STRUCTURE OF SECOND-ORDER SYMMETRIC LORENTZIAN MANIFOLDS

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Abstract. Second-order symmetric Lorentzian spaces, that is to say, Lorentzian manifolds with vanishing second derivative $\nabla^2 R \equiv 0$ of the curvature tensor $R$, are characterized by several geometric properties, and explicitly presented. Locally, they are a product $M = M_1 \times M_2$ where each factor is uniquely determined as follows: $M_2$ is a Riemannian symmetric space and $M_1$ is either a constant-curvature Lorentzian space or a definite type of plane wave generalizing the Cahen-Wallach family. In the proper case (i.e., $\nabla R \neq 0$ at some point), the curvature tensor turns out to be described by some local affine function which characterizes a globally defined parallel lightlike direction. As a consequence, the corresponding global classification is obtained, namely: any complete second-order symmetric space admits as universal covering such a product $M_1 \times M_2$. From the technical point of view, a direct analysis of the second-symmetry partial differential equations is carried out leading to several results of independent interest relative to spaces with a parallel lightlike vector field —the so-called Brinkmann spaces.

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1. Introduction

A venerable result in Differential Geometry (Nomizu and Ozeki [34], Tanno [43]) states that, for a Riemannian manifold \((M,g)\), the vanishing of the \(r\)-th covariant derivative of its curvature tensor \(R\),
\[
\nabla^r R (= \nabla (\cdots \nabla R)) \equiv 0, \quad r \geq 2,
\]
implies the vanishing of the first one, i.e., that \((M,g)\) is locally symmetric. As a consequence, the standard generalization of Riemannian locally symmetric spaces are the semi-symmetric spaces, introduced by Cartan [13] and defined by the commutativity of the covariant derivatives applied to \(R\):
\[
R(X,Y)R := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}R)R = 0 \quad \text{for all vector fields } X, Y.
\]
Their structure was determined by Szabó locally in [41] and globally later in [42]. However, when \((M,g)\) is a Lorentzian manifold, the equality (1) does not imply \(\nabla R = 0\). In fact, the irreducible character of de Rham decomposition, which was an essential ingredient in the Riemannian result, fails for Lorentzian metrics. Thus, a ladder of logical generalizations of Lorentzian locally symmetric spaces is given by (1). We call these semi-Riemannian manifolds \(r\)th order symmetric (or \(r\)th-symmetric for short) spaces\(^1\). The purpose of the present article is to determine the simplest of these new classes explicitly: the Lorentzian 2nd-symmetric spaces\(^2\).

The classification of Riemannian locally symmetric spaces is known since Cartan’s work [12] (see also [25, 7]), and the classification of the Lorentzian simply-connected symmetric spaces was carried out by Cahen and Wallach in [10]. Extensions to other signatures and to non-simply-connected cases are also available, see Cahen and Parker [9], Neukirchner [33] and specially Kuth and Olbrich [26, 27]. Lorentzian semi-symmetric spaces have also been studied in the literature, see for instance their classification in four dimensions [16, 22] and references therein. Nevertheless, prior to the paper [37] by one of the authors, the 2nd-symmetric spaces had not been studied systematically. As pointed out in this reference, simple examples of proper \(r\)th-symmetric — \(r\)th-symmetric but not \((r-1)\)th-symmetric — Lorentzian spaces can be constructed within the class of \(n\)-dimensional plane waves, see subsection 3.3 below. They constitute a straightforward generalization of the locally symmetric Cahen-Wallach spaces [10] and, as we will prove, they essentially exhaust the whole class of proper 2nd-symmetric Lorentzian spaces.

It is worth pointing out that the 2nd-symmetric spaces are appealing from the viewpoint of the local group of symmetries of the manifold, because the condition \(\nabla^2 R = 0\) can be expressed in terms of the infinitesimal holonomy algebra of the manifold (as this algebra is generated by the image of the curvature two-form and its first derivative see, for example, [28, Th. 9.2 Ch. III]); in fact, the main result in [37] is a property of this holonomy group. With the help of this property and the well-established results on locally symmetric spaces by Cahen and Wallach, our proof will be completely self-contained, by solving the equations of 2nd-symmetry crudely\(^3\).

Specifically, the main result we will prove is:

**Theorem 1.1.** An \(n\)-dimensional proper 2nd-symmetric Lorentzian space \((M,g)\) is locally isometric to a direct product \((M_1 \times M_2, g_1 \oplus g_2)\) where \((M_2, g_2)\) is a non-flat Riemannian symmetric space and \((M_1, g_1)\) is a proper generalized Cahen-Wallach space of order 2, defined as \(M_1 = \mathbb{R}^{d+2}\)

---

\(^1\)3th-order symmetric spaces were introduced in [37] and termed as \(r\)-symmetric for short. However, a different notion of \(3\)-symmetric space, introduced by Gray [20], was already available and somehow spread in the literature (for example, see the recent article [21]). Thus, we have preferred to use the ordinal (3rd-symmetric, say) to avoid any possible confusion.

\(^2\)The results of the present paper were announced in particular in the Spanish Relativity Meetings celebrated in Bilbao, 7-11 September ’09 (only the four dimensional case, see [5]) and in Granada, 6-10 September ’10 where the general \(n\)-dimensional case was considered (see [6]).

\(^3\)While the present paper was being finished, a different local approach fully based in holonomy groups (including landmarks such as the classification of all the holonomy groups in Lorentzian signature [30]) has been developed in [1], [19]. By using it, the crucial result in [37, Theorem 4.2] is revisited, and a partial version of the Theorem 1.1 below is provided.
(d ≥ 0) endowed with the metric
\[ g_1 = -2du \left( dv + du \sum_{i,j=2}^{d+1} p_{ij}(u)x^i x^j \right) + \sum_{i=2}^{d+1} (dx^i)^2, \]
where \((u, v, x^2, \ldots, x^{d+1})\) are the natural coordinates of \(\mathbb{R}^{d+2}\) and each function \(p_{ij}\) is affine: \(p_{ij}(t) = \alpha_{ij}u + \beta_{ij}\) for some \(\alpha_{ij}, \beta_{ij} \in \mathbb{R}\) with at least one of the \(\alpha_{ij}\) non vanishing, and all \(i, j = 2, \ldots, d + 1\).

Moreover, if \((M, g)\) is also geodesically complete and simply connected, then \((M, g)\) is globally isometric to one such direct product.

Very roughly, the idea of the proof is the following. The starting point is a significant result obtained by one of the authors ([37, Theorem 4.2]): any simply-connected Lorentzian proper 2nd-symmetric space \((M, g)\) admits a parallel lightlike vector field \(K\). Lorentzian spaces with such a \(K\) were obtained by Brinkmann [8] and will be studied in Sec.4, where local bases associated to what we call Brinkmann charts \(\{u, v, x^1\}\) will be introduced. The fact that, for any such chart, the slices with constant \(u\) and \(v\) happen to be locally symmetric suggests a reduction of the equations for 2nd-symmetry. This reduction is carried out by exploiting the integrability conditions in full, and by applying some technical algebraic properties. Then, Eisenhart-type decompositions can be used to transform the original equations in \((M, g)\) into the 2nd-symmetry equations of two simpler spacetimes from which the stated parts \((M_1, g_1)\) and \((M_2, g_2)\) emerge locally. Some geometric elaborations yield the global result. In fact, along the proof it will become apparent that \((M, g)\) admits a globally defined parallel lightlike direction. Thereupon, the global requirements complete the proof easily.

Summing up, we will prove that the implicit symmetries in the equations of 2nd-symmetry turn out to be sufficient to solve them, and actually to find their general solution explicitly. Incidentally, different technical tools for some classes of partial differential equations, which may have interest in its own right, will be developed.

1.1. Outline of the paper. This article is organized as follows. In Section 2, we fix all our conventions and explain the notation. We are specially careful with the latter, for we will use a powerful combination of intrinsic expressions and tensor-component computations.

In Section 3, we review some results on locally symmetric spaces (Subsection 3.1), and compare them with the known results on 2nd-symmetric ones (Subsection 3.2). We also describe the properties of the generalized Cahen-Wallach family of Lorentzian manifolds (Subsection 3.3). They will turn out to be the non-trivial part of the proper 2nd-symmetric spaces (Proposition 3.9). Judicious interpretations alongside some technical results (Corollary 3.8, Lemma 3.10) will imply that the global counterpart of Theorem 1.1 can be obtained from the local one. Accordingly, in the following sections we will work locally, except when otherwise stated explicitly.

Section 4 is devoted to a local study of Brinkmann spaces. It has a technical nature, but it may have interest on its own right. In Subsection 4.1 we revisit the known procedure to find a Brinkmann chart \(\{u, v, x^1\}\) associated to a Brinkmann decomposition \(\{u, v\}\), so that the metric of the manifold is written as:
\[ g = -2du dv + H(u, x^k) du + W_i(u, x^k) dx^i + g_{ij}(u, x^k) dx^i dx^j, \quad k = 2, \ldots, n - 1. \]  
(see Section 2 for notation). These yield an associated spacelike \((n - 2)\)-foliation \(\mathcal{M}\) characterized by constant values of \(u\) and \(v\), and a non-necessarily orthogonal timelike \(2\)-foliation \(\mathcal{U}\) generated by \(\partial_u, \partial_v\), as well as other distributions of interest. For the sake of clarity, the relations among tensor fields on these distributions are briefly explained. In Subsection 4.2, three operators \(\nabla, \overline{\nabla}\) and \(\partial\) adapted to the foliations \(\mathcal{M}\) and \(\mathcal{U}\) are introduced and related to the connection \(\nabla\), the exterior differential \(d\) and the geometry of the foliation \(\mathcal{M}\). In Subsection 4.3, computations à la Cartan on local vector-field bases introduced in (10) yield manageable expressions of the connection one-forms and the first two covariant derivatives of the curvature tensor \(R\). To help on simplifying the resulting expressions, a fourth differential operator \(D_0\) with a transverse nature, which complements the three previous ones, is introduced. The dependence of all mentioned
objects on the Brinkmann chart and their behavior under changes of charts is emphasized and explicitly controlled. Then, a version of a classical Eisenhart theorem [17] adapted to our problem, which involves \( u \)-dependent metrics on the leaves of a foliation endowed with the intrinsic \( \nabla \) and the transverse \( D_0 \) derivatives, is provided in Theorem 4.29 of Subsection 4.4. This allows us to obtain sufficient conditions such that the \( u \)-family of Riemannian metrics \( \bar{g} = g_{ij}(u,x^k)dx^idx^j \) in (3) are simultaneously reducible.

In Section 5 we solve the equations of 2nd-symmetry on Brinkmann spaces in several steps. Firstly, we prove that the equations of 2nd-symmetry imply that the foliation \( \mathcal{M} \) must be locally symmetric for all Brinkmann charts \( \{u,v,x^i\} \) associated to a fixed decomposition \( \{u,v\} \), hence severe restrictions on the curvature must hold. Secondly, using some auxiliary algebraic results on vector spaces, those restrictions are explicitly determined in Propositions 5.6 and 5.7. Moreover, a two-covariant tensor field \( \tilde{A} \) on \( \mathcal{M} \), associated to the Brinkmann decomposition \( \{u,v\} \) only but not to the other coordinates of the Brinkmann chart, is defined in Corollary 5.2. All the information of the tensor field \( \nabla R \) is codified in \( \tilde{A} \) and the leaves of \( \mathcal{M} \). These results are summarized in Theorem 5.1 and its corollary of Subsection 5.1. Thirdly, a reorganization of the 2nd-symmetry equations into two independent blocks associated to different Brinkmann manifolds \( (M^{[m]}, g^{[m]}) \) with \( m \in \{1,2\} \) is proven in Subsection 5.2 (Theorem 5.16 and Proposition 5.18). This is achieved by applying our version of Eisenhart theorem showing that the metric is suitably reducible into a Ricci-flat part and a non-Ricci-flat one. Actually, the former is flat as a consequence of a known result [2] and the operator \( \tilde{A} \) lives only in this flat part (Proposition 5.12). Finally, the proof of the main result is complete in Subsection 5.3. The first space \( (M^{[1]}, g^{[1]}) \) is directly computable leading to the required generalized Cahen-Wallach expression of the part \( (M_1, g_1) \) in Theorem 1.1. The 2nd-symmetry equations for the second space \( (M^{[2]}, g^{[2]}) \) become equivalent to the equations for local symmetry so that \( (M^{[2]}, g^{[2]}) \) collapses to a locally symmetric Brinkmann space, therefore the Cahen-Wallach classification allows us to determine the Riemannian locally symmetric part \( (M_2, g_2) \) in Theorem 1.1.

To end this introduction, we would like to emphasize that our results open new questions and lines of interest. The first two are obvious, consisting on the study of proper \( r \)-th-symmetric spaces with \( r > 2 \), and the study of 2nd-order symmetric spaces with metrics of index greater than 1. In both cases, our approach is not directly applicable and, in fact, it is not clear how these generalizations would affect even to our starting point ([37, Theorem 4.2]). Moreover, recall that the obtained proper 2nd-symmetric spaces share the symmetries of plane waves (for explicit expressions including the more exotic Kerr-Schild symmetries, see [15]). So, an interesting question may be to find connections between the group of symmetries inherent a priori to \( r \)-th-symmetry, and the actual symmetries of the eventually obtained proper \( r \)-th-symmetric spaces. Lastly, the non-simply connected case and, in particular, the existence of compact quotients of plane waves, becomes also a natural problem in this setting (we acknowledge Professor A. Zeghib, from ENS Lyon, for discussions stressing the importance of this question).

2. Notation and conventions

\( M \) will denote a (connected) \( n \)-dimensional manifold. For simplicity, it will be implicitly assumed to be differentiable of class \( C^k \) with \( k = \infty \), but one only needs \( k = r + 3 \) for \( r \)-th order symmetric spaces, i.e., \( k = 5 \) for most of the paper. Accordingly, all objects will be assumed to be as differentiable as necessary depending on \( k \). For Lorentzian metrics, our convention on the signature is \((-\ldots,+,\ldots,+)\). Indices written in greek small letters \( \alpha, \beta, \lambda, \ldots \) will run from 0 to \( n-1 \), while those in latin small letters starting at \( i \) \((i,j,k,\ldots)\) will run from 2 to \( n-1 \) and the usual summation convention is used.

A chart on the manifold will be indicated simply with its coordinate functions \( \{x^a\} \). When working on a Brinkmann space the coordinates \( x^0 \) and \( x^1 \) will also be written as \( u \) and \( v \), respectively, according to (3). For \( f \in C^\infty(M) \) its partial derivatives are denoted by

\[
\frac{\partial f}{\partial u^i} \quad \text{and} \quad \frac{\partial f}{\partial x^i}.
\]
Most of our computations will be local, namely, in some appropriate neighborhood \( U \) (reduced if necessary) of any point \( p \in M \), but we will not specify the neighborhood — as we have already done with the charts. In general, we will use the notation as if \( U = M \) except if there were some possibility of confusion.

Let \( TM \) and \( T^*M \) be the tangent and the cotangent bundles of \( M \), respectively, and \( \pi : TM \rightarrow M \) the natural projection. A \( \ell \)-distribution of \( M \) will be regarded as a \( \ell \)-subbundle \( E \) of \( TM \) and it will be involutive if \( E = TF \) for some foliation \( F \) of \( M \). In this case, the bundle of all the \( s \)-covariant and \( r \)-contravariant tensors on \( TF \) is denoted by \( T^s_rF \). The sections of a fiber bundle \( \pi_E : E \rightarrow M \) will be written as \( \Gamma(E) \) unless the base is not evident, in which case we use \( \Gamma(M,E) \). In the case of sections of \( s \)-forms the notation will be simplified: \( \Lambda^sE \) denotes the space of all the \( s \)-form sections.

To write tensor equations in components, some local vector field basis \( \{ V_\alpha \} \) on \( TM \), or on some of its subbundles, plus its dual basis \( \{ \xi^\alpha \} \) for \( T^*M \) are used. Of course, these bases are not necessarily holonomic, i.e., associated to specific coordinates \( \{ x^\alpha \} \), for which we use the standard notation \( \{ \partial_\alpha \} \), \( \{ dx^\alpha \} \). We will follow typical notation for covariant derivatives and their components as, for example, in [36, pp. 30-35]. Notice that we denote the components of \( R \) as defined in (2) by \( R^\alpha_{\beta\lambda\mu} \), which agrees with [24, 40] but differs from [36] where the same is written as \( R_{\lambda\beta\alpha\mu} \) (therefore, \( R_{\alpha\beta\lambda\mu} \) differs on a sign).

Sometimes the abstract index notation is also used, see [44] for more information. The symmetrization (respectively anti-symmetrization) of a tensor field \( T \) is denoted by round (respectively square) brackets that enclose the indices to be symmetrized (respectively antisymmetrized). For example, \( T_{(\alpha\beta)\lambda} = (T_{\alpha\beta\lambda} + T_{\beta\alpha\lambda})/2 \) and \( T_{[\alpha\beta]\lambda} = (T_{\alpha\beta\lambda} - T_{\beta\alpha\lambda})/2 \). Furthermore, we write \( 2dx^\alpha \otimes dx^\beta = dx^\alpha \otimes dx^\beta + dx^\beta \otimes dx^\alpha \). For the wedge product, the convention is \( \beta^1 \wedge \ldots \wedge \beta^m = \sum_{\sigma \in S_m} (-1)^{\lvert \sigma \rvert} \beta^{\sigma(1)} \otimes \ldots \otimes \beta^{\sigma(m)} \), where \( \beta^i \) are one-forms, \( i = 1, \ldots, m \) and \( S_m \) denotes the set of all permutations of \( \{ 1, \ldots, m \} \). The metric isomorphism \( \flat : TM \rightarrow T^*M, \vec{v} \mapsto g(\vec{v}, \cdot) \) and its inverse \( \sharp : T^*M \rightarrow TM \), are written in components or abstract index notation so that \( X_\alpha := (X^\alpha)_\alpha = g_{\alpha\beta} X^\beta \) and \( \tau^\alpha := (\tau^\alpha)^\alpha = g^{\alpha\beta} \tau_\beta \) for all \( X \in \Gamma(TM) \), \( \tau \in \Gamma(T^*M) \).

