Interacting Weyl semimetals on a lattice

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Abstract

We perform an exact renormalization group analysis of a fermionic system on a three-dimensional lattice, showing the persistence of the Weyl semimetallic phase up the quantum critical point discriminating between the semimetallic and insulating phases. The interaction produces finite modifications of the anisotropic Fermi velocity and of the overlap of the true (dressed) quasiparticle with the free electron wave function. The optical conductivity is the one of Weyl fermions with anisotropic Fermi velocities replacing the light velocity, provided that the frequencies are below a certain energy scale, which is vanishing at the quantum critical point; above such scale, a different behavior is found, dominated by the quadratic corrections to the dispersion relation.

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1. Introduction

Weyl semimetals are three-dimensional fermionic systems whose Fermi ‘surface’ consists of disconnected points. Their properties, somewhat intermediate between a metal and an insulator, are quite different with respect to systems with extended surfaces. The possibility of Weyl semimetallic behavior has been theoretically predicted to occur in several systems [1–25] and recent experiments in Bi$_{1-x}$Sb$_x$ [26] or in Bi$_2$Ir$_2$O$_7$ [27] found indications of it. In Weyl semimetals the bands touch in correspondence of certain Weyl points and nearby the dispersion relation is approximately linear; it turns out that, neglecting higher order corrections, the elementary excitations can be effectively described in terms of massless Dirac (or Weyl) particles in $D = 3 + 1$. This fact suggests that phenomena typical of high energy physics (quarks and leptons in the standard model are Weyl particles) can have a low energy analogue in these systems. The emerging description in terms of Dirac particles [28, 29] in Weyl semimetals is common to other materials, among which is graphene, in which the excitations are $2 + 1$ massless Dirac particles; the different dimensionality produces however important differences.
The Weyl semimetallic phase is typically present between two insulating phases. For instance in the lattice model introduced in [24] the Weyl semimetallic phase is present when the bands touch at two distinct Weyl points and close to them the dispersion relation is essentially linear (conical). Varying the parameters, the Weyl points move closer to each other and the quadratic corrections become more relevant; at the quantum critical point the two Weyl points coalesce and the dispersion relation becomes purely quadratic (see also [30]). Beyond such points an insulating phase is reached with a gap in the spectrum.

It is of course important to understand the effect of the many body interaction between fermions, which is surely present in real compounds. Previous studies describe the properties of the charge carriers in the Weyl semimetallic phase in terms of massless fermions Dirac fermions interacting with a Coulomb [31] (see also [32, 33]) or short ranged [34] potential. In the first case a class of diverging diagrams in the perturbative expansion is resummed, and logarithmic corrections to the conductivity are found. In the second case the potential is irrelevant according to naive power counting, so that in the weak coupling regime one expects that interactions produce no effects; at strong coupling the possibility of instabilities is argued. The above conclusions are based on the assumption that the quadratic corrections can be neglected, that is that the nonlinear dispersion relation can be replaced with a linear one.

This is however clearly questionable close to the quantum critical point, as the quadratic corrections becomes more and more relevant. If one insists in neglecting the nonlinear terms, some troubles are obviously encountered. In the ‘relativistic’ approximation the expansion parameter is the coupling divided by the velocity components; some of the velocities, close to the boundary of the semimetallic phase, are very small, so that in such effective description the effective coupling is strong, even if it is small in the original lattice model.

Even well inside the semimetallic phase the linear approximation presents problems. Such approximation transforms the system in an interacting Quantum Field Theory in which ultraviolet divergences are present, which were of course absent in the original realistic lattice model. In order to avoid them, an ultraviolet regularization is needed. There are several possible choices, and there is no guarantee that the physical quantities are independent on the chosen cut-off. Indeed in the case of graphene (which is very related to Weyl semimetals) the interaction corrections to conductivity are known to be dependent from the regularization [35–46] and the natural choice (apparently giving the better agreement with experiments) consists in considering the natural cut-off present in the lattice model, instead of adding an artificial cut-off to relativistic fermions. One may of course expect that the relativistic model captures, at least well inside the semimetallic phase, the main features ‘up to finite corrections’: but such finite corrections are of course crucial in a number of issues (for instance in the universality properties of the conductivity).

In this paper we consider for the first time the effect of the interaction in a lattice model for Weyl semimetal, taking fully into account the effects of the nonlinear dispersion relation and in a wide range of parameters, including the quantum critical point discriminating between the semimetallic and insulating phase. This will be done by expansions with an estimated radius of convergence which is non-vanishing in a region of parameters including the quantum critical point; by them, the Weyl semimetallic phase and the transition between the semimetallic and insulating phase will be studied, focusing in particular on the properties of the optical conductivity.

The plan of the paper is the following. In section 2 the model and the physical quantities are defined. In section 3 the properties of the non-interacting model are recalled. In section 4 the main results are described. In section 5 the renormalization Group analysis well inside the semimetallic phase is described, while in section 6 the renormalization group analysis describing the crossover between the semimetallic and insulating phase is developed. In the
appendices, some symmetry properties are established and lowest order computations are presented.

2. The model

2.1. The hopping Hamiltonian

Let us consider a tight binding model defined on a 3D lattice composed of layers of face-centered square lattice (see figure 1) composed by two sublattices: one is $\Lambda_A = A$ with side $L$ given by the points $\vec{\delta} = \begin{pmatrix} 1 \end{pmatrix}$, with $\vec{\delta}_1 = (0, 0, 1)$, $\vec{\delta}_2 = (0, 1, 0)$, $\vec{\delta}_3 = (1, 0, 0)$; the other is $\Lambda_B$ whose points are $\vec{x} + \vec{\delta}_x$ with $\vec{\delta}_x = \begin{pmatrix} \frac{1}{2} \end{pmatrix}$ and $\vec{\delta}_x = \begin{pmatrix} -\frac{1}{2} \end{pmatrix}$. The planar nearest neighbor hopping is denoted by $t$, the next to nearest planar hopping by $t'$ and the interlayer hopping by $t_{\perp}$; an on site energy difference is also introduced. The non-interacting version of this model was introduced in [24], to which we refer for a microscopic derivation (see also [29]).

Introducing fermionic creation and annihilation operators $(a^{\dagger}_{\vec{x}}, b^{\dagger}_{\vec{x}+\vec{\delta}_x})$, the Hamiltonian is

$$H = H_1 + H_3 + H_4 + UV,$$

where $H_1$ describes the hopping between the A and B sublattice

$$H_1 = \frac{1}{2} \sum_{\vec{x} \in A} \left\{ i t \left( a^{\dagger}_{\vec{x}} b_{\vec{x}+\vec{\delta}_x} + b^{\dagger}_{\vec{x}+\vec{\delta}_x} a_{\vec{x}+2\vec{\delta}_x} \right) + h. c. \right\}$$

$$+ \left\{ t \left( a^{\dagger}_{\vec{x}} b_{\vec{x}+\vec{\delta}_x} - b^{\dagger}_{\vec{x}+\vec{\delta}_x} a_{\vec{x}+2\vec{\delta}_x} \right) + h. c. \right\}$$

Figure 1. Graphical representation of the lattice; the small dots belong to the A sublattice, the big dots to the B sublattice.
while $H_2$ contains $AA$ or $BB$ hopping

$$H_2 = \frac{1}{2} \sum_{i \in A} \left\{ t_A \left( a_i^+ a_i - b_i^+ b_i \right) + \text{h.c.} \right\} - t' \sum_{i=1,2} \left\{ a_i^+ a_i - b_i^+ b_i + \text{h.c.} \right\}$$

and $H_3$ takes into account the on site energy difference between the two sublattices

$$H_3 = \frac{\mu}{2} \sum_{i \in A} \left( a_i^+ a_i - b_i^+ b_i + \text{h.c.} \right).$$

The hopping parameters $t, t', t''$ are assumed $O(1)$ and positive. The interaction is

$$V = \sum_{i,j} \nu \left( x_i - x_j \right) \left[ a_i^+ a_j^+ + b_i^+ b_j^+ \right] \left[ a_i^+ a_j^+ + b_i^+ b_j^+ + \text{h.c.} \right]$$

and $\nu(x)$ is a short-range interaction such that $|\nu(x)| \leq e^{-|x|k}$ for a suitable constant $k$.

