ON THE CASSON INVARIANT CONJECTURE OF
NEUMANN–WAHL

ANDRÁS NÉMETHI AND TOMOHIRO OKUMA

Abstract. In the article we prove the Casson Invariant Conjecture of Neumann–Wahl for splice type surface singularities. Namely, for such an isolated complete intersection, whose link is an integral homology sphere, we show that the Casson invariant of the link is one-eighth the signature of the Milnor fiber.

1. Introduction

Almost twenty years ago, Neumann and Wahl formulated the following conjecture:

Casson Invariant Conjecture. Let \((X, o)\) be an isolated complete intersection surface singularity whose link \(\Sigma\) is an integral homology 3–sphere. Then the Casson invariant \(\lambda(\Sigma)\) of the link is one-eighth the signature of the Milnor fiber of \((X, o)\).

The conjecture can be reformulated in terms of the geometric genus \(p_g\) of \((X, o)\) as well (see below in 4.1).

The conjecture is true for Brieskorn hypersurface singularities by a result of Fintushel and Stern. This and additivity properties (with respect to splice decomposition) lead to the verification of the conjecture for Brieskorn complete intersections, done independently by Neumann–Wahl and Fukuhara–Matsumoto–Sakamoto. For suspension hypersurface singularities it was verified in \(\text{[15]}\). Some iterative generalizations, related with cyclic coverings and using techniques of equivariant Casson invariant and gauge theory, were covered by Collin and Saveliev (cf. \(\text{[3, 4, 5]}\)).

Recently, Neumann and Wahl have introduced an important family of complete intersection surface singularities, the splice type singularities. In \(\text{[15]}\) they treated the case when the link is an integral homology sphere (the reader may consult in \(\text{[10, 14, 19]}\) the case of rational homology sphere links too). In \(\text{[15]}\), they have also verified the above conjecture (by a direct computation of the geometric genus) for special splice type singularities (when the nodes of the splice diagram are in a line).

The goal of the present article is to verify the conjecture for an arbitrary splice type singularity:

Theorem. The Casson Invariant Conjecture is true for any splice type singularity with integral homology sphere link.

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This theorem can also be judged in the light of the following expectation/conjecture (cf. [15, 14], see also section 6 in [22]): any complete intersection surface singularity with integral homology sphere link is of splice type.

The article is organized as follows. Section 2 is a review of splice type surface singularities. The proof of the theorem is broken into two parts: section 3 contains the analytic part, while the topological/combinatorial part is in the last section 4.

The proof contains an inductive formula for the geometric genus. The inductive step corresponds to the splice decomposition of the link \( \Sigma \) (cf. [17]). But we wish to emphasize that the study of any analytic invariant with respect to the splice decomposition is rather delicate since the splice decomposition (gluing), rewritten in the language of plumbing, contains a purely topological step (namely the 0-absorption, cf. [17], §22) which cannot be represented in the world of negative definite plumbing graphs. A new construction of section 4 surmounts this difficulty.

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## 2. Splice type surface singularities (review)

This section contains a brief introduction of splice type singularities in terms of “monomial cycles” (cf. [21, §3], [10, §13]). We always assume that all the links are integral homology spheres.

The splice type singularities, which generalize Brieskorn complete intersections, were introduced by Neumann and Wahl (see [10, 13, 19]). To any resolution (or, negative definite plumbing) graph, associated with a fixed plumbed 3–manifold, one first associates a new weighted tree, called the “splice diagram”. From this one writes down the system of (leading forms of) splice diagram equations. They define a splice type singularity whose link is the original 3-manifold. (In the next presentation the splice diagram is almost ‘hidden’, the interested reader may consult the above references for more details.)

### 2.1. Basic notations

Let \( (X, o) \) be a germ of a normal complex surface singularity and \( \pi : \tilde{X} \rightarrow X \) a good resolution, i.e., its exceptional divisor \( E := \pi^{-1}(o) \) has only simple normal crossings. Let \( \{E_v\}_{v \in V} \) denote the set of irreducible components of \( E \). The link \( \Sigma \) of the singularity \( X \) is an integral homology sphere if and only if \( E \) is a tree of rational curves and the intersection matrix \( I(E) := (E_v \cdot E_w)_{v, w \in V} \) is unimodular. Let \( (m_{vw})_{v, w \in V} = -I(E)^{-1} \). Then every \( m_{vw} \) is a positive integer. We call an element of a group \( \sum_{v \in V} ZE_v \) a cycle. For any cycle \( D = \sum_{w \in V} a_w E_w \), we write \( m_{vw}(D) = a_w \).

