SIGMA-CONTINUITY WITH CLOSED WITNESSES

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Abstract. We use variants of the $G_0$ dichotomy to establish a refinement of Solecki’s basis theorem for the family of Baire-class one functions which are not $\sigma$-continuous with closed witnesses.

INTRODUCTION

A subset of a topological space is $F_\sigma$ if it is a union of countably-many closed sets, Borel if it is in the $\sigma$-algebra generated by the closed sets, and analytic if it is a continuous image of a closed subset of $\mathbb{N}^\mathbb{N}$.

A function between topological spaces is $\sigma$-continuous with closed witnesses if its domain is a union of countably-many closed sets on which it is continuous, Baire class one if preimages of open sets are $F_\sigma$, strongly $\sigma$-closed-to-one if its domain is a union of countably-many analytic sets intersecting the preimage of each singleton in a closed set, $F_\sigma$-to-one if the preimage of each singleton is $F_\sigma$, and Borel if preimages of open sets are Borel.

A topological embedding of a topological space $X$ into a topological space $Y$ is a function $\pi: X \to Y$ which is a homeomorphism onto its image, where the latter is endowed with the subspace topology. A topological embedding of a set $A \subseteq X$ into a set $B \subseteq Y$ is a topological embedding $\pi$ of $X$ into $Y$ such that $A = \pi^{-1}(B)$. A topological embedding of a function $f: X \to Y$ into a function $f': X' \to Y'$ is a pair $(\pi_X, \pi_Y)$, consisting of topological embeddings $\pi_X$ of $X$ into $X'$ and $\pi_Y$ of $f(X)$ into $f'(X')$, with $f' \circ \pi_X = \pi_Y \circ f$.

Some time ago, Jayne-Rogers showed that a function between Polish spaces is $\sigma$-continuous with closed witnesses if and only if preimages of closed sets are $F_\sigma$ (see [JR82, Theorem 1]). Solecki later refined this result by providing a two-element basis, under topological embeddability, for the family of Baire-class one functions which do not have this property (see [Sol98, Theorem 3.1]). Here we use variants of the $G_0$ dichotomy (see [KST99]) to establish a pair of dichotomies which together refine Solecki’s theorem.
In §1, we provide a simple characterization of Baire-class one functions that is used throughout the remainder of the paper. As a first application, we use the Lecomte-Zeleny $\Delta^0_0$-measurable analog of the $G_0$ dichotomy theorem (see [LZ14, Corollary 4.5]) to establish that the property of being Baire class one is determined by behaviour on countable sets.

In §2, we use the Hurewicz dichotomy theorem to provide a one-element basis, under topological embeddability, for the family of Baire-class one functions which are not $F_\sigma$-to-one. To be precise, let $\mathbb{N}^{\leq N}$ denote the set $\mathbb{N}^{\leq N}$ equipped with the smallest topology making the sets $N_s^* = \{ t \in \mathbb{N}^{\leq N} \mid s \subseteq t \}$ clopen for all $s \in \mathbb{N}^{< N}$, and fix a function $f_0: \mathbb{N}^{< N} \to \mathbb{R}$ such that $f_0 \upharpoonright \mathbb{N}^N$ has constant value zero and $f_0 \upharpoonright \mathbb{N}^{< N}$ is an injection into $\{1/n \mid n \in \mathbb{N}\}$.

**Theorem 1.** Suppose that $X$ and $Y$ are Polish spaces and $f: X \to Y$ is a Baire-class one function. Then exactly one of the following holds:

1. The function $f$ is $F_\sigma$-to-one.
2. There is a topological embedding of $f_0$ into $f$.

In §3, we use the sequential $\aleph_0$-dimensional analog of the $G_0$ dichotomy theorem (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 21]) to provide a one-element basis, under topological embeddability, for the family of $F_\sigma$-to-one Baire-class one functions which are not $\sigma$-continuous with closed witnesses. To be precise, let $\mathbb{N}^{\leq N}_{**}$ denote the set $\mathbb{N}^{\leq N}$ equipped with the smallest topology making the sets $N_s^* = \{ t \in \mathbb{N}^{\leq N} \mid s \subseteq t \}$ and $\{s\}$ clopen for all $s \in \mathbb{N}^{< N}$, and define $f_1: \mathbb{N}^{< N} \to \mathbb{N}^{\leq N}_{**}$ by $f_1(s) = s$.

**Theorem 2.** Suppose that $X$ and $Y$ are Polish spaces and $f: X \to Y$ is an $F_\sigma$-to-one Baire-class one function. Then exactly one of the following holds:

1. The function $f$ is $\sigma$-continuous with closed witnesses.
2. There is a topological embedding of $f_1$ into $f$.

Theorem 2 trivially yields the following.

**Theorem 3 (Jayne-Rogers).** Suppose that $X$ and $Y$ are Polish spaces, and $f: X \to Y$ is a function with the property that $f^{-1}(C)$ is $F_\sigma$, for all closed subsets $C$ of $Y$. Then $f$ is $\sigma$-continuous with closed witnesses.

And Theorems 1 and 2 trivially yield the following.

**Theorem 4 (Solecki).** Suppose that $X$ and $Y$ are Polish spaces and $f$ is a Baire-class one function. Then exactly one of the following holds:
The function \( f \) is \( \sigma \)-continuous with closed witnesses.

There is a topological embedding of \( f_0 \) or \( f_1 \) into \( f \).

We close the paper with an appendix in which we prove several auxiliary facts needed for the proof of the main results above. Along the way, we use Lecomte’s \( \aleph_0 \)-dimensional analog of the \( \mathcal{G}_0 \) dichotomy theorem (see [Lec09, Theorem 1.6] or [Mil11, Theorem 18]) to give a new proof of a special case of Hurewicz’s dichotomy theorem (see, for example, [Kec95, Theorem 21.18]), yielding the existence of a one-element basis, under topological embeddability, for the family of Borel sets which are not \( F_\sigma \). To be precise, we show that if \( X \) is a Polish space and \( B \subseteq X \) is Borel, then either \( B \) is \( F_\sigma \), or there is a topological embedding \( \pi : \mathbb{N}^* \to X \) of \( \mathbb{N}^* \) into \( B \). We note that the same argument, using the parametrized \( \aleph_0 \)-dimensional analog of the \( \mathcal{G}_0 \) dichotomy theorem (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 31]) in lieu of its non-parametrized counterpart, yields a slight weakening of Saint Raymond’s parametrized analog of Hurewicz’s result (see, for example, [Kec95, Theorem 35.45]). As a corollary, we show that \( F_\sigma \)-to-one Borel functions between Polish spaces are strongly \( \sigma \)-closed-to-one.

1. Baire-class one functions

Throughout the rest of the paper, we will rely on the following characterization of Baire-class one functions.

**Proposition 1.1.** Suppose that \( X \) is a topological space, \( Y \) is a second countable metric space, and \( f : X \to Y \) is a function. Then the following are equivalent:

1. The function \( f \) is Baire class one.
2. For all \( \epsilon > 0 \), there is a cover of \( X \) by countably-many closed subsets whose \( f \)-images have \( d_Y \)-diameter strictly less than \( \epsilon \).

