As a unifying framework for examining several properties that nominally involve eigenvalues, we present a particular structure of the singular values of the Gaussian orthogonal ensemble (GOE): the even-location singular values are distributed as the positive eigenvalues of a Gaussian ensemble with chiral unitary symmetry, while the odd-location singular values, conditioned on the even-location ones, can be algebraically transformed into a set of independent $\chi$-distributed random variables. We discuss three applications of this structure: first, there is a pair of bidiagonal square matrices, whose singular values are jointly distributed as the even- and odd-location ones of the GOE; second, the magnitude of the determinant of the GOE is distributed as a product of simple independent random variables; third, on symmetric intervals, the gap probabilities of the GOE can be expressed in terms of the Laguerre unitary ensemble. We work specifically with matrices of finite order, but by passing to a large matrix limit, we also obtain new insight into asymptotic properties such as the central limit theorem of the determinant or the gap probabilities in the bulk-scaling limit. The analysis in this paper avoids much of the technical machinery (e.g. Pfaffians, skew-orthogonal polynomials, martingales, Meijer $G$-function, etc.) that was previously used to analyze some of the applications.

Keywords: random matrices, GOE, anti-GUE, LUE, singular values, determinants, gap probabilities

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1. Introduction

This paper studies the structure of the singular values of the Gaussian orthogonal ensemble (GOE), using it as a unifying framework for examining several properties that nominally involve eigenvalues. Here, the GOE$_n$ of order $n$ is the ensemble of real symmetric random matrices

$$G = (X + X')/2,$$

where $X$ is an $n \times n$ Gaussian matrix with all entries independent standard normals. Since the singular values of symmetric matrices are the magnitudes of the eigenvalues,
the ensemble of singular values will be briefly denoted by $|\text{GOE}_n|$. Central to our
discussion is the immediate set decomposition,

$$|\text{GOE}_n| = \text{even}\,|\text{GOE}_n| \cup \text{odd}\,|\text{GOE}_n|, \quad (1.1)$$

of the ordered singular values according to the parity of their indices, where the
even-location decimated ensemble even $|\text{GOE}_n|$ is defined by taking the 2nd largest,
4th largest, etc. singular value, and similarly for odd $|\text{GOE}_n|$.

Our first set of main results relates the decomposition (1.1) to the eigenvalues
of a Gaussian ensemble with chiral, or anti-symmetric, unitary symmetry. Namely,
with $X$ as above, the ensemble of real skew-symmetric random matrices

$$A = (X - X')/2$$

will be called the anti-GUE with its (almost surely) different and positive singular
values written briefly as aGUE$_n$ (if $n$ is odd, there is a surplus singular value zero,
which is omitted).\(^a\)

Then, the following structure holds.

**Theorem 1.1.** Denoting equality of the joint distribution by $\overset{d}{=}$, there holds

$$\text{even}\,|\text{GOE}_n| \overset{d}{=} \text{aGUE}_n. \quad (1.2)$$

We will give two proofs that differ in their handling of the odd-location singular values:
one (Section 4) by algebraically transforming them to a set of independent
random variables, each distributed as $\chi^2_2$ and, if $n$ is odd, a surplus $\chi^2_1$; the other (Section 7) by
integrating them out. Both proofs are based on an algebraic factorization (Section 2)
of the joint density of $|\text{GOE}_n|$, where one factor depends only on the even-location
singular values, the other on the odd-location ones. If we recall the superposition
representation, see [8, Eq. (2.6)] or [7, Thm. 1],

$$|\text{GUE}_n| \overset{d}{=} \text{aGUE}_n \cup \text{aGUE}_{n+1},$$

of the singular values of the Gaussian unitary ensemble (GUE), with both ensembles
on the right drawn independently, Theorem 1.1 immediately implies the following
remarkable relation between the singular values of GUE and GOE:

**Corollary 1.1.** With the ensembles on the right drawn independently, there holds

$$|\text{GUE}_n| \overset{d}{=} \text{even}\,|\text{GOE}_n| \cup \text{even}\,|\text{GOE}_{n+1}|. \quad (1.3)$$

The superposition (1.3) bears a striking similarity with a corresponding result for
the eigenvalue distributions, see [10, Eq. (5.9)] and [11, Eq. (6.14)], namely

$$\text{GUE}_n \overset{d}{=} \text{even}\,(\text{GOE}_n \cup \text{GOE}_{n+1}).$$

Our second set of main results (Sections 4/5) sharpens Theorem 1.1 by realizing
$|\text{GOE}_n|$ as the singular values of other matrix models that reveal a rich additional

\[^a\]In this paper, the Gaussian weights are $e^{-\beta x^2/2}$ with $\beta = 1$ for orthogonal and $\beta = 2$ for unitary
symmetry.
structure. A first model (Corollary 4.1), initially identified by comparing moments of the product of the even singular values to known moments of the determinant [1, Eqs. (23) and (24)], is constructed by bordering the skew-symmetric matrix $A$ defining the anti-GUE with an independent standard normal vector $b \in \mathbb{R}^n$: that is to say, the singular values of $G$ and those of

$$H = (b A)$$

are both distributed as $|\text{GOE}_n|$. To the same end, the bordering vector could also be chosen as $b = \tau_n e_1$, where $\tau_n$ is a $\chi_n$-distributed variable, independent of $A$, and $e_1$ denotes the first unit vector. The precise effect of such borderings on the singular values of a matrix is studied in the preparatory Section 3.

Using a technique (Lemma 5.1) that was, in essence, introduced by Dumitriu and Forrester [6], this bordered matrix model is finally (Theorems 5.1/5.2) transformed into a pair $(R^{\text{even}}, R^{\text{odd}})$ of bidiagonal square matrices, whose singular values are jointly distributed as even $|\text{GOE}_{2m}|$ and odd $|\text{GOE}_{2m}|$. Both matrices depend in a very simple fashion on a set of independent $\chi_k$-distributed random variables. Specifically, for $n = 2m$ even (the structure of the odd order case is similar), we get

$$R^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1^2 + 2\xi_{2m}^2} & \xi_{2m-2} & \xi_{2m-4} & \cdots & \xi_5 & \xi_2 & \xi_1 \\ \xi_{2m-1} & \xi_{2m-4} & \cdots & \xi_5 & \xi_2 \end{pmatrix}$$

and

$$R^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 & \xi_{2m-2} \\ \xi_{2m-1} & \xi_{2m-4} \\ \cdots & \cdots \\ \xi_5 & \xi_2 \\ \xi_3 \end{pmatrix},$$

where $\xi_1, \xi_2, \ldots, \xi_{2m}$ are independent random variables, with $\xi_k$ distributed as $\chi_k$. The singular values of $R^{\text{odd}}$ correspond to odd $|\text{GOE}_{2m}|$ and those of $R^{\text{even}}$ to even $|\text{GOE}_{2m}|$, both drawn from the same ensemble.

As a striking application (Corollary 5.1) of this new matrix pair model, we establish that $|\text{det GOE}_n|$ can be expressed explicitly as a product of independent random variables. Specifically, for $n = 2m$ even, the determinant of $M = \sqrt{2}G$ factors as

$$|\text{det } M| = \xi_1 \sqrt{\xi_1^2 + 2\xi_{2m}^2 \cdot \xi_3^2 \cdot \xi_5^2 \cdots \xi_{2m-1}^2},$$

with independent variables $\xi_k$ distributed as $\chi_k$. A similar factorization holds in the odd order case. The form of these variables explains the absence of large prime factors in the moments of the determinant, and leads to a new, simple proof (Section 6) of
the known central limit theorem for \( \log |\det \text{GOE}_n| \), cf. Delannay and Le Caër [4, Section III], Tao and Vu [17, Thm. 4]. While the representation (1.4) of \( |\det M| \) as a product of independent random variables can be found implicitly in the work of Delannay and Le Caër, namely in form of a factorization [4, Eq. (41)] of the Meijer G-function representation of the Mellin transform of \( |\det M| \) into hypergeometric terms, see the discussion of (5.6), Tao and Vu, who approximated the log-determinant by a sum of weakly dependent terms, speculated that such a representation would not be possible [17, p. 78].

Our third set of main results (Section 8) studies the implication of Theorem 1.1 on the inter-relation of gap probabilities, that is, the probabilities \( E(k; J) \) that the interval \( J \) contains exactly \( k \) eigenvalues drawn from a random matrix ensemble. Specifically, for order \( n \), we get

\[
E^n_{\text{GOE}}(2k + \mu - 1; (-s, s)) + E^n_{\text{GOE}}(2k + \mu; (-s, s)) = E^n_{\text{GUE}}(k; (0, s)),
\]

where \( \mu = 0, 1 \) denotes the parity of \( n \). This formula was previously known only in the case \( \mu = 0 \), see Forrester [8, Eq. (1.14)]. We initially used a heuristic argument, see (8.3), to extrapolate the formula to the case \( \mu = 1 \). A substantial portion of the present discussion was derived from attempts to justify the heuristic, after this prediction held up under numerical scrutiny. Taking the bulk scaling limit of both cases provides a new, simpler proof of a remarkable formula previously obtained by Mehta relating the gap probabilities of the GOE and those of the Laguerre unitary ensemble (LUE), see (8.4).

**Notation.** In contrast to the previous analyses mentioned, where either ensembles of odd (e.g., if Pfaffians were used) or of even order (e.g., if Mellin transforms were used) have typically presented considerable technical complications, our treatment of ensembles of even and odd order is nearly identical. The formulae themselves, however, will often depend on the parity \( \mu \) of the underlying order \( n \) and we will, throughout this paper, write

\[
n = 2m + \mu \quad (\mu = 0, 1), \quad \hat{m} = m + \mu, \quad (1.5a)
\]

that is,

\[
m = \lfloor n/2 \rfloor, \quad \hat{m} = \lceil n/2 \rceil, \quad \mu = \lceil n/2 \rceil - \lfloor n/2 \rfloor. \quad (1.5b)
\]

Terms that only appear for \( n \) odd will be written with a factor \( \mu \) in a sum and with an exponent \( \mu \) in a product; etc. This way, without suggesting any natural interpolation between the cases \( \mu = 0 \) and \( \mu = 1 \) (with the notable exception of the usage of the heuristic duality principle (8.3) that started our work), we simply avoid writing out awkward case distinctions.

### 2. Joint Density of the Singular Values

In this section we establish, in two different ways, the joint probability distribution of the singular values \( \sigma_j = |\lambda_j| \) of the GOE induced by the corresponding density
The Singular Values of the GOE

for eigenvalues as given by the symmetric function

\[ p(\lambda_1, \lambda_2, \ldots, \lambda_n) = c_n \prod_{k=1}^{n} e^{-\lambda_k^2/2} \cdot |\Delta(\lambda_1, \lambda_2, \ldots, \lambda_n)| \]  

(2.1)

with some normalization constant \( c_n \) and the Vandermonde determinant

\[ \Delta(\xi_1, \ldots, \xi_n) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_n \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{pmatrix} = \prod_{k>j} (\xi_k - \xi_j). \]

We will frequently use that \( \Delta(\xi_1, \ldots, \xi_n) > 0 \) if the arguments are increasingly ordered, \( \xi_1 \leq \cdots \leq \xi_n \).

