On a free boundary problem and minimal surfaces

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Abstract
From minimal surfaces such as Simons’ cone and catenoids, using refined Lyapunov-Schmidt reduction method, we construct new solutions for a free boundary problem whose free boundary has two components. In dimension 8, using variational arguments, we also obtain solutions which are global minimizers of the corresponding energy functional. This shows that Savin’s theorem [43] is optimal.

1 Introduction

In this paper, we are interested in the following free boundary problem:

\[
\begin{cases}
\Delta u = 0 \text{ in } \Omega := \{ -1 < u < 1 \}, \\
|\nabla u| = 1 \text{ on } \partial \Omega.
\end{cases}
\] (1)

Here the domain $\Omega \subset \mathbb{R}^n$ is a priori unspecified and $\partial \Omega$ is the free boundary. Solutions of (1) arise formally as critical points of the energy functional:

\[
J(u) := \int \left[ |\nabla u|^2 + \chi_{(-1,1)}(u) \right].
\] (2)

In this variational formulation, the boundary condition $|\nabla u| = 1$ should be understood in some weak sense if the free boundary $\partial \Omega$ is not regular enough.
Problem (1) can be regarded as a simplified version of the classical one-phase free boundary problem:

\[
\begin{align*}
\Delta u &= 0 \text{ in } \Omega := \{u > 0\}, \\
|\nabla u| &= 1 \text{ on } \partial \Omega.
\end{align*}
\] (3)

The regularity of the free boundary problems actually has been a subject of extensive studies, pioneered by the work of Caffarelli (see [2] [3] [7, 8, 9] and the references therein). It is now known that in dimension \( n \leq 4 \), the free boundary of a solution to (3) has no singularity, provided that it is an energy minimizer ([2, 10, 30]). In fact, it is conjectured that for \( n \leq 6 \), minimizers should have smooth free boundary. On the other hand, in higher dimensions \( n \geq 7 \), an energy minimizing free boundary may have singularities. To explain this, let us mention that by blow up analysis, the regularity of the free boundary is essentially related to the existence or nonexistence of minimizing cone. Let us consider the cone in \( \mathbb{R}^n \) given by (see [10])

\[
|x_n| < \alpha_n \sqrt{x_1^2 + ... + x_n^2},
\] (4)

where \( \alpha_n \) is a dimensional constant chosen such that on this cone there is a solution to the one-phase free boundary problem. It has been proved ([12]) that in dimension \( n = 7 \) (Actually also for \( n = 9, 11, 13, 15 \) and hopefully for all \( n \geq 7 \)), the solution to (3) corresponding to the cone (4) is a minimizer for the energy functional. For \( 3 \leq n \leq 6 \), this solution is already known to be unstable, thanks to the work of [10]. On the other hand, if a solution to (3) is a minimizer and if the free boundary is a priori a graph, then by the result of [13], this free boundary will be real analytic. It is worth pointing out that all these regularity results are in many respect analogous to that of the minimal surface theory, and these two subjects are closely related.

In \( \mathbb{R}^2 \), Traizet ([47, 48]) proved that there is a one-to-one correspondence between solutions to (1) and (5) and certain type of minimal surfaces in \( \mathbb{R}^3 \). Hence at least in dimension two, this problem is well understood, although even for the minimal surfaces in \( \mathbb{R}^3 \), many questions remain unanswered up to now. We also refer to [27], [28], [37] for related existence and classification results for other types of free boundary problems. Now we emphasize that in higher dimensions, the explicit correspondence between minimal surfaces and free boundary problem is not available. However, in \( \mathbb{R}^3 \), it is proved by Kamburov ([32]) using sub and super solution method that there exists a solution to (1) where the free boundary is close to two copies of the famous Bombieri-De Giorgi-Giusti minimal graph. His result indicates that there should be deeper relation between minimal surface and the free boundary problem (1). Here in this paper we would like to further explore this relation by constructing solutions to (1) based on minimal surfaces.

Notice that problem (1) can be considered as special case of over-determined problems. In recent years the following so-called Serrin’s overdetermined prob-
\[
\begin{aligned}
\Delta u &= f(u) \text{ in } \Omega := \{u > 0\}, \\
u &= 0, \ |\nabla u| = \text{Constant} \text{ on } \partial \Omega.
\end{aligned}
\]

has also received much attention. We refer to [19, 20, 21, 15, 42, 37, 45, 49] and the references therein.

Another motivation for studying (1) is related to De Giorgi’s conjecture. In 1978 De Giorgi conjectured that the only bounded solution to

\[
\Delta u + u - u^3 = 0 \text{ in } \mathbb{R}^n
\]

which is monotone in \(x_n\) must be one dimensional (up to rotation and translation) at least in dimension \(n \leq 8\). De Giorgi’s conjecture is a natural, parallel statement to Bernstein theorem for minimal graphs, which in its most general form, due to Simons [44], states that any minimal hypersurface in \(\mathbb{R}^n\), which is also a graph of a function of \(n - 1\) variables, must be a hyperplane if \(n \leq 8\). Strikingly, Bombieri, De Giorgi and Giusti [6] proved that this fact is false in dimension \(n \geq 9\).

Great advance in De Giorgi’s conjecture has been achieved in recent years, having been fully established in dimensions \(n = 2\) by Ghoussoub and Gui [25] and for \(n = 3\) by Ambrosio and Cabre [4]. A celebrated result by Savin [38] established its validity for \(4 \leq n \leq 8\) under the following additional assumption

\[
\lim_{x_n \to \pm \infty} u(x', x_n) = \pm 1
\]

(See Savin-Sciunzi-Valdinoci [41] and Farina-Valdinoci [22, 23] for generalizations.) Del Pino, Kowalczyk and Wei [17] constructed a counterexample in dimensions \(n \geq 9\).

Replacing the monotonicity assumption by global minimality of energy, Savin proved that in dimensions \(n \leq 7\) all global minimizers to (6) are one-dimensional. We proved that Savin’s result is optimal by constructing global minimizers in dimensional 8 ([35]). (Stable solutions are constructed in Pacard-Wei [36].)

In a recent paper [43], Savin also extended the De Giorgi type conjecture result to problems with more general nonlinearities including

\[
\Delta u = W_\alpha(u)
\]

where \(W = (1 - u^2)\alpha, \alpha \geq 0\). \(\alpha = 0\) corresponds to the problem (1). In particular he proved global minimizers of (1) must be one-dimensional if \(n \leq 7\). One of our results below shows that this is optimal.

The purpose of this paper is to establish the connection between minimal surfaces and problem (1). In particular we shall construct new solutions to (1) by developing new gluing methods for overdetermined problems. We know very little information about the solutions of (1) in dimensions \(n \geq 3\). In dimension 2 Traizet’s characterization [17] reduces the problem to singly minimal surfaces.
in $\mathbb{R}^3$. In dimension 9, Kambrunov’s solution [32] is a monotone solution whose two components are approximately Bombieri-De Giorgi-Giusti graphs. For $3 \leq n \leq 8$ we know no solutions to (1). In this paper we establish a connection between minimal surfaces and solutions to (1) and thereby provide plenty of new solutions to (1). In addition, we shall prove the existence of global minimizers in $\mathbb{R}^8$ and execute the Jerison-Memme program for problem (1).

Rather than considering the most general minimal surfaces, we shall focus on two types of classical minimal surfaces. The first type of minimal surfaces are the area minimizing cones (minimizing hypersurfaces) in $\mathbb{R}^n$ ($n \geq 8$). As an example, let us consider the famous Simons’ cone:

$$S := \{ (x_1, ..., x_8) \in \mathbb{R}^8 : \sum_{i=1}^4 x_i^2 = \sum_{i=5}^8 x_i^2 \}.$$

This is a minimal surface with one singularity at the origin. The fact that Simons’ cone is area minimizing has been proved in the classical work of Bombieri-De Giorgi-Giusti [6]. Using the minimizing property, Hardt-Simon [26] was able to show that there exists a family of foliated minimal surfaces $S^+_\delta$ lying on one side of the cone and is asymptotic to the cone at infinity. Similarly, the other side of the cone is also foliated by a family of minimal surfaces $S^-_\delta$. Due to scaling invariance, this family of surfaces $S^\pm_\delta$ can be obtained simply as homothety of $S^\pm_1$, that is $S^\pm_\delta = \delta^{-1} S^\pm_1$. Actually, Hardt-Simon proved more. They showed that the Simons’ cone is strictly area minimizing which implies that each surface $S^\pm_\delta$ approaches the cone at the slowest possible rate.

As we mentioned before, there should be similarities between the minimal surfaces and free boundary problem. A natural question is whether there are analogous solutions for the free boundary problem (1) as the Simons’ cone and its associated foliation. We answer this affirmatively.

**Theorem 1** For each $\varepsilon$ small enough, there exists domain $\Omega^\varepsilon$ close to the radius one tubular neighbourhood of $S^+_\varepsilon$ and solution $u_\varepsilon$ to the free boundary problem (1). Moreover, $u_\varepsilon$ is stable in the sense that there exists a function $\Phi > 0$ in $\Omega^\varepsilon$, and

$$\begin{cases}
\Delta \Phi = 0, & \text{in } \Omega^\varepsilon, \\
\Phi_\nu + H \Phi = 0, & \text{on } \partial \Omega^\varepsilon.
\end{cases}$$

Here $\nu$ is the outward normal to $\partial \Omega^\varepsilon$ and $H$ is the mean curvature of $\partial \Omega^\varepsilon$.

By this theorem, there are solutions whose nodal set is close to $S^+_\varepsilon$ for $\varepsilon$ small. It is well known that the family of minimal surfaces $S^+_\delta, \delta \in \mathbb{R}$, are all area minimizing. Therefore, it is natural to ask whether the solutions $u_\varepsilon$ are also minimizers of the energy functional $J$. We believe this is true, but here in this paper we shall only give the following.

**Theorem 2** There exists a nontrivial solution (not one dimensional) $U$ to the free boundary problem (1) in $\mathbb{R}^8$ which is also energy minimizing.

With additional efforts, one can actually prove that for each $S^+_\delta$, there exists an energy minimizer whose nodal set is asymptotic to $S^+_\delta$ at infinity. We will
not pursue this in this paper. One can compare this result with a similar result for the Allen-Cahn equation [35].

Using the variational method of Jerison-Monneau [31], we can construct monotone solutions in $\mathbb{R}^9$ using this minimizer $U$. This complements the result of Kamburov [32], where the existence of monotone solutions is established by sub and super solution method.

**Theorem 3** There is a family of solutions in $\mathbb{R}^9$ to (1) which are monotone in the $x_9$ direction.

Our second type of minimal surfaces will be the catenoids, which is a family of classical minimal surfaces with finite total curvature. They are rotationally symmetric and given explicitly by the equation

$$x_1^2 + x_2^2 = \frac{1}{\varepsilon^2} \cosh^2 (\varepsilon x_3).$$

Here $\varepsilon > 0$ is a parameter. In higher dimensions, we have analogous codimension one minimal submanifold which we call higher dimensional catenoids. To be more precise, let $(x_1, ..., x_n)$ be the coordinate in $\mathbb{R}^n$ ($n > 3$). Let $\phi$ be the solution of

$$\begin{cases}
\omega'' + \frac{n-2}{\omega} = 0, \\
\omega(0) = 1, \omega'(0) = 0.
\end{cases}$$

Then the surface $C_1$ in $\mathbb{R}^n$ given by

$$r := \varepsilon \sqrt{x_1^2 + \cdots + x_{n-1}^2} = \omega(x_n)$$

is a minimal surface, called catenoid. We can also write it as

$$x_n = \bar{\omega}(r), r \in [r_0, +\infty).$$

Then there are constants $c_n, c'_n$ such that

$$x_n \sim c_n - c'_n r^{3-n}.$$ 

Actually a homothety of $C_1$ is also a minimal surface, which we denoted by $C_\varepsilon$, which is then described by

$$x_n = \bar{\omega}_\varepsilon(r) := \frac{1}{\varepsilon} \bar{\omega}(\varepsilon r).$$

We refer to [46] for more detailed properties on catenoids, including their Morse index. Here we are interested in $C_\varepsilon$ with $\varepsilon$ small. In this case, the catenoid has a large waist.

