Inhomogeneous Neutrino Degeneracy and Big Bang Nucleosynthesis

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We examine Big Bang nucleosynthesis (BBN) in the case of inhomogenous neutrino degeneracy, in the limit where the fluctuations are sufficiently small on large length scales that the present-day element abundances are homogeneous. We consider two representative cases: degeneracy of the electron neutrino alone, and equal chemical potentials for all three neutrinos. We use a linear programming method to constrain an arbitrary distribution of the chemical potentials. For the current set of (highly-restrictive) limits on the primordial element abundances, homogeneous neutrino degeneracy barely changes the allowed range of the baryon-to-photon ratio $\eta$. Inhomogeneous degeneracy allows for little change in the lower bound on $\eta$, but the upper bound in this case can be as large as $\eta = 1.1 \times 10^{-8}$ (only $\nu_e$ degeneracy) or $\eta = 1.0 \times 10^{-9}$ (equal degeneracies for all three neutrinos).

For the case of inhomogeneous neutrino degeneracy, we show that there is no BBN upper bound on the neutrino energy density, which is bounded in this case only by limits from structure formation and the cosmic microwave background.
I. INTRODUCTION

Although the standard model of Big Bang nucleosynthesis (BBN) is highly successful (for a recent discussion, see references [6,7]) many variations on this model have been proposed [3]. One of the most frequently investigated variations on the standard model is neutrino degeneracy, in which each type of neutrino is allowed to have a non-zero chemical potential [4] - [11], and a number of recent models have been proposed to produce a large lepton degeneracy [12] - [14].

More recently, Dolgov and Pagel have suggested the possibility of inhomogeneous neutrino degeneracy [15]. Their model was proposed to explain the apparent discrepancy between various measurements of the primordial deuterium abundance in high-redshift Lyman-alpha clouds. Here we consider a more mundane possibility: that the neutrino chemical potential is inhomogeneous, but on much smaller scales. In particular, we assume that the amplitude of the inhomogeneities is small on length scales larger than the typical baryonic diffusion scales after nucleosynthesis, so that the element abundances are homogeneous today. We calculate the element abundances for this scenario and compare to observational limits. Using a method similar to that in reference [14], we can simulate arbitrary distributions of the neutrino chemical potential, and so determine the upper and lower bounds on the baryon-to-photon ratio $\eta$ in this model.

In the next section, we discuss our model for inhomogeneous neutrino degeneracy and its physical consequences. In Section 3, we use a linear programming technique to calculate upper and lower bounds on $\eta$ in this model. Our conclusions are summarized in Section 4. We find that when the chemical potential is inhomogeneous, there are no BBN limits on the overall neutrino energy density; the only limits in this case come from other cosmological considerations such as structure formation [14] - [17] or the CMB [18]. Not surprisingly, inhomogeneous neutrino degeneracy allows for a wider range of values for $\eta$ than does homogeneous degeneracy.

II. INHOMOGENEOUS NEUTRINO DEGENERACY

Consider first the case of homogeneous neutrino degeneracy. In this case, each type of neutrino is characterized by a chemical potential $\mu_i$ ($i = e$, $\mu$, $\tau$), which redshifts as the temperature, so it is convenient to define the constant quantity $\xi_i = \mu_i/T_i$. In terms of $\xi_i$, the neutrino and antineutrino number densities are given by

$$\nu_i = \frac{1}{2\pi^2} T^3_\nu \int_0^\infty \frac{x^2 dx}{1 + \exp(x - \xi_i)}, \quad (1)$$

and

$$\bar{\nu}_i = \frac{1}{2\pi^2} T^3_\nu \int_0^\infty \frac{x^2 dx}{1 + \exp(x + \xi_i)}, \quad (2)$$

while the total energy density of the neutrinos and antineutrinos is

$$\rho = \frac{1}{2\pi^2} T^4_\nu \int_0^\infty \frac{x^3 dx}{1 + \exp(x - \xi_i)} + \frac{1}{2\pi^2} T^4_\nu \int_0^\infty \frac{x^3 dx}{1 + \exp(x + \xi_i)} \quad (3)$$

Degeneracy of the electron neutrinos alters the $n \leftrightarrow p$ weak rates relevant for BBN through the number densities given in equations (1) and (2), while the change in the expansion rate due to the altered density in equation (3) affects BBN for degeneracy of any of the three types of neutrinos (see, for example, reference [10] for a more detailed discussion).

