Dirac Operator of a Conformal Surface Immersed in $\mathbb{R}^4$: Further Generalized Weierstrass Relation

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Abstract.
In the previous report (J. Phys. A (1997) 30 4019-4029), I showed that the Dirac operator defined over a conformal surface immersed in $\mathbb{R}^3$ is identified with the Dirac operator which is generalized the Weierstrass-Enneper equation and Lax operator of the modified Novikov-Veselov (MNV) equation. In this article, I determine the Dirac operator defined over a conformal surface immersed in $\mathbb{R}^4$, which is reduced to the Lax operators of the nonlinear Schrödinger and the MNV equations by taking appropriate limits.

§1. Introduction

Investigation on an immersed geometry is current in various fields [1-6,8-25]. A certain class of the immersed surface in three dimensional space $\mathbb{R}^3$ is related to soliton theory. The surfaces, e.g., constant mean curvature surface, constant Gauss curvature surface, Willmore surface and so on, can be expressed by soliton equation and are sometimes called soliton surfaces. In the studies, Bobenko pointed out that an immersed surface in $\mathbb{R}^3$ should be studied through the spin structure over it [1,2]. The auxiliary linear differential operator related to soliton surface should be regarded as an operator acting the spin bundle on the surface.

Recently Konopelchenko discovered a Dirac-type operator defined over a conformal surface immersed in $\mathbb{R}^3$ which completely exhibits the symmetry of the immersed surface itself [3-4]. The Dirac-type differential equation for an conformal surface immersed in $\mathbb{R}^3$ is given as

$$\partial f_1 = pf_2, \quad \bar{\partial} f_2 = -pf_1,$$

where $p := \frac{i}{2} \sqrt{\rho} H$, $H$ is the mean curvature of the surface parameterized by complex $z$ and $\rho$ is the conformal metric induced from $\mathbb{R}^3$. (1-1) is a generalization of the Weierstrass-Enneper equation for the minimal surface [3-6]. This equation is sometimes called as generalized Weierstrass equation. As the geometrical interpretation of the modified Novikov-Veselov (MNV) equation [8], Konopelchenko and Taimanov studied the dynamics of the surface obeying the MNV equation [3-6].

Independently I proposed a construction scheme of a Dirac operator defined over an immersed object [8]. In a series of works [8-16], Burgess and Jensen and I have been studying the Dirac operator in an immersed object in $\mathbb{R}^n$. In refs.[8-13], I showed that the Dirac operator in an immersed curve in $\mathbb{R}^n$ can be regarded as the Lax operator of soliton equation while the extrinsic curvature of the immerse curve obeys the soliton equation. In other words, I obtained a natural spin structure over an immersed curve in $\mathbb{R}^n$. In fact, I proved that the Dirac operator I obtained is a canonical object and its analytic index agrees with the topological index of the immersed space [9,11].
After I proposed the construction scheme [8], Burgess and Jensen immediately applied my construction scheme to Dirac operator on a surface in $\mathbb{R}^3$ [14]. In previous work [15], I show that if I restrict the surface to conformal one, the Dirac operator turns out to be that of Konopelchenko (1-1) [2,3] and by investigation of its quantum symmetry, its functional space also exhibits the surface itself [16]. Hence it can be regarded that my construction scheme of the Dirac operator is canonical and gives structure over the immersed objects as Bobenko pointed out [1,2].

Recently using the Dirac operator in (1-1), Polyakov’s extrinsic string [17] has been studied by several authors [2-6,16,18-21]. As the Dirac operator in (1-1) completely exhibits the symmetry of an immersed surface in $\mathbb{R}^3$, in terms of the operator, one can investigate its symmetry. In other words, the Dirac equation (1-1) can be interpreted as a generalization of the Frenet-Serret relation of a space curve in the sense that the Dirac equation over an space curve can be regarded as ”a square root” of the Frenet-Serret relation [8-13].

Investigation of a space curve itself is of concern as a polymer physics and as a geometrical interpretation of soliton theory [22-28]. Using the Frenet-Serret relation (or the Dirac equation over the space curve) and isometry condition, I proposed a calculation procedure to obtain an exact partition function of an closed space curve (elastica) in two dimensional space $\mathbb{R}^2$ based on a soliton theory [26]. There appears the modified Korteweg-de Vries (MKdV) equation as a virtual equation of motion or thermal (quantized) fluctuation in the calculation of the partition function [26]. For the case of a space curve in $\mathbb{R}^3$, its partition function is related to the complex MKdV equation and the nonlinear Schrödinger (NLS) equation [24,25,27].

Furthermore since there is natural relation between a surface and a space curve, the space curve interests us from the point of view of string theory [17]. Grinevich and Schmidt studied a closed space curve from such viewpoint [28]. In fact the MNV equation can be regarded as a kind of complexification of the MKdV equation and corresponding surface (Willmore surface) in $\mathbb{R}^3$ can be interpreted as a generalization of an elastica [3-7,21]; both codimension are one. Using the correspondence and the Dirac operator in (1-1), I applied the calculation procedure of the partition function of [26] to the Willmore surface which has Polyakov extrinsic action [21]. It means that I quantized the surface (a kind of string in the string theory) in $\mathbb{R}^3$ [21]. In the calculation, I used the Dirac operator in (1-1) to look into the symmetry of the surface. Thus the my calculation procedure of the partition function of an immersed object is closely related to the construction scheme of the Dirac operator [15,16,21].

