Analysis of Euler Characteristic

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Abstract. Usually, people first learn about Euler characteristic in three-dimensional space, related to the famous Euler formula. This invariant is explained in a top-down manner. In particular, there is an invariant first defined on general smooth manifold. Then, looking at different properties of this invariant leads to the discover of coincidence with Euler formula in lower dimension case. This paper extends the definition of Euler characteristic to more general geometries by glueing, and seeks for some possible applications. Result shows that Euler Characteristic is an useful invariant that ties many subject in topology together. By studying Euler Characteristic in various cases, one may found interesting relationship between objects that are not seemingly related.

1. Introduction
In this paper, readers are assumed to be familiar with differential forms, i.e. multilinear map preserves its value multiplied by the signature of a permutation with respect to the permutation of its operand, and p-form are such p-linear maps. We denote $\Lambda^p X^*$ to be all p-forms from $X^p$ to $\mathbb{R}$.

We know that total differential is a map between tangent bundle of manifolds. Consider the tangent bundle as disjoint union of tangent spaces. From linear algebra, we know cotangent spaces are dual of tangent spaces. It is easy to show the sections of $\Lambda^p T^* X$ are p-forms on tangent space of $X$, for all $x \in X$. We call such section differential p-form on $X$; $\Omega^p(X)$. Then the differential of a differential p-form would be a $p + 1$-form. If we consider the set of all differential p-forms on $X$, $\Omega^p(X)$. Then there is an induced map $d : \Omega^p(X) \rightarrow \Omega^{p+1}(X)$. We call it the de Rham differential on $X$.

2. De Rham Cohomology and Mayer-Vietoris Theorem
For a differential p-form, we say it is closed, if its differential $d\omega$ is zero. On the other hand, if $\omega \neq 0$ is the differential of an $p-1$-form, then we call it exact form. Overtly, any form is exact is also closed, since $d \circ d = 0$. Hence, the space of exact forms is a subspace of closed forms. It inspires us to investigate their quotient space $H^p X$, mention that both are linear spaces. We call $H^p X$ the de Rham cohomology group of $X$. By the property of de Rham differential, we have the following long non-exact sequence ("exact" in abstract algebra sense):

$$0 \xrightarrow{d} H^0(X) \xrightarrow{d} H^1(X) \xrightarrow{d} \cdots \xrightarrow{d} H^p(X) \xrightarrow{d} \cdots$$
Observe that $H^pX = 0$ for all $p > \dim(X)$, we obtain finite sequence for finite dimensional manifolds. This sequence is not exact because $d$ may not be surjective. However, following theorem “complete” it to an exact sequence

**Theorem 1** (Mayer-Vietoris). let $f$ and $g$ be induced by inclusions of $U$ and $V$ open subsets of $X$, for each $p \neq 0$, there exists a linear map $\delta : \bigcap H^p(U \cap V) \to H^{p+1}(X)$, such that the following sequence is exact:

$$
\cdots \to H^p(U \cap V) \xrightarrow{\delta} H^p(U) \oplus H^p(V) \xrightarrow{\delta} H^p(U \cap V) \to H^{p+1}(X) \xrightarrow{\delta} \cdots
$$

From Abstract algebra, we know the alternating sum of dimension of spaces in the sequence, usually called Euler Characteristic, is zero. Which give us a method to realize the structure of cohomology groups of a manifold.

**Example 1.** Find the structure of cohomology groups of $S^n$.

**Proof.** For any $n$-sphere, and $N, S$ be its two poles, we denote $U := S^n \setminus N$ and $V := S^n \setminus S$. For $n = 1$, we have following sequence from Mayer-Vietoris, $0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H^1(S^1) \to 0$. Hence, $\dim(H^1(S1)) = \dim(R)$. Since the cohomology group is a linear space. we have $H^1(S1) = R$.