By symmetry, we can establish the joint density of the singular values by restricting ourselves to the cone of increasingly ordered singular values

\[ 0 \leq \sigma_1 \leq \cdots \leq \sigma_n, \]  

(2.2)

this way parametrizing \(|\text{GOE}_n|\). To simplify notation and to avoid case distinctions between odd and even order \( n \) in later parts of the paper, we introduce two further sets of coordinates for this cone. Writing, as detailed in (1.5), \( n = 2m + \mu \) and \( \hat{m} = m + \mu \) with \( \mu = 0, 1 \), the coordinates

\[ x_j = \sigma_{2j-1} \quad (j = 1, \ldots, \hat{m}), \quad y_j = \sigma_{2j} \quad (j = 1, \ldots, m) \]  

(2.3a)

satisfy the interlacing property

\[ 0 \leq x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_{\hat{m}} \leq y_{\hat{m}}, \]  

(2.3b)

formally adding the value \( y_{m+1} = \infty \) if \( \mu = 1 \). With \( x^i \) and \( y^i \) denoting the \( x \) and \( y \) vectors with their components taken in the reverse order, so \( x^i = (x_{\hat{m}}, x_{\hat{m}-1}, \ldots, x_1) \) and \( y^i = (y_m, y_{m-1}, \ldots, y_1) \), we define, depending on the parity of \( n \), the coordinates

\[ (t, s) = (y^i, x^i) \quad (\mu = 0), \quad (t, s) = (x^i, y^i) \quad (\mu = 1), \]  

(2.4a)

satisfying the interlacing property

\[ t_1 \geq s_1 \geq t_2 \geq s_2 \geq \cdots \geq t_{\hat{m}} \geq s_{\hat{m}} \geq 0, \]  

(2.4b)

again formally adding the value \( s_{m+1} = 0 \) if \( \mu = 1 \). A large part of the apparent dependence on parity is the fact that some results, like Theorem 2.1, have stable expressions in terms of the \((x, y)\) coordinates, while others, like Theorem 4.1, are stable in the \((t, s)\) coordinates. Since the mapping from \( \sigma = (\sigma_1, \ldots, \sigma_n) \) to either the pair of coordinates \((x, y)\) or \((t, s)\) is orthogonal, transforming the density between the three sets of coordinates is simply done by inserting new variable names for old ones. Note that the \( s \) variables parametrize the even-location decimated ensemble even \(|\text{GOE}_n|\) while the \( t \)-variables do the same for odd \(|\text{GOE}_n|\). We call them the even and odd singular values.
Supported on the cone defined by (2.2), the joint probability density of the singular values is

\[ q(\sigma_1, \ldots, \sigma_n) = n! \sum_{\epsilon \in \{\pm 1\}^n} p(\epsilon_1 \sigma_1, \ldots, \epsilon_n \sigma_n) = c_n n! \prod_{k=1}^n e^{-\sigma_k^2/2} D(\sigma_1, \ldots, \sigma_n) \]

with

\[ D(\sigma_1, \ldots, \sigma_n) = \sum_{\epsilon \in \{\pm 1\}^n} |\Delta(\epsilon_1 \sigma_1, \ldots, \epsilon_n \sigma_n)|. \]

To determine the signs of the Vandermonde terms it suffices to discuss the case \( \sigma_1 < \cdots < \sigma_n \): we then get, because \( \text{sign}(\epsilon_k \sigma_k - \epsilon_j \sigma_j) = \epsilon_k \) if \( k > j \),

\[ \text{sign} \Delta(\epsilon_1 \sigma_1, \ldots, \epsilon_n \sigma_n) = \prod_{k>j} \epsilon_k = \prod_{k=2}^n \epsilon_k^{-1} = \prod_{k \text{ even}} \epsilon_k. \]

Hence, by continuity, there holds on all of (2.2)

\[ D(\sigma_1, \ldots, \sigma_n) = \sum_{\epsilon \in \{\pm 1\}^n} \theta_0(\epsilon) \Delta(\epsilon_1 \sigma_1, \ldots, \epsilon_n \sigma_n), \quad \theta_0(\epsilon) = \prod_{k \text{ even}} \epsilon_k. \tag{2.5} \]

The form of \( \theta_0(\epsilon) \) suggests we proceed in terms of the \((x, y)\) coordinates introduced in (2.3). With respect to these coordinates, we obtain the following theorem.

**Theorem 2.1.** The joint probability density of \(|\text{GOE}_n|\), supported on the cone (2.2) and expressed in the coordinates (2.3), is given by

\[ c_n n! 2^n \cdot \left( \prod_{k=1}^m e^{-x_k^2/2} \cdot \Delta(x_1^2, \ldots, x_m^2) \right) \cdot \left( \prod_{k=1}^m y_k e^{-y_k^2/2} \cdot \Delta(y_1^2, \ldots, y_m^2) \right), \tag{2.6} \]

where \( c_n \) is the normalization constant of the GOE-density (2.1).

**Remark 2.1.** Despite the fact that the joint density factors on its domain of support, it does not reveal an independence between the underlying variables \( x \) and \( y \). Their dependence is entirely by the interlacing (2.3b).

We will give two different proofs of the theorem. The first uses the determinantal structure of the Vandermonde terms to establish the factorization, while the second uses their polynomial structure and their symmetries. It is our consideration that the first proof is more straightforward, while the second provides additional insight into the structure of the factorization.

**Proof by Determinantal Structure**

We write \( D(x; y) \) for (2.5) when expressed in terms of the \((x, y)\) variables (2.3); it is convenient to split the sign changes \( \epsilon \) into \( \epsilon^x \) and \( \epsilon^y \) accordingly and to use

\[ \theta_0(\epsilon) = \theta(\epsilon^y), \quad \theta(\epsilon^y) = \epsilon_1^y \cdots \epsilon_m^y. \]
We give a second proof of the factorization (2.7) based on the observation that the odd columns and rows occur before the even ones, we express the Vandermonde terms as

\[
\Delta(\epsilon_1, \ldots, \epsilon_n) = \det \begin{pmatrix}
\pi_0^{(n)}(x_1) & \ldots & \pi_0^{(n)}(x_m) & \pi_0^{(n)}(y_1) & \ldots & \pi_0^{(n)}(y_m) \\
\epsilon_1 \pi_1^{(m)}(x_1) & \ldots & \epsilon_m \pi_1^{(m)}(x_m) & \epsilon_1 \pi_1^{(m)}(y_1) & \ldots & \epsilon_m \pi_1^{(m)}(y_m)
\end{pmatrix}
\]

by writing the determinant column-wise with \(\pi_\mu^{(n)}(x) = \begin{pmatrix} x^\mu \\ x^{\mu+2} \\ \vdots \\ x^{\mu+2n-2} \end{pmatrix} \in \mathbb{R}^n \quad (\mu = 0, 1)\).

Now, we calculate

\[
D(x; y) = \sum_{\epsilon^x, \epsilon^y \in \{\pm 1\}^m} \theta(\epsilon^x) \theta(\epsilon^y) \det \begin{pmatrix}
\pi_0^{(n)}(x_1) & \ldots & \pi_0^{(n)}(x_m) & \pi_0^{(n)}(y_1) & \ldots & \pi_0^{(n)}(y_m) \\
\epsilon_1 \pi_1^{(m)}(x_1) & \ldots & \epsilon_m \pi_1^{(m)}(x_m) & \epsilon_1 \pi_1^{(m)}(y_1) & \ldots & \epsilon_m \pi_1^{(m)}(y_m)
\end{pmatrix}
\]

\[
= \sum_{\epsilon^x, \epsilon^y \in \{\pm 1\}^m} \theta(\epsilon^y) \det \begin{pmatrix}
2 \pi_0^{(m)}(x_1) & \ldots & 2 \pi_0^{(m)}(x_m) & \pi_0^{(m)}(y_1) & \ldots & \pi_0^{(m)}(y_m) \\
0 & \ldots & 0 & \epsilon_1 \pi_1^{(m)}(y_1) & \ldots & \epsilon_m \pi_1^{(m)}(y_m)
\end{pmatrix}
\]

\[
= 2^m \det \left( \pi_0^{(m)}(x_1) \ldots \pi_0^{(m)}(x_m) \right) \cdot \sum_{\epsilon^y \in \{\pm 1\}^m} \theta(\epsilon^y)^2 \det \left( \pi_1^{(m)}(y_1) \ldots \pi_1^{(m)}(y_m) \right)
\]

\[
= 2^m \det \left( \pi_0^{(m)}(x_1) \ldots \pi_0^{(m)}(x_m) \right) \cdot 2^m \det \left( \pi_1^{(m)}(y_1) \ldots \pi_1^{(m)}(y_m) \right).
\]

By noting \(\hat{m} + m = n\) and by expressing the result in terms of Vandermonde determinants, we finally get

\[
D(x; y) = 2^n \cdot \Delta(x_1^2, \ldots, x_m^2) \cdot y_1 \cdots y_m \Delta(y_1^2, \ldots, y_m^2).
\]

This factorization establishes Theorem 2.1.

**Proof by Polynomiality**

We give a second proof of the factorization (2.7) based on the observation that the sum in (2.5) defines a polynomial of homogeneous degree at most \(\binom{n}{2}\). We identify the factors by symmetrizing known vanishings of this polynomial. Extending the definition of \(D\) polynomially to all real values of its arguments, we get in particular

\[
D(\epsilon_1 \sigma_1, \ldots, \epsilon_n \sigma_n) = \theta_0(\epsilon) D(\sigma_1, \ldots, \sigma_n) \quad (\epsilon \in \{\pm 1\}^n)
\]
and, inherited from the Vandermonde terms, $D$ is antisymmetric with respect to permutations of either the even or odd indices of $\sigma$ since both sets of permutations leave the factor $d_0(\epsilon)$ invariant.

Now, if $\sigma_j = \sigma_{j+2}$ for $\sigma$ belonging to the cone (2.2), we have $\sigma_j = \sigma_{j+1} = \sigma_{j+2}$ and, hence, by the pigeonhole principle, for each choice of signs $\epsilon$ at least one of
\[ \epsilon_j \sigma_j = \epsilon_{j+1} \sigma_{j+1} \quad \text{or} \quad \epsilon_j \sigma_j = \epsilon_{j+2} \sigma_{j+2} \quad \text{or} \quad \epsilon_{j+1} \sigma_{j+1} = \epsilon_{j+2} \sigma_{j+2} \]
holds. Therefore, each of the Vandermonde terms in (2.5) vanishes. It follows that $\sigma_{j+1} - \sigma_{j+2}$ and by (2.8) also $\sigma_j + \sigma_{j+2}$ divide $D$ for all $j$, thus so does the product $\sigma_j^2 - \sigma_{j+2}^2$. We also note that if $\sigma_2 = 0$, we have $\sigma_1 = \sigma_2 = 0$ and, hence, for each choice of signs $\epsilon_1 \sigma_1 = \epsilon_2 \sigma_2$. Once more each of the Vandermonde terms in (2.5) vanishes and it follows that $\sigma_2$ divides $D$.

