THE SHARP HARDY UNCERTAINTY PRINCIPLE FOR SCHRÖDINGER EVOLUTIONS

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Abstract. We give a new proof of Hardy’s uncertainty principle, up to the
end-point case, which is only based on calculus. The method allows us to ex-
tend Hardy’s uncertainty principle to Schrödinger equations with non-constant
coefficients. We also deduce optimal Gaussian decay bounds for solutions to
these Schrödinger equations.

1. Introduction

In this paper we continue the study initiated in [10], [4], [5] and [6] on unique
continuation properties of solutions of Schrödinger evolutions
\[ \partial_t u = i (\triangle u + V(x, t)u) \quad \text{in} \quad \mathbb{R}^n \times [0,T]. \]

The goal is to obtain sharp and non-trivial sufficient conditions on a solution \( u \),
the potential \( V \) and the behavior of the solution at two different times,
\( t_0 < t_1 \), which guarantee that \( u \equiv 0 \) in \( \mathbb{R}^n \times [t_0, t_1] \).

One of our motivations comes from a well known result due to G. H. Hardy [12,
pp. 131], which concerns the decay of a function \( f \) and its Fourier transform,
\[ \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, dx. \]

If \( f(x) = O(e^{-|x|^2/\beta^2}) \), \( \hat{f}(\xi) = O(e^{-4|\xi|^2/\alpha^2}) \) and \( 1/\alpha\beta > 1/4 \), then \( f \equiv 0 \). Also,
if \( 1/\alpha\beta = 1/4 \), \( f \) is a constant multiple of \( e^{-|x|^2/\beta^2} \).

As far as we know, the only known proof of this result and its variants uses com-
plex analysis (the Phragmén-Lindelöf principle). There has also been considerable
interest in a better understanding of this result and on extensions of it to other
settings: [3], [7], [11], [1] and [2].

This result can be rewritten in terms of the free solution of the Schrödinger
equation in \( \mathbb{R}^n \times (0, +\infty) \), \( i\partial_t u + \triangle u = 0 \), with initial data \( f \),
\[ u(x,t) = (4\pi it)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4it}} f(y) \, dy = (2\pi it)^{-\frac{n}{2}} e^{-\frac{|y|^2}{4t}} e^{-\frac{|x|^2}{4t}} \hat{f} \left( \frac{x}{2t} \right), \]

in the following way:

If \( u(x,0) = O(e^{-|x|^2/\beta^2}) \), \( u(x,T) = O(e^{-|x|^2/\alpha^2}) \) and \( T/\alpha\beta > 1/4 \), then \( u \equiv 0 \).
Also, if \( T/\alpha\beta = 1/4 \), \( u \) has as initial data a constant multiple of \( e^{-(1/\beta^2+i/4T)|y|^2} \).

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The corresponding results in terms of $L^2$-norms, established in [11], are the following:

If $e^{\xi^2/\beta^2} f$, $e^{4|\xi|^2/\alpha^2} \hat{f}$ are in $L^2(\mathbb{R}^n)$ and $1/\alpha \beta \geq 1/4$, then $f \equiv 0$.

If $e^{|\xi|^2/\beta^2} u(x,0)$, $e^{\xi^2/\alpha^2} u(x,T)$ are in $L^2(\mathbb{R}^n)$ and $T/\alpha \beta \geq 1/4$, then $u \equiv 0$.

In [6] we proved a uniqueness result in this direction for bounded potentials $V$ verifying, $V(x,t) = V_1(x) + V_2(x,t)$, with $V_1$ real-valued and

$$\sup_{[0,T]} \|e^{T^2|\xi|^2/(\alpha t + \beta (T-t))^2} V_2(t)\|_{L^\infty(\mathbb{R}^n)} < +\infty$$

or

$$\lim_{R \to +\infty} \int_0^T \|V(t)\|_{L^\infty(\mathbb{R}^n \setminus B_R)} \, dt = 0.$$  

More precisely, we proved that the only solution $u$ to (1.1) in $C([0,T], L^2(\mathbb{R}^n))$, which verifies

$$\|e^{\xi^2/\beta^2} u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\xi^2/\alpha^2} u(T)\|_{L^2(\mathbb{R}^n)} < +\infty$$

is the zero solution, when $T/\alpha \beta > 1/2$ and $V$ verifies one of the above conditions.

This linear result was then applied to show that two regular solutions $u_1$ and $u_2$ of non-linear equations of the type

$$i \partial_t u + \Delta u = F(u, \pi), \text{ in } \mathbb{R}^n \times [0,T]$$

and for very general non-linearities $F$, must agree in $\mathbb{R}^n \times [0,T]$, when $u = u_1 - u_2$ satisfies (1.2). This replaced the assumption that the solutions coincide on large sub-domains of $\mathbb{R}^n$ at two different times, which was studied in [10] and [8], and showed that (weaker) variants of Hardy’s Theorem hold even in the context of non-linear Schrödinger evolutions.

The main results in this paper improve the results in [4], [6] and show that the optimal version of Hardy’s Uncertainty Principle in terms of $L^2$-norms, as established in [11], holds for solutions to (1.1), when $T/\alpha \beta > 1/4$ and for many general bounded potentials, while it fails for some complex-valued potentials in the end-point case, $T/\alpha \beta = 1/4$. As a by product of our argument, sharp improvements of Gaussian decay estimates are also obtained.

**Theorem 1.** Assume that $u$ in $C([0,T], L^2(\mathbb{R}^n))$ verifies

$$\partial_t u = i (\Delta u + V(x,t)u), \text{ in } \mathbb{R}^n \times [0,T],$$

$\alpha$ and $\beta$ are positive, $T/\alpha \beta > 1/4$, $\|e^{\xi^2/\beta^2} u(0)\|_{L^2(\mathbb{R}^n)}$ and $\|e^{\xi^2/\alpha^2} u(T)\|_{L^2(\mathbb{R}^n)}$ are both finite, the potential $V$ is bounded and either, $V(x,t) = V_1(x) + V_2(x,t)$, with $V_1$ real-valued and

$$\sup_{[0,T]} \|e^{T^2|\xi|^2/(\alpha t + \beta (T-t))^2} V_2(t)\|_{L^\infty(\mathbb{R}^n)} < +\infty$$

or

$$\lim_{R \to +\infty} \|V\|_{L^1([0,T], L^\infty(\mathbb{R}^n \setminus B_R))} = 0. \text{ Then, } u \equiv 0.$$

**Theorem 2.** Assume that $T/\alpha \beta = 1/4$. Then, there is a smooth complex-valued potential $V$ verifying

$$|V(x,t)| \lesssim \frac{1}{1 + |x|^2}, \text{ in } \mathbb{R}^n \times [0,T]$$
and a nonzero smooth function $u$ in $C^\infty([0,T], S(\mathbb{R}^n))$ such that
\[ \partial_t u = i(\Delta u + V(x,t)\bar{u}), \text{ in } \mathbb{R}^n \times [0,T], \]
and $\|e^{x^2/\beta^2}u(0)\|_{L^2(\mathbb{R}^n)}$ and $\|e^{x^2/\alpha^2}u(T)\|_{L^2(\mathbb{R}^n)}$ are both finite.

Our proof of Theorem 1 does not use any complex analysis, it only uses calculus! It provides the first proof (up to the end-point) that we know of Hardy's uncertainty principle for the Fourier transform, without the use of complex analysis.

As a by product, we derive the following optimal interior estimate for the Gaussian decay of solutions to (1.1).

**Theorem 3.** Assume that $u$ and $V$ verify the hypothesis in Theorem 1 and $T/\alpha\beta \leq 1/4$. Then,

\[ \sup_{[0,T]} \|e^{a(t)}x^2\|_{L^2(\mathbb{R}^n)} + \|\sqrt{t(T-t)}\|_{L^2(\mathbb{R}^n)} \leq \frac{\alpha\beta RT}{2(1 + R^2)}, \]

where $a(t) = \frac{\alpha\beta RT}{2(1 + R^2)}$, and $N$ depends on $T$, $\alpha$, $\beta$ and the conditions on the potential $V$ in Theorem 1.

Observe that $1/a(t)$ is convex and attains its minimum value in the interior of $[0,T]$, when $\|a - \beta\| < R^2 (\alpha + \beta)$.

