STATIONARY COVARIANCE REGIME FOR AFFINE STOCHASTIC COVARIANCE MODELS IN HILBERT SPACES

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Abstract. We study the long-time behavior of affine processes on positive self-adjoint Hilbert-Schmidt operators which are of pure-jump type, conservative and have finite second moment. For subcritical processes we prove the existence of a unique limit distribution and construct the corresponding stationary affine process. Moreover, we obtain an explicit convergence rate of the underlying transition kernels to the limit distribution in the Wasserstein distance of order $p \in [1, 2]$ and provide explicit formulas for the first two moments of the limit distribution. We apply our results to the study of infinite-dimensional affine stochastic covariance models in the stationary covariance regime, where the stationary affine process models the instantaneous covariance process. In this context we investigate the behavior of the implied forward volatility smile for large forward dates in a geometric affine forward curve model used for the modeling of forward curve dynamics in fixed income or commodity markets formulated in the Heath-Jarrow-Morton-Musiela framework.

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be a real separable Hilbert space and denote by $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ the Hilbert space of all self-adjoint Hilbert-Schmidt operators on $H$ equipped with the trace inner-product $(x, y) := \text{Tr}(yx)$ for $x, y \in \mathcal{H}$. Let $\mathcal{H}^+ \subset \mathcal{H}$ stand for the cone of all positive self-adjoint Hilbert-Schmidt operators on $H$. Given a time-homogeneous Markov process $(X_t)_{t \geq 0}$ with values in $\mathcal{H}^+$, we denote its transition kernels by $(p_t(x, \cdot))_{t \geq 0}$. Following the general terminology of affine processes, we call $(X_t)_{t \geq 0}$ an affine process on $\mathcal{H}^+$, whenever the Laplace transform of $X_t$, for every $t \geq 0$, is of an exponential affine form in the initial value $x \in \mathcal{H}^+$, i.e.

$$
\int_{\mathcal{H}^+} e^{-\langle \xi, u \rangle} p_t(x, d\xi) = e^{-\phi(t,u) - \langle x, \psi(t,u) \rangle}, \quad t \geq 0, u \in \mathcal{H}^+, \quad (1.1)
$$

for some functions $\phi : \mathbb{R}^+ \times \mathcal{H}^+ \to \mathbb{R}^+$ and $\psi : \mathbb{R}^+ \times \mathcal{H}^+ \to \mathcal{H}^+$. Affine processes are widely used for applications in finance due to their analytical tractability, i.e. their Fourier-Laplace transforms given by (1.1) are quasi-explicit up to the functions $\phi$ and $\psi$. Typically the functions $\phi$ and $\psi$ are the solutions of certain differential equations.

In finite dimensions affine processes and their applications were studied by many authors on numerous state spaces including the canonical state space $\mathbb{R}_d^+ \times \mathbb{R}^n$, see, e.g. [21, 42, 19, 24, 44, 40], and the space of positive and symmetric $d \times d$-matrices $\mathbb{S}_d^+$, see [17, 16]. More recently it became increasingly popular to study their infinite-dimensional extensions, see, e.g. [18, 57, 33, 13, 14]. In the present work we contribute to this new direction of research towards affine processes on infinite-dimensional state spaces by studying their long-time behavior.

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The affine processes on the state-space $\mathcal{H}^+$, as considered in this work, have been first introduced and studied in [13]. Their construction is summarized in Theorem 2.3 below, see also [13, Theorem 2.8]. We want to emphasize here that affine processes on $\mathcal{H}^+$ can be considered as the natural infinite-dimensional analog of affine processes on positive and symmetric $d \times d$-matrices $S_d^+$ studied in [17]. Indeed, for $\mathcal{H} = \mathbb{R}^d$ the self-adjoint Hilbert-Schmidt operators on $\mathcal{H}$ are precisely the symmetric matrices equipped with the Frobenius norm and $\mathcal{H}^+ = S_d^+$ holds in this case. Due to their flexibility and tractability, affine processes on $S_d^+$ are widely used to model the instantaneous covariance process in multi-variate stochastic volatility models, see [17, 47, 4, 32]. Analogously, affine processes on the state-space $\mathcal{H}^+$ are well-suited to model the instantaneous covariance process in infinite-dimensional stochastic covariance models, see [14]. The long-time behavior of affine processes, used to model the instantaneous covariance process, plays an important role in the calibration of stochastic covariance models, see [1, 48].

1.1. Contribution and related literature. In this article we deal with the long-time behavior of affine processes on $\mathcal{H}^+$. In particular we are interested in the existence and uniqueness of a limit distribution $\pi$ as the time $t$ tends to infinity. Heuristically, if the transition kernels $(p_t(x, \cdot))_{t \geq 0}$ converge weakly to some probability measure $\pi$ on $\mathcal{H}^+$, then $\pi$ is an invariant measure for $(p_t(x, \cdot))_{t \geq 0}$, i.e.

$$\int_{\mathcal{H}^+} p_t(x, d\xi) \pi(d\xi) = \pi(d\xi), \quad \text{for all } t \geq 0 \text{ and } x \in \mathcal{H}^+.$$ 

Once the existence of an invariant measure is established, we study the construction of the corresponding stationary process. The latter one allows us to introduce stochastic covariance models in the stationary covariance regime, see Section 4.1 of this work. Within this context we model the implied forward volatility smile in a geometric affine covariance model for forward curve dynamics formulated in the Heath-Jarrow-Morton-Musiela (HJMM) modeling framework, see Section 4.2 below. In addition to the above, we are also interested in explicit convergence rate towards the invariant measure.

Below we provide a short version of our main result on the long-time behavior of affine processes on $\mathcal{H}^+$. We call an affine process on $\mathcal{H}^+$ subcritical, if its state-dependent drift is negative (i.e. all eigenvalues have strictly negative real-parts), see Assumption A below.

**Theorem.** Let $(X_t)_{t \geq 0}$ be a subcritical affine process on $\mathcal{H}^+$ with transition kernels $(p_t(x, \cdot))_{t \geq 0}$, the existence of which is guaranteed by Theorem 2.3 below. Then the following holds true:

i) There exists a unique invariant measure $\pi$ of $(p_t(x, \cdot))_{t \geq 0}$ for all $x \in \mathcal{H}^+$.

ii) For every $x \in \mathcal{H}^+$ and $p \in [1, 2]$ the sequence $(p_t(x, \cdot))_{t \geq 0}$ converges exponentially fast to $\pi$ as $t \to \infty$ in the Wasserstein distance of order $p$.

iii) There exists a Markov process $(X^n_t)_{t \geq 0}$, with transition kernels $(p_t(x, \cdot))_{t \geq 0}$, such that the distribution of $X^n_t$ is equal to $\pi$ for all $t \geq 0$.

The long-time behavior of affine processes on the finite dimensional state spaces $(\mathbb{R}_+)^d \times \mathbb{R}^n$ and $S_d^+$ for $d, n \in \mathbb{N}$, is now mostly well-understood. More precisely, based on the representation by strong solutions of stochastic differential equations, ergodicity was studied in [30] for different Wasserstein distances. By using regularity of transition densities with respect to the Lebesgue measure combined with the Meyn-and-Tweedie stability theory the ergodicity in total variation distances has been studied in [3, 27, 38, 37, 53, 29]. Finally, coupling techniques for affine processes are studied in [59, 52]. Unfortunately, these methods implicitly use the dimension of the state-space and hence do not allow for an immediate extension to
infinite dimensional settings. Indeed, for general affine processes on $\mathcal{H}^+$ there does not exist, so far, a pathwise construction. The absence of an infinite-dimensional Lebesgue measure prevents us to effectively use the Meyn-and-Tweedie stability theory (in terms of estimates on the density). Although there exist some extensions of the coupling techniques to infinite dimensional state-spaces (see [51] for measure-valued branching processes), these methods seem to be closely related to the measure-valued structure of the process and hence not suitable for our Hilbert space framework.

The most promising method to study the long-time behavior for affine processes in infinite-dimensional settings is therefore based on the convergence of Fourier-Laplace transforms. The latter one requires, in view of the affine-transform formula (1.1), to study the long-time behavior of the solutions to the generalized Riccati equations $\phi$ and $\psi$. For finite-dimensional state spaces, these ideas have been developed in [31, 43, 45, 39, 55, 28]. In these works, the existence of an invariant distribution (as well as weak convergence of transition probabilities) is obtained from Lévy’s continuity theorem. More recently, such techniques also have been applied in [26] to Dawson-Watanabe superprocesses with immigration which form a specific class of affine processes on the state space of (possibly tempered) measures (so-called measure-valued Markov processes).

Unfortunately, infinite-dimensional analogues of Lévy’s continuity theorem on Hilbert spaces require an additional tightness condition on the transition probabilities (to obtain the existence of a limit distribution and hence invariant measure). For Ornstein-Uhlenbeck processes on Hilbert spaces, this problem can be avoided by taking advantage of their infinite divisibility, see [12]. Note that such processes form a subclass of affine processes, see also Example 3.6 below. Apart from this, the long-time behavior of affine processes in infinite-dimensions has not been investigated in a systematic way. Our work therefore provides a first general treatment of long-time behavior for affine processes on infinite-dimensional Hilbert spaces as a state space.

Our methodology for the proof of our main result builds on the ideas taken from [28], where the long-time behavior of affine processes on $\mathcal{S}_d^+$ was studied. Namely, we show that for subcritical affine processes the limits $\lim_{t \to \infty} \phi(t, u)$ and $\lim_{t \to \infty} \psi(t, u)$ exist for every $u \in \mathcal{H}^+$ and hence the Fourier-Laplace transform of the process (see (1.1)) converges when $t \to \infty$. To overcome the difficulty related to the absence of a full analogue of Lévy continuity theorem, we utilize the generalized Feller semigroup approach for the process (see Appendix A for a definition). More precisely, we provide uniform bounds on the operator norm of the transition semigroup which allows us to prove that $\lim_{t \to \infty} P_t f =: \ell(f)$ has a limit for a sufficiently large class of functions $f$. By showing that the limit $\ell$ is a continuous linear functional, we can apply a variant of Riesz representation theorem for generalized Feller semigroups to show that $\ell$ has representation $\ell(f) = \int_{\mathcal{H}^+} f(y)\pi(dy)$. The measure $\pi$ is the desired unique invariant probability measure. As a byproduct we also obtain weak convergence of transition probabilities in the weak topology on $\mathcal{H}^+$. In the second step we strengthen this convergence by proving estimates on the Wasserstein distance of order $p \in [1, 2]$ of the transition probabilities to the invariant measure. In contrast to the finite-dimensional results in [30, 28], our new bounds are dimension-free and explicit. As a consequence, we conclude that the transition probabilities converge weakly to the invariant measure in the norm topology on $\mathcal{H}^+$. Finally, we show that the invariant measure has finite second moments and compute them explicitly.

1.2. Applications and Examples. Our main motivation for studying the long-time behavior of affine processes on positive Hilbert-Schmidt operators comes from
stochastic covariance modeling in Hilbert spaces, see [7, 8, 9, 14]. Generally speaking, a stochastic covariance model in an Hilbert space consist of two processes: one is a Hilbert space valued process that models some stochastic dynamics, e.g. forward curve dynamics in fixed income or commodity markets formulated in the HJMM framework. The second process models the instantaneous covariance process of the former and takes values in some space of positive and self-adjoint operators, e.g. the cone of positive and self-adjoint Hilbert-Schmidt operators. An affine stochastic covariance model is a stochastic covariance model such that the mixed Fourier-Laplace transform of the joint process satisfies an affine transform formula.

The affine stochastic covariance model proposed in [14] consists of a "price process" determined by a linear SDE in a Hilbert space driven by a Hilbert-valued Brownian motion which is modulated by the square-root of an affine process on positive Hilbert-Schmidt operators as introduced in [13]. Inspired by the univariate case in [43] we introduce the corresponding affine stochastic covariance model in the stationary covariance regime. Namely, we replace the affine instantaneous covariance process by the stationary affine process with the same transition probabilities, the existence of which is guaranteed by Corollary 3.3. In Proposition 4.1 we show that the affine stochastic covariance model in the stationary covariance regime satisfies an affine transform formula, which makes it a very tractable model for, e.g. pricing options written on forwards, see Section 4.2 below. As an example we derive the characteristic function of the operator valued Barndorff–Nielsen-Shepard (BNS) model established in [7] in the stationary covariance regime. This complements the literature on operator valued BNS type models by their long-time behavior. This was already studied for the matrix valued case in [4, 55].

Lastly, we present a geometric affine stochastic covariance model for commodity forward curve dynamics and consider a class of forward-start options written on forwards. We then study the implied forward volatility smile in the geometric affine stochastic covariance model and show in Proposition 4.2, that it converges to the implied spot volatility of a European call option written on the forward, but this time modeled in the stationary covariance regime. This extends a result in [43, Proposition 5.2] for forward-start options on (univariate) affine stochastic volatility (SV) models to an infinite-dimensional setting.

1.3. Layout of the article. In Section 2 we introduce the class of affine processes on $\mathcal{H}^+$ established in [13] and recall some preliminary results. Subsequently, in Section 3 we present and discuss our main results in full detail. Afterwards, in Section 4, we discuss applications of our results in the context of affine stochastic covariance models in Hilbert spaces. Finally, the proofs are contained in Section 5, which is subdivided into several subsections: first we consider the long-time behavior of the solutions of the generalized Riccati equations in Section 5.1, then prove the existence of a unique invariant measure in Section 5.2, derive the convergence rates in Section 5.3, show existence of stationary affine processes in Section 5.4 and lastly prove the moment formulas of the invariant measure in Section 5.5. For the readers convenience we added some background information on generalized Feller semigroups in Appendix A, where in particular we give a version of Kolmogorov’s extension theorem tailored to our needs. In Appendix B we give a convolution property for the Wasserstein distance of order 2.

2. Preliminaries: Affine processes on $\mathcal{H}^+$

We set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$. For a complex number $z \in \mathbb{C}$ we denote its real part by $\Re(z)$ and its imaginary part by $\Im(z)$. For a vector space $X$ and a subset $U \subseteq X$ we denote the linear span of $U$ in $X$ by $\text{lin}(U)$.
For $(X, \tau)$ a topological vector space and a subset $S \subseteq X$ we denote the Borel-$\sigma$-algebra generated by the relative topology on $S$ by $\mathcal{B}(S)$. We write $C_b(S)$ for the space of real-valued bounded functions on $S$ that are continuous with respect to the relative topology. $C_b(S)$ is a Banach space when endowed with the supremum norm $\| \cdot \|_{C_b(S)}$. For a Banach space $X$ with norm $\| \cdot \|_X$ we denote by $\mathcal{L}(X)$ the space of all bounded linear operators on $X$, which becomes a Banach space when equipped with the operator norm $\| \cdot \|_{\mathcal{L}(X)}$.

