Birkhoff theorem and conformal Killing-Yano tensors

Joan Josep Ferrando · Juan Antonio Sáez

Received: date / Accepted: date

Abstract We analyze the main geometric conditions imposed by the hypothesis of the Jebsen-Birkhoff theorem. We show that the result (existence of an additional Killing vector) does not necessarily require a three-dimensional isometry group on two-dimensional orbits but only the existence of a conformal Killing-Yano tensor. In this approach the (additional) isometry appears as the known invariant Killing vector that the $D$-metrics admit.

PACS 04.20.-q · 04.20.Jb

1 Introduction

The pioneering results by Jebsen [1] and Birkhoff [2] are applied to spherically symmetric vacuum solutions. Subsequently, several generalizations have been accomplished concerning either the energy content (two double eigen-values [3]), or the isometry group (plane and hyperbolical on space-like orbits [4], or maximal symmetry on time-like orbits [5]).

Thus, the two hypotheses of the original Jebsen-Birkhoff theorem have been weakened. Our aim here is to show that the present version of the former, the existence of a maximal group of symmetries on two-dimensional non-null orbits, admits a weaker statement.

In this paper we work on an oriented space-time with a metric tensor $g$ of signature $\{-, +, +, +\}$ and metric volume element $\eta$. The Riemann, Ricci and Weyl tensors are defined as given in [6] and are denoted, respectively, by $\text{Riem}$, $\text{Ric}$ and $W$. For the metric product of two vectors we write $(X, Y) = g(X, Y)$. If
A and B are 2-tensors, $A \cdot B$ denotes the 2-tensor $(A \cdot B)_{\alpha \beta} = A^\alpha \beta B^\mu \beta \cdot A^2 = A \cdot A$, $A(X, Y) = A_{\alpha \beta} X^\alpha Y^\beta$, $A(X) = A_{\alpha \beta} X^\beta$, and $(A, B) = 1/2 A_{\alpha \beta} B^\alpha \beta$. For a vector $X$ and a $(p+1)$-tensor $t$, $i(X)t$ denotes the inner product, $[i(X)t]_\mu = X^\alpha t_{\alpha \mu}$, the underline denoting multi-index. And if $\omega$ is a $(p+1)$-form, $\delta \omega$ denotes its exterior codifferential, $(\delta \omega)_{\mu} = -\nabla_\mu \omega^\alpha$.

Note that the existence of a maximal group of symmetries on two-dimensional non-null orbits implies that the metric is conformal to a 2+2 product one with a restricted conformal factor $[7]$, which leads to a 2+2 warped product space-time.

Then, a non null Killing-Yano tensor $A$ exists $[8]$. Now, we impose a weaker condition: the existence of a conformal Killing-Yano (CKY) tensor $A$. Then, the associated self-dual two-form $A = 1/\sqrt{2} (A - i \ast A)$ satisfies the CKY equation $[9]$:

$$3 \nabla A = 2i(\mathcal{Z}) \mathcal{G}, \quad \mathcal{Z} \equiv \delta A,$$  

where $\ast$ is the Hodge dual operator. The 4-tensor $\mathcal{G}$ is the endowed metric on the 3-dimensional complex space of the self-dual two-forms, $\mathcal{G} = \frac{1}{4}(G - i \eta)$, $G$ being the metric on the space of 2-forms, $G_{\alpha \beta \gamma \delta} = g_{\alpha \gamma} g_{\beta \delta} - g_{\alpha \delta} g_{\beta \gamma}$. The self-dual Weyl tensor is $W = \frac{1}{4}(W - i \ast W)$.

2 Main results

We know (see $[10]$ and references therein) that the integrability conditions of the CKY equation (1) lead to constraints on the Petrov-Bel type. Indeed, if $A$ is a non null two-form then the space-time is type O or type D, and in type D case $A$ is an eigen-two-form of the Weyl tensor: $A = e^\lambda U$, $U$ being the simple unitary Weyl eigen-two-form. On the other hand, if $A$ is a null two-form then the space-time is type O or type N, and in type N case $A$ is a null eigen-two-form of the Weyl tensor: the self-dual Weyl tensor writes $W = \mathcal{H} \otimes \mathcal{H}$, and $A = e^\mu \mathcal{H}$.

