A 3-LOCAL CHARACTERIZATION OF \( \text{Co}_2 \)

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Abstract. Conway’s second largest simple group, \( \text{Co}_2 \), is characterized by the centralizer of an element of order 3 and certain fusion data.

1. Introduction

The vistas revealed by Goldschmidt in [13] inspired many investigations of amalgams, particularly in their application to finite groups and their geometries. One such was the fundamental work of Delgado and Stellmacher [8] in which weak \( BN \) pairs were classified. Later Parker and Rowley [27] determined the finite local characteristic \( p \) completions of weak \( BN \) pairs (when \( p \) is odd and excluding the amalgams of type \( \text{PSL}_3(p) \)). However a number of exceptional configurations when \( p \in \{3, 5, 7\} \) required further attention—all but one of them have been addressed in Parker and Rowley [26, 28], Parker [23] and Parker and Weidorn [29]. The last one is run to ground here in our main result which gives a characterization of Conway’s second largest simple group, \( \text{Co}_2 \).

Theorem 1.1. Suppose that \( G \) is a finite group, \( S \in \text{Syl}_3(G) \), \( Z = \text{Z}(S) \) and \( C = \text{C}_G(Z) \). Assume that \( \text{O}_3(C) \) is extraspecial of order \( 3^5 \), \( \text{O}_2(C/\text{O}_3(C)) \) is extraspecial of order \( 2^5 \) and \( C/\text{O}_3,C_2(C) \cong \text{Alt}(5) \). If \( Z \) is not weakly closed in \( S \) with respect to \( G \), then \( G \) is isomorphic to \( \text{Co}_2 \).

The hypothesis on the structure of \( C \) in Theorem 1.1 amounts to saying that \( C \) has shape \( 3^{1+4}.2^{1+4}.\text{Alt}(5) \). Note that no assertion about the types of extension is included and the extraspecial groups could have either \( + \)- or \( - \)-type. We remark, as may be seen from [37] or [7], that \( \text{Co}_2 \) actually satisfies the hypothesis of Theorem 1.1. As a consequence of Theorem 1.1 and earlier work on the exceptional cases arising in [27], we can now see that part (ii) of [27, Theorem 1.5] does not occur. Theorem 1.1 investigates a more general configuration than required to settle [27, Theorem 1.5 (ii) (c)]. Though not immediately

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apparent, this configuration rather quickly gives rise to a subgroup $M$ of shape $3^4\Omega^-_4(3) \cong 3^4\text{Alt}(6)$. This particular subgroup makes appearances in other simple groups such as $\text{SU}_4(3)$, $\text{PSU}_6(2)$ and McL and is the root cause of the exceptional possibilities itemized in [27, Theorem 1.5 (ii)(a), (b) and (c)].

A number of the sporadic simple groups have been characterized in terms of 3-local data. The earliest being a characterization of $J_1$ by Higman [15, Theorem 12]. In [22], O’Nan determined the finite simple groups having an elementary abelian subgroup $P$ of order $3^2$ such that for $x \in P^\#$, $C_G(x)/\langle x \rangle$ is isomorphic to $\text{PSL}_2(q)$, $\text{PGL}_2(q)$ or $\text{PΣL}_2(q)$ ($q$ odd). Thereby he characterized the sporadic simple groups $M_{22}$, $M_{23}$, $M_{24}$, $J_2$, HS and Ru. For the remaining Janko groups, 3-local identifications for $J_3$ were obtained first by Durakov [10] and later by Aschbacher [1], and for $J_4$ by Stroth [36], Stafford [35] and Güloğlu [14]. The groups O’N and He were dealt with, respectively, by Il’inyh [16] and Borovik [5]. All of these results were obtained prior to 1990. Recently there has been a resurgence of interest and activity in 3-local characterizations of finite simple groups partly prompted by the revision project concerning groups of local characteristic $p$ (see, for example, [21]). The sporadic simple groups studied in this renaissance period are $\text{Co}_3$ (Korchagina, Parker and Rowley [18]), $\text{Fi}_{22}$ (Parker [23]), McL (Parker and Rowley [28]), $M_{12}$ (Astill and Parker [4]), Th (Fowler [11]), and $\text{Co}_1$, $\text{Fi}'_{24}$, (Salarian [31, 32] ) and M (Salarian and Stroth [33]).

With a few exceptions, to date, characterization results for finite groups in terms of 3-local data ultimately rely upon identifying the target group(s) via 2-local information. This is the case here, F. Smith’s Theorem [34] providing the final identification. Thus most of this paper is spent manoeuvering into a position where we can use this result. We begin in Section 2 giving background results– F. Smith’s Theorem appearing as Theorem 2.1. Another characterization result appearing in Theorem 2.2 due to Prince, is employed in Lemma 5.4. Lemma 5.4 which is the bridge to the 2-local structure of $G$ ($G$ as in Theorem 1.1), states that $N_G(B) \cong \text{Sym}(3) \times \text{Aut}(\text{SU}_4(2))$ for a certain subgroup $B$ of $G$ of order 3. In $N_G(B)$ there is an involution $t$ inverting $B$ and centralizing $O^3(C_G(B)) \cong \text{Aut}(\text{SU}_4(2))$. Not only does this lemma fill out our knowledge of the 3-local subgroups but it also gives us a toehold in $C_G(t)$. After Lemmas 2.3, 2.8 results which play minor supporting roles, we present Lemmas 2.10, 2.11 and 2.12, which are pivotal for the identification of the normalizer of $J$, the Thompson subgroup of $S$, $S \in \text{Syl}_3(G)$. It turns out that $J$ is elementary abelian of order $3^4$ and these lemmas allow us to assert in Lemma 4.8 that $N_G(J)/J \cong \text{CO}_4^-_4(3)$, the
group of all similitudes of a non-degenerate orthogonal form of \(-\) type in dimension 4. This opens the way for us to use facts about the action of this group on \(J\). The pertinent facts are listed in Lemma \ref{lem:2.13}. This plays an important role in Lemma \ref{lem:5.2} where we show that \(3'\)-signalizers for \(J\) are trivial. Various properties of groups of shape \(2^{1+4}\).Alt(5) are given in Lemmas \ref{lem:2.15}, \ref{lem:2.16} and \ref{lem:2.17}. These results will be applied to bring the structure of \(C_G(Z)\) into sharper focus, where \(Z = Z(S)\). We conclude Section 2 with Lemmas \ref{lem:2.18} and \ref{lem:2.19} which concern the spin module for \(Sp_6(2)\), followed by an elementary result on \(\text{Aut}(SU_4(2))\) in Lemma \ref{lem:2.20}.

The main result of Section 3, Theorem \ref{thm:3.1}, anticipates the end game in our analysis of \(C_G(t)\), \(t\) being the involution mentioned earlier. In fact, Theorem \ref{thm:3.1} will be applied to \(C_G(t)/(t)\).

Section 4 sees us start the proof of Theorem \ref{thm:1.1}. After Lemma \ref{lem:4.1} in which the structure of \(C_G(Z)\) is examined (where \(Z = Z(S)\), \(S \in \text{Syl}_3(G)\)), Lemmas \ref{lem:4.2} and \ref{lem:4.3} look at centralizers and commutators of certain involutions in \(C_G(Z)\). In Lemmas \ref{lem:4.4}, \ref{lem:4.5} and \ref{lem:4.6} it is \(S\) and its subgroups that mostly occupy our attention. Two subgroups of \(S\) that will play central roles in the proof of Theorem \ref{thm:1.1} are \(Z\) and \(J = C_S([Q, S])\) where \(Q = O_3(N_G(Z))\). In Lemmas \ref{lem:4.5} and \ref{lem:4.6} we learn that \(J\) is the Thompson subgroup of \(S\), \(J\) is elementary abelian of order \(3^4\) and that all \(G\)-conjugates of \(Z\) in \(S\) are trapped inside \(J\). Another important subgroup of \(S\), namely \(B\), along with the involution \(t\), already noted earlier, make their entrance after Lemma \ref{lem:4.8}.

In the latter part of Section 4, our attention moves on to \(N_G(Z)\), resulting in structural information about this subgroup in Lemmas \ref{lem:4.11} and \ref{lem:4.12}. Drawing upon the results in Section 4, in Section 5 we determine the structure of \(N_G(B)\). Our last section brings to bear all the earlier results on \(C_G(t)\) eventually yielding that \(C_G(t)/(t)\) satisfies the hypotheses of Theorem \ref{thm:3.1}.

Then using Theorem \ref{thm:3.1} we rapidly obtain the hypotheses of Theorem \ref{thm:2.1} whence we deduce that \(G \cong \text{Co}_2\).

We follow the ATLAS \cite{Atlas} notation and conventions there with a number of variations which we now mention or hope are self explanatory. We shall use Sym\((n)\) and Alt\((n)\) to denote, respectively, the symmetric and alternating groups of degree \(n\) and Dih\((n)\), Q\((n)\) and SDih\((n)\), respectively, to stand for the dihedral group, quaternion group and semidihedral group of order \(n\). Finally \(X \sim Y\) where \(X\) and \(Y\) are groups will indicate that \(X\) and \(Y\) have the same shape.

The remainder of our notation is standard as given, for example, in \cite{Serre} and \cite{SSW}.

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2. Preliminary Results

**Theorem 2.1** (F. Smith). Suppose that $X$ is a finite group with $Z(X) = O_2'(X) = 1$, and $Y$ is the centralizer of an involution in $X$. If $Y/O_2(Y) \cong \text{Sp}_6(2)$ and $O_2(Y)$ is a non-abelian group of order $2^9$ such that the elements of order 5 in $Y$ act fixed point freely on $O_2(Y)/Z(O_2(Y))$, then $X$ is isomorphic to $\text{Co}_2$.

*Proof.* See [34]. □

**Theorem 2.2** (A. Prince). Suppose that $Y$ is isomorphic to the centralizer of a central element of order 3 in $\text{PSp}_4(3)$ and that $X$ is a finite group with a non-trivial element $d$ such that $C_X(d) \cong Y$. Let $P \in \text{Syl}_3(C_X(d))$ and $E$ be the elementary abelian subgroup of $P$ of order 27. If $E$ does not normalize any non-trivial $3'$-subgroup of $X$ and $d$ is $X$-conjugate to its inverse, then either

(i) $|X:C_X(d)| = 2$;
(ii) $X$ is isomorphic to $\text{Aut}(\text{SU}_4(2))$; or
(iii) $X$ is isomorphic to $\text{Sp}_6(2)$.

*Proof.* See [30, Theorem 2]. □

**Lemma 2.3.** Suppose that $X$ is a group of shape $3_+^{1+2}.\text{SL}_2(3)$, $O_2(X) = 1$ and a Sylow 3-subgroup of $X$ contains an elementary abelian subgroup of order 3$^3$. Then $X$ is isomorphic to the centralizer of a non-trivial 3-central element in $\text{PSp}_4(3)$.

*Proof.* See [23, Lemma 6]. □

We will also use the following variation of Lemma 2.3.

**Lemma 2.4.** Suppose that $X$ is a group of shape $3_+^{1+2}.\text{SL}_2(3)$, $O_2(X) = 1$ and the Sylow 3-subgroups of a centralizer of an involution in $X$ are elementary abelian. Then $X$ is isomorphic to the centralizer of a non-trivial 3-central element in $\text{PSp}_4(3)$.

*Proof.* Let $S \in \text{Syl}_3(X)$, $R = O_3(X)$, and $F \leq R$ be a normal subgroup of $S$ of order 9. Let $N = N_X(S)$. If $F$ is not normal in $N$, then there exists $n \in N$ such that $R = F^nF$. But then $S$ centralizes $FF^n/Z(R) = R/Z(R)$ and so $C_X(R/Z(R)) > R$ and this contradicts $O_2(X) \neq 1$. Hence $F$ is normal in $N$. Let $E = C_S(F) = C_N(F)$. Then $E$ is abelian of order 27. Let $u$ be an involution in $N$. Then $u$ normalizes $E$ and, as
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$[S, u] \leq R$, $C_E(u) \not\leq R$. Therefore $E = C_E(u)F$. Since $F$ and $C_E(u)$ are elementary abelian by hypothesis, $E$ is elementary abelian of order $3^3$. Hence Lemma 2.3 applies and yields the result.

Lemma 2.5. Suppose that $p$ is a prime, $X$ is a finite group and $P \in \text{Syl}_p(X)$. If $x, y \in Z(J(P))$ are $X$-conjugate, then $x$ and $y$ are $N_X(J(P))$-conjugate.

Proof. See [2, 37.6].

Lemma 2.6. Suppose that $p$ is a prime, $X$ is a finite group and $P \in \text{Syl}_p(X)$. If $R \leq P$ is not weakly closed in $P$ with respect to $X$, then there exists $x \in X$ such that $R \neq R^x$ and $R$ and $R^x$ normalize each other.

Proof. Suppose that $R$ is not normal in $P$. Let $N = N_P(R)$ and $M = N_P(N)$. Then $M > N$. Choose $x \in M \setminus N$. Then $R \neq R^x$ and, as $R$ and $R^x$ are both normal in $N$, we obtain the lemma. Hence we may assume that $R$ is normal in $P$. Since $R$ is not weakly closed in $P$ with respect to $X$, there exists $y \in X$ such that $R^y \neq R$ and $R^y \leq P$. If $R^y$ is normal in $P$, then $R$ and $R^y$ normalize each other and we take $x = y$. Otherwise, repeating the argument as for $R$ and $R^y$, we find $z \in P$ such that $R^y$ and $R^{xy}$ normalize each other. Taking $x = yzy^{-1}$ completes the proof of the lemma.

Lemma 2.7. Suppose that $X$ is a finite group, $x \in X$ an involution of $X$ and $V$ an elementary abelian normal $2$-subgroup of $X$. Set $C = C_X(x)$. Then the map $(vx)^{VC} \mapsto [(v[V, x])^C]$ is a bijection between $VC$-orbits of the involutions in the coset $Vx$ and the $C$-orbits of the elements of $C_V(x)/[V, x]$. Furthermore, for $vx$ an involution in $Vx$, $|(vx)^{VC}| = |[(v[V, x])^C]| |[V, x]|$.