### 2.2 The currents and the conductivity

The currents are defined as usual via the Peierls substitution, by modifying the hopping parameter along the bond $(x_i, x_{i+1})$ as $U_{x_i, x_{i+1}}(A) = e^{ik} \int_0^1 \tilde{A}(x_i + s \tilde{A}) ds$, where $e$ is the electric charge and $\tilde{A} = (A_1, A_2, A_3)$ is the vector electromagnetic field; the modified Hamiltonian is

$$H(A) = H_1(A) + H_2(A) + H_3 + UV,$$

where

$$H_1(A) = \frac{1}{2} \sum_{x \in A} \left\{ -i \left( a_x^+ U_{x, x+\delta} a_{x+\delta} + b_x^+ b_{x+\delta} \right) + i \left( a_x^+ U_{x, x+\delta} a_{x+\delta} + b_x^+ b_{x+\delta} \right) \right\}$$

and

$$H_2(A) = \frac{1}{2} \sum_{x \in A} \left\{ t_A \left( a_x^+ a_x + b_x^+ b_x \right) + \text{h.c.} \right\} - t' \sum_{i=1,2} \left\{ a_i^+ a_i - b_i^+ b_i + \text{h.c.} \right\}$$

We define $a_+ \equiv \frac{1}{L} \sum_i e^{ikx} a_i^+$ and $b_+ \equiv \frac{1}{L} \sum_i e^{ikx} b_i^+$, where $k = \frac{2\pi}{L} n$. 

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The paramagnetic lattice currents are given by

\[
\hat{j}_{\pm} = -\frac{\partial H(\hat{A})}{\partial A_{\pm,\hat{r}}}
\]

if \( A_\pm = \frac{A_1 + A_2}{2} \): using the notation \( \int d\mathbf{k} = \frac{1}{p^2} \sum_k \) one obtains, if \( \hat{\psi}_k^\pm = (a_k^\pm, b_k^\pm) \)

\[
\hat{j}_{\pm,\hat{r}} = e \int d\mathbf{k} \hat{\psi}_k^\pm \left[ w_{a,k}(\mathbf{k}, \hat{\rho}) \sigma_1 + w_{b,k}(\mathbf{k}, \hat{\rho}) \sigma_3 \right] \hat{\psi}_k^\pm,
\]

\[
\hat{j}_{\pm,\hat{r}} = e \int d\mathbf{k} \hat{\psi}_k^\pm \left[ w_{a,k}(\mathbf{k}, \hat{\rho}) \sigma_1 + w_{b,k}(\mathbf{k}, \hat{\rho}) \sigma_3 \right] \hat{\psi}_k^\pm,
\]

\[
\hat{j}_{\pm,\hat{r}} = e \int d\mathbf{k} \hat{\psi}_k^\pm \left[ w_{a,k}(\mathbf{k}, \hat{\rho}) \sigma_1 + w_{b,k}(\mathbf{k}, \hat{\rho}) \sigma_3 \right] \hat{\psi}_k^\pm,
\]

where

\[
w_{a,k}(\mathbf{k}, \hat{\rho}) = \frac{1}{2} m_{a,k}(\hat{\rho})(e^{(\mathbf{k} + \hat{\rho})\hat{\delta}_x} + e^{-\hat{\delta}_x}),
\]

\[
w_{b,k}(\mathbf{k}, \hat{\rho}) = \frac{1}{2} \sum_{i=1,2} \eta_i(\hat{\rho}) \left[ e^{(\mathbf{k} + \hat{\rho})\hat{\delta}_i} - e^{-\hat{\delta}_i} \right],
\]

\[
w_{3}(\mathbf{k}, \hat{\rho}) = -i \frac{1}{\hbar} \sum_{i=1,2} \eta_i(\hat{\rho}) \left[ e^{(\mathbf{k} + \hat{\rho})\hat{\delta}_i} - e^{-\hat{\delta}_i} \right]
\]

with \( \eta_i(\hat{\rho}) = \frac{1 - e^{-\frac{\hbar}{\hat{\rho}\delta_i}}}{1 + O(\hat{\rho})} \). Moreover \( \sigma_0 = I \) and \( \sigma_i, i = 1, 2, 3 \) are Pauli matrices,

\[
\sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

The diamagnetic tensor is defined as, if \( i = (+, -, 3), j = (+, -, 3) \)

\[
D_{\hat{p},\hat{r}ij} = \frac{\partial^2 H(\hat{A})}{\partial A_{i,\hat{r}} \partial A_{j,\hat{q}}} \bigg|_{\hat{A} = 0}
\]

and the density operator is defined as \( \rho_{\pm} = a_{\perp}^\dagger a_{\perp}^\pm + b_{\perp}^\dagger b_{\perp}^\pm + \hat{\delta}^\perp \).

If \( \Omega = e^{\omega H} \mathcal{O}_x e^{-\omega H} \), with \( x = (x_0, \mathbf{x}) \), we denote by

\[
\left\langle O_{x_1}^{(1)} \ldots O_{x_n}^{(n)} \right\rangle_{\beta} = \lim_{L \to \infty} \mathcal{Z}^{-1} \text{Tr} \left\{ e^{-\beta H} \mathcal{T} \left( O_{x_1}^{(1)} \ldots O_{x_n}^{(n)} \right) \right\}
\]

where \( \mathcal{Z} = \text{Tr} \{ e^{-\beta H} \} \) and \( \mathcal{T} \) is the operator of fermionic time ordering; moreover we denote by \( \left\langle O_{x_1}^{(1)} \ldots O_{x_n}^{(n)} \right\rangle_{\beta} \) the corresponding truncated expectations and by \( \left\langle O_{x_1}^{(1)} \ldots O_{x_n}^{(n)} \right\rangle_{0} \) their zero temperature limit.

We will particularly interested in the two-point Schwinger function \( \left\langle \psi_{x_2}^\dagger \psi_{x_1} \right\rangle_{\beta} \) and in the current–current correlations. In the collisionsless regime \( \omega \gg \beta^{-1} \) information on the optical conductivity can be obtained by Kubo formula, as collisions with thermally excited carriers can be neglected; if \( \left\langle j_{x_2} j_{x_1} \right\rangle, i = (+, -, 3) \), is the current–current correlation and \( \left\langle j_{\hat{p},\hat{r}ij} \right\rangle_{\beta} \) the corresponding Fourier transform, with \( \hat{p} = (\omega, \hat{\rho}) \), the conductivity at zero temperature is defined as

\[
\sigma_{ij}(i\omega) = -\lim_{\beta \to \infty} \frac{1}{\beta \omega} \left\langle j_{\hat{p}}, j_{\hat{r}ij} \right\rangle_{\beta} + \left\langle D_{\hat{p},\hat{r}ij} \right\rangle_{\beta}
\]

(13)
3. Properties of the non-interacting model: Weyl and insulating phases

The Hamiltonian in the non-interacting $U = 0$ case can be easily written in diagonal form

$$H_0 = \frac{1}{L^3} \sum_k \left[ |E(\vec{k})| \hat{a}_k^+ \hat{a}_k^- - \hat{\pi}_k^+ \hat{\pi}_k^- \right], \quad (14)$$

where

$$|E(\vec{k})| = \sqrt{t^2 \left( \sin^2(k_+) + \sin^2(k_-) \right) + \left( \mu + t_\perp \cos k_3 - \frac{1}{2} t' (\cos k_1 + \cos k_2) \right)^2}, \quad (15)$$

where $\hat{a}_{\vec{k}}^\pm, \hat{\pi}_{\vec{k}}^\pm$ are suitable linear combinations of $\hat{a}_{\vec{k}}^\pm, \hat{\pi}_{\vec{k}}^\pm$. The physical properties of the system are then crucially depending from the parameters in the Hamiltonian. We assume for definiteness that

$$\mu + t' > 2t_\perp. \quad (16)$$

The Weyl semimetallic phase is present when the two bands only touch at two distinct points. The functions $\sin (k_+)$ and $\sin (k_-)$ vanish in correspondence of two points $(k_1, k_2) = (0, 0)$ and $(k_1, k_2) = (\pi, \pi)$ and assuming (16) we see that $|E(\vec{k})|$ can vanish only is

$$\left| \mu - t' \right| < t_\perp \quad (17)$$

the bands touch in two Weyl points $\pm \vec{p}_F$, with

$$\vec{p}_F = \left\{ 0, 0, \cos^{-1} \left( \frac{t' - \mu}{t_\perp} \right) \right\}. \quad (18)$$