To understand why Theorem 3 is optimal, observe that

\[ u(x,t) = R^{-\frac{n}{2}} (t - \frac{i}{\beta})^{-\frac{n}{2}} e^{-\frac{|x|^2}{4(t - \frac{i}{\beta})}} = (Rt - i)^{-\frac{n}{2}} e^{-\frac{(Rt - i)^{-\frac{n}{2}}}{4(Rt^2 + \frac{1}{\beta^2})}|x|^2}, \]

is a free wave (i.e. $V = 0$, in (1.1)) satisfying in $\mathbb{R}^n \times [-1,1]$ the corresponding time translated conditions in Theorem 3 with $T = 2$ and

\[ \frac{1}{\beta^2} = \frac{1}{\alpha^2} = \mu = \frac{R}{4(1 + R^2)} \leq \frac{1}{8}. \]

Moreover

\[ \frac{R}{4(1 + R^2t^2)}, \]

is increasing in the $R$-variable, when $0 < R \leq 1$ and $-1 \leq t \leq 1$. See also Lemma 5.

As a direct consequence of Theorem 1 we get the following application concerning the uniqueness of solutions for non-linear equations of the form (1.3).

**Theorem 4.** Let $u_1$ and $u_2$ be strong solutions in $C([0,T], H^k(\mathbb{R}^n))$ of the equation (1.3) with $k \in \mathbb{Z}^+$, $k > n/2$, $F : \mathbb{C}^2 \rightarrow \mathbb{C}$, $F \in C^k$ and $F(0) = \partial_u F(0) = \partial_t F(0) = 0$. If there are $\alpha$ and $\beta$ positive with $T/\alpha\beta > 1/4$ such that

\[ e^{x^2/\beta^2} (u_1(0) - u_2(0)) \text{ and } e^{x^2/\alpha^2} (u_1(t) - u_2(t)) \]

are in $L^2(\mathbb{R}^n)$, then $u_1 \equiv u_2$. 

Notice that the condition, $T/\alpha \beta > 1/4$, is independent of the size of the potential or the dimension and that we do not assume any decay of the gradient, neither of the solutions or of time-independent potentials or any regularity of the potentials.

Our improvement for the results in [4] and [6] comes from a better understanding of the solutions to (1.1), which have Gaussian decay. We started the study of this particular type of solutions in [5], where we considered free waves. The improvement of the latter results is a consequence of the possibility of extending the following outline of a strategy to prove Theorem 1 for free waves to the non-free wave cases:

First, by a suitable change of variables based on the conformal or Appell transform (See Lemma 5), it suffices to prove Theorem 1, when $u \in C([-1, 1], L^2(\mathbb{R}^n))$ is a solution of

$$
\partial_t u - i \Delta u = 0, \text{ in } \mathbb{R}^n \times [-1, 1]
$$

and

$$
\|e^{\mu |x|^2} u(-1)\|_{L^2(\mathbb{R}^n)} + \|e^{\mu |x|^2} u(1)\|_{L^2(\mathbb{R}^n)} < +\infty,
$$

for some $\mu > 0$. Our strategy consists of showing that either $u \equiv 0$ or there is a function $\theta_R : [-1, 1] \rightarrow [0, 1]$ such that

$$
\|e^{\mu |x|^2} \partial_t (\theta_{R} - i \Delta) e^{-\mu |x|^2} u(t)\|_{L^2(\mathbb{R}^n)} \leq \|e^{\mu |x|^2} u(-1)\|_{L^2(\mathbb{R}^n)}^{\theta_{R}(t)} \|e^{\mu |x|^2} u(1)\|_{L^2(\mathbb{R}^n)}^{1-\theta_{R}(t)},
$$

where $R$ is the smallest root of the equation

$$
\mu = \frac{R}{4(1 + R^2)}.
$$

Thus, we obtain the optimal improvement of the Gaussian decay of a free wave verifying (1.6) and we derive that $\mu \leq 1/8$, when $u$ is not zero.

The proof of these facts relies on new logarithmic convexity properties of free waves verifying (1.6) and on those already established in [6]. In [6, Theorem 3], the positivity of the space-time commutator of the symmetric and skew-symmetric parts of the operator,

$$
e^{\mu |x|^2} (\partial_t - i \Delta) e^{-\mu |x|^2},$$

is used to show that $\|e^{\mu |x|^2} u(t)\|_{L^2(\mathbb{R}^n)}$ is logarithmically convex in $[-1, 1]$. In particular, that

$$
\|e^{\mu |x|^2} u(t)\|_{L^2(\mathbb{R}^n)} \leq \|e^{\mu |x|^2} u(-1)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{\theta_R(t)}} \|e^{\mu |x|^2} u(1)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{1-\theta_R(t)}},
$$

when, $-1 \leq t \leq 1$.

Beginning from this fact we set, $a_1 \equiv \mu$, and we begin a constructive procedure, where at the $k$th step, we construct $k$ smooth even functions, $a_i : [-1, 1] \rightarrow (0, +\infty)$, $1 \leq i \leq k$, such that

$$
\mu \equiv a_1 < a_2 < \cdots < a_k, \text{ in } (-1, 1),
$$

$$
F(a_i) > 0, \text{ in } [-1, 1], a_i(1) = \mu, \text{ } i = 1, \ldots, k,
$$

where

$$
F(a) = \frac{1}{a} \left( \hat{a} - \frac{3a^2}{2a} + 32a^3 \right)
$$

and functions $\theta_i : [-1, 1] \rightarrow [0, 1]$, $1 \leq i \leq k$, such that

$$
\|e^{a_i(t) |x|^2} u(t)\|_{L^2(\mathbb{R}^n)} \leq \|e^{\mu |x|^2} u(-1)\|_{L^2(\mathbb{R}^n)}^{\theta_i(t)} \|e^{\mu |x|^2} u(1)\|_{L^2(\mathbb{R}^n)}^{1-\theta_i(t)}, \text{ } -1 \leq t \leq 1.
$$
These estimates are proved from the construction of the functions $a_i$, while the method strongly relies on the following formal convexity properties of free waves:

\begin{equation}
\partial_t \left( \frac{1}{a} \partial_t \log H_b \right) \geq - \frac{2b^2|\xi|^2}{F(a)},
\end{equation}

\begin{equation}
\partial_t \left( \frac{1}{a} \partial_t H \right) \geq \epsilon_a \int_{\mathbb{R}^n} e^{a|x|^2} \left( |\nabla u|^2 + |x|^2 |u|^2 \right) \, dx,
\end{equation}

where

\[ H_b(t) = \|e^{a(t)|x+b(t)|\xi|^2} u(t)\|_{L^2(\mathbb{R}^n)}^2, \quad H(t) = \|e^{a(t)|x|^2} u(t)\|_{L^2(\mathbb{R}^n)}^2, \]

$\xi \in \mathbb{R}^n$ and $a, b : [-1, 1] \rightarrow \mathbb{R}$ are smooth functions with $a > 0$, $F(a) > 0$, in $[-1, 1]$.

Once the $k$th step is completed, we take $a = a_k$ in (1.8) with a certain choice of $b = b_k$, verifying $b(-1) = b(1) = 0$ and then, a certain test is performed. When the answer to the test is positive, it follows that $u \equiv 0$. Otherwise, the logarithmic convexity associated to (1.8) allows us to find a new smooth function $a_{k+1}$ in $[-1, 1]$ with

\[ a_1 < a_2 < \cdots < a_k < a_{k+1}, \text{ in } (-1, 1), \]

and verifying the same properties as $a_1, \ldots, a_k$.

When the process is infinite, we have (1.7) for all $k \geq 1$ and there are two possibilities: either $\lim_{t \to +\infty} a_k(0) = +\infty$ or $\lim_{t \to +\infty} a_k(0) < +\infty$. The first case and (1.7) implies that $u \equiv 0$, while in the second, the sequence $a_k$ is shown to converge to an even function $a$ verifying

\[ \begin{cases} \ddot{a} - \frac{3a^2}{2a} + 32a^3 &= 0, \text{ in } [-1, 1], \\ a(1) &= \mu. \end{cases} \]

Because

\[ \frac{R}{4(1 + R^2)}, \quad R \in \mathbb{R}, \]

are all the possible even solutions of this equation, $a$ must be one of them and

\[ \mu = \frac{R}{4(1 + R^2)}, \]

for some $R > 0$. In particular, $u \equiv 0$, when $\mu > 1/8$.