Let \( \delta_{v} = (E - E_v) \cdot E_v \), the number of irreducible components of \( E \) intersecting \( E_v \). A curve \( E_v \) (or its index \( v \)) is called an end (resp. node) if \( \delta_{v} = 1 \) (resp. \( \delta_{v} \geq 3 \)). We denote by \( E \) (resp. \( N \)) the set of indices of ends (resp. nodes). A connected component of \( E - E_v \) is called a branch of \( v \).

### 2.2. Monomial cycle

For any \( v \in V \), let \( E_{v}^{*} = \sum_{w \in V} m_{vw} E_{w} \). Then \( E_{v}^{*} \cdot E_{w} = -\delta_{vw} \) for every \( w \in V \), where \( \delta_{vw} \) denotes the Kronecker delta. An element of a semigroup \( \sum_{w \in V} \mathbb{Z}_{\geq 0} E_{w}^{*} \), where \( \mathbb{Z}_{\geq 0} \) is the set of nonnegative integers, is called a monomial cycle. Let \( \mathbb{C}\langle z \rangle := \mathbb{C}\langle z_{w}; w \in E \rangle \) be the convergent power series ring in \#\( E \) variables. Then for a monomial cycle \( D = \sum_{w \in E} \alpha_{w} E_{w}^{*} \), we associate a monomial \( z(D) := \prod_{w \in E} z_{w}^{\alpha_{w}} \in \mathbb{C}\langle z \rangle \).

### 2.3. Degree and order associated with \( v \)

We fix \( v \in V \). One defines the \( v \)-weight of any variable \( z_w, w \in E \), by \( m_{vw} \). Note that \( \gcd\{m_{vw}\}_{w \in V} = 1 \). If \( D \) is a monomial cycle, then the \( v \)-degree of \( z(D) \) is equal to \( m_v(D) \).

For any \( f \in \mathbb{C}\langle z \rangle \) write \( f = f_0 + f_1 \in \mathbb{C}\langle z \rangle \), where \( f_0 \) is a nonzero quasihomogeneous polynomial with respect to the \( v \)-weight and \( f_1 \) is a series in monomials of higher \( v \)-degrees. Then we call \( f_0 \) the \( v \)-leading form of \( f \), and denote it by \( \text{LF}_v(f) \). We define the \( v \)-order of \( f \) by \( \nu_{\text{ord}}(f) := \nu_{\text{deg}}(\text{LF}_v(f)) \).
2.4. Monomial Condition. We say that \( E \) satisfies the monomial condition if for any node \( v \) and any branch \( C \) of \( v \), there exists a monomial cycle \( D \) such that \( D - E_v^o \) is an effective cycle supported on \( C \). In this case, \( z(D) \) is called an admissible monomial belonging to the branch \( C \). Note that a chain \( C \) always admits a monomial cycle.

The monomial condition is equivalent to the semigroup and congruence conditions of Neumann-Wahl (see [19, §13]) (although the congruence condition is trivial in our case.)

2.5. Neumann-Wahl system. Assume that the monomial condition is satisfied. For any fixed node \( v \) let \( C_1, \ldots, C_{\delta_v} \) be the branches of \( v \). Suppose \( \{m_1, \ldots, m_{\delta_v}\} \) is a set of admissible monomials such that \( m_i \) belongs to \( C_i \) for \( i = 1, \ldots, \delta_v \). Let \( F = (c_{ij}), c_{ij} \in \mathbb{C}, \) be any \( (\delta_v - 2) \times \delta_v \)-matrix such that all the maximal minors have rank \( \delta_v - 2 \). We define polynomials \( f_1, \ldots, f_{\delta_v - 2} \) by

\[
\begin{pmatrix}
  f_1 \\
  \vdots \\
  f_{\delta_v - 2}
\end{pmatrix} = F \begin{pmatrix}
  m_1 \\
  \vdots \\
  m_{\delta_v}
\end{pmatrix}.
\]

We call the set \( \{f_1, \ldots, f_{\delta_v - 2}\} \) a Neumann-Wahl system at \( v \). If we have a Neumann-Wahl system \( F_v \) at every node \( v \) then we call the set \( \mathcal{F} := \bigcup_{v \in \mathcal{N}} F_v \) a Neumann-Wahl system associated with \( E \). Note that \( \#\mathcal{F} = \#\mathcal{E} - 2 \).

