Abstract

We introduce and study the complexity of Path Packing. Given a graph $G$ and a list of paths, the task is to embed the paths edge-disjoint in $G$. This generalizes the well known Hamiltonian-Path problem.

Since Hamiltonian Path is efficiently solvable for graphs of small treewidth, we study how this result translates to the much more general Path Packing. On the positive side, we give an FPT-algorithm on trees for the number of paths as parameter. Further, we give an XP-algorithm with the combined parameters maximal degree, number of connected components and number of nodes of degree at least three. Surprisingly the latter is an almost tight result by runtime and parameterization. We show an ETH lower bound almost matching our runtime. Moreover, if two of the three values are constant and one is unbounded the problem becomes NP-hard.

Further, we study restrictions to the given list of paths. On the positive side, we present an FPT-algorithm parameterized by the sum of the lengths of the paths. Packing paths of length two is polynomial time solvable, while packing paths of length three is NP-hard. Finally, even the spacial case Exact Path Packing where the paths have to cover every edge in $G$ exactly once is already NP-hard for two paths on 4-regular graphs.

1 Introduction

Packing, covering and partitioning are well researched fields in graph theory. In general, the task is to cover a given graph $G = (V, E)$ with or partition it into smaller substructures, or
to pack given structures into the graph. Besides that these terms are often used, they are not well defined throughout the literature. Thus, it is important to define problems in this field carefully and in detail.

For example, the path partition problem is a well studied problem [2, 4, 6, 13, 19, 22, 27] which is also known as path cover problem. The task is to cover all vertices of a graph with vertex-disjoint paths. This is equivalent to partitioning the graph into vertex-disjoint paths. The smallest number of paths to achieve this is called the path partition number or path cover number. Observe that $G$ has a Hamiltonian path iff the path-partition number is one, thus the problem is NP-complete.

An NP-complete variant of this problem is the $k$-path partition problem [23, 28, 20]. Here the task is to partition a graph $G$ into paths, such that none of the path lengths exceeds $k$. Observe that the 1-path-partition problem corresponds to finding a maximum matching.

Another related problem is the recognition of path-perfect graphs [5, 11, 14, 24, 30], which we denote in this work as Path-Perfect Packing. Instead of partitioning the graph into vertex-disjoint paths, the complete edge set must be partitioned into edge-disjoint paths of ascending length, starting by one. This can also be understood as packing $k$ paths of length 1 to $k$ into $G$ without using an edge twice or leaving one edge uncovered.

This approach of packing smaller subgraphs into a given graph is also well researched [29]. For example, packing edge-disjoint trees into a clique is considered [25]. Since packing edge-disjoint and vertex-disjoint triangles is NP-hard for planar graphs, the parameterized complexity is studied [7].

We generalize the path-perfect graph problem and ask for a given graph $G$ and a list of $k$ paths $P = \{p_1, \ldots, p_k\}$ if they can be embedded into $G$ without using the same edge twice. Note that we define the length of a path equals its number of edges. This problem arises naturally when restricting the path partition problem to edge-disjoint paths instead of vertex-disjoint paths. We denote this problem as Path Packing. Let us formalize what we mean by embedding. An embedding of a graph $H$ into a graph $G$ is an injective mapping $f : V(H) \rightarrow V(G)$ such that for every original edge $(u, v) \in E(H)$ also $(f(u), f(v)) \in E(G)$.

An embedding of a list of graphs $H$ into $G$ is an embedding of each graph $H$ into $G$. Note, that we do not ask to embed the graphs pairwise vertex-disjointly. The embeddings we consider in this work are pairwise edge-disjoint embeddings of paths.

**Path Packing**

Input: A list of paths $P = \{p_1, \ldots, p_k\}$ of length $l_1, \ldots, l_k$. A graph $G = (V, E)$.

Question: Is there an edge-disjoint embedding of $P$ into $G$?

The **Exact Path Packing** problem additionally requires that every edge is covered exactly once.

**Exact Path Packing**

Input: A list of paths $P = \{p_1, \ldots, p_k\}$ of length $l_1, \ldots, l_k$. A graph $G = (V, E)$.

Question: Is there an edge-disjoint embedding of $P$ into $G$ such that each edge $e \in E$ is covered exactly once?

Path Packing is clearly more general than Exact Path Packing, since one can reduce from one to the other by additionally requiring the sum of the path lengths to be equal to the number of edges in the graph. Most of our hardness results are for Exact Path Packing, and therefore translate to Path Packing. Our upper bounds are always regarding more the general Path Packing.
Our Results

The Hamiltonian path problem is a special case of Path Packing. An even though the Hamiltonian path problem is tractable on graphs of bounded treewidth, Path Packing is already NP-complete on subdivided stars. Therefore, we focus on the parameterized complexity to classify this problem on a finer scale. We will analyze the impact of various parameters.

In Section 3, we analyze the parameterized complexity of our packing problems with respect to the number of paths (denoted by $k$). On the one hand, we give an FPT algorithm for Path Packing that solves the problem in time $2^k n^{O(1)}$ on subcubic (i.e. degree at most three) forests (Theorem 3). On the other hand, we show that Exact Path Packing on graphs with treewidth two is $\text{W}[1]$-hard and cannot be solved in time $f(k)n^{o(k/\log k)}$ under ETH (Theorem 11).

In Section 4 we introduce path dependent restrictions. We show that Exact Path Packing is NP-complete even for two paths on 4-regular graphs (Theorem 14). length $i$ is easy for $i = 2$ and NP-complete for $i = 3$ (Theorem 15). If we however parameterize by the summed length of all paths Path Packing is in FPT (Theorem 16).

After parameterizing by the number of paths and their lengths, we further analyze graph dependent parameters in Section 5. We introduce the $\text{bcd}$-number of a graph, which is the maximum of the number of components, the maximal degree, and the number of vertices with degree larger than two. We show that Path Packing can be solved in time $k! k^{O(k^2)}$, where $k$ is the $\text{bcd}$-number (Theorem 20). This is complemented by showing that the problem cannot be solved in $f(k)n^{o(k^2/\log k)}$ under ETH (Theorem 21). We further show that all three $\text{bcd}$ parameters are necessary: If two values are constant and one is unbounded the problem becomes NP-hard (Theorem 1, Corollary 18, Theorem 19).

Note that, one can embed paths $p_1, \ldots, p_k$ as edge-disjoint subgraphs into a graph $G$ if and only if one can embed these paths as vertex-disjoint induced subgraphs into the linegraph of $G$. Therefore, our results yield new insights for the problem of covering a graph with a list of vertex-disjoint induced paths [18]. Especially, our hardness results for certain graph classes transfer to hardness results on the linegraphs of these graph classes.

2 Preliminaries

All graphs are simple (i.e. without multi-edges or self-loops). The length of a path $p$ is denoted by $|p|$ and equals its number of edges.

3 Path Packing on Forests

Our packing problem is a generalization of the Hamiltonian path problem and therefore NP-complete. The Hamiltonian path problem is solvable in polynomial time if the treewidth of the input graph is bounded [9]. We show that (unlike Hamiltonian path) Exact Path Packing is NP-complete on trees. This is done by reducing it to the following NP-complete partitioning problem.

**Multi-Way Number Partition**

Input: A list of weights $w_1, \ldots, w_n \in \mathbb{N}$ encoded in unary, and an integer $k \in \mathbb{N}$.

Question: Is there a partition of $w_1, \ldots, w_n$ into $k$ multi-sets $S_1, \ldots, S_k$ such that $\sum_{i \in S_j} w_i = \frac{1}{k} \sum_{i=1}^{n} w_i$, for every $1 \leq j \leq k$?

We reduce from Multi-Way Number Partition to prove that Exact Path Packing is NP-hard on very simple trees.
The Complexity of Packing Edge-Disjoint Paths

Figure 1: Subdivided Star construed from an instance of the Multi-Way Number Partition.

Theorem 1. Exact Path Packing is NP-complete on subdivided stars.

Proof. We construct an instance of the exact path packing problem from an instance of Multi-Way Number Partition problem as follows. We create for each weight $w_j$ a path $p_j$ of length $w_j \cdot 2^k$, with $1 \leq j \leq n$ and $n$ the number of weights. So, the sum of the path lengths is $2^k \cdot \sum_{j=1}^{n} w_j$. To represent the sets in the Multi-Way Number Partition we construct $k$ paths of length $2 \cdot \sum_{j=1}^{n} w_j$ that share the center vertex $v$. Thus, we obtain a subdivided star where the center vertex $v$ is connected to $2^k$ paths of length $\sum_{j=1}^{n} w_j$.

Figure 1 shows the graph that results from an instance of the Multi-Way Number Partition problem with the weights $2, 3, 4, 6, 7, 8$ that should be partitioned into three sets. The overall sum of the weights is 30 and therefore the graph in Figure 1 has six paths of length 30 connected to one vertex $v$. For each weight, we construct one path with the length equal to the weight multiplied with six. Thus, the graph in Figure 1 needs to be covered by six paths of length $12, 18, 24, 36, 42$ and $48$.

