_\tau + \hat{\tau} \partial_i$$

(13)

and the gradient HV

$$H = \hat{\tau} \partial_\tau + x \partial_i, \quad \psi_H = 1.$$  

(14)

The curvature scalar $R_{(2)}$ of the 2d-metric (11) is calculated to be:

$$R_{(2)} = 2 \frac{\Gamma_{\hat{\tau}\hat{\tau}}}{\Gamma}.$$  

(15)

According to proposition 2, the condition that the $(1+2)$d - metric (10) - and consequently the 3d-metric $ds^2$ - is conformally flat, is that the 2d-metric (12) is a metric of a space of constant curvature. We set $R_{(2)} = \text{const.} = 2c$ and find that this is the case when $\Gamma_{\hat{\tau}\hat{\tau}} = c\Gamma$.

On the other hand when $ds^2$ is of constant curvature then by means of the inverse of proposition 2 the metric $ds^2$ is conformally flat hence the metric $ds^2$ is also conformally flat.

We conclude that the classification of Bianchi I spacetimes which admit CKVs is done in two classes:

Class A: Contains all Bianchi I spacetimes which are conformally flat. According to proposition 2, in this case the 3d-metric $ds^2_{(3)}$ is of constant curvature and the form of the metric functions $A_1(\tau), B_1(\tau)$ is fixed.

Class B: Contains all Bianchi I spacetimes which are not conformally flat therefore the decomposable metric is not conformally flat. According to the inverse of proposition 2, in this class the 3-d metric $ds^2_{(3)}$ is not the metric of a space of constant curvature.

In Class B there are two cases to be considered.

Case B1: The 3d-metric $ds^2_{(3)}$ is not conformally flat in which case the scalar curvature of the 2d-metric $R_{(2)} \neq \text{const.}$

Case B2: The 3d-metric $ds^2_{(3)}$ is conformally flat hence according to proposition 2, the 2d-metric $ds^2_{(2)}$ is of constant curvature i.e. $R_{(2)} = \text{const.}$.

In the following we consider each Class and derive the corresponding Bianchi I spacetimes together with the CKV(s). We ignore the cases $A_1 = B_1 \leftrightarrow \Gamma = \text{const.}$ which lead to LRS spacetimes whose CKVs have already been found in [15].

3 Class A: The conformally flat Bianchi I spacetimes

Demanding that the Weyl tensor of the metric (7) vanishes we find the following conditions on the metric functions $A_1, B_1$:

$$A_1 \dot{B}_1 + B_1 \dot{A}_1 - 2 \dot{A}_1 B_1 = 0$$  

(16)

$$A_1 \ddot{B}_1 - 2B_1 \ddot{A}_1 + \dot{A}_1 B_1 = 0$$  

(17)

$$\ddot{A}_1 B_1 - 2A_1 \ddot{B}_1 + \dot{A}_1 B_1 = 0$$  

(18)

where a dot over a symbol denotes differentiation with respect to coordinate $\tau$. We note that only two of these three equations are independent.
Using (15)-(18) we can prove that the 3-metric (8) is the metric of a 3-space of constant curvature $R_3 = 6\varepsilon a^2$ where $\varepsilon = \pm 1$ and $a \neq 0$ is a constant. There are only two such spacetimes the RT spacetime [16] and the ART spacetime [17] mentioned above.

The RT and the ART spacetimes in isochronous coordinates have the line element

$$ds^2_{RT} = -dt^2 + \sin^2(t/a)dx^2 + \cos^2(t/a)dy^2 + dz^2$$

and

$$ds^2_{ART} = -dt^2 + \sinh^2(t/a)dx^2 + \cosh^2(t/a)dy^2 + dz^2$$

respectively. These spacetimes are 1+3 decomposable spaces whose three dimensional space is a space of constant curvature. They admit a 15 dimensional conformal algebra with a seven dimensional conformal subalgebra, which has been given in [15]. For the completeness of the paper in appendix B we give the conformal algebra of the RT and the ART spacetimes in a convenient form.

4 Class B: The non-conformally flat Bianchi I spacetimes

In this class there are two subcases to be considered depending on $R_{(2)} = const$ and $R_{(2)} \neq const$ where $R_{(2)}$ is the Ricciscalar of the two dimensional space [12].

4.1 Case B.I: $R_{(2)} \neq const$.

In this case we are interested only for the KVs and the HV of $ds^2_{(2)}$ since if there exist a proper CKV which satisfy condition (4) of proposition 1 then the two dimeniona space is of constant curvature. From the CKVs of $ds^2_{(2)} = (-dt^2 + dx^2)$ only the ones which do not contain terms $f(\tau)g(x)\partial_{\tau}$ with $f(\tau) \neq \tau$ can satisfy this property. It is well known that the two dimensional space $ds^2_{(2)}$ admits infinity CKVs. However, the vector fields which do not contain the terms $f(\tau)g(x)\partial_{x}$ with $f(\tau) \neq \tau$ are the two vector fields $P_\tau$ and the $H$.

The conformal factor of $P_\tau$ of the metric $ds^2_{(2)}$ is:

$$\psi(P_\tau) = \Gamma_\tau.$$

If we demand $\psi(P_\tau) = 0$ (the case of KVs) then we get $A_1^2 = B_1^2$, i.e. the LRS case which we ignore. If we demand $\psi(P_\tau) = const$ then we find $\Gamma_{\tau\tau} = 0$ which implies by (15) that $R_{(2)} = 0$ i.e constant which contradicts our assumption. Therefore $P_\tau$ produces nothing relevant.

