Wave Packet Frame in Two Dimensions in General Lattices

To cite this article: Guochang Wu and Lanfeng Wang 2018 IOP Conf. Ser.: Mater. Sci. Eng. 394 052083

View the article online for updates and enhancements.

Related content
- Generation of giant spatially localised Gaussian wave packets in active fibres with saturable inertial nonlinearity
  V.M. Zhuravlev, I.O. Zolotovskii, P.P. Mironov et al.
- Mappings that raise the fundamental dimension
  Yu V Lubenets
- ON THE DIMENSION Dim OF TOPOLOGICAL SPACES
  D P Adnadzevi
Wave Packet Frame in Two Dimensions in General Lattices

Guochang Wu1, * and Lanfeng Wang2

1School of Arts and Science, Sias International College of Zhengzhou University, Xinzheng 451150, P. R. China
2School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, P. R. China

*Corresponding author e-mail: archang-0111@163.com

Abstract. Frame theory is a useful tool for many engineering applications such as in signal processing, image processing, data compression and sampling theory. In this paper, symmetric wave packet frames with several generators about general shift lattices in two dimensions are constructed from any frames given, which generalizes the existing results to the case of general shift lattices.

1. Introduction

Frame theory is an important tool in many engineering applications such as signal processing, image processing, data compression and sampling theory. Its main advantage is the fact that frame is a redundant system which can provide reconstruction formula [1]. Recently, frames are also used to in communication systems to improve the robustness of data transmission [2] and to design high-rate constellation in multiple-antenna code design [3].

Wavelet frame is a kind of important frame, which is obtained by translating and dilating some functions. Wavelets attracted many interest including the mathematical community and many other diverse disciplines. Another important frame is Gabor frame, which are obtained by modulations and translations of some functions.

In paper [4], some authors defined wave packet systems by applying dilations, modulations and translations to the Gaussian function. In paper [5], authors defined general wave packet systems which generalize the Gaussian function to any functions by applying the same operations. Then, we give wave packet systems by the following definition

$$\{D_{l}^{j}E_{al}T_{bk}g^{m} : j \in Z, l, k \in Z^{2}, m = 1,2, L M\}$$

Where

$$D_{l}^{j}f(x) = \sqrt{2}f(2x), T_{a}f(x) = f(x-ak)$$

and

$$E_{al}f(x) = e^{2\pi i x \cdot \lambda}_{a}f(x)$$. In fact, Gabor systems and wavelet systems are special wave packet systems.

In paper [6], authors studied wave packet systems by the theory of shift invariant systems and gave a sufficient condition of wave packet frame and constructed many good examples. In [7], authors provided a way to construct symmetric and antisymmetric wavelet frames. By use of the ways of [7], people began to construct symmetric wave packet systems from any frames given. In [8], symmetric wave packet frames with integer shift are constructed. In [9], authors generalized the above result to the case of several generators. In [10], symmetric wave packet frames with several generators about origin
in two dimensions are constructed from any frames given, which generalizes the above results to higher dimensions.

In this paper, symmetric wave packet frames with several generators about general lattices are studied from any frames given, which generalizes the integer shift in [10] to the case of general shift.

2. Preliminaries
In this paper, \( N, R \) and \( Z \) are referred to the set of natural numbers, the set of the set of integers and real numbers, respectively. \( L^2(R^n) \) is the square-integrable functions’ space in \( n \) dimensions, and \( <\cdot,\cdot>, \|\cdot\| \) denote the inner product and norm in \( L^2(R^n) \), respectively.

Then we give definition of frame.

Definition 2.1 let \( H \) be a separable Hilbert space. A sequence \( \{f_n\}_{n \in N} \) of elements of \( H \) is a frame for \( H \) if there exist constants \( 0 < C \leq D < \infty \) such that for all \( f \in H \), we have

\[
C\|f\|^2 \leq \sum_{n \in N} \langle f, f_n \rangle^2 \leq D\|f\|^2.
\]

The numbers \( C, D \) are called lower and upper frame bounds, respectively (the largest \( C \) and the smallest \( D \) for which (2.1) holds are the optimal frame bounds). Those sequences which satisfy only the upper inequality in (2.1) are called Bessel sequences.