What happens if this degeneracy is not homogeneous, as assumed in almost all previous work, but instead varies with position? Dolgov and Pagel [15] considered such a model in order to explain the discrepancy in observed deuterium abundances at high-redshift. In their model, $\xi$ varies on scales $\sim 100 - 1000$ Mpc, producing an observable inhomogeneity in the present-day element abundances. We make the opposite assumption: we take the variation in $\xi$ to be small on such large scales, and large on much smaller scales, so that elements are well-mixed before the present day, erasing any detectable inhomogeneities.

Although models have been proposed which produce inhomogeneities in $\xi$ (see, for example, the discussion in reference [19]), we will follow the example of reference [13] and keep our discussion as general as possible. In general, one would expect a distribution of fluctuations in $\xi$ over all length scales. However, since the neutrinos are relativistic, they will free-stream and erase any fluctuations on length scales smaller than the horizon at any given time. We will make only two assumptions concerning the fluctuations in $\xi$: that the fluctuations are significant on large enough scales to avoid being erased by free-streaming, and that they become negligible on small enough scales that the resulting element distribution is homogeneous today.

The first of these conditions requires that the fluctuations are significant on scales larger than the horizon scale when the $n \leftrightarrow p$ reactions freeze out at $T \sim 1$ MeV. If this were not the case, then free-streaming would erase all of the fluctuations in $\xi$ before BBN began. This horizon scale corresponds to a comoving length scale $\sim 100$ pc today.

The condition that the element distribution be homogeneous today requires that the fluctuations in $\xi$ decrease sufficiently quickly with length scale that they have no significant effect on nucleosynthesis on scales above the element diffusion length. Although no detailed studies of element diffusion have been performed in connection with inhomogeneous BBN scenarios, it seems safe to assume that complete mixing of the primordial elements will occur on scales well within the nonlinear regime today, $< 1$
Mpc. By requiring the fluctuations in $\xi$ to be negligible above this scale, we can also ignore any constraints from CMB observations, which severely constrain models with fluctuations on larger scales [1].

Given these assumptions, we can assume that BBN takes place in separate horizon volumes, with the value of $\xi$ being homogeneous within each volume. At late times, the elements produced within each volume mix uniformly to produce the observed element abundances today.

Note that with this set of assumptions, it is no longer meaningful to talk about the value of $\xi$ for the neutrinos at the present. Since the neutrinos from different horizon volumes diffuse freely up to the scale of the horizon, the different thermal distributions with different values of $\xi$ will combine to give a highly non-thermal neutrino distribution at late times. Thus, in the inhomogeneous scenario, it is still possible to put constraints on the present value of $\rho_\nu$, but it is meaningless to discuss limits on $\xi$, since the present neutrino distribution will be non-thermal and cannot be characterized by a single value of $\xi$. This effect is present at some level even at the time of nucleosynthesis. The neutrinos remain in thermal equilibrium down to a temperature $T \sim 2 - 3$ MeV, so that the neutrino distribution remains thermal down to this temperature, with a single unique value of $\xi$ in each horizon volume. However, the $n \leftrightarrow p$ rates do not freeze out until $T \sim 1$ MeV, so that neutrino free-streaming after decoupling will tend to produce a somewhat non-thermal background as early as the beginning of nucleosynthesis.

We have neglected this effect, which will be negligible in any case if the neutrino inhomogeneities are confined to comoving scales $> 100$ pc.

We also assume for simplicity that $\eta$ remains uniform in the presence of an inhomogeneous lepton distribution. This need not be the case if, for example, baryogenesis is related in some way to the lepton number [13,14].

### III. THE EFFECT ON BIG BANG NUCLEOSYNTHESIS

Given our discussion in the previous section, we assume that the distribution of each type of neutrino is homogeneous within a given horizon volume during nucleosynthesis, and characterized by a single degeneracy parameter $\xi_i$ ($i = e, \mu, \tau$). Different horizon volumes may have different values of $\xi_i$, so we characterize the distribution of $\xi_i$ by a distribution function $f(\xi_i)$, which gives the probability that a given horizon volume has a value of $\xi_i$ between $\xi_i$ and $\xi_i + d\xi_i$. What form should we choose for $f(\xi_i)$? In analogy with the distribution of primordial density perturbations (and in accordance with the central limit theorem) the most obvious choice is a Gaussian distribution. However, we can consider a more general case than this. Using linear programming techniques like those in reference [1], it is possible to analyze the general case of an arbitrary distribution $f$. Consider first the case where only $\nu_e$ is degenerate, and suppose that $f(\xi_e)$ is an arbitrary distribution. Then all of the element abundances will be functions of $\xi_e$ (for fixed $\eta$), and we can write, for a given nuclide $A$, 

$$\bar{X}_A = \int_{-\infty}^{\infty} X_A(\xi_e) f(\xi_e) d\xi_e,$$  

(4)

where $X_A(\xi_e)$ is the mass fraction of $A$ as a function of $\xi_e$, and $\bar{X}_A$ is the mass fraction of $A$ averaged over all space; after the matter is thoroughly mixed, $\bar{X}_A$ will be the final observed primordial element abundance.