However a string in the string theory [17] might be immersed in ten or twenty-four dimensional space. Further it is expected that it will be quantized. Thus I desire the Dirac operator which expresses a surface immersed in higher dimensional space as a further generalization of the Dirac operator of Konopelchenko or the generalized Weierstrass operator (1-1). In this article, I will give a Dirac operator defined on a conformal surface immersed in four dimensional space $\mathbb{R}^4$ following the construction scheme I proposed in ref.[8]. I conjecture that the Dirac operator which I will obtain might exhibit the symmetry of the immersed object itself. In fact, it becomes the Lax operator of NLS equation [11] and that of the MNV equation [15] by taking appropriate limits.

Organization of this article is following. §2 gives the review of the extrinsic geometry of a conformal surface embedded in $\mathbb{R}^4$ and provides the notations in this article. In §3, I will confine a Dirac particle in the surface and determine the Dirac equation over the surface. Using the complex representation, I will gives more explicit form of the Dirac operator in §4. §5 gives confirmation whether obtained Dirac operator is natural object by taking appropriate limits. I will discuss my result in §6.

§2 Geometry of a Curved Surface Embedded in $\mathbb{R}^3$

In this section, I will set up a geometrical situation of the system which I will discuss, and give the notations used in this article [11,16,21,28]. I will deal with a conformal surface $S$ embedded in
\[ \Sigma \to S \subset \mathbb{R}^4, \quad (2-1) \]

where \( \Sigma = \mathbb{C}/\Gamma; \mathbb{C} \) is the complex plane and \( \Gamma \) is a Fuchsian group. Since the imaginary time and the euclidean quantum mechanics are very useful in the path integral method, I will deal with only the euclidean Dirac field in this article. Furthermore, for sake of simplicity, I assume that such a surface is embedded in \( \mathbb{R}^4 \) rather than immersed for a while.

I assume that a position on a conformal compact surface \( S \) is represented using the affine vector \[ x(q^1, q^2) = (x^1, x^2, x^3, x^4) \] in \( \mathbb{R}^4 \) and normal vectors of \( S \) are denoted by \( e_3 \) and \( e_4 \). Here \( q^1 \) and \( q^2 \) are natural parameters attached in the surface \( S \). The surface \( S \) has the conformal flat metric which is induced from the natural metric of \( \mathbb{R}^4 \),

\[ g_{\alpha \beta} dq^\alpha dq^\beta = \rho \delta_{\alpha \beta} dq^\alpha dq^\beta. \quad (2-2) \]

I assume that the beginning parts of Greek indices \( (q^\alpha, q^\beta, \ldots) \) span from one to two and adopt Einstein convention. \( \delta_{\alpha \beta} \) means the Kronecker delta matrix.

I will employ that the complex parameterization of the surface as,

\[ z := q^1 + iq^2, \quad (2-3) \]

and

\[ \partial := \frac{1}{2}(\partial q^1 - i\partial q^2), \quad \bar{\partial} := \frac{1}{2}(\partial q^1 + i\partial q^2), \quad d^2 q := dq^1 dq^2 := \frac{1}{2}d^2 z := \frac{1}{2}idz d\bar{z}. \quad (2-4) \]

I sometimes express it as \( f = f(q) \) for a real analytic function \( f \) over \( S \) but I will use the notation \( f = f(z) \) for a complex analytic function.

The moving frame over \( S \) is denoted as

\[ e^I_\alpha := \partial_\alpha x^I, \quad e^I_z := \partial x^I, \quad (2-5) \]

where \( \partial_\alpha := \partial_\alpha q^\alpha := \partial / \partial q^\alpha \). Their inverse matrices are denoted as \( e^\alpha_I \) and \( e^z_I \). The induced metric is expressed as

\[ g_{\alpha \beta} = \delta_{IJ} e^I_\alpha e^J_\beta, \quad \frac{1}{4} \rho = \langle e_z, e^z \rangle = \delta_{IJ} e^I_z e^J_z, \quad (2-6) \]

where \( \langle, \rangle \) denotes the canonical inner product in the euclidean space \( \mathbb{R}^4 \).

Since I will constraint a Dirac particle to be on \( S \) by taking an appropriate limit, I can pay attention to only the vicinity of \( S \) or a tubular neighborhood \( T \) of the surface \( S \) even before squeezing limit. I will consider the properties of the tubular neighborhood \( T \) and its affine structure in \( \mathbb{R}^4 \).

