Sensing Optical Cavity Mismatch with a Mode-Converter and Quadrant Photodiode

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We present a new technique for sensing optical cavity mode mismatch and alignment by using a cylindrical lens mode converting telescope, radio-frequency quadrant photodiodes, and a heterodyne detection scheme. The telescope allows the conversion of the Laguerre-Gauss bullseye mode ($LG_{01}$) into the $45^\circ$ rotated Hermite-Gauss ("pringle") mode ($HG_{11}$), which can be easily measured with quadrant photodiodes. We show that we can convert to the $HG$ basis optically, measure mode mismatched and alignment signals using widely produced radio-frequency quadrant photodiodes, and obtain a feedback error signal with heterodyne detection.

I. INTRODUCTION

Optical cavities are ubiquitously used in interferometry and in particular in the Laser Interferometer Gravitational-wave Observatory (LIGO). Optical cavities must be aligned and mode matched to yield the best performance. Alignment hardware and schemes are well developed [1] while mode matching hardware and schemes have not attained the same level of maturity. This leads to a reduction of sensitivity for gravitational-wave detectors such as Advanced LIGO [2]. Monitoring mode matching and dynamically correcting for it will ensure the best performance of future Advanced LIGO upgrades. This is particularly true for the use of non-classical squeezed vacuum states of light [3] currently being commissioned for use in Advanced LIGO, as these states are exponentially sensitive to any optical loss mechanism, including imperfect mode matching.

A theoretical description of misalignment and mode mismatch is done by Anderson [4]. Optical cavity misalignment and mode mismatching generate higher order optical modes. The first relevant modes for cavity misalignment are the well-known Hermite-Gaussian modes $HG_{10}$ and $HG_{01}$, while the dominant mode relevant for mode-mismatching is the Laguerre-Gaussian $LG_{01}$ mode ($LG_{lp}$ where $l$ is the azimuthal mode index and $p$ is the radial mode index). Higher order mode-sensing techniques currently utilize CCD cameras, clipped photodiode arrays [5], or bullseye photodiodes (BPD) [6]. These sensors provide feedback error signals for correcting either the beam waist size or waist location, but also have drawbacks. Some of the drawbacks include slow signal acquisition for CCD sensors, 50% reduction in sensing capabilities for clipped arrays, and expensive custom parts that are difficult to setup for bullseye photodiodes. While sensing mode matching is challenging, alignment sensing is well developed in comparison and relies on easily available RF quadrant photodiodes. By applying a $\frac{\pi}{2}$ mode converter [7], we show that the $LG_{01}$ mode turns into a $45^\circ$-rotated $HG_{11}$ mode, shaped perfectly for a quadrant photodiode. After sensing with a quadrant photodiode (QPD) we are free to use well-known heterodyne detection methods [4–6, 8] to extract a robust mode matching error signal. Thus the mode converter allows using the usually discarded “pringle” quadrant combination (+-+-) in existing alignment schemes for mode-matching feedback (see FIG. 1). This sensing scheme remains valid for large deviations from ideal mode-matching where a number of higher order modes contribute to the error signal (appendix V C).

II. MODELING MODE CONVERSION AND ERROR SIGNALS

A. Mode Converter

To understand how we can convert a Laguerre-Gauss $|LG_{01}\rangle$ mode into a $45^\circ$ rotated Hermite-Gauss $|HG_{11}\rangle$ mode we can decompose the beam in the $|HG_{nm}\rangle$ basis. The $|LG_{01}\rangle$ bullseye mode is the sum of exactly two...
modes

\[ |LG_{01}| = \frac{1}{\sqrt{2}}|HG_{20}| + \frac{1}{\sqrt{2}}|HG_{02}|, \]

(1)

as illustrated in FIG. 2. However, if we instead subtract the HG components instead of adding them, we will find that

\[ |HG_{11}^{45\circ \text{rot}}| = \frac{1}{\sqrt{2}}|HG_{20}| - \frac{1}{\sqrt{2}}|HG_{02}|. \]

(2)

where \( |HG_{11}^{45\circ \text{rot}}| \) is the 45° rotated \( |HG_{11}| \) mode.

This reveals that the only difference between a \( |HG_{11}^{45\circ \text{rot}}| \) mode and a \( |LG_{01}| \) mode is a sign flip along one axis, converting a parabolic wave front into a hyperbolic saddle point wave front.

A \( \frac{\pi}{2} \) mode converter creates a region where Gouy phase is accumulated at different rates for the each transverse axis as seen in FIG. 3. The cylindrical lens focusing axis accumulates \( \frac{\pi}{2} \) more phase than the non focusing axis. Since second order modes accumulate twice the Gouy phase, the \( |HG_{20}| \) and the \( |HG_{02}| \) see a phase accumulation difference of exactly \( \pi \). This flips the sign along one axis via the Euler identity, \( -1 = e^{i\pi} \), and creates the desired effect seen in FIG. 2 and FIG. 11. Designing a mode converter is described in appendix V A 2 and by Beijersbergen [9].