**Theorem 4** For $\varepsilon$ small enough, there exists a rationally symmetric domain $\Omega^\varepsilon$ close to radius one tubular neighbourhood of $C_\varepsilon$ and a solution $u_\varepsilon$ to the free boundary problem (1).
Now let us explain the main ideas of the proof. The proofs of Theorems 1 and 4 are based on the infinite dimensional gluing methods developed in [14, 15]. In [1, 14], entire solutions for the Allen-Cahn equation have been constructed. The zero level sets of the solutions lie close to certain nondegenerate minimal surfaces. To construct these solutions, they used the method of infinite dimensional Lyapunov-Schmidt reduction. More recently, in [15], an over-determined problem was investigated using similar method. Here we develop new gluing methods for (1). There are two main difficulties in performing gluing methods for (1). The first one is that the one-dimensional solution, which is given by

\[ u_0(x_1) = \begin{cases} 
-1, & x_1 \leq -1; \\
 x_1, & -1 < x_1 < 1; \\
 1, & x_1 \geq 1,
\end{cases} \]  

(8)
is only continuous and is not differentiable. This means that one cannot linearize the problem around this one dimensional profile. This is quite different from [1, 14, 15]. The second difficulty is that this is an over-determined problem and we have to adjust two interfaces.

To solve the problem (1), we introduce a pair of unknown functions \((h_1, h_2)\) on a rescaled minimal surface. Using these two functions, we define a perturbed domain \(\Omega_h\) which will be very close to the radius one tubular neighbourhood \(N_1\) of the minimal surface. The functions \(h_1\) and \(h_2\) measures the deviation of \(\Omega_h\) to \(N_1\). Next, we define suitable approximate solutions for (1) on \(\Omega_h\). We analyze in detail the differences between this approximate solution and the harmonic function in \(\Omega\) with Dirichlet boundary condition. In the last step, we use fixed point argument to show that one can find functions \(h_1\) and \(h_2\) such that our problem is solvable and we can get a solution \(u\). In this step, we show that to match the required Neumann boundary condition, we need to analyze the solvability and a priori estimate of a system of equations for the function \(h_1, h_2\). (See (22).) It turns out that one of them reduces to the analyze of the Jacobi operator on the minimal surface

\[ \Delta_M h + |A|^2 h = f \]  

(9)

but the other problem is of fractional differential operator

\[ (-\Delta_M + 1)^{\frac{1}{2}} h = f. \]  

(10)

We remark that the family of solutions constructed from the Simons’ cone are ordered and hence stable, while the solutions arising from catenoids are unstable.

To prove Theorems 2 and 3, we first extend the construction of Jerison-Monneau [31] and follow the variational approach in [35] to construct minimizers in \(\mathbb{R}^8\) and monotone solutions in \(\mathbb{R}^9\). The main difficulty is the regularity of the solutions. To this end, we use axial symmetry of the solutions and also make
use of classical regularity result of Weiss \([51, 52]\) as well as recent regularity results of Jerison-Savin [30].

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## 2 Solutions from Simons’ cone

### 2.1 Preliminary on Simons’ cone and the associated foliation

Let us first of all recall some basic facts about the geometry of the Simons’ cone. Throughout the paper we shall use \( S^k(\rho) \) to denote the radius \( \rho \) sphere in \( \mathbb{R}^{k+1} \). In the manifold \( S^7(1) \), we shall consider the codimension one submanifold

\[
\Lambda := S^3(\rho) \times S^3(\rho),
\]

where

\[
\rho = \sqrt{\frac{1}{2}}.
\]

The induced metric on \( \Lambda \) is given by \( g^* := \rho^2 g_1 + \rho^2 g_2 \), where \( g_1, g_2 \) are the metric on the two copies \( S^3(1) \). The Simons cone is defined to be

\[
S := \{ rX \in \mathbb{R}^8 : r \in (0, +\infty), X \in \Lambda \}.
\]

One can verify that this is a minimal hypersurface in \( \mathbb{R}^8 \). The induced metric tensor on \( S \) is then given by

\[
dr^2 + r^2 g^*.
\]

For a codimension one submanifold \( M \) in \( \mathbb{R}^n \), with the induced metric, we shall use \( J_M \) to denote its Jacobi operator, which explicitly has the form

\[
J_M = \Delta_M + |A|^2,
\]

where \( |A|^2 = \sum_{i=1}^{n-1} k_i^2 \) is the squared norm of the second fundamental form of \( M \), with \( k_i \) being the principle curvatures of \( M \). The Jacobi operator about \( S \) is then given by

\[
J_S = \Delta_S + |A|^2 = \partial_r^2 + \frac{6}{r} \partial_r + \frac{\Delta g^* + 6}{r^2}.
\]

The set \( \mathbb{R}^8 \setminus S \) has two components. Each component is foliated by a family of smooth minimal hypersurfaces \( S^\pm \) which are asymptotic to \( S \) at infinity. We can choose \( S_1 \) to be the surface having the form

\[
S_1 \setminus B_{r_0} = \{ X + \eta_0(X) \nu, X \in S \},
\]
where \( \nu \) is a choice of the unit normal at \( S \), and \( \eta_0 (X) = \lvert X \rvert^{-2} + o \left( \lvert X \rvert^{-2} \right) \). Then \( S_\varepsilon = \varepsilon^{-1} S_1 \).

Let \( x = \sqrt{x_1^2 + \ldots + x_4^2} \), \( y = \sqrt{x_5^2 + \ldots + x_8^2} \). We can write the standard metric on \( \mathbb{R}^8 \) in the polar coordinate as

\[
\begin{align*}
  dx^2 + x^2 d\theta^2 + dy^2 + y^2 d\bar{\theta}^2,
\end{align*}
\]

where \( d\theta^2 \) and \( d\bar{\theta}^2 \) represents the metric tensor on the unit three-dimensional sphere \( S^3(1) \). Suppose in the \((x, y)\) coordinate \( S_\delta \) is described by \( y = \varphi_\delta (x) \) for a monotone function \( \varphi_\delta \), then the metric tensor on \( S_\delta \) is

\[
\begin{align*}
  \left[ 1 + \varphi_\delta^2 (x) \right] dx^2 + \varphi_\delta^2 (x) d\bar{\theta}^2 + x^2 d\theta^2.
\end{align*}
\]

Let us introduce the arc length variable \( l \) by the formula

\[
\begin{align*}
  l = \int_0^x \sqrt{1 + \varphi_\delta^2 (t)} dt.
\end{align*}
\]

Then the metric \( g_\delta \) on \( S_\delta \) also read as

\[
\begin{align*}
  dl^2 + \varphi_\delta^2 (x) d\bar{\theta}^2 + x^2 d\theta^2.
\end{align*}
\]

Note that \( \det g_\delta = \varphi_\delta^6 (x) x^6 \). Let \( \eta \) be a function on \( S_\delta \) which is invariant under the action of the group \( O(4) \times O(4) \). The Laplacian operator on \( S_\delta \) acting on function \( \eta \) has the form

\[
\begin{align*}
  \Delta_{S_\delta} \eta = \frac{1}{\sqrt{\det g_\delta}} \partial_i \left( \sqrt{\det g_\delta} g_\delta^{ij} \partial_j \eta \right)
  \quad &= \frac{d^2 \eta}{dl^2} + \frac{d[\varphi_\delta^3 (x) x^3]}{dl} \frac{d\eta}{dl}
  \quad &= \frac{d^2 \eta}{dl^2} + \left( \frac{3 \varphi_\delta}{x} + \frac{3 \varphi_\delta'}{\varphi_\delta} \right) \frac{dx}{dl} \frac{d\eta}{dl}.
\end{align*}
\]

### 2.2 Analysis of the approximate solutions

We will construct solutions based on the minimal hypersurfaces \( S_\varepsilon \) where \( \varepsilon > 0 \) is sufficiently small. Let us choose a unit normal \( \nu \) for the codimension one manifold \( S_\varepsilon \). Let \( h_1, h_{-1} \in C^{2, \alpha}_{loc} (S_\varepsilon) \), small in certain sense. For each function \( \eta \) defined on \( S_\varepsilon \), we set

\[
\Gamma_\eta := \{ X + \eta(X) \nu(X) : X \in S_\varepsilon \}.
\]

Although \( \Gamma_\eta \) depends also on \( \varepsilon \), we will not make this dependence explicit in the notation. We establish a Fermi coordinate in a tubular neighbourhood of \( S_\varepsilon \). By \( s \) we denote the signed distance of a point to \( S_\varepsilon \). Slightly abusing the notation, define

\[
\Gamma_s := \{ X + s \nu(X) : X \in S_\varepsilon \}.
\]
Note that for \( \varepsilon \) small, this is well defined and \( \Gamma_s \) is smooth, for all \( |s| < 1 \).

Let us consider the region \( \Omega \) trapped between the surfaces \( \Gamma_{-1+h_{-1}} \) and \( \Gamma_{1+h_1} \). For each pair of functions \( h = (h_{-1}, h_1) \), we shall define an approximate solution \( w_h \) in \( \Omega \):

\[
\begin{align*}
    w_h(s, l) &= \frac{s - g(l)}{1 + f(l)},
\end{align*}
\]

where

\[
\begin{align*}
    f &= \frac{h_1 - h_{-1}}{2},
    
    g &= \frac{h_1 + h_{-1}}{2}.
\end{align*}
\]

Note that in the current situation, the range of \( l \) is \([0, +\infty)\). With this definition, \( w_h \) satisfies the boundary condition:

\[
\begin{align*}
    w_h &= \begin{cases} 
        -1, & \text{on } \Gamma_{-1+h_{-1}}, \\
        1, & \text{on } \Gamma_{1+h_1}. 
    \end{cases}
\end{align*}
\]

It will be convenient for us to introduce a new variable

\[
    t = \frac{s - g(l)}{1 + f(l)}.
\]

Then the domain \( \Omega_h \) can be parameterized by \((l, t)\) with \( t \in [-1, 1] \).

Let us use \( H_M \) to denote the mean curvature of a codimension one submanifold \( M \). The formula of Laplacian operator in the Fermi coordinate (see [16]) tells us that

\[
\begin{align*}
    \Delta w_h (s, l) &= \Delta_{\Gamma_s} w_h + \partial_s^2 w_h - H_{\Gamma_s} \partial_s w_h \\
    &= \Delta_{\Gamma_s} w_h - \frac{H_{\Gamma_s}}{1 + f}.
\end{align*}
\]

We need to understand the main order of these terms.