In terms of the $(x, y)$-coordinates (2.3a) we thus see that $x_j^2 - x_{j+1}^2$, $y_j^2 - y_{j+1}^2$ and $y_j$ divide $D(x; y)$. Invoking the antisymmetry with respect to either $x$ or $y$, we see that $D$ is divisible by $x_j^2 - x_k^2$, $y_j^2 - y_k^2$ for every $j \neq k$ and by $y_j$ for every $j$. These factors contribute homogeneous degree
\[ 2 \left( \frac{m}{2} \right) + 2 \left( \frac{m}{2} \right) + m = \left( \frac{n}{2} \right), \]
so $D$ cannot have any other non-unit factors and we get
\[ D(x; y) = d_n \cdot \Delta(x_1^2, \ldots, x_m^2) \cdot y_1 \cdots y_m \Delta(y_1^2, \ldots, y_m) \] (2.9)
with a positive constant $d_n$. This is (2.7) except for identifying $d_n = 2^n$.

**Remark 2.2.** The value of $d_n$ can easily be calculated without resorting to the first proof: a straightforward inspection shows that the expressions (2.5) and (2.9) both induce an asymptotics of the form
\[ D(\sigma_1, \ldots, \sigma_{n-1}, \sigma_n) \sim \kappa_n \sigma_n^{-1} D(\sigma_1, \ldots, \sigma_{n-1}) \quad (\sigma_n \to \infty); \]
the first one gives $\kappa_n = 2$, the second one $\kappa_n = d_n / d_{n-1}$. From $D(\sigma_1) = 2$ we thus get $d_n = 2^n$.

3. Singular Values of Bordered (Skew-)Symmetric Matrices

In preparation for what follows, in this section we study an algebraic device that allows us to untangle the interlacing (2.4b) of even and odd singular values, namely bordering a skew-symmetric matrix $A \in \mathbb{R}^{n \times n}$ with a column vector: $A \mapsto (b \ A)$. By looking at the purely imaginary Hermitian matrix $iA$ we see that each non-zero singular value of $A$ occurs with even multiplicity. That is, with $n = 2m + \mu$, the $n$ singular values of $A$ can be arranged as the sequence $s_1, s_1, s_2, s_2, \ldots, s_m, s_m$ and, if $\mu = 1$, also $s_{m+1} = 0$, decreasingly ordered according to
\[ s_1 \geq s_2 \geq \cdots \geq s_m, \quad \hat{m} = m + \mu. \]
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The results of this section are twofold. First, Lemma 3.1 shows that the singular values of \((b \ A)\) are of the form (with the value \(s_{\hat{m}} = 0\) only formally added to the list of inequalities if \(\mu = 1\))

\[
t_1 \geq s_1 \geq t_2 \geq s_2 \geq \cdots \geq t_{\hat{m}} \geq s_{\hat{m}}. \tag{2.4b}
\]

That is, bordering \(A\) by a column \(b\) splits the double listed pairs \((s_j, t_j)\) of singular values into \((s_j, t_j)\) and, if \(\mu = 1\), modifies the surplus singular value \(s_{\hat{m}} = 0\) into some \(t_{\hat{m}}\) subject to the interlacing (2.4b). They are strictly interlacing if

\[
t_1 > s_1 > t_2 > s_2 > \cdots > t_{\hat{m}} > s_{\hat{m}}.
\]

Second, Lemma 3.2 establishes a coordinate change \((t, s) \rightarrow (r, s)\) through an explicit algebraic map such that strict interlacing of \(t\) with \(s\) corresponds to strict positivity of the components of \(r\).

To begin with, there are orthogonal matrices \(U\) and \(V\) such that (the last row and column of the block partitioning are understood to be \(\mu\)-dimensional, meaning that they are missing if \(\mu = 0\))

\[
UAV' = \begin{pmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U(b \ A) \begin{pmatrix} 1 & 0 & 0 \\ 0 & V' \end{pmatrix} = \begin{pmatrix} u & S & 0 & 0 \\ v & 0 & S & 0 \\ \eta & 0 & 0 & 0 \end{pmatrix}, \tag{3.1}
\]

with \(S = \text{diag}(s_1, \ldots, s_m)\) built from the singular values of \(A\) and the partitioning

\[
Ub = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix}, \quad u, v \in \mathbb{R}^m, \quad \eta \in \mathbb{R}^\mu.
\]

Hence, the singular values of \((b \ A)\) are given by the following lemma.

**Lemma 3.1.** Let \(S = \text{diag}(s_1, \ldots, s_m)\) be a diagonal matrix and \(u, v \in \mathbb{R}^m, \eta \in \mathbb{R}^\mu\) \((\mu = 0, 1)\). Then, with \(\hat{m} = m + \mu\), the singular values of the \((m + \hat{m}) \times (m + \hat{m} + 1)\) block matrix

\[
\begin{pmatrix} u \\ v \\ \eta \end{pmatrix}
\]

are \(s_1, \ldots, s_m, t_1, \ldots, t_{\hat{m}}\), satisfying the interlacing property (2.4b). Here, the \(t_j\) are the singular values of the \(\hat{m} \times (\hat{m} + 1)\) bordered matrix \((r \ \hat{S})\) with \(\hat{S} = \text{diag}(s_1, \ldots, s_{\hat{m}})\) and \(r_j = \sqrt{u_j^2 + v_j^2} \ (j = 1, \ldots, m); \ if \ \mu = 1, \ then \ r_{m+1} = |\eta| \ and \ s_{m+1} = 0. \ Further, \ there \ holds

\[
t_1^2 + \cdots + t_{\hat{m}}^2 = r_1^2 + \cdots + r_{\hat{m}}^2 + s_1^2 + \cdots + s_m^2 \tag{3.2}
\]

and, if \(\mu = 1\),

\[
t_1 \cdots t_{\hat{m}} = r_{\hat{m}} \cdot s_1 \cdots s_m. \tag{3.3}
\]
Proof. It suffices to prove that the two matrices

\[ M_1 = \begin{pmatrix} u & S & 0 & 0 \\ v & 0 & S & 0 \\ \eta & 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & S & 0 & 0 \\ r & 0 & S & 0 \\ \eta & 0 & 0 & 0 \end{pmatrix} \]

with \( r_j = \sqrt{u_j^2 + v_j^2}, j = 1, \ldots, m \), have the same singular values. Using a Givens rotation \( U_j \) with

\[ U_j \begin{pmatrix} u_j \\ v_j \\ 0 \\ s_j \end{pmatrix} = \begin{pmatrix} 0 \\ r_j \\ 0 \\ s_j \end{pmatrix}, \quad (j = 1, \ldots, m) \]

one gets

\[ U_j \begin{pmatrix} u_j & s_j & 0 & 0 \\ v_j & 0 & s_j & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & U_j' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ r_j & s_j \end{pmatrix}. \]

Hence, by successively applying these two-dimensional orthogonal operations to the corresponding rows and columns (and addressing a possible sign change of the last row if \( \mu = 1 \)) one transforms \( M_1 \) into \( M_2 \) while leaving the singular values invariant.

We note that the decreasingly ordered singular values \( s_1, \ldots, s_m \) of a matrix \( \hat{S} \) and those of the bordered matrix \( (r \quad \hat{S}) \), \( t_1, \ldots, t_m \), are generally known [13, Cor. 7.3.6] to be interlacing as in (2.4b).

To finish, (3.2) follows from expressing the Frobenius norm of \( (r \quad \hat{S}) \) in terms of its singular values \( t \) and (3.3) follows from doing the same, if \( \mu = 1 \), for the magnitude of the determinant of that matrix with the last column (which is all zeros then) deleted.

The next lemma shows that one can uniquely solve the inverse problem \( t \mapsto r \) for strict interlacing.

Lemma 3.2. With the notation as in Lemma 3.1, let \( s_1 > s_2 > \cdots > s_m \geq 0 \) with \( s_m = 0 \) if \( \mu = 1 \), and let \( \hat{S} = \text{diag}(s_1, \ldots, s_m) \). Then, the map

\[ \Phi : r \mapsto t = \text{the decreasingly ordered singular values of } (r \quad \hat{S}) \]

defines a diffeomorphism

\[ \Phi : \mathbb{R}^m_{>0} \to \{ t \in \mathbb{R}^m_{>0} : t \text{ is strictly interlacing with } s \}. \]

If \( t \) is strictly interlacing with \( s \), its preimage \( r = \Phi^{-1}(t) \) is the unique positive solution of the system

\[ \sum_{k=1}^m \frac{r_k^2}{r_j^2 - s_k^2} = 1 \quad (j = 1, \ldots, m), \quad (3.4) \]

which is explicitly solved by

\[ r_j^2 = -\frac{\omega_t(s_j^2)}{\omega_s'(s_j^2)}, \quad \omega_s(\xi) = (\xi - s_1^2) \cdots (\xi - s_m^2), \quad \omega_t(\xi) = (\xi - t_1^2) \cdots (\xi - t_m^2). \quad (3.5) \]
The Jacobian of the inverse map $\Phi^{-1}$ is given by

$$\det\left(\frac{\partial r_j}{\partial t_k}\right)_{1 \leq j,k \leq \hat{m}} = \frac{1}{r_1 \cdots r_m} \cdot \frac{(t_1 \cdots t_\hat{m})^{1-\mu} \Delta(t_{\hat{m}}, \cdots, t_1^2)}{(s_1 \cdots s_m)^\mu \Delta(s_{\hat{m}}, \cdots, s_1^2)} \quad (\mu = 0, 1). \quad (3.6)$$

**Proof.** The squares of the singular values $t_1, \ldots, t_{\hat{m}}$ of $(r \hat{S})$ are the eigenvalues of $(r \hat{S})(r \hat{S})' = \hat{S} \hat{S}'+rr' = \text{diag}(s_1^2, \ldots, s_{\hat{m}}^2) + rr'$.

Now, any set of values $t_j^2$ for which $t_j$ satisfies the interlacing property (2.4b) can be obtained in this way, that is, as the eigenvalues of a positive semi-definite rank-one perturbation of $D = \text{diag}(s_1^2, \ldots, s_{\hat{m}}^2)$ (see, e.g., [18, Sect. 2]). Since $rr'$ does not depend on the signs of the individual entries of $r$, we can always choose $r \in \mathbb{R}_{\geq 0}^{\hat{m}}$.

If $r_\nu = 0$ for some $\nu$, then the $\nu$-th row and the $\nu$-th column of $rr'$ are zero which means that $s_\nu^2$ appears among the values of $t_j^2$. Hence, *strictly* interlacing implies $r \in \mathbb{R}_{\geq 0}^{\hat{m}}$.

Given such an $r \in \mathbb{R}_{\geq 0}^{\hat{m}}$, the eigenvalues $t_j^2$ of $D+rr'$ are known ([12, Lemma 8.4.3]) to be *strictly* interlacing with the $s_k^2$ and satisfy the secular equation

$$f(t_j^2) = 0 \quad (j = 1, \ldots, \hat{m}), \quad f(\lambda) = 1 + r'(D - \lambda I)^{-1}r,$$

which is (3.4). Since the determinant (3.7) given below is non-zero and, hence, the Cauchy matrix

$$C = \left(\frac{1}{t_j^2 - s_k^2}\right)_{1 \leq j,k \leq \hat{m}}$$

is non-singular, there is a one-to-one correspondence of $r \in \mathbb{R}_{\geq 0}^{\hat{m}}$, with those $t$ that *strictly* interlace with $s$. Because each of the steps $t \mapsto C \mapsto r$ is smooth, we have therefore proved that $\Phi$ is a diffeomorphism.

