Near-Optimal Recovery of Linear and $N$-Convex Functions on Unions of Convex Sets

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Abstract

In this paper we build provably near-optimal, in the minimax sense, estimates of linear forms and, more generally, “$N$-convex functionals” (the simplest example being the maximum of several fractional-linear functions) of unknown “signal” known to belong to the union of finitely many convex compact sets from indirect noisy observations of the signal. Our main assumption is that the observation scheme in question is good in the sense of [15], the simplest example being the Gaussian scheme where the observation is the sum of linear image of the signal and the standard Gaussian noise. The proposed estimates, same as upper bounds on their worst-case risks, stem from solutions to explicit convex optimization problems, making the estimates “computation-friendly.”

1 Introduction

The simplest version of the problem considered in this paper is as follows. Given access to $K$ independent observations
\[ \omega_t = Ax + \sigma \xi_t, \quad 1 \leq t \leq K \quad [A \in \mathbb{R}^{m \times n}, \xi_t \sim \mathcal{N}(0, I_m)] \]
of “signal” $x$ known to belong to the union $X = \bigcup_{i=1}^{I} X_i$ of convex compact sets $X_i \subset \mathbb{R}^n$, we want to recover $f(x)$, where $f$ is either linear, or, more generally, $N$-convex. Here $N$-convexity means that $f: \mathcal{X} \to \mathbb{R}$ is a continuous function on a convex compact domain $\mathcal{X} \supset X$ such that for every $a \in \mathbb{R}$, each of the two level sets $\{x \in \mathcal{X}: f(x) \geq a\}$ and $\{x \in \mathcal{X}: f(x) \leq a\}$ can be represented as the union of at most $N$ convex compact sets $^1$. Our principal contribution is a provably near-optimal in the minimax sense estimation routine. Our construction is not restricted to the Gaussian observation scheme (1) and deals with good observation schemes (o.s.’s), as defined in [15], primarily with

- Poisson o.s., where $\omega_t$ are independent across $t$ identically distributed vectors with independent across $i \leq m$ entries $|\omega_t| \sim \text{Poisson}(a^T_i x)$, and
- Discrete o.s., where $\omega_t$ are independent across $t$ realizations of discrete random variable taking values $1, \ldots, m$ with probabilities affinely parameterized by $x$.

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1Immediate examples are affine-fractional functions $f(x) = (a^T x + a)/(b^T x + b)$ with denominators positive on $\mathcal{X}$, in particular, affine functions ($N = 1$), and piecewise linear functions like $\max[a^T x + a, \min[b^T x + b, c^T x + c]]$ ($N = 3$). A less trivial example is conditional quantile of a discrete distribution ($N = 2$), see Section 4.2.

2Our main results can be easily extended to the more general case of simple families – families of distributions specified in terms of upper bounds on their moment-generating functions, see [23, 22] for details. Restricting the framework to the case of good observation schemes is aimed at streamlining the presentation.
The problem of (near-)optimal recovery of linear function \( f(x) \) on a convex compact set or a finite union of convex sets \( X \) has received much attention in the statistical literature (see, e.g., [20, 11, 12, 13, 14, 10, 7, 8, 9, 21]). In particular, D. Donoho proved, see [10], that in the case of Gaussian observation scheme (1) and convex and compact \( X \), the worst-case, over \( x \in X \), risk of the minimax optimal affine in observations estimate is within factor 1.2 of the actual minimax risk. Later, in [21], this near-optimality result was extended to other good observation schemes. In [8, 9] the minimax affine estimator was used as “working horse” to build the near-optimal estimator of a linear functional over a finite union \( X \) of convex compact sets in the Gaussian observation scheme. As compared to the existing results, our contribution here is twofold. First, we pass from Gaussian o.s. to essentially more general good o.s.’s, extending in this respect the results of [8, 9]. Second, we relax the requirement of affinity of the function to be recovered to \( N \)-convexity of the function.

It should be stressed that the actual “common denominator” of the cited contributions and of the present work is the “operational nature” of the results, as opposed to typical results of non-parametric statistics which can be considered as descriptive. The traditional results present near-optimal estimates and their risks in a “closed analytical form,” the toll being severe restrictions on the families \( X \) of signals and observation schemes. For instance, in the case of (1) such “conventional” results would impose strong and restrictive assumptions on the interconnection between the geometries of \( X \) and \( A \). In contrast, the approach we advocate here, same as that of, e.g., [10, 21], allows for quite general, modulo convexity, signal sets \( X_i \), for arbitrary matrices \( A \) in the case of (1), etc. As a result, due to this generality, the proposed estimators and their risks are yielded by efficient computation rather than being given in a closed analytical form. All we know in advance is that those computed risks are nearly as low as they can be under the circumstances.

The main body of the paper is organized as follows. Section 2 contains preliminaries, originating from [21, 15], on good o.s.’s. In Section 3 we deal with recovery of linear functions on the unions of convex sets. Finally, recovery of \( N \)-convex functions is the subject of Section 4. It is worth to mention that the construction of near-optimal estimator used in Section 4 is completely different from that employed in [10, 7, 8, 9, 21]. The estimator by multiple testing we use is closely related to the binary search estimator from [11, 12].

Some technical proofs are relegated to Appendix.

2 Preliminaries: good observation schemes

The estimates to be developed in this paper heavily exploit the notion of a good observation scheme introduced in [15]. To make the presentation self-contained we start with explaining this notion here.

2.1 Good observation schemes: definitions

Formally, a good observation scheme (o.s.) is a collection \( \mathcal{O} = (\Omega, P, \{p_\mu(\cdot) : \mu \in \mathcal{M}\}, \mathcal{F}) \), where

- \((\Omega, P)\) is an observation space: \( \Omega \) is a Polish (complete metric separable) space, and \( P \) is a \( \sigma \)-finite \( \sigma \)-additive Borel reference measure on \( \Omega \), such that \( \Omega \) is the support of \( P \);
- \( \{p_\mu(\cdot) : \mu \in \mathcal{M}\} \) is a parametric family of probability densities, specifically, \( \mathcal{M} \) is a convex relatively open set in some \( \mathbb{R}^M \), and for \( \mu \in \mathcal{M} \), \( p_\mu(\cdot) \) is a probability density, taken w.r.t. \( P \), on \( \Omega \). We assume that the function \( p_\mu(\omega) \) is positive and continuous in \((\mu, \omega) \in \mathcal{M} \times \Omega\).

\[^{3}\]In the hindsight, it is interesting to note that the authors of [12] believed their “... estimator not intended to be implemented on a computer...” They considered their construction as purely theoretical and finally oriented their analysis in the “traditional” way, by imposing assumptions allowing to end up with explicit convergence rates in some specific situations.
• $\mathcal{F}$ is a finite-dimensional linear subspace in the space of continuous functions on $\Omega$. We assume that $\mathcal{F}$ contains constants and all functions of the form $\ln(p(\cdot)/p_\nu(\cdot))$, $\mu, \nu \in \mathcal{M}$, and that the function

$$\Phi_{\mathcal{O}}(\phi; \mu) = \ln \left( \int_{\Omega} e^{\phi(\omega)} p_\mu(\omega) P(d\omega) \right)$$

is real-valued on $\mathcal{F} \times \mathcal{M}$ and is concave in $\mu \in \mathcal{M}$; note that this function is automatically convex in $\phi \in \mathcal{F}$. From real-valuedness, convexity-concavity and the fact that both $\mathcal{F}$ and $\mathcal{M}$ are convex relatively open, it follows that $\Phi$ is continuous on $\mathcal{F} \times \mathcal{M}$.

### 2.2 Examples of good observation schemes

As shown in [15] (and can be immediately verified), the following o.s.’s are good:

1. **Gaussian o.s.**, where $P$ is the Lebesgue measure on $\Omega = \mathbb{R}^d$, $\mathcal{M} = \mathbb{R}^d$, $p_\mu(\omega)$ is the density of the Gaussian distribution $\mathcal{N}(\mu, I_d)$ (mean $\mu$, unit covariance), and $\mathcal{F}$ is the family of affine functions on $\mathbb{R}^d$. Gaussian o.s. with $\mu$ linearly parameterized by signal $x$ underlying observations, see (1), is the standard observation model in signal processing;

2. **Poisson o.s.**, where $P$ is the counting measure on the nonnegative integer $d$-dimensional lattice $\Omega = \mathbb{Z}_+^d$, $\mathcal{M} = \mathbb{R}^d_+$, $p_\mu(\omega) = \mu_\omega$, $\omega \in \Omega$ is the density, taken w.r.t. $P$, of random $d$-dimensional vector with independent Poisson$(\mu_i)$ entries, $i = 1, \ldots, d$, and $\mathcal{F}$ is the family of all affine functions on $\Omega$. Poisson o.s. with $\mu$ affinely parameterized by signal $x$ underlying observation is the standard observation model in Poisson imaging, including Positron Emission Tomography [25], Large Binocular Telescope [4, 3], and Nanoscale Fluorescent Microscopy, a.k.a. Poisson Biophotonics [18, 16, 5, 17, 19];

3. **Discrete o.s.**, where $P$ is the counting measure on the finite set $\Omega = \{1, \ldots, d\}$, $\mathcal{M}$ is the set of positive $d$-dimensional probabilistic vectors $\mu = [\mu_1; \ldots; \mu_d] > 0$, $p_\mu(\omega) = \mu_\omega$, $\omega \in \Omega$ is the density, taken w.r.t. $P$, of a probability distribution $\mu$ on $\Omega$, and $\mathcal{F} = \mathbb{R}^d$ is the space of all real-valued functions on $\Omega$;

4. **Direct product of good o.s.’s**. Given good o.s.’s $\mathcal{O}_t = ((\Omega_t, P_t), \{p_{t, \mu} : \mu \in \mathcal{M}_t\}, \mathcal{F}_t)$, $t = 1, \ldots, K$, we can build from them a new o.s. $\mathcal{O}_1 \times \ldots \times \mathcal{O}_K$ with the observation space $\Omega_1 \times \ldots \times \Omega_K$, reference measure $P_1 \times \ldots \times P_K$, family of probability densities $\{p_{t, \mu}(\omega_1, \ldots, \omega_K) = \prod_{i=1}^K p_{t, \mu_i}(\omega_t) : \mu = [\mu_1; \ldots; \mu_K] \in \mathcal{M}_1 \times \ldots \times \mathcal{M}_K\}$ and $\mathcal{F} = \{\phi(\omega_1, \ldots, \omega_K) = \sum_{t=1}^K \phi_t(\omega_t) : \phi_t \in \mathcal{F}_t, t \leq K\}$. The direct product of o.s.’s $\mathcal{O}_t$ is the observation scheme we arrive at when observing collections $\omega^K = (\omega_1, \ldots, \omega_K)$ with independent across $t$ components $\omega_t$ yielded by o.s.’s $\mathcal{O}_t$.

When all factors $\mathcal{O}_t$, $t = 1, \ldots, K$, are identical to each other, we can reduce the direct product $\mathcal{O}_1 \times \ldots \times \mathcal{O}_K$ to its “diagonal,” referred to as $K$-th power $\mathcal{O}^K$, or stationary $K$-repeated version, of $\mathcal{O} = \mathcal{O}_1 = \ldots = \mathcal{O}_K$. Same as in the direct product case, the observation space and reference measure in $\mathcal{O}^K$ are $\Omega^K = \Omega \times \ldots \times \Omega$ and $P^K = P \times \ldots \times P$, the family of densities is $\{p^K_{\mu}(\omega^K) = \prod_{t=1}^K p_t(\omega_t) : \mu \in \mathcal{M}\}$, and the family $\mathcal{F}$ is $\{\phi^K(\omega_1, \ldots, \omega_K) = \sum_{t=1}^K \phi(\omega_t) : \phi \in \mathcal{F}\}$. Informally, $\mathcal{O}^K$ is the observation scheme we arrive at when passing from a single observation drawn from a distribution $p_{\mu}$, $\mu \in \mathcal{M}$, to $K$ independent observations drawn from the same distribution $p_{\mu}$.

It is immediately seen that direct product of good o.s.’s, same as power of good o.s., are themselves good o.s.
3 Recovering linear forms on unions of convex sets

Our objective now is to extend the results of [21] to the situation where $X$ is finite union of convex sets. At the same time, the results of this section can be seen as an extension to more general observation schemes of the constructions of [8, 9].

3.1 The problem

Let $\mathcal{O} = (\Omega, P), \{p_\mu(\cdot) : \mu \in \mathcal{M}\}, \mathcal{F})$ be a good o.s.. The problem we are interested in this section is as follows:

We are given a positive integer $K$ and $I$ nonempty convex compact sets $X_j \subset \mathbb{R}^n$, along with affine mappings $A_j(\cdot) : \mathbb{R}^n \to \mathbb{R}^M$ such that $A_j(x) \in \mathcal{M}$ whenever $x \in X_j, 1 \leq j \leq I$. In addition, we are given a linear function $g^T x$ on $\mathbb{R}^n$.

Given random observation

$$\omega^K = (\omega_1, ..., \omega_K)$$

with $\omega_k$ drawn, independently across $k$, from $p_{A_j(x)}$ with $j \leq I$ and $x \in X_j$, we want to recover $g^T x$. It should be stressed that we do not know neither $j$ nor $x$ underlying our observation.

