Superconducting microwave resonators with non-centrosymmetric nonlinearity

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Abstract
We investigated both theoretically and experimentally open-ended coplanar waveguide resonators with rf SQUIDs embedded in the central conductor at different positions. These rf SQUIDs can be tuned by an external magnetic field and thus may exhibit the non-centrosymmetric nonlinearity of \( \chi^{(2)} \) type with suppressed Kerr nonlinearity. We demonstrated that this nonlinearity allows for efficient mixing of \( \lambda/2 \) and \( \lambda \) modes in the cavity and thus enables various parametric effects with three wave mixing. These effects are the second harmonic generation, the halftone generation, the parametric amplification in both degenerate and non-degenerate regimes and deamplification in degenerate regime.

Keywords: microwave resonator, nonlinearity, rf SQUID, mode coupling, three-wave mixing

1. Introduction
Superconducting coplanar waveguide (CPW) resonators with embedded Josephson junctions and SQUIDs are widely used for parametric amplification [1–3], bifurcation-based quantum detection [4–8], generation of nonclassical states of microwaves [8–11], studying the dynamical Casimir effect and photon field correlations [12, 13], parametric down conversion [14], etc. At the working microwave frequencies (up to approximately 20 GHz), superconducting resonators have very low losses, while the Josephson tunnel junctions facilitate the parametric effects. The operation of these circuits is usually based either on the Kerr nonlinearity of the Josephson inductance [1, 5–9, 15–17] or on a periodic modulation of the dc SQUID inductance by means of an alternating magnetic flux [2, 11–13].

Recently, the toolbox of superconducting quantum technologies [18] has been supplemented with the elements having non-centrosymmetric nonlinearity of \( \chi^{(2)} \) type [19]. These superconducting elements are based either on rf SQUIDs [20–22], asymmetric dc SQUIDs [14], or the multijunction SQUIDs, i.e. the so-called superconducting nonlinear asymmetric inductive elements (SNAILs) [22, 23]. For the optimal constant flux \( \Phi_0 \) applied to the SQUID loop, the current-phase relation in these elements may have the Kerr-free shape [21–23],

\[
I(\varphi) \approx (\varphi - \beta \varphi^2)\varphi_0 L_1^{-1},
\]

where \( \varphi_0 = \Phi_0 / 2\pi \) is the reduced magnetic flux quantum, \( L_1 \) is the linear SQUID inductance, and nonlinear coefficient \( \beta \) is the electrical analog of susceptibility tensor \( \chi^{(2)} \) in optics [19]. The nonlinear relation (1) enables three wave mixing (3WM) including frequency doubling and parametric down-conversion. Optical crystals having nonzero susceptibility \( \chi^{(2)} \) are quite rare in nature [24] and fabrication of optical fibers with non-centrosymmetric nonlinearity suitable for engineering traveling wave amplifiers is not as easy as fabrication of the silica fibers with Kerr nonlinearity [25]. However, the superconducting technology allows the fabrication of the
transmission lines with embedded rf SQUIDs. These circuits can have a nonlinearity given by equation (1) and, thus, enable parametric amplification of traveling microwaves using 3WM [21, 22, 26]. A recent study of the effect of parameter variations [27] confirmed that such a parametric amplifier is a strong candidate for achieving a high gain in a wide frequency band together with a high fabrication yield.

Parametric amplification based on serial arrays of SNAILs inserted in the CPW resonators was recently demonstrated by the Yale group. These Josephson parametric amplifiers with 3WM clearly demonstrated a number of advantages of the Kerr-free operation including an improved dynamic range [28], relatively large saturation power with a widely tunable bandwidth [29, 30], and near-quantum-limited performance [31].

In this paper, we first examined Nb open-ended CPW resonators with single rf SQUIDs embedded in their center, i.e. in the antinode of the fundamental (λ/2) mode. The parameters of rf SQUIDs were close to those of the rf SQUIDs exploited in the Josephson traveling wave parametric amplifiers (JTWPAs) with 3WM [22]. Our primary motivation was the investigation of these tunable nonlinear elements, including the determination of their electric parameters. These parameters were found from the dependence of the resonant frequency on magnetic flux Φ.

The tunability of the rf SQUID inductance has been utilized earlier in microwave circuits for the coupling of, for example, two resonators [32] or a resonator and a phase qubit [33]. Here we investigate the circuit with the intermode coupling based on the nonlinear characteristic of the rf SQUID given by equation (1). For this purpose we designed CPW resonators with the rf SQUID positioned at one third of the open-ended resonator length. Thus, we engineered an artificial medium with a nonlinearity of χ(2) type enabling efficient coupling of primarily the λ/2 and λ modes. In this circuit, we demonstrated a number of parametric 3WM phenomena including the second harmonic generation (SHG), the half tone generation (HTG), the parametric amplification in both degenerate and non-degenerate regimes, etc. The improved design of a circuit with a high degree of intermode coupling is proposed.

2. Design and fabrication

The design of our open-ended superconducting CPW resonators (the microwave analog of a Fabry–Perot cavity) with an embedded rf SQUID is schematically shown in figure 1. The resonant frequencies of λ/2 mode were designed to be around 3.9 GHz. The resonator length was 16.136 mm with a center-conductor width of w = 32 μm and gaps between this conductor and the ground plane conductors of s = 16 μm. The nominal thickness of Nb layer was 200 nm. These dimensions of the CPW waveguide yielded a wave impedance of Z0 = 50 Ω. This impedance was matched with the impedance of the input and output lines. The CPW waveguide parameters were chosen similar to those of the CPW transmission line used in the design of the JTWPAs [22]. The total inductance and the total capacitance of the resonator were Lout = ℓd = 6.454 nH and Cout = cd = 2.582 pF, respectively. The corresponding specific values were ℓ = 0.4 pH μm−1 and c = 0.16 fF μm−1.

