Stress tensor and current correlators of interacting conformal field theories in 2+1 dimensions: Fermionic Dirac matter coupled to $U(1)$ gauge field

Yejin Huh$^{1,2,*}$ and Philipp Strack$^{1,3,†}$

$^1$Department of Physics, Harvard University, Cambridge MA 02138
$^2$Department of Physics, University of Toronto, Ontario M5S 1A7, Canada and
$^3$Institut für Theoretische Physik, Universität zu Köln, D-50937 Cologne, Germany

(Dated: October 9, 2014)

Abstract

We compute the central charge $C_T$ and universal conductivity $C_J$ of $N_F$ fermions coupled to a $U(1)$ gauge field up to next-to-leading order in the $1/N_F$ expansion. We discuss implications of these precision computations as a diagnostic for response and entanglement properties of interacting conformal field theories for strongly correlated condensed matter phases and conformal quantum electrodynamics in 2 + 1 dimensions.
I. INTRODUCTION

A variety of strongly correlated electron systems at quantum critical points or phases in two spatial dimensions are believed to be described by (interacting) conformal field theories in 2+1 dimensions (CFT$_3$’s). The workhorse is the Wilson-Fisher CFT$_3$, also known as the $O(N)$-model of a real-valued vector field with $N$ components [1–3], which describes, among other things, the Ising model for $N = 1$ [4, 5], superfluid-to-insulator transitions for $N = 2$ [6, 7], and quantum magnetic transitions for $N = 3$ [8, 9]. Especially intriguing are gauge theoretical descriptions of condensed matter systems (e.g.: [10] and references therein for an overview) such as of quantum Hall systems (e.g.: [11, 12] and references therein), fractionalized magnets and deconfined critical
points in strongly correlated Mott insulators [13–15], and effective theories for the cuprates [16–19]. There, the relevant dynamics is often provided by emergent or effective degrees of freedom not necessarily present in the bare Hamiltonian. These conformal phases of quantum matter in 2+1 dimensions provide a unique interpolation between the better understood CFT’s in 1+1 dimensions [20] and much studied gauge theories for high energy vacua in 3+1 dimensions [21, 22].

A common feature of CFT’s is the absence of quasi-particles and for condensed matter systems it is of particular interest to understand response properties of interacting CFT’s to externally applied perturbations such as electromagnetic fields or mechanical forces without invoking a quasi-particle picture.

A. Model: $N_F$ Dirac fermions coupled to $U(1)$ gauge field

In this paper, we consider $N_F$ Dirac fermions minimally coupled to a $U(1)$ gauge field. This theory arises in a variety of condensed matter contexts [10, 12, 16, 17, 19]. The Euclidean action,

$$S = \int d^2 r d\tau \bar{\psi}_\alpha \left[ i \gamma^\mu \left( \partial_\mu - i \frac{A_\mu}{\sqrt{N_F}} \right) \psi_\alpha \right],$$  

contains Grassmannian two-component fermion fields $\bar{\psi}_\alpha$ and $\psi_\alpha$, where $\alpha$ is the fermion flavor index, and $\mu$ is the spatial and (imaginary) temporal index in 2+1 dimensions. Repeated indices are summed over. $\gamma^\mu$’s are the Dirac matrices that satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$. We use the same conventions as Kaul and Sachdev for their fermion sector [10].

The gauge field $A_\mu$, a conventional spin-1 boson often dubbed as “emergent photon” in the condensed matter context, ensures fulfillment of a local $U(1)$ gauge symmetry at every point ($\tau, r$) in (Euclidean) space-time. A potential, bare Maxwell term $\frac{1}{2\pi} F_{\mu\nu} F^{\mu\nu}$ is not written in Eq. (1) and is unimportant for the universal constants of interest in this paper. The gauge field gets dynamical by integrating the fermion fields in the large $N_F$ limit. In Landau gauge, the gauge field propagator at $N_F \to \infty$ is purely transverse and takes the characteristic overdamped form (with $p = |p|$)

$$D^{(0)}_{\mu\nu}(p) = \frac{16}{p} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right).$$  

Model Eq. (1) with a bare Maxwell term is also known as QED$_3$ and flows to strong coupling in the infrared and shares its propensity to form fermion bound states “mesons” with QCD in 3+1 dimensions [29–31]. Deforming QED$_3$ toward graphene-type models with instantaneous Coulomb interactions are also interesting [32–35]. It is believed that for sufficiently large $N_F$, Eq. (1) flows
to a strongly coupled conformal phase in the infrared, preserving scale invariance [36] (and references therein). This is the regime of interest in the present paper.

**B. Key results: central charge $C_T$ and $C_J$ up to next-to-leading order in $1/N_F$**

The main result of this paper is an explicit formula and numerical value of the central charge $C_T$ of Eq. (1), defined below as the universal constant appearing in the stress tensor correlator at the interacting conformal fixed point, up to next-to-leading order in the $1/N_F$ expansion:

\[
\frac{C_T}{N_F} = \frac{1}{256} \left( 1 + \frac{1}{N_F} \left( \tilde{C}_T^{(1)} + 4 + \frac{104}{15\pi^2} \right) \right)
\]

\[
= \frac{1}{256} \left( 1 + \frac{4.2870118590002470406}{N_F} \right).
\]

$\tilde{C}_T^{(1)}$ comes from one out of nine Feynman graphs in momentum space computed below in Fig. (5)

\[
\tilde{C}_T^{(1)} = -\frac{4}{45\pi^2} \left( 180Li_2(3 - 2\sqrt{2}) - 720Li_2(-1 + \sqrt{2}) - 398 + 90\pi^2 + 45\log^2(3 - 2\sqrt{2}) 
\]

\[
+ 1146\sqrt{2}\log(3 - 2\sqrt{2}) + 12\left( 191\sqrt{2} + 15\log(3 - 2\sqrt{2}) \right) \sinh^{-1}(1) \right)
\]

\[
= -0.415481680919696150803 ,
\]

where $\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ is the polylogarithm or Jonquiére’s function for $n = 2$. The sum of other eight diagrams evaluate to the remaining term in the innermost bracket, $4 + \frac{104}{15\pi^2}$, in the first line of Eq. (3). We observe from Eq. (3) that $1/N_F$ corrections to the $N_F \rightarrow \infty$ value remain as large as $\approx 50\%$ down to $N_F \approx 8$. Similarly large corrections were also observed (for current correlators) in the $CP^{N-1}$ model and attributed in particular to vertices directly involving the gauge field [28].

