A parametrization of the abstract Ramsey theorem

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Abstract

We give a parametrization with perfect subsets of $2^{\infty}$ of the abstract Ramsey theorem (see [13]). Our main tool is an extension of the parametrized version of the combinatorial forcing developed in [11] and [13], used in [8] to obtain a parametrization of the abstract Ellen
tuck theorem. As one of the consequences, we obtain a parametrized version of the Hales-Jewett theorem. Finally, we conclude that the family of perfectly $S$-Ramsey subsets of $2^{\infty} \times \mathcal{R}$ is closed under the Souslin operation.

Key words and phrases: Ramsey theorem, Ramsey space, parametrization.

1 Introduction

In [13], S. Todorcevic presents an abstract characterization of those topological spaces in which an analog of Ellen
tuck’s theorem [1] can be proven. These are called topological Ramsey spaces and the main result about them is referred to in [13] as abstract Ellen
tuck theorem. In [8], a parametrization with perfect subsets of $2^{\infty}$ of the abstract Ellen
tuck theorem is given, obtaining in this way new proofs of parametrized versions of the Galvin-Prikry theorem [6] (see [9]) and of Ellen
tuck’s theorem (see [12]), as well as a parametrized version of Milliken’s theorem [10]. The methods used in [8] are inspired by those used

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in [5] to obtain a parametrization of the semiselective version of Ellentuck’s theorem.

Nevertheless, topological Ramsey spaces are a particular kind of a more general type of spaces (introduced in [13]), in which the Ramsey property can be characterized in terms of the abstract Baire property. These are called Ramsey spaces. One of such spaces, known as the Hales-Jewett space, is described below (for a more complete description of this – non topological – Ramsey space, see [13]). S. Todorcevic has given a characterization of Ramsey spaces which is summed up in a result known as the abstract Ramsey theorem. It turns out that the abstract Ellentuck theorem is a consequence of the abstract Ramsey theorem (see [13]). Definitions of all these concepts will be given below.

In this work we adapt in a natural way the methods used in [8] in order to obtain a parametrized version of the abstract Ramsey theorem. In this way, we not only generalize the results obtained in [8] but we also obtain, in corollary 1 below, a parametrization of the infinite dimensional version of the Hales-Jewett theorem [7] (see [13]), which is the analog to Ellentuck’s theorem corresponding to the Hales-Jewett space.

In the next section we summarize the definitions and main results related to Ramsey spaces given in [13]. In section 3 we introduce the (parametrized) combinatorial forcing adapted to the context of Ramsey spaces and present our main result (theorem 5 below). Finally, we conclude that the generalization of the perfectly Ramsey property (see [2] and [12]) to the context of Ramsey spaces is preserved by the Souslin operation (see corollary 4 below).

We’ll use the following definitions and results concerning to perfect sets and trees (see [12]). For \( x = (x_n) \in 2^\mathbb{N} \), \( x|_k = (x_0, x_1, \ldots, x_{k-1}) \). For \( u \in 2^{\text{infty}} \), let \( |u| = \{ x \in 2^\mathbb{N} : (\exists k)(u = x|_k) \} \) and denote the length of \( u \) by \( |u| \). If \( Q \subseteq 2^\mathbb{N} \) is a perfect set, we denote \( T_Q \) its asociated perfect tree. For \( u, v = (v_0, \ldots, v_{|v|-1}) \in 2^{<\mathbb{N}} \), we write \( u \sqsubseteq v \) to mean \( (\exists k \leq |v|)(u = (v_0, v_1, \ldots, v_{k-1})) \). Given \( u \in 2^{<\mathbb{N}} \), let \( Q(u) = Q \cap [u(Q)] \), where \( u(Q) \) is defined as follows: \( \emptyset(Q) = \emptyset \). If \( u(Q) \) is already defined, find \( \sigma \in T_Q \) such that \( \sigma \) is the \( \supseteq \)-extension of \( u(Q) \) where the first ramification occurs. Then, set \( (\sigma^\gamma_i)(Q) = \sigma^\gamma_i, i = 1, 0 \). Where “\( \supseteq \)“ is concatenation. Thus, for each \( n, Q = \bigcup\{ Q(u) : u \in 2^n \} \). For \( n \in \mathbb{N} \) and perfect sets \( S, Q \), we write \( S \sqsubseteq_n Q \) to mean \( S(u) \subseteq Q(u) \) for every \( u \in 2^n \). Thus “\( \sqsubseteq_n \)“ is a partial order and, if we have chosen \( S_u \subseteq Q(u) \) for every \( u \in 2^n \), then \( S = \bigcup_u S_u \) is perfect, \( S(u) = S_u \) and \( S \sqsubseteq_n Q \). The property of fusion of this order is: if \( Q_{n+1} \sqsubseteq_{n+1} Q_n \) for \( n \in \mathbb{N} \), then \( Q = \cap_n Q_n \) is perfect and \( Q \sqsubseteq_n Q_n \) for each \( n \).
2 Abstract Ramsey theory

We introduce some definitions and results due to Todorcevic (see [13]). Our objects will be structures of the form \((R, S, \leq, \leq_0, r, s)\) where \(\leq\) and \(\leq_0\) are relations on \(S \times S\) and \(R \times S\) respectively; and \(r, s\) give finite approximations:

\[
r: R \times \omega \rightarrow AR \\
s: S \times \omega \rightarrow AS
\]

we denote \(r_n(A) = r(A, n), s_n(X) = s(X, n)\), for \(A \in R, X \in S, n \in \mathbb{N}\). The following three axioms are assumed for every \((P, p) \in \{(R, r), (S, s)\}\).