The input, Cin, and output, Cout, capacitances of the resonator were realized either as gap capacitances with spacing wG = 2 μm or interdigital capacitances having four fingers (see, e.g. [34]). The finger width and spacing between the fingers were 2 μm, while the finger length varied from 10 to 50 μm. The nominal capacitance values Cin and Cout for Sample 1 were 1.8 and 2.8 pF, respectively, and 2.8 and 14 fF for Sample 2, respectively. In the case of Sample 1 these values result in a coupling quality factor [34] Q = 94 000 for the fundamental λ/2 mode, while the loaded quality factor Q ≪ Qc. The experimental value Q ≈ 4000...8000 was increasing for an increasing drive power −105 dBm...−90 dBm, what can be explained by the saturation of microscopic two-level systems [35, 36] and indicates that Q is mainly determined by the dielectric loss in the deposited SiO2 layer [37] and/or in the Josephson junction barriers [38].

Sample 2 was designed to be critically coupled with Qc = 5100, approximately equal to the internal quality factor, which is an optimum in the compromise of coupling strength and quality factor needed for pronounced nonlinear interactions [34]. For this sake, we created a strongly coupled output port, i.e. Cout was chosen much larger than Cin [8]. The resulting experimental quality factor was Q ≈ 2000 and did not exhibit significant temperature dependence in the range from 20 mK up to 4.2 K.

A local magnetic field was applied to the rf SQUID loops via a control-current line (the thin Nb wire seen very close to the upper ground plane in figure 1(c)). The thin-film rf SQUID inductances had the meander shape with a few turns (see figure 1(c)) giving the nominal values of Lx around 30 pH.

\[ L_{out} = \ell d = 6.454 \text{nH} \quad \text{and} \quad C_{out} = cd = 2.582 \text{pF}, \]

\[ \ell = 0.4 \text{pH} \mu \text{m}^{-1} \quad \text{and} \quad c = 0.16 \text{fF} \mu \text{m}^{-1}. \]
The samples were fabricated using the Nb-trilayer technology with a critical current density \( j_c \) from 200 to 500 A cm\(^{-2} \) on a Si/SiO\(_2\) substrate [39]. The self-capacitance \( C_L \) of the Josephson junctions with a nominal area of 1 \( \mu \)m\(^2\) was in the range from 45 to 50 fF. The details of manufacturing similar CPW resonators with embedded Josephson junctions and dc SQUIDs were described in [40].

### 3. Linear regime: characterization of the rf SQUIDs

The inverse inductance for a small alternating current \( i_{ac} \) (see figure 1(b)) passing through the rf SQUID is given by the expression:

\[
L_{ac}^{-1} = L_L^{-1} \left[ 1 + \beta_L \cos \left( \Phi_{ac} / \phi_0 \right) \right].
\]  

(2)

Here we assumed that the Josephson tunnel junction in the SQUID loop has the sinusoidal (conventional) current-phase relation. The constant flux in the SQUID loop \( \Phi_{ac} \) is found by solving the transcendental equation [41]:

\[
\Phi_{ac} = \Phi_{dc} + \beta_L \phi_0 \sin \left( \Phi_{dc} / \phi_0 \right),
\]

(3)

where \( \Phi_{ac} \) is applied magnetic flux, \( \beta_L = L_L / \phi_0 < 1 \) is the dimensionless screening parameter, \( L_L \) is the SQUID inductance, and \( \phi_0 \) is the Josephson critical current. Embedding the rf SQUID in the CPW resonator causes a shift of the resonant frequency. The flux dependence of the rf SQUID inductance given by equation (2) allows the characterization of the rf SQUID by measuring that frequency shift. This method is conceptually similar to that developed by Rifkin and Deaver [42] for the characterization of rf SQUIDs having inductive coupling to a rf-driven tank circuit [43] (see also [44]).

The resonant frequency of the \( n \)th mode, \( n \lambda / 2 = d \), \( n = 1, 2, \ldots \), is given by the general formula (see equations (A21) and (A22) of appendix):

\[
\frac{\omega_n}{\omega_n^{(0)}} = 1 - \frac{L_L}{L_{tot}} \sin^2 \frac{n \pi x}{d} + \Delta x / d.
\]

(4)

For the first mode, \( n = 1 \), and the central position of the rf SQUID, \( x = d / 2 \), expression (4) takes the form:

\[
\frac{\omega_1}{\omega_1^{(0)}} \approx 1 - \frac{L_L}{L_{tot}} \frac{L_L / \phi_0}{1 + \beta_L \cos \left( \Phi_{dc} / \phi_0 \right)},
\]

(5)

where we used equation (2) and neglected the small constant term \( \Delta x / d \approx 10^{-3} \) because of the small SQUID size \( \Delta x \).

The bare resonant frequency in equation (5) is [45]:

\[
\omega_n^{(0)} = \frac{1}{\sqrt{L_{eff,n} C_{eff,n}}} = \frac{n \pi}{\sqrt{L_{tot} C_{tot}}},
\]

(6)

where the effective \( LC \) parameters for the \( n \)th mode are given by the following expressions [46]:

\[
\frac{1}{L_{eff,n}} = \frac{(k_n d)^2}{2 L_{tot}} \left[ 1 + \sin 2k_n d \right] = \frac{(n \pi)^2}{2 L_{tot}}.
\]

(7)

Figure 2. One period of the normalized dependence of the resonant frequency of the fundamental \( \lambda / 2 \)-mode, \( \omega_n (\Phi_{dc}) / \omega_n^{(0)} \), on the normalized magnetic flux, \( \Phi_{dc} / \Phi_0 \), measured at two temperatures in Sample 1. The resonant frequency is normalized to its maximum value of \( \omega_n (0) = \omega_n^{(0)} \left[ 1 - L_L / \left[ (1 + \beta_L) L_{tot} \right] \right] \). In this sample, the rf SQUID is embedded in the center of the CPW resonator, \( x = 0.5d \). Dashed lines denote the fitting curves using formula (5). Both measurements were performed in a dilution refrigerator.

\[
C_{eff,n} = \frac{C_{tot}}{2} \left[ 1 + \frac{\sin 2k_n d}{2k_n d} \right] = \frac{C_{tot}}{2},
\]

(8)

and

\[
\omega_n = \frac{k_n (n \pi / d). As long as the rf SQUID plasma frequency \( \omega_p \) is sufficiently high, that is \( \omega_p = 1 / \sqrt{L_{tot} C_{tot}} \approx 2 \pi \times 120 \text{ GHz} > \omega_n^{(0)} \), the effect of the Josephson junction self-capacitance \( C_J \) on the resulting resonant frequency \( \omega_n \) can safely be neglected at least for the first 10 modes.