**Lemma 5** We have

\[
\begin{align*}
    \Delta_{\Gamma_0} w_h &= -\Delta_{\Gamma_0} g - t \Delta_{\Gamma_0} f + E_1,
\end{align*}
\]

where

\[
E_1 = -tf \Delta_{\Gamma_0} f + \Delta_{\Gamma_0} (fg) - g \Delta_{\Gamma_0} f + \Delta_{\Gamma_0} \left( (s - g) \frac{f^2}{1 + f} \right).
\]

**Remark 6** \( E_1 \) can be regarded as a perturbation term.

**Proof.** Having in mind that \( f, g \) are small, we write

\[
\begin{align*}
    w_h &= \frac{s - g(l)}{1 + f(l)} = (s - g) \left( 1 - f + \frac{f^2}{1 + f} \right) \\
    &= s - g - sf + gf + (s - g) \frac{f^2}{1 + f}.
\end{align*}
\]
We then compute
\[ \Delta_{\Gamma_0} w_h = -\Delta_{\Gamma_0} g - s \Delta_{\Gamma_0} f + \Delta_{\Gamma_0} (fg) + \Delta_{\Gamma_0} \left[ (s - g) \frac{f^2}{1 + f} \right]. \]

Inserting the relation \( s = t (1 + f) + g \) into the left hand side, we get
\[
\Delta_{\Gamma_0} w_h = -\Delta_{\Gamma_0} g - [t (1 + f) + g] \Delta_{\Gamma_0} f + \Delta_{\Gamma_0} (fg) + \Delta_{\Gamma_0} \left[ (s - g) \frac{f^2}{1 + f} \right] \\
= -\Delta_{\Gamma_0} g - t \Delta_{\Gamma_0} f - tf \Delta_{\Gamma_0} f - g \Delta_{\Gamma_0} f + \Delta_{\Gamma_0} \left[ (s - g) \frac{f^2}{1 + f} \right].
\]
This finishes the proof. \( \blacksquare \)

Let us use \( k_i, i = 1, \ldots, 6 \) to denote the principle curvatures of \( S_\varepsilon \).

**Lemma 7** We have the following formula:
\[
\frac{H_{\Gamma_0}}{1 + f} = t |A|^2 + g |A|^2 + E_2,
\]
where
\[
E_2 = \frac{1}{1 + f} \sum_{i=1}^{6} s^2 k_i^3 - \frac{fg |A|^2}{1 + f}.
\]

**Proof.** By a well known formula (see [16]),
\[
H_{\Gamma_0} = \sum_{i=1}^{6} \frac{k_i}{1 - sk_i} = \sum_{i=1}^{6} k_i + \sum_{i=1}^{6} sk_i^2 + \sum_{i=1}^{6} \frac{s^2 k_i^3}{1 - sk_i}.
\]
Recall that \( \sum_{i=1}^{6} k_i = H_{\Gamma_0} = 0 \). Hence
\[
\frac{H_{\Gamma_0}}{1 + f} = \frac{|A|^2}{1 + f} (t + g) + \frac{1}{1 + f} \sum_{i=1}^{6} \frac{s^2 k_i^3}{1 - sk_i} \\
= t |A|^2 + g |A|^2 + \frac{fg |A|^2}{1 + f} + \frac{1}{1 + f} \sum_{i=1}^{6} \frac{s^2 k_i^3}{1 - sk_i}.
\]
The proof is thus completed. \( \blacksquare \)

We seek a solution \( u \) to the free boundary problem (1) in the form \( u = w_h + \phi \).

Here we require \( \phi = 0 \) on \( \partial \Omega_h \). Let us now analyze the boundary condition \( |\nabla u| = 1 \) on \( \partial \Omega_h \). Suppose in the \((l, \theta, \theta, s)\) coordinate the metric tensor \( g \) in a tubular neighbourhood of \( S_\varepsilon \) has matrix with entries \( g_{ij} \) and its inverse matrix has entries \( g^{ij} \). Since we are working in the Fermi coordinate, the entries in the last column and row are all zero, except the rightmost entry on the last row. We omit the subscript \( h \) in \( w_h \) and write it as \( w \).
Lemma 8  The condition $|\nabla u| = 1$ on $\Gamma_{i+h_i}$ is equivalent to

$$\partial_t \phi - f = E_{3,i}.$$  

Here for $i = -1, 1$, $E_{3,i}$ is defined on $\Gamma_{i+h_i}$ to be

$$-\frac{1}{2} \left(1 + g^{1,1}_i h'_i \right) (\partial_t \phi)^2 + \frac{g^{1,1}_i h'_i}{1 + f} \partial_t \phi + \frac{1}{2} \left(1 + \frac{g^{1,1}_i}{1 + f} \partial_l \phi \right)^2 - \frac{1}{2} \left(1 + \frac{g^{1,1}_i}{1 + f} \partial_l \phi \right)^2.$$  

Proof. We compute the norm of the gradient in the $(s, l)$ coordinate and get the following equation to be satisfied on the boundary $\partial \Omega_h$:

$$|\nabla (w + \phi)|^2 = (\partial_s w + \partial_s \phi)^2 + g^{1,1} (\partial_l w + \partial_l \phi)^2 = 1.$$  \hspace{1cm} (12)

Direct computation yields

$$\partial_s w = \frac{1}{1 + f},$$

and

$$\partial_l w = -\frac{g'}{1 + f} - \frac{(s - g) f'}{(1 + f)^2}.$$  

On the other hand, differentiating the identity $\phi (-1 + h_1, l) = 0$ with respect to $l$, we obtain

$$\partial_l \phi = -\partial_s \phi h'_i$$ on $\Gamma_{-1+h_{-1}}$.  

On $\Gamma_{-1+h_{-1}}$, the right hand side of (12) is equivalent to

$$(1 + g^{1,1}_i h'_i) (\partial_s \phi)^2 + (2 \partial_s w - 2g^{1,1}_i h'_i) \partial_s \phi + (\partial_s w)^2 + g^{1,1} (\partial_l \phi)^2 = 1. \hspace{1cm} (13)$$

Inserting the equation

$$\partial_s \phi = \frac{\partial \phi}{1 + f}$$

into (13), we get

$$(1 + g^{1,1}_i h'_i) (\partial_s \phi)^2 + \left(2 - 2g^{1,1}_i h'_i \right) \partial_s \phi - 2 f - f^2 + g^{1,1} \left(\frac{(s - g) f'}{1 + f} \right)^2 = 0.$$  

This completes the proof.  \hspace{0.5cm} \blacksquare 

The function $\phi$ should also satisfy

$$\Delta \phi = -\Delta w = J_{\Gamma_0} g + \left(\Delta_{\Gamma_0} f + |A|^2 \right) t - E_1 + E_2 + \Delta_{\Gamma_0} w - \Delta_{\Gamma_0} w, \text{ in } \Omega_h.$$  

Here we recall that by $J_{\Gamma_0}$ we denote the Jacobi operator of $\Gamma_0$. Therefore, we are lead to solve the following nonlinear problem for the unknown functions $(f, g, \phi)$:

$$\left\{ \begin{array}{l}
\Delta \phi = J_{\Gamma_0} g + \left(\Delta_{\Gamma_0} f + |A|^2 \right) t - E_1 + E_2 + \Delta_{\Gamma_0} w - \Delta_{\Gamma_0} w, \text{ in } \Omega_h, \\
\phi = 0 \text{ and } \partial_t \phi - f = E_{3,i}, \text{ on } \partial \Omega_h. \end{array} \right. \hspace{1cm} (14)$$
Lemma 9 We have the following estimate for the Laplacian operator acting on functions depending on $s$ and $l$:

$$\Delta_{\Gamma_0} \eta - \partial_l^2 \eta - \frac{3}{l} \partial_l \eta = O \left( \frac{\varepsilon}{1 + \varepsilon l} \right) \partial_l \eta,$$

and

$$\Delta_{\Gamma_s} \eta - \Delta_{\Gamma_0} \eta = O \left( \frac{\varepsilon^2}{(1 + \varepsilon l)^2} \right) \partial_l \eta + O \left( \frac{\varepsilon}{1 + \varepsilon l} \right) \partial_l^2 \eta. \quad (15)$$

Proof. By (11), we have

$$\Delta_{\Gamma_0} \eta - \frac{d^2 \eta}{dl^2} - \frac{3}{l} \frac{d\eta}{dl} = \left[ \left( \frac{3}{x} + \frac{3 \varphi'_x}{\varphi_x} \right) \frac{dx}{dl} - \frac{3}{l} \right] \frac{d\eta}{dl}.$$

We compute

$$\left( \frac{3}{x} + \frac{3 \varphi'_x}{\varphi_x} \right) \frac{dx}{dl} - \frac{3}{l} = \frac{1}{\sqrt{1 + \varphi'_x^2}} \left( \frac{3}{x} + \frac{3 \varphi'_x}{\varphi_x} \right) - \frac{3}{l}$$

$$= \frac{1}{\sqrt{1 + (\varphi'_1(\varepsilon x))^2}} \left( \frac{3}{x} + \frac{3 \varepsilon \varphi'_1(\varepsilon x)}{\varphi_1(\varepsilon x)} \right) - \frac{3}{l}$$

$$= \frac{3}{l} \frac{1}{x} \sqrt{1 + (\varphi'_1(\varepsilon x))^2} + \frac{\varepsilon}{1 + \varepsilon l} \left( \frac{1}{1 + \varepsilon l} \right).$$

Next we prove (15). Let us denote by $g_s$ the metric tensor of $\Gamma_s$. Explicitly,

$$\sqrt{\det g_s} = \sqrt{\det g_0} \prod_{i=1}^{6} (1 - k_i s),$$

where $k_i$ are the principle curvatures of $\Gamma_0 = S_\varepsilon$. Hence, for a function $\eta$ depending on $s$ and $l$,

$$\Delta_{\Gamma_s} \eta = \frac{1}{\sqrt{\det g_s}} \partial_i \left( \sqrt{\det g_s} g_s^{i,j} \partial_j \eta \right)$$

$$= \partial_l \left( \ln \left( \sqrt{\det g_0} \prod_{i=1}^{6} (1 - k_i s) \right) \right) g_s^{1,1} \partial_l \eta + \partial_l \left( g_s^{1,1} \partial_l \eta \right).$$

Consequently,

$$\Delta_{\Gamma_s} \eta - \Delta_{\Gamma_0} \eta = \partial_l \left( \ln \left( \prod_{i=1}^{6} (1 - k_i s) \right) \right) g_s^{1,1} \partial_l \eta$$

$$+ \partial_l \left( \ln \sqrt{\det g_0} \right) \left( g_s^{1,1} - g_0^{1,1} \right) \partial_l \eta$$

$$+ \partial_l \left( \left( g_s^{1,1} - g_0^{1,1} \right) \partial_l \eta \right).$$
Then the desired estimate follows from the fact that

\[ \frac{dk_i}{dt} \leq C \frac{\varepsilon^2}{(1 + \varepsilon t)^2}. \]

By the previous computations, the term \(-E_1 + E_2 + \Delta_{\Gamma_0} w - \Delta_{\Gamma_1} w\) will be small and can be regarded as perturbation terms.

To get a solution \((f, g, \phi)\) for the original problem, let us introduce the functional framework to work with. Let \(\alpha \in (0, 1)\) be a fixed constant. Note that the functions \(f\) and \(g\) are both defined on the minimal surface \(S_{\varepsilon}\). However, we shall work both in functional spaces defined on \(S_{\varepsilon}\) and \(S_1\). Hence we introduce the following

**Definition 10** For \(\mu = 0, 1, 2, \beta \geq 0, \delta > 0\), the space \(\mathcal{B}_{\beta, \mu; \delta}\) consists of those functions \(\eta\) defined on \(S_{\delta}\) such that

\[ \|\eta\|_{\beta, \mu; \delta} := \sup_{|z|=l} \left[ (1 + \delta l)^\beta \|\eta\|_{C^{0, \alpha}(S_{\delta} \cap B_1(z))} \right] < +\infty. \]

**Definition 11** The space \(\bar{\mathcal{B}}_{\beta, 2; \delta}\) consists of those functions \(\eta\) defined on \(S_{\delta}\) such that

\[ \|\eta\|_{\beta, 2; \delta} := \sup_{|z|=l} \left[ (1 + \delta l)^\beta \|\eta\|_{C^{0, \alpha}(S_{\delta} \cap B_1(z))} + (1 + \delta l)^{\beta+1} \|\eta\|_{C^{0, \alpha}(S_{\delta} \cap B_1(z))} \right] \]

\[ + \sup_{|z|=l} \left[ (1 + \delta l)^{\beta+2} \|\eta\|_{C^{0, \alpha}(S_{\delta} \cap B_1(z))} \right] < +\infty. \]

With the above definition, we shall assume a priori \(f \in \bar{\mathcal{B}}_{2, 2; \varepsilon}\). We also assume the rescaled function \(\tilde{g}(\cdot) = g(\cdot \varepsilon) \in \bar{\mathcal{B}}_{\beta_0; 2; 1}\), where \(\beta_0 > 2\) is a fixed constant with \(\beta_0 - 2\) small. On the other hand, the function \(\phi\) is defined on \(\Omega_b\), which depends on \(f\) and \(g\). This turns out to be not very convenient for our later purpose. Hence slightly abusing the notation, we also regard \(\phi\) as the restriction of a function \(T(\phi)\) on \(\Xi := [-1, 1] \times [0, +\infty)\), where \(T(\phi)\) is a function of \(t\) and \(l\) defined for \((t, l) \in \Xi := [-1, 1] \times \mathbb{R}\), even in the variable \(l\).

