Selection on $X_1 + X_2 + \cdots + X_m$ with layer-ordered heaps

Patrick Kreitzberg  
University of Montana  
Department of Computer Science  
32 Campus Drive, Missoula, MT  
United States of America

Kyle Lucke  
University of Montana  
Department of Computer Science  
32 Campus Drive, Missoula, MT  
United States of America

Oliver Serang*  
University of Montana  
Department of Computer Science  
32 Campus Drive, Missoula, MT  
United States of America  
oliver.serang@umontana.edu*

October 29, 2019

Abstract

Selection on $X_1 + X_2 + \cdots + X_m$ is an important problem with many applications in areas such as max-convolution, max-product Bayesian inference, calculating most probable isotopes, and computing non-parametric test statistics, among others. Faster-than-naïve approaches exist for $m = 2$: Johnson & Mizoguchi (1978) find the smallest $k$ values in $A + B$ with runtime $O(n \log(n))$. Frederickson & Johnson (1982) created a method for finding the $k$ smallest values in $A + B$ with runtime $O(n + \min(k, n) \log(\frac{k}{\min(k, n)}))$. In 1993, Frederickson published an optimal algorithm for selection on $A + B$, which runs in $O(n + k)$. In 2018, Kaplan et al. described another optimal algorithm in terms Chazelle’s of soft heaps. No fast methods exist for $m > 2$. Johnson & Mizoguchi (1978) introduced a method to compute the minimal $k$ terms when $m > 2$, but that method runs in $O(m \cdot n^{\frac{2}{m}} \log(n))$ and is inefficient when $m \gg 1$.

In this paper, we introduce the first efficient methods for problems where $m > 2$. We introduce the “layer-ordered heap,” a simple special class of heap with which we produce a new, fast selection algorithm on the Cartesian product. Using this new algorithm to perform $k$-selection on the Cartesian product of $m$ arrays of length $n$ has runtime $\in o(m \cdot n + k \cdot m)$. We also provide implementations of the algorithms proposed and their performance in practice.

*Corresponding author.
1 Introduction

Sorting all values in $A + B$, where $A$ and $B$ are vectors of length $n$ and $A + B$ is the Cartesian product of these vectors under the + operator, is nontrivial. In fact, there is no known approach faster than naively computing and sorting them $\in O(n^2 \log(n^2)) = O(n^2 \log(n))$ [2]; however, Frederickson & Johnson demonstrated in 1982 that the minimal $k$ values in $A + B$ (i.e., selection on $A + B$) can be performed $\in O(n + \min(n, k) \log(k))$ [6] using an approach reminiscent of the median-of-medians algorithm [1]. In 1993, Frederickson published the first optimal selection algorithm on $A + B$, which runs in $O(n + k)$. This method uses a data structure similar to what would later become the soft heap (discovered by Chazelle in 2000 [4]).

In 2018, Kaplan et al. described an optimal method, this time in terms of soft heaps [7]. This method worked by binary heapifying $A$ and $B$, each in $O(n)$ time. If $A$ is in binary heap order, then $A_i \leq A_{2i}$ and $A_i \leq A_{2i+1}$ (note that 1-indexing is used). The minimal value in $A + B$ must be $A_1 + B_1$. Furthermore, we will never use a child of some index $i$ from our binary heapified $A$ or index $j$ from the binary heapified $B$ unless we have already used index $(i, j)$ itself. Kaplan gives a method by which we enumerate every value in $A + B$ without duplicates where children are only considered after their parents in the binary heap ordering. All currently considered values (and their pairwise $(i, j)$ indices) are stored in a min soft heap. This soft heap has constant insertion and removal time; however, it may remove values not in sorted order, instead moving other values towards the front of the queue due to “corruption.” The constant parameter $\epsilon \in (0, 1)$ bounds that the number of corrupted elements at any moment in a soft heap containing $\ell$ items is $\leq \epsilon \cdot \ell$. When index $(i, j)$ is popped from the soft heap as a candidate for the next lowest value, then the the following indices are inserted into the soft heap as new candidates:

\[
\begin{align*}
\{(2i, 1), (2i + 1, 1), (i, 2), (i, 3)\}, & \quad j = 1 \\
\{(i, 2j), (i, 2j + 1)\}, & \quad j > 1.
\end{align*}
\]

After $k$ pop operations, $\epsilon \cdot \ell$ corrupt values, which may or may not be members of the correct results of the $k$-selection, will be emitted along with $k - \epsilon \cdot \ell$ members of the $k$-selection. Each pop operation inserts at most four children, resulting in an increase of the soft heap size by three. After the first $k + \epsilon \cdot 3k$ values are popped from the soft heap (where the new candidates are inserted into the soft heap as necessary), those $k + \epsilon \cdot 3k$ values must contain the smallest $k$ values (even if corruption is as bad as possible). Thus, with $\epsilon = \frac{1}{3}$, we pop the first $2k$ values from the soft heap (again, inserting new candidates as necessary) and are guaranteed that they contain the minimal $k$ values. The minimal $k$ values of these can be computed via a one-dimensional selection via either median-of-medians [1] or via soft heap-based one-dimensional selection.

In this paper, we construct generalizations for selection on $X_1 + X_2 + \cdots + X_m$ when $m \gg 1$, with and without soft heaps. This problem is important for max-convolution [3] (and max-product Bayesian inference [10, 9]), for computing the $k$ best quotes on a supply chain for a business, and for calculating the most abundant isotopes of a molecule where $X_i$ would represent a value and its isotopes and $Y$ represents the full molecule [8].

2 Methods

2.1 SoftTensor: a direct $m$-dimensional generalization of Kaplan et al. 2018

The following recursive function generalizes the proposal scheme from Kaplan et al. 2018 [7]. It accepts a multidimensional index and generates all inferior “children” indices without inserting any
Choosing the smallest total runtime of so the best performance will be achieved using the smallest allowed we have ensure the fastest possible performance, we want the comes from the one-dimensional selection. This can be simplified to \(c\) possibly corrupted values.

In this manner, we simply apply the direct \(m\)-dimensional generalization of the soft heap pairwise method from Kaplan et al. Each pop operation will insert \(\leq 2m\) new indices, because each of the \(m\) axes may advance its binary heap through two children (left and right); thus, we can proceed as described below in SortTensor, popping \(c \cdot k\) values and indices from the soft heap (where \(c > 1\) is a value that guarantees that the minimal \(k\) values will be popped). As each value is popped, its \(\leq 2m\) dependents are generated through the advance routine above. These dependents will be inserted into the soft heap by summing the appropriate \(X_i\) entries (computed naïvely in either \(O(m)\) time or by observing that it has only a single term changed from a previously created index, enabling computation in \(O(1)\) time) and creating their tuple index. In this scheme, the \(m\)-dimensional tuple indices are necessary, because they will be used to feed into the advance routine to generate the subsequent dependent tuple indices. Because of the necessity of storing the tuple indices, each pop operation will require \(\leq 2m\) insertions, which will each cost \(\in O(m)\), and thus each pop operation will cost \(\in O(m^2)\).

### 2.1.1 Space

The space usage is dominated by the number of terms (which each include an \(m\)-dimensional index) in the soft heap. The number of index tuples in the soft heap after \(c \cdot k\) pop operations are performed will be \(\leq 2c \cdot k \cdot m\). Each index tuple has size \(m\), so the space usage will be \(\in O(c \cdot k \cdot m^2)\). Below, we demonstrate a choice of \(c \in O(1)\) that produces an efficient runtime, and thus the total space usage, including the \(n \cdot m\) cost to store the vectors, is \(\in O(n \cdot m + k \cdot m^2)\).

### 2.1.2 Runtime

Since each pop will result in an insertion into the soft heap of \(\leq 2m\) new candidates, the number of corrupt elements popped when popping \(c \cdot k\) elements is \(\leq \epsilon \cdot 2m \cdot c \cdot k\). Thus, we are guaranteed the number of uncorrupted values popped \(\geq c \cdot k - \epsilon \cdot 2m \cdot c \cdot k\). We need this value to be \(\geq k\) in order to perform our subsequent one-dimensional selection to retrieve the minimal \(k\) values from the pool of \(c \cdot k\) possibly corrupted values.

Each pop operation in the soft heap costs \(\in O\left(\frac{1}{\epsilon}\right)\), so the total runtime of retrieving the minimal \(c \cdot k\) values from \(X_1 + X_2 + \cdots + X_m\) will be \(\in O(n \cdot m + c \cdot k \cdot (\log(\frac{1}{\epsilon}) + m^2) + c \cdot k)\), where the final \(c \cdot k\) comes from the one-dimensional selection. This can be simplified to \(\in O(n \cdot m + c \cdot k \cdot (\log(\frac{1}{\epsilon}) + m^2))\). To ensure the fastest possible performance, we want the \(\log(\frac{1}{\epsilon})\) term to not exceed the \(m^2\) term, and so we have \(\log(\frac{1}{\epsilon}) \leq m^2\). This is satisfied by \(\epsilon \geq \frac{1}{e^{m^2}}\). \(c\) will need to satisfy \(c \cdot k - \epsilon \cdot 2m \cdot c \cdot k \geq k\), which implies \(c \geq \frac{1}{1 - \epsilon \cdot 2m}\). If the \(\log(\frac{1}{\epsilon})\) term does not dominate the runtime, then the performance will be as fast as possible when \(c\) is minimal. The lower bound for \(c\) is monotonically decreasing with \(\epsilon\), and so the best performance will be achieved using the smallest allowed \(\epsilon = \frac{1}{e^{m^2}}\). This gives \(c \geq \frac{1}{1 - \frac{2m}{e^{m^2}}}\).

Choosing the smallest \(c\) (and hence most efficient) that satisfies yields \(c = \frac{1}{1 - \frac{2m}{e^{m^2}}} \in O(1)\) and a total runtime of

\[
O\left(n \cdot m + \frac{k \cdot m^2}{1 - \frac{2m}{e^{m^2}}}\right) = O\left(n \cdot m + k \cdot m^2\right).
\]
Practically, this most efficient choice of $\epsilon$ may be numerically unstable. The same $O(\cdot)$ runtime can be achieved by using $c = 2$, which yields the following inequality for the total number of uncorrupted values popped from the soft heap after $2k$ pops: $2k - \epsilon \cdot 2k \cdot 2m \geq k$, which gives $\epsilon \leq \frac{1}{4m}$. This gives a total runtime $\in O(n \cdot m + k \cdot (\log(4m) + m^2)) = O(n \cdot m + k \cdot m^2)$.

### 2.1.3 Implementation

The SoftTensor algorithm is implemented in code listing 3.

#### 2.2 SoftTree: a soft heap-based balanced binary tree method

A balanced binary tree is constructed with nodes where each node performs pairwise selection on $A + B$, where $A$ and $B$ are the left and right child nodes. These pairwise selection problems of the form $A + B$ are solved via the optimal method from Kaplan et al. 2018 via soft heaps; however, it is nontrivial to stack these pairwise soft heap selection nodes in this manner, because they require their inputs $A$ and $B$ to be binary heaps with random access. The output of these pairwise soft heap selection nodes must therefore be made to be in heap order with random access.

**Lemma 1.** Let $a$ be the number of terms of $A$ required to produce the minimal $k$ values of the form $A_i + B_j$. Let $b$ be the number of terms of $B$ required to produce the minimal $k$ values of the form $A_i + B_j$. Then $a + b - 1 \leq k$.

**Proof.** Although the algorithm does not sort, here we refer to $A$ and $B$ in ascending order: $A_i \leq A_{i+1}$ and $B_i \leq B_{i+1}$. Denote the sorted tuples $(A_i + B_j, i,j) \leq (A_{i+1} + B_{j+1}, i+1,j+1)$ sorted in ascending order. Denote $S$ as the minimal $k$ terms: $S = \{(A_i + B_j, i,j) \mid (A_{i+1} + B_{j+1}, i+1,j+1) \}$.

By supposition, $\{i_1,i_2,\ldots i_k\} = \{1,2,\ldots a\}$ and $\{j_1,j_2,\ldots j_k\} = \{1,2,\ldots b\}$. W.l.o.g., for any $i \in \{1,2,\ldots a\}$, $\exists j: (A_i + B_j, i,j) \in S$; therefore, $(A_i + B_1,i,1) \leq (A_i + B_j,i,j)$ and thus $(A_i + B_1,i,1) \in S$. $\{(A_i + B_1,i,1) \mid i \in 1,2,\ldots a\} \cup \{(A_1 + B_j,1,j) \mid j \in 1,2,\ldots b\} = \{(A_i + B_1,i,1) \mid i \in 1,2,\ldots a\} + \{(A_1 + B_j,1,j) \mid j \in 1,2,\ldots b\} - \{(A_1 + B_1,1,1)\} = a + b - 1$. Thus $S$ must contain these $a + b - 1$ terms, and so $a + b - 1 \leq k$.

By lemma[1] the total number of values required from $A$ and $B$ will together be $\leq k + 1$. Thus, it is sufficient to let $a = b = k$. This yields a simple, recursive algorithm that proceeds in a pairwise manner: select the minimal $k$ terms from the left child, select the minimal $k$ terms from the right child, and then use only those values to select the minimal $k$ terms from their sum.

### 2.2.1 Space

Aside from leaves, every node in the tree will generate $k$ values from its Cartesian sum, each in $O(k)$ time via optimal pairwise selection on $A + B$. There are $m$ leaves, $\frac{m}{2}$ nodes on the previous layer, etc., and thus $< 2m$ total nodes in the tree via geometric series. Including the $n \cdot m$ cost of storing the input data, the space of this method is thus $\in O(n \cdot m + k \cdot m)$.

### 2.2.2 Runtime

For reasons similar to the space usage above (each pairwise $A + B$ selection node has a linear runtime in $k$), the runtime is $\in O(n \cdot m + k \cdot m)$. 

4
2.2.3 Implementation

The SoftTree algorithm is implemented in code listing 6.

2.3 SortTensor: a direct $m$-dimensional generalization of the sorting-based $C = A + B$ method

Forming a matrix sorted by both rows and columns can be a fast way to get the minimum $k$ values from two vectors $A, B$. This can be observed by sorting both $A$ and $B$ in ascending order, and then sparsely building a matrix of $A_i + B_j$. If $A'$ and $B'$ represent sorted vectors such that $A'_1 \leq A'_2 \leq \cdots$ and $B'_1 \leq B'_2 \leq \cdots$, then the minimal value of $C$ is $A'_1 + B'_1$. The second smallest value in $C$ is $\text{min}(A'_1 + B'_2, A'_2 + B'_1)$, and so on. Whenever the next minimum value has been obtained its direct neighbors can be inserted into the matrix, the collection of inserted values which have not yet been popped form a “hull.” In practice, it is efficient to use a min-heap to store the values in the hull, then to get the next minimal value simply pop from the hull and then insert the neighbors of the recently popped item.