**B. Mode-Match Error Signal**

As with any Pound-Drever-Hall-style sensing [1, 4–6] scheme we sense the light using RF-demodulated photodiodes. Since, after passing the mode-converter, the mode-matching information is contained in the \( |HG_{11}^{45\circ \text{rot}}| \) mode, we use a quadrant photodiode rotated by a 45° relative to the mode-converter cylindrical lens axis. After demodulation we add the diagonals and subtract them from each other to get the error signal, see FIG. 4. In contrast, for a bullseye photodiode-based scheme we take the inner segment subtracted by the sum of the outer segments. Both schemes also allow sensing alignment and length signals (pitch, yaw and length).

![Mode-Match Error Signal](image)

**C. Maintaining alignment sensing**

Typical optical cavity alignment sensing requires the ability to measure the \( |HG_{01}| \) and \( |HG_{10}| \) modes with a quadrant photodiode [4]. Thus we need to examine what happens to the modes generated by misalignment after they pass through the mode converter. If the nodal lines of the \( |HG_{01}| \) and \( |HG_{10}| \) modes are at 0° or 90° relative to the mode-converter cylindrical lens axis, the two modes are passed unchanged - albeit with a relative phase shift of \( \frac{\pi}{2} \) between the two. In a Pound-Drever-Hall-style alignment sensing scheme these modes beat against the fundamental \( |HG_{00}| \) mode, which passes the mode-converter unchanged (see equations 17 to 21). Thus all alignment signals are still present, but the signals for one axis are shifted by \( \frac{\pi}{2} \) in sensing Gouy phase relative to
from the first photodiode. The optimal location depends on the phase, somewhere between 45° and 90° Gouy phase. Since the orthogonal mode-matching degrees of freedom is separated by 45° and 90° Gouy phase away from the actuator, this would preserve all alignment signals, only requiring a new sensing matrix. At the same time it would provide a sensor for one of the two possible mode-matching degrees of freedom. Since the orthogonal mode-matching degrees of freedom is separated by 45° apart, this setup guarantees that every possible signal is accessible. Installing a mode-converter in front of such a sensing sled would thus preserve all alignment signals, only requiring a new sensing matrix. At the same time it would provide a sensor for one of the two possible mode-matching degrees of freedom. Since the orthogonal mode-matching degrees of freedom is separated by 45° apart, this setup guarantees that every possible signal is accessible.

If sensitivity to only one mode-matching degrees of freedom is required - as is often the case in gravitational-wave interferometer applications - this simple upgrade would suffice, as long as the Gouy phase of the diodes is carefully chosen. If sensing of both mode-matching degrees of freedom is required, one can compromise by placing the second photodiode at an intermediate Gouy phase, somewhere between 45° and 90° Gouy phase away from the first photodiode. The optimal location depends on the sensing noise requirements. Alternatively one can choose to install a third RF-quadrant photodiode at 45° Gouy phase away from diodes one and two.

D. Controlling mode-match in interferometers

Besides optimizing optical gain, the quality of mode-matching between the various optical cavities in a gravitational-wave interferometer matters for two critical reasons. First, gravitational-wave interferometers like Advanced LIGO are now routinely using squeezed vacuum injected from the anti-symmetric port to reduce the quantum noise level [3]. Imperfect mode-matching couples the regular quantum vacuum fluctuations back into the readout, reducing the benefit from using squeezed vacuum. Second, a number of important noise sources, such as for example intensity noise on carrier and sideband, phase noise on the sideband and beam jitter, couple to the gravitational-wave readout through higher-order modes in the interferometer. While both 1st and 2nd order modes are problematic, an alignment system actively cancels 1st-order modes. Thus the largest higher-order modes are typically 2nd-order; they dominate the noise couplings unless an active mode-matching system suppresses them.

Alignment control of gravitational-wave interferometers has been extensively studied [1], [10], [11], [12]. All systems are an extension of the single-cavity Pound-Drever-Hall control scheme. The key differences when going to a more complicated system of coupled cavities are: (i) The beam splitter changes the sensing basis from individual arm cavities to common/differential arm cavities, sensed at the symmetric and anti-symmetric port of the beam splitter. And (ii) alignment signals from mirrors in coupled cavities can be disentangled by using multiple sensors operating with optical sidebands that are resonant in different portions of the coupled cavities. This design philosophy directly translates to mode-matching sensors, except that the system uses the second order transverse modes instead of the first order ones, requiring the type of sensors described in paragraphs II.B and II.C.

III. EXPERIMENTAL DEMONSTRATION

A. Experimental Layout

The adaptive mode matching experiment at Syracuse University was built to study and provide mode matching sensor solutions for Advanced LIGO. FIG. 5 shows the optical layout we used to compare two types of wavefront sensing photodiodes.

A 1064 nanometer wave length Nd:YAG Mephisto S laser beam passes through a 13 MHz locked triangular mode cleaner. The triangular mode cleaner feedback and sensing electronics are not shown, but consist of a typical Pound-Drever-Hall (PDH) loop. The beam then passes through a 25 MHz EOM for PDH locking and wave front sensing. The phase modulated beam propagates to mode matching lenses and then to a four segment thermal lens actuator [13–15]. A telescope is built around the thermal lens actuator such that the beam spot size is as big as possible without clipping on the 1 inch optic. The beam then enters a well-aligned and mode-matched optical cavity. The reflected beam continues through a Gouy phase telescope that also mode matches to a cylindrical lens mode converting telescope. Additionally, this telescope ensures that the beam size at the bullseye photodiode has the correct size and Gouy phase. A radio-frequency bullseye photodiode (BPD) and a radio-frequency quadrant photodiode (QPD) are placed at similar Gouy phases for a sensing comparison. The cavity reflected power is attenuated by a factor of 0.30 and 0.12 on the QPD and BPD respectively by various beam splitters. The optical power is then sensed, demodulated, and sent to a digital data acquisition system. In the digital system, the signals of each segment can then be combined to produce error signals.