The HV $H$ has conformal factor

$$\psi(H) = \Gamma_{\tau} \int \frac{d\tau}{\Gamma} + 1.$$  (21)

The requirement that $H$ is a KV of the 2-metric $ds^2_{(2)}$ gives $\tau\Gamma_{\tau} + \Gamma = 0$, hence $\Gamma(\tau) = \frac{\tau}{\Gamma}$ which implies $R_{(2)} = const.$ and it is excluded. The requirement that $H$ is a HV with conformal factor $\alpha_2(\neq 0)$ gives:

$$\Gamma = c_1 \tau^{\alpha_2 - 1}$$  (22)

where $c_1 = const$.

This HV is acceptable provided that $\alpha_2 \neq 1$ in order to avoid the LRS case. By proposition 1 this gives the following HV for the 1+2 metric (10):

$$H_1 = \alpha_2 x \partial_x + \tau \partial_\tau + x \partial_x$$  (23)
with conformal factor

$$\psi(H_1) = \alpha_2.$$  

(24)

This vector is a non-gradient CKV for the metric (9) with conformal factor:

$$\psi(H_1) = \tau(\ln A_1) + \alpha_2.$$  

(25)

We are interested in KVs and HVs (we show in the Appendix that the gradient CKVs of the form $\lambda(\xi)_{\alpha\beta} = p\lambda(\xi)_{\alpha\beta}$ imply that the 3-metric (9) is of constant curvature) thus we examine possible reductions of this CKV to a KV or a HV.

If $H_1$ is a KV then $\bar{\psi}(H_1) = 0$ and this gives $A_1 = c_2 \hat{\tau}^{-\alpha_2}$. From (12) and (22) we obtain $B_1 = \frac{c_1}{c_2}$ which implies $\hat{\tau} = c_1 e^{\tau/c}$ where $c = c_1c_2$. Thus we have the following KV:

$$X_{B_1} = \alpha_2 y \partial_y + c \partial_\tau + x \partial_x$$  

(26)

for the three dimensional metric:

$$ds^2_{(3)} = -d\tau^2 + c_2^2 c_3^{-2} e^{-2\alpha_2 \tau/c} ds^2 + \left( \frac{c}{c_3} \right)^2 e^{-2\tau/c} dx^2.$$  

(27)

Due to proposition 1 this is also a KV for the metric $ds^2_{1+3} = dz^2 + ds^2_{(3)}$ hence a proper CKV for the metric (7) with conformal factor (note that $\partial_\tau = A_1 \partial_t$)

$$\psi(X_{B_1}) = c(C(t)),.$$  

(28)

The metric $ds^2$ is given in (7) and describes a family of Bianchi I metrics parameterized by the function $C(t)$.

When $H_1$ is a HV from equation (25) we obtain ($\alpha_3 = \text{const.}$):

$$\tau(\ln A_1) + \alpha_2 = \alpha_3 \Leftrightarrow A_1 = c_2^{\alpha_3 - \alpha_2}$$  

(29)

and

$$B_1 = c_1c_2^{2\alpha_3 - 1}$$  

(30)

therefore we have that

$$\hat{\tau} = \left( \frac{\alpha_3}{c_1c_2} \right)^{1/\alpha_3} \tau^{1/\alpha_3}.$$  

(31)

Eventually we have the CKV:

$$X_{B_1} = \alpha_2 y \partial_y + \alpha_3 \tau \partial_\tau + x \partial_x + \alpha_3 z \partial_z$$  

(32)

for the Bianchi I metric:

$$ds^2 = C^2(\tau) \left[ dz^2 - d\tau^2 + c_2^2 \left( \frac{\alpha_3}{c_1c_2} \right)^{\frac{\alpha_3(\alpha_3 - 1)}{\alpha_3}} \tau^{\frac{\alpha_3(\alpha_3 - 1)}{\alpha_3}} dy^2 + c_1^2 c_2^2 \left( \frac{\alpha_3}{c_1c_2} \right)^{\frac{\alpha_3(\alpha_3 - 1)}{\alpha_3}} \tau^{\frac{\alpha_3(\alpha_3 - 1)}{\alpha_3}} dx^2 \right]$$  

(33)

with conformal factor:

$$\psi(X_{B_1}) = \alpha_3 [1 + \tau(\ln |C|),].$$  

(34)
4.2 Case B.II: $R^{(2)} = \text{const}$

We consider the subcases: $R^{(2)} = 0$, and $R^{(2)} \neq 0$.

When $R^{(2)} = 0$ from (15) we have that

$$\Gamma = b_0 \bar{\tau} \Leftrightarrow B_1 = b_0 \bar{\tau} A_1.$$  \hspace{1cm} (35)

Equation (35) implies that the 3-metric (10) has the form (we ignore the unimportant integration constant $b_0$):

$$ds^2_{1+2} = dy^2 - dt^2 + \bar{\tau}^2 dx^2.$$ \hspace{1cm} (36)

The CKVs of the flat 3-metric $ds^2_{1+2}$ are known \cite{24}. Using the transformation $\tau = \bar{\tau} \cosh x, \xi = \bar{\tau} \sinh x, \tilde{\gamma} = y$ we obtain the 3-metric (36) from which we obtain the following conformal algebra (we ignore the KVs $\partial_t, \partial_j$; $i = 1, 2, 3, 4; \alpha = 1, 2, 3,$):

