The Post-Newtonian Limit of Dilaton Gravity

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ABSTRACT

We study the post-Newtonian limit of a generalized dilaton gravity in which gravity is coupled to dilaton and electromagnetic fields. The field equations are derived using the post-Newtonian scheme, and the approximate solution is presented for a point mass with electric and dilaton charges. The result indicates that the dilaton effect can be detected, in post-Newtonian level, using a charged test particle but not a neutral one. We have also checked that the approximate solution is indeed consistent with the weak field expansion of charged dilaton black hole solution in the harmonic coordinate.

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1 Introduction

It is of interest to investigate how the properties of black holes are modified when the low-energy effective actions of superstring theories \([1] - [5]\) are considered. Some new black hole solutions have been obtained in the low-energy string theories in which the Kalb-Ramond field, dilaton field and gauge field are incorporated with gravity \([6] - [32]\). Above all, when non-minimally coupled dilaton and \(U(1)\) gauge field are included, the effective theory is called dilaton gravity. The dilaton gravity theory has been studied extensively in recent years due to many interesting properties in this theory. The non-minimal coupling allows to violate the no-hair theorem which offers the uniqueness of the black hole solutions of the Einstein theory. In the extremal limit, the black hole is on the edge of becoming naked singularity and should be regarded as elementary particles. \([25] [26]\). Moreover, a very useful property of extremal black hole solution is that the metric split into a direct product of 2-d solution with a 2-sphere of constant radius. That naturally leads to a reduced 2-dimensional model, called CGHS model \([27]\). Studying a 2-d dilaton gravity instead of a 4-d theory, makes the problem more tractable. In fact, many works have devoted to studying the evaporation and information puzzle of the 4-dimensional black hole, by investigating CGHS mode \([28]\).

However, gravitational theory is a experimental tested theory, such as general relativity. Recently, Damour and Polyakov \([33]\) studied the violation of the equivalence principle in a string inspired model-dilaton gravity.
They found that the violation is well below the present experimental limit. It means that the high precision tests of the equivalence principle can be viewed as low-energy window on string-scale physics. That provides a new motivation for improving the experimental tests of Einstein’s Equivalence Principle.

It is interested to ask whether the dilaton gravity will imply some new observable effects in the post-Newtonian limit. These studies will improve our understanding in effective string theories. In this report, we will derive the field equation of a generalized dilaton gravity, in which gravity, electromagnetic field and dilaton field are coupled to their source terms individually in the post-Newtonian limit. We also demonstrate an approximate solution for a point mass with electric charge and dilaton charge, and compare it with the weak field expansion of a static, charged dilatonic black hole in the harmonic coordinate.

The plan of this paper is as follows. In Sec. 2, we review the black hole solution in the dilaton gravity. In Sec. 3, field equations of the modified dilaton gravity are developed in the post-Newtonian scheme, and an approximate solution of a point mass with both electric and dilaton charges was demonstrated. We rewrite the electric dilatonic black hole in the harmonic coordinate and compare it with the approximate solution in Sec. 4. Finally, we present some concluding remarks.
2 Charged dilatonic black hole

The dilaton gravity is described by the action,

$$ I = \int d^4x \sqrt{-g} \left[ R - 2(\nabla \phi)^2 - e^{-2\phi} F^2 \right] . $$

(1)

Gibbons and Maede\cite{13} obtained the dyonic black hole solution for the theory in terms of the non-standard metric form. One of us\cite{30} found the same solution in terms of the standard metric form,

$$ ds^2 = -\Delta^2 dt^2 + \frac{\sigma^2}{\Delta^2} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) , $$

(2)

where

$$ \Delta^2 = 1 - \frac{2M}{r^2} \sqrt{r^2 + \rho^2 + \beta r^2} , $$

(3)

$$ \sigma^2 = \frac{r^2}{r^2 + \rho^2} , $$

(4)

and

$$ \rho = \frac{1}{2M} (Q_e^2 e^{2\phi} - Q_m^2 e^{-2\phi}) , $$

(5)

$$ \beta = (Q_e^2 e^{2\phi} + Q_m^2 e^{-2\phi}) . $$

(6)