Next, we need to show that the constructed exact path packing instance has a solution if and only if the Multi-Way Number Partition instance is feasible. Given a solution for the exact path packing problem on the constructed graph, we can compute a solution for the multi-way number problem as follows. A pair of two paths represent one set $S_i$ and the paths $p_j$ that cover the edges correspond to the weights $w_j$ that must be selected by the set $S_i$, for $1 \leq i \leq k$. The pair is unique if a path $p_j$ covers edges of both paths, like the paths of length $36, 42$ and $48$ from the example in Figure 1. Otherwise, there is an even number of edge covering paths that start or end at the center vertex $v$ and these paths can be combined arbitrarily to represent a set $S_i$. There are $k$ many pairs of paths of the same length where all edges are covered. Thus, the resulting sets refer to weights which sum up to $\frac{1}{k} \sum_{j=1}^{n} w_j$

If we have a solution for the multi-way number problem we can choose a pair of paths adjacent to the center vertex $v$ to represent a set $S_i$. The weights $w_j$ with $j \in S_i$ correspond to the paths $p_j$ that cover the chosen path pair. The sum of the path lengths covers exactly one path pair.

$$\sum_{j \in S_i} |p_j| = 2k \cdot \sum_{j \in S_i} w_j = 2 \cdot \sum_{j=1}^{n} w_j.$$
Figure 2 Packing paths of lengths 10, 8, 7, 5, 5, 3 into a subcubic tree. Although the packing looks very loose there is no solution if we replace 3 by 4.

Fast subset convolution

We develop dynamic programming algorithms on subcubic trees whose running time will be $O^*(2^k)$, where $k$ is the number of paths that we want to pack. First we develop a naive and not too complicated algorithm with running time $O^*(3^k)$, whose longer running time is due to some very simple operation that occurs when we combine two dynamic programming tables. Björklund, Husfeldt, Kaski, and Koivisto introduced a technique called fast subset convolution that was used to speed up the computation of Steiner trees with small integer weights [3] and also to speed up some algorithms that do dynamic programming on tree decompositions [26]. We can use this technique to our advantage to significantly speed up the path packing algorithm on trees. The result that we will be using is:

$\triangleright$ Proposition 2. [3] Let $N$ be a set of $n$ natural numbers and $f, g : N \to N$ two functions. Then we can compute $(f \ast g)(S)$ for all $S \subseteq N$ in time $O(2^n n^2)$ if $f$ and $g$ can be evaluated in constant time and where $(f \ast g)(S) = \sum_{T \subset S} f(T)g(S - T)$. Here $f(S) = \sum_{i \in S} f(i)$.

$\triangleright$ Theorem 3. We can solve Path Packing for $k$ paths in time $O^*(2^k)$ on subcubic forests.

Proof. Let us assume that the graph is a subcubic tree $T$, but the proof easily generalizes to subcubic forests. Let $l_1, \ldots, l_k$ be length of the paths that we want to pack into $T$. We can further assume that $T$ is a rooted tree by designating an arbitrary vertex as its root. If $v$ is a vertex of $T$ then let $T(v)$ be the subtree rooted at $v$.

We solve the path packing problem by dynamic programming computing a table for each vertex in a bottom-up order. Such a table is a mapping $L : V \times 2^k \to \{\infty\} \cup \{0\}$. The size of this table is $O(2^k n)$. We interpret the content of the table as follows:

$L(v, P) = r$ with $r \geq 0$ means that it is possible to pack all paths with indices in $P$ (in short all $P$-paths) into the subtree $T(v)$ and additionally a path of length $r$ that ends in $v$.

The special case $L(v, P) = 0$ means that we can pack all $P$-paths into $T(v)$, but however we pack them there is no space left to pack another path that ends in $v$.

If it is not possible to pack all $P$-paths into $T(v)$ at all then let $L(v, P) = -\infty$.

It is quite clear that having computed all tables enables us to find out whether $(T, P)$ is a yes-instance of the path packing-problem. Simply check whether $L(r, \{1, \ldots, k\}) \neq -\infty$.

To compute the tables for all $v$ we distinguish three cases how to compose trees into bigger trees: 1 $v$ is a leaf, 2 $v$ has one child, 3 $v$ has at least two children.
The Complexity of Packing Edge-Disjoint Paths

Figure 3 Left side: Packing paths of lengths $l_1 = 4$, $l_2 = 4$, $l_3 = 2$ into $T(v)$. $L(u, \{1, 2, 3\}) = -\infty$, but $L(u, \{1, 2, 3\} - \{1\}) = l_1 - 1$, so $L(v, \{1, 2, 3\}) = 0$.
Right side: Now $l_1 = 4$, $l_2 = 3$, $l_3 = 2$. $L(u, \{1, 2, 3\}) = 1$, so $L(v, \{1, 2, 3\}) = 1 + 1 = 2$. An additional path of length 2 can be packed into $T(v)$, because an additional path of length 1 can be packed into $T(u)$.

Leaf. If $v$ is a leaf then $L(v, \emptyset) = 0$ and $L(v, P) = -\infty$ if $P \neq \emptyset$ because we cannot pack any path into an empty tree (that has no edges).

One child. If $v$ has one child $u$ then it is also quite easy to compute $L(v, P)$: If $L(u, P) = r$ with $r \geq 0$, then clearly $L(v, P) = r + 1$. The right hand side of Figure 3 shows an example. The more complicated possibility is $L(u, P) = -\infty$, which means that it is completely impossible to pack all $P$-paths into $T(u)$. It might become possible to pack all $P$-paths into $T(v)$ by using the additional edge $uv$. If this is possible, then one path, say the $i$th one with length $l_i$, uses the edge $uv$. Then all paths in $P - \{i\}$ are packed into $T(u)$ and one additional path of length $l_i - 1$ that ends in $u$. We can check this by verifying that $L(u, P - \{i\}) \geq l_i - 1$ for some $1 \leq i \leq k$ (actually, $L(u, P - \{i\}) \geq l_i$ is impossible, because then all $P$-paths could be packed into $T(u)$ and $L(u, P) \neq -\infty$). If we find such an $i$, then all $P$-paths can be packed into $T(v)$, but only by using the edge $uv$. This means that no other path can be packed into $T(v)$ that ends in $v$ and therefore $L(v, P) = 0$. See the left hand side of Figure 3 for an example.

$$L(v, P) = \begin{cases} 
L(u, P) + 1 & \text{if } L(u, P) \geq 0 \\
0 & \text{if } L(u, P) = -\infty \text{ and } L(u, P - \{i\}) = l_i - 1 \text{ for some } i \in P \\
-\infty & \text{otherwise}
\end{cases}$$

Two children. Finally, we assume that $v$ has exactly two children $u_1, u_2$. In that case we can construct for each of them a new tree by attaching new roots $v_1, v_2$ to $T(u_1)$ and $T(u_2)$ and computing the $L$-tables for both of them. To compute the table of $v$ it is sufficient to compute a table for a tree that we get by glueing two trees together by identifying their roots. We just have to glue $v_1$ to $v_2$.

So we can assume that we have two trees with roots $v_1$ and $v_2$ and a tree with root $v$ that we get by identifying $v_1$ and $v_2$ and renaming it to $v$. This is often called a join operation. We have the tables for $v_1$ and $v_2$ and want to compute the table for $v$.

Clearly, $L(v, P) = r$ with $r > 0$ iff some of the $P$-paths can be packed into $T(v_1)$ and the others into $T(v_2)$ and the additional path with length $r$ that ends in $v$ can be packed into $T(v_1)$ or $T(v_2)$. The additional path of length $r$ that ends in $v$ prevents any $P$-path from being packed partially into $T(v_1)$ and $T(v_2)$. That is the case iff there is a bipartition of $P$ into $P_1$ and $P_2$ such that $L(v_1, P_1), L(v_2, P_2) \geq 0$ and $\max\{L(v_1, P_1), L(v_2, P_2)\} = r$. There are $2^{|P|}$ many subsets of $P$. To check all bipartitions for all these subsets $P \subseteq [k]$ means looking at $\sum_{i=0}^{k} \binom{k}{i} 2^i = 3^k$ many cases. Using fast subset convolution lets us speed up the
Again, by using fast subset convolution we can check this in $\mathcal{O}(1)$ steps: Let

$$f_i(S) = \begin{cases} 1 & \text{if } L(v_i, S) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g_i(S) = \begin{cases} 1 & \text{if } L(v_i, S) \geq r \\ 0 & \text{otherwise.} \end{cases}$$

Then $L(v, P) \geq r$ iff $(f_1 * g_2)(P) + (g_1 * f_2)(P) \geq 1$.

