Integrable geodesic flows of non-holonomic metrics

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§1. Introduction

In the present article we show how to produce new examples of integrable dynamical systems of differential geometry origin.

This is based on a construction of a canonical Hamiltonian structure for the geodesic flows of Carnot–Carathéodory metrics ([7, 17]) via the Pontryagin maximum principle. This Hamiltonian structure is achieved by introducing Lagrange multipliers bundles being the phase spaces of these Hamiltonian flows. These bundles are diffeomorphic to cotangent bundles but have another meaning. A transference to this phase space is given by a generalised Legendre transform.

We analyse the geodesic flow of the left-invariant Carnot–Carathéodory metric on the three-dimensional Heisenberg group as a super-integrable Hamiltonian system (Theorem 1). Moreover, its super-integrability explains the foliation of its phase space into one- and two-dimensional invariant submanifolds as it was pointed out in [17].

The geodesic flows of left-invariant Carnot–Carathéodory metrics on Lie groups reduce to equations on Lie algebras in the same manner as the geodesic flows of left-invariant Riemannian metrics reduce to the Euler equations on Lie algebras ([1]) (Theorem 2). Most of these flows are integrable. Moreover, this reduction to equations on Lie algebras gives a Hamiltonian explanation for the description of such flows on three-dimensional Lie groups given in [17] by using Euler–Lagrange equations (the will to explain this in terms of integrability was the starting point for the present work).

In comparison with the case of Riemannian geodesic flows there is another class of invariant flows, corresponding to left-invariant metrics and right-invariant distributions. In §6 we examine the simplest example of such flow on \( \mathbb{H}^3 \) and, in particular, show that this flow is integrable (Theorem 3).

In §7 we consider an observation related to a Hamiltonian structure for the equations for the motion of a heavy rigid body with a fixed point.

For completeness of explanation we discuss in §8 another approach to defining “straight lines” in non-holonomic geometry which does not arrive at Hamiltonian systems but some “straight line” flows have important mechanical meaning (for instance, the Chaplygin top ([6])).

We also discuss some problems concerning dynamics and, in particular, integrability of these systems (see §9. Concluding remarks).

The present article is dedicated to D. V. Anosov on his 60th birthday.

§2. The geodesic flows of Carnot–Carathéodory metrics

A. Carnot–Carathéodory metrics.

Let \( M^n \) be a smooth manifold of dimension \( n \).
A family $\mathcal{F}$ of $k$-dimensional subspaces of the tangent spaces to $M^n$ is called a $k$-dimensional smooth distribution if $\mathcal{F}_x$ is a smooth section of the Grassmann bundle on $M^n$.

In the sequel we suppose that distributions are smooth.

Let $V_\mathcal{F}$ be the linear space spanned by vector fields tangent to $\mathcal{F}$. Denote by $A_\mathcal{F}$ the algebra generated by fields from $V_\mathcal{F}$ via a commutation. A distribution is called non-holonomic if $A_\mathcal{F}$ does not coincide with $V_\mathcal{F}$ as a linear space. Otherwise, a distribution is called holonomic and, by the Frobenius theorem, locally looks like the family of spaces tangent to the leaves of a foliation. It is easily seen that near a generic point the distribution corresponding to $A_\mathcal{F}$ is holonomic. The distribution $\mathcal{F}$ is called completely non-holonomic if the algebra $A_\mathcal{F}$ coincides with the whole algebra of vector fields on $M^n$. In [13] it is said that such distributions satisfy the bracket generating hypothesis.

In the sequel we assume that distributions are completely non-holonomic. This assumption is not very strong because otherwise we may restrict geodesic flows to the leaves of the foliation and consider the restricted distributions as completely non-holonomic. Really, we use this for defining a Carnot–Carathéodory metric as a correct intrinsic metric only.

Thus we assume now that $M^n$ is endowed with a completely non-holonomic distribution $\mathcal{F}$. We also assume that $M^n$ is a complete Riemannian manifold with the metric $\tilde{g}_{ij}$.

A piecewise smooth curve in $M^n$ is called admissible if it is tangent to $\mathcal{F}$. A Carnot–Carathéodory metric $d_{\text{CC}}(x, y)$ is defined as follows. Denote by $\Omega_{x,y}$ the set of admissible curves with ends at the points $x$ and $y$ in $M^n$. Then

$$d_{\text{CC}}(x, y) = \inf_{\gamma \in \Omega_{x,y}} \text{length}(\gamma)$$

with the lengths of curves taken with respect to the metric $\tilde{g}_{ij}$.

By the Chow–Rashevskii theorem, any pair of points in a complete Riemannian manifold endowed with a completely non-holonomic distribution is connected by an admissible curve and, thus, (1) correctly defines an inner metric on $M^n$.

Carnot–Carathéodory metrics are simplest examples of non-holonomic metrics which are defined (1) for different choices of $\Omega_{x,y}$ corresponding to non-integrable constraints. In the case of Carnot–Carathéodory metrics these constraints are linear in velocities.

