Exact Travelling Wave Solution of Fractional Space-time Zakharov–Kuznetsov Equation

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Abstract

Objectives: To find the new exact travelling wave solutions of fractional order space-time Zakharov–Kuznetsov (ZK) equation of ion-acoustic waves. Methods/Statistical: The fractional derivative is defined in the sense of modified Riemann- Liouville derivative to convert the fractional space-time ZK equation to nonlinear ordinary differential equation. The two proposed techniques the (ω'/ω)-expansion and extended (ω'/ω)-expansion methods are employed for constructing the new exact travelling wave solutions of ZK equation. Findings/Results: The obtained new exact travelling wave solutions of ZK equation include the hyperbolic function, rational function, and trigonometric function. Application: The results reveal that the two used methods here are effective and powerful methods and might be accustomed to establish the other solutions of nonlinear fractional partial differential equations that arising in nonlinear science.

Keywords: The (ω'/ω)-Expansion Method, The Extended (ω'/ω)-Expansion Method, The Modified Riemann- Liouville Derivative, Homogeneous Balance, The New Exact Travelling Wave Solutions

1. Introduction

The fractional calculus has a wide array namely, nonlinear optic, solid state physics, fluid flow, plasma physics, control theory, signal processing, systems identifications, biology and so on. During this paper the target of our aim by means that two proposed methods, namely and extended expansions methods to solve the following fractional space-time in nonlinear of ion-acoustic waves arising in plasma physics as

\[ 16 \left( \frac{\partial \phi}{\partial \tilde{u}} - \frac{c \partial \phi}{\partial \tilde{v}} \right) + 30 \phi \frac{\partial^3 \phi}{\partial \tilde{u}^3} + \frac{\partial^3 \phi}{\partial \tilde{v}^3} + \frac{\partial}{\partial \tilde{u}} \left( \frac{\partial^2 \phi}{\partial \tilde{u}^2} + \frac{\partial^2 \phi}{\partial \tilde{v}^2} \right) = 0 \]  

(1.1a)

Eq.(1.1a) can be rewritten in the fractional space-time form as

\[ 16 \left( \frac{\partial^\alpha \phi}{\partial t^\alpha} - \frac{c \partial^\alpha \phi}{\partial x^\alpha} \right) + 30 \phi \frac{\partial^3 \phi}{\partial x^3} + \frac{\partial^3 \phi}{\partial y^3} + \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = 0 \]  

(1.1b)

where, c is constant, consider the fractional transform

\[ \phi(x,y,z,t)=G(\theta), \quad \theta=\frac{\tilde{u}^a}{\Gamma(l+a)} + \frac{\tilde{v}^a}{\Gamma(l+a)} + \frac{\tilde{a}^a}{\Gamma(l+a)} - \frac{\tilde{a}^a}{\Gamma(l+a)} \]  

(1.2)

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where, and are constants to be resolved later on. By using Eq. (1.2) into Eq. (1.1b) can be reduced to

$$16(w - kc)G' + 30G^2 G' + (k^3 + k l^2 + k \rho^2)G'' = 0$$

(1.3)

Integrate Eq. (1.3) and chose the constant of integration to be zero, we deduce

$$16(w - kc)G + 20G^2 + (k^3 + k l^2 + k \rho^2)G'' = 0$$

(1.4)

Using $$G^{\frac{1}{2}} = V$$ Eq.(1.4) becomes

$$16(w - kc)V^2 + 2(V^3 + 2(k^3 + k l^2 + k \rho^2)(V' + V') = 0$$

(1.5)

This paper has been outlined as follows: Section 2 represents the modified Riemann- Liouville derivative and its a few properties. In sections 3A, 3B two proposed techniques are explained. In sections 4, 5 the obtained solutions of the reduced Eq. (1.5) have been presented. In last section the conclusion is given.