Throughout this article we let $(H, \langle \cdot, \cdot \rangle_H)$ be a separable real Hilbert space. We denote by $\mathcal{H}$ the set of all self-adjoint Hilbert-Schmidt operators from $H$ to $H$. This is a Hilbert space when endowed with the trace inner product $\langle A, B \rangle = \sum_{n=1}^{\infty} \langle Af_n, Bf_n \rangle_H$ for $A, B \in \mathcal{H}$ and where $(f_n)_{n \in \mathbb{N}}$ is an orthonormal basis for $H$. Note that $\langle \cdot, \cdot \rangle$ is independent of the choice of the orthonormal basis (see, e.g., [60, Section VI.6]). We denote by $\| \cdot \|$ the norm on $\mathcal{H}$ induced by $\langle \cdot, \cdot \rangle$. In addition, we define by $H^+$ the set of all positive operators in $\mathcal{H}$ i.e. $H^+ := \{ A \in \mathcal{H} : \langle Ah, h \rangle_H \geq 0 \text{ for all } h \in H \}$. Note that $H^+$ is a closed subset of $\mathcal{H}$. Moreover, it is a convex cone in $\mathcal{H}$ i.e. $H^+ + H^+ \subseteq H^+$, $\lambda H^+ \subseteq H^+$ for all $\lambda \geq 0$ and $H^+ \cap (-H^+) = \{ 0 \}$. The cone $H^+$ induces a partial ordering $\leq_{H^+}$ on $\mathcal{H}$, which is defined by $x \leq_{H^+} y$ whenever $y - x \in H^+$. The cone $H^+$ is also generating for $\mathcal{H}$ i.e. $\mathcal{H} = H^+ - H^+$. For a Hilbert space $(V, \langle \cdot, \cdot \rangle_V)$, in this article either $(H, \langle \cdot, \cdot \rangle_H)$ or $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, we denote the adjoint of $A \in \mathcal{L}(V)$ by $A^*$. For two elements $x$ and $y$ in $V$ we define the operator $x \otimes y \in \mathcal{L}(V)$ by $x \otimes y(h) = \langle x, h \rangle_V y$ for every $h \in V$ and write $x^{\otimes 2} := x \otimes x$.

2.1. Admissible parameters and moment conditions. Let $\chi: \mathcal{H} \to \mathcal{H}$ be given by $\chi(\xi) = \xi \mathbb{1}_{2|\xi| \leq 1}(\xi)$. The following definition of an admissible parameter set stems from [13, Definition 2.3].

**Definition 2.1.** An admissible parameter set $(b, B, m, \mu)$ consists of

i) a measure $m: \mathcal{B}(H^+ \setminus \{0\}) \to [0, \infty]$ such that

(a) $\int_{H^+ \setminus \{0\}} \| \xi \|^2 m(d\xi) < \infty$ and

(b) $\int_{H^+ \setminus \{0\}} |\langle \chi(\xi), h \rangle| m(d\xi) < \infty$ for all $h \in \mathcal{H}$ and there exists an element $l_m \in \mathcal{H}$ such that $\langle l_m, h \rangle = \int_{H^+ \setminus \{0\}} \langle \chi(\xi), h \rangle m(d\xi)$ for every $h \in \mathcal{H}$;

ii) a vector $b \in \mathcal{H}$ such that

$$\langle b, v \rangle - \int_{H^+ \setminus \{0\}} \langle \chi(\xi), v \rangle m(d\xi) \geq 0 \quad \text{for all } v \in H^+;$$

iii) a $H^+$-valued measure $\mu: \mathcal{B}(H^+ \setminus \{0\}) \to H^+$ such that the kernel $M(x, d\xi)$, for every $x \in H^+$ defined on $\mathcal{B}(H^+ \setminus \{0\})$ by

$$M(x, d\xi) := \frac{\langle x, \mu(d\xi) \rangle}{\| \xi \|^2}, \quad (2.1)$$

satisfies, for all $u, x \in H^+$ such that $\langle u, x \rangle = 0$,

$$\int_{H^+ \setminus \{0\}} \langle \chi(\xi), u \rangle M(x, d\xi) < \infty; \quad (2.2)$$

iv) an operator $B \in \mathcal{L}(\mathcal{H})$ with adjoint $B^*$ satisfying

$$\langle B^*(u), x \rangle - \int_{H^+ \setminus \{0\}} \langle \chi(\xi), u \rangle \frac{\langle \mu(d\xi), x \rangle}{\| \xi \|^2} \geq 0,$$

for all $x, u \in H^+$ with $\langle u, x \rangle = 0$.

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1See [5] and [50] for the notion of a vector valued measure and integration theory with respect to these.
Remark 2.2. 1) Part (a) of Definition 2.1 i) yields \( \int_{\mathcal{H}^+} \|\xi\|^2 m(d\xi) \leq \int_{\mathcal{H}^+} \|\xi\|^2 m(d\xi) < \infty \). Hence the integral \( \int_{\mathcal{H}^+} \|\xi\|^2 m(d\xi) \) is well-defined in the Bochner sense.

2) Similarly, the map \( u \mapsto \int_{\mathcal{H}^+} \langle \xi, u \rangle \frac{\mu(d\xi)}{\|\xi\|^2} \) is a bounded linear operator on \( \mathcal{H} \). Indeed, for \( v \in \mathcal{H} \) such that \( v = v^+ - v^- \) for \( v^+, v^- \in \mathcal{H}^+ \) we write \( |\langle \mu(d\xi), v \rangle| = |\langle \mu(d\xi), v^+ \rangle + \langle \mu(d\xi), v^- \rangle | \) and see that \( |\langle \mu(d\xi), v \rangle| \) is a positive measure for all \( v \in \mathcal{H} \). We thus have:

\[
\langle \int_{\mathcal{H}^+} \|\xi\|^2 m(d\xi) \rangle \leq \|u\| \left( \int_{\mathcal{H}^+} \langle \xi, u \rangle \frac{\mu(d\xi)}{\|\xi\|^2} \right) \leq \|u\| \langle \mu(\mathcal{H}^+) \rangle \cap \{ \|\xi\| > 1 \}, v \rangle, \]

taking the supremum over all \( v \in \mathcal{H} \) with \( \|v\| = 1 \) on both sides proves the boundedness of the map. Note that \( \|\mu(\mathcal{H}^+) \cap \{ \|\xi\| > 1 \} \| < \infty \), since by Definition 2.1 iii) for every \( A \in \mathcal{B}(\mathcal{H}^+ \setminus \{0\}) \) we have \( \mu(A) \in \mathcal{H}^+ \) and hence \( \|\mu(A)\| < \infty \).

Let \( u \in \mathcal{H} \). We define the constant and linear effective drift terms \( \hat{b} \) and \( \hat{B} \) as follows:

\[
\hat{b} := b + \int_{\mathcal{H}^+} \xi m(d\xi) \quad \text{and} \quad \hat{B}(u) := B^*(u) + \int_{\mathcal{H}^+} \langle \xi, u \rangle \frac{\mu(d\xi)}{\|\xi\|^2}.
\]

By the preceding Remark 2.2 we see that \( \hat{b} \in \mathcal{H} \) and \( \hat{B} \in \mathcal{L}(\mathcal{H}) \) are well-defined.

Now, we recall the main result in [13, Theorem 2.8] which ensures the existence of a broad class of affine processes on \( \mathcal{H}^+ \) associated with the admissible parameter set \( (b, B, m, \mu) \) satisfying the conditions in Definition 2.1.

Theorem 2.3. Let \( (b, B, m, \mu) \) be an admissible parameter set according to Definition 2.1. Then there exist a conservative time-homogeneous \( \mathcal{H}^+ \)-valued Markov process \( X \) with transition kernels \( (p_t(x, \cdot))_{t \geq 0} \), and constants \( K, \omega \in [1, \infty) \) such that

\[
\int_{\mathcal{H}^+} \|\xi\|^2 p_t(x, d\xi) \leq Ke^{\omega t}(\|x\|^2 + 1),
\]

and for all \( t \geq 0 \) and \( x, u \in \mathcal{H}^+ \) we have

\[
\int_{\mathcal{H}^+} e^{-\langle \xi, u \rangle} p_t(x, d\xi) = e^{-\phi(t,u) - \langle x, \psi(t,u) \rangle},
\]

where \( \phi(\cdot, u) \) and \( \psi(\cdot, u) \) are the unique solutions to the following generalized Riccati equations

\[
\begin{align*}
\frac{\partial \phi(t, u)}{\partial t} &= F(\psi(t, u)), & & t > 0, \quad \phi(0, u) = 0, \\
\frac{\partial \psi(t, u)}{\partial t} &= R(\psi(t, u)), & & t > 0, \quad \psi(0, u) = u,
\end{align*}
\]

where \( F: \mathcal{H}^+ \to \mathbb{R} \) and \( R: \mathcal{H}^+ \to \mathcal{H} \) are given by

\[
\begin{align*}
F(u) &= \langle b, u \rangle - \int_{\mathcal{H}^+ \setminus \{0\}} \left( e^{-\langle \xi, u \rangle} - 1 + \langle \chi(\xi), u \rangle \right) m(d\xi), \\
R(u) &= B^*(u) - \int_{\mathcal{H}^+ \setminus \{0\}} \left( e^{-\langle \xi, u \rangle} - 1 + \langle \chi(\xi), u \rangle \right) \frac{\mu(d\xi)}{\|\xi\|^2}.
\end{align*}
\]

Define the transition semigroup \( (P_t)_{t \geq 0} \) by

\[
P_tf(x) = \int_{\mathcal{H}^+} f(\xi)p_t(x, d\xi)
\]
for bounded measurable functions \( f : \mathcal{H}^+ \rightarrow \mathbb{R} \). Then \((P_t)_{t \geq 0}\) is a positive semigroup. Let \( \rho(x) = 1 + \|x\|^2 \) and define
\[
\|f\|_\rho = \sup_{x \in \mathcal{H}^+} \frac{|f(x)|}{\rho(x)}.
\]
Denote by \( B_\rho(\mathcal{H}^+) \) the Banach space of all measurable functions \( f : \mathcal{H}^+ \rightarrow \mathbb{R} \) for which \( \|f\|_\rho \) is finite. Clearly \((P_t)_{t \geq 0}\) extends onto \( B_\rho(\mathcal{H}^+) \) and satisfies for each \( f \in B_\rho(\mathcal{H}^+) \)
\[
|P_t f(x)| \leq \|f\|_\rho \int_{\mathcal{H}^+} \rho(y) p_t(x, dy) \leq \|f\|_\rho (1 + K) e^{\omega t} \rho(x), \quad x \in \mathcal{H}^+, \quad (2.8)
\]
i.e. \((P_t)_{t \geq 0}\) leaves \( B_\rho(\mathcal{H}^+) \) invariant. Let \( \mathcal{H}^+_w \) be the space \( \mathcal{H}^+ \) equipped with the weak topology, and denote by \( C_0(\mathcal{H}^+_w) \) the space of all bounded and weakly continuous functions \( f : \mathcal{H}^+ \rightarrow \mathbb{R} \). Finally let \( B_\rho(\mathcal{H}^+_w) \) be the closure of \( C_0(\mathcal{H}^+_w) \) in \( B_\rho(\mathcal{H}^+) \). It follows from [13] that \((P_t)_{t \geq 0}\) leaves \( B_\rho(\mathcal{H}^+_w) \) invariant and satisfies
\[
\lim_{t \to 0^+} P_t f(x) = f(x) \quad \text{for all } f \in B_\rho(\mathcal{H}^+_w) \text{ and } x \in \mathcal{H}^+_w. \quad \text{From (2.8) we then obtain}
\]
\[
\|P_t\|_{\mathcal{L}(B_\rho(\mathcal{H}^+_w))} \leq (1 + K) e^{\omega t}, \quad t \geq 0.
\]
Hence \((P_t)_{t \geq 0}\) is a generalized Feller semigroup on \( B_\rho(\mathcal{H}^+_w) \) (see the Appendix A for the definition and additional details).

**Remark 2.4.** Given the transition kernels \((p_t(x, \cdot))_{t \geq 0}\), the process \((X_t)_{t \geq 0}\) with initial value \( X_0 = x \in \mathcal{H}^+ \) can be constructed by a version of Kolmogorov’s extension theorem in [18, Theorem 2.11]. Indeed, for every \( x \in \mathcal{H}^+ \) one can show the existence of a unique measure \( \mathbb{P}_x \) on \( \Omega := (\mathcal{H}^+)^{\mathbb{R}^+} \), equipped with the \( \sigma \)-algebra generated by the canonical projections \( X_t : \Omega \rightarrow \mathcal{H}^+ \), given by \( X_t(\omega) = \omega(t) \) for \( \omega \in \Omega \), are measurable. For \( x \in \mathcal{H}^+ \) the probability measure \( \mathbb{P}_x \) is the distribution of \( X \) with \( \mathbb{P}_x(X_0 = x) = 1 \). We denote the expectation with respect to \( \mathbb{P}_x \) by \( \mathbb{E}_x \cdot \cdot \cdot \).

Let \((b, \mathcal{B}, m, \mu)\) be an admissible parameter set according to Definition 2.1 and let us denote by \((X_t)_{t \geq 0}\) the associated affine process on \( \mathcal{H}^+ \). Below we state explicit formulas for the first two moments of the process, see [13, Proposition 4.7] for a proof.

**Proposition 2.5.** Let \((X_t)_{t \geq 0}\) be the affine process associated with the admissible parameter set \((b, \mathcal{B}, m, \mu)\). Then for all \( v, w \in \mathcal{H}^+ \) the following formulas hold true:
\[
\mathbb{E}_x [\langle X_t, v \rangle] = \int_0^t \langle \hat{b}e^{sB}v, ds + \langle x, e^{sB}v \rangle \quad (2.9)
\]
and
\[
\mathbb{E}_x [\langle X_s, v \rangle \langle X_t, w \rangle] = \left( \int_0^t \langle \hat{b}e^{sB}v, ds + \langle x, e^{sB}v \rangle \rangle \left( \int_0^t \langle \hat{b}e^{sB}w, ds + \langle x, e^{sB}w \rangle \right) \right)
\]
\[
+ \int_0^t \int_{\mathcal{H}^+} \langle \xi, e^{sB}v \rangle \langle \xi, e^{sB}w \rangle m(d\xi) ds
\]
\[
+ \int_0^t \int_{\mathcal{H}^+} \langle \hat{b}e^{(s-u)B} \int_{\mathcal{H}^+} \langle \xi, e^{sB}v \rangle \langle \xi, e^{sB}w \rangle m(d\xi) \frac{\mu(d\xi)}{\|\xi\|^2} \rangle du ds
\]
\[
+ \int_0^t \langle x, e^{(t-s)B} \int_{\mathcal{H}^+} \langle \xi, e^{sB}w \rangle \langle \xi, e^{sB}v \rangle \frac{\mu(d\xi)}{\|\xi\|^2} \rangle ds. \quad (2.10)
\]
3. Main results

Let \( V_\tau := (V, \tau) \) be a topological vector space and denote by \( \mathcal{M}(V_\tau) \) the set of all probability measures defined on the Borel-\( \sigma \)-algebra \( \mathcal{B}(V_\tau) \). For the vector space \( \mathcal{H} \) equipped with its weak topology \( \tau_w \) we write \( \mathcal{H}_w = (\mathcal{H}, \tau_w) \). Note that the positive cone \( \mathcal{H}_w^+ \) is also closed in the weak topology and moreover, the Borel-\( \sigma \)-algebras of the strong and weak topology coincide, i.e., \( \mathcal{B}(\mathcal{H}^+) = \mathcal{B}(\mathcal{H}_w^+) \). We say that a measure \( \nu \in \mathcal{M}(\mathcal{H}^+) \) is inner-regular (with respect to the topology \( \tau \)), whenever

\[
\nu(A) = \sup \{ \nu(K) : K \subseteq A, K \text{ is } \tau\text{-compact} \}.
\]

For a sequence \( (\nu_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}(\mathcal{H}^+) \) we write \( \nu_n \rightharpoonup \nu \) as \( n \to \infty \) for the weak convergence of \( (\nu_n)_{n \in \mathbb{N}} \) to \( \nu \) in the strong topology i.e.

\[
\lim_{n \to \infty} \int_{\mathcal{H}^+} f(\xi) \nu_n(\text{d}\xi) = \int_{\mathcal{H}^+} f(\xi) \nu(\text{d}\xi) \quad \text{for all } f \in C_b(\mathcal{H}).
\]

For \( \nu_1, \nu_2 \in \mathcal{M}(\mathcal{H}^+) \) we call a probability measure \( G \), defined on the product Borel-\( \sigma \)-algebra \( \mathcal{B}(\mathcal{H}^+)^* \times \mathcal{B}(\mathcal{H}^+)^* \), a coupling of \( (\nu_1, \nu_2) \), whenever its marginal distributions are given by \( \nu_1 \) and \( \nu_2 \), respectively. We denote the set of all possible couplings of \( (\nu_1, \nu_2) \) by \( \mathcal{C}(\nu_1, \nu_2) \). For \( p \in [1, \infty) \) the Wasserstein distance of order \( p \) between \( \nu_1 \in \mathcal{M}(\mathcal{H}^+) \) and \( \nu_2 \in \mathcal{M}(\mathcal{H}^+) \) is defined as

\[
W_p(\nu_1, \nu_2) = \left( \inf \left\{ \int_{\mathcal{H}^+ \times \mathcal{H}^+} \|x - y\|^p G(\text{d}x, \text{d}y) : G \in \mathcal{C}(\nu_1, \nu_2) \right\} \right)^{1/p}.
\]

For an introduction to Wasserstein distances we refer to [58, Section 6].