Moreover, the integrability conditions of the CKY equation also constrains the Ricci tensor. More precisely, from the Ricci identities for the two-form $A$ and taking into account (1) we obtain:

$$2 \mathcal{L}_\mathcal{Z} \mathcal{G} = 3[A, Ric],$$  

where $\mathcal{L}_\mathcal{Z}$ denotes the Lie derivative with respect the vector field $\mathcal{Z}$, and for two 2-tensors $P, Q$, $[P, Q]$ denotes their commutator, $[P, Q] = P \cdot Q - Q \cdot P$. The commutator $[A, Ric]$ vanishes if, and only if, $[A, Ric] = [\ast A, Ric] = 0$. Then, we can state the following.

**Theorem 1** If a space-time admits a conformal Killing-Yano tensor $A$, then $\mathcal{Z} \equiv \delta A$ is a (complex) Killing vector (or it vanishes) if, and only if, $[A, Ric] = [\ast A, Ric] = 0$. When $A$ is a non null two-form this condition states that the Ricci tensor is of Segré types $[(11)(11)]$ or $[(1111)]$.

When $A$ is a null two-form this condition states that the Ricci tensor is of Segré types $[(211)]$ or $[(1111)]$.

**Remark 1.** As commented above, a CKY tensor is an eigen-two-form of the Weyl tensor. Theorem $[1]$ states that $\mathcal{Z}$ is a Killing vector if, and only if, the Ricci
geometry is also aligned with \( A \). Thus when \( A = e^\lambda U \) is a non null two-form then the Ricci tensor writes \( \text{Ric} = -\kappa \Pi + \Lambda g \), \( \Pi = 2U \cdot \bar{U} \) being the 2+2 structure tensor, and where \( \kappa \) and \( \Lambda \) are two Ricci-invariant scalars. On the other hand, when \( A = e^\lambda \mathcal{H} \) is a null two-form then the Ricci tensor writes \( \text{Ric} = \sigma \ell \otimes \ell + Ag \), where \( \ell \) is the null fundamental vector of \( \mathcal{H} \).

**Remark 2.** Tachibana [11] obtained a similar equation to (2) for a n-dimensional Riemann space, and he concludes that for an Einstein space, \( Z \) is a Killing vector. Theorem 1 above generalizes this result for the case of a four-dimensional space-time by considering all the compatible Ricci tensors. Thus it gives not only a necessary condition but also a sufficient one in order for \( Z \) to be a Killing vector. The particular case when \( A \) is a Killing-Yano tensor has previously been considered [12].

**Remark 3.** Hougston and Sommers [13] proved that the \( \mathcal{D} \)-metrics (vacuum type \( \mathcal{D} \) metrics and their charged counterpart) admit a complex Killing vector given by the divergence of a non null CKY tensor. This Killing vector is invariant since it is a Weyl concomitant. Hougston and Sommers do not quote the Tachibana paper but their result applies to a family of both vacuum and non vacuum solutions which cover the physically significant space-times for which theorem 4 applies. For short, in what follows we denote \( \mathcal{D} \)-metrics the non conformally flat space-times admitting a non null CKY tensor \( A \) such that \([A, \text{Ric}] = 0\). Elsewhere [14] we have shown that the \( \mathcal{D} \)-metrics are the \( \mathcal{D} \)-metrics with \( \Lambda = \text{constant} \), and we have extended some known properties of the \( \mathcal{D} \)-metrics to the \( \mathcal{D} \)-metrics.

The CKY equation (1) for the non null two-form \( A = e^\lambda U \) is equivalent to the umbilical and Maxwellian character of the 2+2 structure defined by \( U \). These two properties mean that its principal directions are geodesic shear-free congruences and they determine a non null solution of the source-free Maxwell equations, and they can be written, respectively, in terms of \( \{ \lambda, U \} \) as [9]:

\[
\nabla U = i(\delta U)[U \otimes U + G], \quad U(\delta U) = d\lambda. \tag{3}
\]

From this latter equation we obtain the following expression for the Killing vector:

\[
\mathcal{Z} \equiv \delta A = \frac{3}{2} e^\lambda \delta U. \tag{4}
\]