Around such points the dispersion relation is essentially conical and the two-point function strongly reminds the one of massless Dirac fermions. Note indeed that $\langle \psi^{\vec{k}}_x \psi^{\vec{k}}_y \rangle_{U=0} \equiv g(\vec{x} - \vec{y})$ is given by

$$g(\vec{x}) = \int d\vec{k} e^{i \vec{k} \vec{x}} \begin{pmatrix} -i k_0 + t_\perp (\cos k_3 - \cos k_F) + E(\vec{k}) & t (\sin k_+ - isink_-) \\ t (\sin k_+ + isink_-) & -i k_0 - t_\perp (\cos k_3 - \cos k_F) - E(\vec{k}) \end{pmatrix}^{-1} \quad (19)$$

with $E(\vec{k}) = t'(\cos k_+ \cos k_- - 1)$. For momenta respectively close $\vec{p}_F$ or $-\vec{p}_F$ ($\vec{p}_F = (0, \vec{p}_F)$) we can write, if $\vec{k} = \vec{k}' \pm \vec{p}_F$

$$t_\perp (\cos (k_3' \pm p_F) - \cos p_F) = \pm \sqrt{2} \sin p_F \sin k_3'$$

$$+ (\cos k_3' - 1) \cos p_F = \mp \sqrt{2} \sin p_F \sin k_3' + \mathcal{O}(k_3'^3). \quad (20)$$

where $\sqrt{2} = t_\perp \sin p_F$; therefore close to the band touching points the two-point function can be written as
\[ \dot{g}(k' \pm p_F) \]

\[ = \begin{cases} 
-ik_0 \pm v_{3,0} k_3 \left( 1 + O\left( \frac{k_3^2}{v_{3,0}} \right) \right) + O\left( \frac{k_3^2}{v_{3,0}} \right) & \quad \text{if } v_{3,0} = 1 \\
\pm \frac{v_\pm 0}{v_{3,0}}(k_+ - ik_-) + O\left( \frac{k_3^2}{v_{3,0}} \right) & \quad \text{if } v_{3,0} = 1
\end{cases} \]

(21)

with \( v_{3,0} = t \).

If the quadratic corrections are neglected, the two \( 2 \times 2 \) matrices \( \dot{g}(k' + p_F) \) and \( \dot{g}(k' - p_F) \) can be combined in a \( 4 \times 4 \) matrix coinciding with the propagator of a massless Dirac (or Weyl) particle, with anisotropic light velocity. It should however be remarked that when \( \mu_{-1} / t_c \) is increased from 0 to 1, the third component of the Fermi velocity is decreasing and the quadratic corrections in (20) becomes more and more important; this means that the relativistic linear behavior is found only for momenta very close to the Weyl points, that is \( |k_3| \ll v_{3,0} \) and \( k_\pm^2 \ll v_{3,0}^2 k_3^2 \).

When \( \mu_{-1} / t_c = 1 \) the two Weyl points coalesce and \( v_{3,0} = 0 \); the dispersion relation is quadratic in the third direction. Finally for \( \mu_{-1} / t_c > 1 \) the two bands do not touch and one has an insulating behavior; the two-point function decays exponentially (the denominator of (19) does not vanish).

In the Weyl semimetallic phase \( \mu_{-1} / t_c < 1 \) for \( \omega \ll v_{3,0} \) the conductivity is approximately equal to the conductivity of Weyl fermions up to a rescaling factor and small corrections; indeed if \( i = +, -, 3 \) the zero temperature conductivity

\[ \sigma_{ii}(i\omega) \bigg|_{\omega=0} = \begin{cases} 
\epsilon^2 \frac{v_{3,0}^2}{(v_{3,0})^2} & \quad \text{if } v_{3,0} = 1 \\
\sigma_{0,\text{weyl}}(i\omega)(1 + O(\omega)) & \quad \text{otherwise}
\end{cases} \]

(22)

where \( \sigma_{0,\text{weyl}}(i\omega) \) is the conductivity of Weyl fermions with light velocity \( c = 1 \) and a momentum cut-off. By analytic continuation \( i\omega \rightarrow \omega + i\varepsilon \) one can verify that the real part of \( \sigma_{0,\text{weyl}}(\omega) \) vanishes as \( O(\varepsilon) \) [31] while the imaginary part vanishes as \( O(\omega \log |\omega|) \) [33].

As we said such relativistic behavior for the conductivity is found for energies \( \omega \ll v_{3,0} \).

At the quantum critical point \( \mu_{-1} / t_c = 1 \) then \( v_{3,0} = 0 \) and dimensional considerations, see [30], say that for small \( \omega \)

\[ \sigma_{++}(i\omega) \bigg|_{\omega=0} = O\left( \omega^{1/2} \right), \quad \sigma_{--}(i\omega) \bigg|_{\omega=0} = O\left( \omega^{1/2} \right), \quad \sigma_{33}(i\omega) \bigg|_{\omega=0} = O\left( \omega^{3/2} \right) \]

(23)

where the different behavior of the conductivity in the third direction is due to the factor \( \sin k_3 \) in the definition of the current \( \langle \bar{c}_i \sigma^{ij} c_j \rangle \). The above behavior is found also in the semimetallic regime close to the quantum critical point for \( \omega \gg v_{3,0} \).

4. The interacting case: main results

The model will be analyzed using the renormalization group methods peculiar to constructive quantum field theory, previously applied to solid state models like graphene [45] or Luttinger liquids [47]. Contrary to field theoretical renormalization group methods, they do not require linear dispersion relations; lattice effects can be fully taken into account. Another important peculiarity is that they are rigorous, in the sense that the convergence of the renormalized expansion can be established. The physical quantities are expressed by a lowest order computation plus a rest which can be rigorously bounded. This should be compared with
usual perturbative approaches, in which higher orders are simply neglected truncating the expansions. The explicit proof of convergence is rather technical and long (but rather similar to the ones for graphene or 1D metals) and it will be published elsewhere; here we assume it and we explore its physical consequences.

(1) We start considering a renormalization group analysis for values of the parameters well inside the semimetallic phase, that is \( \frac{U}{\sqrt{\nu}} \ll \nu_{3,0} \ll \nu_{\perp} \). We show that there exists an \( U_0 \) such that for \( |U| \leq U_0 \) the two-point function has an asymptotic behavior given by

\[
\langle \psi_k^{\dagger} \psi_k \rangle = \begin{pmatrix} v_{z_3} (1 + O(\frac{k^2}{\nu})) + O\left(k^3\right) & v_{z_2} (k_+ + ik_-) + O\left(k^3\right) \\ v_{z_2} (k_+ + ik_-) + O\left(k^3\right) & -ik_0 + v_{z_3} (1 + O(\frac{k^3}{\nu})) + O\left(k^3\right) \end{pmatrix}^{-1},
\]

where \( v_{z_3}, v_{z_2} \) are the dressed velocities, that is

\[
v_{z_1} = v_{z,0} + a_3 U + O\left(U^2\right) \quad v_{z_2} = v_{z,0} + a_2 U + O\left(U^2\right)
\]

(25)

and \( Z = 1 + O(U^2) \); the explicit expression for the coefficients \( a_3, a_2 \) is in appendix B. The non-zero renormalization of the velocities depends only from the presence of the irrelevant terms; if we neglect the quadratic corrections (that is, in the relativistic approximation) there is no renormalization. The interaction also produces a finite overlap of the true (dressed) quasiparticle with the free electron wave function (that is, \( Z \neq 1 \)). Finally \( p_F = p_F + O(U) \), that is also the interaction also changes the location of the Weyl points.