**Proof.** To see (1) \( \implies \) (2), it is sufficient to show that for all real numbers \( \epsilon > 0 \) and open sets \( V \subseteq Y \) of \( d_Y \)-diameter strictly less than \( \epsilon \), the set \( f^{-1}(V) \) is a union of countably-many closed subsets of \( X \). But this follows from the fact that \( f^{-1}(V) \) is \( F_\sigma \).

To see (2) \( \implies \) (1), it is sufficient to show that for all real numbers \( \epsilon > 0 \) and open sets \( V \subseteq Y \), there is an \( F_\sigma \) set \( F \subseteq X \) such that \( f^{-1} (V_\epsilon) \subseteq F \subseteq f^{-1}(V) \), where \( V_\epsilon = \{ y \in Y \mid B_{d_Y} (y, \epsilon) \subseteq V \} \). Towards this end, fix a cover \( (C_n)_{n \in \mathbb{N}} \) of \( X \) by closed sets whose \( f \)-images have \( d_Y \)-diameter strictly
less than $\epsilon$, define $N = \{ n \in \mathbb{N} \mid f(C_n) \cap V_n \neq \emptyset \}$, and observe that the set $F = \bigcup_{n \in N} C_n$ is as desired. \[ \square \]

As a corollary, we obtain the following.

**Theorem 1.2.** Suppose that $X$ and $Y$ are Polish spaces, $d_Y$ is a compatible metric on $Y$, and $f : X \to Y$ is Borel. Suppose further that for all countable sets $C \subseteq X$ and real numbers $\epsilon > 0$, there is a Baire-class one function $g : X \to Y$ with $\sup_{x \in C} d_Y(f(x), g(x)) \leq \epsilon$. Then $f$ is Baire class one.

**Proof.** Suppose, towards a contradiction, that $f$ is not Baire class one, fix a compatible metric $d_Y$ on $Y$, and appeal to Proposition 1.1 to find $\delta > 0$ for which there is no cover of $X$ by countably-many closed subsets whose $f$-images have $d_Y$-diameter at most $\delta$.

A **digraph** on a set $X$ is an irreflexive subset of $X \times X$. A **homomorphism** from a digraph $G$ on $X$ to a digraph $H$ on $Y$ is a function $\phi : X \to Y$ sending $G$-related points to $H$-related points.

Let $G_{\delta,f}$ denote the digraph on $X$ consisting of all $(w, x) \in X \times X$ for which $d_Y(f(w), f(x)) > \delta$. We say that a set $W \subseteq X$ is $G_{\delta,f}$-independent if $G_{\delta,f} \upharpoonright W = \emptyset$. Our choice of $\delta$ ensures that $X$ is not the union of countably-many closed $G_{\delta,f}$-independent sets.

Fix $s^\Delta_0 \in 2^n$ such that $\forall s \in 2^{<\mathbb{N}} \exists n \in \mathbb{N} \ s \subseteq s^\Delta_0$, as well as $z_n \in 2^\mathbb{N}$ for all $n \in \mathbb{N}$. Now define a digraph on $2^\mathbb{N}$ by setting

$$G^\Delta_0 = \{(s^\Delta_0 \upharpoonright (0) \upharpoonright z_n, s^\Delta_0 \upharpoonright (1) \upharpoonright z_n) \mid n \in \mathbb{N} \}.$$

The Lecomte-Zeleny dichotomy theorem characterizing analytic graphs of uncountable $\Delta^0_2$-measurable chromatic number (see [LZ14, Corollary 4.5]) yields a continuous homomorphism $\phi : 2^\mathbb{N} \to X$ from this digraph to $G_{\delta,f}$. Set

$$C = \phi(\{s_n^\Delta_0 \upharpoonright (i) \upharpoonright z_n \mid i < 2 \text{ and } n \in \mathbb{N} \})$$

and $\epsilon = \delta/3$.

It only remains to check that no function $g : X \to Y$ with the property that $\sup_{x \in C} d_Y(f(x), g(x)) \leq \epsilon$ is Baire class one. As $\phi$ is necessarily a homomorphism from the above digraph to the digraph $G_{\epsilon,g}$ associated with such a function, there can be no cover of $X$ by countably-many closed subsets whose $g$-images have $d_Y$-diameter at most $\epsilon$, so one more appeal to Proposition 1.1 ensures that $g$ is not Baire class one. \[ \square \]

2. $F_\sigma$-to-one functions

The proof of Theorem 1 is based on a technical but useful sufficient condition for the topological embeddability of $f_0$. 

Proposition 2.1. Suppose that $Y$ is a Polish space and $f : \mathbb{N}_x^\leq \mathbb{N} \to Y$ is a Baire-class one function for which there exists $y \in Y$ such that $\mathbb{N}^\mathbb{N} = f^{-1}(y)$. Then there is a topological embedding of $f_0$ into $f$.

Proof. Fix a compatible metric $d_Y$ on $Y$.

Lemma 2.2. Suppose that $\epsilon > 0$. Then there is a dense open subset $U$ of $\mathbb{N}_x^\leq \mathbb{N}$ such that $f(U) \subseteq B_{d_Y}(y, \epsilon)$.

Proof. By Proposition 1.1, there is a partition $(C_n)_{n \in \mathbb{N}}$ of $\mathbb{N}_x^\leq \mathbb{N}$ into closed sets whose $f$-images have $d_Y$-diameter strictly less than $\epsilon$. Then for each non-empty open set $V \subseteq \mathbb{N}_x^\leq \mathbb{N}$, there exists $n \in \mathbb{N}$ for which $C_n$ is non-meager in $V$, so there is a non-empty open set $W \subseteq V$ such that $C_n$ is comeager in $W$. As $C_n$ is closed, it follows that $W \subseteq C_n$, thus the diameter of $f(W)$ is strictly less than $\epsilon$. As $W$ necessarily contains a point of $\mathbb{N}^\mathbb{N}$, it follows that $f(W) \subseteq B_{d_Y}(y, \epsilon)$. The union of the non-empty open sets $W \subseteq \mathbb{N}_x^\leq \mathbb{N}$ obtained in this way from non-empty open sets $V \subseteq \mathbb{N}_x^\leq \mathbb{N}$ is therefore as desired.

Fix an injective enumeration $(s_n)_{n \in \mathbb{N}}$ of $\mathbb{N}_x^{<\mathbb{N}}$ with the property that $s_m \subseteq s_n \implies m \leq n$ for all $m, n \in \mathbb{N}$, fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of strictly positive real numbers such that $0 = \lim_{n \to \infty} \epsilon_n$, and for each $n \in \mathbb{N}$, set $m(n + 1) = \max\{m \leq n \mid s_m \subseteq s_{n+1}\}$. Define $u_0 = \emptyset$, and recursively appeal to Lemma 2.2 to obtain sequences $u_{s_{n+1}} \in \mathbb{N}_x^{<\mathbb{N}}$, for all $n \in \mathbb{N}$, with the property that $u_{s_{m(n+1)}}(s_{m(n+1)}) \subseteq u_{s_{n+1}}$ and $f(N^{s_{m(n+1)}}) \subseteq B_{d_Y}(y, \min\{\epsilon_n, d_Y(y, f(u_{s_{m(n)}}))\})$.