The proof of Theorem 1 for non-zero potentials $V$ relies on extending the above convexity properties to the non-free case. The path that goes from the formal level to a rigorous one is not an easy one. In fact in [6, §6], we gave explicit examples of functions $a(t)$ such that $\log H$ is formally convex, when

\[ H(t) = \|e^{a(t)|x|^2} u(t)\|_{L^2(\mathbb{R}^n)}^2 \]

but for which, the corresponding inequalities lead to false statements: all free waves verifying (1.6) for some $\mu > 0$ are identically zero. Therefore most parts of this paper, as those in [6], are devoted to making rigorous the above formal arguments.
2. A Few Lemmas

In the sequel

\[(f, g) = \int_{\mathbb{R}^n} f \overline{g} \, dx , \quad \|f\|^2 = (f, f) .\]

The following formal identities or inequalities appeared or were proved within the proof of [6, Lemma 2].

**Lemma 1.** \(S\) is a symmetric operator, \(A\) is skew-symmetric, both are allowed to depend on the time variable, \(f(x,t)\) is a reasonable function,

\begin{equation}
H(t) = (f, f) , \quad \partial_t S = S_t , \quad D(t) = (S f, f) \quad \text{and} \quad N(t) = \frac{D(t)}{H(t)} .
\end{equation}

Then,

\begin{equation}
\dot{H}(t) = 2 \Re (\partial_t f - S f - A f, f) + 2 (S f, f) ,
\end{equation}

\begin{equation}
\ddot{H} \geq 2 \partial_t \Re (\partial_t f - S f - A f, f) + 2 (S_t f + [S, A] f, f) - \|\partial_t f - A f - S f\|^2 ,
\end{equation}

\begin{equation}
\dot{D}(t) \geq (S_t f + [S, A] f, f) - \frac{1}{2} \|\partial_t f - A f - S f\|^2
\end{equation}

and

\begin{equation}
\dot{N}(t) \geq (S_t f + [S, A] f, f) / H - \|\partial_t f - A f - S f\|^2 / (2H) .
\end{equation}

Lemma 2 shows how to find possible convexity or log-convexity properties of \(H(t)\) with respect to a new and possibly unknown variable \(s\), which is related to the original time variable \(t\) by the ordinary differential equation

\[
\frac{dt}{ds} = \gamma(t).
\]

**Lemma 2.** Assume that \(S\), \(A\) and \(f\) are as above, \(\epsilon > 0\), and \(\gamma : [c, d] \rightarrow (0, +\infty)\) and \(\psi : [c, d] \rightarrow [0, +\infty)\) are smooth functions satisfying

\begin{equation}
(\gamma S_t f(t) + \gamma [S, A] f(t) + \dot{\gamma} S f(t), f(t)) \geq -\psi(t) H(t) , \quad \text{when} \quad c \leq t \leq d.
\end{equation}

Then,

\begin{equation}
H(t) + \epsilon \leq (H(c) + \epsilon)^{\theta(t)} (H(d) + \epsilon)^{1-\theta(t)} e^{2T(t)+M_c(t)+2N_c} , \quad \text{when} \quad c \leq t \leq d,
\end{equation}

where \(T\) and \(M_c\) verify

\[
\partial_t (\gamma \partial_t T) = -\psi , \quad \text{in} \quad [c, d] , \quad T(c) = T(d) = 0 ,
\]

\[
\partial_t (\gamma \partial_t M_c) = -\gamma \|\partial_t f - S f - A f\|^2 / (H + \epsilon) , \quad \text{in} \quad [c, d] , \quad M_c(c) = M_c(d) = 0 ,
\]

\[
N_c = \int_c^d \left| \Re \left( \partial_s f(s) - S f(s) - A f(s), f(s) \right) / H(s) + \epsilon \right| ds
\]

and

\[
\theta(t) = \frac{\int_c^t ds}{\int_c^d ds}.
\]
Moreover,

\[
\partial_t (\gamma \partial_t H - \gamma \Re (\partial_t f - 8f - Af, f)) + \gamma \| \partial_t f - 8f - Af \|^2 \\
\geq 2 (\gamma \dot{S} f + \gamma [S, A] f + \dot{\gamma} S f, f).
\]

**Proof.** From (2.2)

\[
\partial_t \log (H + \epsilon) - 2 \Re \frac{(\partial_t f - 8f - Af, f)}{H + \epsilon} = \frac{2 (S f, f)}{H + \epsilon} = \frac{2D}{H + \epsilon},
\]

The differentiation of the second identity in (2.9) gives

\[
\partial_t^2 \log (H + \epsilon) - 2 \partial_t \Re \frac{(\partial_t f - 8f - Af, f)}{H + \epsilon} = \frac{2 \dot{D}}{(H + \epsilon)^2} + \frac{2H^2 \dot{N}}{(H + \epsilon)^2},
\]

with \(D\) and \(N\) as defined in (2.1), and from (2.4) and (2.5)

\[
\partial_t^2 \log (H + \epsilon) - 2 \partial_t \Re \frac{(\partial_t f - 8f - Af, f)}{H + \epsilon} \geq \frac{2 (\gamma \dot{S} f + \gamma [S, A] f + \dot{\gamma} S f, f)}{H + \epsilon}.
\]

Multiply the first identity in (2.9) by \(\dot{\gamma}\), (2.10) by \(\gamma\) and add up the corresponding identity and inequality to obtain the inequality

\[
\partial_t \left( \gamma \partial_t \log (H + \epsilon) - 2 \gamma \Re \frac{(\partial_t f - 8f - Af, f)}{H + \epsilon} \right) \geq \frac{2 (\gamma S f + \gamma [S, A] f + \dot{\gamma} S f, f)}{H + \epsilon} - \gamma \frac{\| \partial_t f - 8f - Af \|^2}{H + \epsilon}.
\]

This and (2.6) show that

\[
\partial_t \left( \gamma \partial_t \log (H + \epsilon) - 2 \gamma \Re \frac{(\partial_t f - 8f - Af, f)}{H + \epsilon} \right) \geq - 2 \dot{\psi} - \gamma \frac{\| \partial_t f - 8f - Af \|^2}{H + \epsilon}.
\]

Thus,

\[
\partial_t \left( \gamma \left( \partial_t \log (H + \epsilon) - 2 \partial_t T - \partial_t M \right) - 2 \Re \frac{(\partial_t f - 8f - Af, f)}{H + \epsilon} \right) \geq 0,
\]

in \([c, d]\). The monotonicity associated to this inequality shows that

\[
\frac{1}{\gamma (\tau)} \left[ \partial_s \log (H(s) + \epsilon) - 2 \partial_s T(s) - \partial_s M(s) \right] - \frac{2}{\gamma (\tau)} \Re \frac{\partial_s f(s) - 8f(s) - Af(s, f(s))}{H(s) + \epsilon}
\]
≤ \frac{1}{\gamma(s)}[\partial_\tau \log (H(\tau) + \epsilon) - 2\partial_\tau T(\tau) - \partial_\tau M_\epsilon(\tau)]
\left[ - \frac{2}{\gamma(s)} \operatorname{Re} \frac{\partial_\tau f(\tau) - Sf(\tau) - Af(\tau), f(\tau)}{H(\tau) + \epsilon},
\right]

when \( c \leq s \leq \tau \leq d \), and the integration of this inequality for \((s, \tau)\) in \([c, t]\times[t, d]\) and the boundary conditions satisfied by \(T\) and \(M_\epsilon\), imply that the "logarithmic convexity" type inequality (2.7) holds, when \( c \leq t \leq d \).

To derive (2.8) multiply the identity (2.2) by \( \dot{\gamma} \), the inequality (2.3) by \( \gamma \) and add up the corresponding identity and inequality. □

A calculation (See also formulae (2.12), (2.13) and (2.14) in [6]) shows that given smooth functions \( a : [c, d] \rightarrow (0, +\infty)\) and \( b : [c, d] \rightarrow \mathbb{R}^n \),

\[
e^{a(t)|x+b(t)|^2} (\partial_t - i\triangle) e^{-a(t)|x+b(t)|^2} = \partial_t - \mathbf{S} - \mathbf{A},
\]

where \( \mathbf{S} \) and \( \mathbf{A} \) are respectively symmetric and skew-symmetric operators on \( \mathbb{R}^n \), given by the formulae,

\[
\mathbf{S} = -2i (2a(x+b) \cdot \nabla + an) + \dot{a}|x+b|^2 + 2\dot{a}\dot{b} \cdot (x+b),
\]

\[
\mathbf{A} = i (\triangle + 4a^2|x+b|^2).
\]

Moreover,

\[
\mathbf{S}_t + \left[ \mathbf{S}, \mathbf{A} \right] = -8a\triangle - 2i \left( \frac{\dot{a}}{a} (x+b) + 2\dot{a}\dot{b} \right) \cdot \nabla + 2\dot{a}n)
\]

\[
+ \left( \dot{a} + 32a^3 \right)|x+b|^2 + \left( \frac{\dot{a}}{a} + \dot{\gamma} \right)^2 (x+b) + 2a|\dot{b}|^2.
\]

In Lemma 3 we calculate a lower bound for the self-adjoint operator \( \gamma \mathbf{S}_t + \gamma \left[ \mathbf{S}, \mathbf{A} \right] + \gamma \mathbf{S} \), when \( \gamma : [c, d] \rightarrow (0, +\infty) \) is a smooth function.

