A radiative seesaw model in modular $A_4$ symmetry

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Abstract

We study one-loop induced radiative seesaw model applying a modular $A_4$ flavor symmetry, in which the neutrino mass matrix is achieved by two different Yukawa couplings one of which also contributes to positive value of muon anomalous magnetic moment as well as lepton flavor violations. Thanks to the specific mass matrix via $A_4$ symmetry and its modular weight, we find several predictions for lepton sector through our numerical analysis.

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I. INTRODUCTION

Radiative seesaw models provide rich phenomenologies at TeV scale such as dark matter (DM) candidate, lepton flavor violations (LFVs), muon anomalous magnetic moment, electroweak precision tests like Z-boson decays, and collider physics, depending on models. Its representative scenario is known as Ma model [1], which is the first model to correlate the neutrino sector and dark sector inside the neutrino mass loop. Recently, modular flavor symmetries have been proposed [2, 3] to provide more predictions to the quark and lepton sector, because any Yukawa couplings can have a representation of flavor groups as well as modular weight. \(^1\) Their typical groups are found in basis of the \(A_4\) modular group \([3, 12]\), \(S_3\) \([13–16]\), \(S_4\) \([17–19]\), \(A_5\) \([20, 21]\), larger groups \([22]\), multiple modular symmetries \([23]\), and double covering of \(A_4\) \([24]\) in which masses, mixings, and CP phases for quark and lepton are predicted. \(^2\) Furthermore, thanks to the modular weight that is another degree of freedom originated from modular symmetry, this modular weight can be identified as a symmetry to stabilize DM candidate if DM is included in a model. Thus, radiative seesaw models with modular flavor symmetries are well motivated in view of neutrino predictions and DM origin.

In this paper, we apply a modular \(A_4\) flavor symmetry to induce one-loop induced neutrino mass matrix with two different Yukawa couplings, running two singly-charged bosons and six singly-charged-fermions inside the neutrino loop. Apart from the DM candidate this time, we try to derive positive muon anomalous magnetic moment from the same Yukawa coupling that contributes to the neutrino mass matrix as well as obtaining several predictions on the neutrino sector satisfying the LFVs through our numerical analysis.

This paper is organized as follows. In Sec. \(\text{III}\) we explain our model setup under modular \(A_4\) symmetry, and formulate each of exotic mass matrix and mixing, LFVs, muon anomalous magnetic moment, the neutrino mass matrix and some relations to achieve the numerical analysis. In Sec. \(\text{III}\), we show numerical analysis and demonstrate several figures that provide predictions. Finally we conclude and discuss in Sec. \(\text{IV}\).

\(^1\) One notices that Yukawa couplings have to have even number of modular weight only, especially the modular weight starts from 4 if the Yukawa coupling is assigned to be singlets under the flavor symmetry.

\(^2\) Several reviews are helpful to understand whole the ideas \([25–32]\).
### TABLE I: Field contents of fermions and bosons and their charge assignments under $SU(2)_L \times U(1)_Y \times A_4$ in the lepton and boson sector, where $-k$ is the number of modular weight and the quark sector is the same as the SM.

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
 & \text{Fermions} & \text{Bosons} \\
\hline
SU(2)_L & (L_{Le}, L_{L\mu}, L_{L\tau}) & (e_{Re}, e_{R\mu}, e_{R\tau}) & L'_{1,2,3} & e'_{1,2,3} & H & \eta & s^+ \\
\hline
U(1)_Y & 2 & 1 & 2 & 1 & 2 & 2 & 1 \\
\hline
A_4 & (1, 1', 1'') & (1, 1', 1'') & 3 & 3 & 1 & 1 & 1 \\
\hline
-k & 0 & 0 & -1 & -1 & 0 & -1 & -3 \\
\hline
\end{array}
$$

### TABLE II: Modular weight assignments for Yukawa and Higgs couplings, the other couplings are all neutral under the modular symmetry.

$$
\begin{array}{|c|c|c|}
\hline
 & \text{Couplings} \\
\hline
 & Y^{(4)}(1) & Y^{(2)}(3) & Y^{(4)}(3) \\
A_4 & 1 & 3 & 3 \\
-k & 4 & 2 & 4 \\
\hline
\end{array}
$$

