INPUT/OUTPUT-TO-STATE STABILITY OF SWITCHED SYSTEMS UNDER RESTRICTED SWITCHING

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ABSTRACT. This paper deals with input/output-to-state stability (IOSS) of continuous-time switched nonlinear systems. Given a family of systems, possibly containing unstable dynamics, and a set of restrictions on admissible switches between the subsystems and admissible dwell times on the subsystems, we identify a class of switching signals that obeys these restrictions and preserves stability of the resulting switched system. The primary apparatus for our analysis is multiple Lyapunov-like functions. Input-to-state stability (ISS) and global asymptotic stability (GAS) of switched systems under pre-specified restrictions on switching signals fall as special cases of our results when no outputs (resp., also inputs) are considered.

1. Introduction

A switched system has two ingredients — a family of systems and a switching signal. The switching signal selects an active subsystem at every instant of time, i.e., the system from the family that is currently being followed [14, Section 1.1.2]. Switched systems find wide applications in power systems and power electronics, automotive control, aircraft and air traffic control, network and congestion control, etc. [4, p. 5].

A vast body of the existing literature focusses on identifying classes of switching signals that ensure stability of switched systems, see e.g., [5, 21, 20, 17, 11, 13, 12]. These classes of switching signals are typically characterized in terms of point-wise or asymptotic behaviour of the signals such as the number of switches between the subsystems, the dwell times on the subsystems, etc., see e.g., [13, 12] for a survey of and connections between the available results. However, in practice, the admissible transitions between the subsystems and the allowable duration of activation of the subsystems (and hence the number of switches on various intervals of time) are often governed by system specifications and, therefore, restricted. A distinction between admissible and inadmissible transitions captures situations where switches between certain subsystems may be prohibited. For instance, in automobile gear switching, certain switching orders (e.g., from first gear to second gear, etc.) are followed [15, Section III]. A restriction on minimum dwell time on subsystems arises in situations where actuator saturations may prevent switching frequency beyond a certain limit. Also, in order to switch from one component to another, a system may undergo certain operations of non-negligible durations [6, Section I] resulting in a minimum dwell time restriction on subsystems. Restricted maximum dwell time is natural to systems whose components need regular maintenance or replacements, e.g., aircraft carriers, MEMS systems, etc. Moreover, systems dependent on diurnal or seasonal changes, e.g., components of an electricity grid, have inherent restrictions on admissible dwell time [8, Remark 1]. It is, therefore, of interest to study stability of switched systems under classes of switching signals that obey pre-specified restrictions.

In this paper our focus is on input/output-to-state stability (IOSS) of continuous-time switched nonlinear systems under restricted switching. Loosely speaking, IOSS implies
that if the exogenous inputs and the observed outputs are small, then irrespective of the initial value, the state of the switched system eventually becomes small. This property is useful in constructing state-norm estimators for switched systems, see [17] for a detailed discussion.

While IOSS of continuous-time switched nonlinear systems was studied under arbitrary switching in [16] and average dwell time switching with constrained point-wise activation of unstable subsystems in [17], to the best of our knowledge, the setting of pre-specified restrictions on switching signals has been considered only in [8, 9, 10] so far. In [8] we presented an algorithm to design a switching signal that obeys a set of given restrictions on admissible switches between the subsystems and admissible dwell times on the subsystems and preserves IOSS of the resulting switched system. The said design involves finding a class of cycles on a weighted directed graph representation of a switched system that satisfies certain conditions involving multiple Lyapunov-like functions [2] (one corresponding to each subsystem) and the admissible dwell times. This design mechanism was further extended to accommodate small perturbations in dwell times on the subsystems in [9]. In [10] we focused on stability under a class of switching signals that obeys given restrictions on admissible switches between the subsystems and admissible dwell times on the subsystems. We considered common admissible minimum and maximum dwell times on all the subsystems. We showed that if (a) it is allowed to switch from every unstable subsystem to a stable subsystem, and (b) a set of scalars computed from Lyapunov-like functions corresponding to the individual subsystems together with an admissible choice of dwell times on the subsystems satisfy a certain inequality, then every switching signal that dwells on the subsystems for the above admissible durations of time and does not activate unstable subsystems in any two consecutive switching instants, preserves IOSS of the switched system.

In this paper we present a new class of switching signals that obeys pre-specified restrictions on admissible switches between the subsystems and (possibly different) admissible dwell times on the subsystems and preserves IOSS of the resulting switched system. While the minimum and maximum number of switches on any interval of time and the set of switching destinations are governed by the pre-specified restrictions, there is an element of choice associated to the total number of switches where a specific subsystem is activated and/or where a switch from a specific subsystem to another specific subsystem has occurred. We show that if a certain inequality involving (a) a set of scalars in a given range, (b) a set of scalars computed from Lyapunov-like functions corresponding to the individual subsystems, and the admissible transitions between them, and (c) the set of admissible minimum and maximum dwell times on the subsystems, is satisfied, then every switching signal that obeys the given restrictions with the frequency of activation of the subsystems and the frequency of switches between the pairs of subsystems governed by a function of the set of scalars mentioned in (a), preserves IOSS of a switched system. In the absence of outputs (resp., also inputs), our class of switching signals ensures input-to-state stability (ISS) (resp., Global Asymptotic Stability (GAS)) of a switched system. The contents of this paper differ from our earlier works on IOSS of switched systems under restricted switching signals both in terms of generality of the problem statement and the results, see Remark 5 for a detailed discussion.

The remainder of this paper is organized as follows: In Section 2 we formulate the problem under consideration. In Section 3 we catalog a set of preliminaries for our results. Our results appear in Section 4. We also discuss various features of our results and provide examples in this section. We conclude in Section 5 with a brief discussion of future research directions. A proof of our main result is presented in Section 6.

Notation. \( \mathbb{R} \) is the set of real numbers, \( \| \cdot \| \) is the Euclidean norm, and for any interval \( I \subseteq [0, +\infty[ \), \( \| \cdot \|_I \) is the essential supremum norm of a map from \( I \) into some Euclidean
space. For a real number $a$, we denote by $\lceil a \rceil$ the biggest integer that is smaller than or equal to $a$.