It is hard to overestimate the fundamental importance of the central charge in conformal field theory with applications ranging from thermodynamics, quantum critical transport, to quantum information theory [23]. An interesting recent application are explicit formulae for the Rényi entropy for $d$-dimensional flat space CFT’s and we quote here the formula from Perlmutter [24].

\[
S_{q=1}' = -\text{Vol}(\mathbb{H}^{d-1}) \frac{\pi^{d/2+1} \Gamma(d/2)(d-1)}{(d+1)!} C_T
\]

The prime denotes a derivative with respect to $q$ of the Rényi entropy $S_q = \frac{1}{1-q} \log \text{Tr}[\rho^q]$, $\rho$ a reduced density matrix, and $\mathbb{H}^{d-1}$ the hyperboloid entangling surface. Moreover, precision values of $C_T$ may be useful for conformal bootstrap approaches for the 3D-Ising and other models [4] as well as serving as a benchmark for numerical simulations of frustrated quantum magnets [25].

4
Computations of stress tensor correlators in interacting CFT’s (at least without an excessive amount of symmetry such as supersymmetries) in effective dimensionality greater than 2 are extremely scarce and we are not aware of a previous computation of $C_T$ for Eq. (1) in 2+1 dimensions. We quote here related works across the quantum field theory universe we are aware of to date: two papers by Hathrell using loop expansions from 1982, one on scalar fields up to 5-loops [40] and one on QED up to 3 loops [41], an $\epsilon$-expansion around four dimensions for scalar and gauge theories by Cappelli, Friedan and LaTorre in 1991 [42], and a series of papers on the $O(N)$ vector model from 1994-1996 by Petkou [27, 37, 43].

For essentially free field theories, stress tensor amplitudes [39, 44] and Rényi entropies [26] have also been computed. (Multi-point) correlators of the stress tensor are also instrumental for the relation between scale and conformal invariance (e.g.: [45, 46]).

In the present paper, we compute $C_T$ by direct evaluation of Feynman graphs in momentum space fulfilling and using the relation [38, 39],

$$\langle T_{\mu\nu}(-p)T_{\lambda\rho}(p)\rangle = C_T|p|^3 \left( \delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\mu\nu}\delta_{\lambda\rho} + \delta_{\mu\nu}\frac{p_{\lambda}p_{\rho}}{p^2} + \delta_{\nu\lambda}\frac{p_{\mu}p_{\rho}}{p^2} - \delta_{\nu\rho}\frac{p_{\mu}p_{\lambda}}{p^2} - \delta_{\mu\lambda}\frac{p_{\nu}p_{\rho}}{p^2} + \frac{p_{\mu}p_{\nu}p_{\lambda}p_{\rho}}{p^4} \right) \tag{6}$$

generalizing our recently developed technology [15, 28] to Dirac fermions and contractions over stress tensor vertices. We discuss this further in Sec. IV.

The second result of this paper is an (somewhat simpler) computation of the universal constant $C_J$ of the two-point correlator of the conserved flavor current of Eq. (1):

$$J^\ell_\mu = \bar{\psi}^\alpha T^\ell_{\alpha\beta} \gamma_\mu \psi_\beta, \tag{7}$$

where $T^\ell$’s are generators of the SU($N_F$) group normalized to satisfy $\text{Tr}(T^\ell T^m) = \delta^\ell_m$. As the stress tensor $T_{\mu\nu}$, this flavor current is conserved and its two-point correlator depends on one universal constant $C_J$

$$\langle J^\ell_\mu(-p)J^m_\nu(p)\rangle = -C_J|p|^3 \left( \delta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right). \tag{8}$$

For single fermion QED$_3$, $C_J$ describes the universal electrical conductivity in the collisionless regime $\omega \gg T$, with $T$ being the temperature. Depending on the physical context, however, it may also be related to magnetic or other response functions [16]. Our result for $C_J$ to next-to-leading
order in $1/N_F$ is (derived in Sec. II)

$$C_J = \frac{1}{16} \left( 1 + \frac{1}{N_F} \left( \tilde{C}_J^{(1)} - \frac{40}{9\pi^2} \right) \right)$$

$$= \frac{1}{16} \left( 1 + \frac{1}{N_F} \cdot 0.14291062004225554348 \right) \quad (9)$$

with the analytical expression corresponding to one of the graphs being

$$\tilde{C}_J^{(1)} = -\frac{4}{3\pi^2} \left( -34 + 6\pi^2 + \sinh^{-1}(1) \left( 52 \sqrt{2} + 6 \log \left( 17 - 12 \sqrt{2} \right) \right) + 26 \sqrt{2} \log \left( 3 - 2 \sqrt{2} \right) 
+ 3 \log^2 \left( 3 - 2 \sqrt{2} \right) + 24\text{Li}_2 \left( 1 - \sqrt{2} \right) - 24\text{Li}_2 \left( -1 + \sqrt{2} \right) \right)$$

$$= 0.59322699178597897212 \quad (10)$$

This result is seemingly in disagreement with the value computed in the Appendix of Ref. 12 and we compare to their value in detail in Sec. II. As a (positive) cross-check, we have repeated a different calculation of the (non-conserved) staggered spin susceptibility in the Appendix of Rantner and Wen [16] using our approach and found the same logarithmically divergent coefficients.

Note that Eq. (1) has a further conserved “topological” current related to the curl of the gauge field [39] but we do not consider it further here.

