High energy scattering amplitudes in matrix string theory

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Abstract

High energy fixed angle scattering is studied in matrix string theory. The saddle point world sheet configurations, which give the dominant contributions to the string theory amplitude, are taken as classical backgrounds in matrix string theory. A one loop fluctuation analysis about the classical background is performed. An exact treatment of the fermionic and bosonic zero modes is shown to lead to all of the expected structure of the scattering amplitude. The ten-dimensional Lorentz invariant kinematical structure is obtained from the fermion zero modes, and the correct factor of the string coupling constant is obtained from the abelian gauge field zero modes. Up to a numerical factor we reproduce, from matrix string theory, the high energy limit of the tree level, four graviton scattering amplitude.

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1. Introduction

Weakly coupled perturbative string theory is conjectured to be described by the strong coupling limit of the matrix string supersymmetric gauge theory \([1][2][3]\) (see also \([4]\) for a review). In \([3]\) it is observed that the strong coupling limit corresponds to the infra red limit of the theory, and that symmetry more or less constrains the low energy effective action to be that of the light cone type IIA CFT. Matrix string theory, however, is a non-perturbative description of string theory and will contain phenomena, inaccessible to the infra red fixed point reasoning of \([3]\), which can only be investigated by studying the gauge theory. A necessary first step in this direction is to derive, directly from the gauge theory, standard string perturbation theory. Although this program has not yet been realised several pieces of the puzzle have been found.

Matrix string theory has classical solutions corresponding to interacting string world sheets in light-cone gauge \([5]\) (see also \([6]\)). The strings split and join via instanton like field configurations consisting of two regions; a core region, around the interaction point, where the fields do not commute, and an asymptotic region, away from the interaction point, where the fields do commute and where the eigenvalues glue together to form the Riemann surfaces of light-cone string theory \([7]\) (see also \([8]\)). Arguments have been given that in the large \(N\) \([5]\) or the strong coupling limit \([9]\) the effective theory is indeed the type IIA CFT defined on the corresponding Riemann surface. However there are subtleties that mean that these arguments are not completely justified. An analysis of the effective expansion parameter for a perturbative gauge theory loop expansion \([10]\) shows that the loop expansion diverges for most physical scattering processes. There is thus, at present no rigorous approach to calculating the strongly coupled gauge theory in most situations of physical interest.

A simple analysis does, however, lead to an explanation of the factors of the string coupling constant \(g_s\) associated with a string diagram \([9]\). In the strong coupling limit there is, in addition to the Green-Schwarz light cone string action, a completely decoupled abelian gauge theory. The power of \(g_s\) is entirely due to the zero mode structure of this abelian gauge-field; it is just the difference between the number of closed, non-exact one-forms (the gauge field zero modes) and the number of ghost zero modes (exactly one, the constant mode).

In this paper we focus on a scattering process where perturbative gauge theory calculations can be justified. The scattering process chosen is high energy, fixed angle, string
scattering, originally studied, from the string theory point of view, in \([11]\) and re-analyzed from the perspective of matrix string theory in \([7]\). Indeed it was originally pointed out in \([7]\) that this is a process where a perturbative gauge theory calculation might be justified. The scattering amplitude for high energy, fixed angle scattering is dominated by a classical world sheet configuration whose size, in the target space, is proportional to the incoming momenta. In other words high momenta correspond to large (in the target space sense) world sheets. By going to sufficiently high momenta, i.e. by making the world sheets sufficiently large, we can justify using a perturbative gauge theory calculation. This is similar to matrix theory \([13]\) graviton scattering calculations where one pulls the gravitons sufficiently far apart to be able to use a perturbative calculation.

We will study explicitly the tree level contribution to the four ground state scattering amplitude. Starting from the instanton-like classical matrix string configuration for such a world sheet we will estimate the fluctuation determinant and calculate exactly the contribution of the bosonic and fermionic zero modes. We will see that the zero mode calculation leads to all of the expected structure of the scattering amplitude. In addition to the correct power of \(g_s\) (observed in \([7]\)) we will reproduce the precise ten dimensional Lorentz invariant kinematic structure of the amplitude. The fluctuation determinant can not be calculated exactly, but we can nevertheless make some qualitative assessment of its overall form. The final result being that we will reproduce, up to an unknown numerical factor from the determinant, the four graviton scattering amplitude.

Although we explicitly focus on the four graviton scattering amplitude the formalism is general and can be applied to the scattering of any ground state particles. The calculation can also be generalized, in principle without difficulty, to loop scattering amplitudes, although we do not pursue this in this paper.

We begin, in section 2, by covering some of the essential material necessary for the calculations of later sections. We start with a very brief review of matrix string theory. We then turn to saddle point world sheets in high energy string scattering and how they occur in light cone string theory. Next we review the instanton like field configurations corresponding to finite string interactions and how they embed into a solution corresponding to a string worldsheet splitting and joining at the interaction points. Finally we study the kinematics of four particle scattering in light cone gauge in the centre of mass frame and derive a few simple kinematic identities useful for section 8.

In this paper we perform a one loop fluctuation calculation around classical world sheet configurations of matrix string theory. Section 3 discusses the validity of such a
calculation and defines the Euclidean one loop action. In particular the Minkowski space Majorana-Weyl fermions are combined into complex fermionic coordinates and momenta which then allow a Wick rotation into Euclidean space.

Section 4 analyses the fluctuation determinant. Sections 5 and 6 are devoted to the zero modes. In section 5 we focus on the behaviour of the zero modes in the neighbourhood of the instanton-like configurations, the objective being to understand how the core of the instanton, where the fields do not commute, affects the finiteness of the zero mode field configurations. Far from the core of the instanton these field configurations commute with each other and can be glued into global zero mode configurations for the global classical field configurations corresponding to light-cone string diagrams. These global zero modes we construct in section 6.

In section 7 we discuss the incoming and outgoing wavefunctions and show how to construct ground state wavefunctions. In particular we explicitly construct from the fermionic coordinates of section 3 the graviton wavefunction.

Section 8 is then devoted to the four graviton scattering amplitude, we explicitly integrate over all zero modes, the integration over the gauge and ghost zero modes leading to a factor of $g_s^2$ (as argued by [9]) and the integration over the fermion zero modes reproducing the ten dimensional Lorentz invariant kinematic factors for four graviton scattering.

Having focused in previous sections on four graviton scattering we summarize in section 9 the general procedure for calculating high energy scattering amplitudes in matrix string theory, and point out the close connection with standard light cone gauge super-string calculations. We finish section 9 by discussing how the analysis might be performed away from the high energy limit, and speculate on how the calculations could be be given a rigorous basis.

Technical details of the Euclideanization of the fermions and the fermion zero mode integrations are contained in the appendix.

2. Matrix string theory, world sheets and high energy scattering

2.1. Matrix String Theory

Matrix string theory [1][2][3] is equivalent to ten dimensional SYM theory dimensionally reduced to two dimensions:

$$S = \frac{1}{2\pi} \int d\tau d\sigma \text{Tr} \left[ -\frac{1}{2} (D_\alpha X^\alpha)^2 + \frac{i}{2} S^T \mathcal{D} S - \frac{1}{4} F_{\alpha\beta}^2 + \frac{1}{4g_s^2} [X^I, X^J]^2 + \frac{1}{2g_s} S^T \Gamma_I [X^I, S] \right].$$  
(2.1)
A careful derivation of this action (using the ideas of [14][15][16]) from the original matrix theory proposal [13] can be found in [6]. All fields are $N \times N$ hermitean matrices. The index $I$ for the bosonic fields runs from 1 to 8 and corresponds to the transverse directions of the ten dimensional target space. The 16 component fermion fields $S$ split into $S^a$ and $\dot{S}^a$, the 8s and 8c representations of $SO(8)$. The matrices $\Gamma^\mu$, $\mu = 1, \cdots, 9$ are the spin(8) gamma matrices with $\Gamma^0$ in the Dirac operator $\slashed{D}$ equal to the sixteen by sixteen unit matrix. $g_s$ is the string coupling constant and the coordinate $\sigma$ runs from 0 to $2\pi$. The action (2.1) is conjectured to describe non-perturbative type IIA string theory compactified on a light-like circle with $N$ the number of quanta of $p^+$ momenta along the compactified direction.

String world sheets are described in matrix string theory by commuting matrix configurations in which the $N$ eigenvalues of the matrices form a branched covering of the cylinder, and hence form the Riemann surfaces of interacting light-cone string theory. These surfaces are characterized by strings of different length which split and join, the total length of the strings, which corresponds to the light-cone $p^+$ momentum, being preserved. In the limit $N \to \infty$ the moduli space of the branched coverings is equivalent to the moduli space of all possible two-dimensional Riemann surfaces. A crucial ingredient of these configurations is that they are associated with a topologically non-trivial two-dimensional gauge field, which via Wilson lines generates the correct monodromy around all the branch points [5]. Specifically a single valued matrix description of a multivalued branched covering of the cylinder is given by

$$X = U \text{diag}(x_1, \cdots, x_N) U^\dagger, \quad A_\alpha = i g_s U^\dagger (\partial_\alpha U), \quad (2.2)$$

where the eigenvalues $x_i$ form the branched covering and the unitary matrix $U$ (which is multivalued) generates the monodromies around the branch points.

The gauge field configurations however are singular at the branch points and lead to a delta function singularity in the field strength. As was realised in [6] this singularity can be resolved by instanton like field configurations. Far from the core of the instanton the fields commute with each other and their eigenvalues describe a Riemann surface with a branch point. Close to the core of the instanton, however, the fields no longer commute and the Riemann surface interpretation breaks down. It is to be expected that any classical string world sheet will have a corresponding classical solution in matrix string theory.
In [7] it was shown that there exists a physically very interesting class of classical world sheets that preserve a certain amount of symmetry around the branch points and hence allow a simple construction of the instanton like field configurations. These world sheets are the saddle point classical world sheets dominating the amplitudes of high energy, fixed angle string scattering. Before reviewing the construction of the matrix string solutions we briefly recall how these saddle point world sheets appear in light cone string theory.

2.2. High energy scattering and light cone string theory

High energy, fixed angle scattering processes were studied in string theory in [11][12]. String theory simplifies enormously in this limit as the integral over world sheets localizes around a finite number of saddle-point configurations. It was conjectured in [11][12] that the saddle point world sheets of different genus follow the same target space path, up to an overall scaling factor. This simplification permits one to study string perturbation theory out to arbitrarily high orders. One of the results of this analysis is that the perturbation series is extremely divergent, with the genus $g$ world sheets contributing a factor of $g^{9g}$. A non-perturbative description such as matrix string theory should provide a natural cut-off to this divergence. First steps in an analysis along these lines were taken in [7]. We will be less ambitious in this article and focus on retrieving from matrix string theory standard string perturbation theory. For the purposes of this article, we focus on high energy scattering as a means to justify using a perturbative gauge theory calculation. Below we describe how the saddle-point world sheets arise in light cone string theory.

![Fig 1. Light cone string world sheets in matrix string theory](image)
In light cone string theory the Virasoro constraints have been imposed at the classical level to eliminate the non-physical degrees of freedom. Specifically light cone string coordinates, \( X^\pm = 1/\sqrt{2}(X^0 \pm X^9) \) have been eliminated from the dynamics by imposing the gauge
\[
X^+ = \tau, \tag{2.3}
\]
where \( \tau \) is the world sheet time direction. The coordinate \( X^- \) is then a function of the transverse coordinates \( X^I \ (I = 1, \cdots, 8) \) and the fermion fields.

The light cone coordinates \( w = \tau + i\sigma \) are defined using the Mandelstam mapping from the light cone diagram to the Riemann surface with uniformization \( z \)
\[
w = \sum p_i^+ G(z, z_i), \tag{2.4}
\]
where \( G \) is an abelian differential with purely imaginary periods and real part equal to the Greens function. The existence and uniqueness of such a differential for arbitrary genus has been proved in [17][18]. The branch points of the light cone string diagram are situated at the zeros of \( \omega \), i.e. the stationary points of \( w \) as a function of \( z \). In the neighbourhood of a simple zero of \( \omega \) situated at \( z_0 \) we have \( w - w_0 = (z - z_0)^2 \).

For tree level scattering we have
\[
w = \tau + i\sigma = \sum \epsilon_i N_i \log(z - z_i), \tag{2.5}
\]
with \( z \) defined in the complex plane and \( \epsilon_i N_i \) the \( p^+ \) momentum of the \( i \)th string. The \( \epsilon_i \) are equal to +1 for incoming states and −1 for outgoing states. The lengths of the strings are proportional to their \( p^+ \) momenta. The light-cone interaction points are given by the zeros of \( \partial_z w = 0 \), i.e. by the roots of the polynomial equation
\[
\sum \epsilon_i N_i \frac{1}{z - z_i} = 0. \tag{2.6}
\]

The classical solution for a string world sheet is given by
\[
X = \sum \epsilon_i p_i \log|z - z_i|, \tag{2.7}
\]
where \( p_i \) are the transverse momenta and \( z = z(w) \) is defined through (2.5).
For the classical field configurations (2.7) the light cone action is given entirely by boundary terms

\[ S = -\frac{1}{4\pi} \int d^2 w (\partial_\alpha X^I)^2 = \frac{1}{4\pi} \sum_i \epsilon_i \int d\sigma X^I \partial_\tau X^I \bigg|_{\tau = \tau_i}, \quad (2.8) \]

where the \( \tau_i \) are the initial(final) times for the scattering process and the \( \sigma \) integral for the \( i \)th string runs over an interval of length \( N_i \). For \( \tau_i \) very big, i.e. far from the interaction region, equation (2.5) can be inverted perturbatively and substituted into (2.7)(2.8) to obtain the results

\[ X = p_i \frac{N_i}{N_i} \tau_i + \sum_{i \neq j} \epsilon_j [p_j - p_i \frac{N_j}{N_i}] \log |z_i - z_j| + \cdots \]

\[ S = \sum_i \epsilon_i p_i^- \tau_i + \frac{1}{2} \sum_{i \neq j} \epsilon_i \epsilon_j [p_i^I p_j^I - p_i^- N_j - N_i p_j^-] \log |z_i - z_j| + \cdots, \quad (2.9) \]

where the \( p^- \) momentum is given in terms of the transverse momenta \( p_i \) and \( p^+ \) momentum \( N_i \) by

\[ p_i^- = \frac{|p_i^I|^2}{2N_i} \quad (2.10) \]

The dots in (2.9) correspond to exponentially small corrections. The first term in the action (2.9) is a phase factor corresponding to the time evolution of the incoming and outgoing states with respect to light cone Hamiltonian \( p^- \), and cancels out in scattering amplitudes (see [19]). The second term is the physically relevant Lorentz invariant contribution:

\[ \frac{1}{2} \sum_{i \neq j} k_i.k_j \log |z_i - z_j|, \quad (2.11) \]

where the \( k_i \) are ten momenta defined to be incoming to the scattering process, i.e. \( k_i = (N_i, |p_i^I|^2/(2N_i), p_i^I) \) for incoming states and \( k_i = -(N_i, |p_i^I|^2/(2N_i), p_i^I) \) for outgoing states.

The saddle point contribution to the action is found by looking for the stationary point of the above term under variations with respect to the modular parameters \( z_i \).

2.3. Classical world sheets in matrix string theory

The matrix string solutions for a general classical world sheet can be expected to be very complicated and to date no exact solution for a matrix string world sheet with
asymptotic states has been constructed \(^1\). A successful strategy was however developed for
the saddle point world sheet configurations which dominate high energy scattering amplitudes (see \([7]\)). Starting with the saddle point world sheet the authors of \([7]\) first identified
the type of square root (holomorphic and/or anti-holomorphic) involved at the interaction points. They then constructed instanton-like field configurations whose behaviour far from the core matched the square root behaviour of the world sheet. The approach is an
approximation, in that the instanton-like field configurations have a finite size and should
strictly be matched onto more than just the dominant square root dependence on the world
sheet coordinates. The true instanton-like field configuration would thus be expected to
be slightly modified away the core.

In \([7]\) this program was implemented explicitly for the construction of the four string
scattering process. Below we briefly review the main points. In the centre of mass frame
the scattering process describes a two dimensional plane and the corresponding classical
world sheet can be described by just two of the eight transverse coordinates, \(X^1\) and \(X^2\).
For convenience these are combined into a pair of complex fields defined by

\[
X = \frac{1}{\sqrt{2}}(X^1 + iX^2) \quad \text{and} \quad X^* = \frac{1}{\sqrt{2}}(X^1 - iX^2).
\]

(2.12)

The light cone world sheet has two branch points \(z_0^+\) and \(z_0^-\) given by the two roots of
the quadratic equation for four particle scattering of (2.6). The dominant classical world
sheet has the property that the complex scalar field \(X\) is an anti-holomorphic, respectively,
holomorphic function of \(z\) in the neighbourhood of the two branch points. Specifically :

\[
\partial_z X|_{z = z_0^+} = \sum_i \epsilon_i p_i (z - z_i)|_{z = z_0^+} = 0,
\]

(2.13)

and

\[
\partial_{\bar{z}} X|_{z = z_0^-} = \sum_i \epsilon_i p_i (\bar{z} - \bar{z}_i)|_{z = z_0^-} = 0.
\]

(2.14)

This symmetry allows a simple construction for the instanton-like field configurations
around the branch points. The starting point is the four dimensional self dual YM equation

\(^1\) Exact matrix string world sheets have been constructed and studied in depth in \([6,9]\), however
these are for classical world sheets which are entirely holomorphic or entirely anti-holomorphic
functions of the world sheet coordinate and thus cannot represent the matrix string versions of
the classical world sheets \([2,7]\). We thus do not consider them further in this article.

