A Note On
Two Fiber Bundles and The Manifestations Of
“Shtuka”

Esmail Arasteh Rad

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Abstract

In this note we intend to look at the moduli stacks for global $G$-shtukas from a new perspective. We discuss a unifying interpretation of several moduli spaces (stacks) including moduli of global $G$-shtukas and (a variant of the) moduli of Higgs bundles. We view these spaces (stacks) as different fibers of a family over a scheme (stack) locally of finite type. We discuss (a relative version of) the local model theory for this family. We also consider the Hecke stacks over the moduli stack of $G$-shtukas and discuss the corresponding (motivic) Hecke operations.

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Introduction

Let $C$ be a smooth projective geometrically irreducible curve over a perfect field $k$. Let $\mathcal{G}$ be a flat affine group scheme of finite type over $C$. The stack $\mathcal{H}^1(C, \mathcal{G})$ of (principal) $\mathcal{G}$-bundles over $C$, has been considered by various authors, especially due to the key roles that it plays, both in the geometric Langlands program and also in the arithmetic Langlands program over function fields. This stack is an Artin stack, locally of finite type; e.g. see [Beh] when $\mathcal{G}$ is constant split reductive, and [AraHar2] for the general case.

There are interesting although inadequate studies aimed to describe the symmetries of $\mathcal{H}^1(C, \mathcal{G})$. Note that when $\mathcal{G} = \text{GL}_r$, this moduli stack coincides the moduli of vector bundles of rank $r$ over $C$. In $\text{char } k = 0$, and in the presence of stability condition, in [BGM] the authors describe the group of automorphisms of the moduli space of stable vector bundles. They observe that they all come from the obvious ones. This means that they either arise from $\text{Aut}(C)$, or by twisting by a line bundle, or possibly by sending a vector bundle to its dual vector bundle. Note in addition that the stack of morphisms between Artin stacks has been considered, and studied, by several authors, including Aoki, Olson, Hall and Rydh, see [Ao], [Ol2] and [HaRy].

On the other hand, for some purposes, it is useful to consider the moduli of $\mathcal{G}$-bundles over a relative curve $C$ over $S$. In [Wan], Wang proves that the stack $\mathcal{H}^1(C, \mathcal{G})$ remains Artin (over $S$) when we replace the curve $C/k$ with a projective scheme $X$ over a base $S$. Considering the relative version was in fact motivated and proposed for certain applications in the geometric Langlands program. Beyond this, as another remarkable example, to prove the purity of the cohomology of this moduli stack over $\mathbb{F}_p$, in [HeiSch], Heinloth and Schmitt, consider the relative situation over a curve $C$ over a Dedekind domain $R$ over $\mathbb{Z}$, and implement techniques from nearby-vanishing cycles to lift the Atiyah-Bott theory from characteristic zero (in the generic fiber) to the special fiber of this moduli stack.
In this note we introduce a family $\Sigma$ over the stack $\mathcal{E}$ of endomorphisms of $H^1(C, G)$, and we study the geometry of the family and its fibers. Note that after imposing relevant boundedness conditions (and endowing with extra structures), different fibers of this family realize some interesting moduli stacks (spaces), such as (global Schubert varieties inside) Beilinson-Drinfeld affine Grassmannian, (a variant of the) moduli of Higgs bundles, and moduli of (bounded) $\mathcal{G}$-shtukas; see subsection 2.1 and subsection 3.3.

Let us briefly explain the content of each section. In Section 1 after we fix some notation in 1.1, we recall the basic definitions of formal algebraic stacks (and rigid analytic stacks) in Subsection 1.2. In Section 2 we explain the general construction of the family $\Sigma = \Sigma(\mathcal{H} \to \mathcal{Y}) \to \mathcal{E}$, corresponding to a two fiber bundle $\mathcal{H} \to \mathcal{Y}$. Here $\mathcal{E}$ is the stack of endomorphisms of $\mathcal{Y}$. We prove a general local model theorem for a fiber over an endomorphism of $\mathcal{Y}$ which annihilate the tangent bundle; see Theorem 2.5. Then we discuss some applications of this theorem; see Corollaries 2.6 and 2.7.

In paragraph 3.1.1 we recall Heinloth-Schmitt stability condition. We will observe that at least for split reductive case, the stack of endomorphisms of $H^1(C, G)$, after imposing stability condition, is an Artin stack. In paragraph 3.3.1 we recall that the Beilinson-Drinfeld affine Grassmannian also arises in this context, and we further discuss boundedness conditions and functoriality. We further discuss how a boundedness condition gives rise to a Hecke cycle; see Proposition 3.15.

In paragraph 3.3.2 we consider certain symmetries of the moduli stack $H^1(C, G)$ which are encapsulated in itself, in the sense of twisting by torsors. Consequently we see that certain fibers can be regarded as a variant of the moduli stacks of Higgs bundles. We further discuss the (formally) properness of certain restrictions of this family, as well as some lifting properties of the corresponding stacks.

In paragraph 3.3.3 we discuss the corresponding picture for the moduli of $\mathcal{G}$-shtukas. We address similar lifting problems. See Theorem 3.26 and Proposition 3.28.

In Section 4 as another example for a two fiber bundle over an algebraic stack, we introduce a Hecke stack over moduli of $\mathcal{G}$-shtukas. We in particular observe that loop group invariant cycles inside a global affine Grassmannian $GR_m(C, G)$ induce certain homomorphisms between motives of the moduli stacks of $\mathcal{G}$-shtukas; see theorem 4.5.

The later gives certain bivariant classes.

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1 Preliminaries

1.1 Notation and Conventions

S a locally noetherian scheme,
C a smooth projective relative curve over S with geometrically irreducible fibers,
\mathcal{G} a smooth affine group scheme over C,
Sch/S category of schemes over S,
n a positive integer,
\underline{s} an n-tuple of sections \( s_i : T \to C \),
\Gamma_{\underline{s}} the union of the graphs \( \Gamma_{s_i} \subseteq C \times_S T \),

Assume that we have two morphisms \( f, g : X \to Y \) of schemes or stacks. We denote by \( \text{equi}(f, g : X \Rightarrow Y) \) the pull back of the diagonal under the morphism \( (f, g) : X \to Y \times_Z Y \), that is

\[ \text{equi}(f, g : X \Rightarrow Y) := X \times_{(f, g), Y \times_Y \Delta Y} Y \]

where \( \Delta = \Delta_{Y/Z} : Y \to Y \times_Z Y \) is the diagonal morphism.

By an IC-sheaf \( IC(\mathcal{X}) \) on a stack \( \mathcal{X} \), we will mean the intermediate extension of the constant perverse sheaf \( \mathbb{Q}_\ell \) on an open dense substack \( \mathcal{X}^\circ \) of \( \mathcal{X} \) such that the corresponding reduced stack \( \mathcal{X}^\circ_{\text{red}} \) is smooth. The IC-sheaf is normalized so that it is pure of weight zero.

Let \( \widehat{S} \) be a formal scheme. We denote by \( \mathcal{N}ilp_{\widehat{S}} \) the category of schemes over \( \widehat{S} \) on which an ideal of definition of \( \widehat{S} \) is locally nilpotent, equipped with the étale topology.

Let \( H \) be a sheaf of groups (for the étale topology) on a scheme \( X \). By a (right) \( H \)-torsor (also called an \( H \)-bundle) on \( X \) we mean a sheaf \( \mathcal{G} \) for the étale topology on \( X \)
together with a (right) action of the sheaf $H$ such that $G$ is isomorphic to $H$ on a étale covering of $X$. Here $H$ is viewed as an $H$-torsor by right multiplication. We denote by $\mathcal{H}^1(C, \mathcal{G})$ the category fibered in groupoids over $\text{Sch}/S$ with $\mathcal{H}^1(C, \mathcal{G})(T)$ the groupoid of $\mathcal{G}$-bundles over $C_T := C \times_S T$.

When $S = \text{Spec} \mathbb{F}_q$ we follow our notation in [AraHab] and [AraHar2], in particular by $\nabla_n \mathcal{H}^1(C, \mathcal{G})$ we denote the moduli stack whose $T$-points consists of $\mathcal{G}$-shtukas over $T$ with $n$ characteristic sections, e.g. see [AraHar2] Definition 3.3 or [AraHab] 2.0.10.