The situation is different if $L(v, P) = 0$. In that case both edges $u_1v$ and $u_2v$ have to be used when packing all $P$-paths into $T(v)$ because otherwise at least a path of length one that ends in $v$ could additionally be packed into $T(v)$.

In such a packing one path, say the $i$th one with length $l_i$, uses $u_1v$ and $u_2v$. That is possible iff there is a bipartition of $P - \{i\}$ into $P_1$ and $P_2$ such that $L(v_1, P_1) + L(v_2, P_2) \geq l_i$.

Again, by using fast subset convolution we can check this in $2^k\mathcal{O}(1)$ steps.

The above proof does not work any more if we glue together two trees whose roots have degree higher than one. For general trees the dynamic programming is much more complicated and we will need more complicated tables.

Let $T(v)$ be a rooted tree with root $v$ and no restrictions on the degree of vertices (and thus on the number of children). Let us again fix length $l_1, \ldots, l_k \in \mathbb{N}$ of paths that we are going to pack into a tree. We are going to identify a set of paths by a set $P \subseteq [k]$. We speak of $P$-paths as the paths with length $l_i$ for every $i \in P$.

**Definition 4.** Let us fix $l_1, \ldots, l_k \in \mathbb{N}$, $P \subseteq [k]$, and $T$ be a rooted tree. $T(v)$ is the subtree of $T$ with root $v$.

1. Let $M, M' \subseteq \mathbb{N}$ be two multisets. We say that $M' \succ M$ if we can construct $M'$ from $M$ by adding numbers and increasing numbers that are already in $M'$.

   Let $M' \succ M$ iff $M' \neq M$ and $M' \succeq M$.

   Example: $\{3, 3, 5, 5, 7\} \succ \{2, 3, 4, 6\}$, but $\{3, 3, 5, 5, 7\} \nprec \{2, 3, 4, 8\}$.

2. Let $S$ be a set of multisets of natural numbers. Then

   $$K(S) = \{ M \in S \mid \text{there is no } M' \in S \text{ with } M' \succ M \}.$$ 

3. Then we define $L(v, P)$ as a set of multisets of natural numbers as follows:

   Let $M \subseteq \mathbb{N}$ be a multiset of natural numbers. Then $M \in L(v, P)$ iff it is possible to pack all $P$-paths into $T(v)$ such that we can pack additionally all non-empty paths into $T(v)$ that start at $v$ and have lengths given in $M$ and if there is no $M' \in L(v, P)$ with $M' \succeq M$.

   Particularly, $L(v, P) = \emptyset$ iff it is impossible to pack all $P$-paths into $T(v)$ and $L(v, P) = \{\emptyset\}$ iff it is possible to pack all $P$-paths into $T(v)$, but there is no possibility to additionally pack a non-empty path that starts at $v$.
4. If $M \subseteq \mathbb{N}$ then $\max_q(M)$ is the multiset that consists of the $q$ biggest elements in $M$ or of all of them if $M$ contains less than $q$ numbers, e.g., $\max_3(\{5, 5, 4, 4, 3, 2, 1\}) = \{5, 5, 4\}$.

In the following let $l_1, \ldots, l_k$ be fixed.

Lemma 5. Let $T(v)$ be a rooted tree with root $v$ such that $v$ has one child $u$.
1. Assume that $L(u, P) = \{M_1, \ldots, M_m\}$ with $m \geq 1$. Define $L_{\max}(u, P) = \max(M_1 \cup \cdots \cup M_m)$ (where $\max\emptyset = 0$).
   Then $L(v, P) = \{\{L_{\max}(u, P) + 1\}\}$.
2. Assume that $L(u, P) = \emptyset$ and there is an $i \in \{1, \ldots, k\}$ with $L_{\max}(u, P - \{i\}) = l_i - 1$.
   Then $L(v, P) = \emptyset$.
3. Otherwise $L(v, P) = \emptyset$.

Proof. We have to consider exactly two cases. The first case is that it is possible to pack all $P$-paths into $T(u)$. If this is the case, then an additional path of length $r + 1$ can be packed into $T(v)$ starting at $v$ iff an additional path of length $r$ can be packed into $T(u)$ starting at $u$. The latter is the case iff $L_{\max}(u, P) = r$.

The second case is that it is impossible to pack all $P$-paths into $T(u)$ alone. It might still be possible to pack them into $T(v)$, but only if the edge $uv$ is used. This means that there is only space for an additional path of length zero that starts at $v$.

In fact, exactly one path, say the $i$th one, uses the edge $uv$. This is possible iff we can pack all $(P - \{i\})$-paths into $T(u)$ and being able to additionally pack a path of length at least $l_i - 1$ into $T(u)$ starting at $u$. Actually, this path cannot be longer than $l_i - 1$ because then we would be able to pack all $P$-paths, which is a contradiction.

Lemma 6. Let $T(v_1)$ and $T(v_2)$ be two rooted trees with no common vertices, such that $v_2$ has exactly one child. Let $T(v)$ be the tree that we get by identifying $v_1$ with $v_2$ and renaming it to $v$. Then $L(v, P) = K(L_1 \cup L_2)$ where

\[
\begin{align*}
L_1 &= \bigcup_{P_1 \subseteq P, P_2 = P - P_1} \bigcup_{M_1 \in L(v_1, P_1)} \{M_1 \cup M_2\} \\
L_2 &= \bigcup_{P_1 \subseteq P, P_2 = P - P_1} \bigcup_{i \in P} \bigcup_{M_1 \in L(v_1, P_1)} \bigcup_{M_2 \in L(v_2, P_2)} \{M_1 \cup (M_2 - \{r_2\})\} \\
&\quad \bigcup_{r_1 \in M_1, r_2 \in M_2, r_1 + r_2 \geq l_i} \{M_1 \cup (M_2 - \{r_2\})\}
\end{align*}
\]

Proof. “$L(v, P) \supseteq K(L_1 \cup L_2)$”: If $M \in L_1 \cup L_2$ then $M \in L_1$ or $M \in L_2$. Let us first consider the case $M \in L_1$. By the definition of $L_1$ there are $P_1 \subseteq P$, $P_2 = P - P_1$, 

![Figure 5](image-url) In this tree there are nodes with more than two children and paths can “cross.” We pack paths with lengths 13, 9, 9, 7, 6. There is no solution if we replace 6 by 7.
\( M_1 \in L(v_1, P_1) \), and \( M_2 \in L(v_2, P_2) \) such that \( M = M_1 \cup M_2 \). By induction we know that \( P_1 \) can be packed into \( T(v_1) \) as well as additional paths of lengths \( M_1 \) starting at \( v_1 \). The same holds for \( P_2, v_2 \), and \( M_2 \). Using this packing we actually packed \( P \) into \( T(v) \) and additional paths of lengths \( M_1 \cup M_2 = M \) starting at \( v \). By definition then \( M \in L(v, P) \).

The other possibility is \( M \in L_2 \), which is a bit more complicated. If \( M \in L_2 \), then \( M = (M_1 - \{r_1\}) \cup (M_2 - \{r_2\}) \), where \( r_1 \in M_1, r_2 \in M_2, r_1 + r_2 \geq l_i, M_1 \in L(v_1, P_1 - \{i\}), M_2 \in L(v_2, P_2 - \{i\}), P_1 \subseteq P, P_2 = P - P_1 \), and \( i \in P \).

We have to show that it is possible to pack \( P \) into \( T(v) \) and additionally paths with lengths from \( M \) starting at \( v \). By induction we know that we can pack all \( (P_1 - \{i\}) \)-paths into \( T(v_1) \) and all \( (P_2 - \{i\}) \)-paths into \( T(v_2) \). Simultaneously, we can pack additional paths with lengths from \( M_1 \) into \( T(v_1) \) starting at \( v_1 \) and paths with lengths from \( M_2 \) into \( T(v_2) \) starting at \( v_2 \). Hence, we can pack paths with lengths \( M = (M_1 - \{r_1\}) \cup (M_2 - \{r_2\}) \) into \( T(v) \) leaving space for a path of length \( r_1 \) in \( T(v_1) \) and a path of length \( r_2 \) in \( T(v_2) \). We can combine these two paths into one path of length \( r_1 + r_2 \geq l_i \) and pack one additional path of length \( l_i \) into \( T(v) \). Altogether we packed \( P_1, P_2, \{i\} \) and therefore all \( P \)-paths into \( T(v) \).

