Effects of Dean number and curvature on fluid flow through a curved pipe with magnetic field

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Abstract

Numerical study is performed to investigate the Magnetohydrodynamics fluid flow through a curved pipe with circular cross-section under various conditions. Spectral method is applied as a main tool for the numerical technique; where, Fourier series, Chebyshev polynomials, Collocation methods, and Iteration method are used as secondary tools. The Magnetohydrodynamics incompressible viscous steady flow through a curved pipe with circular cross-section is investigated numerically to examine the combined effects of high Dean Number $D_n$, magnetic parameter $M_g$ and non-dimensional curvature $\delta$. The flow patterns have been shown graphically for large Dean Numbers as well as magnetic parameter and a wide range of curvatures $0.01 \leq \delta \leq 0.4$. Two vortex solutions have been found. Axial velocity has been found to increase with the increase of Dean Number and decrease with the increase of curvature and magnetic parameter. For high magnetic parameter, Dean Number and low curvature almost all the fluid particles strength are weak.

Keywords: Dean Number; magnetic parameter and curvature

Nomenclature

| Symbol | Description               |
|--------|---------------------------|
| $D_n$  | Dean Number               |
| $M_g$  | Magnetic Parameter        |
| $\delta$ | Non-dimensional Curvature of the Pipe |
| $\sigma$ | Electrical Conductivity   |
| $J$    | Current density           |
| $H$    | Magnetic field            |
| $\mu_e$ | Magnetic Permeability     |

1. Introduction

Fully developed flow in curved ducts is encountered in various practical processes. In the analysis of fluidic devices, flows in separation processes, heat exchangers, physiological systems are examples of such processes. In the past few
decades, most of the research works have been done on the fully developed flow through curved ducts. Therefore, the fully developed flow phenomena in the curved ducts with magnetic field have drawn a keen attention.

Dean (1927) first formulated the curved duct problem mathematically under the fully developed flow conditions and confirmed the existence of a pair of counter rotating vortices as a secondary flow in the curved pipe. For the fully developed flow in a curved circular pipe, Ito (1951) separately showed the existence of a two-vortex secondary flow patterns by using perturbation method as was done by Dean (1927). Cheng and Akiyama (1970) and Cheng et al. (1975) reported two-vortex secondary flow patterns in a curved duct with square cross-section by using finite difference method. Masliyah (1980) investigated both numerically and experimentally the flow through a semi-circular duct with a flat outer wall. He found the existence of dual solution i.e. both two-vortex and four-vortex secondary flow patterns exist at the same Dean number. Both Nandakumar and Masliyah (1982) and Dennis and Ng (1982) separately obtained dual solution for the flow through a curved tube with circular cross-section. They found that two-vortex solution and four-vortex solution coexist. Later, the stability of the dual solutions of two-vortex and four-vortex secondary flow patterns was studied by Yanase et al. (1989). They found that the two-vortex secondary flow patterns are stable while the four-vortex flow patterns are unstable. In the numerical research works by Shanthini and Nandakumar (1986), Winters (1987) and Daskopolous and Lenhoff (1989), dual solutions for fully developed flow in a curved duct of square cross-section are found. Yang and Wang (2001) studied numerically the bifurcation structure and stability of the solutions of fully developed viscous flow in curved square duct. The governing equations were discretized by using the finite volume method.

Dean’s work was extended by Reid (1958). Dean vortices were observed experimentally by Brewster et al. (1959) in a channel with an aspect ratio of 35 and a curvature ratio of 12.5. Later, Finlay (1989) used a weakly non-linear perturbation analysis to determine the non-linear equation of two dimensional vortices in a curved channel with infinite aspect ratio. Finlay and Nandakumar (1990) investigated the onset of Dean vortices in curved rectangular channels with aspect ratios ranging from 20 to 30. They used two dimensional finite difference methods in this investigation and found vortex pairs in the centre of the channel.