**Definition 12** For \(\mu = 0, 1, 2, \beta \geq 0\), the space \(\mathcal{B}_{\beta, \mu; \ast}\) consists of those functions \(\phi\) such that

\[ \|\phi\|_{\beta, \mu; \ast} := \sup_{t \in \mathbb{R}, |z| = |l|} \left[ (1 + \varepsilon |l|)^\beta \|T(\phi)\|_{C^{0, \alpha}(\Xi \cap B_1(z))} \right] < +\infty. \]

We shall assume \(\phi \in \mathcal{B}_{2, 2; \ast}\). The following invertibility property of the Jacobi operator on \(S_1\) will play an important role in our analysis.

**Lemma 13** For each function \(\xi \in \mathcal{B}_{\beta_0+2, 0; 1}\), there is a solution \(\eta \in \bar{\mathcal{B}}_{\beta_0; 2; 1}\) such that \(J_{S_1}(\eta) = \xi\).

Moreover, it satisfies

\[ \|\eta\|_{\beta_0, 2; 1} \leq C \|\xi\|_{\beta_0+2, 0; 1}. \]
Proof. The proof of this lemma goes in a similar fashion as that of [36], we omit the details.

We would like to solve the nonlinear problem (14) using fixed point arguments.

Lemma 14 For each \( \eta \in B_{\beta,0;*} \), there exists a unique solution \( \phi \in B_{\beta,2;*} \) to the problem

\[
\begin{aligned}
\partial_t^2 \phi + \partial_l^2 \phi + \frac{3}{2} \partial_l \phi &= \eta, \quad \text{in } \Omega_h, \\
\phi &= 0 \quad \text{on } \partial \Omega_h,
\end{aligned}
\]

with \( \|\phi\|_{\beta,2;*} \leq C \|\eta\|_{\beta,0;*} \). This solution will be denoted by \( L_1(\eta) \).

Remark 15 In terms of the \((t,l)\) coordinate, the first equation in (16) actually should be considered in the region \((t,l) \in [-1,1] \times [0, +\infty) \). However, for the sake of notational simplicity, we just write it as in \( \Omega_h \). Similarly, we use the notation \( \partial \Omega_h \) in the second equation of (16).

The proof of Lemma 14 follows from standard arguments.

Next, given two functions \( \gamma_1 \) and \( \gamma_{-1} \) defined on \( S_\varepsilon \), we consider

\[
\begin{aligned}
\partial_t^2 \phi + \partial_l^2 \phi + \frac{3}{2} \partial_l \phi &= J_{\Gamma_0} g + \left( \Delta_{\Gamma_0} f + |A|^2 \right) t, \quad \text{in } \Omega_h, \\
\phi (\pm 1, l) &= 0, \\
\partial_t \phi - f &= \gamma_{-1}, \quad \text{for } t = -1, \\
\partial_t \phi - f &= \gamma_1, \quad \text{for } t = 1.
\end{aligned}
\]

To find the explicit form of the solution \( \phi \) of this problem, we need to introduce some notations. For each fixed \( \xi \in \mathbb{R}^4 \), let us use \( p_{1,\xi}(\cdot) \) to denote the solution of the problem

\[
\begin{aligned}
p''_{1,\xi}(t) - |\xi|^2 p_{1,\xi}(t) &= 1, \\
p_{1,\xi}(-1) &= p_{1,\xi}(1) = 0.
\end{aligned}
\]

We use \( p_{2,\xi}(\cdot) \) to denote the solution of

\[
\begin{aligned}
p''_{2,\xi}(t) - |\xi|^2 p_{2,\xi}(t) &= t, \\
p_{2,\xi}(-1) &= p_{2,\xi}(1) = 0.
\end{aligned}
\]

Note that \( p_{1,\xi} \) is even, while \( p_{2,\xi} \) is odd. For convenience, we collect properties of \( p_{i,\xi} \) in the following

Lemma 16 Explicitly,

\[
\begin{aligned}
p_{1,\xi}(t) &= \frac{\cosh(|\xi| t)}{|\xi|^2 \cosh |\xi|} - \frac{1}{|\xi|^2}, \\
p_{2,\xi}(t) &= \frac{\sinh(|\xi| t)}{|\xi|^2 \sinh |\xi|} - \frac{t}{|\xi|^2}.
\end{aligned}
\]

Moreover,

\[
\frac{1}{p_{1,\xi}(1)} - |\xi| = - \frac{|\xi|}{\tanh |\xi|} - |\xi| = O \left( e^{-\frac{4\xi}{|\xi|^2}} \right), \quad \text{as } |\xi| \to +\infty,
\]
where $l$ to the usual Fourier transform in $\mathbb{R}$

Therefore, to get a solution for (20), it suffices for us to solve the following

\[ f_0 = -\left( \frac{|A|^2}{\xi^2 - p_{2,\xi}^2} \right)^\vee. \]

By the discussion in the next proposition, this definition makes sense.

**Proposition 17** Suppose $\gamma_1 - \gamma_{-1} \in B_{\beta_0 + 2, 1; \varepsilon}$, $\gamma_1 + \gamma_{-1} \in B_{\beta_0, 1; \varepsilon}$. Then the system (17) has a solution $(f, \tilde{g})$ with

\[ \|f - f_0\|_{\beta_0, 2; \varepsilon} \leq C \|\gamma_1 + \gamma_{-1}\|_{\beta_0, 1; \varepsilon}; \quad (18) \]

and

\[ \|\tilde{g}\|_{\beta_0, 2; 1; \varepsilon} \leq C \varepsilon^{-2} \|\gamma_1 - \gamma_{-1}\|_{\beta_0 + 2, 1; \varepsilon}. \quad (19) \]

This solution $(f, \tilde{g})$ will be denoted by $L_2(\gamma_{-1}, \gamma_1)$.

**Proof.** We are lead to the problem

\[
\begin{align*}
\partial_t^2 \hat{\phi} - |\xi|^2 \hat{\phi} &= (J_{\Gamma_0}g)\hat{\phi}, \\
\hat{\phi}(-1, \xi) &= 0, \\
\hat{\phi}_\xi(-1, \xi) - \hat{f}(\xi) &= \hat{\gamma}_{-1}(\xi), \\
\hat{\phi}_\xi(1, \xi) &= \hat{\gamma}_{1}(\xi).
\end{align*}
\]

The solution $\hat{\phi}$ of the first equation in (20) can be written in the form

\[ \hat{\phi}(t, \xi) = (J_{\Gamma_0}g)\hat{p}_{1, \xi}(-1) + \left( \Delta_{\Gamma_0}f + |A|^2 \right)\hat{p}_{2, \xi}(1) \hat{p}_{2, \xi}(t). \]

Therefore, to get a solution for (20), it suffices for us to solve the following problem:

\[
\begin{align*}
(J_{\Gamma_0}g)\hat{p}_{1, \xi}(-1) + \left( \Delta_{\Gamma_0}f + |A|^2 \right)\hat{p}_{2, \xi}(-1) - \hat{f}(\xi) &= \hat{\gamma}_{-1}(\xi), \\
(J_{\Gamma_0}g)\hat{p}_{1, \xi}(1) + \left( \Delta_{\Gamma_0}f + |A|^2 \right)\hat{p}_{2, \xi}(1) - \hat{f}(\xi) &= \hat{\gamma}_{1}(\xi).
\end{align*}
\]

15
Due to the symmetry of $p_{1, \xi}$ and $p_{2, \xi}$, (21) is equivalent to
\[
\begin{cases}
(J_{\Gamma_0} g)^* = \frac{\tilde{\gamma}_1(\xi) - \tilde{\gamma}_2(\xi)}{2p_{1, \xi}'(1)}, \\
(\Delta_{\Gamma_0} f + |A|^2) = \frac{2\hat{f}(\xi) + \tilde{\gamma}_1(\xi) + \gamma(\xi)}{2p_{2, \xi}'(1)}.
\end{cases}
\] (22)

One can perform inverse Fourier transform for the first equation in this system and then use Lemma 13 to get a solution $g$.

We proceed to estimate the norm of $\hat{g}(\cdot) = g(\hat{\cdot})$. Put $\rho = \gamma_1 - \gamma_2$. We would like to show
\[
\int_{|z| > \frac{1}{2}} \frac{\rho(|z|) - \rho(|y|)}{|z - y|^5} dy.
\]

For $|z|$ large, we have
\[
\left| \int_{|z - y| < \frac{1}{2}} \frac{\rho(|z|) - \rho(|y|)}{|z - y|^5} dy \right| \leq C |\rho(|z|)| + \int_{|z - y| > \frac{1}{2}} \frac{|\rho(|y|)|}{|z - y|^5} dy
\]
\[
\leq C |\rho(|z|)| + \frac{1}{|z|} \int_{|z - y| > \frac{1}{2}} |\rho(|y|)| dy
\]
\[
\leq C |\rho(|z|)| + \frac{\|\rho\|_{\beta_0 + 2, 1, \varepsilon}}{1 + \varepsilon^5 |z|^5}.
\] (23)

On the other hand,
\[
\left| \int_{1 < |z - y| < \frac{1}{2}} \frac{\rho(|z|) - \rho(|y|)}{|z - y|^5} dy \right| \leq \frac{C \|\rho\|_{\beta_0 + 2, 1, \varepsilon}}{(1 + \varepsilon^5 |z|)^{\beta_0 + 2}} \int_{1 < |z - y| < \frac{1}{2}} \frac{dy}{|z - y|^5}
\] (24)
\[
\leq \frac{C \|\rho\|_{\beta_0 + 2, 1, \varepsilon}}{(1 + \varepsilon^5 |z|)^{\beta_0 + 2}}.
\]

Furthermore, using the fact that $\rho \in C^{1, \alpha}$, we get
\[
\left| \text{P.V.} \int_{0 < |z - y| < 1} \frac{\rho(|z|) - \rho(|y|)}{|z - y|^5} dy \right| \leq C \|\rho\|_{C^{1, \alpha}(B_1(z))}.
\] (25)
Inequalities (23), (24), (25) give us the required weighted \( C^0 \) estimate of \( \left( |\xi| \cdot \hat{\rho} (\xi) \right)^\vee (z) \). Similarly, one can also get corresponding estimate for the Holder norm. Hence the desired estimate (19) follows.

