On the variable-charged black holes in General Relativity: 
Hawking’s radiation and naked singularities

Ng. Ibohal,
Dept of Mathematics, Manipur University, 
Imphal 795003, Manipur, India
E-mail: ngibohal@rediffmail.com

Abstract

In this paper the variable-charged non-rotating Reissner-Nordstrom as well as rotating Kerr-Newman black holes are discussed. “Such a variable charge \( e \) with respect to the polar coordinate \( r \) in the field equations is referred as an electrical radiation of the black hole”. It is shown that every electrical radiation \( e(r) \) of the non-rotating black hole leads to a reduction of its mass \( M \) by some quantity. If one considers such electrical radiation to take place continuously one after another for a long time, then a continuous reduction of the mass may take place in the black hole body and the original mass of the black hole may be evaporated completely. At that stage, the gravity of the object may depend only on the electromagnetic field, not on the mass. Just after the complete evaporation of the mass, if the next radiation continues, there may be a creation of a new mass leading to the formation of negative mass naked singularity. It appears that this new mass of the naked singularity would never decrease, but might increase gradually as the radiation continues forever. A similar investigation is also discussed in the case of variable-charged rotating Kerr-Newman black hole. It has been shown by incorporating of Hawking’s evaporation of radiating black holes in the form of space-time metrics, every electrical radiation of variable-charged rotating as well as non-rotating black holes may produce a change in the mass of the body without affecting the Maxwell scalar.

PACS number: 0420, 0420J, 0430, 0440N.

1. Introduction

In General Relativity, charged black holes are defined as solutions of Einstein-Maxwell field equations. So, it is well-known that the Kerr-Newman solution is a rotating charged black-hole with three constant parameters \( M, e, a \) of the family; \( M \) representing the gravitational mass, \( aM \) the angular momentum and \( e \) the charge of the body. When \( a = 0 \), the metric reduces to the non-rotating Reissner-Nordstrom solution with the charge \( e \). When \( e = 0 \), the space-time metric reduces to the vacuum Kerr family and when \( a = e = 0 \), the solution becomes the Schwarzschild solution. The charged Kerr-Newman black hole has an external event horizon at \( r_+ = M + \sqrt{(M^2 - a^2 - e^2)} \) and an internal Cauchy horizon at \( r_- = M - \sqrt{(M^2 - a^2 - e^2)} \). The stationary limit surface \( g_{tt} > 0 \) of the rotating black hole i.e. \( r = r_e(\theta) = M + \sqrt{(M^2 - a^2 \cos^2 \theta - e^2)} \) does not coincide with the event horizon at \( r_+ \) thereby producing the ergosphere. This stationary limit coincides with the event horizon at the poles \( \theta = 0 \) and \( \theta = \pi \) [1]. However, in the case of the non-rotating charged Reissner-Nordstrom solution with \( a = 0 \), the event horizon \( r_+ \) coincides with the stationary limit at \( r_e \).
The Hawking radiation \cite{2} suggests that black holes which are formed by collapse, are not completely black, but emit radiation with a thermal spectrum due to quantum effects \cite{3}. That objects radiate and thereby must decrease in size. Hawking \cite{4} stated –“Because this radiation carries away energy, the black holes must presumably lose mass and eventually disappear. If one tries to describe this process of black hole evaporation by a classical space-time metric, there is inevitably a naked singularity when the black hole disappears”. In an introductory survey Hawking and Israel \cite{5} have discussed the black hole radiation in three possibilities with creative remarks –“So far there is no good theoretical framework with which to treat the final stages of a black hole but there seem to be three possibilities: (i) The black hole might disappear completely, leaving just the thermal radiation that it emitted during its evaporation. (ii) It might leave behind a non-radiating black hole of about the Planck mass. (iii) The emission of energy might continue indefinitely creating a negative mass naked singularity”. Boulware \cite{6} suggested that the radiation may be expressed in terms of the stress energy momentum tensor associated with the field whose quanta are being radiated. Thus, the change in the metric due to the radiation may be calculated by using the stress energy tensor in Einstein’s field equations. If one were to interpret $r, \theta, \phi$ as representing spherical polar coordinates, there is a singularity at the origin $r = 0$, whereas when $r = \infty$, the metric approaches the Minkowski flat metric. So the nature of the black holes depends on the polar radius $r$. The question arises what happens when the charge $e$ is considered to be a function of $r$ in the stress energy momentum tensor $T_{ab}$ of the Einstein-Maxwell field equations of the rotating as well as non-rotating charged black holes. Here it is to mention that Vaidya \cite{7} could produce a metric describing a radiative black hole by considering the mass of the Schwarzschild solution, variable with respect to the retarded time $u$.

In this paper we try to incorporate the Hawking radiation effects by considering Boulware’s suggestion that the energy-momentum tensors of electromagnetic field must have different forms from those of Reissner-Nordstrom as well as Kerr-Newman black holes as these two black holes seem not to have any direct Hawking radiation effects. That is, the only idea left is to consider the charge $e$ to be variable with respect to the coordinate $r$ of these black holes. Here, we mean the ‘electrical radiation’ of a charged black hole as the variation of the charge $e$ of the body with respect to the coordinate $r$ in the stress energy momentum tensor of the Einstein-Maxwell field equations. So, for every electrical radiation we consider the charge $e$ to be a function of $r$ in solving the Einstein-Maxwell field equations and we have shown mathematically how the electrical radiation induces to produce the changes of the mass of variable-charged black holes. One may incorporate the idea of losing (or changing) mass at the rate as the electrical energy is radiated from the charged black hole. It is noted that these results, presented in this paper are mainly mathematical. We do not, therefore intend to determine the exact change of the mass numerically, but could certainly observe mathematical result of losing (or changing) mass in the space-time metrics cited below after every electrical radiation.

If the energy momentum tensor is of electromagnetic fields with the charge $e(r)$, the Einstein-Maxwell field equations for both rotating and non-rotating charged black holes yield the Ricci scalar $R (= R_{ab} g^{ab})$ not equal to zero. But for electromagnetic fields this Ricci scalar has to vanish in general. As this Ricci scalar $R=R_{ab} g^{ab}$ for electromagnetic fields vanishes, one would find that
some quantity $m_1$ is being decreased from the original masses of the black holes. However, it is found that the form of the Maxwell scalar $\phi_1 = \frac{1}{2} F_{ab}(\ell^a n^b + \overline{m}^a m^b)$ of the black holes remains unchanged.

