REFINED ENUMERATION OF VERTICES AMONG ALL ROOTED ORDERED $d$-TREES

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Abstract. In this paper we enumerate the cardinalities for the set of all vertices of outdegree $\geq k$ at level $\geq \ell$ among all rooted ordered $d$-trees with $n$ edges. Our results unite and generalize several previous works in the literature.

1. Introduction

For a positive integer $d$, the $n$th $d$-Fuss-Catalan number is given by

$$\text{Cat}^{(d)}_n = \frac{1}{dn+1} \binom{(d+1)n}{n}, \quad \text{for } n \geq 0.$$  

They are generalization of well-known Catalan numbers. Like Catalan numbers, there are several classes which are enumerated by Fuss-Catalan numbers. Most well-known class is Fuss-Catalan paths. A $d$-Fuss-Catalan path of length $(d+1)n$ is a lattice path from $(0,0)$ to $((d+1)n,0)$ using up steps $(1,d)$ and down steps $(1,-1)$ such that it stays weakly above the $x$-axis. Denote by $\mathcal{FC}^{(d)}_n$ the set of $d$-Fuss-Catalan paths of length $(d+1)n$. Another example is dissections of a $(dn+2)$-gon into $(d+2)$-gons by diagonals. There are three more classes which are enumerated by $d$-Fuss-Catalan numbers.

Rooted ordered $d$-trees. A rooted tree can be considered as a process of successively gluing an edge (1-simplex) to a vertex (0-simplex) from the root in a half-plane, where the root is fixed in the line (1-dimensional hyperplane) as the boundary of the given half-plane. In same way, we can define a rooted $d$-tree in $(d+1)$-dimensional lower Euclidean half-space $\mathbb{R}^{d+1}$ as follows: The root $r$ is a $(d-1)$-simplex fixed in the boundary of $\mathbb{R}^{d+1}$. From the root $(d-1)$-simplex $r$, we glue $d$-simplices (as edges) successively to one of previous $(d-1)$-simplices (as vertices) in $\mathbb{R}^{d+1}$. (See [BP69 d-dimensional trees].) By definition, if $d = 1$, a rooted $d$-tree is a rooted tree.

In a rooted tree, we can consider a linear order among all edges having one common vertex by their positions and such a tree is called a rooted ordered tree. Similarly, in higher dimensional cases, we can also give a linear order among $d$-simplices having one common $(d-1)$-simplices naturally by their positions and such a tree is also called a rooted ordered $d$-tree. Jani, Rieper and Zeleke [JRZ02] enumerate ordered $K$-trees, which is obtained in a similar way using $d$-simplices with $d \in K$.

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Table 1. The number of vertices of outdegree $k$ at level $\ell$ among all rooted 3-tuplet trees in $T_3^{(3)}$.

| $\ell \setminus k$ | 0 | 1 | 2 | 3 | $\sum$ |
|---------------------|---|---|---|---|--------|
| 0                   | 0 | 15| 6 | 1 | 22     |
| 1                   | 66| 21| 3 | 0 | 90     |
| 2                   | 72| 9 | 0 | 0 | 81     |
| 3                   | 27| 0 | 0 | 0 | 27     |
| $\sum$              | 165| 45| 9 | 1 | 220    |

Rooted $d$-ary cacti. A cactus is a connected simple graph in which each edge is contained in exactly one elementary cycle. These graphs are also known as “Husimi trees”. They are introduced by Harary and Uhlenbeck [HU53]. If each elementary cycle has exactly $d$ edges, a cactus is called a $d$-ary cactus. Bóna et al. [BBLL00] provide enumerations of various classes of $d$-ary cacti.

