On weakly Gibson $F_\sigma$-measurable mappings

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Abstract

A function $f : X \to Y$ between topological spaces is said to be a weakly Gibson function if $f(U) \subseteq \overline{f(U)}$ for any open connected set $U \subseteq X$. We prove that if $X$ is a locally connected hereditarily Baire space and $Y$ is a $T_1$-space then an $F_\sigma$-measurable mapping $f : X \to Y$ is weakly Gibson if and only if for any connected set $C \subseteq X$ with the dense connected interior the image $f(C)$ is connected. Moreover, we show that each weakly Gibson $F_\sigma$-measurable mapping $f : \mathbb{R}^n \to Y$, where $Y$ is a $T_1$-space, has a connected graph.

1 Introduction

The classical theorem of Kuratowski and Sierpiński states that any Darboux Baire-one function $f : \mathbb{R} \to \mathbb{R}$ has a connected graph.

In 2010 K. Kellum introduced Gibson and weak Gibson properties for a mapping $f$ between topological spaces $X$ and $Y$. He calls $f$ (weakly) Gibson if $f(U) \subseteq \overline{f(U)}$ for an arbitrary open (and connected) set $U \subseteq X$.

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Since every Darboux function has the weak Gibson property [5], it is naturally to ask whether the theorem of Kuratiwski–Sierpiński remains valid if we replace the Darboux property by the weak Gibson property? It was shown in [5] that any weakly Gibson barely continuous mapping (in the sense that for each non-empty closed subspace $F \subseteq X$ the restriction $f|_F$ has a continuity point) defined on a connected and locally connected space $X$ and with values in a topological space $Y$ has a connected graph. It is found out that the condition of barely continuity in the above mentioned result from [5] is not necessary (see Example 4.4).

In this paper we consider weakly Gibson mappings $f : X \to Y$ which are $F_\sigma$-measurable, i.e. the preimage $f^{-1}(V)$ of an open set $V \subseteq Y$ is an $F_\sigma$-set in $X$. Note that in the case when $Y$ is a perfectly normal space, every Baire-one mapping $f : X \to Y$ is $F_\sigma$-measurable (see for instance [7, p. 394]). In Section 2 we introduce the notions of $G$-closed and $\mathcal{W}$-closed sets and prove that the Euclidean space $\mathbb{R}^n$ cannot be written as a union of two non-empty disjoint $F_\sigma$ and $G_\delta \mathcal{W}$-closed subsets as well as a connected and locally connected hereditarily Baire space cannot be written as a union of two non-empty disjoint $F_\sigma$ and $G_\delta G$-closed subsets. Using these facts in Section 3 we prove that each $F_\sigma$-measurable mapping $f$ between a locally connected hereditarily Baire space $X$ and a $T_1$-space $Y$ is weakly Gibson if and only if for any connected set $C \subseteq X$ with the dense connected interior the image $f(C)$ is connected. This generalizes the result of M. Evans and P. Humke [3] who proved the similar theorem for $X = \mathbb{R}^n$ and $Y = \mathbb{R}$. We prove also that each weakly Gibson $F_\sigma$-measurable mapping $f : \mathbb{R}^n \to Y$, where $Y$ is a $T_1$-space, has a connected graph.

## 2 $\mathcal{A}$-closed sets and their properties

Let $X$ be a topological space and let

- $\mathcal{T}(X)$ be the system of all open subsets of $X$,
- $\mathcal{C}(X)$ be the system of all connected subsets of $X$,
- $\mathcal{G}(X)$ be the system of all connected open subsets of $X$,
- $\mathcal{W}(X)$ be the system of all open convex subsets of a topological vector space $X$.

Let $\mathcal{A}(X)$ be a system of subsets of $X$. A subset $E \subseteq X$ is called closed with respect to $\mathcal{A}(X)$ or, briefly, $\mathcal{A}$-closed if for any $A \in \mathcal{A}(X)$ with $A \subseteq E$
we have $\overline{A} \subseteq E$.

**Proposition 2.1.** Let $X$ be a connected and locally connected space and $U$ be an open $G$-closed subset of $X$. Then $U = \emptyset$ or $U = X$.