The measurements of the resonant frequency of the fundamental mode (\( n = 1 \)) were performed (see appendix C for details) in a dilution refrigerator at temperatures \( T = 20 \text{ mK} \pm 0.2 \text{ mK} \) and \( T = 4 \text{ K} \pm 0.1 \text{ K} \). The resonant frequency showed a clear periodic dependence on the control current producing the magnetic flux \( \Phi \). The experimental data and the fits using formula (5) are presented in figure 2. Because the values of the resonant frequency \( \omega_1 (0) \) found in both experiments at zero magnetic flux \( \Phi \) were identical (3.917 GHz), we concluded that resonator inductance \( L_{tot} \) does not depend on temperature.

The participation ratio value, \( L_L / L_{tot} = 0.0036 \), found from fitting the experimental curves turned out to be also temperature independent. The values of the SQUID-parameter \( \beta_L \) for different temperatures, \( \beta_L^{(4 \text{ K})} = 0.263 \) and \( \beta_L^{(20 \text{ mK})} = 0.285 \) (see figure 2), clearly pointed to a temperature dependence of the critical current. Thus, the ratio of the critical current values at the two temperatures is:

\[
\frac{I_c^{(20 \text{ mK})}}{I_c^{(4 \text{ K})}} \approx \beta_L^{(20 \text{ mK})} / \beta_L^{(4 \text{ K})} \approx 1.08.
\]

(9)
Using the Ambegaokar–Baratoff formula [47] for the temperature dependence of the critical current of an ideal tunnel junction between two identical BCS superconductors with critical temperature \( T_c = T_c^{(Nb)} \approx 9.0 \) K (here \( T_c^{(Nb)} \) is the critical temperature of our Nb films) we arrive at the same value of \( \epsilon_{AB} = 1.08 \).

4. Nonlinear effects

Setting the external magnetic flux \( \Phi_x \) in equation (3) such that the constant flux \( \Phi_{dc} = \Phi_0/4 \), that is:

\[
\Phi_x/\Phi_0 = \pi/2 + \beta, \tag{10}
\]

yields the Kerr-free nonlinear inductance of the rf SQUID of the form given by equation (1) [21]. The nonlinear coefficient in the current-phase relation (1), given in the general case by:

\[
\beta = 0.5\beta_L \frac{\sin(\Phi_{dc}/\Phi_0)}{1 + \beta_L \cos(\Phi_{dc}/\Phi_0)}, \tag{11}
\]

then equals \( \beta = 0.5 \beta_L \). For \( \Phi_{dc} = \Phi_0/4 \), the linear inductance of the rf SQUID (2) is equal to its geometrical inductance, \( L_s = L_1 \), while the inductance of the Josephson junction is infinite, \( L_J = \Phi_0/L_c \Phi_{dc}/\Phi_0 \) → \( \infty \).

To realize efficient mixing of the two lowest modes, we moved the rf SQUID out of the resonator center, because at \( x = 0.5d \) mode \( n = 2 \) has a node in the standing-wave of current and, hence, is only weakly coupled to fundamental mode \( n = 1 \). Embedding the rf SQUID at \( x = d/3 \) yields nonzero values of \( \sin(n\pi x/d) = |\sin(n\pi/3)| = \sqrt{3}/2 \), for all integer \( n \) except \( n = 3, 6, 9, \ldots \). Then the three-photon coupling coefficient (see equation (B12) of appendix B) is

\[
B_{3ph} \propto \beta \sin k_s x \sin k_n x \sin k_n x \approx \beta \sin(l\pi/3) \sin(n\pi/3) \sin(n\pi/3). \tag{12}
\]

For modes \( l = 1, m = 1, \) and \( n = 2 \), the value of \( |B_{112}| \) is maximum, while, for example, for modes \( l = 1, m = 2, \) and \( n = 3 \), its value is zero, \( |B_{123}| = 0 \). The latter property is evident, because the rf SQUID position, \( x = d/3 \), coincides with the standing-wave node of mode \( n = 3 \) (see the corresponding dashed curve in figure 3).

4.1. Second harmonic generation

Up-conversion (doubling) of the frequency of a light beam while preserving its quantum state [48] is of particular importance in nonlinear optics. A laser beam in this process is usually passing through a large crystal having non-centrosymmetric nonlinearity [49]. Using of an optical cavity containing a nonlinear crystal and resonant at the second harmonic may enable a source of ultraviolet radiation within this cavity [50]. To demonstrate the intracavity up-conversion of microwaves we applied to the circuit a high frequency drive signal:

\[
I_s(t) = I_0(t) \sin \omega_st, \tag{13}
\]

with a slowly oscillating amplitude having the shape:

\[
I_0(t) = A_0(1 + \mu \sin \omega_m t), \tag{14}
\]

where carrier frequency \( \omega_s \approx \omega_1 \), modulation frequency \( \omega_m \ll \omega_1 \), \( \omega_0 = \text{const} \), and modulation index \( \mu \ll 1 \). For sufficiently small driving signal (13), the fundamental-mode oscillations in the cavity have a shape similar to that of the steady-state oscillations in a driven oscillator (see, for example [51]),

\[
I_1 = I_0(t) \sin(\omega_s t + \delta + \delta_0), \tag{15}
\]

with the amplitude:

\[
I_0(t) \approx \frac{I_0(0)}{\sqrt{1 - (\omega_s/\omega_0)^2} + (\omega_s/\omega_0)^2/|Q|}, \tag{16}
\]

and phase \( \delta_0 \), that is:

\[
\tan \delta = \frac{\omega_s \omega_0}{|Q| (\omega_s^2 - \omega_0^2)}. \tag{17}
\]

Constant phase \( \delta_0 \) is determined by the parameters of the measuring setup.