\(^2\) Scattering processes involving more than four particles would require three or more transverse
coordinates to describe their classical world sheets.
dimensionally reduced to two dimensions. The two possible signs for the self dual equations correspond to the two different holomorphic behaviours of equation (2.13). The instanton solution around the branch point $z_0^+$ is obtained from $F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$ with $\epsilon_{0129} = +1$. In terms of the two dimensional fields the equation reads

$$
F_{w\bar{w}} = -\frac{i}{g_s} [X, X^*]
$$

$$
D_w X = 0
$$

$$
D_{\bar{w}} X^* = 0
$$

(2.15)

where, in addition to the complex scalar fields defined in (2.12) we have introduced complex two dimensional gauge fields and coordinates

$$
A = \frac{1}{\sqrt{2}} (A^0 - iA^9) \quad \text{and} \quad w = \frac{1}{\sqrt{2}} (\sigma^0 + i\sigma^9).
$$

(2.16)

A simple solution to these equations of motion corresponding to a simple branch point involving two eigenvalues is

$$
X = U\hat{X}U^\dagger \quad \text{and} \quad A = ig_s U^\dagger \partial U \quad \text{with} \quad \hat{X} = B\sqrt{\bar{w}}\tau_3, \quad \text{and} \quad U = e^{\frac{i}{8} \ln \frac{w}{\bar{w}}} \tau_1.
$$

(2.17)

As already described this leads to a delta function singularity in the field strength at the interaction points. This singularity can be removed once we are working with complex coordinates $X$, by using a complexified “gauge” transformation $G$ which also has a singularity at the origin tuned in such a way as to leave a singularity free field strength. With this insight the solution can be written in the form

$$
X = UG\hat{X}G^{-1}U^\dagger
$$

$$
A = -ig_s [G^{-1}(\partial_w G) + U^\dagger (\partial_w U)],
$$

(2.18)

where the diagonal matrix $\hat{X}$ and unitary matrix $U$ are as in (2.17) and the matrix $G$ is given by

$$
G = e^{\alpha (w\bar{w})\tau_1}.
$$

(2.19)

This ansatz automatically satisfies the last two equations of (2.13) with the first equation leading to a differential equation for $\alpha$

$$
(\partial_r^2 + \frac{1}{r} \partial_r) \alpha = \frac{8B^2}{g_s^2} r \sinh 2\alpha \quad \text{with} \quad \alpha \to \begin{cases}
0 & \text{for} \quad r \to \infty \\
-\frac{1}{4} \ln r & \text{for} \quad r \to 0
\end{cases}
$$

(2.20)
where \( r = \sqrt{w\bar{w}} \) is the radial distance from the branch point. The boundary conditions are necessary for a finite solution. In particular the second boundary condition ensures that there are no \( \frac{1}{w} \) pole terms in the gauge field \( A \) and hence no delta function singularity in the field strength \( F_{w\bar{w}} \).

There is no explicit solution to this equation but it can easily be solved numerically. In terms of the function \( \alpha \) the expressions for the scalar fields \( X \) and the field strength \( F_{w\bar{w}} \) are given by the simple expressions

\[
X = B \sqrt{w} (\cosh \alpha \tau_3 + i \sinh \alpha \tau_2) \quad \text{and} \quad F_{w\bar{w}} = \frac{2iB^2}{g_s} r \sinh 2\alpha. \tag{2.21}
\]

For simplicity we do not include the final gauge transformation \( U \).

The instanton like solutions (2.21) are embedded into a global solution, defined on the cylinder, corresponding to the classical world sheet. The parameter \( B \) of equations (2.17),(2.20) and (2.21) is determined in terms the four external momenta by

\[
|B|^2 = \frac{|p_1 p_3^* - p_1^* p_3|(N_1 + N_2)}{\sqrt{N_1 N_2 N_3 N_4}}, \tag{2.22}
\]

where the \( p_i \) and \( p_i^* \) are the complex transverse momenta associated with the field \( X \) and its complex conjugate. The \( N_i \) are the \( p^+ \) momenta for the \( i \)th string.

Finally we note that there are two physical scales associated with the instanton. Firstly the differential equation (2.21) can be given a dimensionless form by absorbing the coupling constants into a rescaling of the radial coordinate. In other words the instanton has a natural world sheet scale :

\[
l_{\text{inst}} \sim \left( \frac{g_s^2}{B^2} \right)^{\frac{1}{4}}. \tag{2.23}
\]

Secondly, a simple analysis of the limiting behaviour of the field \( X \) at the origin shows that there is a minimal target space “distance” between the two strings :

\[
d_{\text{inst}} = \sqrt{\text{Tr}(X(0)X^*(0))} \sim g_s^{\frac{1}{4}} |B|^{\frac{1}{2}}. \tag{2.24}
\]

2.4. Kinematic relations for four string scattering

We will explicitly study the four string ground state scattering process. The calculation is performed in the centre of mass frame. In this subsection we give various kinematic identities useful for the calculations of later sections.

All particles are massless. They have transverse momentum lying in the \( X^1, X^2 \) plane specified by the complex numbers \( p_i = 1/\sqrt{2}(p_i^1 + ip_i^2) \). The \( p^+ \) momentum is given by
the length \( N_i \) of the string. To make contact with standard conventions for scattering processes we define all ten-vectors of momenta \( k_i \) to be incoming momenta:

\[
k_i = (p_i^+, p_i^-, p_i, p_i^*) = \begin{cases} (N_i, \frac{|p_i|^2}{N_i}, p_i, p_i^*) & \text{for } i = 1, 2 \\ (-N_i, -\frac{|p_i|^2}{N_i}, -p_i, -p_i^*) & \text{for } i = 3, 4 \end{cases}
\]  

(2.25)

where

\[
|p_i|^2 = p_i p_i^* = \frac{1}{2}((p_i^1)^2 + (p_i^2)^2).
\]  

(2.26)

The dot product of two momenta then reads

\[
k_i . k_j = -p_i^+ p_j^- + p_i^+ p_j^- + (p_i p_j^* + p_j p_i^*).
\]  

(2.27)

In the centre of mass frame conservation of momentum reads

\[
p_1 + p_2 = p_3 + p_4 = 0 \\
N_1 + N_2 = N_3 + N_4 = N \\
|p_1|^2 \frac{N}{N_1 N_2} = |p_3|^2 \frac{N}{N_3 N_4} = p^2,
\]

(2.28)

where in the final line (conservation of \( p^- \) momentum) we have used the first and second lines (conservation of transverse and \( p^+ \) momenta) to simplify the result and to define the quantity \( p^2 \). Inverting the third line we can write \( |p_1|^2 \) and \( |p_3|^2 \) in terms of \( p^2 \)

\[
|p_1|^2 = \frac{N_1 N_2}{N} p^2 \quad \text{and} \quad |p_3|^2 = \frac{N_3 N_4}{N} p^2.
\]  

(2.29)

Finally it is useful to define the ten dimensional Lorentz invariant quantities for the scattering process

\[
s = -(k_1 + k_2)^2 = -2k_1 . k_2 \\
t = -(k_2 + k_3)^2 = -2k_2 . k_3 \\
u = -(k_1 + k_3)^2 = -2k_1 . k_3,
\]

(2.30)

where the sign conventions are, as before, for \( k_1, k_2, k_3 \) and \( k_4 \) incoming momenta. Using the definitions for the \( k_i \) (2.25) along with the identities (2.28) and (2.29) the Lorentz invariants \( s, t \) and \( u \) can be written as

\[
s = 2p^2 \\
t = -2\frac{(N_1 N_3 + N_2 N_4)}{N^2} p^2 - 2q^2 \quad \text{with} \quad q^2 = p_1^* p_3 + p_3 p_1^*.
\]  

(2.31)
\[ p^2 \text{ in the above expressions is given in (2.28)(2.29). It is easy to check that the above expressions satisfy the identity relating } s, t \text{ and } u \text{ for massless particles} \]

\[ s + t + u = 0. \quad (2.32) \]

3. Perturbative calculations and the Euclidean action

There are two standard limits one can take in matrix string theory to recover perturbative string theory. Firstly one can take the limit \( N \to \infty \) followed by \( g_s \to 0 \). By the matrix string theory conjecture this is the limit necessary to recover perturbative type IIA string theory in uncompactified ten dimensional space. Secondly one can hold \( N \) finite and send \( g_s \to 0 \). By Susskind’s conjecture [20] this limit corresponds to string theory compactified on a light-like circle with \( N \) the number of \( p^+ \) quanta around the compact direction.

Since both are strong coupling limits one would not expect perturbative gauge theory calculations to be justified. Indeed, a perturbative gauge theory expansion would involve, through the three and four point vertices, arbitrary powers of \( 1/g_s \) which diverge as \( g_s \to 0 \). However, when deciding whether or not a perturbative calculation is justified it is the effective parameter weighting the loop expansion that is important not the size of the coupling constants. This is determined by both the coupling constants and the masses of the particles exchanged in the propagators. The masses of the quantum fluctuations are determined by the commutator term in the action \( (2.1) \) and are hence proportional to \( 1/g_s \). The mass of the propagators can thus compensate for the strength of the coupling constants. A good analogy to bare in mind is the top quark and Higgs in the standard model. The coupling of the top quark to the Higgs field is enormous (it is this that gives it its enormous mass). This does not mean that perturbative standard model calculations involving a virtual top quark and Higgs are not justified (ask any phenomenologist). There is precise cancellation between the mass in the propagator of the virtual top quark and the coupling of that virtual top quark to the Higgs. An identical cancellation happens in matrix(string) theory.

In the context of matrix string theory an analysis of the balance between these two effects, in the large \( N \) limit, was carried out in [10]. Starting from a bosonic background field there is a systematic expansion in powers of derivatives of the background fields for the effective action that one can calculate. It’s overall form is entirely determined by
dimensional reasoning and the identification of the loop counting parameter \( [21] [22] \). The result is that the effective action can be written in the form

\[
S_{\text{eff}} = \int d^2\sigma \left[ F^2 + \sum_{L=1}^{\infty} \mathcal{L}_L \right] \quad \text{with} \quad \mathcal{L}_L = \sum_{n=2}^{\infty} g_s^{2n-2} \frac{F^{2n}}{X^{4n+2L-4}}.
\]

(3.1)

where \( F^{2n}/X^{2m} \) means bosonic terms with \( 2n \) derivatives in the numerator and \( 2m \) powers of the scalar fields \( X^I \) in the denominator.

The analysis of \([10]\) showed that the two standard limits mentioned above lead to divergent loop expansions. For the case \( N \rightarrow \infty \) before \( g_s \rightarrow 0 \) the tensor structures of the terms \( F^{2n}/X^{4n+2L-4} \) are such that the loop expansion diverges with positive powers of \( N \). The physical cause of the divergence is that neighbouring strips (eigenvalues) of the background long string configuration come arbitrarily close together in the limit \( N \rightarrow \infty \). The off diagonal field variables connecting the neighbouring strips thus become massless in this limit. Holding \( N \) finite and sending \( g_s \rightarrow 0 \) leads to a different kind of problem. Naively from (3.1) the effective action would be well behaved in this limit. However close to the interaction points the instanton-like field configurations depend upon \( g_s \). Taking this into account leads to a loop expansion weighted by inverse powers of \( g_s \), (see \([10]\)) which diverges in the limit \( g_s \rightarrow 0 \). Physically the reason for this divergence is that, as can be seen from (2.24), the “minimal distance” between the two strings tends to zero in the limit \( g_s \rightarrow 0 \). This again leads to massless fluctuations and a divergence. To be able to calculate in these two limits would thus first require the development of direct strong coupling techniques.

In this article we will avoid these problems by studying a third limit which will allow us to calculate a perturbative string theory scattering amplitude by performing a perturbative gauge theory calculation. The limit corresponds to high energy string scattering in the finite \( N \) version of matrix string theory. Specifically we will hold \( N \) fixed, hold \( g_s \) fixed and study the dominant contribution in the limit in which the external momenta, \( p_i \), of the scattering process tend to infinity. Note that \( N \) can be large and \( g_s \) can be small. What is important is that we send \( p_i \rightarrow \infty \) before taking any other limit.

The strategy of this paper is to perform a fluctuation calculation about a classical matrix string theory background corresponding to one of the dominant saddle point world sheets found by Gross and Mende \([11] [12]\) in the \( p_i \rightarrow \infty \) limit. As can be seen directly from (2.7) the target space size of the saddle point backgrounds are proportional to the external momenta. \( p_i \rightarrow \infty \) is thus a limit in which the target space size of the background
becomes infinite. Looking at loop expansion (3.1) we see that there are always more powers of the background field in the numerator than in the denominator. In the limit in which the background becomes infinitely large these terms will thus be scaled away altogether. Physically the limit has the effect of separating, in target space, the individual strips(eigenvalues) of the matrix background, with the result that the off-diagonal elements connecting together the strips become infinitely massive. Again this simple argument could be invalidated by the existence of the interaction points where two eigenvalues are connected by a branch point and come close together. As already stated the instanton-like field configurations lead to their being a “minimal distance” (2.24) between the two eigenvalues. Reading off directly from (2.24)(2.22) we see that this minimal distance tends to infinity if we take the limit $p_i \to \infty$ while holding $N$ and $g_s$ fixed. There is thus no problem of massless fluctuations as there was for the $g_s \to 0$ limit.

To summarize, taking the limit $p_i \to \infty$, before taking any other limit will lead to a loop expansion in which the contributions from higher order loops will be scaled away. It is thus justified in this limit to use a one loop calculation. As will be discussed below the calculation reduces to the evaluation of the determinant for the quadratic fluctuations (by the above arguments this will be effectively equal to one) and to an integration over the zero modes.

A final comment is in order. Asymptotically far from the interaction points the classical background splits into separate blocks for the different strings. Each block is proportional to a unit matrix. The off diagonal elements within a block are thus massless. This is a result of the fact that we are not using true matrix string wavefunctions. Presumably there is some LSZ type reasoning that could be developed to justify replacing the true matrix string wavefunctions by simple diagonal blocks.

3.1. Euclidean fermions

The fermions of the action (2.1) are Majorana Weyl fermions in ten dimensional Minkowski space and consist of sixteen real fermionic components. We will need to work in ten dimensional Euclidean space (dimensionally reduced to two dimensions) where the instanton configuration and functional integral are defined, and where it is not possible to define Majorana Weyl spinors. In addition the Majorana Weyl fermions satisfy the commutation relations

$$\{S^a(\sigma), S^b(\sigma')\} = \pi \delta^{ab} \delta(\sigma - \sigma')$$

and

$$\{\tilde{S}^{\hat{a}}(\sigma), \tilde{S}^{\hat{b}}(\sigma')\} = \pi \delta^{\hat{a}\hat{b}} \delta(\sigma - \sigma'),$$

(3.2)
where \( a, b \) and \( \dot{a}, \dot{b} \) denote respectively the \( 8_s \) and \( 8_c \) representations of \( SO(8) \). In other words the spinor \( S^a \) is simultaneously a coordinate and its conjugate momenta, and similarly for \( \bar{S}^{\dot{a}} \). The spinor variables can however be combined into complex spinor coordinates, which are distinct from their conjugate momenta, and which can be used to define a Euclidean action. The complexification procedure means that the \( SO(8) \) symmetry of the theory will no longer be manifest, although, as we will see, it is restored in the calculation of physical amplitudes. A similar complexification procedure has to be carried out in light cone superstring theory [19].

The calculation of this paper is up to quadratic order only in the background fluctuations. To quadratic order the Dirac operator involves only four gamma matrices, \( \Gamma^0 \) and \( \Gamma^9 \), which couple to the background gauge field and \( \Gamma^1 \) and \( \Gamma^2 \) coupling to the background fields for \( X^1 \) and \( X^2 \). This fact permits a simple definition for the Euclidean action. We start by choosing a basis for the gamma matrices such that the four gamma matrices, \( \Gamma^0, \Gamma^9, \Gamma^1 \) and \( \Gamma^2 \), can be written as four by four blocks (see appendix). We then define four spinor coordinates \( \theta^A, \bar{\theta}^A, \bar{\dot{\theta}}^A \) and four corresponding momenta \( \lambda_A, \bar{\lambda}_A, \bar{\dot{\lambda}}_A, \bar{\bar{\dot{\lambda}}}^A \) by (see appendix)

\[
(v \otimes v \otimes 1)S^a = \begin{pmatrix} \theta^A \\ \bar{\theta}^A \\ \lambda_A \end{pmatrix} \quad \text{and} \quad (v \otimes v \otimes 1)\bar{S}^{\dot{a}} = \begin{pmatrix} \bar{\theta}^{\dot{A}} \\ \dot{\lambda}^{\dot{A}} \end{pmatrix}, \quad \text{with} \quad v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.
\]

The indices \( A \) and \( \dot{A} \) take the values \( A, \dot{A} = 1, 2 \). Note that \( \bar{\theta} \) is not the complex conjugate of \( \theta \), they are distinct variables. The notation is chosen to match standard conventions for spinors in four dimensional space (see [23]). This choice of coordinates breaks the \( spin(8) \) symmetry of the spinors down to \( U(1) \otimes U(1) \otimes SU(2) \otimes SU(2) \). The two \( U(1) \)’s correspond to \( SO(2) \) rotations in the \( X^1, X^2 \), and (with the basis of gamma matrices chosen in the appendix) \( X^3, X^4 \) planes. The \( SU(2) \otimes SU(2) \) corresponds to an \( SO(4) \) for the remaining four dimensional space \( X^5, X^6, X^7 \) and \( X^8 \). The charges of the fermions under the rotations in the \( X^1, X^2 \) and \( X^3, X^4 \) planes as well as the transformations under rotations in the \( X^5, X^6, X^7, X^8 \) can be read off directly from the form of the eight dimensional \( \Gamma^{I\bar{J}} \) (see appendix). The final result of this decomposition is that we have the following bosonic and fermionic coordinates.

\[
X, X^* = \frac{1}{\sqrt{2}}(X^1 \pm iX^2) \quad \theta^A(\frac{1}{2}, \frac{1}{2})
\]

\[
\bar{X}, \bar{X}^* = \frac{1}{\sqrt{2}}(X^3 \pm iX^4) \quad \bar{\theta}^{\dot{A}}(\frac{1}{2}, \frac{1}{2})
\]

\[
X^m \quad \text{for} \quad m = 5, \cdots, 8 \quad \bar{\theta}^{\dot{A}}(\frac{1}{2}, \frac{1}{2})
\]

\[
\bar{\theta}^{\dot{A}}(\frac{1}{2}, \frac{1}{2})
\]

\[
\bar{\theta}^{\dot{A}}(\frac{1}{2}, \frac{1}{2})
\]

\[
\bar{\theta}^{\dot{A}}(\frac{1}{2}, \frac{1}{2})
\]
The first and second arguments of the fermions are their $U(1)$ charges under rotations respectively the $X^1, X^2$ and $X^3, X^4$ planes. The absence or presence of the bar above the fermions indicates how they transform under rotations in the $X^5, X^6, X^7, X^8$ space. Specifically under such an infinitesimal rotation we have

$$\delta X^m = \frac{1}{2} w^{mn} J^{pq} X^q \quad (J^{pq}) = \delta^{mp} \delta^{nq} - \delta^{mq} \delta^{np}$$

$$\delta \theta^A = \frac{1}{4} w^{mn} (\sigma^{mn})^A_B \theta^B \quad \text{with} \quad (\sigma^{mn})^A_B = \frac{1}{2} (\sigma^m \sigma^n - \sigma^n \sigma^m)$$

$$\delta \bar{\theta}^A = \frac{1}{4} w^{mn} (\bar{\sigma}^{mn})^A_B \bar{\theta}^B \quad (\bar{\sigma}^{mn})^A_B = \frac{1}{2} (\bar{\sigma}^m \sigma^n - \bar{\sigma}^n \sigma^m)$$

The indices $mnpq$ run from 5 to 8. The $\sigma$ and $\bar{\sigma}$ matrices are given in terms of the pauli matrices by

$$(\sigma^m)^A_B = (i, \tau_1, \tau_2, \tau_3) \quad \text{and} \quad (\bar{\sigma}^m)^A_B = (-i, \tau_1, \tau_2, \tau_3).$$

Note that this decomposition of the fermions into coordinates and momenta differs from the standard decomposition used in light cone superstring theory [24][19] where one breaks the $spin(8)$ symmetry of the spinors into $U(1) \otimes SU(4)$. The classical background further breaks the $SU(4)$ down to $U(1) \times SU(2) \times SU(2)$.

