Perturbative nonequilibrium dynamics of phase transitions in an expanding universe

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(March 28, 2022)

A complete set of Feynman rules is derived, which permits a perturbative description of the nonequilibrium dynamics of a symmetry-breaking phase transition in $\lambda \phi^4$ theory in an expanding universe. In contrast to a naive expansion in powers of the coupling constant, this approximation scheme provides for (a) a description of the nonequilibrium state in terms of its own finite-width quasiparticle excitations, thus correctly incorporating dissipative effects in low-order calculations, and (b) the emergence from a symmetric initial state of a final state exhibiting the properties of spontaneous symmetry breaking, while maintaining the constraint $\langle \phi \rangle \equiv 0$. Earlier work on dissipative perturbation theory and spontaneous symmetry breaking in Minkowski spacetime is reviewed. The central problem addressed is the construction of a perturbative approximation scheme which treats the initial symmetric state in terms of the field $\phi$, while the state that emerges at later times is treated in terms of a field $\zeta$, linearly related to $\phi^2$. The connection between early and late times involves an infinite sequence of composite propagators. Explicit one-loop calculations are given of the gap equations that determine quasiparticle masses and of the equation of motion for $\langle \phi^2(t) \rangle$ and the renormalization of these equations is described. The perturbation series needed to describe the symmetric and broken-symmetry states are not equivalent, and this leads to ambiguities intrinsic to any perturbative approach. These ambiguities are discussed in detail and a systematic procedure for matching the two approximations is described.

98.80.Cq, 11.10.Wx, 11.30.Qc, 04.62.+v

I. INTRODUCTION

It has long been apparent that the hot, dense matter present in the early universe might undergo a variety of phase transitions $^{\text{[1]}}$. In particular, the “new inflation” scenario of Linde $^{\text{[2]}}$ and Albrecht and Steinhardt $^{\text{[3]}}$ (modifying an earlier proposal of Guth $^{\text{[4]}}$) suggests that a symmetry-breaking phase transition at the GUT scale would lead to a period of quasi-exponential expansion, perhaps with highly desirable cosmological consequences (see e.g. $^{\text{[5]}}$). Variants of this proposal are still of great current interest. Amongst cosmologists, the conventional view is that the state of a quantum field during an inflationary era is adequately described by classical field theory (except for the purpose of estimating density perturbations that are taken to originate from small quantum fluctuations) and, on the basis of classical calculations, that the new inflation scenario does not work in detail. However, a fully quantum-field-theoretic analysis of the dynamics of phase transitions in, say, grand unified theories, has never been given.

Apart from its intrinsic theoretical interest, a solution of this problem is of some importance. If there is a cosmological era in which matter is adequately described by a spontaneously broken gauge theory, then the consequences of the associated phase transition need to be correctly understood, whether they lead to the desired inflationary picture or not. Moreover, the above conclusions drawn from classical calculations are not necessarily secure. One reason is that the effective potential which traditionally appears in the classical equations of motion is not, in principle, an appropriate dynamical tool (for the reasons discussed in $^{\text{[6]}}$ amongst others). Another is that the value of a classical field (say, the expectation value of a quantum field) does not necessarily provide an adequate characterisation of the actual quantum state. In an influential paper $^{\text{[7]}}$, Guth and Pi argue, for the case of an unstable free scalar field theory, that at sufficiently late times the quantum probability distribution evolves in an approximately classical manner, because the minimal quantum uncertainty is negligible. However, as these authors recognise, this does not by any means imply that, say, the expectation value of the energy density $\langle \rho(\phi) \rangle$ that appears in the field equations of semiclassical general relativity is well approximated by $\rho(\langle \phi \rangle)$. Furthermore, the era of quasiclassical evolution arises from a growth in the amplitudes of unstable field modes, which may be considerable (see also $^{\text{[8]}}$) so that the effects of interactions may become significant even in a weakly coupled theory.

The purpose of this paper is to derive a set of Feynman rules, by means of which the nonequilibrium evolution of a quantum field theory may be estimated perturbatively during the course of a symmetry-breaking phase transition. One would like, for example, to be able to solve the semiclassical field equations

$$G_{\mu\nu} = \kappa (T_{\mu\nu}) ,$$  \hspace{1cm} (1.1)

where $T_{\mu\nu}$ is the stress tensor of an appropriate quantum field theory, and the question we address is how the expectation value can be estimated. Here, we study the example of a self-interacting scalar field, but the methods we develop can be generalised to the case where this field belongs to the Higgs sector of a spontaneously
broken gauge theory. For simplicity, we restrict attention to a conformally coupled field in a spatially flat Robertson-Walker universe with the line element \( ds^2 = a^2(t) [dt^2 - d\mathbf{x}^2] \), so that \( t \) is conformal time and the spatial coordinates \( \mathbf{x} \) are comoving. The action for this theory can be expressed as

\[
S(\phi) = \int d^4x \left[ \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} m^2(t) \phi^2 - \frac{\lambda}{4!} \phi^4 \right],
\]

(1.2)

where \( m^2(t) = a^2(t)m_0^2 \) and \( m_0 \) is the bare mass of the corresponding Minkowski-space theory. With \( m_0^2 > 0 \), we would expect the zero-temperature Minkowski-space theory to exhibit spontaneous symmetry breaking, and we take the initial state (at a time that we shall call \( t = 0 \)) to be a state of thermal equilibrium at a temperature \( 1/\beta_0 \) which is high enough for the symmetry to be unbroken. The expectation value of a Heisenberg-picture operator \( \mathcal{O}(t) \) is then

\[
\langle \mathcal{O}(t) \rangle = \frac{\text{Tr} \left[ e^{-\beta_0 H(0)} \mathcal{O}(t) \right]}{\text{Tr} \left[ e^{-\beta_0 H(0)} \right]},
\]

(1.3)

where \( H(t) \) is the Hamiltonian, which depends explicitly on time through the time-dependent mass \( m(t) \). This choice of an initial state is somewhat artificial, and is motivated mainly by the fact that it yields a well-defined problem. It has, in particular, the property that \( \langle \phi(x,0) \rangle = 0 \). This feature will cause considerable difficulty, since it implies that \( \langle \phi(x,t) \rangle = 0 \) at all later times also, so it is worth discussing at the outset. Of course, this initial state is well-defined, and should evolve in a well-defined manner, which it is of theoretical interest to investigate. One may wonder, however, whether the condition \( \langle \phi \rangle = 0 \) is too special to warrant the technical effort needed to deal with it, and we want to argue that it is not. We are supposing that, like the gauge symmetries of more realistic models, the symmetry \( \phi \leftrightarrow -\phi \) is an exact symmetry of nature. This means that any field configuration \( \phi(\mathbf{x}) \) is physically indistinguishable from the configuration \( -\phi(\mathbf{x}) \). There is therefore no physical mechanism that can produce different probabilities for these two configurations, which implies that \( \langle \phi(x,t) \rangle \) vanishes identically at every spacetime point \( (\mathbf{x},t) \). (This does not by any means imply that the state is spatially uniform, since a quantity such as the energy density, which is not constrained by symmetry, may perfectly well have a non-uniform expectation value even when \( \langle \phi \rangle = 0 \).)

In the context of semiclassical gravity, and in the spirit of the Copenhagen interpretation of quantum mechanics, it is reasonable to suppose that the density matrix can depend only on quantities which couple to (and can therefore be “measured” by) the classical spacetime. It might therefore depend on, say, the energy density and pressure, which might be spatially non-uniform, but not, independently of these, on \( \phi \), which cannot be measured either by the spacetime or by any other physical probe. If \( \phi \) belongs to the Higgs sector of a gauge theory, then these qualitative arguments are superfluous, since a well-known theorem of Elitzur assures us that its expectation value must vanish.

For a variety of reasons, some of which we shall have cause to discuss in detail later, perturbation theory is a rather limited tool for treating this problem, but we know of no other approximation scheme which might be used to treat the single scalar field considered here, or the more realistic models of particle physics that one might wish to tackle. The functional Schrödinger picture approach developed in \( \text{[1]} \) seems to be restricted in practice to Gaussian wavefunctionals (and, perhaps, to scalar fields in planar universes). Motivated by the belief that growing unstable modes make perturbation theory completely unreliable, Boyanovsky, Holman and de Vega, with several other collaborators, have studied in considerable detail the case of \( O(N) \)-symmetric scalar field theory in the limit \( N \to \infty \) (see \( \text{[1]} \) and references therein). In this limit, the problem can be solved exactly (up to the numerical solution of a set of integro-differential equations) without the aid of perturbation theory. While significant insights can be gained in this way, the large-\( N \) limit is a rather special one (in which, for example, dissipative effects are absent). Corrections of order \( 1/N \) and beyond seem to be intractable, so this approach does not appear to provided the basis for a non-perturbative treatment of any more realistic models.

In constructing a perturbative means of tackling the problem, we shall need to draw on earlier work which investigated firstly the possibility of treating dissipation by describing a nonequilibrium state in terms of its own quasiparticle excitations and, secondly, how the phenomena normally associated with spontaneous symmetry breaking can be recovered when \( \langle \phi \rangle = 0 \), by dealing instead with \( \langle \phi^2 \rangle \) which is not constrained to vanish. The results of this earlier work are summarised in section \( \text{[II]} \) below. We find, in particular, that two different perturbative expansions are needed to describe the symmetrical state which exists at early times and the broken-symmetry state which exists at later times. Section \( \text{[III]} \) describes a construction of the path integral which facilitates this dual description and derives the complete set of Feynman rules. In section \( \text{[IV]} \), we apply these rules to determine, at the lowest nontrivial order of our approximation scheme, the gap equations for quasiparticle masses and the evolution equation for \( \langle \phi^2 \rangle \). Both this equation of motion and the differential equation obeyed by the late-time propagators require boundary conditions, which we obtain from the continuity of appropriate expectation values. Renormalization of the gap equation, the equation of motion and the boundary conditions which apply to them is considered in section \( \text{[V]} \). Finally, the virtues and shortcomings of the approximation scheme we propose are discussed in section \( \text{[VI]} \).
II. SUMMARY OF PREVIOUS RESULTS

A. Dissipative perturbation theory

Our calculations of nonequilibrium expectation values are based on the closed-time-path formalism \[12\text{-}14\]. More specifically, we adopt the path-integral technique described by Semenoff and Weiss \[17\], in which Green’s functions are obtained from the generating functional

\[ Z(j_a) = \int [d\phi_a] \exp \left[ i\tilde{S}(\phi_a) + i \int d^4x \, j \cdot \phi \right]. \]  

(2.1)

Here, the single quantum field \( \tilde{\phi}(x,t) \) is represented by three path integration variables \( \phi_a(x,t) \) \((a = 1, \ldots, 3)\), which can be envisaged as inhabiting three segments of a contour in the complex time plane. The segment labelled by \( a \) is associated with time-ordered products. It runs from an initial time (which we shall call \( t = 0 \)) to \( t_f - i\epsilon \), where \( t_f \) is the largest time in which we are interested and \( \epsilon \) is infinitesimal. The segment \( a = 2 \), associated with anti-time-ordered products, returns from \( t_f - i\epsilon \) to \( 0 - 2i\epsilon \). Finally, the segment \( a = 3 \), which provides a path-integral representation of the initial equilibrium density operator at \( t = 0 \), runs from \( 0 - 2i\epsilon \) to \( 0 - i\beta \), where \( \beta \) is the inverse of the initial temperature. The action \( \tilde{S}(\phi_a) \) appearing in \( Z(j_a) \) is

\[ \tilde{S}(\phi_a) = \int d^4x \left[ \int_0^{t_f} dt \, L(\phi_1) - \int_0^{t_f} dt \, \mathcal{L}(\phi_2) \right. \]
\[ \left. + i \int_0^\beta d\tau \, \mathcal{L}_E(\phi_3) \right], \]

(2.2)

where \( \mathcal{L}(\phi) \) is the original Lagrangian density (in our case, that given in \[13\]), while \( \mathcal{L}_E \) is the Euclidean version associated with the density operator, namely

\[ \mathcal{L}_E(\phi_3) = \frac{1}{2} \left( \partial_\tau \phi_3 \right)^2 + \frac{1}{2} \left( \nabla \phi_3 \right)^2 - \frac{1}{2} m^2(0) \phi_3^2 + \frac{\lambda}{4!} \phi_3^4. \]

(2.3)

The source term in \[2.1\] is

\[ \int d^4x \, j \cdot \phi = \]
\[ \int d^4x \left\{ \int_0^{t_f} dt \left[ j_1(x,t) \phi_1(x,t) + j_2(x,t) \phi_2(x,t) \right] \right. \]
\[ + \int_0^\beta d\tau \, j_3(x,\tau) \phi_3(x,\tau) \right\}. \]

(2.4)

Of particular importance are the real-time 2-point functions \((\alpha, \beta = 1, 2)\) given by

\[ G_{\alpha\beta}(x,x') = -\frac{\partial}{\partial j_\alpha}(x) \frac{\partial}{\partial j_\beta}(x') \ln Z(j_a) \bigg|_{j_a=0} \]
\[ = \left( \frac{\text{Tr}[\rho T(\hat{\phi}(x)\hat{\phi}(x'))]}{\text{Tr}[\rho \hat{\phi}(x)\hat{\phi}(x)]} \right) \]
\[ \times \left( \frac{\text{Tr}[\rho T(\hat{\phi}(x')\hat{\phi}(x)]}{\text{Tr}[\rho \hat{\phi}(x')\hat{\phi}(x)]} \right) \]  

(2.5)

where \( T \) and \( \hat{T} \) denote respectively time-ordered and anti-time-ordered products of the quantum field operator \( \hat{\phi}(x) \) and \( \rho \) is the initial density operator. Other expectation values can, of course, be obtained from appropriate derivatives of \( Z(j_a) \).

To evaluate these expectation values perturbatively, one splits the action into an unperturbed part \( \tilde{S}_0(\phi_a) \), which is quadratic in \( \phi_a \), and an interaction \( \tilde{S}_{\text{int}}(\phi_a) \):

\[ \tilde{S}(\phi_a) = \tilde{S}_0(\phi_a) + \tilde{S}_{\text{int}}(\phi_a). \]

(2.6)

After an integration by parts, \( \tilde{S}_0 \) can be written in terms of a differential operator \( D_{ab} \) as

\[ \tilde{S}_0(\phi_a) = -\frac{1}{2} \int d^4x \, \phi_a(x) D_{ab}(x, \partial_\mu) \phi_b(x). \]

(2.7)

The lowest-order approximations to the 2-point functions are then the propagators \( g_{ab}(x,x') \), which are solutions of

\[ D_{ab}(x, \partial_\mu) g_{bc}(x,x') = g_{ab}(x,x') D_{bc}(x', -\partial_{\mu'}) \]
\[ = -i\delta_{a,b} \delta(x - x'), \]

(2.8)

subject to appropriate boundary conditions, which arise from continuity of the \( \phi_a \) and their time derivatives around the time path, including the periodicity condition \( \phi_3(x,\beta) = \phi_1(x,0) \). The perturbation series for an expectation value of some product of fields can be represented in terms of the usual Feynman diagrams, in which propagator lines representing \( g_{ab} \) connect vertices arising from \( \tilde{S}_{\text{int}} \). Since we consider only a spatially homogeneous system, we shall normally deal with Fourier transformed Green’s functions

\[ G_{ab}(t, t'; k) = \int d^3(x-x') \, e^{-ik \cdot (x-x')} G_{ab}(x, x') \]

(2.9)

and with propagators \( g_{ab}(t, t'; k) \) defined in the same way.