(A.1) \(p_0(P) = p_0(Q)\), for all \(P, Q \in P\).

(A.2) \(P \neq Q \Rightarrow p_n(P) \neq p_n(Q)\) for some \(n \in \mathbb{N}\).

(A.3) \(p_n(P) = p_m(Q) \Rightarrow n = m\) and \(p_k(P) = p_k(Q)\) if \(k < n\).

In this way we can consider elements of \(R\) and \(S\) as infinite sequences \((r_n(A))_{n \in \mathbb{N}}, (s_n(X))_{n \in \mathbb{N}}\). Also, if \(a \in AR\) and \(x \in AS\) we can think of \(a\) and \(x\) as finite sequences \((r_k(A))_{k < n}, (s_k(X))_{k < m}\) respectively; with \(n, m\) the unique integers such that \(r_n(A) = a\) and \(s_m(X) = x\). Such \(n\) and \(m\) are called the length of \(a\) and the length of \(x\), which we denote \(|a|\) and \(|x|\), respectively.

We say that \(b \in AR\) is an end-extension of \(a \in AR\) and write \(a \sqsubseteq b\), if \((\exists B \in Rb = r_n(B)) \Rightarrow \exists m \leq |b| (a = r_m(B))\). In an analogous way we define the relation \(\sqsubseteq\) on \(AS\).

(A.4) **Finization:** There are relations \(\leq_{fin}\) and \(\leq_{fin}^0\) on \(AS \times AS\) and \(AR \times AS\), respectively, such that:

1. \(\{a: a \leq_{fin}^0 x\}\) and \(\{y: y \leq_{fin} x\}\) are finite for all \(x \in AS\).
2. \(X \leq Y\) iff \(\forall n \exists m s_n(X) \leq_{fin} s_m(Y)\).
3. \(A \leq_{fin}^0 X\) iff \(\forall n \exists m r_n(A) \leq_{fin}^0 s_m(X)\).
4. \(\forall a \in AR \forall x, y \in AS [a \leq_{fin}^0 x \leq_{fin} y \Rightarrow (a \leq_{fin}^0 y)]\).
5. \(\forall a, b \in AR \forall x \in AS [a \sqsubseteq b \text{ and } b \leq_{fin}^0 x \Rightarrow \exists y \sqsubseteq x (a \leq_{fin}^0 y)]\).

We deal with the basic sets

\(\begin{align*}
[a, Y] &= \{A \in R: A \leq_{fin}^0 Y \text{ and } \exists n (r_n(A) = a)\} \\
[x, Y] &= \{X \in S: X \leq Y \text{ and } \exists n (s_n(X) = x)\}
\end{align*}\)
for $a \in \mathcal{AR}$, $x \in \mathcal{AS}$ and $Y \in \mathcal{S}$. Notation:

$$[n, Y] = [s_n(Y), Y]$$

Also, we define the depth of $a \in \mathcal{AR}$ in $Y \in \mathcal{S}$ by

$$\text{depth}_Y (a) = \begin{cases} \min \{k : a \leq s_k(Y)\}, & \text{if } \exists k (a \leq s_k(Y)) \\ -1, & \text{otherwise} \end{cases}$$

The next result is immediate.

**Lemma 1.** If $a \sqsubseteq b$ then $\text{depth}_Y (a) \leq \text{depth}_Y (b)$. ■

Now we state the last two axioms:

(A.5) Amalgamation: $\forall a \in \mathcal{AR}$, $\forall Y \in \mathcal{S}$, if $\text{depth}_Y (a) = d$, then:

1. $d \geq 0 \Rightarrow \forall X \in [d, Y] ([a, X] \neq \emptyset)$.
2. Given $X \in \mathcal{S}$,

$$X \leq Y \text{ and } [a, X] \neq \emptyset \Rightarrow \exists Y' \in [d, Y] ([a, Y'] \subseteq [a, X])$$

(A.6) Pigeon hole principle: Suppose $a \in \mathcal{AR}$ has length $l$ and $\mathcal{O} \subseteq \mathcal{AR}_{l+1} = r_{l+1}(\mathcal{R})$. Then for every $Y \in \mathcal{S}$ with $[a, Y] \neq \emptyset$, there exists $X \in [\text{depth}_Y (a), Y]$ such that $r_{l+1}([a, X]) \subseteq \mathcal{O}$ or $r_{l+1}([a, X]) \subseteq \mathcal{O}^c$.

**Definition 1.** We say that $\mathcal{X} \subseteq \mathcal{R}$ is $\mathcal{S}$-Ramsey if for every $[a, Y]$ there exists $X \in [\text{depth}_Y (a), Y]$ such that $[a, X] \subseteq \mathcal{X}$ or $[a, X] \subseteq \mathcal{X}^c$. If for every $[a, Y] \neq \emptyset$ there exists $X \in [\text{depth}_Y (a), Y]$ such that $[a, X] \subseteq \mathcal{X}$, we say that $\mathcal{X}$ is $\mathcal{S}$-Ramsey null.