Thus, one might summarize the result of the paper in the form of theorems:

**Theorem 1.** Every electrical radiation of ‘variable-charged’ non-rotating Reissner-Nordstrom and rotating Kerr-Newman black holes may produce a change in the mass of the bodies without affecting the Maxwell scalar.

**Theorem 2.** During the radiation process, after the complete evaporation of masses of both ‘variable-charged’ non-rotating Reissner-Nordstrom and rotating Kerr-Newman black holes, the electrical radiation may continue indefinitely creating negative mass naked singularities.

It appears that this theorem 2 may be in favor of Hawking-Israel third possibility quoted above, but a violation of Penrose’s cosmic-censorship hypothesis that no naked singularity can never be created [1]. The theorem 2 has been presented in the form of classical space-time metrics below.

Here, we use the word ‘change in the mass’ rather then ‘loss of mass’ as there may be possible of creation of mass after the exhaustion of the original mass, if one repeats the same process of electrical radiation. This may be seen latter in this paper. Hawking radiation is being incorporated, in the classical General Relativity describing the change in mass appearing in the classical space-time metrics, without quantum mechanical aspect as done by Hawking [2] or the path integral method used by Hurtle and Hawking [8] or thermodynamic viewpoint [9]. We present classical space-time metrics affected by the change in the mass of the variable charged black holes after electrical radiation in Section 2. In Section 3 the properties of the metrics formed after the electrical radiation is discussed in regards with Kerr-Schild form and Chandrasekhar’s relation. The NP version of original Reissner-Nordstrom and Kerr-Newman solutions are presented in an Appendix. We use the language of Newman and Penrose [10] essentially based on a differential form structure [11] throughout the paper.

### 2. Changing masses of variable-charged black holes

In this section, by solving Einstein-Maxwell field equations with the variable-charge $e(r)$, we develop the relativistic aspect of Hawking radiation in classical space-time metrics. The calculation of Newman-Penrose (NP) spin coefficients is being carried out through the technique developed by McIntosh and Hickman [11] in (+, −, −, −) signature. In the formulation of this relativistic aspect of Hawking radiation, we do not impose any condition in the field equations except considering the charge $e$ to be a function of the polar coordinate $r$.

#### 2.1. Variable-charged Reissner-Nordstrom solution

We consider the non-rotating variable-charged solution with the assumption that the charge $e$ of the body is a function of coordinate $r$:

$$
\text{ds}^2 = \left\{1 - \frac{2M}{r} + \frac{e^2(r)}{r^2}\right\} \text{dr}^2 + 2 \text{dr} \text{d}r - r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2).
$$

(2.1)
Initially when $e$ is constant, this metric will reduce to non-rotating charged Reissner-Nordstrom solution. The complex null tetrad vectors for this metric are chosen as

$$\ell^a = \delta_2^a,$$

$$n^a = \delta_1^a - \frac{1}{2} \left\{ 1 - \frac{2M}{r} + \frac{e^2(r)}{r^2} \right\} \delta_2^a,$$

$$m^a = \frac{1}{\sqrt{2}r} \left( \delta_3^a + \frac{i}{\sin \theta} \delta_4^a \right),$$

where $\ell^a$, $n^a$ are real null vectors and $m^a$ is complex null vector. Using these null tetrad vectors one could calculate the spin coefficients, Ricci scalars and Weyl scalars as follows:

$$\kappa = \sigma = \lambda = \pi = \tau = \epsilon = 0, \quad \rho = -\frac{1}{r}, \quad \beta = -\alpha = \frac{1}{2\sqrt{2}r} \cot \theta,$$

$$\mu = -\frac{1}{2r} \left\{ 1 - \frac{2M}{r} + \frac{e^2(r)}{r^2} \right\},$$

$$\gamma = \frac{1}{2r^2} \left\{ M + e(r) e'(r) - \frac{e^2(r)}{r^2} \right\},$$

$$\phi_{11} = \frac{1}{4r^2} \left\{ e'^2(r) + e(r) e''(r) \right\} - \frac{e(r)e'(r)}{r^3} + \frac{e^2(r)}{2r^4},$$

$$\Lambda = -\frac{1}{12r^2} \left\{ e'^2(r) + e(r) e''(r) \right\},$$

$$\psi_2 = \frac{1}{6r^2} \left\{ e'^2(r) + e(r) e''(r) \right\} - \frac{1}{r^3} \left\{ M + e(r)e'(r) - \frac{e^2(r)}{r} \right\},$$

where the Weyl curvature scalar $\psi_2$, Ricci scalars $\phi_{11}$ and $\Lambda$ are defined by

$$\psi_2 \equiv -C_{abcd} \ell^a m^b n^c \tilde{n}^d,$$

$$\phi_{11} \equiv -\frac{1}{4} R_{ab}(\ell^a n^b + m^a \tilde{m}^b), \quad \Lambda \equiv \frac{1}{24} R_{ab} g^{ab}. \quad (2.6)$$

Here, a prime denotes the derivative with respect to $r$. If the energy momentum tensor is of electromagnetic fields, then the Ricci tensor $R_{ab}$ is proportional to the Maxwell stress tensor [8] that is

$$\phi_{AB} = k \phi_A \phi_B, \quad k = 8\pi G/c^2$$

with $A,B = 0,1,2$ and the NP Ricci scalar

$$\Lambda \equiv \frac{1}{24} R_{ab} g^{ab} = 0. \quad (2.8)$$

Hence, vanishing $\Lambda$ in (2.8) with (2.4) leads

$$e^2(r) = 2rm_1 + C$$

(2.9)
where \(m_1\) and \(C\) are real constants. Then the Ricci scalar becomes

\[
\phi_{11} = \frac{C}{2r^4}.
\]

(2.10)