2.6. Splice diagram equations. Consider a finite set of germs

\[
\{f_{v,j_v} \mid v \in \mathcal{N}, j_v = 1, \ldots, \delta_v - 2\} \subset \mathbb{C}(z).
\]

If the set \( \{LF_v(f_{v,j_v}) \mid v \in \mathcal{N}, j_v = 1, \ldots, \delta_v - 2\} \) is a Neumann-Wahl system associated with \( E \) then a system of equations

\[
f_{v,j_v} = 0, \quad v \in \mathcal{N}, \quad j_v = 1, \ldots, \delta_v - 2,
\]

is called the splice diagram equations and the germs \( \{f_{v,j_v}\} \) splice diagram functions. The germ of singularity defined by the splice diagram equations in \((\mathbb{C}^{\#\mathcal{E}}, o)\) is called a splice type singularity (associated with the combinatorics of \( E \)).

We note that a splice type singularity is an equisingular deformation of a singularity defined by a Neumann-Wahl system associated with \( E \) (cf. [21] (4.)).

In fact, all the above construction of the splice equations (including the validity of the monomial condition) depends only on the dual graph associated with \( E \), hence only on a fixed plumbing graph of the link \( \Sigma \).

2.7. Theorem. [15] (2.1)] Suppose that \((Z,o)\) is a splice type singularity associated with \( E \) (or, with a graph of \( \Sigma \)). Then \((Z,o)\) is an isolated complete intersection surface singularity whose link is \( \Sigma \).

2.8. End-Curve Condition. We keep the notations of [21]. We say that \( \breve{X} \) satisfies the end-curve condition if for each \( w \in \mathcal{E} \) there exists an irreducible curve \( H_w \subset \breve{X} \), not an exceptional curve, and a function \( f \in H^0(\mathcal{O}_\breve{X}(\mathbb{Z}E_w)) \) such that \( \text{div}(f) = E_w^o + H_w^o \). We call such \( f \) an end-curve function of \( E_w \). The end-curve condition is equivalent to the fact that for every \( w \in \mathcal{E} \) the linear system of \( \mathcal{O}_\breve{X}(\mathbb{Z}E_w) \) has no fixed component in \( E \).

Notice that if \( \breve{X} \) satisfies the end-curve condition then so do the minimal good resolution and any resolution obtained from \( \breve{X} \) by blowing-up the singular points of \( E \).

If \((X,o)\) is defined by splice diagram equations, then its resolution \( \breve{X} \) satisfies the end-curve condition (since the coordinate functions serve as end-curve functions). The converse is guaranteed by the following:
3.3. Hilbert series. Let $H(t)$ denote the Hilbert series of the graded ring $\mathcal{G}$, i.e., 

$$H(t) = \sum_{i \geq 0} (\dim \mathcal{G}_i) t^i.$$ 

From [20], by a well-known formula, valid for complete intersections, one has:
3.4. Proposition.  

\[ H(t) = \prod_{w \in V} (1 - t^{m_{vw}})^{\delta_w} \]  

From [20], one may also read the $a$–invariant of Goto and Watanabe associated with $G$ (namely $\sum_{w \in N_{\chi} \cup \chi} \nu_{\chi}(L \nu_{\chi} f_{w}) = -\sum_{w \in E} \nu_{\chi}(z_w)$, cf. [10] (3.1)):  

\[ a(G) = \sum_{w \in V} (\delta_w - 2) m_{vw}. \]  

3.5. Let $\pi': \tilde{X} \to X'$ be the morphism which contracts each connected component of the divisor $E - E_0$ to a normal point and $\pi_1: X' \to X$ the natural morphism. Let $E' = \pi'(E_0)$.  

