Second order asymptotics in fixed-length source coding and intrinsic randomness
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Abstract—Second order asymptotics of fixed-length source coding and intrinsic randomness is discussed with a constant error constraint. There was a difference between optimal rates of fixed-length source coding and intrinsic randomness, which never occurred in the first order asymptotics. In addition, the relation between uniform distribution and compressed data is discussed based on this fact. These results are valid for general information sources as well as independent and identical distributions. A universal code attaining the second order optimal rate is also constructed.

Index Terms—Second order asymptotics, Fixed-length source coding, Intrinsic randomness, Information spectrum, Folklore for source coding

I. INTRODUCTION

Many researchers believe that any sufficiently compressed data approaches a uniform random number. This conjecture is called Folklore for source coding (Han [1]). The main reason for this conjecture seems to be the fact that the optimal limits of both rates coincide with the entropy rate: that is, the optimal compression length equals the optimal length of intrinsic randomness (uniform random number generation) in the asymptotic first order. There is, however, no research comparing them in the asymptotic second order even though some researchers treat the second order asymptotics for variable-length source coding [2], [3]. In this paper, taking account of the asymptotic second order, we compare them in the case of the general information source in the fixed-length setting. Especially, we show by application to the case of the independent and identical distribution (i.i.d.), that the size of compression is greater than the one of intrinsic randomness with respect to the asymptotic second order. This fact implies that data generated by the fixed-length source coding is not a uniform random number.

Details of the above discussion are as follows. The size of generated data is one of the main points in data compression and intrinsic randomness. In the asymptotic setting, by approximating the size $M_n$ as $M_n \cong e^{na}$, we usually focus on the exponential component (exponent) $a$. Smaller size is better in data compression, but larger size is better in intrinsic randomness. Both optimal exponents $a$ coincide. However, as will be shown in this paper, the size $M_n$ can be approximated as $M_n \cong e^{na+\sqrt{n}}$. In this paper, we call the issue concerning the coefficient $a$ of $n$ the first order asymptotics, and the issue concerning the coefficient $b$ of $\sqrt{n}$ the second order asymptotics. When the information source is the independent and identical distribution $P^n$ of a probability distribution $P$, the optimal first coefficient is the entropy $H(P)$ in both settings. In this paper, we treat the optimization of the second coefficient $b$ for general information sources. In particular, we treat intrinsic randomness by using half of the variational distance. These two coefficients do not coincide with each other in many cases. In particular, these optimal second order coefficients depend on the allowable error even in the i.i.d. case. (Conversely, it is known that these optimal first order coefficients are independent of the allowable error in the i.i.d. case when the allowable error is constant.) If the allowable error is less than $1/2$, the optimal second order coefficient for source coding is strictly larger than the one for intrinsic randomness. As a consequence, when the constraint error for source coding is sufficiently small, the compressed random number is different from the uniform random number. Hence, there exists a trade-off relation between the error of compression and the error of intrinsic randomness.

However, Han [1], [4], [5] showed that the compressed data achieving the optimal rate is ‘almost’ uniform random at least in the i.i.d. case in the fixed-length compression. Visweswariah et al. [6] and Han & Uchida [7] also treated a similar problem in the variable-length setting. One may think that Han’s result contradicts our result, but there is no contradiction. This is because Han’s error criterion between the obtained distribution and the true uniform distribution is based on normalized KL-divergence [30], and is not as restrictive as our criterion. Thus, the distribution of the compressed data may not be different from the uniform distribution under our criterion even if it is ‘almost’ the uniform distribution under his criterion. Indeed, Han [5] has already stated in his conclusion that if we adopt the variational distance, the compressed data is different from the uniform random number in the case of the stationary ergodic source. However, in this paper, using the results of second order asymptotics, we succeeded in deriving the tight trade-off relation between the variational distance from the uniform distribution and decoding error probability of the fixed-length compression in the asymptotic setting. Further, when we adopt KL-divergence divided by $\sqrt{n}$ instead of normalized KL-divergence, the compressed data is different from the uniform random number. Hence, the speed of convergence of normalized KL-divergence to 0 is essential.

In this paper, we use the information spectrum method mainly formulated by Han[4]. We treat the general information source, which is the general sequence $\{p_n\}$ of probability distributions without structure. This method enables us to characterize the asymptotic performance only with the random variable $\frac{1}{n} \log p_n$ (the logarithm of likelihood) without any
further assumption. In order to treat the i.i.d. case based on the above general result, it is sufficient to calculate the asymptotic behavior of the random variable $\frac{1}{n} \log p_n$. Moreover, the information spectrum method leads us to treat the second order asymptotics in a manner parallel to the first order asymptotics, whose large part is known. That is, if we can suitably formulate theorems in the second order asymptotics and establish an appropriate relation between the first order asymptotics and the second order asymptotics, we can easily extend proofs concerning the first order asymptotics to those of the second order asymptotics. This is because the technique used in the information spectrum method is quite universal. Thus, the discussion of the first order asymptotics plays an important role in our proof of some important theorems in the second order asymptotics. Therefore, we give proofs of some theorems in the first order asymptotics even though they are known. This treatment produces short proofs of main theorems for the second order asymptotics with reference to the corresponding proofs on the first order asymptotics.

While we referred the i.i.d. case in the above discussion, the Markovian case also has a similar asymptotic structure. That is, the limiting distribution of the logarithm of likelihood is equal to normal distribution. Hence, we have the same conclusion concerning Folklore for source coding in the Markovian case. Moreover, we construct a fixed-length source code that attains the optimal rate up to the second order asymptotics, i.e., a universal fixed-length source code. We also prove the existence of a similar universal operation for intrinsic randomness. Further, in Section VI-A, we derived the optimal generation rate of intrinsic randomness under the constant constraint concerning the normalized KL-divergence, which was mentioned as an open problem in Han’s textbook[4].

Finally, we should remark that the second order asymptotics correspond to the central limit theorem in the i.i.d. case while the first order asymptotics corresponds to the law of large numbers. But, in statistics, the first order asymptotics corresponds to the central limit theorem. Concerning variable-length source coding, the second order asymptotics corresponds to the central limit theorem, but its order is $\log n$. As seen in sections VIII and IX, the application of this theorem to variable- and fixed-length source coding is different.

This paper is organized as follows. We explain some notations for the information spectrum method in the first and the second order asymptotics in section II. We treat the first order asymptotics of fixed-length source coding and intrinsic randomness based on variational distance in section III, some of which are known. For the comparisons with several preceding results, we treat several versions of the optimal rate in this section. The second order asymptotics in both settings are discussed as the main result in section IV. We discuss the relation between the second order asymptotics and Folklore for source coding based on variational distance in section V. In addition, we discuss intrinsic randomness based on KL-divergence, and the relation between Han[4]'s criterion and the second order asymptotics in section VI. For comparison with Han[4]'s result, we treat intrinsic randomness under another KL-divergence criterion, in which the input distributions of KL-divergence are exchanged. In section VII, the Markovian case is discussed. A universal fixed-length source code and a universal operation for intrinsic randomness are treated in section VIII. All proofs are given in section IX.

II. NOTATIONS OF INFORMATION SPECTRUM

In this paper, we treat general information source. Through this treatment, we can understand the essential properties of problems discussed in this paper. First, we focus on a sequence of probability spaces $\{\Omega_n\}_{n=1}^\infty$ and a sequence of probability distributions $p \defeq \{p_n\}_{n=1}^\infty$ on them. The asymptotic behavior of the the logarithm of likelihood can be characterized by the following known quantities

$$H(\epsilon|\bar{p}) \defeq \inf_a \left\{ \frac{1}{n} \log p_n(\omega) < a \geq \epsilon \right\}$$

$$= \sup_a \left\{ \frac{1}{n} \log p_n(\omega) < a < \epsilon \right\},$$

$$\overline{H}(\epsilon|\bar{p}) \defeq \inf_a \left\{ \frac{1}{n} \log p_n(\omega) < a \geq \epsilon \right\}$$

$$= \sup_a \left\{ \frac{1}{n} \log p_n(\omega) < a < \epsilon \right\},$$

for $0 < \epsilon \leq 1$, and

$$H_+(\epsilon|\bar{p}) \defeq \inf_a \left\{ \frac{1}{n} \log p_n(\omega) < a > \epsilon \right\}$$

$$= \sup_a \left\{ \frac{1}{n} \log p_n(\omega) < a \leq \epsilon \right\},$$

$$\overline{H}_+(\epsilon|\bar{p}) \defeq \inf_a \left\{ \frac{1}{n} \log p_n(\omega) < a > \epsilon \right\}$$

$$= \sup_a \left\{ \frac{1}{n} \log p_n(\omega) < a \leq \epsilon \right\},$$

for $0 \leq \epsilon < 1$, where the $\omega$ is an element of the probability space $\Omega_n$.

For example, when the probability $p_n$ is the $n$-th independent and identical distribution (i.i.d.) $P^n$ of $P$, the law of large numbers guarantees that these quantities coincide with entropy $H(P) \defeq -\sum_\omega p(\omega) \log p(\omega)$. Therefore, for a more detailed description of asymptotic behavior, we introduce the following quantities.

$$H(\epsilon, a|\bar{p}) \defeq \inf_b \left\{ \frac{1}{n} \log p_n(\omega_n) < a + \frac{b}{\sqrt{n}} \right\} < \epsilon \right\},$$

$$\overline{H}(\epsilon, a|\bar{p}) \defeq \inf_b \left\{ \frac{1}{n} \log p_n(\omega_n) < a + \frac{b}{\sqrt{n}} \right\} > \epsilon \right\}$$

$$= \sup_b \left\{ \frac{1}{n} \log p_n(\omega_n) < a + \frac{b}{\sqrt{n}} \right\}< \epsilon \right\},$$

for $0 < \epsilon \leq 1$. Similarly, $H_+(\epsilon, a|\bar{p})$ and $\overline{H}_+(\epsilon, a|\bar{p})$ are defined for $0 \leq \epsilon < 1$. When the distribution $p_n$ is the i.i.d. $P^n$ of $P$, the central limit theorem guarantees that $\sqrt{n}(-\frac{1}{2} \log P^n(\omega_n) - H(P))$ obeys the normal distribution with expectation 0 and variance $V_P = \sum_\omega p(\omega)(-\log P(\omega) - H(P))^2$. Therefore, by using the distribution function $\Phi$ for the standard normal distribution (with expectation 0 and the variance 1):

$$\Phi(x) \defeq \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx,$$
we can express the above quantities as follows:

\[ H(\epsilon, H(P)|\mathcal{P}) = \overline{H}(\epsilon, H(P)|\mathcal{P}) \]

\[ = H_+(\epsilon, H(P)|\mathcal{P}) = \overline{H}(\epsilon, H(P)|\mathcal{P}) = \sqrt{V_P} \Phi^{-1}(\epsilon), \quad (1) \]

where \( \mathcal{P} = \{p^n\} \).

In the following, we discuss the relation between the above mentioned quantities, fixed-length source coding, and intrinsic randomness.

\[ \Omega_n = \{\omega \in \Omega_n | \psi_n \circ \phi_n(\omega) \neq \omega\} \]

The operation recovering the original output \( \omega \) from the element of \( \mathcal{M}_n \) is described by a map \( \phi_n : \Omega_n \rightarrow \mathcal{M}_n \), which is called encoding.

\[ \epsilon_p_n(\Phi_n) \overset{\text{def}}{=} p_n\{\omega \in \Omega_n | \psi_n \circ \phi_n(\omega) \neq \omega\}. \]

When we do not need to express the distribution of information source \( p_n \), we simplify \( \epsilon_p_n(\Phi_n) \) to \( \epsilon(\Phi_n) \). In order to discuss the asymptotic bound of compression rate under the constant constraint on the error probability, we focus on the following values:

\[ R(\epsilon|\mathcal{P}) \overset{\text{def}}{=} \inf_{\Phi_n} \left\{ \lim_{n} \frac{1}{n} \log |\Phi_n| \mid \lim_{n} \epsilon(\Phi_n) \leq \epsilon, \forall n \right\}, \]

\[ R^l(\epsilon|\mathcal{P}) \overset{\text{def}}{=} \inf_{\Phi_n} \left\{ \lim_{n} \frac{1}{n} \log |\Phi_n| \mid \lim_{n} \epsilon(\Phi_n) \leq \epsilon, \forall n \geq N \right\}, \]

for \( 0 < \epsilon < 1 \).

Further, as intermediate quantities, we define

\[ \tilde{R}(\epsilon|\mathcal{P}) \overset{\text{def}}{=} \inf_{\Phi_n} \left\{ \lim_{n} \frac{1}{n} \log |\Phi_n| \mid \exists N \epsilon(\Phi_n) \leq \epsilon, \forall n \geq N \right\}, \]

\[ \tilde{R}^l(\epsilon|\mathcal{P}) \overset{\text{def}}{=} \inf_{\Phi_n} \left\{ \lim_{n} \frac{1}{n} \log |\Phi_n| \mid \epsilon(\Phi_n) \leq \epsilon \right\}, \]

for \( 0 < \epsilon < 1 \). Here, in order to see the relation with existing results, we defined many versions of the optimal coding length. The following relations follow from their definitions:

\[ R(\epsilon|\mathcal{P}) \leq \tilde{R}(\epsilon|\mathcal{P}) \leq R^l(\epsilon|\mathcal{P}), \quad (2) \]

\[ R^l(\epsilon|\mathcal{P}) \leq \tilde{R}^l(\epsilon|\mathcal{P}) \leq R^l(\epsilon|\mathcal{P}), \quad (3) \]

\[ R(\epsilon|\mathcal{P}) \leq \tilde{R}(\epsilon|\mathcal{P}) \leq R^l(\epsilon|\mathcal{P}), \quad (4) \]

for \( 0 < \epsilon < 1 \).

Concerning these quantities, the following theorem holds. Theorem 1: Han[4, Theorem 1.6.1], Steinberg & Verdú[8], Chen & Alajaji [9], Nagaoka & Hayashi [10] The relations

\[ R(1 - \epsilon|\mathcal{P}) = \overline{H}(\epsilon|\mathcal{P}), \quad (5) \]

\[ R^l(1 - \epsilon|\mathcal{P}) = \overline{H}^l(1 - \epsilon|\mathcal{P}) = \overline{H}(\epsilon|\mathcal{P}), \quad (6) \]

hold for \( 0 \leq \epsilon < 1 \), and the relations

\[ R^l(1 - \epsilon|\mathcal{P}) = \overline{H}^l(1 - \epsilon|\mathcal{P}) = \overline{H}^l(\epsilon|\mathcal{P}), \quad (7) \]

\[ R^l(1 - \epsilon|\mathcal{P}) = \overline{H}^l(1 - \epsilon|\mathcal{P}) = \overline{H}^l(\epsilon|\mathcal{P}), \quad (8) \]

hold for \( 0 < \epsilon \leq 1 \).