Thus, the Maxwell scalar \(\phi_1 = \frac{1}{2}F_{ab}(\ell^a n^b + m^a n^b)\) takes the form, by identifying the real constant \(C = e^2\),

\[
\phi_1 = \frac{1}{\sqrt{2}} e r^{-2},
\]

(2.11)

showing that the Maxwell scalar \(\phi_1\) does not change its form by considering the charge \(e\) to be a function of \(r\) in Einstein-Maxwell field equations. Here, by using equation (2.9) in (2.3) and (2.5), we have the resulting NP quantities

\[
\mu = -\frac{1}{2r} \{1 - \frac{2}{r}(M - m_1) + \frac{e^2}{r^2}\},
\]

(2.12)

\[
\gamma = \frac{1}{2r^2} \{(M - m_1) - \frac{e^2}{r}\},
\]

\[
\psi_2 = -\frac{1}{r^3} \{(M - m_1) - \frac{e^2}{r}\}, \quad \phi_1 = \frac{1}{\sqrt{2}} e r^{-2},
\]

(2.13)

and the metric (2.1) takes the form

\[
ds^2 = \{1 - \frac{2}{r}(M - m_1) + \frac{e^2}{r^2}\} du^2 + 2du dr - r^2(d\theta^2 + \sin^2\theta d\phi^2).
\]

(2.14)

This means that the mass \(M\) of non-rotating black hole (2.1) is lost a quantity \(m_1\) at the end of the first electrical radiation. This loss of mass is agreeing with Hawking’s discovery that the radiating objects must lose its mass [2]. On this losing mass, Wald [12] has pointed that a black hole will lose its mass at the rate as the energy is radiated. If one considers the same process for second time taking \(e\) in (2.14) to be function of \(r\) with the mass \(M - m_1\) in Einstein-Maxwell field equations, then the mass may again be decreased by another constant \(m_2\) (say); that is, after the second time radiation the total mass might become \(M - (m_1 + m_2)\). This is due to the fact, that the Maxwell scalar \(\phi_1\) with condition (2.8) does not change its form after considering the charge \(e\) to be function of \(r\) for the second time as \(\Lambda\) calculated from the Einstein-Maxwell field equations has to vanish for electromagnetic fields with \(e(r)\). Hence, if one repeats the same process for \(n\)-times considering every time the charge \(e\) to be function of \(r\), then one would expect the solution to change gradually and the total mass becomes \(M - (m_1 + m_2 + m_3 + ... + m_n)\) and therefore the metric (2.14) may take the form:

\[
ds^2 = [1 - \frac{2}{r} \mathcal{M} + \frac{e^2}{r^2}] du^2 + 2du dr - r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]

(2.15)

where the mass of the black hole after the radiation of \(n\)-times would be

\[
\mathcal{M} = M - (m_1 + m_2 + m_3 + ... + m_n).
\]

(2.16)

So, accordingly the changed NP quantities are

\[
\mu = -\frac{1}{2r} \{1 - \frac{2}{r} \mathcal{M} + \frac{e^2}{r^2}\},
\]

(2.17)
\[
\gamma = \frac{1}{2r^2} \{ M - \frac{e^2}{r} \},
\]
(2.18)

\[
\psi_2 = -\frac{1}{r^3} \{ M - \frac{e^2}{r} \}.
\]
(2.19)

This suggests that for every electrical radiation, the original mass \( M \) of the non-rotating black hole may lose some quantity. Thus, it seems reasonable to expect that, taking Hawking’s radiation of black holes into account, such continuously lose of mass may lead to evaporate the original mass \( M \). In case the black hole has evaporated down to the Planck mass, the mass \( M \) may not exactly equal to the continuously lost quantities \( m_1 + m_2 + m_3 + \ldots + m_n \). That is, according to the second possibility of Hawking and Israel [5] quoted above, there may left a small quantity of mass, say, Planck mass of about \( 10^{-5} \) \( g \) with continuous electrical radiation. Otherwise, when \( M = m_1 + m_2 + m_3 + \ldots + m_n \) for a complete evaporation of the mass, \( M \) would be zero, rather than, leaving behind a Planck-size mass black hole remnant. At this stage the non-rotating black hole geometry would have the electric charge \( e \) only, but no mass, that the line element would be of the form

\[
ds^2 = (1 + \frac{e^2}{r^2}) \, du^2 + 2du \, dr - r^2(\, d\theta^2 + \sin^2\theta \, d\phi^2).
\]
(2.20)

That is, the black hole might be radiated away all its mass completely just leaving the electrical radiation. Then the classical space-time metric (2.20) may regard to represent a non-rotating zero mass naked singularity as pointed out by Davies [13]. Here we refer such object with zero mass as ‘instantaneous’ naked singularity – a naked singularity that exists for an instant and then continues its electrical radiation to create negative mass. [Syteinmular, King and LoSota [14] refer a spherically symmetric star, which radiates away all its mass as ‘instantaneous’ naked singularity that exists only for an instant and then disappears. This ‘instantaneous’ naked singularity is also mentioned in [15]).

The time taken between two consecutive radiations is supposed to be very short that one may not physically realize how quickly radiations take place. Thus it seems natural to expect the existence of ‘instantaneous’ naked singularity with zero mass only for an instant before continuing its next radiation to create negative mass naked singularity. It may also be possible in the reasonable theory of black holes that, as a black hole is invisible in nature, one may not know that in the universe, a particular black hole has mass or not, but electrical radiation may be detected on the black hole surface. So, there may be some radiating black holes without masses in the universe, where the gravity may depend only on the electric charge, \( i.e., \psi_2= e^2/r^4 \), not on the mass of the black holes. Just after the exhaustion of the mass, if one continues the remaining solution (2.20) to radiate, there may be a formation of new mass \( m_1^* \) (say). If one repeats the electrical radiation further, the new mass might increase gradually and then, the metric (2.20) with the new mass would become

\[
ds^2 = (1 + \frac{2}{r} \mathcal{M}^* + \frac{e^2}{r^2}) \, du^2 + 2du \, dr - r^2(\, d\theta^2 + \sin^2\theta \, d\phi^2),
\]
(2.21)

where the new mass is given by

\[
\mathcal{M}^* = m_1^* + m_2^* + m_3^* + m_4^* + \ldots
\]
(2.22)
Comparing the metrics (2.15) and (2.21) one could observe that the classical space-time (2.21) may describe a non-rotating spherical symmetric star with a negative mass \( M^* \). Such objects with negative masses are referred as naked singularities \([1,4,5]\). The metric (2.21) may be regarded to describe the incorporation of the third possibility of Hawking and Israel \([5]\) in the case of non-rotating singularity. Here it is noted that the creation of negative mass naked singularity is mainly based on the continuous electrical radiation of the variable charge \( e(r) \) in the energy momentum tensor of Einstein-Maxwell equations. This also indicates the incorporation of Boulware’s suggestion \([6]\) that ‘the stress-energy tensor may be used to calculate the change in the metric due to the radiation’.

