Enumeration of lozenge tilings of a hexagon with shamrock hole on boundary

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Abstract

Ciucu and Krattenthaler proved a dual of MacMahon’s classical theorem on plane partitions by enumerating lozenge tilings of a hexagon with a “shamrock” hole at the center (Proc. Natl. Acad. Sci. USA, 2013). We consider a new situation when a similar hole appears on the boundary of a hexagon. We prove that the lozenge tilings of new region are always enumerated by a simple product formula. In addition, we investigate a related problem on $q$-enumeration plane partitions fitting in a connected union of several boxes.

Keywords: perfect matching, plane partition, lozenge tiling, dual graph, graphical condensation.

1 Introduction and main results

Given $k$ positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. A plane partition of shape $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is an array of non-negative integers

\[
\begin{array}{ccccccc}
  n_{1,1} & n_{1,2} & n_{1,3} & \cdots & \cdots & \cdots & n_{1,\lambda_1} \\
  n_{2,1} & n_{2,2} & n_{2,3} & \cdots & \cdots & n_{2,\lambda_2} \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
  n_{k,1} & n_{k,2} & n_{k,3} & \cdots & n_{k,\lambda_k} \\
\end{array}
\]

so that $n_{i,j} \geq n_{i,j+1}$ and $n_{i,j} \geq n_{i+1,j}$ (i.e. all rows and all columns are weakly decreasing from left to right and from top to bottom, respectively).

A plane partition of rectangular shape $(b, b, \ldots, b)$ ($a$ rows) with entries at most $c$ is identified with its 3-D diagram – a stack of unit cubes fitting in an $a \times b \times c$ box. The later in turn corresponds to a lozenge tiling of a semi-regular hexagon of side-lengths $a, b, c, a, b, c$ (in cyclic order) on the triangular lattice. Here, a lozenge (or unit rhombus) is union of any two unit

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equilateral triangles sharing an edge; and a lozenge tiling of a region is a covering of the region by lozenges so that there are no gaps or overlaps. MacMahon [6] proved that the number of plane partitions that fit in an $a \times b \times c$ box is equal to
\[
\frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(b+c)H(c+a)},
\]
(1.1)
where the hyperfactorial function $H(n)$ is defined by $H(n) := 0! \cdot 1! \cdot 2! \ldots (n-1)!$. Equivalently, the formula (1.1) gives the number of lozenge tilings of a semi-regular hexagon of side-lengths $a, b, c, a, b, c$. We denote by $Hex(a, b, c)$ the semi-regular hexagon.

Extending the MacMahon’s classical theorem, Ciucu, Eisenkölbl, Krattenthaler, and Zare [1] proved a simple product formula for the number of tilings of a hexagon of side-lengths $a, b + m, c, a + m, b, c + m$ with a triangular hole of size $m$ at the center (the region was called cored hexagon in [2]). Recently, Ciucu and Krattenthaler generalized further the later result by expending the central triangular hole to a hole consisting of four different equilateral triangles. The new hole has been called shamrock hole, and the corresponding tiling formula was mentioned as a dual of MacMahon’s formula [2]. Precisely, the shamrock $S_{m,a,b,c}$ is the union of four equilateral triangles with sides $m, a, b, c$ on the triangular lattice as in Figure 1.1.

Let us consider a related situation of the cored hexagon in [1] when the triangular hole appears on the boundary (instead of the center). It has been proven that the regions of such type have the number of lozenge tilings given by a simple product formula (see e.g. Proposition 2.1 in [3]). In spirit of the Ciucu and Krattenthaler’s dual MacMahon theorem, we extend the triangular hole on the boundary of the hexagon to a shamrock hole as follows.

We start with a hexagon of side-lengths $z + a + b + c, x + y + m, t + a + b + c, z + m, x + y + a + b + c, t + m$. Next, we remove a shamrock $S_{m,a,b,c}$ from the hexagon so that the lower-left vertex of the $a$-triangle in the shamrock is $x + c$ units to the right of the lower-left vertex of the hexagon. We denote by $Q\left(\begin{array}{ccccc} x & y & z & t \\ m & a & b & c \end{array}\right)$ the resulting region. Figure 1.2 shows the region $Q\left(\begin{array}{ccccc} 4 & 3 & 2 & 3 \\ 4 & 3 & 2 & 1 \end{array}\right)$.

We use the notation $M(R)$ for the number of lozenge tilings of region $R$. The number of lozenge tilings of our new region is given by the theorem stated below.

\footnote{From now on, we always list the side-lengths of a hexagon on the triangular lattice in the clockwise order, starting from the northwest side.}
Theorem 1.1. For non-negative integers $x, y, z, t, m, a, b, c$

\[
M \left( Q \begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix} \right) = \frac{H(m + a + b + c + x + y + z + t)}{H(m + a + b + c + x + y + t) H(m + a + b + c + x + y + z)} \\
\times \frac{H(m + a + b + c + x + t) H(m + a + b + c + x + y) H(m + a + b + c + y + z)}{H(m + a + b + c + z + t) H(m + a + b + c + x) H(m + a + b + c + y)} \\
\times \frac{H(x) H(y) H(z) H(t) H(m)^3 H(a)^2 H(b) H(c) H(m + a + b + c)}{H(x + t) H(y + z) H(m + a + b + c + x) H(m + a + b + c + y) H(m + a + b + c + y + z)} \\
\times \frac{H(m + b + c + z + t) H(m + a + c + x) H(m + a + b + y) H(m + b + y + z)}{H(c + x + t) H(b + y + z) H(m + c + x + t)} \\
\times \frac{H(c + x + t) H(b + y + z) H(a + c + x) H(a + b + y) H(b + c + z + t)}{H(a + c) H(a + b + y) H(b + c + z + t)}.
\]

(1.2)

By letting $b = c = 0$, our region $Q \begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$ becomes a magnet bar (region) $B_{m,a}(x, y, z, t)$ first introduced in [2]. Figure 1.3 shows the magnet bar $B_{2,2}(4, 3, 3, 2)$. Thus, Theorem 1.1 im-
plies the tiling formula of a magnet bar given by the following theorem.

**Theorem 1.2** (Theorem 3.1 in [2]). For non-negative integers \(x, y, z, t, m, a\)

\[
M(B_{m,a}(x,y,z,t)) = \frac{\mathcal{H}(m+a+x+y+z+t)}{\mathcal{H}(m+a+x+y+t) \mathcal{H}(m+a+x+y+z)} \times \frac{\mathcal{H}(m+a+x+t) \mathcal{H}(m+a+x+y) \mathcal{H}(m+a+y+1)}{\mathcal{H}(m+a+z+t) \mathcal{H}(m+a+x) \mathcal{H}(m+a+y)} \times \frac{\mathcal{H}(x) \mathcal{H}(y) \mathcal{H}(z) \mathcal{H}(t) \mathcal{H}(m) \mathcal{H}(a)^2}{\mathcal{H}(a+x) \mathcal{H}(a+y) \mathcal{H}(z+t) \mathcal{H}(m+a)} \times \frac{\mathcal{H}(m+z+t) \mathcal{H}(m+a+x) \mathcal{H}(m+a+y)}{\mathcal{H}(m+y+z) \mathcal{H}(m+x+t)}. \quad (1.3)
\]

Next, we consider a \(q\)-analog of Theorem 1.1. Let \(q\) be an indeterminate. The \(q\)-integer \(\left[ n \right]_q\) is defined by \(\left[ n \right]_q := 1 + q + q^2 + \ldots + q^{n-1}\). We also define \(q\)-factorial by \(\left[ n \right]_q! := \left[ 1 \right]_q \left[ 2 \right]_q \ldots \left[ n \right]_q\), and \(q\)-hyperfactorial function by \(\mathcal{H}_q(n) := [1]_q! \cdot [2]_q! \cdot \ldots \cdot [n]_q!\).

MacMahon [6] actually obtained in a more general result than the formula (1.1). In particular, he proved that

\[
\sum_{\pi} q^{\left| \pi \right|} = \frac{\mathcal{H}_q(a) \mathcal{H}_q(b) \mathcal{H}_q(c) \mathcal{H}_q(a+b+c)}{\mathcal{H}_q(a+b) \mathcal{H}_q(b+c) \mathcal{H}_q(c+a)}, \quad (1.4)
\]

where the sum is taken over all plane partitions \(\pi\) fitting in an \(a \times b \times c\) box, and \(\left| \pi \right|\) is the number of unit cubes in \(\pi\) (i.e. the volume of \(\pi\)). By letting \(q = 1\), (1.4) implies (1.1).