By using the relations (2), (5), and (6), \( \tilde{R}(\epsilon|\mathcal{P}) \), \( \tilde{R}^l(\epsilon|\mathcal{P}) \), and \( \tilde{R}(\epsilon|\mathcal{P}) \) are characterized as follows.

Corollary 1:

\[ \overline{H}(\epsilon|\mathcal{P}) \leq \tilde{R}(1 - \epsilon|\mathcal{P}) \leq \overline{H}^l(\epsilon|\mathcal{P}), \quad (9) \]

\[ \overline{H}(\epsilon|\mathcal{P}) \leq \tilde{R}^l(1 - \epsilon|\mathcal{P}) \leq \overline{H}^l(\epsilon|\mathcal{P}), \quad (10) \]

\[ \overline{H}(\epsilon|\mathcal{P}) \leq \tilde{R}(1 - \epsilon|\mathcal{P}) \leq \overline{H}^l(\epsilon|\mathcal{P}). \quad (11) \]
Remark 1: Historically, Steinberg & Verdú [8] derived (9), and Chen & Alajaji [9] did [9]. Han [4] proved the equation $R(1 - \epsilon | \mathcal{P}) = \overline{H}(\epsilon | \mathcal{P})$. Following these results, Nagaoka & Hayashi [10] proved $R^t(1 - \epsilon | \mathcal{P}) = H^t + (\epsilon | \mathcal{P})$. Other relations are proved for the first time in this paper.

The bounds $R^t(1 | \mathcal{P})$ and $R^t(1 | \mathcal{P})$ are shortest among the above bounds because $R(\epsilon | \mathcal{P})$, $R^t(\epsilon | \mathcal{P})$, $R^t(\epsilon | \mathcal{P})$, and $R^t(\epsilon | \mathcal{P})$ are not defined for $\epsilon = 1$. Hence, the bounds $R^t(1 | \mathcal{P})$ and $R^t(1 | \mathcal{P})$ are used in the discussion concerning strong converse property.

B. Intrinsic randomness

Next, we consider the problem of constructing approximately the uniform probability distribution from a biased probability distribution $p_n$ on $\Omega_n$. We call this problem intrinsic randomness, and discuss it based on (half) the variational distance in this section. Our operation is described by the pair of size $M_n$ of the target uniform probability distribution and the map $\phi_n$ from $\Omega_n$ to $\mathcal{M}_n = \{1, \ldots, M_n\}$.

Fig. 3. Typical operation of intrinsic randomness

The performance of $\Psi_n = (\mathcal{M}_n, \phi_n)$ is characterized by the size $|\Psi_n| \equal{} M_n$ and a half of the variational distance between the target distribution and the constructed distribution:

$$\epsilon_{p_n}(\Psi_n) \equal{} d(p_n \circ \phi_n^{-1}, p_{U|S}),$$

where $d(p, q) \equal{} \frac{1}{n} \sum \log |p(\omega) - q(\omega)|$ and $p_{U|S}$ is the uniform distribution on $S$. When we do not need to express the distribution of information source, $p_n$, we simplify $\epsilon_{p_n}(\Psi_n)$ to $\epsilon(\Psi_n)$. Under the condition that this distance is less than $\epsilon$, the optimal size is asymptotically characterized as follows:

$$S(\epsilon | \mathcal{P}) \equal{} \lim \frac{1}{n} \log |\Psi_n| \lim \epsilon(\Psi_n) < \epsilon,$$

$$S^t(\epsilon | \mathcal{P}) \equal{} \lim \frac{1}{n} \log |\Psi_n| \lim \epsilon(\Psi_n) < \epsilon,$$

$$S^t(\epsilon | \mathcal{P}) \equal{} \lim \frac{1}{n} \log |\Psi_n| \lim \epsilon(\Psi_n) < \epsilon,$$

for $0 < \epsilon < 1$, and

$$S^+(\epsilon | \mathcal{P}) \equal{} \sup \left\{ \lim \frac{1}{n} \log |\Psi_n| \lim \epsilon(\Psi_n) \leq \epsilon \right\},$$

$$S^+(\epsilon | \mathcal{P}) \equal{} \sup \left\{ \lim \frac{1}{n} \log |\Psi_n| \lim \epsilon(\Psi_n) \leq \epsilon \right\},$$

$$S^+(\epsilon | \mathcal{P}) \equal{} \sup \left\{ \lim \frac{1}{n} \log |\Psi_n| \lim \epsilon(\Psi_n) \leq \epsilon \right\},$$

for $0 \leq \epsilon < 1$. As intermediate quantities,

$$\tilde{S}(\epsilon | \mathcal{P}) \equal{} \sup \left\{ \lim \frac{1}{n} \log |\Psi_n| \epsilon(\Psi_n) \leq \epsilon \right\},$$

$$\tilde{S}^t(\epsilon | \mathcal{P}) \equal{} \sup \left\{ \lim \frac{1}{n} \log |\Psi_n| \epsilon(\Psi_n) \leq \epsilon \right\},$$

are defined for $0 < \epsilon < 1$. Similarly, we obtain the following trivial relations:

$$S(\epsilon | \mathcal{P}) \leq \tilde{S}(\epsilon | \mathcal{P}) \leq S^+(\epsilon | \mathcal{P}),$$

$$S^t(\epsilon | \mathcal{P}) \leq \tilde{S}^t(\epsilon | \mathcal{P}) \leq S^+(\epsilon | \mathcal{P}),$$

$$S^t(\epsilon | \mathcal{P}) \leq \tilde{S}^t(\epsilon | \mathcal{P}) \leq S^+(\epsilon | \mathcal{P}),$$

for $0 < \epsilon < 1$.

These quantities are characterized by the following theorem.

Theorem 2: Han[4, Theorem 2.4.2] The relations

$$S(\epsilon | \mathcal{P}) = H(\epsilon | \mathcal{P}), \quad S^t(\epsilon | \mathcal{P}) = \tilde{S}^t(\epsilon | \mathcal{P}) = \overline{H}(\epsilon | \mathcal{P})$$

hold for $0 < \epsilon \leq 1$, and the relations

$$S^+(\epsilon | \mathcal{P}) = H^t + (\epsilon | \mathcal{P}), \quad S^t(\epsilon | \mathcal{P}) = \tilde{S}^t(\epsilon | \mathcal{P}) = \overline{H}^t + (\epsilon | \mathcal{P})$$

hold for $0 \leq \epsilon < 1$.

Similarly, the following corollary holds.

Corollary 2: The relations

$$\overline{H}(\epsilon | \mathcal{P}) \leq \tilde{S}(\epsilon | \mathcal{P}) \leq H^t + (\epsilon | \mathcal{P}),$$

$$\overline{H}(\epsilon | \mathcal{P}) \leq \tilde{S}^t(\epsilon | \mathcal{P}) \leq \overline{H}^t + (\epsilon | \mathcal{P}),$$

hold for $0 < \epsilon < 1$.

Remark 2: Han[4] proved only the first equation of \[16\]. Other equations are proved for the first time in this paper.

In the following, in order to treat Folklore for source coding, we focus on the operation $\Psi_n = (\mathcal{M}_n, \phi_n)$ defined from the code $\Phi_n = (\mathcal{M}_n, \phi_n, \psi_n)$. For fixed real numbers $\epsilon$ and $\epsilon'$ satisfying $0 \leq \epsilon, \epsilon' < 1$, we consider whether there exist codes $\Phi_n = (\mathcal{M}_n, \phi_n, \psi_n)$ such that

$$\lim \epsilon(\Phi_n) \leq \epsilon, \quad \lim \epsilon(\Psi_n) \leq \epsilon'.$$

If there exists a sequence of codes $\{\Phi_n\}$ satisfying the above conditions, the inequalities

$$H(\epsilon' | \mathcal{P}) = S(\epsilon' | \mathcal{P}) \geq \lim \frac{1}{n} \log M_n \geq H(\epsilon | \mathcal{P}) = H(1 - \epsilon | \mathcal{P}),$$

$$\overline{H}(\epsilon' | \mathcal{P}) = S^t(\epsilon' | \mathcal{P}) \geq \lim \frac{1}{n} \log M_n \geq R^t(\epsilon | \mathcal{P}) = \overline{H}(1 - \epsilon | \mathcal{P})$$

Thus, the necessary condition (22) is satisfied in the case of inequalities:

\[ \max H(\epsilon^2, a \mid \mathcal{P}) \geq H(1 - \epsilon^2, a \mid \mathcal{P}), \quad \max H(\epsilon', a \mid \mathcal{P}) \geq H(1 - \epsilon, a \mid \mathcal{P}). \] (23)

Thus, the necessary condition (22) is satisfied in the case of i.i.d. \( P^n \) because these quantities coincide with the entropy \( H(P) \).

However, the above discussion is not sufficient, because, as is shown based on the second order asymptotics, a stronger necessary condition exists.

IV. SECOND ORDER ASYMPTOTICS

Next, we proceed to the second order asymptotics, which is very useful for obtaining the stronger necessary condition than (22). Since these values \( H(\epsilon^2, a \mid \mathcal{P}), H(\epsilon', a \mid \mathcal{P}) \) are independent of \( \epsilon \) in the i.i.d. case, we introduce the following values for treatment of the dependence of \( \epsilon \):

\[
R(\epsilon, a \mid \mathcal{P}) \overset{\text{def}}{=} \inf_{\Phi_n} \left\{ \lim_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{\| \Phi_n \|_{e^{\alpha_n}}}{} \left| \lim_{n \to \infty} \epsilon(\Phi_n) \leq \epsilon \right. \right\},
\]

\[
R^1(\epsilon, a \mid \mathcal{P}) \overset{\text{def}}{=} \inf_{\Phi_n} \left\{ \lim_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{\| \Phi_n \|_{e^{\alpha_n}}}{} \left| \lim_{n \to \infty} \epsilon(\Phi_n) \leq \epsilon \right. \right\},
\]

\[
R^2(\epsilon, a \mid \mathcal{P}) \overset{\text{def}}{=} \inf_{\Phi_n} \left\{ \lim_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{\| \Phi_n \|_{e^{\alpha_n}}}{} \left| \lim_{n \to \infty} \epsilon(\Phi_n) \leq \epsilon \right. \right\},
\]

for \( 0 \leq \epsilon < 1 \), and

\[
S(\epsilon, a \mid \mathcal{P}) \overset{\text{def}}{=} \sup_{\Phi_n} \left\{ \lim_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{\| \Phi_n \|_{e^{\alpha_n}}}{} \left| \lim_{n \to \infty} \epsilon(\Phi_n) \leq \epsilon \right. \right\},
\]

\[
S^1(\epsilon, a \mid \mathcal{P}) \overset{\text{def}}{=} \sup_{\Phi_n} \left\{ \lim_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{\| \Phi_n \|_{e^{\alpha_n}}}{} \left| \lim_{n \to \infty} \epsilon(\Phi_n) \leq \epsilon \right. \right\},
\]

\[
S^2(\epsilon, a \mid \mathcal{P}) \overset{\text{def}}{=} \sup_{\Phi_n} \left\{ \lim_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{\| \Phi_n \|_{e^{\alpha_n}}}{} \left| \lim_{n \to \infty} \epsilon(\Phi_n) \leq \epsilon \right. \right\},
\]

for \( 0 < \epsilon \leq 1 \). While we can define other quantities \( R_+ (\epsilon, a \mid \mathcal{P}), R^1_+ (\epsilon, a \mid \mathcal{P}), R^2_+ (\epsilon, a \mid \mathcal{P}), S_+ (\epsilon, a \mid \mathcal{P}), S^1_+ (\epsilon, a \mid \mathcal{P}), \) and \( S^2_+ (\epsilon, a \mid \mathcal{P}) \), we treat only the above values in this section. This is because the later values can be treated in a similar way. The following theorem holds.

Theorem 3:

\[
S(\epsilon, a \mid \mathcal{P}) = R^1(1 - \epsilon, a \mid \mathcal{P}) = R^2(1 - \epsilon, a \mid \mathcal{P}) = H(\epsilon, a \mid \mathcal{P}), \]

\[
S^1(\epsilon, a \mid \mathcal{P}) = S^2(\epsilon, a \mid \mathcal{P}) = R(1 - \epsilon, a \mid \mathcal{P}) = \max H(\epsilon', a \mid \mathcal{P}).
\]

Especially, in the case of the i.i.d. \( P^n \), as is characterized in (11), these quantities with \( a = H(P) \) depend on \( \epsilon \).

V. RELATION TO FOLKLORE FOR SOURCE CODING

Next, we apply Theorem 3 to the relation between the code \( \Phi_n = (\mathcal{M}_n, \phi_n, \psi_n) \) and the operation \( \Psi_n = (\mathcal{M}_n, \phi_n) \). When

\[
\max \epsilon(\Phi_n) = \epsilon, \quad \max \epsilon(\Psi_n) = \epsilon',
\]

similar to the previous section, we can derive the following inequalities:

\[
H(\epsilon', a \mid \mathcal{P}) \geq H(1 - \epsilon, a \mid \mathcal{P}), \quad \max H(\epsilon', a \mid \mathcal{P}) \geq H(1 - \epsilon, a \mid \mathcal{P}).
\]

Thus, if \( \max H(\epsilon', a \mid \mathcal{P}) \) or \( H(\epsilon', a \mid \mathcal{P}) \) is continuous with respect to \( \epsilon' \) at least as in the i.i.d. case, the above equation yields \( \epsilon' \geq 1 - \epsilon \). That is, the following trade-off holds between the error probability of compression and the performance of intrinsic randomness:

\[
\lim \epsilon(\Phi_n) + \lim \epsilon(\Psi_n) \geq 1. \] (24)

Therefore, Folklore for source coding does not hold with respect to variational distance. In other word, generating completely uniform random numbers requires over compression. Generally, the following theorem holds.

