Opinion Dynamics in the Presence of Increasing Agreement Pressure

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Abstract—In this paper, we study a model of agent consensus in a social network in the presence increasing interagent influence, i.e., increasing peer pressure. Each agent in the social network has a distinct social stress function given by a weighted sum of internal and external behavioral pressures. We assume a weighted average update rule consistent with the classic DeGroot model and prove conditions under which a connected group of agents converge to a fixed opinion distribution, and under which conditions the group reaches consensus. We show that the update rule converges to gradient descent and explain its transient and asymptotic convergence properties. Through simulation, we study the rate of convergence on a scale-free network.

Index Terms—Agents, consensus, control, opinion formation.

I. INTRODUCTION

DISTRIBUTED agent-based systems frequently must converge to a consensus value given external information. Moreover, if these agents represent individuals (e.g., humans) the agent owner may have a distinct preference or bias that is static. The strength of this bias may be based on a specific data source or it may simply be an underlying preference of the individual that is reflected in the agent. This problem is fundamentally similar to the opinion formation and consensus problem. Beginning with DeGroot [1], opinion models have been studied extensively (see [2]–[19]). In these models, opinion (i.e., one’s understanding of a certain popular or commonly known topic/subject) is a dynamic state variable whose evolution in some compact subset of \( \mathbb{R}^n \) is governed by an autonomous dynamical system. Using this formalism, opinion models have been unified with flocking models (see [20], [21]) in [22]. Most recent work on opinion dynamics (and their unification with flocking models) considers the interaction of agents on a graph structure [16]–[19], [22]. When considered on a lattice, these models are share characteristics to continuous variations of Ising models [5].

In addition to the work on (uncontrolled) consensus, there has been substantial work on control and stability in models like these. Jadabaie et al. [23] studied coordination of mobile agents using flocking rules, and Blondel et al. [11], [24] studied consensus and flocking from a control theoretic point of view. Justh and Krishnaprasad [25] studied extremal collective behaviors from the point of view of geometric optimal control and Olfati-Saber et al. [26] studied the problem of controlled consensus, but in a topologically dynamic network.

Recent work [18], [19] considers the evolution of opinion on a social network in which agents are resistant to change because of an innate belief. In particular, Bindel et al. [18] and Bhawalkar et al. [19] used a variant of the model in [2] and studied this problem from a game-theoretic perspective by considering the price of anarchy on the opinion formation process on a connected graph. The existence of innate beliefs, which are hidden but affect (publicly) presented opinion, is supported in recent empirical work by [27]–[29]. While the work in [18] and [19] introduces the concept of the stubborn agent, it does not consider the effect of situationally variant peer pressure on agents’ opinions, though statically weighted user connections are considered. Peer pressure in social networks is well-documented. Adoption of trends [30], [31], purchasing behaviors [32], beliefs and cultural norms [33], privacy behaviors [34], bullying [35], and health behaviors [36]–[39] have all been linked to peer influence.

Within the cybernetics literature, consensus protocols are well studied. Wu et al. [40] studied consensus in multiagent systems with aperiodically sampled data control. He et al. [41] studied consensus under networked systems with bounded noise. In [42]–[44], leader–follower consensus is studied. The work of Tan et al. [45] studies consensus under the mutator-replicator dynamics. Additional work on consensus in the Cybernetics context is presented in [46]–[60].

In this paper, we consider the problem of opinion dynamics on a social network of agents with innate beliefs in which peer-pressure is a dynamically changing quantity, independent of the opinions themselves. Peer pressure intuitively denotes a single measure representing the social considerations of an individual user with respect to a certain belief.
To model peer pressure, we assume that individuals are encouraged to behave in accordance with the beliefs of their social group. In particular, we assume that individuals in a network are exposed to pressure by peers to come to agreement and reach a common belief or popular opinion. If agents are nonhumans, we show in the sequel how modeled peer-pressure can be an effective distributed fully distributed method that ensures consensus.

Note that the resulting model does not assume uniform adoption, but also accounts for agents with relatively varying resistance to changing their innate beliefs. We use a recent result from functional analysis on the composition of (distinct) contraction mappings along with the Sherman–Morrison formula to show the following.

1) Under increasing peer-pressure, the dynamical system converges.
2) If peer-pressure increases in an unbounded way, consensus emerges a weighted average of the innate beliefs of the individuals.
3) The opinion update process converges to a gradient descent, with linear convergence rate.