Similar to the bijection between lozenge tilings of a semi-regular hexagon \(Hex(a,b,c)\) and plane partitions fitting in an \(a \times b \times c\) box, one can view a lozenge tiling of \(Q(x\ y\ z\ t\ m\ a\ b\ c)\) as a stack of unit cubes that fits in a compound box \(\mathcal{B}(x\ y\ z\ t\ m\ a\ b\ c)\), which will be defined particularly in Section 3 (see Figure 1.4). We call the above stacks of unit cubes generalized plane partitions, since they satisfy a similar monotonicity as the ordinary plane partitions (this will also be discussed carefully in Section 3). In spirit of MacMahon’s \(q\)-formula (1.4), we have the following theorem on the similar \(q\)-sum of generalized plane partitions.

**Theorem 1.3.** Let \(m, a, b, c, x, y, z, t\) be non-negative integers. Then

\[
\sum_{\pi} q^{\left| \pi \right|} = \frac{\mathcal{H}_q(m+a+b+c+x+y+z+t)}{\mathcal{H}_q(m+a+b+c+x+y+t) \mathcal{H}_q(m+a+b+c+x+y+z)} \times \frac{\mathcal{H}_q(m+a+b+c+x+t) \mathcal{H}_q(m+a+b+c+x+y) \mathcal{H}_q(m+a+b+c+y)}{\mathcal{H}_q(m+a+b+c+z+t) \mathcal{H}_q(m+a+b+c+x) \mathcal{H}_q(m+a+b+c+y)} \times \frac{\mathcal{H}_q(x) \mathcal{H}_q(y) \mathcal{H}_q(z) \mathcal{H}_q(t) \mathcal{H}_q(m)^3 \mathcal{H}_q(a)^2 \mathcal{H}_q(b) \mathcal{H}_q(c) \mathcal{H}_q(m+a+b+c)}{\mathcal{H}_q(x+t) \mathcal{H}_q(y+z) \mathcal{H}_q(m+a)^2 \mathcal{H}_q(m+b) \mathcal{H}_q(m+c)} \times \frac{\mathcal{H}_q(m+b+c+z+t) \mathcal{H}_q(m+a+c+x) \mathcal{H}_q(m+a+b+y)}{\mathcal{H}_q(m+b+y+z) \mathcal{H}_q(m+c+x+t)} \times \frac{\mathcal{H}_q(c+x+t) \mathcal{H}_q(b+y+z)}{\mathcal{H}_q(a+c+x) \mathcal{H}_q(a+b+y) \mathcal{H}_q(b+c+z+t)}, \quad (1.5)
\]
where the sum is taken over all generalized plane partitions $\pi$ fitting in the compound box $B\left(\begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array}\right)$.

One readily sees that the $q = 0$ specialization of Theorem 1.3 deduces Theorem 1.1.

Our paper is organized as follows. We give simple proof of Theorem 1.1 using Kuo's graphical condensation method [4] in Section 2. That allows ones prove Theorem 1.1 directly instead of following a longer (and more complicated) proof of Theorem 1.3. The rest of the paper is devoted to the proof of Theorem 1.3. In Section 3, we give a detailed description of the correspondence between lozenge tilings of our $Q$-type region and generalized plane partitions fitting in a compound box. Next, we introduce two simple weight assignments on the lozenges in Section 4. These assignments allow us apply Kuo condensation to prove Theorem 1.3. Section 5 presents a bijection between lozenge tilings of a semihexagon with defects and column-strict plane partitions. This also gives a $q$-enumeration for tilings of a semihexagon with defects. Finally, we prove Theorem 1.3 in Section 7.

2 Proof of Theorem 1.1

Let $G$ be a finite simple graph without loops. A perfect matching of $G$ is a collection of disjoint edges covering all vertices of $G$. Let $R$ be a region\footnote{From now on, we use the word region to mean a finite connected region on the triangular lattice.}. The (planar) dual graph of $R$ is the graph whose vertices are unit triangles in $R$ and whose edges connect precisely two unit triangles sharing an edge. One can identify the lozenge tilings of $R$ with the perfect matchings of its dual graph. In the view of this, we denote by $M(G)$ for the number of perfect matchings of a graph $G$.

The following Kuo's Condensation Theorem is the key of our proofs.

**Theorem 2.1** (Kuo [4]). Let $G = (V_1, V_2, E)$ be a (weighted) bipartite planar graph in which $|V_1| = |V_2|$. Assume that $u, v, w, s$ are four vertices appearing in a cyclic order on a face of $G$.
so that \( u, w \in V_1 \) and \( v, s \in V_2 \). Then

\[
M(G)M(G - \{u, v, w, s\}) = M(G - \{u, v\})M(G - \{w, s\}) + M(G - \{u, s\})M(G - \{v, w\})). \tag{2.1}
\]

If a region \( R \) admits a lozenge tiling, then the numbers of up-pointing triangles and down-pointing triangles in \( R \) are equal. Moreover, if a region satisfies the later balancing condition, we say that the region is balanced.

**Lemma 2.2 (Region-splitting Lemma).** Let \( R \) be a balanced region. Assume that a subregion \( S \) of \( R \) satisfies following two conditions:

(i) (Separating Condition) There are only one type of unit triangles (up-pointing or down-pointing) running along each side of the border between \( S \) and \( R - S \).

(ii) (Balancing Condition) \( S \) is balanced.

Then

\[
M(R) = M(S)M(R - S). \tag{2.2}
\]

**Proof.** Let \( G \) be the dual graph of the region \( R \). Then the dual graph \( K \) of the subregion \( S \) is an induced subgraph of \( G \). It is easy to see that \( K \) satisfy the conditions in Lemma 3.6(a) in [5], and the lemma follows. \( \square \)

**Proof of theorem 1.1.** We prove the equality (1.2) by induction on \( y + z + t \). Our base cases are the situations when \( x = 0, y = 0, z = 0 \) or \( t = 0 \).

If \( x = 0 \), we consider the hexagon \( Hex(m, b, a) \) on the west corner of the region \( Q \left( \begin{array}{ccc} x & 0 & z \\ m & a & b \\ 0 & y & t \end{array} \right) \) (see the shaded hexagon in Figure 2.1(a)). One readily sees that \( Hex(a, c, m) \) satisfies the conditions in Region-splitting Lemma 2.2, so we get

\[
M \left( Q \left( \begin{array}{ccc} 0 & y & z \\ m & a & b \\ c & t \end{array} \right) \right) = M \left( Hex(a, c, m) \right) M \left( Q \left( \begin{array}{ccc} 0 & y & z \\ m & a & b \\ c & t \end{array} \right) - Hex(a, c, m) \right). \tag{2.3}
\]

There are several lozenges in the region \( Q \left( \begin{array}{ccc} 0 & y & z \\ m & a & b \\ c & t \end{array} \right) - Hex(b, a, m) \), which are forced to be in any tilings. By removing these forced lozenges, we get a new region having the same number of tilings as the original one. However, the new region is exactly a magnet bar \( B_{b,m}(t + c, a, y, z) \) (rotated \( 60^0 \) clockwise). Thus, we have

\[
M \left( Q \left( \begin{array}{ccc} 0 & y & z \\ m & a & b \\ c & t \end{array} \right) \right) = M \left( Hex(a, c, m) \right) M \left( B_{b,m}(t + c, a, y, z) \right), \tag{2.4}
\]

and (1.2) follows from MacMahon theorem (1.1) and Theorem 1.2.

Similarly, if \( y = 0 \), we have

\[
M \left( Q \left( \begin{array}{ccc} x & 0 & z \\ m & a & b \\ c & t \end{array} \right) \right) = M \left( Hex(m, b, a) \right) M \left( B_{c,m}(a, b + z, t, x) \right) \tag{2.5}
\]

(see Figure 2.1(b)); if \( z = 0 \), we get

\[
M \left( Q \left( \begin{array}{ccc} x & y & 0 \\ m & a & b \\ c \end{array} \right) \right) = M \left( Hex(m, y + b, a) \right) M \left( B_{c,m}(a, b, t, x) \right) \tag{2.6}
\]
Figure 2.1: The four base cases: (a) $x = 0$, (b) $y = 0$, (c) $z = 0$ (b), and $t = 0$. 
(illustrated in Figure 2.1(c)); and finally, if $t = 0$, we obtain

$$M\left(Q\left(\begin{array}{ccc}x & y & z \\
m & a & b \\
c & t & 0 \end{array}\right)\right) = M\left(\text{Hex}(a, x + c, m)\right) M\left(B_{b, m}(c, a, y, z)\right)$$

(2.7)

(shown in Figure 2.1(d)). Again, (1.2) is implied by MacMahon’s theorem (1.1) and Theorem 1.2.