Proof. The given map is easily checked to be a bijection.

Lemma 2.8. Suppose that $Q$ is an extraspecial $p$-group and $\alpha \in \text{Aut}(Q)$. If $A$ is a maximal abelian subgroup of $Q$ and $[A, \alpha] = 1$, then $\alpha$ is a $p$-element.

Proof. The Three Subgroup Lemma implies that $[Q, \alpha] \leq A$. Then $[Q, \alpha, \alpha] \leq [A, \alpha] = 1$ and so $\alpha$ is a $p$-element.

When we are studying signalizers in Lemma 6.9, we shall call on the following lemma repeatedly.

Lemma 2.9. Suppose that $p$ is a prime, $X$ is a group and $P$ is a $p$-subgroup of $X$. If $U \leq O_p(N_X(P))$ and $U$ and $P$ are contained in some soluble subgroup $Y$ of $K$, then $U \leq O_p(Y)$.
Proof. See [20, 8.2.13, pg. 190].

The proof of the next lemma is taken from [19, Lemma 1].

**Lemma 2.10.** Suppose that \( F \) is a field, \( V \) is a finite dimensional vector space over \( F \) and \( X = \text{GL}(V) \). Assume that \( q \) is a quadratic form of Witt index at least 1 and with non-degenerate associated bilinear form \( f \), where, for \( v, w \in V \), \( f(v, w) = q(v + w) - q(v) - q(w) \). Let \( S \) be the set of singular 1-dimensional subspaces of \( V \) with respect to \( q \). Then the stabilizer in \( X \) of \( S \) preserves \( q \) up to similarity.

**Proof.** Let \( Y \) be the subgroup of \( X \) preserving \( q \) up to similarity. Assume that \( g \in X \) stabilizes \( S \) and select \( \langle x \rangle, \langle y \rangle \in S \) such that \( f(x, y) = 1 \). Then \( W = \langle x, y \rangle \) is a hyperbolic plane. Since \( g \) preserves \( S \), \( Wg \) is also a hyperbolic plane. By Witt’s Lemma [2, pg. 81], \( Y \) contains an element mapping \( Wg \) to \( W \) which also maps \( \langle xg \rangle \) to \( \langle x \rangle \) and \( \langle yg \rangle \) to \( \langle y \rangle \). Hence multiplying \( g \) by a suitable element of \( Y \) we may assume that \( xg = x \) and \( yg = \lambda y \) for some \( \lambda \in F \). Let \( z \in W^\perp \) and set \( U = \langle x, z \rangle g = \langle x, zg \rangle \). Since \( f(x, z) = 0 = q(x) \), for \( \mu \in F \) we have \( q(\mu x + z) = q(z) \). So either every one-space of \( \langle x, z \rangle \) is singular, or \( q(z) \neq 0 \), and \( \langle x \rangle \) is the only singular one-space in \( \langle x, z \rangle \). Since \( g \) stabilizes \( S \), it follows that either \( U \) is totally singular, or \( \langle x \rangle \) is the only singular one-space contained in \( U \). Hence, in either case, \( zg \in x^\perp \). A similar argument also shows that \( zg \in y^\perp \). Hence \( zg \in W^\perp \). Since \( z \in W^\perp \), \( z + x - q(x)y \) is a singular vector and thus, as \( g \) maps singular vectors to singular vectors, \( zg + x - q(x)\lambda y \) is also a singular vector. Now, using \( zg \in W^\perp \), we obtain \( q(zg) = \lambda q(z) \). Because \( V = W \oplus W^\perp \) we then conclude that \( q(vg) = \lambda q(v) \) for all \( v \in V \) and so \( g \in Y \) as claimed.

**Lemma 2.11.** Suppose that \( p \) is an odd prime, \( X = \text{GL}_4(p) \) and \( V \) is the natural \( GF(p)X \)-module. Let \( A = \langle a, b \rangle \leq X \) be elementary abelian of order \( p^2 \) and assume that \( [V, a] = C_V(b) \) and \( [V, b] = C_V(a) \) are distinct and of dimension 2. Let \( v \in V \setminus [V, A] \). Then \( A \) leaves invariant a non-degenerate quadratic form with respect to which \( v \) is a singular vector and \( C_V(A) \) is a singular one-space. In particular, \( X \) contains exactly two conjugacy classes of subgroups such as \( A \), one being conjugate to a Sylow \( p \)-subgroup of \( \text{GO}_4^+(p) \) and the other to a Sylow \( p \)-subgroup of \( \text{GO}_4^-(p) \).

**Proof.** Since \( A \) is a \( p \)-group, \( C_V(A) = C_V(a) \cap C_V(b) \) has dimension 1 and \( [V, A] = [V, a] + [V, b] \) has dimension 3. Also note that \( [V, A]/C_V(A) = C_V/C_V(A)(a) = C_V/C_V(A)(b) \). We have \( va \in v + [V, a] \) but \( [v, a] \not\in C_V(A) \). Hence \( va = v + w \) where \( w \in [V, a] \setminus C_V(A) \). Similarly \( vb = v + x \) where \( x \in [V, b] \setminus C_V(A) \). Also \( wa = w + y \) for some \( y \in [V, a] \setminus C_V(A) \).
$C_V(A)^\#$ and then $xb = x + \lambda y$ for some $\lambda \in GF(p)^\#$. Take $\{v, w, x, y\}$ as an ordered basis of $V$. With respect to this basis $a$ corresponds to the matrix $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $b$ corresponds to $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \lambda \end{pmatrix}$. Let $f$ be the symmetric bilinear form on $V$ which has matrix $Y = \begin{pmatrix} 0 & -1/2 & -\lambda/2 & -1 \\ -1/2 & 1 & 0 & 0 \\ -\lambda/2 & 0 & \lambda & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$.

Then $a$ and $b$ preserve $f$ and, since $\det Y = -\lambda \neq 0$, $f$ is non-degenerate. Obviously $v$ is a singular vector and $C_V(A)$ is a singular one-space with respect to $f$. Since the Sylow $p$-subgroups of $GO_4^\pm(p)$ have order $p^2$, the lemma is proven. □

**Lemma 2.12.** Suppose that $p$ is an odd prime, $X = GL_4(p)$, $A \subseteq X$ is elementary abelian of order $p^2$, $V$ is the natural $GF(p)^4X$-module and $v \in V \setminus [V, A]$. Assume that no element of $A$ acts quadratically on $V$ and that $\dim [V, a] = 2$ for all $a \in A^\#$. Then $A$ preserves a quadratic form of $\langle \cdot, \cdot \rangle$-type which has singular 1-spaces $\{C_V(A)\} \cup \{\langle v \rangle a \mid a \in A\}$.

**Proof.** Suppose that $a \in A^\#$. Then $\dim [V, a] = \dim C_V(a) = 2$, $[V, a] = [V, a] + C_V(a)$ and $C_V(A) = [V, a] \cap C_V(a)$. Since $\dim C_V(A) = 1$, for $a, b \in A^\#$ with $\langle a \rangle \neq \langle b \rangle$, $C_V(a) \neq C_V(b)$ and therefore, fixing $a \in A^\#$, there exists unique cyclic subgroup $\langle b \rangle \leq A$ such that $C_V(b) = [V, a]$. Now, as $a$ and $b$ commute, $[V, b, a] = [V, a, b]$ and therefore $[V, a] = C_V(b)$. We now fix $a$ and $b$ as generators of $A$ and apply Lemma 2.11. This shows us that $A$ preserves a non-degenerate quadratic form $q$ and that $q(v) = 0$. Since the Sylow $p$-subgroup of $GO_4^+(p)$ contains elements which act quadratically, we infer that $q$ has $\langle \cdot, \cdot \rangle$-type. In particular, $V$ has $p^2 + 1$ singular vectors with respect to $q$. Since $\{C_V(A)\} \cup \{\langle v \rangle x \mid x \in A\}$ are all singular and $|\{\langle v \rangle x \mid x \in A\}| = p^2$, the result follows. □

**Lemma 2.13.** Suppose that $X = \Omega_7(3)$ and let $V$ be the natural $GF(3)^4X$-module. Then the following hold.

(i) $X$ has three orbits $O_0$, $O_1$ and $O_2$ on the one-dimensional subspaces of $V$. The set $O_0$ consists of singular one-spaces, while $O_1$ and $O_2$ consist of non-singular one-spaces. Furthermore, $|O_0| = 10$ and $|O_1| = |O_2| = 15$. The stabilizers of a member of $O_1$ and of a member of $O_2$ are not conjugate in $X$.

(ii) If $t$ is an involution in $X$, then $\dim C_V(t) = 2$ and $C_V(t)$ is a hyperbolic space. The subspace $C_V(t)$ contains two subspaces from $O_0$ and one each from $O_1$ and $O_2$. Furthermore, $C_X(t) \cong \text{Dih}(8)$ interchanges the two members of $O_0$ in $C_V(t)$ and $|C_X(t)/C_{C_X(t)}(C_V(t))| = 4$.

(iii) If $g \in X$ has order 4, then $C_V(g) = 0$. 
(iv) If $D \in \text{Syl}_3(X)$, then $\dim C_V(D) = \dim V/[V,D] = 1$ and $C_V(D) \in O_0$. 

(v) If $d \in X$ has order 3, then $\dim C_V(d) = \dim [V,d] = 2$ and $d$ is not quadratic on $V$.

(vi) If $D \in \text{Syl}_3(X)$ and $t \in N_X(D)$ is an involution, then $t$ centralizes $C_V(D)$ and $V/[V,D]$.

Proof. This is an elementary calculation. □

Lemma 2.14. Suppose that $X$, $V$ and $O_0$ are as in Lemma 2.13 and assume that $V_0$ is a hyperplane of $V$. Then $V_0$ contains a member of $O_0$.

Proof. Every 3-dimensional subspace of an orthogonal space contains a singular vector. □

Lemma 2.15. Suppose that $V$ is a faithful 4-dimensional $GF(3)X$-module and that $X$ contains a normal subgroup $Y$ with $Y \sim 2^{1+4}.\text{Alt}(5)$. Then $X$ is 2-constrained, $O_2(X) = O_2(Y)$ is extraspecial of $-$-type and either $X = Y$ or $X/O_2(X) \cong \text{Sym}(5)$.

Proof. Let $Q = O_2(Y)$. Then $Q$ is normalized by $X$. Let $Z = C_X(Q)$. Then, as $Q$ acts irreducibly on $V$ and $GF(3)$ is a splitting field for this action, $Z = Z(Q)$ by Schur’s Lemma [2]. It follows that $\text{Aut}(Q)$ contains a subgroup isomorphic to $2^4.\text{Alt}(5)$ and so $Q$ is extraspecial of $-$-type. Hence $\text{Aut}(Q) \cong 2^4.\text{Sym}(5)$ by [9, Theorems 20.8 and 20.9] and this proves the result. □

Lemma 2.16. Suppose that $X \sim 2^{1+4}.\text{Alt}(5)$ is 2-constrained. Let $Q = O_2(X)$ and $T \in \text{Syl}_3(X)$.

(i) If $i \in Q$ is a non-central involution, then $|i^X| = 10$ and $C_X(i) \sim (Q(8) \times 2).\text{Alt}(4)$. In particular, $C_X(i)Q/Q \cong \text{Alt}(4)$; and

(ii) $C_Q(T) \cong \text{Dih}(8)$ and $N_X(T)Q/Q \cong \text{Sym}(3)$.

Proof. We know that $Q$ is the central product of $\text{Dih}(8)$ and $Q(8)$ and so it is straightforward to calculate that there are 10 non-central involutions. They are conjugate in pairs in $Q$ and the element of order 5 in $X$ acts fixed point freely on $Q/Z(Q)$. It is now easy to confirm the details stated in (i). Since elements of order 3 in $X$ centralize a non-central involution and since $C_Q(T)$ is extraspecial, we get $C_Q(T) \cong \text{Dih}(8)$. The second part of (ii) follows from the Frattini Argument. □

Lemma 2.17. Suppose that $V$ is a faithful 4-dimensional $GF(3)Y$-module and that $Y \sim 2^{1+4}.\text{Alt}(5)$. Then the following hold.
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| Conjugacy Classes | $\text{Sp}_6(2)$ | $\text{Aut(SU}_4(2))$ | $|C_X(x)|$ | $|C_Y(x)|$ | $|C_V(x)|$ |
|-------------------|-----------------|----------------------|---------|---------|---------|
| $A_1$             | 2A              | 2C                   | $2^3.3^2.5$ | $2^3.3^2.5$ | $2^4$   |
| $A_2$             | 2B              | 2A                   | $2^3.3^2$ | $2^3.3^2$ | $2^6$   |
| $A_3$             | 2B              | 2B                   | $2^9.3$ | $2^6.3$ | $2^4$   |
| $A_4$             | 2D              | 2D                   | $2^7.3$ | $2^5.3$ | $2^4$   |

Table 1. Involutions in $\text{Aut(SU}_4(2))$ and $\text{Sp}_6(2)$

(i) For $v \in V^\#$, we have $C_Y(v) \cong \text{SL}_2(3)$. In particular, $Y$ operates transitively on $V^\#$.

(ii) Every element of order 2 in $Y$ is contained in $O_2(Y)$.

(iii) If $T \in \text{Syl}_3(Y)$, then $N_Y(T)/T \cong \text{SDih}(16)$.

Proof. Let $Q = O_2(Y)$, $s \in Z(Q)^\#$ and $v \in V^\#$. Then $s$ negates $v$ and so $C_Q(v)$ is a subgroup of $Q$ which does not contain $s$. Since $Q \cong 2^{1+4}$, we get that $C_Q(v)$ has order dividing 2. Hence every orbit of $Y$ on $V$ has order divisible by 16. Since the elements of $Y$ of order 5 centralize only the zero vector, the orbits of $Y$ have length divisible by 5. As there are 80 non-zero vectors it follows that $Y$ acts transitively on $V^\#$, $|C_Q(v)| = 2$ and $C_Y(v)Q/Q \cong \text{Alt}(4)$. Since $Y$ is perfect and is isomorphic to a subgroup of $\text{SL}_4(3)$, the 2-rank of $Y$ is at most 3. By considering $\langle s, C_Y(v) \rangle$ we see that $C_Y(v) \not\cong 2 \times \text{Alt}(4)$ and therefore $C_Y(v)$ is isomorphic to the unique double cover of $\text{Alt}(4)$, namely $\text{SL}_2(3)$. This proves (i).