\("L(v, P) \subseteq K(L_1 \cup L_2)\): Let \( M \in L(v, P) \). Then \( P \) can be packed into \( T(v) \). There are two possibilities:

1. No path corresponding to \( i \in P \) lies partially in \( T(v_1) \) and partially in \( T(v_2) \). Then we can split \( P = P_1 \cup P_2 \) such that \( P_1 \)-paths are packed into \( T(v_1) \) and \( P_2 \)-paths into \( T(v_2) \). The additional path with lengths from \( M \) are also packed into \( T(v_1) \) and \( T(v_2) \). Let us say \( M = M_1 \cup M_2 \), where \( M_1 \) is in \( T(v_1) \) and \( M_2 \) in \( T(v_2) \). Then it is easy to see that \( M \in L_1 \).
2. There is an \( i \in P \) such that all \( (P - \{i\}) \)-paths are packed into \( T(v_1) \) and \( T(v_2) \), but exactly one path with length \( l_i \) is packed into \( T(v) \) using edges from both \( T(v_1) \) and \( T(v_2) \). Note that there can be at most one such path because \( v_2 \) has only one child in \( T(v_2) \). Then all additional paths with lengths in \( M \) that start at \( v \) have to be packed into \( T(v_1) \) alone because the edge in \( T(v_2) \) is not available any more. Let \( r_1 \) be the length of the part of the bridging path of length \( l_i \) that lies in \( T(v_1) \) and \( r_2 \) the length of the part in \( T(v_2) \). Clearly, \( r_1 + r_2 = l_i \). With all these facts we can again easily verify that \( M \in L_2 \).

The following lemma shows that the size of the tables is bounded by a function in \( k \) and the maximal degree. The estimate is quite pessimistic, but we are not trying to optimize the runtime of the dynamic programming algorithm at the moment and are content with proving fixed parameter tractability.

\textbf{Lemma 7. Let} \( T(v) \) \textbf{be a rooted tree and assume that vertex} \( v \) \textbf{has d children}. Then \( |L(v, P)| \leq d^2 k^d \).

\textbf{Proof.} If \( v \) has only one child, then \( |L(v, P)| = 1 \) and the statement is true. Assume next that \( T(v) \) has \( d \) children. Each subtree can receive at most \( 2^k \) different sets of packed paths yielding at most \( 2^k \) different length of the longest path that can be additionally packed. Therefore a set \( M \in L(v, P) \) can have size at most \( d \) and contain up to \( d \) numbers each chosen from a set of size at most \( 2^k \). In total that are at most \( d 2^{kd} \) possibilities for a set \( M \).

\textbf{Theorem 8. Let} \( T \) \textbf{be a rooted tree and} \( P \) \textbf{a multiset of paths}. \textbf{In polynomial time a rooted tree} \( T' \) \textbf{can be computed that has the following properties:}

1. \( P \) \textbf{can be packed into} \( T \) \textbf{iff} \( P \) \textbf{can be packed into} \( T' \),
2. each node in \( T' \) \textbf{has at most} \( 3|P| \) \textbf{children}, \textbf{and (3)} \( T' \) \textbf{is a subtree of} \( T \).

\textbf{Proof.} Let \( l_1(u) \) \textbf{be the length of the longest path in} \( T(u) \) \textbf{that starts in} \( u \) \textbf{and} \( l_2(u) \) \textbf{be the length of the longest path in} \( T(u) \). \textbf{Assume that} \( P \) \textbf{can be packed into} \( T \) \textbf{and} \( v \) \textbf{be an arbitrary vertex in} \( T \). \textbf{Let us fix an edge-disjoint packing of} \( P \).
Let $v$ be an arbitrary node in $T$ and $v_1, \ldots, v_m$ the children of $v$. Let us further assume that $v_1, \ldots, v_{3|P|}$ contain the $|P|$ children with biggest $l_1(v_i)$ and $2|P|$ children with biggest $l_2(v_i)$. Ties can be arbitrarily ordered.

If $m \leq 3|P|$ we do nothing. Otherwise assume that $P$ is packed into $T$ and some path $p \in P$ uses $T(v_i)$ with $i > 3|P|$. There are two possibilities:

(i) $p$ contains $v_i$. Then $p$ is possibly packed partially inside $T(v_i)$ and partially outside. Let $p'$ be the part of $p$ inside $T(v_i)$. Clearly, $p'$ starts at $v_i$. By the pigeonhole principle there must be some $T(v_k)$ that has not been used in the packing of $P$, $l_1(v_k) \geq l_1(v_i)$, and $k \leq 3|P|$. Then we can repack $p$ such that it uses $T(v_k)$ instead of $T(v_i)$.

(ii) $p$ does not contain $v_i$ and is therefore completely packed into $T(v_i)$. Again by the pigeonhole principle we can find an appropriate $T(v_{k})$ with $k \leq 3|P|$ and $l_2(v_k) \geq l_2(v_i)$. We can repack $p$ from $T(v_i)$ into $T(v_k)$.

Repeated repacking in these two ways leads to a packing that uses only the subtrees $T(v_1), \ldots, T(v_{3|P|})$. We can therefore remove all other subtrees without changing a yes-instance into a no-instance. Applying this pruning to all vertices in $T$ leads to a new tree $T'$ that has all properties stated in the theorem. It is also clear that $T'$ can be computed in polynomial time as it is easy to find longest paths in trees.

Combining the above results (with the base case for a leaf $v$: $L(v, P) = \emptyset$ if $P$ contains only empty paths and $L(v, P) = \emptyset$ otherwise) we can prove the following:

\begin{itemize}
  \item \textbf{Theorem 9.} \textit{Path Packing into forests parameterized by the number of paths is in FPT.}
\end{itemize}

\textbf{Proof.} Given a tree $T$ compute a rooted tree $T'$ where each node has at most 3$k$ children and every $P$ (with $|P| = k$) can be packed into $T$ iff it can be packed into $T'$ (Theorem 8). Then use dynamic programming to find out whether the paths can be packed into $T'$. By Lemma 7 and 6 this only takes time $f(k)|T|^{O(1)}$ for some function $f$. ◀

\textbf{Lower bound}

While Path Packing on graphs with treewidth one is in FPT when parameterized by the number of paths, we now show that the problem becomes hard on graphs with treewidth two. As an intermediate step, we reduce from Unary Bin Packing \cite{10} to show hardness of Multi-Way Number Partition. This then leads to hardness results for Exact Path Packing. Remember that for Multi-Way Number Partition the numbers are unary encoded.

\begin{itemize}
  \item \textbf{Lemma 10.} \textit{Multi-Way Number Partition parameterized by the number of sets $k$ is W[1]-hard. Moreover, unless ETH fails there is no algorithm that solves the problem in $f(k)N^{o(k/\log k)}$ time for some function $f$ where $N$ is the input size.}
\end{itemize}

\textbf{Proof.} We give an FPT-reduction from Unary Bin Packing with the number of bins $k$ as parameter. The input to this problem is a a list of weights $w_1, \ldots, w_n$ encoded in unary and $k \in \mathbb{N}$ bins, each with capacity $b$. The task is to decide if the weights can be packed into the $k$ bins. This problem is $W[1]$-hard, and cannot be solved in time $f(k)N^{o(k/\log k)}$ where $N$ is the input size unless ETH fails \cite{10}.

Given an instance of Unary Bin Packing, we construct an instance of Multi-Way Number Partition as follows. If the sum of weights $\sum_{i=1}^{n} w_i$ exceeds the total capacity, $bk$, return a trivial no-instance. Otherwise, let $bk - \sum_{i=1}^{n} w_i = d \geq 0$ be the remaining space after all items are packed (assuming this is possible). If $d \geq \sum_{i=1}^{n} w_i$, note that a
folklore 2-approximation of bin packing guarantees a solution. In this case, we return a trivial yes-instance.

Otherwise, we start constructing the following Multi-Way Number Partition instance: Additionally to the old weights, we add new ‘dummy’ elements with weight one. The number of partitions $k$ is the number of bins. Doing so increases the input size at most by a factor of two, since we add at most $\sum w_i$ weights in unary.

We claim that the constructed Multi-Way Number Partition-instance has a solution if and only if the original Unary Bin Packing has a solution. A solution of Multi-Way Number Partition partitions the weights and ‘dummy’ weights into multi-sets of sum $bk/k = b$. Thus there is a bin packing of the weights (without ‘dummy’ weights) into $b$ bins of size $k$. For the other direction, a solution to the bin packing gives a partition into $k$ sets, each of sum $\leq b$, and total sum $bk - d$. Thus, filling the bins with the $d$ ‘dummy’ weights yields a solution for the Multi-Way Number Partition-instance.

The run time is polynomial in the input size $N$ and the parameter $k$ remains equal. Since Unary Bin Packing is $W[1]$-hard and has an ETH based lower bound of $f(k)N^{o(k/\log k)}$, these hardness results translate to Multi-Way Number Partition.

\begin{theorem}
The Exact Path Packing problem parameterized by the number of paths on graphs with treewidth two is $W[1]$-hard. Moreover, unless ETH fails there is no algorithm that solves the problem in $f(k)N^{o(k/\log k)}$ time for some function $f$ where $k$ is the number of paths and $n$ the number of vertices in the input graph.
\end{theorem}

\begin{proof}
We give an FPT reduction from Multi-Way Number Partition, parameterized by the number of sets. Assume we want to partition weights $w_1, \ldots, w_b$ into $k$ sets. By Lemma 10 it is sufficient to reduce such an Multi-Way Number Partition-instance to an instance of the Exact Path Packing problem with $k$ paths and a series-parallel graph $G$ whose number of vertices will be linear in $N = \sum w_i$. We proceed as follows.