The electric and magnetic fields are

$$ F_{01} = \frac{Q_e}{r^2} e^{2\phi} , $$

(7)

$$ F_{23} = \frac{Q_m}{r^2} . $$

(8)

and the dilaton field obeys

$$ e^{2\phi} = e^{2\phi_0} \left( 1 - \frac{2\rho}{\sqrt{r^2 + \rho^2 + \rho}} \right) . $$

(9)
The solution is characterized by mass $M$, electric charge $Q_e$, magnetic charge $Q_m$ and asymptotic value of the dilaton $\phi_0$. It is obvious that the dual transformation can be represented by changing $(Q_e, Q_m, \phi)$ to $(Q_m, Q_e, -\phi)$ in the solution.

In this form, the properties of black hole are much easier to interpret and to be compared with those of Reissner-Nordström black holes. The structure and thermodynamical properties of dyonic dilaton black hole are similar to those of the conventional charged black hole, except for the pure electric or the pure magnetic cases. For the extremal charged black hole,

$$M = \frac{1}{\sqrt{2}}(|Q_e|e^{\phi_0} + |Q_m|e^{-\phi_0}),$$

the thermodynamical description is inappropriate\cite{15, 29, 30}. When we set $Q_m = 0$ or $Q_e = 0$, the solution is reduced to the pure electric or pure magnetic solutions. Garfinkle et al.\cite{14} also found these solutions in terms of non-standard metric form. The description of the pure electric or pure magnetic black holes as thermal objects break down as the extreme limit is approached. Preskill et al.\cite{25} and Holzhey et al.\cite{26} suggest that these extreme charged solutions should be regarded as elementary particles.

Although, the exact spherically symmetric solutions are found in the massless dilaton gravity. The exact axially symmetric solution is not found, only the approximate solution was presented\cite{21}. And they only found a numerical solution for the massive dilaton black hole\cite{19, 20}. Therefore,
beside to understand the experimental tested effect in dilaton gravity, the
dpost-Newtonian scheme also offers a systematic way to find all kinds of ap-
proximate solutions.

3 The post-Newtonian scheme

Recently, the post-Newtonian limit of the superstring effective action was
discussed by Kaehagias [35]. He studied the post-Newtonian limit and gravi-
tational radiation of the effective action in which gravity is coupled to dilaton
field and antisymmetric tensor field called axion. He found that the theory
is identical to Einstein gravity in the post-Newtonian approximation. Also,
he predicted all possible types of radiation in the weak field approximation.
There exist monopole, dipole, quadrupole, etc. contributions in the radiation
luminosity.

Here, we consider a generalized action of dilaton gravity in which gravity
is coupled to dilaton, electromagnetic field and source terms.

\[
I = \int d^4 x \sqrt{-g} \left\{ \frac{R}{2\kappa^2} - 2(\nabla \phi)^2 - b e^{-2a\phi} F^2 \right\} + \mathcal{L}_{\text{matter}}(g_{\mu\nu}, A_{\mu}, \phi), \quad (11)
\]

with constant parameters \( \kappa^2 = 8\pi G, a = \sqrt{2}\kappa \) and \( b = \frac{\alpha'}{16\pi} \). \( G \) is the
gravitational constant and \( \alpha' \) is the fundamental constant in string theory.
When \( b = \frac{1}{8} \), the action is the effective theory of four-dimensional heterotic
string. When \( a = 0 \), the action reduces to the usual Einstein-Maxwell scalar
theory. The dilaton gravity is the special case when $2\kappa^2 = 1, b = 1$ and $a = 1$. In this paper, we will use the geometrized unit, that is $G = 1$ and $c = 1$, and we adapt the convention of MTW [36].