**B. The geodesic flow of a Carnot–Carathéodory metric.**

By definition, the lengths of admissible curves depend only on the restrictions of the metric $\tilde{g}_{ij}$ onto $\mathcal{F}_x$. Denote these restricted forms by $Q_x$. This family of bi-linear forms on $\mathcal{F}$ enables us to define the canonical mapping

$$g(x) : T^*M^n \to \mathcal{F}_x \subset TM^n$$

(2)

taking for $g(x)\xi \in \mathcal{F}_x$ a vector determined uniquely by the condition

$$Q_x(Y, g(x)\xi) = \langle Y, \xi \rangle \quad \text{for every } Y \in \mathcal{F}_x.$$  

(3)

The symmetric tensor $g^{ij}$ is called a Carnot–Carathéodory metric tensor. It generalises a Riemannian metric tensor into which it degenerates when $\mathcal{F}_x = T_xM^n$.

A curve $\gamma$ in $T^*M^n$ is called a cotangent lift of a curve $\gamma$ in $M^n$ if

$$g(\gamma(t))\xi(t) = \frac{d\gamma(t)}{dt}$$

(4)
where $\tilde{\gamma}(t) = (\gamma(t), \xi(t)), \xi(t) \in T_{\gamma(t)}M^n$. By (3), in the terms of cotangent lifts the length of an admissible curve $\gamma(t)$ in $M^n$ is expressed by

$$L(\gamma) = \int \sqrt{\langle g(\gamma(t))\xi(t), \xi(t) \rangle} dt$$

(5)

and the energy of $\gamma$ equals

$$E(\gamma) = \frac{1}{2} \int \langle g(\gamma(t))\xi(t), \xi(t) \rangle dt.$$ 

(6)

An admissible curve is called a geodesic of the Carnot–Carathéodory metric $g^{ij}$ if locally it is an energy-minimising curve.

The geodesics of the Carnot–Carathéodory metric $g^{ij}$ are described by the Euler–Lagrange equations for a Lagrange function

$$L(x, \dot{x}) = \frac{1}{2} \tilde{g}_{pq} \dot{x}^p \dot{x}^q + \sum_{\alpha=1}^{n-k} \mu_\alpha \langle \dot{x}, \omega^{(\alpha)} \rangle$$

(7)

where $\omega^{(1)}, \ldots, \omega^{(n-k)}$ is a basis for $\mathcal{F}_x^\perp$. Here $\mathcal{F}_x^\perp$ is the annihilator of $\mathcal{F}_x$, i.e., the subset of $T_x^*M^n$ formed by covectors $\xi$ such that $\langle \xi, v \rangle = 0$ for every $v \in \mathcal{F}_x$.

These equations are written as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} =$$

$$\left\{ \frac{\partial \tilde{g}_{pq}}{\partial \dot{x}^p} \dot{x}^q + \tilde{g}_{pq} \dot{x}^q + \sum_\alpha \left( \mu_\alpha \omega_i^{(\alpha)} + \mu_\alpha \frac{\partial \omega_i^{(\alpha)}}{\partial x^p} \dot{x}^p \right) \right\} -$$

$$\left\{ \frac{1}{2} \frac{\partial \tilde{g}_{pq}}{\partial x^i} \dot{x}^p \dot{x}^q + \sum_\alpha \mu_\alpha \dot{x}^p \frac{\partial \omega_i^{(\alpha)}}{\partial x^p} \right\} = 0,$$

$$\frac{\partial L}{\partial \mu_\alpha} = \langle \dot{x}, \omega^{(\alpha)} \rangle = 0.$$ 

(8)

In fact, although the Riemann metric tensor $\tilde{g}^{ij}$ enters these equations, the geodesic flow is determined by the restriction of this tensor onto the distribution, the Carnot–Carathéodory metric tensor $g^{ij}$, only.

§3. The Pontryagin maximum principle and a Hamiltonian structure for the geodesic flows of a Carnot–Carathéodory metric. A generalised Legendre transform

A. The Pontryagin maximum principle and the geodesics of Carnot–Carathéodory metrics.

In [13] Strichartz had shown that the geodesic flow of a Carnot-Carathéodory metric is described by equations derived from the Pontryagin maximum principle.

Explain his idea in brief. First, roughly quote the Pontryagin maximum principle in a weak form sufficient for our study referring to [3] (Theorem 5.1) for the absolutely rigorous statement.

**Pontryagin Theorem.** Consider the minimum problem for a functional

$$I[x(t), u(t)] = \int_{t_1}^{t_2} f^0(x, u) dt$$

(10)
in the class of admissible functions \((x(t), u(t))\) such that
\[ \dot{x}^k = f^k(x, u) \] (11)
and some constraints \(x \in A, u \in U\) hold.

Introduce the functions
\[ \tilde{H}(x, u, \tilde{\lambda}) = \lambda_0 f^0(x, u) + \lambda_1 f^1(x, u) + \ldots + \lambda_n f^n(x, u) \] (12)
and
\[ M(x, \tilde{\lambda}) = \inf_{u \in U} \tilde{H}(x, u, \tilde{\lambda}) \] (13)
where \(\tilde{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_n)\).