**Definition 2.** We say that $\mathcal{X} \subseteq \mathcal{R}$ is $\mathcal{S}$-Baire if for every $[a, Y] \neq \emptyset$ there exists a nonempty $[b, X] \subseteq [a, Y]$ such that $[b, X] \subseteq \mathcal{X}$ or $[b, X] \subseteq \mathcal{X}^c$. If for every $[a, Y] \neq \emptyset$ there exists a nonempty $[b, X] \subseteq [a, Y]$ such that $[b, X] \subseteq \mathcal{X}$, we say that $\mathcal{X}$ is $\mathcal{S}$-meager.

It is clear that every $\mathcal{S}$-Ramsey set is $\mathcal{S}$-Baire and every $\mathcal{S}$-Ramsey null set is $\mathcal{S}$-meager.

Considering $\mathcal{AS}$ with the discrete topology and $\mathcal{AS}^\mathbb{N}$ with the completely metrizable product topology; we say that $\mathcal{S}$ is closed if it corresponds to a closed subset of $\mathcal{AS}^\mathbb{N}$ via the identification $X \rightarrow (s_n(X))_{n \in \mathbb{N}}$. 
Definition 3. We say that \((R, S, \leq, \leq^0, r, s)\) is a Ramsey space if every \(S\)-Baire subset of \(R\) is \(S\)-Ramsey and every \(S\)-meager subset of \(R\) is \(S\)-Ramsey null.

Theorem 1 (Abstract Ramsey theorem). Suppose \((R, S, \leq, \leq^0, r, s)\) satisfies (A.1) ... (A.6) and \(S\) is closed. Then ■

Example: The Hales-Jewett space
Fix a countable alphabet \(L = \sqcup_{n \in \mathbb{N}} L_n\) with \(L_n \subseteq L_{n+1}\) and \(L_n\) finite for all \(n\); fix \(v \notin L\) a "variable" and denote \(W_L\) and \(W_{Lv}\) the semigroups of words over \(L\) and of variable words over \(L\), respectively. Given \(X = (x_n)_{n \in \mathbb{N}} \subseteq W_L \cup W_{Lv}\), we say that \(X\) is rapidly increasing if
\[
|x_n| > \sum_{i=0}^{n-1} |x_i|
\]
for all \(n \in \mathbb{N}\). Put
\[
W_L^{[\infty]} = \{X = (x_n)_{n \in \mathbb{N}} \subseteq W_L : X \text{ is rapidly increasing}\}
\]
\[
W_{Lv}^{[\infty]} = \{X = (x_n)_{n \in \mathbb{N}} \subseteq W_{Lv} : X \text{ is rapidly increasing}\}
\]
By restricting to finite sequences with
\[
r_n : W_L^{[\infty]} \to W_L^{[n]} \quad s_n : W_{Lv}^{[\infty]} \to W_{Lv}^{[n]}
\]
we have rapidly increasing finite sequences of words or variable words. The combinatorial subspaces are defined for every \(X \in W_L^{[\infty]}\) by
\[
[X]_L = \{x_n[\lambda_0] \cdots x_n[k] : n_0 < \cdots < n_k, \lambda_i \in L_n\}
\]
\[
[X]_{Lv} = \{x_n[\lambda_0] \cdots x_n[k] : n_0 < \cdots < n_k, \lambda_i \in L_n, \cup \{v\}\}
\]
where "\(^\sim\)" denotes concatenation of words and \(x[\lambda]\) is the evaluation of the variable word \(x\) on the letter \(\lambda\).
For \(w \in [X]_L \cup [X]_{Lv}\) we call support of \(w\) in \(X\) the unique set \(\text{supp}_X(w) = \{n_0 < n_1 < \cdots < n_k\}\) such that \(w = x_n[\lambda_0] \cdots x_n[k]\) as in the definition of the combinatorial subspaces \([X]_L\) and \([X]_{Lv}\). We say that \(Y = (y_n)_{n \in \mathbb{N}} \in W_{Lv}^{[\infty]}\) is a block subsequence of \(X = (x_n)_{n \in \mathbb{N}} \in W_L^{[\infty]}\) if \(y_n \in [X]_{Lv} \forall n\) and
\[
\max(\text{supp}_X(y_n)) < \min(\text{supp}_X(y_m))
\]
whenever \( n < m \), and write \( Y \leq X \). We define the relation \( \leq^0 \) on \( W^{[\omega]}_L \times W^{[\omega]}_{Lv} \) in the natural way. Then, if \((\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s) = (W^{[\omega]}_L, W^{[\omega]}_{Lv}, \leq, \leq^0, r, s)\) is as before, where \( r, s \) are the restrictions

\[
\begin{align*}
  r_n(X) &= (x_0, x_1, \ldots, x_{n-1}) \\
  s_n(Y) &= (y_0, y_1, \ldots, y_{n-1})
\end{align*}
\]

we have \((A.1)\ldots(A.6)\), particularly, \((A.6)\) is the well known result:

**Theorem 2.** For every finite coloring of \( W_L \cup W_{Lv} \) and every \( Y \in W^{[\omega]}_{Lv} \) there exists \( X \leq Y \) in \( W^{[\omega]}_{Lv} \) such that \([X]_L\) and \([X]_{Lv}\) are monochromatic. ■

And as a particular case of theorem 1, we have (see [7])

**Theorem 3** (Hales–Jewett). The field of \( W^{[\omega]}_{Lv} \)-Ramsey subsets of \( W^{[\omega]}_L \) is closed under the Souslin operation and it coincides with the field of \( W^{[\omega]}_{Lv} \)-Baire subsets of \( W^{[\omega]}_L \). Moreover, the ideals of \( W^{[\omega]}_{Lv} \)-Ramsey null subsets of \( W^{[\omega]}_L \) and \( W^{[\omega]}_{Lv} \)-meager subsets of \( W^{[\omega]}_L \) are \( \sigma \)-ideals and they also coincide. ■