#### II. MODEL

In this section, we explain our model. Here, we introduce three generation of exotic vector-like leptons $L' \equiv [N', E']^T$ and $e'$ under isospin doublet and singlet, where both of them are assigned by $(3, -1)$ under $(A_4, -k)$. Also we introduce an isospin doublet inert boson $\eta \equiv [\eta^+, \eta_0]^T$ and a singly-charged singlet boson $s^+$, where $\eta$ and $s^+$ are respectively assigned by $-1, -3$ under modular weight, but both of them are trivial singlet under $A_4$. The Standard Model (SM) leptons are assigned by $1, 1', 1''$ for $e, \mu, \tau$ under $A_4$ respectively, while they are neutral under modular weight. The SM Higgs is defined by $H$ and its VEV is denoted by $\langle H \rangle \equiv v_H/\sqrt{2}$, where $H$ is trivial singlet and neutral under $A_4$ and modular weight, respectively. The $A_4$ representation and modular weight for fields are given by Tab. I while the ones of Yukawa couplings are respectively given by Tab. II. Under these
symmetries, one writes renormalizable Lagrangian as follows:

\[-\mathcal{L}_{\text{Lepton}} = \sum_{\ell = e, \mu, \tau} y_{\ell} \bar{L}_{\ell} H e_{R_{\ell}} + Y'(\bar{L}_{\ell}^c H e'_{R_{\ell}} + \bar{L}_{\ell}^t H e'_{R_{\ell}} + \bar{L}_{\tau} H e'_{R_{\tau}}) + M'(\bar{L}_{\ell}^t L_{R_{\ell}} + \bar{L}_{\ell}^t L_{R_{\ell}} + \bar{L}_{\tau} L_{R_{\mu}}) + m'(\bar{e}_{L_{\ell}} e'_{R_{\ell}} + \bar{e}_{L_{\mu}} e'_{R_{\ell}} + \bar{e}_{L_{\tau}} e'_{R_{\mu}}) + \alpha_1(Y_3^{(2)} \otimes \bar{L}_{\ell} \otimes e'_{R_{\ell}}) + \beta_1(Y_3^{(2)} \otimes \bar{L}_{\ell} \otimes e'_{R_{\ell}}) + \gamma_1(Y_3^{(2)} \otimes \bar{L}_{\ell} \otimes e'_{R_{\mu}}) + \alpha_2(Y_3^{(4)} \otimes \bar{L}_{\ell}^c \otimes L_{R_{\ell}}) s^+ + \beta_2(Y_3^{(4)} \otimes \bar{L}_{\ell}^c \otimes L_{R_{\ell}}) s^+ + \gamma_2(Y_3^{(4)} \otimes \bar{L}_{\ell} \otimes L_{R_{\ell}}) s^+ + \text{h.c.}, \quad \text{(II.1)}\]

where \(Y', M', m'\) include the invariant modular factor \(1/(\tau + i\tau)\).

The modular forms of weight 2, \((y_1, y_2, y_3)\), transforming as a triplet of \(A_4\) is written in terms of Dedekind eta-function \(\eta(\tau)\) and its derivative [3]:

\[
y_1(\tau) = \frac{i}{2\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'(\tau + 1/3)}{\eta(\tau + 1/3)} + \frac{\eta'(\tau + 2/3)}{\eta(\tau + 2/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right),
\]

\[
y_2(\tau) = -\frac{i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'(\tau + 1/3)}{\eta(\tau + 1/3)} + \omega^2 \frac{\eta'(\tau + 2/3)}{\eta(\tau + 2/3)} \right), \quad \text{(II.2)}
\]

\[
y_3(\tau) = -\frac{i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2 \frac{\eta'(\tau + 1/3)}{\eta(\tau + 1/3)} + \omega \frac{\eta'(\tau + 2/3)}{\eta(\tau + 2/3)} \right).
\]

The overall coefficient in Eq. (II.2) is one possible choice; it cannot be uniquely determined. Then, any couplings of higher weight are constructed by multiplication rules of \(A_4\), and one finds the following couplings:

\[Y_1^{(4)} = y_1^2 + 2y_2y_3, \quad Y_3^{(4)} \equiv \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} y_1^2 - y_2y_3 \\ y_2^2 - y_1y_3 \\ y_3^2 - y_1y_3 \end{bmatrix}, \quad \text{(II.3)}\]