The field equations of the effective string action, eq.(11), are
\[
R_{\mu\nu} = 2\kappa^2 [2\nabla_\mu \phi \nabla_\nu \phi + 2b e^{-2a\phi} (F_{\mu\nu} F_{\mu\nu} - \frac{1}{4} g_{\mu\nu} F^2) + \kappa^2 (T^m_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^m)] ,
\]
and
\[
\nabla_\mu (e^{-2a\phi} F^{\mu\nu}) = -4\pi J^\nu ,
\]
where $T^m_{\mu\nu} = \frac{1}{2} \sqrt{-g} \frac{\partial}{\partial g_{\mu\nu}} (\sqrt{-g} L_{\text{matter}})$, $J^\nu = \frac{1}{16\pi b} \frac{\partial}{\partial A_\mu} (L_{\text{matter}})$ and $\Sigma = \frac{1}{{16\pi}} \frac{\partial}{\partial \phi} (L_{\text{matter}})$ are source terms of gravity, electromagnetic and dilaton field.

The post-Newtonian limit of the theory is determination of the metric tensor to order $O(v^4)$ for $g_{00}$, $O(v^3)$ for $g_{0i}$ and $O(v^2)$ for $g_{ij}$ [37]. Thus the metric can be written as:
\[
\begin{align*}
g_{00} &= -1 + (2) g_{00} + (4) g_{00} + ..., \\
g_{ij} &= \delta_{ij} + (2) g_{ij} + ..., \\
g_{0i} &= (3) g_{0i} + ...
\end{align*}
\]
where $(n)g_{\mu\nu}$ is of order $v^n$, and the Latin indices denote the space components which is running from 1 to 3. The inverse metric has also an expansion which
may be written as

\[ g^{00} = -1 + (2)^0 g^{00} + (4)^0 g^{00} + ..., \]
\[ g^{ij} = \delta^{ij} + (2)^i g^{ij} + ..., \]
\[ g^{0i} = (3)^0 g^{0i} + ..., \] (16)

with

\[ (2)^0 g^{00} = -(2)^0 g_{00}, \]
\[ (2)^i g^{ij} = -(2)^{ij} g_{ij}, \]
\[ (3)^0 g^{0i} = (3)^0 g_{0i}. \]

It is well known that we can choose the harmonic coordinate system which the metric satisfies the harmonic condition [37],

\[ g^{\mu\nu} \Gamma^k_{\mu\nu} = 0. \] (17)

Under these conditions the Ricci tensor were simplified to

\[ (2)^0 R_{00} = -\frac{1}{2} \nabla^2 (2) g_{00}, \] (18)
\[ (2)^0 R_{ij} = -\frac{1}{2} \nabla^2 (2) g_{ij}, \] (19)
\[ (3)^0 R_{0i} = -\frac{1}{2} \nabla^2 (3) g_{0i}, \] (20)
\[ (4)^0 R_{00} = \frac{1}{2} \partial_i (2) g_{00} - \frac{1}{2} \nabla^2 (4) g_{00} - \frac{1}{2} \delta^{ij} \partial_i (2) g_{00} \partial_j (2) g_{00} 
+ \frac{1}{2} (2)^i g_{ij} \partial_i \partial_j (2) g_{00}. \] (21)

Now, we assume the expansion for the dilaton is given by

\[ \phi = (0)^0 \phi + (2)^0 \phi + (4)^0 \phi + ... . \] (22)
should be the asymptotical value $\phi_0$ of the dilaton field. And the expansion for the electromagnetic field is given by

$$F_{oi} = (2) F_{oi} + (4) F_{oi} + \ldots,$$

$$F_{ij} = (3) F_{ij} + \ldots. \quad (23)$$

It means that, when the Lorentz gauge, $\partial_\mu A^\mu = 0$, was taken, the electric potential $A_0$ and vector potential $A_i$ have the expansion

$$A_0 = (2) A_0 + (4) A_0 + \ldots,$$

$$A_i = (3) A_i + \ldots. \quad (24)$$

The reason to assign those orders of the field expansion is that we can easily control the orders of fields.