For a perfect field $k$ and $X$ in $\text{Sch}_k$, let $Ch_i(X)$ and $Ch^i(X)$ denote Fulton’s $i$-th Chow groups and let $Ch_*(X) := \oplus_i Ch_i(X)$ (resp. $Ch^*(X) := \oplus_i Ch^i(X)$).

Finally, to denote the motivic categories over a perfect field $k$, such as
\[ \text{DM}_{gm}(k, \mathbb{Z}), \text{DM}_{eff}^{gm}(k, \mathbb{Z}), \text{DM}_{eff}(k), \text{etc.} \]
and the functors $M(\cdot) : \text{Sch}_k \to \text{DM}_{gm}^{eff}(k, \mathbb{Z})$ and $M^c(\cdot) : \text{Sch}_k \to \text{DM}_{gm}^{eff}(k, \mathbb{Z})$ we use the same notation that was introduced in [VSP]. When $\text{char } k > 0$ we assume coefficients in $\mathbb{Q}$.

For the definition of the geometric motives with compact support in positive characteristic we also refer to [H-K, Appendix B].

1.2 Formal and analytic stacks

Recall that a formal space $\hat{X}$ over a formal scheme $\hat{S}$ is a sheaf of sets on the site $\text{Nilp}_{\hat{S}}$. In addition it is called a formal algebraic space if the diagonal morphism $\hat{X} \to \hat{X} \times_{\hat{S}} \hat{X}$ is representable by a quasi-compact morphism of formal schemes and there is a formal scheme $\hat{X}'$ over $\hat{S}$ and a morphism of formal $\hat{S}$-spaces $\hat{X}' \to \hat{X}$ which is representable by an étale surjective morphism of formal schemes.

Let $\mathcal{X}$ be a stack over a scheme $S$. Let $S_0$ be a locally closed subscheme of $S$. Let $\hat{S}$ denote the formal completion of $S$ along $S_0$. Restricting the fibered functor $\mathcal{X}$ to the category $\text{Nilp}_S$ gives a category $\hat{\mathcal{X}}$ fibred in groupoids over $\text{Nilp}_S$ which inherits the following properties from $\mathcal{X}$

i) for every $V$ in $\text{Nilp}_S$ and $x, y$ in $\hat{\mathcal{X}}(V)$ the presheaf
\[
\text{Isom} : \text{Sch}/V \to \text{Sets}
\]
\[ U \to V \mapsto \text{Hom}_{\hat{\mathcal{X}}(U)}(x_U, y_U), \]
is a sheaf on $\text{Sch}/\mathcal{V}$.

ii) for every covering $\mathcal{V}_i \to \mathcal{V}$ in $\text{Nilp}_{\hat{S}}$ all descent data for this covering are effective.

Furthermore if $\mathcal{X}$ is an algebraic stack in the sense of Artin (resp. Deligne-Mumford (DM)) we have

(a) the diagonal 1-morphism $\hat{\mathcal{X}} \to \hat{\mathcal{X}} \times_{\hat{S}} \hat{\mathcal{X}}$ over $\hat{S}$ is representable by formal algebraic $\hat{S}$-spaces (resp. schemes), separated, and quasi-compact.

(b) there exists a formal algebraic $\hat{S}$-space $\hat{X}$ and a presentation

$$P : \hat{X} \to \hat{\mathcal{X}}$$

of formal $\hat{S}$-stacks which is representable by a smooth (resp. étale) and surjective morphism of formal algebraic $\hat{S}$-spaces.

Let us abstractify the above observation and phrase it in the following way

**Definition 1.1.** A category $\hat{\mathcal{X}}$ fibered in groupoids over $\text{Nilp}_{\hat{S}}$ is called a **formal stack** if it has the properties i) and ii) indicated above. Also we say $\hat{\mathcal{X}}$ is **formal algebraic (Artin) stack** (resp. formal stack of DM-type) if in addition it is subject to a) and b) above.

Let $K$ be a complete discrete valuation field. Let $\mathcal{O}_K$ and $k$ denote the corresponding ring of integers and residue field.

**Remark 1.2.** Let $\mathcal{A}n_K$ denote the category of $K$-analytic spaces, equipped with the étale topology. A $K$-analytic stack $\mathcal{X}$ is a stack in groupoids over the site $\mathcal{A}n_K$. It is called Artin (resp. Deligne Mumford) if similar conditions to the above conditions (a) and (b) hold in this category. One can extend the analytification functor from the category of schemes to obtain the functor $(-)^{an}$ from the category of algebraic stacks locally of finite type over $K$ to the category of $K$-analytic stacks. We refer to [PoYu, Section 6] for details. Similarly, one can produce the special fiber functor $(-)_s$ (resp. the generic fiber functor $(-)_g$) from the category of formal stacks locally finitely presented over $\text{Spf} \mathcal{O}_K$ to the category of algebraic stacks locally of finite type over $k$ (resp. to the category of $K$-analytic stacks).

## 2 Two fiber bundles and the dynamics of the base

In this section we introduce a family $\Sigma$, corresponding to a two fiber bundle $\mathcal{H}$, over certain stacks of endomorphisms. We prove some general statements related to local and global geometry of certain fibers of such families.
2.1 General construction of the family $\Sigma \to \mathcal{E}$

**Definition 2.1.** Let $\mathcal{Y}$ (resp. $\mathcal{Y}'$) be an algebraic stack in the Artin’s sense (resp. formal algebraic stack), locally of finite type over a scheme $S$. Let $X$ be a projective flat scheme over $S$.

(a) Let $\mathcal{H}$ be a stack over $X$ via a morphism $\text{char} : \mathcal{H} \to X$, and furthermore assume that $\mathcal{H}$ is fibred over $\mathcal{Y}$ and $\mathcal{Y}'$ via the following maps

$$pr^{-} : \mathcal{H} \to \mathcal{Y} \quad \text{and} \quad pr^{+} : \mathcal{H} \to \mathcal{Y}'.$$

We call the tuple $\mathcal{H} := (\mathcal{H}, \text{char}, pr^{-}, pr^{+})$, consisting of the above data, a $HR$-tuple. We say that $\mathcal{H}$ is a $CHR$-tuple of degree $m$ if $pr^{+} \times \text{char}$ is proper and of finite type, and $pr^{-}$ is finite type and flat of relative dimension $m$.

(b) Let $\text{Hom}_{0} := \text{Hom}(\mathcal{Y}, \mathcal{Y}')$ denote the corresponding Hom-stack. It is contravariant 2-functor from the category of affine noetherian schemes over $S$ to the 2-category of groupoids given by assigning the groupoid of 1-morphisms $\text{Hom}_{T}(\mathcal{Y} \times_{S} T, \mathcal{Y}' \times_{S} T)$ to a test scheme $T$. When $\mathcal{Y}' = \mathcal{Y}$, then we set $\mathcal{E} := \text{Hom}(\mathcal{Y}, \mathcal{Y})$.

(c) Define $\Sigma(\mathcal{H})$ via the following pull-back diagram

$$\begin{array}{ccc}
\Sigma(\mathcal{H}) & \longrightarrow & \mathcal{H} \\
\varphi \downarrow & & \downarrow (pr^{-}, pr^{+}) \\
\text{Hom} \times \mathcal{Y}' & \longrightarrow & \mathcal{Y} \times \mathcal{Y}'
\end{array}$$

of stacks. The bottom arrow is given by

$$(f : \mathcal{Y} \to \mathcal{Y}', y) \mapsto (y, f(y)).$$

We view $\Sigma(\mathcal{H})$ as a family over $\text{Hom}$. When it is clear from the context we use the shorthand $\Sigma$ to denote $\Sigma(\mathcal{H})$.