The standard choice for \( \tilde{S}_0 \) is simply the quadratic part of \( \tilde{S} \). With this choice (and supposing, temporarily, that \( m^2(t) < 0 \), so that there is no symmetry breaking), one finds that the \( g_{ab}(t, t'; k) \) are composed of mode functions \( f_k(t) \), which are essentially single-particle wavefunctions, together with constants which can be interpreted as the occupation numbers \( n_k \) of the corresponding single-particle modes. Because this perturbation theory has as its lowest-order approximation the theory of a gas of free particles, which do not scatter, the occupation numbers remain fixed at their initial values \( \tilde{n}_k \). In the problem at hand, we deal with an interacting system, driven

*In an expanding universe, even a free quantum field theory
away from equilibrium by a time-dependent Hamiltonian
and, under these circumstances, one would expect the
state of the system to evolve in response to its changing
environment. In standard perturbation theory, this state
is represented at lowest order by the fixed occupation
numbers. One therefore suspects that low-order calcula-
tions of time-dependent expectation values (which are
all that one can realistically hope to obtain) will be ade-
quate only over a period of time which is short compared
with some characteristic relaxation time. The relaxation
effects which would cause occupation numbers (or some
appropriate generalization of these) to evolve with time
in the expected manner arise from the absorptive parts
of higher-order loop diagrams, so the perturbation expan-
sion is improved if these contributions can be partially
resummed so as to appear in the propagators \( g_{\alpha\beta}(t, t'; k) \).

As explained in detail in [20–22], this resummation can
be achieved by a somewhat more sophisticated choice of
the real-time part of \( \bar{S}_0 \), namely

\[
\bar{S}_0(\phi_1, \phi_2) = \bar{S}^{(2)}_0(\phi_1, \phi_2) + \frac{1}{2} \int_{t'}^t dt \int \frac{d^3k}{(2\pi)^3} \phi_\alpha(k, t) M_{\alpha\beta}(k, t, \partial_t) \phi_\beta(-k, t),
\]

(2.10)

where \( \bar{S}^{(2)}_0 \) is the quadratic part of \( \bar{S} \), \( \phi_\alpha(k, t) \) is the
spatial Fourier transform of \( \phi_\alpha(x) \) and \( M_{\alpha\beta} \) is a differen-
tial operator chosen in the following way. The propaga-
tors \( g_{\alpha\beta}(t, t'; k) \) now obey the spatial Fourier transfor-
m of \( (2.8) \) with a differential operator \( D_{\alpha\beta} \) that contains
\( M_{\alpha\beta} \), and we would like them to mimic the full 2-point
functions as nearly as possible. From the hermiticity of
\( \phi \) and of the density operator \( \rho \), it is straightforward to
show that all the real-time 2-point functions can be writ-
en in terms of a single complex function \( H(x, x') \) as

\[
G_{\alpha\beta}(x, x') = H_{\beta}(x, x') \theta(t - t') + H_{\alpha}(x', x) \theta(t' - t),
\]

(2.11)

where \( H_1(x, x') = H(x, x') \) and \( H_2(x, x') = H^*(x, x') \).

Clearly, we want \( g_{\alpha\beta} \) to have the same form, and the
most general choice of \( D_{\alpha\beta} \) for which \( (2.8) \) admits such
a solution is

\[
D_{\alpha\beta} = \left( \begin{array}{cc}
\partial_t^2 + \beta_k - i\alpha_k & \gamma_k \partial_t + i\alpha_k \\
-\gamma_k \partial_t - i\gamma_k & -\partial_t^2 - \beta_k - i\alpha_k
\end{array} \right),
\]

(2.12)

where \( \alpha_k(t) \), \( \beta_k(t) \) and \( \gamma_k(t) \) are undetermined, real func-
tions. These functions can be determined in a self-
consistent manner through an appropriate renormaliza-
tion prescription. Thus, with the choice \( (2.10) \) for \( \bar{S}_0 \),
the interaction \( \bar{S}_{\text{int}} \) contains a counterterm \( -\frac{1}{2} \int \phi M\phi \)
which we require to cancel some part of the higher-order
contributions to \( G_{\alpha\beta} \), thereby optimising \( g_{\alpha\beta} \) as an
approximation to \( G_{\alpha\beta} \). In this work we will, in particular,
use a renormalization prescription such that \( \beta_k(t) \) has the
form \( \beta_k(t) = k^2 + M^2(t) \), corresponding simply to a
mass renormalization, though more general prescriptions
are possible in principle. It is perhaps worth emphasising
that the complete action \( \bar{S} \) is, of course, independent of
\( M_{\alpha\beta} \). The introduction of this counterterm does not
change the full theory, but serves to optimise our choice
of a lowest-order approximation. The structure of \( M_{\alpha\beta} \)
is analogous to that of the effective action obtained, for
example, in [23] by integrating out extra environmental
degrees of freedom. Here, nothing is integrated out, but
we might say that each field mode is treated self-
consistently as interacting with an environment provided
by the remaining modes.

With \( D_{\alpha\beta} \) given by \( (2.12) \), the solution to \( (2.8) \) can be writ-
en as

\[
g_{\alpha\beta}(t, t'; k) = h_{\beta}(t, t'; k) \theta(t - t') + h_{\alpha}(t', t; k) \theta(t' - t),
\]

(2.13)

where \( h_1(t, t'; k) = h(t, t'; k) \) and \( h_2(t, t'; k) = h^*(t', t; k) \),
with

\[
h(t, t'; k) = \frac{1}{\pi} \text{exp} \left( -\frac{1}{2} \int_{t'}^t dt'' \gamma_k(t'') \right)
\times \left[ (N_k(t') + 1) f_k(t) f_k^*(t') + (N_k^*(t') - 1) f_k^*(t) f_k(t') \right].
\]

(2.14)

The mode function \( f_k(t) \) is a complex solution of

\[
[\partial_t^2 + \beta_k(t) - \frac{1}{2} \gamma_k^2(t)] f_k(t) = 0,
\]

(2.15)

satisfying the Wronskian condition

\[
\dot{f}_k(t) f_k^*(t) - f_k(t) \dot{f}_k^*(t) = -i,
\]

(2.16)

while \( N_k(t) \) obeys

\[
[\partial_t + 2i \Omega_k(t) - \frac{\Omega_k(t)}{\Omega_k(t)} + \gamma_k(t)] [\partial_t + \gamma_k(t)] N_k(t) = 2i \alpha_k(t),
\]

(2.17)

where \( \Omega_k(t) = 1/2 f_k(t) f_k^*(t) \). This function is also re-
quired to satisfy the subsidiary condition

\[
\dot{N}_k(t) + N_k^*(t) + 2i \Omega_k(t) \left[ N_k(t) - N_k^*(t) \right] + \gamma_k(t) \left[ N_k(t) + N_k^*(t) \right] = 0,
\]

(2.18)
in order that \( \partial_t \left[ h_k(t, t') - h_k(t', t) \right]_{t=t'} = -i \), which reflects the canonical commutation relation. This condition is preserved by (2.17), so it can be regarded as a constraint on the initial values of \( N_k \) and \( \tilde{N}_k \). These initial values are determined by the boundary conditions.

Indeed, with approximations appropriate to slow evolution, equation (2.17) reduces to a Boltzmann equation \[ \gamma_k(t) \] in which \( \gamma_k(t) \) and \( \alpha_k(t) \) provide the standard scattering integral. In this work we consider only a real scalar field, but the dissipative formalism outlined here can be extended to complex scalar fields \[ \alpha \] and (with somewhat greater difficulty) to spin-\( \frac{1}{2} \) fermions \[ \psi \]. The dressed propagators obtained by somewhat different methods in [26,27] to describe dissipations in systems close to equilibrium (or, at least, to a steady state) seem to be a special case of those given here.

For future use, we note that, associated with \( D_{\alpha\beta} \) is an operator
\[
\hat{d}_{\alpha\beta}(t, k) = \left( \begin{array}{cc} \hat{\partial}_t & \gamma_k(t) \\ -\gamma_k(t) & -\hat{\partial}_t \end{array} \right),
\] which has the property
\[
g(t, t'; k) \hat{d}(t'', k) g(t', t'') = \begin{cases} -ig(t, t'; k), & t > t'' > t' \\ ig(t, t'; k), & t' > t'' > t \\ 0, & \text{otherwise} \end{cases} \tag{2.20}
\]

B. Spontaneously unbroken symmetry

As indicated above, we wish to follow the progress of a phase transition starting from a high-temperature state in which the symmetry \( \phi \leftrightarrow -\phi \) is unbroken and so \( \langle \phi(x) \rangle = 0 \) at each point in space. Since the dynamics is governed by a Hamiltonian which respects this symmetry, it is inevitable that \( \langle \phi(x) \rangle \equiv 0 \) at all subsequent times. We nevertheless expect to encounter states in which the phenomena conventionally associated with spontaneously broken symmetry (that is, with a non-zero value of \( \langle \phi \rangle \)) are realised, and will refer to such states as having “spontaneously unbroken symmetry”. Here, we review the means by which a vacuum state of this kind in Minkowski spacetime can be treated [28]. How such a state may be seen to emerge from a phase transition is the central problem to be addressed in this paper and will be discussed in detail later on.

Heuristically, we can envisage two candidate vacua, say \(|+\rangle \) in which \( \langle \phi \rangle = +\sigma/\sqrt{\lambda} \) and \(|-\rangle \) in which \( \langle \phi \rangle = -\sigma/\sqrt{\lambda} \). The overlap \( \langle + | - \rangle \) of these states vanishes exponentially as the volume of space becomes infinite, so \( \langle + | \mathcal{O} | - \rangle \) also vanishes if \( \mathcal{O} \) is any local operator or the integral over all space of a local operator. Consequently, if we consider a superposition \( |\alpha\rangle = \sqrt{\alpha}|+\rangle + \sqrt{1-\alpha}|-\rangle \), for which \( \langle \alpha | \phi \rangle = (2\alpha - 1)\sigma/\sqrt{\lambda} \), then \( \langle \alpha | \mathcal{O} |\alpha\rangle = \alpha \langle + | \mathcal{O} | + \rangle + (1-\alpha) \langle - | \mathcal{O} | - \rangle \). If the symmetry is exact in Nature, then any experimental probe can couple only to a symmetrical operator, for which \( \langle + | \mathcal{O} | + \rangle = \langle - | \mathcal{O} | - \rangle \). In that case, \( \langle \alpha | \mathcal{O} |\alpha\rangle \) is independent of \( \alpha \), and no experiment can determine the value of \( \langle \phi \rangle \). In statistical mechanics, we need incoherent sums over states such as \(|+\rangle \) and \(|-\rangle \), but the same principle applies.

At the formal level, we need a means of calculating expectation values of symmetrical operators without assuming a non-zero expectation value for \( \phi \). In particular, we need a lowest-order Hamiltonian whose eigenstates yield the Fock space built on \( |\alpha\rangle \) rather than on \(|+\rangle \) or \(|-\rangle \). To this end, we take the state of spontaneously unbroken symmetry to be characterised by the expectation value of \( \phi^2 \), which is not constrained by symmetry. Instead of the conventional field variable \( \psi(x) \), defined by
\[
\phi(x) = \sigma/\sqrt{\lambda} + \psi(x), \tag{2.21}
\]
we deal with a field \( \zeta(x) \) defined by
\[
\phi^2(x) = U^2 + 2U \zeta(x), \tag{2.22}
\]
where \( U^2 = \langle \phi^2(x) \rangle \). Taking \( m^2(t) = m_0^2 \) in (1.2), we find
\[
\mathcal{L} = \frac{1}{2} (1 + 2U^{-1}\zeta)^{-1} \partial_{\mu} \zeta \partial^{\mu} \zeta - \frac{\lambda}{6} U^2 \zeta^2 + U \left( m_0^2 - \frac{\lambda}{6} t^2 \right) \zeta + \frac{i}{2} \delta^d(0) \ln (1 + 2U^{-1}\zeta), \tag{2.23}
\]
where the last term comes from the functional Jacobian of the transformation and an irrelevant constant has been dropped. Since \( \zeta = 0 \), the linear term must vanish to leading order, so we identify \( U = \sqrt{2m_0^2/\lambda} (1 + O(\lambda)) \). The range of \( \zeta \) in the path integral is, of course, \(-U/2 \cdots \infty \). But, since \( U \) is of order \( \lambda^{-1/2} \), the lower limit can be extended to \(-\infty \) at the expense of corrections of order \( e^{-1/\lambda} \), which do not contribute to perturbation theory. On expanding (2.23) in powers of \( \lambda \), we find
\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \zeta \partial^{\mu} \zeta - \frac{1}{2} (2m_0^2) \zeta^2 + \cdots . \tag{2.24}
\]
At leading order, we see that particles created by \( \zeta \) from the spontaneously unbroken vacuum have the same mass \( \sqrt{2m_0} \) as those created from the broken-symmetry vacuum by \( \psi \), and it is not hard to convince oneself [28] that these two types of particle are completely indistinguishable. In this formulation, interactions arise from the kinetic term, and there are an infinite number of vertices (though only a finite number of these appear at a given order in \( \lambda \)). Renormalizability follows from the fact that...
\( \phi^2(x) \) is a multiplicatively renormalizable composite operator in the standard formulation, and the \( \delta^4(0) \) in the Jacobian serves to cancel quartic divergences arising from the derivative interactions.

It will be of some importance in what follows that the above manipulations do not, in themselves, assume that \( \langle \phi \rangle = 0 \), but rather leave \( \langle \phi \rangle \) undetermined. We may therefore consider the case that \( \langle \phi \rangle = \sigma/\sqrt{\lambda} \) and \( \zeta \) is related to \( \psi \) by

\[
\zeta(x) = \frac{1}{2U} \left[ \frac{\sigma^2}{\lambda} - U^2 + 2 \left( \frac{\sigma}{\sqrt{\lambda}} \right) \psi(x) + \psi^2(x) \right].
\]

(2.25)

In particular, we can express the connected two-point function \( \langle \zeta(x) \zeta(y) \rangle_c \) as

\[
\langle \zeta(x) \zeta(y) \rangle_c = \frac{\sigma^2}{4U^2} \langle \psi(x) \psi(y) \rangle_c + \frac{\sigma}{2\sqrt{U^2}} \langle \psi(x) \psi^2(y) \rangle_c + \frac{1}{4U^2} \langle \psi^2(x) \psi^2(y) \rangle_c.
\]

(2.26)

In the Minkowski-space theory, one can easily calculate \( U \) in terms of \( \sigma \) and verify this relation order by order in \( \lambda \). In the nonequilibrium state resulting from a phase transition that we plan to investigate, the field \( \psi \) is not defined. Nevertheless, it will be possible to identify by eye terms corresponding to the various correlators on the right of (2.26).

In what follows, we shall consider only the theory of one real scalar field. It is appropriate to mention, howver, that the analysis of spontaneously broken symmetry outlined here can be extended to the Higgs sector of a spontaneously broken gauge theory [29]. In accordance with Elitzur’s theorem [30], non-zero expectation values need be assigned only to operators which are invariant under the gauge and global symmetries of the theory. The generation of fermion masses can also be accomplished in this way. Consider, for example, a pair of massless fermions \( \psi_L \) and \( \psi_R \) and a complex scalar \( \phi \) with Lagrangian density

\[
\mathcal{L} = \partial_{\mu} \phi^* \partial^{\mu} \phi + \bar{\psi}_L i \gamma_\mu \psi_L + \bar{\psi}_R i \gamma_\mu \psi_R - f \left( \bar{\psi}_L \gamma_\mu \phi + \bar{\psi}_R \gamma_\mu \phi^* \right) + \cdots.
\]

(2.27)

With \( \langle \phi^* \phi \rangle = U^2 \), we write \( \phi(x) = \sqrt{U^2 + 2U \zeta(x)} e^{i\theta(x)} \) and construct the single massive spinor

\[
\psi(x) = e^{-i\theta(x)/2} \psi_L(x) + e^{i\theta(x)/2} \psi_R(x)
\]

(2.28)

to obtain

\[
\mathcal{L} = \partial_{\mu} \zeta \partial^{\mu} \zeta + \bar{\psi}(i \partial \theta - f U) \psi + (U^2 + 2U \zeta) \partial_{\mu} \theta \partial^{\mu} \theta + \bar{\psi} \gamma^\mu \gamma^5 \partial_{\mu} \theta + \cdots.
\]

(2.29)

Since only the derivatives of \( \theta \) appear, we need not assign \( \theta \) an expectation value, and the fermion mass is \( f \sqrt{\langle \phi^* \phi \rangle} \).