Define $\pi_X : \mathbb{N}_x^{\leq \mathbb{N}} \to \mathbb{N}_x^{\leq \mathbb{N}}$ by

$$
\pi_X(s) = \begin{cases} 
  u_s & \text{if } s \in \mathbb{N}^{<\mathbb{N}}, \\
  \bigcup_{n \in \mathbb{N}} u_{s|n} & \text{otherwise.}
\end{cases}
$$

Define $\pi_Y : f_0(\mathbb{N}_x^{\leq \mathbb{N}}) \to f(\mathbb{X})$ by $\pi_Y(0) = y$ and $\pi_Y(f_0(s)) = f(u_s)$, for all $s \in \mathbb{N}^{<\mathbb{N}}$. As both of these functions are continuous injections with compact domains, they are necessarily topological embeddings, thus $(\pi_X, \pi_Y)$ is a topological embedding of $f_0$ into $f$.

Proof of Theorem 1. Proposition A.1 ensures that conditions (1) and (2) are mutually exclusive. To see $\neg(1) \implies (2)$, suppose that there exists $y \in Y$ such that $f^{-1}(y)$ is not $F_\sigma$, and appeal to Theorem A.4 to obtain a topological embedding $\pi : \mathbb{N}_x^{\leq \mathbb{N}} \to X$ of $\mathbb{N}^\mathbb{N}$ into $f^{-1}(y)$. Proposition 2.1 then yields a topological embedding $(\pi_X, \pi_Y)$ of $f_0$ into $f \circ \pi$, and it follows that $(\pi \circ \pi_X, \pi_Y)$ is a topological embedding of $f_0$ into $f$. \qed
3. Sigma-continuous functions

We begin with a technical but useful sufficient condition for the topological embeddability of $f_1$.

**Proposition 3.1.** Suppose that $X$ and $Y$ are metric spaces, $f: X \to Y$, and there is a dense $G_δ$ set $C \subseteq \mathbb{N}^N$, a set $W \subseteq X$ intersecting the $f$-preimage of every singleton in a closed set, and a function $φ: C \to W$, such that both $φ$ and $f \circ φ$ are continuous, which is a homomorphism from $\mathbb{G}_{0,m}^N \mid C$ to the $\aleph_0$-dimensional dihypergraph $G_m$ consisting of all convergent sequences $(x_n)_{n \in \mathbb{N}} \in X^N$ with $f(\lim_{n \to \infty} x_n) \neq \lim_{n \to \infty} f(x_n)$ but \{f(x_n) \mid n \in \mathbb{N}\} \subseteq B_{d_\mathbb{N}}(f(\lim_{n \to \infty} x_n), 1/m)$, for all $m \in \mathbb{N}$. Then there is a topological embedding of $f_1$ into $f$.

**Proof.** Fix dense open sets $U_n \subseteq \mathbb{N}^N$ such that $\bigcap_{n \in \mathbb{N}} U_n \subseteq C$. We will recursively construct sequences $(u_s)_{s \in \mathbb{N}^n}$ of elements of $\mathbb{N}^N$, sequences $(V_s)_{s \in \mathbb{N}^n}$ of open subsets of $Y$, and sequences $(x_s)_{s \in \mathbb{N}^n}$ of elements of $X$, for all $n \in \mathbb{N}$, such that:

1. $∀i \in \mathbb{N} \forall s \in \mathbb{N}^N u_s ⊆ u_{s^\tau(i)}$.
2. $∀i \in \mathbb{N} \forall s \in \mathbb{N}^N N_{u_{s^\tau(i)}} \subseteq U_s$.
3. $∀s \in \mathbb{N}^N (f \circ φ)(N_{u_s}) \cup \{f(x_s)\} \subseteq V_s$.
4. $∀i \in \mathbb{N} \forall s \in \mathbb{N}^N V_s \cap (i) \subseteq V_s$.
5. $∀i \in \mathbb{N} \forall s \in \mathbb{N}^N \text{diam}_X(φ(N_{u_{s^\tau(i)}})) < 1/|s|$.
6. $∀i \in \mathbb{N} \forall s \in \mathbb{N}^N \text{diam}_Y(V_{s^\tau(i)}) < 1/|s|$.
7. $∀i \in \mathbb{N} \forall s \in \mathbb{N}^N φ(N_{u_{s^\tau(i)}}) \subseteq B_{d_\mathbb{N}}(x_s, 1/i)$.
8. $∀s \in \mathbb{N}^N f(x_s) \notin \bigcup_{i \in \mathbb{N}} V_{s^\tau(i)}$.
9. $∀i \in \mathbb{N} \forall s \in \mathbb{N}^N V_{s^\tau(i)} \cap \bigcup_{j \in \mathbb{N} \setminus \{i\}} V_{s^\tau(j)} = \emptyset$.

We begin by setting $u_0 = \emptyset$ and $V_0 = Y$. Suppose now that $n \in \mathbb{N}$ and we have already found $(u_s)_{s \in \mathbb{N}^n}$, $(V_s)_{s \in \mathbb{N}^n}$, and $(x_s)_{s \in \mathbb{N}^n}$. For each $s \in \mathbb{N}^n$, fix $δ_s > 0$ as well as $u'_s \in \mathbb{N}^N$ such that $u_s \subseteq u'_s$, $N_{u'_s} \subseteq U_n$, $\text{diam}_X(φ(N_{u'_s})) < 1/n$, $\text{diam}_Y ((f \circ φ)(N_{u'_s})) < 3/2n$, and $B_{d_\mathbb{N}}((f \circ φ)(N_{u'_s}), δ_s) \subseteq V_s$. Fix a natural number $n_s \geq 1/δ_s$ such that $u'_s \subseteq s_{n_s}^N$, appeal to the Baire category theorem to find $z_s \in \mathbb{N}^N$ with the property that $s_{n_s}^N \cap (i) \cap z_s \subseteq C$ for all $i \in \mathbb{N}$, and define $x_{i,s} = φ(s_{n_s}^N \cap (i) \cap z_s)$ and $y_{i,s} = f(x_{i,s})$ for all $i \in \mathbb{N}$, as well as $x_s = \lim_{i \to \infty} x_{i,s}$. The fact that $f(x_s) \neq \lim_{i \to \infty} y_{i,s}$ ensures the existence of an infinite set $I_s \subseteq \mathbb{N}$ for which $f(x_s) \notin \{y_{i,s} \mid i \in I_s\}$. Note that there can be no infinite set $J \subseteq I_s$ such that $(y_{j,s})_{j \in J}$ is constant, since otherwise the fact that $φ(C) \subseteq W$ would imply that $f(x_s) = y_{j,s}$, for all $j \in J$. So by passing to an infinite subset of $I_s$, we can assume that $(y_{i,s})_{i \in I_s}$ is injective. By passing to a further infinite subset of $I_s$, we can ensure that $(y_{i,s})_{i \in I_s}$ has
at most one limit point. By eliminating this limit point from the sequence if necessary, we can therefore ensure that $y_{i,s} \not\in \{y_{j,s} \mid j \in I_s \setminus \{i\}\}$, for all $i \in I_s$. Similarly, we can assume that $x_s \not\in \{x_{i,s} \mid i \in I_s\}$. By passing one last time to an infinite subset of $I_s$, we can assume that $d_X(x_s, x_{i,k,s,s}) < 1/k$ for all $k \in \mathbb{N}$, where $(i_{k,s})_{k \in \mathbb{N}}$ is the strictly increasing enumeration of $I_s$.