In what follows we restrict ourselves to the space-times with non constant curvature by considering either \( \mathcal{D} \)-metrics or conformally flat metrics with a Ricci tensor \( \text{Ric} = -\kappa \Pi + \Lambda g \), \( \kappa \neq 0 \). Under this assumption \( U \) is a Riemann invariant two-form and then (3) and (4) imply:

\[
\mathcal{L}_Z U = 0, \quad \mathcal{L}_Z \lambda = 0, \quad \mathcal{L}_Z \bar{U} = 0. \tag{5}
\]

From these constraints (see [15] for a similar reasoning) we obtain the following expression for the Killing two-form associated with \( Z \):

\[
d \mathcal{Z} = aU + m\bar{U} + \frac{4}{3} e^{-\lambda} \mathcal{G}(\mathcal{Z} \wedge \bar{\mathcal{Z}}) \cdot \bar{U}, \tag{6}
\]

where for a double two-form \( W \) and a two-form \( F \), \( W(F)_{\alpha\beta} = \frac{1}{2} W_{\alpha\beta\mu\nu} F^{\mu\nu} \). Then, from (4), (5) and (6) we can easily show the following.
Proposition 1 If a non constant curvature space-time admits a non null conformal Killing-Yano tensor $A$ such that $[A, \text{Ric}] = [\ast A, \text{Ric}] = 0$, then $Z_1 \equiv \delta A$ and $Z_2 \equiv \delta \ast A$ are Killing vectors (or they vanish) verifying: (i) The CKY tensor is $Z_1$-invariant: $L_{Z_1} A = L_{Z_2} A = 0$, (ii) If $Z_1 \wedge Z_2 \neq 0$, they define a commutative algebra: $[Z_1, Z_2] = 0$.

Condition $Z_1 \wedge Z_2 = 0$ characterizes the Kerr-NUT solutions in the set of the $D$-metrics \[13\] \[14\]. Their properties can be extended to the $D$-metrics and to the conformally flat case. Indeed, from \[14\] and \[15\] (see \[14\] and \[15\] for a similar reasoning) we obtain the following.

Proposition 2 In a non constant curvature space-time admitting a non null conformal Killing-Yano tensor $A$ such that $[A, \text{Ric}] = [\ast A, \text{Ric}] = 0$, let us consider $Z_1 \equiv \delta A$ and $Z_2 \equiv \delta \ast A$. Then, we have the following equivalent conditions: (i) $Z_1 \wedge Z_2 = 0$, (ii) A constant duality rotation $\theta$ exists such that $F = \cos \theta A + \sin \theta \ast A$ is a Killing-Yano tensor, (iii) $K = F^2$ is a Killing tensor, (iv) The Killing two-form $dZ$ of the Killing vector $Z = \delta \ast F$ is aligned with $F$: $[F, dZ] = 0$.

Moreover, if these conditions hold, $Z$ and $Y = K(Z)$ are Killing vectors (or they vanish) such that $[Z, Y] = 0$.

Remark 4. Some of the results in propositions \[1\] and \[2\] were obtained in \[13\] and \[15\] for the case of the $D$-metrics. In \[14\] we completed and partially extended these results, and here we state that all of them hold for both, the $D$-metrics and the conformally flat case.

In the present version of the generalized Jebsen-Birkhoff theorem the additional symmetry is defined by a hypersurface-orthogonal Killing vector. When do the Killing vectors in theorem \[4\] have this property? We know that a simple Killing-Yano tensor exists in the $A$-metrics and $B$-metrics where the Jebsen-Birkhoff theorem applies. Let us note that $A$ is a simple (rank two) two-form if, and only if, the scalar invariant $(A, *A) = \frac{1}{2} \eta_{\alpha \beta \gamma \delta} A^{\alpha \beta} A^{\gamma \delta}$ vanishes. We show now that this condition for the CKY tensor guarantees the hypersurface-orthogonal character of the Killing vectors defined by $Z = \delta A$.

It is worth remarking that $(A, *A) = 0$ is equivalent to $\lambda$ being a real scalar and, from \[5\], implies that the vector $U(\delta U)$ is real, that is, the $2+2$ structure is integrable (the two 2-planes are foliation) \[9\]. On the other hand, $Z_1 = \delta A$ and $Z_2 = \delta \ast A$ are hypersurface-orthogonal vectors when $I_1 = \ast (Z_1 \wedge d Z_1) = 0$ and $I_2 = \ast (Z_2 \wedge d Z_2) = 0$.