Let \( T_f \) denote the synthesis operator of \( f = \{f_i\}_{i \in N} \), i.e., \( T_f(c) = \sum_i c_i f_i \) for each sequence of scalars \( c = (c_i)_{i \in N} \). Then the frame operator \( S_h = T_f T_f^* (h) \) associated with \( \{f_i\}_{i \in N} \) is a bounded, invertible, and positive operator mapping of \( H \) on itself. This provides the reconstruction formula

\[
h = \sum_{i=1}^\infty <h, f_i> f_i = \sum_{i=1}^\infty <h, f_i> g_i, \quad \forall h \in H. \tag{2.2}
\]

Where \( g_i = S^{-1} f_i \). The family \( \{g_i\}_{i \in N} \) is also a frame for \( H \) and is called the canonical dual frame of \( \{f_i\}_{i \in N} \). If \( \{g_i\}_{i \in N} \) is any sequence in \( H \) which satisfies

\[
h = \sum_{i=1}^\infty <h, g_i> f_i = \sum_{i=1}^\infty <h, f_i> g_i, \quad \forall h \in H, \tag{2.3}
\]

It is called an alternate dual frame of \( \{f_i\}_{i \in N} \).

Then, we will give the definitions of a wave packet frame and the frame wave packet functions.

Definition 2.2 we say that the wave packet system defined by (1.1) is a wave packet frame if it is a frame for \( L^2(R^n) \). Then, the vector functions \( (g^{1'}, g^{2'}, L, g^{M'}) \) are called the frame wave packet functions.

In the following, we will construct symmetric wave packet frames with general lattices from any frames given.

3. Symmetric Wave Packet Frames with General Lattices
Let \( (g^{1'}, g^{2'}, L, g^{M'}) \subset L^2(R^2) \), define new symmetric functions in two dimensions as the following:
\[ g_1^m(x) = \frac{g^m(x) + g^m(-x)}{2}, \quad g_2^m(x) = \frac{g^m(x) - g^m(-x)}{2}, \quad m = 1, 2, L, M. \] (3.1)

So, we will prove Theorem 3.1 if that wave packet system
\[ \{ D_{2j}^m E_{al_{bk}}^m : j \in Z, k \in Z^2, l \in Z^2, m = 1, 2, L, M \} \]

Defined by (1.1) is a frame for \( L^2(R^2) \) with frame bounds \( C_1, C_2 \), then wave packet system
\[ \{ D_{2j}^m E_{al_{bk}}^m \cup D_{2j}^m E_{al_{bk}}^m : j \in Z, k \in Z^2, l \in Z^2, m = 1, 2, L, M \} \] (3.2)

Is also a symmetric frame with same frame bounds, where the functions \( g_1^m(x), g_2^m(x) \) are defined by (3.1).

Proof. Because wave packet system
\[ \{ D_{2j}^m E_{al_{bk}}^m : j \in Z, k \in Z^2, l \in Z^2, m = 1, 2, L, M \} \]

Is a frame for all \( f(x) \in L^2(R^2) \), thus, we have
\[ C_1 \| f \|^2 \leq \sum_{m=1}^{M} \sum_{j \in Z} \sum_{l \in Z^2} \sum_{k \in Z^2} \langle f, D_{2j}^m E_{al_{bk}}^m \rangle^2 \leq C_2 \| f \|^2. \] (3.3)

Then, we discuss the series
\[ \sum_{m=1}^{M} \sum_{j \in Z} \sum_{l \in Z^2} \sum_{k \in Z^2} \langle f, D_{2j}^m E_{al_{bk}}^m \rangle^2 + \sum_{m=1}^{M} \sum_{j \in Z} \sum_{l \in Z^2} \sum_{k \in Z^2} \langle f, D_{2j}^m E_{al_{bk}}^m \rangle^2. \] (3.4)

Also, we have
\[ | \langle f(\cdot), D_{2j}^m E_{al_{bk}}^m \rangle |^2 = | \langle f(\cdot), D_{2j}^m E_{al_{bk}}^m \rangle^2 + \langle f(\cdot), D_{2j}^m E_{al_{bk}}^m \rangle \frac{g^m(\cdot)}{2} + \langle f(\cdot), D_{2j}^m E_{al_{bk}}^m \rangle \frac{g^m(-\cdot)}{2} |^2. \] (3.5)

For any complex numbers \( z_1, z_2 \), we get
\[ | z_1 + z_2 |^2 = | z_1 |^2 + | z_2 |^2 + z_1 \bar{z}_2 + z_2 \bar{z}_1. \] (3.6)

Comparing with (3.5) and (3.6), we get
\[ \langle f(\cdot), D_{2j}^m E_{al_{bk}}^m(\cdot) \rangle \frac{g^m(\cdot)}{2} = \frac{1}{4} | \langle f(\cdot), D_{2j}^m E_{al_{bk}}^m(\cdot) \rangle |^2 + \frac{1}{4} | \langle f(\cdot), D_{2j}^m E_{al_{bk}}^m(\cdot) \rangle |^2 \]
\[ + \frac{1}{4} | \langle f(\cdot), D_{2j}^m E_{al_{bk}}^m(\cdot) \rangle |^2 + \frac{1}{4} | \langle f(\cdot), D_{2j}^m E_{al_{bk}}^m(\cdot) \rangle |^2. \] (3.7)