Now, let \( (b, B, m, \mu) \) be an admissible parameter set and denote the spectrum of \( \hat{B} \), the operator defined in (2.3), by \( \sigma(\hat{B}) \). We introduce the following central assumption:

**Assumption A.** The spectral bound \( s(\hat{B}) := \sup \{ \Re(\lambda) : \lambda \in \sigma(\hat{B}) \} \) of \( \hat{B} \) in (2.3) is strictly negative, i.e. \( s(\hat{B}) < 0 \).

We call an affine process \( (X_t)_{t \geq 0} \) on \( \mathcal{H}^+ \) associated with an admissible parameter set \( (b, B, m, \mu) \) satisfying Assumption A a subcritical affine process on \( \mathcal{H}^+ \). Recall that \( \hat{B} \) is bounded and it generates the operator semigroup \( (e^{t\hat{B}})_{t \geq 0} \) given by \( e^{t\hat{B}} := \sum_{n=0}^{\infty} \frac{t^n \hat{B}^n}{n!} \), where the convergence of the series is understood in the \( \mathcal{L}(\mathcal{H}) \)-norm. It is well known that \( (e^{t\hat{B}})_{t \geq 0} \) is a uniformly continuous semigroup, see [22, Chapter I, Section 3], which implies that the spectral bound \( s(\hat{B}) \) coincides with the growth bound of \( (e^{t\hat{B}})_{t \geq 0} \), see [22, Corollary 4.2.4], i.e.

\[
s(\hat{B}) = \inf \left\{ w \in \mathbb{R} : \exists M_w \geq 1 \text{ s.t. } \|e^{t\hat{B}}\|_{\mathcal{L}(\mathcal{H})} \leq M_w e^{wt} \forall t \geq 0 \right\}.
\]

Therefore whenever Assumption A is satisfied, there exists a \( M \geq 1 \) and \( \delta > 0 \) such that

\[
\|e^{t\hat{B}}\|_{\mathcal{L}(\mathcal{H})} \leq M e^{-\delta t},
\]

in particular we could choose \( \delta = -s(\hat{B}) \). The following theorem is a detailed version of our main result concerning the long-time behavior of affine processes on the state-space \( \mathcal{H}^+ \):

**Theorem 3.1.** Let \( (b, B, m, \mu) \) be an admissible parameter set such that Assumption A is satisfied. Denote the associated subcritical affine process on \( \mathcal{H}^+ \) by \( (X_t)_{t \geq 0} \) and its transition kernels by \( (p_t(x, \cdot))_{t \geq 0} \). Then the following holds true:
i) There exists a unique invariant measure \( \pi \) for \((p_t(x,\cdot))_{t \geq 0}\) and the Laplace transform of \( \pi \) is given by

\[
\int_{\mathcal{H}^+} e^{-\langle u,x \rangle} \pi(dx) = \exp \left( -\int_0^\infty F(\psi(s,u)) \, ds \right), \quad u \in \mathcal{H}^+, \tag{3.2}
\]

where \( F \) and \( \psi(s,u) \) are as in Equation (2.7a) and (2.6b). Moreover, \( \pi \) is an inner-regular measure on \( \mathcal{B}(\mathcal{H}^+_{\text{loc}}) \).

ii) For \( p \in [1,2] \), \( t \geq 0 \) and \( x \in \mathcal{H}^+ \) we have

\[
W_p(p_t(x,\cdot), \pi) \leq C_1 e^{-\delta t} \left( \|x\| + \left( \int_{\mathcal{H}^+} \|g\|^p \pi(dy) \right)^{1/p} \right) \tag{3.3}
+ C_2 e^{-\delta/2t} \left( \|x\|^{1/2} + \left( \int_{\mathcal{H}^+} \|g\|^{p/2} \pi(dy) \right)^{1/p} \right), \tag{3.4}
\]

where \( C_1 = 2M \) and \( C_2 = 2^{1/2}M^{3/2}\delta^{-1/2}\|\mu(\mathcal{H}^+ \setminus \{0\})\|^{1/2} \) for \( M \geq 1 \) and \( \delta > 0 \) as in (3.1). In particular, we have \( p_t(x,\cdot) \Rightarrow \pi \) as \( t \to \infty \).

**Remark 3.2.**

1) For locally compact and second countable Hausdorff spaces, in particular for finite dimensional normed spaces, every probability measure defined on the Borel-\( \sigma \)-algebra is regular. The last assertion in Theorem 3.1 i) states that the invariant measure \( \pi \) is an inner-regular measure on \( \mathcal{B}(\mathcal{H}^+_{\text{loc}}) \), i.e. inner-regular in the weak topology, albeit \( \mathcal{H}^+_{\text{loc}} \) in the infinite dimensional case is not locally compact. We see that the inner-regularity is a non-trivial property of the invariant measure and we actually use it in the proof of Corollary 3.3 below.

2) For the case \( p = 1 \) we can compare the convergence rates obtained in Theorem 3.1 iii) with the ones in [28, Theorem 2.9] for the state space \( \mathbb{S}_d^+ \) (\( d \in \mathbb{N} \)) i.e. \( H = \mathbb{R}^d \) in our case. We see that instead of the square-root of the dimension \( d \in \mathbb{N} \) as it appears in the convergence rate in [28, equation 2.12], we have the additional term (3.4) which also converges exponentially fast as \( t \to \infty \), but with the exponential factor \(-\delta/2\) instead of \(-\delta\). However, the convergence rates here do not depend on the dimension of the state-space, in particular they hold true in infinite dimensions.

As a corollary from Theorem 3.1 i) which ensures the existence of an invariant inner-regular measure \( \pi \) on \( \mathcal{B}(\mathcal{H}^+_{\text{loc}}) \), we assert the existence of a stationary process with stationary measure \( \pi \). The only assertion that is left to prove here is, that we can start an affine process with transition kernels \( p_t(x,\cdot) \) at distribution \( \pi \) instead of \( \delta_v \):

**Corollary 3.3.** There exists a process \((X_t^x)_{t \geq 0}\) on \( \mathcal{H}^+ \) with transition kernels \((p_t(x,\cdot))_{t \geq 0}\) such that the distribution of \( X_t^x \) equals \( \pi \) for all \( t \geq 0 \).

Note here that the \( p \)-th absolute moment of \( \pi \) shows up in the convergence rate in (3.3), where we implicitly assumed that these terms are finite. That this is indeed the case is part of the next proposition, where we also prove explicit formulas for the first two moments of the invariant measures \( \pi \).

**Proposition 3.4.** Under the same conditions as in Theorem 3.1 and by denoting the unique invariant measure of \((p_t(x,\cdot))_{t \geq 0}\) by \( \pi \) we have \( \int_{\mathcal{H}^+} \|y\|^2 \pi(dy) < \infty \),

\[
\lim_{t \to \infty} \mathbb{E}_x [X_t] = \int_{\mathcal{H}^+} y \pi(dy) = \int_0^\infty e^{\delta t} \left( b + \int_{\mathcal{H}^+ \cap \{\|\xi\| > 1\}} \xi \, m(d\xi) \right) \, ds, \tag{3.5}
\]
and
\[
\lim_{t \to \infty} \mathbb{E}_x [X_t \otimes X_t] = \int_{\mathcal{H}^+} y \otimes y \pi(dy)
\]
\[
= \int_0^\infty (e^{B^t + b})^2 ds + \int_{\mathcal{H}^+ \setminus \{0\}} \int_0^\infty (e^{B^t + \xi})^2 m(d\xi) ds
\]
\[
+ \int_0^s \int_0^t \int_{\mathcal{H}^+ \setminus \{0\}} (e^{uB^t + \xi})^2 \hat{\pi}(s-u) B \mu(d\xi) \|\xi\|^2 du ds. \quad (3.6)
\]

**Remark 3.5.** It is well known that convergence in Wasserstein distance of order \(p \in [1, \infty)\) implies weak convergence and the convergence of the \(p\)-th absolute moment, see [58, Theorem 6.9]. However, we want to remark here that the proof of (3.5), given in Section 5.5, does not depend on the convergence in Wasserstein distance of order \(p = 2\) as established by Theorem 3.1 ii). Instead we solely use the generalized Feller property of the transition semigroups \((\hat{P}_t)_{t \geq 0}\) together with Proposition 2.5.

**Example 3.6 (Lévy driven Ornstein-Uhlenbeck processes).** Let \(m\) be a Lévy measure on \(B(\mathcal{H}^+ \setminus \{0\})\) with finite second moment and \(b \in \mathcal{H}^+\) such that Definition 2.1 ii) is satisfied. Let \(\mu = 0\) and \(B \in \mathcal{L}(\mathcal{H})\) be of the form \(B(u) = Gu + uG^*\) for some \(G \in \mathcal{L}(\mathcal{H})\), then Definition 2.1 iv) is satisfied, which can be seen from the fact that for every \(u \in \mathcal{H}^+\) we have \(e^{B^t + b} = e^{Gu + uG^*}\) for all \(t \geq 0\). Hence [49, Theorem 1] implies that \(B\) satisfies Definition 2.1 iv). Thus the tuple \((b, m, B, 0)\) is an admissible parameter set according to Definition 2.1 and the associated affine process \((X_t)_{t \geq 0}\) becomes an Ornstein-Uhlenbeck process driven by a \(\mathcal{H}^+\)-valued Lévy process \((L_t)_{t \geq 0}\) with characteristics \((b, 0, m)\), see [14, Lemma 2.3], i.e.

\[
X_t = e^{G^t} x e^{G^*} + \int_0^t e^{(t-s)G} dL_s e^{(t-s)G^*}, \quad t \geq 0.
\]

Since \(\sigma(B) = \sigma(G) + \sigma(G)\), see [56], and hence \(s(B) \leq s(G)\), we see that whenever the spectral bound \(s(G)\) of the operator \(G\) is negative, the same holds for \(s(B)\) and hence Assumption 3 is satisfied. This provides an explicit and simple sufficient criterion for the Ornstein-Uhlenbeck process \((X_t)_{t \geq 0}\) to be subcritical. By Theorem 3.1 there exists a unique invariant measure \(\pi\) with Laplace transform

\[
\int_{\mathcal{H}^+} e^{-\langle u, x \rangle} \pi(dx) = \exp \left( -\int_0^\infty \varphi_L (e^{sG} e^{sG^*}) ds \right),
\]

where \(\varphi_L : \mathcal{H} \to \mathbb{C}\) denotes the Laplace exponent of the Lévy process \(L\) given by

\[
\varphi_L(u) = \langle b, u \rangle - \int_{\mathcal{H}^+ \setminus \{0\}} (e^{-\langle \xi, u \rangle} - 1 + \langle \chi(\xi), u \rangle) m(d\xi), \quad u \in \mathcal{H}^+. \quad (3.7)
\]

Existence and uniqueness of invariant measures for Ornstein-Uhlenbeck processes were studied in [12], where a similar result follows under the weaker log-moment condition on the Lévy measure \(m\). Following Proposition 3.4 the stronger second moment assumption in our case allows us to deduce explicit formulas for the first and second moments of \(\pi\). Indeed, setting \(\mu = 0\) and \(B(u) = Gu + uG^*\) in (3.5) and (3.6) gives

\[
\lim_{t \to \infty} \mathbb{E}_x [X_t] = \int_0^\infty e^{sG} \left( b + \int_{\mathcal{H}^+ \setminus \{0\} \cap \|\xi\| > 1} \xi m(d\xi) \right) e^{sG^*} ds,
\]

and

\[
\lim_{t \to \infty} \mathbb{E}_x [X_t \otimes X_t] = \int_0^\infty (e^{sG^*} b e^{sG^*})^2 ds + \int_0^\infty \int_{\mathcal{H}^+ \setminus \{0\}} \left( e^{sG} \xi e^{sG^*} \right) ^2 m(d\xi) ds.
\]
4. Applications

In this section we discuss applications of our results in the context of affine stochastic covariance models in Hilbert spaces. In Section 4.1 we introduce an abstract Hilbert valued stochastic covariance model in the so called stationarity covariance regime and derive the stationary affine transform formulas for examples from the literature. In Section 4.2 we then consider a concrete example of an geometric affine stochastic covariance model describing the dynamics of commodity forward curves.

In particular, we show that the implied volatility of forward-start options written on forwards modeled by this model can be related to the implied volatility of plain vanilla options on forwards modeled under the stationary covariance regime.

4.1. Affine SV models in the stationary variance regime. Let \((X_t)_{t \geq 0}\) be an affine process on \(\mathcal{H}^+\) with with initial value \(X_0 = x\) and associated with an admissible parameter set \((b, B, m, \mu)\). Moreover, assume that \((Y_t)_{t \geq 0}\) is the unique (mild) solution to the following stochastic differential equation on some separable Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H})\):

\[ Y_t = S(t)y + \int_0^t S(t-s)G(X_s) \, ds + \int_0^t S(t-s)X_s^{1/2} \, dW_t, \quad t \geq 0, \quad (4.1) \]

where \(G: \mathcal{H} \to \mathcal{H}\) is a continuous affine linear function, \((W_t)_{t \geq 0}\) is a \(H\)-valued Brownian motion, independent of \((X_t)_{t \geq 0}\), with covariance operator \(Q \in \mathcal{L}_2(H)\), and \((S(t))_{t \geq 0}\) is a strongly continuous semigroup on \(H\) with generator \((\mathcal{A}, \text{dom}(\mathcal{A}))\).