2. The Modified Riemann-Liouville Derivative and Some Properties

In this section, the Jumarie's Riemann-Liouville derivative is outlined as follows \(^1\)

$$D^\alpha \eta(\tau) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^\tau (\tau - \eta)^{-\alpha} [g(\eta) - g(0)]d\eta, & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_0^\tau (\tau - \eta)^{-\alpha} [g(\eta) - g(0)]d\eta, & \alpha < 1 \\ (g^{(m)}(\tau))^{(\alpha-m)} & m \leq \alpha < m+1, \; m \geq 1 \end{cases}$$

(2.1)

With the properties:

i) $$D^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \; \gamma > 0,$$

(2.2)

ii) $$D^\alpha_t (a g(t) + b f(t)) = a D^\alpha_t g(t) + b D^\alpha_t f(t),$$

(2.3)

where a, b are constants.

iii) $$D^\alpha g(\eta) = \frac{dg}{d\eta} D^\alpha \eta, \; \eta = g(t),$$

(2.4)

iv) $$D^\alpha c = 0, \; c \text{ is constant}$$

(2.5)

v) $$d^\alpha x(t) = \Gamma(1+\alpha) dx(t).$$

(2.6)

3. Algorithms of the Two Proposed Method

3.1 Methodology of the -Expansion Technique

In what follows, we explain the algorithm of the -expansion method as follows:

Step 1: Assume the general fractional space-time NPDE as

$$H \{v, D^\alpha v, D^\beta v, D^\large\gamma v, D^\alpha D^\beta v, D^\gamma D^\alpha v, \ldots\}, \; 0 < \alpha, \beta < 1,$$

(3.1a)

where, is an unknown function in two independent variables and, is a polynomial in terms of, and its derivatives and including the nonlinear terms.
Step 2: Consider the travelling wave transorm form as:

\( v(x,t) = V(\eta), \quad \eta = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} \).

(3.2a)

Inserting Eq.(3.2a) into (3.1a), we obtain:

\( Z(V', cV', V'', cV'', cV''', V''''', V'''''', ...) = 0, \)

(3.3a)

Step 3: We express the solution of ODE (3.3a) in the form:

\( V(\eta) = \sum_{i=0}^{N} \beta_i \left( \frac{\omega'}{\omega} \right)^i \)

(3.4a)

where, satisfies

\( \omega'' + \lambda \omega' + \mu \omega = 0, \)

(3.5a)

where, \( \beta_i, \lambda \) and \( \mu \) can be specified later on, \( \beta_i \neq 0 \).

Step 4: Inserting Eq.(3.4a) into Eq. (3.3a) with Eq. (3.5a), equating the order of the same order of \( \omega \) to be zero, we get the set of algebraic equations for \( \beta_i, c, \lambda \) and \( \mu \). By solving the system of algebraic equations with the solutions of Eq. (3.5) admits to the solutions of Eq.(3.1a).

3.2 The Extended –Expansion Technique

In what follows, the proposed technique\(^{2}\) is described. For a give general partial differential equation as

\[ H \left( v, D_v^\alpha, D_v^{2\alpha}, D_v^{3\alpha}, ... \right), \quad 0 < \alpha, \beta < 1, \]

(3.1b)

To derive the exact solution of Eq. (3.1b), consider the fractional transformation:

\( v \dddot{\dddot{v}} = V(\eta), \quad \eta = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} \)

(3.2b)

Substituting Eq.(3.2b) into Eq.(3.1a) transfers o

\( Z(V', cV', V'', cV'', cV''', V''''', V'''''', ...) = 0, \)

(3.3b)

We express the solution \( V(\eta) \) as finite series in the form

\( V(\eta) = \sum_{i=0}^{N} \beta_i \left( \frac{\omega'}{\omega} \right)^i, \)

(3.4b)

where, \( N \) is the balancing number which is positive integer, it is evaluated by considering the homogeneous balance. \( \omega=\omega(\eta) \) satisfies

\( \omega'' = D\omega^2 + E\omega' + F(\omega')^2 \)

(3.5b)

where, \( D, E \) and \( F \) are real parameters should be determined later.

4. Application of the \((\omega'/\omega) –\)Expansion Technique to the ZK Equation

Apply the \((\omega'/\omega) –\)expansion technique to extract the exact solution for ZK equation (1.5). Assume that the homogenous balancing between \( V^1 \) with \( VV'' \), we get \( N = 2 \). Then the solutions of Eq. (1.5) is defined as:

\( V(\eta) = \beta_0 + \beta_1 \left( \frac{\omega'}{\omega} \right) + \beta_2 \left( \frac{\omega'}{\omega} \right)^2, \)

(4.1)

Satisfies Eq. (3.5a), are constants should be calculated later on.

