REDUCTIVE GROUPS AND STEINBERG MAPS

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Abstract. This is a preliminary version of the first chapter of a book project on the character theory of finite groups of Lie type. It provides the foundations from the general theory of reductive algebraic groups over a finite field.

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This first chapter is of a preparatory nature; its purpose is to collect some basic results about algebraic groups (with proofs where appropriate) which will be needed for the discussion of characters and applications in later chapters. In particular, one of our aims is to arrive at the point where we can give a precise definition of a “series of finite groups of Lie type” \( \{ G(q) \} \), indexed by a parameter \( q \). We also introduce a number of tools which will be helpful in the discussion of examples.

For a reader who is familiar with the basic notions about algebraic groups, root data and Frobenius maps it may just be sufficient to browse through this chapter on a first reading, in order to see some of our notation. There are, however, a few topics and results which are frequently used in the literature on algebraic groups and finite groups of Lie type, but for which we have found the coverage in standard reference texts (like [Bor91], [Ca85], [DiMi91], [Hum91], [Spr98]) not to be sufficient; these will be treated here in a fairly self-contained manner.

Section 1.1 is purely expository: it introduces affine varieties, linear algebraic groups in general, and the first definitions concerning reductive algebraic groups.

In Section 1.2, we consider in some detail (abstract) root data, the basic underlying combinatorial structure of the theory of reductive algebraic groups. In particular, we present the approach of [BrLu12], in which root data simply appear as factorisations of the Cartan matrix of a root system. This provides, first of all, an efficient procedure for constructing root data from Cartan matrices; secondly, it will be extremely useful for computational purposes and the discussion of examples.

Date: August 4, 2016.

2000 Mathematics Subject Classification. Primary 20C33, Secondary 20G40.

Key words and phrases. Finite groups of Lie type, character theory.
Section 1.3 contains the fundamental existence and isomorphism theorems of Chevalley [Ch55, Ch56-58/05] concerning connected reductive algebraic groups. We also state the more general “isogeny theorem” and present some of its basic applications. (There is now quite a short proof available, due to Steinberg [St99b].) An important class of homomorphisms of algebraic groups to which this more general theorem applies are the Steinberg maps, to be discussed in detail in Section 1.4.

Following [St68], one might just define a Steinberg map of a connected reductive algebraic group $G$ to be an endomorphism whose fixed point set is finite. But it will be important and convenient to single out a certain subclass of such morphisms to which one can naturally attach a positive real number $q$ (some power of which is a prime power) and such that one can speak of the corresponding finite group $G(q)$. The known results on Frobenius and Steinberg maps are somewhat scattered in the literature so we treat this in some detail here, with complete proofs.

In Section 1.5 we illustrate the material developed so far by a number of further basic constructions and examples. In Section 1.6 we show how all this leads to the notion of “generic” reductive groups, in which $q$ will appear as a formal parameter. Finally, Section 1.7 discusses in some detail the first applications to the character theory of finite groups of Lie type: the “Multiplicity–Freeness” Theorem 1.7.15.

1.1. AFFINE VARIETIES AND ALGEBRAIC GROUPS

In this section, we introduce some basic notions concerning affine varieties and algebraic groups. We will do this in a somewhat informal way, assuming that the reader is willing to fill in some details from text books like [Bor91, Ca85, Ge03a, Hum91, MaTe11, Spr98].

1.1.1. Affine varieties. Let $k$ be a field and let $X$ be a set. Let $A$ be a subalgebra of the $k$-algebra $\mathcal{A}(X,k)$ of all functions $f : X \to k$. Using $A$ we can try to define a topology on $X$: a subset $X' \subseteq X$ is called closed if there is a subset $A \subseteq X$ such that $X' = \{x \in X \mid f(x) = 0 \text{ for all } f \in A\}$. This works well, and gives rise to the Zariski topology on $X$, if $A$ is neither too small nor too big. The precise requirement are (see [Car55-56]):

1. $A$ is finitely generated as a $k$-algebra and contains the identity of $\mathcal{A}(X,k)$;
2. $A$ separates points (that is, given $x \neq x'$ in $X$, there exist some $f \in A$ such that $f(x) \neq f(x')$);
3. any $k$-algebra homomorphism $\lambda : A \to k$ is given by evaluation at a point (that is, there exists some $x \in X$ such that $\lambda(f) = f(x)$ for all $f \in A$).

A pair $(X, A)$ satisfying the above conditions will be called an affine variety over $k$; the functions in $A$ are called the regular functions on $X$. We define dim $X$ to be the supremum of all $r \geq 0$ such that there exist $r$ algebraically independent elements in $A$. Since $A$ is finitely generated, dim $X < \infty$. (See [Ge03a, 1.2.18].) If $A$ is an integral domain, then $X$ is called irreducible.

There is now also a natural notion of morphisms. Let $(X, A)$ and $(Y, B)$ be affine varieties over $k$. A map $\varphi : X \to Y$ will be called a morphism if composition with $\varphi$ maps $B$ into $A$ (that is, for all $g \in B$, we have $\varphi^*(g) := g \circ \varphi \in A$); in this case, $\varphi^* : B \to A$ is an algebra homomorphism, and every algebra homomorphism $B \to A$ arises in this way. The morphism $\varphi$ is an isomorphism if there is a morphism $\psi : Y \to X$ such that $\psi \circ \varphi = \text{id}_X$. (Equivalently: the induced algebra homomorphism $\varphi^* : B \to A$ is an isomorphism.)

Starting with these definitions, the basics of (affine) algebraic geometry are developed in [St74], and this is also the approach taken in [Ge03a]. The link with
the more traditional approach via closed subsets in affine space (which, when considered as an algebraic set with the Zariski topology, we denote by $k^n$) is obtained as follows. Let $(X, A)$ be an affine variety over $k$. Choose a set $\{a_1, \ldots, a_n\}$ of algebra generators of $A$ and consider the polynomial ring $k[t_1, \ldots, t_n]$ in $n$ independent indeterminates $t_1, \ldots, t_n$. There is a unique algebra homomorphism $\pi: k[t_1, \ldots, t_n] \to A$ such that $\pi(t_i) = a_i$ for $1 \leq i \leq n$. Then we have a morphism

$$\varphi: X \to k^n, \quad x \mapsto (a_1(x), \ldots, a_n(x)),$$

such that $\varphi^* = \pi$. The image of $\varphi$ is the “Zariski closed” set of $k^n$ consisting of all $(x_1, \ldots, x_n) \in k^n$ such that $f(x_1, \ldots, x_n) = 0$ for all $f \in \ker(\pi)$.

To develop these matters any further, it is then essential to assume that $k$ is algebraically closed, which we will do from now on. One can go a long way towards those parts of the theory which are relevant for algebraic groups, once the following basic result about morphisms is available (see [St74 §1.13], [Ge03a §2.2]):

Let $\varphi: X \to Y$ be a morphism between irreducible affine varieties such that $\varphi(X)$ is dense in $Y$. Then there is a non-empty open subset $V \subseteq Y$ such that $V \subseteq \varphi(X)$ and, for all $y \in V$, we have $\dim \varphi^{-1}(y) = \dim X - \dim Y$.

1.1.2. Algebraic groups. In order to define algebraic groups, we need to know that direct products of affine varieties are again affine varieties. So let $(X, A)$ and $(Y, B)$ be affine varieties over $k$. Given $f \in A$ and $g \in B$, we define the function $f \otimes g: X \times Y \to k$, $(x, y) \mapsto f(x)g(y)$. Let $A \otimes B$ be the subspace of $\mathcal{A}(X \times Y, k)$ spanned by all $f \otimes g$ where $f \in A$ and $g \in B$. Then $A \otimes B$ is a subalgebra of $\mathcal{A}(X \times Y, k)$ (isomorphic to the tensor product of $A, B$ over $k$) and the pair $(X \times Y, A \otimes B)$ is easily seen be an affine variety over $k$. Now let $(G, A)$ be an affine variety and assume that $G$ is an abstract group where multiplication and inversion are defined by maps $\mu: G \times G \to G$ and $\iota: G \to G$. Then we say that $G$ is an affine algebraic group if $\mu$ and $\iota$ are morphisms. The first example is the additive group of $k$ which, when considered as an algebraic group, we denote by $k^+$ (with algebra of regular functions given by the polynomial functions $k \to k$).

Most importantly, the group $GL_n(k)$ ($n \geq 1$), is an affine algebraic group, with algebra of regular functions given as follows. For $1 \leq i,j \leq n$ let $f_{ij}: GL_n(k) \to k$ be the function which sends a matrix $g \in GL_n(k)$ to its $(i,j)$-entry; furthermore, let $\delta: GL_n(k) \to k, g \mapsto \det(g)^{-1}$. Then the algebra of regular functions on $GL_n(k)$ is the subalgebra of $\mathcal{A}(GL_n(k), k)$ generated by $\delta$ and all $f_{ij}$ ($1 \leq i, j \leq n$). In particular, the multiplicative group $k^\times := GL_1(k)$ is an affine algebraic group.

It is a basic fact that any affine algebraic group $G$ over $k$ is isomorphic to a closed subgroup of $GL_n(k)$, for some $n \geq 1$; see [Ge03a 2.4.4]. For this reason, an affine algebraic group is also called a linear algebraic group. When we just write “algebraic group”, we always mean an affine (linear) algebraic group.

1.1.3. Connected algebraic groups. A topological space is connected if it cannot be written as a disjoint union of two non-empty open subsets. A linear algebraic group $G$ can always be written as the disjoint union of finitely many connected components, where the component containing the identity element is a closed connected normal subgroup of $G$, denoted by $G^\circ$; see [Ge03a 1.3.13]. Thus, $G$ is connected if and only if $G = G^\circ$. (Equivalently: $G$ is irreducible as an affine variety; see [Ge03a 1.1.12, 1.3.1].)

What is the significance of this fundamental notion? Every finite group $G$ can be regarded as a linear algebraic group, with algebra of regular functions given by all of $\mathcal{A}(G, k)$. Thus, the study of all linear algebraic groups is necessarily more complicated than the study of the class of all finite groups. But, as Vogan [Vo07] writes,
“a miracle happens” when we consider connected algebraic groups: things actually become much less complicated. One reason is that a connected algebraic group is almost completely determined by its Lie algebra (see 1.1.5 and also 1.1.11 below), and the latter can be studied using linear algebra methods.

Combined with our assumption that \( k \) is algebraically closed, this gives us some powerful tools. For example, matrices over algebraically closed fields can be put in triangular form. An analogue of this fact for an arbitrary connected algebraic group is the statement that every element is contained in a Borel subgroup (that is, a maximal closed connected normal solvable subgroup); see [Ge03a, 3.4.9].

A useful criterion for showing the connectedness of a subgroup of \( G \) is as follows.

Let \( \{ H_i \}_{i \in I} \) be a family of closed connected subgroups in \( G \). Then the (abstract) subgroup \( H = \langle H_i \mid i \in I \rangle \subseteq G \) generated by this family is closed and connected; furthermore, we have \( H = H_i_1 \cdot \cdot \cdot H_i_n \) for some \( n \) and \( i_1, \ldots, i_n \in I \).

The proof uses the result on morphisms mentioned at the end of 1.1.1, see, e.g., [Ge03a, 2.4.6]. Note that, if \( U, V \) are any closed subgroups of \( G \), then the abstract subgroup \( \langle U, V \rangle \subseteq G \) need not even be closed. For example, if \( G = \text{SL}_2(\mathbb{C}) \), then it is well-known that the subgroup \( \text{SL}_2(\mathbb{Z}) \) is generated by two elements of order 4 and 6, but this subgroup is certainly not closed in \( G \). However, if \( V \) is normalised by \( U \), then \( \langle U, V \rangle = U.V \) is closed; see [Ch56-58/05 §3.3, Corollaire].

We will use without further special mention some standard facts (whose proofs also rely on the above-mentioned result on morphisms). For example, if \( f : G \to G' \) is a homomorphism of linear algebraic groups, then the image \( f(G) \) is a closed subgroup of \( G' \) (connected if \( G \) is connected), the kernel of \( f \) is a closed subgroup of \( G \) and we have \( \dim G = \dim \ker(f) + \dim f(G) \). (See, e.g., [Ge03a 2.2.14].)

1.1.4. Classical groups. These form an important class of examples of linear algebraic groups. They are closed subgroups of \( \text{GL}_n(k) \) defined by certain quadratic polynomials corresponding to a bilinear or quadratic form on the underlying vector space \( k^n \). There is an extensive literature on these groups; see, e.g., [Bou07, Dieu74, Gro92, Tay92]. Since our base field \( k \) is algebraically closed, the general theory simplifies considerably and we only need to consider three classes of groups, leading to the Dynkin types \( B, C, D \). First, and quite generally, for any invertible matrix \( Q_n \in M_n(k) \), we obtain a linear algebraic group

\[
\Gamma(Q_n, k) := \{ A \in M_n(k) \mid A^T Q_n A = Q_n \};
\]

note that \( \det(A) = \pm 1 \) for all \( A \in \Gamma(Q_n, k) \). Let us now take \( Q_n \) of the form

\[
Q_n = \begin{bmatrix}
0 & \cdots & 0 & \pm 1 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \pm 1 & \ddots & \\
\pm 1 & 0 & \cdots & 0
\end{bmatrix} \in M_n(k) \quad (n \geq 2)
\]

where the signs are such that \( Q_n^{tr} = \pm Q_n \). Then \( Q_n \) is the matrix of a non-degenerate symmetric or alternating bilinear form on \( k^n \); furthermore, \( Q_n^{-1} = Q_n^{tr} \) and \( \Gamma(Q_n, k) \) will be invariant under transposing matrices.

If \( Q_n^{tr} = -Q_n \) and \( n \) is even, then \( \Gamma(Q_n, k) \) will be denoted \( \text{Sp}_n(k) \) and called the symplectic group. This group is always connected; see [Ge03a 1.7.4].

Now assume that \( Q_n^{tr} = Q_n \) and that all signs in \( Q_n \) are \( + \). Then we also consider the quadratic form on \( k^n \) defined by the polynomial

\[
f_n := \begin{cases}
t_1 t_{2m+1} + t_2 t_{2m} + \ldots + t_m t_{m+2} + t_{m+1}^2 & \text{if } n = 2m + 1 \text{ is odd}, \\
t_1 t_{2m} + t_2 t_{2m-2} + \ldots + t_m t_{m+1} & \text{if } n = 2m \text{ is even},
\end{cases}
\]
(where \(t_1, \ldots, t_n\) are indeterminates). This defines a function \(\tilde{f}_n : k^n \to k\), where we regard the elements of \(k^n\) as column vectors. Then, using the notation in the general orthogonal group is defined as

\[
GO_n(k) := \{ A \in M_n(k) \mid \tilde{f}_n(Av) = \tilde{f}_n(v) \text{ for all } v \in k^n \};
\]

furthermore, \(SO_n(k) := GO_n(k)\) will be called the special orthogonal group. In each case, we have \([GO_n(k) : SO_n(k)] \leq 2\); see [Ge03a, §1.7], [Gro02] for further details. Note also that, if \(\text{char}(k) \neq 2\), then \(GO_n(k) = \Gamma(Q_n, k)\); otherwise, \(GO_n(k)\) will be strictly contained in \(\Gamma(Q_n, k)\). (See also Example 1.5.5 for the case where \(n\) is odd and \(\text{char}(k) = 2\).

The particular choices of \(Q_n\) and \(f_n\) lead to simple descriptions of a BN-pair in \(\text{Sp}_n(k)\) and \(SO_n(k)\); see, e.g., [Ge03a, §1.7] (and also §1.13 below). The Dynkin types and dimensions are given as follows.

| Group Type | Dimension |
|------------|-----------|
| \(SO_{2m+1}(k)\) | \(B_m\) \(2m^2 + m\) |
| \(Sp_{2m}(k)\) | \(C_m\) \(2m^2 + m\) |
| \(SO_{2m}(k)\) | \(D_m\) \(2m^2 - m\) |

1.1.5. Tangent spaces and the Lie algebra. Let \((X, A)\) be an affine variety over \(k\). Then the tangent space \(T_x(X)\) of \(X\) at a point \(x \in X\) is the set of all \(k\)-linear maps \(D: A \to k\) such that \(D(fg) = f(x)D(g) + g(x)D(f)\). (Such linear maps are called derivations.) Clearly, \(T_x(X)\) is a subspace of the vector space of all linear maps from \(A\) to \(k\). Any \(D \in T_x(X)\) is uniquely determined by its values on a set of algebra generators of \(A\). Hence, since \(A\) is finitely generated, we have \(\dim T_x(X) < \infty\). If \(X' \subseteq X\) is a closed subvariety, we have a natural inclusion \(T_x(X') \subseteq T_x(X)\) for any \(x \in X'\). For example, we can identify \(T_x(k^n)\) with \(k^n\) for all \(x \in k^n\) and so, if \(X \subseteq k^n\) is a Zariski closed subset, we have \(T_x(X) \subseteq k^n\) for all \(x \in X\) (see [Ge03a, 1.4.10]). More generally, any morphism \(\varphi: X \to Y\) between affine varieties over \(k\) naturally induces a linear map \(d_x\varphi: T_x(X) \to T_{\varphi(x)}(Y)\) for any \(x \in X\), called the differential of \(\varphi\) at \(x\). (See [Ge03a, §1.4].)

Now let \(G\) be a linear algebraic group and denote \(L(G) := T_1(G)\), the tangent space at the identity element of \(G\). Then

\[
L(G) = L(G^n) \quad \text{and} \quad \dim G = \dim L(G);
\]

see [Ge03a, 1.5.2]. Furthermore, there is a Lie product \([ , ]\) on \(L(G)\) which can be defined as follows. Consider a realisation of \(G\) as a closed subgroup of \(GL_n(k)\) for some \(n \geq 1\). We have a natural isomorphism of \(L(GL_n(k))\) onto \(M_n(k)\), the vector space of all \(n \times n\)-matrices over \(k\); see [Ge03a, 1.4.14]. Hence we obtain an embedding \(L(G) \subseteq M_n(k)\) where \(M_n(k)\) is endowed with the usual Lie product \([A, B] = AB - BA\) for \(A, B \in M_n(k)\). Then one shows that \([L(G), L(G)] \subseteq L(G)\) and so \([ , ]\) restricts to a Lie product on \(L(G)\); see [Ge03a, 1.5.3]. (Of course, there is also an intrinsic description of \(L(G)\) in terms of the algebra of regular functions on \(G\) which shows, in particular, that the product does not depend on the choice of the realisation of \(G\); see [Ge03a, 1.5.4].)

1.1.6. Quotients. Let \(G\) be a linear algebraic group and \(H\) be a closed normal subgroup. We have the abstract factor group \(G/H\) and we would certainly like to know if this can also be viewed as an algebraic group. More generally, let \(X\) be an affine variety and \(H\) be a linear algebraic group such that we have a morphism \(H \times X \to X\) which defines an action of \(H\) on \(X\). The question of whether we can view the set of orbits \(X/H\) as an algebraic variety leads to “geometric invariant
theory”; in general, these are quite delicate matters. Let us begin by noting that there is a natural candidate for the algebra of functions on the orbit set \( X/H \): If \( A \) is the algebra of regular functions on \( X \), then
\[
A^H := \{ f \in A \mid f(h.x) = f(x) \text{ for all } h \in H \text{ and all } x \in X \}
\]
can naturally be regarded as an algebra of \( k \)-valued functions on \( X/H \). However, the three properties in 1.1.1 will not be satisfied in general. There are two particular situations in which this is the case, and these will be sufficient for most parts of this book; these two situations are:

- \( H \) is a finite group, or
- \( X = G \) is an algebraic group and \( H \) is a closed normal subgroup (acting by left multiplication).

(For the proofs, see [Fo69, 5.25] or [Ge03a, 2.5.12] in the first case, and [Fo69, 2.26] or [Spr98, §5.5] in the second case.) Now let us assume that \((X/H, A^H)\) is an affine variety. Then, first of all, the natural map \( X \to X/H \) is a morphism of affine varieties; furthermore, we have the following universal property:

If \( \varphi: X \to Y \) is any morphism of affine varieties which is constant on the orbits of \( H \) on \( X \), then there is a unique morphism \( \bar{\varphi}: X/H \to Y \) such that \( \varphi \) is the composition of \( \bar{\varphi} \) and the natural map \( X \to X/H \).

(Indeed, if \( B \) is the algebra of regular functions on \( Y \), then the induced algebra homomorphism \( \varphi^* : B \to A \) has image in \( A^H \), hence it factors through an algebra homomorphism \( \bar{\varphi}^* : B \to A^H \) for a unique morphism \( \bar{\varphi} : X/H \to Y \).)

For example, if we are in the second of the above two cases, then the universal property shows that the induced multiplication and inversion maps on \( G/H \) are morphisms of affine varieties. Thus, \( G/H \) is an affine algebraic group.

### 1.1.7. Algebraic groups in positive characteristic

The finite groups that we shall study in this book are obtained as
\[
G^F := \{ g \in G \mid F(g) = g \}
\]
where \( F: G \to G \) are certain bijective endomorphisms with finitely many fixed points, the so-called Steinberg maps. (This will be discussed in detail in Section 1.4.) Such maps \( F \) will only exist if \( k \) has prime characteristic. So we will usually assume that \( p \) is a prime number and \( k = \mathbb{F}_p \) is an algebraic closure of the field \( F_p = \mathbb{Z}/p\mathbb{Z} \).

Now, algebraic geometry over fields with positive characteristic is, in some respects, more tricky than algebraic geometry over \( \mathbb{C} \), say (because of the inseparability of certain field extensions; see also 1.1.8 below). However, some things are actually easier. For example, using an embedding of \( G \) into some \( GL_n(k) \) as in 1.1.2, we see that every element \( g \in G \) has finite order. Thus, we can define \( g \) to be semisimple if the order of \( g \) is prime to \( p \); we define \( g \) to be unipotent if the order of \( g \) is a power of \( p \). Then, clearly, any \( g \in G \) has a unique decomposition
\[
g = us = su
\]
where \( s \in G \) is semisimple and \( u \in G \) is unipotent, called the Jordan decomposition of elements. (The proof in characteristic 0 certainly requires more work; see [Spr98] [2.4].) Another example: An algebraic group \( G \) is called a torus if \( G \) is isomorphic to a direct product of a finite number of copies of the multiplicative group \( k^\times \). Then \( G \) is a torus if and only if \( G \) is connected, abelian and consists entirely of elements of order prime to \( p \); see [Ge03a, 3.1.9]. (To formulate this in characteristic 0, one would need the general definition of semisimple elements.)
1.1.8. Some things that go wrong in positive characteristic. Here we collect a few items which show that, when working over a field \( k = \mathbb{F}_p \) as above, things may not work as one might hope or expect. The first item is:

- A bijective homomorphism of algebraic groups \( \varphi : G_1 \to G_2 \) need not be an isomorphism.

The standard example is the Frobenius map \( \mathbb{F}_p \to \mathbb{F}_p, x \mapsto x^p \). (Note that, over \( \mathbb{C} \), a bijective homomorphism between connected algebraic groups is an isomorphism; see [GoWa98, 11.1.16].) A useful criterion is given as follows (see [Ge03a, 2.3.15]):

- A bijective homomorphism of algebraic groups \( \varphi : G_1 \to G_2 \) is an isomorphism if \( G_1, G_2 \) are connected and if the differential \( d_1 \varphi : T_1(G_1) \to T_1(G_2) \) between the tangent spaces is an isomorphism.

The next item concerns the Lie algebra of an algebraic group. Let \( G \) be a linear algebraic group and \( U, H \) be closed subgroups of \( G \). As already noted in 1.1.5 we have natural inclusions of \( L(U), L(H) \) and \( L(U \cap H) \) into \( L(G) \). It is always true that \( L(U \cap H) \subseteq L(U) \cap L(H) \).

- When considering the intersection of closed subgroups \( U, H \) of an algebraic group \( G \), it is not always true that \( L(U \cap H) = L(U) \cap L(H) \).

A good example to keep in mind is as follows. Let \( G = \text{GL}_n(\mathbb{F}_p) \), \( H = \text{SL}_n(\mathbb{F}_p) \) and \( Z \) be the center of \( G \) (the scalar matrices in \( G \)). Then \( Z, H \) are closed subgroups of \( G \). As in 1.1.5 we can identify \( L(G) = M_n(k) \); then \( L(H) \) consists of all matrices of trace 0 and \( L(Z) \) consists of all scalar matrices. (For these facts see, for example, [Ge03a, §1.5].) Assume now that \( p \) divides \( n \). Then, clearly, \( L(Z) \subseteq L(H) \), whereas \( Z \cap H \) is finite and so \( L(Z \cap H) = L((Z \cap H)^\circ) \neq \{0\} \). (This phenomenon can not happen in characteristic 0; see [Bor91, 6.12] or [Hum91, 12.5] ). Closely related to the above item is the next item: semidirect products. Let \( G \) be an algebraic group and \( U, H \) be closed subgroups such that \( U \) is normal, \( G = U \cdot H \) and \( U \cap H = \{1\} \). Following [Bor91, 1.11], we say that \( G \) is the semidirect product (of algebraic groups) of \( U, H \) if the natural map \( U \times H \to G \) given by multiplication is an isomorphism of affine varieties. If this holds, we have an inverse isomorphism \( G \to U \times H \) and the second projection will induce an isomorphism of algebraic groups \( G/U \cong H \).

- In the definition of semidirect products of algebraic groups, the assumption that \( U \times H \to G \) is an isomorphism of affine varieties can not be omitted.

Take again the above example where \( G = \text{GL}_n(\mathbb{F}_p) \), \( H = \text{SL}_n(\mathbb{F}_p) \) and \( Z \) is the center of \( G \) (the scalar matrices in \( G \)). Assume now that \( n = p \). Then \( Z, H \) are closed connected normal subgroups such that \( G = Z \cdot H \) and \( Z \cap H = \{1\} \). However, this is not a semidirect product of algebraic groups! For, if it were, then we would have an induced isomorphism \( \text{SL}_p(\mathbb{F}_p) = H \cong G/Z = \text{PGL}_p(\mathbb{F}_p) \) which does not exist, as we will see later in Example 1.8.9.

1.1.9. The unipotent radical. Let \( G \) be a linear algebraic group over \( k = \mathbb{F}_p \), where \( p \) is a prime number. We can now also define the unipotent radical \( R_u(G) \subseteq G \), as follows. An abstract subgroup of \( G \) is called unipotent if all of its elements are unipotent. Since every element in \( G \) has finite order, one easily sees that the product of two normal unipotent subgroups is again a normal unipotent subgroup of \( G \). If \( G \) is finite, then this immediately shows that there is a unique maximal normal unipotent subgroup in \( G \). (In the theory of finite groups, this is denoted \( O_p(G) \).) In the general case, we define

\[
R_u(G) := \text{subgroup of } G \text{ generated by all } U \in \mathcal{U}_{\text{unip}}(G),
\]
where $\mathcal{U}_{\text{unip}}(G)$ denotes the set of all closed connected normal unipotent subgroups of $G$. It is clear that $R_u(G)$ is an abstract normal subgroup. By the criterion in 1.1.3, $R_u(G)$ is a closed connected subgroup of $G$; furthermore, $R_u(G) = U_1 \cdots U_n$ for some $n \geq 1$ and $U_1, \ldots, U_n \in \mathcal{U}_{\text{unip}}(G)$. As already remarked before, this product will consist of unipotent elements. Thus, $R_u(G)$ is the unique maximal closed connected normal unipotent subgroup of $G$. (The analogous definition also works when $k$ is an arbitrary algebraically closed field, using the slightly more complicated characterisation of unipotents elements in that case.)

We say that $G$ is reductive if $R_u(G) = \{1\}$.

(Thus, connected reductive groups can be regarded as analogues of finite groups $G$ with $O_p(G) = \{1\}$.) These are the groups that we will be primarily concerned with. In an arbitrary algebraic group $G$, we always have the closed connected normal subgroups $R_u(G) \subseteq G^c \subseteq G$, and $G/R_u(G)$ will be reductive. Note also that, clearly, we have the implication

$$G \text{ simple } \Rightarrow G \text{ reductive (and connected)}.$$ 

Here, we say that $G$ is a simple algebraic group if it is connected, non-abelian and if it has no closed connected normal subgroups other than $\{1\}$ and $G$ itself. (So, for example, $\text{SL}_n(k)$ is a simple algebraic group, although in general it is not simple as an abstract group; $\text{GL}_n(k)$ is reductive, but not simple.)

Note that, even if one is mainly interested in studying a simple group $G$, one will also have to look at subgroups with a geometric origin, like Levi subgroups or centralisers of semisimple elements. These subgroups tend to be reductive, not just simple. For example, if $G$ is connected, reductive and $s \in G$ is a semisimple element, then the centralizer $C_G(s)$ will be a closed reductive (not necessarily connected or simple) subgroup; see [Ca85, 3.5.4].

1.1.10. Characters and co-characters of tori. The simplest examples of connected reductive algebraic groups are tori, and it will be essential to understand some basic constructions with them. First, a general definition. A homomorphism of algebraic groups $\lambda : G \to k^\times$ will be called a character of $G$. The set $X = X(G)$ of all characters of $G$ is an abelian group (which we write additively), called the character group of $G$. Similarly, a homomorphism of algebraic groups $\nu : k^\times \to G$ will be called a co-character of $G$. If $G$ is abelian, then the set $Y = Y(G)$ of all co-characters of $G$ also is an abelian group (written additively), called the co-character group of $G$. Now let $G = T$ be a torus over $k$; recall that this means that $T$ is isomorphic to a direct product of a finite number of copies of $k^\times$. It is an easy exercise to show that every homomorphism of algebraic groups of the multiplicative group $k^\times$ into itself is given by $\xi \mapsto \xi^n$ for a well-defined $n \in \mathbb{Z}$. Thus, we have $X(k^\times) = Y(k^\times) \cong \mathbb{Z}$ and this yields

$$X(T) \cong Y(T) \cong \mathbb{Z}^r \quad \text{where} \quad T \cong k^\times \times \cdots \times k^\times \quad (r \text{ factors}).$$

Hence, $X(T)$ and $Y(T)$ are free abelian groups of the same finite rank. Furthermore, we obtain a natural bilinear pairing

$$(\ , \ ) : X(T) \times Y(T) \to \mathbb{Z},$$

defined by the condition that $\lambda(\nu(\xi)) = \xi^{(\lambda, \nu)}$ for all $\lambda \in X(T)$, $\nu \in Y(T)$ and $\xi \in k^\times$. This pairing is a perfect pairing, that is, it induces group isomorphisms

$$X(T) \xrightarrow{\sim} \text{Hom}(Y(T), \mathbb{Z}), \quad \lambda \mapsto (\nu \mapsto (\lambda, \nu)), \quad Y(T) \xrightarrow{\sim} \text{Hom}(X(T), \mathbb{Z}), \quad \nu \mapsto (\lambda \mapsto (\lambda, \nu)).$$
(see [MaTe11, 3.6]). The pair \((X(T), Y(T))\), together with the above pairing, is the simplest example of a so-called “root datum”, which will be considered in more detail in Section 1.2. The assignment \(T \mapsto X(T)\) has the following fundamental property: if \(T'\) is another torus over \(k\), then we have a natural bijection

\[
\{\text{homomorphisms of algebraic groups } T \to T'\} \xleftrightarrow{\sim} \text{Hom}(X(T'), X(T))
\]

where, on the right hand side, Hom just stands for homomorphisms of abstract abelian groups. The correspondence is defined by sending a homomorphism of algebraic groups \(f: T \to T'\) to the map \(\varphi: X(T') \to X(T), \chi' \mapsto \chi' \circ f\). For future reference, we state the following basic properties of this correspondence:

(a) \(f: T \to T'\) is a closed embedding (that is, an isomorphism onto a closed subgroup of \(T'\)) if and only if \(\varphi: X(T') \to X(T)\) is surjective; in this case, we have a canonical isomorphism \(\ker(\varphi) \cong X(T')/f(T)\).

(b) \(f: T \to T'\) is surjective if and only if \(\varphi: X(T') \to X(T)\) is injective; in this case, we have a canonical isomorphism \(X(T')/\varphi(X(T')) \cong X(\ker(f))\) (induced by restriction of characters from \(T\) to \(\ker(f)\)).

See [Bor91, Chap. III, §8] and [S71, §2.6] for proofs and further details. Furthermore, by [Ca85, §3.1], \(T\) can be recovered from \(X(T)\) through the isomorphism

\[
\gamma \mapsto \text{Hom}(X(T), k^\times), \quad t \mapsto (\lambda \mapsto \lambda(t)).
\]

(Here again, Hom just stands for abstract homomorphisms of abelian groups.)

\subsection{1.1.11. Weight spaces.}

Characters of tori play a major role in the following context. Let \(G\) be a linear algebraic group and \(V\) be a finite-dimensional vector space over \(k\). Note that \(V\) is an affine variety with algebra of regular functions given by the subalgebra generated by the dual space \(V^* = \text{Hom}(V, k) \subseteq \mathcal{A}(V, k)\).

Assume that we have a representation of \(G\) on \(V\), that is, we are given a morphism of affine varieties \(G \times V \to V\) which defines a linear action of \(G\) on \(V\). Let \(T \subseteq G\) be a maximal torus. (Any torus of maximum dimension is maximal.) For each character \(\lambda \in X(T)\) we define the subspace

\[
V_\lambda := \{v \in V \mid t.v = \lambda(t)v \text{ for all } t \in T\}.
\]

Let \(\Psi(T, V)\) be the set of all \(\lambda \in X(T)\) such that \(V_\lambda \neq \{0\}\). Since \(T\) consists of pairwise commuting semisimple elements, we have

\[
V = \bigoplus_{\lambda \in \Psi(T, V)} V_\lambda
\]

(see [Ge83a, 3.1.5]); in particular, this shows that \(\Psi(T, V)\) is finite. The characters in \(\Psi(T, V)\) are called \textit{weights} and the corresponding subspaces \(V_\lambda\) called \textit{weight spaces} (relative to \(T\)). Now, we always have the \textit{adjoint representation} of \(G\) on its Lie algebra \(L(G)\), defined as follows. For \(g \in G\), consider the inner automorphism \(\gamma_g\) of \(G\) defined by \(\gamma_g(x) = gxg^{-1}\). Taking the differential, we obtain a linear map \(d_1\gamma_g: L(G) \to L(G)\), which is a vector space isomorphism. Hence, we obtain a linear action of \(G\) on \(L(G)\) such that \(g.v = d_1(\gamma_g)(v)\) for all \(g \in G\) and \(v \in L(G)\). (The corresponding map \(G \times L(G) \to L(G)\) indeed is a representation; see, for example, [Hum91, 10.3].) Then the finite set

\[
R := \Psi(T, L(G)) \setminus \{0\} \subseteq X(T)
\]

is called the set of \textit{roots of \(G\ relative to \(T\); we have the root space decomposition

\[
L(G) = L(G)_0 \oplus \bigoplus_{\alpha \in R} L(G)_\alpha.
\]
This works in complete generality, for any algebraic group $G$. If $G$ is connected and reductive, then it is possible to obtain much more precise information about the root space decomposition. It turns out that then

$$L(G)_0 = L(T), \quad R = -R \quad \text{and} \quad \dim L(G)_\alpha = 1 \quad \text{for all} \quad \alpha \in R.$$ 

So, in this case, the picture is analogous to that in the theory of complex semisimple Lie algebras and, quite surprisingly, it shows that some crucial aspects of the theory do not depend on the underlying field! This fundamental result, first proved in the Séminaire Chevalley [Ch56-58/05], will be discussed in more detail in Section [1.3]

1.1.12. General structure of connected reductive algebraic groups. Let $G$ be a connected linear algebraic group. Denote by $Z = Z(G)$ the center of $G$. Then we have $Z = R_u(Z) \times S$ where $S$ is a torus; see [Ge03a, 3.5.3]. Since $R_u(Z)$ is a characteristic subgroup of $Z$ and $Z$ is a characteristic subgroup of $G$, we see that $R_u(Z)$ is normal in $G$. Hence, if $G$ is reductive, then $Z$ is a torus. In this case, the above-mentioned results about the root space decomposition lead to the following product decomposition of $G$ (see [MnTe11, §8.4], [Spr98, §8.1]):

$$G = Z^\circ.G_1 \cdots G_n \quad \text{where} \quad G_1, \ldots, G_n \quad \text{are closed normal simple subgroups}$$

and $G_i, G_j$ pairwise commute with each other for $i \neq j$; furthermore, this decomposition of $G$ has the following properties.

- The subgroups $\{G_1, \ldots, G_n\}$ are uniquely determined in the sense that every closed normal simple subgroup of $G$ is equal to some $G_i$.
- We have $G_1 \cdots G_n = G_{der} := \text{commutator (or derived) subgroup of } G$.

(Recall from [1.1.9] that simple algebraic groups are assumed to be connected and non-trivial; note also that the commutator subgroup of a connected algebraic group always is a closed connected normal subgroup; see [Ge03a, 2.4.7].) A connected reductive algebraic group $G$ will be called semisimple if $Z^\circ = \{1\}$ (or, equivalently, if the center of $G$ is finite). Thus, in the above setting, $G_{der}$ is semisimple.

The above product decomposition can be used to prove general statements about connected reductive algebraic groups by a reduction to simple algebraic groups; see, for example, Lemma [1.6.7] Theorem [1.7.15]

1.1.13. Algebraic BN-pairs (or Tits systems). The concept of $BN$-pairs has been introduced by Tits [Ti62], and it has turned out to be extremely useful. It applies to connected algebraic groups and to finite groups, and it allows to give uniform proofs of many results, instead of going through a large number of case–by–case proofs. Recall that two subgroups $B, N$ in an arbitrary (abstract) group $G$ form a $BN$-pair (or a Tits system) if the following conditions are satisfied.

- (BN1) $G$ is generated by $B$ and $N$.
- (BN2) $H := B \cap N$ is normal in $N$ and the quotient $W := N/H$ is a finite group generated by a set $S$ of elements of order 2.
- (BN3) $n_sBn_s \neq B$ if $s \in S$ and $n_s$ is a representative of $s$ in $N$.
- (BN4) $n_sBn \subseteq Bn_sBnB \cup BnB$ for any $s \in S$ and $n \in N$.

