Classification and study of a new class of $\xi^{(as)}$-QSO

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Abstract — Many systems are presented using theory of nonlinear operators. A quadratic stochastic operator (QSO) is perceived as a nonlinear operator. It has a wide range of applications in various disciplines, such as mathematics, biology, and other sciences. The central problem that surrounds this nonlinear operator lies in the requirement that behavior should be studied. Nonlinear operators, even QSO (i.e., the simplest nonlinear operator), have not been thoroughly investigated. This study aims to present a new class of $\xi^{(as)}$-QSO defined on 2D simplex and to classify it into 18 non-conjugate (isomorphic) classes based on their conjugacy and the remuneration of coordinates. In addition, the limiting points of the behavior of trajectories for four classes defined on 2D simplex are examined.

1 Introduction

The concept of a quadratic stochastic operator (QSO) was developed by brainchild of S. Bernstein in 1924 [1]. Since then, QSOs have been studied intensively as they emerge in various models in physics [15, 21], biology [1, 9, 27], economics and different branches of mathematics, such as graph theory and probability theory [9, 10, 11, 22].

In the biological context, QSOs can be applied in the area of population genetics. QSO can describe a distribution of the next generation when the initial distribution of the generation is provided. We shall briefly highlight how these operators are used to interpret in population genetics. Consider a biological population, i.e., a community of organisms that is closed with regard to procreation. Assume that every individual in this population belongs to one of the following varying species (traits): $\{1, \cdots, m\}$. Let $x^{(0)} = (x_1^{(0)}, \cdots, x_m^{(0)})$ be a probability distribution of species at an initial state and let the heredity coefficient $p_{ij,k}$ be the conditional probability $p(k \mid i, j)$ that $i^{th}$ and $j^{th}$ species have interbred successfully to produce an individual $k^{th}$. The first generation $x^{(1)} = (x_1^{(1)}, \cdots, x_m^{(1)})$ can be calculated using the total probability $x_k^{(1)} = \sum_{i,j=1}^{m} p(k \mid i, j)P(i,j)$, $k = 1, m$. Given that no difference exists

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between \(i\)th and \(j\)th in any generation, the parents \(i, j\) are independent, i.e., \(P(i,j) = P_i P_j\). This condition suggests that

\[ x_k^{(1)} = \sum_{i,j=1}^{m} P_{i,j,k} x_i^{(0)} x_j^{(0)}, \quad k = 1, m. \]

Consequently, the relation \(x^{(0)} \rightarrow x^{(1)}\) represents a mapping \(V\), which is known as the evolution operator. Starting from the selected initial state \(x^{(0)}\), the population develops to the first generation \(x^{(1)} = V(x^{(0)})\), and then to the second generation \(x^{(2)} = V(x^{(1)}) = V(V(x^{(0)})) = V^{(2)}(x^{(0)})\), and so on. Hence, the discrete dynamical system presents population system evolution states as follows:

\[ x^{(0)}, \quad x^{(1)} = V(x^{(0)}), \quad x^{(2)} = V^{(2)}(x^{(0)}), \quad \ldots. \]

One of the main issues that underlies this theory is finding the limit points of \(V\) for any arbitrary initial point \(x^{(0)}\). Studying the limit points of QSOs is a complicated task even in 2D simplex. This problem has not yet been solved. Numerous researchers have presented a specific class of QSO and have examined their behavior, e.g., F-QSO [18], Volterra-QSO [6, 28, 29], permuted Volterra-QSO [7, 8], \(\ell\)-Volterra-QSO [16, 17], Quasi-Volterra-QSO [4], non-Volterra-QSO [5, 20], strictly non-Volterra-QSO [19], non-Volterra operators, and others produced via measurements [2, 3]. An attempt was made to study the behavior of nonlinear operators, which is regarded as the main problem in nonlinear operators. However this problem has not been studied comprehensively because it depends on a given cubic matrix \((P_{ijk})_{i,j,k=1}^{m}\). Nevertheless, these classes together cannot cover a set of all QSOs.

Recently, the author of [23] introduced \(\xi^{(as)}\)-QSO, which is a new class of QSOs that depend on a partition of the coupled index sets (which have couple traits) \(P_m = \{(i,j) : i < j\} \subset I \times I\) and \(\Delta_m = \{(i,i) : i \in N\} \subset I \times I\). In case of 2D simplex \((m = 3)\), \(P_3\) and \(\Delta_m\) have five possible partitions.

In [23, 26], the \(\xi^{(a)}\)-QSO related to \(|\xi_1| = 2\) of \(P_3\) with point a partition of \(\Delta_3\) was investigated. In [13], the \(\xi^{(a)}\)-QSO related to \(|\xi_1| = 2\) of \(P_3\) with a trivial partition of \(\Delta_3\) was studied. The \(\xi^{(as)}\)-QSO related to \(|\xi_1| = 3\) of \(P_m\) with a point partition of \(\Delta_3\) was examined in [12]. Furthermore, the \(\xi^{(a)}\)-QSO and \(\xi^{(a)}\)-QSO related to \(|\xi_1| = 1\) of \(P_3\) with point and trivial partitions of \(\Delta_3\), respectively, were discussed in [25]. Therefore, some partitions of \(\Delta_3\) which have not yet been studied. The current work describes and classifies the operators generated by \(\xi^{(as)}\)-QSO with a cardinality \(|\xi_1| = 2\) of \(P_3\) and \(\Delta_3\) generated also by \(|\xi_1| = 2\). The rest of this paper is organized as follows. Section 2 establishes a number of preliminary definitions. Section 3 presents the description and classification of \(\xi^{(as)}\)-QSO. Section 4 elucidates the study examines the behavior of \(V_3\) and \(V_{15}\) obtained from classes \(G_3\) and \(G_9\), respectively. Section 5 examines the behavior of \(V_{26}\) and \(V_{25}\) obtained from classes \(G_{13}\) and \(G_{14}\), respectively.

## 2 Preliminaries

Several basic concepts are recalled in this section.

**Definition 1** \(QSO\) is a mapping of the simplex

\[ S^{m-1} = \left\{ x = (x_1, \cdots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} x_i = 1, \quad x_i \geq 0, \quad i = 1, m \right\} \]
into itself with the form

\[ x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = 1, m, \]  

(2)

where \( V(x) = x' = (x'_1, \ldots, x'_m) \), and \( P_{ij,k} \) is a coefficient of heredity that satisfies the following conditions:

\[ P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{m} P_{ij,k} = 1. \]  

(3)

From the preceding definition, we can conclude that each QSO \( V : S^{m-1} \to S^{m-1} \) can be uniquely defined by a cubic matrix \( \mathcal{P} = (P_{ijk})_{i,j,k=1}^{m} \) with conditions (3).

For \( V : S^{m-1} \to S^{m-1} \), we denote the set of fixed points as \( \text{Fix}(V) \). Moreover, for \( x^{(0)} \in S^{m-1} \), we denote the set of limiting points as \( \omega_V(x^{(0)}) \).

Recall that Volterra-QSO is defined by (2), (3), and the additional assumption

\[ P_{ij,k} = 0 \quad \text{if} \quad k \notin \{i, j\}. \]  

(4)

The biological treatment of Condition (4) is clear: the offspring repeats the genotype (trait) of one of its parents. Volterra-QSO exhibits the following form:

\[ x'_k = x_k \left( 1 + \sum_{i=1}^{m} a_{ki} x_i \right), \quad k \in I, \]  

(5)

where

\[ a_{ki} = 2P_{ik,k} - 1 \quad \text{for} \ i \neq k \quad \text{and} \ a_{ii} = 0, \ i \in I. \]  

(6)

Moreover,

\[ a_{ki} = -a_{ik} \quad \text{and} \quad |a_{ki}| \leq 1. \]

This type of operator was intensively studied in [6, 28, 29].

The concept of \( \ell \)-Volterra-QSO was introduced in [16]. This concept is recalled as follows.

Let \( \ell \in I \) be fixed. Suppose that the heredity coefficient \( \{P_{ij,k}\} \) satisfies

\[ P_{ij,k} = 0 \quad \text{if} \quad k \notin \{i, j\} \quad \text{for any} \ k \in \{1, \ldots, \ell\}, \ i, j \in I, \]  

(7)

\[ P_{i0j0,k} > 0 \quad \text{for some} \ (i_0, j_0), \ i_0 \neq k, \ j_0 \neq k, \ k \in \{\ell + 1, \ldots, m\}. \]  

(8)

Therefore, the QSO defined by (2), (3), (7), and (8) is called \( \ell \)-Volterra-QSO.

**Remark 1** Here, we emphasize the following points:

(i) An \( \ell \)-Volterra-QSO is a Volterra-QSO if and only if \( \ell = m \).

(ii) No periodic trajectory exists for Volterra-QSO [6]. However, such trajectories exist for \( \ell \)-Volterra-QSO [16].
In accordance with [23], each element \( x \in S^{m-1} \) is a probability distribution of set \( I = \{1, \ldots, m\} \). Let \( x = (x_1, \cdots, x_m) \) and \( y = (y_1, \cdots, y_m) \) be vectors obtained from \( S^{m-1} \). We say that \( x \) is equivalent to \( y \) if \( x_k = 0 \Leftrightarrow y_k = 0 \). We denote this relation as \( x \sim y \).

Let \( \text{supp}(x) = \{i : x_i \neq 0\} \) be a support of \( x \in S^{m-1} \). We say that \( x \) is singular to \( y \) and denote this relation as \( x \perp y \) if \( \text{supp}(x) \cap \text{supp}(y) = \emptyset \). Notably if \( x, y \in S^{m-1} \), then \( x \perp y \) if and only if \( (x, y) = 0 \), where \((\cdot, \cdot)\) denotes a standard inner product in \( \mathbb{R}^m \).