Now suppose that $y \in Y \setminus Q$ has order 2. Then as $y$ is a noncentral involution in $Y$, $C_V(y) \neq 0$. But then (i) implies $y \in Q$, a contradiction. Hence (ii) holds.

We now claim that $N_Y(T)/T \cong \text{SDih}(16)$. Since $T$ has order 3, we have $\dim C_V(T) \geq 2$. If $\dim C_V(T) = 3$, then as $\text{Alt}(5)$ is generated by two subgroups of order 3, we find that an element of order 5 has fixed points on $V$ and this is impossible. Therefore $\dim C_V(T) = 2$ and $N_Y(T)$ acts upon this subspace. Let $R \in \text{Syl}_2(N_Y(T))$. Then by Lemma 2.16(ii), $|R| = 2^4$ and $R \cap Q \cong \text{Dih}(8)$. By (ii) the elements of $R \setminus Q$ have order at least 4. Since the central involution in $Q$ inverts $V$, we see that $R$ acts faithfully on $C_V(T)$. It follows that $R$ is isomorphic to a Sylow 2-subgroup of $\text{GL}_2(3)$ and this proves (iii).

The group $\text{Sp}_6(2)$ has a unique 8-dimensional irreducible module over $\text{GF}(2)$ as can be seen for example in [17]. This module is usually called the spin module for $\text{Sp}_6(2)$. On restriction to any subgroup of $\text{Sp}_6(2)$ isomorphic $\text{Aut(SU}_4(2))$ the spin module remains irreducible and is the unique irreducible module of dimension 8 over $\text{GF}(2)$ for this group. In Section 3, we shall refer to this module as the natural module.
for Aut(SU_4(2)). The next two lemmas collect information about the action of certain subgroups and elements of these two groups on the spin module for Sp_6(2).

**Lemma 2.18.** Suppose that $X \cong Sp_6(2)$, $Y$ is a subgroup of $X$ with $Y \cong Aut(SU_4(2))$ and $V$ is the GF(2)$X$-spin module. Then the following hold.

(i) There are exactly four conjugacy classes $A_1, A_2, A_3$ and $A_4$ of involutions in $X$ and, for $1 \leq i \leq 4$, $A_i \cap Y$ is a conjugacy class of involutions in $Y$. For each conjugacy class $A_i$, $1 \leq i \leq 4$, and for $x$ an involution in $A_i$, Table 1 gives the ATLAS class name for $A_i$ in both $X$ and $Y$, $|C_X(x)|$, $|C_Y(x)|$ and $|C_V(x)|$.

(ii) If $P$ is a parabolic subgroup of shape $2^5 Sp_4(2)$ in $X$, then $O_2(P)$ contains one involution from $A_1$ and fifteen involutions from each of $A_2$ and $A_3$. Furthermore, as a $P/O_2(P)$-module, $O_2(P)$ is an indecomposable extension of the trivial module by a natural module.

(iii) If $x \in A_2$, then $\langle x \rangle = Z(C_X(x))$ and $C_X(x)$ is a maximal subgroup of $X$.

(iv) If $f \in X$ has order five, then $C_Y(f) = 0$.

(v) For $v \in V$, $|C_Y(v)|$ and $|C_X(v)|$ are divisible by 3.

(vi) For $S \in Syl_2(Y)$, $|C_V(S)| = |C_V/C_Y(S)| = 2$.

(vii) If $S \in Syl_2(X)$ and $x \in N_X(Z(S))$ has order 3, then $x$ acts fixed-point-freely on $V$.

(viii) There are no subgroups of $X$ of order $2^5$ which have all non-trivial elements in class $A_2$.

**Proof.** The facts in (i) regarding involutions classes and their centralizers in $X$ and $Y$ are taken from the ATLAS [71] pgs. 26 and 46]–we determine $|C_V(x)|$ later in the proof. We also immediately see that $C_X(x)$ is a maximal subgroup of $X$ for $x \in A_2$. So (iii) holds.

Let $S \in Syl_2(X)$ and $P_1, P_2$ and $P_3$ be the maximal parabolic subgroups of $X$ containing $S$ with $P_1 \sim 2^5 Sp_4(2)$, $P_2 \sim 2^6 SL_3(2)$ and $|P_3| = 2^9.3^2$. Then the restrictions of $V$ to $P_i$, $i = 1, 2, 3$ are given in [24]. In particular, we have that $[V, O_2(P_1)] = C_V(O_2(P_1))$ has dimension 4 and, as $P/O_2(P_1)$ modules, $V/C_V(O_2(P_1)) \cong C_V(O_2(P_1))$ and both are natural $Sp_4(2)$-modules. Therefore, the elements of order 5 in $X$ act fixed point freely on $V$ which gives (iv).

From the character table of $X$, we read that there are dihedral subgroups of $X$ of order 10 which contain involutions from classes $A_1$, $A_3$ and $A_4$. Therefore $|C_V(x)| = 2^4$ for $x$ in any of these classes. We have that $V$ restricted to a Levi complement $L$ of $P_1$ decomposes as a direct sum of two natural modules and so the transvections in $L$ centralize
a subspace of dimension 6 in $V$. These elements are therefore in class $A_2$. This completes the proof of (i).

Since $C_V(S)$ is normalized by $P_2$, we calculate that $Y$ has two orbits on $V^\#$, one of length 135 and the other of length 120. In particular (v) holds.

Since $Z = Z(S)$ contains elements from classes $A_1$, $A_2$ and $A_3$ which we denote by $z_a$, $z_b$ and $z_c$ respectively, $N_X(Z) = C_X(Z) \leq C_X(z_c) \leq P_1 \cap P_3 \leq C_X(z_a) \cap C_X(z_b) \leq C_X(Z)$. It follows that $N_X(Z) \not\leq P_2$ and hence the elements $d$ of order 3 in $N_X(Z)$ have $C_V(d) = 0$. Thus (vii) holds.

From Table 1 we have that $Z(S) \leq O_2(P_1)$ contains elements from each of the classes $A_1$, $A_2$ and $A_3$. As $P_1$ centralizes an element $z$ of $Z(S)$ in class $A_1$ and since $P_1$ acts transitively on the non-trivial elements of $O_2(P_1)/(z)$. The first part of (ii) holds. The final part of (ii) is well known and can be, for example, verified by using the Chevalley commutator formula to calculate that $|[O_2(P), S]| = 2^4$ where $S \in \text{Syl}_2(P)$.

Suppose that $B$ is an elementary abelian subgroup of $X$ of order $2^5$ in which every involution is in $A_2$. By considering the restriction of $V$ to $P_1$, we see that $|BO_2(P_1)/O_2(P_1)| \leq 2$. Thus $B \cap O_2(P_1)$ contains all the $A_2$-involutions of $O_2(P_1)$ and is consequently $P_1$ invariant. This contradicts (ii), so proving part (viii).

We prove (vi). Let $P$ be the parabolic subgroup of $\text{Aut}(\text{SU}_4(2))$ of shape $2^4 : \text{Sym}(5)$, $R = O_2(P)$ and $S \in \text{Syl}_2(P)$. Then as the elements of order 5 in $P$ act fixed point freely on $V$, $C_V(R) = [V, R]$ has dimension 4. Furthermore, $C_V(R)$ is an irreducible $P/R$-module and from this we obtain $C_V(S) = C_{C_V(R)}(S)$ and $C_{C_V(R)/C_V(S)}(S)$ have dimension 1. Since $[S, S] \cap R$ has order $2^3$ and $R$ contains only 5 elements in class $A_2$, we deduce that $[S, S]$ contains an involution that is not in class $A_2$. As the preimage of $C_{C_V(S)}(S)$ is centralized by $[S, S]$, we see that $C_{C_V/S}(S) = C_{C_V(R)/C_V(S)}(S)$ and (vi) follows.

**Lemma 2.19.** Suppose that $X \cong \text{Sp}_6(2)$ and $V$ is the GF(2)$X$-spin module. If $F \leq X$, $[V, F, F] = 0$ and $|V/C_V(F)| \leq |F|$, then there exists $f \in F^\#$ which is not in class $A_2$.

**Proof.** First of all we note that, as $V$ is self-dual, $|[V, F]| = |V/C_V(F)| \leq |F|$.

Assume that every non-trivial element of $F$ is in class $A_2$. Then $2^4 \geq |F| > 2$ by Lemma 2.15 (i) and (viii). If $|F| = 2^2$, then for $f_1, f_2 \in F^\#$ with $f_1 \neq f_2$ we have $C_V(f_1) = C_V(f_2) = C_V(F)$. But then $C_V(F)$ is invariant under $\langle C_X(f_1), C_X(f_2) \rangle = X$ as $C_X(f_1)$ is a maximal
such that $F$ is a subgroup of $X$ by Lemma 2.18(iii). Therefore $|V : C_V(F)| \geq 2^3$ and $|F| \geq 2^3$.

Assume that $P_1$ is a parabolic subgroup of $X$ of shape $2^4\cdot\text{Sp}_4(2)$ such that $F \leq P_1$. Set $E = F \cap O_2(P_1)$. Suppose that $|E| \geq 2^3$. If $|E| = 2^4$, then $E$ contains all the $A_2$-elements of $O_2(P_1)$ and hence is invariant under the action of $P_1$. This contradicts Lemma 2.18(ii) and so we conclude that $|E| = 2^3$. Let $P \leq P_1$ be the parabolic subgroup of $P_1$ which normalizes $EZ(P_1)$. Since $E$ contains all the $A_2$-elements of $EZ(P_1)$, $P$ normalizes $E$. Also, since $P$ normalizes $EZ(P_1)$, $P$ normalizes $Z(S)$ for any $S \in \text{Syl}_2(P)$. Hence $P$ only normalizes subspaces of even dimension by Lemma 2.18(vii). Consequently, as $P$ normalizes $C_V(E)$ and $|C_V(E)| \leq 2^5$, we deduce that $C_V(E) = C_V(O_2(P_1))$ has order $2^4$. Since $E$ acts quadratically on $V$, $[V, E] = C_V(E)$ and thus $C_V(F) = C_V(E)$. So $|F| = 2^4$ and hence, as $|E| = 2^3$, $F \not\leq O_2(P_1)$. But then $C_V(F) < C_V(E)$ which is a contradiction. Hence $|E| \leq 2^3$. Because $O^2(P_1) \setminus O_2(P_1)$ contains no $A_2$-elements, we have $|F| \leq 2^3$ and so $|F| = 2^3$. Finally, $[V, F] \geq [V, E] + [V, f]$ for some $f \in F \setminus O_2(P_1)$ and so, as $[V, f] \not\leq [V, O_2(P_1)]$ and $[V, E] \leq [V, O_2(P_1)]$ with $|[V, E]| \geq 2^3$, we have $|[V, F]| > |F|$, and this is our final contradiction. \hfill \ensuremath{\square}

**Lemma 2.20.** Suppose that $X \cong \text{Aut}(\text{SU}_4(2))$ and $x$ is an involution of $X$ with $C_X(x) \cong 2 \times \text{Sym}(6)$. Let $F \in \text{Syl}_3(C_X(x))$. If $T \in \text{Syl}_3(X)$ and $F \leq T$, then $F \leq J(T)$.

**Proof.** Note that $J(T)$ is elementary abelian of order $3^3$. If $Z(T) \leq F$, then $x \in C_X(Z(T)) \leq X'$ by [7, pg. 26] whereas $x \not\in X'$. Thus $Z(T) \not\leq F$. Hence $Z(T)F$ is elementary abelian of order $3^3$ and so $Z(T)F = J(T)$, and the lemma holds. \hfill \ensuremath{\square}

3. A 2-local subgroup

As intimated in Section 1, the raison d’être for Theorem 3.1 is to assist in uncovering the structure of an involution centralizer in a group satisfying the hypothesis of Theorem 1.1. The main thrust of the proof of Theorem 3.1 is to show that $Q$ is a strongly closed 2-subgroup of $T$ with respect to $G$ where $T \in \text{Syl}_2(H)$. Goldschmidt’s classification of groups with a strongly closed abelian 2-subgroup [12] quickly concludes the proof. We use the simultaneous notation for conjugacy classes in the groups $\text{Sp}_6(2)$ and $\text{Aut}(\text{SU}_4(2))$ given in Table 1. In the next theorem we use $(3 \times \text{SU}_4(2)) : 2$ to indicate the split extension of $3 \times \text{SU}_4(2)$ by an involution which inverts the normal subgroup of order 3 and acts as a non-trivial outer automorphism on the normal subgroup isomorphic to $\text{SU}_4(2)$. The case where $H/Q \cong (3 \times \text{SU}_4(2)) : 2$ does not arise in this paper; however it will find application in work in preparation by
Parker and Stroth which characterizes automorphism groups related to
PSU$_6(2)$.

**Theorem 3.1.** Suppose that $G$ is a finite group, $Q$ is a subgroup of $G$
and $H = N_G(Q)$. Assume that the following hold
(i) $H/Q \cong \text{Aut}(SU_4(2))$, $(3 \times SU_4(2)):2$ or $Sp_6(2)$;
(ii) $Q = C_G(Q)$ is a minimal normal subgroup of $H$ and is elemen-
tary abelian of order $2^8$;
(iii) $H$ controls fusion of elements of $H$ of order 3; and
(iv) if $g \in G \setminus H$ and $d \in H \cap H^g$ has order 3, then $C_Q(d) = 1$.
Then $G = H O_2\langle G \rangle$.

**Proof.** Let $T \in \text{Syl}_2(H)$. To begin with we note that as a GF(2) $H$-
module, $Q$ is isomorphic to the $Sp_6(2)$ spin-module when $H/Q \cong Sp_6(2)$
and to the natural $\text{Aut}(SU_4(2))$-module when $H/Q \cong \text{Aut}(SU_4(2))$. If
$H/Q \cong (3 \times U_4(2)):2$, then letting $H_0$ be the subgroup of index 3 in
$H$, $Q$ is isomorphic to the natural $H_0/Q$-module.