### C. Organization of paper

The remainder of the paper is organized as follows: in Sec. II, we define the Feynman rules for Eq. (1) and the current vertex, and evaluate the 3 graphs renormalizing the current-current correlator. In Sec. III, we briefly recapitulate the main elements of the Tensoria technology for the momentum integrals. In Sec. IV, we define the stress tensor vertex and evaluate the 9 graphs renormalizing the stress tensor correlator. In the conclusions, we summarize and point toward potential future directions where our technology could be applied to.
II. FLAVOR CURRENT CORRELATOR $\langle JJ \rangle$

In this section, we compute the SU($N_F$) flavor current-current correlator and compare it to the two previous computations also using the $1/N_F$ expansion that we are aware of [12, 16]. We begin by stating the Feynman rules, compute the leading $N_F \to \infty$ graph in some detail, and then the more complicated self-energy and vertex corrections at order $1/N_F$. We will separate the contributions into longitudinal and transverse projections and show that all longitudinal and logarithmically singular corrections mutually cancel as they should for a conserved, transverse quantity.

A. Feynman rules and graphs in momentum space

The Feynman rules for $N_F$ Dirac fermions coupled to $U(1)$ gauge field in Eq. (1) contain the relativistic fermion propagator

$$G_{\psi}(k) = \frac{k_a \gamma_a}{k^2},$$

(11)

the gauge field propagator in Eq. (2), and the photon-fermion vertex drawn in Fig. 1. The current vertex in Fig. 2 involves one generator of the SU($N_F$) but the traces over them in the actual diagrams are innocuous and just give $\delta$-functions in the flavor indices.

Using the Feynman rules explained above, Fig. 3 exhibits the 3 contractions to the current correlator to order $1/N_F$. Each of the expressions in Eq. 12 contain a minus sign due to the trace over fermions, a (trivial) trace over flavor indices, a trace over the Dirac matrices, and one (1-loop graph) or two (the two 2-loop graphs) $2 + 1$ dimensional momentum integrals $\int \equiv \int \frac{d^3 k}{8\pi^3}$. We get:

\[ G_{\psi} \quad D^{(0)}_{\mu\nu} \quad \gamma_{\mu}/\sqrt{N_F} \]

FIG. 1: Feynman rules for $N_F$ Dirac fermions coupled to $U(1)$ gauge field in Eq. (1).
\[ J^\ell_{\mu
u}(p) = -\text{Tr} \left[ \int \gamma^\nu T^m \frac{k_a \gamma_a}{k^2} \gamma^\mu T^\ell (p + k)_b \gamma_b \frac{(p + k)^2}{(p + k)^2} \right] \quad (12) \]

\[ J^m_{\mu
u}(p)^{(0)} = -\text{Tr} \left[ \int_{k} \frac{\gamma^\nu k a \gamma_a}{k^2} \frac{\gamma^\mu T^\ell (p + k)_b \gamma_b}{(p + k)^2} \frac{(p + k)^2}{(p + k)^2} \gamma^\rho k_d \gamma_d \frac{16}{q} \left( \delta_{\mu\rho} - \frac{q \rho q}{q^2} \right) \right] \]

\[ J^m_{\mu
u}(p)^{(1)} = -\text{Tr} \left[ \int_{k,q} \frac{\gamma^\nu T^m (k + p)_a \gamma_a}{(k + p)^2} \frac{k_b \gamma_b}{\sqrt{N_F}} \frac{(k + q)_c \gamma_c}{(k + q)^2} \frac{\gamma^\mu T^\ell (k + q)_b \gamma_b}{(k + q)^2} \frac{\gamma^\rho k_d \gamma_d}{\sqrt{N_F}} \frac{16}{q} \left( \delta_{\mu\rho} - \frac{q \rho q}{q^2} \right) \right] \]

\[ J^m_{\mu
u}(p)^{(2)} = -\text{Tr} \left[ \int_{k,q} \frac{\gamma^\nu T^m (k + p)_a \gamma_a}{(k + p)^2} \frac{k_b \gamma_b}{\sqrt{N_F}} \frac{\gamma^\mu T^\ell (k + q)_b \gamma_b}{(k + q)^2} \frac{\gamma^\rho k_d \gamma_d}{\sqrt{N_F}} \frac{16}{q} \left( \delta_{\mu\rho} - \frac{q \rho q}{q^2} \right) \right] . \]

These expressions are now evaluated in the following way using our “Tensoria” technology [28]: We first perform the trace over the Dirac indices, collecting the contracted expressions in the numerator. Especially for the more complicated expressions it is helpful to automate it and use the Feyncalc MATHEMATICA package for this [50]. Then we replace the integrals of momentum written in components as described in the next section and in the Appendix of Ref. 28. Finally, we separate out the transverse \( I_T^{(i)} \) and longitudinal \( I_L^{(i)} \) momentum projections in the following form:

\[ \langle J^\ell_{\mu}(p)J^m_{\nu}(p) \rangle = \delta^{\ell m} \sum_{i=0}^{2} a_i J^{(i)}_{\mu\nu}(p) \equiv \sum_{i=0}^{2} a_i \left( I_T^{(i)}(p) \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + I_L^{(i)}(p) \frac{p_\mu p_\nu}{p^2} \right) . \quad (13) \]

FIG. 2: Feynman rule for the current vertex. \( T^\ell \) is a generator of the SU(\( N_F \)).

FIG. 3: Feynman diagrams contributing to the current current correlator to order \( 1/N_F \). Diagram (0) is the leading order contribution and the only one that survives the \( N_F \to \infty \) limit. Diagram (1) is the vertex correction, diagram (2) the self-energy correction that comes with a factor of \( a_2 = 2 \).
B. Leading order \( N_F \to \infty \) graph for \( C_J \)

To illustrate the procedure with a simple example, let us evaluate the leading order graph:

\[
J_{\mu\nu}^{(0)}(p) = -\text{Tr} \left[ \int \gamma_{\nu} T_{\mu} \frac{k_{\mu} p_{\rho}}{k^2} \gamma_{\rho} T_{\nu} \frac{(p + k)_{\rho}}{(p + k)^2} \right] = \delta_{\mu\nu} \int \frac{d^3 k}{8\pi^3} \frac{2 k^2 \delta_{\mu\nu} + 2 \delta_{\mu\nu} k_{\alpha} p_{\alpha} - 4 k_{\mu} k_{\nu} - 2 k_{\nu} p_{\mu} - 2 k_{\mu} p_{\nu}}{k^2 (p + k)^2}. \tag{14}
\]