### 3. Locally symmetric versus 2nd-symmetric semi-Riemannian manifolds

#### 3.1. Generalities on local symmetry

Let \((M, g)\) be a semi-Riemannian manifold and \( p \in M \). The local geodesic symmetry \( s_p \) with respect to \( p \) is the diffeomorphism \( s_p : N_p \rightarrow N_p \), defined on a sufficiently small normal neighborhood \( N_p \) of \( p \), which maps each \( q = \gamma(1) \in N_p \) into \( s_p(q) = \gamma(-1) \), where \( \gamma \) is the univocally determined geodesic in \( N_p \) from \( p \) to \( q \). We collect in the following proposition some characterizations of locally symmetric semi-Riemannian manifolds, i.e., manifolds satisfying \( \nabla R = 0 \), for comparison with 2nd-symmetric spaces. See [35, pp.219-223] for further details and results.

**Proposition 3.1.** For a semi-Riemannian manifold \((M, g)\) the following conditions are equivalent:

(i) \((M, g)\) is locally symmetric.

(ii) If \( L : T_pM \rightarrow T_pM \) is a linear isometry that preserves curvature (i.e., for any \( X_p, Y_p, Z_p \in T_pM \), \( L(R(X_p, Y_p)Z_p) = R(L(X_p), L(Y_p))L(Z_p) \)) there exist small normal neighborhoods \( N_p \) of \( p \) and \( N_q \) of \( q \) and a unique isometry \( \phi : N_p \rightarrow N_q \) such that \( d\phi|_p = L \).

(iii) The local geodesic symmetry \( s_p \) is an isometry at any \( p \in M \).

(iv) If \( X, Y \) and \( Z \) are parallel vector fields along a curve \( \alpha \), then the vector field \( R(X, Y)Z \) is also parallel along \( \alpha \).

(v) The sectional curvature is invariant under parallel translation: for any non-degenerate plane, its parallel transport along any curve has constant sectional curvature.

Another important known result is [25, 35]:

**Proposition 3.2.** Any semi-Riemannian symmetric space is an analytic, (geodesically) complete, homogeneous space \( G/H \). Moreover, the universal covering of a complete connected locally symmetric space is symmetric.

For the Riemannian case, we will need the following.
Proposition 3.3. Let \((M, g)\) be a locally symmetric Riemannian manifold. Then,

1. \((M, g)\) is locally isometric to the direct product of a finite number of irreducible locally symmetric spaces and a Euclidean space of dimension \(d \geq 0\).
2. If \((M, g)\) is irreducible then it is an Einstein manifold, i.e., \(\text{Ric} = cg\).
3. If \((M, g)\) is Ricci-flat (that is, Einstein with \(c = 0\)), then it is flat.

Proof. (1) This a consequence of the classical de-Rham decomposition of \(M\), as any irreducible part must be locally symmetric.

(2) As the Ricci tensor in a locally symmetric space is parallel, the result follows from a classical result by Eisenhart (Theorem 4.28 below).

(3) By hypothesis, \((M, g)\) is locally isometric to a Ricci-flat symmetric space and, by a result in [2], this space must be flat\(^4\).

The classification of Lorentzian simply-connected symmetric spaces by Cahen and Wallach [10] can be summarized as follows:

Theorem 3.4. Any simply-connected Lorentzian symmetric space \((M, g)\) is isometric to the product of a simply-connected Riemannian symmetric space and one of the following Lorentzian manifolds:

(a) \((\mathbb{R}, -dt^2)\)
(b) the universal cover of \(d\)-dimensional de Sitter or anti-de Sitter spaces, \(d \geq 2\),
(c) A Cahen-Wallach space \(\text{CW}^d(A) = (\mathbb{R}^d, g_A)\), \(d \geq 2\), where \(A = (A_{ij})\) is a \((d-2) \times (d-2)\) symmetric constant matrix, and the metric is written

\[
g_A = -2du (dv + A_{ij} x^i dx^j) + \delta_{ij} dx^i dx^j,\]

where \(\delta_{ij}\) is the Kronecker delta (observe that \(\text{CW}^2 = \mathbb{L}^2\), as \(A\) necessarily vanishes).

Therefore, if a Lorentzian symmetric space admits a parallel lightlike vector field, then it is locally isometric to the product of a \(d\)-dimensional Cahen-Wallach space and an \((n-d)\)-dimensional Riemannian symmetric space with \(d \geq 2\).

3.2. Characterization of 2nd-symmetric spaces. There is no a priori reason to think that the characterization for 2nd-symmetric Lorentzian manifolds may include properties on local geodesic symmetries, as in Proposition 3.1(iii), or any other similar semi-local property. Of course, this does not mean that such a property might not be found a posteriori.

For any geodesic \(\gamma\) such that \(p = \gamma(0)\) and tangent plane \(\Pi \subset T_p M\), consider the parallelly transported plane \(\tau \mapsto \Pi_\tau \subset T_{\tau(\gamma)} M\). Using that the scalar product between parallelly propagated vector fields is constant and that the sectional curvature \(K(\Pi_\tau)\) on non-degenerate planes \(\Pi_\tau\) determines the full curvature tensor, one easily derives the next Lemma.

Lemma 3.5. The following three conditions are equivalent:

(i) For any non-degenerate tangent plane \(\Pi\), its parallel transport \(\Pi_\gamma\) along any geodesic \(\gamma\) satisfies that \(\frac{d}{d\tau}(K(\Pi_\tau))\) remains constant along \(\gamma\).
(ii) For any parallelly propagated vector fields \(X, Y, Z\) along any geodesic \(\gamma\), the vector field \(\nabla_{\gamma(R)}(X, Y)Z\) is itself parallelly propagated along \(\gamma\).
(iii) \(\nabla_X(\nabla_Y R) - \nabla_{\nabla_Y X} R\) is skew-symmetric in \(X, Y\).

Moreover, if these conditions hold, then the following property follows:

(S) if \(\Pi\) is a lightlike plane with radical spanned by \(\vec{v} \in T_p M\), and \(V\) and \(\Pi_\gamma\) are the parallel transports of \(\vec{v}\) and \(\Pi\) along any geodesic \(\gamma\), respectively, then \(\frac{d}{d\tau}(K_V(\Pi_\tau))\) remains constant along \(\gamma\), where \(K_V(\Pi_\tau)\) denotes the null sectional curvature along \(\gamma\) and is defined

\(^4\)In fact, Alekseevskii and Kimelfeld [2] proved that any Ricci-flat homogeneous Riemannian space is flat. It is worth pointing out that, as a difference with the locally symmetric case, the locally homogeneous spaces may be non-regular, that is, non-locally isometric to some homogeneous space [29]. However, Spiro [39] showed that all locally homogeneous spaces with non-positive Ricci curvature are regular, hence the result in [2] can be extended to the locally homogeneous case. Nevertheless, it cannot be extended to the Lorentzian case: it is easy to find a counterexample in the Cahen-Wallach spaces below.
Proposition 3.6. The following statements are equivalent for a Lorentzian manifold \((M, g)\):

(i) \(\nabla \nabla R = 0\), i.e., \((M, g)\) is 2nd-symmetric.

(ii) If \(V, X, Y, Z\) are parallelly propagated vector fields along any curve \(\alpha\), then \((\nabla_V R)(X, Y)Z\) is itself parallelly propagated along the curve.

(iii) \((M, g)\) is semi-symmetric (i.e., the curvature tensor fulfills (2)) and satisfies the equivalent conditions in Lemma 3.5 for any geodesic \(\gamma\).

Proof. As (ii) characterizes when the tensor \(\nabla R\) is parallel, this condition is equivalent to (i). Also, both conditions imply (iii) trivially. For the converse, notice that, by semi-symmetry, \(\nabla \nabla R\) is symmetric in \(U_1, U_2\), and, by (ii), it vanishes by applying the condition (iii) of Lemma 3.5. \(\square\)

Remark 3.7. For any Lorentzian manifold that satisfies the equivalent conditions in Lemma 3.5 the (null) sectional curvature of a plane parallelly propagated along a geodesic varies as an affine function on the affine parameter of the geodesics; in other words, the curvature along geodesics grows linearly. Recall that the constancy of the sectional curvature characterized locally symmetric spaces (Proposition 3.1). On the other hand, there are obvious situations in which its non-constant linear growth can be excluded. For example, assume that \(\gamma : [0, \infty) \to M\) is a complete lightlike half-geodesic such that its velocity is imprisoned in a compact subset \(C \subseteq TM\). Then, for any lightlike plane II with radical spanned by \(\tilde{\eta} = \gamma'(0)\), the derivative of \(K_V(\Pi_s)\) must vanish. (There exists a sequence \(\tau_n \not\to \infty\) such that \(\{\gamma(\tau_n)\} \to q \in M\), \(\{\gamma'(\tau_n)\} \to \tilde{x}_q \in T_q M\), \(\{\Pi_s(\tau_n)\} \to \Pi_q \subseteq T_q M\), hence \(K_V(\Pi_s(\tau_n)) \to K_{\tilde{x}_q}(\Pi_q)\) and this is incompatible with a linear growth of the curvature.)

This sort of properties, together with Proposition 3.6, suggest global obstructions to the existence of compact proper 2nd-symmetric spaces.

Corollary 3.8. Let \((M, g)\) be a proper (connected) 2nd-symmetric space. Then, \(\nabla R \neq 0\) holds everywhere, and \((M, g)\) admits a unique parallel lightlike direction.

Proof. Obviously, if \((\nabla R)_p \neq 0\) at some \(p \in M\), the (parallel tensor) \(\nabla R\) cannot vanish at any point. To prove the existence of the parallel lightlike direction recall that, as we have already mentioned, around each point \(p\) there exists a parallel lightlike vector \(K\) [37, Theorem 4.2]. Moreover, there cannot exist a second such \(K'\) that is independent of \(K\) at \(p\), as otherwise a parallel timelike vector field \(T\) could be constructed as a linear combination of \(K\) and \(K'\); hence, the metric would split around \(p\) as a product \(-dt^2 \oplus g_R\), with \(T = \nabla t\) and \(g_R\) a Riemannian metric, which should be locally symmetric, in contradiction with \((\nabla R)_p \neq 0\). Thus, the corresponding parallel lightlike directions locally generated by \(K\) and \(K'\) must agree, so that they match in a single global one. \(\square\)

3.3. Generalized Cahen-Wallach spaces of order \(r\). In this section we introduce the archetypes for \(r\)th-symmetric Lorentzian manifolds, which are generalizations of the \(d\)-dimensional Cahen-Wallach spaces \(CW^d(A)\) introduced in Theorem 3.4.

Consider the larger family \(CW^d_r(A)\) of all the \(d\)-dimensional generalized Cahen-Wallach spaces of order \(r\) \(CW^d_r(A) = (\mathbb{R}^d, g_A)\), defined by using the same expression (5) but letting now \(A\) to depend on \(u\) as a matrix of polynomials of degree less or equal than \(r - 1\)

\[
A \equiv A(u) = A^{(r-1)}u^{r-1} + \ldots + A^{(1)}u + A^{(0)}
\]

where each \(A^{(l)}\) is a constant symmetric \((d-2) \times (d-2)\) matrix for all \(l \in \{0, \ldots, r-1\}\). As before, a generalized Cahen-Wallach space is proper if at least one of the polynomials has degree \(r - 1\), that is, if \(A^{(r-1)} \neq 0\). Notice that all the spaces \(CW^d_r(A)\) contain a parallel lightlike vector field (see Section 4.1), and that \(CW^d_r(A) = CW^d(A)\). In addition
Proposition 3.9. Any proper generalized Cahen-Wallach space is analytic, (geodesically) complete and proper rth-symmetric.

Proof. The analyticity is obvious and the completeness follows from [11, Proposition 3.5], where a more general type of plane waves was treated. The last property was already mentioned in [37] and follows by computing the derivatives of the curvature tensor in the basis \( \{E_a\} = \{\partial_u - H\partial_v, \partial_v, \partial_\epsilon - W_i \partial_\epsilon\} \) (see Section 4.3). The only non-vanishing components of \( \nabla^l R \) for \( l \in \{0, \ldots, r - 1\} \) (for \( l \geq r \), \( \nabla^l R = 0 \)) are

\[
\nabla_0 \ldots \nabla_0 R^1_{i0j} = \frac{d^l A_{ij}}{du^l} = \sum_{k=0}^{l-1} \frac{k!}{(k-l)!} A^{(k)}_{ij} u^{k-l}
\]

which leads to the result immediately. □

The following lemma will be used to reduce the global version of Theorem 1.1 to the local one.

Lemma 3.10. Let \((M, g)\) be a complete simply-connected Lorentzian manifold which is locally isometric to the product of some generalized Cahen-Wallach space with a Riemannian symmetric space. Then, \((M, g)\) is in fact globally isometric to one of such products.

Proof. By assumption, \((M, g)\) is locally isometric to an analytical manifold due to Propositions 3.2 and 3.9 and, thus, it is analytical too. The result follows from the fact that, for any two complete simply-connected analytic semi-Riemannian manifolds \((M, g)\) and \((M', g')\), every isometry defined between connected open subsets of \(M\) and \(M'\) can be uniquely extended to an isometry for the entire \(M\) and \(M'\) (see, for example [28, Cor. 6.4 Ch. VI] for the Riemannian case, and [28, Th. 6.1 Ch. VI], [35, Cor. 7.29] for its generalization to the semi-Riemannian one). □

4. BRINKMANN SPACES

A Lorentzian manifold is called a Brinkmann space if and only if it admits a parallel lightlike vector field. Brinkmann spaces have deserved increasing attention in recent years, see e.g. [3],[4],[18]. In this section, we derive some of their properties relevant to our problem.

4.1. Basics on Brinkmann spaces. In what follows, the lightlike parallel vector field \(K\) of a Brinkmann space \((M, g)\) will always be assumed to be fixed. Then, it is well-known that any point \(p \in M\) admits a coordinate chart \(\{x^u\} = \{u, v, x^i\}\), that we call a Brinkmann chart, in such a way that the metric takes the form (3) and \(K = -\partial_u\) [8], see also [45]. Without loss of generality, we will assume that the range of the coordinates includes \(|u|, |v|, |x^i| < \epsilon\) for some \(\epsilon > 0\).

Brinkmann charts can be obtained by means of the following process (see Figure 1). As \(K\) is parallel, choose a function \(u\) such that \(K = \nabla u\) and with the value \(0\) in the image of \(u\). Each level set \(\Sigma_u = u^{-1}(u)\) is a lightlike integral manifold of the distribution \(K^\perp\) orthogonal to \(K\). Set \(\Sigma = \Sigma_0\) and choose a hypersurface \(\Omega\) which is transverse to both \(\Sigma\) and \(K\). The function \(u\) will serve as a coordinate for \(M\) as well as for \(\Omega\). In \(\Omega\) we choose a coordinate neighborhood \(\{u, x^i_0\}\) completing \(u\). The coordinate \(v\) on \(M\) is defined by using the flow \(\phi\) of \(K\) to move each point \(p\) to the point \(\phi_v(p) \in \Sigma_u \cap \Omega\), and then we put \(x^i = x^i_0(\phi_v(p))\). Observe that we have chosen to move \(p\) by using the flow of \(K = -\partial_u\) so that \(1 = du(\partial_u) = g(\nabla u, \partial_u) = g(K, \partial_u) = -g(\partial_v, \partial_u)\).

Conversely, the expression (3) selects \(K = -\partial_u\), and \(\Sigma\) and \(\Omega\) are the hypersurfaces \(u = 0\) and \(v = 0\) respectively. Locally, a pair \((\Sigma, \Omega)\) determines a Brinkmann decomposition, i.e., a pair of functions \(\{u, v\}\) constructed as above, which may serve as the two first coordinates for different Brinkmann charts.