Theorem 4: We define the distance from the uniform distribution as follows:

\[
\delta(p_n) \overset{\text{def}}{=} \min_{S \subset \Omega_n} d(p_n, p_{U,S}). \] (25)

Then the following inequality holds:

\[
\epsilon(\Phi_n) + \epsilon(\Psi_n) \geq \delta(p_n). \] (26)

Especially, in the i.i.d. case, the quantity \( \delta(p_n) \) goes to 1. In such a case, the trade-off relation

\[
\lim \epsilon(\Phi_n) + \epsilon(\Psi_n) \geq 1 \] (27)

holds. Furthermore, the above trade-off inequality (27) is rigid as is indicated by the following theorem.

Theorem 5: When the convergence \( \lim_{n \to \infty} p_n \left\{ -\frac{1}{n} \log p_n(\omega_n) < a + \frac{b + \gamma}{\sqrt{n}} \right\} \) is uniform concerning \( \gamma \) in an enough small neighbourhood of 0 and the relation

\[
\lim_{\gamma \to 0} \lim_{n \to \infty} p_n \left\{ -\frac{1}{n} \log p_n(\omega_n) < a + \frac{b + \gamma}{\sqrt{n}} \right\} = \epsilon
\]

holds, there exists a sequence of codes \( \Phi_n = (\mathcal{M}_n, \phi_n, \psi_n) \) \( (\Psi_n = (\mathcal{M}_n, \phi_n)) \) satisfying the following conditions:

\[
\lim \epsilon(\Phi_n) \leq 1 - \epsilon, \quad \lim \epsilon(\Psi_n) \leq \epsilon,
\]

\[
\lim \frac{1}{\sqrt{n}} \log \frac{\| \Phi_n \|_{e^{\alpha_n}}}{} = b. \] (29)

VI. INTRINSIC RANDOMNESS BASED ON KL-DIVERGENCE CRITERION

A. First order asymptotics

Next, we discuss intrinsic randomness based on KL-divergence. Since Han [1] discussed Folklore for source coding based on KL-divergence criterion, we need this type of discussion for comparing our result and Han’s result. The first work on intrinsic randomness based on KL-divergence was done by Venmi & Verdù[11]. They focused on the normalized KL-divergence:

\[
\frac{1}{n} D(p_n \circ \phi_n^{-1} \mid p_{U,M_n}), \] (30)

where \( D(p \mid q) \) is the KL-divergence \( \sum \log \frac{p(\omega)}{q(\omega)} \). Han [1] called the sequence of distributions \( p_n \circ \phi_n^{-1} \) ‘almost’ uniform random if the above value goes to 0.

Proposition 1: Vembo & Verdù[11, Theorem 1]

\[
S^1(\mathcal{P}) \overset{\text{def}}{=} \sup_{\Psi_n} \left\{ \lim_{n \to \infty} \frac{1}{\sqrt{n}} \log \| \Psi_n \| \right\}, \quad \lim_{n \to \infty} \frac{1}{n} D(p_n \circ \phi_n^{-1} \mid p_{U,M_n}) = 0
\]

\[
= \max H(\epsilon', a \mid \mathcal{P}) \overset{\text{def}}{=} \sup_a \left\{ \lim_n \log p_n \left\{ -\frac{1}{n} \log p_n(\omega) < a \right\} = 0 \right\}. \] (31)
In a thorough discussion of the above proposition, Han [1] worked out the following proposition concerning Folklore for source coding.

**Proposition 2:** Han[1, Theorem 31] The following three conditions for the sequence \(\mathbf{p} = \{p_n\}\) are equivalent:

- When a sequence of codes \(\Phi_n = (\mathcal{M}_n, \phi_n, \psi_n, \omega_n)\) satisfies
  \(\varepsilon(\Phi_n) \rightarrow 0, \frac{1}{n} \log |\mathcal{M}_n| \rightarrow \overline{H}(\mathbf{p})\) then the value (30) goes to 0.
- There exists a sequence of codes \(\Phi_n = (\mathcal{M}_n, \phi_n, \psi_n)\) satisfying \(\varepsilon(\Phi_n) \rightarrow 0, \frac{1}{n} \log |\mathcal{M}_n| \rightarrow \overline{H}(\mathbf{p})\) and the value (30) goes to 0.
- The sequence \(\mathbf{p} = \{p_n\}\) satisfies the strong converse property:

\[
\bar{H}(\mathbf{p}) = \overline{H}(\mathbf{p}) \equiv \inf_{a} \{ a \mid \lim p_n \left\{ -\frac{1}{n} \log p_n(\omega) < a \right\} = 1 \},
\]

(32)

In order to discuss Folklore for source coding in KL-divergence criterion, we need to generalize Vembu & Verdù’s theorem as follows.

**Theorem 6:** Assume that \(\overline{H}(\epsilon; \mathbf{p}) = \overline{H}(\epsilon; \mathbf{p})\). We define the probability distribution function \(F\) by

\[
\int_{0}^{\overline{H}(\epsilon; \mathbf{p})} F(dx) = \epsilon.
\]

Then, the inequality

\[
\lim_{n} \frac{1}{n} D(p_n \circ \phi_n^{-1} \| p_{U,M_n} ) \geq \int_{0}^{a} (a-x)F(dx)
\]

holds, where \(a = \lim_{n} \frac{1}{n} \log M_n\). Furthermore, when \(\overline{H}(1-\epsilon; \mathbf{p}) = a\), there exists a sequence of codes \(\{\Phi_n\}\) attaining the equality of (33) and satisfying \(\lim \varepsilon(\Phi_n) = \epsilon\). Here, we remark that the inequality (33) is equivalent to the inequality:

\[
\lim_{n} \frac{1}{n} H(p_n \circ \phi_n^{-1}) \leq \int_{0}^{a} xF(dx) + a(1 - F(a)).
\]

(34)

Note that the following equation follows from the above theorem:

\[
S^*_\epsilon(\mathbf{p}) \equiv \sup_{\{\Phi_n\}} \left\{ \lim_{n} \frac{1}{n} \log |\Psi_n| |\lim_{n} \frac{1}{n} D(p_n \circ \phi_n^{-1} \| p_{U,M_n} ) \leq \delta \right\}
\]

\[
= \sup_{a} \left\{ a \mid \int_{0}^{a} (a-x)F(dx) \leq \delta \right\}.
\]

**Remark 3:** The characterization \(S^*_\epsilon(\mathbf{p})\) as a function of \(\delta\) was treated as an open problem in Han’s textbook [4].

In the i.i.d. case of probability distribution \(P\), since

\[
\int_{0}^{a} (a-x)F(dx) = \begin{cases} a - H(P) & a \geq H(P) \\ 0 & a < H(P) \end{cases}
\]

we obtain

\[
S^*_\epsilon(\mathbf{p}) = H(P) + \delta.
\]

Next, we focus on the opposite criterion:

\[
D(p_{U,M_n} \| p_n \circ \phi_n^{-1}),
\]

and define the following quantities:

\[
S^*_1(\delta; \mathbf{p}) \equiv \sup_{\{\Psi_n\}} \left\{ \lim_{n} \frac{1}{n} \log |\Psi_n| |\lim_{n} \frac{1}{n} D(p_{U,M_n} \| p_n \circ \phi_n^{-1}) < \delta \right\},
\]

\[
S^*_2(\delta; \mathbf{p}) \equiv \sup_{\{\Psi_n\}} \left\{ \lim_{n} \frac{1}{n} \log |\Psi_n| |\lim_{n} \frac{1}{n} D(p_{U,M_n} \| p_n \circ \phi_n^{-1}) < \delta \right\}.
\]

Then, they are characterized as follows:

**Theorem 7:**

\[
S^*_1(\delta; \mathbf{p}) = H(1 - e^{-\delta; \mathbf{p}}).
\]

(35)

If the limit

\[
\sigma(a) \equiv \lim_{n} \frac{1}{n} \log p_n \{ -\frac{1}{n} \log p_n(\omega) \geq a \}
\]

converges, the relation

\[
S^*_2(\delta; \mathbf{p}) = \sup_{a} \{ a - \sigma(a) \mid \sigma(a) < \delta \}
\]

holds for \(\forall \delta > 0\).

**Remark 4:** Indeed, Han [4] proved a similar relation concerning the fixed-length source coding with the constraint for error exponent:

\[
\inf_{\{\Phi_n\}} \left\{ \lim_{n} \frac{1}{n} \log |\Psi_n| |\lim_{n} \frac{1}{n} \log \varepsilon(\Phi_n) \geq r \right\}
\]

\[
= \sup_{a} \{ a - \mathfrak{g}(a) \mid \mathfrak{g}(a) < r \},
\]

(37)

where

\[
\mathfrak{g}(a) \equiv \lim_{n} \frac{1}{n} \log p_n \{ -\frac{1}{n} \log p_n(\omega) \geq a \}.
\]

Moreover, Nagaoka and Hayashi [10] proved that equation (37) holds when we define \(\mathfrak{g}(a)\) by

\[
\mathfrak{g}(a) \equiv \lim_{n} \frac{1}{n} \log p_n \{ -\frac{1}{n} \log p_n(\omega) > a \}.
\]

(38)

Hence, when the limit \(\sigma(a)\) exists, equation (36) holds with replacing \(\sigma(a)\) by (38). Further, Hayashi [12] showed that when the limit \(\sigma(a)\) exists, \(\sup_{a} \{ a - \sigma(a) \mid \sigma(a) \leq r \}\) is equal to the bound of gener-ation rate of maximally entangled state with the exponential constraint for success probability with the correspondence of each probability to the square of the Schmidt coefficient.

In the i.i.d. case of \(P\), these quantities are calculated as

\[
S^*_1(\delta; \mathbf{p}) = H(P),
\]

\[
S^*_2(\delta; \mathbf{p}) = \min_{0 < s \leq 1} \frac{s\delta + \psi(s)}{1 - s}, \quad \psi(s) \equiv \log \sum_{\omega} P(\omega)^s,
\]

(39)

where we use the known value of the left hand side of (37), in the calculation (39).

**Remark 5:** As is discussed in Theorem 3 of Hayashi [12], when the limit \(\psi(s) \equiv \lim_{n} \frac{1}{n} \log \sum_{\omega} p_n(\omega)^s\) and its first and second derivatives \(\psi'(s)\) and \(\psi''(s)\) exist for \(s \in (0, 1)\), the relation

\[
S^*_2(\delta; \mathbf{p}) = \min_{0 < s \leq 1} \frac{s\delta + \psi(s)}{1 - s}
\]

(40)
Therefore, we obtain

**B. Second order asymptotics**

Similar to the variational distance criterion, in order to more deeply discuss Folklore for source coding, we need to treat the second order asymptotics. For this purpose, we focus on the following values:

\[
S^*(\delta, a|\mathcal{P}) \quad \text{def} \quad \sup_{\{\psi_n\}} \left\{ \lim_{n} \frac{1}{\sqrt{n}} \log \left| \frac{\psi_n}{e^{\alpha n}} \right| \right\} \lim_{n} \frac{1}{\sqrt{n}} D(p_n \circ \phi_n^{-1} || p_{U,\mathcal{M}_n}) \leq \delta \},
\]

Concerning the first value, the following theorem holds.

**Theorem 8:** Assume that the condition (52) and the equation \( \mathcal{H}(\epsilon, \mathcal{H}(\mathcal{P})|\mathcal{P}) = \mathcal{H}(\epsilon, \mathcal{H}(\overline{\mathcal{P}})|\overline{\mathcal{P}}) \) hold. Define the probability distribution function \( F \) by

\[
\int_{0}^{\mathcal{H}(\epsilon, \mathcal{H}(\mathcal{P})|\mathcal{P})} F(dx) = \epsilon.
\]

Then, the inequality

\[
\lim_{n} \frac{1}{\sqrt{n}} D(P_n \circ \phi_n^{-1} || p_{U,\mathcal{M}_n}) \geq \int_{-\infty}^{b} (b-x) F(dx)
\]

holds, where \( b = \lim_{n} \frac{1}{\sqrt{n}} \log \frac{M_n}{M_0} \). Furthermore, when \( \mathcal{H}(1-\epsilon, \mathcal{H}(\mathcal{P})|\mathcal{P}) = b \), there exists a sequence of codes \( \{\Phi_n\} \) attaining the equality (41) and satisfying \( \lim \epsilon(\Phi_n) = \epsilon \). Finally, we remark that the inequality (41) is equivalent to the inequality:

\[
\lim_{n} \frac{1}{\sqrt{n}} (H(p_n \circ \phi_n^{-1}) - nH(\mathcal{P})) \leq \int_{-\infty}^{b} x F(dx) + b(1 - F(b)).
\]

Therefore, we obtain

\[
S^*(\delta, \mathcal{H}(\mathcal{P})|\mathcal{P}) = \sup_{b} \left\{ b \left| \int_{-\infty}^{b} (b-x) F(dx) \leq \delta \right. \right\}.
\]

Concerning the opposite criterion, the following theorem holds.

**Theorem 9:**

\[
S^*_1(\delta, a|\mathcal{P}) = \mathcal{H}(1 - e^{-\epsilon}, a|\mathcal{P}).
\]

In the i.i.d. case of \( P \), these quantities are simplified to

\[
S^*(\delta, H(P)|\mathcal{P}) = \sup_{b} \left\{ b \left| \sqrt{V_P} \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \leq \delta \right. \right\}
\]

\[
S^*_1(\delta, H(P)|\mathcal{P}) = \sqrt{V_P} \Phi^{-1}(1 - e^{-\delta}).
\]

Especially, when we take the limit \( \delta \rightarrow 0 \), the relations

\[
S^*(\delta, H(P)|\mathcal{P}) \rightarrow -\infty, \quad S^*_1(\delta, H(P)|\mathcal{P}) \rightarrow -\infty
\]

hold. On the other hand, Theorem 5 guarantees that \( R^1(\epsilon, a|\mathcal{P}) = \mathcal{H}(1 - \epsilon, a|\mathcal{P}) \), and \( \lim_{\epsilon \rightarrow 0} \mathcal{H}(1 - \epsilon, a|\mathcal{P}) = +\infty \).