We may also expand the energy momentum $T_{\mu \nu}^m$, current $J^\mu$ and dilaton source $\Sigma$ in powers of $v$

$$T_{m}^{00} = (0) T_{m}^{00} + (2) T_{m}^{00} + \ldots,$$

$$T_{m}^{0i} = (1) T_{m}^{0i} + \ldots,$$

$$T_{m}^{ij} = (2) T_{m}^{ij} + \ldots, \quad (25)$$

$$J_0 = (0) J_0 + (2) J_0 + \ldots,$$

$$J_i = (1) J_i + \ldots, \quad (26)$$

$$\Sigma = (0) \Sigma + (2) \Sigma + \ldots. \quad (27)$$
Plugging all the expansions into the field equations, and comparing both side of the equations order by order, we obtain a set of Poisson equations in the post-Newtonian limit,

\begin{align*}
\nabla^2 (2) g_{00} &= -\kappa^2 (0) T^0_m, \\
\nabla^2 (2) g_{ij} &= -\kappa^2 \delta_{ij} T^0_m, \\
\nabla^2 (2) g_{0i} &= \kappa^2 (1) T^0_i, \\
\nabla^2 (4) g_{00} &= -\kappa^2 (2) T^0_m - (2) T^{ii} - 2 (2) g^{(0)}_{00} T^0_m + 4 b e^{-2 a \phi_0} F_{oi} (2) F_{oi} \\
&\quad + \partial_0^2 (2) g_{00} - \partial_i^2 (2) g_{00} \partial_i^2 g_{00} + (2) g_{ij} \partial_i \partial_j^2 g_{00}, \\
\nabla^2 (2) \phi &= -4 \pi (0) \Sigma, \\
\nabla^2 (4) \phi &= -4 \pi (2) \Sigma + \partial_0^2 (2) \phi \\
&\quad + (2) g_{ij} \partial_i \partial_j^2 \phi + a b e^{-2 a \phi_0} (2) F_{oi} (2) F_{oi}, \\
\nabla^2 (2) A_0 &= -4 \pi e^{-2 a \phi_0} (0) J_0, \\
\nabla^2 (3) A_i &= -4 \pi e^{-2 a \phi_0} (1) J_i, \\
\nabla^2 (4) A_0 &= -4 \pi e^{-2 a \phi_0} (2) J_0 + \partial_0^2 (2) A_0 + 2 a (2) \phi \nabla^2 (2) A_0 \\
&\quad - 2 a \partial_0^2 (2) \phi (2) F_{oi} + \partial_i^2 (2) g_{00} (2) F_{oi}.
\end{align*}

In order to avoid some obscure energy density terms in $O(v^4)$ equations which will result in divergences after the integration. The metric, dilaton and electromagnetic fields are usually rewritten as

\begin{align*}
(2) g_{00} &= -2 U, \\
(2) g_{ij} &= -2 U \delta_{ij}.
\end{align*}
\( (2) \phi = \Theta, \quad \) (39)  

\( (2) A_0 = \Phi, \quad \) (40)  

\( (3) g_{0i} = \xi_i, \quad \) (41)  

\( (3) A_i = \zeta_i, \quad \) (42)  

\( (4) g_{00} = -2\Psi - 2U^2 - 2b\kappa^2 e^{-2\alpha\phi_0} \Phi^2, \quad \) (43)  

\( (4) \phi = \Xi + \frac{1}{2} ab e^{-2\alpha\phi_0} \Phi^2, \quad \) (44)  

\( (4) A_0 = \Upsilon + a\Theta\Phi + U\Phi. \quad \) (45)  

After replacing eqs (37)-(45) into eqs. (28)-(36), we end up with a set of simpler field equations,

\[
\nabla^2 U = \frac{\kappa^2}{2} T_{m}^{00},
\]

\( (46) \)

\[
\nabla^2 U \delta_{ij} = \frac{\kappa^2}{2} \delta_{ij} T_{m}^{00},
\]