### 3 The parametrization

We will denote the family of perfect subsets of \( 2^{\omega} \) by \( \mathbb{P} \) and define

\[
\mathcal{A}R[X] = \{ b \in \mathcal{A}R : [b, X] \neq \emptyset \}
\]

also we'll use this notation

\[
M \in \mathbb{P} \upharpoonright Q \iff (M \in \mathbb{P}) \land (M \subseteq Q)
\]

From now on we assume that \((\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)\) is an Ramsey space; that is, \((A.1)\ldots(A.6)\) hold and \( \mathcal{S} \) is closed. The following are the abstract versions of perfectly-Ramsey sets and the \( \mathbb{P} \times \text{Exp}(\mathcal{R}) \)-Baire property as defined in [5].

**Definition 4.** \( \Lambda \subseteq 2^{\omega} \times \mathcal{R} \) is **perfectly \( \mathcal{S} \)-Ramsey** if for every \( Q \in \mathbb{P} \) and \([a, Y] \neq \emptyset\), there exist \( M \in \mathbb{P} \upharpoonright Q \) and \( X \in [\text{depth}(a), Y] \) with \([a, X] \neq \emptyset\) such that \( M \times [a, X] \subseteq \Lambda \) or \( M \times [a, X] \subseteq \Lambda^c \). If for every \( Q \in \mathbb{P} \) and \([a, Y] \neq \emptyset\), there exist \( M \in \mathbb{P} \upharpoonright Q \) and \( X \in [\text{depth}(a), Y] \) with \([a, X] \neq \emptyset\) such that \( M \times [a, X] \subseteq \Lambda^c \); we say that \( \Lambda \) is **perfectly \( \mathcal{S} \)-Ramsey null**
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Definition 5. $\Lambda \subseteq 2^\infty \times R$ is perfectly $S$-Baire if for every $Q \in \mathbb{P}$ and $[a, Y] \neq \emptyset$, there exist $M \in \mathbb{P} \upharpoonright Q$ and $[b, X] \subseteq [a, Y]$ such that $M \times [b, X] \subseteq \Lambda$ or $M \times [b, X] \subseteq \Lambda^c$. If for every $Q \in \mathbb{P}$ and $[a, Y] \neq \emptyset$, there exist $M \in \mathbb{P} \upharpoonright Q$ and $[b, X] \subseteq [a, Y]$ such that $M \times [b, X] \subseteq \Lambda^c$; we say that $\Lambda$ is perfectly $S$-meager.

Now, the natural extension of combinatorial forcing will be given. From now on fix $\mathcal{F} \subseteq 2^{<\infty} \times \mathcal{A}R$ and $\Lambda \subseteq 2^\infty \times R$.

Combinatorial forcing 1 Given $Q \in \mathbb{P}$, $Y \in \mathcal{S}$ and $(u, a) \in 2^{<\infty} \times \mathcal{A}R[Y]$; we say that $(Q, Y)$ accepts $(u, a)$ if for every $x \in Q(u)$ and for every $B \in [a, Y]$ there exist integers $k$ and $m$ such that $(x_{|k}, r_m(B)) \in \mathcal{F}$.

Combinatorial forcing 2 Given $Q \in \mathbb{P}$, $Y \in \mathcal{S}$ and $(u, a) \in 2^{<\infty} \times \mathcal{A}R[Y]$; we say that $(Q, Y)$ accepts $(u, a)$ if $Q(u) \times [a, Y] \subseteq \Lambda$.

For both combinatorial forcings we say that $(Q, Y)$ rejects $(u, a)$ if for every $M \in \mathbb{P} \upharpoonright Q(u)$ and for every $X \leq Y$ compatible with $a$; $(M, X)$ does not accept $(u, a)$. Also, we say that $(Q, Y)$ decides $(u, a)$ if it accepts or rejects it.

The following lemmas hold for both combinatorial forcings.

Lemma 2. a) If $(Q, Y)$ accepts (rejects) $(u, a)$ then $(M, X)$ also accepts (rejects) $(u, a)$ for every $M \in \mathbb{P} \upharpoonright Q(u)$ and for every $X \leq Y$ compatible with $a$.

b) If $(Q, Y)$ accepts (rejects) $(u, a)$ then $(Q, X)$ also accepts (rejects) $(u, a)$ for every $X \leq Y$ compatible with $a$.

c) For all $(u, a)$ and $(Q, Y)$ with $[a, Y] \neq \emptyset$, there exist $M \in \mathbb{P} \upharpoonright Q$ and $X \leq Y$ compatible with $a$, such that $(M, X)$ decides $(u, a)$.

d) If $(Q, Y)$ accepts $(u, a)$ then $(Q, Y)$ accepts $(u, b)$ for every $b \in r_{|a|+1}([a, Y])$.

e) If $(Q, Y)$ rejects $(u, a)$ then there exists $X \in [\text{depth}_Y(a), Y]$ such that $(Q, Y)$ does not accept $(u, b)$ for every $b \in r_{|a|+1}([a, X])$.

f) $(Q, Y)$ accepts (rejects) $(u, a)$ iff $(Q, Y)$ accepts (rejects) $(v, a)$ for every $v \in 2^{<\infty}$ such that $u \subseteq v$.

