Abstract. Let \( \hat{g} \) be an affine Lie algebra of the type \( A^{(1)}_\ell \). We find a combinatorial basis of Feigin-Stoyanovsky’s type subspace \( W(\Lambda) \) given in terms of difference and initial conditions. Linear independence of the generating set is proved inductively by using coefficients of intertwining operators. A basis of \( L(\Lambda) \) is obtained as an “inductive limit” of the basis of \( W(\Lambda) \).

1. Introduction

Vertex-operator construction of affine Lie algebras has been used to prove and to give a representation-theoretic interpretation of Rogers-Ramanujan-type combinatorial identities. This approach was initiated by J.Lepowsky and R.Wilson ([LW]), and was continued in the works of Lepowsky, M.Primc, A.Meurman and others (cf. [LP], [MP]). An important part of this program was to find monomial bases of standard modules for affine Lie algebras, or some of its subspaces. Knowledge of bases was then used to calculate the character of these spaces, which gave the sum side in the Rogers-Ramanujan-type partition identities.

Later, B.Feigin and A.Stoyanovsky considered what they called a principal subspace of the basic \( \hat{sl}(2, \mathbb{C}) \)-module ([FS]). They have constructed a combinatorial basis of this subspace, which again gave the sum side of a Rogers-Ramanujan identity. G.Georgiev extended character formulas obtained by Feigin and Stoyanovsky to a family of standard \( \hat{sl}(\ell + 1, \mathbb{C}) \)-modules ([G]). He explicitly constructed combinatorial bases for these subspaces and in the proof of linear independence, he used intertwining operators from [DL]. More recently, S.Capparelli, J.Lepowsky and A.Milas laid out a program to interpret and obtain Rogers-Ramanujan-type recursions in the setting of vertex-operator algebras and affine Lie algebras. They used intertwining operators to construct exact sequences between different principal subspaces for \( \hat{sl}(2, \mathbb{C}) \) and in this way, obtained Rogers-Ramanujan and Rogers-Selberg recursions for characters of these subspaces ([CLM1,CLM2]). As a continuation of this program, C.Calinescu obtained Rogers-Ramanujan-type recursions for some classes of standard modules for \( \hat{sl}(\ell + 1, \mathbb{C}) \) ([C1,C2]), and Calinescu, Lepowsky and Milas provided new proofs of presentation theorems for principal subspaces for \( \hat{sl}(2, \mathbb{C}) \) ([CalLM1,CalLM2]).

In parallel with these developments, M.Primc studied similar subspaces of standard modules for different affine Lie algebras ([P1,P2]), which he later called Feigin-Stoyanovsky’s type subspaces ([P3]). He used bases of these subspaces to construct from them bases of the whole standard modules. For \( \hat{sl}(3, \mathbb{C}) \), these bases were parameterized by \( (k, \ell + 1) \)-admissible configurations, a combinatorial objects that were introduced and further studied in [FJLMM] and [FJMMT]. In [P3], Primc

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proved linear independence of the spanning set by using Capparelli-Lepowsky-Milas’ approach via intertwining operators and a description of the basis from [FJLMM]. These operators and a description of basis were used by M.Jerkević to obtain exact sequences of Feigin-Stoyanovsky’s type subspaces and recurrence relations for the corresponding characters ([J1]). By solving these relations in the $sl(3, \mathbb{C})$-case he was able to generalize character formulas from [FJLMM].

In our previous paper ([T]) we have used ideas of Georgiev, Capparelli, Lepowsky and Milas, and of Primc to construct and prove linear independence of the spanning set for a Feigin-Stoyanovsky’s type subspace for all basic modules for $sl(\ell+1, \mathbb{C})$. In this paper we generalize this result to higher-level standard modules for $sl(\ell+1, \mathbb{C})$.

Let $g = sl(\ell+1, \mathbb{C})$ be a simple complex Lie algebra of type $A_\ell$, $\mathfrak{h} \subset g$ its Cartan subalgebra, $R$ the corresponding root system. Then one has a root decomposition $g = \mathfrak{h} + \sum_{\alpha \in R} g_\alpha$. Fix root vectors $x_\alpha \in g_\alpha$. Let $\langle \cdot, \cdot \rangle$ be a normalized invariant bilinear form on $g$, and by the same symbol denote the induced form on $g^*$. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a basis of the root system $R$, and $\{\omega_1, \ldots, \omega_\ell\}$ the corresponding set of fundamental weights. We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ in the usual way and fix a fundamental weight $\omega = \omega_m$. Set

$$\Gamma = \{\gamma \in R \mid \langle \gamma, \omega \rangle = 1\} = \{\gamma_{ij} \mid i = 1, \ldots, m; j = m, \ldots, \ell\},$$

where $\gamma_{ij} = \alpha_i + \cdots + \alpha_m + \cdots + \alpha_j$. Set

$$g_{\pm 1} = \sum_{\alpha \in \pm \Gamma} g_\alpha, \quad g_0 = \mathfrak{h} \oplus \sum_{\langle \alpha, \omega \rangle = 0} g_\alpha.$$ 

Then

$$(1) \quad g = g_{-1} \oplus g_0 \oplus g_{1}$$

is a $\mathbb{Z}$-gradation of $g$. The set $\Gamma$ is called the set of colors. For $\gamma \in \Gamma$, we say that a fixed basis element $x_\gamma \in g_\gamma$ is of the color $\gamma$. The set of colors $\Gamma$ can be pictured as a rectangle with row indices $1, \ldots, m$ and column indices $m, \ldots, \ell$ (see figure 1).

**Figure 1.** The set of colors $\Gamma$

Affine Lie algebra associated with $g$ is $\tilde{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $c$ is the canonical central element, and $d$ is the degree operator. Elements $x_\alpha(n) = x_\alpha \otimes t^n$ are fixed real root vectors. The $\mathbb{Z}$-gradation (1) of $g$ induces analogous $\mathbb{Z}$-gradation of $\tilde{g}$:

$$\tilde{g} = \tilde{g}_{-1} \oplus \tilde{g}_0 \oplus \tilde{g}_{1},$$

where $\tilde{g}_{1} = g_1 \otimes \mathbb{C}[t, t^{-1}]$ is a commutative Lie subalgebra with a basis

$$\{x_\gamma(j) \mid j \in \mathbb{Z}, \gamma \in \Gamma\}.$$
Let \( L(\Lambda) \) be a standard \( \tilde{\mathfrak{g}} \)-module of level \( k = \Lambda(c) \), with a fixed highest weight vector \( v_\Lambda \). A Feigin-Stojanovsky’s type subspace is a \( \tilde{\mathfrak{g}}_1 \)-submodule of \( L(\Lambda) \) generated with \( v_\Lambda \),

\[
W(\Lambda) = U(\tilde{\mathfrak{g}}_1) : v_\Lambda \subset L(\Lambda).
\]

We find a basis of the Feigin-Stoyanovsky’s type subspace \( W(\Lambda) \) consisting of monomial vectors

\[
\{ x_{\gamma_1}(-r_1) \cdots x_{\gamma_n}(-r_n) v_\Lambda \mid n \in \mathbb{Z}_+; \gamma_j \in \Gamma, r_j \in \mathbb{N} \}
\]

whose monomial parts

\[
(2) \quad x_{\gamma_1}(-r_1) \cdots x_{\gamma_n}(-r_n)
\]

satisfy certain combinatorial conditions, called difference and initial conditions.

We’ll say that a monomial \( (2) \) satisfies difference conditions on \( L(\Lambda) \), if exponents of its factors satisfy the following family of inequalities:

\[
a_{i_{s+1}}^{r_{s+1}} + \cdots + a_{i_{s+1}j_{s+1}}^{r_{s+1}} + \cdots + a_{i_{t+1}}^{r_{t+1}} \leq k,
\]

\[
1 \leq i_1 \leq \cdots \leq i_s \leq i_{s+1} \leq \cdots \leq i_t \leq m,
\]

\[
\ell \geq j_1 \geq \cdots \geq j_s \geq j_{s+1} \geq \cdots \geq j_t \geq m
\]

\[
(i_\nu, j_\nu) \neq (i_{\nu+1}, j_{\nu+1}), \text{ for } \nu = 1, \ldots, s - 1,
\]

\[
\ell \geq j_1 \geq \cdots \geq j_s \geq j_{s+1} \geq \cdots \geq j_t \geq m
\]

where \( a_{ij}^r \) is an exponent of \( x_{\gamma_i}(-r) \). We can reformulate this by saying that for any configuration of colors of elements of degree \(-r\) and \(-r-1\) of the type pictured on the figure 2, the sum of corresponding exponents must be less than \( k + 1 \).

**Figure 2. Difference conditions**

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Similarly, we’ll say that a monomial \( (2) \) satisfies initial conditions on \( L(\Lambda) \) if

\[
a_{i_{j_1}}^1 + a_{i_{j_2}}^1 + \cdots + a_{i_{j_t}}^1 \leq k_0 + k_1 + \cdots + k_{i_s-1} + k_{j_{s+1}} + \cdots + k_\ell,
\]

\[
1 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq m,
\]

\[
\ell \geq j_1 \geq \cdots \geq j_s \geq j_{s+1} \geq \cdots \geq j_t \geq m
\]

\[
(i_\nu, j_\nu) \neq (i_{\nu+1}, j_{\nu+1}).
\]

Difference conditions on monomials are obtained by observing relations between fields \( x_{\gamma}(z), \gamma \in \Gamma \) on \( L(\Lambda) \), while initial conditions are consequences of relations for some modules of lower level.

In the case \( k = 1 \), these conditions are equivalent to the conditions we have obtained in [T]. Since a standard module of level \( k \) can be found inside a \( k \)-fold tensor product of modules of level 1, it is natural to ask can a monomial \( (2) \) be factorized in such a way so that each factor satisfies difference and initial conditions on the corresponding level 1 module. By combinatorial arguments, we show that the answer
to this question is affirmative. This enables us to use tensor products of coefficients of intertwining operators that were constructed in [T] and to inductively prove the linear independence of the set of monomial vectors $x_{a_1}(-r_1) \cdots x_{a_n}(-r_n)v_{\Lambda}$ whose monomial parts satisfy difference and initial conditions. Thus, this set is a basis of $\mathcal{W}(\Lambda)$.

Following the approach of Primc ([P1],[P2]) we construct the basis of the whole standard module $L(\Lambda)$ as an “inductive limit” of the basis of $W(\Lambda)$.

2. AFFINE LIE ALGEBRAS

For $\ell \in \mathbb{N}$, let

$$\mathfrak{g} = \mathfrak{sl}(\ell + 1, \mathbb{C}),$$

a simple Lie algebra of the type $A_\ell$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$ and $R$ the corresponding root system. Fix a basis $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ of $R$. Then we have the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. By $R_+$ and $R_-$ we denote sets of positive and negative roots, and let $\theta$ be the maximal root. Let $\langle x, y \rangle = \text{tr} xy$ be a normalized invariant bilinear form on $\mathfrak{g}$; via $\langle \cdot, \cdot \rangle$ we have an identification $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^\ast$. For each root $\alpha$ fix a root vector $x_{\alpha} \in \mathfrak{g}_\alpha$.

Let $\{\omega_1, \ldots, \omega_\ell\}$ be the set of fundamental weights of $\mathfrak{g}$, $\langle \omega_i, \alpha_j \rangle = \delta_{ij}, i, j = 1, \ldots, \ell$. Denote by $Q = \sum_{i=1}^\ell \mathbb{Z}\omega_i$ the root lattice, and by $P = \sum_{i=1}^\ell \mathbb{Z}\omega_i$ the weight lattice of $\mathfrak{g}$.