The direct generalization of the $A + B$ method is straightforward: Instead of a matrix, we have an $\mathbb{R}^m$ tensor, where the hull is composed of a collection of $m$-dimensional indices. In each iteration, we pop the minimal value from the hull and append that to the result vector. Let this minimal value come from index $(i_1, i_2, \ldots, i_m)$. Now we insert the $m$ values from the neighbors of index $(i_1, i_2, \ldots, i_m)$ into the heap holding the hull: $(i_1 + 1, i_2, \ldots, i_m), (i_1, i_2 + 1, \ldots, i_m), \ldots, (i_1, i_2, \ldots, i_m + 1)$. As with the two-dimensional method, it is possible to store not only the $X_{1,i_1} + X_{2,i_2} + X_{3,i_3} + \cdots + X_{m,i_m}$ in the heap, but also store the index tuple from which it came.

Note that the selection of the minimal $k$ values in $X_1 + X_2 + \cdots + X_m$ will be reported in sorted order.

2.3.1 Space

The size of the hull, denoted using $h$, grows by $m – 1$ values (excluding the first iteration which adds $m$ values). Each of these values will be accompanied by an $m$-dimensional index. Generalizing the 2D case, the maximal size of $h$ is $O(n^{m-1})$, because it will have largest size as a diagonal hyperplane across all $m$ axes. Thus, after $k$ values are retrieved the hull will take storage space $\in O(\text{min}(k \cdot m^2, n^{m-1}))$. Storing the input vectors costs $n \times m$, so this method has space usage $\in O(n \cdot m + \text{min}(k \cdot m^2, n^{m-1}))$.

2.3.2 Runtime

Initially, the input vectors are heapified which costs $\in O(n \cdot m)$.

In each iteration, there are $O(m)$ objects inserted into the hull’s heap (one advancement per axis) and a single removal. Note that each of the objects inserted includes an $m$-dimensional index. Using a binomial heap (which has amortized $O(1)$ insert), the time to insert each of these $O(m)$ objects is $O(1)$, but it takes $O(m)$ steps to construct each $m$-dimensional tuple index. Thus, the insertion time per iteration is in $O(m^2)$; therefore, over $k$ iterations, it costs $O(k \cdot m^2)$.

In each iteration the hull grows in size by $O(m)$ values, so in the first iteration the cost to pop from the hull is $O(\log(m))$; in the second iteration the cost is $O(\log(2m))$, etc. Over all $k$ iterations the cost of popping is $O(\log(m) + \log(2m) + \cdots + \log(k \cdot m)) = O(\log(m \cdot (2 \cdot m) \cdot (3 \cdot m) \cdots (k \cdot m))) = O(\log(m^k \cdot k!)) = O(k \log(m) + k \log(k))$.

In each iteration, the margin of at most one axis is advanced (see invariant below). Advancing that margin requires popping the min value from that input heap. This costs $\in O(\log(n))$ per
iteration.

If \( k \in o(n^{m-1}) \), then the total runtime of \( k \) iterations of the SortTensor method is \( \in O(n \cdot m + k \cdot m^2 + k \log(m) + k \log(k) + k \log(n)) \).  

\[
= O(n \cdot m + k \cdot m^2 + k \log(k \cdot n)).
\]

### 2.3.3 Implementation

The SortTensor algorithm is implemented in code listing 8.

### 2.4 SortTree: A sorting-based balanced binary tree

In lemma 1 we saw that it is not necessary to generate more than \( k + 1 \) values total from children \( A \) and \( B \) combined; however, it is not trivial to discern how many values we must generate from each child. One approach to resolve this is to keep the \( A \) and \( B \) values from each child. One approach to resolve this is to keep the \( A \) and \( B \) values in sorted order, generating one at a time. Furthermore, it is not necessary for \( A \) and \( B \) to be sorted vectors; instead, it is sufficient that \( A \) and \( B \) simply be heap data structures, from which we can repeatedly request the next smallest value.

The SortTree method runs similarly to the two-dimensional case: the top-left corner of \( A + B \) is computed via the minimal value of \( A \) and the minimal value of \( B \). This inserts two values into the hull: either the sum of the smallest value in \( A \) and the second-smallest value in \( B \) or the sum of the second-smallest value in \( A \) and the smallest value in \( B \). Neither the full, sorted contents of \( A \) nor the full, sorted contents of \( B \) are needed.

We thus construct a balanced binary tree of these heap-like structures. Except for the leaves (which are binary heaps of the \( m \) arrays of length \( n \)), each heap-like structure is of the form \( A + B \), where \( A \) and \( B \) are heap-like structures (figure 1).

For a node in the tree we call the set of values which are available along each sorted axis the “margin” (these correspond to the gray boxes in the child node corresponding to that axis in Figure 1). The size of the vertical margin, \( f_v \), is the largest sorted index requested from the left child (in the figure, \( f_v = 6 \)). Similarly, the size of the horizontal margin, \( f_h \), is the largest sorted index requested from the right child (in the figure, \( f_h = 4 \)).

The only time a parent node needs to request the next smallest value from a child node is when the value of either \( f_h \) or \( f_v \) is incremented. We have an invariant: after the first iteration, at any node, the margin of at most one axis is incremented at a time. We demonstrate by contradiction: To advance both margins simultaneously, we have to pop cell \((f_v, f_h)\) from the hull’s heap and select its value as the next smallest. This adds indices \((f_v + 1, f_h)\) and \((f_v, f_h + 1)\) to the hull, which would require popping from both the left and right children (i.e., advancing both margins); however, for \((f_v, f_h)\) to be in the hull, either \((f_v - 1, f_h)\) or \((f_v, f_h - 1)\) must have already been popped. These pop operations would have pushed either \((f_v, f_h)\) and \((f_v - 1, f_h + 1)\) or \((f_v + 1, f_h - 1)\) and \((f_v, f_h)\), respectively. Either of these would have advanced the vertical or horizontal margins, making it impossible for popping cell \((f_v, f_h)\) to advance both margins simultaneously. The exception is when both \( f_v - 1 \) is not a valid row and \( f_h - 1 \) is not a valid column; in that case, we have \( f_v = f_h = 1 \), and thus both margins can be advanced only in the first iteration.

From this invariant, we see that we will asymptotically propagate only to up to one child per iteration. The constant per-node cost of the first iteration in all nodes (where we may visit both children) can be amortized out during construction of the balanced binary tree: there are \( O(m) \) nodes in the tree. In a balanced binary tree, the longest path from root to leaves that includes only one child per node is simply a path from the root to a leaf, and thus propagates to \( \leq \log(m) + 1 \) nodes.
Figure 1: **Illustration of the SortTree method.** Problems of the form $A + B$ are assembled into a balanced binary tree. The gray squares in each 2D matrix represent values which have already been popped from a heap of $A + B$ at the request of a parent node of $A + B$. When a value in a child is popped at the request of a parent, it advances the corresponding margin along one axis of the parent’s matrix. The blue squares are values in the hull, but which have not yet been popped. The child on the left has popped six values and currently has four values in its hull; the row axis of the parent has six values that have been realized thus far so $f_{vertical} = 6$. The child on the right has popped four values and currently has four values in its hull; the column axis of the parent has four values that have been realized thus far so $f_{horizontal} = 4$. 
Note that the selection of the minimal $k$ values in $X_1 + X_2 + \cdots + X_m$ will be reported in sorted order.

The $m$-dimensional indices from which these minimal $k$ values were drawn can be optionally computed via depth-first search in $O(k \cdot m)$ total.

### 2.4.1 Space

The space needed to store the input vectors is $O(m \cdot n)$. From the invariant above, in each iteration, only $O(\log(m))$ nodes will be visited and they will add $O(1)$ new values into some hull; thus, the space of all the hulls combined grows by $\log(m)$ per iteration. The total space usage is $\in O(m \cdot n + k \log(m))$. For $k \leq \frac{m \cdot n}{\log(m)}$ the space usage is $\in O(m \cdot n)$, which is the space required just to store the original data.

If the $m$-dimensional indices are to be calculated and stored this adds $O(k \cdot m)$ storage space to become $O(m \cdot n + k \cdot m + k \log(m))$. If $k \leq n$ then the space to store the indices is $\in O(n \cdot m)$, which is the space to store the original data.

### 2.4.2 Runtime

Initially, the input vectors are heapified which costs $O(n \cdot m)$. To find the cost of popping the minimal value from the hull we have to figure out the maximum size of the hull. For the generalized case, $Y = X_1 + X_2 + \cdots + X_m$, the second layer (the layer above the leaves) will have matrix size $n^2$, the next layer will have matrix size $n^4$, etc. For each layer, the size of the matrix is squared from the size of a child’s matrix and there will be $\leq \log(m) + 1$ layers so the layer with the largest matrix (the root) will have matrix size $n^{2 \log(m)} = n^m$. With a matrix of size $n^m$, the hull is at most $n^{\frac{m}{2}}$ which means popping from the hull is $\in O(\log(n^{\frac{m}{2}}))$. In the worst case, $k = n^m$ so the time to pop from the hull is $\in O(\log(k))$.

Since both horizontal and vertical margins of a node cannot advance simultaneously (except for the very first value, which is amortized into the cost of tree construction), at most one child of each parent will have to pop from its hull so there will be $O(\log(m))$ pops per iteration. In the second iteration, the hulls will be of size 2 so the time to pop from all hulls is $\in O(\log(2) \log(m))$, similarly, for the third iteration the time to pop from all hulls is $\in O(\log(3) \log(m))$, etc. After $k$ iterations the total time spent popping from hulls is $\in O(\log(k!) \log(m)) = O(k \log(k) \log(m))$.

In total the runtime of the SortTree method is $\in O(n \cdot m + k \log(k) \log(m))$. If the $m$-dimensional indices are to be calculated and stored (each length-$m$ tuple index is computed via an in-order traversal of the tree, which costs $O(m)$ per iteration), we add $O(k \cdot m)$ to the time to become $\in O(n \cdot m + k \log(k) \log(m) + k \cdot m)$.

If the distribution on the $X_i$’s lead to a fixed probability, $p < 1$, of advancing the margin in either child in the tree, then the probability of descending to layer $\ell$ of the tree is $\in O(p^{-\ell})$. Thus for such inputs, the expected runtime is a geometric series. In that case, the $\log(m)$ coefficient disappears and the total runtime without calculating the indices is $\in O(n \cdot m + k \log(k))$ and total runtime with calculating the indices is $\in O(n \cdot m + k \log(k) + k \cdot m)$.

### 2.4.3 Implementation

The SortTree algorithm is implemented in code listing 9.
2.5 FastSoftTree: an improved soft heap-based balanced binary tree method

From lemma 1, we saw that the total number of values generated from children $A$ and $B$ combined need only be at most $k + 1$. As we saw with the SortTree method, sorting was one way to discern which of the children should grow. This was essential to the SortTree method achieving a runtime $\in O(m \cdot n + k \log(k) o(m))$; however, this came at a price: comparison-based sorting involves a decision tree narrowing $k!$ unsorted lists to 1, and thus has a depth of $\log(k!)$; therefore, any sorting-based method must be $\in \Omega(\log(k!)) = \Omega(k \log(k))$. Thus, a sorting-based method cannot achieve a runtime $\in O(m \cdot n + o(k \log(k)) \cdot m)$.

We will first demonstrate a means by which a bound on $k$ cannot achieve a runtime $\in O(m \cdot n + o(k \log(k)) \cdot m)$. Let $\gamma = -A_1 - B_1$. select$(A + B, k)$ denote the minimal $k$ terms of the form $A_i + B_j$. For any finite $\gamma$, selection on $A + B$ is bijective to selection on $A + B + \gamma$: select$(A + B, k) = select(A + B + \gamma, k) - \gamma$.

Let $\gamma = -A_1 - B_1$. select$(A + B, k) = select(A + B + \gamma, k) - \gamma$. select$(A + B + \gamma, k) = select(A', B', k)$, where $A'_i = A_i - A_1$ and $B'_j = B_j - B_1$. Let every $A'_1, A'_2, \ldots, A'_a$ and $B'_1, B'_2, \ldots, B'_b$ be used at least once in some $A'_i + B'_j$ in select$(A', B', k)$, where $a$ and $b$ are unknown.

Every term $A'_i + B'_j \geq A'_i + B'_1, A'_i + B'_j$ and so all $A'_i + B'_1, i \in \{1, 2, \ldots a\}$ and $A'_i + B'_j, j \in \{1, 2, \ldots b\}$ must be used in the selection results. Thus, $T = select(A'_1 + B'_1, A'_2 + B'_1, A'_3 + B'_1, \ldots A'_a + B'_1, A'_1 + B'_2, A'_1 + B'_3, \ldots A'_1 + B'_b, k) \subseteq select(A' + B', k)$. $T = select(0, A'_2 + 0, A'_3 + 0, \ldots A'_a + 0, 0 + B'_2, 0 + B'_3, \ldots 0 + B'_b, k) = select((A', B'), k + 1) = (A'_1, B'_1), select((A'_2, B'_2, B'_3, \ldots, B'_b), k - 1)$.

Thus bounds $s$ and $t$ can be found such that $a \leq s$ and $b \leq t$ with $s + t - 1 \leq k$ by concatenating $A'|B'$ and joining $A'_1, B'_1$ with the $k - 1$ selection on the concatenation $A'|B'$.

Now it is necessary to efficiently perform selection on the concatenation $A|B$; however, this is more difficult than it may seem because in a tree where each node represents a problem of the form $A + B$, generating every term in $A$ or $B$ will be combinatorial and thus could be exponentially difficult. For this reason, we must perform the selection on $A|B$ without generating every term in $A$ or $B$.

One way to perform selection on the concatenation of $A$ and $B$ without generating many unnecessary values of either $A$ or $B$ would be to generate values from $A$ and $B$ in sorted order in a manner reminiscent of the SortTree method; however, as we saw above, sorting cannot achieve a runtime with $k$ term $\in o(k \log(k))$.