In section 8 we will use the transformation properties of the spinors $\theta, \bar{\theta}, \tilde{\theta}$ and $\tilde{\bar{\theta}}$ to construct incoming and outgoing wavefunctions transforming under $SO(8)$ rotations. Specifically we will construct the combinations of $\theta$ and $\bar{\theta}$ corresponding to the $X, X^*, \bar{X}, \bar{X}^*$ and $X^m$ components of an $SO(8)$ vector and similarly for $\tilde{\theta}$ and $\tilde{\bar{\theta}}$. These “left” and “right” $SO(8)$ vectors will form incoming and outgoing graviton states. We will then explicitly calculate the four graviton scattering amplitude.

3.2. The Euclidean action

It is convenient in this section to use ten dimensional notation for the bosonic fields. Indices $\mu$ run from $\mu = 0, \cdots, 9$ and split into indices 0,9 for the two dimensional cylindrical coordinates on which the fields depend and $I = 1, \cdots , 8$. All bosonic fields are denoted by $A^\mu$ with $A^I = X^I$. We then split the fields into a background part, $A$, corresponding to the classical matrix string solution and a fluctuation part $V$:

$$A_{total} = A + V,$$

with $V^I = Y^I$. To calculate the effect of quantum fluctuations about the classical configuration the action needs to be gauge fixed. We use the standard background covariant gauge fixing term

$$\mathcal{L}_{gf} = (D_\mu V_\mu)^2 \quad \text{with} \quad D_\mu = \partial_\mu + i \frac{g_s}{2} [A_\mu , ].$$
The Euclidean Lagrangian for the quadratic fluctuations reads

$$\mathcal{L}_{Eu} = \text{Tr} \left[ - (D_\mu V^\mu)^2 - 2 \frac{i}{g_s} V^\mu [F^{\mu\nu}, V^\nu] + \lambda^T \overline{\partial} \theta + c^* D^2 c \right], \quad (3.9)$$

where $V$ are the bosonic fluctuations and $\theta$ and $\lambda$ the fermionic coordinates and momenta defined in (3.3). The fields $c$ are the ghosts. $D$ is the background covariant derivative given in (3.8). Using the definition of gamma matrices given in the appendix the fermionic part of the Lagrangian reads

$$\lambda^T \overline{\partial} \theta = \left( \lambda_A \tilde{\lambda}_B \lambda_{\bar{A}} \bar{\lambda}_{\bar{B}} \right) \begin{pmatrix} D_w & \frac{i}{g_s} [X^*, ] \\ -\frac{1}{g_s} [X, ] & D_{\bar{w}} \end{pmatrix} \begin{pmatrix} \theta^A \\ \bar{\theta} \end{pmatrix} \quad (3.10)$$

4. The fluctuation determinant

The fermionic coordinates $\lambda$ act as Lagrange multipliers and integrating over them leads to $\delta(\overline{\partial} \theta)$. In other words the functional integral for the fermionic coordinates $\theta$ is projected down onto their zero modes. This is discussed in more detail in the context of light cone string theory in [25]. The Jacobian factor from this projection is $\det(\overline{\partial} \partial)$. In this section we focus on the contribution from the non zero modes i.e. on the determinants. For the four string scattering background considered in this paper both the fermion term and the boson term consist of four by four non-trivial blocks. Canceling part of the trivial block of the bosonic determinant with the ghost determinant the total fluctuation determinant $J$ reads

$$J = \frac{\det(\overline{\partial} \partial)^{4 \times 4}}{\det(D^2)^{2 \times 2} \det \frac{1}{2} \left( D^2 + 2 \frac{i}{g_s} F_{\rho\sigma}, \right)^{4 \times 4}}, \quad (4.1)$$

where the indices $\rho$, $\sigma$ take the values 0, 9, 1, 2, and zero modes have been excluded from the determinants. The fermion determinant is in four by four spinor space and the boson determinant in four by four $\rho$, $\sigma$ space.

It is well known that the fermionic and bosonic fluctuation determinants precisely cancel in a self dual background. Using the definition of gamma matrices given in the appendix it is only a short calculation to verify that this is indeed the case. Specifically for the instanton background of section 2.2 we have

$$F_{\rho\sigma} = \begin{pmatrix} F_{w\bar{w}} & D_w X^* & 0 \\ -D_{\bar{w}} X & -F_{w\bar{w}} & 0 \\ 0 & -F_{w\bar{w}} & D_{\bar{w}} X \end{pmatrix} \quad (4.2)$$
where we have used complex indices, i.e. rows labeled from top to bottom by \( w, x, \bar{w}, \bar{x} \) and columns labeled from left to right by \( \bar{w}, \bar{x}, w, x \). Using (3.10) we have

\[
\mathcal{D}^\dagger \mathcal{D} = \frac{1}{2} \left( D^2 + \frac{2i}{g_s} [\tilde{F}, \mathcal{J}] \right)
\]

with

\[
\tilde{F} = \begin{pmatrix}
0 & F_{w\bar{w}} & D_w X^* \\
-F_{w\bar{w}} & -D_w X & -F_{w\bar{w}}
\end{pmatrix},
\]

leading to \( J = 1 \).

In other words if the background was self dual everywhere the fluctuation determinant would be equal to one. The matrix string background corresponding to four string scattering is however only locally self dual around the branch points and its fluctuation determinant would be corrected by perturbations from the non self dual part away from the interaction points. The important point to note is that, in the regime where it is justified to use a one loop calculation, these are just small corrections and to lowest order can be dropped. The conclusion from this analysis is that the interaction points do not lead to a singular contribution to the determinant. This is true even in the limit where \( g_s \) and hence (via (2.23)) the size of the instanton, tend to zero.

It is interesting to compare this result with what one would expect from a CFT fluctuation determinant around a branch point. In light cone superstring theory \[25\] \( \theta \) and \( \lambda \) have the conformal weights 1 and 0, respectively. Potentially there is a singular contribution coming from the square root cut point which hides a world sheet curvature singularity. Smoothing this out over some cut off distance \( \epsilon \) leads to a singular contribution \( 1/\epsilon^{17} \) coming from the induced Liouville action for the bosons and fermions. However for the choice of weights given above the bosonic and fermionic central charges cancel and there is no singular contribution. This can also be seen directly from a comparison of the determinants (see [26]).

The conclusion of this section is that there are no singular contributions to the determinant coming from the interaction points and that this is consistent with the light cone superstring calculation. This does not mean, however, that the fluctuation determinant will be equal to precisely one. We will return to this point in section 8. The remaining part of the functional integral is an integration over a finite number of zero modes and collective coordinates of the background configuration. In the next two sections we construct these modes.
5. **Instanton zero modes, local considerations**

In this section we focus on the field configurations in the neighbourhood of the instanton. The objective is to understand how the core of the instanton, where the fields do not commute, effects the finiteness of the zero mode field configurations. Far from the core of the instanton the fields commute and the zero modes correspond to those defined on a Riemann surface in the neighbourhood of a branch point. In particular we will see that there are zero modes which, from the Riemann surface point of view, would appear to diverge at the branch point, but which are rendered finite by the non-commuting core of the instanton.

![Diagram showing finite field configurations for “singular” zero modes](image)

**Fig 2.** Finite field configurations for “singular” zero modes shown in the $w$ plane and the $z = \sqrt{w}$ plane.

These “singular” modes will play a crucial role in the next section where we construct global zero modes. The instanton is embedded into a global classical solution corresponding to a classical string world sheet. The instanton zero modes will thus be embedded into global zero modes which tend, asymptotically far down the strings, to constant values. From the point of view of the world sheet these modes can be written as holomorphic and/or anti-holomorphic functions of the cylindrical coordinates. The singularities at the branch points are important since it is only with singularities that one can construct non-trivial global modes.

The global zero modes will include the moduli for the classical string world sheet along with their superpartners and zero modes for the abelian gauge theory defined on the string world sheet.

For the purposes of this section we will treat the instanton as being embedded into the complex plane. Below we will explicitly construct the bosonic and fermionic zero modes, imposing the condition that they must be finite everywhere (except at infinity).
5.1. Bosonic zero modes and collective coordinates

The classical solution for the instanton \((2.18)(2.19)(2.20)\) involves only the two dimensional gauge field \(A\) and two of the 8 possible scalar fields, \(X_1\) and \(X_2\). There are three types of bosonic zero modes for the fluctuations about this configuration characterized by which fields they tend to asymptotically. Firstly there are the two-dimensional gauge field zero modes. These come from the fact that the gauge fixing term doesn’t completely fix the gauge. In other words there are local gauge transformations that are zero modes of the gauge fixing term. The second correspond to translations and deformations of the instanton solution, and asymptotically involve only the scalar fields \(X_1\) and \(X_2\). Finally there are modes involving \(X_3, \cdots, X_8\). All three types are zero modes of the eigenvalue equation for quadratic fluctuations about the classical instanton background:

\[
\left[ D_\rho D_\rho \eta_{\mu\nu} + \frac{2i}{g} [F_{\mu\nu}, \ ] \right] V^{(n)}_\nu = \lambda^{(n)} V^{(n)}_\nu. 
\] (5.1)

In this equation the \(V^{(n)}_\nu\) are the fluctuation modes and \(\lambda^{(n)}\) their eigenvalues. For compactness the equation has been written in its ten dimensional form. The ten dimensional indices \(\mu, \nu = 0, \cdots, 9\) split into the two dimensional indices \(\alpha, \beta = 0, 9\), corresponding to the cylindrical coordinates, and the indices \(I = 1, \cdots, 8\) corresponding to the scalar matrix fields. The covariant derivatives \(D_\mu\) are covariant with respect to the background fields, i.e. \(D_\mu = \partial_\mu + i/g[A_\mu, \ ]\), and \(F_{\mu\nu}\) is the background field strength. The standard Feynman background gauge fixing term \((D_\mu A_\mu)^2\) has been used.

5.2. Gauge field zero modes

For the gauge field zero modes we search for gauge-fields, \(V_\mu\), that are locally pure gauge,

\[
V_\mu = D_\mu \Lambda, 
\] (5.2)

and which satisfy equation\((5.1)\). Substituting \((5.2)\) into \((5.1)\) and commuting the covariant derivative to the right through the \(D^2\) term cancels the \(F_{\mu\nu}\) term and leads us to look for modes \(\Lambda\) that satisfy

\[
D_\rho D_\rho \Lambda = 0, 
\] (5.3)

i.e. to look for zero modes of the gauge fixing term. The resulting gauge fields \(V_\mu\) must be finite everywhere (except possibly at infinity) and must tend to some function times \(\tau_3\) at infinity so that they commute with the bosonic fields. Note that equation\((5.3)\) is just the
equation of motion for the complex bosonic field $X$. So we can immediately write down a candidate zero mode:

$$\lambda^{(\frac{1}{2})} = X^*. \quad (5.4)$$

with $X$ given by (2.21). The gauge field zero mode $V^{(-\frac{1}{2})} = D_w \lambda^{(\frac{1}{2})}$ is given by

$$V^{(-\frac{1}{2})} = \frac{1}{w^{\frac{1}{2}}} [(\frac{1}{2} \cosh \alpha + r \partial_r \alpha \sinh \alpha) \tau_3 + i(\frac{1}{2} \sinh \alpha + r \partial_r \alpha \cosh \alpha) \tau_2]. \quad (5.5)$$

One still has to check that $V^{(-\frac{1}{2})}$ is finite at the origin, since it potentially diverges as $1/r$. Using the fact that for small $r \alpha(r) = -1/2 \log r + a + \mathcal{O}(r^2)$, we see that it is indeed finite at the origin:

$$V^{(-\frac{1}{2})} \to \frac{r^{\frac{1}{2}}}{w^{\frac{1}{2}}} \frac{1}{2} e^{-a} [\tau_3 + i \tau_2] \quad \text{as} \quad r \to \infty, \quad (5.6)$$

and is thus a valid zero mode.

The most general solution for the gauge zero modes can be found by writing equation (5.3) in terms of the two-dimensional fields and covariant derivatives:

$$D_{\rho} D_{\rho} \Lambda = D_w D_{\bar{w}} \Lambda + D_{\bar{w}} D_w \Lambda - \frac{1}{g^2} ([X, [X^*, \Lambda]] + [X^*, [X, \Lambda]]) = 0. \quad (5.7)$$

It is straightforward using (2.13) to see that it is satisfied by

$$\Lambda^{(n)} = \lambda^{(n)} + (\lambda^{(n)})^*, \quad (5.8)$$

with

$$\lambda^{(n)} = \begin{cases} w^{n-\frac{1}{2}} X^* & \text{for} \quad n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \\ w^{n} 1_{2 \times 2} & \text{for} \quad n = 1, 2, 3, \ldots \end{cases} \quad (5.9)$$

and

$$V^{(n)} = D_w \lambda^{(n-1)}. \quad (5.10)$$

All these solutions are of course written in the singular gauge in which asymptotically far from the instanton all fields are diagonal.

The physical meaning of these zero modes is given by their behaviour far from the non-commuting core of the instanton where they can be interpreted as zero modes of an abelian gauge field defined on a Riemann surface in the neighbourhood of a branch point. They are zero modes of the abelian gauge-fixing term $(\partial_\alpha V_\alpha)^2$. We see that we have the set of abelian pure gauge zero modes,

$$V_w = w^{-\frac{1}{2}}, 1, w^{\frac{1}{2}}, w, w^{\frac{3}{2}}, w^2, \ldots, \quad (5.11)$$

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in the complex $w$ plane. In particular there is a mode $V^{(-\frac{1}{2})}$ which is finite at the origin of the instanton but asymptotically behaves as $w^{-\frac{1}{2}}$. Unwinding the branch cut $w = z^2$, $V_z = 2\sqrt{w}V_w$ we see that we have the full set of abelian gauge field zero modes,

$$V_z = 1, z, z^2, z^3, z^4, \ldots,$$

(5.12)
defined in the complex $z$ plane. In section 4 we study the zero modes about the classical solutions corresponding to light-cone Riemann surfaces. In other words we embed the zero modes constructed above into global abelian zero modes defined on the light cone Riemann surfaces. The fact that we have the full set of zero modes (5.12) means that we will be able to construct arbitrary abelian zero modes on the Riemann surface which are not in any way pinned to, or constrained by, the interaction points.

The conclusion from this section is that for the construction of the global zero modes for the complex gauge field $V$ we are allowed a $(w - w_0^+)^{-\frac{1}{2}} \sim 1/(z - z_0^+)$ singularity at $z = z_0^+$ and a $(w - w_0^-)^{-\frac{1}{2}} \sim 1/(z - z_0^-)$ singularity at $z = z_0^-$.  

5.3. Zero modes for the ghosts and for $X^3, \ldots, X^8$

The zero mode equation for the ghosts is identical to equations (5.3)(5.7). This allows to automatically write down all their zero modes. They are given by the solutions for $\Lambda$ (equations (5.8)(5.9)) of the previous section. The important point is that there is no mode that asymptotically behaves as $w^{-\frac{1}{2}}$ and is finite at the origin, so there will be no non-trivial global zero mode we can construct.

Since the background only depends on the scalar fields $X^1$ and $X^2$ the zero mode equation for the scalar fields $X^3, \ldots, X^8$, contains no $F_{\mu\nu}$ term and is thus also identical to equations(5.3)(5.7). So again there will be no non-trivial zero modes.