This new mass \( M^* \) would never decrease, rather might increase gradually as the radiation continues forever. Then the spin coefficients for the metric (2.21) are

\[
\mu = -\frac{1}{2r}\left(1 + \frac{2}{r}M^* + \frac{e^2}{r^2}\right), \quad (2.23)
\]

\[
\gamma = -\frac{1}{2r^2}\left(M^* + \frac{e^2}{r}\right), \quad (2.24)
\]

\[
\psi_2 = \frac{1}{r^3}\left(M^* + \frac{e^2}{r}\right), \quad (2.25)
\]

and the unchanged Maxwell scalar \( \phi_1 \) is given in (2.11). Thus, one has seen the changes in the mass of the non-rotating charged black hole after every radiation. Hence, it follows the theorem cited above in the case of non-rotating variable-charged black hole.

### 2.2. Variable-charged Kerr-Newman solution

Here, one may incorporate the Hawking radiation, how the rotating variable-charged black hole affects in the classical space-time metric when the electric charge \( e \) is taken as a function of \( r \) in the Einstein-Maxwell field equations. The line element with \( e(r) \) is

\[
\begin{align*}
ds^2 &= \left[1 - R^{-2}\{2Mr - e^2(r)\}\right]du^2 + 2du\,dr \\
&\quad + 2aR^{-2}\{2Mr - e^2(r)\}\sin^2\theta\,du\,d\phi - 2a\sin^2\theta\,dr\,d\phi \\
&\quad - R^2d\theta^2 - \{(r^2 + a^2)^2 - \Delta^*a^2\sin^2\theta\}R^{-2}\sin^2\theta\,d\phi^2,
\end{align*}
\]

(2.26)

where

\[
R^2 = r^2 + a^2\cos^2\theta, \quad \Delta^* = r^2 - 2Mr + a^2 + e^2(r).
\]

(2.27)

This metric will also reduce to rotating Kerr-Newman solution when \( e \) becomes constant. The null tetrad vectors are chosen as

\[
\begin{align*}
\ell^a &= \delta_2^a, \quad n^a = \frac{1}{R^2}\left[\left(r^2 + a^2\right)\delta_1^a - \frac{\Delta^*}{2}\delta_2^a + a\delta_4^a\right], \\
m^a &= \frac{1}{\sqrt{2R}}\left[i\theta a\sin\theta\delta_1^a + \delta_3^a + \frac{i}{\sin\theta}\delta_4^a\right],
\end{align*}
\]

(2.28)
where $R = r + ia \cos \theta$. Then we solve the Einstein-Maxwell field equations for the metric (2.26) and write only the changed NP quantities

$$\mu = -\frac{1}{2RR^2} \{r^2 - 2Mr + a^2 + e^2(r)\},$$

(2.29)

$$\gamma = \frac{1}{2RR^2} \left[ \{r - M + e(r)e'(r)\} - \{r^2 - 2Mr + a^2 + e^2(r)\} \right],$$

(2.30)

$$\psi_2 = \frac{1}{RRR^2} \left[ -MR + e^2(r) - e(r)e'(r) \right] + \frac{1}{6R^2} \left\{ e'^2(r) + e(r)e''(r) \right\},$$

(2.31)

$$\phi_{11} = \frac{1}{2RR^2} \left\{ e^2(r) - 2re(r)e'(r) \right\} + \frac{1}{4R^2} \left\{ e'^2(r) + e(r)e''(r) \right\},$$

(2.32)

$$\Lambda = -\frac{1}{12 R^2} \left\{ e'^2(r) + e(r)e''(r) \right\},$$

(2.33)

where a prime denotes the derivative with respect to $r$.

Now, like equation (2.8) in non-rotating black hole, the scalar $\Lambda$ must vanish for this rotating metric. Thus, vanishing $\Lambda$ of the equation (2.33) implies that

$$e^2(r) = 2rm_1 + C$$

(2.34)

where $m_1$ and $C$ are real constants of integration. Then, using this result in equation (2.32) we obtain the Ricci scalar

$$\phi_{11} = \frac{C}{2R^2 R^2}.$$  

(2.35)

Accordingly, the Maxwell scalar may become, after identifying the constant $C = e^2$,

$$\phi_1 = \frac{e}{\sqrt{2RR}}.$$  

(2.36)

This is the same Maxwell scalar of the electrovac Kerr-Newman black hole (see appendix) where the charge $e$ is constant. Hence, from the Einstein-Maxwell equations we have the changed NP quantities

$$\mu = -\frac{1}{2RR^2} \{r^2 - 2r(M - m_1) + a^2 + e^2\},$$

(2.37)

$$\gamma = \frac{1}{2RR^2} \left[ \{r - (M - m_1)\} - \{r^2 - 2r(M - m_1) + a^2 + e^2\} \right],$$

(2.38)

$$\psi_2 = \frac{1}{RRR^2} \left\{ -(M - m_1)R + e^2 \right\},$$

(2.39)

Thus, the rotating solution with a new constant $m_1$ takes the following form

$$ds^2 = \left[ 1 - R^{-2} \{2r(M - m_1) - e^2\} \right] du^2 + 2 du dr + 2a R^{-2} \{2r(M - m_1) - e^2\} \sin^2 \theta \ du \ d\phi - 2a \sin^2 \theta \ dr \ d\phi.$$
\[- R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta\} R^{-2} \sin^2 \theta \, d\phi^2, \quad (2.40)\]

where

\[ \Delta^* = r^2 - 2r(M - m_1) + a^2 + e^2. \quad (2.41) \]