Our induction step is based on Kuo’s condensation Theorem 2.1.

Assume that $x, y, z, t \geq 1$ and that (1.2) holds for any $Q$-type regions in which the sum of the $y$-, $z$- and $t$-parameters is strictly less than $y + z + t$.

Let $G$ be the dual graph of the region $Q := Q\left(\begin{array}{ccc}x & y & z \\
m & a & b \\
c & t & 0 \end{array}\right)$. Each vertex of $G$ corresponds to a unit triangle in $Q$. We will apply Kuo’s Theorem 2.1 to $G$ with the four vertices $u, v, w$ and $s$ chosen as in Figure 2.2(b). In particular, $u$ corresponds to the rightmost shaded unit triangle, and $v, w$ and $s$ correspond to the next shaded unit triangles as we go counter-clockwise from the rightmost one (see Figure 2.2 for the case $x = 4, y = 3, z = 2, t = 3, m = 4, a = 2, b = 2, c = 1$).

We notice that there are two hidden conditions for the above choice of the four vertices $u, v, w, s$: $x + y + m \geq 2$ and $t + a + b + c \geq 2$. However, we are assuming that $x, y, z, t \geq 1$, the first condition holds here. Moreover, without loss of generality, we can assume that $b + c \geq 1$ (otherwise, our region is a magnet bar whose number of tilings is given by Theorem 1.2), thus the second condition also holds.

Consider the graph $G - \{u, v\}$. This corresponds to the region in Figure 2.2(c). After removing the lozenges forced by two black unit triangles, we get the region of the same type as the original one. Precisely, we get here the region $Q\left(\begin{array}{ccc}x & y - 1 & z \\
m & a & b \\
c & t & 0 \end{array}\right)$ and obtain

$$M(G - \{u, v\}) = M\left(Q\left(\begin{array}{ccc}x & y - 1 & z \\
m & a & b \\
c & t & 0 \end{array}\right)\right).$$

(2.8)

Similarly, we have

$$M(G - \{w, s\}) = M\left(Q\left(\begin{array}{ccc}x & y & z - 1 \\
m & a & b \\
c & t - 1 & 0 \end{array}\right)\right)$$

(see Figure 2.2(d)),

$$M(G - \{u, s\}) = M\left(Q\left(\begin{array}{ccc}x & y - 1 & z + 1 \\
m & a & b \\
c & t - 1 & 0 \end{array}\right)\right)$$

(see Figure 2.2(e)),

$$M(G - \{v, w\}) = M\left(Q\left(\begin{array}{ccc}x & y & z - 1 \\
m & a & b \\
c & t & 0 \end{array}\right)\right)$$

(see Figure 2.2(f)),

$$M(G - \{u, v, w, s\}) = M\left(Q\left(\begin{array}{ccc}x & y - 1 & z - 1 \\
m & a & b \\
c & t & 0 \end{array}\right)\right)$$

(see Figure 2.2(b)).

(2.12)

Substituting the above five identities into the equation (2.1) in Kuo Theorem 2.1, we have the
Figure 2.2: Obtaining the recurrence on the numbers of tilings by using Kuo’s condensation.
Note that all the regions in the above recurrence, except for the first one, have the sum of their y-, z- and t-parameters strictly less than y + z + t.

If we denote by $\Phi \left( \frac{x \ y \ z \ t}{m \ a \ b \ c} \right)$ the expression on the right-hand side of (1.2), we only need to show that the function $\Phi$ satisfies also the recurrence (2.13), i.e.

$$
\Phi \left( \frac{x \ y \ z \ t}{m \ a \ b \ c} \right) \Phi \left( \frac{x \ y - 1 \ z \ t - 1}{m \ a \ b \ c} \right) = \Phi \left( \frac{x \ y - 1 \ z \ t}{m \ a \ b \ c} \right) \Phi \left( \frac{x \ y \ z \ t - 1}{m \ a \ b \ c} \right) + \Phi \left( \frac{x \ y - 1 \ z + 1 \ t - 1}{m \ a \ b \ c} \right) \Phi \left( \frac{x \ y \ z - 1 \ t}{m \ a \ b \ c} \right).
$$

Equivalently, we need to show that

$$
\frac{\Phi \left( \frac{x \ y - 1 \ z \ t}{m \ a \ b \ c} \right) \Phi \left( \frac{x \ y \ z \ t - 1}{m \ a \ b \ c} \right)}{\Phi \left( \frac{x \ y \ z \ t}{m \ a \ b \ c} \right) \Phi \left( \frac{x \ y - 1 \ z - 1 \ t - 1}{m \ a \ b \ c} \right)} + \frac{\Phi \left( \frac{x \ y \ z \ t - 1}{m \ a \ b \ c} \right) \Phi \left( \frac{x \ y - 1 \ z + 1 \ t - 1}{m \ a \ b \ c} \right)}{\Phi \left( \frac{x \ y \ z \ t}{m \ a \ b \ c} \right) \Phi \left( \frac{x \ y - 1 \ z + 1 \ t - 1}{m \ a \ b \ c} \right)} = 1. 
$$

Let us simplify the first term on the left-hand side of (2.15). We notice that the two $\Phi$-functions in the numerator and the denominator of the first fraction in the first term are different only at their y-parameters. Cancelling out all terms having no y-parameter and using the trivial fact $H(n + 1)/H(n) = n!$, we get

$$
\frac{\Phi \left( \frac{x \ y - 1 \ z \ t}{m \ a \ b \ c} \right)}{\Phi \left( \frac{x \ y \ z \ t}{m \ a \ b \ c} \right)} = \frac{(y + z - 1)!(a + b + y - 1)!(m + b + y + z - 1)!}{(y - 1)!(b + y + z - 1)!(m + a + b + y - 1)!} 
\times \frac{(m + a + b + c + y - 1)!(m + a + b + c + x + y + t - 1)!(m + a + b + c + x + y + z - 1)!}{(m + a + b + c + x + y - 1)!(m + a + b + c + x + y + z - 1)!(m + a + b + c + x + y + z + t - 1)!}. 
$$

Doing similarly to the second fraction of the first term, we obtain

$$
\frac{\Phi \left( \frac{x \ y \ z \ t - 1}{m \ a \ b \ c} \right)}{\Phi \left( \frac{x \ y - 1 \ z \ t - 1}{m \ a \ b \ c} \right)} = \frac{(y - 1)!(b + y + z - 1)!(m + a + b + y - 1)!}{(y + z - 1)!(a + b + y - 1)!(m + b + y + z - 1)!} 
\times \frac{(m + a + b + c + x + y - 1)!(m + a + b + c + y + z - 1)!(m + a + b + c + x + y + z - 1)!(m + a + b + c + x + y + z + t - 2)!}{(m + a + b + c + y - 1)!(m + a + b + c + x + y + t - 2)!(m + a + b + c + x + y + z - 1)!}. 
$$
This implies that the first term on the left-hand side of (2.15) can be simplified as

\[
\Phi \left( \begin{array}{cccc} x & y & 1 - z & t \\ m & a & b & c \end{array} \right) / \Phi \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right) = \frac{m + a + b + c + x + y + t - 1}{m + a + b + c + x + y + z + t - 1}. \tag{2.18}
\]

We simplify the second term on the left-hand side of (2.15) in the same way (the numerator and denominator in each fraction are now different at their z-parameters). We get

\[
\Phi \left( \begin{array}{cccc} x & y & z & 1 - t \\ m & a & b & c \end{array} \right) / \Phi \left( \begin{array}{cccc} x & y & 1 & t \\ m & a & b & c \end{array} \right) = \frac{z}{m + a + b + c + x + y + z + t - 1}. \tag{2.19}
\]

By (2.18) and (2.19), the equality (2.15) becomes the following obvious identity

\[
\frac{m + a + b + c + x + y + t - 1}{m + a + b + c + x + y + z + t - 1} + \frac{z}{m + a + b + c + x + y + z + t - 1} = 1. \tag{2.20}
\]

This completes our proof.

\[\square\]

3 Generalized plane partitions fitting in a compound box

In this section, we describe precisely the bijection (illustrated in Figure 1.4 at the end of Section 1) between the lozenge tilings of the region \( Q := Q \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right) \) and a stack of unit cubes that fit in a certain box \( B := B \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right) \).