The group $W$ is called the corresponding Weyl group. We have a length function on $W$, as follows. We set $l(1) = 0$. If $w \neq 1$, we define $l(w)$ to be the length of a shortest possible expression of $w$ as a product of generators in $S$. (Note that we don’t have to take into account inverses, since $s^2 = 1$ for all $s \in S$.) Thus, any $w \in W$ can be written in the form $w = s_1 \cdots s_p$ where $p = l(w)$ and $s_i \in S$ for all $i$. Such an expression (which is by no means unique) will be called a reduced expression for $w$. 


Furthermore, for any $w \in W$, we set $C(w) := Bn_wB$ where $n_w \in N$ is a representative of $w$ in $N$. Since any two representatives of $w$ lie in the same coset of $H \subseteq B$, we see that $C(w)$ does not depend on the choice of the representative. The double cosets $C(w)$ are called \textit{Bruhat cells} of $G$. Then the above axioms imply the fundamental \textit{Bruhat decomposition} (see [Bou68, Chap. IV, n° 2.3]):

$$G = \bigsqcup_{w \in W} Bn_wB.$$ 

As Lusztig [Lu10] writes, by allowing one to reduce many questions about $G$ to questions about the Weyl group $W$, the Bruhat decomposition is indispensable for the understanding of both the structure and representations of $G$. A key role in this context will be played by the Iwahori-Hecke algebra (introduced in [Iw64]); this is a deformation of the group algebra of $W$ whose definition is based on the Bruhat decomposition. (We will come back to this in a later section on Hecke algebras.)

Now let $G$ be a linear algebraic group over $k$ and let $B, N$ be closed subgroups of $G$ which form an algebraic $BN$-pair. Following [Ca85] §2.5, we shall say that this is an \textit{algebraic $BN$-pair} if $H = B \cap N$ is abelian and consists entirely of semisimple elements, and we have an abstract semidirect product decomposition $B = U.H$ where $U$ is a closed normal unipotent subgroup of $B$ such that $U \cap H = \{1\}$. (If $B$ is connected, then this is automatically a semidirect product of algebraic groups as in [Ca85] see [Spr98] 6.3.5.) We do not assume that $G$ is connected, so the definition can apply in particular to finite algebraic groups. We now have:

\textbf{Proposition 1.1.14.} Let $G$ be a linear algebraic group and $B, N$ be subgroups which form an algebraic $BN$-pair in $G$, where $B = U.H$ as above. Assume that $H, U$ are connected and that $C_G(H) = H$. Then the following hold.

(a) $G$ is connected and reductive.

(b) $B$ is a Borel subgroup (that is, a maximal closed connected solvable subgroup of $G$); we have $B = N_G(U)$ and $[B, B] = U = R_u(B)$.

(c) $H$ is a maximal torus of $G$ and we have $N = N_G(H)$.

(See [Ca85] §2.5 and [Ge03a] 3.4.6, 3.4.7.) As in [Ge03a] 3.4.5, a $BN$-pair satisfying the conditions in Proposition 1.1.14 will be called a \textit{reductive $BN$-pair}.

Much more difficult is the converse of the above result, which comes about as the culmination of a long series of arguments. Namely, if $G$ is a connected reductive algebraic group, then $G$ has a reductive $BN$-pair in which $B$ is a Borel subgroup and $N$ is the normaliser of a maximal torus contained in $B$. (We will discuss this in more detail in Section 1.3.) For our purposes here, the realisation of connected reductive algebraic groups in terms of algebraic $BN$-pairs as above is sufficient for many purposes. For example, if $G$ is a “classical group” as in [Ca85] then algebraic $BN$-pairs as above are explicitly described in [Ge03a] §1.7. In these cases, one can always find an algebraic $BN$-pair in which $B$ consists of upper triangular matrices and $H$ consists of diagonal matrices. See also the relevant chapters in [GLS94], [GLS96], [GLS98].

\section*{1.2. Root data}

We now introduce abstract root data and prove some basic properties of them. As we shall see in later sections, these form the combinatorial skeleton of connected reductive algebraic groups, that is, they capture those features which do not depend on the underlying field $k$. (A reader who wishes to see a much more systematic discussion of root data is referred to [DG70/11 Exposé XXI].)

1.2.1. Let $X,Y$ be free abelian groups of the same finite rank; assume that there is a bilinear pairing $(\ ,\ ) : X \times Y \to \mathbb{Z}$ which is perfect, that is, it induces group isomorphisms $Y \cong \text{Hom}(X,\mathbb{Z})$ and $X \cong \text{Hom}(Y,\mathbb{Z})$ (as in [1.1.10]). Furthermore, let $R \subseteq X$ and $R' \subseteq Y$ be finite subsets. Then the quadruple $\mathcal{R} = (X,R,Y,R')$ is called a root datum if the following conditions are satisfied.

(R1) There is a bijection $R \to R'$, $\alpha \mapsto \alpha'$, such that $(\alpha,\alpha') = 2$ for all $\alpha \in R$.
(R2) For every $\alpha \in R$, we have $2\alpha \notin R$.
(R3) For $\alpha \in R$, we define endomorphisms $w_\alpha : X \to X$ and $w'_\alpha : Y \to Y$ by

$$w_\alpha(\lambda) = \lambda - (\lambda,\alpha')\alpha \quad \text{and} \quad w'_\alpha(\nu) = \nu - (\alpha,\nu')\alpha'$$

for all $\lambda \in X$ and $\nu \in Y$. Then we require that $w_\alpha(R) = R$ and $w'_\alpha(R') = R'$ for all $\alpha \in R$.

We shall see in 1.2.5 that the concept of root data is, in a very precise sense, an enhancement of the more traditional concept of root systems (related to finite reflection groups; see [Bou68]). First, we need some preparations.

The defining formula immediately shows that $w_\alpha^2 = \text{id}_X$ and $(w'_\alpha)^2 = \text{id}_Y$. Hence, we have $w_\alpha \in \text{Aut}(X)$ and $w'_\alpha \in \text{Aut}(Y)$ for all $\alpha \in R$. We set

$$W := \langle w_\alpha \mid \alpha \in R \rangle \subseteq \text{Aut}(X) \quad \text{and} \quad W' := \langle w'_\alpha \mid \alpha \in R \rangle \subseteq \text{Aut}(Y);$$

these groups are called the Weyl groups of $R$ and $R'$, respectively. By (R3), we have an action of $W$ on $R$ and an action of $W'$ on $R'$.

1.2.2. Let $\mathcal{R} = (X,R,Y,R')$ and $\mathcal{R}' = (X',R',Y',R'')$ be root data. Let $\varphi : X' \to X$ be a group homomorphism. The corresponding transpose map $\varphi^{tr} : Y' \to Y'$ is uniquely defined by the condition that

$$(\varphi(\lambda'),\nu) = (\lambda',\varphi^{tr}(\nu))' \quad \text{for all } \lambda' \in X' \text{ and } \nu \in Y',$$

where $(\ ,\ )$ is the bilinear pairing for $\mathcal{R}$ and $(\ ,\ )'$ is the bilinear pairing for $\mathcal{R}'$. We say that $\varphi$ is a homomorphism of root data if $\varphi$ maps $R'$ bijectively onto $R$ and $\varphi^{tr}$ maps $R''$ bijectively onto $R'$. It then follows automatically that $\varphi^{tr}(\varphi(\beta')) = \beta'$ for all $\beta \in R'$; see [DG70/11] XXI, 6.1.2. If $\varphi$ is a bijective homomorphism of root data, we say that $\mathcal{R}$ and $\mathcal{R}'$ are isomorphic.

Lemma 1.2.3. Let $\mathcal{R} = (X,R,Y,R')$ be a root datum.

(a) There is a unique group isomorphism $\delta : W \to W'$ such that $\delta(w) = w'$ for all $\alpha \in R$; we have

$$\langle w^{-1}(\lambda),\nu \rangle = (\lambda,\delta(w)(\nu)) \quad \text{for all } w \in W, \lambda \in X, \nu \in Y.$$

(b) The quadruple $(Y,R',X,R)$ also is a root datum, with pairing $(\ ,\ )^* : X \times Y \to \mathbb{Z}$ defined by $(\nu,\lambda)^* := (\lambda,\nu)$ for all $\nu \in Y$ and $\lambda \in X$.

(c) For any $\lambda \in X$ and $w \in W$, we have $\lambda - w(\lambda) \in \mathbb{Z}R$.

The root datum in (b) is called the dual root datum of $\mathcal{R}$.

Proof. (a) For any group homomorphism $\varphi : X \to X$, consider its transpose $\varphi^{tr} : Y \to Y$, as defined above. Clearly, we have $\text{id}_X^{tr} = \text{id}_Y$ and $(\varphi \circ \psi)^{tr} = \psi^{tr} \circ \varphi^{tr}$ if $\psi : X \to X$ is a further group homomorphism. Thus, $W^{tr} := \{w^{tr} \mid w \in W\}$ is a subgroup of $\text{Aut}(Y)$ and the map $\delta : W \to W^{tr}$, $w \mapsto (w^{-1})^{tr}$, is an isomorphism. Now, using the defining formulae in (R3), one immediately checks that

$$\langle w_\alpha(\lambda),\nu \rangle = (\lambda,w'_\alpha(\nu)) \quad \text{for all } \alpha \in R, \lambda \in X, \nu \in Y.$$
Hence, we have \( w_{\alpha}^{tr} = \bar{w}_{\alpha}^\vee \) for all \( \alpha \in R \) and so \( W^{tr} = W^\vee \). This yields (a).

(b) This is a straightforward verification.

(c) The defining formula shows that this is true if \( w = w_{\alpha} \) for \( \alpha \in R \). But then it also follows in general, since \( W \) is generated by the \( w_{\alpha} \) (\( \alpha \in R \)).

**Lemma 1.2.4.** Let \( \mathcal{R} = (X, R, Y, R^\vee) \) be a root datum. We set \( X_0 := \{ \lambda \in X \mid \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in R \} \). Then

\[
X_0 \cap ZR = \{0\} \quad \text{and} \quad |X/(X_0 + ZR)| < \infty.
\]

Consequently, \( W \) is a finite group and the action of \( W \) on \( R \) is faithful (that is, if \( w \in W \) is such that \( w(\alpha) = \alpha \) for all \( \alpha \in R \), then \( w = 1 \)).

**Proof.** Let us extend scalars from \( Z \) to \( Q \). We denote \( X_Q = Q \otimes_Z X \) and \( Y_Q = Q \otimes_Z Y \). Then \( \langle , \rangle \) extends to a non-degenerate \( Q \)-bilinear form on \( X_Q \times Y_Q \) which we denote by the same symbol. Since \( X, Y \) are free \( Z \)-modules, we can naturally regard \( X \) as a subset of \( X_Q \) and \( Y \) as a subset of \( Y_Q \). Similarly, we can regard \( W \) as a subgroup of \( GL(X_Q) \) and \( W^\vee \) as a subgroup of \( GL(Y_Q) \). So, in order to show the statements about \( X_0 \) and \( ZR \), it is sufficient to show that

\[
X_Q = X_{0,Q} \oplus QR \quad \text{where} \quad X_{0,Q} := \{ x \in X_Q \mid \langle x, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in R \}.
\]

For this purpose, following [DG70/11], XXI, §1.2], we consider the linear map

\[
f : X_Q \to Y_Q, \quad x \mapsto \sum_{\alpha \in R} \langle x, \alpha^\vee \rangle \alpha^\vee.
\]

Let \( \beta \in R \). Using (R3), Lemma [12.3](a) and the fact that \( (w_{\beta}^\vee)^2 = id_Y \), we obtain

\[
(f \circ w_{\beta})(x) = \sum_{\alpha \in R} \langle w_{\beta}(x), \alpha^\vee \rangle \alpha^\vee = \sum_{\alpha \in R} \langle x, w_{\beta}^\vee(\alpha^\vee) \rangle \alpha^\vee = (w_{\beta}^\vee \circ f)(x)
\]

for all \( x \in X_Q \). This identity in turn implies that, for any \( \beta \in R \), we have:

\[
f(\beta) = -f(w_{\beta}(\beta)) = -w_{\beta}^\vee(f(\beta)) = -\sum_{\alpha \in R} \langle \beta, \alpha^\vee \rangle w_{\beta}^\vee(\alpha^\vee)
\]

\[
= -\sum_{\alpha \in R} \langle \beta, \alpha^\vee \rangle (\alpha^\vee - \langle \beta, \alpha^\vee \rangle \beta^\vee) = -f(\beta) + \left( \sum_{\alpha \in R} \langle \beta, \alpha^\vee \rangle^2 \right) \beta^\vee.
\]

Noting that \( \langle \beta, f(\beta) \rangle = \sum_{\alpha \in R} \langle \beta, \alpha^\vee \rangle^2 \), we deduce that

\[
2f(\beta) = \langle \beta, f(\beta) \rangle \beta^\vee \quad \text{and} \quad \langle \beta, f(\beta) \rangle > 0 \quad \text{for all } \beta \in R.
\]

This shows that \( f(QR) = QR^\vee \) and so \( \dim QR \geq \dim QR^\vee \). By the symmetry expressed in Lemma [12.3](b), the reverse inequality also holds and so \( \dim QR = \dim QR^\vee \). Thus, \( f \) restricts to an isomorphism \( f : QR \to QR^\vee \). Now, we clearly have \( X_{0,Q} \subseteq \ker(f) \), whence \( X_{0,Q} \cap QR = \{0\} \). Since \( \langle , \rangle \) extends to a non-degenerate bilinear form on \( X_Q \times Y_Q \), we have \( \dim X_Q = \dim X_{0,Q} + \dim QR^\vee \). Since also \( \dim QR^\vee = \dim QR \), we conclude that \( X_Q = X_{0,Q} \oplus QR \), as desired.

Now we show that the action of \( W \) on \( R \) is faithful. Let \( w \in W \) be such that \( w(\alpha) = \alpha \) for all \( \alpha \in R \). Then \( w \) acts as the identity on the subspace \( QR \subseteq X_Q \). On the other hand, the defining equation shows that all \( w_{\alpha}, \alpha \in R \), act as the identity on \( X_{0,Q} \), so \( W \) is trivial on \( X_{0,Q} \). Hence, \( w = 1 \) since \( X_Q = X_{0,Q} + QR \). Since \( R \) is finite, it follows that \( W \) must be finite, too. \( \square \)
1.2.5. Let \( R = (X, R, Y, R') \) be a root datum. As in the above proof, we extend scalars from \( \mathbb{Z} \) to \( \mathbb{Q} \) and set \( X_Q = \mathbb{Q} \otimes_\mathbb{Z} X \). Following [Bou68 Chap. VI, §1, Prop. 3], we define a symmetric bilinear form \((\ , \ ) : X_Q \times X_Q \to \mathbb{Q}\) by
\[
(x, y) := \sum_{\alpha \in R} \langle x, \alpha' \rangle \langle y, \alpha' \rangle \quad \text{for all } x, y \in X_Q.
\]
Using (R3) and Lemma [1.2.3(a)], we see that \((\ , \ )\) is \( W \)-invariant, that is, \((w(x), w(y)) = (x, y)\) for all \( w \in W \) and all \( x, y \in X_Q \). Clearly, we have \((x, x) \geq 0\) for all \( x \in X_Q \); furthermore, \(\langle \beta, \beta \rangle > 0\) for all \( \beta \in R \) (since \(\langle \beta, \beta' \rangle = 2 > 0\)). By a standard argument (see [Bou68 Chap. VI, §1, Lemme 2]), this yields that
\[
2 \langle \alpha, \beta \rangle \langle \beta, \beta' \rangle = \langle \alpha, \beta' \rangle \in \mathbb{Z} \quad \text{for all } \alpha, \beta \in R.
\]
We claim that the restriction of \((\ , \ )\) to \( Q R \times Q R \) is positive-definite. Indeed, assume that \((x, x) = 0\) where \( x \in Q R \). Then \(0 = (x, x) = \sum_{\alpha \in R} \langle x, \alpha' \rangle^2\) and so \(\langle x, \alpha' \rangle = 0\) for all \( \alpha \in R \). Hence, Lemma [1.2.4] shows that \( x = 0 \), as desired.

Thus, we see that \( R \) is a crystallographic root system in the subspace \( Q R \) of \( X_Q \); see [Bou68 Chap. VI, §1, Déf. 1]. The Weyl group of \( R \) is \( W \); see Lemma [1.2.4]. Furthermore, \( R \) is reduced, in the sense that
\[
R \cap Q \alpha = \{ \pm \alpha \} \quad \text{for all } \alpha \in R.
\]
(This is an easy consequence of (R2); see [Bor91 14.7].) Similarly, \( R' \) is a reduced crystallographic root system in \( Q R' \), by the symmetry in Lemma [1.2.3(b)].

1.2.6. Keeping the above notation, we now recall some standard results on root systems (see, e.g., [MaTe11 App. A]). There is a subset \( \Pi \subseteq R \) such that:
(a) \( \Pi \) is linearly independent in \( Q R \) and
(b) every \( \alpha \in R \) can be written as \( \alpha = \sum_{\beta \in \Pi} x_\beta \beta \) where \( x_\beta \in \mathbb{Q} \) and either \( x_\beta \geq 0 \) for all \( \beta \in \Pi \) or \( x_\beta \leq 0 \) for all \( \beta \in \Pi \).

We call \( \Pi \) a base for \( R \). The corresponding set of positive roots \( R^+ \subseteq \Pi \) consists of those \( \alpha \in R \) which can be written as \( \alpha = \sum_{\beta \in \Pi} x_\beta \beta \) where \( x_\beta \in \mathbb{Q} \) and \( x_\beta \geq 0 \) for all \( \beta \in \Pi \). The roots in \( R^- := -R^+ \) are called the corresponding negative roots. Furthermore, if (a) and (b) hold, then we also have:
(c) For every \( \alpha \in R \), there exists some \( w \in W \) such that \( w(\alpha) \in \Pi \).
(d) Every \( \alpha \in R \) is a \( \mathbb{Z} \)-linear combination of the roots in the base \( \Pi \). (That is, the coefficients \( x_\beta \) in (b) are always integers.)
(e) \( W \) is a Coxeter group, with generators \( \{ w_\beta \mid \beta \in \Pi \} \) and defining relations \( (w_\beta w_\gamma)^{m_{\beta \gamma}} = 1 \) for all \( \beta, \gamma \in \Pi \), where \( m_{\beta \gamma} \geq 1 \) is the order of \( w_\beta w_\gamma \in W \); furthermore, we have \( 4 \cos^2(\pi/m_{\beta \gamma}) = \langle \gamma, \beta' \rangle \langle \beta, \gamma' \rangle \) for all \( \beta, \gamma \in \Pi \).

Finally, any two bases of \( R \) can be transformed into each other by a unique element of \( W \). In particular, \( r := |\Pi| \) is well-defined and called the rank of \( R \); furthermore, writing \( \Pi = \{ \beta_1, \ldots, \beta_r \} \), the matrix
\[
C := \left( \langle \beta_i, \beta_j' \rangle \right)_{1 \leq i, j \leq r}
\]
is uniquely determined by \( \mathcal{R} \) (up to reordering the rows and columns); it is called the Cartan matrix of \( \mathcal{R} \). We say that two root data \( \mathcal{R}, \mathcal{R}' \) have the same Cartan type if the corresponding Cartan matrices are the same (up to choosing a bijection between the associated bases \( \Pi, \Pi' \)). Thus, \( \mathcal{R} \) and \( \mathcal{R}' \) have the same Cartan type if and only if \( R \subseteq X_Q \) and \( R' \subseteq X_Q' \) are isomorphic root systems (see [Bou68 Chap. VI, n° 1.5]).
We associate with \( C \) a Dynkin diagram, defined as follows. It has vertices labelled by the elements in \( \Pi = \{ \beta_1, \ldots, \beta_r \} \). If \( i \neq j \) and \( |(\beta_j, \beta_i^\vee)| \geq |(\beta_i, \beta_i^\vee)| \), then the corresponding vertices are joined by \( |(\beta_j, \beta_i^\vee)| \) lines; furthermore, these lines are equipped with an arrow pointing towards the vertex labeled by \( \beta_i \) if \( |(\beta_j, \beta_i^\vee)| > 1 \). (Note that, in this case, we automatically have \( (\beta_i, \beta_i^\vee) = -1 \) by (e).)

We say that \( C \) is an indecomposable Cartan matrix if the associated Dynkin diagram is a connected graph; otherwise, we say that \( C \) is decomposable. Clearly, any Cartan matrix can be expressed as a block diagonal matrix with diagonal blocks given by indecomposable Cartan matrices. The classification of indecomposable Cartan matrices is well-known (see [Bou68, Chap. VI, §4]); the corresponding Dynkin diagrams are listed in Table 1.1. (The Cartan matrices of type \( C_n \), \( G_2 \), \( F_4 \), \( E_6 \), \( E_7 \), \( E_8 \) are printed explicitly in Examples 1.2.19, 1.3.7, 1.5.5 below.) See Kac [Kac83, Chap. 4] for a somewhat different approach to this classification.

**Table 1.1.** Dynkin diagrams of indecomposable Cartan matrices

\[
\begin{align*}
A_n & \quad n \geq 1 \\
B_n & \quad n \geq 2 \\
D_n & \quad n \geq 3 \\
G_2 & \\
F_4 & \\
E_6 & \\
E_7 & \\
E_8 & 
\end{align*}
\]

(This labeling will be used throughout this book; it is the same as in CHEVIE [GHLMP], [MICHS]. Note that \( B_2 = C_2 \) and \( D_3 = A_3 \), up to re-labeling the vertices.)

We have the following general characterisation of Cartan matrices.

**Proposition 1.2.7** (Cf. [Bou68, Chap. VI, §4]). Let \( S \) be a finite set and \( C = (c_{st})_{s,t \in S} \) be a matrix with integer entries. Then \( C \) is the Cartan matrix of a reduced crystallographic root system if and only if the following conditions hold:

(C1) We have \( c_{ss} = 2 \) and, for \( s \neq t \) we have \( c_{st} \leq 0 \); furthermore, \( c_{st} \neq 0 \) if and only if \( c_{ts} \neq 0 \).

(C2) For \( s, t \in S \), let \( m_{st} \in \mathbb{Z}_{\geq 1} \) be defined by the condition that \( c_{st}c_{ts} = 4\cos^2(\pi/m_{st}) \). (Thus, we have \( m_{ss} = 1 \) and \( m_{st} \in \{2,3,4,6\} \) for \( s \neq t \). Then the matrix \( (-\cos(\pi/m_{st}))_{s,t \in S} \) is positive-definite.

**Remark 1.2.8.** Let \( C = (c_{st})_{s,t \in S} \) be a Cartan matrix. Let \( \Omega \) be the free abelian group with basis \( \{ \omega_s \mid s \in S \} \). Let \( \mathbb{Z}C \subseteq \Omega \) be the subgroup generated by the columns of \( C \), that is, by all vectors of the form \( \sum_{s \in S} c_{st} \omega_s \) for \( t \in S \). Then

\[
\Lambda(C) := \Omega/\mathbb{Z}C
\]

is called the fundamental group of \( C \). (This agrees with the definitions in [MaTe11, 9.14] or [Spr98, 8.1.11], for example.) If \( C \) is indecomposable, then the groups \( \Lambda(C) \) are easily computed and listed in Table 1.2.
In Section 1.2, we have defined what it means for two root data to be isomorphic. We shall also need the following, somewhat more general notion.

**Definition 1.2.9.** Let \( R = (X, R, Y, R^\vee) \) and \( R' = (X', R', Y', R'^\vee) \) be root data. We fix an integer \( p \) such that either \( p = 1 \) or \( p \) is a prime number. Then a group homomorphism \( \varphi : X' \to X \) is called a \( p \)-isogeny of root data if there exist a bijection \( R \to R' \), \( \alpha \mapsto \alpha^\dagger \), and positive integers \( q_\alpha > 0 \), each an integral power of \( p \), such that \( \varphi \) and its transpose \( \varphi^{tr} : Y \to Y' \) satisfy the following conditions.

1. \( \varphi \) and \( \varphi^{tr} \) are injective.
2. We have \( \varphi(\alpha^\dagger) = q_\alpha \alpha \) and \( \varphi^{tr}(\alpha^\vee) = q_\alpha (\alpha^\dagger)^\vee \) for all \( \alpha \in R \).

The conditions (I) and (II) appear in [Ch56-58/05, §18.2]; following Chevalley, we call the numbers \( \{q_\alpha\} \) the root exponents of \( \varphi \). Note that \( \alpha \mapsto \alpha^\dagger \) and the numbers \( \{q_\alpha\} \) are uniquely determined by \( \varphi \) (since \( R \) is reduced).

Let \( W \subseteq \text{Aut}(X) \) be the Weyl group of \( R \). Then one easily sees that, for any \( \alpha \in R \) and \( w \in W \), we have \( q_{w(\alpha)} = q_\alpha \); see [Spr98, 9.6.4]. Hence, by 1.2.6(c), the map \( \alpha \mapsto q_\alpha \) is determined by its values on a base of \( R \). We also see that \( \varphi \) is an isomorphism of root data if and only if \( \varphi \) is a bijective isogeny where \( q_\alpha = 1 \) for all \( \alpha \in R \). Finally note that if \( p = 1 \), then \( q_\alpha = 1 \) for all \( \alpha \in R \).

A simple example of a \( p \)-isogeny of a root datum into itself is given by \( \varphi : X \to X \), \( \lambda \mapsto p\lambda \) (scalar multiplication with \( p \)); this will be continued in Example 1.3.17.

**Remark 1.2.10.** Keep the notation in the above definition. Let \( W \subseteq \text{Aut}(X) \) be the Weyl group of \( R \) and \( W' \subseteq \text{Aut}(X') \) be the Weyl group of \( R' \). Then one easily sees that a \( p \)-isogeny \( \varphi : X' \to X \) induces a unique group isomorphism

\[
\sigma : W \to W' \quad \text{such that} \quad \varphi \circ \sigma(w) = w \circ \varphi \quad (w \in W).
\]

We have \( \sigma(w_\alpha) = w_{\alpha^\dagger} \) for all \( \alpha \in R \) where \( w_\alpha \in W \) is the reflection corresponding to \( \alpha \in R \) and \( w_{\alpha^\dagger} \in W' \) is the reflection corresponding to \( \alpha^\dagger \in R' \). (See [Ch56-58/05, §18.3] for further details; see also [L2.18] below.) In particular, if \( \varphi : X \to X \) is a \( p \)-isogeny of \( R \) into itself, then \( \varphi \circ W = W \circ \varphi \) and so \( \varphi \) normalizes \( W \subseteq \text{Aut}(X) \).

**Remark 1.2.11.** Let \( R = (X, R, Y, R^\vee) \) and \( R' = (X', R', Y', R'^\vee) \) be root data. Let us fix a base \( \Pi \) of \( R \) and a base \( \Pi' \) of \( R' \); see 1.2.6. Let \( \varphi : X' \to X \) be a group homomorphism which defines a \( p \)-isogeny of root data. Then (I2) shows that \( \Pi^\dagger := \{\alpha^\dagger \mid \alpha \in \Pi\} \) also is a base of \( R' \), where \( \alpha \mapsto \alpha^\dagger \) denotes the bijection \( R \to R' \) associated with \( \varphi \). As already mentioned in 1.2.6 there exists a unique \( w \) in the Weyl group \( W' \) of \( R' \) such that \( \Pi^\dagger = w(\Pi') \). Now \( w \in W' \subseteq \text{Aut}(X') \) certainly is an isomorphism of \( R' \) into itself. Hence, the composition \( \varphi' := \varphi \circ w : X' \to X \) will also be a \( p \)-isogeny of root data and the bijection \( R \to R' \) associated with \( \varphi' \) will map \( \Pi \) onto \( \Pi' \). This shows that, replacing \( \varphi \) by \( \varphi \circ w \) for a suitable \( w \in W' \) if

| Type of \( C \) | \( \text{A}(C) \) |
|----------------|----------------|
| \( A_{n-1} \)   | \( \mathbb{Z}/n\mathbb{Z} \) |
| \( B_n, C_n \)  | \( \mathbb{Z}/2\mathbb{Z} \) |
| \( D_n \)  | \( \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ even} \\ \mathbb{Z}/4\mathbb{Z} & \text{if } n \text{ odd} \end{cases} \) |
| \( G_2, F_4, E_8 \) | \( \{0\} \) |
| \( E_6 \)  | \( \mathbb{Z}/3\mathbb{Z} \) |
| \( E_7 \)  | \( \mathbb{Z}/2\mathbb{Z} \) |

**Table 1.2.** Fundamental groups of indecomposable Cartan matrices


necessary, we can always assume that the bijection $R \to R'$ will preserve the given bases $\Pi \subseteq R$ and $\Pi' \subseteq R'$.

**Remark 1.2.12.** Let $\mathcal{R} = (X, R, Y, R')$ be a root datum and $W \subseteq \text{Aut}(X)$ be the corresponding Weyl group. Let $\Pi$ be a base of $R$. By (1.2.6(e)), $W$ is a Coxeter group with generating set $S = \{w_\alpha \mid \alpha \in \Pi\}$. We denote by $l_W: W \to \mathbb{Z}_{\geq 0}$ the corresponding length function. To unify the notation, we shall use $S$ as an indexing set for $\Pi$, that is, $\Pi = \{\alpha_s \mid s \in S\}$ where $\alpha_s$ is the root of the reflection $s$.

Using the isomorphism $\delta$ in Lemma 1.2.3(a), we can identify $W^\vee = W = \langle S \rangle$. Under this identification, $W$ will act on both $X$ and $Y$, and we have

$$\langle w^{-1} \lambda, \nu \rangle = \langle \lambda, w \nu \rangle$$

for all $w \in W$, $\lambda \in X$, $\nu \in Y$.

Recall that the action of $s \in S$ on $X$ is given by $s \lambda = \lambda - (\lambda, \alpha_s^\vee)\alpha_s$ for all $\lambda \in X$; the action of $s \in S$ on $Y$ is given by $s \nu = \nu - (\alpha_s, \nu)\alpha_s^\vee$ for all $\nu \in Y$.

We now turn to the question of actually constructing root data and $p$-isogenies for a Cartan matrix $C$. The key idea is contained in the following remark.

**Remark 1.2.13.** Let $\mathcal{R} = (X, R, Y, R')$ be a root datum and let $\Pi$ be a base of $R$. Let $C$ be the corresponding Cartan matrix; see 1.2.6. Let $\{\lambda_i \mid i \in I\}$ be a $\mathbb{Z}$-basis of $X$, where $I$ is some finite indexing set. (We have $|I| \geq |\Pi|$.) Let $\{\nu_i \mid i \in I\}$ be the corresponding factorisation.

We also note that the dual root system $(Y, R'^\vee, X, R)$ (see Lemma 1.2.3) has Cartan matrix $C'^\vee$, with corresponding factorisation $C'^\vee = B \cdot B'^\vee$ where $B = \tilde{A}$ and $B'^\vee = A$.

Following [BrLu12] §2.1, we shall now reverse this argument and show that every factorisation of a Cartan matrix $C$ as above leads to a root datum.

1.2.14. Let $S$ be a finite set and $C = (c_{st})_{s,t \in S}$ be a Cartan matrix, that is, a matrix satisfying the conditions in Proposition 1.2.3. Let $I$ be another finite index set, with $|I| \geq |S|$, and assume that we have a factorisation

$$C = \tilde{A} \cdot A^\text{tr}$$

where $A = (a_{si})_{s \in S, i \in I}$ and $\tilde{A} = (\tilde{a}_{si})_{s \in S, i \in I}$.

Note that all of $R \subseteq X$ and $R'^\vee \subseteq Y$ are uniquely determined by $C, A$ and $\tilde{A}$. (This follows from 1.2.6(c) and the fact that $W = \langle w_\beta \mid \beta \in \Pi \rangle$; see 1.2.6(e).) We also note that the dual root system $(Y, R'^\vee, X, R)$ has Cartan matrix $C'^\vee$, with corresponding factorisation $C'^\vee = B \cdot B'^\vee$ where $B = \tilde{A}$ and $B'^\vee = A$.

For $s \in S$, we define endomorphisms $w_s: X \to X$ and $w_s^\vee: Y \to Y$ by

$$w_s(\lambda) = \lambda - (\lambda, \alpha_s^\vee)\alpha_s$$

and

$$w_s^\vee(\nu) = \nu - (\alpha_s, \nu)\alpha_s^\vee$$

for all $s \in S$. 

for all \( \lambda \in X \) and \( \nu \in Y \). Then \( w_s^2 = \text{id}_X \) and \((w_t^\nu)^2 = \text{id}_Y \). Hence, we have \( w_s \in \text{Aut}(X) \) and \( w_t^\nu \in \text{Aut}(Y) \) for all \( s \in S \). Let

\[
W := \langle w_s \mid s \in S \rangle \subseteq \text{Aut}(X) \quad \text{and} \quad W^\nu := \langle w_s^\nu \mid s \in S \rangle \subseteq \text{Aut}(Y).
\]

Finally, let \( R := \{ w(\alpha_s) \mid w \in W, s \in S \} \) and \( R^\nu := \{ w(\alpha_s^\nu) \mid w^\nu \in W^\nu, s \in S \} \).

Lemma 1.2.15. The quadruple \( \mathscr{Q} := (X, R, Y, R^\nu) \) in [1.2.14] is a root datum with Cartan matrix \( C \), where \( \{ \alpha_s \mid s \in S \} \) is a base of \( R \) and \( \{ \alpha_s^\nu \mid s \in S \} \) is a base of \( R^\nu \). Furthermore, we have \( W = W \) and \( W^\nu = W^\nu \) (with the notation of [1.2.1]). The bijection \( R \to R^\nu, \alpha \mapsto \alpha^\nu \), is determined as follows. If \( w \in W \) and \( s \in S \) are such that \( \alpha = w(\alpha_s) \), then \( \alpha^\nu = \delta(w)(\alpha_s^\nu) \) (with \( \delta \) as in Lemma 1.2.3(a)).

Proof. Let \( V \subseteq \mathbb{Q} \otimes \mathbb{R} X \) be the subspace spanned by \( \{ \alpha_s \mid s \in S \} \). Then \( w_s(\alpha_t) = \alpha_t - c_{st} \alpha_s \) for all \( t \in S \). So \( w_s(V) \subseteq V \) for all \( s \in S \). Let \( W_R \subseteq \text{GL}(V) \) be the group generated by the restrictions \( w_s : V \to V \); then \( R = \{ w(\alpha_s) \mid w \in W_R, s \in S \} \subseteq V \). Thus, we are in the setting of [GePf00] [1.1].

The matrix \( C \) is symmetric, that is, there exist positive numbers \( \{ d_s \mid s \in S \} \) such that \( (d_s c_{st})_{s,t \in S} \) is a symmetric matrix. (This easily follows from the fact that there are no closed paths in the Dynkin diagram of \( C \); see also [Kac85] [4.6].) Then we can define a \( W_R \)-invariant symmetric bilinear form on \( V \) by \( (\alpha_s, \alpha_t) = d_s c_{st}/2 \) for \( s, t \in S \). The \( W_R \)-invariance implies that

\[
c_{st} = 2 \frac{(\alpha_s, \alpha_t)}{(\alpha_s, \alpha_s)} \quad \text{for all } s, t \in S;
\]

see [GePf00] 1.3.2]. By (C2), this form is positive-definite; furthermore, each \( w_s : V \to V \) is an orthogonal reflection with root \( \alpha_s \). So \( W_R \) and \( R \) are finite; see [Bou68] Chap. V, [8] or [GePf00] 1.3.8]. In fact, \( R \) is a root system in \( V \) with Weyl group \( W_R \), with \( \{ \alpha_s \mid s \in S \} \) as base and \( C \) as Cartan matrix; see [GePf00] 1.1.10]. Also note that \( R \) is reduced, that is, \( (R2) \) holds; see [GePf00] 1.3.7]. Let \( R^+ \) be the set of positive roots in \( R \) defined by the base \( \{ \alpha_s \mid s \in S \} \).

Similarly, let \( V^\nu \subseteq \mathbb{Q} \otimes \mathbb{R} Y \) be the subspace spanned by \( \{ \alpha_s^\nu \mid s \in S \} \). Then \( w_s^\nu(\alpha_t^\nu) = \alpha_t^\nu - c_{st}^\nu \alpha_s^\nu \) for all \( t \in S \). Let \( W_{R^\nu} \subseteq \text{GL}(V^\nu) \) be the group generated by the restrictions \( w_s^\nu : V^\nu \to V^\nu \). Again, we are in the setting of [GePf00] [1.1] (with respect to \( C^\nu \), the transpose of \( C \)) and so we can repeat the previous argument. Consequently, \( W_{R^\nu} \) is also finite; furthermore, \( R^\nu \) is a reduced root system in \( V^\nu \) with Weyl group \( W_{R^\nu} \), with \( \{ \alpha_s^\nu \mid s \in S \} \) as base and \( C^\nu \) as Cartan matrix. Let \( (R^\nu)^+ \) be the set of positive roots in \( R^\nu \) defined by the base \( \{ \alpha_s^\nu \mid s \in S \} \).

Next, we define a linear map \( f : V \to V^\nu \) by \( f(\alpha_s) = \frac{1}{2}(\alpha_s, \alpha_s)\alpha_s^\nu \) for all \( s \in S \). (This is analogous to the definition in the proof of Lemma 1.2.4] Clearly, \( f \) is bijective. One immediately checks that \( w_s^\nu \circ f = f \circ w_s \) for all \( s \in S \). So the map \( w \mapsto f \circ w \circ f^{-1} \) defines a group isomorphism \( \delta : W_R \cong W_{R^\nu} \) such that \( \delta(w_s) = w_s^\nu \) for all \( s \in S \). Consequently, we can define a bijection \( R \to R^\nu, \alpha \mapsto \alpha^\nu \), as follows. First, let \( \alpha \in R^+ \). By definition, \( \alpha = w(\alpha_s) \) for some \( w \in W, s \in S \). Then

\[
f(\alpha) = f(w(\alpha_s)) = \delta(w)(f(\alpha_s)) = \frac{1}{2}(\alpha_s, \alpha_s)\delta(w)(\alpha_s^\nu).
\]

Since \( \alpha_s^\nu \in R^\nu \), we have \( \delta(w)(\alpha_s^\nu) \in R^\nu \) by definition. Thus, \( f(\alpha) \in V^\nu \) is a positive scalar multiple of some element of \( R^\nu \); since \( R^\nu \) is reduced, there is a unique positive root with this property, denoted \( \alpha^\nu \), and the above computation shows that \( \alpha^\nu = \delta(w)(\alpha_s^\nu) \). The definition of \( \alpha^\nu \) for negative \( \alpha \) is analogous; we then have \( (-\alpha)^\nu = -\alpha^\nu \) for all \( \alpha \in R \). Consequently, we obtain a map \( R \to R^\nu \), \( \alpha \mapsto \alpha^\nu \), which is easily seen to be bijective. Once this is established, the maps
In particular, we obtain orthogonal reflection with root isomorphism. Thus, given a Cartan matrix data of Cartan type, we can think of the various root data of Cartan type $C$ simply as factorisations $C = \tilde{A} \cdot A^{tr}$, where $A, \tilde{A}$ are integer matrices of the same size. (This observation, in this explicit form, appears in [BrLu12]. It is also implicit in [Lu89, §1], [Lu09d, §§1–3].)