Proof: (a) and (b) follow from the inclusion: $M(u) \times [a, X] \subseteq Q(u) \times [a, Y]$ if $X \leq Y$ and $M \subseteq Q(u)$.
Suppose that we have \((Q, Y)\) such that for every \(M \in \mathbb{P} \) and every \(X \leq Y\) compatible with \(a\), \((M, X)\) does not decide \((u, a)\). Then \((M, X)\) does not accept \((u, a)\) if \(M \in \mathbb{P} \mid Q(u)\); i.e. \((Q, Y)\) rejects \((u, a)\).

(d) Follows from: \(a \subseteq b\) and \([a, Y] \subseteq [b, Y]\), if \(b \in r_{|a|+1}([a, X])\).

(e) Suppose \((Q, Y)\) rejects \((u, a)\) and define \(\phi: \mathcal{AR}_{|a|+1} \rightarrow 2\) by \(\phi(b) = 1\) if \((Q, Y)\) accepts \((u, b)\). By \((A.6)\) there exist \(X \in [\text{depth}_Y(a), Y]\) such that \(\phi\) is constant in \(r_{|a|+1}([a, X])\). If \(\phi(r_{|a|+1}([a, X])) = 1\) then \((Q, X)\) accepts \((u, a)\), which contradicts \((Q, Y)\) rejects \((u, a)\) (by part (b)). The result follows.

(f) \((\Leftarrow)\)Obvious.

\((\Rightarrow)\) Follows from the inclusion: \(Q(v) \subseteq Q(u)\) if \(u \subseteq v\). 

We say that a sequence \([n_k, Y_k]\) \(k \in \mathbb{N}\) is a fusion sequence if:

1. \((n_k)_{k \in \mathbb{N}}\) is nondecreasing and converges to \(\infty\).
2. \(X_{k+1} \in [n_k, X_k]\) for all \(k\).

Note that since \(S\) is closed, for every fusion sequence \([n_k, Y_k])_{k \in \mathbb{N}}\) there exist a unique \(Y \in S\) such that \(s_{n_k}(Y) = s_{n_k}(X_k)\) and \(Y \in [n_k, X_k]\) for all \(k\). \(Y\) is called the fusion of the sequence and is denoted \(\text{lim}_k X_k\).

**Lemma 3.** Given \(P \in \mathbb{P}, Y \in S\) and \(N \geq 0\); there exist \(Q \in \mathbb{P} \mid P\) and \(X \leq Y\) such that \((Q, X)\) decides every \((u, a)\) \(\in 2^{<\infty} \times \mathcal{AR}[X]\) with \(N \leq \text{depth}_X(a) \leq |u|\).

**Proof:** We build sequences \((Q_k)_k\) and \((Y_k)_k\) such that:

1. \(Q_0 = P, Y_0 = Y\).
2. \(n_k = N + k\).
3. \((Q_{k+1}, Y_{k+1})\) decides every \((u, b)\) \(\in 2^{n_k} \times \mathcal{AR}[Y_k]\) with \(\text{depth}_Y(b) = n_k\).

Suppose we have defined \((Q_k, Y_k)\). List \(\{b_0, \ldots, b_r\} = \{b \in \mathcal{AR}[Y_k]: \text{depth}_Y(b) = n_k\}\) and \(\{u_0, \ldots, u^{n_k-1}\} = 2^{n_k}\). By lemma 1(c) there exist \(Q_k^{0,0} \in \mathbb{P} \mid Q_k(u_0)\) and \(Y_k^{0,0} \in [n_k, Y_k]\) compatible with \(b_0\) such that \((Q_k^{0,0}, Y_k^{0,0})\) decides \((u_0, b_0)\). In this way we can obtain \((Q_k^{i,j}, Y_k^{i,j})\) for every \((i, j) \in \{0, \ldots, 2^{n_k} - 1\} \times \{0, \ldots, r\}\), which decides \((u_i, b_j)\) and such that \(Q_k^{i,j+1} \in \mathbb{P} \mid Q_k^{i,j}(u_i), Y_k^{i,j+1} \leq Y_k^{i,j}\) is compatible with \(b_{j+1}, Q_k^{i+1,0} \in \mathbb{P} \mid Q_k(u_{i+1})\) and \(Y_k^{i+1,0} \leq Y_k^{i,r}\).
Define
\[ Q_{k+1} = \bigcup_{i=0}^{2^{n_k} - 1} Q_{i,r}^{i+1} \quad \text{and} \quad Y_{k+1} = Y_{k}^{2^{n_k} - 1,r} \]

Then, given \((u, b) \in 2^{n_k} \times AR[Y_{k+1}]\) with \(\text{depth}_{Y_{k+1}}(b) = n_k = \text{depth}_X(b)\), there exist \((i, j) \in \{0, \ldots, 2^{n_k} - 1\} \times \{0, \ldots, r\}\) such that \(u = u_i\) and \(b = b_j\). So \((Q_{k}^{i,j}, Y_{k}^{i,j})\) decides \((u, b)\) and, since
\[ Q_{k+1}(u_i) = Q_{k}^{i,r} \subseteq Q_{k}^{i,j}(u_i) \subseteq Q_{k}^{i,j} \quad \text{and} \quad Y_{k+1} \subseteq Y_k \]
we have \((Q_{k+1}, Y_{k+1})\) decides \((u, b)\) (by lemma 1(a)) We claim that \(Q = \cap_k Q_k\) and \(X = \lim_k Y_k\) are as required: given \((u, a) \in 2^{\leq \infty} \times AR[X]\) with \(N \leq \text{depth}_X(a) \leq |u|\), we have \(\text{depth}_X(a) = n_k = \text{depth}_Y(a)\) for some \(k\). Then, if \(|u| = n_k\), \((Q_{k+1}, Y_{k+1})\) from the construction of \(X\) decides \((u, a)\) and hence \((Q, X)\) decides \((u, a)\). If \(|u| > n_k\) \((Q, X)\) decides \((u, a)\) by lemma 1(f).