We first investigate the structure of the box \( B \). Figure 3.1(b) gives us a 3-D picture of the box \( B \) by showing the empty stack. Our box consists of 6 non-overlapping small boxes, which we call rooms. Each room has four walls (faces perpendicular to \( \overrightarrow{O_i} \) or \( \overrightarrow{O_j} \)), a ceiling and a floor (faces perpendicular to \( \overrightarrow{O_k} \)). The floor of these rooms are labelled by 1, 2, ..., 6 in Figure 3.1(b). If two rooms share a portion of their walls, we remove this portion to make them connected. The such two rooms are called adjacent rooms, and the box \( B \) is called a compound box. One readily sees that rooms \( i \) and \( j \) in \( B \) are adjacent if and only if \( |i - j| = 1 \).

We consider the projective diagram of the box \( B \) on the \( O_i j \) plane. In the diagram each room is represented by a rectangle of the same sides as its floor. Moreover, in each rectangle we record a pair of integers \((a, b)\), where \( a \) is the level of the floor and \( b \) is the height of the corresponding room. We always assume that the floor of room 1 is on level 0. We call this diagram the floor plan of the box; and it determines our box. It means that we now can define officially \( B \) to be the unique compound box, which has the floor plan given by Figure 3.1(a).

We note that the rectangles corresponding to rooms 1 and 3 are overlapped (the intersection is indicated by shaded area in Figure 3.1(a)). However, the two rooms are not overlapped since the floor of room 3 is above the ceiling of room 1 (as \( a + b \geq a \)).

Next, we consider the stacks of unit cubes fitting in the compound box \( B \).

Similar to the ordinary plane partitions, the stacks corresponding to the lozenge tilings of the region \( Q \) consists of several columns of unit cubes. The stacks satisfy the following monotonicity:
the levels of the tops of its columns are weakly decreasing along \( \overrightarrow{Oi} \) and \( \overrightarrow{Oj} \). To precise, two adjacent columns are two columns in the same room or in two adjacent rooms so that their projections on the \( Oij \) plane are two unit squares sharing an edge. Then our monotonicity is that the top of a column does not exceed the tops of its adjacent columns on the left and behind. In the view of this, we call our stacks generalized plane partitions. We notice that one should not compare the heights of the columns (as in the case of ordinary plane partitions) since our columns may stay on different levels.

In summary, we have a bijection between lozenge tilings of the region \( Q(\begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array}) \) and generalized plane partitions fitting in the compound box \( B(\begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array}) \).

4 Two simple weight assignments

Lozenges of a region \( R \) can carry weights. \( M(R) \) is now the sum of weights of all lozenge tilings of \( R \), where the weight of a tiling is the product of weights of all its constituent lozenges. We call \( M(R) \) the tiling generating function of \( R \). Similarly, we can define the matching generating function \( M(G) \) of a weighted graph \( G \).

Lozenges in a region \( R \) come with three different orientations: left, right, and vertical lozenges (see Figure 4.1). Next, we consider two simple weight assignments of lozenges in our region \( Q := Q(\begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array}) \) as follows:

1. **Assignment 1.** The weights of left and vertical lozenges are all 1. The weight of a right lozenge is \( q^l \), where \( l \) is the distance between the left side of the lozenge and the southeast side of the region \( Q \). We use notation \( wt_1 \) for this weight assignment (see Figure 4.2(a)).
Figure 4.1: Three orientations of lozenges.

Figure 4.2: Two weights assignments on a sample tiling of region $Q$: (a) Assignment 1, (b) Assignment 2. The right lozenges with label $x$ are weighted by $q^x$.

(2) Assignment 2. All left and vertical lozenges are still weighted by 1. However, a right lozenge is now weighted by $q^n$, where $n$ is the distance between the top of the lozenge and the base of the region $Q$. This assignment is denoted by $wt_2$ (see Figure 4.2(b)).

Let $T$ be a tiling of $Q$. We denote by $wt_1(T)$ and $wt_2(T)$ weights of the tiling $T$ with respect to the weight assignments $wt_1$ and $wt_2$. We also denote by $M_1(Q)$ and $M_2(Q)$ the tiling generating functions of $Q$ corresponding to the weight assignments $wt_1$ and $wt_2$.

View the hexagon $Hex(a, b, c)$ as a special case of the region $Q := Q \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right)$ with an empty shamrock hole. The two tiling generating functions $M_1 \left( Hex(a, b, c) \right)$ and $M_2 \left( Hex(a, b, c) \right)$ are different from MacMahon’s $q$-formula (1.4) by only a power of $q$.

**Proposition 4.1.** For non-negative integers $a, b, c$

\[
M_1 \left( Hex(a, b, c) \right) = q^{ab(b+1)/2} \frac{H_q(a) H_q(b) H_q(c) H_q(a+b+c)}{H_q(a+b) H_q(b+c) H_q(c+a)}
\]

(4.1)

and

\[
M_2 \left( Hex(a, b, c) \right) = q^{ba(a+1)/2} \frac{H_q(a) H_q(b) H_q(c) H_q(a+b+c)}{H_q(a+b) H_q(b+c) H_q(c+a)}
\]

(4.2)

**Proof.** Let $T$ be an arbitrary lozenge tiling of the hexagon $Hex(a, b, c)$. We have a plane partition $\mu_T$ corresponding to $T$ (see Figure 4.3(a)). View the right lozenges of $T$ as the tops of columns of unit cubes in $\mu_T$. Assign each right lozenge a weight $q^t$, where $t$ is the number of unit cubes in the corresponding column; and all left and vertical lozenges are still weighted by 1. This gives
Figure 4.3: Comparing the two weights assignment of tilings of a the hexagon. The lozenge with label $x$ has weight $q^x$.

us a new weight assignment $wt_0$ on the lozenges of the hexagon. Moreover, by the definition, we have $wt_0(T) = q^{\mu_T}$, where $wt_0(T)$ is the weight of $T$ with respect to the weight assignment $wt_0$.

Next, we compare the weights $wt_1(T)$ and $wt_2(T)$ to $wt_0(T)$.

Encode the tiling $T$ of $Hex(a, b, c)$ as a $b$-tuple of disjoint lozenge-paths connecting the top and bottom of the hexagon (indicated by the dotted paths in Figure 4.3(b)). One readily sees that each right lozenge in the path $i$ (from right to left) has weight $q^{i+l}$, where $q^l$ is the weight of the lozenge in the weight assignment $wt_0$. Since each path here has exactly $a$ right lozenges, we have $wt_1(T) = q^{ab(b+1)/2}wt_0(T)$. Thus, (4.1) follows from MacMahon $q$-formula (1.4).

We now encode $T$ as an $a$-tuple of disjoint lozenge-paths connecting the northwest and southeast sides of the hexagon (see Figure 4.3(c)). Divide the weight of each right lozenge on the path $i$ (from bottom to top) by $q^i$. This way, we get back the weight assignment $wt_0$. Thus, we have $wt_2(T) = q^{ba(a+1)/2}wt_0(T)$, and (4.2) follows again from (1.4).

We define two functions

$f \left( \frac{x}{m}, \frac{y}{a}, \frac{z}{b}, \frac{t}{c} \right) := m \binom{y + b + 1}{2} + z \binom{y + 1}{2} + m(z + b)(y + a + b) + (z + b) \binom{m + 1}{2} + x(z + b + c)(y + m + a + b + c) + (z + b + c) \binom{x + 1}{2} + a(x + c)(y + a + b) + \left( x + c + 1 \right) \binom{a + 1}{2}$

and

$g \left( \frac{x}{m}, \frac{y}{a}, \frac{z}{b}, \frac{t}{c} \right) := (y + b) \binom{m + 1}{2} + myz + y \binom{z + 1}{2} + m(z + b)(m + a) + m \binom{z + b + 1}{2} + x(m + a)(z + b + c) + x \binom{z + b + c + 1}{2} + (x + c) \binom{a + 1}{2}.$
Figure 4.4: The partial-partition corresponding to the room 3.

**Proposition 4.2.** For any non-negative integers $m, a, b, c, x, y, z, t$

$$M_1 \left( Q \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right) \right) = q \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right) \sum_{\pi} q^{|\pi|}$$

and

$$M_2 \left( Q \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right) \right) = q \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right) \sum_{\pi} q^{|\pi|},$$

(4.5)

(4.6)

where the sums on the right-hand sides are taken over all generalized plane partitions $\pi$ fitting in the compound box $B \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right)$.

**Proof.** We use the following shorthand notations in this proof: $f := f \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right)$, $g := g \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right)$, $B := B \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right)$, and $Q := Q \left( \begin{array}{cccc} x & y & z & t \\ m & a & b & c \end{array} \right)$.