(3.1.1) Suppose that $g \in G$ and $y \in (Q^g \cap H) \setminus Q$. Then $C_H(y)$ is a
$3'$-group.

Let $y \in (Q^g \cap H) \setminus Q$ and suppose that 3 divides $|C_H(y)|$, $S \in
\text{Syl}_3(C_H(y))$ and $x = y^{g^{-1}}$. Then $x \in Q$ and $|C_H(x)|$ is divisible by
3 by Lemma 2.18 (v). Let $P \in \text{Syl}_3(C_H(x))$. If $P \not\subseteq \text{Syl}_3(C_G(x))$, then $N_{C_G(x)}(P) \not\subseteq H$ and so there exists $n \in N_{C_G(x)}(P) \setminus H$ such that
$P \leq H \cap H^n$. Since, for $d \in P$ of order 3, $x \in C_Q(d)$, this contradicts as-
sumption (iv). Hence $P \in \text{Syl}_3(C_G(x))$ and therefore $P^n \in \text{Syl}_3(C_G(y))$.
Since $S$ is a 3-subgroup of $C_G(y)$, there is an $k \in C_G(y)$ such that
$P^{g_k} \geq S$. By assumption (iii), $H$ controls fusion of elements of order
3 in $H$. Hence, as each element of $S$ is $G$-conjugate to an element of
$P$, each element of $S$ is $H$-conjugate to an element of $P$. Now, as
$x \in C_Q(P)$ and $Q$ is normal in $H$, for all elements of $s \in S$ we have
$C_Q(s) \neq 1$. Since $S \leq H \cap H^{g_k}$, we then get $g_k \in H$ by (iv). Thus
$y = x^{g_k} \in Q^{g_k} = Q$ and we have a contradiction as $y \not\in Q$. Therefore,
3 does not divide $|C_H(y)|$ as claimed.

(3.1.2) Let $g \in G$ and suppose $y \in (Q^g \cap H) \setminus Q$. Then $yQ$ is an
$A_2$-involution in $H/Q$ and $C_H(y)Q \in \text{Syl}_2(H)$. Furthermore, $H/Q \not\cong
(3 \times SU_4(2)):2$.

If $yQ$ is not in the $A_2$-class of $H/Q$, then, by Lemma 2.18(i), $C_Q(y) =
[Q, y]$ and so Lemma 2.7 gives $C_H(y)Q/Q = C_{H/Q}(y)$. Thus $C_H(y)$ is
not a $3'$-group by Lemma 2.18(ii) again, and this is contrary to (3.1.1).
Hence $yQ$ is in the $A_2$-class of $H/Q$. Let $D$ be the full preimage of
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In particular, by Lemma 2.18(i), \( |D : C_H(y)| = 2^6 \) and \(|D/Q|\) is divisible by 9. In particular, \( |I| = 64 \). By Lemma 2.18(i), \( |D : C_H(y)| \) is divisible by 9 and, by Lemma 2.18(i), \( |Q : C_Q(y)| = 2^2 \). Therefore \(|D : C_H(y)|\) is divisible by 36. Since \( D \) obviously cannot have an orbit of length 72 on a set of 64 elements, we conclude that \(|D : C_H(y)| = 36\). If \( H/Q \cong (3 \times SU_4(2)) : 2 \), then in fact 27 divides \(|D|\) and we conclude that \(|C_H(y)|\) is divisible by 3, contrary to (3.1.1). Thus \( H/Q \cong (3 \times SU_4(2)) : 2 \). If \( H/Q \cong Aut(SU_4(2)) \), we get \(|C_H(y)| = 2^{15}\) and, if \( H/Q \cong Aut(SU_4(2)) \), we get \( 2^{13} \). Therefore, as \(|Q : C_Q(y)| = 2^2, C_H(y)Q \in \text{Syl}_2(H)\). So (3.1.2) holds. 

We note that (3.1.2) applies equally well to show that involutions in \((Q \cap H^g)Q^g/Q^g\) are in the \( A_2 \)-class of \( H^g/Q^g\).

(3.1.3) \( Q \) is weakly closed in \( H \) with respect to \( G \). In particular, \( T \in \text{Syl}_2(G) \).

Suppose that (3.1.3) is false. Then, by Lemma 2.6, there exists \( g \in G \setminus H \) such that \( Q^g \) and \( Q \) normalize each other. In particular, \( Q^g \leq H \). Hence we may assume that \(|Q : C_Q(Q^g)| \leq |Q^gQ/Q|\). By (3.1.2) the non-trivial elements of \( Q^gQ/Q \) are all in \( H/Q \) class \( A_2 \). These two facts together contradict Lemma 2.19. Therefore \( Q \) is weakly closed in \( H \) with respect to \( G \) and consequently \( \text{Syl}_2(H) \subseteq \text{Syl}_2(G) \). 

Aiming for a contradiction we now suppose that \( Q \) is not strongly closed in \( T \) with respect to \( G \).

(3.1.4) We can select \( g \in G \) and \( y \in (Q^g \cap H) \setminus Q \) so that \( C_H(y) \leq H^g \).

Since \( Q \) is not strongly closed in \( T (\leq H) \), there exists \( g \in G \) and \( y \in (Q^g \cap H) \setminus Q \). Clearly \( Q^g \leq C_G(y) \) and so we may select a Sylow 2-subgroup \( T_1 \) of \( C_G(y) \) such that \( T_1 \) contains \( Q^g \). Since \( C_H(y) \) is a 2-group by (3.1.2) there exists a Sylow 2-subgroup \( T_2 \) of \( C_G(y) \) which contains \( C_H(y) \). Thus there is an \( f \in C_G(y) \) such that \( T_1^f = T_2 \). Because \( Q \) is weakly closed in \( H \) and \( Q^g \leq T_2, C_H(y) \leq T_2 \leq N_G(Q^g) = H^g \). Since \( f \in C_G(y), y \in (Q^g \cap H) \setminus Q \). Thus we may replace \( g \) by \( gf \) and we have proved (3.1.4). 

Choosing \( g \) and \( y \) as in (3.1.4) we set \( W = C_H(y)Q^g \).

(3.1.5) There exists a Sylow 2-subgroup \( T_0 \) of \( H^g \) which normalizes \( Q \cap Q^g \) and contains \( W \). Furthermore, \(|T_0 : W| \leq 2 \).

Since \( C_H(y)Q \in \text{Syl}_2(H) \) by (3.1.2) and \( C_H(y)Q \) normalizes \( Q \cap Q^g \) by (3.1.4). \( N_{H}(Q \cap Q^g) \) contains a Sylow 2-subgroup of \( G \) by (3.1.3). Since \( W \) normalizes \( Q \cap Q^g \), there is a \( T_0 \in \text{Syl}_2(N_G(Q \cap Q^g)) \) with
\(T_0 \geq W\). Therefore, as \(Q^g\) is weakly closed in \(W, T_0 \leq H^g\). Since
\(|Q : C_Q(y)| = 4\), we have \(|T_0 : W| \leq 4\) by [3.1.2]. If \(|T_0 : W| = 4\), then we must have \(Q^g \leq C_H(y)\) which contradicts \(Q\) being weakly closed in \(H\) and \(Q \neq Q^g\). Hence \(|T_0 : W| \leq 2\.

\[\star\]

Let \(Z_2(T_0)\) be the second centre of \(T_0\) where \(T_0\) is as in [3.1.5]. Then, as \(|Z_2(T_0)| = 4\) by Lemma 2.18(vi) and \(Q \cap Q^g\) is normal in \(T_0\), we either have \(|Q \cap Q^g| \leq 2\), or \(Z_2(T_0) \leq Q \cap Q^g\). Since \(|T_0 : W| \leq 2\), \(C_{Q^g}(W) \leq Z_2(T_0)\). From \(y \in C_{Q^g}(W) \leq Z_2(T_0)\) and \(y \not\in Q\), we must have \(|Q \cap Q^g| \leq 2\). Since \(yQ\) is in \(H/Q\) class \(A_2\), we have \(|C_Q(y)| = 2^6\). Hence \(|C_Q(y)Q^g/Q^g| = |C_Q(y) : Q \cap Q^g| \geq 2^5\) and, by [3.1.2], all the involutions of \(C_Q(y)Q^g/Q^g\) are in \(H^g/Q^g\) class \(A_2\), which contradicts Lemma 2.18(viii). We have therefore shown that \(Q\) is strongly closed in \(T\) with respect to \(G\).

Set \(M = \langle Q^G\rangle\). If \(M \neq QO_2^+(G)\), then \(|M : Q|\) is even and hence we have \(T \cap M > Q\) by [3.1.3]. But then \(\langle(T \cap M)^H\rangle\) has index at most 2 in \(H\) and is contained in \(M\). Finally, applying Goldschmidt’s Theorem [12], we see that the possible composition factors of \(M/O_2^+(M)\) do not involve either \(SU_4(2)\) or \(Sp_6(2)\). Thus \(M = QO_2^+(G)\) and the Frattini Argument completes the proof of the theorem.

\[\square\]

4. Part of the 3-local structure

Having now gathered together our prerequisite results, we are ready to begin the proof of Theorem 1.11. Thus for the remainder of this article we assume that \(G\) is a finite group with \(S\) a Sylow 3-subgroup of \(G\) and \(Z = Z(S)\). Additionally, we assume that \(Z\) is not weakly closed in \(S\) with respect to \(G\) and \(C_G(Z)\) has shape \(3^{1+4}2^{1+4}\), \(Alt(5)\) as described in the hypothesis of Theorem 1.11. We set \(L = N_G(Z)\), \(L* = C_G(Z)\), \(Q = O_3(L)\) and let \(P \in Syl_2(O_{3,2}(L*))\). So \(P\) and \(Q\) are extraspecial of order \(2^5\) and \(3^5\) respectively and \(O_{3,2}(L_*) = PQ\). Furthermore, \(O_3(L_*) = Q\). Let \(\langle u \rangle = Z(P)\).

We begin by fleshing out the structure and embeddings of these groups. In the next proof we use the fact that \(Sp_4(3)\) contains no subgroup isomorphic to \(Alt(5)\). This is easy to see as the 2-rank of both \(Sp_4(3)\) and \(Alt(5)\) is 2 whereas \(Alt(5)\) has no non-trivial central elements.

\textbf{Lemma 4.1.} 
(i) \(Z = Z(Q)\) has order 3.
(ii) \(L_*\) and \(L\) are 3-constrained.
(iii) \(L_*/Q\) is 2-constrained, acts irreducibly on \(Q/Z\) and \(P \cong 2_+^{1+4}\).
(iv) \(Q\) is extraspecial of + -type.
Proof. Since $Z$ is normal in $L^*$, $Z \leq O_3(L^*) = Q$ and so, as $Q$ is extraspecial, $Z = Z(Q)$ has order 3. This is (i).

Suppose that $C_L(Q) \not\leq Q$. Then $C_L(Q)Q/Q$ is a non-trivial normal subgroup of $L_*/Q$. Let $D \in \text{Syl}_3(C_L(Q))$. Then $|D| \leq 9$ and hence is abelian. If $D > Z$, then $DQ = S$ and hence $D \leq Z(S) = Z$ which is a contradiction. Thus $D = Z \leq Q$ by (i). The assumed structure of $L_*$ now indicates that $C_L(Q) \leq QP$. In particular, $L_*/C_L(Q)$ has a composition factor isomorphic to $\text{Alt}(5)$. As $Q$ is extraspecial, the commutator map defines a symplectic form on $Q/Z$ and so $\text{Out}(Q)$ is isomorphic to a subgroup of $\text{GSp}_4(3)$. Since $\text{Sp}_4(3)$ has no subgroups isomorphic to $\text{Alt}(5)$, $C_L(Q) < QP$. If $C_L(Q)Q = \langle u \rangle Q$, then $PC_L(Q)/Q$ has 2-rank 4, contrary to the 2-rank of $\text{Sp}_4(3)$ being 2. Thus $\langle u \rangle Q/Q < C_L(Q)Q/Q < QP/Q$. In this case, $C_{L_*/Q}(PQ/Q)$ must contain a component $L_1$ isomorphic to $\text{Alt}(5)$ or $\text{SL}_2(5)$. The former case being impossible, we get $L_1 \cong \text{SL}_2(5)$. Since $L_1 \cap PQ/Q$ is normal of order 2 we deduce that $L_1 \geq \langle u \rangle Q/Q$, and once again we have $L_1C_L(Q)Q/Q \cong \text{Alt}(5)$ which is our final contradiction. Hence $C_L(Q) = Z$ and (ii) holds.

Part (iii) follows from Lemma 2.15 since $L_*/Q$ acts faithfully on $Q/Z$ and $PQ/Q$ is extraspecial.

Finally (iv) is a consequence of (iii) and [25, Lemma 2.8].

\textbf{Lemma 4.2.} Suppose that $s$ is an involution of $L_*$ with $sQ \neq uQ$. Then the following hold.

(i) $s \in PQ$.
(ii) $C_{L^*}(s)PQ/PQ \cong \text{Alt}(4)$.
(iii) $Q = C_Q(s)[Q,s], [C_Q(s), [Q,s]] = 1$ and $C_Q(s) \cong [Q,s] \cong 3_+^{1+2}$.
(iv) $C_{PQ}(s) \sim 3_+^{1+2}.(Q(8) \times 2)$ and $O^*_3(C_{PQ}(s)) = \langle s \rangle$.
(v) $C_{L^*}(u)/O_2(C_{L^*}(u)) \cong 3 \times \text{Alt}(5)$.

Proof. Part (i) follows from Lemma 2.17(ii) and part (ii) comes from Lemma 2.16(i).