**Remark 2.2.** Let $\mathcal{H}$ be a $CHR$-tuple of degree $m$. Then one may define the following Hecke-type operation, from the category of perverse sheaves on $\mathcal{Y}$ to the derived category of sheaves on $\mathcal{Y}' \times_{S} X$ given by the formula

$$(pr^{-} \times \text{char})_{*} \circ pr^{+*}(-).$$

**Notation-Remark 2.3.** One can mimic the above construction also in the category of formal stacks over a formal scheme $\hat{S}$. We then use the notation $\hat{\mathcal{Y}}, \hat{\mathcal{Y}'}, \hat{\mathcal{H}}, \hat{\mathcal{H}}, \hat{\mathcal{E}}, \hat{\Sigma} := \hat{\Sigma}(\hat{\mathcal{H}})$ and etc. to denote the corresponding formal stacks.
Remark 2.4. Assume that $\mathcal{Y}$ is a quasi-projective scheme over $S$. Then there is an obvious morphism $\mathcal{E} \to \text{Hilb}_{\dim(\mathcal{Y})}(\mathcal{Y} \times_S \mathcal{Y})$, defined by sending $f$ to its graph $\Gamma_f \subseteq \mathcal{Y} \times_S \mathcal{Y}$. This identifies $\mathcal{E}$ with an open subscheme of $\text{Hilb}_{\dim(\mathcal{Y})}(\mathcal{Y} \times_S \mathcal{Y})$. In particular each connected component of $\mathcal{E}$ is of finite type, and these components form a countable set. The automorphism group scheme $\text{Aut}(\mathcal{Y})$ is open in $\mathcal{E}$ by [Gro, p. 267] (see also [Kol, Lemma I.1.10.1]). If $\mathcal{Y}$ is a projective variety, then $\text{Aut}(\mathcal{Y})$ is also closed in $\mathcal{E}$, according to [Br2, Lemma 4.4.4]; thus, $\text{Aut}(\mathcal{Y})$ is a union of connected components of $\mathcal{E}$. Note further that this method can not be implemented to treat algebraic stacks. This is because the construction of Hilb scheme for algebraic stacks is problematic. The reason is that the graph $\Gamma_f$ of a morphism $\mathcal{Y} \to \mathcal{Y}'$ of algebraic stacks is not closed in general, and requiring this is in fact too much restrictive. For example one may observe that this assumption for the graph of diagonal $\Delta : \mathcal{Y} \to \mathcal{Y} \times_S \mathcal{Y}$ implies that $\mathcal{Y}$ is representable by an algebraic space. To handle the case of algebraic stacks one needs to make use of more sophisticated techniques such as deformation theory of 1-morphisms of algebraic stacks and etc.; e.g. see [Ao], [Ol2] and [HaRy].

2.2 Local Model For $\Sigma_\varsigma$

Recall that for a stack $\mathcal{Y}$ over $S$, one defines the tangent bundle $T_\mathcal{Y}$ via the following functor

$$ T/S \mapsto \mathcal{Y}(T[\varepsilon]), $$

where $T[\varepsilon] := T \times_Z \mathbb{Z}[\varepsilon]$ with $\varepsilon^2 = 0$.

One can see that when $\mathcal{Y}$ is an Artin stack then the same holds for $T_\mathcal{Y}$. The stack $T_\mathcal{Y}$ is equipped with the projection morphism $T_\mathcal{Y} \to \mathcal{Y}$ and zero section $\mathcal{Y} \to T_\mathcal{Y}$, that are induced by $\mathbb{Z}[\varepsilon] \to \mathbb{Z}$, $\varepsilon \mapsto 0$, and inclusion $\mathbb{Z} \to \mathbb{Z}[\varepsilon]$ respectively. Consider the groupoid $\mathcal{E}(T_\mathcal{Y}) := \text{Hom}(T_\mathcal{Y}, T_\mathcal{Y})$ of endomorphisms of $T_\mathcal{Y}$. The projection morphism and zero section define a projection morphism $\mathcal{E}(T_\mathcal{Y}) \to \mathcal{E}$ and an obvious section $\mathcal{E} \to \mathcal{E}(T_\mathcal{Y})$. Consider the following morphism

$$ d : \mathcal{E} \to \mathcal{E}(T_\mathcal{Y}) $$

given by sending $\varsigma$ to its derivation $d_\varsigma$. We let $\mathcal{E}_{d=0}$ denote the pull back $\mathcal{E} \times_{d,\mathcal{E},\mathcal{E}(T_\mathcal{Y})} \mathcal{E}(T_\mathcal{Y})$.

**Theorem 2.5.** Let $\mathcal{H} := (\mathcal{H}, \text{char}, \text{pr}^+, \text{pr}^-)$ be a HR-tuple, see [2.1]. Let $\Sigma_\varsigma$ denote the fiber of $\Sigma(\mathcal{H})$ over $\varsigma \in \mathcal{E}$. Let $z$ be a point in $\Sigma_\varsigma$, and set $y = \text{pr}^+(z)$. Assume that

(a) $\mathcal{Y}$ is an Artin stack, smooth at $y$, which admits an étale neighborhood $\mathcal{U}_y \to \mathcal{Y}$ at $y$,  
(b) $y$ lies in the vanishing locus of $d_\varsigma$, and
(c) there exist a family $\mathcal{F}$ over $X$ and an étale neighborhood $U_{(y, x)}$ of $(y, x)$ that trivializes $\mathcal{H}$ over $\mathcal{Y} \times X$ in the following sense

$$\mathcal{H} \times_{\mathcal{Y} \times X} U_{(y, x)} \rightarrow (\mathcal{F} \times \mathcal{Y}) \times_{\mathcal{Y} \times X} U_{(y, x)}.$$ 

Then there is an étale neighborhood $U_z$ of $z$ and a roof of étale morphisms

$$U_z \xrightarrow{\Sigma_\zeta} \mathcal{F},$$

Proof. Set $U = U(y, x)$ and

$$U'' := H|_U \cong \mathcal{F} \times \mathcal{Y}|_U =: U''.'$$

Consider the following diagram

We need to check that the composition of the maps

$$\tilde{U}_z \rightarrow U' \xrightarrow{\cong} U'' \rightarrow \tilde{\mathcal{F}} \times \mathcal{Y} \xrightarrow{pr_1} \mathcal{F}$$

is étale. Note that we may realize $\tilde{U}_z$ as a substack of $U'$ given by

$$\tilde{U}_z := \text{equi}(\zeta \circ f, g; U' \Rightarrow \mathcal{Y}).$$
Here \( f \) is the morphism induced by the projection \( pr^- : \mathcal{H} \to \mathcal{Y} \) and \( g \) is given by \( U' \cong U'' \to \mathcal{Y} \times \mathcal{Y} \) followed by the projection to the second factor.

Let \( v \in \tilde{U} \) be a point and let \( w \in U' \) and \( y = \xi \circ f(w) = g(w) \in \mathcal{Y} \), as well as \( t \in \mathcal{Y} \) be its images. By smoothness of \( \mathcal{Y} \), we may further reduce to the case that \( \mathcal{Y} \) is an affine space.

Namely take an affine open neighborhood \( Y' \) of \( y \) in \( \mathcal{Y} \) which admits an étale morphism \( \pi : Y' \to \tilde{Y} \) to some affine space \( \tilde{Y} = \mathbb{A}^m_S = \text{Spec} \mathcal{O}_S[z_1, \ldots, z_m] \), and consider an affine neighborhood \( T' \) of \( t \) which we write as a closed subscheme of some \( \tilde{T} = \mathbb{A}^t_O \). Replace \( Y' \) by the affine neighborhood \( Y'' \) of \( y \) and \( U' \) by an affine neighborhood \( W' \) contained in \( (\xi \circ f)^{-1}(Y') \cap g^{-1}(Y') \cap \iota^{-1}(T' \times Y') \). Then \( \tilde{U}' := W' \times_{U'} \tilde{U} = W' \times_{(\sigma \xi, f, g)} Y' \times Y', \Delta \) \( Y'' \) is an open neighborhood of \( v \) in \( \tilde{U} \). We may extend the étale morphism \( \iota : W' \to T' \times Y' \) to an étale morphism \( \tilde{\iota} : \tilde{W} \to \tilde{T} \times Y' \) with \( \tilde{W} \times_{\tilde{T}} T' = W' \). We also extend \( \pi \circ f : W' \to \tilde{Y} \) to a morphism \( \tilde{f} : \tilde{W} \to \tilde{Y} \) and \( g \) to \( \tilde{g} : \tilde{W} \to \tilde{Y} \), and set \( \tilde{U} := \text{equi}(\xi \circ \tilde{f}, \tilde{g}) : \tilde{W} \rightrightarrows \tilde{Z} \) = \( \tilde{W} \times_{(\sigma \xi, f, g)} \tilde{Z} \). Since \( \Delta : Y' \to Y' \times Y' \) is an open immersion, also the natural morphism

\[
\tilde{U} \to \tilde{U} \times \tilde{T} \times Y' = W' \times_{(\sigma \xi, f, g)} \tilde{Y} \times Y', \Delta \tilde{Y} = W' \times_{(\sigma \xi, f, g)} Y' \times Y', \Delta (Y' \times \tilde{Y} \times Y')
\]
is an open immersion. Since \( \tilde{W} \) is smooth over \( \tilde{T} \) of relative dimension \( m \) and \( \tilde{U} \) is given by \( m \) Artin-Schreier type equations

\[
\tilde{g}^*(z_j) - \xi^*(\tilde{f}^*(z_j)),
\]
and \( y \) lies in the vanishing locus of \( d\xi \) and thus the above equations have linearly independent differentials \( d\tilde{g}^*(z_j) \), we observe that \( \tilde{U} \rightrightarrows \tilde{T} \) is étale according to the Jacobi-criterion [BLR §2.2, Proposition 7].