In order to test all possible distribution functions $f$, we can divide the range in $\xi_e$ into discrete bins (not necessarily all of the same size), and approximate the integral in equation (4) as a sum:

$$\bar{X}_A = \sum_j X_{Aj} f_j \Delta \xi_{ej},$$  

(5)

where the dependence of $X_A$ and $f$ on $\xi_e$ is expressed through their dependence on the bin number $j$.

For each of the elements of interest ($^4$He, D, and $^7$Li) we have an upper and a lower observational bound. Thus, for each of these three elements, we can write down equations of the form:

$$X_{\text{lower bound}} < \sum_j X_{Aj} f_j \Delta \xi_{ej} < X_{\text{upper bound}}.$$  

(6)

Furthermore, $f(\xi_e)$ is normalized to unity, so

$$\sum_j f_j \Delta \xi_{ej} = 1.$$  

(7)

If we now define

$$p_j = f_j \Delta \xi_{ej},$$  

(8)

then equations (5) and (6) become:

$$X_{\text{lower bound}} < \sum_j X_{Aj} p_j < X_{\text{upper bound}},$$  

(9)

and

$$\sum_j p_j = 1.$$  

(10)

If we put an upper and lower cutoff on these sums, so that we retain only a finite number of terms, then equations (5) and (6) are in the form of the constraint equations in a linear programming problem, with the $N$ independent variables being the $p_j$’s. In reference [13], the variable under consideration was $\eta$ rather than $\xi$, so that the final quantity which needed to be maximized or minimized was the mean value of $\eta$. In our case, we wish to determine, for a given value of $\eta$, whether there is a
solution to equations (8) and (11). Since there are non-BBN limits on \( \rho_\nu \), we have chosen to take the quantity \( \rho_\nu / \rho_\nu \) as our objective function, where \( \rho_\nu \) is the final mean total neutrino density in the degenerate case, and \( \rho_\nu \) is the neutrino density in the absence of degeneracy. (These densities include all three neutrinos and antineutrinos). We then determine whether a solution exists to our constraint equations for a given value of \( \eta \), and scan through the allowed range of \( \eta \) until we reach an upper and lower value of \( \eta \) for which a solution no longer exists. At these limiting values for \( \eta \), our linear programming routine gives the minimum possible value of \( \rho_\nu / \rho_\nu \), which we can compare to other constraints.

We consider two representative cases of interest: first, the case where \( \xi_\mu \neq 0 \) and \( \xi_\mu = \xi_\tau = 0 \), which is equivalent to \( \xi_\mu \gg \xi_\mu, \xi_\tau \), and the case \( \xi_\mu = \xi_\mu = \xi_\tau \). The latter is probably the most physically realistic case \( 1 \). Although we have discussed our linear programming procedure only for the case of \( \nu_\mu \) degeneracy, it generalizes in an obvious way for the case where \( \xi_\mu = \xi_\mu = \xi_\tau \). We have not considered the most general possible case, in which all three degeneracy parameters vary independently. However, as we shall see, arbitrary inhomogeneity in \( \xi_\mu \) alone allows absurdly large values of \( \eta \) to be compatible with BBN, so there is nothing further to be gained in considering the most general case.

We use for our limits on the element abundances the values in the recent review in reference \( 2 \). For the primordial helium-4 mass fraction, \( Y_P \), we take

\[
0.228 \leq Y_P \leq 0.248. \tag{11}
\]

The limits on the number ratios of deuterium and lithium-7 to hydrogen are:

\[
2.9 \times 10^{-5} \leq D/H \leq 4.0 \times 10^{-5}, \tag{12}
\]

and

\[
1.3 \times 10^{-10} \leq ^7\text{Li}/H \leq 2.0 \times 10^{-10}. \tag{13}
\]

However, a BBN calculation with these limits alone yields no single value of \( \eta \) consistent with all three sets of limits. One can argue either that the theoretical uncertainties are large enough to account for this discrepancy \( 2 \), or that one of these sets of limits (most likely lithium) does not represent the true primordial abundance \( 4 \). We have chosen the former approach. Folding in the theoretical uncertainties in the BBN predictions from reference \( 2 \), we take the following limits on \( D/H \) and \( ^7\text{Li}/H \):

\[
2 \times 10^{-5} \leq D/H \leq 5 \times 10^{-5}, \tag{14}
\]

and

\[
1 \times 10^{-10} \leq ^7\text{Li}/H \leq 4 \times 10^{-10}. \tag{15}
\]

We have ignored the theoretical uncertainty in helium-4 because it represents a much smaller fractional change in \( Y_P \).