**Lemma 3.** Let \( a, \gamma \) and \( b \) be as above and assume that

\[
F(a, \gamma) = \gamma \left( \frac{\dot{a}}{a} + 32a^3 - \frac{3\dot{a}^2}{2a} - \frac{a}{2} \left( \frac{\dot{a}}{a} + \frac{\dot{\gamma}}{\gamma} \right)^2 \right) > 0, \text{ in } [c, d].
\]

Then, if \( \gamma \) denotes the identity operator,

\[
\gamma \mathbf{S}_t + \gamma \left[ \mathbf{S}, \mathbf{A} \right] + \gamma \mathbf{S} \geq -\frac{\gamma^2a^2|\dot{b}|^2}{F(a, \gamma)},
\]

for each time \( t \) in \([c, d]\).

**Proof.** From (2.11) , (2.12) and the identity

\[
(\gamma \mathbf{S}_t f + \gamma \left[ \mathbf{S}, \mathbf{A} \right] f + \gamma \mathbf{S} f, f) = \operatorname{Re}(\gamma \mathbf{S}_t f + \gamma \left[ \mathbf{S}, \mathbf{A} \right] f + \gamma \mathbf{S} f, f),
\]
we have

\[(\gamma S_t f + \gamma [S, A] f + \gamma S f, f) = \int_{\mathbb{R}^n} \left(32\gamma a^3 + \gamma \ddot{a} + \gamma \dot{a}\right) |x + b|^2 f^2 \, dx
\]

\[+ \int_{\mathbb{R}^n} \left(4\gamma \dot{a} \ddot{b} + 2\gamma \dddot{a} + 2\gamma \dot{b}\right) \cdot (x + b) + 2\gamma a |b|^2 |f|^2 \, dx
\]

\[+ \int_{\mathbb{R}^n} 8\gamma a - i\nabla f |f|^2 + 2\text{Re} \left( -i\nabla f \cdot \left(4\gamma \dot{a} \ddot{b}\right) \right)
\]

\[+ \int_{\mathbb{R}^n} 2\text{Re} \left( -i\nabla f \cdot ((2\gamma a + 4\gamma \dot{a})(x + b) f) \right) \, dx,
\]

when \(f\) is in \(S(\mathbb{R}^n)\). Completing the square corresponding to the first and second terms in the third line above, we get

\[(\gamma S_t f + \gamma [S, A] f + \gamma S f, f) = \int_{\mathbb{R}^n} \left(32\gamma a^3 + \gamma \ddot{a} + \gamma \dot{a}\right) |x + b|^2 f^2 \, dx
\]

\[+ \int_{\mathbb{R}^n} 2\gamma \dddot{a} \cdot (x + b) |f|^2 \, dx
\]

\[+ \int_{\mathbb{R}^n} 8\gamma a - i\nabla f + \frac{b}{2} f |f|^2 + 2\text{Re} \left( -i\nabla f + \frac{b}{2} f \right) \cdot ((2\gamma a + 4\gamma \dot{a})(x + b) f) \, dx.
\]

Rewriting \(-i\nabla f\) in the second term in the third line above as

\[\left( -i\nabla f + \frac{b}{2} f \right) - \frac{b}{2} f,
\]

gives the formula

\[(\gamma S_t f + \gamma [S, A] f + \gamma S f, f) = \int_{\mathbb{R}^n} \left(32\gamma a^3 + \gamma \ddot{a} + \gamma \dot{a}\right) |x + b|^2 f^2 \, dx
\]

\[+ \int_{\mathbb{R}^n} 2\gamma \dddot{a} \cdot (x + b) |f|^2 \, dx
\]

\[+ \int_{\mathbb{R}^n} 8\gamma a - i\nabla f + \frac{b}{2} f |f|^2 + 2\text{Re} \left( -i\nabla f + \frac{b}{2} f \right) \cdot ((2\gamma a + 4\gamma \dot{a})(x + b) f) \, dx.
\]

Next, we complete the square corresponding to the two terms in the third line above and find that

\[(\gamma S_t f + \gamma [S, A] f + \gamma S f, f) = \int_{\mathbb{R}^n} 8\gamma a - i\nabla f + \frac{b}{2} f + \left(\frac{\dot{a}}{a} + \frac{\dot{\gamma}}{\gamma}\right) (x + b) f |f|^2 \, dx
\]

\[+ \int_{\mathbb{R}^n} \left[ F(a, \gamma) |x + b|^2 + 2\gamma \dddot{a} \cdot (x + b) \right] |f|^2 \, dx,
\]

where

\[F(a, \gamma) = \ddot{a} + 32\gamma a^3 + \gamma \dot{a} - \frac{\gamma a}{2} \left(\frac{2\ddot{a}}{a} + \frac{\dot{\gamma}}{\gamma}\right)^2
\]

\[= \gamma \left(\ddot{a} + 32\gamma a^3 - 3\dddot{a} + \frac{\dddot{a}}{2} + \frac{\dot{\gamma}}{\gamma}\right)^2.
\]
Finally, we complete the square corresponding to the terms in the second line of (2.14) to obtain that

\begin{equation}
(\gamma \frac{\partial}{\partial t} f + \gamma [S,A] f + \gamma S f, f) = \int_{\mathbb{R}^n} 8\gamma a - i\nabla f + \frac{b}{2} f + \left(\frac{\partial}{\partial t} + \frac{\gamma a}{\sqrt{2}}\right) (x + b) f \, dx
\end{equation}

\begin{equation*}
+ \int_{\mathbb{R}^n} F(a,\gamma) [x + b + \frac{\alpha b}{F(a,\gamma)}] (e^2 f) \, dx - \frac{\gamma^2 a^2|\tilde{b}|^2}{F(a,\gamma)} \int_{\mathbb{R}^n} |f|^2,
\end{equation*}

when \( f \) is in \( S(\mathbb{R}^n) \) and \( c \leq t \leq d \), which proves Lemma 3. \( \Box \)

In the sequel we set

\[ F(a) = F(a, \frac{1}{a}) = \frac{1}{a} \left( \bar{a} + 32a^3 - \frac{3\bar{a}^2}{2a} \right). \]

The main Lemma used here in justifying the formal calculations is the following.

**Lemma 4.** Assume that \( u \) in \( C([-1, 1], L^2(\mathbb{R}^n)) \) verifies

\begin{equation}
\partial_t u = i (\triangle u + V(x,t) u), \text{ in } \mathbb{R}^n \times [-1, 1], \sup \|e^{\mu|\xi|^2} u(t)\| < +\infty,
\end{equation}

for some \( \mu > 0 \) and some bounded complex-valued potential \( V \). Also assume that \( a : [-1, 1] \rightarrow (0, +\infty) \) and \( b : [-1, 1] \rightarrow \mathbb{R}^n \) are smooth, \( b(-1) = b(1) = 0 \), \( a \) is even, \( \dot{a} \leq 0 \) in \( [0,1] \), \( a(1) = \mu \), \( a \geq \mu \) and \( F(a) > 0 \) in \( [-1, 1] \), and that

\begin{equation}
\sup_{[-1,1]} \|e^{\alpha(t) - \epsilon|\xi|^2} u(t)\| < +\infty,
\end{equation}

for all \( \epsilon > 0 \). Then,

\begin{equation}
\|e^{\alpha(t) + \beta(t)} u(t)\| \leq e^{T(t)} + 2\|V\|_\infty + 4\|V\|_{(\infty)}^2 \sup_{[-1,1]} \|e^{\alpha|\xi|^2} u(t)\|, \quad -1 \leq t \leq 1
\end{equation}

where \( T \) verifies

\begin{equation}
\begin{cases}
\partial_t \left( \frac{1}{a} \partial_T \right) = -\frac{|\tilde{b}|^2}{F(a)}, \text{ in } [-1, 1], \\
T(-1) = T(1) = 0.
\end{cases}
\end{equation}