**Proof.** Consider a component $C$ of $U$. The locally connectedness of $U$ implies that $C$ is clopen in $U$, consequently, $C$ is open in $X$. Since $U$ is $G$-closed, $\overline{C} \subseteq U$. Therefore, $\overline{C} = C$ provided $C$ is a component. Hence, $C$ is clopen in a connected space $X$. Therefore, $C = \emptyset$ or $C = X$. Since $U$ is a union of all components, $U = \emptyset$ or $U = X$. \hfill $\square$

We need the following auxiliary fact.

**Lemma 2.2.** [7, p. 136] Let $A$ and $B$ be subsets of a topological space $X$ such that $A$ is connected and $A \cap B \neq \emptyset \neq A \setminus B$. Then $A \cap \text{fr}B \neq \emptyset$.

For a point $x_0$ of a normed space $X$ and for $\varepsilon > 0$ by $B(x_0, \varepsilon) / B[x_0, \varepsilon]$ we denote an open /closed/ ball with the center at $x_0$ and radius $\varepsilon$.

If a subset of a topological space is simultaneously $F_\sigma$ and $G_\delta$, then it is said to be ambiguous.

**Theorem 2.3.** Let $X$ be a hereditarily Baire space, $X_1$ and $X_2$ are ambiguous disjoint $A$-closed subsets of $X$ such that $X = X_1 \cup X_2$. If

1. $X$ is a connected and locally connected space and $A(X) = G(X)$, or
2. $X = \mathbb{R}^n$, $n \geq 1$, and $A(X) = W(X)$,

then $X_1 = X$ or $X_2 = X$.

**Proof.** To obtain a contradiction, suppose that $X_1 \neq X$ and $X_2 \neq X$. Let $F = \overline{X}_1 \cap \overline{X}_2$. Since $X$ is connected, $F \neq \emptyset$. We show that $X_1 \cap F$ is dense in $F$. Conversely, choose a point $x_0 \in F$ and an open neighborhood $U$ of $x_0$ in $X$ such that

$$U \cap F \subseteq X_2.$$  

Then $x_0 \in \overline{X}_1 \cap \overline{X}_2$.

1). Since $X$ is locally connected, we may assume that $U$ is connected. Note that $U \cap X_1 \neq \emptyset$ and take $a \in U \cap X_1$. Then $a \not\in \overline{X}_2$. Let $G$ be a component of $X \setminus \overline{X}_2$ which contains $a$. Then $G$ is open in $X$. Remark that $U \cap G \neq \emptyset \neq U \setminus G$. Lemma 2.2 implies that $U \cap \text{fr}G \neq \emptyset$. Since $G$ is closed in $X \setminus \overline{X}_2$, $\text{fr}G \subseteq \overline{X}_2$. Moreover, $G \subseteq X_1$. Therefore, $\text{fr}G \subseteq F$. Choose $b \in U \cap \text{fr}G$. Then $b \in X_2$. Since $X_1$ is $G$-closed, $b \in \overline{G} \subseteq X_1$, which is impossible.
2). We may suppose that $U = B(x_0, \varepsilon)$. Take an arbitrary $a \in B(x_0, \varepsilon/2) \cap X_1$. Let
\[ R = \sup \{ r : B(a, r) \subseteq X_1 \}. \]
Note that $R \leq \varepsilon/2$, since $x_0 \in X_2$. We have
\[ d(x, x_0) \leq d(x, a) + d(a, x_0) < R + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon \]
for all $x \in B[a, R]$. Hence, $B[a, R] \subseteq U$. It is not hard to verify that $B[a, R] \cap \overline{X_2} \neq \emptyset$, provided $B[a, R]$ is compact. Therefore, there is $b \in B[a, R] \cap \overline{X_2}$. Since $B(a, R)$ is open and convex and $X_1$ is $W$-closed, $b \in X_1$. But $b \in U \cap F$, which implies that $b \in X_2$. Thus, $b \in X_1 \cap X_2$ which is impossible.

Hence, $X_1 \cap F$ is dense in $F$. It can be proved similarly that $X_2 \cap F$ is dense in $F$. Then $X_1 \cap F$ and $X_2 \cap F$ are disjoint dense $G_\delta$-subsets of a Baire space $F$, which implies a contradiction. Therefore, $X_1 = X$ or $X_2 = X$.