A heap order of $A$ or $B$ is less strict than computing $A$ or $B$ in sorted order; however, the minimal $k$ values in a heap will not necessarily be confined to any location in the heap. Thus, a heap is insufficient to perform selection on the concatenation of $A$ and $B$ while still generating few terms from both $A$ and $B$ together.
For this reason, we introduce a slightly stricter form of a heap structure: the layer-ordered heap (definition 1). However, layer-ordered heaps are still less informative than sorting and perhaps more surprisingly, actually useful; for this reason they can potentially be used to achieve a runtime with $k$ term $\in o(k \log(k))$.

**Definition 1.** Define a layer-ordered heap of rank $\alpha$ to be a stricter form of heap structure. Where in a standard heap, $\forall y \in \text{children}(x), x \leq y$, in a layer-ordered heap, the maximum value in every layer is $\leq$ to the minimum value in all subsequent layers.

Given $\alpha > 1$, let $c_i$ be the number of values in layer $i$:

$$c_i = \begin{cases} 1, & i = 1 \\ \lceil \alpha \cdot c_{i-1} \rceil, & \text{else.} \end{cases}$$

Let $L_i$ denote layer $i$ of the layer-ordered heap (where $|L_i| = c_i$) and $L_{i,j}$ denote the $j^{th}$ element of that layer: $\forall j_1, j_2, L_{i,j_1} \leq L_{i+1,j_2}$, which we denote as $L_i \leq L_{i+1}$.

If $\alpha < 2$, indexing of children is slightly more complicated than for a binary heap (where the children of index $j$ are found at indices $2j, 2j+1$). We resolve this by having the first nodes in a layer have two children and subsequent nodes in that layer have only one child. In this manner, all nodes in a layer have children that are in the layer immediately following. The index of a node relative to the start of the layer to which it belongs is its "offset." The number of nodes in layer $i$ with one child will be $d_{i,1} = 2c_i - c_{i+1}$, and the number of nodes with two children $d_{i,2}$ will satisfy $d_{i,1} + d_{i,2} = c_i$, allowing us to determine how many (and which) nodes in a particular layer have two children (or one child). If a node in layer $i$ at offset $j$ has two children, those children are at offsets $2j, 2j+1$ in the next layer. Otherwise, the single child of a node at layer $i$ and at offset $j$ will occur in the next layer at an offset of $j - d_{i,2} + 2d_{i,2} = j + d_{i,2}$. Rather than attempt to compute closed forms, these values are cached by using the fact that we will never visit layer $i$ before visiting every offset from every layer to come before it.

A layer-ordered heap may be generated online or it may be constructed in situ via an array. In the latter case, layer-ordered heapification of an array of length $n$ can be performed in $O\left(\frac{n}{\alpha-1}\right)$ using linear-time selection: Let $c_\ell$ be the largest layer with $c_1 + c_2 + \cdots c_\ell \leq n$. The $c_\ell$-selection of the array is performed in $O(n)$. The remaining values that are not the largest $c_\ell$ are retrieved during the selection and recursively selected to find the top $c_{\ell-1}$ values, etc. Because the problems shrink exponentially, the total runtime is $\in O(n)$. Note that the choice of $\alpha$ introduces a significant constant. This will be discussed below.

By using a layer-ordered heap, we can perform selection on the $A|B$ while asymptotically generating at most $\alpha^2 \cdot k$ terms of both $A$ and $B$ combined (theorem 1). Furthermore, this method generalizes to perform online selection on $A|B$ so that successively larger selections $k_1 \leq k_2 \leq k_3 \leq \cdots \leq k$ can be performed in $O(\alpha^2 \cdot k)$ and while asymptotically generating at most $\alpha^2 \cdot k$ terms of both $A$ and $B$ combined (corollary 1).

**Theorem 1.** Consider two layer-ordered heaps $A$ and $B$ of rank $\alpha$, whose layers are generated dynamically an entire layer at a time (smallest layers first). Selection on the concatenation of $A,B$, (i.e., $\text{select}(A_1, A_2, \ldots, B_1, B_2, \ldots, k)$) can be performed by generating at most $\approx \alpha^2 \cdot k$ values of $A$ and $B$ combined.

**Proof.** Begin by generating the first layers, $A_1$ and $B_1$. If $\max(A_1) \leq \max(B_1)$, generate $A_2$; otherwise, generate $B_2$. Proceed iteratively in this manner: if $A_1, A_2, \ldots, A_x$ and $B_1, B_2, \ldots, B_y$ are extant layers, then generate layer $A_{x+1}$ if $\max(A_x) \leq \max(B_y)$; otherwise, generate layer $B_{y+1}$. Continue in this fashion until $u = c_1 + c_2 + \cdots + c_x + c_1 + c_2 + \cdots + c_y \geq k$. 

10
W.l.o.g., let $A_x$ be the layer whose generation resulted in $u \geq k$, breaking the iterative process described above. Before running the iteration during which this process terminates, we must have $u < k$ or we would have terminated at the end of the previous iteration. Only one new layer, $A_x$, is generated. Thus we have $u' = c_1 + c_2 + \cdots + c_{x-1} + c_1 + c_2 + \cdots + c_y < k$ and $u = c_1 + c_2 + \cdots + c_x + c_1 + c_2 + \cdots + c_y \geq k$. The magnitude of $u$ compared to $u'$ is $u' + \varepsilon_x/u = 1 + \varepsilon_x/u < 1 + \frac{c_y}{c_1 + c_2 + \cdots + c_{x-1}}$. Because of the iterated ceilings, $c_x \leq \lceil \alpha \rceil \cdot \alpha \cdots \leq (\lceil \alpha \rceil \cdot \alpha + 1) \cdot \alpha + \cdots = \alpha^x + \alpha^{x-1} + \cdots + 1 = \frac{\alpha^x - 1}{\alpha - 1}$.

From this we have $\frac{u}{u'} < \frac{\frac{\alpha^x - 1}{\alpha - 1}}{c_1 + c_2 + \cdots + c_{x-1}} < \frac{\frac{\alpha^x - 1}{\alpha - 1}}{\sum_{t=0}^{x-2} \alpha^t} = \frac{\alpha^x - 1}{\alpha - 1} \approx \frac{\alpha^x - 1}{\alpha - 1} \approx \alpha$ for large problems. $u' < k$, so $u < \frac{u}{u'} \cdot k$, and thus $u < \alpha \cdot k$.

By the stopping criteria, there are at least $k$ values from $A_1, A_2, \ldots, A_x, B_1, B_2, \ldots, B_y$ that have been generated. Furthermore, by the layer-ordered property, $A_1, A_2, \ldots, A_x, B_1, B_2, \ldots, B_y \leq \max(\max(A_x), \max(B_y))$. Thus, $\max(\max(A_x), \max(B_y))$ is an upper bound on the value threshold that performs $k$-selection on $A, B$.

First, let's consider the case where $\max(A_x) < \max(B_y)$. Since $\max(B_y)$ is an upper bound on the value threshold performing $k$-selection on $A, B$ and since the layer-ordered property dictates $\max(B_y) \leq B_{y+1} \leq B_{y+2} \leq \cdots$, no values in $B_{y+1}, B_{y+2}, \ldots$ will reach the $k$-selection of $A, B$. Thus, any values in $k$-select of $A, B$ that are not yet in $A_1, A_2, A_x, B_1, B_2, \ldots, B_y$ may be required. Because $A_x$ was generated after $B_y$, the state of the layers when $B_y$ was generated were of the form $A_1 \leq A_2 \leq \cdots \leq A_x \leq B_1 \leq B_2 \leq \cdots \leq B_{y-1}$, where $s < x$. $B_y$ was generated rather than $A_{s+1}$ meaning that $\max(B_{y-1}) < \max(A_s)$. Combined with the layer-ordered property, we get $B_1 \leq \cdots \leq B_{y-2} \leq B_{y-1} \leq \max(B_{y-1}) < \max(A_s) \leq A_{s+1} \leq \cdots \leq A_x \leq A_{x+1} \leq \cdots$. Thus, although the $k$-selection may require more values generated from $A_{x+1}, A_{x+2}, \ldots$, these values cannot displace any values from $B_1, B_2, \ldots, B_{y-1}$; therefore, all values in $B_1, B_2, \ldots, B_{y-1}$ are already in the $k$-selection. By the layer-ordered property, values generated from $A_{x+1}, A_{x+2}, \ldots$ may likewise not displace any values from $A_1, A_2, \ldots, A_x$; therefore, new values from $A_{x+1}, A_{x+2}, \ldots$ may only displace in the $k$-selection extant values from $B_y$. Thus, at most $c_y$ additional values generated from $A_{x+1}, A_{x+2}, \ldots$ may be required.

In the case where $\max(B_y) < \max(A_x)$, $\max(A_x)$ is an upper bound on the value threshold used for $k$-selection on $A, B$; thus, no values of $A_{x+1}, A_{x+2}, \ldots$ need be considered by the $k$-selection. In this case, we need only consider additional layers generated in $B_{y+1}, B_{y+2}, \ldots$. We can exploit the fact that $\max(A_{x-1}) \leq \max(B_y)$ (resulting in the generation of $A_x$ that halted the process above): by the layer-ordered property, $A_1 \leq A_2 \leq \cdots \leq A_{x-1} \leq \max(A_{x-1}) \leq \max(B_y) \leq B_{y+1} \leq B_{y+2} \leq \cdots$, and thus new values generated from $B_{y+1}, B_{y+2}, \ldots$ may not displace the $k$-selection values from $A_1, A_2, \ldots, A_{x-1}$; therefore, all values in $A_1, A_2, \ldots, A_{x-1}$ are already in the $k$-selection of $A, B$. Because of the layer-ordered property, $B_1 \leq B_2 \leq \cdots \leq B_{y} \leq B_{y+1} \leq \cdots$, and so values generated from $B_{y+1}, B_{y+2}, \ldots$ may not displace values from $B_1, B_2, \ldots, B_y$; these values are likewise in the $k$-selection of $A, B$. The only values that can be displaced by $B_{y+1}, B_{y+2}, \ldots$ are from $A_x$. Thus, at most $c_y$ additional values generated from $B_{y+1}, B_{y+2}, \ldots$ may be required.

In the final case, $\max(A_x) = \max(B_y)$. Like the two cases before, $\max(\max(A_x), \max(B_y))$ is an upper bound on the value threshold for $k$-selection on $A, B$; therefore, no larger values need be fetched to perform the $k$-selection. By the layer-ordered property, $A_x \leq A_{x+1} \leq \cdots$ and $B_y \leq B_{y+1} \leq \cdots$, and so any further layers of neither $A$ nor $B$ need be generated: by the termination of the iterative process above, we already have $\geq k$ values $\max(\max(A_x), \max(B_y))$, and no values
smaller than \( \leq \max(\max(A_x), \max(B_y)) \) can exist in \( A_{x+1}, A_{x+2}, \ldots, B_{y+1}, B_{y+2}, \ldots \); therefore, we already have all values necessary to perform the \( k \) selection on \( A, B \).

In the cases where either \( \max(A_x) \neq \max(B_y) \), additional layers may be generated with the total number of values \( \leq c_x \) where \( x \) is the index of a layer in either \( A \) or \( B \) that has the maximal value after \( u \) was made to be \( \geq k \). W.l.o.g., these values will be generated sequentially from some layers \( B_t, B_{t+1}, \ldots B_{t+\ell} \) such that \( c_t + c_{t+1} + \cdots c_{t+\ell} \geq c_x \). By the same reasoning as the computation on \( \frac{u}{w} \) above, on large problems, this series will overestimate \( c_x \) by at most a factor of \( \alpha \).

The total number of values generated in both layer-ordered heaps was \( u < \alpha \cdot k \). In the worst-case scenario, \( c_x \) is as large as possible and \( y \) must be as small as possible (i.e., \( y = 1 \)); therefore, \( u = c_1 + c_1 + c_2 + \cdots + c_{x-1} + c_{x} \). After adding additional layers, we have generated at most \( u + \alpha \cdot c_x \) where \( \frac{u + \alpha \cdot c_x}{w'} = \frac{u + \alpha \cdot (u - u')}{u'} \approx \alpha + \alpha \cdot (\alpha - 1) = \alpha^2 \), for large problems.

\( \square \)

**Corollary 1.** Consider two layer-ordered heaps \( A \) and \( B \) of rank \( \alpha \), whose layers are generated dynamically an entire layer at a time (smallest layers first). Successive selections of \( k_1 \leq k_2 \leq k_3 \leq \cdots k \) on the concatenation of \( A, B \) can be performed in an online manner by generating at most \( \alpha^2 \cdot k \) values of \( A \) and \( B \) combined.

Proof. Consider cases where \( \max(A_x) \neq \max(B_y) \) and where the algorithm from theorem 1 adds additional layers after \( u > k \). W.l.o.g., let \( \max(B_y) < \max(A_x) \), and thus layers may be added to \( B \). The algorithm from theorem 1 finishes adding layers to \( B \) when these accumulated layers \( B_t, B_{t+1}, \ldots B_{t+\ell} \) have total size exceeding that of \( A_x \): \( c_t + c_{t+1} + \cdots c_{t+\ell} \geq c_x \).

Modify the selection algorithm from theorem 1 so that it also terminate adding layers to \( B \) at any point when the most recent layer added to \( B, B_t \), has \( \max(B_t) \geq \max(A_x) \). This does not alter the correctness of the algorithm, because \( \max(A_x) \) was an upper bound on the value threshold for performing a \( k_1 \)-selection on \( A, B \); therefore, no layers beyond \( B_t \) where \( \max(A_x) \leq \max(B_t) \leq B_{t+1} \leq B_{t+2} \leq \cdots \) may be included in the \( k_1 \)-selection.

In this modified algorithm, any layers added to \( B \) in this scheme would have been added regardless in a subsequent \( k_2 \)-selection (wherein each iteration generated a new layer for the layer-ordered heap with a smaller max generated thus far) if \( k_2 > k_1 \).

\( \square \)

If the runtime to generate terms in the layer-ordered heaps \( A \) and \( B \) (both of rank \( \alpha \)) is constant per term generated (i.e., linear overall), then the runtime (including the constant from \( \alpha \)) to generate \( k \) terms will be \( \in O\left(\frac{\alpha^2 \cdot k}{\alpha - 1}\right) \) (corollary 2).

**Corollary 2.** Given two processes that generate layers for layer-ordered heaps \( A \) and \( B \) of rank \( \alpha \) where the runtime to generate terms of the layer-ordered heaps is linear in the total number of terms generated, \( k \)-selection on \( A + B \) can be run in \( O\left(\frac{\alpha^2 \cdot k}{\alpha - 1}\right) \).