Let \((x(t), u(t))\) be a solution to this minimum problem. Then there exists an absolutely continuous vector function \(\tilde{\lambda}(t)\) such that
(i) \(\lambda_0 = \text{const}, \lambda_0 \geq 0,\) and
\[ \frac{d\lambda_i}{dt} = -\frac{\partial \tilde{H}(x(t), u(t), \tilde{\lambda}(t))}{\partial x^i}; \] (14)
(ii) \(\tilde{H}(x(t), u(t), \tilde{\lambda}(t)) = M(x(t), \tilde{\lambda}(t))\) for every \(t \in [t_1, t_2]\);
(iii) \(M(x(t), \tilde{\lambda}(t)) = \text{const}.\)

Now it suffices only to notice that in the case of the geodesic flows of Carnot–Carathéodory metrics we have the minimum problem on the set of the cotangent lifts of admissible curves on \(M^n\) where we consider the variables \(\xi_i\) as the control functions \(u_i\). In this event
\[ f^0(x, u) = \frac{1}{2} g^{ij} u_i u_j, \] (15)
\[ f^i(x, u) = g^{ij} u_j, \] (16)
and
\[ \tilde{H}(x, u, \tilde{\lambda}) = \frac{\lambda_0}{2} g^{ij} u_i u_j + g^{ij} \lambda_i u_j. \] (17)

Thus, we have
\[ M(x, \tilde{\lambda}) = \begin{cases} -\frac{1}{2\lambda_0} g^{ij} \lambda_i \lambda_j, & \text{for } \lambda_0 \neq 0 \\ 0, & \text{for } \lambda_0 = 0 \text{ and } g^{ij} \lambda_j \equiv 0 \\ -\infty, & \text{otherwise} \end{cases} \] (18)

B. A Hamiltonian structure for the geodesic flows of Carnot–Carathéodory metrics.

Let consider the bundle \(\Lambda M^n \rightarrow M^n\) diffeomorphic to the cotangent bundle \(T^* M^n\) via the diffeomorphism \((x, p) \leftrightarrow (x, \lambda)\) but having another sense. We call it the Lagrange multipliers bundle on \(M^n\). As well as the cotangent bundle this bundle is endowed with the natural symplectic structure generated by the form
\[ \Omega = \sum_{i=1}^{n} d\lambda_i \wedge dx^i. \] (19)

Consider now on this symplectic manifold a Hamiltonian flow with a Hamiltonian function
\[ H(x, \lambda) = -\frac{1}{2} g^{ij} \lambda_i \lambda_j. \] (20)
Definition. A geodesic of a Carnot–Carathéodory metric is called normal if $\lambda_0 \neq 0$.

Otherwise, if $g^{ij}\lambda_j = 0$ then $M(x, \lambda) = 0$ and the Pontryagin theorem gives nothing.

Theorem HS (on a Hamiltonian structure for normal geodesic flow). The projections of trajectories of the Hamiltonian flow on $\Lambda M^n$ with the Hamiltonian function (20) are exactly the naturally-parametrised normal geodesics of the Carnot–Carathéodory metric $g^{ij}$. Moreover, $|\dot{x}|^2 = -2H(x, \lambda)$.

Proof of Theorem HS.

From (18), and the homogeneity of $\tilde{H}(x, u, \tilde{\lambda})$ we infer that after changing of the parameter on an extremal to a multiple one, if that needs, we obtain an extremal $\tilde{\gamma}$ with $\lambda_0 = 1$.

By (13), we derive

$$\frac{\partial \tilde{H}(x, u, \tilde{\lambda})}{\partial u_i} = 0$$

which is equivalent to

$$g^{ij}u_j = -g^{ij}\lambda_j.$$  \hspace{1cm} (21)

Now it follows from (21) that

$$g^{ij}u_iu_j = g^{ij}\lambda_i\lambda_j.$$  \hspace{1cm} (22)

Remind that, by (3), we have

$$g^{ij}u_j = \dot{x}^i.$$  \hspace{1cm} (23)

and together with (22) this implies that

$$|\dot{x}|^2 = \tilde{g}_{ij}\dot{x}^i\dot{x}^j = g^{ij}u_iu_j = |u|^2.$$  \hspace{1cm} (24)

Now it follows from (22) and the statement (iii) of the Pontryagin theorem that the extremal $\tilde{\gamma}$ is naturally-parametrised.

Since (3) and (24), we have that

$$\dot{x}^i = \frac{\partial H(x, \lambda)}{\partial \lambda_i}$$  \hspace{1cm} (25)

and we may regard (15) as

$$\dot{\lambda}_i = -\frac{\partial H(x, \lambda)}{\partial x^i}.$$  \hspace{1cm} (26)

It is easily seen that the equations (25–26) form a system of Hamilton equations on $\Lambda M^n$ for the Hamiltonian function (20).

Thus we prove that the normal geodesics are the projections of trajectories of this Hamiltonian flow.

For the converse we refer to [8], where a comprehensive examination of analytical properties of the energy functional for Carnot–Carathéodory metrics is given.

This completes the proof of the theorem.

