ON THE NONLINEAR WAVE EQUATION WITH TIME PERIODIC POTENTIAL

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Abstract. It is known that for some time periodic potentials \( q(t, x) \geq 0 \) having compact support with respect to \( x \) some solutions of the Cauchy problem for the wave equation \( \partial_t^2 u - \Delta_x u + q(t, x)u = 0 \) have exponentially increasing energy as \( t \to \infty \). We show that if one adds a nonlinear defocusing interaction \( |u|^r u, 2 \leq r < 4 \), then the solution of the nonlinear wave equation exists for all \( t \in \mathbb{R} \) and its energy is polynomially bounded as \( t \to \infty \) for every choice of \( q \). Moreover, we prove that the zero solution of the nonlinear wave equation is instable if the corresponding linear equation has the property mentioned above.

1. Introduction

Our goal in this paper is to show that a defocusing nonlinear interaction may improve, in a certain sense, the long time properties of the solutions of the wave equation with a time periodic potential.

Consider the Cauchy problem for the following potential perturbation of the classical wave equation in the Euclidean space \( \mathbb{R}^3 \)

\[
\partial_t^2 u - \Delta_x u + q(t, x)u = 0, \quad u(0, x) = f_1(x), \quad \partial_t u(0, x) = f_2(x),
\]

(1.1)

where \( 0 \leq q(t, x) \in C^\infty(\mathbb{R} \times \mathbb{R}^3) \) is periodic in time \( t \) with period \( T > 0 \) and has a compact support with respect to \( x \) included in \( \{ x \in \mathbb{R}^3 : |x| \leq \rho \} \), for some positive \( \rho \). It is easy to show that the Cauchy problem (1.1) is globally well-posed in \( \mathcal{H} = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \). The analysis of the long time behavior of the solution of (1.1) may be quite intricate (see e.g. [6, 1]). A slight adaptation of the arguments presented in [1] leads the following result.

Theorem 1. There exist \( q \) and \( (f_1, f_2) \in \mathcal{H} \) such that the solution of (1.1) satisfies :

\[
\exists C > 0, \exists \alpha > 0 \quad \text{such that} \quad \forall t \geq 0, \quad \| u(t, \cdot) \|_{H^1(\mathbb{R}^3)} \geq C e^{\alpha t}.
\]

(1.2)

The above result has been established in [1] for the Cauchy problem with initial data in \( H = H_D(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \). In fact we show that the propagator of (1.1)

\[
V(T, 0) : \mathcal{H} \ni (f_1(x), f_2(x)) \mapsto (u(T, x), u_t(T, x)) \in \mathcal{H}
\]

has an eigenvalue \( y, |y| > 1 \) which implies (1.2).

Our purpose is to show that adding a nonlinear perturbation to (1.1) forbids the existence of solutions satisfying (1.2). Consider therefore the following Cauchy problem

\[
\partial_t^2 u - \Delta_x u + q(t, x)u + |u|^r u = 0, \quad u(0, x) = f_1(x), \quad \partial_t u(0, x) = f_2(x),
\]

(1.3)

where \( 2 \leq r < 4 \). We have the following statement.

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Theorem 2. For any choice of $q$ the Cauchy problem (1.3) is globally well-posed in $\mathcal{H}$. Moreover, for every $(f_1, f_2) \in \mathcal{H}$ there exists a constant $C > 0$ such that for every $t \in \mathbb{R}$, the solution of (1.3) satisfies the polynomial bound

\[
\|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^3)} + \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq 2 \left( X(0) \frac{r^2}{C t} + C |t| \right)^{\frac{r+2}{r+\tau}},
\]

\[
\|u(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq \|f_1\|_{L^2} + 2|t| \left( X(0) \frac{r^2}{C t} + C |t| \right)^{\frac{r+2}{r+\tau}},
\]

where

\[
X(t) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla_x u|^2 + \frac{1}{2} q |u|^2 + \frac{1}{r+2} |u|^{r+2} \right) dx
\]

and $C > 0$ depends only on $q$ and $r$.

By global well-posedness we mean the existence, the uniqueness and the continuous dependence with respect to the data. The proof of Theorem 2 is based on the equality

\[
X'(t) = \frac{1}{2} \text{Re} \int_{\mathbb{R}^3} (\partial_t q)|u|^2 dx
\]

and the estimate

\[
|X'(t)| \leq CX^{1-\frac{\tau}{r+\tau}}(t).
\]

It is classical to expect that the result of Theorem 1 implies the instability of the zero solution of (1.3). More precisely, we have the following instability result.

Theorem 3. With $q$ as in Theorem 1 the following holds true. There is $\eta > 0$ such that for every $\delta > 0$ there exists $(f_1, f_2) \in \mathcal{H}$, $\|(f_1, f_2)\|_{\mathcal{H}} < \delta$ and there exists $n = n(\delta) > 0$ such that the solution of (1.3) satisfies $\|(u(nT, \cdot), \partial_t u(nT, \cdot))\|_{\mathcal{H}} > \eta$.

We are not aware of any nontrivial choice of $(f_1, f_2) \in \mathcal{H}$ such that the solution $u(t, x)$ of (1.3) and $u_t(t, x)$ remain uniformly bounded in $\mathcal{H}$ for all $t \geq 0$. The paper is organized as follows. In the next section, we prove Theorem 1. The third section is devoted to the proof of Theorem 2. First we obtain a local existence and uniqueness result on intervals $[0, t^*]$ with $t^* = c(1 + \|(f_1, f_2)\|_{\mathcal{H}})^{-\gamma}$ with constants $c > 0$ and $\gamma > 0$ independent on $f$. Next we establish (1.4) for solutions

\[
u(t, x) \in C([0, A], H^2(\mathbb{R}^3)) \cap C^1([0, A], H^1(\mathbb{R}^3)) \cap L_t^{\frac{2r+2}{r+\tau}}([0, A], L_x^{2r+2}(\mathbb{R}^3))
\]

and finally, by a local approximation in small intervals we justify (1.4) for every fixed $A > 0$ and $0 \leq t \leq A$. In the fourth section, we prove Theorem 3 passing to a system

\[
w_{n+1} = \mathcal{F}(w_n), \quad n \geq 0,
\]

where $\mathcal{F} = \mathcal{U}(0, T)$ is the propagator of the nonlinear equation. In the fifth section we discuss the generalizations concerning the nonlinear equations

\[
\partial_t^2 u - \Delta_x u + |u|^r u + \sum_{j=0}^{r-1} q_j(t, x)|u|^j u = 0, \quad r = 2, 3
\]

with time-periodic functions $q_j(t + T_j, x) = q_j(t, x) \geq 0$, $j = 0, 1, r-1$ having compact support with respect to $x$. 