Proof. By theorem 1, \( \alpha^2 \cdot k \) terms are generated from \( A \) and \( B \) combined. The amortization constant for growing an array by factor \( \alpha \) is \( \frac{\text{previous work}}{\text{work in final layer}} = \frac{u'}{c_x} = \left(\frac{c_x}{w}\right)^{-1} = \left(\frac{u - u'}{w'}\right)^{-1} = \frac{1}{\alpha - 1} \cdot 1 + \alpha^2 \cdot k \) is used to perform linear-time \( k \)-selection on the final results (e.g., using median-of-medians 1 or soft-select 4). Thus the runtime is \( \in O\left(\frac{\alpha^2 \cdot k}{\alpha - 1} + k + \alpha^2 \cdot k\right) = O\left(\frac{\alpha^2 \cdot k}{\alpha - 1}\right) \).

\( \square \)
The FastSoftTree algorithm works as an improved version of the SoftTree method: instead of generating \( k \) values from both children \( A \) and \( B \) and subsequently performing soft heap-based selection on \( A+B \), the FastSoftTree dynamically generates values from \( A \) and \( B \) using selection on \( A|B \). This limits the total number of values generated by \( A \) and \( B \) to be asymptotically \( \leq \alpha^2 \cdot k \). If we denote \( A = C + D \), then as layers of \( A \) are generated during the selection on \( A|B \), \( A \) is generating layers in its own layer-ordered heap of \( C + D \). This propagates down the tree.

The pairwise selection on \( A+B \) scheme used in the SoftTree approach requires that \( A \) and \( B \) be arranged in a binary heap where \( A_i \leq A_{2i}, A_{2i+1} \). In that scheme, selection of a term \( A_i + B_j \) may insert the candidates \( A_{2i} + B_j, A_{2i+1} + B_j \) for consideration; however, in a standard binary heap, it may be that a child’s child’s child is superior to a child’s sibling, and thus in that naïve scheme, it may be necessary to generate exponentially many values of \( A \) and \( B \). With a layer-ordered heap, we know that a child’s child is the inferior of a child’s sibling.

Thus, we use the same proposal scheme as the pairwise soft heap-based method for selection on \( A + B \), but after \( A \) and \( B \) are generated for the current selection (via the selection on \( A|B \)), any out-of-bounds indices proposed for either \( A \) or \( B \) are not considered. Instead, they are placed into a purgatory list, whose contents will only be considered after \( A \) or \( B \) is resized in a subsequent selection (again, during the selection on \( A|B \), which is responsible for adding layers to \( A \) and \( B \) in anticipation of a selection on \( A + B \)). We use three purgatory lists: one to consider when \( A \) has been resized, one to consider when \( B \) has been resized, and one to consider only after a layer has been added to both \( A \) and \( B \). Importantly, from the layer ordering, we know that the children of some value from layer \( i \) must be in layer \( i+1 \). Thus, after a resize of the relevant layer-ordered heap, it must be that either any \((i, j)\) index pair in the purgatory list is now accessible or that all layers in the layer-ordered heap have been generated and an out-of-bounds \((i, j)\) index pair is never to be used.

### 2.5.1 Space

At any internal node, time to fetch is linear with the number of \( A + B \) selected from that node. Since these values and nothing substantially more are stored, the time and space are comparable. Thus, using the argument for the runtime below, we have a space usage \( \in O(n \cdot m + k \cdot \log(m)) \).

### 2.5.2 Runtime

By corollary 2, the runtime to perform a \( k \)-selection on \( A|B \) is \( \in O\left(\frac{\alpha^2 k}{\alpha - 1}\right) \). Once the possibly used layers of \( A \) and \( B \) are generated by selection on \( A|B \), the soft heap-based selection on \( A+B \) has a runtime \( \in O(k) \).

Note that the total problem size at the root is \( k \). In the next layer, the total problem size will be asymptotically \( \leq \alpha^2 \cdot k \). Since \( \alpha > 1 \), this is a leaf-heavy recurrence, and thus the total runtime will be dominated by the work done at the leaves. Thus, the runtime of propagation in the tree is \( \in O\left(\frac{k \cdot n^2 \log_2(m)}{\alpha - 1}\right) = O\left(\frac{k \cdot m^{\log_2(\alpha^2)}}{\alpha - 1}\right) \). Choosing \( \alpha = \sqrt{2} \) may in the worst case lead to doubling the problem size in each successive layer (leading to an algorithm with propagation time \( \in \Omega(k \cdot m) \), equivalent to or worse than SoftTree). Likewise, \( \alpha = 1 \) grows too slowly, as seen by the infinite amortization constant.

We now seek to find a good value of \( \alpha \). This could be done by computing \( \frac{\partial}{\partial \alpha} \) and finding the \( \alpha \) value that achieves a zero in the partial derivative and thus an extremum. When minimizing \( \frac{m \cdot n + k \cdot n^2 \log_2(m)}{\alpha - 1} \) with respect to \( \alpha \), Mathematica 11 could not solve the resulting equation for \( \alpha \).

However, we can consider two cases: In the first case, propagation through the tree dominates. In the second case, layer-ordered heapification of \( m \) arrays each of length \( n \), dominates.
In the first case, we can choose the optimal \( \alpha \) by focusing only on the relevant term that dominate the runtime:

\[
\frac{k \cdot \alpha^{2 \log_2(m)}}{\alpha - 1} \rightarrow \alpha^{*} = \frac{\log(2m)}{\log(m)},
\]

which yields the following runtime of that dominant term:

\[
O(k \cdot m^{\log_2\left(\frac{\log(2m)}{\log(m)}\right)} \cdot \log(m)).
\]

The \( m^{\log_2\left(\frac{\log(2m)}{\log(m)}\right)} \) term approaches \( e \) as \( m \to \infty \). Hence, the dominant term is \( k \cdot \log(m) \).

In the second case, if the recursive method described here is used for layer-ordered heapification, then it will be favored by choosing \( \alpha \gg 1 \) (but still < 2). An alternative to using that recursive algorithm would be to simply sort all of the \( X_i \) arrays in \( O(m \cdot n \log(n)) \) total time; sorted order is stricter than layer ordering, and so sorting necessarily performs layer-ordered heapification.

Note that when \( n \gg k \), the cost of layer-ordered heapification on the \( m \) input vectors of length \( n \) could be lessened by first performing \( k + m - 1 \)-selection on the concatenated \( X_i \) vectors (by naïvely concatenating them and using linear-time one-dimensional selection in \( O(m \cdot n + k) \) time); by lemma \( 2 \) this finds only the values that may contribute to the final result. Because each array \( X_i \) must have at least \(|X_i| \geq 1\), then there are \( k - 1 \) free values that may be distributed between the arrays. If linear-time one-dimensional selection is used to layer-ordered heapify the resulting, trimmed \( X_i \), then the runtime will be linear in each and so \( O\left(\frac{\sum_{i=1}^{m} |X_i|}{\alpha - 1}\right) = O\left(\frac{k + m}{\alpha - 1}\right) \) . If sorting is used to layer-ordered heapify the trimmed \( X_i \) arrays, then the runtime is worst when w.l.o.g. \(|X_1| = |X_2| = \cdots = |X_{m-1}| = 1, |X_m| = k - 1\), because comparison sort is superlinear. This has a runtime \( \in O(m + k \log(k)) \).

If we assume that both terms weigh equally in the final runtime (in order to prevent either from being dominant), then we can solve for \( \alpha \) to get

\[
\alpha^{*} = \sqrt{2^{1 - \log_m\left(\frac{k}{m}\right)}},
\]

this will only be applicable when \( \frac{k}{m} < m \). When \( \frac{k}{m} \gg m \), then this is the first case described above: the layer-ordered heapification will be much less expensive than the propagation through the tree. When applicable, that approach gives a runtime of

\[
\frac{m \cdot n \cdot t}{\sqrt{2} - t},
\]

where \( t = 2^{\log_m\left(\frac{k}{m}\right)} \).

Regardless of which term dominates, the FastSoftTree can easily perform selection \( \in o(m \cdot n + k \cdot m) \) by simply setting \( \alpha = 1.05 \). This results in a runtime \( \leq 100m \cdot n + k \cdot \alpha^{2 \log_2(m)} = 20m \cdot n + k \cdot m^{2 \log_2(\alpha)} = 20m \cdot n + k \cdot m^{0.140\ldots} \in O(m \cdot n + k \cdot m^{0.140\ldots}) \). Thus, the FastSoftTree method has the best theoretical runtime of all methods introduced here.

### 2.5.3 Implementation

The FastSoftTree algorithm is implemented in code listing 16.

### 3 Results

#### 3.1 Memory use and runtime

Figure 2 plots how memory usage grows relative to \( k \) and relative to \( m \) when each \( X_i \) is uniformly distributed and exponentially distributed. Figure 3 plots how runtime grows relative to \( k \) and relative to \( m \) when each \( X_i \) is uniformly distributed and exponentially distributed.
Figure 2: Memory usage. Memory usage for each method is plotted. When $k$ is varied, it is $\in \{4, 8, 16, \ldots, 256\}$ with $n = 32$ and $m = 128$. When $m$ is varied, it is $\in \{4, 8, 16, \ldots, 256\}$ with $n = 32$ and $k = 128$. We measure the average over five replicates with error bars showing the minimum and maximum.
Figure 3: **Runtime.** Runtime for each method is plotted. When $k$ is varied, it is $\in \{4, 8, 16, \ldots 256\}$ with $n = 32$ and $m = 128$. When $m$ is varied, it is $\in \{4, 8, 16, \ldots 256\}$ with $n = 32$ and $k = 128$. We measure the average over five replicates with error bars showing the minimum and maximum.
Table 1: Data is from varying \( k \) with a uniform distribution and \( m = 64, n = 1024 \). This table shows the average number of times a node was asked to pop a value based on its depth in the SortTree. The left-most column represents the root, as the columns move to the right they traverse down the tree until they hit the root.

| \( k \) | Root | L1 | L2 | L3 | L4 | L5 | Leaves |
|-------|------|----|----|----|----|----|--------|
| 64    | 64.0 | 33.50 | 18.20 | 10.62 | 6.812 | 4.875 | 3.900 |
| 128   | 128.0 | 65.50 | 34.25 | 18.62 | 10.75 | 6.843 | 4.890 |
| 512   | 512.0 | 257.5 | 130.2 | 66.62 | 34.81 | 18.87 | 10.89 |
| 1024  | 1024.0 | 513.5 | 258.2 | 130.6 | 66.68 | 34.84 | 18.82 |
| 2048  | 2048.0 | 1025.0 | 502.7 | 252.9 | 66.18 | 34.59 | 18.70 |

Table 2: Data is from varying \( k \) with an exponential distribution and \( m = 64, n = 1024 \). This table shows the average number of times a node was asked to pop a value based on its depth in the SortTree. The left-most column represents the root, as the columns move to the right they traverse down the tree until they hit the root.

| \( k \) | Root | L1 | L2 | L3 | L4 | L5 | Leaves |
|-------|------|----|----|----|----|----|--------|
| 64    | 64.0 | 33.50 | 18.20 | 10.62 | 6.812 | 4.875 | 3.900 |
| 128   | 128.0 | 65.50 | 34.25 | 18.62 | 10.75 | 6.843 | 4.890 |
| 512   | 512.0 | 257.5 | 130.2 | 66.62 | 34.81 | 18.87 | 10.89 |
| 1024  | 1024.0 | 513.5 | 258.2 | 130.6 | 66.68 | 34.84 | 18.82 |
| 2048  | 2048.0 | 1025.0 | 502.7 | 252.9 | 66.18 | 34.59 | 18.70 |

3.2 Propagation depth in SortTree

The SortTree method does not always need to fetch a value from either children (\( A \) or \( B \)) in order to compute the new value in \( A + B \). As a result, the expected overall runtime may sometimes be \( \in O(m \cdot n + k \log(k)) \) rather than \( \in O(m \cdot n + k \log(k) \log(m)) \).

Table 1 and table 2 demonstrate the in-practice propagation depth when \( k \) is varied and when the \( X_i \) are uniformly and exponentially distributed, respectively. Table 3 and table 4 demonstrate the in-practice propagation depth when \( m \) is varied and when the \( X_i \) are uniformly and exponentially distributed, respectively. For both the uniformly and exponentially distributed data, the average number of pops from each leaf is sublinear with respect to \( k \) and grows inversely proportional to \( m \).

4 Discussion

Although the FastSoftTree method has the best theoretical runtime of all methods here (e.g., when \( \alpha = 1.05 \)), the SortTree method performed best in practice for these experiments. One reason

| \( m \) | Root | L1 | L2 | L3 | L4 | L5 | L6 | L7 | L8 | L9 | L10 | L11 |
|-------|------|----|----|----|----|----|----|----|----|----|------|------|
| 64    | 512.0 | 49.00 | 12.00 | 5.625 | 3.813 | 3.343 | 3.125 |
| 128   | 512.0 | 48.50 | 10.75 | 5.500 | 3.813 | 3.281 | 3.078 | 2.961 |
| 256   | 512.0 | 54.50 | 12.00 | 5.500 | 3.688 | 3.250 | 3.078 | 2.992 | 2.945 |
| 512   | 512.0 | 71.00 | 14.50 | 6.000 | 4.125 | 3.281 | 3.109 | 2.977 | 2.945 | 2.928 |
| 1024  | 512.0 | 45.50 | 14.25 | 5.875 | 4.063 | 3.313 | 3.094 | 2.961 | 2.938 | 2.930 | 2.913 |
| 2048  | 512.0 | 46.50 | 10.75 | 4.625 | 3.500 | 3.156 | 3.047 | 2.961 | 2.930 | 2.928 | 2.921 | 2.913 |

Table 3: Data is from varying \( m \) with a uniform distribution and \( k = 512 \) and \( n = 1024 \). This table shows the average number of times a node was asked to pop a value based on its depth in the SortTree. The left-most column represents the root, as the columns move to the right they traverse down the tree until they hit the root. As \( m \) grows so does the depth of the tree, so the right-most column which has entries always represent the leaves.
### Table 4: Data is from varying $m$ with an exponential distribution and $k = 512$ and $n = 1024$. This table shows the average number of times a node was asked to pop a value based on its depth in the SortTree. The left-most column represents the root, as the columns move to the right they traverse down the tree until they hit the root. As $m$ grows so does the depth of the tree, so the right-most column which has entries always represent the leaves.

| $m$  | Root | L1    | L2    | L3    | L4    | L5    | L6    | L7    | L8    | L9    | L10   | L11   |
|------|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 64  | 512.0| 257.5 | 130.3 | 66.63 | 34.81 | 18.91 | 10.93 | 6.914 |       |       |       |       |
| 128 | 512.0| 257.5 | 130.3 | 66.63 | 34.75 | 18.81 | 10.87 | 6.906 | 4.914 | 3.914 |       |       |
| 512 | 512.0| 257.5 | 130.3 | 66.63 | 34.75 | 18.84 | 10.87 | 6.914 | 4.902 | 3.919 | 3.418 |       |
| 1024| 512.0| 257.5 | 130.3 | 66.63 | 34.81 | 18.88 | 10.90 | 6.914 | 4.902 | 3.919 | 3.418 | 3.418 |
| 2048| 512.0| 46.50 | 10.75 | 4.630 | 3.500 | 3.156 | 3.046 | 2.960 | 2.929 | 2.927 | 2.920 | 2.912 |

for the practical efficiency of the SortTree is the fact that it’s unlikely that every layer between the root and leaves are visited when computing each next value in the sort of $X_1 + X_2 + \cdots + X_m$ (tables 1, 2, 3, & 4).