Denote by $\widehat{\mathfrak{g}}$ the associated affine Lie algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Set $x(j) = x \otimes t^j$ for $x \in \mathfrak{g}, j \in \mathbb{Z}$. Commutation relations are then given by

$$[c, \mathfrak{g}] = 0,$$

$$[d, x(j)] = jx(j),$$

$$[x(i), y(j)] = [x, y](i + j) + i(x, y)\delta_{i+j, 0}c.$$

Set $\mathfrak{h}^\ast = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$, $\widehat{\mathfrak{n}} = \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}] \oplus \mathfrak{n}_\pm$. Then we also have the triangular decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \mathfrak{h}^\ast \oplus \widehat{\mathfrak{n}}_+$. Let $\widehat{\Pi} = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\} \subset (\mathfrak{h}^\ast)^\ast$ be the set of simple roots of $\widehat{\mathfrak{g}}$. Usual extensions of bilinear forms $\langle \cdot, \cdot \rangle$ onto $\mathfrak{h}^\ast$ and $(\mathfrak{h}^\ast)^\ast$ are denoted by the same symbols (we take $\langle c, d \rangle = 1$). Define fundamental weights $\Lambda_i \in (\mathfrak{h}^\ast)^\ast$ by $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij} \Lambda_i(d) = 0, i, j = 0, \ldots, \ell$.

Let $V$ be a highest weight module for affine Lie algebra $\widehat{\mathfrak{g}}$. Then $V$ is generated by a highest weight vector $v_{\Lambda}$ such that

$$h \cdot v_{\Lambda} = \Lambda(h)v_{\Lambda}, \quad \text{for } h \in \mathfrak{h}^\ast,$$

$$x \cdot v_{\Lambda} = 0, \quad \text{for } x \in \widehat{\mathfrak{n}}_+,$$

for $\Lambda \in (\mathfrak{h}^\ast)^\ast$. Module $V$ is a direct sum of weight subspaces $V_\mu = \{v \in V \mid h \cdot V = \mu(h)v \text{ for } h \in \mathfrak{h}^\ast, \mu \in \mathfrak{h}^\ast\}$.

Standard (i.e. integrable highest weight) $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ is an irreducible highest weight module, with the highest weight $\Lambda$ being dominant integral, i.e.

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell,$$

where $k_i \in \mathbb{Z}_+, i = 0, \ldots, \ell$. The central element $c$ acts on $L(\Lambda)$ as multiplication by scalar

$$k = \Lambda(c) = k_0 + k_1 + \cdots + k_\ell,$$

which is called the level of the module $L(\Lambda)$. 
3. FEIGIN-STOYANOVSKY’S TYPE SUBSPACE

Vector \( v \in \mathfrak{h} \) is said to be \textit{cominuscule} if

\[
\{ \alpha(v) \mid \alpha \in R \} = \{-1, 0, 1\}.
\]

Similarly, weight \( \omega \in P \) is said to be \textit{minuscule} if

\[
\{ \langle \omega, \alpha \rangle \mid \alpha \in R \} = \{-1, 0, 1\}.
\]

A dominant integral weight \( \omega \in P^+ \) is minuscule if and only if

\[
\langle \omega, \theta \rangle = 1.
\]

There is a 1–1 correspondence between cominuscule vectors and minuscule weights, and there exists finitely many minuscule weights. More precisely, a vector \( v \in \mathfrak{h} \) is cominuscule if and only if it is dual to some minuscule fundamental weight \( \omega \), in the sense that

\[
v = \nu^{-1}(\omega),
\]

for some choice of positive roots.

Fix a cominuscule vector \( v \in \mathfrak{h} \). For \( g = \mathfrak{sl}(\ell + 1, \mathbb{C}) \), all fundamental weights are minuscule. So we can assume that the cominuscule vector \( v \) is dual to a fundamental weight \( \omega = \omega_m \), for some \( m \in \{1, \ldots, \ell\} \). Set

\[
\Gamma = \{ \alpha \in R \mid \alpha(v) = 1 \} = \{ \alpha \in R \mid \langle \omega, \alpha \rangle = 1 \}.
\]

Then we have the induced \( \mathbb{Z} \)-gradation of \( g \):

\[
g = g_{-1} \oplus g_0 \oplus g_1,
\]

where

\[
g_0 = \mathfrak{h} \oplus \sum_{\alpha(v) = 0} g_\alpha,
\]

\[
g_{\pm 1} = \sum_{\alpha \in \pm \Gamma} g_\alpha.
\]

Subalgebras \( g_1 \) and \( g_{-1} \) are commutative, and \( g_0 \) acts on them by adjoint action. The subalgebra \( g_0 \) is reductive with semisimple part \( \mathfrak{l}_0 = [g_0, g_0] \) of the type \( A_{m-1} \times A_{\ell-m} \); as a root basis one can take \( \{\alpha_1, \ldots, \alpha_{m-1}\} \cup \{\alpha_{m+1}, \ldots, \alpha_\ell\} \), and the center is equal to \( \mathbb{C} v \).

Basis of the subalgebra \( g_1 \) can be identified with the set of roots \( \Gamma \). We will call elements \( \gamma \in \Gamma \) \textit{colors} and the set \( \Gamma \) \textit{the set of colors}. For \( \omega = \omega_m \), the set of colors is

\[
\Gamma = \{ \gamma_{ij} \mid i = 1, \ldots, m; j = m, \ldots, \ell \}
\]

where

\[
\gamma_{ij} = \alpha_i + \cdots + \alpha_m + \cdots + \alpha_j.
\]

The maximal root \( \theta \) is equal to \( \gamma_{1\ell} \).

We picture the set of colors \( \Gamma \) as a rectangle with row-indices \( 1, \ldots, m \) and column-indices \( m, \ldots, \ell \), like in the figure [1].

Similarly, one also has the induced \( \mathbb{Z} \)-gradation of affine Lie algebra \( \tilde{g} \):

\[
\tilde{g}_0 = g_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} c \oplus \mathbb{C} d,
\]

\[
\tilde{g}_{\pm 1} = g_{\pm 1} \otimes \mathbb{C}[t, t^{-1}],
\]

\[
\tilde{g} = \tilde{g}_{-1} + \tilde{g}_0 + \tilde{g}_1.
\]

As above, \( \tilde{g}_{-1} \) and \( \tilde{g}_1 \) are commutative subalgebras, and \( \tilde{g}_1 \) is a \( \tilde{g}_0 \)-module.
For a dominant integral weight $\Lambda$, we define a Feigin-Stoyanovsky’s type subspace

$$W(\Lambda) = U(\hat{g}_1) \cdot v_\Lambda \subset L(\Lambda).$$

Our objective is to find a combinatorial basis of $W(\Lambda)$. Set

$$\hat{g}_1^+ = \hat{g}_1 \cap \hat{n}_+, \quad \hat{g}_1^- = \hat{g}_1 \cap \hat{n}_-.$$  

Then we have

$$W(\Lambda) = U(\hat{g}_1^+) \cdot v_\Lambda.$$  

By Poincaré-Birkhoff-Witt theorem, we have a spanning set of $W(\Lambda)$ consisting of monomial vectors

$$\{ x_{\gamma_t} (-r_t) \cdots x_{\gamma_2} (-r_2) x_{\gamma_1} (-r_1) v_\Lambda \mid t \in \mathbb{Z}_+; \gamma_j \in \Gamma, r_j \in \mathbb{N} \}.$$  \hspace{1cm} (5)

In the end, we’ll say a few words about notation. Elements of the spanning set can be identified with monomials from $U(\hat{g}_1) = S(\hat{g}_1)$. Because of this we often refer to elements of $\{ x_\gamma (-r) \mid \gamma \in \Gamma, r \in \mathbb{Z} \}$ in $\hat{g}_1$ as to variables, elements or factors of a monomial.

Monomials from $S(\hat{g}_1)$ can be identified with colored partitions. Let $\pi : \{ x_\gamma (-r) \mid \gamma \in \Gamma, r \in \mathbb{Z} \} \rightarrow \mathbb{Z}_+$ be a colored partition (cf. [P1], section 3). The corresponding monomial $x(\pi) \in S(\hat{g}_1)$ is

$$x(\pi) = x_{\gamma_t} (-r_t) \pi(x_{\gamma_t} (-r_t)) \cdots x_{\gamma_2} (-r_2) \pi(x_{\gamma_2} (-r_2)) \cdots x_{\gamma_1} (-r_1) \pi(x_{\gamma_1} (-r_1)).$$

From this identification we take notation $x(\pi)$ for the monomials from $S(\hat{g}_1)$. It will be convenient to define some new monomials by using this identification. Also, our combinatorial conditions for the basis elements will be written in terms of exponents $\pi(x_\gamma (-r))$, which gives a parametrization of the basis by a certain generalization of the notion of $(k, \ell + 1)$-admissible configurations from [FJLMM].

4. ORDER ON THE SET OF MONOMIALS

We introduce a linear order on the set of monomials.

On the weight and root lattice, we have an order $<$ defined in the standard way: for $\mu, \nu \in P$ set $\mu < \nu$ if $\mu - \nu$ is an integral linear combination of simple roots $\alpha_i, i = 1, \ldots, \ell$, with non-negative coefficients.

Next, we define a linear order $<$ on the set of colors $\Gamma$ which is an extension of the order $\prec$. For elements of $\Gamma$, $\gamma_i' < \gamma_ij$ is equivalent to saying that $i' \geq i$ and $j' \leq j$. The order $<$ on $\Gamma$ is defined in the following way:

$$\gamma_i' < \gamma_ij \quad \text{if} \quad \begin{cases} i' > i, \\ i' = i, \ j' < j. \end{cases}$$

It is clear that this is a linear order on the set of colors.

On the set of variables $\{ x_\gamma (-r) \mid \gamma \in \Gamma, r \in \mathbb{Z} \} \subset \hat{g}_1$ we define a linear order $<$ so that we compare degrees first, and then colors of variables:

$$x_\alpha (-r) < x_\beta (-r') \quad \text{if} \quad \begin{cases} -r < -r', \\ r = r' \quad \text{and} \quad \alpha < \beta. \end{cases}$$

Since the algebra $\hat{g}_1$ is commutative, we can assume that the variables in monomials from $S(\hat{g}_1)$ are sorted ascendingly from left to right. The order $<$ on the set of monomials is defined as a lexicographic order, where we compare variables from right to left (from the greatest to the lowest one). If $x(\pi)$ and $x(\pi')$ are two monomials,

$$x(\pi) = x_{\gamma_t} (-r_t) x_{\gamma_{t-1}} (-t_{t-1}) \cdots x_{\gamma_2} (-r_2) x_{\gamma_1} (-r_1),$$

$$x(\pi') = x_{\gamma_t'} (-r_t') x_{\gamma_{t'-1}}' (-r_{t'-1})' \cdots x_{\gamma_2'}' (-r_2')' x_{\gamma_1'}' (-r_1'),$$

then $x(\pi) < x(\pi')$ if there exist $i_0 \in \mathbb{N}$ so that $x_{\gamma_{i_0}} (-r_{i_0}) = x_{\gamma_{i_0}'}' (-r_{i_0}')$, for all $i < i_0$, and either $i_0 = t + 1 \leq s$ or $x_{\gamma_{i_0}} (-r_{i_0}) < x_{\gamma_{i_0}} (-r_{i_0}')$. 

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This monomial order is compatible with multiplication:

**Proposition 1 ([T]).** Let

\[ x(\pi_1) \leq x(\mu_1) \quad \text{and} \quad x(\pi_2) \leq x(\mu_2). \]

Then

\[ x(\pi_1)x(\pi_2) \leq x(\mu_1)x(\mu_2), \]

and if one of the first two inequalities is strict, then the last one is also strict.

