Parallel transport on non-Abelian flux tubes

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I propose a way of unambiguously parallel transporting fields on non-Abelian flux tubes, or strings, by means of two gauge fields. One gauge field transports along the tube, while the other transports normal to the tube. Ambiguity is removed by imposing an integrability condition on the pair of fields. The construction leads to a gauge theory of mathematical objects known as Lie 2-groups, which are known to result also from the parallel transport of the flux tubes themselves. The integrability condition is also shown to be equivalent to the assumption that parallel transport along nearby string configurations are equal up to arbitrary gauge transformations. Attempts to implement this condition in a field theory leads to effective actions for two-form fields.

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Just as a gauge field couples to the world line of a charged particle, a two-form field couples to the world sheet of a string. But unlike particles with non-Abelian charge, which couple to gauge fields, non-Abelian strings or flux tubes pose a mathematical hurdle. The crux of the problem is to consistently define a notion of surface holonomy, i.e., to assign elements of a group to surfaces swept out by parallel transport of these strings. Simply put, the holonomy of a surface is trivial unless the holonomy group is Abelian [1], essentially because there is no canonical notion of surface ordering. Recently there have been fresh attempts to overcome this problem by circumventing it. The proposal is that the proper geometrical setting for non-Abelian string dynamics is a category theoretic generalization of principal fiber bundles called gerbes [2]. Further, while the symmetry transformations of fields which describe particles form groups, for non-Abelian strings a more generalized notion of symmetry is needed, encoded in structures known as Lie 2-groups or Lie crossed modules. Different authors have given descriptions of these structures [3, 4, 5, 6, 7, 8, 9].

In these discussions, categorical groups and gerbes have been associated with two-form gauge fields. For example, in the 2-group construction of surface holonomy, a one-form field and a two-form field, valued in the two Lie algebras, are introduced for parallel transport of a charged string. The two-form couples to the face, while the one-form couples to the edges, for a holonomy valued in the Lie 2-group. This construction leads to a solution of surface holonomy problem [4, 5]. Non-Abelian two-form fields also appear in string theory when two or more D-branes coincide. The geometrical setting for these fields is provided by gerbes, which are like principal fiber bundles of categorical groups [4]. Another relevant example is that of discrete torsion. The action of the orbifold group on the two-form field in string theory gives rise to discrete torsion. The contribution of this to the string partition function comes from the holonomy of the two-form [10].

In this note I point out that Lie 2-groups, which at first sight may seem to be rather esoteric mathematical objects, appear very naturally in any theory with charged one-dimensional objects like flux tubes or strings. It is not necessary to start with a two-form field and construct parallel transport of strings, i.e. surface holonomy, as has been done so far. My approach will be to consider instead parallel transport of fields, both along and between, nearby flux tube configurations. I will show that the usual notion of holonomy along paths, applied to paths along and between flux tubes, give rise to Lie 2-groups. Since fields and their parallel transport by gauge fields are ubiquitous in physics, this construction may be useful in many different contexts where flux tubes and strings appear in field theory, not only in string theory, and not only if a two-form field is present a priori.

Even though I do not consider parallel transporting a string itself, a two-form field will appear naturally as a Lagrange multiplier in this construction. A simple argument, typical in quantum theory, then produces an action for this field. I will also show that the composition of parallel transport of fields living on strings follows the rules of a Lie 2-group, or more precisely those of a Lie crossed module, in exactly the same manner as the composition of parallel transport of the strings themselves. In what follows, I will use the terms Lie 2-group and Lie crossed module interchangeably.

What is a Lie crossed module? In a simplified definition, it consists of two Lie groups \( G \) and \( H \), with a homomorphism \( t : H \to G \) and an action of \( G \) on \( H \), which may be written as \( g h g^{-1} \), where \( g \in G \) and \( h \in H \). The homomorphism is required to preserve this action, i.e.

\[
t(ghg^{-1}) = gt(h)g^{-1}, \quad t(h)h't(h)^{-1} = hh'h^{-1}.
\]

Roughly speaking, the idea is to associate elements of
$G$ and $H$ to string configurations such that an element $g \in G$ is ‘along’ the string configuration in some sense, while an element $h \in H$ ‘compares’ two configurations between the same endpoints. I will loosely call this a string ‘carrying’ $(h, g)$, keeping in mind that actually two configurations are needed to define $(h, g)$. Then composition of paths along the string (horizontal composition) corresponds to multiplication in $G$, while composition of paths in the space of string configurations with fixed endpoints (vertical composition) corresponds to group composition in $H$. \[ (h_1, g_1) \cdot (h_2, g_2) = (h_1 h_2 g_1^{-1}, g_1 g_2). \] (2)

Vertical composition of one configuration carrying $(h, g)$ with another between the same endpoints and carrying $(h', g')$, which is defined if and only if $g' = t(h) g$, is given by

\[ (h, g) \circ (h', g') = (h'h, g). \] (3)

The transition from geometry to physics is natural when associating paths to group elements. This is done by taking a gauge field, or connection, valued in the corresponding Lie algebra, and calculating its (untraced) holonomy, or ‘path-ordered exponential’, along the path. In order to use this procedure for strings, I will think of horizontal composition as composition of holonomies along the string, and vertical composition as composition for holonomies between two string configurations.