Given reliability tolerance $\epsilon \in (0, 1)$, we quantify the performance of a candidate estimate – a Borel function $\hat{g}(\cdot) : \Omega^K \to \mathbb{R}$ – by the worst case, over $j$ and $x$, width of $(1 - \epsilon)$-confidence interval. Specifically, we say that $\hat{g}(\cdot)$ is $(\rho, \epsilon)$-reliable, if

$$\forall (j \leq I, x \in X_j) : \text{Prob}_{\omega \sim p_{A_j(x)}} \{|\hat{g}(\omega) - g^T x| > \rho\} \leq \epsilon.$$  

We define $\epsilon$-risk of the estimate as the smallest $\rho$ such that $\hat{g}$ is $(\rho, \epsilon)$-reliable:

$$\text{Risk}_\epsilon[\hat{g}] = \inf \{\rho : \hat{g} \text{ is } (\rho, \epsilon)\text{-reliable}\}.$$  

3.2 The estimate

Let us associate with a pair $(i, j)$, $1 \leq i, j \leq I$, the functions

$$\Phi_{ij}(\alpha, \phi; x, y) = \frac{1}{2} K \alpha \left[\Phi_\mathcal{O}(\phi/\alpha; A_i(x)) + \Phi_\mathcal{O}(-\phi/\alpha; A_j(y))\right] + \frac{1}{2} g^T[y - x] + \alpha \ln(2I/\epsilon):$$

$$\{\alpha > 0, \phi \in \mathcal{F}\} \times [X_i \times X_j] \to \mathbb{R},$$

$$\Psi_{ij}(\alpha, \phi) = \max_{x \in X_i, y \in X_j} \Phi_{ij}(\alpha, \phi; x, y) = \frac{1}{2} \left[\Psi_{i,+}(\alpha, \phi) + \Psi_{j,-}(\alpha, \phi)\right]: \{\alpha > 0\} \times \mathcal{F} \to \mathbb{R},$$

where

$$\Psi_{i,+}(\beta, \psi) = \max_{x \in X_i} \left[K \beta \Phi_\mathcal{O}(\psi/\beta; A_i(x)) - g^T x + \beta \ln(2I/\epsilon)\right]: \{\beta > 0, \psi \in \mathcal{F}\} \to \mathbb{R},$$

$$\Psi_{j,-}(\beta, \psi) = \max_{x \in X_j} \left[K \beta \Phi_\mathcal{O}(-\psi/\beta; A_j(x)) + g^T x + \beta \ln(2I/\epsilon)\right]: \{\beta > 0, \psi \in \mathcal{F}\} \to \mathbb{R}$$

and $\Phi_\mathcal{O}$ is given by (2). Note that the function $\alpha \Phi_\mathcal{O}(\phi/\alpha; A_i(x))$ is obtained from continuous convex-concave function $\Phi_\mathcal{O}(\cdot, \cdot)$ by projective transformation in the convex argument, and affine substitution in the concave argument, so that the former function is convex-concave and continuous on the domain $\{\alpha > 0, \phi \in \mathcal{F}\} \times X_i$. By similar argument, the function $\alpha \Phi_\mathcal{O}(-\phi/\alpha; A_j(y))$ is convex-concave and continuous on the domain $\{\alpha > 0, \phi \in \mathcal{F}\} \times X_j$. These observations combine with compactness of $X_i$ and $X_j$ to imply that $\Psi_{ij}(\alpha, \phi)$ is real-valued continuous convex function on the domain

$$\mathcal{F}^+ = \{\alpha > 0\} \times \mathcal{F}.$$
Observe that functions $\Psi_{ij}(\alpha, \phi)$ are nonnegative on $\mathcal{F}^+$. Indeed, selecting somehow $\bar{x} \in X_t$, and setting $\mu = A_t(\bar{x})$, we have

$$
\Psi_{ij}(\alpha, \phi) \geq \Phi_{ij}(\alpha, \phi; \bar{x}, \bar{x}) = \frac{1}{2} K \Phi_O(\phi/\alpha: \mu) + \Phi_O(-\phi/\alpha; \mu) + 2 \ln(2I/\epsilon)
$$

where the infimum is taken over all $\mu = A_t(\bar{x})$, we have

$$
\frac{1}{2} K \ln \left( \int \exp\{\phi(\omega)/\alpha\} p_\mu(\omega) P(d\omega) \right) \geq \alpha \ln(2I/\epsilon) > 0.
$$

Functions $\Psi_{ij}$ give rise to convex and feasible optimization problems

$$
\text{Opt}_{ij} = \text{Opt}_{ij}(K) = \min_{\alpha, \phi} \left\{ \Psi_{ij}(\alpha, \phi) : (\alpha, \phi) \in \mathcal{F}^+ \right\}.
$$

By its origin, $\text{Opt}_{ij}$ is either a real, or $-\infty$; by the observation above, $\text{Opt}_{ij}$ are nonnegative. Our estimate is as follows.

1. For $1 \leq i, j \leq I$, we select somehow feasible solutions $\alpha_{ij}, \phi_{ij}$ to problems (3) (the less the values of the corresponding objectives, the better) and set

$$
\rho_{ij} = \Psi_{ij}(\alpha_{ij}, \phi_{ij}) = \frac{1}{2} [\Psi_{ij,+}(\alpha_{ij}, \phi_{ij}) + \Psi_{ij,-}(\alpha_{ij}, \phi_{ij})]
$$

$$
x_{ij} = \frac{1}{2} [\Psi_{ij,-}(\alpha_{ij}, \phi_{ij}) - \Psi_{ij,+}(\alpha_{ij}, \phi_{ij})]
$$

$$
g_{ij}(\omega^K) = \sum_{k=1}^K \phi_{ij}(\omega_k) + x_{ij}
$$

2. Given observation $\omega^K$, we specify the estimate $\hat{g}(\omega^K)$ as follows:

$$
\hat{g}(\omega^K) = \frac{1}{2} \left[ \min_{i \leq I} r_i + \max_{j \leq I} c_j \right] \text{ with } r_i = \max_{j \leq I} g_{ij}(\omega^K), c_j = \min_{i \leq I} g_{ij}(\omega^K)
$$

**Proposition 3.1** For $i \in \{1, \ldots, I\}$, let $\rho_i = \max_{1 \leq j \leq I} \max[\rho_{ij}, \rho_{ji}]$, and let

$$
\rho = \max_i \rho_i = \max_{1 \leq i, j \leq I} \rho_{ij}.
$$

Assume that the density $p$ taken w.r.t. $P^K$, of the distribution of the $K$-repeated observation $\omega^K$ is $p^K_{A_t(x)}$ for some $\ell \leq I$ and $x \in X_t$ (i.e., the signal $x$ underlying observation satisfies $x \in X_t$ for some $\ell \in \{1, \ldots, I\}$). Then

$$
\Pr_{\omega^K \sim p^K_{A_t(x)}} \left\{ |\hat{g}(\omega^K) - g^T(x) > \rho_i \right\} \leq \epsilon.
$$

As a result, the $\epsilon$-risk of the estimate we have built satisfies

$$
\text{Risk}_\epsilon[\hat{g}(\cdot)] \leq \rho.
$$

See Section A.1 for the proof.

Observe that by properly selecting $\phi_{ij}$ and $\alpha_{ij}$ we can make, in a computationally efficient manner, the upper bound $\rho$ on the $\epsilon$-risk of the above estimate arbitrarily close to

$$
\text{Opt}(K) = \max_{1 \leq i, j \leq I} \text{Opt}_{ij}(K).
$$

We are about to show that the quantity $\text{Opt}(K)$ “nearly lower-bounds” the minimax optimal $\epsilon$-risk

$$
\text{Risk}_\epsilon^*(K) = \inf_{\hat{g}(\cdot)} \text{Risk}_\epsilon[\hat{g}],
$$

where the infimum is taken over all $K$-observation Borel estimates. The precise statement is as follows:
Proposition 3.2 In the situation of this Section, let \( \epsilon \in (0, 1/2) \) and \( \bar{K} \) be a positive integer. Then for every integer \( K \) satisfying
\[
K > \frac{2\ln(2I/\epsilon)}{\ln([4\epsilon(1-\epsilon)]^{-1})} \bar{K}
\]
one has
\[
\text{Opt}(K) \leq \text{Risk}^*_K(\bar{K}).
\]
In addition, in the special case where for every \( i, j \) there exists \( \bar{x}_{ij} \in X_i \cap X_j \) such that \( A_i(\bar{x}_{ij}) = A_j(\bar{x}_{ij}) \) one has
\[
K \geq \bar{K} \Rightarrow \text{Opt}(K) \leq \frac{2\ln(2I/\epsilon)}{\ln([4\epsilon(1-\epsilon)]^{-1})} \text{Risk}^*_K(\bar{K}).
\]
See Section A.2 for the proof.

3.3 Illustration

We illustrate our construction by applying it to the simplest possible example in which the observation scheme is Gaussian and \( X_i = \{x_i\} \) are singletons in \( \mathbb{R}^n \), \( i = 1, \ldots, I \). Setting \( y_i = A_i(x_i) \in \mathbb{R}^m \), the observation components \( \omega_k, 1 \leq k \leq K \), stemming from signal \( x_i \), are drawn independently of each other from the normal distribution \( \mathcal{N}(y_i, I_m) \). Recall that in the Gaussian o.s. \( \mathcal{F} \) is comprised of affine functions \( \phi(\omega) = \phi_0 + \sum_{i=1}^n \phi_i \omega_i =: \phi_0 + \varphi^T \omega \) on the observation space (which now is \( \mathbb{R}^m \)), and, as is immediately seen,
\[
\Phi_{\mathcal{O}}(\phi; \mu) = \phi_0 + \varphi^T \mu + \frac{1}{2} \varphi^T \varphi : (\mathbb{R} \times \mathbb{R}^m) \times \mathbb{R}^m \to \mathbb{R}.
\]
A straightforward computation shows that in the case in question, using the notation \( \theta = \ln(2I/\epsilon) \), we get
\[
\begin{align*}
\Psi^-_{i,+(\alpha, \phi)} &= K\phi_0/\alpha + \frac{1}{2} \varphi^T y_i/\alpha + \frac{1}{2} \varphi^T \varphi/\alpha^2 + \alpha \theta - g^T x_i = K\phi_0 + K\varphi^T y_i - g^T x_i + K \frac{1}{2} \varphi^T \varphi + \alpha \theta \\
\Psi^{+}_{i,-(\alpha, \phi)} &= -K\phi_0 - K\varphi^T y_i + g^T x_i + K \frac{1}{2} \varphi^T \varphi + \alpha \theta \\
\text{Opt}_{i,j} &= \inf_{\alpha>0,\phi} \frac{1}{2} \left[ \Psi^{+}_{i,+(\alpha, \phi)} + \Psi^{-}_{j,-(\alpha, \phi)} \right] \\
&= \frac{1}{2} g^T [x_j - x_i] + \inf_{\varphi \in \mathbb{R}^m} \left[ K \frac{1}{2} \varphi^T [y_i - y_j] + \inf_{\alpha>0} \left[ K \frac{1}{2} \varphi^T \varphi + \alpha \theta \right] \right] \\
&= \frac{1}{2} g^T [x_j - x_i] + \inf_{\varphi} \left[ K \varphi^T [y_i - y_j] + \sqrt{2K\theta} \|\varphi\|_2 \right] \\
&= \begin{cases} \\
\frac{1}{2} g^T [x_j - x_i], & \|y_i - y_j\|_2 \leq 2\sqrt{2\theta/K}, \\
-\infty, & \|y_i - y_j\|_2 > 2\sqrt{2\theta/K}.
\end{cases}
\end{align*}
\]
We see that we can safely set \( \phi_0 = 0 \), and that setting
\[
\mathcal{I} = \{(i, j) : \|y_i - y_j\|_2 \leq 2\sqrt{2\theta/K}, \}
\]
Opt_{i,j}(K) is finite when \( (i, j) \in \mathcal{I} \) and is \( -\infty \) otherwise; in both cases, the optimization problem specifying \( \text{Opt}_{i,j} \) has no optimal solution. Indeed, this clearly is the case when \( (i, j) \notin \mathcal{I} \); when \( (i, j) \in \mathcal{I} \), a minimizing sequence is, e.g., \( \phi \equiv 0, \alpha_i \to 0 \), but its limit is not in the minimization domain (on this domain, \( \alpha \) should be positive). \(^4\) In the considered example, the simplest way to overcome the difficulty is to restrict the optimization domain \( \mathcal{F}^+ \) in (3) with its compact subset \( \{\alpha \geq 1/R, \phi_0 = 0, \|\varphi\|_2 \leq R\} \) with a large \( R \) (e.g. \( R = 10^{10} - 10^{20} \)). Therefore, we specify the entities participating in (4) as
\[
\phi_{ij}(\omega) = \varphi_{ij}^T \omega, \quad \varphi_{ij} = \begin{cases} \\
0, & (i, j) \in \mathcal{I}, \\
-R[y_i - y_j]/\|y_i - y_j\|_2, & (i, j) \notin \mathcal{I}.
\end{cases} \quad \alpha_{ij} = \begin{cases} \\
1/R, & (i, j) \in \mathcal{I}, \\
\sqrt{\frac{1}{2\theta}} R, & (i, j) \notin \mathcal{I}.
\end{cases}
\]
\(^4\)Dealing with this case was exactly the reason why in our construction we required from \( \phi_{ij}, \alpha_{ij} \) to be feasible, and not necessary optimal, solutions to the optimization problems in question.
resulting in

\[
\begin{align*}
\kappa_{ij} &= \frac{1}{2} [\Psi_{j,-}(\alpha_{ij}, \phi_{ij}) - \Psi_{i,+}(\alpha_{ij}, \phi_{ij})] = \frac{1}{2} g^T [x_i + x_j] - \frac{K}{2} \varphi_{ij}^T [y_i + y_j] \\
\rho_{ij} &= \frac{1}{2} [\Psi_{i,+}(\alpha_{ij}, \phi_{ij})] = \frac{K}{2 \alpha_{ij}} \varphi_{ij}^T + \alpha_{ij} \theta + \frac{1}{2} g^T [x_j - x_i] + \frac{K}{2} \varphi_{ij}^T [y_i - y_j] \\
&= \begin{cases} \\
\frac{1}{2} g^T [x_j - x_i] + R^{-1} \theta, & (i, j) \in \mathcal{I} \\
\frac{1}{2} g^T [x_j - x_i] + \sqrt{2 K \theta} - \frac{K}{2} \| y_i - y_j \|_2^2 R, & (i, j) \notin \mathcal{I}
\end{cases}
\end{align*}
\] (11)

In the numerical experiments we report below we use \( n = 20, m = 10, \) and \( I = 100, \) with \( x_i, i \leq I, \) drawn independently of each other from \( \mathcal{N}(0, I_n), \) and \( y_i = A x_i \) with randomly generated matrix \( A \) (namely, matrix with independent \( \mathcal{N}(0, 1) \) entries normalized to have unit spectral norm). The linear form to be recovered is the first coordinate of \( x, \) the confidence parameter is set to \( \epsilon = 0.01, \) and \( R = 10^{20}. \) The results of typical experiment are presented in Figure 1.

Figure 1: Empirical distributions, over 20 random estimation problems, of the upper 0.01-risk bounds \( \max_{1 \leq i, j \leq 100} \rho_{ij} \) as in (11) for different observation sample sizes \( K. \)

## 4 Recovering \( N \)-convex functions on unions of convex sets

### 4.1 Preliminaries: testing convex hypotheses in good o.s.

What follows is a summary of relevant to our goals results of [15]. Assume that \( \omega^K = (\omega_1, ..., \omega_K) \) is a stationary \( K \)-repeated observation in a good o.s. \( \mathcal{O} = (\Omega, P, \{p_\mu : \mu \in \mathcal{M}\}, \mathcal{F}), \) so that \( \omega_1, ..., \omega_K \) are, independently of each other, drawn from a distribution \( p_\mu \) with some \( \mu \in \mathcal{M}. \) Given \( \omega^K \) we want to decide on the hypotheses \( H_1 \) and \( H_2, \) with \( H_X, \chi = 1, 2, \) stating that \( \omega_t \sim p_\mu \) for some \( \mu \in M_\chi, \) where \( M_\chi \) is a nonempty convex compact subset of \( \mathcal{M}. \) In the sequel, we refer to hypotheses of this type, parameterized by nonempty convex compact subsets of \( \mathcal{M}, \) as to convex hypotheses in the good o.s. in question.

The principal “building block” of our subsequent constructions is a test \( T^K \) for this problem which is as follows:

- Given convex compact sets \( M_\chi, \chi = 1, 2, \) we solve the optimization problem

\[
\text{Opt} = \max_{\mu \in M_1, \nu \in M_2} \ln \left( \int_{\Omega} \sqrt{p_\mu(\omega)p_\nu(\omega)} P(d\omega) \right)
\] (12)
It is shown in [15] that in the case of good o.s., problem (12) is a solvable convex problem (convexity meaning that the objective to be maximized is a concave continuous function of \( \mu, \nu \)).