The effects of the magnetic field on fluid flow have been studied primarily for straight pipes by Chang and Lundgren (1961), Shercliff (1956), Walker (1986) and Holroyd (1978). Shercliff (1956) solved the problem of flow in circular pipes under transverse magnetic fields in an approximate manner for large Hartmann numbers assuming walls of zero and small conductivity. The effect of wall conductivity was also studied by Chang and Lundgren (1961). Pressure drop in thin walled circular straight ducts was studied by Holroyd and Walker [9], neglecting the inertial effects and induced magnetic field. Recently, Walker (1986) developed solutions to MHD flow equations by asymptotic analysis for circular straight ducts under strong transverse magnetic fields.

At present, our aim is to obtain a detail results on the Dean numbers as well as magnetic parameter at curvatures (δ) = 0.01 and 0.2. In this present study, the magnetic field has been imposed along the centre line of a curved pipe.

2. Governing equation

The basic equations for steady laminar flow are

Continuity: \( \nabla \cdot \mathbf{q} = 0 \) \hspace{1cm} (1)

Navier-Stoke’s equation for incompressible fluid is

\[ \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} = - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{q} \] \hspace{1cm} (2)

Now, in the equation of motion appears a body force \( \frac{1}{\rho} \mathbf{J} \wedge \mathbf{B} \) per unit volume of electro- magnetic origin. If there is no other body force then the equation (2) becomes

\[ \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} = - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{q} + \frac{1}{\rho} \mathbf{J} \wedge \mathbf{B} \] \hspace{1cm} (3)

Let us consider a curved pipe with circular cross-section containing incompressible inviscid fluid. Let the radius of the pipe
be \( L \), radius of cross-section be \( a \)(Fig. 1). To reach the point \((x,y,z)\) we have to travel \(L + r\) unit \((0 \leq r \leq a)\) along \(x\)-axis, then turn an angle \(\theta\) considering the origin as centre in the \(xy\) plane, then turn again an angle \(\alpha\) in the plane of cross-section. Then \((x, y, z)\) and \((r, \alpha, \theta)\) are related as,

\[
x = (L + r \cos \alpha) \cos \theta, \quad y = (L + r \cos \alpha) \sin \theta, \quad z = r \sin \alpha
\]

Let \( q = q_x i + q_y j + q_z k = q_x e_x + q_a e_a + q_\theta e_\theta \) be the velocity of a particle of the fluid when it is supposed to a constant pressure gradient force only. So the equation of continuity takes the form,

\[
\nabla q = \frac{\partial}{\partial r} \left\{ r (L + r \cos \alpha) q_r \right\} + \frac{\partial}{\partial \alpha} \left\{ (L + r \cos \alpha) q_\alpha \right\} + \frac{\partial}{\partial \theta} \left\{ r q_\theta \right\} = 0
\]

(4)

By using Ohm's law and Maxwell's equation the Navier-Stokes equation in the radial, circumferential and axial direction becomes.

Radial direction (from the coefficient of \( e_r \))

\[
\frac{\partial q_r}{\partial t} + \left( q_r \frac{\partial}{\partial r} + q_a \frac{\partial}{\partial \alpha} + \frac{q_\theta}{r} \frac{\partial}{\partial \theta} \right) q_r + \frac{q_a^2 \cos \alpha}{(L + r \cos \alpha)} = \frac{1}{\rho} \frac{\partial}{\partial r} \left( \rho \left( \frac{\partial q_r}{\partial r} + \frac{q_a}{r} \frac{\partial q_\alpha}{\partial \alpha} \right) \right) - \frac{\sigma \mu^2}{\rho} q_\alpha H_\theta^2
\]