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2. Proof of Theorem 1

2.1. The linear wave equation with time periodic potential. Let \( H_D(\mathbb{R}^3) \) be the closure of \( C_0^\infty(\mathbb{R}^3) \) with respect to the norm \( \| f \|_{H_D} = \| \nabla_x f \|_{L^2(\mathbb{R}^3)} \). Define the (energy) space
\[
H = H_D(\mathbb{R}^3) \times L^2(\mathbb{R}^3)
\]
with norm
\[
\| f \|_0 = (\| f_1 \|_{H_D}^2 + \| f_2 \|_{L^2}^2)^{1/2}, \quad f = (f_1, f_2).
\]
Let \( u(t, x; s) \) be the solution of the Cauchy problem
\[
\partial_t^2 u - \Delta_x u + q(t, x) u = 0, \quad u(s, x) = f_1(x), \quad \partial_t u(s, x) = f_2(x)
\]
with \( f = (f_1, f_2) \in H \). Therefore the operator
\[
H \ni f \mapsto U(t, s)f = (u(t, x; s), \partial_t u(t, x; s)) \in H
\]
is called the propagator (monodromy operator) of (2.1) and here exist \( C > 0 \) and \( \alpha \geq 0 \) so that
\[
\| U(t, s)f \|_0 \leq Ce^{\alpha|t-s|}\| f \|_0.
\]
Let \( U_0(t-s)f = (u_0(t, x; s), \partial_t u_0(t, x; s)) \), where \( u_0 \) solves \( \partial_t^2 u_0 - \Delta_x u_0 = 0 \) with initial data \( f \) for \( t = s \). Then we have
\[
U(t, s)f - U_0(t-s)f = -\int_s^t U_0(t-\tau)Q(\tau)U(\tau, s)f\,d\tau,
\]
where
\[
U_0(t) = \begin{pmatrix} \cos(t\sqrt{-\Delta}) & \sin(t\sqrt{-\Delta}) \\ -\sqrt{-\Delta}\sin(t\sqrt{-\Delta}) & \cos(t\sqrt{-\Delta}) \end{pmatrix}, \quad Q(t) = \begin{pmatrix} 0 & 0 \\ q(t, x) & 0 \end{pmatrix}.
\]
Using the relation (2.3) and the compact support of \( q \) allows us to obtain the estimate
\[
\| U(t, s)f - U_0(t-s)f \|_{H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} \leq C\|U(t, s)f\|_0.
\]
Moreover the support property of \( q \) also yields
\[
\operatorname{supp}_x (U(t, s)f - U_0(t-s)f) \subset \{ |x| \leq \rho + |t-s| \}.
\]
Consequently \( U(t, s) \) is a compact perturbation of the unitary operator \( U_0(t-s) \).

Now consider the space \( \mathcal{H} = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \subset H \) with norm
\[
\| f \|_1 = (\| f_1 \|_{H^1}^2 + \| f_2 \|_{L^2}^2)^{1/2}, \quad \| f_1 \|_{H^1}^2 = \| \nabla_x f_1 \|_{L^2}^2 + \| f_1 \|_{L^2}^2.
\]
The map \( U_0(t) \) is not unitary in \( \mathcal{H} \). However, one easily checks that
\[
\|U_0(t)f\|_1 \leq C(1 + |t|)\| f \|_1, \quad \forall t \in \mathbb{R}
\]
with a constant \( C > 0 \) independent of \( t \). Consequently, the spectral radius of the operator \( U_0(T) : \mathcal{H} \to \mathcal{H} \) is not greater than 1.

By using (2.3), it is easy to show by a fixed point theorem that for small \( t_0 > 0 \) and \( s \leq t \leq s + t_0 \) we have a local solution \( (v(t, x; s), \partial_t v(t, x; s)) \in \mathcal{H} \) of the Cauchy problem (2.1) with initial data \( f \in \mathcal{H} \). For this solution one deduces
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \left( |\partial_t v(t, x; s)|^2 + |\nabla_x v(t, x; s)|^2 + |v(t, x; s)|^2 \right) dx = -2\text{Re} \int_{\mathbb{R}^3} q\overline{v}\partial_t v dx + 2\text{Re} \int_{\mathbb{R}^3} v\overline{\partial_t v} dx
\]
which yields
\[
\frac{d}{dt}\| (v(t, x; s), \partial_t v(t, x; s)) \|_1^2 \leq C_1\| (v(t, x; s), \partial_t v(t, x; s)) \|_1^2
\]
with a constant \( C_1 > 0 \) independent of \( f \) and \( s \). The last inequality implies an estimate
\[
\| (v(t, x; s), \partial_t v(t, x; s)) \|_1 \leq C_2e^{\beta|t-s|}\| f \|_1, \quad s \leq t \leq s + t_0, \beta \geq 0.
\]
By a standard argument this leads to a global existence of a solution of (2.1). Introduce the propagator
\[ \mathcal{H} \ni f \mapsto V(t, s)f = (v(t, x; s), \partial_k v(t, x; s)) \in \mathcal{H} \]
corresponding to the Cauchy problem (1.1) with initial data \( f \in \mathcal{H} \). As in Section 5 in [9] it is easy to see that we have the following properties
\[ U(t, s) \circ U(s, r) = U(t, r), \quad U(s, s) = \text{Id}, \quad U(t + T, s + T) = U(t, s), \quad t, s, r \in \mathbb{R}. \]
The same properties hold for the propagator \( V(t, s) \). In particular, \( V(T, 0) = V((k + 1)T, kT), k \in \mathbb{N} \) and \( V(nT, 0) = (V(T, 0))^n \).
As above notice that \( V(t, s) - U_0(t - s) \) is a compact operator in \( \mathcal{L}(\mathcal{H}) \). For \( |z| \gg 1 \) we have
\[ (V(T, 0) - zI)^{-1} = (U_0(T) - zI)^{-1} - (U_0(T) - zI)^{-1} (V(T, 0) - U_0(T)) (V(T, 0) - zI)^{-1} \]
and it is not clear if the eigenfunction \( \phi \) of \( V \) has the form \( \chi \).

The number \( \eta \) is fixed sufficiently small and the propagator \( K^\delta(T) \) related to the equation
\[ \partial_t^2 u - \Delta_x u + q(t)\chi^\delta(x) u = 0, \quad t \geq 0, \quad |x| < L \]
with Dirichlet boundary conditions on \(|x| = L\) has an eigenvalue \( z_1, |z_1| > 1 \) with eigenfunction \( \varphi \in H_0^1(|x| \leq L) \), that is \( K^\delta(T)\varphi = z_1\varphi \). Let \( S^\epsilon(T) : H \to H \) be the propagator corresponding to the Cauchy problem for the equation
\[ \partial_t^2 u - \Delta_x u + V^\epsilon(t, x)u = 0, \quad t \geq 0, \quad x \in \mathbb{R}^3 \]
and let \( W^\epsilon(T) : H \to H \) be the propagator for the same problem with initial data in \( \mathcal{H} \). The problem is to show that for \( \epsilon > 0 \) sufficiently small \( W^\epsilon(T) \) has an eigenvalues \( y, |y| > 1 \) (Here \( S^\epsilon(T), W^\epsilon(T) \) correspond to our notations \( U(T, 0), V(T, 0) \) and these operators have domains \( H \) and \( \mathcal{H} \), respectively).