To find the solution \( f \) for the second equation in (22), we first consider the equation

\[
\left( f'' + \frac{3}{\ell} f' + |A|^2 \right)^\vee = \frac{2 \hat{f} (\xi) + \hat{\gamma}_{-1} (\xi) + \hat{\gamma}_1 (\xi)}{2 \rho^2_{2,\xi} (1)}.
\] (26)

This can be written as

\[
\hat{f} (\xi) = -\left( \frac{|A|^2}{|\xi|^2 - \frac{1}{\rho^2_{2,\xi} (1)}} \right)^\vee + \frac{\hat{\gamma}_{-1} (\xi) + \hat{\gamma}_1 (\xi)}{2 \left( |\xi|^2 \rho^2_{2,\xi} (1) - 1 \right)}.
\] (27)

We may take inverse Fourier transform on both sides of (27). Let

\[
K_1 = \left( \frac{1}{|\xi|^2 - \frac{1}{\rho^2_{2,\xi} (1)}} \right)^\vee, \quad K_2 = \left( \frac{1}{|\xi|^2 \rho^2_{2,\xi} (1) - 1} \right)^\vee.
\]

In view of the explicit formula of \( \rho_{2,\xi} (1) \), we know \( |\xi|^2 - \frac{1}{\rho^2_{2,\xi} (1)} \) and \( |\xi|^2 \rho^2_{2,\xi} (1) - 1 \) are positive and real analytic. This implies that \( K_1 \) and \( K_2 \) decay fast enough at infinity. On the other hand,

\[
\frac{1}{|\xi|^2 - \frac{1}{\rho^2_{2,\xi} (1)}} \sim \frac{1}{|\xi|^2}, \quad \frac{1}{|\xi|^2 \rho^2_{2,\xi} (1) - 1} \sim \frac{1}{|\xi|}, \text{ as } |\xi| \to +\infty.
\]

Observe that the inverse Fourier transform of \( |\xi|^{-1} \) is \( c_1 |x|^{-3} \) (see [24]). It follows that \( K_2 \) has a singularity of the order \( O \left( |x|^{-3} \right) \) near origin. The estimate (18) for solution \( f \) of (27) then follows from routine calculation in potential theory. Since by Lemma 9, \( \Delta \Gamma_0 f \) is a small perturbation of \( f'' + \frac{3}{\ell} f' \), then we can use a perturbation argument to show the same estimate for solution \( f \) of the second equation in (22). This finishes the proof. 

With the model linear problem understood, we proceed to solve the nonlinear problem. Let \( \phi_0 \) be the solution of the problem

\[
\begin{cases}
\partial_t^2 \phi_0 + \partial_t \phi_0 + \frac{4}{\ell} \partial_t \phi_0 = t |A|^2, & \text{in } \Omega_h, \\
\phi_0 = 0 & \text{on } \partial \Omega_h.
\end{cases}
\]

**Lemma 18** Suppose \( \| f - f_0 \|_{\beta_0,2\varepsilon} \leq C\varepsilon^2, \| \tilde{g} \|_{\beta_0,2,1} \leq C\varepsilon, \) and \( \| \phi - \phi_0 \|_{\beta_0,2,\varepsilon} \leq C\varepsilon^2. \) There holds

\[
\begin{align*}
\| E_{3,1} - E_{3,-1} \|_{\beta_0 + 2,1} & \leq C\varepsilon^3, \\
\| E_{3,1} + E_{3,-1} \|_{\beta_0,1} & \leq C\varepsilon^3.
\end{align*}
\]
Let us introduce the notation

$$E_{3,i} = -\frac{1}{2} \left( 1 + g^{1,1} h_i^2 \right) (\partial_t \phi)^2 + \frac{g^{1,1} h_i'}{1 + f} \partial_t \phi + \frac{1}{2} f^2 - \frac{1}{2} g^{1,1} \left( g' + tf' \right)^2.$$  

Using the boundedness of $g^{1,1}$, taking into account of the fact that

$$\|g'\|_{3,1; \varepsilon} \leq C \varepsilon^2, \|f\|_{2,2; \varepsilon} \leq C \varepsilon^2, \|\partial_t \phi (\pm 1, l) - \partial_t \phi_0 (\pm 1, l)\|_{\beta_0,2; \varepsilon} \leq C \varepsilon^2,$$

we find that

$$\left\| g^{1,1} h_i^2 (\partial_t \phi)^2 \right\|_{\beta_0+2,1; \varepsilon} + \left\| g^{1,1} (g')^2 \right\|_{\beta_0+2,1; \varepsilon} + \left\| g^{1,1} f' \right\|_{\beta_0+2,1; \varepsilon} \leq C \varepsilon^3.$$

Now we subtract $E_{3,1}$ with $E_{3,-1}$, the term $f^2$ will be cancelled. Additionally, using the asymptotic expansion of $g^{1,1}$, we know

$$\left\| \left( t^2 g^{1,1} f' \right) \mid_{t=-1} - \left( t^2 g^{1,1} f' \right) \mid_{t=1} \right\|_{\beta_0+2,1; \varepsilon} \leq C \varepsilon^3.$$

Furthermore, observing that $\|f_0'\|_{3,1; \varepsilon} \leq C \varepsilon^2$, we get

$$\left\| \left( g^{1,1} h_i' \partial_t \phi \right) \mid_{t=-1} - \left( g^{1,1} h_i' \partial_t \phi \right) \mid_{t=1} \right\|_{\beta_0+2,1; \varepsilon} \leq C \varepsilon^3 + C \|(f_0' \partial_t \phi_0) \mid_{t=-1} + (f_0' \partial_t \phi_0) \mid_{t=1} \|_{\beta_0+2,1; \varepsilon} \leq C \varepsilon^3.$$

Hence we get

$$\left\| E_{3,1} - E_{3,-1} \right\|_{\beta_0+2,1; \varepsilon} \leq C \varepsilon^3 + \frac{1}{2} \left\| \partial_t \phi_0 (1,l)^2 - \partial_t \phi_0 (1,l)^2 \right\|_{\beta_0+2,1; \varepsilon} \leq C \varepsilon^3.$$

The proof of $\|E_{3,1} + E_{3,-1}\|_{\beta_0,1; \varepsilon} \leq C \varepsilon^3$ is similar. ■

To proceed, let us consider the nonlinear problem

$$\left\{ \begin{array}{l}
\Delta \phi = J_{1,0} g + \left( \Delta_{1,0} f + \|A\|^2 \right) t - E_1 + E_2 + \Delta_{1,w} w - \Delta_{1,v} w, \text{ in } \Omega_h, \\
\phi = 0 \text{ on } \partial \Omega_h.
\end{array} \right. \quad (28)$$

Let us introduce the notation

$$P(f, \bar{g}, \phi) := -E_1 + E_2 + \Delta_{1,w} w - \Delta_{1,v} w + \delta_t^2 \phi + \delta_t^2 \phi + \frac{3}{t} \delta_t \phi - \Delta \phi. \quad (29)$$

We will investigate the Lipschitz dependence of $P$ on $f$ and $\bar{g}$.

**Lemma 19** For $f_i \in B_{2,2; \varepsilon}, \bar{g}_i \in \bar{B}_{\beta_0,2;1}$, with $\|f_i - f_0\|_{\beta_0,2; \varepsilon} \leq C \varepsilon^2, \|\bar{g}_i\|_{\beta_0,2,1; \varepsilon} \leq C \varepsilon$, $i=1, 2$, we have

$$\left\| P(f_1, \bar{g}_1, \phi) - P(f_2, \bar{g}_2, \phi) \right\|_{\beta_0+2,0; \varepsilon} = O (\varepsilon^2) \left\| f_1 - f_2 \right\|_{\beta_0,2; \varepsilon} + O (\varepsilon^3) \|\bar{g}_1 - \bar{g}_2\|_{\beta_0,2,1; \varepsilon}.$$
Proof. Let us consider the terms in \[ 30 \]. Recall that

\[
E_1 (f, g) = -tf \Delta \gamma_0 f + \Delta \gamma_0 (fg) - g \Delta \gamma_0 f + \Delta \gamma_0 \left( (s - g) \frac{f^2}{1 + f} \right).
\]

We compute directly that

\[
f_1 \Delta \gamma_0 f_1 - f_2 \Delta \gamma_0 f_2 = f_1 \Delta \gamma_0 (f_1 - f_2) + \Delta \gamma_0 f_2 (f_1 - f_2).
\] (30)

Next, since

\[
\Delta \gamma_0 (fg) - g \Delta \gamma_0 f = 2f'g' + f \Delta \gamma_0 g,
\]
we have

\[
[\Delta \gamma_0 (f_1 g_1) - g_1 \Delta \gamma_0 f_1] - [\Delta \gamma_0 (f_2 g_2) - g_2 \Delta \gamma_0 f_2]
= 2 (f'_1 - f'_2) g'_1 + 2 f'_2 (g'_1 - g'_2)
+ \Delta \gamma_0 g_1 (f_1 - f_2) + f_2 \Delta \gamma_0 (g_1 - g_2).
\] (31)

Now combining (30), (31) and performing a similar computation for the term

\[
\Delta \gamma_0 \left( (s - g) \frac{f^2}{1 + f} \right),
\]
we obtain

\[
\| E_1 (f_1, g_1) - E_1 (f_2, g_2) \|_{\beta_0 + 2, 0; \varepsilon} = O (\varepsilon^2) \| f_1 - f_2 \|_{\beta_0; 2; \varepsilon} + O (\varepsilon^3) \| g_1 - g_2 \|_{\beta_0; 2; 1}.
\]

For the term

\[
E_2 (f, g) = \frac{1}{1 + f} \sum_{i=1}^{6} \frac{s^2 k_i^3}{1 - sk_i} - \frac{fg |A|^2}{1 + f},
\]
we have

\[
E_2 (f_1, g_1) - E_2 (f_2, g_2) = -|A|^2 \left( \frac{f_1 g_1}{1 + f_1} - \frac{f_2 g_2}{1 + f_2} \right) + \frac{f_2 - f_1}{(1 + f_1)(1 + f_2)} \sum_{i=1}^{6} \frac{s^2 k_i^3}{1 - sk_i}.
\]

Since \(|A|^2 = O \left( \frac{\varepsilon^2}{(1 + l)^2} \right)\), we obtain

\[
\| E_2 (f_1, g_1) - E_2 (f_2, g_2) \|_{\beta_0 + 2, 0; \varepsilon} = O (\varepsilon^2) \| f_1 - f_2 \|_{\beta_0; 2; \varepsilon} + O (\varepsilon^3) \| g_1 - g_2 \|_{\beta_0; 2; 1}.
\]

It remains to analyze the term \( \Delta \gamma_0 w - \Delta \gamma_0 w \). To handle it, we simply note that by Lemma \[ 9 \] the following expansion holds:

\[
\Delta \gamma_0 w - \Delta \gamma_0 w = O \left( \frac{\varepsilon^2}{(1 + \varepsilon l)^2} \right) \partial_l w + O \left( \frac{\varepsilon}{1 + \varepsilon l} \right) \partial^2_l w
= O \left( \frac{\varepsilon^2}{(1 + \varepsilon l)^2} \right) \left( -g' (1 + f) - (s - g) f' \right)
+ O \left( \frac{\varepsilon}{1 + \varepsilon l} \right) \left( -g' (1 + f) - (s - g) f' \right)^\prime.
\]
which yields the desired estimate:

\[ \left\| (\Delta r_0 w - \Delta r, w) \right\|_{(f_1, g_1)} - (\Delta r_0 w - \Delta r, w) \right\|_{(f_2, g_2)} \|_{\beta_0 + 2, 2\varepsilon} = O \left( \varepsilon^2 \right) \| f_1 - f_2 \|_{\beta_0, 2\varepsilon} + O \left( \varepsilon^3 \right) \| g_1 - g_2 \|_{\beta_0, 2; 1}. \]

The proof is thus completed.