Note that for basic good o.s.’s problem (12) reads

\[
\begin{align*}
\text{Opt} &= \max_{\mu \in M_1, \nu \in M_2} \begin{cases} \\
-\frac{1}{8} \| \mu - \nu \|^2_2, & \text{Gaussian o.s.} \\
-\frac{1}{2} \sum_{i=1}^d [\sqrt{\mu_i} - \sqrt{\nu_i}]^2, & \text{Poisson o.s.} \\
\ln \left( \sum_{i=1}^d \mu_i \nu_i \right), & \text{Discrete o.s.}
\end{cases}
\end{align*}
\]

(13)

• An optimal solution \( \mu^*, \nu^* \) to (12) induces detectors

\[
\begin{align*}
\phi_*(\omega) &= \frac{1}{2} \ln(\frac{p_{\mu^*}(\omega)}{p_{\nu^*}(\omega)}): \Omega \to \mathbb{R}, \\
\phi_*^{(K)}(\omega^K) &= \sum_{t=1}^K \phi_*(\omega^K_t): \Omega \times \ldots \times \Omega \to \mathbb{R}
\end{align*}
\]

(14)

Given a stationary \( K \)-repeated observation \( \omega^K \), the test \( T^K \) accepts hypothesis \( H_1 \) and rejects hypothesis \( H_2 \) whenever \( \phi_*^{(K)}(\omega^K) \geq 0 \), otherwise the test rejects \( H_1 \) and accepts \( H_2 \). The risk of \( T^K \) – the maximal probability to reject a hypothesis when it is true – does not exceed \( \epsilon_*^{(K)} \), where

\[ \epsilon_* = \exp(\text{Opt}). \]

In other words, whenever observation \( \omega^K \) stems from a distribution \( p_{\mu} \) with \( \mu \in M_1 \cup M_2 \),

– the \( p_{\mu^*} \)-probability to reject \( H_1 \) when the hypothesis is true (i.e., when \( \mu \in M_1 \)) is at most \( \epsilon_*^{(K)} \), and

– the \( p_{\mu^*} \)-probability to reject \( H_2 \) when the hypothesis is true (i.e., when \( \mu \in M_2 \)) is at most \( \epsilon_*^{(K)} \).

The test \( T^K \) possesses the following optimality properties:

A. The associated detector \( \phi_*^{(K)} \) and the risk \( \epsilon_*^{(K)} \) form an optimal solution and the optimal value in the optimization problem

\[
\begin{align*}
\min_{\phi} \max_{\mu \in M_1} \int_{\Omega^K} e^{-\phi(\omega^K)} p_{\mu^*}(\omega^K) P^K(d\omega^K), \\
\max_{\nu \in M_2} \int_{\Omega^K} e^{\phi(\omega^K)} p_{\nu^*}(\omega^K) P^K(d\omega^K)
\end{align*}
\]

where the minimum is taken w.r.t. all Borel functions \( \phi(\cdot): \Omega^K \to \mathbb{R} \);

B. Let \( \epsilon \in (0, 1/2) \), and suppose that there exists a test which, using a stationary \( K \)-repeated observation, decides on the hypotheses \( H_1, H_2 \) with risk \( \leq \epsilon \). Then

\[ \epsilon_* \leq [2\sqrt{\epsilon(1-\epsilon)}]^{1/K} \]

(15)

and the test \( T^K \) with

\[
K = \left\lfloor \frac{2 \ln(1/\epsilon)}{\ln(\ln(\epsilon^{-1}))} \right\rfloor
\]

decides on the hypotheses \( H_1, H_2 \) with risk \( \leq \epsilon \) as well. Note that \( K = 2(1+o(1))K \) as \( \epsilon \to +0 \).
“Inferring colors:” testing multiple hypotheses in good o.s. As shown in [15], the just outlined near-optimal pairwise tests deciding on pairs of convex hypotheses in good o.s.’s can be used as building blocks when constructing near-optimal tests deciding on multiple convex hypotheses. In the sequel, we will repeatedly use one of these constructions, namely, as follows.

Assume that we are given a good o.s. $\mathcal{O} = (\Omega, P), \{p_\mu : \mu \in \mathcal{M}\}, \mathcal{F}$ and two finite collections of nonempty convex compact subsets $B_1, \ldots, B_b$ (“blue sets”) and $R_1, \ldots, R_r$ (“red sets”) of $\mathcal{M}$. Our objective is, given a stationary $K$-repeated observation $\omega^K$ stemming from a distribution $p_\mu$, $\mu \in \mathcal{M}$, to infer the color of $\mu$, that is, to decide on the hypothesis $\mu \in B := B_1 \cup \ldots \cup B_b$ vs. the alternative $\mu \in R := R_1 \cup \ldots \cup R_r$. To this end we act as follows:

1. For every pair $i, j$ with $i \leq b$ and $j \leq r$, we solve the problem (13) with $B_i$ in the role of $M_1$ and $R_j$ in the role of $M_2$; we denote $\text{Opt}_{ij}$ the associated optimal values. The corresponding optimal solutions $\mu_{ij}$ and $\nu_{ij}$ give rise to the detectors

$$\phi_{ij}(\omega) = \frac{1}{2} \ln \left( \frac{p_{\mu_{ij}}(\omega)}{p_{\nu_{ij}}(\omega)} \right) : \Omega \to \mathbb{R}, \quad \phi^{(K)}_{ij} = \sum_{t=1}^{K} \phi_{ij}(\omega_t) : \Omega^K \to \mathbb{R}$$  \hspace{1cm} (16)

(cf. (14)) and risks

$$\epsilon_{ij} = \exp(\text{Opt}_{ij}) = \int_\Omega \sqrt{p_{\mu_{ij}}(\omega)p_{\nu_{ij}}(\omega)} P(d\omega).$$  \hspace{1cm} (17)

2. We build the entrywise positive $b \times r$ matrix $E^{(K)} = [e^{(K)}_{ij}]_{1 \leq i \leq b, 1 \leq j \leq r}$ and symmetric entrywise non-negative $(b + r) \times (b + r)$ matrix $E_K = \left[ \frac{1}{\sqrt{\mu_{ij}}\nu_{ij}} e^{(K)}_{ij} \right]$. Let $\epsilon_K$ be the spectral norm of the matrix $E^{(K)}$ (equivalently, spectral norm of $E_K$), and let $e = [g; h]$ be the Perron-Frobenius eigenvector of $E_K$, so that $e$ is a nontrivial nonnegative vector such that $E_K e = \epsilon_K e$. Note that from entrywise positivity of $E^{(K)}$ it immediately follows that $\epsilon > 0$, so that the quantities

$$\alpha_{ij} = \ln(h_j/g_i), \quad 1 \leq i \leq b, 1 \leq j \leq r$$

are well defined. We set

$$\psi^{(K)}_{ij}(\omega^K) = \phi^{(K)}_{ij}(\omega^K) - \alpha_{ij} = \sum_{t=1}^{K} \phi_{ij}(\omega_t) - \alpha_{ij} : \Omega^K \to \mathbb{R}, \quad 1 \leq i \leq b, 1 \leq j \leq r$$  \hspace{1cm} (18)

3. Given observation $\omega^K \in \Omega^K$ with $\omega_t$, $t = 1, \ldots, K$, drawn, independently of each other, from a distribution $p_\mu$, we claim that $\mu$ is blue (equivalently, $\mu \in B$), if there exists $i \leq b$ such that $\psi_{ij}(\omega^K) \geq 0$ for all $j = 1, \ldots, r$, and claim that $\mu$ is red (equivalently, $\mu \in R$) otherwise.

The main result about the just described “color inferring” test is as follows

**Proposition 4.1** [15, Proposition 3.2] Let the components $\omega_t$ of $\omega^K$ be drawn, independently of each other, from distribution $p_\mu \in B \cup R$. Then the just defined test, for every $\omega^K$, assigns $\mu$ with exactly one color, blue or red, depending on the observation. Moreover,

- when $\mu$ is blue (i.e., $\mu \in B$), the test makes correct inference “$\mu$ is blue” with $p_\mu$-probability at least $1 - \epsilon_K$;
- similarly, when $\mu$ is red (i.e., $\mu \in Rb$), the test makes correct inference “$\mu$ is red” with $p_\mu$-probability at least $1 - \epsilon_K$.
4.2 Problem’s setting

In the sequel, we deal with the situation as follows. Given are:

1. good o.s. $\mathcal{O} = ((\Omega, P), \{p_\mu(\cdot) : \mu \in \mathcal{M}\}, \mathcal{F})$,

2. convex compact set $\mathcal{X} \subset \mathbb{R}^n$ along with a collection of $I$ convex compact sets $X_i \subset \mathcal{X}$,

3. affine “encoding” $x \mapsto A(x) : \mathcal{X} \to \mathcal{M}$,

4. a continuous function $f(x) : \mathcal{X} \to \mathbb{R}$ which is $N$-convex, meaning that for every $a \in \mathbb{R}$ the sets $\mathcal{X}_a^{\geq} = \{x \in \mathcal{X} : f(x) \geq a\}$ and $\mathcal{X}_a^{\leq} = \{x \in \mathcal{X} : f(x) \leq a\}$ can be represented as unions of at most $N$ closed convex sets $\mathcal{X}_a^{\geq}, \mathcal{X}_a^{\leq}$:

$$\mathcal{X}_a^{\geq} = \bigcup_{\nu=1}^{N} \mathcal{X}_a^{\geq} \nu, \mathcal{X}_a^{\leq} = \bigcup_{\nu=1}^{N} \mathcal{X}_a^{\leq}.$$ (19)

For some unknown $x$ known to belong to $X = \bigcup_{i=1}^{I} X_i$, we have at our disposal observation $\omega^K = (\omega_1, ..., \omega_K)$ with i.i.d. $\omega_i \sim p_{\lambda_i(\cdot)}$, and our goal is to estimate from this observation the quantity $\hat{f}(\omega^K)$.

The $\epsilon$-risk of a candidate estimate $\hat{f} (\omega^K)$ is defined in the same way it was done in Section 3.1. Specifically, given tolerances $\rho > 0$, $\epsilon \in (0, 1)$, we call $\hat{f}(\omega^K) \rho, \epsilon$-reliable, if for every $x \in \mathcal{X}$, $|\hat{f}(\omega^K) - f(x)| \leq \rho$ with the $p_{\lambda_i(\cdot)}$-probability at least $1 - \epsilon$. Then $\epsilon$-risk of $\hat{f}(\omega^K)$ is the smallest $\rho$ such that $\hat{f}(\cdot)$ is $\rho, \epsilon$-reliable.

**Examples of N-convex functions.** In the above problem setting we allow for rather “complex” sets $X$ – finite unions of convex sets – and a specific class of functions $f$, assumed to be $N$-convex. Being rather restrictive, this class comprises, along with linear functions, some interesting examples, which we discuss below.

**Example 4.1** [Minima and Maxima of linear-fractional functions] *Every function which can be obtained from linear-fractional functions $g_{\nu}(\cdot)$, $h_{\nu}$ are affine functions on $\mathcal{X}$, and $h_{\nu}$ are positive on $\mathcal{X}$) by taking maxima and minima is N-convex for appropriately selected $N$ due to the following immediate observations:*

- linear-fractional function $g(x) \over h(x)$ with positive on $\mathcal{X}$ denominator is $1$-convex;
- if $f(x)$ is $N$-convex, so is $-f(x)$;
- if $f_i(x)$ is $N_i$-convex, $i = 1, 2, ..., I$, then $f(x) = \max_i f_i(x)$ is $\max\prod_i N_i, \sum_i N_i$-convex, due to

$$\{ x \in \mathcal{X} : f(x) \leq a \} = \bigcap_{i=1}^{I} \{ x \in \mathcal{X} : f_i(x) \leq a \}, \{ x \in \mathcal{X} : f(x) \geq a \} = \bigcup_{i=1}^{I} \{ x \in \mathcal{X} : f_i(x) \geq a \}.$$ 

Note that the first set is the intersection of $I$ unions of convex sets with $N_i$ components in $i$-th union, and thus is the union of $\prod_i N_i$ convex sets; the second set is the union of $I$ unions, $N_i$ components in $i$-th of them, of convex sets, and thus is the union of $\sum_i N_i$ convex sets.
happen only in the case when both tested hypotheses are wrong. In contrast, when neither hypothesis is true, the accepted hypothesis can be wrong as well, but this can only happen if both hypotheses are incorrect.

In the sequel, we sometimes use pairwise hypothesis tests in the situation where neither hypothesis is true. For this reason, we start its presentation with an informal outline, which exposes some simple ideas underlying the construction.

### Example 4.2 (Conditional quantile)

Let \( S = \{s_1 < s_2 < \ldots < s_M\} \subset \mathbb{R} \), \( T \) be a finite set, and let \( X\) be a convex compact set in the space of nonvanishing probability distributions on \( S \times T \). We identify \( x \in X\) with the array \( \{x(\mu, t)\}_{1 \leq \mu \leq M, t \in T} \), where \( x(\mu, t) \) is the \( x\)-probability of the point \((s_\mu, t)\) in \( S \times T \). Given \( \alpha \in (0, 1) \) and \( \tau \in T \), we define the conditional \( \alpha \)-quantile \( q_{\alpha}[x]\) of a distribution \( x \in X\) as follows. For a nonvanishing probability distribution \( r = [r_1; \ldots; r_M]\) on \( S \), let the distribution \( \bar{r}\) on \( [s_1, s_M]\) be obtained by assigning mass \( r_1\) to the point \( s_1\), and spreading uniformly over \([s_\mu, s_{\mu+1}]\) the masses \( r_{\mu+1}, 1 \leq \mu < M - 1\). The regularized \( \alpha \)-quantile of \( r\) is defined as the usual \( \alpha \)-quantile of \( \bar{r}\):

\[
q_{\alpha}[r] = \min \{ s \in [s_1, s_M] : \bar{r}([s_1, s]) \geq \alpha \}.
\]

Finally, the conditional \( \alpha \)-quantile \( q_{\alpha}[x]\) of a distribution \( x \in X\) is the regularized \( \alpha \)-quantile of the conditional by \( t = \tau\) distribution \( x(\cdot | \tau)\) on \( S\) induced by \( x\):

\[
q_{\alpha}[x] = \min \{ s \in [s_1, s_M] : \bar{r}([s_1, s]) \geq \alpha \}, \text{ where } r_\mu = x(\mu | \tau) := \frac{x(\mu, \tau)}{\sum_{\nu=1}^{M} x(\nu, \tau)}, 1 \leq \mu \leq M.
\]

Function \( q_{\alpha}[x] : X \to \mathbb{R} \) turns out to be 2-convex, see Appendix B.

### 4.3 Bisection Estimate

As we have already mentioned, the proposed estimation procedure is a “close relative” of the binary search algorithm of [12], but is not identical to that algorithm. Though the bisection estimator is, in a nutshell, quite simple, its formal description turns out to be rather involved. For this reason, we start its presentation with an informal outline, which exposes some simple ideas underlying the construction.