The variables $\lambda_1, \ldots, \lambda_n$ have no physical meanings in comparison with the momenta $u_1, \ldots, u_n$. The correspondence between them and velocities is given by (25) and is one-to-one in the case when the form $g^{ij}$ is non-degenerate, i.e., for Riemannian metrics only. In this case the existence of a Hamiltonian formalism for the geodesic flows of Riemannian metrics follows from both the Legendre transform.
and the Pontryagin theorem. Thus the transference to the new variables \((x, \lambda)\) has to be regarded as a generalized Legendre transform.

Theorem HS is contained implicitly in \([13]\). However, this fact did not attract attention of specialists on integrable systems because mostly the problem of regularity of geodesics and that of exponential maps were in study \((8, 16)\). From the analytical point of view this also coincides with the introducing a Hamiltonian in the "vakonomic mechanics" \([2]\) where the momenta are introduced by the implicit theorem procedure and the Hamiltonian system is also regarded as a system on a cotangent bundle. However, the difference between \(T^* M^n\) and \(\Lambda M^n\) is essential for applications to mechanics and physics because, at least, the variables \(\lambda_1, \ldots, \lambda_n\) are not observable.

Although for a long time it had being assumed that all geodesics are normal, recently Montgomery had showed that abnormal geodesics exist \([10]\). Geodesics found by him do not admit end-point \(F\)-tangent perturbations and, thus, are solutions to any variation problem on the space of admissible curves.

But if the space \(\mathcal{V}_F + [\mathcal{V}_F, \mathcal{V}_F]\) coincides with the whole algebra of vector fields on \(M^n\) then every geodesic is normal \([13]\).

In the sequel speaking about geodesic flows we shall mean by them normal geodesic flows.

§4. Integrability of the geodesic flow of the left-invariant Carnot–Carathéodory metric on the three-dimensional Heisenberg group

We mean by a left-invariant Carnot–Carathéodory metric a left-invariant metric restricted onto a left-invariant distribution. It is known that such metric on the three-dimensional Heisenberg group is unique up to isomorphism \([17]\).

This flow is described in \([3, 8, 17, 19]\), somewhere with its generalizations. However, it is nowhere regarded as a completely integrable Hamiltonian system.

The three-dimensional Heisenberg group \(H^3\) is the group of matrices

\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\]  

(27)

with respect to a multiplication, where \(x, y, z \in \mathbb{R}\). Its Lie algebra \(\mathcal{L}\) is spanned by the following elements

\[
e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]  

(28)

Denote by \(\mathcal{L}_0\) the linear subspace spanned by \(e_1\) and \(e_2\).

The group \(H^3\) acts on itself by the left and right translations:

\[
L_g : H^3 \to H^3 : \quad L_g(h) = gh, \\
R_g : H^3 \to H^3 : \quad R_g(h) = hg.
\]

The left-invariant distribution generated by \(\mathcal{L}_0\) consists of the 2-planes \(\mathcal{F}_x = L_g \mathcal{L}_0\).

Since the following commutation relations

\[ [e_1, e_2] = e_3, \quad [e_1, e_3] = [e_2, e_3] = 0 \]
hold, this distribution is completely non-holonomic.

Consider the left-invariant metric on $H^3$ which at the unit of the group takes the form

$$(e_i, e_j) = \delta_{ij}. \quad (29)$$

Identify $H^3$ with $R^3$ by the diffeomorphism which assigns to the matrix (27) the point in $R^3$ with the coordinates $(x, y, z)$. Thus we identify the tangent space at every point of $H^3$ with the vector space generated by the matrices (28). In this event the left translations act on $TH^3$ as follows:

$$L_{g^*}(e_1) = e_1, \quad L_{g^*}(e_2) = e_2 + xe_3, \quad L_{g^*}(e_3) = e_3 \quad (30)$$

where $g$ is the element of $H^3$ given by the matrix (27). It follows from (30) that in these coordinates the mapping $L_{g^*} : T_eH^3 \rightarrow T_gH^3$ is written as

$$L_{g^*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix} \quad (31)$$

The left-invariant Riemannian metric on $H^3$ in these coordinates takes the form

$$[\tilde{g}^{ij}(x, y, z)] = (L_{g^*})^{**} \cdot [g^{ij}(0, 0, 0)] \cdot (L_{g^*})^* \quad (32)$$

and, since we put

$$g^{ij}(0, 0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (33)$$

we derive

$$[g^{ij}(x, y, z)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & x^2 \end{pmatrix}. \quad (34)$$

The left-invariant Riemannian metric on $H^3$ in these coordinates takes the form

$$[g_{ij}(x, y, z)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1 + x^2) & -x \\ 0 & -x & 1 \end{pmatrix}. \quad (35)$$

Now, it follows from Theorem HS that the geodesic flow of the left-invariant Carnot–Carathéodory metric corresponding to the Riemannian metric (35) and the distribution $L_{g^*}L_0$ is Hamiltonian on $\Lambda H^3$ with the following Hamiltonian function

$$H(q, \lambda) = \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + x^2\lambda_3^2 + 2x\lambda_2\lambda_3) \quad (36)$$

where $q = (x, y, z)$.