Example 1.2.16. Let $C$ be a Cartan matrix. We have just seen that any factorisation $C = \tilde{A} \cdot A^{tr}$ as in [1.2.14] gives rise to a root datum $\mathcal{R} = (X, R, Y, R^{\vee})$. Obviously, there are two natural choices for such a factorisation, namely,

- either $A$ is the identity matrix and, hence, $\tilde{A} = C$;
- or $\tilde{A}$ is the identity matrix and, hence, $A = C^{tr}$.

In the first case, we denote the corresponding root datum by $\mathcal{R} = \mathcal{R}_{ad}(C)$. We have $X = \mathbb{Z}R$ in this case; any root datum satisfying $X = \mathbb{Z}R$ will be called a root datum of adjoint type. In the second case, we denote the corresponding root datum by $\mathcal{R} = \mathcal{R}_{sc}(C)$. We have $Y = \mathbb{Z}R^{\vee}$ in this case; any root datum satisfying $Y = \mathbb{Z}R^{\vee}$ will be called a root datum of simply-connected type.

Thus, $\mathcal{R}_{ad}(C)$ and $\mathcal{R}_{sc}(C)$ may be regarded as the standard models of root data of adjoint type and simply-connected type, respectively. (See also Example 1.2.21.)

The relevance of these notions will become clearer when we consider semisimple algebraic groups in Section 1.5.

Example 1.2.17. There is an obvious notion of direct product of root data. Indeed, if $\mathcal{R}_{i} = (X_{i}, R_{i}, Y_{i}, R_{i}^{\vee})$ for $i = 1, \ldots, n$ are root data, then we obtain a new root datum $\mathcal{R} = (X, R, Y, R^{\vee})$ as follows. We set

$$X := X_{1} \oplus \cdots \oplus X_{n}, \quad R := R_{1} \cup \cdots \cup R_{n},$$

$$Y := Y_{1} \oplus \cdots \oplus Y_{n}, \quad R^{\vee} := R_{1}^{\vee} \cup \cdots \cup R_{n}^{\vee},$$

where, for each $i$, we let $R_{i} \subseteq X$ denote the image of $R_{i}$ under the natural embedding $X_{i} \hookrightarrow X$; similarly, $R_{i}^{\vee} \subseteq Y$ denotes the image of $R_{i}^{\vee}$ under the natural embedding $Y_{i} \hookrightarrow Y$. Furthermore, the perfect bilinear pairings for the various $\mathcal{R}_{i}$ define a unique perfect bilinear pairing for $\mathcal{R}$ in a natural way. Also note that, if $\Pi_{i}$ is a base of $R_{i}$ for $i = 1, \ldots, n$, then $\Pi := \Pi_{1} \cup \cdots \cup \Pi_{n}$ is a base of $R$.

In terms of the matrix language of 1.2.14, the situation is described as follows. Each $\mathcal{R}_{i}$ is determined by a factorisation $C_{i} = \tilde{A}_{i} \cdot A_{i}^{tr}$ where $C_{i}$ is the Cartan matrix of $R_{i}$ with respect to a base $\Pi_{i}$ of $R_{i}$. Then $\mathcal{R}$ is determined by the factorisation $C = \tilde{A} \cdot A^{tr}$, where $C, A$ and $\tilde{A}$ are block diagonal matrices with diagonal blocks given by $C_{i}, A_{i}$ and $\tilde{A}_{i}$, respectively. The matrix $C$ is the Cartan matrix of $R$ with respect to the base $\Pi = \Pi_{1} \cup \cdots \cup \Pi_{n}$.

We now translate the conditions in Definition 1.2.9 into the matrix language of Remark 1.2.13. This will be an extremely efficient tool for constructing isogenies, as it reduces the conditions to be checked to the verification of simple matrix identities.
1.2.18. Let \( \mathcal{R} = (X, R, Y, R') \) and \( \mathcal{R}' = (X', R', Y', R') \) be root data. Assume that \( X \) and \( X' \) have the same rank and that \( R \) and \( R' \) have bases indexed by the same set \( S \). Denote these bases by \( \Pi = \{ \alpha_s \mid s \in S \} \) and \( \Pi' = \{ \beta_s \mid s \in S \} \), respectively. Let \( C \) and \( C' \) be the corresponding Cartan matrices. Let us also fix a \( \mathbb{Z} \)-basis \( \{ \lambda_i \mid i \in I \} \) of \( X \) and a \( \mathbb{Z} \)-basis \( \{ X_j' \mid j \in J \} \) of \( X' \). Then \( \mathcal{R} \) and \( \mathcal{R}' \) are determined by factorisations as in Remark 1.2.13.

\[
C = \tilde{A} \cdot A^{tr} \quad \text{where} \quad A = (a_{s,i})_{s \in S, i \in I} \quad \text{and} \quad \tilde{A} = (\tilde{a}_{s,i})_{s \in S, i \in I},
\]
\[
C' = \tilde{B} \cdot B^{tr} \quad \text{where} \quad B = (b_{s,j})_{s \in S, j \in J} \quad \text{and} \quad \tilde{B} = (\tilde{b}_{s,j})_{s \in S, j \in J}.
\]

(Here, \(|I| = |J| \), since \( X, X' \) have the same rank.) Now giving a linear map \( \varphi : X' \rightarrow X \) is the same as giving a matrix \( P = (p_{ij})_{i \in I, j \in J} \) with integer coefficients:

\[
\varphi(X_j') = \sum_{i \in I} p_{ij} \lambda_i \quad \text{for all} \quad j \in J.
\]

Assume now that \( \varphi : X' \rightarrow X \) is a linear map which is “base preserving”, in the sense that there is a permutation \( S \rightarrow S, s \mapsto s' \), such that

\[
\varphi(\beta_s) = q_s \alpha_s \quad \text{where} \quad 0 \neq q_s \in \mathbb{Z} \quad \text{for all} \quad s \in S.
\]

We encode this in a monomial matrix \( P^o = (p^o_{ss'})_{s \in S} \) where \( p^o_{ss'} = q_s \) for \( s \in S \). Let \( p = 1 \) or \( p \) be a prime number and assume that \( \varphi \) is a \( p \)-isogeny. Then the conditions in Definition 1.2.9 immediately imply that the following conditions hold.

(M1) \( P^o \) is a monomial matrix whose non-zero entries are all powers of \( p \).

(M2) \( P \) is invertible over \( \mathbb{Q} \); furthermore, \( P \cdot B^{tr} = A^{tr} \cdot P^o \) and \( P^o \cdot \tilde{B} = \tilde{A} \cdot P \).

Conversely, it is straightforward to check that \( \text{any} \) pair of integer matrices \( (P, P^o) \) satisfying (M1) and (M2) defines a \( p \)-isogeny of root data. The argument is similar to the proof of Lemma 1.2.15; let us just briefly sketch it. Let \( \varphi : X' \rightarrow X \) be the linear map with matrix \( P \). Condition (1) holds since \( P \) is invertible over \( \mathbb{Q} \). Since \( P^o \) is monomial, there is a permutation \( S \rightarrow S, s \mapsto s' \), such that

\[
q_s := p^o_{ss'} \neq 0 \quad \text{for all} \quad s \in S.
\]

Then (M2) means that \( \varphi(\beta_s) = q_s \alpha_s \) and \( \varphi^\vee(\alpha_s^\vee) = q_s \beta_s^\vee \) for all \( s \in S \). Thus, \( (I2) \) holds for simple roots and coroots. To see that \( (I2) \) holds for all roots and coroots, note that (M2) implies that \( C \cdot P^o = P^o \cdot C' \). Consequently, using Lemma 1.2.6(e) for \( W \) and for \( W' \), there is a unique group isomorphism

\[
\sigma : W \rightarrow W' \quad \text{such that} \quad w_{\alpha_s} \mapsto w_{\beta_s} \quad (s \in S).
\]

Using Lemma 1.2.5, one shows that this implies that

\[
\varphi \circ \sigma(w) = w \circ \varphi \quad \text{for all} \quad w \in W.
\]

We can now define a bijection \( R \rightarrow R', \alpha \mapsto \alpha' \), with the required properties, as follows. Let \( \alpha \in R \) and write \( \alpha = w(\alpha_s) \) for some \( w \in W \) and \( s \in S \). Then we set \( \alpha' := \sigma(w)(\beta_s) \in R' \). Now, we have

\[
\varphi(\alpha') = \varphi(\sigma(w)(\beta_s)) = w(\varphi(\beta_s)) = q_s w(\alpha_s) = q_s \alpha.
\]

Since \( \varphi \) is injective, \( \alpha' \) is uniquely determined by \( \alpha \) (and does not depend on the choice of \( w \) and \( s \)); furthermore, the first of the two identities in \( (I2) \) holds, where \( q_s = q_s \). The argument for the second identity is similar, using the bijection \( R \rightarrow R' \) (see Lemma 1.2.15). Thus, \( (I2) \) is seen to hold for all roots and coroots.
Example 1.2.19 (Cf. [Ch56-58/05 §§21.5, 22.4, 23.7]). Let \( C = (c_{ij})_{1 \leq i, j \leq r} \) (\( r = 2 \) or \( 4 \)) be a Cartan matrix of type \( C_2 \), \( G_2 \) or \( F_4 \); see Table 1.1. Explicitly:

\[
C_2 : \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad F_4 : \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.
\]

(Note that \( C_2 = B_2 \) up to relabeling the two vertices of the Dynkin diagram.)

We set \( p = 2 \) if \( C \) is of type \( C_2 \) or \( F_4 \), and \( p = 3 \) if \( C \) is of type \( G_2 \). Let us consider the corresponding root datum \( \mathcal{R} = \mathcal{R}_{\text{ad}}(C) = (X, R, Y, R^\vee) \) as in Example 1.2.16; we have \( C = \hat{A} \cdot A^{rt} \), where \( A \) is the identity matrix and \( C = \hat{A} \). For any \( m \geq 0 \), we define two matrices \( P_m^o \) and \( P_m \) as follows:

\[
C_2 : \quad P_m = P_m^o := \begin{pmatrix} 0 & 2m+1 \\ 2m+1 & 0 \end{pmatrix},
\]

\[
G_2 : \quad P_m = P_m^o := \begin{pmatrix} 0 & 3m+1 \\ 3m+1 & 0 \end{pmatrix},
\]

\[
F_4 : \quad P_m = P_m^o := \begin{pmatrix} 0 & 0 & 0 & 2m \\ 0 & 0 & 2m+1 & 0 \\ 0 & 2m & 0 & 0 \\ 2m+1 & 0 & 0 & 0 \end{pmatrix}.
\]

Now, in the setting of 1.2.18 let \( C' = C \), \( B = A \), \( \hat{B} = \hat{A} \). Then \( P_m, P_m^o \) satisfy (MI1), (MI2) and, hence, the pair \( (P_m, P_m^o) \) defines a group homomorphism \( \varphi_m : X \to X \) which is a \( p \)-isogeny of \( \mathcal{R} \) into itself, such that \( \varphi_m^2 = p^{2m+1} \text{id}_X \).

See [Ca72] 12.3, 12.4 for a more detailed discussion of these "exceptional" isogenies; they give rise to the finite Suzuki and Ree groups (see Example 1.4.21). Another instance of such an "exceptional" isogeny will be considered in Example 1.5.5.

Lemma 1.2.20. Let \( C \) be a Cartan matrix. Let \( \mathcal{R} = (X, R, Y, R^\vee) \) and \( \mathcal{R}' = (X', R', Y', R'^\vee) \) be root data which are both of Cartan type \( C \). Choosing bases for \( R, R' \) and for \( X, X' \), let \( C = \hat{A} \cdot A^{rt} = \hat{B} \cdot B^{rt} \) be the corresponding factorisations as in Remark 1.2.19 (where \( A, \hat{A} \) correspond to \( \mathcal{R} \) and \( B, \hat{B} \) correspond to \( \mathcal{R}' \)). Then \( \mathcal{R}, \mathcal{R}' \) are isomorphic root data if and only if there exist square matrices \( P, P^o \) with integer coefficients such that \( P^o \) is a permutation matrix, \( P \) is invertible over \( \mathbb{Z} \) and we have \( P \cdot B^{rt} = A^{rt} \cdot P^o, P^o \cdot \hat{B} = \hat{A} \cdot P \).

Proof. Recall from Definition 1.2.9 that a \( p \)-isogeny \( \varphi : X' \to X \) is an isomorphism of root data if and only if \( \varphi \) is bijective and \( q_\alpha = 1 \) for all \( \alpha \in R \). Furthermore, replacing \( \varphi \) by \( \varphi \circ w \) for some \( w \in W' \) (the Weyl group of \( \mathcal{R}' \)) if necessary, we may assume that \( \varphi \) sends the chosen base of \( R' \) to the chosen base of \( R \) (see Remark 1.2.11). Hence, in the setting of 1.2.18 \( \varphi \) is an isomorphism if and only if \( P^o \) is a permutation matrix and \( P \) is invertible over \( \mathbb{Z} \). \( \square \)

Example 1.2.21. Assume that \( C \) is a Cartan matrix of type \( G_2 \), \( F_4 \) or \( E_8 \). Then \( C \) is invertible over \( \mathbb{Z} \) and, hence, \( \mathcal{R}_{\text{ad}}(C) \) and \( \mathcal{R}_{\text{sc}}(C) \) are isomorphic root data. Indeed, \( \mathcal{R}_{\text{ad}}(C) \) corresponds to the factorisation \( C = \hat{A} \cdot A^{rt} \) where \( A \) is the identity matrix, while \( \mathcal{R}_{\text{sc}}(C) \) corresponds to the factorisation \( C = \hat{B} \cdot B^{rt} \) where \( \hat{B} \) is the identity matrix. Then the conditions in Lemma 1.2.20 hold, where \( P = C^{-1} \) and \( P^o \) is the identity matrix.
1.3. Chevalley’s classification theorems

Throughout this section, let $k$ be an algebraically closed field and $G$ be a linear algebraic group over $k$. We can now explain how one can naturally attach to $G$ a root datum, when $G$ is connected reductive.

1.3.1. Assume that $G$ is connected reductive. Let $T \subseteq G$ be a maximal torus, $X = X(T)$ and $L(G)$ be the Lie algebra. Recall from 1.1.11 that there is a finite subset $R \subseteq X$ and a corresponding root space decomposition of $L(G)$:

$$L(G) = L(G)_0 \oplus \bigoplus_{\alpha \in R} L(G)_\alpha.$$

As already mentioned in 1.1.11 we have $L(G)_0 = L(T)$, $R = -R$ and $\dim L(G)_\alpha = 1$ for all $\alpha \in R$; in particular,

$$\dim G = \dim L(G) = \dim T + |R|.$$

The roots can be directly characterised in terms of $G$, as follows. Let $\alpha \in X$. Then $\alpha$ is a root if and only if there exists a homomorphism of algebraic groups $u_\alpha : k^+ \to G$ such that $u_\alpha$ is an isomorphism onto its image and we have

$$t u_\alpha(\xi) t^{-1} = u_\alpha(\alpha(t) \xi) \quad \text{for all } t \in T \text{ and } \xi \in k.$$

Thus, $U_\alpha := \{ u_\alpha(\xi) \mid \xi \in k \} \subseteq G$ is a one-dimensional closed connected unipotent subgroup normalized by $T$. It is uniquely determined by $\alpha$ and called the root subgroup corresponding to $\alpha$. Conversely, every one-dimensional closed connected unipotent subgroup normalized by $T$ is equal to $U_\alpha$ for some $\alpha \in R$. We have

$$G = \langle T, U_\alpha \mid \alpha \in R \rangle.$$

Now consider also the co-character group $Y = Y(T)$; we wish to define a finite subset $R^\vee \subseteq Y$. Recall from 1.1.10 that $X, Y$ are free abelian groups of the same (finite) rank and that there is a natural pairing $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$. The Weyl group of $G$ with respect to $T$ is defined as $W(G, T) := N_G(T)/T$. Since $N_G(T)$ acts on $T$ by conjugation, we have induced actions of $W(G, T)$ on $X$ and on $Y$ via

$$(w.\lambda)(t) = \lambda(w^{-1} t w) \quad (\lambda \in X, t \in T),$$

$$(w.\nu)(\xi) = \nu w(\xi) w^{-1} \quad (\nu \in Y, \xi \in k^\times),$$

where, for any $w \in W(G, T)$, we denote by $w$ a representative in $N_G(T)$. Using these actions, we can identify $W(G, T)$ with subgroups of Aut$(X)$ and of Aut$(Y)$. Now let $\alpha \in R$. Then $G_\alpha := C_G(\ker(\alpha)^\circ) = \langle T, U_\alpha, U_{-\alpha} \rangle$ is a closed connected reductive subgroup of $G$; its Weyl group $W(G_\alpha, T) := N_{G_\alpha}(T)/T$ has order 2. Let $w_\alpha \in W(G_\alpha, T)$ be the non-trivial element and $\hat{w}_\alpha$ be a representative of $w_\alpha$ in $N_{G_\alpha}(T) \subseteq N_G(T)$. Then there exists a unique $\alpha^\vee \in Y$ such that

$$w_\alpha.\lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \quad \text{for all } \lambda \in X.$$

Following, e.g., [Con14, 1.2.8], this element $\alpha^\vee$ can also be determined as follows. The maps $u_{\pm \alpha} : k^+ \to U_{\pm \alpha}$ can be chosen such that the assignment

$$ \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \mapsto u_\alpha(\xi), \quad \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \mapsto u_{-\alpha}(\xi) \quad (\xi \in k)$$

defines a homomorphism of algebraic groups $\varphi_\alpha : SL_2(k) \to G$. Then we have

$$\alpha^\vee(\xi) = \varphi_\alpha \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \in T \quad \text{for all } \xi \in k^\times.$$
Thus, we obtain a well-defined finite subset $R^\vee = \{ \alpha^\vee \mid \alpha \in R \} \subseteq Y$; we have

$$W(G, T) = \langle w_\alpha \mid \alpha \in R \rangle.$$

Complete proofs of the above statements can be found in [Bor91], [Hum91], and, of course, the original source [Ch56-58/05]. A thorough guide through this argument, with indications of the proofs and many worked-out examples, can be found in [MaTe11] §8. (See also [Al09], [Jan03 Chap. II].)

With this notation, we can now state the following result which shows that we are exactly in the situation described by Proposition 1.1.14.

**Theorem 1.3.2.** The quadruple $\mathcal{R} = (X(T), R, Y(T), R^\vee)$ in 1.3.1 is a root datum as defined in 1.2.1 with Weyl group $W(G, T)$ (identified with a subgroup of $\text{Aut}(X(T))$) as above). Furthermore, let $R^+ \subseteq R$ be the set of positive roots with respect to a base $\Pi \subseteq R$. Then $B := \langle T, U_\alpha \mid \alpha \in R^+ \rangle \subseteq G$ is a Borel subgroup and $B, N_G(T)$ form a reductive BN-pair in $G$ where $C_G(T) = T = B \cap N_G(T)$.

**Proof.** In its essence, this is due to Chevalley [Ch56-58/05], but the notion of BN-pairs was not yet available at that time. A proof of the fact that $\mathcal{R}$ is a root datum can be found, for example, in [MaTe11 9.11], [Spr98 7.4.3]. The BN-pair axioms are shown in [Bor91 14.15], [MaTe11 11.16]. For the equality $C_G(T) = T$, see [MaTe11 8.13] or [Spr98 7.6.4].

**Theorem 1.3.3.** Assume that $G$ is connected reductive. Then $G$ acts transitively (by simultaneous conjugation) on the set of all pairs $(T, B)$ where $T \subseteq G$ is a maximal torus and $B \subseteq G$ is a Borel subgroup such that $T \subseteq B$. In particular, the root data (as in Theorem 1.3.2) with respect to any two maximal tori of $G$ are isomorphic in the sense of 1.2.2.

**Proof.** The conjugacy results are due to Borel [Bor91 10.6, 11.1]; see also [Spr98 6.2.7, 6.3.5]. A somewhat more elementary proof of the conjugacy of Borel subgroups is given in [St77]. (See also [Ge03a §3.4].) Once these conjugacy results are shown, the assertion about the isomorphism between root data is clear.

**Remark 1.3.4.** Let $G, T, \mathcal{R}, B$ as in Theorem 1.3.2 let $W := W(G, T)$. Then the set of all Borel subgroups of $G$ containing $T$ is described as follows. Let $B_1$ be any Borel subgroup of $G$ containing $T$. By Theorems 1.3.2, 1.3.3 and the BN-pair axioms, there is a unique $w \in W$ such that $B_1 = \hat{w}^{-1}B\hat{w}$. Now, the base $\Pi \subseteq R$ used to define $B$ is transformed under $w$ to a new base $\Pi_1$ of $R$. Consequenty, we have $B_1 = \langle T, U_\alpha \mid \alpha \in R^+_1 \rangle$ where $R^+_1 \subseteq R$ is the set of positive roots with respect to $\Pi_1$. Further recall from 1.2.6 that any two bases of $R$ can be transformed into each other by a unique element of $W$. Thus, we obtain bijective correspondences

$$\{\text{Borel subgroups containing } T\} \quad \cong \quad W \quad \cong \quad \{\text{bases of } R\}.$$ 

**Remark 1.3.5.** Assume that $G$ is connected reductive. In 1.1.12 we have defined $G$ to be semisimple if $|Z| < \infty$ where $Z = Z(G)$ denotes the center of $G$; alternatively, $G$ is semisimple if and only if $G = G_{\text{der}}$. We also have the following characterisation in terms of the root datum $\mathcal{R} = (X, R, Y, R^\vee)$ (with respect to a maximal torus $T \subseteq G$). By [MaTe11 8.17(h)], [Spr98 8.1.8], we have

(a) $Z = \{ t \in T \mid \alpha(t) = 1 \text{ for all } \alpha \in R \}$

and the isomorphism $T \cong \text{Hom}(X(T), k^\times)$ in 1.1.10 restricts to an isomorphism

(b) $Z \cong \text{Hom}(X(T)/ZR, k^\times)$. 
Thus, we obtain the equivalences:
\[(c) \quad |Z| < \infty \iff |X/ZR| < \infty \iff |Y/ZR^\vee| < \infty.\]

If we consider the factorisation \(C = \tilde{A} \cdot A^r\) determined by \(\mathcal{R}\) as in Remark 1.2.13 then \(G\) is semisimple if and only if \(A, \tilde{A}\) are square matrices.

**Remark 1.3.6.** Assume that \(G\) is connected reductive. As in \[1.1.12\] we have \(G = Z^0 \cdot G_{\text{der}}\); furthermore, \(G_{\text{der}} = G_1 \cdots G_n\) where \(G_1, \ldots, G_n\) are the closed normal simple subgroups of \(G\); they commute pairwise with each other. These subgroups have the following description in terms of the root datum \(\mathcal{R} = (X, R, Y, R^\vee)\) (with respect to a maximal torus \(T \subseteq G\)) and the corresponding root subgroups \(U_\alpha\) \((\alpha \in R)\). First note that

\[G_{\text{der}} = \langle U_\alpha \mid \alpha \in R \rangle,\]

see \[MaTe11, 8.21\]. Now let \(C\) be the Cartan matrix of the root system \(R\), with respect to a base \(\Pi\) of \(R\). Then \(C\) can be expressed as a block diagonal matrix where the diagonal blocks are indecomposable Cartan matrices, \(C_1, \ldots, C_n\) say. (Thus, \(C_1, \ldots, C_n\) correspond to the connected components of the Dynkin diagram of \(C\).) Let \(\Pi = \Pi_1 \cup \ldots \cup \Pi_n\) be the corresponding partition of \(\Pi\). Then we also have \(R = R_1 \cup \ldots \cup R_n\) where \(R_i\) consists of all roots in \(R\) which can be expressed as linear combinations of simple roots in \(\Pi_i\). Then we have

\[G_i = \langle U_\alpha \mid \alpha \in R_i \rangle \subseteq G \quad \text{for } i = 1, \ldots, n.\]

A maximal torus of \(G_i\) is given by \(T_i := G_i \cap T\) where \(T\) is a fixed maximal torus of \(G\). (See \[Bor91, Chap. IV, \S11\], \[MaTe11, \S8.4\], \[Spr98, \S8.1\] for further details.)

Before continuing with the general theory, we give three concrete examples. We shall see that the point of view in \[1.2.13\] where root data are described in terms of factorisations of Cartan matrices, provides a particularly efficient and convenient way of encoding the information involved in these examples.

**Example 1.3.7.** Let \(G = GL_n(k)\). Let \(B \subseteq G\) be the subgroup of all upper triangular matrices and \(N \subseteq G\) the subgroup of all monomial matrices. It is well-known that these groups form a BN-pair; see \[Bou68, Chap. IV, \S2.2\]. For further details see \[Ge03a, 1.6.10, 3.4.5\], where it is also shown that this is an algebraic BN-pair satisfying the conditions in Proposition \[1.1.14\] in particular, \(G\) is connected reductive. Let us describe the root datum of \(G\) with respect to the maximal torus \(T = B \cap N\) consisting of all diagonal matrices in \(G\).

It will be convenient to introduce some notation concerning matrices. For \(1 \leq i \leq n - 1\), let \(n_i\) be the matrix which is obtained by interchanging the \(i\)-th and the \((i + 1)\)-th row in the identity matrix, which we denote by \(I_n\). More generally, if \(w \in \mathfrak{S}_n\) is any permutation, let \(n_w\) be the matrix which is obtained by permuting the rows of \(I_n\) as specified by \(w\). (Thus, if \(\{e_1, \ldots, e_n\}\) denotes the standard basis of \(k^n\), then \(n_w(e_i) = e_w(i)\) for \(1 \leq i \leq n\); we have \(n_ww' = n_u n_w w'\) for all \(w, w' \in \mathfrak{S}_n\).) Then \(N = \{hn_w \mid h \in T, w \in \mathfrak{S}_n\}\) and so we have an exact sequence

\[1 \to T \to N \to \mathfrak{S}_n \to 1,\]

where \(N \to \mathfrak{S}_n\) sends \(n_w\) to \(w\). Next, for \(1 \leq i < j \leq n\) let \(E_{ij}\) be the “elementary” matrix with coefficient 1 at the position \((i, j)\) and 0 otherwise. We define

\[U_{ij} = \{I_n + \xi E_{ij} \mid \xi \in k\} \quad \text{where } 1 \leq i, j \leq n, i \neq j.\]

All of these are one-dimensional, closed connected subgroups of \(G\). Finally, if \(\xi_1, \ldots, \xi_n\) are non-zero elements of \(k\), we denote by \(h(\xi_1, \ldots, \xi_n) \in T\) the diagonal
matrix with $\xi_1, \ldots, \xi_n$ along the diagonal. Then the map
\[(k^\times)^n \to T, \quad (\xi_1, \ldots, \xi_n) \mapsto h(\xi_1, \ldots, \xi_n),\]
certainly is an isomorphism of algebraic groups. Hence $X = X(T)$ is the free abelian group with basis $\lambda_1, \ldots, \lambda_n$ where $\lambda_i(h(\xi_1, \ldots, \xi_n)) = \xi_i$ for all $i$.

Each subgroup $U_{ij}$ is normalised by $T$. Let $u_{ij} : k^\times \to G$ be the homomorphism given by $u_{ij}(\xi) = I_n + \xi E_{ij}$ for $\xi \in k^\times$. Then $U_{ij}$ is the image of this homomorphism, $u_{ij}$ is an isomorphism onto its image and we have
\[tu_{ij}(\xi) t^{-1} = u_{ij}(\xi \xi_j^{-1})\]
where $t = h(\xi_1, \ldots, \xi_n) \in T$ and $\xi \in k^\times$.

Hence, $\alpha_{ij} := \lambda_i - \lambda_j \in X$ is a root and $U_{ij}$ is the corresponding root subgroup. To see that these are all the roots, we can use the formula $\dim G = \dim T + |R|$ in Remark 1.3.1. Thus, since $\dim G = n^2$ and $\dim T = n$, we conclude that $R = \{\alpha_{ij} \mid 1 \leq i, j \leq n, i \neq j\}$ is the root system of $G$ with respect to $T$. We also see that
\[\Pi := \{\alpha_{i,i+1} = \lambda_i - \lambda_{i+1} \mid 1 \leq i \leq n-1\} \subseteq R\]
is a base of $R$ and that $B$ is the Borel subgroup associated with this base (as in Remark 1.3.4). Now consider coroots. The dual basis of $Y$ is given by the co-characters $\nu_j : k^\times \to T$ such that $\nu_j(\xi)$ is the diagonal matrix with coefficient $\xi$ at position $j$, and coefficient 1 otherwise. Now, for $i \neq j$, we have a unique embedding of algebraic groups $\varphi_{ij} : SL_2(k) \hookrightarrow G$ such that
\[\varphi_{ij} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} = u_{ij}(\xi) \quad \text{and} \quad \varphi_{ij} \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} = u_{ji}(\xi) \quad \text{for all} \ \xi \in k.\]
Hence, $\varphi_{ij}$ satisfies the condition in 1.3.1 and so we obtain $\alpha_{ij}^{\vee} \in Y$ such that
\[\alpha_{ij}^{\vee}(\xi) = \varphi_{ij} \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \in T\]
is the diagonal matrix with coefficient $\xi$ at position $i$ and coefficient $\xi^{-1}$ at position $j$. Thus, we have $R^\vee = \{\alpha_{ij}^{\vee} = \nu_i - \nu_j \mid 1 \leq i, j \leq n, i \neq j\}$. We also see that
\[\Pi^\vee := \{\alpha_{i,i+1}^{\vee} = \nu_i - \nu_{i+1} \mid 1 \leq i \leq n-1\} \subseteq R^\vee\]
is a base of $R^\vee$. The corresponding Cartan matrix $C = (c_{ij})_{1 \leq i, j \leq n-1}$ is given by
\[
C = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots \\
0 & -1 & 2 & -1 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -1 & 2 \\
0 & \cdots & 0 & -1 & 2
\end{pmatrix}
\]
Thus, $C$ is of type $A_{n-1}$. The factorisation in Remark 1.2.13 is given by
\[C = \tilde{A} \cdot A^{\text{tr}} \quad \text{where} \quad A = \tilde{A} = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 \\
0 & 1 & -1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & -1 \\
0 & \cdots & 0 & -1 & 1
\end{pmatrix}.
\]
(Here, $A = \tilde{A}$ has $n - 1$ rows and $n$ columns.)
Example 1.3.8. Let $n \geq 2$ and $G' = \text{SL}_n(k)$, the special linear group. We keep the notation $G = \text{GL}_n(k)$, $U_{ij}$, $B$, $N$, $T$, $X = X(T)$, $Y = Y(T)$ from the previous example. Then an algebraic $BN$-pair satisfying the conditions in Proposition 1.1.13 is given by the subgroups $B' := B \cap G'$ and $N' := N \cap G'$; see [Bou68, Chap. IV, §2, Exercise 10], [Ge03a, 1.6.11, 3.4.5]. Let us describe the root datum of $G'$ with respect to the maximal torus $T' = T \cap G'$. Let

$$X' = X(T') \quad \text{and} \quad Y' = Y(T').$$

For $1 \leq i, j \leq n$, $i \neq j$, the subgroup $U_{ij}$ of $G$ is already contained in $G'$. So, if $\alpha_{ij}'$ denotes the restriction of $\alpha_{ij}$ to $X$ to $T'$, then $\alpha_{ij}' \in X'$ and $\alpha_{ij}'$ is a root of $G'$ with corresponding root subgroup $U_{ij} \subseteq G'$. Since $\dim G' = n^2 - 1$ and $T' = n - 1$, it follows as above that $R' = \{\alpha_{ij}' \mid 1 \leq i, j \leq n, i \neq j\}$ is the root system of $G'$ with respect to $T'$ and that

$$\Pi' = \{\alpha_{i,i+1}' \mid 1 \leq i \leq n - 1\}$$

is a base for $R'$.

Furthermore, the image of the embedding $\varphi_{ij} : \text{SL}_2(k) \to G$ is clearly contained in $G'$. Consequently, any coroot $\alpha_{ij}'' \in Y$ also is a coroot in $Y'$. Thus, we have $R'' = R'' = \{\alpha_{ij}'' \mid 1 \leq i, j \leq n, i \neq j\}$ and

$$\Pi'' = \{\alpha_{i,i+1}'' \mid 1 \leq i \leq n - 1\}$$

is a base for $R''$.

In particular, we obtain the same Cartan matrix $C$ of type $A_{n-1}$ as in Example 1.3.7. Now note that we have an isomorphism of algebraic groups

$$(k^x)^{n-1} \to T', \quad (\xi_1, \ldots, \xi_n) \mapsto h(\xi_1, \ldots, \xi_n, (\xi_1 \cdots (n-1)^{-1}).$$

(Its inverse is given by sending $h(\xi_1, \ldots, \xi_n) \in T'$ to $(\xi_1, \ldots, \xi_n) \in (k^x)^{n-1}$.) Hence, if we define co-characters $\nu'_{ij} : k^{x} \to T'$ (for $1 \leq j \leq n - 1$) such that $\nu'_{ij}(\xi)$ is the diagonal matrix with $\xi$ at position $j$ and $\xi^{-1}$ at position $n$, then $\{\nu_1', \ldots, \nu_{n-1}'\}$ is a $Z$-basis of $Y'$. Then both $\Pi'^\vee$ also is a $Z$-basis of $Y'$. If we consider the corresponding dual basis of $X'$, then the factorisation in Remark 1.2.13 is given by

$$C = A \cdot A'^\vee \quad \text{where} \quad \bar{A} = I_{n-1} \quad \text{and} \quad A = C'^\vee.$$ 

Thus, $G'$ is semisimple and the root datum of $\text{SL}_n(k)$ is of simply-connected type (see Example 1.2.10).

Example 1.3.9. Let $G = \text{GL}_n(k)$ and $Z \subseteq G$ be the center of $G$, consisting of all non-zero scalar matrices. Assume that $n \geq 2$ and let $\bar{G} = \text{PGL}_n(k) := G/Z$, the projective linear group. (This is a linear algebraic group as discussed in 1.3.8.) Let us denote the canonical map $G \to \bar{G}$ by $g \mapsto \bar{g}$. In particular, we obtain subgroups $\bar{B}$ and $\bar{N}$ in $\bar{G}$ which form a $BN$-pair since $Z \subseteq \bar{B}$; see [Bou68, Chap. IV, §2, Exercise 2]. One easily checks that this is an algebraic $BN$-pair satisfying the conditions in Proposition 1.1.14. Let us describe the root datum of $\bar{G}$ with respect to the maximal torus $\bar{T}$ of $\bar{G}$. Let

$$\bar{X} = X(T) \quad \text{and} \quad \bar{Y} = Y(T).$$

For every root $\alpha$ of $G$, we clearly have $Z \subseteq \ker(\alpha)$. So, using the universal property of quotients, there is a well-defined $\bar{\alpha} \in \bar{X}$ such that $\alpha(t) = \bar{\alpha}(t)$ for all $t \in T$. Now, for $1 \leq i, j \leq n$, $i \neq j$, the image $U_{ij}$ of the subgroup $U_{ij} \subseteq G$ in $\bar{G}$ is still closed, connected, isomorphic to $k^x$ and normalised by $T$. Hence, $\bar{\alpha}_{ij}$ is a root with corresponding root subgroup $\bar{U}_{ij} \subseteq \bar{G}$. As above, it follows that $\bar{R} = \{\bar{\alpha}_{ij} \mid 1 \leq i, j \leq n, i \neq j\}$ is the root system of $\bar{G}$ with respect to $\bar{T}$ and that

$$\bar{\Pi} = \{\bar{\alpha}_{i,i+1} \mid 1 \leq i \leq n - 1\}$$

is a base for $\bar{R}$. 
On the other hand, we obtain homomorphisms of algebraic groups \( \varphi_{ij} : \text{SL}_2(k) \to G \), simply by composing \( \varphi_{ij} : \text{SL}_2(k) \to G \) with the canonical map \( G \to G \). Thus, every coroot \( \alpha^Y \) of \( G \) determines a coroot \( \bar{\alpha}^Y \in \bar{Y} \). Consequently, we have \( \bar{R}^Y = \{ \bar{\alpha}_{ij}^Y \mid 1 \leq i, j \leq n, i \neq j \} \)

\[
\Pi^Y = \{ \bar{\alpha}_{i,i+1}^Y \mid 1 \leq i \leq n - 1 \}
\]

is a base for \( \bar{R}^Y \).

In particular, we obtain the same Cartan matrix \( C \) of type \( A_{n-1} \) as in Example 1.3.7. Now consider the homomorphism of algebraic groups

\[
T \to (k^\times)^{n-1}, \quad h(\xi_1, \ldots, \xi_n) \mapsto (\xi_1\xi_n^{-1}, \ldots, \xi_{n-1}\xi_n^{-1}).
\]

It has \( Z \) in its kernel so there is an induced homomorphism of algebraic groups \( \bar{T} \to (k^\times)^{n-1} \). The latter homomorphism is an isomorphism: its inverse is given by sending \( (\xi_1, \ldots, \xi_{n-1}) \in (k^\times)^{n-1} \) to the image of \( h(\xi_1, \ldots, \xi_{n-1}, 1) \in T \) in \( \bar{T} \). It follows that

\[
\{ \bar{\alpha}_{i,n} \mid 1 \leq i \leq n - 1 \}
\]

is a \( \mathbb{Z} \)-basis of \( \bar{X} \).

But then \( \Pi \) also is a \( \mathbb{Z} \)-basis of \( X \). If we consider the corresponding dual basis of \( Y \), then the factorisation in Remark 1.2.13 is given by

\[
C = \bar{A} \cdot A^\text{tr}
\]

where \( \bar{A} = C \) and \( A = I_{n-1} \).

