Nori’s construction and the second Abel-Jacobi map

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1 Introduction

Let $k$ be a subfield of $\mathbb{C}$. Nori constructs an abelian category of mixed motives over $k$. One of the fundamental facts in his construction is the following ([6]):

**Theorem 1.1 (Basic Lemma).** Let $X$ be an affine scheme of finite type over $k$. Let $n$ be the dimension of $X$. Let $F$ be a weakly constructible sheaf on $X(\mathbb{C})$ for the usual topology. Then there is a Zariski open $U$ in $X$ with the properties below, where $j : U \to X$ denotes the inclusion.

1. $\dim Y < n$ where $Y = X - U$.
2. $H^q(X(\mathbb{C}), j_! j^* F) = 0$ for $q \neq n$.

Here a sheaf $F$ on $X$ is weakly constructible if $X$ is the disjoint union of finite collection of locally closed subschemes $Y_i$ defined over $k$ such that the restrictions $F|_{Y_i}$ are locally constant. Beilinson ([1], Lemma 3.3) proves this fact in all characteristics of the base field. Based on Theorem 1.1 Nori shows that affine $k$-varieties have a kind of “cellular decomposition”. In section 2 of this note we give an exposition of the outcome if we apply this construction to étale cohomology. It can also be viewed as a partial exposition of $l$-adic realization of Nori’s category of motives. The main result (Theorem 2.2) says that $Rf_* \mathbb{Z}_l(a)$ for a variety $X \to \text{Spec} k$ is a complex each component of which comes from a mixed motive. This gives an answer to a question asked by Jannsen in a remark in [4]. In [2] a similar result for perverse sheaves is proven. We learned about Nori’s category in [5].

In section 3, we give a simple description of the second $l$-adic Abel-Jacobi map for certain algebraic cycles on a smooth projective variety. We briefly
recall the definition of the \( l \)-adic Abel-Jacobi map. Let \( X \xrightarrow{f} \text{Spec} \ k \) be a smooth projective variety of dimension \( n \). We denote the absolute Galois group of \( k \) by \( G_k \). For an algebraic cycle \( z \) on \( X \) of codimension \( i \) the class
\[
[z] \in H^{2i}_{\text{cont}}(X, \mathbb{Z}_l(i))
\]
is defined. Here \( H^{2i}_{\text{cont}}(X, \mathbb{Z}_l(i)) \) is continuous etale cohomology. The usual cycle class \( cl(z) = cl^0(z) \) is the image of \([z]\) under the edge homomorphism
\[
H^{2i}_{\text{cont}}(X, \mathbb{Z}_l(i)) \to H^{2i}(\overline{X}, \mathbb{Z}_l(i)).
\]
The Hochschield-Serre spectral sequence
\[
E_2^{p,q} = H^p(G_k, H^q(\overline{X}, \mathbb{Z}_l(i))) \Longrightarrow H^{2i}_{\text{cont}}(X, \mathbb{Z}_l(i))
\]
induces higher classes
\[
cl^j : \ker cl^{j-1} \to H^j(G_k, H^{2i-j}(\overline{X}, \mathbb{Q}_l(i))).
\]
We refer the reader to [4] for more details. See also [8].

Let \( z \in CH^i(X) \) be an algebraic cycle such that \( 0 = cl^1(z) \in H^1(G_k, H^{2i-1}(\overline{X}, \mathbb{Z}_l(i))) \). In Theorem 3.1 we give a simple description of the push-out of \( cl^2(z) \in H^2(G_k, H^{2i-2}(\overline{X}, \mathbb{Q}_l(i))) \) by a quotient map \( H^{2i-2}(\overline{X}, \mathbb{Q}_l(i)) \to H^{2i-2}(\overline{X}, \mathbb{Q}_l(i))/H^{2i-2}_{\text{H}}(\overline{X}, \mathbb{Q}_l(i)) \) for a certain multiple hypersurface section \( H \) of \( X \).

G. Welters([8]) also gives a description of the second Abel-Jacobi map for zero-cycles. It would be interesting to compare these two descriptions.

Acknowledgements
Theorem 3.1 used to be only about zero cycles. The author is grateful to Spencer Bloch for pointing out the possibility of generalizing it to cycles of other dimensions. He also thanks Masanori Asakura for his comments on an earlier version.

## 2 Nori’s construction

Let \( k \) be a subfield of \( \mathbb{C} \). In this note a variety is an integral separated scheme of finite type over \( k \). All schemes and morphisms between them are defined over \( k \). For a variety \( X \) \( X \) denotes \( X \times_{\text{Spec} \ k} \text{Spec} \ k \) where \( k \) is an algebraic closure of \( k \).
Theorem 2.1. Let $X$ be an affine scheme of finite type of dimension $n$. Let $F$ be a constructible sheaf on $X(C)$ for the usual topology. Then there is a Zariski open $U$ in $X$ with the properties below, where $j : U \to X$ denotes the inclusion.