For composition of strings carrying two groups, I could then think of two gauge fields attached to the string. One of the gauge fields parallel transports along the string, while the other parallel transports between string configurations. This is different from the picture involving Lie crossed modules, in which the recent attempts to define parallel transport associated an element of $H$ with the surface between two string configurations, by introducing a two-form gauge field which transports along surfaces. Then constructing a gauge theory for Lie crossed modules is the same as giving a definition of surface holonomy, because composition of elements of Lie 2-groups can be made equivalent to composition of holonomies along both surfaces and edges. It will be seen that the holonomies of the two gauge fields produce the same structure of a Lie 2-group. For the moment, I will ignore the 2-group construction of surface holonomy, and derive results in terms of gauge fields, coming back to the similarity at the end.

I will restrict my considerations to infinitesimal parallel transports, i.e. holonomies along infinitesimal paths. Parallel transports are done along the edges of an infinitesimal square, in which the top and bottom edges belong to infinitesimally close configurations of the string, while the side edges are infinitesimal paths connecting the configurations. Corresponding group elements are of the generic form $\exp(\epsilon A) \sim 1 + \epsilon A$, where $\epsilon$ is the length (vector) of an edge. Parallel transport from one corner of the square to another corner should be unambiguous.

If there were only one gauge field present, all parallel transports would be done with the help of this field $A$. Then parallel transporting a field around an infinitesimal square produces a term proportional to the curvature $F = dA + A \wedge A$. If $P_{ab}$ denotes the operator for parallel transport from $a$ to $b$ in Fig. 1, $P_{24}P_{12} = P_{34}P_{13}$ implies that $F = 0$. This is the well-known integrability condition — removing the ambiguity in parallel transport forces the connection $A$ to be flat. An equivalent way of stating this is that if it does not matter where the string is, i.e., if parallel transporting between two points along one path gives the same result as parallel transporting along an infinitesimally near path, $A$ must be flat. Then the corresponding gauge theory is one of flat connections only. However, motivated by the earlier discussion that non-Abelian symmetries of strings has the structure of Lie 2-groups, I can relax my assumptions slightly.

![FIG. 1: Reference loop for doing parallel transports.](image)

Given a smooth string, it is possible to distinguish edges which lie along it from edges which are normal to it. Let parallel transports along the string be done with the help of the gauge field $A$, and let parallel transports normal to the string be done with a different gauge field $\bar{A}$. To begin with, suppose the two Lie algebras are identical. Then I can write $\bar{A} = A + V$, where $V$ is some one-form belonging to the same Lie algebra as $\bar{A}$. The effect of parallel transporting around an infinitesimal square is then no longer only the curvature. A simple calculation shows that removing the ambiguity in parallel transporting around an infinitesimal square requires

\[ F + \frac{1}{2} d_A V \equiv F + \frac{1}{2} (dV + A \wedge V + V \wedge A) = 0. \] (4)

Another way of thinking about this result is to consider, as before, parallel transport along the bottom edge of the square. A weaker version of the integrability condition is that going around the other three edges gives the same result, but only up to gauge transformations. In the notation used earlier, this condition can be written as $(P_{24})^{-1}P_{34}P_{13} = a^{-1}P_{12}a$, for some group element $a$, where now all the parallel transports are done with the
same connection $A$. Since all the paths are infinitesimal, and since $a$ should be the identity when the side edges are collapsed so that the top and bottom edges coincide, I can write $a \sim 1 + \epsilon \cdot V$. This leads to Eq. (4), which determines $A$ in terms of $V$ in principle. In this picture, $V$ is not a connection, but an arbitrary one-form field valued in the Lie algebra.

Note that even though the two connections $A$ and $\hat{A}$ (or the fields $A$ and $V$ in the alternative picture above) turned out to be related, the reason for working with two connections has not disappeared. If only one connection $A$ were used to describe the parallel transport, I would get $F = 0$ everywhere, making parallel transport a trivial operation. However, there is a priori no reason to use only one connection, as there are two inequivalent directions on the ‘soap bubble’ between two configurations of the flux tube. So there is no reason why parallel transport must be trivial, since I can now use two different connections in the two directions. Then the parallel transport is encoded in the pair $(h, g)$ which comes from path-ordered exponentials of these two connections along the two directions. The collection of these pairs form a Lie 2-group, as will be discussed in detail later. The ‘alternative picture’ above has a slightly different justification, that parallel transport along two configurations should be gauge equivalent rather than equal. This leads to the same expression for parallel transport.