We denote sets of coupled indexes as

\[
P_m = \{(i, j) : i < j\} \subset I \times I, \quad \Delta_m = \{(i, i) : i \in I\} \subset I \times I.
\]

For a given pair \((i, j) \in P_m \cup \Delta_m\), a vector \( P_{ij} = (P_{ij,1}, \cdots, P_{ij,m}) \) is set. Evidently, \( P_{ij} \in S^{m-1} \).

Let \( \xi_1 = \{A_i\}_{i=1}^N \) and \( \xi_2 = \{B_i\}_{i=1}^M \) be fixed partitions of \( P_m \) and \( \Delta_m \), respectively, i.e., \( A_i \cap A_j = \emptyset, B_i \cap B_j = \emptyset, \bigcup_{i=1}^N A_i = P_m, \bigcup_{i=1}^M B_i = \Delta_m \), where \( N, M \leq m \).

**Definition 2** [23] QSO \( V : S^{m-1} \rightarrow S^{m-1} \) is given by [2], [3], is considered a \( \xi^{(as)} \)-QSO w.r.t. partitions \( \xi_1 \) and \( \xi_2 \) if the following conditions are satisfied:

(i) For each \( k \in \{1, \ldots, N\} \) and any \((i, j), (u, v) \in A_k\), \( P_{ij} \sim P_{uv} \) is considered.

(ii) For any \( k \neq \ell, k, \ell \in \{1, \ldots, N\} \) and any \((i, j) \in A_k\) and \((u, v) \in A_\ell\), \( P_{ij} \perp P_{uv} \) is considered.

(iii) For each \( d \in \{1, \ldots, M\} \) and any \((i, i), (j, j) \in B_d\), \( P_{ii} \sim P_{jj} \) is considered.

(iv) For any \( s \neq h, s, h \in \{1, \ldots, M\} \) and any \((u, u) \in B_s\) and \((v, v) \in B_h\), \( P_{uu} \perp P_{vv} \) is considered.

3 Classification of \( \xi^{(as)} \)-QSO operators

This section presents the description and classification of \( \xi^{(as)} \)-QSO in 2D simplex when \( m = 3 \) and the cardinality of the potential partitions of \( P_m \) and \( \Delta_m \) are equal to 2. Therefore, the potential partitions of \( P_3 \) are listed as follows:

\[
\begin{align*}
\xi_1 : & = \{(1, 2), (1, 3), (2, 3)\}, |\xi_1| = 3, \\
\xi_2 : & = \{(2, 3), (1, 2, 1, 3)\}, |\xi_2| = 2, \\
\xi_3 : & = \{(1, 3), (1, 2, 2, 3)\}, |\xi_3| = 2, \\
\xi_4 : & = \{(1, 2), (1, 3), (2, 3)\}, |\xi_4| = 2, \\
\xi_5 : & = \{(1, 2), (1, 3), (2, 3)\}, |\xi_5| = 1.
\end{align*}
\]

The potential partitions of \( \Delta_3 \) are listed as follows:
\[ \xi_1 : = \{(1,1), (2,2), (3,3)\}, |\xi_1| = 3, \]
\[ \xi_2 : = \{(1,1), (2,2), (3,3)\}, |\xi_2| = 1, \]
\[ \xi_3 : = \{(1,1), (2,2), (3,3)\}, |\xi_3| = 2, \]
\[ \xi_4 : = \{(3,3), (1,1), (2,2)\}, |\xi_4| = 2, \]
\[ \xi_5 : = \{(2,2), (1,1), (3,3)\}, |\xi_5| = 2. \]

**Proposition 1** For a class of \(\xi^{(as)}\)-QSO generated from the possible partitions of \(\mathbf{P}_3\) and \(\Delta_{3}\) with cardinals equal to 2, we determine the following:

(a) A class of all \(\xi^{(as)}\)-QSO that correspond to partition \(\xi_3\) of \(\mathbf{P}_3\) and partition \(\xi_5\) of \(\Delta_{3}\) is conjugate to a class of all \(\xi^{(as)}\)-QSO that correspond to partition \(\xi_2\) of \(\mathbf{P}_3\) and partition \(\xi_3\) of \(\Delta_{3}\).

(b) A class of all \(\xi^{(as)}\)-QSO that correspond to the partition \(\xi_4\) of \(\mathbf{P}_3\) and partition \(\xi_4\) of \(\Delta_{3}\) is conjugate to a classes of all \(\xi^{(as)}\)-QSO that correspond to partition \(\xi_2\) of \(\mathbf{P}_3\) and partition \(\xi_3\) of \(\Delta_{3}\).

**Proof.** (a) In accordance with the general form of QSO given by \(\mathbf{[2,3]}\), the coefficients \((P_{ij,k})^3_{i,j,k=1}\) of operator \(V\) in \(\xi^{(as)}\)-QSO that correspond to partition \(\xi_5 = \{(2,2), (1,1), (3,3)\}\) of \(\Delta_{3}\) and partition \(\xi_3 = \{(1,3), (1,2), (2,3)\}\) of \(\mathbf{P}_3\) satisfy the following conditions:

i. \(\mathbb{P}_{11} \sim \mathbb{P}_{33}\) and \(\mathbb{P}_{22} \perp \mathbb{P}_{mm}, m = 1,3;\) ii. \(\mathbb{P}_{12} \sim \mathbb{P}_{23}\) and \(\mathbb{P}_{13} \perp (\mathbb{P}_{12}, \mathbb{P}_{23});\)

where \(\mathbb{P}_{ij} = (p_{ij,1}, p_{ij,2}, p_{ij,3}).\)

To perform \(V_\pi = \pi V \pi^{-1}\) transformation on operator \(V\), where permutation \(\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\).

\[ V_\pi : x'_k = \sum_{i,j=1}^{3} P_{\pi_{ij,k}^{\pi}} x_i x_j, \quad k = \overline{1,3}, \]

such that \(P_{\pi_{ij,k}^{\pi}} = P_{\pi(i),\pi(j),\pi(k)}\) for any \(i, j, k \in \overline{1,3}\). Equivalently, \(P_{\pi_{ij}^{\pi}} = \pi P_{\pi_{i,j}^{(j)}}\) (in vector form) for any \(i, j = 1,2,3\).

Subsequently, operator \(V_\pi\) that corresponds to partitions \(\xi_3\) of \(\Delta_{3}\) and \(\xi_2\) of \(\mathbf{P}_3\) is presented by applying the permutation \(\pi\) for the coefficient of \(V\) that corresponds to partition \(\xi_5\) of \(\Delta_{3}\) and \(\xi_3\) of \(\mathbf{P}_3\). The following relations are derived:

i. \(\mathbb{P}_{11} \sim \mathbb{P}_{33}\) and \(\mathbb{P}_{22} \perp (\mathbb{P}_{11}, \mathbb{P}_{33}).\) Given that \(P_{11}^\pi = P_{33}, P_{22}^\pi = P_{11},\) and \(P_{33}^\pi = P_{22},\) we obtain \(\mathbb{P}_{33} \sim \mathbb{P}_{22}\) and \(\mathbb{P}_{11} \perp (\mathbb{P}_{22}, \mathbb{P}_{33}).\)

ii. \(\mathbb{P}_{12} \sim \mathbb{P}_{23}\) and \(\mathbb{P}_{13} \perp (\mathbb{P}_{12}, \mathbb{P}_{23}).\) Given that \(P_{12}^\pi = P_{13}, P_{23}^\pi = P_{23},\) and \(P_{23}^\pi = P_{12},\) we obtain \(\mathbb{P}_{12} \sim \mathbb{P}_{13}\) and \(\mathbb{P}_{23} \perp (\mathbb{P}_{12}, \mathbb{P}_{13}).\)

Similarly, we can prove b by choosing permutation \(\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.\) This process completes the proof. \(\square\)
The preceding discussion shows that any $\xi^{(as)}$-QSO obtained from the class that corresponds to partitions $\xi_5$ of $\Delta_3$ and $\xi_3$ of $P_3$ or $\xi_4$ of $\Delta_3$ and $\xi_4$ of $P_3$ is conjugate to certain $\xi^{(as)}$-QSO obtained from the class that corresponds to partitions $\xi_3$ of $\Delta_3$ and $\xi_2$ of $P_3$.

To investigate the operators of class $\xi^{(as)}$-QSO that correspond to partitions $\xi_2$ of $P_3$ and $\xi_3$ of $\Delta_3$, coefficient $(P_{ij,k})^3_{i,j,k=1}$ in special forms is selected as shown in Tables (a) and (b).

| Case | $P_{11}$ | $P_{22}$ | $P_{33}$ |
|------|----------|----------|----------|
| $I_1$ | $(\alpha, \beta, 0)$ | $(0, 0, 1)$ | $(0, 0, 1)$ |
| $I_2$ | $(\alpha, 0, \beta)$ | $(0, 1, 0)$ | $(0, 1, 0)$ |
| $I_3$ | $(\beta, \alpha, 0)$ | $(0, 0, 1)$ | $(0, 0, 1)$ |
| $I_4$ | $(\beta, 0, \alpha)$ | $(0, 1, 0)$ | $(0, 1, 0)$ |
| $I_5$ | $(0, \alpha, \beta)$ | $(1, 0, 0)$ | $(1, 0, 0)$ |
| $I_6$ | $(0, \beta, \alpha)$ | $(1, 0, 0)$ | $(1, 0, 0)$ |

where $\alpha, \beta \in [0, 1]$. Moreover, $\alpha + \beta = 1$.

| Case | $P_{12}$ | $P_{13}$ | $P_{23}$ |
|------|----------|----------|----------|
| $II_1$ | $(1, 0, 0)$ | $(1, 0, 0)$ | $(0, 0, 1)$ |
| $II_2$ | $(1, 0, 0)$ | $(1, 0, 0)$ | $(0, 1, 0)$ |
| $II_3$ | $(0, 1, 0)$ | $(0, 1, 0)$ | $(1, 0, 0)$ |
| $II_4$ | $(0, 1, 0)$ | $(0, 0, 1)$ | $(1, 0, 0)$ |
| $II_5$ | $(0, 0, 1)$ | $(0, 0, 1)$ | $(0, 1, 0)$ |