This indicates both that our formalism can in principle be extended to more realistic particle-physics models and also that our earlier remarks concerning the unobservability of \( \langle \phi \rangle \) are not invalidated by the linear Yukawa coupling of \( \phi \) to fermions.

### III. DERIVATION OF THE FEYNMAN RULES

The central purpose of this paper is to derive a set of Feynman rules which will permit us to follow the evolution of the state of our system as the symmetry becomes spontaneously unbroken, in the sense described in the last section. As an intuitive guide to the considerations involved, we offer in Figures 1a and 1b an artist’s impression of a time-dependent effective potential which might be thought to govern the evolution of \( \phi \). At times earlier than, say, \( t_1 \), it has a single minimum at \( \phi = 0 \) while at times later than \( t_1 \) it has two symmetrically placed minima at \( \phi = \pm \phi_0(t) \). This is no more than an intuitive guide, because we have given no precise definition of \( V_{\text{eff}}(\phi) \). There are several conventional definitions of effective potentials, all of which refer to equilibrium situations, and do not necessarily fit the dynamical role which is often forced upon them.

Figures 1c and 1d show an artist’s impression of the probability density \( P(\phi, t) \) for the field at the spatial point \( \mathbf{x} \) to have the value \( \phi \) at time \( t \). Because of spatial homogeneity, this probability is independent of \( \mathbf{x} \). Again, this gives only an intuitive expectation. Perturbation theory gives no ready access to \( P(\phi, t) \) and its structure might well be more complicated than that sketched in the figure (and, of course, we are ignoring the problems of renormalization that would be encountered in trying to construct a well-defined \( P(\phi, t) \)). Also, \( P(\phi, t) \) does not by any means give a complete characterization of the state of the system. For that, one would need the vastly more complicated probability density for field configurations over all space. The intuition illustrated is that before some time \( t_0 \), the most likely value of \( \phi \) is zero, whereas after \( t_0 \), a broken-symmetry state emerges in which the most likely values are \( \pm \sigma(t)/\sqrt{\lambda} \).

These heuristic considerations do not provide a formal basis for the calculational scheme we wish to propose. The foregoing discussion is intended to motivate the assumption we do make, namely that there is some time \( t_0 \) before which the most likely field values are near zero, and the appropriate field variable for perturbative calculations is \( \phi \), while after \( t_0 \) the most likely values are near \( \pm \sigma(t)/\sqrt{\lambda} \) and the appropriate field variable is \( \zeta \). For brevity, we shall refer to these two varieties of perturbation theory as the “\( \phi \)-theory” and the “\( \zeta \)-theory”. This situation presents two difficulties. One is that in the \( \phi \)-theory the perturbative expansion of \( \langle \phi^2 \rangle \) has a leading term of order \( \lambda^0 \), while in the \( \zeta \)-theory the leading term is of order \( \lambda^{-1} \). Thus, although both perturbation series are expansions in powers of \( \lambda \), they are not the same.
expansion. We will propose a concrete means of dealing with this later on. The second difficulty is that we need to make the change of variable \( \frac{2}{22} \) only for times later than \( t_0 \), and thus to evaluate a path integral of the form

\[
\int_{t_1}^{t_2} [d\phi] \int_{t_1}^{t_2} [d\zeta] e^{iS(\phi, \zeta)}. \tag{3.1}
\]

This does not factorise into two independent integrals, on account of the boundary condition that \( \phi^2(t_0) = U^2(t_0) + 2U(t_0)\zeta(t_0) \). Indeed, the nonequilibrium state of \( \zeta \) at time \( t_0 \) is determined by the evolution of \( \phi \) between \( t = 0 \) and \( t = t_0 \), and must be incorporated through a correct handling of the boundary conditions at \( t_0 \). The evaluation of this path integral is a hazardous undertaking, and the method we propose is not entirely rigorous. We shall encounter various ill-defined quantities, and sensible prescriptions for dealing with these will be needed. We therefore begin by explaining our strategy in the context of a toy model, for which we obtain what are manifestly the right answers.

A. A toy path integral

The toy model in question is simply a free scalar field theory in Minkowski spacetime. The real-time part of the action (including a source \( J_0(t) \)) is

\[
\hat{S} = \int_0^{t_f} dt \left[ \frac{1}{2} \left( \phi_1^2(t) - \phi_2^2(t) \right) + \left( J_1(t)\phi_1(t) + J_2(t)\phi_2(t) \right) \right]. \tag{3.2}
\]

For notational clarity, we do not indicate explicitly the spatial integral or the gradient and mass terms, which play no direct role in this part of our analysis. We will evaluate the generating functional given by standard methods as

\[
Z(J_0) = \int [d\phi] e^{i\hat{S}(\phi, J)}.
\]

by integrating first over \( \phi_2(t) \) for \( t > t_0 \) and then independently over the remaining fields. We write the sources as \( J_0(t) = J_0(t)\phi(t_0 - t) + J_1(t)\phi(t - t_0) \) and in the first instance set \( J_0(t) = 0 \). The propagator \( g_{\alpha\beta}(t, t') \) now obeys \( \frac{2}{28} \) in the form \( \epsilon_{\alpha\beta}\partial_2^2 g_{\gamma\beta} = -i\delta_{\alpha\beta}(t - t') \), where \( \epsilon_{11} = -\epsilon_{22} = 1 \) and \( \epsilon_{12} = \epsilon_{21} = 0 \), and we have again suppressed the spatial derivative and mass terms. We define

\[
L_{\alpha}(t) = \int_{t_0}^{t_f} dt' g_{\alpha\beta}(t, t')l_{\beta}(t'), \tag{3.4}
\]

which clearly satisfies \( \epsilon_{\alpha\beta}\partial_2^2 L_{\beta}(t) = -iL_{\alpha}(t) \), and make the change of integration variable

\[
\phi_{\alpha}(t) \rightarrow \phi_{\alpha}(t) + iL_{\alpha}(t)\theta(t - t_0). \tag{3.5}
\]

After an integration by parts using the boundary condition \( \phi_1(t_f) = \phi_2(t_f) \), which also entails \( l_1(t_f) = l_2(t_f) \), the action \( \frac{2}{2} \) becomes

\[
\tilde{S} = \frac{1}{2} \int_0^{t_f} \epsilon_{\alpha\beta}\phi_{\alpha}(t)\dot{\phi}_{\beta}(t)dt + i\epsilon_{\alpha\beta}L_{\alpha}(t)\partial_1\phi_{\beta}(t) \bigg|_{t=t_0} + \frac{1}{2} \int_{t_0}^{t_f} l_{\alpha}(t)g_{\alpha\beta}(t, t')l_{\beta}(t')dt dt' - \frac{1}{2} \left[ L^2(t_0) - L^2(t_0) \right] \delta(0), \tag{3.6}
\]

provided that we set \( \theta(0) = \frac{1}{2} \). Here, the fields for \( t > t_0 \) are decoupled from the sources, so we can integrate them out. To do this, we note that \( \exp\left[i\hat{S}(\phi_1, \phi_2)\right] \) arises from the product of two time evolution operators:

\[
\int [d\phi] e^{i\hat{S}(\phi_1, \phi_2)} = \text{Tr} \left[ \cdots U^{-1}(0, t_f)U(0, t_f) \right]
\]

\[
= \text{Tr} \left[ \cdots U^{-1}(0, t_f)U^{-1}(t_0, t_f)U(t_0, t_f)U(t_0, 0) \right]. \tag{3.7}
\]

where \( U_i \) is the time evolution operator in the presence of the source \( J_i \), and that the derivation of the path integral implies the boundary conditions \( \phi_1(t_f) = \phi_2(t_f) \) and \( \dot{\phi}_1(t_f) = \dot{\phi}_2(t_f) \). In the absence of sources between \( t_0 \) and \( t_f \), the product \( U_2^{-1}(t_0, t_f)U(t_0, t_f) \) is the identity. Consequently, the path integration for \( t_0 < t \leq t_f \) yields a factor of 1 together with the boundary conditions \( \phi_1(t_0) = \phi_2(t_0) \) and \( \dot{\phi}_1(t_0) = \dot{\phi}_2(t_0) \) on the remaining integral.

Before evaluating the remaining integral, we reinstate a non-zero source \( j(t) \) for times before \( t_0 \). In the present example, this could have been retained throughout, at the expense only of additional terms in our equations which were irrelevant until now. When dealing with the dynamics of symmetry breaking, however, we will have more cogent reasons for introducing \( j(t) \) only at this point. The quantity still to be integrated is now \( \exp[i\hat{S}'(\phi, J)] \), where

\[
\tilde{S}' = \int_0^{t_0} \left[ \frac{1}{2} \epsilon_{\alpha\beta}\phi_{\alpha}(t)\dot{\phi}_{\beta}(t) + \left( j_0(t) + \hat{L}_\alpha(t) \right) \dot{\phi}_\alpha(t) \right] dt. \tag{3.8}
\]

The quantity \( \hat{L}_\alpha(t) \) is a distribution concentrated at \( t = t_0 \) such that for any pair of test functions \( f_\alpha(t) \)

\[
\int_0^{t_0} \hat{L}_\alpha(t)f_\alpha(t) = i\epsilon_{\alpha\beta}L_\alpha(t)\partial_1f_\beta(t) \bigg|_{t=t_0}. \tag{3.9}
\]

Evaluating the integral in the standard way, we obtain \( \text{const} \times \exp[-\frac{1}{2}J] \), with
\[
J = \int_{t_0}^{t_f} \left[ j_\alpha(t) + \tilde{L}_\alpha(t) \right] \tilde{g}_{\alpha\beta}(t, t') \\
\times \left[ \tilde{\phi}_\beta(t') + \tilde{\phi}_\beta(t') \right] dt dt'.
\] (3.10)

The propagator \( \tilde{g}_{\alpha\beta}(t, t') \) is a solution of the same equation as \( g_{\alpha\beta}(t, t') \). If we assume that these are the same solutions (which is true if they and their first derivatives coincide when \( t = t_0 \) or \( t' = t_0 \)), then we can use (2.20) in the form

\[
g_{\alpha\gamma}(t, t'') \epsilon_{\gamma \delta} \partial_t \epsilon_{t'} g_{\beta\delta}(t'', t) = -i g_{\alpha\beta}(t, t') \\
\times \left[ \theta(t - t'') \theta(t' - t') - \theta(t' - t'') \theta(t'' - t) \right]
\] (3.11)
to find

\[
\int_{t_0}^{t_f} dt dt' \tilde{L}_\alpha(t) g_{\alpha\beta}(t, t') j_\beta(t') = \int_{t_0}^{t_f} dt \int_{t_0}^{t_f} dt' l_{\alpha}(t) g_{\alpha\beta}(t, t') j_\beta(t')
\] (3.12)
and

\[
\int_{t_0}^{t_f} dt dt' \tilde{L}_\alpha(t) g_{\alpha\beta}(t, t') \tilde{\phi}_\beta(t') = -i \left[ L_1^1(t_0) - L_2(t_0) \right] \delta(0).
\] (3.13)

Combining these results with those in (3.4), we recover the expected result (3.5). In particular, the terms proportional to \( \delta(0) \) cancel. It is straightforward to verify that these manipulations also work if we replace the free field theory (3.2) with the dissipative approximate theory (2.10), provided that the operator \( \epsilon_{\alpha\beta} \frac{\partial_t}{t} \) in (3.4) and (3.5) is replaced by \( \tilde{d}_{\alpha\beta} \) as defined in (2.19).

**B. Path integral for a symmetry-breaking phase transition**

We are finally ready to undertake our central piece of analysis, which is to derive a perturbative means of calculating the generating functional \( Z(j_\alpha, l_\alpha) \) of Green’s functions which involve \( \phi_\alpha(x, t) (\alpha = 1, 2, 3) \) for \( t < t_0 \) and \( \zeta_\alpha(x, t) (\alpha = 1, 2) \) for \( t_0 < t < t_f \). The derivation is quite lengthy, and we set it out in several steps.

**Step 1: The generating functional.** In the first instance, we set \( j_\alpha = 0 \) and define

\[
Z(0, l_\alpha) = \int [d\phi] \exp \left[ iS(\phi_\alpha) + i \int_{t_0}^{t_f} dt \int d^3 x l_\alpha(x, t) \zeta_\alpha(x, t) \right],
\] (3.14)

where

\[
\zeta_\alpha(x, t) = [2U(t)]^{-1} \left[ \phi_\alpha^2(x, t) - U^2(t) \right]
\]

and \( U^2(t) = \langle \phi_\alpha^2(x, t) \rangle \). This is the exact expectation value which must, of course, be determined self-consistently by demanding that \( \langle \zeta_\alpha(x, t) \rangle = 0 \). The reason for setting the source \( j_\alpha \) for \( \phi_\alpha \) equal to zero is that this source breaks the exact symmetry on which our manipulations depend. We shall introduce \( j_\alpha \) at a later stage, and subsequently discuss the exact meaning of the generating functional obtained by this route.

**Step 2: A change of variable.** We now change the integration variables from \( \phi_\alpha(x, t) \) to \( \zeta_\alpha(x, t) \) on the whole closed time path, so as to avoid difficulties over the time derivatives of fields at \( t = t_0 \). The transformation is formally legitimate even for \( t < t_0 \), but for these times we shall eventually have to undo the transformation in order to construct a sensible perturbation theory. The resulting Lagrangian is rather complicated, owing to the fact that \( U(t) \) is now time dependent, and we give only the real-time part (depending on \( \phi_1 \) and \( \phi_2 \)) of which we shall make explicit use. For later convenience, we write it as the sum of several terms:

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}}^{(1)} + \mathcal{L}_{\text{ct}}^{(2)} + \mathcal{L}_{\text{ct}}^{(3)} + \mathcal{L}_{\text{ct}}^{(4)} + \mathcal{L}_{\text{ct}}^{(5)} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{jac}}.
\] (3.15)

Of these, the first is

\[
\mathcal{L}_0 = -\frac{1}{2} \zeta_\alpha D_{\alpha\beta} \zeta_\beta + \epsilon_\alpha \frac{d}{dt} \left[ \frac{1}{U} \frac{\partial^2}{\partial x^2} \right] \zeta_\alpha.
\] (3.16)

where \( D_{\alpha\beta} \) has the form shown in (2.12), but with \( \beta_k(t) = k^2 + M^2(t) \). Here and below, we deal with the spatial Fourier transforms of fields, propagators, etc., but will generally not indicate explicitly their dependence on \( k \) or the associated momentum integrals. The total derivative term, in which \( \epsilon_1 = -\epsilon_2 = 1 \), combines with the \( \partial_t^2 \) terms in \( -\frac{1}{2} \zeta_\alpha D_{\alpha\beta} \zeta_\beta \) to produce \( \frac{i}{2} \epsilon_\alpha \zeta_\beta \zeta_\beta \).