By using \[4\] and \[5\] we can compute the (real) vectors $I_1$ and $I_2$ and the codifferential of the complex vectors $U(Z)$ and $U(\bar{Z})$, and we obtain:

\[ I_1 + I_2 = 2i \{ m U(Z) \cdot a U(Z) + \frac{2}{3} e^{-\lambda}(Z, \bar{Z}) U(Z) - (Z, \bar{Z}) \bar{U}(Z) \} - c.c. \]  \[ I_1 - I_2 = 2i \{ m \bar{U}(\bar{Z}) \cdot a U(\bar{Z}) + \frac{2}{3} e^{-\lambda}(\bar{Z}, Z) U(\bar{Z}) - (\bar{Z}, Z) \bar{U}(\bar{Z}) \} - c.c. \] \[ \delta U(Z) = a + \frac{2}{3} e^{-\lambda}(Z, \bar{Z}), \quad \delta \bar{U}(\bar{Z}) = m + \frac{2}{3} e^{-\lambda}(Z, \bar{Z}). \]

Let us suppose that $A$ is a simple CKY tensor and $[A, \text{Ric}] = 0$. From theorem \[5\] this second condition implies that $Z_1$ and $Z_2$ are Killing vectors. Moreover, as commented above, the first one implies that $\lambda$ and $U(\delta U)$ are real, and \[6\] and \[7\] impose $U(Z)$ and $U(\bar{Z})$ are real too. Then \[8\] imposes $a$ and $m$ to be real scalars.
Using all these constraints in (7) we obtain \( I_1 = I_2 = 0 \) and consequently \( Z_1 \) and \( Z_2 \) are hypersurface-orthogonal vectors.

Let us suppose now that \( Z_1, Z_2 \) are hypersurface-orthogonal Killing vectors. Thus \( I_1 = I_2 = 0 \). Moreover, from theorem 1 we have \([A, \text{Ric}] = 0\). Then \( Z_1 \) and \( Z_2 \) lie on the principal planes defined by the Ricci and/or the Weyl tensors (which are defined by \( U \)). This condition imposes \( U(Z) \) and \( \bar{U}(Z) \) to be real when \( Z_1 \wedge Z_2 \neq 0 \), and then from (7) and (8) we can show that \( \lambda \) is real and then \((A, \ast A) = 0\). When \( Z_1 \wedge Z_2 = 0 \), we can consider the Killing-Yano tensor \( F \) and \( Z \equiv \delta \ast F \) of proposition 2, and then a similar reasoning implies \((F, \ast F) = 0\) if \((Z, Z) \neq 0\). Moreover \((Z, Z) = 0\) implies \( U(Z) = Z \), and then 3 leads to \( \nabla Z = 0 \), that is, the space-time is a pp-wave if \( Z \neq 0 \). Otherwise, when \( Z = 0 \), \( \ast F \) is also a Killing-Yano tensor and the space-time is a product one. Consequently we have proven the following.

**Theorem 2** In a non constant curvature space-time admitting a non null conformal Killing-Yano tensor \( A \) such that \([A, \text{Ric}] = [\ast A, \text{Ric}] = 0\), let us consider \( Z_1 \equiv \delta A \) and \( Z_2 \equiv \delta \ast A \). Then, the following statements hold:

1. If \( Z_1 \wedge Z_2 \neq 0 \), then \( Z_1 \) and \( Z_2 \) are hypersurface-orthogonal Killing vectors if, and only if, \((A, \ast A) = 0\).
2. If \( Z_1 \wedge Z_2 = 0 \), let \( F = \cos \theta A + \sin \theta \ast A \) be the Killing-Yano tensor given in proposition 3 and \( Z \equiv \delta \ast F \neq 0 \). Then: (i) when \((Z, Z) \neq 0\), \( Z \) is a hypersurface-orthogonal Killing vector if, and only if, \((F, \ast F) = 0\); (ii) when \((Z, Z) = 0\), \( Z \) is a hypersurface-orthogonal Killing vector and the space-time is a pp-wave.

The space-time has a product metric if, and only if, \( Z_1 = Z_2 = 0 \).