Rooted $d$-tuplet trees. Instead of $d$-simplices used in rooted ordered $d$-trees, we may use $(d+1)$-gons. A root is a vertex fixed in the bounding hyperplane of a half-plane. One can glue $(d+1)$-gons to a vertex from the root. A tree obtained in this way is called a rooted $d$-tuplet tree and the $(d+1)$-gons are called $d$-tuplets. As there is a linear order on the vertices in a tuplet, one can show that there is a one-to-one correspondence between rooted ordered $d$-trees with $n$ edges and rooted $d$-tuplet trees with $n$ tuplets. Thus rooted ordered $d$-trees and rooted $d$-tuplet trees are essentially the same. Note that the underlying graph of a $d$-tuplet tree is a $(d+1)$-ary cactus.

Let $T_n^{(d)}$ be the set of rooted $d$-tuplet trees with $n$ tuplets. It is easy to see that the cardinality of $T_n^{(d)}$ is the $n$th $d$-Fuss-Catalan number $\text{Cat}_n^{(d)}$. For example, there are 22 rooted 3-tuplet trees with 3 tuplets, see Figure 1. Clearly the number of vertices among rooted $d$-tuplet tree with $n$ tuplets in $T_n^{(d)}$ is

$$
(dn + 1) \text{Cat}_n^{(d)} = \binom{(d + 1)n}{n}.
$$

In a rooted $d$-tuplet tree, the degree of a vertex is the number of tuplets it connects. We can have the notion of the outdegree of a vertex $v$, which is the number of tuplets starting at $v$ and pointing away from the root. The level of a vertex $v$ in a rooted $d$-tuplet tree is the distance (number of tuplets) from the root to $v$. Table 1 shows the number of vertices of outdegree $k$ at level $\ell$ among all rooted 3-tuplet trees in $T_3^{(3)}$. For example, there are 9 vertices of outdegree 1 at level 2 in $T_3^{(3)}$, see Figure 1.

In a rooted $d$-tuplet tree, there exists the unique vertex $u$ in each tuplet such that its level is less than levels of the other vertices $v_1, \ldots, v_d$. Here, $u$ is called the parent of $v_i$’s and each $v_i$ is called a child of $u$. For each vertex $v$ (except the root), there exists the unique tuplet containing $v$ toward the root, called the tuplet of $v$. Vertices with the same parent are called siblings. For two siblings $v$ and $w$, if $v$ is on the left of $w$, $v$ is called an elder sibling of $w$ and $w$ is called a younger sibling of $v$.

Recently Eu, Seo, and Shin [ESS17] gave a formula for the number of vertices among all trees in the set of rooted ordered trees under some conditions.
Figure 1. All rooted 3-tuplet trees with 3 tuplets in $T_3^{(3)}$

**Theorem 1** (Eu, Seo, and Shin, 2017). Given $n \geq 1$, for any nonnegative integers $k$ and $\ell$, the number of all vertices of outdegree $\geq k$ at level $\geq \ell$ among all rooted ordered trees with $n$ edges is

$$\binom{2n - k}{n + \ell}.$$  \hfill (2)

We give a generalization of the formula (2) for $T_n^{(d)}$ by generalizing their bijection.

**Theorem 2** (Main Result). Given $n \geq 1$, for any nonnegative integers $k$ and $\ell$, the number of all vertices of outdegree $\geq k$ at level $\geq \ell$ among all rooted $d$-tuplet trees with $n$ tuplets is

$$d^\ell \binom{(d + 1)n - k}{dn + \ell}.$$ \hfill (3)

We also find a refinement of the formula (3).
Theorem 3. Given $n \geq 1$, for any two nonnegative integers $i, j$, one nonnegative integer $k$ which is a multiple of $d$, and one positive integer $\ell$, the number of all vertices among all rooted $d$-tuplet trees with $n$ tuplets such that

- having at least $i$ elder siblings,
- having at least $j$ younger siblings,
- having at least $k$ children,
- at level $\geq \ell$

is

$$d^\ell \left(1 - \frac{\beta}{d} \frac{dn + \ell}{(d + 1)n - \alpha} \right) \left(\frac{(d + 1)n - k}{dn + \ell} \right),$$

(4)

where $\alpha$ and $\beta$ are nonnegative integers satisfying $i + j + k = \alpha d + \beta$ and $0 \leq \beta < d$.