We wish to emphasize that our general results are fairly insensitive to small changes in the limits quoted above. Since we are exploring a rather radical change to the standard model, we make no effort to perform an ultra-high-precision calculation.

We used the procedure discussed above to determine the largest and smallest values of \( \eta \) which are consistent with the limits on \( Y_P, D/H, \) and \( ^7\text{Li}/H \) in equations (11), (14), and (15). Our mixing procedure requires the use of mass fractions, rather than ratios to hydrogen, so we have made this conversion in our calculation.

Consider first the “standard model” with no degeneracy. For the limits quoted above, we obtain bounds on \( \eta \) of \( 3.7 \times 10^{-10} \leq \eta \leq 5.3 \times 10^{-10} \). Now what happens if we add a homogeneous neutrino degeneracy? We have calculated the bounds on \( \eta \) for the case in which \( \xi_\mu \) (only) can have an arbitrary value, and for the case where \( \xi_\mu = \xi_\mu = \xi_\tau \) can be set to any desired value. For both cases, we find that the bounds on \( \eta \) are almost unchanged. (The lower and upper limits are both enlarged by less than 2%). While this might seem surprising in light of earlier similar calculations \( 1 \), it is a consequence of the increasingly narrower limits on the primordial element abundances. With such sharp limits as those considered here, even a free variation in \( \xi_\mu \) or in \( \xi_\mu = \xi_\mu = \xi_\tau \) cannot significantly alter the limits on \( \eta \). (We could obtain a larger range in \( \eta \) by allowing \( \nu_e \) and either \( \nu_\mu \) or \( \nu_\tau \) to vary independently, but we would still expect a narrower allowed range than in reference \( 4 \) because of the improved observational limits).

Now we proceed to the case of inhomogeneous degeneracy. As we have noted previously, there is no well-defined mean final value of \( \xi \) in this case, since the neutrinos mix at late times to produce a non-thermal distribution. However, the mean final value of \( \rho_\nu \) is still well-defined, so we can attempt to constrain it with BBN. Consider first the case of \( \xi_\mu \gg \xi_\mu, \xi_\tau \). In this case, all of the element abundances go to zero in the limit of large \( \xi_\mu \). Thus, if we take \( f(\xi_\mu) \) to have the form \( f(\xi_\mu = 0) \approx 1 \), and \( f(\xi_\mu = \xi_\mu) = f_0 \ll 1 \), where \( f_0 \) is sufficiently small, then as we take the limit where \( \xi_\mu \to \infty \), the element abundances approach their values in the standard nondegenerate model, while \( \rho_\nu \) goes to infinity. Thus, in the case of inhomogeneous \( \nu_\mu \) degeneracy, there is no BBN limit on \( \rho_\nu \). Of course, there are other cosmological limits on \( \rho_\nu \) in this case, from the requirement that structure formation not be disrupted by the extra radiation \( 1 \) and that the extra radiation not distort the CMB fluctuation spectrum \( 18 \). Our argument also applies to the case where all three neutrinos have equal chemical potentials.

There are still interesting limits to be placed on \( \eta \). To determine these limits, we calculated the BBN element abundances for a grid of values of \( \xi \). We took \( \xi \) in steps of \( \Delta \xi = 1.0 \) between \( \xi = -60 \) and \( \xi = 10 \). We embedded a smaller grid between \( \xi = -1.0 \) and \( \xi = 1.0 \) in steps of \( \Delta \xi = 0.05 \). In calculating the element abundances for
the degenerate case, we used the approximation given in reference [14] for the decrease in the neutrino temperature at large $\xi$. Although rough, this approximation is adequate for our purposes.

For the case $\xi_\epsilon \gg \xi_\nu, \xi_\tau$, we find acceptable solutions for $\eta$ in the range

$$3.0 \times 10^{-10} \leq \eta \leq 1.1 \times 10^{-8},$$

(16)

while for the case $\xi_\epsilon = \xi_\nu = \xi_\tau$, we have

$$3.1 \times 10^{-10} \leq \eta \leq 1.0 \times 10^{-9}.$$

(17)

The actual $\xi$ values of the non-zero bins, along with the corresponding values for $p_j$ and $\rho'_\nu/\rho_\nu$ are given in Tables I and II. Note that our linear programming method will always yield a final optimal distribution for the $p_j$’s in which at most seven of the bins are non-zero (since equations (4) and (14) correspond to a total of seven constraint equations); effectively, this corresponds to a final distribution for $f(\xi)$ which is a sum of at most seven delta functions (see references [16,20] for a more detailed discussion).