\section{Applications of $\mathcal{A}$-closed sets}

We say that a mapping $f : X \to Y$ has a 	extit{Gibson property with respect to a system $\mathcal{A}(X)$}, or $f$ is an $\mathcal{A}$-Gibson if for any $A \in \mathcal{A}(X)$ we have
\[ f(A) \subseteq f(\overline{A}). \]

It $\mathcal{A}(X) = T(X)$ then $f$ is said to be a 	extit{Gibson mapping}, and if $\mathcal{A}(X) = G(X)$ then $f$ is a 	extit{weakly Gibson mapping} (see [6]).

A mapping $f : X \to Y$ is 	extit{strongly Gibson with respect to a system $\mathcal{A}(X)$}, or $f$ is strongly $\mathcal{A}$-Gibson if for any $x \in X$ and $A \in \mathcal{A}(X)$ such that $x \in \overline{A}$ we have
\[ f(x) \in f(\overline{A \cap U}) \]
for an arbitrary neighborhood $U$ of $x$ in $X$.

\textbf{Theorem 3.1.} Let $X$ be a topological space, $Y$ a $T_1$-space and let $f : X \to Y$ be a mapping such that for any connected set $C \subseteq X$ with the dense connected interior the set $f(C)$ is connected. Then $f$ is a weakly Gibson mapping.

If, moreover, $X$ is a locally convex space then $f$ has the strong Gibson property with respect to the system $W(X)$.

\textit{Proof.} Fix an arbitrary open connected set $U \subseteq X$, a point $x_0 \in \overline{U}$ and an open neighborhood $V$ of $f(x_0)$ in $Y$. Denote $C = U \cup \{x_0\}$. Then the
inclusions \( U \subseteq C \subseteq \overline{U} \) imply that \( f(C) \) is a connected set. Assume \( f(U) \cap V = \emptyset \). Then

\[
f(C) = f(U \cup \{x_0\}) = f(U) \cup \{f(x_0)\} \subseteq (Y \setminus V) \cup \{f(x_0)\},
\]

which contradicts to the connectedness of \( f(C) \).

Now let \( X \) be a locally convex space. Fix a set \( G \in \mathcal{W}(X) \), a point \( x_0 \in \overline{G} \), an open convex neighborhood \( W \) of \( x_0 \) in \( X \) and an open neighborhood \( V \) of \( f(x_0) \) in \( Y \). Denote \( U = W \cap G \). Clearly, \( U \in \mathcal{G}(X) \). The rest of the proof runs as before.

\[\square\]

The converse proposition is true for \( F_\sigma \)-measurable mappings defined on a locally connected hereditarily Baire space.

**Theorem 3.2.** Let \( X \) be a locally connected hereditarily Baire space, \( Y \) a topological space and let \( f : X \to Y \) be a weakly Gibson \( F_\sigma \)-measurable mapping. Then for any connected set \( C \subseteq X \) with the dense connected interior the set \( f(C) \) is connected.

**Proof.** Let \( C \in C(X), U = \text{int} \, C \) and \( C \subseteq \overline{U} \).

We first prove that \( f(U) \) is a connected set. Suppose, contrary to our claim, that \( f(U) = W_1 \cup W_2 \), where \( W_1 \) and \( W_2 \) are non-empty disjoint open subsets of \( f(U) \). Set \( g = f|_U \). Evidently, \( g : U \to f(U) \) is a weakly Gibson \( F_\sigma \)-measurable mapping. Let \( A_i = g^{-1}(W_i) \) for \( i = 1, 2 \). Then every set \( A_i \) is \( \mathcal{G} \)-closed in \( U \), provided \( g \) is weakly Gibson. Moreover, every \( A_i \) is ambiguous set in \( U \), \( U = A_1 \cup A_2 \) and \( A_1 \cap A_2 = \emptyset \). Taking into account that \( U \) is a hereditarily Baire connected and locally connected space, we obtain that \( A_1 = U \) or \( A_2 = U \) according to Theorem 2.3(1). Then \( W_1 = \emptyset \) or \( W_2 = \emptyset \), a contradiction. Therefore, \( f(U) \) is a connected set.

