A NEW SOLUTION METHOD FOR NONLINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

SERTAN ALKAN
Department of Mathematics, Mustafa Kemal University
31000, Hatay, Turkey

Abstract. The aim of this paper is to obtain approximate solution of a class of nonlinear fractional Fredholm integro-differential equations by means of sinc-collocation method which is not used for solving them in the literature before. The fractional derivatives are defined in the Caputo sense often used in fractional calculus. The important feature of the present study is that obtained results are stated as two new theorems. The introduced method is tested on some nonlinear problems and it seems that the method is a very efficient and powerful tool to obtain numerical solutions of nonlinear fractional integro-differential equations.

1. Introduction. Fractional calculus is one of the most popular calculus types having a vast range of applications in many different area of scientific and engineering [18]. Fractional calculus and fractional differential equations are frequently used in the modeling of many sorts of scientific phenomena including image processing, earthquake engineering, physics, viscoelasticity, electrochemistry, diffusion processes, control theory, heat conduction, electricity, mechanics (see some references at the end, e.g., [2, 7, 15, 17]).

Since finding the exact or analytical solutions of fractional order differential equations is not an easy task, several different numerical solution techniques have been developed for the approximate solutions of these types of equations. Some well-known methods are summarized as follows, but not limited to: Adomian decomposition method [11, 12], Taylor expansion method [5], fractional differential transform method [14, 1] and homotopy perturbation method [19, 13].

Since sinc methods give a much better rate of convergence and more efficient results than classical polynomial methods in the presence of singularities [8], they are used for solving (systems of) the integer order integro-differential equations with Fredholm and Volterra type by several authors in the literature (see [9, 10, 21, 4, 22]).

The main purpose of the present paper is to obtain approximate solutions of the fractional order nonlinear Fredholm integro-differential equation with homogeneous boundary conditions, for \( m, s \in \mathbb{Z} \)

\[
y'' + p(x)y' + q(x)_a^C D_x^\alpha y + r(x)y^m = f(x) + \lambda \int_a^b K(x, t)y^s(t)dt, \quad 0 < \alpha < 1
\]

\( y(a) = 0, y(b) = 0 \)  \hspace{1cm} (1)

2010 Mathematics Subject Classification. Primary: 41A55, 41A30; Secondary: 65D15, 65D32.

Key words and phrases. Nonlinear fractional Fredholm integro-differential equation, sinc-collocation method, Caputo derivative, quadrature rule, mathematica.
where \( C_a^\alpha D_x \) is the Caputo fractional derivative operator, \( p(x), q(x), r(x) \) and \( f(x) \) are analytic functions in an open interval \((a, b)\) and may be singular in \( a \) or \( b \) or both. Also \( K(x, t) \) is analytic function, \( \lambda \) is a parameter, \( a \) and \( b \) are real constant.

The rest of this paper is organized as follows. In section 2, we have given some definition and theorems for fractional calculus and sinc-collocation method. In section 3, we present two new theorems and sinc-collocation method to obtain an approximate solution of a class of nonlinear fractional integro-differential equation. In section 4, some test problems are given to show the ability of present method by using tables and graphics. Finally, in section 5, we have completed the paper with a conclusion.

2. Fractional derivative and sinc functions. In this section, we recall notations and definitions of the sinc function and Caputo fractional derivative, also we derive useful formulas that are important for this paper.

**Definition 2.1.** See [[20], Definition 2.1]. Let \( f : [a, b] \to \mathbb{R} \) be a function, \( \alpha \) a positive real number, \( n \) the integer satisfying \( n - 1 \leq \alpha < n \), and \( \Gamma \) the Euler gamma function. Then, the left Caputo fractional derivative of order \( \alpha \) of \( f(x) \) is given as

\[
C_a^\alpha D_x f(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x (x - t)^{n-\alpha-1} f^{(n)}(t) dt. \tag{2}
\]

**Definition 2.2.** See [[20], Definition 2.3]. The Sinc function is defined on the whole real line \(-\infty < x < \infty\) by

\[
sinc(x) = \begin{cases} 
\frac{\sin(\pi x)}{\pi x} & x \neq 0 \\
1 & x = 0.
\end{cases}
\]