The power spectra measured in the vicinities of frequencies \( \omega_s \) and \( 2\omega_s \) are shown in figure 4. One can see the generated frequency triplet consisting of the double frequency carrier at \( \omega = 2\omega_s \) with power \( P_2 \) and two sidebands at frequencies \( 2\omega_0 \pm \omega_m \) with powers \( (\mu^2/4)P_2 \) (see panel (b)). In the time domain, this signal has the shape:

\[
I_2 \propto (1 + \mu \sin \omega_m t) \sin(2\omega_s t + \vartheta), \tag{18}
\]

where \( \vartheta \) is the relative phase of the generated signal which is coupled to the total phase of the fundamental mode, \( \delta + \delta_0 \). The relatively small output power \( P_2 \) can be explained by rather large frequency mismatch, \( (2\omega_s - \omega_0)/2\pi = 37.1 \text{ MHz} \gg \omega_2/(2\pi Q_2) \approx 3.86 \text{ MHz} \), thus the generated signal is off-resonant for the second mode.

![Figure 3](image-url)
4.2. Half tone generation

Applying an intensive harmonic signal with frequency \( \omega_p \) close to the double resonant frequency of the fundamental mode, \( \omega_1 \approx \omega_2 \), may result in HTG, or, equivalently, the oscillation period doubling. (Note that the period tripling was earlier observed in a superconducting resonator with the Kerr nonlinearity by Svensson et al [52]). The period doubling is possible within a finite range of signal frequencies \( \omega_s \) around \( 2\omega_1 \), i.e. \( \delta \omega \approx |\omega_1/2 - \omega_1| \). Frequency range \( \delta \omega \) depends on the losses for the halftone mode and the signal power that should be sufficient for compensating these losses [51]. In this case, the zero state of mode \( \omega_1 \) becomes unstable and the circuit switches in one of the oscillating states, both with frequency \( \omega_1/2 \) and the relative phase difference of \( \pi \) [53].

Figure 5(a) shows HTG in the case of an amplitude-modulated input signal (13) with \( \omega_s \approx 2\omega_1 \), or \( \delta \omega \approx 0 \), whose power spectrum is presented in figure 5(b). The output spectrum measured around the half signal frequency, \( \omega \approx 0.5\omega_1 \) (shown in panel (a)) mimics the spectrum of the input signal (panel (b)). It consists of the carrier frequency \( (=0.5\omega_1) \) and two sideband peaks. Due to a very small modulation frequency \( \omega_m \ll \omega_2/Q \) the input signal can be considered as a harmonic signal with a slowly-varying amplitude. Due to the down-conversion (coupling coefficient \( Q \) is \( \neq 0 \)) and a sufficiently small modulation index, \( \mu \ll 1 \), the output signal at frequency \( \omega = 0.5\omega_1 \) also has an amplitude which is slowly-varying with frequency \( \omega_m \). Thus, its spectrum presents a triplet, where the small sideband peaks are positioned at \( 0.5\omega_1 \pm \omega_m \).

Relative phase \( \vartheta \) of the generated tone, \( I_{1/2} \propto \sin(0.5 \omega_1 t + \vartheta) \), takes randomly one of the two values, \( \vartheta = \vartheta_{1.2} \). In the general case, these values depend on the power \( (\propto I_p^2) \) and the dissipation in the circuit, but always have a fixed difference, \( |\vartheta_1 - \vartheta_2| = \pi \) (see, for example, equation (19) in [53]).

4.3. Parametric amplification

To demonstrate the operation of the circuit in the regime of a Josephson parametric amplifier (JPA) with 3WM in the degenerate mode, we applied a small harmonic signal \( \omega_s \) of frequency \( \omega_1 \approx \omega_2 \), and large pump, \( |I_p| \gg |I_t| \), at the double frequency, \( \omega_p = 2\omega_1 \), with relative phase difference \( \theta \),

\[
I_{+} + I_{p} = I_{0} \sin(\omega_1 t + \vartheta) + I_{0} \sin(2\omega_1 t).
\]  (19)

Figure (6) shows the measured amplification/deamplification versus phase \( \vartheta \). The maximum gain of 17 dB is comparable with the figure reported by Yamamoto et al [2] for the flux-driven Josephson parametric amplifier based on a \( \lambda/4 \)-cavity terminated by a dc SQUID. The 4WM amplifier of Castellanos-Beltran and Lehnert [1], based on a serial array of dc SQUIDs embedded in a \( \lambda/4 \) CPW resonator, showed a gain slightly above 20 dB. The largest deamplification (about \(-8 \) dB) measured in our sample is weaker than that achieved in [2, 10] (ca. \(-20 \) dB). This may be associated with relatively large background noise in our experiment. Still, the observed deamplification can be interpreted as a fingerprint of background-noise squeezing [9, 20].

Using the obvious advantage of 3WM, that the frequencies of an intensive pump and a small signal belong to different modes, we have realized the non-degenerate mode of operation of our JPA. The gain in this case is phase preserving. Fixing a small harmonic signal of frequency \( \omega_s \approx \omega_1 \) slightly above the half of the pump frequency, \( (\omega_p - \omega_p)/2 \pi = 50 \text{ kHz} \ll \omega_1/Q \), signal gain \( G = P_{\text{out}}/P_{\text{in}} \) of about 9 dB was observed. This direct gain was accompanied by the cross-gain \( G_{\text{cross}} = P_{\text{cross}}/P_{\text{in}} \) (approximately 8 dB), where the idler at frequency \( \omega_i = \omega_p - \omega_s \ll \omega_s \) has, according to the Manley-Rowe relation [54], the power \( P_i \lesssim P_{\text{in}} \) (see figure 7). Both the direct gain \( G \) and the cross-gain \( G_{\text{cross}} \) are proportional to nonlinear coefficient \( \beta^2 \) and, hence, are sensitive to the setting of magnetic flux \( \Phi_e \).