If \(\{u', v', x'^i\}\) denotes a second Brinkmann chart which overlaps \(\{u, v, x^i\}\), the corresponding \(\Sigma'\) will also be a level set of \(u\), and \(\Omega'\) can be regarded as a graph on \(\Omega\). So, the change of coordinates can be written as

\[
\begin{align*}
u' &= u - u_0, \quad v' = v + F(u, x^j); \quad x'^i = x'^i(u, x^j) \quad (7)
\end{align*}
\]
LORENTZIAN MANIFOLDS WITH $\nabla^2 R = 0$

Consequently, the relations between $H, W_i, g_{ij}$ in the original coordinates $\{u, v, x^i\}$ and $H', W_i', g'_{ij}$ in the new ones $\{u', v', x'^i\}$ are:

$$H = H' + F + W_i' \dot{x}^i - \frac{1}{2} g'_{ij} \dot{x}^i \dot{x}^j$$

$$W_i = W_j \frac{\partial x^j}{\partial x^i} + F_i - \frac{1}{2} g'_{ik} \left( \frac{\partial x^j}{\partial x^i} \dot{x}^k + \frac{\partial x^k}{\partial x^i} \dot{x}^j \right)$$

$$g_{ij} = g'_{kl} \frac{\partial x^k}{\partial x^i} \frac{\partial x^l}{\partial x^j}$$

The freedom in the choice of $\Omega$ makes it possible to obtain a Brinkmann decomposition $\{u, v\}$ with a Brinkmann chart such that $H \equiv 0 \equiv W_i$, in particular $\partial_u$ is lightlike and geodesic in the associated chart, see e.g. [38]. For the sake of completeness, let us construct such a coordinate chart. Choose some $\Sigma$ as above and take any $(n-2)$-submanifold $\Sigma \hookrightarrow \Sigma$ which is transverse to $K$. All such $\Sigma$ are locally isometric, as $K$ is a (parallel) lightlike direction in $\Sigma$. Now, consider for each $x \in \Sigma$ the unique lightlike direction $\vec{L}_x$ orthogonal to $\Sigma$ and linearly independent of $K_x$. In a small neighborhood, construct $\Omega$ by taking the geodesics with initial velocity $\vec{L}_x$ for all $x \in \Sigma$. Complete the chart by choosing some local coordinates $\{x^i_\Omega\}$ in $\Sigma$ and defining the coordinates $x^i_\Omega$ at each $y \in \Omega$ by $x^i_\Omega(y) := x^i_M(x_y)$, where $x_y$ is the unique point in $\Sigma$ which lies on the same lightlike geodesic as $y$. Notice that $\Omega$ is a lightlike hypersurface, and the corresponding coordinate vector field $\partial_u$ spans its radical, i.e., $H = 0 = W_i$, as required. We emphasize that the value of $H$ and $W_i$ depend on the choice of the coordinates $\{x^i\}$. Moreover, given a Brinkmann decomposition $\{u, v\}$, there exists a Brinkmann chart $\{u, v, x^i\}$ with $H = 0 = W_i$ if and only if the hypersurface $\Omega$ obtained as $v = 0$ is lightlike (when $\Omega$ is lightlike, the integral curves of its radical must be lightlike geodesics of $M$, because of the local maximizing properties of these curves). In spite of its simplicity, such a Brinkmann decomposition will not be especially relevant for our study. It might simplify some intermediate computations, but they are not well adapted to the generalized Cahen-Wallach spaces of order 2 which will turn out to be, as already announced, the essential part of proper 2nd-symmetric Lorentzian manifolds.

Given a Brinkmann chart, each integral curve of $\partial_v$ is labelled by $\{u = u_0, x^i = x^i_0\}$, and each hypersurface $\Omega_{v_0} \equiv \{v = v_0\}$ is a general pseudo-Riemannian hypersurface (possibly signature-changing, as studied systematically in [31]) and isometric to $\Omega$. We will repeatedly use the two natural transverse foliations associated to each Brinkmann decomposition, namely:
(1) The $(n-2)$-dimensional foliation $\overline{M}$ with leaves $\overline{M} = \Sigma \cap \Omega$ and the submanifolds obtained by moving $\overline{M}$ with the flows of $\partial_\alpha$ and $\partial_\nu$. Each leaf is defined by $\{u = u_0, v = v_0\}$ and represented by $\overline{M}_{(u_0,v_0)}$. The induced metric will be denoted by $\overline{g} (\overline{g} : T\overline{M} \times T\overline{M} \to \mathbb{R})$, so that $\overline{g}_{ij} = g_{ij}$ in any Brinkmann chart. When necessary, $\overline{g}$ will be regarded as a metric on a single leaf. This foliation depends only on the Brinkmann decomposition $(\Sigma, \Omega)$, or equivalently on the chosen functions $\{u, v\}$ only.

(2) The 2-dimensional foliation $\mathcal{U}$ whose leaves are the surfaces obtained by moving each single point with the flows of $\partial_\alpha$ and $\partial_\nu$. Each leaf is given by $\{x^i = c^i_0\}$ for some constants $c^i_0$ and the induced metric is $-2dudv + H\,dv$. This foliation depends on the expression of $\partial_\alpha$ and, thereby, on the coordinates chosen on $\Omega$ for the Brinkmann chart.

Any Brinkmann chart allows for the decomposition of the tensor bundles in different ways, some of them to be detailed here. By a partly null frame $\{E_\alpha\}$ for a Lorentzian manifold we mean a local basis of vector fields such that

$$g(E_0, E_0) = 0; \quad g(E_1, E_1) = 0; \quad g(E_0, E_1) = -1;$$

for some functions $a_{ij} = a_{ji}$, which must obviously define a positive definite metric. Its dual basis will be denoted by $\{\theta^\alpha\}$. Now, fix a Brinkmann chart and consider the associated foliations $\overline{M}$, $\mathcal{U}$ and distributions $T\overline{M}$, $T\mathcal{U}$ and $(T\mathcal{U})^\perp$. The (canonical) partly null frame of the Brinkmann chart is given by

$$\{E_\alpha\} = \{\partial_\alpha - H\partial_\nu, \partial_\nu, -W_i \partial_\nu + \partial_i\}, \quad \{\theta^\alpha\} = \{du, dv + H\,du + W_j\,dx^j, dx^i\}, \quad i = 2, \ldots, n-1 \quad (10)$$

Notice that $(T\mathcal{U})^\perp$ is spanned by the vector fields $\{E_\alpha\}$, and the brackets $[E_\alpha, E_\beta] = (\partial_\alpha W_\beta - \partial_\beta W_\alpha) \partial_\nu \in \Gamma(T\mathcal{U})$ measure its lack of involutivity.

The decomposition $TM = T\mathcal{U} \oplus T\overline{M}$ associated to any Brinkmann chart yields, on the one hand, the projection

$$P_{\overline{M}} : \Gamma(TM) \to \Gamma(T\overline{M}), \quad X \mapsto \overline{X}$$

so that $P_{\overline{M}}(\partial_\nu) = P_{\overline{M}}(\partial_\nu) = 0$ and $P_{\overline{M}}(E_\alpha) = \partial_\alpha$, and on the other hand the natural inclusion

$$\mathcal{I}_{\overline{M}} : \Gamma(T\overline{M}) \to \Gamma(TM)$$

with the induced dual map by restriction

$$\mathcal{I}_{\overline{M}}^* : \Gamma(T^*M) \to \Gamma(T^*\overline{M}), \quad \beta \mapsto \overline{\beta}.$$

Clearly, $\{E_\alpha\} = \{\partial_\alpha\}$ is a basis in $T\overline{M}$ whose cobasis is $\{\theta^\alpha = dx^\alpha\}$. If $X = X^\alpha E_\alpha \in \Gamma(TM)$ and $\beta = \beta^\alpha\theta^\alpha \in \Gamma(T^*M)$, then $X^\overline{\alpha} = X^\alpha$ and $\overline{\beta}_\downarrow = \beta_\downarrow$ in the given bases.

We also introduce two linear homomorphisms between spaces of sections defined as:

1. $\cdot : \Gamma(T^s_\alpha \overline{M}) \to \Gamma(T^s \overline{M})$, which maps each $T = T^{\beta_1 \ldots \beta_s}_j E_\alpha \otimes \ldots \otimes E_{\alpha'} \otimes \theta^{\beta_1} \otimes \ldots \otimes \theta^{\beta_s}$ into $\overline{T} = T^{j_1 \ldots j_s}_i \overline{E}_{i_1} \otimes \ldots \otimes \overline{E}_{i_s} \otimes \overline{\theta}^{j_1} \otimes \ldots \otimes \overline{\theta}^{j_s}$,

2. $\cdot : \Gamma(T^s \overline{M}) \to \Gamma(T^s_\alpha \overline{M})$, which maps each $T = T^{i_1 \ldots i_s}_j \overline{E}_{i_1} \otimes \ldots \otimes \overline{E}_{i_s} \otimes \theta^{j_1} \otimes \ldots \otimes \theta^{j_s}$ into $\overline{T} = T^{j_1 \ldots j_s}_i E_{i_1} \otimes \ldots \otimes E_{i_s} \otimes \theta^{i_1} \otimes \ldots \otimes \theta^{i_s}$.

In general, $\overline{T} = T$ but $\overline{T} \neq T$, because in the first case $T \in \Gamma(T^s_\alpha \overline{M})$ necessarily, but not in the second.

**Remark 4.1.** The previous maps are $C^\infty(M)$ linear homomorphisms between spaces of sections, they commute with contractions, tensor products and, when applicable, wedge products in a natural way; moreover, they leave invariant the $C^\infty(M)$ functions considered as $(0,0)$-tensor fields: $\overline{f} = f = f$ for all $f \in C^\infty(M)$. Both homomorphisms $\cdot$ and $\cdot$ have trivial expressions in the introduced bases. Essentially, all components of the tensor fields and their images remain equal except for the fact that, in the case of $T \to \overline{T}$, the components aligned with any of $E_0$, $E_1$, $\theta^0$ or $\partial^i$ must be dropped while, in the case of $T \to \overline{T}$, these components must be restored with vanishing value. This simplifies our subsequent work in components substantially, as we will not need to distinguish notationally among different tensor fields derived from a single one: it will be enough to realize which space of sections is being considered.
As important illustrative examples, observe that the functions \( W_i \) in the Brinkmann expression (3) can be regarded as the components in the basis \( \{ E_i \} = \{ \partial_i \} \) of a one-form section in \( \mathcal{M} \), namely: \( W_i = W_i^j \in \Gamma(T^* \mathcal{M}) \), as well as the components in the basis \( \{ E_a \} \) of the one-form section \( \bar{W} \) in \( M \). Analogously, \( g_{ij} \) can be regarded as the components of the projection \( P_{\mathcal{M}}(g) = \bar{g} \in \Gamma(T^*_2 \mathcal{M}) \), which is the inherited metric on \( \mathcal{M} \). Therefore, the metric \( g \) can be rewritten as
\[
 g = -2du(dv + Hdu + W) + \bar{g}.
\]
One should also keep in mind that, when a single leaf \( \mathcal{M}_{(u,v)} \) is considered, \( g_{ij} \) will also denote the components of the induced metric \( \bar{g} \) on the leaf, with no explicit mention to \( (u,v) \) or the underlying Brinkmann decomposition.

4.2. Three differential operators adapted to \( \mathcal{M} \) and \( \nu \)-invariant sections. By using the previously obtained vector bundle decomposition associated to each Brinkmann chart, three differential operators \( \partial, \nabla \) and \( \tilde{\partial} \) ("dot") are introduced next. The operator \( \nabla \) depends on the Brinkmann decomposition \( \{ u, v \} \) only, while the other two depend on the whole Brinkmann chart \( \{ u, v, x^i \} \).

4.2.1. \( \nu \)-invariant tensor fields. The variation on \( \mathcal{M} \) of \( T \in \Gamma(T^*_x \mathcal{M}) \) under displacements along the direction \( E_1 = \partial_v \) will be rather irrelevant for us. In fact, if \( E_1 = -K \) is parallel and, therefore, all the interesting objects to be used here will also be invariant by its flow. More precisely:

**Definition 4.2.** A tensor field \( T \in \Gamma(T^*_x \mathcal{M}) \) on \( \mathcal{M} \) is \( \nu \)-invariant if \( \tilde{T} \) is Lie-parallel along the flow of \( E_1 \), that is, \( \mathcal{L}_{E_1} \tilde{T} = 0 \).

**Remark 4.3.** (1) As \( E_1 = \partial_v \), a tensor field \( T = T_{ij...s}^i...j...s \partial_i \otimes ... \partial_s \otimes \partial_x^j \otimes ... \partial_x^s \in \Gamma(T^*_x \mathcal{M}) \) is \( \nu \)-invariant if and only if \( \partial_v (T_{ij...s}^i...j...s) = 0 \). For example, \( \bar{g}, H \) and \( W \) are \( \nu \)-invariant.

(2) Obviously, if \( T \) is \( \nu \)-invariant then it is determined by its value on any of the hypersurfaces \( \Omega_{x_0} = \{ v = v_0 \} \), and any section \( T_{\Omega_{x_0}} \in \Gamma(\Omega_{x_0}, T^*_x \mathcal{M}) \), defined only on \( \Omega_{x_0} \), can be extended to a unique \( \nu \)-invariant section \( \mathcal{T}_{\mathcal{M}} \in \Gamma(\mathcal{T}^* \mathcal{M}) \).

(3) As it will become obvious later, the derivatives of a \( \nu \)-invariant section with respect to any of the three operators \( \partial, \nabla \) and \( \dot{\partial} \) to be defined below are also \( \nu \)-invariant. Hence, these operators can be naturally defined for sections in \( \Gamma(\Omega_{x_0}, T^*_x \mathcal{M}) \) (for instance, \( \nabla T_{\Omega_{x_0}} \) is the restriction of \( \nabla T_{\mathcal{M}} \) to \( \Omega_{x_0} \)).

There is an alternative definition of \( \nu \)-invariance.

**Proposition 4.4.** \( T \in \Gamma(T^*_x \mathcal{M}) \) is \( \nu \)-invariant if and only if \( \tilde{T} \) is parallel in the direction \( E_1 \).

**Proof.** As \( E_1 \) is parallel, \( \nabla_E_1 Q = \mathcal{L}_{E_1} Q \) for any \( Q \in \Gamma(T^*_x M) \). □

4.2.2. The \( \mathcal{M} \) exterior derivative \( \partial \).

**Definition 4.5.** The \( \mathcal{M} \) exterior derivative \( \partial : \Lambda^s \mathcal{M} \to \Lambda^{s+1} \mathcal{M} \) associated to a Brinkmann chart \( \{ u, v, x^i \} \) is defined by:
\[
 \partial \beta = \partial \beta \quad \forall \beta \in \Lambda^s \mathcal{M},
\]
where \( \partial \) is the usual exterior derivative on the manifold \( M \).

It is straightforward to check that, if \( \beta \in \Lambda^s M \), then \( \partial \partial \beta = \partial \partial \beta \). In particular, \( \partial f = \partial f \) for \( f \in C^\infty(M) \) and thus \( \partial f = \partial dx^i = \partial x^i \). This allows us to use the expressions \( \partial \partial \beta \) instead of \( \partial \beta \) and \( \partial \beta \), respectively. Moreover, using the notation introduced in (4),
\[
 df = \partial f du + \partial v dv + \partial \beta .
\]

**Proposition 4.6.** Let \( \beta \in \Lambda^s \mathcal{M} \). The differential \( \partial \beta \) satisfies the following properties:

1. It is linear and \( \partial (\beta \wedge \tau) = \partial \beta \wedge \tau + (-1)^s \beta \wedge \partial \tau \) for all \( \tau \in \Lambda^s \mathcal{M} \).
2. \( \partial (\partial \beta) = 0 \).
3. If \( \beta = \frac{1}{s!} \beta_{i_1...i_s} dx^{i_1} \wedge ... \wedge dx^{i_s} \), then \( \partial \beta = \frac{1}{s!} \partial_k (\beta_{i_1...i_s}) dx^k \wedge dx^{i_1} \wedge ... \wedge dx^{i_s} \).
4.2.3. The covariant derivative $\nabla$ and its curvature tensor $\mathcal{R}$ on $\mathcal{M}$.

**Definition 4.7.** The covariant derivative $\nabla$ on $\mathcal{M}$ associated to a Brinkmann decomposition $\{u,v\}$ is the map

$$\nabla : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M})$$

defined at each point $p \in \mathcal{M}$ by

$$(\nabla_X Y)_p := (\nabla^\tilde{g}_{\tilde{X}}Y)_p$$

where $\tilde{X}$ and $\tilde{Y}$ are the restrictions of $X,Y$ to the leaf $\mathcal{M}(u(p),v(p))$ and $\nabla^\tilde{g}$ the Levi-Civita connection of the first fundamental form $\tilde{g}$ on that leaf.

**Remark 4.8.**

a) Clearly, $\nabla$ satisfies the formal properties of a symmetric covariant derivative, as well as $\nabla^\tilde{g} = 0$.

b) The Christoffel symbols $\Gamma^i_{jk}$ of $\nabla$ in the basis $\{\partial_i\}$ are defined by the relation $\nabla_{\partial_j}\partial_i = \Gamma^k_{ij}\partial_k$.

Trivially, they are smooth and invariant by the flow of $\partial_i$.

c) The covariant derivative on $\mathcal{M}$ can be extended to sections in $T\mathcal{M}$. For the coordinate basis $\{\partial_i\}$ and any $T \in \Gamma(T\mathcal{M})$ one has:

$$\nabla_m T^{i_1 \ldots i_s}_{j_1 j_2 \ldots j_s} = \partial_m (T^{i_1 \ldots i_s}_{j_1 j_2 \ldots j_s}) + \sum_{a=1}^r \Gamma^a_{km} T^{i_1 \ldots i_{a-1} k i_{a+1} \ldots i_r}_{j_1 j_2 \ldots j_s} - \sum_{b=1}^s \Gamma^b_{km} T^{i_1 \ldots i_r}_{j_1 j_2 \ldots j_{b-1} k j_{b+1} \ldots j_s}$$

d) For $X,Y \in \Gamma(TM)$, $\nabla_X Y \neq \nabla_Y X$ in general (for example, if $X = E_0$ then $\tilde{X} = 0$).

The covariant derivative $\nabla$ yields a natural curvature tensor $\mathcal{R}$ of the foliation $\mathcal{M}$ defined formally as the usual curvature of $\nabla$:

$$\mathcal{R}(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} )Z \in \Gamma(T\mathcal{M}), \forall X,Y,Z \in \Gamma(T\mathcal{M})$$  \quad (11)$$

and also its derived Ricci tensor $\mathcal{R}^{\text{c}}$ and scalar curvature $\mathcal{S}$ of $\mathcal{M}$. All of them satisfy the standard symmetries corresponding to a curvature tensor.