The integral over the first term in the numerator \( 2 k^2 \delta_{\mu\nu} \) is a power-law divergence in the UV and can be dropped. The second, third, fourth and fifth term in the numerator can be integrated using the identities

\[
\int \frac{d^3 k}{8\pi^3} \frac{k_{\mu}}{k^2 (p + k)^2} = -\frac{p_{\mu}}{16p},
\]

\[
\int \frac{d^3 k}{8\pi^3} \frac{k_{\mu} k_{\nu}}{k^2 (p + k)^2} = \left( 3 \frac{p_{\mu} p_{\nu}}{p^2} - \delta_{\mu\nu} \right) \frac{p}{64}.
\tag{15}
\]

with the abbreviation for the modulus \( p = |p| \) and interchangably \( p^2 = p^2 \). The result

\[
J_{\mu\nu}^{(0)}(p) = -\frac{p}{16} \left( \delta_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2} \right). \tag{16}
\]

comes out purely transverse, leading to \( C_J^{N_F \to \infty} = 1/16 \). Note that in order to compute \( \langle TT \rangle \) (in the next section), Tensoria performs momentum integrals of the type Eq. (15) containing up to six different momentum indices in the numerator and four propagators in the denominator.

C. 1/\( N_F \) corrections to \( C_J \) and discussion

We evaluate the vertex correction and self-energy correction graphs (1) and (2) in Eq. 12 algorithmically and the results are in Table I. As expected for a conserved quantity, the log-singularities of each individual graph cancel when taking the sum, so does the longitudinal part. As announced in the Introduction, our result Eq. (9) seems to disagree with Chen et al.[12] who computed \( C_J \) for QED\(_3\) to order 1/\( N_F \). The relevant 1/\( N_F \) correction is given in Eq. (A17) in the appendix of their paper. Mapping to our conventions we take \( g = 1 \) and \( A = 16 \) and an overall minus sign. These authors obtained

\[
C_J^{\text{Chen, et al.}} = \frac{1}{16} \left( 1 + \frac{16}{N_F (2\pi)^2} \right) = \frac{1}{16} \left( 1 + \frac{1}{N_F} 1.216 \right), \tag{17}
\]

The sign of their 1/\( N_F \) correction match but the value seem to be different from Eq. (9).
TABLE I: Evaluated contributions to the current-current correlator. The sum of longitudinal components off all the graphs add to 0 and the transverse parts add up to Eq. (9). The analytic expression for $I_T^{(1)}(p)$ (multiplied by $-16$) is in Eq. (10) The log-singularities mutually cancel. The self-energy correction graph (2) comes with a factor of $a_2 = 2$.

| Diagram | $I_T^{(1)}(p)$ | $I_L^{(1)}(p)$ | Log-Singularity (transverse) | Factor $a_i$ |
|---------|----------------|----------------|-------------------------------|-------------|
| 0       | $-\frac{1}{16} p$ | 0              | 0                            | 1           |
| 1       | $-\frac{p}{N_f} 0.0370767$ | $\frac{p}{N_f} \frac{1}{3\pi^2}$ | $\frac{p}{N_f} \frac{2}{3\pi^2} \log \frac{\Lambda}{p}$ | 1           |
| 2       | $\frac{p}{N_f} \frac{5}{36\pi^2}$ | $-\frac{p}{N_f} \frac{1}{6\pi^2}$ | $-\frac{p}{N_f} \frac{1}{3\pi^2} \log \frac{\Lambda}{p}$ | 2           |

III. TENSORIA TECHNOLOGY: MINI-RECAP

Before proceeding, let us briefly recapitulate our algorithm to evaluate the tensor-valued momentum integrals as described in more detail in the Appendices of [28, 49]. At the heart of the algorithm are Davydychev permutation [47, 48] relations to perform integrals of the form:

$$J_{\mu_1...\mu_M}(p_1, p_2, p_3; n; \nu_1) = \int d^d k \frac{k_{\mu_1}...k_{\mu_M}}{(k + p_1)^{2\nu_1} (k + p_2)^{2\nu_2} (k + p_3)^{2\nu_3}}.$$  \hspace{1cm} \text{(18)}$$

After the Dirac traces, the integrals can all be brought into this form. After the first momentum integration, we temporarily introduce a UV-momentum cutoff that formally breaks symmetries such as conformal invariance. Using this cutoff as a sorter, all power-law divergences are discarded as they would be absent in a gauge-invariant regularization schemes such as dimensional regularization. The remaining finite and logarithmically divergent terms can be integrated analytically graph-by-graph and the log-singularities are seen to cancel exactly.

We close this recap by noting that despite the exact cancellations of the log-singularities as a strong consistency check, and the many additionally performed checks of all sub-routines in Tensoria, at the moment we have no proof that of the exactness to $O(1/N_f)$ of our results and it would be very desirable to compare results to another method.
IV. STRESS ENERGY TENSOR CORRELATOR $\langle TT \rangle$

In this section, we extend our technology to compute the stress tensor correlator of Eq. (1) to next-to-leading order in $1/N_F$. We first define the stress tensor itself and write down the Feynman rules for the stress tensor vertices. Then, we first illustrate in some detail the calculation of the leading $N_F \to \infty$ graph before evaluating the remaining 8 graphs with Tensoria. The two major complications here are: (i) the gauge field can connect directly to the stress tensor vertex leading to a vertex involving 3 lines, and (ii) four 3-loop graphs, including those of the Azlamasov-Larkin type, appear. As in the $\langle JJ \rangle$ computation, we explicitly show that all log-singularities cancel when summing all graphs to ensure to conserved nature of $T_{\mu\nu}$ in accordance with symmetries.