**Remark 5.** When \( Z_1 \wedge Z_2 \neq 0 \), the hypersurface-orthogonal nature of the Killing vectors \( Z_1 \) and \( Z_2 \) is guaranteed if \((A, \ast A) = 0\), that is, when the scalar \( \lambda \) is real. Nevertheless, if the imaginary part of \( \lambda \) is a non vanishing constant (and then \((A, \ast A) \neq 0\)), a constant duality rotation leads to a simple CKY tensor \( A' = \cos \theta A + \sin \theta \ast A \), and the first statement in theorem 2 applies.

**Remark 6.** When a Killing tensor \( F \) exists and \( Z \equiv \delta \ast F \neq 0 \) is a null Killing vector \((Z, Z) = 0\), then \( \nabla Z = 0 \) and the space-time is a pp-wave. In this case \( F \) has a constant eigenvalue and \( F \) is simple only when this eigenvalue vanishes. The only Ricci tensor of the considered type \([(11)(11)]\) that is compatible with the integrability conditions of \( \nabla Z = 0 \) has a vanishing eigen-value associated with the time-like eigen-plane. Consequently, these metrics have an unclear physical meaning.

**Remark 7.** Theorems 1 and 2 imply the existence of one or two symmetries if the space-time has some specific geometric properties. But these symmetries do not exist when both, \( Z_1 = \delta A \) and \( Z_2 = \delta \ast A \), vanish. In this case both, \( A \) and \( \ast A \), are Killing-Yano tensors and, as stated in the third point of theorem 2 the space-time metric is a product one. It is worth remarking that this is, precisely, the case where the generalized Jebsen-Birkhoff theorem does not apply (see 5 and 7).

On the other hand, a simple Killing-Yano tensor \( A \) exists in the space-times where the Jebsen-Birkhoff theorem applies. Moreover, for a Killing-Yano tensor we have \([A, \text{Ric}] = 0\). Then we obtain the following.
Corollary 1 If a non constant curvature space-time admits a non null Killing-Yano tensor $A$ and $Z \equiv \delta \star A \neq 0$, then:

When $(Z,Z) \neq 0$, then $Z$ is a hypersurface orthogonal Killing vector if, and only if, $[\star A, \text{Ric}] = 0$ and $(A, \star A) = 0$.

When $(Z,Z) = 0$, then $Z$ is a hypersurface orthogonal Killing vector if, and only if, $[\star A, \text{Ric}] = 0$.

Plainly, the generalized Birkhoff theorem follows from this corollary. Indeed, a metric $g$ admitting a maximal group of symmetries on two-dimensional non-null orbits admits the canonical form of a 2+2 warped product [7]:

\begin{equation}
\begin{aligned}
g &= v(x^0, x^1) + \phi^2(x^0, x^1)h(x^2, x^3),
\end{aligned}
\end{equation}

where $\phi$ is a function, $v$ is a Lorentzian (or Riemannian) metric and $h$ is a Riemannian (or Lorentzian) metric of constant curvature. Then $A = \phi H$ is a simple Killing-Yano tensor for the metric [9], $H$ being the metric volume element of the metric $\phi h$ [8]. On the other hand, the Ricci tensor of the metric [9] has $\{x^2, x^3\}$ as an eigenplane. Consequently, a Ricci with two double eigen-values implies its alignment with $\star A$. Moreover, we have $Z \equiv \delta \star A = -\star (d \phi \wedge H)$, which vanishes if, and only if, $d \phi = 0$. Thus, we can apply corollary 1 and we recover the generalized Birkhoff theorem.

Corollary 2 If a space-time admits a maximal group of symmetries acting on two-dimensional non-null orbits and the Ricci tensor is of types \([1111]\) or \([1111]\) then it admits an additional hypersurface-orthogonal Killing vector provided that $d \phi \neq 0$. This Killing vector is given by $Z = -\star (d \phi \wedge H)$, where $H$ is the volume element of the group orbits.

3 Ending comments.

Corollary generalizes the Jebsen-Birkhoff theorem because the hypothesis of a three-dimensional group of isometries on two-dimensional orbits has been weakened by considering the existence of a simple Killing Yano tensor. Note that no symmetries are required a priori. Nevertheless, if we consider the non-conformally flat case with $\Lambda = \text{constant}$, we obtain the charged A-metrics and B-metrics, and all of them admit maximal symmetry on non null two-dimensional orbits. But these symmetries are not a hypothesis but a consequence of the field equations.