(2) In the same region of the parameters the interacting conductivity is given by

\[
\sigma_{\pm,\pm}(i\omega) = e^2 \left[ \frac{Z_\pm}{Z} \right]^2 \frac{1}{(v_{z,0})^3} \sigma_{ii,\text{weyl}}(i\omega) \left(1 + O(\omega)\right) + UR_{\pm}(\omega),
\]

\[
\sigma_{3,3}(i\omega) = e^2 \left[ \frac{Z_3}{Z} \right]^2 \frac{1}{(v_{z,0})^3} \sigma_{ii,\text{weyl}}(i\omega) \left(1 + O(\omega)\right) + UR_3(\omega),
\]

where

\[
Z_\pm = 1 + b_2 U + O\left(U^2\right) \quad Z_3 = 1 + b_3 U + O\left(U^2\right)
\]

(27)

and the coefficients \( b_2, b_3 \) can be found in appendix B; moreover

\[
|R_i(\omega)| \leq C |\omega|.
\]

(28)

The term \( R_i(\omega) \) is a correction with respect to the non-interacting result, coming from the contribution of higher orders in the renormalized expansion. Note that while the linear behavior of the free conductivity has logarithmic corrections, they are absent in the correction term. The constants \( Z_\pm, Z_3 \) represents finite renormalizations of the current; they are non-universal coefficients, depending the irrelevant terms. Remarkably, as consequence of Ward identities, the following relations are true

\[
\frac{Z_\pm}{Z} = v_{z_\pm} \quad \frac{Z_3}{Z} = v_{z_3}
\]

(29)

which are valid at any order of the expansion, and can be explicitly checked at lowest
order (see appendix B). Inserting (29) in (26) we get

\[ \sigma_{\omega}(1) = e^2 \frac{v_f^2}{v_3^2} \sigma_{\text{Weyl}}(1 + O(\omega)) + UR_\pm(\omega). \]

Despite the fact the the velocities and the renormalizations get finite renormalizations, the conductivity in the interacting case is identical, up to subleading corrections, to the one of Weyl fermions, provided that the bare velocities are replaced by the dressed ones.

(3) Finally we extend the analysis for values of the parameters \(|v_{3,0}| \lesssim \frac{\nu}{\sqrt{2}}\), that is in a region including the quantum critical point. The multiscale analysis used for \(|v_{3,0}| \gtrsim \frac{\nu}{\sqrt{2}}\) produces an expansion with an estimated radius of convergence \(U_0\) proportional to \(v_{3,0}\); therefore it cannot be used to get information close to the quantum critical point, where \(v_{3,0} = 0\). We have then to develop a different renormalization group scheme, based on two different multiscale analysis in two regions of the energy momentum space; in the smaller energy region the effective relativistic description is valid while in the larger energy region the quadratic corrections due to the lattice are dominating. In both regimes the interaction is irrelevant but the scaling dimensions are different; after the integration of the first regime one gets gain factors which compensate exactly the velocities at the denominator produced in the second regime. Even if the estimated radius of convergence \(U_0\) is not optimal (we expect convergence in a larger domain), the crucial point is that it is non-zero in a wide region of the parameters including the critical point, where \(v_{3,0} = 0\). The semimetallic phase persists up to the quantum critical point also in presence of the interaction; in particular, the conductivity has the relativistic behavior given by (30) for \(\omega \ll v_3\), that is in a range of energy smaller and smaller approaching the quantum critical point; on the contrary for \(\omega \gg v_3\) it has the nonlinear behavior given by (23).

5. Interaction effects well inside the semimetallic phase

5.1. Renormalization group analysis

The physical observables can be expressed as usual in terms of Grassmann integrals. We denote by \(\psi_x = (\psi_x^+, b_{x+1}^-)\), \(x = (\nu_0, x)\), \(d = (0, \delta)\) a set of Grassmann variables; with abuse of notation, we denote them by the same symbol as the fermionic fields. As we expect that the location of the Weyl points will be in general modified by the presence of the interaction, we find convenient to fix them to their non-interacting value by replacing \(\mu\) with \(I_\perp \cos p_F + \nu\), where \(\nu\) is a counterterm to be suitably chosen as function of \(U\); a non-vanishing \(\nu\) means that the location of the Weyl points is shifted. We introduce the generating functional, \(A = (A_0, \bar{A})\)

\[ e^{iW(A, \phi^*, \phi)} = \int P(\text{d}\psi) e^{i\mathcal{V}(\psi) + iA(\bar{\psi}, \psi)} \]

where \(P(\text{d}\psi)\) is the fermionic integration with propagator (19) and \(\mathcal{V}\) is the interaction

\[ \mathcal{V} = \nu N + U\nu, \]

\[ \nu = \frac{1}{\sqrt{2}}, \]

\[ \mathcal{V} = \nu N + U\nu, \]
where, if \( \int dx = \int d\mathbf{x} \sum \mathbf{r} \)

\[
N = \int dx_1 \psi_+^+ \gamma_3 \psi_+^-
\]

\[
V = \int dx_1 dy \psi_+^+ (x-y) \left( \psi_+^+ \gamma_0 \psi_+ \right) \left( \psi_+^+ \gamma_0 \psi_+^+ \right)
\]

(33)

if \( v(x) = \{ d(x) \} v(x) \). Moreover \((\psi, \phi) = \int dx \{ \psi_+^+ \gamma_0 \phi_+^+ + \psi_+^+ \gamma_0 \phi_+ \}\) and

\[
B(A, \psi) = \int dx_1 \lambda_0 (x) \psi_+^+ \gamma_0 \phi_+^+ + B_1 (A, \psi) + B_2 (A, \psi),
\]

(34)

where the explicit expression of \( B_1 (A, \psi) \) and \( B_2 (A, \psi) \) is obtained by \( H_1 (A) \) and \( H_2 (A) \) (see (7)) by replacing \( \sum \mathbf{r} \) with \( \int d\mathbf{x} \sum \mathbf{r} \), the Fermi operators \( \mathbf{a} \) and \( \mathbf{b} \), with the Grassmann variables \( a_\mathbf{a} \) and \( b_\mathbf{b} \).

In order to investigate the effect of the electron–electron interaction on the physical behavior we perform a renormalization group analysis of the generating functional (62).

We consider first the behavior well inside the semimetallic phase, that is for values of the parameters \( \mu', \perp_t, \) such that \( \mu > \perp_t \), and

\[
\pi_p \leq \perp \leq \pi_p^2
\]

(35)

where \( \perp = \mu - \perp_t \). This condition ensures that the Fermi velocity \( v_{3,0} = \sin \perp_t \) is not too small, that is \( \perp > \perp_t \).

The starting point is the decomposition of the frequency-energy space in circular sectors of radius and width \( O(2^h) \); more technically, one introduces a decomposition of the unity \( \mathbf{a} \) and \( \mathbf{b} \), where \( f_h (k) \) is non-vanishing in \( \mathbf{a} \) and \( \mathbf{b} \). The first step of the renormalization group analysis of the generating function (31) is the integration of the positive ultraviolet scales; due to the presence of the lattice, making the dispersion relation \( \mathbf{a} \) bounded, the ultraviolet integration poses no problems and the results of the integration of \( \psi (0) \) is an expression similar to (31) with \( \psi (0) \) replaced by \( \psi (0) \leq (0) \) sum of monomials of any order in \( \psi (0) \) and \( A \). The propagator \( \h_0 (k) \) is non-vanishing in a range of frequency and energies verifying \( \sqrt{k_0^2 + \ell (k)}^2 \leq 2a \). We can choose the constant \( a \) so that such region corresponds to two disconnected regions in \( k \) space, centered around the Fermi points and labeled by \( \epsilon = \pm \). This means that the Grassmann fields can be conveniently written as sum of two independent fields

\[
\psi_x (0) = \sum_{\epsilon = \pm} \psi_{\epsilon, x} (0)
\]