The above theorem has the following immediate corollaries that partly describe global and local geometry of \( \Sigma_\xi \) for \( \xi \) in \( \mathcal{E}_{d=0} \).

Let \( S = \text{Spec} \ k \). Let \( \mathcal{H} \) be a HR-tuple. Assume further that there is \( y_0 \) in \( \mathcal{Y}(k) \) and that \( \mathcal{H} \to \mathcal{Y} \times X \) is a fiber bundle with fiber \( \mathcal{F} := \mathcal{H}_{y_0} \) for the étale topology on \( \mathcal{Y} \times X \). Suppose \( \mathcal{H} \) admits a stratification \( \{ \mathcal{H}_\lambda \}_\lambda \), that further induces stratification \( \{ (\Sigma_\xi)_\lambda \}_\lambda \) of \( \Sigma_\xi \). One can easily observe that

**Corollary 2.6.** Keep the notation and assumptions in Theorem 2.5 together with the above notation and assumptions. Let \( \xi \) be in \( \mathcal{E}_{d=0} \). The IC-sheaf \( IC(\Sigma_\xi) \) is the restriction of \( IC(\mathcal{H}) \) up to some shift and Tate twist.

**Proof.** Let \( \Sigma_\xi \) denote the fiber above the constant morphism \( \xi : \mathcal{Y} \to \mathcal{Y} \), sending \( \mathcal{Y} \) to the point \( y_0 \). According to 2.5 we may replace \( \mathcal{F} \) by \( \Sigma_\xi \). Therefore we may assume that \( \mathcal{F} \)
is equipped with a natural stratification induced by \( \{ H_\lambda \}_\lambda \) of \( H \). Consider the following diagram

\[
\begin{array}{ccccccccc}
 & & i & & & & & & \\
\Sigma_\zeta & \rightarrow & \mathcal{H} & \rightarrow & \mathcal{H}|_U \cong \mathcal{F} \times \mathcal{Y}|_U & \overset{pr_1}{\rightarrow} & \mathcal{F} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{Y} \times X & \overset{(id, \varsigma) \times id}{\rightarrow} & \mathcal{Y} \times \mathcal{Y} \times X & \rightarrow & \mathcal{Y} \times X & \leftarrow & U \\
\end{array}
\]

see the proof of the above theorem 2.5. The stratification on \( \Sigma_\zeta \) is induced by that of \( H \). Moreover by Theorem 2.5 the smooth open stratum \( \Sigma_\zeta^o \) lies inside the pull back of the open smooth stratum \( H^o \) of \( H \), the statement is obvious over the open stratum. By the Theorem 2.5 we have \( IC(U_z) = (pr_1 \circ i)^* IC(\mathcal{F}) \) which equals \( i^* IC(H|_U) \) up to shift and Tate twist by \( \dim \mathcal{Y} \). By étaleness of \( i' \) the later coincides \( i^* i'^* IC(H) \) which equals the restriction of \( IC(H) \) to \( U_z \).

\[\square\]

**Corollary 2.7.** Keep the notation and assumptions in 2.6. The stalks \( \hat{\mathcal{O}}_{\Sigma_\zeta,z} \) at the point \( z \) satisfy Serre’s condition \( S_i \) (resp. \( R_i \)) if the points of the fiber \( \Sigma_\zeta \) satisfy \( S_i \) (resp. \( R_i \)).

### 2.3 Lifting of Fibers

Let \( S \) be the formal spectrum of a complete discrete valuation ring \( R \), with special point \( s \) and generic point \( \eta \). Let \( k = \kappa(s) \) (resp. \( \kappa(\eta) \)) denote the residue fields at \( s \) (resp. \( \eta \)). Set \( \overline{S} := \text{Spec } k \). Choose a separable closure \( \overline{\eta} \) of \( \eta \), and let \( \overline{\eta} \) be the residue field of the normalization \( S \) of \( S \) in \( \kappa(\eta) \).

**Lemma 2.8.** Let \( \mathcal{W} \rightarrow \text{Hom} \) be a \( S \)-morphism from a smooth stack \( \mathcal{W} \). Let \( \overline{S} \) be a \( \overline{S} \)-point of \( \text{Hom} \) which comes from a \( \mathcal{S} \)-point of \( \mathcal{W} \). Then there is a stack \( \overline{\Sigma}_\zeta \) over \( \text{Spf } R \) which lifts \( \Sigma_\zeta \) over \( \text{Spf } R \). Assuming further that \( \mathcal{H} \) and \( \mathcal{Y} \) are proper, then there is a \( \text{Gal}(\overline{\eta}/\eta) \)-equivariant isomorphism

\[
H^i(\overline{\Sigma}_\zeta, \overline{\mathcal{F}}) \cong H^i(\Sigma_\zeta, R\Psi(\overline{\mathcal{F}})).
\]

Here \( R\Psi(\overline{\mathcal{F}}) \) denotes the corresponding sheaf of nearby cycles.
Proof. The point $ς : S \to \text{Hom}$ comes from $S$-point of the smooth stack $W$. By infinitesimal criterion of smoothness it gives a compatible set of morphisms $ς_n : Y_n := \mathcal{Y} \times_R \mathfrak{m}_R^n \to \mathcal{Y}_n$. This consequently yields a compatible system $\Sigma_{\varsigma_n} \hookrightarrow \Sigma_{\varsigma_{n+1}}$ of closed immersions of algebraic stacks, which accordingly define a formal algebraic stack $\hat{\Sigma}_\varsigma$ over $\text{Spf} \, R$. Let $\hat{\mathcal{H}} := \hat{\mathcal{Y}} \times \text{Spf} \, R$ and $\hat{\mathcal{Y}} := \hat{\mathcal{Y}} \times \text{Spf} \, R$. Assuming that $\mathcal{H}$ and $\mathcal{Y}$ are proper we see that $\hat{\mathcal{H}} \to \hat{\mathcal{Y}} \times \hat{\mathcal{Y}}$ and also $\Sigma_{\varsigma}$ are proper, thus we see by Grothendieck existence theorem that $\hat{\Sigma}_\varsigma$ lifts to a projective family over $\text{Spec} \, R$. The isomorphism

$$H^i(\hat{\Sigma}_\varsigma, \mathcal{U}_\ell) \cong H^i(\Sigma_{\varsigma}, RΨ(\mathcal{U}_\ell))$$

of the cohomology groups follows from basic properties of the sheaf of nearby cycles; see for example [AEK, Theorem 10.1]. For the Grothendieck existence theorem in the context of algebraic stacks see [Ol1, Theorem 1.4].

3 The family $\Sigma \to E$ arising from Hecke stack

In this section we focus on particular examples of the construction we discussed in Section 2.1. Before doing this, we need to recall some further preliminary materials.