Extend \( \varphi \) as 0 outside \(|x| \geq L \) and denote the new function \( \varphi \in \mathcal{H} \) again by \( \varphi \). Let
\[ \gamma = \{ z \in \mathbb{C} : |z - z_1| = \eta > 0 \} \subset \{ z : |z| > 1 \} \]
be a circle with center $z_1$ such that $K^\delta(T) - zI$ is analytic on $\gamma$ and $z_1$ is the only eigenvalue of $K^\delta(T)$ in $|z - z_1| \leq \eta$. If $W^\epsilon(T)$ has an eigenvalues on $\gamma$ the problem is solved. Assume that $W^\epsilon(T)$ has no eigenvalues on $\gamma$. It is easy to see that 
\[
(W^\epsilon(T) - zI)^{-1}\varphi = (S^\epsilon(T) - zI)^{-1}\varphi \in H, \ z \in \gamma.
\]
Indeed,
\[
(W^\epsilon(T) - zI)^{-1}\varphi = (S^\epsilon(T) - zI)^{-1}\varphi + (S^\epsilon(T) - zI)^{-1}(S^\epsilon(T) - W^\epsilon(T))(W^\epsilon(T) - zI)^{-1}\varphi
\]
and 
\[
(S^\epsilon(T) - W^\epsilon(T))(W^\epsilon(T) - zI)^{-1}\varphi = 0.
\]
Our purpose is to study 
\[
(\varphi, (W^\epsilon(T) - zI)^{-1}\varphi)_H = (\varphi, (S^\epsilon(T) - zI)^{-1}\varphi)_H,
\]
where $(.,.)_H$ denotes the scalar product in $H$ and $(.,.)_H$ denotes the scalar product in $H$. It was proved in [1] that for $z \in \gamma$ one has the weak convergence in $H$
\[
(S^\epsilon(T) - zI)^{-1}\varphi \xrightarrow{\epsilon \to 0} (K^\delta(T) - zI)^{-1}\varphi.
\]
so
\[
(\varphi, (S^\epsilon(T) - zI)^{-1}\varphi)_H \rightarrow (\varphi, (K^\delta(T) - zI)^{-1}\varphi)_H.
\]
Here we have used the fact that $\varphi = 0$ for $|x| > L$. Let $\varphi = (\varphi_1, \varphi_2)$. We claim that as $\epsilon \to 0$ we have 
\[
(\varphi_1, (S^\epsilon(T) - zI)^{-1}\varphi_1)_{L^2} \rightarrow (\varphi_1, ((K^\delta(T) - zI)^{-1}\varphi_1)_{L^2}.
\]
To prove this write 
\[
\varphi_1 = -\Delta \psi \text{ with } \psi = \left(\frac{1}{4\pi|x|} \ast \varphi_1\right).
\]
The main point is the following 

**Lemma 1.** We have $\psi \in H_D(\mathbb{R}^3)$.

**Proof.** Since 
\[
|\partial_x \psi(x)| = \left|\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x_j - y_j)\varphi_1(y)}{|x - y|^3} \, dy\right| \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|\varphi_1(y)|}{|x - y|^2} \, dy,
\]
we can apply the Hardy-Littlewood-Sobolev inequality. More precisely, by using Theorem 4.3 of [5] with $n = 3$, $\lambda = 2$, $\gamma = 2$, $p = 6/5$, we obtain that 
\[
\|\partial_{x_j} \psi(x)\|_{L^2(\mathbb{R}^3)} \leq C\|\varphi_1(x)\|_{L^{6/5}(\mathbb{R}^3)}.
\]
Now using that $\varphi_1(x)$ is with compact support and the Hölder inequality, we obtain that 
\[
\|\varphi_1(x)\|_{L^{6/5}(\mathbb{R}^3)} \leq C_1\|\varphi_1(x)\|_{L^2(\mathbb{R}^3)}.
\]
This completes the proof of Lemma 1. \hfill \square

Therefore 
\[
(-\Delta \psi, ((S^\epsilon(T) - zI)^{-1}\varphi)_1)_{L^2} = \left(\langle \nabla_x \psi, \nabla_x ((S^\epsilon(T) - zI)^{-1}\varphi) \rangle_1\right)_{L^2}
\]
\[
\rightarrow_{\epsilon \to 0} \left(\langle \nabla_x \psi, \nabla_x ((K^\delta(T) - zI)^{-1}\varphi) \rangle_1\right)_{L^2} = (-\Delta \psi, ((K^\delta(T) - zI)^{-1}\varphi)_1)_{L^2}
\]
which proves the claim (2.4). Consequently, 
\[
(\varphi, (W^\epsilon(T) - zI)^{-1}\varphi)_H \rightarrow (\varphi, (K^\delta(T) - zI)^{-1}\varphi)_H. \quad (2.5)
\]
Moreover, Proposition 4.2 in [1] says that with a constant $C_0 > 0$ we have uniformly for $z \in \gamma$ the norm estimate 
\[
\|(S^\epsilon(T) - zI)^{-1}\|_H \leq C_0, \ \forall \epsilon \in [0, \epsilon_0].
\]
Since
\[ \| (S^\epsilon(T) - zI)^{-1} \varphi \|_{L^2(|x| \leq L)} \leq C_1 \| (S^\epsilon(T) - zI)^{-1} \varphi \|_H, \]
the sequence \((\varphi, (W^\epsilon(T) - zI)^{-1} \varphi)_H\) is bounded for \(z \in \gamma\). Repeating the argument of Section 5 in [1], one deduces
\[ (\varphi, \frac{1}{2\pi i} \int_\gamma (W^\epsilon(T) - zI)^{-1} \varphi dz)_H \rightarrow (\varphi, \frac{1}{2\pi i} \int_\gamma (K^\epsilon(T) - zI)^{-1} \varphi dz)_H = \| \varphi \|_H^2 \neq 0. \]
This completes the proof that for small \(\epsilon\) the operator \(W^\epsilon(T)\) has an eigenvalue \(y, |y| > 1\).

3. Proof of Theorem 2

3.1. Local well-posedness. Consider the linear problem
\[ \partial_t^2 u - \Delta_x u + q(t, x) u = F, \ u(s, x) = f_1(x), \ \partial_t u(s, x) = f_2(x). \quad (3.1) \]
By using the argument in [6], one may show that the solution of (3.1) satisfies the same local in time Strichartz estimates as in the case \(q = 0\). Notice that for these local Strichartz estimates we don’t need a global control of the local energy and we can establish them without a condition on the cut-off resolvent \(\varphi(V(T, 0) - z)^{-1} \varphi\). More precisely for every finite \(a, b > 0\) and \(f = (f_1, f_2) \in \mathcal{H}, F \in L^1([s, s + a]; L^2(\mathbb{R}^3))\) we have that the solution of (3.1) satisfies
\[ \| (u, \partial_t u) \|_{C([s, s + a], \mathcal{H})} + \| u \|_{L^q_t([s, s + a], L^r_x(\mathbb{R}^3))} \leq C(a)(\| (f_1, f_2) \|_{\mathcal{H}} + \| F \|_{L^1([s, s + a]; L^2(\mathbb{R}^3))}), \quad (3.2) \]
provided \(\frac{1}{p} + \frac{3}{q} = \frac{1}{2}, p > 2\) (the constant \(C(a)\) in (3.2) depends on \(a, p, q, t, x\)). Moreover, if \((f_1, f_2) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\) and \(F \in L^1([s, s + a]; H^1(\mathbb{R}^3))\), we have
\[ \| (u, \partial_t u) \|_{C([s, s + a], H^2 \times H^1)} + \| \nabla_x u \|_{L^2_t([s, s + a]; L^2_x(\mathbb{R}^3))} \leq C(a)(\| (f_1, f_2) \|_{H^2 \times H^1} + \| F \|_{L^1([s, s + a]; H^1(\mathbb{R}^3))}). \quad (3.3) \]
A standard application of (3.2), (3.3) is the following local well-posedness result for the nonlinear wave equation
\[ \partial_t^2 u - \Delta_x u + q(t, x) u + |u|^r u = 0, \ u(s, x) = f_1(x), \ \partial_t u(s, x) = f_2(x), \quad 2 \leq r < 4. \quad (3.4) \]