2. Problem statement

We consider a family of continuous-time systems

\begin{align}
\dot{x}(t) &= f_p(x(t), u(t)), \\
y(t) &= h_p(x(t)),
\end{align}

(1)

where $x(t) \in \mathbb{R}^d$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the vectors of states, inputs and outputs at time $t$, respectively, and $\mathcal{P} = \{1, 2, \ldots, N\}$ is an index set. Let $\sigma : [0, +\infty] \rightarrow \mathcal{P}$ be a switching signal — it is a piecewise constant function that selects at each time $t$, the index of the active subsystem, i.e., the system from the family (1) that is currently being followed. By convention, $\sigma$ is assumed to be continuous from right and having limits from the left everywhere. Let $\mathcal{S}$ denote the set of all switching signals. The switched system generated by a family of systems (1) and a switching signal $\sigma$ is given by

\begin{align}
\dot{x}(t) &= f_{\sigma(t)}(x(t), u(t)), \\
y(t) &= h_{\sigma(t)}(x(t)),
\end{align}

(2)

We assume that for each $p \in \mathcal{P}$, $f_p$ is locally Lipschitz, $f_p(0, 0) = 0$ and $h_p$ is continuous, $h_p(0) = 0$; the exogenous inputs are Lebesgue measurable and essentially bounded. Thus, a solution to the switched system (2) exists in Carathéodory sense for some non-trivial time interval containing $0$ [14, Chapter 2].

In this paper we focus on IOSS of the switched system (2).

**Definition 1.** [14] Appendix A.6] The switched system (2) is input/output-to-state stable (IOSS) for a given switching signal $\sigma \in \mathcal{S}$ if there exist class $\mathcal{K}_\infty$ functions $\alpha$, $\chi_1$, $\chi_2$ and a class $\mathcal{K}_\mathcal{L}$ function $\beta$ such that for all inputs $u$ and initial states $x_0$, we have

\begin{align}
\alpha(||x(t)||) &\leq \beta(||x_0||, t) + \chi_1(||u||_{0:t}) + \chi_2(||y||_{0:t})
\end{align}

(3)

for all $t \geq 0$. If $\chi_2 \equiv 0$, then (3) reduces to input-to-state stability (ISS) of (2) [14, Appendix A.6], and if also $u \equiv 0$, then (3) reduces to global asymptotic stability (GAS) of (2) [14 Appendix A.1].

Let $\mathcal{P}_S$ and $\mathcal{P}_U$ denote the sets of indices of IOSS and non-IOSS subsystems, respectively, $\mathcal{P} = \mathcal{P}_S \cup \mathcal{P}_U$. Let $E(\mathcal{P})$ denote the set all pairs $(p, q)$ such that it is allowed to switch from subsystem $p$ to subsystem $q$, $p, q \in \mathcal{P}$, $p \neq q$. We let $0 = t_0 < t_1 < \cdots$ be the switching instants; these are the points in time where $\sigma$ jumps. Let $\delta_p$ and $\Delta_p$ denote the admissible minimum and maximum dwell time on subsystem $p \in \mathcal{P}$, respectively.

**Definition 2.** A switching signal $\sigma \in \mathcal{S}$ is called admissible if it obeys

\begin{align}
(\sigma(t), \sigma(t+1)) &\in E(\mathcal{P}), \quad i = 0, 1, \ldots,
\end{align}

(4)

and

\begin{align}
t_{i+1} - t_i &\in [\delta_p(t), \Delta_p(t)], \quad i = 0, 1, 2, \ldots
\end{align}

(5)

$^1$\(\mathcal{K} := \{\phi : [0, +\infty] \rightarrow [0, +\infty] \mid \phi \text{ is continuous, strictly increasing, } \phi(0) = 0\}, \mathcal{KL} := \{\phi : [0, +\infty]^2 \rightarrow [0, +\infty] \mid \phi(s, t) \in \mathcal{K} \text{ for each } s \text{ and } \phi(t, r) \downarrow 0 \text{ as } r \rightarrow +\infty \text{ for each } r\}, \mathcal{KL} := \{\phi \in \mathcal{KL} \mid \phi(t) \rightarrow +\infty \text{ as } t \rightarrow +\infty\}.\)

$^2$Throughout this paper we will assume that the set of admissible switches is such that for every subsystem $p \in \mathcal{P}$, there is at least one subsystem $q \in \mathcal{P}$ such that $(p, q) \in E(\mathcal{P})$. Notice that if there exists a subsystem $p \in \mathcal{P}$ such that there is no subsystem $q \in \mathcal{P}$ with $(p, q) \in E(\mathcal{P})$, then a switching signal, $\sigma$, that activates the subsystem $p$ at time $t'$, is undefined beyond time $t' + \Delta$.\)
Let $S_R$ denote the set of all admissible switching signals $\sigma$. We will solve the following problem:

**Problem 1.** Given a family of systems (1), the set of admissible switches between the subsystems, $E(P)$, and the admissible minimum and maximum dwell times, $\delta_p$ and $\Delta_p$, on the subsystems $p \in P$, identify a class of admissible switching signals $\tilde{S}_R \subseteq S_R$ that ensures IOSS of the switched system (2).

In the sequel we will often refer to the elements of the set $\tilde{S}_R$ as stabilizing switching signals. We note the following:

- Even though we are allowing IOSS dynamics in the family (1), the existence of even one stabilizing switching signal $\sigma \in S_R$ is not guaranteed. Indeed, the trivial choice $\sigma(t) = p$ for all $t \geq 0$ with a fixed $p \in P$ is excluded by the maximum dwell time restriction on all subsystems. Beyond this choice, it is well-known that even stability of all the subsystems is not sufficient for stability of a switched system. Our aim is to identify, if exists, a subset of $S_R$ whose elements are stabilizing.
- Our setting is different from the classical problem of IOSS of the switched system (2) under (average) dwell time switching signals. In case of the latter, minimum dwell times on the stable subsystems and maximum dwell times on the unstable subsystems such that a switched system generated with a given family of (stable and unstable) systems is stable, are identified. More precisely, a set of elements of $S$ that preserve IOSS of the switched system (2) are identified. In contrast, in Problem 1 we consider the admissible switches between the subsystems and admissible minimum and maximum dwell times on the subsystems to be “given” and aim to identify a subset of $S_R \subseteq S$ whose elements are stabilizing. Further, there always exists a non-empty subset of $S$ whose elements are stabilizing. Indeed, consider the set of all switching signals $\sigma \in S$ that satisfies $\sigma(t) = p$ for all $t \geq 0$ with a fixed $p \in P$. In contrast, the existence of a non-empty subset of $S_R$ whose elements are stabilizing, is not guaranteed.
- Since the estimates of (average) dwell time on the subsystems (more specifically, the minimum dwell time on the stable subsystems and the maximum dwell time on the unstable subsystems) for IOSS of the switched system (2) is known in the literature, see e.g., [17], an intuitive way to solve Problem 1 is to compute those estimates and match them with the given admissible dwell times. However, recall that the classical estimates of (average) dwell time on the subsystems are only sufficient for stability of a switched system and do not imply instability under a set of dwell times different from the estimated ones. It is, therefore, of interest to identify new classes of switching signals that obey the given restrictions and preserve stability of a switched system beyond the above procedure. More specifically, instead of identifying a subset $S'$ of $S$ whose elements are stabilizing and checking if the intersection of the sets $S'$ and $S_R$ is non-empty, we opt to restrict our attention to $S_R$ and look for its subsets that cater to our needs.

