NON-COMMUTATIVE FUNCTIONS AND NON-COMMUTATIVE FREE LEVY-HINCIN FORMULA

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ABSTRACT. The paper is discussing infinite divisibility in the setting of operator-valued boolean, free and, more general, c-free independences. Particularly, using Hilbert bimodules and non-commutative functions techniques, we obtain analogues of the Levy-Hincin integral representation for infinitely divisible real measures.

1. Introduction and notations

The paper presents some results concerning infinite divisibility in the framework of operator-valued non-commutative probability.

In probability theory - classic and non-commutative - limit theorems play a central role. The “most general” limit theorems involve so-called infinitesimal arrays, and the limit distributions are usually identified with infinitely divisible distributions. There is a consistent literature about infinitely divisible measures in classical probability (see [9]), dating back to Kolmogorov ([10]), P. Levy ([14]) etc. Similar results have been found for non-commutative independences, such as [3] and [4] for free independence, [23] for boolean probability and [12], [31] for conditionally free probability. In the operator-valued case, when states are replaced by positive conditional expectations or, more general, by completely positive maps between C*- or operator algebras, very little was known about operator-valued infinite divisibility; the only exception we know of was Speicher’s work [24]. One of the obstructions is that while in the scalar case important results characterizing infinite divisibility are coming from Nevalinna-Pick representation properties of the functions that linearize additive convolutions (such as the log of the Fourier transform in the classic case or the Voiculescu’s R- and φ-transforms in the free case), such analytic tools are not yet available in the operator-valued case. The new topic of non-commutative ([11]) or completely matricial ([29]) functions may possibly fill this gap. In particular, the results from Section 5 of the present paper indicate that results similar to the Nevalinna-Pick representation hold also for non-commutative functions.

The paper is organized in five sections. First section presents the introduction and some notations. Sections 2-4 are aimed towards results characterizing infinite divisibility in operator-valued non-commutative probability using combinatorial and operator-algebras methods and constructions. More precisely, in Section 2 we use a non-commutative version of the “Boolean Fock space” construction from [1] to prove that, as in the scalar case, boolean infinite divisibility is trivial in the operator-valued case; particularly, any completely positive map between two C*-algebras is boolean infinitely divisible. Section 3 is describing infinite divisibility with respect to free independence over a positive conditional expectation in terms of maps satisfying a condition of complete positivity. Section 4 is utilizing the
techniques from the Boolean case (Section 2) to extend the results of Section 3 from positive conditional expectations to completely positive maps. In particular, we present a construction of the non-commutative version of the conditionally free R-transform of Bozejko, Leinert and Speicher in terms of creation, annihilation and preservation operators on a certain inner-product bimodules. In the scalar-valued case this construction gives a new, combinatorial proof of the main result from [12] (see also [31]) characterizing conditionally free infinite divisibility. In Section 5 we use the tools from the theory of non-commutative functions (see [11], [29]) to define the non-commutative R- (also constructed in [29]), cR- and B-transforms. Reformulated in terms of these transforms, the results from Sections 2, 3 and 4 are very similar to the free and conditionally free versions of the Levy-Hincin formula from [3], respectively [12] and [31]. The present material is using the notions detailed in [11] (work in progress), but it is self-contained in this regard, the needed material on non-commutative functions is briefly discussed in Section 5.2.

We will introduce now several notations. Throughout the paper B will be a unital C* algebra. We will denote by B⟨X⟩ the ∗-algebra freely generated by B and the selfadjoint symbol X. Unless otherwise explicitly stated, we do not suppose that B commutes with X. We will also use the notations B⟨X⟩0 for the ∗-subalgebra of B⟨X⟩ of all polynomials without a free term, and the notation B⟨X1, X2,...⟩ for the ∗-algebra freely generated by B and the non-commutating self-adjoint symbols X1, X2,...

In several instances we will identify T(B), the tensor algebra over B, to the subalgebra of B⟨X⟩ spanned by \{Xb1Xb2...Xbn : n ∈ N, b1,...,bn ∈ B\} via

\[ b_1 \otimes b_2 \otimes ... \otimes b_n \mapsto Xb_1Xb_2...Xb_n. \]

The set of all positive conditional expectations from B⟨X⟩ to B will be denoted by \( Σ_B \).

For B ⊆ D a unital inclusion of C* algebras, we denote by \( Σ_{B,D} \) the set of all unital, positive B-bimodule maps \( µ : B⟨X⟩ \rightarrow D \) with the property that for all positive integers n and all \( \{f_i(X)\}^n_{i=1} \subset B⟨X⟩ \) we have that:

\[ [\mu(f_1(X)) \ast f_1(X))]^n_{j=1} \geq 0 \text{ in } M_n(D). \]  

Remark that \( Σ_B = Σ_{B,B} \), as an easy consequence of Exercise 3.18 from [16]. We will denote by \( Σ^0_{B,D} \), respectively \( Σ^0_{B,D} \), the set of all \( µ \in Σ_B \) (respectively \( µ \in Σ_{B,D} \) whose moments do not grow faster than exponentially, that is there exists some \( M > 0 \) such that for all \( b_1,...,b_n \in B \), we have

\[ ||µ(Xb_1Xb_2...Xb_n X)|| < M^{n+1} ||b_1||...||b_n||. \]  

We will also use the following definition (see [13]):

**Definition 1.1.** Let A be a C*-algebra. A semi-inner-product A-module is a linear space E which is a right A-module together with a map \( ⟨x,y⟩ : E \times E \rightarrow A \) such that \( (x,y,z) ∈ E, a ∈ A, α, β ∈ C \):

(i) \( ⟨αx + βy, z⟩ = α⟨x, y⟩ + β⟨y, z⟩ \)

(ii) \( ⟨xa, y⟩ = ⟨x, y⟩a \)

(iii) \( ⟨y, x⟩ = ⟨y, x⟩^* \)

(iv) \( ⟨x, x⟩ ≥ 0 \)

The set of all adjointable maps \( T : E \rightarrow E \) will be denoted by \( L(E) \). Since \( ⟨\cdot, \cdot⟩ \) is not strictly positive, the adjoint of a map from \( L(E) \) is in general not unique. If
\[ E \text{ is a Hilbert } A\text{-module}, \text{ that is the inequality at (iv) is strict and } E \text{ is complete with respect to the norm } \| \xi \| \equiv \| (\xi, \xi) \|^{1/2}, \text{ then the adjoint is unique and the bounded elements of } L(E) \text{ form a } C^\ast\text{-algebra.} \]

If \( B \subseteq A, B \subseteq D \) are unital inclusions of \( C^\ast\)-algebras, \( \phi : A \to D \) is a unital positive \( B\)-bimodule map and \( a \) is a selfadjoint element of \( A \), we will denote by \( B\langle a \rangle \) the \( *\)-algebra generated in \( A \) by \( B \) and \( a \) and by \( \mu_a, \) “the \( D\)-distribution” of \( a \), that is the positive \( B\)-bimodule map \( \phi_a : B\langle X \rangle \to D \) defined by \( \phi_a = \phi \circ \tau_a \) where \( \tau_a : B\langle X \rangle \to A \) is the unique homomorphism such that \( \tau_a(X) = a \) and \( \tau_a(b) = b \) for all \( b \in B \). The set of elements from \( \Sigma_{B;D} \) that can be realized in such way is \( \Sigma_{B;D}^0, \) more precisely we have the following property:

**Proposition 1.2.** Let \( \mu \in \Sigma_{B;D} \). Then \( \mu \in \Sigma_{B;D}^0 \) if and only if there exist a \( C^\ast\)-algebra \( A \) containing \( B \) as a \( C^\ast\)-subalgebra, a completely positive \( B\)-bimodule map \( \phi : A \to D \) and a self-adjoint element \( a \in A \) such that \( \mu = \phi_a \).

Moreover, the condition (2) is equivalent to the existence of \( M > 0 \) such that for all \( b_1, \ldots, b_n \in M_m(B) \) we have

\[
\| (\text{Id}_m \otimes \mu)(X \cdot b_1 X \cdot b_2 \cdots X \cdot b_n X) \| \leq M^{n+1} \| b_1 \| \cdots \| b_n \|,
\]

where \( X \) acts on \( M_m(\mathbb{C}) \otimes B\langle X \rangle \) by multiplication on each entry (that is we identify \( X \) to \( \text{Id}_m \otimes X \)).

**Proof.** If \( \mu = \phi_a \) as above, then the result is trivial.

Suppose now that \( \mu \in \Sigma_{B;D}^0 \). We first prove that \( \mathcal{N}_0 = \{ f \in B\langle X \rangle : \mu(f^* f) = 0 \} \) is a left ideal of \( B\langle X \rangle \). It suffices to prove that if \( f \in \mathcal{N}_0 \) then \( X \cdot f \in \mathcal{N}_0 \) and \( b \cdot f \in \mathcal{N}_0 \) for all \( b \in B \).

Since \( b^* b \leq \| b^* b \| \) (in the \( C^\ast\)-algebra \( B \)), the positivity of \( \mu \) implies

\[
\mu(f^* (\| b^* b \| - b^* b) f) \geq 0,
\]

that is \( \mu(f^* b^* b f) \leq \| b^* b \| \mu(f^* f) = 0 \), hence \( b \cdot f \in \mathcal{N}_0 \).

For \( g = b_0 X b_1 X \cdots X b_n \), a monomial in \( B\langle X \rangle \), define \( p(g) = M^n \| b_0 \| \cdots \| b_n \| \) (in particular, condition (2) states that \( \| \mu(g) \| \leq p(g) \)). Consider

\[
B\langle \langle X \rangle \rangle_{\mu} = \left\{ \sum_{n=0}^{\infty} f_n : f_n = \text{monomials in } B\langle X \rangle \text{ such that } \sum_{n=0}^{\infty} p(f_n) < \infty \right\}.
\]

\( B\langle \langle X \rangle \rangle_{\mu} \) is a \(*\)-algebra (with the structure inherited from \( B\langle X \rangle \)) and \( \mu \) extends to a positive map \( \bar{\mu} : B\langle \langle X \rangle \rangle_{\mu} \to D \) via \( \bar{\mu}(\sum_{n=0}^{\infty} f_n) = \sum_{n=0}^{\infty} \mu(f_n) \).

Take now \( g_n = (2n)! \left[(1 - 2n)(n!)^2 (2M)^{2n}\right]^{-1} X^{2n}, n \geq 0 \). Then \( g_n = g_n^* \) and \( p(g_n) \leq 4^{-n}, \) so \( g = \sum_{n=0}^{\infty} g_n \) is also a selfadjoint element of \( B\langle \langle X \rangle \rangle_{\mu} \).

Since \( g^2 = 1 - [(2M)^{-1} X]^2 \), we have that

\[
0 \leq \bar{\mu}(f^* g^* g f) = \bar{\mu}(f^* [1 - (2M)^{-2} X^2] \cdot f) = \mu(f^* f) - (2M)^{-2} \mu((X f)^* X f)
\]

hence \( \mu((X f)^* X f) \leq 4M^2 \mu(f^* f) \), so \( X f \in \mathcal{N}_0 \).

Consider now \( K = B\langle X \rangle \otimes_{B} D \) with the right \( D\)-module structure given by \( (f(X) \otimes d_1)(g(X) \otimes d_2) = d_2^* \mu(g(X)^* f(X)) d_1 \). Also, since \( \mu \) satisfies the condition (1), we can define a \( D\)-valued sesquilinear inner-product structure on \( K \) (see [13], page 40) via

\[
(f(X) \otimes d_1, g(X) \otimes d_2) = d_2^* \mu(g(X)^* f(X)) d_1.
\]
Let now $\mathcal{N} = \{ \eta \in K : \langle \eta, \eta \rangle = 0 \}$. From the above argument on $\mathcal{N}_0$, we have that $\mathcal{N}$ is a left $\mathcal{B}(\mathcal{X})$-module. Finally, take $E$ the completion of $\mathcal{K}/\mathcal{N}$ in the norm induced by the inner-product structure (see [13]) and let $\xi = 1 \otimes 1 + \mathcal{N}$. The multipliers with polynomials from $\mathcal{B}(\mathcal{X})$ are in the $C^*$-algebra of bounded maps from $\mathcal{L}(E)$, since condition (2) ensures the boundness. Moreover, if $\phi(\cdot) = \langle \xi, \xi \rangle$ and $a$ is the right multiplier with $\mathcal{X}$, then $\mu = \phi_a$.

Condition (3) is implied by the equality $\|\text{Id}_m \otimes a\| = \|a\|$, where the first norm is in the $C^*$-algebra $M_m(A)$ and the second norm in the $C^*$-algebra $A$.

\[ \square \]

2. Infinite divisibility: the boolean case

The main result of this section is the non-commutative analogue of the following theorem (see [24]):

**Theorem 2.1.** Any compactly supported real measure is infinitely divisible with respect to boolean convolution.

We will use the following notion of boolean independence over a $C^*$-algebra $\mathcal{B}$ (see [18]):

**Definition 2.2.** Let $\mathcal{B}$ be a unital $C^*$-algebra, $\mathcal{B} \subseteq \mathcal{D}$, $\mathcal{B} \subseteq \mathcal{A}$ be unital inclusions of $*$-algebras and $\phi : \mathcal{A} \rightarrow \mathcal{D}$ be a unital $\mathcal{B}$-bimodule map. A family $\{\alpha_{ij}\}_{i,j} \subseteq \mathcal{B}$ of selfadjoint elements from $\mathcal{A}$ is said to be boolean independent with respect to $\phi$ if

$$\phi(\alpha_{i1}\alpha_{i2}\alpha_{i3} \cdots) = \phi(\alpha_{1})\phi(\alpha_{2})\phi(\alpha_{3}) \cdots$$

for all $A_k \in \mathcal{B}(a_{\epsilon(k)})_0$ (the $*$-algebra spanned by non-commutative polynomials in $a_{\epsilon(k)}$ and coefficients in $\mathcal{B}$ without a free term), with $\epsilon(k) \in I$ and $\epsilon(k) \neq \epsilon(k + 1)$.