The second contribution

\[
\mathcal{L}_{\text{int}} = \frac{i}{4} \left[ \ln \left( 1 + \frac{2}{U} \zeta \right) - \frac{2}{U} \zeta \right] \alpha \beta \epsilon_\alpha \zeta_\beta
\] (3.17)

contains the principal interactions. There are derivative interactions of the kind already encountered in section II B, and these have been expressed in terms of the operator \( D_{\alpha\beta} \), so that we can use any of (2.5) in computing Feynman diagrams. There are also non-derivative interactions arising from the time dependence of \( U(t) \). The linear counterterm

\[
\mathcal{L}_{\text{ct}}^{(1)} = -\left[ \frac{1}{U} - m^2(t) U + (\lambda/6) U^3 \right] \epsilon_\alpha \zeta_\alpha
\] (3.18)

appears because we have taken \( U^2(t) \) to be the exact expectation value of \( \phi_\alpha^2 \), which means that \( \langle \zeta_\alpha \rangle = 0 \). This requirement will be implemented self-consistently by requiring the counterterm to cancel all higher-order contributions to \( \langle \zeta_\alpha \rangle \). Further counterterms, denoted by
\[ \mathcal{L}_{ct}^{(1)} = \frac{1}{2U} \left[ \dot{U} + M^2 U - \frac{\lambda}{3} U^3 \right] \epsilon_{\alpha\beta} \zeta_\alpha \zeta_\beta \]
\[ + \frac{1}{4} U \ln \left( 1 + \frac{2}{U} \zeta \right) ^{2} \mathcal{M}_{\alpha\beta} \zeta_\alpha \zeta_\beta, \quad (3.19) \]

arise from the fact that we used a renormalized mass \( M(t) \) and the dissipative coefficients \( \alpha_k(t) \) and \( \gamma_k(t) \) in \( \mathcal{L}_0 \). Again, these quantities are to be determined self-consistently by using the counterterms to cancel appropriate parts of higher-order contributions to the 2-point functions. The dissipative counterterms
\[ \mathcal{M}_{\alpha\beta} = \left( \begin{array}{cc}
-\frac{i\alpha}{2} & 2 \gamma + \frac{i\alpha}{2} \\
-\gamma \partial_t - \frac{i\alpha}{2} & -i\alpha
\end{array} \right) \quad (3.20) \]
differs from the \( \mathcal{M}_{\alpha\beta} \) of section I.A only insofar as it excludes \( M(t) \), which is treated separately. The two additional counterterms
\[ \mathcal{L}_{tdc} = X_\alpha D_{\alpha\beta} \zeta_\beta - \zeta_\alpha D_{\alpha\beta} X_\beta + \frac{d}{dt} \left[ Y_\alpha D_{\alpha\beta} \zeta_\beta^2 \right], \quad (3.21) \]
where \( X_\alpha(t) \) and \( Y_\alpha(t, k) \) are as yet undetermined functions, are total time derivatives. They will be used to facilitate the handling of boundary terms arising from integrations by parts in the evaluation of certain Feynman diagrams. More terms of this kind might be needed for calculations at higher orders than we consider explicitly in this paper. The sum of terms given so far differs from the original Lagrangian by a total time derivative, namely
\[ \mathcal{L}_{td} = \frac{d}{dt} \left\{ \frac{1}{4} \left[ \ln \left( 1 + \frac{2}{U} \zeta \right) - \frac{2}{U} \zeta \right] \right\} \epsilon_{\alpha\beta} \left( \dot{U} \zeta_\beta - \dot{U} \zeta_\beta \right) 
\[ + \epsilon_\alpha \dot{U} \zeta_\alpha - X_\alpha \dot{\delta}_{\alpha\beta} \zeta_\beta - Y_\alpha \dot{\delta}_{\alpha\beta} \zeta_\beta. \quad (3.22) \]

At this point, \( \mathcal{L}_{td} \) could be integrated round the whole time contour to yield zero, when use is made of the boundary conditions at \( t = 0 \) and \( t = t_f \). We do not do this, because we want to insert an extra boundary at \( t_0 \) and to indicate explicitly how we treat the boundary conditions there. Finally, the Jacobian of the transformation is provided by
\[ \mathcal{L}_{jac} = \frac{i}{2} \delta^4(0) \left[ \ln \left( 1 + \frac{2}{U} \zeta_1 \right) + \ln \left( 1 + \frac{2}{U} \zeta_2 \right) \right]. \quad (3.23) \]

Step 3: The path integral for \( t_0 < t < t_f \). Our next task is to evaluate the \( \zeta \) integral for times after \( t_0 \). As usual, it is necessary to extract the interactions and counterterms as derivatives with respect to the source. Thus, we write
\[ Z(0, \alpha) = \exp \left[ i V_\zeta (\zeta) \right] Z_1(\alpha), \quad (3.24) \]
where \( V_\zeta = \int_{t_0}^{t_f} (\mathcal{L} - \mathcal{L}_0) dt \)

\[ Z_1(\alpha) = \int [d\zeta] \exp \left[ i \int_{t_0}^{t_f} \mathcal{L} dt + i \int_{t_0}^{t_f} (\mathcal{L}_0 + \zeta \zeta) dt \right] . \quad (3.25) \]
The notation \( \int_{t_0}^{t_f} dt \) indicates that the integral is over both the real-time segments of the time path for \( 0 < t < t_0 \) and the imaginary time segment. We shall assume that the contribution to \( V_\zeta \) from \( \mathcal{L}_{tdc} \) can be neglected. The integral of this contribution over the closed time path does indeed vanish, provided that the boundary conditions to be explained in Step 4 below are valid. We shall later argue on heuristic grounds that the total derivatives in \( \mathcal{L}_{tdc} \) (which would also integrate to zero) ought nevertheless to be retained. Now the manipulations leading to (3.6) can be repeated, with the result
\[ Z(0, \alpha) = \exp \left[ i V_\zeta (\zeta) \right] \exp \left[ -\frac{1}{2} \mathcal{J}(\zeta) \right] \int [d\zeta] e^{i S_1} \]. \quad (3.26) \]
Here, \( \mathcal{J}(\zeta) \) is
\[ \mathcal{J}(\zeta) = \int_{t_0}^{t_f} \zeta_1(t, \alpha) g_{\alpha\beta}(t, \alpha) \zeta_\beta dt, \quad (3.27) \]
where \( g_{\alpha\beta} \) is the propagator for the field \( \zeta \), which we shall need to distinguish from that for \( \phi \), and
\[ \zeta_1(t, \alpha) = \int_{t_0}^{t_f} g_{\alpha\beta}(t, \alpha) \zeta_\beta dt. \quad (3.28) \]
The action in the remaining path integral is
\[ S_1 = \int_{t_0}^{t_f} \mathcal{L} dt + \int_{t_0}^{t_f} \mathcal{L}_0 dt + i \left. \zeta_\alpha(t) \right|_{t=t_f} \left. \zeta_\alpha(t) \right|_{t=t_0} \quad (3.29) \]
provided, as we assume, that any effect of the shift analogous to (3.5) on the interaction terms in \( \mathcal{L} \) can be neglected. There is, indeed, no effect if these interaction terms are taken to exist at all times less than, but not equal to \( t_0 \), but we are unable to prove that this is really legitimate. As in the toy calculation, the path integral for \( t_0 < t < t_f \) can now be performed, yielding a factor of 1.

Step 4: The path integral for \( t < t_0 \). We now transform the path integration variables from \( \zeta \) back to \( \phi \). In making this transformation, we assume that the boundary conditions \( \zeta_1(t_0) = \zeta_2(t_0) \) and \( \zeta_1(t_f) = \zeta_2(t_f) \) translate into \( \phi_1(t_0) = \phi_2(t_0) \) and \( \phi_1(t_f) = \phi_2(t_f) \). Indeed, we have already implicitly assumed that the same is true at \( t_f \). At the heuristic level, this seems justified if we regard the path integral as a sum over sufficiently smooth functions \( \phi_1(t) \) or \( \zeta_1(t) \) for which \( d\phi_1^2(t)/dt = 2\phi_1(t)\phi_2(t) \). The action in the remaining path integral is just the standard action for \( \phi \), except for the boundary term in (3.29).
At this point, we add sources $j_\alpha(t)$ for the fields $\phi_\alpha(t)$, thereby defining
\[ Z(j_\alpha, l_\alpha) = \exp \left[ i V^\phi (-i \delta/\delta a_\alpha) + i V^\zeta (-i \delta/\delta j_\alpha) \right] \times \left[ \exp \left( -\frac{1}{2} \mathcal{J} \int_{t<0} [d\phi] e^{i \bar{S}_2} \right) \right], \] 
with
\[ \bar{S}_2 = \int_{t_0}^{t} dt \left\{ \mathcal{L} + j_\alpha(t) \phi_\alpha(t) + \frac{\lambda}{2} \bar{L}_\alpha(t) \left[ \phi_\alpha^2(t) - U^2(t) \right] \right\}, \] 
where now $\bar{L}_\alpha(t)$ is a distribution such that
\[ \int_{t_0}^{t} L_\alpha(t) f_\alpha(t) dt = L_\beta(t) \delta_{\beta\alpha}(t) \left( \frac{f_\alpha(t)}{U(t)} \right) \bigg|_{t=t_0}. \] 
It should be apparent that $Z(j_\alpha, l_\alpha)$ has the following significance: (i) $Z(0, l_\alpha)$ correctly generates the expectation values we seek for $t_0 < t < t_f$; (ii) $Z(j_\alpha, 0)$ correctly generates expectation values for $0 < t < t_0$, which we also need; (iii) however, because of the way in which $j_\alpha$ has been introduced, $Z(j_\alpha, l_\alpha)$ does not yield correctly the expectation values that would exist after $t_0$ if a real, physical source $j(t)$ had been present at earlier times. This does not concern us, since for us $j_\alpha(t)$ is merely a technical device for generating the expectation values of the source-free theory.

Formally, the remaining path integral can be evaluated by standard methods, with the result
\[ Z(j_\alpha, l_\alpha) = \exp \left[ i V^\zeta (-i \delta/\delta l_\alpha) + i V^\phi (-i \delta/\delta j_\alpha) \right] \times Z_0(j_\alpha, l_\alpha), \] 
where $V^\phi$ represents the interactions of $\phi$ for $t < t_0$. Here, dissipative effects are taken into account by a counterterm $-\frac{1}{2} \phi_\alpha \bar{M}_{\alpha\beta} \phi_\beta$ in which $\bar{M}_{\alpha\beta}$ has the same structure as $\bar{M}_{\alpha\beta}$, but with different coefficients $\alpha_\beta(t)$ and $\gamma_\beta(t)$ associated with the behaviour of the $\phi$ correlator. The functional $Z_0(j_\alpha, l_\alpha)$ is conveniently written as $\exp \left( -\frac{1}{2} \mathcal{K} \right)$, with
\[ \mathcal{K} = \mathcal{K}_\zeta + \mathcal{K}_{\phi\zeta} + \mathcal{K}_{\zeta \rightarrow \tau} + \mathcal{K}_\delta + \mathcal{K}_0. \] 
The various terms arise from different stages of the foregoing calculation as follows. The first,
\[ \mathcal{K}_\zeta = \int_{t_0}^{t_f} dt_\zeta \int_0^{t_\zeta} dl_\alpha(t_\zeta) g^\phi_{\alpha\beta}(t_\zeta, t') l_\beta(t') dt' \] 
provides just the $\zeta$ propagator as in (3.27). The second,
\[ \mathcal{K}_{\phi\zeta} = \int_{t_0}^{t_f} dt_\zeta \int_0^{t_\zeta} dl_\alpha(t_\zeta) g^\phi_{\alpha\beta}(t_\zeta, t') j_\beta(t') dt' \] 
arises in the same way from the path integral over $\phi$. However, the boundary term in (3.31), being quadratic in $\phi$, leads to a modified differential operator $\bar{D}_\alpha^\phi(t; l)$, whose real-time components
\[ \bar{D}_\alpha^\phi(t; l) = D_\alpha^\phi(t) - i \left( \bar{L}_1(t) \begin{pmatrix} 0 \\ \bar{L}_2(t) \end{pmatrix} \right), \] 
depend on the source $l_\alpha(t)$ for $\zeta(t)$. The propagator $g^\phi_{\alpha\beta}(t, t'; l)$ is a solution of
\[ \bar{D}_\alpha^\phi(t; l) g^\phi_{\alpha\beta}(t, t'; l) = -i \delta_{\alpha\beta} \delta(t - t'), \] 
and is therefore also a functional of $l_\alpha(t)$. For this reason too, the final Gaussian path integral that remains after the extraction of $K_\zeta$ and $K_{\phi\zeta}$ depends on $l_\alpha(t)$ and on evaluating it we find
\[ K_{\tau r} = \text{Tr} \ln \bar{D}_\zeta^\phi(l). \] 
Finally,
\[ K_\delta = i \epsilon_{\alpha\beta} L_\alpha(t_0) L_\beta(t_0) \delta(0) \] 
is the last term of (3.27) and
\[ K_0 = - \sum_\alpha \int_{t_0}^{t_f} dt \bar{L}_\alpha(t) U^2(t) \] 
\[ = - \left[ L_1(t) - L_2(t) \right] \left( \bar{\delta} + \gamma^\zeta(t) \right) U(t) \bigg|_{t=t_0} \] 
is the last term of (3.31). The superscript on $\gamma^\zeta(t)$ indicates that this is the damping rate for $\zeta$ rather than for $\phi$.

C. The Feynman rules

In the usual way, perturbation theory now consists in expanding the interactions $V^\phi$ and $V^\zeta$ in (3.33) in powers of $\lambda$, and we shall shortly describe the diagrammatic rules through which the terms of the perturbation series can be represented. First, however, we must confront a difficulty which was postponed earlier. Namely, the perturbation theory for $t < t_0$, based on the field $\phi$ yields an expectation value for $\phi^2$ which is of order $\lambda^0$, whereas the scheme based on $\zeta$ for $t > t_0$ yields a leading term of order $\lambda^{-1}$. Indeed, a perturbative treatment of the $\zeta$-theory is possible only if we take $U(t) = \sqrt{i \delta^2(t)} = O(\lambda^{-1/2})$.

We propose to resolve this conflict in the following way. For $t > t_0$, we define $v(t) = \lambda^{1/2} U(t)$. Then the expectation value of a quantity $A(\zeta)$ can be estimated in the form
\[ \langle A(\zeta) \rangle = \lambda^n \left[ a_0(v) + \lambda a_1(v) + \lambda^2 a_2(v) + \cdots \right], \] 
which is a power series in $\lambda$, provided that $v$ is formally regarded as being of order $\lambda^0$. For $v(t)$, we will have an equation of motion roughly of the form
\[ \ddot{v} = f_0(v) + \lambda f_1(v) + \lambda^2 f_2(v) + \cdots, \]
whose right hand side is again a formal power series in \( \lambda \). The desired expectation value will then be obtained by substituting into (3.42) the solution of (3.43) which satisfies the appropriate initial conditions. The initial values \( \psi(t_0) \) and \( \dot{\psi}(t_0) \) are calculable in the \( \phi \)-theory as power series in \( \lambda \), but are of order \( \lambda^{1/2} \). In principle, each of these power series can be truncated at any desired order (or some partial resummation technique might perhaps be devised). However, the solution of (3.42) with the appropriate initial conditions cannot be expressed as a power series in \( \lambda \) and nor, therefore, can the final result for \( A(\zeta) \). Nevertheless, this scheme leads to a sequence of approximations which in principle can be systematically pursued to arbitrary orders. In view of the evident complexity of the nonequilibrium theory, though, a determination of the convergence or summability properties of this sequence is well beyond the analytical powers of the present author.

With this approximation scheme in mind, we now describe the Feynman rules for constructing the required power series. For \( t < t_0 \), the rules are just the standard ones for \( \lambda \phi^4 \) theory, with propagators constructed according to the prescription described in section 1.A. For \( t \geq t_0 \), things are more complicated: we discuss first the propagators and then the vertices.

1. The propagators

In standard perturbation theory, one has a lowest-order generating functional \( Z_0(\lambda, \rho) = \exp\left[ -\frac{1}{\lambda} K \right] \) in which \( K \) is a quadratic functional of the sources, giving rise to propagator lines which connect vertices, and there is one such propagator for each particle species. Here, \( K \) is the nonlinear functional defined by (3.34) - (3.41) and this gives rise to an additional infinite set of propagators which connect times before and after \( t_0 \). In effect, these propagators represent the nonequilibrium density matrix \( \rho(t_0) \) which, in the Schrödinger picture, would result from evolving the initial state from \( t = 0 \) to \( t = t_0 \).