The experimental setup is very similar to our computer simulation described in appendix V D and in FIG. 12. Though both model and experiment conclude that a mode converter, paired with QPDs, is equivalent to the use of BPDs there are a few subtle differences. The model uses four wavefront sensors. BPD2 and QPD2 are placed at an effective 0° Gouy phase from the actuator. The second set, BPD1 and QPD1, are placed at an effective 45° Gouy phase from the actuator. In our experimental demonstration we place one BPD at 283° Gouy phase, which is an effective 39° from our actuator after phase wrapping. Additionally, we place one QPD at 278° Gouy phase, which after phase wrapping is at an effective 34°
from our actuator. Note that the Gouy phases for the QPDs are reported with respect to the non-focusing axis of the cylindrical lenses. The Gouy phase along the cylindrical lens focusing axis is an additional $90^\circ$. Also, in the model we changed the input beam complex beam parameter to simulate either waist size or waist location only.

In practice, our lens actuator caused a change in both waist size and waist location at the same time.

### B. Thermal Lens Actuator Telescope

The thermal lens actuating telescope is composed of the first five lenses noted in FIG.6, FIG.7, and Table 1. The first two lenses expand and collimate the beam into the thermal lens actuator while the last two mode-match into the optical cavity. A larger beam on the thermal lens will provide better actuation range. The power overlap of the Gaussian beam before and after a thermal lens with focal length $f$ is given by

$$|I|^2 = 1 - \left(\frac{\pi w^2(z)}{2f\lambda}\right)^2 + O\left(\frac{w^4}{f^4\lambda^4}\right).$$

Thus a large beam spot size is needed for effective actuation. Furthermore, an annually heated thermal lens with power $P_h$ produces a power overlap of

$$|I|^2 = 1 - \left(\frac{w(z)}{R_{\text{optic}}\cdot\Delta}\right)^4 \cdot \left(\frac{FOM\cdot P_h}{4\lambda}\right)^2$$

where $FOM$ is obtained from [13]. This means that the two competing terms are the beam size and optic radius.

Incorporating these principles into a design yielded a thermal lens actuating telescope that produced mode-matching between 100% and slightly below 10%. Though
significant mode mismatch can be generated, wavefront sensors are best suited for measuring small amounts of misalignment or mode mismatch. This means that for relatively low input heating power, less than 5 watts, our thermal lens actuator telescope could measurably mismatch the beam into the optical cavity. The thermal lens actuation is further explained with FIG. 8.

In addition to mode mismatching, this thermal lens actuator also had the capability to create pitch and yaw misalignment. This was used to verify the preservation of alignment waveform sensing.

C. Wavefront Sensor Calibration

In this subsection we discuss how we calibrated the wavefront sensors and thermal lens actuator. The field mode mismatch $\epsilon = (q' - q)/(q - q')$, generated by our actuator, is ultimately converted to digital counts (cts) in the data acquisition system as follows. The power mode mismatch $|\epsilon|^2$ was monitored via a DC photodiode in the transmission of our optical cavity. The power drop percentage is proportional to the power mode mismatch $|\epsilon|^2$ as described by Anderson [4]. Our thermal lens actuator was set up so that we could degrade the optical cavity mode matching from 100% to just below 10% as seen in FIG 8. Though we had a wide range for mode matching, we chose to induce between 100% and 91% mode matching or 9% mode mismatch.

We next calculated from first principles the expected reflected RF power due to mode mismatch. As stated in equation 39 from the appendix a certain amount of mode mismatch $\epsilon$ will induce the following reflected optical power in watts peak for the quadrant photodiode

$$P_{\text{watts peak QPD}} = 4\Psi_{SSB}\Psi_C\sin^2(\epsilon D\phi_G/2\pi)$$

and the following reflected optical power for the bullseye photodiode

$$P_{\text{watts peak BPD}} = 4\Psi_{SSB}\Psi_C\sin^2(\epsilon D\phi_G/2\pi).$$

Note that $\Psi_C$ is the carrier field extracted from directly measuring the optical cavity transmitted power $\Psi_C = \sqrt{P_{\text{cavtrans}}}$. It should also be noted that $\Psi_{SSB}$ is the single-sideband field back-calculated from measured cavity transmitted power, cavity input power, cavity mirror measured transmissivity, and also includes a 0.95% intra-cavity loss term. The Gouy phase between the actuator and sensors $\Delta\phi_G$ can be read from the telescope table I above for both the BPD and QPD. The Gouy phase separation between the BPD sensor and the thermal lens actuator is 39° while the Gouy phase separation between the QPD sensor and the thermal lens actuator is 34°.