Let now $N \in \mathbb{N}$ and $\{\nu_j\}_{j=1}^N$ be a family of elements from $\Sigma_{\mathcal{B},\mathcal{D}}$. We define their additive boolean convolution as follows. Consider the symbols $\{X_j\}_{j=1}^N$ such that $\mu_j : \mathcal{B}(X_j) \cong \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{B}$ coincides to $\mu_j$. Then consider the $*$-algebra $\mathcal{B}(X_1, X_2, \ldots, X_N)$ with the conditional expectation $\mu$ such that its restrictions to $\mathcal{B}(X_j)$ coincide to $\mu_j$ and the mixed moments of $X_1, X_2, \ldots, X_N$ are calculated via the rule from Definition 2.2. The additive boolean convolution of $\{\nu_j\}_{j=1}^N$ is the unital $\mathcal{B}$-bimodule map

$$\nu_{j=1}^N \mu_j = \mu_{X_1 + X_2 + \ldots + X_N} : \mathcal{B}(X_1 + X_2 + \ldots + X_N) \cong \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{D}$$

**Remark 2.3.** If $\mu_j \in \Sigma_{\mathcal{B},\mathcal{D}}$ for all $j = 1, \ldots, N$, then $\nu_{j=1}^N \mu_j$ is also in $\Sigma_{\mathcal{B},\mathcal{D}}$.

**Proof.** For each $j$, consider the right-$\mathcal{D}$ module $\mathcal{K}_j = \mathcal{B}(X_j) \otimes \mathcal{D}$ as in Proposition 1.2.

Note that the $\mathcal{D}$-submodule $1 \otimes \mathcal{D}$ is complemented in each $\mathcal{K}_j$, since, for all $f(X) \in \mathcal{B}(\mathcal{X})$, we have that $1 \otimes \mu_j(f(X_j)) \in 1 \otimes \mathcal{D}$ and

$$\langle 1 \otimes 1, f(X_j) \otimes 1 - 1 \otimes \mu_j(f(X_j)) \rangle_j = \mu_j(f(X_j)^*) - \mu_j(f(X_j))^* = 0$$

Denote by $\mathcal{K}_j^0$ the complement of $1 \otimes \mathcal{D}$ in $\mathcal{K}_j$ and let

$$\mathcal{K} = (1 \otimes \mathcal{D}) \oplus \bigoplus_{j=1}^N \mathcal{K}_j^0.$$ 

On $\mathcal{L}(\mathcal{K})$ we consider the map $\phi : T \mapsto \langle T1 \otimes 1, 1 \otimes 1 \rangle$. 

\( B(X_j) \) can be seen as a subalgebra of linear maps on \( \mathcal{K}_j \) via
\[ f(X_j)[g(X_j) \otimes d] = (f(X_j)g(X_j)) \otimes d. \]

Since \( f(X_j)^* \) is adjoint to \( f(X_j) \) we have \( B(X_j) \subseteq \mathcal{L}(\mathcal{K}) \). moreover, by setting the restrictions of \( B(X_j)_0 \) to each \( \mathcal{K}_j^0 \) to the 0 if \( j \neq j \), we can see \( B(X_j) \) as a subalgebra of \( \mathcal{L}(\mathcal{K}) \). Note that \( \phi_{B(X_j)_0} = \mu_j \).

If \( j \neq l \) and \( A_1 \in B(X_j)_0 \) while \( A_2 \in B(X_l)_0 \), then
\[ A_1A_2(1 \otimes 1) = A_1[\phi(A_2)(1 \otimes 1) + \xi], \]
where \( \xi = A_2(1 \otimes 1) - \phi(A_2)1 \otimes 1 \in \mathcal{K}_j^0 \)
\[ = A_1\phi(A_2)(1 \otimes 1). \]

Iterating, we obtain that for \( \phi_{B(X_j)_0} \) with \( \epsilon(k) \neq \epsilon(k+1) \)
\[ \phi_1 \cdots A_m = \phi(A_1\phi(A_2) \cdots \phi(A_m)) = \phi(A_1) \cdots \phi(A_m). \]

that is \( X_j \)'s are boolean independent in \( \mathcal{L}(\mathcal{K}) \) with respect to \( \phi \), and since the restrictions of \( \phi \) to \( B(X_j) \) are \( \mu_j \), we have that \( \psi_{j=1}^n \mu_j = \phi_{B(X_1 + \cdots + X_N)} \) so q.e.d.

**Definition 2.4.** An element \( \mu \in \Sigma_{B,D} \) is said to be \( \{\} \)-infinite divisible if for any positive integer \( n \) there exist some \( \mu_n \in \Sigma_{B,D} \) such that \( \mu \) is the additive boolean convolution of \( n \) copies of \( \mu_n \).

The main result of this section is the following

**Theorem 2.5.** Any element \( \mu \) from \( \Sigma_{B,D} \) is boolean infinitely divisible.

To prove [2.5] we will need the following generalization of the notion of scalar boolean cumulants from [23]:

**Definition 2.6.** In the above setting, let \( X \) be a selfadjoint element from \( A \). The boolean cumulants of \( X \) are defined as the multilinear maps \( \{B_n(X)\}_{n \geq 1} \), with \( B_n(X) : B^n \to D \), given by the following recurrence:
\[ \phi(Xb_nXb_{n-1} \cdots Xb_1) = \sum_{k=1}^n \phi(Xb_n \cdots Xb_{k+1})B_k(X(b_k, \cdots, b_1)). \]

As shown in [13], the \( D \)-valued boolean cumulants defined above have the same additivity property as their scalar analogues (see [13], Corollary 4.6):

**Proposition 2.7.** If, in the above setting, \( X \) and \( Y \) are boolean independent over \( \phi \), then, for all positive integers \( n \), we have that
\[ B_{n,X+Y} = B_{n,X} + B_{n,Y}. \]

Definition 2.4 can be reformulated in terms of Proposition 2.7. Namely, for \( \mu \in \Sigma_{B,D} \) we define the \( n \)-th boolean cumulant of \( \mu \) as the multilinear map \( B_{n,\mu} : B^n \to D \) given by the recurrence:
\[ \mu(Xb_1Xb_2 \cdots Xb_n) = \sum_{k=1}^n B_{k,\mu}(b_1, \ldots, b_k)\mu(Xb_{k+1} \cdots Xb_n). \]

From Proposition 2.7 we have the following

**Remark 2.8.** An element \( \mu \in \Sigma_{B,D} \) is \( \{\} \)-infinite divisible if for any positive integer \( n \) there exist \( \mu_n \in \Sigma_{B,D} \) such that for all positive integers \( m \) we have that
\[ B_{m,\mu} = nB_{m,\mu_n}. \]
Before proving the main result, i.e., Theorem 2.5, we will first need the following result.

**Lemma 2.9.** Let $B \subseteq A$ be a unital inclusion of $C^*$-algebras, $\mathcal{H}$ be a semi-inner product $A$-bimodule which is also a left $B$-module and $\Omega$ a symbol that commutes with $A$ such that $\langle \Omega, \Omega \rangle = 1$ and $\langle \Omega, \Omega \rangle : \mathcal{L}(\mathcal{H} \oplus \Omega A) \rightarrow A$ is a $B$-bimodule map.

Let $T, \Lambda \in \mathcal{L}(\mathcal{H} \oplus \Omega A)$ be selfadjoint operators such that $T(\Omega A) = 0, T(\mathcal{H}) \subseteq \mathcal{H}$, $\Lambda(\mathcal{H}) = 0, \Lambda(\Omega A) \subseteq \Omega A$. For $\xi \in \mathcal{H}$ define the operators $a_\xi, a_\xi^* \in \mathcal{L}(\mathcal{H} + \Omega A)$ given by

$$
\begin{align*}
\left\{ \begin{array}{ll}
a_\xi\alpha = 0 & \alpha \in A \\
a_\xi\eta = \langle \eta, \xi \rangle & \eta \in \mathcal{H}
\end{array} \right.
\quad \left\{ \begin{array}{ll}
a_\xi^*\alpha = \xi\alpha & \alpha \in A \\
a_\xi^*\eta = 0 & \eta \in \mathcal{H}
\end{array} \right.
\end{align*}
$$

Then $a_\xi, a_\xi^*$ are adjoint to each other and the boolean cumulants $\{B_n, V\}_n$ of $V = a_\xi + a_\xi^* + T + \Lambda$ with respect to the map $\langle \Omega, \Omega \rangle$ are given by:

$(i)$ $B_{1, V}(b_1) = \langle \Lambda b_1 \Omega, \Omega \rangle$

$(ii)$ $B_{n, V}(b_1, \ldots, b_n) = \langle a_\xi^* b_n T b_{n-1} \cdots T b_2 a_\xi^* b_1 \Omega, \Omega \rangle$ if $n \geq 2$.

**Proof.** The fact that $a_\xi$ and $a_\xi^*$ are adjoint to each other is just a trivial computation. For (i), note that $b_1 \Omega \in \Omega A$, hence $\langle V b_1 \Omega, \Omega \rangle = \langle \Lambda b_1 \Omega, \Omega \rangle$.

For (ii), remark first that

$$
a_\xi^* b_n T b_{n-1} \cdots T b_2 a_\xi^* b_1 \Omega \subseteq \Omega A
$$

for all $b_1, \ldots, b_n \in B$, since $a_\xi^* b_1 \Omega = \xi b_1 \in \mathcal{H}$, also $b_n T b_{n-1} \cdots T b_2 \mathcal{H} \subseteq \mathcal{H}$ and $a_\xi \mathcal{H} \subseteq \Omega A$.

We have that

$$
\langle V b_n \cdots V b_1 \Omega, \Omega \rangle = \sum_{V_j \in \{a_\xi, a_\xi^*; T, \Lambda\}} \langle V_n b_n \cdots V_1 b_1 \Omega, \Omega \rangle.
$$

Let us suppose that the term $\langle V_n b_n \cdots V_1 b_1 \Omega, \Omega \rangle$ does not cancel. Since $a_\xi(\Omega A) = T(\Omega A) = 0$, it follows that $V_1 \in \{a_\xi^*, \Lambda\}$. If $V_1 = a_\xi^*$, then $V_1 b_1 \Omega \in \mathcal{H}$ and, since

$$a_\xi^* \mathcal{H} = \Lambda \mathcal{H} = 0$$

it follows that $V_2 \in \{a_\xi, T\}$.

Also, since $T \mathcal{H} \subseteq \mathcal{H}$, if $V_2 = V_3 = \cdots = V_p = T$, then $V_{p+1} \in \{a_\xi, T\}$. Finally, note that $T b_n \cdots T b_2 a_\xi^* \xi b_1 \Omega \in \mathcal{H}$, henceforth

$$
\langle V b_n \cdots V b_1 \Omega, \Omega \rangle = \langle V b_n \cdots V b_2 \Lambda b_1 \Omega, \Omega \rangle
$$

$$
+ \sum_{p=2}^{n} \langle V b_n \cdots V b_{p+1} a_\xi b_p T b_{p-1} \cdots T b_2 a_\xi^* \xi b_1 \Omega, \Omega \rangle
$$

$$
= \langle V b_n \cdots V b_2 \Omega, \Omega \rangle \cdot \langle \Lambda b_1 \Omega, \Omega \rangle
$$

$$
+ \sum_{p=2}^{n} \langle V b_n \cdots V b_{p+1} \Omega, \Omega \rangle \cdot \langle a_\xi b_p T b_{p-1} \cdots T b_2 a_\xi^* \xi b_1 \Omega, \Omega \rangle
$$

and the conclusion follows from (i) and Definition 2.6. \hfill \Box

**Proof of the Theorem 2.5**
Consider $\mathcal{K} = \mathcal{B}(\mathcal{X}) \otimes_B \mathcal{D}$ as for Remark 2.3. Denote by $\mathcal{K}^0$ the complement of $1 \otimes_D$ in $\mathcal{K}$ and let $\xi = \mathcal{X} \otimes 1 - 1 \otimes \mu(\mathcal{X}) \in \mathcal{K}^0$. Define the operator $a_\xi : \mathcal{K} \longrightarrow \mathcal{K}$ by

$$a_\xi(1 \otimes 1) = 0$$

$$a_\xi \eta = 1 \otimes \langle \eta, \xi \rangle \text{ for } \eta \in \mathcal{K}^0$$

Remark that $a_\xi$ is $\mathcal{D}$-linear, adjointable, and its adjoint is given by

$$a_\xi^*(1 \otimes 1) = \xi 1$$

$$a_\xi^* \eta = 0 \text{ for } \eta \in \mathcal{K}^0$$

Let also let $\mathcal{I}_\mathcal{X} : \mathcal{K} \longrightarrow \mathcal{K}$ be the selfadjoint map given by $f(\mathcal{X}) \otimes d \mapsto \mathcal{X} f(\mathcal{X}) \otimes d$.

We will identify $\mathcal{B}$ with a subalgebra of $\mathcal{L}(\mathcal{K})$ via

$$b[f(\mathcal{X}) \otimes d] = [b f(\mathcal{X})] \otimes d.$$  

Note that $\langle 1 \otimes 1, 1 \otimes 1 \rangle$ is a $\mathcal{B}$-bimodule map.