For each $k \in \mathbb{N}$, fix $\epsilon^X_{k,s} > 0$ strictly less than $1/k - d_X(x_s, x_{i,k,s,s})$, and fix $\epsilon^Y_{k,s} > 0$ strictly less than $d_Y(f(x_s), y_{i,k,s,s})/2$ and $d_Y(y_{i,s}, y_{i,k,s,s})/3$, for all $i \in I_s \setminus \{i_{k,s}\}$. Set $V_{s^{-}(k)} = B_{d_Y(y_{i,k,s,s}, \epsilon^Y_{k,s})} \cap V_s$, and fix an initial segment $u_{s^{-}(k)}$ of $s_{n_s}^N \cap (i_{k,s}) \setminus z_s$ of length at least $n_s + 1$ with the property that $\phi(N_{u_{s^{-}(k)}}) \subseteq B_{d_X(x_{i,k,s,s}, \epsilon^X_{k,s})}$ and $(f \circ \phi)(N_{u_{s^{-}(k)}}) \subseteq V_{s^{-}(k)}$. Our choice of $u_s'$ ensures that conditions (1), (2), and (5) hold, and along with the fact that $\phi$ is a homomorphism, that condition (3) holds as well. Condition (4) holds trivially, and the remaining conditions follow from our upper bounds on $\epsilon^X_{k,s}$ and $\epsilon^Y_{k,s}$. This completes the recursive construction.

By condition (1), we obtain a continuous function $\psi: \mathbb{N}^N \to \mathbb{N}^N$ by setting $\psi(s) = \bigcup_{n \in \mathbb{N}} u_s|n$. Condition (2) ensures that $\psi(\mathbb{N}^N) \subseteq C$. Set $x_s = (\phi \circ \psi)(s)$ for $s \in \mathbb{N}^N$, and define $\pi_X: \mathbb{N}_{\leq}^N \to X$ and $\pi_Y: \mathbb{N}_{\leq}^N \to Y$ by $\pi_X(s) = x_s$ and $\pi_Y = f \circ \pi_X$. We will show that $(\pi_X, \pi_Y)$ is a topological embedding of $f_1$ into $f$.

**Lemma 3.2.** Suppose that $s \in \mathbb{N}^{<N}$. Then $\pi_X(N_s^*) \subseteq \overline{\phi(N_{u_s})}$.

**Proof.** Simply observe that

$$\pi_X(N_s^*) = (\phi \circ \psi)(N_s) \cup \{x_t \mid t \in N_s^* \setminus N_s\}$$

$$\subseteq \phi(N_{u_s}) \cup \bigcup_{t \in N_s^* \setminus N_s} \phi(N_{u_t})$$

$$\subseteq \phi(N_{u_s}),$$

by conditions (1) and (7). □

**Lemma 3.3.** Suppose that $s \in \mathbb{N}^{<N}$. Then $\pi_Y(N_s^*) \subseteq V_s$.

**Proof.** Simply observe that

$$\pi_Y(N_s^*) = (f \circ \phi \circ \psi)(N_s) \cup \{f(x_t) \mid t \in N_s^* \setminus N_s\}$$

$$\subseteq (f \circ \phi)(N_{u_s}) \cup \{f(x_t) \mid t \in N_s^* \setminus N_s\}$$

$$\subseteq V_s \cup \bigcup_{t \in N_s^* \setminus N_s} V_t$$

$$\subseteq V_s,$$

by conditions (3) and (4). □

To see that $\pi_X$ and $\pi_Y$ are injective, it is enough to check that the latter is injective. Towards this end, suppose that $s, t \in \mathbb{N}_{\leq}^N$ are distinct. If there
is a least \( n \leq \min\{|s|, |t|\} \) with \( s \upharpoonright n \neq t \upharpoonright n \), then condition (9) ensures that \( V_s[n] \) and \( V_t[n] \) are disjoint, and since Lemma 3.3 implies that \( \pi_Y(s) \) is in the former and \( \pi_Y(t) \) is in the latter, it follows that they are distinct. Otherwise, after reversing the roles of \( s \) and \( t \) if necessary, we can assume that there exists \( n < |t| \) for which \( s = t \upharpoonright n \). But then condition (8) ensures that \( \pi_Y(s) \notin V_t[(n+1)] \), while Lemma 3.3 implies that \( \pi_Y(t) \in V_t[(n+1)] \), thus \( \pi_Y(s) \neq \pi_Y(t) \).

To see that \( \pi_X \) is a topological embedding, it only remains to show that it is continuous (since \( \mathbb{N}_s^{\leq N} \) is compact). And for this, it is enough to check that for all \( n \in \mathbb{N} \) and \( s \in \mathbb{N}_s^{\leq N} \), there is an open neighborhood of \( s \) whose image under \( \pi_X \) is a subset of \( \mathcal{B}_{d_X}(\pi_X(s), 1/n) \). Towards this end, observe that if \( s \in \mathbb{N}_s^{N} \), then Lemma 3.2 ensures that \( \pi_X(N_s^*[(n+1)]) \subseteq \overline{\phi(N_{\pi^{-1}(s)}^{\pi^{-1}(n+1)})} \), so condition (5) implies that \( N_{\pi^{-1}(n+1)}^* \) is an open neighborhood of \( s \) whose image under \( \pi_X \) is a subset of \( \mathcal{B}_{d_X}(\pi_X(s), 1/n) \). And if \( s \in \mathbb{N}_s^{<N} \), then Lemma 3.2 ensures that

\[
\pi_X(N_s^* \setminus \bigcup_{i<n} N_s^{*(i)}) = \pi_X(\{s\} \cup \bigcup_{i\geq n} N_s^{*(i)}) \subseteq \{\pi_X(s)\} \cup \bigcup_{i\geq n} \phi(N_{\pi^{-1}(s)}^{\pi^{-1}(i)}),
\]

so condition (7) implies that \( N_s^* \setminus \bigcup_{i<n} N_s^{*(i)} \) is an open neighborhood of \( s \) whose image under \( \pi_X \) is a subset of \( \mathcal{B}_{d_X}(\pi_X(s), 1/n) \).