Circumferential direction (from coefficient of \( e_\alpha \))

\[
\frac{\partial q_\alpha}{\partial t} + \left( q_r \frac{\partial}{\partial r} + q_a \frac{\partial}{\partial \alpha} + \frac{q_\theta}{r} \frac{\partial}{\partial \theta} \right) q_\alpha + \frac{q_a q_\theta}{r \cos \alpha} = \frac{1}{\rho} \frac{\partial}{\partial \alpha} \left( \rho \left( \frac{\partial q_\alpha}{\partial \alpha} + \frac{q_r}{r} \frac{\partial q_r}{\partial r} - \frac{q_\theta}{r \cos \alpha} \right) \right) - \frac{\sigma \mu^2}{\rho} q_a H_\theta^2
\]
Axial direction (from coefficient of $e_\theta$)

$$
\frac{\partial q_\theta}{\partial t} + \left( q_r \frac{\partial}{\partial r} + \frac{q_\alpha}{r} \frac{\partial}{\partial \alpha} + \frac{q_\theta}{L + r \cos \alpha} \frac{\partial}{\partial \theta} \right) q_\theta + \frac{\cos \alpha}{L + r \cos \alpha} q_\alpha q_\theta - \frac{\sin \alpha}{L + r \cos \alpha} q_\alpha q_\theta
$$

$$
= \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \left( \frac{1}{L + r \cos \alpha} \frac{\partial}{\partial \alpha} \right) q_\theta \cos \alpha + \frac{1}{r^2} \frac{\partial^2 q_\theta}{\partial \alpha^2} - \frac{1}{r \cos \alpha} \left[ q_\alpha \sin \alpha \left( \frac{1}{L + r \cos \alpha} \right) \right]
$$

3. Toroidal coordinates

Let us defined the following non-dimensional variables

$$
u' = \frac{q_r}{\nu}, \quad v' = \frac{q_\alpha}{\nu}, \quad w' = \frac{q_\theta}{\nu} \frac{2a}{L}, \quad r' = \frac{r}{a}, \quad S' = \frac{L \theta}{a}, \quad a = \delta, \quad p' = \frac{p}{\rho \left( \frac{v}{a} \right)^2}
$$

where $u', v', w'$ are non-dimensional velocities along the radial, circumferential and axial direction respectively. $r'$ is non-dimensional radius, $s'$ is the non-dimensional axial variable, $\delta$ is non-dimensional curvature and $p'$ non-dimensional pressure.

Constant pressure gradient force is applied along the axial direction through the centre of cross section. At the centre of the cross-section $r = 0$ and at the boundary of the cross-section $r = a$, where all the velocity components are zero. In dimensionless form this reduces to $r' = 0$ at the centre of cross-section and $r' = 1$ at the boundary of the cross-section. With the help of the above dimensionless variables and the boundary conditions the radial and circumferential equation of motion reduces to the following form:

$$
\frac{1}{r^2} \left( \frac{\partial^2 \psi}{\partial \alpha^2} - \frac{\partial^2 \psi}{\partial S^2} \right) + \Delta \psi + w' \left( \sin \alpha \frac{\partial \omega'}{\partial r'} + \frac{\cos \alpha}{r'} \frac{\partial \omega'}{\partial \alpha} \right) - M \mu \psi = 0 \quad (5)
$$

and the axial equation becomes

$$
\frac{1}{r^2} \left( \frac{\partial^2 \psi}{\partial \alpha^2} - \frac{\partial^2 \psi}{\partial S^2} \right) + \Delta \omega' + D_s = 0 \quad (6)
$$

where, $\Delta \equiv \frac{\partial^2}{\partial r'^2} + \frac{1}{r'} \frac{\partial}{\partial r'} + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2}, \quad G = \frac{\partial p}{\partial S}, \quad D_s = \frac{a^2}{\mu \nu} \frac{2a}{L} G$ and $M = \sigma' \mu \nu H^2 \mu$. Here, $\psi$ is the stream function defined by, $u' = \frac{1}{r'} \frac{\partial \psi}{\partial \alpha}, \quad v' = -\frac{\partial \psi}{\partial r'}, \quad G$ is the constant pressure gradient force, $\mu$ is the viscosity, $\nu$ is the kinematic viscosity, $D_s$ is the Dean number and $M$ is the magnetic parameter. Equation (5) and (6) are called secondary and axial flow respectively. The dimensionless flux $\kappa$ is given by, $\kappa = \frac{\sqrt{3}}{\pi} \int_0^1 \int_0^{2\pi} w' d\alpha dr'$.  