(36)

with \( \psi_{\epsilon, x} (0) \) with propagator \( \mathbf{g}_0 (x) = \int d\mathbf{k} e^{i \mathbf{k} \cdot \mathbf{x}} \mathbf{g}_0 (k) \). The integration of the infrared scales \( h = 0 \) is done iteratively. Assume that we have integrated \( \psi^{(\geq 1)} \), \( \psi^{(0)} \), \( \psi^{(1)} \), obtaining

\[
e^{W} = \int \prod_{\epsilon = \pm} P \left( \psi_{\epsilon}^{(\geq 1)} \right) e^{\psi^{(0)} (\mathcal{F}_0 \psi^{(0)} \psi^{(0)}) + B^{(0)} (A, \mathcal{F}_0 \psi^{(0)} \psi^{(0)})},
\]

(37)
where $P(\psi^{(\phi)})$ has propagator given by

$$g^{(\phi)}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x}} \chi(\mathbf{k}) Z_h \left( -i k_0 + v_{3,h} \sin k_+ E_k \right) \left( v_{2,h} \left( \sin k_+ - \sin k_- \right) - i k_0 - v_{3,h} \sin k_- E_k \right)^{-1} \right.$$ (38)

with $\chi(\mathbf{k}) = \sum_{h} \theta(h k_+ + p_F) f_0(\mathbf{k} + \epsilon p_F)$ is non-vanishing in a region $O(2^h)$ around $\epsilon p_F$, and $E_k = E(k) + \cos p_F (\cos k_+ - 1)$. Moreover $\mathbf{Y}^{(\phi)}$, the effective potential, is sum over monomials of $e^{\mathbf{p} \cdot \mathbf{p}^{(\phi)}}$ multiplied by kernels $W_{h,0}^{(\phi)}$, similarly the effective source (at $\phi = 0$ for definiteness) is sum of monomials with $n$ fields $e^{\mathbf{p} \cdot \mathbf{p}^{(\phi)}}$ and $m A$-fields multiplied by kernels $W_{h,m}$. Note that the Fermi velocities and the wave function renormalization are modified by interaction and depend on the energy scale.

At large distances the single scale propagator decays faster than any power; that is for any $N$

$$\left| g^{(\phi)}(\mathbf{x}) \right| \leq C_N \frac{2^{3^h}}{1 + \left[ 2^h |\mathbf{x}| \right]^N}$$ (39)

for a suitable constant $C_N$. As consequence of (39), the scaling dimension of the monomials in the effective potential and in the effective source with $n \psi$-fields and $m A$-field is

$$D_1 = 4 - \frac{3}{2} n - m.$$ (40)

The scaling dimension says that the only non-irrelevant terms are the ones with $(n, m) = 2, 0$ ($D_1 = 1$ relevant) or $(n, m) = (2, 1)$ ($D_1 = 0$ marginal). As in the case of graphene (Dirac fermions in $d = 2 + 1$) with short range interaction, in which the scaling dimension is $D = 3 - n - m$, the terms with four or more fermionic fields are irrelevant; this is in sharp contrast to what happens in 1D metals (Dirac fermions in $1 + 1$ dimension), where $D = 2 - n/2 - m$ so that the quartic terms are marginal. Also in systems with an extended Fermi surface, the quartic terms are typically marginal.

Before integrating the the single scale field field $\psi^{(\phi)}$ one has to renormalize the relevant and marginal terms. This consists in rewriting $W_{2,0}^{(\phi)}(\mathbf{k})$ as its Taylor expansion around the Fermi point $\mathbf{k} = \epsilon p_F$ up to the first order plus a rest; one then moves the first order terms in the free integration, where they produce a renormalization of the wave function and the velocities, while the zeroth order terms contribute to the running coupling constant $\nu^{(\phi)}$ expressing the possible shift of the Fermi points. The fact that the terms generated by the integration of higher energy fields are of the same kind of the terms present originally is a consequence of symmetries, see appendix A. Similarly we rewrite $W_{2,1}^{(\phi)}(\mathbf{k}, \mathbf{p})$ as its Taylor expansion around $\epsilon p_F$ up to zeroth order and a rest, and the zeroth-term contribution to the renormalization of the currents $Z_{\mu,h}$, $\mu = 0, \pm, 3$. We can therefore write (37) as

$$\int \prod_{\mathbf{r}} \hat{P}(\psi^{(\phi)}) e^{\phi} \left( \sqrt{Z_{-1,h} \psi^{(\phi)}} \right) + \hat{B}^{(\phi)} \left( \sqrt{Z_{-1,h} \psi^{(\phi)},A,\phi} \right),$$

where $\hat{P}(\psi^{(\phi)})$ has a propagator similar to $g^{(\phi)}$ (38) with $Z_h, v_{2,h}, v_{3,h}$ replaced by $Z_{h-1}, v_{2,h-1}, v_{3,h-1}$. Moreover
\[ \psi^{(b)}(\nu) = 2^d \nu \sum_{\nu = \pm} \int dK \psi_{+\nu}^{\nu} \sigma_{\nu} \psi_{-\nu}^{\nu} + \text{Irr. Terms} \]  

(41)

and Irr.Terms are terms with negative scaling dimension, for instance terms like \( \nu^\nu \partial \phi \partial \phi \) or monomials with \( n > 2 \) fermionic fields. Similarly the effective source is given by

\[ B^{(b)}( \nu, \sqrt{Z_{h-1}} \psi^{(b)}, 0) = \sum_{\nu = \pm} \int dK dp \psi_{+\nu}^{(h)} \sigma_{\nu} \psi_{-\nu}^{(h)} \]  

\[ + Z_{-h} A_{-}(p) \sigma \psi_{-h}^{(h)} \]  

The relation between the renormalized parameters at scale \( h \) and \( h+1 \) is the following

\[ \nu_{h-1} = \frac{Z_{h}}{Z_{h+1}} \left( \nu_{h} + \gamma^{-h} \tilde{W}_{h}^{(h)}(\epsilon p) \right), \]  

(42)

\[ Z_{h+1} = 1 + \partial_{0} \tilde{W}_{h}^{(h)}, \]  

\[ v_{\alpha,h-1} = \frac{Z_{h}}{Z_{h+1}} \left( v_{\alpha,h} + \partial_{0} \tilde{W}_{h}^{(h)}(\epsilon p) \right), \quad \alpha = \pm, 3, \]  

(43)

\[ v_{\mu,h-1} = \frac{Z_{h}}{Z_{h+1}} \left( 1 + \tilde{W}_{h}^{(h)}(\epsilon p) \right), \quad \mu = 0, \pm, 3. \]  

We can write \( \chi_{h}(k^2) = \chi_{h-1}(k^2) + \delta h_{h}(k^2) \) and \( \tilde{P}(d\psi_{h}^{(h)}) = P(d\psi_{h}^{(h-1)}) P(d\psi_{h}^{(h)}), \) where \( P(d\psi_{h}^{(h-1)}) \) and \( P(d\psi_{h}^{(h)}) \) have propagator given by \( g^{(h-1)} \) and \( g^{(h)}, \) similar to (38) with respectively \( Z_{h-1} \) and \( f_{h} \) replacing \( \chi_{h} \). The integration of the single scale propagator is then performed, and one obtains an expression similar to (37) with \( h \) replaced by \( h-1 \); the procedure can be then iterated.

It can be shown that there exists a non-zero \( U_{0} \) such that, for \( |U| \leq U_{0} \), for \( 0 < \theta < 1 \) (the proof of this statement is rather technical and it will be given elsewhere) the following bound is valid

\[ \frac{1}{L^{2\beta}} \int d\xi \left| W_{n,m}^{(h)} \right| \leq C |U|^{2 \left( \frac{3}{2} + \frac{3}{2} - m \right) h} \]  

(44)

Note that in addition to the factor \( 2^{2 \beta - \frac{1}{2} - m \cdot h} \) corresponding to the scaling dimension there is a dimensional gain \( 2^{h} \) due to the irrelevance of the effective electron–electron interaction: every contribution in perturbation theory involving an effective scattering in the infrared is suppressed thanks to the irrelevance of the kernels with four or more fields.