To see that \( \pi_Y \) is continuous, it is sufficient to check that for all \( n \in \mathbb{N} \) and \( s \in \mathbb{N}_s^{\leq N} \), there is an open neighborhood of \( s \) whose image under \( \pi_Y \) is contained in \( \mathcal{B}_{d_Y}(\pi_Y(s), 1/n) \). Towards this end, observe that if \( s \in \mathbb{N}_s^{N} \), then Lemma 3.3 ensures that \( \pi_Y(N_s^*[(n+1)]) \subseteq V_s[(n+1)] \), so condition (6) implies that \( N_s^{*(n+1)} \) is an open neighborhood of \( s \) whose image under \( \pi_Y \) is contained in \( \mathcal{B}_{d_Y}(\pi_Y(s), 1/n) \). And if \( s \in \mathbb{N}_s^{<N} \), then \( \{s\} \) is an open neighborhood of \( s \) whose image under \( \pi_Y \) is a subset of \( \mathcal{B}_{d_Y}(\pi_Y(s), 1/n) \).

Before showing that \( \pi_Y \) is a topological embedding, we first establish several lemmas.

**Lemma 3.4.** Suppose that \( s \in \mathbb{N}_s^{<N} \). Then \( \pi_Y(N_s^*) = \overline{V_s} \cap \pi_Y(\mathbb{N}_s^{\leq N}) \).

**Proof.** Lemma 3.3 ensures that \( \pi_Y(N_s^*) \subseteq \overline{V_s} \cap \pi_Y(\mathbb{N}_s^{\leq N}) \), so it is enough to show that \( \pi_Y(\mathbb{N}_s^{\leq N} \setminus N_s^*) \cap \overline{V_s} = \emptyset \). Towards this end, note that if \( t \in \mathbb{N}_s^{\leq N} \setminus N_s^* \), then either there exists a least \( n \leq \min\{|s|, |t|\} \) for which \( s \upharpoonright n \) and \( t \upharpoonright n \) are incompatible, or \( t \subseteq s \). In the former case, condition (9) implies that \( \overline{V_s[n]} \) and \( V_t[n] \) are disjoint, and since Lemma 3.3 implies that \( \pi_Y(t) \) is in the latter, it is not in the former. But then it is also not in \( V_s \), by condition (4). In the latter case, set \( n = |t| \), and appeal to condition (8) to see that \( \pi_Y(t) \) is not in \( \overline{V_s[(n+1)]} \). But then it is also not in \( V_s \), by condition (4). \( \square \)
Lemma 3.5. Suppose that $s \in \mathbb{N}^{<\mathbb{N}}$. Then

$$
\pi_Y(N^*_s) = \pi_Y(N^{<\mathbb{N}}_s) \setminus \left( \bigcup_{t \perp s} V_t \cup \{ \pi_Y(t) \mid t \sqsubset s \} \right).
$$

Proof. To see that $\pi_Y(N^*_s) \cap \left( \bigcup_{t \perp s} V_t \cup \{ \pi_Y(t) \mid t \sqsubset s \} \right) = \emptyset$, note that if $t \perp s$, then there is a maximal $n < \min\{|s|, |t|\}$ with the property that $s \upharpoonright n = t \upharpoonright n$, in which case $t$ is an extension of $(s \upharpoonright n) \leadsto (j)$, for some $j \in \mathbb{N} \setminus \{s(n)\}$. Condition (4) therefore ensures that

$$
\bigcup_{t \perp s} V_t = \bigcup_{n < |s|} \bigcup_{j \in \mathbb{N} \setminus \{s(n)\}} V(s[n] \leadsto (j)).
$$

As Lemma 3.3 implies that $\pi_Y(N^*_s) \subseteq V_s$, and condition (4) ensures that $V_s \subseteq V_{s[(n+1)]}$ for all $n < |s|$, it follows from condition (9) that $\pi_Y(N^*_s) \cap \bigcup_{t \perp s} V_t = \emptyset$.

To see that $\pi_Y(N^{<\mathbb{N}}_s \setminus N^*_s) \subseteq \bigcup_{t \perp s} V_t \cup \{ \pi_Y(t) \mid t \sqsubset s \}$, note that if $t \in N^{<\mathbb{N}}_s \setminus N^*_s$ and $t \not\sqsubseteq s$, then there exists $n \leq \min\{|s|, |t|\}$ such that $s \upharpoonright n$ and $t \upharpoonright n$ are incompatible, so Lemma 3.3 ensures that $\pi_Y(t) \in \bigcup_{t \perp s} V_t$. □

Lemma 3.6. Suppose that $s \in \mathbb{N}^{<\mathbb{N}}$. Then $\pi_Y(s)$ is the unique element of

$$
\pi_Y(N^{<\mathbb{N}}_s) \setminus \left( \bigcup_{t \perp s} V_t \cup \{ \pi_Y(t) \mid t \sqsubset s \} \right).
$$

Proof. As $\bigcup_{t \perp s} V_t = \bigcup_{t \perp s} \bigcup_{j \in \mathbb{N} \setminus \{s(n)\}} V(s[n] \leadsto (j))$ by condition (4), Lemma 3.5 ensures that we need only show that $\pi_Y(s)$ is the unique element of $\pi_Y(N^*_s) \setminus \bigcup_{t \perp s} V_t$. Condition (8) ensures that $\pi_Y(s)$ is in this set, while Lemma 3.3 implies that the other points of $\pi_Y(N^*_s)$ are not. □

It remains to show that $\pi_Y(N^*_s)$ and $\{ \pi_Y(s) \}$ are clopen in $\pi_Y(N^{<\mathbb{N}}_s)$, for all $s \in \mathbb{N}^{<\mathbb{N}}$. The former is a consequence of Lemmas 3.4 and 3.5, while the latter follows from Lemma 3.6. □

Proposition 3.7. Suppose that $X$ and $Y$ are Polish spaces, $f: X \to Y$, $G$ is the $\kappa_0$-dimensional dihypergraph on $X$ consisting of all convergent sequences $(x_n)_{n \in \mathbb{N}}$ such that $f(\lim_{n \to \infty} x_n) \neq \lim_{n \to \infty} f(x_n)$, and $W \subseteq X$ is $G$-independent. Then $\overline{W}$ is $G$-independent.