The Hamilton equations for (36) look simply:

$$\dot{x} = \frac{\partial H}{\partial \lambda_1} = \lambda_1, \quad \dot{y} = \frac{\partial H}{\partial \lambda_2} = \lambda_2 + x\lambda_3, \quad \dot{z} = \frac{\partial H}{\partial \lambda_3} = x\lambda_2 + x^2\lambda_3, \quad \lambda_1 = -\frac{\partial H}{\partial x} = -x\lambda_3^2 - \lambda_2\lambda_3, \quad (37)$$
\[ \lambda_2 = -\frac{\partial H}{\partial y} = 0, \quad \lambda_3 = -\frac{\partial H}{\partial z} = 0, \]

and we immediately infer from (37) that this Hamiltonian system is completely integrable because it has three first integrals

\[ I_1 = H, \quad I_2 = \lambda_2, \quad I_3 = \lambda_3 \]  

(38)

which are in involution and functionally independent almost everywhere. In particular, these integrals are functionally independent in the domain \( H \neq 0 \).

Moreover, since, by (37),

\[ \lambda_3 = \text{const}, \quad \dot{z} = x \dot{y} \]  

(39)

along trajectories of the flow, we may restrict this flow onto the level set \( \{ \lambda_3 = C = \text{const} \} \) and project this restriction of the flow onto the plane \((x, y)\). Denote this system by \( P_C \) and notice that it is defined on the 4-dimensional symplectic manifold \( M_C \) diffeomorphic to the cotangent bundle to the 2-plane with the coordinates \((x, y)\) but with another Poisson structure.

Introduce the new variables on \( M_C \)

\[ u = \lambda_1, \quad v = \lambda_2 + x\lambda_3. \]  

(40)

Then, by (19) and (40), the Poisson structure on \( M_C \) induced from \( \Lambda H^3 \) is written as

\[ \{x, u\} = \{y, v\} = 1, \quad \{u, v\} = -C \quad (= -\lambda_3), \]

\[ \{x, v\} = \{y, u\} = \{x, y\} = 0 \]  

(41)

in the coordinates \((x, y, u, v)\). The flow \( P_C \) is also a Hamiltonian system with the following Hamiltonian function

\[ H(x, y, u, v) = \frac{u^2 + v^2}{2}. \]  

(42)

It is easily seen that the flow \((11)\) describes nothing else but the motion of a charged particle on the Euclidean plane \((x, y)\) in the constant magnetic field \( F = -\lambda_3 dx \wedge dy \) \((11)\). This system has three first integrals which are functionally independent almost everywhere

\[ \tilde{I}_1 = H, \quad \tilde{I}_2 = \lambda_3 x - v, \quad \tilde{I}_3 = \lambda_3 y + u, \]  

(43)

and, thus, we conclude that these systems are super-integrable, i.e. have more first integrals than \( \dim M_C/2 \). This is also true for the main flow.

Hence, we conclude

**Theorem 1.** 1) The geodesic flow of the left-invariant Carnot–Carathéodory metric, on \( H^3 \), corresponding to the left-invariant Riemannian metric \((23)\) and the left-invariant distribution \( L_\alpha L_0 \) is a Hamiltonian system on \( \Lambda H^3 \) which is integrable in the Liouville sense via the first integrals \((38)\) which are in involution and functionally independent almost everywhere.

Moreover, since this flow possesses the fourth first integral \( I_4 = \lambda_3 y + \lambda_1 \) functionally independent on \( (23) \), it is super-integrable and the subset \( \{ \lambda_3 \neq 0 \} \) of its phase space is foliated into 2-dimensional invariant Liouville tori \( S^1 \times \mathbb{R} \).

2) Let restrict this geodesic flow onto the level set \( \{ \lambda_3 = C \} \) and project this restriction of the flow onto the plane \((x, y)\). The flow \( P_C \) constructed by this procedure
is equivalent to the Hamiltonian system describing the motion of a charged particle on the Euclidean \( \mathbb{R}^2 \)-plane \((x, y)\) in the constant magnetic field \( F = -\lambda_3 dx \wedge dy \). This flow is super-integrable and for \( \lambda_3 \neq 0 \) its phase space \( \mathcal{M}_C \) is foliated into closed trajectories.

§ 5. Left-invariant Carnot–Carathéodory metrics on Lie groups and their geodesic flows

Let \( \mathcal{G} \) be a Lie group, let \( G \) be its Lie algebra, and let \( G_0 \) be a subspace, of \( G \), generating \( G \). Take a scalar product \( \mathcal{J} \) in \( G \) and decompose \( G \) into a direct sum of \( G_0 \) and its orthogonal complement:

\[
G = G_0 \oplus G_0^\perp.
\]

Take another bi-linear from \( \mathcal{J}_0 \) on \( G \) uniquely defined by the following conditions

\[
\mathcal{J}_0(x, y) = 0 \quad \text{for every} \quad x \in G_0^\perp, y \in G,
\]

\[
\mathcal{J}_0(x, y) = \mathcal{J}(x, y) \quad \text{for every} \quad x, y \in G_0.
\]

Now, take the left-invariant distribution \( \mathcal{L}_{g_0} G_0 \) on \( G \) and the left-invariant Riemannian metric generated by \( \mathcal{J} \). To this pair there is uniquely assigned a left-invariant Carnot–Carathéodory metric.