Note also that $a$ need not be in the same group as the holonomies of $A$, but can live in a subgroup of the latter. If this is a normal subgroup, generators of this group commute with the remaining generators of the Lie algebra. Then Eq. (4) is satisfied only in this subalgebra, while the connection is flat along the other generators. In this case, and even in general, I ought to consider isomorphisms of one algebra with an invariant subalgebra of the other, rather than blithely equating objects in different spaces. That is, I really ought to write $dt(V)$ instead of $V$ in Eq. (4) and $(t(h), g)$ instead of $(h, g)$ for the parallel transport. This notation will allow comparison with the literature easier, but I will continue to keep the isomorphism implicit to avoid cluttering equations.

The condition in Eq. (4) is a constraint on the fields in the background on which the flux tube propagates. Any effective field theory of these fields will have to incorporate the constraint. So it helps to think of the constraint as a statement of gauge equivalence of parallel transports along nearby configurations of a flux tube. For, now the integrability condition can be stated as follows. Parallel transports along two nearby string configurations, which share the same endpoints, are equal up to arbitrary gauge transformations. The adjoint vector field $V$, which exponentiates to the group element $a$, may have any configuration allowed by Eq. (4). Since there is no reason a priori to prefer any one configuration, I should sum over all possible configurations of $V$.

Different field configurations can be summed over in a path integral of the form

$$Z = \int \mathcal{D}A \mathcal{D}V \delta \left[ F + \frac{1}{2} d_A V \right] e^{iI}.$$  \hspace{1cm} (5)

Here the $\delta$-functional enforces the constraint, and $I$ is some action which can involve both $V$ and the gauge field $A$. To begin with, let me rewrite the $\delta$-functional using a Lagrange multiplier field $B$, to get

$$Z = \int \mathcal{D}A \mathcal{D}V \mathcal{D}B \ e^{iI + i \int B \wedge (F + \frac{1}{2} d_A V)}, \hspace{1cm} (6)$$

The field $B$ is a two-form in four dimensions, and integrating over $V$ will lead to an effective action for the non-Abelian two-form $B$. Geometrical considerations, as discussed so far, do not lead to a specific form of the action $I$. Different choices for the action $I$ will obviously lead to different field theories involving $B$.

For example, suppose I choose the simplest possible action, namely that of pure gauge theory. Then the path integral takes the form

$$Z = \int \mathcal{D}A \mathcal{D}V \mathcal{D}B \ e^{-i \int \frac{1}{4} F^2 - B \wedge (F + \frac{1}{2} d_A V)}, \hspace{1cm} (7)$$

where $F^2 = F \wedge F$, and gauge indices have been summed over and hidden. $V$ can be integrated out from this, producing a constraint $d_A B = 0$. The Lagrangian of the resulting effective theory is $\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^2 - B \wedge F \wedge F$. This is the usual Lagrangian for topological field theory in four dimensions, as $F$ can be set to zero using the equation of motion for $B$, and the constraint $d_A B = 0$ is then the equation of motion for $A$. The same effective action results from setting $V = 0$.

Another interesting effective action appears if I note that instead of summing over all configurations of $V$ with equal weight, I could choose to take a Gaussian average. That is, I could assume that $V$ is peaked around zero, so that the connection for parallel transporting a field away from the string is peaked around the value of the connection for parallel transporting along the string. The averaging is done by adding a term proportional to $V^2$ to the exponent in the path integral. Then the partition function is

$$Z = \int \mathcal{D}A \mathcal{D}V \mathcal{D}B \ e^{-i \int \frac{1}{4} F^2 + \frac{1}{2} m^2 V^2 - B \wedge (F + \frac{1}{2} d_A V)}, \hspace{1cm} (8)$$

where I have introduced a constant $m$ of mass dimension one so that all terms have the same dimension. Integrating over $V$ shows that this is the partition function for the action

$$I_{\text{eff}} = \int -\frac{1}{2} F \wedge * F - \frac{1}{2} H \wedge * H + mB \wedge F, \hspace{1cm} (9)$$

where $B$ has been redefined as $mB$, and $H = d_A B$.