Because $Q = C_Q(s)[Q,s]$ the Three Subgroup Lemma shows that $[C_Q(s), [Q,s]] = 1$. Thus, as $sQ \neq uQ$, $[Q,s] < Q$ and so, as $s$ does not centralize $Q$, we deduce that $C_Q(s) \cong [Q,s] \cong 3_+^{1+2}$ from Lemma 4.1(iv). Part (iv) follows from Lemma 2.16(i) and part (iii). Since $C_{L^*}(Q) = Z$, and $L^*$ acts irreducibly on $Q/Z$, $u$ inverts $Q/Z$. Therefore, $C_{Q}(u) = Z$ and by the Frattini Argument, $C_{L^*}(u)Q = L^*$. Now $C_{L^*}(u)/O_2(C_{L^*}(u))$ has shape $3 \times \text{Alt}(5)$ and hence is isomorphic to $3 \times \text{Alt}(5)$ as the Schur multiplier of $\text{Alt}(5)$ has order 2. Thus (v) holds.

\qed
The next lemma shines a light on the structure of $C_{L^*}(s)$ for $s \in L^*$ an involution with $sQ \neq uQ$.

**Lemma 4.3.** Suppose that $s$ is an involution of $L^*$ with $sQ \neq uQ$. Then the following hold.

(i) $[O_2(C_{L^*}(s)), O_3(C_{L^*}(s))] = 1$.
(ii) $O_2(C_{L^*}(s)) = O_3(C_{L^*}(s)) \cong Q(8)$.
(iii) $C_{L^*}(s)/O_2(C_{L^*}(s)) \cong 3_+^{1+2}.SL_2(3)$ is isomorphic to the centralizer of a non-trivial 3-central element in $PSp_4(3)$.
(iv) If $b \in C_{L^*}(s)$ has order 3 and $b \notin Q$, then $C_{O_2(C_{L^*}(s))}(b) = \langle s \rangle$.

**Proof.** Part (i) is trivial (and is included as it illuminates the structure of $C_{L^*}(s)$). Set $Y = C_{L^*}(s)$, $W = C_Q(s) = Q \cap Y$ and select an involution of $Qu$ which centralizes $s$ and, for convenience, call it $u$. Then, by Lemma 4.2 (iv), $W \cong 3_+^{1+2}$. Therefore $Y/C_Y(W)$ embeds into $\text{Aut}(3_+^{1+2}) \cong 3_+^{2}.GL_2(3)$. As $W$ is extraspecial, $WC_Y(W)/C_Y(W) \cong 3^2$. Let $X = C_Y(W)$. Since $(QP \cap Y)/Q/Q \cong Q(8) \times 2$ by Lemma 4.2 (iv) and since $u$ inverts $W/Z$, $C_{QYP}(W) = C_W(W)\langle s \rangle = Z\langle s \rangle$. Hence, as $X$ is normal in $Y$, we have

$$[X, C_{QYP}(s)] \leq X \cap C_{QYP}(s) = Z\langle s \rangle.$$ 

As the elements of order 3 in $Y \setminus W$ act non-trivially on $(QP \cap Y)Q/Q$, we get $X \leq C_{FP}(s)$ where $F \in \text{Syl}_2(Y)$. Additionally, as $Y/Q$ is 2-closed, we have $Y/C_Y(W) \cong 3_+^{2}.SL_2(3)$ and $C_Y(W)$ has order $2^3.3$. It follows that $|O_2(Y)| = 2^3$. Noting that $O_2(Y)$ and $u$ are in a common Sylow 2-subgroup of $Y$, $[Q, s] = C_Q(su)$ and that $O_2(Y)$ acts faithfully on $[Q, s]$ by the 3-constraint of $L^*$. By applying the above conclusions to the involution $su$, we obtain $O_2(Y) \cong Q(8)$. As $O_2(Y) = O_3(Y)$, (ii) holds.

Now we have $Y/O_2(Y) \cong 3_+^{1+2}.SL_2(3)$ and $O_2(Y/O_2(Y)) = 1$. From Lemma 4.2 (v), $C_{L^*}(u)$ has elementary abelian Sylow 3-subgroups. It follows that the Sylow 3-subgroups of $C_{Y/O_2(Y)}(uO_2(Y))$ are elementary abelian. So, using Lemma 2.3 the conclusion in (iii) holds.

Since by Lemma 4.2 (ii), $C_{L^*}(s) P/Q \cong \text{Alt}(4)$ and $b \notin Q$, we have $C_{O_2(C_{L^*}(s))}(b) \leq PQ$. Thus (iv) follows from Lemma 4.2 (v).

Another, less precise, way of recording Lemma 4.3 is to say that $C_{L^*}(s)$ has shape $(3_+^{1+2} \times Q(8)).SL_2(3)$.

**Lemma 4.4.** $C_{Q/Z}(S) = [Q/Z, S]$ has order $3^2$ and $[Q, S]$ is elementary abelian of order $3^3$. In particular $C_Q([Q, S]) = [Q, S]$.

**Proof.** Since $L_*/Q \sim 2_+^{1+4}.\text{Alt}(5)$, Lemma 4.2 (ii) implies that there is an involution $s \in PQ$ which centralizes $S/Q$ and satisfies $sQ \neq O_3(G)$. As $O_2(G) = O_2(C_Q(s)) = Q(8)$, by Lemma 4.2 (iv), $O_2(C_Q(s)) \cong 3_+^{2}.SL_2(3)$, and the conclusion is immediate.
Therefore $A = S$, and conclude that $J$ acts quadratically on $S$. Hence $S$ normalize $Q_1 = C_Q(s)$ and $Q_2 = [Q, s]$. Thus, by Lemma 4.2(iii), $C_Q(S) = C_{Q_1}/Z(S)C_{Q_2}/Z(S)$. By Lemma 2.16(ii), $sQ$ and $suQ$ are conjugate in $N_{PQ}(S)$ by $fQ$ say. Since $u$ inverts $Q/Z$ by Lemma 4.1(iii), we get that $Q^u = Q$. Thus $|C_{Q_1}/Z(S)| = |C_{Q_2}/Z(S)|$. Therefore, as $L_*$ is 3-constrained by Lemma 4.1(ii), $|C_{Q_3}/Z(S)| = 3^2$. Since, for $i = 1, 2$, $[Q_i/Z, S] \leq C_{Q_i/Z}(S)$, we get that $C_{Q_i/Z}(S) = [Q/Z, S]$ has order $3^2$ as claimed. The Three Subgroup Lemma and $Q$ being of exponent 3 shows that $[Q, S]$ is elementary abelian. Finally, noting that $Z \leq [Q, S]$ we have $|[Q, S]| = 3^3$. In particular, $[Q, S]$ is a maximal abelian subgroup of $Q$. 

We now put $J = C_S([Q, S])$, and start the investigation of the 3-local subgroup $M = N_G(J)$. Set $M_* = O^{3'}(M)$.

**Lemma 4.5.** The following hold.

(i) $J = J(S)$ is elementary abelian of order $3^4$;
(ii) $S = JQ$;
(iii) no element of $S \setminus J$ is acts quadratically on $J$; and
(iv) every element of order 3 in $S$ is contained in $J \cup Q$.

**Proof.** From Lemma 4.4 we have that $C_Q([Q, S]) = [Q, S]$. It follows that $|J| \leq 3^4$. Let $b \in C_S(u) \setminus Q$. Then, by Lemma 4.2(v), $b$ has order 3. Also, as $[Q, S]$ is abelian and $u$ inverts $Q/Z$ and centralizes $Z$, we have $[[Q, S], u] \cap Z = 1$ and $[Q, S] = [[Q, S], u] Z$. Since $[[[Q, S], u], b] \leq [[Q, S], u] \cap [Q, S, S] = [[Q, S], u] \cap Z$, we see that $b$ centralizes $[Q, S]$ and conclude that $J$ is elementary abelian of order 3. Suppose that $A$ is an abelian subgroup of $S$ of order at least $3^4$. Then $3^3 \geq |A \cap Q| \geq 3^3$. Therefore $A \cap Q$ has order $3^3$ and $[S, A \cap Q] = [AQ, A \cap Q] \leq [Q, Q] = Z$. Hence $A \cap Q = [Q, S]$ by Lemma 4.4. But then $A \leq J$ and we have $A = J$. Thus $J = J(S)$.

Since $J \leq Q$, (ii) is obvious.

We get $N_{L_3}(S)/S \cong SDih(16)$ from Lemma 2.17(iii), and so $N_{L_3}(S)$ acts transitively on the elements of $S/J$. Thus if any element of $S/J$ acts quadratically on $J$, then they all do. So suppose that $s \in PQ$ with $sU \neq uQ$, $[J, s] \leq Q$ and $x \in C_Q(s)/J/J$ is non-trivial and acts quadratically on $J$. Then $1 = [J, x, x] = [J, C_Q(s), C_Q(s)]$. In particular, $[J, x] \leq Z(C_Q(s))$. By Lemma 4.2(iii), $C_Q(s)$ is extraspecial, and hence $[J, C_Q(s)] \leq Z(C_Q(s)) = Z$. Now using Lemma 4.4 we have $C_Q(s) \leq [Q, S]$. Since the former group is extraspecial and the latter group is abelian, we have a contradiction. This proves (iii).

For (iv) assume for a contradiction that $x \in S = JQ$ has order 3 and that $x \notin J \cup Q$. Then $x = jq$ where $j \in J \setminus Q$ and $q \in Q \setminus J$. Since
The following hold.

Proof. Suppose that $X \leq Q$ has order 3 and $Z \neq X$. Then, by Lemma 2.17(i), we may assume that $Z$ is normal in $S$. Let $T = C_S(ZX)$. Using Lemma 2.17(i) again we get $[C_T(ZX), O_{3,2}(C_T(ZX))]$ is extraspecial of order 27 and so $X \not\leq [C_T(ZX), O_{3,2}(C_T(ZX))]$. It follows that $X \not\leq C_T(ZX)'$. This is (i).

Suppose that $T \leq T'$ and $X \leq Q$. Let $T = C_T(ZX)$. Then $T \leq T'$ and $X \not\leq T'$ by (i). On the other hand $T \leq C_T(X)$ and so, as $Z$ and $X$ are $G$-conjugate, it follows that $Z \leq O_3(C_T(X))$. But then, the situation is symmetric and so $X \leq T'$ and this is a contradiction. Hence (ii) holds.

The third statement follows from part (ii) and Lemma 4.5(iv). □

Lemma 4.7. The following hold:

(i) $L \cap M = N_G(S)$.

(ii) $C_G(J) = C_G([Q, S]) = J$.

Proof. We have $N_G(S)$ normalizes $Z(S) = Z(Q)$ and $J = J(S)$. Hence $N_G(S) \leq L \cap M$. Since $L \cap M$ normalizes $S = QJ$, (i) holds.

From $Z \leq [Q, S], C_G([Q, S]) \leq L$, and also, by Lemma 2.8, $C_G([Q, S])$ is a 3-group. Since $C_G([Q, S]) \cap Q = [Q, S]$, we have $|C_G([Q, S])| \leq 3^4$ and hence $C_G([Q, S]) = J$ as claimed in (iii). □

Lemma 4.8. (i) There are exactly ten $G$-conjugates of $Z$ in $J$.

(ii) $|L/L_s| = 2, L \sim 3^{1+4}.2^{1+4}.\text{Sym}(5)$.

(iii) $M/J \cong CO_4^-(3)$ the group of all similitudes of a non-degenerate quadratic form of $-\text{type}, M_s/J \cong \Omega_4^-(3)$ and $M/M^* \cong \text{Dih}(8)$.

Proof. Since $Z$ is not weakly closed in $S$, Lemma 4.6(ii) and (iii) imply that there exists $g \in G$ such that $X = Z^g \leq J$ and $X \neq Z$. Since $J$ is abelian, $J$ centralizes $ZX$ and, by Lemma 4.4, $N_S(ZX) = [Q, S]X = J$. Thus there are nine $S$-conjugates of $X$ in $J$. This shows that the number of $G$-conjugates of $Z$ in $J$ is congruent to 1 modulo 9. Since, by
Lemmas 2.3 and 4.5 (i), $M$ controls $G$-fusion in $J$, all the $G$-conjugates of $Z$ in $J$ are conjugate in $M$. Because there is a unique conjugate of $Z$ in $J \cap Q$ by Lemma 4.6(ii), we deduce that $|Z^M| \leq 28$. Since $M/J$ acts faithfully on $J$ by Lemma 4.7(i), we have that $M/J$ is isomorphic to a subgroup of $GL_4(3)$. Now $|GL_4(3)|$ is not divisible by either 7 or 19 and so there is no choice other than $|Z^M| = 10$. Hence (i) holds.

Since $J$ is characteristic in $S$, $N_{L*}(S) \leq M$. Thus, as $X^S = Z^M \setminus \{Z\}$ and $N_{L*}(S)$ normalizes $Z$, $N_{N_{L*}(S)}(X)/N_{L*}(S)$. In particular, $X$ is normalized by a Sylow 2-subgroup $T$ of $N_{L*}(S)$. Since $XQP/QP$ is inverted in $L*/QP$, we must have that $X$ is inverted by an element in $T$. Hence $L > L*$ and now (ii) follows from Lemma 2.15.

From (ii) we have $|N_{L*}(S)/J| = 2^5 \cdot 3^2$. Therefore $|M/J| = 2^6 \cdot 3^2 \cdot 5$ by (i). By Lemmas 2.12 and 4.5 we have that $Z^M$ is the set of singular 1-spaces of a quadratic form of $−$-type and by Lemma 2.10 we have that $M$ is isomorphic to a subgroup of $CO_4^-(3)$. Since the latter group has order $2^6 \cdot 3^2 \cdot 5$, this proves (iii).

Define $M_0 = M classified by $O_{3,2}(M)$ and let $t \in N_P(S)$ be an involution with $t \neq u$. Finally set $M_1 = \langle t \rangle M_0$ and $B = [J,t]$. Note that $M_0/J \cong 2 \times \Omega_4^-(3) = SO_4^-(3)$.

**Lemma 4.9.** $C_S(t) \in Syl_3(C_{L*}(t))$.

**Proof.** This follows directly from Lemma 4.3(iii).

**Lemma 4.10.** $B \leq Q$, $|B| = 3$ and $|C_J(t)| = 3^3$. In particular, $t$ acts as a reflection on the quadratic space $J$.