We will not plunge into the details but only mention that for they same reasoning as the geodesic flow of a left-invariant Riemannian metric on a Lie group reduces to equations on its Lie co-algebra, Euler equations ([1], the following theorem holds.

**Theorem 2.** The geodesic flow of a left-invariant Carnot–Carathéodory metric reduces to the following equations on \( G^* \)

\[
\dot{M} = ad_{\omega^*}^* M
\]

where \( \omega = L_{g^{-1}} \dot{g} \in G \) and \( M = \mathcal{J}_0 \omega \). Here \( \mathcal{J}_0 \) is regarded as an operator \( \mathcal{J}_0 : G \rightarrow G^* \) acting as \( \mathcal{J}_0(x, y) = \langle \mathcal{J}_0 x, y \rangle \).

If there exists an invariant non-degenerate bi-linear form on \( G \) we may identify \( G \) and \( G^* \) and the equations (44) take the form

\[
\dot{M} = [\omega, M], \quad M = \mathcal{J}_0 \omega.
\]

Since these flows possess commutation representations ([13]), they give a lot of new examples of integrable Hamiltonian systems and the well-known methods of integrating Euler equations on Lie algebras are immediately generalised for them.

Consider the simplest example. Let \( \mathcal{G} = SO(3) \), let \( e_1, e_2, e_3 \) be its generators satisfying the commutation relations

\[
[e_i, e_j] = \varepsilon_{ijk} e_k,
\]

let \( \mathcal{J} = \text{diag}(1, 1, 1) \), and let \( G_0 \) be spanned by \( e_1 \) and \( e_2 \). In this case, the equations ([13]) has two first integrals \( \langle \mathcal{J} x, x \rangle \) and \( \langle \mathcal{J}_0 x, x \rangle \) and, thus, are completely integrable.

We would like to notice that passing from the Lagrange equations ([8]) to the Hamiltonian equations ([14]) simplifies the research of left-invariant Carnot–Carathéodory geodesic flows and explains the integrable behaviour of such flows on three-dimensional Lie algebras given in ([17]).
§6. The geodesic flow of the Carnot-Carathéodory metric on $\mathcal{H}^3$
corresponding to a left-invariant metric and a right-invariant
distribution

Since a Carnot–Carathéodory metric is defined by a pair of objects (a metric
and a distribution, there are other classes of invariant Carnot–Carathéodory metrics
corresponding to metrics and distributions invariant with respect to different actions
of a Lie group $G$.

Examine, for instance, the geodesic flow of the Carnot–Carathéodory metric
on $\mathcal{H}^3$ which corresponds to the left-invariant metric (35) and the right-invariant
distribution $R_g^*\mathcal{L}_0$ and show that it is integrable in the Liouville sense.

By (3) and (35), we have

$g^{11} = (1 + x^2)V, \quad g^{12} = g^{21} = xyV,$

$g^{22} = (1 + y^2)V, \quad g^{13} = g^{31} = y(1 + x^2)V,$

$g^{23} = g^{32} = xy^2V, \quad g^{33} = y^2(1 + x^2)V, \quad V = \frac{1}{1 + x^2 + y^2}.$

This system has two obvious first integrals: $I_1 = H$ and $I_2 = \lambda_3$. As in §4,
restrict this flow onto the level set $\{\lambda_3 = C = \text{const}\}$ and successively project
this restriction of the flow onto the plane $(x, y)$. Thus we obtain a Hamiltonian system
$\mathcal{R}_C$ on the 4-dimensional symplectic manifold $\mathcal{M}_C$. Introduce the new variables

$u = \lambda_1 + y\lambda_3, \quad v = \lambda_2.$

Then the Poisson structure on $\mathcal{M}_C$ is given by

$\{x, u\} = \{y, v\} = 1, \quad \{u, v\} = C = \lambda_3,$

$\{x, v\} = \{y, u\} = \{x, y\} = 0.$

The Hamiltonian functions are written as

$H(x, y, u, v) = \frac{1}{2(1 + x^2 + y^2)}((1 + x^2)u^2 + 2xyuv + (1 + y^2)v^2).$

By the same reasoning as in the proof of Theorem 2 we conclude that this flow
is equivalent to a Hamiltonian system describing the motion of a charged particle
on the 2-plane with the Riemannian metric

$(1 + y^2)dx^2 - 2xydxdy + (1 + x^2)dy^2$

in the constant magnetic field $F = \lambda_3 dx \wedge dy$.