Of the contributions to \( K \) listed in (3.34), the first, \( K_{\phi} \), yields just the propagator \( g_{\alpha\beta}(t, t') \), which is indicated by the solid line of Figure 2(a). The second contribution, \( K_{\rho} \), involves \( g_{\alpha\beta}(t, t', l) \), which is a solution of (3.38). This solution can be written as

\[
g_{\alpha\beta}(t, t', l) = \sum_{n=0}^{\infty} (-1)^n \int dt_1 \cdots dt_n g_{\alpha\alpha}(t, t_1) \tilde{L}_{\alpha\beta}(t_1) \times g_{\beta\gamma}(t_1, t_2) \tilde{L}_{\gamma\delta}(t_2) \cdots g_{\delta\alpha}(t_n, t'),
\]

(3.44)

where \( \tilde{L}_{\alpha\beta} \) is the array with \( \tilde{L}_{11} = \tilde{L}_1, \tilde{L}_{22} = \tilde{L}_2 \) and \( \tilde{L}_{12} = \tilde{L}_{21} = 0 \). The first term of this series \( (n = 0) \) is just the \( \phi \) propagator \( g_{\alpha\beta}(t, t') \), depicted by the broken line of Figure 2(b). The term \( n = 1 \) is equal to

\[
\int_{t_0}^{t'} g_{\alpha\beta}(t, t', t'') \tilde{L}_{\gamma\delta}(t'') dt''.
\]

The new propagator

\[
g_{\alpha\beta}(t, t', t'') = \sum_{\alpha\beta} \left[ \frac{\tilde{g}_{\alpha\alpha}(t_1, t_2)}{U(t_1)} \right] \right|_{t_1 = t_i}^{t_2 = t_f}
\]

(3.45)

is depicted in Figure 2(c), where the open circle denotes \( U(t_1)^{-1} \tilde{d}(t_1)|_{t_1 = t_0} \). The terms \( n = 2 \) and \( n = 3 \) are shown in Figures 2(d) and 2(e). The dotted lines represent \( g^n(t_i, t_j) \) in which, after performing the derivatives in \( \tilde{d} \), \( t_i \) and \( t_j \) are set equal to \( t_0 \). Explicit expressions for these propagators are straightforwardly obtained, but are somewhat cumbersome and will not be reproduced here.

In the same way, \( K_{\rho} \) defined in (3.38) is given (apart from an irrelevant constant) by

\[
K_{\rho} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int dt_1 \cdots dt_n \text{tr} \left[ \tilde{L}(t_1) g^0(t_1, t_2) \tilde{L}(t_2) \times g^0(t_2, t_3) \tilde{L}(t_3) \cdots g^0(t_n, t_1) \right],
\]

(3.46)

The propagators corresponding to the first few terms of this series are depicted in Figures 2(g) - 2(i). For example, the first term is \( \int_{t_0}^{t_f} g_{\alpha\beta}(t, t_\alpha(t)) dt \), where

\[
g_{\alpha}(t) = -\left[ \frac{h_{\alpha}(t_1)}{U(t_1)} \right] \left[ \partial t_1 - \gamma(t_1) \right]
\]

\[
\times \left[ g_{\alpha\alpha}(t_1, t) - g_{\beta\alpha}(t_1, t) \right]|_{t_1 = t_0}.
\]

(3.47)

Here, use has been made of the fact that \( g_{\alpha\beta}(t, t') = h_{\alpha}(t, t') \) is independent of \( \alpha \) and \( \beta \), as is easily checked from (2.13) and (2.14). In this propagator, the two time arguments of \( g^0(t, t') \) are set equal before acting with the derivatives in \( \tilde{d} \). Later terms in the series for both \( K_{\rho} \) and \( K_{\rho} \) involve derivatives of \( g^0(t, t') \) whose time arguments are to be set equal after differentiation, and these equal-time limits are, unfortunately, not unambiguously defined. They can, of course, be made well-defined by specifying the order in which limits are to be taken, but we are not able to offer a well-motivated general prescription for how this should be done. We shall, however, describe examples of low-order calculations in which the appropriate prescription can be ascertained with reasonable plausibility.

The contribution \( K_{\rho} \) to \( K \) defined in (3.41) can be written as \( \int_{t_0}^{t_f} g_{\alpha\beta}(t, t_\alpha(t)) dt \), with

\[
g_{\alpha}(t) = U(t_1) \left[ \partial t_1 - \gamma(t_1) \right] \left[ g_{\alpha\alpha}(t_1, t) - g_{\beta\alpha}(t_1, t) \right]|_{t_1 = t_0},
\]

(3.48)
and is represented in Figure 2(f). It evidently has a structure similar to that of (3.47), and the significance of this will become clear below.

Compared with diagrammatic rules of the conventional kind, most of the propagators shown in Figure 2 look like vertices, and it is perhaps worth emphasising that they actually are propagators. They must be used to connect the vertices described below in order to form a valid diagram. For example, if the object in Figure 2(e) were a vertex, it could be combined with the propagator of Figure 2(b) to form a diagram with the topology of Figure 2(g), but this does not result in a valid expression. To make this apparent at the visual level, we envisage the propagators as “cables” terminated by “plugs” (indicated by the arrows in Figure 2) and will depict genuine vertices as possessing “sockets” that accept such plugs. Then, a valid diagram can be wired up by plugging cables into sockets, but not by tying cables together. Formally, plugs correspond to the sources $j_a$ or $l_a$ which accompany the propagators in $Z_0(j_a, l_a)$, while sockets correspond to the derivatives in $V^\phi(-i\partial/\partial j)$ and $V^\zeta(-i\partial/\partial l)$. Once a valid diagram is formed, account must also be taken of the spatial momenta which we have not indicated explicitly. These appear in the standard way, being conserved both at genuine vertices and at the circled vertices internal to the composite propagators. Each of these circled vertices contains the operator $\delta(t; k)$ whose momentum $k$ is that of the single $\zeta$ propagator emerging from it.

2. The vertices

The interactions $V^\phi$ and $V^\zeta$ in (3.33) are, of course, represented by vertices. Interactions of $\phi$ exist only before $t_0$ while those of $\zeta$ exist only after $t_0$ and there are no vertices involving both fields. For $\phi$, we introduce a renormalized mass $\mu(t)$ and a dissipative counterterm $\mathcal{M}_\phi(t)$ as described in section IIA. The interaction part of the Lagrangian density is then

$$L_{\text{int}}^\phi = \frac{1}{2} \left[ m^2(t) + \mu^2(t) \right] \phi_\alpha \phi_\beta \delta_{\alpha\beta} + \frac{1}{2} \phi_\alpha \mathcal{M}_\phi(t) \phi_\beta - \frac{\lambda}{4!} (\phi_1^4 - \phi_2^4).$$

(3.49)

The vertices corresponding to the quartic interaction in (3.49) are shown in Figures 3(a), 3(f) and 3(g) respectively.

The corresponding vertices are those shown in Figures 3(b) - 3(e), where a stroke on one of the legs indicates the action of $\mathcal{D}_\alpha$ on the propagator attached to that leg. The linear counterterm (3.18) becomes

$$L_{\text{ct}}^{(1)} = -\lambda^{-1/2} \left( \bar{\psi} - m^2 \psi + \frac{1}{3} \psi^3 \right) \epsilon_\alpha \zeta_\alpha$$

(3.51)

and is represented by Figure 3(h). The counterterms contained in (3.19) include a mass renormalization

$$L_{\text{ct}}^{(\text{mass})} = \frac{1}{2} \bar{\psi}^{-1} \left( \bar{\psi} + M^2 \psi - \frac{1}{3} \psi^3 \right) (\zeta_1^2 - \zeta_2^2),$$

(3.52)

depicted in Figure 3(k), of which we shall make explicit use, together with counterterms associated with $\mathcal{M}_\zeta$. The first two of these are shown in Figures 3(l) and 3(p), but we shall not make explicit use of them. Finally, the two total derivatives in (3.21) are represented by Figures 3(i) and 3(m), while the first two of the infinite sequence of terms representing the Jacobian in (3.23) are shown in Figures 3(j) and 3(n).

IV. EQUATIONS OF MOTION AND CONTINUITY CONDITIONS AT $t_0$

We now discuss several calculations which serve the dual purpose of completing the specification of the Feynman rules derived in the last section and of illustrating their use. The Feynman rules are so far incompletely specified for two reasons. One is that they involve the expectation value $v^2(t) = \lambda (\phi^2(t))$, the renormalized masses $\mu(t)$ and $M(t)$ and the dissipative coefficients $\alpha_\alpha^\phi(t)$, $\alpha_\alpha^\zeta(t)$, $\gamma_\alpha^\phi(t)$ and $\gamma_\alpha^\zeta(t)$ for which we have no concrete expressions in hand. The second is that the propagators $g_{ab}^\phi(t, t'; k)$ and $g_{ab}^\zeta(t, t'; k)$ are solutions of (2.8) (with the appropriate operator $\mathcal{D}_{ab}$ in each case), but the appropriate solutions must be identified through suitable boundary conditions. When these propagators are represented in the form of (2.13) and (2.14), the remaining ambiguity resides in the functions $N_\alpha^\phi(t)$ and $N_\alpha^\zeta(t)$. These in turn are solutions of (2.17) and it is the initial conditions for these functions that are needed.

For the unbroken-symmetry state prior to $t_0$, the renormalization and boundary conditions which determine $\mu(t)$, $\alpha_\alpha^\phi(t)$, $\gamma_\alpha^\phi(t)$ and $N_\alpha^\phi(t)$ are described in (21)(22) and we shall not repeat the discussion here. The determination of $\alpha_\alpha^\zeta(t)$ and $\gamma_\zeta(t)$ presents no new difficulty beyond that of computing the required integrals and will also not be discussed. It is therefore the determination of $v(t)$, $M(t)$ and the initial conditions on $N_\alpha^\zeta(t)$ which are principally of interest. Also of concern is the fact that, in evaluating the path integral for $Z(j_a, l_a)$, we were unable to treat the boundary conditions at $t_0$ in a fully rigorous manner, with the result that certain ambiguities remain. We will show how these ambiguities can be resolved at the lowest non-trivial order of our approximation scheme.
A. The condition $\langle \zeta(t) \rangle = 0$

We start with the fact that $U^2(t) = \lambda^{-1} \rho^2(t)$ was defined to be the exact expectation value of $\langle \phi^2(t) \rangle$ and is therefore to be determined self-consistently from the requirement that $\langle \zeta(t) \rangle = 0$. It is sufficient to ensure that the one-particle-irreducible contribution $\langle \zeta(t) \rangle_{\text{1PI}}$ vanishes. As illustrated in Figure 4, this quantity can be separated into two parts,

$$\langle \zeta(t) \rangle_{\text{1PI}} = \langle \zeta(t) \rangle^{(\text{anchored})}_{\text{1PI}} + \langle \zeta(t) \rangle^{(\text{free})}_{\text{1PI}}. \quad (4.1)$$

Diagrams contributing to the first part contain at least one of the circled vertices internal to one of the composite propagators, which we will describe as “anchoring” the corresponding time argument at $t = t_0$, whereas all the vertices in the second part are free to range between $t_0$ and $t_f$. In Figure 4(a), the diagram labelled (ii) is just the propagator of Figure 2(g), while that labelled (iii) is constructed from the propagator of Figure 2(c) together with the ordinary $\phi$ vertex of Figure 3(a). These are the first two of a sequence of diagrams whose sum reproduces $\langle \phi^2(t_0) \rangle$ as calculated from the $\phi$-theory. More precisely, on combining these with diagram (i) (which arises from Figure 2(f)), we obtain the expression

$$-\frac{i}{2} \left[ g_{\alpha_1}(t, t_1) - g_{\alpha_2}(t, t_1) \right] \left( \partial_{t_1} + \gamma^{\alpha}(t_1) \right) \times \left[ U(t_1) - \frac{\langle \phi^2(t_1) \rangle}{U(t_1)} \right] \bigg|_{t_1 = t_0}. \quad (4.2)$$

With the natural requirement that $\langle \phi^2(t) \rangle$ and its first derivative should be continuous at $t_0$, this vanishes identically. The fact that contributions from several different terms of (3.34) conspire to give this satisfactory result is somewhat reassuring. We now turn to the second part of $\langle \zeta(t) \rangle_{\text{1PI}}$ illustrated in Figure 4(b). First, we dispose of the diagram labelled (v). In this diagram, $D_{\alpha \beta}^\zeta$ acts on the internal propagator to produce $\delta(0)$, and this contribution is precisely cancelled by the Jacobian counterterm of diagram (iii). In order to set $\langle \zeta(t) \rangle_{\text{1PI}}$ equal to zero, we would like the remaining contributions to sum to an expression of the form $\int z_\alpha(t') g_{\alpha \beta}(t', t) dt'$, so that $z_\alpha(t)$ can be set to zero. The obstacle to this is diagram (vi), in which $D_{\alpha \beta}^\zeta$ acts on the external propagator to produce $\delta(t - t')$. Now, this problem could be solved through an integration by parts to make $D_{\alpha \beta}^\zeta$ act on the bubble to its left, were it not for the boundary terms at $t' = t_0$, whose interpretation is a little unclear. It was to facilitate the handling of this integration by parts that we introduced the counterterm proportional to $X_\alpha(t)$ in (3.24), which is a total derivative and contributes diagram (ii) of Figure 4(b). In effect, we can now perform the integration by parts without incurring boundary terms by choosing

$$X_1(t) = X_2(t) = -\frac{\lambda^{1/2}}{2v(t)} I_1^I(t), \quad (4.3)$$

where

$$I_1^I(t) = \int \frac{d^3k}{(2\pi)^3} K(t, t; k) \quad (4.4)$$

is (with the spatial momentum now made explicit) the bubble contained in diagrams (iv) and (vi). With this choice, the term $X_\alpha D_{\alpha \beta}^\zeta \zeta_\beta$ cancels diagram (iv) while $-\zeta_\alpha D_{\alpha \beta}^\zeta X_\beta$ supplies the result of the integration by parts. The counterterm was, of course, subtracted off again in $\mathcal{C}_{\text{1PI}}$ (equation (4.22)), which we subsequently assumed could be neglected. This assumption is now seen to amount to a prescription for handling the present integration by parts and other similar ones. In the present case, we take this prescription to be justified a posteriori by the fact that we can now set $\langle \zeta(t) \rangle = 0$ and hence obtain a sensible equation of motion for $v(t)$. Including the remaining diagram (i) (which represents the counterterm $\mathcal{C}_{\text{1PI}}$), the equation of motion at this order is

$$\left( \ddot{v} - m^2 v + \frac{1}{6} v^3 \right) + \frac{3}{2} \lambda \left( \ddot{v} + M^2 v \right) \left( \frac{\dot{\xi}}{v^2} \right) - \frac{1}{2} \lambda \left( \partial_t^2 + M^2 \right) \left( \frac{\dot{\xi}}{v} \right) = 0. \quad (4.5)$$

B. The 2-point function $\langle \phi^2_\alpha(t) \phi^2_\beta(t') \rangle$

It is now necessary to address several issues concerning the connected 2-point function $\langle \phi^2_\alpha(t) \phi^2_\beta(t') \rangle_c$. Clearly, if $t < t_0$ and $t' > t_0$, this object is equal to $2U(t') \langle \phi^2_\alpha(t) \phi_\beta(t') \rangle$, while if $t$ and $t'$ are both greater than $t_0$, it is equal to $4U(t) U(t') \langle \phi_\alpha(t) \phi_\beta(t') \rangle$. Continuity of this function at $t_0$ will supply the initial conditions needed to specify $g_{\alpha \beta}(t, t')$ completely, and we also require a suitable definition of the renormalized effective mass $M(t)$. We shall deal with these issues at the lowest order of our approximation scheme, but first a technical question must be addressed.