- Four KVs

$$X_1 = \cosh x \partial_t - \frac{\sinh x}{\tau} \partial_j,$$

$$X_2 = \sinh x \partial_t + \frac{\cosh x}{\tau} \partial_j,$$

$$X_3 = y \sinh x \partial_t - y \cosh x \frac{\partial_t}{\tau} + \tau \sinh x \partial_j,$$

$$X_4 = y \cosh x \partial_t - y \frac{\sinh x}{\tau} \partial_t + \tau \cosh x \partial_j.$$  \hspace{1cm} (37)

- One gradient HV

$$X_7 = \tau \partial_t + y \partial_j, \hspace{0.5cm} \psi(X_7) = 1$$

- Three special CKVs

$$X_8 = (\gamma^2 + \bar{\tau}^2) \cosh x \partial_t + \frac{\tau^2 - y^2}{\tau} \sinh x \partial_j + 2y \bar{\tau} \cosh x \partial_j,$$

$$X_9 = (\gamma^2 + \bar{\tau}^2) \sinh x \partial_t + \frac{\tau^2 - y^2}{\bar{\tau}} \cosh x \partial_j + 2y \bar{\tau} \sinh x \partial_j,$$

$$X_{10} = 2 \tau \gamma \partial_t + (\gamma^2 + \bar{\tau}^2) \partial_j.$$  \hspace{1cm} (38)

with corresponding conformal factors:

$$\psi(X_8) = 2 \tau \cosh x, \hspace{0.5cm} \psi(X_9) = 2 \tau \sinh x, \hspace{0.5cm} \psi(X_{10}) = 2y.$$  \hspace{1cm} (39)

These vectors are also CKVs for the metric (9) but with conformal factors:

$$\psi'(X_A) = X_A (\ln A) + \psi(X_A)$$

where $A = 1, 2, ..., 10$. The possible vectors $X_A$ which give $\psi'(X_A) = \text{const.}$ are the KVs and the HV which do not contain terms of $f(\tau)g(x)\partial_x$. The only such vector is the HV $X_7$.

The case that $X_7$ is a KV for the metric (9) gives $B_1 = \text{const.}$ and we ignore it. We set $\psi'(X_A) = \alpha_A$ and we obtain, after standard calculations, that the vector $X_7 = \alpha_4 \tau \partial_t + y \partial_j$ is a HV for the 3-metric:

$$ds^2 = -d\tau^2 + b_1^2 \left( \frac{\alpha_4}{b_1} \right)^{\frac{\alpha_4 - 1}{\alpha_4}} \tau^{\frac{\alpha_4 - 1}{\alpha_4}} dy^2 + \alpha_4^2 \tau^2 dx^2.$$

\hspace{1cm} (40)
with conformal factor $\alpha_4$. This vector is extended to a HV for the 1+3 metric $ds_{1+3}^2 = dz^2 + ds_{(3)}^2$ which is of the form:

$$X_{B_1} = \alpha_4 \tau \partial_\tau + y \partial_y + \alpha_4 z \partial_z.$$  \hfill (41)

The Bianchi I metric (40) and the CKV (41) are obtained from the metric (33) and the CKV (32) if we set $a_1 = 0$ and interchange the coordinates $x, y$. Therefore it is not a new case.

A detailed study of the subcase $R_{(2)} \neq 0$ shows that there are no more new Bianchi I metrics which admit CKVs. The calculations are rather standard and similar to the ones above and are omitted.

We conclude that there are two families of metrics in B.II class parameterized by the function $C(\tau)$. Each family admits one proper CKV and have as follows:

Metrics $B_1$ with $(\alpha_1 \neq 0, 1, c \neq 0)$

$$ds^2 = C^2(\tau) \left[ -d\tau^2 + e^{-\frac{2c}{\alpha_1}} dx^2 + e^{-\frac{2c}{\alpha_1}} dy^2 + dz^2 \right]$$  \hfill (42)

and corresponding CKV

$$X_{B_1} = c \partial_\tau + x \partial_x + \alpha_1 y \partial_y.$$  \hfill (43)

$$\psi (X_{B_1}) = c (\ln |C|)_\tau.$$  \hfill (44)

Metrics $B_2$ with $(\alpha_2 \neq 0, 1)$ and $(\alpha_1 \neq \alpha_2)$

$$ds^2 = C^2(\tau) \left[ -d\tau^2 + \tau^{2^{\alpha_2 - 1}} dx^2 + \tau^{2^{\alpha_2 - 1}} dy^2 + dz^2 \right]$$  \hfill (45)

and corresponding CKV

$$X_{B_2} = \alpha_2 \tau \partial_\tau + \alpha_1 y \partial_y + x \partial_x + \alpha_2 z \partial_z.$$  \hfill (46)

with conformal factor

$$\psi (X_{B_2}) = \alpha_2 [1 + \tau (\ln |C|)_\tau].$$  \hfill (47)

We observe that the CKV $X_{B_1}$ of the metric $B_1$ becomes a HV when $(\ln |C|)_\tau = \psi_0$, i.e. $C(\tau) = e^{\psi_0 \tau}$. In that case the metric (42) becomes $(e^{\psi_0 \tau} = t)$

$$ds^2 = -dt^2 + t^{\frac{2c}{\alpha_1}} dx^2 + t^{\frac{2c}{\alpha_1}} dy^2 + dz^2$$  \hfill (48)

where we substitute $e^{\psi_0 \tau} = t$. Furthermore the metric $B_2$ admits a HV when $C(\tau) = \tau^{\psi_0 - 1}$. In that case the line element (45) becomes

$$ds^2 = -dt^2 + t^{\frac{2c}{\alpha_1}} dx^2 + t^{\frac{2c}{\alpha_1}} dy^2 + t^{\frac{2c}{\alpha_1}} dz^2$$  \hfill (49)

Therefore from the spacetimes (48) and (49) we have that the Bianchi I spacetimes (1) which admit a proper HV are the spacetimes with power law coefficients. As it has been noted in the introduction all vacuum Bianchi I spacetimes which admit a Homothetic vector have been determined in [21].