We have that

$$\langle \mathcal{I}_\mathcal{X} b_1 \cdots \mathcal{I}_\mathcal{X} b_n (1 \otimes 1), 1 \otimes 1 \rangle = \langle \mathcal{X} b_1 \mathcal{X} b_2 \cdots \mathcal{X} b_n \otimes, 1 \otimes 1 \rangle = \mu(\mathcal{X} b_1 \mathcal{X} b_2 \cdots \mathcal{X} b_n)$$

and hence the distribution of $\mathcal{I}_\mathcal{X}$ with respect to $\langle 1 \otimes 1, 1 \otimes 1 \rangle$ coincides with $\mu$.

Consider the selfadjoint map $\Lambda_\mu \in \mathcal{L}(\mathcal{K})$ given by

$$\begin{cases} 
\Lambda_\mu(1 \otimes d) = 1 \otimes \mu(\mathcal{X}) d, \\
\Lambda_\mu \eta = 0, 
\end{cases} \quad d \in \mathcal{D} \quad \eta \in \mathcal{K}^0$$

and define $T \in \mathcal{L}(\mathcal{K})$ as the map $T = \mathcal{I}_\mathcal{X} - (a_\xi + a_\xi^*) - \Lambda_\mu$. We have that $T$ is selfadjoint, $T \mathcal{K}^0 \subseteq \mathcal{K}^0$ and $T(1 \otimes \mathcal{D}) = 0$.

Since $\mathcal{I}_\mathcal{X} = a_\xi + a_\xi^* + \Lambda_\mu + T$ and its distribution with respect to $\langle 1 \otimes 1, 1 \otimes 1 \rangle$ is $\mu$ from Lemma 2.9 we have that the boolean cumulants of $\mu$ are given by

$$B_{1, \mu}(b_1) = \langle \Lambda_\mu b_1 (1 \otimes 1), 1 \otimes 1 \rangle$$

$$B_{n, \mu}(b_n, \ldots, b_1) = \langle a_\xi b_n T b_{n-1} \cdots T b_2 a_\xi^* b_1 (1 \otimes 1), 1 \otimes 1 \rangle \text{ if } n \geq 2.$$  

Fix $N$ a positive integer. Let $\xi_N = \frac{1}{\sqrt{N}} \xi$, $\Lambda_N = \frac{1}{N} \Lambda_\mu$ and $Y_N \in \mathcal{L}(\mathcal{K})$ be the selfadjoint operator

$$Y_N = a_{\xi_N} + a_{\xi_N^*} + \Lambda_N + T.$$  

Define $\mu_N \in \Sigma_{\mathcal{B}, \mathcal{D}}$ via

$$\mu_N(f(\mathcal{X})) = \langle f(Y_N) (1 \otimes 1), 1 \otimes 1 \rangle.$$  

From Lemma 2.9 the boolean cumulants of $\mu_N$ are given by:

$$B_{1, \mu_N}(b_1) = \langle \Lambda_N b_1 (1 \otimes 1), 1 \otimes 1 \rangle = \frac{1}{N} \langle \Lambda_\mu b_1 (1 \otimes 1), 1 \otimes 1 \rangle = \frac{1}{N} B_{1, \mu}(b_1),$$

$$B_{n, \mu_N}(b_n, \ldots, b_1) = \langle a_{\xi_N} b_1 T b_2 \cdots T b_{n-1} a_{\xi_N}^* b_n (1 \otimes 1), 1 \otimes 1 \rangle = \frac{1}{N} \langle a_{\xi} b_1 T b_2 \cdots T b_{n-1} a_{\xi}^* b_n (1 \otimes 1), 1 \otimes 1 \rangle = \frac{1}{N} B_{n, \mu}(b_1, \ldots, b_n)$$

and the conclusion follows from Remark 2.8.
3. Infinite divisibility: the Free case

3.1. Preliminaries.

**Definition 3.1.** (see [27]) Let $\mathcal{B}$ be a unital $C^*$-algebra, $\mathcal{B} \subseteq \mathcal{A}$ be a unital inclusion of $*$-algebras and $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a positive conditional expectation. A family $\{X_i\}_{i \in I}$ of selfadjoint elements from $\mathcal{A}$ is said to be free with respect to $\phi$ if

$$\phi(A_1 A_2 \cdots A_n) = 0$$

whenever $A_j \in \mathcal{B} \langle X_{\epsilon(j)} \rangle$ with $\epsilon(k) \neq \epsilon(k + 1)$ and $\phi(A_j) = 0$.

In the above setting, let $X$ be a selfadjoint element from $\mathcal{A}$. The free cumulants of $X$ are the multilinear functions $\kappa_{n,X} : \mathcal{B}^n \rightarrow \mathcal{B}$ given by the recurrence:

$$\phi(X_{b_1} X_{b_2} \cdots X_{b_n}) = \sum_{p=1}^n \sum_{1 < j_1 < \cdots < j_{p-1}} \kappa_{p,X}((b_{j_1} \phi(X_{b_{j_2}} \cdots X_{b_{j_{p-1}}}), (b_{j_1+1} \phi(X_{b_{j_2}+1} \cdots X_{b_{j_{p-1}}})), \ldots, (b_{j_p} \phi(X_{b_{j_p+1}} \cdots X_{b_n}))).$$

More intuitively, the above relation can be graphically illustrated by the picture below, where the boxes stand for the application of $\phi$ and the oblique lines signify that each $Y_s = \phi(X_{b_{j_s}+2} \cdots X_{b_{j_s+1}})$ are multiplied with $b_{j_s+1}$ in the arguments of the free cumulants:

![Free cumulant diagram]

The free cumulants have the following additivity property (see [24], [17]):

**Proposition 3.2.** If $X, Y$ are free in the sense of Definition 3.1, then

$$\kappa_{n,X+Y} = \kappa_{n,X} + \kappa_{n,Y}.$$

Let now $N \in \mathbb{N}$ and $\{\mu_j\}_{j=1}^N$ be a family of elements from $\Sigma_B$. We define their additive free convolution similarly to be boolean case: Consider the symbols $\{X_j\}_{j=1}^N$; on the algebra $\mathcal{B}(X_1, X_2, \ldots, X_N)$ take the conditional expectation $\mu$ such that $\mu \circ \tau_{X_j} = \mu_j$ and the mixed moments of $X_1, \ldots, X_n$ are computed via the rules from Definition 3.1. The free additive convolution of $\{\mu_j\}_{j=1}^N$ is the conditional expectation

$$\boxplus_{j=1}^N \mu_j = \mu \circ \tau_{X_1 + X_2 + \cdots + X_N} : \mathcal{B}(X_1 + X_2 + \cdots + X_N) \cong \mathcal{B}(X) \rightarrow \mathcal{B}.$$

We have that $\boxplus_{j=1}^N \mu_j$ is also an element of $\Sigma_B$: in [24], Theorem 3.5.6, it is shown that, defined as above, is a positive conditional expectation, therefore so is $\mu \circ \tau_{X_1 + X_2 + \cdots + X_N}$.

**Definition 3.3.** An element $\mu \in \Sigma_B$ is said to be $\boxplus$-infinite divisible if for any positive integer $N$ there exists $\mu_N \in \Sigma_B$ such that $\mu$ is the free additive convolution of $N$ copies of $\mu_N$. 
3.2. Free cumulants and $\boxplus$-infinite divisibility.

Definition 3.4. Let $\mu \in \Sigma_\mathcal{B}$. Using the relations from Section 3.1, we define the free cumulants of $\mu$ as the multilinear maps $\kappa_{n,\mu} : \mathcal{B}^n \rightarrow \mathcal{B}$ given by the recurrence:

$$
\mu(Xb_1Xb_2 \cdots Xb_n) = \sum_{p=1}^n \sum_{1<j_1<\cdots<j_{p-1}} \kappa_{p,\mu}((b_1 \mu(Xb_2 \cdots Xb_{j_1-1}), (b_{j_1+1} \mu(Xb_{j_2+2} \cdots Xb_{j_2})), \ldots, (b_{j_p} \mu(Xb_{j_p+1} \cdots Xb_n))).
$$

Remark 3.5. Using Proposition 3.2, we can reformulate Definition 3.4 in terms of free cumulants. More precisely, $\mu \in \Sigma_\mathcal{B}$ is $\boxplus$-ininitely divisible if for any positive integer $N$ there exists some $\mu_N \in \Sigma_\mathcal{B}$ such that for all $m$

$$
\kappa_{m,\mu} = N \cdot \kappa_{m,\mu_N}.
$$

For $\mu \in \Sigma_\mathcal{B}$, define the conditional expectation $\rho_\mu : \mathcal{B}(X) \rightarrow \mathcal{B}$ generated by

$$
\rho_\mu(Xb_1Xb_2 \cdots Xb_n) = \kappa_{n,\mu}(b_1, b_2, \ldots, b_n).
$$

Proposition 3.6. If $\mu \in \Sigma_\mathcal{B}$ is $\boxplus$-ininitely divisible then the restriction of $\rho_\mu$ to $\mathcal{B}(X)_0$ is positive.

Proof. Fix $N$ and suppose that $\mu$ is the free additive convolution of $N$ copies of $\mu_N$. Note that, for $n \leq 1$,

$$
(6) \quad \mu_N(Xb_1 \cdots Xb_n) = \frac{1}{N} \kappa_{n,\mu}(b_1, \ldots, b_n) + O\left(\frac{1}{N^2}\right).
$$

The assertion is trivial for $n = 1$. Suppose that (6) is true for $n < m$. Since the free cumulants are multilinear, for all $1 = l_1 < l_2 < \cdots < l_{p+1} = m + 1$ and $Y_s = b_{l_s} \mu_N(Xb_{l_s+1} \cdots Xb_{l_{s+1}-1})$, $(1 \leq s \leq p)$ we have that

$$
Y_s = \left\{ \begin{array}{ll}
    b_{l_s} & \text{if } l_{s+1} = l_s + 1 \\
    \mathcal{O}\left(\frac{1}{N}\right) & \text{if } l_{s+1} = l_s + 1
\end{array} \right.
$$

hence

$$
\kappa_{p,\mu_N}(Y_1, \ldots, Y_p) = \frac{1}{N} \kappa_{p,\mu}(Y_1, \ldots, Y_p) = \mathcal{O}\left(\frac{1}{N^2}\right) \text{ unless } l_{s+1} = l_s + 1 \text{ for all } s \in \{1, \ldots, p\}, \text{ i. e. } p = m.
$$

Definition 3.4 gives

$$
\mu_N(Xb_1 \cdots Xb_m) = \kappa_{m,\mu_N}(b_1, \ldots, b_m) + \mathcal{O}\left(\frac{1}{N^2}\right) = \frac{1}{N} \kappa_{m,\mu}(b_1, \ldots, b_m) + \mathcal{O}\left(\frac{1}{N^2}\right)
$$

that is (6). It follows that

$$
(7) \quad \lim_{N \rightarrow \infty} N \cdot \mu_N(Xb_1 \cdots Xb_n) = \kappa_{n,\mu}(b_1, \ldots, b_n)
$$

Fix now $f(X) \in \mathcal{B}(X)_0$. Then

$$
\rho_\mu(f(X)^* f(X)) = \lim_{N \rightarrow \infty} N \cdot \mu_N(f(X)^* f(X)) \geq 0 \text{ since all } \mu_N \text{ are positive}.
$$

Lemma 3.7. Let $\mathcal{A}$ be a unital $C^*$-algebra and $\mathcal{H}$ a semi-inner-product $\mathcal{A}$-module which is also a left $\mathcal{A}$-module. Let $\mathcal{T}(\mathcal{H})$ be the full Fock $\mathcal{A}$-module over $\mathcal{H}$, that is

$$
\mathcal{T}(\mathcal{H}) = \mathcal{A} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes_\mathcal{B} \mathcal{H}) \oplus (\mathcal{H} \otimes_\mathcal{B} \mathcal{H} \otimes_\mathcal{B} \mathcal{H}) \oplus \ldots
$$
Fix $\xi \in \mathcal{H}$, $\beta \in A$, $T \in \mathcal{L}(\mathcal{H})$, $T$ and $\beta$ selfadjoint, and define the maps $(\alpha \in A, \eta_1, \ldots, \eta_n \in \mathcal{H})$

\[
\begin{cases}
a_\xi \alpha = 0 \\
 a_\xi \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n = \langle \eta, \xi \rangle \eta_2 \otimes \cdots \otimes \eta_n \\
a_\xi^* \alpha = \xi \alpha \\
a_\xi \eta_1 \otimes \cdots \otimes \eta_n = \xi \otimes \eta_1 \otimes \cdots \otimes \eta_n \\
p(T) \alpha = 0 \\
p(T) \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n = T(\eta_1) \otimes \eta_2 \otimes \cdots \otimes \eta_n
\end{cases}
\]

Then $a_\xi$, $a_\xi^*$ are adjoint to each other, $p(T)$ is selfadjoint and the cumulants of $X = a_\xi + a_\xi^* + T + \beta \chi_A$ with respect to the conditional expectation $\langle 1,1 \rangle$ are given by

\[
\begin{align*}
\kappa_{1,X}(b_1) &= \beta b_1 \\
\kappa_{n,X}(b_1, \ldots, b_n) &= \langle a_\xi b_1 p(T)b_2 \cdots p(T)b_{n-1} a^*\xi b_1, 1 \rangle.
\end{align*}
\]

Proof. For $\kappa_{1,X}$ the assertion is trivial. To prove the relation for higher order free cumulants, let us fix $N > 1$ and consider \{(\mathcal{H}_{j,N}, T_{j,N}, \xi_{j,N})\}_{j=1}^N$ to be a set of $N$ identical copies of $(\mathcal{H}, T, (\frac{1}{\sqrt{N}}\xi))$ from above. Let

\[
\begin{align*}
\mathcal{H}_N &= \mathcal{H}_{1,N} \oplus \mathcal{H}_{2,N} \oplus \cdots \oplus \mathcal{H}_{N,N} \\
\xi_N &= \xi_{1,N} \oplus \xi_{2,N} \oplus \cdots \oplus \xi_{N,N} \in \mathcal{H}_N \\
T_N &= T_{1,N} \oplus \cdots \oplus T_{N,N} \in \mathcal{L}(\mathcal{H}_N) \\
X_N &= a_{\xi,N} + a_{\xi,N}^* + p(T_N) + \beta \chi_A \in \mathcal{L}(\mathcal{H}_N) \\
X_{j,N} &= a_{\xi_{j,N}} + a_{\xi_{j,N}}^* + p(T_{j,N}) + \frac{1}{N^2} \beta \chi_A \in \mathcal{L}(\mathcal{H}_N).
\end{align*}
\]

First, note that $X$ and $X_N$ are identically distributed with respect to $\langle 1,1 \rangle$. Then $X_N = X_{1,N} + \cdots + X_{N,N}$, $X_{j,N}$ are identically distributed and free. To see that, consider $A_1, \ldots, A_m, A_k \in A(\mathcal{H}_{k,N})$ with $\langle A_k 1, 1 \rangle = 0$ and $\epsilon(k) \neq \epsilon(k+1)$. We have to show that $\langle A_1 \cdots A_{1,1} 1 \rangle = 0$.