We call the joint process \((Y_t, X_t)_{t \geq 0}\) an affine stochastic covariance model on \(H\). Examples for stochastic covariance models in a Hilbert space setting can be found in [9, 7, 14, 8]. Note that the joint process \((Y_t, X_t)_{t \geq 0}\) can be considered as a stochastic process on the filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_x) := (\Omega^1 \times \Omega^2, (\mathcal{F}^1 \otimes \mathcal{F}^2), (\mathcal{F}^1_t \otimes \mathcal{F}^2_t)_{t \geq 0}, \mathbb{Q} \otimes \mathbb{P}_x)\), where \((\Omega^2, \mathcal{F}^2, (\mathcal{F}^2_t)_{t \geq 0}, \mathbb{P}_x)\) denotes the filtered probability space accommodating the affine process \((X_t)_{t \geq 0}\), see also Remark 2.4, and \((\Omega^1, \mathcal{F}^1, (\mathcal{F}^1_t)_{t \geq 0}, \mathbb{Q})\) is another filtered probability space, that carries a \(\mathbb{Q}\)-Wiener process \(W: [0, \infty) \times \Omega \to \mathcal{H}\) and the solution process \((Y_t)_{t \geq 0}\) such that \(\mathbb{Q}(Y_0 = y) = 1\). From now on we write \((Y_t^y)_{t \geq 0}\) where the superscript \(y\) indicates the initial value of the process \((Y_t)_{t \geq 0}\). Moreover, we denote the expectation with respect to the product measure \(\mathbb{Q}_x\) by \(\mathbb{E}_x[\cdot]\).

Heuristically, the joint process \((Y_t^y, X_t)_{t \geq 0}\) satisfies a similar transform formula for its mixed Fourier-Laplace transform as the process \((X_t)_{t \geq 0}\) does for the Laplace transform in (1.1), see also [7, 14], which justifies the name affine stochastic covariance model. If moreover Assumption \(A\) is satisfied, then by Theorem 3.1 there exists a unique invariant measure \(\pi\) for \((p_t(x, \cdot))_{t \geq 0}\) and by Corollary 3.3 the existence of the stationary process \((X_t^\pi)_{t \geq 0}\) is ensured. Now, if there exists a mild solution \((\tilde{Y}_t)_{t \geq 0}\) of (4.1) for \(y = 0\) and where the process \((X_t)_{t \geq 0}\) is replaced by \((X_t^\pi)_{t \geq 0}\), then we call the joint process \((\tilde{Y}_t, X_t^\pi)_{t \geq 0}\), defined on \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}_\pi)\) with \(\mathbb{Q}_\pi = \mathbb{Q} \otimes \mathbb{P}_\pi\) and the expectation with respect to \(\mathbb{Q}_\pi\) denoted by \(\mathbb{E}_\pi[\cdot]\) an affine stochastic covariance model on \(H\) in the stationarity covariance regime. This terminology is inspired by the univariate setting in [43, Section 3].

Below we consider a particular class of models constructed in [14] which we call SV model with affine pure-jump variance. Namely, let \((Y_t^u, X_t)_{t \geq 0}\) be the process with first component given by (4.1) and the second component given by the affine process \((X_t)_{t \geq 0}\), where we assume that it has càdlàg paths and that there exists a positive and self-adjoint \(D \in \mathcal{L}(H)\) such that \(X_t^{1/2}QX_t^{1/2} = D^{1/2}X_tD^{1/2}\) for all \(t \geq 0\), compare with [14, Assumption \(B\) and \(C\)]. It was shown that under these conditions the stochastic covariance model \((Y_t^u, X_t)_{t \geq 0}\) is well defined for
every initial value \((y, x) \in H \times H^+\). In the following proposition we give an affine transform formula for this model in the stationary covariance regime:

**Proposition 4.1.** Assume that \((Y_t^\psi, X_t)_{t \geq 0}\) is an affine stochastic covariance model satisfying the assumptions above and let \((\tilde{Y}_t, X_t^\psi)_{t \geq 0}\) be the model in the stationary covariance regime. Then for every \(T \geq 0\) and \(u = (u_1, u_2) \in iH \times H^+\) we have

\[
E_\pi \left[ e^{\langle \tilde{Y}_t, u_1 \rangle - \langle X_t^\psi, u_2 \rangle} \right] = e^{-\Phi(t, u_1, u_2) - \int_0^t F(\psi_1(s, 0, \psi_2(t, u_1, u_2))) \, ds}, \quad t \in [0, T],
\]

where \(\Phi(\cdot, u_1, u_2), \psi_1(\cdot, u_1, u_2)\) and \(\psi_2(\cdot, u_1, u_2)\) are the unique solutions on \([0, T]\) of the following differential equations:

\[
\frac{d\Phi}{dt}(t, u) = F(\psi_2(t, u)), \quad \Phi(0, u) = 0, \quad (4.3a)
\]

\[
\psi_1(t, u) = u_1 - iA^* \left( \int_0^t \psi_1(s, u) \, ds \right), \quad \psi_1(0, u) = u_1, \quad (4.3b)
\]

\[
\frac{d\psi_2}{dt}(t, u) = R(\psi_1(t, u), \psi_2(t, u)), \quad \psi_2(0, u) = u_2, \quad (4.3c)
\]

where \(F\) is as in (2.7a), \(R(h, u) := R(u) - \frac{1}{2}D^{1/2}h \otimes D^{1/2}h\) with \(R\) as in (2.7b), and \((A^*, D(A^*))\) denotes the adjoint operator of the generator \((A, D(A))\) of \((S(t))_{t \geq 0}\).

**Proof.** Let \(T \geq 0\) and let \((Y_t^\psi, X_t)_{t \in [0, T]}\) be the mild solution to (4.1) on \([0, T]\) satisfying the assumptions above. From [14, Theorem 3.3] we recall the following affine transform formula for the mixed Fourier-Laplace transform of \((Y_t^\psi, X_t)\) for \(t \in [0, T]\) and \(u = (u_1, u_2) \in iH \times H^+\):

\[
E_x \left[ e^{\langle Y_t^\psi, u_1 \rangle - \langle X_t^\psi, u_2 \rangle} \right] = e^{-\Phi(t, u_1, u_2) - \langle x, \psi_2(t, u_1, u_2) \rangle}, \quad x \in H^+,
\]

where \(\Phi(\cdot, u_1, u_2)\) and \(\psi_2(\cdot, u_1, u_2)\) are the unique strong solutions to (4.3a) and (4.3c), respectively, and \(\psi_1(\cdot, u_1, u_2)\) is the unique mild solution of (4.3b). Note that for every \(F^2\)-measurable and bounded function \(f\) we have

\[
\int_{\Omega_2} f(\omega_2) \, dP_x(\omega_2) = \int_{H^+} \left( \int_{\Omega_2} f(\omega_2) \, dP_x(\omega_2) \right) \pi(dx).
\]

From this and (4.4) we conclude

\[
E_\pi \left[ e^{\langle \tilde{Y}_t, u_1 \rangle - \langle X_t^\psi, u_2 \rangle} \right] = \int_{\Omega_2} \left( \int_{\Omega_1} e^{\langle Y_t, u_1, u_2 \rangle - \langle X_t^\psi, u_2 \rangle} \, dQ(x_1) \right) \, dP_\pi(\omega_2)
\]

\[
= \int_{H^+} E_x \left[ e^{\langle Y_t, u_1 \rangle - \langle X_t, u_2 \rangle} \right] \pi(dx)
\]

\[
= \int_{H^+} e^{-\Phi(t, u_1, u_2) - \langle x, \psi_2(t, u_1, u_2) \rangle} \pi(dx). \quad (4.5)
\]

From Theorem 3.1(i) it then follows that

\[
\int_{H^+} e^{-\langle x, \psi_2(t, u_1, u_2) \rangle} \pi(dx) = \exp \left( - \int_0^\infty F(\psi(s, \psi_2(t, u_1, u_2))) \, ds \right), \quad (4.6)
\]

where \(\psi(\cdot, u)\) is given by (2.6b). From (4.3c) and the definition of \(R\) we see that \(\psi_2(t, 0, u_2) = \psi(t, u_2)\) for every \(u_2 \in H^+\), hence multiplying both sides of (4.6) by \(e^{-\Phi(t, u_1, u_2)}\) together with (4.5) yields the desired formula (4.2). \(\square\)

In a more specific setting, the authors in [7] proposed an operator Barndorff-Nielsen-Shepard (BNS) stochastic covariance model which can be described by the following pair of Hilbert valued SDEs:

\[
\begin{align*}
\frac{dY_t}{dt} &= A(Y_t) \, dt + X_t^{1/2} \, dW_t, \quad Y_0 = y \in H, \\
\frac{dX_t}{dt} &= B(X_t) \, dt + dL_t, \quad X_0 = x \in H^+,
\end{align*}
\]

(4.7)
where $B$ and $(L_t)_{t \geq 0}$ are as in Example 3.6. Note that $(Y^y_t)_{t \geq 0}$ is as in (4.1) with $G = 0$ written in the differential form, where $(S(t))_{t \geq 0}$ is the strongly continuous semigroup generated by $(\mathcal{A}, \text{dom}(\mathcal{A}))$. For the $\mathcal{H}^+$-valued Ornstein-Uhlenbeck process $(X_t)_{t \geq 0}$ we already showed the existence of a unique invariant measure $\pi$ of $(X_t)_{t \geq 0}$ in Example 3.6. Hence we may consider the operator BNS model in the stationary covariance regime and denote it by $(\tilde{Y}_t, X^y_t)_{t \geq 0}$. In [14] it was shown that the operator valued BNS model is a particular case of the SV models with affine pure-jump variance as introduced above. Hence we obtain from Proposition 4.1 applied to this particular case

$$\mathbb{E}_{\pi}\left[e^{\tilde{Y}_t, u_1} (X^y_t, u_2) | \pi\right] = \exp\left(-\int_0^t \varphi_L\left(\psi(s, u_1, u_2)\right) \, ds - \int_0^\infty \varphi_L\left(e^{sB} u_2\right) \, ds\right),$$

for every $(u_1, u_2) \in \mathbb{R} \times \mathcal{H}^+$, where $\varphi_L$ is given by (3.7) and $\psi(t, u_1, u_2)$ is explicitly known as

$$\psi(t, u_1, u_2) = e^{sB} u_2 + \frac{1}{2} \int_0^s \varphi_s (e^{(s-\tau)B}(D^{1/2}S^\ast(\tau)u_1)\otimes^2 \, d\tau.$$}

### 4.2. Forward curve dynamics and forward-start options on forwards.

In this section we consider a concrete example of a geometric affine stochastic covariance model on a Hilbert space which describes the dynamics of forward curves in fixed income or commodity markets. Then, in Proposition 4.2 we study the long-time behavior of the forward implied volatility smile in this model. First, we recall from [23, 25] the class of Hilbert spaces consisting of forward curves: for $\beta > 0$ we denote by $H_\beta$ the space of all absolutely continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that $\|f\|_\beta := (\int_{\mathbb{R}^+} e^{\beta|\tau|} |f'(x)|^2 \, dx)^{1/2} < \infty$. The space $H_\beta$ is a separable Hilbert space when equipped with the inner product

$$(f, g)_\beta = f(0)g(0) + \int_{\mathbb{R}^+} e^{\beta|\tau|} f'(x)g'(x) \, dx.$$}

Moreover, we note that the left-shift semigroup, denoted by $(S(t))_{t \geq 0}$, is strongly continuous on $H_\beta$ and for every $t \in \mathbb{R}^+$ the point evaluation maps $\delta_t : H_\beta \to \mathbb{R}$ are continuous linear functionals, see [25, Theorem 2.1]. Throughout this section we shall identify the point evaluation maps $\delta_t (t \in \mathbb{R}^+)$ with an element $u_t \in H_\beta$ such that $(f, u_t)_\beta = \delta_t (f)$.

Let $0 \leq T \leq \hat{T}$ and denote by $F(T, \hat{T})$ the forward price at time $T$ with delivery/maturity date $\hat{T}$, e.g. $F(T, \hat{T})$ denotes the price at time $T$ for the delivery of one unit of an underlying spot commodity at time $\hat{T}$. We follow the HJMM approach and model the dynamics of $(F(T, \hat{T}))_{T \leq \hat{T}}$ directly by means of a (transformed) stochastic differential equation on $H_\beta$. Namely, we let $(\tilde{Y}^y_t, X^y_t)_{t \geq 0}$ be an affine stochastic covariance model as in (4.1) on the Hilbert space $H_\beta$ and where $(S(t))_{t \geq 0}$ is the left-shift semigroup on $H_\beta$. Then for $0 \leq T \leq \hat{T}$ we set

$$F(T, \hat{T}) := \delta_{\hat{T}-T}(\exp(\tilde{Y}^y_T)) = \exp((\tilde{Y}^y_T, u_{\hat{T}-T})_\beta). \quad (4.8)$$

A geometric forward curve model of the type in (4.8) was proposed in [8] to model the dynamics of forward curve dynamics in commodity markets. In their case, the underlying stochastic covariance model $(Y^y_t, X^y_t)_{t \geq 0}$ is the Hilbert valued BNS model from (4.7) with an additional leverage term. The consideration of a geometric model has the advantage of producing positive forward curves which is a crucial characteristic in many forward markets. Here, we extend the geometric Hilbert valued BNS model to the larger class of $SV$ models with affine pure-jump variance as introduced in Section 4.1. Moreover, we make the assumption that we already model under a risk-neutral measure $\tilde{Q}$, i.e. we assume that $(F(T, \hat{T}))_{T \leq \hat{T}}$ is a $\tilde{Q}$-martingale for all $\hat{T} \geq 0$. We refer to [8, Proposition 6.8] for a sufficient
condition on the function $G$ that ensures the existence of a risk-neutral measure. In the following we focus on forwards in commodity markets, see also [6].

A forward-start option with forward-start date $\tau \geq 0$, forward maturity $T$ and strike $e^K$ written on a forward with maturity date $\hat{T}$ is defined as an European option with pay-off at time $\tau + T$ given by

$$
\left( \frac{F(\tau + T, \tau + \hat{T})}{F(\tau, \tau + T)} - e^K \right)^+. \tag{4.9}
$$

A forward-start option is a contract on the relative price difference of a forward contract at two times, $\tau$ and $\tau + T$, in the future. In practice, it is used to price future volatility of the underlying asset. Forward-start options are very common in commodity forward markets and more complex derivatives such as Cliquet options are building up on these see, e.g. [15]. Forward-start options on stocks are discussed in, e.g. [35, 36, 46, 43]. Here we restrict ourselves to the ratio-type pay-off functions in (4.9), but similar results can be obtained for the difference-type pay-offs, i.e. $(F(\tau + T, \tau + T) - K)F(\tau, \tau + T))^+$, see also [46].