In the following section we study the CKVs of some well known exact solutions of Einstein field equations in a Bianchi I spacetime.
5 Exact Bianchi I solutions and conformal symmetries

One can apply the results of the last section to determine if a given Bianchi I metric admits or not CKVs and at the same time determine the exact form of the CKVs and their conformal factors. The method of work is simple and consists of the following steps.

From the given Bianchi metric one computes the traceless projection tensor \( \Delta_{ab} = g_{ab} - \frac{1}{4} \delta_{ab} \) and demands that \( \Delta_{ab} X_{c,d} = 0 \) where \( X_c \) is any of the CKVs defined in (77), (78), (82), (83) (conformally flat case) and (43), (46) (non-conformally flat case). If this condition cannot be satisfied for any values of the parameters of the metric then the metric does not admit a CKV otherwise it does. It is possible that the conformal factors are constants in which case the CKVs reduce to HVs.

Before one proceeds with the above it is convenient to compute the Weyl tensor and examine if the space is conformally flat or not. If it is not there is no need to consider the vectors (77), (78), (82), (83) whereas if it is there is no need to consider the vectors (43), (46).

In the following section we apply the above method to various anisotropic Bianchi I metrics which we have traced in the literature. We present the derivation of the results for the Kasner type metrics in some detail whereas the for rest of the metrics we give only the results of the calculations.

5.1 Kasner type metrics

The Kasner type metrics are defined by the line element:

\[
d s^2 = -dt^2 + t^{2p} dx^2 + t^{2q} dy^2 + t^{2r} dz^2
\]

where \( p, q, r \) are different constants (otherwise the metric reduces to an LRS metric (two of the constants equal) or to a FRW metric (all constants equal). The well known Kasner spacetime - which has been used extensively in the literature in the discussion of anisotropies of the Universe - is a vacuum solution of Einstein’s field equations with the parameters \( p, q, r \) restricted by the relations:

\[
\begin{align*}
p + q + r &= 1 \\
p^2 + q^2 + r^2 &= 1.
\end{align*}
\]

Kasner spacetime is vacuum so if conformally flat it is flat therefore we have a non-conformally flat case. Condition \( \Delta_{ab} X_{c,d} = 0 \) for the vector fields (43), (46) yields in turn:

\( X_{B_1} \):

We find \( r = 1, c = 1, \alpha_1 = \frac{q - 1}{p - 1}, \tau d = \frac{1}{p - 1} td t \) (\( p \neq 1 \) otherwise we have an LRS spacetime) from which follows that the Kasner type metric:

\[
d s^2 = -dt^2 + t^{2p} dx^2 + t^{2q} dy^2 + t^{2r} dz^2
\]

admits the HV (1):

\[
X_{B_1} = \frac{1}{1 - p} t \partial_t + \frac{q - 1}{p - 1} x \partial_y + x \partial_z; \quad \psi(X_{B_1}) = \frac{1}{1 - p}
\]
We find $r \neq 1$, $\alpha_1 = \frac{q-1}{p-1}$, $\alpha_2 = \frac{q-1}{p-1}$ ($p \neq 1$) from which we conclude that the Kasner type metric (50) with $r \neq 1$, $p \neq 1$ admits the HV:

$$X_{B_2} = \frac{r-1}{p-1} \xi \partial_x + x \partial_y + \frac{q-1}{p-1} y \partial_z; \quad \psi(X_{B_2}) = \frac{1}{1-p}$$  (54)

We emphasize that due to conditions (51) the Kasner spacetime admits only the HV $X_{B_2}$.

These results agree with those of [21].

5.2 Bianchi I shear free spacetimes

This class contains many well known solutions of the field equations. The general form of the spacetime metric is

$$ds^2 = -dt^2 + S^2(t) f^2(t) dx^2 + S^2(t) f^2(t) dy^2 + S^2(t) f^2(t) dz^2$$  (55)

where the functions $S(t), f(t)$ are general functions. The various known solutions of this form are perfect fluid solutions with vanishing and non-vanishing cosmological constant $\Lambda$.

These solutions are:

a. Dust solution $\Lambda = 0$ [22].

$$S^3(t) = \frac{9}{2} Mt(t + \Sigma); f(t) = \frac{t^{2/3}}{S(t)}; p = 2 \sin \alpha, q = 2 \sin(\alpha + \frac{2\pi}{3}); r = 2 \sin(\alpha + \frac{4\pi}{3})$$  (56)

The constant $\alpha$ is the angle where the anisotropy is maximal ($-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$) and $\Sigma, M$ are constants with $\Sigma > 0$.