For a monomials \( x(\pi) \in S(\hat{\mathfrak{g}}_1) \), we also define a degree and a shape of \( x(\pi) \). A **degree** of a monomial is equal to the sum of degrees of its variables. For

\[ x(\pi) = x_{\gamma_1}(-r_1)x_{\gamma_{i-1}}(-r_{i-1}) \cdots x_{\gamma_2}(-r_2)x_{\gamma_1}(-r_1), \]

its degree is equal to \(-r_1 - r_2 - \cdots - r_i\). A **shape** of a monomial is gotten from its colored partition by forgetting colors and considering only degrees of factors. More precisely, for a monomial \( x(\pi) \) and its partition \( \pi : \{ x_{\gamma}(-r) \mid \gamma \in \Gamma, r \in \mathbb{Z} \} \to \mathbb{Z}_+ \), the corresponding shape will be

\[ s_\pi : \mathbb{Z} \to \mathbb{Z}_+, \]

\[ s_\pi(r) = \sum_{\gamma \in \Gamma} \pi(x_{\gamma}(-r)). \]

A linear order can also be defined on the set of shapes; we’ll say that \( s_\pi < s_\pi' \) if there exists \( r_0 \in \mathbb{Z} \) such that \( s_\pi(r) = s_\pi'(r) \) for \( r < r_0 \) and either \( s_\pi(r_0) < s_\pi'(r_0) \) and \( s_\pi(r') \neq 0 \) for some \( r' > r_0 \), or \( s_\pi(r_0) > s_\pi'(r_0) \) and \( s_\pi(r) = 0 \) for \( r > r_0 \).

In the end, for the sake of simplicity, we introduce the following notation:

\[ x_{ij}(-r) = x_{\gamma_{ij}}(-r), \]

for \( \gamma_{ij} \in \Gamma, r \in \mathbb{N} \).

5. **Vertex operator construction**

We use the vertex operator algebra construction of basic \( \hat{\mathfrak{g}} \)-modules (i.e. standard \( \hat{\mathfrak{g}} \)-modules of level 1). We’ll sketch this construction here, details can be found in [FLM], [DL] or [LL]; see also [FK], [S].

Consider tensor products

\[ V_P = M(1) \otimes \mathbb{C}[P]; \]

\[ V_Q = M(1) \otimes \mathbb{C}[Q]; \]

where \( M(1) \) is the Fock space for the Heisenberg subalgebra \( \hat{\mathfrak{h}} = \sum_{n \in \mathbb{Z}} \{ \mathfrak{h} \otimes t^n \} \oplus \mathbb{C}c \) and \( \mathbb{C}[P] \) and \( \mathbb{C}[Q] \) are group algebras of the weight and root lattice with bases consisting of \( \{ e^\lambda \mid \lambda \in P \} \), and \( \{ e^\alpha \mid \alpha \in Q \} \), respectively. We identify group elements \( e^\lambda = 1 \otimes e^\lambda \in V_P \).

Space \( V_Q \) has a natural structure of vertex operator algebra and \( V_P \) is a module for this algebra. Vertex operators are defined as follows:

\[ Y(e^\lambda, z) = E^-(\lambda, z)E^+(\lambda, z) \otimes e^\lambda z^\lambda \epsilon(\lambda, \cdot), \]

where \( e^\lambda = 1 \otimes e^\lambda \) is a multiplication operator, \( e_\lambda = 1 \otimes \epsilon(\lambda, \cdot) \) and \( \epsilon(\cdot, \cdot) \) is a 2-cocycle (cf. [DL]), operator \( z^\lambda = 1 \otimes z^\lambda \), \( z^\lambda \cdot e^\mu = e^\mu z^\lambda(\lambda, \mu) \) and

\[ E^\pm(\lambda, z) = \exp \left( \sum_{m \geq 1} \lambda(\pm m) \frac{z^m}{\pm m} \right), \]

for \( \lambda \in P \).

By using vertex operators, one can define the structure of \( \hat{\mathfrak{g}} \)-module on \( V_P \). For \( \alpha \in R, j \in \mathbb{Z} \) set

\[ x_\alpha(z) = Y(e^\alpha, z), \]

for \( \alpha \in R \) and \( j \in \mathbb{Z} \).
for a properly chosen root vector \( x_\alpha \). Heisenberg subalgebra acts on the Fock space \( M(1) \) and \( e \) acts as identity. In this way \( V_Q \) and \( V_Qe^{x_\gamma i} \), \( j = 1, \ldots, \ell \) become standard \( \tilde{\mathfrak{g}} \)-modules of level 1 with highest weight vectors \( v_0 = 1 \) and \( v_j = e^{x_\gamma i}, j = 1, \ldots, \ell \),

\[
L(\Lambda_0) \cong V_Q \quad \text{and} \quad L(\Lambda_j) \cong V_Qe^{x_\gamma i}, j = 1, \ldots, \ell
\]

and

\[
V_P \cong L(\Lambda_0) \oplus L(\Lambda_1) \oplus \cdots \oplus L(\Lambda_\ell).
\]

We will also be using intertwining operators \( \mathcal{Y} \). More precisely we’ll be needing operators

\[
\mathcal{Y}(e^\lambda, z) = Y(e^\lambda, z)e^{i\pi\lambda}c(\cdot, \lambda),
\]

for \( \lambda \in P \), where \( c(\cdot, \lambda) \) is a commutator map (cf. [DL]).

Restrictions of \( \mathcal{Y}(e^\lambda, z) \) are in fact maps between standard modules of level 1: if \( \lambda + \omega_i \equiv \omega_j \mod Q \), then

\[
\mathcal{Y}(e^\lambda, z) : L(\Lambda_i) \to L(\Lambda_j)\{z\},
\]

where \( L(\Lambda_j)\{z\} \) is a space of formal series with coefficients in \( L(\Lambda_j) \). Here, for convenience, we’ve set \( \omega_0 = 0 \). Also, for a suitable choice of \( \mu \in P \), the operators \( \mathcal{Y}(e^\mu, z) \) will commute with \( \tilde{\mathfrak{g}}_1 \) (cf. [T]).

Standard modules of level \( k > 1 \) can be viewed, by the complete reducibility, as submodules of tensor products of basic modules;

\[
L(\Lambda) \subset L(\Lambda_0)^{\otimes k_0} \otimes \cdots \otimes L(\Lambda_\ell)^{\otimes k_\ell},
\]

if \( \Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell \), \( k = k_0 + k_1 + \cdots + k_\ell \). Highest weight vector of \( L(\Lambda) \) is

\[
v_\Lambda = v_0^{\otimes k_0} \otimes \cdots \otimes v_\ell^{\otimes k_\ell}.
\]

This all can be imbedded into \( V_P^{\otimes k} \). One can also define vertex operators corresponding to elements

\[
u_1 \otimes \cdots \otimes u_k \in V_P^{\otimes k}
\]

as tensor products of vertex operators on the appropriate tensor factors:

\[
Y(u_1 \otimes \cdots \otimes u_k, z) = Y(u_1, z) \otimes \cdots \otimes Y(u_k, z).
\]

Then \( V_P^{\otimes k} = L(\Lambda_0)^{\otimes k} \) becomes vertex operator algebra, and \( V_P^{\otimes k} \) with its subspaces \( L(\Lambda_0)^{\otimes k_0} \otimes \cdots \otimes L(\Lambda_\ell)^{\otimes k_\ell} \) become modules for this algebra.

6. OPERATOR \( e(\omega) \)

For \( \lambda \in P \), \( e^\lambda \) denotes multiplication operator \( 1 \otimes e^\lambda \) in \( V_P = M(1) \otimes \mathbb{C}[P] \). Set

\[
e(\lambda) = e^\lambda c(\cdot, \lambda), \quad e(\lambda) : V_P \to V_P,
\]

Clearly, \( e(\lambda) \) is a linear bijection. Its restrictions on basic modules are bijections from one basic module \( L(\Lambda_i) \) onto another basic module \( L(\Lambda_j) \). From the definition of vertex operators \( Y(e^\alpha, z), \alpha \in R \) one gets the following commutation relation

\[
Y(e^\alpha, z)e(\lambda) = e(\lambda)z^{(\lambda, \alpha)}Y(e^\alpha, z),
\]

or, in terms of components,

\[
x_\alpha(n)e(\lambda) = e(\lambda)x_\alpha(n + (\lambda, \alpha)), \quad n \in \mathbb{Z}.
\]

For standard modules of level \( k > 1 \), one defines operator \( e(\lambda) \) on the tensor product of basic modules as a tensor product of the appropriate operators

\[
e(\lambda) = e(\lambda) \otimes \cdots \otimes e(\lambda) : \otimes_{s=1}^k L(\Lambda_i_s) \to \otimes_{s=1}^k L(\Lambda_i_s).
\]

Operator \( e(\lambda) \) is again a linear bijection, and relation (8) still holds.

For \( \lambda = \omega \) and \( \gamma \in \Gamma \), the relation (8) becomes

\[
x_\gamma(n)e(\omega) = e(\omega)x_\gamma(n + 1).
\]
More generally, for a monomial $x(\pi) \in S(\tilde{g}_1)$,

$$x(\pi)e(\omega) = e(\omega)x(\pi^+),$$

where $x(\pi^+) \in S(\tilde{g}_1)$ denotes a monomial corresponding to partition

(9) \[ \pi^+(x, (n+1)) = \pi(x,(n)). \]

7. Case $k = 1$

Here we’ll briefly recall the main results from [T] concerning the basis of a Feigin-Stoyanovsky’s type subspace of the standard module $L(\Lambda_i)$. It was described in terms of difference and initial conditions.

A monomial $x(\pi)$ satisfies difference conditions for $L(\Lambda_i)$ if the following holds:

- if $x(\pi)$ contains elements $x_{pq}(-r)$ and $x_{p'q'}(-r)$, and $\gamma_{p'q'} \leq \gamma_{pq}$, then $p' > p$ and $q' < q$.
- if $x(\pi)$ contains elements $x_{pq}(-r)$ and $x_{p'q'}(-r-1)$, then $p' > p$ or $q' < q$.

From this we can conclude that colors of the elements of the same degree $-r$ inside $x(\pi)$ make a descending sequence as pictured below; appropriate row-indices strictly increase, while column-indices strictly decrease. Colors of elements of degree $-r-1$ also form a decreasing sequence which is placed below or on the left of the minimal color of elements of degree $-r$ (see figure 3).

![Figure 3. Difference conditions - level 1 case](image)

A monomial $x(\pi)$ satisfies initial conditions for $L(\Lambda_i)$ if it doesn’t contain any element $x_{pq}(-1) \in \tilde{g}_1$ such that $x_{pq}(-1)v_i = 0$. In the case $0 \leq i \leq m$, $x_{pq}(-1)v_i = 0$ if $p \leq i$; in the case $m \leq i \leq \ell$, $x_{pq}(-1)v_i = 0$ if $q \geq i$. Hence, initial conditions imply that the sequence of colors of elements of degree $-1$ lies below the $i$-th row (if $0 \leq i \leq m$), or on the left of the $i$-th column (for $m \leq i \leq \ell$); see figure 4.

8. Difference and initial conditions

8.1. Relations. As in the case $k = 1$ (cf. [T]) we will first find relations between fields $x_\gamma(z), \gamma \in \Gamma$ on the standard module $L(\Lambda)$. By equating coefficients of powers of $z$ in these relations, we’ll get relations between monomials. We’ll then identify the minimal monomial among these, the so called leading term of a relation, and exclude from the spanning set all monomials that contain leading terms. Difference conditions combinatorially describe such monomials in terms of exponentials of factors $x_{rs}(j) \in \tilde{g}_1$.

To obtain relations, we start from a consequence of Frenkel-Kac-Segal formula for $L(\Lambda)$:

$$x_\theta(z)^{k+1} = 0.$$
Since $\tilde{g}_1$ is commutative, the product of fields on the left side of is a vertex-operator corresponding to the element $x_\theta(-1)^{k+1}1$ in $L(k\Lambda_0) \subset L(\Lambda_0)^{\otimes k}$, and the relation above is equivalent to

$$x_\theta(-1)^{k+1}1 = 0.$$ 

By acting on this relation with $y \in \mathfrak{l}_0$, one gets

$$0 = y \cdot x_\theta(-1)^{k+1}1 = [y, x_\theta(-1)^{k+1}]1 + x_\theta(-1)^{k+1}y \cdot 1 = [y, x_\theta(-1)^{k+1}]1.$$ 

A commutator $[y, x_\theta(-1)^{k+1}]$ is again an element of $S(g_1(1)) \subset S(\tilde{g}_1)$ and hence from the above equality we obtain another relation between fields $x_\gamma(z), \gamma \in \Gamma$ on $L(\Lambda)$, by adjoint action of $\mathfrak{l}_0$ we would obtain a new relation between these fields.