#### 4.3.1 Outline

Let us consider a simple situation where the signal space \( X\) is a convex set in \( \mathbb{R}^2\), as presented in Figure 2, and suppose that our objective is to estimate the value of a linear function \( f(x) = x_1\) at \( x = [x_1 ; x_2] \in X\) given a Gaussian observation \( \omega\) with mean \( A(x)\), where \( A(\cdot)\) is a given affine mapping, and known covariance. Observe that hypotheses \( f(x) \geq b \) and \( f(x) \leq a\) translate into convex hypotheses on the expectation of the observed Gaussian r.v., so that we can use the hypothesis testing machinery of Section 4.1 to decide on hypotheses of this type and to localize \( f(x)\) in a (hopefully, small) segment by a bisection-type process. Before describing the process, let us make a terminological agreement. In the sequel, we sometimes use pairwise hypothesis tests in the situation where neither of the hypotheses is true. In this case, we say that the outcome of a test is correct, if the rejected hypothesis indeed is wrong; in this case, the accepted hypothesis can be wrong as well, but this can happen only in the case when both tested hypotheses are wrong.

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Figure 2: Bisection via hypothesis testing. (a): set \( X\) of signals and initial localizer \([a, b]\) for the value of \( f(x) = x_1\); (b): blue hypothesis \( H_1 = \{x \in X_1\}\) and red hypothesis \( H_2 = \{x \in X_2\}\); (c): blue hypotheses \( H'_1 = \{x \in X'_1\}\) and red hypothesis \( H'_2 = \{x \in X'_2\}\).
Let $\epsilon \in (0, 1)$ and let $L$ be a positive integer. The estimation procedure is organized in steps. At the beginning of the first step $\Delta_1 = [a, b]$ with $a = \min_{x \in X} x_1$, and $b = \max_{x \in X} x_1$, is the current localizer for the value of $f(x) = x_1$, see Figure 2, and let $c = \frac{1}{2}(a + b)$. To compute the new localizer, we run a pair of Blue vs. Red tests $T$ and $T'$, such that

- $T$ decides upon the “left hypothesis” $H_1 = \{x \in X : x_1 \leq \ell\}$ (blue) vs. $H_2 = \{x \in X : x_1 \geq c\}$ (red), where $\ell < c$ is as close to $c$ as possible under the restriction that $T$ decides on $H_1$, $H_2$ with risk $\leq \frac{\epsilon}{2L}$;
- $T'$ decides upon the “right hypothesis” $H'_1 = \{x \in X : x_1 \leq c\}$ (blue) vs. $H'_2 = \{x \in X : x_1 \geq r\}$ (red), where $r > c$ is as close to $c$ as possible under the restriction that $T'$ decides on $H'_1, H'_2$ with risk $\leq \frac{\epsilon}{2L}$.

Assuming that both tests rejected wrong hypotheses (this happens with probability at least $1 - \frac{\epsilon}{L}$), the results of the tests allow for the following conclusions:

- when both tests reject red hypotheses from the corresponding pairs, it is certain that $x_1 \leq c$ (since otherwise in the first test the rejected hypothesis were in fact true, contradicting the assumption that both tests make no wrong rejections);
- when both tests reject blue hypotheses from the corresponding pairs, it is certain that $x_1 \geq c$ (for the same reasons as in the previous case);
- when the tests “disagree,” rejecting hypotheses of different colors, $x_1 \in [\ell, r]$. Indeed, otherwise either $x_1 \leq \ell$ (and thus $x$ is “colored blue” in both pairs of hypotheses), or $x_1 \geq r$ (and $x$ is “colored red” in both pairs). Since we have assumed that in both tests no wrong rejections took place, in the first case both tests must reject red hypotheses, and both should reject blue ones in the second, while none of these events took place.

In the first two cases we take the right or the left half of the initial segment $\Delta_1 = [a, b]$ as a new localizer for $f(x) = x_1$ (and the corresponding cut $X \cap \{x_1 \geq c\}$ or $X \cap \{x_1 \leq c\}$ as a new localizer for $x$). In the last case, we take the segment $[\ell, r]$ as a new localizer for $x_1$, terminate the process and output $\hat{f} = \frac{1}{2}(\ell + r)$ as estimate of $f(x)$ – the $\epsilon/L$-risk of this estimate is equal to $\frac{1}{2}(r - \ell)$ and is already small! In Bisection, we iterate the outlined procedure, replacing current localizers with twice smaller ones until terminating either due to running into “disagreement,” or due to reaching a prescribed number $L$ of steps. Upon termination, we return the last localizer as a confidence set for $f(x) = x_1$, and its midpoint – as the estimate of $f(x)$.

Note that, unlike the binary search procedure of [12]), in our procedure the “search trajectory” – the sequence of pairs of hypotheses participating in the tests – is not random, it is uniquely defined by the value of $f(x)$, provided no wrong rejections happen. Indeed, with no wrong rejections prior to termination, the sequence of localizers produced by the procedure is exactly the same as if we were running deterministic bisection algorithm, that is, were updating subsequent localizers $\Delta_{\ell}$ for $f(x)$ according to the rules

- $\Delta_1 = [a, b]$, the obvious initial segment $f(x)$,
- $\Delta_{\ell+1}$ is precisely the half of $\Delta_{\ell}$ containing $f(x)$ (say, the left half in the case of a tie).

In the above argument we neglected the possibility of wrong rejection by one of the tests we run. Since, by construction, the risks of every one of these tests do not exceed $\frac{\epsilon}{2L}$ and, by the above, with no wrong rejections, the sequence of tests we run depends solely on the value $f(x)$, not on the observations (observations can affect only the number of steps before termination), the probability of
wrong rejection in course of running the algorithm is \( \leq \epsilon \). Note that the risks of “individual tests” define, in turn, the allowed width of separators – segments \([\ell, c]\) and \([c, r]\) in Figure 2.b (“uncertainty zone” of the corresponding test), and thus – the accuracy to which \( f(x) \) can be estimated. It should be noted that the number \( L \) of steps of Bisection always is a moderate integer. Indeed, otherwise the width of the separators at the concluding bisection steps (which is of order of \( 2^{-L} \)), would be too small to allow for deciding on the concluding pairs of our hypotheses with risk \( \frac{\epsilon}{2L} \).

From the above sketch of our construction, it is clear that all that matters is our ability to decide, given \( \ell < r \), on the pairs of hypotheses \( \{x \in X : f(x) \leq \ell\} \) and \( \{x \in X : f(x) \geq r\} \) via observation drawn from \( p_A(x) \). In our outline, these were convex hypotheses in Gaussian o.s., and in this case we can use detector-based pairwise tests presented in Section 4.1. Applying the machinery developed in the latter section, we could also handle the case when the sets \( \{x \in X : f(x) \leq \ell\} \) and \( \{x \in X : f(x) \geq r\} \) are unions of a number of convex sets (which is the case when \( f \) is \( N \)-convex and \( X \) is the union of a number of convex sets), the o.s. in question still being good, and this is the situation we intend to consider.

4.3.2 Building the Bisection estimate: preliminaries

While the construction we present below admits numerous refinements, we focus here on its simplest version as follows (for notation, see Section 4.2).

Upper an lower feasibility/infeasibility, sets \( Z_i^{a,\geq} \) and \( Z_i^{a,\leq} \). Let \( a \) be a real. We associate with \( a \) the collection of upper \( a \)-sets defined as follows: we look at the sets \( X_i \cap \mathcal{X}_I^{a,\geq} \), \( 1 \leq i \leq I \), \( 1 \leq \nu \leq N \), and arrange the nonempty sets from this family into a sequence \( Z_i^{a,\geq} \), \( 1 \leq i \leq I_{a,\geq} \). Here \( I_{a,\geq} = 0 \) if all sets in the family are empty; in the latter case, we refer to \( a \) as upper-infeasible, otherwise upper-feasible.

Similarly, we associate with \( a \) the collection of lower \( a \)-sets \( Z_i^{a,\leq} \), \( 1 \leq i \leq I_{a,\leq} \), by arranging into a sequence all nonempty sets from the family \( X_i \cap \mathcal{X}_I^{a,\leq} \). We say that \( a \) is lower-feasible or lower-infeasible depending on whether \( I_{a,\leq} \) is positive or zero. Note that upper and lower \( a \)-sets are nonempty convex compact sets, and

\[
X_i^{a,\geq} := \{x \in X : f(x) \geq a\} = \bigcup_{1 \leq i \leq I_{a,\geq}} Z_i^{a,\geq}, \quad X_i^{a,\leq} := \{x \in X : f(x) \leq a\} = \bigcup_{1 \leq i \leq I_{a,\leq}} Z_i^{a,\leq}.
\] (20)

Right tests. Given a segment \( \Delta = [a, b] \) of positive length with lower-feasible \( a \), we associate with this segment right test – a function \( T^K_{\Delta, s}(\omega^K) \) taking values red and blue, and risk \( \sigma_{\Delta, s} \geq 0 \) – as follows:

1. if \( b \) is upper-infeasible, \( T^K_{\Delta, s}(\cdot) \equiv \text{blue} \) and \( \sigma_{\Delta, s} = 0 \).

2. if \( b \) is upper-feasible, the collections \( \{A(Z_i^{b,\geq})\}_{1 \leq i \leq I_{b,\geq}} \) ("red sets"), \( \{A(Z_j^{b,\leq})\}_{1 \leq j \leq I_{b,\leq}} \) ("blue sets"), are nonempty, and the test is the associated with these sets Inferring Color test from Section 4.1 as applied to the stationary \( K \)-repeated version of \( \mathcal{O} \) in the role of \( \mathcal{O} \), specifically,

- for \( 1 \leq i \leq I_{b,\geq}, 1 \leq j \leq I_{a,\leq} \), we build the detectors \( \phi_{ij\Delta}(\omega^K) = \sum_{t=1}^K \psi_{ij\Delta}(\omega_t) \), with \( \phi_{ij\Delta}(\omega) \) given by

\[
\begin{align*}
\phi_{ij\Delta}(\omega) &= \frac{1}{2} \ln \left( \frac{p_{A(r_{ij\Delta})(\omega)}(\omega)}{P_{A(s_{ij\Delta})(\omega)}} \right), \\
&\in \text{Argmin}_{r_{ij\Delta} \in \mathcal{Z}^{b,\geq}, s_{ij\Delta} \in \mathcal{Z}^{b,\leq}} \ln \left( \int_{\mathcal{O}} \sqrt{p_{A(r_{ij\Delta})(\omega)}(\omega)} P(d\omega) \right), \quad (21)
\end{align*}
\]
\[ \epsilon_{ij} = \int_{\Omega} \sqrt{p_A(r_{ij}\Delta)}(\omega)p_A(s_{ij}\Delta)(\omega)P(d\omega) \]  

(22)

and build the \( I_{b\leq} \times I_{a\geq} \) matrix \( E_{\Delta,r} = [\epsilon_{ij}^K\Delta]_{1\leq i \leq t_{b\leq} ; 1\leq j \leq t_{a\geq}} \).

- \( \sigma_{\Delta,r} \) is defined as the spectral norm of \( E_{\Delta,r} \). We compute the Perron-Frobenius eigenvector \( [g^{\Delta,r}, h^{\Delta,r}] \) of the matrix \( \begin{bmatrix} E_{\Delta,r} & E_{\Delta,r} \end{bmatrix} \), so that (see Section 4.1)

\[ g^{\Delta,r} > 0, h^{\Delta,r} > 0, \sigma_{\Delta,r} g^{\Delta,r} = E_{\Delta,r} h^{\Delta,r}, \sigma_{\Delta,r} h^{\Delta,r} = E_{\Delta,r}^T g^{\Delta,r}. \]

Finally, we define the matrix-valued function

\[ D_{\Delta,r}(\omega^K) = [\phi_{ij}^K(\omega^K) + \ln(h^{\Delta,r}) - \ln(g^{\Delta,r})]_{1\leq i \leq t_{b\leq} ; 1\leq j \leq t_{a\geq}}. \]

Test \( T_{\Delta,r}(\omega^K) \) takes value \textbf{red} iff the matrix \( D_{\Delta,r}(\omega^K) \) has a nonnegative row, and takes value \textbf{blue} otherwise.

Given \( \delta > 0 \) and \( \kappa > 0 \), we call segment \( \Delta = [a, b] \) \( \delta \)-good (right), if \( a \) is lower-feasible, \( b > a \), and \( \sigma_{\Delta,r} \leq \delta \). We call a \( \delta \)-good (right) segment \( \Delta = [a, b] \) \( \kappa \)-maximal, if the segment \( [a, b - \kappa] \) is not \( \delta \)-good (right).

**Left tests.** The “mirror” version of the above is as follows. Given a segment \( \Delta = [a, b] \) of positive length with upper-feasible \( b \), we associate with this segment left test – a function \( T_{\Delta,l}(\omega^K) \) taking values \textbf{red} and \textbf{blue}, and risk \( \sigma_{\Delta,l} \geq 0 \) – as follows:

1. if \( a \) is lower-infeasible, \( T_{\Delta,l}(\cdot) \equiv \textbf{red} \) and \( \sigma_{\Delta,l} = 0 \).

2. if \( a \) is lower-feasible, we set \( T_{\Delta,l}^K \equiv T_{\Delta,r}^K \), \( \sigma_{\Delta,l} = \sigma_{\Delta,r} \).

Given \( \delta > 0 \), \( \kappa > 0 \), we call segment \( \Delta = [a, b] \) \( \delta \)-good (left), if \( b \) is upper-feasible, \( b > a \), and \( \sigma_{\Delta,l} \leq \delta \). We call a \( \delta \)-good (left) segment \( \Delta = [a, b] \) \( \kappa \)-maximal, if the segment \( [a + \kappa, b] \) is not \( \delta \)-good (left).

**Remark:** note that when \( a < b \) and \( a \) is lower-feasible, \( b \) is upper-feasible, so that the sets

\[ X^{a\leq} = \{ x \in X : f(x) \leq a \}, \quad X^{b\geq} = \{ x \in X : f(x) \geq b \} \]

are nonempty, the right and the left tests \( T_{\Delta,r}^K, T_{\Delta,l}^K \) are identical and coincide with the Color Inferring test, built as explained in Section 4.1, deciding, via stationary \( K \)-repeated observations, on the “color” of the distribution \( p_A(x) \) underlying observations – whether this color is blue (“blue” hypothesis stating that \( x \in X \) and \( f(x) \leq a \), whence \( A(x) \in \bigcup_{1\leq i \leq t_{b\geq}} A(Z_{i}^{a\leq}) \)), or red (“red” hypothesis, stating that \( x \in X \) and \( f(x) \geq b \), whence \( A(x) \in \bigcup_{1\leq i \leq t_{b\geq}} A(Z_{i}^{b\geq}) \)). When \( a \) is lower-feasible and \( b \) is not upper-feasible, the red hypothesis is empty, and the left test associated with \([a, b] \), naturally, always accepts the blue hypothesis. Similarly, when \( a \) is lower-infeasible and \( b \) is upper-feasible, the right test associated with \([a, b] \) always accepts the red hypothesis.

A segment \([a, b] \) with \( a < b \) is \( \delta \)-good (left), if the corresponding to the segment “red” hypothesis is nonempty, and the left test \( T_{\Delta,l}^K \) associated with \([a, b] \) decides on the “red” and the “blue” hypotheses with risk \( \leq \delta \), that is,
• whenever \( A(x) \in \bigcup_{1 \leq i \leq l_n} A(Z_i^{b, \geq}) \), the \( p_{A(x)} \)-probability for the test to output red is \( \geq 1 - \delta \), and

• whenever \( A(x) \in \bigcup_{1 \leq i \leq l_n} A(Z_i^{b, \leq}) \), the \( p_{A(x)} \)-probability for the test to output blue is \( \geq 1 - \delta \).