$\Lambda \neq 0$ [1]

$$S^3(t) = \begin{cases} a \sinh \omega t + \frac{M}{\omega} (\cosh \omega t - 1) & \text{for } \Lambda > 0 \\ a \sin \omega t + \frac{M}{\omega} (\cos \omega t - 1) & \text{for } \Lambda < 0 \end{cases}$$  (57)

$$f(t) = \begin{cases} \frac{\cosh \omega t - 1}{S(t)} & \text{for } \Lambda > 0 \\ \frac{1 - \cos \omega t}{S(t)} & \text{for } \Lambda < 0 \end{cases}.$$  (58)

b. Perfect fluid solutions with an equation of state $p = (\gamma - 1) \mu$ [3, 11]

$$S^3(t) = \begin{cases} c \sinh \omega t & \text{for } \Lambda > 0 \\ \sqrt{3(3 + M)} t & \text{for } \Lambda = 0 \\ c \sin \omega t & \text{for } \Lambda < 0 \end{cases}$$  (59)

$$f(t) = \begin{cases} (\tanh \omega t)^b & \text{for } \Lambda > 0 \\ b & \text{for } \Lambda = 0 \\ (\tan \omega t)^b & \text{for } \Lambda < 0 \end{cases}.$$  (60)

where $b = \left( \frac{3 + M}{3} \right)^{1/2}$ and $c = \left( \frac{3 + M}{\Lambda} \right)^{1/2}$.

For $\mu = 0$ we take the vacuum solutions for $\Lambda = -, >, < 0$. In [11] one can find the form of the solutions for various values of $\gamma$. 


Table 1 Exact solutions of Bianchi I spacetimes which admit CKVs

| Spacetime          | CKVs | Conformal factor |
|--------------------|------|------------------|
| Datta Solution     | $\frac{\partial}{\partial t}$ | $\frac{\partial}{\partial \tau}$ |
| Rosen Solution     | $\frac{\partial}{\partial t}$ | $\frac{\partial}{\partial \tau}$ |
| Kasner-type        | $\frac{\partial}{\partial \tau}$, $\frac{\partial}{\partial \lambda}$ | constant |
| for $\Lambda > 0$  | $\frac{\partial}{\partial \lambda}$ | $\frac{\partial}{\partial \lambda}$ for $\Lambda > 0$ |
| Shear free spacetimes | $\frac{\partial}{\partial \tau}$, $\frac{\partial}{\partial \lambda}$ | constant |
| for $\Lambda < 0$  | $\frac{\partial}{\partial \lambda}$ | $\frac{\partial}{\partial \lambda}$ for $\Lambda < 0$ |
| Dust solution      | $\frac{\partial}{\partial \tau}$ | $\frac{\partial}{\partial \lambda}$ |

5.3 Einstein-Maxwell solutions

We have found two solutions describing cosmological models with an electromagnetic field satisfying the Rainich conditions. These are:

**Data solution [26]:**

$$ds^2 = A^{-1}(-dt^2 + A^2 dx^2 + ABdy^2 + ACdz^2)$$ (61)

where:

$$A = c_1 t^\mu + c_2 t^{-\mu}$$

$$AB = t^\lambda$$ and $$AC = t^{2-\lambda}$$

and $c_1, c_2, \mu, \lambda$ are constants with $c_1 c_2 \neq 0$.

**Rosen solution [27]:**

$$ds^2 = \frac{b_1^2 (\tan \frac{1}{2} t)^{2(b_2+b_3)}}{\sin^4 t} dt^2 + \sin^2 t dx^2 + \frac{(\tan \frac{1}{2} t)^{2b_2}}{\sin^2 t} dy^2 + \frac{(\tan \frac{1}{2} t)^{2b_3}}{\sin^2 t} dz^2$$ (62)

where $b_1, b_2, b_3$ are constants and $b_2 b_3 = 1$.

Using the criterion $\Delta_{ab} X_{cd} = 0$ for each of the above spacetimes, we find, after standard but lengthy computations, the results of Table 1.

6 Discussion

In this work we studied the CKVs of proper (that is the LRS case is excluded) Bianchi I spacetimes. We have shown that there are only four families of Bianchi type I spacetimes which admit CKVs. Two of these families concern conformally flat spacetimes and two non-conformally flat spacetimes. The non-conformally flat families, to the best of our knowledge, are new.

One important aspect of these metrics is the symmetry inheritance of the CKVs by the 4-velocity $u^a = \delta_0^a$ of the comoving observers. This property is important because it assures that Lie dragging along the CKVs, fluid flow lines transform onto fluid flow lines thus giving rise to dynamical conservation laws [7,8,10,11,12,13].

The application of the general results of this work to the widely known Bianchi I metrics (52) and (55) has shown that these spacetimes do not belong to the solutions we have found. More specifically the Kasner type spacetimes (52) and (56) admit at most a HV while the Bianchi type I dust solution (55) does not admit even a HV.
The families of Bianchi I metrics we have found contain many anisotropic matter solutions which was not possible to be found before due to the complexity of the conformal equations for Bianchi I spacetimes. It is hoped that these new solutions will have at least equally interesting properties as the classical Bianchi I metrics and will make possible the production of new results mainly at the kinematical level where CKVs play a significant role.

A final remark concerns the Lie and the Noether point symmetries of differential equations. Indeed it has been shown that for a general class of second order partial differential equations the Lie point symmetries are related to the conformal algebra of the underlying geometry \[29\]. This class of equations contains among others the heat equation and the Klein Gordon equation. Therefore one is possible to use the CKVs we have determined and construct conservation laws or to solve explicitly this type of differential equations in the corresponding Bianchi I spacetimes.

Acknowledgements. We would like to thank the anonymous referee for helpful comments which have improved the manuscript. AP acknowledge financial support of INFN.

A Proof of Proposition \[2\]

In this appendix we give the direct and the inverse proof of Proposition \[2\].