Thus we have to study a subrepresentation $V \subset S(\tilde{g}_1(-1))$ of $\mathfrak{l}_0$ generated by a singular vector $x_\theta(-1)^{k+1}$,

$$V = U(\mathfrak{l}_0) \cdot x_\theta(-1)^{k+1} \subset S(\tilde{g}_1(-1)) \subset U(\tilde{g}_1).$$

Action of $\mathfrak{l}_0$ is defined by the adjoint action of $\mathfrak{l}_0$ on $\tilde{g}_1$ (i.e. on $g_1$).

Generally, the algebra $\mathfrak{l}_0$ is a direct sum of two simple subalgebras, $\mathfrak{l}_0 = \mathfrak{l}_0' \oplus \mathfrak{l}_0''$, with the first one being of type $A_{m-1}$, and the second one of type $A_{\ell-m}$. The Chevalley basis of $g$ can be chosen such that the adjoint action of $\mathfrak{l}_0$ is given by:

$$[x_{-\alpha_i}, x_{\gamma_pq}] = \delta_{ip}x_{\gamma_p+1, q},$$

if $i = 1, \ldots, m-1$,

$$[x_{\alpha_i}, x_{\gamma_pq}] = \delta_{i,p-1}x_{\gamma_p-1, q},$$

if $i = m + 1, \ldots, \ell$. One can say that the first subalgebra $\mathfrak{l}_0'$ acts by changing the row-index, and the second subalgebra $\mathfrak{l}_0''$ by changing the column-index of elements $x_\gamma, \gamma \in \Gamma$.

By a simple calculation, one sees that the vector $x_\theta(-1)^{k+1}$ is the highest weight vector in $V$ of weight $(k + 1)\theta = (k + 1)(\omega_1 + \omega_\ell)$ for $\mathfrak{l}_0$. The highest weight representation of $\mathfrak{l}_0$ can be obtained in another way, by taking tensor products of highest weight representations its simple subalgebras. Let $V_1$ be a highest weight
representation of \( \ell'_0 \) of highest weight \((k + 1)\omega_1\). It can be realized as a subspace of homogenous polynomials of degree \(k + 1\) in \(m\) variables,

\[
V_1 = S^{k+1}(x_1, \ldots, x_m) \subset S(x_1, \ldots, x_m).
\]

The action of \( \ell'_0 \) is given on the generators \( \{x_{-\alpha_i}, x_{\alpha_i} \mid i = 1, \ldots, m-1\} \) by

\[
x_{-\alpha_i} \mapsto x_{i+1} \frac{\partial}{\partial x_i}, \\
x_{\alpha_i} \mapsto \frac{1}{x_i} \frac{\partial}{\partial x_{i+1}}.
\]

Similarly, let \( V_2 \) a highest weight representation of \( \ell''_0 \) of highest weight \((k + 1)\omega_\ell\). It can be realized as homogenous polynomials of degree \(k + 1\) in \(\ell - m + 1\) variables

\[
V_2 = S^{k+1}(x_m, \ldots, x_\ell) \subset S(x_m, \ldots, x_\ell),
\]

where the action is defined on the generators \( \{x_{-\alpha_i}, x_{\alpha_i} \mid i = m + 1, \ldots, \ell\} \) by

\[
x_{-\alpha_i} \mapsto x_{i-1} \frac{\partial}{\partial x_i}, \\
x_{\alpha_i} \mapsto \frac{1}{x_i} \frac{\partial}{\partial x_i-1}.
\]

Then

\[
V \cong V_1 \otimes V_2.
\]

A highest weight vector in \( V_1 \) is \( x_1^{k+1} \). From the character formula (cf. [H]) one sees that monomials

\[
x_{i_1}x_{i_2} \cdots x_{i_{k+1}}, \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{k+1} \leq m,
\]

constitute a basis of \( V_1 \) made of weight vectors. Similarly, a monomial \( x_\ell^{k+1} \) is a highest weight vector of \( V_2 \), and the basis constitutes of monomials

\[
x_{j_1}x_{j_2} \cdots x_{j_{k+1}}, \quad \ell \geq j_1 \geq j_2 \geq \cdots \geq j_{k+1} \geq m.
\]

Hence, the basis of \( V_1 \otimes V_2 \) is constituted by tensor products

\[
x_{i_1}x_{i_2} \cdots x_{i_{k+1}} \otimes x_{j_1}x_{j_2} \cdots x_{j_{k+1}},
\]

\[
1 \leq i_1 \leq i_2 \leq \cdots \leq i_{k+1} \leq m, \\
\ell \geq j_1 \geq j_2 \geq \cdots \geq j_{k+1} \geq m.
\]

The goal is to determine the corresponding basis of \( V \subset S(\tilde{g}_1(-1)) \) which will then give us relations on \( L(\Lambda) \). We’ll show that to the vector above corresponds a linear combinations of all possible products of \(k+1\) elements \( x_{pq}(-1) \) such that the (multi)set of row-indexes is equal to \( \{i_1, i_2, \ldots, i_{k+1}\} \), and the (multi)set of column-indexes is equal to \( \{j_1, j_2, \ldots, j_{k+1}\} \),

\[
\sum_{\{p_1, \ldots, p_{k+1}\} = \{i_1, \ldots, i_{k+1}\}, q_1, \ldots, q_{k+1}= \{j_1, \ldots, j_{k+1}\}} C_{pq}x_{p_1q_1}(-1)x_{p_2q_2}(-1) \cdots x_{p_{k+1}q_{k+1}}(-1),
\]

and the coefficients \( C_{pq} \) are positive integers.

One can first act on the highest weight vectors by the \( x_{-\alpha_i} \)'s, \( i \in \{m + 1, \ldots, \ell\} \), that would send \( x_1^{k+1} \otimes x_\ell^{k+1} \in V_1 \otimes V_2 \) to

\[
x_1^{k+1} \otimes x_{j_1}x_{j_2} \cdots x_{j_{k+1}}.
\]

Acting with the same \( x_{-\alpha_i} \)'s on \( x_1^{k+1}(-1) \in V \) will give

\[
x_{i_1j_1}(-1)x_{i_2j_2}(-1) \cdots x_{i_{k+1}j_{k+1}}(-1),
\]
or, more precisely, some multiple of that monomial with positive integer coefficient. Next, one would act on these vectors with \( x_{-\alpha_{k+1}} \ldots x_{-\alpha_1} \); on one side, one gets
\[
x_k^k x_{i_{k+1}} \otimes x_{j_1}, x_{j_2} \ldots x_{j_{k+1}} \in V_1 \otimes V_2,
\]
and on the other
\[
\sum_{r=1}^{k+1} x_{1j_1} (-1) \cdots x_{i_{k+1}j_r} (-1) \cdots x_{1j_{k+1}} (-1) \in V.
\]
In the second vector for every occurrence of index 1 at the first place in (15), we have a monomial where this index was changed to \( i_{k+1} \). Monomials
\[
x_{1j_1} (-1) \cdots x_{1j_{r-1}} (-1)x_{1j_{r+1}} (-1) \cdots x_{1j_{k+1}} (-1)
\]
correspond to vectors
\[
x_k^k \otimes x_{j_1} \cdots x_{j_{r-1}} x_{j_{r+1}} \cdots x_{j_{k+1}}
\]
from the appropriate module \( V'_1 \otimes V'_2 \) of the highest weight \( k\theta \). Next, one would act on the vector and monomials above by \( x_{-\alpha_{k+1}} x_{-\alpha_{k-1}} \cdots x_{-\alpha_1} \) that would change one occurrence of index 1 into \( i_k \). Since \( x_{-\alpha_{i_k}} x_{i_{k+1}} = 0 \) for \( s < i_{k+1} \), the proof follows by induction on \( k \).

Like in (10), from (14) we obtain
\[
\sum_{\{p_1, \ldots, p_{k+1}\} = \{i_1, \ldots, i_{k+1}\}} \sum_{\{q_1, \ldots, q_{k+1}\} = \{j_1, \ldots, j_{k+1}\}} C_{pq} x_{p_1, q_1} (-1)x_{p_2, q_2} (-1) \cdots x_{p_{k+1}, q_{k+1}} (-1) = 0
\]
in \( L(k\Lambda_0) \subset L(\Lambda_0)^{\otimes k} \).

From this we obtain the following family of relations between corresponding vertex-operators on \( L(\Lambda) \):
for \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{k+1} \leq m \) and \( \ell \geq j_1 \geq j_2 \geq \cdots \geq j_{k+1} \geq m \),
\[
\sum_{\{p_1, \ldots, p_{k+1}\} = \{i_1, \ldots, i_{k+1}\}} \sum_{\{q_1, \ldots, q_{k+1}\} = \{j_1, \ldots, j_{k+1}\}} C_{pq} x_{p_1, q_1} (z)x_{p_2, q_2} (z) \cdots x_{p_{k+1}, q_{k+1}} (z) = 0,
\]
where the coefficients \( C_{pq} \) are positive integers.

8.2. Leading terms. Fix one choice
\[
1 \leq i_1 \leq i_2 \leq \cdots \leq i_{k+1} \leq m,
\ell \geq j_1 \geq j_2 \geq \cdots \geq j_{k+1} \geq m
\]
and observe the corresponding relation (17). For every \( n \geq k + 1 \), coefficients of powers \( z^{n-k-1} \) are infinite sums of monomials:
\[
\sum_{\{p_1, \ldots, p_{k+1}\} = \{n_1, \ldots, n_{k+1}\}} C_{pq} x_{p_1, q_1} (-n_1)x_{p_2, q_2} (-n_2) \cdots x_{p_{k+1}, q_{k+1}} (-n_{k+1}) = 0.
\]
In each such sum, we identify the minimal monomial in the lexicographical order defined in section 3. We call this monomial the leading term of the relation. Because of the minimality, every monomial that contains a leading term can be excluded from the spanning set (cf. Proposition 2).

All monomials that appear in (18) are of the same length \( k + 1 \) and of the same total degree \( -n \). Hence we can consider only those monomials that are of the minimal shape, i.e. the ones in which degrees of factors differ for at most 1. The others will be greater then these.
Consider first the case \( n = r(k + 1) \), for some \( r \in \mathbb{N} \). In this case, monomials of minimal shape will have all \( k + 1 \) factors of the same degree \(-r\). So we need to find the minimal possible monomial

\[
x_{p_1,q_1}(-r)x_{p_2,q_2}(-r) \cdots x_{p_{k+1},q_{k+1}}(-r)
\]
such that

\[
\{p_1, \ldots, p_{k+1}\} = \{i_1, \ldots, i_{k+1}\},
\{q_1, \ldots, q_{k+1}\} = \{j_1, \ldots, j_{k+1}\}.
\]

Since all factors are of the same degree, the minimal monomial will be the one that has the minimal configuration of colors of its factors. If we assume that factors of a monomial are sorted ascendingly from left to right, this means that we have to choose the smallest possible color \( \gamma_{p_{k+1},q_{k+1}} \) (the greatest color in a monomial), next the smallest possible color \( \gamma_{p_1,q_1} \), and so on.