In the same way, we can get
\[ \langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle^2 = \frac{1}{4} \langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle^2 + \frac{1}{4} \langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle^2 \]

\[ - \frac{1}{4} \langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle \langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle \]

From (3.7) and (3.8), we get

\[ \sum_{m=1}^{M} \sum_{j \in Z} \sum_{i \in Z^2} \sum_{k \in Z^2} |\langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle|^2 \geq \frac{1}{2} \sum_{m=1}^{M} \sum_{j \in Z} \sum_{i \in Z^2} \sum_{k \in Z^2} |\langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle|^2 \]

\[ + \frac{1}{2} \sum_{m=1}^{M} \sum_{j \in Z} \sum_{i \in Z^2} \sum_{k \in Z^2} |\langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle|^2. \]

Thus, we obtain

\[ \sum_{m=1}^{M} \sum_{j \in Z} \sum_{i \in Z^2} \sum_{k \in Z^2} |\langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle|^2 \leq \sum_{m=1}^{M} \sum_{j \in Z} \sum_{i \in Z^2} \sum_{k \in Z^2} |\langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle|^2. \]  \( (3.10) \)

From (3.3), we get

\[ C_1 \| f(\cdot) \| \leq \sum_{m=1}^{M} \sum_{j \in Z} \sum_{i \in Z^2} \sum_{k \in Z^2} |\langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle|^2 \leq C_2 \| f(\cdot) \|^2, \]

\( (3.11) \)

From (3.10), (3.11) and \( \| f(\cdot) \|^2 = \| f(\cdot) \|^2 \), we obtain

\[ C_1 \| f(\cdot) \|^2 \leq \sum_{m=1}^{M} \sum_{j \in Z} \sum_{i \in Z^2} \sum_{k \in Z^2} |\langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle|^2 \leq C_2 \| f(\cdot) \|^2, \forall f(x) \in L^2(R^2). \]

\( (3.12) \)

At last, by use of (3.3), (3.9) and (3.12), we deduce

\[ C_1 \| f(\cdot) \|^2 \leq \sum_{m=1}^{M} \sum_{j \in Z} \sum_{i \in Z^2} \sum_{k \in Z^2} |\langle f(\cdot), D^2_2 E_{alT_{hk} g_1^m(\cdot)} \rangle|^2 \leq C_2 \| f(\cdot) \|^2. \]  Thus, we have completed the proof.

References
[1] R. Duffin, A. Schaeffer, A class of nonharmonic Fourier series. Trans. Amer. Math. Soc. 72 (1952) 341 - 366.
[2] V. Goyal, J. Kovacevic, J. Kelner, Quantized frames expansions with erasures. Appl. Comput. Harmon. Anal., Vol. 10 (2001), p. 203 - 233.
[3] B. Hassibi, B. Hochwald, A. Shokrollahi, W. Sweldens, Representation theory for high-rate multiple-antenna code design, IEEE Trans. Inform. Theory, Vol. 47 (2001), p. 2335 - 2367.
[4] A. Cordoba and C. Fefferman. Wave packets and Fourier integral operators. Comm. Partial
Differential Equations 3 (1978) 979 - 1005.

[5] O. Christensen and A. Rahimi. Frame properties of wave packet systems in $L^2(R^d)$. Adv. Comput. Math. 29 (2008) 101-111.

[6] E. Hernandez, D. Labate, G. Weiss and E. Wilson. Oversampling, quasi affine frames and wave packets. Appl. Comput. Harmon. Anal. 16 (2004) 111 - 147.

[7] S. Goh, Z. Lim, Z. Shen, Symmetric and antisymmetric tight wavelet frames, Appl. Comput. Harmon. Anal. 20 (2006) 411 - 421.

[8] G. Wu and H. Cao, Symmetric wave packet frames, Applied Mechanics and Materials 227 - 429 (2013) 1528 - 1531.

[9] J. Z. Li and X. Q. Yang, Symmetric Fusion Frames with Several Generators, Applied Mechanics and Materials 889 - 890 (2014) 575 - 578.

[10] Z. Wang and G. Wu, Symmetric wave packet frames in higher dimensions, IOP Conf. Series: Earth and Environmental Science 108 (2018) 052118.