We can now compare the RF power in watts peak calculated from first principles to the RF power measured from calibrated electronics. The reflected beam first travels through several beam splitters which attenuate the beam by a factor of $A_{BPD} = .300$ for the bullseye and $A_{QPD} = .119$ for the quadrant. The optical power is then converted to current at the photodiode. All the electronics were calibrated by injecting voltage signals and measuring the output. The response of the quadrant

\begin{table}[h]
\begin{tabular}{|c|c|c|c|}
\hline
 & $f$ [m] & $d$ [m] & Gouy phase \ 
\hline
PMC & N/A & 0.0000 & 0° \ 
Lens1 & -0.574 & 0.7805 & 62° \ 
Lens2 & +2.291 & 0.9230 & 63° \ 
TL & - inf to -10 & 1.2375 & 64° \ 
Lens4 & +1.179 & 1.2630 & 64° \ 
Lens5 & -0.574 & 1.4012 & 65° \ 
IC & 0.33 m RoC & 1.8462 & 92° \ 
FP & N/A & 2.0102 & 137° \ 
OC & 0.33 m RoC & 2.1742 & 182° \ 
Lens6 & 0.45767 & 2.5022 & 122° \ 
Lens7 & inf & 2.6302 & 123° \ 
CL1 & +0.100 & 3.4274 & 186° \ 
CL2 & +0.100 & 3.5688 & 227° \ 
QPD & N/A & 4.3058 & 278° \ 
BPD & N/A & 4.7358 & 283° \ 
\hline
\end{tabular}
\end{table}

FIG. 8. Power mode overlap $P = \left|\frac{2\sqrt{\sin(\epsilon D\phi_G/2\pi)}}{\sin(\epsilon D\phi_G)}\right|^2$ between the optical cavity and the input beam is shown as the contour lines in percentage. The thermal lens actuator path is seen in red. As the thermal lens actuator changes the input beam into the cavity, the power mode overlap decreases. Mode mismatching can reach well below 10%.

\begin{align}
\Psi_C = \sqrt{P_{\text{cavtrans}}} & \\
\Psi_{SSB} = \Psi_C \sin(\epsilon D\phi_G/2\pi) & \\
\epsilon D\phi_G & = 39° \\
\Delta\phi_G & = 34°
\end{align}
photodiode is 0.03 Amps/Watt at 1064 nm wavelength and has a transimpedance 10,000 Volts/Amp. The response of the bullseye photodiode is 0.20 Amps/Watt at 1064 nm wavelength and has a transimpedance of 7,100 Volts/Amp. These RF voltages are then demodulated with our LIGO-built wavefront sensing electronic crate. The wavefront sensing crate demodulates the RF signal and contributes a factor of 6.7 gain. This gain was measured by injecting a 25 MHz sine wave at 12.7mV peak-to-peak. The demodulated signal was not constantly in phase so a 200 mHz wave at 190mV peak-to-peak was observed. If the injection was perfectly in phase we would see a DC voltage of 190mV/2=85 mV. From this we calculate the factor of 6.7 gain by 6.7 = 85mV/12.7mV. Now the demodulated signals are relatively low frequency and are sent to the digital system. The digital system has low pass filters, but do not alter the demodulated signals. We injected a known voltage into the digital data acquisition system and obtained a conversion of \( \frac{1V_{01}}{126 ct} \). Combining the beam splitter attenuation and all electronic gains leads to a direct conversion from cts to radio frequency optical watts peak at 25 MHz.

For the quadrant photodiode we have

\[
P_{\text{watts peak QPD}} = \frac{\text{cts}}{1326} \cdot \frac{7}{10} \cdot 6.7 \cdot \frac{0.1A}{0.000V} \cdot \frac{6W}{0.034A}
\]

and for the bullseye photodiode we have

\[
P_{\text{watts peak BPD}} = \frac{\text{cts}}{1326} \cdot \frac{7}{10} \cdot 6.7 \cdot \frac{0.1A}{0.000V} \cdot \frac{1W}{0.2A}
\]

We compress this whole calibration into a term \( C_Q = \frac{1V}{1326} \cdot \frac{6.7}{10} \cdot \frac{0.1A}{0.000V} \cdot \frac{6W}{0.034A} \) for the quadrant photodiode and similarly for the bullseye photodiode \( C_B = \frac{1V}{1326} \cdot \frac{6.7}{10} \cdot \frac{0.1A}{0.000V} \cdot \frac{1W}{0.2A} \).

We then solve for mode mismatch \( \epsilon \) and have a fully calibrated expression in terms of counts (cts).

\[
\epsilon_Q = \frac{\text{cts}_Q \cdot C_Q}{4A_Q \Psi S \psi C 2e^{-1}(\cos(2\pi \Delta \phi_G))}
\]

\[
\epsilon_B = \frac{\text{cts}_B \cdot C_B}{4A_B \Psi S \psi C 2e^{-1}(\cos(2\pi \Delta \phi_G))}
\]

D. Experimental Results

The results show good agreement between the bullseye photodiode (BPD) and the mode-converted quadrant photodiode (QPD) as seen in FIG. 9. Additionally, both QPD and BPD measured 9% mode mismatch which is consistent with the 9% mode mismatch induced by the thermal lens actuator. Note that the photodiode placement was chosen to reduce the number of lenses needed and to be relatively far away from a beam focal point, such that the beam size could easily match the photodiode size. This however resulted in a sub-optimal readout.

Gouy phase choice (QPD: 34° + n·90°, BPD: 39° + n·90°, where 45° + n·90° would be orthogonal.) Though this was a sub-optimal design choice, our results still clearly demonstrate the robustness of the heterodyne detection scheme. An ideal effective Gouy phase accumulation between an actuator and sensor should be a multiple of 90°.