The choices for Cases $(I_j, II_j)$, where $i, j = 1, \cdots, 6$, provide 36 operators. These operators are defined as follows:

$$V_1 := \begin{cases} 
    x' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\
    y' = \beta(x^{(0)})^2 \\
    z' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)} 
\end{cases}$$

$$V_2 := \begin{cases} 
    x' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\
    y' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\
    z' = (y^{(0)})^2 + (z^{(0)})^2 \end{cases}$$

$$V_3 := \begin{cases} 
    x' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\
    y' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\
    z' = (y^{(0)})^2 + (z^{(0)})^2 \end{cases}$$

$$V_4 := \begin{cases} 
    x' = \alpha(x^{(0)})^2 \\
    y' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\
    z' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases}$$
\[
V_5 := \begin{cases} 
  x' &= \alpha (x^{(0)})^2 + 2y^{(0)}z^{(0)} \\
  y' &= \beta (x^{(0)})^2 \\
  z' &= (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} 
\end{cases}
\]

\[
V_6 := \begin{cases} 
  x' &= \alpha (x^{(0)})^2 \\
  y' &= \beta (x^{(0)})^2 + 2y^{(0)}z^{(0)} \\
  z' &= (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} 
\end{cases}
\]

\[
V_7 := \begin{cases} 
  x' &= \alpha (x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\
  y' &= (y^{(0)})^2 + (z^{(0)})^2 \\
  z' &= \beta (x^{(0)})^2 + 2y^{(0)}z^{(0)} 
\end{cases}
\]

\[
V_8 := \begin{cases} 
  x' &= \alpha (x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\
  y' &= (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)} \\
  z' &= \beta (x^{(0)})^2 
\end{cases}
\]

\[
V_9 := \begin{cases} 
  x' &= \alpha (x^{(0)})^2 + 2y^{(0)}z^{(0)} \\
  y' &= (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\
  z' &= \beta (x^{(0)})^2 
\end{cases}
\]

\[
V_{10} := \begin{cases} 
  x' &= \alpha (x^{(0)})^2 \\
  y' &= (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\
  z' &= \beta (x^{(0)})^2 + 2y^{(0)}z^{(0)} 
\end{cases}
\]

\[
V_{11} := \begin{cases} 
  x' &= \alpha (x^{(0)})^2 + 2y^{(0)}z^{(0)} \\
  y' &= (y^{(0)})^2 + (z^{(0)})^2 \\
  z' &= \beta (x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} 
\end{cases}
\]

\[
V_{12} := \begin{cases} 
  x' &= \alpha (x^{(0)})^2 \\
  y' &= (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)} \\
  z' &= \beta (x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} 
\end{cases}
\]

\[
V_{13} := \begin{cases} 
  x' &= \beta (x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\
  y' &= \alpha (x^{(0)})^2 \\
  z' &= (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)} 
\end{cases}
\]

\[
V_{14} := \begin{cases} 
  x' &= \beta (x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\
  y' &= \alpha (x^{(0)})^2 + 2y^{(0)}z^{(0)} \\
  z' &= (y^{(0)})^2 + (z^{(0)})^2 
\end{cases}
\]
\begin{align*}
V_{15} & := \begin{cases} 
x' &= \beta (x^{(0)})^2 + 2y^{(0)} z^{(0)} \\
y' &= \alpha (x^{(0)})^2 + 2x^{(0)} y^{(0)} + 2x^{(0)} z^{(0)} \\
z' &= (y^{(0)})^2 + (z^{(0)})^2 
\end{cases} \\
V_{16} & := \begin{cases} 
x' &= \beta (x^{(0)})^2 \\
y' &= \alpha (x^{(0)})^2 + 2x^{(0)} y^{(0)} + 2x^{(0)} z^{(0)} \\
z' &= (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)} z^{(0)} 
\end{cases} \\
V_{17} & := \begin{cases} 
x' &= \beta (x^{(0)})^2 + 2y^{(0)} z^{(0)} \\
y' &= \alpha (x^{(0)})^2 \\
z' &= (z^{(0)})^2 + (y^{(0)})^2 + 2x^{(0)} y^{(0)} + 2x^{(0)} z^{(0)} 
\end{cases} \\
V_{18} & := \begin{cases} 
x' &= \beta (x^{(0)})^2 \\
y' &= \alpha (x^{(0)})^2 + 2y^{(0)} z^{(0)} \\
z' &= (z^{(0)})^2 + (y^{(0)})^2 + 2x^{(0)} y^{(0)} + 2x^{(0)} z^{(0)} 
\end{cases} \\
V_{19} & := \begin{cases} 
x' &= \beta (x^{(0)})^2 + 2x^{(0)} y^{(0)} + 2x^{(0)} z^{(0)} \\
y' &= (y^{(0)})^2 + (z^{(0)})^2 \\
z' &= \alpha (x^{(0)})^2 + 2y^{(0)} z^{(0)} 
\end{cases} \\
V_{20} & := \begin{cases} 
x' &= \beta (x^{(0)})^2 + 2x^{(0)} y^{(0)} + 2x^{(0)} z^{(0)} \\
y' &= (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)} z^{(0)} \\
z' &= \alpha (x^{(0)})^2 
\end{cases} \\
V_{21} & := \begin{cases} 
x' &= \beta (x^{(0)})^2 + 2y^{(0)} z^{(0)} \\
y' &= (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)} y^{(0)} + 2x^{(0)} z^{(0)} \\
z' &= \alpha (x^{(0)})^2 
\end{cases} \\
V_{22} & := \begin{cases} 
x' &= \beta (x^{(0)})^2 \\
y' &= (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)} y^{(0)} + 2x^{(0)} z^{(0)} \\
z' &= \alpha (x^{(0)})^2 + 2y^{(0)} z^{(0)} 
\end{cases} \\
V_{23} & := \begin{cases} 
x' &= \beta (x^{(0)})^2 + 2y^{(0)} z^{(0)} \\
y' &= (y^{(0)})^2 + (z^{(0)})^2 \\
z' &= \alpha (x^{(0)})^2 + 2x^{(0)} y^{(0)} + 2x^{(0)} z^{(0)} 
\end{cases} \\
V_{24} & := \begin{cases} 
x' &= \beta (x^{(0)})^2 \\
y' &= (y^{(0)})^2 + (z^{(0)})^2 + 2z^{(0)} y^{(0)} \\
z' &= \alpha (x^{(0)})^2 + 2x^{(0)} y^{(0)} + 2x^{(0)} z^{(0)} 
\end{cases}
\end{align*}
\[ \begin{align*}
V_{25} := & \begin{cases} 
x' = (y^{(0)})^2 + (z^{(0)})^2 + 2x(0)y^{(0)} + 2x^{(0)}z^{(0)} \\
y' = \alpha(x^{(0)})^2 \\
z' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} 
\end{cases} \\
V_{26} := & \begin{cases} 
x' = (y^{(0)})^2 + (z^{(0)})^2 + 2x(0)y^{(0)} + 2x^{(0)}z^{(0)} \\
y' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\
z' = \beta(x^{(0)})^2 
\end{cases} \\
V_{27} := & \begin{cases} 
x' = (y^{(0)})^2 + (z^{(0)})^2 + 2z^{(0)}y^{(0)} \\
y' = \alpha(x^{(0)})^2 + 2x(0)y^{(0)} + 2x^{(0)}z^{(0)} \\
z' = \beta(x^{(0)})^2 
\end{cases} \\
V_{28} := & \begin{cases} 
x' = (y^{(0)})^2 + (z^{(0)})^2 \\
y' = \alpha(x^{(0)})^2 + 2x(0)y^{(0)} + 2x^{(0)}z^{(0)} \\
z' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} 
\end{cases} \\
V_{29} := & \begin{cases} 
x' = (y^{(0)})^2 + (z^{(0)})^2 + 2z^{(0)}y^{(0)} \\
y' = \alpha(x^{(0)})^2 \\
z' = \beta(x^{(0)})^2 + 2x(0)y^{(0)} + 2x^{(0)}z^{(0)} 
\end{cases} \\
V_{30} := & \begin{cases} 
x' = (y^{(0)})^2 + (z^{(0)})^2 \\
y' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\
z' = \beta(x^{(0)})^2 + 2x(0)y^{(0)} + 2x^{(0)}z^{(0)} 
\end{cases} \\
V_{31} := & \begin{cases} 
x' = (y^{(0)})^2 + (z^{(0)})^2 + 2x(0)y^{(0)} + 2x^{(0)}z^{(0)} \\
y' = \beta(x^{(0)})^2 \\
z' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} 
\end{cases} \\
V_{32} := & \begin{cases} 
x' = (y^{(0)})^2 + (z^{(0)})^2 + 2x(0)y^{(0)} + 2x^{(0)}z^{(0)} \\
y' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\
z' = \alpha(x^{(0)})^2 
\end{cases} \\
V_{33} := & \begin{cases} 
x' = (y^{(0)})^2 + (z^{(0)})^2 + 2z^{(0)}y^{(0)} \\
y' = \beta(x^{(0)})^2 + 2x(0)y^{(0)} + 2x^{(0)}z^{(0)} \\
z' = \alpha(x^{(0)})^2 
\end{cases} \\
V_{34} := & \begin{cases} 
x' = (y^{(0)})^2 + (z^{(0)})^2 \\
y' = \beta(x^{(0)})^2 + 2x(0)y^{(0)} + 2x^{(0)}z^{(0)} \\
z' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} 
\end{cases} 
\end{align*} \]
We can derive the other classes by following the same procedure. This process completes the proof.

\[ V_{35} := \begin{cases} 
  x' = (y^{(0)})^2 + (z^{(0)})^2 + 2z^{(0)}y^{(0)} \\
  y' = \beta(x^{(0)})^2 \\
  z' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)}
\end{cases} \]

\[ V_{36} := \begin{cases} 
  x' = (y^{(0)})^2 + (z^{(0)})^2 \\
  y' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\
  z' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)}
\end{cases} \]

Evidently, class \( \xi^{(as)} \)-QSO contains 36 operators, which operators are too numerous to explore individually. Therefore, we classify such operators into small classes and examine only the operators within these classes.