The extra factor \( 2^{h} \) in (44) has a crucial role in the study of the flow of the effective velocities and renormalizations; for instance by using (44) in (42) we get \( v_{+h-1} = v_{+h} + O(U2^{h}) \) so that by iteration \( v_{+h-1} = v_{+h} + \sum_{k} O(U2^{k}) = v_{0} + O(U) \). Therefore
where the explicit form of the coefficients is in appendix B. Finally note that $\nu_h$ is a relevant coupling so that it would increase at each RG iteration; however we can choose the counterterm $\nu$ so that $\nu_h$ is $\Theta(2)$ for any $h$.

As a consequence of the previous analysis we get that the interacting two-point function is given by (24). Similarly, for $\mu = 0, \pm 3$ and $|p| \ll |k - e p_F| \ll 1$

$$\left\langle \hat{j}_{\mu, p} ; \hat{\psi}_k^+ \hat{\psi}_k^- + p \right\rangle = e Z_{\mu} \left( \left\langle \hat{\psi}_k^- \hat{\psi}_k^+ \right\rangle \delta_{\mu} \left( \left\langle \hat{\psi}_k^- + p \hat{\psi}_k^+ + p \right\rangle \right) + O \left( \left| k - e p_F \right| \right)^2 \right) \right\rangle,$$

(46)

where $Z_{\mu}$ is given by (45) and $\delta_{\epsilon} = \epsilon_1, \delta_\epsilon = \epsilon_2$, while $\delta_0 = \epsilon_0$ and $\delta_3 = \epsilon_3$.

5.2. Ward identities

As in lattice gauge theory, by the change of variables $a^z \to e^{i \epsilon_0} a^z$, $b^z \to e^{i \epsilon_0} b^z$, and using the relation $U_{k, x + d} = e^{i \epsilon_0} e^{-i \epsilon_0} U(A)$ one obtains

$$W \left( A + \delta \alpha, \phi^* e^{i \epsilon_0}, \phi e^{-i \epsilon_0} \right) = W \left( A, \phi^*, \phi^- \right),$$

(47)

where the fact that the Jacobian of the transformation is equal to 1 has been exploited; due to the presence of the lattice, no anomalies are present. From (47) we get the following identity

$$\partial_\alpha W \left( A + \delta \alpha, \phi^* e^{i \epsilon_0}, \phi e^{-i \epsilon_0} \right) = 0$$

(48)

from which by differentiating with respect to the external fields $\alpha, \phi$ an infinite number of Ward identities connecting correlation functions is obtained.

In particular, if $p = (\omega, \vec{p})$

$$- i \omega \left\langle \hat{j}_p \hat{\psi}_k^+ \hat{\psi}_k^- + p \right\rangle + \sum_{l=x, y, z} p_l \left\langle \hat{j}_{l, p} \hat{\psi}_k^+ \hat{\psi}_k^- + p \right\rangle = \left\langle \hat{\psi}_k^+ \hat{\psi}_k^- \right\rangle - \left\langle \hat{\psi}_k^- + p \hat{\psi}_k^+ + p \right\rangle,$$

(49)

where the currents $j_x, j_y, j_z$ are given by (9). Similarly we can derive equation for the current–current correlation

$$- i \omega \left\langle \hat{j}_p \hat{\psi}_k^+ \hat{\psi}_k^- + p \right\rangle + \sum_{l=x, y, z} p_l \left\langle \hat{j}_{l, p} \hat{\psi}_k^+ \hat{\psi}_k^- + p \right\rangle = 0,$$

(50)
where we have used that
\[
\frac{\partial^2 W(A)}{\partial A_{i,p} \partial A_{i,-p}} \bigg|_{0} = \langle \hat{j}_{i,p} \hat{j}_{i,-p} \rangle \quad i \neq l
\]
\[
\frac{\partial^2 W(A)}{\partial A_{i,p} \partial A_{i,-p}} \bigg|_{0} = \langle \hat{j}_{i,p} \hat{j}_{i,-p} \rangle + \langle D_{p,-p,i,i} \rangle. \tag{51}
\]

From (50) we get the following equality
\[
\frac{\alpha^2}{p_i^2} \langle \hat{\rho}_{p} \hat{\rho}_{-p} \rangle = \langle \hat{j}_{i,p} \hat{j}_{i,-p} \rangle + \langle D_{p,-p,i,i} \rangle, \tag{52}
\]
where \( p \) is obtained from \( q \) setting \( p_i = 0, l \neq (0, i) \). From the above equation we get, differentiating with respect to \( \omega \)
\[
\frac{2\omega}{p_i^2} \langle \hat{\rho}_{p} \hat{\rho}_{-p} \rangle + \frac{\alpha^2}{p_i^2} \partial_\omega \langle \hat{\rho}_{p} \hat{\rho}_{-p} \rangle = \partial_\omega \langle \hat{j}_{i,p} \hat{j}_{i,-p} \rangle \tag{53}
\]
from which we get
\[
\langle \hat{j}_{i,p} \hat{j}_{i,-p} \rangle \bigg|_{\omega = 0} + \langle D_{p,-p,i,i} \rangle = 0 \quad \partial_\omega \langle \hat{j}_{i,p} \hat{j}_{i,-p} \rangle \bigg|_{\omega = 0} = 0. \tag{54}
\]

From (54) we see that the properties of the conductivity (13) depend crucially on the continuity and differentiability of the Fourier transform of the current–current correlations. Indeed if \( \langle \hat{j}_{i,p} \hat{j}_{i,-p} \rangle \) is continuous in \( p \) then from the first of (54) we get
\[
\sigma_\omega(i\omega) = -\frac{1}{\omega} \left[ \langle \hat{j}_{i,0,0} \hat{j}_{i,0,0} \rangle - \langle j_{i,0,0} \hat{j}_{i,0,0} \rangle \right] \tag{55}
\]
and if the derivative is continuous then vanishing by the second of (54). In the non-interacting case \( \langle \hat{j}_{i,x} \hat{j}_{i,y} \rangle \) decays as \( O(|x-y|^{-6}) \) at large distance; therefore the Fourier transform is continuous and with continuous first derivative, hence \( \sigma_0(i\omega) \) vanishes as \( \omega \to 0 \). However the second derivative is not continuous, and this explains the logarithmic correction to the linear behavior of the free conductivity.

As we noticed above the velocities are renormalized by the interaction and they have a non-universal value function of all the microscopic details. In addition, the current renormalizations are also dressed and not constrained to be equal by Lorentz symmetry. Ward identities imply exact relations among such renormalized quantities. Inserting (24) and (46) in the Ward identity (49) one finds the identities (29). At lowest order, the above identities simply says that \( a_1 = b_1 \) and \( a_2 = b_2 \), and this can be explicitly checked, see appendix B. However, (29) is a non-perturbative identity, implying an infinite series of identities between Feynman graphs of arbitrary order.

5.3. The conductivity

The current–current correlation is given by, \( i = \pm, 3 \)
\[
\langle \hat{j}_{i,p} \hat{j}_{i,-p} \rangle = e^2 \left[ \frac{Z^2}{Z} \right] \sum_{\ell=\pm} \int dk \text{Tr} \left( \hat{c}_{\ell,g_{\text{rel},\epsilon}^{(4)}}(k) \hat{d}_{\ell,g_{\text{rel},\epsilon}^{(4)}}(k + p) \right) + \hat{H}_i(p). \tag{56}
\]
where

\[ \hat{g}_{\text{rel}, \epsilon}^{(\leq 0)}(k) = \chi_0(k) \left( -i k_0 \pm v_3 k_3 \right) \left( \frac{v_+ - i k_-}{v_2} \right) \left( \frac{v_+ + i k_-}{v_2} \right)^{-1} \]

The first term in the above expression is equal to the conductivity of free fermions, with dressed velocities and current renormalizations, while \(|\hat{H}(p)|\) contains the higher order contributions. In the coordinate space

\[ |H_s(x)| \leq C|x|^{6-\theta} \tag{56} \]

while the first term of the current–current correlations decays as \(O(|k|^{-6})\); therefore \(H_s(x)\) has continuous second derivative (while the first term has not continuous second derivative) and we can develop in Taylor series

\[ \hat{H}_1(\omega, \vec{0}) - \hat{H}_0(0, \vec{0}) = \omega \partial_\omega \hat{H}_0(0, \vec{0}) + \frac{1}{2} \omega^2 \partial_2^2 \hat{H}_0(0, \vec{0}) + \omega^2 R_i(\omega, \vec{0}). \tag{57} \]

Using the second of (54), we have \(\partial_\omega \hat{H}_0(0, \vec{0}) = 0\); therefore, by the equality (29) following from the Ward identity, we finally get (30). The optical conductivity is equal to the one of Weyl fermions with renormalized velocities, up to subdominant corrections.