\( (47) \)

\[
\nabla^2 \xi_i = -\kappa^2 T_{m}^{0i},
\]

\( (48) \)

\[
\nabla^2 \psi = -\frac{\kappa^2}{2} T_{m}^{00} + (2) T_{m}^{ii} + 16\pi be^{-4\alpha\phi_0} \Phi^{(0)} J_0 + \frac{2}{\kappa^2} \partial^2 U, \]

\( (49) \)

\[
\nabla^2 \Theta = -4\pi \Sigma,
\]

\( (50) \)

\[
\nabla^2 \Xi = -4\pi (2) \Sigma - 2U^{(0)} \Sigma + ab e^{-2\alpha\phi_0} \Phi^{(0)} J_0 - \frac{1}{4\pi} \partial^2 \Theta, \]

\( (51) \)

\[
\nabla^2 \Phi = -4\pi e^{-2\alpha\phi_0} (0) J_0, \]

\( (52) \)

\[
\nabla^2 \zeta_i = -4\pi e^{-2\alpha\phi_0} (1) J_i, \]

\( (53) \)

\[
\nabla^2 \Upsilon = -4\pi (e^{-2\alpha\phi_0} (2) J_0 + a e^{-2\alpha\phi_0} \Theta^{(0)} J_0 - e^{-2\alpha\phi_0} U^{(0)} J_0 + a \Phi^{(0)} \Sigma - \frac{\kappa^2}{8\pi} \Phi^{(0)} T_{m}^{00} - \frac{1}{4\pi} \partial^2 \Phi). \]

\( (54) \)
Integrating equations (46)-(54), we find out the solutions for this system:

\[
U(x, t) = -\frac{\kappa^2}{8\pi} \int \frac{d^3x'}{|x - x'|} (0) T_{m}^{00}(x', t),
\]

\[
\xi_i(x, t) = -\frac{\kappa^2}{4\pi} \int \frac{d^3x'}{|x - x'|} (1) T_{m}^{0i}(x', t),
\]

\[
\Psi(x, t) = -\frac{\kappa^2}{8\pi} \int \frac{d^3x'}{|x - x'|} [(2) T_{m}^{00}(x', t) + (2) T_{m}^{ii}(x', t) \]
\[+ 16\pi be^{-4a\phi_0} \Phi(x', t)(0) J_0(x', t) + \frac{2}{\kappa^2} \partial_0^2 U(x', t)],
\]

\[
\Theta(x, t) = \int \frac{d^3x'}{|x - x'|} (0) \Sigma(x', t),
\]

\[
\Xi(x, t) = \int \frac{d^3x'}{|x - x'|} [(2) \Sigma(x', t) - 2U(x', t)(0) \Sigma(x', t) \]
\[+ abe^{-2a\phi_0} \Phi(x', t)(0) J_0(x', t) - \frac{1}{4\pi} \partial_0^2 \Theta(x', t)],
\]

\[
\Phi(x, t) = \int \frac{d^3x'}{|x - x'|} e^{2a\phi_0(0)} J_0(x', t),
\]

\[
\zeta_i(x, t) = \int \frac{d^3x'}{|x - x'|} e^{2a\phi_0(1)} J_i(x', t),
\]

\[
\Upsilon(x, t) = \int \frac{d^3x'}{|x - x'|} [e^{-2a\phi_0(2)} J_0(x', t) + ae^{-2a\phi_0} \Theta(x', t)(0) J_0(x', t) \]
\[+ e^{-2a\phi_0} U(x', t)(0) J_0(x', t) + a\Phi(x', t)(0) \Sigma(x', t) \]
\[+ \frac{\kappa^2}{8\pi} \Phi(x', t)(0) T_{m}^{00}(x', t) - \frac{1}{4\pi} \partial_0^2 \Phi(x', t)].
\]