3.6. Lemma.  

(1) There exists $k \in \mathbb{Z}_{>0}$ such that $O_X(-kE^*_v)$ is $\pi$–generated.  

(2) For any $n \in \mathbb{Z}_{>0}$, $I_n = H^0(O_X((-nE')) = H^n(\tilde{O}_X([-n\pi^*E'])))$.  

(3) $-E'$ is $\pi_1$-ample and $\pi_1$ coincides with the filtered blowing up  

\[ \text{Projan}_X \left( \bigoplus_{n \geq 0} \pi_1^{*} O_X((-nE')) \right) \to X. \]  

(4) $H^1(O_X((-nE')) = 0$ for $n > a(G)$.  

Proof. First we treat (1). Let $C$ be one of the branches of $v$, and $E_v$ be that component of $C$ which intersects $E_0$. Fix any end $E_v$ of $E$ in $C$, and set $A := -\sum_{v \in C} \det(-C) I(C)_{v_{\chi}} E_{v}$. This is an effective cycle supported by $C$ such that $m_v(A)E_v^* + A$ is a monomial cycle, namely $\det(-C)E_v^*$. Therefore, if we fix two branches $C_i$ (i = 1, 2), we can construct two positive integers $k_i$ and effective cycles $A_i$ supported by $C_i$ such that $k_iE_v^* + A_i$ is monomial. Set $k = k_1k_2$ and $f_i \in H^0(\tilde{O}_X)$ the image of the monomial $z(kE_v^* + k_jA_i)$, where $i, j = 1, 2$ and $i \neq j$. Then it is easy to see that $f_1, f_2 \in H^0(\tilde{O}_X(-kE_v^*))$ generate $O_X(-kE_v^*)$.  

The proof of (2)–(4) is same as the proof of [20] (3.3–3.4) based on part (1) and filtered ring theory [24]. \hfill \Box  

Since $m_{vw}\pi^*E' = E_v^*$ is integral, from (2) we get for any $m \in \mathbb{Z}_{>0}$  

\[ I_{mE_v} = H^n(\tilde{O}_X((-mE_v^*)). \]  

3.7. $p_g$-formula. Let $(X_i, x_i)$ denote the singularity germ obtained by contracting the branch $C_i$ (hence $X_i \subset X'$). Fix $k \in \mathbb{Z}_{>0}$ such that $\pi_v^*(kE_v^*)$ is Cartier. Its existence is guaranteed by the proof of [20] (1) (namely by the existence of functions $f_i$).  

3.8. Theorem. Set $P(n) = \sum_{i=0}^{n-1} (\dim G_i)$ for $n \in \mathbb{Z}_{>0}$. Then for any integer $m > a(G)/km_{vw}$  

\[ p_g(X, o) = P(mk_{\nu}) - \chi(mE_v^*) + \sum_{i=1}^{\delta_v} p_g(X, x_i). \]  

Proof. We write $D = mkE_v^*$ and $D' = mk_{\nu}E'$. Consider the exact sequence  

\[ 0 \to \tilde{O}_X(-D) \to \tilde{O}_X \to \tilde{O}_D \to 0. \]  

By equation [3.1] one has  

\[ \dim \mathbb{C} H^0(\tilde{O}_X)/H^0(\tilde{O}_X(-D)) = \dim \mathbb{C} I_0/I_{mk_{\nu}E_v} = P(mk_{\nu}). \]  

Therefore it suffices to show that  

\[ h^1(\tilde{O}_X(-D)) = \sum_{i=1}^{\delta_v} p_g(X, x_i). \]  