Work herein is complementary to [17]–[19] in that we consider a dynamic (increasing) peer-pressure coefficient with variable weights on initial belief. Additionally, we analyze the convergence rate of the dynamical system to the fixed point, while Bindel et al. [18] and Bhawalkar et al. [19] focused on the model from a game-theoretic perspective.

The remainder of this paper is organized as follows. In Section II, we present the basic model. In Section III, we prove convergence of the model and that increasing peer pressure depends on their stubbornness (si) and the relative weights of influence by their neighbors (wij). Thus each user experiences a distinct peer-pressure effect. As noted in [18], under these assumptions, the first order necessary conditions are sufficient for minimizing Ji(x^(k), x^(k-1), k). The optimal state for agent i at time k is then

\[ x_i^{(k)} = \frac{s_i x_i^+ + \rho(k) \sum_{j=1}^{n} w_{ij} x_j^{(k-1)}}{s_i + \rho(k) d_i} \]

where \( d_i = \sum_{j=1}^{n} w_{ij} \) is the weighted degree of vertex i. The updated opinion of agent i is then the weighted average of the innate beliefs of its neighbors.

We say that the agents converge to consensus \( \hat{x} \) if there is some N such that \( \| x - \hat{x} \| < \epsilon \) for some small \( \epsilon > 0 \). This represents meaningful compromise on the issue under consideration.

### II. Problem Statement and Model

We model a network of agents in which each user communicates with its neighbors, but not necessarily the entire network. Assume that the agents' network is represented by a simple graph \( G = (V, E) \), where vertexes \( V \) are agents and edges \( E \) are the connections (e.g., communications) between them. It is clear that disconnected sections of the graph are independent, so we assume that \( G \) is connected. For the remainder of this paper, let \( V = \{1, 2, \ldots, n\} \), so \( E \) is a subset of the two-element subsets of \( V \). The state of agent i at time k is a continuous value \( x_i^{(k)} \in [0,1] \) that represents disclosed opinion or position on a bivalent topic. Each agent has a constant preference \( x_i^+ \in [0,1] \) representing its inherent position on the topic. This may differ from the opinion disclosed to the public. The value \( x_i^+ \) represents inherent agent bias. Further, agent i is assigned a non-negative vertex weight \( s_i \) and positive edge weights \( w_{ij} \), respectively, for \( (i,j) \in E \). The weight \( s_i \), termed stubbornness [17], models the tendency of agent i to maintain its (private) position \( x_i^+ \) in public. The edge weights \( w_{ij} \) represent friendship affinity. The set of all disclosed opinions is denoted by the vector \( x^{(k)} \) while the set of constant private preferences is \( x^+ \). For the remainder of this paper, we refer to publicly disclosed opinions simply as opinions.

Agent i's state is updated by minimizing its social stress

\[ J_i(x_i^{(k)}, x^{(k-1)}, k) = s_i (x_i^{(k)} - x_i^+)^2 + \rho(k) \sum_{j=1}^{n} w_{ij} (x_j^{(k)} - x_j^{(k-1)})^2. \]

Here \( \rho(k) \) is the peer-pressure coefficient. In the sequel, we assume \( \rho(k) \) is an increasing function of k. Intuitively, we assume that as more users change their beliefs, the value of peer pressure will increase accordingly. For simplicity, we assume users are exposed to the same degree of pressure, but their adoption rate depends on their stubbornness (si) and the relative weights of influence by their neighbors (wij). Thus each user experiences a distinct peer-pressure effect. As noted in [18], under these assumptions, the first order necessary conditions are sufficient for minimizing \( J_i(x_i^{(k)}, x^{(k-1)}, k) \). The optimal state for agent i at time k is then

\[ x_i^{(k)} = \frac{s_i x_i^+ + \rho(k) \sum_{j=1}^{n} w_{ij} x_j^{(k-1)}}{s_i + \rho(k) d_i} \]

where \( d_i = \sum_{j=1}^{n} w_{ij} \) is the weighted degree of vertex i. The updated opinion of agent i is then the weighted average of the innate beliefs of its neighbors.

We say that the agents converge to consensus \( \hat{x} \) if there is some N such that \( \| x - \hat{x} \| < \epsilon \) for some small \( \epsilon > 0 \). This represents meaningful compromise on the issue under consideration.

### III. Convergence

In this section, we consider the update rule in (2) as a sequence of contraction mappings each with its own fixed point. We then show that all these fixed points converge to a weighted average.