### A. Singly-charged exotic fermion mass matrix

After the electroweak spontaneous symmetry breaking, singly-charged exotic fermion mass matrix is found in basis of \((e', E')_R\) as

\[\mathcal{M}_E = \begin{bmatrix} mP_{23} & m_{e'L'}P_{23} \\ m_{e'L'}^tP_{23} & M'P_{23} \end{bmatrix}, \quad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{(II.4)}\]

where \(m_{e'L'} \equiv v_H Y'/\sqrt{2}\). Then, \(\mathcal{M}_E\) is diagonalized by a unitary mixing matrix; \(D_E \equiv V^\dagger \mathcal{M}_E V^T\), where two set of three degenerate mass eigenstates are given by this mass matrix,
and the mass eigenstates are related to the flavor eigenstates $\psi$ as follows: $(e^-, E^-)^{T}_{L,R} \equiv V^T \psi_{L,R}^-.$

B. Neutral exotic fermion mass matrix

Similar to the singly-charged exotic fermion mass matrix, its form is found as

$$\mathcal{M}_N = M' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$ (II.5)

Then, $\mathcal{M}_N$ is diagonalized by the unitary mixing matrix; $D_N \equiv V_N^* \mathcal{M}_N V_N^T$, where three degenerate mass eigenstates are given by this mass matrix, and the mass eigenstates are related to the flavor eigenstates $N'$ as follows: $N'^{T}_{L,R} \equiv V_N^T \psi_{L,R}^0.$

C. Singly-charged bosons

The singly-charged bosons mix each other through the term of $Y_1^{(4)}(\eta^T \cdot H)s^+$, where $\sigma_2$ is the second Pauli matrix. Here, we define as follows:

$$s^\pm = c_\alpha H_1^\pm + s_\alpha H_2^\pm, \quad \eta^\pm = -s_\alpha H_1^\pm + c_\alpha H_2^\pm,$$ (II.6)

where $s_\alpha(c_\alpha)$ is the short-hand symbol of $\sin \alpha(\cos \alpha)$, and $\cdot \equiv i\sigma_2$.

D. Neutrino mass matrix and lepton flavor violations

Neutrino mass matrix and LFVs are originated from the terms of $L_L \eta e_R'$ and $L_L^C L_L' s^+$, and their explicit forms are given by

$$\mathcal{L} = \sum_{i=1}^{3} \sum_{a=1}^{3} \sum_{b=1}^{6} \left[ \bar{\nu}_{L_i}(y_\eta)_{i,a} (V^T)_{a,b} \psi_{R_b}^- (-s_\alpha H_1^+ + c_\alpha H_2^+) + \bar{\nu}_{L_i}^C (y_s)_{i,a} (V^T)_{a+3,b} \psi_{L_b}^- (c_\alpha H_1^+ + s_\alpha H_2^+) \right]$$

$$+ \sum_{i=1}^{3} \sum_{a=1}^{3} \sum_{b=1}^{6} \bar{\ell}_{L_i}(y_\eta)_{i,a} (V^T)_{a,b} \psi_{R_b}^- \eta_0 + \sum_{i=1}^{3} \sum_{j=1}^{3} \bar{\ell}_{L_i}^C (y_s V_N^T)_{i,j} \psi_{L_j}^0 (c_\alpha H_1^+ + s_\alpha H_2^+) + \text{h.c.,}$$

(II.7)
which will be imposed in our numerical calculation. Muon g-2 is also given by

\[ y_\eta = \begin{bmatrix} \alpha_\eta & 0 & 0 \\ 0 & \beta_\eta & 0 \\ 0 & 0 & \gamma_\eta \end{bmatrix}, \quad y_s = \begin{bmatrix} \alpha_s & 0 & 0 \\ 0 & \beta_s & 0 \\ 0 & 0 & \gamma_s \end{bmatrix}. \]  

(II.8)