This introduction of constant \( m_1 \) in the metric (2.40) suggests that the first electrical radiation of rotating black hole may reduce the original gravitational mass \( M \) by a quantity \( m_1 \). If one considers another radiation by taking \( e \) in (2.40) to be a function of \( r \) with the mass \( M - m_1 \), then the Einstein-Maxwell field equations yield to reduce this mass by another constant quantity \( m_2 \); i.e., after the second radiation, the mass may become \( M - (m_1 + m_2) \). Here again, the Maxwell scalar \( \phi_1 \) remains the same form after the second radiation also. Thus, if one considers the \( n \)-time radiations taking every time the charge \( e \) to be function of \( r \), the Maxwell scalar \( \phi_1 \) will be the same, but the metrics may be of the following form:

\[
ds^2 = \left[ 1 - R^{-2}\{2rM - e^2\} \right] du^2 + 2du \, dr \\
+ 2ae^2 R^{-2} \sin^2 \theta \, du \, d\phi - 2a \sin^2 \theta \, dr \, d\phi \\
- R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta\} R^{-2} \sin^2 \theta \, d\phi^2, \quad (2.42)\]

where the total mass of the black hole, after the \( n \)-time radiations may take the form

\[ M = M - (m_1 + m_2 + m_3 + m_4 + \ldots + m_n). \quad (2.43) \]

Taking Hawking’s radiation of black holes, one might expect that the total mass of black hole may be radiated away just leaving \( M \) equivalent to Planck mass of about \( 10^{-5} \text{g} \), that is, \( M \) may not be exactly equal to \( m_1 + m_2 + m_3 + m_4 + \ldots + m_n \), but has a difference of about Planck-size mass, as in the case of non-rotating black hole. Otherwise, the total mass of black hole may be evaporated completely after continuous radiation when \( M = 0 \), that is, \( M = m_1 + m_2 + m_3 + m_4 + \ldots + m_n \). Here one may regard that the rotating variable-charged black hole might be radiated completely away all its mass just leaving the electrical charge \( e \) only. One could observe this situation in the form of classical space-time metric as

\[
ds^2 = (1 + e^2 R^{-2}) \, du^2 + 2du \, dr \\
+ 2ae^2 R^{-2} \sin^2 \theta \, du \, d\phi - 2a \sin^2 \theta \, dr \, d\phi \\
- R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta\} R^{-2} \sin^2 \theta \, d\phi^2, \quad (2.44)\]

with the charge \( e \), but no mass, where \( \Delta^* = r^2 + a^2 + e^2 \). The metric (2.44) may describe a rotating ‘instantaneous’ naked singularity with zero mass. At this stage, the Weyl scalar \( \psi_2 \) takes the form

\[ \psi_2 = \frac{e^2}{RRR} \quad (2.45) \]

showing the gravity on the surface of the remaining solution depending only on the electric charge \( e \); however, the Maxwell scalar \( \phi_1 \) remains the same as in (2.36). For future use, we mention the changed NP spin coefficients

\[ \mu = -\frac{1}{2RRR} \{r^2 + a^2 + e^2\}, \quad (2.46) \]
\[ \gamma = \frac{1}{2R R^2} [r R - \{r^2 + a^2 + e^2\}], \]

It suggests that there may be rotating black holes in the universe whose masses are completely radiated; their gravity depend only on the electric charge of the body and their metrics look like the one given in the equation (2.44). It appears that the idea of this evaporation of masses of radiating black holes may be agreed with that of Hawking’s evaporation of black holes. Unruh [16] has examined various aspects of black hole evaporation based on Schwarzschild metric. It is worth studying the nature of such black holes (2.44) or in the case of non-rotating (2.20). This might give a different nature, which one has not yet come across so far in the reasonable theory of black holes.

Here, one may consider again the charge \( e \) to be function of radial coordinate \( r \) for next radiation in (2.44), so that one must get from the Einstein’s field equations the scalar \( \Lambda \) as given in equation (2.33). Then the vanishing of this \( \Lambda \) for electromagnetic field, there may be creation of a new mass (say \( m_1^* \)) in the remaining space-time geometry. If this radiation process continues forever, the new mass may increase gradually as

\[ \mathcal{M}^* = m_1^* + m_2^* + m_3^* + m_4^* + \ldots \]  \hspace{1cm} (2.47)

However, it appears that this new mass would never decrease. Then the space-time geometry may take the form

\[
\begin{align*}
\text{ds}^2 &= [1 + R^{-2}\{2r\mathcal{M}^* + e^2\}] \, du^2 + 2du \, dr \\
&\quad + 2aR^{-2}\{2r\mathcal{M}^* + e^2\} \sin^2 \theta \, du \, d\phi - 2a \sin^2 \theta \, dr \, d\phi \\
&\quad - R^2 \, d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta\} \, R^{-2} \sin^2 \theta \, d\phi^2 ,
\end{align*}
\]  \hspace{1cm} (2.48)

where

\[ \Delta^* = r^2 + 2r\mathcal{M}^* + a^2 + e^2 . \]  \hspace{1cm} (2.49)

The Weyl scalar \( \psi_2 \) and other NP coefficients are calculated from the Einstein-Maxwell field equations as

\[ \psi_2 = \frac{1}{R R R^2} \{\mathcal{M}^* R + e^2\} . \]  \hspace{1cm} (2.50)

\[ \mu = -\frac{1}{2R R^2} \{r^2 + 2r\mathcal{M}^* + a^2 + e^2\} , \]  \hspace{1cm} (2.51)

\[ \gamma = \frac{1}{2R R^2} \{[(r + \mathcal{M}^*) R - \{r^2 + 2r\mathcal{M}^* + a^2 + e^2\}]\}, \]

with \( \phi_1 \) given in (2.36). The metric (2.48) may be regarded to describe a rotating negative mass naked singularity. We have presented the possible changes in the mass of the rotating charged black hole without affecting the Maxwell scalar \( \phi_1 \) and accordingly, metrics are cited for future use. Thus, this completes the proofs of other parts of the theorems for the rotating charged black hole.