By relating Cauchy matrices with Lagrangian polynomial interpolation, Schechter [16, Eq. (16)] gave a short and simple proof of the explicit formula (3.5). Differentiation with respect to $t_k$ gives

$$J_{jk} = \frac{\partial r_j}{\partial t_k} = \frac{r_j t_k}{s_j^2 - t_k^2}.$$

Hence,

$$J = \text{diag}(r_1, \ldots, r_{\hat{m}})C \text{diag}(t_1, \ldots, t_{\hat{m}}),$$

which implies $\det J = t_1 \cdots t_{\hat{m}} r_1 \cdots r_{\hat{m}} \det C$. Now, using the explicit determinantal formula [16, Eq. (4)]

$$\det C = \frac{\prod_{j<k}(t_j^2 - t_k^2)(s_j^2 - s_k^2)}{\prod_{j<k}(t_j^2 - s_k^2)} \quad (3.7)$$

and

$$r_1^2 \cdots r_{\hat{m}}^2 = (-1)^{\hat{m}} \frac{\omega_1(s_1^2) \cdots \omega_\hat{m}(s_{\hat{m}}^2)}{\omega'(s_1^2) \cdots \omega'(s_{\hat{m}}^2)}.$$
together with the following straightforward evaluations of the product terms

\((-1)^{\hat{m}} \omega_t (s_1^2) \cdots \omega_t (s_{\hat{m}}^2) = \prod_{j, k} (t_j^2 - s_k^2), \quad \omega_s (s_1^2) \cdots \omega_s (s_{\hat{m}}^2) = \prod_{j \neq k} (s_j^2 - s_k^2),\)

one gets

\[
\det J = \frac{t_1 \cdots t_{\hat{m}}}{r_1 \cdots r_{\hat{m}}} \prod_{j < k} (t_j^2 - t_k^2) \cdot \prod_{j < k} (s_j^2 - s_k^2).
\]

With

\[
\prod_{j < k} (t_j^2 - t_k^2) = \Delta(t_{\hat{m}}^2, \ldots, t_1^2),
\]

\[
\prod_{j < k} (s_j^2 - s_k^2) = \Delta(s_{\hat{m}}^2, \ldots, s_1^2) = (s_1 \cdots s_m)^{2\mu} \Delta(s_1^2, \ldots, s_m^2),
\]

one finally gets the expression (3.6) by using (3.3) if \(\mu = 1\).

\[\square\]

4. Random Matrix Models for the Odd and Even Singular Values

Because of interlacing, the factorization of the joint density stated in Theorem 2.1 does not reveal an independence between the \(x\) and the \(y\) components of the singular values, or to the same end, between the \(t\) and the \(s\) components. If we change, however, the \((t, s)\) coordinates to the \((r, s)\) coordinates introduced in Lemma 3.2, the interlacing is replaced by just a positivity condition on the \(r\) components. The following theorem, which sharpens Theorem 1.1, shows that not only are the \(r\) and the \(s\) components independent of each other but both sets of components have so much additional structure that they can be completely described in terms of known distributions.

**Theorem 4.1.** Applying the transform \((t, s) \mapsto (r, s)\) of Lemma 3.2 to the \((t, s)\) parametrization (2.4) of the decimated ensembles odd \(|\text{GOE}_n|\) and even \(|\text{GOE}_n|\) defines a set of random variables \(r_k\), which are distributed as \(\chi_2^2\) for \(k = 1, \ldots, m\), and, if \(\mu = 1\), distributed as \(\chi_1^2\) for \(k = m + 1\). They are independent of each other and of the even singular values \(s\), which are jointly distributed as even \(|\text{GOE}_n| \overset{d}{=} \text{aGUE}_n\).

**Proof.** In terms of the \((t, s)\) coordinates, the joint density (2.6) of the singular values of the GOE can be recast in the form

\[
q(s; t) = c_n 2^n n! \cdot (s_1 \cdots s_m)^\mu (t_1 \cdots t_{\hat{m}})^{1-\mu} \cdot \Delta(s_{\hat{m}}^2, \ldots, s_1^2) \Delta(t_{\hat{m}}^2, \ldots, t_1^2) \cdot e^{-\sum_{j=1}^{m-1} s_j^2 - \sum_{j=1}^{\hat{m}} t_j^2},
\]

where the case distinction between even \((\mu = 0)\) and odd \((\mu = 1)\) orders \(n\) has been expressed in terms of powers. If we apply the coordinate change \((t, s) \mapsto (r, s)\) of
Lemma 3.2, which is a diffeomorphism up to an exceptional set of zero probability, the density with respect to the \((r, s)\) is

\[
\begin{align*}
&\left(\det \left( \frac{\partial r_j}{\partial t_k} \right)_{1 \leq j, k \leq m} \right)^{-1} \cdot q(s; t) = r_1 \cdots r_m \cdot \frac{(s_1 \cdots s_m)^\mu \Delta(s_2^2, \ldots, s_m^2)}{(t_1 \cdots t_m)^{1-\mu} \Delta(t_2^2, \ldots, t_m^2)} q(s; t) \\
&= \left( \prod_{j=1}^m r_j e^{-r_j^2/2} \right) \cdot \left( \sqrt{\frac{2}{\pi}} e^{-r_\hat{m}^2/2} \right)^\mu \cdot \left( \delta_\mu c_n 2^n n! \cdot \prod_{j=1}^m s_j^{2\mu} e^{-s_j^2} \cdot \Delta(s_2^2, \ldots, s_m^2)^2 \right)
\end{align*}
\]

with \(\delta_\mu = \left( \frac{\pi}{2} \right)^{\mu/2}\). Here we used expression (3.6) for the Jacobian and simplified the exponential functions according to (3.2). On their supporting domains, the first \(m\) factors of the resulting density are a \(\chi_2\)-density each, the next one is a \(\chi_1\)-density if \(\mu = 1\) (disappearing if \(\mu = 0\)), and the last one is the joint density of the anti-GUE of order \(n\), see [15, Sect. 13.1] or [9, Ex. 1.3.5(iv)].

**Remark 4.1.** As a side product, the proof shows that the normalization constant \(a_n\) of the joint density of the anti-GUE, if extended by symmetry to be supported on \([0, \infty)^m\), is given by

\[
a_n = c_n \left( \frac{\pi}{2} \right)^{\mu/2} \frac{2^n n!}{m!} \quad (n = 2m + \mu, \mu = 0, 1).
\]

This is consistent with the explicit formulae for \(c_n^{-1}\) and \(a_n^{-1}\) given in [9, Eq. (1.163)]/Eq. (4.157)].

The proof of Theorem 4.1 shows that the joint density \(p(t|s)\) of the \(t\) variables conditioned on \(s\) is given by the expression

\[
p(t|s) = \frac{1}{\delta_\mu} \frac{(t_1 \cdots t_m)^{1-\mu} \Delta(t_2^2, \ldots, t_m^2)}{(s_1 \cdots s_m)^\mu \Delta(s_2^2, \ldots, s_m^2)} e^{-\sum_{j=1}^m t_j^2/2 + \sum_{j=1}^m s_j^2/2}.
\]

(4.2)

This is just a particular case of a general result by Forrester and Rains [11, Cor. 3], which gives the probability \(p(t|s)\) if the \(t_j\) are the solutions of the secular equation (3.4) with parameters \(r_j\) being independently gamma distributed. In retrospect, we could thus have proved Theorem 4.1, based on Theorem 2.1, starting with the Forrester–Rains formula (4.2) and working backwards.

Now, Theorem 4.1 and Lemma 3.1 yield a new random matrix model for \(|\text{GOE}_n|\) which amounts to the singular values of certain bordered skew-symmetric Gaussian matrices.

**Corollary 4.1.** Let \(X \in \mathbb{R}^{n \times n}\) be a random matrix with independent standard normal entries. Denote by \(G = (X + X')/2\) its symmetric and by \(A = (X - X')/2\) its skew-symmetric part. Let \(\tau_n\) be a \(\chi_n\)-distributed random variable that is independent of \(X\). Then, both the singular values of \(G\) and the singular values of the bordered matrix (with \(e_1\) denoting the first unit vector)

\[
H = (\tau_n e_1 \ A)
\]

(4.3)
are jointly distributed as those of the GOE of order \(n\). The same holds if the matrix \(H\) is obtained from bordering \(A\) with an independent standard normal vector.

**Proof.** Note that the singular values of the symmetric part \(G = (X + X')/2\) are by definition jointly distributed as the singular values of the GOE of order \(n\).

As discussed in the derivation of (3.1), the singular value decomposition of \(A\) takes the form

\[
UAV' = \begin{bmatrix}
S & 0 & 0 \\
0 & S & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad S = \text{diag}(s_1, \ldots, s_m),
\]

where the last row and columns are missing if \(n\) is odd. By symmetry, the orthogonal matrix \(U\) is Haar distributed—indeed of \(\{s_1, \ldots, s_m\}\), which are jointly distributed as the anti-GUE of order \(n\), cf. [15, Sect. 13.1] or [9, Ex. 1.3.5(iv)]. Applying \(U\) to the first column \(H_1\) of \(H\) defines

\[
UH_1 = \tau U_1 = \begin{bmatrix}
u \\
\eta
\end{bmatrix}, \quad u, v \in \mathbb{R}^m, \quad \eta \in \mathbb{R}^\mu.
\]

Since the first column \(U_1\) of \(U\) is uniformly distributed on the sphere \(S^{n-1}\) and \(\tau\) is independently \(\chi\)-distributed, we see that \(UH_1\) is a standard normal vector, see, e.g., [5, Sect. V.4]; the same conclusion holds for standard normal \(H_1\). Hence, the variables

\[
\forall j = 1, \ldots, m \quad r_j = \sqrt{u^2_j + v^2_j}
\]

are independently \(\chi\)-distributed and, if \(\mu = 1\), \(r_{m+1} = |\eta|\) is independently \(\chi_1\)-distributed. Comparing the results of Lemma 3.1 and of Theorem 4.1 finishes the proof. \(\square\)

The random matrix model of the last corollary can easily be turned into the following sparse model that separates the even and odd singular values. Note that just one of the two matrices is square, the other is rectangular.

**Corollary 4.2.** Let \(n = 2m + \mu\), \(\mu = 0, 1\) and let \(\tau_1, \ldots, \tau_n\) be independent random variables, with \(\tau_k\) distributed as \(\chi_k\). The union of the singular values of both the bidiagonal matrix

\[
B_{\mu}^{\text{odd}} = (\tau_n e_1 \ B_{\mu}^{\text{even}}) \in \mathbb{R}^{(m+\mu+1)\times(m+1)}
\]

and the bidiagonal matrix \(B_{\mu}^{\text{even}} \in \mathbb{R}^{(m+\mu)\times m}\), defined by

\[
B_{0}^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\tau_2 m - 1 \\
\tau_2 m - 2 \ 	au_2 m - 3 \\
\vdots \\
\tau_2 \\
\tau_1
\end{pmatrix}, \quad B_{1}^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\tau_2 m \\
\tau_2 m - 1 \tau_2 m - 2 \\
\vdots \\
\tau_3 \tau_2 \\
\tau_1
\end{pmatrix},
\]
is jointly distributed as $|\text{GOE}_n|$. Here, the singular values of $B^{\text{odd}}_\mu$ correspond to odd $|\text{GOE}_n|$ and the singular values of $B^{\text{even}}_\mu$ correspond to even $|\text{GOE}_n|$, both drawn from the same ensemble.