The rest of the paper is organized as follows. In Section 2, we show the Theorem 2 bijectively. In Section 3, we give a combinatorial proof of the Theorem 3. In Section 4, we present corollaries induced from Theorem 2 and 3.

2. A bijective proof of Theorem 2

Henceforth, a tree is assumed to be a rooted $d$-tuplet tree. Let $V$ be the set of pairs $(T, v)$ such that $v$ is a vertex of outdegree $\geq k$ at level $\geq \ell$ in $T \in \mathcal{T}_n^{(d)}$. Let $P$ be the set of sequences in $\{0, \ldots, d - 1\}$ of length $\ell$. Let $L$ be the set of lattice paths of length $((d + 1)n - k)$ from $(k, dk)$ to $((d + 1)n, -(d + 1)\ell)$, consisting of $(n - k - \ell)$ up-steps along the vector $(1, d)$ and $(dn + \ell)$ down-steps along the vector $(1, -1)$. In order to show Theorem 2 it is enough to construct a bijection $\Phi$ between $V$ and $P \times L$, due to

$$\#P = d^\ell, \quad \#L = \binom{(d + 1)n - k}{n - k - \ell}, \frac{dn + \ell}{(d + 1)n - k}.$$ 

Three bijections $\varphi$, $\overline{\varphi}$, and $\psi$. Let a reverse $d$-Fuss-Catalan path of length $(d + 1)n$ be a lattice path from $(0, 0)$ to $((d + 1)n, 0)$ using up steps $(1, d)$ and down steps $(1, -1)$ such that it stays weakly below the $x$-axis. Denote by $\overline{\mathcal{FC}}_n^{(d)}$ the set of reverse $d$-Fuss-Catalan paths of length $(d + 1)n$.

Before constructing the bijection $\Phi$, we introduce three bijections

$$\varphi : \mathcal{T}_n^{(d)} \rightarrow \overline{\mathcal{FC}}_n^{(d)}, \quad \overline{\varphi} : \mathcal{T}_n^{(d)} \rightarrow \overline{\mathcal{FC}}_n^{(d)}, \quad \psi : \mathcal{T}_n^{(d)} \rightarrow \mathcal{FC}_n^{(d)}.$$ 

The bijection $\varphi$ corresponds a tree to a lattice path weakly above the $x$-axis by recording the steps when the tree is traversed in preorder: whenever we go down a side of a tuplet, record an up-step along the vector $(1, d)$ and whenever we go right or up a side of a tuplet, record a down-step along the vector $(1, -1)$. Similarly, the bijection $\overline{\varphi}$ corresponds a tree to a lattice path weakly below the $x$-axis by recording the steps when the tree is traversed in preorder: whenever we go down or right a side, record a down-step along the vector $(1, -1)$ and whenever we go up a side, record a up-step along the vector $(1, d)$. An example of two bijections $\varphi$ and $\overline{\varphi}$ is shown in Figure 2.

The bijection $\psi$ corresponds a tree to a lattice path weakly above the $x$-axis by recording the steps when the tree is traversed in preorder: whenever we meet a vertex of outdegree $m$, except the last leaf, record $m$ up-steps and 1 down-step. An example of the bijection $\psi$ is shown in Figure 3.
Step 1. Given \((T,v) \in \mathcal{V}\), let \(D_v\) be the subtree consisting of \(v\) and all its descendants in \(T\), say the \textit{descendant subtree} of \(v\). Letting \(\ell'(\geq \ell)\) be the level of \(v\), consider the path from \(v\) to the root \(r\) of \(T\):
\[
v(=v_0) \to v_1 \to \cdots \to v_\ell \to \cdots \to v_{\ell'-1} \to r (=v_{\ell'}).\]
Record the number \(p_i\) of elder siblings of \(v_i\) in the tuplet of \(v_i\) for all \(0 \leq i \leq \ell - 1\). For all \(0 \leq i \leq \ell - 1\), if \(w_i\) is the youngest sibling of \(v_i\) in the tuplet of \(v_i\), we exchange two subtrees \(D_{v_i}\) and \(D_{w_i}\) and we obtain the tree \(T'\).