I wish that geometry in the tubular neighborhood \( T \) is expressed using the variables of \( S \). I will introduce a general coordinate system \( (q^1, q^2, q^3, q^4) \) besides \( (q^1, q^2) \) as a coordinate system of \( T \). The surface \( S \) will be expressed as \( q^3(x) = q^3(x) = 0 \).

Let the middle parts of the Greek indices \( (q^\mu, q^\nu, \ldots) \) used as curved system, \( \mu = 1, 2, 3, 4 \). Further I assume that the beginning parts of the Greek indices with dot \( (q^\alpha, q^\beta, \ldots) \) run from three to four.

I will assume that the normal structure is trivial; the metric of the normal plane is induced from \( \mathbb{R}^4 \) and vectors \( e_3 \) and \( e_4 \) are orthonormal: \( g_{\alpha \beta} = \delta_{\alpha \beta} \). For later convenience, I also introduce the complex structure over the normal plane,

\[ \xi := q^3 + iq^4, \quad \bar{\xi} := q^3 - iq^4. \quad (2-7) \]

The relation between the affine vector \( X := (X^1, X^2, X^3, X^4) \) in \( T \) and natural coordinate systems \( q^\mu \) is given by

\[ X(q^\mu) = x(q^\alpha) + q^\alpha e_\alpha. \quad (2-8) \]
This relation is uniquely determined if \( q^\dot{\alpha} \) is sufficiently small.

Next I will consider the extrinsic geometry of \( S \) itself. Using the moving frame \( e^I_\alpha \), I divide the ordinary derivative along \( S \) into the horizontal and vertical part; the horizontal part is written by \( \nabla_\alpha \) defined as

\[
\nabla_\alpha b := \partial_\alpha b - \langle \partial_\alpha b, e_\dot{\alpha} \rangle e_\dot{\alpha},
\]

for a vector field \( b \). The two-dimensional Christoffel symbol \( \gamma^\gamma_{\beta\alpha} \) attached on \( S \) is thus defined as

\[
\nabla_\alpha e_\beta = \gamma^\gamma_{\beta\alpha} e_\gamma,
\]

(2-9)

The second fundamental form is denoted as,

\[
\gamma^{\dot{\alpha}}_{\beta\alpha} := \langle e_\dot{\alpha}, \partial_\alpha e_\beta \rangle, \quad \gamma^\xi_{\beta\alpha} = \frac{1}{2}(\gamma^3_{\beta\alpha} + i\gamma^4_{\beta\alpha}).
\]

(2-10)

On the other hand, the Weingarten map, \( -\gamma^{\alpha}_{\dot{\beta}\dot{\alpha}} e_\alpha \), is defined by

\[
\gamma^{\alpha}_{\dot{\beta}\dot{\alpha}} e_\alpha := \nabla_\beta e_\dot{\alpha}, \quad \gamma^{\alpha}_{\dot{\beta}\dot{\alpha}} = \langle e_\alpha, \partial_\beta e_\dot{\alpha} \rangle.
\]

(2-12)

Because of \( \partial_\alpha < e_\beta, e_\dot{\alpha} > = 0 \), \( \gamma^{\alpha}_{\dot{\beta}\dot{\alpha}} \) is associated with the second fundamental form through the relation,

\[
\gamma^{\dot{\alpha}}_{\beta\alpha} = -\gamma^{\dot{\alpha}}_{\alpha\beta} g_{\alpha\beta}.
\]

(2-13)

Here I will choose the normal vectors \( e_\dot{\alpha} \) satisfied with,

\[
\partial_\alpha < e_\dot{\alpha}, e_\beta > = 0.
\]

(2-14)

The derivative of a more general normal orthonormal base \( \tilde{e}_\dot{\alpha} \) is given as

\[
\partial_\alpha \tilde{e}_\dot{\alpha} = \gamma^{\beta}_{\dot{\alpha}\alpha} \tilde{e}_\beta + \tilde{\gamma}^{\beta}_{\dot{\alpha}\beta} \tilde{e}_\dot{\beta}
\]

(2-15)

instead of (2-14). From the orthonormal condition \( \partial_\alpha < \tilde{e}_\dot{\alpha}, \tilde{e}_\dot{\beta} > = 0 \), the relations are obtained,

\[
\tilde{\gamma}^{\beta}_{\dot{\alpha}\alpha} = -\tilde{\gamma}^{\dot{\alpha}}_{\beta\alpha}, \quad \tilde{\gamma}^{\dot{\alpha}}_{\dot{\alpha}\alpha} \equiv 0 \ (\text{not summed over } \dot{\alpha}).
\]

(2-16)

In other words, there are only two parameters \( \tilde{\gamma}^{3}_{4\alpha} \) for \( \alpha = 1, 2 \). Thus I will employ a SO(2) transformation so that I hold the relation (2-14);

\[
\begin{pmatrix}
  e_3 \\
  e_4
\end{pmatrix} = \begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
  \tilde{e}_3 \\
  \tilde{e}_4
\end{pmatrix},
\]

(2-17)

where

\[
\theta := \int^{q_1} dq^1 \tilde{\gamma}^{3}_{41} + \int^{q_2} dq^2 \tilde{\gamma}^{3}_{42}.
\]

(2-18)

This transformation is sometimes called as Hashimoto transformation \([11,22,30]\).