Since $\langle A_1 1, 1 \rangle = 0$, we have that $A_1 1 \in T(\mathcal{H}_{(1),N}) \ominus A$. Also, $\langle A_2 1, 1 \rangle = 0$, so $A_2 A_1 1 \in (T(\mathcal{H}_{(2),N}) \ominus A) \otimes (T(\mathcal{H}_{(1),N}) \ominus A)$. Iterating, we obtain

\[
A_m \cdots A_2 A_1 1 \in (T(\mathcal{H}_{(m),N}) \ominus A) \otimes \cdots \otimes (T(\mathcal{H}_{(1),N}) \ominus A)
\]

so $\langle A_m \cdots A_2 A_1, 1 \rangle = 0$.

Using (1) and denoting $\mathcal{V} = \left\{ \frac{1}{\sqrt{N}} a_\xi, \frac{1}{\sqrt{N}} a_\xi^*, p(T), \frac{1}{N} \beta \chi_A \right\}$ we have $(n \geq 2)$:

\[
\begin{align*}
\kappa_{n,X}(b_1, \ldots, b_{n}) &= \kappa_{n,X_N}(b_1, \ldots, b_{n}) \\
&= \lim_{N \to \infty} N \langle X_N b_1 \cdots X_N b_n, 1 \rangle \\
&= \sum_{V \in \mathcal{V}} \langle V_1 b_1 \cdots V_n b_n, 1 \rangle
\end{align*}
\]

But $\langle V_1 b_1 \cdots V_n b_n, 1 \rangle = 0$ unless $V_n \in \left\{ \frac{1}{\sqrt{N}} a_\xi, \frac{1}{N} \beta \chi_A \right\}$ and $V_1 \in \left\{ \frac{1}{\sqrt{N}} a_\xi^*, \frac{1}{N} \beta \chi_A \right\}$, therefore

\[
\langle X_N b_1 \cdots X_N b_n, 1 \rangle = \langle \frac{1}{\sqrt{N}} a_\xi b_1 p(T)b_2 \cdots p(T)b_{n-1} a_\xi^* b_1, 1 \rangle + O(\frac{1}{N^2})
\]

hence the conclusion. \qed
Theorem 3.8. The conditional expectation \( \mu \in \Sigma_B \) is \( \boxplus \)-ininitely divisible if and only if the restriction of \( \rho_\mu \) to \( B(\mathcal{X})_0 \) is positive.

Proof. Suppose that \( \rho_\mu|B(\mathcal{X})_0 \) is positive. Then (see [3], pag. 42) \( B(\mathcal{X})_0 \) is a semi-inner-product \( B \)-module with respect to the pairing

\[
\langle f(\mathcal{X}), g(\mathcal{X}) \rangle = \rho_\mu(g(\mathcal{X})^* f(\mathcal{X})).
\]

Consider the selfadjoint map \( T : B(\mathcal{X})_0 \to B(\mathcal{X})_0 \) given by \( T(f(\mathcal{X})) = Xf(\mathcal{X}) \) and denote by \( V \) the map from \( \mathcal{L}(T(B(\mathcal{X})_0)) \) defined as

\[
V = a_X + a_X^* + \bar{T} + \frac{1}{N} \mu(\mathcal{X}) \text{Id}.
\]

From Lemma 3.7 we have that the free cumulants of \( V \) with respect to \( \langle \cdot, 1 \rangle \) are given by

\[
\kappa_{1,V}(b_1) = \mu(X)(b_1) = \kappa_{1,\mu}(b_1)
\]

\[
\kappa_{n,V}(b_1, \ldots, b_n) = \langle a_X b_1 \bar{T} b_2 \cdots \bar{T} b_{n-1} a_X b_{n} 1, 1 \rangle = \langle b_1 X b_2 \cdots X b_n, \mathcal{X} \rangle = \kappa_{\mu}(b_1, \ldots, b_n)
\]

Fix a positive integer \( N \). From Remark 3.5 it suffices to find a selfadjoint element from \( \mathcal{L}(T(B(\mathcal{X})_0)) \) whose free cumulants are proportional to the free cumulants of \( V \) by a factor of \( \frac{1}{N} \). Define

\[
V_N = \frac{1}{\sqrt{N}} a_X + \frac{1}{\sqrt{N}} a_X^* + \bar{T} + \frac{1}{N} \mu(\mathcal{X}) \text{Id}.
\]

Applying again Lemma 3.7 the free cumulants of \( V_N \) are

\[
\kappa_{1,V_N}(b_1) = \frac{1}{N} \mu(X)(b_1) = \frac{1}{N} \kappa_{1,V}(b_1)
\]

\[
\kappa_{n,V_N}(b_1, \ldots, b_n) = \frac{1}{\sqrt{N}} \langle a_X b_1 \bar{T} b_2 \cdots \bar{T} b_{n-1} a_X b_{n} 1, 1 \rangle = \frac{1}{N} \kappa_{n,V}(b_1, \ldots, b_n)
\]

The converse implication is Proposition 3.6.

\[ \square \]

4. Infinite divisibility: The c-free case

In this section we aim to extend the results from Section 3 to the case \( \mu \in \Sigma_{B,D} \). First we will need a suitable definition for the free additive convolution of elements from \( \Sigma_{B,D} \); in this setting, if \( B \) is simply replaced by \( D \) in Definition 3.1 the resulting relation does not uniquely determine the joint moments of \( X_1, \ldots, X_n \). As shown in [5], [8], a more suitable approach is the conditional freeness (see also [17], [8]).

Definition 4.1. Let \( B \) be a unital C*-algebra \( B \subseteq A \), \( B \subseteq D \) be unital inclusions of \( * \)-algebras, \( \varphi : A \to B \) a conditional expectation and \( \theta : A \to D \) be a unital \( B \)-bimodule map.

The family \( \{X_i\}_{i \in I} \) of selfadjoint elements from \( A \) is said to be c-free with respect to \( (\varphi, \theta) \) if

(i) the family \( \{X_i\}_{i \in I} \) is free with respect to \( \varphi \)

(ii) \( \theta(A_1 A_2 \cdots A_n) = \theta(A_1) \theta(A_2) \cdots \theta(A_n) \) for all \( A_i \in B(X_{\iota(i)}) \) such that \( \varphi(A_i) = 0 \) and \( \epsilon(k) \neq \epsilon(k + 1) \).

\[ \square \]
Let $X$ be a selfadjoint element from $A$. The $c$-free cumulants of $X$ are the multilinear functions $c_{\kappa_{n,X}} : \mathcal{B}^n \rightarrow \mathcal{D}$ given by the recurrence:

$$\theta(X b_1 X b_2 \cdots X b_n) = \sum_{p=1}^n \sum_{1 < j_1 < \cdots < j_p}^{i_p = n-1} \theta(X b_1 \cdots X b_{j_1}) \cdot c_{\kappa_{p,X}}(b_{j_1+1} \varphi(X b_{j_1+2} \cdots X b_{l_1}), \ldots, b_{l_p-1+1} \varphi(X b_{l_p-1+2} \cdots X b_{l_p}), b_n)$$

As in the previous section, the above equation can be represented more intuitively by the picture below, where the dark boxes stand for the application of $\theta$, the light ones for the application of $\varphi$ and the oblique lines signify that each $Y_s = \phi(X b_{l_1+2} \cdots X b_{l_1+1})$ are multiplied with $b_{l_1+1}$ in the arguments of the $c$-free cumulants, except for $b_n$.

The $c$-free cumulants have the following additivity property (see [17], [15]):

**Proposition 4.2.** If $X, Y$ are $c$-free with respect to $(\theta, \varphi)$ in the sense of Definition 4.1, then

$$\kappa_{n,X+Y} = \kappa_{n,X} + \kappa_{n,Y}$$

$$c_{\kappa_{n,X+Y}} = c_{\kappa_{n,X}} + c_{\kappa_{n,Y}}$$

where $\kappa_{n,X}$ is the $n$-th free cumulant of $X$ with respect to the conditional expectation $\varphi$.

Let now $N$ be a positive integer and $\{(\mu_i, \nu_i)\}_{i=1}^N$ be a family from $\Sigma_{\mathcal{B} : \mathcal{D}} \times \Sigma_{\mathcal{B}}$. We define their additive $c$-free convolution similarly to be boolean and free case: Consider the selfadjoint symbols $\{X_i\}_{i=1}^N$ and the mappings

$$\mu : \mathcal{B}\langle X_1, X_2, \ldots, X_N \rangle \rightarrow \mathcal{D}$$

$$\nu : \mathcal{B}\langle X_1, X_2, \ldots, X_N \rangle \rightarrow \mathcal{B}$$

such that such that $\mu \circ \tau_{X_i} = \mu_i$ and $\nu \circ \tau_{X_i} = \nu_i$ for all $i = 1, \ldots, N$ and the mixed moments of $\mu$ and $\nu$ are computed according to Definition 4.1. The $c$-free additive convolution of $\{(\mu_i, \nu_i)\}_{i=1}^N$ is the pair $(\mu_c, \nu_c) = \bigotimes_{i=1}^N (\mu_i, \nu_i)$, where

$$\nu_c = \nu \circ \tau_{X_1+X_2+\cdots+X_N} = \bigoplus_{i=1}^N \nu_i \in \Sigma_{\mathcal{B}}$$

$$\mu_c = \mu \circ \tau_{X_1+X_2+\cdots+X_N} : \mathcal{B}\langle X_1 + X_2 + \cdots + X_N \rangle \cong \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{D}$$

**Definition 4.3.** A pair $(\mu, \nu) \in \Sigma_{\mathcal{B} : \mathcal{D}} \times \Sigma_{\mathcal{B}}$ is said to be $c$-infinite divisible if for any positive integer $N$ there exists $(\mu_N, \nu_N) \in \Sigma_{\mathcal{B} : \mathcal{D}} \times \Sigma_{\mathcal{B}}$ such that $(\mu, \nu)$ is the $c$-free additive convolution of $N$ copies of $(\mu_N, \nu_N)$.

4.1. C-free cumulants and infinite divisibility.
Definition 4.4. The c-free cumulants of the pair \((\mu, \nu) \in \Sigma_{\mathcal{B}, \mathcal{D}} \times \Sigma_{\mathcal{B}}\) are the multilinear functions \(\kappa_{\mu, \nu} : \mathcal{B}^n \to \mathcal{D}\) given by the recurrence:

\[
\kappa_{\mu, \nu}(X_{b_1} \cdots X_{b_n}) = \sum_{p=1}^n \sum_{l_p=n-1}^{n} \mu(X_{b_1} \cdots X_{b_{l_p}}) \cdot \kappa_{\mu, \nu}(X_{b_{l_p+1}} \cdots X_{b_n})
\]

Remark 4.5. As in the previous section, can reformulate Definition 4.4 in terms of free and c-free cumulants. More precisely, the pair \((\mu, \nu)\) is \(\mathcal{C}\)-infinitely divisible if for any positive integer \(N\) there exists some \((\mu_N, \nu_N) \in \Sigma_{\mathcal{B}, \mathcal{D}} \times \Sigma_{\mathcal{B}}\) such that for all \(m\) we have that \(\kappa_{m, \nu_N} = N \kappa_{m, \nu_N}\) and \(\kappa_{m, \nu} = N^{\epsilon} \kappa_{m, \nu}\).

Define the map \(\rho_{\mu, \nu} : \mathcal{B}(\mathcal{X}) \to \mathcal{D}\) as the \(\mathcal{B}\)-bimodule extension of

\[
\rho_{\mu, \nu}(X_{b_1} \cdots X_{b_n}) = \kappa_{\mu, \nu}(b_1, \ldots, b_n).
\]

Proposition 4.6. Suppose that \((\mu, \nu) \in \Sigma_{\mathcal{B}, \mathcal{D}} \times \Sigma_{\mathcal{B}}\) is \(\mathcal{C}\)-infinitely divisible. Then the restriction of \(\rho_{\mu, \nu}\) to \(\mathcal{B}(\mathcal{X})_0\) satisfies property (\(\square\)) (see Introduction).

