Some algebraic Rhotrices using a method of spanning

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Abstract  Rhotrix has been extensively studied in recent time due to their applicability, as theoretic tools in other field such as engineering, physicist, and other related fields. The aim of this research is to identify the relationship between Rhotrices and some other fields in algebra, and classes of Rhotrices as abstract structures, such as Rings, fields, groups, and then further extended to generate some results using the idea of spanning.

1.0 INTRODUCTION

Mathematical arrays that are between two dimensional vectors and $2 \times 2$ dimensional matrices where suggested by [1]. As extension to this idea, [2] introduced an object which lies between $2 \times 2$ matrices and $3 \times 3$ dimensional matrices called a Rhotrix.

The definition of Rhotrix was later generalized by [3] to include any finite dimension $2z + 1$. Thus, by a rhotrix $R$ we mean a rhomboidal array of dimension $n$ can be written as:

$$ R = \begin{pmatrix} n & o & p & q \\ o & p & q & r \end{pmatrix} : n, o, p, q, r \in \mathbb{R} \quad (1) $$

The dimension of rhotrices can be increase in size by retaining all the properties are discussed below. It should be noted that rhotrix is always of odd dimension, hence a rhotrix with dimension $n \left( n, \text{odd} \right)$ where $n \in 2 \mathbb{Z} + 1$. The dimensions can be extended in various ways, but in this paper the researcher is interested in applying $3 \times 3$ Rhotrix in spanning to test the linear combination between scalar and vectors Rhotrix.

Heart of Rhotrix: The element of the perpendicular intersection of the diagonal of a Rhotrix $h(R)$, could be discussed below

$$ R = \begin{pmatrix} n \\ o & h(R) & q \\ q & r \end{pmatrix} $$

It is noteworthy to mention that, the development of all the further abstract structures are based on multiplication $(\phi)$ [2]. Also the definitions of relevant abstract structures [4] were applied in the
development of the following new abstract structures of Rhotrices with respect to the binary operations of addition (+) and multiplication (⊙).

After the introduction the researcher is going to review some literature work, methodology, results and discussions, conclusions and recommendations.

**Definitions:**

1.1 Addition of two Rhotrices:
Two Rhotrices \( R_1 \) and \( R_2 \) can be added (+) ie \( R_1 + R_2 \)
\( \forall R_1 (n \times m) \text{ and } R_2 (j \times k) \text{ iff } n = j, \ m = k \)

\[
R_1 = \begin{pmatrix}
  n & h(p) & q \\
  r & \end{pmatrix}
\quad \text{and} \quad
R_2 = \begin{pmatrix}
  c & h(e) & f \\
  g & \end{pmatrix}
, \text{ then}
\]

\[
R_1 + R_2 = \begin{pmatrix}
  n + c & h(p) + h(e) & q + f \\
  r + g & \end{pmatrix}
\tag{2}
\]

1.2 Transpose of a Rhotrix
For any Rhotrix
\( R = (n \times n) \)

Transpoe of Rhotrix can be defined as \( R^T \)
\( i.e. \) interchanging rows with column and vice versa.

\[
R = \begin{pmatrix}
  n & o & p & q \\
  r & \end{pmatrix}
, \text{ then } R^T = \begin{pmatrix}
  o & n & p & r \\
  q & \end{pmatrix}
\tag{3}
\]

1.3 Multiplication of two Rhotrices:
Multiplication of Rhotrix can be defined by “⊙” as follows:

\[
R_1 \circ R_2 \text{ i.e. if } R_1 = \begin{pmatrix}
  n & o & h(p) & q \\
  r & \end{pmatrix}
\text{ and } R_2 = \begin{pmatrix}
  c & d & h(e) & f \\
  g & \end{pmatrix}
, \text{ then}
\]

\[
R_1 \circ R_2 = \begin{pmatrix}
  nh(e) + ch(p) & oh(e) + dh(p) & h(p)h(e) & qh(e) + fh(p) \\
  rh(e) + gh(p) & \end{pmatrix}
\tag{4}
\]
1.4. Identity element of rhotrix

Given rhotrix $I = G$ and rhotrix $2 = I$, then $(G \circ I) = G$ then there is identity element i.e.

\[
\begin{bmatrix}
  o & h(G) & q \\
  r & & \\
\end{bmatrix} \circ \begin{bmatrix}
  c & & \\
  d & h(I) & f \\
  g & & \\
\end{bmatrix} = \begin{bmatrix}
  o & h(G) & q \\
  r & & \\
\end{bmatrix}
\]

\[
\begin{align*}
oh(I) + dh(G) &= o \\
h(G)h(I) &= h(G) \\
qh(I) + fh(G) &= q \\
rh(I) + gh(G) &= r
\end{align*}
\]

It follows that $c = d = g = f = 0$, $h(I) = 1$

\[
I = \begin{bmatrix}
  0 \\
  1 \\
  0 \\
\end{bmatrix}
\]  \tag{5}

1.5 Zero rhotrix is given by

\[
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
\end{bmatrix}
\]  \tag{6}

1.6 Spanning

Spanning Sets:

- For a set of vectors $S = \{V_1, V_2, \ldots, V_r\}$, the subspace $\text{span} \ (S) = \{\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_r V_r\}$ generated by forming all linear combinations of vectors from $S$ is called the space spanned by $S$.
- If $V$ is a vector space such that, $V = \text{span} \ (S)$ we say $S$ is a spanning set for $V$. In other words, $S$ spans $V$ whenever each vector in $V$ is a linear combination of vectors from $S$. Carl (2000) [5].