\textbf{Lemma 4.} Given \(P \in \mathbb{P}, Y \in \mathcal{S}, (u, a) \in 2^{\leq \infty} \times AR[Y]\) with \(\text{depth}_Y(a) \leq |u|\) and \((Q, X)\) as in lemma 2 with \(N = \text{depth}_Y(a)\); if \((Q, X)\) rejects \((u, a)\) then there exist \(Z \leq X\) such that \((Q, Z)\) rejects \((v, b)\) if \(u \sqsubset v, a \sqsubset b\) and \(\text{depth}_Z(b) \leq |v|\).

\textbf{Proof:} Let’s build a fusion sequence \([n_k, Z_k]_k\), with \(n_k = |u| + k\). Let \(Z_0 = X\). Then \((Q, Z_0)\) rejects \((u, a)\) (and by lemma 1(f) it rejects \((v, a)\) if \(u \sqsubset v\)). Suppose we have \((Q, Z_k)\) which rejects every \((v, b)\) with \(v \subseteq 2^{n_k}\) extending \(u, a \sqsubset b\) and \(\text{depth}_{Z_k}(b) \leq n_k\). List \(\{b_0, \ldots, b_r\} = \{b \in AR[Z_k]: a \sqsubset b\} \) and \(\text{depth}_{Z_k}(b) \leq n_k\}\} and \(\{u_0, \ldots, u_s\}\) the set of all \(v \subseteq 2^{n_k+1}\) extending \(u\). By lemma 1(f) \((Q, Z_k)\) rejects \((u_i, b_j)\), for every \((i, j) \in \{0, \ldots, s\} \times \{0, \ldots, r\}\). Use lemma 1(e) to find \(Z_{k,0}^0 \subseteq [n_k, Z_k]\) such that \((Q, Z_{k,0}^0)\) rejects \((u_0, b)\) if \(b \in r_{|b_0|+1}([b_0, Z_{k,0}^0])\). In this way, for every \((i, j) \in \{0, \ldots, s\} \times \{0, \ldots, r\}\), we can find \(Z_{k,i,j}^0 \subseteq [n_k, Z_k]\) such that \(Z_{k+1} = Z_{k,i,j}^0 \subseteq [n_k, Z_k]^0\}\}) and \((Q, Z_{k,i,j}^0)\) rejects \((u_i, b)\) if \(b \in r_{|b_j|+1}([b_j, Z_{k,i,j}^0])\). Define \(Z_{k+1} = Z_{k,i,j}^{r+1}\). Note that if \((v, b) \in 2^{\leq \infty} \times AR[Z_{k+1}], a \sqsubset b, u \sqsubset v\) and \(\text{depth}_{Z_{k+1}}(b) = n_k + 1\) then \(v = u_i\) for some \(i \in \{0, \ldots, s\}\) and \(b = r_{|b|}(A)\), \(a = r_{|a|}(A)\) for some \(A \subseteq Z_{k+1}\); by (A.4)(5) there exist \(m \leq n_k\) such that \(b' = r_{|b|+1}(A) \leq s_m(Z_{k+1})\), so \(\text{depth}_{Z_{k+1}}(b') \leq n_k\), i.e. \(b' = b_j\) for some \(j \in \{0, \ldots, r\}\). Then \(b \in r_{|b|+1}([b_j, Z_{k+1}^0])\). Hence, by lemma 1(f), \((Q, Z_{k+1})\) rejects \((v, b)\). Then \(Z = \lim_k Z_k\) is as required: given \((v, b)\) with \(u \sqsubset v, a \sqsubset b\) and \(\text{depth}_{Z}(b) \leq |v|\) then \(\text{depth}_{Z}(b) = \text{depth}_{Y}(a) + k \leq n_k\) for some \(k\) and \(b \in r_{|b|+1}([b_j, Z_{k+1}^j])\) for some \(j \in \{0, \ldots, r\}\) from the construction of \(Z\) (again, by (A.4)(5)). So


$(Q, Z_k)$ (from the construction of $Z$) rejects $(v, b)$ and, by lemma 1(a), $(Q, Z)$ also does it. ■

The following theorem is an extension of theorem 3 [8] and its proof is analogous.

**Theorem 4.** For every $F \subseteq 2^{<\omega} \times AR$, $P \in P$, $Y \in S$ and $(u, a) \in 2^{<\omega} \times AR$ there exist $Q \in \mathbb{P} \upharpoonright P$ and $X \leq Y$ such that one of the following holds:

1. For every $x \in Q$ and $A \in [a, X]$ there exist integers $k, m > 0$ such that $(x|_k, r_m(A)) \in F$.

2. $(T_Q \times AR[X]) \cap F = \emptyset$.