**Definition 2.3.** See [[16], Definition 2.1]. For \( h > 0 \) and \( k = 0, \pm 1, \pm 2, \ldots \) the translated sinc function with space node are given by:

\[
S(k, h)(x) = sinc\left(\frac{x - kh}{h}\right) = \begin{cases}
\frac{\sin(\pi x)}{\pi x} & x \neq kh \\
1 & x = kh.
\end{cases}
\]

**Definition 2.4.** See [[6], Definition 2.1]. If \( f(x) \) is defined on the real line, then for \( h > 0 \) the series

\[
C(f, h)(x) = \sum_{k=-\infty}^{\infty} f(kh) sinc\left(\frac{x - kh}{h}\right)
\]

is called the Whittaker cardinal expansion of \( f \) whenever this series converges.

In general, approximations can be constructed for infinite, semi-infinite and finite intervals. To construct approximation on the interval \((a, b)\) the conformal map

\[
\phi(z) = \ln\left(\frac{z - a}{b - z}\right) \tag{3}
\]

is employed. This map carries \( D_E \) the eye-shaped domain in the \( z \)-plane

\[
D_E = \left\{ z = x + iy : |\arg\left(\frac{z - a}{b - z}\right)| < d \leq \frac{\pi}{2} \right\}.
\]

onto the infinite strip \( D_S \)

\[
D_S = \left\{ w = u + iv : |v| < d \leq \frac{\pi}{2} \right\}.
\]
The basis functions on the interval \((a, b)\) are derived from the composite translated sinc functions

\[ S_k(z) = S(k, h)(z) = \phi(z) = \sin\left(\frac{\phi(z) - kh}{h}\right), \]

for \(z \in D_E\). The inverse map of \(w = \phi(z)\) is

\[ z = \phi^{-1}(w) = \frac{a + bern}{1 + e^{-w}}. \]

The sinc grid points \(z_k \in (a, b)\) in \(D_E\) will be denoted by \(x_k\) because they are real. For the evenly spaced nodes \(\{kh\}_{k=-\infty}^{\infty}\) on the real line, the image which corresponds to these nodes is denoted by

\[ x_k = \phi^{-1}(kh) = \frac{a + bern}{1 + e^{-kh}}, \quad k = 0, \pm 1, \pm 2, \ldots \]

**Definition 2.5.** See [[6], Definition 2.1]. An open set \(S \subseteq \mathbb{C}\) is called connected if it cannot be written as the union of two disjoint open sets \(A\) and \(B\) such that both \(A\) and \(B\) intersect \(S\). An open set \(S \subseteq \mathbb{C}\) is called simply connected if \(\mathbb{C} \setminus S\), where \(\mathbb{C}\) is the extended complex plane denoted \(\mathbb{C} \cup \{\infty}\), is connected.

**Definition 2.6.** See [[3], Definition 2.1]. Let \(D_E\) be a simply connected domain in the complex plane \(\mathbb{C}\), and let \(\partial D_E\) denote the boundary of \(D_E\). Let \(a, b\) be points on \(\partial D_E\) and \(\phi\) be a conformal map \(D_E\) onto \(D_S\) such that \(\phi(a) = -\infty\) and \(\phi(b) = \infty\). If the inverse map of \(\phi\) is denoted by \(\varphi\), define

\[ \Gamma = \{\phi^{-1}(u) \in D_E : -\infty < u < \infty\} \]

and \(z_k = \varphi(kh), k = 0, \pm 1, \pm 2, \ldots \)

**Definition 2.7.** See [[3], Definition 2.2]. Let \(B(D_E)\) be the class of functions \(F\) that are analytic in \(D_E\) and satisfy

\[ \int_{\psi(L+u)} |F(z)|dz \to \text{as} \quad u = \mp \infty, \]

where

\[ L = \{iy : |y| < d \leq \frac{\pi}{2}\}, \]

and those on the boundary of \(D_E\) satisfy

\[ T(F) = \int_{\partial D_E} |F(z)dz| < \infty. \]