As I prepared the languages to express the geometry of \( T \), I will gives it. In terms of \( e^I_\alpha \), the moving frame of \( T \) is described as,

\[
E^I_\mu := \partial_\mu X^I,
\]

(2-19)

\[
E^I_\alpha = e^I_\alpha + q^{\dot{\alpha}}_{\beta\dot{\alpha}} \gamma^{\beta}_{\alpha\beta} e^I_\beta, \quad E^I_{\dot{\alpha}} = e^I_{\dot{\alpha}}.
\]

(2-20)

Its inverse matrix is denoted by \( (E^\mu_I) \). The induced metric of \( T \) is given as

\[
G_{\mu\nu} = \delta_{IJ} E^I_\mu E^J_\nu,
\]

(2-21)
and is explicitly expressed as
\[
G_{\alpha\beta} = g_{\alpha\beta} + [\gamma^\gamma_{\alpha\alpha}g_{\gamma\beta} + g_{\alpha\gamma}\gamma^\gamma_{\alpha\beta}]q^\alpha + [\gamma^\gamma_{\alpha\alpha}g_{\beta\gamma} \gamma^\gamma_{\beta\beta}]q^\beta, \\
G_{\alpha\alpha} = G_{\alpha\dot{\alpha}} = 0, \\
G_{\dot{\alpha}\dot{\beta}} = g_{\dot{\alpha}\dot{\beta}} = \delta_{\dot{\alpha}\dot{\beta}}.
\]
(2-22)

The determinant of the metric \(G := \det(G_{\mu\nu})\) becomes
\[
G = g^\zeta, \quad \zeta := (1 + \text{tr}_2(\gamma^\alpha_{3\beta})q^3 + \text{tr}_2(\gamma^\alpha_{4\beta})q^4 + K(q^3, q^4)),
\]
where \(g := \det_2(g_{\alpha\beta})\) and \(K(q^3, q^4) = \mathcal{O}((q^3)^2, q^3q^4, (q^4)^2)\). Here \(\text{tr}_2\) is the two-dimensional trace over \(\alpha\) and \(\beta\). I will denote them as
\[
H_1 := -\frac{1}{2}\text{tr}_2(\gamma^\alpha_{3\beta}), \quad H_2 := -\frac{1}{2}\text{tr}_2(\gamma^\alpha_{4\beta}),
\]
(2-24)
and introduce the "complex mean curvature",
\[
H_\zeta = H_1 + iH_2.
\]
(2-25)

For an appropriate limit, it becomes ordinary mean curvature of one-codimensional [15] case and for another limit, it becomes complex curvature of a space curve in \(\mathbb{R}^3\) which is known as curvature given by the Hashimoto transformation [11,22-24,27].

Using (2-22), the Christoffel symbols associated with the coordinate system of the tubular neighborhood \(T\) is given as
\[
\Gamma^\mu_{\nu\rho} := \frac{1}{2}G^{\mu\tau}(G_{\nu\tau,\rho} + G_{\rho,\tau,\nu} - G_{\nu,\rho,\tau}).
\]
(2-26)
In terms of (2-26), the covariant derivative \(\nabla_\mu\) in \(T\) is defined by,
\[
\nabla_\mu B_\nu := \partial_\mu B_\nu - \Gamma^\lambda_{\mu\nu}B_\lambda,
\]
(2-27)
for a covariant vector \(B_\mu\).

§3 Dirac Field on a Surface Embedded in \(\mathbb{R}^4\)

As I set up the geometrical situation of my system, in this section I will consider the Dirac field \(\Psi = (\Psi^1, \Psi^2)^T\) defined in the tubular neighborhood \(T\) and confine it into \(S\) by taking a limit [8-16].

I will start with the original lagrangian given by,
\[
\mathcal{L}d^4x = \bar{\Psi}(x)i(\Gamma^I \partial_I - m_{\text{conf}}(q^I))\Psi(x) d^4x,
\]
(3-1)
where \(\Gamma^I\) is the gamma matrix in the \(\mathbb{R}^4\), \(\partial_I := \partial/\partial x^I\) and \(\bar{\Psi} = \Psi^I \Gamma^I\). I assume that \(m_{\text{conf}}\) has the form, \(m_{\text{conf}}(q^I) := \sqrt{\mu_0^2 + \omega^2((q^1)^2 + (q^2)^2)}\) for a large \(\omega\) and \(\mu_0\) with \(\mu_0 \gg \sqrt{\omega}\) [13,15].