However, some of the performance of SortTree here was likely from the fact that SortTree uses Python’s `heapq`, which has a better runtime constant than native Python code. Soft heaps are implemented with linked lists; these are already poor for cache performance (as opposed to Python’s contiguous array-based heaping), and suffer in a Python implementation (e.g., they frequently use the `.operator to lookup the next in the linked list, and each of these lookups requires a dictionary access in Python). It would be interesting to compare optimized C++ implementations of both methods on large problems, while noting that the log($k$) term present in SortTree’s runtime but not in FastSoftTree will always be < 270 because the particles in the universe number roughly $2^{270}$ [5].

Finding the optimal $\alpha$ for any $n, m, k$ problem could also lead to better performance in practice. It may even be possible to mix different $\alpha$ values throughout layers of the tree.

If it were possible to efficiently implement the pairwise $A + B$ selection without soft heaps (or reminiscent data structure), then the practical performance of the FastSoftTree would likely be much better. Layer-ordered heaps are essentially vectors iteratively partitioned (as from quicksort); they can be stored in a contiguous manner and with low overhead overall (including a fast implementation of median-of-medians instead of soft-select for the one-dimensional selection).

Although they are quite simple, layer-ordered heaps can be used to describe other problems: For instance, given a fast, in-place method to construct a layer-ordered heap of an array, it is possible to implement an efficient one-dimensional selection. This is accomplished as a simplified form of the algorithm from theorem 2. A layer-ordered heap is built on the full array, and then layers are added (small layers first) until the selection threshold is exceeded. At this point, the layer of the layer-ordered heap that resulted in exceeding the selection threshold is itself partitioned into a layer-ordered heap, and selection on the remaining values is performed only within that layer. This is continued until the exact selection threshold is met. Assuming a linear-time layer-ordered heapification algorithm, the runtime of this one-dimensional $k$-selection will be characterized by the recurrence $r(k) = k + r((\alpha - 1) \cdot k)$; the final layer will be added only if we have $< k$ so far, and we will overshoot by at most a factor of $\alpha$ (and thus the final layer has size $\alpha \cdot k - k$. For any $\alpha < 2$ and not close to 1, this achieves linear-time one-dimensional selection by essentially achieving a more and more sorted ordering at the selection threshold.

It is likely that algorithms like these using layer-ordered heaps will be useful on other problems.
5 Availability

Python source code for all methods is available at https://bitbucket.org/seranglab/cartesian-selection-with-layer-ordered-heap (MIT license, free for both academic and commercial use).

6 Acknowledgment

This work was supported by grant number 1845465 from the National Science Foundation.

7 Supplemental information

7.1 Python code

Listing 1: SoftHeap.py: An implementation of Chazelle’s soft heap.

```python
import math

class SoftHeap:
    class ilcell:
        def __init__(self, key, data):
            self.key = key
            self.data = data
            self.next = None

        def get_key(self):
            return self.key

        def set_next(self, value):
            assert type(value) is SoftHeap.ilcell, "Error passed into set_next must be of type ilcell; received value of type " + str(type(value)) + "."
            self.next = value

        def get_next(self):
            return self.next

    class node:
        def __init__(self):
            self.ckey = 0
            self.rank = 0
```

19
# node
self.next = None
# node
self.child = None
# ilcells
self.il = None
# ilcells
self.il_tail = None

def set_ckey(self, value):
    self.ckey = value

def get_ckey(self):
    return self.ckey

def set_rank(self, value):
    self.rank = value

def get_rank(self):
    return self.rank

def set_next(self, value):
    assert type(value) is SoftHeap.node or value == None, "Error@node.set_next:Value passed into set_next must be of type node; received value " + str(value) + " of type " + str(type(value)) + "."
    self.next = value

def get_next(self):
    return self.next

def set_child(self, value):
    assert type(value) is SoftHeap.node or value == None, "Error@node.set_child:Value passed into set_child must be of type node; received value " + str(value) + " of type " + str(type(value)) + "."
    self.child = value

def get_child(self):
    return self.child

def set_il(self, value):
    assert type(value) is SoftHeap.ilcell or value == None, "Error@node.set_il:Value passed into set_il must be of type ilcell; received value " + str(value) + " of type " + str(type(value)) + "."
    self.il = value

def get_il(self):
    return self.il

def set_il_tail(self, value):
    assert type(value) is SoftHeap.ilcell or value == None, "Error@node.set_il_tail:Value passed into set_il_tail must be of type ilcell; received value " + str(value) + " of type " + str(type(value)) + "."
value) + "of type" + str(type(value)) + "."

self.il_tail = value

def get_il_tail(self):
    return self.il_tail

class head:
    def __init__(self):
        # node
        self.queue = None
        # head
        self.next = None
        # head
        self.prev = None
        # head
        self.suffix_min = None
        # int
        self.rank = 0

def set_queue(self, value):
    assert type(value) is SoftHeap.node, "Error\n\nhead.set_queue: Value passed into method must be of type node; received value" + str(value) + "of type" + str(type(value)) + "."

    self.queue = value

def get_queue(self):
    return self.queue

def set_next(self, value):
    assert type(value) is SoftHeap.head, "Error\n\nhead.set_next: Value passed into method must be of type head; received value" + str(value) + "of type" + str(type(value)) + "."

    self.next = value

def get_next(self):
    return self.next

def set_prev(self, value):
    assert type(value) is SoftHeap.head, "Error\n\nhead.set_prev: Value passed into method must be of type head; received value" + str(value) + "of type" + str(type(value)) + "."

    self.prev = value

def get_prev(self):
    return self.prev

def set_suffix_min(self, value):
    assert type(value) is SoftHeap.head, "Error\n\nhead.set_suffix_min: Value passed into method must be of type head; received value" + str(value) + "of type" + str(type(value)) + "."
self.suffix_min = value

def get_suffix_min(self):
    return self.suffix_min

def set_rank(self, value):
    self.rank = value

def get_rank(self):
    return self.rank

def __len__(self):
    return self._size

def __init__(self, epsilon):
    # head
    self.header = self.head()
    # head
    self.tail = self.head()

    self.tail.set_rank(math.inf)
    self.header.set_next(self.tail)

    self.tail.set_prev(self.header)

    # int
    self.epsilon = epsilon

    # fixme: should this be log base 2 or base e?

    # Chazelle's paper says log(...) (assumed to be base e); however,
    # immediately before logs are used, he specifies information in
    # bits (base 2). Using base 2 because it is conservative relative
    # to base e.

    #self.r = 2 + 2*int(math.ceil(math.log(1/epsilon)))
    self.r = 2 + 2*int(math.ceil(math.log2(1/epsilon)))

    self._num_inserted = 0
    self._size = 0

def insert(self, key, data):
    self._size += 1
    self._num_inserted += 1

    # ilcell
    l = self.ilcell(key, data)

    q = self.node()
    q.set_rank(0)
    q.set_ckey(key)
    q.set_Il(l)
    q.set_Il_tail(l)
    self.meld(q)
def number_insertions(self):
    return self._num_inserted

def meld(self, q):  # q is node

    assert type(q) is SoftHeap.node, "Error @ SoftHeap.meld: Value 'q' passed into method must be of type node; received value " + str(q) + " of type " + str(type(q)) + "."

    # head
    h = self.head()
    # head
    prevhead = self.head()
    # head
    tohead = self.header.get_next()
    # node
    top = self.node()
    # node
    bottom = self.node()

    while q.get_rank() > tohead.get_rank():
        tohead = tohead.get_next()
        prevhead = tohead.get_prev()
    while q.get_rank() == tohead.get_rank():
        if tohead.get_queue().get_ckey() > q.get_ckey():
            top = q
            bottom = tohead.get_queue()
        else:
            top = tohead.get_queue()
            bottom = q

    q = self.node()
    q.set_ckey(top.get_ckey())
    q.set_rank(top.get_rank() + 1)
    q.set_child(bottom)
    q.set_next(top)
    q.set_il(top.get_il())
    q.set_il_tail(top.get_il_tail())
    tohead = tohead.get_next()

    if prevhead == tohead.get_prev():
        h = self.head()
    else:
        h = prevhead.get_next()

    h.set_queue(q)
    h.set_rank(q.get_rank())
    h.set_prev(prevhead)
    h.set_next(tohead)
    prevhead.set_next(h)
    tohead.set_prev(h)

    self.fix_minlist(h)
def fix_minlist(self, h):  # head
    assert type(h) is SoftHeap.head, "Error/SoftHeap.fix_minlist: Value 'h' passed into method must be of type head; received value", str(h) + ", type", str(type(h)) + "."
    if h.get_next() == self.tail:
        tmpmin = h
    else:
        tmpmin = h.get_next().get_suffix_min()
    while h != self.header:
        if h.get_queue().get_ckey() < tmpmin.get_queue().get_ckey():
            tmpmin = h
        h.set_suffix_min(tmpmin)
        h = h.get_prev()

def sift(self, v):
    assert type(v) is SoftHeap.node, "Error/SoftHeap.sift: Value 'v' passed into method must be of type node; received value", str(v) + ", type", str(type(v)) + "."
    v.set_il(None)
    v.set_il_tail(None)
    if v.get_next() == None and v.get_child() == None:
        v.set_ckey(math.inf)
        return v
    else:
        v.set_next(self.sift(v.get_next()))
        if v.get_next().get_ckey() > v.get_child().get_ckey():
            tmp = v.get_child()
            v.set_child(v.get_next())
            v.set_next(tmp)
            v.set_il(v.get_next().get_il())
            v.set_il_tail(v.get_next().get_il_tail())
            v.set_ckey(v.get_next().get_ckey())
            if v.get_rank() > self.r and (v.get_rank() % 2 == 1 or v.get_child().get_rank() < v.get_rank() - 1):
                v.set_next(self.sift(v.get_next()))
                if v.get_next().get_ckey() > v.get_child().get_ckey():
                    tmp = v.get_child()
                    v.set_child(v.get_next())
                    v.set_next(tmp)
                    #I (Patrick) Added and v.get_Il() != None: for debugging
                    if v.get_next().get_ckey() != math.inf and v.get_next().get_Il() != None and v.get_Il() != None:
                        v.get_next().get_Il_tail().set_next(v.get_Il())
                        v.set_Il(v.get_next().get_Il())
                    if v.get_Il_tail() == None:
                        #
v.set_il_tail(v.get_next().get_il_tail())
v.set_ckey(v.get_next().get_ckey())

if v.get_child().get_ckey() == math.inf:
    if v.get_next().get_ckey() == math.inf:
        v.set_child(None)
v.set_next(None)
    else:
        v.set_child(v.get_next().get_child())
v.set_next(v.get_next().get_next())
return v

def deletemin(self):
    assert(len(self) > 0)

    # head
    h = self.header.get_next().get_suffix_min()
    while h.get_queue().get_il() == None:
        tmp = h.get_queue()
        # int
        childcount = 0
        while tmp.get_next() != None:
            tmp = tmp.get_next()
            childcount += 1
        if childcount < int(h.get_rank() / 2):
            h.get_prev().set_next(h.get_next())
h.get_next().set_prev(h.get_prev())
self.fix_minlist(h.get_prev())
        tmp = h.get_queue()
        while tmp.get_next() != None:
            self.meld(tmp.get_child())
            tmp = tmp.get_next()
    else:
        h.set_queue(self.sift(h.get_queue()))
        if h.get_queue().get_ckey() == math.inf:
            h.get_prev().set_next(h.get_next())
h.get_next().set_prev(h.get_prev())
    h = h.get_prev()
    self.fix_minlist(h)
    h = self.header.get_next().get_suffix_min()

    min_item = h.get_queue().get_il()
    h.get_queue().set_il(h.get_queue().get_il().get_next())
    if h.get_queue().get_il() == None:
        h.get_queue().set_il_tail(None)
    self._size -= 1
    return (min_item.key, min_item.data)

# Note: This function is only for debugging. It is slow, and it may
# alter the soft heap as it empties and re-fills (e.g., it may
# corrupt it more than rebuilding from scratch)
def __iter__(self):
    all_items = []
while len(self)>0:
    key,data = self.deletemin()
    all_items.append((key,data))
for key,data in all_items:
    self.insert(key, data)
return all_items.__iter__()

Listing 2: one_dimensional_selection.py: Routines for linear-time one-dimensional selection.

from SoftHeap import *
import numpy as np

def left_child(index):
    return 2*index+1

def right_child(index):
    return 2*index+2

def check_list_is_in_min_heap_order(x):
    # Input should be heapified in min-heap form
    # To check, for x_i, x_2i+1 and x_2i+2 should be greater than x
    for i in range(len(x)):
        if left_child(i) < len(x) and x[left_child(i)] < x[i]:
            return False
        if right_child(i) < len(x) and x[right_child(i)] < x[i]:
            return False
    return True

def sort_select(v, k):
    result = sorted(v)[:k]
    np.random.shuffle(result)
    return result

def sort_select_into_good_and_bad(v, k):
    result = sorted(v)
    good = result[:k]
    bad = result[k:]
    np.random.shuffle(good)
    np.random.shuffle(bad)
    return good, bad

def select_k_leq_key_and_remainders(key_data_pairs, key_value, k):
    selected = []
    not_selected = []
    leftovers = []

    for key, data in key_data_pairs:
        if key < key_value:
            selected.append((key, data))
        elif key > key_value:
            not_selected.append((key, data))
        else:
            leftovers.append((key, data))

    number_left_to_select = k-len(selected)
    selected.extend( leftovers[:number_left_to_select] )
    not_selected.extend(leftovers[number_left_to_select:])