Moreover, there is \( \eta_0 > 0 \), such that

\begin{equation}
\|\sqrt{1 - t^2} \nabla (e^{\alpha(t) + \beta(t)} |\xi|^2 u)\|_{L^2(\mathbb{R}^n \times [-1,1])} + \eta_0 \|\sqrt{1 - t^2} e^{\alpha(t)} |\xi|^2 u\|_{L^2(\mathbb{R}^n \times [-1,1])} \leq e^{2\|V\|_\infty + 4\|V\|_{(\infty)}^2} \sup_{[-1,1]} \|e^{\alpha|\xi|^2} u(t)\|.
\end{equation}

**Proof.** Extend \( u \) to \( \mathbb{R}^{n+1} \) as \( u \equiv 0 \), when \( |t| > 1 \). For \( \epsilon > 0 \), set

\[ a_\epsilon(t) = a(t) - \epsilon, \quad g_\epsilon(x,t) = e^{a_\epsilon(t)|\xi|^2} u(x,t), \quad f_\epsilon(x,t) = e^{a_\epsilon(t)} |x + b(t)|^2 u(x,t). \]

Then, \( f_\epsilon \) is in \( L^\infty([-1, 1], L^2(\mathbb{R}^n)) \) and verifies

\begin{equation}
\partial_t f_\epsilon - S_\epsilon f_\epsilon - A_\epsilon f_\epsilon = iV(x,t)f_\epsilon,
\end{equation}

in the sense of distributions in \( \mathbb{R}^n \times (-1, 1) \), with \( S_\epsilon, A_\epsilon \) defined by formulae (2.11) but with \( a \) replaced by \( a_\epsilon \):

\begin{equation}
\int f_\epsilon (\partial_\xi \bar{\xi} - S_\epsilon \xi + A_\epsilon \xi) \, dyds = i \int Vf_\epsilon \bar{\xi} \, dyds, \text{ for all } \xi \in C_0^\infty(\mathbb{R}^n \times (-1,1)).
\end{equation}
Let $\theta$ in $C^\infty(\mathbb{R}^{n+1})$ be a standard mollifier supported in the unit ball of $\mathbb{R}^{n+1}$ and for $0 < \delta \leq \frac{1}{4}$ set, $g_\epsilon, \theta = g_\epsilon \ast \theta_\delta$, $f_\epsilon, \theta = f_\epsilon \ast \theta_\delta$ and
\[
\theta_\delta^w(y, s) = \delta^{-n-1}\theta\left(\frac{y - s}{\delta}, \frac{t - \delta}{\delta}\right).
\]
Then, $f_\epsilon, \theta$ is in $C^\infty([-1, 1], \mathcal{S}(\mathbb{R}^n))$ and
\[
(2.23) \quad (\partial_t f_\epsilon, \theta - \delta \epsilon f_\epsilon, \theta - A_\epsilon f_\epsilon, \theta)(x, t) = \int f_\epsilon (-\partial_t \theta_\delta^w - \delta \epsilon \theta_\delta^w + A_\epsilon \theta_\delta^w)dyds
\]
\[
+ \int f_\epsilon \left([\hat{a}_\epsilon(s) + 4ia_\epsilon^2(s)] |y + b(s)|^2 - (\hat{a}_\epsilon(t) + 4ia_\epsilon^2(t)) |x + b(t)|^2\right) \theta_\delta^wdyds
\]
\[
+ \int f_\epsilon \left[2a_\epsilon(s) \hat{b}(s) \cdot (y + b(s)) - 2a_\epsilon(t) \hat{b}(t) \cdot (x + b(t))\right] \theta_\delta^wdyds
\]
\[
+ \int f_\epsilon \left[4i(a_\epsilon(s)(y + b(s)) - a_\epsilon(t)(x + b(t))) \cdot \nabla_y \theta_\delta^w\right]dyds
\]
\[
+ \int 2i\epsilon f_\epsilon(a_\epsilon(s) + a_\epsilon(t)) \theta_\delta^wdyds,
\]
when $x$ is in $\mathbb{R}^n$ and $-1 + \delta \leq t \leq 1 - \delta$. The last identity and $(2.22)$ give,
\[
(2.24) \quad (\partial_t f_\epsilon, \theta - S_\epsilon f_\epsilon, \theta - A_\epsilon f_\epsilon, \theta)(x, t) = i(Vf_\epsilon) \ast \theta_\delta(x, t) + A_\epsilon, \theta(x, t) + B_\epsilon, \theta(x, t),
\]
in $\mathbb{R}^n \times [-1 + \delta, 1 - \delta]$, where $A_\epsilon, \theta$ and $B_\epsilon, \theta$ denote respectively the sum of the second and third integrals and of the fourth and fifth in $(2.23)$. Moreover, there is $N_{a, b, \epsilon}$ such that
\[
|A_\epsilon, \theta(x, t)| \leq \delta^{-n}N_{a, b, \epsilon} \int_{t-\delta}^{t+\delta} \int_{B_{\delta}(x)} |e^{(a(s) - \delta)}|u|^2 |u(y, s)| dyds,
\]
\[
|B_\epsilon, \theta(x, t)| \leq \delta^{-1-n}N_{a, b, \epsilon} \int_{t-\delta}^{t+\delta} \int_{B_{\delta}(x)} |e^{(a(s) - \delta)}|u|^2 |u(y, s)| dyds,
\]
when $(x, t)$ is in $\mathbb{R}^n \times [-1 + \delta, 1 - \delta]$, which implies that
\[
(2.25) \quad \sup_{[-1+\delta, 1-\delta]} \|A_\epsilon, \theta(t)\|_{L^2(\mathbb{R}^n \times [-1+\delta, 1-\delta])} \leq \delta N_{a, b, \epsilon} \sup_{[-1, 1]} \|e^{(a(t) - \frac{\delta}{2})}|u(t)|\|
\]
\[
(2.26) \quad \|B_\epsilon, \theta\|_{L^2(\mathbb{R}^n \times [-1+\delta, 1-\delta])} \leq N_{a, b, \epsilon} \sup_{[-1, 1]} \|e^{(a(t) - \frac{\delta}{2})}|u(t)|\|
\]
We also have,
\[
(2.27) \quad \sup_{[-1, 1]} \|(Vf_\epsilon) \ast \theta_\delta(t)\| \leq \|V\|_{\infty} \sup_{[-1, 1]} \|e^{(a(t) - \frac{\delta}{2})}|u(t)|\|
\]
Clearly, $g_\epsilon, \theta$ also verifies $(2.24)$ with the corresponding $A_\epsilon, \theta$ and $B_\epsilon, \theta$ verifying $(2.25)$ and $(2.26)$. Just set $b \equiv 0$ in the definitions of $S_\epsilon, A_\epsilon$ and replace $f_\epsilon$ by $g_\epsilon$ in $A_\epsilon, \theta$ and $B_\epsilon, \theta$. We can also replace $f_\epsilon$ by $g_\epsilon$ in $(2.27)$.

From the hypothesis on $a$ and $b$, there is $\epsilon_a > 0$ such that
\[
(2.28) \quad F(a_\epsilon) \geq \frac{1}{2} F(a), \text{ in } [-1, 1], \text{ when } 0 < \epsilon \leq \epsilon_a
\]
Analogously, (2.33) \( \gamma \) and derive that

\[
| - 2 | \left| t \right| \leq 1.
\]

Moreover, the identity (2.15) in Lemma 3 with [c, d] = \([-1 + \delta, 1 - \delta]\), \( \gamma = \frac{1}{a^2} \), \( S = S_c \), \( A = A_c \), \( H_{c, \delta}(t) = \| g_{c, \delta}(t) \|^2 \) and we get

\[
\left( \frac{1}{a^2} \right) S \left( \begin{array}{c} \partial_t \left( \frac{1}{a^2} \partial_t H_{c, \delta} - \frac{1}{a^2} \text{Re} \left( \partial_t g_{c, \delta} - \bar{S}_c g_{c, \delta} - A_c g_{c, \delta} \right) \right) + \frac{1}{a^2} \| \partial_t g_{c, \delta} - \bar{S}_c g_{c, \delta} \right. \\
\left. + A_c g_{c, \delta} \| ^2 \\
\geq 2 \left( \frac{1}{a^2} S \partial_t g_{c, \delta} + \frac{1}{a^2} [S_c, A_c] g_{c, \delta} - \frac{a^2}{3 \eta} \bar{S}_c g_{c, \delta} \right) \right)
\]