We now claim $H^p(S^n) = \begin{cases} \mathbb{R} & p \in \{0, n\} \\ 0 & \text{otherwise} \end{cases}$

For $S^{n+1}$, We see $U \cap V$ is homotopy equivalent to $S^n$, and $U, V$ are diffeomorphic to $\mathbb{R}^{n+1}$. Thus we have following long exact sequence,

$0 \to \mathbb{R} \to \mathbb{R} \oplus \to \mathbb{R} \to H^1(S^{n+1}) \to 0 \to \cdots \to 0 \to H^p(S^{n+1}) \to 0 \to \cdots$

We can extract the short exact sequences from this sequence and observe the claim is true for $S^{n+1}$. By induction, the claim is proved.

### 3. Euler Characteristic and Properties

In general, the alternating sum of dimension of de Rham cohomology groups may not be zero, since the sequence is not exact. It suggest we to extend the definition of Euler characteristic to this sequence, and study the relationship between its value and associated manifold.

**Definition 1.** Let $X$ be a smooth manifold, and $H^\bullet(X)$ be the De Rham cohomology groups of $X$. Call

$$
\chi(X) = \sum_p (-1)^p \dim(H^p(X))
$$

the Euler Characteristic of $X$.

Remark. Since the dimension of $H^\bullet(X)$ is homotopic invariant, their alternating sum, $\chi(X)$, is also homotopic invariant.

**Proposition 1.** Let $X$ be a smooth manifold. $U, V$ are submanifold of $X$ s.t. $U \cup V = X$, and $U \cap V = \emptyset$. Then $\chi(X) = \chi(U) + \chi(V )$.

**Proof.** It suffices to prove dimension of homology groups are additive. Let $i_U$ and $i_V$ be inclusions. $\forall \omega \in \Omega p(X)$, if pull-backs $i_U \ast \omega = 0$ and $i_V \ast \omega = 0$, we have $\omega$ itself is zero. Hence, the inclusions induce an isomorphism of forms. Then, $\forall \omega \in \Omega p(U)$ and $\omega' \in \Omega p(V)$, we have the element of direct product $(\omega, \omega') \in \Omega p(X)$. Naturally, there is an isomorphism form $H^p(X)$ to $H^p(U) \times H^p(V)$, and $\dim H^p(X) = \dim H^p(U) + \dim H^p(V)$.

**Corollary 1.** Let $X$ be a smooth manifold. $\{U_i\}_{i=1}^n$ are submanifold of $X$
s.t. \( \cup_i U_i = X \), and \( U_i \cap U_j = \emptyset \) \( \forall i \neq j \). Then \( \chi(X) = \sum_i \chi(U_i) \)

**Corollary 2.** (Inclusion-exclusion principle). Let \( X \) be a smooth manifold, and \( U, V \) are submanifolds of \( X \). Then \( \chi(X) = \chi(U) + \chi(V) - \chi(U \cap V) \).

Proof. It is easy to see, \( X = U \cup (V \setminus (U \cap V)) \), and the proof follows from the proposition.

### 4. Morse Theory and Alternative Definition on Finite Complexes

We now introduce some basics in Morse theory, and seek its relationship with Euler Characteristic in special case.

**Definition 2.** A Morse function on \( M \) is a function \( f : M \to \mathbb{R} \) s.t. all critical point non-degenerate, i.e. has invertible Hessian matrices at all critical points.

**Theorem 2** (First Fundamental Theorem of Morse Theory). Let \( M \) be a smooth manifold, and let \( f \) be a Morse function on \( M \). Denote \( M_y \) to be the preimage of \( f \) that maps to value less or equal to \( y \), \( M_y = \{ x \in M | f(x) \leq y \} \).

Then for \( a < b \), \( M_a \) is diffeomorphic to \( M_b \) if the interval \([a, b] \) contains no critical value. In fact, \( M_a \) is a deformation retract of \( M_b \).

**Remark.** \( M_a \) has the same Euler characteristic as \( M_b \).

**Definition 3.** The index of a critical point \( x_0 \) of \( f \) is the number of negative eigen-values of Hessian of \( f \) evaluating at \( x_0 \). Denote \( C_{\lambda(k)} \) to be the set of critical points of \( f \) that has index \( k \).