I will denote a thin tubular neighborhood by \(T_0\). I can approximately regarded \(T_0\) as a trivial disk bundle; \(T_0 \cong \mathcal{D}_{1/\omega} \times S, \mathcal{D}_{1/\omega} := \{(y_1, y_2) \mid |(y_1, y_2)| < 1/\omega\}\). As well as in refs.[8-16,31-33], by paying my attention only on the ground state, the Dirac particle is approximately confined in the thin tubular neighborhood \(T_0\). Even though there is a \(U(1)(\approx \mathcal{D}_{1/\omega} - \{0\})\) symmetry around \(S\), I will choose a trivial rotation or constant function as a section of a sphere bundle \(U(1) \times S\) because non-trivial rotational component of the energy is expected as order of \(\omega\). After taking squeezed limit, I can realize the quasi-two-dimensional subspace in \(\mathbb{R}^4\) and, by integrating the Dirac field along the normal direction, express the system using the two-dimensional parameters \((q^1, q^2)\). Then I will interpret the Dirac field as that over the surface \(S\) itself [8-16].
Thus I will express the lagrangian in terms of the curved coordinate system $q^\mu$. For the coordinate transformation (2-8), the Dirac operator becomes $[8,13,15]$ 

$$i\Gamma^\mu \partial_\mu = i\Gamma^\mu \partial_\mu, \quad (3-2)$$

and the spinor representation of the coordinate transformation is given as

$$\Psi(q) = e^{-\Sigma^{IJ}\Omega_{IJ}}\Psi(x), \quad (3-3)$$

where $\Sigma^{IJ}$ is the spin matrix,

$$\Sigma^{IJ} := \frac{1}{2}[\Gamma^I, \Gamma^J], \quad (3-4)$$

and $\Sigma^{IJ}\Omega_{IJ}$ is a solution of the differential equation,

$$\partial_\mu (\Sigma^{IJ}\Omega_{IJ}) = \Omega_\mu, \quad \Omega_\mu := \frac{1}{2}\Sigma^{IJ}E_\mu^\nu(\nabla_\mu E_{J\nu}). \quad (3-5)$$

Hence the lagrangian density (3-1) becomes

$$\mathcal{L} \, d^4x = \bar{\Psi}(q)i(\Gamma^\mu D_\mu - m_{\text{conf}}(q^\dot{\alpha}))\Psi(q) \sqrt{G} \, d^4q, \quad (3-6)$$

where $\Gamma^\mu := \Gamma^I E_\mu^\mu$ and $D_\mu$ denotes the spin connection,

$$D_\mu := (\partial_\mu + \Omega_\mu). \quad (3-7)$$

Here I will note the relation for the normal direction,

$$D_\dot{\alpha} = \partial_\dot{\alpha} \quad \text{module} \ q^{\dot{\beta}}. \quad (3-8)$$

Since the measure on the curved system is given as,

$$d^4x = \sqrt{G} \cdot d^4q, \quad (3-9)$$

and $-iD_\dot{\alpha}$ is not hermite nor a momentum operator, I redefine the field as $[8-16,30-33]$,

$$\Psi = \zeta^{1/2}\Psi. \quad (3-10)$$

Then the lagrangian density (3-6) changes as,

$$\mathcal{L} \, d^4x = \bar{\Psi}(q)i(\Gamma^\mu \mathbb{D}_\mu - m_{\text{conf}}(q^\dot{\alpha}))\Psi(q) \sqrt{\bar{g}} \, d^4q, \quad (3-11)$$

where

$$\mathbb{D}_\alpha := D_\alpha - \frac{1}{4}\partial_\alpha \log \zeta, \quad \mathbb{D}_{\dot{\alpha}} := D_{\dot{\alpha}} + \frac{H_{\dot{\alpha}} - 2H_1q^3 - 2H_2q^4 + K(q^3, q^4)}{1 - 2H_1q^3 - 2H_2q^4 + K(q^3, q^4)}. \quad (3-12)$$

Due to (3-10), in the deformed Hilbert space spanned by $\Psi$, $-iD_\dot{\alpha}$ is a generator of the translation along the normal direction and thus it is the momentum operator. Thus after squeezing limit, the normal direction is independent space and can be regarded as a inner space.
More mathematical speaking, the quantum physics is based on tangent bundle of a manifold rather than the manifold itself. The original differential operator should be interpreted as an operator defined over \( T\mathbb{R}^4 \approx \mathbb{R}^8 \) rather than \( \mathbb{R}^4 \) \([33]\). More precisely speaking, in the case of the Dirac particle, the Dirac operator acts upon the spin bundle over \( \mathbb{R}^4 \), \( \text{Spin}(\mathbb{R}^4) \) and its fiber has the structure of Clifford algebra induced from the tangent bundle \( T\mathbb{R}^4 \) \([34]\). The differential operator \(-i\partial\) is a generator of the translation \( \mathbb{R}^4 \) and should be regarded as a base of the fiber space of \( T\mathbb{R}^4 \) or \( \text{Spin}(\mathbb{R}^4) \). Hence after coordinate transformation from \( x^i \) to \( q^\mu \) by restricting the manifold \( \mathbb{R}^4 \) to \( T_0 \), I interpret the Dirac operator as an operator over the spin bundle \( \text{Spin}(T_0) \) induced from the tangent bundle \( T\mathbb{T}_0 \). The action of (3-11) decomposes the function space over the tangent space of \( T\mathbb{T}_0 \) to that over the tangential and that over the normal direction for \( S \). By (3-8) and (3-10), the normal part is asymptotically equivalent with that of \( T\mathbb{R}^2 \) in the sense of real analytic function. In other words, \( D_\alpha \) becomes the base of the fiber space and due to the confinement potential, the function space along the normal direction is effectively restricted as a set of compact support functions over the fiber space \( \mathfrak{D}_{1/\omega} \approx \mathbb{R}^2 \) of \( T_0 \). In the limit, \( D_\alpha \) behaves like \( \partial_\alpha \) which is a translation operator of flat space \( \mathbb{R}^4 \) or \( \text{Spin}(\mathbb{R}^4) \) is a generator of the translation \( \mathbb{R}^4 \)]. Hence after squeezing limit, \( T\mathbb{T}_0 \) is reduced to \( TS \times T\mathfrak{D}_{1/\omega} \) as a trivial bundle over \( TS \). Then I can regard the normal direction \( T\mathfrak{D}_{1/\omega} \) as an inner space over \( TS \) \([8-16,31-33]\).