Situation with a \( \delta \)-good (right) segment \([a, b] \) is completely similar.

### 4.3.3 Bisection estimate: construction

The control parameters of the Bisection estimate are

1. positive integer \( L \) – the maximum allowed number of bisection steps,
2. tolerances \( \delta \in (0, 1) \) and \( \kappa > 0 \).

The estimate of \( f(x) \) (\( x \) is the signal underlying our observations: \( \omega_t \sim p_{A(x)} \)) is given by the following recurrence run on the observation \( \omega^K = (\omega_1, ..., \omega_K) \) which we have at our disposal:

1. **Initialization.** We suppose that a valid upper bound \( b_0 \) on \( \max_{u \in X} f(u) \) and a valid lower bound \( a_0 \) on \( \min_{u \in X} f(u) \) are available; we assume w.l.o.g. that \( a_0 < b_0 \), otherwise the estimation is trivial. We set \( \Delta_0 = [a_0, b_0] \) (note that \( f(a) \in \Delta_0 \)).

2. **Bisection Step** \( \ell, 1 \leq \ell \leq L \). Given \( \text{localizer} \Delta_{\ell-1} = [a_{\ell-1}, b_{\ell-1}] \) with \( a_{\ell-1} < b_{\ell-1} \), we act as follows:

   a. Set \( c_\ell = \frac{1}{2}[a_{\ell-1} + b_{\ell-1}] \).

      If \( c_\ell \) is not upper-feasible, we set \( \Delta_\ell = [a_{\ell-1}, c_\ell] \) and pass to 2e, and if \( c_\ell \) is not lower-feasible, we set \( \Delta_\ell = [c_\ell, b_{\ell-1}] \) and pass to 2e.

      **Note:** In the latter case, the set \( \Delta_\ell \setminus \Delta_{\ell-1} \) does not intersect with \( f(X) \); in particular, in these cases \( f(x) \in \Delta_\ell \) provided that \( f(x) \in \Delta_{\ell-1} \).

   b. When \( c_\ell \) is both upper- and lower-feasible, we check whether the segment \([c_\ell, b_{\ell-1}] \) is \( \delta \)-good (right). If it is not the case, we terminate and claim that \( f(x) \in \bar{\Delta} := \Delta_{\ell-1} \), otherwise find \( v_\ell, c_\ell < v_\ell \leq b_{\ell-1} \), such that the segment \( \Delta_{\ell, v_\ell} = [c_\ell, v_\ell] \) is \( \delta \)-good (right) \( \kappa \)-maximal.

      **Note:** In terms of the outline of our strategy presented in Section 4.3.1, termination when the segment \([c_\ell, b_{\ell-1}] \) is not \( \delta \)-good (right) corresponds to the case where the current localizer is too small to allow for a separator wide enough to ensure low-risk decision on the blue and the red hypotheses.

      To find \( v_\ell \), we check the candidates with \( v_k^\ell = b_{\ell-1} - k\kappa, k = 0, 1, ... \) until arriving for the first time at segment \([c_\ell, v_k^\ell] \) which is not \( \delta \)-good (right), and take, as \( v_\ell \), the quantity \( v_k^\ell \) (the resulting value of \( v_\ell \) is well defined and clearly meets the above requirements as we clearly have \( k \geq 1 \)).

   c. Similarly, we check whether the segment \([a_{\ell-1}, c_\ell] \) is \( \delta \)-good (left). If it is not the case, we terminate and claim that \( f(x) \in \bar{\Delta} := \Delta_{\ell-1} \), otherwise we find \( u_\ell, a_{\ell-1} \leq u_\ell < c_\ell \), such that the segment \( \Delta_{\ell, u_\ell} = [u_\ell, c_\ell] \) is \( \delta \)-good (left) \( \kappa \)-maximal.

      **Note:** The rules for building \( u_\ell \) are completely similar to those for \( v_\ell \).

   d. We compute \( T^K_{\Delta, \ell, v_\ell}(\omega^K) \) and \( T^K_{\Delta, \ell, u_\ell}(\omega^K) \). If \( T^K_{\Delta, \ell, v_\ell}(\omega^K) = T^K_{\Delta, \ell, u_\ell}(\omega^K) \) ("consensus"), we set

\[
\Delta_\ell = [a_\ell, b_\ell] = \begin{cases} 
[c_\ell, b_{\ell-1}], & T^K_{\Delta, \ell, v_\ell}(\omega^K) = \text{red}, \\
[a_{\ell-1}, c_\ell], & T^K_{\Delta, \ell, v_\ell}(\omega^K) = \text{blue} 
\end{cases}
\]
and pass to 2e. Otherwise (“disagreement”) we terminate and claim that \( f(x) \in \tilde{\Delta} = \left[ u_\ell, v_\ell \right] \).

(e) When \( \ell < L \), we pass to step \( \ell + 1 \), otherwise we terminate and claim that \( f(x) \in \tilde{\Delta} := \Delta_L \).

3. **Output** of the estimation procedure is the segment \( \tilde{\Delta} \) built upon termination and claimed to contain \( f(x) \), see rules 2c, 2b, 2d; the midpoint of this segment is the estimate of \( f(x) \) yielded by our procedure.

### 4.3.4 Bisection estimate: Main result

**Proposition 4.2** Consider the situation described in the beginning of Section 4.2, and let \( \epsilon \in (0, 1/2) \) be given. Then

(i) [reliability] for every positive integer \( L \) and every \( \kappa > 0 \), Bisection with control parameters \( \delta = \frac{\kappa}{2L}, \) and \( \kappa \) is \((1 - \epsilon)\)-reliable: for every \( x \in X \), the \( p_{A(x)} \)-probability of the event

\[
\left( \bar{\Delta} \subseteq \tilde{\Delta} \right)
\]

\( (\bar{\Delta} \text{ is the output of Bisection as defined above}) \) is at least \( 1 - \epsilon \).

(ii) [near-optimality] Let \( \rho > 0 \) and positive integer \( \bar{K} \) be such that in the nature there exists a \((\rho, \epsilon)\)-reliable estimate \( \bar{f}(\cdot) \) of \( f(x) \), \( x \in X := \bigcup_{k \leq L} X_k \), via stationary \( \bar{K} \)-repeated observation \( \bar{\omega} = \omega_k \) with \( \omega_k \sim p_{A(x)} \). 1 \( \leq k \leq \bar{K} \). Given \( \varrho > 2\rho \), the Bisection estimate utilizing stationary \( \bar{K} \)-repeated observations, with

\[
K = \left\lfloor \frac{2 \ln(2LN\epsilon)}{\ln(4(1 - \epsilon)^{-1})} \bar{K} \right\rfloor, \quad (24)
\]

the control parameters of the estimate being

\[
L = \left\lfloor \log_2 \left( \frac{b_0 - a_0}{2\varrho} \right) \right\rfloor, \quad \delta = \frac{\epsilon}{2L}, \quad \kappa = \varrho - 2\rho, \quad (25)
\]

is \((\varrho, \epsilon)\)-reliable.

For proof, see Section A.3.

Note that the running time \( K \) of Bisection estimate as given by (24) is just by (at most) logarithmic in \( N, I, L \) and \( \epsilon^{-1} \) factor larger than \( \bar{K} \), and that \( L \) is just logarithmic in \( 1/\rho \). Assume, for instance, that for some \( \gamma > 0 \) “in the nature” there exist \((\epsilon^\gamma, \epsilon)\) reliable estimates, parameterized by \( \epsilon \in (0, 1/2) \), with \( \bar{K} = \bar{K}(\epsilon) \). Then Bisection with the volume of observation and control parameters given by (24) (25), where \( \bar{\rho} = 3\rho = 3\epsilon^\gamma \), and \( \bar{K} = \bar{K}(\epsilon) \), is \((3\epsilon^\gamma, \epsilon)\)-reliable and requires \( K = K(\epsilon) \)-repeated observations with \( \lim_{\epsilon \to 0} K(\epsilon)/\bar{K}(\epsilon) \leq 2 \).

### 4.4 Illustration: estimating survival rate

Let \( \xi \in \mathbb{R}_+ \) be a random variable representing lifetime. Suppose that our objective is, given \( K \) independent indirect observations of \( \xi \) and a value \( \tau \in \mathbb{R} \), estimate the corresponding hazard rate \( s_\tau = f_\xi(\tau)/(1 - F_\xi(\tau)) \) where \( f_\xi \) and \( F_\xi \) are, respectively, density and cumulative distribution function of \( \xi \). Suppose that the density \( f_\xi \) is smooth with bounded second derivative, and that observations are subjected to “mixed” multiplicative censoring (see, e.g. [24, 1, 6, 2]): the exact value of \( \xi_k \) is observed with probability \( 0 \leq \theta \leq 1 \), and with complementary probability, the available observation is \( \eta_k \xi_k \), where \( \eta_k \) is uniformly distributed over \([0, 1]\).

We assume that after an appropriate discretization, the estimation problem can be reformulated as follows: let \( x \) be the distribution of the (discrete-valued) lifetime taking values in \( S = \{1, 2, ..., M\} \).
We define the corresponding hazard rate $s_j(x)$ (the conditional probability of the lifetime to be exactly $j$ given that it is at least $j$) according to

$$s_j(x) = \frac{x_j}{\sum_{i=j}^{M} x_i}, \quad 1 \leq j \leq M.$$ 

Our objective is to estimate $s_j(x)$, given $K$ independent observations $\omega_k$ with distribution $\mu = Ax$, where $A \in \mathbb{R}^{M \times M}$ is a given column-stochastic matrix.

We use the following setup:

- $X = \{x \in \mathbb{R}^m : x_i \geq (3M)^{-1}; \sum_{i=1}^{M} x_i = 1; |x_{i-1} - 2x_i + x_{i+1}| \leq 2M^{-2}, 1 < i < M\}$;
- $A = \theta I_M + (1 - \theta)R$, where $R$ is upper-triangular matrix with $i$-th column $(i^{-1}, \ldots, i^{-1}, 0, \ldots, 0)^T$.

For various combinations of $\theta$ and $K$ we carried out 100 simulations of bisection estimation. In each simulation, we first selected $x \in X$ at random, drew $K$ observations $\omega_t, t = 1, \ldots, K$, from the distribution $Ax$, and then ran Bisection on these observations. Plots in Figure 3 illustrate some typical results of our experiments.

![Figure 3](image)

**Figure 3:** Empirical error distribution of Bisection estimate over 100 random estimation problems. (a) For $K = 10,000$, estimation error as function of $\theta \in \{0, 0.25, 0.5, 0.75, 1\}$; (b) estimation error as function of $K$ for $\theta = 0.9$. In these experiments, the initial risk – half-width of the initial localizer is equal to 0.0524.

**References**

[1] K. E. Andersen and M. B. Hansen. Multiplicative censoring: density estimation by a series expansion approach. *Journal of Statistical Planning and Inference*, 98(1-2):137–155, 2001.

[2] D. Belomestny and A. Goldenschluger. Nonparametric density estimation from observations with multiplicative measurement errors. *arXiv preprint arXiv:1709.00629*, 2017.

[3] M. Bertero and P. Boccacci. Application of the os-em method to the restoration of lbt images. *Astronomy and Astrophysics Supplement Series*, 144(1):181–186, 2000.
[4] M. Bertero and P. Boccacci. Image restoration methods for the large binocular telescope (lbt). *Astronomy and Astrophysics Supplement Series*, 147(2):323–333, 2000.

[5] E. Betzig, G. H. Patterson, R. Sougrat, O. W. Lindwasser, S. Olenych, J. S. Bonifacino, M. W. Davidson, J. Lippincott-Schwartz, and H. F. Hess. Imaging intracellular fluorescent proteins at nanometer resolution. *Science*, 313(5793):1642–1645, 2006.

[6] E. Brunel, F. Comte, and V. Genon-Catalot. Nonparametric density and survival function estimation in the multiplicative censoring model. *Test*, 25(3):570–590, 2016.

[7] T. T. Cai and M. G. Low. A note on nonparametric estimation of linear functionals. *Annals of statistics*, pages 1140–1153, 2003.

[8] T. T. Cai and M. G. Low. Minimax estimation of linear functionals over nonconvex parameter spaces. *The Annals of statistics*, 32(2):552–576, 2004.

[9] T. T. Cai, M. G. Low, et al. On adaptive estimation of linear functionals. *The Annals of Statistics*, 33(5):2311–2343, 2005.

[10] D. L. Donoho. Statistical estimation and optimal recovery. *The Annals of Statistics*, 22(1):238–270, 1994.

[11] D. L. Donoho and R. C. Liu. Geometrizing rates of convergence, i. Technical Report 137, Dept. of Statist., University of California, Berkeley.

[12] D. L. Donoho and R. C. Liu. Geometrizing rates of convergence, ii. *The Annals of Statistics*, pages 633–667, 1991.

[13] D. L. Donoho and R. C. Liu. Geometrizing rates of convergence, iii. *The Annals of Statistics*, pages 668–701, 1991.

[14] D. L. Donoho and M. G. Low. Renormalization exponents and optimal pointwise rates of convergence. *The Annals of Statistics*, pages 944–970, 1992.

[15] A. Goldenshluger, A. Juditsky, and A. Nemirovski. Hypothesis testing by convex optimization. *Electronic Journal of Statistics*, 9(2):1645–1712, 2015.

[16] S. W. Hell. Toward fluorescence nanoscopy. *Nature biotechnology*, 21(11):1347, 2003.

[17] S. W. Hell. Microscopy and its focal switch. *Nature methods*, 6(1):24, 2009.

[18] S. W. Hell and J. Wichmann. Breaking the diffraction resolution limit by stimulated emission: stimulated-emission-depletion fluorescence microscopy. *Optics letters*, 19(11):780–782, 1994.

[19] S. T. Hess, T. P. Girirajan, and M. D. Mason. Ultra-high resolution imaging by fluorescence photoactivation localization microscopy. *Biophysical journal*, 91(11):4258–4272, 2006.

[20] I. A. Ibragimov and R. Z. Khas minskii. On nonparametric estimation of the value of a linear functional in gaussian white noise. *Theory of Probability & Its Applications*, 29(1):18–32, 1985.

[21] A. Juditsky and A. Nemirovski. Nonparametric estimation by convex programming. *The Annals of Statistics*, 37(5a):2278–2300, 2009.

[22] A. Juditsky and A. Nemirovski. Estimating linear and quadratic forms via indirect observations. *arXiv preprint arXiv:1612.01508*, 2016.
A Proofs

A.1 Proof of Proposition 3.1

Proof. Let the common distribution \( p \) of independent across \( k \) components \( \omega_k \) of \( \omega^K \) be \( p_{\A_k(u)} \) for some \( \ell \leq I \) and \( u \in X_\ell \). Let us fix these \( \ell \) and \( u \), let \( \mu = A_\ell(u) \), and let \( p^K \) stand for the distribution of \( \omega^K \).