**Lemma 1 ([61, Ch. 13]):** If \( L = D - A \) is the weighted graph Laplacian, then \( L \) has an eigenvalue 0 with multiplicity 1 and a corresponding eigenvector \( \mathbf{1} \), where \( \mathbf{1} \) is the vector of all 1s.

**Lemma 2:** For any \( \mathbf{S} + \rho(k) \mathbf{L} \) is invertible.

**Proof:** By definition, the graph Laplacian is a positive semidefinite symmetric matrix. In addition, the only eigenvector with eigenvalue 0 is the vector of all 1s, written \( \mathbf{1} \).

Since \( \mathbf{S} \) is symmetric and \( s_i \geq 0 \), \( \mathbf{S} + \rho(k) \mathbf{L} \) is positive semidefinite as well. Choose \( x \in \mathbb{R}^n \) such that \( x^T (\mathbf{S} + \rho(k) \mathbf{L}) x = 0 \). Then \( x^T (\mathbf{S} + \rho(k) \mathbf{L}) \mathbf{x} = x^T \mathbf{Sx} + \rho(k) x^T \mathbf{Lx} \). Since \( \mathbf{S} \) and \( \mathbf{L} \) are positive semidefinite and \( \rho(k) > 0 \), this implies that \( x^T \mathbf{Sx} = x^T \mathbf{Lx} = 0 \).
Since $L$ is symmetric, by the spectral theorem it has is an orthonormal basis of eigenvectors $\{b_1, \ldots, b_n\}$ with associated eigenvalues $[\lambda_1, \ldots, \lambda_n]$. Because $L$ is positive semidefinite $\lambda_i \geq 0$, that $x^T L x = \sum_{i=1}^n \lambda_i (x^T b_i)^2$. And, because $x^T L x = 0$, if $\lambda_i \neq 0$, $x^T b_i = 0$.

It follows that $x$ is an eigenvector of $L$ with eigenvalue 0; that is, $x = c1$ for some constant $c$, and therefore $x^T S x = c^2 \sum_{i=1}^n s_i$. Since $s_i \geq 0$ and not all $s_i$ are zero, we must have $c = 0$, so $x = 0$. Following, $S + \rho(k) L$ is positive definite, and therefore invertible.

Define

$$F_k(x) = \left( S + \rho(k) L \right)^{-1} \left( Sx^T + \rho(k) Ax \right)$$

and let

$$G_k = F_k \circ F_{k-1} \circ \cdots \circ F_1.$$ (3)

Then $x^{(k)} = F_k(x^{(k-1)})$ and $x^{(0)} = G_k(x^{(0)})$. That is, iterating these $F_k$ captures the evolution of $x^{(k)}$. We show that for each $k$, $F_k$ is a contraction and therefore has a fixed point by the Banach fixed point theorem [62]. We use this result in the proof of Theorem 2.

Lemma 3: For all $k$, $F_k$ is a contraction map with fixed point given by $\overline{x}^{(k)} = (S + \rho(k) L)^{-1} Sx^T$.

Proof: Let $B$ be the $(n+1) \times (n+1)$ matrix given by adding a row and column to $(S + \rho(k) L)^{-1} A$ as follows:

$$B = \begin{bmatrix} \rho(k) (S + \rho(k) L)^{-1} A & (S + \rho(k) L)^{-1} S1 \\ 0 & 1 \end{bmatrix}.$$ The rows of $B$ sum to 1. To see this, replace $x^T$ and $x^{(k-1)}$ in (2) with 1. Thus $B$ is a stochastic matrix for a Markov process with a single absorbing state. Since $G$ is connected and not all $s_i$ are equal to 0, a transition exists from each state to the steady state; thus from any starting state, convergence to the steady state is guaranteed. This means that $\lim_{k \to \infty} \rho(k) (S + \rho(k) L)^{-1} A = 0$, so $A$ is a convergent matrix. Equivalently, if $\| \|$ denotes the matrix operator norm, then $\| \rho(k) (S + \rho(k) L)^{-1} A \| < 1$. Therefore for any $x, y \in [0, 1]^n$

$$\| F_k(x) - F_k(y) \| = \| (S + \rho(k) L)^{-1} \rho(k) A (x - y) \| \leq \| (S + \rho(k) L)^{-1} \rho(k) A \| \| x - y \|.$$ That is, $F_k$ is a contraction map on a compact set, so by the Banach fixed point theorem, it has a unique fixed point $\overline{x}^{(k)}$.

Let $\overline{x}^{(k)}$ be that fixed point. Then $\overline{x}^{(k)} = F_k(\overline{x}^{(k)})$.