Proposition 1. There exist \(C > 0, c > 0\) and \(\gamma > 0\) such that for every \((f_1, f_2) \in \mathcal{H}\) there is a unique solution \((u, \partial_t u) \in C([s, s + \tau], H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3))\) in (3.4) on \([s, s + \tau]\) with \(\tau = c(1 + \| (f_1, f_2) \|_{\mathcal{H}})^{-7} \). Moreover, the solution satisfies
\[ \| (u, \partial_t u) \|_{C([s, s + \tau], \mathcal{H})} + \| u \|_{L^{2r+2}_t([s, s + \tau], L^{2r+2}_x(\mathbb{R}^3))} \leq C(\| (f_1, f_2) \|_{\mathcal{H}}). \quad (3.5) \]
If in addition \((f_1, f_2) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\), then \((u, \partial_t u) \in C([s, s + \tau]; H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3))\).

Remark 1. In the case \(r = 2\) the Strichartz estimates are not needed because one may only rely on the Sobolev embedding \(H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)\).

Let us recall the main step in the proof of Proposition 1. One may construct the solutions as the limit of the sequence \((u_n)_{n \geq 0}\), where \(u_0 = 0\) and \(u_{n+1}\) solves the linear problem
\[ \partial_t^2 u_{n+1} - \Delta u_{n+1} + q(t, x) u_{n+1} + |u_n|^r u_n = 0, \ u(s, x) = f_1(x), \ \partial_t u(s, x) = f_2(x), \quad (3.6) \]
where \(t \in [s, s + \tau]\). Set
\[ \| u \|_{S} := \| (u, \partial_t u) \|_{C([s, s + \tau], \mathcal{H})} + \| u \|_{L^{2r+2}_t([s, s + \tau], L^{2r+2}_x(\mathbb{R}^3))}. \]

Using (3.2) for \(2 < r < 4\) with
\[ \frac{1}{p} = \frac{r - 2}{2r + 2}, \quad \frac{1}{q} = \frac{1}{2r + 2}, \quad (3.7) \]
we obtain
\[ \|u_{n+1}\|_S \leq C\|f_1, f_2\|_H + C\|u_n\|_{L^{r+1}(\{s, s+\tau]: L^{2r+2}(\mathbb{R}^3))}. \]

Now using the Hölder inequality in time, we can write
\[ \|u_n\|_{L^{r+1}(\{s, s+\tau]: L^{2r+2}(\mathbb{R}^3))} \leq \frac{4-r}{2} \|u_n\|_{L^{2r+2}(\{s, s+\tau]: L^{2r+2}(\mathbb{R}^3))} \leq \frac{4-r}{2} \|u_n\|_S. \]

Therefore, we arrive at the bound
\[ \|u_{n+1}\|_S \leq C\|f_1, f_2\|_H + C\tau \frac{4-r}{2} \|u_n\|^{r+1}_S. \quad (3.8) \]

Assume that we have the estimate
\[ \|u_n\|_S \leq 2C\|f_1, f_2\|_H. \]

Applying (3.8), and choosing \( \tau \) so that
\[ \tau \frac{4-r}{2} (2C)^{r+1} \|f_1, f_2\|_H \leq 1, \]
we obtain the same bound for \( \|u_{n+1}\|_S \). By recurrence we conclude that
\[ \|u_{n+1}\|_S \leq 2C\|f_1, f_2\|_H, \quad \forall n \geq 0. \]

Next, let \( w_n = u_{n+1} - u_n \) be a solution of the problem
\[ \partial_t^2 w_n - \Delta w_n + q(t, x)w_n = |u_n|^r u_n - |u_{n+1}|^r u_{n+1}, \quad w_n(0, x) = \partial_t w_n(0, x) = 0. \]

By using the inequality
\[ \|v|^r v - |w|^r w \| \leq D_r |v - w| (|v|^r + |w|^r), \]
with constant \( D_r \) depending only on \( r \), we can similarly show that
\[ \|u_{n+1} - u_n\|_S \leq \frac{1}{2} \|u_n - u_{n-1}\|_S \]
which implies the convergence of \( (u_n)_{n\geq0} \) with respect to the \( \| \cdot \|_S \) norm.

Now assume that \( (f_1, f_2) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \) and introduce the norm
\[ \|u\|_{S_1} := \|(u, \partial_t u)\|_{C([s, s+\tau]; H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3))} + \|\nabla_x u\|_{L^{2r+2}(\{s, s+\tau]: L^{2r+2}(\mathbb{R}^3))}. \]

Therefore the sequence \( (u_n)_{n\geq0} \) satisfies the estimate
\[ \|u_{n+1}\|_{S_1} \leq C\|f_1, f_2\|_{H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} + C\|u_n|^r u_n\|_{L^1([s, s+\tau]; H^1(\mathbb{R}^3))} \]
and we have
\[ \sum_{n=0}^\infty \|u_n|^r u_n\|_{L^1([s, s+\tau]; H^1(\mathbb{R}^3))} \leq C(r) \sum_{n=0}^\infty \|u_n\|_{S_1^1}, \]
which leads to
\[ \|u_{n+1}\|_{S_1} \leq C_1\|f_1, f_2\|_{H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} + C_1 \tau \frac{4-r}{2} \|u_n\|_{S_1}. \quad (3.9) \]

Indeed, we can write
\[ |u_n|^r u_n = u_n^{r/2} \frac{1}{u_n^{r/2}} u_n^{r/2} \]
and therefore
\[ \partial_{x_j} (u_n^{r/2} \frac{1}{u_n^{r/2}} u_n^{r/2}) = (r/2 + 1) \partial_{x_j} u_n u_n^{r/2} \frac{1}{u_n^{r/2}} + \frac{r}{2} \partial_{x_j} u_n u_n^{r/2} \frac{1}{u_n^{r/2}} - 1 \]
yields
\[ |\nabla_x (|u_n|^r u_n)| \leq C_r |\nabla_x u_n| |u_n|^r. \]

Applying the Hölder inequality, one obtains
\[ |\nabla_x (|u_n|^r u_n)|_{L^2_x} \leq C_1 \|\nabla_x u_n\|_{L^{2r+2}(\mathbb{R}^3)} \|u_n|^r\|_{L^{2r+2}(\mathbb{R}^3)} = C_1 \|\nabla_x u_n\|_{L^{2r+2}(\mathbb{R}^3)} \|u_n|^r\|_{L^{2r+2}(\mathbb{R}^3)}. \]
Increasing, if it is necessary, the constant $C > 0$ we may arrange that (3.8) and (3.9) hold with the same constant. Therefore we obtain a local solution $u(t, x) \in C([s, s + \tau], H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3))$ in the same interval $[s, s + \tau]$.