**Definition 4.9.** Fixed a Brinkmann decomposition $\{u,v\}$. The foliation $\mathcal{M}$ is called $u$-Einstein if $\mathcal{R}^{\text{c}} = \mu \tilde{g}$ for some function $\mu$ such that $d\mu \wedge du = 0$. In particular, when $\mu$ is constant, $\mathcal{M}$ is called Einstein, and when $\mu \equiv 0$ we say that $\mathcal{M}$ is Ricci-flat.

In the case $\mathcal{R} = 0$ (respectively $\nabla \mathcal{R} = 0$), the foliation $\mathcal{M}$ is said to be flat (respectively locally symmetric).

Some simple properties follow immediately.

**Proposition 4.10.** Let $(M,g)$ be a Brinkmann space with a fixed Brinkmann decomposition $\{u,v\}$.

1) If $\nabla \mathcal{R} = 0$ for some $r > 1$, then $\mathcal{M}$ is a locally symmetric foliation.

2) If $\mathcal{M}$ is locally symmetric and Ricci-flat, then it is flat.
4.2.4. The Brinkmann decomposition admits a chart \( \{ u, v, y^i \} \) such that the metric \( g \) becomes:

\[
g = -2du(dv + H du + W_i dy^i) + \delta_{ij} dy^i dy^j.
\]

Proof. (1) For each \( (u, v) \), the Riemannian result which states that \( \nabla^\gamma R = 0 \) implies local symmetry can be applied to each leaf \( \mathcal{M}_{(u,v)} \).

(2) Apply Proposition 3.3(3) to each leaf of \( \mathcal{M} \).

(3) We sketch a procedure to obtain the required new coordinates \( y^i = y^i(u, x^j) \), for the sake of completeness. Let \( \gamma \) be an integral curve of \( \partial_u \) along the hypersurface \( \Omega = \{ v = 0 \} \), and put for small \( U \subset \mathbb{R}^{n-2} \), \( 0 \in U :\)

\[
\varphi : [-\epsilon, \epsilon] \times U \to M, \quad (u, w^i) \mapsto \exp_{\gamma(u)}(w^i \partial_i |_{\gamma(u)}),
\]

where \( \exp_{\gamma(u)} \) denotes the exponential at \( \gamma(u) \) on the leaf \( \mathcal{M}_{(\gamma(u),0)} \) for the (flat) metric \( g \). The classical Cartan theorem shows that each map \( \varphi_u := \varphi(u, \cdot) \) is an affine transformation from \( U \) to the leaf. So, regarding \( \{ w^i \} \) as coordinates in \( \mathbb{R}^{n-2} \),

\[
\varphi_u^* g = \omega_{ij}(u) du^i du^j
\]

for some constants \( \omega_{ij}(u) \) on each leaf (which define an Euclidean metric and depend smoothly on \( u \)). Now, \( \varphi \) allows to consider \( \{ u, w^i \} \) as coordinates in \( \Omega \), with each coordinate vector field \( \partial/\partial u \) parallel on each leaf of \( \mathcal{M}_{(u,0)} \). By using the Gram-Schmidt procedure, an orthonormal basis \( \{ V_j = B^j_i \frac{\partial}{\partial w^i} \} \) in \( T\mathcal{M} \) is obtained. Indeed, by construction the transition matrix \( \{ B^j_i \} \) depends smoothly only on \( u \) and, thus, by Remark 4.3 (3) \( \nabla V_j = \partial(B^j_i) \frac{\partial}{\partial w^i} = 0 \). Therefore, Proposition 4.6 (4) implies that \( (V^j_i)^{\flat} = dy^j_i \) for some functions \( \{ y^j_i(u, w^i) \} \) on some open subset of \( \Omega \). The required functions \( \{ y^j_i \} \) are obtained by extending \( \{ y^j_i(u, w^i) \} \) to a neighborhood of \( M \) in a \( v \)-invariant way according to Remark 4.3 (2). □

4.2.4. The ‘(dot) derivative’. We introduce the following simple derivative:

**Definition 4.11.** The dot derivative \( \dot{T} \in \Gamma(T^*_x \mathcal{M}) \) of a tensor field \( T \in \Gamma(T^*_x \mathcal{M}) \) is defined as \( \dot{T} = (\partial_\gamma T) \).

The components of \( \dot{T} \) in the coordinate basis \( \{ \partial_i \} \) are \( \dot{T}^{i_1 \ldots i_s} = \partial_{u} (T^{i_1 \ldots i_s}_{j_1 \ldots j_s}) \). Indeed, the first usage of the dot for functions was the definition (4).

4.3. Some geometrical objects and the operator \( D_0 \). Our aim in this section is to obtain explicit expressions for the curvature and its derivatives adapted to a Brinkmann chart. We start by computing the curvature two-forms for a partly null frame. Then, we introduce a new operator \( D_0 \) which, together with those introduced in the last section, will allow us to simplify calculations and to provide manageable formulae for the derivatives of the curvature. This will be very helpful in order to solve the equations of 2nd-symmetry.

4.3.1. The connection 1-forms and the curvature tensor. We compute the connection and curvature forms by using Cartan techniques, according to the conventions in [32][14]. In a standard manner, we define the connection one-forms \( \omega^i_j \in \Lambda^1(\mathcal{M}) \) and the corresponding curvature two-forms \( \Omega^i_j \in \Lambda^2(\mathcal{M}) \) of the foliation \( \mathcal{M} \) for the given coordinate basis \( \{ \partial_i \} \) as:

\[
\nabla_X \partial_i = \omega^i_j(X) \partial_j, \quad \nabla(Y, X) \partial_i = \nabla^j_i(X, Y) \partial_j, \quad \forall X, Y \in \Gamma(T\mathcal{M}).
\]

Observe that \( \omega^i_j = \Gamma^i_{jk} dx^k \), where \( \Gamma^i_{jk} \) are the Christoffel symbols of \( \nabla \) as introduced in Remark 4.8. Then, a simple computation shows that the first and second Cartan’s equations for \( \mathcal{M} \) still hold:

\[
0 = \omega^i_j \wedge dx^j, \quad \Omega^i_j = d \omega^i_j + \omega^i_k \wedge \omega^k_j.
\]
In order to compute the connection and curvature of the Brinkmann spaces we introduce two tensor fields on \( \mathcal{M} \) in terms of the adapted differential \( \mathcal{d} \) and the dot derivative, which will be especially relevant in the computations. We define \( h \in \Lambda^1(\mathcal{M}) \) and \( t \in T^0_2(\mathcal{M}) \) as
\[
\begin{align*}
    h &= \mathcal{d} H - \dot{W}, \\
    t &= -\frac{1}{2} (\mathcal{g} + \mathcal{d} W).
\end{align*}
\]
Observe that the symmetric and skew-symmetric parts of \( t \) are precisely \( -\mathcal{g}/2 \) and \(-\mathcal{d} W/2\), respectively. We emphasize that \( h \) and \( t \) depend on the Brinkmann chart. Equipped with these sections, one can check that the non-vanishing connection one-forms associated to the a partly null frame \( \{E_\alpha\} \) of any Brinkmann chart are
\[
\begin{align*}
    \omega^i_1 &= \omega^i_0 g_{ij} = h_i \theta^0 - t_{ij} \theta^j, \\
    \omega^i_j &= -g^{kj} t_{kj} \theta^0 + \tilde{\omega}^i_j.
\end{align*}
\]
where \( h_i = (H_i - \dot{W}_i) \) and \( t_{ij} = \frac{1}{2} (-\dot{g}_{ij} + W_{i,j} - W_{j,i}) \) are the components of the tensors defined in (12) and (13). It should be noted that \( \omega^i_j = \tilde{\omega}^i_j \), but \( \tilde{\omega}^i_j \neq \omega^i_j \). Then, the non-vanishing curvature two-forms associated to the partly null frame \( \{E_\alpha\} \) of any Brinkmann chart read
\[
\begin{align*}
    \Omega^i_1 &= - (\nabla_j h_i + i_{ij} + t^k i_{kj}) \theta^0 \wedge \theta^j + \frac{1}{2} (\nabla_k t_{ij} - \nabla_j t_{ik}) \theta^j \wedge \theta^k, \\
    \Omega^i_j &= (\nabla_k t^i_j + \tilde{\Gamma}^i_{jk}) \theta^0 \wedge \theta^k + \tilde{\Omega}^i_j,
\end{align*}
\]
so that again \( \tilde{\Omega}^i_j = \tilde{\Omega}^i_j \), but \( \tilde{\omega}^i_j \neq \omega^i_j \).

From (14-15) it is immediate to obtain the components \( \gamma^\alpha_{\beta\lambda} \) of the connection one-forms \( \omega^\alpha_{\beta} \), defined by \( \omega^\alpha_{\beta} = \gamma^\alpha_{\beta\lambda} \theta^\lambda \). These will have to be used later. Recall also the identity \( \nabla_{E_\beta} E_\alpha = \gamma^\lambda_{\alpha\beta} E_\lambda \).

This together with (14) provides the simple formula
\[
    h(X) = g(\dot{X}, \nabla_{E_0} E_0), \quad \forall X \in \Gamma(T\mathcal{M}).
\]
Similarly, the non-vanishing components \( R^\alpha_{\beta\gamma\delta} \) of the curvature tensor \( R \), defined by \( \Omega^0_\beta = \frac{1}{2} R^\alpha_{\beta\gamma\delta} \theta^\gamma \wedge \theta^\delta \) can be read off from (16-17):
\[
\begin{align*}
    R^1_{i0j} &= - (\nabla_j h_i + i_{ij} + t^k i_{kj}), \\
    R^1_{ijk} &= \nabla_k t_{ij} - \nabla_j t_{ik}, \\
    R^i_{jk} &= \tilde{\Omega}^i_{jk}, \\
    R^i_{j} &= -g^{ij} R^1_{krij} = \nabla_k t^i_j + \tilde{\Gamma}^i_{jk}
\end{align*}
\]
where, in the last expression, the dot acts on the functions \( \Gamma^i_{jk} \). From here, the non-vanishing components of the Ricci tensor can be easily computed (note that \( R^1_{\alpha\beta\mu} = -R_{0\alpha\beta\mu} \)):
\[
\begin{align*}
    R_{00} &= R^1_{00} = -g^{ij} R^1_{j0i} = \nabla_i h^i + g^{ij} i_{ji} + t^k t_{ki}, \\
    R_{0i} &= \nabla_i t^i_j - \nabla_j t^i_i, \\
    R_{ij} &= \tilde{R}_{ij},
\end{align*}
\]
and the scalar curvature of \( M \) turns out to be equal to that of \( \mathcal{M} \)
\[
    S = \mathcal{S}.
\]
Therefore, we have proven the following result.

**Proposition 4.12.** The curvature tensor \( \overline{\mathcal{R}} \) of the foliation \( \mathcal{M} \) as defined in (11), and its associated Ricci tensor \( \overline{\text{Ric}} \) and scalar curvature \( \overline{S} \) satisfy that
\[
\overline{\mathcal{R}} = \mathcal{R}; \quad \overline{\text{Ric}} = \text{Ric}; \quad \overline{S} = \mathcal{S},
\]
where \( \mathcal{R}, \text{Ric} \) and \( \mathcal{S} \) are the projections by the homomorphism \( \mathcal{g} \) of the curvature tensor \( R \), the Ricci tensor \( \text{Ric} \) and the scalar curvature \( S \) of the Brinkmann space \( (M, g) \), respectively.
Using this result, from now on there will be no need to distinguish between these objects and we will use only the notation \( \overline{R}, \overline{Ric} \) and \( \overline{S} \) for them.

**Remark 4.13.** About the consistency of the tensor equations on \( M \):

1. If, say, the first index of \( t \) and \( \dot{t} \) is raised, then the same expression \( t^i \dot{j} \) is obtained, as the isomorphisms \( \vec{\theta} \) and \( \vec{\theta}^\ast \) commute with the homomorphism \( T \mapsto \dot{T} \) \((t^i \dot{j} \equiv g^{i\alpha} t_{\alpha j} = g^i r t_j)\).
2. Observe that \( \overline{R} = \overline{\overline{R}} \) is related to \( \overline{R} \) by means of the following formula:

\[
\overline{R}(X,Y)Z = \overline{R}(X,Y)\dot{Z}, \quad \forall X,Y,Z \in \Gamma(\overline{\nabla T\overline{M}}).
\]

4.3.2. The \( D_0 \) derivation.

**Definition 4.14.** The \( D_0 \) operator associated to any Brinkmann chart is defined as:

\[
D_0 : \quad \Gamma(T^*\overline{M}) \rightarrow \Gamma(T^*\overline{M})
\]

\[
T \rightarrow D_0T = (\overline{\nabla E_0} T)
\]

Consequently, \( T \in \Gamma(T^*\overline{M}) \) is said to be \( D_0 \)-parallel if \( D_0T = 0 \).

\( D_0 \) measures the variation, projected to \( T^*\overline{M} \), of tensor fields \( T \in \Gamma(T^*\overline{M}) \) under displacements along the direction \( E_0 \), which is transverse to the leaves of \( \overline{M} \).

**Proposition 4.15.** \( D_0 \) satisfies the formal properties of a tensor derivation on \( \overline{M} \):

i) \( \mathbb{R} \)-linearity: \( D_0(aA + bB) = aD_0A + bD_0B, \forall a,b \in \mathbb{R}, \forall A,B \in \Gamma(T^*\overline{M}). \)

ii) Leibniz rule: \( D_0(\Theta A \Theta B) = (D_0 \Theta A) \Theta B + A \Theta (D_0 B), \forall A,B \in \Gamma(T^*\overline{M}). \)

iii) Commutativity with contractions: \( D_0(C_j^i(A)) = C_j^i(D_0A), \forall A \in \Gamma(T^*\overline{M}), \) where \( C_j^i \) denotes the contraction of the \( j \)-th contravariant slot with the \( i \)-th covariant one.

**Proposition 4.16.** For \( X \in \Gamma(T\overline{M}) \), the decomposition of \( \nabla E_0 \dot{X} \) in \( TU \otimes (TU)^\perp \) is

\[
\nabla E_0 \dot{X} = h(X)E_1 + \overline{\overline{\nabla E_0} \dot{X}}.
\]

*Proof.* Putting \( \nabla E_0 \dot{X} = X_1 + X_2 \), with \( X_1 \in \Gamma(TU) \) and \( X_2 \in \Gamma((TU)^\perp) \), the definition of \( D_0 \) gives immediately \( D_0X = X_2 \). For \( X_1 \), using that \( g(\dot{X},E_0) = g(\dot{X},E_1) = 0 \), that \( E_1 \) is parallel and formula (18) we get:

\[
X_1 = g(\nabla E_0 \dot{X},E_0)E_0 - g(\nabla E_0 \dot{X},E_0)E_1 = g(\dot{X},\nabla E_0 E_1)E_0 + g(\dot{X},\nabla E_0 E_0)E_1 = h(X)E_1. \quad \square
\]

**Proposition 4.17.** \( \overline{\overline{\Theta}} \) is \( D_0 \)-parallel, that is to say:

\[
D_0 \overline{\overline{\Theta}} = 0.
\]

*Proof.* By definition, \( D_0 \overline{\overline{\Theta}} \equiv \nabla E_0 \dot{\Theta} \) and \( \Theta = -\theta^0 \otimes \theta^1 - \theta^1 \otimes \theta^0 + \Theta \). Since \( \theta^0 \) is parallel and \( \nabla E_0 \theta^0 = 0 \),

\[
-\theta^0 \otimes \nabla E_0 \theta^1 - \nabla E_0 \theta^1 \otimes \theta^0 + \nabla E_0 \dot{\Theta} = 0.
\]

Therefore, as \( \left( \theta^0 \otimes \nabla E_0 \theta^1 + \nabla E_0 \theta^1 \otimes \theta^0 \right) = 0 \), we have that \( \nabla E_0 \dot{\Theta} = 0. \quad \square
\]

**Remark 4.18.** It is not difficult to show that, for arbitrary \( \omega \in \Gamma(T^*\overline{M}) \), the \( D_0 \)-derivative acts such that

\[
(D_0\omega)(X) = E_0(\omega)(X) + t(\omega^\sharp, X), \quad \forall X \in \Gamma(T\overline{M})
\]

while, with the same notation

\[
\omega(D_0X) = \omega(E_0(X)) - t(\omega^\sharp, X).
\]

These formulas can then be extended to arbitrary sections \( T \in \Gamma(T^*\overline{M}) \), and can be expressed in the local basis \( \{\partial_i\} \) of \( \Gamma(T\overline{M}) \) by means of:

\[
(D_0X)^i \equiv D_0X^i = (\partial_a - H\partial_c)(X^i) - t^i_j X^j,
\]

\[
(D_0\omega)_i \equiv D_0\omega_i = (\partial_a - H\partial_c)(\omega_i) + t^i_j \omega_j.
\]
Its generalization to any section $T \in \Gamma(T_{\mathcal{M}})$ is

$$(D_0 T)_{j_1 \ldots j_x}^{i_1 \ldots i_y} = D_0 T_{j_1 \ldots j_x}^{i_1 \ldots i_y} = (\partial_u - H \partial_v)(T_{j_1 \ldots j_x}^{i_1 \ldots i_y}) - \sum_{a=1}^r t_{a}^k T_{j_1 \ldots j_x}^{i_1 \ldots i_y (a-1) k(a+1) \ldots i_y} + \sum_{b=1}^s t_{b}^k T_{j_1 \ldots j_x}^{i_1 \ldots i_y}.$$

For $\nu$-invariant tensor fields the expression $D_0 T$ simplifies ($D_0 T = \overline{\nabla} u_0 T$) such that

$$(D_0 T)_{j_1 \ldots j_x}^{i_1 \ldots i_y} = \dot{T}_{j_1 \ldots j_x}^{i_1 \ldots i_y} - \sum_{a=1}^r t_{a}^k T_{j_1 \ldots j_x}^{i_1 \ldots i_y (a-1) k(a+1) \ldots i_y} + \sum_{b=1}^s t_{b}^k T_{j_1 \ldots j_x}^{i_1 \ldots i_y}.$$  \hfill (22)

Next, we collect some elementary properties of the $D_0$-parallel transport to be used later. First, let $\eta$ be an integral curve of $E_0$ and take a vector field $X_\eta$ along $\eta$ which is everywhere tangent to $\mathcal{M}$, i.e., $X_\eta \in \Gamma(T_\eta \mathcal{M})$. The derivative $D_0(X_\eta)$ of $X_\eta$, as well as its $D_0$-parallelism, makes an obvious sense.