Proof. Fix \(N > 1\) and suppose that \((\mu, \nu)\) is the c-free additive convolution of \(n\) copies of \((\mu_N, \nu_N)\). As in the proof of Proposition 3.6, we will first show that

\[
\mu_N(X_{b_1} \cdots X_{b_n}) = \frac{1}{N} \kappa_{\mu_N, \nu_N}(b_1, \ldots, b_n) + O\left(\frac{1}{N^2}\right).
\]

For \(n = 1\) the assertion is trivial. Suppose that \((\square)\) holds true for \(n < m\). Since the c-free cumulants of are multilinear, for all \(1 = l_1 < l_2 < \cdots < l_{p+1} < m\) and \(Y_s = \nu_N(X_{b_{l_s+1}} \cdots X_{b_{l_{s+1}-1}}), (1 \leq s \leq p)\) we have that

\[
Y_s = \begin{cases} 
    b_{l_s} & \text{if } l_{s+1} = l_s + 1 \\
    O\left(\frac{1}{N}\right) & \text{if } l_{s+1} = l_s + 1
\end{cases}
\]

hence \(\kappa_{\mu_N, \nu_N}(Y_1, \ldots, Y_p) = \frac{1}{N} \kappa_{\mu_N, \nu}(Y_1, \ldots, Y_p) = O\left(\frac{1}{N^2}\right)\), unless \(l_{s+1} = l_s + 1\) for all \(s \in \{1, \ldots, p\}\).

Definition 4.4 gives

\[
\mu_N(X_{b_1} \cdots X_{b_m}) = \kappa_{\mu_N, \nu_N}(b_1, \ldots, b_m) + \sum_{s=1}^{m-1} \kappa_{\mu_N, \nu_N}(b_1, \ldots, b_s) \mu_N(X_{b_{s+1}} \cdots X_{b_m}) + O\left(\frac{1}{N^2}\right)
\]

hence \((\square)\) follows from the induction hypothesis. Therefore

\[
\lim_{N \to \infty} N \mu_N(X_{b_1} \cdots X_{b_n}) = \kappa_{\mu_N, \nu_N}(b_1, \ldots, b_n)
\]

Fix now a family \(\{f_i(\mathcal{X})\}_{i=1}^n\) in \(\mathcal{B}(\mathcal{X})_0\). Then

\[
[\rho_{\mu, \nu}(f_j(\mathcal{X})^* f_i(\mathcal{X}))]^n_{i,j=1} = \lim_{N \to \infty} [N \mu_N(f_j(\mathcal{X})^* f_i(\mathcal{X}))]^n_{i,j=1} \geq 0.
\]

since each \(\mu_N\) satisfies \((\square)\). \(\square\)
Lemma 4.7. Let $\mathcal{B} \subseteq \mathcal{D}$ be unital inclusion of $C^*$-algebras, $\mathcal{K}$ be a semi-inner-product $\mathcal{D}$-bimodule and $\mathcal{H}$ be a semi-inner-product $\mathcal{B}$-bimodule. Consider

$$
\mathcal{E} = (\mathcal{T}(\mathcal{H}) \otimes_{\mathcal{B}} \mathcal{D}) \oplus (1_{\mathcal{B}} \otimes_{\mathcal{B}} \omega \mathcal{D}) \oplus (\mathcal{T}(\mathcal{H}) \otimes_{\mathcal{B}} \mathcal{K}).
$$

Fix $\xi \in \mathcal{H}$, $\eta \in \mathcal{K}$ and $t \in \mathcal{L}(\mathcal{H})$, $T \in \mathcal{L}(\mathcal{K})$ selfadjoints. Define the maps $a_\xi, a_\xi^*, A_\eta, A_\eta^*$, $p(t)$, $P(T)$ from $\mathcal{L}(\mathcal{E})$ given by:

\[
\begin{align*}
\{a_\xi|_{1_{\mathcal{B}} \otimes \omega \mathcal{D}} & = 0 \\
a_\xi(f_1 \otimes \ldots \otimes f_n) \otimes d = ((f_1, \xi f_2 \otimes \ldots \otimes f_n) \otimes d \\
a_\xi(f_1 \otimes \ldots \otimes f_n) \otimes \eta = ((f_1, \xi f_2 \otimes \ldots \otimes f_n) \otimes \eta & = 0 \\
a_\xi(f_1 \otimes \ldots \otimes f_n) \otimes d = ((f_1, \xi f_2 \otimes \ldots \otimes f_n) \otimes d \\
a_\xi(f_1 \otimes \ldots \otimes f_n) \otimes \eta = ((f_1, \xi f_2 \otimes \ldots \otimes f_n) \otimes \eta & = 0 \\
p(t)(f_1 \otimes \ldots \otimes f_n) \otimes d = (tf_1) \otimes f_2 \otimes \ldots \otimes f_n) \otimes d \\
p(t)(f_1 \otimes \ldots \otimes f_n) \otimes \eta = (tf_1) \otimes f_2 \otimes \ldots \otimes f_n) \otimes \eta & = 0 \\
A_\eta(1 \otimes \zeta) = 1 \otimes \mathcal{O}(\zeta, \eta) & = 0 \\
A_\eta(1 \otimes \Omega) = 1 \otimes \mathcal{O}(\zeta, \eta) & = 0 \\
P(T)(1 \otimes \zeta) = 1 \otimes T\zeta & = 0 \\
P(T)(1 \otimes \Omega) = 1 \otimes T\zeta & = 0
\end{align*}
\]

Define also $I_1 = Id_{(\mathcal{T}(\mathcal{H}) \otimes \mathcal{D}) \oplus (\mathcal{T}(\mathcal{H}) \otimes \mathcal{K})} \oplus 0 \in \mathcal{L}(\mathcal{E})$ and $I_2 = Id_{1_{\mathcal{B}} \otimes \omega \mathcal{D}} \oplus 0 \in \mathcal{L}(\mathcal{E})$. Consider $X = a_\xi + a_\xi^* + p(t) + \lambda_1 I_1 + A_\eta + A_\eta^* + \lambda_2 I_2$ where $\lambda_1 \in \mathcal{B}$, $\lambda_2 \in \mathcal{D}$ selfadjoint. Then the free and $c$-free cumulants of $X$ with respect to $(\theta, \varphi) = (\langle 1 \otimes \Omega, 1 \otimes \Omega \rangle, \langle 1 \otimes 1, 1 \otimes 1 \rangle)$ are given by the following relations:

\[
\begin{align*}
k_{1, X}(b_1) = & \lambda_1 b_1 \\
k_{n, X}(b_1, \ldots, b_n) = & \langle a_\xi b_1 p(t) b_2 \ldots p(t) b_{n-1} a_\xi^* b_1 (1 \otimes 1), 1 \otimes 1 \rangle \\
k_{1, X}(b_1) = & \lambda_2 b_1 \\
k_{n, X}(b_1, \ldots, b_n) = & \langle A_\eta b_1 P(T) b_2 \ldots P(T) b_{n-1} A_\eta^* b_1 (1 \otimes 1), 1 \otimes 1 \rangle
\end{align*}
\]

Proof. The results are trivial for $k_{1, X}$ and $k_{1, X}^c$. For $k_{n, X}$ note that $a_\xi, a_\xi^*, p(t)$ map $\mathcal{T}(\mathcal{H}) \otimes 1$ in $\mathcal{T}(\mathcal{H}) \otimes 1$ and, since $1 \otimes 1 \in \mathcal{T}(\mathcal{H}) \otimes 1$, the result reduces to Lemma 4.7.

To prove the formula for $k_{n, X}^c$, let us first note $V_0 = a_\xi + a_\xi^* + p(t) + \lambda_1 I_1$, $V_1 = A_\eta$, $V_2 = A_\eta$, $V_3 = P(T)$, $V_4 = \lambda_2 I_2$ and

\[
J(n) = \{ \bar{u} = (u_n, \ldots, u_1) : 0 \leq u_k \leq 4 \}.
\]

Finally, for $b_1, \ldots, b_n \in \mathcal{B}$, denote $\epsilon(\bar{u}) = \theta(V_{u_n} b_n \cdots V_{u_1} b_1)$.

Note that $V_{k|\mathcal{T}(\mathcal{H})} \equiv 0 \ (1 \leq k \leq 4)$ and $V_{0|1 \otimes \omega \mathcal{D}} \equiv 0$. The latest implies that $\epsilon(\bar{u}) = 0$ unless $V_{u_1} \in \{ A_\eta^*, \lambda_2 I_2 \}$, hence

\[
\theta(X b_n \cdots X b_1) = \sum_{\bar{u} \in J(n)} \epsilon(\bar{u}) = \sum_{\bar{u} \in J(n)} \epsilon(\bar{u}) + \sum_{\bar{u} \in J(n)} \epsilon(\bar{u})
\]

\[
= \sum_{\bar{u} \in J(n)} \epsilon(\bar{u}) + \sum_{\bar{u} \in J(n-1)} \theta(V_{u_{n-1}} b_n \cdots V_{u_1} b_2) \lambda_2
\]
Suppose that $V_{u_1} = A_\eta^*$; then $V_{u_1} b_1 (1 \otimes \Omega) = 1 \otimes \eta b_1$, hence $\epsilon (\vec{u})$ cancels unless $V_{u_2} \in \{ V_0, A_\eta, P(T) \}$. Let $p = \min \{ s : s > 1, u_s \neq 0 \}$. It follows that $V_{u_p} \in \{ A_\eta, P(T) \}$. Since the restrictions of $A_\eta, P(T)$ to $T(\mathcal{H}) \otimes K$ are 0, we have

$$V_{u_p} b_p \cdots V_{u_1} b_1 (1 \otimes \Omega) = V_{u_p} b_p (V_0 b_{p-1} \cdots V_0 b_2) V_{u_1} b_1 (1 \otimes \Omega) = V_{u_p} b_p \varphi (X b_{p-1} \cdots X b_2) V_{u_1} b_1 (1 \otimes \Omega).$$

If $V_{u_p} = P(T)$ and $s = \min \{ q : p < q \leq n, u_q \neq 0 \}$, from a similar argument as above we have that $u_{p+1} = \cdots = u_{q-1} = 0$ and

$$V_{u_q} b_q \cdots V_{u_1} b_1 (1 \otimes \Omega) = P(T) b_q \varphi (X b_{q-1} \cdots X b_{p+1}) P(T) b_p \varphi (X b_{p-1} \cdots X b_2) A_\eta^* b_1 (1 \otimes \Omega)$$

Note also that, for all $b_1, \ldots, b_m \in B$, one has

$$T b_m \cdots T b_2 A_\eta^* b_1 (1 \otimes \Omega) \in 1 \otimes K$$

$$A_\eta b_m T b_{m-1} \cdots T b_2 A_\eta^* b_1 (1 \otimes \Omega) \in 1 \otimes D,$$

hence $\epsilon (\vec{u})$ cancels unless there exists some $j > 1$ such that $V_{u_j} = A_\eta$.

Using the results above, we have that

$$\sum_{\vec{u} \in (J(n), u_1 = 4)} \epsilon (\vec{u}) = \sum_{s=1}^{n} \sum_{\substack{p_1 < \cdots < p_s \leq n \\ u_1 = 4, u_{p_s} = 2 \\ u_p = 1, \forall p \neq s}} \epsilon (\vec{u})$$

$$= \sum_{s=1}^{n} \sum_{\substack{p_1 < \cdots < p_s \leq n \\ u_p = 1, \forall p \neq s}} \theta (X b_n \cdots X b_{p_s}) \cdot \theta (A_\eta b_{p_s+1} \varphi (X b_{p_s-1} \cdots X b_n))$$

$$= \sum_{s=1}^{n} \sum_{\substack{p_1 < \cdots < p_s \leq n \\ u_p = 1, \forall p \neq s}} \theta (X b_n \cdots X b_{p_s}) \cdot \theta (A_\eta b_{p_s+1} \varphi (X b_{p_s-1} \cdots X b_n)) \cdot P(T) b_{p_s+1} \varphi (X b_{p_s-1} \cdots X b_2) A_\eta^* b_1)$$

Comparing these relations with the definition of the c-free cumulants, we have q.e.d.

\( Theorem 4.8. \) The pair $(\mu, \nu) \in \Sigma_{B,D} \times \Sigma_B$ is infinitely divisible if and only if $\nu$ is infinitely divisible and the restriction of $\epsilon \rho_{\mu,\nu}$ to $B(\mathcal{X})_0$ satisfies property (I).

\( Proof. \) Suppose that $\nu$ is infinitely divisible (hence, from Theorem 4.8, the restriction of $\rho_\nu$ to $B(\mathcal{X})_0$ is positive) and that the restriction of $\epsilon \rho_{\mu,\nu}$ to $B(\mathcal{X})_0$ satisfies (I).