Since the row and column-indexes of colors of monomial are fixed (\( \{i_1, \ldots, i_{k+1}\} \) and \( \{j_1, \ldots, j_{k+1}\} \), resp.) the greatest color will lie in the \( i_1 \)-th row, and the smallest possible of them is \( \gamma_{i_1,j_{k+1}} \). The second greatest color lies in the \( i_2 \)-th row, the smallest possible being \( \gamma_{i_2,j_k} \). We proceed in the same manner, and obtain a monomial

\[
x_{i_{k+1},j_1}(-r) \cdots x_{i_2,j_k}(-r)x_{i_1,j_{k+1}}(-r).
\]

Consider now the configuration of the colors in (19). Each color \( \gamma_{i_{t+1}j_{k-t+1}} \) is placed on the right of, or below, or diagonally on the right and below color \( \gamma_{i_tj_{k-t+2}} \) (see figure 5).

**Figure 5.** Colors of leading terms

Consequently, we conclude that colors of the leading term lie on a diagonal path as pictured in the figure 6.

Finally, we can describe the leading terms via exponents of factors. Let \( a^r_{i,j} = \pi(x_{ij}(-r)) \) be an exponent of \( x_{ij}(-r) \) in \( x(\pi) \). Then the leading terms correspond to solutions of equations

\[
a^r_{i_1,j_1} + a^r_{i_2,j_{t-1}} + \cdots + a^r_{i_t,j_1} = k + 1,
\]

\( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq m, \)

\( \ell \geq j_1 \geq j_2 \geq \cdots \geq j_t \geq m, \)

\( (i_\nu,j_\nu) \neq (i_{\nu+1},j_{\nu+1}), \)

\( i_1 \neq i_2, \) then \( \gamma_{i_1,j_{k+1}} < \gamma_{i_2,j_k} \), but this shouldn’t concern us, because in this way we’ll certainly obtain the smallest possible choice of colors from the \( i_1 \)-th row.
Consider next the case \( n = r(k + 1) + s \). Monomials of minimal shape in this case are built of \( s \) factors of degree \(-r-1\) and \( k-s+1 \) factors of degree \(-r\). Since factors of degree \(-r\) are greater than factors of degree \(-r-1\), the leading term of \((19)\) can be obtained so that first one chooses the smallest possible \((-r)\)-part of a monomial, and then the smallest possible \((-r-1)\)-part. Hence factors of degree \(-r\) are placed in the last \( k-s+1 \) rows and the first \( k-s+1 \) columns, and in the rest are placed factors of degree \(-r-1\):

\[
\begin{align*}
(-r)\text{-part} & \quad \Rightarrow (i_{s+1}, i_{s+2}, \ldots, i_{k+1}), (j_{s+1}, j_{s+2}, \ldots, j_{k+1}), \\
(-r-1)\text{-part} & \quad \Rightarrow (i_1, i_2, \ldots, i_s), (j_1, j_2, \ldots, j_s).
\end{align*}
\]

We proceed as in the first case; the colors of elements of the same degree will lie on a diagonal path as before, and the configuration of colors is of the type pictured on the figure 2.

In terms of exponents, this leading terms correspond to solutions of equations

\[
\begin{align*}
a_i^{-1} + \cdots + a_i^{-1} + a_{i+1}^{-1} + \cdots + a_{i+1}^{-1} &= k + 1, \\
1 \leq i_1 \leq \cdots \leq i_s &\leq i_{s+1} \leq \cdots \leq i_t \leq m, \\
\ell \geq j_1 \geq \cdots \geq j_s &\geq j_{s+1} \geq \cdots \geq j_t \geq m \\
(i_\nu, j_\nu) &\neq (i_{\nu+1}, j_{\nu+1}), \text{ for } \nu = 1, \ldots, s-1, \\
s+1, \ldots, \ell
\end{align*}
\]

where \( a_{ij}^r = \pi(x_{ij}(-r)) \), for all \( r \in \mathbb{N} \).

In the end, observe that solutions of (20) are also solutions of some equation of this type. Since the rectangles in which colors of elements of degree \(-r\) and \(-r-1\) lie, intersect in the upper-right and the lower-left corner, one can take for the \((-r)\)-part to lie in the whole \( \Gamma \), and for the \((-r-1)\)-part to lie in upper-right square, on the position \( \ell \).

We’ll say that a monomial \( x(\pi) \in S(\tilde{\mathfrak{g}}^-) \) satisfies difference conditions on \( L(\Lambda) \), or shortly, that \( x(\pi) \) satisfies DC on \( L(\Lambda) \), if it doesn’t contain a leading term. More precisely, \( x(\pi) \) satisfies difference conditions if

\[
\begin{align*}
a_i^{-1} + \cdots + a_i^{-1} + a_{i+1}^{-1} + \cdots + a_{i+1}^{-1} &\leq k, \\
1 \leq i_1 \leq \cdots \leq i_s &\leq i_{s+1} \leq \cdots \leq i_t \leq m, \\
\ell \geq j_1 \geq \cdots \geq j_s &\geq j_{s+1} \geq \cdots \geq j_t \geq m \\
(i_\nu, j_\nu) &\neq (i_{\nu+1}, j_{\nu+1}), \text{ for } \nu = 1, \ldots, s-1, \\
s+1, \ldots, \ell
\end{align*}
\]

where \( a_{ij}^r = \pi(x_{ij}(-r)) \), for all \( r \in \mathbb{N} \).
The following proposition was essentially proved in [P1]. The statement follows from the proposition \([\text{P1}]\) and the minimality of leading terms.

**Proposition 2** ([P1]). The set

\[(22) \{ x(\pi)v_\Lambda \mid x(\pi) \text{ satisfies DC} \}\]

spans \(W(\Lambda)\).

8.3. **Initial conditions.** By difference conditions, in a monomial \(x(\pi)\), factors of degree \(-r\), for \(r > 1\), are restricted by factors of degree \(-r - 1\) and \(-r + 1\). Exceptions are factors of degree \(-1\), which are restricted only "from below", by factors of degree \(-2\). Initial conditions will play the part of restrictions "from above" on factors of degree \(-1\).

In the case \(k = 1\) there were no relations between monomials consisting of factors of degree \(-1\), other than those already used for difference conditions. Thus, the initial conditions demanded only that there are no \((-1)\)-factors that annihilate highest weight vector. However, when the level \(k > 1\), generally, there are other relations between such monomials. These relations amount to relations for difference conditions but for modules of level lesser than \(k\).

We’ll say that a monomial \(x(\pi) \in S(\tilde{\mathfrak{g}}_1^-)\) satisfies initial conditions on \(L(\Lambda)\), or shortly, that \(x(\pi)\) satisfies IC on \(L(\Lambda)\), if

\[(23) a_{i_1j_1}^1 + a_{i_2j_{r-1}}^1 + \cdots + a_{i_tj_t}^1 \leq k_0 + k_1 + \cdots + k_{i_t-1} + k_{j_t+1} + \cdots + k_\ell,
1 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq m,
\ell \geq j_1 \geq j_2 \geq \cdots \geq j_t \geq m,
(i_\nu, j_\nu) \neq (i_{\nu+1}, j_{\nu+1}),\]

where \(a_{ij}^1 = \pi(i_j(1))\). The sum on the right side of inequality is a sum of multiplicities of those modules of level 1 on which at least one \(x_{i_\nu j_\nu - 1}(-1)\) doesn’t act as 0.

**Proposition 3.** The set

\[ \{ x(\pi)v_\Lambda \mid x(\pi) \text{ satisfies IC and DC} \}\]

spans \(W(\Lambda)\).

**Proof:** Assume that \(x(\pi)\) doesn’t satisfy some inequality of the type \((23)\) and set \(d = k_0 + k_1 + \cdots + k_{i_t-1} + k_{j_t+1} + \cdots + k_\ell + 1\). Then \(x(\pi)\) contains a monomial \(x(\pi')\) consisting only of factors of degree \(-1\), such that it also doesn’t satisfy that inequality. Furthermore, one can assume that the length of \(x(\pi')\) is equal to \(d\).

We’ll show that we can find monomials \(x(\pi'_1), \ldots, x(\pi'_{\ell})\) such that \(x(\pi') < x(\pi'_i)\), \(x(\pi'_i)\)'s are of the same degree as \(x(\pi')\) and

\[ x(\pi')v_\Lambda = \alpha_1x(\pi'_1)v_\Lambda + \cdots + \alpha_\ell x(\pi'_{\ell})v_\Lambda, \]

for some \(\alpha_i \in \mathbb{C}\). Upon multiplying them with the rest of \(x(\pi)\), we’ll obtain monomials \(x(\pi_i)\) of the same degree as \(x(\pi)\), such that \(x(\pi) < x(\pi_i)\) (cf. Proposition \([\text{P1}]\) and

\[ x(\pi)v_\Lambda = \alpha_1x(\pi_1)v_\Lambda + \cdots + \alpha_\ell x(\pi_{\ell})v_\Lambda. \]

The proof then follows in the same way as the proof of Proposition \([\text{P2}]\) (cf. [P1,P2]).

Let

\[ x(\pi') = x_{i_1j_1}(-1)\cdots x_{i_\nu j_\nu}(-1) \]

be as above. There are 2 possibilities:

(i) \(d = k + 1\).

In this case the initial condition is equivalent to difference condition and the proof follows from the analogous proof for difference conditions (cf. Proposition \([\text{P2}]\)).
\( (ii) \; d \leq k. \)

Then in the tensor product \( v_\Lambda = v_{i_0}^{\otimes k_0} \otimes v_{i_1}^{\otimes k_1} \otimes \cdots \otimes v_{i_\ell}^{\otimes k_\ell} \) there is at least one \( v_i \) that is annihilated by all factors of \( x(\pi') \), and \( k_i > 0 \). Group \( \Lambda_i \)'s in the following way

\[
\Lambda' = \sum_{i=0}^{j_d-1} k_i \Lambda_i + \sum_{i=j_d+1}^\ell k_i \Lambda_i,
\]

\[
\Lambda'' = \sum_{i=j_d}^j k_i \Lambda_i = \Lambda - \Lambda'.
\]

Denote by \( v_{\Lambda'} \) and \( v_{\Lambda''} \) highest weight vectors of standard modules \( L(\Lambda') \) and \( L(\Lambda'') \). Then, by the complete reducibility,

\[
L(\Lambda) \subset L(\Lambda') \otimes L(\Lambda''),
\]

\[
v_\Lambda = v_{\Lambda'} \otimes v_{\Lambda''}.
\]

Since all factors of \( x(\pi') \) annihilate \( v_{\Lambda''} \), we have

\[
x(\pi') v_\Lambda = (x(\pi') v_{\Lambda'}) \otimes v_{\Lambda''}.
\]

Note that \( L(\Lambda') \) is a module of level \( k' < k \) and \( d = k' + 1 \). From relations [13] for the module \( L(\Lambda'') \) we obtain monomials \( x(\pi'_1), \ldots, x(\pi'_j) \) of the same length and degree, such that \( x(\pi') v_{\Lambda'} = \alpha_1 x(\pi'_1) v_{\Lambda'} + \cdots + \alpha_j x(\pi'_j) v_{\Lambda'} \), \( \alpha_i \in \mathbb{C} \), and \( x(\pi') < x(\pi'_i) \). Also, from these relations we see that colors of factors of monomials \( x(\pi'_i) \) lie in the same rows and columns as colors of \( x(\pi') \). Hence, all factors of \( x(\pi'_i) \) also act as 0 on \( v_{\Lambda''} \). Consequently,

\[
x(\pi') v_\Lambda = \alpha_1 x(\pi'_1) v_\Lambda + \cdots + \alpha_j x(\pi'_j) v_\Lambda.
\]

\[\blacksquare\]

9. Difference and Initial Conditions II

We’ll show how difference and initial conditions for modules of level \( k > 1 \) can be restated in terms of difference and initial conditions for modules of level 1. We are going to prove

**Theorem 4.** Let \( L(\Lambda) \) be a standard module of level \( k \) with highest weight vector \( v_\Lambda = v_{i_1} \otimes \cdots \otimes v_{i_k} \), where \( v_{i_j} \) are highest weight vectors of corresponding modules \( L(\Lambda_{i_j}) \) of level 1. If a monomial \( x(\pi) \in S(\tilde{\mathfrak{g}}_1) \) satisfies difference and initial conditions on \( L(\Lambda) \), then there exists a factorization

\[
x(\pi) = x(\pi^{(1)}) \cdots x(\pi^{(k)})
\]

such that \( x(\pi^{(j)}) \) satisfies difference and initial conditions on \( L(\Lambda_{i_j}) \).