The latter property together with the periodic dependence of the nonlinear coefficient \( \beta \) on the external flux \( \Phi_e \) with the period of \( \Delta \Phi_e = \Phi_0 \), can be used for the evaluation of SQUID-parameter \( \beta_1 \). Using relations (3) and (11) one can
find the values of magnetic flux $\Phi_e$, giving the maximum of $\beta^2$. Within one period, $0 \leq \Phi_e < \Phi_0$, these two values are:

$$\Phi_{e1} = \varphi_0 \arccos (-\beta_L) + \varphi_0 \beta_L \sqrt{1 - \beta_L^2},$$

and

$$\Phi_{e2} = \Phi_0 - \Phi_{e1},$$

respectively. Note, that the magnetic flux value given by equation (20) is somewhat larger than the value given by equation (10) for the Kerr-free case, although the difference between these two values is vanishingly small for $\beta_L \ll 1$. As long as flux $\Phi_e$ is proportional to the control current strength, the ratio $(\Phi_{e2} - \Phi_{e1})/\Phi_0 = (\Phi_{e2} - \Phi_{e1})/(\Phi_{e2} + \Phi_{e1})$ can be found from the corresponding ratio of the control currents. The above ratio allows finding parameter $\beta_L$. Thus, $\beta_L$ found in Sample 2 from the parametric cross-gain measurements is 0.446, while the value found from the resonance measurements of this sample is $\beta_L = 0.44$.

Finally, we note that our open-ended cavity can also operate as a two-mode parametric frequency converter [51]. (Compare with the ring modulator based on the Wheatstone bridge configuration of four Josephson junctions enabling mixing of the waves in three attached resonators [55].) For example, applying a large pump ($\omega_p$) and a small signal ($\omega_s$), both with frequencies around $\omega_1$, gave rise to generating the sum frequency signal (SFG), $\omega_{sum} = \omega_1 + \omega_p \approx \omega_2$. Similarly, the difference frequency generation (DFG), $\omega_{diff} = \omega_2 - \omega_p \approx \omega_1$, was also possible, where input signal frequency $\omega_2 \approx \omega_2$, while pump frequency $\omega_p \approx \omega_1$.

5. Discussion and outlook

In the open-ended cavity with finite coupling capacitances $C_m$ and $C_{cont}$ and nonzero participation ratio $\alpha$ the resonant frequencies of the modes are not strictly equidistant [34]. One can, however, modify the circuit design such that for modes $l = m = 1$ and $n = 2$ the ratio of the resonant frequencies is 1:2. Then, 3WM condition, $\omega_1 + 2\omega_m = \omega_p$, is strictly fulfilled for the first two resonant frequencies of the cavity. A possible modification of the circuit is shown schematically in figure 8, where the array of $N$ dc SQUIDs [15] and a relatively large ground capacitance $C_0$ [56] are embedded in the center of the CPW resonator. As long as the critical current of the Josephson junctions in these dc SQUIDs is relatively large, their Kerr nonlinearity is small. However, the SQUID chain inductance and, thus, the resonant frequencies of all odd modes including $n = 1$ (having a current antinode at $x = d/2$) can be varied in a wide range with the help of dc control current $I_{cont}$ [15]. In a similar manner, ground capacitance $C_0$ reduces the resonant frequencies of the even modes (having a voltage antinode at $x = d/2$). For a sufficiently large inductance of the dc SQUID chain and a sufficiently large capacitance $C_0$, equidistance of all cavity modes with $n \geq 3$ is destroyed, while the frequency ratio for the two lowest modes, $\omega_2/\omega_1 = 2$, can be kept fixed by adjusting $I_{cont}$.

The system of two modes tuned in the resonance of type $2\omega_1 = \omega_2$, and having nonlinear coupling of $\chi^{(2)}$ type can be of particular interest in physics [57]. Our CPW cavity with the improved design or its lumped-element analog (see figure 9) could be a such a system implemented with the help of superconducting elements and microwave frequencies. For example, if energy losses are negligibly small and a drive is applied to mode $\omega_1$ the energy may be periodically transferred from mode $\omega_1$ to mode $\omega_2$ and vice versa. This behavior mimics the beating-like dynamics of two weakly-coupled identical pendulums [51]. However, there is a principal difference between these two phenomena; in the case

\[ \text{Figure 6. Phase-sensitive parametric amplification and deamplification in Sample 2 (figure 3). The input signal frequency and the power are } \omega_s/2\pi = 3.875 \text{ GHz and } -140 \text{ dBm, respectively. The pump frequency and the power are } \omega_p/2\pi = 7.75 \text{ GHz and } -70 \text{ dBm, respectively. The relative phase between these two phase-locked signals is } \theta (19). \text{ Dashed line shows the level of the output signal power when the pump is off. The maximum signal gain is 17 dB, while the maximum deamplification is about } -8 \text{ dB. The measurements were performed in a dilution refrigerator at temperature } T = 18 \text{ mK.} \]

\[ \text{Figure 7. Parametric amplification of signal tone } f_s = 3.88125 \text{ GHz by means of } 3WM \text{ in the non-degenerate case (pump frequency } f_p = 7.7624 \text{ GHz and idler frequency } f_i = f_p - f_s = 3.88115 \text{ GHz). The pump power is } P_p \approx -65 \text{ dBm. The measurements were performed in a helium bath at } T = 4.2 \text{ K.} \]
Figure 8. Array of N dc SQUIDs and ground capacitor $C_0$ embedded in the central part of the CPW resonator, $x = d/2$. Tuning the reduction of the resonant frequencies of the odd modes is possible by means of constant control current $I_{\text{cont}}$ creating a local magnetic field and thus varying the dc SQUID array inductance. A necessary reduction of the resonant frequencies of the even modes occurs due to the ground capacitance $C_0$.