Thus, if a sequence of codes \( \Phi_n = (\mathcal{M}_n, \phi_n, \psi_n) \) satisfies that \( \epsilon(\Phi_n) \rightarrow 0 \), it does not satisfy

\[
\frac{1}{\sqrt{n}} D(p_n \circ \phi_n^{-1} || p_{U,\mathcal{M}_n}) \rightarrow 0
\]

nor

\[
D(p_{U,\mathcal{M}_n} || p_n \circ \phi_n^{-1}) \rightarrow 0.
\]

Therefore, even if we focus on KL-divergence, if we adopt the criterion (44) or (45), Folklore of source coding does not hold.

Furthermore, combining Theorem 8 we obtain the following corollary.

**Corollary 3:** Assume the same assumption as Theorem 8

If the function \( \epsilon \mapsto \overline{\mathcal{H}}(\epsilon, \mathcal{H}(\mathcal{P})|\mathcal{P}) \) is continuous, then

\[
\inf_{\{\phi_n\}} \left\{ \lim_{n} \frac{1}{\sqrt{n}} D(p_n \circ \phi_n^{-1} || p_{U,\mathcal{M}_n}) \left| \lim \epsilon(\Phi_n) \leq \epsilon \right. \right\} 
\]

\[
\leq \inf_{\{\phi_n\}} \left\{ \delta | S^*(\delta, \mathcal{H}(\mathcal{P})|\mathcal{P}) \geq R^1(\epsilon, \mathcal{H}(\mathcal{P})|\mathcal{P}) \right\}
\]

\[
= \int_{-\infty}^{\Phi^{-1}(1-\epsilon)} (\Phi^{-1}(1-\epsilon) - x) F(dx)
\]

\[
\leq \inf_{\{\phi_n\}} \left\{ \delta | S^*_1(\delta, \mathcal{H}(\mathcal{P})|\mathcal{P}) \geq R^1(\epsilon, \mathcal{H}(\mathcal{P})|\mathcal{P}) \right\}
\]

\[
= -\log \epsilon.
\]

In the i.i.d. case, the r. h. s. of (46) equals

\[
\sqrt{V_P} \int_{-\infty}^{\Phi^{-1}(1-\epsilon)} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \int_{-\infty}^{\Phi^{-1}(1-\epsilon)} \Phi^{-1}(1-\epsilon) - x e^{-x^2/2} dx.
\]

Finally, we compare the topologies defined by the following limits:

\[
d(p_n \circ \phi_n^{-1}, p_{U,\mathcal{M}_n}) \rightarrow 0
\]

\[
\frac{1}{\sqrt{n}} D(p_n \circ \phi_n^{-1} || p_{U,\mathcal{M}_n}) \rightarrow 0
\]

\[
D(p_{U,\mathcal{M}_n} || p_n \circ \phi_n^{-1}) \rightarrow 0.
\]

The relations

\[
\text{[50]} \Rightarrow \text{[49]} \Rightarrow \text{[48]}.
\]

\[
\text{[50]} \Rightarrow \text{[47]} \Rightarrow \text{[48]}
\]

\[
\text{[51]} \Rightarrow \text{[47]}
\]

hold. The relation [52] is trivial, the first relation of [53] and the relation [54] is trivial from Pinsker's inequality. For the second one of [53], see Appendix.

That is, (48) gives the weakest topology among the above topologies. Thus, there is no contradiction, even if Folklore for source coding holds in (48), but does not hold in (49), (51), or (47).
VII. MARKOVIAN CASE

Now, we proceed to the Markovian case with irreducible transition matrix $Q_{j,i}$, where $i$ indicates the input signal and $j$ does the output signal. When the initial distribution is the stationary distribution $P_i$, which is the eigen vector of $Q_{j,i}$ with eigen value 1, the average $H_n(Q)$ of the normalized likelihood can be calculated as

$$H_n(Q)$$

$$= -E_{i_1,...,i_n} \frac{1}{n} \log Q_{i_n,i_{n-1}} \cdots Q_{i_2,i_1} P_{i_1}$$

$$= -\frac{1}{n} \sum_{i_{n-1},i_n} P_{i_{n-1}} \log Q_{i_n,i_{n-1}} + \cdots$$

$$+ \sum_{i_1,i_2} P_{i_1} Q_{i_2,i_1} \log Q_{i_2,i_1} + \sum_{i_1} P_{i_1} \log P_{i_1}$$

$$= -\frac{1}{n} \sum_i P_i \log P_i - \frac{n-1}{n} \sum_{j,i} P_{j,i} \log Q_{j,i}$$

$$\rightarrow H(Q) := -\sum_{j,i} P_{j,i} \log Q_{j,i},$$

where $E_{i_1,...,i_n}$ is the expectation concerning the distribution $Q_{i_n,i_{n-1}} \cdots Q_{i_2,i_1} P_{i_1}$.

In order to treat the limit distribution of the normalized likelihood, we calculate the second cumulant as

$$E_{i_1,...,i_n} \left( \frac{-\log Q_{i_n,i_{n-1}} \cdots Q_{i_2,i_1} P_{i_1} - nH_n(Q)}{\sqrt{n}} \right)^2$$

$$=E_{i_1,...,i_n} \left( \frac{X(i_n,i_{n-1}) + \cdots + X(i_2,i_1) + Y(i_1)}{\sqrt{n}} \right)^2$$

$$=E_{i_1,...,i_n} \frac{1}{n} \left( X(i_n,i_{n-1})^2 + \cdots + X(i_2,i_1)^2 + Y(i_1)^2 + 2X(i_n,i_{n-1})X(i_{n-1},i_{n-2}) + \cdots$$

$$+ 2X(i_3,i_2)X(i_2,i_1) + 2X(i_2,i_1)Y(i_1) \right)$$

$$\rightarrow V(Q),$$

where $X(i_{k+1},i_k) := -\log Q_{i_{k+1},i_k} - H(Q), Y(i_1) := P_{i_1} - H(P), and$

$$V(Q) := \sum_{j,i} Q_{j,i} P_j (-\log Q_{j,i} - H(Q))^2$$

$$+ 2 \sum_{k,j,i} Q_{k,j} Q_{j,i} P_i (-\log Q_{k,j} - H(Q))(-\log Q_{j,i} - H(Q))$$

The limit of the third cumulant is calculated as

$$E_{i_1,...,i_n} \left( \frac{-\log Q_{i_n,i_{n-1}} \cdots Q_{i_2,i_1} P_{i_1} - nH_n(Q)}{\sqrt{n}} \right)^3$$

$$=E_{i_1,...,i_n} \left( \frac{X(i_n,i_{n-1}) + \cdots + X(i_2,i_1) + Y(i_1)}{\sqrt{n}} \right)^3$$

$$+ 3(\frac{X(i_n,i_{n-1})^2 X(i_{n-1},i_{n-2}) + \cdots}{\sqrt{n}}$$

$$+ X(i_3,i_2)^2 X(i_2,i_1) + X(i_2,i_2)^2 Y(i_1))$$

$$+ 3(\frac{X(i_n,i_{n-1})X(i_{n-1},i_{n-2})^2 + \cdots}{\sqrt{n}}$$

$$+ X(i_3,i_2)X(i_2,i_1)^2 + X(i_2,i_1)Y(i_1)^2$$

$$+ 2(\frac{X(i_n,i_{n-1})X(i_{n-1},i_{n-2})X(i_{n-2},i_{n-3}) + \cdots}{\sqrt{n}}$$

$$+ X(i_4,i_3)X(i_3,i_2)X(i_2,i_1) + X(i_3,i_2)X(i_2,i_1)Y(i_1)) \right) \rightarrow 0.$$
of information source, but gives a code depending on this information source. In contrast, universal code assumes on the independent and identical information source, (or Markovian source), but depends only on the coding rate not on the information source. As is stated in the following theorem, there exists a universal fixed-length source code attaining the second order optimal rate.

**Theorem 10:** Assume that \(|\Omega|\) is a finite number \(d\), then there exists a fixed-length source code \(\Phi_n\) on \(\Omega^n\) such that

\[
\lim \frac{1}{\sqrt{n}} |\Phi_n| = b
\]  

(58)

and

\[
\lim \varepsilon p_n(\Phi_n) = \begin{cases} 
0 & H(P) < a \\
1 - \Phi\left(\frac{b}{\sqrt{n}}\right) & H(P) = a.
\end{cases}
\]  

(59)

The error probability of the universal fixed-length source code had not been discussed when the rate equaled the entropy of the information source. But, this theorem clarifies asymptotic behavior of the error probability in such a special case by treating the second order asymptotics.

Concerning intrinsic randomness, while Oohama and Sugano [15] proved that there exists an operation universally attaining the first order optimal rate, we can also prove the existence of a universal operation achieving the second order optimal rate.

**Theorem 11:** Assume that \(|\Omega|\) is a finite number \(d\), then there exists an operation \(\Psi_n\) on \(\Omega^n\) such that

\[
\lim \frac{1}{\sqrt{n}} |\Psi_n| = b
\]  

(60)

and

\[
\lim \varepsilon p_n(\Psi_n) = \begin{cases} 
0 & H(P) < a \\
\Phi\left(\frac{b}{\sqrt{n}}\right) & H(P) = a.
\end{cases}
\]  

(61)

IX. PROOF OF THEOREMS

First, we give proofs of Theorems 1 and 2 which are partially known. Following these proofs, we give our proof of Theorem 3 which is the main result of this paper. This is because the former are preliminaries to our proof of Theorem 3. After these proofs, we give proofs of Theorems 4 and 5.

A. Proof of Theorem 1

**Lemma 1:** Han [4, Lemma 1.3.1] For any integer \(M_n\), there exists a code \(\Phi_n\) satisfying

\[
1 - \varepsilon(\Phi_n) \geq p_n\{p_n(\omega) > \frac{1}{M_n}\}, \ |\Phi_n| \leq M_n.
\]  

(62)

**Lemma 2:** Han [4, Lemma 1.3.2] Any integer \(M'_n\) and any code \(\Phi_n\) satisfy the following condition:

\[
1 - \varepsilon(\Phi_n) \leq p_n\{p_n(\omega) > \frac{1}{M'_n}\} + \frac{|\Phi_n|}{M'_n}.
\]

By using these lemmas and the following expressions of the quantities \(R(1 - \varepsilon(\varpi)), R^\dagger(1 - \varepsilon(\varpi))\) and \(R^\ddagger(1 - \varepsilon(\varpi))\), we will prove Theorem 1

\[
R(1 - \varepsilon(\varpi)) = \inf_{\langle \Phi_n \rangle} \left\{ \lim_{n} \frac{1}{n} \log |\Phi_n| \right\} \lim_{n} 1 - \varepsilon(\Phi_n) \geq \varepsilon, \]

\[
R^\dagger(1 - \varepsilon(\varpi)) = \inf_{\langle \Phi_n \rangle} \left\{ \lim_{n} \frac{1}{n} \log |\Phi_n| \right\} \lim_{n} 1 - \varepsilon(\Phi_n) \geq \varepsilon, \]

\[
R^\ddagger(1 - \varepsilon(\varpi)) = \inf_{\langle \Phi_n \rangle} \left\{ \lim_{n} \frac{1}{n} \log |\Phi_n| \right\} \lim_{n} 1 - \varepsilon(\Phi_n) \geq \varepsilon.
\]

**Proof of direct part:** For any real number \(a > H(\varepsilon(\varpi))\), by applying Lemma 1 to the case of \(M_n = e^{na}\), we can show

\[
\lim \frac{1}{n} \log p_n(\omega) > \frac{1}{M_n} = \lim \frac{1}{n} \log p_n(\omega) < a \geq \varepsilon,
\]  

(63)

which implies that \(a \geq R(1 - \varepsilon(\varpi))\). Thus, we obtain

\[
H(\varepsilon(\varpi)) \geq R(1 - \varepsilon(\varpi)).
\]

By replacing the limit \(\lim\) in (63) by \(\lim\), we can show

\[
H(\varepsilon(\varpi)) \geq R^\dagger(1 - \varepsilon(\varpi)).
\]

Finally, by choosing \(M_n\) satisfying

\[
\lim \frac{1}{n} \log p_n(\omega) < a \geq \varepsilon
\]

\[
\lim \frac{1}{n} \log M_n = a > H(\varepsilon(\varpi)),
\]

we can prove

\[
H(\varepsilon(\varpi)) \geq R^\ddagger(1 - \varepsilon(\varpi)).
\]

The direct part of (7) and (8) can be proved by replacing \(\geq \varepsilon\) by \(> \varepsilon\) in the above proof.

**Proof of converse part:** First, we prove

\[
H(\varepsilon(\varpi)) \leq R(1 - \varepsilon(\varpi)),
\]  

(64)

Assume that \(a = \lim \frac{1}{n} \log |\Phi_n|, \ \lim 1 - \varepsilon(\Phi_n) \geq \varepsilon.\) For any real number \(\delta > 0\), we apply Lemma 2 to the case of \(M_n' = e^{n(a + \delta)}\). Then, we obtain

\[
p_n\left\{ -\frac{1}{n} \log p_n(\omega) < a + \delta \right\} \geq 1 - \varepsilon(\Phi_n) - \frac{|\Phi_n|}{e^{n(a + \delta)}}.
\]  

(65)

Taking the limit \(\lim\), we can show

\[
\lim p_n\{ -\frac{1}{n} \log p_n(\omega) < a + \delta \} \geq \varepsilon.
\]

From this relation, we obtain \(a + \delta \geq H(\varepsilon(\varpi))\), which implies (64).

Similarly, taking the limit \(\lim\) at (63), we can prove

\[
H(\varepsilon(\varpi)) \leq R^\ddagger(1 - \varepsilon(\varpi)).
\]

Finally, we focus on a subsequence \(n_k\) satisfying \(a = \lim \frac{1}{n_k} \log |\Phi_n| = \lim \frac{1}{n_k} \log |\Phi_n|\). By using (65), we obtain

\[
\lim p_n\{ -\frac{1}{n_k} \log p_n(\omega) < a - \delta \} \leq \lim p_{n_k}\{ -\frac{1}{n_k} \log p_{n_k}(\omega) < a - \delta \} \leq \lim 1 - \varepsilon(\Phi_{n_k}).
\]

Taking account into the above discussions, we can prove

\[
H(\varepsilon(\varpi)) \leq R^\ddagger(1 - \varepsilon(\varpi)).
\]

Similarly, the converse part of (7) and (8) can be proved by replacing \(\geq \varepsilon\) by \(> \varepsilon\) in the above proof.
B. Proof of Theorem 2

Lemma 3: Han [4, Lemma 2.1.1] For any integers $M_n'$ and $M_n$, there exists an operation $Ψ_n = (M_n, φ_n)$ satisfying

$$ε(Ψ_n) ≤ p_n(p_n(ω)) > \frac{1}{M_n'} + \frac{M_n}{M_n'}, |Ψ_n| = M_n.$$ \hspace{1cm} (66)

Lemma 4: Han [4, Lemma 2.1.2] Any integer $M_n'$ and any operation $Ψ_n$ satisfy

$$ε(Ψ_n) ≥ p_n(p_n(ω)) > \frac{1}{M_n'} - |Ψ_n|.$$ \hspace{1cm} (67)

By using these lemmas, we prove Theorem 2.