**Lemma 4.19.** Let $p \in M$, $\vec{v} \in T_p \mathcal{M}$ and $\eta$ the integral curve of $E_0$ with $\eta(0) = p = (u_p, v_p, x^i_p)$. Then, there exists a unique vector field $X_\eta$ obtained as the $D_0$-parallel transport along $\eta$ such that $X_\eta(0) = \vec{v}$.

A standard reasoning leads to:

**Proposition 4.20.** The map which sends each $\vec{v} \in T_p \mathcal{M}$ to its unique $D_0$-parallel transport $X_\eta(\tau) \in \Gamma(T_{\eta(\tau)} \mathcal{M})$ along $\eta$ is a linear isometry from $T_p \mathcal{M}$ to $T_{\eta(\tau)} \mathcal{M}$.

**Proof.** The result follows from (21), since $(\mathcal{F}(X,Y))|_\eta$ depends only on the parameter $u$ of $\eta$. □

Our last result yields an extension to all the leaves of any vector field on one leaf.

**Proposition 4.21.** Let $X_{\mathcal{M}}$ be a vector field on a leaf $\mathcal{M}(u_0,v_0)$ of $\mathcal{M}$. Then, there exists a unique $\nu$-invariant and $D_0$-parallel vector field $X_{\mathcal{M}} \in \Gamma(T \mathcal{M})$ which extends $X_{\mathcal{M}}$.

**Proof.** Consider the hypersurface $\Omega_{v_0} = \{v = v_0\}$ and extend $X_{\Omega_{v_0}}$ to a vector field $X_{\Omega_{v_0}}$ on $\Omega_{v_0}$ by taking each integral curve $\eta$ of $E_0$ which starts at some $p \in \mathcal{M}(u_0,v_0)$, and defining $X_{\Omega_{v_0}} \circ \eta$ as the $D_0$-parallel transport of $X_p$. Then, extend $X_{\Omega_{v_0}}$ to a $\nu$-invariant vector field according to Remark 4.3(2). □

4.3.3. Derivatives of the curvature tensor $R$. As $E_1$ is parallel, the curvature tensor will be determined on each Brinkmann chart by its value on quadruples of vectors tangent to $\mathcal{M}$, plus some extra tensors which take care of the remaining components (partly) along the $\theta^1$ or $E_0$ directions. We start by defining two such tensor fields on $\mathcal{M}$.

**Definition 4.22.** For any Brinkmann chart and its associated partly null frame $\{E_a\}$ we define $A \in \Gamma(T_2^2 \mathcal{M})$ and $B \in \Gamma(T_3^3 \mathcal{M})$ as $A := \theta^1(R(E_0,\cdot))$ and $B := \theta^1(R)$, that is to say

$$A(X,Y) = \theta^1(R(E_0,\dot{Y})\dot{X}), \quad B(X,Y,Z) = \theta^1(R(\dot{Y},\dot{Z})\dot{X}) \quad \forall X,Y,Z \in \Gamma(T \mathcal{M}).$$

From the symmetries of the curvature tensor it is obvious that $A$ is symmetric

$$A(X,Y) = A(Y,X) \quad \forall X,Y \in \Gamma(T \mathcal{M})$$

and that $B$ is skew-symmetric in its last two slots and it satisfies a cyclic identity

$$B(X,Y,Z) = -B(X,Z,Y), \quad B(X,Y,Z) + B(Y,Z,X) + B(Z,X,Y) = 0 \quad \forall X,Y,Z \in \Gamma(T \mathcal{M}).$$

**Remark 4.23.** In the given basis $\{\partial_i\}$, $A_{ij} = A(\partial_i, \partial_j) = R^{l}_{i0j}$ and $B_{ijk} = B(\partial_i, \partial_j, \partial_k) = R^{l}_{ijk}$. Then, the previous properties can be expressed using this notation as $A_{ij} = A_{ij}$, $B_{ijk} = B_{ijk}$, and $B_{ijk} = 0$. In what follows, and for the sake of brevity, we will resort to using index notation in many cases, which is sufficient to illustrate these properties and reveals itself as very helpful in the required complicated calculations for 2nd-symmetry. As a starting example, note that we additionally have for instance

$$B_{ij}^{k} (= g^{kr}B_{ijr}) = R^{k}_{j0i}.$$
A direct computation of $\nabla R$, for instance in the basis $\{E_\alpha\}$, provides the following formulae:

$$\begin{align*}
\theta^1(\nabla_{E_0} R(E_0, \cdot)) &= D_0 A + 2S [C_3^1(h^2 \otimes B)] \\
\theta^1(\nabla R(E_0, \cdot)) &= \nabla A - 2S [C_{15}(t \otimes B)] \\
\theta^1(\nabla_{E_0} R) &= D_0 B + h(\tilde{R}) \\
\nabla_{E_0} R &= D_0 \tilde{R} \\
\nabla R &= \nabla \tilde{R}
\end{align*}$$

where $S[T]$ gives the symmetric part of any covariant section $T \in \Gamma(T^0_0 \mathcal{M})$ in its last two slots, and $C_{ij}$ denotes the contraction of the $i^{th}$ and $j^{th}$ covariant indices (via the metric $\bar{g}^{-1}$).

We give names to the lefthand sides of these relations (except for the last one), thereby defining five tensor fields on $\mathcal{M}$ which will allow for simpler expressions when computing $\nabla^2 R$.

**Definition 4.24.** For any Brinkmann chart and its associated partly null frame $\{E_\alpha\}$ we define $\tilde{A} \in \Gamma(T^2_0 \mathcal{M})$, $\tilde{A}, \tilde{B} \in \Gamma(T^3_0 \mathcal{M})$, $\tilde{B} \in \Gamma(T^3_0 \mathcal{M})$ and $\tilde{R} \in \Gamma(T^3_0 \mathcal{M})$, for all $X, Y, Z, V \in \Gamma(T \mathcal{M})$ by:

$$\begin{align*}
\tilde{A}(X,Y) &= \theta^1 ((\nabla_{E_0} R)(E_0, \cdot) \tilde{X}), \\
\tilde{A}(X,Y,Z) &= \theta^1 ((\nabla_{E_0} R)(E_0, \cdot) \tilde{Y}), \\
\tilde{B}(X,Y,Z) &= \theta^1 ((\nabla_{E_0} R)(E_0, \cdot) \tilde{Z}), \\
\tilde{B}(X,Y,Z, V) &= \theta^1 ((\nabla_{E_0} R)(E_0, \cdot) \tilde{V}),
\end{align*}$$

Observe that, therefore, one has $\tilde{R} = D_0 \tilde{R}$, $\tilde{B} = D_0 B + h(\tilde{R})$, etcetera in agreement with the previous formulas.

**Remark 4.25.** In the given basis $\{\partial_i\}$, $\tilde{A}_{ij} = \tilde{A}(\partial_i, \partial_j) = \nabla_0 R^i \partial_j$, $\tilde{A}_{sij} = \tilde{A}(\partial_s, \partial_i, \partial_j) = \nabla_s R^i \partial_{ij}$, $\tilde{B}_{ijk} = \tilde{B}(\partial_i, \partial_j, \partial_k) = \nabla_i R^j \partial_{k}$, $\tilde{B}_{sijk} = \tilde{B}(\partial_s, \partial_i, \partial_j, \partial_k) = \nabla_s R^j \partial_{k}$ and $\tilde{R}^{i j kl} = \partial_{i j kl} \tilde{R}(\partial_k, \partial_i, \partial_j) = \nabla_0 R^i \partial_{jkl}$. Besides, from the symmetries of the curvature tensor it is obvious that:

1. $\tilde{A}$ and $\tilde{A}$ are symmetric in the last two indices; $\tilde{A}_{ij} = \tilde{A}_{ij}$; $\tilde{A}_{sij} = \tilde{A}_{sij}$.
2. $\tilde{B}, \tilde{B}, \tilde{R}$ are skew-symmetric in their last two indices: $\tilde{B}_{ijk} = \tilde{B}_{jki}; \tilde{B}_{sijk} = \tilde{B}_{sikj}; \tilde{R}_{sijk} = \tilde{R}_{sikj}$.
3. $\tilde{B}, \tilde{B}, \tilde{R}$ satisfy a cyclic identity: $\tilde{R}_{sijkl} = 0; \tilde{B}_{sijkl} = 0; \tilde{R}_{sijkl} = 0$.
4. $\tilde{R}$ also satisfies that $\tilde{R}_{ijkl} = -\tilde{R}_{jikl}$ and $\tilde{R}_{ijkl} = \tilde{R}_{klij}$, so that it has all the symmetries of a Riemann tensor.

One can prove that, in addition, $\tilde{B}_{ij} k = \nabla_0 R^i j_0 i$ and $\tilde{B}_{sij} k = \nabla_s R^i j_0 i$. We also point out the following basic relations:

$$\begin{align*}
\tilde{R}_{ijkl} &= -2\tilde{B}_{[ij]kl} \\
\tilde{B}_{[ij]} &= 2\tilde{A}_{[ij]} \\
\tilde{B}_{[i][jk]} &= 0
\end{align*}$$

They follow by direct application of the second Bianchi identity $\nabla [\alpha R_{\beta \sigma}]_{\mu \nu} = 0$ in the partly null frame $\{E_\alpha\}$: the first two relations follow by taking $\{\alpha, \beta, \sigma\} = \{0, i, j\}$ and the last one with $\{\alpha, \beta, \sigma\} = \{i, j, k\}$. 


Using all the above, another direct computation of $\nabla \nabla R$ leads to the following formulae:

\begin{align*}
\nabla \nabla R &= \nabla \nabla R, \\
\nabla \nabla E_0 R &= \nabla \tilde{R} + C_{13}(t \otimes \nabla R), \\
\tilde{\theta}^1(\nabla \nabla R) &= \nabla \tilde{B} - C_{1}^1(t \otimes \nabla R), \\
\tilde{\theta}^1(\nabla \nabla E_0 R) &= \nabla \tilde{B} - C_{1}^1(t \otimes \tilde{R}) + C_{13}(t \otimes \tilde{B}), \\
\tilde{\theta}^1(\nabla \nabla E_0 R(E_0, \cdot)) &= \nabla \tilde{A} - 2S[C_{16}(t \otimes \tilde{B})], \\
\tilde{\theta}^1(\nabla \nabla R(E_0, \cdot)) &= \nabla \tilde{A} + 2S[C_{16}(h^s \otimes \tilde{B})],
\end{align*}

(25a) \quad (25b) \quad (25c) \quad (25d) \quad (25e) \quad (25f) \quad (25g)

Recall that, for example, the equations in the first row can be written as:

\begin{align*}
\nabla_m \nabla_s R^{ijkl} &= \nabla_m \nabla_s R^{ijkl}, \\
\nabla_0 \nabla_s R^{ijkl} &= D_0 \nabla_s R^{ijkl}.
\end{align*}

(26)

4.4. Reducibility and a generalized Eisenhart theorem. The proof of our main Theorem 1.1 will be carried out by means of some local decompositions of the Brinkmann space. To that end, we need an appropriate notion of reducibility and a version of a classical theorem by Eisenhart adapted to Brinkmann spaces.

**Definition 4.26.** A Brinkmann decomposition $\{u, v\}$ of a Brinkmann space $(M, g)$ is spatially reducible if there exists a Brinkmann chart $\{u, v, x^r\}$ around each point and a partition of the indices $I_1 = \{2, \ldots, d + 1\}$, $I_2 = \{d + 2, \ldots, n - 1\}$ for some $d \in \{1, \ldots, n - 3\}$ such that $\tilde{g}_{aar} = 0$ and $\partial_a \tilde{g}_{ab} = 0$, where the unprimed indices $a, b$ always belong to the same subset $I_m$ ($m \in \{1, 2\}$) and the primed ones $a', b'$ to the other one.

For such a Brinkmann chart, we say that $T \in \Gamma(T_s^* \overline{M})$ is reducible whenever $T = T^{(1)} + T^{(2)}$, with $T^{(m)} = T^{(m)}_{b_1 \ldots b_r}(u, v^c) \partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \otimes \tilde{d}x^{b_1} \otimes \ldots \otimes \tilde{d}x^{b_s}$, and $a_1, \ldots, a_r, b_1, \ldots, b_s, c \in I_m$. We will denote such decomposition as $T = T^{(1)} \oplus T^{(2)}$.

**Remark 4.27.** (1) Observe that, if a Brinkmann decomposition is spatially reducible, then the metric $\tilde{g}$ on $\overline{M}$ is reducible as a tensor field.

(2) Spatial reducibility implies the following property, which provides a more intrinsic expression for some of its consequences. For some Brinkmann decomposition $\{u, v\}$ there exist two foliations $\overline{M}^{(1)}, \overline{M}^{(2)}$ of $M$ such that $T \overline{M} = T \overline{M}^{(1)} \oplus T \overline{M}^{(2)}$, and this sum is orthogonal with respect to $\tilde{g}$. In this case, we write $\overline{M} = \overline{M}^{(1)} \times \overline{M}^{(2)}$ and $\tilde{g} = \tilde{g}^{(1)} \oplus \tilde{g}^{(2)}$, according to the notation in Definition 4.26. Then, the metric $g$ on $M$ can be written as:

$$g = -2du(dv + Hdu + \tilde{W}) + \tilde{g}^{(1)} \oplus \tilde{g}^{(2)}$$

(recall that $H$ and $W$ depend on the chosen coordinates $\{x^i\}$). In this context, a tensor field $T$ is reducible if and only if $T = T^{(1)} + T^{(2)}$ where each $T^{(m)}$ is invariant by the flow of vectors in $\overline{M}^{(m)}$ ($\mathcal{L}_X T^{(m)} = 0, \forall X \in \Gamma(T \overline{M}^{(m)})$) and it vanishes when applied on any element of $T \overline{M}^{(m)}$ and $T^* \overline{M}^{(m)}$. In particular, this happens for $\tilde{g}$. The Riemannian manifold $(\overline{M}, \tilde{g})$ can be written as the product of two manifolds which will also be denoted, abusing the notation, by $(\overline{M}^{(1)}, \tilde{g}^{(1)})$ and $(\overline{M}^{(2)}, \tilde{g}^{(2)})$, each $\overline{M}^{(m)}$ generating $\overline{M}^{(m)}$ as $\overline{M}$ generated $\overline{M}$, see Section 4. Thus, the equality $\tilde{g} = \tilde{g}^{(1)} \oplus \tilde{g}^{(2)}$ may refer either to the metric decomposition in a leaf $\overline{M}^{(m)}_{(u_0, v_0)}$ or in $\overline{M}$. The latter depends on $u$ and is $v$-invariant. Even though this ambiguity is harmless, we will always refer to decompositions in $\overline{M}$ except if otherwise is specified.

(3) Sometimes, a Brinkmann chart may admit a partition $I_1, \ldots, I_s, s \geq 2$ of the indices $\{2, \ldots, n - 1\}$ so that $\tilde{g} \in \Gamma(T^*_s \overline{M})$ satisfies the properties given in Definition 4.26 for each
i, j ∈ I_m, k′, l′ ∈ I_{m′} and m ≠ m′. In this case, the notation for orthogonal decomposition is naturally extended:

\[ \mathcal{M} = \mathcal{M}^{(1)} \times \ldots \times \mathcal{M}^{(s)}, \quad \mathcal{M} = \mathcal{M}^{(1)} \times \ldots \times \mathcal{M}^{(s)} \times \mathcal{\tilde{g}} = \mathcal{g}^{(1)} \oplus \ldots \oplus \mathcal{g}^{(s)} \]

and notions as being Einstein or flat are also extended to each \( \mathcal{M}^{(m)}, \mathcal{M}^{(m)} \) in a trivial way. However, the following caution must be kept in mind. The given decomposition of \( \mathcal{\tilde{M}} \) induces also an orthogonal decomposition at each leaf \( \mathcal{M}^{(u,v)}, \) in particular at \( \mathcal{M} \). Nevertheless, as \( \mathcal{\tilde{g}} \) is “\( u \)-dependent” such a decomposition may be irreducible in the sense of the traditional de Rham’s theorem for some leaves, but reducible for other ones. In principle, we will not care about this possible “spatial irreducibility” of the metrics \( \mathcal{\tilde{g}}^{(m)} \) on the leaves. Eventually, though, we will arrive at a decomposition of \( \mathcal{\tilde{g}} \) which will induce an irreducible decomposition of all the leaves, independent of \( u \).

Let us turn to the Eisenhart theorem. Its classical version [17] states:

**Theorem 4.28.** If a Riemannian manifold \((N, g_N)\) admits a symmetric two-covariant tensor field \( L \in \Gamma(T^2_N) \) not proportional to the metric \( g_R \) such that \( \nabla g_R L = 0 \), then

- \( g_R \) is reducible: \( g_R = g_R^{(1)} \oplus g_R^{(2)} \oplus \ldots \oplus g_R^{(s)} \) (with each \( g_R^{(m)} \) not necessarily irreducible).
- \( L = \sum_{m=1}^{s} \lambda_m g_R^{(m)} \) for some constants \( \lambda_m \).