3. Conclusion
In the above section, it has shown the changes in the mass of charged black holes after every radiation. However, according to Hawking evaporation rate \[2\], the quantities \(m'_1\)'s to be reduced from the mass of the black hole, may not be equal i.e., \(m_1 \neq m_2 \neq m_3 \neq \ldots\). It appears from the results discussed in this paper that the black hole radiation process is mainly based on the electrical radiation of the variable-charged \(e(r)\) in the energy momentum tensor describing the change in mass in classical space-time metrics. The creation of negative mass naked singularities is also due to the continuous electrical radiation. This clearly suggests that an electrical radiating black hole, non-rotating or rotating would not disappear completely, rather, would form a negative mass naked singularity with regards to the nature of rotation. However, the disappearance of such black hole during the radiation process may occur only for an instant, just at the time of formation of ‘instantaneous’ naked singularity with zero mass. The formation of naked singularity of negative mass is also Hawking’s suggestion mentioned in the introduction above \[4\]. This suggests that, if one excepts the continuous electrical radiation to lead the complete evaporation of the original mass of black holes, then the same radiation might also lead the creation of new mass to form negative mass naked singularities. The classical space-time metrics, rotating or non-rotating discussed above, would describe the possible life style of electrically radiating black holes during their radiation process.

Also, we know from the above that the change in the mass of black holes takes place due to the Maxwell scalar \(\phi_1\), remaining unchanged in the field equations. So, if the Maxwell scalar \(\phi_1\) is absent from the space-time geometry, there will be no radiation and consequently, no reduction in the mass of the black hole. Therefore, one could not expect, theoretically to observe such ‘relativistic change’ in the mass in the case of uncharged Schwarzschild as well as Kerr black holes. Here, a brief description of the metrics formed after radiation is being presented.

Under the transformation \[1\]

\[du = dt - \frac{(r^2 + a^2)}{\Delta^*} dr, \quad d\phi' = d\phi - \frac{a}{\Delta^*} dr,\]  

(3.1)

the metric (2.44) is written in the Boyer-Lindquist coordinates \((t, r, \theta, \phi)\)

\[ds^2 = \frac{(1 + e^2 R^{-2}) dt^2 - R^2 \Delta^* dr^2 - R^2 d\theta^2}{\Delta^*} - \frac{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta}{\Delta^*} R^{-2} \sin^2 \theta d\phi^2 - 2ae^2 R^{-2} \sin^2 \theta dt d\phi.\]

(3.2)

Clearly, one might see that the equation

\[\Delta^* \equiv r^2 + a^2 + e^2 = 0\]

(3.3)

has no solution for the real radial coordinate \(r\) unless \(a^2 + e^2 < 0\). Hence, the metric (3.2) has no event horizon and may not describe a black hole for \(a^2 + e^2 > 0\). Also there is no stationary limit for this metric as \(g_{tt} > 0\) has no real solution for \(r\).

The metric (2.48) could be written in the Boyer-Lindquist coordinates \((t, r, \theta, \phi)\)

\[ds^2 = \left[1 + R^{-2} \{2R \mathcal{M}^* + e^2\} \right] dt^2 - \frac{R^2}{\Delta^*} dr^2 - R^2 d\theta^2\]
\[ \{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta \} R^{-2} \sin^2 \theta \, d\phi^2 \]
\[ - 2a(2r M^* + e^2) R^{-2} \sin^2 \theta \, dt \, d\phi. \] (3.4)

where
\[ \Delta^* = r^2 + 2r M^* + a^2 + e^2. \] (3.5)

If one expects the rotating charged metric (3.4) to be a black hole, then one might find \( r_{\pm} = -M^* \pm \sqrt{(M^*2 - a^2 - e^2)} \) as the roots of the equation \( \Delta^* = 0 \). It appears that the event horizon at \( r_{+} \) for this metric would be very small comparative to the mass \( M^* \) given in (2.47). Also the stationary limit at \( r_{e} = -M^* \pm \sqrt{(M^*2 - a^2 \cos^2 \theta - e^2)} \) for the surface \( g_{tt} > 0 \) is also quite small.

So the true singularity at \( R_2 = r^2 + a^2 \cos^2 \theta = 0 \) may be the only singularity of the coordinate components (3.4). Thus, it seems reasonable to conclude that the metric (3.4) might describe a negative mass naked singularity. However, this solution certainly characterizes a rotating electrovac Petrov type \( D \) space-time as \( \psi_2 \neq 0 \), with the charge \( e \).

The metric (2.48) can be written in the coordinates \((t, x, y, z)\)
\[ ds^2 = dt^2 - dx^2 - dy^2 - dz^2 + \frac{(2r M^* + e^2) r^2}{(r^4 + a^2 z^2)} \{ dt - \frac{1}{(r^2 + a^2)} \{ r(xdx + ydy) + a(xdy - ydx) \} - \frac{1}{r} zdz \}^2 \] (3.6)

where \( r \) is defined, with a sign difference in terms of \((x, y, z)\) [1]
\[ r^4 - (x^2 + y^2 + z^2 - a^2) r^2 - a^2 z^2 = 0, \] (3.7)

and the \((x, y, z)\) have the following relations
\[ x = (r \cos \phi + a \sin \phi) \sin \theta, \quad y = (r \sin \phi - a \cos \phi) \sin \theta, \]
\[ z = r \cos \theta, \quad x^2 + y^2 = (r^2 + a^2) \sin^2 \theta. \] (3.8)

Then, the above metric (3.6) is the Kerr-Schild form with
\[ g_{ab} = \eta_{ab} + 2 H(x, y, z) L_a L_b, \] (3.9)

where \( \eta_{ab} \) is the flat metric and
\[ H(x, y, z) = \frac{(2r M^* + e^2) r^2}{2 (r^4 + a^2 z^2)} \] (3.10)
\[ L_a dx^a = dt - \frac{1}{(r^2 + a^2)} \{ r(xdx + ydy) + a(xdy - ydx) \} - \frac{1}{r} zdz. \] (3.11)

From the Kerr-Schild form metric (3.6), one clearly sees that when \( M^* = 0 \), it reduces to the Kerr-Schild form of the metric (3.2) and it again comes to flat metric with \( e = 0 \). Here, one may note the difference from the Kerr-Newman metric [1] that there is a change in sign before the square parenthesis of (3.6) for the metric (2.48) and also for the metric (2.44) with \( M^* = 0 \).