Rearranging the terms yields

$$\left( S + \rho(k) L \right) \overline{x}^{(k)} - \rho(k) A \overline{x}^{(k)} = \left( S + \rho(k) L \right) \overline{x}^{(k)} = Sx^T.$$ Therefore,

$$\overline{x}^{(k)} = \left( S + \rho(k) L \right)^{-1} Sx^T.$$ (4)

This completes the proof.

The following lemma will allow us to consider the matrices $(S + \rho(k) L)^{-1}$ for $k \in \{1, 2, \ldots \}$ in $GL_n(\mathbb{R})$ (the Lie group of invertible $n \times n$ real matrices) as perturbations. This enables effective approximations of asymptotic behaviors.

Lemma 4: Let $\{b_1, \ldots, b_n\}$ an orthonormal basis of $\mathbb{R}^n$. Also let $M : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible symmetric linear transformation (invertible square matrix) and $\{u_1, \ldots, u_n\}$ be a set of unit vectors such that for some value real $\beta$ and a small constant $\delta$, $M^{-1} b_i = \beta b_i + O(\delta) u_1$ and $M^{-1} b_j = O(\delta) u_j$ for $j \neq i$.

Then if $\|v\| = 1$, and $s \in \mathbb{R}$, then unless $(M + svv^T)$ is not invertible, there exists a set of unit vectors $\{u'_1, \ldots, u'_n\}$ such that $(M + svv^T)^{-1} b_i = [(\beta)/1 + \beta s(v^T b_i)] b_i + O(\delta) u'_i$ and $(M + svv^T)^{-1} b_j = O(\delta) u'_j$ for $j \neq i$.

Proof: Since $\{b_1, \ldots, b_n\}$ is an orthonormal basis, $v = \sum_{i=1}^n a_i b_i$ where $a_i = v^T b_i$. This means that $M^{-1} v = \sum_{j=1}^n a_j M^{-1} (b_j) = \beta a_1 b_i + O(\delta) \sum_{j=1}^n a_j u_i$. By Cauchy–Schwartz, $|a_1| \leq \|v\| = 1$, so by the triangle inequality, $\| \sum_{i=1}^n a_i u_i \| \leq n$. Then letting $u = [(\sum_{i=1}^n a_i u_i)/(n)]$, we have that $M^{-1} v = \beta a_1 b_i + O(\delta) u_i$, where $\|u\| \leq 1$.

By the Sherman–Morrison formula

$$(M + svv^T)^{-1} = M^{-1} - M^{-1} v v^T M^{-1} / (1 + sv^T v).$$

Using this, and choosing each $u'_i$ to be an appropriate rescaling of the $(\delta)$ terms yields

$$(M + svv^T)^{-1} (b_1) = M^{-1} b_1 - M^{-1} v v^T M^{-1} b_1 = \beta b_1 + O(\delta) u_1 - M^{-1} v v^T M^{-1} b_1 = \beta b_1 + O(\delta) u_1 - \beta a_1 b_i + O(\delta) u_i$$

Furthermore, for $j \neq 1$

$$(M + svv^T)^{-1} (b_j) = M^{-1} b_j - M^{-1} v v^T M^{-1} b_j = O(\delta) u_1 - M^{-1} v v^T M^{-1} b_1.$$

This completes the proof.
{λ₁, ..., λₙ}. Since G is connected, only a single eigenvalue λ₁ = 0 and the associated unit eigenvector is b₁ = (1/√n)1.

Since every vector is an eigenvector of the identity matrix I, {b₁, ..., bₙ} are orthonormal basis of eigenvectors for I + ρ(k)L with eigenvalues {1, 1 + ρ(k)λ₂, ..., 1 + ρ(k)λₙ}. But then (I + ρ(k)L)⁻¹ has the same basis of eigenvectors, with eigenvalues {1, [(1)/(1 + ρ(k)λ₂)], ..., [(1)/(1 + ρ(k)λₙ)]}.

As ρ(k) → ∞, [(1)/(1 + ρ(k)λ₂)] → 0 for each j ≠ 1. In particular, for any δ > 0, there is a k such that for each i = 2, ..., n

\[ \frac{1}{1 + \rho(k)λ_i} < δ \]

since ρ(k) is increasing. Moreover

\[ (I + \rho(k)L)^{-1}b_i = b_i \]
\[ (I + \rho(k)L)^{-1}b_i = \frac{1}{1 + \rho(k)λ_i}b_i \quad i = 2, \ldots, n. \]

Consequently, I + ρ(k)L satisfies the conditions of Lemma 4 with β = 1 and uᵢ = bᵢ for all i. This value of β will form the basis of a recurrence relation used to complete the proof.