Thus, \( \text{PGL}_n(k) \) is semisimple and the root datum of \( \text{PGL}_n(k) \) is of adjoint type (see Example 1.2.10). In particular, we see that the root data of \( \text{PGL}_n(k) \) and \( \text{SL}_n(k) \) are not isomorphic. (In the former, \( Z\mathbb{R} = X \); in the latter, \( Z\mathbb{R} \neq X \).) So, by Theorem 1.3.9, \( \text{PGL}_n(k) \) and \( \text{SL}_n(k) \) are not isomorphic as algebraic groups. (This question was raised at the end of 1.1.8)

Now, a key feature of the whole theory is the fact that a connected reductive algebraic group is uniquely determined by its root datum up to isomorphism. This follows from a more general result, the so-called isogeny theorem. As a preparation, we cite the following general results concerning surjective homomorphisms of algebraic groups, which will be useful at several places below.

1.3.10. Let \( f : G \to G' \) be a surjective homomorphism of connected algebraic groups over \( k \). Then we have the following preservation results.

(a) \( f \) maps a Borel subgroup of \( G \) onto a Borel subgroup of \( G' \), and all Borel subgroups of \( G' \) arise in this way; a similar statement holds for maximal tori. (See [Bor91 11.14] for more.)

(b) \( f \) maps the unipotent radical of \( G \) onto the unipotent radical of \( G' \); in particular, if \( G \) is reductive, then so is \( G' \). (See [Bor91 11.14] for more.)

(c) If \( G \) is reductive, then \( f \) maps the center of \( G \) onto the center of \( G' \). (This follows from (a) and the fact that the center of a reductive group is the intersection of all its maximal tori; see [Bor91 11.11] for more.)

(d) Assume that \( G, G' \) are reductive. Let \( T \) be a maximal torus of \( G \); by (a), \( T' := f(T) \) is a maximal torus of \( G' \). Then \( f \) induces a surjective homomorphism \( W(G,T) \to W(G',T') \), and this is an isomorphism if \( \ker(f) \) is contained in the center of \( G \). (See [Bor91 11.20] for more.)

1.3.11. Let \( G \) and \( G' \) be connected and reductive. Let \( f : G \to G' \) be an isogeny, that is, a surjective homomorphism of algebraic groups such that \( \ker(f) \) is finite. Note that then \( \ker(f) \) is automatically contained in the center of \( G \). Further to the properties in 1.3.10, \( f \) also preserves the root and coroot structure of \( G \). More precisely, this works as follows, where we refer to [Ch56-58/05 §18.2], [MaTe11 §11], [Spr98 §9.6], [Sl99b §1] for further details.
Let $T$ be a maximal torus of $G$; then $\ker(f) \subseteq T$ and $T' = f(T)$ is a maximal torus of $G'$. Let $(X, R, Y, R')$ and $(X', R', Y', R'')$ be the corresponding root data. The map $f$ induces a homomorphism $\varphi : X' \to X$ such that $\varphi(\lambda') = \lambda' \circ f|_T$ for all $\lambda' \in X'$. Then it follows that $\varphi$ is a $p$-isogeny of root data as in Definition 1.2.9 where $p$ is the characteristic exponent of $k$. (Recall that the characteristic exponent of $k$ is 1 in case $\text{char}(k) = 0$ and is equal to $\text{char}(k)$ otherwise.) The numbers $\{q_\alpha \mid \alpha \in R\}$ and the bijection $R \to R', \alpha \mapsto \alpha^t$, in (I2) come about as follows.

Let $\alpha \in R$ and consider the corresponding root subgroup $U_\alpha \subseteq G$; see 1.3.1. Then $f(U_\alpha)$ is a one-dimensional closed connected unipotent subgroup of $G'$ normalised by $T'$. Hence, there is a well-defined $\alpha^t \in R'$ such that $f(U_\alpha)$ equals the root subgroup $U'_\alpha$ in $G'$. Let $u_\alpha : k^+ \to U_\alpha$ and $u'_\alpha : k^+ \to U'_\alpha$ be the corresponding isomorphisms. Then the map $f : U_\alpha \to U'_\alpha$, has the following form. There is some $c_\alpha \in k^\times$ such that

$$f(u_\alpha(\xi)) = u'_\alpha(c_\alpha \xi^{q_\alpha}) \quad \text{for all } \xi \in k^+.$$ 

In this situation, the numbers $\{q_\alpha\}$ will also be called the root exponents of $f$. The above discussion shows that an isogeny of connected reductive groups induces a $p$-isogeny of root data. Conversely, we have the following fundamental result.

**Theorem 1.3.12** (Isogeny Theorem). Let $G$ and $G'$ be connected reductive algebraic groups over $k$, let $T \subseteq G$ and $T' \subseteq G'$ be maximal tori, and let $\varphi : X(T') \to X(T)$ be a $p$-isogeny of their root data (see Definition 1.2.9), where $p$ is the characteristic exponent of $k$. Then there exists an isogeny $f : G \to G'$ which maps $T$ onto $T'$ and induces $\varphi$. If $f' : G \to G'$ is another isogeny with these properties, then there exists some $t \in T$ such that $f'(g) = f(tg^{-1})$ for all $g \in G$.

See [St99b] for a recent, quite short proof of this fundamental result which, for semisimple groups, is one of the main results of the Séminaire Chevalley; see [Ch56-58/05, §18.2]. As a first consequence, we have:

**Corollary 1.3.13** (Isomorphism Theorem). In the setting of Theorem 1.3.12, assume that $\varphi$ is bijective. Then $f : G \to G'$ is an isomorphism of algebraic groups.

**Proof.** We use the notation in 1.3.11. First note that, since $\varphi$ is bijective, we must have $q_\alpha = 1$ for all $\alpha \in R$. Hence, the inverse map $\varphi^{-1} : X \to X'$ also defines an isogeny of root data. By Theorem 1.3.12 there exist isogenies $f : G \to G'$ and $g : G' \to G$ corresponding to $\varphi$ and $\varphi^{-1}$. Then $g \circ f$ induces the identity isogeny of the root datum of $G$ and hence equals the inner automorphism $u_t$ for some $t \in T$. Thus $g' \circ f = \text{id}_G$ with $g' = u_t^{-1} \circ g$, and then $f \circ g' \circ f = f$ and $f \circ g' = \text{id}_{G'}$ because $f$ is surjective. Hence $f$ is an isomorphism with $g'$ as its inverse. □

The general theory is completed by the following existence result.

**Theorem 1.3.14** (Existence Theorem). Let $R = (X, R, Y, R')$ be a root datum. Then there exists a connected reductive algebraic group $G$ over $k$ and a maximal torus $T \subseteq G$ such that $R$ is isomorphic to the root datum of $G$ relative to $T$.

For semisimple groups, this is originally due to Chevalley; see [Ch55] and the comments in [Ch56-58/05, §24]. See [Ca72, St67, §5, Theorem 6] where this is explained in detail, following and extending Chevalley’s original approach. The general case can be reduced to this one; see [Spr98, §10.1] and [DG70/11, Exposé XXV]. Only recently, Lusztig [Lu09c] found a new approach to the general case based on the theory of “canonical bases” of quantum groups.
Example 1.3.15. Let us see what the above results mean in the simplest non-trivial case where $R = (X, R, Y, R^\vee)$ is a root datum of Cartan type $A_1$. Let $G$ be a corresponding connected reductive algebraic group over $k$. Now, since $C = (2)$ is the Cartan matrix in this case, $R$ is determined by an equation

$$2 = \sum_{1 \leq i \leq d} a_i a_i$$

where $d = \text{rank } X = \text{rank } Y$ and $a_i \in \mathbb{Z}$ for all $i$;

see Remark 1.2.13. Up to isomorphism (where isomorphisms are determined by an invertible matrix $P$ over $\mathbb{Z}$ as in Lemma 1.2.20), there are three possible cases:

1. $(a_1, \ldots, a_d) = (2, 0, \ldots, 0)$ and $(\bar{a}_1, \ldots, \bar{a}_d) = (1, 0, \ldots, 0)$, in which case $G \cong \text{SL}_2(k) \times (k^\times)^{d-1}$.
2. $(a_1, \ldots, a_d) = (1, 0, \ldots, 0)$ and $(\bar{a}_1, \ldots, \bar{a}_d) = (2, 0, \ldots, 0)$, in which case $G \cong \text{PGL}_2(k) \times (k^\times)^{d-1}$.
3. $d \geq 2$, $(a_1, \ldots, a_d) = (1, 1, 0, \ldots, 0)$ and $(\bar{a}_1, \ldots, \bar{a}_d) = (1, 1, 0, \ldots, 0)$, in which case $G \cong \text{GL}_2(k) \times (k^\times)^{d-2}$.

This is contained in [S99a, 2.2]; we leave it as an exercise to the reader. In particular, for $d = 1$ (that is, $G$ semisimple), we have either $G \cong \text{SL}_2(k)$ or $G \cong \text{PGL}_2(k)$.

Besides its fundamental importance for the classification of connected reductive algebraic groups, the Isogeny Theorem is an indispensable tool for showing the existence of homomorphisms with prescribed properties. Here are the first examples.

Example 1.3.16. Let $G$ be a connected reductive algebraic group over $k$ and $T$ be a maximal torus of $G$. Let $(X, R, Y, R^\vee)$ be the corresponding root datum. Then there exists an automorphism of algebraic groups $\tau : G \to G$ such that

$$\tau(t) = t^{-1} \quad (t \in T) \quad \text{and} \quad \tau(U_\alpha) = U_{-\alpha} \quad (\alpha \in R).$$

Indeed, $\varphi : X \to X, \lambda \mapsto -\lambda$, certainly is a $p$-isogeny, where $q_\alpha = 1$ for all $\alpha \in R$.

Hence, since $\varphi$ is bijective, Corollary[1.3.13] shows that there exists an automorphism $\tau : G \to G$ such that $\tau(T) = T$ and such that $\varphi$ is the map induced by $\tau$ on $X$. Now, as discussed in [1.1.10], there is a natural bijection between group homomorphisms of $X$ into itself and algebraic homomorphisms of $T$ into itself. Under this bijection, $\varphi$ clearly corresponds to the map $t \mapsto t^{-1}$ on $T$. Hence, $\tau$ has the required properties.

Example 1.3.17. Let $p$ be a prime number and $G$ be a connected reductive algebraic group over $k = \mathbb{F}_p$. Let $T$ be a maximal torus of $G$ and $(X, R, Y, R^\vee)$ be the corresponding root datum. Then $\varphi : X \to X, \lambda \mapsto p\lambda$, certainly is a $p$-isogeny of root data, where $q_\alpha = p$ for all $\alpha \in R$. Hence, by Theorem [1.3.12] there exists an isogeny $F_p : G \to G$ such that $F_p(T) = T$ and such that $F_p$ induces $\varphi$ on $X$. Arguing as in the previous example, it follows that

$$F_p(U_\alpha) = U_{-\alpha} \quad (\alpha \in R) \quad \text{and} \quad F_p(t) = t^p \quad (t \in T).$$

For example, $F_p : \text{GL}_n(k) \to \text{GL}_n(k), (a_{ij}) \mapsto (a_{ij}^p)$, is an isogeny satisfying the above conditions.

We shall see in Section[1.4] that the fixed point set of $G$ under $F_p$ is a finite group. More generally, we shall consider isogenies $F : G \to G$ such that $F^d = F_p^m$ for some $d, m \geq 1$. The finite groups arising as fixed point sets of connected reductive groups under such isogenies are the finite groups of Lie type; see Definition[1.4.7].

Example 1.3.18. Let $R_i = (X_i, R_i, Y_i, R_i^\vee)$ (for $i = 1, \ldots, n$) be root data. Let $R = (X, R, Y, R^\vee)$ be the direct sum of these root data; see Example 1.2.14. For
i = 1, . . . , n, let \( G_i \) be a connected reductive algebraic group with root datum isomorphic to \( \mathcal{R}_i \) (relative to a maximal torus \( T_i \subseteq G_i \)). Then, using Corollary 1.3.13, one easily sees that the direct product \( \mathcal{G} := G_1 \times \cdots \times G_n \) has root datum isomorphic to \( \mathcal{R} \) (relative to the maximal torus \( T := T_1 \times \cdots \times T_n \) of \( \mathcal{G} \)).

Example 1.3.19. Let \( G = \text{GL}_n(k) \), with root datum \( \mathcal{R} = (X, R, Y, R') \) as in Example 1.3.7. It is given by a factorisation \( C = \tilde{A} \cdot A^r \) where \( C = (c_{ij})_{1 \leq i, j \leq n-1} \) is the Cartan matrix of type \( A_{n-1} \) and \( A = \tilde{A} \) is a certain matrix of size \( (n-1) \times n \).

Then, by the procedure in 1.2.18, we obtain an isogeny \( \varphi : X \rightarrow X \) via the pair of matrices \( (P, P') = (J_n, J_{n-1}) \) where, for any \( m \geq 1 \), we set

\[
J_m := \begin{pmatrix}
0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & 1 & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{pmatrix} \in M_m(k).
\]

Then \( \varphi \) has order 2. So there is a corresponding automorphism of algebraic groups \( \gamma : \text{GL}_n(k) \rightarrow \text{GL}_n(k) \) which maps the maximal torus \( T \) into itself and induces \( \varphi \) on \( X \). Concretely, the map given by

\[
\gamma : \text{GL}_n(k) \rightarrow \text{GL}_n(k), \quad g \mapsto J_n (g^{tr})^{-1} J_n,
\]

is an automorphism with this property.

Remark 1.3.20. Let \( f : G \rightarrow G' \) be an isogeny of connected reductive algebraic groups over \( k \). In the setting of 1.3.14, let \( \{ q_\alpha \mid \alpha \in R \} \) be the root exponents of \( f \). Following [Spr98, 9.6.3], we say that \( f \) is a central isogeny if \( q_\alpha = 1 \) for all \( \alpha \in R \).

The terminology is justified as follows. Consider the corresponding homomorphism of Lie algebras \( d_1 f : L(G) \rightarrow L(G') \). Then, by [Bor91, 22.4], \( f \) is a central isogeny if and only if the kernel of \( d_1 f \) is contained in the center of \( L(G) \). For example, the isogeny in Example 1.3.16 is central while that in Example 1.3.17 is not.

There are extensions of the Isogeny Theorem to the case where we consider homomorphisms whose kernel is still central but not finite: We shall only formulate the following version here. (This will be needed, for example, in Section 1.7.)

1.3.21. Let \( G, G' \) be connected reductive algebraic groups over \( k \), and \( f : G \rightarrow G' \) be a homomorphism of algebraic groups.

(a) Following [Boi06, Chap. I, 3.A], we say that \( f \) is an isotopy if \( \ker(f) \subseteq Z(G) \) and \( G'_{\text{der}} \subseteq f(G) \). If this is the case, then we have \( G' = f(G).Z(G') \), \( f(G_{\text{der}}) = G'_{\text{der}} \) and \( f \) restricts to an isogeny \( G_{\text{der}} \rightarrow G'_{\text{der}} \).

(b) Now let \( T \subseteq G \) and \( T' \subseteq G' \) be maximal tori such that \( f(T) \subseteq T' \). Then \( f \) induces a group homomorphism \( \varphi : X(T') \rightarrow X(T) \), \( \lambda \mapsto \lambda \circ f|_T \). In analogy to Remark 1.3.20, we say that \( f \) is a central isotopy if \( \varphi \) is a homomorphism of root data as in 1.2.2. (Note that, as pointed out in the remarks following [Jan03 II, Prop. 1.14], a central isotopy is automatically an isotopy.)

Theorem 1.3.22 (Extended Isogeny Theorem; see [Jan03 II, 1.14, 1.15], [St99II §5]). Let \( G \) and \( G' \) be connected reductive algebraic groups over \( k \), let \( T \subseteq G \) and \( T' \subseteq G' \) be maximal tori, and let \( \varphi : X(T') \rightarrow X(T) \) be a homomorphism of their root data (see 1.2.2). Then there exists a central isotopy \( f : G \rightarrow G' \) such that \( f(T) \subseteq T' \) and \( f \) induces \( \varphi \). Furthermore, the following hold.

(a) If \( f' : G \rightarrow G' \) is another central isotopy inducing \( \varphi \), then there exists some \( t \in T \) such that \( f'(g) = f(tg^{-1}) \) for all \( g \in G \).
(b) If \( f|_T : T \to T' \) is an isomorphism, then so is \( f : G \to G' \).

**Proof.** Let \( \Pi \) be a base of the root system \( R \subseteq X(T) \). For \( \alpha \in \Pi \), consider the subgroup \( G_\alpha = \langle T, U_\alpha, U_{-\alpha} \rangle \subseteq G \) defined in \( \ref{St99b, §3} \). Then \( G_\alpha \cap G_\beta = T \) for \( \alpha \neq \beta \) in \( \Pi \). As in \( \text{Jan03} \) II, §1.13, one sees that there exists a map

\[
f : \bigcup_{\alpha \in \Pi} G_\alpha \to G'
\]

which is a homomorphism on each \( G_\alpha \) and is such that \( f \) maps \( T \) into \( T' \) and induces \( \alpha \). Now, \( U_\alpha \) and \( U_{-\beta} \) certainly commute with each other for all \( \alpha \neq \beta \) in \( \Pi \) (by Chevalley’s commutator relations; see \( \text{MaTe11, 11.8} \)). Hence, by \( \text{St99b} \) Theorem 5.3, \( f \) extends to a homomorphism of algebraic groups from \( G \) to \( G' \). The uniqueness statement in (a) is proved as in the case of Theorem \( \ref{1.3.12} \) see \( \text{St99b} \) §3. Finally, (b) holds by \( \text{Jan03} \) II, §1.15.

1.4. **Frobenius maps and Steinberg maps**

We assume in this section that \( k = \mathbb{F}_p \) is an algebraic closure of the finite field with \( p \) elements, where \( p \) is a prime number. We consider a particular class of isogenies in this context, the so-called Steinberg maps. This will be treated in some detail, where one aim is to work out explicitly some useful characterisations of Steinberg maps in terms of isogenies of root data. In particular, in Proposition \( \ref{1.4.17} \) we recover the set-up in Example \( \ref{1.3.17} \). We also establish a precise characterisation of Frobenius maps among all Steinberg maps; see Proposition \( \ref{1.4.27} \).

**Definition 1.4.1.** Let \( X \) be an affine variety over \( k \). Let \( q \) be a power of \( p \) and \( \mathbb{F}_q \subseteq k \) be the finite field with \( q \) elements. We say that \( X \) has an \( \mathbb{F}_q \)-rational structure (or that \( X \) is defined over \( \mathbb{F}_q \)) if there exists some \( n \geq 1 \) and an isomorphism of affine varieties \( \iota : X \to X' \) where \( X' \subseteq k^n \) is Zariski closed and stable under the standard Frobenius map

\[
F_q : k^n \to k^n, \quad (\xi_1, \ldots, \xi_n) \mapsto (\xi_1^q, \ldots, \xi_n^q).
\]

In this case, there is a unique morphism of affine varieties \( F : X \to X \) such that \( \iota \circ F = F_q \circ \iota \); it is called the Frobenius map corresponding to the \( \mathbb{F}_q \)-rational structure of \( X \). Note that \( F_q \) is a bijective morphism whose fixed point set is \( F_q^q \). Consequently, \( F \) is a bijective morphism such that

\[
|X^F| < \infty \quad \text{where} \quad X^F := \{ x \in X \mid F(x) = x \}.
\]

**Example 1.4.2.** A Zariski closed subset \( X \subseteq k^n \) is called \( \mathbb{F}_q \)-closed if \( X \) is defined by a set of polynomials in \( \mathbb{F}_q[T_1, \ldots, T_n] \). If this holds, then \( X \) is stable under \( F_q \) and so \( X \) has an \( \mathbb{F}_q \)-rational structure, as defined above; the fixed point set \( X^{F_q} \) consists precisely of those \( x \in X \) which have all their coordinates in \( \mathbb{F}_q \). Conversely, if \( F_q(X) \subseteq X \), then \( X \) is \( \mathbb{F}_q \)-closed. (The proof uses the fact that \( k \supseteq \mathbb{F}_q \) is a separable field extension; see \( \text{Ge03a, 4.1.6} \), \( \text{Bor91, AG.14.4} \). In general, the discussion of rational structures is much more complicated.]

**Remark 1.4.3.** Let \( X \) be an affine variety over \( k \) and assume that \( X \) is defined over \( \mathbb{F}_q \), with Frobenius map \( F : X \to X \). Here are some basic properties of \( F \). First note that \( F^2, F^3, \ldots \) are also Frobenius maps. Furthermore, for any \( x \in X \), we have \( F^m(x) = x \) for some \( m \geq 1 \). Hence,

\[
X = \bigcup_{m \geq 1} X^{F^m} \quad \text{where} \quad |X^{F^m}| < \infty \quad \text{for all} \ m \geq 1.
\]
(Note that every element of \( k \) lies in a finite subfield of \( k \)). Finally, it is also clear that, if \( X' \subseteq X \) is a closed subset such that \( F(X') \subseteq X' \), then \( X' \) is defined over \( \mathbb{F}_q \), with Frobenius map given by the restriction of \( F \) to \( X' \).

**Remark 1.4.4.** Let \( X \) be an affine variety over \( k \) and let \( A \) be the algebra of regular functions on \( X \). There is an intrinsic characterisation of Frobenius maps in terms of \( A \), as follows. A morphism \( F : X \to X \) is the Frobenius map corresponding to an \( \mathbb{F}_q \)-rational structure of \( X \) if and only if the following two conditions hold for the associated algebra homomorphism \( F^* : A \to A \):

- (a) \( F^* \) is injective and \( F^*(A) = \{ a^q \mid a \in A \} \).
- (b) For each \( a \in A \), there exists some \( e \geq 1 \) such that \( (F^*)^e(a) = a^{q^e} \).

One easily sees that (a) and (b) hold for the standard Frobenius map \( F_q : k^n \to k^n \).

This implies that (a) and (b) hold for any \( F_q \)-stable closed subset \( X \subseteq k^n \) as in Example 1.4.2. The converse requires a bit more work; see [Ge03a, §4.1] or [Sr79, Chap. II] for details. One advantage of this characterisation of Frobenius maps is, for example, that it provides an easy proof of the following statement:

- (c) If \( F \) is a Frobenius map (with respect to \( \mathbb{F}_q \), as above) and \( \gamma : X \to X \) is an automorphism of affine varieties of finite order which commutes with \( F \), then \( \gamma \circ F \) also is a Frobenius map on \( X \) (with respect to \( \mathbb{F}_q \), same \( q \)).

(See, for example, [Ge03a, Exercise 4.4].) The above characterisation is also equivalent to the definition of an “abstract affine algebraic \((\mathbb{F}_q, k)\)-set” in [Car55-56].

In the sequel, \( G \) will always be a linear algebraic group over \( k = \mathbb{F}_p \).

1.4.5. Assume that, as an affine variety, \( G \) is defined over \( \mathbb{F}_q \) with corresponding Frobenius map \( F \). Then we say that \( G \) (as an algebraic group) is defined over \( \mathbb{F}_q \) if \( F \) is a group homomorphism. In this case, the set of fixed points \( G^{F_q} \) is a finite group. There is a more concrete description, similar to Definition 1.4.1. We have the standard Frobenius map

\[ F_q : \text{GL}_n(k) \to \text{GL}_n(k); \quad (a_{ij}) \mapsto (a_{ij}^q). \]

Then \( G \) is defined over \( \mathbb{F}_q \) if and only if there is a homomorphism

\[ \iota : G \to \text{GL}_n(k) \quad \text{(for some } n \geq 1) \]

of algebraic groups such that \( \iota \) is an isomorphism onto its image and the image is stable under \( F_q \); in this case, the corresponding Frobenius map \( F : G \to G \) is defined by the condition that \( \iota \circ F = F_q \circ \iota \). (See [Ge03a, 4.1.11] for further details.) In particular, if \( G \subseteq \text{GL}_n(k) \) is a closed subgroup defined by a collection of polynomials with coefficients in \( \mathbb{F}_q \), then \( F_q \) restricts to a Frobenius map on \( G \).

**Example 1.4.6.** Let \( T \subseteq G \) be an abelian subgroup consisting of semisimple elements (e.g., a torus). We claim that there always exists some Frobenius map \( F : G \to G \) (with respect to an \( \mathbb{F}_q \)-rational structure on \( G \)) such that \( T \) is \( F \)-stable and \( F(t) = t^q \) for all \( t \in T \).

Indeed, we can realise \( G \) as a closed subgroup \( G \subseteq \text{GL}_n(k) \) for some \( n \geq 1 \). Since \( T \) consists of commuting semisimple elements, we can assume that then \( T \) consists of diagonal matrices. Now, the defining ideal of \( G \) (as an algebraic subset of \( \text{GL}_n(k) \)) is generated by a finite set of polynomials with coefficients in \( k \). So there is some \( q = p^m \) \((m \geq 1) \) such that all these coefficients lie in \( \mathbb{F}_q \). Then \( G \) is stable under the standard Frobenius map \( F_q \) on \( \text{GL}_n(k) \). So \( F_q \) restricts to a Frobenius map \( F : G \to G \). Since any \( t \in T \) is a diagonal matrix, we have \( F(t) = t^q \).
Definition 1.4.7. Let $F: G \rightarrow G$ be an endomorphism of algebraic groups. Then $F$ is called a Steinberg map if some power of $F$ is the Frobenius map with respect to an $\mathbb{F}_q$-rational structure on $G$, for some power $q$ of $p$. Note that, in this case, $F$ is a bijective homomorphism of algebraic groups and $G^F$ is a finite group. If $G$ is connected and reductive, then $G^F$ will be called a finite group of Lie type or a finite reductive group.

The following result is the key tool to pass from properties of $G$ to properties of the finite group $G^F$; we shall see numerous applications in what follows.

Theorem 1.4.8 (Lang [La56], Steinberg [St68, 10.1]). Assume that $G$ is connected and let $F: G \rightarrow G$ be a Steinberg map (or, more generally, any endomorphism such that $|G^F| < \infty$). Then the map $\mathcal{L}: G \rightarrow G, g \mapsto g^{-1}F(g)$, is surjective.

Proof. If $F$ is a Steinberg map (and this is the case that we are mainly interested in), then [Mu03] gives a quick proof, as follows. The group $G$ acts on itself (on the right) where $g \in G$ sends $x \in G$ to $g^{-1}xF(g)$. Any action of an algebraic group on an affine variety has a closed orbit; see [Ge03a, 2.5.2]. Let $\Omega$ be such a closed orbit and let $x \in \Omega$. Since $G$ is connected, it will be sufficient to show that $\dim G = \dim \Omega$ (because then $G = \Omega$ and so $1 \in \Omega$). We have $\dim \Omega = \dim G - \dim \text{Stab}_G(x)$ (see [Ge03a, 2.5.3]), so it will be sufficient to show that $\text{Stab}_G(x)$ is finite. Now, an element $g \in G$ belongs to this stabiliser if and only if $g^{-1}xF(g) = x$, which is equivalent to $f(y) = g$, where $f(y) := xF(g)x^{-1}$. Let $m \geq 1$ be such that $F^m$ is a Frobenius map and $F^m(x) = x$ (see Remark 1.4.3). Let $r \geq 1$ be the order of the element $xF(x)F^2(x) \cdots F^{m-1}(x) \in G$. Then $f^{mr}(g) = F^{mr}(g)$ for all $g \in G$. So $f^{mr}(g) = g$ has only finitely many solutions in $G$, hence $f(g) = g$ has only finitely many solutions in $G$. \hfill $\Box$

For various parts of the subsequent discussion it would be sufficient to work with endomorphisms of $G$ whose fixed point set is finite. However, we will just formulate everything in terms of Steinberg maps, as defined above. We note that the discussion in [St68, 11.6] in combination with Proposition 1.4.17 below implies that an endomorphism of a simple algebraic group with a finite fixed point set is automatically a Steinberg map; see also Example 1.4.19 below.

Here is the prototype of an application of the above theorem.

Proposition 1.4.9. Assume that $G$ is connected and let $F: G \rightarrow G$ be a Steinberg map. Let $X$ be an abstract set on which $G$ acts transitively; let $F': X \rightarrow X$ be a map such that $F'(g,x) = F(g,F(x))$ for all $g \in G$ and $x \in X$. Then there exists some $x_0 \in X$ such that $F'(x_0) = x_0$.

Proof. Take any $x \in X$. Since $G$ acts transitively, we have $F'(x) = g^{-1}.x$ for some $g \in G$. By Theorem 1.4.8 we can write $g = h^{-1}F(h)$. Then one immediately checks that $x_0 := h.x$ is fixed by $F'$. \hfill $\Box$

Example 1.4.10. Assume that $G$ is connected and let $F: G \rightarrow G$ be a Steinberg map. Let $C$ be a conjugacy class of $G$ such that $F(C) = C$. Then $G$ acts transitively on $C$ by conjugation; let $F'$ be the restriction of $F$ to $C$. Applying Proposition 1.4.9 yields that there exists an element $x \in C$ such that $F(x) = x$.

Similarly, there exists a pair $(T, B)$ where $T$ is an $F$-stable maximal torus of $G$ and $B$ is an $F$-stable Borel subgroup such that $T \subseteq B$. (Just note that, by Theorem 1.3.3, all these pairs are conjugate in $G$ and, by 1.3.10(a), $F$ preserves the set of all these pairs.) An $F$-stable maximal torus of $G$ which is contained in an $F$-stable Borel subgroup of $G$ will be called a maximally split torus.
\textbf{Proposition 1.4.11 (Cf. [St68, 10.10]).} Let $G$ be connected reductive and $F: G \to G$ be a Steinberg map. Then all maximally split tori of $G$ are $G^F$-conjugate. More precisely, all pairs $(T, B)$ consisting of an $F$-stable maximal torus $T$ and an $F$-stable Borel subgroup $B$ such that $T \subseteq B$ are conjugate in $G^F$.

\textbf{Proof.} Let $(T, B)$ and $(T_1, B_1)$ be two pairs as above. By Theorem 1.4.8, there exists some $x \in G$ such that $xBx^{-1} = B_1$ and $xTx^{-1} = T_1$. Since $B, B_1$ are $F$-stable, this implies that $x^{-1}F(x) \in N_G(B) = B$, where the last equality holds by [Bor91, 11.16] or [Spr98, 6.4.9]. Similarly, since $T, T_1$ are $F$-stable, we have $x^{-1}F(x) \in N_G(T)$. Hence, $x^{-1}F(x) \in B \cap N_G(T) = T$ (see Theorem 1.3.2). Applying Theorem 1.4.8 to the restriction of $F$ to $T$, we obtain an element $t \in T$ such that $x^{-1}F(x) = t^{-1}F(t)$. Then $g := xt^{-1} \in G^F$ and $g$ simultaneously conjugates $B$ to $B_1$ and $T$ to $T_1$. \hfill $\square$

The following result deals with a subletty concerning Steinberg maps: A surjective homomorphism of algebraic groups will not necessarily induce a surjective map on the level of the fixed point sets under Steinberg maps. But one can say precisely what happens in this situation:

\textbf{Proposition 1.4.12 (Cf. [St68, 4.5]).} Let $f : G \to G'$ be a surjective homomorphism of connected algebraic groups such that $K := \ker(f)$ is contained in the center of $G$. Let $F : G \to G$ and $F' : G' \to G'$ be Steinberg maps such that $F' \circ f = f \circ F$. We denote $G = G^K$ and $G' = G'^K$. Then the following hold.

(a) Let $L' : G \to G'$, $g \mapsto g^{-1}F(g)$. Then $L(K)$ is a normal subgroup of $K$.

(b) $f(G) \subseteq G'$ is a normal subgroup and $G'/f(G) \cong K/L'(K)$. In particular, if $K$ is connected, then $L'(K) = K$ and $f(G) = G'$.

(c) If $K$ is finite (that is, $f$ is an isogeny), then $|G| = |G'|$.

\textbf{Proof.} First note that $K$ is $F$-stable. Since $K$ is contained in the center of $G$, this implies (a). Now Steinberg [St68, 4.5] states a general result (about arbitrary groups) which shows that $f : G \to G'$ induces a long exact sequence

$$
\{1\} \to K^F \to G \xrightarrow{f} G' \xrightarrow{\delta} (L(G) \cap K)/L'(K) \to \{1\},
$$

where $\delta$ is given as follows. Let $g' \in G'$ and choose $g \in G$ such that $f(g) = g'$. Then $g^{-1}F(g) \in K$ and $\delta(g')$ is the image of $g^{-1}F(g)$ in $(L(G) \cap K)/L'(K)$.

Since $G$ is connected, we have $L(G) = G$ by Theorem 1.4.8. Now let $\ker(L(K)) = \{z \in K \mid z^{-1}F(z) = 1\} = K^F = \ker(f|_G)$. So, if $K$ is finite, then $|K/L'(K)| = |\ker(f|_G)|$ and, hence, $|G| = |f(G)||K/L'(K)|$. But, by (b), $K/L(K)$ and $G'/f(G)$ have the same order and so $|G| = |G'|$, that is, (c) holds. \hfill $\square$

\textbf{Lemma 1.4.13 (Cf. [St68, 10.9]).} Assume that $G$ is connected and let $F : G \to G$ be a Steinberg map. Let $y \in G$ and define $F' : G \to G$ by $F'(g) = yF(g)y^{-1}$ for all $g \in G$. Then $F'$ is a Steinberg map and we have $G^{F'} \cong G^F$.

Furthermore, if $F$ is a Frobenius map corresponding to an $\mathbb{F}_q$-rational structure, then so is $F'$ (with the same $q$).

\textbf{Proof.} Since $G$ is connected, Theorem 1.4.8 shows that we can write $y = x^{-1}F(x)$ for some $x \in G$. Then $F'(g) = x^{-1}F(xgx^{-1})x$ for all $g \in G$. Thus, we have $F' = t_x^{-1} \circ F \circ t_x$ where $t_x$ denotes the inner automorphism of $G$ defined by $x$. This formula shows that $F'(g) = g$ if and only if $xgx^{-1} \in G^F$. Hence, conjugation with $x$ defines a group isomorphism $G^{F'} \cong G^F$.

Now we show that $F'$ is a Steinberg map. For $m \geq 1$, we have $F'^m(g) = x^{-1}F^m(xgx^{-1})x$ for all $g \in G$. By Remark 1.4.9 (and the definition of Steinberg
maps), there exists some $m \geq 1$ such that $F^m(x) = x$. For this $m$, we have $F^m(g) = F^m(\ell)$ for all $g \in G$. Thus, $F^m$ is a Steinberg map.

Finally, assume that $F$ is a Frobenius map. We use the characterisation in Remark 1.4.4 to show that $F'$ is a Frobenius map. Since $F'$ is the conjugate of $F$ by an automorphism of $G$, we have that $F'^* = (F^*)^e$ is the conjugate of $F^*$ by an algebra automorphism of $A$. This implies that $F'^*$ is injective and $F'^*(A) = \{a^q \mid a \in A\}$. On the other hand, we have $F^m = F^1$. So, if $a \in A$ and $e \geq 1$ are such that $(F^*)^e(a) = a^q$, then $(F'^*)^e(a) = (F^*)^e(a) = a^{q^m}$, as required.

Lemma 1.4.14. Assume that $G$ is connected reductive. Let $F: G \to G$ be a Steinberg map and $T$ be an $F$-stable maximal torus of $G$. Let $F': G \to G$ be a further isogeny of $G$ such that $F'(T) = T$. If $F$ and $F'$ induce the same map on $X(T)$, then there exists some $y \in T$ such that $F'(g) = yF(g)y^{-1}$ for all $g \in G$. In particular, the conclusions of Lemma 1.4.13 apply to $F'$.

Proof. Since $F$, $F'$ induce the same map on $X(T)$, Theorem 1.3.12 implies that there exists some $y \in T$ such that $F'(g) = yF(g)y^{-1}$ for all $g \in G$. □

Example 1.4.15. Assume that $G$ is connected reductive and let $T \subseteq G$ be a maximal torus, with associated root datum $\mathcal{R} = (X, R, Y, R^\vee)$. Let $F_p: G \to G$ be an isogeny as in Example 1.3.17 such that

$$F_p(U_\alpha) = U_\alpha \quad (\alpha \in R) \quad \text{and} \quad F_p(t) = t^p \quad (t \in T).$$

(Note that $F_p$ is only unique up to composition with inner automorphisms defined by elements of $T$.) Now, if one is willing to appeal to a stronger version of Theorem 1.3.12 (involving fields of definition), then one can find an $F_p$ as above such that $F_p$ is the Frobenius map with respect to an $F_p$-rational structure on $G$: see [Spr98, 16.3.3], [DG70/11] Exposé XXV, (Alternatively, one could use Lusztig’s approach [Lu98], as pointed out in [DG70/11] Exposé XXV, footnote 1.) If $G$ is semisimple, then this is also contained in [St67] Theorem 6 (p. 58), [Bor70] Part A, §3.3 and §4.3. Once some $F_p$ is known to be a Frobenius map, Lemma 1.4.13 shows that any $F_p$ satisfying the above conditions is a Frobenius map.

In any case, for most of our purposes here, it will be sufficient to know that $F_p$ is a Steinberg map; this is easily seen as follows. By Example 1.4.10 there exists a Frobenius map $F: G \to G$ such that $F(t) = t^q$ for all $t \in T$, where $q = p^m$ for some $m \geq 1$. Then $F$ induces multiplication with $q$ on $X$. Hence, $F$ induces the same map on $X$ as $F_p^m$. So Lemma 1.4.14 shows that $F_p$ is a Steinberg map.

Lemma 1.4.16. Let $T$ be a torus over $k$ and $F: T \to T$ be the Frobenius map corresponding to an $F_q$-rational structure on $T$, where $q$ is a power of $p$. Then the map induced by $F$ on $X = X(T)$ is given by $q\psi_0$ where $\psi_0: X \to X$ is an invertible endomorphism of finite order.