Such inner space comes from the symmetry of the euclidean space \( \mathbb{R}^4 \) and is restricted by an immersed object. Hence at least, in the cases of a space curve in \( \mathbb{R}^n \) \( n > 1 \), and a surface in \( \mathbb{R}^3 \), the Dirac operator completely exhibits the embedded symmetry \([8-16]\).

After squeezing limit, due to the confinement potential \( m_{\text{conf}} \), the Dirac field for the normal direction is asymptotically factorized and can be expressed by the modes classified by a non-negative integer \( n \) \([8-15]\). Then there exists a mode with the lowest energy for normal direction and trivial rotation such that

\[
\Psi(q^1, q^2, q^3, q^4) \sim \sqrt{\delta(|q^\alpha|)}\psi(q^1, q^2)
\]

and

\[
(\Gamma^\alpha \partial_\alpha - m_{\text{conf}}(q^\alpha))\sqrt{\delta(q^\alpha)}\psi(q^1, q^2) = m_0 \sqrt{\delta(|q^\alpha|)}\psi(q^1, q^2).
\]

By restricting the function space of the Dirac field to the ground state for the normal direction \((n = 0)\), I will define the lagrangian density on a surface \( S \) \([15]\),

\[
\mathcal{L}_S^{(0)} \sqrt{g}d^2q := \left( \int dq^3dq^4 \mathcal{L} \sqrt{G} \right)_{n=0} \quad d^2q,
\]

and then it has the form,

\[
\mathcal{L}_S^{(0)} \sqrt{g}d^2q = i\bar{\psi}(\gamma^1 D_1 + \gamma^2 D_2 + H_1 \gamma^3 + H_2 \gamma^4 + m_0)\psi \sqrt{g}d^2q,
\]

where I rewrite the quantities as

\[
e^I_\alpha \equiv E^I_\alpha|_{q^\alpha = 0}, \quad \gamma^\mu(q^1, q^2) := \Gamma^\mu(q^1, q^2, q^\alpha \equiv 0),
\]

\[
\omega_\alpha(q^1, q^2) := \Omega_\alpha(q^1, q^2, q^\alpha \equiv 0), \quad \mathcal{D}_\alpha := \partial_\alpha + \omega_\alpha.
\]

Since a confined space is expressed by the two-dimensional parameter \((q^1, q^2)\) and can be regarded as a two-dimensional space, I can identify the confined space with the surface \( S \) itself and then the differential operator in (3-16) is interpreted as a Dirac operator in a surface \( S \) embedded in \( \mathbb{R}^4 \). Hence the spin connection can be written as

\[
\omega_\alpha := \frac{1}{2} \sum_{ij} e^i_\alpha (\nabla_\alpha e_j),
\]

(3-18)
where the indices of $i, j$ run from 1 to 2.

It should be noted that the Dirac operator in (3-16) is not hermite in general as that in a space curve immersed in $\mathbb{R}^4$ is neither [8-16]. It is natural because the extra term in the corresponding Schrödinger equation in the surface $S$ behaves the negative potential [30-33]; roughly speaking the square root of the negative potential appears as a pure imaginary extra field in the Dirac equation [8-16].

As I argued in ref.[15], an immersion can be approximately regarded as embedding an object in more higher dimensional space; A loop soliton in $\mathbb{R}^2$ is mathematically realized as an immersed curve in $\mathbb{R}^2$ but is physically realized as a thin rod on a plane in $\mathbb{R}^3$, whose overlap is negligible and their dimensions can be approximately regarded as one and two of an immersed curve in $\mathbb{R}^2$. Thus I can generalize this result of an embedding surface in $\mathbb{R}^4$ to that of immersed case.