4. Numerical technique

The Spectral method which is a very useful numerical tool for solving Navier-Stokes equation(Gottelib and Orszag 1977) has been used to solve the equations (1) and (2). Fourier series and Chebyshev polynomials are used in circumferential and radial directions respectively. Assuming that steady solution is symmetric with respect to the horizontal line of the cross-
section, $\psi$ and $w'$ are expanded as,

$$\psi(r', \alpha) = \sum_{n=1}^{N} f_n(r') \sin n\alpha + \sum_{n=1}^{N} f_n(r') \cos n\alpha$$

and $w'(r', \alpha) = \sum_{n=1}^{N} w_n(r') \sin n\alpha + \sum_{n=1}^{N} w_n(r') \cos n\alpha$

The collocation points are taken to be, $R = \cos\left(\frac{N+2-i}{N+2}\right) \pi [1 \leq i \leq N+1]$. Then we get non-linear equations for $W_{mx}, W_{nm}, F_{mx}, F_{nm}$. The obtained non-linear algebraic equations are solved under an iteration method with under-relaxation. Convergence of this solution is taken up to five decimal places by taking $\epsilon_p < 10^{-5}$. Where

$$\epsilon_p = \sum_{n=1}^{M} \sum_{m=0}^{N} \left[ (F_{mx}^{(n)} - F_{mx}^{(n+1)})^2 + (W_{mx}^{(n)} - W_{mx}^{(n+1)})^2 \right] + \sum_{n=0}^{N} \sum_{m=0}^{M} \left[ (F_{nm}^{(n)} - F_{nm}^{(n+1)})^2 + (W_{nm}^{(n)} - W_{nm}^{(n+1)})^2 \right]$$

Here, $p$ is the iteration number. The values of $M$ and $N$ are taken to be 60 and 35 respectively for better accuracy.

5. Results and discussions

Flow through a curved pipe of circular cross section with Magnetic field has been considered. The flow is governed by two non-dimensional parameters: the Dean number ($D_n$) and the magnetic parameter ($M_g$). In this paper, steady laminar flow for viscous incompressible fluid has been analyzed under the action Dean numbers as well as magnetic parameter at curvatures $\delta = 0.01$ and 0.2.

The results have been shown through stream line, vector plots of the secondary flow and contour plots of the axial flow. The stream line, vector and contour plots of the flow development have been shown at different magnetic parameters for Dean number $D_n = 800, 1500$ and $2000$ at non-dimensional curvature $\delta = 0.01$ and 0.2 which are arranged in a column form left to right. In each figure, six columns have been produced. Among them the first three columns shows the stream line, vector plots of the secondary flow and contour plots of the axial flow behaviour for non-dimensional curvature $\delta = 0.01$ at different Dean number. Similarly, the last three columns shows the stream line, vector plots of the secondary flow and contour plots of the axial flow behaviour for non-dimensional curvature $\delta = 0.2$ at different Dean number as well as magnetic parameter.

In Fig. 2 non-dimensional flux ($\kappa$) has been plotted against Dean number at $\delta = 0.01$ and 0.2 for different magnetic parameter respectively. And for each figure it is clear that the flux increases with the increase of Dean number as well as magnetic parameter. But if the magnetic parameter increases continuously the rate of change of flux is negligible. It is found that if the magnetic parameter increased then total flow will automatically be increased for each figure. And finally after a comprehensive survey over the parametric space, steady solution curve has been obtained in Fig. 3-Fig. 5.
The stream line, vector plots of the secondary flow and axial flow for Dean Number $D_n = 800$, $1500$, $2000$ at non-dimensional curvature $\delta = 0.01$ and $0.2$ have been shown at the first, second and third column respectively in Fig. 3- Fig. 5. The highest values of magnetic parameter, increment in axial velocity ($\Delta w$), increment in constant $\psi$-lines ($\Delta \psi$) have been given on the left side.