Proof. (Cf. [DiMi91, p. 40], [Sa71, §1.2.4].) Let $A$ be the algebra of regular functions on $T$. Let $\ell \in X$. Composing $\ell$ with the inclusion $k^\times \to k$, we can regard $\ell$ as a regular function on $T$, that is, $\ell \in A$. By Remark 1.4.3 we have $F^*(A) = \{a^q \mid a \in A\}$. Hence, $\ell^q = F^*(\ell^*)$ for some $\ell^* \in A$. Then we have

$$\ell^*(F(t)) = \ell(t)^q = \ell(t^q) \quad \text{for all } t \in T.$$ 

Since $F: T \to T$ is a bijective group homomorphism, $\ell^*$ is uniquely determined by $(\ast)$; furthermore, $\ell^*(T) \subseteq k^\times$ and $\ell^*: T \to k^\times$ is a group homomorphism. Hence, $\ell^* \in X$. We also see that the map $\psi: X \to X, \lambda \mapsto \ell^*$, is linear. Finally, $(\ast)$ implies that $(\psi^m(\lambda))(F^m(t)) = \ell(t^{qm})$ for all $m \geq 1$. Now, by Example 1.4.6 we
can find some \( m \geq 1 \) such that \( F^m(t) = t^{q^m} \) for all \( t \in T \). For any such \( m \), we then have \( \psi^m(\lambda) = \lambda \) for all \( \lambda \in X \). Hence, \( \psi \) is an endomorphism of \( X \) of order dividing \( m \). Setting \( \psi_0 := \psi^{-1} \), the map on \( X \) induced by \( F \) is given by \( q_0 \psi_0 \). \( \square \)

We now obtain the following characterisation of Steinberg maps.

**Proposition 1.4.17.** Let \( G \) be connected reductive, \( F: G \to G \) be an isogeny and \( T \subseteq G \) be an \( F \)-stable maximal torus. Then the following are equivalent.

(i) \( F \) is a Steinberg map.

(ii) There exist integers \( d, m \geq 1 \) such that the map induced by \( F^d \) on \( X = X(T) \) is given by scalar multiplication with \( p^m \).

(iii) There is an isogeny \( F_p \) as in Example 1.4.15 such that some positive power of \( F \) equals some positive power of \( F_p \).

**Proof.** “(i) \( \Rightarrow \) (ii)” Let \( d_1 \geq 1 \) be such that \( F^{d_1} \) is a Frobenius map with respect to some \( \mathbb{F}_q \)-rational structure on \( G \) where \( q_0 \) is a power of \( p \). Let \( \varphi: X \to X \) be the map induced by \( F \). By Lemma 1.4.16 we have \( \varphi^{d_1} = q_0 \psi_0 \) where \( \psi_0: X \to X \) has finite order, \( e \geq 1 \) say. Then \( \varphi^{d_1 e} = q_0^e \text{id}_X \).

“(ii) \( \Rightarrow \) (iii)” Assume that the map induced by \( F^d \) on \( X \) is given by scalar multiplication with \( p^m \). Let \( F_p \) be as in Example 1.4.15. Then \( F^d \) and \( F_p^m \) induce the same map on \( X \) and so there is some \( y \in T \) such that \( F^d(y) = y F_p^m(g) y^{-1} \) for all \( g \in G \); see Lemma 1.4.14. By Theorem 1.4.8 we can write \( y = x^{-1} F_p^m(x) \) for some \( x \in T \). As in the proof of Lemma 1.4.13, we have \( F^d = \iota_x^{-1} \circ F_p^m \circ \iota_x \). But then we also have \( F^d = (\iota_x^{-1} \circ F_p \circ \iota_x)^m \) and it remains to note that \( F_p^m := \iota_x^{-1} \circ F_p \circ \iota_x \) is a map satisfying the conditions in Example 1.4.15.

“(iii) \( \Rightarrow \) (i)” This is clear by definition, since \( F_p \) is known to be a Steinberg map (see Example 1.4.15). \( \square \)

**Proposition 1.4.18.** Assume that \( G \) is connected. Let \( F: G \to G \) be a Steinberg map. Let \( q \) be the positive real number defined by \( q^d = q_0 \), where \( d \geq 1 \) is an integer such that \( F^d \) is a Frobenius map with respect to some \( \mathbb{F}_q \)-rational structure on \( G \) (where \( q_0 \) is a power of \( p \)). Then \( q \) does not depend on \( d, q_0 \). Furthermore, the following hold for every \( F \)-stable maximal torus \( T \) of \( G \).

(a) We have \( \det(\varphi) = \pm q^{\text{rank } X} \) where \( \varphi: X(T) \to X(T) \) is the linear map induced by \( F \).

(b) The map induced by \( F \) on \( X_R := \mathbb{R} \otimes \mathbb{Z} X(T) \) is of the form \( q \varphi_0 \) where \( \varphi_0 \in \text{GL}(X_R) \) has finite order.

**Proof.** The independence of \( q \) of \( d, q_0 \) is clear, once (a) is established. So let \( T \) be any \( F \)-stable maximal torus of \( G \) (which exists by Example 1.4.10). Let \( X = X(T) \) and \( \varphi: X \to X \) be the linear map induced by \( F \).

(a) By Remark 1.4.3 the restriction of \( F^d \) to \( T \) is a Frobenius map with respect to an \( \mathbb{F}_q \)-rational structure on \( T \). So, by Lemma 1.4.16 we have \( \varphi^d = q_0 \psi_0 \) where \( \psi_0: X \to X \) has finite order. Then \( \det(\varphi^d) = q_0^{\text{rank } X} \det(\psi_0) \). Since \( \det(\varphi) \) is an integer and \( \det(\psi_0) \) is a root of unity, we must have \( \det(\varphi^d) = \pm q_0^{\text{rank } X} \) and, hence, \( \det(\varphi) = \pm q^{\text{rank } X} \).

(b) Denote by \( \varphi_\mathbb{R} \) the canonical extension of \( \varphi \) to \( X_\mathbb{R} \). Then \( \varphi_0 := q^{-1} \varphi_\mathbb{R} \) is a linear map such that \( \varphi_0^d = \psi_0 \). Hence, \( \varphi_0 = q \varphi_0 \) where \( \varphi_0 \) has finite order. \( \square \)

Having defined \( q \), one may also write \( G(q) \) instead of \( G^F \) if there is no danger of confusion. An alternative characterisation of \( q \) will be given in Remark 1.6.9(a). The defining formula in Proposition 1.4.18 shows that \( q \) is a \( d \)-th root of a prime power. The examples below will show that all such roots do actually occur.
Example 1.4.19. This example is just meant to give a simple illustration of the difference between Steinberg maps and arbitrary isogenies with a finite fixed point set. Let $q, q'$ be two distinct powers of $p$. Let $G = \text{SL}_2(k) \times \text{SL}_2(k)$ and define $F: G \to G$ by $F(x, y) = (F_q(x), F_{q'}(y))$ where $F_q$ and $F_{q'}$ denote the standard Frobenius maps with respect to $q$ and $q'$, respectively. Then $F$ is a bijective homomorphism of algebraic groups and $G^F = \text{SL}_2(q) \times \text{SL}_2(q')$ certainly is finite. Let $T \cong k^\times$ be the standard maximal torus of $\text{SL}_2(k)$. Then $T \times T$ is an $F$-stable maximal torus of $G$ and we can identify $X(T \times T)$ with $\mathbb{Z}$. Under this identification, the map induced by $F$ is given by $(n, m) \mapsto (qn, q'm)$ for all $(n, m) \in \mathbb{Z}^2$. Thus, Proposition 1.4.17(b) shows that $F$ is not a Steinberg map.

Example 1.4.20. Assume that $G$ is connected reductive. Let $T \subseteq G$ be a maximal torus and $\mathcal{B} = (X, R, Y, R^\vee)$ be the root datum relative to $T$. Assume that we have an automorphism $\varphi_0: X \to X$ of finite order such that $\varphi_0(R) = R$ and $\varphi_0(R^\vee) = R^\vee$. (In particular, $\varphi_0$ is an isogeny of root data with all root exponents equal to 1.) Let $q = p^m$ for some $m \geq 1$. Then $q\varphi_0$ is a $p$-isogeny and so, by Theorem 1.3.12 there is a corresponding isogeny $F: G \to G$ such that $F(T) = T$. Now $F$ is a Steinberg map by Proposition 1.4.17 the number $q = p^m$ satisfies the conditions in Proposition 1.4.18 if $G$ is semisimple, then $G = G^F$ is an untwisted ($\varphi_0 = \text{id}_X$) or twisted Chevalley group; see Steinberg’s lecture notes [St67, §11] for further details. We discuss the various possibilities in more detail in Section 1.6.

Let us just give one concrete example. Let $G = \text{GL}_n(k)$. If $\varphi_0 = \text{id}_X$, then we obtain a Frobenius map $F: \text{GL}_n(k) \to \text{GL}_n(k)$ such that $\text{GL}_n(k)^F = \text{GL}_n(q)$, the finite general linear group over $\mathbb{F}_q$. On the other hand, the bijective isogeny $\varphi_0: X \to X$ of order 2 defined in Example 1.3.19 also satisfies the above conditions. The corresponding isogeny $F': \text{GL}_n(k) \to \text{GL}_n(k)$ is a Steinberg map such that $\text{GU}_n(q) = \text{GL}_n(k)^F$ is the finite general unitary group. Similarly, we have $\text{SL}_n(k)^F = \text{SL}_n(q)$ and $\text{SL}_n(k)^{F'} = \text{SU}_n(q)$.

Example 1.4.21. Assume that $G$ is connected reductive and that the root datum $\mathcal{B} = (X, R, Y, R^\vee)$ relative to a maximal torus $T \subseteq G$ is as in Example 1.2.19 where $p = 2$ or 3. For any $m \geq 0$, we have a $p$-isogeny $\varphi_m$ on $X$ such that $\varphi_m^2 = p^{2m+1} \text{id}_X$. Let $F: G \to G$ be the corresponding isogeny such that $F(T) = T$. Then Proposition 1.4.17 shows that $F$ is a Steinberg map; the number $q$ in Proposition 1.4.18 is given by $q = \sqrt[p]{p^{2m+1}}$. In these cases $G = G^F$ is the Suzuki group $^2B_2(q^2) = ^2C_2(q)$, the Ree group $^2G_2(q^2)$ or the Ree group $^2F_4(q^2)$, respectively. See [Ca72, Chap. 13] or Steinberg’s lecture notes [St67, §11] for further details.

Example 1.4.22. Let $F: G \to G$ be the Frobenius map corresponding to some $\mathbb{F}_q$-rational structure on $G$ where $q_0$ is a power of $p$. Consider the direct product $G'' = G \times \cdots \times G$ (with $r$ factors) and define a map

$$F': G' \to G'', \quad (g_1, g_2, \ldots, g_r) \mapsto (F(g_1), g_2, \ldots, g_{r-1}).$$

Then $F'$ is a homomorphism of algebraic groups and we have $(F')^r(g_1, \ldots, g_r) = (F(g_1), F(g_2), \ldots, F(g_r))$ for all $g_i \in G$. Clearly, the latter map is a Frobenius map on $G'$. Thus, $F'$ is a Steinberg map. The number $q$ in Proposition 1.4.18 is given by $q = \sqrt[r]{q_0}$. Note also that we have a group isomorphism

$$G'^F \cong G^F, \quad (g_1, g_2, \ldots, g_r) \mapsto g_1.$$

\[\text{In finite group theory, it is common to write } ^2B_2(q^2) \text{ etc., although this is not entirely consistent with the general setting of algebraic groups where the notation should be } ^2B_2(q).\]
We have the following extension of the Isogeny Theorem 1.3.12, taking into account the presence of Steinberg maps.

**Lemma 1.4.23.** In the set-up of Theorem 1.3.12 assume, in addition, that there are Steinberg maps $F: G \to G$ and $F': G' \to G'$ such that $F(T) = T'$, $F'(T') = T'$ and $\Phi \circ \varphi = \varphi \circ \Psi'$ where $\Phi: X(T) \to X(T)$ and $\Psi': X(T') \to X(T')$ are the maps induced by $F$ and $F'$. Then there exists an isogeny $f: G \to G'$ which maps $T$ onto $T'$ and induces $\varphi$, and such that $f \circ F = F' \circ f$.

**Proof.** By Theorem 1.3.12 there exists an isogeny $f': G \to G'$ which maps $T$ onto $T'$ and induces $\varphi$. Then $f' \circ F$ and $F' \circ f'$ both induce $\Phi \circ \varphi = \varphi \circ \Psi'$. Hence, by Lemma 1.3.12 there exists some $t \in T'$ such that $(F' \circ f')(g) = (f' \circ F)(t^{-1}gt)$ for all $g \in G$. Then $f'(F(t)) = t'$ and so, by Theorem 1.4.8 we can write $f'(F(t)) = x^{-1}F'(x)$ for some $x \in T'$. We define $f: G \to G'$ by $f(g) = x^{-1}F'(x)x^{-1}$ for all $g \in G$. Then $f$ is an isogeny which maps $T$ onto $T'$ and also induces $\varphi$. Furthermore,

$$(F' \circ f)(g) = F'(x)(f' \circ F)(t^{-1}gt)F'(x)^{-1} = x(f' \circ F)(g)x^{-1} = (f \circ F)(g)$$

for all $g \in G$, as required.

**Example 1.4.24.** Assume that $G$ is connected reductive. Let $F: G \to G$ be a Steinberg map and $T$ be an $F$-stable maximal torus of $G$.

(a) Lemma 1.4.23 immediately shows that an automorphism $\tau: G \to G$ as in Example 1.3.16 can be chosen such that we also have $\tau \circ F = F \circ \tau$.

(b) Consider an isogeny $F_p: G \to G$ as in Example 1.4.13 and let $\varphi$ be the map induced on $X = X(T)$ by $F$. Since $F_p$ is a Steinberg map, Lemma 1.4.23 shows that there is an isogeny $F': G \to G$ which maps $T$ onto $T'$ and induces $\varphi$, and such that $F' \circ F_p = F_p \circ F'$. Since $F, F'$ induce the same map on $X$, we have that $F'$ is a Steinberg map such that $G^F = G^{F'}$; see Lemma 1.4.14. (Thus, replacing $F$ by $F'$ if necessary, we can always assume that $F \circ F_p = F_p \circ F$, that is, we are in the setting of [Lus74 §2.1].)

**Lemma 1.4.25.** Assume that $G$ is connected reductive. Let $K$ be a closed normal subgroup of $G$. Then $K$ is reductive and $\bar{G} := G/K$ is connected and reductive. If, furthermore, $F: G \to G$ is a Steinberg map such that $F(K) = K$, then the map $\bar{F}: G \to \bar{G}$ induced by $F$ is a Steinberg map.

**Proof.** Since the unipotent radical in a linear algebraic group is invariant under any automorphism of algebraic groups, it is clear that every closed normal subgroup of $G$ is reductive. Now consider $\bar{G} = G/K$. First recall from 1.1.6 that $\bar{G}$ is a linear algebraic group; it is also connected since it is the quotient of a connected group. Finally, $\bar{G}$ is reductive by 1.3.10 (b).

Now let $\bar{F}: \bar{G} \to \bar{G}$ be a Steinberg map and assume that $F(K) \subseteq K$. Then we obtain an induced (abstract) group homomorphism $\bar{F}: G \to \bar{G}$, which is bijective. By the universal property of quotients, $\bar{F}$ is a homomorphism of algebraic groups. Let $T \subseteq G$ be an $F$-stable maximal torus. Let $\bar{T}$ be the image of $T$ in $\bar{G}$. By 1.3.10 (a), $T$ is a maximal torus of $G$; we also have $\bar{F}(T) = \bar{T}$. Let $X = X(T)$, $\bar{X} = X(\bar{T})$ and $\varphi: X \to X$ be the map induced by the canonical map $f: G \to \bar{G}$; note that $\varphi$ is injective. Let $\bar{\psi}: X \to X$ denote $\varphi$. Since $\bar{F} \circ f = f \circ \bar{F}$, we have $\varphi \circ \bar{\psi} = \bar{\psi} \circ \varphi$ and so $\varphi \circ \bar{\psi}^m = \bar{\psi}^m \circ \varphi$ for all $m \geq 1$. Now there is some $m \geq 1$ such that $\bar{\psi}^m$ is given by scalar multiplication with a power of $p$. Since $\varphi$ is injective, this implies that $\bar{\psi}^m$ is also given by scalar multiplication with a power of $p$. Hence, $\bar{F}$ is a Steinberg map by Proposition 1.3.14. \(\square\)
Finally, we address the question of characterising Frobenius maps among all Steinberg maps on $G$. The results are certainly well-known to the experts and are contained in more advanced texts on reductive groups (like [BoTi65], [Sa71]), where they appear as special cases of general considerations of rationality questions. Since in our case the rational structures are given by Frobenius maps, one can give more direct proofs. The key property is contained in the following result.

**Lemma 1.4.26.** Let $G$ be connected reductive and $F: G \to G$ be a Frobenius map with respect to some $\mathbb{F}_q$-rational structure on $G$. Let $T$ be an $F$-stable maximal torus. Then the root exponents of $F$ (relative to $T$) are all equal to $q$.

**Proof.** (Cf. [BoTi65] 6.2, [Sa71] §II.2.1.) Let $R$ be the set of roots with respect to $T$. Let $\alpha \in R$ and $u_\alpha: k^+ \to G$ be the corresponding homomorphism with image $U_\alpha \subseteq G$. We have $F(U_\alpha) = U_{\alpha^q}$, where $\alpha^q \in R$; see [1.3.11]. In order to identify the root exponents, we need to exhibit a homomorphism $u_{\alpha^q}: k^+ \to G$ whose image is $U_\alpha$ and such that $u_{\alpha^q}$ is an isomorphism onto its image. This is done as follows. Let $A$ be the algebra of regular functions on $G$. The algebra of regular functions on $k^+$ is given by the polynomial ring $k[c]$ where $c$ is the identity function on $k^+$. Since $u_\alpha$ is an isomorphism onto its image, the induced algebra homomorphism $u_{\alpha^q}: A \to k[c]$ is surjective (see [Ge03a, 2.2.1]). Now consider the standard Frobenius map $F_1: k^+ \to k^{+q}$, $\xi \mapsto \xi^q$. By Remark 1.4.4, we have $F^*(A) = \{a^q \mid a \in A\}$ and $F_1^*(k[c]) = k[c^q]$. Hence, the composition $u_{\alpha^q} \circ F^*$ sends $A$ onto $k[c^q]$. Since $F_1^*: k[c] \to k[c^q]$ is a group homomorphism, we obtain an algebra homomorphism $\gamma: A \to k[c]$ by setting $\gamma := (F_1^*)^{-1} \circ u_{\alpha^q} \circ F^*$; note that $\gamma$ is surjective. Let $u_{\alpha^q}: k^+ \to G$ be the morphism of affine varieties such that $u_{\alpha^q} = \gamma$. Then $F \circ u_\alpha = u_{\alpha^q} \circ F_1$ and so

$$u_{\alpha^q}(\xi^q) = (u_{\alpha^q} \circ F_1)(\xi) = (F \circ u_\alpha)(\xi) = F(u_\alpha(\xi)) \quad \text{for all } \xi \in k.$$ 

This shows, first of all, that $u_{\alpha^q}$ is a group homomorphism with image $F(U_\alpha) = U_{\alpha^q}$. Furthermore, since $\gamma$ is surjective, $u_{\alpha^q}$ is an isomorphism onto its image (see again [Ge03a] 2.2.1)). For all $t \in T$ and $\xi \in k$, we have

$$F(t)u_{\alpha^q}(\xi^q)F(t)^{-1} = F(tu_\alpha(\xi)^{q^{-1}}) = F(u_\alpha(\alpha(t)\xi)) = u_{\alpha^q}(\alpha(t)q^q\xi^q),$$

which shows that $\alpha^q(F(t)) = \alpha(t)^q$ for all $t \in T$, as desired. \qed

In the following result, the proof of the implication “(iii) $\Rightarrow$ (i)” relies on the fact that $F_p$ as in Example 1.4.13 is a Frobenius map.

**Proposition 1.4.27.** Assume that $G$ is connected reductive. Let $F: G \to G$ be an isogeny and $T \subseteq G$ be an $F$-stable maximal torus. Let $\varphi$ be the map induced by $F$ on $X = X(T)$. Then the following conditions are equivalent.

(i) $F$ is a Frobenius map (corresponding to a rational structure on $G$ over a finite subfield of $k$).

(ii) We have $\varphi = p^m\varphi_0$ where $m \in \mathbb{Z}_{\geq 1}$ and $\varphi_0: X \to X$ is an automorphism of finite order such that $\varphi_0(R) = R$ and $\varphi_0^q(R^{q^k}) = R^{q^k}$. (In particular, $\varphi_0$ is an isogeny of root data with all root exponents equal to 1.)

(iii) There exists an automorphism of algebraic groups $\gamma: G \to G$ of finite order such that $\gamma(T) = T$ and some $m' \geq 1$ such that $F = \gamma \circ F_p^m = F_p^m \circ \gamma$, where $F_p$ is an isogeny as in Example 1.4.13.

If these conditions hold, then $m$ (as in (ii)) equals $m'$ (as in (iii)) and $F$ is the Frobenius map with respect to an $\mathbb{F}_q$-rational structure where $q = p^m$. Furthermore, all root exponents of $F$ are equal to $q$ and $q$ is the number defined in Proposition 1.4.18.
**Proof.** “(i) ⇒ (ii)” Let $F$ be a Frobenius map corresponding to an $\mathbb{F}_q$-rational structure on $G$ where $q = p^m$ for some $m \geq 1$. By assumption, $T$ is $F$-stable so $T$ is also defined over $\mathbb{F}_q$; see Remark 1.4.10. Hence, we can apply Lemma 1.4.14 and so $\varphi = q\varphi_0$ where $\varphi_0 : X \to X$ has finite order. Furthermore, using Lemma 1.4.26 one sees that $\varphi_0(\alpha^1) = \alpha$ and $\varphi_0^\vee(\alpha^\vee) = (\alpha^1)^\vee$ for all $\alpha \in R$. Thus, (I1) and (I2) hold for $\varphi_0$, where the root exponents of $\varphi_0$ are all equal to 1.

“(ii) ⇒ (iii)” Let $F_p : G \to G$ be as in Example 1.4.15. Then $F_p^m$ is a Steinberg map and the map induced by $F_{p,m}^m$ on $X$ is scalar multiplication with $p^m$. So, by Lemma 1.4.23 there exists an isogeny $\psi : G \to G$ which maps $T$ onto itself and induces $\varphi_0$, and such that $f \circ F_p = F_p^m \circ f$. Now $\varphi_0$ has finite order, say $d$. Then $F_{m}^{d}$ induces the identity on $X$. Hence, by Theorem 1.3.12 there exists some $x \in T$ such that $F_{m}^{d}(g) = tgt^{-1}$ for all $g \in G$. Since $t$ also has finite order, we conclude that some positive power of $F_{m}^{d}$ is the identity. Hence, $f$ itself has finite order. Now, $F$ and $F' := f \circ F_p$ are isogenies which induce the same map on $X$. Hence, by Theorem 1.3.12 there exists some $y \in T$ such that $F'(g) = yF(g)y^{-1}$ for all $g \in G$. As in the proof of Lemma 1.4.18 there exists some $x \in T$ such that $F' = \iota_x^{-1} \circ F \circ \iota_x$, where $\iota_x$ denotes the inner automorphism of $G$ defined by $x$. Then

$$F = \iota_x \circ F' \circ \iota_x^{-1} = (\iota_x \circ f \circ \iota_x^{-1}) \circ (\iota_x \circ F_p \circ \iota_x^{-1})^m$$

(and the two factors still commute). Now, since $x \in T$, the isogeny $F_p' := \iota_x \circ F_p \circ \iota_x^{-1}$ also satisfies the conditions in Example 1.4.15. Furthermore, $\gamma := \iota_x \circ f \circ \iota_x^{-1}$ is an automorphism of finite order such that $\gamma(T) = T$. Thus, (c) holds.

“(iii) ⇒ (i)” As already mentioned in Example 1.4.15, we can assume that $F_p$ is the Frobenius map corresponding to an $\mathbb{F}_p$-rational structure on $G$. Then $F_p^m$ is the Frobenius map corresponding to an $\mathbb{F}_q$-rational structure on $G$ where $q = p^m$. Hence so is $F = \gamma \circ F_p^m$ by Remark 1.4.1(c).

Finally, assume that (i), (ii), (iii) hold. Then the above arguments show that $m = m'$. Furthermore, (ii) shows that $\det(\varphi) = \pm (p^m)^{\text{rank}X}$ and so $q = p^m$ satisfies the conditions in Proposition 1.4.18.

The following example indicates that Steinberg maps can be much more complicated than Frobenius maps.

**Example 1.4.28.** (a) In the setting of Example 1.4.22 one easily sees that neither the conclusion of Lemma 1.4.10 nor that of Lemma 1.4.26 hold for $F'$. Hence, although $F$ is a Frobenius map, the map $F'$ is not.

(b) Let $G = SL_2(k) \times PGL_2(k)$ and $p = 2$. Then $G$ is semisimple of type $A_1 \times A_1$, with Cartan matrix $C = 2I_2$ where $I_2$ denotes the identity matrix. The root datum of $G$ is determined by the factorisation

$$C = \tilde{A} \cdot A^\vee \quad \text{where} \quad A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{A} = \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \end{pmatrix}.$$

For a fixed $m \geq 1$, we define

$$P = \begin{pmatrix} 0 & 2^m \\ 2^m & 0 \end{pmatrix} \quad \text{and} \quad P^o = \begin{pmatrix} 0 & 2^{m-1} \\ 2^{m+1} & 0 \end{pmatrix}.$$

Then the pair $(P, P^o)$ satisfies the conditions in 1.2.18 and so there is a corresponding isogeny $F : G \to G$, with root exponents $2^{m+1}, 2^{m-1}$. Since $P^2 = 4I_2$, we know that $F$ is a Steinberg map (see Proposition 1.4.17). Furthermore, we have $P = 2^m P_0$ where $P_0 \in M_2(\mathbb{Z})$ has order 2; thus, the conclusion of Lemma 1.4.10 holds for $F$ where $q = 2^m$. Note also that the two projections (on the first and on the second factor) define isomorphisms of finite groups $G^F \cong SL_2(q) \cong PGL_2(q)$. 

On the other hand, since the root exponents are not all equal, Lemma 1.4.26 shows that \( F \) is not a Frobenius map! One easily checks directly that there is no matrix \( P_0^0 \) such that the pair \((P_0, P_0^0)\) satisfies the conditions in 1.2.18. Thus, \( P_0 \) does not come from an isogeny of root data.

### 1.5. Working with isogenies and root data; examples

We now discuss some applications and examples of the theory developed so far. We start with some basic material about semisimple groups. Up until Proposition 1.5.10, \( k \) may be any algebraically closed field.

#### 1.5.1. Let us fix a Cartan matrix \( C = (c_{st})_{s,t \in S} \). Let \( \Lambda(C) \) be the finite abelian group defined in Remark 1.2.8. Then the semisimple algebraic groups with a root datum of Cartan type \( C \) are classified in terms of subgroups of \( \Lambda(C) \). This works as follows. We have \( \Lambda(C) := \Omega/\mathbb{Z}C \) where \( \Omega \) is the free abelian group with basis \( \{\omega_s \mid s \in S\} \) and \( \mathbb{Z}C \) is the subgroup generated by \( \{\sum_{s \in S} c_{st} \omega_s \mid t \in S\} \). Thus, giving a subgroup of \( \Lambda(C) \) is the same as giving a lattice \( L \) such that \( \mathbb{Z}C \subseteq L \subseteq \Omega \).

Such a lattice \( L \) is free abelian of the same rank as \( \Omega \). We choose a set of free generators \( \{x_s \mid s \in S\} \) of \( L \). Since \( \mathbb{Z}C \subseteq L \), we have unique expressions

\[
\sum_{s \in S} c_{st} \omega_s = \sum_{u \in S} a_{tu} x_u \quad (t \in S),
\]

where \( A = (a_{tu})_{t,u \in S} \) is a square matrix with integer coefficients. We also write \( x_u = \sum_{s \in S} \tilde{a}_{su} \omega_s \) where \( \tilde{A} = (\tilde{a}_{su})_{s,u \in S} \) is a square matrix with integer coefficients. Substituting this into \((*)\) and comparing coefficients, we obtain \( C = \tilde{A} \cdot A^\ast \). As in 1.2.14 such a factorisation of \( C \) determines a root datum \( \mathcal{R}_L = (X,R,Y,R') \) of Cartan type \( C \), where \( R \) has a base given by \( \alpha_t := \sum_{s \in S} a_{ts} x_s \) for \( t \in S \). We have \( |X/\mathbb{Z}R| < \infty \) since \( \tilde{A}, A \) are square matrices. If we choose a different set of generators of \( L \), say \( \{y_t \mid t \in S\} \), then we obtain another factorisation \( C = \tilde{B} \cdot B^\ast \) where \( B, \tilde{B} \) are square integer matrices. Writing \( y_t = \sum_{u \in S} p_{tu} x_u \) where \( P = (p_{tu})_{u,t \in S} \) is invertible over \( \mathbb{Z} \), we have \( PP^\ast = A^\ast \) and \( \tilde{B} = \tilde{A} P \). Hence, the root data defined by \( C = \tilde{A} \cdot A^\ast \) and by \( C = \tilde{B} \cdot B^\ast \) are isomorphic; see Lemma 1.2.20. Thus, every lattice \( L \) such that \( \mathbb{Z}C \subseteq L \subseteq \Omega \) determines a root datum \( \mathcal{R}_L \) as above, which is unique up to isomorphism. By Theorem 1.3.14 there exists a corresponding connected reductive algebraic group \( G_L \) over \( k \) (unique up to isomorphism by Corollary 1.3.13). The group \( G_L \) is semisimple since \( |X/\mathbb{Z}R| < \infty \); see Remark 1.3.5.

**Proposition 1.5.2.** Let \( G \) be a semisimple algebraic group over \( k \) with root datum \( \mathcal{R} = (X,R,Y,R') \) (relative to some maximal torus of \( G \)). Let \( C = (c_{st})_{s,t \in S} \) be the Cartan matrix of \( \mathcal{R} \). Then there exists a lattice \( L \) as in 1.5.1 such that \( G \cong G_L \).

We have \( X/\mathbb{Z}R \cong L/\mathbb{Z}C \) and, hence, \( Z(G) \cong \text{Hom}(L/\mathbb{Z}C,k^\times) \).

**Proof.** Let \( \Pi \) be a base of \( R \); we have \( |\Pi| = \text{rank } X \) since \( G \) is semisimple and, hence, \( X/\mathbb{Z}R \) is finite. Also choose a \( \mathbb{Z} \)-basis of \( X \). By Remark 1.2.13 we have a corresponding factorisation \( C = \tilde{A} \cdot A^\ast \) where \( \tilde{A}, A \) are square integral matrices. In particular, we can use \( S \) as an indexing set for both the rows and the columns of these matrices. Then let \( \tilde{L} \) be the sublattice of \( \Omega \) spanned by the elements

\[
x_t := \sum_{s \in S} \tilde{a}_{st} \omega_s \quad (t \in S), \quad \text{where} \quad \tilde{A} = (\tilde{a}_{st})_{s,t \in S}.
\]

We have \( \mathbb{Z}C \subseteq \tilde{L} \subseteq \tilde{\Omega} \), since \( C = \tilde{A} \cdot A^\ast \). Applying the construction in 1.5.1 to \( \tilde{L} \), we obtain a group \( G_{\tilde{L}} \). Then Corollary 1.3.13 shows that \( G \cong G_{\tilde{L}} \). Finally, 1.5.1\((*)\) implies that \( X/\mathbb{Z} \cong L/\mathbb{Z}C \) and this yields \( Z(G) \); see Remark 1.3.5. \( \square \)
Example 1.5.3. Let \( C = (c_{st})_{s,t \in S} \) be a Cartan matrix and consider the group \( \Lambda(C) = \Omega/\mathbb{Z}C \), as above.

(a) If \( L = \mathbb{Z}C \), then we choose the generators \( \{x_s \mid s \in S\} \) of \( L \) to be the given generators of \( \mathbb{Z}C \). So \( A \) in [1.5.1(*)] is the identity matrix and \( \tilde{A} = C \).

Hence, in this case, \( \mathcal{R}_L \) is the root datum \( \mathcal{R}_\text{ad}(C) \) in Example 1.2.16. The corresponding group \( G_L \) will be denoted by \( G_{\text{ad}} \); we have \( Z(G_{\text{ad}}) = \{1\} \).

(b) If \( L = \Omega \), then we can take \( x_s = \omega_s \) for all \( s \in S \). So \( A = C^{\text{ad}} \) and \( \tilde{A} \) is the identity matrix. Hence, in this case, \( \mathcal{R}_L \) is the root datum \( \mathcal{R}_\text{sc}(C) \) in Example 1.2.16. The corresponding group \( G_L \) will be denoted by \( G_{\text{sc}} \); we have \( Z(G_{\text{sc}}) \cong \text{Hom}(\Lambda(C), \mathbb{R}^\times) \).

The groups \( G_{\text{sc}} \) and \( G_{\text{ad}} \) have some important universal properties which will be discussed in further detail below. We shall call \( G_{\text{sc}} \) the semisimple group of simply-connected type \( C \) and \( G_{\text{ad}} \) the semisimple group of adjoint type \( C \).

Example 1.5.4. Assume that \( C \) is an indecomposable Cartan matrix. The isomorphism types of \( \Lambda(C) \) are listed in Remark 1.2.8

(a) If \( C \) is of type \( A_{n-1} \), then \( \Lambda(C) \cong \mathbb{Z}/n\mathbb{Z} \). Hence, for each divisor \( d \) of \( n \), we have a unique lattice \( L_d \subseteq \Omega \) such that \( |L_d/\mathbb{Z}C| = d \); let \( G(d) \) be the corresponding group. We have \( G(1) = G_{\text{ad}} \cong \text{PGL}_n(k) \) and \( G(n) = G_{\text{sc}} \cong \text{SL}_n(k) \); see Examples 1.3.9 and 1.3.8. The remaining groups \( G(d) \) are explicitly constructed in [Ch56-58/05 §20.3], as images of \( \text{SL}_n(k) \) under certain representations.

(b) If \( C \) is of type \( B_n, C_n, E_6 \) or \( E_7 \), then \( \Lambda(C) \) is cyclic of prime order. Hence, either \( L = \mathbb{Z}C \) or \( L = \Omega \). So, in this case, the only possible groups are \( G_{\text{ad}} \) and \( G_{\text{sc}} \).

In type \( B_n \), we have \( G_{\text{ad}} \cong \text{SO}_{2n+1}(k) \) and \( G_{\text{sc}} \cong \text{Spin}_{2n+1}(k) \).

In type \( C_n \), we have \( G_{\text{ad}} \cong \text{PCSp}_{2n}(k) \) and \( G_{\text{sc}} \cong \text{Sp}_{2n}(k) \).

(See the references in [1.1.4] for the precise definitions of these groups.)

(c) If \( C \) is of type \( D_n \), then \( \Lambda(C) \) has order 4 and there are 3 (for \( n \) odd) or 5 (for \( n \) even) possible lattices \( L \). We have \( G_{\text{ad}} \cong \text{PGO}_{2n}(k) \) and \( G_{\text{sc}} \cong \text{Spin}_{2n}(k) \). Using the labelling in Table 1.1 the group \( \text{SO}_{2n}(k) \) corresponds to the unique \( L \) of index 2 in \( \Omega \) which is invariant under the involution of \( \Omega \) obtained by exchanging \( \omega_1 \) and \( \omega_2 \). In terms of our matrix language in Section 1.2 the root datum of \( \text{SO}_{2n}(k) \) is given by the factorisation \( C = \tilde{A} \cdot A^{\text{ad}} \) where

\[
A = \tilde{A} = \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 \\
-1 & 1 & 0 & \cdots & 0 \\
0 & -1 & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -1 & 1 \\
0 & \cdots & 0 & \cdots & 1
\end{pmatrix}.
\]

(See [Spr98 Exercise 7.4.7].) If \( n \) is even, then there are two further lattices of index 2, which both give rise to the half-spin group \( \text{HSpin}_{2n}(k) \).

(d) Finally, if \( C \) is of type \( G_2, F_4 \) or \( E_8 \), then \( \Lambda(C) = \{0\} \). So, in this case, all semisimple algebraic groups over a fixed field \( k \) with a root datum of Cartan type \( C \) are isomorphic to each other; in particular, \( G_{\text{sc}} \cong G_{\text{ad}} \).

We refer to [Grob98 §1.7], [Spr98 §7.4], for further details about the various groups of classical type \( B_n, C_n \) and \( D_n \).
Example 1.5.5. The Cartan matrices of type $B_n$ and $C_n$ are the $n \times n$-matrices given by

$$
B_n : \begin{pmatrix}
2 & -2 & 0 & \ldots & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 \\
0 & -1 & 2 & -1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & -1 & 2 & -1 \\
0 & \ldots & 0 & 0 & -1 & 2
\end{pmatrix},
$$

$$
C_n : \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 \\
-2 & 2 & -1 & 0 & \ldots & 0 \\
0 & -1 & 2 & -1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & -1 & 2 & -1 \\
0 & \ldots & 0 & 0 & -1 & 2
\end{pmatrix},
$$

respectively. Let $C$ denote the second matrix, and $C'$ the first. Let $P^0$ be the diagonal matrix of size $n$ with diagonal entries $1, 2, 2, \ldots, 2$. Then $CP^0 = P'C'$. Thus, if we also set $P = P^0$, then the two conditions in Example 1.5.6 are satisfied and so the pair $(P, P^0)$ defines a 2-isogeny from $R_{sc}(C)$ to $R_{sc}(C')$ (see Example 1.2.10). Let $k$ be an algebraically closed field of characteristic 2. Let $G_{sc}$ and $G'_{sc}$ be the semisimple algebraic groups over $k$ corresponding to $R_{sc}(C)$ and $R_{sc}(C')$, respectively. We have $G_{sc} \cong \text{Spin}_{2n+1}(k)$ and $G'_{sc} \cong \text{Sp}_{2n}(k)$. Then Theorem 1.3.12 yields the existence of an isogeny $f : G_{sc} \to G'_{sc}$. This is one of Chevalley’s exceptional isogenies considered at the end of [Ch56-58/05, §23.7].

Example 1.5.6. Let $G$ be connected reductive over $k$ with root datum $\mathcal{R} = (X, R, Y, R^\vee)$, relative to some maximal torus $T$ of $G$. Dually to the isomorphism in [1.1.10(c)], we have a canonical isomorphism (see [Ca85 §3.1])

$$Y(T) \otimes_k k^\times \xrightarrow{\sim} T, \quad \nu \otimes \xi \mapsto \nu(\xi).$$

Hence, if $\{\nu_1, \ldots, \nu_n\}$ is a $\mathbb{Z}$-basis of $Y(T)$, then every element $t \in T$ can be written uniquely in the form $t = \nu_1(\xi_1) \cdots \nu_n(\xi_n)$ where $\xi_1, \ldots, \xi_n \in k^\times$.