5.4. Zero modes for $X^1$ and $X^2$

The most obvious zero mode to write down for the fields $X = 1/\sqrt{2}(X^1 + iX^2)$ and $X^*$ is that corresponding to the translation of the instanton configuration. To construct this mode one proceeds as for the four dimensional instanton by taking the derivative of all the fields and at the same time shifting them by a gauge transformation so that the background gauge fixing condition, $D_\mu V_\mu$, is preserved. The result being that the zero modes of (5.1) that correspond to translations are given by

$$V_\mu = \partial_\alpha A_\mu - D_\mu A_\alpha = F_{\alpha\mu}.$$ 

(5.13)
The first term corresponds to translation in the $\alpha$ direction, the second term is the gauge transformation with gauge parameter $A_\alpha$. Substituting (5.13) into (5.1) and using the equations of motion and the Bianchi identity this is indeed seen to be a zero mode of (5.1). Written in terms of the two dimensional complex fields of section 2 the solution reads

$$
\begin{pmatrix}
V \\
Y
\end{pmatrix} = 
\begin{pmatrix}
-F_{w\bar{w}} \\
D_{\bar{w}}X
\end{pmatrix} \rightarrow \tau_3 \begin{pmatrix} 0 \\
\bar{w}^{-1/2} \end{pmatrix},
$$

(5.14)

Its asymptotic behaviour, indicated on the r.h.s, shows that from the world sheet point of view it is singular at the origin. The non-commutative core of the instanton, however, renders the field configuration finite at the origin (see equation (5.6)). As already stressed the fact that it is a singular zero mode from the world sheet point of view will allow us to construct a non-trivial global zero mode. It will correspond to deforming the classical world sheet by moving a branch point.

It is important to check to see if there are any other singular zero modes. The only other possibility would be a mode asymptotically behaving as $w^{-\frac{1}{2}}$. We write equation (5.1) in terms of the two dimensional covariant derivatives, field strengths and scalar fields:

$$
\left\{ D_\rho D_\rho + 2i \frac{g}{2} \begin{pmatrix}
[F_{w\bar{w}}, ] & [D_w X^*, ] \\
[D_{\bar{w}} X, ] & [-F_{w\bar{w}}, ]
\end{pmatrix}\right\} \begin{pmatrix} V \\
Y \end{pmatrix} = 0,
$$

(5.15)

where $D_\rho D_\rho$, in terms of two dimensional covariant derivatives and fields, has already been given in (5.7). It is possible to find simple generalizations of (5.14), satisfying (5.15), for arbitrary half-integer powers of $\bar{w}$:

$$
\begin{pmatrix} V \\
Y \end{pmatrix} = \begin{pmatrix} -\bar{w}^n F_{w\bar{w}} \\
D_{\bar{w}}(\bar{w}^n X) \end{pmatrix} \rightarrow \tau_3 \begin{pmatrix} 0 \\
\bar{w}^{n-1/2} \end{pmatrix} \quad n \geq 0
$$

(5.16)

along with the trivial solution

$$
Y = \bar{w}^n 1_{2x2} \quad n \geq 0.
$$

(5.17)

However there is no obvious, simple generalization of (5.16) for powers of $w$. It would seem natural, however, that there would NOT be a mode behaving as $w^{-\frac{1}{2}}$ which is also finite at the origin.

We will thus assume that this is the case (to prove it would mean analyzing carefully the coupled differential equations of (5.15)). With this assumption the conclusion from this section then is that for the construction of the global zero modes we are allowed a $(\bar{w} - \bar{w}_0^\mp)^{-\frac{1}{2}} \sim 1/(z^\mp - z_0^\mp)$ singularity at $z = z_0^+$ and a $(w - w_0^-)^{-\frac{1}{2}} \sim 1/(z - z_0^-)$ singularity at $z = z_0^-$. 

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5.5. Fermionic zero modes

The fermion zero mode equation reads
\[ \mathcal{D} \theta = 0, \tag{5.18} \]
with \( \mathcal{D} \theta \) given explicitly in (3.10). The most important zero mode is that coming from the Euclidean version of the infinitesimal Minkowski space supersymmetry transformation,
\[ \theta = F_{\mu \nu} \Gamma_{\mu \nu} \epsilon, \tag{5.19} \]
where \( \epsilon \) is a constant ten-dimensional Majorana-Weyl spinor. This can be shown to be a solution of (5.18) by using the gamma matrix identity
\[ \Gamma_\rho \Gamma_{\mu \nu} = \Gamma_{\rho \mu \nu} + \frac{1}{2} (\eta_{\rho \mu} \Gamma_\nu - \eta_{\rho \nu} \Gamma_\mu), \]
followed by the Bianchi identity and the equation of motion for \( F_{\mu \nu} \).

It is easy to see from (3.10) that the Euclidean fermion zero mode equivalent to (5.19) has the form \( \theta = \tilde{\theta} = 0 \) and
\[ \left( \begin{array}{c} \bar{\theta} \\ \theta \end{array} \right) = \left( \begin{array}{cc} 0 & D_w X \\ -F_{w \bar{w}} & 0 \end{array} \right) \left( \begin{array}{c} \bar{\theta} \\ \theta \end{array} \right) \rightarrow \left( \begin{array}{c} \bar{\theta} \\ \theta \end{array} \right) \left( \begin{array}{c} \bar{\theta} \tilde{\theta} \\ \tilde{\theta} \end{array} \right), \tag{5.20} \]
where \( \bar{\theta} \) and \( \tilde{\theta} \) are two component constant, complex spinor variables. The asymptotic behaviour on the r.h.s. of (5.20) shows that from the world sheet point of view the mode is singular at the origin. As for the bosonic zero modes the field configuration is in fact finite at the origin.

Again it is important to check that there are no other “singular” zero modes. Using the explicit expression for \( \mathcal{D} \) of (3.10) it is relatively simple to see that \( \theta = \tilde{\theta} = 0 \) and
\[ \left( \begin{array}{c} \bar{\theta} \\ \theta \end{array} \right) = \left( \begin{array}{cc} 0 & D_w X \\ -F_{w \bar{w}} & 0 \end{array} \right) \left( \begin{array}{c} \bar{\theta} \\ \theta \end{array} \right) \rightarrow \left( \begin{array}{c} \bar{\theta} \\ \theta \end{array} \right) \left( \begin{array}{c} \bar{\theta} \tilde{\theta} \\ \tilde{\theta} \end{array} \right), \tag{5.21} \]
for \( \bar{\theta} \) and \( \tilde{\theta} \) we have already found the “singular” zero mode (3.19) and there are no other possibilities (we cannot, for example, have \( \bar{\theta} \) behaving as \( w^{-\frac{1}{2}} \) since asymptotically \( \bar{\theta} \) must be an anti-holomorphic function of \( w \)).

The fermion zero mode (5.20) is for the self-dual configuration around the branch point \( w = w_0^+ \). The zero mode around the anti-self-dual configuration\(^3\) of the \( w_0^- \) branch point can also be read off from (3.10). It is given by \( \bar{\theta} = \bar{\theta} \tilde{\theta} = 0 \) and
\[ \left( \begin{array}{c} \theta \\ \bar{\theta} \end{array} \right) = \left( \begin{array}{cc} F_{w \bar{w}} & D_{\bar{w}} X^* \\ D_w X & F_{w \bar{w}} \end{array} \right) \left( \begin{array}{c} \tilde{\theta} \\ \theta \end{array} \right) \rightarrow \left( \begin{array}{c} \theta \\ \bar{\theta} \end{array} \right) \left( \begin{array}{c} \theta \\ \tilde{\theta} \end{array} \right) \left( \begin{array}{c} \theta \tilde{\theta} \\ \tilde{\theta} \end{array} \right). \tag{5.21} \]

The conclusion is that for the construction of global fermion zero modes in the next section \( \theta, \tilde{\theta}, \bar{\theta} \) and \( \tilde{\theta} \) can respectively have the singularities \( 1/(\bar{z} - \bar{z}_0^-) \), \( 1/(z - z_0^-) \), \( 1/(\bar{z} - \bar{z}_0^+) \), and \( 1/(z - z_0^+) \).

\(^3\) For the anti-self-dual instanton we have \( D_w X^* = 0, D_{\bar{w}} X = 0 \) and \( F_{w \bar{w}} + \frac{i}{g_s} [X^*, X] = 0 \).
6. Zero modes, global considerations

In the previous section we analyzed in detail the bosonic and fermionic zero modes in the neighbourhood of the instanton configuration for the branch points of the matrix string world sheet. In particular we identified zero mode field configurations which are finite at the branch points but from the world sheet point of view appear to have a singularity at the branch point. In this section we take these zero modes and embed them into non-trivial zero modes for the gauge, boson and fermion fields for global classical solutions corresponding to light cone Riemann surfaces.

We will construct global zero modes which tend to constant values asymptotically far down the strings. Specifically we will construct modes satisfying the boundary conditions

\[ V_\tau = \partial_\tau V_\sigma = 0, \quad \partial_\tau Y^I = 0, \quad \partial_\tau \theta = 0 \quad \text{for} \quad \tau = \tau_i \text{ or } \tau_f, \]  

(6.1)

where \( \tau_i \) and \( \tau_f \) are the initial and final times.

From the world sheet point of view all zero modes can be decomposed as holomorphic and/or anti-holomorphic functions of \( z \), with possible simple pole singularities at the branch points. At the end of sections 4.2, 4.3, 4.4 and 4.5 we enumerated the possible pole structures for the bosonic and fermionic modes. In the following sections we use these singularities to explicitly construct non-trivial global zero modes satisfying the boundary conditions (6.1).

6.1. Abelian gauge and ghost field zero modes

The two dimensional gauge field \( V = 1/\sqrt{2}(V_\tau - iV_\sigma) \) must be a holomorphic function of \( w \) or equivalently of \( z \). We have already seen in the previous section that it can have the poles \( 1/(z - z_0^+) \) at the branch point \( z_0^+ \) and \( 1/(z - z_0^-) \) at the branch point \( z_0^- \). This allows us to construct the following zero mode

\[ V = i \sum_i \epsilon_i \frac{a_i}{z - z_i} \rightarrow \frac{a_i}{N_i} \quad \text{for} \quad z \to z_i, \]  

(6.2)

with the \( a_i \) real numbers satisfying

\[ \sum_i \epsilon_i a_i = 0. \]  

(6.3)

Note from equation (2.6) that the numerator of (6.2) has zeros at the branch points and hence gives the correct pole structure.

Asymptotically far down a string (\( z \to z_i \)) the boundary conditions (6.1) are satisfied with \( V_\sigma \) tending to the constant value \( a_i/N_i \) as indicated in (6.2). A Wilson loop encircling the string thus generates the \( U(1) \) element \( e^{i\epsilon_\sigma a_i} \).

The mode (6.2) is the most general abelian differential defined on the string diagram Riemann surface.

For the ghost field zero modes no singularities are allowed at the branch points and there is thus just a single, globally constant zero mode.

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6.2. Bosonic field zero modes

The non-trivial bosonic zero modes satisfying the boundary conditions (6.1) involve only the bosonic field \( X = 1/\sqrt{2}(X_1 + iX_2) \) and its complex conjugate. From the analysis of section 4.4 the zero modes can have can have a \( 1/(\bar{z} - \bar{z}_0^+ - \bar{z}_0^-) \) pole at \( z = \bar{z}_0^+ \) and a \( 1/(z - z_0^-) \) pole at \( z = z_0^- \).

The most general zero mode, with such singularities, satisfying (6.1) is given by a simple generalization of (6.2): 

\[
Y = -b \sum_i \epsilon_i \frac{p_i}{z - z_i} - \tilde{b}^* \sum_i \epsilon_i \frac{p_i}{\bar{z} - \bar{z}_i} \rightarrow -(b + \tilde{b}^*) \frac{p_i}{N_i} \quad \text{for} \quad z \rightarrow z_i, \quad (6.4)
\]

where the \( b \) and the \( \tilde{b} \) are complex numbers. The numerators of (6.4) have zeros at \( z = z_0^+ \) and \( z = z_0^- \) (see equations (2.6)). The denominators of the holomorphic and anti-holomorphic parts have zeros at respectively \( z = z_0^+ \) and \( \bar{z} = \bar{z}_0^+ = \bar{z}_0^- \) (see equations (2.13)(2.14)). So in total the zero mode (6.4) has the correct pole structure. Choosing the standard positions for three of the incoming and outgoing strings \( z_1 = 0, \ z_2 = 1, \ z_4 = \infty \) and \( z_3 = \lambda \) we can write the zero mode as the simpler expression

\[
Y = -(b + \tilde{b}^*) \frac{p_4}{N_4} + \left( \frac{p_1}{N_1} - \frac{p_4}{N_4} \right) \left( b \frac{z_0^-}{z - z_0^-} + \tilde{b}^* \frac{\bar{z}_0^+}{\bar{z} - \bar{z}_0^-} \right). \quad (6.5)
\]

We could, of course, have written down the overall form of this mode directly from the constraints on its singularities specified above. The advantage, of the more complicated form (6.4) is that we can read off immediately its behaviour asymptotically far down the strings which we indicate on the r.h.s of (6.4).

The holomorphic term in (6.3) corresponds to translation of the branch point at \( z_0^- \) and the anti-holomorphic term to the translation of the branch point \( z_0^+ \). To see this more explicitly note that in the neighbourhood of one of the \( z_0 \)'s we can write

\[
w = w_0 + \frac{1}{2} \alpha (z - z_0)^2 + \cdots \quad \text{and} \quad X = X(z_0) + \beta (z - z_0) + \cdots \quad (6.6)
\]

Inverting the first equation to write \( z - z_0 = \sqrt{2/\alpha(w - w_0)} \) and substituting into the second we arrive at an expression for \( X \) in terms of \( w \). We can then differentiate with respect to the branch point position \( w_0 \) to find

\[
\partial_{w_0} X = \frac{\beta}{\alpha} \frac{1}{z - z_0^-} + \cdots. \quad (6.7)
\]
with \( \alpha \) and \( \beta \) given by
\[
\alpha = \partial_z^2 w |_{z = z_0} \quad \text{and} \quad \beta = \partial_z X |_{z = z_0}. \tag{6.8}
\]
Evaluating \( \alpha \) and \( \beta \) explicitly one finds
\[
\partial_{w_0^-} X = \left( \frac{p_1}{N_1} - \frac{p_4}{N_4} \right) \frac{z_0^-}{z - z_0^-} + \cdots \quad \text{and} \quad \partial_{w_0^+} X = \left( \frac{p_1}{N_1} - \frac{p_4}{N_4} \right) \frac{z_0^+}{\bar{z} - \bar{z}_0^+} + \cdots. \tag{6.9}
\]
The final outcome of this analysis being that one can identify the shift in the branch points \( \delta w_0^- \) and \( \delta w_0^+ \) with the parameters \( b \) and \( \tilde{b} \) via
\[
b = \delta w_0^- \quad \text{and} \quad \tilde{b} = \delta w_0^+. \tag{6.10}
\]
Out of the four parameter space two parameters translate the whole classical solution in the \( \tau \) and \( \sigma \) directions, and two change the relative \( \tau \) and \( \sigma \) separations of the branch points.

Finally, in addition to the non-trivial bosonic zero modes constructed above, there are eight constant modes corresponding to globally translating the world sheet in the eight dimensional transverse target space.

6.3. Fermion zero modes

From section 4.5 the pole structures allowed for the fermion zero modes are a \( 1/(z - z_0^-) \) pole for \( \theta \), a \( 1/(z - z_0^+) \) pole for \( \bar{\theta} \), a \( 1/(\bar{z} - \bar{z}_0^-) \) pole for \( \bar{\tilde{\theta}} \) and a \( 1/(\bar{z} - \bar{z}_0^+) \) for \( \tilde{\theta} \).

The construction of the fermion zero modes is thus virtually identical to that for the bosons; they are given by
\[
\theta = \sum_i \frac{\epsilon_i p_i}{z - z_i} \theta^p = \left( \frac{p_4}{N_4} - \left( \frac{p_1}{N_1} - \frac{p_4}{N_4} \right) \frac{z_0^-}{z - z_0^-} \right) \theta^p \rightarrow \frac{p_i^4}{N_i} \theta^p, \tag{6.11}
\]
\[
\bar{\theta} = \sum_i \frac{\epsilon_i N_i}{z - z_i} \bar{\theta}^p = \left( \frac{p_4}{N_4} - \left( \frac{p_1}{N_1} - \frac{p_4}{N_4} \right) \frac{z_0^-}{z - z_0^-} \right) \bar{\theta}^p \rightarrow \frac{p_i}{N_i} \bar{\theta}^p, \tag{6.12}
\]
\[
\bar{\tilde{\theta}} = \sum_i \frac{\epsilon_i p_i}{\bar{z} - \bar{z}_i} \bar{\tilde{\theta}}^p = \left( \frac{p_4}{N_4} - \left( \frac{p_1}{N_1} - \frac{p_4}{N_4} \right) \frac{z_0^+}{\bar{z} - \bar{z}_0^+} \right) \bar{\tilde{\theta}}^p \rightarrow \frac{p_i}{N_i} \bar{\tilde{\theta}}^p, \tag{6.13}
\]
\[
\tilde{\theta} = \sum_i \frac{\epsilon_i N_i}{\bar{z} - \bar{z}_i} \tilde{\theta}^p = \left( \frac{p_4}{N_4} - \left( \frac{p_1}{N_1} - \frac{p_4}{N_4} \right) \frac{z_0^+}{\bar{z} - \bar{z}_0^+} \right) \tilde{\theta}^p \rightarrow \frac{p_i^4}{N_i} \tilde{\theta}^p, \tag{6.14}
\]
where \( \theta^p, \bar{\theta}^p, \bar{\tilde{\theta}}^p \) and \( \tilde{\theta}^p \) are constant two component complex spinor variables. The simpler expressions are written for the standard choice \( z_1 = 0, z_2 = 1, z_4 = \infty \) and \( z_3 = \lambda \). Asymptotically far down a string \( (z \to z_i) \) the fermion zero modes tend to the constant values indicated on the r.h.s. of the above expressions. As we will see in section 8 these asymptotic values determine the kinematic structure of the scattering amplitudes.