We now define the implied forward volatility smile of the model (4.8). First, let us denote the price of a forward-start option with pay-off (4.9) by $C_{\text{fws}}(\tau, T, \hat{T}, K)$. Then, as a reference model for the forward prices $F(T, \hat{T})$ we take Black’s model, see [10], and denote by this model by $F^B(T, \hat{T})$. We assume that the following spot-forward relation holds:

$$
F^B(T, \hat{T}) = s_T e^{(\hat{T} - T)}, \quad 0 \leq T \leq \hat{T}, \tag{4.10}
$$

where $r \geq 0$ denotes the risk-free interest rate and $(s_t)_{t \geq 0}$ denotes the spot price process of the underlying commodity, which is given by a geometric Brownian motion with volatility parameter $\sigma$. We denote by $C_{\text{fws}}^B(\tau, T, \hat{T}, K, \sigma)$ the price of a forward-start option with identical pay-off function as in (4.9) in Black’s model and define the implied forward volatility smile $\sigma(\tau, T, \hat{T}, K)$ as the unique solution to $C_{\text{fws}}^B(\tau, T, \hat{T}, K, \sigma(\tau, T, \hat{T}, K)) = C_{\text{fws}}(\tau, T, \hat{T}, K)$. In the following proposition we show that $\sigma(\tau, T, \hat{T}, K)$ exists for all $\tau, K \geq 0$ and study its long-time behavior as $\tau \to \infty$:

**Proposition 4.2.** Let $0 \leq T \leq \hat{T}$ and denote by $F(T, \hat{T})$ the forward price at time $T$ with maturity date $\hat{T}$ given by (4.8), where $(Y^\nu_t, X^\nu_t)_{t \geq 0}$ is an affine stochastic covariance model on $H_\beta$ as defined in Section 4.1 with $(S(t))_{t \geq 0}$ the left-shift semigroup on $H_\beta$. Moreover, let $(Y^\nu_t, X^\nu_t)_{t \geq 0}$ be the model in the stationary covariance regime and define $\tilde{F}(T, \hat{T}) := \exp((\bar{Y}_T, u_{\hat{T} - T})\beta)$. Suppose we model directly under the pricing measure $\tilde{Q}$ such that $(\tilde{F}(T, \hat{T}))_{T \leq \hat{T}}$ is a $\tilde{Q}$-martingale for all $\hat{T} \geq 0$. Then for all $\tau, K \geq 0$ the implied forward volatility smile $\sigma(\tau, T, \hat{T}, K)$ exists and we have

$$
\lim_{\tau \to \infty} \sigma(\tau, T, \hat{T}, K) = \hat{\sigma}(T, \hat{T}, K), \tag{4.11}
$$

where $\hat{\sigma}(T, \hat{T}, K)$ denotes the implied volatility of a European call option with pay-off function $(\tilde{F}(T, \hat{T}) - K)^+$. 

**Proof.** First, we show a certain relation between the price of a European call option $C^B(T, \hat{T}, K, \sigma)$, where $0 \leq T \leq \hat{T}$ and $K \geq 0$, and the forward-start call option $C_{\text{fws}}^B(\tau, T, \hat{T}, K, \sigma)$ with forward start date $\tau \geq 0$ in Black’s model. Namely, let $Q$ denote the unique risk-neutral measure in Black’s model and recall $C_{\text{fws}}^B(\tau, T, \hat{T}, K, \sigma)$ the price of a forward-start option at time zero with forward-start date $\tau$ written on
the forward with maturity $\hat{T}$. Inserting (4.10) into the pay-off function and by risk-neutral pricing we have

$$C_{fws}^{B} (\tau, T, \hat{T}, K, \sigma) = e^{-r(\tau+T)}E_{\mathbb{Q}}\left[ \frac{S_{\tau+T}}{S_{\tau}} e^{r\tau} - e^{K} \right]^{+}.$$ 

It is known that in the Black-Scholes model the forward-start option and the European call option satisfy: $e^{-r(\tau+T)}E_{\mathbb{Q}}\left[ \left( \frac{S_{T}}{S_{\tau}} - K \right)^{+} \right] = e^{-r(\tau+T)}E_{\mathbb{Q}}\left[ (s_{T} - K)^{+} \right]$, see also [43], and hence

$$C_{fws}^{B} (\tau, T, \hat{T}, K, \sigma) = e^{-r(\tau+T)} e^{-rT}E_{\mathbb{Q}}\left[ \frac{S_{\tau+T}}{S_{\tau}} - e^{K} \right]^{+}$$

$$= e^{-r(\tau+T)} e^{-rT}E_{\mathbb{Q}}\left[ (s_{T} - e^{K})^{+} \right]$$

$$= e^{-r(\tau+T)}C_{fws}^{BS} (T, K', \sigma),$$

where $K' = K + rT$ and by the superscript BS we indicate that the underlying model for $(s_{t})_{t \geq 0}$ is the Black-Scholes model. From this and the definition of the implied forward volatility smile $\sigma(\tau, T, \hat{T}, K)$ we have

$$C_{fws}^{BS} (T, K', \sigma(\tau, T, \hat{T}, K)) = e^{r(\tau+T)}C_{fws}^{B} (\tau, T, \hat{T}, K, \sigma(\tau, T, \hat{T}, K))$$

$$= e^{r(\tau+T)}C_{fws} (\tau, T, \hat{T}, K). \quad (4.12)$$

Next, we compute the right-hand side of (4.12). Recall that for every $t \in \mathbb{R}$ and $f \in H_{\beta}$ we use the identification $b_{t}(f) = (f, u_{t})$ for the evaluation functional with $u_{t} \in H_{\beta}$. Moreover, we denote the expectation with respect to the pricing measure $\mathbb{Q}$ by $E_{\mathbb{Q}}[\cdot]$ (here we suppress the initial value $x$ compared to $Q_{x}$ above). The payoff function of the forward-start option is given by (4.9), hence by risk-neutral pricing and inserting our model (4.8) we have

$$C_{fws} (\tau, T, \hat{T}, K) = e^{-r(\tau+T)}E_{\mathbb{Q}}\left[ \left( \frac{F(\tau+T, \tau + \hat{T})}{F(\tau, \tau + \hat{T})} - e^{K} \right)^{+} \right]$$

$$= e^{-r(\tau+T)}E_{\mathbb{Q}}\left[ (e^{Y_{\tau+T, u_{\tau+T}} - (Y_{\tau} u_{\tau})} - e^{K})^{+} \right]. \quad (4.13)$$

Note that by definition of $Y_{\tau}$ we have

$$\langle S(T)Y_{\tau}, u_{\tau-T} \rangle_{\beta} = \langle S(T+\tau)y, u_{\tau-T} \rangle_{\beta} + \langle \int_{0}^{\tau} S(T + \tau - s)G(X_{s}) \, ds, u_{\tau-T} \rangle_{\beta}$$

$$+ \langle \int_{0}^{\tau} S(T + \tau - s)X_{s}^{1/2} \, dW_{s}, u_{\tau-T} \rangle_{\beta},$$

and the left-shift $S(T)$ satisfies $\langle S(T)Y_{\tau}, u_{\tau-T} \rangle_{\beta} = \langle Y_{\tau} u_{\tau} \rangle_{\beta}$, thus we conclude that

$$\langle Y_{\tau+T}^{y}, u_{\tau-T} \rangle_{\beta} - \langle Y_{\tau} u_{\tau} \rangle_{\beta} = \langle \int_{\tau}^{\tau+T} S(T + \tau - s)G(X_{s}) \, ds, u_{\tau-T} \rangle_{\beta}$$

$$+ \langle \int_{\tau}^{\tau+T} S(T + \tau - s)X_{s}^{1/2} \, dW_{s}, u_{\tau-T} \rangle_{\beta}. \quad (4.14)$$

By the independent increments property and the Markov property of $(X_{t})_{t \geq 0}$ the sum of the integrals inside the inner products on the right-hand side of (4.14) has the same distribution as $Y_{\tau}^{y} = \int_{0}^{T} S(T - s)G(X_{\tau+s}) \, ds + \int_{0}^{T} S(T - s)X_{\tau+s}^{1/2} \, dW_{s}$. Hence for the expectation on the right-hand side in (4.13) we obtain

$$E_{\mathbb{Q}}\left[ (e^{Y_{\tau+T, u_{\tau-T}} - (Y_{\tau} u_{\tau})} - e^{K})^{+} \right] = E_{\mathbb{Q}}\left[ E \left[ (e^{Y_{\tau} u_{\tau}} - e^{K})^{+} \mid X_{\tau} \right] \right],$$
and thus we conclude that the left-hand side of (4.12) is given by
\[ C^{\text{BS}}(T, K', \sigma(\tau, T, \hat{T}, K)) = E_\tilde{Q} \left[ e^{\left((e^{Y_{T, u_{\hat{T}-\tau}}}_{T} - e^{K'})^+\right) X_{\tau}} \right]. \]

Now taking the limit \( \tau \to \infty \) and since \( \tilde{\Phi}(T, \hat{T}) = e^{(Y_{T, u_{\hat{T}-\tau}} - \sigma)^+} \) we obtain
\[ \lim_{\tau \to \infty} C^{\text{BS}}(T, K', \sigma(\tau, T, \hat{T}, K)) = E_\tilde{Q} \left[ e^{(Y_{T, u_{\hat{T}-\tau}} - \sigma)^+} \right] = E_\tilde{Q} \left[ (\tilde{\Phi}(T, \hat{T}) - e^{K'})^+ \right]. \quad (4.15) \]

The term \( E_\tilde{Q} \left[ (\tilde{\Phi}(T, \hat{T}) - e^{K'})^+ \right] \) on the right-hand side of (4.15) is precisely the price of an European call option remunerated by \( e^{\tau T} \) and we have
\[ \lim_{\tau \to \infty} C^{\text{BS}}(T, K', \sigma(\tau, T, \hat{T}, K)) = C^{\text{BS}}(T, K', \lim_{\tau \to \infty} \sigma(\tau, T, \hat{T}, K)), \]
from which we conclude (4.11), since equation (4.15) has a unique solution in terms of the Black-Scholes implied volatility. \( \square \)

**Remark 4.3.** By Proposition 4.2 we can approximate the option prices of forward-start option for large forward dates \( \tau \) by European plain-vanilla options modeled in the stationary covariance regime. The relevance of this result becomes evident when noting that option pricing in affine stochastic covariance models (in particular in the stationary covariance regime) can be conducted with a reasonable computational effort. Indeed, since an affine transform formula is provided by Proposition 4.1, the option prices can be computed quasi-explicitly via Fourier inversion, see e.g. [11, 41, 34]. In an ongoing work we study option pricing in general affine stochastic covariance models on Hilbert spaces, derive the quasi-explicit option pricing formulas for popular option types, e.g. European call or Spread options in commodity forward markets, and conduct a numerical analysis of the pricing procedure.

5. **Proof of the main results**

Throughout this section we assume that \((h, B, m, \mu)\) is an admissible parameter set according to Definition 2.1. We denote the unique subcritical affine process associated with \((h, B, m, \mu)\), through Theorem 2.3, by \((X_t)_{t \geq 0}\) and its family of transition kernels by \(\langle p_t(x, \cdot)\rangle_{t \geq 0}\). We set \(Rf := \int_{\mathcal{H}} f(\xi) p_t(\cdot, d\xi)\) for all measurable functions \(f\) such that the integral exists. Recall from [13] that the transition semigroup \(\{P_t\}_{t \geq 0}\) is a generalized Feller semigroup on the space \(\mathcal{B}_b(\mathcal{H}_+^+),\) see also Appendix A.

5.1. **Some properties of the generalized Riccati equations** (2.6a)-(2.6b).

In this section we consider the long-time behavior of the solutions \(\phi(\cdot, u)\) and \(\psi(\cdot, u)\) of the generalized Riccati equations in (2.6a)-(2.6b). We recall from [13, Section 3] that for every \(u \in \mathcal{H}^+\) there exists a unique and global solution \(\psi(\cdot, u) \in C^1(\mathbb{R}^+, \mathcal{H}^+)\) to (2.6b). Given \(\psi(\cdot, u)\) we solve the first equation (2.6a) by mere integration and obtain \(\phi(\cdot, u) \in C^1(\mathbb{R}^+, \mathcal{H}^+)\) given by \(\phi(t, u) = \int_0^t F(\psi(s, u)) \, ds\). This means that we can write the affine transform formula (2.5) as
\[ \int_{\mathcal{H}^+} e^{-\langle z, u \rangle} p_t(x, d\xi) = \exp \left( -\int_0^t F(\psi(s, u)) \, ds - \langle x, \psi(t, u) \rangle \right). \]
Moreover, we recall that the unique solution \(\psi(\cdot, u)\) to (2.6b) satisfies the flow equation:
\[ \psi(t + s, u) = \psi(t, \psi(s, u)). \quad (5.1) \]

In the next lemma we show that \(F\) and \(R\) are continuous functions on \(\mathcal{H}^+\) and grow at most quadratically:
Lemma 5.1. Let \((b,B,m,\mu)\) be an admissible parameter set according to Definition 2.1 and let \(F\) and \(R\) be given by (2.7a) and (2.7b), respectively. Then \(F\) and \(R\) are continuous on \(\mathcal{H}^+\) and for all \(u \in \mathcal{H}^+\) we have

\[
\|F(u)\| \leq \left(\|b\| + \int_{\mathcal{H}^+ \setminus \{0\}} \|\xi\|^2 \, m(d\xi) \right) (\|u\| + \|u\|^2),
\]
and

\[
\|R(u)\| \leq (\|B\|_{\mathcal{L}(\mathcal{H})} + \|\mu(\mathcal{H}^+ \setminus \{0\})\|) (\|u\| + \|u\|^2).
\]

Proof. Note that for all \(\xi, u \in \mathcal{H}^+\) we have

\[
\left| e^{-\langle \xi, u \rangle} - 1 + \langle \chi(\xi), u \rangle \right| \leq \frac{1}{2} \langle \xi, u \rangle^2 \mathbf{1}_{\{\|\xi\| \leq 1\}} + \langle \langle \xi, u \rangle \mathbf{1}_{\{\|\xi\| > 1\}} \right|
\]

\[
\leq \frac{1}{2} \langle \xi, u \rangle^2 \mathbf{1}_{\{\|\xi\| \leq 1\}} + \|\xi\| \|u\| \mathbf{1}_{\{\|\xi\| > 1\}},
\]
from which (5.2), (5.3), and the continuity of \(F, R\) readily follows (by dominated convergence).

Assumption \(A\) implies that the semigroup \((e^{tB})_{t \geq 0}\) satisfies (3.1), that is \((e^{tB})_{t \geq 0}\) is uniformly exponential stable. This has the following consequence on the solution \(\psi(\cdot, u)\) of the generalized Riccati equation (2.6b):

Lemma 5.2. Let \((b,B,m,\mu)\) be an admissible parameter set according to Definition 2.1 and for \(u \in \mathcal{H}^+\) let \(\psi(\cdot, u)\) be the unique solution to (2.6b). Then

\[
\|\psi(t, u)\| \leq \|e^{tB}\|_{\mathcal{L}(\mathcal{H})} \|u\|, \quad \forall t \geq 0.
\]

If moreover Assumption \(A\) is satisfied, then \(\lim_{t \to \infty} \psi(t, u) = 0\).

Proof. First note that whenever \((b,B,m,\mu)\) is an admissible parameter set according to Definition 2.1, then so is \((0, B, 0, \mu)\). Therefore the existence of an affine Markov process \((Y_t)_{t \geq 0}\) associated to \((0, B, 0, \mu)\) and initial value \(Y_0 = x\) is guaranteed by Theorem 2.3. Note that the unique solution \(\psi(\cdot, u)\) to (2.6b) is \(\mathcal{H}^+\)-valued. Let \(u \in \mathcal{H}^+\) and note that due to the convexity of the exponential function and Jensen’s inequality we have

\[
e^{-\mathbb{E}_x[\langle u, Y_t \rangle]} \leq \mathbb{E}_x \left[e^{-\langle \psi(t, u), x \rangle} \right] = \exp(-\langle \psi(t, u), x \rangle).
\]

Then for all \(u, x \in \mathcal{H}^+\):

\[
\langle \psi(t, u), x \rangle \leq \mathbb{E}_x [\langle u, Y_t \rangle] = \langle x, e^{tB}u \rangle \\
\leq \|x\| \|u\| \|e^{tB}\|_{\mathcal{L}(\mathcal{H})}.
\]

For fixed \(t \geq 0\) and \(u \in \mathcal{H}^+\) we choose \(x = \psi(t, u) \in \mathcal{H}^+\) and obtain

\[
\|\psi(t, u)\|^2 \leq \|\psi(t, u)\| \|u\| \|e^{tB}\|_{\mathcal{L}(\mathcal{H})},
\]

which proves the first statement. If Assumption \(A\) is satisfied, then from (5.5) and (3.1) it follows that \(\|\psi(t, u)\| \leq M e^{-\delta t} \|u\|\) and hence \(\lim_{t \to \infty} \psi(t, u) = 0\).