Proof. Fix compatible metrics $d_X$ and $d_Y$ on $X$ and $Y$, respectively. We must show that if $x = \lim_{n \to \infty} \overline{w}_n$ and each $\overline{w}_n$ is in $\overline{W}$, then $f(x) = \lim_{n \to \infty} f(\overline{w}_n)$. For each $n \in \mathbb{N}$, write $\overline{w}_n = \lim_{m \to \infty} w_{m,n}$, where each $w_{m,n}$ is in $W$. The fact that $W$ is $G$-independent then ensures that $f(\overline{w}_n) = \lim_{m \to \infty} f(w_{m,n})$. Fix $m_n \in \mathbb{N}$ with the property that both $d_X(w_{m,n}, \overline{w}_n)$ and $d_Y(f(w_{m,n}), f(\overline{w}_n))$ are at most $1/n$. It follows that $x = \lim_{n \to \infty} w_{m,n}$, so one more appeal to the fact that $W$ is $G$-independent yields that $f(x) = \lim_{n \to \infty} f(w_{m,n}) = \lim_{n \to \infty} f(\overline{w}_n)$. □
Proof of Theorem 2. Clearly, if (1) holds then $f^{-1}(C) = F$ for every closed set $C \subseteq Y$. Hence conditions (1) and (2) are mutually exclusive. To see that at least one of them holds, let $G$ denote the $\aleph_0$-dimensional dihypergraph on $X$ consisting of all convergent sequences $(x_n)_{n \in \mathbb{N}}$ such that $f(\lim_{n \to \infty} x_n) \neq \lim_{n \to \infty} f(x_n)$.

As $f$ is continuous on a closed set if and only if the set in question is $G$-independent, Proposition 3.7 ensures that if $X$ is a union of countably-many $G$-independent sets, then $f$ is $\sigma$-continuous with closed witnesses. We can therefore focus on the case that $X$ is not a union of countably-many $G$-independent sets. While it is not difficult to see that condition (1) fails in this case, simply applying the dichotomy for $\aleph_0$-dimensional analytic dihypergraphs of uncountable chromatic number (see [Lec09, Theorem 1.6] or [Mil11, Theorem 18]) will not yield the sort of homomorphism we require. So instead, we will use our further assumptions to obtain a homomorphism with stronger properties.

Fix a compatible metric $d_Y$ on $Y$, and for each $\epsilon > 0$, let $G_\epsilon$ denote the $\aleph_0$-dimensional dihypergraph on $X$ consisting of all sequences $(x_n)_{n \in \mathbb{N}} \in G$ with $\{f(x_n) \mid n \in \mathbb{N}\} \subseteq B_{d_Y}(f(\lim_{n \to \infty} x_n), \epsilon)$. Note that if $B \subseteq X$ is a $G_\epsilon$-independent set and $C \subseteq X$ is a closed set whose $f$-image has $d_Y$-diameter strictly less than $\epsilon$, then $B \cap C$ is $G$-independent. As Proposition 1.1 ensures that $X$ is a union of countably-many closed sets whose $f$-images have $d_Y$-diameter strictly less than $\epsilon$, it follows that every $G_\epsilon$-independent set is a union of countably-many $G$-independent sets.

We say that a set $W \subseteq X$ is eventually $(G_\epsilon)_{\epsilon > 0}$-independent if there exists $\epsilon > 0$ for which it is $G_\epsilon$-independent. As $X$ is not a union of countably-many $G$-independent sets, it follows that it is not a union of countably-many eventually $(G_\epsilon)_{\epsilon > 0}$-independent sets. Again, however, simply applying the sequential $\aleph_0$-dimensional analog of the $\mathcal{G}_0$ dichotomy theorem (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 21]) will not yield the sort of homomorphism we require, and we must once more appeal to our further assumptions.

Theorem A.6 ensures that $X$ is a union of countably-many analytic sets whose intersection with the $f$-preimage of each singleton is closed. As $X$ is not a union of countably-many eventually $(G_\epsilon)_{\epsilon > 0}$-independent sets, it follows that there is an analytic set $A \subseteq X$, whose intersection with the $f$-preimage of each singleton is closed, that is not a union of countably-many eventually $(G_\epsilon)_{\epsilon > 0}$-independent sets.
At long last, we now appeal to the sequential $\aleph_0$-dimensional analog of the $G_0$ dichotomy theorem (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 21]) to obtain a dense $G_\delta$ set $C \subseteq \omega$ and a continuous function $\phi: C \to A$ which is a homomorphism from $\omega_0^n \upharpoonright C$ to $G_{1/n}$, for all $n \in \omega$. In fact, by first replacing the given topology of $X$ with a finer Polish topology consisting only of Borel sets but with respect to which $f$ is continuous (see, for example, [Kec95, Theorem 13.11]), we can ensure that $f \circ \phi$ is continuous as well. An application of Proposition 3.1 therefore yields the desired topological embedding of $f_1$ into $f$. □

**Appendix: $F_\sigma$ sets**

We begin the appendix with a straightforward observation.

**Proposition A.1.**

(a) The set $\omega$ is not an $F_\sigma$ subspace of $\omega^\omega$.

(b) The set $\omega$ is a closed subspace of $\omega^\omega$.

**Proof.** To see (a), note that a subset of a topological space is $G_\delta$ if it is an intersection of countably-many open sets. As $\omega^\omega$ is countable and $\omega^\omega$ is dense in $\omega^\omega$, it follows that $\omega^\omega$ is a dense $G_\delta$ subspace of $\omega^\omega$. As $\omega^\omega$ is also dense in $\omega^\omega$, the Baire category theorem (see, for example, [Kec95, Theorem 8.4]) ensures that it is not a $G_\delta$ subspace of $\omega^\omega$, thus $\omega$ is not an $F_\sigma$ subspace of $\omega^\omega$.

To see (b), note that $\{s\}$ is clopen in $\omega^\omega$ for all $s \in \omega^\omega$, so $\omega^\omega$ is open in $\omega^\omega$, thus $\omega$ is closed in $\omega^\omega$. □

An $\aleph_0$-dimensional dihypergraph on a set $X$ is a set of non-constant elements of $X^\omega$. A homomorphism from an $\aleph_0$-dimensional dihypergraph $G$ on $X$ to an $\aleph_0$-dimensional dihypergraph $H$ on $Y$ is a function $\phi: X \to Y$ sending elements of $G$ to elements of $H$.

Fix sequences $s_n^N \in \omega^n$ such that $\forall s \in \omega^\omega \exists n \in \omega \ s \subseteq s_n^N$, and define $\aleph_0$-dimensional dihypergraphs on $\omega^\omega$ by setting

$$\omega_0^n = \{(s_n \upharpoonright (i) \upharpoonright z)_{i,z} \in \omega^n \mid z \in \omega^\omega\},$$

for all $n \in \omega$, and $\omega_0^\omega = \bigcup_{n \in \omega} \omega_0^n$.

We now establish a technical but useful sufficient condition for the topological embeddability of $\omega^\omega$.

**Proposition A.2.** Suppose that $X$ is a metric space, $Y \subseteq X$ is a set, and there are a dense $G_\delta$ set $C \subseteq \omega^\omega$ and a continuous homomorphism $\phi: C \to Y$ from $\omega_0^n \upharpoonright C$ to the $\aleph_0$-dimensional dihypergraph

$$G = \{(y_n)_{n \in \omega} \in Y^\omega \mid \exists x \in X \setminus Y \ x = \lim_{n \to \infty} y_n\}.$$
Then there is a topological embedding \( \pi: \mathbb{N}^\subseteq \rightarrow X \) of \( \mathbb{N}^\subseteq \) into \( Y \).