**Theorem 2.8.** See [[3], Theorem 2.1]. Let \(\Gamma\) be \((0, 1), F \in B(D_E), \) then for \(h > 0\) small enough,

\[ \int_{\Gamma} F(z)dz - h \sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\phi'(z_j)} = \frac{i}{2} \int_{\partial D} \frac{F(z)k(\phi, h)(z)}{\sin(\pi\phi(z)/h)}dz \equiv I_F \quad (4) \]

where

\[ |k(\phi, h)|_{z \in \partial D} = \left| e^{\frac{\pi\phi(z)}{h} \text{sgn}(\text{Im}\phi(z))}\right|_{z \in \partial D} = e^{-\frac{\pi d}{h}}. \]

For the term of fractional in 1, the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.
Theorem 2.9. See [3, Theorem 2.2]. If there exist positive constants $\alpha, \beta$ and $C$ such that
\[
\begin{align*}
|F(x)| \leq C \left\{ e^{-\alpha|\phi(x)|} & \quad x \in \psi((-\infty, \infty)) \\
|\phi'(x)| & \quad x \in \psi((0, \infty))
\right. \tag{5}
\end{align*}
\]
then the error bound for the quadrature rule 4 is
\[
\begin{align*}
\left| \int_{\Gamma} F(x) \, dx - h \sum_{j=-M}^{N} \frac{F(x_j)}{\phi'(x_j)} \right| & \leq C \left( e^{-\alpha Mh} + e^{-\beta Nh} \right) + |I_F|. \tag{6}
\end{align*}
\]

The infinite sum in 4 is truncated with the use of 5 to arrive at the inequality 6. Making the selections
\[
h = \sqrt{\frac{\pi d}{\alpha M}} \\
N \equiv \left[ \frac{\alpha M}{\beta} + 1 \right]
\]
where $\left[ \ldots \right]$ is floor function that gives an integer part of the statement and $M$ is the integer value which specifies the grid size, then
\[
\int_{\Gamma} F(x) \, dx = h \sum_{j=-M}^{N} \frac{F(x_j)}{\phi'(x_j)} + O \left( e^{-(\pi \alpha d M)^{1/2}} \right). \tag{7}
\]

We used these theorems to approximate the kernel integral and the arising integral in the formulation of the term fractional in 1.

Lemma 2.10. See [20, Lemma 3.1]. Let $\phi$ be the conformal one-to-one mapping of the simply connected domain $D_E$ onto $D_S$, given by 3. Then
\[
\begin{align*}
\delta_{jk}^{(0)} &= [S(j, h) \circ \phi(x)]_{x=x_k} \left\{ \begin{array}{ll} 1 & j = k \\
0 & j \neq k \end{array} \right. \\
\delta_{jk}^{(1)} &= h \frac{d}{d\phi} [S(j, h) \circ \phi(x)]_{x=x_k} \left\{ \begin{array}{ll} 0 & j = k \\
(-1)^{k-j} \frac{k-j}{k-j} & j \neq k \end{array} \right. \\
\delta_{jk}^{(2)} &= h^2 \frac{d^2}{d\phi^2} [S(j, h) \circ \phi(x)]_{x=x_k} \left\{ \begin{array}{ll} -\frac{\pi^2}{2} & j = k \\
-2(-1)^{k-j} & j \neq k \end{array} \right.
\end{align*}
\]

3. The sinc-collocation method. We assume an approximate solution for $y(x)$ in problem 1 by the finite expansion of sinc basis functions
\[
y_n(x) = \sum_{k=-M}^{N} c_k S_k(x), \quad n = M + N + 1 \tag{8}
\]
where $S_k(x)$ is the function $S(k, h) \circ \phi(x)$. The unknown coefficients $c_k$ in 8 are determined by sinc-collocation method. For this purpose, the first and second derivatives of $y_n(x)$ are given by
\[
\begin{align*}
\frac{d}{dx} y_n(x) &= \sum_{k=-M}^{N} c_k \phi'(x) \frac{d}{d\phi} S_k(x) \tag{9} \\
\frac{d^2}{dx^2} y_n(x) &= \sum_{k=-M}^{N} c_k \left( \phi''(x) \frac{d}{d\phi} S_k(x) + (\phi')^2 \frac{d^2}{d\phi^2} S_k(x) \right) \tag{10}
\end{align*}
\]
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Similarly, \( \alpha \) order derivative of \( y_n(x) \) for \( 0 < \alpha < 1 \) is given by the following theorem.