From the spectral sequence  

\[ E_2^{ij} = H^1(R^i\pi_v^*\tilde{O}_X(-D)) \Rightarrow H^0(\tilde{O}_X(-D)) \]
one gets the exact sequence
\[ H^1(\pi'_*O_X(-D)) \to H^1(O_X(-D)) \xrightarrow{\alpha} H^0(R^1\pi'_*O_X(-D)) \to 0. \]
Since \( \pi'_*D = D' \) is Cartier and \( \pi''E' = E'_v/m_{vv} \), by projection formula and \( 3.6 \) (4),
\[ H^1(\pi'_*O_X(-D)) = H^1(\pi'_*O_X(-\pi''D')) = H^1(O_{X'}(-D')) = 0, \]
hence \( \alpha \) is an isomorphism. Clearly the support of \( R^1\pi'_*O_X(-D) \) is in the set \( \{x_i\} \). Since
\( D = \pi''D' \) and \( D' \) is Cartier, we have
\[ (R^1\pi'_*O_X(-D))_{x_i} \cong (R^1\pi'_*O_X \otimes O_{X'}(-D'))_{x_i} \cong (R^1\pi'_*O_X)_{x_i}. \]
This proves \( 3.2. \)

3.9. Periodic constants. In the formula of \( 3.8 \), \( P(mkmn) - \chi(\text{mk}E^*_v) \) is independent of \( m \gg 0 \). Let us formulate this property. Let \( F(t) = \sum_{i \geq 0} a_i t^i \) be a formal power
series. Set \( P_F(n) = \sum_{i=0}^{n-1} a_i \). Suppose that \( P_F(kn) \) is a polynomial function of \( n \) for some \( k \in \mathbb{Z}_{>0} \). Then for any \( k' \in \mathbb{Z}_{>0} \) satisfying this property, the constant terms of \( P_F(kn) \)
and \( P_F(k'n) \) are the same. We call this constant the periodic constant of \( F(t) \) and denote
it by \( F|_{pc} \). For example, the formula of Theorem \( 3.8 \) is
\[ p_g(X, a) = H|_{pc} + \sum_{i=1}^{\delta_u} p_g(X, x_i). \]
Note that if two formal power series \( F_1(t) \) and \( F_2(t) \) have periodic constants then
\[ (F_1 + F_2)|_{pc} = F_1|_{pc} + F_2|_{pc}. \]

3.10. Remarks. (1) The message of the above theorem \( 3.8 \) is the following. Fix a negative definite resolution graph \( \Gamma \) (or its plumbed 3-manifold \( \Sigma \)) which satisfies the monomial condition. Then there exists a splice type singularity whose link is \( \Sigma \) (via Neumann-Wahl system/construction), and the geometric genus of any such analytic structure is independent of the choice of the Neumann-Wahl system (by 4.3 of \( 21 \) and 10.1 of \( 19 \)), hence it depends only on the combinatorics of \( \Gamma \) — it will be denoted by \( p_g(\Gamma) \). Moreover, the geometric genus of any such analytic structure satisfies the inductive formula \( 3.8 \).

(2) The end-curve condition and splice diagram equations are considered for any surface singularity with rational homology sphere links. For this case, and under the assumption that \( v \) is a node, a \( p_v \)-formula is established in \( 20 \).

4. The proof of the main theorem

4.1. In this section we prove the main Theorem stated in the introduction. For this, we will apply theorem \( 3.8 \) in several different situations.

We start with \( (X, o) \), a splice type singularity (whose minimal good resolution \( \bar{X} \) satisfies the end-curve condition), and whose link \( \Sigma \) is an integral homology sphere. \( \Gamma \) will denote the dual resolution graph associated with \( \bar{X} \); \( \mathcal{V} \) stays for the set of vertices of \( \Gamma \).

It is well-known that in the conjecture one may replace the signature \( \sigma \) of the Milnor fiber by the geometric genus \( p_g \) of \( (X, o) \) via a formula of Durfee and Laufer \( \sigma = 8p_g + c(\Gamma) = 0 \), where \( c(\Gamma) := K^2 + b_2(\bar{X}) \) (here \( K \) denotes the canonical class and \( b_2 \) the second Betti number) depends only on the combinatorics of \( \Gamma \).