A. Feynman rules and graphs in momentum space

The stress tensor operator for Eq. (1) depends on both the fermions and the gauge fields via the gauge covariant derivative $D_{\mu} = \partial_{\mu} - iA_{\mu}/\sqrt{N_F}$ [39]

$$T_{\mu\nu} = \sum_{\alpha=1}^{N_F} \frac{i}{4} \left( \bar{\psi}_\alpha \gamma_{\mu} (D_{\nu} \psi_\alpha) + \bar{\psi}_\alpha \gamma_{\nu} (D_{\mu} \psi_\alpha) - (D^*_{\mu} \bar{\psi}_\alpha) \gamma_{\nu} \psi_\alpha - (D^*_{\nu} \bar{\psi}_\alpha) \gamma_{\mu} \psi_\alpha \right).$$

leading to the stress tensor vertices shown in Fig. 4. The eight graphs and their analytical expressions shown in Figs. 5, 6 contribute to order $1/N_F$ and we again denote their sum by

$$\langle T_{\mu\nu}(-p)T_{\rho\lambda}(p) \rangle = \sum_{i=0}^{7} a_i T^{(i)}_{\mu\nu\rho\lambda}(p).$$

![Diagram](https://example.com/diagram4.png)

**FIG. 4:** Feynman rules for the stress tensor vertices.
FIG. 5: Feynman diagrams contributing to the stress energy tensor correlator to next-to-leading order in $1/N_F$. Only diagram (0) survives in the $N_F \to \infty$ limit. Diagrams (2) and (4) come with a factor of $a_2 = 2$, $a_4 = 2$, respectively. The factors for the other graphs are unity $a_i = 1$. The numerical values and logarithmic singularities for each of these graphs are exhibited in Table II.
\( T_{\mu \nu \lambda \rho}^{(0)}(p) = -N_F \text{Tr} \left[ \int \frac{1}{4} \gamma_\lambda(2k + p)_\mu \frac{(k + p)_{\alpha} \gamma_\alpha 1}{(k + p)^2} 4 \gamma_\mu(2k + p)_\nu \frac{k_{\beta} \gamma_{\beta}}{k^2} \right] + \text{perm1} \)

\( T_{\mu \nu \lambda \rho}^{(1)}(p) = -N_F \text{Tr} \left[ \int \frac{1}{4} \gamma_\lambda(2k + p)_\mu \frac{(k + p)_{\alpha} \gamma_\alpha e}{(k + p)^2} \sqrt{N_F} \frac{(k + p + q)^2 \gamma_{\beta} b \gamma_{\beta} 1}{(k + p + q)^2} 4 \gamma_\mu(2k + 2q + p)_\nu \frac{(k + q)_{\epsilon} \gamma_{\epsilon} c \gamma_{\epsilon} k_{d} \gamma_{d}}{(k + q)^2} \sqrt{N_F} \frac{k_{d} \gamma_{d}}{k^2} \right] 
\frac{16}{q} \left( \delta_{ee} - \frac{q_{a} q_{b}}{q^2} \right) + \text{perm1} \)

\( T_{\mu \nu \lambda \rho}^{(2)}(p) = -N_F \text{Tr} \left[ \int \frac{1}{4} \gamma_\lambda(2k + p)_\mu \frac{(k + p)_{\alpha} \gamma_\alpha 1}{(k + p)^2} 4 \gamma_\mu(2k + p)_\nu \frac{k_{b} \gamma_{b} e}{(k + q)^2} \sqrt{N_F} \frac{(k + q)^2 \gamma_{c} c \gamma_{c} k_{d} \gamma_{d}}{k^2} \sqrt{N_F} \frac{k_{d} \gamma_{d}}{q} \left( \delta_{ee} - \frac{q_{a} q_{b}}{q^2} \right) \right] + \text{perm1} \)

\( T_{\mu \nu \lambda \rho}^{(3)}(p) = -N_F \text{Tr} \left[ \int \frac{1}{4} \gamma_\lambda(2k + p)_\mu \frac{(k + p + q)_{\alpha} \gamma_\alpha \mu_{\alpha} k_{b} \gamma_{b} 16}{(k + p + q)^2} \frac{\gamma_{\mu} k_{c} \gamma_{c} 16}{(k + p + q)^2} \frac{(p + q)_{\lambda}(p + q)_{\lambda}}{(p + q)^2} \right] + \text{perm1} \)

\( T_{\mu \nu \lambda \rho}^{(4)}(p) = -N_F \text{Tr} \left[ \int \frac{1}{4} \gamma_\lambda(2k + p)_\mu \frac{(k + p + q)_{\alpha} \gamma_\alpha e}{(k + p + q)^2} \sqrt{N_F} \frac{(k + p + q + l)^2 \gamma_{b} b \gamma_{b} 16}{(k + p + q + l)^2} \frac{\gamma_{\mu} k_{c} \gamma_{c} 16}{(k + p + q + l)^2} \frac{(p + q)_{\lambda}(p + q)_{\lambda}}{(p + q)^2} \right] + \text{perm2} \)

\( T_{\mu \nu \lambda \rho}^{(5)}(p) = \int \frac{N_F}{q} \text{Tr} \left[ \int \frac{1}{4} \gamma_\mu(p + 2l)_{\lambda} \gamma_{\alpha} c \gamma_{\alpha} l_{d} \gamma_{d} \frac{(p + q + l)_{\alpha} l_{d} \gamma_{d} \gamma_{c} 16}{(p + q + l)^2} \frac{\gamma_{\mu}(k + q + l)_{\epsilon} \gamma_{\epsilon} k_{d} \gamma_{d}}{(p + q + l)^2} \right] + \text{perm1} \)

\( T_{\mu \nu \lambda \rho}^{(6)}(p) = \int \frac{N_F}{q} \text{Tr} \left[ \int \frac{1}{4} \gamma_\mu(p + 2l)_{\lambda} \gamma_{\alpha} c \gamma_{\alpha} l_{d} \gamma_{d} \frac{(p + q + l)_{\alpha} l_{d} \gamma_{d} \gamma_{c} 16}{(p + q + l)^2} \frac{\gamma_{\mu}(k + q + l)_{\epsilon} \gamma_{\epsilon} k_{d} \gamma_{d}}{(p + q + l)^2} \right] + \text{perm2} \)