5.2. Invariant measure for affine processes on \(\mathcal{H}^+\). For two measures \(\nu_1, \nu_2 \in \mathcal{M}(\mathcal{H}^+)\) we denote the convolution of \(\nu_1\) and \(\nu_2\) by \(\nu_1 * \nu_2\). In the following lemma we give an important convolution property of the transition kernels \(p_t(x, \cdot)\).

Lemma 5.3. Let \((Y_t)_{t \geq 0}\) be the unique affine process associated with the admissible parameter set \((0, B, 0, \mu)\) and denote its transition kernels by \((q_t(x, \cdot))_{t \geq 0}\). Then for every \(t \geq 0\) and \(x \in \mathcal{H}^+\) we have

\[
p_t(x, \cdot) = p_t(0, \cdot) * q_t(x, \cdot).
\]
Proof. Since $b = 0$ and $m = 0$ the function $F$ in (2.6a) vanishes, see also (2.7a), and thus $\phi(t, u) = 0$ for all $t \geq 0$. Hence for every $t \geq 0$ the affine-transform formula (2.5) for $Y_t$ takes the form

\[
\int_{\mathcal{H}^+} e^{-(u, \xi)} q_t(x, d\xi) = \exp \left(-\langle \psi(t, u), x \rangle \right), \quad \text{for } u \in \mathcal{H}^+. \tag{5.7}
\]

Now, let $(X_t)_{t \geq 0}$ denote the unique affine process associated with the admissible parameter set $(b, B, m, \mu)$ and denote its transition kernels by $p_t(x, \cdot)$. Let $t \geq 0$ arbitrary and $u \in \mathcal{H}^+$, then

\[
\int_{\mathcal{H}^+} e^{-(u, \xi)} p_t(0, \cdot) * q_t(x, \cdot)(d\xi) = \int_{\mathcal{H}^+} \left( \int_{\mathcal{H}^+} e^{-(u, \xi_1 + \xi_2)} p_t(0, d\xi_1) \right) q_t(x, d\xi_2)
\]

\[
= e^{-\phi(t, u)} \int_{\mathcal{H}^+} e^{-(u, \xi_2)} q_t(x, d\xi_2)
\]

\[
= e^{-\phi(t, u)} e^{-(\psi(t, u), x)},
\]

which completes the proof thanks to (2.5) and the fact that the functions $x \mapsto e^{-(u, x)}$ characterize measures, see [14, Lemma A.1]. □

In the next proposition we show that the Laplace transform of a subcritical affine process converges pointwise as the time $t$ tends to infinity.

**Proposition 5.4.** Let $(X_t)_{t \geq 0}$ be an affine process associated with the admissible parameter set $(b, B, m, \mu)$ satisfying Assumption A. Then for all $u \in \mathcal{H}^+$ and for all $x \in \mathcal{H}^+$ we have

\[
\lim_{t \to \infty} \mathbb{E}_x \left[ e^{-(u, X_t)} \right] = \exp \left(-\int_0^\infty F(\psi(s, u)) \, ds \right) \in [0, \infty). \tag{5.8}
\]

Proof. Let $u \in \mathcal{H}^+$ and $x \in \mathcal{H}^+$, then by Lemma 5.2 and (3.1) we have

\[
|\psi(t, u), x| \leq \|\psi(t, u)\| \|x\| \leq \|e^B\|_{L(\mathcal{H})} \|x\| \|u\| \leq M e^{-\delta t} \|x\| \|u\|.
\]

Lemma 5.1 gives

\[
|F(\psi(t, u))| \leq C \left( \|\psi(t, u)\| + \|\psi(t, u)^2\| \right) \leq C M^2 e^{-\delta s} (\|u\| + \|u\|^2), \tag{5.9}
\]

with $C = \|b\| + \int_{\mathcal{H}^+ \setminus \{0\}} \|\xi\|^2 \, m(d\xi)$. For every $u \in \mathcal{H}^+$ this implies

\[
\int_0^\infty |F(\psi(s, u))| \, ds \leq \frac{CM^2}{\delta} (\|u\| + \|u\|^2) < \infty,
\]

and hence the limit $\lim_{t \to \infty} \phi(t, u) = \int_0^\infty F(\psi(s, u)) \, ds$ exists for every $u \in \mathcal{H}^+$. This, the continuity of the exponential function and the fact that by Lemma 5.2 $\langle \psi(t, u), x \rangle \to 0$ for all $x, u \in \mathcal{H}^+$ as $t \to \infty$ imply (5.8). □

The next lemma asserts uniform boundedness in time of the transition semigroup $(P_t)_{t \geq 0}$ in the operator norm on $\mathcal{B}_p(\mathcal{H}^+)$. 

**Lemma 5.5.** Let $(X_t)_{t \geq 0}$ be an affine process associated with the admissible parameter set $(b, B, m, \mu)$ satisfying Assumption A and denote its transition semigroup by $(P_t)_{t \geq 0}$. Then we have

\[
\sup_{t \geq 0} \|P_t\|_{L(\mathcal{B}_p(\mathcal{H}^+))} < \infty. \tag{5.10}
\]

Proof. Recall that $\rho(x) = 1 + \|x\|^2$ and note that for every $f \in \mathcal{B}_p(\mathcal{H}^+) \setminus \{0\}$ we have $|f(y)| \leq \|f\| \|\rho\| \|f\|_{\mathcal{B}_p}$ and hence

\[
\|P_t f\|_{\mathcal{B}_p(\mathcal{H}^+)} = \sup_{x \in \mathcal{H}^+} \rho(x)^{-1} \left| \int_{\mathcal{H}^+} f(y) p_t(x, d\xi) \right| \leq \|f\|_{\mathcal{B}_p(\mathcal{H}^+)} \|P_t \rho\|_{\mathcal{B}_p(\mathcal{H}^+)},
\]
which yields \( \sup_{t \geq 0} \| P_t \|_{\mathcal{L}(\mathcal{B}_s(\mathcal{H}^+))} \leq \sup_{t \geq 0} \| P_t \rho \|_{\mathcal{B}_s(\mathcal{H}^+)} \). Let \((e_i)_{i \in \mathbb{N}}\) be an orthonormal basis of \(\mathcal{H}\) and recall that by [13, Remark 4.6] we have \( P_t \rho(x) = E_x [\rho(X_t)] \) for all \( t \geq 0 \). Hence by Parseval’s identity we conclude

\[
0 \leq P_t \rho(x) = 1 + \mathbb{E}_x [\|X_t\|^2] = 1 + \sum_{i=1}^{\infty} \mathbb{E}_x [(X_t, e_i)^2].
\]

Using (2.10) with \( v = w = e_i \) for \( i \in \mathbb{N} \) we find

\[
\mathbb{E}_x [(X_t, e_i)^2] = \left( \int_0^t \langle \hat{b}, e^{s\hat{B}} e_i \rangle \, ds + \left( \int_0^t \int_{\mathcal{H}^+ \setminus \{0\}} \langle \xi, e^{s\hat{B}} e_i \rangle^2 \frac{m(d\xi)}{\|\xi\|^2} \right) \, ds \right) + \int_0^t \left( \int_0^t \int_{\mathcal{H}^+ \setminus \{0\}} \langle \xi, e^{s\hat{B}} e_i \rangle \frac{\mu(d\xi)}{\|\xi\|^2} \right) \, du \, ds + \int_0^t \left( \int_0^t \int_{\mathcal{H}^+ \setminus \{0\}} \langle \xi, e^{s\hat{B}} e_i \rangle \frac{\mu(d\xi)}{\|\xi\|^2} \right) \, ds,
\]

and hence

\[
\sum_{i=1}^{\infty} \mathbb{E}_x [(X_t, e_i)^2] \leq 2 \left( \int_0^t \|e^{s\hat{B}} \hat{b}\|^2 \, ds \right)^2 + 2 \|e^{t\hat{B}} x\|^2 + \int_0^t \int_{\mathcal{H}^+ \setminus \{0\}} \|e^{s\hat{B}} \xi\|^2 \frac{m(d\xi)}{\|\xi\|^2} \, ds + \int_0^t \int_0^s \int_{\mathcal{H}^+ \setminus \{0\}} \|e^{s\hat{B}} \xi\|^2 \frac{\mu(d\xi)}{\|\xi\|^2} \, du \, ds + \int_0^t \int_0^s \int_{\mathcal{H}^+ \setminus \{0\}} \|e^{s\hat{B}} \xi\|^2 \frac{\mu(d\xi)}{\|\xi\|^2} \, ds. \tag{5.11}
\]

In the following we show that every term on the right-hand side of (5.11)-(5.14) converges as \( t \to \infty \) uniformly in \( x \), which then yields (5.10).

Note first that the adjoint semigroup \((e^{t\hat{B}})_{t \geq 0}\) generated by \(\hat{B}^*\), the adjoint of \(\hat{B}\), is also uniformly stable as \( \| e^{t\hat{B}} \|_{\mathcal{L}(\mathcal{H})} = \| e^{t\hat{B}} \|_{\mathcal{L}(\mathcal{H})} \) for all \( t \geq 0 \). For the first term on the right-hand side of (5.11) we have \( \int_0^t \left\| e^{s\hat{B}} \hat{b} \right\|^2 \, ds \leq \frac{M^2}{2} \left\| \hat{b} \right\|^2 \). The second term in (5.11) vanishes as \( t \to \infty \), since \((e^{t\hat{B}})_{t \geq 0}\) is uniformly stable. Note that \( s \mapsto M^2 e^{-2\delta s} \int_{\mathcal{H}^+ \setminus \{0\}} \|\xi\|^2 \, m(d\xi) \) is an integrable majorant for the term in (5.12) and thus the integral converges for \( t \to \infty \). For (5.13) note that \( \langle \hat{b}, e^{(s-u)\hat{B}} \frac{\mu(d\xi)}{\|\xi\|^2} \rangle \geq 0 \) for every \( s, u \in \mathbb{R}^+ \), which follows from the admissible parameter conditions, which imply that \( \hat{b} \in \mathcal{H}^+ \) and \( e^{(s-u)\hat{B}}(\mathcal{H}^+) \subseteq \mathcal{H}^+ \), whenever \( s \geq u \). Hence we have

\[
\int_0^\infty \int_0^s \int_{\mathcal{H}^+ \setminus \{0\}} \|e^{r\hat{B}} \xi\|^2 \left( \langle \phi, e^{(s-u)\hat{B}} \frac{\mu(d\xi)}{\|\xi\|^2} \rangle \right) \, du \, dr \leq \frac{3M^2}{2} \sup_{s \geq u} \left\| \hat{b} \right\|^2 \|\mu(\mathcal{H}^+ \setminus \{0\})\|.
\]

Finally note that \( \int_0^t e^{-2\delta s} e^{-\delta(t-s)} \, ds = \frac{1}{\delta}(e^{-\delta t} - e^{-2\delta t}) \) and hence the last term in (5.14) vanishes as \( t \to \infty \), which can be seen from

\[
\int_0^t \int_{\mathcal{H}^+ \setminus \{0\}} \|e^{r\hat{B}} \xi\|^2 \left( \langle \phi, e^{(t-s)\hat{B}} \frac{\mu(d\xi)}{\|\xi\|^2} \rangle \right) \, ds \leq M^3 \|\mu(\mathcal{H}^+ \setminus \{0\})\| \left\| \phi \right\| \int_0^t e^{-2\delta s} e^{-\delta(t-s)} \, ds \leq \frac{M^3}{\delta} \|\mu(\mathcal{H}^+ \setminus \{0\})\| \left\| \phi \right\| (e^{-\delta t} - e^{-2\delta t}). \tag{5.15}
\]
Thus we proved that $\sup_{t \geq 0} \sup_{x \in H^+} E_x [\rho(X_t)] < \infty$, which proves the statement. \hfill \square

In the next proposition we show first that for every $f \in B_0(H_w^+)$ the transition semigroup $(P_t)_{t \geq 0}$ converges in $B_0(H_w^+)$ as $t \to \infty$ and subsequently use this to define a continuous linear functional on $B_0(H_w^+)$ given by the limits.

**Proposition 5.6.** For all $f \in B_0(H_w^+)$ the limit $\lim_{t \to \infty} P_t f$ exists in $B_0(H_w^+)$ and $\pi(f) := \lim_{t \to \infty} P_t f(x)$ defines a continuous linear functional on $B_0(H_w^+)$.\[\]

**Proof.** By Proposition 5.4 we know that for every $u \in H^+$

$$
\lim_{t \to \infty} (P_t e^{-(u,\cdot)})(x) = e^{-\int_0^\infty F(\psi(s,u)) \, ds}, \quad \forall x \in H^+.
$$

Define, for $u \in H^+$, $\pi_u := e^{-\int_0^\infty F(\psi(s,u)) \, ds} \mathbf{1}$ where $\mathbf{1}$ denotes the constant one function. We claim that the sequence $(P_t e^{-(u,\cdot)})_{t \geq 0}$ converges in $B_0(H_w^+)$ to the constant function $\pi_u \in B_0(H_w^+)$. Indeed, we have

$$
\|P_t e^{-(u,\cdot)} - \pi_u\| = \sup_{x \in H^+} \frac{\left| e^{-\int_0^t F(\psi(s,u)) \, ds} - e^{-\int_0^\infty F(\psi(s,u)) \, ds} \right|}{\rho(x)} \leq \frac{\left| \int_t^\infty F(\psi(s,u)) \, ds \right|}{\rho(x)} \leq \int_{\infty}^\infty |F(\psi(s,u))| \, ds \sup_{x \in H^+} \frac{\|x\|}{\rho(x)},
$$

where we have used $\rho(x) = 1 + \|x\|^2$. The first term converges to zero due to $(5.9)$, while the second term tends to zero by Lemma 5.2.

Let $D := \left\{ e^{-(u,\cdot)} : u \in H^+ \right\}$ and define $\pi$ as the linear extension of $\pi_u$ onto $D$. In particular, we have $\lim_{t \to \infty} P_t f = \pi(f)$ in $B_0(H_w^+)$ for every $f \in D$. In view of Proposition 5.5 we have $\sup_{t \geq 0} \|P_t\|_{L(B_0(H_w^+))} < \infty$ and hence $|\pi(f)| \leq \sup_{t \geq 0} \|P_t\|_{L(B_0(H_w^+))} \|f\|_{B_0(H_w^+)}$, i.e. $\pi$ is bounded on $D$. Since $D$ is dense in $B_0(H_w^+)$, see [13, Lemma 4.7], this means that there exists a unique extension of $\pi$ to a continuous linear functional on $B_0(H_w^+)$, which we also denote by $\pi$. We thus proved the existence of $\pi \in L(B_0(H_w^+), \mathbb{R})$ and it is only left to show that $P_t f \to \pi(f)$ as $t \to \infty$ for all $f \in B_0(H_w^+)$. The latter one is an immediate consequence of an $\varepsilon/3$-argument using $\sup_{t \geq 0} \|P_t\|_{L(B_0(H_w^+), \mathbb{R})} < \infty$ and $\overline{D} = B_0(H_w^+)$. Thus we conclude the assertion. \hfill \square

In the following lemma we prove that the functional $\pi$ is represented by a unique probability measure on $B(H^+)$:

**Lemma 5.7.** Let $\pi$ denote the continuous linear functional from Proposition 5.6. Then there exists a unique probability measure $\nu$ on $H^+$ such that

$$
\pi(f) = \int_{H^+} f(\xi) \, \nu(d\xi) \quad \text{for all } f \in B_0(H_w^+),
$$

and $\nu$ is inner-regular on $B(H^+)$ when $H^+$ is equipped with the weak topology.