Now assume that $G$ is semisimple of simply-connected type. Then $Y(T) = \mathbb{Z}R^\vee$. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a base for $R$ and $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ be the corresponding base for $R^\vee$. Hence, we have

$$T = \{h(\xi_1, \ldots, \xi_n) := \alpha_1^\vee(\xi_1) \cdots \alpha_n^\vee(\xi_n) \mid \xi_1, \ldots, \xi_n \in k^\times\}.$$

In this setting, one can now explicitly determine the center of $G$ as a subset of $T$. Indeed, using Remark 1.3.5 and the above description of $T$, we obtain

$$Z(G) = \{h(\xi_1, \ldots, \xi_n) \in T \mid \xi_{\mathfrak{s}_1}^{(\alpha_j, \alpha_i^\vee)} \cdots \xi_{\mathfrak{s}_1}^{(\alpha_j, \alpha_n^\vee)} = 1 \text{ for } 1 \leq i, j \leq n\}.$$

Now the numbers $c_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$ ($1 \leq i, j \leq n$) are just the entries of the Cartan matrix of $\mathcal{R}$, so this yields an explicit system of $n$ equations which we need to solve for $\xi_1, \ldots, \xi_n$. Let us describe this explicitly in all cases, where we refer to the labelling of the simple roots in Table 1.1 (This will also be relevant in Section 1.7)

- $A_n$: $G \cong \text{SL}_{n+1}(k)$ and $Z(G) = \{h(\xi, \xi^2, \xi^3, \ldots, \xi^n) \mid \xi^{n+1} = 1\}$; this is contained in the subtorus $S := \{h(\xi, \xi^2, \xi^3, \ldots, \xi^n) \mid \xi \in k^\times\} \subseteq T$.

- $B_n$: $G \cong \text{Spin}_{2n+1}(k)$. If $n \geq 2$ is even, then

$$Z(G) = \{h(1, \xi, 1, \xi, 1, \xi, 1, \ldots) \mid \xi^2 = 1\}$$

is contained in $S := \{h(1, \xi, 1, \xi, 1, \xi, 1, \ldots) \mid \xi \in k^\times\}$. For $n \geq 3$ odd,

$$Z(G) = \{h(\xi, 1, \xi, 1, \xi, 1, \xi, 1, \ldots) \mid \xi^2 = 1\}$$

is contained in $S := \{h(\xi, 1, \xi, 1, \xi, 1, \xi, 1, \ldots) \mid \xi \in k^\times\}$.

- $C_n$: $G \cong \text{Sp}_{2n}(k)$ and $Z(G) = \{h(\xi, 1, 1, 1, \ldots) \mid \xi^2 = 1\}$; this is contained in $S := \{h(\xi, 1, 1, 1, \ldots) \mid \xi \in k^\times\}$.
In each case, we obtain these descriptions, we did not have to rely on explicit realisations of groups of classical type as matrix groups: the abstract setting in terms of root data has 1.5.7. Following remarks will be useful.

\[ \begin{align*}
T &\text{ (relative to some maximal torus } T \text{) is of Cartan type } C. \text{ Let } G_{sc} \text{ and } G_{ad} \text{ be of Cartan type } C, \text{ as in Example 1.5.3, relative to } T \subseteq B \subseteq G_{sc} \text{ and } T' \subseteq B' \subseteq G_{ad}. \text{ Then there exist central isogenies }
\end{align*} \]

\[ \begin{align*}
\tilde{f} : G_{sc} &\longrightarrow G \quad \text{and} \quad f' : G \longrightarrow G_{ad},
\end{align*} \]

such that \( \tilde{f}(T) = T, \tilde{f}(B) = B, f'(T) = T', f'(B) = B' \).
where (\(G_1, T_1, B_1\)) = (G_{sc}, T, B) and (\(G_2, T_2, B_2\)) = (G, T, B). The root datum of \(G_1\) is given by a factorisation of \(C\) as above where \(A_1 = C^{sc}\) and \(\tilde{A}_1 = I\) (identity matrix). The only extra information about the root datum of \(G_2\) is that, in the factorisation \(C = \tilde{A}_2 \cdot A_2^{sc}\), both \(A_2, \tilde{A}_2\) are square matrices (since \(G\) is semisimple).

In order to find \(f: G_1 \to G_2\), we need to specify a pair of (square) integral matrices \((\tilde{P}, \tilde{P}^o)\) where \(\tilde{P} A_2^{sc} = C \tilde{P}^o\), \(\tilde{P} \tilde{A}_2 = P\) and \(\tilde{P}^o\) is a monomial matrix whose non-zero entries are powers of \(p\). (Note that \(\tilde{P}\) will be automatically invertible over \(\mathbb{Q}\).) There is a natural choice for such a pair, namely, \((\tilde{P}, \tilde{P}^o) := (\tilde{A}_2, I)\). Thus, the correspondence in [1.5.7] yields the existence of \(\tilde{f}\). The root exponents of \(\tilde{f}\) (which are the non-zero entries of \(\tilde{P}^o = I\)) are all equal to 1, hence \(\tilde{f}\) is a central isogeny.

Now consider \(G_{ad}\). We argue as above, where now \((G_1, T_1, B_1) = (G, T, B)\) and \((G_2, T_2, B_2) = (G_{ad}, T', B')\). The root datum of \(G_2\) is given by \(C = \tilde{A}_2 \cdot A_2^{sc}\) where \(A_2 = I\) and \(\tilde{A}_2 = C\). We need to specify a pair of (square) integral matrices \((P', \tilde{P}^o)\) where \(P' = A_1^{sc} P^o\), \(P^o C = \tilde{A}_1 P'\) and \(P^o\) is a monomial matrix whose non-zero entries are powers of \(p\). Again, there is a natural choice for such a pair, namely, \((P', \tilde{P}^o) := (A_1^{sc}, I)\). As above, this yields the existence of \(f'\).

\[\text{Proof.}\]

First consider \(G_{sc}\). We place ourselves in the setting of [1.5.7] where \((G_1, T_1, B_1) = (G_{sc}, T, B)\) and \((G_2, T_2, B_2) = (G, T, B)\). The root datum of \(G_1\) is given by a factorisation of \(C\) as above where \(A_1 = C^{sc}\) and \(\tilde{A}_1 = I\) (identity matrix). The only extra information about the root datum of \(G_2\) is that, in the factorisation \(C = \tilde{A}_2 \cdot A_2^{sc}\), both \(A_2, \tilde{A}_2\) are square matrices (since \(G\) is semisimple).

Let \(\tilde{f}\) be as in Proposition [1.5.8]. Assume, furthermore, that \(F: G \to G\) is an isogeny. Then the following hold.

(a) The isogeny \(F\) lifts to \(G_{sc}\); more precisely, there exists a unique isogeny \(\tilde{F}: G_{sc} \to G_{sc}\) such that \(F \circ \tilde{f} = \tilde{f} \circ \tilde{F}\).

(b) The isogeny \(F\) descends to \(G_{ad}\); more precisely, there exists a unique isogeny \(F': G_{ad} \to G_{ad}\) such that \(F' \circ f' = f' \circ F\).

In both cases, the root exponents of \(\tilde{F}\) and of \(F'\) are equal to those of \(F\). If, moreover, \(F\) is a Steinberg map, then so are \(F'\) and \(\tilde{F}\).

\[\text{Proof.}\]

First note that \(F'\), if it exists, is clearly unique. The uniqueness of \(\tilde{F}\) (if it exists) is shown as follows. Let \(F_1: G_{sc} \to G_{sc}\) be another isogeny such that \(F \circ \tilde{f} = f \circ F_1\). Then the map sending \(g \in G_{sc}\) to \(\tilde{F}(g) F_1(g)^{-1}\) is a homomorphism of algebraic groups from \(G_{sc}\) to the center of \(G_{sc}\). Since \(G_{sc}\) is connected and the center of \(G_{sc}\) is finite, that map must be constant and so \(\tilde{F}(g) F_1(g)^{-1} = 1\) for all \(g \in G_{sc}\). We now turn to the problem of showing the existence of \(\tilde{F}\) and \(F'\).

Let \(T \subseteq B \subseteq G, T' \subseteq B' \subseteq G_{ad}\) be as in Proposition [1.5.8]. We consider the corresponding root data of \(G, G_{sc}, G_{ad}\), and write \(X = X(T), X' = X(T')\). Then \(\tilde{f}\) induces a p-isogeny \(\tilde{\varphi}: X' \to X\) and \(f'\) induces a p-isogeny \(\varphi': X' \to X\). Thus, we are in the setting of [1.5.7]. Now consider the isogeny \(F: G \to G\). As already pointed out in the proof of [St06, 9.16], one easily sees that (a) and (b) hold for \(F\) if and only if (a) and (b) hold for \(\iota_g \circ F\), where \(\iota_g\) is an inner automorphism of \(G\) (for any \(g \in G\)). Hence, using Theorem 1.3.3 and replacing \(F\) by \(\iota_g \circ F\) for a suitable \(g\), we may assume without loss of generality that \(F(T) = T\) and \(F(B) = B\). Then \(F\) induces a p-isogeny \(\Phi: X \to X\) and, again, we are in the setting of [1.5.7]. Now, if we can show that there exist p-isogenies \(\Phi: X \to X\) and \(\Phi': X' \to X'\) such that

\[\tilde{\varphi} \circ \Phi = \tilde{\Phi} \circ \tilde{\varphi}\]

and

\[\varphi' \circ \Phi' = \Phi \circ \varphi',\]

then the existence of \(\tilde{F}\) and \(F'\) follows from a general result about isogenies, which can already be found in [Ch56-58/05, §18.4] and which is a step in the proof of the Isogeny Theorem [1.3.12] (see also [St99, 3.2, 3.3]). In order to see how \(\tilde{\Phi}\) and
\( \Phi' \) can be constructed, we use the correspondence in \textbf{1.5.7} to describe everything in terms of pairs of square integral matrices. (The following part of the proof is somewhat different from the original proof in \cite{St68}.)

Let \((\tilde{P}, \tilde{P}^o)\) and \((P', P'^o)\) be the pairs of matrices corresponding to \( \tilde{\varphi} \) and \( \varphi' \), respectively. Furthermore, let \((Q, Q^o)\) be the pair of matrices corresponding to \( \Phi \). Now recall that the root data of \( G_{sc} \) and \( G_{ad} \) are given by the factorisations \( C = I \cdot (C')^{tr} \) and \( C = C \cdot I^{tr} \), respectively. Let \( \tilde{A}, \tilde{\varphi} \) be square integral matrices such that the root datum of \( \tilde{G} \) is given by the factorisation \( \tilde{C} = \tilde{A} \cdot \tilde{A}' \). Then the conditions in \textbf{1.2.18} imply that

\[
\tilde{P} = \tilde{P}^o \tilde{A}, \quad P' = A^{tr} P'^o, \quad QA^{tr} = A^{tr} Q^o, \quad Q^o \tilde{A} = \tilde{A} Q^o,
\]

where \( \tilde{P}^o \) and \( P'^o \) are monomial matrices all of whose non-zero entries are equal to 1 (since \( \tilde{f} \) and \( f' \) are central).

Assume first that \( \tilde{\varphi} \) exists and let \((\tilde{Q}, \tilde{Q}^o)\) be the corresponding pair of matrices. Since the root datum of \( G_{sc} \) is given by the factorisation \( C = I \cdot (C')^{tr} \), the conditions in \textbf{1.2.18} imply that \( \tilde{Q} = \tilde{Q}^o \). Since \( \tilde{\varphi} \circ \Phi = \tilde{\varphi} \circ \tilde{\varphi} \), we must have \( \tilde{P} Q = \tilde{Q} \tilde{P} \). Using the above relations, we deduce that

\[
\tilde{Q} = \tilde{Q}^o = \tilde{P}^o Q^o (\tilde{P}^o)^{-1}.
\]

Thus, if \( \tilde{\varphi} \) exists, then \((\tilde{Q}, \tilde{Q}^o)\) is determined by \( Q^o \) and \( \tilde{P}^o \); in particular, the root exponents of \( \tilde{\varphi} \) are equal to those of \( \tilde{\varphi} \). Conversely, it is straightforward to check that the map \( \Phi : X \to \tilde{X} \) defined by the matrix \( Q \) given by the above formula has the required properties. Thus, (a) is proved.

Similarly, assume first that \( \Phi' \) exists and let \((Q', Q'^o)\) be the corresponding pair of matrices. Then one deduces that

\[
Q' = Q'^o = (P'^o)^{-1} Q^o P'^o.
\]

Conversely, one checks that the map \( \Phi' : X' \to \tilde{X} \) defined by the matrix \( Q' \) given by the above formula has the required properties. Thus, (b) is proved.

Finally, assume that \( F \) is a Steinberg map. Then, using the characterisation in Proposition \textbf{1.4.17} one easily sees that \( F' \) and \( \tilde{F} \) are also Steinberg maps. \( \square \)

**Proposition 1.5.10** (Cf. \cite{St67} p. 46). Let \( C \) be a Cartan matrix and \( G_{sc}, G_{ad} \) be as in Example \textbf{1.5.3}. Assume that \( C \) is a block diagonal matrix, with diagonal blocks \( C_1, \ldots, C_n \). Then we have

\[
G_{sc} = G_1 \times \ldots \times G_n \quad \text{and} \quad G_{ad} = G'_1 \times \ldots \times G'_n,
\]

where \( G_i \subseteq G_{sc} \) and \( G'_i \subseteq G_{ad} \) are the normal subgroups corresponding to the various diagonal blocks \( C_i \), as in Remark \textbf{1.3.6}. For each \( i \), the group \( G_i \) is simple of simply-connected type \( C_i \) and \( G'_i \) is simple of adjoint type \( C_i \).

**Proof.** The definition of \( R_{sc}(C) \) shows that this root datum is the direct sum of the root data \( R_{sc}(C_1), \ldots, R_{sc}(C_n) \). Hence, the assertion concerning \( G_{sc} \) immediately follows from Remark \textbf{1.3.6} The argument for \( G_{ad} \) is analogous. \( \square \)

**1.5.11.** We shall assume from now on that \( G \) is connected reductive over \( k = \overline{\mathbb{F}}_p \) (where \( p \) is a prime number) and \( F : G \to G \) is a Steinberg map. Let \( Z \) be the center of \( G \) and \( G_{der} \) be the derived subgroup of \( G \). Clearly, we have \( F(Z) = Z \) and \( F(G_{der}) = G_{der} \). Since \( G = Z^o G_{der} \) and \( Z^o \cap G_{der} \) is finite, we obtain isogenies

\[
Z^o \times G_{der} \longrightarrow G \quad \text{and} \quad G \longrightarrow G / G_{der} \times G / Z^o \]

\[
(z, g) \mapsto zg \quad \text{and} \quad g \mapsto (gG_{der}, gZ^o).
\]
(Note that these are maps between groups of the same dimension; the first map is clearly surjective and, hence, has a finite kernel; the second map has a finite kernel and, hence, is surjective.) Recall from Lemma 1.1.12 that $G_{\text{der}}$ is semisimple. The group $G/G_{\text{der}}$ is a torus. (For, it is connected, abelian and consists of elements of order prime to $p$; see Lemma 1.1.1.) Furthermore, $G_{\text{ss}} := G/Z^0$ is reductive (see Lemma 1.4.25) with a finite center and, hence, is semisimple. Using also the isogenies in Proposition 1.5.8 we obtain isogenies

$$Z^0 \times (G_{\text{der}})_{\text{ss}} \to G \to G/G_{\text{der}} \times (G_{\text{ss}})_{\text{ad}}.$$ 

Now, by Lemma 1.4.25, we have induced Steinberg maps on $G_{\text{ss}}$ and on $G/G_{\text{der}}$. By Proposition 1.5.9 there are also induced Steinberg maps on $(G_{\text{der}})_{\text{ss}}$ and on $(G_{\text{ss}})_{\text{ad}}$. Since all these maps are induced and uniquely determined by $F$, we will now simplify our notation and denote all these induced maps by $F$. Using Proposition 1.4.12(c), we conclude that $|G_{\text{der}}^F| = |(G_{\text{der}})_{\text{ss}}^F|$, $|G_{\text{ss}}^F| = |(G_{\text{ss}})_{\text{ad}}^F|$ and $|G^F| = |(Z^0)^F| |G_{\text{der}}^F||G_{\text{ss}}^F|$. 

Also note that the natural map $G \to G_{\text{ss}}$ induces a surjective map $G^F \to G_{\text{ss}}^F$ (since the kernel of $G \to G_{\text{ss}}$ is connected).

**Remark 1.5.12.** By a slight abuse of notation, we shall denote $(G_{\text{ss}})_{\text{ad}}$ simply by $G_{\text{ad}}$. Thus, as above, we obtain a central isogeny $G_{\text{ss}} \to G_{\text{ad}}$ which commutes with the action of $F$ on both sides. Composing this isogeny with the natural map $G \to G_{\text{ss}} = G/Z^0$, we obtain a surjective, central isotypy

$$\pi_{\text{ad}} : G \to G_{\text{ad}} \quad \text{with} \quad \ker(\pi_{\text{ad}}) = Z,$$

which commutes with the action of $F$ on both sides and which we call an adjoint quotient of $G$. We certainly have an inclusion $\pi_{\text{ad}}(G^F) \subseteq G^F_{\text{ad}}$ but, in general, equality will not hold. By Proposition 1.4.12, $\pi_{\text{ad}}(G^F)$ is a normal subgroup of $G^F_{\text{ad}}$ and we have an isomorphism

$$G^F_{\text{ad}}/\pi_{\text{ad}}(G^F) \cong Z/\mathcal{L}(Z),$$

where $\mathcal{L} : G \to G$, $g \mapsto g^{-1}F(g)$. One easily sees that $Z/\mathcal{L}(Z) = (Z/Z^0)^F$, where the subscript $F$ denotes “$F$-coinvariants”, that is, the largest quotient on which $F$ acts trivially. The above isomorphism is explicitly obtained by sending $g \in G^F_{\text{ad}}$ to $\hat{g} = g^{-1}F(\hat{g}) \in Z$ where $\hat{g} \in G$ is any element satisfying $\pi_{\text{ad}}(\hat{g}) = g$.

Also note that each $g \in G^F_{\text{ad}}$ defines an automorphism $\alpha_g : G^F \to G^F$, $g_1 \mapsto g_1\hat{g}^{-1} \hat{g}$ (where, as above, $\hat{g} \in G$ is such that $\pi_{\text{ad}}(\hat{g}) = g$; the map $\alpha_g$ obviously does not depend on the choice of $\hat{g}$). In this way, we obtain a group homomorphism

$$G^F_{\text{ad}} \to \text{Aut}(G^F), \quad g \mapsto \alpha_g.$$ 

The automorphisms $\alpha_g$ are called diagonal automorphisms of $G^F$.

**Remark 1.5.13.** Again, by a slight abuse of notation, we shall denote $(G_{\text{der}})_{\text{ss}}$ simply by $G_{\text{ss}}$. Thus, as above, we obtain a central isotypy

$$\pi_{\text{ss}} : G_{\text{ss}} \to G \quad \text{with} \quad \pi_{\text{ss}}(G_{\text{ss}}) = G_{\text{der}},$$

which commutes with the action of $F$ on both sides and which we call a simply-connected covering of the derived subgroup of $G$. We certainly have $\pi_{\text{ss}}(G_{\text{ss}}) \subseteq G_{\text{der}}$ but, again, equality will not hold in general. In fact, we have:

(a) $\pi_{\text{ss}}(G_{\text{ss}}^F) = \langle u \in G^F \mid u \text{ unipotent} \rangle \subseteq G_{\text{der}}^F$ (cf. [St68, 12.6]), and

(b) $G^F/\pi_{\text{ss}}(G_{\text{ss}}^F)$ is abelian of order prime to $p$ ([DeLu76, 1.23]).
Note that $\pi_{sc}(G^{F}_{sc})$ is a characteristic subgroup of $G^{F}$; furthermore, we have the inclusions $[G^{F}_{sc}, G^{F}_{sc}] \subseteq \pi_{sc}(G^{F}_{sc}) \subseteq G^{F}_{der}$ but each of these may be strict, as can be seen already in the example where $G = G_{der} = \text{PGL}_2(k)$.

Following [St67, p. 45], [St68, 9.1], we call $\ker(\pi_{sc})$ the fundamental group of $G$.

For example, if $G$ is simple of simply-connected type, then the fundamental group of $G$ is trivial (but the converse is not necessarily true). If $G$ is simple of adjoint type, with corresponding Cartan matrix $C$, then $\ker(\pi_{sc}) \cong \text{Hom}(\Lambda(C), k^{\times})$ where $\Lambda(C)$ is the fundamental group of $C$; see Remark 1.2.8 and Example 1.5.3.

1.5.14. We keep the above notation. As already stated in 1.1.12, we have $G_{der} = G_{1} \ldots G_{n}$ where $G_{1}, \ldots, G_{n}$ are the closed normal simple subgroups of $G$. (They elementwise commute with each other.) Now, one complication of the theory arises from the fact that, in general, this product decomposition is not stable under the Steinberg map $F$: $G_{der} \rightarrow G_{der}$. What happens is the following. Consider the set-up in Remark 1.3.6 with partitions $\Pi = \Pi_{1} \cup \ldots \cup \Pi_{n}$ and $R = R_{1} \cup \ldots \cup R_{n}$; then $G_{i} = \langle U_{\alpha} \mid \alpha \in R_{i} \rangle$ for $i = 1, \ldots, n$. Now, $F$ will permute the simple subgroups $G_{i}$ and, correspondingly, the permutation $\alpha \mapsto \alpha^{\dagger}$ of $R$ (induced by $F$) will permute the subsets $R_{i}$. Hence, there is an induced permutation $\rho$ of $\{1, \ldots, n\}$ such that, for all $i = 1, \ldots, n$, we have $R_{\rho(i)} = \{\alpha^{\dagger} \mid \alpha \in R_{i}\}$ and

(a) \[ F(G_{i}) = \langle F(U_{\alpha}) \mid \alpha \in R_{i} \rangle = \langle U_{\alpha}^{\dagger} \mid \alpha \in R_{i} \rangle = G_{\rho(i)}. \]

Following [St68, p. 78], we say that $G_{der}$ is $F$-simple if $\rho$ is a cyclic permutation (it has a single orbit). Thus, by grouping together the factors in the various $\rho$-orbits on $\{1, \ldots, n\}$, we can write $G_{der}$ as a product of various $F$-stable and $F$-simple semisimple groups. An analogous statement holds for $G_{ss}$. Indeed, for each $i$, let $G_{i}$ be the image of $G_{i}$ under the canonical map $G \rightarrow G_{ss}$. Then we have $G_{ss} = G_{1} \ldots G_{n}$ and the induced Steinberg map $F$: $G_{ss} \rightarrow G_{ss}$ permutes the factors $G_{i}$ according to the permutation $\rho$.

Now consider the isogenies in 1.5.11 and the groups $G_{sc}, G_{ad}$. By Proposition 1.5.10, we have direct product decompositions

(b) \[ G_{sc} = G_{1} \times \ldots \times G_{n} \quad \text{and} \quad G_{ad} = G'_{1} \times \ldots \times G'_{n}, \]

such that, under the isogeny $G_{sc} \rightarrow G_{der}$, the factor $G_{i}$ is mapped to $G_{i}$ and, under the isogeny $G_{ss} \rightarrow G_{ad}$, the factor $G_{i}$ is mapped to $G'_{i}$. By the compatibility of all the above isogenies with the various Steinberg maps involved, it follows that

(c) \[ F(G_{i}) = G_{\rho(i)}' \quad \text{and} \quad F(G'_{i}) = G'_{\rho(i)} \quad \text{for} \quad i = 1, \ldots, n. \]

The following result deals with $F$-simple semisimple groups. (If $F$ is a Frobenius map with respect to some $F_{q}$-rational structure, then the construction below is related to the operation “restriction of scalars”; see [Spr98, §11.4].)

Lemma 1.5.15. Assume that $G$ is semisimple and $F$-simple, as defined in 1.5.14. Then $G = G_{der} = G_{1} \ldots G_{n}$ as above. Assume that this product is an abstract direct product. Then $F^{n}(G_{1}) = G_{1}$ and

i: $G_{1} \rightarrow G, \quad g \mapsto gF(g) \cdots F^{n-1}(g),$

is an injective homomorphism of algebraic groups which restricts to an isomorphism $G_{1}^{F_{n}} \cong G^{F}$. Furthermore, if $q$ is the positive real number attached to $(G, F)$ (see 1.6.7), then $q^{n}$ is the positive real number attached to $(G_{1}, F^{n})$.

Proof. By assumption, $F$ cyclically permutes the factors $G_{i}$. So we can choose the labelling such that $F^{i}(G_{1}) = G_{i+1}$ for $i = 1, \ldots, n-1$ and $F^{n}(G_{1}) = G_{1}$. The map $i$ clearly is a morphism of affine varieties, which is injective since the product
is direct. This map is a group homomorphism because the groups $G_i$ elementwise commute with each other. For the same reason, we have $ι(G_i^{F_{ni}}) \subseteq G_i^F$. Since we have a direct product, every $g \in G$ can be written uniquely as $g = g_1 \cdots g_n$ with $g_i \in G_i$. Then $F(g) = g$ if and only if $F_i(g_i) = g_{i+1}$ for $i = 1, \ldots, n - 1$ and $F_n(g_n) = g_1$. Thus, $F(g) = g$ if and only if $g = ι(g_1)$ and $F_n(g_1) = g_1$. The statement concerning $q$, $q^n$ immediately follows from the definition of these numbers in Proposition 1.4.18.

**Corollary 1.5.16.** Assume that $G$ is connected reductive; let $G = Z \cdot G_{der}$ and $G_{der} = G_1 \cdots G_n$, as above. Let $F: G \rightarrow G$ be a Steinberg map and $I \subseteq \{1, \ldots, n\}$ be a set of representatives of the $ρ$-orbits on $\{1, \ldots, n\}$, with $ρ$ as in 1.5.14(a). For each $i \in I$, let $n_i$ be the length of the corresponding $ρ$-orbit. Then we have isomorphisms (of abstract finite groups)

$$G_{sc}^F \cong \prod_{i \in I} G_i^{F_{ni}} \quad \text{and} \quad G_{ad}^F \cong \prod_{i \in I} G_i^{F_{ni}},$$

where $G_{sc} = G_1 \times \cdots \times G_n$ and $G_{ad} = G_1' \times \cdots \times G_n'$ are as in 1.5.14(b).

**Proof.** This is immediate from 1.5.14 and Lemma 1.5.15. Also recall our identification in 1.5.11 of the various Steinberg maps involved.

See Example 1.4.28 for a good illustration of the above result; the group $G = \text{SL}_2(k) \times \text{PGL}_2(k)$ (with $\text{char}(k) = 2$) considered there is $F$-simple! In particular, the simple factors of an $F$-simple semisimple group need not all be isomorphic to each other as algebraic groups. (They are isomorphic as abstract groups.)

To complete this section, we discuss another fundamental construction involving the formalism of root data: “dual groups”.

**Definition 1.5.17** ([Del1976 5.21; see also Ca85 §4.3, Lu84a 8.4]). Consider two pairs $(G, F)$ and $(G^*, F^*)$ where $G, G^*$ are connected reductive and $F: G \rightarrow G$, $F^*: G^* \rightarrow G^*$ are Steinberg maps. We say that $(G, F)$ and $(G^*, F^*)$ are in duality if there is a maximally split torus $T_0 \subseteq G$ and a maximally split torus $T_0^* \subseteq G^*$ such that the following conditions hold, where $⋆ = (X, R, Y, R^⋆)$ is the root datum of $G^*$ (with respect to $T_0$) and $⋆^* = (X^*, R^*, Y^*, R^⋆)$ is the root datum of $G^*$ (with respect to $T_0^*$):

(a) There is an isomorphism $δ: X \rightarrow Y^*$ such that $δ(R) = R^⋆$ and

$$\langle λ, α^ś \rangle = \langle α^⋆, δ(λ) \rangle \quad \text{for all } λ \in X(T_0) \text{ and } α \in R,$$

where $α^⋆ \in R$ is defined by $δ(α) = α^⋆$.

(b) If $λ: T_0 \rightarrow k^\times$ (an element of $X = X(T_0)$) and $ν: k^\times \rightarrow T_0^*$ (an element of $Y^* = Y(T_0^*)$) correspond to each other under $δ$, then $λ \circ F: T_0 \rightarrow k^\times$ and $F^* \circ ν: k^\times \rightarrow T_0^*$ also correspond to each other under $δ$.

The relation of being in duality is symmetric: the above conditions on $δ: X \rightarrow Y^*$ are equivalent to analogous conditions concerning a map $ε: Y \rightarrow X^*$ obtained by transposing $δ$; see Ca85 4.2.2, 4.3.1. In particular, $δ$ defines an isomorphism of root data between $⋆^*$ and the dual root datum considered in Lemma 1.2.3(b).

Thus, connected reductive groups in duality have dual root data. Note also that the dual of $(G^*, F^*)$ can be naturally identified with $(G, F)$.

1.5.18. It may be worthwhile to reformulate the above definition in terms of the matrix language of Section 1.2; the following discussion will also show that, for any given pair $(G, F)$, there exists a corresponding dual pair $(G^*, F^*)$. 
So let $G$ be connected reductive and $F: G \to G$ be a Steinberg map. Let $T_0 \subseteq G$ be a maximally split torus and $B_0 \subseteq G$ be an $F$-stable Borel subgroup such that $T_0 \subseteq B_0$. Let $\mathcal{A} = (X, R, Y, R^\circ)$ be the root datum of $G$ with respect to $T_0$; let $\Pi$ be the base for $R$ determined by $B_0$ (see Remark 1.3.4). As in Remark 1.2.13 we choose a $\mathbb{Z}$-basis of $X$. Then $\mathcal{A}$ determines a factorisation

\[ C = \hat{A} \cdot A^{tr} \quad \text{where} \quad C = \text{Cartan matrix of } \mathcal{A}. \]

Furthermore, as discussed in 1.5.7 the isogeny $F: G \to G$ determines a pair of integer matrices $(P, P^\circ)$ satisfying the conditions (MI1), (MI2) in 1.2.18. Now we notice that $C^{tr}$ also is a Cartan matrix (see Remark 1.2.13). Hence, transposing the above matrix equation, we obtain a factorisation

\[ C^{tr} = \hat{B} \cdot B^{tr} \quad \text{where} \quad \hat{B} := A \quad \text{and} \quad B := \hat{A}. \]

Furthermore, setting $Q := P^{tr}$ and $Q^\circ := (P^\circ)^{tr}$, the pair $(Q, Q^\circ)$ satisfies the conditions (MI1), (MI2) with respect to $C^{tr} = \hat{B} \cdot B^{tr}$. Now the latter factorisation of $C^{tr}$ determines a root datum $\mathcal{A}^*$ and a base for its root system (see Lemma 1.2.15). One easily sees that these are independent of the choice of the $\mathbb{Z}$-basis for $X$. Hence, going the above argument backwards, there exists a connected reductive group $G^*$ such that $\mathcal{A}^*$ (together with the base of its root system) is isomorphic to the root datum of $G^*$ with respect to a maximal torus $T_0^* \subseteq G^*$ and a Borel subgroup $B_0^* \subseteq G^*$ containing $T_0^*$. The pair of matrices $(Q, Q^\circ)$ determines an isogeny $F^* : G^* \to G^*$ such that $T_0^*$ and $B_0^*$ are $F$-stable. Since $F$ is a Steinberg map, the characterisation in Proposition 1.4.17 immediately shows that $F^*$ also is a Steinberg map. The number $q$ in Proposition 1.4.18 is the same for $F$ and for $F^*$.

Remark 1.5.19. Let $T, T^*$ be tori and assume that we are given Steinberg maps $F : T \to T$ and $F^* : T^* \to T^*$. Then $T, T^*$ are connected reductive groups (with empty root systems). Hence, $(T, F)$ and $(T^*, F^*)$ are in duality if and only if there exists an isomorphism $\delta : X(T) \to Y(T^*)$ such that condition (b) in Definition 1.5.17 holds. For future reference we note that, if $(T, F)$ and $(T^*, F^*)$ are in duality as above, then there is a corresponding (non-canonical) isomorphism of abelian groups

\[ T^{*F} \xrightarrow{\sim} \text{Irr}(T^F), \quad s \mapsto \theta_s, \]

where $\text{Irr}(T^F)$ denotes the set of complex irreducible characters of $T^F$ (which is a group under the tensor product of characters, since $T^F$ is abelian); see [Ca85, 4.4.1]. The construction of the above isomorphism depends on some choices, e.g., the choice of an embedding $\hat{\mathbb{F}}_p^\times \to \mathbb{C}^\times$. (This will be discussed later in further detail.)

We shall have to say more about groups in duality in later sections. It will be useful and important to know how properties of $G$ translate or connect to properties of $G^*$. The lemma below contains just one example. (See Lusztig [Lu09] where a number of such “bridges” of a much deeper nature are discussed.)

Lemma 1.5.20. Let $(G, F)$ and $(G^*, F^*)$ be two pairs in duality, as in Definition 1.5.17. Then the following conditions are equivalent.

(i) The center of $G$ is connected.
(ii) The abelian group $X/\mathbb{Z}R$ has no $p'$-torsion.
(iii) The fundamental group of $G^*$ (see Remark 1.5.13) is trivial.

Proof. For the equivalence of (i) and (ii), see [Ca85, 4.5.1]. The equivalence of (ii) and (iii) is shown in [Ca85, 4.5.8]. \qed
Example 1.5.21. (a) Let $G = \text{GL}_n(k)$. Then $G$ has Cartan type $A_{n-1}$ and the Cartan matrix $C$ factorises as $C = \hat{A} \cdot A^\text{tr}$ where $A = \hat{A}$; see Example 1.3.7. Hence the discussion in 1.5.18 immediately shows that $G$ is dual to itself.

(b) Assume that $G$ is semisimple of adjoint type. In the setting of 1.5.18, this means that the Cartan matrix factorises as $C = \hat{A} \cdot A^\text{tr}$ where $A$ is the identity matrix and $\hat{A} = C$. Hence, $C^\text{tr} = \hat{B} \cdot B^\text{tr}$ where $\hat{B}$ is the identity matrix and $B = C$. Thus, $G^*$ is seen to be semisimple of simply-connected type. If $C$ is symmetric, then Proposition 1.5.8 yields a central isogeny $G^* \to G$.

(c) The examples in (a), (b) seem to indicate that dual groups are related in quite a strong way. However, as pointed out in the introduction of [1.0], dual groups in general are related only through a very weak connection (via their root system); in particular there is no direct, elementary construction which produces $G^*$ from $G$. Perhaps the most striking example is the case where $G = \text{SO}_{2n+1}(k)$. Then $G$ is simple of adjoint type, with Cartan matrix $C$ of type $B_n$. As in (b), $G^*$ will be simple of simply-connected type. However, since $C^\text{tr}$ has type $C_n$, we see that $G^* \cong \text{Sp}_{2n}(k)$. If $\text{char}(k) \neq 2$, then there is not even any abstract non-trivial group homomorphism between $G$ and $G^*$!

1.6. Generic finite reductive groups

Recall from Definition 1.4.7 that a finite group of Lie type is a finite group of the form $G = G^F$, where $G$ is a connected reductive algebraic group over $k = \mathbb{F}_p$ and $F: G \to G$ is a Steinberg map. Then it is common to speak of the (twisted or untwisted) “type” of $G^F$: for example, we say that the finite general linear groups are of untwisted type $A_{n-1}$, the finite unitary groups are of twisted type $A_{n-1}$ (denoted $2A_{n-1}$; see Example 1.4.20), or that the Suzuki groups are of “very twisted” type $B_2$ (denoted $2B_2$; see Example 1.4.21). Here, the superscript (as in $2A_{n-1}$) indicates the order of the automorphism of the Weyl group of $G$ which is induced by $F$; in particular, $G^F$ is of “untwisted” type if $F$ induces the identity map on the Weyl group. Using the machinery developed in the previous sections, we can now give a somewhat more precise definition, as follows.

1.6.1. Assume that $G$ is connected reductive and let $F: G \to G$ be a Steinberg map. Then we can canonically attach to $G$ and $F$ a pair

$$\mathcal{C}(G, F) := (C, P^o)$$

consisting of a Cartan matrix $C = (c_{st})_{s,t \in S}$ and a monomial matrix $P^o = (p^o_{st})_{s,t \in S}$ whose non-zero entries are positive powers of $p$ and such that $CP^o = P^o C$. Let us recall how this is done. First, we choose a \emph{maximally split} torus $T_0 \subseteq G$. Recall from Example 1.4.10 that this means that $T_0$ is an $F$-stable maximal torus of $G$ which is contained in an $F$-stable Borel subgroup $B_0 \subseteq G$. (By Proposition 1.4.11 the pair $(T_0, B_0)$ is unique up to conjugation by elements of $G^F$.) Let $\mathcal{R} = (X, R, Y, R^+)$ be the root datum of $G$ relative to $T_0$ and $\varphi: X \to X$ be the $p$-isogeny induced by $F$. By Remark 1.3.4 there is a unique base $\Pi$ of $\mathcal{R}$ such that $B_0 = (T_0, U_\alpha \mid \alpha \in R^+)$ where $R^+$ are the positive roots with respect to $\Pi$. Let us write $\Pi = \{s, t \in S\}$ and let $C = (c_{st})_{s,t \in S}$ be the corresponding Cartan matrix.

Since $\varphi$ is a $p$-isogeny, there is a permutation $\alpha \mapsto \alpha^\dagger$ of $\mathcal{R}$ such that $\varphi(\alpha^\dagger) = q_\alpha \alpha$ for all $\alpha \in R$. The fact that $B_0$ is $F$-stable implies that this permutation leaves $R^+$ invariant. Hence, this permutation will also leave the base $\Pi$ invariant and so there is an induced permutation $S \to S$, $s \mapsto s^\dagger$, such that $s^\dagger = s\alpha^\dagger$ for all $s \in S$. Thus, $\varphi$ is “base preserving” as in 1.2.18 and we have a corresponding monomial
matrix $P^o = (p^o_{s,t})_{s,t \in S}$ whose non-zero entries are given by $p^o_{s,t} = q_s := q_{\alpha_s}$ for all $s \in S$. The condition (M2) implies that $CP^o = P^oC$, which means that

(a) $q tc^s = q_sc_{s,t}c^t$ for all $s, t \in S$.