**Definition 4.** An n-cell is a topological space homeomorphic to the open ball \( B^n \).

**Definition 5.** Let \( M \) be a topological space and let \( f_\delta : S^{n-1} \to M \) be a continuous map. The quotient space \( M \cup f_\delta B^n := M \cup B^n / \sim \) is referred as the space obtained from \( M \) by attaching an n-cell and \( f_\delta \) is the attaching map, where \( x \sim f_\delta(x) \) \( \forall x \in \partial B^n \).

**Theorem 3** (Second Fundamental Theorem of Morse Theory). Let \( M \) be a smooth manifold, and \( f \) be a Morse function on \( M \) with critical point \( x_0 \) of index \( \lambda \). Let \( c = f(x_0) \) and assume for \( \delta > 0 \) sufficiently small. There is no critical point other than \( x_0 \) belongs to interval \([x_0 - \delta, x_0 + \delta] \). Then, for \( 0 < \varepsilon < \delta \), \( \varepsilon \) sufficiently small, \( M_{x_0 + \varepsilon} \) is homotopic equivalent to \( M_{x_0 - \varepsilon} \) with a \( \lambda \)-cell attached.

**Definition 6.** A topological space \( M \) is said to have a CW structure if there are subspaces \( M(0) \subseteq M(1) \subseteq \cdots \subseteq M = \bigcup_{n \in \mathbb{Z}^+} M(n) \) s.t.

1. \( M(0) \) is a discrete set of points
2. \( M(n+1) \) is obtained from \( M(n) \) by attaching \((n+1)\)-cells for all \( n \geq 0 \)
3. A subset \( V \subset M \) is closed if and only if \( V \cap M(n) \) is closed for all \( n \geq 0 \)

Such a topological space is called a CW complex and the subspaces \( M(n) \) is the n-skeleton of \( M \).

A finite complex is a CW complex with only finitely many cells attached.

**Proposition 2.** Let \( M \) be a smooth finite complex, and \( f \) be a Morse function on \( M \). Then

\[
\chi(M) = \sum_{i=0}^{\infty} (-1)^i \left| C_{\lambda(i)}(f) \right|
\]

Proof. Since \( M \) is a finite complex, the increasing sequence \( \{M(n)\} \) terminates. Hence the sum is well-defined. By Morse Theory, \( M \) is homotopic equivalent to the disjoint union of cells which relates to critical points. By the property of Euler Characteristic, we have \( \chi(M) = \sum_i \chi(B^n_{\lambda(i)}) \), where \( \{x_i \} \) is the set of critical points of \( f \), and \( \lambda_i \) is the index of \( x_i \). It remains 4 to show that \( \chi(\lambda\text{-cell}) = (-1)^\lambda \). We know that \( B^n \) is diffeomorphic to a punctured \( S^n \) and

\[
H^p(S^n) = \begin{cases} \mathbb{R} & p \in \{0, n\} \\ 0 & \text{otherwise} \end{cases}
\]

Thus, by definition, \( \chi(S^n) = 1 + (-1)^n \). On the other hand, \( \chi(S^n) = \chi(B^n) + \chi(B_0) \). But \( \chi(B_0) = 1 \). Naturally, \( \chi(B^n) = (-1)^n \).

Remark. Since Euler characteristic is homotopic invariant, the proposition does not depend on \( f \).
5. Simplex and Euler Characteristic

Before this point, we only define Euler characteristic on smooth manifold. It is plausible to look for its generalization to random geometries, at least geometries that is not smooth or with singularities.

**Definition 7.** A standard n-simplex is

\[ \Delta_n := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\} \]

A k-face of \( \Delta_n \) is defined as \([e_{i_0}, \ldots, e_{i_k}]\) with \(0 \leq i_0 \leq \cdots \leq i_k \leq n\), where \(e_0, \ldots, e_n\) denotes the standard basis. The i-th face map of \( \Delta_n \) is defined to be the map

\[ F_i^n := [e_0, \ldots, \hat{e}_i, \ldots, e_n] : \Delta_{n-1} \to \Delta_n \]

**Definition 8.** Let \( X \) be a topological space. Then a singular n-simplex of \( X \) is a continuous map \( \sigma : \Delta_n \to X \). The i-th n-face of \( \sigma \) is given by \( \sigma_i := \sigma \circ F_i^n \), and the boundary of \( \sigma \) is defined as

\[ \partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma_i \]

Remark. The factor \((-1)^i\) takes into account the orientation of faces.