Lepton flavor violations also arises from \( y_D \) as \( [37, 38] \)

\[
\text{BR}(\ell_i \rightarrow \ell_j \gamma) \approx \frac{48\pi^3 \alpha_{em} C_{ij}}{G_F^2 (4\pi)^4} |\mathcal{M}_{i,j}|^2,
\]

\[
\mathcal{M}_{i,j} \approx -\sum_{a=1}^3 \sum_{b=1}^6 \sum_{c=1}^3 (y_\eta)_{j,a} (V^T)_{a,b} (V^*)_{b,c} (y^\dagger_\eta)_{c,i} F(\psi_b^-, \eta_0)
\]

\[
+ \sum_{a=1}^3 \sum_{b=1}^6 \sum_{c=1}^3 (y_s)_{i,a} (V^T)_{a+3,b} (V^*)_{b+3,c} (y^\dagger_s)_{c,j} \left[ c^2 F(\psi_b^0, H^1) + s^2 F(\psi_b^0, H^2) \right], \quad (II.9)
\]

\[
F(a, b) \approx \frac{2m^6_a + 3m^4_a m^2_b - 6m^2_a m^4_b + m^6_b + 12m^4_a m^2_b \ln \left( \frac{m_b}{m_a} \right)}{12(m^2_a - m^2_b)^4}, \quad (II.10)
\]

where \( C_{21} = 1, \; C_{31} = 0.1784, \; C_{32} = 0.1736, \; \alpha_{em}(m_Z) = 1/128.9, \; \) and \( G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2} \). The experimental upper bounds are given by \( [39, 41] \)

\[
\text{BR}(\mu \rightarrow e\gamma) \lesssim 4.2 \times 10^{-13}, \quad \text{BR}(\tau \rightarrow e\gamma) \lesssim 3.3 \times 10^{-8}, \quad \text{BR}(\tau \rightarrow \mu\gamma) \lesssim 4.4 \times 10^{-8}, \quad (II.11)
\]

which will be imposed in our numerical calculation. Muon g-2 is also given by

\[
\Delta a_\mu \approx -\frac{2m^2_\mu}{(4\pi)^2} \mathcal{M}_{2,2}. \quad (II.12)
\]

It implies that \( y_\eta \) term provides positive contribution, while \( y_s \) does negative contribution, since the experimental result suggests positive anomaly, \( y_\eta >> y_s \) is expected.

Neutrino mass matrix is given at one-loop level, and its form is found as

\[
m_{\nu_{ij}} \approx s_\alpha c_\alpha \sum_{a=1}^{3} \frac{Y_{\nu a} D_{Ea} Y^\dagger_{\nu a} + Y^\dagger_{\nu a} D_{Ea} Y_{\nu a}^T}{(4\pi)^2} \left( \frac{m^2_1}{m^2_1 - D_{Ea}^2} \ln \left[ \frac{m^2_1}{D_{Ea}^2} \right] - \frac{m^2_2}{m^2_2 - D_{Ea}^2} \ln \left[ \frac{m^2_2}{D_{Ea}^2} \right] \right), \quad (II.13)
\]

where \( (Y_{\nu})_{i,j} \equiv \sum_{a=1}^{3} (y_\eta)_{i,a} (V^T)_{a,j}, \; (Y_s)_{i,j} \equiv \sum_{a=1}^{3} (y_s)_{i,a} (V^T)_{a+3,j}, \; m_{1,2} \) is the mass of \( H^\pm_{1,2} \). Then the neutrino mass matrix is diagonalized by an unitary matrix \( U_\nu \) as \( U_\nu m_{\nu} U_\nu^T = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) \equiv D_\nu, \) where \( \text{Tr}[D_\nu] \lesssim 0.12 \text{ eV} \) is given by the recent cosmological data \( [42] \). Then, one finds \( U_{PMNS} = V_{\nu L}^T U_\nu \). Each of mixing is given in terms of
the component of $U_{MNS}$ as follows:

$$\sin^2 \theta_{13} = |(U_{PMNS})_{13}|^2, \quad \sin^2 \theta_{23} = \frac{|(U_{PMNS})_{23}|^2}{1 - |(U_{PMNS})_{13}|^2}, \quad \sin^2 \theta_{12} = \frac{|(U_{PMNS})_{12}|^2}{1 - |(U_{PMNS})_{13}|^2}. \quad (II.14)$$

Also, the effective mass for the neutrinoless double beta decay is given by

$$m_{ee} = |D_{\nu_1} \cos^2 \theta_{12} \cos^2 \theta_{13} + D_{\nu_2} \sin^2 \theta_{12} \cos^2 \theta_{13} e^{i\alpha_{21}} + D_{\nu_3} \sin^2 \theta_{13} e^{i(\alpha_{31} - 2\delta_{CP})}|, \quad (II.15)$$

where its observed value could be measured by KamLAND-Zen in future [43].

