Direct walk on hierarchic trees with continuous branching: a renormalization group approach

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We investigate the directed random walk on hierarchic trees. Two cases are investigated: random variables on deterministic trees with a continuous branching, and random variables on the trees constructed through the random branching process. We derive renormalization group (partial differential) equations for the branching models with binomial, Poisson and compound Poisson distributions of random variables on the links of tree. These renormalization group equations are new class of reaction-diffusion equations in 1-dimension.

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I. INTRODUCTION

Models on hierarchic trees are rather popular in statistical physics [1]. Hierarchic tree models with random quenched disorder [2-6] are especially popular object of research, they are connected with Random Energy Model. Due to special geometry, these models could be solved through recursive equations [1]. Considering random walks (direct polymers) on trees, the recursive equation for these models have been approximated with Kolmogorov-Petrovsky-Piscounov (KPP) equation.

The investigation of statistical physics models with quenched disorder on hierarchic trees with continuous branching are the subject of extensive studies. The models with continuous branching are especially popular in financial market literature [7],[8], as well as in turbulence. In [8] and in any level of hierarchy there are q links from every node, and on every link a random variable $\epsilon_i$ is defined. The variable $y_i$ entering the definition of the partition function Eq.(1) is associated with a unique path going from the origin O to the i-th tree endpoint and amounts to summing up variables $\epsilon_i$ along this path,

$$ y_i = \sum_{l} \epsilon_l $$. (2)

One can calculate mean free energy $\ln Z$ using the identity [4]

$$ < \ln Z > = \int_0^\infty \left( e^{-p} - \frac{< e^{-pZ} >}{p} \right) dp = \Gamma'(1) + \int_0^\infty \ln pd < e^{-tZ} > $$ (3)

Thus we need to calculate

$$ g(p, \beta) = G_K(x) = < \exp[-p \sum_i e^{-\beta y_i}] > |_{\epsilon_i}, \quad p = e^{-\beta x} $$ (4)

for the average $< \exp[-p \sum_i e^{-\beta y_i}] > |_{\epsilon_i}$ over the configurations on the K-level hierarchic tree.

Having the expression for $g(p, \beta)$, we can calculate the moments of $Z^{-n}$:

$$ < Z^{-n} > = \int_0^\infty dpp^n g(p, \beta) $$ (5)

In Ref. [2] an equivalent problem has been considered, and recursive relations have been introduced. Following [1], we define

$$ G_0(x) = \exp[\frac{-e^{-\beta x}}{\beta x}] $$ (6)

and then other $K$ functions $G_K(x) = < \exp[-e^{-\beta x} \sum_i e^{-\beta y_i}] > |_{\epsilon_i}$ for the model on the...
trees with $l$ level of hierarchy, $0 \leq l \leq K$. We identify $g(p, \beta) \equiv G_K(-\ln p/\beta)$.

Consider the $l$-level hierarchic tree. It could be fractured into $q$ trees with hierarchy level $l-1$ each. The $y_i$ at the endpoints of $l$-level tree can be derived from the $y_i$ of the $(l-1)$ level tree adding a random variable $\epsilon$. A simple consideration gives a recursive relation [2, 5]:

$$G_l(x) = \int dc \ G_{l-1}(x+\epsilon)\rho(\epsilon)^q$$  

(7)

Considering recursive relations for $1 \leq l \leq K$ with the boundary condition by Eq. (6), we can calculate $G_K$.

Eq. (7), derived in [2, 5] is an exact recursive relation. Similar recursive equations have been considered in the research of real space renormalization approach to quantum disorder in d-dimensional space, see Eq.(32) in [20]. The latter is some approximation, while Eq.(7) is an exact relation. The more serious difference- our random variables $\epsilon$ have independent distributions for different hierarchy levels $l$, while at quantum disorder point [20] there is some correlation of noise at different hierarchy levels.

We take $\hat{\rho}(x) \equiv \rho(1, x)$. A concrete case of the latter distribution with $\ln \cosh(h)$ has been used while considering the diluted REM [19]. Here we will consider another distributions, popular in turbulence and financial theory.