**Theorem 3.1.** If \( \xi \) is a conformal map for the interval \([a, x]\), then \( \alpha \) order derivative of \( y_n(x) \) for \( 0 < \alpha < 1 \) is given by

\[
\frac{C}{a} D_x^\alpha(y_n(x)) \approx \sum_{k=-M}^{N} c_k R(x)
\]

where

\[
R(x) = \frac{h_L}{\Gamma(1-\alpha)} \sum_{r=-L}^{L} \frac{(x-x_r)S'_k(x_r)}{\xi'(x_r)}
\]

**Proof of Theorem 3.1.** If we use the definition of Caputo fractional derivative given in 2, it is written that

\[
\frac{C}{a} D_x^\alpha(y_n(x)) = \sum_{k=-M}^{N} c_k \frac{C}{a} D_x^\alpha(S_k(x))
\]

where

\[
\frac{C}{a} D_x^\alpha(S_k(x)) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{x} (x-t)^{-\alpha} S'_k(t) dt
\]

Now we use quadrature rule given in 7 to compute the above integral which is divergent on the interval \([a, x]\). For this purpose, a conformal map and its inverse image that denotes the sinc grid points are given by

\[
\xi(t) = \ln \left( \frac{t-a}{x-t} \right)
\]

and

\[
x_r = \xi^{-1}(rh_L) = \frac{a + xe^{rh_L}}{1 + e^{rh_L}}
\]

where \( h_L = \pi/\sqrt{L} \). Then, according to equality 7, we write

\[
\frac{C}{a} D_x^\alpha(S_k(x)) \approx \frac{h_L}{\Gamma(1-\alpha)} \sum_{r=-L}^{L} \frac{(x-x_r)S'_k(x_r)}{\xi'(x_r)}
\]

This completes the proof.

Application of 7 to the kernel integral in 1 gives the following lemma.

**Lemma 3.2.** See [9], Lemma 3.1] The following relation holds

\[
\int_{a}^{b} K(x, t)y^s(t) dt \approx h \sum_{k=-M}^{N} \frac{K(x, t_k)}{\phi'(t_k)} y^s_k
\]

where \( y_k \) denotes an approximate value of \( y(t_k) \).

Replacing each term of 1 with the approximation given in 8, 9, 10, 12 and 13 multiplying the resulting equation by \( ((1/\phi')^2)^2 \) and setting \( x = x_j \), we obtain the following nonlinear system

\[
\sum_{k=-M}^{N} c_k \left\{ \frac{d^2}{d\phi^2} S_k + \left[ p \left( \frac{1}{\phi'} \right) - \left( \frac{1}{\phi'} \right)' \right] \frac{d}{d\phi} S_k + q \left( \frac{1}{\phi'} \right)^2 R \right\}(x_j)
\]
+ c_j^n \left( r \left( \frac{1}{\phi'} \right)^2 \right)(x_j) - \lambda h \sum_{k=-M}^N c_k^s \frac{K(x_j, t_k)}{\phi'(t_k)} \left( \frac{1}{\phi'(x_j)} \right)^2 \\
= \left( f \left( \frac{1}{\phi'} \right)^2 \right)(x_j), \quad j = -M, \ldots, N

By using Lemma 2.10, we know that
\delta^{(0)}_{jk} = \delta^{(0)}_{kj}, \quad \delta^{(1)}_{jk} = -\delta^{(1)}_{kj}, \quad \delta^{(2)}_{jk} = \delta^{(2)}_{kj}
then we obtain the following theorem.