Our aim now is to prove a version of this theorem adapted to the spatial reducibility of Definition 4.26 for Brinkmann decompositions. In our generalized version, the reduced metrics \( \mathcal{\tilde{g}}^{(m)} \) will be dependent on \( u \) but the \( \lambda_m \) will still be constants, independent of \( u \).

**Theorem 4.29.** Let \((M, g)\) be a Brinkmann space and fix a Brinkmann chart \( \{u, v, x\} \). Assume that there exists a symmetric \( v \)-invariant, \( \nabla \)-parallel and \( D_0 \)-parallel section \( \mathcal{L} \in \Gamma(T^2_M) \) which is not proportional to \( \mathcal{\tilde{g}} \). Then, the Brinkmann decomposition \( \{u, v\} \) is spatially reducible and \( \mathcal{L} \) is reducible. Furthermore, the decomposition \( \{u, v\} \) admits a Brinkmann chart \( \{u, v, y^i\} \) such that:

1. \( \mathcal{\tilde{g}} = \mathcal{g}^{(1)} \oplus \ldots \oplus \mathcal{g}^{(s)} \) for some \( s \geq 2 \).
2. \( \mathcal{L} = \sum_{m=1}^{s} \lambda_m \mathcal{\tilde{g}}^{(m)} \) for some constants \( \lambda_m \in \mathbb{R} \).

**Proof.** Let \( p \) be any point of the chart. We construct an orthonormal basis of eigenvector fields \( V_i \in T_p \mathcal{M} \) defined on the hypersurface \( \Omega_{v_p} = \{v = v_p\} \) as we describe now: consider the eigenvalue problem of \( \mathcal{L}_p \) with respect to \( \mathcal{\tilde{g}}_p \) on the vector space \( T_p \mathcal{M} \), i.e.,

\[ \mathcal{L}_p(\cdot, \bar{v}) - \lambda \mathcal{\tilde{g}}_p(\cdot, \bar{v}) = 0, \quad (27) \]

and take an orthonormal basis \( \{\tilde{v}_i\}_{i=2}^{n+1} \) of eigenvectors of \( \mathcal{L}_p \) in \( T_p \mathcal{M}_{(v_p, v_p)} \). Extend this basis to a normal neighborhood \( \mathcal{U} \) of \( p \) in the leaf \( \mathcal{M}_{(u_p, v_p)} \) by defining \( V_q \) at each \( q \in \mathcal{U} \) as the vector obtained by \( \nabla \)-parallelly transporting \( \tilde{v}_i \) along the unique geodesic \( \gamma_q : [0, 1] \to \mathcal{U} \) from \( p \) to \( q \). Clearly, if \( \tilde{v}_i \) is a \( \lambda \)-eigenvector, then \( V_q \) is an eigenvector of \( \mathcal{L}_q \) with the same eigenvalue \( \lambda \), because the one-forms on \( \gamma_q \) defined as \( \tau \mapsto \mathcal{L}_{\gamma_q(\tau)}(\cdot, V_{\gamma_q(\tau)}) \) and \( \tau \mapsto \lambda \mathcal{\tilde{g}}_{\gamma_q(\tau)}(\cdot, V_{\gamma_q(\tau)}) \) are parallel and coincide at \( p \) due to (27). Besides, \( \{V_i\} \) is an orthonormal basis on \( \mathcal{U} \). Now, obtain the sought basis \( \{V_i\} \) on \( \Omega_{v_p} \) by propagating each \( V_i \) in a \( D_0 \)-parallel manner along the integral curve of \( E_0 \) at \( q \in \mathcal{U} \). Since \( \mathcal{L} \) is \( D_0 \)-parallel, \( \{V_i\} \) is still an orthonormal basis of eigenvector fields of \( \mathcal{L} \) and the eigenvalues of \( \mathcal{L} \) are constant on \( \Omega_{v_p} \).

From \( v \)-invariance, the eigenvalues and the dimension of each eigenspace remain constant all over \( M \). Therefore, if we denote any \( \lambda \)-eigenvector field as \( V^{(\lambda)} \), and \( \lambda_1, \ldots, \lambda_s \) are the eigenvalues of \( \mathcal{L} \) with corresponding multiplicities \( m_1, \ldots, m_s \), we can reorder the vector fields so that

\[ \{V^{(\lambda_1)}_1, \ldots, V^{(\lambda_s)}_1, V^{(\lambda_1)}_2, \ldots, V^{(\lambda_s)}_2, \ldots, V^{(\lambda_1)}_m, \ldots, V^{(\lambda_s)}_m\} \]

is an orthonormal basis of \( T \mathcal{M} \).

Let \( \lambda \) be one of the eigenvalues and let us prove that the distribution \( S_\lambda \) generated by its eigenvectors is involutive. Taking the \( \nabla_V \) covariant derivative of \( \mathcal{L}(\cdot, V^{(\lambda)}_i) = \lambda \mathcal{\tilde{g}}(\cdot, V^{(\lambda)}_i) \) for any \( V \in \Gamma(T \mathcal{M}) \) and using that \( \mathcal{L} \) is \( \nabla \)-parallel, we have that \( \nabla_V V^{(\lambda)}_i \) lies in \( S_\lambda \), and so does
Clearly, under the hypotheses of Theorem 4.29, Remark 4.30. Remark 4.3 thereby including the coordinate $v$ construct a chart on $M$ under a change of the partly null frame associated to a transformation of the type (7). In particular, $\lambda_0 = 0$, which is part of (2), and recalling that $R = D_0 \tilde{\gamma}$ it follows that $0 = D_0 \tilde{\gamma}$, so that $\tilde{\gamma} = \gamma$ is constant, as required. We are now equipped with all the elements that will allow us to solve the problem of 2nd-symmetry. The only remaining task is to solve the equations for 2nd-symmetry, given by setting all the expressions in (25) equal to zero. We are going to do this in several steps.

5. Proper 2nd-symmetric Lorentzian manifolds

We are now equipped with all the elements that will allow us to solve the problem of 2nd-symmetry. The only remaining task is to solve the equations for 2nd-symmetry, given by setting all the expressions in (25) equal to zero. We are going to do this in several steps.

5.1. Reduction of the equations. We start by proving a fundamental simplification of 2nd-symmetric Brinkmann spaces, that is, of all proper 2nd-symmetric spaces, interesting in its own right.

**Theorem 5.1.** Let $(M, g)$ be a Brinkmann space with a fixed Brinkmann decomposition $\{u, v\}$. Then, if $(M, g)$ is a 2nd-symmetric manifold, it follows that:

1. the foliation $\mathcal{F}$ is locally symmetric, i.e., $\nabla R = 0$

2. for any associated Brinkmann chart, the tensor fields $\hat{B}, \tilde{R}, \hat{A}, \widetilde{B}$ in Definition 4.24 vanish.

3. the scalar curvature $S$ of the manifold is constant.

The proof of this theorem will be carried out in two steps, one involving the intrinsic geometry of $\mathcal{F}$ (beginning of Section 5.1.1), and the other involving the integrability equations (Propositions 5.6 and 5.7 in Section 5.1.3). An important consequence of this theorem, to be used later, is:

**Corollary 5.2.** Let $(M, g)$ be a 2nd-symmetric Brinkmann space with a fixed Brinkmann decomposition $\{u, v\}$. Then, the section $\tilde{A}$ is a tensor field on $\mathcal{F}$, independent of the chosen Brinkmann chart. Moreover, $(M, g)$ is proper 2nd-symmetric if and only if $\tilde{A} \neq 0$.

**Proof.** From Theorem 5.1, the only non-zero components of $\nabla R$ in any partly null frame associated to a Brinkmann chart $\{u, v, x^i\}$ are $\nabla_0 \hat{R}^i_j = \hat{A}_{ij}$. Then, it is straightforward to check that

$$\hat{A}_{ij} = \frac{\partial x^i}{\partial x^\alpha} \frac{\partial x^j}{\partial x^\beta} \hat{A}_{ij}$$

under a change of the partly null frame associated to a transformation of the type (7). In particular, $\hat{A}$ behaves as a tensor field on $\mathcal{F}$ for the given Brinkmann decomposition $\{u, v\}$.

5.1.1. First step: local symmetry of $\mathcal{F}$. To prove (1) of Theorem 5.1, observe that the first in (26) (when set to zero) together with Proposition 4.10 imply the result immediately. At this stage, we can also prove (3) if we assume (2): given that $S = \gamma$ and due to $\nabla R = 0$ the function $\gamma$ depends only on $u$. Thus, assuming $\tilde{R} = 0$, which is part of (2), and recalling that $\hat{R} = D_0 \gamma$ it follows that $0 = D_0 \gamma$, so that $\gamma = \tilde{\gamma}$ is constant, as required.

Using (25) and Theorem 5.1(1), the equations of 2nd-symmetry thus become
\[ \nabla_n \hat{B}_{sijk} = 0, \]
\[ \nabla_r \hat{R}_{ijkl} = 0, \]
\[ \nabla_k \hat{A}_{sij} = 2t^r k \hat{B}_{s(ij)r}, \]
\[ \nabla_s \hat{B}_{ijk} = t^r s (\hat{R}_{rijk} - \hat{B}_{rijk}), \]
\[ \nabla_s \hat{A}_{ij} = t^r k (2 \hat{B}_{ijr} - \hat{A}_{rij}), \]
\[ D_0 \hat{A}_{ij} = -h^r (2 \hat{B}_{ijr} - \hat{A}_{rij}), \]
\[ \nabla_s \hat{R}_{ijkl} = 0, \]

where, for completeness, we include the expressions for all these objects in this notation:
\[ \tilde{R}_{ijkl} = D_0 \hat{R}_{ijkl}, \]
\[ \check{B}_{sijk} = \nabla_s B_{ijk} - t_r s \hat{R}^r_{ijkl}, \]
\[ \check{B}_{ijk} = D_0 B_{ijk} + h_i \hat{R}^r_{ijkl}, \]
\[ \hat{A}_{sij} = \nabla_s A_{ij} - 2t^k s B_{(ij)k}, \]
\[ \check{A}_{ij} = D_0 A_{ij} + 2h^k B_{(ij)k}, \]
\[ B_{ijkl} = \nabla_{kit} - \nabla_j t_{ik}, \]
\[ A_{ij} = t_i t^r j - \nabla_j h_i - D_0 t_{ij}, \]
\[ \hat{R}_{ijkl} = R^l_{ijkl}. \]

5.1.2. Auxiliary algebraic results. As an interlude, before starting with the second step we prove a couple of technical results about tensors with particular properties on a real vector space \( V \) of finite dimension. Tensors will be denoted in abstract index form.

**Proposition 5.3.** Let \( V \) be an \( l \)-dimensional vector space with a positive definite inner product and \( T_{ijk} \) a three-covariant tensor such that:

(a) it is skew-symmetric in the last two indices: \( T_{[ijk]} = T_{i[jk]} \),

(b) it satisfies a cyclic identity: \( T_{ijk} + T_{jki} + T_{kij} = 0 \)

If \( T_{(ijk)} \) satisfies
\[ T_{(ij)^r} T_{(rnn)} = 0 \]
then \( T_{ijk} = 0 \).

**Proof.** Observe that, on using (a), (b) can be rewritten as \( T_{[ijk]} = 0 \). We use arguments inspired in [37, Lemma 4.1]. Suppose first that there does not exist a non-vanishing vector \( v \in V \) such that \( v^r T_{rnm} = 0 \). Define the set of vectors \( \{Q^r(i,j)\}_{i,j=1}^l \) by \( Q^r(i,j) = T_{(ij)^r} \). Then, equation (29) can be rewritten as \( Q^r(i,j) T_{rnm} = 0 \), that is, \( Q^r(i,j) = 0 \) for all \( i, j \) by our assumption. Consequently, \( T_{(ij)k} = 0 \), i.e., \( T_{ijk} = T_{[ijk]} \). Using this fact and (a), \( T_{ijk} \) is totally skew-symmetric, and so it vanishes by (b).

Suppose then that there does exist a non-vanishing vector \( v \in V \) such that \( v^r T_{rnm} = 0 \). Without loss of generality, we can assume that \( ||v|| = 1 \). The orthogonal splitting of \( T_{ijk} \) with respect to \( v \) is, then:
\[ T_{ijk} = a_{ijk} + b_{ij} v_k - b_{ik} v_j \]
where
\[ a_{[ijk]} = 0, \quad a_{i[jk]} = a_{ijk}; \quad a_{ijk} v^i = a_{ijk} v^j = 0, \quad b_{ij} = T_{ijk} v^k; \quad b_{ij} v^i (= T_{ijk} v^k v^i) = 0, \quad b_{ij} v^j = 0 \]
Since (b) implies \( v^r (T_{ijk} + T_{jki} + T_{kij}) = 0 \), we obtain by (30) that \( b_{[ijk]} = 0 \), i.e., \( b_{ij} \) is a symmetric two-covariant tensor. But (29) implies \( v^r v^m T_{(ij)^r} T_{rnm} = 0 \), which in turn implies \( b_{ij} v^r b_{rnm} = 0 \) on using (30) once more. Contracting the indices \( j, n \) and using that \( b_{ij} \) is symmetric, we have that
Let \( T_{ijkl} \) be a four-covariant tensor such that:

(a) it is skew-symmetric in the last two indices: \( T_{st[jk]} = 0 \),
(b) it satisfies the cyclic identity: \( T_{ijkl} + T_{iklj} + T_{ijlk} = 0 \)

If \( T_{s(ij)} T_{Trnm} = 0 \) holds, then \( T_{stjk} = 0 \).

**Proof.** Let us define for any \( v \in \mathcal{V} \) the tensor \( v^1 T_{ijkl} \). This tensor satisfies all the conditions in Proposition 5.3, so \( v^1 T_{ijkl} = 0 \) for all \( v \in \mathcal{V} \), that is, \( T_{ijkl} = 0 \). □

5.1.3. **Second step: integrability equations.** Obviously, 2nd-symmetry implies semi-symmetry. As a matter of fact, the equations of semi-symmetry happen to be some of the integrability conditions for the equations of 2nd-symmetry. To check this, and in order to exploit the integrability condition in full, we introduce the expressions for the commutativity of \( \nabla \) and \( D_0 \).

**Proposition 5.5.** Given a fixed Brinkmann chart \( \{ u, v, x^1 \} \):

1. For any \( T \in \Gamma(T^r_s \mathcal{M}) \), the Ricci identity reads:

\[
(\nabla_n \nabla_s - \nabla_s \nabla_n) T^{i_1 \ldots i_r} = \sum_{b=1}^s R^{i_1 \ldots i_r}_{j_1 \ldots j_b} T^{j_1 \ldots j_b} - \sum_{a=1}^r R^{i_1 \ldots i_a - 1 k_{i_a+1} \ldots i_r}_{j_1 \ldots j_a} \tag{31}
\]

2. The commutation of \( D_0 \) with \( \nabla \) for any \( F \in \Gamma(T^1_1 \mathcal{M}) \) (trivially extendable to arbitrary \( T \in \Gamma(T^r_s \mathcal{M}) \)) is:

\[
(\nabla_k D_0 - D_0 \nabla_k) F^i_j = (H_k)(\partial_v F^i_j) + F^i_r B_{kj}^r - F^r_j B_{kr}^i - t^r_k \nabla_r F^i_j.
\]

And, when \( F \) is \( v \)-invariant this simplifies to:

\[
(\nabla_k D_0 - D_0 \nabla_k) F^i_j = F^i_r B_{kj}^r - F^r_j B_{kr}^i - t^r_k \nabla_r F^i_j.
\]

**Proof.** (1) follows from the Ricci identities for \( \nabla \) on each leaf of \( \mathcal{M} \). To prove (2), using (22) and Remark 4.8 we get

\[
\nabla_k (D_0 F^i_j) = \nabla_k (\partial_v F^i_j - H \partial_v F^i_j - t^r_k F^r_j + t^r_j F^i_r),
\]

\[
D_0 (\nabla_k F^i_j) = D_0 (\partial_v F^i_j + \varGamma^i_{rk} F^r_j - \varGamma^r_{jk} F^i_r).
\]

The result follows by expanding these two expressions, and taking into account \( \partial_v \varGamma^i_{jk} = 0 \) (Remark 4.8) plus formula (20). □

Now, we can easily prove the following statement:

**Proposition 5.6.** Let \((M, g)\) be a Brinkmann space with a fixed Brinkmann chart \( \{ u, v, x^1 \} \). If \((M, g)\) is a 2nd-symmetry manifold, the sections \( \hat{B} \) and \( \hat{R} \) on \( \mathcal{M} \) vanish.

**Proof.** By 2nd-symmetry, the integrability conditions associated to the first in (28a) and in (28c) read, respectively:

\[
\nabla_n [\nabla_m] \hat{B}_{ijkl} = 0 \quad \Rightarrow \quad \hat{R}^{snm}_{jkn} \hat{B}_{rjkl} + \hat{R}^{inm}_{jkn} \hat{B}_{srjk} + \hat{R}^{jmnm}_{jkn} \hat{B}_{stlr} + \hat{R}^{knn}_{jkn} \hat{B}_{sijr} = 0, \tag{33}
\]

\[
\nabla_n [\nabla_m] \hat{A}_{sij} = B^r_{mn} \hat{B}_{s(ijr)} \quad \Rightarrow \quad \hat{R}^{snm}_{jkn} \hat{A}_{rjij} + \hat{R}^{r_{nm}}_{jkn} \hat{A}_{srj} + \hat{R}^{jmnm}_{jkn} \hat{A}_{sir} = 2B^r_{mn} \hat{B}_{s(ijr)}. \tag{34}
\]

In both cases we used (31), and for the last case we also used the first in (28a) and in (28c), as well as (28m). If we differentiate (34) with respect to \( \nabla_k \), using the same information as before plus (28g), (28i) and (33), we obtain

\[
\hat{B}_{s(ij)} T_{Trnm} = 0.
\]

Therefore, \( \hat{B} \) satisfies all the hypotheses in Corollary 5.4 at each leaf of \( \mathcal{M} \), so that \( \hat{B} = 0 \) and, by the identity (23), \( \hat{R} = 0 \) too. □
This actually implies, due to (28h), (28g), (28d) and (28c), that the three sections $R$, $\tilde{R}$ and $\hat{A}$ are $\nabla$-parallel and $D_0$-parallel.