A. Renormalization-group equation for Hierarchic trees with continuous branching

While Eq.(7) is derived for integer $q$, we can consider the equation for any positive value of $q$ as well. Let us consider the model, where the branching number is close to 1. We can now derive exact differential equation instead of iteration equation Eq. (7). We have $K = 1/\Delta v$ levels of branching on our tree with $q = e^{L\Delta v}$, where $L\Delta v \ll 1$. We identify

$$G_l(x) \equiv G(x, v),$$

$$v = \frac{1}{K},$$

$$g(p, \beta) = G(-\ln p/\beta)$$  

(8)

To define a distribution for random variable $\epsilon_l$, let us start with some random distribution $\hat{\rho}(x)$. In the next step we calculate the distribution for the sum of $L$ random variables with such distributions. For our purposes it is convenient to use the following representation for $\hat{\rho}$:

$$\hat{\rho}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dh \ \exp[\phi(ih) - ihx]$$  

(9)

Then for the $\epsilon$, a sum of $L$ random variables $x$, we get the compose distribution $\rho(L, \epsilon)$ just multiplying $\phi$ in the exponent of Eq. (9) by $L$:

$$\rho(L, \epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dh \ \exp[L\phi(ih) - ihe]$$  

(10)

To ensure the probability balance condition, there is a constraint

$$\phi(0) = 0$$  

(11)

We can consider the Taylor expansion

$$\phi(ik) = \sum_{l \geq 1} b_l(ik)^l$$  

(12)

According to our notations $\hat{\rho}(x) \equiv \rho(1, x)$. A concrete case of the latter distribution with $\ln \cosh(h)$ has been used while considering the diluted REM [19]. Here we will consider another distributions, popular in turbulence and financial theory.

We take $\hat{\rho}(L, y_i)$ as a distribution of $y_i$. The $L$ in the exponent of Eq.(10) gives correct scaling for the $< e^{\beta y_i} >$, where $e^L$ is the number of end-points of hierarchic tree.

As $y_i$ is a sum of $1/\Delta v$ random variables $\epsilon$ on our tree, we define the following distribution for the distribution of random variables on the links $\rho(\epsilon)$:

$$\rho(\epsilon) \equiv \hat{\rho}(L \Delta v, \epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dh \ \exp[\phi(ih)L \Delta v - ihe]$$  

(14)

For the $L\Delta v \ll 1$ we expand in the exponent the terms with $b_1, l > 1$, and derive:

$$\int dG(x + \epsilon, v)\rho(\epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dh \ \exp[b_1 L\Delta v - ihe]$$

$$\times (1 + L\Delta v \sum_i b_i(1+b_i^2))G(x + \epsilon, v) = \int dc \ [\delta(\epsilon - b_1 L\Delta v)$$

$$+ L\Delta v \sum_{l \geq 2} b_l(1+b_l^2)^l \frac{1}{2\pi} \int_{-\infty}^{\infty} dh \ \exp(-ihe)]G(x + \epsilon, v)$$  

(15)

We assume a smooth behavior of $G(x, v)$ at the infinity $x \to \pm \infty$, and $\partial^n G(x, v) |_{x = \pm \infty} = 0$ for $n \geq 1$.

Integrating by parts we derive:

$$\int dG(x + \epsilon, v)\rho(\epsilon) = G(x, v) + L\Delta v \phi(\frac{\partial}{\partial x})G(x, v)$$  

(16)

Thus we obtain:

$$G(x, v) \to \left[ (G + L\Delta v \phi(\partial G(x, v)) \right] ^{1+L\Delta v} G(x, v) + \Delta v L(\phi(\partial G(x, v) + G(x, v) \ln G(x, v))$$  

(17)
or
\[
\frac{\partial G(x,v)}{\partial v} = \phi(\partial x)G(x,v) + G(x,v)\ln G(x,v)
\] (18)

where \(0 \leq v \leq 1\) plays the role of time of reaction-diffusion equation, and we should solve the equation with the initial distribution
\[
G(x,0) = \exp[-e^{-\beta x}]
\] (19)

Due to proper choose of the scaling in the exponent of Eq.(14), there is no \(L\) dependence in Eq.(18).

Contrary to the KPP equation in the case of models of Ref. [2], Eq. (18) is an exact equation.

Solving Eq.(18) for the initial distribution by Eq.(19), we can calculate \(\rho\) of Ref. [2], Eq. (18) is an exact equation.

Due to proper choose of the scaling in the exponent of diffusion equation, and we should solve the equation with the initial distribution
\[
G(x,0) = \exp[-e^{-\beta x}]
\] (19)

We get for \(\phi(k)\): \[
\phi(k) = \gamma \ln(1 + k/\gamma)
\] (26)

While we can formulate formally a PDE in this case,
\[
\frac{\partial G(x,v)}{\partial v} = \gamma \ln[1 + \frac{1}{\gamma} \frac{\partial}{\partial x}] G(x,v) + G \ln G,
\] (27)

the equation can be better formulated in Fourier space.