Let $\mathcal{H}$ be the left $\mathcal{B}$-bimodule $B(\mathcal{X})_0$ with the pairing

$$\langle f(\mathcal{X}), g(\mathcal{X}) \rangle_\mathcal{H} = \rho_\nu (g(\mathcal{X})^* f(\mathcal{X}))$$

and $\mathcal{K}$ be the left $\mathcal{D}$-module $B(\mathcal{X})_0 \otimes_B \mathcal{D}$ with the pairing

$$\langle f(\mathcal{X}) \otimes d_1, g(\mathcal{X}) \otimes d_2 \rangle_\mathcal{K} = d_1 \cdot \epsilon \rho_{\mu,\nu} (g^* (\mathcal{X}) f(\mathcal{X})) d_1.$$
As in Lemma 4.7, consider
\[ E = (T(H) \otimes_B D) \oplus (1_B \otimes_B \Omega D) \oplus (T(H) \otimes_B K) \]
and \( V \in \mathcal{L}(E) \) given by
\[ V = a_X + a_X^* + p(t) + \lambda_1 I_1 + A_X \otimes 1 + A_X^* \otimes 1 + \lambda_2 I_2. \]

Denoting \( \varphi(\cdot) = \langle (1 \otimes 1), 1 \otimes 1 \rangle \) and \( \theta(\cdot) = \langle (1 \otimes \Omega), 1 \otimes \Omega \rangle \) and applying Lemma 4.7 we have that the free and c-free cumulants of \( V \) with respect to \( (\theta, \varphi) \) are given by:
\[ \kappa_{1, X}(b_1) = \lambda_1 b_1 = \rho_\nu(X) \]
\[ \kappa_{n, X}(b_1, \ldots, b_n) = \langle a_X b_1 P(t) b_2 \cdots P(t) b_{n-1} a_X^* b_n (1 \otimes 1), 1 \otimes 1 \rangle \]
\[ \kappa_{n, \nu}(b_1, \ldots, b_n) = \langle \rho_\nu(X b_1 \cdots X b_n) \otimes 1, 1 \otimes 1 \rangle \]
respectively by
\[ \kappa_{1, V}(b_1) = \lambda_2 b_1 = \kappa_{1, \nu}(b_1) \]
\[ \kappa_{n, V}(b_1, \ldots, b_n) = \langle A_X \otimes 1 P(T) b_2 \cdots P(T) b_{n-1} A_X^* \otimes 1 b_1 (1 \otimes \Omega), 1 \otimes \Omega \rangle \]
\[ \kappa_{n, \nu}(b_1, \ldots, b_n) = \langle \kappa_{n, \nu}(X b_1 \cdots X b_n) \otimes \Omega, 1 \otimes \Omega \rangle \]

Fix \( N > 0 \) and define
\[ V_N = \frac{1}{\sqrt{N}} a_X + \frac{1}{\sqrt{N}} a_X^* + p(t) + \frac{1}{N} \lambda_1 I_1 + \frac{1}{\sqrt{N}} A_X \otimes 1 + \frac{1}{\sqrt{N}} A_X^* \otimes 1 + P(T) + \frac{1}{N} \lambda_2 I_2. \]

Using again Lemma 4.7 similar computations as above give \( (n \geq 1) \):
\[ \kappa_{n, V_N} = \frac{1}{N^{\rho_\nu(X)}} \kappa_{n, V} \]
\[ \kappa_{n, V_N} = \frac{1}{N} \kappa_{n, V} \]

hence q.e.d..

The converse implication is Proposition 4.6.

5. THE NON-COMMUTATIVE R- AND cR-TRANSFORMS AND NON-COMMUTATIVE FREE LEVY-HINCIN FORMULA

5.1. The R- and cR- transforms and free infinite divisibility: scalar case.
Definition 5.1. Let \((\mu, \nu)\) be a pair of compactly supported measures on \(\mathbb{R}\). If \(M_\mu(z), M_\nu(z)\) are the moment-generating series for \(\mu\), respectively \(\nu\), that is
\[
M_\mu(z) = \sum_{n=0}^{\infty} \int_{\mathbb{R}} t^n d\mu(t) z^n \\
M_\nu(z) = \sum_{n=0}^{\infty} \int_{\mathbb{R}} t^n d\nu(t) z^n
\]
then the \(R\)-transform of \(\nu\), respectively the \(cR\)-transform of \((\mu, \nu)\) are the analytic functions \(R_\nu, cR_{\mu,\nu}\) given by
\[
R_\nu(z) - 1 = R_\nu(z M_\nu(z)) \\
(M_\mu(z) - 1) M_\nu(z) = M_\mu(z) \cdot cR_{\mu,\nu}(zM_\nu(z)).
\]
(10) 
(11)

If \(A\) is a \(\ast\)-algebra and \(\varphi : A \to \mathbb{C}\) is a positive linear functional, then a selfadjoint element \(a \in A\) determines a compactly supported real measure \(\mu_a\) via
\[
\int_{\mathbb{R}} t^n d\mu_a(t) = \varphi(a^n).
\]
For convenience, we will also use the notation \(R_a\) for \(R_{\mu_a}\), respectively the notation \(cR_a\) for \(cR_{\mu_a,\nu_a}\) for the case that the the \(\ast\)-algebra \(A\) endowed with two positive linear functionals.

The key property of the \(R\) and \(cR\)-transforms is the linearization of the free convolution \([30, 22]\), respectively c-free additive convolution: if \(a\) and \(b\) are free, respectively c-free, selfadjoint elements from \(A\), then
\[
R_{a+b}(z) = R_a(z) + R_b(z) \\
cR_{a+b} = cR_a(z) + cR_b(z)
\]
(12) 
(13)

The goal of this material is to give a non-commutative analogue for the following theorem \([22\), Theorem 13.16 and \([12\), Theorem 5]:

\textbf{Theorem 5.2.} Let \(\nu\) be a compactly supported supported probability measure on \(\mathbb{R}\) and let \(R_\nu(z) = \sum_{n=1}^{\infty} \kappa_n z^n\) be the Taylor expansion of its \(R\)-transform. Then the following statements are equivalent:

1. \(\nu\) is infinitely divisible.
2. The sequence \(\{\kappa_n\}_n \geq 2\) is positive definite, i.e. there exists some real measure \(\sigma\) such that \(\kappa_n = \int_{\mathbb{R}} t^{n-2} d\sigma\)
3. The \(R\)-transform of \(\nu\) is of the form
\[
\frac{1}{z} R_\nu(z) = \kappa_1 + \int_{\mathbb{R}} \frac{z}{1-tz} d\rho(t),
\]
for some finite measure \(\rho\) on \(\mathbb{R}\) with compact support.

Moreover, if \((\mu, \nu)\) is a pair of compactly supported measures on \(\mathbb{R}\), then \((\mu, \nu)\) is \(c\)-free infinitely divisible if and only if \(\nu\) is infinitely divisible and \(cR_{\mu,\nu}\) satisfies the condition (3) from above.
5.2. **Non-Commutative functions.** For stating our main result, we introduce the language of noncommutative [11] or fully matricial [29] functions, see also the pioneering work [25, 26]. For a vector space $V$ over $\mathbb{C}$, we let $V^{n \times m} = V \otimes \mathcal{M}_{n \times m}(\mathbb{C})$ denote $n \times m$ matrices over $V$ (in literature - for example in [11] - on $V^{n \times n}$ is used the algebra structure induced by the tensor algebra $T(V)$ over $V$; in order to avoid confusion, when $V$ is an algebra, we will use the notation $M_n(V)$ for the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$). We define the noncommutative space over $V$ by $V_{nc} = \bigcap_{n=1}^{\infty} V^{n \times n}$. We call $\Omega \subseteq V_{nc}$ a noncommutative set if it is closed under direct sums. Explicitly, denoting $\Omega_n = \Omega \cap V^{n \times n}$, we have

$$a \oplus b = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \Omega_{n+m}$$

for all $a \in \Omega_n$, $b \in \Omega_m$. Notice that matrices over $\mathbb{C}$ act from the right and from the left on matrices over $V$ by the standard rules of matrix multiplication.

A noncommutative set $\Omega \subseteq V_{nc}$ is called upper admissible if for all $a \in \Omega_n$, $b \in \Omega_m$ and all $c \in V^{n \times m}$, there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that

$$\begin{bmatrix} a & \lambda c \\ 0 & b \end{bmatrix} \in \Omega_{n+m}.$$

This notion is crucial since it is used to define the (right) noncommutative difference-differential operators by evaluating a noncommutative function on block upper triangular matrices. We will encounter only the following upper admissible noncommutative sets:

1. The set $\text{Nilp}(V) = \bigcup_{n=1}^{\infty} \text{Nilp}(V; n)$ of nilpotent matrices over $V$. Here the set $\text{Nilp}(V; n)$ of nilpotent $n \times n$ matrices over $V$ consists of all $a \in V^{n \times n}$ such that $a^r = 0$ for some $r$, where we view $a$ as a matrix over the tensor algebra $T(V)$ of $V$ over $\mathbb{C}$. This is equivalent to $tat^{-1}$ being strictly upper triangular for some $t \in M_n(\mathbb{C})$ (the equivalence follows from Engel’s Theorem — notice that we can restrict ourselves to the finite dimensional subspace of $V$ spanned by the elements of $a$).

2. A noncommutative ball $B(A, \rho) = \{ a \in A_{nc} : \| a \| < \rho \}$ centered in zero and of radius $\rho > 0$ over a $C^*$-algebra $A$ ($A$ could have been replaced by any operator space with the corresponding operator space norm).

3. The upper and lower fully matricial half-planes $\mathbb{H}^+(A_{nc})$ and $\mathbb{H}^-(A_{nc})$ over a $C^*$-algebra $A$, where if $C$ is a $C^*$-algebra, then

$$\mathbb{H}^+(C) = \{ a \in C, \exists \alpha = \frac{a - \alpha^*}{2} > 0 \}$$

$$\mathbb{H}^-(C) = \{ a \in C, \exists \alpha = \frac{a - \alpha^*}{2} < 0 \}$$

and $\mathbb{H}^+(A_{nc}) = \bigcup_{n=1}^{\infty} \mathbb{H}^+(M_n(A))$.

Let $V$ and $W$ be vector spaces over $\mathbb{C}$, and let $\Omega \subseteq V_{nc}$ be a noncommutative set. A mapping $f : \Omega \rightarrow W_{nc}$ with $f(\Omega_n) \subseteq W^{n \times n}$ is called a noncommutative function if $f$ satisfies the following two conditions:

- $f$ respects direct sums: $f(a \oplus b) = f(a) \oplus f(b)$ for all $a, b \in \Omega$.

- $f$ respects similarities: if $a \in \Omega_n$ and $s \in \mathbb{C}^{n \times n}$ is invertible with $sas^{-1} \in \Omega_n$, then $f(sas^{-1}) = sf(a)s^{-1}$.
We will denote \( f_n = f|_{\Omega_n} : \Omega_n \to \mathcal{W}^{n \times n} \). While we will not need this fact, it is important to notice that the two conditions in the definition of a noncommutative function can be actually replaced by a single one: a mapping \( f : \Omega \to \mathcal{W}_{nc} \) with \( f(\Omega_n) \subseteq \mathcal{W}^{n \times n} \) respects direct sums and similarities if and only if it respects intertwinnings: for any \( a \in \Omega_n, b \in \Omega_m \), and \( t \in \mathbb{C}^{n \times m} \) such that \( at = tb \), one has \( f(a)t = tf(b) \). This last condition goes back to [23, 26].

Let \( \alpha : \mathcal{V}^k \to \mathcal{W} \) be a multilinear mapping; we set

\[
\tilde{\alpha} = \tilde{\alpha}_n \overset{\text{def}}{=} \alpha \otimes \text{Id}_n : (\mathcal{V}^\otimes k)^{n \times n} \to \mathcal{W}^{n \times n}
\]

for each \( n \in \mathbb{N} \). It is clear that the mapping \( a \mapsto \tilde{\alpha}(a^k) \), where we view \( a \) as a matrix over \( T(\mathcal{V}) \), respects direct sums and similarities, so that it defines a noncommutative function from \( \mathcal{V}_{nc} \) to \( \mathcal{W}_{nc} \). Therefore for any linear mapping \( \phi : T(\mathcal{V}) \to \mathcal{W} \), we obtain a noncommutative function defined by

\[
f(a) = \tilde{\phi} (\langle I - a \rangle^{-1}) = \sum_{k=0}^{\infty} \tilde{\phi}(a^k)  \quad (14)
\]

(whence the notation \( \langle I \rangle \) should be understood as \( \text{Id}_n \) componentwise, i.e. \( f(a) = \tilde{\phi}_n (\langle \text{Id}_n - a \rangle^{-1}) \) for \( a \in \mathcal{V}_{nc}^{n \times n} \)), except that we have to make sense of the infinite sum on the right-hand side.

1. If \( a \in \text{Nilp}(\mathcal{V}) \) then the sum is finite, so that \( f \) is always a noncommutative function on \( \text{Nilp}(\mathcal{V}) \).

2. If \( \phi : T(\mathcal{A}) \to \mathcal{C} \), where \( \mathcal{A} \) and \( \mathcal{C} \) are \( C^* \) algebras, and we have an exponential growth estimate: \( \| \phi \|_{\mathcal{A}^\otimes k \to \mathcal{C}} \|_{cb} \leq \alpha \beta^k \) (where \( \mathcal{A}^\otimes k \) is considered with the Haagerup tensor norm (see [16], Chapter 17) and \( \| \cdot \|_{cb} \) denotes the completely bounded norm), then the series defining \( f \) converges absolutely and uniformly on any noncommutative ball \( \mathcal{B}(\mathcal{A}, r) \) over \( \mathcal{A} \) of radius \( r < 1/\beta \), so that \( f \) is a noncommutative function on the noncommutative ball \( \mathcal{B}(\mathcal{A}, 1/\beta) \).

There is — in a sense — a converse to this construction that we briefly describe, though we will make no real use of it here. A noncommutative function \( f \) admits a series expansion

\[
f(a) = \sum_{k=0}^{\infty} \Delta^k_R f(\underbrace{0, \ldots, 0}_{k+1}) (a^\otimes k).  \quad (15)
\]

More precisely:

1. If \( f \) is a noncommutative function on \( \text{Nilp}(\mathcal{V}) \), then the sum is finite and the equality holds everywhere.

2. If \( f \) is a noncommutative function on a noncommutative ball \( \mathcal{B}(\mathcal{A}, \rho) \) over a \( C^* \) algebra \( \mathcal{A} \) with values in a noncommutative space \( \mathcal{C}_{nc} \) over a \( C^* \) algebra \( \mathcal{C} \), which is bounded on noncommutative balls of radius less then \( \rho \), then the series converges to \( f \), uniformly on every noncommutative ball of radius less than \( \rho \). (The convergence still holds if \( f \) is only assumed to be locally bounded in every matrix dimension separately, but then it is no longer uniform across matrix dimensions.)