**Theorem 1** Let \( \{V_1, \cdots, V_{36}\} \) be the \( \xi^{(as)} \)-QSO presented above. Then, these operators are divided into 18 non-isomorphic classes:

- \( G_1 = \{V_1, V_5\} \)
- \( G_2 = \{V_2, V_7\} \)
- \( G_3 = \{V_3, V_{11}\} \)
- \( G_4 = \{V_4, V_{12}\} \)
- \( G_5 = \{V_5, V_9\} \)
- \( G_6 = \{V_6, V_{10}\} \)
- \( G_7 = \{V_{13}, V_{20}\} \)
- \( G_8 = \{V_{14}, V_{19}\} \)
- \( G_9 = \{V_{15}, V_{23}\} \)
- \( G_{10} = \{V_{16}, V_{24}\} \)
- \( G_{11} = \{V_{17}, V_{21}\} \)
- \( G_{12} = \{V_{18}, V_{22}\} \)
- \( G_{13} = \{V_{25}, V_{32}\} \)
- \( G_{14} = \{V_{26}, V_{31}\} \)
- \( G_{15} = \{V_{27}, V_{35}\} \)
- \( G_{16} = \{V_{28}, V_{36}\} \)
- \( G_{17} = \{V_{29}, V_{33}\} \)
- \( G_{18} = \{V_{30}, V_{34}\} \)

**Proof.** Evidently, the partitions \( \xi_2 \) of \( P_3 \) and \( \xi_3 \) of \( \Delta_3 \) are invariant only under the permutation \( \pi = \begin{pmatrix} x^{(0)} & y^{(0)} & z^{(0)} \\
  x^{(0)} & y^{(0)} & z^{(0)} \end{pmatrix} \). Therefore, the given operators should be classified with respect to the remuneration of their coordinates. Consequently, we have to perform \( \pi V \pi^{-1} \) transformation on all the operators.

Starting with \( V_1 \) as the first operator, we obtain

\[ V_1 \left( \pi^{-1}(x^{(0)}, y^{(0)}, z^{(0)}) \right) = V_1(x^{(0)}, z^{(0)}, y^{(0)}) = (\alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)}, \beta(x^{(0)})^2, \right) \]

\[ (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)}). \] Thus,

\[ \pi V_1 \pi^{-1} = \left( \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)}, (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)}, (a-1)\beta(x^{(0)})^2 \right) = V_8. \]

We can derive the other classes by following the same procedure. This process completes the proof.
4 Dynamics of classes $G_3$ and $G_9$

This section explores the dynamics of $\xi^{(as)}$-QSO $V_{3,15} : S^2 \to S^2$ selected from $G_3$ and $G_9$. To begin, $V_3$ is rewritten as follows:

$$V_3 := \begin{cases} x' = \alpha x(0)^2 + 2y(0)z(0) \\ y' = (1 - \alpha) x(0)^2 + 2x(0)(1 - x(0)) \\ z' = (z(0))^2 + (y(0))^2 \end{cases} \quad (9)$$

The operator $V_3$ can be redrafted as a convex combination $V_3 = \alpha W_1 + (1 - \alpha) W_2$, where

$$W_1 := \begin{cases} x' = x(0)^2 + 2yz \\ y' = 2x(0)(1 - x(0)) \\ z' = (z(0))^2 + (y(0))^2 \end{cases} \quad (10)$$

and

$$W_2 := \begin{cases} x' = 2y(0)z(0) \\ y' = 2x(0) - (x(0))^2 \\ z' = (z(0))^2 + (y(0))^2 \end{cases} \quad (11)$$

**Theorem 2** Let $W_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (10) and $x_1^{(0)} = (x(0), y(0), z(0)) \notin Fix(W_1)$ be any initial point in the simplex $S^2$. Then, the following statements are true:

(i) $Fix(W_1) = \{e_1, e_3, (3 - \sqrt{3}/4, \sqrt{3}/4, 1/4)\}$,

(ii) $\omega_{W_1}(x_1^{(0)}) = \{(3 - \sqrt{3}/4, \sqrt{3}/4, 1/4)\}$.

**Proof.** Let $W_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (10), $x_1^{(0)} \notin Fix(W_1)$ be any initial point in simplex $S^2$, and $\{W_1^{(n)}\}_{n=1}^{\infty}$ be a trajectory of $W_1$ starting from point $x_1^{(0)}$.

(i) The set of fixed points of $W_1$ are obtained by finding the solution for the following system of equations:

$$\begin{cases} x = x^2 + 2yz \\ y = 2x(1 - x) \\ z = y^2 + z^2 \end{cases} \quad (12)$$

By depending on the first equation in system (12), we derive $x - x^2 = 2yz$. Subsequently, the last equation is multiplied by 2, and the new equation is substituted into the second equation in system (12). We obtain $y(1 - 4z) = 0$ and find $y = 0$ or $z = \frac{1}{4}$. If $y = 0$, then $x = 0$ or $x = 1$ can be easily found; hence, the fixed points are $e_1 = (1, 0, 0)$ and $e_3 = (0, 0, 1)$. If $z = \frac{1}{4}$, then $y = \frac{\sqrt{3}}{4}$ and $x = \frac{3 - \sqrt{3}}{4}$ can be found by using the first and third equation in system (12). Therefore, the fixed point is $\left(\frac{3 - \sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{1}{4}\right)$. 


(ii) To investigate the dynamics of $W_1$, the following regions are introduced:

$$
A_1 : = \{ x_1^{(0)} \in S^2 : 0 \leq x^{(0)}, y^{(0)}, z^{(0)} \leq \frac{1}{2}, \}
$$
$$
A_2 : = \{ x_1^{(0)} \in S^2 : \frac{1}{2} < x^{(0)} < 1, \}
$$
$$
A_3 : = \{ x_1^{(0)} \in S^2 : \frac{1}{2} < y^{(0)} < 1, \}
$$
$$
A_4 : = \{ x_1^{(0)} \in S^2 : \frac{1}{2} < z^{(0)} < 1, \}
$$
$$
A_5 : = \{ x_1^{(0)} \in S^2 : 0 < z^{(0)} \leq x^{(0)} < y^{(0)} < \frac{1}{2}, \}
$$
$$
A_6 : = \{ x_1^{(0)} \in S^2 : 0 \leq y^{(0)} \leq z^{(0)} \leq x^{(0)} \leq \frac{1}{2}, \}
$$
$$
A_7 : = \{ x_1^{(0)} \in S^2 : 0 \leq x^{(0)} \leq y^{(0)} \leq z^{(0)} \leq \frac{1}{2}, \}
$$
$$
A_8 : = \{ x_1^{(0)} \in S^2 : 0 < z^{(0)} \leq x^{(0)} \leq \frac{1}{3}, \quad \frac{1}{3} < y^{(0)} \leq \frac{1}{2}. \}
$$

Subsequently, the behavior of $W_1$ across all the aforementioned regions is explored. Then, the behavior of $W_1$ will be described. To achieve this objective, the following results should be shown:

1. Let $x_1^{(0)} \in A_1$. Then, $0 \leq x^{(0)}, y^{(0)}, z^{(0)} \leq \frac{1}{2}$. Evidently $-1 \leq 3x^{(0)} - 1 \leq \frac{1}{2}$ by squaring and adding $-3(y^{(0)} - z^{(0)})^2$. The last inequality becomes $0 \leq (3x^{(0)} - 1)^2 - 3(y^{(0)} - z^{(0)})^2 \leq 1$, and $9(x^{(0)})^2 - 6x^{(0)} + 1 - 3(y^{(0)} - z^{(0)})^2 \leq 1$ is obtained. Dividing the previous inequality by three after adding two to both parts of the inequality will derive

$$
3(x^{(0)})^2 - 2x^{(0)} + 1 - (y^{(0)} - z^{(0)})^2 \leq 1.
$$

Therefore,

$$
2(x^{(0)})^2 + (y^{(0)})^2 + (z^{(0)})^2 - (y^{(0)} - z^{(0)})^2 \leq 1.
$$

Then, $2(x^{(0)})^2 + 4y^{(0)}z^{(0)} \leq 1$, which implies that $x' \leq \frac{1}{2}$. To show that $y' \leq \frac{1}{2}$, one can check that $y' \leq \frac{1}{2}$ for all $x^{(0)}$. Evidently see that $0 \leq (y^{(0)})^2, (z^{(0)})^2 \leq \frac{1}{4}$, which implies that $z' \leq \frac{1}{2}$. Hence, $A_1$ is an invariant region.

2. The second coordinate of $W_1$ is less than $\frac{1}{2}$ at any initial point $x_1^{(0)}$, thereby indicating that $A_3$ is not an invariant region. Then, we intend to show that $A_2$ is also not an invariant region. To achieve this objective, we suppose that $A_2$ is an invariant region, which indicates that $y' \leq x'$ and $z' \leq x'$. However,

$$
x' = (x^{(0)})^2 + 2y^{(0)}z^{(0)} \leq (x^{(0)})^2 + (y^{(0)})^2 + (z^{(0)})^2 \leq x^{(0)}(x^{(0)} + y^{(0)} + z^{(0)}) = x^{(0)}.
$$

Then $\frac{x'}{x^{(0)}} < 1$, which implies that the first coordinate is a decreasing bounded sequence that converges to zero, thereby contradicting our assumption. Hence, if $x_1^{(0)} \in A_2 \cup A_3$, then $n_{k_1}, n_{k_2} \in \mathbb{N}$, such that the sequences $x^{(n_{k_1})}$ and $y^{(n_{k_2})}$ tend toward the invariant region $A_1$.