Since \( f \) is weakly Gibson, \( f(U) \subseteq f(C) \subseteq f(\overline{U}) \subseteq f(\overline{U}) \). Consequently, the set \( f(C) \) is connected. \[\square\]

For a mapping \( f : X \to Y \) we define \( \gamma_f : X \to X \times Y \),

\[
\gamma_f(x) = (x, f(x)).
\]

Remark that if \( X \) is a connected and locally connected hereditarily Baire space and \( \gamma_f \) is an \( F_\sigma \)-measurable weakly Gibson mapping then Theorem 3.2 implies that \( f \) has a connected graph \( \Gamma \), provided \( \Gamma = \gamma_f(X) \). It is not hard to prove that \( \gamma_f \) remains to be weakly Gibson for any weakly Gibson mapping \( f : \mathbb{R} \to \mathbb{R} \). But Example 4.2 shows that \( \gamma_f \) need not be weakly Gibson for a weakly Gibson \( F_\sigma \)-measurable mapping \( f : \mathbb{R}^2 \to \mathbb{R} \).
**Theorem 3.3.** Let \( X = \mathbb{R}^n \) with \( n \geq 1 \) and let \( Y \) be a \( T_1 \)-space. If \( f : X \to Y \) is a weakly Gibson \( F_\sigma \)-measurable mapping then \( f \) has a connected graph.

**Proof.** We first observe that by Theorem 3.2 for any \( U \in \mathcal{G}(X) \) and for any \( C \) with \( U \subseteq C \subseteq \overline{C} \) the set \( f(C) \) is connected. Then \( f \) has the strong Gibson property with respect to the system \( \mathcal{W}(X) \) according to Theorem 3.1. It is easy to see that \( \gamma_f \) is also \( \mathcal{W} \)-strongly Gibson.

We show that \( \gamma_f : X \to X \times Y \) is \( F_\sigma \)-measurable. Let \( \{B_k : k \in \mathbb{N}\} \) be a base of open sets in \( X \) and \( W \) be an arbitrary open set in \( X \times Y \). Put

\[
V_k = \bigcup \{V : V \text{ is open in } Y \text{ and } B_k \times V \subseteq W \}.
\]

Then \( W = \bigcup_{k=1}^{\infty} (B_k \times V_k) \). Since \( \gamma_f^{-1}(W) = \bigcup_{k=1}^{\infty} (B_k \cap f^{-1}(V_k)) \), \( \gamma_f^{-1}(W) \) is an \( F_\sigma \)-subset of \( X \).

Now assume that \( Y_0 = \gamma_f(X) \) is not connected and choose open disjoint non-empty subsets \( W_1 \) and \( W_2 \) of \( Y_0 \) such that \( Y_0 = W_1 \cup W_2 \). Let \( X_i = \gamma_f^{-1}(W_i) \) for \( i = 1, 2 \). It is easy to check that \( X_1 \) and \( X_2 \) are \( \mathcal{W} \)-closed ambiguous subsets of \( X \). Moreover, \( X_1 \cap X_2 = \emptyset \) and \( X = X_1 \cup X_2 \). Then \( X_1 = X \) or \( X_2 = X \) by Theorem 2.3 (2). Consequently, \( W_1 = \emptyset \) or \( W_2 = \emptyset \), a contradiction. \( \square \)

The following question is open.

**Question 3.4.** Let \( X \) be a normed space, \( Y \) a \( T_1 \)-space and let \( f : X \to Y \) be a weakly Gibson \( F_\sigma \)-measurable mapping. Is the graph of \( f \) a connected set?

## 4 Examples

Our first example shows that the class of all \( F_\sigma \)-measurable Darboux mappings is strictly wider than the class of all Baire-one Darboux mappings.

**Example 4.1.** There exist a connected subset \( Y \subseteq \mathbb{R}^2 \) and an \( F_\sigma \)-measurable Darboux function \( f : \mathbb{R} \to Y \) which is not a Baire-one function.

**Proof.** Let \( \mathbb{Q} = \{r_n : n \in \mathbb{N}\} \) be the set of all rational numbers. For every \( n \in \mathbb{N} \) we consider the function \( \varphi_n : \mathbb{R} \to \mathbb{R}, \)

\[
\varphi_n(x) = \begin{cases} 
\sin \frac{1}{x-r_n}, & x \neq r_n, \\
0, & x = r_n.
\end{cases}
\]

\( \square \)
Define the function $g : \mathbb{R} \to \mathbb{R}$,

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \varphi_n(x).$$

Let

$$Y = \{(x, y) \in \mathbb{R}^2 : y = g(x)\} \quad \text{and} \quad f = \gamma_g.$$

Observe that for every $n$ the function $g_n(x) = \sum_{k=1}^{n} \frac{1}{2^k} \varphi_k(x)$ is a Baire-one Darboux function. Since the sequence $(g_n)_{n=1}^{\infty}$ is uniformly convergent to $g$ on $\mathbb{R}$, $g$ is a Baire-one Darboux function [1, Theorem 3.4]. Consequently, the graph of $g|_C$ is connected for every connected subset $C \subseteq \mathbb{R}$ according to [1, Theorem 1.1]. Therefore, $f : \mathbb{R} \to Y$ is a Darboux function. Moreover, $f : \mathbb{R} \to \mathbb{R}^2$ is a Baire-one mapping, which implies that $f : \mathbb{R} \to Y$ is $F_\sigma$-measurable.