1. $\dim Y < n$ where $Y = X - U$.
2. $H^q(X(C), j_! j^* F) = 0$ for $q \neq n$.

Let $X'$ be the largest open subset of $X$ such that $X'$ is smooth and $F|_{X'}$ is locally constant. As in Remark 1.1 in [6] the open set $U$ depends only on the open $X'$ and not on $F$.

Fix an integer $a$ and a prime $l$. We are going to use Theorem 2.1 in the case where $F$ is an étale sheaf of the form $j_! j^* Z/l^m Z(a)$ for an open set $V \to X$.

In this case $H^q(X(C), j_! j^* F)$ in the assertion 2 of Theorem 2.1 is isomorphic to $H^q(X_{et}, j_! j^* F)$.

Let $X$ be an affine variety and let $Z$ be a proper closed subset of $X$. Let $j$ be the inclusion $X - Z \to X$. By applying Theorem 2.1 to $F = j_! j^* Z/l^m Z(a)$ we obtain the following:

**Corollary 2.1.** Let $X$ be an affine variety of dimension $n$ and let $Z$ be a closed subset of dimension $< n$. Then there exists a closed subset $Y$ of $X$ of dimension $< n$ which contains $Z$ such that

$$H^q(X, Y, Z/l^m Z(a)) = 0$$

for all $m \geq 1$.

By Corollary 2.1 there is a filtration by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$$

such that $H^q(X_i, X_{i-1}, Z/l^m Z(a)) = 0$ for $q \neq i$ and for $m \geq 1$.

Let $f : X \to \text{Spec} k$ be the structure morphism. We consider $Rf_* Z_l(a)$ in $D^b(Sh(\text{Spec} k_{et})_{Z_l})$. Here for a variety $X$ $Sh(X_{et})_{Z_l}$ denotes the category of $l$-adic sheaves of Jannsen ([2], (6.9)). $Sh(X_{et})_{Z_l}$ has enough injectives.

**Proposition 2.1.** Let $X$ be an affine variety of dimension $n$ and take a filtration

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$$

by closed subsets as above.
1. Let $D_n$ be the complex

$$0 \to H^0(X_0, Z_l(a)) \to \cdots \to H^i(X_i, X_{i-1}, Z_l(a)) \to \cdots \to H^n(X, X_{n-1}, Z_l(a)) \to 0$$

where the maps between components are the boundary map of cohomology multiplied by $-1$. Let $f : X \to \text{Spec} k$ be the structure morphism. Then there is a natural isomorphism

$$\mathbb{R}f_* Z_l(a) \simeq D_n$$

in $D^b(\text{Sh} (\text{Spec} k \mathbb{Z}_l))$.

2. Let $Y_j \hookrightarrow X$ be an affine open subset of $X$ and

$$\emptyset = Y_{-1} \subset Y_0 \subset \cdots \subset Y_n = Y$$

be a filtration by closed subsets such that $H^q(Y_i, Y_{i-1}, Z_l/m Z_l(a)) = 0$ for $q \neq i$ and for $m \geq 1$. Assume that $Y_i \subset X_i$ for each $i$. Let $D_{nY}$ be the complex

$$0 \to H^0(Y_0, Z_l(a)) \to \cdots \to H^i(Y_i, Y_{i-1}, Z_l(a)) \to \cdots \to H^n(Y, Y_{n-1}, Z_l(a)) \to 0$$

Let $g = f \circ j_Y : Y \to \text{Spec} k$. Then the isomorphisms $\mathbb{R}f_* Z_l(a) \simeq D_n$ and $\mathbb{R}g_* Z_l(a) \simeq D_{nY}$ are compatible with the pull-back $j_Y^*$.

Proof. 1. For $0 \leq a \leq n$ let $X^a = X_a - X_{a-1}$, $j_a : X^a \hookrightarrow X_a$ and $i_a : X_a \hookrightarrow X$. Let $B^m_a = Rf_* i_a_* Z_l/m Z_l(a)|_{X_a}$. We have exact triangles

$$Rf_* i_a_* (j_a! Z_l/m Z_l(a)|_{X_a}) \to B^m_a \to B^m_{a-1} \to .$$

We denote the term on the left end by $A^m_a$.