In addition to the non-trivial fermion zero modes constructed above, there are eight, globally constant fermion modes \( \theta^N, \bar{\theta}^N, \bar{\tilde{\theta}}^N \) and \( \tilde{\theta}^N \) which take the same constant value on each of the four strings.
7. Fermion zero modes and graviton wavefunctions

In this section we will focus on the fermionic zero modes for a single string. This is equivalent to studying the zero modes of the previous section asymptotically far down one of the strings. We will construct ground state wave functions from these modes.

The decomposition of the fermionic variables (3.3) into coordinates and momenta explicitly breaks the spin(8) invariance down to $U(1) \otimes U(1) \otimes SU(2) \otimes SU(2)$. Using the transformation properties of the $\theta$ under $SO(8)$ rotations however one can build up from combinations of the $\theta$ the components of the $\mathbf{8_s}, \mathbf{8_c}$ and $\mathbf{8_v}$ representations of $SO(8)$. We will restrict our attention here to the vector representation $\mathbf{8_v}$ and will construct one set of vector components from $\theta^A$ and $\bar{\theta}^\dagger$ for the left moving sector and another set of vector components from $\bar{\theta}^A$ and $\tilde{\theta}$ for the right moving sector. Tensored together they will then form the ground state quantum numbers for the $SO(8)$ polarizations of the graviton $G^{IJ}$, antisymmetric tensor field $B^{IJ}$ and the dilaton $\Phi$. Specifically, with polarization tensor $\epsilon^{IJ}$ the wavefunction reads

$$\Psi(\epsilon^{IJ}, \theta, \bar{\theta}) = \Psi^I(\theta, \bar{\theta}) \tilde{\Psi}^J(\bar{\theta}, \tilde{\theta}) \epsilon^{IJ}, \quad (7.1)$$

with $\Psi$ and $\tilde{\Psi}$ denoting respectively the left and right moving parts of the wavefunction. We will just focus on the left moving sector, since (before integration over the moduli) the left and right sectors are completely decoupled and the analysis for the right movers is identical to that for the left movers. As a book keeping device we will contract the left moving components with a polarization vector $\xi$. Using the $SO(8)$ transformation properties of $\theta^A$ and $\bar{\theta}^\dagger$ enumerated in section 3 and the appendix the $SO(8)$ vector separates into three parts corresponding to the subspaces $(X^1, X^2)$, $(X^3, X^4)$ and $(X^5, X^6, X^7, X^8)$

$$\Psi(\xi^I, \theta, \bar{\theta}) = \Psi^I(\theta, \bar{\theta}) \xi^I$$

$$= \Psi_{(12)}(\xi, \xi^*, \theta, \bar{\theta}) + \Psi_{(34)}(\bar{\xi}, \bar{\xi}^*, \theta, \bar{\theta}) + \Psi_{(5678)}(\xi^m, \theta, \bar{\theta}), \quad (7.2)$$

where the indices indicate the corresponding subspaces. The $\Psi_{(12)}$, $\Psi_{(34)}$ and $\Psi_{(5678)}$ are given by

$$\Psi_{(12)}(\xi, \xi^*, \theta, \bar{\theta}) = ip^+(\xi^* \theta^2 + \xi \bar{\theta}^2)$$

$$\Psi_{(34)}(\bar{\xi}, \bar{\xi}^*, \theta, \bar{\theta}) = \bar{\xi} + (p^+)^2 \xi^* \theta^2 \bar{\theta}^2$$

$$\Psi_{(5678)}(\xi^m, \theta, \bar{\theta}) = p^+ \xi^m \theta \sigma^m \bar{\theta},$$

with

$$\xi, \xi^* = \frac{1}{2}(\xi^1 \pm i\xi^2)$$

$$\bar{\xi}, \bar{\xi}^* = \frac{1}{2}(\xi^3 \pm i\xi^4). \quad (7.3)$$
The standard definitions for $\theta^2$, $\bar{\theta}^2$ and $\theta\sigma^m\bar{\theta}$ are given in the appendix. Note that to obtain the correct $U(1)$ charges for the components of the polarization vector $\xi$ the wavefunction $\Psi$ must be assigned the $U(1)$ charges $(0, 1)$ with respect to rotations in respectively the $X^1, X^2$ and $X^3, X^4$ planes. Each $\theta$ is weighted with a factor of $\sqrt{p^+}$, where $p^+$ is measured incoming to the process. In other words for the two strings 1 and 2 $p^+$ is replaced respectively by $N_1$ and $N_2$ whereas for the two outgoing strings 3 and 4 $p^+$ is replaced by $-N_3$ and $-N_4$. The powers of $p^+$ are needed to reproduce Lorentz invariant results.\(^4\)

Finally note that in light cone gauge an arbitrary physical polarization vector $\xi$ can be constructed by specifying only its components in the transverse space. The equations of motion impose the constraint

$$k: \xi = 0.$$  \hspace{1cm} (7.4)

There is a remaining gauge degree of freedom which leaves the physical polarization condition (7.4) unchanged and corresponds to adding a multiple of momenta $k$ to $\xi$. By gauge choice one can set the $+$ component of the polarization to zero. We thus have the physical polarization given by

$$\xi = (\xi^+, \xi^-, \xi, \xi^*, \bar{\xi}, \bar{\xi}^*, \xi^m),$$  \hspace{1cm} (7.5)

with

$$\xi^+ = 0 \text{ and } \xi^- = \frac{1}{N}(\xi p^* + \xi^* p).$$  \hspace{1cm} (7.6)

Likewise an arbitrary physical polarization tensor $\epsilon$ can be specified by its transverse components.

8. The four graviton scattering amplitude

The calculation involves integrating over all modes. The non zero modes lead to determinants, whereas the zero modes must be soaked up into the wavefunctions, or, in the case of the gauge field zero mode, will be integrated over a finite interval. In this section we study in detail the integration over the zero modes. At the end we return to the fluctuation determinant and combine the results to obtain the four graviton scattering amplitude.

\(^4\) We deduce the necessary factors, by observing that the resulting scattering amplitude is Lorentz invariant. A more rigorous way of determining them, however, would be to construct the $SO(8)$ rotation generators out of the $\theta$ and $\lambda$ and verify that, with the relative factors of (7.3), $\Psi^I$ is indeed an $SO(8)$ vector. We do not pursue this here.
8.1. Integration over bosonic zero modes

In section 6.2 we constructed four non-trivial zero modes for the bosonic fields. In addition to these modes there are eight constant bosonic zero modes corresponding to shifting the whole string configuration through the eight dimensional transverse space. Integration over these zero modes leads to transverse momentum conservation.

The non-trivial bosonic zero modes of (6.4) correspond to translations of the branchpoints. Two of the possible four modes, \( b = \tilde{b} = \delta \tau \) and \( b = \tilde{b} = i \sigma \) correspond to translation of the whole classical solution in the \( \tau \) and \( \sigma \) directions respectively. Integration over the first of these modes leads to conservation of the light cone energy, \( p^- \). Integration over the second leads to invariance under shift of \( \sigma \).

The two others correspond to the relative displacement of the branch points. The integral over these modes is not suppressed by a gaussian term in the action. We thus have to integrate over them carefully.

Let us denote by \( w_r = \tau_r + i \sigma_r \) the point in moduli space of the four string scattering diagram. It specifies the relative displacement of the branchpoints with respect to each other. The point \( w_r = 0 \) is the saddle point world sheet. For small \( w_r \) we can identify \( w_r \) with the parameters \( b \) and \( \tilde{b} \) of (6.10) i.e. \( w_r = b = -\tilde{b} \). Denote \( X_c(w, w_r) \) the classical matrix string field configuration corresponding to the point \( w_r \). Varying \( w_r \) from \( w_r \) to \( w_r + \delta w_r \) takes us from one classical configuration to another. The change of fields corresponding to this deformation are, by the equations of motion, zero modes of the quadratic operator:

\[
Y_\tau(w, w_r) = \partial_{\tau_r} X_c(w, w_r) \quad \text{and} \quad Y_\sigma(w, w_r) = \partial_{\sigma_r} X_c(w, w_r). \tag{8.1}
\]

However, zero modes which tend to a non-zero constant value asymptotically far down a string will not decouple from the action. The background field configuration (2.4) is linear in \( \tau \) and in integrating the kinetic term \( (\partial (X_c + Y))^2 \) by parts one picks up a boundary term, which couples the zero mode linearly to the classical background. For the saddle point world sheet configuration the zero mode corresponding to relative displacement of the branch points tends to zero asymptotically down the strings, and hence does decouple. This is another way of seeing that the background is a stationary point of the action. It also indicates that one needs to switch from integrating over this mode to integrating over the modular parameter \( w_r \).

A general field configuration can be decomposed in two ways. We can set \( w_r = 0 \), i.e. start with the saddle point classical configuration, and expand using a complete set
of modes about this background. Alternatively we can keep \( w_r \) as a free complex variable and expand in the set of modes excluding the zero modes \( Y_\tau \) and \( Y_\sigma \) of (8.1). We thus have the identity

\[
X(w) = X_c(w, 0) + \sum_n y_n Y_n(w, 0) = X_c(w, w_r) + \sum_{n \neq \tau, \sigma} \tilde{y}_n Y_n(w, 0). \tag{8.2}
\]

We know the integration measure for the first decomposition. It is determined by the condition.

\[
\int (dY) e^{-\pi<Y|Y>} = 1 \quad \text{with} \quad <Y|Y'> = \frac{1}{2\pi^2} \int d^2w \text{Tr}[\bar{Y}(w)Y'(w) + \bar{Y}'(w)Y(w)]. \tag{8.3}
\]

Substituting in the first decomposition of (8.2) we find the zero mode measure

\[
\int (dY)_0 = \int \prod_q dy_q |\det <Y_q|Y_r>|^{\frac{1}{2}}. \tag{8.4}
\]

To obtain the integration measure for the second decomposition we need to calculate the jacobian for passing from integration variables \( x_\tau, x_\sigma \) and \( x_n \) to integration variables \( \tau_r, \sigma_r \) and \( \tilde{x}_n, (n \neq \tau, \sigma) \).

From (8.2) we have the identity

\[
\sum_n y_n Y_n(w) = X(w, w_r) - X(w, 0) + \sum_{n \neq \tau, \sigma} \tilde{y}_n Y_n(w), \tag{8.5}
\]

where the sums are over all modes. Taking inner products we can relate the two sets of variables

\[
\begin{align*}
y_\tau &= \frac{<Y_\tau|X(w, w_r) - X(w, 0)>}{<Y_\tau|Y_\tau>} \\
y_\sigma &= \frac{<Y_\sigma|X(w, w_r) - X(w, 0)>}{<Y_\sigma|Y_\sigma>} \\
y_n &= \frac{<Y_n|X(w, w_r) - X(w, 0)>}{<Y_n|Y_n>} + \tilde{y}_n \quad \text{for} \quad n \neq \tau, \sigma
\end{align*} \tag{8.6}
\]

The jacobian for passing from one set of variables to the other is seen to be given by the finite dimensional determinant

\[
\mathcal{J} = \det \begin{pmatrix}
\partial_{\tau_r} y_\tau & \partial_{\tau_r} y_\sigma \\
\partial_{\sigma_r} y_\tau & \partial_{\sigma_r} y_\sigma
\end{pmatrix} \sim 1, \tag{8.7}
\]

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where we have used (8.5)(8.1) to see that, to lowest order, $J = 1$. Corrections to this approximation form a power series in $\tau$ and $\sigma$. In the domain of validity of the one loop approximation, where the momenta $p$ are very large, these terms can be dropped since the integrals over $\tau$ and $\sigma$ are entirely dominated by their quadratic corrections to the minimal action. The conclusion is that the integration measure for the collective coordinates $\tau$ and $\sigma$ and for the other zero modes is exactly as for (8.4) except with $dy_\tau$ replaced by $d\tau_r$ and $dy_\sigma$ replaced by $d\sigma_r$:

$$\int (dY)_0 = \int d\tau_r d\sigma_r \prod_{q \neq \tau, \sigma} dy_q |\det < Y_q | Y_r >|^{\frac{1}{2}}.$$  \hspace{1cm} (8.8)

By the choice of the decomposition (8.2) the only dependence of the amplitude on the relative displacements $\tau_r$ and $\sigma_r$ is through the classical action. Expanding the classical action up to quadratic order in $\tau_r$ and $\sigma_r$ the integrations over $\tau_r$ and $\sigma_r$ can then be performed. The integrals have to be evaluated by the method of steepest descent (see [12]) leading to a factor of $i$, the final result being

$$\int d\tau_r d\sigma_r e^{S(w_r)} = i \frac{c}{s u t} e^{-\frac{1}{4}(s \log s + t \log t + u \log u)},$$  \hspace{1cm} (8.9)

with $s, t$ and $u$ the Lorentz invariants defined in (2.30)(2.31). The non-Lorentz invariant factor $c$ is given, up to numerical factors, by

$$c = s^2 N_4^2 |z_0^+ - z_0^-|^2,$$  \hspace{1cm} (8.10)

where $z_0^+$ and $z_0^-$ are the two branch point positions in the complex $z$ plane. They are determined by the quadratic equation (2.4) written with the standard choice $z_1 = 0, z_2 = 1, z_3 = \lambda, z_4 = \infty$, with $\lambda$ set to the value corresponding to a stationary point of the classical action (see [7] for details).

The non-Lorentz invariant factor $c$ might at first appear problematic. However precisely this factor occurs in a light-cone gauge calculation for the same amplitude in superstring theory or bosonic string theory where it is well known that light-cone gauge produces Lorentz invariant results. It is thus instructive to see how Lorentz invariance is restored in the case of light cone string theory. Described in detail in chapter 11 (see in particular appendix 11A) of Green Schwarz Witten [19] is a careful evaluation of the fluctuation determinant (involving a regularization of the branch points) using the trace anomaly. The result being that the determinant contains precisely the right Jacobean factor to convert the integral $\int d\tau_r d\sigma_r$ into the integral $\int d^2 \lambda$ where $\lambda$ is the standard complex moduli for
the four string scattering process already introduced in the previous paragraph. In terms of the complex moduli \( \lambda \) the classical action (2.11) reads

\[
S = k_1 k_3 \log |\lambda| + k_2 k_3 \log |1 - \lambda|.
\] (8.11)

Integration over \( \lambda \) clearly leads to a Lorentz invariant result.

In the light of the above discussion it seems plausible that the matrix string determinant for the amplitude studied in this paper could likewise generate the same Jacobean factor. To show this explicitly, however, would involve finding some generalization of the the trace anomaly techniques applicable to the non-abelian determinant of (4.1). We do not address this question more fully here.

8.2. Integration over gauge field and ghost zero modes

There are three nontrivial gauge boson modes (6.2), corresponding to the fact that a generic gauge field zero mode takes on four different values, \( \frac{a_i}{N_i} \), down the four strings, subject to the constraint \( \sum_i \epsilon_i a_i = 0 \).

As for the bosonic zero modes of the previous section their integration measure includes a determinant of their inner products.

\[
\int (dV)_0 = \prod_{i=1}^{4} \int_0^{2\pi g_s} da_i \left| \det < V_i|V_j > \right|^\frac{1}{2} \delta(a_1 + a_2 - a_3 - a_4),
\] (8.12)

where the determinant is a three by three determinant and can be calculated by choosing any linearly independent basis for the \( V_i \). One such basis would be to say that \( V_i \) (\( i = 1, 2, 3 \)) is the mode which tends to 1 down the \( i \)th string, tends to \( \pm 1 \) down the fourth string and tends to zero down the others. The integrals only go up to \( a_i = g_s \) since there is a globally well defined gauge transformation that identifies \( a_i = g_s \) with \( a_i = 0 \). Specifically for the string \( i \) the globally defined unitary matrix is \( U = e^{-i\sigma/N_i} \).

Integrating over the gauge field zero modes gives rise to the contribution

\[
(2\pi g_s)^3 \left| \det < V_i|V_j > \right|^\frac{1}{2}.
\] (8.13)

There is a single ghost field zero mode, the constant mode. It comes from the fact that there is a globally constant \( U(1) \) gauge transformation. To understand how to deal with this mode, let us look carefully at the Fadeev Popov representation of the gauge fixing determinant. We insert into the functional integral the factor of 1 written as

\[
1 = \Delta \int (dU) \delta'(f - D_\mu V^{(U)}_\mu),
\] (8.14)
where \( \int (dU) \) is the Haar measure for the integral over the unitary group and \( V_\alpha^{(U)} \) is the
gauge transformation of \( V_\alpha \) with respect to \( U \). The prime on the functional delta function
means that the constant mode is excluded. By the standard argument, once this factor is
inserted into the functional integral, a gauge transformation allows one to factor out the
volume factor \( \int (dU) \) which one can then divide out, and one is left with just the Jacobian
factor \( \Delta \). \( f \) in (8.14) is some arbitrary function which can be integrated over with gaussian
weight to induce the standard gauge fixing term (3.8).

To evaluate \( \Delta \) we expand the unitary integral about the point where the argument of
the delta function is zero. We then have, under a gauge transformation,

\[
\delta V_\alpha = i g_s U U^\dagger D_\alpha U = D_\alpha \Lambda \quad \text{with} \quad U = e^{-\frac{i}{g_s} \Lambda}.
\]

We now write \( \Lambda \), the unitary integral \( \int (dU) \) and the functional delta function in terms of
modes.

\[
\Lambda(\sigma) = \sum_n \lambda_n \Lambda_n(\sigma) \quad \text{with} \quad \lambda_n = \frac{\langle \Lambda_n | \Lambda \rangle}{\langle \Lambda_n | \Lambda_n \rangle},
\]

\[
\int (dU) = \int (d\Lambda) = \int \prod_n d\lambda_n |\Lambda_n| = \delta'(A - B) = \prod_{n\neq 0} \delta(a_n - b_n) \frac{1}{|\Lambda_n|}.
\]

We have used the flat space definition of the measure (8.4) for the functional integral over
\( \Lambda \). The normalization factors \(|\Lambda_n|\) are given by

\[
|\Lambda_n| = <\Lambda_n | \Lambda_n >^{\frac{1}{2}}.
\]

In the expression for the delta function of (8.16) \( A \) and \( B \) are arbitrary functions which
we have expanded in terms of the modes \( \Lambda_n \), i.e. \( A(\sigma) = \sum_n a_n \Lambda_n(\sigma) \) and similarly for
\( B \). Note that there is nothing special about the decomposition into the modes \( \Lambda_n \), any
(orthogonal) set of modes can be used.