**Proof.** By an application of the Riesz-representation theorem in [18, Theorem 2.4] there exists a unique finite signed Radon measure $\nu$ on $B(H^+)$ such that (5.16) and

$$
\int_{H^+} (1 + \|x\|^2) |\nu|(d\xi) = \|\pi\|_{L(B_0(H_w^+), \mathbb{R})}
$$

hold. Here $|\nu|$ denotes the total variation measure of $\nu$. Note that $\nu$ is a Radon measure with respect to the weak topology on $H^+$, which implies the statement on
the inner-regularity. It is left to prove that $\nu$ is a probability measure. Note that since $\lim_{t \to \infty} P_t 1(x) = 1$ we have $\pi(1) = 1$ and hence $\nu(H^+) = 1$. Moreover, as $P_t f \geq 0$ for all non-negative $f \in C_b(H^+_w)$ and all $t \geq 0$, we have $\lim_{t \to \infty} P_t f(x) \geq 0$ for all $x \in H^+$ and hence $\int_{H^+} f(\xi) \nu(d\xi) \geq 0$ for all non-negative $f \in C_b(H^+_w)$, which implies that the measure $\nu$ is also non-negative and hence it is a probability measure on $B(H^+)$. \hfill \Box

In the following we identify the linear functional $\pi$ with the measure $\nu$ given by Lemma 5.7 and write $\pi$ instead of $\nu$. Finally we show that $\pi$ is, indeed, the unique invariant measure of $(p_t(x, \cdot))_{t \geq 0}$.

**Proposition 5.8.** Let $(b, B, m, \mu)$ be an admissible parameter set such that Assumption $\mathcal{A}$ is satisfied and denote the associated subcritical affine Markov process on $H^+$ by $(X_t)_{t \geq 0}$ and its transition kernels by $(p_t(x, d\xi))_{t \geq 0}$. Then there exists a unique invariant measure $\pi$ for $(p_t(x, \cdot))_{t \geq 0}$. Moreover, for every $x \in H^+$ we have

$$\lim_{t \to \infty} \int_{H^+} f(\xi) p_t(x, d\xi) \to \int_{H^+} f(\xi) \pi(d\xi), \quad \forall f \in C_b(H^+_w),$$

and the Laplace transform of $\pi$ is given by (3.2).

**Proof.** In Proposition 5.6 and the subsequent arguments, we have already shown the existence of the Borel measure $\pi$ such that (5.18) holds. It is left to show that $\pi$ is the unique invariant measure. We have

$$\int_{H^+} e^{-(u, \xi)} \left( \int_{H^+} p_t(x, d\xi) \pi(dx) \right) = \int_{H^+} \left( \int_{H^+} e^{-\langle u, \xi \rangle} p_t(x, d\xi) \right) \pi(dx)$$

$$= e^{-\phi(t, u)} \int_{H^+} e^{-\langle x, \psi(t, u) \rangle} \pi(dx).$$

Note that by (5.1) we have $\psi(t + s, u) = \psi(t, \psi(s, u))$ and hence for every $u \in H^+$ we have

$$e^{-\phi(t, u)} \int_{H^+} e^{-\langle x, \psi(t, u) \rangle} \pi(dx) = e^{-\phi(t, u)} e^{-\int_0^\infty F(\psi(s, \psi(t, u))) \, ds}$$

$$= e^{-\phi(t, u)} e^{-\int_0^\infty F(\psi(t+s, u)) \, ds}$$

$$= e^{-\phi(t, u)} e^{-\int_0^\infty F(\psi(s, u)) \, ds}$$

$$= e^{-\int_0^\infty F(\psi(s, u)) \, ds}$$

$$= \int_{H^+} e^{-\langle x, u \rangle} \pi(dx).$$

This proves the invariance of $\pi$. Next, we prove that $\pi$ is the unique invariant measure. Suppose there exists a $\pi' \in \mathcal{M}(H^+)$ which is invariant for $p_t(x, d\xi)$, then for every $u \in H^+$ and $t \geq 0$ we have:

$$\int_{H^+} e^{-\langle x, u \rangle} \pi'(dx) = \int_{H^+ \setminus \{0\}} e^{-\langle u, \xi \rangle} \left( \int_{H^+ \setminus \{0\}} p_t(x, d\xi) \pi'(dx) \right)$$

$$= \int_{H^+ \setminus \{0\}} e^{-\phi(t, u)} - \langle x, \psi(t, u) \rangle \pi'(dx),$$

now by letting $t \to \infty$ we find that

$$\int_{H^+ \setminus \{0\}} e^{-\langle u, x \rangle} \pi'(dx) = \exp \left( - \int_0^\infty F(\psi(s, u)) \, ds \right).$$

The Laplace transform is measure determining for measures on $B(H^+)$ and hence $\pi = \pi'$. \hfill \Box
Remark 5.9. The convergence in (5.18) is weak convergence of $p_t(x, \cdot)$ to $\pi$ as $t \to \infty$ in the weak topology on $\mathcal{H}$. Even though the Borel algebras of $\mathcal{H}$ equipped with the norm topology and weak topology coincide, the weak convergence is different in general. We say that $p_t(x, \cdot) \rightharpoonup \pi$ as $t \to \infty$ weakly in the weak topology on $\mathcal{H}^+$, whenever $P_t f(x) \to \int_{\mathcal{H}^+} f(\xi) \pi(d\xi)$ for all $f \in C_b(\mathcal{H}^+)$.

If the stronger assumption $P_t f(x) \to \int_{\mathcal{H}^+} f(\xi) \pi(d\xi)$ for all $f \in C_b(\mathcal{H}^+)$ holds, we speak of the usual weak convergence, i.e., $p_t(x, \cdot) \Rightarrow \pi$ as $t \to \infty$. By [54, Theorem 1 and 2] we know that weak convergence in the weak topology together with

$$\lim_{N \to \infty} \sup_{n \in \mathbb{N}} p_t(x, A_{N,n}) = 0, \quad \text{for all } \varepsilon > 0,$$

where $A_{N,n} := \{ \sum_{i=1}^{\infty} (x, e_i)^2 \geq \varepsilon \}$ for $N, n \in \mathbb{N}$, implies $p_t(x, \cdot) \Rightarrow \pi$ as $t \to \infty$. Note that in our main Theorem 3.1 we assert weak convergence in the strong topology, which will be shown below.

5.3. Proof of Theorem 3.1. Proposition 5.8 ensures the existence of a unique invariant measure $\pi$ of $(p_t(x, \cdot))_{t \geq 0}$ with Laplace transform (3.2). We also proved weak convergence of $(p_t(x, \cdot))_{t \geq 0}$ to $\pi$ as $t \to \infty$ in the weak topology. What is left to show is the convergence rates in Wasserstein distance of order $p$ for $p \in [1, 2]$ as in (3.3). Then convergence in Wasserstein distance of some order $p \in [1, \infty)$ implies weak convergence (in the strong topology) and convergence of the $p$-th absolute moment, see [58, Theorem 6.9]. This implies the last assertion of Theorem 3.1. In the remainder we prove the convergence rates (3.3).

Let $p \in [1, 2]$ and as before we denote by $q_t(x, d\xi)$ the transition kernel of an affine process associated with the admissible parameter set $(0, B, 0, \mu)$. Let $t \geq 0$, $x \in \mathcal{H}^+$ and $G \in \mathcal{C}(\delta_x, \pi)$ i.e. $G$ is a coupling with marginals $\delta_x$ and $\pi$. Note that

$$p_t(x, dy) = \int_{\mathcal{H}^+} p_t(z, dy) \delta_x(dz) = \int_{\mathcal{H}^+ \times \mathcal{H}^+} p_t(z, dy) 1(z') G(dz, dz')$$

and by the invariance of $\pi$ we also have

$$\pi(dy) = \int_{\mathcal{H}^+} p_t(z', dy) \pi(dz') = \int_{\mathcal{H}^+ \times \mathcal{H}^+} p_t(z', dy) 1(z) G(dz, dz').$$

Thus by the convexity property in [58, Theorem 4.8] and since $W_p \leq W_2$ for $p \in [1, 2]$ we have

$$W_p(p_t(x, \cdot), \pi) = W_p\left(\int_{\mathcal{H}^+} p_t(z, \cdot) \delta_x(dz), \int_{\mathcal{H}^+} p_t(y, \cdot) \pi(dy)\right) \leq \left(\int_{\mathcal{H}^+ \times \mathcal{H}^+} W_2(p_t(z, \cdot), p_t(y, \cdot))^p G(dz, dy)\right)^{1/p}. \quad (5.19)$$

By Lemma 5.3 we have $p_t(z, \cdot) = q_t(z, \cdot) * p_t(0, \cdot)$ for every $t \geq 0$. Thus for $H \in \mathcal{C}(q_t(z, \cdot), q_t(y, \cdot))$ we obtain by Lemma B.1 that

$$W_2(p_t(z, \cdot), p_t(y, \cdot))^p \leq W_2(q_t(z, \cdot), q_t(y, \cdot))^p \leq \left(\int_{\mathcal{H}^+ \times \mathcal{H}^+} \|\tilde{x} - \tilde{y}\|^2 H(d\tilde{x}, d\tilde{y})\right)^{p/2} \leq \left(2 \int_{\mathcal{H}^+ \times \mathcal{H}^+} (\|\tilde{x}\|^2 + \|\tilde{y}\|^2) H(d\tilde{x}, d\tilde{y})\right)^{p/2} = \left(2 \int_{\mathcal{H}^+ \times \mathcal{H}^+} \|\tilde{x}\|^2 q_t(z, d\tilde{x}) + 2 \int_{\mathcal{H}^+ \times \mathcal{H}^+} \|\tilde{y}\|^2 q_t(y, d\tilde{y})\right)^{p/2}. \quad (5.20)$$
Now, recall from (5.11) that
\[
\int_{\mathcal{H}^+} \|\tilde{x}\|^2 q(z, d\tilde{x}) \leq 2\|e^{\tilde{B}^*}z\|^2 + \int_0^t \int_{\mathcal{H}^+ \setminus \{0\}} \|e^{\tilde{B}^*}\xi\|^2 (z, e^{(t-s)\tilde{B}} \frac{\mu(d\xi)}{\|\xi\|^2}) \, ds,
\]
while all the other terms vanish as \( \hat{b} = 0 \) and \( m = 0 \). By the same estimations as in (5.15) we conclude that
\[
\int_{\mathcal{H}^+} \|\tilde{x}\|^2 q(z, d\tilde{x}) \leq 2\|e^{\tilde{B}^*}z\|^2 + \int_0^t \int_{\mathcal{H}^+ \setminus \{0\}} \|e^{\tilde{B}^*}\xi\|^2 (z, e^{(t-s)\tilde{B}} \frac{\mu(d\xi)}{\|\xi\|^2}) \, ds
\leq 2M^2 e^{-2\delta t} \|z\|^2 + \frac{M^3}{\delta} \|\mu(\mathcal{H}^+ \setminus \{0\})\| e^{-\delta t} \|z\|.
\]
Inserting this back into (5.20) and using the sub-additivity of \( x \mapsto x^{p/2} \) for \( p \in [1, 2] \), we obtain
\[
W_2(p_t(z, \cdot), p_t(y, \cdot))^p \leq (C_1 e^{-\delta t} \|z\|)^p + (C_2 e^{-\delta t/2} \|z\|^{1/2})^p
+ (C_1 e^{-\delta t} \|y\|)^p + (C_2 e^{-\delta t/2} \|y\|^{1/2})^p,
\]
for \( C_1 = 2M \) and \( C_2 = 2^{1/2} M^{1/2} \delta^{-1/2} \|\mu(\mathcal{H}^+ \setminus \{0\})\|^{1/2} \). Now, plugging (5.21) back into (5.19) and again by the sub-additivity of \( x \mapsto x^{1/p} \), we obtain the desired (3.3).

5.4. Proof of Corollary 3.3. For every \( x \in \mathcal{H}^+ \) let \( (p_t(x, \cdot))_{t \geq 0} \) be the transition kernels associated to the admissible parameter set \((b, B, m, \mu)\) by Theorem 2.3. Moreover, let \( \pi \) be the unique invariant distribution of \( (p_t(x, \cdot))_{t \geq 0} \) (which is independent of \( x \in \mathcal{H}^+ \). From Theorem 3.1 i) we know that \( \pi \) is inner-regular. Thus from Proposition A.5 we conclude the existence of a unique Markov process \((X_t^\pi)_{t \geq 0}\) such that for all \( f \in B_p(\mathcal{H}_1^w) \) we have \( \mathbb{E}_\pi [f(X_t^\pi)] = \int_{\mathcal{H}^+} f(x) \pi(dx) \). Moreover, since \( \pi \) is the invariant measure we have for each \( t \geq 0 \)
\[
\int_{\mathcal{H}^+} p_t f(x) \pi(dx) = \int_{\mathcal{H}^+} \left( \int_{\mathcal{H}^+} f(\xi)p_t(x, \xi) \right) \pi(dx) = \int_{\mathcal{H}^+} f(\xi) \pi(d\xi),
\]
which implies that for all \( t \geq 0 \) the random variable \( X_t^\pi \) has distribution \( \pi \).

5.5. Proof of Proposition 3.4. Let us denote the space of all Hilbert-Schmidt operators on \( \mathcal{H} \) by \( \mathcal{L}_2(\mathcal{H}) \) and note that \((e_i \otimes e_j)_{i,j \in \mathbb{N}}\) is an orthonormal basis of \( \mathcal{L}_2(\mathcal{H}) \). For every \( y \in \mathcal{H} \) the operator \( y \otimes y : \mathcal{H} \to \mathcal{H} \) defined by \( y \otimes y(x) = (x, y) \) for every \( x \in \mathcal{H}^+ \) is a Hilbert-Schmidt operator on \( \mathcal{H} \) and we can write \( y \otimes y = \sum_{i,j=1}^\infty (y, e_i) (y, e_j) e_i \otimes e_j \). Note that by (5.17) we have \( \int_{\mathcal{H}^+} \rho(\xi) \pi(d\xi) = \|\pi\|_{\mathcal{L}(B_p(\mathcal{H}_1^w), \mathbb{R})} < \infty \) and hence the absolute second moment of \( \pi \) is finite, which implies
\[
\int_{\mathcal{H}^+} \|y \otimes y\|_{\mathcal{L}_2(\mathcal{H})} \pi(dy) \leq \int_{\mathcal{H}^+} \text{Tr}(y \otimes y) \pi(dy) \leq \int_{\mathcal{H}^+} \|y\|^2 \pi(dy) < \infty,
\]
and hence the integral \( \int_{\mathcal{H}^+} y \otimes y \pi(dy) \) is well-defined in the Bochner sense. Thus it remains to compute the first two moments of the invariant distribution.