1. Anchored contributions to $\langle \zeta_\alpha(t) \zeta_\beta(t') \rangle$

The functional $K$ which gives rise to our various propagators contains a contribution $K_{\bar{\delta}}$, given in (3.40) which is not included in Figure 2. In the toy calculation of section II A the analogous quantity was found to cancel exactly another singular contribution (equation (3.13)), provided that the propagators satisfied continuity conditions which were expected on other grounds. We now find a similar cancellation involving the composite propagator of Figure 2(h), which is the lowest-order anchored contribution to $\langle \zeta_\alpha(t) \zeta_\beta(t') \rangle$. The internal lines in this propagator correspond to the expression
\[
\chi_{\alpha\beta}(t, t'; k) = \int \frac{d^3k'}{(2\pi)^3} g_{\alpha\beta}^\phi(t, t'; k') g_{\alpha\beta}^\phi(t, t'; k' + k),
\]
(4.6)

and \(2\chi_{\alpha\beta}(t, t'; k)\) is precisely the lowest-order contribution to \(\langle \phi_2^\phi(t)\phi_2^\phi(t') \rangle\). Suppressing the momentum argument, the expression represented by Figure 2(h) is

\[
g_{\alpha\gamma}(t, t_1) \frac{d}{dt_\gamma(t_1)} \chi_{\delta\rho}^\phi(t_1, t_2) \frac{d}{dt_\rho(t_2)} g_{\alpha\beta}^\phi(t_2, t'),
\]
(4.7)

As indicated above, this expression is ill-defined, because it involves the quantity \(\partial_t \partial_\gamma \chi_{\alpha\beta}^\phi(t_1, t_2)\) at \(t_1 = t_2 = t_0\). However, we can use (3.11) (with \(\epsilon_{\alpha\beta\gamma} \partial_\gamma\) replaced by \(\frac{d}{dt_\gamma}\)) to find that this contribution to \(K_{t_2}\) is exactly cancelled by \(K_{t_1}\), provided that the boundary conditions on \(g_{\alpha\beta}^\phi(t, t')\) satisfy

\[
2\chi_{\alpha\beta}(t, t') = 0 \quad U(t) \frac{d}{dt} U(t) g_{\alpha\beta}^\phi(t, t'),
\]
(4.8)

The notation \(\alpha = 0\) here indicates that the quantities on the left and right, together with their derivatives with respect to \(t\) and \(t'\) are equal at \(t = t' = t_0\). If this prescription (which is consistent with the boundary conditions to be discussed below) is adopted, then we may delete the propagator of Figure 2(h). This matter will, however, need to be reconsidered when we come to discuss renormalization in section 4.

2. Continuity of the 2-point function

At the lowest order of approximation, the 2-point function \(\langle \phi_2^\phi(t)\phi_2^\phi(t') \rangle\) is represented by one of the diagrams shown in Figure 5, depending on whether its time arguments are greater or smaller than \(t_0\). The anchored vertex in diagram 5(b) involves \(U(t_0)\) and \(\dot{U}(t_0)\), which we expand in powers of \(\lambda\), retaining only the leading terms

\[
\left[U(0)^\prime(t)\right]^2 = I_1^\phi(t) \equiv \int \frac{d^3k}{(2\pi)^3} h^\phi(t, t),
\]

\[
U_0 \equiv U(0)(t_0); \quad \dot{U}_0 = \frac{d}{dt} U(0)(t) \bigg|_{t=t_0}.
\]

Because of the propagator structure exhibited in (2.13), this diagram can be expressed in the form

\[
g_{\alpha\beta}^{\phi\phi}(t, t'; k) = -i \int \frac{d^3k'}{(2\pi)^3} \left[ h^\phi_{\alpha}(t_1, t; k') h^\phi_{\beta}(t_1, t; k' + k) \right] \frac{1}{U(0)(t_1)} \cdot
\]
\[
\times \left( \frac{\partial}{\partial t_1} - \gamma^\phi(t_1) \right) \left[ h^\phi(t', t_1; k) - h^\phi(t', t_1; k) \right] \bigg|_{t_1=t_0}.
\]
(4.10)

In order to display the boundary conditions on the 2-point function succinctly, we define

\[
R_{\alpha}(t; k) = \int \frac{d^3k'}{(2\pi)^3} h^\phi_{\alpha}(t_0, t; k') h^\phi_{\alpha}(t_0, t; k' + k),
\]
\[
S_{\alpha}(t; k) = \partial_t \int \frac{d^3k'}{(2\pi)^3} h^\phi_{\alpha}(t_1, t; k') h^\phi_{\alpha}(t_1, t; k' + k) \bigg|_{t_1=t_0},
\]
\[
P(t; k) = h^\phi(t, t_0; k) - h^\phi(t, t_0; k),
\]
\[
Q(t; k) = \left( \partial_t - \gamma^\phi(t_1) \right) \left[ h^\phi(t, t_1; k) - h^\phi(t, t_1; k) \right] \bigg|_{t_1=t_0},
\]

which satisfy, in particular,

\[
P(t_0) = 0, \quad Q(t_0) = i
\]
(4.11)
\[
P(t_0) = -i, \quad Q(t_0) = 0
\]
(4.12)
\[
R_0(t_0) = R_0(t_0; k) = R_0^*(t_0; k),
\]
\[
S_0(t_0) = S_0(t_0; k) = S_0^*(t_0; k).
\]
(4.13)

In terms of these, we have

\[
g_{\alpha\beta}^{\phi\phi}(t, t') = -i \frac{U_0}{U_0} \left[ R_{\alpha}(t) Q(t') \right.
\]
\[
\left. + \frac{U_0}{U_0} R_{\alpha}(t) - S_\alpha(t) \right] P(t'),
\]
(4.15)
\[
\langle \phi_2^\phi(t)\phi_2^\phi(t') \rangle = 2R_{\alpha}(t),
\]
\[
\partial_t \langle \phi_2^\phi(t)\phi_2^\phi(t') \rangle \bigg|_{t'=t_0} = 2S_\alpha(t).
\]
(4.16)

In principle, the function \(\langle \phi_2^\phi(t)\phi_2^\phi(t') \rangle\) should be perfectly smooth as its arguments pass through \(t_0\). We will actually attempt to impose the rather weaker conditions that the function itself and its first derivatives are continuous at \(t_0\). At lowest order, these conditions read

\[
g_{\alpha\beta}^{\phi\phi}(t, t_0) = \frac{1}{2U_0} \langle \phi_2^\phi(t)\phi_2^\phi(t_0) \rangle,
\]
\[
g_{\alpha\beta}^{\phi\phi}(t_0, t') = 2U_0 g_{\alpha\beta}^{\phi\phi}(t_0, t'),
\]
\[
\partial_t g_{\alpha\beta}^{\phi\phi}(t, t') \bigg|_{t'=t_0} = \partial_t \left[ \langle \phi_2^\phi (t)\phi_2^\phi(t') \rangle \right] \frac{1}{2U(0)(t')} \bigg|_{t'=t_0},
\]
\[
\partial_t g_{\alpha\beta}^{\phi\phi}(t, t') \bigg|_{t=t_0} = \partial_t \left[ 2U(0)(t) g_{\alpha\beta}^{\phi\phi}(t, t') \right] \bigg|_{t=t_0}.
\]
(4.18)

Of these, (4.18) and (4.20) are satisfied identically, because of (4.11) and (4.12). The condition (4.19) is satisfied provided that

\[
\text{Im} S_0 = -U_0^2
\]
(4.22)

and

\[
N^\phi(t_0) = \frac{i}{U_0^2} \left[ \text{Re} S_0 - \left( Z - \frac{1}{2} \gamma^\phi(t_0) \right) R_0 \right],
\]
(4.23)
where

\[ Z = \frac{\tilde{f}^*}{f^*} \left( t_0 \right) + \frac{U_0}{\bar{U}_0} \]  
(4.24)

and \( f_k^*(t) \) is the mode function introduced in (2.15) that appears in \( g_{\alpha \beta}^\zeta \). Explicit calculation shows that (4.22) is indeed true and (4.23) of course provides one of the initial values that we seek. The last condition (4.21) is satisfied if

\[ \hat{N}^\zeta(t_0) = i \frac{1}{U_0^2} \left[ \text{Re} \hat{S}_0 - 2Z \text{Re} \hat{S}_0 + \left( Z^2 - \frac{1}{4} \gamma^2(t_0) \right) R_0 \right], \]
(4.25)

with \( \hat{S}_0 = \hat{S}_1(t_0) = \hat{S}_2(t_0) \), and if also

\[ \text{Im} \hat{S}_0 = -U_0^2 \gamma^\zeta(t_0). \]  
(4.26)

The second of our initial values for the \( \zeta \) propagator is, of course provided by (1.22), but the status of (1.26) is less clear. Evaluating this equation explicitly, we obtain a moderately plausible relation between the damping rates for \( \phi \) and \( \zeta \), namely

\[ \int \frac{d^3k'}{(2\pi)^3} j^\phi_k(t_0) h^\phi(t_0, t_0; k' + k) = U_0^2 \gamma^\zeta(t_0). \]  
(4.27)

However, this relation is not automatically satisfied. Apparently, it implies a constraint on the prescriptions used to define these damping rates, but we are not able to say whether a prescription of the kind described in [21] would naturally satisfy this constraint. In any case, when we come to consider renormalization, we shall find that the boundary conditions derived here require a significant modification. We note, though, that in this unrenormalized form, the initial values \( N^\zeta(t_0) \) and \( \hat{N}^\zeta(t_0) \) do satisfy the condition (2.13).

3. The 2-point function \( \langle \zeta_\alpha(t) \zeta_\beta(t') \rangle \)

The last step required to specify the Feynman rules completely is to find a suitable definition of the renormalized \( \zeta \) mass \( M(t) \). To do this, we calculate the 2-point function \( \langle \zeta_\alpha(t) \zeta_\beta(t') \rangle \) at first order in \( \lambda \). It is the sum of the eigenshows diagrams shown in Figure 6 (recalling from our earlier discussion that the composite propagator of Figure 2(h) can be ignored). The computation of these diagrams is straightforward, except for Figure 6(s), in which \( D_{\alpha \beta} \) acts twice on the same internal propagator. In order to evaluate this diagram we have (i) ignored some contributions from the dissipative coefficients \( \alpha^\zeta \) and \( \gamma^\zeta \) which can formally be regarded as of higher order; (ii) adopted the following prescription for the equal-time limit of derivatives of the propagator:

\[ \partial_t g^\zeta_{\alpha \alpha}(t, t') \bigg|_{t=t'} = \frac{1}{2} \left[ \lim_{t \to t'} \lim_{t' \to t} \right] \partial_t g^\zeta_{\alpha \alpha}(t, t') \]

\[ = \frac{1}{2} \left[ \partial_t h^\zeta(t, t') + \partial_t h^\zeta(t', t') \right] \bigg|_{t=t'} \]

and (iii) identified the second counterterm in (3.21) as

\[ Y_1(t) = Y_2(t) = \frac{\lambda}{8} I_2^\zeta(t), \]  
(4.28)

where \( I_2^\zeta(t) \) was defined in (4.4), so as to facilitate an integration by parts. The result we obtain is consistent, at this order of approximation, with the structure displayed in (2.20). Specifically, we write it as

\[ G^\zeta_{\alpha \beta}(t, t') = \frac{\sigma(t)}{v(t)} G^{(\psi\psi)}_{\alpha \beta}(t, t') \frac{\sigma(t')}{v(t')} \]

\[ + \frac{\lambda}{2} \frac{1}{v(t)} G^{(\psi\sigma\psi)}_{\alpha \beta}(t, t') \frac{\sigma(t)}{v(t')} \]

\[ + \frac{\lambda}{4} \frac{1}{v(t)} G^{(\psi\sigma\sigma)}_{\alpha \beta}(t, t') \frac{1}{v(t')}, \]  
(4.29)

where the quantity

\[ \sigma(t) = v(t) \left[ 1 - \frac{\lambda}{2} I_2^\zeta(t) \right] + O(\lambda^2) \]  
(4.30)

might loosely be interpreted as corresponding to the peaks of a probability density of the kind sketched in Figure 1(d). In the first term of (4.29), with which we are principally concerned, we have

\[ G^{(\psi\psi)}_{\alpha \beta}(t, t') \]

\[ = g^\zeta_{\alpha \beta}(t, t') + i \int_{t_0}^{t_f} dt'' g^\zeta_{\alpha \gamma} \epsilon_{\gamma \delta} A(t'') g^{\zeta}_{\delta \beta}(t', t') \]

\[ + \int_{t_0}^{t_f} dt'' dt''' g^\zeta_{\alpha \gamma} \epsilon_{\gamma \delta} B_{\delta \gamma}(t'', t''') \epsilon_{\lambda \delta} g^\zeta_{\lambda \beta}(t'''', t'), \]  
(4.31)

with auxiliary functions \( A(t) \) and \( B_{\alpha \beta}(t, t') \) which will be specified shortly. It will be seen, for example, that the term \( \langle \sigma/v \rangle g^{\zeta}_{\alpha \beta}(t, t') \langle \sigma/v \rangle \) arises from the sum of the diagrams (a), (g) and (h) of Figure 6, together with a 2-loop diagram which is easily seen to be present at order \( \lambda^2 \). As always in the closed-time-path formalism, the signs \( \epsilon_{\alpha \beta} \) attached to the vertices and the structure (2.13) of the propagators ensure causality, in the sense that the integrands in (4.31) vanish whenever \( t'' \) or \( t''' \) is greater than both \( t \) and \( t' \). The auxiliary functions are given by

\[ A(t) = \left( \frac{\bar{v} + M^2 v}{v^3} \right) t + \frac{3\lambda}{2} \left( \frac{\bar{v} + M^2 v}{v^3} \right) t I_1^\zeta(t) \]

\[ - \frac{\lambda}{2v(t)} \left( \partial_t^2 + M^2 \right) \frac{I_2^\zeta(t)}{v(t)} + \frac{9\lambda}{2} \left( \frac{\bar{v} + M^2 v}{v^2} \right)^2 t I_2^\zeta(t), \]  
(4.32)
and

\[
E_{\alpha\beta}(t, t') = -\frac{9\alpha}{2} \left( \frac{\ddot{v} + M^2 v}{v^2} \right)_t \\
\times \left[ I_2(t, t')_{\alpha\beta} + i\epsilon_{\alpha\beta\gamma}(t - t') I_2 \right] \left( \frac{\ddot{v} + M^2 v}{v^2} \right)_{t'},
\]

(4.33)

with

\[
I_2(t, t'; k)_{\alpha\beta} = \int \frac{d^4k'}{(2\pi)^4} \bar{g}^\zeta_{\alpha\beta}(t, t'; k') g_{\alpha\beta}(t, t'; k' + k).
\]

(4.34)

The terms proportional to \( I_2 \) cancel out in (4.31) and have been inserted for the purpose of renormalization, which is considered below. For completeness, we record the following functions are

\[
G_{\alpha\beta}^{(\psi^2)}(t, t') = -3i\lambda^{1/2} \int_{t_0}^{t_f} dt'' g_{\gamma\gamma}(t, t'') \epsilon_{\gamma\delta} \\
\times \left( \frac{\ddot{v} + M^2 v}{v^2} \right)_{t''} I_2(t'', t')_{\delta\beta}.
\]

\[
G_{\alpha\beta}^{(\psi^2)}(t, t') = -3i\lambda^{1/2} \int_{t_0}^{t_f} dt'' \epsilon_{\gamma\delta} g_{\gamma\gamma}(t, t'')_{\alpha\gamma} \\
\times \left( \frac{\ddot{v} + M^2 v}{v^2} \right)_{t''} I_2(t'', t')_{\delta\beta}.
\]

\[
G_{\alpha\beta}^{(\psi^2)}(t, t') = 2I_2(t, t')_{\alpha\beta}.
\]

These are easily seen to correspond at lowest order to the correlators indicated in (2.26).