Moreover, the identity (2.15) in Lemma 3 with \( \gamma = \frac{1}{a^2} \), \( b \equiv 0 \) and (2.28) show that

\[
\left( \frac{1}{a^2} S \partial_t f + \frac{1}{a^2} [S_c, A_c] f - \frac{a^2}{3 \eta} S_c f, f \right)
\]

\[
\geq \int_{\mathbb{R}^n} | - 2 | \left| x \right| \leq 1 \text{ } \text{dx}
\]

\[
\geq \int_{\mathbb{R}^n} \left| \nabla (e^{\frac{a^2}{4 \eta} |x|^2}) f \right| \text{dx} + \eta a \int_{\mathbb{R}^n} | \nabla f |^2 + | x |^2 \left| f \right|^2 \text{dx},
\]

when \( f \in S(\mathbb{R}^n) \). The multiplication of the inequality (2.30) by \((1 - \delta) - t^2\), integration by parts, (2.25), (2.26), (2.27) and the fact that \( V \) is bounded in \( L^\infty \) imply that

\[
\| \sqrt{(1 - \delta) - t^2} \nabla g_{c, \delta} \|_{L^2(\mathbb{R}^n \times [1 + \delta, 1 - \delta])} \leq \frac{N_{a,c}}{.}
\]

Analogously,

\[
\| \sqrt{(1 - \delta) - t^2} \nabla f_{c, \delta} \|_{L^2(\mathbb{R}^n \times [1 + \delta, 1 - \delta])} \leq \frac{N_{a,b,c}}{.}
\]

The latter makes it possible to write the error term \( B_{c, \delta} \) as

\[
\int [4i \left( a_c(t) (x + b(t)) - a_c(s) (y + b(s)) \right) \cdot \nabla_y f_c + 2i a_c (t - a_c(s)) f_c] \theta_{\delta, \delta}^p \text{d}y \text{d}s
\]

and derive that

\[
\| B_{c, \delta} \|_{L^2(\mathbb{R}^n \times [1 + \delta, 1 - \delta])} \leq \delta \frac{N_{a,b,c}}{.}, \text{ when } 0 < \delta \leq \delta_c.
\]

Recalling Lemma 3, apply now the estimate (2.7) on logarithmic convexity to \( f_{c, \delta} \), with \( H_{c, \delta}(t) = \| f_{c, \delta}(t) \|^2 \), \( [c, d] = [-1 + \delta, 1 - \delta] \), \( \gamma = \frac{1}{a^2} \), \( S = S_c \) and \( A = A_c \), and from (2.28) and (2.29), we get

\[
H_{c, \delta}(t) \leq \left( \sup_{[1, -1]} \| e^{\mu |x|^2} u(t) \| + \epsilon \right)^2 e^{2T_c(t) + M_{c, \delta}(t) + 2N_{c, \delta}}, \text{ when } |t| \leq 1 - \delta_c,
\]

where \( T_c \) and \( M_{c, \delta} \) verify

\[
\begin{align*}
\left\{ \partial_t \left( \frac{1}{a^2} \partial_t T_c \right) \right. & = - \frac{\delta^2}{F(\delta)} \text{, in } [-1 + \delta, 1 - \delta], \\
T_c(-1 + \delta_c) & = T_c(1 - \delta_c) = 0,
\end{align*}
\]
\[
\begin{align*}
\left\{ \partial_t \left( \frac{1}{a} \partial_a M_{e, \delta} \right) \right\} &= -\frac{1}{a e} \frac{\|\partial_a f_e - S_{e, \delta} f_e - A_{e, \delta} \|^2}{H_{e, \delta} + \epsilon}, \quad \text{in } [-1 + \delta_e, 1 - \delta_e], \\
M_{e, \delta}(-1 + \delta_e) &= M_{e, \delta}(1 - \delta_e) = 0,
\end{align*}
\]
and
\[
N_{e, \delta} = \int_{-1 + \delta_e}^{1 + \delta_e} \frac{\|\partial_a f_e(s) - S_{e, \delta}(s) - A_{e, \delta}(s)\|}{\sqrt{H_{e, \delta}(s) + \epsilon}} ds.
\]

The equation (2.21) verified by \(f_e\), (2.17), (2.33) and the formulae (2.11) show that \(f_e\) is in \(C((-1, 1), L^2(\mathbb{R}^n))\) and \(H_{e, \delta}\) converges uniformly on compact sets of \((-1, 1)\) to \(H_e(t) = \|f_e(t)\|^2\). From (2.25), (2.34) and letting first \(\delta\) tend to zero and then \(\epsilon\) tend to zero in (2.35), we get
\[
\|e^{a(t)|x+b(t)|^2} u(t)\|^2 \leq \sup_{[-1, 1]} \|e^{\mu|x|^2} u(t)\|^2 e^{2T(t) + M(t) + 4\|V\|_\infty}, \quad \text{when } |t| \leq 1,
\]
where \(T\) was defined in (2.19) and
\[
\begin{align*}
\left\{ \partial_t \left( \frac{1}{a} \partial_a M \right) \right\} &= -\frac{1}{a} \|V\|^2, \\
M(-1) &= M(1) = 0.
\end{align*}
\]
Because \(M\) is even,
\[
M(t) = \|V\|^2 \int_0^1 \int_0^s \frac{a(s)}{a(\tau)} d\tau ds, \quad \text{in } [0, 1],
\]
and the monotonicity of \(a\) in \([0, 1]\) implies that \(M(t) \leq \|V\|^2/2\), in \([-1, 1]\), which proves (2.18). The inequality
\[
\|e^{a(t)|x|^2} u(t)\| \leq e^{2\|V\|_\infty + \frac{3}{4} \|V\|^2} \sup_{[-1, 1]} \|e^{\mu|x|^2} u(t)\|, \quad -1 \leq t \leq 1
\]
and the reasoning leading to (2.32) can be repeated again, but now using (2.36), together with (2.25), (2.34) and (2.31), to show that
\[
\begin{align*}
\|\sqrt{(1 - \delta_e)^2 - t^2} \nabla (e^{\frac{\mu|x|^2}{2}} g_{e, \delta})\|_{L^2(\mathbb{R}^n \times [-1 + \delta_e, 1 - \delta_e])} \\
+ \eta_n \|\sqrt{(1 - \delta_e)^2 - t^2} \nabla g_{e, \delta}\|_{L^2(\mathbb{R}^n \times [-1 + \delta_e, 1 - \delta_e])} \\
+ \eta_n \|\sqrt{(1 - \delta_e)^2 - t^2} x g_{e, \delta}\|_{L^2(\mathbb{R}^n \times [-1 + \delta_e, 1 - \delta_e])} \\
\leq e^{2\|V\|_\infty + \frac{3}{4} \|V\|^2} \sup_{[-1, 1]} \|e^{\mu|x|^2} u(t)\| + \delta N_{n, e}.
\end{align*}
\]
Letting first \(\delta\) tend to zero, and then \(\epsilon\) tend to zero, we get (2.20), which proves Lemma 4.

\[\square\]

3. Proofs of Theorems 1, 2 and 3

We first recall the following Lemma proved in [6, Lemma 5]. It is useful to reduce the case of different Gaussian decays at two distinct times to the the case of the same Gaussian decay.
Lemma 5. Assume that $\alpha$ and $\beta$ are positive, $\gamma \in \mathbb{R}$ and that $u$ in $C([0, 1], L^2(\mathbb{R}^n))$ verifies
\[ \partial_t u = i(\triangle u + V(y, s)u) , \text{ in } \mathbb{R}^n \times [0, 1]. \]

Set
\[ \tilde{u}(x, t) = \left( \frac{\sqrt{\gamma}}{\alpha(1-t)+\beta} \right)^\frac{\alpha}{2} u \left( \frac{\sqrt{\gamma}}{\alpha(1-t)+\beta}, \frac{\beta t}{\alpha(1-t)+\beta} \right) e^{(\alpha-\beta)|x|^2/4i(\alpha(1-t)+\beta)}. \]
Then, $\tilde{u}$ is in $C([0, 1], L^2(\mathbb{R}^n))$ and verifies
\[ \partial_t \tilde{u} = i\left( \triangle \tilde{u} + \tilde{V}(x, t)\tilde{u} \right) , \text{ in } \mathbb{R}^n \times [0, 1], \]
with
\[ \tilde{V}(x, t) = \left( \frac{\sqrt{\gamma}}{\alpha(1-t)+\beta} \right)^\frac{\alpha}{2} V \left( \frac{\sqrt{\gamma}}{\alpha(1-t)+\beta}, \frac{\beta t}{\alpha(1-t)+\beta} \right). \]

Moreover,
\[
\|e^{\gamma|x|^2/2} \tilde{u}(t)\| = \|e^{\gamma|\alpha\beta|\beta^2/(\alpha^2 + \beta(1-t))^2} u(s)\|,
\]
when $s = \frac{\beta \alpha t}{\alpha(1-t)+\beta}$.