3.1 The moduli stack $\mathcal{H}^1(C, \mathcal{G})$

Let $X \to S$ be a projective flat morphism of schemes. Notice that we later restrict ourselves to the case that $X$ is a (relative) curve $C$ (over $S$). Let $\mathcal{H}^1(X, \mathcal{G})$ be the stack classifying $\mathcal{G}$-bundles on $X$. Assume that $\mathcal{G}$ admits a representation $\iota : \mathcal{G} \to \text{GL}(V_0)$, where $V_0$ is a vector bundle over $X$, and such that it fulfills the following requirement

there is a scheme $Y$ affine and of finite type over $X$ with an action $\mathcal{G} \times_X Y \to Y$ (3.1) of $\mathcal{G}$ and a $\text{GL}(V_0)$-equivariant open immersion $\text{GL}(V_0)/\mathcal{G} \hookrightarrow Y$.

To see up to what extent the above condition can be served see [AraHar2].

The representation $\iota$ induces a morphism $\iota : \mathcal{H}^1(X, \mathcal{G}) \to \mathcal{H}^1(X, \text{GL}(V_0))$ of $S$-stacks. For a scheme $T$ and a morphism $T \to \mathcal{H}^1(X, \text{GL}(V_0))$, corresponding to $\text{GL}(V_0)$-torsor $\mathcal{G}$, one forms the following 2-Cartesian diagram

$$
\begin{array}{ccc}
\pi_*(\mathcal{G}/\mathcal{G}) & \longrightarrow & T \\
\downarrow & & \downarrow \varphi \\
\mathcal{H}^1(X, \mathcal{G}) & \longrightarrow & \mathcal{H}^1(X, \text{GL}(V_0))
\end{array}
$$


The above condition ensures that \( \pi_*(G/\mathcal{G}) \) is a quasi-affine \( S \)-scheme of finite presentation; see [AraHar2, Theorem 2.6]. This way one reduces the study of \( \mathcal{H}^1(X, \mathcal{G}) \) to the well-known case where \( \mathcal{G} = \text{GL}(V_0) \).

Let us state the following basic result.

**Theorem 3.1.** Let \( X \to S \) be a projective flat morphism of schemes. Let \( \mathcal{G} \) be as above. Then \( \mathcal{H}^1(X, \mathcal{G}) \) is an algebraic \( S \)-stack locally of finite presentation.

**Proof.** The theorem is well known when \( \mathcal{G} \) is constant, for a split reductive group \( G \), and \( X = C \), where \( C \) is a smooth projective curve over a perfect field \( k \). When \( \mathcal{G} \) is a parahoric group scheme over a smooth projective curve \( C \) over a perfect field \( k \) see Heinloth [Hei, Proposition 1]. For more general case where \( \mathcal{G} \) is a flat affine group scheme of finite type over \( C \), a proof is given in [AraHar2, Theorem 2.5]. The idea is using the method discussed above and showing that a flat affine group scheme of finite type over the curve \( C \) satisfies the above condition. The statement for the general relative case is similar, except that we do not have the relative version of the [AraHar2, Proposition 2.2] which ensures the existence of the representation \( \iota : \mathcal{G} \to \text{GL}(V_0) \) that satisfies the above condition. Note however that this is obvious for the constant reductive case. Also for the relative case see [Wan, Theorem 1.0.1].

**Remark 3.2.** When \( X = C \), where \( C \) is a smooth projective family of curves with geometrically reduced, connected fibres over \( S \), and \( \mathcal{G} \) is smooth, then the stack \( \mathcal{H}^1(C, \mathcal{G}) \) admits an open covering \( \{U_\alpha\}_{\alpha \in I} \) by smooth algebraic substacks of finite presentation over \( S \); e.g. see [Wan, Theorem 1.0.1] or [AraHar2, Theorem 2.5]. Furthermore it’s diagonal morphism is schematic, affine and of finite presentation.

### 3.1.1 Heinloth-Schmitt \( a \)-stability condition

Note that in [HeiSch] the authors establish this theory for the case that \( S = \text{Spec } k \) for a finite field \( k \) and \( \mathcal{G} \) is constant, i.e. \( \mathcal{G} = G \times_{\mathbb{F}_q} C \) for a split reductive group \( G \) over \( k \). Then they explain that their theory carries over to the case where \( G \) is a reductive group over an integral ring \( R \), finitely generated over \( \mathbb{Z} \), up to some modifications, see [HeiSch, Remark 3.2.4]. This is essential for the techniques they implement in their article, see proof of [HeiSch, Corollary 3.3.4 and Theorem 3.3.5]. We don’t know up to what extent they may remain valid for more general \( \mathcal{G} \). According to this, whenever we make use of the stability condition, we implicitly assume that \( \mathcal{G} \) is constant for a reductive group \( G \) over \( S \).

Here for the convenience of the reader we briefly recall the \( a \)-(semi)stability result of Heinloth-Schmitt from [HeiSch].

**Definition-Remark 3.3.** a) To recall the \( a \)-stability condition we recall the definition of the \( G \)-bundles with flagging of type \( (x, P) \). Here \( \underline{x} = (x_i)_{i=1,\ldots,b} \) denote a
finite set of distinct $k$-rational points of $C$, and $\underline{P} = (P_i)_{i=1,\ldots,b}$ denotes a tuple of parabolic subgroups of $G$. A principal $G$-bundle with a flagging of type $(\underline{x}, \underline{P})$ is a tuple $(\mathcal{P}, \underline{s})$ that consists of a principal $G$-bundle $\mathcal{P}$ on $C$ and a tuple $\underline{s} = (s_1, \ldots, s_b)$ of sections $s_i : x_i \to (\mathcal{P} \times_C \{x_i\})/P_i$. The category of principal $G$-bundles with a flagging of type $(\underline{x}, \underline{P})$ form the smooth algebraic stack $\mathcal{H}^1(C, \mathfrak{G}, \underline{x}, \underline{P})$; see [HeiSch, Lemma 3.2.2].

b) For an algebraic group $P$ let $X^*(P)$ denote the corresponding group of characters. Let $X^*(P)_{Q,+}$ denote the set of all elements $a \in X^*(P)_Q$ such that $a(\det P' \otimes \det P) < 0$ for all $P \subseteq P'$.

Let $(\mathcal{P}, \underline{s})$ be a flagged principal $G$-bundle and $\mathcal{P}_{x_i,P_i}$ the $P_i$-torsor over $x_i$ defined by $s_i$, $i = 1, \ldots, b$. Set $P_s := \text{Aut}_P(\mathcal{P}_{x_i,P_i}) \subseteq \text{Aut}_G(\mathcal{P}_{x_i})$ the corresponding parabolic subgroup. There are canonical isomorphisms $X^*(P_i)_Q \cong X^*(P_s)_Q$ and $\check{X}^*(P_i)_{Q,+} = \check{X}^*(P_s)_{Q,+}$, $i = 1, \ldots, b$. For $a_i \in X^*(P_i)_{Q,+}$ we let $a_i$ denote the corresponding element in $\check{X}^*(P_s)_{Q,+}$. For a parabolic subgroup $Q$ of $G$, a character $\chi$ of $Q$, and a reduction $\mathcal{P}_Q$ of $\mathcal{P}$ to $Q$, we get in each point $x_i$ a parabolic subgroup $Q_i$ in $\text{Aut}(\mathcal{P}_{x_i})$ and a character $\chi_{s_i}$ of that parabolic subgroup, $i = 1, \ldots, b$. Fix $\underline{a} \in \prod_i X^*(P_i)$.

For a parabolic subgroup $Q$ of $G$ and a reduction $\mathcal{P}_Q$ of $\mathcal{P}$ to $Q$, define

\[
\underline{a} - \text{deg}(\mathcal{P}_Q) : X^*(Q) \to \mathbb{Q} \\
\chi \mapsto \text{deg}(\mathcal{P}_Q(\chi)) + \sum_i \langle \chi_{s_i}, a_i \rangle.
\]

Here $\mathcal{P}_Q(\chi)$ is the line bundle on $C$ given by pushing forward the principal $Q$-bundle $\mathcal{P}_Q$ via the character $\chi$, and $\chi_{s_i}$ is a character of $Q_i \subseteq \text{Aut}(\mathcal{P}_{x_i})$ in each point $x_i$. To see that the pairing $\langle \chi_{s_i}, a_i \rangle$ is well-defined see [HeiSch, Remark 4.1.2 iii)].

c) A flagged principal $G$-bundle $(\mathcal{P}, \underline{s})$ is called $\underline{a}$-semistable (resp. stable), if for any parabolic subgroup $Q \subseteq G$ and any reduction $\mathcal{P}_Q$ of $\mathcal{P}$ to $Q$, the following holds

\[
\underline{a} - \text{deg}(\mathcal{P}_Q) \leq 0
\]

(resp. $\underline{a} - \text{deg}(\mathcal{P}_Q) < 0$).