Our renormalized mass \( M(t) \) will now be defined by requiring \( A(t) = 0 \). In the nonequilibrium theory, there is no clear analogue of the pole of a zero-temperature propagator in Minkowski spacetime, which unambiguously identifies the mass of a stable particle. Here, our rationale is, rather, to optimise \( g_{\alpha\beta} \) as an approximation to the full 2-point function by using the mass counterterm (5.52) to cancel as much as possible of the higher order corrections. Although the function \( A(t) \) was identified by eye from our explicit \( O(\lambda) \) result, we assume that this condition actually makes sense to all orders, so the definition of \( M(t) \) is an implicit one, which can be realised in practice only to a given order of approximation. Setting the expression (4.32) to zero appears to give a second equation of motion for \( v(t) \) which is similar, but not identical, to that already found in (4.3) by requiring \( \langle \zeta(t) \rangle = 0 \). In fact, these two equations together yield both the equation of motion for \( v(t) \) and the relation between \( M(t) \) and the bare mass \( m(t) \). Consistently ignoring terms of order \( \lambda^2 \), we obtain the equation of motion

\[
(\partial_t^2 + M^2) \left[ v - \frac{\lambda I_2}{2} \frac{v}{v^2} + O(\lambda^2) \right] = \frac{v^3}{3} \left[ 1 - \frac{3\lambda I_2}{2} \frac{v^2}{v^2} - \frac{3}{2} \lambda I_2 + O(\lambda^2) \right]
\]

and the gap equation

\[
m^2(t) = -M^2(t) + \frac{1}{2}v^2(t) \left[ 1 - \lambda I_2 \right] + O(\lambda^2).
\]

This gap equation expresses the bare mass \( m(t) \), which occurs only in the counterterm (5.51), as a function of \( M(t) \), \( v(t) \) and \( \lambda \). For a given \( m(t) \), these are two equations to be solved simultaneously for \( v(t) \) and \( M(t) \). In the specific application to an expanding universe, we have \( m^2(t) = a^2(t)m_0^2 \), and one must simultaneously solve Einstein’s field equations for the scale factor \( a(t) \).

V. RENORMALIZED EQUATIONS OF MOTION AND INITIAL CONDITIONS

The equations of motion have just derived in-\ involvement integrals \( I_2(t) \) and \( I_2(t, t'; k)_{\alpha\beta} \) which are ultraviolet divergent, and many more divergent integrals are of course to be expected at higher orders. Of necessity, the nonequilibrium theory is formulated in terms of quantities which are defined implicitly as the solutions of differential equations or of self-consistency relations. This system of equations, while ultimately susceptible of numerical solution when truncated at some (no doubt low) order of perturbation theory, is far too complicated to admit of any exact analytical solution, and a proof that they can be renormalized so as to remove ultraviolet divergences at all orders is currently beyond the ingenuity of this author. A numerical investigation at the order of approximation we have considered explicitly ought to be feasible, provided that a suitable renormalization prescription can be given. Even at this level, we have in hand only formal expressions for the propagators \( g_{\alpha\beta}(t, t') \) in terms of mode functions \( f_k(t) \) and generalized “occupation numbers” represented by the functions \( N_k(t) \), which must be found from numerical solution of (2.15) and (2.17), so an unambiguous proof of renormalizability is rather difficult. Here, we suggest a renormalization prescription which might be applied in the context of a numerical computation involving a spatial momentum cutoff \( \Lambda \), which reflects the renormalizations that are well known to work in the Minkowski-space theory. Since ultraviolet divergences are a feature of the short-distance and short-time properties of the theory, one expects that they should be essentially independent of the nonequilibrium state, and that these renormalizations should be adequate. In practice, one hopes that numerical computations would approach finite limits when \( \Lambda \to \infty \), and we shall offer circumstantial grounds for optimism that this should indeed be so.
A. Large-$k$ behaviour of propagators and 1-loop integrals

We need to estimate the large-$k$ behaviour of our propagators, and do so using a method similar to that described, for example, in [30]. The usual propagators of the equilibrium theory are obtained from our $g_{\alpha\beta}(t,t')$, by setting $N_k(t) = 2n_k + 1$, where $n_k$ is the Bose-Einstein distribution, which falls off exponentially at large $k$. When time evolution is sufficiently slow, the dissipative coefficients $\alpha_k(t)$ and $\gamma_k(t)$ are given by scattering integrals [28], which are convolutions of the $n_k$, and also exponentially small at large $k$. For present purposes therefore, we assume that it is sufficient to set $N_k(t) = 1$ and $\alpha_k(t) = \gamma_k(t) = 0$. Then, from (2.14), we need only to estimate $h(t,t';k) \approx \frac{1}{2} f_k(t) f_k(t')$, with mode functions obeying

$$[\partial^2_t + k^2 + M^2(t)] f_k(t) = 0$$

and the Wronskian condition (2.11). The general solution may be written as

$$f_k(t) = \frac{e^{i\theta(k)}}{\sqrt{2}\Omega_k(t)} \exp \left[-i\int_0^t \Omega_k(t') dt'\right],$$

(5.1)

where $\theta(k)$ is an arbitrary, but time-independent, function of $k$ which does not affect the propagator and will therefore be ignored. We will choose (as is always possible) a time-dependent frequency which behaves as $\Omega_k(t) = k + O(k^0)$ for large $k$. In this case, we obtain the large-$k$ expansion

$$f_k(t) f_k(t') = \frac{e^{-ik(t-t')}}{2k} \times \left[1 - \frac{i}{2k} W(t,t') - \frac{1}{4k^2} X(t,t') + O(k^{-3})\right],$$

(5.2)

where

$$W(t,t') = \int_{t'}^t M^2(t'')dt''$$

and $X(t,t') = M^2(t) + M^2(t') + \frac{1}{2} W(t,t')$. This expansion can be used to isolate the ultraviolet divergences of the two integrals $\bar{I}_1(t)$ and $\bar{I}_2(t,t';k)$. We impose a cutoff $\Lambda$ on physical momenta $k_{\text{phys}} = |k_{\text{phys}}| < \Lambda$. Up to this point, our analysis has been entirely in terms of comoving coordinates and momenta and, in principle, we should translate all of the above results into physical coordinates in order to impose this cutoff systematically. However, both $\bar{I}_1(t)$ and the divergent part of $\bar{I}_2(t,t';k)$ are local in time and, for our present purposes, it is sufficient simply to convert the physical cutoff to a comoving cutoff $k < a(t)\Lambda$. Specifically, we find

$$\bar{I}_1(t) = \frac{1}{8\pi^2} \left[(a(t)\Lambda)^2 - M^2(t) \ln(a(t)\Lambda)\right] + \cdots,$$

(5.3)

$$\bar{I}_2(t,t';k)_{\alpha\beta} = -i\delta_{\alpha\beta}\epsilon_\alpha \delta(t-t') \tilde{I}_2(t) + \cdots,$$

(5.4)

where the ellipsis represents ultraviolet-finite contributions. The quantity $\bar{I}_2(t)$ here is that appearing in (4.32) and (4.33) and we see, in particular, that the net integral in (4.33) is finite.

B. Renormalized equation of motion and gap equation

We hope, of course, that both the equation of motion (4.35) and the gap equation (4.36) can be expressed in a form which is free of ultraviolet divergences. To achieve this, we introduce the physical particle mass $m$, which locates the pole of the Minkowski-space propagator in the broken-symmetry vacuum. It is related to the bare mass $m_0$ by

$$m^2 = \frac{\lambda}{16\pi^2} \Lambda^2 + \frac{1}{2} \hat{m}^2 a^2 \left(1 + \frac{\lambda R}{16\pi^2} \hat{c}\right) + O(\Lambda^2),$$

(5.5)

with $c = 1 + \ln 2 - \sqrt{\pi}/2$. (The 1-loop integrals used to obtain this relation are, of course, Minkowski-space integrals involving a physical 3-momentum $k_{\text{phys}}$, whose magnitude is cut off at the value $\Lambda$.) In the Minkowski-space theory, Green’s functions involving the operator $\phi^2$ are rendered finite by a combination of additive and multiplicative renormalizations. Guided by the standard theory of these renormalizations, we surmise that a renormalized expectation value $\langle \phi^2 \rangle_R = \langle \phi^2 \rangle_R$ can be defined, at the order of approximation we are using, by

$$\langle \phi^2 \rangle = \frac{\langle a(t) \rangle^2}{8\pi^2} + \frac{1}{2} \hat{m}^2 a^2 \left(1 + \frac{\lambda R}{16\pi^2} \hat{c}\right) \tilde{I}_2 + Z_{\phi^2} \left(\frac{\langle \phi^2 \rangle_R}{\lambda R}\right),$$

(5.6)

where $Z_{\phi^2} = 1 - \frac{1}{2} \lambda R \tilde{I}_2 + O(\lambda R^2)$ is the multiplicative renormalization factor and $\hat{c}(t) = c - \ln(a(t)\hat{m})$. The renormalized coupling constant $\lambda_R$ is defined by $\lambda = Z_{\lambda} \lambda_R$, with $Z_{\lambda} = 1 + \frac{3}{2} \lambda R \tilde{I}_2 + O(\lambda R^2)$. With these definitions, we indeed obtain a renormalized equation of motion

$$\left[\partial^2_t + M^2\right] v_R - \frac{\lambda R}{2} \frac{\tilde{I}_2}{v_R} + O(\lambda R^2) = \frac{v_R^2}{3} \left[1 - \frac{3\lambda R}{2} \frac{\tilde{I}_2}{v_R} + O(\lambda R^2)\right],$$

(5.7)

and a renormalized gap equation

$$\langle \phi^2 \rangle_R = \frac{1}{2} v_R^2 - \frac{1}{2} \hat{m}^2 a^2 \left(1 + \frac{\lambda R}{16\pi^2} \hat{c}\right) + O(\lambda R^2),$$

(5.8)
where
\[ \tilde{I}_1^\phi(t) = I_1^\phi(t) - \left[ (a(t)\Lambda)^2 - M^2(t) \ln(a(t)\Lambda) \right]/(8\pi^2). \]
(5.12)

If our assumptions about the large-\( k \) behaviour of the integrals are correct, then these are finite relations between renormalized quantities. (To simplify matters, we have neglected a term in (5.10) involving the time derivatives of \( I_2 \) which is finite, but probably an artifact of our regularization procedure.)

For the symmetric state which exists prior to \( t_0 \), the renormalized mass \( \mu(t) \) is defined in a similar manner to \( M(t) \) and satisfies the gap equation
\[ \mu^2(t) = \frac{\lambda R}{2} \tilde{\delta}(t) - \frac{1}{2}a(t)^2 m^2 \left( 1 + \frac{\lambda R}{16\pi^2} \right) + O(\lambda R^2), \]
(5.13)
where \( \tilde{\delta}(t) \) is defined in the same way as \( I_1^\phi(t) \), but using the \( \phi \) propagator and mass \( \mu(t) \).

C. Initial conditions at \( t = t_0 \)

The equation of motion (5.10) needs, of course, the initial values \( \nu_R(t_0) \) and \( \nu_R(t_0) \) and these are obtained from the continuity of \( \langle \phi^2(t) \rangle \) and \( d\langle \phi^2(t) \rangle/dt \) at \( t = t_0 \). This expectation value is given by \( \langle \phi^2(t) \rangle = I_0^\phi + O(\lambda) \) just before \( t_0 \) and by definition it is equal to \( \lambda^{-1}v^2(t) \) just after \( t_0 \). In terms of \( v_R \), the continuity conditions are (in the notation of (4.18))
\[ v_R^2(t_0) = \lambda_R \tilde{\delta}(t_0) + O(\lambda_R^2). \]
(5.14)
As a consequence, we find that \( M(t_0) = \mu(t_0) \) at this order, suggesting the satisfactory possibility that the excitations of the state at \( t_0 \) described by \( \phi \) and by \( \zeta \) are essentially the same.

Our final task is to investigate how the initial conditions (4.22) and (4.23) for the generalized occupation numbers \( N_k^\phi(t) \) and \( N_k^\zeta(t) \) might be renormalized. It turns out that a far-reaching modification of the strategy used to obtain these conditions is needed. Indeed, (4.23) has a potentially disastrous consequence, since it implies that
\[ g_{\alpha\beta}^\phi(t_0, t_0; k) = \frac{1}{2} f_\alpha^* k (t_0) f_\beta k (t_0) \left[ N_k^\phi(t_0) + N_k^\phi(t_0) \right] = \frac{R_0(k)}{2U_0^\phi}. \]
(5.15)
On integrating this relation we discover that
\[ I_1^\phi(t_0) = \int \frac{d^3k}{(2\pi)^3} g_{\alpha\beta}^\phi(t_0, t_0; k) = \frac{1}{2U_0^\phi} \int \frac{d^3k}{(2\pi)^3} R_0(k) = \frac{1}{2} I_1^\phi(t_0). \]
(5.16)
The renormalization programme we have outlined so far clearly requires that the ultraviolet divergences of \( I_1^\phi \) and \( I^\phi \) should be the same, which is incompatible with (5.16).

The problem arises from the fact that we need to match two different perturbation series at \( t_0 \), whereas a consistent renormalization scheme can be implemented only within a single systematic expansion. In view of this, we replace \( g_{\alpha\beta}^\phi(t, t') \) in (4.19) and (4.22) with a function \( \tilde{g}_{\alpha\beta}^\phi(t, t') \) obtained by summing those contributions to the 2-point function for \( \zeta \) which are of order \( \lambda^0 \), where \( v \) is regarded as of order \( \lambda^{1/2} \), as it is in the \( \phi \)-theory. This function is
\[ \tilde{g}_{\alpha\beta}^\phi(t, t') = \frac{\tilde{\sigma}(t)}{v(t)} g_{\alpha\beta}^\phi(t, t') \frac{\tilde{\sigma}(t')}{v(t')} + \frac{\lambda}{2} \frac{1}{v(t)} I_1^\phi(t, t') \frac{1}{v(t)}, \]
(5.17)
where
\[ \tilde{\sigma}(t) = \left[ v^2(t) - \lambda I_1^\phi(t) \right]^{1/2} \]
(5.18)
and \( v(t) \) is understood to be evaluated at lowest order. On evaluating the modified version of (4.19) at \( t' = t_0 \), we find
\[ \tilde{\sigma}^2(t_0) g_{\alpha\beta}^\phi(t_0, t_0; k) = \frac{\lambda}{2} \left[ I_0^\phi(t_0, t_0; k) - I_2^\phi(t_0, t_0; k) \right], \]
(5.19)
and we want to find initial conditions for \( g^\phi \) for which this condition holds. It clearly will hold if \( \tilde{\sigma}(t_0) = 0 \) and \( I_2^\phi(t_0, t_0; k) = I_2^\phi(t_0, t_0; k) \), and this is in fact the only possibility. On integrating (5.19) and using in (5.18) the fact that \( v^2(t_0) = \lambda I_1^\phi(t_0) \), we obtain
\[ \left[ I_0^\phi(t_0) - I_1^\phi(t_0) \right] I_1^\phi(t_0) = \frac{1}{2} \left[ I_0^\phi(t_0)^2 - I_1^\phi(t_0)^2 \right], \]
(5.20)
which implies that \( I_1^\phi(t_0) = I_1^\phi(t_0) \) and hence \( \tilde{\sigma}(t_0) = 0 \). Now, we found above that the \( \phi \) and \( \zeta \) masses \( \mu(t) \) and \( M(t) \) are equal (at lowest order) at \( t = t_0 \), so we can choose the same mode functions \( f_k(t) \) for both propagators near \( t = t_0 \). Evidently, (5.13) is satisfied if we also choose \( N_k^\phi(t_0) = N_k^\phi(t_0) \) and, in fact, the modified versions of (4.18) - (4.21) all hold at \( t = t_0 \) if we choose \( N_k^\phi(t_0) = N_k^\phi(t_0) \). We thus reach the apparently satisfactory conclusion that the propagators \( g_{\alpha\beta}^\phi(t, t') \) and \( g_{\alpha\beta}^\phi(t, t') \) and their first derivatives are the same at \( t_0 \), so that the \( \phi \) and \( \zeta \) excitations are essentially the same. The requirement (4.18) is of course satisfied for \( N_k^\phi(t) \), since it is already satisfied for \( N_k^\phi(t) \). The new boundary conditions do, of course, spoil the exact cancellation between the composite propagator of figure 2(h) and the unwanted \( K_\phi \) discussed in section IV B 1. This propagator can be calculated explicitly using the representation (4.13) for the internal lines and setting the anchored time.
arguments equal to $t_0$ after the required derivatives have been performed. We find that derivatives of $\theta(t-t')$ produce terms proportional to $\delta(0)$ which are again exactly cancelled by $\mathcal{K}_\lambda$, but there are now residual terms. These contribute to the function $\mathcal{B}_{\alpha\beta}$ in (4.31) and do not enter into our explicit calculations.