Since all pairs $(T_0, B_0)$ as above are conjugate by elements of $G^F$, the pair $(C, P^o)$ is uniquely determined by $G, F$ up to relabeling the elements of $S$.

Now consider the Weyl group $W$ of $R$. Recall from Remark 1.2.12 that we identify $S$ with a subset of $W$ via $s \mapsto w_\alpha$; thus, we have $W = \langle S \rangle$. Let $t: W \to Z_{\geq 0}$ be the corresponding length function. By Remark 1.2.10, the $p$-isogeny $\varphi$ induces a group automorphism $\sigma: W \to W$ such that

(b) $\sigma(s) = s^t$ ($s \in S$) and $\varphi \circ \sigma(w) = w \circ \varphi$ ($w \in W$).

In Theorem 1.3.2, we have seen that we can naturally identify $W = N_G(T_0)/T_0$. (Under this identification, the reflection $w_\alpha \in W$ corresponds to the element $w_\alpha \in N_G(T_0)$.) Since $T_0$ and, hence, $N_G(T_0)$ are $F$-stable, $F$ naturally induces an automorphism $\sigma_F: W \to W$, $qT_0 \mapsto F(q)T_0$ ($q \in N_G(T_0)$). It is straightforward to check that all of the above constructions and identifications are compatible, that is, we have $\sigma_F(w) = \sigma(w)$ for all $w \in W$.

Finally, the numbers $\{q_s\}$ satisfy the following conditions. If $S_1, \ldots, S_r$ are the orbits of the permutation $s \mapsto s^t$ on $S$, then

(c) $q^{[S_i]} = \prod_{s \in S_i} q_s$ ($i = 1, \ldots, r$),

where $q > 0$ is the real number defined in Proposition 1.4.17. (This easily follows from the equation $\varphi(\alpha_s) = q_s \alpha_s$ for all $s \in S$; see [St68, 11.17].) Hence, we also have $q^{[S]} = \prod_{s \in S} q_s$, which provides an alternative characterisation of $q$.

1.6.2. As in [Lus84a, 3.1], we say that $\dagger$ (or $\sigma: W \to W$) is ordinary if the following condition is satisfied: whenever $s \neq t$ in $S$, then the order of the product $st$ is 2 or 3. With this notion, we have the following distinction of cases. The group $G^F$ is

- either “untwisted”, that is, $\dagger$ is the identity (and $q_s = q$ for all $s \in S$);
- or “twisted”, that is, $\dagger$ is not the identity but ordinary (as defined above);
- or “very twisted”, otherwise.

The typical examples to keep in mind are: the finite general linear groups (untwisted), the finite general unitary groups (twisted) and the finite Suzuki groups (very twisted). Note that these are notions which depend on $G$ and $F$ used to define $G^F$, not just on the finite group $G^F$: In Example 1.4.28, there is a realisation of $SL_2(q)$ (where $q$ is a power of 2) as a twisted (but not very twisted) group.

Remark 1.6.3. Assume that $F$ is a Frobenius map, with respect to some $F$-rational structure on $G$. Then, as pointed out by Lusztig [Lus84a, 3.4.1], the induced automorphism $\sigma: W \to W$ is ordinary in the sense defined above.

This is seen as follows. Let $s \neq t$ in $S$ be in the same $\dagger$-orbit. Replacing $F$ by a power of $F$ if necessary, we can assume without loss of generality that $t = s^t$. Now, applying $\dagger$ repeatedly to $\{s, t\}$, we obtain a whole $\dagger$-orbit of pairs $\{s', t'\}$ where $s' \neq t'$ are in $S$. Then 1.6.1(a) shows that $c_{s't'} \neq 0$ for all these pairs. So, if this $\dagger$-orbit of pairs had more than one element, then we would obtain a closed path in the Dynkin diagram of $C$, which is impossible (see Table 1.1 p. 15). Hence, we must have $t = s^t$ and $s = t^s$. But then 1.6.1(a) implies that $q_tc_{st} = q_sc_{s,t}c^t = q_sc_{st}$. However, by Lemma 1.4.28, we have $q_s = q_t = q$. Hence, $c_{st} = c_{ts}$ and so $c_{st} \in \{0, -1\}$, which means that $st$ has order 2 or 3, as required.
1.6.4. We now have all the ingredients to state the order formula for $G^F$. Since $B_0$ is $F$-stable, the unipotent radical $U_0 = R_u(B_0)$ is also $F$-stable. Since $B_0 = U_0 \cdot T_0$ where $U_0 \cap T_0 = \{1\}$, we obtain

$$|G^F| = |U_0^F| \cdot |T_0^F| \cdot |G^F / B_0^F|.$$ 

The factor $|T_0^F|$ is evaluated as follows. As in Proposition 1.4.18 we extend scalars from $Z$ to $\mathbb{R}$ and consider the induced linear map $\varphi_\mathbb{R} : X_\mathbb{R} \to X_\mathbb{R}$ where $X_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} X$. We have $\varphi_\mathbb{R} = q \cdot \varphi_0$ where $\varphi_0 \in GL(X_\mathbb{R})$ is a linear map of finite order. Then

$$|T_0^F| = \varepsilon_\varphi \det(\varphi - \text{id}_X) = \det(q \text{id}_{X_\mathbb{R}} - \varphi_0^{-1}),$$

where $\varepsilon_\varphi \in \{ \pm 1 \}$ is the sign such that $\varepsilon_\varphi \det(\varphi) > 0$. In particular, this shows that the order of $T_0^F$ is obtained from the characteristic polynomial of $\varphi_0^{-1}$ by evaluation at $q$. This is discussed in detail in [Ca85, §3.3, MaTe11 §25.1].

Further evaluation of the remaining two factors in the above expression for $|G^F|$ leads to the following formula, due to Chevalley [Ch55] (in the case where $G$ is semisimple and $\dagger$ is simple, explicit formulæ for the various possibilities are given as in Table 1.3.

**Theorem 1.6.5** (Order formula). With the above notation, we have

$$|G^F| = q^{|R|/2} \det(q \text{id}_{X_\mathbb{R}} - \varphi_0^{-1}) \sum_{w \in W^0} q^{\ell(w)},$$

where $W^0 = \{ w \in W \mid \sigma(w) = w \}$ (which is a finite Coxeter group). When $G$ is simple, explicit formulæ for the various possibilities are given as in Table 1.6.

The fact that the list in Table 1.3 exhausts all the possible pairs $(G, F)$ where $G$ is simple and $F : G \to G$ is a Steinberg map is shown in [St68 §11.6]; note that, in this case, we have $|G^F| = |G^F_{ad}| = |G^F_{sc}|$ (see 1.5.11).

**Table 1.3.** Order formulæ for $|G^F|$ when $G$ is simple

| Type | $|G^F|$ |
|------|--------|
| $A_{n-1}$ | $q^{(n-1)/2}(q^2-1)(q^3-1) \cdots (q^n-1)$ |
| $B_n$ | $q^{2n}(q^2-1)(q^4-1) \cdots (q^{2n}-1)$ |
| $C_n$ | same as $B_n$ |
| $D_n$ | $q^{2n}(q^2-1)(q^4-1) \cdots (q^{2n}-1)$ |
| $G_2$ | $q^6(q^2-1)(q^6-1)$ |
| $F_4$ | $q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$ |
| $E_6$ | $q^{36}(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)$ |
| $E_7$ | $q^{42}(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)$ |
| $E_8$ | $q^{72}(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)$ |
| $2A_{n-1}$ | $q^{(n-1)/2}(q^2-1)(q^3+1) \cdots (q^n-(-1)^n)$ |
| $2D_n$ | $q^{2n}(q^2-1)(q^4-1) \cdots (q^{2n}-1)$ |
| $3D_4$ | $q^{12}(q^2-1)(q^6-1)(q^8+q^4+1)$ |
| $2E_6$ | $q^{36}(q^2-1)(q^6+1)(q^8-1)(q^{10}+1)(q^{12}-1)$ |
| $3B_2$ | $q^4(q^2-1)(q^4+1)$ (when $q = \sqrt{2^{m+1}}$) |
| $2F_4$ | $q^{24}(q^2-1)(q^8+1)(q^{10}-1)(q^{12}+1)$ (when $q = \sqrt{3^{2m+1}}$) |
| $3G_2$ | $q^6(q^2-1)(q^6+1)$ (when $q = \sqrt{5^{2m+1}}$) |

(The first 9 are “untwisted”, the next 4 “twisted” and the last 3 “very twisted”.)
Remark 1.6.6. (a) The formula shows that \( q^{\frac{R_l}{2}} \) is the \( p \)-part of the order of \( G^F \); this provides a further characterisation of the number \( q \). We also note that the above expression for \( |G^F| \) can be interpreted as a polynomial in one variable evaluated at \( q \), where the polynomial only depends on the root datum of \( G \) and the maps \( \varphi_0, \sigma \) derived from \( F \). This will be formalised in Definition 1.6.8 below.

(b) Steinberg [St68, 14.14] shows that the number of \( F \)-stable maximal tori of \( G \) is equal to \( q^{\frac{R_l}{2}} \). This equality is in fact equivalent to the following Molien series formula:

\[
\frac{q^{\frac{R_l}{2}}}{|G^F|} = \sum_{w \in W} \frac{1}{\det(q^0 X_w - w^\varphi_0^{-1} X_w)}.
\]

see [Ca88] §3.4, [MaTe11] Exc. 30.15 for further details. One advantage of this expression is that it does not involve the length function on \( W \) or the induced automorphism \( \sigma \) of \( W \). The inverse of the sum on the right hand side can be further expressed as a product of various cyclotomic polynomials; in this way, one obtains the formal formulae for \( G^F \) when \( G \) is simple; see Table 1.3.

Also note that, instead of considering the maps \( w \circ \varphi_0^{-1} : X_R \to X_R \) in the above formula, one can take their transposes \( (w \circ \varphi_0^{-1})^\text{tr} : Y_R \to Y_R \) where \( Y_R = R \otimes \mathbb{Z} Y \): the characteristic polynomials will certainly remain the same. The same is true when we replace \( w \circ \varphi_0^{-1} \) by \( \varphi_0^{-1} \circ w \).

(c) If \( F \) is a Frobenius map, then Theorem 1.6.5 can be proved by a general argument; see [Ge03a, 4.2.5]. The general case (where the root exponents \( q_\alpha \) may not all be equal) is treated in [St68, §11] (see also [MaTe11, §24.1]), assuming that \( G \) is semisimple and “\( F \)-simple” (see 1.6.12). But, as already noted in [St68, p. 78], the case of an arbitrary connected reductive \( G \) can be easily recovered from this case. As an illustration of the methods developed in the previous section, let us explicitly work out the reduction argument.

Lemma 1.6.7. Suppose that the formula in Theorem 1.6.4 is known to hold when \( G \) is simple of adjoint type. Then the formula holds in general.

Proof. Since the formula for \( |T_0^F| \) in 1.6.4 is already known to hold in general, it will be sufficient to consider the cardinality of \( G^F / T_0^F \). It will be convenient to slightly rephrase this as follows. Let us consider the set of cosets \( G / T_0 = \{ g T_0 \mid g \in G \} \) (just as an abstract set, we don’t need the notion of a quotient variety here). Since \( T_0 \) is \( F \)-stable, we have an induced action of \( F \) on \( G / T_0 \). Consequently, we have a natural injective map \( G^F / T_0^F \to (G / T_0)^F, \quad g T_0^F \mapsto g T_0 \). Now the connected group \( T_0 \) acts transitively on \( g T_0 \) by right multiplication. Hence, if \( g T_0 \) is \( F \)-stable, then Proposition 1.4.4 shows that \( g T_0 \) contains a representative fixed by \( F \). It follows that the above map is surjective and so \( |G^F / T_0^F| = |(G / T_0)^F| \).

Thus, it will now be sufficient to consider the identity:

\[(*) \quad |(G / T_0)^F| = \mathbb{O}(W, \sigma, q) \quad \text{where} \quad \mathbb{O}(W, \sigma, q) := q^{\frac{R_l}{2}} \sum_{w \in W^\sigma} q^{l(w)}.
\]

Since the formula in Theorem 1.6.5 is assumed to hold when \( G \) is simple of adjoint type, the same is true of the formula (\( * \)). We must deduce from this that (\( * \)) holds in general. We do this in two steps.

1) First assume that \( G \) is semisimple of adjoint type. As in 1.5.14(b), we have a direct product decomposition \( G = G_1 \times \ldots \times G_n \) where each \( G_i \) is simple of adjoint type. Furthermore, there is a permutation \( \rho \) of \( \{1, \ldots, n\} \) such that \( F(G_i) = G_{\rho(i)} \) for \( i = 1, \ldots, n \). Now note that \( T_0 = T_1 \times \ldots \times T_n \) where each \( T_i \) (for \( i = 1, \ldots, n \))
is a maximal torus of $G$, such that $F(T_i) = T_{p(i)}$. Hence, we can also identify $G/T_0 = G_1/T_1 \times \cdots \times G_n/T_n$. Furthermore, if $I \subseteq \{1, \ldots, n\}$ and $n_i (i \in I)$ are as in Corollary 1.5.10, then $F^{n_i}(G_i/T_i) = G_i/T_i$ for all $i \in I$ and

\[(1a) \quad |(G/T_0)^F| = \prod_{i \in I} |(G_i/T_i)^{F^{n_i}}|.
\]

It remains to show that there is a similar factorisation of the right hand side of $(\ast)$. Recall from 1.5.14 that we have a partition $R = R_1 \sqcup \cdots \sqcup R_n$. Consequently, we also have a direct product decomposition

\[W = W_1 \times \cdots \times W_n \quad \text{where} \quad W_i := \langle w_\alpha \mid \alpha \in R_i \rangle;
\]

furthermore, $\sigma(W_i) = W_{p(i)}$ for $i = 1, \ldots, n$. Here, $W_i$ is the Weyl group of the factor $G_i$ (relative to $T_i \subseteq G_i$). Now, if $w \in W$ and $w = w_1 \cdots w_n$ with $w_i \in W_i$ for all $i$, then $l(w) = l(w_1) + \cdots + l(w_n)$. Using this formula, it is straightforward to verify that the expression for $\Omega(W, \sigma, q)$ is compatible with the above product decomposition, that is, we have $\sigma^n(W_i) = W_i$ for all $i \in I$ and

\[(1b) \quad \Omega(W, \sigma, q) = \prod_{i \in I} \Omega(W_i, \sigma^n, q^{n_i}).
\]

By assumption and Lemma 1.5.13, we have $|(G_i/T_i)^{F^{n_i}}| = \Omega(W_i, \sigma^n, q^{n_i})$ for all $i \in I$. Hence, comparing $(1a)$ and $(1b)$, we see that $(\ast)$ holds for $G$ as well.

2) Now let $G$ be arbitrary (connected and reductive). As in Remark 1.5.12, we consider an adjoint quotient $\pi_{ad} : G \to G_{ad}$ with kernel $Z = Z(G)$. By Remark 1.3.5(a), we have $Z \subseteq T_0$; furthermore, $T' := \pi_{ad}(T_0)$ is an $F$-stable maximal torus of $G_{ad}$ (see 1.3.10(a)). So we get a bijective map

\[G/T_0 \to G_{ad}/T', \quad gT_0 \mapsto \pi_{ad}(g)T',
\]

which is compatible with the action of $F$ on $G/T_0$ and on $G_{ad}/T'$. In particular, $|(G/T_0)^F| = |(G_{ad}/T')^F|$ and so the left hand side of $(\ast)$ does not change when we pass from $G$ to $G_{ad}$. On the other hand, by 1.3.10(d), $\pi$ induces an $F$-equivariant isomorphism from the Weyl group of $G$ (relative to $T_0$) onto the Weyl group of $G_{ad}$ (relative to $T'$). Hence, the right hand side of $(\ast)$ does not change either when we pass from $G$ to $G_{ad}$. Thus, if $(\ast)$ holds for $G_{ad}$, then $(\ast)$ also holds for $G$. □

Following [BrMa92], we now formally introduce “series of finite groups of Lie type”. This relies on the following definition, which is a slight modification of that in [BrMa92] §1. (See Example 1.6.9 below for further comments on this.)

**Definition 1.6.8.** Let $\mathcal{R} = (X, R, Y, R')$ be a root datum, with Weyl group $W \subseteq \text{Aut}(X)$. We set $X_\mathbb{R} := \mathbb{R} \otimes \mathbb{R} X$. We can canonically regard $X$ as a subset of $X_\mathbb{R}$; thus, we also have $W \subseteq \text{GL}(X_\mathbb{R})$. Let $\varphi_0 \in \text{GL}(X_\mathbb{R})$ be an invertible linear map of finite order which normalises $W$, and assume that

\[\mathcal{P} = \mathcal{P}_G := \left\{ q \in \mathbb{R}_{>0} \mid q\varphi_0(X) \subseteq X \text{ and the corresponding map} \ q\varphi_0 : X \to X \text{ is a p-isogeny for some prime p} \right\}
\]

is non-empty. We form the coset $\varphi_0 W \subseteq \text{GL}(X_\mathbb{R})$. Then

\[G = (X, R, Y, R', \varphi_0 W)
\]

is called a complete root datum or a generic finite reductive group. We define a corresponding rational function $|G| \in \mathbb{R}(y)$ (where $y$ is an indeterminate) by

\[
\frac{y^{\frac{1}{|G|}}}{|G|} = \frac{1}{|W|^n |G|} \sum_{w \in W} \frac{1}{\det(y \text{id}_{X_\mathbb{R}} - w\varphi_0^{-1})}.
\]
We call \(|G|\) the order polynomial of \(G\); this will be justified in Remark 1.6.13 below. Note that \(\mathcal{P}\) is an infinite set: If \(q \in \mathcal{P}\) and \(q \varphi_0\) is a \(p\)-isogeny (where \(p\) is a prime), then \(p^m q \in \mathcal{P}\) for all integers \(m \geq 1\).

**Example 1.6.9.** Let \(G\) be connected reductive and \(F: G \to G\) be a Steinberg map. Then we obtain a corresponding complete root datum by taking the root datum of \(G\) (relative to an \(F\)-stable maximal torus \(T \subseteq G\)) together with the linear map \(\varphi_0\) defined in Proposition 1.4.18(b). In particular, this includes all the cases discussed in Examples 1.4.20, 1.4.21, 1.4.22. This shows that the above Definition 1.6.8 is somewhat more general than that in [BrMa92], in which cases like those in Example 1.4.22 are not included.

Let us now fix a complete root datum \(G = ((X, R, Y, R^\vee), \varphi_0 W)\).

**Remark 1.6.10.** Let \(q \in \mathcal{P}\) and set \(\varphi := q \varphi_0\). Then \(\varphi\) is a \(p\)-isogeny (for some prime \(p\)) and we have \(\varphi^d = q^d \text{id}_X\), where \(d \geq 1\) is the order of \(\varphi_0\). In particular, this implies that \(q^d = p^m\) for some \(m \geq 1\). Let \(G\) be a connected reductive algebraic group over \(k = \mathbb{F}_p\) whose root datum (relative to a maximal torus \(T \subseteq G\)) is isomorphic to \((X, R, Y, R^\vee)\). Then \(\varphi\) gives rise to an isogeny \(F: G \to G\) which is a Steinberg map by Proposition 1.4.17. We write \(G(q) := G^F\). Thus, we obtain a family of finite groups

\[ \{ G(q) \mid q \in \mathcal{P} \} \]

which we call the series of finite groups of Lie type defined by \(G\). There are some choices involved in the definition of \(G(q)\) but we shall see in the remarks below that different choices lead to isomorphic finite groups.

**Remark 1.6.11.** Since \(\varphi_0\) normalises \(W\), we obtain a group automorphism \(\sigma: W \to W\) such that

\[ \sigma(w) = \varphi_0^{-1}w\varphi_0 \quad \text{for all } w \in W. \]

Note that this is compatible with Remark 1.2.10. For any \(q \in \mathcal{P}\), the automorphism of \(W\) induced by the \(p\)-isogeny \(q \varphi_0\) (where \(p\) is a prime) is given by \(\sigma\). Let us now see what happens when we replace \(\varphi_0\) by another map in the coset \(\varphi_0 W\). First note that \(\varphi_0 w\) has finite order for any \(w \in W\). Furthermore, if \(q \in \mathcal{P}\) and \(q \varphi_0\) is a \(p\)-isogeny (where \(p\) is a prime), then \((q \varphi_0)w\) also is a \(p\)-isogeny. Thus, if \(\varphi_0\) satisfies the defining conditions for a complete root datum, then so does \(\varphi_0 w\) for any \(w \in W\). We also note the following identity:

\[ (\varphi_0 w)^m = \varphi_0^n \cdot (\sigma^{m-1}(w) \cdots \sigma(w) \sigma(w)) \quad \text{for all } m \geq 1. \]

Now let \(q \in \mathcal{P}\) and let \(G, T, F\) be as in Remark 1.6.10. Then \(\varphi := q \varphi_0\) is the linear map induced on \(X \cong X(T)\) by \(F\). Let \(w \in W\) and \(\hat{w}\) be a representative of \(w\) in \(N_G(T)\). We define \(F': G \to G\) by \(F'(g) := \hat{w}^{-1}F(g)\hat{w}\) for \(g \in G\). By Lemma 1.4.13, \(F'\) also is a Steinberg map and we have \(G^{F'} \cong G^F\). Now \(T\) is \(F'\)-stable and one easily sees that \(\varphi w: X \to X\) is the linear map induced by \(F'\). (See, for example, [Ca85 3.3.4].) This shows that, if we replace \(\varphi_0\) by \(\varphi w\) for some \(w \in W\), then \(F\) changes to \(F'\) but we obtain isomorphic finite groups.

**Remark 1.6.12.** Let \(q \in \mathcal{P}\) and set \(\varphi := q \varphi_0\). Then \(\varphi\) is a \(p\)-isogeny (where \(p\) is a prime) and so there is a corresponding permutation \(\alpha \mapsto \alpha^q\) of \(R\); we have \(\varphi(\alpha^q) = q_{\alpha} \varphi_0\alpha\) for all \(\alpha \in R\), where \(\{q_{\alpha}\}\) are the root exponents of \(\varphi\). Let \(\sigma: W \to W\) be the group automorphism in Remark 1.6.11. By Remark 1.2.10 we have \(\sigma(w) = w_{\alpha^q}\) for all \(\alpha \in R\); also recall that the root exponents are positive. Hence, we conclude that the permutation \(\alpha \mapsto \alpha^q\) only depends on \(\varphi_0\), but not on \(q\).
Remark 1.6.13. Let us fix a base \( \Pi \) of \( R \). Since any two bases can be transformed into each other by a unique element of \( W \), there is a unique \( w \in W \) such that, if we replace \( \varphi_0 \) by \( \varphi_0^{\prime} := \varphi_0 w \), then \( \Pi^{\prime} = \Pi \) where \( \alpha \mapsto \alpha^{\dagger} \) is the permutation induced by \( \varphi_0^{\prime} \); see Remarks 1.2.11 and 1.6.12. Assume now that this is the case. Let \( q \in \mathcal{P} \) and \( G, T, F \) be as in Remark 1.6.11. Then \( T \) lies in the \( F \)-stable Borel subgroup \( B = \langle T, U_\alpha \mid \alpha \in R^+ \rangle \) (where \( R^+ \) are the positive roots with respect to \( \Pi \)) and we are in the setting of 1.6.11, where \( T_0 := T \). So, by Remark 1.6.6 we have \( |G^F| = |G|(q) \) and this is also equal to the expression in Theorem 1.6.5. Since this holds for all \( q \in \mathcal{P} \), we obtain an identity of rational functions in \( y \):

\[
|G| = y^{R/2} \det(y \text{id}_{X_R} - \varphi_0^{-1}) \sum_{w \in W^s} y^{l(w)}.
\]

Thus, the rational function \( |G| \) actually is a polynomial in \( y \) such that \( |G^F| = |G|(q) \). This provides the justification for calling \( |G| \) the order polynomial of \( G \) (see also [BrMa92 1.12]). Now let \( K \supseteq \mathbb{R} \) be a subfield such that \( \det(y \text{id}_{X_R} - \varphi_0^{-1}) \in K[y] \). Since \( \varphi_0 \) has finite order, all eigenvalues of this polynomial are roots of unity. By [St08 2.1], an analogous result is also true for the term \( \sum_w y^{l(w)} \) in (a). So there is a factorisation

\[
|G| = y^{R/2} \times \text{product of cyclotomic polynomials in } K[y].
\]

If \( G \) is simple, then such factorisations can be seen explicitly in Table 1.13 (p. 63).

Remark 1.6.14. Let \( \Pi = \{ \alpha_s \mid s \in S \} \) be a base of \( R \) and \( C = \langle c_{st} \rangle_{s,t \in S} \) be the corresponding Cartan matrix; also choose a \( \mathbb{Z} \)-basis of \( X \). Then \( \mathcal{P} \) is determined by a factorisation \( C = \tilde{A} \cdot A^{\text{tr}} \) as in Remark 1.6.13. Assume that \( \varphi_0 \) is chosen such that the permutation of \( R \) induced by \( \varphi_0 \) leaves \( \Pi \) invariant (which is possible by Remark 1.6.13). Let \( Q \) be the matrix of \( \varphi_0 : X_R \to X_R \) (with respect to the chosen basis of \( X \)). If \( q \in \mathcal{P} \), then \( q \varphi_0 \) is a \( p \)-isogeny (for some prime \( p \)) and the conditions (MI1), (MI2) in 1.2.18 show that \( qQA^{\text{tr}} = A^{\text{tr}}P^o \) and \( P^o \tilde{A} = qAQ \), where \( P^o \) is a monomial matrix whose non-zero entries are all powers of \( p \). It follows that

\[
QA^{\text{tr}} = A^{\text{tr}}Q^o \quad \text{and} \quad Q^o \tilde{A} = \tilde{A}Q,
\]

where \( Q^o := q^{-1}P^o \) is a monomial matrix; each non-zero entry of \( Q^o \) is a positive real number such that some positive power of it is an integral power of \( p \). Now note that, although the pair \( (P, P^o) \) is used in the construction, \( Q^o \) is uniquely determined by \( Q \) and, hence, independent of \( (P, P^o) \). There are two cases:

(I) All non-zero entries of \( Q^o \) are equal to 1. Then \( \varphi_0 \) is a 1-isogeny, as in Example 1.4.20. Consequently, the set \( \mathcal{P} \) consists of all prime powers. This is what is called the “cas général” in [BrMa92 §1].

(II) Otherwise, there is a unique prime number \( p \) such that each non-zero entry of \( Q^o \) has the property that some positive power of it is a positive integral power of \( p \). In this case, \( \mathcal{P} \) will only consist of positive real numbers \( q \) such that \( q \varphi_0 \) is a \( p \)-isogeny for this prime \( p \). For example, consider the root datum of Cartan type \( A_1 \times A_1 \) in Example 1.4.28(b), where \( \varphi_0 \) is determined by a certain matrix of order 2, denoted \( P_0 \). Then

\[
Q = P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Q^o = \begin{pmatrix} 0 & 1/2 \\ 2 & 0 \end{pmatrix}.
\]

So we are in case (II) where \( p = 2 \) and \( \mathcal{P} = \{ 2^m \mid m \geq 1 \} \). Similarly, the complete root data of the Suzuki and Ree groups in Example 1.4.21 are of type (II), where \( \mathcal{P} = \{ \sqrt{2^{2m+1}} \mid m \geq 0 \} \) (for \( 2B_2, 2F_4 \)) or \( \mathcal{P} = \{ \sqrt{3^{2m+1}} \mid m \geq 0 \} \) (for \( 3G_2 \)).
**Definition 1.6.15** (See [BrMa92, p. 250], [BMM93 1.5]). The **ennola dual** of a complete root datum \( G = ((X, R, Y, R^\vee), \varphi_0 W) \) is defined by
\[
G^- := ((X, R, Y, R^\vee), -\varphi_0 W).
\]

(Note that \( G^- \) is a complete root datum since, for any \( p \)-isogeny of root data \( \varphi: X \to X \), the map \( -\varphi \) also is a \( p \)-isogeny of root data; in particular, \( \mathcal{P}_G^- = \mathcal{P}_G \).)

In this situation, we write \( G(-q) := G^-(q) \) for any \( q \in \mathcal{P}_G \). We have
\[
|G^-|(y) = (-1)^{\text{rank}_X} |G|(y).
\]

For the origin of the name “ennola dual”, see Example 1.6.16 below.

**Example 1.6.16.** (a) Assume that \(-\text{id}_X \in W\). Then, clearly, we have \( G^- = G \) and \( |G^-| = |G| \in \mathbb{R}[y] \).

(b) Let \( G = \text{GL}_n(k) \) and \( T_0 \subseteq G \) be the maximal torus consisting of the diagonal matrices in \( G \). We have described the corresponding root datum in Example 1.3.7.

If we set \( \varphi_0 = \text{id}_X \) and denote by \( G \) the corresponding complete root datum, then \( G(q) \equiv \text{GL}_n(q) \) for all prime powers \( q \). We claim that
\[
G(-q) \equiv \text{GU}_n(q) \quad \text{for all } q \in \mathcal{P}_G.
\]

This is seen as follows. Let \( \tau: G \to G \) be the automorphism which sends an invertible matrix to its transpose inverse. Then \( \tau(T_0) = T_0 \) and the induced map on \( X \) is \(-\text{id}_X\). (Thus, \( \tau \) is a concrete realisation of the isogeny in Example 1.3.16.)

Let \( F_q: G \to G \) be the standard Frobenius map (raising every matrix entry to its \( q \)-th power). Then \( \tau \) commutes with \( F_q \) and so \( F' = \tau \circ F_q \) is a Frobenius map on \( G \); see Remark 1.4.4(c). We have \( F'(T_0) = T_0 \) and the induced map on \( X \) is given by \(-q \times \text{id}_X\). Thus, \((G, F')\) gives rise to the complete root datum \( G^- \); finally, note that \( G^{F'} \equiv \text{GU}_n(q) \). (The difference between this realisation of \( \text{GU}_n(q) \) and the one in Example 1.4.20 is that, here, \( T_0 \) is not a maximally split torus for \( F' \).)

Now the identity \( |G^-|(y) = (-1)^{\text{rank}_X} |G|(y) \) gives an a priori explanation for the fact that the order formula for \( \text{GU}_n(q) \) in Table 1.3 (p. 53) is obtained from that of \( \text{GL}_n(q) \) by simply changing \( q \) to \(-q \) (and fixing the total sign). Ennola observed that a similar statement should even be true for the irreducible characters of these groups; we will discuss this in further detail at a later stage.

**Example 1.6.17.** The **dual complete root datum** of \( G = ((X, R, Y, R^\vee), \varphi_0 W) \) is defined by
\[
G^* := ((Y, R^\vee, X, R), \varphi^*_0 W),
\]
where \( \varphi^*_0: Y_\mathbb{R} \to Y_\mathbb{R} \) is the transpose map defined, as in 1.2.2 through the canonical extension of the pairing \((\cdot, \cdot): X \times Y \to \mathbb{Z}\) to a pairing \( X_\mathbb{R} \times Y_\mathbb{R} \to \mathbb{R} \). Here, we also use the identification of \( W^\vee \subseteq \text{Aut}(Y) \) with \( W \), as in Remark 1.2.12. We have \( \mathcal{P}_G \equiv \mathcal{P}_{G^*} \); furthermore, by Remark 1.6.6(b), we also have \( |G^*(y)| = |G|(y) \).

Now, for each \( q \in \mathcal{P}_G \), we obtain a finite group \( G(q) \) (arising from a pair \((G, F)\) as in Remark 1.6.10) and a finite group \( G^*(q) \) (arising from an analogous pair \((G^*, F^*)\)). We then see that \((G, F)\) and \((G^*, F^*)\) are in duality as in Definition 1.5.17.

**Definition 1.6.18** (See [BrMa92, 1.1]). Let \( G = ((X, R, Y, R^\vee), \varphi_0 W) \) be a complete root datum. For any \( w \in W \), the complete root datum
\[
G_w := ((X, \varnothing, Y, \varnothing), \varphi_0 w^{-1})
\]
is called a **maximal toric sub-dataum** of \( G \). (We choose \( w^{-1} \) here in order to have consistency with the order formulae below and the notation in [Lu84a 2.1].) In general, \( G \) is said to be a **toric datum** if \( R = \varnothing \); in this case, a corresponding connected reductive algebraic group is a torus.
1.6.19. Let \( G = (X, R, Y, Y'), \varphi_0 W \) be a complete root datum. We assume that \( \varphi_0 \) is chosen such that the permutation of \( R \) induced by \( \varphi_0 \) leaves a base of \( R \) invariant (which is possible by Remark 1.6.13). Thus, if \( q \in \mathcal{P} \) and \( G, T, F \) are as in Remark 1.6.10 then we are in the setting in 1.6.14 where \( T_0 := T \).

Now let \( w \in W \) and consider the maximal toric sub-datum \( \mathbb{G}_w \) in Definition 1.6.13. The corresponding order polynomial is just given by

\[
|\mathbb{G}_w| = \det(g \operatorname{id}_{X_\alpha} - w\varphi_0^{-1}) \in \mathbb{R}[y].
\]

Let \( q \in \mathcal{P} \) and \( G, T_0, F \) be as above. Let \( \dot{w} \) be a representative of \( w \) in \( N_G(T_0) \). By Theorem 1.4.8 (Lang–Steinberg), we can write \( \dot{w} = g^{-1}F(g) \) for some \( g \in G \).

Then \( T' := gT_0g^{-1} \) is an \( F \)-stable maximal torus of \( G \); we say that \( T' \) is a torus of type \( w \). Now conjugation with \( g \) defines an isomorphism of algebraic groups from \( T' \) onto \( T_0 \). This isomorphism sends \( T'^F \) onto the subgroup

\[
T_0[w] := \{ t \in T_0 \mid F(t) = \dot{w}^{-1}tw \} \subseteq T_0.
\]

(Another common notation for this subgroup is \( T_0^{wF} \).) Note that \( T_0[w] \) only depends on \( w \), but not on the choice of \( \dot{w} \). Furthermore, the isomorphism \( T' \cong T_0 \) induces an isomorphism of abelian groups \( X(T') \cong X(T_0) \). One easily checks that, under this isomorphism and the identification \( X = X(T_0) \), the map induced by \( F \) on \( X(T') \) corresponds to the map \( \varphi_0w^{-1} \): \( X \to X \). (See [Ca85] 3.3.4 for further details.) By [Ca85] 3.3.5, this implies that

\[
|T'^F| = |T_0[w]| = |\mathbb{G}_w|(q) = \det(g \operatorname{id}_{X_\alpha} - w\varphi_0^{-1}).
\]

Thus, \( T'^F \cong T_0[w] \) is a member of the series of finite groups of Lie type defined by the complete root datum \( \mathbb{G}_w \). Since \( |T'^F| \) divides \( |G^F| \) and since this holds for all \( q \in \mathcal{P} \), we conclude that

\[
|\mathbb{G}_w| \text{ divides } |G| \text{ in } \mathbb{R}[y].
\]

Conversely, since all maximal tori are conjugate in \( G \), an arbitrary \( F \)-stable maximal torus is of the form \( gT_0g^{-1} \) for some \( g \in G \) such that \( g^{-1}F(g) \in N_G(T_0) \). If \( w \) denotes the image of \( g^{-1}F(g) \) in \( W \), then \( gT_0g^{-1} \) is a torus of type \( w \), as above. This discussion shows that the subgroups of \( G^F \) arising from \( F \)-stable maximal tori can all be realized as subgroups of the form \( T_0[w] \subseteq T_0 \), for various \( w \in W \).

In later chapters, we will see that various other classes of subgroups of \( G^F \) fit into the framework of complete root data. The general formalism is further developed in [BMM93], [BMM99], [BMM12]. Note that, even with our slightly more general definition, any complete root datum as above defines a reflection datum as in [BMM12] Def. 2.6], over a suitable subfield \( K \subseteq \mathbb{R} \).

1.7. Regular embeddings

Lusztig’s work [Lu84a], [Lu88] (to be discussed in more detail in later chapters) shows that the character theory of finite groups of Lie type is considerably easier when the center of the underlying algebraic group is connected. Thus, when trying to prove a result about a general finite group of Lie type, it often happens that one first tries to establish this result in the case where the center is connected. The concept of “regular embedding” provides a technical tool in order to reduce a general statement to the connected center case. A major result on representations in this context will be stated in Theorem 1.7.16.
Definition 1.7.1. Let $G$, $G'$ be connected reductive algebraic groups over $k = \mathbb{F}_p$ and $F: G \to G$, $F': G' \to G'$ be Steinberg maps. Let $i: G \to G'$ be a homomorphism of algebraic groups such that $i \circ F = F' \circ i$. Following [Lu88], we say that $i$ is a regular embedding if $G'$ has connected center, $i$ is an isomorphism of $G$ with a closed subgroup of $G'$ and $i(G), G'$ have the same derived subgroup.

Note that $G'_{der} \subseteq i(G)$ and so $i(G)$ is normal in $G'$ with $G'/i(G)$ abelian. Then the finite group $i(G)^F = i(G)^{F'}$ contains the derived subgroup of the finite group $G'^{F'}$ and so $i(G)^F$ is normal in $G'^{F'}$ with $G'^{F'}/i(G)^F$ abelian. Thus, as far as the representation theory of $G^F$ and of $G'^{F'}$ is concerned, we are in a situation where Clifford theory (with abelian factor group) applies.

Example 1.7.2. (a) Let $G$ be a connected reductive algebraic group with a connected center and $F: G \to G$ be a Steinberg map. Then $G_{der}$ is semisimple and, clearly, $G_{der} \subseteq G$ is a regular embedding. A standard example is given by $G = GL_n(k)$ where $G_{der} = SL_n(k)$; note that this works for both the Frobenius maps in Example 1.4.20, where either $G^F = GL_n(q)$ or $G^F = GU_n(q)$.

(b) Let $G = SL_n(k)$ and $F: G \to G$ be a Frobenius map. Without using $GL_n(k)$ directly, we can construct a regular embedding $i: G \to G'$ as follows. Let $G' := \{(A, \xi) \in M_n(k) \times k^\times \mid \xi \det(A) = 1\}$; then $Z(G') = \{\xi I_n, \xi^{-n} \} \subseteq k^\times$ is connected and $\dim Z(G') = 1$. For $A \in G$ we set $i(A) := (A, 1) \in G'$; then $i$ is a closed embedding. A Frobenius map $F': G' \to G'$ is defined by $F'(A, \xi) = (F(A), \xi^q)$ if $G^F = SL_n(q)$, and by $F'(A, \xi) = (F(A), \xi^{-q})$ if $G^F = GU_n(q)$. In Lemma 1.7.3 below, the basic idea of this construction is generalised to an arbitrary connected reductive group $G$.