The multilinear forms \( \Delta^k_R f(\underbrace{0, \ldots, 0}_{k+1}) : \mathcal{V}^k \to \mathcal{W} \) are the values at \( (0, \ldots, 0) \) of the \( k \)th order noncommutative difference-differential operators applied to \( f \). They
are uniquely determined, and can be calculated directly by evaluating \( f \) on upper triangular matrices:

\[
f \begin{pmatrix}
  0 & a_1 & 0 & \cdots & 0 \\
  0 & 0 & a_2 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & a_k \\
  0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  f(0) & \Delta_R f(0,0)(a_1) & \cdots & \Delta_R^k f(0,\ldots,0)(a_0,\ldots,a_k) \\
  0 & f(0) & \cdots & \Delta_R^{k-1} f(0,\ldots,0)(a_2,\ldots,a_k) \\
  \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \vdots & \ddots \\
  0 & \cdots & \cdots & f(0) & \Delta_R f(0,0)(a_k) \\
  & 0 & \cdots & \cdots & 0 & f(0)
\end{pmatrix}.
\]

5.3. The non-commutative \( B_\ast \), \( R \) and \( cR \)-transforms.

In particular, we can apply the construction (14) to \( \mu \in \Sigma_{\mathcal{B} \cdot \mathcal{D}} \), and define

\[
M_\mu(b) = \bar{\mu}((1 - \mathcal{X}b)^{-1}) = \sum_{k=0}^{\infty} \bar{\mu}((\mathcal{X}b)^k),
\]

where the notation \( \mathbb{1} \) is the one from the previous section and, as before, we identify the monomials \((\mathcal{X}b)^n\) from \( \mathcal{B}(\mathcal{X})_0 \) to \( b^\otimes n \) from the tensor algebra \( \mathbf{T}(\mathcal{B}) \). \( M_\mu \) is a noncommutative function with values in \( \mathcal{D}_{nc} \). It is always defined on \( \text{Nilp}(\mathcal{B}) \). If \( \mathcal{D} \) is a C*-algebra and \( \mu \in \Sigma_{\mathcal{B} \cdot \mathcal{D}} \), that is there is some \( C > 0 \) such that \( \| \mu_k \|_{cb} \leq C^{k+1} \), where \( \mu_k \) is the restriction of \( \mu \) to the subspace of \( \mathcal{B}(\mathcal{X}) \) spanned by words with exactly \( k \) occurrences of the symbol \( \mathcal{X} \) (that can be identified with \( \mathcal{B}^{\otimes k} \)), then \( M_\mu \) is also defined on a noncommutative ball in \( \mathcal{B} \) with sufficiently small radius. Remark that (14) also implies \( \Delta_R^k M_\mu(0,\ldots,0) = \mu_k \).

Consider now \( \nu \in \Sigma_{\mathcal{B}} \) and \( \mu \in \Sigma_{\mathcal{B} \cdot \mathcal{D}} \). We define the linear maps \( \rho_\nu, c\rho_{\mu,\nu} \) as in Section 4 respectively Section 3 that is the \( \mathcal{B} \)-bimodule extensions of

\[
\rho_\nu(\mathcal{X}b_1 \cdots \mathcal{X}b_p) = \kappa_{\nu,p}(b_1,\ldots,b_p), \quad c\rho_{\mu,\nu}(\mathcal{X}b_1 \cdots \mathcal{X}b_p) = c\kappa_{\mu,\nu,p}(b_1,\ldots,b_p)
\]

and the map \( \beta_\mu \) as the \( \mathcal{B} \)-bimodule extension of

\[
\beta_\mu(\mathcal{X}b_1 \cdots \mathcal{X}b_p) = B_{\mu,p}(b_1,\cdots,b_p).
\]

Using again (14), we define the noncommutative functions \( R_\nu \) (with values in \( \mathcal{B}_{nc} \)) and \( B_\nu, cR_{\mu,\nu} \) (with values in \( \mathcal{D}_{nc} \)) via the equations:

\[
R_\nu(b) = \tilde{\rho}_\nu((1 - \mathcal{X}b)^{-1}) - \mathbb{1},
\]

\[
cR_{\mu,\nu}(b) = c\tilde{\rho}_{\mu,\nu}((1 - \mathcal{X}b)^{-1}) - \mathbb{1},
\]

\[
B_\mu(b) = \tilde{\beta}_\mu((1 - \mathcal{X}b)^{-1}) - \mathbb{1}.
\]

Remark again that (14) implies \( \Delta_R^p R_\nu(0,\ldots,0) = \kappa_{\nu,p}, \Delta_R^p cR_{\mu,\nu}(0,\ldots,0) = c\kappa_{\mu,\nu,p} \) and \( \Delta_R^p B_\mu(0,\ldots,0) = B_{\mu,p} \).
Proposition 5.3. For all \((\mu, \nu) \in \Sigma_{\mathcal{B}_1} \times \Sigma_{\mathcal{B}_2}\) we have that

\[
\begin{align*}
\Id_n \otimes \rho_\nu &= \rho_{\rho_\nu} \\
\Id_n \otimes \epsilon \rho_{\mu, \nu} &= \epsilon \rho_{\rho_\mu, \rho_\nu} \\
\Id_n \otimes \beta_\mu &= \beta_\rho_\mu.
\end{align*}
\]

Proof. For the first property, it suffices to show that for all positive integers \(n\), all \(1 \leq i_k, j_k \leq n\) and all \(b_1, \ldots, b_m \in \mathcal{B}\) we have that

\[
\Id_n \otimes \rho_\nu(\mathcal{X}B_1 \cdots \mathcal{X}B_m) = \rho_{\rho_\nu}(\mathcal{X}B_1 \cdots \mathcal{X}B_m),
\]

where \(B_k = e_{i_k, j_k} \otimes b_k\) for \(e_{i_k, j_k}\) the complex \(n \times n\) matrix with the \((i_k, j_k)\) entry 1 and all others 0.

For \(m = 1\) the property is trivial. Suppose now that the assertion holds true for all \(m < N\). The definition of free cumulants gives

\[
\rho_{\rho_\nu}(\mathcal{X}B_1 \cdots \mathcal{X}B_N) = \rho_\nu(\mathcal{X}B_1 \cdots \mathcal{X}B_N) - \sum_{p=1}^{N-1} \sum_{s_1 < \cdots < s_p} \rho_{\rho_\nu}(\mathcal{X}B_1 \rho_\nu(\mathcal{X}B_2 \cdots \mathcal{X}B_{s_p-1}) \cdots \mathcal{X}B_{s_p} \rho_\nu(\mathcal{X}B_{s_p+1} \cdots \mathcal{X}B_N))
\]

From the induction hypothesis, the right-hand side cancels unless \(i_k = J_{k-1}\) for all \(1 < k \leq N\); in this case, the equation above becomes

\[
\rho_{\rho_\nu}(\mathcal{X}B_1 \cdots \mathcal{X}B_N) = \epsilon_{i_1, j_N} \otimes \rho_\nu(\mathcal{X}B_1 \cdots \mathcal{X}B_N) - \sum_{p=1}^{N-1} \sum_{s_1 < \cdots < s_p} \rho_{\rho_\nu}(\epsilon_{i_1, j_N} \otimes [\mathcal{X}B_1 \rho_\nu(\mathcal{X}B_2 \cdots \mathcal{X}B_{s_p-1}) \cdots \mathcal{X}B_{s_p} \rho_\nu(\mathcal{X}B_{s_p+1} \cdots \mathcal{X}B_N)])
\]

\[
= \epsilon_{i_1, j_N} \otimes \rho_\nu(\mathcal{X}B_1 \cdots \mathcal{X}B_N)
\]

\[
= \Id_n \otimes \rho_\nu(\mathcal{X}B_1 \cdots \mathcal{X}B_N)
\]

hence the conclusion.

The argument for \(\epsilon \rho_{\mu, \nu}\) and \(\beta_\mu\) is analogous. \(\square\)

Proposition 5.3 implies that

\[
R^{(n)}_\nu(b) = \sum_{n=1}^{\infty} \kappa_{n, \rho_\nu}(b, \ldots, b)
\]

and the analogous relations for the components of \(\epsilon \rho_{\mu, \nu}\) and \(B_\mu\). From the moment-cumulant recursions, we have then the following

Corollary 5.4. The non-commutative functions \(R_\nu\), \(B_\nu\) and \(\epsilon \rho_{\mu, \nu}\) satisfy the equations:

\[
\begin{align*}
(16) \quad M_\mu(b) - 1 &= B_\mu(b) \cdot M_\mu(b) \\
(17) \quad M_\nu(b) - 1 &= R_\nu(b M_\nu(b)) \\
(18) \quad (M_\mu(b) - 1) \cdot M_\nu(b) &= M_\mu(b) \cdot \epsilon \rho_{\mu, \nu}(b M_\nu(b))
\end{align*}
\]

Remark 5.5. If \((\mu, \nu) \in \Sigma_{\mathcal{B}_1} \times \Sigma_{\mathcal{B}_2}\) then \(R_\nu\), \(\epsilon \rho_{\mu, \nu}\) and \(B_\mu\) are well-defined on a non-commutative ball (as described in Section 5.2) centered in 0.
Proof: For $B_\mu$, the assertion is trivial since $B_\mu(b)$ is well-defined if $M_\mu$ is invertible. For $R_\nu$ and $\epsilon R_{\mu, \nu}$ we will use combinatorial techniques, namely the Moebius inversion formula for the partially ordered set of non-crossing partitions. First, we need the following general property (the Moebius inversion formula).

Proposition: Let $P$ be a finite partially ordered set and $K$ a complex vector space. Then there exist a map $\text{moeb} : P \times P \to \mathbb{R}$ such that if the maps $f, g : P \to K$ have the property

$$f(\pi) = \sum_{\sigma \in P, \sigma \leq \pi} g(\sigma), \pi \in P$$

then

$$g(\pi) = \sum_{\sigma \in P, \sigma \leq \pi} f(\sigma) \cdot \text{moeb}(\sigma, \pi).$$

We will apply the above result on the partially ordered set $NC$ of non-crossing partitions. By a partition on the ordered set $\langle n \rangle = \{1, 2, \ldots, n\}$ we will understand a collection of mutually disjoint subsets of $\langle n \rangle$, $\gamma = (B_1, \ldots, B_q)$ called blocks whose union is the entire set $\langle n \rangle$. A crossing is a sequence $i < j < k < l$ from $\langle n \rangle$ with the property that there exist two different blocks $B_r$ and $B_s$ such that $i, k \in B_r$ and $j, l \in B_s$. A partition that has no crossings will be called non-crossing. The set of all non-crossing partitions on $\langle n \rangle$ will be denoted by $NC(n)$. For $\gamma \in NC(n)$ a block $B = (i_1, \ldots, i_k)$ of $\gamma$ will be called interior if there exists another block $D \in \gamma$ and $i, j \in D$ such that $i < i_1, i_2, \ldots, i_k < j$. A block will be called exterior if is not interior. Each $NC(n)$ has a lattice structure with respect to block refinement with biggest element $1_n$; the partition with a single block; the correspondent Moebius function satisfies $|\text{moeb}(\pi, 1_n)| \leq 4^n$ (see [22], Lecture 13). Define $NC = \bigsqcup_{n=1}^{\infty} NC(n)$.

Finally, let $(\mu, \nu) \in \Sigma_{N_{B} \cdot D} \times \Sigma_{B}$, $b \in B$. We define $f, \rho : NC \to D$ as follows:

(a) $f(\pi_1 \cdot \pi_2) = f(\pi_1) \cdot f(\pi_2)$ and $F(\pi_1 \cdot \pi_2) = F(\pi_1) \cdot F(\pi_2)$, where $\pi_1 \cdot \pi_2 \in NC(m + n)$ is obtained by juxtaposing $\pi_1 \in NC(n)$ and $\pi_2 \in NC(m)$.

(b) $f(1_m) = \nu((\lambda b)^m)$ and $F(1_m) = \mu((\lambda b)^m)$, where $1_m \in NC(m)$ is the partition with a single block $(1, 2, \ldots, m)$.

(c) $f([\pi_1 \pi_2 \ldots \pi_q]) = \nu(\lambda b \cdot f(\pi_1) \cdot \lambda b \cdot f(\pi_2) \cdots \lambda b \cdot f(\pi_q) \cdot \lambda b)$

$F([\pi_1 \pi_2 \ldots \pi_q]) = \mu(\lambda b \cdot f(\pi_1) \cdot \lambda b \cdot f(\pi_2) \cdots \lambda b \cdot f(\pi_q) \cdot \lambda b)$

where $[\pi_1 \pi_2 \ldots \pi_q]$ is the partition with a single exterior block with $q + 1$ elements and with restrictions between the elements of the exterior block $\pi_1, \ldots, \pi_q$.