**Direct Proof:** First recall the decomposition of the curvature tensor \[1\]:

\[
R_{abcd} = C_{abcd} + \frac{2}{n-2} \left( R_{a[c} R_{b]d} + R_{a[c} R_{b]d} \right) - \frac{R}{(n-1)(n-2)} R_{abcd} \tag{63}
\]

where \(R_{abcd} = g_{ac} R_{bd} - g_{ad} R_{bc}\) and the dimension of space is \(n \geq 4\). Furthermore in a \(1 + (n-1)\) decomposable space holds that \[1\]:

\[
R_{abcd} = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c R_{abcd} = \delta^a_c \delta^b_d R_{ab} = \delta^a_c \delta^b_d R = n-1 R \tag{64}
\]

We consider cases.

**Case 1:** \(n \geq 5\)

Assume the metric \(g_{ab}\) to be conformally flat; then \(C_{abcd} = 0\). Replacing \(R_{abcd}, R_{ab}, R\) in \[63\] and taking into account that \(C_{abcd} = 0\) we find

\[
R_{a[b} \gamma_{c]} = \frac{2}{n-2} \left( g_{[b} R_{c]} + g_{c]} R_{[b} \right) - \frac{n-1}{(n-1)(n-2)} g_{a[b} \gamma_{c]} \tag{65}
\]

where \(g_{a[b} \gamma_{c]}\) is defined similarly to \(g_{abcd}\).

From \[63\] we conclude that \(n^{-1} g_{a[b} \gamma_{c]} = 0\) (because \(n-1 \geq 4\)) therefore the \(n-1\) space is conformally flat. Contracting with \(g^{\alpha\gamma}\) we get:

\[
R_{a[b} \gamma_{c]} = \frac{n-1}{n-1} g_{a[b} \gamma_{c]} \tag{66}
\]

and the \((n-1)\) space is also an Einstein space. We conclude that the \(n-1\) space is a space of constant curvature \[31\].

In order to compute the constant \(p\) we insert \[66\] back to \[63\] and find:

\[
R_{a[b} \gamma_{c]} = \frac{n-1}{(n-1)(n-2)} g_{a[b} \gamma_{c]} \tag{67}
\]

The \(n\) space being conformally flat admits CKVs. According to proposition \[1\] these vectors are found from the gradient CKVs of the \((n-1)\) space of the form \[4\]. Ricci identity for the CKV \(\psi_{\mu}\) gives:

\[
\psi_{[\mu} \rho_{\nu]} - \psi_{\rho [\mu} \rho_{\nu]} = \frac{n-1}{R} \rho_{\alpha \mu} \rho_{\nu} \psi. \tag{68}
\]
Using (67) and (1) in equation (68) we obtain:

\[
\left[ \frac{n-1}{(n-1)(n-2)} \right] R_{ab\beta\delta} \psi^\beta = 0, \tag{69}
\]

from which follows:

\[
\frac{n-1}{R} = -p \frac{(n-1)(n-2)}{(n-1)(n-2)} \tag{70}
\]

and

\[
p = \frac{n-1}{R(n-1)(n-2)}. \tag{71}
\]

Case 2: \( n = 4 \)

In this case relation (63) still applies and (65) becomes:

\[
\frac{1}{3} R_{\gamma\beta\delta} = \left( \epsilon_{\gamma[\alpha} R_{\beta] \delta} + \epsilon_{\delta[\alpha} R_{\beta] \gamma} \right) - \frac{1}{3} R_{\gamma\beta\delta}. \tag{71}
\]

where now the Greek indices take the values 1,2,3. Contracting with \( g_{\alpha\gamma} \) we find

\[
\frac{1}{3} R_{\beta\delta} = \frac{1}{3} R_{\gamma\beta\delta}. \tag{72}
\]

which implies that the 3d space is an Einstein space. Although the 3d space is an Einstein space of curvature \( R = \text{const.} \) we cannot conclude that it is a space of constant curvature before we prove that it is conformally flat. The condition for this is that the Cotton - York tensor

\[
C_{\beta\delta} = 2 \epsilon_{\alpha\gamma\delta} \left( \frac{3}{6} R_{\beta\gamma} - \frac{1}{3} g_{\beta\gamma} R \right)
\]

vanishes \( \text{[31]} \). Replacing \( R_{\beta\delta} \) from (72) we find

\[
C_{\beta\delta} = \frac{1}{6} \epsilon_{\alpha\gamma\delta} g_{\beta\gamma} R_{\beta\delta}. \tag{73}
\]

We replace \( R_{\beta\delta} \) from (72) in (71) and find

\[
\frac{1}{3} R_{\alpha\beta\gamma\delta} = \left( \epsilon_{\gamma[\alpha} R_{\beta] \delta} + \epsilon_{\delta[\alpha} R_{\beta] \gamma} \right) - \frac{1}{3} R_{\gamma\beta\delta} = \frac{1}{3} g_{\alpha\beta\gamma\delta}. \tag{74}
\]

Ricci identity for the gradient CKV \( \psi_\mu \) gives:

\[
\psi_{\mu\nu\sigma} - \psi_{\mu\sigma\nu} = \frac{3}{3} R_{\gamma\nu\mu\delta} \psi^\delta = \frac{1}{3} \epsilon_{\mu\nu\sigma} \psi^\delta \tag{75}
\]

Using (67) and (1) in equation (68) we obtain:

\[
\left[ \frac{1}{3} \frac{R}{6} + p \right] \epsilon_{\alpha\beta\gamma\delta} \psi^\delta = 0 \tag{76}
\]

from which follows \( R_{\beta\delta} = 0 \) hence \( C_{\beta\delta} = 0 \), which completes the proof.