1°. We have

\[
\Psi_{\ell,+(\alpha, \phi)} = \max_{x \in X_\ell} \left\{ K\alpha \phi \Omega(x) \Omega(x) - g^T x + \alpha \ln (2I/\epsilon) \right\} - g^T u + \alpha \ln (2I/\epsilon)
\]

so that

\[
\alpha \ln \left( \Pr_{\alpha, \phi} \{ g_{\ell, \phi} > g^T u + \rho \} \right) \leq \Psi_{\ell,+(\alpha, \phi)} + \rho - \rho \leq \alpha \ln (2I/\epsilon)
\]

and we arrive at

\[
\Pr_{\alpha, \phi} \{ g_{\ell, \phi} > g^T u \} \leq \frac{\epsilon}{2I}.
\]  

Similarly,

\[
\Psi_{\ell,-(\alpha, \phi)} = \max_{y \in X_\ell} \left\{ K\alpha \phi \Omega(-\phi \Omega(y) + \alpha \ln (2I/\epsilon) \right\} + g^T y + \alpha \ln (2I/\epsilon)
\]

implying that

\[
\alpha \ln \left( \Pr_{\alpha, \phi} \{ g_{\ell, \phi} < g^T u - \rho \} \right) \leq \Psi_{\ell,-(\alpha, \phi)} - \rho + \alpha \ln (2I/\epsilon)
\]

and we conclude that

\[
\Pr_{\alpha, \phi} \{ g_{\ell, \phi} < g^T u - \rho \} \leq \frac{\epsilon}{2I}.
\]
2. Let
\[ \mathcal{E} = \{ \omega^K : g_{ij}(\omega^K) \leq g^{T}u + \rho_{ij}, \; g_{ij}(\omega^K) \geq g^{T}u - \rho_{ij}, \; 1 \leq i, j \leq I \}. \]

From (26), (27) and the union bound it follows that \( p^K \)-probability of the event \( \mathcal{E} \) is \( \geq 1 - \epsilon \). As a result, all we need to complete the proof of Proposition is to verify that for all \( \omega^K \in \mathcal{E} \),

\[ |\hat{g}(\omega^K) - g^{T}u| \leq \rho_{\ell}. \quad (28) \]

Indeed, let us fix \( \omega^K \in \mathcal{E} \), and let \( E \) be the \( I \times I \) matrix with entries \( E_{ij} = g_{ij}(\omega^K), \; 1 \leq i, j \leq I \). The quantity \( r_{\ell} \), see (5), is the maximum of entries in \( i \)-th row of \( E \), and the quantity \( c_{j} \) is the minimum of entries in \( j \)-th column of \( E \). In particular, \( r_{\ell} \geq E_{ij} \geq c_{j} \) for all \( i, j \), implying that \( r_{\ell} \geq c_{\ell} \) and \( c_{j} \leq r_{\ell} \) for all \( i, j \). Now, since \( \omega^K \in \mathcal{E} \), we have for all \( j \):

\[ E_{ij} = g_{ij}(\omega^K) \leq g^{T}u + \rho_{ij} \leq g^{T}u + \rho_{\ell}, \]

implying that \( r_{\ell} = \max_{j} E_{ij} \leq g^{T}u + \rho_{\ell} \). Similarly, \( \omega^K \in \mathcal{E} \) implies that for all \( i \)

\[ E_{ij} = g_{ij}(\omega^K) \geq g^{T}u - \rho_{ij} \geq g^{T}u - \rho_{\ell}, \]

so that \( c_{\ell} = \min_{i} E_{ij} \geq g^{T}u - \rho_{\ell} \). We have \( r_{\ell} := \min_{i} r_{ij} \leq r_{\ell} \), and, as we have already seen, \( r_{\ell} \geq c_{\ell} \), implying that \( r_{\ell} \) belongs to \( \Delta_{\ell} = [g^{T}u - \rho_{\ell}, g^{T}u + \rho_{\ell}] \). By similar argument, \( c_{\ell} := \max_{j} c_{j} \in \Delta_{\ell} \) as well. Finally, \( \hat{g}(\omega^K) = \frac{1}{2}[r_{\ell} + c_{\ell}], \) that is, \( \hat{g}(\omega^K) \in \Delta_{\ell} \), and (28) follows. □

### A.2 Proof of Proposition 3.2

1. Observe that \( \text{Opt}_{ij}^{K}(K) \) is the saddle point value in the convex-concave saddle point problem:

\[ \text{Opt}_{ij}^{K}(K) = \inf_{\alpha > 0, \phi \in \mathcal{F}} \max_{x \in X, y \in X} \frac{1}{2} K \alpha \{ \Phi_{\mathcal{O}}(\phi/\alpha; A_{x}(x)) + \Phi_{\mathcal{O}}(-\phi/\alpha; A_{y}(y)) \} + \frac{1}{2} g^{T} \ln(2I/\epsilon) + \frac{1}{2} g^{T} \ln(2I/\epsilon). \]

The domain of the maximization variable is compact and the cost function is continuous on its domain, whence, by Sion-Kakutani Theorem, we have also

\[ \text{Opt}_{ij}^{K}(K) = \max_{x \in X, y \in X} \Theta_{ij}(x, y), \]

\[ \Theta_{ij}(x, y) = \inf_{\alpha > 0, \psi \in \mathcal{F}} \frac{1}{2} K \alpha \{ \Phi_{\mathcal{O}}(\psi/\alpha; A_{x}(x)) + \Phi_{\mathcal{O}}(-\psi/\alpha; A_{y}(y)) \} + \alpha \ln(2I/\epsilon) + \frac{1}{2} g^{T} \ln(2I/\epsilon). \quad (29) \]

We have

\[ \Theta_{ij}(x, y) = \inf_{\alpha > 0, \psi \in \mathcal{F}} \frac{1}{2} K \alpha \{ \Phi_{\mathcal{O}}(\psi/\alpha; A_{x}(x)) + \Phi_{\mathcal{O}}(-\psi/\alpha; A_{y}(y)) \} + \alpha \ln(2I/\epsilon) + \frac{1}{2} g^{T} \ln(2I/\epsilon) \]

\[ = \inf_{\alpha > 0} \frac{1}{2} K \alpha \inf_{\psi \in \mathcal{F}} \{ \Phi_{\mathcal{O}}(\psi; A_{x}(x)) + \Phi_{\mathcal{O}}(-\psi; A_{y}(y)) \} + \alpha \ln(2I/\epsilon) + \frac{1}{2} g^{T} \ln(2I/\epsilon) \]

Given \( x \in X, y \in X \) and setting \( \mu = A_{x}(x), \nu = A_{y}(y) \), we obtain

\[ \inf_{\psi \in \mathcal{F}} \{ \Phi_{\mathcal{O}}(\psi; A_{x}(x)) + \Phi_{\mathcal{O}}(-\psi; A_{y}(y)) \} = \inf_{\psi \in \mathcal{F}} \ln \left( \int \exp\{\psi(\omega)\} p_{\mu}(\omega) P(d\omega) \right) \]

\[ + \ln \left( \int \exp\{-\psi(\omega)\} p_{\nu}(\omega) P(d\omega) \right). \]
Since $O$ is a good o.s., the function $\tilde{\psi}(\omega) = \frac{1}{2} \ln(p_\nu(\omega)/p_\mu(\omega))$ belongs to $F$, and

$$
\inf_{\psi \in F} \left[ \ln \left( \int \exp\{\psi(\omega)\} p_\mu(\omega) P(d\omega) \right) + \ln \left( \int \exp\{-\psi(\omega)\} p_\nu(\omega) P(d\omega) \right) \right] 
= \inf_{\delta \in F} \left[ \ln \left( \int \exp\{\tilde{\psi}(\omega) + \delta(\omega)\} p_\mu(\omega) P(d\omega) \right) + \ln \left( \int \exp\{-\tilde{\psi}(\omega) - \delta(\omega)\} p_\nu(\omega) P(d\omega) \right) \right] 
= \inf_{\delta \in F} \left[ \ln \left( \int \exp\{\delta(\omega)\} \sqrt{p_\mu(\omega)p_\nu(\omega)} P(d\omega) \right) + \ln \left( \int \exp\{-\delta(\omega)\} \sqrt{p_\mu(\omega)p_\nu(\omega)} P(d\omega) \right) \right].
$$

Observe that $f(\delta)$ clearly is a convex and even function of $\delta \in F$; as such, it attains its minimum over $\delta \in F$ when $\delta = 0$. The bottom line is that

$$
\inf_{\psi \in F} [\Phi_O(\psi; A_i(x)) + \Phi_O(-\psi; A_j(y))] = 2 \ln \left( \int \sqrt{p_{A_i(x)}(\omega)p_{A_j(y)}(\omega)} P(d\omega) \right),
$$

and

$$
\Theta_{ij}(x, y) = \inf_{\alpha > 0} \left[ K \ln \left( \int \sqrt{p_{A_i(x)}(\omega)p_{A_j(y)}(\omega)} P(d\omega) \right) + \ln(2I/\epsilon) \right] + g^T[y - x] 
= \begin{cases} 
\frac{1}{2} g^T[y - x], & K \ln \left( \int \sqrt{p_{A_i(x)}(\omega)p_{A_j(y)}(\omega)} P(d\omega) \right) + \ln(2I/\epsilon) \geq 0, \\
-\infty, & \text{otherwise}.
\end{cases}
$$

This combines with (29) to imply that

$$
\text{Opt}_{ij}(K) = \max_{x, y} \left\{ \frac{1}{2} g^T[y - x] : x \in X_i, y \in X_j, \left[ \int \sqrt{p_{A_i(x)}(\omega)p_{A_j(y)}(\omega)} P(d\omega) \right]^{K} \geq \frac{\epsilon}{2I} \right\}.
$$

2\textsuperscript{0}. We claim that under the premise of Proposition, for all $i, j$, $1 \leq i, j \leq I$, one has

$$
\text{Opt}_{ij}(K) \leq \text{Risk}_v^*(\tilde{K}),
$$

implying the validity of (7). Indeed, assume that for some pair $i, j$ the opposite inequality holds true:

$$
\text{Opt}_{ij}(K) > \text{Risk}_v^*(\tilde{K}),
$$

and let us lead this assumption to a contradiction. Under our assumption optimization problem in (31) has a feasible solution $(\bar{x}, \bar{y})$ such that

$$
r := \frac{1}{2} g^T[\bar{y} - \bar{x}] > \text{Risk}_v^*(\tilde{K}),
$$

implying, due to the origin of $\text{Risk}_v^*(\tilde{K})$, that there exists an estimate $\tilde{g}(\omega^{\tilde{K}})$ such that for $\mu = A_i(\bar{x})$, $\nu = A_j(\bar{y})$ it holds

$$
\text{Prob}_{\omega^{\tilde{K}} \sim P^{\tilde{K}}_\nu} \left\{ \tilde{g}(\omega^{\tilde{K}}) - g^T[\bar{y} - \bar{x}] \right\} \leq \epsilon,
$$

so that we can decide on two simple hypotheses stating that observation $\omega^{\tilde{K}}$ obeys distribution $P^{\tilde{K}}_\mu$, resp., $P^{\tilde{K}}_\nu$, with risk $\leq \epsilon$. Therefore,

$$
\int \min \left[ P^{\tilde{K}}_\mu(\omega^{\tilde{K}}), P^{\tilde{K}}_\nu(\omega^{\tilde{K}}) \right] P(\omega^{\tilde{K}}) \leq 2\epsilon.
$$
Indeed, the solution being such is concave. Besides this, the optimization problem in (33) is feasible whenever \( \{s_0 \} \geq w \) is feasible for the optimization problem specifying \( \{s_0 \} \) is continuous on \( \mathbb{N} \), that is, \( \bar{s} \in \mathbb{N} \). At this feasible solution we have \( g \leq 2 \) for \( s \geq 0 \). Observe also that from concavity of \( H(x,y) \) it follows that \( w_{ij}(s) \) is concave on the ray \( \{ s \geq 0 \} \). Finally, we claim that

\[
\tag{34}
\label{34}
w_{ij}(\bar{s}) \leq \text{Risk}^*_A(\bar{K}), \quad \bar{s} = -\ln(2\sqrt{\epsilon(1-\epsilon)}).
\]

Indeed, \( w_{ij}(s) \) is nonnegative, concave and bounded (since \( X_i, X_j \) are compact) on \( \mathbb{R}_+ \), implying that \( w_{ij}(s) \) is continuous on \( \{ s > 0 \} \). Assuming, on the contrary to our claim, that \( w_{ij}(\bar{s}) > \text{Risk}^*_A(\bar{K}) \), there exists \( s' \in (0, \bar{s}) \) such that \( w_{ij}(s') > \text{Risk}^*_A(\bar{K}) \) and thus there exist \( \bar{x} \in X_i, \bar{y} \in X_j \) such that \((\bar{x}, \bar{y})\) is feasible for the optimization problem specifying \( w_{ij}(s') \) and (32) takes place. We have seen in item 2\( \text{b} \) that the latter relation implies that for \( \mu = A_i(\bar{x}), \nu = A_j(\bar{y}) \) it holds

\[
\left[ \int \sqrt{p_\mu(\omega)p_\nu(\omega)} P(\omega) \right]^{K} \leq 2\sqrt{\epsilon(1-\epsilon)},
\]

that is,

\[K \ln \left( \int \sqrt{p_\mu(\omega)p_\nu(\omega)} P(\omega) \right) + \bar{s} \leq 0,
\]

Consequently,

\[
\left[ \int \sqrt{p_\mu(\omega)p_\nu(\omega)} P(\omega) \right]^{K} \leq 2\sqrt{\epsilon(1-\epsilon)}^{K/K} < \frac{\epsilon}{21},
\]

which is the desired contradiction (recall that \( \mu = A_i(\bar{x}), \nu = A_j(\bar{y}) \) and \((\bar{x}, \bar{y})\) is feasible for (31)).