Proof of Theorems 1 and 3. Let $u$ satisfy the conditions in Theorem 1 and set $u_T(x, t) = T^{\frac{\alpha}{\beta}} u(\sqrt{T}x, Tt)$. We have,
\[ \partial_t u_T = i(\triangle u_T + V_T(x, t)) , \text{ in } \mathbb{R}^n \times [0, 1], \]
with $V_T(x, t) = TV(\sqrt{T}x, Tt)$ and
\[ \|e^{T|x|^2/\beta^2} u(0)\| + \|e^{T|x|^2/\alpha^2} u(T)\| = \|e^{T|x|^2/\beta^2} u_T(0)\| + \|e^{T|x|^2/\alpha^2} u_T(t)\|. \]

From [6, Theorem 3], when $V$ verifies the first condition or [6, Theorem 5], when $V$ verifies the second condition in Theorem 1, we know that
\[ \sup_{[0,T]} \|e^{T|x|^2/(\alpha+\beta(T-t))^2} u_T(t)\| = \sup_{[0,1]} \|e^{T|x|^2/(\alpha+\beta(1-t))^2} u_T(t)\| < +\infty. \]

In fact, there it is shown that
\[
(3.2) \sup_{[0,1]} \|e^{T|x|^2/(\alpha+\beta(1-t))^2} u_T(t)\| \leq N \left( \|e^{T|x|^2/\beta^2} u_T(0)\| + \|e^{T|x|^2/\alpha^2} u_T(T)\| \right),
\]
where $N$ depends on $\alpha$, $\beta$ and the conditions imposed on the potential $V$ in Theorem 1. From Lemma 5
\[ \tilde{u}(x, t) = \left( \frac{\sqrt{\gamma}}{\alpha(1-t)+\beta} \right)^\frac{\alpha}{2} u_T \left( \frac{\sqrt{\gamma}}{\alpha(1-t)+\beta}, \frac{\beta t}{\alpha(1-t)+\beta} \right) e^{(\alpha-\beta)|x|^2/4i(\alpha(1-t)+\beta)}, \]
verifies
\[ \partial_t \tilde{u} = i(\triangle \tilde{u} + \tilde{V}_T(x, t)\tilde{u}) , \text{ in } \mathbb{R}^n \times [0, 1], \]
with
\[ \tilde{V}_T(x, t) = \left( \frac{\sqrt{\gamma}}{\alpha(1-t)+\beta} \right)^\frac{\alpha}{2} V_T \left( \frac{\sqrt{\gamma}}{\alpha(1-t)+\beta}, \frac{\beta t}{\alpha(1-t)+\beta} \right), \]
and from (3.1),
\[ \sup_{[0,1]} \|e^{T|x|^2/\alpha\beta} \tilde{u}(t)\| = \sup_{[0,1]} \|e^{T|x|^2/(\alpha+\beta(1-t))^2} u_T(t)\| < +\infty. \]

Finally, $v(x, t) = 2^{-\frac{n}{2}} \tilde{u} \left( \frac{x}{\sqrt{T}}, \frac{1+t}{T} \right)$, verifies
\[ \partial_t v = i(\triangle v + V(x, t)v) , \text{ in } \mathbb{R}^n \times [-1, 1], \]
Observe that
\[
\dot{b} = 0 \quad \text{in} \quad [0, 1], \quad F(a_{j}) > 0, \quad \text{in} \quad [-1, 1], \quad a_{j}(1) = \mu,
\]
and
\[
\sup_{[-1,1]} \|e^{\alpha t|x|^2} u(t)\| \leq e^{2\|V\|_{\infty} + \frac{1}{2}\|V\|_{\infty}^2} \sup_{[-1,1]} \|e^{\mu t|\xi|^2} u(t)\|,
\]
where \(j = 1, \ldots, k\). The case \(k = 1\) follows from (3.3) and Lemma 4. Assume now, that \(a_{1}, \ldots, a_{k}\) have been constructed and set \(c_{j} = a_{j}^{-\frac{1}{2}}\). We have,
\[
F(a_{j}) = 2c_{j}^{-1} (16 c_{j}^{-3} - \tilde{c}_{j}), \quad \text{for} \quad j = 1, \ldots, k.
\]
Let \(b_{k} : [-1, 1] \to \mathbb{R}\) be the solution to
\[
\begin{cases}
\dot{b}_{k} = -2c_{k} \left(16 c_{k}^{-3} - \tilde{c}_{k}\right), & \text{in} \quad [-1, 1], \\
\text{and} \quad \dot{b}_{k} < 0 \quad \text{in} \quad (0, 1].
\end{cases}
\]
and \(\dot{b}_{k} < 0 \quad \text{in} \quad (0, 1].\) Apply now (2.18) in Lemma 4, with \(a = a_{k}\) and \(b = b_{k} \xi\), \(\xi \in \mathbb{R}^{n}\). We get
\[
\|e^{\alpha_{k}(t)|\xi|^2} u(t)\| \leq e^{T_{k}(t)|\xi|^2 + \frac{1}{2}\|V\|_{\infty}^2} \sup_{[-1,1]} \|e^{\mu t|\xi|^2} u(t)\|, \quad -1 \leq t \leq 1,
\]
with
\[
\begin{cases}
\partial_{t} \left(\frac{1}{a_{k}} \partial_{\tau} T_{k}\right) = -\frac{\dot{b}_{k}}{r_{(a_{k})}}, & \text{in} \quad [-1, 1], \\
T_{k}(-1) = T_{k}(1) = 0.
\end{cases}
\]
Because \(T_{k}\) is even, the monotonicity of \(a_{k}, (3.7)\) and (3.8), we get
\[
T_{k}(t) = 2 \int_{0}^{1} \int_{0}^{s} \frac{a_{k}(s)}{a_{k}(\tau)} c_{k}(\tau) \left(16 c_{k}^{-3}(\tau) - \tilde{c}_{k}(\tau)\right) d\tau ds \leq b_{k}(t),
\]
in $[-1,1]$, and from (3.9)
\begin{equation}
(3.10) \int_{\mathbb{R}^n} e^{2\alpha_k(t)|x|^2} - 2\xi|\xi|^2 b_k(t)(1-\alpha_k(t)b_k(t)) + 4\alpha_k(t)b_k(t)\xi \cdot u(t) \, dx \\
\leq e^{|V|+\frac{1}{2}|V|^2} \sup_{[-1,1]} \|e^{\alpha_k(t)}u(t)\|^2, \text{ when } -1 \leq t \leq 1.
\end{equation}

The latter implies that $u \equiv 0$, when $1 - \alpha_k(0)b_k(0) \leq 0$, a case in which the process stops. Otherwise, the monotonicity of $a_k$ and $b_k$ implies that $1 - \alpha_k b_k > 0$ in $[-1,1]$. Multiply then (3.10) by $e^{-2\alpha_k(t)|\xi|^2}$, $\epsilon > 0$, and integrate the corresponding inequality with respect to $\xi$ in $\mathbb{R}^n$. It gives,
\begin{equation}
(3.11) \sup_{[-1,1]} \|e^{\alpha_k(t)+\epsilon(|\xi|^2)} u(t)\| \leq \left(1 + \frac{1}{\epsilon}\right)^\frac{3}{2} e^{|\epsilon|+\frac{2}{3}|V|^2+2V}\sup_{[-1,1]} \|e^{\alpha_k(t)}u(t)\|
\end{equation}

with
\begin{equation}
a_{k+1}^\epsilon = \frac{(1 + \epsilon) a_k}{1 - \epsilon a_k b_k}.
\end{equation}
Set then
\begin{equation}
a_{k+1} = \frac{a_k}{1 - a_k b_k}, \quad c_{k+1} = a_{k+1}^{-\frac{3}{2}}.
\end{equation}
We get that $a_{k+1}$ is even, $a_{k+1}(1) = \mu$, $a_k < a_{k+1}$, in $(-1,1)$, $\dot{a}_{k+1} \leq 0$, in $[0,1]$ and $F(a_{k+1}) > 0$, in $[-1,1]$. To verify the latter, recall that
\begin{equation}
F(a_{k+1}) = 2a_{k+1}^{-1} \left(16 c_{k+1}^{-3} - \check{c}_{k+1}\right).
\end{equation}
From (3.12) and (3.7), $c_{k+1} = (c_k^2 - b_k)^\frac{3}{2}$ and
\begin{equation}
\check{c}_{k+1} = c_{k+1}^{-3} \left(16 - \frac{b_k^2}{4} + c_k \check{c}_k b_k - c_k^2 b_k - 16 c_k^{-2} b_k\right).
\end{equation}
Moreover, from (3.4) and (3.8), $\check{c}_k b_k \leq 0$ and $16 b_k c_k^{-2} + \check{c}_k^2 > 0$, in $[-1,1]$. Thus, $\check{c}_{k+1} < 16 c_{k+1}^{-3}$, in $[-1,1]$.