If \( \Gamma \) has only one node then Theorem follows from \( 13 \) (7.7)]. A different argument runs as follows: The above \( p_g \) formula (namely \( p_g = H|_{pc} \), cf. \( 3.8 \)) is valid for any splice type analytic structure (supported by the same topological type), hence we may assume that \( (X, o) \) admits a good \( \mathbb{C}^* \)-action. Then \( (X, o) \) is automatically a Brieskorn complete intersection (cf. \( 13 \)), hence we may apply \( 15 \) too. Therefore, in the sequel we assume that \( \Gamma \) has at least two nodes.
4.2. In our inductive procedure the following fact is crucial (cf. also with §4.10): for any splice type singularity with (not necessarily minimal) fixed good resolution graph $\Gamma$, consider any connected subgraph $\Gamma' \subset \Gamma$. Let $E'$ denote the reduced connected cycle corresponding to $\Gamma'$. Then a neighborhood of $E'$ satisfies the end-curve condition (see also (2.15)); hence $E'$ (or $\Gamma'$) satisfies the monomial condition by (2.3). Therefore if $(X', x')$ denotes the normal surface singularity obtained by contracting $E'$, then $p_g(X', x')$ is topological computable from $\Gamma'$. We write $p_g(X', x') = p_g(\Gamma')$.

4.3. Starting the induction. Let $v_1 \in N$ be an end-node of $\Gamma$ and $v_2 \in N$ the node which is nearest to $v_1$. Write $\Gamma' \subset \Gamma$ for the branch of $v_1$ containing $v_2$, and denote by $w$ that vertex in $\Gamma'$ which is connected by $v_1$ in $\Gamma$. Let $H_{\Gamma,v_1}(t)$ be the Hilbert series of the associated graded ring of the filtration with respect to $E_{v_1}$. Since $p_g(\Gamma'') = 0$ for all the branches $\Gamma''$ of $v_1$ which are chains, by (3.8) we have the following.

4.4. Proposition. $p_g(\Gamma) = H_{\Gamma,v_1,pc} + p_g(\Gamma')$.

4.5. One needs to modify this inductive step, since, e.g., in general, $\Gamma'$ is not unimodular. The next goal is to fit $\Gamma'$ into a ‘good’ inductive procedure which is compatible with the splice decomposition of $\Sigma$ (see also §3.1.1 for more motivation).

Let $\Delta$ be the splice diagram of $\Sigma$. For the correspondence between $\Gamma$ and $\Delta$ see [7, §22]. We decompose $\Delta$ as the splice of two splice diagrams $\Delta_1$ and $\Delta_2$, where $\Delta_i \ni v_i$.

Let $\Gamma_i$ be the minimal plumbing graph associated with $\Delta_i$ ($i = 1, 2$) (cf. [7]). Next we review how one can recover from $\Delta_1$ and $\Delta_2$ the maximal chain $\Gamma_0$ in $\Gamma$ which connects $v_1$ and $v_2$, and we relate $\Gamma_2$ with $\Gamma'$. Set $a := \prod_{i=1}^{r} a_i$ and $d := \prod_{i=1}^{s} d_i$. Express $a/b$ as a continued fraction $[\alpha_0, \ldots, \alpha_m]$, i.e. $a/b = \alpha_0 - 1/(\alpha_1 - 1/(\ldots - 1/\alpha_m))$ with $\alpha_i \in \mathbb{Z}_{>0}$ and $\alpha_i \geq 2$ for $i \geq 1$. Similarly, write $d/c = [\beta_0, \ldots, \beta_n]$, and consider the chain:

$$\Gamma_0 : \quad v_1 \quad -\alpha_m \quad \ldots \quad -\alpha_0 \quad -\beta_3 \quad \ldots \quad -\beta_n \quad v_2$$

This is not a ‘minimal’ graph, and it is equivalent via (topological) plumbing calculus with the (minimal) chain $\Gamma_0$. This calculus runs as follows: by the positivity of the edge determinant $bc > ad$ (cf. §1.5 §1]), $\alpha_0 = 1$ or $\beta_0 = 1$. Then one successively blows down the vertices whose weight are $-1$, and at some moment inevitably a vertex with weight 0 appears; then one makes a 0-absorption by the rule:

$$\begin{array}{cccc}
-\varepsilon_1 & 0 & -\varepsilon_2 & \rightarrow \quad -\varepsilon_1 - \varepsilon_2
\end{array}$$