Let $D^m_n$ be the complex

$$0 \to H^0(X_0, Z_l/m Z_l(a)) \to \cdots \to H^i(X_i, X_{i-1}, Z_l/m Z_l(a)) \to \cdots \to H^n(X, X_{n-1}, Z_l/m Z_l(a)) \to 0$$
We need to construct a chain of quasi-isomorphisms between $D_m^n$ and $B_m^n$. We fix notation and the signs. The mapping cone of $f : A^i \to B^i$ between two cohomological complexes is given in degree $i$ by

$$A^{i+1} \oplus B^i, \quad \text{with differential} \quad d(a, b) = (-da, db - f(a)).$$

We have a quasi-isomorphism

$$\text{cone}(B_m^{n-1} \to \text{cone}(B_m^n \to B_m^{n-1}))[-1] \to B_m^n.$$  

By smooth base change theorem $R^q f_* j_n^! \mathbb{Z}/l^m \mathbb{Z}(a) = 0$ for $q \neq n$ and $R^m f_* j_{n!} \mathbb{Z}/l^m \mathbb{Z}(a) = H^n(X, X_{n-1}, \mathbb{Z}/l^m \mathbb{Z}(a))$. Here we regard an etale sheaf on Spec as a $G_k$ module. So

$$\text{cone}(B_m^n \to B_m^{n-1}) \overset{\text{qis}}{\sim} A_n[1] \overset{\text{qis}}{\sim} H^n(X, X_{n-1}, \mathbb{Z}/l^m \mathbb{Z}(a))[1-n].$$

So there is a quasi-isomorphism

$$\text{cone}(B_m^{n-1} \to \text{cone}(B_m^n \to B_m^{n-1}))[-1] \to \text{cone}(B_m^n \to H^n(X, X_{n-1}, \mathbb{Z}/l^m \mathbb{Z}(a))[1-n])[1-n].$$

We have a quasi-isomorphism

$$\text{cone}(B_m^{n-2} \to \text{cone}(B_m^{n-1} \to B_m^{n-2}))[-1] \to B_m^{n-1}.$$  

So there is a quasi-isomorphism

$$\left[ \text{cone}(B_m^{n-2} \to \text{cone}(B_m^{n-1} \to B_m^{n-2}))[-1] \right. \to \left. H^n(X, X_{n-1}, \mathbb{Z}/l^m \mathbb{Z}(a))[1-n] \right][1-n].$$

We can repeat this process until we get the complex $D_m^n$. All the maps appearing here are compatible with the transition map $\mathbb{Z}/l^{m+1} \mathbb{Z}(a) \to \mathbb{Z}/l^m \mathbb{Z}(a)$.

2. All the maps in the construction of the isomorphism $R f_* \mathbb{Z}/l^m \mathbb{Z}(a) \simeq D_n$ are compatible with the pull-back $j_Y^*$.  

In general we use Čeck construction. Let $X$ be a variety of dimension $n$. Let $I$ be a finite set $\{1, \ldots, s\}$ and let $U_i (i \in I)$ be a covering of $X$ by affine open subsets. For any finite sets of indices $i_0, \ldots, i_p \in I$ we denote the intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U_{i_0 \cdots i_p}$. We denote the open immersion
The total complex associated to this double complex is isomorphic to $D_{i_0 \ldots i_p} \hookrightarrow X$ by $j_{i_0 \ldots i_p}$. For $U_{i_0 \ldots i_p} = \bigcap_{i \in I} U_i$ choose a filtration by closed subsets $\emptyset = C_{-1}^{i_0 \ldots i_p} \subset C_{0}^{i_0 \ldots i_p} \subset C_{1}^{i_0 \ldots i_p} \subset \cdots C_{n-1}^{i_0 \ldots i_p} \subset C_{n}^{i_0 \ldots i_p} = U_{i_0 \ldots i_p}$. We have the Abel-Jacobi map $\eta$.

Theorem 2.2. The complex $Rf_* Z_l(a)$ is isomorphic to the total complex associated to the double complex $0 \to \bigoplus_{i \in I} D_i \to \bigoplus_{i_0 < i_1} D_{i_0 i_1} \to \cdots \to D_{1 \ldots 0} \to 0$ in $D^b(Sh(Spec_k)^l Z_l)$. By the assertion 1 of Proposition 2.1 $R(f \circ j_{i_0 \ldots i_p})_* Z_l(a)$ is isomorphic to

$$0 \to H^0(C_{m}^{i_0 \ldots i_p}, Z_l(a)) \to H^1(C_{m-1}^{i_0 \ldots i_p}, C_{m}^{i_0 \ldots i_p}, Z_l(a)) \to \cdots \to H^n(U_{i_0 \ldots i_p}, C_{n-1}^{i_0 \ldots i_p}, Z_l(a)) \to 0.$$ 

We denote this complex by $D_{i_0 \ldots i_p}$. By the assertion 2 of Proposition 2.1 the isomorphism $D_{i_0 \ldots i_p} \to R(f \circ j_{i_0 \ldots i_p})_* Z_l(a)$ is compatible with the restriction $R(f \circ j_{i_0 \ldots i_p})_* Z_l(a) \to R(f \circ j_{i_0 \ldots i_{p+1}})_* Z_l(a)$. So we obtain the following.