3. Thereafter, we intend to show that if $x_1^{(0)} \in A_4$, then $n_k \in \mathbb{N}$, such that the sequence $z^{(n_k)}$ returns to region $A_1$. To achieve this objective, $A_4$ is supposed as an invariant region;
hence, \( z' \geq y' + x' \) and \( x', y' \leq \frac{1}{2} \). Evidently \( x' \leq y' \). By using the last inequality and the first coordinate of \( W_1 \), we obtain \( y' \geq 2y(0)z(0) \). That is, \( z' \leq \frac{1}{2} \), which repudiates our assumption. Hence, region \( A_4 \) is not invariant.

(4) Given that \( y', x' \leq \frac{1}{2} \), we can easily conclude \( x' \leq y' \) thereby indicating \( A_6 \) is impossible to be invariant region. Subsequently, we intend to verify whether \( A_7 \) is an invariant region. Let \( x_1(0) \in A_7 \). Then,

\[
z' = (z(0))^2 + (y(0))^2 \leq (x(0))^2 + (y(0))^2 + (z(0))^2 \leq x(0)z(0) + y(0)z(0) + (z(0))^2 = z(0)(x(0) + y(0) + z(0)) = z(0).
\]

We determine that \( \frac{x'}{z(0)} < 1 \), which indicates that \( z(n) \) is a decreasing bounded sequence, i.e., \( z(n) \) converges to the fixed point zero, thereby negating our presumption. Thus, region \( A_7 \) is not invariant. Then, we consider a new sequence \( x' + z' = 2(x(0))^2 - 2x(0) + 1 \). The new sequence has a minimum value of \( \frac{1}{2} \), which indicates that all coordinates are greater than zero and less than \( \frac{1}{2} \). Hence, if \( x_1(0) \in A_6 \cup A_7 \cup A_1 \), then \( n_{k_1}, n_{k_2}, n_{k_3} \in \mathbb{N} \), such that the sequences \( x(n_{k_1}), y(n_{k_2}), \) and \( z(n_{k_3}) \) return to invariant region \( A_5 \).

(5) Let \( x(0) \leq \frac{1}{3} \). Whether the maximum value of the first coordinate \( x' = (x(0))^2 + 2y(0)(1 - x(0) - y(0)) \) occurs when \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\) can be easily checked. Thus, \( x(n) \leq \frac{1}{3} \) and \( z(n) \leq \frac{1}{3} \). Given that all coordinates are equal to one, we conclude that \( y(n) \geq \frac{1}{3} \). Therefore, if \( x_1(0) \in A_5 \), then \( n_k \in \mathbb{N} \), such that \( W_1(n_k) \) returns to \( A_8 \). Hence, \( A_8 \) is an invariant region.

We have proven that if \( x_1(0) \in A_i, i \in \{1, \ldots, 7\} \), then the trajectory \( \{W_1(n)\}_{n=1}^\infty \) goes to invariant region \( A_8 \). Thus, exploring the dynamics of \( W_1 \) over region \( A_8 \) is adequate. Evidently, \( y(n) \) is a bounded increasing sequence. Given that \( y(n) + x(n) \) is a bounded decreasing sequence and \( x(n) = y(n) - y(n) + x(n) \), we conclude that \( x(n) \) is a decreasing bounded sequence that converges to \( \frac{3 - \sqrt{3}}{4} \). Thus, we have \( y(n) \) converging to \( \frac{\sqrt{3}}{4} \). Therefore,

\[
\omega_{W_1}(x_1(0)) = \left\{ \left( \frac{3 - \sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{1}{17} \right) \right\},
\]

which is the desired conclusion. \( \square \)

**Theorem 3** Let \( W_2 : S^2 \to S^2 \) be a \( \xi^{(as)} \)-QSO given by \( \{1\} \) and \( x_1(0) = (x(0), y(0), z(0)) \notin \text{Fix}(W_2) \) be any initial point in simplex \( S^2 \). Then, the following statements are true:

(i) \( \text{Fix}(W_2) = \{e_3, (x', y', z')\} \),

where \( x' = -\frac{1}{6} \sqrt{7} - \frac{8}{3} \sqrt{\frac{1}{4}} \), \( y' = -\frac{1}{6} \sqrt{7} - \frac{8}{3} \sqrt{\frac{1}{4}} + 2 \sqrt{7} - 21 \sqrt{7} - 5 \sqrt{7} - 8 \), \( z' = -\frac{1}{6} \sqrt{7} - \frac{8}{3} \sqrt{\frac{1}{4}} + 2 \sqrt{7} - 11 \sqrt{7} - 17 \sqrt{7} + 6 \sqrt{7} - 10 \),

and \( t = (98 + 18 \sqrt{17}) \).
Therefore, solutions for the following equation:

\[ x^2 - y = 0 \]

First, we start when \( y \) the following solution:

\[ (x, y, 0, 1 - y^o), \text{if } y(0) = 0 \]

Evidently, the second coordinate of \( W \) iteration of \( \omega \) points in set \( S \)

\[ \omega_\omega(x_1(0)) = \begin{cases} (x', y', z') & \text{if } x_1(0) \in \text{int} (S^2) \\ (x^o, 0, 1 - x^o), (0, y^o, 1 - y^o) & \text{if } x_1(0) \in \text{int} (S^2) \\ e_3 & \text{if } x(0), y(0) = 1 \end{cases} \]

\[ \omega_{\omega_\omega}(x_1(0)) = \begin{cases} (x', y', z') & \text{if } x_1(0) \in \text{int} (S^2) \\ (x^o, 0, 1 - x^o), (0, y^o, 1 - y^o) & \text{if } x_1(0) \in \text{int} (S^2) \\ e_3 & \text{if } x(0), y(0) = 1 \end{cases} \]

**Proof.** Let \( W_2 : S^2 \to S^2 \) be a \( \xi^{(a)} \)-QSO given by (11), \( x_1(0) \notin \text{Fix}(W_2) \) be any initial point in \( S^2 \), and \( \{ W_2^{(n)} \}_{n=1}^\infty \) be a trajectory of \( W_2 \) starting from point \( x_1(0) \).

(i) The set of fixed points of \( W_2 \) is obtained by finding the solution for the following system of equations:

\[ \begin{align*}
  x &= 2yz \\
  y &= 2x - x^2 \\
  z &= z^2 + y^2
\end{align*} \] (15)

On the basis of the first equation in system (15), we have \( z = \frac{x}{2y} \). By using \( z = 1 - y - x \) and the second equation in system (15), we obtain \( 3x - 14x^2 + 10x^3 - 2x^4 = 0 \). Thus, the roots of the previous equation are \( \{0, x'\} \). By compensating for the values of \( x \), namely, \( x = 0 \) and \( x = x' \) in the second equation in system (15), we obtain \( y = 0 \) and \( z = 1 \) or \( y = y' \) and \( z = z' \). Therefore, the fixed points of \( W_2 \) are \( e_3 = (0, 0, 1) \) and \( (x', y', z') \).

(ii) To find \( 2 \)-periodic points of \( W_2 \), we should prove that \( W_2 \) has no any order periodic points in set \( S^2 \setminus L_1 \cup L_2 \), where \( L_1 = \{ x_1(0) \in S^2 : x(0) = 0 \} \) and \( L_2 = \{ x_1(0) \in S^2 : y(0) = 0 \} \). Evidently, the second coordinate of \( W_2 \) increases along the iteration of \( W_2 \) in set \( S^2 \setminus L_2 \). Consider a new sequence \( x' + y' = 2x(0) - (x(0))^2 + 2y(0)(1 - x(0) - y(0)) \). Whether \( x' + y' \) is a decreasing sequence can be easily checked, thereby indicating that sequence \( x^{(n)} \) is decreasing because \( x^{(n)} = y^{(n)} - y^{(n)} + x^{(n)} \). Thus, the first coordinate of \( W_2 \) decreases along the iteration of \( W_2 \) in set \( S^2 \setminus L_1 \), which indicates that \( W_2 \) has no any order \( 2 \)-periodic points in set \( S^2 \setminus L_1 \cup L_2 \). Therefore, finding \( 2 \)-periodic points of \( W_2 \) in \( L_1 \cup L_2 \) is sufficient. To find \( 2 \)-periodic points, the succeeding system of equations should be solved:

\[ \begin{align*}
  x &= 2(x - x^2)(y^2 + z^2) \\
  y &= 4y + 4y^2z^2 \\
  z &= (2x - x^2)(y^2 + z^2)^2
\end{align*} \] (16)

First, we start when \( x = 0 \). Then, we find the solution for \( y = 4y - 8y^2 + 8y^3 - 4y^4 \). We obtain the following solution: \( y = 0 \) or \( y = y^o \). If \( y = 0 \), then \( z = 1 \). If \( y = y^o \), then \( z = 1 - y^o \). Therefore, \( e_3 \) and \( (0, y^o, 1 - y^o) \) are \( 2 \)-periodic points. On the other hand, if \( y = 0 \), then the solutions for the following equation: \( x = 2(2 - x - x^2)(1 - x)^2 \) are \( x = 0 \) or \( x = x^o \). If \( x = 0 \),
(iii) To investigate the dynamics of $W_2$, the following regions are introduced:

\[
\ell_1 : = \{ x_1^{(0)} \in \text{int} \left( S^2 \right) : 0 < x_1^{(0)}, y_1^{(0)}, z_1^{(0)} \leq \frac{1}{2} \};
\]

\[
\ell_2 : = \{ x_1^{(0)} \in \text{int} \left( S^2 \right) : 0 < x_1^{(0)} \leq z_1^{(0)} \leq y_1^{(0)} \leq \frac{1}{2} \}.
\]