Here, when the source distributions are given, all the post-Newtonian expansion of metric, dilaton and electric fields can be carried out by straightforward integrations.
Let us illustrate this scheme in a simple example, the approximate solution for a point mass $\mathcal{M}$ which carries electric $Q$ and dilaton charges $D$. For this case, the source terms are:

\begin{align*}
^{(0)}T_{m}^{00} &= M\delta^{3}(x), \\
^{(0)}\Sigma &= -D\delta^{3}(x), \\
^{(0)}J_{0} &= Q\delta^{3}(x), \\
^{(2)}T_{m}^{00} &= ^{(1)}T_{m}^{0i} = ^{(2)}T_{m}^{ij} = 0, \quad ^{(2)}J_{0} = ^{(1)}J_{i} = 0, \quad ^{(2)}\Sigma = 0.
\end{align*}

Putting into the integral, finally, we get the results:

\begin{align*}
U(x) &= -\frac{M}{|x|}, \\
\xi_{i}(x) &= 0, \\
\Psi(x) &= 0, \\
\Theta(x) &= -\frac{D}{|x|}, \\
\Xi(x) &= 0, \\
\Phi(x) &= \frac{Qe^{2\alpha\phi_{0}}}{|x|}, \\
\zeta_{i}(x) &= 0, \\
\Upsilon(x) &= 0,
\end{align*}

where $M = \frac{e^{2}}{8\pi} \mathcal{M}$.

(64)
Therefore the post-Newtonian expansion of metric, dilaton and electric field strengths are

\[
\begin{align*}
g_{00} &= -\left\{ 1 - \frac{2M}{|x|} + \frac{1}{|x|^2} (2M^2 + 2\kappa^2 b Q^2 e^{2\phi_0}) + \ldots \right\} , \\
g_{ij} &= \delta_{ij} + \frac{2M}{|x|} \delta_{ij}, \\
g_{0i} &= 0. \\
F_{0i} &= \frac{x_i}{|x|^3} Q e^{2\phi_0} - \frac{x_i}{|x|^4} Q e^{2\alpha \phi} (2M + 2\alpha D) + \ldots , \\
\phi &= \phi_0 - \frac{D}{|x|} + \frac{ab Q^2 e^{2\phi_0}}{2 |x|^2} + \ldots .
\end{align*}
\]

Besides the asymptotical background, the lowest order of potential expansions, \( (2) g_{00}, (2) A_0 \) and \( (2) \phi \), are order of \( v^2 \). It means that all the charges \( Q, D \) and mass \( M \) are same order as \( \phi_0 \approx 0 \) in this example.

4 An electric dilatonic black hole in the harmonic coordinate

In order to make sure those results presented in the last section are correct. We will compare the approximate solution of a point mass which carries electric and dilaton charges to the pure, electric dilatonic black hole solution of dilaton gravity.

From the exact solution, eqs (2)-(9), we know that the electric dilatonic
black hole is also characterized by a dilaton charge \[14\]

\[
D = \frac{1}{4\pi} \oint_S d^2S^\mu \nabla_\mu \phi = \frac{Q^2 e^{2\phi_0}}{2M} . 
\]

(68)

Due to the dilaton source is free in the dilaton gravity, eq(1), \( D \) is not a new free parameter in this case. Once \( \phi_0 \) is fixed, it is determined by \( M \) and \( Q \).

Here we have set \( Q = Q_e \).

For convenience, we rewrite the charged black hole solution, eq.(2), in terms of dilaton charge, electric charge and mass, as followings:

\[
ds^2 = -\Delta^2 dt^2 + \frac{\sigma^2}{\Delta^2} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) ,
\]

(69)

where

\[
\Delta^2 = 1 - \frac{2M}{r^2} \sqrt{r^2 + D^2} + \frac{Q^2 e^{2\phi_0}}{r^2} ,
\]

(70)

\[
\sigma^2 = \frac{r^2}{r^2 + D^2} ,
\]

(71)

The electric field is

\[
F_{01} = \frac{Q}{r^2} e^{2\phi} ,
\]