Note that for every \( u \in \mathcal{H} \) the linear functional \( \langle u, \cdot \rangle : \mathcal{H} \to \mathbb{R} \) satisfies the following two properties:

i) for every \( R > 0 \) we have \( \langle u, \cdot \rangle \in C_0(K_u^R) \) where the set
\[
K_u^R := \{ x \in \mathcal{H}^+: \|x\|^2 + 1 \leq R \}
\]
is compact in \( \mathcal{H}^+ \) equipped with the weak topology and

ii) \( \lim_{R \to \infty} \sup_{x \in \mathcal{H}^+ \setminus K_u^R} \|\langle u, x\rangle(1 + \|x\|^2)^{-1} = 0. \)
which by [20, Theorem 2.7] implies \((u, \cdot) \in \mathcal{B}_1(\mathcal{H}_u^+)\) for all \(u \in \mathcal{H}\). By Proposition 5.6 we have \(P_t f \to \pi(f)\) as \(t \to \infty\) for all \(f \in \mathcal{B}_p(\mathcal{H}_u^+)\) and hence also \(P_t \langle u, \cdot \rangle \to \int_{\mathcal{H}^+} \langle u, \xi \rangle \pi(d\xi)\) as \(t \to \infty\). Let \((e_i)_{i \in \mathbb{N}}\) be an orthonormal basis of \(\mathcal{H}\). Then by (2.9) for \(u = e_i\) for \(i \in \mathbb{N}\) we have

\[
\lim_{t \to \infty} P_t \langle e_i, \cdot \rangle = \lim_{t \to \infty} \left( \int_0^t \langle \hat{b}, e^{sB} e_i \rangle \, ds + \langle x, e^{tB} e_i \rangle \right) = \int_0^\infty \langle \hat{b}, e^{tB} e_i \rangle \, ds
\]

and since \(\xi = \sum_{i=1}^\infty \langle \xi, e_i \rangle e_i\) it follows that

\[
\lim_{t \to \infty} \int_{\mathcal{H}^+} \xi p_t(x, d\xi) = \lim_{t \to \infty} \sum_{i=1}^\infty \int_{\mathcal{H}^+} \langle \xi, e_i \rangle e_i \, p_t(x, d\xi)
= \sum_{i=1}^\infty \left( \int_0^\infty \langle \hat{b}, e^{sB} e_i \rangle \, ds \right) e_i
= \int_0^\infty e^{sB} \hat{b} \, ds,
\]

where we have used \(\lim_{t \to \infty} \sum_{i=1}^\infty \left( \int_0^t \langle \hat{b}, e^{sB} e_i \rangle \, ds \right) e_i = \sum_{i=1}^\infty \left( \int_0^\infty \langle \hat{b}, e^{sB} e_i \rangle \, ds \right) e_i\), which is justified if \(\lim_{N \to \infty} \sup_{t \geq 0} \sum_{i=1}^N \| \int_0^t (e^{sB} \hat{b}, e_i) e_i \, ds \| = 0\). The latter one follows from \(\sup_{t \geq 0} \sum_{i=1}^N \| \int_0^t (e^{sB} \hat{b}, e_i) e_i \, ds \| \leq \int_0^\infty \sum_{i=1}^N \| (e^{sB} \hat{b}, e_i) \| \, ds\) and

\[
\int_0^\infty \sum_{i=1}^\infty \| (e^{sB} \hat{b}, e_i) \| \, ds \leq \int_0^\infty \| e^{sB} \hat{b} \| \, ds \leq M \| \hat{b} \| \delta^{-1} < \infty.
\]

Recalling that \(\hat{b} = b + \int_{\mathcal{H}^+ \cap \{\| \xi \| > 1\}} \xi \, m(d\xi)\) yields (3.5).

Next we prove the desired formula for the second moments of \(\pi\). For \(i, j \in \mathbb{N}\) we set \(g^{i,j} = \langle \cdot, e_i \rangle \langle \cdot, e_j \rangle\). From (2.10) and analogous arguments as we used in Lemma 5.5 (to show that the integrals on the right-hand side of (5.22) below exists and are finite), we find that

\[
\lim_{t \to \infty} P_t g^{i,j}(x) = \left( \int_0^\infty \langle \hat{b}, e^{sB} e_i \rangle \, ds \right) \left( \int_0^\infty \langle \hat{b}, e^{sB} e_j \rangle \, ds \right) + \int_0^\infty \int_{\mathcal{H}^+ \setminus \{0\}} \langle \xi, e^{sB} e_i \rangle \langle \xi, e^{sB} e_j \rangle \, m(d\xi) \, ds + \int_0^\infty \int_0^s \langle \hat{b}, e^{(s-n)B} \int_{\mathcal{H}^+ \setminus \{0\}} \langle \xi, e^{nB} e_i \rangle \langle \xi, e^{nB} e_j \rangle \frac{H(d\xi)}{\| \xi \|^2} \rangle \, ds \, ds.
\]

(5.22)

holds for all \(i, j \in \mathbb{N}\). The second moment formula (3.6) then follows from this and \(y \otimes y = \sum_{i,j=1}^\infty \langle y, e_i \rangle \langle y, e_j \rangle e_i \otimes e_j\), once we have shown that

\[
\lim_{t \to \infty} P_t g^{i,j}(x) = \int_{\mathcal{H}^+} g^{i,j}(y) \, \pi(dy), \quad i, j \in \mathbb{N}.
\]

(5.23)

Since the function \(g^{i,j}\) does not belong to \(\mathcal{B}_p(\mathcal{H}_u^+)\), we cannot obtain (5.23) directly from Proposition 5.6. However, since we have \(P_t \langle \cdot, e_i \rangle \langle \cdot, e_j \rangle(x) \leq P_t \rho(x) < \infty\) for all \(t \geq 0\) and \(x \in \mathcal{H}^+\), we see that the function is in the larger space \(\mathcal{B}_p(\mathcal{H}_u^+)\) and we deduce the assertion by an additional approximation argument. Namely, define
We define $g_n^{i,j} := g^{i,j} \wedge n$ for $n \in \mathbb{N}$. Then $g_n^{i,j} \in B_\rho(\mathcal{H}_c^+)$ and we find that
\[
\left| P_t g^{i,j}(x) - \int_{\mathcal{H}_c^+} g_n^{i,j}(x) \pi(dx) \right| \leq |P_t g^{i,j}(x) - P_t g_n^{i,j}(x)| \\
+ \left| P_t g_n^{i,j}(x) - \int_{\mathcal{H}_c^+} g_n^{i,j}(x) \pi(dx) \right| \\
+ \left| \int_{\mathcal{H}_c^+} g_n^{i,j}(x) \pi(dx) - \int_{\mathcal{H}_c^+} g^{i,j}(x) \pi(dx) \right|.
\]

Let $\varepsilon > 0$. Take $n \in \mathbb{N}$ large enough so that $\left| \int_{\mathcal{H}_c}(g_n^{i,j}(x) - g^{i,j}(x)) \pi(dx) \right| < \varepsilon$. Next, note that
\[
\lim_{n \to \infty} \sup_{t \geq 0} |P_t g^{i,j}(x) - P_t g_n^{i,j}(x)| \leq \lim_{n \to \infty} \sup_{t \geq 0} \mathbb{E} \left[ \|X_t\|^2 \mathbb{R}(\|X_t\|^2 > n) \right] = 0
\]
where the last identity follows from the characterization of convergence in the Wasserstein distance (see [58, Section 6]). Hence we find $n$ large enough such that $|P_t g^{i,j}(x) - P_t g_n^{i,j}(x)| < \varepsilon$ holds uniformly in $t \geq 0$. Finally, for this fixed choice of $n$, we may choose in view of Proposition 5.6 $t$ large enough so that $|P_t g^{i,j}(x) - \int_{\mathcal{H}_c} g^{i,j}(x) \pi(dx)| < \varepsilon$. Combining all these estimates proves (5.23).

This completes the proof of Proposition 3.4.

**Appendix A. Generalized Feller semigroups**

Let $(Y, \tau)$ be a completely regular Hausdorff topological space. We define:

**Definition A.1.** A function $\rho: Y \to (0, \infty)$ such that for every $R > 0$ the set $K_R := \{x \in Y : \rho(x) \leq R\}$ is compact is called an admissible weight function. The pair $(Y, \rho)$ is called weighted space.

Let $\rho: Y \to (0, \infty)$ be an admissible weight function and assume that $\rho(x) \geq 1$ for every $x \in \mathcal{H}_c$. For $f: Y \to \mathbb{R}$ we define $\|f\|_\rho := \sup_{x \in Y} \frac{|f(x)|}{\rho(x)}$. (A.1)

Note that $\|\cdot\|_\rho$ defines a norm on the vector space $B_\rho(Y) := \{f: Y \to \mathbb{R} : \|f\|_\rho < \infty\}$ which renders $(B_\rho(Y), \|\cdot\|_\rho)$ a Banach space. Recall that $C_b(Y)$ denotes the space of bounded $\mathbb{R}$-valued $\tau$-continuous functions on $Y$. As the admissible weight function $\rho$ satisfies $\inf_{x \in Y} \rho(x) > 0$, we have that $C_b(Y) \subseteq B_\rho(Y)$.

**Definition A.2.** We define $B_\rho(Y)$ to be the closure of $C_b(Y)$ in $B_\rho(Y)$.

The following useful characterization of $B_\rho(Y)$ is proven in [20, Theorem 2.7]:

**Theorem A.3.** Let $(Y, \rho)$ be a weighted space. Then $f \in B_\rho(Y)$ if and only if $f|_{K_R} \in C(K_R)$ for all $R > 0$ and
\[
\lim_{R \to \infty} \sup_{x \in Y \setminus K_R} \frac{|f(x)|}{\rho(x)} = 0. \tag{A.2}
\]

Next we recall the definition of a generalized Feller semigroup, as introduced in [20, Section 3].

**Definition A.4.** A family of bounded linear operators $(P_t)_{t \geq 0}$ in $\mathcal{L}(B_\rho(Y))$ is called a generalized Feller semigroup (on $B_\rho(Y)$), if
1. $P_0 = I$, the identity on $B_\rho(Y)$,
2. $P_{s+t} = P_s P_t$ for all $t, s \geq 0$,
3. $\lim_{t \to 0^+} P_t f(x) = f(x)$ for all $f \in B_\rho(Y)$ and $x \in Y$,
4. there exist constants $C \in \mathbb{R}$ and $\varepsilon > 0$ such that $\|P_t\|_{\mathcal{L}(B_\rho(Y))} \leq C$ for all $t \in [0, \varepsilon]$. 


(5) \((P_t)_{t \geq 0}\) is a positive semigroup, i.e., \(P_tf \geq 0\) for all \(t \geq 0\) and for all \(f \in \mathcal{B}_\rho(Y)\) satisfying \(f \geq 0\).

We make use of the following adapted version of the Kolmogorov extension theorem presented in [18, Theorem 2.11].

**Proposition A.5.** Let \((Y, \rho)\) be a weighted space and let \((P_t)_{t \geq 0}\) be a generalized Feller semigroup on \(\mathcal{B}_\rho(Y)\) with \(P_t1 = 1\) for \(t \geq 0\). Then for every \(\nu \in \mathcal{M}(Y)\) which is inner-regular, there exists a probability space \((\Omega, \mathcal{F}, \mathbb{P}_\nu)\), filtered by a right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\), and a Markov process \((X_t)_{t \geq 0}\) with values in \(Y\) such that \(\mathbb{P}_\nu(X_0 = x) = \nu(A)\) for every \(A \in \mathcal{B}(Y)\) and

\[
\mathbb{E}_{\mathbb{P}_\nu}[f(X_t)] = \int_Y P_t f(\xi) \, \nu(d\xi), \quad t \geq 0, f \in \mathcal{B}_\rho(Y).
\]

**Proof.** In [18, Theorem 2.11] this version of the Kolmogorov extension theorem is proven for \(\nu = \delta_x\) and \(x \in Y\). In the proof of [18, Theorem 2.11] it is shown that the transition kernels \(p_t(x, \cdot)\) for \(x \in Y\) and \(t \geq 0\) given through the relation

\[
P_t f(x) = \int_Y f(\xi) \, p_t(x, d\xi),
\]

form a Kolmogorov consistent family according to [2, Section 15.6]. It is then concluded from a Kolmogorov extension theorem given by [2, Theorem 2.13] that there exists a probability measure \(\mathbb{P}_{\delta_x}\) such that the assertions of the proposition hold. We draw the same conclusion for any other probability measures \(\nu\) in \(\mathcal{M}(Y)\) satisfying

\[
\nu(A) = \sup \{\mu(K) : K \subseteq A, A \in \mathcal{K} \land A \in \mathcal{B}(Y) \cap \mathcal{K}\}, \quad (A.3)
\]

where \(\mathcal{K}\) is a compact class in \(Y\). Note that every weighted space \(Y\) is a Hausdorff topological space and hence the family \(\mathcal{K}\) of all compact sets of \(Y\) forms a compact class, see [2, Theorem 2.31]. We thus see that every inner-regular probability measure \(\nu \in \mathcal{M}(Y)\) satisfies \((A.3)\) and hence the assertion of the proposition follows from this, Kolmogorov extension theorem [2, Theorem 15.23] and analogous arguments as in [18, Theorem 2.11] \(\square\).

**Appendix B. A property of the Wasserstein distance**

**Lemma B.1.** Let \(W_2\) be the Wasserstein distance on \(\mathcal{H}^+\). Let \(\mu, \nu, \rho\) be Borel probability measures on \(\mathcal{H}^+\). Then \(W_2(\rho * \mu, \rho * \nu) \leq W_2(\mu, \nu)\).

**Proof.** Let \(G\) be any coupling of \((\mu, \nu)\) and let \(G'\) be any coupling of \((\rho, \rho)\). For each \(f, g : \mathcal{H} \to \mathbb{R}_+\) we find that

\[
\int_{\mathcal{H}^+ \times \mathcal{H}^+} (f(x) + g(y)) (G' * G)(dx, dy)\\n= \int_{\mathcal{H}^+ \times \mathcal{H}^+} \int_{\mathcal{H}^+ \times \mathcal{H}^+} (f(x + z) + g(y + z')) G'(dz, dz') G(dx, dy)\\n= \int_{\mathcal{H}^+ \times \mathcal{H}^+} f(x + z) \rho(dz) \mu(dx) + \int_{\mathcal{H}^+ \times \mathcal{H}^+} g(y + z') \rho(dz') \nu(dy)
\]

where \(\mathcal{H}\) is the state space and \(\rho, \mu, \nu\) are probability measures on \(\mathcal{H}\).
which shows that \( G' \ast G \) is a coupling of \( (\rho \ast \mu, \rho \ast \nu) \). Hence
\[
W_2(\rho \ast \mu, \rho \ast \nu)^2 \\
\leq \int_{H^+ \times H^+} \| x - y \|^2 (G' \ast G)(dx, dy) \\
= \int_{H^+ \times H^+} \int_{H^+ \times H^+} \|(x + z) - (y + z')\|^2 G'(dz, dz') G(dx, dy) \\
= \int_{H^+ \times H^+} \int_{H^+ \times H^+} (\|x - y\|^2 + 2 \langle x - y, z - z' \rangle) G'(dz, dz') G(dx, dy) \\
= \int_{H^+ \times H^+} \|x - y\|^2 G(dz, dz') + \int_{H^+ \times H^+} \|z - z'\|^2 G'(dz, dz')
\]
where the last inequality follows from the fact that \( G' \) has the same marginals so that \( \int_{H^+ \times H^+} \langle x - y, z - z' \rangle G'(dz, dz') = 0 \). Letting now \( G' \) be the specific coupling determined by \( G'(A \times B) = \rho(\{z \in H^+ : z \in A \cap B\}) \) with \( A, B \in \mathcal{B}(H^+) \), shows that \( \int_{H^+ \times H^+} \|z - z'\|^2 G'(dz, dz') = 0 \). Since \( G \) was arbitrary, the assertion is proved. 

\[ \square \]

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