To achieve numerical analysis, we derive several relations. First of all, we define the normalized neutrino mass matrix as follows:

$$\tilde{m}_{\nu_{ij}} \equiv \frac{m_{\nu_{ij}}}{s_\alpha c_\alpha} \approx \sum_{a=1}^{6} Y_{\eta a} F_a Y_{\eta a},$$

$$F_a \equiv D_{E_a} \left( \frac{m_1^2}{m_1^2 - D_{E_a}^2} \ln \left[ \frac{m_1^2}{D_{E_a}^2} \right] - \frac{m_2^2}{m_2^2 - D_{E_a}^2} \ln \left[ \frac{m_2^2}{D_{E_a}^2} \right] \right). \quad (II.16)$$

Then the normalized neutrino mass eigenvalues are given in terms of neutrino mass eigenvalues; $\text{diag}(\tilde{m}_{\nu_1}^2, \tilde{m}_{\nu_2}^2, \tilde{m}_{\nu_3}^2) = \text{diag}(m_{\nu_1}^2, m_{\nu_2}^2, m_{\nu_3}^2)/(s_\alpha c_\alpha)^2$. It is found that $s_\alpha$ is given by

$$s_\alpha^2 (1 - s_\alpha^2) = \frac{\Delta m_{\text{atm}}^2}{\tilde{m}_{\nu_3}^2 - \tilde{m}_{\nu_1}^2}, \quad (II.17)$$

where normal hierarchy is assumed and $\Delta m_{\text{atm}}^2$ is the atmospheric neutrino mass difference square. The solar neutrino mass difference square is also found as

$$\Delta m_{\text{sol}}^2 = \Delta m_{\text{atm}}^2 \frac{\tilde{m}_{\nu_2}^2 - \tilde{m}_{\nu_1}^2}{\tilde{m}_{\nu_3}^2 - \tilde{m}_{\nu_1}^2}, \quad (II.18)$$

In numerical analysis, this value should be within the experimental result, while $\Delta m_{\text{atm}}^2$ is expected to be input parameter.

III. NUMERICAL ANALYSIS

Here, we show numerical analysis to satisfy all of the constraints that we discussed above, where we assume the neutrino mass ordering is normal hierarchy and $m_{\eta^z} \approx m_2$ to avoid the oblique parameters simply.
FIG. 1: The sum of neutrino masses $\sum m(\equiv \text{Tr}[D_{\nu}])$ versus $\sin^2 \theta_{12}$ (red color) and $\sin^2 \theta_{23}$ (blue color). Here, the horizontal black solid lines are the best fit values, the green dotted lines show 3$\sigma$ range.

Then, we provide the experimentally allowed ranges for neutrino mixings and mass difference squares at 3$\sigma$ range [44] as follows:

$$\Delta m_{\text{atm}}^2 = [2.431 - 2.622] \times 10^{-3} \text{ eV}^2, \quad \Delta m_{\text{sol}}^2 = [6.79 - 8.01] \times 10^{-5} \text{ eV}^2,$$

$$\sin^2 \theta_{13} = [0.02044 - 0.02437], \quad \sin^2 \theta_{23} = [0.428 - 0.624], \quad \sin^2 \theta_{12} = [0.275 - 0.350].$$

The range of absolute value of the complex dimensionless parameters $\alpha_\eta, \beta_\eta, \gamma_\eta$ are taken to be [0.1-1], while $\alpha_s, \beta_s, \gamma_s$ are taken to be $[10^{-10}, 10^{-5}]$ with real parameter. \footnote{The large hierarchies between $\alpha_\eta, \beta_\eta, \gamma_\eta$ and $\alpha_s, \beta_s, \gamma_s$ are expected to induce positive muon $g-2$ as discussed in the part of $\Delta a_\mu$.} The mass parameters $m_{1,2}, m', m_{e',\nu'}, M'$ are [0.1, 10] TeV.