At first, we construct a graph $G$: We add vertices $v_0, \ldots, v_b$. Additionally, every vertex pair $v_{j-1}, v_j$ with $1 \leq j \leq b$ we add $k - 1$ paths of length 2 and one path of length $w_j + 2$ connecting them. At last, we add a set $S$ of $k$ private neighbors to $v_0$ and a set $T$ of $k$ private neighbors to $v_b$. Figure 6 shows the graph constructed from a Multi-Way Number Partition-instance with weights $3, 10, 8, 2, 6, 4, 7, 5, \ldots$ and $k = 4$. We further specify $k$ paths of length $2(b + 1) + N/k$ each. Obviously, $G$ is series-parallel and its number of vertices is bounded by $O(N)$. It remains to show that the $k$ paths of length $2(b + 1) + N/k$ can be embedded into $G$ if and only if $w_1, \ldots, w_b$ can be partitioned into $k$ sets of size $N/k$.

Since every feasible packing has to cover every edge exactly once, the $k$ paths all have to start in $S$, visit $v_0, \ldots, v_b$ in this order and end in $T$. Assume we have embedded paths $p_1, \ldots, p_b$ of arbitrary length into $G$ such that they start in $S$, visit $v_0, \ldots, v_b$ in this order and end in $T$. Between vertices $v_{j-1}$ to $v_j$, for $1 \leq j \leq b$, these paths have options: Exactly $k - 1$ paths have to take the direct path of length 2 and one path has to take the detour which is $w_j$ steps longer than the direct path.

We define a partition of $w_1, \ldots, w_b$ into multi-sets $S_1, \ldots, S_k$ as follows: If path $p_i$ takes the detour between $v_{j-1}$ to $v_j$, we add $w_j$ to $S_i$. Now the length of $p_i$ equals $2(b + 1) + |S_i|$ for $1 \leq i \leq k$. We can now see that it is possible to choose $p_1, \ldots, p_b$ of length $2(b + 1) + N/k$ each if and only if it is possible to partition $w_1, \ldots, w_b$ into multi-sets $S_1, \ldots, S_k$ of size $N/k$ each.
\end{proof}
4 Path Packing Parametrized by Path Dependent Attributes

In the previous section we solved Path Packing on forests. Since Path Packing is NP-hard even for graphs with treewidth 2, we try to find some path dependent parameters to cope with its difficulty. At first, we will restrict the number of paths, then we will bound the length of each path and finally we consider the sum of the lengths of all paths.

Number of Paths

We denote the number of paths of an instance by \( k \). We start with \( k = 1 \). Consider an instance where the length of the single path corresponds to the number of vertices in a complete graph \( G \).

\[ \text{Observation 12. Since Hamiltonian Path is NP-hard, also path packing for } k = 1 \text{ is NP-hard} \]

On the other side, for \( k = 1 \) the special case of Exact Path Packing becomes easy.

\[ \text{Observation 13. Exact Path Packing is solvable in polynomial time for } k = 1 \text{ by deciding if the input graph is a path of length } l_1. \]

Unfortunately, for fixed \( k \geq 2 \) restricting the number of paths is not enough to gain a polynomial time algorithm. This holds for Exact Path Packing and therefore also for Path Packing.

\[ \text{Theorem 14. Let } k \geq 2. \text{ Exact Path Packing with } k \text{ paths is NP-complete on 4-regular graphs.} \]

\[ \text{Proof. We show that Exact Path Packing is NP-complete for } k = 2 \text{ by a reduction from the Hamiltonian Circuit problem in 3-regular graphs [12]. Let } G = (V_G, E_G) \text{ denote a 3-regular graph with } |V_G| = n \text{ of an Hamiltonian Circuit Instance, where one arbitrary but fixed edge } e_0 \in E_G \text{ should be contained in the circuit. We construct a graph } H \text{ such that } G \text{ has a Hamiltonian Circuit if and only if } H \text{ has an exact path cover with two equal-length paths. We construct } H \text{ as follows: We replace every vertex } v_i \in V_G \text{ by three vertexes } V_H^i = \{v_{i,1}, v_{i,2}, v_{i,3}\} \text{ and add three edges such that they form a clique } K_3 \text{ (see filled vertexes/thick edges in Figures 7/8). For every edge } e_l = (v_i, v_j) \in E_G \setminus e_0 \text{ we introduce an additional vertex } v_{e_l}. \text{ We add four incident edges to } v_{e_l}, \text{ such that two edges are incident to two different vertexes of } V_H^i \text{ as well as } V_H^j \text{ and no vertex } v \in V_H \text{ has degree greater} \]
than four (see unfilled vertexes). For edge $e_0 = (v_i, v_j) \in G$ we introduce four vertexes $V_H^0 = \{v_{e_0,0}, v_{e_0,1}, v_{e_0,2}, v_{e_0,3}\}$ instead of one vertex $v_{e_0}$ and connect each vertex with one edge to the corresponding remaining vertexes of $V_H^1$ and $V_H^2$ (see Figure 9 and 10), such that each edge $e \in V_H^0$ is a terminal node.

Assume that $G$ has a Hamiltonian Circuit. Since $G$ is 3-regular, one incident edge to every vertex is not covered by this circuit. Observe that these edges correspond to a perfect matching in $G$. In the remaining proof we will denote the edges that are covered by the matching with $E_G^M$ and the ones covered by the circuit $E_G^C$. We rename the vertexes for all $V_H^i$ such that the vertexes which are incident to the matching are those with the smallest index. We construct two paths $A$ and $B$ in $H$. Consider $e_1 = (v_i, v_j) \in E_G^M$. Without loss of generality we assign the subpath $(v_{e_0,1}, v_{e_0,2}, v_{e_0,3})$ to $A$, and $(v_{e_0,1}, v_{e_0,2}, v_{e_0,3})$ to $B$. To cover the cliques $V_H^i$ and $V_H^j$ completely we assign $(v_{e_0,1}, v_{e_0,2}, v_{e_0,3})$ to $A$ and $(v_{e_0,1}, v_{e_0,2}, v_{e_0,3})$ to $B$ (see Figure 11). For $e_1 = (v_i, v_j) \in E_G^C$ we connect the ends of all the subpaths in the cliques via $v_{e_0}$ such that all subpaths of $A$ and $B$ are connected. Due to our construction of $H$ this is always possible. Note that $A$ as well as $B$ correspond to a Hamiltonian Circuit in case you ignore the special structure for edge $e_0$.

For edge $e_0$ we have to adapt our strategy. Since $v_{e_0}$ is replaced by four vertexes with degree one, $A$ and $B$ have to start and finish in one of these vertexes. If $e_0 \in E_G^M$, the outgoing edges of both endpoints of both paths are incident to the same clique, if $e_0 \in E_G^C$, one edge is incident to each clique respectively (see Figure 12 and 13). The resulting two paths form an exact path cover of $H$.

Assume $H$ has an exact path cover with two paths. If the endpoints of the paths are incident to the same clique, the corresponding edge in $G$ is not contained in the Hamiltonian Circuit and vice versa. All other vertexes have degree four, so they have to be traversed by both paths. Let $H^i$ denote the subgraph that contains the clique that corresponds to vertex $v_i \in G$ and the three vertexes that represent the incident edges. $H^i$ has to be entered and left once by each path. So for every subgraph one incident edge-vertex is only traversed by
The Complexity of Packing Edge-Disjoint Paths

Figure 11 Paths A/B. Figure 12 Edge $e_0 \in E_G^0$. Figure 13 Edge $e_0 \in E_G^1$

one path. Since $H$ has an exact path cover, there exists a second subgraph where this vertex is also contained in the other path. Thus the corresponding edge in $G$ is not contained in the Hamiltonian Circuit. Since $G$ is 3-regular, all remaining edges form a Hamiltonian Circuit. For $k > 2$ one can add $k - 2$ paths of length one to $P$ as well as $k - 2$ isolated edges to $G$. Since these edges can only be covered by the new paths the proof above also holds for all $k > 2$.

Paths with bounded length

Observe that all hardness proofs that we have seen so far somehow involve paths of a certain length. Thus, we analyze the complexity of \textsc{Path Packing} based on the length of the paths we want to pack.

\textbf{Length-}i \textsc{Exact Edge Packing}

\textbf{Input:} A set of paths $P = \{p_1, \ldots, p_k\}$ of length $l_1 = \ldots = l_k = i$, a graph $G = (V, E)$.