As the metric (2.48) or (3.4) describes an electrovac Petrov type \( D \) space-time \( (\psi_2 \neq 0) \), it may be better to mention that the spin coefficients of the metric (2.48) satisfy Chandrasekhar’s relation. That, Chandrasekhar [17] has established a relation of spin coefficients \( \rho, \mu, \tau, \pi \) in the case of an
affinely parameterized geodesic vector, generating an integral which is constant along the geodesic in a vacuum Petrov type \(D\) space-time

\[
\frac{\rho}{\tau} = \frac{\mu}{\pi} = \frac{\tau}{\pi} = \frac{\pi}{\tau}.
\]

The original derivation of this relation is purely based on the vacuum Petrov type \(D\) space-time with \(\psi_2 \neq 0, \psi_0 = \psi_1 = \psi_3 = \psi_4 = 0\) and \(\phi_{01} = \phi_{02} = \phi_{10} = \phi_{20} = \phi_{12} = \phi_{21} = \phi_{00} = \phi_{22} = \phi_{11} = \Lambda = 0\). However, after the introduction of Killing-Yano scalar \(\chi_1\) in the above relation as \[18\]

\[
\frac{\rho}{\tau} = \frac{\mu}{\pi} = \frac{\tau}{\pi} = \frac{\pi}{\tau} = -\frac{\chi_1}{\chi_1},
\]

where \(\chi_1 = \frac{1}{2} f_{ab} (\ell^a n^b + m^a m^b)\) with Killing-Yano tensor \(f_{ab}\) satisfying the Killing-Yano equations

\[
f_{abc} + f_{acb} = 0,
\]

it is found that Chandrasekhar’s relation holds true for non-vacuum Petrov type \(D\) space-times. The importance of KY tensor in General Relativity seems to lie on Carter’s remarkable result [19] that the separation constant of Hamilton-Jacobi equation (for charged orbits) in the Kerr space-time gives a fourth constant. In fact, this constant arises from the scalar field \(K_{ab} v^a v^b\) which has vanishing divergent along a unit vector \(v^a\) tangent to an orbit of charged particle. Here \(K_{ab} = f_{ma} f^m_b\). For examples, (i) in Kerr-Newman solution where the source of gravitational field is electromagnetic field, the relation (3.13) takes

\[
\frac{\rho}{\tau} = \frac{\mu}{\pi} = \frac{\tau}{\pi} = \frac{\pi}{\tau} = \frac{R}{\chi_1},
\]

where \(R = r + i a \cos \theta\) and the spin coefficients \(\rho, \mu, \tau, \pi\) are given in Appendix. The Killing-Yano scalar \(\chi_1\) is obtained as [18]

\[
\chi_1 = i C (r - i a \cos \theta)
\]

where \(C\) is a real constant. Here the Killing-Yano tensor for Kerr-Newman metric is

\[
f_{ab} = 4 a \cos \theta n_{[a} \ell_{b]} + 4 i r m_{[a} m_{b]},
\]

and accordingly the Killing tensor \(K_{ab} = f_{ma} f^m_b\)

\[
K_{ab} = -8 \{ a^2 \cos^2 \theta \ell_{(a} n_{b)} + r^2 m_{(a} m_{b)} \}.
\]

(ii) in Kantowski-Sachs metric, where the source of gravitation is dust i.e. \(T_{ab} = \rho^* u_a u_b\) with \(\rho^*\) being the density of the material dust particles, \(u_a\) the unit time like vector, the relation (3.13) becomes [18]

\[
\frac{\rho}{\tau} = \frac{\mu}{\pi} = -\frac{\chi_1}{\chi_1} = 1.
\]

Here, the Killing-Yano scalar is found as \(\chi_1 = \frac{1}{2} i C Y(t)\), where \(C\) is constant and \(Y(t)\) is the function appeared in the Kantowski-Sachs metric. Then the Killing-Yano tensor and the Killing tensor for this metric take the form

\[
f_{ab} = 2 i Y(t) m_{[a} m_{b]} \quad \text{and} \quad K_{ab} = 2 Y^2(t) m_{(a} m_{b)}.
\]
From the above examples, it seems reasonable to refer the relation (3.12) as ‘Chandrasekhar’s identity’ as mentioned by Fernandes and Lun [20]. Thus, it concludes that the metric (2.48) admits the relation (3.13) with the same \( \chi_1 \) as in (3.16).

Taking Hawking radiation into account, one might mention here that the time taken for one radiation to another would be very short that one may not, practically realize after losing \( m_1 \) from the mass how quickly \( m_2 \) is being reduced and so on, as seen above in Section 2. It appears that the metric (2.20) or (2.44) without mass may occur only for a very short period, as the radiation continues further. Thus, we have incorporated Hawking radiation in relativistic viewpoint in curved space-time geometry. One may expect that the metrics with mass \( M^* \) in equations (2.21) and (2.48) might have different nature from Reissner-Nordstrom as well as Kerr-Newman solutions.