Also, (3.11) implies that
\begin{equation}
\sup_{[-1,1]} \|e^{(\alpha_{k+1}(t)-\epsilon)|\xi|^2} u(t)\| < +\infty, \text{ for all } \epsilon > 0,
\end{equation}
and Lemma 4 now shows that (3.4), (3.5) and (3.6) hold up to $j = k + 1$.

When the process is infinite, we have (3.5) and (3.6) for all $j \geq 1$ and there are two possibilities: either $\lim_{k \to +\infty} a_k(0) = +\infty$ or $\lim_{k \to +\infty} a_k(0) < +\infty$. The first case and (3.5) implies, $u \equiv 0$, while in the second, the sequence $a_k$ verifies,
\begin{equation}
\mu \equiv a_1 \leq a_2 \leq \ldots \leq a_k \leq \cdots \leq \lim_{k \to +\infty} a_k(0), \text{ in } [-1,1],
\end{equation}
and if $a(t) = \lim_{k \to +\infty} a_k(t)$, set $c = a^{-\frac{1}{2}}$. From (3.12),
\begin{equation}
b_k = \frac{a_{k+1} - a_k}{a_k a_{k+1}},
\end{equation}
$\{b_k\}$ is uniformly bounded and $\lim_{k \to +\infty} b_k(t) = 0$, in $[-1,1]$. From (3.4) and the evenness of $c_k$, we have, $\check{c}_k \leq 16 c_k^{-3}$, in $[-1,1]$ and
\begin{equation}
0 \leq \dot{c}_k(t) \leq 16 \int_0^t c_k^{-3}(s) \, ds, \text{ in } [0,1].
\end{equation}
Thus, \( \{\dot{c}_k\} \) is uniformly bounded in \([-1,1]\). From (3.8) and (3.7)

\[
\dot{b}_k(t) = -\int_0^t 32 c_k^{-2}(\tau) + 2 \dot{c}_k^2(\tau) \, d\tau + \dot{c}_k^2(t),
\]

\[
b_k(t) + \frac{1}{\mu} = \int_t^1 \int_0^s 32 c_k^{-2}(\tau) + 2 \dot{c}_k^2(\tau) \, d\tau \, ds + c_k^2(t),
\]

in \([0,1]\), and (3.15) shows that \( \{\dot{b}_k\} \) is uniformly bounded in \([-1,1]\), while the uniform boundedness of \( \{\ddot{c}_k\} \) follows from (3.13). Letting \( k \) tend to infinity in (3.16), we find that

\[
\frac{1}{\mu} = e^2(t) + \int_t^1 \int_0^s 32 c^{-2}(\tau) + 2 \dot{c}^2(\tau) \, d\tau \, ds, \text{ in } [0,1],
\]

which implies that

\[
\begin{cases}
\dot{c} = 16 e^{-3}, \text{ in } [-1,1], \\
\dot{c}(0) = 0, e(1) = \frac{1}{\sqrt{\mu}}.
\end{cases}
\]

In particular, \( a \) is even and

\[
\begin{cases}
\ddot{a} - \frac{32}{2\alpha} + 32a^3 = 0, \text{ in } [-1,1], \\
a(1) = \mu.
\end{cases}
\]

Because

\[
\frac{R}{4(1 + R^2 t^2)}, \quad R \in \mathbb{R},
\]

are all the possible even solutions of this equation, \( a \) must be one of them and

\[
\mu = \frac{R}{4(1 + R^2 t^2)},
\]

for some \( R > 0 \). In particular, \( u \equiv 0 \), when \( \mu > 1/8 \).

The bounds,

\[
\sup_{[-1,1]} \|e^{a(t)|x|^2}u(t)\| \leq e^{2\|V\|_\infty + \frac{1}{4} \|V\|_\infty^2} \sup_{[-1,1]} \|e^{a|t|^2}u(t)\|,
\]

\[
\|\sqrt{1 - t^2 \nabla(e^{a(t)\frac{32}{2\alpha}|x|^2})^2} u\|_{L^2(\mathbb{R}^n \times [-1,1])} + \eta \|\sqrt{1 - t^2 e^{a(t-\epsilon)|x|^2} \nabla u\|_{L^2(\mathbb{R}^n \times [-1,1])} \\
\leq e^{2\|V\|_\infty + \frac{1}{4} \|V\|_\infty^2} \sup_{[-1,1]} \|e^{a|t|^2}u(t)\|,
\]

follow from (3.6).

The application of the constructive process to the free wave \( u_R \) in (1.4) shows that \( \lim_{k \to +\infty} a_k(0) < +\infty \), when \( \mu < \frac{1}{8} \) and that the final limit \( a \) is determined by the smallest root of the equation (3.17). The same holds, when \( \mu = \frac{1}{8} \), as it follows by applying the process to the counterexample in the proof of Theorem 2.

Theorem 3 follows after undoing the changes of variables at the beginning of the proof and from the bounds in [6, Theorem 3], when \( V \) verifies the first condition or [6, Theorem 5], when \( V \) verifies the second condition in Theorem 1. The relation between the original \( u \) and \( v \) is the following:

\[
v(x,t) = \left(\frac{\sqrt{2\alpha^2 T}}{\alpha(1-t)+\beta(1+t)}\right)^\frac{3}{2} u \left(\frac{\sqrt{2\alpha^2 T} x}{\alpha(1-t)+\beta(1+t)}, \frac{\beta(1+t) T}{\alpha(1-t)+\beta(1+t)}\right) e^{(\alpha-\beta)|x|^2/4(\alpha(1-t)+\beta(1+t))}.
\]
Proof of Theorem 2. Set
\[ u(x, t) = (1 + it)^{-2k - \frac{n}{2}} \left( 1 + |x|^2 \right)^{-k} e^{-\frac{1}{4(1+it)} |x|^2}, \]
for some \( k > \frac{n}{2} \). Then,
\[ \|e^{\frac{|x|^2}{8}} u(\pm 1)\| < +\infty \quad \partial_t u = i(\triangle u + V(x, t) u), \]
in \( \mathbb{R}^{n+1} \), with
\[ V(x, t) = \frac{1}{1 + |x|^2} \left( \frac{2k}{1 + it} + 2kn - \frac{4k(1 + k)}{1 + |x|^2} \right), \]
and
\[ |V(x, t)| \lesssim \frac{1}{1 + |x|^2}, \]
in \( \mathbb{R}^n \times [-1, 1] \). What remains follows by modifying the above counterexample with the changes of variables in Lemma 5. \( \square \)

Remark 1. The above arguments show that the following also holds under the conditions in Theorem 3: given \( \epsilon > 0 \) there is \( \eta_\epsilon \) such that
\[ \eta_\epsilon \| \sqrt{t(T - t)} e^{(a(t) - \epsilon)|x|^2} \nabla u \|_{L^2(\mathbb{R}^n \times [0, T])} \]
\[ \leq N \left( \| e^{\frac{|x|^2}{\alpha^2}} u(0) \|_{L^2(\mathbb{R}^n)} + \| e^{\frac{|x|^2}{\alpha^2}} u(T) \|_{L^2(\mathbb{R}^n)} \right), \]
with \( a \) and \( N \) as in Theorem 3. We cannot make \( \epsilon = 0 \) in the exponent of (3.18) because at the end of the process \( F(a) \) is identically zero in \([-1, 1] \) and we loose the control of \( \| x e^{a(t)|x|^2} u(t) \| \) in (2.31).


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