\textbf{Question:} Is there an edge-disjoint embedding of $P$ into $G$ such that each edge $e \in E$ is covered exactly once?

\textbf{Length-2 Exact Edge Packing} is solvable in polynomial time by reformulating it as a matching problem on the line graph. We show that \textbf{Length-3 Exact Edge Packing} is already \textbf{NP}-hard via a reduction from the 3-dimensional matching problem, one of Karp’s original 21 \textbf{NP}-complete problems. The 3-dimensional matching problem takes as input sets $X, Y, Z$ of size $n$ and $T \subseteq X \times Y \times Z$. The question is whether there exists a set $M \subseteq T$ of size $n$ such that for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in M$, $x_1 \neq x_2$, $y_1 \neq y_2$, $z_1 \neq z_2$. The reduction is similar to the P2-packing reduction in [21].

\textbf{Theorem 15.} \textbf{Length-3 Exact Edge Packing} is \textbf{NP}-hard.

\textbf{Proof.} We reduce from 3-dimensional matching. Let $X, Y, Z, T$ be a 3-dimensional matching instance. For each $t \in X \cup Y \cup Z$ we add two vertices $t, t'$ and an edge $(t, t')$. Furthermore, for every $(t_1, t_2, t_3) \in T$, we attach a gadget to the vertices $t_1, t_2, t_3$ as depicted in Figure 14 and 15. We show that the resulting graph $G$ admits a packing of $P$ such that all edges are covered if and only if $X, Y, Z, T$ admits a 3-dimensional matching.

Consider a packing of $P$ in $G$ such that all edges are covered. Every $p_j \in P$ contains edges from at most one gadget. There are only two ways in which a gadget can be covered. The first way (left) covers $(t_1, t_1'), (t_2, t_2'), (t_3, t_3')$ and the second way (right) does not cover $(t_1, t_1'), (t_2, t_2'), (t_3, t_3')$. We define $M \subseteq T$ to be the set of all tuples $(t_1, t_2, t_3)$ such that there exists a gadget whose paths cover $(t_1, t_1'), (t_2, t_2'), (t_3, t_3')$. Every edge in $G$ is covered exactly once, therefore $M$ is a 3-dimensional matching.

On the other hand, assume $M$ to be a 3-dimensional matching. We construct a packing as follows: Let $(t_1, t_2, t_3) \in T$. We consider the gadget which is attached to $t_1, t_2, t_3$. If $(t_1, t_2, t_3) \in M$, we pack the paths as depicted on the left. Otherwise, we pack the paths as depicted on the right. In the resulting packing every edge is covered exactly once. ▶
Bounded sum of path lengths

The two previous results show that Path Packing is NP-hard even if the number of paths or the maximal length of the paths is bounded by a constant. At last, we the problem is in FPT when parameterized by the number of paths and their length. We give a randomized FPT algorithm using color coding \cite{8,1} that can easily be derandomized using perfect hash families \cite{8}.

\textbf{Theorem 16.} Path Packing parameterized by the summed length of all paths is in FPT.

\textbf{Proof.} Let $G, p_1, \ldots, p_k$ be a path packing instance. The parameter $l$ is the summed length of $p_1, \ldots, p_k$. We color the edges of $G$ with $k$ colors. Let $G_i$ be $G$ induced on all edges with color $i$. We return yes if for $i \in \{1, \ldots, k\}$, $p_i$ can be embedded into $G_i$. This can be done in FPT time \cite{8}. If $G, p_1, \ldots, p_k$ is a no-instance then this algorithm returns no. If $G, p_1, \ldots, p_k$ is a yes-instance then $p_1, \ldots, p_k$ can be embedded edge-disjoint into $G$. With probability at least $l^{-l}$ the embedding of $p_i$ is colored with color $i$ and the algorithm returns yes. We repeat the procedure $\Omega(l^l)$ times to obtain the correct answer with probability at least $2/3$. This yields a randomized FPT algorithm for path packing. \hfill ▶

For packing vertex-disjoint paths similar results are known: The $P_2$-packing problem takes a graph $G$ and an integer $k$ and asks if there is a set of $k$ vertex-disjoint $P_2$ in $G$. This problem is NP-complete \cite{17,21}. Fernau and Raible given FPT algorithm parameterized by the number of paths \cite{10}.

5 Path Packing Parametrized by Graph Dependent Attributes

Earlier (Theorem 1), we showed that Exact Path Packing is NP-hard even on a single subdivided star. So even for trees where there is only one node of degree higher than two the problem becomes NP-hard. In this section we study further restrictions to forests and finally identify a polynomial time solvable case. We do so by considering restrictions to the following three parameters: number of vertices of degree at least three, the maximal degree, and the number of connected components. For an easier notation we define this combined parameter as the ‘bcd’ of graph $G$. It is a bound on branching nodes, connected components and maximum degree.

\textbf{Definition 17.} Let $G$ be a graph. Then $\text{bcd}(G)$ is the minimal $k \in \mathbb{N}$ such that $G$ has at most $k$ nodes of degree larger than two, at most $k$ connected components, and a maximal degree of at most $k$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures/figure14.png}
\caption{Optimal path packing if $(t_1, t_2, t_3)$ is part of the matching.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures/figure15.png}
\caption{Optimal path packing if $(t_1, t_2, t_3)$ is not part of the matching.}
\end{figure}
The above mentioned reduction showing NP-hardness for a subdivided star constructs a graph with unbounded degree. What is the complexity if we limit the vertex degree to a constant, but in turn allow multiple components? Unfortunately even for a forest of paths, thus a maximum degree of two, the problem remains NP-hard. NP-hardness follows by an easy adaption of the proof of Theorem 1. The constructed subdivided star has an even number $2m$ of legs of length $\ell$. Instead one could also use $m$ disjoint paths of length $2\ell$. Thus we can follow NP-hardness of Exact Path Packing even for forests of paths.

**Corollary 18.** **Exact Path Packing** is NP-hard even on forests of paths.

Thus, if we drop either the degree or the number of components as parameters, the problem becomes NP-complete, even if the remaining parameters are bounded by a constant. Thus the remaining question is: What is the complexity if we limit the vertex degree to a constant, limit the number of connected components to a constant, but in turn allow arbitrary many vertices of degree at least three? We show hardness in this scenario even for the more restricted problem of packing paths of ascending length, denoted by Path-Perfect Packing.

**Path-Perfect Packing**

**Input:** A graph $G = (V, E)$ where $|E(G)| = 1 + 2 + \cdots + n$ for some $n \in \mathbb{N}$.

**Question:** Does there exist an edge-disjoint embedding of paths $p_1, \ldots, p_n$ with lengths $\ell_1 = 1, \ldots, \ell_n = n$ into $G$ such that each edge $e \in E$ is covered exactly once?

We show NP-hardness of this restricted problem on subdivisions of a caterpillar with vertex degree at most eight. We reduce from the following unary version of 3-partition. This version is slightly non-standard since we require that no numbers occur twice and relax the condition to put exactly three elements into each partition.

**Unary 3-Partition**

**Input:** A set of integers $A = \{a_1, \ldots, a_{3s}\} \subseteq \mathbb{N}$ in unary encoding.

**Question:** Is there a partition of $A$ into $s$ sets of equal sum?

Hulett et al. show that the above problem is NP-hard if we require each of the $s$ partitions to contain exactly three elements [15]. We get NP-hardness without the extra condition by increasing each number in $A$ by adding a big number (for example $\sum_{i=1}^{3s} a_i$).

We sketch how to reduce from Path-Perfect Packing on caterpillars with vertex degree at most eight to Unary 3-Partition. Note that, each partition of a Unary 3-Partition instance must have size $m = \frac{1}{s} \sum_{i=1}^{3s} a_i$. Consider the paths $\{p_i \mid 1 \leq i \leq m, \ i \in A\}$ whose length occurs in $A$. We translate a partition of $A$ into an exact packing of these paths. However, we have to account for the paths $\{p_i \mid 1 \leq i \leq m, \ i \notin A\}$ whose length occurs not as an integer in set $A$. For each of these we introduce a path where it fits in precisely, and by an exchange argument we may assume it is packed there. Now, roughly what remains to do is to construct a large caterpillar of low maximum degree where all these paths can be packed in.

**Theorem 19.** **Path-Perfect Packing** is NP-hard even for subdivided caterpillars with vertex degree at most eight.

**Proof.** We show NP-hardness by a polynomial time reduction from Unary 3-Partition.

Let the set $A = \{a_1, \ldots, a_{3s}\} \subseteq \mathbb{N}$ be a Unary 3-Partition instance. Let $m := \frac{1}{s} \sum_{i=1}^{3s} a_i$ be the ‘bucket size’, in other words the target size of the sets of the partition. We may assume that $s \geq 2$ and $a_i \leq m$ for every $i \in \mathbb{N}$. We construct a Path-Perfect Packing instance with a maximal path length of $n := 8m$. Let us denote by $p_i$ a path of length $i$ for $i \in \mathbb{N}$. In our construction, we use paths of lengths equal to the integers that do
not occur in $A$, which is $P_A = \{p_i \mid 1 \leq i \leq m, i \notin A\}$. Further, we use a set of paths whose length is longer than any integer in $A$, more specifically $P = \{p_{m+1}, p_{m+2}, \ldots, p_n\}$. Moreover, let $P_M$ be a set of $s$ many paths of length $m$. Note, that since $a_1, \ldots, a_3$, are distinct, $s < m$.