The integrability equations associated to (28e-28g), (28k), (28i) and the first in (28d) simplified via using Proposition 5.6 are written, on using (28m) for the expressions containing $\nabla t$, as:

$$2\nabla \lbrack n \nabla \lbrack m \tilde{B}_{ijk} = 0, \quad 2\nabla \lbrack n \nabla \lbrack m \tilde{A}_{ij} = B^{r} mn(2\tilde{B}_{(ij)r} - \hat{\tilde{A}}_{rij}), \quad \nabla \lbrack n \nabla \lbrack m \hat{R}_{ijkl} = 0,$$

$$2\nabla \lbrack n \nabla \lbrack m B_{ijk} = B^{r} mn\tilde{R}_{r(ij)k}, \quad \nabla \lbrack n \nabla \lbrack m \tilde{A}_{ij} = B^{r} mnB_{(ij)r},$$

which via (31) provide

$$\tilde{R}^{r} inm\tilde{B}_{rjk} + \tilde{R}^{r} jnm\tilde{B}_{irk} + \tilde{R}^{r} knm\tilde{B}_{ijr} = 0,$$  \hspace{1cm} (35)

$$\tilde{R}^{r} inm\hat{A}_{rj} + \tilde{R}^{r} jnm\hat{A}_{ir} = B^{r} mn(2\tilde{B}_{(ij)r} - \hat{\tilde{A}}_{rij}),$$  \hspace{1cm} (36)

$$\tilde{R}^{r} snm\tilde{R}_{rijk} + \tilde{R}^{r} inm\tilde{R}_{srjk} + \tilde{R}^{r} jnm\tilde{R}_{sirk} + \tilde{R}^{r} knm\tilde{R}_{sijr} = 0,$$  \hspace{1cm} (37)

$$\tilde{R}^{r} inmB_{rjk} + \tilde{R}^{r} jnmB_{irk} + \tilde{R}^{r} knmB_{ijr} = B^{r} mn\tilde{R}_{rijk},$$  \hspace{1cm} (38)

$$\tilde{R}^{r} inm\hat{A}_{rj} + \tilde{R}^{r} jnm\hat{A}_{ir} = 2B^{r} mnB_{(ij)r}.$$

On the other hand, the integrability conditions derived from (32) applied to $\tilde{A}, B$ and $\hat{A}$ become

$$B_{ms}^{r} \hat{A}_{rij} + B_{mi}^{r} \hat{A}_{irn} + B_{mj}^{r} \hat{A}_{irm} = 0,$$  \hspace{1cm} (40)

$$B_{rjk}B_{mi}^{r} + B_{irk}B_{mj}^{r} + B_{ijr}B_{mk}^{r} = \tilde{R}^{r} ijkA_{rm},$$  \hspace{1cm} (41)

$$B_{mi}^{r} \hat{A}_{rj} + B_{mj}^{r} \hat{A}_{ir} = (2\tilde{B}_{(ij)r} - \hat{\tilde{A}}_{rij})A^{r} m.$$  \hspace{1cm} (42)

where we have used that $\tilde{B}$ and $\hat{A}$ are $D_0$- and $\nabla$-parallel together with (28n), (28f-28e) for $\tilde{A}$ and (28i) reduced via Proposition 5.6 for $B$.

Using these integrability equations we can prove the following further improvement of Proposition 5.6, which then completes the proof of Theorem 5.1.

**Proposition 5.7.** Under the hypotheses of Proposition 5.6 the sections $\tilde{B}$ and $\hat{B}$ vanish.

**Proof:** We use the results of Proposition 5.6 throughout the proof to simplify the formulas to be combined in the calculations. Start by applying $D_0$ to (38), and use (28h), (28j), (37) and (35) in order to get

$$\tilde{B}^{r} mk\tilde{R}_{sijr} = 0.$$  \hspace{1cm} (43)

Apply $\nabla$ to (41) and use (28g), (28i), (28k) and (38) to obtain

$$\hat{A}^{r} mk\tilde{R}_{sijr} = 0.$$  \hspace{1cm} (44)

Applying $D_0$ to (36) and using (34), the second in (28c) and (28d), (28f), (28h), (28j) and (35) we get

$$\tilde{B}^{r} mk(2\tilde{B}_{(ij)r} - \hat{\tilde{A}}_{rij}) = 0.$$  \hspace{1cm} (45)

However, if we apply $D_0$ to (39) and use (28h), (28j), (28i) and (36) and (38) we derive $2\tilde{B}^{r} mkB_{(ij)r} + \tilde{B}^{r} mk\hat{A}_{rij} = 0$. By applying $D_0$ once more to this last expression and using the second in (28c) and (28d), (28j), (43) and (44) we also get

$$\tilde{B}^{r} mk(2\tilde{B}_{(ij)r} + \hat{\tilde{A}}_{rij}) = 0.$$  \hspace{1cm} (46)

In conclusion, comparing (45) and (46), we have that

$$\tilde{B}^{r} mk\tilde{B}_{(ij)r} = 0$$

so that $\tilde{B}$ satisfies all the hypotheses in Proposition 5.3 for each leaf of $\overline{M}$, and thus $\tilde{B} = 0$. By the identity (24), we also get $\tilde{A}_{(ijk)l} = 0$, i.e., $\tilde{A}_{ijk} = \tilde{A}_{(ijk)k}$. $D_0$-differentiating (42), using (40) and putting $\tilde{B} = 0$ in (28f), (28g) and (36), we get $\tilde{A}_{rkn}A^{r} ij = 0$. Contracting all the indices and using that $\tilde{A}$ is symmetric in the first two, we arrive at $0 = \tilde{A}_{rkn}\tilde{A}^{rnk} = \hat{\tilde{A}}_{rnk}\tilde{A}^{rnk}$, i.e., $\tilde{A} = 0$. $\square$

Therefore,

**Corollary 5.8.** $\tilde{A}$ is also a $D_0$- and $\nabla$-parallel section.
Remark 5.9. We simply remark that all the equations written in this section, be them 2nd-symmetry ones or their integrability conditions, are satisfied for any Brinkmann chart: if we perform a new decomposition of type (7), the aforementioned equations must be satisfied in the new Brinkmann chart too.

5.2. Transformation into two independent Lorentzian problems.

5.2.1. Reducibility of $\mathcal{M}$, $\overline{Ric}$ and $\tilde{A}$. For the following application of the generalized Eisenhart Theorem recall Definitions 4.9 and 4.26.

**Proposition 5.10.** Let $(M,g)$ be a 2nd-symmetric Brinkmann space. Then, any Brinkmann decomposition $\{u,v\}$ is spatially reducible

$$\mathcal{M} = \mathcal{M}^{(1)} \times \mathcal{M}^{(2)}, \quad \mathcal{g} = \mathcal{g}^{(1)} \oplus \mathcal{g}^{(2)}$$

where $\mathcal{M}^{(1)}$ is a $d$-dimensional flat foliation and $\mathcal{M}^{(2)}$ is a $d'$-dimensional locally symmetric foliation with $d, d' \geq 0$, $d + d' = n - 2$. Furthermore, $\mathcal{M}^{(2)}$ is itself the product of $s_2$ locally symmetric (non-Ricci flat) Einstein foliations:

$$\mathcal{M}^{(2)} = \mathcal{M}^{(2,1)} \times \ldots \times \mathcal{M}^{(2,s_2)}, \quad \mathcal{g}^{(2)} = \mathcal{g}^{(2,1)} \oplus \ldots \oplus \mathcal{g}^{(2,s_2)}$$

Proof. As $\overline{R}$ is $\nabla$- and $D_0$-parallel it follows that $D_0 \overline{Ric} = 0$ and $\nabla \overline{Ric} = 0$. Then, if the foliation is $u$-Einstein, the result is trivial (put $\mathcal{M} = \mathcal{M}^{(1)}$, $\mathcal{g} = \mathcal{g}^{(1)}$ for the Ricci-flat case; otherwise, from $D_0 \overline{Ric} = 0$ the foliation is Einstein and $\mathcal{M} = \mathcal{M}^{(2)}$, $\mathcal{g} = \mathcal{g}^{(2)}$). If it is not $u$-Einstein, apply Theorem 4.29 to $\overline{D} = \overline{Ric}$ and, if $\overline{Ric}$ happens to have a vanishing eigenvalue, choose $\mathcal{M}^{(1)}$ as the corresponding Ricci-flat part which, by virtue of Proposition 4.10(2), it is actually flat. $\square$

Therefore, under the conditions of Proposition 5.10 with $d, d' > 0$, the partition of the indices corresponds to the reducibility of $\mathcal{g}$ according to Definition 4.26, where $I_1$ yields the indices of the flat foliation and $I_2$ the indices of the locally symmetric one. Moreover, the associated curvature tensors satisfy $\overline{R}^{(1)} = 0$ and $\overline{R}^{(2)} \neq 0$, so that $\overline{R}^{(2)} = \overline{R}^{(2)}$ and $\nabla \overline{R}^{(2)} = 0$. Besides,

$$\overline{Ric} = \sum_{m=1}^{s_2} \mu_m \mathcal{g}^{(2,m)}, \quad \mu_m \in \mathbb{R} - \{0\}$$

and the flat coordinates $\{x^2, \ldots, x^{d+1}\}$ correspond to the zero eigenvalue $\mu_0 = 0$.

**Convention 5.11.** From now on:

1. When dealing with a proper 2nd-symmetric Brinkmann manifold and using Proposition 5.10, we will restrict ourselves to spatially reducible Brinkmann decompositions $\{u,v\}$ so that $\mathcal{M} = \mathcal{M}^{(1)} \times \mathcal{M}^{(2)}$ as in the proposition. This includes the limit case when $\mathcal{M}$ is Einstein, so that either $\mathcal{M} = \mathcal{M}^{(1)}$ ($d' = 0$) or $\mathcal{M} = \mathcal{M}^{(2)}$ ($d = 0$).

2. The indices $a, b, c, \ldots$ will run from 2 to $d + 1$ and the indices $a', b', c', \ldots$ will run from $d + 2$ to $n - 1$.

3. We will denote by $\mu^*$ any of the non-zero eigenvalues of $\overline{Ric}$.

Next, our aim is to prove that $\tilde{A}$ is reducible and it admits a similar decomposition to $\overline{Ric}$. Nevertheless, as a difference with $\overline{Ric}$, the non-trivial part of $\tilde{A}$ lies on $\mathcal{M}^{(1)}$.

**Proposition 5.12.** Choose any spatially-reducible Brinkmann decomposition of a proper 2nd-symmetric Brinkmann space. Then, $\tilde{A}$ is reducible as $\tilde{A}^{(1)} \oplus \tilde{A}^{(2)}$ with $\tilde{A}^{(2)} = 0$. In addition, there exists a Brinkmann chart $\{u,v,x^i\}$ such that $\mathcal{g}^{(1)} = \delta_{ik}dx^a dx^b$ and the matrix of components $(\tilde{A}^{(1)}_{ij})$ is constant and diagonal.
Proof. Applying $D_0$ to (41) and using (28l), (28j) and Proposition 5.7 together with the fact that $\overline{R}$ is $D_0$-parallel one derives

$$\overline{R}'_{ijk} \overline{A}_{rm} = 0$$

from where we get $\overline{R}'_{ij} \overline{A}_{rj} = 0$ for all $i, j$. However, $\overline{A} \neq 0$ by Corollary 5.2 so that, in the Brinkmann chart associated to the spatial reduction of Proposition 5.10, if we set $i \in I_2$, i.e., $i = b'$, we get that $\mu' \overline{A}_{b'} = 0$, that is, $\overline{A}_{b'} = 0$ for all $j$ and

$$\overline{A} = \overline{A}(u, x^i)_{bc} \overline{A}^b \overline{A}^c.$$ 

Now, from Corollary 5.8 we have $0 = \nabla_a \overline{A}_{bc} = \partial_a \overline{A}_{bc}$ (last equality because $\gamma_{ba'} = 0$), that is, $\overline{A}$ is reducible with $\overline{A}(2) = 0$ and satisfies the hypotheses of Theorem 4.29. Moreover, the coordinate vector fields $\{\partial_{a+2}, \ldots, \partial_{a-1}\}$ span a subspace $S$ of the eigenspace $S_0$ associated to the zero eigenvalue of $\overline{A}$. Therefore, one can follow the steps of the proof of Theorem 4.29, but working just on $S^\perp$, $(X(x^a) = 0$, for all $X \in S$) to obtain

- $\overline{g}^{(1)} = \overline{g}^{(1,1)} \oplus \ldots \oplus \overline{g}^{(1,s_1)}$.
- $\overline{A} = \sum_{m=1}^{s_1} \lambda_m \overline{g}^{(1,m)}$.

As $\overline{g}^{(1)}$ is a flat metric, applying Proposition 4.10(3) to each of the mutually orthogonal $\overline{g}^{(1,m)}$ a further change of coordinates in the flat block yields $\overline{g}^{(1)} = \delta_{ab} dx^a dx^b$ (for example, use the proof of that proposition on the restriction to the block of coordinates associated to each $\overline{A}(1,m)$).

In conclusion, in this Brinkmann chart the matrix of the tensor field $\overline{A}$ is a diagonal matrix of constants. □

Remark 5.13. Observe that the Brinkmann chart obtained in Proposition 5.12:

- (1) maintains the Brinkmann decomposition $\{u, v\}$ of Proposition 5.10. As the tensor fields $\overline{R}ic$ and $\overline{A}$ (Corollary 5.2) depend only on the decomposition, the conclusions obtained for them are independent on the remainder coordinates of the Brinkmann chart.
- (2) is such that $\overline{A}$ and $\overline{R}ic$ are orthogonal and occasionally they may vanish simultaneously for a subbundle on $\overline{M}(1)$.
- (3) has a flat metric $\overline{g}^{(1)} = \delta_{ab} dx^a dx^b$ so that $\nabla_a = \partial_a$. Moreover, $\nabla_a T^a_b = \partial_a (T^a_b)$.

5.2.2. The building blocks of the metric $g$. Until now, we have proven the reducibility of the metric $\overline{g}$ for any Brinkmann decomposition $\{u, v\}$. Our aim is to prove that, in fact, the 2nd-symmetry induces the existence of two simpler 2nd-symmetric Brinkmann spaces associated to the original Brinkmann space $(M, g)$.

Lemma 5.14. For a Brinkmann chart $\{u, v, x^i\}$ as in Proposition 5.12, the sections $B$ and $A$ in Definition 4.22 satisfy that:

$$B_{ba'b'} = 0; \quad B_{a'ab} = 0; \quad B_{a'b'a} = 0; \quad B_{aa'a'} = 0; \quad A_{aa'} = 0.$$

Proof. The commutation property (32) applied to the $\nabla_\cdot$ and $D_0$ parallel section $\overline{R}$ provides the following integrability condition

$$B_{ms} \overline{R}_{rijk} + B_{mi} \overline{R}_{srjk} + B_{mj} \overline{R}_{sirk} + B_{mk} \overline{R}_{sijr} = 0.$$

(i) Contracting here $i$ and $k$ and

- taking $m = a$, $s = b$, $j = a'$, one gets: $B_{ab} \overline{R}_{ra'} = B_{aba'} \mu' = 0$ that is, $B_{aba'} = 0$,
- taking $m = a'$, $s = b'$, $j = a$, one gets: $B_{a'a} \overline{R}_{ba'} = B_{a'a'b'} \mu' = 0$ that is, $B_{a'a'b'} = 0$.

(ii) Using now $B_{[ijk]} = 0$, and $B_{[ijk]} = 0$, (i) above and:

- taking $i = a'$, $j = b'$ and $k = a$, one has that $B_{aa'b'} = 0$,
- taking $i = a'$, $j = a$ and $k = b$, one has that $B_{a'ab} = 0$.