Note that the space $Y$ is punctiform (i.e., $Y$ does not contain any continuum of cardinality larger than one), since $g$ is discontinuous on everywhere dense set $\mathbb{Q}$ (see [8]). Then each continuous mapping between $\mathbb{R}$ and $Y$ is constant. Therefore, $f : \mathbb{R} \to Y$ is not a Baire-one mapping.

**Example 4.2.** For all $(x, y) \in \mathbb{R}^2$ define

$$f(x, y) = \begin{cases} \sin \frac{1}{x}, & x > 0, \\ 1, & x \leq 0. \end{cases}$$

Then $f : \mathbb{R}^2 \to \mathbb{R}$ is an $F_\sigma$-measurable weakly Gibson function, but $\gamma_f$ is not weakly Gibson.

**Proof.** Show that $f$ is weakly Gibson. It is sufficient to check that $f$ is weakly Gibson at each point of the set $\{0\} \times \mathbb{R}$. Fix $y_0 \in \mathbb{R}$ and an open connected set $U \subseteq \mathbb{R}^2$ such that $p_0 = (0, y_0) \in \overline{U} \setminus U$. Take an arbitrary neighborhood $V$ of $f(p_0)$ in $\mathbb{R}$. Clearly, $f(p) \in V$ for all $p \in U \cap ((-\infty, 0] \times \mathbb{R})$. Consider the case $U \subseteq (0, +\infty) \times \mathbb{R}$. Since $p_0 \in \overline{U}$ and $U$ is connected, there exists $n \in \mathbb{N}$ such that $U \cap \left(\left\{\frac{1}{\pi/2 + 2n\pi}\right\} \times \mathbb{R}\right) \neq \emptyset$. Let $y \in \mathbb{R}$ with $p = \left(\frac{1}{\pi/2 + 2n\pi}, y\right) \in U$. Then $f(p) = 1$ and $f(p) \in V$. Hence, $f$ is weakly Gibson.

Consider open connected set $U = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } |y - \sin \frac{1}{x}| < x\}$ and let $C = U \cup \{(0, 0)\}$. Then $U \subseteq C \subseteq \overline{U}$. Note that $\gamma_f : \mathbb{R}^2 \to \mathbb{R}^3$ is $F_\sigma$-measurable. One easily checks that $\gamma_f(C)$ is not connected. Therefore, $\gamma_f$ is not weakly Gibson by Theorem 3.2.

Finally, we give an example of a space $Y$ and an $F_\sigma$-measurable Darboux mapping $f : \mathbb{R} \to Y$ which is not barely continuous.
We need first some definitions and auxiliary facts. For a topological space \( Y \) by \( \mathcal{F}(Y) \) we denote the space of all non-empty closed subsets of \( Y \) equipped with the Vietoris topology. A multivalued mapping \( F : X \to Y \) is said to be upper (lower) continuous at \( x_0 \in X \) if for any open set \( V \) in \( Y \) such that \( F(x_0) \subseteq V \) (\( F(x_0) \cap V \neq \emptyset \)) there exists a neighborhood \( U \) of \( x_0 \) in \( X \) such that for every \( x \in U \) we have \( F(x) \subseteq V \) (\( F(x) \cap V \neq \emptyset \)). A multivalued mapping \( f \) which is upper and lower continuous at \( x_0 \) is called continuous at \( x_0 \).

**Lemma 4.3.** There exists a continuous mapping \( f_0 : \mathbb{R} \to \mathcal{F}(\mathbb{R}) \) such that for all \( x \in [0, 1] \) and \( p \in P = \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{ 0 \} \) there are \( n_p \in \mathbb{N} \), strictly increasing unbounded sequence \( (v_n)_{n \geq n_p} \) of reals \( v_n > 0 \) and strictly decreasing unbounded sequence \( (u_n)_{n \geq n_p} \) of reals \( u_n < 0 \) such that

\[
f_0(u_n) = f_0(v_n) = \{ p \} \cup \bigcup_{k=1}^{n} [k, k+x] \cup \bigcup_{k>n} \{ k \}
\]

for all \( n \geq n_p \).