§4 Dirac Operator on a Complex Surface Immersed in $\mathbb{R}^4$

Generally $m_0$ does not vanish, but I will neglect the mass $m_0$ hereafter as in refs.[8-16]. I am interested in the properties of the Dirac operator itself and I will investigate the high energy behavior of the field with $m_0$, i.e., the behavior of the high energy for surface direction and the lowest energy of the normal direction using the independency of the normal modes.

Since the surface $S$ is conformal, I will explicitly express (3-16). Then the Christoffel symbol is calculated as [17],

$$\gamma^\alpha_{\beta\gamma} = \frac{1}{2} \rho^{-1} (\delta^\alpha_\beta \partial_\gamma \rho + \delta^\alpha_\gamma \partial_\beta \rho - \delta^\alpha_\beta \partial_\gamma \rho). \tag{4-1}$$

I will denote a natural euclidean inner space by $y^a, y^b, \cdots, (a, b = 1, 2)$. Then the moving frame is written as,

$$e^a_\alpha = \rho^{1/2} \sigma^a_\alpha, \tag{4-2}$$

and the gamma matrix is connected to the Pauli matrices $\sigma^a$,

$$\gamma^\alpha = e^a_\alpha \sigma^1 \otimes \sigma^a. \tag{4-3}$$

Thus the spin connection becomes

$$\omega_\alpha = -\frac{1}{4} \rho^{-1} (\sigma^a_{\delta a} \rho \delta^a_{\rho b} - \sigma^a_{\rho b} \delta^a_{\rho a}), \tag{4-4}$$

where $\sigma^{ab} := 1 \otimes [\sigma^a, \sigma^b]/2$. The Dirac operator in (3-16) can be expressed as

$$\gamma^\alpha D_\alpha = \sigma^1 \otimes \sigma^a \delta^a_{\rho b} [\rho^{-1/2} \partial_\alpha + \frac{1}{2} \rho^{-3/2} (\partial_\alpha \rho)]. \tag{4-5}$$

Similar to (3-10), I will redefine the Dirac field in the surface $S$ as

$$\psi := \rho^{1/2} \psi \tag{4-6}$$

and then the Dirac operator (4-5) becomes simpler [15,17],

$$\gamma^\alpha D_\alpha \psi = \rho^{-1} \sigma^a \delta^a_{\rho b} \partial_\alpha \psi. \tag{4-7}$$

On the other hand, due to the properties of the Dirac matrix and noting the fact that the normal direction should be regarded as an inner space, $\gamma^\alpha$ is represented as,

$$\gamma^3 = \sigma^1 \otimes \sigma^3, \quad \gamma^4 = \sigma^2 \otimes 1. \tag{4-8}$$
Since $\bar{\psi} \gamma^1 \psi \sqrt{g} d^2 q$ is the charge density, I expect the relation,

$$\bar{\psi} = \psi^\dagger \gamma_1 = \psi^\dagger \sigma^1 \otimes \sigma^1 \rho^{1/2}. \quad (4-9)$$

Accordingly the lagrangian density (3-16) is reduced to,

$$L^{(0)}_S \sqrt{g} d^2 q = i \bar{\psi} \rho - \frac{1}{2} \left( \psi_1^+ \bar{\psi}_1^+ \sigma^1 \otimes \sigma^1 \rho^{1/2} \right)$$

$$(4-10)$$

Here I will denote the field as

$$\psi = \begin{pmatrix} \psi_1^+ \\ \psi_1^- \\ \psi_2^+ \\ \psi_2^- \end{pmatrix}. \quad (4-11)$$

Using the complex parameterization of the surface (2-3) and (2-4), the lagrangian density (4-10) can be explicitly expressed. Then the equation of motion of the Dirac field in the conformal surface $S$ immersed in $\mathbb{R}^4$ is derived as

$$\begin{pmatrix} \frac{\partial}{\partial \bar{p}_c} & \partial \\ \frac{\partial}{\partial p_c} & -\bar{p}_c \end{pmatrix} \begin{pmatrix} \psi_1^+ \\ \psi_1^- \\ \psi_2^+ \\ \psi_2^- \end{pmatrix} = 0,$$

$$(4-12)$$

where the "external" field $p_c$ is defined as,

$$p_c := \frac{1}{2} \rho^{1/2} H_{\xi}. \quad (4-13)$$

(4-12) is the Dirac equation canonically defined over the conformal surface $S$ immersed in $\mathbb{R}^4$. This equation is a generalization of that of a space curve immersed in $\mathbb{R}^3$ [8-13] and that of a conformal surface immersed in $\mathbb{R}^3$ [14-16] following my procedure [8-16].

### §5 Some Limits

In this section, I will check whether the obtained Dirac operator in (4-12) naturally behaves after taking some limits.

#### §5-1 Reduction to a space curve.

First I will consider a space curve $C$ in $\mathbb{R}^3$, which is parameterized by $q^1 \in S^1$. Its codimension is two. Due to the codimensionality, this situation resembles a surface immersed in $\mathbb{R}^4$, which I dealt with until the previous section. I can extend $C \subset \mathbb{R}^3$ to $S \subset \mathbb{R}^4$ by multiplying a trivial line manifold $\mathbb{R}$; $C \times \mathbb{R}$ can be regarded as a surface $S$. This surface is not compact and is contradict in my assumption in (2-1). However it is not difficult to see that the assumption can be extended to this case.