## 3. Preliminaries

### 3.1. The family of systems (1)

We are interested in the temporal behaviour of Lyapunov-like functions for the systems in family (1) along the corresponding system trajectories.

**Assumption 1.** There exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}, \gamma_1, \gamma_2$ continuously differentiable functions $V_p : \mathbb{R}^d \rightarrow [0, +\infty[$, $p \in P$, and constants $\mathbb{R} \ni \lambda_p > 0$, $p \in P_S$ and $\mathbb{R} \ni \lambda_p < 0$, $p \in P_U$, such that for all $\xi \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^n$, the following hold:

\begin{equation}
\underline{\alpha}(\|\xi\|) \leq V_p(\xi) \leq \overline{\alpha}(\|\xi\|), \quad p \in P,
\end{equation}

and

\begin{equation}
\frac{\partial V_p}{\partial \xi} f_p(\xi, \eta) \leq -\lambda_p V_p(\xi) + \gamma_1(\|\eta\|) + \gamma_2\left(\|h_p(\xi)\|\right), \quad p \in P.
\end{equation}
Assumption 2. There exist \( \mu_{pq} > 0 \), \((p, q) \in E(\mathcal{P})\) such that the IOSS-Lyapunov-like functions are related as follows:

\[
V_q(\xi) \leq \mu_{pq}V_p(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^d.
\]

The functions \( V_p, p \in \mathcal{P} \), are called the (multiple) IOSS-Lyapunov-like functions. Condition (7) is equivalent to the IOSS property for IOSS subsystems \([7, 18]\) and the unboundedness observability property for the non-IOSS subsystems \([1, 19]\). The scalars \( \lambda_p, p \in \mathcal{P}_S \) (resp., \( \lambda_p, p \in \mathcal{P}_U \)) provide a quantitative measure of stability (resp., instability) of the subsystems \( p \in \mathcal{P} \). Condition (5) restricts the class of Lyapunov-like functions to be linearly comparable.

Notice that given a family of systems \([1]\), the choice of IOSS-Lyapunov-like functions, \( V_p \), corresponding to the subsystems, \( p \in \mathcal{P} \), and consequently, the scalars, \( \lambda_p, p \in \mathcal{P} \) and \( \mu_{pq}, (p, q) \in E(\mathcal{P}) \) is not unique. Let us consider that the functions \( V_p, p \in \mathcal{P} \) and their corresponding scalars \( \lambda_p, p \in \mathcal{P} \) and \( \mu_{pq}, (p, q) \in E(\mathcal{P}) \) are “given”. We define \( E_-(\mathcal{P}) = \{(p, q) \in E(\mathcal{P}) | \ln \mu_{pq} < 0\} \) and \( E_+(\mathcal{P}) = \{(p, q) \in E(\mathcal{P}) | \ln \mu_{pq} > 0\} \).

3.2. Switching signals. We employ the following notations to express various parameters related to a switching signal \( \sigma \). Fix an interval \([s, t] \subseteq [0, +\infty[\) of time.

- Let \( N(s, t) \) denote the total number of switches on \([s, t]\).
- Let \( N_p(s, t) \) denote the total number of switches on \([s, t]\) where the subsystem \( p \) is activated.
- Let \( N_{pq}(s, t) \) denote the total number of switches on \([s, t]\) where a switch from subsystem \( p \) to subsystem \( q \) has occurred.
- Let \( T_p(s, t) \) denote the total duration of activation of the subsystem \( p \) on \([s, t]\).

We suppress the dependence of the above quantities on \( \sigma \) for notational simplicity. Further,

- the quantity \( \frac{N_{pq}(s, t)}{N(s, t)} \) gives the frequency of activation of subsystem \( p \in \mathcal{P} \) on \([s, t]\), and
- the quantity \( \frac{N_{pq}(s, t)}{N(s, t)} \) gives the frequency of switches from subsystem \( p \) to subsystem \( q \) on \([s, t]\), \((p, q) \in E(\mathcal{P})\).

We are now in a position to present our solution to Problem 1.

4. Main results

We assume that

Assumption 3. There exist \( \rho_p^S \in [0, 1], p \in \mathcal{P}_S, \rho_p^U \in [0, 1], p \in \mathcal{P}_U \) and \( \rho_{pq}^+ \in [0, 1], (p, q) \in E_+(\mathcal{P}), \rho_{pq}^- \in [0, 1], (p, q) \in E_-(\mathcal{P}) \) that satisfy

\[
-\frac{1}{\Delta_{\max}} \left( \sum_{p \in \mathcal{P}_S} |\lambda_p^S| \rho_p^S \delta_p + \sum_{(p, q) \in E_-(\mathcal{P})} |\ln \mu_{pq}| \rho_{pq}^- \right)
+\frac{1}{\Delta_{\min}} \left( \sum_{p \in \mathcal{P}_U} |\lambda_p^U| \rho_p^U \Delta_p + \sum_{(p, q) \in E_+(\mathcal{P})} |\ln \mu_{pq}| \rho_{pq}^+ \right) < 0,
\]

where \( \delta_p = \min_{p \in \mathcal{P}} \delta_p, \Delta_{\max} = \max_{p \in \mathcal{P}} \Delta_p \), and the scalars \( \lambda_p, p \in \mathcal{P}, \mu_{pq}, (p, q) \in E(\mathcal{P}) \) are as described in Section 3.

Assumption 3 requires the existence of four sets of scalars: (i) \( \lambda_p^S, p \in \mathcal{P}_S \), (ii) \( \lambda_p^U, p \in \mathcal{P}_U \), (iii) \( \lambda_{pq}, (p, q) \in E_+(\mathcal{P}) \) and (iv) \( \lambda_{pq}^+, (p, q) \in E_+(\mathcal{P}) \) that together with (a) the scalars \( \mu_{pq}, (p, q) \in E(\mathcal{P}) \) obtained from the IOSS-Lyapunov-like functions of the individual subsystems \( p \in \mathcal{P} \) and (b) the admissible minimum and maximum dwell
times, \( \delta_p \) and \( \Delta_p \), on the subsystems \( p \in \mathcal{P} \), respectively, satisfy the inequality (9). We will use the scalars \( \rho_p^U, p \in \mathcal{P}_S, \rho_p^U, p \in \mathcal{P}_U, \rho_{pq}^U, (p, q) \in E_-(\mathcal{P}) \) and \( \rho_{pq}^U, (p, q) \in E_+\mathcal{P} \) to bound the frequency of activation of the stable subsystems, the frequency of activation of the unstable subsystems and the frequency of switches between the pairs of subsystems, respectively.