Proof of direct part: For any real numbers $a < \mathcal{H}(ε|\mathcal{P})$ and $δ > 0$, we apply Lemma 3 to the case of $M_n = e^{n(α−δ)}$, $M_n' = e^{nα}$ as follows:

$$\lim p_n(p_n(ω) > \frac{1}{M_n'}) = \lim p_n\{p_n(ω) < a\} < ε.$$ \hspace{1cm} (68)

Since $\frac{M_n'}{M_n} → 0$, we obtain $\lim p_n(ω) < a$, which implies $a−δ ≤ S(ε|\mathcal{P})$. Thus, the inequality $\mathcal{H}(ε|\mathcal{P}) ≤ S(ε|\mathcal{P})$ holds. Moreover, by replacing the limit in (68) by lim, we can prove

$$\mathcal{H}(ε|\mathcal{P}) ≤ S^{1}(ε|\mathcal{P})$$

Finally, by choosing $M_n'$ satisfying

$$\lim p_n\{p_n(ω) < a\} ≤ ε,$$

we obtain

$$\mathcal{H}(ε|\mathcal{P}) ≤ S^{1}(ε|\mathcal{P}).$$

The direct part of (17) can be proved by replacing $< ε$ by $≤ ε$ in the above proof.

Proof of converse part: First, we prove

$$\mathcal{H}(ε|\mathcal{P}) ≥ S(ε|\mathcal{P})$$

Assume that $a = \lim \frac{1}{n} log |Ψ_n|, \lim e(Ψ_n) < ε$. For any real number $δ > 0$, we apply Lemma 4 to the case of $M_n = e^{n(α−δ)}$. Then, we obtain

$$p_n\{p_n(ω) < a−δ\} ≤ ε(Ψ_n) + \frac{e^{n(α−δ)}}{|Ψ_n|}. \hspace{1cm} (69)$$

Taking the limit lim, we can show that

$$\lim p_n\{p_n(ω) < a−δ\} < ε.$$ \hspace{1cm} (70)

Thus, we obtain $a−δ ≤ \mathcal{H}(ε|\mathcal{P})$, which implies (68).

Similarly, by taking the limit lim at the inequality (69), we obtain

$$\mathcal{H}(ε|\mathcal{P}) ≥ S^{1}(ε|\mathcal{P}).$$

Moreover, by focusing on a subsequence $n_k$ satisfying $a = \lim \frac{1}{n} log |Ψ_n| = \lim k\frac{1}{n_k} log |Ψ_{n_k}|$, we can show the following relations from (69):

$$\lim p_n\{p_n(ω) < a−δ\} ≤ \frac{1}{n} log p_n(ω) < a−δ ≤ ε(Ψ_{n_k}),$$

which implies that

$$\mathcal{H}(ε|\mathcal{P}) ≥ S^{1}(ε|\mathcal{P}).$$

Similarly, the converse part of (17) can be proved by replacing $< ε$ by $≤ ε$ in the above proof. 

C. Proof of Theorem 3

For any real number $b > \mathcal{H}(ε, a|\mathcal{P})$, by applying Lemma 4 to the case of $M_n = e^{na+V_{n0}}$, we can show

$$\lim p_n\{p_n(ω) > \frac{1}{M_n}\} = \lim p_n\{p_n(ω) > a + \frac{b}{√n}\} ≥ ε,$$

which implies $b ≥ R(1−ε, a|\mathcal{P})$. Thus, we obtain

$$\mathcal{H}(ε, a|\mathcal{P}) ≥ R(1−ε, a|\mathcal{P}).$$

Similar to Proof of Theorem 2, we can show

$$\mathcal{H}(ε, a|\mathcal{P}) ≥ R^{(1−ε, a|\mathcal{P})}, \mathcal{H}(ε, a|\mathcal{P}) ≥ R^{(1−ε, a|\mathcal{P})}. \hspace{1cm} (70)$$

Next, we prove

$$\mathcal{H}(ε, a|\mathcal{P}) ≥ R(1−ε, a|\mathcal{P}).$$

Assume that $b = \lim \frac{1}{n} log |Ψ_n|, \lim 1−ε(Ψ_n) ≥ ε$. For any real number $δ > 0$, we apply Lemma 2 to the case of $M_n' = e^{nα+√(b+δ)}$. Then, we obtain

$$p_n\{p_n(ω) < a + \frac{b+δ}{√n}\} ≥ 1−ε(Ψ_n) - \frac{|Ψ_n|}{e^{na+√(b+δ)}}.$$

Taking the limit lim, we obtain

$$\lim p_n\{p_n(ω) < a + \frac{b+δ}{√n}\} ≥ ε,$$

which implies $b + δ ≥ \mathcal{H}(ε, a|\mathcal{P})$. Thus, we obtain (70). Therefore, similar to our proof of Theorem 2, we can show

$$\mathcal{H}(ε, a|\mathcal{P}) ≥ R^{(1−ε, a|\mathcal{P})}, \mathcal{H}(ε, a|\mathcal{P}) ≥ R^{(1−ε, a|\mathcal{P})}.$$

Next, we prove

$$\mathcal{H}(ε, a|\mathcal{P}) ≤ S(ε|\mathcal{P}).$$ \hspace{1cm} (71)

For any real numbers $b < \mathcal{H}(ε, a|\mathcal{P})$ and $δ > 0$, we apply Lemma 3 to the case of $M_n = e^{na+√(b−δ)}, M_n' = e^{na+√b}$. Since

$$\sum p_n\{p_n(ω) > \frac{1}{M_n}\} = \lim p_n\{p_n(ω) > a + \frac{b}{√n}\} < ε$$

and $\frac{M_n'}{M_n} → 0$, we obtain $\lim ε(Ψ_n) < ε$ which implies $a−δ ≤ S(ε|\mathcal{P})$. Thus, we obtain (11).
Similar to our proof of Theorem 2, we can show
\[ H(e, a|\bar{p}) \leq S^1(e, a|\bar{p}), \quad H(e, a|\bar{p}) \leq S^1(e, a|\bar{p}). \]
Finally, we prove
\[ H(e, a|\bar{p}) \geq S(e|\bar{p}). \tag{72} \]

Assume that \( b \overset{\text{def}}{=} \lim \frac{1}{n} \log \frac{|\Psi_n|}{|\mathcal{M}_n|} \) and \( \lim \epsilon(\Psi_n) < \epsilon \). For any real number \( \delta > 0 \), we apply Lemma 4 to the case of \( M_n' = e^{na + \sqrt{M}b - \delta} \). Then, the inequality
\[ p_n\left(-\frac{1}{n} \log p_n(\omega) < a + \frac{b - \delta}{\sqrt{n}} \right) \leq \epsilon(\Psi_n) + \frac{e^{na + \sqrt{M}b - \delta}}{|\Psi_n|} \]
holds. Taking the limit \( \lim n \), we obtain
\[ \lim_{n \to \infty} p_n\left(-\frac{1}{n} \log p_n(\omega) < a + \frac{b - \delta}{\sqrt{n}} \right) < \epsilon, \]
which implies \( b - \delta \leq H(e, a|\bar{p}) \). Thus, the relation (72) holds.

Similar to our proof of Theorem 2, the inequalities
\[ H(e, a|\bar{p}) \geq S^1(e, a|\bar{p}), \quad H(e, a|\bar{p}) \geq S^1(e, a|\bar{p}) \]
are proved.

D. Proof of Theorem 4

We define the subset \( M_n' \) of \( M_n \) as
\[ M_n' \overset{\text{def}}{=} \{ i \in M_n | \psi_n(i) \in \phi^{-1}_n(i) \}. \]

Since the relation \( \phi^{-1}_n(i) \cap \phi^{-1}_n(j) = \emptyset \) holds for any distinct integers \( i, j \), the map \( \psi_n \) is injective on \( M_n' \). Thus, \( p_n \) can be regarded as a probability distribution on \( M_n' \cup (\Omega_n \setminus \psi_n(M_n')) \subset M_n \cup (\Omega_n \setminus \psi_n(M_n')) \). Similarly, \( p_n \circ \phi^{-1}_n \) also can be regarded as a probability distribution on \( M_n \cup (\Omega_n \setminus \psi_n(M_n')) \).

Then, the relation
\[ d(p_n, p_{U', M_n'}) \leq d(p_n, p_{U', M_n}) \]
holds. The definition of \( \delta(p_n) \) guarantees that
\[ \delta(p_n) \leq d(p_n, p_{U', M_n}). \]

The axiom of distance yields that
\[ d(p_n, p_{U', M_n}) = d(p_n, p_n \circ \phi^{-1}_n) + d(p_n \circ \phi^{-1}_n, p_{U', M_n}). \]

Furthermore, the quantity \( \epsilon(\Phi_n) \) has another expression:
\[ \epsilon(\Phi_n) = p_n(\Omega_n \setminus \psi_n(M_n')). \]

Since the set \( (\Omega_n \setminus \psi_n(M_n')) \) coincides with the set of the element of \( M_n \cup (\Omega_n \setminus \psi_n(M_n')) \) such that the probability \( p_n \) is greater than the probability \( p_n \circ \phi^{-1}_n \), the equation
\[ d(p_n, p_n \circ \phi^{-1}_n) = p_n(\Omega_n \setminus \psi_n(M_n')) = \epsilon(\Phi_n) \]
holds.

Combining the above relations, we obtain
\[ \delta(p_n) \leq \epsilon(\Phi_n) + \epsilon(\Psi_n). \]

E. Proof of Theorem 5

First, we construct a sequence of codes \( \Phi_n = (M_n, \phi_n, \psi_n) \) satisfying (28) and (29) as follows. We assume that \( S_n(a, b) \overset{\text{def}}{=} \{ -\frac{1}{n} \log p_n(\omega) < a + \frac{b}{\sqrt{n}} \} \)
and denote the one-to-one map from \( S_n(a, b) \) to \( M_n \overset{\text{def}}{=} \{ 1, \ldots, M_n \} \) by \( \phi_n \).

Then, the inequality \( \tilde{M}_n \leq e^{na + \sqrt{M}b} \) holds. Next, we define \( \epsilon_n \overset{\text{def}}{=} p_n(S_n(a, b)) \) and focus on the probability distribution \( p_n(\omega) \overset{\text{def}}{=} p_n(S_n(a, b)) \) on \( S_n(a, b) \). Then, we apply Lemma 5 to the case of \( M_n' = M_n' \overset{\text{def}}{=} (1 - \epsilon_n)e^{na + \sqrt{M}b + 2\gamma_n} \), \( M_n = M_n = (1 - \epsilon_n)e^{na + \sqrt{M}b + 2\gamma_n} \), \( \gamma_n = 1/n^{1/4} \), and denote the transformation satisfying the condition of Lemma 6 by \( \hat{\phi}_n \), where the range of \( \hat{\phi}_n \) is \( \{ M_n + 1, \ldots, M_n + M_n \} \). Half of the variational distance between \( \tilde{p}_n \circ \phi_n^{-1} \) and the uniform distribution is less than
\[ \tilde{p}_n\left(-\frac{1}{n} \log p_n(\omega) < a + \frac{b + 2\gamma_n}{\sqrt{n}} \right) + e^{-\sqrt{n}\gamma_n}. \tag{73} \]

Next, we define a code \( \Phi_n = (M_n, \phi_n, \psi_n) \) with the size \( M_n = e^{na + \sqrt{M}b + 2\gamma_n} \) as follows. The encoding \( \phi_n \) is defined by \( \hat{\phi}_n \) and \( \tilde{\phi}_n \). The decoding \( \psi_n \) is defined as the inverse map on the subset \( M_n \subset M_n \), and is defined as an arbitrary map on the compliment set \( M_n \). Since
\[ 1 - \epsilon(\Phi_n) \geq p_n(S_n(a, b)) = \epsilon_n, \tag{74} \]
we obtain the first inequality of (28).

Since the variational distance equals the sum of that on the range of \( \phi_n \) and that on the compliment set of the range, \( \epsilon(\Psi_n) \) can be evaluated as follows:
\[ \epsilon(\Psi_n) \leq (1 - \epsilon_n) \left( p_n\left(-\frac{1}{n} \log p_n(\omega) < a + \frac{b + 2\gamma_n}{\sqrt{n}} \right) + p_n(S_n(a, b)) \right) + p_n(S_n(a, b)) = (1 - \epsilon_n)e^{-\sqrt{n}\gamma_n} + p_n(S_n(a, b)) \]
\[ = (1 - \epsilon_n)e^{-\sqrt{n}\gamma_n} + p_n\left(-\frac{1}{n} \log p_n(\omega) < a + \frac{b + 2\gamma_n}{\sqrt{n}} \right) \]
\[ = (1 - \epsilon_n)e^{-\sqrt{n}\gamma_n} + p_n\left(-\frac{1}{n} \log p_n(\omega) < a + \frac{b + 2\gamma_n}{\sqrt{n}} \right) + \epsilon, \]
where we use the relation \( S_n(a, b) \subset \{ -\frac{1}{n} \log p_n(\omega) < a + \frac{b + 2\gamma_n}{\sqrt{n}} \} \). Since the definition of \( M_n \) guarantees the condition (29), the proof is completed.