Substituting in the expressions (8.16) into (8.14) we find

\[
1 = \Delta \int d\lambda_0 |\Lambda_0| \prod_{n \neq 0} \frac{<\Lambda_n | \Lambda_n >}{<\Lambda_n | D^2 | \Lambda_n >}.
\]

The integral over \( \lambda_0 \) is decoupled from the functional delta function and hence is not
localized. It corresponds to a global \( U(1) \) rotation and from (8.15) we see that it must
be integrated over the finite interval \([0, 2\pi g_s]\). The product(determinant) in (8.18) can be
represented in the standard way by introducing a ghost action and integrating over all ghost modes except the zero mode. We thus have

\[
\Delta = \frac{1}{2\pi g_s} \left< \Lambda_0 | \Lambda_0 > \right|^\frac{1}{2} \int (d\bar{c}dc)' e^{\int Tr[\bar{c}D^2c]},
\]

(8.19)

where the prime indicates that the integral excludes the zero mode. The combined contribution from the ghost and gauge zero modes is thus

\[
(2\pi g_s)^2 \left[ \det \left< V_i | V_j > \right> \right]^\frac{1}{2},
\]

(8.20)

The factor of \(g_s^2\) is the correct \(g_s\) dependence expected from string theory with each branch point contributing a factor of \(g_s\).

As was originally argued in [9] the zero mode structure of the abelian gauge and ghost fields will always generate the correct \(g_s\) weight for the topology of the string diagram.

8.3. Integration over fermion zero modes

In section 6 we constructed eight non-trivial fermion zero modes which tended to constant but different values asymptotically far down each of the four strings ((6.11) (6.12) (6.13) (6.14)). In addition there are eight constant fermion zero modes taking the same value on each string, \(\theta^N\). We thus see that the fermion zero mode down the ith string can be written as

\[
\begin{pmatrix}
\theta_i \\
\bar{\theta}_i
\end{pmatrix}
= \frac{1}{N_i}
\begin{pmatrix}
p_i^* \theta^p + N_i \theta^N \\
p_i \bar{\theta}^p + N_i \bar{\theta}^N
\end{pmatrix}
\]

(8.21)

We now integrate over the fermion zero modes including a graviton wavefunction for each of the four strings. Since the left and right moving sectors of the fermionic integrals are identical we only need focus on one of them. We have the following zero mode integral for the left moving sector

\[
\mathcal{N} \int d^8 \theta \prod_{i=1}^{4} \Psi^i(\xi_i, \theta_i, \bar{\theta}_i) \quad \text{with} \quad \int d^8 \theta = \int d^2 \theta^p \ d^2 \bar{\theta}^p \ d^2 \theta^N \ d^2 \bar{\theta}^N \Rightarrow \mathcal{N} = |\det < \theta^p | \theta^p >|^{-\frac{1}{2}}.
\]

(8.22)

\(\mathcal{N}\) is the normalization for the zero mode integral. It follows from the fermionic equivalent of (8.3). The wavefunctions are given in (7.2) and the zero modes \(\theta_i\) and \(\bar{\theta}_i\) are given in terms of \(\theta^p, \bar{\theta}^p, \theta^N, \text{and} \ \bar{\theta}^N\) through equation (8.21). The fermionic integrals will pick out
from the product of wavefunctions precisely two $\theta^p$, two $\bar{\theta}^p$, two $\theta^N$ and two $\bar{\theta}^N$. Using the fact that the wavefunctions break into three subspaces, the fermionic integrals pick out of the product of wavefunctions six different types of contribution. In other words, under the fermionic integrals we have the identity

$$
\prod_{i=1}^{4} \Psi^i(\xi_i, \theta_i, \bar{\theta}_i) = \prod_{i=1}^{4} \Psi^i_{(12)} + \prod_{i=1}^{4} \Psi^i_{(34)} + \prod_{i=1}^{4} \Psi^i_{(5678)}
$$

$$
+ \sum_{i\neq j \neq k \neq l} (\Psi^i_{(12)} \Psi^j_{(12)} \Psi^k_{(34)} \Psi^l_{(34)} + \Psi^i_{(12)} \Psi^j_{(12)} \Psi^k_{(5678)} \Psi^l_{(5678)}
$$

$$
+ \Psi^i_{(34)} \Psi^j_{(34)} \Psi^k_{(5678)} \Psi^l_{(5678})
$$

(8.23)

Each of these six contributions has to be evaluated separately. The calculations are lengthy but straightforward. The appendix lists the identities necessary to derive the results and presents the first calculation in detail. It also gives essential intermediate results to aid verification of the five other calculations. Although the calculations are not particularly enlightening in themselves, the reader is nevertheless urged to turn to the appendix to convince him(her)self that it is highly non-trivial that the fermion zero mode integrations produce the ten dimensionally Lorentz invariant results enumerated below. In this section we will simply state the results, referring the reader to the appendix for all details of the calculations.

We start by considering the case in which all polarization vectors are in the $X^5, X^6, X^7, X^8$ space. Performing the fermionic integrations we find, after several pages of algebra, (see appendix) that we can express the final result in terms of Lorentz invariant quantities. Specifically we find

$$
\int d^8 \theta \prod_{i=1}^{4} \Psi^i_{(5678)} = -\left[ ut(\xi_1, \xi_2)(\xi_3, \xi_4) + st(\xi_1, \xi_3)(\xi_2, \xi_4) + su(\xi_1, \xi_4)(\xi_2, \xi_4) \right],
$$

(8.24)

where $s$, $t$ and $u$ are the ten dimensional Lorentz invariants given in (2.30)-(2.31). This is of the form expected for all four polarization vectors orthogonal to the momenta. (see for example [27][19])

The next case considered is that in which all four polarization tensors lie in the $X^3, X^4$ space. Integrating over the grassman variables one finds, after several pages of algebra (see
the appendix for essential details) that the result can also be expressed in terms of Lorentz invariants. The final result reads

$$\int d^8 \theta \prod_{i=1}^{4} \Psi_{(34)}^{i} = 4 \left[ s^2 \bar{\xi}_1 \bar{\xi}_2 \bar{\xi}_3 \bar{\xi}_4 + u^2 \bar{\xi}_1 \bar{\xi}_3 \bar{\xi}_2 \bar{\xi}_4 + t^2 \bar{\xi}_1 \bar{\xi}_4 \bar{\xi}_2 \bar{\xi}_3 + u^2 \bar{\xi}_2 \bar{\xi}_4 \bar{\xi}_1 \bar{\xi}_3 + s^2 \bar{\xi}_3 \bar{\xi}_4 \bar{\xi}_1 \bar{\xi}_2 \right].$$

(8.25)

Using the identity (2.32) and the definitions for $\xi$ and $\tilde{\xi}$ of (7.3) it is straightforward to show that this is again of the form (8.24).

Focusing next on the case with two polarizations in the $X^5,X^6,X^7,X^8$ space and two in the $X^3,X^4$ space we find that there are six types of contribution depending on the way the pairs of polarizations are divided amongst the four strings. Performing the fermionic integrations we find after a long calculation (see the appendix for essential results) that the results can be expressed in the Lorentz invariant form

$$\int d^8 \theta \psi_{(34)}^1 \psi_{(34)}^2 \psi_{(5678)}^3 \psi_{(5678)}^4 = -2ut(\bar{\xi}_1 \tilde{\xi}_2 + \bar{\xi}_2 \tilde{\xi}_1)(\xi_3 \cdot \xi_4)$$

$$\int d^8 \theta \psi_{(34)}^1 \psi_{(34)}^3 \psi_{(5678)}^2 \psi_{(5678)}^4 = -2st(\bar{\xi}_1 \tilde{\xi}_3 + \bar{\xi}_3 \tilde{\xi}_1)(\xi_2 \cdot \xi_4)$$

$$\int d^8 \theta \psi_{(34)}^1 \psi_{(34)}^4 \psi_{(5678)}^2 \psi_{(5678)}^3 = -2su(\bar{\xi}_1 \tilde{\xi}_4 + \bar{\xi}_4 \tilde{\xi}_1)(\xi_2 \cdot \xi_3)$$

$$\int d^8 \theta \psi_{(34)}^2 \psi_{(34)}^3 \psi_{(5678)}^1 \psi_{(5678)}^4 = -2su(\bar{\xi}_2 \tilde{\xi}_3 + \bar{\xi}_3 \tilde{\xi}_2)(\xi_1 \cdot \xi_4)$$

$$\int d^8 \theta \psi_{(34)}^2 \psi_{(34)}^4 \psi_{(5678)}^1 \psi_{(5678)}^3 = -2st(\bar{\xi}_2 \tilde{\xi}_4 + \bar{\xi}_4 \tilde{\xi}_2)(\xi_1 \cdot \xi_3)$$

$$\int d^8 \theta \psi_{(34)}^3 \psi_{(34)}^4 \psi_{(5678)}^1 \psi_{(5678)}^2 = -2ut(\bar{\xi}_3 \tilde{\xi}_4 + \bar{\xi}_4 \tilde{\xi}_3)(\xi_1 \cdot \xi_2),$$

(8.26)

where, $s,t$ and $u$ are the Lorentz invariants defined in (2.31). This is again of the form (8.24). In conclusion we see that for arbitrary polarization vectors lying in the $X^3,X^4,X^5,X^6,X^7,X^8$ space, i.e. transverse to the momenta, the result can always be written in the form (8.24).

We still have to evaluate the fermion integrals for the three cases where two or more polarizations are in the $X^1,X^2$ plane. We begin with the case in which all four polarizations lie in the $X^1,X^2$ plane. Evaluating the fermion integrals we find a relatively simple result (see the appendix). The result can then be shown to be equal to the Lorentz invariant
The vector products are the Lorentz invariant vector products, with $k_i$ the ten vectors of momenta (2.23) and $\xi$ the ten vectors of polarization which include a $\xi^-$ part given by equation (7.6). To verify equation (8.27) one substitutes into the right hand side the momentum ten vectors (2.23) and polarization ten vectors (7.5)(7.6). The complicated form of the right hand side of (8.27) can then be shown to reduce down to the simple result given in the appendix. It is recommended to use a computer algebra program such as Mathematica to check this.

The result (8.27) is precisely the four graviton kinematic factor calculated in string theory [27][19]. It is totally symmetric under interchange of the strings and is furthermore gauge invariant. In other words if one replaces any one polarization tensor by the corresponding momenta the result is zero.

Next we treat the cases in which two polarizations lie in the $X^1,X^2$ plane and two in the $X^3,X^4$ plane. There are six different possible ways of distributing the two types of polarization amongst the four strings. Computing the fermionic integrals one arrives at some relatively simple expressions (see the appendix). It is then straightforward to show
that they can be expressed in the Lorentz invariant form listed below.

\[ \int d^8 \theta \Psi^1\Psi^2\Psi^3\Psi^4 = -[ut(\xi_1,\xi_2) - 2t(\xi_1,k_3)(\xi_2,k_4) - 2u(\xi_1,k_4)(\xi_2,k_3)](\xi_3,\xi_4) \]

\[ \int d^8 \theta \Psi^1\Psi^2\Psi^3\Psi^4 = -[st(\xi_1,\xi_3) - 2s(\xi_1,k_4)(\xi_3,k_2) - 2t(\xi_3,k_4)(\xi_1,k_2)](\xi_2,\xi_4) \]

\[ \int d^8 \theta \Psi^1\Psi^2\Psi^3\Psi^4 = -[us(\xi_1,\xi_4) - 2s(\xi_1,k_3)(\xi_4,k_2) - 2u(\xi_1,k_2)(\xi_4,k_3)](\xi_2,\xi_3) \]

\[ \int d^8 \theta \Psi^1\Psi^2\Psi^3\Psi^4 = -[us(\xi_2,\xi_3) - 2s(\xi_2,k_4)(\xi_3,k_1) - 2u(\xi_3,k_4)(\xi_2,k_1)](\xi_1,\xi_4) \]

\[ \int d^8 \theta \Psi^1\Psi^2\Psi^3\Psi^4 = -[st(\xi_2,\xi_4) - 2s(\xi_2,k_3)(\xi_4,k_1) - 2t(\xi_2,k_1)(\xi_4,k_3)](\xi_1,\xi_3) \]

\[ \int d^8 \theta \Psi^1\Psi^2\Psi^3\Psi^4 = -[ut(\xi_3,\xi_4) - 2t(\xi_3,k_1)(\xi_4,k_2) - 2u(\xi_3,k_2)(\xi_4,k_1)](\xi_1,\xi_2) \]

These are again of the form (8.27).

Finally for the case with two polarizations in the \( X^1,X^2 \) plane and two in the subspace \( X^5,X^6,X^7,X^8 \) we find (see appendix) identical results to those of (8.28) with the wavefunctions \( \Psi_{(34)} \) replaced by \( \Psi_{(5678)} \).

The result of this analysis is that an arbitrary physical polarization vector, determined by its components in the eight dimensional transverse space, leads to the Lorentz invariant result of equation (8.27). The restoration of the full \( SO(8) \) invariance of the theory was to be expected from a correct quantization of the theory, but the extension of this invariance to ten dimensional Lorentz invariance seems to be a very non-trivial test of matrix string theory.

8.4. The four graviton scattering amplitude

Combining the integrals over the left and right moving fermion zero modes, the bosonic and gauge zero mode integrals and the determinant from the non zero modes of section 4 we arrive at the final result for the four string scattering amplitude:

\[ \mathcal{A} = \mathcal{I} \frac{g}{s,t,u} e^{-\frac{1}{2}(s+2u)log t + u log u} K_{\mu_1\mu_2\mu_3\mu_4} K_{\nu_1\nu_2\nu_3\nu_4} \epsilon^{\mu_1\nu_1}_1 \epsilon^{\mu_2\nu_2}_2 \epsilon^{\mu_3\nu_3}_3 \epsilon^{\mu_4\nu_4}_4, \] (8.29)

where the tensor \( K_{\mu_1\mu_2\mu_3\mu_4} \) is the coefficient of \( \xi_1^{\mu_1} \xi_2^{\mu_2} \xi_3^{\mu_3} \xi_4^{\mu_4} \) in equation (8.27). The factor \( \mathcal{I} \) contains the finite determinants from the zero mode normalizations along with the fluctuation determinant \( \mathcal{J} \) of section 4. Up to an overall numerical factor it is given by

\[ \mathcal{I} = c \mathcal{J} \left[ \frac{<\Lambda_0,|\Lambda_0> \det<\theta^n,|\theta^n>}{\det<V_i|V_j> \det<Y_q|Y_r>} \right]^{\frac{1}{2}}, \] (8.30)
where $c$ is the non-Lorentz invariant quantity defined in (8.10). We have shown in section 4 that the fluctuation determinant $J$ does not receive any singular contributions from the branch points. We further argued that up to small corrections the bosonic and fermionic determinants would cancel, but we do not know how to evaluate $J$ more precisely. We can, however, take some inspiration from conformal field theory. In CFT it is well known that the finite determinants for the zero modes combine with determinants from the non-zero modes to give a result (in the critical dimension) that is conformally invariant. The heat kernel methods used to show this (see for example [28]) do not generalise in any obvious way to the determinants calculated in this paper. It seems likely, however, that a similar mechanism is at work for the combination of determinants of (8.30).\footnote{I thank Herman Verlinde for emphasising the importance of this effect in CFT and for suggesting that a similar mechanism could play a role here.} As already discussed at the end of section (8.1) it is also quite possible that the non-Lorentz invariant factor $c$ would also be absorbed in a natural way into the determinant (again this is precisely what happens in the string theory calculation for the same amplitude).

If this is the case then, up to an overall numerical factor, we have reproduced from matrix string theory the high energy limit of the string theory scattering amplitude for four gravitons \cite{27, 19, 11, 12}.

9. Discussion and conclusions

In the previous section we have performed an analysis of graviton scattering. This formalism can however be extended in a straightforward way to wavefunctions involving fermionic ground states for the left and/or right moving sectors. These would be constructed from odd numbers of $\theta$, and would permit the construction of the complete multiplet of massless states of the type IIA superstring.

9.1. Summary of the general calculational procedure

We study scattering amplitudes which in string theory are dominated by the points in moduli space corresponding to saddle points of the classical action. We focused on the tree level contribution, but in string theory there is a whole tower of saddle point world sheets of higher topology that also contribute.\footnote{In the original analysis of Gross and Mende \cite{11, 12} it was found that the perturbation series was strongly divergent with genus $g$ world sheets giving a contribution proportional to $g^{9g}$.} The methods developed in this paper should...
be directly applicable to these higher genus contributions. Let us summarize the general procedure.

The classical matrix string solutions corresponding to the saddle point world sheets are used as background field configurations. The quantum fluctuations around the backgrounds are treated by a one loop calculation. This consists of a zero mode part and a fluctuation determinant for the non-zero modes. All the essential structure of the amplitude is contained in the zero mode integrations which can be calculated exactly. To find the zero modes one first uses the fact that around the interaction points there are instanton like field configurations. Locally these break the translation symmetry and part of the supersymmetry. Each broken symmetry has a corresponding zero mode. They are given by equations (5.13)(5.19)

\[ V_\mu = F_\alpha\mu \quad \text{and} \quad \theta = F_{\mu\nu} \Gamma_{\mu\nu} \epsilon. \quad (9.1) \]

where \( F_{\mu\nu} \) is the background field strength written in ten dimensional notation (see section 3.2). In addition there are zero modes for the two dimensional gauge field. They can be written as

\[ V_\mu = D_\mu X_I, \quad (9.2) \]

where \( X_I \) is any of the non zero background bosonic fields. The zero mode field configurations (9.1)(9.2) are finite at the interaction points. Since, however, the background bosonic field configurations \( X_I \) correspond to world sheets branch points, the zero modes (9.1)(9.2) will behave, asymptotically far from the branch point, like singular string world sheet zero modes, with the singularity sitting at the branch point. These modes can be glued into global world sheet zero modes.