At this point, then, we have a fully renormalized equation of motion and gap equation and finite initial conditions for both the equation of motion and the propagator $\hat{g}_{\alpha\beta}$. To the first non-trivial order of our approximation scheme, all the ingredients are to hand for the calculation of the expectation values we want, which must necessarily be completed by numerical methods. This apparently satisfactory state of affairs conceals, however, an uncomfortable feature of perturbation theory which must now be exposed. If we define

$$\sigma_R^2(t) = \nu_R^2(t) - \lambda_R \hat{R}_1^2(t) + O(\lambda_R^2),$$

then (5.10) and (5.11) are consistent at this order with the pair of equations

$$\dot{\sigma}_R + M^2(t)\sigma_R = \frac{1}{\nu_R} \delta_R^3 + O(\lambda_R^2),$$

$$M^2(t) = \frac{1}{2} \sigma_R^2(t) + \frac{1}{2} \lambda_R \hat{R}_1^2(t)$$

$$- \frac{1}{2} \hat{m}\nu_2^2(t) \left[ 1 + \frac{1}{\lambda_R} \hat{c}(t) \right] + O(\lambda_R^2).$$

Suppose, for the sake of argument, that $a(t)$ approaches the constant value $a = 1$ at late times. There are two steady-state solutions to (5.22) and (5.23) to which the final state might correspond. The first is $\sigma_R = \sqrt{3} M + O(\lambda_R^2)$ and $M^2 = \hat{m}^2 - \frac{1}{2} \lambda_R \hat{R}_1^2 + O(\lambda_R^2)$. This corresponds as in section 1 to a superposition of states in which $\langle \phi \rangle = \pm \sigma_R$, and is the state that we might hope to see emerging. The second solution is $\sigma_R = 0$ and $M^2 = -\frac{1}{2} \hat{m}^2 + \frac{1}{2} \lambda_R \hat{R}_1^2 + O(\lambda_R^2)$. In terms of the conventional description of spontaneous symmetry breaking in Minkowski spacetime, this of course corresponds to the ill-defined perturbation theory with $\langle \phi \rangle = 0$. Which of these situations will emerge from our dynamical description depends crucially on how we treat our perturbative expansions. Using (5.24) as it stands, the initial conditions we have derived would imply that $\sigma_R = 0$ for all times ($\sigma_R$ being essentially a renormalized version of the $\tilde{\sigma}$ introduced in (5.13)), and hence that $\nu_R^2 = \lambda_R \hat{R}_1^2 = \lambda_R \hat{R}_1^2$ at all times. Although we have only first-order results in hand, this strongly suggests that the $\zeta$-theory would then simply reproduce the $\phi$-theory, but by a more complicated route. This perturbation theory is unsatisfactory when $\nu_R$ becomes of order 1, as it eventually must do. On the other hand, if we use the actual $O(\lambda_R)$ result in (5.10) to define

$$\sigma_R = \nu_R - \frac{\lambda_R \hat{R}_1^2}{2} + O(\lambda_R^2)$$

and truncate the expansion at this order, then $\sigma_R$ and $\dot{\sigma}_R$ receive nonzero initial values at $t_0$ and should evolve qualitatively in the expected manner. This ambiguity is inherent in any perturbative approach to our problem, which necessarily requires us to match the two different expansions about the states indicated schematically in figures 1(c) and 1(d). Intuitively, we would like to interpret the definition (5.24) as yielding $\sigma_R = \langle \phi \rangle = 0$ and (5.22) as approximating the locations of the most probable values of $\phi$ in figure 1(d). In practical terms, the most appropriate strategy seems to be that set out earlier. Namely, we make no reference to $\sigma_R$, but deal exclusively with $\nu_R$, which can be calculated both in the $\phi$-theory and in the $\zeta$-theory. Within the $\zeta$-theory, we adhere to a strict series expansion of (5.10) and (5.11) in powers of $\lambda_R$, treating the initial values obtained from the $\phi$-theory as purely numerical input to these equations, even though the numbers are obtained within the $\phi$-theory from an expansion in powers of $\lambda_R$. In principle, at least, this yields a sequence of approximations which can be pursued systematically to arbitrarily high orders.

VI. DISCUSSION

In this paper, we have proposed a perturbative approximation scheme through which quantum-field-theoretic expectation values might be estimated during the course of a symmetry-breaking phase transition. The scheme has two distinctive features, both of which seem to be essential to this problem. First, in order that low-order calculations adequately reflect the evolving nonequilibrium state, this state is described in terms of its own quasiparticle excitations. Lowest-order propagators describe the propagation of these excitations and incorporate approximations to their mass (or, more generally, their dispersion relation) and damping rate, which can be obtained as the solutions of appropriate self-consistency conditions. Second, because we deal with an exact symmetry $\phi \leftrightarrow -\phi$, which is a symmetry both of the Hamiltonian that governs time evolution and of the initial state, the scalar field $\phi$ can never acquire a nonzero expectation value. In fact, the symmetry can never, properly speaking, be broken at all. In Minkowski spacetime, nevertheless, the effects conventionally associated with a nonzero value of $\langle \phi \rangle$ can be recovered in perturbation theory by dealing instead with $\langle \phi^2 \rangle$ and we have here generalised this description to accommodate the state emerging dynamically from the phase transition. (As noted in section 4, the value of $\langle \phi \rangle$ does not in itself provide a full characterisation of the nonequilibrium state. Nor, of course, does the value of $\langle \phi^2 \rangle$, but this quantity plays an indispensable role in the perturbative description of this state.)

Inherent in any attempt to apply perturbation theory to this problem is the difficulty that $\langle \phi^2 \rangle$ is formally of order $\lambda^{-1}$ in the early, symmetrical state, but of order $\lambda^{-1}$ in the state that emerges from the transition. This necessarily means that two inequivalent perturbation series
must be used, before and after a time that we have called $t_0$. Intuitively, we suppose that before $t_0$, the most probable value of $\phi$ is zero, while after $t_0$ the most probable values are, say, $\pm \sigma/\sqrt{\lambda}$. However, perturbation theory gives no clear indication of the instant at which this bifurcation occurs, so in practice an ad hoc choice of $t_0$, perhaps optimising in some way the matching of the two approximations, would be required. More importantly, perhaps, the above difficulty leads to two specific shortcomings of our analysis which deserve some emphasis. One is that we had to perform in section II A a somewhat singular transformation of the path integral from which expectation values are computed, which led to ambiguities concerning the treatment of total time derivatives and the meaning of derivatives of propagators when their time arguments coincide at $t_0$. We succeeded in resolving these ambiguities sufficiently for the immediate purposes of the explicit calculations described in sections IV and V, but have not obtained any well-founded general resolution. It is possible that a sufficiently careful derivation might avoid these ambiguities, but we have not yet been able to achieve this. The second shortcoming is that discussed in section V C, namely that our perturbative estimate of $v_R(t)$ from the equation of motion (5.10) and the initial conditions (5.14) depend crucially on how the perturbation series is organised. We gave a prescription which should apparently produce the expected type of behaviour, and can be pursued systematically, but it is not unique. We do not think that any significantly better strategy is available within perturbation theory. Possibly, some non-perturbative means of improving the matching between the two expansions might be found, but we have not yet been able to do this.

Quite extensive numerical calculations are needed to extract concrete results from the perturbation theory we have constructed, and we hope to report such results in future publications. In the absence of further approximations beyond the perturbation expansion, the numerical problem involves the simultaneous solution of several coupled nonlinear equations: the equation of motion (5.10); the gap equation (5.11); the equations (2.15) and (2.17) for mode functions and generalized occupation numbers, and two equations we have not given explicitly for the dissipative coefficients $\alpha_k(t)$ and $\gamma_k(t)$, which involve at least 2-loop integrals, as detailed in [21]. If the scale factor $a(t)$ is also to be determined self-consistently, then there are also two independent field equations contained in (5.13). The nature of the solution is rather difficult to forecast, but we wish to speculate briefly on some particular features, partly in the light of investigations reported by Boyanovsky et al. for the somewhat simpler case of the large-$N$ model. These authors studied a phase transition roughly of the kind envisaged here in II A using a fixed de Sitter background and in [11] by solving the Friedmann equation simultaneously with the field theory problem. The field-theory equations they solve have, to some extent, a similar structure to (5.22) and (5.23). There are trivial differences arising from the fact that, in order to obtain the action (5.23) (which is equivalent to a Minkowski-space theory with a time-dependent mass simply proportional to $a(t)$) we have used $\phi(x) = a(t)\Phi(x)$, where $\Phi(x)$ is a genuine scalar field under general coordinate transformations, taken this field to be conformally (rather than minimally) coupled to gravity and used a conformal time coordinate $t$. There are, however, more fundamental differences. Most obviously, the one-loop equations are exact for the large-$N$ model, whereas those we have given explicitly are just the lowest non-trivial order of our perturbative approximation scheme. In the large-$N$ model, the quantity denoted in (11.3) by $\eta(t)$, which is analogous to our $\sigma_\mu(t)$ is the expectation value of one of the $N$ fields. The evolution of this field is entirely classical, while the quantum modes contributing to the self-energy integral $I_1$ arise purely from the $N - 1$ transverse fields. As a consequence, the mode equation corresponding to (5.2) is identical for $k = 0$ to the equation of motion for $\eta(t)$, while here it coincides with (5.22) only when $\sigma_\mu(t)$ is small. In the absence of explicit symmetry breaking, one has $\eta(t) \equiv 0$ and the calculation is analogous to using our $\phi$-theory for all times. The large-$N$ limit produces, of course, a constrained Gaussian model, in which there is no scattering, so the dissipative effects treated in section II A are entirely absent.

Suppose, then, that our $\phi$-theory were used for all times. The relevant gap equation is then (5.13) and the mode functions contained in $I_1^\phi(t)$ obey (5.1) with $M(t)$ replaced by $\mu(t)$. Initially, $\mu^2$ is positive and $I_1^\phi(t)$ should vary only slowly with time. Since $a(t)$ increases with time, $\mu^2$ becomes negative, at which point the mode functions $f_k(t)$ for which $k^2 + \mu^2 < 0$ begin to grow, and so therefore does $I_1^\phi(t)$. If this growth is fast enough, then $\mu^2$ might again become positive and, perhaps, a final state might emerge in which $\mu^2$ remains positive. In the large-$N$ model, this is more or less what happens, except that the quantity analogous to $\mu^2$ approaches a vanishingly small value, corresponding to a universe populated entirely by Goldstone bosons. In our $N = 1$ model, a late-time state with $\mu^2 \geq 0$ would make little sense (though this does not rule it out as a solution to our 1-loop equations!). In such a state, the thermal contribution to the effective mass is still large enough for the symmetry to remain unbroken (in the conventional sense): in the (not strictly appropriate) language of equilibrium field theory, the system would have reheated to above its critical temperature. In order to obtain a sensible final state, it is essential to move to the $\zeta$ description at some point during the growth of the unstable modes. From (5.22) and (5.23), it is clear that when $\sigma_R$ is positive, it grows more rapidly than any of the unstable modes (and the same should be true of $v_R$ in our preferred approximation scheme), so that $M^2$ is more likely than $\mu^2$ to become positive at late times, as it should.

Finally, we comment briefly on two significant issues raised by the growth of unstable modes. First, these
modes give rise to large propagators which threaten to make perturbation theory useless. However, as indicated above, the growth does not continue indefinitely, and the period of growth will become shorter if the coupling is made stronger. In fact, the results presented in [11] show a quantity analogous to \( \Lambda_{t}(t) \) growing to a value of 1 in the large-N model and to about 1/3 in a Hartree approximation which is more similar to our 1-loop approximation. While these numbers are not small, they seem to offer hope that perturbation theory may not become grossly unreliable. In our \( \zeta \)-theory, moreover, vertices involving \( \zeta^{n} \) carry factors proportional to \( v^{2-n} \) (see [15]). Since each factor of \( \zeta \) in \( L_{\text{cut}}^{*} \) corresponds roughly to one mode function \( f_{k}(t) \) inside a Feynman diagram, the growth of these mode functions will be largely offset by the growth of \( v_{R} \). The second issue concerns the nature of the state at late times. In [1], the small-\( k \) modes of the large-\( \phi \) model are interpreted as constituting an effective classical field \( \phi_{\text{eff}} \sim \sqrt{\langle \phi^{2} \rangle_{R}} \). Here, this role is explicitly assumed by \( v_{R} \), but the remaining quantum field \( \zeta \) still contains the growing modes. Once these modes have acquired large amplitudes, the growth does not continue indefinitely, and the perturbative description of the thermalized state one might expect finally to emerge. One would expect to describe this state in terms of mode functions with amplitudes of order \( (k^{2} + M^{2})^{-1/4} \), and the presence of large amplitude modes seems to suggest coherent behaviour that ought to have been absorbed in the expectation value \( v_{R} \). However, the decomposition of our propagators \( g_{\alpha \beta} \) into mode functions and generalized occupation numbers \( N_{k}(t) \) is not unique. At late times, it will be possible to reexpress these propagators in terms of small-amplitude mode functions and large occupation numbers for the previously unstable modes. We then see that the dissipative mechanisms of section (5.1) should cause the expected thermalization (though this does not depend on how we choose to write the propagators down!). Indeed, dissipation will be present during the period of instability as well. Its effects are difficult to forecast, but some inhibition of the growth of unstable fluctuations seems likely.

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FIG. 1. Schematic representation of a time dependent effective potential and of the probability density for $\phi$.

FIG. 2. Diagrammatic representations of the propagators. Solid lines (a) represent $g_{\alpha\beta}^\zeta$, dashed lines (b) represent $g_{ab}^\phi$ and dotted lines are $\phi$ propagators whose time arguments are both equal to $t_0$.

FIG. 3. Vertices representing interactions in the $\phi$ and $\zeta$ theories.

FIG. 4. Low-order diagrams contributing to $\langle \zeta(t) \rangle$. Anchored contributions (a) contain composite propagators with vertices fixed at $t_0$ while the free contributions (b) contain none of these vertices.

FIG. 5. Leading-order contributions to $\langle \phi^2(t) \phi^2(t') \rangle$ when (a) both times are earlier than $t_0$, (b) one time is later than $t_0$ and (c) both times are later than $t_0$.

FIG. 6. Low-order contributions to the connected two-point function $\langle \zeta_\alpha(t) \zeta_\beta(t') \rangle$. 
Figure 1
Figure 2
Figure 3
Lawrie perturbative nonequilibrium dynamics... Figure 3
\[ \langle \zeta(t) \rangle_{\text{IPI}}^{(\text{anchored})} = \quad + \quad + \quad + \quad + \cdots \]

(i) (ii) (iii)

(a)

\[ \langle \zeta(t) \rangle_{\text{IPI}}^{(\text{free})} = \quad + \quad + \quad + \quad + \quad + \cdots \]

(i) (ii) (iii) (iv) (v) (vi)

(b)

Figure 4
Figure 5
\[ \langle \zeta_{\alpha}(t) \zeta_{\beta}(t') \rangle = \] 

(a) + (b) + (c) + (d) + (e) + (f) + (g) + (h) + (i) + (j) + (k) + (l) + (m) + (n) + (p) + (q) + (r) + (s) + \ldots

**Figure 6**