Fig. 1 shows the sum of neutrino masses $\sum m(\equiv \text{Tr}[D_{\nu}])$ versus $\sin^2 \theta_{12}$ (red color) and $\sin^2 \theta_{23}$ (blue color). Here, the horizontal black solid lines are the best fit values, the green dotted lines show 3$\sigma$ range. Three of the mixing angles satisfy whole the allowed region at 3 $\sigma$ interval, but $\sin^2 \theta_{23}$ tends to be in favor of the lower range [0.428-0.522], while $\sin^2 \theta_{12}$ tends to be in favor of the upper range [0.312-0.350]. The sum of neutrino masses are within the range of $[0.060-0.064, 0.067-0.072]$, which is below the cosmological bound 0.12. Here, $\sin^2 \theta_{13}$ runs whole the range of the allowed region in Eq. (III.1) without any favored regions.

Fig. 2 shows phases of $\delta_{CP}$ in terms of $\alpha_{31}$. This figure implies that the Dirac CP phase is favored in the range of [70-120, 240-280][deg], and $\alpha_{31}$ is favored in the range of [120-
FIG. 2: Phases of $\delta^\ell_{CP}$ in terms of $\alpha_3$, where $\alpha_2$ is found to be zero.

Fig. 3 demonstrates the lightest neutrino mass versus the effective mass for the neutrinoless double beta decay. It suggests that $m_1$ is in the range of [0.0015-0.035, 0.0065-0.0085] eV while $m_1$ is in the range of [0.0045-0.0065, 0.008-0.010] eV. The other remarks are in order:

1. The typical region of modulus $\tau$ is found in narrow space as

$$1.85 \lesssim \text{Re}[\tau] \lesssim 1.95 \text{ and } 1.65 \lesssim \text{Im}[\tau] \lesssim 1.85.$$  

2. The upper bound of our LFVs are as follows:

$$\text{BR}(\mu \to e\gamma) \lesssim 4.2 \times 10^{-13}, \quad \text{BR}(\tau \to e\gamma) \lesssim 1.2 \times 10^{-13}, \quad \text{BR}(\tau \to \mu\gamma) \lesssim 4 \times 10^{-13}.$$  

These imply that the future experiment of $\mu \to e\gamma$ is promising to test our model.
3. The muon anomalous magnetic moment is positively obtained by the scale cannot be so large enough to reach the experimentally expected value by five order magnitude. Our upper bound of $\Delta a_\mu$ is found to be

$$\Delta a_\mu \lesssim 6 \times 10^{-14}.$$

IV. CONCLUSION AND DISCUSSION

We have constructed a predictive lepton model with modular $A_4$ symmetry in framework of one-loop induced radiative seesaw model, running singly-charged bosons and singly-charged-fermions. We have the term that can contribute to the muon anomalous magnetic moment with positive value as well as the one with negative value, and these terms also provide us the structure of neutrino mass matrix. The each structure of mass matrix for charged-leptons and neutrinos is uniquely determined by the $A_4$ symmetry, and inert property is assured by modular weight. Especially, the charged-lepton mass matrix can always be diagonal in the stage of flavor eigenstate thanks to the $A_4$ symmetry, once we assign each of field $(e, \mu, \tau)$ to $(1, 1', 1'')$. Due to the $A_4$ symmetry and their assignments, we have found the exotic fermion mass structures are very simple and two sets of three degenerate charged-fermion mass eigenvalues, and three degenerate neutral heavy fermion mass eigenvalues. In numerical analysis, we have found only $\mu \rightarrow e\gamma$ reached the current upper bound and future experiment will provides more restriction for our model. Even though we have gotten the positive muon $g - 2$, but the scale is very small compared to the experimentally expected result. Our maximum value is around $6 \times 10^{-14}$ that is as tiny as the experimental value by five order magnitude. In our numerical analyses, we have found several remarkable predictions on neutrino sector as follows:

1. Three of the mixing angles satisfy whole the allowed region at 3 $\sigma$ interval, but $\sin^2 \theta_{23}$ tends to be in favor of the lower range [0.428-0.522], while $\sin^2 \theta_{12}$ tends to be in favor of the upper range [0.312-0.350]. The sum of neutrino masses are within the range of [0.060-0.064, 0.067-0.072], which is below the cosmological bound 0.12.

2. The Dirac CP phase is favored in the range of [70-120, 240-280][deg], and $\alpha_{31}$ is favored in the range of [120-280][deg], where $\alpha_{21}$ is found to be zero. Since the Dirac CP phase
of $3\pi/2(=270\,\text{[deg]})$ is favored by T2K experiment, our model could be well predicted and tested further.

These predictions will be tested in the near future.

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