Denote by $\mathcal{H}^1(C, \mathfrak{G}, \underline{x}, \underline{P})^{\underline{a}(s)}$, the substack of the moduli stack $\mathcal{H}^1(C, \mathfrak{G}, \underline{x}, \underline{P})$ parametrizing $\underline{a}$-(semi)stable flagged principal $\mathfrak{G}$-bundles of type $(\underline{x}, \underline{P})$. We further use the notation $\mathcal{H}^1(C, \mathfrak{G})^{\underline{a}(s)}$ to denote the (scheme theoretic) image of $\mathcal{H}^1(C, \mathfrak{G}, \underline{x}, \underline{P})^{\underline{a}(s)}$ under the projection

\[
\mathcal{H}^1(C, \mathfrak{G}, \underline{x}, \underline{P})^{\underline{a}(s)} \to \mathcal{H}^1(C, \mathfrak{G});
\]
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see [Stacks, Tag 0CMH, Lemma 98.37.3.] for existence of the scheme theoretic image. Note that for the scheme theoretic image $Z$ of a quasi-compact morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$, one can observe that $|Z|$ is the closure of the image of $|f|$; see [Stacks, Tag 0CMH, Lemma 98.37.6.].

Let us now state the following important result of Heinloth and Schmitt which was the key point in their proof of the purity of $H^*(\mathcal{H}^1(C, \mathcal{G}), \mathbb{Q}_\ell)$.

**Theorem 3.4.** For any substack $\mathcal{U}$ of $\mathcal{H}^1(C, \mathcal{G})$, and any integer $i > 0$, there is a type $(x, P)$ and stability condition $a$ such that $\mathcal{H}^1(C, \mathcal{G})_{(x, P)^{a-(s)s}}$ is smooth proper, and it’s image contains $\mathcal{U}$ (in particular the closure $\overline{\mathcal{U}}$ in $\mathcal{H}^1(C, \mathcal{G})$ is proper). Furthermore the codimension of $\mathcal{H}^1(C, \mathcal{G})_{(x, P)^{a-(s)s}}$ in $\mathcal{H}^1(C, \mathcal{G})_{(x, P)}$ is $> i$.

**Proof.** This is [HeiSch, Theorem 3.2.3].

**Proposition 3.5.** Fix open substacks $U, V \subseteq \mathcal{H}^1(C, \mathcal{G})$ of finite type. The stack

$$\text{Hom}(\mathcal{H}^1(C, \mathcal{G})_{(x, P)^{a-(s)s}}, \mathcal{H}^1(C, \mathcal{G}))$$

(resp. $\mathcal{E}_{a-ss} : = \text{Hom}(\mathcal{H}^1(C, \mathcal{G})_{(x, P)_{a-(s)s}}, \mathcal{H}^1(C, \mathcal{G})_{(x, P)_{a-(s)s}}$), resp. $\text{Hom}(U, V)$) is an algebraic stack for appropriate choice of $(x, P)$. Furthermore, $\text{Hom}(U, V)$, for affine charts $U \to \mathcal{H}^1(C, \mathcal{G})$ and $V \to \mathcal{H}^1(C, \mathcal{G})$ (resp. $U \to \mathcal{H}^1(C, \mathcal{G})_{(x, P)_{a-(s)s}}$ and $V \to \mathcal{H}^1(C, \mathcal{G})_{(x, P)_{a-(s)s}}$), is smooth.

**Proof.** The algebraicity of these stacks follow from the above theorem 3.4 and the main result of [HaRy]; see also [Ol2]. The smoothness follows from infinitesimal criterion for smoothness and the fact that $\mathcal{H}^1(C, \mathcal{G}, x, P)$ is a locally trivial bundle over the smooth Artin stack $\mathcal{H}^1(C, \mathcal{G})$ whose fibers are isomorphic to $\prod_{i=1}^b (G/P_i)$.

**3.2 Construction of $\Sigma \to \mathcal{E}$ for $\mathcal{H} = \text{Hecke}_n(C, \mathcal{G})$**

In this subsection we first recall the definition of the Hecke stack. We view it as a two fiber bundle over $\mathcal{H}^1(C, \mathcal{G})$. This yields a family $\Sigma \to \mathcal{E}$ via the construction explained in section 2. We then discuss certain fibers of this family and we recall the notion of boundedness condition from [AraHar1], [AraHar2] and [AraHab]. We further observe that a boundedness condition gives rise to certain Hecke classes in Chow group; see Proposition 3.15. We finally construct a local model roof for the family $\Sigma \to \mathcal{E}$, see Proposition 3.20.

**Definition 3.6.** For a natural number $n$, let $\text{Hecke}_n(C/S, \mathcal{G})$ be the stack fibered in groupoids over the category of $S$-schemes, whose category of $T$-valued points consists of tuples $(\mathcal{G}, \mathcal{G}', \Sigma, \tau)$, where
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- $\mathcal{G}$ and $\mathcal{G}'$ are in $\mathcal{H}^1(C, \mathfrak{G})(T)$,
- $s := (s_i)_i \in C^n(T)$ is an n-tuple of sections, and
- $\tau: \mathcal{G}'_{|c_T \setminus \Gamma} \xrightarrow{\sim} \mathcal{G}_{|c_T \setminus \Gamma}$ is an isomorphism.

Forgetting the isomorphism $\tau$ defines a morphism

$$\text{Hecke}_n(C/S, \mathfrak{G}) \to \mathcal{H}^1(C, \mathfrak{G}) \times_S \mathcal{H}^1(C, \mathfrak{G}) \times_S C^n.$$  

(3.2)

**Proposition 3.7.** Let $\mathfrak{G}$ be a flat affine group scheme of finite type over $C$. Furthermore assume that it admits a faithful representation $\rho: \mathfrak{G} \to \text{SL}(V_0)$, for a vector bundle $V_0$ over $C$, with affine (resp. quasi-affine) quotient $\text{SL}(V_0)/\mathfrak{G}$. The stack $\text{Hecke}_n(C/S, \mathfrak{G})$ is ind-algebraic stack, ind-projective (resp. ind-quasi-projective) over $C^n \times \mathcal{H}^1(C, \mathfrak{G})$; see [AraHar2, Propostion 3.9]. Furthermore it is ind-projective if $\mathfrak{G}$ is parahoric.

**Proof.** See [AraHar2, Proposition 3.9]. Note however that the argument given in loc. cit. is stated for $S = \text{Spec } k$, but nevertheless, one can literally follow the same lines to prove the more general statement.

\[\square\]

**Notation-Remark 3.8.** (a) Consider the $HR$-tuple $\mathcal{H}_n(C, \mathfrak{G}) := (\mathcal{H}_n(C, \mathfrak{G}), \text{char}, pr^{\leftarrow}, pr^{-})$, where

- $\mathcal{H}_n(C, \mathfrak{G}) := \text{Hecke}_n(C/S, \mathfrak{G})$,
- $\text{char}: \mathcal{H}_n(C, \mathfrak{G}) \to C^n (\mathcal{G}, \mathcal{G}', s, \tau) \mapsto s$,
- $\mathcal{Y}(C, \mathfrak{G}) := \mathcal{H}^1(C, \mathfrak{G})$,
- $pr^{\leftarrow}: \mathcal{H}_n(C, \mathfrak{G}) \to \mathcal{Y}(C, \mathfrak{G})$ (resp. $pr^{-}: \mathcal{H}_n(C, \mathfrak{G}) \to \mathcal{Y}(C, \mathfrak{G})$) the projection to the first (resp. second) factor, see 3.2.

For $HR$-tuple $\mathcal{H} := \mathcal{H}_n(C, \mathfrak{G}) := (\mathcal{H}_n(C, \mathfrak{G}), \text{char}, pr^{\leftarrow}, pr^{-})$ as above, we set $\Sigma_n(C, \mathfrak{G}) := \Sigma(\mathcal{H})$; see Definition 2.1.