Step 2. For all \(1 \leq i \leq \ell - 1\) and \(i = \ell'\), let \(R_i\) be the subtree consisting \(v_i\) and all its descendants on the right of the tuplet of \(v_{i-1}\) in \(T'\). We obtain the tree \(L\) by cutting the \(\ell + 1\) subtrees \(D_v, R_1, \ldots, R_{\ell-1}, R_{\ell'}\) from the tree \(T'\), see Figure 4.

A construction of the bijection \(\Phi\). We will construct the bijection \(\Phi\) between \(\mathcal{V}\) to \(\mathcal{P} \times \mathcal{L}\). Given \((T,v) \in \mathcal{V}\), let \(k'(\geq k)\) be the outdegree of \(v\) in \(T\) and let \(\ell'(\geq \ell)\) be the level of \(v\) in \(T\).

We separate two cases:

Case 1. If \(v\) is not the root of \(T\), i.e., \(\ell' > 0\). We obtain the sequence \(p = (p_0, \ldots, p_{\ell-1}) \in \mathcal{P}\) in Step 1 and \((\ell + 2)\) trees \(D_v, R_1, R_2, \ldots, R_{\ell-1}, R_{\ell'}, L\) after Step 2 as Figure 4.

Let \(\rho\) be the mapping on the set of lattice paths defined by
\[
\rho(s_1s_2\cdots s_n) = s_2\cdots s_ns_1,
\]
where each \(s_i\) is a step. Note that \(\rho^m\) means to apply \(\rho\) recursively \(m\) times.

Clearly, the outdegree of the root of \(D_v\) is \(k'\). In the tree \(L\), there are no younger siblings of \(v\) in the tuplet of \(v\) and the outdegree of vertex \(v\) is 0. Thus the lattice path \(\rho^{a+\ell}(\varphi(L))\) ends with one down-step and \(\ell\) consecutive up-steps, where \(a\) is the number of vertices of \(L\) which precede \(v\) in preorder.

We define a lattice path \(P\) from \((0,0)\) to \(((d+1)n+(\ell+1), -(\ell+1))\) by
\[
P = \psi(D_v) \searrow \varphi(R_1) \searrow \varphi(R_2) \searrow \cdots \searrow \varphi(R_{\ell-1}) \searrow \varphi(R_{\ell'}) \searrow \rho^{a+\ell}(\varphi(L))\]
where \( \downarrow \) means a down-step.

**Case II.** If \( v \) is the root of \( T \), i.e., \( \ell' = 0 \). We define a sequence \( p = () \in P \) and a lattice path 
\[ P = \psi(T) \downarrow . \]

In all cases, the lattice path \( P \) always starts with at least \( k \) (precisely \( k' \)) consecutive up-steps and ends with one down-step and \( \ell \) consecutive up-steps as red segments in Figure 5.

By removing the first \( k \) steps and the last \((\ell + 1)\) steps from \( P \), we obtain the lattice path \( \hat{P} \) of length \((d + 1)n - k\) from \((k,dk)\) to \(((d + 1)n, -(d + 1)\ell)\), consisting of \((n-k-\ell)\) up-steps along the vector \((1,d)\) and \((dn+\ell)\) down-steps along the vector \((1,-1)\), so \( \hat{P} \) belongs to \( L \).

Hence the map \( \Phi : V \to P \times L \) is defined by 
\[ \Phi(T,v) = (p, \hat{P}). \]

**A description of the bijection \( \Phi^{-1} \).** In the Case I of the construction of the bijection \( \Phi \), given a lattice path \( P \) from \((0,0)\) to \(((d + 1)n + (\ell + 1), -(\ell + 1))\), we decompose \( P \) into \((\ell + 2)\)
paths \(P_D, P_1, \ldots, P_{\ell-1}, P_{\ell'}, P_L\) by removing the leftmost down-steps from height \(-i\) to height \(-(i+1)\) for \(0 \leq i \leq \ell\). Some of those paths may be empty.