**Remark 2.** We work in the complex setting, but if $(f_1, f_2)$ is real valued, then the solution remains real valued. Indeed, if $u$ is a solution of (3.4) then so is $\overline{u}$ and we may apply the uniqueness to conclude that $u = \overline{u}$.

3.2. **Global well-posedness and polynomial bounds.** Fix $(f_1, f_2) \in \mathcal{H}$. Let $u$ be the local solution of (3.3) obtained in Proposition 1 (with $s = 0$). First we prove the following

**Lemma 2.** The solutions

$$u(t, x) \in C([0, A], H^2(\mathbb{R}^3)) \cap C^1([0, A], H^1(\mathbb{R}^3)) \cap L^{2r + 2}_{t}([0, A], L^2_{x}(\mathbb{R}^3))$$

of (3.4) satisfy the relation

$$\frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla_x u|^2 + \frac{1}{2} q |u|^{r+2} \right) dx = \frac{1}{2} \text{Re} \int_{\mathbb{R}^3} (\partial_t q) |u|^2 dx, \ 0 \leq t \leq A. \quad (3.10)$$

**Remark 3.** We show that (3.10) holds in the sense of distributions $\mathcal{D}'([0, A])$. Since the right hand side of (3.10) is continuous in $]0, A[$ the derivative of the left hand side can be taken in the classical sense.

**Proof.** Let us first remark that $\int_{\mathbb{R}^3} |u|^{j+2}(t, x) dx \leq \|u(t, x)\|^{j+2}_{H^1(\mathbb{R}^3)}$ for $0 \leq j < 4$, thanks to the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^{j+2}(\mathbb{R}^3)$. Moreover, from our assumption it follows that $u(t, x) \in C([0, A], L^\infty_x(\mathbb{R}^3))$ and this implies

$$|u|^r(t, x)u(t, x) \in C([0, A], L^2_x(\mathbb{R}^3)).$$

Therefore, from the equation (3.3) we deduce $\partial_t^2 u(t, x) \in C([0, A], L^2_x(\mathbb{R}^3))$.

To verify (3.10), notice that

$$\text{Re} \left( \int_{\mathbb{R}^3} (\partial_t^2 u - \Delta_x u + |u|^r u) \overline{\partial_t u} dx \right) = -\text{Re} \left( \int_{\mathbb{R}^3} q(t, x) u \overline{\partial_t u} dx \right)$$

$$= -\frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}^3} q |u|^2 dx \right) + \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t q) |u|^2 dx$$

and the integrals

$$\int_{\mathbb{R}^3} (\partial_t^2 u - \Delta_x u) \overline{\partial_t u} dx, \ \int_{\mathbb{R}^3} |u|^r u \overline{\partial_t u} dx$$

are well defined. After an approximation with smooth functions and integration by parts we deduce

$$\text{Re} \int_{\mathbb{R}^3} (\partial_t^2 u - \Delta_x u) \overline{\partial_t u} dx = \frac{d}{dt} \int_{\mathbb{R}^3} \frac{1}{2} (|\partial_t u|^2 + |\nabla_x u|^2) dx.$$

On the other hand,

$$(r/2 + 1)(u^2 \overline{\partial_t u}^2 + u^2 \partial_t u) + u^{r+2} \partial_t \overline{u} = (\partial_t (u^{2+1}) \overline{u}^{2+1} + \partial_t (u^{2+1}) \overline{u}^{2+1})$$

and hence

$$\text{Re} \int_{\mathbb{R}^3} |u|^r u \overline{\partial_t u} dx = \frac{1}{r + 2} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^r+2 dx.$$

Thus (3.10) holds for $0 < t < A$ and by continuity one covers the interval $[0, A]$. \[\square\]

We need the following simple lemma.
Lemma 3. Let \( 0 < \gamma < 1 \) and let \( X(t) : [0, \infty) \to [0, \infty) \) be a derivable function such that for some \( A > 0 \),
\[
|X'(t)| \leq CX^{1-\gamma}(t), \quad 0 \leq t \leq A.
\]
Then
\[
X(t) \leq (X^\gamma(0) + C\gamma t)^{\frac{1}{\gamma}}, \quad 0 \leq t \leq A.
\]

Proof. First assume that \( X(t) > 0 \) for all \( 0 \leq t \leq A \). We have
\[
\left| \frac{d}{dt}(X^\gamma(t)) \right| = \gamma X^{\gamma-1}(t)X'(t) \leq C\gamma.
\]
Hence
\[
X^\gamma(t) = \int_0^t (X^\gamma(\tau)) \, d\tau + X^\gamma(0) \leq X^\gamma(0) + C\gamma t
\]
and we obtain the assertion for \( X(t) > 0 \). In the general case, we apply the previous argument to \( X(t) + \epsilon, \epsilon > 0 \) and we let \( \epsilon \to 0 \). This completes the proof. \( \square \)

Let \( u(t, x) \in C([0, A], H^2(\mathbb{R}^3) \cap C^1([0, A], H^1(\mathbb{R}^3)) \cap L^{2r+2}_t([0, A], L^{2r+2}(\mathbb{R}^3)) \) be a solution of (3.11) and let
\[
X(t) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla_x u|^2 + \frac{1}{2} q|u|^2 + \frac{1}{r+2} |u|^{r+2} \right) \, dx.
\]
The support property \( q(t, x) = 0 \) for \( |x| > \rho \) and the Hölder inequality imply
\[
\int_{\mathbb{R}^3} (\partial_t q)|u|^2 \, dx \leq C\|u(t, \cdot)\|_{L^2(|x| \leq \rho)}^2 \leq C_1\|u(t, \cdot)\|_{L^{r+2}(|x| \leq \rho)}^2.
\]
Therefore
\[
|X'(t)| \leq C_2X^{\frac{r}{r+2}}(t) = C_2X^{1-\frac{2}{r+2}}(t)
\]
and applying Lemma 3 we deduce
\[
X(t) \leq \left( X^{\frac{r}{r+2}}(0) + \frac{C_2r}{r+2} t \right)^{\frac{r+2}{2r}} \quad 0 \leq t \leq A. \quad (3.11)
\]