Proposition [6] will imply the converse of the theorem. Hence, we’ll have an equivalence:

**Corollary 5.** With notation as above, a monomial \( x(\pi) \in S(\tilde{\mathfrak{g}}_1) \) satisfies difference and initial conditions on \( L(\Lambda) \) if and only if there exists a factorization

\[
x(\pi) = x(\pi^{(1)}) \cdots x(\pi^{(k)})
\]

such that \( x(\pi^{(j)}) \) satisfies difference and initial conditions on \( L(\Lambda_{i_j}) \).
9.1. Difference conditions can be written in another way. We first prove theorem 4 in the special case \( \Lambda = k\Lambda_0 \) and later for general \( \Lambda \). For \( \Lambda = k\Lambda_0 \), initial conditions are contained in difference conditions, so we’re only considering difference conditions on monomials.

Define another order on the set of variables:

\[
(24) \quad x_{ij}(-r) \sqsubset x_{i'j'}(-r') \quad \text{if} \quad \begin{cases} 
-r \leq -r' - 2, \\
-r = -r' - 1; \quad i > i' \text{ or } j < j', \\
-r = -r; \quad i > i' \text{ and } j < j'.
\end{cases}
\]

This is equivalent to saying that \( x_{ij}(-r) < x_{i'j'}(-r') \) and \( x_{ij}(-r)x_{i'j'}(-r') \) satisfies difference conditions on modules of level 1. By proposition 3 from [T], difference conditions for modules of level 1 are conditions “at distance 1”, i.e. relation \( \sqsubset \) is transitive, and hence it is a (strict) partial order on the set of variables.

We look at monomials \( x(\pi) \in S(\tilde{\Phi}_1) \) as (multi)sets. Then we have the following characterization of monomials satisfying difference conditions on level \( k \) modules:

**Proposition 6.** A monomial \( x(\pi) \) satisfies difference conditions on modules of level \( k \) if and only if every subset of \( x(\pi) \) in which there are no two elements comparable in the sense of \( \sqsubset \), has at most \( k \) elements.

**Proof:** Let \( x_{ij}(-r), x_{i'j'}(-r') \in \tilde{\Phi}_1 \) be two variables and assume \( r \geq r' \). By (24), they are incomparable if and only if

\[
(25) \quad \begin{cases} 
-r = -r'; \quad i \leq i', j \leq j', \\
-r = -r'; \quad i \geq i', j \geq j', \\
-r = -r' - 1; \quad i \leq i', j \geq j'.
\end{cases}
\]

It is now clear that elements whose colors lie on a diagonal path that was considered in (21) are mutually incomparable in the sense of \( \sqsubset \). Hence, if \( x(\pi) \) doesn’t satisfy difference conditions, then it has a subset of at least \( k + 1 \) mutually incomparable elements. Conversely, consider a subset of \( x(\pi) \) in which all elements are mutually incomparable. By the relation (24), degrees of its elements can differ for at most 1. Assume that they are of degrees \(-r\) and \(-r - 1\). Since the elements of the same degree are incomparable, their colors must all lie on a diagonal path like in (20). Finally, since elements of a different degree aren’t comparable these two paths are related like in (21).

Notice that by proposition 3 from [T], if \( \{ x_{\gamma_1}(-r_1), x_{\gamma_2}(-r_2), \ldots, x_{\gamma_t}(-r_t) \} \) is a linearly ordered subset,

\[
x_{\gamma_1}(-r_1) \sqsubset x_{\gamma_2}(-r_2) \sqsubset \cdots \sqsubset x_{\gamma_t}(-r_t),
\]

then the monomial

\[
x_{\gamma_1}(-r_1)x_{\gamma_2}(-r_2) \cdots x_{\gamma_t}(-r_t)
\]

satisfies difference conditions on modules of level 1. Thus theorem 4 will be proved when we show that there exists a partition of \( x(\pi) \) into \( k \) linearly ordered subsets.

9.2. **Proof of theorem 4** Let \( S \) be a finite set, \( |S| = n \). Let \( \sqsubset \) be a (strict) partial order on \( S \). For a subset \( X \subset S \), we say that \( X \) is totally disordered or discretely ordered if elements of \( X \) are mutually incomparable, i.e. if the restriction \( \sqsubset \mid_{X \times X} \) is an empty set.

Theorem 4 now follows from the following combinatorial lemma:

**Lemma 7.** Let \( (S, \sqsubset) \) be a finite set with a strict partial order \( \sqsubset \). If every totally disordered subset of \( X \) consists of at most \( k \) elements, then there exists a partition of \( S \) into at most \( k \) linearly ordered subsets.
Proof:
Let \( l \) be the maximal cardinality of a totally disordered subset of \( S \); \( l \leq k \). We’re going to show that there is a partition of \( S \) into \( l \) linearly ordered subsets.

We prove this by induction on \( l \) and on the number of elements of \( S \), \( n = |S| \).

Distinguish 2 cases:

(i) There exists a subset \( \{ a_1, \ldots, a_l \} \subset S \) consisting of mutually incomparable elements, such that \( a_1, \ldots, a_l \) aren’t all maximal elements of \( S \), or all minimal elements of \( S \).

Because of the maximality of \( l \), every element of \( S \) is comparable to some element of \( \{ a_1, \ldots, a_l \} \).

Define subsets
\[
G = \{ x \in S, x \sqsubseteq a_i \text{ for some } i \},
\]
\[
D = \{ x \in S, x \sqsupseteq a_i \text{ for some } i \}.
\]

Since by the hypothesis \( a_1, \ldots, a_l \) are not all maximal elements of \( S \), nor all minimal elements of \( S \), sets \( G \) and \( D \) are nonempty. Then we have a partition
\[
S = G \cup D \cup \{ a_1, \ldots, a_l \}.
\]

Set
\[
G' = G \cup \{ a_1, \ldots, a_l \},
\]
\[
D' = D \cup \{ a_1, \ldots, a_l \}.
\]

Sets \( G' \) and \( D' \) have less than \( n \) elements, so by the induction hypothesis they can be partitioned into linearly ordered subsets. The set \( \{ a_1, \ldots, a_l \} \) is at the same time the set of minimal elements of \( G' \), and the set of maximal elements of \( D' \). Hence, linearly ordered subsets of \( G' \) end with some of the \( a_1, \ldots, a_l \), while linearly ordered subsets of \( D' \) start with some of the \( a_1, \ldots, a_l \). By “gluing” appropriate pairs together, we get a partition of \( S \) into \( l \) linearly ordered subsets.

(ii) The only sets with \( l \) mutually incomparable elements is either the set of minimal, or the set of maximal elements of \( S \). In this case we cannot construct a partition as we did earlier because either \( G \) or \( D \) would be empty.

Consider 2 cases:

(a) Assume that the only totally disordered subset with \( l \) elements is the set of maximal elements of \( S \) (analogously for minimal elements). Denote them by \( a_1, \ldots, a_l \). Choose a linearly ordered subset \( \{ x_1, x_2, \ldots, x_r \} \subset S \) that starts with \( a_1 \),
\[
a_1 = x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_r.
\]

The set \( S \setminus \{ x_1, \ldots, x_r \} \) has totally disordered subsets of at most \( l - 1 \) elements, so by the induction hypothesis it can be partitioned into \( l - 1 \) linearly ordered subsets. Together with \( \{ x_1, x_2, \ldots, x_r \} \), this gives a partition of \( S \) into \( l \) linearly ordered subsets.

(b) Assume that \( a_1, \ldots, a_l \) are all maximal, and \( b_1, \ldots, b_l \) all minimal elements of \( S \).

Choose a linearly ordered subset \( \{ x_1, x_2, \ldots, x_r \} \subset S \) that starts with \( a_1 \) and ends with some of the \( b_l \)'s,
\[
a_1 = x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_r = b_l.
\]

Like in the previous case, the set \( S \setminus \{ x_1, \ldots, x_r \} \) has totally disordered subsets of at most \( l - 1 \) elements. By the induction hypothesis it can be partitioned into \( l - 1 \) linearly ordered subsets, which together with \( \{ x_1, x_2, \ldots, x_r \} \) gives a partition of \( S \) into \( l \) linearly ordered subsets.

\[\blacksquare\]
9.3. Initial conditions. We now prove the theorem 4 in the general case when 
\[ \Lambda = k_0 \Lambda_0 + \cdots + k_\ell \Lambda_\ell. \]

First, let us recall initial conditions for a level 1 module \( L(\Lambda_i), i = 1, \ldots, \ell \). A monomial \( x(\pi) \) satisfies initial conditions for \( L(\Lambda_i) \) if colors of elements of degree \(-1\) lie below the \( i\)-th row (for \( 1 \leq i \leq m \)), or on the left of the \( i\)-th column (for \( m \leq i \leq \ell \)). Note that these conditions can be understood as difference conditions if we add some imaginary elements of degree 0 to \( x(\pi) \): for \( 1 \leq i \leq m \) add \( x_{im}(0) \) to \( x(\pi) \), and for \( m \leq i \leq \ell \) add \( x_{mi}(0) \) to \( x(\pi) \). Then \( x(\pi) \) satisfies difference and initial conditions for \( L(\Lambda_i) \) if and only if this new monomial satisfies difference conditions for \( L(\Lambda_i) \).

This observation generalizes to any level \( k \). Let 
\[ \Lambda = k_0 \Lambda_0 + \cdots + k_\ell \Lambda_\ell, \quad k = k_0 + \cdots + k_\ell. \]
For every \( i = 1, \ldots, \ell \), we add \( k_i \) elements of degree 0 of the appropriate color to \( x(\pi) \). Concretely, denote by \( x(\pi') \) the monomial 
\[ x(\pi') = x(\pi) \cdot x_{1m}(0)x_{2m}(0)\cdots x_{mm}(0)x_{m+1m}(0)\cdots x_{m\ell}(0). \]
Colors of \((0)\)-elements of \( x(\pi') \) lie on a diagonal path as pictured on the figure 7. Consider difference conditions on \( x(\pi') \) for elements of degrees \(-1\) and 0. Assume that \( x(\pi) \) (and \( x(\pi') \)) contains elements of degree \(-1\) whose colors \( \gamma_{i_1, j_1}, \ldots, \gamma_{i_t, j_t} \) lie on a diagonal path as on the figure 7. Let \( a_{i_1, j_1}, \ldots, a_{i_t, j_t} \) be the exponents of these elements. For every such choice of \((-1)\)-elements, consider \((0)\)-elements of \( x(\pi') \) whose colors lie below the \((i_1 - 1)\)-st row and on the left of the \((j_1 + 1)\)-st column - these are the elements \( x_{i_1m}(0), x_{i_1+1,m}(0), \ldots, x_{mm}(0), x_{m+1m}(0), \ldots, x_{m\ell}(0), \) with exponents \( k_{i_1}, k_{i_1+1}, \ldots, k_m, \ldots, k_{j_1}, \) respectively. By difference conditions (21) for \( x(\pi') \), we have 
\[ a_{i_1, j_1} + \cdots + a_{i_t, j_t} + k_{i_1} + \cdots + k_{j_t} \leq k. \]
Then 
\[ a_{i_1, j_1} + \cdots + a_{i_t, j_t} \leq k - k_{i_1} - \cdots - k_{j_t}. \]
Hence 
\[ a_{i_1, j_1} + \cdots + a_{i_t, j_t} \leq k_0 + k_1 + \cdots + k_{i_1-1} + k_{j_1+1} + \cdots + k_{j_t}. \]
So, we’ve obtained initial conditions for \( W(\Lambda) \) (cf. (23)). We’ve proved

**Proposition 8.** Let \( x(\pi) \) and \( x(\pi') \) be as above. Then \( x(\pi) \) satisfies difference and initial conditions for \( W(\Lambda) \) if and only if \( x(\pi') \) satisfies difference conditions.