**Proof.** Fix dense open sets \( U_n \subseteq \mathbb{N}^\subseteq \) such that \( \bigcap_{n \in \mathbb{N}} U_n \subseteq C \). We will recursively construct sequences \( (u_s)_{s \in \mathbb{N}^a} \) of elements of \( \mathbb{N}^\subseteq \) and sequences \( (x_s)_{s \in \mathbb{N}^a} \) of elements of \( X \), for all \( n \in \mathbb{N} \), such that:

1. \( \forall i \in \mathbb{N} \forall s \in \mathbb{N}^\subseteq \) \( u_s \sqsubseteq u_{s\neg(i)} \).
2. \( \forall i \in \mathbb{N} \forall s \in \mathbb{N}^\subseteq \) \( \mathcal{N}_{u_{s\neg(i)}} \subseteq U_{\lfloor s \rfloor} \).
3. \( \forall i \in \mathbb{N} \forall s \in \mathbb{N}^\subseteq \) \( \text{diam}_{\mathcal{N}}(\phi(\mathcal{N}_{u_{s\neg(i)}})) < 1/|s| \).
4. \( \forall i \in \mathbb{N} \forall s \in \mathbb{N}^\subseteq \) \( \phi(\mathcal{N}_{u_{s\neg(i)}}) \subseteq B_{\mathcal{N}}(x_s, 1/i) \).
5. \( \forall i \in \mathbb{N} \forall s \in \mathbb{N}^\subseteq \) \( x_s \notin \phi(\mathcal{N}_{u_{s\neg(i)}}) \).
6. \( \forall i, j \in \mathbb{N} \forall s \in \mathbb{N}^\subseteq \) \( i \neq j \implies \phi(\mathcal{N}_{u_{s\neg(i)}}) \cap \phi(\mathcal{N}_{u_{s\neg(j)}}) = \emptyset \).

We begin by setting \( u_\emptyset = \emptyset \). Suppose now that \( n \in \mathbb{N} \) and we have already found \( (u_s)_{s \in \mathbb{N}^\subseteq} \) and \( (x_s)_{s \in \mathbb{N}^\subseteq} \). For each \( s \in \mathbb{N}^a \), fix \( u'_s \in \mathbb{N}^\subseteq \) such that \( u_s \sqsubseteq u'_s \), \( \mathcal{N}_{u'_s} \subseteq U_n \), and \( \text{diam}_{\mathcal{N}}(\phi(\mathcal{N}_{u'_s})) < 1/n \), fix \( n_s \in \mathbb{N} \) for which \( u'_s \sqsubseteq s_{n_s} \), and appeal to the Baire category theorem to find \( z_s \in \mathbb{N}^\subseteq \) with the property that \( s_{n_s} \smallsetminus (i) \sqsubseteq z_s \in C \), for all \( i \in \mathbb{N} \). Set \( y_{i,s} = \phi(s_{n_s} \smallsetminus (i) \smallsetminus z_s) \) for all \( i \in \mathbb{N} \), as well as \( x_s = \lim_{n \rightarrow \infty} y_{i,s} \). As \( x_s \notin \{ y_{i,s} \mid i \in \mathbb{N} \} \), there is an infinite set \( I_s \subseteq \mathbb{N} \) for which \( (y_{i,s})_{i \in I_s} \) is injective. By passing to an infinite subset of \( I_s \), we can assume that \( d_X(x_s, y_{i,k,s}) < 1/k \) for all \( k \in \mathbb{N} \), where \((i,k,s)_{k \in \mathbb{N}} \) is the strictly increasing enumeration of \( I_s \). For each \( k \in \mathbb{N} \), fix \( \epsilon_{k,s} > 0 \) strictly less than \( 1/k - d_X(x_s, y_{i,k,s}) \), \( d_X(x_s, y_{i,k,s}) \), and \( d_X(y_{i,s}, y_{i,k,s})/2 \) for all \( i \in I_s \setminus \{i_k,s\} \), and fix an initial segment \( u_{s\neg(k)} \) of \( s_{n_s} \smallsetminus (i_k,s) \smallsetminus z_s \) of length at least \( n_s + 1 \) with the property that \( \phi(\mathcal{N}_{u_{s\neg(k)}}) \subseteq B_{\mathcal{N}}(y_{i,k,s}, \epsilon_{k,s}) \). Our choice of \( u'_s \) ensures that conditions (1) – (3) hold, and our strict upper bounds on \( \epsilon_{k,s} \) yield the remaining conditions. This completes the recursive construction.

Condition (1) ensures that we obtain a function \( \psi: \mathbb{N}^\subseteq \rightarrow \mathbb{N}^\subseteq \) by setting \( \psi(s) = \bigcup_{n \in \mathbb{N}} u_s[n] \), and condition (2) implies that \( \psi(\mathbb{N}^\subseteq) \subseteq C \). Set \( x_s = (\phi \circ \psi)(s) \) for \( s \in \mathbb{N}^\subseteq \), and define \( \pi: \mathbb{N}^\subseteq \rightarrow X \) by \( \pi(s) = x_s \). We will show that \( \pi \) is a topological embedding of \( \mathbb{N}^\subseteq \) into \( Y \).

**Lemma A.3.** Suppose that \( s \in \mathbb{N}^\subseteq \). Then \( \pi(\mathcal{N}_s) \subseteq \overline{\phi(\mathcal{N}_u)} \).

**Proof.** Simply observe that

\[
\pi(\mathcal{N}_s) = (\phi \circ \psi)(\mathcal{N}_s) \cup \{ x_t \mid t \in \mathcal{N}_s \setminus \mathcal{N}_s \}
\ \subseteq \phi(\mathcal{N}_u) \cup \bigcup_{t \in \mathcal{N}_s \setminus \mathcal{N}_s} \overline{\phi(\mathcal{N}_u)}
\ \subseteq \overline{\phi(\mathcal{N}_u)},
\]

by conditions (1) and (4). \( \square \)
To see that \( \pi \) is injective, suppose that \( s, t \in \mathbb{N}^{\leq \omega} \) are distinct. If there is a least \( n \leq \min\{|s|, |t|\} \) with \( s \upharpoonright n \neq t \upharpoonright n \), then condition (6) ensures that \( \phi(N_{u_{i(n+1)}}) \) and \( \phi(N_{u_{i(n+1)}}) \) are disjoint, and since Lemma A.3 implies that \( \pi(s) \) is in the former and \( \pi(t) \) is in the latter, it follows that they are distinct. Otherwise, after reversing the roles of \( s \) and \( t \) if necessary, we can assume that there exists \( n < |t| \) for which \( s = t \upharpoonright n \). But then condition (5) ensures that \( \pi(s) \notin \phi(N_{u_{i(n+1)}}) \), while Lemma A.3 implies that \( \pi(t) \in \phi(N_{u_{i(n+1)}}) \), thus \( \pi(s) \neq \pi(t) \).