**Lemma 20** Given \( f, g \), with \( \| f - f_0 \|_{\beta_0, 2\varepsilon} \leq C\varepsilon^2, \| g \|_{\beta_0, 2; 1} \leq C\varepsilon \), problem (28) has a unique solution \( \Phi (f, g) \).

If we write this solution as \( \Phi (f_1, g_1) \), then

\[ \| \Phi (f_1, g_1) - \Phi (f_2, g_2) \|_{\beta_0 + 1, 2; \varepsilon} \leq C \| f_1 - f_2 \|_{\beta_0, 2\varepsilon} + C\varepsilon^2 \| g_1 - g_2 \|_{\beta_0, 2; 1}. \]

**Proof.** We may recast (28) as

\[ \phi = L_1 \left[ J_{r_0} g + \left( \Delta r_0 f + |A|^2 \right) t \right] + L_1 \left[ P \left( f, g, \phi \right) \right], \]

where \( \phi = \phi_0 + \phi^* \), \( \phi^* \in \mathcal{B}_{\beta_0 + 1, 2; \varepsilon} \). In other words,

\[ \phi^* = \bar{L}_1 \left( f, g, \phi^* \right) := L_1 \left[ J_{r_0} g + \left( \Delta r_0 f + |A|^2 \right) t \right] + L_1 \left[ P \left( f, g, \phi_0 + \phi^* \right) \right] - \phi_0, \]

We regard it as a fixed point problem of \( \phi^* \) for the map \( \bar{L}_1 \). Observe that although \( \phi_0 \) only belongs to \( \mathcal{B}_{2, 2; \varepsilon} \), the function \( P \left( f, g, \phi_0 + \phi^* \right) \) actually lies in \( \mathcal{B}_{\beta_0 + 1, 0; \varepsilon} \). Now we show \( \bar{L}_1 \) is a contraction map. Indeed, by Lemma 7

\[ \Delta \phi = \partial^2 f \phi + \Delta r, \phi - H_{r_0} \partial \phi \]

\[ = \frac{1}{(1 + f)^2} \partial^2 f \phi + \Delta r_0 \phi + O \left( \frac{\varepsilon}{(1 + \varepsilon)^2} \right) \partial_t \phi \]

\[ + O \left( \frac{\varepsilon}{1 + \varepsilon} \right) \partial^2 \phi + O \left( \sum k_i^2 \right) \partial \phi. \]

Using this expansion, we can verify that

\[ \| \bar{L}_1 \left( f, g, \phi^* \right) - \bar{L}_1 \left( f, g, \phi^*_1 \right) \|_{\beta_0 + 1, 2; \varepsilon} \leq C\varepsilon \| \phi^*_1 - \phi^*_2 \|_{\beta_0 + 1, 2; \varepsilon}. \]

This implies that \( \bar{L}_1 \) is a contraction mapping provided that \( \varepsilon \) is small enough. It follows that (28) has a solution.

To see the Lipschitz dependence of \( \Phi \) on \( f, g \), we subtract the equations satisfied by \( \Phi \left( f_1, g_1 \right) \) and \( \Phi \left( f_2, g_2 \right) \). Then one can use the explicit expression for \( E_1, E_2 \) to get the desired estimate. ■

If we write \( \Phi \left( f, g \right) = \phi_1 + L_1 \left( P \left( f, g, \Phi \left( f, g \right) \right) \right) \), then our original nonlinear problem will be transformed into

\[
\begin{align*}
\partial_t^2 \phi_1 + \partial^2 f \phi_1 + \frac{\varepsilon}{(1 + f)^2} \partial_t \phi_1 &= J_{r_0} g + \left( \Delta r_0 f + |A|^2 \right) t + P \left( f, g, \phi_1 \right), \text{ in } \Omega_h, \\
\phi_1 &= 0 \text{ and } \partial_t \phi_1 - f = E_{3, i} - \partial_t \left[ L_1 \left( P \left( f, g, \Phi \left( f, g \right) \right) \right) \right], \text{ on } \Gamma_{i+1}. \end{align*}
\]

(32)
With all these preparations, we are now ready to prove Theorem 1.

**Proof of Theorem 1.** Let us set $f = f_0 + f$. Using Proposition 17, we find that to solve (32), it suffices to get a solution for the following fixed point problem for $(\bar{f}, \bar{g})$:

$$(\bar{f}, \bar{g}) = L_2 (\bar{f}, \bar{g}) := L_2 (T_1 - T_1) - (f_0, 0),$$

where

$$T_i = E_3, i - \partial_t [L_1 (P (f, \Phi (f, g)))] |_{t = i}, i = \pm 1.$$ 

Let us define the space

$$B := \{ (\bar{f}, \bar{g}) | (\bar{f}, \bar{g}) \in B_{\beta_2;2, \varepsilon} \times B_{\beta_2;2, \varepsilon} \},$$

equipped with the norm

$$\| (\bar{f}, \bar{g}) \| := \| \bar{f} \|_{\beta_2;2, \varepsilon} + \varepsilon^2 \| \bar{g} \|_{\beta_2;2, \varepsilon}.$$ 

We claim that $L_2$ is a contraction mapping in the set

$$B_1 := \{ (\bar{f}, \bar{g}) \in B : \| (\bar{f}, \bar{g}) \| \leq C_0 \varepsilon \},$$

where $C_0$ is a fixed large constant. Indeed, let

$$\eta_{\pm} (f, g) := \partial_t [L_1 (P (f, \Phi (f, g)))] |_{t = -1} \pm \partial_t [L_1 (P (f, \Phi (f, g)))] |_{t = 1},$$

and

$$f_1 = f_0 + \bar{f}_1, f_2 = f_0 + \bar{f}_2.$$ 

Using Proposition 17 we can show

$$\| \eta_+ (f_1, g_1) - \eta_+ (f_2, g_2) \|_{\beta_2;2, \varepsilon} + \| \eta_- (f_1, g_1) - \eta_- (f_2, g_2) \|_{\beta_2;2, \varepsilon} = O (\varepsilon^2) \| f_1 - f_2 \|_{\beta_2;2, \varepsilon} + O (\varepsilon^3) \| g_1 - g_2 \|_{\beta_2;2, \varepsilon}.$$ 

It then follows from Proposition 17, Lemma 19 and Lemma 20 that

$$\| L_2 (\bar{f}_1, \bar{g}_1) - L_2 (\bar{f}_2, \bar{g}_2) \| \leq C \varepsilon \| (\bar{f}_1, \bar{g}_1) - (\bar{f}_2, \bar{g}_2) \|.$$ 

This proves the claim.

To prove the existence of a fixed point for $L_2$, it remains to show that $L_2 (B_1) \subset B_1$. Since $(\bar{f}, \bar{g}) \in B_1$, we have

$$\| \bar{f} \|_{\beta_2;2, \varepsilon} \leq C_0 \varepsilon^2, \| \bar{g} \|_{\beta_2;2, \varepsilon} \leq C_0 \varepsilon.$$ 

Observe that due to the presence of the term $|A|^t$ and $t^3 \sum k_1^3$, the function $L_1 (P (f, \Phi (f, g))) |_{t = 1}$ does not have enough decay and only belongs to $B_{\infty;2, \varepsilon} \ast$. However, since these two terms are odd, their contribution to the boundary derivative at $t = \pm 1$ cancel and therefore

$$\| \eta_+ \|_{\beta_2;2, \varepsilon} \leq C \varepsilon^2, \| \eta_- \|_{\beta_2;2, \varepsilon} \leq C \varepsilon^3.$$ 

21
Hence by Proposition 17,
\[ \overline{L}_2 \overline{\left( \tilde{f}, \tilde{g} \right)} \leq C\varepsilon^3, \]
which implies that \( \overline{L}_2 (B_1) \subset B_1 \), provided that \( C_0 \) is chosen large enough.

The solution \( w_h + \phi \) depends smoothly on \( \varepsilon \). Let us take the derivatives of \( w_h + \phi \) with respect to \( \varepsilon \). Note that the main order of \( w_h + \phi \) is \( s - g_1 + f \), where \( s \) is the Fermi coordinate around the minimal hypersurface \( S_\varepsilon \). Using the fact that \( S_\varepsilon \) is a minimal foliation associated to the Simons’ cone, we find that \( \frac{d(w_h + \phi)}{d\varepsilon} \) is positive and satisfy the system (7) (see [30]). This proves that our solution of the free boundary problem is stable. This finishes the proof of Theorem 1.

3 Existence of an energy minimizer in \( \mathbb{R}^8 \)—Proof of Theorem 2

In the previous section, we have shown that if \( \varepsilon_0 > 0 \) is small enough, then for each \( \varepsilon < \varepsilon_0 \), we have a solution for the free boundary problem whose nodal set is asymptotic to \( S_\varepsilon^+ \). By symmetry, one also has solutions whose nodal sets are asymptotic to \( S_\varepsilon^- \). We denote these two continuous families of solutions by \( u_+^{\varepsilon_0} \) and \( u_-^{\varepsilon_0} \), with \( u_-^{\varepsilon_0} < u_+^{\varepsilon_0} \). In this section, we will use variational arguments to show the existence of an energy minimizer \( U \) in \( \mathbb{R}^8 \), lying between \( u_+^{\varepsilon_0} \) and \( u_-^{\varepsilon_0} \).

Let us consider the minimizing problem
\[
\min_{\eta - b_a \in B_0 (B_a)} J (\eta). \tag{33}
\]

**Lemma 21** The minimizing problem (33) has a solution \( u_a \) which is invariant under \( O (4) \times O (4) \).

**Proof.** The existence of a minimizer \( u \) for (33) follows from standard arguments. The point is that we need to prove the existence of a minimizer which is additionally invariant under \( O (4) \times O (4) \).

Since \( u \) solves the free boundary problem, it is continuous. We define
\[
w_1 (x) = \min \left\{ u (gx) : g \in O (4) \times O (4) \right\},
\]
\[
w_2 (x) = \max \left\{ u (gx) : g \in O (4) \times O (4) \right\}.
\]

Then \( w_1 \) and \( w_2 \) are invariant under \( O (4) \times O (4) \). We claim that \( w_1 \) and \( w_2 \) are also minimizers. Indeed, for each \( k \in \mathbb{N} \) and a finite set \( \{ g_1, \ldots, g_k \} \in O (4) \times O (4), \) let
\[
\bar{w}_k = \min \left\{ u (g_i x) : g_i \in O (4) \times O (4), i = 1, \ldots, k \right\}.
\]
Then \( \bar{w}_k \) is a minimizer. We cover \( O(4) \times O(4) \) by finitely many balls with radius \( \varepsilon \). Denote by \( n \varepsilon \) the number of balls. In each ball, let us choose a \( g_i \in O(4) \times O(4) \). We will define

\[
q_\varepsilon (x) := \min \{ u (g_i x) : i = 1, \ldots, n \varepsilon \}.
\]

Then \( q_\varepsilon \) is also a minimizer. We observe that by the continuity of a minimizer,

\[
w_1 (x) = \lim_{\varepsilon \to 0} q_\varepsilon (x).
\]

On the other hand, let \( \{ \varepsilon_k \} \) be a sequence converge to 0. Then standard arguments yield that \( q_{\varepsilon_k} (x) \) converges a.e. to minimizer \( q \). This \( q \) must be \( w_1 \). This proves that \( w_1 \) is also a minimizer. Similarly, \( w_2 \) is also a minimizer.

### 3.1 Regularity of the free boundary

We would like to analyze the regularity property of the free boundary of the solution \( u_a \).

**Lemma 22** The free boundary of \( u_a \) is smooth in \( B_a \setminus \{0\} \).

**Proof.** We shall use the standard arguments in the regularity theory: Blow up analysis around a free boundary point, cf. \([51, 52]\). Let \( x_0 \in B_a \) be a point on the free boundary of \( u \). Suppose \( x_0 \neq 0 \) and \( u_a (x_0) = 1 \). We distinguish three cases.