6. Interaction effects close to the quantum critical point

6.1. Renormalization group analysis in the first regime

The multiscale analysis described in the previous section is valid for \(|U| \leq U_0\), where \(U_0\) is an estimate of the convergence radius which turns out to be proportional to \(v_{3,0}\) when \(v_{3,0}\) is small; therefore such analysis is useful only inside the semimetallic phase, where the \(v_{3,0} \geq \frac{v_F}{2} \). To get information outside such region, up to the quantum critical point, where \(v_{3,0} = 0\), one needs to modify the integration procedure. Let us introduce the parameters \(\epsilon\) defined as

\[ \epsilon = 1 - \frac{v' - \mu}{v_L} = 1 - \cos p_F \tag{58} \]

and we consider \(\epsilon\) small (the case \(v' - \mu \sim -1\) can be treated similarly); when \(\epsilon = 0\) the Weyl points coalesce and the Fermi velocity vanishes, that is

\[ v_{3,0} = O(\sqrt{\epsilon}) \quad p_F = O(\sqrt{\epsilon}). \tag{59} \]

As before, we decompose frequency-energy space in circular sectors of radius and width \(O(2^\theta)\). Two different regimes are naturally identified, separated by an energy scale

\[ a \tilde{\hbar} \sim \epsilon. \tag{60} \]

For scales greater than \(\tilde{\hbar}\) the dispersion relation in the third direction is essentially quadratic and \(\cos k - 1 + \epsilon \sim \frac{k^2}{2}\); the behavior in this region is essentially the same as at the quantum critical point. It is only at smaller scales, that is \(h \leq \tilde{\hbar}\), that the dispersion relation in the third direction becomes linear around the Weyl points, and the fermions acquire an extra label corresponding to their closeness to \(p_F\) or \(-p_F\).

We start describing the integration of the scales \(h \geq \tilde{\hbar}\). As before the integration is described inductively; assume that we have integrated the fields \(\psi^{(0)}, \psi^{(-1)}, ..., \psi^{(h)}\), \(h \geq \hat{h}\), obtaining
\[
\int \mathcal{P}(d\psi^{(\epsilon)}(\epsilon)) e^{i \int \left( \frac{\partial}{\partial \psi^{(\epsilon)}} + H^{(\epsilon)} \right) + B^{(\epsilon)}(\psi^{(\epsilon)}(A, \phi)),
\]

where \( \mathcal{P}(d\psi^{(\epsilon)}) \) has propagator given by

\[
g^{(\epsilon)}(x) = \int dk e^{ikx} \frac{Z_k(k)}{Z_a}
\left(-ik_0 + \bar{v}_{3,\lambda}(\cos(k_3 - 1) + \epsilon + E(\bar{k}) - v_{\pm,\lambda} \sin k_+ - \text{isink}_-) - i\bar{k}_0 - \bar{v}_{3,\lambda}(\cos(k_3 - 1) - \epsilon - E(\bar{k})) \right)^{-1}
\]

with \( \bar{v}_{3,\lambda} = t_\lambda(1 + O(U)), \) \( v_{\pm,\lambda} = t(1 + O(U)) \). Choosing (say) \( a 2^{\frac{\lambda}{2}} = 10\epsilon \) one has \( t_\lambda(\cos k_3 - 1 + \epsilon) \leq a 2^{\lambda} \) so that \( k_3 \) is smaller than \( O(2^{\lambda}) \); therefore the single scale propagator behaves asymptotically as

\[
\left| g^{(0)}(x) \right| \leq C_N \frac{2^{\frac{\lambda}{2}}}{1 + \left[ 2^\lambda \left( |x_0| + |x_+| + |x_3| + 2^2|x_3| \right) \right]^N}
\]

and correspondingly the scaling dimension is

\[
D_2 = \frac{7}{2} - \frac{5}{4}n - m.
\]

The scaling dimension is radically different with respect to the one found in the previous case, for the quadratic behavior of the dispersion relation in the third direction. The non-irrelevant terms are only the ones with \( (n, m) = (2, 0) \) \( (D_2 = 1) \) and \( (n, m) = (2, 0) \) \( (D_2 = 0) \).

Before integrating the single scale field \( \psi^{(\epsilon)} \) one has renormalize the relevant and marginal terms. This consists in rewriting \( \bar{W}_{2,0}^{(\lambda)}(k) \) as its Taylor expansion around \( k = 0 \) up to the first order \( (D_2 = 1) \) is its dimension) in \( k_0, k_+, k_- \) and up to second order in \( k_3 \) (remember that \( k_3 \sim 2^{\lambda/2} \) so first order in \( k_3 \) is not sufficient to make the dimension negative) plus a rest; one moves the first and second order terms in the free integration, where they produce a renormalization of the wave function and the velocities, while the zeroth order terms contribute to the running coupling constant \( \nu_{\lambda} \). Note that the Taylor expansion is now around \( k = 0 \) and not around the Fermi points; moreover \( \partial_k \bar{W}_{2,0}^{(\lambda)}(0) = 0 \) by parity. Similarly we rewrite \( \bar{W}_{2,1}^{(\lambda)}(k, p) \) as its Taylor expansion around \( k = 0 \) up to zeroth order \( (D_2 = 0) \) is their dimension) and a rest, and the zeroth term contribute to the renormalization of the currents \( Z_{\mu,\lambda}, \mu = 0, \pm \); note that \( Z_{3,\lambda} = 0 \) by symmetry so that the current in the third direction is \( -A_3 \psi^* \partial_\psi \), hence irrelevant according to dimensional arguments. The relation between the effective renormalizations at scale \( h \) and \( h - 1 \) is

\[
\nu_{\lambda,h-1} = \frac{Z_\lambda}{Z_{\lambda,h}} \left( \nu_{\lambda} + \gamma^{-h} \bar{W}_{2}^{(h)}(0) \right),
\]

\[
\frac{Z_{\lambda,h-1}}{Z_{\lambda,h}} = 1 + \partial_h \bar{W}_{2}^{h}(0),
\]

(64)
\[ v_{\pm,h-1} = \frac{Z_{\pm}}{Z_{\pm-1}} \left( v_{\pm,h} + \partial_{\pm} \widetilde{W}^{(h)}_{2,0}(0) \right) \quad \alpha = \pm, \]
\[ \widetilde{v}_{3,h-1} = \frac{Z_{3}}{Z_{3-1}} \left( \widetilde{v}_{3,h} + a_{3}^{2} \widetilde{W}^{(h)}_{2,0}(0) \right), \]
\[ \frac{Z_{\mu,h-1}}{Z_{\mu-1}} = \frac{Z_{\mu}}{Z_{\mu-1}} \left( 1 + \widetilde{W}^{(h)}_{2,1}(0,0) \right) \quad \mu = 0, \pm. \] (65)

As before, we can choose \( \nu = O(U) \) so that \( \nu_h \) is bounded uniformly in \( h \); moreover a bound like (44) holds, with \( 2^{(4-2\nu-m)h/3^h} \) in the rhs replaced by \( 2^{2^{-1/2}-m/2^h} \), again the presence of the extra factor \( 2^h \) implies that the limiting \( Z_6, v_{\pm,\nu}, v_{3,\nu}, Z_{\mu,\nu} \) remain close \( O(U) \) to their initial value. By using (62), and the extra \( \sin \ell_3 \) in the definition of the third component of the current, we get the nonlinear behavior of the conductivity (23).