Inserting Eq. (4.1) into Eq. (1.5) and equating the same power of \( (\omega'/\omega) \) to be zero. We get the set of algebraic equations for \( \beta_0, \beta_1, \beta_2, k, l, w \). By solving this set of algebraic equations, we have
\[ \beta_i = -k^i \mu - k^i \nu - k^i \rho^i \mu, \quad \beta_i = -k^i \lambda - k^i \nu - k^i \rho^i \lambda, \quad \beta_i = -k^i - k^i \rho^i, \]
\[ w = \frac{(4k^i \mu - k^i \lambda^2 - k^i \lambda^2 + 4 \mu^2 + 4 \nu^2 + 4 \rho^i + 4k^i)}{4}, \quad k = k, \quad l = l \]

(4.2)

By inserting Eq. (4.2) into Eq. (4.1), admits the following new three exact solutions as follows:

**Case I:** If \( \Delta = \lambda^2 - 4\mu > 0 \), then the hyperbolic function solution is:

\[ V_i = -\alpha k (\mu + \delta \lambda^2 + \delta^2), \quad (4.3) \]

where,

\[ \alpha_i = k^2 + l^2 + \rho^2, \quad c_i = \frac{\sqrt{\Delta}}{2}. \]

\[ \delta = c_i \left( \frac{h_1 \sinh(c \eta) + h_2 \cosh(c \eta)}{h_1 \cosh(c \eta) + h_2 \sinh(c \eta)} \right) - \frac{\lambda}{2} \]

**Case II:** If \( \Delta = \lambda^2 - 4\mu < 0 \), then the trigonometric function solution is:

\[ V_i = -\alpha k (\mu + \lambda \delta^2 + \delta^2), \quad (4.4) \]

where,

\[ \alpha_i = k^2 + l^2 + \rho^2, \]

\[ c_1 = \frac{\sqrt{\Delta}}{2}, \quad c_2 = \frac{\sqrt{-\Delta}}{2}, \]

\[ \delta = c_2 \left( \frac{-h_1 \sin(c \eta) + h_2 \cos(c \eta)}{h_1 \cos(c \eta) + h_2 \sin(c \eta)} \right) - \frac{\lambda}{2} \]

**Case III:** If \( \Delta = \lambda^2 - 4\mu = 0 \), then the rational function solution is:

\[ V_i = -\alpha (\mu + \lambda \delta^2 + \delta^2), \quad (4.5) \]

where,

\[ \alpha_i = k^2 + l^2 + \rho^2, \]

\[ \delta^2 = \left( \frac{k_2}{k + k_2 \eta} \right) \frac{\lambda}{2}, \]

\[ G^2(\eta) = V(\eta), \quad \eta = \frac{kx}{1+\alpha} + \frac{l}{1+\alpha} + \frac{\rho^i}{1+\alpha} - \frac{w t}{1+\alpha} \]

5. Application of Extended – Expansion Method to the ZK Equation

Here, to solve Eq.(1.5) via the extended –expansion method. Consider the balancing with, wed educe \( N = 2 \). Therefore, the solution of Eq.(1.5) becomes

\[ V(\eta) = \beta_0 + \beta_1 \left( \frac{\omega^i}{\omega} \right) + \beta_2 \left( \frac{\omega^i}{\omega} \right)^2, \quad (5.1) \]

Substituting Eq. (5.1) into Eq.(1.5), we obtain the values of \( \beta_0, \beta_1, \beta_2, k, l, w \)

\[ \beta_0 = -k^i D^i F^i - k^i D^i D^i F^i + k^i D^i D^i + k^i D^i D^i, \]
\[ \beta_1 = k^i E^i + k^i F^i + k^i E^i - k^i \rho^i E^i - k^i \rho^i E^i + k^i \rho^i E^i, \]
\[ \beta_2 = -k^i T^i - k^i L^i T^i - k^i F^i + k^i D^i D^i F^i - k^i D^i D^i + k^i D^i D^i - k^i D^i D^i D^i - k^i D^i D^i + 4k^i), \]
\[ w = \frac{(4k^i D^i F^i + 4D^i D^i F^i - 4k^i D^i D^i - 4k^i D^i D^i D^i - 4k^i D^i D^i - k^i D^i D^i D^i + 4k^i)}{4}, \]
\[ k = k, \quad l = l, \quad c = c, \quad D = D \]