**Definition 3.** Let \( S_R \) be the set of all switching signals, \( \sigma \in \mathcal{S} \), that obey the following properties on every interval \([s, t] \subseteq [0, +\infty)\) of time:

\[
\begin{align*}
(10) & \quad \frac{t - s}{\Delta_{\text{max}}} \leq N_p(s, t) \leq \frac{t - s}{\delta_{\text{min}}}, \\
(11) & \quad N_p(s, t) \geq \left\lfloor \rho_p^S N(s, t) \right\rfloor, \quad p \in \mathcal{P}_S, \\
(12) & \quad N_p(s, t) \leq \left\lceil \rho_p^U N(s, t) \right\rceil, \quad p \in \mathcal{P}_U, \\
(13) & \quad N_{pq}(s, t) \geq \left\lfloor \rho_{pq}^U N(s, t) \right\rfloor, \quad (p, q) \in E_-(\mathcal{P}), \\
(14) & \quad N_{pq}(s, t) \leq \left\lceil \rho_{pq}^U N(s, t) \right\rceil, \quad (p, q) \in E_+(\mathcal{P}), \\
(15) & \quad \sum_{p \in \mathcal{P}_S} N_p(s, t) + \sum_{p \in \mathcal{P}_U} N_p(s, t) = N(s, t), \\
(16) & \quad \sum_{(p, q) \in E_-(\mathcal{P})} N_{pq}(s, t) + \sum_{(p, q) \in E_+(\mathcal{P})} N_{pq} = N(s, t).
\end{align*}
\]

Here \( \rho_p^U, p \in \mathcal{P}_S \) and \( \rho_{pq}^U, (p, q) \in E_+(\mathcal{P}) \) are positive integers, and the scalars \( \rho_p^S, p \in \mathcal{P}_S, \rho_p^U, p \in \mathcal{P}_U, \rho_{pq}, (p, q) \in E_+(\mathcal{P}), \rho_{pq}, (p, q) \in E_-(\mathcal{P}) \) are as described in Assumption 2.

Notice that the elements of \( S_R \) obey, on every interval of time, restrictions on (a) the total number of switches (condition (10)), (b) the minimum number of switches when stable subsystems \( p \) are activated (condition (11)), (c) the maximum number of switches when unstable subsystems \( p \) are activated (condition (12)), (d) the minimum number of switches from subsystems \( p \) to subsystems \( q \) such that \( (p, q) \in E_-(\mathcal{P}) \) (condition (13)), (e) the maximum number of switches from subsystems \( p \) to subsystems \( q \) such that \( (p, q) \in E_+(\mathcal{P}) \) (condition (14)), (f) the total number of switches where a stable (resp., unstable) subsystem is activated (condition (15)), and (g) the total number of switches between the admissible pairs of subsystems (condition (16)). We observe that

**Observation 1.** \( S_R \subseteq S_R \).

Since condition (5) implies condition (10) and no property of \( \sigma \in S_R \) violates condition (4), the above fact follows at once.

**Remark 1.** Notice that: (a) The inequality (11) provides a lower bound on the frequency of activation of the subsystems in \( \mathcal{P}_S \). Indeed, \( N_p(s, t) \geq \left\lfloor \rho_p^S N(s, t) \right\rfloor \geq (\rho_p^S - 1)N(s, t) \) implies \( \frac{N_p(s, t)}{N(s, t)} \geq \frac{\rho_p^S}{\rho_p^S} = \rho_p^S - 1, \quad p \in \mathcal{P}_S \). (b) The inequality (12) provides an upper bound on the frequency of activation of the subsystems in \( \mathcal{P}_U \). Indeed, \( N_p(s, t) \leq \left\lceil \rho_p^U N(s, t) \right\rceil \leq \rho_p^U + \rho_p^U N(s, t) \) implies \( \frac{N_p(s, t) - \rho_p^U}{N(s, t)} \leq \rho_p^U, \quad p \in \mathcal{P}_U \). (c) The inequality (13) provides a lower bound on the frequency of switches from subsystems \( p \) to subsystems \( q \) such that \( (p, q) \in E_-(\mathcal{P}) \). Indeed, \( N_{pq}(s, t) \geq \left\lceil \rho_{pq}^U N(s, t) \right\rceil \geq (\rho_{pq}^U - 1)N(s, t) \) implies \( \frac{N_{pq}(s, t)}{N(s, t)} \geq \rho_{pq}^U - 1, \quad (p, q) \in E_-(\mathcal{P}) \). (d) The inequality (14) provides an upper bound on the frequency of switches from subsystems \( p \) to subsystems \( q \) such that
Remark 2. Notice that without the integers \( \tilde{\rho}^+_{pq} \), \( p \in \mathcal{P}_U \) and \( \tilde{\rho}^+_{pq} \) \((p, q) \in E_+(\mathcal{P})\), we can never activate subsystems \( p \in \mathcal{P}_U \) and/or switch from subsystems \( p \) to subsystems \( q \) with \((p, q) \in E_+(\mathcal{P})\). Indeed, consider an interval \([s, t]\), where \(N(s, t) = 10\) is an admissible number of switches. Suppose that \(\tilde{\rho}^+_{pq} = 0\), \( p \in \mathcal{P}_U \). Let us try to activate subsystem \( p' \in \mathcal{P}_U \) at a time \(s' \in [s, t]\). However, on an interval \([s' - \epsilon, s' + \epsilon]\), \( \epsilon > 0 \) (small enough) such that \(N(s' - \epsilon, s' + \epsilon) = 0\), we are not allowed to activate subsystem \( p' \) as \(N_p(s' - \epsilon, s' + \epsilon) \leq 0\). This limitation does not apply to the activation of subsystems \( p \in \mathcal{P}_S \) and/or switches from subsystems \( p \) to subsystems \( q \) with \((p, q) \in E_+(\mathcal{P})\) because of the usage of “at least (≤)” relation with the total number of switches on any interval of time.

Remark 3. On every interval of time, \((11)-(14)\) are inequality conditions and involve minimum (resp., maximum) number of activation of subsystem \( p \) (resp., switches between subsystems \( p \) and \( q \)), and \((15)-(16)\) are equality conditions and involve the (exact) total number of activation of subsystem \( p \) (resp., switches between subsystems \( p \) and \( q \)) with a fixed \(N(s, t)\) obeying \((10)\). In other words, \((11)-(14)\) are concerned with bounds and \((15)-(16)\) are concerned with “how” close/far from the tight version of the bounds the frequencies could be chosen such that a given value of \(N(s, t)\) is obeyed.

We will show that the elements of \(S'_R\) are stabilizing.

Theorem 1. Consider a family of systems \((1)\). Let the admissible minimum and maximum dwell times, \(\delta_p\) and \(\Delta_p\), on the subsystems \( p \in \mathcal{P}_R \), be given. Suppose that Assumptions \((7),(2)\) hold. Then the switched system \((2)\) is input/output-to-state stable (IOSS) under every switching signal \(\sigma \in S'_R\).