26
return (selected, not_selected)

def partition(key_data_pairs, pivot_val):
    i = -1
    j = len(key_data_pairs)
    while True:
        # do while
        i += 1
        key, data = key_data_pairs[i]
        while i < len(key_data_pairs) and key < pivot_val:
            i += 1
            key, data = key_data_pairs[i]
        # do while
        j -= 1
        key, data = key_data_pairs[j]
        while j >=0 and key > pivot_val:
            j -= 1
            key, data = key_data_pairs[j]
        if i>=j:
            return j
        key_data_pairs[i], key_data_pairs[j] = key_data_pairs[j], key_data_pairs[i]

def soft_select_helper(key_data_pairs, k):
    if k == 0:
        return min(key_data_pairs)
    if k == len(key_data_pairs):
        return max(key_data_pairs)
    n = len(key_data_pairs)
    epsilon = 1./3.
    soft_heap = SoftHeap(epsilon)
    for i in range(n):
        key, data = key_data_pairs[i]
        soft_heap.insert(key, data)
    max_popped_uncorrupted_key = -math.inf
    maximum_num_corrupted_items = int(math.ceil(epsilon*n))
    all_popped_items_from_soft_heap = []
    for i in range(maximum_num_corrupted_items):
        key, data = soft_heap.deletemin()
        max_popped_uncorrupted_key = max(max_popped_uncorrupted_key, key)
    partition_index = partition(key_data_pairs, max_popped_uncorrupted_key) + 1
    lower = key_data_pairs[:partition_index]
    upper = key_data_pairs[partition_index:]
    if k <= partition_index:
        return soft_select_helper(lower, k)
    return soft_select_helper(upper, k - partition_index)

def soft_select(key_data_pairs, k):
    if k > len(key_data_pairs):
        return key_data_pairs
    key_data_at_k = soft_select_helper(key_data_pairs, k)
    key, data = key_data_at_k
    good_items, bad_items = select_k_leq_key_and_remainders(key_data_pairs, key, k)
    return good_items
def soft_select_and_return_unused_values(key_data_pairs, k):
    if k > len(key_data_pairs):
        return (key_data_pairs, [])
    if k > len(key_data_pairs):
        return (key_data_pairs, [])
    key_data_at_k = soft_select_helper(key_data_pairs, k)
    key, data = key_data_at_k
    good_items, bad_items = select_k_leq_key_and_remainders(key_data_pairs, key, k)
    assert(len(good_items) + len(bad_items) == len(key_data_pairs))
    return (good_items, bad_items)

def soft_select_value_at_k(key_data_pairs, k):
    if k > len(key_data_pairs):
        return max(key_data_pairs[0])
    key_data_at_k = soft_select_helper(key_data_pairs, k)
    return key_data_at_k

Listing 3: SoftTensor.py: This method uses a soft heap (listing 1) and linear-time one-dimensional selection (listing 2). It runs $\in O(m \cdot n + k \cdot m^2)$.

```python
import heapq
from SoftHeap import *
from one_dimensional_selection import *

def left_child(index):
    return 2*index+1

def right_child(index):
    return 2*index+2

class SoftTensor:
    def __init__(self, vectors):
        vectors_and_indices = [ [(b,a) for a,b in enumerate(v)] for v in vectors ]
        self._m = len(vectors_and_indices)
        self.ascending_vectors_and_unsorted_indices = [ sorted(v) for v in vectors_and_indices ]
        min_value = sum([a for a,b in [v[0] for v in self.ascending_vectors_and_unsorted_indices]])
        zero_tup = (0,)*self._m
        self._size = 1
        for i in range(self._m):
            self._size *= len(self.ascending_vectors_and_unsorted_indices[i])

        self._epsilon = 1./(4.0 * self._m)
        # instead start it empty and call
        self._soft_heap = SoftHeap(self._epsilon)
        self._insert_if_in_bounds(tuple([0]*self._m))

    def select(self, k):
        k = min(k, len(self))
        candidate_results = []
        number_pops_to_guarantee_k_non_corrupt = k + int(np.ceil(self._epsilon*self._soft_heap.number_insertions()))
        for i in range(number_pops_to_guarantee_k_non_corrupt):
```
if len(self._soft_heap) > 0:
    key_and_data_pair = self._pop_next_key_and_data()
    candidate_results.append( key_and_data_pair )
return [ key for key, val in soft_select(candidate_results, k) ]

def __len__(self):
    if self._size > 2**32-1:
        return 2**32-1
    return self._size

# Note: not guaranteed to pop in sorted order because of corruption
def _pop_next_key_and_data(self):
    key, data = self._soft_heap.deletemin()
    next_terms_to_insert = self._advance(data)
    for term in next_terms_to_insert:
        self._insert_if_in_bounds(term)
    return (key, data)

def _advance(self, a_b_c_1dots_z):
    # r(m) = r(m-1) + 2m = 2 + 4 + 6 + 8 + .... + 2m
    # 2m is quadratic
    # Base case
    if len(a_b_c_1dots_z) == 2:
        i = a_b_c_1dots_z[0]
        j = a_b_c_1dots_z[1]
        if j == 0:
            return [[2*i+1, j], [2*i+2, j], [i, 1], [i, 2]]
        return [[i, 2*j+1], [i, 2*j+2]]
    new_terms = []
    last_index = a_b_c_1dots_z[-1]
    if last_index == 0:
        # Advance all other indices
        advance_other_indices = self._advance( a_b_c_1dots_z[:-1] )
        for term in advance_other_indices:
            term.append(last_index)
            new_terms.append( term )
    # Advance last index in heap order
    left_heap_child = list(a_b_c_1dots_z)
    left_heap_child[-1] = 2*left_heap_child[-1] + 1
    next_terms = left_heap_child
    new_terms.append( next_terms )
    right_heap_child = list(a_b_c_1dots_z)
    right_heap_child[-1] = 2*right_heap_child[-1] + 2
    next_terms2 = right_heap_child
    new_terms.append( next_terms2 )
    return new_terms

def _insert_children_into_soft_heap(self, input_index):
    next_heap_input_indices = self._advance(list(input_index))
    for index in next_heap_input_indices:
        self._insert_if_in_bounds(tuple(index), parent=input_index)

def _insert_if_in_bounds(self, heap_ordered_indices, parent=None):
    insert_key = 0
for i in range(len(heap_ordered_indices)):
    if heap_ordered_indices[i] >= len(self.
        ascending_vectors_and_unsorted_indices[i]):
        return
    else:
        index = heap_ordered_indices[i]
        key, data = self.ascending_vectors_and_unsorted_indices[i][
            heap_ordered_indices[i]]
        insert_key += key
        self._soft_heap.insert(insert_key, heap_ordered_indices)

Listing 4: SoftPairwiseSelect.py: This method solves selection on $A+B$ (i.e., $m = 2$) in $O(n+k)$.

```python
import heapq
from SoftHeap import *
from one_dimensional_selection import *
import time

class SoftPairwiseSelect:
    def __init__(self, a_lst, b_lst):
        heapq.heapify(a_lst)
        heapq.heapify(b_lst)
        self._a_lst = a_lst
        self._b_lst = b_lst
        self._size = len(a_lst)*len(b_lst)
        self._epsilon = 1./8.
        self._soft_heap = SoftHeap(self._epsilon)
        self._insert_if_in_bounds(0, 0)

    def select(self, k):
        k = min(k, len(self))
        candidate_results = []
        # Need to pop k + amount of corruption
        # Are we sure this works? The size (and hence the corruption)
        # grows as we pop. (See comments in FastSoftTreeNode.py.)
        number_pops_to_guarantee_k_non_corrupt = k + int(np.ceil(self._epsilon*self.
            _soft_heap.number_insertions()))
        for i in range(number_pops_to_guarantee_k_non_corrupt):
            if len(self._soft_heap) > 0:
                key_and_data_pair = self._pop_next_key_and_data()
                candidate_results.append( key_and_data_pair )
        good_items, bad_items = soft_select_and_return_unused_values(candidate_results
            , k)

        # Rebuild soft heap from scratch: This is better than only
        # inserting items that we want to put back, because corruption
        # is bounded in terms of insertions, not size:
        remaining_items = []
        while len(self._soft_heap) > 0:
            remaining_items.append( self._soft_heap.deletemin() )
        self._soft_heap = SoftHeap(self._epsilon)

        # re-insert items which were popped but will not be returned
        for key, data in bad_items:
            self._soft_heap.insert(key, data)
        for key, data in remaining_items:
```

30
def _soft_heap.insert(key, data):
    return [(key, data) for key, data in good_items]

def __len__(self):
    if self._size > 2**32-1:
        return 2**32-1
    return self._size

# Note: not guaranteed to pop in sorted order because of corruption
def _pop_next_key_and_data(self):
    key, data = self._soft_heap.deletemin()
    (heap_i, heap_j) = data
    self._insert_children_into_soft_heap(heap_i, heap_j)
    return (key, data)

def _insert_children_into_soft_heap(self, i, j):
    if j == 0:
        self._insert_children_when_y_is_0(i)
    else:
        self._insert_children_when_y_greater_than_zero(i, j)

def _insert_if_in_bounds(self, i, j):
    if i < len(self._a_lst) and j < len(self._b_lst):
        key_a, data_a = self._a_lst[i]
        key_b, data_b = self._b_lst[j]
        key = key_a + key_b
        data = (i, j)
        self._soft_heap.insert(key, data)

def _insert_children_when_y_is_0(self, i):
    self._insert_if_in_bounds(left_child(i), 0)
    self._insert_if_in_bounds(right_child(i), 0)
    self._insert_if_in_bounds(i, 1)
    self._insert_if_in_bounds(i, 2)

def _insert_children_when_y_greater_than_zero(self, i, j):
    self._insert_if_in_bounds(i, left_child(j))
    self._insert_if_in_bounds(i, right_child(j))

Listing 5: SoftTreeNode.py: This wrapper for SoftPairwiseSelect generates $k$ terms from each of $A$ and $B$ and then performs the $k$-selection on $A + B$. This will be used in SoftTree (listing 6).

import heapq
from SoftPairwiseSelect import *

class SoftTreeNode:
    sum_of_all_selections = 0

    def __init__(self, child_x, child_y):
        self._child_x_results = []
        self._child_y_results = []

        if type(child_x) is list:
            assert(check_list_is_in_min_heap_order(child_x))
            self._child_x_results = child_x
        if type(child_y) is list:
            assert(check_list_is_in_min_heap_order(child_y))

            self._child_y_results = child_y
self._child_y_results = child_y

self._child_x = child_x
self._child_y = child_y
self._size = len(self._child_x) * len(self._child_y)

def __len__(self):
    if self._size <= 2**32 - 1:
        return self._size
    return 2**32-1

def select(self, k):
    SoftTreeNode.sum_of_all_selections += k

    if type(self._child_x) is list:
        data_a = self._child_x
    else:
        data_a = self._child_x.select(k)

    if type(self._child_y) is list:
        data_b = self._child_y
    else:
        data_b = self._child_y.select(k)

    scps = SoftPairwiseSelect(data_a, data_b)
    return scps.select(k)

Listing 6: SoftTree.py: This method builds a balanced binary tree of the pairwise wrappers from listing 5. The method runs in $O(m \cdot n + k \cdot m)$.

```python
import heapq
from SoftTreeNode import *

class SoftTree:
    def __init__(self, vectors):
        vectors = [ list(vec) for vec in vectors ]
        for vec in vectors:
            heapq.heapify(vec)

        key_and_data_vectors = [ [(b,a) for a,b in enumerate(v)] for v in vectors ]
        self._m = len(key_and_data_vectors)

        current_layer = key_and_data_vectors
        while len(current_layer) > 1:
            next_layer = []
            for i in range(len(current_layer) // 2):
                next_layer.append( SoftTreeNode(current_layer[2*i], current_layer[2*i+1]) )
            if len(current_layer) % 2 == 1:
                next_layer.append( current_layer[-1] )
            current_layer = next_layer
        self._root = current_layer[0]

    def select(self, k):
        select_from_root = self._root.select(k)
        return [ key for key, val in select_from_root ]
```
Listing 7: MinIndexHeap.py: A wrapper for Python’s heapq class, which gives indices as well as values. This will be used in listing 8 and listing 9.

```python
import heapq

class MinIndexHeap:
    def __init__(self, values_and_indices=[]):
        self.min_heap = list(values_and_indices)
        heapq.heapify(self.min_heap)
        self.descending_values_and_indices = []
        self.num_id_heaps = 1

    def insert(self, val_and_index):
        # Note: does not guard against double insertion of an index
        val, index = val_and_index
        heapq.heappush(self.min_heap, (val, index))

    def pop_min_and_index(self):
        val, index = heapq.heappop(self.min_heap)
        self.descending_values_and_indices.append((val, index))
        return (val, index)

    def get_value_and_index_at_rank(self, rank):
        assert(rank >= 0 and rank < len(self.descending_values_and_indices))
        return self.descending_values_and_indices[rank]

    def __len__(self):
        return len(self.min_heap)
```

Listing 8: SortTensor.py: This method uses an \( m \)-dimensional tensor with sorted axes. The runtime is in \( O(m \cdot n + k \log(k) \cdot m^2) \).

```python
from MinBinomialIndexHeap import *

class SortTensorBinomialSumHeap:
    def __init__(self, vectors):
        vectors_and_indices = [[(b,a) for a,b in enumerate(v)] for v in vectors]
        self.m = len(vectors_and_indices)
        self.ascending_vectors_and_unsorted_indices = [sorted(v) for v in 
                                                    vectors_and_indices]

        min_value = sum([a for a,b in [v[0] for v in self.ascending_vectors_and_unsorted_indices]])
        zero_tup = (0,)*self.m

        # The binomial heap will break ties by using the data tuple.
        # This means possibly popping the wrong value.
        # To get around this we add the sum of the tuple to the heap input
        # this works because we want to pop (i,j) before (i+1, j) or (i, j+1) and
        # since we have a MIN heap we will pop (i, j) first.
        self.fringe = MinBinomialIndexHeap([((min_value, sum(zero_tup)), zero_tup)])
        self.sorted_indices_in_fringe = set([zero_tup])

    def pop_min_and_index(self):
        val, index = self.fringe.pop_min_and_index()
        self.sorted_indices_in_fringe.remove(index)
        # insert neighbors from fringe if not already visited:
        for i in range(self.m):
```
new_index = list(index)
new_index[i] += 1
new_index = tuple(new_index)

asec_vec_i = self.ascending_vectors_and_unsorted_indices[i]

if new_index[i] < len(asec_vec_i) and new_index not in self.
  sorted_indices_in_fringe:
  old_val_at_index = asec_vec_i[new_index[i]-1][0]
  new_val_at_index = asec_vec_i[new_index[i]][0]
  new_val = val - old_val_at_index + new_val_at_index
  self.fringe.insert( ((new_val, sum(new_index)), new_index) )
  self.sorted_indices_in_fringe.add( new_index )

unsorted_index = tuple([ self.ascending_vectors_and_unsorted_indices[i][j][1]
  for i,j in enumerate(index) ])
return (val, unsorted_index)

def pop_min(self):
  val, unsorted_index = self.pop_min_and_index()
  return val

def select(self, k):
  return [ self.pop_min() for i in range(k) ]