As \( \mathbb{N}^{\leq \omega} \) is compact, it only remains to check that \( \pi \) is continuous. And for this, it is enough to check that for all \( n \in \mathbb{N} \) and \( s \in \mathbb{N}_{s_{\mathbb{N}}}^{\leq \omega} \), there is an open neighborhood of \( s \) whose image under \( \pi \) is a subset of \( B_{d_{\chi}}(\pi(s), 1/n) \). Towards this end, note first that if \( s \in \mathbb{N}^{\omega} \), then Lemma A.3 ensures that \( \pi(N_{s_{\mathbb{N}}(n+1)}) \subseteq \phi(N_{u_{i(n+1)}}) \), so condition (3) implies that \( N_{s_{\mathbb{N}}(n+1)} \) is an open neighborhood of \( s \) whose image under \( \pi \) is a subset of \( B_{d_{\chi}}(\pi(s), 1/n) \). On the other hand, if \( s \in \mathbb{N}^{\leq \omega} \), then Lemma A.3 ensures that

\[
\pi(N_{s_{\mathbb{N}}} \setminus \bigcup_{i \leq n} N_{s_{\mathbb{N}}^*(i)}) = \pi(\{s\} \cup \bigcup_{i \geq n} N_{s_{\mathbb{N}}^*(i)}) \\
\subseteq \{\pi(s)\} \cup \bigcup_{i \geq n} \phi(N_{u_{s_{\mathbb{N}}^*(i)}}),
\]

so condition (4) implies that \( N_{s_{\mathbb{N}}} \setminus \bigcup_{i \leq n} N_{s_{\mathbb{N}}^*(i)} \) is an open neighborhood of \( s \) whose image under \( \pi \) is a subset of \( B_{d_{\chi}}(\pi(s), 1/n) \).

As a corollary, we obtain the following dichotomy theorem characterizing the family of Borel sets which are \( F_\sigma \).

**Theorem A.4** (Hurewicz). Suppose that \( X \) is a Polish space and \( B \subseteq X \) is Borel. Then exactly one of the following holds:

1. The set \( B \) is \( F_\sigma \).
2. There is a topological embedding \( \pi : \mathbb{N}^{\leq \omega} \rightarrow X \) of \( \mathbb{N}^{\omega} \) into \( B \).

**Proof.** Proposition A.1 ensures that conditions (1) and (2) are mutually exclusive. To see that at least one of them holds, let \( G \) denote the \( \aleph_0 \)-dimensional dihypergraph consisting of all sequences \((y_n)_{n \in \mathbb{N}}\) of points of \( B \) converging to a point of \( X \setminus B \). We say that a set \( W \subseteq X \) is \( G \)-independent if \( G \upharpoonright W = \emptyset \). Note that the closure of every such subset of \( B \) is contained in \( B \). In particular, it follows that if \( B \) is a union of countably-many \( G \)-independent sets, then it is \( F_\sigma \). Otherwise, Lecomte’s dichotomy theorem for \( \aleph_0 \)-dimensional dihypergraphs of uncountable chromatic number (see [Lec09, Theorem 1.6] or [Mil11, Theorem 18]) yields a dense \( G_\delta \) set \( C \subseteq \mathbb{N}^{\omega} \) for which there is a continuous homomorphism \( \phi : C \rightarrow B \) from \( G_\delta \upharpoonright C \) to \( G \),
in which case Proposition A.2 yields a topological embedding \( \pi : N^e_\omega \to X \) of \( N^N \) into \( B \).

There is also a parametrized form of this theorem.

**Theorem A.5** (Saint Raymond). Suppose that \( X \) and \( Y \) are Polish spaces and \( R \subseteq X \times Y \) is a Borel set with \( F_\sigma \) horizontal sections. Then \( R \) is a union of countably-many analytic subsets with closed horizontal sections.

**Proof.** The parametrized form of our earlier dihypergraph is given by

\[
G = \{(x_n)_{n \in \mathbb{N}, y} \in (R^y)^N \times Y \mid \exists x \in X \setminus R^y x = \lim_{n \to \infty} x_n\}.
\]

We say that a set \( S \subseteq X \times Y \) is \( G \)-independent if \( S^y \) is \( G^y \)-independent, for all \( y \in Y \). Note that the closure of every horizontal section of every such subset of \( R \) is contained in the corresponding horizontal section of \( R \). Moreover, if \( S \subseteq R \) is analytic, then so too is the set \( \{(x, y) \in X \times Y \mid x \in \overline{S^y}\} \). In particular, it follows that if \( R \) is a union of countably-many \( G \)-independent analytic subsets, then it is a union of countably-many analytic subsets with closed horizontal sections. Otherwise, the parametrized form of the dichotomy theorem for \( \omega \)-dimensional dihypergraphs of uncountable Borel chromatic number (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 31]) yields a dense \( G_\delta \) set \( C \subseteq N^N \) and \( y \in Y \) for which there is a continuous homomorphism \( \phi : C \to R^y \) from \( G_0^N \upharpoonright C \) to \( G^y \), in which case Proposition A.2 yields a continuous embedding \( \pi : N^e_\omega \to X \) of \( N^N \) into \( R^y \). As the latter is \( F_\sigma \), this contradicts Proposition A.1. \( \square \)

Our use of this result is via the following corollary.

**Theorem A.6.** Suppose that \( X \) and \( Y \) are Polish spaces and \( f : X \to Y \) is an \( F_\sigma \)-to-one Borel function. Then \( f \) is strongly \( \sigma \)-closed-to-one.

**Proof.** As the set \( R = \text{graph}(f) \) is Borel (see, for example, [Kec95, Proposition 12.4]) and has \( F_\sigma \) horizontal sections, an application of Theorem A.5 ensures that it is a union of countably-many analytic sets with closed horizontal sections. As the projections of these sets onto \( X \) intersect the preimage of each singleton in a closed set, it follows that \( f \) is strongly \( \sigma \)-closed-to-one. \( \square \)

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References

[JR82] J. E. Jayne and C. A. Rogers, *First level Borel functions and isomorphisms*, J. Math. Pures Appl. (9) 61 (1982), 177–205.

[Kec95] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.

[KST99] A. S. Kechris, S. Solecki, and S. Todorcevic, *Borel chromatic numbers*, Adv. Math. 141 (1999), 1–44.

[Lec09] D. Lecomte, *A dichotomy characterizing analytic digraphs of uncountable Borel chromatic number in any dimension*, Trans. Amer. Math. Soc. 361 (2009), 4181–4193.

[LZ14] D. Lecomte and M. Zeleny, *Baire-class $\xi$ colorings: the first three levels*, Trans. Amer. Math. Soc. 366 (2014), 2345–2373.

[Mil11] B. D. Miller, *Dichotomy theorems for countably infinite dimensional analytic hypergraphs*, Ann. Pure Appl. Logic 162 (2011), 561–565.

[Mil12] B. D. Miller, *The graph-theoretic approach to descriptive set theory*, Bull. Symbolic Logic 18 (2012), 554–575.

[Sol98] S. Solecki, *Decomposing Borel sets and functions and the structure of Baire class 1 functions*, J. Amer. Math. Soc. 11 (1998), 521–550.

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