Let I + ρ(k)L = M₀. Then, for each l up to n, let Mₗ = (Mₗ₋₁ + (sₗ - 1)eᵢᵀₑᵢ) where eᵢ is the lth vector of the standard basis. Since eᵢeᵢᵀ is the zero matrix with a one in the lth place on the diagonal, \( \sum_{i=1}^{n}(s_i - 1)e_i e_i^T = S - I \) and therefore \( M_n = (I + \rho(k)L) + \sum_{i=1}^{n}(s_i - 1)e_i e_i^T = S + \rho(k)L \).

By iterating Lemma 4 with s = s₁ = 1 and v = eᵢ, we have that for each l there is a βᵢ such that \( M_τ⁻¹b_i = β_i b_i + O(δ)u_i^{(τ)} \) and \( M_τ⁻¹b_j = O(δ)u_j^{(τ)} \) for j ≠ 1.

Since eᵢbᵢ = (1/√n), Lemma 4 gives the recurrence

\[ β_i = \frac{β_{i-1}}{1 + β_{i-1}s_{i-1}}. \]

Solving this recurrence with β₀ = 1 yields

\[ M_τ⁻¹b_i = \frac{n}{n + \sum_{k=1}^{τ}(s_k - 1)}b_i + O(δ)u_i^{(τ)}. \]

Since \( \sum_{k=1}^{τ}(s_k - 1) = tr(S) - n \), it is clear that

\[ M_τ⁻¹b_i = \frac{n}{tr(S)}b_i + O(δ)u_i^{(τ)}. \]

Therefore, for \( u = \sum_{i=1}^{n}b_i^TSx^u_i^{(n)} \)

\[ \bar{x}(k) = \frac{n}{tr(S)}b_i^TSx^u_i b_i + O(δ)\sum_{i=1}^{n}b_i^TSx^u_i^{(n)} \]
\[ = \frac{1^TSx^u}{tr(S)}1 + O(δ)u \]
\[ = \frac{\sum_{i=1}^{n}s_iX_i^}{\sum_{i=1}^{n}s_i}1 + O(δ)u. \]

Since δ → 0 as ρ(k) → ∞, if \( \lim_{k→∞}ρ(k) = ∞ \), then

\[ \lim_{k→∞}\bar{x}(k) = \frac{\sum_{i=1}^{n}s_iX_i^}{\sum_{i=1}^{n}s_i}1. \]

This completes the proof.

Since peer pressure increases in each step, no single \( F_k \) [see (3)] is sufficient to model the process of convergence. We use the following result from [63] and [64].

**Lemma 5** ([64, Th. 1] and [63, Th. 2]): Let \( \{f_n\} \) be a sequence of analytic contractions in a domain \( D \) with \( f_n(D) ⊆ E ⊆ D \) for all n. Then \( F_n = f_n ∘ f_{n-1} ∘ ... ∘ f_1 \) converges uniformly in \( D_0 \) and locally uniformly in \( D \) to a constant function \( F(c) = c ∈ E \). Furthermore, the fixed points of \( f_n \) converge to the constant c.

The following corollary is now immediate from Lemmas 3 and 5 and the fact that \( F_k \) is an analytic contraction in each dimension.

**Corollary 1:** From (3), let \( G_k = F_k ∘ G_{k-1} = F_k ∘ F_{k-1} ∘ ... ∘ F_1 \) for each k ≥ 0. Then \( G = \lim_{k→∞}G_k \) is a constant function.

We now have the following theorem, which follows immediately from Corollary 1 and Theorem 1.

**Theorem 2:** If \( ρ(k) → ∞ \), then

\[ \lim_{k→∞}\bar{x}(k) = \frac{\sum_{i=1}^{n}s_iX_i^}{\sum_{i=1}^{n}s_i}1. \]

This means that in the case of increasing and unbounded peer pressure, all the agents’ opinions always converge to consensus. In addition, the value of this consensus is the average of their preferences weighted by their stubbornness. This is irrespective of the weighting of the edges in the network, so long as the network is connected.

We illustrate opinion consensus on a simple graph with 15 vertices in Fig. 1. The vertices are organized into three connected cliques. Each clique was initialized with a distinct range of opinions in [0, 1]. Initial stubbornness was set randomly and is shown by relative vertex size. It is worth noting that care must be taken when simulating this model; numerical instability occurs rapidly because of its structure and care must be taken to control it. The opinion trajectories for this example are shown in Fig. 2.