The small discrepancy between the amplitude of the BPD and QPD error signals in FIG. 9 may be due to the in phase (I) and quadrature phase (Q) manual tuning. In the tuning we manually adjust the gain until the quadrature signal is extinguished. However, the quadrature signal does not always go exactly to zero. The computer simulation in the appendix is better suited for comparing ideal BPD and ideal mode converted QPD error signals. It should be noted that even the idealized simulation contains some gain discrepancy which is due to the geometry of the photodiodes.

IV. CONCLUSION

We theoretically derived the mode-matching error signal for bullseye photodiode (BPD) and mode-converted quadrant photodiode (QPD) wave front sensing. We showed that a mode-converted quadrant photodiode preserves the ability to measure alignment whilst enabling the ability to measure mode-match. We proposed a sensing scheme usable by any heterodyne optical setup directed towards Advanced LIGO, and experimentally...
demonstrated a side-by-side comparison of bullseye photodiode and mode-converted quadrant photodiode sensing. We should also point out that using a mode-converted quadrant photodiode shifts the difficulties in setting up a bullseye photodiode Gouy phase telescope with a specific beam size to the placement of the mode converter lenses, which is much easier to fine adjust.

We conclude that this mode-converter-based sensing scheme could yield a non-invasive, inexpensive mode-matching upgrade to terrestrial gravitational-wave detectors such as Advanced LIGO, Advanced Virgo and KAGRA. All RF quadrant photodiodes used for interferometer alignment in those detectors could be upgraded by redesigning their respective Gouy phase telescopes to include cylindrical lenses.

V. APPENDIX

A. Hermite-Gaussian modes with two complex q parameters

1. Complex Beam parameters

The complex beam parameter of a Gaussian beam with Rayleigh range \( z_R \), at a distance \( z \) from its waist, is defined as

\[
q = z + iz_R.
\]

Beam size \( w \) and phase front radius of curvature \( R \) are then given by

\[
\frac{1}{q} = \frac{1}{R} - i \frac{\lambda}{\pi w^2},
\]

where \( \lambda = 2\pi/k \) is the wave length of the light. It allows expressing the Gaussian beam in a simple form:

\[
\Psi(x, y, q) = A(x, y, q)e^{-ikz}
\]

\[
A(x, y, q) = \frac{A}{q} e^{-i k \frac{z^2 + w^2}{2w^2}}
\]

where \( A \) is a complex constant (amplitude). It can be helpful to introduce the field amplitude on the optical axis, \( \psi = A/q \), which now evolves along the z-axis due to the Gouy phase evolution, but is unaffected when passing through a thin lens. Thus, for any given location on the optical axis \( z \), the Gaussian beam is completely described by the two complex parameters \( \psi \) and \( q \). The main advantage of this formalism becomes apparent when using ray-transfer matrices \( M \) defined in geometric optics (e.g. Saleh, Teich) to represent the action of a full optical system. The two complex parameters \( (q_f, \psi_f) \) after the system are given in terms of the initial parameters \( (q_i, \psi_i) \) by

\[
M \begin{pmatrix} \frac{1}{\psi_i} \\ \frac{1}{q_i} \end{pmatrix} = \begin{pmatrix} \frac{1}{\psi_f} \\ \frac{1}{q_f} \end{pmatrix},
\]

and the change of the Gouy phase through the system, \( \Delta \phi \), is given by

\[
e^{i\Delta \phi} = \sqrt{\frac{\psi_f}{\psi_i}} \frac{\psi_i}{\psi_f} e^{i\pi R^2}.
\]

This expression is consistent with the usual definition of local Gouy phase for a Gaussian beam as \( \phi = \arctan z/z_R \), but preserves the Gouy phase when propagating through a lens. To prove expressions 13 and 14 it is sufficient to verify them for a pure free-space propagation and a pure lens.

If we now introduce astigmatism, either intentionally with cylindrical lenses or accidentally through imperfections, cylindrical symmetry around the beam axis will be lost. As long as we introduce this astigmatism along a pre-determined axis (say the x-axis), we can simply proceed by introducing separate q-parameters for the x- and y-axis, \( q_x \) and \( q_y \). Since ray-transfer matrices are introduced with only 1 transverse axis, the propagation of \( q_x \) and \( q_y \) is done with ray-transfer matrices defined for the corresponding transverse axis. Thus we now have a separately-defined beam size \( w_x, w_y \), phase front radius of curvature \( R_x, R_y \), Rayleigh range \( z_{Rx}, z_{Ry} \) and Gouy phase \( \phi_x \) and \( \phi_y \) for each of the two transverse directions. The corresponding fundamental Gaussian beam is given by

\[
\Psi(x, y, q_x, q_y) = A(x, y, q_x, q_y)e^{-ikz}
\]

\[
A(x, y, q_x, q_y) = \frac{A}{\sqrt{q_x q_y}} e^{-i k \frac{x^2}{2w_x} - i k \frac{y^2}{2w_y}}
\]

where \( A \) is again a complex amplitude. Next we introduce the Hermite-Gaussian basis set corresponding to the fundamental Gaussian beam. In the literature this is typically done only relative to a single q-parameter, but it directly generalizes to the case with separate \( q_x \) and \( q_y \) parameters:

\[
\Psi_{nm}(x, y, q_x, q_y) = A_{nm}(x, y, q_x, q_y)e^{-ikz}
\]

\[
A_{nm}(x, y, q_x, q_y) = N A_n(x, q_x) A_m(y, q_y)
\]

\[
A_p(\xi, q\xi) = e^{ipq\xi} \sqrt{\frac{1}{2\pi q\xi}} H_p(\sqrt{2\xi}) e^{-i k \frac{\xi^2}{2q\xi}}
\]

\[
\psi_\xi = \sqrt{\frac{2}{\pi}} \frac{e^{i\phi_\xi}}{w_\xi} = \sqrt{\frac{2zR}{\lambda q_\xi}} \frac{i}{q_\xi}
\]

\[
H_0(\eta) = 1, \ H_{p+1}(\eta) = 2\eta H_p(\eta) - \frac{d}{d\eta} H_p(\eta)
\]
\[ \int |\Psi_{nm}|^2 dx \, dy = |N|^2. \] That equations 15 and 16 are of the same form as equations 17 to 21 can be seen by using the identity \( iz \varphi / q = e^{i \varphi} w / w \). Furthermore we defined \( \varphi \) in analog to the field amplitude \( \psi \) introduced after equation 12, that is the field amplitude on the optical axis of the fundamental mode. It thus evolves, together with \( q \), according to equations 13 and 14. Note though that there is an extra Gouy phase term for the higher order modes that is explicitly excluded from the definition of \( \varphi \). As a result, the overall Gouy phase evolution of \( \Psi_{nm}(x, y, q_x, q_y) \) is proportional to \( e^{i(n+1/2) \varphi + i(m+1/2) \varphi} \).

As expected, these modes still solve the paraxial Helmholtz equation

\[ (\Delta_n - 2ik \frac{\partial}{\partial z}) A_{nm}(x, y, q_x, q_y) = 0 \] (22)

exactly. Finally, in the main text we use the simplified bra-ket notation for readability:

\[ |HG_{nm}\rangle = |\Psi_{nm}(x, y, q_x, q_y)\rangle. \] (23)

Specializing to the non-astigmatic \( q_x = q_y \) we also use the two identities

\[ |LG_{01}\rangle = \frac{1}{\sqrt{2}} |HG_{20}\rangle + \frac{1}{\sqrt{2}} |HG_{02}\rangle, \] (24)

\[ |HG_{11}^{\text{rot}}\rangle = \frac{1}{\sqrt{2}} |HG_{20}\rangle - \frac{1}{\sqrt{2}} |HG_{02}\rangle. \] (25)

Equation 24 relates the Hermite-Gaussian basis to the Laguerre-Gaussian basis (see e.g. [7]) , while equation 25 directly follows from equations 17 to 21 under a 45° rotation around the beam axis.

2. Design of the \( \frac{\pi}{2} \) mode-converter

Equations 24 and 25 highlight that the key requirements for a mode-converter capable of converting a \( |LG_{01}\rangle \) into a \( |HG_{11}^{\text{rot}}\rangle \) mode: We need a difference of \( \pi \) in phase evolution between the two 2nd order modes \( |HG_{20}\rangle \) and \( |HG_{02}\rangle \), leading to a relative sign flip. We thus require a telescope consisting of at least two cylindrical lenses that

1. has a x-Gouy phase \( \Delta \phi_x \) and y-Gouy phase \( \Delta \phi_y \) evolution that differs by exactly \( \frac{\pi}{2} \) between the first and last cylindrical lens (\( \Delta \phi_x - \Delta \phi_y = \frac{\pi}{2} \)), and

2. again matches the x- and y- Gaussian parameters \( q_x \) and \( q_y \) after the last cylindrical lens. Note that technically the quadrant photo detector (QPD) could be placed at the location of, and instead of the last cylindrical lens. But that would make any further downstream adjustment of the sensing Gouy phase of the QPD impossible.

While there are an infinite number of solutions that fit conditions 1) and 2) above, there is only one symmetric solution with two cylindrical lenses with the same focal length \( f \) and the waist exactly in the middle between the two lenses. For this symmetric case, condition 2) requires the x- and y- beam size to be identical at the lenses:

\[ \Im \left( \frac{1}{q_x} - \frac{1}{q_y} \right) = \Im \left( \frac{1}{\frac{d}{2} + iz_{Rx}} - \frac{1}{\frac{d}{2} + iz_{Ry}} \right) = 0, \] (26)

where \( d \) is the separation between the lenses, \( z_{Rx}, z_{Ry} \) are the Rayleigh ranges for the x- and y- Gaussian beam profile, and \( 3 \) denotes the imaginary part. Excluding the trivial solution \( z_{Rx} = z_{Ry} \), this implies the condition

\[ \frac{d}{2z_{Rx}}, \frac{d}{2z_{Ry}} = \tan \frac{\Delta \phi_x}{2} \cdot \tan \frac{\Delta \phi_y}{2} = 1. \] (27)

This is equivalent to

\[ \cos \frac{\Delta \phi_x + \Delta \phi_y}{2} = 0. \] (28)

Using \( \Delta \phi_x - \Delta \phi_y = \frac{\pi}{2} \) from condition 1., we thus find

\[ \Delta \phi_x = \frac{3\pi}{4}, \Delta \phi_y = \frac{\pi}{4}. \] (29)

Finally, since \( \tan \frac{\pi}{8} = \frac{1}{\sqrt{2} + 1} \) and \( \tan \frac{3\pi}{8} = \frac{1}{\sqrt{2} - 1} \), we get for the cylindrical focal length \( f \) of both lenses and the lens separation \( d \)

\[ f = \frac{z_0}{1 + \frac{1}{\sqrt{2}}}, \quad d = \sqrt{2} f, \] (30)

where \( z_0 = z_{Ry} = \frac{\pi w^2}{\lambda} \) is the Rayleigh range of the incoming beam (no lens in y-direction).