Theorem 1 is our solution to Problem 1. Given a family of systems \((1)\), possibly containing unstable dynamics, a set of admissible switches between the subsystems and admissible minimum and maximum dwell times on the subsystems, we present a class of switching signals, \(S'_R\), that obeys the pre-specified restrictions and preserves stability of the resulting switched system. Our precise characterization of \(S'_R\) is the following: if (a) the subsystems admit linearly comparable Lyapunov-like functions (Assumptions \((7),(2)\)), and (b) the rates of decay (resp., growth) of Lyapunov-like functions corresponding to stable (resp., unstable) subsystems, the linear comparison factor between these functions, the admissible dwell times on the subsystems, and a set of scalars together satisfy a certain inequality (Assumption \((5)\)), then every admissible switching signal \(\sigma \in S'_R\) for which the frequency of activation of various subsystems and frequency of switches between admissible pairs of subsystems, are governed by the set of scalars mentioned in (b) above (à la Definition \((3)\)) preserves IOSS of the switched system \((2)\). More specifically, a choice of \(S'_R\) described in Problem 1 is \(S'_R\) described above. A proof of Theorem 1 is presented in Section 6. In the absence of outputs (resp., also inputs), the switched system \((2)\) is ISS (resp., GAS) under the elements of \(S'_R\).

Remark 4. It will follow from our proof of Theorem 1 that the functions \(\alpha, \beta, \chi_1\) and \(\chi_2\) can be chosen independent of \(\sigma \in S'_R\). Consequently, IOSS of the switched system \((2)\) is uniform over the elements of \(\sigma \in S'_R\) in the above sense.

Remark 5. We now compare the results presented in this paper with our earlier works on IOSS of continuous-time switched nonlinear systems under pre-specified restrictions on admissible switches between the subsystems and admissible dwell times on the subsystems.

- In \([8]\) we presented an algorithm to design a switching signal that obeys pre-specified restrictions and preserves stability of a switched system. In \([9]\) we extended the technique...
In the current paper we present a class of switching signals that obeys pre-specified restrictions and preserves stability of a switched system. We consider common admissible dwell times on all subsystems, i.e., \( \delta_p = \delta \) and \( \delta_u = \Delta \) for all \( p \in \mathcal{P} \).

The elements of \( \hat{S}_R \) do not activate unstable subsystems on any two consecutive switching instants.

The elements of \( \hat{S}_R \) dwell on all subsystems for certain (sub)-ranges of the given range of admissible dwell times. In particular, the dwell time on stable subsystems is in the interval \( [\delta, \Delta] \) and the dwell time on unstable subsystems is in the interval \( [\delta, \hat{\Delta}] \), where \( \delta, \hat{\Delta} \in [\delta, \Delta] \) satisfy a certain inequality involving the rate of decay (resp., growth) of Lyapunov-like functions corresponding to stable (resp., unstable) subsystems, the linear comparison factor between these functions and the admissible dwell times on the subsystems.

Our analysis does not distinguish between different stable and unstable subsystems in terms of their measures of (in)stability and switches between different pairs of subsystems. In particular, we consider \( \lambda_p = \lambda \) for all \( p \in \mathcal{P}_S \), \( \lambda_p = \lambda_u \) for all \( p \in \mathcal{P}_U \), \( \mu_{pq} = \mu \) for all \( (p, q) \in E(\mathcal{P}) \).

In the current paper we present a class of switching signals that obeys pre-specified restrictions and preserves stability of a switched system.

We allow the admissible dwell times on various subsystems to be different. This leads to a more general problem setting than [10].

The elements of \( \hat{S}_R \) do not have any restriction on consecutive activation of unstable subsystems as long as conditions (10)–(16) are satisfied.

The elements of \( \hat{S}_R \) are allowed to dwell on any subsystem for any admissible duration of time. In particular, the dwell time on subsystem \( p \in \mathcal{P} \) is allowed to be any value in the interval \( [\delta_p, \Delta_p] \).

Our analysis distinguishes between different stable and unstable subsystems in terms of their measures of (in)stability and switches between different pairs of subsystems (by using different rate of decay (resp., growth) of the Lyapunov-like functions corresponding to the individual subsystems and the jump between these functions corresponding to the admissible switches).

**Remark 6.** As already noted in Section 3 the choice of Lyapunov-like functions, \( V_p \), for the individual subsystems, \( p \in \mathcal{P} \), and their corresponding scalars, \( \lambda_p, \mu_{pq}, (p, q) \in E(\mathcal{P}) \), is not unique. Consequently, non-satisfaction of (9) with a certain choice of Lyapunov-like functions and their corresponding scalars does not imply non-existence of Lyapunov-like functions and their corresponding scalars that satisfy condition (9). In general, the design of linearly comparable Lyapunov-like functions for a set of nonlinear systems is a numerically difficult problem. Further, our stability condition is only sufficient. Thus, non-satisfaction of the Assumptions [11] does not imply the absence of stabilizing elements in \( \hat{S}_R \).

We next present a set of scenarios where condition (9) is satisfied.

**Example 1.** Consider \( \mathcal{P}_S = \{1\}, \mathcal{P}_U = \{2\}, E_+(\mathcal{P}) = \{(1, 2)\} \) and \( E_-(\mathcal{P}) = \{(2, 1)\} \).

Suppose that there exists \( \rho \in [0, 1] \) such that

\[
- \rho \frac{\Delta_{\max}}{\Delta_{\min}} (|\lambda_1| \Delta_1 + |\ln \mu_1|) + \frac{1 - \rho}{\Delta_{\min}} (|\lambda_1| \Delta_2 + |\ln \mu_2|) < 0.
\]

Then condition (9) holds with \( \rho_1 = \rho_2^- = \rho, \rho_1^+ = \rho_2^+ = 1 - \rho \). Let \( \rho_1^U = \rho_2^U = 1 \). A stabilizing switching signal \( \sigma \) involves alternate activation of subsystems 1 and 2 or vice-versa and corresponding dwell times on the intervals \([\delta_1, \Delta_1]\) and \([\delta_2, \Delta_2]\).
Example 2. Consider \( \mathcal{P}_S = \{1, 2\} \), \( \mathcal{P}_U = \{3\} \). \( E_e(\mathcal{P}) = \{(1, 2), (2, 1), (3, 1)\} \) and \( E_u(\mathcal{P}) = \{(1, 3), (2, 3)\} \). Suppose that there exists \( \rho \in [0, 1[ \) such that

\[
\frac{-\rho}{\Delta_{\text{max}}} (|\lambda_1| \delta_1 + |\lambda_2| \delta_2 + |\ln \mu_{12}| + \frac{1}{2} |\ln \mu_{21}| + \frac{1}{2} |\ln \mu_{31}|) \\
+ \frac{1 - \rho}{\delta_{\text{min}}} (|\lambda_3| \Delta_3 + \frac{1}{2} |\ln \mu_{31}| + \frac{1}{2} |\ln \mu_{32}|) < 0.
\]