**Theorem 3.3.** If the assumed approximate solution of boundary value problem 1 is \( s \), then the discrete sinc-collocation system for the determination of the unknown coefficients \( \{c_k\}_{k=-M}^N \) is given by

\[
\sum_{k=-M}^N c_k \left\{ \frac{1}{h^2} \delta_{jk}^{(2)} + \frac{1}{h} \left( \frac{1}{\phi'} \right) \left( \frac{1}{\phi'} \right)' - p \left( \frac{1}{\phi'} \right) \right\} (x_j) \delta_{jk}^{(1)} + \left( q \left( \frac{1}{\phi'} \right)^2 R \right)(x_j) \right\} \\
+ c_j^n \left( r \left( \frac{1}{\phi'} \right)^2 \right)(x_j) - \lambda h \sum_{k=-M}^N c_k^s \frac{K(x_j, t_k)}{\phi'(t_k)} \left( \frac{1}{\phi'(x_j)} \right)^2 \\
= \left( f \left( \frac{1}{\phi'} \right)^2 \right)(x_j), \quad j = -M, \ldots, N
\]

Now we define some notations to represent in the matrix-vector form for system 14. Let \( D(y) \) denotes a diagonal matrix whose diagonal elements are \( y(x_{-M}), y(x_{-M+1}), \ldots, y(x_N) \) and non-diagonal elements are zero, let \( G = R(x_j) \) and

\[
E = \frac{K(x_j, t_k)}{(\phi'(x_j))^2 \phi'(t_k)}
\]
denote a matrix and also let \( I^{(i)} \) denotes the matrices

\[
I^{(i)} = [\delta_{jk}^{(i)}], \quad i = 0, 1, 2
\]

where \( D, G, E, I^{(0)}, I^{(1)} \) and \( I^{(2)} \) are square matrices of order \( n \). Particularly, \( I^{(0)}, I^{(1)} \) and \( I^{(2)} \) are the identity matrix, the skew-symmetric matrix and the symmetric matrix, respectively. In order to calculate unknown coefficients \( c_k \) in nonlinear system 14, we rewrite this system by using the above notations in matrix-vector form as

\[
A_1 C + A_2 C^n + A_3 C^s = B
\]

where

\[
A_1 = \frac{1}{h^2} I^{(2)} + \frac{1}{h} D \left( \left( \frac{1}{\phi'} \right)' \right) I^{(1)} + D \left( q \left( \frac{1}{\phi'} \right)^2 \right) G \\
A_2 = D \left( r \left( \frac{1}{\phi'} \right)^2 \right) I^{(0)}
\]
Table 1. Numerical results for Example 1 when $L = 5, M = 5$

| $x$ | Exact Sol. | Approx. Sol. | Error |
|-----|------------|--------------|-------|
| 0   | 0          | 0            | 0     |
| 0.1 | 0.26182    | 0.27037      | $8.54 \times 10^{-3}$ |
| 0.2 | 0.50365    | 0.49528      | $8.36 \times 10^{-3}$ |
| 0.3 | 0.72548    | 0.70814      | $1.73 \times 10^{-2}$ |
| 0.4 | 0.92731    | 0.91201      | $1.52 \times 10^{-2}$ |
| 0.5 | 1.10914    | 1.10067      | $8.47 \times 10^{-3}$ |
| 0.6 | 1.27097    | 1.27018      | $7.89 \times 10^{-4}$ |
| 0.7 | 1.41280    | 1.41870      | $5.90 \times 10^{-3}$ |
| 0.8 | 1.53463    | 1.54557      | $1.09 \times 10^{-2}$ |
| 0.9 | 1.63645    | 1.65073      | $1.42 \times 10^{-2}$ |
| 1   | 0          | 0            | 0     |

$$A_3 = -h\lambda E$$

$$B = \left( \left( \frac{1}{\phi'} \right)^2(x - M), \left( \frac{1}{\phi'} \right)^2(x - M + 1), \ldots, \left( \frac{1}{\phi'} \right)^2(x_N) \right)^T$$

$$C = (c_{-M}, c_{-M + 1}, \ldots, c_N)^T$$

$$C^m = (c_{-M}^m, c_{-M + 1}^m, \ldots, c_N^m)^T$$

$$C^s = (c_{-M}^s, c_{-M + 1}^s, \ldots, c_N^s)^T$$

Now we have nonlinear system of $n$ equations in the $n$ unknown coefficients given by 15. When it is solved by Newton’s method, we can obtain the unknown coefficients $c_k$ that are necessary for approximate solution in 8.