**Proof.** Using the notation of Corollary 4.1, a Householder tridiagonalization of $A$, rescaling rows and columns by $-1$ as necessary, yields $UAV' = T$ with orthogonal matrices $U, V$ and

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \tau_{n-1} & \tau_{n-2} \\ \tau_{n-1} & 0 & \tau_{n-3} \\ \tau_{n-2} & \tau_{n-3} & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & \tau_1 \\ \tau_2 & \cdots & \tau_1 & 0 \\ \tau_1 & \cdots & \tau_1 & 0 \end{pmatrix},$$

where the entries $\tau_k$ are jointly distributed as independent $\chi_k$-variables with degrees of freedom $k = 1, \ldots, n$, see [6, Sect. II]. Since Householder triagonalizations do not operate on the first row and column, we have $Ue_1 = e_1$ and, hence,

$$U \cdot (\tau_n e_1 \ A) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} V' = (\tau_n e_1 \ T).$$

Thus the matrices $(\tau_n e_1 \ A)$ and $(\tau_n e_1 \ T)$ have the same singular values. By a simultaneous row and column permutation of $T$ so that the odd columns and rows occur before the even ones, we see that the matrices $(\tau_n e_1 \ T)$ and (with the length of the first unit vector $e_1$ adjusted)

$$\begin{pmatrix} \tau_n e_1 & 0 & B^{\text{even}}_\mu \\ 0 & (B^{\text{even}}_\mu)' & 0 \end{pmatrix}$$

have the same singular values. Interlacing shows that the singular values of $B^{\text{even}}_\mu$ correspond to the even ones of $(\tau_n e_1 \ A)$ and the singular values of $(\tau_n e_1 \ B^{\text{even}}_\mu)$ correspond to the odd ones.

**Remark 4.2.** The mapping from the singular values of a sample drawn from the GOE to $\tau$ coordinates is deterministic and can be made explicit (the same remark applies to the construction of the $\xi$ variables in the next section): starting with $(r, s)$ coordinates, the $\tau$ are obtained by applying appropriate orthogonal row and column transformations to the matrix

$$\begin{pmatrix} 0 & 0 & S' \\ r & -S & 0 \end{pmatrix}, \quad \text{where} \quad S = \begin{pmatrix} s_1 \\ \vdots \\ s_m \\ 0 \cdots 0 \end{pmatrix};$$

if $\mu = 0$, the last row of $S$ is missing.
5. Square Bidiagonal Matrix Models and the Determinant

To study the distribution of determinants we turn the bidiagonal random matrix model of Corollary 4.2 into one with square matrices only. Key to this transformation is the following variant of a result by Dumitriu and Forrester [6, Claim 6.5].

Lemma 5.1. Let the variables $\tau_k$ ($k = 1, \ldots, 2m - 1$) be distributed as $\chi_k$, with the distribution of $\tau_{2m}$ arbitrary such that $\tau_1, \ldots, \tau_{2m}$ are independent of each other. Then the singular values of the $m \times (m + 1)$ bidiagonal matrix

$$B = \begin{pmatrix} \tau_{2m} & \tau_{2m-1} \\ \tau_{2m-2} & \tau_{2m-3} \\ \vdots & \vdots \\ \tau_2 & \tau_1 \end{pmatrix}$$

(5.1)

are the same as those of the $m \times m$ bidiagonal matrix

$$R = \begin{pmatrix} \xi_{2m+1} & \xi_{2m-2} \\ \xi_{2m-1} & \xi_{2m-4} \\ \vdots & \vdots \\ \xi_5 & \xi_2 \\ \xi_3 \end{pmatrix}$$

(5.2)

constructed by the normalized reduced RQ-decomposition\(^b\) $B = RQ$ with a row-orthogonal matrix $Q$, that is, by the almost surely positive solution of the set of equations

$$\xi_{2k+1}^2 + \xi_{2k-2}^2 = \tau_{2k}^2 + \tau_{2k-1}^2 \quad (k = 1, \ldots, m),$$

(5.3a)

$$\xi_{2k+1} \xi_{2k} = \tau_{2k+1} \tau_{2k} \quad (k = 1, \ldots, m - 1).$$

(5.3b)

The variables $\xi_2, \ldots, \xi_{2m-1}$ are distributed as $\chi_2, \ldots, \chi_{2m-1}$; they are independent of each other and of $\tau_{2m}$. The variable $\xi_{2m+1}$ is of the form

$$\xi_{2m+1} = \sqrt{\xi_1^2 + \tau_{2m}^2}, \quad \text{where} \quad \xi_1 = \sqrt{\tau_{2m-2}^2 - \xi_{2m-2}^2}$$

is distributed as $\chi_1$ and is also independent of $\xi_2, \ldots, \xi_{2m-1}$ and of $\tau_{2m}$.

Proof. A well-known result [5, Thm. IX.3.1] about the $\chi^2$-distribution states that the involution

$$\phi(X, Y, Z) = \left( Z \frac{X}{X+Y}, Z \frac{Y}{X+Y}, X+Y \right)$$

(5.4)

maps a set of mutually independent random variables $X, Y, Z$ distributed as $\chi_1^2, \chi_2^2$ and $\chi_r^2$ to a new set of mutually independent random variables of exactly the

\(^b\)The $R$-factor can equivalently be obtained from the Cholesky-type decomposition $RR' = BB'$.  

same type. Starting with $\tau_{1,1} = \tau_1$, the system (5.3) is recursively solved for the variables $\xi_2, \ldots, \xi_{2m-1}$ by

$$(\tau_{1,k+1}^2, \xi_{2k}^2, \tau_{2k+1}^2) = \phi(\tau_{1,k}^2, \tau_{2k}^2, \tau_{2k+1}^2), \quad (k = 1, \ldots, m - 1).$$

Hence, the variable $\xi_1 = \tau_{1,m}$ and the thus constructed $\xi_2, \ldots, \xi_{2m-1}$ are independent of each other and of the not yet used variable $\tau_{2m}$; they are distributed as $\chi_k$ ($k = 1, \ldots, 2m - 1$). Because $\phi$ is an involution, there is

$$\tau_{2m-1}^2 = \tau_{1,m}^2 + \xi_{2m-2}^2 = \xi_1^2 + \xi_{2m-2}^2.$$

Hence, the yet to be used $k = m$ case of Eq. (5.3a) finally implies the asserted form of $\xi_{2m+1}$.

**Remark 5.1.** The use of the involution (5.4) has been motivated by the observation [7, Lemma 1] that for $2 \times 2$ matrices the $R$-factor of the $RQ$-decomposition of a lower triangular matrix

$$L = \begin{pmatrix} z & 0 \\ y & x \end{pmatrix} = RQ, \quad R = \begin{pmatrix} \xi & \eta \\ 0 & \zeta \end{pmatrix},$$

is induced by the transformation $(\xi^2, \eta^2, \zeta^2) = \phi(x^2, y^2, z^2)$. Basically, the $R$-factor (5.2) of the bidiagonal matrix (5.1) is then obtained by successively applying this transformation along the diagonal from the lower right to the upper left.

Application of this lemma to Corollary 4.2 yields sparse random matrix models for the odd and even singular values of the GOE in terms of bidiagonal square matrices. We begin with the case of even order $n = 2m$.

**Theorem 5.1.** Let $\xi_1, \ldots, \xi_{2m}$ be independent random variables, with $\xi_k$ distributed as $\chi_k$. The union of the singular values of the two bidiagonal square matrices

$$R_0^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1^2 + 2\xi_{2m}^2} & \xi_{2m-2} & \cdots & \xi_3 \\ \xi_{2m-1} & \xi_{2m-4} & \cdots & \xi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_5 & \xi_4 & \cdots & \xi_1 \end{pmatrix}$$

and

$$R_0^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 & \xi_{2m-2} & \cdots & \xi_2 \\ \xi_{2m-1} & \xi_{2m-4} & \cdots & \xi_3 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_5 & \xi_4 & \cdots & \xi_1 \end{pmatrix}$$

is jointly distributed as $|\text{GOE}_{2m}|$. Here, the singular values of $R_0^{\text{odd}}$ correspond to odd $|\text{GOE}_{2m}|$ and the singular values of $R_0^{\text{even}}$ correspond to even $|\text{GOE}_{2m}|$, both drawn from the same ensemble.
Proof. Let \( \tau_1, \ldots, \tau_{2m} \) be independent random variables, with \( \tau_k \) distributed as \( \chi_k \). Prepending a zero column to the second matrix, Corollary 4.2 shows that the singular values of the two matrices

\[
B_{0}^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_{2m} & \tau_{2m-1} \\ \tau_{2m-2} & \tau_{2m-3} \\ \vdots & \vdots \\ \tau_2 & \tau_1 \end{pmatrix}
\]

and

\[
B_{0}^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_{2m} & \tau_{2m-1} \\ \tau_{2m-2} & \tau_{2m-3} \\ \vdots & \vdots \\ \tau_2 & \tau_1 \end{pmatrix},
\]

where

\[
\tau_{2m}^{\text{odd}} = \sqrt{2} \tau_{2m}, \quad \tau_{2m}^{\text{even}} = 0,
\]

are jointly distributed as \( |\text{GOE}_{2m}| \). Here, the singular values of \( B_{0}^{\text{odd}} \) correspond to odd \( |\text{GOE}_{2m}| \) and the singular values of \( B_{0}^{\text{even}} \) correspond to even \( |\text{GOE}_{2m}| \), both drawn from the same ensemble. If we apply the construction of Lemma 5.1 to both matrices \( B_{0}^{\text{odd}} \) and \( B_{0}^{\text{even}} \) simultaneously, we obtain the \( R \)-factors \( R_{0}^{\text{odd}} \) and \( R_{0}^{\text{even}} \) with one and the same set of variables \( \xi_1, \ldots, \xi_{2m} \) subject to the asserted properties and additionally

\[
\xi_{2m+1}^{\text{odd}} = \sqrt{\xi_1^2 + (\tau_{2m}^{\text{odd}})^2} = \sqrt{\xi_1^2 + 2 \tau_{2m}^2}, \quad \xi_{2m+1}^{\text{even}} = \sqrt{\xi_1^2 + (\tau_{2m}^{\text{even}})^2} = \xi_1.
\]

By defining \( \xi_{2m} = \tau_{2m} \) we thus get the asserted form of \( R_{0}^{\text{odd}} \) and \( R_{0}^{\text{even}} \).

Remark 5.2. The matrices \( R_{0}^{\text{even}} \) of Theorem 5.1 and \( B_{0}^{\text{even}} \) of Corollary 4.2 are superficially related, up to a different sample of the independent random variables, by a transposition followed by a cyclic permutation of their diagonals. Such a transformation would not, in general, preserve singular values, but depends instead on properties of the \( \chi \) distributions.

The case of odd order exhibits a similar structure. The equivalence between the models \( B_{1}^{\text{even}} \) and \( R_{1}^{\text{even}} \) used in the following theorem is also noted, for the anti-GUE, in [7, Sect. 2] and [6, Clm. 6.5].