Figure 9. Schematic of a lumped-element circuit comprising two resonators with the ratio of resonant frequencies $\omega_2/\omega_1 = 2$ and having nonlinear coupling of $\chi^{(2)}$-type. In the quantum case, this circuit enables SFG with single photons, $h\omega_a + h\omega_b = h\omega_c$, where input frequencies $\omega_a \approx \omega_b \approx \omega_1$. Under a drive at frequency $\omega_c \approx \omega_2$, a destruction of photon $h\omega_c$ gives rise to creation of the twin photons with frequencies $\omega_a \approx \omega_b (\approx \omega_1)$.

of the circuit with parametric coupling the energy exchange may occur between two oscillators with frequencies being in the ratio of 2 to 1. Similar behavior associated with nonlinear coupling may occur, for example, with the pitch and roll modes of ship motions when the ratio of these frequencies is close to two due to unequal moments of inertia for these two modes [58].

The superconducting cavity with the predominant coupling of modes $n = 1$ and $n = 2$ allows for efficient generation of nonclassical states of microwaves. Due to strict phase relations between the input and the output signals, a coherent transfer of the source phase from one part of the spectrum to another is possible. For example, SHG, SFG, and DFG, on the one hand, and HTG and JPA, on the other hand, give rise to quantum frequency conversion (QFC) [19] and spontaneous parametric down-conversion (SPDC) [59], respectively.

In the case of sufficiently low photon-loss rate, $\kappa \ll g$, SFC (with quantum relation $h\omega_a + h\omega_b = h\omega_c$) and SPDC ($h\omega_c = h\omega_a + h\omega_b$) are described by the second quantization Hamiltonian (B13) with the relevant part:

$$H_{\text{ad}} = \hbar g(a_\alpha a_\beta a_\gamma^\dagger + a_\alpha^\dagger a_\beta^\dagger a_\gamma),$$

(22)

These processes are schematically shown in figure 9. In the case of SFC, the input signals have frequencies $\omega_a$ and $\omega_b$ around $\omega_1$, while the frequency of the output signal, $\omega_c \approx 2\omega_1$. In the case of SPDC, the second mode driven at frequency $\omega_c \approx \omega_2$, gives rise to generating entangled photon pairs within the first mode bandwidth. Thus, the double-mode circuit with $\chi^{(2)}$ nonlinearity can serve as a source of nonclassical microwave light, including squeezed states [60]. Note that the three-photon SPDC was recently observed in a microwave-driven superconducting cavity with Kerr nonlinearity [14].

6. Conclusion

We have demonstrated that the rf SQUIDs embedded in superconducting coplanar waveguide resonators enabled the observation of a number of remarkable parametric effects. The degree of the non-centrosymmetric nonlinearity, enabling these effects, is governed by the rf SQUID parameter $\beta_L$, which value we extracted from resonant frequency modulation. The asymmetric position of this nonlinear element, i.e. at one third of the length of the open-ended resonator, allowed efficient coupling of three waves within the two lowest modes, $n = 1$ and $n = 2$. The range of observed 3WM processes include SHG, HTG, SFG, DFG, JPA in both degenerate and non-degenerate modes of operation.

The obvious advantage of a resonator with nonlinearity of a $\chi^{(2)}$ type over its counterpart with the Kerr nonlinearity [7] is that the former circuit can provide a principally stronger coupling. This property is particularly important for the open-ended resonators, where the intermode coupling is proportional to $\alpha^2$ and $\alpha^3$, respectively.

In conclusion, we believe that the superconducting cavities with $\chi^{(2)}$ nonlinearity are suitable not only for experiments with classical signals, but also for generating and converting nonclassical states of microwave fields on the level of single photons. In perspective, these circuits may enable nondemolition quantum measurements and quantum computing operations without conventional qubits. The proposed circuit will definitely extend the range of quantum information experiments with microwave photons and thus will have impact on quantum communication technologies.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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Appendix A. Resonant frequency of a cavity with embedded inductance

A particular solution of a wave equation for the phase variable $\phi(z,t)$ in the lossless open-ended resonator (the coupling capacitances, $C_{\text{in}}$ and $C_{\text{out}}$, are assumed to be negligibly small), having boundary conditions:

$$\frac{\partial \phi(0, t)}{\partial z} = \frac{\partial \phi(d, t)}{\partial z} = 0$$

(A1)

and embedded inductor $L_s$ (positioned not necessarily in the center, as shown in figure 1) has the form of a standing-wave,

$$\phi(z, t) = \psi(z)e^{-i\omega_n t}.$$  

(A2)

Here $\omega_n$ is the frequency of the $n$th mode and wave amplitude $\psi(z)$ is a piecewise function whose values for the left and the right segments of the transmission line are given by the harmonic (cosine) functions,

$$\psi(z) = \begin{cases} 
\psi_l(z) = \psi_0 \cos k_0 z, & 0 \leq z < x_l, \\
\psi_r(z) = \psi_0 \cos[k_n(d - z)], & x_r < z \leq d. 
\end{cases}$$

(A3)

Thus the outer boundary conditions equation (A1) are fulfilled.