The aim of this research is to identify the relationship between Rhotrices and some other fields in algebra, and classes of Rhotrices as abstract structures, such as Rings, fields, groups, and then further extended to generate some results using the idea of spanning.

The objectives of this research can be adumbrated as below:

1- To testify the properties of matrices using the idea of Rhotrices,
2- To overview some theorems and their proofs as the abstract structure of Rhotrix.
3- To consider the basis $R$ which spanned by the vector Rhotrix and to determine the application of Rhotrices.
2.0 REVIEW OF RELATED WORK

The review of some publications related to this topic:

2.1. [6] made a remark on the classification of Rhotrices as abstract structures of groups, semi-groups, Monoid sand Boolean algebra, which has an interest to stimulate ideas on how to imagine the new forms of mathematical rhomboid arrays and their classifications as abstract structure, and his purpose was to outline the development of abstract structures, ring, field, integral domain (ID), Principal Idea Domain (PID), and unique Factorization Domain (UFD) of Rhotrices as a remarks to the earlier works. Muhammed also applied the definitions of relevant abstract structures [2] in the development of the following new abstract structures of Rhotrices with respect to the binary operations of addition (+) and multiplication (⊙) respectively [6].

2.2. [7] made his contribution towards conversion of a Rhotrix to a coupled matrix. The method of converting a Rhotrix to a special form of matrix termed a 'couple matrix' was introduced. The special matrix can be used in solving various problems involving \( n \times n \) and \( (n-1)\times(n-1) \) matrices simultaneously.

It is clear that to convert any Rhotrix to matrix, there by transposing any Rhotrix or matrix is equivalent to rotating its columns through 90° in an anti-clockwise direction. In the same way, if columns of a Rhotrix are related through 45°, will get a special form of matrix with missing values. For \( R_3 \) rhotrix for instance,

\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    e_{11} & e_{12} & e_{13} \\
    a_{21} & a_{22} & a_{23} \\
    e_{21} & e_{22} & e_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

Indicates a rotation through 45° in an anticlockwise direction and may be termed as ‘half transpose. The special matrix above is like coupling a \( 3\times3 \) matrix with a \( 2\times2 \) matrix that is why is called a ‘coupled matrix’ [7].

2.3. [8] enrols on the Rhotrix on the construction of involuntary Rhotrices, which he defines involuntary matrix as a matrix that is its own inverse. He also extends the involution to Rhotrices and presents their properties, and provide the method of constructing involuntary Rhotrices. Usaini also defined involuntary Rhotrix as a Rhotrix that is its own inverse. That is Rhotrix \( R \) is involution if and only if \( R^2 = I \).

3.0 ABSTRACT STRUCTURE OF RHOTRIX

In this section, the researcher apply the definitions of relevant abstract structures [4] in the development of the new abstract structures of rhotrices with respect to the binary operations of addition (+) and multiplication (⊙) given by equations (2) and (4) respectively.

Ring of rhotrices: Let \( R^* = \langle R, +, \circ \rangle \) be an abstract structure consisting of the set \( R \) of all real rhotrices of the same dimension as given by equation (1), together with the operations of addition (+) and multiplication (⊙) then \( R \) is a commutative ring of rhotrices.

The unity element of the ring \( R^* \) is the identity rhotrix \( I \), given by equation (5). The zero element of ring \( R^* \) is the zero rhotrix 0 given by equation (6). The units’ elements in the ring \( R^* \) are the nonzero heart rhotrices (that is the invertible rhotrices) given by the set:
\[ \bigcup_1 = \left\{ \begin{pmatrix} n & o & p & q \\ r \end{pmatrix} : n, o, p, q, r \in \mathbb{R} \text{ and } p \neq 0 \right\} \] .............(7)

**Field of rhotrice:** Let \( \bigcup_1 \) be as given by equation (7)

\[ F^* = \left\langle \bigcup_1 \cup \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 \end{pmatrix}, +, \circ \right\} \right\rangle \]

Let the above be an abstract structure consisting of some real rhotrices of the same dimension, together with the operations of addition (+) and multiplication (\( \circ \)) then \( F^* \) is a Field of real rhotrices.

### 3.1 TECHNIQUES AND METHODS OF SOLVING THE RHOTRICES

This chapter introduces the systematic study of methods employed in solving the abstract structures based on Rhotrices, and some other analysis. It consists of Rings, Fields, Groups, Semi-groups and so on based on Rhotrices and considered to be abstract structures of Rhotrices.