Let $T$ be any lozenge tiling of the region $Q$, and $\pi$ the generalized plane partition corresponding to $T$ fitting in $B$. We only need to show that

$$\frac{wt_1(T)}{q^{|\pi|}} = q^f \quad \text{and} \quad \frac{wt_2(T)}{q^{|\pi|}} = q^g.$$  

(4.7)

Assume that each room $i$ of box $B$ has size $a_i \times b_i \times c_i$ (for $1 \leq i \leq 6$). The floor of room $i$ is pictured as a parallelogram $P_i$ in Figure 3.1(b). We assume in addition that the left side of $P_i$ is $x_i$ units to the left of the southeast side of the region $Q$, and the bottom of $P_i$ is $y_i$ units above the bottom of region $Q$.

The generalized plane partition $\pi$ can be divided into 6 disjoint sub-partitions $\pi_i$ ($1 \leq i \leq 6$) fitting in the room $i$ of the compound box $B$. Each partial-partition $\pi_i$ in turn gives a lozenge...
tiling $T_i$ of the semi-regular hexagon $Hex(a_i, b_i, c_i)$. Figure 4.4 shows the partial-partition $\pi_3$ of the generalized plane partition $\pi$ in Figure 1.4(a), as well as the relative positions of the floor $P_3$ to the bottom and the southeast side of the region $Q$.

Apply the weight assignment $wt_1^{(i)}$ to the whole tiling $T$ of the region $Q$. This yields a local weight assignment $wt_1^{(i)}$ for the lozenges in the tiling $T_i$ of hexagon $Hex(a_i, b_i, c_i)$. Precisely, each right lozenge in $T_i$ is now weighted by $q^{x_i+t}$, where $l$ is the distance between the left side of the lozenge and the southeast side of the hexagon $Hex(a_i, b_i, c_i)$. Similar to the proof of Proposition 4.1, we encode the tiling $T_i$ as a $b_i$-tuple of disjoint lozenge-paths connecting the top and the bottom of the hexagon. We now divide the weight of each right lozenge on the path $j$ (from right to left) by $q^{x_i+j}$ to get back the weight assignment $wt_0$ on $T_i$ (where $wt_0$ is defined as in the proof of Proposition 4.1). Thus, we have

$$\frac{wt_1^{(i)}(T_i)}{wt_0(T_i)} = \frac{wt_1^{(i)}(T_i)}{q^{x_i+t}} = q^{a_i b_i x_i + a_i b_i (b_i + 1)/2}.$$ 

Multiplying all above equations for $i = 1, 2, \ldots, 6$, we get

$$\frac{wt_1(T)}{q^{x_i+t}} = q^{\sum_{i=1}^{6} a_i b_i x_i + a_i b_i (b_i + 1)/2}. \quad (4.8)$$

Similarly, we have

$$\frac{wt_2(T)}{q^{x_i+t}} = q^{\sum_{i=1}^{6} a_i b_i y_i + b_i a_i (a_i + 1)/2}. \quad (4.9)$$

Obtaining the formulas of $a_i, b_i, x_i, y_i$ in terms of $a, b, c, x, y, z, t$ from Figure 3.1, we get $f = \sum_{i=1}^{6} a_i b_i x_i + a_i b_i (b_i + 1)/2$ and $g = \sum_{i=1}^{6} a_i b_i y_i + b_i a_i (a_i + 1)/2$. This finishes our proof. □

We note that the powers $q^f$ and $q^g$ in the above proposition are exactly the weights $wt_1(T_0)$ and $wt_2(T_0)$ of the tiling $T_0$ corresponding to the empty stack in Figure 3.1(b).

## 5 Semi-hexagon with dents

A column-strict plane partition is a plane partition having columns strictly decreasing. A semi-hexagon $SH_{a,b}$ is the upper half of a lozenge hexagon $Hex(a, b, a)$. We are interested in the lozenge tilings of the semi-hexagon $SH_{a,b}$, where $a$ up-pointing triangles at the positions $1 \leq s_1 < s_2 < \cdots < s_a \leq a + b$ have been removed from the base. Denote by $SH_{a,b}(s_1, s_2, \ldots, s_a)$ the resulting semi-hexagon with dents (see Figure 5.1 for the region $SH_{6,5}(1, 3, 5, 8, 10, 11)$ (tiled)). There is a bijection between lozenge tilings of $SH_{a,b}(s_1, s_2, \ldots, s_a)$ and the column-strict plane partitions of shape $(s_a - a, s_{a-1} - a + 1, \ldots, s_1 - 1)$ with positive entries at most $a$. We will show how the bijection works in the next paragraph.

Let $T$ be any lozenge tiling of the region (see Figure 5.1(a) for a sample tiling). We add $a$ forced vertical lozenges to $T$ at the positions of the dents. Encode the resulting tiling as a family of disjoint lozenge path connecting the northwest side of the region and the positions of the dents as in Figure 5.1(b). Going up from the bottom of each path, we label each right lozenge by the number of vertical lozenges appearing before it along the path. The labels of the right lozenges give the entries of a column-strict plane partition of shape $(s_a - a, s_{a-1} - a + 1, \ldots, s_1 - 1)$ with positive entries at most $a$ (see Figure 5.1(d)). More precise, the labels in the $i$-th lozenge path (from top to bottom) are the entries of the $i$-th row of the plane partition. Next, we encode the

$$\frac{wt_1^{(i)}(T_i)}{wt_0(T_i)} = \frac{wt_1^{(i)}(T_i)}{q^{x_i+t}} = q^{a_i b_i x_i + a_i b_i (b_i + 1)/2}.$$ 

Multiplying all above equations for $i = 1, 2, \ldots, 6$, we get

$$\frac{wt_1(T)}{q^{x_i+t}} = q^{\sum_{i=1}^{6} a_i b_i x_i + a_i b_i (b_i + 1)/2}. \quad (4.8)$$

Similarly, we have

$$\frac{wt_2(T)}{q^{x_i+t}} = q^{\sum_{i=1}^{6} a_i b_i y_i + b_i a_i (a_i + 1)/2}. \quad (4.9)$$

Obtaining the formulas of $a_i, b_i, x_i, y_i$ in terms of $m, a, b, c, x, y, z, t$ from Figure 3.1, we get $f = \sum_{i=1}^{6} a_i b_i x_i + a_i b_i (b_i + 1)/2$ and $g = \sum_{i=1}^{6} a_i b_i y_i + b_i a_i (a_i + 1)/2$. This finishes our proof. □

We note that the powers $q^f$ and $q^g$ in the above proposition are exactly the weights $wt_1(T_0)$ and $wt_2(T_0)$ of the tiling $T_0$ corresponding to the empty stack in Figure 3.1(b).

## 5 Semi-hexagon with dents

A column-strict plane partition is a plane partition having columns strictly decreasing. A semi-hexagon $SH_{a,b}$ is the upper half of a lozenge hexagon $Hex(a, b, a)$. We are interested in the lozenge tilings of the semi-hexagon $SH_{a,b}$, where $a$ up-pointing triangles at the positions $1 \leq s_1 < s_2 < \cdots < s_a \leq a + b$ have been removed from the base. Denote by $SH_{a,b}(s_1, s_2, \ldots, s_a)$ the resulting semi-hexagon with dents (see Figure 5.1 for the region $SH_{6,5}(1, 3, 5, 8, 10, 11)$ (tiled)). There is a bijection between lozenge tilings of $SH_{a,b}(s_1, s_2, \ldots, s_a)$ and the column-strict plane partitions of shape $(s_a - a, s_{a-1} - a + 1, \ldots, s_1 - 1)$ with positive entries at most $a$. We will show how the bijection works in the next paragraph.

Let $T$ be any lozenge tiling of the region (see Figure 5.1(a) for a sample tiling). We add $a$ forced vertical lozenges to $T$ at the positions of the dents. Encode the resulting tiling as a family of disjoint lozenge path connecting the northwest side of the region and the positions of the dents as in Figure 5.1(b). Going up from the bottom of each path, we label each right lozenge by the number of vertical lozenges appearing before it along the path. The labels of the right lozenges give the entries of a column-strict plane partition of shape $(s_a - a, s_{a-1} - a + 1, \ldots, s_1 - 1)$ with positive entries at most $a$ (see Figure 5.1(d)). More precise, the labels in the $i$-th lozenge path (from top to bottom) are the entries of the $i$-th row of the plane partition. Next, we encode the
Figure 5.1: Bijection between lozenge tilings of semihexagon with dents and column-strict plane partitions.

tiling $T$ in a different way as a $b$-tuple of lozenge paths starting from the top and finishing at the bottom of the region (see Figure 5.1(b)). The labels in each new lozenge path are the entries of a column of the above plane partition. Figure 5.1 shows the bijection for the case $a = 6$, $b = 5$, $s_1 = 1$, $s_2 = 3$, $s_3 = 5$, $s_4 = 8$, $s_5 = 10$, $s_6 = 11$; the vertical unit interval at the bottom of the plane partition in Figure 5.1(d) indicates the row of length 0.