**The Poisson distribution.** It is popular in financial mathematics literature. We have integer values of \(\epsilon\)
\[
\hat{\rho}(\epsilon) = e^{-\gamma \epsilon^\gamma}/\epsilon!
\] (28)

Eq. (9) gives \(\phi(k) = \gamma(-1 + e^k)\). Then from Eq. (18) we derive:
\[
\frac{\partial G(x,v)}{\partial v} = \gamma[\gamma + G(x+1)] + G \ln G
\] (29)

This renormalization group equation differs from Eq. (21) in that the second order derivative is replaced by finite difference.

**Compound Poisson distribution.** The cascade processes with such distribution are also rather popular in literature.

Now we have Poisson distribution for integers \(n\) given by Eq.(28) i.e. \(p_1(n) = e^{-\gamma n}/n!\) and define \(\epsilon\) as a sum of \(n\) random variables \(x_i\) with some random distribution \(p(x)\), defined through the representation:
\[
p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dh \exp[\alpha ih - i\epsilon]
\] (30)

Thus
\[
\alpha(k) = \ln \int_{-\infty}^{\infty} p(y)dy \exp[ky]
\] (31)

A simple calculation gives the function \(\phi(\epsilon)\) for the distribution of \(\epsilon\)
\[
\phi(\epsilon) = \gamma(e^{\alpha(k)} - 1)
\] (32)

Thus we get the following PDE for \(G(x,v)\):
\[
\frac{\partial G(x,v)}{\partial v} = \gamma[e^{\alpha(k)} - 1]G(x,v) + G \ln G
\] (33)

or, using Eq.(31),
\[
\frac{\partial G(x,v)}{\partial v} = \gamma[\int dy p(y)G(x+y,v) - G(x,v)] + G \ln G
\] (34)
C. The phase transition point

General case of distribution. The free energy of directed polymer model Eq.(7) for any q is equivalent to the free energy of corresponding REM with the same number of energy configurations and distribution of i. This issue is well discussed in REM literature [2, 3, 5, 6].

We have $M = e^L$ endpoints of the tree, and any of $y_i$ has a distribution by $\rho(L, y)$, see Eq. (10). Let us consider REM with $M$ energy level with independent distribution of energy by Eq.(10).

At high temperature phase, assuming $< \ln Z > = \ln < Z >$, we have [3],

$$< \ln Z > = L(1 + \phi(\beta))$$

(35)

The phase transition is at the point where the entropy of free energy by Eq.(35) disappears [3],

$$1 + \phi(\beta_c) - \beta_c \phi'(\beta_c) = 0$$

(36)

Binomial distribution. Now we have the following expression for the free energy in the high temperature phase

$$\ln \frac{Z}{L} = \ln \left[ \frac{\alpha_2}{\alpha_1 + \alpha_2} e^{\beta(1-\alpha_1)} + \frac{\alpha_1}{\alpha_1 + \alpha_2} e^{\beta(1+\alpha_2)} \right]$$

(37)

Eq.(36) has no solution in this case, therefore system is always in the high temperature phase.

Gamma distribution. Now we have the following expression for the free energy in the high temperature phase

$$\ln \frac{Z}{L} = 1 + \gamma(1 + \frac{\beta}{\gamma})$$

(38)

Eq.(36) has no solution in this case, therefore system is always in the high temperature phase.

The Poisson distribution. For this case we derive for the free energy of high temperature phase and transition point:

$$\ln \frac{Z}{L} = 1 + \gamma(e^\beta - 1);$$

$$1 + \gamma[e^{\beta_c}(1 - \beta_c) - 1] = 0$$

(39)

In the next sections we will use the travelling wave analysis to investigate the paramagnetic phase solution of our equations and identify the transition point. For investigation of the spin-glass like solutions one needs in advanced mathematics of [17].

Compound Poisson distribution. Now we have for the high temperature phase the following expression for the free energy

$$\ln \frac{Z}{L} = 1 + \gamma[e^{\alpha(\beta)} - 1] =$$

$$1 + \gamma(\int dx p(x)e^{\beta x} - 1)$$

(40)

For the critical point we get an equation:

$$1 + \gamma[\int dx p(x)e^{\beta x} - 1 - \beta \int dx p(x)e^{\beta x}] = 0$$

(41)

D. KPP versus new renormalization group equation

Let us try to describe our hierarchical tree model using KPP equation

$$\frac{\partial Q(x, v)}{\partial v} = \frac{\partial^2 Q(x, v)}{\partial x^2} + Q - Q^2$$

(42)

In [2] has been derived Eq.(42) as an approximation of the Eq.(7) for the special case of Eq.(20). Later Eq.(42) has been derived in [14] as an approximate renormalization group equation for the models of random disorder with logarithmic correlations. While in [2, 3, 5] has already been considered the general distribution, the KPP has been derived only for the normal distribution. In [20], while using general distribution of random variables, the authors does not use KPP. The reason is clear: for the general distribution of random variables on the tree, the KPP equation gives wrong results even for transition point, contrary our new renormalization group equation which is exact.