Theorem 5.2. Let \( \xi_1, \ldots, \xi_{2m+1} \) be independent random variables, with \( \xi_k \) distributed as \( \chi_k \). The union of the singular values of the two bidiagonal square matrices

\[
R_{1}^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} \xi_1 & \sqrt{2} \xi_{2m} \\ \xi_{2m+1} & \xi_{2m-2} \\ \vdots & \vdots \\ \xi_5 & \xi_2 \\ \xi_3 \end{pmatrix},
\]
and

\[
R_1^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\xi_{2m+1} & \xi_{2m-2} \\
\xi_{2m-1} & \xi_{2m-4} \\
\vdots & \ddots \\
\xi_5 & \xi_2 & \xi_3
\end{pmatrix}
\]

is jointly distributed as $|\text{GOE}_{2m+1}|$. Here, the singular values of $R_1^{\text{odd}}$ correspond to odd $|\text{GOE}_{2m+1}|$ and the singular values of $R_1^{\text{even}}$ correspond to even $|\text{GOE}_{2m+1}|$, both drawn from the same ensemble.

**Proof.** Let $\tau_1, \ldots, \tau_{2m+1}$ be independent random variables, with $\tau_k$ distributed as $\chi_k$. By transposing both matrices and prepending a zero column to the first one, Corollary 4.2 shows that the singular values of the two matrices

\[
B_1^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{2}\tau_{2m+1} \\
\tau_{2m} & \tau_{2m-1} \\
\vdots & \ddots \\
\tau_2 & \tau_1 \end{pmatrix}
\]

and

\[
B_1^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_{2m} & \tau_{2m-1} \\
\tau_{2m-2} & \tau_{2m-3} \\
\vdots & \ddots \\
\tau_2 & \tau_1 \end{pmatrix}
\]

are jointly distributed as $|\text{GOE}_{2m+1}|$. Here, the singular values of $B_0^{\text{odd}}$ correspond to odd $|\text{GOE}_{2m+1}|$ and the singular values of $B_0^{\text{even}}$ correspond to even $|\text{GOE}_{2m+1}|$, both drawn from the same ensemble. If we apply the construction of Lemma 5.1 simultaneously to $B_1^{\text{odd}}$ and to $B_1^{\text{even}}$ and to

\[
\tilde{B}_1^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_{2m+2}^{\text{odd}} & \tau_{2m+1}^{\text{odd}} \\
\tau_{2m} & \tau_{2m-1}^{\text{odd}} \\
\vdots & \ddots \\
\tau_2 & \tau_1^{\text{odd}} \end{pmatrix}, \quad \tau_{2m+2}^{\text{odd}} = 0,
\]

which is $B_1^{\text{odd}}$ with its first row rescaled, we obtain the $R$-factors $R_1^{\text{even}}$ and

\[
\tilde{R}_1^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_{2m+3}^{\text{odd}} & \xi_{2m} \\
\xi_{2m+1} & \xi_{2m-2} \\
\vdots & \ddots \\
\xi_5 & \xi_2 & \xi_3
\end{pmatrix}
\]
with one and the same set of variables $\xi_1, \ldots, \xi_{2m+1}$ subject to the asserted properties and additionally

$$
\xi_{2m+3}^{\text{odd}} = \sqrt{\xi_1^2 + (\tau_{2m+2}^{\text{odd}})^2} = \xi_1.
$$

By restoring the proper scaling of the first row, we thus get the asserted form of $R_1^{\text{even}}$ and $R_1^{\text{odd}}$.

Remark 5.3. This theorem implies that, for odd order $n = 2m + 1$, the product $\det R_1^{\text{odd}}$ of the odd singular values of $\text{GOE}_n$ and the product $\det R_1^{\text{even}}$ of the even ones are related by

$$
\det R_1^{\text{odd}} = \xi_1 \det R_1^{\text{even}},
$$

where $\xi_1$ is a random variable distributed as $\chi_1$ which is independent of even $|\text{GOE}_n|$. This is nothing but (3.3), recalling that by the proof of Theorem 4.1 the variable $r_{m+1}$ is distributed as $\chi_1$.

As an immediate consequence of the preceding theorems, stated in the following corollary, the magnitude of the determinant of the GOE can be expressed as a product of independent random variables. Even though Tao and Vu speculated that such a representation does not seem to be possible [17, p. 78], a precursor of this result was recognized implicitly by Delannay and Le Caër who noted in [4, p. 1531] that the Meijer $G$-function they used to describe the distribution of the determinant when $n$ is odd could be sampled as a product of independent gamma-distributed random variables. They did not, however, have any interpretation for these variables in terms of the underlying ensemble, nor did they recognize the possibility of sampling when $n$ is even.

Corollary 5.1. Let $G_n$ be drawn from the GOE of order $n = 2m + \mu$ with parity $\mu = 0, 1, \hat{m} = m + \mu$. Then the determinant of $M_n = \sqrt{2}G_n$ factors into independent random variables of the form

$$
|\det M_n| = \eta_n^{(1)} \cdot \xi_3^2 \cdot \xi_5^2 \cdots \xi_{2\hat{m}-1}^2,
$$

with

$$
\eta_n^{(1)} = \xi_1 \cdot \sqrt{\xi_1^2 + 2\xi_n^2} \quad (\mu = 0), \quad \eta_n^{(1)} = \sqrt{2} \xi_1 \quad (\mu = 1).
$$

Here, the $\xi_k$ are mutually independent random variables distributed as $\chi_k$.

Proof. The assertion follows from the observation that $|\det M_n|$ is the product of the singular values of $M_n$ and therefore, by Theorem 5.1 and 5.2, distributed as

$$
\det(\sqrt{2}R_1^{\text{even}}) \cdot \det(\sqrt{2}R_1^{\text{odd}}).
$$

Multiplication of the diagonal terms of the bidiagonal factors finishes the proof. \hfill \square

In retrospect, once we know that $|\det \text{GOE}_n|$ is distributed as a product of $\hat{m}$ independent random variables, all the factors can be readily identified in the Mellin
transforms computed by Delannay and Le Caër in [4, Eqs. (26/41)], although, for even \( n = 2m \), the Mellin transform of the factor

\[ \eta_{2m}^{(1)} = \xi_1 \sqrt{\xi_1^2 + 2\xi_2^2} \]

into the expression

\[ 2^{(s-1)/2} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(s + m - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2} + m\right)} 2F_1\left(\frac{s}{2}, \frac{1-s}{2}; \frac{s}{2} + m; \frac{1}{2}\right) \]

with the hypergeometric function \( 2F_1 \) may not be familiar to most observers. This aspect of their paper has been missed by several commentators, with Mehta, in [15, §26.6], omitting their expression for even \( n = 2m \) since the inverse Mellin transform of (5.6) cannot be readily written down.

**Remark 5.4.** In the case of odd order \( n = 2m + 1 \), the density of \( \text{det GOE}_{2m+1} \) is necessarily odd, since the eigenvalue density (2.1) is even, but \( \text{det}(G) = -\text{det}(-G) \). It follows that in this case the sign of the determinant is statistically independent of its magnitude, and we can obtain the distribution of the determinant by replacing \( \xi_1 \) by a standard normal variable. No corresponding result is available for even order \( n = 2m \), although the factored presentation of the odd moments of \( \text{det GOE}_{2m} \) in [1, Eq. (23)] suggests that the distribution of the determinant should involve many of the same factors.

### 6. Central Limit Theorem for the Determinant

Delannay and Le Caër used an explicit computation of the Mellin transform of the even part of the distribution of \( \text{det GOE}_n \) to derive the cumulants of the potential \( V = \log|\text{det GOE}_n| \), and to show that \( V \) is asymptotically Gaussian [4, Section III]. Tao and Vu extended this log-normality to determinants of a wider class of Wigner matrices, and provided an alternate proof in the case of Gaussian matrices in [17]: based on analyzing tridiagonal sparse models for the GOE and GUE eigenvalues, they found a way to approximate the log-determinant as a sum of weakly dependent terms, which then yields the asymptotic log-normality by stochastic calculus and the martingale central limit theorem. In this section we present yet another, much simpler proof based on the factorization of the magnitude of the determinant into independent random variables. In particular, our proof elucidates the difference between the GOE and the GUE in the scaling of the central limit theorem.

We start by recalling that, parallel to the factorization given in Corollary 5.1 for the GOE (\( \beta = 1 \), Edelman and La Croix [7, Thm. 2] obtained a factorization for the GUE (\( \beta = 2 \)): with \( G_n \) drawn from the GUE of order \( n \), the determinant of \( M_n = \sqrt{2}G_n \) factors into independent random variables of the form

\[ |\text{det } M_n| = \eta_n^{(2)} \cdot \xi_3 \xi_5 \cdots \xi_{2n-1} \xi_{2n} \]  

with

\[ \eta_n^{(2)} = \xi_1 \xi_{n+1} \quad (\mu = 0), \quad \eta_n^{(2)} = \xi_1 \quad (\mu = 1). \]
Here $\xi_1, \ldots, \xi_n, \tilde{\xi}_3, \ldots, \tilde{\xi}_{2m-1}$ are mutually independent random variables with both $\xi_k$ and $\tilde{\xi}_k$ being distributed as $\chi_k$. Note that, except for the (asymptotically irrelevant) change in the factor $\eta_n(\beta)$, the transition from the GOE to the GUE just amounts for splitting the terms $\xi_k^2$ into the products $\xi_k \tilde{\xi}_k$ of independent factors. It is precisely this split which causes the appearance of $\beta$ in the denominator of the central limit theorem when written in the following form.

**Theorem 6.1 (Tao and Vu [17, Thm. 4]).** With the notation as above there holds, as $n \to \infty$, the central limit theorem

$$\frac{\log |\det M_n| - \frac{1}{2} \log n! + \frac{1}{4} \log n}{\sqrt{\frac{1}{\beta} \log n}} \xrightarrow{d} N(0, 1) \quad (\beta = 1, 2), \quad (6.2)$$

where $\xrightarrow{d}$ denotes convergence in distribution.

To prove this theorem for $\beta = 1$ and $\beta = 2$ in parallel, we split

$$\log |\det M_n| = Y_n^{(\beta)} + Z_n^{(\beta)} \quad (\beta = 1, 2)$$

into the random variables $Y_n^{(\beta)} = \log \eta_n^{(\beta)}$ and $Z_n^{(\beta)}$ defined by

$$Z_n^{(1)} = 2 \log \xi_3 + 2 \log \xi_5 + \cdots + 2 \log \xi_{2m-1},$$

$$Z_n^{(2)} = \left( \log \xi_3 + \log \tilde{\xi}_3 \right) + \left( \log \xi_5 + \log \tilde{\xi}_5 \right) + \cdots + \left( \log \xi_{2m-1} + \log \tilde{\xi}_{2m-1} \right).$$

We immediately observe the relations

$$E\left(Z_n^{(1)}\right) = E\left(Z_n^{(2)}\right), \quad \text{Var}\left(Z_n^{(1)}\right) = 2 \text{Var}\left(Z_n^{(2)}\right). \quad (6.3)$$

Note that the factor of two between the variances is caused, in the transition from GOE to GUE, by the above mentioned split of $\xi_k^2$ into the product $\xi_k \tilde{\xi}_k$.

Now, while proving the central limit theorem in the $\beta = 2$ case, Edelman and La Croix [7, Cor. 2] obtained, in passing, the following result.