The important point about the construction of these global zero modes is that they only depend on the existence of a finite instanton like classical solution around the branch point, not on the precise details of the field configuration.

For any world sheet it should be possible to identify, using (9.1)(9.2), the allowed singularities of the zero modes at the branch points and hence to construct the global zero modes.

Integration over the gauge field zero modes, along with the single, constant, ghost field zero mode, leads to the correct power of \( g_s \) for the string world sheet [9]. The global bosonic zero modes correspond to the collective coordinates for the positions of the branch points. Integration over the global fermion zero modes gives the ten dimensionally Lorentz invariant kinematic structure of the scattering amplitude. This we have has explicitly verified for the case of four graviton scattering.
9.2. Connection with light cone superstring calculations

The matrix string calculations described above have a strong resemblance to the functional integral methods used in light cone superstring calculations (see the review article [25]). Below we discuss the similarities and differences between the two.

In light cone superstring calculations the functional integral reduces down to an integration over zero modes. There are, however, no zero modes corresponding to the translation of the branch point and no abelian gauge field and so no gauge or ghost field zero modes. The inclusion of factors of $g_s$ and the integrals over the string moduli are added by hand.

To perform the Euclidean functional integral the Minkowski space Majorana-Weyl fermions are combined into complex fermionic coordinates and their conjugate momenta. These can be defined by the same formula as for matrix string theory i.e. equation (3.3). This breaks the $\text{spin}(8)$ symmetry down to $U(1) \otimes SU(4)$. The $\theta$ and $\bar{\theta}$ of (3.3) form a $4$ of $SU(4)$ and the $\tilde{\theta}$ and $\bar{\tilde{\theta}}$ form a $\overline{4}$.

In the matrix string theory calculation, there is a classical background, that through the non-abelian nature of the theory, interacts with the fermions and further breaks the $SU(4)$ symmetry. In the four string scattering process discussed in this paper the original $\text{spin}(8)$ symmetry is broken down to $U(1) \otimes U(1) \otimes SU(2) \otimes SU(2)$. Other classical backgrounds, for example five or six string scattering, would break the $\text{spin}(8)$ symmetry even further.

In light cone superstring calculations the fermionic coordinates $\theta$ and momenta $\lambda$ have the conformal weights 1 and 0 respectively. The functional integral over the fermions leads to a determinant and to an integration over the $\theta$ zero modes. These are given by the complete set of abelian differentials defined on the string diagram Riemann surface. In terms of the light cone world sheet coordinate $w$, they consist of all the modes satisfying

$$\partial_w \theta = 0. \quad (9.3)$$

We denote by $\theta$ in (1.3) the $4$ of $SU(4)$ made from $\theta$ and $\bar{\theta}$ of (3.3). There is also the complex conjugate equation for the $\overline{4}$. The modes satisfying (1.3) are allowed to have $1/\sqrt{w - w_0}$ singularities at each simple branch point $w_0$, and have the boundary condition that they tend to constant values asymptotically far down the strings. The generic fermion zero mode thus has a singularity at each branch point. This is in contrast to the matrix string zero modes of section 6.3 which each had a single singularity even though there were
two branch points. In other words, for this process, there are twice as many fermion zero modes in the light cone superstring calculation as there are in the matrix string calculation.

This mismatch is resolved by the final element of light cone superstring calculations: the addition by hand, at each branch point of an operator \( V \) needed to preserve supersymmetry. The precise form of this operator for simple branch points (i.e. where two strings interact) was calculated in [24]. It can be written in the form

\[
V = -\frac{4}{|2\partial^2_w w(z)|} \Psi(\partial_z X^I, \partial_z \theta, \partial_z \bar{\theta}) \bar{\Psi}(\partial_{\bar{z}} X^I, \partial_{\bar{z}} \bar{\theta}, \partial_{\bar{z}} \theta). \tag{9.4}
\]

where \( \Psi(\ ,\ ,\ ) \) is as in (7.2) except with \( p^+ \) replaced by \( 1/(2\partial^2_w w(z)) \), and in the right moving part \( \bar{\Psi}(\ ,\ ,\ ) p^+ \) is replaced by \( 1/(2\partial^2_{\bar{z}} w(z))^* \). \( w \) is the light cone coordinate expressed, through the Mandelstam mapping, in terms of the uniformization \( z \) (for tree level scattering see equation (2.5)). The bosonic fields \( X \) in the operators \( V \) are contracted both with the external wavefunctions and with the other \( V \). The fermions \( \theta \) are the fermion zero modes. It is straightforward to see that the two interaction operators used for the four string scattering amplitude soak up the excess zero modes.

9.3. Conclusions and future directions

The matrix string zero mode calculation of this paper is considerably simpler than that for the light cone superstring, where an interaction operator has to be introduced by hand at each interaction point. So far, however, the matrix string calculations have only been carried out about a classical background corresponding to a saddle point of the classical action. It is important to extend them to other backgrounds. In the context of the four-string scattering calculation, it is clear that there is a problem when the world sheet is no longer a stationary point of the action. In this case, the world sheet is not (anti)holomorphic around the branch points and the corresponding instanton solution will break all of the supersymmetry leading to extra fermion zero modes. There must be a simple mechanism that soaks up these extra zero modes. It is important to understand this as it could permit the calculations to be done at an arbitrary point in moduli space.

The semi-classical analysis of this paper has been justified by focusing on very high energy scattering processes. Away from this limit we do not know how to calculate. There will however be an effective action describing the low energy fluctuations, even if we do not at present have the technical tools to obtain it. The zero mode analysis of this paper only depends on the existence of a finite field configuration for the interaction points, and should also be applicable to this effective field theory.
Finally it is interesting to speculate on how the semi-classical reasoning of this paper could be made more rigorous. In simpler situations, semiclassical methods have recently been shown to be in precise agreement with exact results. Using quasi-classics, the partition function for matrix string theory on a torus was calculated \[29\] and shown to agree, on extrapolation to the small torus limit, with the exact results for the completely reduced SYM theory \[30\]. These latter results were obtained by mapping the original SYM theory to a cohomological field theory (CohFT) using the methods of \[31\]. Recently CohFT methods have been applied directly to the matrix string partition function \[32\] beautifully confirming the semi-classical results of \[29\]. A key aspect of CohFT is that the functional integral localizes on configurations of the classical vacua. Although it is not possible to apply the twisting procedures used in \[31\]\[30\]\[32\] to matrix string scattering processes it is tempting to speculate that some kind of localization mechanism is also behind the success of the calculations presented in this paper.

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Appendix 1. Gamma matrix definitions

We choose as the basis of real spin(8) gamma matrices $\Gamma^I$ and Majorana-Weyl fermions $S$

\[
\Gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \Gamma^I = \begin{pmatrix} 0 & \gamma^I_{\dot{a}a} \\ \gamma^I_{a\dot{a}} & 0 \end{pmatrix}, \Gamma^9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} S^a \\ S^{\dot{a}} \end{pmatrix}, \tag{1.1}
\]

where the indices $a$ and $\dot{a}$ which run from one to eight label the $8_s$ and $8_c$ representations of $spin(8)$ respectively. The $\gamma^I_{a\dot{a}}$ (which are the transposes of the $\gamma^I_{\dot{a}a}$) are given by

\[
\begin{align*}
\gamma^1 &= 1 \otimes 1 \otimes 1 & \gamma^2 &= \epsilon \otimes 1 \otimes 1 \\
\gamma^3 &= \tau_1 \otimes \tau_3 \otimes \epsilon & \gamma^4 &= \tau_1 \otimes \tau_1 \otimes \epsilon \\
\gamma^5 &= \tau_1 \otimes \epsilon \otimes 1 & \gamma^6 &= \tau_3 \otimes \epsilon \otimes \tau_1 \\
\gamma^7 &= -\tau_3 \otimes 1 \otimes \epsilon & \gamma^8 &= \tau_3 \otimes \epsilon \otimes \tau_3, \\
\end{align*} \tag{1.2}
\]

where $\epsilon = i\tau_2$ and $\tau_1, \tau_2$ and $\tau_3$ are the pauli matrices. With this basis the gamma matrices $\Gamma^0, \Gamma^9, \Gamma^1$ and $\Gamma^2$ are given by

\[
\begin{align*}
\Gamma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \Gamma^9 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \Gamma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \Gamma^2 &= \begin{pmatrix} 0 & i\tau_2 \\ -i\tau_2 & 0 \end{pmatrix}, \tag{1.3}
\end{align*}
\]
where all entries correspond to 8x8 blocks with the pauli matrix $\tau_2$ being written as 4x4 blocks.

With respect to these four gamma matrices the sixteen component Majorana-Weyl spinors thus decompose into four blocks of four. We will be working in Euclidean space where ten dimensional Majorana-Weyl spinors do not exist. Furthermore, as discussed in section 3 each Majorana-Weyl fermionic variable is simultaneously a fermionic coordinate and its conjugate momenta. We thus have to combine the fermionic variables into complex fermionic coordinates and their distinct conjugate momenta. It is thus convenient to work with a complex basis of gamma matrices defined by

$$\gamma \rightarrow (v \otimes v \otimes 1)\gamma(v^\dagger \otimes v^\dagger \otimes 1) \quad \text{with} \quad v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \quad (1.4)$$

Written out explicitly they are given by

$$\begin{align*}
\gamma^1 &= 1 \otimes 1 \otimes 1 \\
\gamma^3 &= -i\tau_2 \otimes \tau_1 \otimes \tau_2 \\
\gamma^5 &= i\tau_2 \otimes \tau_3 \otimes 1 \\
\gamma^7 &= -\tau_1 \otimes 1 \otimes \tau_2 \\
\gamma^2 &= -i\tau_3 \otimes 1 \otimes 1 \\
\gamma^4 &= i\tau_2 \otimes \tau_2 \otimes \tau_2 \\
\gamma^6 &= -i\tau_1 \otimes \tau_3 \otimes \tau_1 \\
\gamma^8 &= -i\tau_1 \otimes \tau_3 \otimes \tau_3,
\end{align*} \quad (1.5)$$

In the basis of (1.5) the four gamma matrices that appear in the background covariant Dirac operator are identical to those of (1.3) except for $\Gamma^2$ where $\tau_2$ is replaced by $-\tau_3$. Finally it is useful for the study of the fermion zero modes and determinant to put the Dirac operator into block diagonal form by mixing the two spin(8) representations. Specifically, writing the sixteen by sixteen $\Gamma$ matrices as four by four blocks, we interchange the second and third rows and columns. This leads to the following block diagonal form for the gamma matrices $\Gamma^0, \Gamma^9, \Gamma^1$ and $\Gamma^2$

$$\begin{align*}
\Gamma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\Gamma^9 &= \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \\
\Gamma^1 &= \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \\
\Gamma^2 &= \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix},
\end{align*} \quad (1.6)$$

where again all entries are 8x8 blocks. This transformation also interchanges the $\bar{\theta}_A, \lambda_A$ with $\tilde{\theta}^A, \tilde{\lambda}^A$ in (2.1) of appendix 2 and leads to the block diagonal form for the Euclidean fermion action given in (3.10).
Appendix 2. Fermionic coordinates and momenta

Under the change of basis (1.3) the Majorana-Weyl fermions split into fermionic coordinates $\theta$ and momenta $\lambda$ as follows

\[ S^a \rightarrow (v \otimes v \otimes 1)S^a = \begin{pmatrix} \theta^A \\ \bar{\lambda}^A \\ \bar{\theta}^\lambda \\ \lambda^A \end{pmatrix} \quad \text{and} \quad S^\dagger \rightarrow (v \otimes v \otimes 1)S^\dagger = \begin{pmatrix} \bar{\theta}^A \\ \bar{\lambda}^A \\ \theta^\lambda \\ \lambda^A \end{pmatrix}, \quad (2.1) \]

where the indices $A, \bar{A}$ take the values 1, 2 with the only non-zero anticommutators being

\[ \{ \theta^A, \lambda_B \} = \delta^A_B \quad \text{and} \quad \{ \bar{\theta}^A, \bar{\lambda}^B \} = \delta^A_B \quad (2.2) \]

where we have not explicitly included the spatial delta functions. The transformation of the fermion coordinates and momenta under rotations in the $X^1, X^2$ and $X^3, X^4$ planes and in the four dimensional space $X^5, X^6, X^7$ and $X^8$ can be read off from the explicit form for the rotation generators in the basis (1.3). Specifically under an infinitesimal rotation described by $w^{IJ}$ we have

\[ \delta S^a = \frac{1}{4} w^{IJ} \gamma^{IJ}_{ab} S^b \quad \text{with} \quad \gamma^{IJ}_{ab} = \frac{1}{2} (\gamma^I (\gamma^J)^\dagger - \gamma^J (\gamma^I)^\dagger) \]

\[ \delta S^\dagger = \frac{1}{4} w^{IJ} \tilde{\gamma}^{IJ}_{ab} S^b \quad \text{with} \quad \tilde{\gamma}^{IJ}_{ab} = \frac{1}{2} ((\gamma^I)^\dagger \gamma^J - (\gamma^J)^\dagger \gamma^I) \quad (2.3) \]

We can thus draw up the following table for the action of $\gamma^{IJ}$ on $\theta^A, \bar{\theta}^\lambda, \lambda^A$ and $\bar{\lambda}^\bar{A}$

| $\gamma^{IJ}$ | $\theta^A$ | $\bar{\theta}^\lambda$ | $\lambda^A$ | $\bar{\lambda}^\bar{A}$ |
|---------------|-------------|-----------------|----------|-----------------|
| $\gamma_{12}$ | $\tau_3 \otimes 1 \otimes i$ | $i$ | $-i$ | $-i$ | $i$ |
| $\gamma_{34}$ | $1 \otimes \tau_3 \otimes i$ | $i$ | $i$ | $-i$ | $-i$ |
| $\gamma_{56}$ | $\tau_3 \otimes 1 \otimes i \tau_1$ | $i \tau_1$ | $-i \tau_1$ | $-i \tau_1$ | $i \tau_1$ |
| $\gamma_{57}$ | $\tau_3 \otimes \tau_3 \otimes i \tau_2$ | $i \tau_2$ | $-i \tau_2$ | $i \tau_2$ | $-i \tau_2$ |
| $\gamma_{58}$ | $\tau_3 \otimes 1 \otimes i \tau_3$ | $i \tau_3$ | $-i \tau_3$ | $-i \tau_3$ | $i \tau_3$ |
| $\gamma_{67}$ | $1 \otimes \tau_3 \otimes i \tau_3$ | $i \tau_3$ | $i \tau_3$ | $-i \tau_3$ | $-i \tau_3$ |
| $\gamma_{68}$ | $1 \otimes 1 \otimes -i \tau_2$ | $-i \tau_2$ | $-i \tau_2$ | $-i \tau_2$ | $-i \tau_2$ |
| $\gamma_{78}$ | $1 \otimes \tau_3 \otimes i \tau_1$ | $i \tau_1$ | $i \tau_1$ | $-i \tau_1$ | $-i \tau_1$ |

There is an almost identical table for the action of $\tilde{\gamma}^{IJ}$ on $\bar{\theta}^\lambda, \bar{\lambda}^\bar{A}$ and $\bar{\lambda}^\bar{A}$. The only difference from the above table being that all entries of the first line change sign. We can read of directly from (2.4) the $U(1)$ charges for rotations in the $X^1, X^2$ and $X^3, X^4$ planes.
Further more we can read off the effect of infinitesimal $SO(4)$ rotations in the $X^5, X^6, X^7$ and $X^8$ subspace. Using the notations of [23] for four four dimensional spinors we have

\[ \delta \theta^A = \frac{1}{4} w^{mn} (\sigma^{mn})^A_B \theta^B \quad \delta \lambda_A = -\frac{1}{4} w^{mn} (\sigma^{mn})^B_A \theta_B \]
\[ \delta \bar{\theta}^\bar{A} = \frac{1}{4} w^{mn} (\bar{\sigma}^{mn})^\bar{A}_\bar{B} \bar{\theta}^\bar{B} \quad \delta \bar{\lambda}^\bar{A} = -\frac{1}{4} w^{mn} (\bar{\sigma}^{mn})^\bar{B}_\bar{A} \bar{\lambda}^\bar{B}. \] (2.5)

where $(\sigma^{mn})^A_B$ and $(\bar{\sigma}^{mn})^\bar{A}_\bar{B}$ are defined by

\[ (\sigma^{mn})^A_B = \frac{1}{2} (\sigma^m \sigma^n - \sigma^n \sigma^m) \quad \text{and} \quad (\bar{\sigma}^{mn})^\bar{A}_\bar{B} = \frac{1}{2} (\bar{\sigma}^m \sigma^n - \bar{\sigma}^n \sigma^m). \] (2.6)

with $\sigma^m$ and $\bar{\sigma}^m$ defined through the Pauli matrices

\[ (\sigma^m)^A_B = (i, \tau_1, \tau_2, \tau_3) \quad \text{and} \quad (\bar{\sigma}^m)^\bar{A}_\bar{B} = (-i, \tau_1, \tau_2, \tau_3). \] (2.7)

2.1. Identities for two component 4d spinors

We use the conventions of [23], modified by a factor of $i$ for the zeroth component since the space $X^5, X^6, X^7, X^8$ has Euclidean metric.