**Definition 9.** Let \( X \) be a topological space, and \( \{\sigma^k\} \) be a set of singular simplex of \( X \). \( \{\sigma^k\} \) is a simplexization of \( X \), if \( X = \bigcup_k \text{Im} \sigma^k \); and for \( m \leq n \), \( \sigma^l \) be a singular m-simplex, \( \sigma^r \) be a singular n-simplex, \( \text{Im} \sigma^l \cap \text{Im} \sigma^r = \emptyset \) if and only if

\[ \text{Im} \sigma^l \circ F_{j}^{m} = \text{Im} \sigma^r \circ F_{j}^{m} \]

- The interior \( \Delta^n \) is homotopic to a n-cell.
- A k-face of a singular simplex is diffeomorphic to \( \Delta_k \).

![Figure 1: A cone [6] Figure 2: Simplexized cone](image)

- Let \( X \) be a CW complex with its simplexization, the union of all k-faces of singular simplexes s.t. \( k \leq n \), forms the n-skeleton of \( X \).

**Definition 10.** Let \( X \) be a finite complex with simplexization, then

\[ \chi(X) = \sum_{i=0}^{\infty} (-1)^i |\{\text{singular } i\text{-simplexes}\}| \]

Remark. Naturally, the definition respect the additive propertive of \( \chi \) and extends the definition on smooth manifolds.

**Example 2.** For 2-dimensional surfaces, above definition implies the famous formula \( \chi = V - E + F \).
The cone $z^2 = x^2 + y^2$, $z \in [-1, 1]$ could be simplexized into 6 singular 2-simplexes that is homotopic to the figure 2. Hence, this cone has Euler Characteristic 1.

Example 3. Convex polyhedrons are simpliziation of a ball. Consider the height function of the ball which has one critical point of index 0, and one of index 2. It provides that all convex polyhedrons have $\chi = V - E + F = 2$.

Figure 3: An Icosahedron [4]   Figure 4: A ball [6]

Actually, when we are working with finite complexes, we can always simplexize the manifold $X$ by looking at the critical points of a Morse function $f$. In other words, we connect an edge from $C_{r_n}(f)$ to all points in $C_{r_i}(f)$, for all $i < n$. Then identify $k$-faces by $k$-submanifolds flows from $C_{r_n}(f)$ to $C_{r_i}(f)$, for all $i \leq n - k$. and we do it for all $0 < n \leq \dim(X)$.

6. Effect of Glueing

In this section we would explore how to create singularities by glueing geometries, and investigate how it would change the Euler characteristic.

**Definition 11.** Let $X$ and $Y$ be topological spaces, and $f : D \subset X \rightarrow Y$ be subjective. Then $f$ is a glueing on $X$, and $X/ \sim f$ is the space $X$ with $D$ glued to $Y$ under $f$, where $x \sim y \iff f(x) = f(y)$.

**Proposition 3.** Let $X$ and $Y$ be topological spaces, and $D$ be a subspace of $X$. If $\chi(X) = C_X$, $\chi(Y) = C_Y$, $\chi(D) = CD$, then for subjective $f : D \rightarrow Y$, $\chi(X/ \sim f) = C_X + C_Y - CD$.

**Proof.** Consider $D$ itself, then $f(D) = Y$ and $\chi(D/ \sim f) = C_Y$. Now think $X$ as a disjoint union $X = D \sqcup (X \setminus D)$, then $(X \setminus D)$ is unchanged under $f$.

So we have $\chi(X/ \sim f) = C_Y + \chi(X \setminus D)$. According to the inclusion-exclusion principal, $\chi(X/ \sim f) = C_Y + \chi(X) - \chi(D) = C_X + C_Y - C_D$.