As a consequence of (3.11) we get
\[
\left( \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla_x u(t, \cdot)\|_{L^2(\mathbb{R}^3)}^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \left( X^{\frac{r}{r+2}}(0) + \frac{C_2r}{r+2} t \right)^{\frac{r+2}{2r}}
\]
and therefore
\[
\|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R}^3)} + \|\nabla_x u(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq 2 \left( X^{\frac{r}{r+2}}(0) + \frac{C_2r}{r+2} t \right)^{\frac{r+2}{2r}}.
\]
On the other hand,
\[
X(0) \leq A_r\|(u, u_t)(0, x)\|_1^2 \left( 1 + \|(u, u_t)(0, x)\|_1^2 \right)
\]
with a constant \( A_r \) depending on \( r \). Hence from (3.11) we get
\[
\|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R}^3)} + \|\nabla_x u(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq 2 \left( X^{\frac{r}{r+2}}(0) + \frac{C_2r}{r+2} t \right)^{\frac{r+2}{2r}}
\]
\[
\leq 2 \left( A_r^{\frac{r}{r+2}} \|(u, u_t)(0, x)\|_1^2 \right)^{\frac{2r}{r+2}} \left( 1 + \|(u, u_t)(0, x)\|_1^2 \right)^{\frac{r+2}{r+2}} + \frac{C_2r}{r+2} t^{\frac{r+2}{2r}}, \quad 0 \leq t \leq A. \quad (3.12)
\]
Finally, from
\[
u(t, x) = u(0, x) + \int_0^t \partial_t u(\tau, x) \, d\tau
\]
one deduces
\[ \|u(t, x)\|_{L^2} \leq \|u(0, x)\|_{L^2} + 2t \left( X^\frac{r}{2} (0) + \frac{C_2 r}{r + 2} t^\frac{r+2}{2} \right). \]
This yields a polynomial bound for the solutions
\[ u(t, x) \in C([0, A], H^2(\mathbb{R}^3)) \cap C^1([0, A], H^1(\mathbb{R}^3)) \cap L_t^{2r+2} (\mathbb{R}^3). \]

Now we pass to the global existence of solution of (3.4). We will deal with the case \( 2 < r < 4 \), while the case \( r = 2 \) can be covered by the Sobolev embedding theorem. We fix a number \( a > 0 \) and our purpose is to show that (3.4) has a solution for \( t \in [0, a] \) with initial data \( f \in H \). We fix \( p, q \) by (3.7) and let the Strichartz estimate (3.2) holds in the interval \([0, a]\) with a constant \( C_a > 0 \). The above argument yields a local solution \( u(t, x) \) with initial data \( f = (f_1, f_2) \in H \) for \( t \in [s, s + \tau] \). Recall that \( \tau = c(1 + \|f\|_H)^{-\gamma} \). Introduce the number
\[ B_a := \|f\|_H + a(B_1 + B_2 a)^{\frac{r+2}{2}}, \]
where \( B_1 > 0 \) and \( B_2 > 0 \) depend only on \( \|f\|_H \) and \( r \). This number should be a bound of the energy of the solution \( u(t, x) \) in \([0, a]\) with initial data \( f \in H \) if the above argument based on Lemma 2 and Lemma 3 works. However, the proof of Lemma 2 cannot be applied directly for functions \( u(t, x) \in C([0, a], H^2(\mathbb{R}^3)) \cap C^1([0, a], L^2(\mathbb{R}^3)) \).

Define \( \tau(a) := c(1 + B_a)^{-\gamma} < 1 \) with the constants \( c > 0, \gamma > 0 \) of Proposition 1 and observe that the local existence theorem can be applied in the interval \([s, s + \tau(a)] \subset [0, a]\) if the norm of the initial data for \( t = s \) is bounded by \( B_a \). To overcome the difficulty connected with Lemma 2 and since we did not prove in Proposition 1 the continuous dependence with respect to the initial data in \( H \), we need to apply an approximation argument in \([s, s + \epsilon(a)]\), where the number \( 0 < \epsilon(a) \leq \tau(a) \) will be defined below. For simplicity we treat the case \( s = 0 \) below.

By the local existence let \( u(t, x) \) be the solution of (3.4) in \([0, \tau(a)]\) with initial data \( f = (f_1, f_2) \in H \). Choose a sequence \( g_n = ((g_n)_1, (g_n)_2) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \) converging in \( H \) to \((f_1, f_2) \in H \) as \( n \to \infty \) and let \( w_n(t, x) \) be the solution of the problem (3.4) in the same interval \([0, \tau(a)]\) with initial data \( g_n \). Then by Proposition 1
\[ w_n(t, x) \in C([0, \tau(a)], H^2(\mathbb{R}^3) \cap C^1([0, \tau(a)], H^1(\mathbb{R}^3)) \cap L_t^{2r+2} ([0, \tau(a)], L_x^{2r+2} (\mathbb{R}^3)). \]
Set \( v_n = w_n - u \). We claim that for \( n \to \infty \) we have
\[ \|v_n, (v_n)_t\|_{C([0, \epsilon(a)], H)} + \|v_n\|_{L^p_t([0, \epsilon(a)], H^q_x(\mathbb{R}^3))} \to 0 \]
with \( 0 < \epsilon(a) \leq \tau(a) \) defined below. Clearly, \( v_n \) is a solution of the equation
\[ \partial_t^2 v_n - \Delta v_n + g(t, x)v_n = |u|^r u - |w_n|^r w_n. \]
Applying (3.2), one obtains
\[ \|v_n, (v_n)_t\|_{C([0, \epsilon(a)], H)} + \|v_n\|_{L_t^{2r+2}([0, \epsilon(a)], L_x^{2r+2}(\mathbb{R}^3))} \leq C_a \|g_n - f\|_H + C_a \||u|^r u - |w_n|^r w_n\|_{L^1([0, \epsilon(a)], L^2_x(\mathbb{R}^3))} \]
and
\[ \||u|^r u - |w_n|^r w_n\|_{L^2} \leq C \|v_n(t, .)\|_{L^{2r+2}_t} \left( \|u(t, .)\|_{L^{2r+2}_x} + \|w_n(t, .)\|_{L^{2r+2}_x} \right). \]
Since \( \frac{1}{p} + \frac{r+1}{2} = 1 \), by the generalized Hölder inequality in the integral with respect to \( t \) in (3.13) for large \( n \geq n_0 \) we get
\[ C_a \||u|^r u - |w_n|^r w_n\|_{L^1([0, \epsilon(a)], L^2_x(\mathbb{R}^3))} \]
Consequently, we have
\[ \frac{\partial}{\partial t}w_n(t,x) \leq D_r C_2 \varepsilon(a) \left( 1 - \frac{\varepsilon(a)}{p} \right) \| w_n \|_{L^p([0,\varepsilon(a)],L^2)} + \| w_n \|_{L^p([0,\varepsilon(a)],L^3)} \]
\[ \leq 2D_r C_2^{r+1}(\| f \|_H + 1) \varepsilon(a) \left( 1 - \frac{\varepsilon(a)}{p} \right) \| w_n \|_{L^p([0,\varepsilon(a)],L^3)} . \]

Here \( D_r \) is a constant depending only on \( r \) and we used that by Proposition 1
\[ \| w_n \|_{L^{\frac{2p+2}{2}}([0,\varepsilon(a)],L^2)} \leq C_a \| g_n \|_H \leq C_a(\| f \|_H + 1), \ n \geq n_0 \] (3.14)
with a similar estimate for \( \| u \|_{L^{\frac{2p+2}{2}}([0,\varepsilon(a)],L^2)} \). Clearly, \( 1 - \frac{\varepsilon(a)}{p} = 2 - \frac{r}{2} > 0 \) and we choose \( 0 < \varepsilon(a) \leq \tau(a) \), so that
\[ 2D_r C_2^{r+1}(B_a + 1) \varepsilon(a) \left( 1 - \frac{\varepsilon(a)}{p} \right) \leq \frac{1}{2}. \]