Let $x_1^{(0)} \notin \text{Fix}(W_2) \cup \text{Per}_2(W_2)$ and $x_1^{(0)} \in \text{int} \left( S^2 \right)$ be the initial points where $\text{int} \left( S^2 \right) = \{ x_1^{(0)} \in S^2 : x_1^{(0)}y_1^{(0)}z_1^{(0)} > 0 \}$. Evidently, $y' = 2x_1^{(0)} - (x_1^{(0)})^2 \geq x_1^{(0)}$ and $x' = 2y_1^{(0)}z_1^{(0)} \leq (y_1^{(0)})^2 + (z_1^{(0)})^2 = z'$, which indicates that $x_1^{(n)} \leq z_1^{(n)}$ and $x_1^{(n)} \leq y_1^{(n)}$. Subsequently, we are going to prove that $\ell_1$ is an invariant region. To achieve this objective, we start with $y_1^{(n)}$. Suppose that $y_1' \geq \frac{1}{2}$ by using the first coordinate of $W_2$. Then we have $x_1' = 2y_1^{(0)}z_1^{(0)}$, which implies that $x_1' \geq z'$. This relation is a contradiction because $x_1' \leq z'$. Thus, $y_1^{(n)} \leq \frac{1}{2}$. By performing the same process used to prove $y_1^{(n)} \leq \frac{1}{2}$, we prove that $z' \leq \frac{1}{2}$. Suppose that $z' \geq \frac{1}{2}$. By using the first coordinate in $W_2$, we obtain $x_1' = y_1'$, which is another contradiction. Therefore, $\ell_1$ is an invariant region. Moreover, if $x_1^{(0)} \notin \text{Fix}(W_2) \cup \text{Per}_2(W_2)$, $x_1^{(0)} \in \text{int} \left( S^2 \right)$, and $x_1^{(0)} \in \mathfrak{I}_1$, then $n_{k} \in \mathbb{N}$, such that $W_2^{n_{k}}$ returns to invariant region $\ell_1$. Let us complete proving that $\ell_2$ is an invariant region. To achieve this objective, suppose that $y_1' \leq z'$, which indicates that $z' = (z_1^{(0)})^2 + (y_1^{(0)})^2 \leq 2(z_1^{(0)})^2$. Then, $z_1' = \frac{y_1^{(n)} \leq 1$. Therefore, $z_1^{(n)}$ is a decreasing bounded sequence. That is $z_1^{(n)}$ converges to the fixed point zero. Moreover, $y_1^{(n)}$ is an increasing bounded sequence. Thus, $y_1^{(n)}$ converges to zero. Whether $y_1^{(n)}$ converges to zero if $x_1^{(n)}$ converges to zero can be checked. The result implies that the limiting point for $W_2$ is empty, which is a contradiction. Thus, $n_{k} \in \mathbb{N}$, such that $z_1^{(n_{k})}$ returns to invariant region $z' \leq y_1'$, which proves that $\ell_2$ is an invariant region. Moreover, if $x_1^{(0)} \in \ell_1$, then $n_{k} \in \mathbb{N}$, such that $W_2^{n_{k}}$ returns to invariant region $\ell_2$.

Accordingly, the behavior of $W_2$ can be described. As discussed in proof part 2 of this theorem, we determine that the first and second coordinates, namely, $x_1^{(n)}$ and $y_1^{(n)}$, are decreasing and increasing sequences respectively. Thus, $x_1^{(n)}$ and $y_1^{(n)}$ converge to certain fixed point. The first and second coordinates of $W_2$ are converging; thus, the third coordinate also converges. Between the two fixed points, the aforementioned properties of $W_2$ are only satisfied by point $(x', y', z')$. Therefore, the limiting point is $\omega_{W_2}(x_1^{(0)}) = (x', y', z') \forall x_1^{(0)} \in \text{int} \left( S^2 \right)$.
when \( z^{(0)} = 0 \), which indicates that \( n_k \in \mathbb{N} \), such that the behavior of \( W_2 \) when \( z^{(0)} = 0 \) case will be on the \( yz \)-plane if \( n \) is an odd iteration and on the \( xz \)-plane if \( n \) is an even iteration. Therefore, studying two cases when \( x^{(0)} = 0 \) and \( y^{(0)} = 0 \) are sufficient. Starting with \( x^{(0)} = 0 \), consider the following function:

\[
y^{(2)} = \nu(y^{(0)}) = 4y^{(0)} - 8(y^{(0)})^2 + 8(y^{(0)})^3 - 4(y^{(0)})^4,
\]

where \( y^{(0)} \in (0, 1) \). \( \text{Fix}(\nu) \cap (0, 1) = \{ y^o \} \) can be shown. Through simple calculations, \( \nu\left((0, \frac{1}{2})\right) \subseteq \left[\frac{1}{2}, 1\right) \) can be found. Thus, we conclude that \( [\frac{1}{2}, 1) \) is sufficient to study the dynamics of \( \nu \) at interval \( (0, 1) \).

To study the behavior of \( \nu \), interval \( [\frac{1}{2}, 1) \) is divided into three intervals as follows: \( I_1 = \left[\frac{1}{2}, y^o\right] \), \( I_2 = \left[y^o, \frac{1}{2} + \frac{1}{4}\sqrt{2 - 1}\right] \), and \( I_3 = \left[\frac{1}{2} + \frac{1}{4}\sqrt{2 - 1}, 1\right) \). Evidently, \( \nu(\nu(y^{(0)})) \geq y^{(0)} \) when \( y^{(0)} \in I_1 \) and \( \nu(\nu(y^{(0)})) \leq y^{(0)} \) when \( y^{(0)} \in I_2 \). Therefore, two cases should be discussed separately.

(a) For any \( n \in \mathbb{N} \), \( \nu^{(2n+2)}(y^{(0)}) \geq \nu^{(2n)}(y^{(0)}) \) \( \forall y^{(0)} \in I_1 \) can be easily shown. Thus, \( \nu^{(2n)}(y^{(0)}) \) is an increasing bounded sequence. Furthermore, \( \nu^{(2n)}(y^{(0)}) \) converges to a fixed point of \( \nu^{(2)} \). \( y^o \) is also a fixed point of \( \nu^{(2)} \), and it is the only possible point of the convergence trajectory. Hence, sequence \( y^{(2n)} \) converges to \( y^o \).

(b) Similarly, \( \nu^{(2n+2)}(y^{(0)}) \leq \nu^{(2n)}(y^{(0)}) \) \( \forall y^{(0)} \in I_2 \). Thus, \( \nu^{(2n)}(y^{(0)}) \) a decreasing bounded sequence. Furthermore, \( \nu^{(2n)}(y^{(0)}) \) converges to a fixed point of \( \nu^{(2)} \). \( y^o \) is also a fixed point of \( \nu^{(2)} \), and it is the only possible point of the convergence trajectory. Hence, sequence \( y^{(2n)} \) converges to \( y^o \).

To explore the behavior of \( \nu \), when \( y^{(0)} \in I_3 \), the following claim is required:

**Claim 1** Let \( y^{(0)} \in I_3 \). Then, \( n_k \in \mathbb{N} \), such that \( \nu^{(n_k)} \in I_1 \cup I_2 \).

**Proof.** Let \( y^{(0)} \in I_3 \). Suppose that the interval \( I_3 \) is an invariant interval, which indicates that \( y^{(n)} \in I_3 \) for any \( n \in \mathbb{N} \). Evidently, \( \nu^{(n+1)}(y^{(0)}) \leq \nu^{(n)}(y^{(0)}) \), which results in \( \nu^{(n)} \) being a decreasing bounded sequence and converging to a fixed point of \( \nu \). However, \( \text{Fix}(\nu) \cap I_3 = \emptyset \), which is a contradiction. Hence, \( n_k \in \mathbb{N} \), such that \( \nu^{(n_k)} \in I_1 \cup I_2 \). \( \square \)

In accordance with the claim, \( y^{(2n)} \) will go to \( I_1 \cup I_2 \) after several iterations. Thus, sequence \( (0, y^{(2n)}, x^{(2n)}) \) converges to \( (0, y^o, 1 - y^o) \) whenever \( x^{(0)} = 0 \).

Let \( y^{(0)} = 0 \) and consider the following function:

\[
x^{(2)} = \vartheta(x^{(0)}) = 4x^{(0)} - 10(x^{(0)})^2 + 8(x^{(0)})^3 - 2(x^{(0)})^4,
\]

where \( x^{(0)} \in (0, 1) \). \( \text{Fix}(\vartheta) \cap (0, 1) = \{ x^o \} \) can be easily shown. Through simple calculations, we determine \( \vartheta\left([0, 1 - \frac{1}{4}\sqrt{2}]\right) \subseteq [1 - \frac{1}{4}\sqrt{2}, 1) \) and conclude that \( [1 - \frac{1}{4}\sqrt{2}, 1) \) is sufficient to study the dynamics of \( \vartheta \) on \( (0, 1) \).

To study the behavior of \( \vartheta \), invariant interval \( [1 - \frac{1}{4}\sqrt{2}, 1) \) is divided into three intervals as follows: \( I_1 = \left[1 - \frac{1}{4}\sqrt{2}, x^o\right], I_2 = \left[x^o, 1\right], \) and \( I_3 = \left[\frac{1}{2}, 1\right) \). Thus, we have two separate cases:
(a) Let \( x(0) \in I_1 \), then \( \vartheta(x(0)) \in I_2 \) and, \( \vartheta(2)(x(0)) \in I_1 \). \( \vartheta(2n+2)(x(0)) \leq \vartheta(2n)(x(0)) \) whenever \( x(0) \in I_1 \) can be easily checked. Therefore, \( \vartheta(2n) \) is a decreasing bounded sequence that converges to a fixed point of \( \vartheta(2) \). \( x^o \) is a fixed point of \( \vartheta(2) \) and the only possible point of the convergence trajectory. Hence, \( \vartheta(2) \) converges to \( x^o \).