\( T_{\mu \nu \lambda \rho}^{(7)}(p) = \int \frac{N_F}{q} \text{Tr} \left[ \int \frac{1}{4} \gamma_\mu(p + 2q + 2l)_{\lambda} \gamma_{\alpha} c \gamma_{\alpha} l_{d} \gamma_{d} \frac{(p + q + l)_{\alpha} l_{d} \gamma_{d} \gamma_{c} 16}{(p + q + l)^2} \frac{\gamma_{\mu}(k + q + l)_{\epsilon} \gamma_{\epsilon} k_{d} \gamma_{d}}{(p + q + l)^2} \right] + \text{perm2} \)

\( T_{\mu \nu \lambda \rho}^{(8)}(p) = \int \frac{N_F}{q} \text{Tr} \left[ \int \frac{1}{4} \gamma_\mu(p + 2q + 2l)_{\lambda} \gamma_{\alpha} c \gamma_{\alpha} l_{d} \gamma_{d} \frac{(p + q + l)_{\alpha} l_{d} \gamma_{d} \gamma_{c} 16}{(p + q + l)^2} \frac{\gamma_{\mu}(k + q + l)_{\epsilon} \gamma_{\epsilon} k_{d} \gamma_{d}}{(p + q + l)^2} \right] + \text{perm1} \)

\( T_{\mu \nu \lambda \rho}^{(9)}(p) = \int \frac{N_F}{q} \text{Tr} \left[ \int \frac{1}{4} \gamma_\mu(p + 2q + 2l)_{\lambda} \gamma_{\alpha} c \gamma_{\alpha} l_{d} \gamma_{d} \frac{(p + q + l)_{\alpha} l_{d} \gamma_{d} \gamma_{c} 16}{(p + q + l)^2} \frac{\gamma_{\mu}(k + q + l)_{\epsilon} \gamma_{\epsilon} k_{d} \gamma_{d}}{(p + q + l)^2} \right] + \text{perm2} \)

\( T_{\mu \nu \lambda \rho}^{(10)}(p) = \int \frac{N_F}{q} \text{Tr} \left[ \int \frac{1}{4} \gamma_\mu(p + 2q + 2l)_{\lambda} \gamma_{\alpha} c \gamma_{\alpha} l_{d} \gamma_{d} \frac{(p + q + l)_{\alpha} l_{d} \gamma_{d} \gamma_{c} 16}{(p + q + l)^2} \frac{\gamma_{\mu}(k + q + l)_{\epsilon} \gamma_{\epsilon} k_{d} \gamma_{d}}{(p + q + l)^2} \right] + \text{perm1} \)

\( T_{\mu \nu \lambda \rho}^{(11)}(p) = \int \frac{N_F}{q} \text{Tr} \left[ \int \frac{1}{4} \gamma_\mu(p + 2q + 2l)_{\lambda} \gamma_{\alpha} c \gamma_{\alpha} l_{d} \gamma_{d} \frac{(p + q + l)_{\alpha} l_{d} \gamma_{d} \gamma_{c} 16}{(p + q + l)^2} \frac{\gamma_{\mu}(k + q + l)_{\epsilon} \gamma_{\epsilon} k_{d} \gamma_{d}}{(p + q + l)^2} \right] + \text{perm2} \)

FIG. 6: Analytical expressions for the 8 graphs in Fig. 5. Here, “perm1” indicate permutations \((\mu \leftrightarrow \nu)\), \((\lambda \leftrightarrow \rho)\), and \((\mu \leftrightarrow \nu, \lambda \leftrightarrow \rho)\) and “perm2” indicate \((\mu \leftrightarrow \lambda, \nu \leftrightarrow \rho)\), switched as a pair, in addition to permutations indicated by “perm1”. “perm2” will increase the number of terms by a factor of 8.
In order to compute the “central charge” $C_T$, we will project it out from the evaluated graphs using the relation Eq. (6):

$$C_T = \frac{1}{4|p|^3} \delta_{\mu\lambda} \delta_{\nu\rho} \langle T_{\mu\nu}(-p)T_{\lambda\rho}(p) \rangle .$$

(21)

We note here that a number of previous analyses [3, 27, 37] have been conducted in real space, where the invariance of correlators under the full set of conformal transformations are transparent but the analysis to work out the constants for an interacting CFT is quite involved.

**B. Leading order $N_F \to \infty$ graph for $C_T$**

Let us evaluate the leading order graph, the first line in Fig. 6. Including the index permutations described in the caption of the figure, we have

$$\frac{C_{N_F \to \infty}}{N_F} = \frac{1}{4|p|^3} \delta_{\mu\lambda} \delta_{\nu\rho} T_{\mu\nu}^{(0)}(p)$$

$$= \frac{1}{4|p|^3} \frac{1}{2} \int \frac{d^3 k}{8\pi^3} \frac{k_\alpha k_\beta p_\alpha p_\beta - k^2 p^2}{k^2 (k + p)^2}$$

$$= \frac{1}{4|p|^3} \frac{1}{2} \frac{1}{2} p_\alpha p_\beta \left( 3 \frac{p_\alpha p_\beta}{p^2} - \delta_{\alpha\beta} \right) \frac{p}{64}$$

$$= \frac{1}{256}$$

(22)

where we dropped the second term in the numerator in the second line because it is a power-law divergence in the UV, absent in dimensional regularization. We can also check that without immediately contracting the graph, the uncontracted terms fulfill the index structure of Eq. (6).

**C. $1/N_F$ corrections for $C_T$ and discussion**

Tensoria computes the $1/N_F$ corrections algorithmically and Table II collects the results.