Corollary 1 not only generalizes the Jebsen-Birkhoff theorem by weakening its hypothesis, but it also offers two other new improvements. On the one hand, it is stated as a necessary and sufficient condition. On the other hand, it gives the explicit expression of the hypersurface-orthogonal Killing vector in terms of the magnitudes which appear in the hypothesis (the simple Killing tensor). This advance can be also found in the statement of the Jebsen-Birkhoff theorem given in corollary 2.

Moreover, theorem 2 is a wider generalization than corollary 1 because it establishes the existence of one or two hypersurface-orthogonal Killing vectors under weaker conditions. Besides, theorem 1 includes theorem 2 as a particular case and it shows the close relationship between the Jebsen-Birkhoff theorem and the results by Tachibana [11] and Houghton and Sommers [13].
It is worth remarking that the different cases we found in our study correspond with extensions of the invariant classes of vacuum type D solutions presented in [15]. For each of these classes there is the charged counterpart in the set of the $\mathcal{D}$-metrics, and with similar invariant definitions we can consider the same classes in the $\tilde{\mathcal{D}}$-metrics and in the conformally flat case. Thus, when $Z_1 \wedge Z_2 \neq 0$ we have C-like metrics admitting the subclass where $Z_1$ and $Z_2$ are hypersurface-orthogonal vectors (containing the strict Ehlers and Kundt C-metrics). When $Z_1 \wedge Z_2 = 0$ we have Kerr-NUT-like metrics, with a regular case characterized by $Z \wedge K(Z) \neq 0$. Note that in these three quoted cases a two-dimensional commutative group of isometries exists (see propositions [1] and [2]). However, when $Z_1 \wedge Z_2 = Z \wedge K(Z) = 0$ we obtain A-NUT-like and B-NUT-like metrics and only a real Killing vector $Z$ is defined by the CKY tensor $A$. When $Z$ is a hypersurface-orthogonal vector we have the subfamily where corollary [1] applies.

Theorem [1] also holds when $A$ is a null CKY tensor. Nevertheless, the other results presented herein only apply when $A$ is a non null CKY tensor, and their possible generalization to the null case is not evident. This study and its relationship with previously known Birkhoff-like results for null orbits [5] [15] will be considered elsewhere.

Acknowledgements This work has been supported by the Spanish “Ministerio de Economía y Competitividad”, MICINN-FEDER project FIS2012-33582.

References

1. Jepsen J. T.: Ark. Mat. Astron. Fys. 15, 1 (1921). Reprinted in Gen. Rel. Grav. 37, 2253 (2005)
2. Birkhoff G. D.: Relativity and Modern Physics (Hardward U P, Cambridge) (1923)
3. Cahen M. and Debever R.: C R Acad Sci Paris 260, 815 (1965)
4. Goener H.: Commun. Math. Phys. 16, 34 (1970)
5. Barnes A.: Commun. Math. Phys. 33, 75 (1973)
6. Stephani E., Kramer H., McCallum M. A. H., Hoenselaers C. and Hertl E.: Exact Solutions of Einstein’s Field Equations (Cambridge University Press, Cambridge) (2003)
7. Bona C.: Math. Phys. 29, 1448 (1988)
8. Ferrando J. J. and Sáez J. A.: Class. Quantum Grav. 27, 205023 (2010)
9. Ferrando J. J. and Sáez J. A.: Gen. Relativ. Gravit. 35, 1191 (2003)
10. Glass E. N. and Kress J.: J. Math. Phys. 40, 309 (1999)
11. Tachibana S.: Tohoku Math. J. (2) 21, 56 (1969)
12. Dietz W. and Rüdiger R.: Proc. R. Soc. London Ser. A 375, 361 (1981)
13. Houghton L. P. and Sommers P.: Commun. Math. Phys. 33, 129 (1973)
14. Ferrando J. J. and Sáez J. A.: J. Math. Phys. 48, 102504 (2007)
15. Ferrando J. J. and Sáez J. A.: Gen. Relativ. Gravit. 46, 1073 (2014)
16. Collinson C. D. and Smith P. N.: Commun. Math. Phys. 56, 277 (1977)
17. Stephani H.: Gen. Relativ. Gravit. 9, 789 (1978)
18. Barnes A.: J. Phys. A: Math. Gen. 12, 1493 (1979)