In the polar coordinates $(r, \varphi)$, where $x = r \cos \varphi$ and $y = r \sin \varphi$, the metric (49)
is written as

$dr^2 + (r^2 + r^4)d\varphi^2$

and we infer that the flow $\mathcal{R}_C$ is Hamiltonian,

$\frac{df}{dt} = \{f, H\},$

with the Hamiltonian function

$H(r, \varphi, p_r, p_\varphi) = \frac{1}{2}(p_r^2 + \frac{p_\varphi^2}{r^2 + r^4}).$
and the following Poisson structure on $\mathcal{M}_C$
\begin{align}
\{r, p_r\} &= \{\varphi, p_{\varphi}\} = 1, \quad \{p_r, p_{\varphi}\} = C, \\
\{r, p_{\varphi}\} &= \{\varphi, p_r\} = \{r, \varphi\} = 0.
\end{align}

This flow is defined on the four-dimensional symplectic manifold $\mathcal{M}_C$ and has two functionally independent first integrals $\tilde{I}_1 = H$ and $\tilde{I}_2 = p_{\varphi} + Cr^2/2$. It is clear that these functions are also first integrals of the main geodesic flow. In the initial coordinates the integral $\tilde{I}_2$ takes the form
\[ \tilde{I}_2 = C x^2 + y^2 + xv - yu. \]

We conclude

**Theorem 3.** 1) The geodesic flow, the Carnot–Carathéodory metric corresponding to the left-invariant Riemannian metric (33) and the right-invariant distribution $R_g, L_0$, is a Hamiltonian system on $\Lambda H^3$ with the following Hamiltonian function
\begin{align}
H(q, \lambda) &= \frac{1}{2(1 + x^2 + y^2)}((1 + x^2)\lambda_1^2 + (1 + y^2)\lambda_2^2 + \\
y^2(1 + x^2)\lambda_3^2 + 2xy\lambda_1\lambda_2 + 2y(1 + x^2)\lambda_1\lambda_3 + 2xy^2\lambda_2\lambda_3)
\end{align}
where $q = (x, y, z)$;

2) This flow possesses the first integrals
\[ I_1 = H, \quad I_2 = \lambda_3, \quad I_3 = \lambda_3 \frac{x^2 - y^2}{2} + x\lambda_2 - y\lambda_1 \]
which are involutive and functionally independent almost everywhere. Hence, this flow is integrable in the Liouville sense;

2) Let restrict this flow onto the level set $\{\lambda_3 = \text{const}\}$ and project the restriction of the flow onto the plane $(x, y)$. The flow $\mathcal{R}_C$ constructed by this procedure is equivalent to a Hamiltonian system describing the motion of a charged particle on the 2-plane with the Riemannian metric (44) in the constant magnetic field $\lambda_3 dx \wedge dy$.

§7. The equations for the motion of a heavy rigid body with a fixed point

First, remind that the Lie algebra $e(3)$ of the group of motions of the three-dimensional Euclidean space, $E(3)$, is spanned by the elements $e_1, e_2, e_3, f_1, f_2$, and $f_3$ meeting the following commutation relations
\begin{align}
[e_i, e_j] &= \varepsilon_{ijk}e_k, \quad [e_i, f_j] = \varepsilon_{ijk}f_k, \quad [f_i, f_j] = 0.
\end{align}

Denote by $m_i$ and $\gamma_j$ the adjoint basis in the co-algebra $e^*(3)$. The relations (54) determine the Lie–Poisson structure on the space of functions on $e(3)$ as follows
\begin{align}
\{m_i, m_j\} &= \varepsilon_{ijk}m_k, \quad \{m_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0.
\end{align}

As in the case of the geodesic flows of left-invariant metrics on Lie groups, any Lagrangian system corresponding to a left-invariant metric and a “left-invariant” potential field on the group $E(3)$ reduces to a Hamiltonian system
\[ \frac{df}{dt} = \{f, H\} \]
on the algebra $e(3)$ with a Hamiltonian

$$H(m, \gamma) = \frac{1}{2} a^{ij} m_i m_j + \frac{1}{2} b^{ij} (m_i \gamma_j + m_j \gamma_i) + \frac{1}{2} c^{ij} \gamma_i \gamma_j + V(m, \gamma)$$

(56)

where the matrices $a^{ij}, b^{ij},$ and $c^{ij}$ are symmetric and $V$ is a linear function in $m$ and $\gamma$. Here we call a potential field $V(q)$ “left-invariant” if its gradient is left-invariant. Thus, denoting by $x^i$ and $y^j$ the local coordinates corresponding to $e_i$ and $f_j$, we conclude that

$$V(m, \gamma) = \sum_{i=1}^{3} \left( \frac{\partial V(0)}{\partial x^i} m_i + \frac{\partial V(0)}{\partial y^j} \gamma_j \right)$$

and a “left-invariant” potential field is determined uniquely by its gradient at the unit of the group.

The Kirhhoff equations for the free motion of a rigid body in a liquid correspond to $V \equiv 0$ ([11]). In this case the configuration space is the whole group $E(3)$.

By Theorem HS, in the Hamiltonian formalism constraints contribute to Hamiltonian equations via a Hamiltonian function. Moreover, Theorem HS also holds for holonomic constraints.

Hence, starting from the problem of the free motion of a heavy body in a potential field with the configuration space $E(3)$ we pose holonomic constraints by fixing a point of the body. In this case the configuration space is homeomorphic to $SO(3)$ but the Euler equations (see §5) are still written for the algebra $e(3)$ and correspond to the Hamiltonian function

$$H(m, \gamma) = \frac{(I m, m)}{2} + (r, \gamma).$$

(57)

Thus, we obtain the well-known Hamiltonian formalism for this problem.