Clearly all the paths \(P_D, P_1, \ldots, P_{\ell-1}, P_{\ell'}\) are \(d\)-Fuss-Catalan path. By moving all the steps after the leftmost highest vertex in the lattice path \(P_L\) to the beginning, we obtain a reverse \(d\)-Fuss-Catalan path \(\overline{P}_L\) from \(P_L\). Since \(\phi, \overline{\phi}, \text{and } \psi\) are bijections, we can restore trees \(D_v, R_1, \ldots, R_{\ell-1}, R_{\ell'}, L\) from \(P_D, P_1, \ldots, P_{\ell-1}, P_{\ell'}, \overline{P}_L\).

Therefore, \(\Phi\) is a bijection between \(\mathcal{V}\) and \(\mathcal{P} \times \mathcal{L}\) since all the remaining processes are also reversible.

3. Proof of Theorem 3

For any three nonnegative integers \(i, j, k\) and one positive integer \(\ell\), denote by \(\mathcal{V}_n^{(d)}(i, j, k; \ell)\) the set of pairs \((T, v)\) whose tree \(T\) in \(\mathcal{T}_n^{(d)}\) and vertex \(v\) in \(T\) such that

- \(v\) has at least \(i\) elder siblings in \(T\),
- \(v\) has at least \(j\) younger siblings in \(T\),
- \(v\) has at least \(k\) children in \(T\),
- \(v\) is at level \(\geq \ell\) in \(T\).

We show the following lemma, which is a special case of Theorem 3 that is, \(i\) and \(j\) are multiples of \(d\).

Lemma 4. Given \(n \geq 1\), for any three nonnegative integers \(i, j, k\), all of which are multiples of \(d\), and one positive integer \(\ell\), the cardinality of \(\mathcal{V}_n^{(d)}(i, j, k; \ell)\) is

\[
d^{\ell} \left( \frac{(d+1)n - \alpha}{dn + \ell} \right),
\]

where \(\alpha\) is the nonnegative integer satisfying \(i + j + k = ad\).
Proof. That a vertex \( v \) has at least \( i \) elder (or younger resp.) siblings means that there exists at least \( i/d \) (or \( j/d \) resp.) \( d \)-tuplets directly connected from the parent of \( v \) on its left (or right resp.).

A pair \((T, v)\) in \( V_n^{(d)}(i, j, k, \ell)\) corresponds to a pair \((T', v)\) in \( V_n^{(d)}(0, 0, i + j + k, \ell)\) under a cut-and-paste bijection \( \gamma_{i,j} : (T, v) \mapsto (T', v) \) which cuts the leftmost \( i/d \) tuplets connected from the parent \( p \) of \( v \) and pastes them at \( v \) on the left and does again the rightmost \( j/d \) tuplets connected from the parent \( p \) of \( v \) on the right, as Figure 6.

Since that \( v \) has at least \( i + j + k \) children means that the outdegree of \( v \) greater than or equal to \( \alpha = i + j + k \), this case corresponds to \( k \leftarrow \alpha \) of Theorem 2.

In Theorem 3, what to find is the cardinality of \( V_n^{(d)}(i, j, k; \ell) \) for any two nonnegative integers \( i, j \), one nonnegative integer \( k \) which is a multiple of \( d \), and one positive integer \( \ell \).