**Proof.** Let \( P = \{ p_n : n \in \mathbb{N} \} \). Choose a continuous function \( \varphi_0 : \mathbb{R} \to [0, 1] \) with \( \varphi_0(x) = p_k \) if \( |x| \in [n + \frac{2k-1}{2n}, n + \frac{k}{n}] \), where \( n \in \mathbb{N} \) and \( k = 1, \ldots, n \). For every \( n \in \mathbb{N} \) define a continuous function \( \varphi_n : \mathbb{R} \to [n, n+1] \),

\[
\varphi_n(x) = \begin{cases} n, & |x| \leq n, \\ n + \sin(4\pi k|x|), & |x| \in (k, k+1], k \geq n. \end{cases}
\]

Let \( f_0(x) = \{ \varphi_0(x) \} \cup \bigcup_{n=1}^{\infty} [n, \varphi_n(x)] \). Since all the functions \( \varphi_n \) are continuous and \( \varphi_k(x) = k \) for \( x \in [-n, n] \) and \( k \geq n \), \( f_0 \) is continuous.

Fix \( p = p_n \in P \) and \( x \in [0, 1] \). Denote \( n_p = n \). For all \( k \geq n \) choose \( u_k + k + \frac{1}{2k} \) such that \( \sin(4\pi ku_k) = x \). Then for every \( k \) satisfies the condition of the lemma. It remains to set \( u_k = -v_k \) for all \( k \in \mathbb{N} \).

**Example 4.4.** There exists a Baire-one \( F_\sigma \)-measurable Darboux mapping \( f : \mathbb{R} \to \mathcal{F}(\mathbb{R}) \) such that the restriction \( f|_C \) of \( f \) on the Cantor set \( C \subseteq \mathbb{R} \) is everywhere discontinuous and \( f(\mathbb{R}) \) is hereditarily Lindelöf (in particular, \( f(\mathbb{R}) \) is perfectly normal).

**Proof.** Let \( \mathbb{R} \setminus C = \bigcup_{n=1}^{\infty} I_n \), where \( I_n = (a_n, b_n) \). Set \( A = \{ a_n : n \in \mathbb{N} \} \) and \( B = C \setminus A \). For every \( n \in \mathbb{N} \) we choose a homeomorphism \( \psi_n : I_n \to \mathbb{R} \).
Define
\[ f(x) = \begin{cases} f_0(\psi_n(x)), & x \in I_n, \\ \{\frac{1}{n}\} \cup \bigcup_{k=1}^{\infty} [k, k + x], & n \in \mathbb{N}, x = a_n, \\ \{0\} \cup \bigcup_{k=1}^{\infty} [k, k + x], & x \in B, \end{cases} \]
where \( f_0 \) is the function from Lemma 4.3.

Show that \( f \) is a Baire-one mapping. For every \( n \in \mathbb{N} \) applying Lemma 4.3 we find a number \( m_n \), strictly increasing sequence \((v_k^{(n)})_{k \geq m_n}\) of \( v_k^{(n)} \in (\frac{a_n + b_n}{2}, b_n)\) and strictly decreasing sequence \((u_k^{(n)})_{k \geq m_n}\) of \( u_k^{(n)} \in (\frac{a_n a_n + b_n}{2})\) such that
\[ f_0(\psi_n(u_k^{(n)})) = \{\frac{1}{n}\} \cup \bigcup_{i=1}^{k} [i, i + a_n] \cup \bigcup_{i>k} \{i\} \]
and
\[ f_0(\psi_n(v_k^{(n)})) = \{0\} \cup \bigcup_{i=1}^{k} [i, i + a_n] \cup \bigcup_{i>k} \{i\} \]
for all \( i \geq m_n \).