**Proof.** From the choice of $t$, we have that $B = [J,t] \leq J \cap Q = [Q,S]$. Since $t \in L*$, $t$ centralizes $Z$ and so $|B| \leq 9$. If $|B| = 9$, then $[Q,S] = [[Q,S],u]Z = BZ$ and so $[[Q,S],u]t \leq Z$. Since $ut$ centralizes $Z$ and $J/[Q,S]$, we reason that $ut$ centralizes $J$ and, because $C_J(J) = J$ by Lemma 4.7(ii), this means that $u = t$ contrary to the choice of $t$. Thus $|B| = 3$ and $|C_J(t)| = 3^3$ as claimed. In particular, $t$ acts as a reflection on $J$.

**Lemma 4.11.** $M_1/J \cong GO_4^-(3) \cong 2 \times Sym(6)$.

**Proof.** Since $t$ acts as a reflection, this is clear.

**Lemma 4.12.** $M$ has two orbits on the subgroups of $J$ of order 3. One is $Z^M$ and has length 10 and the other is $B^M$ and has length 30. Furthermore, $N_M(Z)/J \sim (2 \times 2^2).SDih(16)$ and $N_M(B)/J \cong 2 \times 2 \times Sym(4)$.
**Lemma 5.2.** By Lemmas 2.14 and 4.12, each \( H \) is normalized by \( Z \). The structure of \( N_M(Z)/J \) can be extracted from Lemma 4.8(ii).

Suppose that \( X \) is a subgroup of \( J \) of order 3 which is not in \( Z^M \). Then \( X \) is not 3-central and therefore corresponds to a non-singular subspace. Since \( CO_{3}^{-}(3) \) is transitive on such subgroups, we have that \( |X^M| = 30 \), as claimed. Furthermore, in \( CO_{3}^{-}(3) \), the subgroup of index 30 is contained in \( GO_{3}^{-}(3) \). Thus Lemma 4.14 implies that \( N_M(X)/J \cong 2 \times GO_{3}(3) \cong 2 \times 2 \times Sym(4) \). Finally we note that \( B \leq J \cap Q \) and \( B \neq Z \) and so \( B^M = X^M \) by Lemma 4.6(ii).

**5. The centralizer of \( B \)**

In this brief section we uncover the structure of \( C_G(B) \). We maintain the notation of the previous section. So \( t \in N_P(S) \) is an involution with \( t \neq u \) and \( B = [J, t] \).

**Lemma 5.1.** \( \mathcal{U}_{L_*}(J, 3') = \{1\} \).

**Proof.** Suppose that \( R \in \mathcal{U}_{L_*}(J, 3') \). Then, as \( R \) is normalized by \( J \) and normalizes \( Q \), \( R \) centralizes \( Q \cap J = [Q, S] \). Hence \( R \leq J \) by Lemma 4.7(ii) and so \( R = 1 \).

We now extend the scope of the last lemma to the whole of \( G \).

**Lemma 5.2.** \( \mathcal{U}_{G}(J, 3') = \{1\} \).

**Proof.** Suppose that \( R \in \mathcal{U}_{G}(J, 3') \). Then \( R = \langle C_R(H) \mid |J : H| = 3 \rangle \).

By Lemmas 2.14 and 4.12 each \( H \) with \( |J : H| = 3 \) contains a \( M \)-conjugate of \( Z \). Thus

\[
R = \langle C_R(Y) \mid Y \leq J \text{ and } Y \text{ is } M \text{-conjugate of } Z \rangle.
\]

Since, for each \( Y \in Z^M \), \( C_R(Y) \in \mathcal{U}_{C_G(Y)}(J, 3') \), Lemma 5.1 implies that \( C_R(Y) = 1 \). Thus \( R = 1 \) and the lemma holds.

**Lemma 5.3.** We have that \( C_{L_*}(B)/B \) is isomorphic to the centralizer of a non-trivial 3-central element in \( PSp_4(3) \). Furthermore, \( C_{L}(B)/B \) inverts \( ZB/B \).

**Proof.** Since \( Q \) is extraspecial of exponent 3, we have \( C_Q(B) \cong 3 \times 3^{1+2} \).

From Lemma 2.17(i), we have that \( C_{L_*}(B)Q/Q \cong SL_2(3) \). Thus \( C_{L_*}(B)/B \sim 3^{1+2}SL_2(3) \).

Let \( U = O_2(C_{L_*}(B)) \). Then \( Q \geq C_Q(U) \geq C_Q(B) \). Thus \( |Q : C_Q(U)| \leq 3 \).

Since in \( Sp_4(3) \) the subgroup centralizing a hyperplane of the natural \( GF(3)Sp_4(3) \)-module has order 3, we get \( U = 1 \). Since \( J \leq C_{L_*}(B) \) and \( J/B \) is elementary abelian of order \( 3^3 \), \( C_{L_*}(B)/B \) satisfies the hypothesis of Lemma 2.3 and so \( C_{L_*}(B)/B \)
is isomorphic to the centralizer of a non-trivial 3-central element in $\text{PSp}_4(3)$. 

By Lemma 2.17 (i), $L_s$ acts transitively on $(Q/Z)^+$ and so, as $Q$ is extraspecial, $L_s$ acts transitively on $Q \setminus Z$. Consequently $C_L(B) > C_{L_s}(B)$ and so $ZB/B$ is inverted by $C_L(B)$. \hfill \qed 

Lemma 5.4. We have $C_G(B) \cong 3 \times \text{Aut}(\text{SU}_4(2))$, $N_G(B) \cong \text{Sym}(3) \times \text{Aut}(\text{SU}_4(2))$ and $t$ centralizes $O^3(C_G(B))$. 

Proof. Lemmas 5.2 and 5.3 imply that $C_G(B)/B$ satisfies the hypotheses of Theorem 2.2. Furthermore by Lemma 4.12, $N_B(B) \sim 3^4.(2 \times 2 \times \text{Sym}(4))$ which is not a subgroup of $L$. Therefore $C_G(B) \neq C_L(B)$ and hence Theorem 2.2 gives $C_G(B)/B \cong \text{Aut}(\text{SU}_4(2))$ or $\text{Sp}_6(2)$. By Lemma 4.6 (i) $B \nsubseteq C_L(B)$ and $C_L(B)$ contains a Sylow 3-subgroup of $C_G(B)$. Hence, by the Gaschütz Splitting Theorem, $E = O^3(C_G(B)) \cong \text{Aut}(\text{SU}_4(2))$ or $\text{Sp}_6(2)$. As $t$ inverts $B$, $N_G(B)/E \cong \text{Sym}(3)$. Since $t$ centralizes $E \cap J$ which is elementary abelian of order $3^3$ and since this subgroup is self-centralizing in $E$, we infer that $B/t = C_{N_G(B)}(E) \cong \text{Sym}(3)$. Thus the lemma will be proved once we have eliminated the possibility that $E \cong \text{Sp}_6(2)$.

Suppose that $E \cong \text{Sp}_6(2)$. Then $E$ contains a subgroup $F$ with $F \cong \text{Sp}_2(2) \times \text{Sp}_4(2) \cong \text{Sym}(3) \times \text{Sym}(6)$. Since there is a unique conjugacy class of elementary abelian subgroups of order 27 in $\text{Sp}_6(2)$, we may choose $F$ so that $J \cap E \in \text{Syl}_3(F)$. Note that $t$ centralizes $F$. Let $R_1 \in \text{Syl}_3(N_B(J \cap E))$. Then $R_1 \cong 2 \times 2 \times \text{Dih}(8) \leq N_F(J)$ and $R_1$ contains $t$ which inverts $B$. Let $x \in R_1^*$ be an involution. Then $x \in F'' \cong \text{Alt}(6)$ and $x$ inverts $J\cap F''$ and centralizes $O_3(F)B$. On the other hand, by Lemma 4.11, $R_1 \leq M \sim 3^4.(2 \times \text{Sym}(6))$ and so $R_1^* \leq M_*$. But then $C_j(x)$ contains 3-central elements of $G$ by Lemma 2.13 (ii). Hence $O_3(F)B$ contains a 3-central element of $G$, say $e$. However this means that $\text{Alt}(6) \cong F'' \cong C_G(e) \sim 3^{4+4}.2^{1+4}.\text{Alt}(5)$, which is absurd. Hence $E \not\cong \text{Sp}_6(2)$ and the lemma is proven. \hfill \qed

Now set $E = O^3(C_G(B))$, $K = C_G(t)$, $E_L = E \cap L$, $E_M = E \cap M$ and $J_K = J \cap K$.

Lemma 5.5. $E_L \sim 3^{1+2}.\text{GL}_2(3)$ and $E_M = N_E(J_K) \sim 3^3.(2 \times \text{Sym}(4))$. 

Proof. We have that $E = C_G(\langle t, B \rangle)$ and so $Z$ and $J_K$ are contained in $E$. That $Z$ is a 3-central subgroup of $E$ follows from Lemma 5.3. Hence, as $E \cong \text{Aut}(\text{SU}_4(2)) \cong \text{PGSp}_4(3)$, we get $E_L \sim 3^{1+2}.\text{GL}_2(3)$ and, since a Sylow 3-subgroup of $E$ contains a unique elementary abelian subgroup of order 27, $E_M = N_E(J_K) \sim 3^3.(2 \times \text{Sym}(4))$ (see for example \cite[pg. 26]). \hfill \qed
6. The centralizer of $t$

We now start our investigation of the centralizer of the involution $t$. We continue with the notation of the last section. In particular, $K = C_G(t)$. By Lemma 5.4, $K$ contains $E = O^3(C_G(B)) \cong \text{Aut} (\text{SU}_4(2))$. Our first lemma asserts that we already see the Sylow 3-subgroup of $K$ in $C_L(t)$.

**Lemma 6.1.** $C_S(t)$ is a Sylow 3-subgroup of $K$. In particular, $|K|_3 = 3^4$ and $E$ contains a Sylow 3-subgroup of $K$.

**Proof.** Let $F = C_S(t)$. Then Lemmas 4.3(iii) and 4.9 imply that $Z(F) = Z$ and $F \in \text{Syl}_3(C_L(t))$. If $F_1 \in \text{Syl}_3(K)$ and $F \leq F_1$, then $N_{F_1}(F)$ normalizes $Z$ and is consequently contained in $L$. Thus $N_{F_1}(F) = F$ and so $F = F_1$. □

**Lemma 6.2.** The involutions $t$ and $u$ are not $G$-conjugate and $u \in M_*$.

**Proof.** Choose an element $s$ of order 2 in $N_{M_*}(S)$. Then $s$ inverts $S/J$. Using Lemma 2.13(ii) and (vi) we see that $s$ centralizes $J/(J \cap Q)$ and $Z$, and inverts $(Q \cap J)/Z$. Since $s$ normalizes $Q$ by Lemma 4.7(i), we deduce that $\langle s \rangle Q = \langle u \rangle Q$. In particular, $u \in M_*$ and so we have that $C_S(u) = C_J(u)$ contains exactly two 3-central subgroups by Lemma 2.13(ii). Let $F = C_S(u)$. Suppose that $F_1 \in \text{Syl}_3(C_G(u))$ with $F \leq F_1$. If $F_1 > F$, then $|Z^{N_{F_1}(F)}| = 3$ which is not the case. Thus $F_1 = F$ has order 9 and consequently, using Lemma 6.1 we see that $t$ and $u$ are not $G$-conjugate. □

**Lemma 6.3.** Suppose that $x$ is an involution of $M$ with $|C_J(x)| = 3^3$. Then $x$ is $M$-conjugate to $t$.

**Proof.** The involutions $xJ \in M/J$ with $|C_J(x)| = 3^3$ are reflections on $J$. Since the two reflection classes are fused in $\text{CO}_4^- (3)$, we have that all such involutions $xJ$ are conjugate. But then $M$ has exactly one class of such involutions. □

Recall that $J_K = J \cap K = C_J(t)$ is elementary abelian of order $3^3$.

**Lemma 6.4.** We have that

(i) $J_K = J(C_S(t))$;

(ii) $N_G(J_K) \leq M$;

(iii) $C_G(J_K) = J \langle t \rangle$;

(iv) $N_K(J_K)/C_K(J_K) \cong \text{GO}_3(3) \cong 2 \times \text{Sym}(4)$; and

(v) $N_K(J_K) \leq \langle t \rangle E$.

**Proof.** Since $C_S(t)$ is isomorphic to a Sylow 3-subgroup of $\text{PSp}_4(3)$ by Lemma 4.3(iii), (i) holds.
Let \( Y = N_G(J_K) \). Then \( Y \) normalizes \( C_Y(J_K) \) which contains \( J \).
Hence the Frattini Argument implies that \( Y = N_Y(J)C_Y(J_K) \). Since \( Z \leq J_K, C_G(J_K) \leq L \). Because \( J_K \) centralizes \( t \), \( J_K \notin Q \) and so \( C_G(J_K) \leq N_L(S) \) is 3-closed. It follows that \( C_Y(J_K) \) normalizes \( J = J(S) \). So (ii) holds.

Since \( J_K = C_J(t) \), we have that \( J_K \) and \( [J,t] \) are orthogonal. In particular, \( N_K(J_K)/J_K \cong 2 \times \GO_3(3) \cong 2 \times 2 \times \Sym(4) \) and \( C_G(J_K) = J(t) \). This is (iii) and (iv). As \( N_G(J_K) \leq N_M(B) \) part (iii) also holds.

\[ \square \]

**Lemma 6.5.** \( K \) contains a subgroup isomorphic to \( \Sym(3) \times \Sym(6) \).

**Proof.** Let \( t_1 \in E \) be such that \( C_E(t_1) \cong 2 \times \Sym(6) \). Then \( C_G(t_1) \geq B(t) \times C_E(t_1) \cong \Sym(3) \times 2 \times \Sym(6) \) and so it suffices to show that \( t \) and \( t_1 \) are \( G \)-conjugate.

We make our initial choice of \( t_1 \) so that there exists \( F \in \syl_3(C_E(t_1)) \) such that \( F \leq C_S(B) \). Then by Lemma 2.20 \( F \) is contained in the Thompson subgroup of \( S \cap E \) which is \( J_K \). Hence \( BF \leq J \).