3\( \text{b} \). Now let us prove that under the premise of Proposition, (8) takes place. To this end let us set

\[
\tag{33}
\label{33}
w_{ij}(s) = \max_{x \in X_i, y \in X_j} \left\{ \frac{1}{2} g^T[y - x] : |\ln \left( \int \sqrt{p_{A_i(x)}(\omega)p_{A_j(y)}(\omega)} P(\omega) \right) + s \geq 0 \right\}.
\]

As we have seen in item 1\( \text{b} \), see (30), one has

\[
H(x,y) = \inf_{\psi \in \mathcal{F}} \frac{1}{2} \left[ \Phi_\mathcal{O}(\psi; A_i(x)) + \Phi_\mathcal{O}(-\psi, A_j(y)) \right],
\]

that is, \( H(x,y) \) is the infimum of a parametric family of concave functions of \((x,y) \in X_i \times X_j \) as such is concave. Besides this, the optimization problem in (33) is feasible whenever \( s \geq 0 \), a feasible solution being \( y = x = x_{ij} \). At this feasible solution we have \( g^T[y - x] = 0 \), implying that \( w_{ij}(s) \geq 0 \) for \( s \geq 0 \). Observe also that from concavity of \( H(x,y) \) it follows that \( w_{ij}(s) \) is concave on the ray \( \{ s \geq 0 \} \). Finally, we claim that

\[
\tag{34}
\label{34}
w_{ij}(\bar{s}) \leq \text{Risk}^*_A(\bar{K}), \quad \bar{s} = -\ln(2\sqrt{\epsilon(1-\epsilon)}).
\]

Indeed, \( w_{ij}(s) \) is nonnegative, concave and bounded (since \( X_i, X_j \) are compact) on \( \mathbb{R}_+ \), implying that \( w_{ij}(s) \) is continuous on \( \{ s > 0 \} \). Assuming, on the contrary to our claim, that \( w_{ij}(\bar{s}) > \text{Risk}^*_A(\bar{K}) \), there exists \( s' \in (0, \bar{s}) \) such that \( w_{ij}(s') > \text{Risk}^*_A(\bar{K}) \) and thus there exist \( \bar{x} \in X_i, \bar{y} \in X_j \) such that \((\bar{x}, \bar{y})\) is feasible for the optimization problem specifying \( w_{ij}(s') \) and (32) takes place. We have seen in item 2\( \text{b} \) that the latter relation implies that for \( \mu = A_i(\bar{x}), \nu = A_j(\bar{y}) \) it holds

\[
\left[ \int \sqrt{p_\mu(\omega)p_\nu(\omega)} P(\omega) \right]^{K} \leq 2\sqrt{\epsilon(1-\epsilon)},
\]

that is,

\[K \ln \left( \int \sqrt{p_\mu(\omega)p_\nu(\omega)} P(\omega) \right) + \bar{s} \leq 0,
\]

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whence
\[ K \ln \left( \int \sqrt{p_\mu(\omega)p_\nu(\omega)}P(d\omega) \right) + s' < 0, \]
contradicting the fact that \((\bar{x}, \bar{y})\) is feasible for the optimization problem specifying \(w_{ij}(s')\).

It remains to note that (34) combines with concavity of \(w_{ij}(\cdot)\) and the relation \(w_{ij}(0) \geq 0\) to imply that
\[ w_{ij}(\ln(2I/\epsilon)) \leq \vartheta w_{ij}(\bar{s}) \leq \vartheta \text{Risk}^*_i(K), \quad \vartheta = \ln(2I/\epsilon)/\bar{s} = \frac{2 \ln(2I/\epsilon)}{\ln([4\epsilon(1-\epsilon)]^{-1})}. \]

Invoking (31), we conclude that
\[ \text{Opt}_{ij}(\bar{K}) = w_{ij}(\ln(2I/\epsilon)) \leq \vartheta \text{Risk}^*_i(K) \forall i, j. \]

Finally, from (31) it immediately follows that \(\text{Opt}_{ij}(K)\) is nonincreasing in \(K\) (since as \(K\) grows, the feasible set of the right hand side optimization problem in (31) shrinks), that is,
\[ K \geq \bar{K} \Rightarrow \text{Opt}(K) \leq \text{Opt}(\bar{K}) = \max_{i,j} \text{Opt}_{ij}(\bar{K}) \leq \vartheta \text{Risk}^*_i(K), \]
and (8) follows. \(\square\)

### A.3 Proof of Proposition 4.2

#### A.3.1 Proof of Proposition 4.2(i)

We call step \(\ell\) constructive, if at this step rule 2d is invoked.

**1.** Let \(x \in X\) be the true signal underlying our observation \(\omega^K\), so that \(\omega_1, ..., \omega_K\) are independently of each other drawn from the distribution \(p_A(x)\). Consider the “ideal” Bisection given by exactly the same rules as the procedure described in Section 4.3.3 (in the sequel, we refer to the latter as to the “actual” one), up to the fact that tests \(T^*_{\Delta_{\text{rg}}}(\cdot), T^*_{\Delta_{\text{lt}}}(\cdot)\) in rule 2d are replaced by the rules
\[
T^*_{\Delta_{\text{rg}}} = T^*_{\Delta_{\text{lt}}} = \begin{cases} \text{red}, & f(x) > c_\ell \\ \text{blue}, & f(x) \leq c_\ell \end{cases}
\]
Marking by \(^*\) the entities produced by the resulting deterministic procedure, we arrive at a sequence of nested segments \(\Delta^*_\ell = [a^*_\ell, b^*_\ell], 0 \leq \ell \leq L^* \leq L\), along with subsegments \(\Delta^*_{\text{rg}} = [c^*_\ell, v^*_\ell], \Delta^*_{\text{lt}} = [u^*_\ell, c^*_\ell]\) of \(\Delta^*_{\ell-1}\), defined for all \(^*\)-constructive steps \(\ell\), and the output segment \(\Delta^*\) claimed to contain \(f(x)\). Note that the ideal procedure cannot terminate due to a disagreement, and that \(f(x)\), as is immediately seen, is contained in all segments \(\Delta^*_\ell, 0 \leq \ell \leq L^*\), same as \(f(x) \in \Delta^*\).

Let \(L^*\) be the set of all \(^*\)-constructive values of \(\ell\). For \(\ell \in L^*\), let the event \(E_\ell[x]\) parameterized by \(x\) be defined as follows:
\[
E_\ell[x] = \begin{cases} \{\omega^K: T^*_{\Delta^*_{\text{rg}}}(\omega^K) = \text{red or } T^*_{\Delta^*_{\text{lt}}}(\omega^K) = \text{red}\}, & f(x) \leq u^*_\ell \\ \{\omega^K: T^*_{\Delta^*_{\text{rg}}}(\omega^K) = \text{red}\}, & u^*_\ell < f(x) \leq c^*_\ell \\ \{\omega^K: T^*_{\Delta^*_{\text{rg}}}(\omega^K) = \text{blue}\}, & c^*_\ell < f(x) < v^*_\ell \\ \{\omega^K: T^*_{\Delta^*_{\text{lt}}}(\omega^K) = \text{blue or } T^*_{\Delta^*_{\text{lt}}}(\omega^K) = \text{blue}\}, & f(x) \geq v^*_\ell \end{cases}
\]
20. Observe that by construction and in view of Proposition 4.1 we have
\[ \forall \ell \in \mathcal{L}^*: \text{Prob}_{\omega^K \sim p_{A(x)} \times \ldots \times p_{A(x)}} \{ \mathcal{E}_\ell[x] \} \leq 2\delta. \] (36)

Indeed, let $\ell \in \mathcal{L}^*$.

- When $f(x) \leq u_\ell^*$, we have $x \in X$ and $f(x) \leq u_\ell^* \leq c_\ell^*$, implying that $\mathcal{E}_\ell[x]$ takes place only when either the left test $T^K_{\Delta^*_{t,rg}}$, or the right test $T^K_{\Delta^*_{t,rg}}$, or both, did not accept true – blue – hypotheses from the pairs of red and blue hypotheses the tests were applied to. Since the corresponding intervals ($[u_\ell^*, c_\ell^*]$ for the left side test, $[c_\ell^*, v_\ell^*]$ for the right side one) are $\delta$-good left/right, respectively, the risks of the tests do not exceed $\delta$, and the $p_{A(x)}$-probability of the event $\mathcal{E}_\ell[x]$ is at most $2\delta$;

- when $u_\ell^* < f(x) \leq c_\ell^*$, the event $\mathcal{E}_\ell[x]$ takes place only when the right test $T^K_{\Delta^*_{t,rg}}$ does not accept true – blue – hypothesis; similarly to the above, this can happen with $p_{A(x)}$-probability at most $\delta$;

- when $c_\ell < f(x) \leq v_\ell$, the event $\mathcal{E}_\ell[x]$ takes place only when the left test $T^K_{\Delta^*_{t,lf}}$ does not accept true – red – hypothesis, which, again, happens with $p_{A(x)}$-probability $\leq \delta$;

- finally, when $f(x) > v_\ell$, the event $\mathcal{E}_\ell[x]$ takes place only when either the left test $T^K_{\Delta^*_{t,lf}}$, or the right test $T^K_{\Delta^*_{t,rg}}$, or both, does not accept the true – red – hypothesis from the pair of red and blue hypotheses the test was applied to; same as above, this can happen with $p_{A(x)}$-probability at most $2\delta$.

30. Let $\bar{L} = \bar{L}(\omega^K)$ be the last step of the “actual” estimating procedure as run on the observation $\omega^K$. We claim that the following holds true:

**Lemma A.1** Let $\mathcal{E} := \bigcup_{\ell \in \mathcal{L}^*} \mathcal{E}_\ell[x]$, so that the $p_{A(x)}$-probability of the event $\mathcal{E}$, the observations stemming from $x$, is at most $2\delta L = \epsilon$

by (36). Assume that $\omega^K \notin \mathcal{E}$. Then $\bar{L}(\omega^K) \leq L^*$, and just two cases are possible:

(A) The actual estimating procedure did not terminate by disagreement. In this case $\bar{L}(\omega^K) = L^*$, and the trajectories of the ideal and the actual Bisections are identical (same localizers, same constructive steps, same output segments, etc.); in particular, $f(x) \in \bar{\Delta}$;

(B) The actual estimating procedure terminated due to a disagreement. Then $\Delta_\ell = \Delta^*_\ell$ for $\ell < \bar{L}$, and $f(x) \in \bar{\Delta}$.

In view of (A) and (B), the $p_{A(x)}$-probability of the event $f(x) \in \bar{\Delta}$ is at least $1 - \epsilon$, as claimed in Proposition 4.2.

**Proof of the lemma.** Note that the actions at step $\ell$ in the ideal and the actual procedures depend solely on $\Delta_{\ell-1}$ and on the outcome of rule 2d. Taking into account that $\Delta_0 = \Delta^*_0$, all we need to verify is the following:

(!) Let $\omega^K \notin \mathcal{E}$, and let $\ell \leq L^*$ be such that $\Delta_{\ell-1} = \Delta^*_{\ell-1}$, whence also $u_\ell = u^*_\ell$, $c_\ell = c^*_\ell$ and $v_\ell = v^*_\ell$. Assume that $\ell$ is constructive (given that $\Delta_{\ell-1} = \Delta^*_{\ell-1}$, this may happen if and only if $\ell$ is $^*$-constructive as well). Then either

- at step $\ell$ the actual procedure terminates due to disagreement, in which case $f(x) \in \bar{\Delta}$,
– there was no disagreement at step \( \ell \), in which case \( \Delta_\ell \) as given by (23) is identical to \( \Delta^*_\ell \) as given by the ideal counterpart of (23) in the case of \( \Delta^*_{\ell-1} = \Delta_{\ell-1} \), that is, by the rule

\[
\Delta^*_\ell = \begin{cases} 
[c_\ell, b_{\ell-1}], & f(x) > c_\ell, \\
[a_{\ell-1}, c_\ell], & f(x) \leq c_\ell
\end{cases}
\]  

(37)

Let \( \omega^K \) and \( \ell \) satisfy the premise of (!). Note that due to \( \Delta_{\ell-1} = \Delta^*_{\ell-1} \) we have \( u_\ell = u^*_\ell \), \( c_\ell = c^*_\ell \), and \( v_\ell = v^*_\ell \), and thus also \( \Delta^*_{\ell,li} = \Delta_{\ell,ii}, \Delta^*_{\ell,rg} = \Delta_{\ell,rg} \). Let us consider first the case where the actual estimation procedure terminates due to a disagreement at step \( \ell \), so that \( T^K_{\Delta^*_{\ell,ri}}(\omega^K) \neq T^K_{\Delta^*_{\ell,ri}}(\bar{\omega}^K) \).

Assuming for a moment that \( f(x) < u_\ell = u^*_\ell \), the relation \( \bar{\omega}^K \notin \mathcal{E}_\ell[x] \) combines with (35) to imply that \( T^K_{\Delta^*_{\ell,rg}}(\bar{\omega}^K) = T^K_{\Delta^*_{\ell,ri}}(\bar{\omega}^K) = \text{blue} \), which is impossible under disagreement. Assuming \( f(x) > v_\ell = v^*_\ell \), the same argument results in \( T^K_{\Delta^*_{\ell,rg}}(\bar{\omega}^K) = T^K_{\Delta^*_{\ell,ri}}(\bar{\omega}^K) = \text{red} \), which again is impossible. We conclude that in the case in question \( u_\ell \leq f(x) \leq v_\ell \), i.e., \( f(x) \in \Delta \), as claimed.

Now, assume that there is a consensus at the step \( \ell \) in the actual Bisection. When \( \bar{\omega}^K \notin \mathcal{E}_\ell[x] \) this is only possible when

1. \( T^K_{\Delta^*_{\ell,rg}}(\bar{\omega}^K) = \text{blue} \) when \( f(x) \leq u_\ell = u^*_\ell \),
2. \( T^K_{\Delta^*_{\ell,rg}}(\bar{\omega}^K) = \text{blue} \) when \( u_\ell < f(x) \leq c_\ell = c^*_\ell \),
3. \( T^K_{\Delta^*_{\ell,ri}}(\bar{\omega}^K) = \text{red} \) when \( c_\ell < f(x) \leq v_\ell = v^*_\ell \),
4. \( T^K_{\Delta^*_{\ell,ri}}(\bar{\omega}^K) = \text{red} \) when \( v_\ell \leq f(x) \).

In situations 1 and 2, and due to consensus at the step \( \ell \), (23) means that \( \Delta_\ell = [a_{\ell-1}, c_\ell] \), which combines with (37) and \( v_\ell = v^*_\ell \) to imply that \( \Delta_\ell = \Delta^*_\ell \). Similarly, in situations 3-4 and due to consensus at the step \( \ell \), (23) says that \( \Delta_\ell = [c_\ell, b_{\ell-1}] \), which combines with \( u_\ell = u^*_\ell \) and (37) to imply that \( \Delta_\ell = \Delta^*_\ell \). \( \square \)

A.3.2 Proof of Proposition 4.2(ii)

There is nothing to prove when \( \frac{b_0-a_0}{2} \leq \rho \), since in this case the estimate \( \frac{a_0+b_0}{2} \) which does not use observations at all is \((\rho, 0)\)-reliable. From now on we assume that \( b_0 - a_0 > 2\rho \), implying that \( L \) is positive integer.

1. Observe, first, that if \( a, b \) are such that \( a \) is lower-feasible, \( b \) is upper-feasible, and \( b-a > 2\rho \), then for every \( i \leq I_b \) and \( j \leq I_a \), there exists a test, based on \( K \) observations, which decides upon the hypotheses \( H_1, H_2 \), stating that the observations are drawn from \( p_A(x) \) with \( x \in Z_j^{a \leq} (H_1) \) and with \( x \in Z_j^{a \leq} (H_2) \) with risk at most \( \varepsilon \). Indeed, it suffices to consider the test which accepts \( H_1 \) and rejects \( H_2 \) when \( \hat{f}(\omega^K) \geq \frac{a+b}{2} \) and accepts \( H_2 \) and rejects \( H_1 \) otherwise.

2. With parameters of Bisection chosen according to (25), by Lemma A.1 we have

\( (E.1) \) For every \( x \in X \), the \( p_A(x) \)-probability of the event \( f(x) \in \Delta \), \( \Delta \) being the output segment of our Bisection, is at least \( 1 - \varepsilon \).
3. We claim that

\[(F.1)\] Every segment \(\Delta = [a, b]\) with \(b - a > 2\rho\) and lower-feasible \(a\) is \(\delta\)-good (right),

\[(F.2)\] Every segment \(\Delta = [a, b]\) with \(b - a > 2\rho\) and upper-feasible \(b\) is \(\delta\)-good (left),

\[(F.3)\] Every \(\kappa\)-maximal \(\delta\)-good (left or right) segment has length at most \(2\rho + \kappa = \varrho\). As a result, for every constructive step \(\ell\), the lengths of the segments \(\Delta_{\ell,rg}\) and \(\Delta_{\ell,lf}\) do not exceed \(\varrho\).