The paper wish to state some structures, theorems and define some abstract structures based on Rhotrices. Some propositions and theorems are stated with their proofs

**Definition 3.1.** If \( f(z) = A_0 + A_1 z + \ldots + A_{m-1} z^{m-1} + A_m z^m = 0 \)

Then \( f(z) \) is called the polynomial of Rhotrices.

**Theorem 3.2.** Let \( R \) be a Rhotrix, and \( x \) be a number, then \( (R[x],+) \) is a commutative group.

**Proof:** The additive identity in the zero polynomial and the additive inverse of \( F = x^i \) is \( -x^i = -f \)

**Theorem 3.3.** Prove that \( (R[x],\ast) \) is a semi group.

**Proof:** This follows from the fact that \( R \) is also a semi group under multiplication of Rhotrices.

**Theorem 3.4.** Prove that \( (R[x],+,*\ast) \) is a ring.

**Proof:** This follows from the definition of polynomials of Rhotrices (i.e. 3.1), and the theorem of communicative group and the distributivity of \( R \). the multiplicative inverse of the polynomials of \( f \) is \( g \) such that \( fg = 1 \).

**Theorem 3.5.** \( (R[x],+) \) is a commutative ring.

**Proof:** This follows from the commutativity of \( R \).

**Theorem 3.6.** \( (R[x],+) \) is not an integral domain.

**Proof:** Let \( f = \), and \( g = \) such that \( f \) may not be equal to \( g \neq 0R \).

Also let \( c \neq 0R \).

But \( cf = cg = 0R \) and \( f \neq g \)

**Theorem 3.7.** \( (R[x],+) \) is a commutative group.

**Proof:** The additive identity is the zero of \( R \) and the additive inverse of \( R[F] \) is \( -R[F] \), i.e.

\[ R[F] = is -R[F] = \]
Theorem 3.8. \((R[X], +, \cdot)\) is a ring.

**Proof:** This also follows from the fact that \(R\) is distributive. Hence the multiplicative inverse of \(R[F]\) is \(R[F]^{-1}\), that is, \(R[F] = h(B) = h(C)/h(A)\), \(R[F_i] = \)

Theorem 3.9. The group of all invertible Rhotrices is isomorphic to the group of all \(2 \times 2\) non-singular matrices.

**Proof:** Let \((G, \cdot)\) and \((G, \cdot)\) be the group of invertible Rhotrices, and the group of \(2x2\) non-singular real matrices respectively. We define a group homomorphism: \(GG'\) as follows: \(\varphi = h(R)\) is a homomorphism, since: \(\varphi(RQ) = \varphi = h(R)h(Q)\)

Which is also 1-1, since \(Ker \varphi=\) which is the identity.

Definition 3.10. Let \(AB = C\) be a linear system. Where \(A, B\) and \(C\) are the base Rhotrices which can be written as: \(AB = C\) Which is equivalent to: \(= = \)

Proposition 3.11. Let \(A, B\) and \(C\) be Rhotrices over \(R\). then the system \(AB = C\) has a unique solution if and only if \(h(A)0\) and \(h(C)\)

**Proof:** Suppose \(h(A)0\) and \(h(C)\), it follows that \(h(A)0\) and \(h(C)\) if and only if \(h(B) = h(C)/h(A)\), and \(b_i = c_j h(A) - a_j h(C)/h(A)\), \(i = 1,2,3,4\)

4.0 RESULT

The paper considers the basis \(R\) which spanned by the vector Rhotrix and determine the application of Rhotrices using spanning as shown below:

**Spanning of Rhotrix 4.1.** Consider the basis of \(R\) which spanned by the following vectors:

\[
V = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, W = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, X = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, Y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, Z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\]

Thus, these Rhotrices can be written as a linear combination, that are linearly independent. Hence the set \(S = \{V, W, X, Y, Z\}\) forms a basis for \(R\), therefore, any Rhotrix \(R\) can be written as linear combination of the Rhotrix in \(S\) written as:

\[
R = nW + oX + h(R)V + qY + rZ
\]

Hence the equation (i) can be proved, where by checking that \(V \cdot V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\)

Hence the equation (i) can be proved, where by checking that

\[
V \cdot W = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

Similarly \((V \cdot X) = X,(V \cdot Y) = Y\) and \((V \cdot Z) = Z\) In general, \(\forall V \in R, V \cdot R(R \in R) = R\) Iff \(V \in h(R)\)
The summary of the products for the vectors which spanned on rhotrix R is given below:

| \( \circ \) | V | W | X | Y | Z |
|---|---|---|---|---|---|
| V | V | W | X | Y | Z |
| W | W | 0 | 0 | 0 | 0 |
| X | X | 0 | 0 | 0 | 0 |
| Y | Y | 0 | 0 | 0 | 0 |
| Z | Z | 0 | 0 | 0 | 0 |

Hence, the above Rhotrix vector spanned.

5.0 CONCLUSION:
Rhotrices, as it is in the new mathematical concept which was established by some scholars, it could be of a great importance in nowadays contemporary topics. However, Rhotrices like matrices can be used to solve so many problems regarding to the abstract fields and other related problems.

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