For every \( n \in \mathbb{N} \) denote \( M_n = \{k \leq n : m_k \leq n\} \). Clearly, \( M_n \subseteq M_{n+1} \) for all \( n \) and \( N = \bigcup_{n=1}^{\infty} M_n \). Choose a sequence of continuous functions \( g_n : \mathbb{R} \to [0, 1] \) which is pointwise convergent to the function
\[ g(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus A, \\ \frac{1}{n}, & n \in \mathbb{N}, x = a_n. \end{cases} \]
Without loss of generality, we assume that \( g_n(u_k^{(n)}) = \frac{1}{n} \) and \( g_n(v_k^{(n)}) = 0 \) if \( n \in M_k \). Now for every \( k \in \mathbb{N} \) define
\[ f_k(x) = \begin{cases} f_0(\psi_n(x)), & x \in [u_k^{(n)}, v_k^{(n)}], n \in M_k, \\ \{g_k(x)\} \cup \bigcup_{i=1}^{k} [i, i + x] \cup \bigcup_{i>k} \{i\}, & x \in \mathbb{R} \setminus \left( \bigcup_{n \in M_k} [u_k^{(n)}, v_k^{(n)}] \right). \end{cases} \]
It is easy to see that each \( f_k \) is continuous and \( \lim_{k \to \infty} f_k(x) = f(x) \) for all \( x \in \mathbb{R} \).

We now prove that \( f \) has the Darboux property. Let \( I \subseteq \mathbb{R} \) be a connected set of cardinality larger than one. If \( I \subseteq I_n \) for some \( n \in \mathbb{N} \) then \( f(I) \) is connected, provided the restriction \( f|_{I_n} \) is continuous. Suppose \( I \not\subseteq I_n \) for every \( n \in \mathbb{N} \). Let \( M = \{n \in \mathbb{N} : J_n = I_n \cap I \neq \emptyset\} \). Note that the set \( G = \bigcup_{n \in M} J_n \) is dense in \( I \). Set \( f(I) = U \cup V \), where \( U \) and \( V \) are disjoint clopen sets in \( f(\mathbb{R}) \). Denote \( K = \{n \in M : f(J_n) \subseteq U\} \) and \( L = \{n \in M : f(J_n) \subseteq V\} \). Since the restriction of \( f \) on each set \( J_n \) is
continuous, $G = G_1 \cup G_2$, where $G_1 = \bigcup_{n \in K} J_n$, $G_2 = \bigcup_{n \in L} J_n$ and $G_1 \cap G_2 = \emptyset$. Lemma 4.3 implies that $f(G_i) \subseteq f(G_i)$ for $i = 1, 2$. Hence, $f(G_1) \subseteq U$ and $f(G_2) \subseteq V$. Therefore, $I = G_1 \cup G_2$ and $G_1 \cap G_2 = \emptyset$. Consequently, $G_1 = \emptyset$ or $G_2 = \emptyset$. Thus, $U = \emptyset$ or $V = \emptyset$.

To show that $Y = f(\mathbb{R})$ is hereditarily Lindelöf it is sufficient to prove that $Y_1 = f(\mathbb{R} \setminus C)$ and $Y_2 = f(C)$ are hereditarily Lindelöf. Note that $Y_1 = f_0(\mathbb{R})$ is hereditarily Lindelöf, since $Y_1$ is a continuous image of $\mathbb{R}$ under the continuous mapping $f_0$ with values in Hausdorff space $F(\mathbb{R})$. Since $f(a_n) \cap [0, 1] = \{\frac{1}{n}\}$ for every $n \in \mathbb{N}$ and $f(b) \cap [0, 1] = \{0\}$ for each $b \in B$, the space $f(A)$ is countable discrete subspace of $Y_2$. Moreover, for each $b \in B$ the sets $f((b - \varepsilon, b] \cap C)$, where $\varepsilon > 0$, form a base of neighborhoods of $f(x_0)$ in $Y_2$. Since an arbitrary union of sets of the form $(u, v]$ is a union of a sequence $(u_n, v_n]$, $Y_2$ is hereditarily Lindelöf. Hence, $X$ is hereditarily Lindelöf, consequently, $X$ is perfectly normal.

Since $Y$ is perfectly normal and $f$ is a Baire-one function, $f$ is $F_\sigma$-measurable [7, p. 394]. It remains to prove that the restriction $f|_C$ of $f$ on the Cantor set $C$ is everywhere discontinuous. Note that $f|_C$ is discontinuous everywhere on $A$, since $f(A)$ is discrete in $Y_2$. Moreover, for every $b \in B$ sets of the form $(b - \varepsilon, b] \cap C$ is not a neighborhood of $b$ in $C$. Therefore, $f|_C$ is discontinuous at each point $b \in B$.

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