(72)

and the dilaton field obeys

\[
e^{2\phi} = e^{2\phi_0} \left( 1 - \frac{2D}{\sqrt{r^2 + D^2} + D} \right) .
\]

(73)

Moreover, we change the standard spherical coordinates \((t, r, \theta, \varphi)\) to harmonic coordinates \((t, x_i)\), which satisfy the harmonic condition \[37\],

\[
g^{\mu\nu} \nabla_\mu \nabla_\nu x_i = 0 .
\]

(74)

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If we choose the transformations between the harmonic coordinates and the standard coordinates to be

\[ x_1 = R(r)\sin\theta\cos\varphi, \]
\[ x_2 = R(r)\sin\theta\sin\varphi, \]
\[ x_3 = R(r)\cos\varphi, \] (75)

then the transformation function \( R(r) = |x| \) is determined by

\[ \frac{d}{dr} \left( r^2 \Delta^2 \frac{dR}{dr} \right) - 2\sigma R = 0. \] (76)

In the far region, the approximate solution of \( R(r) \) is

\[ R(r) = r\left(1 - \frac{M}{r} + \frac{1}{2} \frac{D^2}{r^2} + \ldots \right). \] (77)

After some complicate calculation, we get the weak field expansions of the metric in the harmonic coordinate,

\[ g_{00} = -\{1 - \frac{2M}{|x|} + \frac{1}{|x|^2}(2M^2 + Q^2 e^{2\phi_0}) + \ldots \} \]
\[ g_{ij} = \delta_{ij} + \frac{2M}{|x|} \delta_{ij}, \]
\[ g_{0i} = 0. \] (78)

The electric and dilaton field expansion are

\[ F_{0i} = \frac{x_i}{|x|^3} Q e^{2\phi_0} - 2 \frac{x_i}{|x|^4} Q e^{2\phi_0} (M + D) + \ldots, \] (79)
\[ \phi = \phi_0 - \frac{D}{|x|} + \frac{1}{2} \frac{Q^2 e^{2\phi_0}}{|x|^2} + \ldots. \] (80)
Obviously, these expansions are consistent with the approximate solution of
dilaton gravity, \( (2\kappa^2 = a = b = 1) \), which was found in the post-Newtonian
scheme.

5 Concluding remarks

we derived the field equation of the generalized dilaton gravity in the post-
Newtonian limit. We also demonstrated an approximate solution for a point
mass with electric charge and dilaton charge, and compare it with the weak
field expansion of a charged, dilatonic black hole in the harmonic coordinate.

According to eqs(43),(57) or (65), the dilaton charge does not give any
contribution to metric up to post-Newtonian terms, \(^{(4)}g_{00}\). Therefore, a neu-
tral test particle, which obey the equation of motion

\[
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0, \tag{81}
\]

can not detect the dilaton effect in the post-Newtonian level. But, the dilat-
on charge gives post-Newtonian contribution to the electric field, \(^{(4)}F_{0i}\), see
eqs (45),(60),(62) or (66). Whereas a charged test particle, which obey the
equation of motion

\[
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = \frac{e}{m} F^\mu_\nu \frac{dx^\nu}{d\tau}, \tag{82}
\]

can detect the effects of the dilaton charge.
However, a charged particles test is hard to be assembled [34][36], we still need to calculate the post-post-Newtonian terms for the neutral particle test. The post-post-Newtonian limit is under our current investigation. We are also interesting in studying the post-Newtonian approximate solutions for the more general dilaton gravity in which a mass term $m\phi^2$ or potential term $V(\phi)$ are included.

**Acknowledgements**

We thank Prof. W.-T. Ni, Prof C.-C. Chen and Prof.T.-H. Cho for useful discussion. This work was supported in part by the National Science Council of the Republic of China under grants No. NSC 84-2112-M006-010, No. NSC 85-2112-M006-005 and No. NSC 84-2112-M194-003.
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