F. Proof of Theorem 6

Proof of inequality 59:

**Lemma 5:** The following relation holds for any operation \( (M_n, \phi) \):
\[ H(p_n \circ \hat{\phi}_n^{-1}) \leq H(M_n, p_n) \]
\[ + p_n\left(-\frac{1}{M_n} \log M_n - \log p_n(\omega) \leq \frac{1}{M_n} \right) \]
where
\[
H(M_n, p_n) \overset{\text{def}}{=} -\sum_{p_n(\omega) > \frac{1}{M_n}} p_n(\omega) \log p_n(\omega). \tag{75}
\]

Proof: Define the set \(M'_n\) and the map \(\phi'_n\) from \(\Omega_n\) to \(M'_n\) as follows:
\[
M'_n = M_n \cup \{p_n(\omega) > \frac{1}{M_n}\},
\]
\[
\phi'_n(\omega) = \begin{cases} \phi_n(\omega) & p_n(\omega) \leq \frac{1}{M_n} \\ \omega & p_n(\omega) > \frac{1}{M_n}. \end{cases}
\tag{76}
\]
Since
\[
\sum_{i=1}^{M_n} p_n \circ \phi'_n^{-1}(i) \log p_n \circ \phi'_n^{-1}(i) \leq p_n(p_n(\omega) \leq \frac{1}{M_n})(\log M_n - \log p_n(p_n(\omega) \leq \frac{1}{M_n})),
\]
the inequality
\[
H(p_n \circ \phi'_n^{-1}) \leq H(M_n, p_n)
\]
holds. When the map \(\phi''_n\) from \(M'_n\) to \(M_n\) is defined by
\[
\phi''_n(\omega) = \begin{cases} \omega & \omega \in M_n \\ \phi_n(\omega) & \omega \in \{p_n(\omega) > \frac{1}{M_n}\}, \end{cases}
\]
the relation \(\phi_n = \phi''_n \circ \phi'_n\) holds. Thus, \(\phi_n^{-1} = \phi'_n^{-1} \circ \phi''_n^{-1}\).

Generally, any map \(f\) and any distribution \(Q\) satisfies
\[
H(Q \circ f^{-1}) = -\sum_{y: y = f(x)} Q(x) \log \left( \sum_{x': y = f(x')} Q(x') \right) \leq -\sum_{y: y = f(x)} Q(x) \log Q(x) = H(Q).
\]
Hence,
\[
H(p_n \circ \phi_n^{-1}) = H(p_n \circ \phi'_n^{-1} \circ \phi''_n^{-1}) \leq H(p_n \circ \phi'_n^{-1}).
\]
Therefore, the proof is completed.

We define the probability distribution function \(F_n\) on the real numbers \(\mathbb{R}\) as:
\[
F_n(x) \overset{\text{def}}{=} p_n\{-\frac{1}{n} \log p_n(\omega) < x\} \tag{78}
\]
for a probability distribution \(p_n\). Then, the relation
\[
\frac{1}{n} H(M_n, p_n) = \int_0^{\frac{1}{n} \log M_n} x F_n(dx)
\]
holds. Thus, Lemma 5 yields the inequality
\[
\frac{1}{n} H(p_n \circ \phi_n^{-1}) \leq \int_0^{\frac{1}{n} \log M_n} x F_n(dx)
\]
\[
+ \frac{1}{n} p_n(p_n(\omega) \leq \frac{1}{M_n})(\log M_n - \log p_n(p_n(\omega) \leq \frac{1}{M_n})).
\]
Taking the limit, we obtain 5.

Proof of the existence part:

Lemma 6: Han[4, Equation (2.2.4)] For any integers \(M_n\) and \(M'_n\), there exists an operation \(\Psi_n\) such that
\[
D(p_n \circ \psi_n^{-1} || p_n, M_n) \leq \log M_n (M'_n/M_n + p_n \{p_n(\omega) > 1/M_n\}),
\]

\(|\Psi_n| = M_n\).

Remark 6: Han [4] derived the above inequality in his proof of Proposition 4.

In the following, by using Lemma 6 we construct the code \(\Phi_n = (M_n, \phi_n, \psi_n)\) satisfying the equality of \(\text{def}\)
and \(\lim e(\Phi_n) = \epsilon\) as follows. Assume that \(S_n(a) \overset{\text{def}}{=} \{-\frac{1}{n} \log p_n(\omega) < \alpha\}, M_n \overset{\text{def}}{=} |S_n(a)|\) and let \(\phi_n\) be the one-to-one map from \(S_n(a)\) to \(M_n \overset{\text{def}}{=} \{1, \ldots, M_n\}\). Then, we can prove that \(M_n \leq e^{na}\). Moreover, we let \(\phi_n\) be a map satisfying the condition of Lemma 6 for the probability distribution \(p_n(\omega) \overset{\text{def}}{=} p_n(\omega)/e^{\alpha}\) on the set \(S_n(a)\) in the case of \(M_n = M_n \overset{\text{def}}{=} (1 - \epsilon_n) e^{na}\) and \(M'_n = \sqrt{M_n}\), where \(\epsilon_n \overset{\text{def}}{=} p_n(S_n(a))\) and the domain of \(\phi_n\) is \(\{M_n + 1, \ldots, M_n + M_n\}\).

Thus,
\[
D(p_n \circ \phi_n^{-1} || p_n, M_n) \leq \log((1 - \epsilon_n) e^{na})(p_n(-\frac{1}{n} \log p_n(\omega) < a) + \frac{2}{\sqrt{M_n}}).
\]
Since any element of \(S_n(a)\) does not satisfy the condition \(-\frac{1}{n} \log p_n(\omega) < a\), the inequality
\[
H(p_n \circ \phi_n^{-1}) \geq na + \log(1 - \epsilon_n) - \frac{2(na + \log(1 - \epsilon_n))}{\sqrt{M_n}}
\]
holds.

We define the code \(\Phi_n = (M_n, \phi_n, \psi_n)\) with the size \(M_n = M_n + M_n\) as follows: The encoding \(\phi_n\) is defined from \(\phi_n\) and \(\phi_n\). The decoding \(\psi_n\) on the subset \(M_n\) of \(M_n\) is the inverse map of \(\phi_n\). Then, we evaluate \(H(p_n \circ \phi_n^{-1})\) as
\[
H(p_n \circ \phi_n^{-1}) = H(e^{na}, p_n) + (1 - \epsilon_n)(H(p_n \circ \phi_n^{-1}) - \log(1 - \epsilon_n))
\]
\[
\geq H(e^{na}, p_n)
\]
\[
+ (1 - \epsilon_n) \left( \frac{na - 2(na + \log(1 - \epsilon_n))}{\sqrt{M_n}} \right)
\]
\[
= \int_0^n x F_n(dx) + na(1 - F_n(a))
\]
\[
- \frac{2(1 - \epsilon_n)(na + \log(1 - \epsilon_n))}{\sqrt{M_n}}.
\]
Dividing both sides by \(n\) and taking the limit, we obtain the opposite inequality of \(\text{41}\), which implies the inequality of \(\text{41}\). Similar to Theorem 5 we can prove that this code satisfies the condition \(\lim e(\Phi_n) = \epsilon\).

G. Proof of 55 in Theorem 4

Proof of direct part: For any real numbers \(\epsilon > 0\) and \(a\) satisfying
\[
a < H(1 - e^{-\epsilon} || p),
\]

(80)
we construct a sequence \( \Psi_n = (M_n, \phi_n) \) such that
\[
\liminf_{n \to \infty} D(p_{U,M_n} \| p_n \circ \phi_n^{-1}) < \delta
\]
\[
\lim_{n \to \infty} \frac{1}{n} \log |\Psi_n| = a - \epsilon.
\]

We define the probability distribution \( \hat{p}_n(\omega) \) on \( S_n(a)^c \) such that \( p_n(S_n(a)) \) satisfies
\[
\begin{align*}
\hat{p}_n(\omega) &\leq \frac{1}{n} \log p_n(\omega) \quad \text{for all } \omega \in S_n(a)^c, \\
\hat{p}_n(\omega) &\geq \frac{1}{n} \log p_n(\omega) \quad \text{for all } \omega \in S_n(a).
\end{align*}
\]

Moreover,
\[
\begin{align*}
\frac{1}{M_n} - \frac{1}{M_n} &\leq \frac{1}{M_n} (1 - e^{-na}).
\end{align*}
\]

Next, we define a map \( \phi_n \) from \( \Omega_n \) to \( M_n = \{1, \ldots, M_n, M_n + 1\} \) by \( \phi_n|_{\hat{S}_n(a)} = \hat{\phi}_n \) and \( \phi_n(S_n(a)) = M_n + 1. \) Then,
\[
D(p_{U,M_n} \| p_n \circ \phi_n^{-1})
\]
\[
= -\frac{1}{M_n + 1} \log \hat{M}_n + 1 + \frac{\hat{M}_n}{M_n + 1} \left( D(p_{U,M_n} \| p_n \circ \phi_n^{-1})
\right.
\]
\[
\left. + \frac{\hat{M}_n}{M_n + 1} - \log p_n(S_n(a)^c) \right). \quad (82)
\]

Since
\[
\lim p_n(S_n(a)) < 1 - e^{-\delta}, \quad (83)
\]
we have the inequality (80) that guarantees
\[
\lim D(p_{U,M_n} \| p_n \circ \phi_n^{-1}) = \lim -\log p_n(S_n(a)^c) = \lim -\log(1 - p_n(S_n(a))) < \delta.
\]

Moreover,
\[
\lim_{n \to \infty} \frac{1}{n} \log |M_n| = \lim_{n \to \infty} \frac{1}{n} \log (\hat{M}_n + 1)
\]
\[
= \lim_{n \to \infty} \frac{1}{n} \log \frac{e^{n(a - \epsilon)}}{p_n(S_n(a)^c)} = a - \epsilon.
\]

**Proof of converse part:** Assume that a sequence \( \Psi_n = (M_n, \phi_n) \) satisfies
\[
\lim_{n \to \infty} \frac{1}{n} \log |\Psi_n| = R
\]
\[
\lim D(p_{U,M_n} \| p_n \circ \phi_n^{-1}) < \delta.
\]

For any \( \epsilon' > 0 \), we define
\[
M'_n \overset{\text{def}}{=} \left\lfloor \frac{1}{n} \log p_n \circ \phi_n^{-1}(i) < R - \epsilon' \right\rfloor
\]
\[
\epsilon_n \overset{\text{def}}{=} p_n \circ \phi_n^{-1} \left\lfloor \frac{1}{n} \log p_n \circ \phi_n^{-1}(i) < R - \epsilon' \right\rfloor
\]
\[
\geq p_n \left\lfloor \frac{1}{n} \log p_n(\omega) < R - \epsilon' \right\rfloor. \quad (85)
\]

Information processing inequality of KL-divergence guarantees that
\[
D(p_{U,M_n} \| p_n \circ \phi_n^{-1}) \geq M'_n \left( \log \frac{M'_n}{|\Psi_n|} - \log \epsilon_n \right)
\]
\[
+ \left( 1 - \frac{M'_n}{|\Psi_n|} \right) \left( \log \left( 1 - \frac{M'_n}{|\Psi_n|} \right) - \log(1 - \epsilon_n) \right) .
\]

Since \( M_n \leq e^{n(R - \epsilon)} \) and (82),
\[
\frac{M'_n}{|\Psi_n|} \to 0, \quad \frac{M'_n}{|\Psi_n|} \log \frac{M'_n}{|\Psi_n|} \to 0.
\]

Therefore, taking the limit \( \lim \), we have
\[
\delta > \lim \sup_{n \to \infty} D(p_{U,M_n} \| p_n \circ \phi_n^{-1}) \geq -\log(1 - \epsilon_n),
\]

which implies
\[
\lim \epsilon_n < 1 - e^{-\delta}.
\]

Thus, inequality (85) yields
\[
\lim p_n \left\lfloor \frac{1}{n} \log p_n(\omega) < R - \epsilon' \right\rfloor < 1 - e^{-\delta}.
\]

Therefore,
\[
R - \epsilon' \leq H(1 - e^{-\delta}|\bar{p}).
\]

Since \( \epsilon' \) is arbitrary, we obtain
\[
S_1^*(\delta|\bar{p}) \leq H(1 - e^{-\delta}|\bar{p}).
\]

**H. Proof of (36) in Theorem 7**

First, by using the following two lemmas, we will prove (56).

**Lemma 7:** When three sequences of positive numbers \( a_n, b_n, \) and \( c_n \) satisfy
\[
a_n \leq b_n + c_n,
\]
then
\[
\lim_{n \to \infty} \frac{1}{n} \log a_n \leq \max \{ \lim_{n \to \infty} \frac{1}{n} \log b_n, \lim_{n \to \infty} \frac{1}{n} \log c_n \}.
\]

**Lemma 8:**
\[
\sup_{a \in \alpha} \{ a - \sigma(a) | \sigma(a) < \delta \} \geq \sup_{a \in \alpha} \{ \overline{\xi}(a) | a - \overline{\xi}(a) < \delta \}, \quad (86)
\]

where \( \overline{\xi}(a) \) is defined as:
\[
\overline{\xi}(a) \overset{\text{def}}{=} \lim \frac{1}{n} \log |\{ \frac{1}{n} \log p_n(\omega) < a \}|.
\]

**Proof of direct part:** We will prove
\[
S_2^*(\delta|\bar{p}) \geq \sup_{a \in \alpha} \{ a - \sigma(a) | \sigma(a) < \delta \}.
\]

That is, for any real numbers \( \epsilon > 0 \) and \( a \) satisfying \( \sigma(a) < \delta \), we construct a sequence \( \Psi_n = (M_n, \phi_n) \) such that
\[
\lim _{n \to \infty} \frac{1}{n} \log |\Psi_n| = a - \sigma(a) - \epsilon.
\]
Similar to the proof of (35), we define \( \hat{p}_n(\omega), S_n(a)^c, S_n(a) \) and \( \phi_n \).