(b) Similarly, let \( x(0) \in I_2 \), then \( \vartheta(x(0)) \in I_1 \) and \( \vartheta(2)(x(0)) \in I_2 \). \( \vartheta(2n+2)(x(0)) \geq \vartheta(2n)(x(0)) \) whenever \( x(0) \in I_2 \) can be easily checked. Therefore, \( \vartheta(2n) \) is an increasing bounded sequence that converges to a fixed point of \( \vartheta(2) \). \( x^o \) is a fixed point of \( \vartheta(2) \) and the only possible point of the convergence trajectory. Hence, \( \vartheta(2n) \) converges to \( x^o \).

To explore the behavior of \( \vartheta \), when \( x(0) \in I_3 \), the following claim is required:

**Claim 2** Let \( x(0) \in I_3 \). Then, \( n_k \in \mathbb{N} \), such that \( \vartheta(n_k) \in I_1 \cup I_2 \).

**Proof.** Let \( x(0) \in I_3 \). Suppose that interval \( I_3 \) is invariant, which indicates that \( x(n) \in I_3 \) for any \( n \in \mathbb{N} \). Evidently, \( \vartheta(n+1)(x(0)) \leq \vartheta(n)(x(0)) \), which results in sequence \( \vartheta(n) \) being a decreasing bounded and converging to a fixed point of \( \vartheta \). However, \( \text{Fix}(\vartheta) \cap I_3 = \emptyset \), which is contradiction. Hence, \( n_k \in \mathbb{N} \), such that \( \vartheta(n_k) \in I_1 \cup I_2 \).

accordance with the claim, \( x(n) \) will go to \( I_1 \cup I_2 \) after several iterations. Thus, sequence \( (x(2n), 0, z(2n)) \) converges to \( (x^o, 0, 1 - x^o) \) whenever \( y(0) = 0 \). In another way, if \( x(0) = 0 \), then

\[
V(n)(W_2) = \begin{cases} 
(0, y^o, 1 - y^o) & , if \ n = 2k \\
(0, 0, 1 - x^o) & , if \ n = 2k + 1 
\end{cases}
\]

and if \( y(0) = 0 \), then

\[
V(n)(W_2) = \begin{cases} 
(0, y^o, 1 - y^o) & , if \ n = 2k + 1 \\
(0, 0, 1 - x^o) & , if \ n = 2k 
\end{cases}
\]

From the preceding, we observe that if \( x(0) = 0 \) and \( n \) is an even, then the behavior of \( W_{2(2n)} \) occurs in \( (0, y^o, 1 - y^o) \), which is equal to the behavior of \( W_2 \) when \( y(0) = 0 \) and \( n \) is an odd iteration. If \( y(0) = 0 \) and \( n \) is an even iteration, then the behavior of \( W_2 \) occurs in \( (x^o, 0, 1 - x^o) \), which is equal to the behavior of \( W_2 \) when \( x(0) = 0 \) and \( n \) is an odd iteration. Therefore, the limiting point of \( W_2 \) consists of \( (x^o, 0, 1 - x^o) \) and \( (0, y^o, 1 - y^o) \) whenever \( x(0) \notin \text{int}(S^2) \). If \( x(0) = 1 \), then the behavior of \( W_2 \) reaches fixed point \( e_3 \) after three iterations; if \( y(0) = 1 \), then the behavior of \( W_2 \) reaches fixed point \( e_3 \) after one iteration. Therefore, the limiting point in this case includes \( e_3 \), which is the desired conclusion.

Subsequently, the behavior of operator \( V_{15} \) selected from class \( G_9 \) is explored:

\[
V_{15} := \begin{cases} 
x' = (1 - \alpha)(x(0))^2 + 2y(0)z(0) \\
y' = \alpha(x(0))^2 + 2y(0)(1 - x(0)) \\
z' = (z(0))^2 + (y(0))^2 
\end{cases}
\]

(21)

The operator \( V_{15} \) can be redrafted as a convex combination \( V_{15} = (1 - \alpha) W_1 + \alpha W_2 \), where
Let $W_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by \( (22) \), and $x_1^0 = (x(0), y(0), z(0)) \notin \text{Fix}(W_1)$ be any initial point in simplex $S^2$. Then, the following statements are true:

(i) $\text{Fix}(W_1) = \{ e_1, e_3, \frac{3 - \sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{1}{4} \}$

(ii) $\omega_{w_1}(x_1^0) = \{ \frac{3 - \sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{1}{4} \}$

For $W_2$ let $W_2 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by \( (23) \) and $x_1^0 = (x(0), y(0), z(0)) \notin \text{Fix}(W_2)$ be any initial point in simplex $S^2$. Then, the following statements are true:

(i) $\text{Fix}(W_2) = \{ e_3, (x', y', z') \}$, \[ x' = \frac{-1}{6} \sqrt{1 - \frac{6}{3 \sqrt{11}}} \frac{1}{3(1 + \sqrt{3})^3}, y' = \frac{-1}{6(1 + 2 \sqrt{3})^2} + \frac{1}{3(1 + 3 \sqrt{3})} \frac{1}{3}, \quad z' = \frac{-1}{6(46 + 6 \sqrt{71})^3} - \frac{2}{3(46 + 6 \sqrt{71})^3} + \frac{1}{36} \]

and $t = (98 + 18 \sqrt{11})$.

(ii) $\text{Per}_2(W_2) = \{ e_3, (x^0, 0, 1 - x^0) , \text{if} \quad x(0) = 0 \}$

\[ e_3, (x^0, 0, 1 - x^0) , \text{if} \quad y(0) = 0 \]

where, $y^0 = \frac{1}{6} \left( 1 + 3 \sqrt{3} \right)^{\frac{1}{3}} - \frac{4}{3(1 + 3 \sqrt{3})^3} + \frac{1}{3(1 + 3 \sqrt{3})^3} + \frac{1}{3}$.

\[ (x^0, 0, 1 - x^0), (0, y^0, 1 - y^0) , \text{if} \quad x_1^0 \notin \text{int} \ (S^2) \]

\[ e_3, \quad \text{if} \quad x(0), y(0) = 1 \]

5 Dynamics of classes $G_{13}$ and $G_{14}$

In this section, we study the dynamics of $V_{26,25} : S^2 \to S^2$ selected from $G_{14}$ and $G_{13}$. To start, $V_{26}$ is rewritten as follows:

\[ V_{26} := \begin{cases} x' = (y(0))^2 + (z(0))^2 + 2x(0) (1 - x(0)) \\ y' = \alpha (x(0))^2 + 2y(0) z(0) \\ z' = (1 - \alpha) (x(0))^2 \end{cases} \]
The operator $V_{26}$ can be redrafted as a convex combination $V_{26} = \alpha W_1 + (1 - \alpha) W_2$, where

$$W_1 := \begin{cases} x' = (y(0))^2 + (z(0))^2 + 2x(0) (1 - x(0)) \\ y' = (x(0))^2 + 2y(0)z(0) \\ z' = 0 \end{cases} \quad (27)$$

and

$$W_2 := \begin{cases} x' = (y(0))^2 + (z(0))^2 + 2x(0) (1 - x(0)) \\ y' = 2y(0)z(0) \\ z' = (x(0))^2 \end{cases} \quad (28)$$

**Theorem 4** Let $W_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (27) and $x_1^{(0)} = (x(0), y(0), z(0)) \notin \text{Fix}(W_1) \cup \text{Per}_2(W_1)$ be any initial point in simplex $S^2$. Then, the following statements are true:

(i) $\text{Fix}(W_1) = \left\{ \left( \frac{\sqrt{5}}{2} - \frac{1}{2}, \frac{3}{2} - \frac{\sqrt{5}}{2}, 0 \right) \right\}$.

(ii) $\text{Per}_2(W_1) = \left\{ e_1, e_2, \left( \frac{\sqrt{5}}{2} - \frac{1}{2}, \frac{-1+\sqrt{5}}{4}, 0 \right) \right\}$.

(iii) $\omega_{W_1}(x_1^{(0)}) = \{ e_1, e_3 \}$.

**Proof.** Let $W_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (27), $x_1^{(0)} \notin \text{Fix}(W_1) \cup \text{Per}_2(W_1)$ be any an initial point in simplex $S^2$, and $\left\{ W_1^{(n)} \right\}_{n=1}^{\infty}$ be a trajectory of $W_1$ starting from point $x_1^{(0)}$.

1. The set of the fixed points of $W_1$ are obtained by finding the solution for the following system of equations:

$$\begin{cases} x = y^2 + z^2 + 2x(1 - x) \\ y = x^2 + 2yz \\ z = 0 \end{cases} \quad (29)$$

By substituting the second and third equations (29) to the first equation, then the first equation in system (29) becomes $x^4 - 2x^2 + x$, then $x = 0$, $x = 1$, and $x = \frac{\sqrt{5}}{2} - \frac{1}{2}, \frac{\sqrt{5}}{2} - \frac{1}{2}$ is verified as the only solution that satisfies system (29). Hence, the fixed point is only \( \left( \frac{\sqrt{5}}{2} - \frac{1}{2}, \frac{3}{2} - \frac{\sqrt{5}}{2}, 0 \right) \).

2. Let $x_1^{(0)} = (1, 0, 0)$ be the initial point. $V^{(1)} \left( x^0, y^0, z^0 \right) = (0, 1, 0)$ and $V^{(2)} \left( x^0, y^0, z^0 \right) = (1, 0, 0)$, which indicates the presence of 2-periodic points. To find all the points, the following system of equations should be solved:

$$\begin{cases} x = 2x^2 - x^4 \\ y = (1 - (1 - y)^2)^2 \\ z = 0 \end{cases} \quad (30)$$

From the first equation in system (30), $x \in \left\{ 0, 1, \frac{\sqrt{5}}{2} - \frac{1}{2} \right\}$, then $y \in \left\{ 1, 0, \frac{-1+\sqrt{5}}{4} \right\}$. Therefore, $\text{Per}_2(W_1) = \left\{ e_1 = (1, 0, 0), e_2 = (0, 1, 0), (\frac{\sqrt{5}}{2} - \frac{1}{2}, \frac{-1+\sqrt{5}}{4}, 0) \right\}$. 