**Proof:** Without loss of generality, we can assume $(u, a) = (\langle \rangle, \emptyset)$. Consider combinatorial forcing 1. Let $(Q, X)$ as in lemma 3 ($N = 0$). If $(Q, X)$ accepts $(\langle \rangle, \emptyset)$, part (1) holds. If $(Q, X)$ rejects $(\langle \rangle, \emptyset)$, use lemma 4 to obtain $Z \leq X$ such that $(Q, X)$ detects $(u, a) \in 2^{<\omega} \times AR[Z]$. If $(t, b) \in (T_Q \times AR[Z]) \cap F$, find $u_t \in 2^{<\omega}$ such that $Q(u_t) \subseteq Q \cap [t]$. Thus, $(Q, Z)$ accepts $(u, b)$. In fact: for $x \in Q(u_t)$ and $B \in [b, Z]$ we have $(x|_k, r_m(A)) = (t, b) \in F$ if $k = |t|$ and $m = |b|$. By lemma 2(f), $(Q, Z)$ accepts $(v, b)$ if $u_t \sqsubset v$ and $depth_Z(b) \leq |v|$. But this contradicts the choice of $Z$. Hence, $(T_Q \times AR[X]) \cap F = \emptyset$. ■

The next theorem is our main result and its proof is analogous to theorem 3 [8].

**Theorem 5.** For $\Lambda \subseteq 2^{<\omega} \times R$ we have:

1. $\Lambda$ is perfectly $S$-Ramsey iff it is perfectly $S$-Baire.

2. $\Lambda$ is perfectly $S$-Ramsey null iff it is perfectly $S$-meager.

**Proof:** (1) We only have to prove the implication from right to left. Suppose that $\Lambda \subseteq 2^{<\omega} \times R$ is perfectly $S$-Baire. Again, without loss of generality, we can lead with a given $Q \times [0, Y]$. Using combinatorial forcing and lemma 3, we have the following:

**Claim 1.** Given $\hat{\Lambda} \subseteq 2^{<\omega} \times R$, $P \in P$ and $Y \in S$, there exists $Q \in \mathbb{P} \upharpoonright P$ and $X \leq Y$ such that for each $(u, b) \in 2^{<\omega} \times AR[X]$ with $depth_X(b) \leq |u|$ one of the following holds:

i) $Q(u) \times [b, X] \subseteq \hat{\Lambda}$

ii) $R \times [b, Z] \not\subseteq \hat{\Lambda}$ for every $R \subseteq Q(u)$ and every $Z \leq X$ compatible with $b$. 

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By applying the claim to $\Lambda$, $P$ and $Y$, we find $Q_1 \subseteq P \upharpoonright P$ and $X_1 \leq Y$ such that for each $(u, b) \in 2^{<\infty} \times AR[X_1]$ with $\text{depth}_{X_1}(b) \leq |u|$ one of the following holds:

- $Q_1(u) \times [b, X_1] \subseteq \Lambda$ or
- $R \times [b, Z] \not\subseteq \Lambda$ for every $R \subseteq Q_1(u)$ and every $Z \leq X_1$ compatible with $b$.

For each $t \in T_{Q_1}$, choose $u_1^t \in 2^{<\infty}$ with $u_1^t(Q_1) \upharpoonright t$. If we define the family

$$F_1 = \{(t, b) \in T_{Q_1} \times AR[X_1]: Q_1(u_1^t) \times [b, X_1] \subseteq \Lambda\}$$

then we find $S_1 \subseteq Q_1$ and $Z_1 \leq X_1$ as in theorem 4. If (1) of theorem 4 holds, we are done. If part (2) holds, apply the claim to $\Lambda^c$, $S_1$ and $Z_1$ to find $Q_2 \subseteq P \upharpoonright P$ and $X_2 \leq Y$ such that for each $(u, b) \in 2^{<\infty} \times AR[X_2]$ with $\text{depth}_{X_2}(b) \leq |u|$ one of the following holds:

- $Q_2(u) \times [b, X_2] \subseteq \Lambda^c$ or
- $R \times [b, Z] \not\subseteq \Lambda^c$ for every $R \subseteq Q_2(u)$ and every $Z \leq X_2$ compatible with $b$.

Again, for each $t \in T_{Q_2}$, choose $u_2^t \in 2^{<\infty}$ with $u_2^t(Q_2) \upharpoonright t$. Define the family

$$F_2 = \{(t, b) \in T_{Q_2} \times AR[X_2]: Q_2(u_2^t) \times [b, X_2] \subseteq \Lambda\}$$

and find $S_2 \subseteq Q_2$ and $Z_2 \leq X_2$ as in theorem 4. If (1) holds, we are done and part (2) is not possible since $\Lambda$ is perfectly $S$-Baire (see [8]). This proves (1).

To see part (2), notice that, as before, we only have to prove the implication from right to left, which follows from part (1) if $\Lambda$ is perfectly $S$-meager.

**Corollary 1** (Parametrized infinite dimensional Hales-Jewett theorem). For $\Lambda \subseteq 2^{<\infty} \times W^{[\infty]}_L$ we have:

1. $\Lambda$ is perfectly Ramsey iff it has the $P \times W^{[\infty]}_L$-Baire property.
2. $\Lambda$ is perfectly Ramsey null iff it is $P \times W^{[\infty]}_L$-meager.

Making $R = S$ in $(\mathcal{R}, \leq, R, S, \leq, [0, r, s])$, we obtain the following:

**Corollary 2** (Mijares). If $(\mathcal{R}, \leq, (p_n)_{n \in \mathbb{N}})$ is a topological Ramsey space then:

1. $\Lambda \subseteq \mathcal{R}$ is perfectly Ramsey iff has the $P \times \text{Exp}(\mathcal{R})$-Baire property.
Corollary 3 (Pawlikowski). For $\Delta \subseteq 2^\omega \times N[\omega]$ we have:

1. $\Lambda$ is perfectly Ramsey iff it has the $P \times \text{Exp}(N[\omega])$-Baire property.