Theorem 2.2. The complex $Rf_* Z_l(a)$ is isomorphic to the total complex associated to the double complex $0 \to \bigoplus_{i \in I} D_i \to \bigoplus_{i_0 < i_1} D_{i_0 i_1} \to \cdots \to D_{1 \ldots 0} \to 0$ in $D^b(Sh(Spec_k)^l Z_l)$.

3 A simple description of the second $l$-adic Abel-Jacobi map

Let $X$ be a smooth projective variety of dimension $n$. Let $z \in CH^i(X)$ be an algebraic cycle which is homologous to $0$. Let $cl^j(z)$ be the image of $z$
under the $l$-adic Abel-Jacobi map

$$cl^1 : CH^i(X)_{\text{hom}} \to H^1(G_k, H^{2i-1}(\mathcal{X}, \mathbb{Z}_l(i))).$$

Assume further that the cycle $z$ satisfies the following condition:

Let $q = 2i - 1 - n$. Then there exists a smooth multiple hypersurface section $H$ of $X$ of codimension $q$ which supports the cycle $z$ such that $z$ is homologous to 0 on $H$.

For example if $i = n$ then by Proposition 4.8 of \[3\] (see also Proposition 5.3 of \[3\]) such a $H$ always exists. Let $|z|$ be the support of $z$ and let $Y = X - |z|$. Let $U = X - H$ and let $j : U \subset Y$ be the inclusion.

Let $g : Y \to \text{Spec} k$ be the structure morphism. The 2-extension

$$0 \to H^{2i-2}(\mathcal{Y}, \mathbb{Q}_l(i)) \to Rg_*\mathbb{Q}_l(i)^{2i-2} \to \partial H^{2i-1} \to H^{2i-1}(\mathcal{Y}, \mathbb{Q}_l(i)) \to 0$$

is denoted $\chi_{2i-2}(Y)$ in \[4\]. Assume that $0 = cl^1(z) \in H^1(G_k, H^{2i-1}(\mathcal{X}, \mathbb{Z}_l(i))).$ Then by Theorem 1 in \[4\] the class $-cl^2(z) \in \text{Ext}^2_{\mathbb{Q}_l}(H^{2i-2}(\mathcal{Y}, \mathbb{Q}_l(i)))$ is the pull-back of $\chi_{2i-2}(Y)$ by the splitting $cl^1(z) : \mathbb{Q}_l \to H^{2i-1}(\mathcal{Y}, \mathbb{Q}_l(i))$.

Let $\mathcal{C}$ be the complex

$$0 \to H^{2i-1}(\mathcal{X}, \mathbb{Q}_l(i)) \to H^{2i-1}(\mathcal{Y}, \mathbb{Q}_l(i)) \to H^{2i}_{|z|}(\mathcal{X}, \mathbb{Q}_l(i)) \to 0$$

and let $\mathcal{C}_H$ be the complex

$$0 \to H^{2(i-q)-1}(\mathcal{H}, \mathbb{Q}_l(i-q)) \to H^{2(i-q)-1}(|Y \cap \mathcal{Y}, \mathbb{Q}_l(i-q)) \to H^{2(i-q)}_{|z|}(\mathcal{H}, \mathbb{Q}_l(i-q)) \to 0.$$

There is the Gysin map $i_{H*} : \mathcal{C}_H \to \mathcal{C}$. From the definition of $q$ the map $i_{H*} : H^{2(i-q)-1}(\mathcal{H}, \mathbb{Q}_l(i-q)) \to H^{2i-1}(\mathcal{X}, \mathbb{Q}_l(i))$ is surjective by hard Lefschetz theorem. So the cycle class $cl^1(z) \in \text{Ext}^1_{\mathbb{Q}_l}(\mathbb{Q}_l, H^{2i-1}(\mathcal{X}, \mathbb{Q}_l(i)))$ is the image of $cl^1(z) \in \text{Ext}^1_{\mathbb{Q}_l}(\mathbb{Q}_l, H^{2(i-q)-1}(\mathcal{H}, \mathbb{Q}_l(i-q)))$ under the Gysin map. It is also equal to the push-out by the quotient $H^{2(i-q)-1}(\mathcal{H}, \mathbb{Q}_l(i-q)) \to \partial H^{2i-2}(\mathcal{U}, \mathbb{Q}_l(n))$. So there is a splitting