**Lemma 6.1.** The random variable $Z_n^{(\beta)}$ satisfies, as $n \to \infty$, a central limit theorem of the form

$$\tilde{Z}_n^{(\beta)} = \frac{Z_n^{(\beta)} - \frac{1}{2} \log n! + \frac{1-\mu}{2} \log n + \frac{1}{4} \log n}{\sqrt{\frac{1}{\beta} \log n}} \xrightarrow{d} N(0, 1) \quad (\beta = 1, 2). \quad (6.4)$$

**Proof.** The proof of [7, Cor. 2] proceeds, first, by establishing asymptotic expansions based on explicit calculations of the mean and variance of $\log \chi$-distributed variables, namely,

$$E\left(Z_n^{(2)}\right) = \frac{1}{2} \log(2\hat{m} - 1)! - \frac{1-\mu}{2} \log(2\hat{m} - 1) - \frac{1}{4} \log(2\hat{m} - 1) + O(1),$$

$$\text{Var}\left(Z_n^{(2)}\right) = \frac{1}{2} \log(2\hat{m} - 1) + O(1),$$

where $\hat{m} = \sqrt{\frac{n}{\beta}}$.
and, next, by showing that $Z_n^{(2)}$ satisfies a Lyapunov condition of order four. Hence, the Lindeberg–Feller central limit theorem can then be applied to $Z_n^{(2)}$ and gives, by noting that

$$\log(2m - 1)! = \log n! - (1 - \mu) \log n, \quad \log(2m - 1) = \log n + O(1),$$

the asserted limit (6.4). By realizing that the sums $Z_n^{(1)}$ and $Z_n^{(2)}$ basically share the same Lyapunov condition, the central limit theorem for $Z_n^{(1)}$ can be induced from that of $Z_n^{(2)}$ by means of (6.3).

The difference between the central limit theorems of $\log |\det M_n|$ and of $Z_n^{(\beta)}$ enjoys the following strong convergence result.

**Lemma 6.2.** The random variable $Y_n^{(\beta)}$ satisfies, as $n \to \infty$,

$$\tilde{Y}_n^{(\beta)} = \frac{Y_n^{(\beta)} - \frac{1 - \mu}{2} \log n}{\sqrt{\frac{1}{\beta} \log n}} \overset{a.s.}{\longrightarrow} 0 \quad (\mu = 0, 1), \quad (6.5)$$

where $\overset{a.s.}{\longrightarrow}$ denotes almost sure convergence.

**Proof.** The case $\mu = 1$ is trivial, since in that case $Y_n^{(\beta)}$ is independent of $n$. Applied to a sum of squares of independent standard Gaussians, the strong law of large numbers gives, as $n \to \infty$,

$$n^{-1} \xi_n^2 \overset{a.s.}{\longrightarrow} E(\xi_1^2) = 1, \quad \text{and, hence,} \quad n^{-1}(\xi_n^2 + 2\xi_n^2) \overset{a.s.}{\longrightarrow} 2.$$ 

Taking the logarithm gives

$$\log \xi_n - \frac{1}{2} \log n \overset{a.s.}{\longrightarrow} 0, \quad \log \sqrt{\xi_n^2 + 2\xi_n^2} - \frac{1}{2} \log n \overset{a.s.}{\longrightarrow} \frac{1}{2} \log 2,$$

which implies the assertion for $\mu = 0$. \qed

Now, adding (6.4) and (6.5) gives, by Slutsky’s theorem,

$$\frac{\log |\det M_n| - \frac{1}{2} \log n! + \frac{1}{4} \log n}{\sqrt{\frac{1}{\beta} \log n}} = \tilde{Y}_n^{(\beta)} + \tilde{Z}_n^{(\beta)} \overset{d}{\longrightarrow} N(0, 1) \quad (\beta = 1, 2),$$

which finishes the proof of the central limit theorem (6.2).

**7. Integrating Out the Odd or Even Singular Values**

Here, we present another proof of Theorem 1.1. If we are interested only in the distribution of the even singular values, then it is possible to proceed from the joint probability density (4.1) by integrating out the odd ones. While not exposing any additional structure, such as in Theorem 4.1, this approach is conceptually more straightforward, and offers the advantage that it can also be used to establish the determinantal formula (7.4) for the probability density of the odd singular values.
This is of interest in its own right, since we constructed separate sparse random matrix models for the odd singular values in Corollary 4.2 and in Theorems 5.1 and 5.2. Moreover, the technique extends to the symmetric Jacobi and to the Cauchy ensembles [3].

7.1. Integrating out the odd singular values

Recalling (1.5), we rewrite the expression (4.1) of the joint density in the form

\[ q(s,t) = cn^2n! \cdot g_\mu(s_1, \ldots, s_m) \cdot g_{1-\mu}(t_1, \ldots, t_m) \]

with functions

\[ g_\mu(z_1, \ldots, z_m) = \prod_{k=1}^m z_k^\mu e^{-z_k^2/2} \cdot \Delta(z_m^2, \ldots, z_1^2). \]  

(7.1)

Corollary 7.1 below shows that integrating out the odd singular values \( t \) subject to the interlacing (2.4b) gives the following marginal density of the even singular values with \( \delta_\mu = (\pi/2)^{\mu/2} \):

\[ q_{\text{even}}(s_1, \ldots, s_m) = \delta_\mu cn^2n! \cdot g_\mu(s_1, \ldots, s_m)^2 \]

\[ = \delta_\mu cn^2n! \cdot \prod_{k=1}^m s_k^{2\mu} e^{-s_k^2} \cdot \Delta(s_m^2, \ldots, s_1^2)^2. \]  

(7.2)

Since the last expression is the joint density of the anti-GUE of order \( n \), see [15, Sect. 13.1] or [9, Ex. 1.3.5(iv)], this is nothing but Theorem 1.1 spelled out in terms of densities.

The integration is based on the following lemma and its first Corollary 7.1.

Lemma 7.1. Let

\[ e^{(n)}_\kappa(x) = \begin{pmatrix} x^{\kappa} e^{-x^2/2} \\ x^{\kappa+2} e^{-x^2/2} \\ \vdots \\ x^{\kappa+2n-2} e^{-x^2/2} \end{pmatrix} \in \mathbb{R}^n \quad (\kappa = -1, 0, 1), \]

with the understanding that, instead of \( x^{-1} e^{-x^2/2} \), the first entry of \( e^{(n)}_{-1}(x) \) is the expression

\[ \eta_{-1}(x) = -\sqrt{\frac{\pi}{2}} \text{erf} \left( \frac{x}{\sqrt{2}} \right). \]

Then, for \( \kappa = 0, 1 \), there holds the integration formula

\[ \int_{x_1}^{x_2} d\xi_1 \cdots \int_{x_n}^{x_{n+1}} d\xi_n \det \left( e^{(n)}_\kappa(\xi_1) \cdots e^{(n)}_\kappa(\xi_n) \right) = \det \begin{pmatrix} e^{(n)}_\kappa(x_1) \cdots e^{(n)}_\kappa(x_{n+1}) \\ 1 \cdots 1 \end{pmatrix}. \]
Proof. Integration by parts yields the three-term recurrence of antiderivatives
\[ \int_x^z e^{-\xi^2/2} \, d\xi = -\eta_1(x), \]
\[ \int_x^z \xi^{k+1} e^{-\xi^2/2} \, d\xi = -x^k e^{-x^2/2} + k \int_x^z \xi^{k-1} e^{-\xi^2/2} \, d\xi \quad (k = 0, 1, 2, \ldots), \]
and, hence, by simplifying notation to \( e_\kappa(x) = e_\kappa^{(n)}(x) \),
\[ \int_x^z e_\kappa(\xi) \, d\xi = L_\kappa e_{\kappa-1}(x) \quad (\kappa = 0, 1) \]
with a lower triangular matrix \( L_\kappa \in \mathbb{R}^{n \times n} \) having \(-1\) all along its main diagonal.

We thus calculate
\[
\int_{x_1}^{x_2} d\xi_1 \cdots \int_{x_n}^{x_{n+1}} d\xi_n \det(e_\kappa(\xi_1) \cdots e_\kappa(\xi_n))
\]
\[ = \det \left( \int_{x_1}^{x_2} e_\kappa(\xi_1) d\xi_1 \cdots \int_{x_n}^{x_{n+1}} e_\kappa(\xi_n) d\xi_n \right) = \det(L_\kappa) \det \left( e_{\kappa-1}^{x_2} \cdots e_{\kappa-1}^{x_{n+1}} \right) \]
\[ = \det \left( \begin{array}{cccc}
1 & 0 & \cdots & 0 \\
e_{\kappa-1}(x_1) & e_{\kappa-1}(x_2) & \cdots & e_{\kappa-1}(x_{n+1}) \\
1 & 1 & \cdots & 1
\end{array} \right) \]
\[ = \det \left( \begin{array}{cccc}
1 & 0 & \cdots & 0 \\
e_{\kappa-1}(x_1) & e_{\kappa-1}(x_2) & \cdots & e_{\kappa-1}(x_{n+1}) \\
1 & 1 & \cdots & 1
\end{array} \right). \]

In the last step we added the first column to the second, then the second to the third, etc. \( \square \)

Corollary 7.1. Let \( \mu \) be as in (7.1) and put \( s_m = 0 \) if \( \mu = 1 \). Then, one has the integration formula
\[ \int_{s_1}^{s_2} dt_1 \int_{s_2}^{s_1} dt_2 \cdots \int_{s_\mu}^{s_{\mu-1}} dt_\mu g_{\mu-\mu}(t_1, \ldots, t_\mu) = \delta_\mu g_\mu(s_1, \ldots, s_\mu) \quad (\mu = 0, 1) \]
with \( \delta_\mu = (\pi/2)^{\mu/2} \).

Proof. Using the notation of Lemma 7.1, we first observe that
\[ g_\mu(z_1, \ldots, z_\mu) = \det \left( e_\mu^{(m)}(z_m) \cdots e_\mu^{(m)}(z_1) \right). \quad (7.3) \]
Now, Lemma 7.1 yields, first using $e_0^{(m)}(\infty) = 0$, that for $\mu = 0$

$$
\int_{s_1}^{\infty} dt_1 \int_{s_2}^{s_1} dt_2 \cdots \int_{s_m}^{s_{m-1}} dt_m \det \left( e_1^{(m)}(t_m) \cdots e_1^{(m)}(t_1) \right)
$$

$$
= \det \left( e_0^{(m)}(s_m) \cdots e_0^{(m)}(s_1) \right) = \det \left( e_0^{(m)}(s_m) \cdots e_0^{(m)}(s_1) \right)
$$

and then, using $e_{-1}^{(m+1)}(0) = 0$ and $e_{-1}^{(m+1)}(\infty) = -\pi/2, 0)$, that for $\mu = 1$

$$
\int_{s_1}^{\infty} dt_1 \int_{s_2}^{s_1} dt_2 \cdots \int_{s_m}^{s_{m+1}} dt_m \det \left( e_0^{(m+1)}(t_m+1) \cdots e_0^{(m+1)}(t_1) \right)
$$

$$
= \det \left( 0 e_{-1}^{(m+1)}(s_m) \cdots e_{-1}^{(m+1)}(s_1) e_{-1}^{(m+1)}(\infty) \right)
$$

$$
= (-1)^m \det \left( e_{-1}^{(m+1)}(s_m) \cdots e_{-1}^{(m+1)}(s_1) e_{-1}^{(m+1)}(\infty) \right)
$$

which finishes the proof of the corollary. \qed

### 7.2. Integrating out the even singular values

The following second corollary of Lemma 7.1 will allow us to integrate out the even singular values from the density $g(s; t)$.