Listing 9: SortTree.py: This method uses a balanced binary tree of pairwise problems that each generated sorted terms from $A + B$. The total runtime of this method is in $O(m \cdot n + k \log(k) \log(m))$.

from CartesianSumPairHeap import *

class SortTree:
  def __init__(self, vectors):
    vectors_and_indices = [ [(b,a) for a,b in enumerate(v)] for v in vectors ]
    self.m = len(vectors_and_indices)

    # build a balanced binary tree and store the root:
    current_layer = [ MinIndexHeap(v) for v in vectors_and_indices ]
    while len(current_layer) > 1:
      next_layer = []
      for i in range(len(current_layer) // 2):
        next_layer.append( CartesianSumPairHeap(current_layer[2*i], current_layer
          [2*i+1]) )
      if len(current_layer)%2 == 1:
        next_layer.append( current_layer[-1] )
      current_layer = next_layer
    self.root = current_layer[0]

  def pop_min_and_index(self):
    val,matrix_index = self.root.pop_min_and_index()
    full_index = [None]*self.m
    self.root.fill_full_index_from_matrix_index(matrix_index, full_index, 0)
    if type(self.root) == MinIndexHeap:
      return (val, (full_index,) )
    else:
      return (val, tuple(full_index) )

  def select(self, k):
    return [ self.pop_min() for i in range(k) ]
def pop_min(self):
    val, matrix_index = self.root.pop_min_and_index()
    return val

Listing 10: LayerArithmetic.py: This file builds a singleton with which we can compute sizes of layers, etc. of a layer-ordered heap of rank \( \alpha \). If layer \( j \) will only be accessed in a program when the program also accesses all layers \( i < j \), then every operation is in amortized \( O(1) \). This class is used by FastSoftTree (listing 16).

```python
import numpy as np
from one_dimensional_selection import *

class LayerArithmeticSingleton:

class __LayerArithmetic:
    def __init__(self, alpha=None):
        # Note: alpha can be >2, but it would defeat the purpose of the
        # algorithms that use layer-ordered heaps in this paper.
        assert(alpha > 1.0 and alpha < 2.0)
        print('ALPHA: \( \alpha \)', alpha)
        self.alpha = alpha
        self._index_to_layer_number = []
        self._layer_to_layer_size = []
        self._layer_to_cumulative_layer_sizes = []

        # all functions amortized in \( O(1) \) if we visit all cells in previous
        # layers before next layer is used

    def get_num_nodes_in_layer(self, layer_number):
        while len(self._layer_to_layer_size) <= layer_number:
            if len(self._layer_to_layer_size) > 0:
                prev_layer_size = self._layer_to_layer_size[-1]
                next_layer_size = int(np.ceil(self.alpha * prev_layer_size))
            else:
                next_layer_size = 1
            self._layer_to_layer_size.append(next_layer_size)
        return self._layer_to_layer_size[layer_number]

    def sum_of_layers_leq(self, layer_number):
        if layer_number < 0:
            return 0
        if len(self._layer_to_cumulative_layer_sizes) == 0:
            sum_of_previous_layer_sizes = 0
        else:
            sum_of_previous_layer_sizes = self._layer_to_cumulative_layer_sizes[-1]

        while len(self._layer_to_cumulative_layer_sizes) <= layer_number:
            new_layer_index = len(self._layer_to_cumulative_layer_sizes)
            next_layer_size = self.get_num_nodes_in_layer(new_layer_index)
            sum_of_previous_layer_sizes += next_layer_size
            self._layer_to_cumulative_layer_sizes.append(sum_of_previous_layer_sizes)

        return self._layer_to_cumulative_layer_sizes[layer_number]

    def _get_layer_from_flat_index(self, ind):
        # peel off all layers of sizes 1, alpha, alpha**2, ... until the
        # index is a valid offset:
```
while ind >= len(self._index_to_layer_number):
    for i in range(self.get_num_nodes_in_layer(self._layer_to_flat_index_is_computed_at_layer)):
        self._index_to_layer_number.append(self._layer_to_flat_index_is_computed_at_layer)
        self._layer_to_flat_index_is_computed_at_layer += 1
return self._index_to_layer_number[ind]

def _get_layer_and_offset_from_flat_index(self, ind):
    layer = self._get_layer_from_flat_index(ind)
    return (layer, ind - self.sum_of_layers_leq(layer - 1))

def has_right_child(self, flat_index):
    layer, offset = self._get_layer_and_offset_from_flat_index(flat_index)
    layer_size = self.get_num_nodes_in_layer(layer)
    next_layer_size = self.get_num_nodes_in_layer(layer + 1)
    # alpha < 2:
    num_nodes_with_one_child = 2 * layer_size - next_layer_size
    num_nodes_with_two_children = layer_size - num_nodes_with_one_child
    return offset < num_nodes_with_two_children

def _get_flat_index_from_layer_and_offset(self, layer, offset):
    assert(offset < self.get_num_nodes_in_layer(layer))
    return self.sum_of_layers_leq(layer - 1) + offset

def get_child_indices(self, flat_index):
    layer, offset = self._get_layer_and_offset_from_flat_index(flat_index)
    layer_size = self.get_num_nodes_in_layer(layer)
    next_layer_size = self.get_num_nodes_in_layer(layer + 1)
    if self.has_right_child(flat_index):
        return [self._get_flat_index_from_layer_and_offset(layer + 1, offset * 2),
                self._get_flat_index_from_layer_and_offset(layer + 1, offset * 2 + 1)]
    else:
        # alpha < 2:
        num_nodes_with_one_child = 2 * layer_size - next_layer_size
        num_nodes_with_two_children = layer_size - num_nodes_with_one_child
        rank_of_nodes_with_one_child = offset - num_nodes_with_two_children
        return [self._get_flat_index_from_layer_and_offset(layer + 1,
                                                    rank_of_nodes_with_one_child + num_nodes_with_two_children * 2)]

def _get_layer_as_list(self, layer_ordered_heap_lst, layer):
    max_layer, offset = self._get_layer_and_offset_from_flat_index(len(layer_ordered_heap_lst) - 1)
    if layer == max_layer:
        # perhaps not a full layer
        return layer_ordered_heap_lst[self._get_flat_index_from_layer_and_offset(layer, 0):]
    else:
        return layer_ordered_heap_lst[self._get_flat_index_from_layer_and_offset(layer, 0):self._get_flat_index_from_layer_and_offset(layer + 1, 0)]

def get_layer_start_and_end(self, layer):
    if layer == 0:
        return (0, 0)
    start = self.sum_of_layers_leq(layer - 1)
    end = self.sum_of_layers_leq(layer) - 1
    return (start, end)
# Achieves linear time by selecting largest first, meaning the
# selected amount and total length are both comparable. Since
# sel(n,k) in O(k+n), we have n + n/alpha + n/alpha + n/alpha^2 +
# ... propto 2n / (alpha-1).

def partition_into_layer_ordered(self, lst):
    len_lst = len(lst)
    largest_layer, offset = self._get_layer_and_offset_from_flat_index(len_lst - 1)
    # final layer may not be full; compute how much is actually used:
    final_layer_size = offset + 1
    lst, result = soft_select_and_return_unused_values(lst, len(lst) - final_layer_size)
    # reverse each layer added so that layers are in roughly sorted
    # order (aesthetic only):
    result = result[::-1]
    # from now on, layers are full:
    for layer in range(largest_layer - 1):
        layer_size = self.get_num_nodes_in_layer(layer)
        lst, next_largest = soft_select_and_return_unused_values(lst, len(lst) - layer_size)
        # reverse each layer added so that layers are in roughly sorted
        # order (aesthetic only):
        result.extend(next_largest[::-1])
    assert(len(result) == len_lst)
    return result[::-1]

def print_layer_ordered_heap(self, loh_array):
    list_of_layers = []
    total_seen = 0
    layer_index = 0
    while total_seen < len(loh_array):
        start, end = self.get_layer_start_and_end(layer_index)
        layer_list = loh_array[start:end+1]
        list_of_layers.append(layer_list)
        total_seen += self.get_num_nodes_in_layer(layer_index)
        layer_index += 1
    print(list_of_layers)

# Note: This is slow. It is for testing only.
def verify_layer_ordered_heap(self, lst):
    max_layer, offset = self._get_layer_and_offset_from_flat_index(len(lst) - 1)
    for layer in range(max_layer):
        max_key, max_data = max(self._get_layer_as_list(lst, layer))
        min_key, min_data = min(self._get_layer_as_list(lst, layer + 1))
        if not max_key <= min_key:
            print('FAIL in verify_layer_ordered_heap', lst[:10] )
            print('FAILED on layers', layer, layer + 1)
            start_11, end_11 = self.get_layer_start_and_end(layer)
            start_12, end_12 = self.get_layer_start_and_end(layer + 1)
            print(lst[start_11:end_11+1] )
            print(lst[start_12:end_12+1] )
            print('max, min')
            print('layer1', max_key, max_data)
            print('layer2', min_key, min_data)
            assert(max_key <= min_key )

# Basically says if an instance of the class already exists then point to the
# same instance of
Listing 11: LayerOrderedHeapGenerator.py: A base class for generating layer-ordered heaps online.

class LayerOrderedHeapGenerator(object):
    def __init__(self):
        self._lordh_array = []
        self._current_layer_index = None
        self._la = LayerArithmeticSingleton()
        self._max_key_in_lordh_array = None

    def __getitem__(self, i):
        if type(i) == slice:
            return [self.__getitem__(j) for j in range(i.start, i.stop, i.step)]
        else:
            assert(self.index_is_available(i))
            return self._lordh_array[i]

    def are_more_layers_available(self):
        return self.get_number_of_elements_fetched_so_far() < len(self)

    def index_is_available(self, i):
        return i >= 0 and i < self.get_number_of_elements_fetched_so_far()

    def index_will_ever_be_available(self, i):
        return i >= 0 and i < len(self)

    def get_number_of_elements_fetched_so_far(self):
        return len(self._lordh_array)

    def max_key_generated(self):
        # cache max value
        return self._max_key_in_lordh_array

    def size_of_most_recent_layer_generated(self):
        # if we aren’t out of elements, then are_more_layers_available() will be True
        # and most recent layer is full of elements:
        if self.are_more_layers_available():
            return self._la.get_num_nodes_in_layer(self._current_layer_index)
        else:
            # otherwise, compute how many elements we added to final layer:
            potential_size_of_final_layer = self._la.get_num_nodes_in_layer(self._current_layer_index)
            return potential_size_of_final_layer
Listing 12: ArrayLayerOrderedHeapGenerator.py: A subclass of LayerOrderedHeapGenerator (listing 11), which wraps an array and accesses it as a layer-ordered heap.

```python
from LayerOrderedHeapGenerator import *

class ArrayLayerOrderedHeapGenerator(LayerOrderedHeapGenerator):
    def __init__(self, lst):
        super(ArrayLayerOrderedHeapGenerator, self).__init__()
        self._entire_lordh_array = [(b, a) for a, b in enumerate(lst)]
        self._entire_lordh_array = self._la.partition_into_layer_ordered(self._entire_lordh_array)
        self._lordh_array = [self._entire_lordh_array[0]]
        self._current_layer_index = 0
        self._max_key_in_lordh_array = self._lordh_array[0]

    def compute_next_layer_if_available(self):
        if self.are_more_layers_available():
            self._current_layer_index += 1
            start, end = self._la.get_layer_start_and_end(self._current_layer_index)
            new_data = self._entire_lordh_array[start:end+1]
            self._lordh_array.extend(new_data)
            self._max_key_in_lordh_array = max(new_data)

    def __len__(self):
        return len(self._entire_lordh_array)
```

Listing 13: CartesianProductLayerOrderedHeapGenerator.py: A subclass of LayerOrderedHeapGenerator (listing 11), which has two LayerOrderedHeapGenerator children and generates a layer-ordered heap based on their Cartesian product.

```python
from LayerOrderedHeapGenerator import *

# for A+B
# and A|B
class CartesianProductLayerOrderedHeapGenerator(LayerOrderedHeapGenerator):
    def __init__(self, lordh_a, lordh_b):
        super(CartesianProductLayerOrderedHeapGenerator, self).__init__()
        assert( isinstance(lordh_a, LayerOrderedHeapGenerator) )
        assert( isinstance(lordh_b, LayerOrderedHeapGenerator) )

        self._lordh_a = lordh_a
        self._lordh_b = lordh_b

        min_a_key, min_a_index = self._lordh_a[0]
        min_b_key, min_b_index = self._lordh_b[0]

        self._current_layer_index = 0
        self._max_key_in_lordh_array = [max(min_a, min_b) for min_a, min_b in zip(self._lordh_a, self._lordh_b)]

    def compute_next_layer_if_available(self):
        if self.are_more_layers_available():
            self._current_layer_index += 1
            start, end = self._la.get_layer_start_and_end(self._current_layer_index)
            new_data = self._entire_lordh_array[start:end+1]
            self._lordh_array.extend(new_data)
            self._max_key_in_lordh_array = max(new_data)

    def __len__(self):
        return len(self._entire_lordh_array)
```