Then condition (9) holds with \( \rho_1^S = \rho_2^S = \rho \), \( \rho_1^U = 1 - \rho \), \( \rho_{12} = \rho \), \( \rho_{21} = \rho_{31} = \frac{6}{7} \), \( \rho_{13}^U = \rho_{23}^U = \frac{1}{2} \). Let \( \rho = 4 \). Suppose that on the interval \([0, t]\), an admissible choice of \( N(0, t) \) is 10. Then a switching signal \( \sigma \in S^U_{N} \) is: \( \sigma(t_0) = 1 \), \( \sigma(t_1) = 2 \), \( \sigma(t_2) = 1 \), \( \sigma(t_3) = 3 \), \( \sigma(t_4) = 1 \), \( \sigma(t_5) = 2 \), \( \sigma(t_6) = 3 \), \( \sigma(t_7) = 1 \), \( \sigma(t_8) = 2 \), \( \sigma(t_9) = 1 \), \( \sigma(t_{10}) = 2 \), where \( t_{k+1} - t_k \in [\delta_{\sigma(t_k)}, \Delta_{\sigma(t_k)}] \), \( k = 0, 1, \ldots, 9 \).

5. Concluding remarks

In this paper we identified a class of switching signals that guarantees IOSS of a continuous-time switched nonlinear system under restrictions on admissible switches between the subsystems and admissible dwell times on the subsystems. Our stabilizing switching signals are characterized based on the frequency of activation of various subsystems and the frequency of switches between various pairs of subsystems. The proposed techniques extend to the discrete-time setting under standard assumptions. Future work will focus on design of state-norm estimators (à la [17, §4]) under the proposed class of stabilizing switching signals.

6. Proof of Theorem[1]

We will use the following auxiliary results in our proof of Theorem[1]

Lemma 1. For every interval \([s, t]\) \( \subset [0, +\infty[ \) of time, we have \( T_p(s, t) \geq N_p(s, t) \delta_p \), \( p \in \mathcal{P}_S \).

Proof. Let \( \tau_1 < \tau_2 < \cdots < \tau_n \) be the switching instants on \([s, t]\). Now,

\[
T_p(s, t) = \sum_{\tau_{i+1} = \tau_i}^n (\tau_{i+1} - \tau_i) + (\tau_1 - s) \geq \sum_{\tau_{i+1} = \tau_i}^n (\tau_{i+1} - \tau_i) \\
= N_p(\tau_1, t) \delta_p = N_p(s, t) \delta_p.
\]

\( \square \)

Lemma 2. For every interval \([s, t]\) \( \subset [0, +\infty[ \) of time, we have \( T_p(s, t) \leq N_p(s, t) \Delta_p + \Delta_p \), \( p \in \mathcal{P}_U \).

Proof. Let \( \tau_1 < \tau_2 < \cdots < \tau_n \) be the switching instants on \([s, t]\). Now,

\[
T_p(s, t) = \sum_{\tau_{i+1} = \tau_i}^n (\tau_{i+1} - \tau_i) + (\tau_1 - s) \leq N_p(\tau_1, t) \Delta_p + \Delta_p \\
= N_p(s, t) \Delta_p + \Delta_p.
\]

\( \square \)

We are now ready to present a proof of Theorem[1]
Proof.   (Proof of Theorem 1) Consider a switching signal $\sigma \in S_\infty$. We will show that the switched system (2) is IOSS under $\sigma$.

Fix $t > 0$. Recall that $0 = \tau_0 < \tau_1 < \cdots < \tau_{N(0,t)}$ are the switching instants before (and including) $t$. In view of (7), we have that

$$V_{\sigma(t)}(x(t)) \leq \exp\left(-\lambda_{\sigma(\tau_{N(0,t)})}(t - \tau_{N(0,t)})\right)V_{\sigma(t)}(x(\tau_{N(0,t)}))$$

$$+ \left(g_1(\|v\|_{0,t}) + g_2(\|y\|_{0,t})\right) \times \int_{\tau_{N(0,t)}}^{t} \exp\left(-\lambda_{\sigma(\tau_{N(0,t)})}(t - s)\right)ds.$$

Applying (8) and iterating the above, we obtain the estimate

$$V_{\sigma(t)}(x(t)) \leq \psi_1(t)V_{\sigma(0)}(x(0)) + \left(g_1(\|v\|_{0,t}) + g_2(\|y\|_{0,t})\right)\psi_2(t),$$

where

$$\psi_1(t) := \exp\left(-\sum_{i=0}^{N(0,t)} \lambda_{\sigma(\tau_i)}(\tau_{i+1} - \tau_i) + \sum_{i=0}^{N(0,t)-1} \ln \mu_{\sigma(\tau_i)\sigma(\tau_{i+1})}\right),$$

and

$$\psi_2(t) := \sum_{i=0}^{N(0,t)} \exp\left(-\sum_{j=i+1}^{N(0,t)} \lambda_{\sigma(\tau_j)}(\tau_{j+1} - \tau_j) + \sum_{j=i+1}^{N(0,t)-1} \ln \mu_{\sigma(\tau_j)\sigma(\tau_{j+1})}\right) \times \frac{1}{\lambda_{\sigma(\tau_i)}} \left(1 - \exp\left(-\lambda_{\sigma(\tau_i)}(\tau_{i+1} - \tau_i)\right)\right).$$

In view of (6), we rewrite the estimate (17) as

$$\sigma(\|x(t)\|) \leq \psi_1(t)\overline{\pi}(\|x(0)\|) + \left(g_1(\|v\|_{0,t}) + g_2(\|y\|_{0,t})\right)\psi_2(t).$$

By Definition 1 for IOSS of (2), we need to show that

i) $\overline{\pi}(\cdot)\psi_1(\cdot)$ can be bounded above by a class $KL$ function, and

ii) $\psi_2(\cdot)$ is bounded above by a constant.

We first verify i). Towards this end, we already see that $\overline{\pi} \in KL$ from Assumption 1. Therefore, it remains to show that $\psi_1(\cdot)$ is bounded above by a function in class $KL$.