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1 All figures created with Mathematica.
In the case of increasing but bounded peer pressure, we have

$$\lim_{k \to \infty} \rho^{(k)} \leq \rho^*.$$

Further, this limit always exists by monotone convergence. Intuitively, this means the influence of others is limited, and that personal preferences will always slightly skew the opinions of others [18], [65].

**Theorem 3:** Suppose \( \rho^{(k)} \) is increasing and bounded and that

$$\lim_{k \to \infty} \rho^{(k)} = \rho^*$$

then

$$\lim_{k \to \infty} \mathbf{x}^{(k)} = (\mathbf{S} + \rho^* \mathbf{L})^{-1} \mathbf{Sx}^*.$$

**Proof:** Since \( \rho^{(k)} \) is increasing and bounded, it converges to a finite number \( \rho^* \) by monotone convergence. From Lemma 2, \( (\mathbf{S} + \rho^* \mathbf{L}) \) is defined and invertible. Since matrix inversion is continuous in \( \text{GL}_n(\mathbb{R}) \), by Theorem 1

$$\lim_{k \to \infty} \mathbf{x}^{(k)} = \lim_{k \to \infty} \mathbf{x}^{(k)} = \lim_{k \to \infty} (\mathbf{S} + \rho^{(k)} \mathbf{L})^{-1} \mathbf{Sx}^* = (\mathbf{S} + \rho^* \mathbf{L})^{-1} \mathbf{Sx}^*.$$

The above theorem tells us that if peer pressure is increasing and bounded, the agents’ opinions converge to a fixed distribution, which may not be a consensus, but is easily computable from the initial preferences. In this case, the shape of the network is important for determining the limit distribution, as the edge weights factor into the Laplacian. This result is similar to the convergence point given in [18] where stubbornness coefficients are not presented and peer pressure is constant.

**IV. CONVERGENCE RATE**

We analyze the convergence rate of the algorithm and obtain a secondary result on efficiency. Define the utility of these convergent points to be the sum of the stress of the agents when the state \( \mathbf{x} \) is constant. Formally

$$U^{(k)}(\mathbf{x}) = \sum_{i} J_i(x_i, \mathbf{x}, k) = \sum_{i=1}^{n} s_i (x_i - x_i^+)^2 + \rho^{(k)} \left( \sum_{i,j} w_{ij} (x_i - x_j)^2 \right) = (\mathbf{x} - \mathbf{x}^+)^T \mathbf{S}(\mathbf{x} - \mathbf{x}^*) + 2\rho^{(k)} \mathbf{x}^T \mathbf{Lx} = \mathbf{x}^T (\mathbf{S} + 2\rho^{(k)} \mathbf{L}) \mathbf{x} - 2\mathbf{x}^T \mathbf{Sx}^* + (\mathbf{x}^*)^T \mathbf{Sx}^*. \tag{5}$$

Define the limiting utility \( U(\mathbf{x}) \) as

$$U(\mathbf{x}) = \lim_{k \to \infty} \frac{1}{\rho^{(k)}} U^{(k)}(\mathbf{x}). \tag{6}$$

The following lemma is immediately clear from the construction of the functions \( J_i \), the fact that \( U^{(k)} \) is a strictly convex function and \( U \) is the limit of these strictly convex functions.

**Lemma 6:** The global utility function \( U(\mathbf{x}) \) is convex. Furthermore, the fact that: 1) \( U^{(k)} \) is smooth on its entire domain and 2) \( U^{(k)}(\mathbf{x})/\rho^{(k)} \) converges uniformly to \( U(\mathbf{x}) \), implies that \( U(\mathbf{x}) \) is both differentiable and its derivative can be computed as the limit of the derivatives of \( U^{(k)}(\mathbf{x})/\rho^{(k)} \).