First, it was derived in physical terms. Here $I$ is the inertia tensor, $m$ is the angular momentum, $\gamma$ is the vector in the direction of gravity, and $r$ is the centre of mass. All coordinates are taken with respect to the orthogonal frame attached to the body with the fixed point the centre of coordinates.

§8. Another definition of “straight lines” and problems of mechanics

There is another way to define “straight lines” in non-holonomic geometry. We discuss only the case of constraints linear in velocities.

Roughly speaking, geodesics of Carnot–Carathéodory metric are solutions of the following variation problem. Let $L(x, \dot{x}) = \tilde{g}_{ij} \dot{x}^i \dot{x}^j dt$ be an energy functional on a suitable space of curves (periodic or with fixed end-points) in manifold $M^n$ and $\tilde{g}_{ij}$ be a Riemannian metric tensor. A geodesic $\gamma(t)$ of the Carnot–Carathéodory metric corresponding to $\tilde{g}_{ij}$ and a distribution $\mathcal{F}$ are local extremals of this functional with respect to the set of variations of the form $\gamma(t, u)$ where $\gamma(t, o) = \gamma(t)$ and

$$\frac{\partial \gamma}{\partial t}(t, u) \in \mathcal{F}_{\gamma(t, u)}.$$  

(58)

Of course, all variations belong to the space of curves in study.

But the condition (58) can be changed to another one:

$$\frac{\partial \gamma}{\partial u}(t, 0) \in \mathcal{F}_{\gamma(t, o)}.$$  

(59)
In this case the equations for “straight lines” are written as

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = \sum_\alpha \mu_\alpha \omega^{(\alpha)}_i \]  

(60)

where

\[ \langle \omega^{(\alpha)}, \dot{x} \rangle = \omega^{(\alpha)}_i \dot{x}^i = 0 \]  

(61)

are the constraints and the coefficients \( \mu_\alpha \) are derived from the condition that the relations (61) hold. One can easily see the difference of these equations from the equations (60).

Notice also that these “straight lines” are determined by the Lagrange function defined on the whole tangent space \( T^*M \) but not only on \( \mathcal{F} \).

The most famous example of such system is the Chaplygin top, the dynamically-asymmetric ball rolling on the horizontal plane and with the center of mass coinciding with its geometric centre ([6]).

This system was integrated by Chaplygin by no means of the Liouville integrability theorem. Its algebraic origin was clarified by Veselov and Veselova who regarded it such a flow of “straight lines” on \( E(3) \), the Lie group of the motions of the three-dimensional Euclidean space, endowed with a left-invariant metric and a right-invariant non-holonomic distribution ([18]). They also generalise this system on arbitrary Lie groups and succeed in generalising the Chaplygin integration method for three-dimensional groups.

§ 9. Concluding remarks

1) The procedure of constructing a Hamiltonian structure is generalised for non-holonomic systems with Lagrange functions of the form

\[ L(x, \dot{x}) = \frac{1}{2} |\dot{x}|^2 + U(x) \]

in the usual manner. We already mentioned this in §7.

2) We did not find yet considerably new examples of manifolds which admit the integrable geodesic flow of a Carnot–Carathéodory metric and have no Riemannian metrics with integrable geodesic flows. Nevertheless, we would like to mention that all methods of finding topological obstructions to integrability of the geodesic flows of Riemannian metrics ([12], see also [13]) fail in the case of Carnot–Carathéodory metrics. The reason for this is clear. These methods use a compactness of the level set of a Hamiltonian function which can not be compact for Carnot–Carathéodory metrics. Thus one can expect that the class of manifolds admitting integrable Carnot–Carathéodory geodesic flows is wider than the class of manifolds admitting integrable Riemannian geodesic flows.

The lack of compactness of the level sets of a Hamiltonian function also obstructs us to define entropy characteristics of such flows in the usual manner.

However, in §5 we give an example of an integrable flow on \( SO(3) \) with compact level sets of the first integral \( (Jx, x) \) which is not a Hamiltonian function. This situation is typical for the geodesic flows of left-invariant metrics on compact Lie groups but it is not generic as one can see from the example given in §6.

3) Consider the simplest example of degeneration of integrable Riemannian geodesic flows into an integrable Carnot–Carathéodory geodesic flow.
Take the Lie group $SO(3)$ and denote by $e_1, e_2,$ and $e_3$ the generators of its Lie algebra $so(3)$ with the following commutation relations

$$[e_i, e_j] = \varepsilon_{ijk} e_k.$$ Denote by $\tilde{\mathcal{L}}_0$ the subspace of $so(3)$ spanned by $e_1$ and $e_2$. Consider the family of left-invariant metrics generated by the metrics on $so(3)$ of the form

$$G_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D \end{pmatrix}.$$ The geodesic flows of these metrics are integrable. Tending $D$ to infinity, $D \to \infty$, we arrive at the geodesic flow of the Carnot–Caratheodory metric on $SO(3)$ corresponding to the Riemannian metric $G_1$ and the left-invariant distribution $L_{g^*}\tilde{\mathcal{L}}_0$ (\cite{7,9}). It is showed in §5 that this flow is integrable.

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