These actions should not come as a surprise. Some theory of two-form fields is expected in the effective
field theory of string backgrounds from general considerations. For example, the effective action for ‘global’
vortex strings is that of a free two-form field, while cosmic strings attached to magnetic monopoles give rise to
an effective action of the form Eq. (11). Similar actions also arise from generalizing the notion of a
fiber bundle, which describes the geometry of interactions of point particles, to a category expected to play
a similar role for strings. It is possible to introduce the pair \((A, B')\) as the generalized connection on a ‘princi-
pal 2-bundle’ \([6]\), where \(A\) is valued in the algebra \(\mathcal{G}\) and \(B'\) is valued in the algebra \(\mathcal{H}\). The one-form \(A\) provides
holonomies along edges, while the two-form \(B'\) provides holonomies for surfaces. Then one can write actions simi-
lar to the ones above. Here I have shown that at least in four dimensions, even if transport of strings is not in-
cluded in the original construction, an effective action for a non-Abelian two-form appears naturally. I have used a
prime to distinguish between the \(B\)-field in that picture from the one described above. It is important to note
that the \(B\)-fields in the two pictures do not seem to be related by a local transformation.

\[ \begin{array}{c}
\text{1} \\
\text{2}
\end{array} \xrightarrow{\text{dragging down}} \begin{array}{c}
\text{1} \\
\text{2}
\end{array} \]

FIG. 2: Diagonal composition as horizontal composition.

So far in my construction, the references to Lie 2-
groups, Lie crossed modules and surface holonomy may
have looked somewhat arbitrary or far-fetched. It is time
to establish a closer relationship. First note that par-
allel transport around an infinitesimal square is defined
by two objects, one for the horizontal transport and one
for the vertical transport. Eq. (4) ensures that parallel
transport of a field from any corner to any other is unam-
biguous. When two squares are joined, any paths may be
taken from one corner to another, but some paths make
it easier to compare the construction given here with the
Lie 2-group appearing in surface holonomy.

Let me associate a group element \(g \in \mathcal{G}\) to dragging
left along the top edge and \(h \in \mathcal{H}\) to dragging down along
the left edge. Then parallel transport from the top right
corner to the bottom left corner of a square with \((h, g)\)
requires the combination \(hg\), which is well-defined if the
two Lie algebras are isomorphic, or if one is a subalge-
bra of the other. If two squares are joined at a corner as
in Fig. 2 parallel transport from the top right corner to
the bottom left corner is done by \(h_1 g_1 h_2 g_2\), which is the
same as that for a square with \((h_1 g_1 h_2 g_1^{-1}, g_1 g_2)\). This
operation of joining at a corner corresponds to horizontal
composition of bi-gons as described in Refs. [6, 8]. If two
squares are joined along an edge, say a horizontal edge as
in Fig. 3 dragging left and down requires a parallel trans-
port of \(h_2 h_1 g_1\), same as a rectangle with \((h_2 h_1, g_1)\). This
operation corresponds to vertical composition of bi-gons.
Comparison with Eq. (2) and (3) shows that the con-
struction in terms of two connections is indeed a ‘gauge
theory’ of Lie 2-groups. In fact, these pictures and corre-
sponding composition rules are the same as those given in
Refs. [6, 8, 9], although the gauge theories obtained here
are not exactly the same as those found there, where the
two-form was coupled directly to the area element.

As I mentioned at the beginning, the total holonomy
for a surface is always trivial, or Abelian, for all these con-
structions because otherwise surface ordering becomes
ambiguous. The real issue is to find a way of constructing
a non-Abelian surface holonomy in which the gauge field
is not pure gauge. In the construction where a two-form
\(B'\) couples to the area element and a 1-form \(A\) couples to the
edges, the total holonomy for transporting a string
along an infinitesimal rectangle is \(F + B'\), which has to
vanish if surface ordering is to be unambiguous. In the
construction with two connections, the total holon-
omy for transporting a field around the same rectangle is
\(F + \frac{1}{2}d_A V\), which can vanish even when the connection
\(A\) is not flat.

To sum up, I have constructed a gauge theory of Lie
2-groups, or Lie crossed modules, by considering parallel
transport of fields around an infinitesimal rectangle with
distinguished edges (string configurations). Fields are
transported along the string and away from the string
using different connections \(A\) and \(A\), but the result of transpor-
ting around a loop is still unambiguous provided
Eq. (4) is satisfied. The same condition is required if
parallel transport along two infinitesimally near config-
urations of an infinitesimal string are gauge equivalent.
Summing over contributions from all such pairs of con-
nections (or gauge transformations) leads to an effective
field theory of non-Abelian two-forms. In the absence of
strings, there are no distinguished edges, thus no integra-

FIG. 3: Vertical composition.
This suggests that in the presence of strings or flux tubes of usual gauge theory, the effective degrees of freedom are encoded in the gauge theory of a Lie crossed module.

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