2. $\Lambda$ is perfectly Ramsey null iff $\Lambda$ is $P \times \text{Exp}(N[\omega])$-meager.

Now we will prove that the family of perfectly $S$-Ramsey and perfectly $S$-Ramsey null subsets of $2^\omega \times \mathcal{R}$ are closed under the Souslin operation. Recall that the result of applying the Souslin operation to a given $(\Lambda_{\alpha})_{\alpha \in \mathcal{A}}$ is

$$\bigcup_{\alpha \in \mathcal{A}} \bigcap_{n \in \mathbb{N}} \Lambda_{\alpha_n}(A)$$

Proposition 1. The perfecty $S$-Ramsey null subsets of $2^\omega \times \mathcal{R}$ form a $\sigma$-ideal.

Proof: This proof is also analogous to its corresponding version in [8] (lemma 3). So we just expose the main ideas. Given an increasing sequence of perfectly $S$-Ramsey null subsets of $2^\omega \times \mathcal{R}$ and $P \times [\emptyset, Y]$, we proceed as in lemma 3 to build fusion sequences $(Q_n)_n$ and $[n+1, X_n]$ such that

$$Q_n \times [b, X_n] \cap \Lambda_n = \emptyset$$

for every $n \in \mathbb{N}$ and $b \in \mathcal{A}R[X_n]$. Thus, if $Q = \cap_n Q_n$ and $X = \lim_n X_n$, we have $Q \times [\emptyset, X] \cap \bigcup_n \Lambda_n = \emptyset$.

Recall that given a set $X$, two subsets $A, B$ of $X$ are "compatibles" with respect to a family $\mathcal{F}$ of subsets $X$ if there exists $C \in \mathcal{F}$ such that $C \subseteq A \cap B$. And $\mathcal{F}$ is $M$-like if for $\mathcal{G} \subseteq \mathcal{F}$ with $|\mathcal{G}| < |\mathcal{F}|$, every member of $\mathcal{F}$ which is not compatible with any member of $\mathcal{G}$ is compatible with $X \setminus \bigcup \mathcal{G}$. A $\sigma$-algebra $A$ of subsets of $X$ together with a $\sigma$-ideal $A_0 \subseteq A$ is a Marczewski pair if for every $A \subseteq X$ there exists $\Phi(A) \in \mathcal{A}$ such that $A \subseteq \Phi(A)$ and for every $B \subseteq \Phi(A) \setminus A$, $B \in A \Rightarrow B \in A_0$. The following is a well known fact:

Theorem 6 (Marczewski). Every $\sigma$-algebra of sets which together with a $\sigma$-ideal is a Marczewski pair, is closed under the Souslin operation.

Let’s denote $\mathcal{E}(\mathcal{S}) = \{[n, Y] : n \in \mathbb{N}, Y \in \mathcal{S}\}$.

Proposition 2. If $|\mathcal{S}| = 2^{\aleph_0}$, then the family $\mathcal{E}(\mathcal{S})$ is $M$-like.
Proof: Consider $B \subseteq \mathcal{E}(S)$ with $|B| < |\mathcal{E}(S)| = 2^{\aleph_0}$ and suppose that $[a, Y]$ is not compatible with any member of $B$, i.e., for every $B \in B$, $B \cap [a, Y]$ does not contain any member of $\mathcal{E}(S)$. We claim that $(Q, Y)$ is compatible with $\mathcal{R} \setminus \bigcup B$. In fact:

Since $|B| < 2^{\aleph_0}$, $\bigcup B$ is $\mathcal{S}$-Baire (it is $\mathcal{S}$-Ramsey). So, there exist $[b, X] \subseteq [a, Y]$ such that:

1. $[b, X] \subseteq \bigcup B$ or
2. $[b, X] \subseteq \mathcal{R} \setminus \bigcup B$

(1) is not possible because $[a, Y]$ is not compatible with any member of $B$. And (2) says that $[a, Y]$ is compatible with $\mathcal{R} \setminus \bigcup B$.

As consequences of the previous proposition and theorem 6, the following facts hold.

**Corollary 4.** If $|S| = 2^{\aleph_0}$, then the family of perfectly $\mathcal{S}$-Ramsey subsets of $2^\omega \times \mathcal{R}$ is closed under the Souslin operation.

**Corollary 5.** The field of perfectly $W^\omega_{L^\omega}$-Ramsey subsets of $2^\omega \times W^\omega_{L^\omega}$ is closed under the Souslin operation.

Finally, making $\mathcal{R} = S$ in $(\mathcal{R}, S, \leq, \leq^0, r, s)$, we obtain the following:

**Corollary 6** (Mijares). If $(\mathcal{R}, \leq, r)$ satisfies (A.1) . . . (A.6), $\mathcal{R}$ is closed, and $|\mathcal{R}| = 2^{\aleph_0}$ then the family of perfectly Ramsey subsets of $2^\omega \times \mathcal{R}$ is closed under the Souslin operation.

**Corollary 7** (Pawlikowski). The field of perfectly Ramsey subsets of $2^\omega \times \mathbb{N}^{[\omega]}$ is closed under the Souslin operation.

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