Case 3: \( n = 3 \)

In this case the space \( 3 - 1 = 2 \) is conformally flat and admits gradient CKVs hence the curvature scalar is a constant and the space is a space of constant curvature.

**Inverse Proof:** Suppose the \( (n-1) \) space of the \( 1 + (n-1) \) space \( \text{[2]} \) is a space of constant curvature. Then it is conformally flat and by (63) (64) and (65) the \( 1 + (n-1) \) space is conformally flat.

This completes the proof of proposition [2].
B The conformal algebra of RT and ART spacetimes

The eight proper CKVs of the RT spacetime are

\[ X_{(2)\mu} = a^2 A_4, \quad X_{(3)\tau} = -a^2 A_2, \]

\[ X_{(4)4} = a^2 B_4, \quad X_{(4)\mu} = -a^2 B_2, \]

where \( \mu = t, x, y \) and the corresponding conformal factors are

\[ \psi_{\xi_4} = A_4, \quad \psi_{\xi_{1-4}} = B_4 \]

where the fields \( A_4, B_4 \) are given by the expressions

\[ A_4 = \cos(\frac{\xi}{a}) \left\{ \cos(\frac{\xi}{a}) \left[ \sin(\frac{\xi}{a}), \cos(\frac{\xi}{a}) \right], \sinh(\frac{\xi}{a}) \left[ \sin(\frac{\xi}{a}), \cos(\frac{\xi}{a}) \right] \right\} \]

\[ B_4 = \sin(\frac{\xi}{a}) \left\{ \cos(\frac{\xi}{a}) \left[ \sin(\frac{\xi}{a}), \cos(\frac{\xi}{a}) \right], \sinh(\frac{\xi}{a}) \left[ \sin(\frac{\xi}{a}), \cos(\frac{\xi}{a}) \right] \right\}. \]

The eight proper CKVs of the ART spacetime are

\[ Y_{(2)\mu} = -a^2 \bar{A}_4, \quad Y_{(3)\tau} = a^2 \bar{A}_2, \]

\[ Y_{(4)4} = -a^2 \bar{B}_4, \quad Y_{(4)\mu} = a^2 \bar{B}_2, \]

where \( \mu = t, x, y \) and the corresponding conformal factors are

\[ \psi_{\bar{Y}_4} = \bar{A}_4, \quad \psi_{\bar{Y}_{1-4}} = \bar{B}_4 \]

where the fields \( \bar{A}_4, \bar{B}_4 \) are given by

\[ \bar{A}_4 = \cos(\frac{\xi}{a}) \left\{ \cos(\frac{\xi}{a}) \left[ \sin(\frac{\xi}{a}), \cos(\frac{\xi}{a}) \right], \sinh(\frac{\xi}{a}) \left[ \sin(\frac{\xi}{a}), \cos(\frac{\xi}{a}) \right] \right\} \]

\[ \bar{B}_4 = \sin(\frac{\xi}{a}) \left\{ \cos(\frac{\xi}{a}) \left[ \sin(\frac{\xi}{a}), \cos(\frac{\xi}{a}) \right], \sinh(\frac{\xi}{a}) \left[ \sin(\frac{\xi}{a}), \cos(\frac{\xi}{a}) \right] \right\}. \]

For easy reference in tables (6) and (8), we give the explicit form of the CKVs for the RT spacetime and the ART spacetime respectively.

Furthermore, the RT spacetime (19) admits a seven dimensional Killing algebra, the three vector fields are the CKVs \( \delta_3, \delta_4, \delta_5 \) and the four extra CKVs are

\[ \xi_{4,RT} = \sinh(\frac{\xi}{a}) \cos(\frac{\xi}{a}) \delta_2 - \cot(\frac{\xi}{a}) \sinh(\frac{\xi}{a}) \cos(\frac{\xi}{a}) \delta_3 + \tan(\frac{\xi}{a}) \sin(\frac{\xi}{a}) \cos(\frac{\xi}{a}) \delta_5, \]

\[ \xi_{5,RT} = \sinh(\frac{\xi}{a}) \sinh(\frac{\xi}{a}) \delta_2 - \cot(\frac{\xi}{a}) \sin(\frac{\xi}{a}) \sinh(\frac{\xi}{a}) \delta_3 + \tan(\frac{\xi}{a}) \sin(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_5, \]

\[ \xi_{6,RT} = \cos(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_2 - \coth(\frac{\xi}{a}) \sinh(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_3 + \tanh(\frac{\xi}{a}) \sin(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_5, \]

\[ \xi_{7,RT} = \cosh(\frac{\xi}{a}) \sinh(\frac{\xi}{a}) \delta_2 - \sinh(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_3 + \tanh(\frac{\xi}{a}) \sin(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_5. \]

Similarly for the ART spacetime (20), the four extra CKVs are

\[ \xi_{4,ART} = \sinh(\frac{\xi}{a}) \cos(\frac{\xi}{a}) \delta_2 - \coth(\frac{\xi}{a}) \sinh(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_3 + \tanh(\frac{\xi}{a}) \sin(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_5, \]

\[ \xi_{5,ART} = \sinh(\frac{\xi}{a}) \sin(\frac{\xi}{a}) \delta_2 - \coth(\frac{\xi}{a}) \sinh(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_3 + \tanh(\frac{\xi}{a}) \sin(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_5, \]

\[ \xi_{6,ART} = \cos(\frac{\xi}{a}) \cos(\frac{\xi}{a}) \delta_2 - \cos(\frac{\xi}{a}) \sinh(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_3 + \tanh(\frac{\xi}{a}) \sin(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_5, \]