Since \( BF \) is a maximal subgroup of \( J \), \( BF \) contains a conjugate of \( Z \) by Lemma 2.14. Conjugating by a suitable element of \( M \) we may then suppose that \( Z \leq BF \leq J \) and \( t_1 \) centralizes \( BF \). Thus we may view the entire configuration in \( L_* \). By Lemma 4.2(i), \( t_1 \in QP \). Therefore, either \( t_1 \) is conjugate to \( u \) or \( t_1 \) is conjugate to \( t \). Since \( |C_{L_*}(u)|_3 = 3^2 \) by Lemma 4.2(v), we have that \( t_1 \) is conjugate to \( t \) as claimed.

\[ \square \]

For \( n \in \{0,1,2,3,4\} \), \( Z_n \) denotes the set of subgroups of \( J_K \) of order 9 containing precisely \( n \) subgroups which are \( G \)-conjugate to \( Z \).

**Lemma 6.6.**

(i) \( J_K \) contains exactly 4 subgroups \( G \)-conjugate to \( Z \) and the remaining subgroups of \( J_K \) of order 3 are all \( G \)-conjugate to \( B \).

(ii) The \( N_K(J_K) \) orbits, under conjugation, of the subgroups of \( J_K \) of order 9 are \( Z_0 \), \( Z_1 \) and \( Z_2 \). Further, \( |Z_0| = 3 \), \( |Z_1| = 4 \) and \( |Z_2| = 6 \).

**Proof.** From Lemma 6.4 (iii), we have \( N_K(J_K)/C_K(J_K) \cong \GO_3(3) \cong 2 \times \Sym(4) \). Since \( J_K \) is irreducible as an \( N_K(J_K) \)-module, the centre of \( N_K(J_K)/C_K(J_K) \) inverts \( J_K \) and thus has no effect on the orbits of \( N_K(J_K) \) on subgroups of \( J_K \). Since \( J_K \) can be identified as a non-degenerate orthogonal module and \( N_K(J_K)/C_K(J_K) \) can be identified with \( \GO_3(3) \), we see that \( J_K \) has exactly four subgroups of order 3 which correspond to singular one spaces and these are \( Z^{N_K(J_K)} \). The other subgroups of \( J_K \) of order 3 are conjugate to \( B \).

When \( N_K(J_K) \) acts on subgroups \( A \) of order 9 in \( J_K \), we have three possibilities: \( A \) could be hyperbolic, there are six of these, definite,
there are three of these, or degenerate of which there are four. By Witt’s Lemma the respective types are fused in \( N_K(J_K) \). Therefore \( Z_0 \) consists of definite spaces, \( Z_1 \) of degenerate spaces and \( Z_2 \) of hyperbolic spaces.

**Lemma 6.7.** Let \( A \in Z_1 \) and \( a \in A^\# \) be 3-central. Then \( A = J_K \cap O_3(C_G(a)) \).

*Proof.* By Lemma 4.6 (ii), we have that \( J_K \cap O_3(C_G(a)) \in Z_1 \). The result is now verified as, by Lemma 6.6 there are exactly four \( N_K(J_K) \)-conjugates of \( \langle a \rangle \) in \( J_K \) and \( |Z_1| = 4 \).

**Lemma 6.8.** Suppose that \( |J_K : A| = 3 \). Then \( J_K \in \text{Syl}_3(C_K(A)) \). In particular, setting \( E_b = O^3(C_G(b)) \), either \( C_{E_b}(t) \cong 2 \times \text{Sym}(6) \) or \( C_{E_b}(t) \cong 2_+^{1+4}.3^2.2^2 \).

*Proof.* Since \( J_K \) is abelian and \( J_K \leq K, J_K \leq C_K(A) \). By Lemma 6.6, there exists \( b \in A \) which is not 3-central. Now \( C_G(b) = 3 \times \text{Aut}(SU_4(2)) \) by Lemma 5.4. Since \( t \) centralizes \( J_K \cap E_b \) which has order 9, from Table 2 we read that \( |C_{E_b}(t)|_3 = 3^2 \). Now we may further deduce the possible structures of \( C_{E_b}(t) \) as listed.

**Lemma 6.9.** Suppose that \( |J_K : A| = 3 \).

(i) If \( A \in Z_1 \) and \( a \in A^\# \) is 3-central, then \( O_3'(C_K(A)) = O_3'(C_K(a)) \cong Q(8) \). Also, for \( b \in A^\# \) with \( b \) not 3-central in \( G \),

\[
O_3'(C_K(A)) \leq O_3'(C_K(b)) \cong 2_+^{1+4}.
\]

(ii) If \( A \in Z_0 \cup Z_2 \), then \( O_3'(C_K(A)) = \langle t \rangle \).

(iii) If \( T \in \text{u}_{C_G(A)}(J_K, 3') \), then \( T \leq O_3'(C_K(A)) \).

*Proof.* Assume that \( A \in Z_1 \). Let \( a \in A^\# \) be a 3-central element and \( b \in A \setminus \langle a \rangle \). Then \( C_G(a) \sim 3^{1+4}.2^{1+4}.\text{Alt}(5) \). Since every element of order 2 in \( C_G(a) \) is contained in \( O_{3,2}(C_G(a)) \) by Lemma 2.17(ii), we have that \( t \in O_{3,2}(C_G(a)) \). As \( t \) is not conjugate to the elements in \( Z(C_G(a)/O_3(C_G(a))) \) by Lemma 6.2, we have \( O_3'(C_G(a)(t)) \cong Q(8) \) by Lemma 4.3(ii). By Lemma 6.7, \( A = J_K \cap O_3(C_G(a)) \leq C_{O_3(C_G(a))}(t) \). Thus Lemma 4.3(i) and (ii) imply that \( O_3'(C_K(a)) = O_3'(C_K(A)) \cong Q(8) \) which is the first claim in (i). We now focus on \( b \). Using Lemmas 6.6 (i) and 5.4, we have \( C_G(b) \cong 3 \times \text{Aut}(SU_4(2)) \). Let \( E_b = O^3(C_G(b)) \). Then, as \( t \) centralizes \( b, t \in E_b \). Now \( C_{C_G(b)}(t) \) contains \( O_3'(C_K(A)) \cong Q(8) \). Hence, as \( 2 \times \text{Sym}(6) \) doesn’t contain a subgroup isomorphic to \( Q(8) \), we may use Lemma 6.8 to deduce that \( t \in E_b \) and that \( C_K(b) \sim 3 \times 2_+^{1+4}.3^2.2 \). Thus \( O_3'(C_K(b)) \cong 2_+^{1+4} \). Now using
the fact that $C_K(b)$ is soluble and applying Lemma 2.9 we get that $O_{3'}(C_K(A)) \leq O_{3'}(C_K(b))$. Thus (i) holds.

Assume that $A \in Z_2$ and just as above let $a \in A^#$ be a 3-central element. By Lemma 6.7, $A \not\leq O_3(C_G(a))$. Let $b \in A \setminus O_3(C_G(a))$. Again by Lemma 6.2 $t$ is not conjugate to an element of the inverse image of $Z(C_G(a))/O_3(C_G(a))$. Hence using Lemmas 2.17(ii) and 4.3(iv) we get $C_{O_3(C_G(a))}(b) = \{t\}$. In particular, using Lemma 2.9 again (ii) holds for $A \in Z_2$.

Suppose that $A \in Z_0$. Let $b \in A^#$. Then $C_G(b) \cong 3 \times \text{Aut}(\text{SU}_4(2))$ by Lemma 6.6(i). Recall that $E_b = O^3(C_G(b))$. Then from Lemma 6.8, we have $C_{C_G(b)}(t) \sim 3 \times 2^{1+4} \cdot 3.2^2$ or $C_{C_G(b)}(t) \cong 3 \times 2 \times \text{Sym}(6)$. In the latter case the centralizer in $C_{O_3(b)}(t)$ of any further element of order 3 has shape $2 \times 3 \times 3 \times \text{Sym}(3)$ and so (ii) holds if this possibility arises. So assume the former possibility occurs. Then, as $O^2(C_{E_b}(t))$ is isomorphic to the central product $SL_2(3) \circ SL_2(3)$, $C_{O_2(C_{E_b}(t))}(A \cap E_b)$ either has order 8 or 2. In the former case we deduce from centralizer orders that $A \cap E$ is 3-central in $E$ and consequently 3-central in $G$, a contradiction. Thus $C_{O_2(C_{E_b}(t))}(A \cap E) = \{t\}$ and so (ii) holds when $A \in Z_0$.

By Lemma 6.8, $J_K \in \text{Syl}_3(C_K(A))$ and so, as $C_K(A)$ is soluble, $J_K \cong 3 \times \text{Aut}(\text{SU}_4(2))$. Therefore any $3'$-subgroup of $C_K(A)$ which is normalized by $J_K$ centralizes $J_K \cong 3 \times \text{Aut}(\text{SU}_4(2))$. Hence, as $C_K(A)$ is soluble, (iii) follows from Lemma 2.9

Define $R = \langle O_{3'}(C_K(A)) \mid A \in Z_1 \rangle$. Notice, that by Lemma 6.9(i) and (ii), we also have that $R = \langle O_{3'}(C_K(A)) \mid |J_K : A| = 3 \rangle$.

**Lemma 6.10.** $R \cong 2^{1+4}$ and $\mathcal{N}_K(J_K, 3') = \{R\}$.

**Proof.** As $J_K \leq C_K(A)$ for all $A \in Z_1$, $R$ as defined is normalized by $J_K$. Let $Z_i = \{A_1, A_2, A_3, A_4\}$. Then, by Lemma 6.9(i), for $1 \leq i \leq 4$, $O_{3'}(C_K(A_i)) \cong Q(8)$. Additionally, for $1 \leq i < j \leq 4$, $A_i \cap A_j$ is a $G$-conjugate of $B$ by Lemmas 6.6(i) and 6.7. Thus $O_2(C_K(A_i \cap A_j)) \cong 2^{1+4} \cong Q(8) \circ Q(8)$ by Lemma 6.9(i). Note that $2^{1+4}$ contains exactly two subgroups isomorphic to $Q(8)$ and that these subgroups commute. Assume that $O_{3'}(C_K(A_i)) = O_{3'}(C_K(A_j))$, then this subgroup is centralized by $\langle A_i, A_j \rangle = J_K$. Since $Z_0 \cup Z_2 \neq \emptyset$, this contradicts Lemma 6.9(ii) and (iii). Thus $[O_{3'}(C_K(A_i)), O_{3'}(C_K(A_j))] = 1$. It follows now that $R$ is a central product of four subgroups each isomorphic to $Q(8)$ and so $R \cong 2^{1+4}$. In particular, $R \in \mathcal{N}_K(J_K, 3')$.

Suppose that $R_0 \in \mathcal{N}_K(J_K, 3')$. Then $R_0 = \langle C_{R_0}(A) \mid |J_K : A| = 3 \rangle$. Since, for $|J_K : A| = 3$, $C_{R_0}(A) \in \mathcal{N}_{C_G(A)}(J_K, 3')$, we have $C_{R_0}(A) \leq
Lemma 6.11. Suppose that \( A \in \mathcal{Z}_1 \). Then
\[
R = \langle O_3'(C_K(b)) \mid b \in A^#, b \text{ not 3-central in } G \rangle.
\]

Proof. We have \( C_R(A) \cong \mathbb{Q}(8) \) by Lemma 6.9 (i). By Lemma 6.10, \( R/C_R(A) \) is elementary abelian of order \( 2^6 \) and \( O_3'(C_K(b)) \leq R \). Since for \( b \in A^# \) such that \( b \) is not 3-central in \( G \), we have \( |O_3'(C_K(b))/C_R(A)| = 2^2 \) by Lemma 6.9 (i), we infer that
\[
R = \langle O_3'(C_K(b)) \mid b \in A^#, b \text{ not 3-central in } G \rangle.
\]

Lemma 6.12. \( N_K(R) \geq RE \) and \( C_K(R) = \langle t \rangle \).

Proof. By Lemma 5.5, \( E_L \sim 3^{1+2}.\text{GL}_2(3) \) and \( E_M \sim 3^3.(2 \times \text{Sym}(4)) \). Furthermore, \( O_3(E_M) = J_K \). Since \( R \) is the unique member of \( \mathcal{W}_1^* (J_K, 3^2) \), \( E_M \) normalizes \( R \). Let \( T = O_3(E_L) \). Then \( T \cap J_K = \mathcal{Z}_1 \) by Lemma 4.6 (ii). Let \( x \in E_L \setminus E_M \) and set \( A = (T \cap J_K)^x \). Note that \( A \leq T^x = T \), so \( A \) normalizes \( R \) and \( R^x \). Now, using Lemma 6.11 applied to the action of \( A \) on \( R^x \) we have,
\[
R^x = \langle O_3'(C_K(b)) \mid b \in A^#, b \text{ not 3-central in } G \rangle.
\]

Next we consider the action of \( A \) on \( R \). By coprime action we have \( R = C_R(Z)([C_R(b), C_R(Z)] \mid b \in A \setminus Z) \). By Lemmas 6.9 and 6.11, \( C_R(Z) = O_3'(C_K(Z)) = O_3'(C_K(Z))^x = C_R^x(Z) \). Let \( b \in A \setminus Z \). Then \( b \) is not 3-central in \( G \) and consequently \( C_R(b) \leq C_K(b) \) which has shape \( 3 \times 2^{1+4}.3^2.2 \). Therefore any 2-subgroup of \( O_3'(C_K(b)) \) is contained in \( O_3'(C_K(b)) \). Hence \( [C_R(b), Z] \leq O_3'(C_K(b)) \leq R^x \). It follows that \( R \leq R^x \) and so \( R = R^x \). Thus \( R \) is normalized by \( \langle E_M, x \rangle = E \).

Let \( C = C_K(R) \). Then, as \( E \) contains a Sylow 3-subgroup of \( K \) by Lemma 6.1 and \( E \) acts non-trivially on \( R \), \( C_K(R) \) is a 3'-group which is normalized by \( E \) and hence by \( J_K \). Thus \( C_K(R) \leq R \) by Lemma 6.10.