**Corollary 7.2.** Let $g_\mu$ be as in (7.1) and put $t_{m+1} = 0$ if $\mu = 0$. Then, for $\mu = 0, 1$, one has the integration formula

$$
\int_{t_2}^{t_1} ds_1 \int_{s_3}^{s_2} ds_2 \cdots \int_{s_{m+1}}^{s_m} ds_m g_\mu(s_1, \ldots, s_m) = \det \left( e_{1-\mu}^{(m-1)}(t_{m+1}) \cdots e_{1-\mu}^{(m-1)}(t_1) \right)
$$

with $\delta_0(t) = 1$ and $\delta_1(t) = \sqrt{\pi/2} \operatorname{erf}(t/\sqrt{2})$.

**Proof.** Using (7.3) and Lemma 7.1 we obtain

$$
\int_{t_2}^{t_1} ds_1 \cdots \int_{s_{m+1}}^{s_m} ds_m g_\mu(s_1, \ldots, s_m)
$$

$$
= \int_{t_2}^{t_1} ds_1 \cdots \int_{s_{m+1}}^{s_m} ds_m \det \left( e_\mu^{(m)}(s_m) \cdots e_\mu^{(m)}(s_1) \right)
$$

$$
= \det \left( e_{\mu-1}^{(m)}(t_{m+1}) \cdots e_{\mu-1}^{(m)}(t_1) \right),
$$
which is already the assertion for \( \mu = 1 \). For \( \mu = 0 \), the assertion follows from further calculating
\[
\det \begin{pmatrix} e_1^{(m)}(t_{m+1}) & e_1^{(m)}(t_m) & \cdots & e_1^{(m)}(t_1) \\ 1 & 1 & \cdots & 1 \end{pmatrix} = \det \begin{pmatrix} 0 & e_1^{(m)}(t_m) & \cdots & e_1^{(m)}(t_1) \\ 1 & 1 & \cdots & 1 \end{pmatrix}
\]
\[
= (-1)^m \det \begin{pmatrix} e_1^{(m)}(t_m) & \cdots & e_1^{(m)}(t_1) \\ 0 & \cdots & (-1)^m \delta_1(t_1) \end{pmatrix} = (-1)^m \det \begin{pmatrix} -\delta_1(t_m) & \cdots & -\delta_1(t_1) \\ e_1^{(m-1)}(t_m) & \cdots & e_1^{(m-1)}(t_1) \end{pmatrix}
\]
\[
= \det \begin{pmatrix} e_1^{(m-1)}(t_m) & \cdots & e_1^{(m-1)}(t_1) \\ \delta_1(t_m) & \cdots & \delta_1(t_1) \end{pmatrix}
\]
which finishes the proof.

Now, by means of this corollary, the marginal density of the odd singular values supported on \( t_1 \geq t_2 \geq \cdots \geq t_{\tilde{m}} \geq 0 \) is given as
\[
g_{\text{odd}}(t_1, \ldots, t_{\tilde{m}}) = c_n n! 2^n \cdot g_{1-\mu}(t_{\tilde{m}}, \ldots, t_1) \cdot \det \begin{pmatrix} e_1^{(m-1)}(t_{\tilde{m}}) & \cdots & e_1^{(m-1)}(t_1) \\ \delta_1-\mu(t_{\tilde{m}}) & \cdots & \delta_1-\mu(t_1) \end{pmatrix}
\[
= c_n n! 2^n \cdot \det \begin{pmatrix} e_1^{(m-1)}(t_{\tilde{m}}) & \cdots & e_1^{(m-1)}(t_1) \\ \gamma_{1-\mu}(t_{\tilde{m}}) & \cdots & \gamma_{1-\mu}(t_1) \end{pmatrix} \cdot \det \begin{pmatrix} e_1^{(m-1)}(t_{\tilde{m}}) & \cdots & e_1^{(m-1)}(t_1) \\ \delta_1-\mu(t_{\tilde{m}}) & \cdots & \delta_1-\mu(t_1) \end{pmatrix}
\]
(7.4)

with
\[
\gamma_{\mu}(t) = e^{\mu+2\tilde{m}-2} e^{-t^2/2}, \quad \delta_{\mu}(t) = \begin{cases} 1 & \text{if } \mu = 0, \\ \frac{1}{\sqrt{\pi}} \text{erf} \left( \frac{t}{\sqrt{2}} \right) & \text{if } \mu = 1. \end{cases}
\]
Note that the two determinantal factors differ just in their last rows. It is this difference that prevents the expression from becoming a perfect square, which is in marked contrast with the marginal density of the even singular values.

### 8. Gap Probabilities

Theorem 1.1 has an interesting implication in terms of gap probabilities, that is, in terms of the probabilities
\[
E_{n}^{\mu}(k; J), \quad \hat{E}_{n}^{\mu}(k; J),
\]
that the interval \( J \) contains exactly \( k \) eigenvalues drawn from the random matrix ensemble RMT of finite order \( n \), or in some scaling limit. Here, RMT will be the GOE, the aGUE or the LUE with parameter \( a \).

To begin with, by a simple change of coordinates, see [7, p. 8], there holds
\[
E_{2m+\mu}^{\mu}(k; (0, s)) = E_{LUE}^{\mu}(k; (0, s^2)) \bigg|_{a=\mu-\frac{1}{2}} \quad (\mu = 0, 1). \quad (8.1)
\]
By looking at pairs of consecutive values it is easy to see that the event that exactly $k$ values of the decimated ensemble even $|\text{GOE}_n|$, $n = 2m + \mu$, are contained in $(0, s)$ is given by the union of the events that exactly $2k + \mu - 1$ or that exactly $2k + \mu$ values of $|\text{GOE}_n|$ are in that interval. Since these two events are mutually exclusive and since the singular values of $\text{GOE}$ contained in $(0, s)$ correspond to the eigenvalues in $(-s, s)$, we thus get from (1.2) and (8.1) proof of

\[
E_{\text{GOE}}^{2m+\mu}(2k + \mu - 1; (-s, s)) + E_{\text{GOE}}^{2m+\mu}(2k + \mu; (-s, s)) = E_{\text{aGUE}}^{2m+\mu}(k; (0, s))
\]

For even order ($\mu = 0$), a first proof of this formula was given by Forrester [8, Eq. (1.14)]. For odd order ($\mu = 1$), Forrester communicated to us further proof of the $k = 0$ case, a remarkable tour de force extending the techniques from [8] based on generating functions, Pfaffian calculus, and Fredholm determinants—later he was able to use the same approach to establish the general $k$ case; for this and the extension to the symmetric Jacobi and to the Cauchy ensembles see [3].

We first identified the $\mu = 1$ form of (8.2) via a heuristic duality principle based on three observations. First, the LUE of order $m$ and parameter $a = p - m \in \mathbb{N}$ is modeled by the eigenvalues of $m \times m$-Wishart matrices $W = X'X$, where the random $p \times m$-matrices $X$ have independent complex standard normal entries. Second, the eigenvalues of $\tilde{W} = XX'$ are those of $W$ padded with $a = p - m$ zeros; that is, the $(k + a)$-th eigenvalue of $\tilde{W}$ is distributed as the $k$-th eigenvalue of $W$. Last, since $\tilde{W}$ is constructed the same way as $W$, but with dimension $\tilde{m} = m + a$ and parameter $\tilde{a} = -a$, we are thus led, at least formally, to the duality principle

\[
E_{\text{LUE}}^{m+\alpha}(k + \alpha; (0, t)) \big|_{a = -\alpha} = E_{\text{LUE}}^{m}(k; (0, t)) \big|_{a = a}.
\]

Extrapolated to general $\alpha > -1$, it can be taken as a natural definition of an otherwise undefined expression. Now, formally evaluating the $\mu = 0$ form of (8.2) at half-integer values of $m$ and $k$, and invoking the heuristic duality principle (8.3), led us to predict the $\mu = 1$ form. Since it held up under numerical scrutiny, trying to prove this prediction was a key motivation to our present work.

As already noted by Forrester [8, Eq. (1.16)], the bulk scaling of GOE and the hard-edge scaling of LUE allow us to turn (8.2), as $n \to \infty$, into the limit relation

\[
E_{\text{GOE}}^{\text{bulk}}(2k - 1 + \mu; (-s, s)) + E_{\text{GOE}}^{\text{bulk}}(2k + \mu; (-s, s)) = E_{\text{LUE}}^{\text{hard}}(k; (0, \pi^2 s^2), \mu - \frac{1}{2}) \quad (\mu = 0, 1);
\]

a remarkable formula previously established by Mehta [15, Eqs. (7.5.27/29), (20.1.20/21)] using two different, but much more involved methods. In contrast to (8.4), which offers many advantages for the numerical calculation of gap probabilities of the GOE in the bulk scaling limit [2, Sect. 5], the finite dimensional version (8.2) is not yet a closed recursion that would allow us to calculate the
The Singular Values of the GOE

Fig. 1. Fluctuation statistics (densities) of 100 000 samples of GOE

The gap probabilities of the GOE on symmetric intervals: a complimentary expression evaluating

\[ E_{\text{GOE}}^{2m+\mu}(2k - \mu; (-s, s)) + E_{\text{GOE}}^{2m+\mu}(2k + 1 - \mu; (-s, s)) \quad (\mu = 0, 1) \]

is still missing. By the same arguments that justify (8.2) such an expression would establish the gap probabilities of the decimation ensemble odd \([\text{GOE}_n]\), whose joint distribution is given by (7.4).

To finish the paper, it is amusing to note that all three cases of the Tracy–Widom distributions

\[ F_\beta(1; s) = E_\beta^{\text{soft}}(0; (s, \infty)) \quad (\beta = 1, 2, 4), \]

can be sampled from the soft-edge scaling limit of the spectrum of just the GOE, i.e., the \(\beta = 1\) case (the case \(\beta = 2\) corresponds to the GUE, \(\beta = 4\) to the GSE), see Fig. 1. First, let \(\Lambda_1, \Lambda_2\) denote the largest and second-largest soft-edge scaled eigenvalues of the GOE. In the large-matrix limit, as \(n \to \infty\), they are asymptotically distributed as

\[ \Lambda_1 \overset{d}{\sim} F_1(1; s), \quad \Lambda_2 \overset{d}{\sim} F_4(1; s). \]
The first assertion is the definition of the distribution $F_1(1; s)$, while the second follows from the decimation relation, see [10, Thm. 5.2] and [11, p. 44],

$$\text{GSE}_m = \text{even GOE}_{2m+1}.$$ Second, let $\Sigma_1, \Sigma_2$ denote the largest and second-largest scaled singular values of the GOE. They are asymptotically distributed as

$$\Sigma_1 \sim F_1(1; s)^2, \quad \Sigma_2 \sim F_2(1; s).$$

Here, the first assertion follows from the asymptotic independence of the extreme eigenvalues of the GOE and the second follows from (1.2) as follows: $\Sigma_2$ behaves like the largest scaled eigenvalue of the anti-GUE which, like that of the GUE, is governed by the Tracy–Widom distribution $F_2(1; s)$.

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