In this way one gets $\Gamma_0$. On the other hand, the corresponding maximal chain (with determinant $c$) in $\Gamma_2$ has the following form:

$$\Gamma_2 : \quad -\beta_1 \quad \ldots \quad -\beta_n \quad v_2$$

4.6. Lemma. Consider the following resolution graph $\tilde{\Gamma}_2$ which satisfies (1)–(4) below:

$$\tilde{\Gamma}_2 : \quad \begin{array}{c}
\Gamma^0 \quad \ldots \quad \Gamma^m \quad -1 \quad v_1' \\
v_2 \quad \Gamma'
\end{array}$$

(1) the subgraph on the left hand side of $w$ is a chain;
(2) $\Gamma^0$ consists of $(\alpha_0 - 1)$-vertices; the right-end has weight $-3$, all the others $-2$;
Proof. Glue the chain from the left hand side of \( w \) with \( \tilde{\Gamma}_0 \), blow down the \((-1)\)-vertices and use the 0-absorption which identifies \( 0 - \beta_0 \) with \( \Gamma_2 \).

4.7. Corollary. There exists a (non-minimal) resolution graph, namely \( \tilde{\Gamma}_2 \), which represents the splice diagram \( \Delta_2 \), it has \( \Gamma' \) as a subgraph, and supports a splice type singularity.

Proof. We only have to show that \( \tilde{\Gamma}_2 \) satisfies the monomial condition. Let \( u \) be a node of \( \Gamma_2 \), let \( C'' \) a branch of it which contains the distinguished chain involved in the splicing (otherwise the condition is trivial). Let \( C \) be the branch of \( u \) in \( \Gamma \) which contains \( C'' \). Then there exists an effective cycle \( A \) supported on \( C \) so that \( E^*_\alpha + A \) is monomial in \( \Gamma \). Then the restriction \( A|\Gamma' \) to \( \Gamma' \) has the property that \( E^*_\alpha (\Gamma') + A|\Gamma' \) is monomial and has negative intersection with \( E_\sigma \). Then \( A|\Gamma' \) can be extended easily on the chain considered in \( \Delta_0 \) to an effective cycle \( B \) on \( \Gamma_2 \) such that \( E^*_\alpha (\Gamma_2) + B \) is monomial in \( \Gamma_2 \).

Let \( H_{\tilde{\Gamma}_2,v'_1}(t) \) be the Hilbert series of associated graded ring of the filtration with respect to \( E_{v'_1} \) of \( \tilde{\Gamma}_2 \). Then, again by \( \Box \), we have:

4.8. Proposition. \( p_\sigma (\Gamma_2) = H_{\tilde{\Gamma}_2,v'_1} |_{pc} + p_\sigma (\Gamma') \).

4.9. Finally we consider the star–shaped graph \( \Gamma_1 \) (which automatically satisfies the monomial condition), hence (again by Theorem 3.8):

4.10. Proposition. \( p_\sigma (\Gamma_1) = H_{\Gamma_1,v_1} |_{pc} \).

4.11. As a consequence, from the above three propositions one gets:

\[
(4.3) \quad p_\sigma (\Gamma) = p_\sigma (\Gamma_1) - p_\sigma (\Gamma_2) = (H_{\Gamma,v_1} - H_{\tilde{\Gamma}_2,v'_1} - H_{\Gamma_1,v_1}) |_{pc}.
\]

The main point in this ‘additivity’ formula is the following. Two terms in the expression \( \sigma + 8p_\sigma + c(\Gamma) = 0 \) (cf. 4.1) satisfy some additivity properties with respect to the splicing. Namely, the additivity \( \lambda (\Gamma) = \lambda (\Gamma_1) + \lambda (\Gamma_2) \) (here we identify a plumbing graph with its plumbed 3-manifold) was proved independently by Akbulut-McCarthy, Boyer-Nicas and Fukuhara-Maruyama (according to [2]). Let \( b_1 (G_i) \) be the first Betti number of the fiber \( G_i \) of the fibered knot determined by \( \Gamma_i \) and the splicing data (\( i = 1,2 \)). Then, by \( \Box \) (6.4) (cf. also with \( \Box \) (5.20)), one has \( c(\Gamma) = c(\Gamma_1) + c(\Gamma_2) - 2b_1 (G_1)b_2 (G_2) \). In particular, in order to run the induction of our proof, we only need to verify

\[
p_\sigma (\Gamma) = p_\sigma (\Gamma_1) + p_\sigma (\Gamma_2) + b_1 (G_1)b_2 (G_2)/4,
\]

a fact already noticed in \( \Box \) (6.3)]. Therefore the following proposition, together with equation (4.3), implies the theorem of the introduction.