Case 1. \( x_0 \) is not on the \( x \) or \( y \) axis.

In this case, standard arguments, based on Weiss monotonicity formula \([51, 52]\), tell us that the sequence \( w_k := \frac{u_a (x_0 + \rho_k)}{\rho_k} \), with \( \rho_k \to 0 \), has a subsequence converges in suitable sense to a minimizing cone \( \mathcal{C} \) in \( \mathbb{R}^8 \). We observe that \( u_a \) is invariant under \( O(4) \times O(4) \). Hence \( \mathcal{C} \) reduces to a minimizing cone in \( \mathbb{R}^2 \). Therefore it must be a trivial cone. This implies that around \( x_0 \), the free boundary is flat and the regularity theory implies that actually it is smooth (analytic).

Case 2. \( x_0 \) is on the \( x \) or \( y \) axis.

In this case, the cone \( \mathcal{C} \) reduces to a minimizing cone in \( \mathbb{R}^5 \) which is invariant under the \( O(4) \) action of the last four coordinates. If this cone were not trivial, it would be unstable, due to the classification of stable cones by Jerison and Savin in the axial symmetric case (see \([30]\)). This contradicts with the fact that \( u_a \) is a minimizer.

With this regularity at hand, we now want to prove that these minimizers are bounded by \( u^+_a \) and \( u^-_a \), by sweeping the family of ordered solutions \( u^+_a \) and \( u^-_a \), similarly as in \([34]\). By our previous construction, for \( \varepsilon \) sufficiently small, we have

\[
\begin{cases}
  u_a \leq u^+_a, \text{ in } B_a, \\
  u_a < u^+_a, \text{ in } \Lambda := \{ X : |u_a (X)| < 1 \}.
\end{cases}
\]

(34)
We show that actually (34) holds for all $\varepsilon \leq \varepsilon_0$. To see this, we continuously increase the value of $\varepsilon$. Assume to the contrary that there existed a $\delta < \varepsilon_0$, which were the first value where we have

$$u_a \leq u_\delta^+ \text{ in } B_a, \quad \text{and } u_a(X) = u_\delta^+(X) \text{ for some } X \in \bar{\Lambda}. \quad (35)$$

Maximum principle tells us that this $X$ must be on $\partial B_a$. By the results in [33], the free boundary approaches the fixed boundary tangentially, this contradicts with the choice of $\delta$, which is the smallest value satisfying (35). This finishes the proof.

**Proof of Theorem 2.** For each $a$ large, we have a solution $u_a$ with $u_\delta^− < u_a < u_\delta^+$. Sending $a$ to infinity, we can find a subsequence of $u_a$ which converges to a nontrivial solution $U$ of (1). This solution $U$ must be an energy minimizer of $J$, since each $u_a$ is minimizing.$\blacksquare$

4 From minimizers in $\mathbb{R}^8$ to monotone solutions in $\mathbb{R}^9$–Proof of Theorem 3

We have obtained a minimizer of the energy functional in dimension 8. Now we would like to construct monotone solutions in $\mathbb{R}^9$ from $U$, following the arguments of Jerison-Monneau ([31]). We use $(x', x_9)$ to denote the coordinate of a point in $\mathbb{R}^9$, where $x' \in \mathbb{R}^8$. We will still use minimizing argument and work directly in the class of functions which is invariant w.r.t $O(4) \times O(4)$ action on the first eight variables.

We denote by $v_1$ the global minimizer in $\mathbb{R}^8$ we constructed in the last section. We also consider the solution $v_2$ which in the $(x, y)$ coordinate is given by

$$v_2(x, y) = -v_1(y, x).$$

Since $v_1$ is constructed using minimizing argument, we can assume without loss of generality that $v_1 \leq v_2$.

**Proposition 23** Either there exists a nontrivial solution $u : \mathbb{R}^9 \to \mathbb{R}$ monotone in the $x_9$ direction, or for each $\delta \in [v_1(0), v_2(0)]$, there exists a nontrivial global minimizer $v$ in $\mathbb{R}^8$ with $v(0) = \delta$.

**Proof.** Let $\rho$ be a smooth decreasing cutoff function which satisfies

$$\rho(s) = \begin{cases} 
1, & s < 1, \\
0, & s > 2.
\end{cases}$$

Define the function $w(x', x_9) = \rho(x_9) v_1(x') + (1 - \rho(x_9)) v_2(x')$. For each cylinder $C_{R', l} = B_{R'} \times [-l, l]$, consider the minimization problem which equals $w$ on $\partial B_{R'} \times [-l, l]$ and equals $v_1$ on $B_{R'} \times \{-l\}$, equals $v_2$ on $B_{R'} \times \{l\}$, in the class of functions which are invariant under $O(4) \times O(4)$ with respect to the first eight variables. We can find a minimizer $u_{R', l}$ that is monotone in
the $x_9$ direction with this boundary condition. By the gradient bound of De Silva-Jerison (13), the free boundary is smooth in the interior of the cylinder.

Let $l \to +\infty$, we get a solution $u_{R'}$ on the whole cylinder $B_{R'} \times \mathbb{R}$, still monotone in $x_9$ and invariant under $O (4) \times O (4)$. We observe that

$$
\lim_{x_9 \to +\infty} u_{R'} = v_2, \quad \lim_{x_9 \to -\infty} u_{R'} = v_1,
$$

(36) otherwise it will contradict with the fact that $v_1$ and $v_2$ are global minimizer. Now fix a $a \in (v_1 (0), v_2 (0))$. By (36), there exists $h_{R'}$ such that

$$
u_{R'} (x', h_{R'}) = a.$$

Let $\bar{u}_{R'} (x', x_9) = u_{R'} (x', x_9 - h_{R'})$. Then $\bar{u}_{R'} (x', 0) = a$. Let $R' \to +\infty$, we get a solution $u$ monotone in $x_9$, invariant under $O (4) \times O (4)$, and

$$u (0) = a, v_1 \leq u \leq v_2.
$$

If $u$ is independent on $x_9$, then $u$ is a global minimizer in $\mathbb{R}^8$. This proves the proposition. 

Finally we are ready to prove Theorem 24

**Theorem 24** There exists a solution $u$ to our free boundary problem such that $u$ is invariant w.r.p to $O (4) \times O (4)$, monotone in $x_9$ and $u$ is not one dimensional.

**Proof.** Suppose the second possibility of Proposition 23 occurs. Then we can assume there is a global minimizer $v$ in $\mathbb{R}^8$, invariant under $O (4) \times O (4)$ and $-1 < v (0) < 1$.

By $\Theta$ we shall denote the standard one dimensional solution to our free boundary problem:

$$
\Theta (x) = \begin{cases} x, & x \in [-1, 1], \\
1, & x > 1, \\
-1, & x < -1.
\end{cases}
$$

Note that $\Theta$ is monotone, but not strictly monotone. We would like to pose suitable boundary condition on the cylinder $C_{R', l}$. For each $t \in [0, 1]$, let

$$
\Theta_t (x', x_9) = \Theta (tv (x') + (1 - t) x_9).
$$

Then $\Theta_1 (x', x_9) = \Theta (v (x')) = v (x')$. $\Theta_t$ is a connection between $\Theta$ and $v$. Certainly, $\Theta_t (x', x_9) \in [-1, 1]$. We check that $\Theta_t$ is continuous and monotone in the $x_9$ direction, since $\Theta$ itself is monotone. Consider those points where

$$tv (x') + (1 - t) x_9 = 1.
$$

(37)

For each fixed $x'$, there is a unique point $x_9$ satisfying (37).

Let $U_{t, R', l}$ be the minimizer of $J$ in the symmetric (invariant under $O (4) \times O (4)$ action) class of functions defined on $C_{R', l}$ with boundary condition

$$U_t |_{\partial C_{R', l}} = \Theta_t |_{\partial C_{R', l}}.$$
After a possible translation in the $x_9$ direction, we can assume that

$$U_{l,R'}(0) = v(0).$$

For each $R'$, letting $l \to +\infty$, $U_{l,R'}(0)$ converges pointwisely to a solution $U_{l,R'}$, defined on the infinite cylinder $C_{R',+\infty}$. $U_{l,R'}$ is monotone in $x_9$ on the boundary of $C_{R',+\infty}$. Then one can show that $U_{l,R'}$ is monotone in $x_9$ in $C_{R',+\infty}$, with

$$U_{l,R'}(0) = v(0).$$

We claim that the map $t \to \partial_{x_9}U_{l,R'}(0)$ is a continuous map. We first show that it is continuous at the points where $t \neq 1$. In this case, let $t_n \to t$. Then the sequence $U_{t_n,R'}$ converges to a monotone solution $W$. This $W$ must be equal to $U_{l,R'}$. Indeed, since $w$ and $U_{l,R'}$ are equal to each other on the boundary of the cylinder and the boundary value are monotone in the $x_9$ direction, we can infer that $W \geq U_{l,R'}$ and $W \leq U_{l,R'}$ by the sliding method.

The continuity at $t = 1$ also follows from similar arguments as that of Jerison-Monneau [31]. The proof is thus completed.

5 Solutions from Catenoids

In this section, we shall construct solutions of the free boundary problem starting from another type of minimal surfaces—Catenoids. Since most of the arguments are similar to the Simons’ cone case, we will only sketch the proof and point out the difference if necessary.

We remark that it is possible to do the construction for more general minimal surfaces, but this is beyond the scope of this paper.

5.1 The geometry of the catenoids

To begin with, let us choose an “arc-length” parametrization for the catenoid, this choice of coordinate will simplify the computation. Let $(x_1, ..., x_n)$ be the coordinate in $\mathbb{R}^n$. Let $(r, \theta)$ be the polar coordinate in $\mathbb{R}^{n-1}$, where $\theta$ is the coordinate on the unit sphere $S^{n-2}$ in $\mathbb{R}^{n-1}$. As we mentioned before, the generalized catenoid $C_\varepsilon$ in $\mathbb{R}^n$ can be described by

$$x_n = \bar{\omega}_\varepsilon(r), \ r \in [r_0, +\infty).$$

Introduce

$$l = l(r) := \int_{r_0}^r \sqrt{1 + \bar{\omega}_\varepsilon'(s)^2} \, ds.$$

Then locally the catenoid can also be described by the coordinate $(l, \theta)$. We would like to write the Laplacian-Beltrami operator $\Delta_{C_\varepsilon}$ on $C_\varepsilon$ in this coordinate. In the $(r, \theta)$ variable, the metric tensor on $C$ is given by

$$\left[1 + \bar{\omega}_\varepsilon'(r)^2\right] \, dr^2 + r^2 \, d\theta^2.$$
It follows that the metric $g$ in the $(l, \theta)$ coordinate is $dl^2 + r^2 d\theta^2$. Observe that $\det g = r^{2(n-2)}$. For rotationally symmetric function $\varphi = \varphi(l)$, the Laplacian-Beltrami operator is given by

$$\Delta_{C_l} \varphi = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial l} \left( \sqrt{\det g} g^{ij} \partial_j \varphi \right)$$

$$= \varphi''(l) + \frac{n-2}{r} \varphi'(l)$$

$$= \varphi''(l) + O \left( \frac{\varepsilon}{1 + \varepsilon l} \right) \varphi'(l).$$  \hspace{1cm} (38)

Using $s$ to denote the signed distance of a point $P$ to $C_\varepsilon$. Then we can write

$$P = X + sv(X),$$

where $X = X(l, \theta)$ designates a point on the $C_\varepsilon$, $\nu(\cdot)$ is the unit normal of $C_\varepsilon$ at $X$. We also put

$$\Gamma_s := \{ X + sv(X) : X \in C_\varepsilon \}.$$  

Note that actually $\Gamma_s$ depends on $\varepsilon$, although it is not explicit in the notation.