Let us verify (F.1) (verification of F.2 is completely similar, and (F.3) is an immediate consequence of (F.1) and (F.2)). Let \([a, b]\) satisfy the premise of (F.1). It may happen that \(b\) is upper-infeasible, whence \(\Delta = [a, b]\) is 0-good (right), and we are done. Now let \(b\) be upper-feasible. As we have already seen, whenever \(i \leq I_{b,\geq}\) and \(j \leq I_{a,\leq}\), the hypotheses stating that \(\omega_k \sim p_{A(x)}(x)\) for some \(x \in Z_{b,\geq}\), resp., for some \(x \in Z_{j,\leq}\), can be decided upon with risk \(\leq \epsilon\), implying by (15) that

\[
\epsilon_{ij,\Delta} \leq \left[2\sqrt{\epsilon(1-\epsilon)}\right]^{1/K}.
\]

Hence, taking into account that the column and the row sizes of \(E_{\Delta,r}\) do not exceed \(NI\),

\[
\sigma_{\Delta,r} \leq NI \max_{i,j} \epsilon_{ij,\Delta} \leq NI \left[2\sqrt{\epsilon(1-\epsilon)}\right]^{K/K} \leq \frac{\epsilon}{2L} = \delta
\]

(we have used (25)), So, \(\Delta\) indeed is \(\delta\)-good (right).

4. Let us fix \(x \in X\) and consider a trajectory of Bisection, the \(K\)-repeated observation \(\omega^K\) being drawn from \(p^K_{A(x)}\). The output \(\bar{\Delta}\) of the procedure is given by one of the following options:

1. At some step \(\ell\) of Bisection, the process terminated by 2b or 2c. In the first case, the segment \([c_{\ell}, b_{\ell-1}]\) has lower-feasible left endpoint and is not \(\delta\)-good (right), implying by F.1 that the length of this segment (which is 1/2 of the length of \(\Delta = \Delta_{\ell-1}\)) is \(\leq 2\rho\), so that the length \(|\Delta|\) of \(\bar{\Delta}\) is at most \(4\rho \leq 2\varrho\). By completely similar argument, the same conclusion holds true when the process terminated at step \(\ell\) by 2c.

2. At some step \(\ell\) of Bisection, the process terminated due to disagreement. In this case, by (F.3), we have \(|\bar{\Delta}| \leq 2\varrho\).

3. Bisection terminated at step \(L\), and \(\bar{\Delta} = \Delta_L\). In this case, termination clauses of 2b, 2c and 2d were never invoked, clearly implying that \(|\Delta_s| \leq \frac{1}{2}|\Delta_{s-1}|\), 1 \(\leq s \leq L\), and thus \(|\bar{\Delta}| = |\Delta_L| \leq \frac{1}{2^L}|\Delta_0| \leq 2\varrho\) (see (25)).

Thus, along with (E.1) we have

\[(E.2)\] It always holds \(|\bar{\Delta}| \leq 2\varrho\),

implying that whenever the signal \(x \in X\) underlying observations and the output segment \(\bar{\Delta}\) are such that \(f(x) \in \bar{\Delta}\), the error of the Bisection estimate (which is the midpoint of \(\bar{\Delta}\)) is at most \(\varrho\). Invoking (E.1), we conclude that the Bisection estimate is \((\varrho, \epsilon)\)-reliable. \(\square\)
2-convexity of conditional quantile

1°. Let $\mathcal{Q}$ be the family of non-vanishing probability distributions on $S = \{s_1 < s_2 < \ldots < s_M\} \subset \mathbb{R}$. For $r \in \mathcal{Q}$, let

$$F_m(r) = \sum_{i=1}^{m} r_i, \quad 1 \leq m \leq M,$$

so that $0 < F_1(r) < \ldots < F_M(r) = 1$.

Given $\alpha \in [0, 1]$, let us define (regularized) $\alpha$-quantile of $r \in \mathcal{Q}$, $q_\alpha[r]$, as follows:

- if $F_1(r) = r_1 \geq \alpha$, we set $q_\alpha[r] = s_1$;
- otherwise, there exists $m \in \{1, \ldots, M-1\}$ such that $F_m(r) \leq \alpha \leq F_{m+1}(r)$. We select an $m$ with this property, put

$$\beta = \frac{\alpha - F_m(r)}{F_{m+1}(r) - F_m(r)},$$

so that $\beta \in [0, 1]$, $\beta F_{m+1}(r) + (1 - \beta) F_m(r) = \alpha$, and set

$$q_\alpha[r] = \beta s_{m+1} + (1 - \beta) s_m.$$

Note that for some $\alpha$, the above $m$ is not uniquely defined; this happens if and only if $F_k(r) = \alpha$ for some $k$, $1 < k < M$. In this case there are exactly two choices of $m$, one $m = k$, and another $m = k - 1$. The first choice results in

$$\beta = \frac{\alpha - F_k(r)}{F_{k+1}(r) - F_k(r)} = 0, \quad \Rightarrow \quad \beta s_{m+1} + (1 - \beta) s_m = s_k.$$

The choice $m = k - 1$ results in

$$\beta = \frac{\alpha - F_{k-1}(r)}{F_k(r) - F_{k-1}(r)} = \frac{F_k(r) - F_{k-1}(r)}{F_k(r) - F_{k-1}(r)} = 1 \quad \Rightarrow \quad \beta s_{m+1} + (1 - \beta) s_m = s_k.$$

Thus, $q_\alpha[r]$ is well defined in spite of the fact that $m$ is not always uniquely defined by $\alpha$ and $r$.

2°. We conclude that the relation $q_\alpha[r] = s$ for $s_1 < s \leq s_M$ is exactly equivalent to the relation

(1) For some $m \in \{1, \ldots, M-1\}$, we have $F_m(r) \leq \alpha \leq F_{m+1}(r)$, and

$$\frac{[\alpha - F_m(r)] s_{m+1} + [F_{m+1}(r) - \alpha] s_m}{F_{m+1}(r) - F_m(r)} = s$$

(recall that, by definition, $q_\alpha[r] = s_1 \Leftrightarrow r_1 \leq \alpha$).

Now, let $\theta_r(s)$, $s_1 \leq s \leq s_M$, be the piecewise linear version to the cumulative distribution of $r$, that is, the continuous piecewise linear function on $\Delta = [s_1, s_M]$ with derivative breakpoints at $s_1, \ldots, s_M$ and such that $\theta_r(s_m) = F_m(r)$, $1 \leq m \leq M$. This is a strictly increasing function mapping $\Delta$ onto $\Delta^+ := [F_1(r), 1]$ for given $r$, $q_\alpha[r]$, as a function of $\alpha \in [0, 1]$, is obtained from the inverse of $\theta_r(\cdot)$ by extending this inverse from its domain $\Delta^+ \subset [0, 1]$ to the entire $[0, 1]$ by assigning the value $s_1$ to $\alpha < r_1$. As a consequence, it is immediately seen that $q_\alpha[r]$ is continuous in $(\alpha, r) \in [0, 1] \times \mathcal{Q}$. Note that $q_\alpha[r]$ takes all its values in $\Delta = [s_1, s_M]$.

Note that we have just proved the equivalence of the definition of $q_\alpha[r]$ via “spreading masses” used in Example 4.2 in Section 4.2, and the definition we introduced in 1°.
30. Let us fix $\alpha \in (0, 1)$. Given $s \in \Delta$, let us look at the set $Q_s^- := \{r \in Q : q_s[r] \leq s\}$. This set is as follows:

1. When $s = s_1$, by $1^0$ we have $Q_s^- = \{r \in Q : F_1(r) \geq \alpha\}$.

2. Now, let $s_1 < s \leq s_M$. Then for some $k = k(s) \in \{1, \ldots, M - 1\}$ we have $s_k < s \leq s_{k+1}$. We claim that the set $Q_s^-$ is the union of two convex sets:

$$Q_s^- = A_s \cup B_s$$

$$A_s = \{r \in Q : F_k(r) \geq \alpha\}$$

$$B_s = \left\{ r \in Q : F_k(r) \leq \alpha \leq F_{k+1}(r), \frac{\alpha - F_k(r)}{F_{k+1}(r) - F_k(r)} s_{k+1} + \frac{F_{k+1}(r) - \alpha}{F_{k+1}(r) - F_k(r)} s_k \leq s \right\}.$$  \hfill (38)

Indeed, if $r \in A_s$, then, we either have $q_s[r] = s_1 < s$, or $m$ in $1^0$ can be chosen to be $< k$, implying, by $1^0$, that $q_s[r] \leq s_k < s$. Thus, $A_s \subset Q_s^-$. Now let $r \in B_s$. From the first two inequalities in the definition of $B_s$, by (!), we conclude that

$$q_s[r] = \frac{[\alpha - F_k(r)] s_{k+1} + [F_{k+1}(r) - \alpha] s_k}{F_{k+1}(r) - F_k(r)}.$$  \\

The latter quantity, by the third inequality in the definition of $B_s$, is $\leq s$, implying that $r \in Q_s^-$. Thus, $B_s \subset Q_s^-$, and $A_s \cup B_s \subset Q_s^-$. To prove the inverse inclusion, let $r \in Q$ be such that $q_s[r] \leq s$, and let us prove that $r \in A_s \cup B_s$. It may happen that $F_k(r) \geq \alpha$, in which case $r \in A_s$, and we are done. Now, let $F_k(r) < \alpha$. We claim that $F_{k+1}(r) \geq \alpha$. Indeed, assume, on the contrary, that $F_{k+1}(r) < \alpha$. Then, by $1^0$, $q_s[r] \geq s_{k+1}$; the equality $q_s[r] = s_{k+1}$ is possible only when $m$ in $1^0$ can be chosen as $k$, and $\beta$, as defined in $1^0$, is equal to 1, that is, $\alpha = F_{k+1}(r)$, what is assumed not to be the case. Thus, we have $q_s[r] > s_{k+1}$, which is the desired contradiction due to $s_{k+1} \geq s$ and $q_s[r] \leq s$. Therefore, we are in the case where $F_{k+1}(r) \geq \alpha > F_k(r)$, that is, the first two inequalities in the description of $B_s$ hold true. The latter, by (!), implies that

$$q_s[r] = \frac{[\alpha - F_k(r)] s_{k+1} + [F_{k+1}(r) - \alpha] s_k}{F_{k+1}(r) - F_k(r)};$$

what combines with $q_s[r] \leq s$ to imply that $r$ satisfies the last inequality in the description of $B_s$, that is, $r \in B_s$. We conclude that $Q_s^-$ is indeed the union of two closed in $Q$ convex sets, $A_s$ and $B_s$.

Now let us consider the set $Q_s^+ = \{r \in Q : q_s[r] \geq s\}, s \in \Delta$. This set is as follows:

1. When $s = s_1$, by $1^0$, $Q_s^+ = Q$.

2. Now let $s_1 < s \leq s_M$, so that for some $k \in \{1, \ldots, M - 1\}$ we have $s_k < s \leq s_{k+1}$. We claim that now the set $Q_s^+$ is the union of two convex sets:

$$Q_s^+ = A'_s \cup B'_s$$

$$A'_s = \{r \in Q : F_{k+1}(r) \leq \alpha\}$$

$$B'_s = \left\{ r \in Q : F_k(r) \leq \alpha \leq F_{k+1}(r), \frac{\alpha - F_k(r)}{F_{k+1}(r) - F_k(r)} s_{k+1} + \frac{F_{k+1}(r) - \alpha}{F_{k+1}(r) - F_k(r)} s_k \geq s \right\}.$$  \hfill (39)
The proof of (39) is completely analogous to the proof of decomposition (38).

The bottom line is that $Q^+_s$ is the union of two closed in $Q$ convex sets, $A'_s$ and $B'_s$.

Let now $S = \{s_1 < s_2 < \ldots < s_M\}$ be a finite subset of $\mathbb{R}$, $T$ be a finite set, and $\mathcal{P}$ be the set of non-vanishing probability distributions on $\Omega = S \times T$. Given $\tau \in T$ and $\alpha \in (0, 1)$, let $q_{\alpha|\tau}[x] : \mathcal{P} \to [s_1, s_M]$ be the regularized $\alpha$-quantile of the conditional distribution $x(\cdot|\tau)$ on $S$ induced by a distribution $x \in \mathcal{P}$ and the condition $t = \tau$:

$$x(\mu|\tau) = \frac{x(\mu, \tau)}{\sum_{\nu=1}^M x(\nu, \tau)}, \mu = 1, \ldots, M.$$ 

By applying (38) and (39) to $r = x(\cdot|\tau)$ we arrive at the following

**Proposition B.1** In the just described situation, the function $q_{\alpha|\tau}[x]$ is 2-convex on $\mathcal{P}$: for every $s \in (s_1, s_M]$, selecting $k \in \{1, \ldots, M - 1\}$ in such a way that $s_k < s \leq s_{k+1}$, we have

$$\{x \in \mathcal{P} : q_{\alpha|\tau}[x] \leq s\} = \bigcup \left\{x \in \mathcal{P} : F_k(x; \tau) - \alpha F(x; \tau) \leq F_{k+1}(x; \tau), F_{k+1}(x; \tau) - F_k(x; \tau) s_k + 1 + \frac{F_{k+1}(x; \tau) - F_k(x; \tau) s_k}{s_k} \leq s \right\},$$

$$\{x \in \mathcal{P} : q_{\alpha|\tau}[x] \geq s\} = \bigcup \left\{x \in \mathcal{P} : F_k(x; \tau) - \alpha F(x; \tau) \leq F_{k+1}(x; \tau), F_{k+1}(x; \tau) - F_k(x; \tau) s_k + 1 + \frac{F_{k+1}(x; \tau) - F_k(x; \tau) s_k}{s_k} \geq s \right\},$$

where

$$F_m(x; \tau) = \sum_{i=1}^m x(i, \tau), \quad F(x; \tau) = \sum_{i=1}^M x(i, \tau),$$

and

$s < s_1 \Rightarrow \{x \in \mathcal{P} : q_{\alpha|\tau}[x] \geq s\} = \mathcal{P}, \{x \in \mathcal{P} : q_{\alpha|\tau}[x] \leq s\} = \emptyset,$

$s = s_1 \Rightarrow \{x \in \mathcal{P} : q_{\alpha|\tau}[x] \geq s\} = \mathcal{P}, \{x \in \mathcal{P} : q_{\alpha|\tau}[x] \leq s\} = \{x \in \mathcal{P} : F_1(x; \tau) \geq \alpha F(x; \tau)\},$

$s > s_M \Rightarrow \{x \in \mathcal{P} : q_{\alpha|\tau}[x] \geq s\} = \emptyset, \{x \in \mathcal{P} : q_{\alpha|\tau}[x] \leq s\} = \mathcal{P}.$