Using (81) and (82), we have

\[
\lim_{n} -\frac{1}{n} \log p_n(S_n(a)^c) = \sigma(a) < \delta.
\]

Moreover,

\[
\lim_{n} -\frac{1}{n} \log p_n(S_n(a)^c) = -\frac{1}{n} \log p_n(S_n(a)^c) = a - \epsilon - \sigma(a).
\]

**Proof of converse part:** We will prove

\[
S_2^2(\delta|\bar{p}) \leq \sup_a \{a - \sigma(a)|\sigma(a) < \delta\}.
\] (87)

That is, for any sequence \( \Psi_n = (M_n, \phi_n) \) satisfying \( \lim_{n} \frac{1}{n} D(p_{U,M_n} \| p_n \circ \phi_n^{-1}) \leq \delta \), we will prove that

\[
R \overset{\text{def}}{=} \lim_{n} -\frac{1}{n} \log |M_n| \leq \sup_a \{a - \sigma(a)|\sigma(a) < \delta\}.
\]

Let \( \{n_k\} \) be a subsequence such that \( \lim_{k} \frac{1}{n_k} \log |M_{n_k}| = \lim_{n} -\frac{1}{n} \log |M_n| \). We choose the real number \( a_0 \)

\[
a_0 \overset{\text{def}}{=} \inf_a \left\{a \mid \lim_{k} p_{U,M_{n_k}} \left\{\frac{1}{n_k} \log p_{n_k} \circ \phi_{n_k}^{-1}(i) < a\right\} > 0\right\}.
\]

For any real number \( \epsilon_0 > 0 \), the relation \( \lim_{k} p_{U,M_{n_k}} \left\{\frac{1}{n_k} \log p_{n_k} \circ \phi_{n_k}^{-1}(i) < a_0 - \epsilon_0\right\} = 0 \) holds. Since

\[
n(a_0 - \epsilon_0)p_{U,M_{n_k}} \left\{\frac{1}{n_k} \log p_{n_k} \circ \phi_{n_k}^{-1}(i) \geq a_0 - \epsilon_0\right\}
\]

we have

\[
n(a_0 - \epsilon_0)p_{U,M_{n_k}} \left\{\frac{1}{n_k} \log p_{n_k} \circ \phi_{n_k}^{-1}(i) \geq a_0 - \epsilon_0\right\}
\]

Thus,

\[
(a_0 - \epsilon_0) - R
\]

\[
= \lim_{k} \left(\frac{1}{n_k} \log p_{n_k} \circ \phi_{n_k}^{-1}(i) \geq a_0 - \epsilon_0\right)
\]

\[
\leq \lim_{k} \frac{1}{n_k} D(p_{U,M_{n_k}} \| p_{n_k} \circ \phi_{n_k}^{-1}) < \delta.
\]

Taking the limit \( \epsilon_0 \to 0 \),

\[
a_0 - R \leq \lim_{k} \frac{1}{n_k} D(p_{U,M_{n_k}} \| p_{n_k} \circ \phi_{n_k}^{-1}) < \delta.
\]

Next, we choose a real number \( \epsilon \) such that

\[
0 < \epsilon < \delta - (a_0 - R).
\] (88)

Then, there exists a real number \( \alpha > 0 \) such that

\[
\lim_{n_k} p_{U,M_{n_k}} \left\{\frac{1}{n_k} \log p_{n_k} \circ \phi_{n_k}^{-1}(i) < a_0 + \epsilon\right\} > \alpha.
\]

Thus,

\[
|\left\{\frac{1}{n_k} \log p_{n_k} \circ \phi_{n_k}^{-1}(i) < a_0 + \epsilon\right\}| > \alpha M_{n_k}
\]

for sufficiently large \( n_k \). Since

\[
p_n \left\{\frac{1}{n_k} \log p_{n_k} \circ \phi_{n_k}^{-1}(i) < a_0 + \epsilon\right\}
\]

\[
\phi_{n_k} \left\{\frac{1}{n_k} \log p_{n_k}(\omega) < a_0 + \epsilon\right\}
\]

we can evaluate

\[
\leq p_n \left\{\frac{1}{n_k} \log p_{n_k}(\omega) \geq a_0 + \epsilon\right\}.
\]

Thus,

\[
\alpha M_{n_k} < |\left\{\frac{1}{n_k} \log p_{n_k} \circ \phi_{n_k}^{-1}(i) < a_0 + \epsilon\right\}|
\]

\[
\leq p_n \left\{\frac{1}{n_k} \log p_{n_k}(\omega) \geq a_0 + \epsilon\right\} + |\left\{\frac{1}{n_k} \log p_{n_k}(\omega) < a_0 + \epsilon\right\}|
\]

Using Lemma 7 we have

\[
\max\left\{\bar{\zeta}(a_0 + \epsilon), (a_0 + \epsilon) - \sigma(a_0 + \epsilon)\right\} \geq R. \tag{89}
\]

If \( \bar{\zeta}(a_0 + \epsilon) \geq (a_0 + \epsilon) - \sigma(a_0 + \epsilon) \), by combining (88) and (89), we can show

\[
(a + \epsilon) - \bar{\zeta}(a_0 + \epsilon) < \delta.
\]

Therefore, we obtain

\[
R \leq \sup_a \{a - \sigma(a)|\sigma(a) < \delta\} \leq \sup_a \{a - \sigma(a)|a - \bar{\zeta}(a) < \delta\}.
\]

If \( \bar{\zeta}(a_0 + \epsilon) < (a_0 + \epsilon) - \sigma(a_0 + \epsilon) \), combining (88) and (89), we can show

\[
\bar{\zeta}(a_0 + \epsilon) < (a_0 + \epsilon) - \sigma(a_0 + \epsilon) < \delta.
\]

Therefore, we obtain

\[
R \leq \sup_a \{a - \sigma(a)|\sigma(a) < \delta\}.
\]

**Proof of Lemma 7** Since

\[
b_n + c_n \leq \max\{2b_n, 2c_n\},
\]
we have
\[ \frac{1}{n} \log a_n \leq \max \{ \frac{\log 2}{n}, \frac{1}{n} \log b_n, \frac{\log 2}{n}, \frac{1}{n} \log c_n \}. \]

Taking the limit \( \lim \), we obtain
\[ \lim \frac{1}{n} \log a_n \leq \max \{ \lim \frac{1}{n} \log b_n, \lim \frac{1}{n} \log c_n \}. \]

**Proof of Lemma 8** In this proof, the following lemma plays an important role.

**Lemma 9:** Hayashi[12, Lemma 13] If two decreasing functions \( f \) and \( g \) satisfy
\[ f(a) + a \geq g(b) \text{ if } f(a) > f(b), \] (90)
then
\[ \sup_a \{ a - g(a) | g(a) < \delta \} \geq \sup_a \{ f(a) | a - f(a) < \delta \}. \]

**Remark 7:** This lemma is essentially the one obtained by Hayashi[12]. But, this statement is a little different from Hayashi[12]’s.

**Proof:** We prove Lemma 9 by reduction to absurdity. Assume that there exists a real number \( a_0 \) such that
\[ a_0 - f(a_0) < \delta, \]
\[ f(a_0) > \sup_a \{ a - g(a) | g(a) \leq r \} . \] (91) (92)
We define \( a_1 := \inf \{ a | f(a) = f(a_0) \} \) and assume that \( a_0 > a_1 \). For any real number \( \epsilon < \delta < a_0 - a_1 \), the inequality \( f(a_1 + \epsilon) < f(a_1 + \epsilon) \) holds. Using (89), we have
\[ g(a_1 - \epsilon) \leq f(a_1 - \epsilon) + a_1 + \epsilon = -f(a_0) + a_1 + \epsilon < \delta + (a_1 - a_0) + \epsilon < \delta \]
Thus,
\[ \sup_a \{ a - g(a) | g(a) < \delta \} \geq a_1 - \epsilon - g(a_1 - \epsilon) \]
\[ \geq a_1 - \epsilon - (a_1 + \epsilon) - f(a_1 + \epsilon) = f(a_0) - 2\epsilon. \]

Taking the limit \( \epsilon \to 0 \), we obtain \( \sup_a \{ a - g(a) | g(a) < r \} \geq f(a_0) \), which contradicts (92).

Next, we treat the case \( a_0 = a_1 \). The inequality \( f(a_0) > f(a_0 - \epsilon) \) holds for \( \epsilon > 0 \). Using (89), we have \( g(a_0 - \epsilon) \leq -f(a_0) + a_0 - \epsilon \). Thus,
\[ \sup_a \{ a - g(a) | g(a) \leq r \} \geq a_0 - \epsilon - g(a_0 - \epsilon) \]
\[ \geq a_0 - \epsilon - a_0 + f(a_0) = -\epsilon + f(a_0). \]
This also contradicts (92).

Since
\[ (p_n - e^{na}) \{ p_n - e^{na} \leq 0 \} \leq (p_n - e^{na}) \{ p_n - e^{nb} \leq 0 \} \]
By adding \( e^{na} \) to both sides, we have
\[ p_n \{ p_n - e^{na} \leq 0 \} + e^{na} \{ p_n - e^{na} > 0 \} \]
\[ \leq p_n \{ p_n - e^{nb} \leq 0 \} + e^{na} \{ p_n - e^{nb} > 0 \} , \]
which implies
\[ \{ p_n - e^{na} > 0 \} \]
\[ \leq e^{-na} p_n \{ p_n - e^{nb} \leq 0 \} + \{ p_n - e^{nb} > 0 \} . \]
Thus, Lemma 7 guarantees that
\[ \xi(a) \leq \max \{ -a - \sigma(b), \xi(b) \} . \]
Using this relation, we obtain
\[ \xi(a) \leq -a - \sigma(b) \text{ if } \xi(b) < \xi(a) . \]
Therefore, by applying Lemma 9 to the case of \( f = \xi, g = \sigma \), we can show (90).

**I. Proof of Theorem 8**

**Proof of inequality:** We define the probability distribution function \( F_n \) on the real numbers \( \mathbb{R} \) as:
\[ F_n(x) \overset{\text{def}}{=} p_n \{ -\frac{1}{n} \log p_n(\omega) < \mathbb{H}(\mathbb{P}) + \frac{x}{\sqrt{n}} \} \] (93)
for a probability distribution \( p_n \). Then, the relation
\[ H(M_n, p_n) = \int_0^{b_n} (\sqrt{n}x + n\mathbb{H}(\mathbb{P}))F_n(dx) \] (94)
holds, where \( b_n \overset{\text{def}}{=} \frac{1}{\sqrt{n}}(\log M_n - n\mathbb{H}(\mathbb{P})) \). Thus, Lemma 8 yields the inequality
\[ H(p_n \circ \phi_n^{-1}) \]
\[ \leq \int_0^{b_n} (\sqrt{n}x + n\mathbb{H}(\mathbb{P}))F_n(dx) \]
\[ + p_n \{ p_n(\omega) \leq \frac{1}{M_n} \} \]
\[ \times (\sqrt{n}b_n + n\mathbb{H}(\mathbb{P}) - \log p_n \{ p_n(\omega) \leq \frac{1}{M_n} \}) \]
\[ = \sqrt{n} \int_0^{b_n} xF_n(dx) + n\mathbb{H}(\mathbb{P}) \]
\[ + p_n \{ p_n(\omega) \leq \frac{1}{M_n} \} (\sqrt{n}b_n - \log p_n \{ p_n(\omega) \leq \frac{1}{M_n} \}) \]
Therefore, the inequality
\[ \frac{1}{\sqrt{n}} (H(p_n \circ \phi_n^{-1}) - n\mathbb{H}(\mathbb{P})) \]
\[ \leq \int_0^{b_n} xF_n(dx) \]
\[ + p_n \{ p_n(\omega) \leq \frac{1}{M_n} \} (\sqrt{n}b_n - \log p_n \{ p_n(\omega) \leq \frac{1}{M_n} \}) \]
holds. Taking the limit \( \lim \), we obtain (92), which is equivalent with (91).

**Proof of the existence part:** In the following, by using Lemma 9 we construct the code \( \phi_n = (M_n, \phi_n, \psi_n) \), satisfying the inequality at (92) and \( \lim \epsilon(\Phi_n) = \epsilon \) as follows. Let \( \phi_n \) be the one-to-one map from
\[ S_n(\mathbb{H}(\mathbb{P}), b) \overset{\text{def}}{=} \{ -\frac{1}{n} \log p_n(\omega) < \mathbb{H}(\mathbb{P}) + \frac{b}{\sqrt{n}} \} \]
to \( \tilde{M}_n \overset{\text{def}}{=} \{ 1, \ldots, M_n \} \), where \( \tilde{M}_n \overset{\text{def}}{=} |S_n(\mathbb{H}(\mathbb{P}), b)| \). Then, the inequality \( \tilde{M}_n \leq e^{n\mathbb{H}(\mathbb{P}) + b\sqrt{n}} \) holds. Furthermore, we define \( \phi_n \) as a map satisfying the condition of Lemma 9 for the probability distribution \( \tilde{p}_n(\omega) \overset{\text{def}}{=} \frac{p_n(\omega)}{1-\epsilon_n} \) on the set \( S_n(\mathbb{H}(\mathbb{P}), b)^c \) in the case of \( M_n = \tilde{M}_n \overset{\text{def}}{=} (1-\epsilon_n)e^{n\mathbb{H}(\mathbb{P}) + b\sqrt{n}} \).
and $M_n' = \sqrt{M_n}$ where $\epsilon_n \overset{\text{def}}{=} \frac{1}{n} \log p_n(S_n(\mathcal{P}, b))$ and the domain of $\hat{\phi}_n$ is $\{M_n + 1, \ldots, M_n + M_n\}$. Thus, 

$$D(p_n \circ \hat{\phi}_n^{-1} || p_{U,M_n})$$ 

$$\leq \log (1 - \epsilon_n e^{n\mathcal{P} + b + \sqrt{M_n}}).$$ 

$$\left(\hat{p}_n \left\{ - \frac{1}{n} \log p_n(\omega) < \mathcal{P} + \frac{b}{\sqrt{n}} \right\} + \frac{1}{\sqrt{M_n}} \right).$$ 

Because no element of $S_n(\mathcal{P}, b)^c$ satisfies the condition $-\frac{1}{n} \log p_n(\omega) < \mathcal{P} + \frac{b}{\sqrt{n}}$, the inequality 

$$H(p_n \circ \hat{\phi}_n^{-1}) \geq \log (1 - \epsilon_n) + n\mathcal{P} + \sqrt{nb}$$ 

$$- (n\mathcal{P} + \sqrt{nb}) + \log (1 - \epsilon_n))$$ 

holds.