Wave number $k_n$ for mode $n = 1, 2, \ldots$, in the resonator with embedded inductance $L_s$ is:

$$k_n = k_n^{(0)} - \delta k_n,$$

(A4)

where the wave number of the bare resonator is [45]:

$$k_n^{(0)} = n\pi/d,$$

(A5)

and $\delta k_n$ is the change of the wave number due to inductance $L_s$. The continuous current conditions on the left and the right terminals of inductance $L_s$ read:

$$\frac{1}{L_s} \psi'_l(x_l) = -\frac{1}{L_s} [\psi_l(x_l) - \psi_r(x_l)] = \frac{\varphi}{L_s},$$

(A6)

and

$$\frac{1}{L_s} \psi'_r(x_r) = -\frac{1}{L_s} [\psi_l(x_l) - \psi_r(x_r)] = \frac{\varphi}{L_s},$$

(A7)

where $\varphi$ is the phase drop on inductance $L_s$. Using equation (A3) the derivatives on the left-hand-sides of equations (A6) and (A7) are expressed as:

$$\psi'_l(x_l) = -k_n \psi_0 \sin k_0 x_l,$$

(A8)

and

$$\psi'_r(x_r) = k_n \psi_0 \sin[k_n(d - x_r)],$$

(A9)

respectively.

Inserting relations (A8) and (A9) into equations (A6) and (A7) gives the set of two homogeneous linear equations for wave amplitudes $\psi_a$ and $\psi_b$,

$$a_{11}\psi_a + a_{12}\psi_b = 0,$$

(A10)

$$a_{21}\psi_a + a_{22}\psi_b = 0.$$

(A11)

Here coefficients $a_{ij}$ ($i, j = 1, 2$) are:

$$a_{11} = -\alpha k_n d \sin k_0 x_l + \cos k_0 x_l,$$

(A12)

$$a_{12} = -\cos[k_n(d - x_r)],$$

(A13)

$$a_{21} = \cos k_0 x_l,$$

(A14)

$$a_{22} = \alpha k_n d \sin[k_n(d - x_r)] - \cos[k_n(d - x_r)],$$

(A15)

where the inductance participation ratio is:

$$\alpha = L_s/\ell d = L_s/L_{\text{tot}}.$$  

(A16)

The condition that the corresponding matrix has a zero determinant and, hence, that a nonzero solution $(\psi_a, \psi_b)$ of equations (A10) and (A11) exists,

$$\Delta = a_{11}a_{22} - a_{12}a_{21} = 0,$$

(A17)

yields the transcendent equation for wave number $k_n$,

$$\alpha k_n d \sin k_0 x_l \sin[k_n(d - x_r)] = \sin[k_n(d - x_r + x_l)].$$

(A18)

Taking into account the fact that both the participation ratio $\alpha$ and the relative size of inductor, $(x_r - x_l)/d = \Delta x/d$ (see figure 1(a)), are small $(\ll 1)$ and, thus, the wave-number change in equation (A4) is also small, $|\delta k_n| \ll n\pi/d$, equation (A18) is linearized. The resulting simplified equation is:

$$(-1)^{n+1} \alpha k_n^{(0)} \sin^2 k_n^{(0)} x = \sin(n\pi - \delta k_n d - k_n^{(0)} \Delta x)$$

(A19)

or

$$\alpha \sin^2(n\pi x/d) = \delta k_n k_n^{(0)} + \Delta x/d.$$  

(A20)

Thus the relative shift of the wave number and thus of the resonant frequency for mode $n$ is:

$$\frac{\delta k_n}{k_n^{(0)}} = -\frac{\omega_n - \omega_n^{(0)}}{\omega_n^{(0)}} = \alpha_n \equiv \alpha \sin^2 \frac{n\pi x}{d} - \frac{\Delta x}{d}.$$  

(A21)

Here the trigonometric factor $\sin^2(n\pi x/d)$ gives the dependence on the inductor position $x$, while term $\Delta x/d$ describes
the effective reduction of the resonator length, \( d \to d - \Delta x \).

The resulting eigenfrequencies are:

\[
\omega_n = \omega_n^{(0)} (1 - \alpha_n^a), \quad n = 1, 2, 3, \ldots \tag{A22}
\]

and wave numbers are:

\[
k_n = n \pi d (1 - \alpha_n^a). \tag{A23}
\]

The corresponding eigensolutions of the homogeneous wave equation in the range \( \theta \leq z \leq d \) have the form:

\[
\psi_n = [1 - \alpha \pi n (1)] \Theta(x_1 - z) \cos k_n z + \Theta(z - x_r) \cos [k_n (d - z)] \approx \cos k_n z, \tag{A24}
\]

where \( \Theta(z) \) is the Heaviside step function.

**Appendix B. Two-mode Hamiltonian of the cavity with \( \chi^{(2)} \) nonlinearity**

The rf SQUID potential energy in the case of the optimal flux bias, \( \Phi_{dc} = \Phi_0 / 4 \) (or \( \Phi_{dc} = \Phi_0 / 2 = \pi / 2 \)), and small ac phase difference on the Josephson junction, \( |\varphi| \ll \pi / 2 \), is given by:

\[
E_{s} = \varphi_{0} \int_{\varphi_{0}}^{\varphi} I(\varphi) d\varphi = E_{00} \left( \frac{\varphi^2}{2} - \beta \varphi^3 \right), \tag{B1}
\]

where energy \( E_{00} = \varphi_{0}^2 / L_{c} \). The first term (\( \propto \varphi^2 \)) on the right hand side of this relation contributes only to the eigenmode frequency equation (A22). Omitting, for a moment, the nonlinearity of the rf SQUID, the Hamiltonian of the resonator can be presented using the second quantization formalism in the standard form,

\[
H_0 = \sum_{n=1}^{\infty} \hbar \omega_n a_n^\dagger a_n, \tag{B2}
\]

where \( a_n^\dagger \) and \( a_n \) are creation and annihilation operators of the \( n \)-th mode, respectively.