$$cl^1(z) : \mathbb{Q}_l \to \frac{H^{2(i-q)-1}(\mathcal{H} \cap \mathcal{Y}, \mathbb{Q}_l(1))}{\partial H^{2i-2}(\mathcal{U}, \mathbb{Q}_l(n))}.$$
Theorem 3.1. The push-out of \(-c^2(z) \in \text{Ext}^2_{\text{cl}}(\mathbb{Q}_l, H^{2i-2}(\overline{Y}, \mathbb{Q}_l(i)))\) by the quotient \(H^{2i-2}(\overline{Y}, \mathbb{Q}_l(i)) \to \frac{H^{2i-2}(\overline{Y}, \mathbb{Q}_l(i))}{H^{2i-2}(\overline{Y}-U, \mathbb{Q}_l(i))}\) is given by the pull-back of the 2-extension

\[
0 \to \frac{H^{2i-2}(\overline{Y}, \mathbb{Q}_l(n))}{H^{2i-2}(\overline{Y}-U, \mathbb{Q}_l(i))} \to H^{2i-2}(\overline{U}, \mathbb{Q}_l(i)) \to H^{2i-1}(\overline{Y}, \mathbb{Q}_l(i)) \to 0
\]

by \(c^1(z) : \mathbb{Q}_l \to \frac{H^{2i-1}(\overline{Y}, \mathbb{Q}_l(i))}{\partial H^{2i-2}(\overline{U}, \mathbb{Q}_l(i))}\).

Remark.
When \(i\) is equal to \(n\), \(H^{2n-2}(\overline{Y}, \mathbb{Q}_l(n))\) is generated by the cohomology class of \(H(1)\). So we do not lose too much information by the push-out.

**Proof.** We have an exact triangle

\[
Rg_i R_h^i \mathbb{Q}_l(i) \xrightarrow{i_{|U}} Rg_i \mathbb{Q}_l(i) \xrightarrow{j^*_\circ g^*_\circ j_\circ} R(g \circ j)_\circ \mathbb{Q}_l(i)|_U \to .
\]

We denote this triangle by \(A \xrightarrow{i} B \xrightarrow{j} C \to .\) The 2-extension \(\chi_{2i-2}(Y)\) is given by

\[
0 \to H^{2i-2}(\overline{Y}) \xrightarrow{i_{|U}} \frac{B^{2i-2}}{\partial_B^{2i-3}(B^{2i-3})} \xrightarrow{j^*_\circ g^*_\circ j_\circ} \text{Ker} \partial_B^{2i-1} \to H^{2i-1}(\overline{Y}) \to 0.
\]

Let \(C_2\) be the complex

\[
0 \to H^{2i-2}(\overline{Y}) \xrightarrow{(j^*_\circ g^*_\circ j_\circ)^{-1} \text{Ker} \partial_C^{2i-2} + i_{|U}(A^{2i-2}) \partial_B^{2i-2}} \frac{H^{2i-1}(\overline{Y})}{\partial(H^{2i-2}(\overline{U}))} \to 0
\]

Let \(C_3\) be the complex

\[
0 \to \frac{H^{2i-2}(\overline{Y})}{\text{Im}(i_{|U}(\text{Ker} \partial_A^{2i-2}))} \xrightarrow{(j^*_\circ g^*_\circ j_\circ)^{-1} \text{Ker} \partial_C^{2i-2} + i_{|U}(A^{2i-2}) \partial_B^{2i-2}} \frac{H^{2i-1}(\overline{Y})}{\partial(H^{2i-2}(\overline{U}))} \to 0
\]
Let $C_4$ be the complex

$$
0 \to \frac{H^{2i-2}(\mathcal{Y})}{H_{Y-U}^{2i-2}(\mathcal{Y})} \to H^{2i-2}(\mathcal{U}) \to H_{Y-U}^{2i-1}(\mathcal{Y}) \to \frac{H_{Y-U}^{2i-1}(\mathcal{Y})}{\partial(H^{2i-2}(\mathcal{U}))} \to 0
$$

There are natural maps of complexes

$$
\chi_{2i-2}(\mathcal{Y}) \leftarrow C_2 \to C_3 \to C_4.
$$

So there are natural maps between the pull-backs of these complexes by the splittings given by $cl^1(z)$. Since $C_2$ is exact this completes the proof. \hfill \square

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