We define the code $\Phi_n = (M_n, \phi_n, \Psi_n)$ with the size $M_n = \hat{M}_n + M_n$ similar to the proof of Theorem 1 Then, 

$$H(p_n \circ \hat{\phi}_n^{-1}) = H(e^{n\mathcal{P} + b + \sqrt{M_n}}, p_n) + (1 - \epsilon_n)(H(p_n \circ \hat{\phi}_n^{-1}) - \log (1 - \epsilon_n))$$ 

$$\geq H(e^{n\mathcal{P} + b + \sqrt{M_n}}, p_n) + (1 - \epsilon_n) \left( n\mathcal{P} + \sqrt{nb} \right.$$ 

$$- (n\mathcal{P} + \sqrt{nb} + \log (1 - \epsilon_n) \left) \right)$$ 

$$= \sqrt{n} \int_0^b x F_n(dx) + n\mathcal{P} + \sqrt{nb}(1 - F_n(b))$$ 

$$- \frac{1}{\sqrt{M_n}}(1 - \epsilon)(n\mathcal{P} + \sqrt{nb} + \log (1 - \epsilon_n))$$ 

By subtracting $n\mathcal{P}$ from both sides, dividing both by $\sqrt{n}$, and taking the limit, we obtain the opposite inequality of (42), which implies the inequality of (43). Similar to Theorem 5 we can prove that this code satisfies the condition $\lim \epsilon(\Phi_n) = \epsilon$.

\section*{J. Proof of Theorem 7}

\textbf{Proof of direct part:} For any real numbers $\epsilon > 0$ and $\alpha$ satisfying 

$$b < H(1 - e^{-\delta}, a, \mathcal{P}),$$ 

we construct a sequence $\Psi_n = (M_n, \phi_n(a))$ such that 

$$\lim \frac{1}{\sqrt{n}} \log \frac{\epsilon_n}{e^{n\alpha}} = b - \epsilon.$$ 

We define the probability distribution $\hat{p}_n(\omega) = \frac{p_n(\omega)}{p_n(S_n(a,b)^c)}$ on $S_n(a,b)^c \overset{\text{def}}{=} \left\{ - \frac{1}{n} \log p_n(\omega) \geq a + \frac{b}{\sqrt{n}} \right\}(S_n(a,b) \overset{\text{def}}{=} \left\{ - \frac{1}{n} \log p_n(\omega) < a + \frac{b}{\sqrt{n}} \right\})$. Then, for any $\epsilon > 0$, similar to our proof of (43) in Theorem 7 there exists an operation $\hat{\phi}_n$ from $S_n(a,b)^c$ to $\hat{M}_n = e^{n\alpha + \sqrt{M_n}(b - \epsilon)}p_n(S_n(a,b)^c)$ such that 

$$D(p_{U,M_n} \| \hat{p}_n \circ \hat{\phi}_n^{-1}) \leq -\log (1 - e^{-\epsilon \sqrt{M_n}}) \to 0.$$ 

Next, we define a map $\phi_n$ from $\Omega_n$ to $M_n = \{1, \ldots, M_n, M_n + 1\}$ by $\phi_n(S_n(a,b)) = \hat{\phi}_n$ and $\phi_n(S_n(a,b)) = M_n + 1$. Then, we obtain 

$$D(p_{U,M_n} \| \hat{p}_n \circ \hat{\phi}_n^{-1})$$ 

$$= - \frac{1}{M_n + 1} \log (M_n + 1) + \frac{M_n}{M_n + 1} \left( D(p_{U,M_n} \| \hat{p}_n \circ \hat{\phi}_n^{-1}) \right.$$ 

$$+ \log \frac{M_n}{M_n + 1} - \log p_n(S_n(a,b)^c) \right).$$ 

Since the inequality (45) guarantees 

$$\lim \frac{1}{\sqrt{n}} \log p_n(S_n(a,b)) < 1 - e^{-\delta},$$ 

we have 

$$\lim \frac{1}{\sqrt{n}} \log |\Psi_n| = \lim \frac{1}{\sqrt{n}} \log (M_n + 1)$$ 

$$= \lim \frac{1}{\sqrt{n}} \log \frac{e^{n(a - \epsilon)}}{p_n(S_n(a,b)^c)} = b - \epsilon.$$ 

\textbf{Proof of converse part:} Assume that a sequence $\Psi_n = (M_n, \phi_n)$ satisfies 

$$\lim \frac{1}{\sqrt{n}} \log \frac{|\Psi_n|}{e^{n\alpha}} = R$$ 

$$\lim \frac{1}{\sqrt{n}} \log \frac{|\Psi_n|}{e^{n\alpha}} = b - \epsilon.$$ 

For any $\epsilon' > 0$, we define 

$$M_n' = \left\{ \left\{ - \frac{1}{n} \log p_n \circ \phi_n^{-1}(i) < a + \frac{R - \epsilon'}{\sqrt{n}} \right\} \right\}$$ 

$$\epsilon_n = \frac{1}{n} \log p_n \circ \phi_n^{-1}(i) < a + \frac{R - \epsilon'}{\sqrt{n}} \right\}$$ 

$$\geq p_n \left\{ - \frac{1}{n} \log p_n(\omega) < a + \frac{R - \epsilon'}{\sqrt{n}} \right\}.$$ 

Information processing inequality of KL-divergence guarantees that 

$$D(p_{U,M_n} \| \hat{p}_n \circ \hat{\phi}_n^{-1})$$ 

$$\geq M_n' \left( \log \frac{M_n'}{|\Psi_n|} - \log \epsilon_n \right)$$ 

$$+ \left( 1 - \frac{M_n'}{|\Psi_n|} \right) \left( \log \left( 1 - \frac{M_n'}{|\Psi_n|} \right) - \log (1 - \epsilon_n) \right).$$ 

Since $M_n' \leq e^{n\alpha + \sqrt{M_n}(b - \epsilon)}$ and (47), 

$$M_n' \rightarrow 0, \quad \frac{M_n'}{|\Psi_n|} \rightarrow 0.$$ 

Therefore, taking the limit $\lim$, we have 

$$\delta \geq \lim \frac{1}{\sqrt{n}} \log (p_{U,M_n} \| \hat{p}_n \circ \hat{\phi}_n^{-1}) \geq \lim \frac{1}{\sqrt{n}} \log (1 - \epsilon_n)$$ 

$$= - \log (1 - \lim \epsilon_n).$$
which implies 
\[ \lim \epsilon_n < 1 - e^{-\delta}. \]

Thus, the inequality (98) yields 
\[ \lim p_n \left( \frac{-1}{n} \log p_n(\omega) < R - \epsilon' \right) < 1 - e^{-\delta}. \]

Therefore, 
\[ R - \epsilon' \leq H(1 - e^{-\delta}, a|p). \]

Since \( \epsilon' \) is arbitrary, we obtain 
\[ S_1^{\ast}(\delta, a|p) \leq H(1 - e^{-\delta}, a|p). \]

K. Proof of Theorem 10

This theorem is proved by the type method. Let \( T_n \) be the set of \( n \)-th types, i.e., the set of empirical distributions of \( n \) observations. We denote the set of elements \( \Omega^n \) corresponding to \( P \) by \( T^n_P \subset \Omega^n \), and define a subset \( T_n(a, b) \) of the set \( \Omega^n \) as 
\[ T_n(a, b) \overset{\text{def}}{=} \{ p \in T_n : |p - \phi(a, b)| < \epsilon n \} \]

Using this notation, we define the encoding \( \psi_n \) from \( \Omega^n \) to \( T_n(a, b) \) as follows. The map \( \psi_n \) is defined as the map 
\[ \omega \mapsto \begin{cases} \omega & \text{if } \omega \in T_n(a, b) \\ 0 & \text{if } \omega \notin T_n(a, b) \end{cases}. \]

We also define the decoding \( \psi_n \) such that \( \psi_n(\omega) = \omega, \forall \omega \in T_n(a, b) \). The relation 
\[ \varepsilon_p(\Phi_n) = 1 - P^n(T_n(a, b)) \]
holds. Then, the type counting lemma guarantees that 
\[ |T_n(a, b)| \leq (n + 1)^d e^{-\alpha + b \sqrt{n}}, \]
which implies 
\[ \lim \inf \frac{1}{\sqrt{n}} \log \frac{|T_n(a, b)|}{e^n \delta} \leq b. \] (99)

On the other hand, the set \( \{ - \log P^n(\omega) < na + b \sqrt{n} \} \) can be expressed as 
\[ \{ - \log P^n(\omega) < na + b \sqrt{n} \} = \bigcup_{P' \in T_n : P^n(\omega) > e^{-na-b \sqrt{n}}} T^n_{P'}. \]

Hence, when a type \( P' \in T_n \) satisfies \( P^n(\omega) > e^{-na-b \sqrt{n}} \) for \( \omega \in T^n_{P'} \), the inequality \( P^n(\phi_n) \leq 1 \) yields 
\[ |T^n_{P'}| \leq P^n(\omega)^{-1} \leq e^{na+b \sqrt{n}}. \]

Thus, 
\[ \{ - \log P^n(\omega) < na + b \sqrt{n} \} \subset T_n(a, b). \]

Therefore, if the probability distribution \( P \) satisfies \( H(P) = a \), then 
\[ \Phi(\frac{b}{\sqrt{n}}) = \lim P^n \{ - \log P^n(\omega) < na + b \sqrt{n} \} \leq \lim P^n(T_n(a, b)) = 1 - \lim \varepsilon_p(\Phi_n), \]
i.e., 
\[ \lim \varepsilon_p(\Phi_n) \leq 1 - \Phi(\frac{b}{\sqrt{n}}). \] (100)

Since the r.h.s. of (100) is optimal under the condition (99), the inequality of (100) holds. Conversely, Since the r.h.s. of (99) is optimal under the condition (100), the inequality of (99) holds. Thus, we obtain (59).

In the universal variable-length source code, the order of the second term regarding expected coding length is \( \log n \). But, as discussed in the above proof, this term is negligible concerning the second order asymptotics of fixed-length source coding.

Thus, in the variable-length and fixed-length source coding, the central limit theorem plays an important role, while its applications to the respective problems are different.

L. Proof of Theorem 11

Using the type method, we define a map \( \phi_n \) from \( \Omega^n \) to \( M_n \overset{\text{def}}{=} \{ 1, \ldots, 1 \} \) as follows. The map \( \phi_n \) maps any element of \( T_n(a, b) \) to 1. On the other hand, the map \( \psi \) restricted to a subset \( T^n_{P'} \subset T_n(a, b) \) is defined as the map from \( T^n_{P'} \) to \( M_n \) satisfying the conditions Lemma 3 in the case of \( M'_n = |T^n_{P'}| \).

Then, the equality of (100) guarantees 
\[ \varepsilon_p(\phi_n) \]
\[ \leq \sum_{T^n_{P'} \subset T_n(a, b) \cap |T^n_{P'}| \geq e^{n + b \sqrt{n}}} P^n(T^n_{P'}) + \sum_{T^n_{P'} \subset T_n(a, b) \cap |T^n_{P'}| \leq e^{n + b \sqrt{n}}} P^n(T^n_{P'}) \leq P^n(T_n(a, b)) \leq \frac{1}{n} + P^n(T_n(a, b)) \]
\[ \leq 0 \quad \begin{cases} H(P) \quad \text{if } H(P) = a \\ \Phi(\frac{b}{\sqrt{n}}) \quad \text{if } H(P) = a. \end{cases} \]

Therefore, we obtain (61).

X. CONCLUDING REMARKS AND FUTURE STUDY

We proved that Folklore for source coding does not hold for the variational distance criterion (12) nor the KL-divergence criterion (44) nor (45). Of course, since our criteria (12), (44) and (45) are more restrictive than Han’s criterion (30), there is no contradiction. But, it is necessary to discuss which criterion is more suitable for treating Folklore for source coding. This is left to future research.

While we focused on the relation between source coding and intrinsic randomness only in the fixed-length case, the compression scheme used in practice is variable-length. In the variable-length setting, if we use the code whose coding length is decided only from the empirical distribution (this code is called Lynch-Davisson code) in the i.i.d. case, the conditional distribution of the obtained data is the uniform distribution. That is, in the variable-length setting, there exists a code attaining the entropy rate with no error in both settings. Thus, a result different from the fixed-length setting can be expected in the the variable-length setting.

Furthermore, this type second order asymptotics can be extended to other topics in information theory. Indeed, in the case of channel coding, resolvability, and simple hypothesis
testing, lemmas corresponding to Lemmas 11-14 have been obtained by Han [4]. Thus, it is not difficult to derive theorems corresponding to Theorem 3. However, in channel coding it is difficult to calculate the quantities corresponding to $H(\epsilon, \alpha | p)$ and $H(\epsilon, a | p)$ even in the i.i.d. case. On the other hand, similar to fixed-length source coding and intrinsic randomness, we can treat the second order asymptotics concerning the other two problems in the i.i.d. case. Especially, when we discuss simple hypothesis testing with hypothesis $p$ and $q$ from the second order asymptotics viewpoint, we optimize the second order coefficient $b$ of the first error $e^{-nD(p||q) - \sqrt{\epsilon}b}$ under the constraint that the second error probability is less than the fixed constant $\epsilon$. There is no difficulty in this problem. However, there is considerable difficulty in the quantum setting of this problem.

In addition, third order asymptotics is expected, but it seems difficult. In this extension of the i.i.d. case, our issue is the difference of $\sqrt{n}(-\frac{1}{n} \log P^n - H(P))$ from the normal distribution. If the next order is a constant term of $\log P^n$, we cannot use methods similar to those described in this paper. This is an interesting future problem.

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APPENDIX

Proof of (47) ⇒ (48): The relations

$$D(p_n \circ \phi_n^{-1} \| p_{U,M_n} \circ \phi_n^{-1}) = \log M_n - H(p_n \circ \phi_n^{-1})$$

$$= H(p_{U,M_n}) - H(p_n \circ \phi_n^{-1})$$

hold.

If $d(p_n \circ \phi_n^{-1}, p_{U,M_n}) \leq 1/4$, Fannes’ inequality [16] (see also Csiszár and Körner [14]) implies

$$\left| H(p_{U,M_n}) - H(p_n \circ \phi_n^{-1}) \right| \leq - d(p_n \circ \phi_n^{-1}, p_{U,M_n}) \log(d(p_n \circ \phi_n^{-1}, p_{U,M_n})/M_n).$$

Dividing the above by $n$, we have

$$\frac{1}{n} D(p_n \circ \phi_n^{-1} \| p_{U,M_n})$$

$$\leq d(p_n \circ \phi_n^{-1}, p_{U,M_n}) \frac{1}{n} \left( \log M_n - \log(d(p_n \circ \phi_n^{-1}, p_{U,M_n})).$$

Since $\lim_{n \to \infty} \frac{1}{n} \log M_n < \infty$, we obtain (47) ⇒ (48).

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