To understand the Laplacian-Beltrami operator $\Delta_{\Gamma_s}$, we need to analyze the metric on the surface $\Gamma_s$. Let $\nu_1 = \partial l \nu, \nu_2 = \partial \theta \nu$, and $X_1 = \partial l X, X_2 = \partial \theta X$.

Define the matrix $B_0 = [X_1 + s \nu_1, X_2 + s \nu_2]$ and

$$B := [X_1 + sv_1, X_2 + sv_2, v].$$

Then the matrix of the induced metric $g$ in a tubular neighbourhood of $C$ in $(l, \theta, s)$ coordinate has the form

$$B^T B = \begin{bmatrix} B_0^T B_0 & 0 \\ 0 & 1 \end{bmatrix}.$$  

For more details, we refer to [16].

5.2 Proof of Theorem 4

In this part, we sketch the proof of Theorem 4.

Let $h_{-1}, h_1 \in C_{locl}^{2,\alpha}(C_\varepsilon)$, small in certain sense. As before, define an approximate solution $w_h$ in $\Omega_h$, which is a region trapped between $\Gamma_{-1+h_{-1}}$ and $\Gamma_{1+h_1}$:

$$w_h(s, l) = \frac{s - g(l)}{1 + f(l)},$$

where

$$f = \frac{h_1 - h_{-1}}{2}, g = \frac{h_1 + h_{-1}}{2}.$$  

Still set

$$t = \frac{s - g(l)}{1 + f(l)}.$$
The solution \( u \) we are looking for will have the form \( u = w_h + \phi \).

We have the same formulas as in Lemma 5, Lemma 7, and Lemma 8 and will not restate them in this section again.

**Lemma 25** We have the following estimate for the Laplacian operator acting on functions depending on \( s \) and \( l \):

\[
\Delta_{\Gamma_0} \eta = \partial_t^2 \eta = O \left( \frac{\varepsilon}{1 + \varepsilon l} \right) \partial_t \eta,
\]

and

\[
\Delta_{\Gamma_1} \eta - \Delta_{\Gamma_0} \eta = O \left( \frac{\varepsilon^2}{(1 + \varepsilon l)^2} \right) \partial_t \eta + O \left( \frac{\varepsilon}{1 + \varepsilon l} \right) \partial_t^2 \eta.
\]

**Proof.** The first equation has already been proved in (38). The proof of the second equation is same as that of Lemma 9.

Let us introduce the functional framework to work with. Let \( \alpha \in (0, 1) \) be a fixed constant.

**Definition 26** For \( \alpha = 0, 1, 2, \beta \geq 0, \delta > 0 \), the space \( \mathcal{E}_{\beta, \mu; \delta} \) consists of those functions \( \eta \) defined on \( C_3 \) such that

\[
\sup_{|z| = l} \left[ (1 + \delta l)^\beta \| \eta \|_{C^\alpha, \delta(S_l \cap B_1(z))} \right] < +\infty.
\]

Same as before, we also regard \( \phi \) as the restriction of a function \( \mathcal{T} (\phi) \) on \( \Xi := [-1, 1] \times [0, +\infty) \), where \( \mathcal{T} (\phi) \) is a function of \( t \) and \( l \) defined for \((t, l) \in \Xi := [-1, 1] \times \mathbb{R} \), even in the variable \( l \).

**Definition 27** For \( \alpha = 0, 1, 2, \beta \geq 0 \), the space \( \mathcal{E}_{\beta, \mu; \alpha} \) consists of those functions \( \phi \) such that

\[
\| \phi \|_{\beta, \mu; \alpha} := \sup_{t \in \mathbb{R}, z \in \Xi, |z| = |l|} \left[ (1 + \varepsilon |l|)^\beta \| \mathcal{T} (\phi) \|_{C^\alpha, \delta(S_l \cap B_1(z))} \right] < +\infty.
\]

Let \( v (\cdot) \) be an even smooth function such that

\[
v (l) = \begin{cases} |l|^{3-n}, |l| > 2, \\ 0, |l| < 1. \end{cases}
\]

The one dimensional space spanned by this function will be denoted by \( \mathcal{D} \). Let \( \tilde{g} (\cdot) = g (\cdot) \). If \( n \geq 4 \), we shall assume a priori \( g \in \mathcal{E}_{2n-6, 2, 1} \oplus \mathcal{D}, f \in \mathcal{E}_{2n-4, 2, \varepsilon} \), with \( \| g \|_{\mathcal{E}_{2n-6, 2, 1} \oplus \mathcal{D}} \leq C \varepsilon, \| f \|_{\mathcal{E}_{2n-4, 2, \varepsilon}} \leq C \varepsilon^2 \). For notational simplicity, the norm of \( \mathcal{E}_{2n-6, 2, 1} \oplus \mathcal{D} \) will be denoted by \( \| \cdot \| \). In the case \( n = 3 \), we assume \( g \in \mathcal{E}_{2, 2, 1} \oplus \mathcal{D}, f \in \mathcal{E}_{4, 2, \varepsilon} \), with \( \| g \|_{\mathcal{E}_{2, 2, 1} \oplus \mathcal{D}} \leq C \varepsilon, \| f \|_{\mathcal{E}_{4, 2, \varepsilon}} \leq C \varepsilon^2 \), and in this case, the norm of \( \mathcal{E}_{2, 2, 1} \oplus \mathcal{D} \) will also be denoted by \( \| \cdot \| \).

With these choice of function spaces, we can verify that \( \| \Delta w \|_{2n-4, 2, \varepsilon} \leq C \varepsilon^2 \) if \( n \geq 4 \); while \( \| \Delta w \|_{4, 2, \varepsilon} \leq C \varepsilon^2 \) if \( n = 3 \).
Recall that the Jacobi operator on $C_1$ is given by

$$J_{C_1}(\eta) = \Delta_{C_1}\eta + |A|^2\eta.$$ 

Here $|A|^2 = \sum k_i^2$ is the squared norm of the second fundamental form. Using the asymptotic behavior of $\bar{\omega}$, we deduce $|A|^2 = O\left(\frac{1}{(1+l)^{2n-4}}\right)$ as $l \to +\infty$. We need the following lemma, which states that the Jacobi operator on the catenoid $C_1$ is invertible in suitable functional spaces.

**Lemma 28** For each function $\xi \in E_{2n-4,2,1}$, there is a solution $\eta \in E_{2n-6,2,1} \oplus D$ such that

$$J_{C_1}(\eta) = \xi,$$

with

$$\|\eta\| \leq C \|\xi\|_{2n-4,0,1}.$$

**Proof.** Detailed analysis of the Jacobi operator on the higher dimensional catenoid can be found in [1]. The proof of this Lemma follows from similar arguments there. The basic idea is using variation of parameter formula to get the desired estimates. 

With this functional framework at hand, we now deal with the corresponding linear theory for our nonlinear problem. Given functions $\gamma_1, \gamma_{-1}$, consider the problem

$$\begin{cases}
\partial_t^2 \phi + \partial_l^2 \phi = J_{C_0}g + \left(\Delta_{C_0}f + |A|^2\right)t, & \text{in } \Omega_h, \\
\phi = 0 & \text{on } \partial \Omega_h, \\
\partial_t \phi - f = \gamma_{-1}, & \text{on } \Gamma_{-1-h_{-1}}, \\
\partial_t \phi - f = \gamma_1, & \text{on } \Gamma_{1+h_1}.
\end{cases}$$

(39)

**Proposition 29** Suppose $\gamma_1 \pm \gamma_{-1}$ is in $E_{2n-4,1,\varepsilon}$ for $n \geq 4$ and in $E_{4,1,\varepsilon}$ for $n = 3$. Then the system (39) has a solution $(f, \tilde{g})$ such that

$$\|f\|_{2n-4,2,\varepsilon} \leq C \|\gamma_1 + \gamma_{-1}\|_{2n-4,1,\varepsilon} + C \|\gamma_1 + \gamma_{-1}\|_{4,1,\varepsilon} + C \|A|^2\|_{4,1,\varepsilon}, n \geq 4,$n= 3, \|f\|_{2n-4,2,\varepsilon} \leq C \|\gamma_1 + \gamma_{-1}\|_{4,1,\varepsilon} + C \|\gamma_1 + \gamma_{-1}\|_{4,1,\varepsilon} + C \|A|^2\|_{4,1,\varepsilon}, n = 3,$$

and

$$\|\tilde{g}\| \leq C \varepsilon^{-2} \|\gamma_1 - \gamma_{-1}\|_{2n-4,1,\varepsilon}, n \geq 4,$$

$$\|\tilde{g}\| \leq C \varepsilon^{-2} \|\gamma_1 - \gamma_{-1}\|_{4,1,\varepsilon}, n = 3.$$

**Proof.** By even reflection, we can regard (39) as a problem in $(t, l) \in [-1, 1] \times \mathbb{R}$. Take the Fourier transform

$$\hat{\eta}(t, \xi) := \int_{\mathbb{R}} e^{-2\pi i \xi l} \eta(t, l) dl.$$
It is worth mentioning that here $\xi \in \mathbb{R}$, unlike the Simons’ cone case where the Fourier transform is taken in $\mathbb{R}^4$. We are lead to the problem

$$
\begin{align*}
\partial_t^2 \hat{\phi} - |\xi|^2 \hat{\phi} &= (J_{c_*} g)^\ast + \left( \Delta_{c_*} f + |A|^2 \right)^\ast \hat{t}, \quad t \in [-1, 1], \\
\hat{\phi}(-1, \xi) &= \hat{\phi}(1, \xi) = 0, \\
\partial_t \hat{\phi}(-1, \xi) - \hat{f}(\xi) &= \hat{\gamma}_{-1}(\xi), \\
\partial_t \hat{\phi}(1, \xi) - \hat{f}(\xi) &= \hat{\gamma}_1(\xi).
\end{align*}
$$

(40)

The solution $\hat{\phi}$ of the first equation in (40) can be written in the form

$$
\hat{\phi}(t, \xi) = (J_{c_*} g)^\ast p_{1, \xi}(t) + \left( \Delta_{c_*} f + |A|^2 \right)^\ast p_{2, \xi}(t).
$$

This implies that

$$
\begin{align*}
(J_{c_*} g)^\ast &= \frac{\hat{\gamma}_1(\xi) - \hat{\gamma}_{-1}(\xi)}{2p_{1, \xi}(1)}, \\
\left( \Delta_{c_*} f + |A|^2 \right)^\ast &= \frac{2\hat{f}(\xi) + \hat{\gamma}_{-1}(\xi) + \hat{\gamma}_1(\xi)}{2p_{2, \xi}(1)}.
\end{align*}
$$

Observe that $\frac{1}{p_{1, \xi}(1)} - \xi \tanh \xi$ is real analytic and of the order $O\left(e^{-|\xi|^2}\right)$ as $|\xi| \to +\infty$. According to the proof of Lemma [17] one need to estimate the inverse Fourier transform of $\xi \tanh \xi \left[\hat{\gamma}_1(\xi) - \hat{\gamma}_{-1}(\xi)\right]$. To do this, we can apply the fact that the Fourier transform of $x \tanh(\pi x)$ is equal to $\frac{-\cosh(\pi \xi)}{2 \sinh(\pi \xi)}$, which has a singularity of order $O\left(\xi^{-2}\right)$ near the origin. The estimate of $\hat{f}$ is similar as before. 

Once we have established the functional framework and the linear solvability theory, we can proceed in the same way as the Simons’ cone case.

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