4. Computational examples. In this section, three problems that have homogeneous boundary conditions will be tested by using the present method via *Mathematica* 10 on a personal computer. In all the examples, we take $d = \pi/2$, $\alpha = \beta = 1/2$, $N = M$.

**Example 1.** Consider nonlinear fractional Fredholm integro-differential equation in the following form

$$y''(x) - C D_0^{0.5} y(x) + y^3(x) = f(x) - 2 \int_0^1 K(x, t)y^3(t)dt$$

subject to the homogeneous boundary conditions

$$y(0) = 0, \quad y(1) = 0$$

where $f(x) = x^2(e-x)^2 + \frac{\psi^9}{1.5^4}(x-1) + \frac{\pi}{1.5^2}x^{1.5} - \frac{\pi}{1.5^2}x^{0.5} - 2$ and $K(x, t) = t^2(x-1)$. The exact solution of this problem is $y(x) = x(e-x)$. The numerical solutions which are obtained by using the present method for this problem are presented in Table 1 and Table 2. Additionally, the graphics of the exact and approximate solutions for different values of $L$ and $M$ are given in Figure 1 and Figure 2.

**Example 2.** Consider the following nonlinear fractional Fredholm integro-differential equation

$$y''(x) + \frac{1}{x}\frac{C}{x} D_0^{0.3} y(x) + \frac{1}{x - 1} y^3(x) = f(x) + \int_0^1 K(x, t)y^2(t)dt$$
Table 2. Numerical results for Example 1 when $L = 30, M = 50$

| $x$ | Exact Sol. | Approx. Sol. | Error |
|-----|------------|--------------|-------|
| 0   | 0          | 0            | 0     |
| 0.1 | 0.26182    | 0.26185      | $2.89 \times 10^{-5}$ |
| 0.2 | 0.50365    | 0.50371      | $5.58 \times 10^{-5}$ |
| 0.3 | 0.72548    | 0.72556      | $8.03 \times 10^{-5}$ |
| 0.4 | 0.92731    | 0.92741      | $1.01 \times 10^{-4}$ |
| 0.5 | 1.10914    | 1.10926      | $1.20 \times 10^{-4}$ |
| 0.6 | 1.27097    | 1.27110      | $1.34 \times 10^{-4}$ |
| 0.7 | 1.41280    | 1.41294      | $1.43 \times 10^{-4}$ |
| 0.8 | 1.53463    | 1.53477      | $1.48 \times 10^{-4}$ |
| 0.9 | 1.63645    | 1.63660      | $1.47 \times 10^{-4}$ |
| 1   | 0          | 0            | 0     |

Figure 1. The graphics of the exact and approximate solutions for Example 1 when $L = 5, M = 5$

Figure 2. The graphics of the exact and approximate solutions for Example 1 when $L = 30, M = 50$
subject to the homogeneous boundary conditions

\[ y(0) = 0, \quad y(1) = 0 \]

where \( f(x) = \frac{1}{x^2} (x^{12} - 3x^{11} + 3x^{10} - x^9) + \frac{24}{\Gamma(4.7)} x^{2.7} + 12x^2 - \frac{6}{\Gamma(3.7)} x^{1.7} + \frac{1513}{252} x + \frac{1}{300} \)

and \( K(x,t) = x - t \). The exact solution of this problem is \( y(x) = x^3(x-1) \). The numerical solutions which are obtained by using the present method for this problem are presented in Table 3 and Table 4. Additionally, the graphics of the exact and approximate solutions for different values of \( L \) and \( M \) are given in Figure 3 and Figure 4.