To compare we take the case by Eqs.(28) with $\gamma = 1$. The KPP version, having the same variance of distribution $\rho(\epsilon)$, gives

$$\frac{\partial G(x, v)}{\partial v} = \frac{\partial G(x, v)}{\partial x} + \frac{\partial^2 G(x, v)}{\partial x^2} + G - G^2$$

(43)

Using the mapping:

$$G(v, x) = Q(v, x + t)$$

(44)

we return to the KPP equation by Eq.(42). This equation is well investigated in literature. It describes a transition between spin-glass and paramagnetic phases at

$$\beta_c = \sqrt{2}$$

(45)

and free energy at paramagnetic phase is

$$< \ln Z > = L(1 + \beta + \beta^2)$$

(46)

The correct transition point of the model and free energy are:

$$\beta_c = 1$$

$$< \ln Z > = Le^\beta$$

(47)

Thus the KPP wrongly describes the model, and we need in new exact renormalization group equation for the general form of distribution of random variables on the tree.

Let us investigate our Eq.(29). $G(v, x)$ is a monotonic function of $x$ according to the definition by Eqs.(4),(7),

$$G(v, x) \to 1, \quad for \quad x \to \infty,$$

$$G(v, x) \to 0, \quad for \quad x \to -\infty$$

(48)
Consider the case of small $p$ in Eqs.(4),(7). The direct calculations give

$$G(v, x) = \exp[-p \exp \beta x + (v(1 + \gamma(e^\beta - 1))],$$

(49)

The latter expression is not a solution of Eq.(43). Nevertheless, it is the solution of Eq.(29) when

$$p \exp[\beta x + v(1 + \gamma(e^\beta - 1))] \ll 1$$

(50)

Let us investigate the behavior of the asymptotic solution for the more general situation than the case of Eq.(50).

Following to [2, 5, 14], we assume a traveling wave like asymptotic solution

$$G(v, x) = g(x + cv)$$

(51)

We get ODE:

$$cg'(x) = [-g(x) + g(x + 1)] + g(x) \ln g(x)$$

(52)

Eq.(52) describes a traveling wave with a front at

$$x = -cv$$

(53)

We assume that for large negative $x$ there is a solution

$$g(x) = \exp[-e^{\beta x}]$$

(54)

Putting the latter expression to Eq.(52), we get the following equation for $c$:

$$c(\beta) = \frac{[e^{\beta} - 1] \gamma + 1}{\beta}$$

(55)

Eqs.(51),(54),(55) give the free energy expression by Eq.(46).

We can identify the phase transition point as the value of $\beta$ giving the maximum velocity, looking the equation:

$$c'(\beta_c) = 0$$

(56)

Eq.(56) gives the result of Eq.(47), $\beta_c = 1$ for the case $\gamma = 1$.

At high value of $\beta$ there is another solution for $c(\beta)$, but for its derivation one needs in advanced mathematical approach of [17].

Thus the investigation of our renormalization group equation, using the qualitative approach of [2, 5, 14], gives exact expression for the free energy.

### III. RANDOM BRANCHING MODEL

In the previous section we derived new renormalization group equation, using continuous deterministic branching, later considering $q \to 1$ limit. Let us derive a similar renormalization group equation, using less abstract model of random branching [2].

We again have a tree. It starts at some point $O$, and grows down, where the vertical coordinate measures the time $v$. We put random variable $\epsilon_l$ with distribution by Eq.(8) on the links (between two adjacent nodes of the tree), replacing $\Delta v$ by the difference of time coordinate between two nodes of the branch. During period of time $dv$ there is a branching with a probability $dt$.

Consider the dynamics of the partition sum $Z(\beta, v)$ defined as

$$Z(\beta, v) = \sum_j e^{\beta y_j},$$

(57)

Here the index $j$ numerates the endpoints of our tree at the time $v$.

$Z(\beta, v)$ has a deterministic behavior at $v = 0$:

$$Z(\beta, 0) = 1$$

(58)

$Z(\beta, v)$ is a random variable at $v > 0$ both due to the randomness of branching and randomness of $\epsilon_l$.