The length of arrow indicates the ratio of the stream velocity to the axial velocity and the direction of the flow in vector plots are always indicated by an arrowhead, no matter how small the flow is. Thus, the relative strength of the flow is not resolved for areas of a very weak secondary flow. In Fig. 3 the vector plots of the secondary flow show the direction of the fluid particles and the strength of the vortex is shifted towards outer half of the cross-section as magnetic parameter increases.

![Fig. 3. The secondary flow, vector plots of the secondary flow and axial flow for different values of magnetic parameter at Dean Number $D_n = 800$](image)

In case of secondary flow behaviour, symmetric contour plots have been found which are shown Fig 3. As magnetic parameter increases there originate a secondary flow and only 2-vortex solution has been found for the secondary flow. The two vortexes are of same strength but rotating in counter clockwise direction. In Fig. 3 the axial flow is greater in magnitude than secondary flow and it varies a great deal with magnetic parameter. As a result the difference between two consecutive contours line of the axial flow have been taken different for different magnetic parameters. For high magnetic parameter, Dean number and low curvature, the axial flow is shifted towards the centre of the pipe as a result almost all the fluid particles strength are weak.

In case of axial flow behaviour (Fig. 4), the axial flow is also symmetric. The fluid particles are shifted towards the outer wall of the cross section and form a low velocity band inside the outer wall of the cross-section in Fig. 4. As magnetic parameter decreases the magnitude of the axial flow gets higher. The axial flow decreases with the increase of Magnetic parameter. Also the maximum axial flow is shifted to the centre from the wall of the cross section as Magnetic parameter increases. With the development of the flow, the magnitude of the secondary velocity decreases up to a certain limit. When the flow is fully developed, it does not change further.
Two vortex secondary flows have been found which is symmetric about the horizontal line passing through the centre of the cross section with the presence of magnetic field in Fig. 4. Most of the particle gets the radial velocity. The two vortexes are of same strength but rotating in opposite direction. Where the upper vortex is rotating anti-clock wise and the lower vortex is rotating clock wise. At $M_g = 500$ and $D_n = 2000$ axial flow contours are nearly circular and are eccentric with the centres shifted towards the outer wall of the tube pipe in Fig 5. At $M_g = 25000$ a strong magnetic field has been found to confine the secondary flow streamlines to a thin layer near the tube wall. The secondary flow rate in the near wall boundary is increased by the magnetic field. The stable solution zone initially increases with the increase of curvature.

In Fig. 5 the largest magnetic parameter to give stable solution as well as an extra circular zone is $M_g = 25000$ for $D_n = 2000$. The contour plots of the axial velocity have been shown in Fig 5. for magnetic parameter $M_g = 20, 150, 1500, 3000, 25000$ respectively at $D_n = 2000$.

As the flow enters the pipe boundary layer begins to develop. Boundary layer near the inner wall develops faster than that at the outer wall. Just after the entrance, the axial velocity of the particle in the inner half is lower for small curvature. But as the flow precedes downstream the particles in outer half attains higher velocity.

Due to the effect of magnetic field a bracelet has been originates from the right corner of the duct and expands at $M_g = 1500$ in Fig. 5. This bracelet gradually increases with the increase of magnetic parameter and shifted to the centre. In Fig. 5, the bracelet finally dropped to the centre at $M_g = 25000$. In the case, of vector plots of secondary flow (Fig. 5) a clock wise rotating vortex is set up after the entrance at $M_g = 20$. Also an anti-clock wise rotating vortex originates from the top and expands. On the other hand the secondary velocity of the particles around the centre of cross-section decreases starting from the particles above the centre to the particles below the centre in Fig. 5. Finally a strong magnetic field is found to confine the secondary flow streamlines to a thin layer near the tube wall.
6. Conclusions