Given \((T, v)\) in \( V_n^{(d)}(i, j, k; \ell)\), let \( w \) be the \( j \)th younger sibling of \( v \). By exchanging two subtrees \( D_v \) and \( D_w \), we obtain \((T', v)\) in \( V_n^{(d)}(i + j, 0, k; \ell)\) from \((T, v)\) in \( V_n^{(d)}(i, j, k; \ell)\). Let \( \alpha \) and \( \beta \) be the quotient and the remainder when \( i + j + k \) is divided by \( d \), that is,

\[
i + j + k = \alpha d + \beta.
\]

By applying the cut-and-paste bijection \( \gamma_{i+j-\beta,0} \), we obtain \((T'', v)\) in \( V_n^{(d)}(\beta, 0, \alpha d; \ell)\) from \((T', v)\) in \( V_n^{(d)}(i + j, 0, k; \ell)\). One can show that the values

\[
\#V_n^{(d)}(i, 0, \alpha d; \ell) - \#V_n^{(d)}(i + 1, 0, \alpha d; \ell)
\]

are the same for all \( 0 \leq i \leq d - 1 \) under exchanging two descendant subtrees of two sibling in the same tuplet. By telescoping, we get the formula

\[
\#V_n^{(d)}(0, 0, \alpha d; \ell) - \#V_n^{(d)}(\beta, 0, \alpha d; \ell)
\]

\[= \frac{\beta}{d} \left[ \#V_n^{(d)}(0, 0, \alpha d; \ell) - \#V_n^{(d)}(d, 0, \alpha d; \ell) \right].\]
By Lemma 4, we have
\[
\#V_n^{(d)}(0,0,\alpha d; \ell) = d^\ell \binom{(d+1)n - \alpha}{dn + \ell}, \\
\#V_n^{(d)}(d,0,\alpha d; \ell) = d^\ell \binom{(d+1)n - \alpha - 1}{dn + \ell}.
\]
Thus we get the cardinality of $V_n^{(d)}(\beta, 0, \alpha d; \ell)$ and the desired formula (4).

4. FURTHER RESULTS

From Theorem 2, we can obtain the following result.

**Corollary 5.** Given $n \geq 1$, for any two nonnegative integers $k$ and $\ell$, the number of all vertices of outdegree $k$ at level $\ell$ among $d$-trees in $T_n^{(d)}$ is
\[
d^\ell dk + (d + 1)\ell \binom{(d+1)n - k}{dn + \ell}.
\] (5)

**Proof.** By the sieve method with (3), we obtain the formula (5) from
\[
d^\ell \binom{(d+1)n - k}{dn + \ell} - d^{\ell+1} \binom{(d+1)n - k - 1}{dn + \ell + 1} + d^{\ell+1} \binom{(d+1)n - k - 1}{dn + \ell + 1}.
\]
\[\square\]

The next result follows from Theorem 3 for $d = 1$.

**Corollary 6.** Given $n \geq 1$, for any three nonnegative integers $i$, $j$, $k$, and one positive integer $\ell$, the number of all vertices among trees in $T_n$ such that
- having at least $i$ elder siblings,
- having at least $j$ younger siblings,
- having at least $k$ children,
- at level $\geq \ell$

is
\[
\binom{2n - i - j - k}{n + \ell}.
\]

ACKNOWLEDGEMENTS

For the third author, this work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (No. 2017R1C1B2008269).

REFERENCES

[BBLL00] Miklós Bóna, Michel Bousquet, Gilbert Labelle, and Pierre Leroux. Enumeration of $m$-ary cacti. *Adv. in Appl. Math.*, 24(1):22–56, 2000.

[BP69] L. W. Beineke and R. E. Pippert. The number of labeled $k$-dimensional trees. *J. Combinatorial Theory*, 6:200–205, 1969.

[ESS17] S.-P. Eu, S. Seo, and H. Shin. Enumerations of vertices among all rooted ordered trees with levels and degrees. *ArXiv e-prints*, May 2017.
Frank Harary and George E. Uhlenbeck. On the number of Husimi trees. I. *Proc. Nat. Acad. Sci. U. S. A.*, 39:315–322, 1953.

Mahendra Jani, Robert G. Rieper, and Melkamu Zeleke. Enumeration of K-trees and applications. *Ann. Comb.*, 6(3-4):375–382, 2002.

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