Then one has the following recursive equation for the random partition function [2],

$$Z(\beta, v + dv) = Z(\beta, v)e^{-\beta \epsilon}$$

(59)

with probability $1 - \Delta v$, and

$$Z(\beta, v + dv) = (Z^1(\beta, v) + Z^2(\beta, v))e^{-\beta \epsilon}$$

(60)

with probability $\Delta v$. Here $Z^1(\beta, v)$ and $Z^2(\beta, v)$ are random independent variables with the same probability distribution us $Z(\beta, v)$.

We identify $Z(0, v)$ with the number of endpoints of the tree, and Eqs.(59)-(60) give the intuitive result

$$< Z(0, v) >= \exp(t),$$

(61)

confirming the self-consistence of the choice Eqs.(59)-(61). Actually the random process $Z(t)$ is defined completely trough Eqs.(58)-(60), and we can work with these equations missing any reference to the branching trees.

We define now, following [2]

$$G(v, x) = < \exp[-e^{-\beta x}Z(v)] >$$

(62)

Then we have an equation

$$G(v + dv, x) = (1 - dv) \int d\rho(e)G(v, x + e) +$$

$$dvG(v, x)^2$$

(63)

Eventually we get the following equation:

$$\frac{\partial G(x, v)}{\partial v} = \phi(\partial x)G(x, v) - G(x, v)(1 - G(x, v))$$

(64)

For the binomial distribution we get:

$$\frac{\partial G(x, v)}{\partial v} = -G(1 - G) + (1 - \alpha_1) \frac{\partial G(x, v)}{\partial x} +$$

$$\ln\left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right) + \sum_n \left(\frac{(-1)^n}{n!} \frac{\alpha_2}{\alpha_1 + \alpha_2} G(x, n(\alpha_1 + \alpha_2)]\right)$$

(65)
For the Poisson distribution we get
\[ \frac{\partial G(x,v)}{\partial v} = \gamma [\partial G + G(x+1)] + G(x,v)^2 - G(x,v) \]

We can repeat the derivations of the section II-D, and again get the same solutions for the free energy of a paramagnetic phase and the critical temperature.

For the compound Poisson distribution we get:
\[ \frac{\partial G(x,v)}{\partial v} = \gamma \int dy p(y) G(x+y,v) - G(x,v)] + G(G-1) \]

IV. CONCLUSIONS

We provided an exact (renormalization group) partial differential equation, for the analysis of models on hierarchic tree with continuous branching and general distribution of random variables on the tree. We derived new classes of reaction-diffusion equation Eq. (18), which sometimes has finite difference Eqs. (24), (29), (34). While recursive equations for the hierarchical models with general distribution of random variables on the tree have been well known [2, 20], our article is the first result to replace these recursive equation with correct PDE.

Our derivation of renormalization-group equation is quite rigorous. The renormalization group equation is exact, including finite size corrections, while the renormalization group equation in [14] is an approximation and could be applied to calculate only bulk characteristics of corresponding models.

The KPP equation, used in [2, 5, 14], correctly (exact in the thermodynamic limit) describes the bulk free energy and critical temperature for the hierarchic tree models with normal distribution of random variable [2, 5], as well as for the finite dimensional models of disorder with logarithmic correlations [14]. As we checked, KPP equation failed to describe correctly the hierarchic models with non-normal distributions. That’s why the general case of hierarchical model, given by recursive equation (7) derived in [2], never has been investigated using KPP in [2, 3]. To verify our new renormalization group equations, we used the idea of traveling wave and methods of [3, 14]. Analyzing our equations, we found correct expression for the free energy in paramagnetic phase and the phase transition temperature.

An open problem is to construct the finite dimensional models of disorder, which could be described by our new equations, as the models of [14] are described by KPP. We hope to succeed using dynamic stochastic processes in 1 dimensional case.

Our approach (continuous branching) could be applied to investigate the quantum disorder in d-dimensional space. In case of continuous branching we derived exact PDE equations which could be solved numerically, while in alternative approach of [20] has been derived approximate renormalization group equations and there are errors O(1) in the expression of free energy (bulk term \( \sim L \)).

The normal distribution version Eq. (21) has a deep mathematical meaning: in some sense it describes the p-adic space with \( p = 1 \) [18]. We hope that the investigation of these new reaction-diffusion equations will be further encouraged.

Our renormalization group equations could also be applied to calculate also the complicated correlation function, as has been done in Ref. [11] for the string case. This work was supported by Academia Sinica, National Science Council in Taiwan with Grant Number NSC 99-2911-I-001-006, and National Center for Theoretical Sciences (Taipei Branch).