Then we may absorb the term on right hand side of (3.13) involving \( w_n, u \) and letting \( n \to \infty \), we prove our claim. Moreover, for almost all \( t \in [0,\varepsilon(a)] \), taking into account (3.14), we have
\[ \left| \int_{\mathbb{R}^3} \left( |u(t,x)|^{r+2} - |w_n(t,x)|^{r+2} \right) dx \right| \]
\[ \leq D_r \| u(t,x) - w_n(t,x) \|_{L^2(\mathbb{R}^3)} \left( \| u(t,x) \|_{L^{r+2}(\mathbb{R}^3)}^{r+1} + \| w_n(t,x) \|_{L^{r+2}(\mathbb{R}^3)}^{r+1} \right) dx \to_{n \to \infty} 0. \]

Consequently, we have
\[ \int_{\mathbb{R}^3} \left( \frac{1}{2} (|\partial_t w_n|^2 + |\nabla_x w_n|^2 + q |u|^2) + \frac{1}{r+2} |w_n|^{r+2} \right) dx \]
\[ \to_{n \to \infty} \int_{\mathbb{R}^3} \left( \frac{1}{2} (|\partial_t u|^2 + |\nabla_x u|^2 + q |u|^2) + \frac{1}{r+2} |u|^{r+2} \right) dx \]
in the sense of distributions \( \mathcal{D}'(0,\varepsilon(a)) \). The equality (3.10) for \( 0 \leq t \leq \varepsilon(a) \) holds for \( w_n \) and passing to a limit in the sense of distributions, we conclude that (3.10) holds for \( u(t,x) \) for \( 0 < t < \varepsilon(a) \) and hence for \( 0 \leq t \leq \varepsilon(a) \). The right hand side of (3.11) is continuous with respect to \( t \), hence the derivative with respect to \( t \) is taken in a classical sense. Thus we are in position to apply Lemma 3 for the \( u(t,x) \). Finally, we deduce (3.12) for the solution \( u(t,x) \) and the norm \( \| (u, u_t(t,\cdot)) \|_H \) for \( t \in [0,\varepsilon(a)] \) is bounded by \( B_a \) introduced above.

Now we pass to the second step in the interval \( [\epsilon(a),2\epsilon(a)] \subset [0,a] \). As it was mentioned above, we have a bound \( B_a \) for the norm of the initial data \( (u(\epsilon(a),x),u_t(\epsilon(a),x)) \). By the local existence we have solution in \( [\epsilon(a),2\epsilon(a)] \) and \( u(t,x) \) is defined in \( [0,2\epsilon(a)] \). On the other hand, we may approximate the initial data \( (u(\epsilon(a),x),u_t(\epsilon(a),x)) \) by functions \( g_n(2) \in H^2 \times H^1 \) and by the above argument the solution \( u(t,x) \) with \( \epsilon(a),2\epsilon(a) \) is approximated by solutions \( w_n(2)(t,x) \) for which (3.11) holds for \( \epsilon(a) \leq t \leq 2\epsilon(a) \). Thus (3.10) is satisfied for \( u(t,x) \) for \( \epsilon(a) \leq t < 2\epsilon(a) \) and combining this with the first step, one concludes that the same is true for \( 0 \leq t \leq 2\epsilon(a) \). This makes possible to apply Lemma 3 for \( 0 \leq t \leq 2\epsilon(a) \) and deduce (3.12) with uniform constants leading to a bound by \( B_a \). We can iterate this procedure, since \( \tau(a),\epsilon(a) \) depend only on \( \| f \|_H,C_a \) and \( r \), while \( B_a \) depends on \( \| f \|_H,a \) and \( r \). The solution \( u(t,x) \) will be defined globally in a interval \( [0,\alpha(a)] \) with \( 0 < a - \alpha(a) < \epsilon(a) \). Since \( \alpha(a) > a - \epsilon(a) > a - 1 \) and \( a \) is arbitrary, we have a global solution \( u(t,x) \) defined for \( t \geq 0 \). An application of Lemma 3 justifies the bound (3.12) for \( u(t,x) \) and for all \( t \geq 0 \) with constants depending only on \( \| f \|_H \) and \( r \). A similar analysis holds for negative times \( t \).

**Remark 4.** It is likely that in the case \( r = 2 \) by using the approach of [8] one may obtain polynomial bounds on the higher Sobolev norms \( H^\sigma(\mathbb{R}^3) \times H^{\sigma-1}(\mathbb{R}^3), \sigma > 1 \), of the solutions of (3.1).
3.3. A uniform bound. As a byproduct of the (semi-linear) global well-posedness, we have the following uniform bound on the solutions of (3.4).

**Proposition 2.** Let $R > 0$ and $A > 0$. Then there exists a constant $C(A, R) > 0$ such that for every $(f_1, f_2) \in \mathcal{H}$ such that $\|(f_1, f_2)\|_{\mathcal{H}} < R$ the solution $u(t, x)$ of (3.4) satisfies

$$
\|u\|_{L^{r+1}(0, 1; L^2, 2^r(\mathbb{R}^3))} \leq C(A, R)\|(f_1, f_2)\|_{\mathcal{H}}. 
$$

(3.15)

**Proof.** Thanks to the global bounds on the solutions, we obtain that there exists $R' = R'(R, A)$ such that if $\|(f_1, f_2)\|_{\mathcal{H}} < R$ then the corresponding solutions satisfies

$$
\sup_{0 \leq t \leq A} \|(u(t, \cdot), \partial_t u(t, \cdot))\|_{\mathcal{H}} \leq R'.
$$

Denote by $\tau = \tau(A, R') > 0$ the local existence time for initial data having $\mathcal{H}$ norm $\leq R'$, i.e. $\tau = c(1 + R')^{-\gamma}$ with the notations of Proposition 1. Next we split the interval $[0, A]$ into intervals of size $\tau$. In every interval $[k\tau, (k+1)\tau]$ we apply the estimate (3.2) with $F = 0$ and constant $C_A$ independent on $k$. Thus we obtain a bound

$$
\|u(t, x)\|_{L^{2r+2}((k\tau, (k+1)\tau); L^2, 2^r(\mathbb{R}^3))} \leq C_A\|(f_1, f_2)\|_{\mathcal{H}}, 1 \leq k + 1 \leq A/\tau.
$$

By using the Hölder inequality for the integral with respect to $t$, we obtain easily (3.15). \qed

4. PROOF OF THEOREM 3

Let

$$
\mathcal{H} \ni f \rightarrow U(t, s) f = (v(t, x; s), v_t(t, x; s)) \in \mathcal{H}
$$

be the monodromy operator corresponding to the Cauchy problem (3.3) with initial data $f$ for $t = s$. For $U(t, s)$ we have the representation

$$
U(t, s) f = V(t, s) f - \int_s^t V(t, \tau) Q_0 (|U(\tau, s) f|^r U(\tau, s) f) d\tau, 
$$

(4.1)

where

$$
Q_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
$$

Therefore we can write $U(t + T, s + T) f$ as

$$
V(t + T, s + T) f - \int_{s+T}^{t+T} V(t + T, \tau) Q_0 (|U(\tau, s + T) f|^r U(\tau, s + T) f) d\tau
$$

which in turn can be written as

$$
V(t, s) f - \int_s^t V(t, \tau) Q_0 (|U(\tau + T, s + T) f|^r U(\tau + T, s + T) f) d\tau.
$$