There are two pairs of two component spinors, $\theta^A$ and $\bar{\theta}^\bar{A}$ with $A, \bar{B} = 1, 2$. Indices are raised and lowered using the antisymmetric tensor $\epsilon$:

\[ \theta^A = \epsilon^{AB} \theta_B, \quad \theta_A = \epsilon_{BA} \theta^B, \quad \bar{\theta}^\bar{A} = \epsilon^{\bar{B}\bar{A}} \bar{\theta}_\bar{B}, \quad \bar{\theta}_\bar{A} = \epsilon_{\bar{A}\bar{B}} \bar{\theta}^\bar{B}, \] (2.8)

where the $\epsilon$ tensor is defined to be

\[ \epsilon_{AB} = \epsilon^{AB} = -\epsilon_{\bar{A}\bar{B}} = -\epsilon^{\bar{A}\bar{B}} \quad \text{with} \quad \epsilon_{AB} = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right). \] (2.9)

We define $\theta^2$ and $\bar{\theta}^2$ by

\[ \theta^2 = \theta^A \theta_A \quad \text{and} \quad \bar{\theta}^2 = \bar{\theta}^\bar{A} \bar{\theta}_{\bar{A}}. \] (2.10)

Integrals over $\theta$ and $\bar{\theta}$ are defined by

\[ \int d^2 \theta = \int d\theta_1 d\theta_2 \quad \text{and} \quad \int d^2 \bar{\theta} = \int d\bar{\theta}_2 d\bar{\theta}_1. \] (2.11)

We define the four matrices $(\sigma^m)^{C\bar{D}}$ with $m = 5, 6, 7, 8$ and their barred partners by

\[ \sigma^m = (i, \tau), \quad \text{and} \quad \bar{\sigma}^m = (-i, \tau). \] (2.12)
Finally we contract the $\sigma^m$ with $\theta$ and $\bar{\theta}$ to produce a four vector in the space $X^5, X^6, X^7$ and $X^8$:

$$\theta\sigma^m\bar{\theta} = \theta_A(\sigma^m)^{AB}\bar{\theta}_B.$$  \hspace{1cm} (2.13)

Using the above definitions we can derive identities for calculating the fermionic integrals of section 8. Below we list the complete set of identities needed.

$$\epsilon^{AB}\epsilon_{CB} = \delta^A_C \quad \text{and} \quad \epsilon^{\bar{A}\bar{B}}\epsilon_{\bar{C}B} = \delta^{\bar{A}}_{\bar{C}},$$  \hspace{1cm} (2.14)

$$\theta^2 = -\epsilon^{AB}\theta_A\theta_B \quad \text{and} \quad \bar{\theta}^2 = -\epsilon^{\bar{A}\bar{B}}\bar{\theta}_A\bar{\theta}_B,$$  \hspace{1cm} (2.15)

$$\int d^2\theta\left(\theta^2\theta_A\theta_B\right) = \begin{pmatrix} 2 \\
-\epsilon_{AB} \end{pmatrix} \quad \text{and} \quad \int d^2\bar{\theta}\left(\bar{\theta}^2\bar{\theta}_A\bar{\theta}_B\right) = \begin{pmatrix} 2 \\
-\epsilon^{\bar{A}\bar{B}} \end{pmatrix},$$  \hspace{1cm} (2.16)

$$(\bar{\sigma}^m)^{\bar{B}A} = (\sigma^m)^C\bar{D}\epsilon_{CA}\epsilon^{\bar{D}\bar{B}},$$  \hspace{1cm} (2.17)

$$(\sigma^m)^{\bar{A}\bar{B}}(\bar{\sigma}^n)^{\bar{B}C} + (\sigma^n)^{\bar{A}\bar{B}}(\bar{\sigma}^m)^{\bar{B}C} = 2\delta^{nm}\delta^A_C,$$  \hspace{1cm} (2.18)

and

$$\text{Tr}[\sigma^m\bar{\sigma}^n] = 2\delta^{nm}.$$  \hspace{1cm} (2.19)

**Appendix 3. Fermion zero mode integrals**

Using the identities of the previous section the calculations of the fermion integrals of section 8, are straightforward but somewhat tedious. In this section we present one calculation in detail. The other calculations can be evaluated in a similar manner and so for them we just give some essential intermediate results.

For all four polarizations lying in the $X^5, X^6, X^7, X^8$ space we have

$$\int d^8\theta\prod_{i=1}^4 \Psi^i_{(5678)} = \frac{\xi_{m1}\xi_{m2}\xi_{m3}\xi_{m4}}{N_1N_2N_3N_4} \mathcal{I}^{m_1m_2m_3m_4},$$  \hspace{1cm} (3.1)

where

$$\mathcal{I}^{m_1m_2m_3m_4} = \int d^2\theta^p d^2\bar{\theta}^p d^2\theta^N d^2\bar{\theta}^N \prod_{i=1}^4 (p_i^*\theta^p + N_i\theta^N)(\sigma^{m_i})(p_i\bar{\theta}^p + N_i\bar{\theta}^N).$$  \hspace{1cm} (3.2)

The integrals over the $\theta$ only gives a non-zero result for the terms from the product in which there are two $\theta^p$, two $\bar{\theta}^p$, two $\theta^N$ and two $\bar{\theta}^N$. In other words each non-zero contribution will consist of two factors of transverse momenta $p$, two factors of transverse momenta
$p^*$ and four factors of $p^+$ momenta $N$. We thus see that there are three different types of integrals to be calculated according to how the four $p$’s and four $N$’s are distributed amongst the four wavefunctions. Specifically we have

$$I = I_1 + I_2 + I_3 \quad (3.3)$$

where $I_1$ consists of 6 ($=4!/2!2!$) contributions of the form $|p_i|^2|p_j|^2N_k^2N_l^2$, $I_2$ consists of 24 ($=4!$) contributions of the form $|p_i|^2p_j^*p_kN_jN_k^2$ and $I_3$ consists of 6 ($=4!/2!2!$) contributions of the form $p_i^*p_j^*p_kp_lN_iN_jN_kN_l$. For these three different types of term the $\theta$ integrals lead to the following contributions

$$|p_i|^2|p_j|^2N_k^2N_l^2 \int d^8 \theta \quad \theta^p \sigma^{mi} \bar{\theta}^p \theta^{pi} \sigma^{mj} \bar{\theta}^p \theta^{Nj} \sigma^{mk} \bar{\theta}^N \theta^{Nj} \sigma^{ml} \bar{\theta}^N = 4|p_i|^2|p_j|^2N_k^2N_l^2 \theta^{mij} \delta^{mkm_l}$$

$$|p_i|^2p_j^*p_kN_jN_kN_l^2 \int d^8 \theta \quad \theta^p \sigma^{mi} \bar{\theta}^p \theta^{pi} \sigma^{mj} \bar{\theta}^p \theta^{Nj} \sigma^{mk} \bar{\theta}^N \theta^{Nj} \sigma^{ml} \bar{\theta}^N = -2|p_i|^2p_j^*p_kN_jN_kN_l^2 (\delta^{mij} \delta^{mkm_l} + \delta^{mkm} \delta^{mjm_l} - \delta^{mim_l} \delta^{mjm_k})$$

$$p_i^*p_j^*p_kp_lN_iN_jN_kN_l \int d^8 \theta \quad \theta^p \sigma^{mi} \bar{\theta}^N \theta^{pi} \sigma^{mj} \bar{\theta}^N \theta^{Nj} \sigma^{mk} \bar{\theta}^N \theta^{Nj} \sigma^{ml} \bar{\theta}^N = 4p_i^*p_j^*p_kp_lN_iN_jN_kN_l \theta^{mij} \delta^{mkm_l}. \quad (3.4)$$

Adding together all the different contributions of a particular type and using the identities (2.28), (2.29) and (2.31) we find

$$I_1 = 4\frac{p^4}{N^4} N_1N_2N_3N_4[2N_1N_2N_3N_4A + (N_2^2N_4^2 + N_3^2N_4^2)B + (N_2^2N_3^2 + N_1^2N_4^2)C], \quad (3.5)$$

$$I_2 = 4\frac{p^2}{N^2} N_1N_2N_3N_4 \left[ \frac{p^2}{N^2}(N_1^2 + N_2^2)N_3N_4 + (N_3^2 + N_4^2)N_1N_2) (-A + B + C) - q^2(N_1N_4 + N_2N_3)(A - B + C) \right. \quad (3.6)$$

$$\left. - q^2(N_1N_3 + N_2N_4)(A + B - C) \right],$$

and

$$I_3 = 4N_1N_2N_3N_4[q^4A + 2\frac{p^4}{N^4} N_1N_2N_3N_4(-A + B + C)]. \quad (3.7)$$

where we have defined the tensors $A, B$ and $C$ by

$$A = \delta^{m_1m_2}\delta^{m_3m_4}, \quad B = \delta^{m_1m_3}\delta^{m_2m_4}, \quad \text{and} \quad C = \delta^{m_1m_4}\delta^{m_2m_3}. \quad (3.8)$$
Adding these three results together to obtain $\mathcal{I}$ (3.3), substituting into (3.1) and using the definitions (2.31) we obtain the result (3.24).

For all four polarizations lying in the $X^3,X^4$ space we have

$$\int d^8 \theta \prod_{i=1}^{4} \Psi^i_{(34)} = \frac{\tilde{\xi}_1 \tilde{\xi}_2 \tilde{\xi}_3 \tilde{\xi}_4}{N^2 N^2} \mathcal{I}(1;2;3,4) + 5 \text{ other permutations},$$

(3.9)

where

$$\mathcal{I}(i; j; k, l) = \int d^8 \theta \prod_{m=k,l} (p_m^* \theta^p + N_m \theta^N)^2 (p_m \bar{\theta}^p + N_m \bar{\theta}^N)^2.$$  

(3.10)

Evaluating the fermionic integrals (there are five different types of contribution) we find

$$\mathcal{I}(i; j; k, l) = 16 \left[ |p_k|^4 N_l^4 - 2 |p_k|^2 (p_k p_l^* + p_l^* p_k) N_k N_l^3 + p_k^2 (p_l^*)^2 N_k^2 N_l^2 \right] + (k \leftrightarrow l) + 64 |p_k|^2 |p_l|^2 N_k^2 N_l^2.$$  

(3.11)

Using the identities (2.28), (2.29) and (2.31) to evaluate the above result for the six possible permutations of (3.9) we obtain the result (8.24).

For two polarizations lying in the $X^3,X^4$ plane and two lying in the $X^5,X^6,X^7,X^8$ space we have

$$\int d^8 \theta \Psi^i_{(34)} \Psi^j_{(5678)} \Psi^k_{(5678)} = \frac{\tilde{\xi}_i \tilde{\xi}_j (\xi_k \xi_l)}{N^2 N^2} \mathcal{I}(i; j|k,l) + \text{ c.c.}$$

(3.12)

where

$$\mathcal{I}(i; j|k,l) \delta^{m_k m_l} =$$

$$\epsilon \int d^8 \theta \ (p_j^* \theta^p + N_j \theta^N)^2 (p_j \bar{\theta}^p + N_j \bar{\theta}^N)^2 \prod_{n=k,l} (p_n^* \theta^p + N_n \theta^N)(\sigma^{m_n})(p_n \bar{\theta}^p + N_n \bar{\theta}^N),$$

(3.13)

where the sign factor $\epsilon$ is +1 if $k, l = 1, 2$ or 3, 4 and −1 otherwise. This factor comes from the sign of the $p^+$ momenta in the definitions of the wavefunctions (7.3). Integrating over the $\theta$ (there are ten different types of contribution) leads to the result

$$\mathcal{I}(i; j|k,l) = -8 \epsilon \left[ |p_j|^4 N_k^2 N_l^2 + N_j^4 |p_k|^2 |p_l|^2 + |p_j|^2 |p_k|^2 N_j^2 N_l^2 + |p_j|^2 |p_l|^2 N_j^2 N_k^2 + |p_j|^2 |p_k^* p_l| N_j^2 N_k N_l + (p_j^2 p_k^* p_l + (p_j^*)^2 p_k p_l) N_j^2 N_k N_l - |p_j|^2 |p_k|^2 p_l^* p_k N_l - |p_l|^2 |p_j|^2 p_j^* p_l N_k \right]$$

(3.14)
Using the identities (2.28), (2.29) and (2.31) to evaluate the above result for the six possible combinations of pairs of polarizations leads to the results listed in (8.26).

For all four polarizations lying in the $X^1, X^2$ plane we have to evaluate the integral

$$
\int d^8 \theta \prod_{i=1}^{4} \Psi^i_{(12)} = \frac{\xi_1 \xi_2 \xi_3 \xi_4}{N_1 N_2 N_3 N_4} \mathcal{I}(1, 2; 3, 4) + 5 \text{ other permutations},
$$

where

$$
\mathcal{I}(i, j; k, l) = \int d^8 \theta \prod_{m=i,j} \prod_{n=k,l} (p^*_n \theta^p + N_n \theta^N)^2(p^*_n \bar{\theta}^p + N_n \bar{\theta}^N)^2.
$$

Computing the fermionic integrals and using the identities (2.28), (2.29) and (2.31) leads to the following expressions for the $\mathcal{I}(i, j; k, l)$:

$$
\mathcal{I}(1, 2; 3, 4) = 16 N_4^4 (p^*_1)^2 p_3^2
$$
$$
\mathcal{I}(1, 3; 2, 4) = 16 \left[ 6 \frac{p^4}{N^4}(N_1 N_2 N_3 N_4)^2 + (p^*_1)^2 p_3^2 N_2 N_3^2 + p_1^2 (p^*_3)^2 N_1^2 N_4^2
- 4 \frac{p^2}{N^2} N_1 N_2 N_3 N_4 (p^*_1 p_3 N_2 N_3 + p_1 p^*_3 N_1 N_4) \right]
$$
$$
\mathcal{I}(1, 4; 2, 3) = 16 \left[ 6 \frac{p^4}{N^4}(N_1 N_2 N_3 N_4)^2 + (p^*_1)^2 p_3^2 N_2 N_4^2 + p_1^2 (p^*_3)^2 N_1^2 N_3^2
+ 4 \frac{p^2}{N^2} N_1 N_2 N_3 N_4 (p^*_1 p_3 N_2 N_4 + p_1 p^*_3 N_1 N_3) \right].
$$

The other three $\mathcal{I}(i, j; k, l)$ can be obtained by taking complex conjugates of the above results.

Substituting in the ten vectors for the momenta and polarizations into the Lorentz invariant expression (8.27) one can show that the expression (8.27) is equal (3.15)(3.17).

It is crucial, to obtain this equality, to include in the polarization vectors the non-zero $\xi^-$ part (see equation (7.6)).

Turning to the case with two polarizations in the $X^1, X^2$ plane and two in the $X^3, X^4$ plane we have

$$
\int d^8 \theta \Psi^i_{(12)} \Psi^j_{(12)} \Psi^k_{(34)} \Psi^l_{(34)} = \frac{\xi_i \xi_j \xi_k \xi_l}{N_i N_j N_k N_l} \mathcal{I}(i; j | k; l) + 3 \text{ other permutations},
$$

where

$$
\mathcal{I}(i; j | k; l) = -\epsilon \int d^8 \theta \prod_{m=j,l} \prod_{n=i,l} (p^*_m \theta^p + N_m \theta^N)^2(p^*_n \bar{\theta}^p + N_n \bar{\theta}^N)^2,
$$

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where again $\epsilon$ is a sign factor which is equal to $+1$ if $i, j = 1, 2$ or $3, 4$ and $-1$ otherwise. It comes from the sign of the $p^+$ momenta in the definitions of the wavefunctions \((7.3)\). Evaluating the fermionic integrals and using the identities \((2.28), (2.29)\) and \((2.31)\) one finds that

$$\mathcal{I}(i; j|k;l) = \mathcal{I}(i; j|l;k) \quad \text{and} \quad \mathcal{I}(i; j|k;l) = \mathcal{I}^*(j; i|k;l), \quad (3.20)$$

with the $6 (= 4!/(2!)^2)$ different ways of distributing the two types of polarization amongst the four wavefunctions being given by

$$\mathcal{I}(1; 2|3; 4) = 16 \left[ -\frac{p^4}{N_4} N_1 N_2 (N_3^2 - 4N_3 N_4 + N_4^2) + 2 \frac{p^2}{N_2} (N_3 - N_4)(N_1 p_1 p_3^* - N_2 p_1^* p_3) \right. \nonumber$$

$$- \left. \frac{N_1}{N_2} p_1^2 (p_3^*)^2 - \frac{N_2}{N_1} (p_1^*)^2 p_3^2 \right].$$

By substituting in the ten vectors for the momenta and polarizations into the Lorentz invariant expressions \((8.28)\) one can show that the expressions of \((8.28)\) are equal to those of \((3.18)-(3.21)\). Again it is crucial to include in the polarization vectors the non-zero $\xi^-$ part (see equation \((7.4)\)) to obtain agreement.

Finally we turn to the case with two polarizations in the $X^1, X^2$ plane and two in the $X^5, X^6, X^7, X^8$ subspace. For this case the fermionic integrals read

$$\int d^8 \theta \Psi^{i}_{(12)}(\xi_{(12)}) \Psi^{j}_{(5678)}(\xi_{(5678)}) = \frac{\xi_i \xi_j^* (\xi_k \cdot \xi_l)}{N_1 N_2 N_3 N_4} \mathcal{I}(i; j) \quad + \quad \text{c.c.,} \quad (3.22)$$

where

$$\mathcal{I}(i; j) \delta^{m_k m_l} = - \int d^8 \theta \left( (p_j^* \theta^p + N_j \theta^N)^2 (p_i \bar{\theta}^p + N_i \bar{\theta}^N)^2 \right. \nonumber$$

$$\left. \prod_{n=k,l} (p_n^* \theta^p + N_n \theta^N)(\sigma^m_n)(p_n \bar{\theta}^p + N_n \bar{\theta}^N) \right). \quad (3.23)$$
Evaluating the integrals the results are found to be identical to those of (3.21), i.e. we have

\[ I(i; j) = \frac{1}{2} I(i; j|k; l), \quad (3.24) \]

where the \( I(i; j|k; l) \) (which have \( k, l \neq i, j \)) are given by the expressions of (3.21).
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