We see that allowing for a free distribution of the degeneracies significantly increases the upper bound on $\eta$, particularly for the case of $\xi_\epsilon \gg \xi_\nu, \xi_\tau$, but decreases the lower bound only slightly. Furthermore, the minimum increase in the neutrino density needed to achieve these lower bounds is inconsistent with both structure formation considerations [9] and CMB observations [13]. On the other hand, the value of $\rho'_\nu/\rho_\nu$ needed to achieve the upper bounds on $\eta$ is well within the regime allowed by both structure formation and the CMB.

**IV. DISCUSSION**

Our results indicate that, not surprisingly, the introduction of inhomogeneous neutrino degeneracy allows for a much wider range of $\eta$ within the constraints of BBN. Current limits on the primordial element abundances are so tight that even models with homogeneous degeneracy are tightly constrained. Similarly, inhomogeneous neutrino degeneracy does not allow for a significant decrease in $\eta$, and such models tend to give a neutrino energy density in conflict with other cosmological limits. It is quite impressive that even with the radical model discussed here, the limits on the primordial element abundances have become so tight that a significantly lower value of $\eta$ cannot be achieved. On the other hand, inhomogeneous neutrino degeneracy can increase the upper bound on $\eta$ to quite large values: up to $\eta = 1.0 \times 10^{-9}$ for the case of equal degeneracies in all three neutrinos, and $\eta = 1.1 \times 10^{-8}$ if only $\nu_\epsilon$ is degenerate. These correspond to $\Omega_\nu h^2 = 0.036$ and 0.40, respectively.

The distributions of $\xi$ which produce these extreme values for $\eta$ do not correspond to physically likely models. The usefulness of our linear programming calculation is that it allows us to establish upper and lower bounds on $\eta$ for arbitrary distributions of $\xi$, while at the same time giving the smallest value for $\rho'_\nu/\rho_\nu$ corresponding to a given value of $\eta$. Any other distribution $f(\xi)$ is guaranteed to give values of $\eta$ which lie inside of our bounds.

This work does not exhaust the possible models with spatially-varying $\xi$. It is possible to use our methodology to investigate models in which two or three neutrino degeneracies are independent. In addition, if baryogenesis is related to the lepton number in some way [13], then one would expect a correlation between $\eta$ and $\xi$ at each point in space. Given a specification for this correlation, such models could also be examined within the framework we have outlined here. Less general but more physically realistic distributions for $f(\xi)$ (e.g. a Gaussian distribution) could also be considered.

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TABLE I. The nonzero values of $\xi_j$, along with the corresponding $p_j$ and total $\rho_\nu'/\rho_\nu$ for the case where $\xi = \xi_e = \xi_\mu = \xi_\tau$.

| $\eta$          | $\xi_j$ | $p_j$ | $\rho_\nu'/\rho_\nu$ |
|-----------------|--------|------|---------------------|
| $3.0 \times 10^{-10}$ | -0.8  | 0.49 | 31                  |
|                 | 8      | 0.03 |                     |
|                 | 9      | 0.48 |                     |
| $1.1 \times 10^{-8}$ | -37   | $7.3 \times 10^{-4}$ | 4.3              |
|                 | -2     | 0.23 |                     |
|                 | 3      | 0.30 |                     |
|                 | 4      | 0.47 |                     |

TABLE II. The nonzero values of $\xi_j$, along with the corresponding $p_j$ and total $\rho_\nu'/\rho_\nu$ for the case where $\xi = \xi_e = \xi_\mu = \xi_\tau$.

| $\eta$          | $\xi_j$ | $p_j$ | $\rho_\nu'/\rho_\nu$ |
|-----------------|--------|------|---------------------|
| $3.1 \times 10^{-10}$ | -0.7  | 0.52 | 103                  |
|                 | 9      | 0.30 |                     |
|                 | 10     | 0.18 |                     |
| $1.0 \times 10^{-9}$ | -25   | $9.3 \times 10^{-5}$ | 3.2              |
|                 | -1     | 0.37 |                     |
|                 | 2      | 0.45 |                     |
|                 | 3      | 0.18 |                     |

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