Let \( x_1^{(0)} \notin \text{Fix}(W_1) \cup \text{Per}_2(W_1) \). \( L_3 \) is an invariant line under \( W_1 \) where \( L_3 = \{ x_1^{(0)} \in S^2 : z^{(0)} = 0 \} \). Thus, the behavior of \( W_1 \) is explored over this line. Let \( x_1^{(0)} \in L_3 \). Then, \( W_1 \) becomes:

\[
\begin{align*}
    x' &= (y^{(0)})^2 + 2x^{(0)}(1 - x^{(0)}) \\
    y' &= (x^{(0)})^2 \\
    z' &= 0
\end{align*}
\]

(31)

In this case, the first coordinate of \( W_1 \) exhibits the form \( x' = \varphi(x^{(0)}) = (1 - x^{(0)})^2 + 2x^{(0)}(1 - x^{(0)}) \). Clearly, the function \( \varphi \) is decreasing on \([0, 1]\) and the function \( \varphi^{(2)} \) is increasing on \([0, 1]\). From the previous two steps, \( \text{Fix}(\varphi) \cap [0, 1] = \left\{ \frac{\omega}{2}, \frac{1}{2} \right\} \) and \( \text{Fix}(\varphi^{(2)}) \cap [0, 1] = \left\{ 0, \frac{\sqrt{5} - 1}{2}, 1 \right\} \), which indicate that intervals \([0, \frac{\sqrt{5} - 1}{2}] \) and \([\frac{\sqrt{5} - 1}{2}, 1]\) are invariant under the function \( \varphi^{(2)} \). Evidently, \( \varphi^{(2)}(x^{(0)}) \leq x^{(0)} \) for any \( x^{(0)} \in [0, \frac{\sqrt{5} - 1}{2}] \) and \( \varphi^{(2)}(x^{(0)}) \geq x^{(0)} \) for any \( x^{(0)} \in [\frac{\sqrt{5} - 1}{2}, 1] \). If \( x^{(0)} \in [0, \frac{\sqrt{5} - 1}{2}] \), then \( \omega_{\varphi^{(2)}} = \{ 0 \} \); if \( x^{(0)} \in [\frac{\sqrt{5} - 1}{2}, 1] \), then \( \omega_{\varphi^{(2)}} = \{ 1 \} \). In another way,

\[
V_n(W_1) = \begin{cases} 
    (\varphi^{(2k)}(x^{(0)}), 1 - \varphi^{(2k)}(x^{(0)}), 0) & \text{if } n = 2k \\
    (\varphi^{(2k)}(\varphi(x^{(0)})), 1 - \varphi^{(2k)}(\varphi(x^{(0)})), 0) & \text{if } n = 2k + 1
\end{cases}
\]

(32)

Therefore, the limiting point is \( \omega_{W_1}(x_1^{(0)}) = \{ e_1, e_2 \} \). 

\[ \square \]

**Theorem 5** Let \( W_2 : S^2 \to S^2 \) be a \( \xi^{(as)} \)-QSO given by \( (28) \) and \( x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin \text{Fix}(W_2) \cup \text{Per}_2(W_2) \) be any initial point in simplex \( S^2 \). Then, the following statements are true:

(i) \( \text{Fix}(W_2) = \emptyset \). Moreover, \( \text{Per}_2(W_2) = \{ e_1, e_3, (\frac{\sqrt{5} - 1}{2}, 0, \frac{1}{16}(\sqrt{5} - 3)^2(\sqrt{5} + 1)^2) \} \).

(ii) \( \omega_{W_2}(x_1^{(0)}) = \{ e_1, e_3 \} \).

**Proof.** Let \( W_2 : S^2 \to S^2 \) be a \( \xi^{(as)} \)-QSO given by \( (28) \), \( x_1^{(0)} \notin \text{Fix}(W_2) \cup \text{Per}_2(W_2) \) be any initial point in simplex \( S^2 \), and \( \{ W_2^{(n)} \}_{n=1}^{\infty} \) be a trajectory of \( W_2 \) starting from point \( x_1^{(0)} \).

(1) The set of fixed points of \( W_2 \) are obtained by finding the solution for the following system of equations:

\[
\begin{align*}
    x &= y^2 + z^2 + 2x(1 - x) \\
    y &= 2yz \\
    z &= x^2
\end{align*}
\]

(33)

The system provided by \( (33) \) has no solution on \([0, 1]\). Therefore, the set of fixed points is \( \emptyset \). The second coordinate of \( W_2 \) increases if \( z^{(n)} \geq \frac{1}{2} \) and decreases if \( z^{(n)} \leq \frac{1}{2} \). In both cases, \( W_2 \) has no any order periodic points in set \( W_2 \setminus L_2 \) because the second coordinate of \( W_2 \) increases or decreases along the iteration of \( W_2 \setminus L_2 \). Therefore, finding 2–periodic points of \( W_2 \) over \( L_2 \) is sufficient. To find 2–periodic points of \( W_2 \), the following system of equations should be solved:

\[
\begin{align*}
    x &= x^4 + 2x^2(1 - x^2) \\
    y &= 0 \\
    z &= (1 - (1 - z)^2)^2
\end{align*}
\]

(34)
The solution for the first equation in system (34) is easy to find. Therefore, the periodic points of $W_2$ are $e_1 = (1,0,0), e_3 = (0,0,1),$ and $(\frac{\sqrt{5} - 1}{2},0,\frac{1}{16}(\sqrt{5} - 3)^2(\sqrt{5} + 1)^2)$.

(2) Let $x_1^{(0)} \notin \text{Fix}(W_2) \cup \text{Per}_2(W_2)$ and $y^{(0)} = 0$. The first coordinate of $W_2$ can be rewritten as $x' = (1 - x^{(0)})^2 + 2x^{(0)}(1 - x^{(0)})$ because the second coordinate is invariant over $L_2$. The first coordinate is equal to the first coordinate of $W_1$, which has been proven in the previous thereom. Hence, we derive

$$V^{(n)}(W_2) = \begin{cases} (\varphi^{(2k)}(x^{(0)}), 0, 1 - \varphi^{(2k)}(x^{(0)})) & \text{if } n = 2k \\ (\varphi^{(2k)}(\varphi(x^{(0)})), 0, 1 - \varphi^{(2k)}(\varphi(x^{(0)}))) & \text{if } n = 2k + 1 \end{cases}$$

(35)

Therefore, we determine that $\omega_{W_2}(x^{(0)}) = \{e_1, e_3\}$. Let $y^{(0)} \notin L_2$ and $x^{(n)} < \frac{1}{2}$, which indicate that $z^{(n)} < \frac{1}{2}$ and yields $y^{(n+1)} < y^{(n)}$. If $x^{(n)} < \frac{1}{2}$, then the third coordinate $z^{(n)}$ is also less than $\frac{1}{2}$, which indicates that $y^{(n+1)} < y^{(n)}$. In the two previous cases, we conclude that $\frac{y^{(n+1)}}{y^{(n)}} \leq 1$, thereby making $y^{(n+1)}$ is a decreasing bounded sequence that converges to zero, which indicates that studying the dynamics of $W_2$ over $L_2$ was enough. Therefore, $\omega_{W_2}(x_1^{(0)}) = \{e_1, e_3\}$ for any initial point $x_1^{(0)}$ in $S^2$.

Subsequently, we explore the behavior of $V_{25}$, which is selected from class $G_9$.

$$V_{25} := \begin{cases} x' = (y^{(0)})^2 + (y^{(0)})^2 + 2x^{(0)}(1 - x^{(0)}) \\ y' = \alpha(x^{(0)})^2 \\ z' = (1 - \alpha)(x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases}$$

(36)

We rewrite $V_{25}$ as a convex combination $V_{25} = \alpha W_1 + (1 - \alpha) W_2$, where

$$W_1 := \begin{cases} x' = (y^{(0)})^2 + (y^{(0)})^2 + 2x^{(0)}(1 - x^{(0)}) \\ y' = (x^{(0)})^2 \\ z' = 2y^{(0)}z^{(0)} \end{cases}$$

(37)

and

$$W_2 := \begin{cases} x' = (y^{(0)})^2 + (y^{(0)})^2 + 2x^{(0)}(1 - x^{(0)}) \\ y' = 0 \\ z' = (x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases}$$

(38)

**Corollary 2** Let $W_1 : S^2 \rightarrow S^2$ given by (37) be a $\xi^{(as)}$-QSO. Then, the following statements are true:

(i) $\text{Fix}(W_1) = \emptyset$. Moreover, $\text{Per}_2(W_1) = \{e_1,e_2, (\frac{\sqrt{5} - 1}{2}, \frac{1}{16}(\sqrt{5} - 3)^2(\sqrt{5} + 1)^2, 0)\}$.

(ii) $\omega_{W_1}(x_1^{(0)}) = \{e_1,e_2\}$

For $W_2$, let $W_2 : S^2 \rightarrow S^2$ given by (38) be a $\xi^{(as)}$-QSO. Then, the following statements are true:

(i) $\text{Fix}(W_2) = \{ (\frac{\sqrt{5}}{2} - \frac{1}{2}, 0, \frac{3}{2} - \frac{\sqrt{5}}{2}) \}$
(ii) $\text{Per}_2(W_2) = \{e_1, e_3\}$

(iii) $\omega_{w_2}(x_1(0)) = \{e_1, e_2\}$

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