As before, we observe an exact cancellation of the logarithmic singularities of each graph in accordance with symmetry requirements. Summing the graphs leads to Eq. (3) in the Introduction. In addition to the discussion in the Introduction, we mention here that the components of the stress tensor correlator also yield the shear viscosity particularly relevant for strongly interacting quantum field theories at finite temperature [51, 52]. In order to resolve the collisional physics, however, it is necessary to solve a Boltzmann equation or invoke the AdS-CFT correspondence (see e.g.: Refs. 52, 53 and references therein).
| Diagram | $C_T^{(i)}$ | Log-Singularity Factor $a_i$ |
|---------|-------------|-----------------------------|
| 0       | $\frac{N_F}{256}$ | 0                           | 1 |
| 1       | $-0.00162$ | $-\frac{7}{120\pi^2}p^3 \log \frac{\Lambda}{\rho}$ | 1 |
| 2       | $\frac{1}{576\pi^2}$ | $\frac{1}{48\pi^2}p^3 \log \frac{\Lambda}{\rho}$ | 2 |
| 3       | 0             | 0                           | 1 |
| 4       | $\frac{19}{288\pi^2}$ | $\frac{1}{24\pi^2}p^3 \log \frac{\Lambda}{\rho}$ | 2 |
| 5       | 0             | 0                           | 1 |
| 6       | $\frac{1}{128} - \frac{19}{144\pi^2}$ | $-\frac{1}{12\pi^2}p^3 \log \frac{\Lambda}{\rho}$ | 1 |
| 7       | $\frac{1}{256}$ | 0                           | 1 |
| 8       | $\frac{1}{256} + \frac{17}{720\pi^2}$ | $\frac{1}{60\pi^2}p^3 \log \frac{\Lambda}{\rho}$ | 1 |

TABLE II: Evaluated contributions to the stress tensor correlator and the log-singularities. The log-singularities cancel exactly after summing all graphs. The analytic expression for $C_T^{(1)}$ (multiplied by 256) is given in Eq. (4).

V. CONCLUSIONS

The aim of this paper was to provide precision computations of the “central charge” $C_T$ and universal conductivity $C_J$ of interacting conformal field theories in $2 + 1$ dimensions. We considered $N_F$ Dirac fermions coupled to an “emergent photon” motivated by frequent occurrence of this field theory in a variety of condensed matter systems. The low-energy sector is also equivalent to many-flavor QED$_3$ in the conformal phase.

Our hope is that our results could become a useful diagnostic for numerical evaluations of entanglement properties of CFT$_3$’s, conformal bootstrap approaches, or application of the AdS-CFT correspondence. Going forward, our technology may also complement explicit computations of conformal correlators in the context of dualities of Large $N$ Chern-Simons Matter Theories [54, 55].
Acknowledgments

We thank Andrea Allais, Holger Gies, Zohar Komargodski, Jan M. Pawlowski, and Silviu Pufu for discussions and Subir Sachdev for guidance, collaboration on related projects, and critically reading the manuscript. This research was supported by the Leibniz prize of A. Rosch, and the NSF grant DMR-1360789. This research was also supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development & Innovation.

[1] K. G. Wilson, and M. E. Fisher, Critical Exponents in 3.99 Dimensions, Phys. Rev. Lett. 28, 240 (1971).
[2] R. Abe, Critical exponent $\eta$ up to $1/N^2$ for the Three-Dimensional System with Short-Range Interaction, Prog. of Theor. Phys., 49, 6 (1973).
[3] H. Osborn, and A. Petkou, $C_T$ and $C_J$ up to next-to-leading order in $1/N$ in the conformally invariant $O(N)$ vector model for $2 < d < 4$, Phys. Lett. B 359, 101 (1995).
[4] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffins, and A. Vichi, Solving the 3D Ising model with the conformal bootstrap, Phys. Rev. D 86, 025022 (2012).
[5] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffins, and A. Vichi, Solving the 3D Ising model with the conformal bootstrap II. c-Minimization and Precise Critical Exponents, arXiv:1403.4545 (2014).
[6] M. C. Cha, M. P. A. Fisher, S. M. Girvin, M. Wallin, and A. P. Young, Universal conductivity of two-dimensional films at the superconductor-insulator transition, Phys. Rev. B 44, 6883 (1991).
[7] R. Fazio and D. Zappala, $\epsilon$ expansion of the conductivity at the superconductor-Mott-insulator transitions, Phys. Rev. B 53, R8885 (1996).
[8] S. Chakravarty, B. I. Halperin, and D. R. Nelson, Two-dimensional quantum Heisenberg antiferromagnet at low temperatures, Phys. Rev. B 39, 2344 (1994).
[9] A. V. Chubukov, S. Sachdev, Theory of two-dimensional quantum Heisenberg antiferromagnet with a nearly critical ground state, Phys. Rev. B 49, 11919 (1994).
[10] R. K. Kaul, and S. Sachdev, Quantum criticality of $U(1)$ gauge theories with fermionic and bosonic
matter in two spatial dimensions, Phys. Rev. B 77, 155105 (2008).

[11] W. Chen, G. W. Semenoff, and Y.-S. Wu, Two-loop analysis of non-Abelian Chern-Simons theory, Phys. Rev. D 46, 5521 (1992).

[12] W. Chen, M. P. A. Fisher, and Y.-S. Wu, Mott transition in an anyon gas, Phys. Rev. B 48, 13749 (1993).

[13] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, Deconfined Quantum Criticality, Science 303, 1490 (2004).

[14] A. W. Sandvik, Evidence for deconfined quantum criticality in a two-dimensional Heisenberg model with four-spin interactions, Phys. Rev. Lett. 98, 227202 (2007).

[15] Y. Huh, P. Strack, and S. Sachdev, Vector Boson Excitations Near Deconfined Quantum Critical Points, Phys. Rev. Lett. 111, 166401 (2013).

[16] W. Rantner, and X.-G. Wen, Spin correlations in the algebraic spin liquid: Implications for high-$T_c$ superconductors, Phys. Rev. B 66, 144501 (2002).

[17] M. Franz, Z. Tesanovic, and O. Vafek, QED$_3$ theory of pairing pseudogap in cuprates: From d-wave superconductor to antiferromagnet via an algebraic Fermi liquid, Phys. Rev. B 66, 054535 (2002).

[18] M. Franz, T. Pereg-Barnea, D. E. Sheehy, and Z. Tesanovic, Gauge-invariant response functions in algebraic Fermi liquids, Phys. Rev. B 68, 024508 (2003).

[19] R. K. Kaul, Y.-B. Kaim, S. Sachdev, and T. Senthil, Algebraic charge liquids, Nature Physics 4, 28 (2007).

[20] J. Cardy, Conformal Field Theory and Statistical Mechanics, arXiv:0807.3472 (2008).