# Note: internally, two indices are used (left,right). But
# externally, only one index is used: (key, index_in_lordh_of_self)
best_key = min_a_key + min_b_key
self._lordh_array = [(best_key, (0, 0))]
self._current_layer_index = 0
self._size = len(self._lordh_a) * len(self._lordh_b)
self._max_key_in_lordh_array = best_key

def __len__(self):
    # a terrible hack due to python's maximum size:
    if self._size > 2**32-1:
        return 2**32-1
    return self._size

def compute_next_layer_if_available(self):
    pass

Listing 14: AConcatBLayerOrderedHeapGenerator.py: A subclass of CartesianProductLayerOrderedHeapGenerator (listing 13), which is used to generate layers from children A and B with the smallest value so that all values needed for a subsequent k-selection on A + B have been found.

from CartesianProductLayerOrderedHeapGenerator import *

class AConcatBLayerOrderedHeapGenerator(CartesianProductLayerOrderedHeapGenerator):
    def __init__(self, lordh_a, lordh_b):
        super(AConcatBLayerOrderedHeapGenerator, self).__init__(lordh_a, lordh_b)

def compute_next_layer_if_available(self):
    raise Exception('Not needed for this paper')

def generate_axes_with_total_length_at_least(self, k):
    max_possible_size = len(self._lordh_a)+len(self._lordh_b)
    if k > max_possible_size:
        k = max_possible_size

        # _lordh_a and _lordh_b both start with 1 value generated
        total_generated = self._lordh_a.get_number_of_elements_fetched_so_far() + self._lordh_b.get_number_of_elements_fetched_so_far()
        while total_generated <= k:
            if not self._lordh_b.are_more_layers_available():
                self._lordh_a.compute_next_layer_if_available()
                total_generated += self._lordh_a.size_of_most_recent_layer_generated()
            elif not self._lordh_a.are_more_layers_available():
                self._lordh_b.compute_next_layer_if_available()
                total_generated += self._lordh_b.size_of_most_recent_layer_generated()
            else:
                # Both could be extended
                if self._lordh_a.max_key_generated() <= self._lordh_b.max_key_generated():
                    # extend a
                    self._lordh_a.compute_next_layer_if_available()
                    total_generated += self._lordh_a.size_of_most_recent_layer_generated()
                else:
                    # extend b
                    self._lordh_b.compute_next_layer_if_available()
                    total_generated += self._lordh_b.size_of_most_recent_layer_generated()

        # must have retrieved at least k+1:
if self._lordh_a.max_key_generated() == self._lordh_b.max_key_generated():
    # Success
    return

swapped = False
if self._lordh_b.max_key_generated() < self._lordh_a.max_key_generated():
    self._lordh_a, self._lordh_b = self._lordh_b, self._lordh_a
    swapped = True

# w.l.o.g., we will only need to extend child A.
size_of_most_recent_layer_of_b = self._lordh_b.size_of_most_recent_layer_generated()
additional_values_added_from_a = 0
while additional_values_added_from_a < size_of_most_recent_layer_of_b and self._lordh_a.are_more_layers_available():
    self._lordh_a.compute_next_layer_if_available()
    additional_values_added_from_a += self._lordh_a.size_of_most_recent_layer_generated()

if self._lordh_a.max_key_generated() >= self._lordh_b.max_key_generated():
    # No further values of A can help at the moment
    break

if swapped:
    self._lordh_a, self._lordh_b = self._lordh_b, self._lordh_a

    # Success

def lordh_a_most_recent_layer_size(self):
    return self._lordh_a.size_of_most_recent_layer_generated()

def lordh_b_most_recent_layer_size(self):
    return self._lordh_b.size_of_most_recent_layer_generated()

Listing 15: APlusBLayerOrderedHeapGenerator.py: A subclass of CartesianProductLayerOrderedHeapGenerator (listing 13), which is used to generate layers from children A and B (efficiently performed via AConcatBLayerOrderedHeapGenerator, listing 14) to perform selection on A + B. The selection results on A + B are generated as layers of its own layer-ordered heap, thereby enabling a tree of these to be built.

class APlusBLayerOrderedHeapGenerator(CartesianProductLayerOrderedHeapGenerator):
    def __init__(self, lordh_a, lordh_b):
        super(APlusBLayerOrderedHeapGenerator, self).__init__(lordh_a, lordh_b)
        self._concat_a_b_generator = AConcatBLayerOrderedHeapGenerator(lordh_a, lordh_b)
        self._epsilon = 1./8. # 1/m
        self._soft_heap = SoftHeap(self._epsilon)
        self._lordh_a_purgatory = []
        self._lordh_b_purgatory = []
        self._both_lordh_purgatory = []
        self._la = LayerArithmeticSingleton()
        self._lordh_a_layer_at_last_purgatory_insert = self._concat_a_b_generator._lordh_a._current_layer_index
        self._lordh_b_layer_at_last_purgatory_insert = self._concat_a_b_generator._lordh_b._current_layer_index
def select_from_root(self, k):
    while len(self._lordh_array) < k and self.are_more_layers_available():
        self.compute_next_layer_if_available()
    good_items, bad_items = soft_select_and_return_unused_values(self._lordh_array, k)
    return [key for key, data in good_items]

def compute_next_layer_if_available(self):
    if self.are_more_layers_available():
        self._current_layer_index += 1
        num_elements_unseen = len(self) - self.get_number_of_elements_fetched_so_far()
        num_values_in_next_layer = min(self._la.get_num_nodes_in_layer(self._current_layer_index), num_elements_unseen)
        num_elements_unseen_after_layer_added = len(self) - self.get_number_of_elements_fetched_so_far() + num_values_in_next_layer
        # generates all necessary axes for upcoming selection on A+B:
        total_values_after_new_layer_added = self.get_number_of_elements_fetched_so_far() + num_values_in_next_layer
        self._concat_a_b_generator.generate_axes_with_total_length_at_least(total_values_after_new_layer_added)
        self._insert_purgatories_if_children_grew()
        # Maximum amount of corruption in Chazelle soft heap is epsilon * # number of inserted items. Each pop may insert up to 4 times. So we are solving:
        # Define x as the number of pops to perform.
        # x - epsilon*(total insertions after x popped) >= k
        # x - epsilon*(current_size + 4*x) >= k
        # x - epsilon*current_size - 4*epsilon*x >= k
        # x * (1-4*epsilon) - epsilon*current_size >= k
        # x >= (k + epsilon*current_size) / (1-4*epsilon)
        num_pops_to_get_k_uncorrupted_items = num_values_in_next_layer + int(self._epsilon*len(self._soft_heap) / (1-4*self._epsilon)) + 1 # +1 since int()
        candidates = []
        for i in range(num_pops_to_get_k_uncorrupted_items):
            if len(self._soft_heap) > 0:
                results = self._pop_min_and_insert_children()
                if type(results) is not type(None):
                    key, data = results
                    candidates.append((key, data))
        good_items, bad_items = soft_select_and_return_unused_values(candidates, num_values_in_next_layer)
        # Rebuild soft heap from scratch: This is better than only inserting items that we want to put back, because corruption is bounded in terms of insertions, not size:
        remaining_items = []
        while len(self._soft_heap) > 0:
remaining_items.append(self._soft_heap.deletemin())

self._soft_heap = SoftHeap(self._epsilon)
# re-insert items which were popped but will not be returned
for key, data in bad_items:
    self._soft_heap.insert(key, data)
for key, data in remaining_items:
    self._soft_heap.insert(key, data)

new_layer = [(key, data) for key, (data, insert_children) in good_items]
self._max_key_in_lordh_array = max(new_layer)[0]
selse._lordh_array.extend(new_layer)

def _pop_min_and_insert_children(self, root=False):
    results = self._soft_heap.deletemin()
    if type(results) is not type(None):
        key, data = results
        (i, j), insert_children = data
        if insert_children:
            insert_children = False
            data = [(i, j), insert_children]
        self._insert_children_into_soft_heap(i, j)
        return (key, data)
    return None

def _insert_children_into_soft_heap(self, i, j):
    if j == 0:
        self._insert_children_when_y_is_0(i)
    else:
        self._insert_children_when_y_greater_than_zero(i, j)

def _in_bounds(self, i, j):
    return self._lordh_a.index_is_available(i) and self._lordh_b.index_is_available(j)

def _insert_if_in_bounds(self, i, j):
    if self._in_bounds(i, j):
        a_key, a_data = self._lordh_a[i]
        b_key, b_data = self._lordh_b[j]
        key = a_key + b_key
        insert_children = True
        data = [(i, j), insert_children]
        self._soft_heap.insert(key, data)
    else:
        # Out of bounds, put into purgatory if they will ever be in bounds
        if self._lordh_a.index_will_ever_be_available(i) and self._lordh_b.
            index_will_ever_be_available(j):
            if not self._lordh_a.index_is_available(i) and not self._lordh_b.
                index_is_available(j):
                self._both_lordh_purgatory.append((i, j))
            elif not self._lordh_a.index_is_available(i):
                self._lordh_a_purgatory.append((i, j))
            else:
                self._lordh_b_purgatory.append((i, j))

    def _insert_children_when_y_is_0(self, i):
        for i_child in self._la.get_child_indices(i):
def _insert_children_when_y_greater_than_zero(self, i, j):
    for j_child in self._la.get_child_indices(j):
        self._insert_if_in_bounds(i, j_child)

def _insert_purgatories_if_children_grew(self):
    if self._lordh_a_layer_at_last_purgatory_insert != self._concat_a_b_generator._lordh_a._current_layer_index and self._lordh_b_layer_at_last_purgatory_insert != self._concat_a_b_generator._lordh_b._current_layer_index:
        # insert purgatory for both children
        for i, j in self._both_lordh_purgatory:
            key_a, data_a = self._lordh_a[i]
            key_b, data_b = self._lordh_b[j]
            key = key_a + key_b
            insert_children = True
            data = [(i, j), insert_children]
            self._soft_heap.insert(key, data)
        self._both_lordh_purgatory = []

    if self._lordh_a_layer_at_last_purgatory_insert != self._concat_a_b_generator._lordh_a._current_layer_index:
        # insert for child a
        for i, j in self._lordh_a_purgatory:
            key_a, data_a = self._lordh_a[i]
            key_b, data_b = self._lordh_b[j]
            key = key_a + key_b
            insert_children = True
            data = [(i, j), insert_children]
            self._soft_heap.insert(key, data)
        self._lordh_a_purgatory = []

    self._lordh_a_layer_at_last_purgatory_insert = self._concat_a_b_generator._lordh_a._current_layer_index

    if self._lordh_b_layer_at_last_purgatory_insert != self._concat_a_b_generator._lordh_b._current_layer_index:
        # insert for child b
        for i, j in self._lordh_b_purgatory:
            key_a, data_a = self._lordh_a[i]
            key_b, data_b = self._lordh_b[j]
            key = key_a + key_b
            insert_children = True
            data = [(i, j), insert_children]
            self._soft_heap.insert(key, data)
        self._lordh_b_purgatory = []

    self._lordh_b_layer_at_last_purgatory_insert = self._concat_a_b_generator._lordh_b._current_layer_index

Listing 16: FastSoftTree.py: This builds a balanced binary tree with internal nodes of type APlusBLayerOrderedHeapGenerator (listing 15) and leaves of type ArrayLayerOrderedHeapGenerator (listing 12). By generating a layer-ordered heap of the root, we arrive at a layer-ordered heap of values from $X_1 + X_2 + \cdots + X_m$, and can thus perform selection.
from ArrayLayerOrderedHeapGenerator import *
from APlusBLayerOrderedHeapGenerator import *
from LayerArithmetic import *
import numpy as np
from time import time

class FastSoftTree:
    def __init__(self, vectors):
        self._m = len(vectors)
        self.initial_values = []
        for i in range(len(vectors)):
            self.initial_values.append( min(vectors[i]) )
        for j in range(len(vectors[i])):
            vectors[i][j] -= self.initial_values[-1]

        ALPHA = np.log(2*self._m)/np.log(self._m)
        ALPHA = 1.1
        print( 'NOTE:ALPHA IS', ALPHA)
        la = LayerArithmeticSingleton(ALPHA)
        current_layer = [ ArrayLayerOrderedHeapGenerator(v) for v in vectors ]
        while len(current_layer) > 1:
            next_layer = []
            for i in range(len(current_layer) // 2):
                next_layer.append( APlusBLayerOrderedHeapGenerator(current_layer[2*i],
                    current_layer[2*i+1]) )
            if len(current_layer)%2 == 1:
                next_layer.append( current_layer[-1] )
            current_layer = next_layer
        self._root = current_layer[0]

    def select(self, k):
        start_time = time()
        results = self._root.select_from_root(k)
        end_time = time()

        total_take_out = sum(self.initial_values)
        for i in range(len(results)):
            results[i] += total_take_out

        return results

References

[1] Manuel Blum, Robert W. Floyd, Vaughan R. Pratt, Ronald L. Rivest, and Robert Endre Tarjan. Time bounds for selection. *Journal of Computer and System Sciences*, 7(4):448–461, 1973.

[2] D. Bremner, T. M. Chan., E. D. Demaine, J. Erickson, F. Hurtado, J. Iacono, S. Langerman, and P. Taslakian. Necklaces, convolutions, and $X+Y$. In *Algorithms–ESA 2006*, pages 160–171. Springer, 2006.

[3] M. Bussieck, H. Hassler, G. J. Woeginger, and U. T. Zimmermann. Fast algorithms for the maximum convolution problem. *Operations research letters*, 15(3):133–141, 1994.

[4] Bernard Chazelle. The soft heap: an approximate priority queue with optimal error rate. *Journal of the ACM (JACM)*, 47(6):1012–1027, 2000.
[5] A.S. Eddington. *Mathematical Theory of Relativity*. Cambridge University Press, London, 1923.

[6] G. N. Frederickson and D. B. Johnson. The complexity of selection and ranking in $X+Y$ and matrices with sorted columns. *Journal of Computer and System Sciences*, 24(2):197–208, 1982.

[7] H. Kaplan, L. Kozma, O. Zamir, and U. Zwick. Selection from heaps, row-sorted matrices and $x+y$ using soft heaps. *arXiv preprint arXiv:1802.07041v1*, 2018.

[8] M. K. Łącki, M. Startek, D. Valkenborg, and A. Gambin. Isospec: Hyperfast fine structure calculator. *Analytical Chemistry*, 89(6):3272–3277, 2017.

[9] J. Pfeuffer and O. Serang. A bounded $p$-norm approximation of max-convolution for sub-quadratic Bayesian inference on additive factors. *Journal of Machine Learning Research*, 17(36):1–39, 2016.

[10] O. Serang. A fast numerical method for max-convolution and the application to efficient max-product inference in Bayesian networks. *Journal of Computational Biology*, 22:770–783, 2015.