The last term on the right hand side of equation (B1), that is:

\[
E_{nL} = -\frac{1}{2} \beta E_{00} \varphi^3 = -\frac{1}{2} \beta \alpha^3 E_{00} d^3 (\phi^3)^3, \tag{B3}
\]

describes the energy associated with the rf SQUID nonlinearity. Here the inductance participation ratio is \( \alpha = L_{c} / L_{tot} \ll 1 \) and phase \( \varphi \) is expressed via derivative \( \phi' \) using equations (A6) and (A7). Using the normal-mode decomposition of phase \( \phi \) [17],

\[
\phi = \sum_{n=1}^{\infty} A_n \psi_n = \sum_{n=1}^{\infty} A_n \cos k_n x, \tag{B4}
\]

the nonlinear part of the rf SQUID energy takes the form:

\[
E_{nl} = \sum_{l,m,n=1}^{\infty} B_{lmn} A_l A_m A_n, \tag{B5}
\]

where

\[
B_{lmn} = \frac{1}{3} \beta \alpha^3 E_{00} \prod_{j \in \{l,m,n \}} k_j \sin k_j x. \tag{B6}
\]

In the quantum case, the coefficients in the normal-mode decomposition equation (B4) are operators, \( \hat{A}_n \to \hat{A}_n \). They can be expressed via annihilation \( (\hat{a}_n) \) and creation \( (\hat{a}_n^\dagger) \) boson operators [46], that is:

\[
\hat{A}_n = \varphi_{0}^{-1} \sqrt{\hbar / 2 \omega_n} C_{eff,n}, \tag{B7}
\]

where the prefactor is associated with the magnitude of zero-point fluctuations [17, 61],

\[
\varphi_{0}^{-1} \sqrt{\hbar / 2 \omega_n} C_{eff,n} \tag{B8}
\]

Using the bare cavity values of frequencies \( \omega_n \) (6) and capacitances \( C_{eff,n} \) (8) we have:

\[
\varphi_{0}^{-1} \sqrt{\hbar / 2 \omega_n} C_{eff,n} \approx \frac{2Z_0}{nR_Q} \approx 0.124 \sqrt{n}, \tag{B9}
\]

where impedance \( Z_0 = \sqrt{L_{tot} / C_{tot}} = 50 \ \Omega \) and resistance quantum \( R_Q = \hbar / 4e^2 \approx 6.45 \ k\Omega \). Inserting equation (B7) in expression (B5) yields nonlinear part of Hamiltonian \( \hat{H}_{nl} \) in terms of operators \( \hat{a}_n \) and \( \hat{a}_n^\dagger \).

Applying a rotating wave approximation and thus omitting all oscillating terms of type \( \hat{a}_n \hat{a}_m \hat{a}_n^\dagger \hat{a}_n^\dagger \), etc we keep only terms \( \hat{a}_n \hat{a}_m \hat{a}_n^\dagger \hat{a}_n^\dagger \) and \( \hat{a}_n^\dagger \hat{a}_m^\dagger \hat{a}_n \) with the frequencies obeying the three photon (3WM) relation,

\[
\omega_1 + \omega_m - \omega_0 = 0, \quad \text{or} \quad l + m - n = 0, \tag{B10}
\]

we obtain the total Hamiltonian, \( \hat{H} = \hat{H}_0 + \hat{H}_{nl} \), in the form:

\[
\frac{H}{\hbar} = \sum_{n=1}^{\infty} \omega_n \hat{a}_n^\dagger \hat{a}_n + \sum_{l+m-n=0} \sum_{l+m-n=0} g_{lmn} (\hat{a}_n \hat{a}_m \hat{a}_n^\dagger \hat{a}_n^\dagger + \hat{a}_n^\dagger \hat{a}_m^\dagger \hat{a}_n \hat{a}_n), \tag{B11}
\]

where the mode coupling coefficient is:

\[
\frac{g_{lmn}}{\omega_1} = \frac{3}{2} \beta \alpha^3 \left( \frac{3 \sqrt{Z_0}}{R_Q} \right) E_{00} \tag{B12}
\]

Assuming that relation (B10) is fulfilled only for the two lowest modes, i.e. \( \omega_2 = 2 \omega_1 \), Hamiltonian (B11) takes the form:

\[
\frac{H}{\hbar} = \sum_{n=1}^{2} \omega_n \hat{a}_n^\dagger \hat{a}_n + h g (\hat{a}_1 \hat{a}_1 \hat{a}_2^\dagger + \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2), \tag{B13}
\]

where

\[
h g = 3h g_{112} = \beta \alpha^3 \left( \frac{3 \sqrt{Z_0}}{R_Q} \right) E_{00} \tag{B14}
\]

or

\[
\frac{g}{\omega_1} = \frac{3 \beta \alpha^3}{2 \pi^2} \sqrt{\frac{3 \sqrt{Z_0}}{R_Q}} \tag{B15}
\]
Here we used the relations $g_{121} = g_{121} = g_{211} \equiv g/3$ and $\sin k_1 d = \sin k_2 d \approx \sqrt{3}/2$ and the commutative property of the operators associated with different modes, $[a_1, a_2^\dagger] = [a_1^\dagger, a_2] = 0$.

Appendix C. Sample measurements

The samples were characterized using a vector network analyzer (VNA) R&S ZVA 40 by measuring transmission through the CPW resonator (parameters S21 and h2) as a function of frequency and/or microwave power. In the mixing experiments a microwave power supply was made using the sources R&S SMF100A, while an output power was measured by a signal analyzer (R&S FSV 30). In the experiments in liquid helium, the chips were bonded on a printed circuit board with the SMA connectors and mounted in a metallic box. In the dilution fridge unit, the attenuators in the input rf line mounted at different temperature stages gave an attenuation of ~50 dB in total (including attenuation in the cables). In the output rf line, two circulators (with the bandwidth of 3.2–8.1 GHz) were installed at the mixing chamber level. The input and output rf lines (coaxial cables) were directly bonded to the chip. The flux bias lines were supplied with the on-chip low-pass filters comprising the spiral-shape Nb coils with inductance about 5.5 nH. In both setups the sample was protected by a cryoperm shield. The readout of the output signal was done using a semiconductor low-noise amplifier from Low Noise Factory, model LNF-LNC0.3-14A. It operated in the range of 0.3–14 GHz with a gain up to 40 dB and a noise temperature of about 5 K. This cryogenic amplifier was installed at the 4 K stage of the dilution fridge unit.

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