(c) Let $n \geq 2$ and $G \subseteq GL_n(k)$ be one of the classical groups in 1.1.4. Then one can apply a similar construction as in (b). In each case, $Z(G)$ consists of the scalar matrices in $G$. If $G = SO_n(k)$, then $G = Z(G) \cong k^\times$; otherwise, we have $Z(G) = \{\pm I_n\}$ where $I_n$ is the identity matrix. So let us now assume that $\text{char}(k) \neq 2$ and $Z(G) = \{I_n\}$. Then $G = \Gamma(Q_n, k)$ where $Q_n^\times = \pm Q_n$. We set $G' = C\Gamma(Q_n, k) := \{(A, \xi) \in M_n(k) \times k^\times \mid A^nQ_nA = \xi Q_n\}$; this is called the conformal group corresponding to $G$. Then $G'$ is a linear algebraic group such that $Z(G') = \{\xi I_n, \xi^2 \} \subseteq k^\times$ is connected and $\dim Z(G') = 1$. Consider the closed subgroup $G_1 = \{(A, 1) \mid A \in G \subseteq G'\}$. Then we have an injective homomorphism of algebraic groups $i: G \to G_1$, $A \mapsto (A, 1)$, with inverse given by $(A, 1) \mapsto A$. Hence, $i$ is a closed embedding.

Let $F: GL_n(k) \to GL_n(k)$ be the standard Frobenius map (raising each entry of a matrix to its $q$th power). Then $F$ restricts to a Frobenius map on $G$. The map $F': G' \to G'$, $(A, \xi) \mapsto (F(A), \xi^q)$, is easily seen to be a Frobenius map such that $i \circ F = F' \circ i$. Thus, $i$ is a regular embedding.

If $n$ is even and $G = SO_n(k)$, then we also have a “twisted” Frobenius map $F_1: G \to G$, $A \mapsto t_n^{-1}F(A)t_n$, where

$$t_n := \begin{bmatrix} I_{m-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{m-1} \end{bmatrix} \in GO_{2m}(k) \quad (m = n/2).$$

This gives rise to the finite “non-split” orthogonal group $G^F_1 = SO_n^+(q)$; see [Ge03a, 4.1.10(d)]. Again, we have a Frobenius map $F_1': G' \to G'$, $(A, \xi) \mapsto (F_1(A), \xi^q)$,
such that \( i \circ F_1 = F'_1 \circ i \). Thus, \( i \) also is a regular embedding with respect to \( F_1 \).

(These examples already appeared in [Lu77a §8.1].)

We have the following general existence result.

**Lemma 1.7.3** (Cf. [DeLu76, 1.21]). Let \( G \) be connected reductive and \( F : G \to G \) be a Steinberg map. Let \( Z \) be the center of \( G \) and \( S \subseteq G \) be an \( F \)-stable torus such that \( Z \subseteq S \). (For example, one could take any \( F \)-stable maximal torus of \( G \).) Let \( G' \) be the quotient of \( G \times S \) by the closed normal subgroup \( \{(z, z^{-1}) \mid z \in Z\} \). Let \( S' \) be the image of \( \{1\} \times S \subseteq G \times S \) in \( G' \). Then the map \( F' : G' \to G' \) induced by \( F \) is a Steinberg map and the map \( i : G \to G' \) induced by \( G \to G \times S, g \mapsto (g, 1) \), is a regular embedding, where \( S' \) is the center of \( G' \).

**Proof.** By Lemma 1.4.27 \( \mathcal{G}' \) is reductive and, by Lemma 1.4.25 \( F' : G' \to G' \) is a Steinberg map. Furthermore, one easily sees that \( i \) is injective, that \( i \circ F = F' \circ i \) and that \( S' = Z( \mathcal{G}' ) \). (Thus, the center of \( \mathcal{G}' \) indeed is connected.) Let \( \mathcal{Z}' := \{(z, z^{-1}) \mid z \in Z\} \) and \( H := \{i(G) = (g \times z)/Z' \subseteq \mathcal{G}' \}. \) We have \( \mathcal{G}' = H \mathcal{S}' \).

Since \( \mathcal{S}' = Z( \mathcal{G}' ) \), it follows that \( \mathcal{G}'_{der} = H_{der} = i( \mathcal{G}_{der} ) \). Now we claim that \( i_1 : G \to H, g \mapsto i(g) \), is an isomorphism of algebraic groups. To see this, consider the homomorphism \( \pi : G \times Z \to G, (g, z) \mapsto gbz \). Since \( \mathcal{Z}' \subseteq \ker(\pi) \), we have an induced homomorphism \( \pi : H \to G \), which is obviously inverse to \( i_1 : G \to H \). Thus, the claim is proved, and it follows that \( i \) is a regular embedding. \( \square \)

**Example 1.7.4.** Let \( G \) be simple of simply-connected type and \( F : G \to G \) be a Steinberg map. Assume that \( Z = Z( G ) \) is non-trivial. Let \( T_0 \subseteq G \) be a maximally split torus and \( S \subseteq T_0 \) a subtorus as in Example 1.5.6. Then \( Z \subseteq S \) and we would like to perform the construction in Lemma 1.7.3 using \( S \). For this purpose, we need to check that \( S \) is \( F \)-stable.

To see this, let \( \mathcal{A} = (X, R, Y, R') \) be the root datum of \( G \) with respect to \( T_0 \). Let \( \Pi = \{\alpha_1, \ldots, \alpha_n\} \) be a base for \( R \), with a labelling as in Table 11. Then \( Y = ZR' \) and \( \{\alpha_1', \ldots, \alpha_n'\} \) is a \( Z \)-basis of \( Y \). Now \( F \) induces a linear map \( \varphi : X \to X \) which is a \( p \)-isogeny of \( \mathcal{A} \). Hence, there is a permutation \( i \mapsto i' \) of \( \{1, \ldots, n\} \) and there are integers \( q_i > 0 \) (each an integral power of \( p \)) such that

\[
\varphi'^i(\alpha_i') = q_i \alpha_i' \quad \text{for } 1 \leq i \leq n.
\]

(Note: The permutation \( \alpha_i \mapsto \alpha_i' \) is inverse to the permutation \( \alpha_i \mapsto \alpha_{i'} \) in 1.1.5.1.)

Now consider the isomorphism \( Y \otimes_\mathbb{Z} k^\times \to T_0, \xi \otimes \nu \mapsto \nu(\xi) \). Then, for each \( \xi \in k^\times \) and \( \nu \in Y \), the element \( \xi \otimes \varphi'^i(\nu) \) corresponds to \( F( \nu(\xi) \). (See [Ca73] §3.2.) Thus, in terms of the notation \( T_0' = \{h(\xi_1, \ldots, \xi_n) \mid \xi_1, \ldots, \xi_n \in k^\times \} \) in Example 1.5.6.

The action of \( F \) on \( T_0 \) is given by

\[
F(h(\xi_1, \xi_2, \ldots, \xi_n)) = h(\xi_1'^i, \xi_2'^i, \ldots, \xi_n'^i) \quad \text{for all } \xi_1, \ldots, \xi_n \in k^\times.
\]

First note that we do not need to consider the case where \( F \) is a Steinberg map but not a Frobenius map. For, this case only occurs in types \( B_2, G_2, F_4 \) and, in all three cases, we have \( Z = \{1\} \) (since \( \text{char}(k) = 2 \) in type \( B_2 \)). So let now \( F \) be a Frobenius map. Then all \( q_i \) are equal, to \( q \) say, and \( i \mapsto i' \) determines a symmetry of the Dynkin diagram of \( \mathcal{A} \). If \( i \mapsto i' \) is the identity, then

\[
F(h(\xi_1, \xi_2, \ldots, \xi_n)) = h(\xi_1'^q, \xi_2'^q, \ldots, \xi_n'^q) \quad \text{for all } \xi_1, \ldots, \xi_n \in k^\times.
\]

The cases where there exists a non-trivial permutation \( i \mapsto i' \) are as follows.

\[ A_n: \ i' = n + 1 - i \quad \text{for } 1 \leq i \leq n \quad \text{and so} \]

\[
F(h(\xi_1, \xi_2, \ldots, \xi_n)) = h(\xi_n'^q, \xi_{n-1}'^q, \ldots, \xi_1'^q) \quad \text{for all } \xi_1, \ldots, \xi_n \in k^\times.
\]
This completes the proof of (a).

If \( S \) is simple and let \( G \) be a Steinberg map. Then there exists a regular embedding \( i: G \rightarrow G' \) with the following properties.

(a) If \( G \) is of simply-connected type \( D_n \) with \( n \) even, \( \text{char}(k) \neq 2 \), and \( G^F \) is \( "\text{untwisted}" \), then \( \dim Z(G') = 2 \) and there is a surjective map \( G^F/i(G^F) \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

(b) If \( i(G') = G' \) is cyclic. We want to show that \( G^F \) is of simply-connected type. Let \( S \) be the torus in Example 1.5.6. We have \( Z(G) \subseteq S \) and \( S \) is \( F \)-stable by the discussion in Example 1.3.1. So, applying Lemma 1.7.2, we obtain a regular embedding \( i: S \rightarrow G' \). Since \( G = G_{\mathrm{der}} \), we have \( G_{\mathrm{der}} = i(G) \). Then \( i(G^F) = i(G)^{F'} = G_{\mathrm{der}}^{F'} \) and so Proposition 1.4.12(b) implies that

\[
G^F/i(G^F) \cong K^{F'}
\]

where \( K := G'/G_{\mathrm{der}}^{F'} \).

(We have an induced action of \( F' \) on \( K \) by Lemma 1.4.12.) Since \( G' = i(G) \cdot S' \) and \( S' = Z(G') \), the inclusion \( S' \hookrightarrow G' \) induces an isogeny \( S' \rightarrow K \). Composition with the map \( S \rightarrow S' \) from Lemma 1.7.3 yields an isogeny \( f: S \rightarrow K \) such that \( f \circ F = F' \circ f \) and \( \ker(f) = Z(G) \). In particular, \( K \) is a torus and \( \dim K = \dim S = 2 \).

If \( \dim K = 0 \), then \( i(G^F) = G^{F'} \). If \( \dim K = 1 \), then \( K \cong k^\times \) and so \( K^{F'} \) is isomorphic to a finite subgroup of \( k^\times \); hence, \( K^{F'} \) is cyclic in this case.

Finally, assume that \( \dim K = 2 \). This case only occurs in type \( D_n \) with \( n \) even, where \( G = \text{Spin}_{2n}(k) \) and \( Z(G) = \{ t \in S \mid t^2 = 1 \} \). Since \( Z(G) \neq \{ 1 \} \), we have \( \text{char}(k) \neq 2 \) and \( Z(G) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). If \( F \) is \( "\text{untwisted}" \), then Remark 1.7.6(c) below will show that \( G^F/i(G^F) \) has a factor group isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). This completes the proof of (a).

It remains to consider the case where \( F \) is \( "\text{twisted}" \), with all root exponents equal to \( q \). By the description in Example 1.7.4, there is an isomorphism of algebraic groups \( S \cong k^\times \times k^\times \) such that the action of \( F \) on \( S \) corresponds to the map \( (s_1, s_2) \mapsto (s_2^q, s_1^q) \) on \( k^\times \times k^\times \). Consequently, \( S^F \cong \mathbb{F}_q^\times \) is cyclic. We want to show that a similar argument works for \( K \). To see this, let \( \{ \varepsilon_1, \varepsilon_2 \} \) be a \( \mathbb{Z} \)-basis of \( X(S) \). Then \( F \) induces the linear map \( \varphi: X(S) \rightarrow X(S) \) such that \( \varphi(\varepsilon_2) = q\varepsilon_2 \) and \( \varphi(\varepsilon_1) = q\varepsilon_1 \). Now consider the isogeny \( f: S \rightarrow K \) mentioned above. Since it has kernel \( Z(G) \), and since \( \text{char}(k) \neq 2 \), the correspondences in 1.1.10 show that

\[
X(S)/f^*(X(K)) \cong X(Z(G)) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]

Hence, we must have \( f^*(X(K)) = 2X(S) \). For \( i = 1, 2 \), let \( \delta_i \in X(K) \) be such that \( f^*(\delta_i) = 2\varepsilon_i \). Then \( \{ \delta_1, \delta_2 \} \) is a \( \mathbb{Z} \)-basis of \( X(K) \). Let \( \beta: K \rightarrow X(K) \) be the
linear map induced by $F'$. Since $f \circ F = F' \circ f$, we also have $\varphi \circ f^* = f^* \circ \beta$. Hence, we must have $\beta(\delta_1) = q\delta_2$ and $\beta(\delta_2) = q\delta_1$. Then $K \to k^x \times k^x$, $t \mapsto (\delta_1(t), \delta_2(t))$, is an isomorphism of algebraic groups such that the action of $F'$ on $K$ corresponds to the map $(t_1, t_2) \mapsto (t_1^\varphi, t_2^\varphi)$ on $k^x \times k^x$. Consequently, $K'^F \cong F'_q$ is also cyclic.

This settles all cases where $G$ is simple of simply-connected type. Now let $G_1$ be simple and $F_1: G_1 \to G_1$ be a Steinberg map. We can assume that $Z(G_1) \neq \{1\}$ and that $G_1$ is not of simply-connected type. By Example \[1.5.4\] there are only two cases to consider: $G_1$ of type $A_n$ or $D_n$.

Let $G$ be simple of simply-connected type such that $G, G_1$ have the same Cartan type. By Proposition \[1.5.9\] we can find an isogeny $f: G \to G_1$ and a Steinberg map $F': G' \to G$ such that $f \circ F = F_1 \circ f$. We have $\ker(f) \subseteq Z(G)$ and $f(Z(G)) = Z(G_1)$. If $G_1$ is not of type $D_n$ with $n$ even, then let $S$ be an $F$-stable torus in $G$ with $Z(G) \subseteq S$ and $\dim S \leq 1$, as above. Then $S_1 := f(S) \subseteq G_1$ is an $F_1$-stable maximal torus such that $Z(G_1) \subseteq S_1$ and $\dim S_1 \leq 1$. Performing the construction in Lemma \[1.7.3\] on $G_1$ using $S_1$, we obtain a regular embedding $i_1: G_1 \to G_1'$ such that $G_1'^F/i_1(G_1'^F)$ is cyclic, by the same argument as above.

It remains to consider the case where $G_1$ is of type $D_n$ with $n$ even. Since we are also in the case where $Z(G_1) \neq \{1\}$ and $G_1$ is not of simply-connected type, we must have $\text{char}(k) \neq 2$ and $|Z(G_1)| = |\ker(f)| = 2$. Assume first that $G_1'^F$ is untwisted. Then we can argue as follows. Recall that

$$\ker(f) \subseteq Z(G) = \{h(\xi_1, \xi_2, 1, \xi_1\xi_2, 1, \xi_1\xi_2, \ldots) \mid \xi_1^2 = \xi_2^2 = 1\}.$$

Let $\tilde{S}$ be one of the following 1-dimensional subtori of $S$:

$$\{h(\xi, 1, 1, \xi, 1, \ldots) \mid \xi \in k^x\} \text{ or } \{h(1, \xi, 1, \xi, 1, \ldots) \mid \xi \in k^x\}.$$

Each of these is $F$-stable, and we can choose $\tilde{S}$ such that $Z(G) \subseteq \ker(f) \tilde{S}$. But, in this case, $Z(G_1) \subseteq f(\tilde{S})$ and so the same construction as above, using $S_1 = f(\tilde{S})$, yields the desired conclusion. Finally, if $G_1'^F$ is twisted, then $G_1 \cong SO_{2n}(k)$ (see Example \[1.5.4\]) and so a regular embedding with the desired properties is obtained as in Example \[1.7.2\], using the corresponding conformal group. \qed

The following remark contains a number of useful, purely group-theoretical properties of a regular embedding.

**Remark 1.7.6.** Let $i: G \to G'$ be a regular embedding. To simplify notation, we identify $G$ with its image in $G'$ and use the symbol $F$ for both Steinberg maps; thus, $F' = F$, $G \subseteq G'$ and $G_{\text{der}} = G'_{\text{der}}$. Let $Z$ denote the center of $G$ and $Z'$ denote the center of $G'$. Then it is rather straightforward to prove the following results (see \textcite{Leh78} \S1 for details).

(a) We have $Z = Z' \cap G$ and $Z'^F = Z'^F \cap G^F$. Furthermore, the inclusion $G \subseteq G'$ induces isomorphisms $G/Z \cong G'/Z'$ and $(G/Z)^F \cong G^F/Z^F$. (b) Let $T$ be an $F$-stable maximal torus of $G$. Then $T' := T \cap Z'$ is an $F$-stable maximal torus of $G'$, and every $F$-stable maximal torus of $G'$ is of this form. In this situation, we have

$$T = G \cap T', \quad G'^F = G^F, T'^F, \quad T^F = G^F \cap T'^F.$$

Furthermore, the inclusion $T \subseteq T'$ induces an isomorphism $N_G(T)/T \cong N_{G'}(T')/T'$ which is compatible with the action of $F$ on both sides.
(c) Let \( T, T' \) be as in (b). Then there are canonical exact sequences

\[
\{1\} \to G^F.Z^F \to G^F \to (Z/Z^o)_F \to \{1\}
\]

where \((Z/Z^o)_F\) is defined in Remark 1.3.12; the map \( G^F \to (Z/Z^o)_F \) is given by sending \( g' \in G^F \) to \( g^{-1}F(g) \) where \( g \in G \) is such that \( g \in g'Z' \) (which exists by (a)). The map \( T^F \to (Z/Z^o)_F \) is given similarly.

It follows from (c) that, if \( Z \) is connected, then \( G^F = G^F.Z^F \).

**Lemma 1.7.7** (cf. [BN88, p. 164]). Let \( G \subseteq G' \) be a regular embedding (notation as in Remark 1.7.6) and \( \pi_{ad} : G \to G_{ad} \) be an adjoint quotient (see Remark 1.5.12). Then \( \pi_{ad} \) has a unique extension to an abstract group homomorphism \( \tilde{\pi}_{ad} : G' \to G_{ad} \), and this induces a surjective homomorphism

\[
G^F/G' \to G_{ad}^F/\pi_{ad}(G^F).
\]

**Proof.** Suppose that there exists an abstract homomorphism \( \tilde{\pi}_{ad} : G' \to G_{ad} \) extending \( \pi_{ad} \). Then \( \tilde{\pi}_{ad}(Z') \subseteq Z(G_{ad}) = \{1\} \) and so \( Z' \subseteq \ker(\tilde{\pi}_{ad}) \). Since \( G' = G.Z' \), it follows that \( \tilde{\pi}_{ad}(g.z') = \pi(g) \) for all \( g \in G \) and \( z' \in Z' \). Thus, \( \tilde{\pi}_{ad} \) is uniquely determined (if it exists). Conversely, it is straightforward to check that this formula defines an abstract homomorphism \( G' \to G_{ad} \): note that \( \ker(\pi_{ad}) = Z = Z' \cap G \) (see Remark 1.7.6(a) for the last equality). Furthermore, \( \tilde{\pi}_{ad} \) commutes with the action of \( F \) on both sides and we have \( \ker(\tilde{\pi}_{ad}) = Z' \). Hence, since \( \tilde{\pi}_{ad} \) extends \( \pi_{ad} \), we obtain an induced homomorphism \( G^F/G' \to G^F_{ad}/\pi_{ad}(G^F) \) and all that remains to show is that \( \tilde{\pi}_{ad}(G^F) = G_{ad} \). This is seen as follows. Let \( g \in G^F_{ad} \). Then \( \tilde{\pi}_{ad}(g) \) is an \( F \)-stable coset of \( \ker(\tilde{\pi}_{ad}) = Z' \). Now the connected group \( Z' \) acts transitively on this coset by multiplication and Proposition 1.4.9 shows that \( \tilde{\pi}_{ad}^{-1}(g)^F \neq \emptyset \).

In order to obtain further properties of regular embeddings, it will be useful to characterise these maps entirely in terms of root data. In particular, this will allow us to show how regular embeddings relate to dual groups.

**Lemma 1.7.8.** Let \( G, G' \) be connected reductive groups over \( k \) and \( f : G \to G' \) be an isotypy (see 1.3.21). Let \( T \subseteq G \) and \( T' \subseteq G' \) be maximal tori such that \( f(T) \subseteq T' \). Let \( \varphi : X(T') \to X(T) \), \( \chi' \mapsto \chi \circ f|_T \), be the induced homomorphism. Then the following two conditions are equivalent.

(i) \( f \) is an isomorphism of \( G \) onto a closed subgroup of \( G' \).

(ii) \( f \) is a central isotypy and \( \varphi \) is surjective.

**Proof.** First note that the assumptions imply that \( G' = f(G).Z(G') \) and \( f(G)_{der} = G'_{der} \). Let \( G_1 := f(G) \subseteq G' \); this is a closed subgroup which is connected and reductive (see Lemma 1.4.29); furthermore, \( G'_{der} = f(G)_{der} = (G_1)_{der} \). Let \( T_1 := f(T) \subseteq T' \); then \( T_1 \) is a maximal torus of \( G_1 \) (see 1.3.10(a)) and we have \( T_1 = G_1 \cap T' \). (We have \( G_1 \cap T' \subseteq G(T_1) = T_1 \) where the last equality holds since \( G_1 \) is connected reductive; the inclusion “\( \subseteq \)” is clear.) Thus, we have \( f = i \circ f_1 \) where \( f_1 : G \to G_1 \) is the restricted map and \( i : G_1 \to G' \) is the inclusion; it is clear that \( i \) is a central isotypy. (Note that \( (G_1)_{der} \) contains all the root subgroups of \( G_1 \); see Remark 1.3.6) Correspondingly, we have a factorisation \( \varphi = \varphi_1 \circ \varepsilon \) where \( \varphi_1 : X(T_1) \to X(T) \) is induced by \( f_1 \) and \( \varepsilon : X(T') \to X(T_1) \) is given by restriction. Note that \( \varepsilon \) is surjective; see 1.4.10.

Now suppose that (i) holds, that is, \( f_1 : G \to G_1 \) is an isomorphism of algebraic groups. Then the composition \( f = i \circ f_1 \) will be a central isotypy. Furthermore,
\(\phi_1: X(T_1) \rightarrow X(T)\) is an isomorphism of abelian groups. Since \(\varepsilon\) is surjective, it follows that \(\varphi = \phi_1 \circ \varepsilon\) must be surjective. Thus, (ii) holds.

Conversely, assume that (ii) holds. Then \(\varphi_1\) is also surjective. So the correspondences in \([1.1.10]\) show that \(f_1: T \rightarrow T_1\) is a closed embedding. But, \(\dim T_1 = \dim f_1(T) = \dim T - \dim \ker(f_1|_T) = \dim T\) and so \(f_1: T \rightarrow T_1\) is an isomorphism. Then Theorem \([1.3.22]\) shows that \(f_1: G \rightarrow G_1\) is also an isomorphism. \(\square\)

**Corollary 1.7.9.** Let \(G, G'\) be connected reductive and \(F: G \rightarrow G, F': G' \rightarrow G'\) be Steinberg maps. Let \(i: G \rightarrow G'\) be a homomorphism of algebraic groups such that \(i \circ F = F' \circ i\) and \(i(T) \subseteq T'\), where \(T\) is an \(F\)-stable maximal torus of \(G\) and \(T'\) is an \(F'\)-stable maximal torus of \(G'\). Then \(i\) is a regular embedding if and only if the following three conditions hold.

1. \(i\) is a central isotypy, i.e., the induced map \(\varphi: X(T') \rightarrow X(T)\) is a homomorphism of root data;
2. the map \(\varphi: X(T') \rightarrow X(T)\) is surjective; and
3. \((X(T'))/\mathbb{Z}R'\) has no \(p'\)-torsion, where \(R'\) are the roots relative to \(T'\).

**Proof.** Suppose that \(i\) is a regular embedding. Since \(i(G_{\text{der}}) = G'_{\text{der}}\), we have \(G' = i(G)Z(G')\). So the general assumptions of Lemma \([1.7.8]\) plus condition (i) are satisfied. Hence, the first two conditions hold; the third one holds because \(Z(G')\) is connected (see Lemma \([1.5.20]\)). Conversely, if the above three conditions are satisfied, then \(Z(G')\) is connected and Lemma \([1.7.8]\) shows that \(i\) is an isomorphism of \(G\) onto a closed subgroup of \(G'\). Since \(i\) is central, we have \(G' = i(G)Z(G')\) which implies that \(G'_{\text{der}} = i(G_{\text{der}})\). Hence, \(i\) is a regular embedding. \(\square\)

**1.7.10.** Assume that \((G, F)\) and \((G', F')\) are in duality (see Definition \([1.5.17]\)), with respect to maximally split tori \(T_0 \subseteq G\) and \(T_0' \subseteq G'\). Furthermore, assume that \((G', F')\) and \((G'', F'')\) are in duality, with respect to maximally split tori \(T_0' \subseteq G'\) and \(T_0'' \subseteq G''\). Let \(f: G \rightarrow G'\) be a central isotypy such that \(f \circ F = F' \circ f\) and \(f(T_0) \subseteq T_0'\). Thus, the induced map \(\varphi: X(T_0') \rightarrow X(T_0)\) is a homomorphism of root data as in \([1.2.2]\). But then the transpose map \(\varphi^{tr}: Y(T_0) \rightarrow Y(T_0')\) defines a homomorphism of the dual root data. Using the isomorphisms \(\delta^{tr}: Y(T_0) \rightarrow X(T_0')\) and \(\delta^{nr}: Y(T_0') \rightarrow X(T_0'')\) from Definition \([1.5.17]\), we obtain a map \(\hat{\varphi}: X(T_0') \rightarrow X(T_0'')\) which is a homomorphism between the root data of \(G''\) and \(G'\). Hence, by Theorem \([1.3.22]\) (extended isogeny theorem), there exists a central isotypy \(f^*: G'' \rightarrow G'\) which maps \(T_0''\) into \(T_0'\) and induces \(\hat{\varphi}\). Arguing as in Lemma \([1.4.23]\) one shows that \(f^*\) can be chosen such that \(f^* \circ F'' = F' \circ f^*\).

In this situation, we say that the two central isotypies \(f: G \rightarrow G'\) and \(f^*: G'' \rightarrow G'\) correspond to each other by duality.

(This relation is symmetric.) With this notation, we can now state:

**Lemma 1.7.11.** Let \(f: G \rightarrow G'\) and \(f^*: G'' \rightarrow G'\) correspond to each other by duality, as above. Assume that \(f: G \rightarrow G'\) is an isomorphism with a closed subgroup of \(G'\). Then the following hold.

1. \(f^*: G'' \rightarrow G'\) is surjective and \(\ker(f^*)\) is a central torus.
2. \((G'/f(G))\) is a torus and the pairs \((\ker(f^*), F''), (G'/f(G), F')\) are in duality, where \(F': G'/f(G) \rightarrow G'/f(G)\) is induced by \(F'\).
3. The restricted map \(f^*: (G'')^{F''} \rightarrow (G')^{F'}\) is surjective.

**Proof.** (a) We follow [Bo96] 2.5. Let \(T_0, T'_0, T''_0, T'_0\) be as in \([1.7.10]\). By restriction, \(f\) yields a closed embedding \(f: T_0 \rightarrow T'_0\); let \(\varphi: X(T'_0) \rightarrow X(T_0)\) be the induced map. Then \([1.7.10]\) (a) implies that \(\varphi\) is surjective and \(\ker(\varphi) \cong \)
an induced isomorphism Steinberg map. Then there exists a connected reductive group depends on the same choices as in Remark 1.5.19.)

Lemma 1.7.13 arguments; it appears in the unpublished manuscript [As].

is cited in [Lu84a, 8.8], [Lu88, 8.1], [Lu92b, 0.1] in relation to certain reduction in duality, so it remains to use the isomorphism in Corollary 1.5.19.

corresponding dual homomorphism, as in 1.7.10. Then there is an induced (non-canonical) isomorphism of algebraic groups \( T_0' \to T_0 \) and \( f^*(T_0^*') = T_0^* \), we also have \( f^*(G^*) = G^* \).

(b) We follow [Bo06, 2.6]. First note that the inclusion \( T_0' \subseteq G' \) induces an isomorphism of algebraic groups \( T_0' / f(T_0) \to G' / f(G) \); in particular, \( G' / f(G) \) is a torus. Furthermore, from the above proof of (a), we deduce that there is an isomorphism \( X(\ker(f^*)) \to Y(T_0' / f(T_0)) \), and one easily checks that this is compatible with the actions of \( F' \) and \( F^* \).

(c) Since \( \ker(f^*) \) is connected, this follows from Proposition 1.4.12 (b). \( \square \)

Corollary 1.7.12. Let \( i: G \to G' \) be a regular embedding and \( i^*: G^* \to G^* \) be a corresponding dual homomorphism, as in 1.7.10. Then there is an induced (non-canonical) isomorphism \( \ker(i^*) F^* \to \text{Irr}(G^*(F') / i(G^F)) \), \( z \mapsto \theta_z \). (This isomorphism depends on the same choices as in Remark 1.5.12.)

Proof. As in the proof of Lemma 1.7.11, the inclusion \( T_0' \subseteq G' \) induces an isomorphism \( T_0' / (i(T_0)) \to G' / (i(G)) \). Since \( i(T_0) \) and \( i(G) \) are connected, we have an induced isomorphism \( T_0'^F / (i(T_0)^F) \cong G^F / (i(G^F)) \) and, hence, an isomorphism \( \text{Irr}(T_0'^F / (i(T_0)^F)) \cong \text{Irr}(G^F / (i(G^F))) \). By Lemma 1.7.11 \( \ker(i^*) \) and \( G^F / (i(G)) \) are tori in duality, so it remains to use the isomorphism in Corollary 1.5.19. \( \square \)

The above results will play a certain role in the study of “Lusztig series” of characters of finite groups of Lie type; see a later section. The following result is cited in [Lu84a, 8.8], [Lu88, 8.1], [Lu92b, 0.1] in relation to certain reduction arguments; it appears in the unpublished manuscript [As].

Lemma 1.7.13 (Asai [As §2.3]). Let \( G \) be connected reductive and \( F: G \to G \) a Steinberg map. Then there exists a connected reductive group \( \tilde{G} \), a Steinberg map \( \tilde{F}: \tilde{G} \to G \) and a homomorphism of algebraic groups \( f: \tilde{G} \to G \), such that:

(a) \( \tilde{G}_{\text{der}} \) is semisimple of simply-connected type,

(b) \( f \) is surjective and \( F \circ f = \tilde{F} \circ f \),

(c) \( \ker(f) \) is a central torus of \( \tilde{G} \).

In particular, \( f \) induces a surjective homomorphism of finite groups \( \tilde{G}^F \to G^F \). Furthermore, if \( G \) has a connected center, then \( \tilde{G} \) has a connected center, too.

Proof. Asai [As] shows this by explicitly constructing the appropriate root datum for \( \tilde{G} \) and then using Theorem 1.3.22 (extended isogeny theorem). Here is a more direct argument. Let \( \pi_{\text{sc}}: (G_{\text{der}})_{\text{sc}} \to G_{\text{der}} \) be a simply-connected covering of the derived group of \( G \), as in Remark 1.5.19. Assume first that \( \pi_{\text{sc}} \) is bijective. Let \( Z \) be the center of \( G \). We have an isogeny \( f_1: (G_{\text{der}})_{\text{sc}} \times Z^\circ \to G \), \( (g, z) \mapsto \pi_{\text{sc}}(g) z \).
Let \( \tilde{G} := ((G_{\text{der}})_{\text{sc}} \times \mathbb{Z}^o)/\ker(f_1) \). Then \( f_1 \) induces a bijective morphism of algebraic groups \( f: \tilde{G} \to G \) and one easily verifies that (a), (b), (c) hold. Furthermore, since \( f \) is bijective, the center of \( \tilde{G} \) is connected if and only if \( Z \) is connected.

Now consider the general case, where \( \pi_{\text{sc}} \) may not be bijective. By Lemma 1.7.3, there exists a regular embedding \( i: G^* \to H^* \). By duality, we obtain a homomorphism of algebraic groups \( i^*: H^* \to G^* \); note that, as remarked in Definition 1.5.17, we can identify \( (G^*)^o \) with \( G^* \). By Lemma 1.5.11, \( i^* \) is surjective and \( \ker(i^*) \) is a central torus of \( H^* \); furthermore, by Lemma 1.5.20 the simply-connected covering \( (H^*_{\text{der}})_{\text{sc}} \to H^*_{\text{der}} \) is bijective. By the previous argument, there exists a bijective homomorphism of algebraic groups \( f_1: \tilde{G} \to H^* \) such that (a), (b), (c) hold. Then (a), (b), (c) hold for the composition \( f = i^* \circ f_1: \tilde{G} \to G \). Finally, assume that \( Z \) is connected. Now the derived subgroup of \( H \) is isomorphic to that of \( G^* \). Hence, Lemma 1.5.20 implies that the center of \( H^* \) is connected as well. Since \( f_1 \) is bijective, it follows that \( \tilde{G} \) also has a connected center. \( \square \)

**Example 1.7.14.** Assume that \( G \) is semisimple and let \( i: G \to G' \) be a regular embedding. Applying Lemma 1.7.13 to \( G' \), we obtain a homomorphism of algebraic groups \( f: \tilde{G} \to G' \) satisfying the above three conditions. Furthermore, since \( Z(G') \) is connected, we have that \( Z(\tilde{G}') \) is connected, too. Now \( f(\tilde{G}'_{\text{der}}) = G_{\text{der}} = G \) and so, by restriction, we obtain an isogeny \( \tilde{f}: \tilde{G}'_{\text{der}} \to G \) which is a simply-connected covering of \( G \). We have a commutative diagram:

\[
\begin{array}{ccc}
G & \to & G' \\
\uparrow & & \uparrow \\
\tilde{G}'_{\text{der}} & \to & \tilde{G}'
\end{array}
\]

In this sense, a simply-connected covering of \( G \) can always be chosen to be compatible with the given regular embedding \( i: G \to G' \). (This remark appears in [Lu88, 8.1(d)]; it will be relevant in the discussion of Lusztig series in a later section.)

The following result was first stated (for \( K \) of characteristic 0) by Lusztig [Lu88, Prop. 10], together with a reduction argument which reduces the proof to the case where \( G \) is simple of simply-connected type. As far as such groups are concerned, one can use Proposition 1.7.5, which shows that type \( D_n \) with \( n \) even is the most complicated case to deal with. A full proof for this case appeared only much later, first in [CE04, Chap. 16], and then in [Lu08a, §5].

**Theorem 1.7.15** (Multiplicity-Freeness Theorem). Let \( i: G \to G' \) be a regular embedding and \( K \) be any algebraically closed field. Then the restriction of every simple \( KG^F \)-module to \( G^F \) (via \( i \)) is multiplicity-free.

**Proof.** We can only sketch the general strategy here, and highlight where the principal difficulty of the proof lies. First we note that the reduction argument described in the proof of [Lu88, Prop. 10] works for simple modules over any algebraically closed field \( K \), not just for \( \text{char}(K) = 0 \). (Some adjustments of a different kind are required, since Lusztig considers Frobenius maps, not Steinberg maps in general.) Hence, it suffices to prove the theorem in the case where \( G \) is simple of simply-connected type. Furthermore, the reduction argument shows that it is sufficient to consider only one particular regular embedding \( i: G \to G' \), namely, one satisfying the conditions in Proposition 1.7.5. So let us now assume that these conditions are satisfied.
If $G^F_i/G^F$ is cyclic, then a standard result on representations of finite groups shows that the desired assertion holds; see, e.g., [Pei82] Theorem III.2.14. (This uses that $K$ is algebraically closed, but works without any assumption on char($K$).)

It remains to consider the case where $G$ is of type $D_n$ with $n$ even, char($k$) ≠ 2 and $F$ is “untwisted”. Let us identify $G$ with $i(G)$ and use the notational conventions in Remark 1.7. Writing $G = G^F$, $G' = G^F$, $H := G.Z'$, we have

$$H \leq G' \quad \text{and} \quad G'/H \cong (Z/Z')_F = Z \cong Z/2Z \times Z/2Z.$$ 

Let $V$ be a simple $KG'$-module and denote by $V_H$ its restriction to $H$. Since $H = G.Z'$ and $Z'$ is contained in the center of $G'$, it is sufficient to show that $V_H$ is multiplicity-free. (To see this, one only needs to show that non-isomorphic simple $H$-submodules of $V_H$ remain simple and non-isomorphic upon restriction to $G$. And this easily follows, for example, by the argument in [Ge93a, p. 265].) Now, if char($K$) = 2, then $V_H$ is multiplicity-free by some general results on representations of finite groups; see, e.g., [KlTi09] Lemma 3.14. If char($K$) = char($k$) = $p$, then $V_H$ is even simple by [BrLu12] Lemma 3.4.

So, finally, assume that char($K$) ≠ char($k$) and char($K$) ≠ 2. In particular, char($K$) is either 0 or a prime not dividing the index of $H$ in $G'$. By Clifford’s Theorem (see [Hn82] Theorem VII.9.18), $V_H$ is semisimple and there are two possibilities: either $V_H$ is multiplicity-free (with 1, 2 or 4 irreducible constituents) or the direct sum of 2 copies of a simple $KH$-module. In the case where char($K$) = 0, it is shown by an elaborate counting argument (first published in [CE04]; see also [Lu08a]) that the second type does not occur. This argument involves:

- knowledge of the action (by tensor product) of the four 1-dimensional representations of $G'/H$ on the simple $KG'$-modules;
- counting conjugacy classes and simple modules for Spin$_{2n}$(q). (As noted in [Lu88] §13, this is “very long and unpleasant”.)

Finally, it is shown in [Ge93a] §3, using the results on basic sets of Brauer characters in [Ge91], that Lusztig’s argument can be adapted to work as well when char($K$) > 0 (but still char($K$) ≠ char($k$) and char($K$) ≠ 2).

It would be highly desirable to find a more conceptual proof of this result which does not rely on a case-by-case analysis and the counting arguments for Spin$_{2n}$(q).

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