As magnetic parameter as well as magnetic parameter increases there originate a symmetric contour plot of secondary flow and only 2-vortex solution has been found for the secondary flow. The two vortices are of same strength but rotating in counter clockwise direction. The strength of the vortices is shifted to the outer half from the inner half with the increase of Dean number and curvature. For high magnetic parameter, Dean number and low curvature, the axial flow is shifted towards the centre of the pipe as a result almost all the fluid particles strength are week. And finally due to the combined effect of the Dean number and curvature a bracelet has been originates from the right corner of the duct and expands at $D_n = 2000$.

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References

[1] Akiyama, M. and Cheng, K. C. Graetz problem in curved pipe with uniform wall Heat; 1970
[2] Akiyama, M. and Cheng, K. C. Graetz problem in curved pipes with uniform wall Heat flux, Applied Science Research, Vol. 29, pp. 401-418; 1975
[3] Brewster, D. B., Grosberg, P. and Nissan, A. H. The stability of viscous flow between horizontal concentric cylinders, Proceeding of Royal Society of London A, Vol. 251, pp. 76-91; 1959
[4] Cheng, K. C., Lin, R. and Ou, J. W. Graetz problem in curved square channels, Trans. ASME Journal of heat transfer, Vol. 97, pp. 244-248; 1975
[5] C. C. Chang and T. S. Lundgren, Z, Angew. Math. Phys. 12, 100; 1961
[6] Daskopoulos, P. and Lenhoff, A. M. Flow in curved ducts: bifurcation structure for stationary ducts, *Journal of Fluid Mechanics*, Vol. 203, pp. 125-148; 1989
[7] Dean, W. R. Note on the motion of fluid in a curved pipe, *Philosophical magazine and Journal of Science*, Vol. 4(20), pp. 208-223; 1927
[8] Dennis, S. C. R. and Ng, M. Dual solutions for steady laminar flow through a curved tube, *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 35, pp.305-324; 1982
[9] Gottlieb, D. and Orszag, S. A. Numerical Analysis of Spectral Methods, *Society for Industrial and Applied Mathematics, Philadelphia*; 1977
[10] Ito, H. Theory on laminar flows through curved pipes of elliptic and rectangular cross-section, The report of the institute of high speed Mechanics, Tohoku University, Sendai, Japan, Vol. 1, pp. 1-16; 1951
[11] J. A. Shercliff, *J. Fluid Mech.*, 1, 644; 1956
[12] J. S. Walker, *J. Fluid Mech.*, 167, 199;1986
[13] Masliyah, J. H. On laminar flow in curved semicircular ducts, *Journal of Fluid Mechanics*, Vol. 99, pp. 469-479; 1980
[14] Nandakumar, K. and Masliyah, J. H. Swirling flow and heat transfer in coiled and twisted pipes, *Advances in Transport Processes*, Vol.4, pp 49-112; 1982
[15] Nandakumar, K., Mees, P. A. J. and Masliyah, J. H. Multiple two-dimensional solutions to the Dean problem in triangular ducts, *Physics of Fluids A*, Vol. 5 (5),pp. 1182-1187; 1993
[16] Reid, W. H. On stability of viscous flow in curved channel, *Proceeding of Royal Society of London A*, Vol. 244, 186-198; 1958
[17] R. J. Holroyd and J. S. Walker, *J. Fluid Mech.*,84, 47; 1978
[18] Yanase, S., Goto, N. and Yamamoto, K. Dual solutions of the flow through a curved tube, *Fluid Dynamics Research*, Vol. 5, pp. 191-201; 1989
[19] Yang, T. and Wang, L. Solution Structure and Stability of Viscous Flow in Curved Square Ducts, *ASME Journal of Fluids Engineering*, Vol. 123, pp. 863-868; 2001