4.12. Proposition. \( (H_{\Gamma,v_1} - H_{\tilde{\Gamma}_2,v'_1} - H_{\Gamma_1,v_1}) |_{pc} = b_1 (G_1)b_2 (G_2)/4 \).

Proof. Note that in (4.12), each periodic constant independently is a very complicated (Dedekind sum) expression. Nevertheless, their combination provides the simple expression from the right hand side. The proof is based on an explicit computation of the Hilbert series using \( \Box \). The corresponding entries \( m_{v_0v} \) might be determined from the splice diagrams using identities of \( \Box \) [10], or \( \Box \) (9.1)]. Note that the ‘unreduced’ splice diagram associated with \( \tilde{\Gamma}_2 \), having \( v'_1 \) as a node, is the following:

\[
\Delta_{\tilde{\Gamma}_2} : \quad a \quad b \quad \rightarrow \quad v'_1 \quad \leftarrow \quad v_2
\]
We will use the notation of (4.5) and let $V_2$ be the set of vertices of $\Gamma_2$. Then $m_{v_i}, v_i = ab$ and $m_{v_i}, w_i = ab/a_i$. Consider the integers $p_i := m_{v_i}, w_i/b = a/a_i$ and $q_i := m_{v_i}/a$ for all $1 \leq i \leq r$ and $w \in V_2$ respectively. Finally, set $g(t) = \prod_{w \in V_2} (1 - t^{p_i}b)^{k_w - 2}$. Then

$$H_{\Gamma,v_i}(t) = \frac{g(t^{p_i}) (1 - t^{p_i}b)^{r - 1}}{\prod_{i=1}^{r} (1 - t^{p_i}b)}.$$  

Since for any $w \in V_2$, $m_{v_i}, w_i$ in $\Gamma$ and $m_{v_i}, w_i$ in $\Gamma_2$ are the same, one has:

$$H_{\Gamma_2,v_i}(t) = \frac{g(t^{p_i})}{1 - t^p}.$$  

Clearly

$$H_{\Gamma_2,v_i}(t) - H_{\Gamma_1,v_i}(t) = Q_1(t^b)Q_2(t^a) - \frac{1}{(1 - t^a)(1 - t^b)}.$$  

On the other hand, consider the characteristic polynomial $P_i(t) = \det(I - tH_1, i)$ (where $I$ denotes the identity matrix) of the monodromy $h_{1,i} : H^1(G_i) \to H^1(G_i)$ of the fibered knot with Milnor fiber $G_i$ ($i = 1, 2$). Using A’Campo’s formula [1], or [7] §11–12 one has

$$P_1(t) = (1 - t) \cdot \frac{(1 - t^{a})^{r - 1}}{\prod_{i=1}^{r} (1 - t^{p_i})},$$

$$P_2(t) = (1 - t)g(t).$$

It is known for fibered links in integral homology 3-spheres (or using A’Campo’s formula’s is easy to check) that $P_1(1) = 1$ and the derivative also satisfies $P_1'(1) = b_1(G_1)/2$.

Notice also that $Q_i(t) = (P_i(t) - 1)/(1 - t)$, hence $Q_i(1) = -P_i'(1)$. Therefore, since $Q_i$’s are polynomials, $(Q_1(t^b)Q_2(t^a))|_{pc} = Q_1(1)Q_2(1) = b_1(G_1)b_1(G_2)/4$. We end the proof with the remark that $(1 - t^a)^{-1}(1 - t^b)^{-1}|_{pc} = 0$ (since $gcd(a,b) = 1$).

\[\square\]

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**Department of Mathematics, Ohio State University, Columbus, OH 43210; and**

**Rényi Institute of Mathematics, Budapest, Hungary**

*E-mail address:* nemethi@math.ohio-state.edu; nemethi@renyi.hu

**Department of Education, Yamagata University, Yamagata 990-8560, Japan.**

*E-mail address:* okuma@e.yamagata-u.ac.jp