B. Comparison to sensing with a bull’s-eye detector

We use the term bull’s-eye photo-diode (BPD) for a photodiode with a center segment of radius \( r \), and additional outer segments arranged in a ring around the central segment. Typically there are three outer segments to still get alignment information from the detector (see figure 4, right side).

When sensing mode mismatch with a BPD, matching the center segment radius \( r \) to the Gaussian Beam spot size \( w \) via \( w = \sqrt{2} r \) maximizes the mode-mismatch small signal sensing gain, because at that radius the \( |LG_{01}\rangle \) mode has a node. However, for this choice we find that any residual length fringe deviation will couple directly into the mode-mismatch error signal because

\[ \langle HG_{00}|BPD|HG_{00}\rangle = 1 - 2e^{-1} \approx 0.2642 \neq 0, \] (31)

where \( BPD \) is equal to 1 on the central segment \( (x^2 + y^2 < r) \), and -1 on the outer segments \( (x^2 + y^2 > r) \).
This coupling can be reduced to zero by choosing \( r' = w\sqrt{0.5\ln 2} \) as central segment radius, at the cost of some optical gain (see below). Either way though the BPD has to be matched in size to the Gaussian beam. This often makes adjusting the sensing Gouy phase of a BPD a bit awkward, since it is not possible to simply slide the detector across the optical axis. Furthermore, the amount of clipping on the bull’s-eye photo-diode is set at the time of manufacturing by the size of the outer ring segments.

In contrast, a quadrant photo-diode (QPD) placed after a \( \frac{\pi}{2} \) mode-converter has none of these beam size constraints. Instead, the reference beam size is set by the choice of the mode-converter through equation 30, and can be changed by replacing the cylindrical lenses. The QPD can be moved freely to optimize the sensing Gouy phase and clipping, while any residual length fringe deviation does not couple to first order, since for a well-centered beam we find

\[
(H G_{00} | Q P D | H G_{00}) = 0.
\]

Here we chose \( Q P D = \text{sign}(x^2 - y^2) \).

### C. Signal Gain for Sensing Mode-Mismatch

Since we want to sense a mode-mismatched Gaussian beam \( |H G_{00}^q\rangle \) with beam parameter \( q' \), we can expand this beam in the unperturbed basis (\( q \)) as

\[
|H G_{00}^q\rangle = e^{-i3\pi} \sqrt{1 - \epsilon^2} |H G_{00}^{q'}\rangle + \epsilon |L G_{01}^q\rangle + O(\epsilon^2),
\]

where \( q \) denotes the imaginary part and \( \epsilon \) encodes the waist size change \( \Delta w_0 \) and waist displacement \( \Delta z \) of the Gaussian beam via

\[
\epsilon = \frac{q' - q}{q - q^*} = \frac{\Delta w_0}{w_0} - i \frac{\Delta z}{2z_R}.
\]

Equation 33 includes enough \( O(\epsilon^2) \) terms such that the power coupling is accurately given to 2nd order by

\[
|\langle H G_{00}^q | H G_{00}^{q'}\rangle|^2 = 1 - |\epsilon|^2 + O(\epsilon^3).
\]

To calculate the small signal gain for a mode-sensing scheme we need the matrix element

\[
\gamma_B = \langle H G_{00}^q | B P D | L G_{01}^q\rangle = -2e^{-1} e^{2i\phi} \approx -0.7358 e^{2i\phi},
\]

where \( \phi \) is the Gouy phase at the BPD. The minus sign is an artifact of the definition of Laguerre-Gaussian modes [7]. Here the central element radius of the BPD is \( r = w/\sqrt{2} \). For a BPD with central segment radius \( r' = w\sqrt{0.5\ln 2} \) the numerical pre-factor drops to \(-ln(2) \approx -0.6931 \). See section VB for a discussion.

The equivalent matrix element for a QPD, after converting the \( |L G_{01}^q\rangle \) mode into a \( |H G_{11}^{45\text{rot}}\rangle \) mode, is

\[
\gamma_Q = \langle H G_{00}^q | Q P D | H G_{11}^{45\text{rot}}\rangle = \frac{2}{\pi} e^{2i\phi} \approx 0.6366 e^{2i\phi}.
\]

If we use this approach to sense the matching of a cavity (beam parameter \( q' \)) to its input beam using the Pound-Drever-Hall (PDH) approach, we will use an up-front RF phase modulation (modulation index \( \Gamma \)) with a sideband frequency that is not resonant in the cavity. The Gaussian beam reflected from this cavity has the structure

\[
|\Psi_{in}\rangle = |H G_{00}^q\rangle c + \frac{i\Gamma}{2} |H G_{00}^{q'}\rangle + \frac{i\Gamma}{2} |H G_{00}^{q'}\rangle - \frac{O(\Gamma^2)}{3},
\]

where the indices \( C, + \) and \( - \) indicate carrier, upper and lower sideband. We can sense this beam with either a BPD or a QPD behind a mode-converter, and demodulate the signal’s I quadrature. We find in first order of \( \Gamma \) and \( \epsilon \)

\[
I = P \Gamma \Im(\gamma \epsilon),
\]

where \( P \) is the effective power on the photo diode - that is ignoring any power that does not contribute the RF signal, \( \Gamma \) is the modulation index, \( \Im \) denotes the imaginary part, \( \gamma \) is the matrix from equation 36 or 37, and \( \epsilon \) is defined through equations 33, 34, 35.