Now,

$$\begin{align*}
- \sum_{i=0}^{N(0,t)} \lambda_{\sigma(\tau_i)}(\tau_{i+1} - \tau_i) + \sum_{i=0}^{N(0,t)-1} \ln \mu_{\sigma(\tau_i)\sigma(\tau_{i+1})} &= \sum_{p \in F_S} |\lambda_p| T_p(0,t) - \sum_{(p,q) \in E_+} |\ln \mu_{pq}| N_{pq}(0,t) \\
&+ \sum_{p \in F_U} |\lambda_p| T_p(0,t) + \sum_{(p,q) \in E_-(P)} |\ln \mu_{pq}| N_{pq}(0,t).\end{align*}$$

In view of Lemmas 1 and 2 the above expression is at most equal to

$$\begin{align*}
- \sum_{p \in F_S} |\lambda_p| N_p(0,t)\delta_p + \sum_{p \in F_U} |\lambda_p| (N_p(0,t)\Delta_p + \Delta_p) \left(\sum_{(p,q) \in E_+(P)} |\ln \mu_{pq}| N_{pq}(0,t) + \sum_{(p,q) \in E_-(P)} |\ln \mu_{pq}| N_{pq}(0,t)\right)\end{align*}$$

\[4 L := \{ y : [0, +\infty] \rightarrow [0, +\infty] \mid y \text{ is continuous and } y(s) \searrow 0 \text{ as } s \nearrow +\infty \}.\]
\[\dot{\mathcal{E}} + \left( - \sum_{p \in \mathcal{P}_s} |\lambda_p| \left[ \rho_p^S N(0, t) \right] \delta_p + \sum_{p \in \mathcal{P}_U} \left| \lambda_p \right| \left[ \rho_p^U N(0, t) \right] \Delta_p \right. \]
\[\left. - \sum_{(p, q) \in E_-(\mathcal{P})} \ln \mu_{pq} \left| \rho_{pq}^* N(0, t) \right| + \sum_{(p, q) \in E_+(\mathcal{P})} \ln \mu_{pq} \left| \rho_{pq}^* N(0, t) \right| \right) \]
\[\leq \mathcal{E} + \left( - \sum_{p \in \mathcal{P}_s} |\lambda_p| \left( \rho_p^S N(0, t) - 1 \right) \delta_p + \sum_{p \in \mathcal{P}_U} \left| \lambda_p \right| \rho_p^U \Delta_p \right) \]
\[+ \sum_{(p, q) \in E_-(\mathcal{P})} \ln \mu_{pq} \rho_{pq}^* N(0, t) - \sum_{(p, q) \in E_+(\mathcal{P})} \ln \mu_{pq} \rho_{pq}^* N(0, t) - 1) \]
\[\leq \mathcal{E} + \left( - \sum_{p \in \mathcal{P}_s} |\lambda_p| \left( \rho_p^S \delta_p \left( \frac{t}{\Delta_{\max}} - 1 \right) + \sum_{p \in \mathcal{P}_U} \left| \lambda_p \right| \rho_p^U \Delta_p \frac{t}{\delta_{\min}} \right. \]
\[\left. - \sum_{(p, q) \in E_-(\mathcal{P})} \ln \mu_{pq} \rho_{pq}^* \left( \frac{t}{\Delta_{\max}} - 1 \right) + \sum_{(p, q) \in E_+(\mathcal{P})} \ln \mu_{pq} \rho_{pq}^* \frac{t}{\delta_{\min}} \right) \]
\[= \mathcal{E} + \left( - \sum_{p \in \mathcal{P}_s} |\lambda_p| \rho_p^S \frac{\delta_p}{\Delta_{\max}} + \sum_{(p, q) \in E_-(\mathcal{P})} \ln \mu_{pq} \rho_{pq}^* \frac{1}{\Delta_{\max}} \right. \]
\[\left. + \sum_{p \in \mathcal{P}_U} \left| \lambda_p \right| \rho_p^U \frac{\Delta_p}{\delta_{\min}} + \sum_{(p, q) \in E_+(\mathcal{P})} \ln \mu_{pq} \rho_{pq}^* \frac{1}{\delta_{\min}} \right) t, \]

where \( \mathcal{E} = \sum_{p \in \mathcal{P}_s} |\lambda_p| \rho_p^S + \sum_{(p, q) \in E_-(\mathcal{P})} \ln \mu_{pq} \rho_{pq}^* \), \( c = \dot{\mathcal{E}} + \sum_{p \in \mathcal{P}_s} |\lambda_p| \delta_p + \sum_{(p, q) \in E_+(\mathcal{P})} \ln \mu_{pq} \), and \( \mathcal{T} = c + \sum_{p \in \mathcal{P}_s} |\lambda_p| \rho_p^S + \sum_{(p, q) \in E_-(\mathcal{P})} \ln \mu_{pq} \rho_{pq}^* \).

By (9), we have that the above expression is at most equal to \( c_1 \leq \mathcal{T} \) for some \( c_1 \geq \mathcal{T} \) and \( c_2 > 0 \). Therefore, from (18), we obtain that \( \psi_1(t) \leq \exp(c_1 - c_2 t) \).

Notice that the right-hand side of the above inequality decreases as \( t \) increases and tends to 0 as \( t \to +\infty \). Consequently, \( i \) holds.

We now verify \( ii \). We have that the function \( \psi_2(t) \) is

\[\sum_{p \in \mathcal{P}_s} \frac{1}{|\lambda_p|} \left( N(0, t) - \sum_{k \in \mathcal{P}_s} |\lambda_k| T_k(\tau_{i+1}, t) + \sum_{k \in \mathcal{P}_U} |\lambda_k| T_k(\tau_{i+1}, t) \right. \]
\[\left. - \sum_{(k, l) \in E_-(\mathcal{P})} \ln \mu_{kl} \left( N_{kl}(\tau_{i+1}, t) + \sum_{(k, l) \in E_+(\mathcal{P})} (\ln \mu_{kl}) N_{kl}(\tau_{i+1}, t) \right) \times \right. \]
\[\left. \left( 1 - \exp(-|\lambda_j| (\tau_{i+1} - \tau_{i})) \right) \right) \]
\[- \sum_{p \in P} \frac{1}{|P|} \sum_{\sigma(\tau_i) = p \atop S(\tau_i) = t} N(0,t) \left( \exp \left( - \sum_{k \in P_S} |\lambda_k| T_k(\tau_{i+1}, t) + \sum_{k \in P_U} |\lambda_k| T_k(\tau_{i+1}, t) \right) \right. \]

\[- \sum_{(k,t) \in E_u(P)} |\ln \mu_{kt}| N_{kt}(\tau_{i+1}, t) + \sum_{(k,t) \in E_s(P)} (\ln \mu_{kt}) N_{kt}(\tau_{i+1}, t) \right) \times \left( 1 - \exp(-|\lambda_p| (\tau_{i+1} - \tau_i)) \right) \]

\[- \sum_{p \in P} \frac{1}{|P|} \sum_{\sigma(\tau_i) = p \atop S(\tau_i) = t} N(0,t) \left( \exp \left( - \sum_{k \in P_S} |\lambda_k| T_k(\tau_i, t) + \sum_{k \in P_U} |\lambda_k| T_k(\tau_i, t) \right) \right. \]

\[- \sum_{(k,t) \in E_u(P)} |\ln \mu_{kt}| N_{kt}(\tau_i, t) + \sum_{(k,t) \in E_s(P)} (\ln \mu_{kt}) N_{kt}(\tau_i, t) \right) \].