When the curve $C$ and the group $\mathfrak{G}$ are obvious from the context, we remove $(C, \mathfrak{G})$ from our notation. Also when $n$ is clear from the context we write $\mathcal{H} = \mathcal{H}_n$ and $\Sigma = \Sigma_n$.

(b) We view $\Sigma = \Sigma(C, \mathfrak{G})$ as a family $\Sigma \to \mathcal{E} := \mathcal{E}(C, \mathfrak{G}) := \text{Hom}(\mathcal{Y}, \mathcal{Y})$. and we denote by $\Sigma_\zeta$, the fiber above $\zeta \in \mathcal{E}$. We call an object $\mathcal{G}$ in $\Sigma_\zeta(S)$ a (global) $\zeta$-shtuka over $S$. 
(c) For a relative Cartier divisor $D$ on $C$, we denote by $\mathcal{Y}_D := \mathcal{H}^1_D(C, \mathfrak{G})$ the stack over $\mathcal{H}^1(C, \mathfrak{G})$, whose $T$-points classifies $\mathfrak{G}$-bundles on $C_T$ together with $D$-level structures, i.e. a trivialization $\mathcal{G}|_{D \times T} \to \mathfrak{G} \times_C C_T|_{D \times T}$. Using this one can further equip the other constructions with level $D$-structure in an obvious way. We use the subscript $D$ in our notation $\mathcal{H}_D$, $\Sigma_D$ and etc. to illustrate that the objects parameterized by corresponding moduli stacks are equipped with $D$-level structures.

(d) Assume that $\mathfrak{G} = G \times_S C$ for split reductive group $G$ over $S$. Fix a type $(x, P)$; see subsection 3.1.1. We similarly use the notation $\mathcal{H}^1_{(x, P)}$ for the restriction of $\mathcal{H}$ to $\mathcal{Y}^{\mathrm{ss}}_{(x, P)} := \mathcal{H}^1(C, \mathfrak{G})|_{(x, P)}$ under $pr^*: \mathcal{H} \to \mathcal{Y}$. Regarding the procedure described in subsection 2.1 we obtain the family $\Sigma^{\mathrm{ss}}_{(x, P)}$ over

$$\text{Hom}(\mathcal{H}^1(C, \mathfrak{G})|_{(x, P)}, \mathcal{H}^1(C, \mathfrak{G})).$$

Note that the latter stack is an algebraic stack for appropriate choice of $(x, P)$, see Proposition 3.3.

(e) One may alternatively require that the $\mathfrak{G}$-bundles occurring in the above parts are semistable in the sense of [BaPa]. For this one requires $G$ to be semi-simple. We set $\mathcal{H}^{ss} = \text{Hecke}^{ss}, \mathcal{Y}^{ss}, \mathcal{E}^{ss}$ and etc. for the corresponding moduli spaces. Note that the moduli space $\mathcal{Y}^{ss}$ is projective, according to [BaPa] and [BaSe]. Consequently, $\mathcal{E}^{ss}$ is representable by a scheme locally of finite type. See Remark 2.4 and also [Br1] for some details on the structure of the endomorphisms of projective varieties.

### 3.3 Specific fibers of $\Sigma \to \mathcal{E}$

As we will see bellow, some interesting moduli spaces appear in the fibers of the family $\Sigma \to \mathcal{E}$.

#### 3.3.1 Global affine Grassmannian, Boundedness Conditions and Local Model

**Definition 3.9.** Fix a $\mathfrak{G}$-bundle $\mathcal{G}_0$ in $\mathcal{Y} := \mathcal{H}^1(C, \mathfrak{G})(S)$. Take $\varsigma$ to be the constant morphism

$$\Theta_{\mathcal{G}_0} : \mathcal{Y} \to \mathcal{Y},$$

$$y \mapsto \mathcal{G}_0.$$

We denote by $\Sigma_{\Theta_{\mathcal{G}_0}}$ the fiber of the family $\Sigma \to \mathcal{E}$. The stack $\mathcal{Y}$ has an especial $S$-point corresponding to the trivial $\mathfrak{G}$-bundle, we denote the corresponding constant morphism by $\Theta$. The fiber $\Sigma_{\Theta}$ is called the (relative) Beilinson-Drinfeld affine Grassmannian. We also use the notation $G \mathcal{R}_n(C, \mathfrak{G})$ for $\Sigma_{\Theta}$.
Proposition 3.10. Let $\mathcal{G}$ be a flat affine group scheme of finite type over $C$. Furthermore assume that it admits a faithful representation $\rho : \mathcal{G} \to \text{SL}(V_0)$, for a vector bundle $V_0$ over $C$, with affine (resp. quasi-affine) quotient $\text{SL}(V_0)/\mathcal{G}$. Then the fiber $\Sigma_{\mathcal{G}_0}$ is an ind-scheme ind-projective (resp. ind-quasi-projective) over $C^n$. In particular when $\mathcal{G}$ is parahoric the Beilinson-Drinfeld affine Grassmannian $GR_n(C, \mathcal{G})$ is an ind-scheme ind-projective (resp. ind-quasi-projective) over $C^n$.

Proof. This follows from Proposition 3.7.

Definition-Remark 3.11. The (relative) loop group $L_n\mathcal{G}$ (resp. positive loop group $L^+_n\mathcal{G}$) is the space corresponding to the following functor $T \mapsto \{(s_1, \ldots, s_n, \alpha); s_i \in C(T) \text{ and } \alpha \in \mathcal{G}(\mathcal{D}(\Gamma_s))\}$ (resp. $T \mapsto \{(s_1, \ldots, s_n, \alpha); s_i \in C(T) \text{ and } \alpha \in \mathcal{G}(\mathcal{D}(\Gamma_s))\}$).

According to gluing lemma of Beauville-Laszlo, [BeLa], one observes that $L_n\mathcal{G}$ and $L^+_n\mathcal{G}$ operate on $\Sigma_{\mathcal{G}}$ via changing the trivialization.

Note that one can use Beilinson-Drinfeld affine Grassmannian $GR_n(C, \mathcal{G})$ to locally trivialize the family $\mathcal{H} \to C^n \times \mathcal{Y}$.

Proposition 3.12. Consider the stacks $\mathcal{H}$ and $GR_n(C, \mathcal{G}) \times \mathcal{Y}$ as families over $C^n \times \mathcal{Y}$, via the projections $(\mathcal{G}, \mathcal{G}', g, \tau) \mapsto (g, \mathcal{G}')$ and $(\mathcal{G}, g, \tau) \times \mathcal{G}' \mapsto (g, \mathcal{G}')$ respectively. They are locally isomorphic with respect to the étale topology on $C^n \times \mathcal{Y}$.

Proof. For the proof in the case $S = \text{Spec } k$ see [AraHab, Proposition 2.0.11] and [Var, Lemma 4.1]. The proof for the relative situation over a base scheme $S$ is similar.

As we mentioned before, the moduli stack $Hecke_n(C, \mathcal{G})$ is an ind-algebraic stack locally of ind-finite type. To provide a moduli stack which is (locally) of finite type, one may proceed by introducing boundedness conditions. There are various methods to establish such conditions. For a split reductive group $G$, Varshavsky uses an $n$-tuple $\mu := (\mu_i)$ of dominant coweights of $G$ to control the relative position of $\mathcal{G}'$ and $\mathcal{G}$ through $\tau : \mathcal{G}' \to \mathcal{G}$; see [Var, Definition 2.4]. Here we briefly recall the boundedness conditions that have been considered in [AraHar2] and [AraHab] for a flat group scheme $\mathcal{G}$ over $C$.

Definition-Remark 3.13. (a) Fix a faithful representation $\rho : \mathcal{G} \to \text{SL}(V_0) \subseteq \text{GL}(V_0)$ for some vector bundle $V_0$ of rank $r$ with quasi-affine quotient. Consider the induced morphism of stacks:

$$
\mathcal{H}^1(C, \mathcal{G}) \xrightarrow{\rho_*} \mathcal{H}^1(C, \text{SL}(V_0)) \xrightarrow{V(-)} \mathcal{H}^1(C, \text{GL}(V_0)) \cong \text{Vect}^r_C.
$$
Let \( \omega := (\omega_i) \) be an n-tuple of coweights of \( \text{SL}_r \) given as

\[
\omega_i : x \mapsto \text{diag}(x^{\omega_i,1}, \ldots, x^{\omega_i,r}),
\]

for integers \( \omega_{i,1} \geq \cdots \geq \omega_{i,r} \) with \( \sum \omega_{i,\ell} = 0 \).