(5.2)

By inserting Eq. (5.2) into Eq. (5.1), we conclude that a new four exact solutions to Eq.(1.5) as follows:
Case I: If \( E \neq 0 \) and \( \Delta = E^2 + 4D - 4DF \geq 0 \), then the solution is:

\[
V_1 = -k\alpha_i(-D(F-1) + E^2\gamma_1 + E^3\gamma_1')
\]  

\[ (5.3) \]

where,

\[
\alpha_i = k^2 + l^2 + \rho^2, \quad c_i = \frac{\sqrt{\Delta}}{2},
\]

\[
\gamma_1 = \frac{1}{2} + c_1 \left( \frac{h_i \exp(c, \eta) + h_i \exp(-c, \eta)}{h_i \exp(c, \eta) - h_i \exp(-c, \eta)} \right)
\]

Case II: If \( E \neq 0 \) and \( \Delta = E^2 + 4D - 4DF < 0 \), then the trigonometric function solution is:

\[
V_2 = -kD\alpha_i(-D(F-1) + kE^2\alpha_1\gamma_2 + \gamma_2^2E')
\]  

\[ (5.4) \]

where,

\[
\alpha_i = k^2 + l^2 + \rho^2, \quad c_i = \frac{\sqrt{\Delta}}{2}, \quad c_2 = \frac{\sqrt{-\Delta}}{2},
\]

\[
\gamma_2 = \frac{1}{2} + c_2 \left( \frac{i h_i \cos(c, \eta) - h_i \sin(c, \eta)}{i h_i \sin(c, \eta) + h_i \cos(c, \eta)} \right)
\]

Case III: If \( E = 0 \) and \( \Delta = D(1-F) \geq 0 \), then the trigonometric function solution is:

\[
V_3 = -k\alpha_i(D(F-1) - \gamma_3'),
\]  

\[ (5.5) \]

where,

\[
\alpha_i = k^2 + l^2 + \rho^2, \quad c_3 = \sqrt{\Delta},
\]

\[
\gamma_3 = c_3 \left( \frac{h_1 \cos(c, \eta) + h_2 \sin(c, \eta)}{h_1 \sin(c, \eta) - h_2 \cos(c, \eta)} \right)
\]

Case IV: If \( E = 0 \) and \( \Delta = D(1-F) < 0 \), then the hyperbolic function solution is:

\[
V_4 = -k\alpha(D(F-1) - \gamma_4')
\]  

\[ (5.6) \]

where,

\[
\alpha_i = k^2 + l^2 + \rho^2, \quad c_i = \sqrt{-\Delta},
\]

\[
\gamma_4 = c_i \left( \frac{i h_i \cosh(c, \eta) - h_i \sinh(c, \eta)}{i h_i \sinh(c, \eta) - h_i \cosh(c, \eta)} \right)
\]

where,

\[
G^\alpha(\eta) = \frac{Kx^\alpha}{\Gamma(1+\alpha)} + \frac{ly^\alpha}{\Gamma(1+\alpha)} + \frac{\rho z^\alpha}{\Gamma(1+\alpha)} - \frac{w t^\alpha}{\Gamma(1+\alpha)}
\]

6. Conclusion

In this study, the expansion and the extended-expansion methods have been applied for obtaining the new exact solutions of ZK equation of fractional order arising in physics. The obtained solutions can be expressed as trigonometric, hyperbolic and rational functions. The fractional transformation is used to convert the partial differential equation into the ordinary differential equation. The solutions reported here have not been published elsewhere.

Finally, the two planned methods are direct, concise, elementary, and effective and might be used for solving other nonlinear partial differential equation fractional order. This is often our duty in the future.

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