We now set \( H = N_G(R) \). Notice that as \( R \) is extraspecial, we have that \( H \) centralizes \( t \) and so \( H = N_K(R) \). Our next goal is to show that \( G, H \) and \( R \) satisfy the hypothesis of Theorem 3.1.

Lemma 6.13. \( H/R \cong \text{Aut} (SU_4(2)) \) or \( \text{Sp}_6(2) \).

Proof. We have that \( Z \leq E \leq N_G(R) \) by Lemma 6.12. From the definition of \( R \) and Lemma 4.8 (iii), \( O_2(C_L(t)) \leq R \). Thus \( C_L(t)R/R \cong C_L(t)/O_2(C_L(t)) \) is isomorphic to the centralizer of a 3-central element of order 3 in \( \text{PSp}_4(3) \). Since \( ER/R \geq C_L(t)R/R \) we infer

\[O_3'(C_K(A))\text{ by Lemma 6.9 (iii). But then by Lemma 6.9 (i) and (ii), }\]
\[R_0 \leq R.\text{ Hence } \mathcal{W}_1^*(J_K, 3') = \{ R \}.\]

\[\square\]
that \(ZR/R\) is inverted by its normalizer in \(H/R\). By Lemma 6.10 the assumptions of Theorem 2.2 are fulfilled and we have \(H/R \cong \mathrm{Aut}(\mathrm{SU}_4(2))\) or \(\mathrm{Sp}_6(2)\).

**Lemma 6.14.** \(C_H(R) \leq R\) and \(R/\langle t \rangle\) is a minimal normal subgroup of \(H/\langle t \rangle\) of order \(2^8\).

*Proof.* Lemma 6.12 ensures that \(C_H(R) \leq R\). Also as \(R\) is extraspecial of order \(2^9\), \(R/\langle t \rangle\) has order \(2^8\). Suppose that \(R_1\) is a normal subgroup of \(H\) contained in \(R\) with \(\langle t \rangle \leq R_1 \leq R\). Now \(J_KR/R\) is elementary abelian of order 27 and the 3-rank of \(\mathrm{GL}_3(2)\) is 2, and therefore either \(R/R_1\) or \(R_1\) is centralized by \(O^2(H/R)\) and hence by \(J_K\). However \(C_{G}(J_K) = J(t)\) by Lemma 6.4(iii) and so we see that either \(R = R_1\) or \(R_1 = \langle t \rangle\). Thus \(R/\langle t \rangle\) is a minimal normal subgroup of \(H/\langle t \rangle\). \(\Box\)

**Lemma 6.15.** The following hold.

(i) \(C_K(Z) \leq H\).

(ii) \(ER\) controls fusion of elements of order 3 in \(K\).

(iii) \(B^G \cap K = B^K_1 \cup B^K_2\) where \(B_1\) is conjugate to a subgroup of \(J_K\) which together with \(Z\) forms a subgroup in \(Z_1\).

(iv) If \(B_1 \leq J_K\), then \(C_K(B_1) \leq ER\).

*Proof.* Looking in \(E\), we see \(C_E(Z) \sim 3^{1+2}.\mathrm{SL}_2(3)\). From Lemma 6.10, we have \(C_R(Z) \cong \mathbb{Q}(8)\). Since \(|C_K(Z)| = 2^6.3^4\) by Lemma 4.3(iii), part (i) holds.

Since \(J_K\) is torus in \(E \cong \mathrm{SU}_4(2)\) (or using \([7\), pg. 26\]), we have that every element of order 3 in \(E\) is \(E\)-conjugate to an element of \(J_K\). Since \(E\) contains a Sylow 3-subgroup of \(K\) and \(N_K(J_K)\) controls \(K\)-fusion of 3-elements in \(J_K\) by Lemma 2.5, we have (ii).

As \(J_K\) is an orthogonal module for \(N_K(J_K)/C_K(J_K) \cong \mathrm{GO}_3(3)\), Lemma 6.4(iv) implies \(K\) has three conjugacy classes of elements of order 3 and just one 3-central class. Thus (iii) follows from (ii).

Now consider the class \(B^K_1\). We may suppose that \(B_1Z \in Z_1\). Then \(C_R(B_1) \cong 2_+^{1+4}\) by Lemma 6.9(i). It follows that \(t\) is an involution contained in \(O^3(C_G(B_1))' \cong \mathrm{SU}_4(2)\) with \(C_{C_G(B_1)}(t) \sim 3 \times 2_+^{1+4}.3^2.2^2\). In particular, \((C_G(B_1) \cap K)/R/R\) normalizes \(J_KR/R\) and so (iv) follows from Lemma 6.4(v). \(\Box\)

**Lemma 6.16.** Continuing the notation of Lemma 6.15, we have cyclic groups in the same \(H\)-class as \(B_2\) act fixed-point-freely on \(R/\langle t \rangle\).

*Proof.* Since \(B_2\) is not contained in any member of \(Z_1\), we have that \(B_2\) acts faithfully on \(O_3'(C_G(A))\) for each \(A \in Z_1\). Thus, as \(R = \prod_{A \in Z_1} O_3'(C_G(A))\), we have that \(B_2\) acts fixed-point-freely on \(R/\langle t \rangle\). \(\Box\)
Lemma 6.17. If \( k \in K \setminus H \) and \( d \in H \cap H^k \) has order \( 3 \), then \( C_R(d) = \langle t \rangle \).

Proof. We begin by noting that \( R = O_2(H) \) and so \( N_K(H) = H \). Hence if there exists \( k \in K \setminus H \), then \( H \cap H^k \neq H \).

Suppose for a moment that a conjugate of \( J_K \) is contained in \( H \cap H^k \). Then we may assume that \( J_K \leq H^k \). Thus \( J_K \) and \( J_K^{-1} \) are both contained in \( H \). Hence there exists \( h \in H \) such that \( J_K = J_K^{-1}h \). But then \( k^{-1}h \in N_K(J_K) \leq ER \leq H \) by Lemmas 6.4 (v) and 6.12 whence \( k \in H \) and we have a contradiction.

Let \( T \in \text{Syl}_3(H \cap H^k) \) and assume \( T \neq 1 \). Suppose that \( T \) contains a \( K \)-conjugate \( Y \) of \( Z \) or \( B_1 \). Then, as \( H \) controls fusion of elements of order 3 in \( K \) by Lemma 6.15 (ii), we may suppose that either \( Y = Z \) or \( Y = B_1 \). Hence Lemma 6.15 (i) and (iv) gives that \( C_K(Y) \leq H \). However then \( C_{H^k}(Y) \) contains a subgroup \( X \) of \( H^k \) which is conjugate to \( J_K \) as every element of order 3 in \( H \) is fused to an element of \( J_K \) in \( H \). But this means \( X \leq C_K(Y) \leq H \) by Lemma 6.15 (i) and (v) and this contradicts the observation in paragraph two of the proof. It follows that if \( d \in H \cap H^k \) has order 3 and \( k \not\in H \), then \( d \) is conjugate to an element of \( B_2 \). The claim in the lemma now follows from Lemma 6.16.

Proof of Theorem 1.1. Let \( \overline{K} = K/\langle t \rangle \) and set \( \overline{H} = N_{\overline{K}}(R) \). Lemmas 6.13, 6.14, 6.15 (ii) and 6.17 together show that the hypotheses of Theorem 3.1 are satisfied. Therefore \( \overline{K} = O_2(\overline{K}) \overline{H} \). Now \( \overline{H} \) contains a Sylow 3-subgroup of \( \overline{K} \) and so \( O_2(\overline{K}) \leq O_3(\overline{K}) \). Since \( N_K(J_K,3') = \{ R \} \), we infer that \( O_2(\overline{K}) \leq \overline{R} \). Thus \( K = H \). Since, by Lemma 6.13, \( K \) contains a subgroup isomorphic \( \text{Sym}(3) \times \text{Sym}(6) \) whereas \( \text{Aut}(\text{SU}_4(2)) \) does not, we now get that \( H/R \cong \text{Sp}_6(2) \). Since \( O_3(G) = 1 \), Lemma 5.2 implies that \( O_2(G) = Z(G) = 1 \). Since \( R/\langle t \rangle \) is the spin-module for \( H/R \), Lemma 2.18(iv) implies that the elements of order 5 in \( H \) act fixed point freely on \( R/\langle t \rangle \). Hence, at last, Theorem 2.1 gives us that \( G \) is isomorphic to \( \text{Co}_2 \). \( \square \)

References

[1] Aschbacher, Michael. The existence of \( J_3 \) and its embeddings in \( E_6 \). Geom. Dedicata 35 (1990), no. 1-3, 143–154.
[2] Aschbacher, M. Finite group theory. Second edition. Cambridge Studies in Advanced Mathematics, 10. Cambridge University Press, Cambridge, 2000.
[3] Astill, Sarah. 3-local identifications of some finite simple groups. University of Birmingham, MPhil(Qual) thesis, 2007.
[4] Astill, Sarah; Parker, Chris. A 3-local characterization of \( M_{12} \) and \( \text{SL}_3(3) \). Arch. Math. (Basel) 92 (2009), no. 2, 99–110.
[5] Borovik, A. V. 3-local characterization of the Held group. (Russian) Algebra i Logika 19 (1980), no. 4, 387–404.
[6] Burnside, W. Theory of groups of finite order. 2nd ed. Dover Publications, Inc., New York, 1955.
[7] Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; Wilson, R. A. Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray. Oxford University Press, Eynsham, 1985.
[8] Delgado, A.; Goldschmidt, D.; Stellmacher, B. Groups and graphs: new results and methods. With a preface by the authors and Bernd Fischer. DMV Seminar, 6. Birkhäuser Verlag, Basel, 1985.
[9] Doerk, Klaus; Hawkes, Trevor. Finite soluble groups. de Gruyter Expositions in Mathematics, 4. Walter de Gruyter & Co., Berlin, 1992.
[10] Durakov, B. K. Characterization of some finite simple groups by a centralizer of elements of order three. (Russian) Mat. Sb. (N.S.) 109(151) (1979), no. 4, 533–554.
[11] Fowler, Rachel. A 3-local characterization of the Thompson sporadic simple group. PhD thesis, University of Birmingham, 2007.
[12] Goldschmidt, David M. 2-fusion in finite groups. Ann. of Math. (2) 99 (1974), 70–117.
[13] Goldschmidt, David M. Automorphisms of trivalent graphs. Ann. of Math. (2) 111 (1980), no. 2, 377–406.
[14] Güloğlu, Ismail Şüayip. A characterization of the simple group $J_4$. Osaka J. Math. 18 (1981), no. 1, 13–24.
[15] Higman, Graham. Odd characterizations of simple groups, Lecture Notes, University of Michigan, Ann Arbor, Michigan, 1968.
[16] Il’inyh, A. P. Characterization of the simple O’Nan-Sims group by the centralizer of an element of order three. (Russian) Mat. Zametki 24 (1978), no. 4, 487–497.
[17] Jansen, Christoph; Lux, Klaus; Parker, Richard; Wilson, Robert. An atlas of Brauer characters. Appendix 2 by T. Breuer and S. Norton. London Mathematical Society Monographs. New Series, 11. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
[18] Korchagina, I. A.; Parker, C. W.; Rowley, P. J. A 3-local characterization of $Co_3$. European J. Combin. 28 (2007), no. 2, 559–566.
[19] King, Oliver. On subgroups of the special linear group containing the special orthogonal group. J. Algebra 96 (1985), no. 1, 178–193.
[20] Kurzweil, Hans; Stellmacher, Bernd. The theory of finite groups. An introduction. Translated from the 1998 German original. Universitext. Springer-Verlag, New York, 2004.
[21] Meierfrankenfeld, Ulrich; Stellmacher, Bernd; Stroth, Gernot. Finite groups of local characteristic $p$: an overview. Groups, combinatorics & geometry (Durham, 2001), 155–192, World Sci. Publ., River Edge, NJ, 2003.
[22] O’Nan, Michael E. Some characterizations by centralizers of elements of order 3. J. Algebra 48 (1977), no. 1, 113–141.
[23] Parker, Chris. A 3-local characterization of $U_6(2)$ and $Fi_{22}$. J. Algebra 300 (2006), no. 2, 707–728.
[24] Parker, Christopher; Röhrle, Gerhard. The restriction of minuscule representations to parabolic subgroups. Math. Proc. Cambridge Philos. Soc. 135 (2003), no. 1, 59–79.

[25] Parker, Christopher; Rowley, Peter. Symplectic Amalgams. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2002.

[26] Parker, Christopher; Rowley, Peter. A characteristic 5 identification of the Lyons group. J. London Math. Soc. (2) 69 (2004), no. 1, 128–140.

[27] Parker, Christopher; Rowley, Peter. Local characteristic p completions of weak BN-pairs. Proc. London Math. Soc. (3) 93 (2006), no. 2, 325–394.

[28] Parker, Christopher; Rowley, Peter. A 3-local identification of the alternating group of degree 8, the McLaughlin simple group and their automorphism groups. J. Algebra, 319, no. 4, 2008, 1752–1775.

[29] Parker, C. W.; Wiedorn, C. B. A 7-local identification of the Monster. Nagoya Math. J. 178 (2005), 129–149.

[30] Prince, A. R. A characterization of the simple groups PSp(4, 3) and PSp(6, 2). J. Algebra 45 (1977), no. 2, 306–320.

[31] Salarian, M. R. An identification of Co₁. Preprint 2007.

[32] Salarian, M. R. A 3-local characterization of M(24)’ Preprint 2007.

[33] Salarian, M. R and Stroth, G. An identification of the monster group. Preprint 2007.

[34] Smith, Fredrick L. A characterization of the .2 Conway simple group. J. Algebra 31 (1974), 91–116.

[35] Stafford, Richard M. A characterization of Janko’s simple group J₄ by centralizers of elements of order 3. J. Algebra 57 (1979), no. 2, 555–566.

[36] Stroth, G. An odd characterization of J₄. Israel J. Math. 31 (1978), no. 2, 189–192.

[37] Wilson, Robert A. The maximal subgroups of Conway’s group ·2. J. Algebra 84 (1983), no. 1, 107–114.

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