**Remark.** Intuitively, we can think the glueing as union of two spaces with the intersection as domain of $f$. Naturally, $f^{-1}(y_0) \cap f^{-1}(y_1) = \emptyset$, for $y_0 = y_1$.

And

$$\bigsqcup_{y \in Y} f^{-1}(y) = D, \text{ while } \bigsqcup y = Y$$

**Corollary 3.** Let $X$ be a topological space with $\chi(X) = C_X$, and $D \subset X$ be homotopic equivalent to $S^1$. Let $f_1 : D \rightarrow \{0\}$, and $f_2 : D \rightarrow [0, 1]$. Then $\chi(X/ \sim f_1) = \chi(X/ \sim f_2) = C_X + 1$. We call $f_1$ a collapsing on $X$, and call $f_2$ a zipping on $X$.

**Remark.** This corollary can be easily extended to $S^n$, as we know $\chi(S^n) = 1 + (-1)^n$.

**Example 4.** Figure 5 is a torus with a simple loop collapsing, call it $T$. We know that $\chi(T) = 0$, and by the corollary we have $\chi(T) = C_X + 1 = 1$. At the same time, $T$ can be obtained by identifying a pair of antipodal points on $S^2$. Thus, $\chi(T) = \chi(S^2) + 1 - 2 = 1$. 

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It is also interesting to glue multiple space together. We can complete this by first glue each space and then glue their intersection together. Or we can also consider the disjoint union of spaces as a whole and glue at once.

Furthermore, for manifolds, gluing may produce singularity like figure 5 does. But the singularity may be a manifold on itself. We can do gluing recursively and produce singularities in singularities. However, the Euler characteristic will follow from the proposition.

Example 5. Figure 6 is 5 toruses glued together. We can simply calculate $\chi$ by inclusion-exclusion principal. Figure 7 is a torus zipped twice, but it is also the result of two cylinders L glued together. Either way, we compute

$$\chi(\mathbb{T}) + 2 = 2 = 4 + 2 - 4 = 2\chi(L) + 2\chi([0, 1]) - 4\chi(B_2)$$

7. Variation of Gauss-Bonnet Theorem and more

There are many subjects associate with Euler characteristic. For example, the Gauss-Bonnet Theorem stated as following:

**Theorem 4** (Gauss-Bonnet). Suppose $M$ is a compact two-dimensional Riemann manifold with boundary $\partial M$. Let $K$ be the Gaussian curvature of $M$, and let $k_g$ be the geodesic curvature of $\partial M$. Then

$$\int_M K \, dA + \int_{\partial M} k_g \, ds = 2\pi \chi(M)$$

**Remark.** The integral $\int_{\partial M} k_g \, ds$ may be omitted for manifolds without boundary. By Properties of Euler characteristic, a corollary follows immediately.

**Corollary 4.** Let $M, N, E \subset M, F \subset N$ and $G$ be compact two-dimensional Riemann manifolds without boundary. Suppose $M, N$ are imbedded s.t. $E$ and $F$ are glued to $G$ under $f$. Then

$$\int_{M+N+G-E-F} K \, dA + \int_{\partial G-\partial E-\partial F} k_g \, ds = 2\pi \chi((M \cup N)/ \sim_f)$$

Furthermore, Chern theorem generalizes Gauss-Bonnet theorem to any closed smooth orientable n-manifolds. Using Chern theorem, we can derive similar corollary (may not be in form of integrals) for topological spaces with more complicate singularities. Intuitively, we may disassemble the parts with singularity recursively, and partition the space into individual smooth manifolds(under certain imbedding). In the inverse way how we gluing spaces recursively.
8. Conclusion

We first defined Euler Characteristic using De Rham Cohomology. From there, we discovered an alternative definition using Morse Theory, and extended it to non-smooth cases which coincides with Euler’s Formula. Besides, we elaborated on manifolds with singularities by applying glueing to its submanifold. Moreover, we discussed the application of Euler Characteristic and glueing with Gauss-Bonnet Theorem. In short, Euler Characteristic is an useful invariant that ties many subject in topology together. By studying Euler Characteristic in various cases, one may found interesting relationship between objects that are not seemingly related.

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