Now, by an application of a similar set of arguments as employed in the verification of i), we obtain that

\[- \sum_{k \in P_S} |\lambda_k| T_k(\tau_i, t) + \sum_{k \in P_U} |\lambda_k| T_k(\tau_i, t) \]

\[- \sum_{(k,t) \in E_u(P)} |\ln \mu_{kt}| N_{kt}(\tau_i, t) + \sum_{(k,t) \in E_s(P)} (\ln \mu_{kt}) N_{kt}(\tau_i, t) \]

\[ \leq c' + \left( - \sum_{k \in P_S} |\lambda_k| \rho_k^S (\frac{t - \tau_i}{L_{\text{max}}} - 1) \delta_k + \sum_{k \in P_U} |\lambda_k| \rho_k^U \frac{t - \tau_i}{\delta_{\text{min}}} \right) \]

\[- \sum_{(k,t) \in E_u(P)} |\ln \mu_{kt}| \rho_{kt}^- \left( \frac{t - \tau_i}{L_{\text{max}}} - 1 \right) + \sum_{(k,t) \in E_s(P)} (\ln \mu_{kt}) \rho_{kt}^+ \frac{t - \tau_i}{\delta_{\text{min}}} \right) \]

\[ = c'' + \left( - \sum_{k \in P_S} |\lambda_k| \rho_k^S \frac{t - \tau_i}{L_{\text{max}}} \Delta_k \delta_k \right) + \sum_{(k,t) \in E_u(P)} (\ln \mu_{kt}) \rho_{kt}^U \frac{t - \tau_i}{\delta_{\text{min}}} \]

where \( c' = \sum_{k \in P_U} |\lambda_k| \rho_k^U \Delta_k \delta_k + \sum_{(k,t) \in E_u(P)} |\ln \mu_{kt}| \rho_{kt}^+ \Delta_k + \sum_{(k,t) \in E_s(P)} |\ln \mu_{kt}| \rho_{kt}^- \Delta_k \delta_k + \sum_{(k,t) \in E_s(P)} |\ln \mu_{kt}| \rho_{kt}^- \frac{1}{\delta_{\text{min}}} \), and

\( c'' = c' + \sum_{k \in P_S} |\lambda_k| \rho_k^S \Delta_k \delta_k + \sum_{(k,t) \in E_u(P)} |\ln \mu_{kt}| \rho_{kt}^- \Delta_k \delta_k + \sum_{(k,t) \in E_s(P)} |\ln \mu_{kt}| \rho_{kt}^- \frac{1}{\delta_{\text{min}}} \). In view of (2), the above expression is at most equal to \( \Delta_1 - \Delta_2 (t - \tau_i) \) for some \( \Delta_1 \geq c'' \) and \( \Delta_2 > 0 \). Similarly, the expression

\[- \sum_{k \in P_S} |\lambda_k| T_k(\tau_{i+1}, t) - \sum_{(k,t) \in E_s(P)} (\ln \mu_{kt}) N_{kt}(\tau_{i+1}, t) \]
Applying the set of arguments as above, we also obtain that

\[ \text{is at most equal to } \tilde{e}_1 - \tilde{e}_2(t - t_{i+1}) \text{ for some } \tilde{e}_1 \geq c'' \text{ and } \tilde{e}_2 > 0. \]

Now, from (20), we have that

\[ \psi_2(t) \leq \sum_{p \in P_S} \frac{1}{|P|} \sum_{i=0}^{N(0,t)} \exp(-\tilde{e}_2(t - t_i)) + \sum_{p \in P_U} \frac{1}{|P|} \sum_{i=0}^{N(0,t)} \exp(-\tilde{e}_2(t - t_i)) \]

\[ \leq \sum_{p \in P_S} \frac{1}{|P|} \sum_{i=0}^{N(0,t)} \exp(-\tilde{e}_2(t - t_i)) + \sum_{p \in P_U} \frac{1}{|P|} \sum_{i=0}^{N(0,t)} \exp(-\tilde{e}_2(t - t_i)). \]

We will now show that both the sums are \( \sum_{i=0}^{N(0,t)} \exp(-\tilde{e}_2(t - t_i)) \) and \( \sum_{i=0}^{N(0,t)} \exp(-\tilde{e}_2(t - t_i)) \) are bounded above by constants. We have

\[ \sum_{i=0}^{N(0,t)} \exp(-\tilde{e}_2(t - t_i)) \leq \sum_{i=0}^{N_{\min}} \exp(-\tilde{e}_2(t - t_i)) \]

\[ \leq \left( \exp(-\tilde{e}_2(t - 0)) + \exp(-\tilde{e}_2(t - t_1)) + \cdots + \exp(-\tilde{e}_2(t - t_{\Delta_{\max}})) \right) \]

\[ \leq \exp(-\tilde{e}_2 t) + \exp(-\tilde{e}_2 \frac{t}{\Delta_{\max}}) \delta_{\min} + \cdots + \exp(-2\tilde{e}_2 \delta_{\min}) + \exp(-\tilde{e}_2 \delta_{\min}) \]

\[ \leq 1 + \exp(-\tilde{e}_2 \delta_{\min}) \frac{1 - \exp(-\tilde{e}_2 \delta_{\min})}{1 - \exp(-\tilde{e}_2 \delta_{\min}) + 1} \]

\[ \leq 1 + \frac{\exp(-\tilde{e}_2 \delta_{\min})}{1 - \exp(-\tilde{e}_2 \delta_{\min})} = 1 + \frac{1}{\exp(-\tilde{e}_2 \delta_{\min}) - 1}. \]

Applying the set of arguments as above, we also obtain that

\[ \sum_{i=0}^{N(0,t)} \exp(-\tilde{e}_2(t - t_i)) \leq \frac{1}{\exp(-\tilde{e}_2 \delta_{\min}) - 1}. \]

In view of (21)–(22), we arrive at ii). Indeed,

\[ \psi_2(t) \leq \bar{\psi}_2 = \sum_{p \in P_S} \frac{1}{|P|} \exp(\tilde{e}_1) + \sum_{p \in P_U} \frac{1}{|P|} \exp(\tilde{e}_2) \]

\[ \leq \frac{1}{\exp(-\tilde{e}_2 \delta_{\min}) - 1} \]

We conclude ISS of the switched system \( (S) \) under \( \sigma \).

Recall that \( \sigma \in S'_R \) was chosen arbitrarily. Consequently, the assertion of Theorem 1 follows. In particular, condition (3) holds with \( \alpha(r) = r, \beta(r,s) = \overline{m}(r) \exp(e_1 - e_2 s), \chi_1(r) = \gamma_1(r) \overline{\psi}_2 \) and \( \chi_2(r) = \gamma_2(r) \overline{\psi}_2 \).

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