Similarly, we define $g, G : NC \to D$ as above, replacing $\mu$ with $\rho_{\mu, \nu}$ and $\nu$ with $\rho_\nu$. With this notations, the moment-cumulant relation from Definition 3.3 amounts to

$$F(\pi) = \sum_{\sigma \in NC, \sigma \leq \pi} G(\sigma)$$

and applying the Moebius function property to equation (19) we get

$$G(\pi) = \sum_{\sigma \in NC, \sigma \leq \pi} F(\sigma) \cdot \text{moeb}(\sigma, \pi).$$
Suppose now that \((\mu, \nu) \in \Sigma_{B, D}^0 \times \Sigma_B^0\) and that that \(||\mu_n||, ||\nu_n|| < M^{n+1}\) for all \(n > 0\), henceforth \(||F(\sigma)|| \leq M^{n+1}||b||^m\) for all \(\sigma \in NC(m)\). Since \(c\kappa_{\nu,\nu,m}(b) = G(1,m)\), we have

\[
\|c\kappa_{\nu,\nu,m}(b)\| \leq \sum_{\sigma \in NC(m)} \|F(\sigma)\| \cdot |\text{moeb}(\sigma, 1,m)|
\]

\[
< (\|\mu\| NC(m)) \cdot M^{m+1} ||b||^m \cdot 4^m
\]

\[
< \cdot M^{m+1} ||b||^m \cdot 16^m, \text{ since } 2^{NC(m)} < 4^m.
\]

Finally, \(\kappa_{\nu,m}(b) = c\kappa_{\nu,\nu,m}(b)\), so the assertion is proven also for the R-transform. \(\square\)

5.4. Main results. The following property of the \(B\)-transform is a non-commutative analogue of Proposition 3.1. from [23], (i.e. the Nevalinna-Pick representation for the self-energy function of a real measure):

**Theorem 5.6.** Let \(B\) be a \(C^*\)-algebra, \(B \subset D\) be a unital inclusion of \(C^*\)-algebras and \(\mu \in \Sigma_{B, D}\). Then there exists a selfadjoint \(\alpha \in D\) and a \(C\)-linear map \(\sigma : B(\mathcal{X}) \rightarrow D\), satisfying property (1) such that

\[B_{\mu}(b) = [\alpha \cdot 1 + \bar{\sigma}(b(1 - \mathcal{X}b)^{-1})] \cdot b.\]

Moreover, if \(\mu \in \Sigma_{B, D}^0\), then the moments of \(\nu\) do not grow faster than exponentially (i.e. \(\nu\) satisfies property (2)).

**Proof.** Define the map \(\beta_{\mu} : B(\mathcal{X}) \rightarrow D\) as the \(B\)-bimodule extension of

\[\beta_{\mu}(\mathcal{X}b_1 \mathcal{X} \cdots \mathcal{X}b_n) = B_{n, \mu}(b_1, \ldots, b_n).\]

Let \(K = B(\mathcal{X}) \otimes_B D\). As shown in the proof of Theorem 2.5, equation (5), there exists some \(\xi \in K\) and a selfadjoint map \(T \in L(K)\) such that

\[B_{n, \mu}(b_1, \ldots, b_n) = \langle b_1 T b \cdots T b_{n-1}(\xi b_n), \xi \rangle\]

If \(b_n = 1\), the above relation becomes

\[\langle b_1 T b \cdots T b_{n-1}(\xi), \xi \rangle = B_{n, \mu}(b_1, \ldots, b_{n-1}, 1) = \beta_{\mu}(\mathcal{X}b_1 \mathcal{X} \cdots \mathcal{X}b_{n-1} \mathcal{X})\]

Hence for all \(f(\mathcal{X}) \in B(\mathcal{X})\) we have that \(\beta_{\mu}(\mathcal{X}f(\mathcal{X})\mathcal{X}) = \langle f(T)\xi, \xi \rangle\). It follows that the map \(\sigma : B(\mathcal{X}) \rightarrow D\) given by \(\sigma(f(\mathcal{X})) = \beta_{\mu}(\mathcal{X}f(\mathcal{X})\mathcal{X})\) satisfies property (1) and

\[B_{n, \mu}(b, \ldots, b) = \sigma([\mathcal{X}b]^{n-1}) \cdot b,\]

that is the conclusion for \(\alpha = B_{1, \mu}\). The last part is a trivial consequence of the fact that \(\mu \in \Sigma_{B, D}^0\) implies that \(\beta_{\mu} \in \Sigma_{B, D}^0\). \(\square\)

**Remark 5.7.** If \(\mu \in \Sigma_{B, D}^0\), the above result can be more explicit formulated in terms of the generalized resolvent of \(\mu\) from [28] and [2]. More precisely, for \(\mu \in \Sigma_{B, D}^0\), its generalized resolvent or operator-valued Cauchy transform is defined via

\[G_{\mu}(b) = \tilde{\mu}([b - 1 \cdot \mathcal{X}]^{-1}).\]

As shown in [28] and [2], \(G_{\mu}\) is a non-commutative function, well-defined on \(\mathbb{H}^+(B_{nc})\) and \(G_{\mu}(\mathbb{H}^+(M_n(B))) \subseteq \mathbb{H}^-(M_n(D))\). Hence, its reciprocal, \(b \mapsto [G_{\mu}(b)]^{-1}\) is also
a non-commutative function, well defined on \( \mathbb{H}^+(B_{nc}) \). Moreover, identifying the coefficients, we have that

\[
1 - \left[ G_{\mu}(b) \right]^{-1} \cdot b^{-1} = B_{\mu}(b^{-1})
\]

for \( b \in \mathbb{H}^+(B_{nc}) \) with \( ||b^{-1}|| < M \) for some \( M > 0 \).

Using equation (20) and Theorem 5.6, we obtain that, for \( b \) as above,

\[
b - \left[ G_{\mu}(b) \right]^{-1} = \alpha \cdot 1 + \sigma(b^{-1}[1 - X \cdot b^{-1}]^{-1}) = \alpha \cdot 1 + \sigma((1 - X \cdot b^{-1}) \cdot b^{-1}) = \alpha \cdot 1 + \sigma(\sigma([b - 1 \cdot X]^{-1})).
\]

The map \( b \mapsto \sigma([b - 1 \cdot X]^{-1}) \) extends to \( \mathbb{H}^+(B_{nc}) \) (see again (28) and (29)) therefore we obtained that the operator-valued selfenergy function of \( \mu (b \mapsto b - \left[ G_{\mu}(b) \right]^{-1}) \) is the translate with a selfadjoint of the operator-valued Cauchy transform of some \( \mathbb{C} \)-linear map from \( B(X) \) to \( D \) satisfying properties 1 and 2.

For the main result of this section, Theorem 5.10, we will first need the following lemma:

**Lemma 5.8.** Let \( B \subset D \) be a unital inclusion of \( C^* \)-algebras and \( \rho : B(X) \rightarrow D \) be a unital \( B \)-bimodule map. Then the restriction of \( \rho \) to \( B(X)_0 \) satisfies property \( \mathbb{I} \) if and only if there exists some \( \langle \mathbb{C} \rangle \)-linear map \( \sigma = \sigma(\rho) : B(X) \rightarrow D \) satisfying property \( \mathbb{II} \) such that \( \sigma(f(X)) = \rho(X f(X) X) \) for all \( f(X) \in B(X) \).

**Proof.** Suppose that first that the restriction of \( \rho|_{B(X)_0} \) satisfies \( \mathbb{I} \) and define \( \sigma \) via \( \sigma(f(X)) = \rho(X f(X) X) \). If \( \{f_j(X)\}_{j=1}^n \) is some family from \( B(X)_0 \), then

\[
\left[ \sigma(f_j(X)^* f_i(X)) \right]_{i,j=1}^n = \left[ \rho(X f_j(X)^* f_i(X) X) \right]_{i,j=1}^n = \left[ \rho([f_j(X) X]^* [f_i(X) X]) \right]_{i,j=1}^n \geq 0.
\]

For the converse, note that \( \sigma \) satisfies \( \mathbb{II} \) implies (cf [10], pag. 42) that \( B(X) \otimes D \) is a semi-inner product \( D \)-module with respect to the pairing generated by

\[
(f(X) \otimes d_1, g(X) \otimes d_2) = d_2^* \sigma(g(X)^* f(X)) d_1.
\]

Fix now a family \( \{f_j(X)\}_{j=1}^n \) from \( B(X)_0 \). Each \( f_j(X) \) can be written as

\[
f_j(X) = \sum_{k=1}^{N(j)} g_{k,j}(X) \cdot X \cdot \alpha_{k,j}
\]

with \( g_{k,j}(X) \in B(X) \) and \( \alpha_{k,j} \in B \). Denote \( \eta_j = \sum_{k=1}^{N(j)} g_{k,j}(X) \alpha_{k,j} \in B(X) \otimes D \). Then

\[
\left[ \rho(f_j(X)^* f_i(X)) \right]_{i,j=1}^n = \left[ \sum_{k=1}^{N(j)} \sum_{l=1}^{N(i)} \rho(\alpha_{k,j}^* \cdot X \cdot g_{k,j}(X)^* g_{l,i}(X) \cdot X \cdot \alpha_{l,i}) \right]_{i,j=1}^n\]

\[
= \left[ \sum_{k=1}^{N(j)} \sum_{l=1}^{N(i)} \alpha_{k,j}^* \cdot \sigma(g_{k,j}(X)^* g_{l,i}(X) \cdot \alpha_{l,i}) \right]_{i,j=1}^n
\]

\[
= \left[ \left( \sum_{l=1}^{N(i)} g_{l,i}(X) \otimes \alpha_{l,i} \cdot \sum_{k=1}^{N(j)} g_{k,j}(X) \otimes \alpha_{k,j} \right) \right]_{i,j=1}^n
\]

\[
= \left[ \langle \eta_i, \eta_j \rangle \right]_{i,j=1}^n \geq 0.
\]
An immediate consequence of Theorem 5.8 and the above Lemma is the following

**Corollary 5.9.** If \( \mu \in \Sigma_{B,D} \) then \( \beta_{\mu[B]}(X)_{lo} \) satisfies property (1).

**Theorem 5.10.** As before, \( B \subseteq D \) will be a unital inclusion of unital C*-algebras.

(i) Let \( \mu \in \Sigma_B \). Then \( \mu \) is \( \mathbb{N} \)-infinitely divisible if and only if there exist some selfadjoint \( \alpha \in B \) and some \( C \)-linear map \( \sigma : B(\mathcal{X}) \to B \) satisfying property (1), such that

\[
R_{\mu}(b) = \left[ \alpha \cdot \mathbb{1} + \tilde{\sigma}(b(\mathbb{1} - \mathcal{X}b)^{-1}) \right] \cdot b,
\]

(ii) Let \( (\mu, \nu) \in \Sigma_{B,D} \times \Sigma_B \). Then \( (\mu, \nu) \) is \( \mathbb{N} \)-infinitely divisible if and only if \( \nu \) is \( \mathbb{N} \)-infinitely divisible and there exist some selfadjoint \( \alpha \in B \) and some \( C \)-linear map \( \sigma : B(\mathcal{X}) \to D \) satisfying property (1) such that

\[
^cR_{\mu,\nu}(b) = \left[ \alpha \cdot \mathbb{1} + \tilde{\sigma}(b(\mathbb{1} - \mathcal{X}b)^{-1}) \right] \cdot b.
\]

Moreover, if \( \mu \in \Sigma^0_{B,D} \), respectively \( (\mu, \nu) \in \Sigma^0_{B,D} \times \Sigma^0_B \), then the moments of the corresponding maps \( \sigma \) do not grow faster than exponentially.

**Proof.** Since \( \Sigma_{B,B} = \Sigma_{B,D} \), it suffices to prove the assertions for \( ^cR_{\mu,\nu} \).

Since \( ^c\rho_{\mu,\nu}(\mathcal{X}b_1 \cdots \mathcal{X}b_n) = \kappa_{\mu,\nu}(b_1, \ldots, b_n) \), Proposition 5.3 implies that

\[
^cR_{\mu,\nu}(b) = M_{\rho_{\mu,\nu}}(b) - \mathbb{1}.
\]

for all \( b \in Nilp(B) \) or for \( b \) or sufficiently small norm if \( (\mu, \nu) \in \Sigma^0_{B,D} \times \Sigma^0_B \).

Identifying \( \mathcal{X} \) to \( \mathbb{1} \cdot \mathcal{X} \), we have that

\[
^cR_{\mu,\nu}(b) = \sum_{n=1}^{\infty} c_{\rho_{\mu,\nu}}(b^{(\mathcal{X})^n}) = \sum_{n=2}^{\infty} c_{\rho_{\mu,\nu}}((\mathcal{X}b)^{n-1} \mathcal{X}) \cdot b
\]

From Theorem 4.8 and Lemma 5.8 \( (\mu, \nu) \) is \( \mathbb{C} \)-infinitely divisible if and only if there exists \( C \)-linear map \( \sigma : B(\mathcal{X}) \to D \) that satisfy property (1) such that \( \sigma(f(\mathcal{X})) = c_{\rho_{\mu,\nu}}(X f(X) \mathcal{X}) \) for all \( f(\mathcal{X}) \in B(\mathcal{X}) \). That is

\[
^cR_{\mu,\nu}(b) = \left[ \mu(\mathcal{X}) \cdot \mathbb{1} + \sum_{l=0}^{\infty} \tilde{\sigma}(b^{l} \mathcal{X}^{l}) \right] \cdot b
\]

hence the conclusion.

For the last part, if \( (\mu, \nu) \in \Sigma^0_{B,D} \times \Sigma^0_B \), then Remark 5.5 implies that \( ^c\rho_{\mu,\nu} \) is also in \( \Sigma^0_{B,D} \times \Sigma^0_B \), so they satisfy the condition (2) for some \( M > 0 \). The representation of \( ^cR_{\mu,\nu} \) gives (for \( b_1, \ldots, b_n \in M_m(B) \)) and the identification \( \mathcal{X} = \text{Id}_m \otimes \mathcal{X} ):

\[
\|\tilde{\sigma}(\mathcal{X} b_1 \mathcal{X} b_2 \cdots b_n \mathcal{X})\| = \|c_{\rho_{\mu,\nu}}(\mathcal{X}^2 b_1 \mathcal{X} b_2 \cdots b_n \mathcal{X}^2)\|
\]

\[
< M^{n+3} \|b_1\| \cdots \|b_n\|
\]

\[
< (M^2 + 1)^{n+1} \|b_1\| \cdots \|b_n\|.
\]
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