Using the global utility function, we can analyze the convergence rate of the update rule. From (2), we can compute

$$\Delta x_i^{(k-1)} = x_i^{(k-1)} - x_i^{(k-1)} = \frac{s_i (x_i^+ - x_i^{(k-1)}) + \sum_{j=1}^{n} (x_j^{(k-1)} - x_j^{(k-1)})}{s_i + \rho^{(k)} \sum_{j=1}^{n} w_{ij}}, \tag{7}$$

Let

$$\alpha_i^{(k)} = \frac{1}{s_i + \rho^{(k)} \sum_{j=1}^{n} w_{ij}} \tag{8}$$

and define \( \mathbf{H}^{(k)} = (1/2) \text{diag}(\alpha_1^{(k)}, \ldots, \alpha_n^{(k)}) \). Computing the gradient of \( U^{(k)} \) yields

$$\Delta \mathbf{x} = -\mathbf{H}^{(k)} \nabla U^{(k)}(\mathbf{x}^{(k-1)}). \tag{9}$$

We conclude the update rule, (2) can be written as

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} - \mathbf{H}^{(k)} \nabla U^{(k)}(\mathbf{x}^{(k-1)}). \tag{10}$$

Necessarily, \( \mathbf{H}^{(k)} \) is always positive definite and therefore \( -\mathbf{H}^{(k)} \nabla U^{(k)}(\mathbf{x}^{k-1}) \) is always a descent direction.
for $U^{(k)}$. Moreover, $(\nabla U_k)^T \nabla U > 0$ and consequently $-H^{(k)} \nabla U^{(k)}(x^{k-1})$ is a descent direction for $U(x)$. Thus, the update rule is a descent algorithm, which explains the initial fast convergence toward the average (see Fig. 2). When the descent direction converges to a Newton step, a descent algorithm can be shown to converge superlinearly [66]. However, these steps do not converge to Newton steps. As $\rho^{(k)}$ grows large, $a_i^{(k)} \to 0$ and $U_k/\rho^{(k)} \to U$ and consequently for large $k$

$$\frac{1}{\rho^{(k)}} H^{(k)} \nabla U^{(k)}(x^{k-1}) \approx \epsilon \nabla U(x^{k-1})$$

for $\epsilon \sim 1/\rho^{(k)}$. Thus, the update rule approaches a simple gradient descent. We show that a consequence of this is a linear convergence rate.

Let

$$x^* = \frac{\sum_{i=1}^n S_i x_i^+}{\sum_{i=1}^n s_i}$$

and define

$$y^{(k)} = x^{(k)} - x^*.$$  

From (10), we compute

$$\frac{\|x^{(k+1)} - x^*\|}{\|x^{(k)} - x^*\|} = \frac{\|y^{(k)} - H^{(k+1)} \nabla U^{(k+1)}(x^{(k)})\|}{\|y^{(k-1)} - H^{(k)} \nabla U^{(k)}(x^{(k-1)})\|}.$$  

(11)

Expanding the gradient using (5), we obtain

$$\frac{\|x^{(k+1)} - x^*\|}{\|x^{(k)} - x^*\|} = \frac{\|y^{(k)} - H^{(k+1)} [(S + 2\rho^{(k+1)}L)x^{(k)} - 2Sx^*]\|}{\|y^{(k-1)} - H^{(k)} [(S + 2\rho^{(k)}L)x^{(k-1)} - 2Sx^*]\|}. $$

Expanding and taking the limit as $\rho^{(k)} \to \infty$ and simplifying, we see that

$$\lim_{k \to \infty} \frac{\|x^{(k+1)} - x^*\|}{\|x^{(k)} - x^*\|} = \frac{\lim_{k \to \infty} 2\frac{\rho^{(k+1)}}{\rho^{(k)}} \|H^{(k+1)}Lx^{(k)}\|}{\|2\|H^{(k)}Lx^{(k-1)}\|}.$$  

(12)

As $\rho^{(k)} \to \infty$, we see that

$$H^{(k)} \to \frac{1}{2\rho^{(k)}} D^{-1}$$

where $D$ is the diagonal weighted degree matrix. Then

$$\lim_{k \to \infty} \frac{2\rho^{(k+1)}}{\rho^{(k)}} \|H^{(k+1)}Lx^{(k)}\| = \frac{\lim_{k \to \infty} (\rho^{(k+1)}) \|D^{-1}Lx^{(k)}\|}{\|\rho^{(k)} \|D^{-1}Lx^{(k-1)}\|} = 1.$$  

Thus, we have shown the following.

**Theorem 4:** The convergence rate of the update rule given in (2) is linear. In particular

$$\lim_{k \to \infty} \frac{\|x^{(k+1)} - x^*\|}{\|x^{(k)} - x^*\|} = 1.$$  

(13)

We illustrate the slow convergence on a larger example with 500 vertices organized into a scale-free graph using the Barabási-Albert (BA) [67] graph construction algorithm implemented Mathematica. In each round of the BA algorithm, one new vertex and one new edge is added. Edges are created between existing and new vertices at random with probability proportional to the degree of the incumbent vertices. Histograms of opinion evolution through time are shown in Fig. 3. The graph used and initial opinion distribution are shown in Fig. 4. We show the opinion trajectories for the 500 vertex scale-free network in Fig. 5(a) and illustrate (13) in Fig. 5(b). Notice the ratio $\|x^{(k+1)} - x^*\|/\|x^{(k)} - x^*\|$ approaches 1 as expected.