Hence I can regard this situation as a special case of the situation which I argued until the previous section. I explicitly represent the "mean curvature" in (2-24) and metric in (2-6),

$$H_1 = -\frac{1}{2} \text{tr}_2 (\gamma^1_{31}), \quad H_2 = -\frac{1}{2} \text{tr}_2 (\gamma^1_{41}), \quad \rho = 1. \quad (5-1)$$

They are a half of curvatures under the Hashimoto representation of the space curve and $H_{\xi}$ is a complex curvature. Vortex soliton obeys the NLS equation and whose dynamics is expressed in terms of $H_{\xi}$ [22-24],

$$i \partial_t H_{\xi} + \frac{1}{2} \partial^2_{\xi} H_{\xi} + |H_{\xi}|^2 H_{\xi} = 0. \quad (5-2)$$
Now I will consider (4-12) in this case. Then there is a trivial solution $\psi = \psi(q^1)$ and $\partial_2 \psi = 0$. Noting $\partial f(q^1) = \partial_1 f(q^1)/2$, (4-12) reduces to

$$L = \left( \begin{array}{cc} H_\xi & \partial_1 \\ \partial_1 & -H_\xi \end{array} \right).$$

(5-3)

which is one of its Lax operator of the NLS equation [34]. I obtained (5-3) by considering the Dirac operator over an immersed curve in $\mathbb{R}^3$ in refs.[10,11]. Further (5-3) can be regarded as the Frenet-Serret equation by inverse of the Hashimoto transformation [10,11]. It is a remarkable fact that the Dirac operator in (4-12) reproduces the Dirac operator in refs.[10,11] and is associated with the vortex soliton.

§ 5-2 Reduction to a surface in $\mathbb{R}^3$.

Next I will constrain the surface $S$ to be immersed in $\mathbb{R}^3$, which Konopelchenko and Taimanov dealt in refs.[3-6] and I considered in the previous paper [15].

I will suppose that the direction $e_4$ is identified with the direction of $x^4$ and $e_4$ geometry becomes trivial;

$$H_2 = 0$$

(5-4)

Then the external field is obtained as,

$$p_c \equiv p_1 := \frac{1}{2} \sqrt{\rho H_1}$$

(5-5)

and the equation (4-12) is reduced to the Konopelchenko’s Dirac equation (1-1) [3-6], which represents the symmetry of the immersed surface in $\mathbb{R}^3$;

$$\left( \begin{array}{cc} p_1 & \partial \\ \partial & p_1 \end{array} \right) \left( \begin{array}{c} \psi_{a+} \\ \psi_{a-} \end{array} \right) = 0.$$

(5-6)

This is one of the Lax operator of MNV equation [3-7]. Thus the Dirac operator in (4-12) contains my previous result in ref.[15] as its special case.

When I use the complex parameterization of the part of the euclidean space,

$$Z := x^1 + ix^2,$$

(5-7)

we have special solutions of (5-6) [3-6],

$$2i(\psi_{a+})^2 := -\partial \bar{Z}, \quad 2i(\psi_{a-})^2 := \partial \bar{Z}, \quad -2\psi_{a+}\psi_{a-} = \partial x^3,$$

(5-8)

Hence the Dirac operator in (4-12) partially exhibits the symmetry of immersed surface.

§ 5-3 Reduction to a minimal surface in $\mathbb{R}^3$.

Provided that $H_\xi = 0$, the equation of motion (4-12) becomes the original Weierstrass-Enneper formula [1-6,15] which was obtained as a scheme to obtain a minimal surface in last century and the lagrangian (4-10) becomes that in the ordinary (classical) conformal field theory [17].

Thus the Dirac operator in (4-12) contains the original Weierstrass-Enneper formula.

§ 6 Discussion

In this article, I obtained the Dirac operator defined over a conformal surface $S$ immersed in $\mathbb{R}^4$. As I argued in §5, the Dirac operator in (4-12) partially exhibits the symmetries of the immersed object; it recovers the Lax operators of the integrable surface and curve by taking appropriate limits. Thus I believe that it is natural and can be partially regarded as a generalization of the Dirac operator of Konopelchenko or that of the generalized Weierstrass equation. Consequently I conjecture that it would exhibit the symmetry of such immersion in $\mathbb{R}^4$. This conjecture could not be proved in this stage but if true, using this operator, one can quantize a surface in $\mathbb{R}^4$ as I did in ref.[21].
Finally I will comment upon the higher dimensional case. As I did in ref.[10], it is not so difficult that the argument in this article can be generalized to a case of a surface in more higher dimensional space $\mathbb{R}^n$ $n > 3$. Thus it is expected that further studies on this Dirac operator might be important for the immersion and quantization of a surface in $\mathbb{R}^{10}$ or $\mathbb{R}^{24}$.

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