[21] A. M. Polyakov, Gauge Fields and Strings (Harwood Academic, Chur, 1987).

[22] S. Coleman, Aspects of Symmetry (Cambridge University Press, Cambridge, UK, 1988).

[23] J. Cardy, The ubiquitous ‘c’: from the Stefan-Boltzmann Law to Quantum Information, J. Stat. Mech. 1010:P10004 (2010).

[24] E. Perlmutter, A universal feature of CFT Rényi entropy, JHEP 03, 117 (2014).

[25] R. K. Kaul, R. G. Melko, and A. W. Sandvik, Bridging Lattice-Scale Physics and Continuum Field Theory with Quantum Monte Carlo Simulations, Annu. Rev. Condens. Matter Phys. 4, 179 (2013).

[26] I. R. Klebanov, S. S. Pufu, S. Sachdev, and B. R. Safdi, Rényi entropies for free field theories, JHEP 04, 074 (2012).

[27] A. Petkou, Conserved Currents, Consistency Relations, and Operator Product Expansions in the Conformally Invariant O(N) Vector Model, Ann. of Phys. 249, 180 (1996).
[28] Y. Huh, P. Strack, and S. Sachdev, *Conserved current correlators of conformal field theories in 2 + 1 dimensions*, Phys. Rev. B 88, 155109 (2013).

[29] T. W. Appelquist, M. Bowick, D. Karabali, and L. C. R. Wijewardhana, *Spontaneous chiral-symmetry breaking in three-dimensional QED*, Phys. Rev. D 33, 3704 (1986).

[30] T. Appelquist, D. Nash, and L.C.R. Wijewardhana, *Critical Behavior in (2+1)-Dimensional QED*, Phys. Rev. Lett. 60, 2575 (1988).

[31] D. Nash, *High-Order Corrections in (2+1)-Dimensional QED*, Phys. Rev. Lett. 62, 3024 (1989).

[32] D. T. Son, *Quantum critical point in graphene approached in the limit of infinitely strong Coulomb interaction*, Phys. Rev. B 75, 235423 (2007).

[33] V. Juricic, O. Vafek, and I. F. Herbut, *Conductivity of interacting massless Dirac particles in graphene: Collisionless regime*, Phys. Rev. B 82, 235402 (2010).

[34] I. F. Herbut, and V. Mastropietro, *Universal conductivity of graphene in the ultrarelativistic regime*, Phys. Rev. B 87, 205445 (2013).

[35] E. Barnes, E. H. Hwang, R. E. Thockmorton, and S. Das Sarma, *Effective field theory, three-loop perturbative expansion, and their experimental implications in graphene many-body effects*, Phys. Rev. B 89, 235431 (2014).

[36] J. Braun, H. Gies, L. Janssen, D. Roscher, *Phase structure of many-flavor QED3*, Phys. Rev. D 90, 036002 (2014).

[37] H. Osborn, and A. Petkou, *Implications of Conformal Invariance in Field Theories for General Dimensions*, Ann. Phys. 231, 311 (1994).

[38] J. Cardy, *Anisotropic corrections to correlation functions in finite-size systems*, Nucl. Phys. B 290, 355 (1987).

[39] D. Chowdhury, S. Raju, S. Sachdev, A. Singh, and P. Strack, *Multipoint correlators of conformal field theories: Implications for quantum critical transport*, Phys. Rev. B 87, 085138 (2013).

[40] S. J. Hathrell, *Trace Anomalies and λφ^4 Theory in Curved Space*, Annals of Physics 139, 136 (1982).

[41] S. J. Hathrell, *Trace Anomalies and QED in Curved Space*, Annals of Physics 142, 34 (1982).

[42] A. Cappelli, D. Friedan, and J. I. LaTorre, *c-Theorem and spectral representation*, Nucl. Phys. B 352, 616 (1991).

[43] A. Petkou, *C_T and C_J up to next-to-leading order in 1/N in the conformally invariant O(N) vector model for 2 < d < 4*, Phys. Lett. B, 101 (1995).

[44] J.M. Maldacena and G.L. Pimentel, *On graviton non-gaussianities during inflation*, JHEP 09, 045
[45] A. Dymarsky, Z. Komargodski, A. Schwimmer, and S. Theisen, *On Scale and Conformal Invariance in Four Dimensions*, arXiv:1309.2921 (2013).

[46] A. Bzowski and K. Skenderis, *Comments on scale and conformal invariance in four dimensions*, arXiv:1402.3208 (2014).

[47] A.I. Davydychev, *A simple formula for reducing Feynman diagrams to scalar integrals*, Phys. Lett. B 263, 107 (1991).

[48] A.I. Davydychev, *Recursive algorithm for evaluating vertex-type Feynman integrals*, J. Phys. A: Math. Gen. 25, 5587 (1992).

[49] A. Bzowski, P. McFadden, and K. Skenderis, *Holographic predictions for cosmological 3-point functions*, JHEP 03, 091 (2012).

[50] *Tools and Tables for Quantum Field Theory Calculations*, URL: http://www.feyncalc.org/.

[51] P. K. Kovtun, D. T. Son, and A. O. Starinets, *Viscosity in Strongly Interacting Quantum Field Theories from Black Hole Physics*, Phys. Rev. Lett. 94, 111601 (2005).

[52] T. Enss, R. Haussmann, and W. Zwerger, *Viscosity and scale invariance in the unitary Fermi gas*, Ann. of Phys. 326, 770 (2011).

[53] E. Katz, S. Sachdev, E. S. Sorensen, and W. Witczak-Krempa, *Conformal field theories at finite temperature: operator product expansions, Monte Carlo, and holography*, arXiv:1409:3841 (2014).

[54] O. Aharony, G. Gur-Ari, and R. Yacoby, *Correlation Functions of Large N Chern-Simons-Matter Theories and Bosonization in Three Dimensions*, JHEP 12, 028 (2012).

[55] O. Aharony, S. Giombi, G. Gur-Ari, J. Maldacena, and R. Yacoby, *The thermal free energy in large N Chern-Simons-matter theories*, JHEP 03, 121 (2013).