We construct a Path-Perfect Packing instance by combining the paths $P_M, P_A$ and $P$ into a subdivided caterpillar $T$ of degree at most eight. This then is a polynomial time reduction since the integers in $A$ are given in unary.

For a yes-instance of Unary 3-Partition, paths $\{p_i \mid 1 \leq i \leq m, i \in A\}$ may be embedded into $P_M$ covering every edges exactly once. Then the paths $P$ and $P_A$ may be straight-forwardly embedded into the remaining paths in $T$ of length $1, \ldots, n$. Thus a yes-instance of Unary 3-Partition implies a path-perfect packing of $T$, independent of the precise construction.

Let us specify the construction. Let path $p_n$ consists of the vertex sequence $v_0, \ldots, v_n = v_{8m}$. In the following we identify a certain vertex $v$ of $p_n$ with certain vertex $u$ of another path; thereby constructing a larger graph where $u = v$. We denote by the middle of a path an arbitrary vertex that maximizes the distance to its leaves (which is either unique or there are two choices). Identify vertex $v_{4m}$ of path $p_n$ with the middle of path $p_{n-1}$. Identify vertex $v_{4m}$ of $p_n$ with the middle of $p_{n-2}$ and the middle of $p_{n-3}$, then $v_{m+1}$ with the middle of $p_{n-4}$ and the middle of $p_{n-5}$ and so on; In other words, for $i = n-2, \ldots, m+1$, identify the middle of $p_i$ with $v_{j_i}$ where $j_i := 4m + \lfloor \frac{(n-2)-i}{2} \rfloor$. Then, for $p_i \in P_A$, identify the middle of $p_i$ with $v_{4m-i}$. Finally, identify the middle vertices of the paths $P_M$ with $v_{3m-1}, \ldots, v_{3m-s}$, respectively. Note that $3m-s \geq 2m$. This way we constructed a caterpillar $T$ with maximum degree eight at vertex $v_{4m}$ and using the paths $P_M, P_A$ and $P$.

It remains to show that if the constructed Path-Perfect Packing instance has a perfect packing of paths $q_1, \ldots, q_n$ of length $1, \ldots, n$ respectively, then also $A$ has a partition into $s$ sets each of sum $m$. We claim that path $q_n$ may only cover the edges of $p_n$ and $p_{n-1}$. Any other path $p_i$ for $i < n-1$ has length $< n$, and either side of path $p_i$ from $v_j$ has length $\leq \lfloor \frac{i}{2} \rfloor$. The longest path starting in $v_j$ is the path $v_{j_i}, v_{j_i-1}, \ldots, v_0$ of length $j_i$. It is easy to see that $j_i + \lfloor \frac{i}{2} \rfloor < n$, thus path $q_n$ does not fit into $p_i$.

Because of symmetry, we may assume that $q_n$ and $q_{n+1}$ exactly cover the edges of $p_n$ and $p_{n-1}$. Inductively, for $i = n-2, n-4, \ldots, 3s+1$, paths $q_i, q_{i-1}$ may only cover the edges of $p_i, p_{i-1}$. What remains to be covered are $P_A$ and $P_M$ by paths $q_1, \ldots, q_n$.

Consider the longest path $p_j \in P_A$ that is not covered by the path of equal length $q_j$ but instead by at least two paths, say by the paths $q'_1, q'_2, \ldots$. Then path $p_j$ has to occur completely in a path $p$ of $P_M$. We may exchange path $p_j$ with paths $q'_1, q'_2, \ldots$, such that $p_j$ is covered by $q_j$ and $q'_1, q'_2, \ldots$ occur in $p$. Repeat this exchange argument, until $P_A$ is covered exactly by $\{q_i \mid 1 \leq i \leq m, i \notin A\}$. Then the paths $Q_A := \{q_i \mid 1 \leq i \leq m, i \in A\}$ must exactly cover $P_M$. The mapping of $Q_A$ on $s$ many paths $P_M$, each of length $m$, defines a partition of $A$ into $s$ sets, each of size $m$. Thus the set $A$ is a yes-instance of Unary 3-Partition.

The maximum degree eight in the theorem above was chosen to simplify the proof. One can show NP-hardness even for smaller maximum vertex degree.

**XP-algorithm for parameter bcd($G$)**

We saw that if two bcd-parameters are constant and one bcd-parameter is unbounded then Exact Path Packing is NP-complete. We further study the complexity when parameterized by all three parameters. We give an XP-algorithm for Path Packing parametrized by
bcd\( (G) \). This means for every fixed \( k \), there is a polynomial time algorithm for graphs with \( \text{bcd}(G) \leq k \).

**Theorem 20.** There is a \( k!^{k}(n + k^2)^{O(k^2)} \)-time algorithm for Path Packing with \( k = \text{bcd}(G) \).

**Proof.** We give an algorithm, that given a graph \( G \) and a list \( P \) of paths \( p_1, \ldots, p_k \), decides if there is an edge-disjoint embedding of \( p_1, \ldots, p_k \) into \( G \). To do so, we guess a partition of \( G \) into eventually a set of vertex-disjoint paths \( X \). Then it suffices to find an embedding of \( P \) into such a set of vertex-disjoint paths \( X \). The remaining problem then is just a generalized bin-packing problem with \( O(k^2) \) bins, but encoded in unary; thus solvable in time \( n^{O(k^2)} \).

Most technicality lies in guessing the vertex-disjoint paths \( X \). First we guess a partition into a bounded number of walks \( W \). Later we need to partition \( W \) further resulting in vertex-disjoint paths \( X \).

Let \( V_1 \) be the set of vertices of degree two. Let \( V_2^* \) consist of a vertex of every connected component that is a circle. Let \( V_2 \) be the set of vertices of degree two that are not in \( V_2^* \), and let \( V_{\geq 3} \) be the vertices of degree at least three including \( V_2^* \). This seemingly odd definition allows us to work with walks starting and ending in \( V_1 \cup V_{\geq 3} \) that cover every edge, in particular those in a circle. Because there are at most \( k \) connected components, \( |V_2^*| \leq k \). Then since there are at most \( k \) vertices of degree at least three, we have \( |V_{\geq 3}| \leq 2k \).

Assuming a yes-instance, there is an edge-disjoint embedding of paths \( P \) into graph \( G \). At every vertex \( v \in V_{\geq 3} \) every path of \( P \) contains at most two of the incident edges of \( v \). Thus at every vertex \( v \in V_{\geq 3} \) there is a maximal matching \( M_v \) of \( v \)'s incident edges such that no path in \( P \) contains two unmatched edges.

We consider `direct` walks between `neighboring` vertices \( V_1 \cup V_{\geq 3} \): Let \( Q \) be the set of walks between \( u, v \in V_1 \cup V_{\geq 3} \) with inner vertices from \( V_2 \), and further where no vertex among \( V_2 \) is repeated (though possibly \( u = v \)). We join these walks \( Q \) to a set of walks \( W \) according to matchings \( M_v \) for \( v \in V(G) \). Whenever two walks \( w_1, w_2 \) end at some edges \( uv \) respectively \( u'v' \), and \( w \) is matched to \( u'v' \) by \( M_v \), then join walks \( w_1 \) and \( w_2 \) at edges \( uv, v'u' \). This procedure terminates and yields a well defined set of walks \( W \).

Note that every edge is covered by a walk \( Q \) and thus also every edge is covered by a walk \( W \). We further claim that every path \( p \) of \( P \) is a subsequence of edges of some walk \( w \in W \). Assuming otherwise, there are walks \( w_1, w_2 \) ending at edges \( uv \) and \( u'v' \). Then \( v \) is not a leaf, and thus \( v \in V_{\geq 3} \). Then matching \( M_v \) matches edges \( uv, u'v' \), and thus \( w_1, w_2 \) had to be joined to a single walk.

Thus for a yes-instance there is at least one set of matchings \( M_v, v \in V_{\geq 3} \) which determines walks \( W \) such that \( P \) may be embedded into \( W \). An algorithm may try the possible partition of edges into such a set of walks \( W \) as follows. Guess for each vertex \( v \in V_{\geq 3} \) a maximal matching \( M_v \) of its incident edges. There are at most \( k \) high degree vertices \( V_{\geq 3} \setminus V_2^* \), each with at most \( k \) incident edges. (Also we have a matching for \( V_2^* \), though since there are only two incident edges, there is only one possible matching.) Thus the algorithm tries at most \( k!^k \) possibilities. Then combine the paths \( Q \) to walks \( W \) according to the matchings, which is possible in polynomial time.