We notice that if any right lozenge of label $x$ in the above bijection is weighted by $q^x$, we have exactly the weight assignment $wT_2$ on the lozenges of $T$. We still use the notation $M_2$ for the corresponding tiling generating function of the semihexagon with dents.

**Proposition 5.1.** For non-negative integers $a, b$ and $1 \leq s_1 < s_2 < \ldots < s_a \leq a + b$

$$M_2 \left( SH_{a,b}(s_1, s_2, \ldots, s_a) \right) = q^{\sum_{i=1}^{a} (s_i - i)} \prod_{1 \leq i < j \leq a} \frac{q^{s_j} - q^{s_i}}{q^j - q^i}. \quad (5.1)$$

**Proof.** Let $T$ be any lozenge tiling of $SH_{a,b}(s_1, s_2, \ldots, s_a)$ (see Figure 5.1(a) for a sample tiling). By the above bijection, the weight $wT_2(T)$ of $T$ is exactly $q^{\mu_T}$, where $\mu_T$ is the column-strict plane partition corresponding to $T$. Taking the sum over all tilings $T$ of the semihexagon, we have

$$M_2 \left( SH_{a,b}(s_1, s_2, \ldots, s_a) \right) = \sum_{\mu} q^{\mu},$$

where the sum on the right-hand side is taken over all column-strict plane partitions $\mu$ of shape $(s_a - a, s_{a-1} - a + 1, \ldots, s_1 - 1)$ with positive entries at most $a$. However, the later weighted sum of plane partitions is exactly the expression on the right hand side of (5.1) (see e.g. [7], page 375). This completes our proof. \qed
Proposition 5.1 deduces a $q$-enumeration of the lozenge tilings of a hexagon with a triangular hole on the base $K_a(x, y, z, t)$ (defined as the region restricted by the bold contour in Figure 5.2).

Corollary 5.2. For non-negative $a, x, y, z, t$

$$M_2 \left( K_a(x, y, z, t) \right) = q^{y(z+1)+y(a+1)} \frac{H_q(a) H_q(x) H_q(y) H_q(z) H_q(t)}{H_q(x+t) H_q(a+x) H_q(a+y) H_q(y+z)} \times \frac{H_q(a+x+t) H_q(a+x+y) H_q(a+y+z) H_q(a+x+y+z+t)}{H_q(a+x+y+t) H_q(a+x+y+z) H_q(a+t+z)}. \quad (5.2)$$

Proof. The region $K_a(x, y, z, t)$ is obtained by removing forced vertical lozenges from the semihexagon $SH_{a+z+t,x+y}$ with dents at positions $\{1, 2, \ldots, t\} \cup \{t+x+1, t+x+2, \ldots, t+x+a\} \cup \{t+x+a+y+1, t+x+a+y+2, \ldots, t+x+a+y+z\}$. Thus, the corollary follows from Proposition 5.1. \hfill $\Box$

6 Two $q$-enumerations of magnet bar regions

In this section, we ($q$-)enumerate the lozenge tilings of the magnet bar $B_{m,a}(x, y, z, t)$ (see Figure 1.3).

Proposition 6.1. For non-negative integers $m, a, x, y, z, t$

$$M_2 \left( B_{m,a}(x, y, z, t) \right) = q^{y\left(\frac{m+1}{2}\right)+(m+x+y)\left(\frac{z+1}{2}\right)+myz+(m+a)(x+m)z+x^2} \times \frac{H_q(m+a+x+y+z+t)}{H_q(m+a+x+y+z)} \times \frac{H_q(m+a+x+t) H_q(m+a+x+y) H_q(m+a+y+z)}{H_q(m+a+x) H_q(m+a+y) H_q(m+a)} \times \frac{H_q(x) H_q(y) H_q(z) H_q(t) H_q(m) H_q(a)^2}{H_q(a+x) H_q(a+y) H_q(z+t) H_q(m+a)} \times \frac{H_q(m+z+t) H_q(m+a+x) H_q(m+a+y)}{H_q(m+y+z) H_q(m+x+t)}. \quad (6.1)$$
**Proof.** We prove (6.1) by induction on \(y + z + t\). Our base cases are the situations when \(m = 0, a = 0, y = 0, z = 0\) or \(t = 0\).

Assume that our region is weighted by \(wt_2\).

If \(m = 0\), then our region becomes the region \(K_a(x, y, z, t)\) in Corollary 5.2, and (6.1) follows.

If \(a = 0\), by removing forced lozenges along the base of the region \(B_{m,0}(x, y, z, t)\), we get the weighted hexagon \(Hex(z, x + y + m, t)\) in which a right lozenge is weighted by \(q^{m+z}\), where \(l\) is the distance from the top of the lozenge to the bottom of the hexagon (see Figure 6.1(e)). By dividing the weight of each right lozenge by \(q^m\), we get back the weight assignment \(wt_2\) on the hexagon. Moreover, we note that by removing some forced lozenges from a weighted region, the tiling generating function is changed by a factor equal to the product of weights of the forced lozenges. Since the product of weights of the forced lozenges in Figure 6.1(e) is \(q^{(m+1)}\), we get

\[
M_2(B_{m,0}(x, y, z, t)) = q^{\frac{(m+1)}{2}}q^{mz(x+y+m)}M_2(Hex(z, x + y + m, t)),
\]

(6.2)

where the factor \(q^{mz(x+y+m)}\) comes from the weight division in the hexagon \(Hex(z, x + y + m, t)\). Then (6.1) follows also from Proposition 4.1.

If \(y = 0\), after removing forced vertical lozenges, we get a new weighted region \(R\) (the region restricted by the bold contour in Figure 6.1(b)). By rotating the new region \(60^0\) clockwise and reflecting about a vertical line, we get the region \(K_m(z, a, x, t)\) weighted by \(wt_1\). Thus, we have

\[
M_2(B_{m,a}(x, 0, z, t)) = M_1(K_m(z, a, x, t)),
\]

(6.3)

and (6.1) follows from Proposition 4.2 and Corollary 5.2 (note that any \(K\)-type region is exactly a \(Q\)-type region with the \(m\)-, \(b\)- and \(c\)-parameters equal to 0, so we can apply Proposition 4.2 here).

If \(z = 0\), by applying Region-splitting Lemma 2.2, we have

\[
M_2(B_{m,a}(x, y, 0, t)) = M_2(Hex(m, y, a))M_2(B_{m,a}(x, y, 0, t) - Hex(m, y, a)).
\]

(6.4)

By removing forced left lozenges from the region \(B_{m,a}(x, y, 0, t) - Hex(m, y, a)\) as in Figure 6.1(c), we get the hexagon \(Hex(a, x, t + m)\) weighted by \(wt_2\). Thus, we get

\[
M_2(B_{m,a}(x, y, 0, t)) = M_2(Hex(m, y, a))M_2(Hex(a, x, t + m)),
\]

(6.5)

and (6.1) follows from Proposition 4.1.

If \(t = 0\), by the same pattern, we obtain

\[
M_2(B_{m,a}(x, y, z, 0)) = M_2(Hex(z + m, y, a))M_2(B_{m,a}(x, y, z, 0) - Hex(z + m, y, a)).
\]

(6.6)

We also get the hexagon \(Hex(a, x, m)\) (weighted by \(wt_2\)) after removing forced lozenges from the region \(B_{m,a}(x, y, z, 0) - Hex(z + m, y, a)\). However, our forced lozenges are now right lozenges, which have product of weights equal to \(q^{(m+a)(x+m)z+(x+m)\left(\frac{z+1}{2}\right)}\). Thus, we get

\[
M_2(B_{m,a}(x, y, z, 0) - Hex(z + m, y, a)) = q^{(m+a)(x+m)z+(x+m)\left(\frac{z+1}{2}\right)}M_2(Hex(a, x, m)),
\]

so

\[
M_2(B_{m,a}(x, y, z, 0)) = q^{(m+a)(x+m)z+(x+m)\left(\frac{z+1}{2}\right)}M_2(Hex(z + m, y, a))M_2(Hex(a, x, m)).
\]

(6.7)
Figure 6.1: The base cases in the proofs of Propositions 6.1 and 6.2: (a) \( x = 0 \), (b) \( y = 0 \), (c) \( z = 0 \), (d) \( t = 0 \), and (e) \( a = 0 \).