(iii) Contracting $i$ and $s$ in the equation (39) and taking $j = a$, $m = a'$, one derives $\overline{R}_{ra'} A^r_a = 2B_{(aa')} \mu A^r_a$. Using here (i) and (ii) above one arrives at $\mu A^r_a = 0 \Rightarrow A_{aa'} = 0$. □

Proposition 5.15. For a Brinkmann chart $\{u, v, x^i\}$ as in Proposition 5.12, the sections $t$ and $h$ are reducible.
Proof. We have to prove that
\[ t_{aa'} = 0; \quad t_{a'a} = 0; \quad \partial_a t_{ab} = 0; \quad \partial_{a'} t_{a'b} = 0; \quad \partial_a h_{a'} = 0; \quad \partial_{a'} h_a = 0 \]
The fact that \( R \) is \( D_0 \)-parallel can be rewritten using (22) as
\[ R_{ijkl} + t^{r}_{i} R_{rjkl} + t^{r}_{j} R_{rakl} + t^{r}_{k} R_{rijl} + t^{r}_{r} R_{ijkr} = 0. \]
Taking \( i = a \) one gets that \( t^{r}_{a} R_{rjkl} = 0 \), so that contracting \( j \) and \( l \) and taking \( k = a' \) one gets
\[ 0 = t_{a'a} = -t_{aa'}. \quad (47) \]
By (28m) and Lemma 5.14 it follows that \( 0 = B_{a'ba} = \nabla_a t_{ab} - \nabla_b t_{aa'} \), which implies by Remark 5.13(2) and (47) that \( \partial_a t_{ab} = 0 \). Anologously, \( 0 = B_{a'b'a'} = \nabla_a t_{a'b'} - \nabla_{a'} t_{aa'} \) implies \( \partial_{a'} t_{a'b'} = 0 \). Lemma 5.14 and (28a) yield \( 0 = A_{aa'} = t_{aa} - \nabla_a h_{aa} - D_{g} t_{aa'} \), so sustituting (47) and using (22) and Remark 5.13(2) again one obtains \( \partial_{a'} h_{aa'} = \partial_{aa'} h_{a} = 0. \)

Theorem 5.16. For a proper 2nd-symmetric Brinkmann space, there exists a spatially reducible Brinkmann decomposition \( \{ u', v' \} \) such the function \( H \) and the one-form section \( W \) in (3) are reducible in the associated Brinkmann chart \( \{ u', v', x^i \} \). This chart is related to that obtained in Proposition 5.12 by \( \{ u', v', x^b \} = \{ u, v + f(u, x^i), x^i \} \) for some function \( f \).

Proof. We must prove that there exists a Brinkmann chart \( \{ u', v', x^b \} \) such that
\[ H(u', x^b) = H^{(1)}(u', x^a) + H^{(2)}(u', x^{a'}), \]
\[ W_a = W_{a'}(u', x^b)(= W_{a}^{(1)}(u', x^b))\quad W_{a'} = W_{a'}(u', x^{a'})(= W_{a'}^{(2)}(u', x^{a'})). \]
Let \( \{ u, v, x^i \} \) be the Brinkmann chart obtained in Proposition 5.12, so that by Proposition 5.15 the functions \( h \) and \( \partial_t \) are reducible.

Simplification of \( W \): \( \partial_{a'} t_{ab} = 0 \) implies \( W_{a'b} - W_{b'a} \) depends only on the coordinates \( \{ u, x^a \} \), hence there exists a function \( f_1(u, x^i) \) such that \( W_{a'}(u, x^i) = f_{1,a}(u, x^i) + w_{a}(u, x^b) \). Analogously, \( \partial_{a'} t_{a'b} = 0 \) implies the existence of a function \( f_2(u, x^i) \) such that \( W_{a'}(u, x^i) = f_{2,a}(u, x^i) + w_{a'}(u, x^b) \). Then, (47) tells us that \( f_{1,a,a'} = f_{2,a,a'} \), so that \( f_{1}(u, x^i) - f_{2}(u, x^i) = F(u, x^a) + G(u, x^b) \) for some functions \( F \) and \( G \) which can be absorbed into \( w_{a}(u, x^b) \) and \( w_{a'}(u, x^b) \), respectively. Consequently, there exists a function \( f(u, x^i) \) such that
\[ W_{a}(u, x^b) = f_{a}(u, x^i) + w_{a}(u, x^b), \quad W_{a'}(u, x^b) = f_{a'}(u, x^i) + w_{a'}(u, x^b). \quad (48) \]

Simplification of \( H \): using (12), \( \partial_{a'} h_{aa} = \partial_{aa'} h_{a} = 0 \) provide \( H_{a}(u, x^i) = h_{a}(u, x^b) + W_{a}(u, x^i) \) and \( H_{a'}(u, x^i) = h_{a'}(u, x^b) + W_{a'}(u, x^i) \), which become after use of (48), \( H_{a}(u, x^i) = h_{a}(u, x^b) + \dot{f}_{a}(u, x^i) + \dot{w}_{a}(u, x^b) \) and \( H_{a'}(u, x^i) = h_{a'}(u, x^b) + \dot{f}_{a'}(u, x^i) + \dot{w}_{a'}(u, x^b) \), from where it is easy to deduce that
\[ H(u, x^i) = \dot{f}(u, x^i) + H^{(1)}(u, x^a) + H^{(2)}(u, x^{a'}) \]
for some functions \( H^{(1)}, H^{(2)} \).

By choosing now the new Brinkmann decomposition defined by \( v' = v + f(u, x^i) \), the thesis follows on using (8) and (9). \( \square \)

Remark 5.17. (1) What we have proven is that there exists a Brinkmann chart \( \{ u, v, x^i \} \) and a partition of the indices \( I_1 = \{ 2, \ldots, d + 1 \}, I_2 = \{ d + 2, \ldots, n - 1 \} \) for some \( d \in \{ 0, \ldots, n - 2 \} \) (recall Convention 5.11), such that \( g, H \) and \( W \) are simultaneously reducible in the sense given of Definition 4.26.

(2) Recall that if \( \{ u, v \} \) is spatially reducible, there exist two foliations \( M^{(1)}, g^{(1)} \) with associated leaves \( (M^{(1)}, g^{(1)}) \) such that \( g = g^{(1)} \otimes g^{(2)} \) (Remark 4.27). Observe then that the metric \( g \) can be written as
\[ g = -2du dv + (H^{(1)} + H^{(2)}) du + W^{(1)} + W^{(2)} + g^{(1)} \otimes g^{(2)}, \]
so that to any 2nd-symmetric Brinkmann space we can associate a pair of lower-dimensional Brinkmann spaces \( (M^{[m]}, g^{[m]}), m \in \{ 1, 2 \} \) by \( M^{[m]} = \mathbb{R}^2 \times M^{(m)} \) and
\[ g^{[m]} = -2du dv + H^{(m)} du + W^{(m)} + g^{(m)}. \]
Of course, such a pair of spaces may be non-unique. \((M^{[m]}, g^{[m]})\) are the building blocks of any proper 2nd-symmetric Lorentzian manifold, and they are extremely helpful in the resolution of the equations for 2nd-symmetry because they actually simplify to the equations corresponding to each of the two simpler Brinkmann spaces \((M^{[m]}, g^{[m]})\), \(m \in \{1, 2\}\) as the next proposition proves.

**Proposition 5.18.** The pair \((M^{[m]}, g^{[m]})\) of Brinkmann spaces associated to any 2nd-symmetric Brinkmann space \((M, g)\) according the previous Remark are, themselves, 2nd-symmetric.

**Proof.** Note first of all that \(\overline{R}, h\) and \(t\) are reducible, so that \(A, B\) and their \(\nabla\)- and \(D_\xi\)-derivations are reducible too, from where \(\hat{A}\) is also reducible. Therefore, for \(m \in \{1, 2\}\), the reduced sections \(\overline{R}^{(m)}, h^{(m)}, \ t^{(m)}, \ A^{(m)}, \ B^{(m)}\) correspond to the geometrical objects for the Brinkmann spaces \((M^{[m]}, g^{[m]})\) as defined in (11), (12), (13) and Definition 4.22 respectively. It is then straightforward to check that, in the Brinkmann chart of Theorem 5.16, the equations of 2nd-symmetry for \((\hat{M}, \hat{g})\) are exactly identical with the combination of the equations of 2nd-symmetry for the two building blocks \((M^{[m]}, g^{[m]})\). \(\square\)

**Remark 5.19.** In conclusion, applying Proposition 5.18 to the Brinkmann decompositions \(\{u', v'\}\) of Theorem 5.16, we can reorganize the equations of 2nd-symmetry in two simpler sets:

- The equations associated to the Brinkmann space \((M^{[1]}, g^{[1]})\) with coordinates \(\{u, v, x^a\}\), such that: \(\overline{g}^{(1)} = \delta_{ab} dx^a dx^b\), \(\overline{R}^{(1)} = 0\) (ergo \(\overline{\text{Ric}}^{(1)} = 0\)) and \(\overline{A}^{(1)} = \sum_{\lambda=1}^{n_1} \lambda \overline{g}^{(1, \lambda)}\) with some \(\lambda \neq 0\).
- Those associated to \((M^{[2]}, g^{[2]})\) with coordinates \(\{u, v, x^a\}\), such that: \(\overline{g}^{(2)} = \overline{g}^{(2,1)} \oplus \ldots \oplus \overline{g}^{(2,s_2)}\), \(\nabla \overline{R}^{(2,1)} = 0\) but \(\overline{R}^{(2,s_2)} \neq 0\), \(\overline{\text{Ric}}^{(2)} = \sum_{\lambda=1}^{s_2} \mu \overline{g}^{(2, \lambda)}\) with each \(\mu \neq 0\) and \(\overline{A}^{(2)} = 0\).

5.3. **Proof of the main Theorem 1.1.** As a consequence of Corollary 3.8, we can apply Lemma 3.10 and, consequently, only the local version of the result must be proven. We will start the computations in the Brinkmann chart \(\{u', v', x^a\}\) of Theorem 5.16 but dropping the primes for the sake of clarity in the notation.

By Remark 5.19, we consider first the equations for \((M^{[2]}, g^{[2]})\). Since \(\overline{A}^{(2)} = 0\), Corollary 5.2 informs us that this is a locally symmetric Lorentzian manifold with a parallel lightlike vector field. Therefore, Theorem 3.4 implies that it is locally isometric to the product of a symmetric (non-flat) Riemannian space with \(L^2 = (\mathbb{R}^2, -2du dv)\). In particular, up to a change of coordinates of type

\[
u' = u, \quad v' = v + F(u, x^a) ; \quad y^{a'} = y^a(u, x^b) \tag{49}\]

\((M^{[2]}, g^{[2]})\) has \(H^{(2)} = 0, W^{(2)} = 0\) and \(\overline{g}^{(2)} = 0\).

Consider now the 2nd-symmetry equations associated to \((M^{[1]}, g^{[1]})\) according to Remark 5.19. Using \(\overline{R}^{(1)} = 0\) in (39) we have that \(B_{(ab)c} B^c de = 0\), hence Proposition 5.3 leads us to

\[
B^{(1)} = 0. \tag{50}
\]

From (28m) follows that \(\nabla_t t_{ab} - \nabla_b t_{ac} = 0\), which together with (13) and Remark 5.13(2) provides \(\partial_t t_{ab} - \partial_b t_{ac} = \partial_a t_{eb} = 0\). We conclude that \(t^{(1)}\) is \(\nabla\)-parallel. We have two consequences of this fact:

- \(W_{ab} - W_{ba} = 2t_{ab}(u)\) depend only on \(u\). Consequently, \(W^{(1)}\) can be written as
  \[
  W_a(u, x^b) = f_a(u, x^b) + t_{ac}(u)x^c
  \]
  for some function \(f(u, x^b)\).
- Setting \(\hat{A} = 0\) in (28k) and by Lemma 5.14 and (50) we have that \(\partial_a A_{ab} = 0\), which from (28n) and Proposition 5.15 implies that \(\partial_a \partial_b h = 0\), that is, \(h^{(1)}\) takes the form
  \[
  h_a(u, x^b) = \Lambda_{ac}(u)x^c + B_a(u)
  \]
  for some functions \(\Lambda_{ac}\) and \(B_a\). From (12) we then have
  \[
  H_a(u, x^b) = \Lambda_{ac}(u)x^c + B_a(u) + \tilde{W}_a(u, x^b)
  = \Lambda_{ac}(u)x^c + B_a(u) + \tilde{f}_a(u, x^b) + \tilde{t}_{ac}(u)x^c.
  \]
Observe that the symmetry of $H_{a,b}$ implies that $\Lambda_{ab} + \dot{t}_{ab} = \Lambda_{ba} + \dot{t}_{ba} = \Lambda_{(ab)}$ where the antisymmetric character of $\dot{t}$ has been used in the last equality.

In conclusion, the metric for $M^{[1]}$ becomes

\[
g^{[1]} = -2du \left( dv + H^{(1)}(u, x^a)du + \dot{W}^{(1)} \right) + \delta_{ab}dx^a dx^b
\]

with

\[
W^{(1)} = W_a(u, x^b) \delta x^a = [f_a(u, x^b) + t_{ac}(u)x^c] \delta x^a;
\]

\[
H^{(1)}(u, x^a) = \dot{f}(u, x^a) + \frac{1}{2}\Lambda_{(bc)}(u)x^b x^c + B_c(u)x^c + C(u).
\]

Next, we use the following claim, to be proven later.

**Claim 5.20.** For $(M^{[1]}, g^{[1]})$, there exists a change of Brinkmann chart of type

\[
u' = u, \quad v' = v + \chi(u, x^a), \quad y^a = R^a_b(u)x^b + D^a(u),
\]

such that the metric becomes

\[
g^{[1]} = -2du' (dv' + H'(u', y^a)du') + \delta_{ab}dy^a dy^b.
\]

where $H'(u', y^a) = -\Lambda_{ab}(u')y^a y^b$, $\Lambda_{ab}$ being the components of the section $\Lambda$ in Definition 4.22 associated to the Brinkmann chart $\{u', v', y'^1\}$.

Combining the change of coordinates in this claim and the one given in (49), there exists a change of coordinates in the entire Brinkmann space $(M, g)$ of type

\[
u' = u, \quad \nu' = v + F(u, x^a) + \chi(u, x^a),
\]

\[
y'^a = y^a'(u, x^{b'}), \quad y^a = R^a_b(u)x^b + D^a(u)
\]

such that the metric of $(M, g)$ becomes

\[
g = -2du' (dv' + H'(u', y^a)du') + \delta_{ab}dy^a dy^b + \tilde{g}_{a'b'}(y'^c)dy'^a dy'^b
\]

where $H'(u', y^a) = -\Lambda_{ab}(u')y^a y^b$. To end the proof, note that $\tilde{A} = \tilde{A}$ and therefore Corollary 5.8 gives, via $D_0\tilde{A} = 0, \tilde{A}_{ab}(u) = 0$. Of course, we need $\tilde{A}_{ab}\tilde{a}(u) \neq 0$ for some $a_0, b_0$ in order for the manifold not to be locally symmetric, due to Corollary 5.2.

**Proof of the claim.** In order to find the required $\chi$, $D^a$ and $R^a_b$, put $H'(u', y^a) = -\Lambda_{ab}(u')y^a y^b$, substitute (52) in (53) and require that the obtained expression equals (51). Then, the following equations arise:

\[
\dot{\chi} = H^{(1)} - H' + \frac{1}{2}\delta_{ab}(\dot{R}^c_a x^c + \dot{D}^a)(\dot{R}^d_b x^d + \dot{D}^b),
\]

\[
\chi_{,a} = W_a + \frac{1}{2}\delta_{bc} (\dot{R}^d_b x^d + \dot{D}^b)\dot{R}^c_a + (\dot{R}^c_a x^d + \dot{D}^c) R^d_b),
\]

\[
\delta_{ab} R^a_b R^b_a = \delta_{cd}.
\]

Here, the known data are $t_{ab}(u), B_c(u)$ and $\Lambda_{(ab)}(u)$, while the unknowns are $R^a_b(u)$, $D^a(u)$ and $\chi(u, x^a)$. The integrability conditions (given by the cross derivatives) of the first two expressions yield

\[
t_{cd} = \frac{1}{2}\delta_{ab}(\dot{R}^c_a R^d_b - R^d_b \dot{R}^c_a),
\]

\[
\Lambda_{(cd)} = -2A_{bc} R^c_b R^d_a - \frac{1}{2}\delta_{ab}(\dot{R}^c_a \dot{R}^d_b + \dot{R}^b_a \dot{R}^c_a),
\]

\[
B_c = -2A_{bc} R^c_b D^c + \delta_{ab} D^a R^b_a.
\]

To prove that these equations have solutions, proceed as follows. Define $R^a_b(u)$ as a solution of the ODE

\[
\dot{R}^c_b = -\delta^{cd}(R^{-1})^d_a t_{ab}
\]

for the given $t_{ab}(u)$. Then, equation (55) is automatically satisfied. Concerning (54), note that, for such a solution, the derivative of $\delta_{cd} R^c_a R^d_a$ vanishes due to the anti-symmetry of $t_{ab}(u)$. Therefore,
by imposing as initial condition that $R^a_0(0)$ is any orthogonal matrix, the necessary condition (54) holds for all $u$ (that is, $R^a_0(u)$ is a curve of rotation matrices, which can be chosen by using $d(d-1)/2$ free parameters codified in $R^a_0(0)$). Once $R^a_0(u)$ is determined, equation (56) fixes $A_{ab}(u)$ and, using this, the existence of $D^b(u)$ (which will depend on two new constant vectors, that is, on $2d$ new parameters) and, a posteriori, of $\chi$, is ensured by (57) plus, again, standard results in differential equations. \hfill $\Box$

**Remark 5.21.** As $(M_1,g_1)$ in Theorem 1.1 is a proper Cahen-Wallach space of order 2, let $O^j_i$ be the matrix of rotations that diagonalizes the symmetric matrix $A^{(1)}$ in (6). Then, performing a change of coordinates of type $u' = u - u_0$ and $y' = O^j_i x^j$ we can write $A^{(1)}$ as a diagonal matrix and cancel one of the elements of the symmetric matrix $A^{(0)}$. In conclusion, the number of essential parameters of a proper 2nd-symmetric Lorentzian manifold with fixed $K = -\partial_v$ is given by $d - 1 + d(d + 1)/2$ (observe that $A^{(1)}$ is a $d \times d$ square matrix). If $K$ is not fixed, by a change of coordinates of type $u' = \varepsilon u - u_0$ with $v' = v/\varepsilon$ for some constant $\varepsilon \neq 0$ and $y' = O^j_i x^j$, the same simplifications can be achieved and furthermore one of the non-zero eigenvalues of $A^{(1)}$ can be set to $\pm 1$, in which case the number of essential parameters is one less.

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