**Figure 7.** Initial conditions in terms of difference conditions
If $x(\pi')$ satisfies difference conditions then there exists a partition of $x(\pi')$ into $k$ linearly ordered subsets. Elements of degree 0 are mutually incomparable, so they will lie in different subsets of the partition. More precisely, they will be the maximal elements of the corresponding subsets. By removing these (0)-elements from subsets of the partition, we get a partition of $x(\pi)$ into $k$ linearly ordered subsets. Moreover, subsets that’ve contained (0)-elements corresponding to modules $L(\Lambda_i)$, $i = 0, \ldots, \ell$, satisfy initial conditions on these modules. This finishes the proof of the theorem 4 in the general case.

10. Proof of linear independence

10.1. Intertwining operators. In the level 1 case, the main technical tool in the proof of linear independence of the spanning set of $W(\Lambda)$ was the following proposition (cf. [T])

**Proposition 9.** Suppose that a monomial $x(\pi)$ satisfies difference and initial conditions for a level 1 standard module $L(\Lambda_i)$. Write $x(\pi) = x(\pi_1)x(\pi_2)$, where $x(\pi_1)$ is a $(−1)$-part of a monomial, and $x(\pi_2)$ the rest of the monomial. Then there exists a coefficient $w(\mu)$ of an intertwining operator $\mathcal{Y}(\epsilon^\mu, z)$,

$$w(\mu): L(\Lambda_i) \to L(\Lambda_{i'})$$

for some $i' \in \{0, \ldots, \ell\}$, such that:

- $w(\mu)$ commutes with $\tilde{\mathcal{Y}}$,
- $w(\mu)x(\pi_1)v_i = Ce(\omega)v_i$, $C \in \mathbb{C}^\times$,
- $x(\pi_2)$ satisfies IC and DC for $L(\Lambda_{i'})$,
- if $x(\pi')$ has a $(−1)$-part $x(\pi_1')$ greater than $x(\pi_1)$, then $w(\mu)x(\pi')v_i = 0$.

By using theorem 4, we are able to generalize this proposition for higher level standard modules. Let $L(\Lambda)$ be a standard module of level $k$, with the highest weight vector $v_\Lambda = v_1 \otimes \cdots \otimes v_k$.

Fix a monomial $x(\pi)$ that satisfies difference and initial conditions for $L(\Lambda)$. Let

$$x(\pi) = x(\pi_1)x(\pi_2)$$

be a factorization of $x(\pi)$ such that $x(\pi_1)$ is a $(−1)$-part, and $x(\pi_2)$ is the rest of the monomial $x(\pi)$. By theorem 4 there exists a factorization

$$x(\pi) = x(\pi^{(1)}) \cdots x(\pi^{(k)})$$

such that $x(\pi^{(j)})$ satisfies difference and initial conditions for $W(\Lambda_{i_j})$. Furthermore, this induces the corresponding factorizations of $x(\pi_1)$ and $x(\pi_2)$:

$$x(\pi_1) = x(\pi_1^{(1)}) \cdots x(\pi_1^{(k)})$$
$$x(\pi_2) = x(\pi_2^{(1)}) \cdots x(\pi_2^{(k)})$$

By the proposition 4 there exist coefficients of intertwining operators $w(\mu_j)$, $j = 1, \ldots, k$ such that

$$x(\pi_1^{(j)})v_{i_j} \xrightarrow{w(\mu_j)} C(\pi^{(j)})e(\omega)v_{i'_{j}}, C \in \mathbb{C}^\times.$$  

Let $\Lambda' = \Lambda_{i_1'} + \cdots + \Lambda_{i_k'}$, and define an operator $w : L(\Lambda) \to L(\Lambda')$ with

$$w = w(\mu_1) \otimes \cdots \otimes w(\mu_k).$$

Let

$$v_{\Lambda'} = v_{i_1'} \otimes \cdots \otimes v_{i_k'}$$

be the highest weight vector of $L(\Lambda')$. Then

$$x(\pi_1^{(1)})v_{i_1} \otimes \cdots \otimes x(\pi_1^{(k)})v_{i_k} \xrightarrow{w} Ce(\omega)v_{\Lambda'}, C \in \mathbb{C}^\times.$$
Since by the proposition \[ x(\pi_2^{(j)}) \] satisfy difference and initial conditions for \( L(\Lambda'_j) \), then, by corollary \[ x(\pi_2^{(j)}) \] also satisfies difference and initial conditions for \( L(\Lambda') \).

Since \( x(\pi_1^{(1)})v_{i_1} \otimes \cdots \otimes x(\pi_1^{(k)})v_{i_k} \) is only one of the summands that we get by acting with \( x(\pi_1) \) on the tensor product \( v_{\Lambda} = v_{i_1} \otimes \cdots \otimes v_{i_k} \), we need to see what happens with other summands of \( x(\pi_1)v_{\Lambda} \) when we act on them with the operator \( w \)? The other summands of \( x(\pi)v_{\Lambda} \) come from other factorizations of \( x(\pi) \). Let

\[ x(\pi) = x(v^{(1)}) \cdots x(v^{(k)}) \]

be another factorization of \( x(\pi) \), with induced factorizations of \( x(\pi_1), x(\pi_2) \). Assume that \( w \) doesn’t annihilate \( x(v^{(1)})v_{i_1} \otimes \cdots \otimes x(v^{(k)})v_{i_k} \). Since by proposition \[ x(\pi_1^{(j)}) \] is maximal for \( w(\mu_j) \), we have

\[ x(v^{(j)}) \leq x(\pi_1^{(j)}), \quad j = 1, \ldots, k. \]

If some of these inequalities were strict, then by the proposition \[ \text{we would have} \] \( x(\pi) < x(\pi) \), which is a contradiction. Hence, all the factors must be equal,

\[ x(v^{(j)}) = x(\pi_1^{(j)}), \quad j = 1, \ldots, k. \]

We conclude that the operator \( w \) will not annihilate only those summands of \( x(\pi_1)v_{\Lambda} \) that are equal to \( x(\pi_1^{(1)})v_{i_1} \otimes \cdots \otimes x(\pi_1^{(k)})v_{i_k} \), and furthermore,

\[ w \cdot x(\pi_1)v_{\Lambda} = C \cdot (w(\mu_1)x(\pi_1^{(1)})v_{i_1} \otimes \cdots \otimes w(\mu_k)x(\pi_1^{(k)})v_{i_k}) = C' e(\omega)v_{\Lambda'}, \]

for some \( C, C' \in \mathbb{C}^\times \).

In a similar fashion we show that the operator \( w \) annihilates all \( x(\nu)v_{\Lambda} \) whose \((-1)\)-part \( x(\nu_1) \) is greater than \( x(\pi_1) \). If \( w \) doesn’t annihilate \( x(\nu)v_{\Lambda} \), then there exists a factorization

\[ x(\nu) = x(v^{(1)}) \cdots x(v^{(k)}) \]

such that \( x(v^{(j)})v_{i_j} \) aren’t annihilated by \( w(\mu_j) \). We also have the induced factorization of the \((-1)\)-part \( x(\mu_1) \). By the proposition \[ \text{we have} \]

\[ x(v^{(j)}) \leq x(\pi_1^{(j)}), \quad j = 1, \ldots, k, \]

and by the proposition \[ \text{we conclude} \]

\[ x(v_1) \leq x(\pi_1). \]

We have proved a generalization of the proposition \[ x(\pi) \] satisfy difference and initial conditions for \( L(\Lambda) \). Then there exists an operator \( w : L(\Lambda) \to L(\Lambda') \), where \( L(\Lambda') \) is another standard module of the same level, such that:

- \( w \) commutes with \( \tilde{g}_1 \),
- \( w \cdot x(\pi_1)v_{\Lambda} = C e(\omega)v_{\Lambda'}, \quad C \in \mathbb{C}^\times \),
- \( x(\pi_1^{(j)}) \) satisfies IC and DC for \( L(\Lambda') \),
- \( x(\pi_1) \) is maximal for \( w \), i.e. all monomials \( x(\pi') \) such that \( w(\mu)x(\pi')v_{\Lambda} \neq 0 \), have their \((-1)\)-part \( x(\pi_1') \) smaller or equal to \( x(\pi_1) \).  

10.2. Proof of linear independence. Before we proceed with the proof of linear independence, we’ll introduce some more notation, and change a bit the existing one.

For a monomial \( x(\pi) \), set \( x(\pi_r) \) to be the \((-r)\)-part of \( x(\pi) \), and \( x(\pi_r) = 1 \) if \( x(\pi) \) doesn’t contain any element of degree \(-r\). Then

\[ x(\pi) = x(\pi_r)x(\pi_{r-1}) \cdots x(\pi_1). \]
if \( x(\pi) \) consists of elements of degree greater than or equal to \(-n\). Note that the order on the set of monomials is compatible with the order on the "homogenous parts:

\[
x(\pi) = x(\pi_n) x(\pi_{n-1}) \cdots x(\pi_1),
\]

\[
x(\pi') = x(\pi'_n) x(\pi'_{n-1}) \cdots x(\pi'_1),
\]

then \( x(\pi') < x(\pi) \) if and only if

\[
x(\pi'_1) = x(\pi_1)
\]

\[
\vdots
\]

\[
x(\pi'_r) = x(\pi_r)
\]

\[
x(\pi'_{r+1}) < x(\pi_{r+1})
\]

for some \( r \).

Denote by \( x(\pi^+) \) a monomial obtained from \( x(\pi) \) by raising a degree by \( r \) for all elements of degree \(-j < -r\), and omitting elements of degree \(-j \leq -r\). Instead of \( x(\pi^+) \), we can also write \( x(\pi^+) \), for short. Note that this new definition of \( x(\pi^+) \) coincides with the old one if degrees of elements of \( x(\pi) \) are less or equal to \(-2\) (cf. (29)).

We prove linear independence by induction. Let

\[
\sum c_\pi x(\pi) v_\Lambda = 0.
\]

Assume that all monomials in (26) have elements of degree greater or equal to \(-n\). Fix a monomial \( x(\pi) \) in (26) and assume that

\[
c_{\pi'} = 0 \quad \text{for} \quad x(\pi') < x(\pi).
\]

We need to prove that \( c_{\pi'} = 0 \).

By the proposition 10, there exists an operator \( w_1 : L(\Lambda) \to L(\Lambda') \) such that

- \( w_1 \) commutes with \( \hat{g}_1 \),
- \( w_1 \cdot x(\pi_1) v_\Lambda = C_1 e(\omega) v_{\Lambda'} \), \( C_1 \in \mathbb{C}^\times \),
- \( x(\pi^+) \) satisfies DC and IC for \( L(\Lambda') \),
- \( w_1 \cdot x(\pi^+) v_\Lambda = 0 \quad \text{for} \quad x(\pi'_1) > x(\pi_1) \).