**Appendix**

(a) **Reissner-Nordstrom solution**: This is a spherically symmetric solution

\[
ds^2 = (1 - \frac{2M}{r} + \frac{e^2}{r^2}) \, du^2 + 2du \, dr - r^2 \, (d\theta^2 + \sin^2 \theta \, d\phi^2),
\]

(A1)

where \( M \) and \( e \) are the mass and charge respectively. For this metric one chooses the null tetrad vectors

\[
\ell^a = \delta_2^a, \quad n^a = \delta_1^a - \frac{1}{2} \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) \delta_2^a, \quad m^a = \frac{1}{\sqrt{2r}} (\delta_3^a + \frac{i}{\sin \theta} \delta_4^a).
\]

(A2)

The NP quantities are

\[
\kappa = \sigma = \nu = \lambda = \pi = \tau = \epsilon = 0, \quad \rho = -\frac{1}{r}, \quad \beta = -\alpha = \frac{1}{2\sqrt{2r}} \cot \theta, \quad \mu = -\frac{1}{2r} \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right), \quad \gamma = \frac{1}{2r^3} (Mr - e^2), \quad \psi_2 = -(Mr - e^2) r^{-4}, \quad \phi_1 = \frac{1}{\sqrt{2}} \sqrt{er^{-2}}.
\]

(A3)

(b) **Kerr-Newman solution**: The line element is

\[
ds^2 = \left\{1 - R^{-2}(2Mr - e^2)\right\} du^2 + 2du \, dr + 2aR^{-2}(2Mr - e^2) \sin^2 \theta \, du \, d\phi - 2a \sin^2 \theta \, dr \, d\phi - R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta\} R^{-2} \sin^2 \theta \, d\phi^2,
\]

(A4)

where \( R^2 = r^2 + a^2 \cos^2 \theta \), \( \Delta^* = R^{-2} - 2Mr + a^2 + e^2 \). The null tetrad vectors are

\[
\ell^a = \delta_2^a, \quad n^a = \frac{1}{R^2} \left[(r^2 + a^2)^2 \delta_1^a - \frac{\Delta^*}{2} \delta_2^a + a \delta_4^a\right],
\]

\[
m^a = \frac{1}{\sqrt{2R}} \left[i a \sin \theta \delta_1^a + \delta_3^a + \frac{i}{\sin \theta} \delta_4^a\right].
\]

(A5)

where \( R = r + ia \cos \theta \). Then the NP quantities are

\[
\kappa = \sigma = \nu = \lambda = \epsilon = 0, \quad \rho = -\frac{1}{R}, \quad \mu = -\frac{\Delta^*}{2R R^2}.
\]
\[ \alpha = \frac{(2ai - R \cos \theta)}{2\sqrt{2R R \sin \theta}}, \quad \beta = \frac{\cot \theta}{2\sqrt{2R}}, \]
\[ \pi = \frac{i a \sin \theta}{\sqrt{2R R}}, \quad \tau = -\frac{i a \sin \theta}{\sqrt{2R^2}}, \quad \gamma = \frac{1}{2R R^2} \left[ (r - M)R - \Delta^2 \right], \]
\[ \phi_0 = \phi_2 = 0, \quad \phi_1 = \frac{e}{\sqrt{2R R}}, \quad \psi_0 = \psi_1 = \psi_3 = \psi_4 = 0, \quad \psi_2 = -\frac{M}{R^2} + \frac{e^2}{2R R^3}. \]

From the above quantities one can deduce (a) Kerr vacuum solution \((e = 0, a \neq 0)\), (b) Reissner-Nordstrom solution \((e \neq 0, a = 0)\) and (c) Schwarzschild solution \((e = a = 0)\).

**Acknowledgement:** The author acknowledges his appreciation for hospitality received from Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India during the preparation of this paper. I am thankful the unknown referees for remarkable suggestions which lead to significant improvement of an earlier version.

**References**

[1] Chandrasekhar S 1983 *The Mathematical Theory of Black Holes* Clarendon Press, Oxford.

[2] Hawking S W 1974 *Nature* 248 30; 1975 *Commun. math. Phys.* 43 199.

[3] Zhang Z 1991 Physics of Black Holes : Classical, Quantum and Astrophysical in *Black Hole Physics* eds. V De Sabbata and Z Zhang, Kluwer Academic Publishers, Dordrecht.

[4] Hawking S W 1976 *Phys. Rev. D* 14 2460

[5] Hawking S W and Israel W 1980 “An Introductory survey” in *General Relativity: An Einstein Centenary Survey* eds. by S W Hawking and W Israel, Cambridge University Press.

[6] Boulware D G 1976 *Phys. Rev. D* 13 2169.

[7] Vaidya P C 1951 *Proc. Indian Acad. Sci.* A33 264, Reprinted 1999 *Gen. Rel. Grav.* 31 119.

[8] Hartle J B and Hawking S W 1976 *Phys. Rev. D* 13 2188.

[9] Bekenstein J D 1973 *Phys. Rev. D* 7 2333; Nieuwenhuizen T M 1998 *Phys. Rev. D* 81 2201; Sivaram C 2000 *Phys. Rev. Lett.* 84 3209; 2001 *Gen. Rel. Grav.* 33 175, MacGibbon J H 1991 *Phys. Rev. D* 44 376.

[10] Newman E T and Penrose R 1962 *J. Math. Phys.* 3 566.
[11] McIntosh C B G and Hickman M S 1985 *Gen. Rel. Grav.* **17** 111; and Debever R, McLeneghan R G and Tariq N 1979 *Gen. Rel. Grav.* **10** 853.

[12] Wald R M 1975 *Commun. Math.Phys.* **45** 9; 1991 “Black holes and Thermodynamics” in *Black Hole Physics* eds. V De Sabbata and Z Zhang Kluwer Academic Publishers, Dordrecht.

[13] Davies P C W 1980 “Quantum field in Curved Space” in *General Relativity and Gravitation*, Vol 2. ed by A Held, Plenum Press, New York.

[14] Steinmular B, King A R and LoSota J P 1975 *Phys Lett* **51A** 191

[15] Tipler F J, Clerke C J S and Ellis G F R 1980 “Singularities and Horizons – A review article” in *General Relativity and Gravitation: One Hundred Year After the birth of Albert Einstein*, Vol 2, ed A Held, Plenum Press, New York.

[16] Unruh W G 1976 *Phys. Rev. D* **14** 870.

[17] Reference [1] p 324.

[18] Ibohal Ng 1997 *Astrophys and SpaceSc* **249** 73.

[19] Carter B 1968 *Phys. Rev.* **174** 1559 and 1968 *Commun. Math. Phys.* **10** 280; 1973 Black Hole Equilibrium States in C Dewitt and B C Dewitt eds. *Black Holes* Gordon and Breach Sci. Publ. New York.

[20] Fernandes J F Q and Lun A W C 1997 *J. Math. Phys.* **38** 330.