We claim that \( W \) has at most \( k^2 \) walks. Every walk in \( W \) has two endpoints, and the endpoints are among \( V_1 \cup V_{\geq 3} \). Clearly, at every leaf \( v \in V_1 \) at most one path ends. Further, there are at most \( k^3 \) leaves in the input graph of at most \( k \) vertices of degree \( \geq 3 \) and maximal degree of \( k \). If at a vertex \( v \in V_{\geq 3} \) two walks \( w_1, w_2 \) end, there are edges \( uv \) of \( w_1 \) and \( u'v' \) of \( w_2 \) unmatched by \( M_v \), in contradiction to a maximal matching \( M_v \). Thus also at every vertex \( v \in V_{\geq 3} \) at most one walk ends. Then there are at most \( k^2 + 2k \) endpoints of walks, and thus there are at most \( \lceil (k^2 + 2k)/2 \rceil \leq k^2 \) walks in \( W \).
Consider a walk \( w \in \mathcal{W} \) where a vertex \( v \) occurs more than once. Recall that the embedding of a path \( p \in P \) of a yes-instance is injective, thus no vertex \( v \in V(G) \) occurs twice in the same path. A naive approach would be to now solve the bin-packing problem of ‘weights’ \( P \) and ‘bins’ \( \mathcal{W} \). Then, however, a solution to the bin-packing would may potentially translate to an embedding of a path where a vertex occurs twice. Therefore let us guess a partition into paths with multiple occurrence of vertices, as follows.

Between two occurrences of \( v \) on walk \( w \) there must be vertex \( u \) (possibly an occurrence of \( v \) itself) which is the endpoint of two different paths. Therefore there is a partition of the walks \( \mathcal{W} \) into vertex-disjoint paths \( \mathcal{X} \), where still paths \( P \) have an embedding into \( \mathcal{X} \). We may describe this partition by ‘cuts’ of \( \mathcal{W} \) specified by a vertex \( v \) in the union of walks from \( \mathcal{W} \). Note, that in the union of walks \( \mathcal{W} \), each high degree vertex \( v \in V_{\geq 3} \setminus V'_2 \) occurs \( \deg(v) \leq k \) times. Thus there are to up to \( n + k^2 \) potential cut vertices.

We claim that at most \( k^2 \) cuts \( C \) are necessary to cut the walks \( \mathcal{W} \) into vertex-disjoint paths \( \mathcal{X} \). Assume, that there is a cut vertex \( v \in C \) which is on an inner vertex of a path between \( V_1 \) and \( V_{\geq 3} \). Then joining its incident vertex-disjoint paths results in a vertex-disjoint path. Thus we may assume that the cuts \( C \) are at vertices from walks of \( \mathcal{Q} \) between vertices among \( V_{\geq 3} \). Let \( Q_{\geq 3} \) be the set of paths \( \mathcal{Q} \) with endpoints in \( V_{\geq 3} \). Consider the multi-graph with loops on vertex set \( V_{\geq 3} \) with an edge between \( u,v \in V_{\geq 3} \) for every path \( Q_{\geq 3} \) with endpoints \( u \) and \( v \). Since \( |Q_{\geq 3}| \leq k \) and the degree of every vertex \( v \in Q_{\geq 3} \) is at most \( k \), this multi-graph has at most \( k^2 \) edges. Then also there are at most \( k^2 \) paths \( Q_{\geq 3} \). Consider a set of more than \( k^2 \) vertices \( C \subseteq V \) that cut \( \mathcal{Q} \) into vertex-disjoint paths \( \mathcal{X} \). Then there is a path of \( Q_{\geq 3} \) containing distinct cut vertices \( u,v \in C \). Let \( x \in \mathcal{X} \) be the path between \( u \) and \( v \). Joining them with the incident path at, say \( u \), results in a vertex-disjoint path. Thus cutting \( \mathcal{W} \) at vertices \( C \setminus \{ u \} \) still results in a set of vertex-disjoint paths. Therefore at most \( k^2 \) cuts of walks \( \mathcal{W} \) are necessary to yield vertex-disjoint paths \( \mathcal{X} \).

Let us utilize this observation in the design of our algorithm. Guess up to \( k^2 \) cuts \( C \) from the \( n + k^2 \) potential cuts. Cut the previously guessed walks \( \mathcal{W} \) into subpaths according to cuts \( C \). We may force exactly \( k^2 \) cut vertices by allowing \( C \) to be a multi-set containing also leaves, whose cut has no effect. This way, we try another \( (n + k^2)^{k^2} \) possibilities of cut vertices \( C \). Then cut the previously guessed \( \leq k^2 \) walks \( \mathcal{X} \) at the \( k^2 \) cut positions. If the resulting set of walks is not vertex-disjoint, discard this guess. Otherwise we obtain \( \leq 2k^2 \) vertex disjoint paths \( \mathcal{X} \), since every cut increases the number of paths by one. This resembles a bin-packing problem in unary encoding with \( k^2 \) bins of different sizes and total capacity \( n \). We may apply standard dynamic programming technique to test in \( n^{O(k^2)} \) time whether the sizes of the paths \( P \) fit into the bins in the sizes of \( \mathcal{X} \). If the paths \( P \) fit in some guessed paths \( \mathcal{X} \), then corresponding partition of the edges in \( G \) yields paths \( P \). Thus there is an edge-disjoint embedding of \( P \) into \( G \). For the other direction, if the edges of \( G \) can be partitioned into paths \( P \), then as argued before there is a set \( \mathcal{X} \) according to this partition and the there is a solution to the dynamic problem. The runtime is \( k!^k (n + k^2)^{O(k^2)} \cdot \text{poly}(n) = k!^k n^{O(k^2)} \cdot \text{poly}(n) \) where poly is a polynomial.

Can we achieve a better runtime than \( k!^k n^{k^2 + O(1)} \), in particular decrease the dependence on \( k \) in the exponent of \( n \)? Not significantly unless ETH fails, as the following reduction from Multi-Way Number Partition shows.

\begin{itemize}
  \item \textbf{Theorem 21.} There is no algorithm that decides \textsc{Path Packing} in time \( n^{o(k^2/\log k)} \) with \( k = \text{bed}(G) \) unless ETH fails.
\end{itemize}

\textbf{Proof.} We give an FPT reduction from Multi-Way Number Partition parameterized by the number of sets. Assume we have an instance \( I \) with \( k \) sets. Without loss of generality,
we can assume $\sqrt{k} \in \mathbb{N}$. We construct an equivalent instance of the path packing problem of polynomial size with $\sqrt{k}$ components, maximal degree $2\sqrt{k}$ and $\sqrt{k}$ vertices of degree $\geq 3$. By Lemma 10, this is sufficient.

Each weight $w_j$ of $I$ becomes a path $p_j$ with length $|p_j| = w_j \cdot 2k$. We construct $\sqrt{k}$ subdivided stars, each with degree $2\sqrt{k}$. Each leg of a subdivided star has length $\frac{1}{2k} \sum_{j=1}^{n} |p_j|$.

Let $S_1, \ldots, S_k$ be a feasible solution for $I$. The sum of weights for each set $S_i$ is $\sum_{j \in S_i} w_j = \frac{1}{k} \sum_{j=1}^{n} w_j$. The corresponding paths can be packed exactly into two legs of a subdivided star because they have a summed length of

$$\sum_{j \in S_i} |p_j| = 2k \sum_{j \in S_i} w_j = 2 \sum_{j=1}^{n} w_j = \frac{1}{k} \sum_{j=1}^{n} |p_j|.$$

A solution for the path packing instance can easily be transferred to a solution for MULTI-WAY NUMBER PARTITION. Each pair of two legs in a subdivided star is one set. Identifying such a pair is easy: A path is either part of both legs because it contains the center vertex of the star or we choose two legs where the paths end at the center vertex. The number of these legs must be even.

Because we have $\sqrt{k}$ subdivided stars with degree $2\sqrt{k}$, we can construct $k$ leg pairs which represent the sets. Thus, the parameter in our FPT reduction becomes quadratic. If the MULTI-WAY NUMBER PARTITION can not be solved in $O(n^{k/\log k})$, the path packing problem, parameterized by the number of components, maximal degree and vertices of high degree, cannot be solved in $O(n^{k^2/\log k})$. □

6 Conclusion

We showed that edge-disjoint packing of paths into a graph is a very hard problem. Even if the input graph is a subdivided star or a linear forest the problem is hard. If we parameterize the problem by the number of paths, the problem remains hard even for input graphs with treewidth two. However, it becomes fixed parameter tractable on forests. A natural open problem is to not embed paths, but more general graphs such as trees or cycles.

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