For large mode deviations the power coupling from equation 35 is given by the exact expression

\[
|\langle H G_{00}^q | H G_{00}^{q'}\rangle|^2 = \left| \frac{2i\sqrt{(3q')^2-(3q)^2}}{q' - q^*} \right|^2,
\]

where \( 3 \) denotes the imaginary part, and the sensing signal from equation 39 generalizes to

\[
I = P \Gamma \Im(\langle H G_{00}^q | T^\dagger \text{ PD } T | H G_{00}^{q'}\rangle),
\]

where \( PD \) is either the BPD or the QPD. Here \( T \) is the action of both mode-converter telescope (for the QPD) and Gouy phase telescope. Since we know the action of both telescopes on the two-parameter Hermite-Gaussian beams introduced in section VA, we can write the matrix element of equation 41 as

\[
\sum_{n, m} e^{i\phi(n+m)} \langle H G_{00}^q \rangle |B P D | H G_{nm}^{q'}\rangle \langle H G_{nm}^q | H G_{00}^{q'}\rangle.
\]

and

\[
\sum_{n, m} i^n e^{i\phi(n+m)} \langle H G_{00}^q \rangle |Q P D | H G_{nm}^{q'}\rangle \langle H G_{nm}^q | H G_{00}^{q'}\rangle.
\]

These expressions are plotted in figure 10 with \( \phi = 0 \) for waist location variations and \( \phi = \pi/4 \) for waist size variations, taking into account modes up to \( n, m = 20 \). BPD and QPD have comparable, although not identical large signal gains.

### D. Error Signal Model

A computer simulation provided a convenient way for testing our prediction before performing the experiment.
A combination of MATLAB and FINESSE [16] was used to arrive at the mode mismatch error signal. FINESSE uses ray transfer matrices while our MATLAB model uses the Fourier optic representation of lenses and beams. FINESSE was previously used by Bond [17] to study optical cavity mode mismatch. That study served as a basis for comparison.

The optical layout seen in FIG. 12 was constructed to compare the error signals generated by bullseye photodiodes and quadrant photodiodes. The input beam was varied in waist size and waist location. This produced mode mismatching which was calculated in the reflected field. Higher order modes beat against the fundamental sidebands yielding an error signal. At this point, the field is then segmented into quadrants and the diagonals are summed and subtracted from the orthogonal diagonal. This can be better understood by seeing the error signal combination in FIG. 4.

FIG. 11 shows the transverse electric field before and after it passes through a \( \frac{\pi}{2} \) mode converter telescope. The MATLAB model uses a heterodyne detection scheme to measure the beat between the fundamental sidebands and higher order mode mismatch modes [6]. The beam is phase-modulated at 25 MHz, and the photodiode output is demodulated with the same frequency. The cavity is kept locked on resonance.

The simulation results can be seen in FIG. 13 and show that we can generate a beam waist size and beam waist position error signal. We isolate beam waist size and beam waist position with both the bullseye and mode converted quadrant photodiode. The cavity input beam size is varied and results in the two error signal to the left. Notice that only the BPD and QPD placed at \( 45^\circ \) Gouy phase are sensitive to this kind of offset while the other two photodiodes see virtually no change. If instead we look at the second plot where the beam input beam waist position is shifted, we see that the opposite is true. Now the BPD and QPD placed at \( 0^\circ \) Gouy phase are sensitive to this kind of offset while the other photodiodes are not. This is the optimal placement for sensing mode mismatch. In practice we will want to also measure misalignment and thus we’ll have to move the 2nd photodiode to somewhere between \( 45^\circ \) and \( 90^\circ \) Gouy phase, depending on the sensing noise requirements for alignment and mode-matching. This simulation is a direct
FIG. 12. A 1 watt laser produces a beam at 1064 nanometer wave length. The beam passes through an Electro Optic Modulator (EOM) resonant at 9 MHz. The beam then passes through a beam splitter then into a hemispherical resonant optical cavity. The beam reflected from the cavity is then directed back to the beam splitter where now the reflected beam is directed to two paths. The first path contains two radio-frequency bullseye photodiodes (RFBPD) of varying radii. FINESSE automatically changes the bullseye photodetector size to match the beam incident on it. Secondly the beam passes through a beam shaping telescope then to a mode converter before finally arriving at two radio-frequency quadrant photodiodes (RFQPD). Each style of photodiode has one photodiode that measures the beam at 0° Gouy phase and a second photodiode that measures the beam at 45° Gouy phase. This Gouy phase separation is ideal for measuring both beam waist size and beam waist location.

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FIG. 13. Mode mismatch error signals generated by the FINESSE with MATLAB simulation. See appendix V D for more details.