6.2. Renormalization group analysis in the second regime

We have now to discuss the integration of the scales \( \leq \hbar \); after a finite number of integrations again the region \( k_0^2 + |\xi(k)|^2 \leq 2^\hbar \) corresponds to two disconnected regions in moment space \( k \), centered around the Fermi points, so that the Grassmann fields can be conveniently written as sum of two independent fields

\[ \psi^{(h)}_x = \sum_{\epsilon = \pm} \epsilon^{i p_j x_j} \psi^{(h)}_{\epsilon, x}, \] (66)

with \( \psi^{(h)}_{\epsilon, x} \). If \( k = k + \epsilon p_j \), in this case \( v_3 k \sim 2^h \); therefore the asymptotic behavior of the single scale propagator is \( (v_3 = v_3,0(1 + O(U))) \)

\[ |g^{(h)}(x)| \leq C_N \frac{1}{v_3} \left( 1 + \frac{2^h}{|x|^N} \right), \] (67)

where \( \check{x} \) is equal to \( x \) with \( x_3 \) replaced by \( v_3 x_3 \). The renormalization group analysis proceed essential as in section 4; the starting point is a functional integral with integration \( P(\psi^{(h)}_x) \) and effective potential given by \( W^{(h)}_m \), sum of monomials of any degree in \( \psi^{(h)}_x \). The scaling dimensions are the same and the only difference is that each term of the renormalized expansion contributing to \( W^{(h)}_{n,m} \) obtained contracting \( m_i \) vertices with \( i \)-fields has an extra factor \( v_3^{-P} \), where

\[ P = \sum_i \frac{im_i}{2} - n + 1 - \frac{n}{2} \] (68)

and \( n = \sum m_i \). For large \( n \) one as \( P = O(n) \) and positive; as \( v_3 = O(\sqrt{\epsilon}) \) this factor seems to destroy the convergence of the renormalized expansion for \( \epsilon \) small enough (unless \( U \) is not chosen vanishing with \( \epsilon \)). However the scaling dimension of the second regime \( D_2 \) is different from the dimension of the first regime, and the difference is

\[ D_2 - D_1 = -\frac{1}{2} + \frac{n}{4}. \] (69)

Therefore each term in \( W^{(h)} \) has an extra factor \( 2^D(D_2-D_1) \) and therefore one has an extra \( \epsilon^{P} \) at each order of the renormalized expansion, where
\[ Q = -\frac{\tilde{\alpha}}{2} + \sum_{i} \frac{i m_i}{4}. \] (70)

Therefore, as \( v_3 = \mathcal{O}(\sqrt{\epsilon}) \) the total factor (with respect to the analysis in section 4) at each order of the renormalized expansion is

\[ \epsilon^{-\frac{p}{2}} Q = \epsilon^{-\frac{p}{4}}. \] (71)

This says that the small denominators due to the vanishing of the Fermi velocity \( v_3 \) are exactly compensated (there is no dependence from the order of the expansion) by the extra factors due to the difference in the scaling dimensions; convergence is achieved up to the quantum critical point. The semimetallic phase persists up to the quantum critical point also in presence of the interaction; in particular, the conductivity has the relativistic behavior given by (30) for \( \omega \ll v_3 \), that is in a range of energy smaller and smaller approaching the quantum critical point; on the contrary for \( \omega \gg v_3 \) it has the nonlinear behavior given by (23), following from (62). Similarly the two-point function has a linear relativistic only for moments smaller than \( \mathcal{O}(v_3) \), and for higher momenta nonlinear corrections are dominating.

**Appendix A. Symmetry properties**

In the effective action there are no bilinear terms \( \psi_{e}^+ \psi_{e}^- \) by conservation of momentum. Note that the propagator verify the following symmetry properties, calling \( \mathbf{k}^\pm = (k_0, -k_1, -k_2, k_3) \)

\[
\begin{align*}
\hat{g}_{1,1}(\mathbf{k}) &= \hat{g}_{1,1}(\mathbf{k}^+), \\
\hat{g}_{2,2}(\mathbf{k}) &= \hat{g}_{2,2}(\mathbf{k}^+), \\
\hat{g}_{1,2}(\mathbf{k}) &= -\hat{g}_{1,2}(\mathbf{k}^+), \\
\hat{g}_{2,1}(\mathbf{k}) &= -\hat{g}_{2,1}(\mathbf{k}^+).
\end{align*}
\] (A.1)

Moreover the kernels of the currents verify

\[
\begin{align*}
w_{a,\pm}(\mathbf{\tilde{k}}^\pm, 0) &= w_{a,\pm}(\mathbf{\tilde{k}}, 0), \\
w_{b,\pm}(\mathbf{\tilde{k}}^\pm, 0) &= -w_{b,\pm}(\mathbf{\tilde{k}}, 0), \\
w_{3}(\mathbf{\tilde{k}}^\pm, 0) &= w_{3}(\mathbf{\tilde{k}}, 0).
\end{align*}
\] (A.2)

By using the above symmetry properties it is easy to check that

1. The non-diagonal terms \( \tilde{W}_{z,0}^{(h)}(\epsilon \mathbf{p}_F) \) are vanishing by (A.1) as they contain an odd number of non-diagonal propagators, by (A.1).
2. The non-diagonal terms contributing to \( \partial_0 \tilde{W}_{z,0}^{(h)}(\epsilon \mathbf{p}_F) \) or \( \partial_3 \tilde{W}_{z,0}^{(h)}(\epsilon \mathbf{p}_F) \) are vanishing as they contain an odd number of non-diagonal contributions; similarly diagonal terms contributing to \( \partial_0 \tilde{W}_{z,0}^{(h)}(\epsilon \mathbf{p}_F) \) or \( \partial_3 \tilde{W}_{z,0}^{(h)}(\epsilon \mathbf{p}_F) \).
3. The diagonal contributions to \( \tilde{W}_{z,2,1}(\epsilon \mathbf{p}_F, 0) \) are vanishing; indeed the terms containing \( w_{b,\pm} \) contains an even number of non-diagonal propagators, hence they are vanishing; the terms containing \( w_{a,\pm} \) contains instead an odd number of non-diagonal propagators.
4. The non-diagonal contributions to \( \tilde{W}_{y,1,2}^{(h)}(\epsilon \mathbf{p}_F, 0) \) are vanishing as they contain an odd number of non-diagonal propagators.
Appendix B. Lowest order computations

The explicit value of the coefficients in (45) is

\[ a_3 \sigma_3 = \int \text{d}k \hat{v} (k + \mathbf{p}_F) \partial_{\delta^3} \hat{g} (k), \]
\[ a_s \sigma_1 = \int \text{d}k \hat{v} (k) \partial_s \hat{g} (k), \]
\[ b_3 \sigma_3 = \int \text{d}k \hat{v} (k + \mathbf{p}_F) w_3 \left( \tilde{k}, 0 \right) \hat{g} (k) \sigma_3 \hat{g} (k), \]
\[ b_s \sigma_1 = \int \text{d}k \hat{v} (k) \left[ w_{as} \left( \tilde{k}, 0 \right) \hat{g} (k) \sigma_1 \hat{g} (k) + w_{bs} \left( \tilde{k}, 0 \right) \hat{g} (k) \sigma_3 \hat{g} (k) \right]. \] (B.1)

In agreement with (29) \( a_3 = b_3, a_+ = b_+ \) as they can be easily checked from the relations, \( \mathbf{p}_3 = (0, \tilde{p}, \tilde{\delta}_3), \mathbf{p}_+ = (0, \tilde{p}, \tilde{\delta}_3) \),

\[ g^{-1}(k) - g^{-1}(k + \mathbf{p}_3) = p_3 w_3 \left( \tilde{k}, 0 \right) \sigma_3 + O \left( \mathbf{p}^2 \right), \]
\[ g^{-1}(k) - g^{-1}(k + \mathbf{p}_+) = p_+ \left[ w_{as} \left( \tilde{k}, 0 \right) \sigma_1 + w_{bs} \left( \tilde{k}, 0 \right) \sigma_3 \right] + O \left( \mathbf{p}^2 \right). \] (B.2)

Note also that \( a_3 \) and \( b_3 \) are \( O(p_F) \) for small \( p_F \), that is the velocity renormalization is proportional to the bare one.

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