\[ \xi_{7,ART} = \cos(\frac{\xi}{a}) \sinh(\frac{\xi}{a}) \delta_2 - \cosh(\frac{\xi}{a}) \sinh(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_3 + \tanh(\frac{\xi}{a}) \sin(\frac{\xi}{a}) \cosh(\frac{\xi}{a}) \delta_5. \]
Table 2 Proper CKVs of the RT spacetime (19)

| X1 | X2 | X3 | X4 | Conformal factor ψ |
|----|----|----|----|---------------------|
| a sin(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | 0 | a sinh(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | -a cos(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | cos(\(\frac{\tau}{a}\)) cosh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) |
| a sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | 0 | a sin(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | a cos(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | cos(\(\frac{\tau}{a}\)) cosh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) |
| a sin(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | 0 | a sinh(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | -a cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) |
| a sin(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | 0 | a sinh(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | a cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) |
| a cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | 0 | a sinh(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | -a cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) |
| a cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | 0 | a sinh(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | a cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) |

Table 3 The proper CKVs of the ART spacetime (20)

| X1 | X2 | X3 | X4 | Conformal factor ψ |
|----|----|----|----|---------------------|
| a sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) | 0 | a sinh(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | acosh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | a sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) |
| a sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | 0 | a sinh(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | acosh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | a sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) |
| a sin(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | 0 | a sinh(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | acosh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | a sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) |
| a sin(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | 0 | a sinh(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | acosh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | a sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) |
| a cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | 0 | a sinh(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | acosh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | a sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) |
| a cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | 0 | a sinh(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) | acosh(\(\frac{\tau}{a}\)) sin(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) | a sinh(\(\frac{\tau}{a}\)) cos(\(\frac{\tau}{a}\)) sinh(\(\frac{\tau}{a}\)) |

References
1. D. Kramer, H. Stephani, M. MacCallum and E. Herlt, Exact Solutions of Einstein's Field Equations (Cambridge, Cambridge University Press, 1980)
2. R.M. Wald, General Relativity (Chicago University Press, 1984)
3. K.C. Jacobs, Astrophys J 151 (1968) 431; Astrophys J 153 (1968) 661
4. P. Hoyle and J.V. Narlikar, Proc R Soc A273 (1963) 1; C.W. Misner, Ap. J. 151 (1968) 431
5. A.Z. Petrov, Einstein Spaces Pergamon, (Oxford University Press, 1969)
6. M. Tsamparlis, Class.Quantum Grav 15 (1998) 2901
7. R. Maartens, D.P. Mason and M. Tsamparlis, J. Math. Phys. 27 (1986) 2987
8. D.P. Mason and R.J Maartens, Math. Phys. 28 (1986) 2511
9. R. Maartens, S.D. Maharaj and B.O.J. Tupper Class.Quantum Grav 12(1995) 2577; R. Maartens, S.D. Maharaj and B.O.J. Tupper Class.Quantum Grav 13 (1996) 317
10. A.A. Coley and B.O.J. Tupper, Class.Quantum Grav 7 (1990) 1961
11. A.A. Coley and B.O.J. Tupper, Class.Quantum Grav 7 (1990) 2195
12. A.A. Coley and B.O.J. Tupper, Math. Phys. 33 (1992)1754
13. A.A. Coley and B.O.J. Tupper, Class.Quantum Grav 11 (1994) 2553
14. L. Herrera, J. Ponce de Leon J.Math.Phys 26 (1985) 2332; L. Herrera, J. Ponce de Leon J.Math.Phys 26 (1985) 2018
15. P.S. Apostolopoulos and M. Tsamparlis, Class. Quantum Grav. 18 (2001) 3775-3790
16. M.J.Rebouças and J. Tiomno, Phys. Rev. D 28 (1983) 1251
17. M.J. Rebouças and A.F.F. Teixeira, J.Math.Phys 33(1992) 2855
18. L. Defrise-Carter, Commun.Math.Phys. 40 (1975) 273
19. G.S. Hall and J.D. Steele, J.Math.Phys 1991 32 (1991) 1847
20. R. Maartens and C. M. Mellin Class.Quantum Grav 13 (1996) 1571.
21. C.B.G. McIntosh and J.D. Steele, Class. Quantum Grav. 8 (1991) 1173
22. S.W. Hawking and G.F.R. Ellis The large scale structure of space-time (Cambridge University Press, Cambridge 1973)
23. M. Tsamparlis, D. Nikolopoulos and P. S. Apostolopoulos, Class. Quantum Grav. 15 (1998) 2909
24. Y. Choquet-Bruhat, C. DeWitt-Morette and M. Dillard-Bleick Analysis, Manifolds and Physics (Amsterdam: North Holland 1977)
25. G.F.R. Ellis in Cargese Lectures in Physics, Vol 6, edited by E.Schatzman (Gordon and Breach, New York, 1971)
26. B.K. Datta, Nuovo Cimento 36 (1965) 109
27. G.J. Rosen, Math Phys 3 (1962) 313
28. M. Tsamparlis and A. Paliathanasis, Gen. Relativ. Gravit. 42 (2010) 2957
29. A. Paliathanasis and M. Tsamparlis, Int. J. Geom. Methods Mod. Phys. 11 (2014) 1450037
30. M. Tsamparlis, A. Paliathanasis and A. Qadir, Int. J. Geom. Methods Mod. Phys. 15 (2015) 1550003
31. L.P. Eisenhart Riemannian Geometry (Princeton University Press, Princeton 1964)