Again, (6.1) is implied by Proposition (4.1).

For the induction step, we assume that \( m, a, y, z, t \geq 1 \) and that (6.1) holds for any magnet bar regions, which have the sum of the \( y \)-, \( z \)- and \( t \)-parameters strictly less than \( y + z + t \).

We apply Kuo Theorem 2.1 to the dual graph \( G \) of the magnet bar region \( B_{m,a}(x, y, z, t) \) weighted by \( wt_2 \). We pick the four vertices \( u, v, w, s \) as in Figure 6.2 (b). The four shaded unit triangles indicate the ones corresponding to the four vertices. In particular, the shaded unit triangle corresponding to \( u \) is the lowest one, and \( v, w, s \) correspond to the next shaded unit triangles as we move counter-clockwise from the lowest one. We notice that the north side of the region has length \( x + y + m \geq y + m \geq 2 \) and the northeast side has length \( t + a \geq 2 \), so the four vertices \( u, v, w, s \) are well-defined.

By removing lozenges forced by the shaded unit triangles, we get back new \( B \)-type regions weighted by \( wt_2 \). By collecting the weights of forced lozenges in Figure 6.2, we get

\[
M(G - \{u, v\}) = q^{\left(\frac{z+m+1}{2}\right)} M_2 \left(B_{m,a}(x, y - 1, z, t)\right),
\]

(6.8)

\[
M(G - \{w, s\}) = q^{\left(x+y+m-2(z+t+m+a)\right)} M_2 \left(B_{m,a}(x, y, z, t - 1)\right),
\]

(6.9)

\[
M(G - \{u, s\}) = q^{\left(\frac{z+m+1}{2}\right)} M_2 \left(B_{m,a}(x, y - 1, z + 1, t - 1)\right),
\]

(6.10)

\[
M(G - \{v, w\}) = q^{\left(x+y+m-1(z+t+m+a)\right)} M_2 \left(B_{m,a}(x, y, z - 1, t)\right),
\]

(6.11)

and

\[
M(G - \{u, v, w, s\}) = q^{\left(\frac{z+m+1}{2}\right)+(x+y+m-2(z+t+m+a)} M_2 \left(B_{m,a}(x, y - 1, z, t - 1)\right).
\]

(6.12)
Figure 6.2: Obtaining the recurrence for the numbers tilings of magnet bar regions.
Plugging the above identities into the equation (2.1) in Kuo Condensation Theorem 2.1, we obtain

\[
M_2 \left( B_{m,a}(x, y, z, t) \right) M_2 \left( B_{m,a}(x, y - 1, z, t - 1) \right) = M_2 \left( B_{m,a}(x, y, z, t) \right) M_2 \left( B_{m,a}(x, y - 1, z, t) \right) + q^{z+m+a} M_2 \left( B_{m,a}(x, y - 1, z + 1, t - 1) \right) M_2 \left( B_{m,a}(x, y, z - 1, t) \right).
\] (6.13)

All regions in the above equation, except for the first one, have the sum of their $y$-, $z$- and $t$-parameters strictly less than $y + z + t$. Thus, by the induction hypothesis, those regions have their numbers of tilings given by (6.1). By substituting these formulas into the above equation and working on some simplifications, one readily gets $M_2 \left( B_{m,a}(x, y, z, t) \right)$ equal exactly to the expression on the right-hand side of (6.1). This finishes our proof. \(\square\)

We need another $q$-enumeration of the tilings of a magnet bar as follows.

Assume that we now give all right and left lozenges in the magnet bar $B_{m,a}(x, y, z, t)$ a weight 1. Next, we give a vertical lozenge a weight $q^l$, where $l$ is the distance between the northeast side of the lozenge and the southwest side of the region. We denote by $wt_3$ the new weight assignment, and $M_3$ the corresponding tiling generating function.

**Proposition 6.2.** For non-negative integers $m, a, x, y, z, t$

\[
M_3 \left( B_{m,a}(x, y, z, t) \right) = q^{m(a+1)+t(z+m+a)(x+a)+(z+m+1)} H_q(m + a + x + y + z) \times H_q(m + a + x + t) H_q(m + a + x + y) H_q(m + a + y + z) \times H_q(m + a + z + t) H_q(m + a + x) H_q(m + a + y + z) \times H_q(m + z + t) H_q(m + a + x) H_q(m + a + y) \times H_q(m + z + x) H_q(m + y + z) H_q(m + x + t).
\] (6.14)

**Proof.** The equality (6.14) can be treated similarly to (6.1) in the Proposition 6.1 by induction on $y + z + t$. The base cases are the situations when $a = 0$, $x = 0$, $y = 0$, $z = 0$ or $t = 0$.

Assume our region is weighted by $wt_3$.

If $a = 0$, then we also get a weighted version of the hexagon $Hex(z, x + y + m, t)$ by removing forced lozenges as in Figure 6.1(e). We rotate the hexagon $60^0$ clockwise and reflect over a vertical line, we get the hexagon $Hex(z, t, x + y + m)$ weighted by $wt_2$, and (6.14) follows from Proposition 4.1.

If $x = 0$, after removing forced vertical lozenges (whose product of weights is $q^{(t+m)(a+1)}$) as in Figure 6.1(a), we get a weighted region $R$. Next, we rotate $R$ $60^0$ counter-clockwise and reflect about a vertical line. This way, we get the weighted region $K_m(a, t, z, y)$ in which a right lozenge is weighted by $q^{az+l}$, where $l$ is the distance from the top of the lozenge to the bottom of the region. Dividing the weights of right lozenges by $q^a$, we get back the weight assignment $wt_2$ in the region $K_m(a, t, z, y)$. Thus,

\[
M_3 \left( B_{m,a}(0, y, z, t) \right) = q^{(t+m)(a+1)} q^{az+2(z+m)} M_2 \left( K_m(a, t, z, y) \right),
\] (6.15)

where the factor $q^{az+a^2(z+m)}$ comes from the weight division. Then (6.14) follows from Corollary 5.2.
If $y = 0$, by removing forced lozenges (whose product of weights is $q^{(x+a)(z+m)+a(z+m+1)}$) and rotating $60^\circ$ clockwise, we get a weighted version of region $K_m(a, z, t, x)$ in which a right lozenge is weighted by $q^{m+a+x+z+1-l}$, where $l$ is the distance from the left side of the lozenge and the southeast side of the region (see Figure 6.1(b)). By dividing the weight of each right lozenge by $q^{m+a+x+z+1}$, we get back the weight assignment $wt_1$, where $q$ is replaced by $q^{-1}$. Thus, (6.14) follows from Proposition 4.2, Corollary 5.2 and the simple fact $[n]_{q^{-1}} = [n]/q^{(n-1)}$.

If $z = 0$, we apply Region-splitting Lemma 2.2 (and remove forced lozenges) to split our region in two hexagons as in Figure 6.1(c). Next, we rotate the right hexagon $60^\circ$ counterclockwise and reflect about a vertical line to get the hexagon $Hex(a, t + m, x)$ weighted by $wt_2$. For the left hexagon, we also rotate it $60^\circ$ counter-clockwise, reflect about a vertical line and divide the weight of each right lozenge by $q^{x+a}$ to get the hexagon $Hex(m, a, y)$ weighted by $wt_2$. Then we get (6.14) from Proposition 4.1. The case $t = 0$, can be treated similarly to the case $z = 0$, based on Figure 6.1(d).

The induction step is completely analogous to that of the proof of Proposition 6.1. We still follow the Figure 6.2. After removing lozenges forced by the shaded unit triangles, we get back new $B$-type regions weighted by $wt_3$. The Figure 6.2 tells us that the product of $M_3$-generating functions of two regions on the top is equal to the product of the $M_3$-generating functions of two regions in the middle plus the product of $M_2$ generating functions of two regions on the bottom. To precise, we get the following recurrence

$$M_3(B_{m,a}(x, y, z, t)) M_3(B_{m,a}(x, y - 1, z, t - 1)) = M_3(B_{m,a}(x, y - 1, z, t)) M_3(B_{m,a}(x, y, z, t - 1)) + q^{m+a+x+y+z} M_3(B_{m,a}(x, y - 1, z + 1, t - 1)) M_3(B_{m,a}(x, y, z - 1, t)), \quad (6.16)$$

and the proposition follows from the induction hypothesis.

\[\square\]

### 7 Proof of Theorem 1.3

By Proposition 4.2, we only need to show that

$$M_2\left( Q\left( \begin{array}{c} x & y & z \\ m & a & b & c & t \end{array} \right) \right) = \begin{array}{c} q \\ \frac{H_q(m + a + b + c + x + y + z + t)}{H_q(m + a + b + c + x + y + t) H_q(m + a + b + c + x + y + z)} \times H_q(m + a + b + c + x + t) H_q(m + a + b + c + x + y) H_q(m + a + b + c + y + z) \times H_q(x + t) H_q(y + z) H_q(m + a + b + c + x + t) H_q(m + a + b + c + x) H_q(m + a + b + c + y) \times H_q(m + a + b + c + z + t) H_q(m + a + c + x) H_q(m + a + b + y) \times H_q(m + a + b + y + z) H_q(m + c + x + t) \times H_q(c + x + t) H_q(b + y + z) \times H_q(a + c + x) H_q(a + b + y) H_q(b + c + z + t) \end{array} \right) \quad (7.1)$$

Similar to the Theorem 1.1, we prove (7.1) by induction on $y + z + t$. The base cases here are still the situations when at least one of four parameters $x, y, z, t$ equals to 0.