By acting with the operator \( w_1 \) on the relation (26), we get

\[
0 = w_1 \sum c_\pi' x(\pi') v_\Lambda
\]

\[
= w_1 \sum_{\pi'_1 > \pi_1} c_\pi' x(\pi') v_\Lambda + w_1 \sum_{\pi'_1 < \pi_1} c_\pi' x(\pi') v_\Lambda + w_1 \sum_{\pi'_1 = \pi_1} c_\pi' x(\pi') v_\Lambda
\]

The first sum is annihilated by \( w_1 \) because of the maximality of \( x(\pi_1) \) for \( w_1 \), the second sum is equal to 0 by the induction hypothesis. We obtain

\[
0 = w_1 \sum_{\pi'_1 = \pi_1} c_\pi' x(\pi') v_\Lambda
\]

\[
= \sum_{\pi'_1 = \pi_1} c_{\pi'} x(\pi'_1) \cdots x(\pi'_2) C_1 e(\omega) v_{\Lambda'}
\]

\[
= C_1 e(\omega) \sum_{\pi'_1 = \pi_1} c_{\pi'} x(\pi'^+) v_{\Lambda'}
\]

Since \( e(\omega) \) is an injection, we get

\[
(27) \quad \sum_{\pi'_1 = \pi_1} c_{\pi'} x(\pi'^+) v_{\Lambda'} = 0.
\]

Now, for \( x(\pi^+) \) there exists an operator \( w_2 : L(\Lambda') \to L(\Lambda'') \) such that
• $w_2$ commutes with $\hat{g}_1$,
• $w_2 \cdot x(\pi_2^+)v_{\Lambda'} = C_2 e(\omega)v_{\Lambda''}$, $C_2 \in \mathbb{C}^\times$,
• $x(\pi^{+2})$ satisfies DC and IC for $L(\Lambda'')$,
• $w_2 \cdot x(\pi^{+2})v_{\Lambda'} = 0$ if $x(\pi_2^2) > x(\pi_2)$.

Upon acting with $w_2$ on the relation (27), we get

$$0 = w_2 \sum_{\pi_1} c_{\pi'} x(\pi') v_{\Lambda'}$$

$$= w_2 \sum_{\pi_1} c_{\pi'} x(\pi') v_{\Lambda'} + w_2 \sum_{\pi_2 < \pi_2} c_{\pi'} x(\pi') v_{\Lambda'} + w_2 \sum_{\pi_2 = \pi_2} c_{\pi'} x(\pi') v_{\Lambda'}$$

As before, the first two sums are equal to 0 because of the action of $w_2$ and of the induction hypothesis. We obtain

$$0 = w_2 \sum_{\pi_1} c_{\pi'} x(\pi') v_{\Lambda'}$$

$$= \sum_{\pi_1} c_{\pi'} x(\pi_1') \cdots x(\pi_2') C_2 e(\omega) v_{\Lambda''}$$

$$= C_2 e(\omega) \sum_{\pi_1} c_{\pi'} x(\pi^{+2}) v_{\Lambda''}.$$ 

Since $e(\omega)$ is an injection, we get

$$\sum_{\pi_1} c_{\pi'} x(\pi^{+2}) v_{\Lambda''} = 0.$$ 

We proceed inductively; after $n$ steps, we obtain

$$0 = \sum_{\pi_1} c_{\pi'} x(\pi^{+n}) v_{\Lambda^{(n)}} = c_{\pi} x(\pi^{+n}) v_{\Lambda^{(n)}} = c_{\pi} v_{\Lambda^{(n)}}$$

and we can conclude that $c_{\pi} = 0$.

Hence we have proved

**Theorem 11.** The set

$$\{x(\pi)v_{\Lambda} \mid x(\pi) \text{ satisfies DC and IC for } L(\Lambda)\}$$

is a basis for $W(\Lambda)$.

### 11. Basis of a Standard Module

Feigin-Stoyanovsky’s type subspace $W(\Lambda)$ was implicitly introduced and studied in [P1] and [P2] where the basis of the whole standard module $L(\Lambda)$ was constructed from a basis of this subspace. We have used this approach in [T] to construct a basis of standard modules of level 1, for any possible choice of $\mathbb{Z}$-gradation (1). By using corollary 4 we are able to extend this proof to standard modules of higher level.

Set

$$e = \prod_{\gamma \in \Gamma} e_{\gamma} = e^{\sum_{\gamma \in \Gamma} \gamma}.$$ 

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Then
\[ e = e^{(\ell+1)\omega} \]
(cf. relation (26) in [T]). The following proposition was proven by Primc (cf. Theorem 8.2. in [P1] or Proposition 5.2. in [P2])

**Proposition 12.** Let \( L(\Lambda)_\mu \) be a weight subspace of \( L(\Lambda) \). Then there exists an integer \( n_0 \) such that for any fixed \( n \leq n_0 \) the set of vectors
\[ e^n x_{\gamma_1}(r_1) \cdots x_{\gamma_n}(r_n) v_\Lambda \in L(\Lambda)_\mu, \]
where \( s \geq 0, \gamma_1, \ldots, \gamma_s \in \Gamma, r_1, \ldots, r_s \in \mathbb{Z}, \) is a spanning set of \( L(\Lambda)_\mu \). In particular,
\[ L(\Lambda) = \langle e \rangle U(\hat{\mathfrak{g}}) v_\Lambda. \]

We’ll use our results on the basis of \( W(\Lambda) \) to prove

**Theorem 13.** Let \( L(\Lambda)_\mu \) be a weight subspace of a standard \( \hat{\mathfrak{g}} \)-module \( L(\Lambda) \). Then there exists \( n_0 \in \mathbb{Z} \) such that for any fixed \( n \leq n_0 \) the set of vectors
\[ e^n x(\pi) v_\Lambda \in L(\Lambda)_\mu, \quad x(\pi) \text{ satisfies IC and DC for } L(\Lambda), \]
is a basis of \( L(\Lambda)_\mu \). Moreover, for two choices of \( n_1, n_2 \leq n_0 \), the corresponding two bases are connected by a diagonal matrix.

We have proven this theorem in [T] for standard modules of level 1. The first part of the theorem directly follows from proposition 12 and theorem 11. For the second part of the theorem in [T] we have considered a monomial \( x(\mu) \in U(\hat{\mathfrak{g}}) \) which was defined as the maximal monomial satisfying difference and initial conditions for \( L(\Lambda) \) such that its factors are of degree greater or equal to \(-r \frac{\ell+1}{m(\ell-m+1)}\), where \( r \) is equal to the smallest common multiple of \( m \) and \( \ell - m + 1 \). For simplicity, set
\[ f = r \frac{\ell+1}{m(\ell-m+1)}. \]
We’ve shown that the following holds
\[ (i) \quad e(\omega)^f v_\Lambda = C x(\mu) v_\Lambda, \quad \text{for some } C \in \mathbb{N}, \]
\[ (ii) \quad f \text{ divides } \ell + 1, \]
\[ (iii) \quad \text{if a monomial } x(\pi) \text{ satisfies difference and initial conditions for } L(\Lambda), \text{ then so does a monomial } x(\pi^{-f}) x(\mu), \quad \text{where } \pi^{-f} \text{ is a partition defined by} \]
\[ \pi^{-f}(x_\gamma(\ell - n)) = x(\gamma(\ell - n)), \quad \gamma \in \Gamma, n \in \mathbb{Z}. \]

Then we’ve had
\[ e(\omega)^f x(\pi) v_\Lambda = x(\pi^{-f}) e(\omega)^f v_\Lambda = C e(\pi^{-f}) x(\mu) v_\Lambda. \]

Since \( e(\omega) x(\pi) v_\Lambda \) and \( e(\omega) x(\pi) v_\Lambda \) are proportional, the second part of the theorem followed.

To prove the theorem for higher levels, it is enough to construct a monomial \( x(\mu) \in U(\hat{\mathfrak{g}}) \) that satisfies properties (i) and (iii). Let \( L(\Lambda) \) be a standard module of level \( k \), with the highest weight vector \( v_\Lambda = v_{i_1} \otimes \cdots \otimes v_{i_k} \). For each \( L(\Lambda_i) \), let \( x(\mu^{(j)}) \) be as in the previous paragraph. Set
\[ x(\mu) = x(\mu^{(1)}) \cdots x(\mu^{(k)}). \]

As in the proof of proposition 11 we have that \( x(\mu) \) is a maximal monomial satisfying difference and initial conditions for \( L(\Lambda) \) such that its factors are of degree greater or equal to \(-f\) and
\[ x(\mu) v_\Lambda = C \cdot x(\mu^{(1)}) v_{i_1} \otimes \cdots \otimes x(\mu^{(k)}) v_{i_k}. \]

Because of the property (i), we have
\[ x(\mu) v_\Lambda = C \cdot e(\omega)^f v_{i_1} \otimes \cdots \otimes e(\omega)^f v_{i_k} = C e(\omega)^f v_\Lambda, \]

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for some $C \in \mathbb{N}$. Finally, by the property (iii) and the corollary it follows that if a monomial $x(\pi)$ satisfies difference and initial conditions for $L(\Lambda)$, then so does a monomial $x(\pi - f)x(\mu)$.

12. Presentation of $W(\Lambda)$

By definition Feigin-Stoyanovsky’s type subspace $W(\Lambda)$ is a submodule of $L(\Lambda)$ generated by the highest-weight vector $v_\Lambda$, 

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1) \cdot v_\Lambda.$$ 

Since the $\tilde{\mathfrak{g}}_1$ is commutative, $U(\tilde{\mathfrak{g}}_1) \cong S(\tilde{\mathfrak{g}}_1)$, we have 

$$W(\Lambda) = \mathbb{C}[x_\gamma (\pi) \mid \gamma \in \mathbb{N}, \pi \in \Gamma] \cdot v_\Lambda.$$ 

For $1 \leq i < m$ and $m < j \leq \ell$ let $\mathfrak{g}_{ij} \subset \mathfrak{g}_0$ be the subalgebra generated by the elements $x_{\pm \alpha_i}$, where either $1 \leq t \leq i - 1$ or $j + 1 \leq t \leq \ell$.

Consider a polynomial algebra $A = \mathbb{C}[x_\gamma (\pi) \mid \gamma \in \mathbb{N}, \pi \in \Gamma]$, which is also a $\mathfrak{g}_0$-module. Let $J \subset A$ be the ideal generated by the following sets 

$$U(\mathfrak{g}_0) \cdot \left( \sum_{r_1, \ldots, r_{k+1} \leq -1} x_\theta(r_1) \cdots x_\theta(r_{k+1}) \right), \quad \text{for all } n \in \mathbb{Z}_{<0},$$

and 

$$U(\mathfrak{g}_{ij}) \cdot (x_\theta(-1)^{k_0 + \cdots + k_{i-1} + k_{j+1} + \cdots + k_j}), \quad \text{for all } i = 1, \ldots, m - 1; \quad j = m + 1, \ldots, \ell.$$ 

Then we have the following presentation result:

**Theorem 14.** As a vector space, $W(\Lambda)$ is isomorphic to the quotient $A/J$.

**Proof:** Define a mapping 

$$\varphi_0 : A \rightarrow W(\Lambda),$$

$$\varphi_0 : x(\pi) \mapsto x(\pi) \cdot v_\Lambda.$$ 

Since the ideal $J$ lies in the kernel of $\varphi_0$, we can factorize $\varphi_0$ to a quotient map 

$$\varphi : A/J \rightarrow W(\Lambda).$$ 

The map $\varphi$ is clearly a surjection, since $\varphi_0$ is a surjection. We’ll show that $\varphi$ is also an injection. Consider a set 

$$B = \{ x(\pi) \mid x(\pi) \text{ satisfies DC and IC for } L(\Lambda) \} \subset A/J.$$ 

As in the proof of the proposition we see that this set spans $A/J$. Since $\varphi$ bijectively maps this set onto 

$$\{ x(\pi)v_\Lambda \mid x(\pi) \text{ satisfies DC and IC for } L(\Lambda) \} \subset W(\Lambda),$$

which is a basis of $W(\Lambda)$, we see that $B$ is also linearly independent. Hence $\varphi$ maps a basis of $A/J$ onto a basis of $W(\Lambda)$ and therefore $\varphi$ is a bijection. 

This kind of presentation of $W(\Lambda)$ was used in [FJMMT] in order to obtain fermionic formulas for the character of $W(\Lambda)$. Also, presentation of the Feigin-Stoyanovsky’s principal subspace was used in [C1,C2,CalLM1,CalLM2,CLM2], for construction of exact sequences between different principal subspaces from which they obtained recurrence relations for the characters of these spaces.
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