**Example 3.** Consider nonlinear fractional Fredholm integro-differential equation in the following form

\[ y''(x) - x^{2a} D_x^{0.7} y(x) + x y^2(x) = f(x) - \int_0^1 K(x,t)y^2(t)dt \]

subject to the homogeneous boundary conditions

\[ y(0) = 0, \quad y(1) = 0 \]
where $f(x) = x^9 - 2x^7 + x^5 - 12x^2 + \frac{16x}{315} + \frac{1378}{693} + \frac{24}{\Gamma(4.3)}x^{5.3} - \frac{2}{\Gamma(2.3)}x^{3.3}$ and $K(x, t) = 2x - t^2$. The exact solution of this problem is $y(x) = x^2(1 - x^2)$. The numerical solutions which are obtained by using the present method for this problem are presented in Table 5 and Table 6. Additionally, the graphics of the exact and approximate solutions for different values of $L$ and $M$ are given in Figure 5 and Figure 6.

5. Conclusion. In this paper, sinc-collocation method is used to obtain approximate solution of a general nonlinear fractional integro differential equation with boundary conditions. In order to illustrate the accuracy and effective of the method, it is applied to some examples and obtained results are compared with the exact ones. The comparisons in table and graphical forms show that the approximate solutions converge the exact ones when it is increased that the number of sinc grid points $N$ and the present method is a powerful tool for solving nonlinear fractional integro-differential equations with boundary conditions.
Table 5. Numerical results for Example 3 when $L = 5, M = 5$

| $x$ | Exact Sol. | Approx. Sol. | Error       |
|-----|------------|--------------|-------------|
| 0   | 0          | 0            | 0           |
| 0.1 | 0.0099     | 0.007064     | $2.83 \times 10^{-3}$ |
| 0.2 | 0.0384     | 0.039409     | $1.00 \times 10^{-3}$ |
| 0.3 | 0.0819     | 0.087675     | $5.77 \times 10^{-3}$ |
| 0.4 | 0.1344     | 0.138711     | $4.31 \times 10^{-3}$ |
| 0.5 | 0.1875     | 0.185335     | $2.16 \times 10^{-3}$ |
| 0.6 | 0.2304     | 0.221648     | $8.75 \times 10^{-3}$ |
| 0.7 | 0.2499     | 0.240049     | $9.85 \times 10^{-3}$ |
| 0.8 | 0.2304     | 0.227344     | $3.05 \times 10^{-3}$ |
| 0.9 | 0.1539     | 0.156566     | $2.66 \times 10^{-3}$ |
| 1   | 0          | 0            | 0           |

Table 6. Numerical results for Example 3 when $L = 30, M = 50$

| $x$ | Exact Sol. | Approx. Sol. | Error       |
|-----|------------|--------------|-------------|
| 0   | 0          | 0            | 0           |
| 0.1 | 0.0099     | 0.009900     | $7.91 \times 10^{-7}$ |
| 0.2 | 0.0384     | 0.038401     | $1.46 \times 10^{-6}$ |
| 0.3 | 0.0819     | 0.081901     | $1.51 \times 10^{-6}$ |
| 0.4 | 0.1344     | 0.134399     | $2.89 \times 10^{-7}$ |
| 0.5 | 0.1875     | 0.187494     | $5.99 \times 10^{-6}$ |
| 0.6 | 0.2304     | 0.230382     | $1.75 \times 10^{-5}$ |
| 0.7 | 0.2499     | 0.249865     | $3.49 \times 10^{-5}$ |
| 0.8 | 0.2304     | 0.230347     | $5.25 \times 10^{-5}$ |
| 0.9 | 0.1539     | 0.153847     | $5.33 \times 10^{-5}$ |
| 1   | 0          | 0            | 0           |

Figure 5. The graphics of the exact and approximate solutions for Example 3 when $L = 5, M = 5$
Figure 6. The graphics of the exact and approximate solutions for Example 3 when $L = 30, M = 50$

Acknowledgments. The authors are very grateful to the anonymous referees for their careful reading of this paper.

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Received April 2015; revised August 2015.

*E-mail address: sertanalkan@mku.edu.tr*