We assume that our region is weighted by $wt_2$.

If $x = 0$, by applying Region-splitting Lemma 2.2, we split the into two parts (and some forced lozenges) as in Figure 2.1(a). The left part is the hexagon $Hex(a, c, m)$ weighted by $wt_2$. 

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We rotate the right part $60^\circ$ counter-clockwise and reflect about a vertical line. This way we get the magnet bar $B_{b,m}(a,t+c,z,y)$ weighted by $wt_3$. Thus, we have

$$M_2 \left( Q \begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix} \right) = M_2 \left( Hex(a,c,m) \right) M_3 \left( B_{b,m}(a,t+c,z,y) \right),$$

(7.2)

and (7.1) follows from Propositions 4.1 and 6.2. The case $t = 0$ can be treated similarly to the case $x = 0$. The only difference is that our forced lozenges have product of weights equal to $q^{x(a+m)(z+b+c)+x+z}/2$ (see Figure 2.1(d)). Thus, we get

$$M_2 \left( Q \begin{pmatrix} x & y & z & 0 \\ m & a & b & c \end{pmatrix} \right) = M_2 \left( Hex(a+c,m) \right) M_3 \left( B_{b,m}(a,t+c,z,y) \right).$$

(7.3)

Again, (7.1) follows from the Propositions 4.1 and 6.2.

If $y = 0$, again, by Region-splitting Lemma 2.2, we get

$$M_2 \left( Q \begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix} \right) = M_2 \left( Hex(m,b,a) \right) M_2 \left( Q \begin{pmatrix} x & y & z & 0 \\ m & a & b & c \end{pmatrix} - Hex(m,b,a) \right)$$

(7.4)

(see Figure 2.1(b)). We also remove forced lozenges from the second region on the right-hand side to get a region $R'$. Next, we rotate $R'$ clockwise and reflect about a vertical line to get the magnet bar $B_{c,m}(b+z,a,x,t)$ of weight $wt_1$. Thus, we have

$$M_2 \left( Q \begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix} \right) = M_2 \left( Hex(m,b,a) \right) M_1 \left( B_{c,m}(b+z,a,x,t) \right),$$

(7.5)

and (7.1) follows from Propositions 4.1, 4.2 and 6.1. The case $z = 0$ can be obtained in the same way, based on Figure 2.1(c).

For induction step, we also assume that $x, y, z, t \geq 1$ and that (7.1) holds for any $Q$-type regions in which the sum of the $y$-, $z$- and $t$-parameters strictly less than $y + z + t$.

Similar to the proof of Theorem 1.1, we now apply Kuo condensation to the dual graph $G$ of the region $Q \begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$ weighted by $wt_2$ as in Figure 2.2. The only difference is that the forced lozenges have weights in this case. By collecting weights of forced lozenges shown in Figure 2.2, we get the following weighted versions of the equations (2.8)–(2.12) in Theorem 1.1:

$$M(G - \{u, v\}) = q^{(x+y+1)/2} M_2 \left( Q \begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix} \right),$$

(7.6)

$$M(G - \{w, s\}) = q^{(x+y+m-2)(z+t+m+a+b+c)} M_2 \left( Q \begin{pmatrix} x & y & z & t \end{pmatrix} - 1 \\ m & a & b & c \end{pmatrix} \right),$$

(7.7)

$$M(G - \{u, s\}) = q^{(x+y+1)/2} M_2 \left( Q \begin{pmatrix} x & y & z & t \end{pmatrix} - 1 \\ m & a & b & c \end{pmatrix} \right),$$

(7.8)
\[
M(G - \{v, w\}) = q^{(x+y+m-1)(z+t+m+a+b+c)} M_2 \left( Q \left( \frac{x}{m} \frac{y}{a} \frac{z-1}{b} \frac{t}{c} \right) \right), \quad (7.9)
\]

and
\[
M(G - \{u, v, w, s\}) = q^{(x+y+m-2)(z+t+m+a+b+c)} M_2 \left( Q \left( \frac{x-1}{m} \frac{y}{a} \frac{z}{b} \frac{t-1}{c} \right) \right).
\]

Substituting (7.6)–(7.10) into equation (2.1) in Kuo’s Theorem 2.1, we get
\[
M_2 \left( Q \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t}{c} \right) \right) M_2 \left( Q \left( \frac{x}{m} \frac{y-1}{a} \frac{z}{b} \frac{t-1}{c} \right) \right)
= M_2 \left( Q \left( \frac{x}{m} \frac{y-1}{a} \frac{z}{b} \frac{t}{c} \right) \right) M_2 \left( Q \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t-1}{c} \right) \right)
+ q^{z+t+m+a+b+c} M_2 \left( Q \left( \frac{x}{m} \frac{y-1}{a} \frac{z+1}{b} \frac{t-1}{c} \right) \right) M_2 \left( Q \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t-1}{c} \right) \right).
\]

Finally, if we denote by \( \Psi \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t}{c} \right) \) the expression on the right-hand side of (7.1), we only need to show that \( \Psi \) satisfies also the recurrence (7.11). Equivalently, we need to verify that
\[
\frac{\Psi \left( \frac{x}{m} \frac{y-1}{a} \frac{z}{b} \frac{t}{c} \right)}{\Psi \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t}{c} \right)} + q^{z+t+m+a+b+c} \frac{\Psi \left( \frac{x}{m} \frac{y-1}{a} \frac{z+1}{b} \frac{t-1}{c} \right)}{\Psi \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t-1}{c} \right)} = 1. \quad (7.12)
\]

Let \( \Psi' \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t}{c} \right) := q^{-g \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t}{c} \right)} \Psi \left( \frac{x}{m} \frac{y-1}{a} \frac{z}{b} \frac{t}{c} \right) \). By definition of the function \( g \), we get
\[
g \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t}{c} \right) + g \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t-1}{c} \right) = g \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t}{c} \right) + g \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t-1}{c} \right) \quad (7.13)
\]

and
\[
g \left( \frac{x}{m} \frac{y}{a} \frac{z-1}{b} \frac{t}{c} \right) + g \left( \frac{x}{m} \frac{y}{a} \frac{z-1}{b} \frac{t}{c} \right)
= (m+x+y-z-1) + g \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t}{c} \right) + g \left( \frac{x}{m} \frac{y}{a} \frac{z}{b} \frac{t-1}{c} \right). \quad (7.14)
\]
By the above two identities, (7.12) is equivalent to

\[
\left[ \frac{\Psi'(x \ y \ z \ t \ m \ a \ b \ c)}{\Psi'(x \ a \ b \ c)} \right] = 1.
\]

We now can apply the simplifying process at the end of the proof of Theorem 1.1 to the function \( \Psi' \) (with the hyperfactorial functions are now replaced by the corresponding \( q \)-hyperfactorial functions). Similar to (2.18) and (2.19), we have

\[
\left[ \frac{\Psi'(x \ y \ z \ t \ m \ a \ b \ c)}{\Psi'(x \ a \ b \ c)} \right] = \frac{[m + a + b + c + x + y + t - 1]}{[m + a + b + c + x + y + z + t - 1]} 
\]

(7.15)

and

\[
\left[ \frac{\Psi'(x \ y \ z \ t \ m \ a \ b \ c)}{\Psi'(x \ a \ b \ c)} \right] = \frac{[z]}{[m + a + b + c + x + y + z + t - 1]}. 
\]

(7.17)

Therefore, (7.15) becomes the following equation

\[
\frac{[m + a + b + c + x + y + t - 1]}{[m + a + b + c + x + y + z + t - 1]} + q^{m+a+b+c+x+y+t-1} [z] = 1,
\]

which follows directly from the definition of \( q \)-integer. This completes our proof.

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