**V. Cost of Anarchy**

In any agent system, understanding the cost of decentralized coordination is critical. Bindel et al. [18] observed that simultaneous minimization of (1) is a game-theoretic problem and compare the total social utility in a centralized solution to a decentralized solution (Nash equilibrium); i.e., they computed a price of anarchy [18], [68]. To analyze the price of anarchy of this system, it is more convenient to use the total utility function $U_T(x) = \lim_{k \to \infty} U^{(k)}(x)$ to compute the cost of anarchy.
Theorem 5: The convergent point \( \lim_{k \to \infty} x^{(k)} \) minimizes total utility if and only if \( \lim_{k \to \infty} \rho^{(k)} = \infty \).

Proof: If \( \rho^{(k)} \) converges to a finite number \( \rho^* \), then the total utility is

\[
U_T(x) = x^T (S + 2 \rho^* L)x - 2x^T S x^* + (x^*)^T S x^*.
\]

Note that this is identical to the work in [18], except with edge weights multiplied by \( \rho^* \). We note that \( \lim_{k \to \infty} x^{(k)} \) is the Nash Equilibrium used in [18]. From the work in [18] we may conclude the convergent point is not optimal for finite \( \rho^* \).

If \( \lim_{k \to \infty} \rho^{(k)} = \infty \), then if \( x \neq c1 \) for some constant \( c \), then \( x^T L x > 0 \), so \( U^{(k)}(x) \) grows without bound. However, for any \( k \), we have that

\[
U^{(k)}(c1) = \sum_{i=1}^{n} s_i (c - x_{i^+})^2, \quad U_T(c1) = \sum_{i=1}^{n} s_i (c - x_{i^+})^2.
\]

By first order necessary conditions of optimality

\[
\sum_{i=1}^{n} s_i x_{i^+} = \sum_{i=1}^{n} s_i
\]

minimizes \( U(x) \), and thus \( \lim_{k \to \infty} x^{(k)} \) is optimal. \( \square \)

This gives the following trivial corollary, which is consistent with the work in [18].

Corollary 2: The cost of anarchy is 1 if and only if \( \lim_{k \to \infty} \rho^{(k)} = \infty \).

For the behavior when \( \rho^{(k)} \to \rho^* \), the results from [18] show the price of anarchy is bounded above by 9/8, thus completely characterizing the system.

VI. Conclusion

In this paper, we study an opinion formation model under the presence of increasing peer pressure. As in earlier work, we consider agents whose opinion is affected by unchanging innate beliefs. In this paper, the relative strength of these innate beliefs may vary from agent to agent. We show that in the case of unbounded peer-pressure, the opinion converges to a weighted average of innate beliefs is ensured. We also consider the case when peer-pressure is increasing, but bounded. Simulation suggests a numerically slow convergence, which is explained by showing the system dynamics converge to gradient descent applied to a certain convex function. Using this observation we show that convergence is linear.

We note that the assumption of a nonconstant (and increasing) peer-pressure coefficient can help mitigate the fast initial convergence of this class of models. However, by varying peer-pressure, the gradient descent can be controlled to modify transient behaviors in the agents.

In future work, the limitation that the network is undirected and symmetric should be removed to account for asymmetric social influence. In addition, the network is assumed to remain static during the convergence process, with connections independent of the agents’ opinions. Sufficiently different opinions could cause enough stress between agents so as to cause them to reduce influence or even sever the tie between them. A dynamic network model as in [69] could accommodate this kind of network update. Finally, it would be interesting to study corresponding control problems, in which we are given a \( x \), the desired convergence point and we can control a subset of agents reporting values \( (x^{(k)}) \), stubbornness \( (s_i) \) or initial value \( (x_{i^+}) \) to determine conditions under which opinion steering is possible. This problem becomes more interesting if the other agents attempt to determine whether certain agents are intentionally attempting to manipulate the opinion value \( x^{(k)} \). Of equal interest is the transient control problem in which \( x^{(k)} \) is steered through a set \( X \subset \mathbb{R}^n \) under the assumption that external factors will prevent convergence in the long-run.

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Authors’ photographs and biographies not available at the time of publication.