By the uniqueness of the solution of the equation

$$
U(t, s) f = V(t, s) f - \int_s^t V(t, \tau) Q_0 (|U(\tau, s) f|^r U(\tau, s) f) d\tau,
$$

one deduces $U(t + T, s + T) = U(t, s)$. Moreover, one has the property

$$
U(p, r) = U(p, s) \circ U(s, r), \quad p, r, s \in \mathbb{R}.
$$

For the solution $u(t, x; 0)$ of (3.3) (with $s = 0$) with initial data $f \in \mathcal{H}$, set

$$
w_n = (u(nT, x; 0), \partial_t u(nT, x; 0)) = U(nT, 0) f, \quad n \in \mathbb{N}.
$$

Therefore

$$
w_{n+1} = U((n + 1)T, 0) f = U((n + 1)T, nT) \circ U(nT, 0) f = U(T, 0) w_n.
$$

(4.2)
Setting $\mathcal{F} = \mathcal{U}(T, 0)$, we obtain a system
\[ w_{n+1} = \mathcal{F}(w_n), \quad n \geq 0. \tag{4.3} \]
with a nonlinear map $\mathcal{F} : \mathcal{H} \to \mathcal{H}$. Consider the linear map $L = V(T, 0) : \mathcal{H} \to \mathcal{H}$. Our purpose is to show that $L$ is the Fréchet derivative of $\mathcal{F}$ at the origin in the Hilbert space $\mathcal{H}$. We use the representation
\[ \mathcal{F}(h) = Lh - \int_0^T V(T, \tau)Q_0(\|u(t, x; h)\|^ru(t, x; h))d\tau, \]
where $u(t, x; h)$ is the solution of (4.3) with $s = 0$ and initial data $h$ at time 0. Using the Strichartz estimate and Proposition 2, we obtain for $\|h\|_1 \leq 1$ the bound
\[ \sup_{0 \leq t \leq T} \|\mathcal{F}(h) - Lh\|_1 \leq C\|u(t, x; h)\|^{r+1}_{L^{r+1}} \leq C\|h\|^{r+1}_1, \]
where $C > 0$ depends on $T$ but is independent of $h$. This implies immediately that $L$ is the Fréchet derivative of $\mathcal{F}$ at the origin.

For the exponential instability at $u = 0$ we use following definition (see [2]).

**Definition 1.**

(i) The equilibrium $u = 0$ is unstable if there exists $\epsilon > 0$ such that for every $\delta > 0$ one can find a sequence $\{u_n\}$ of solution of (4.3) such that $0 < \|u_0\|_1 \leq \delta$ and $\|u_n\|_1 \geq \epsilon$ for some $n \in \mathbb{N}$.

(ii) The equilibrium $u = 0$ is exponentially unstable at rate $\rho > 1$ if there exist $\epsilon > 0$ and $C > 0$ such that for every $\delta > 0$ one can find a sequence $\{u_n\}$ of solution of (4.3) satisfying $0 < \|u_0\|_1 \leq \delta$ and $\|u_N\|_1 \geq C\rho^N\|u_0\|_1$ for any $N$ for which we have
\[ \max\{\|u_0\|_1, \ldots, \|u_N\|_1\} \leq \epsilon. \]

Clearly, the exponential instability implies instability. We consider the case when the spectral radius $r(L)$ of $L$ is greater than 1. The analysis in Section 2 shows that there exist positive potentials $q(t, x) \geq 0$ for which $r(L) > 1$. We will apply the Rutman-Dalecki theorem or a more general version due to D. Henry (Theorem 5.1.5 in [4]). For this purpose we need the condition
\[ \|\mathcal{F}(u) - Lu\|_1 \leq b\|u\|_1^{1+p} \text{ whenever } \|u\|_1 \leq a \tag{4.4} \]
for some $a > 0$, $b > 0$ and $p > 0$. In our case the condition (4.4) holds with $p = r$ and $a = 1$. Thus we obtain the following

**Theorem 4.** Assume that the linear operator $L$ has spectral radius $r(L) > 1$. Then $\mathcal{F}$ is exponentially unstable at $u = 0$ with rate $r(L)$.

It remains to observe that Theorem 4 implies Theorem 3.

**Remark 5.** The above argument showing nonlinear instability crucially relies on the fact that we deal with a semi-linear problem, i.e. the solution map of (3.3) is of class $C^1$ on $\mathcal{H}$. It is worth to mention that there are examples of problems which are not semi-linear (the solution map is not of class $C^1$) for which one can still get the nonlinear instability of some particular solutions (known to be linearly unstable). In such cases a “more nonlinear approach” is needed. We refer to [3, 9] for more details on this issue.

5. **Generalizations**

We can consider more general nonlinear equations
\[ \partial_t^r u - \Delta u + |u|^r u + \sum_{j=0}^{r-1} q_j(t, x)|u|^j u = 0, \quad r = 2, 3 \tag{5.1} \]
with time-periodic functions \( q_j(t + T_j, x) = q_j(t, x) \geq 0, j = 0, \cdots, r - 1 \) having compact support with respect to \( x \). For solutions

\[
u(t, x) \in C([0, \tau], H^2(\mathbb{R}^3)) \cap C^1([0, \tau], H^1(\mathbb{R}^3)) \cap L^{\frac{2r+2}{r+2}}_t([0, A], L^{r+2}_x(\mathbb{R}^3))
\]

we obtain

\[
\text{Re} \left( \int_{\mathbb{R}^3} (\partial_t^2 u - \Delta u + |u|^r u) \bar{u}_t dx \right) = -\text{Re} \left( \int_{\mathbb{R}^3} \sum_{j=0}^{r-1} q_j(t, x) |u|^{j+2} u_t dx \right)
\]

\[
= \frac{d}{dt} \sum_{j=0}^{r-1} \left( \int_{\mathbb{R}^3} \frac{1}{j+2} q_j |u|^{j+2} dx \right) + \sum_{j=0}^{r-1} \frac{1}{j+2} \int_{\mathbb{R}^3} (q_j)_t |u|^{j+2} dx
\]

and

\[
\frac{1}{j+2} \left| \int_{\mathbb{R}^3} (q_j)_t |u|^{j+2} dx \right| \leq C_j \left( \int_{\mathbb{R}^3} |u|^{r+2} dx \right)^{\frac{1}{1 + \frac{r+2}{r}}} , j = 0, \cdots, r - 1.
\]

Setting

\[
X(t) \equiv \int_{\mathbb{R}^3} \left( \frac{1}{2} |u_t|^2(t, x) + \frac{1}{2} |\nabla_x u|^2(t, x) + \sum_{j=0}^{r-1} \frac{1}{j+2} q_j |u|^{j+2}(t, x) + \frac{1}{r+2} |u|^{r+2}(t, x) \right) dx, 0 \leq t \leq A,
\]

one deduce

\[
|X'(t)| \leq B_r \sum_{j=0}^{r-1} X(t)^{1 - \frac{r+2}{r+2}} \leq B_r (1 + X(t))^{1 - \frac{1}{r+2}}.
\]

Therefore we can apply Lemma 3 to the quantity \( Y(t) = 1 + X(t) \) which implies, as before, the global existence and the polynomial bounds for the Cauchy problem for (5.1).

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