We say that a morphism \( \tau : G \to G' \) between \( \mathfrak{G} \)-bundles \( G \) and \( G' \) over \( C_T \), defined outside graph of the sections \( s_i : T \to C \), is bounded by \( \omega \) if

\[
\wedge C_T \tau^{-1}(\mathcal{V}(g_*\mathcal{G})) \subseteq \wedge C_T \mathcal{V}(g_*\mathcal{G})(\sum_{j=1}^{\ell} \omega_{i,j})\Gamma_{s_i},
\]

with equality when \( \ell = r \). We denote by \( \text{Hecke}^{\omega}(C, \mathfrak{G}) \) the corresponding stack obtained by imposing the above boundedness condition. This further induces a boundedness condition on \( \Sigma \). Similarly we use the notation \( \Sigma^\omega \) for the resulting bounded moduli stack.

(b) The above boundedness condition is not intrinsic as it depends to the choice of a representation. To provide an intrinsic boundedness condition, in [AraHar2] and [AraHab], we discussed another method. Namely, according to this method, a boundedness condition is given by a class of closed \( \mathcal{L}_n^+ \mathfrak{G} \)-stable subschemes \( Z \subseteq \Sigma_{\mathfrak{G}} \times_{C^n} \bar{C} \), where \( \bar{C} \) is a smooth projective curve which is finite over \( C \). Such a class of subschemes determine a minimal curve \( C_Z \), which is called reflex curve, over which the bounded moduli stack is defined; see [AraHab, Definition 3.1.3 and 4.3.2]. To avoid the complications arising in the general set up, we assume that the reflex curve is \( C \) itself, and the boundedness condition is given by \( \mathcal{L}_n^+ \mathfrak{G} \)-stable closed subschemes \( Z \subseteq GR_n \). We in addition assume that \( Z \) is proper and flat over \( C^n \).

We say that a morphism \( \mathcal{G}' \to \mathcal{G} \) defined over \( C_T \setminus \Gamma_s \) is bounded by \( Z \), if for every trivialization of \( \mathcal{G} \), the induced morphism \( T \to GR_n \) factors through \( Z \). This gives boundedness condition \( Z \) on \( \text{Hecke}_n(C/S, \mathfrak{G}) \), which further induces boundedness condition \( Z \) on the moduli stack \( \Sigma \). We denote the corresponding moduli spaces (stacks) obtained by imposing the boundedness condition \( Z \), by \( \mathcal{H}_{n,D}^Z = \text{Hecke}_{n,D}^Z, \Sigma_D^Z \) and etc.

Remark 3.14. When \( \mathfrak{G} \) is parahoric then \( GR_n(C, \mathfrak{G}) \) is ind-proper. In particular a closed subscheme \( Z \) of \( GR_n(C, \mathfrak{G}) \) is automatically projective; see Proposition 3.10. But still flatness of \( Z \) over \( C^n \) is not obvious.

Proposition 3.15. Assume that \( \mathfrak{G} \) is smooth and \( S = \text{Spec} k \). Let \( Z \) be as in Definition-Remark 3.13(b). Then the tuple

\[
\mathcal{H}_{D}^Z := (\mathcal{H}_{n,D}^Z, \text{char}, pr^\leftarrow, pr^\rightarrow)
\]
is a CHR-tuple. Furthermore it induces a cycle \( \varphi_{D, \alpha} \) in \( Ch_{d+d'}(\mathcal{H}_D^1(C, \mathcal{G})_\alpha \times S, \mathcal{H}_D^1(C, \mathcal{G})_\alpha \times S (C \smallsetminus D)^n) \) for \( D \) sufficiently large. Here \( \mathcal{H}_D^1(C, \mathcal{G})_\alpha := \mathcal{H}_D^1(C, \mathcal{G}) \times_{\mathcal{H}^1(C, \mathcal{G})} \mathcal{U}_\alpha \), see Remark 3.18 and \( d' = \dim \mathcal{H}_D^1(C, \mathcal{G}) ).

Proof. The first statement follows from Proposition 3.12 [AraHar2, Proposition 3.12] and basic properties of the functor \( M^c(-) \).

We can take the divisor \( D \) sufficiently large such that \( \mathcal{H}_D^1(C, \mathcal{G})_\alpha \) becomes representable by a quasi-projective scheme, e.g. see [AraHar2, Remark 2.9]. Composing the canonical morphism \( M(\mathcal{H}_D^1(C, \mathcal{G})_\alpha) \to M^c(\mathcal{H}_D^1(C, \mathcal{G})_\alpha) \) and the morphism \( M^c(\mathcal{H}_D^1(C, \mathcal{G})_\alpha)(d)[2d] \to M^c(\mathcal{H}_D^1(C, \mathcal{G})_\alpha) \) induced by \( \mathcal{H}_D^2 \) gives a morphism in

\[
\text{Hom}_{DM(k, \mathbb{Q})}(M(\mathcal{H}_D^1(C, \mathcal{G})_\alpha)(d)[2d], M^c(\mathcal{H}_D^1(C, \mathcal{G})_\alpha \times (C \smallsetminus D)^n)).
\]

Note that \( \mathcal{H}_D^1(C, \mathcal{G})_\alpha \to \mathcal{H}^1(C, \mathcal{G})_\alpha \) is a \( \mathcal{G}_D \)-torsor, and since \( \mathcal{G} \) is smooth, by Remark 3.2 and [Kel] (\( M(\mathcal{H}_D^1(C, \mathcal{G})_\alpha) \) is dualizable, therefore the above is isomorphic to \( \text{Hom}(\mathbb{Q}[d+d'], M^c(\mathcal{H}_D^1(C, \mathcal{G})_\alpha \times \mathcal{H}_D^2(C, \mathcal{G})_\alpha \times C^n)) \). Thus we obtain a cycle \( \varphi_{D, \alpha} \) in \( Ch_{d+d'}(\mathcal{H}_D^1(C, \mathcal{G})_\alpha \times \mathcal{H}_D^2(C, \mathcal{G})_\alpha \times (C \smallsetminus D)^n) \), e.g. see [CD, Theorem 8.4].

Remark 3.16. In the above proposition, it is not necessary to take the coefficients in \( \mathbb{Q} \). Namely, when \( \text{char } k = 0 \) (resp. \( \text{char } k = p \)) one can simply work with \( \mathbb{Z} \) (resp. \( \mathbb{Z}[1/p] \)) as the corresponding ring coefficients.

Remark 3.17. When \( \mathcal{G} \) is constant for a split reductive group \( G \) over \( k \), imposing semi-stability condition, in the sense of 3.8(e) we get a proper scheme \( \mathcal{H}^1(C, \mathcal{G})^{ss} \), then according to Proposition 3.15 we can observe that there is a morphism

\[
\mathcal{C}_d^L \mathcal{G}(GR_n) \to Ch_{d+d'}(\mathcal{H}^1(C, \mathcal{G})^{ss} \times_k \mathcal{H}^1(C, \mathcal{G})^{ss} \times_k C^n)
\]

Here \( \mathcal{C}_d^L \mathcal{G}(GR_n) \) denote the \( \mathbb{Z} \)-module generated by closed \( d \)-equidimensional \( L^a \mathcal{G} \)-equivariant subschemes of \( GR_n \).

Remark 3.18. When \( \mathcal{G} \) is a constant for a split reductive group \( G \) over \( k \), there is a conjectural description of the motive with compact support \( M^c(\mathcal{H}^1(C, \mathcal{G})) \) of the moduli of \( \mathcal{G} \)-bundles according to [BeDh]. For \( \mathcal{G} = GL_n \), the formula has been proved inside Voevodsky’s motivic categories in [HoLe].

Definition-Remark 3.19 (Functoriality). (a) Consider a morphism \( \varrho : \mathcal{G}' \to \mathcal{G} \) of algebraic groups. This induces a 1-morphism \( \varrho_* : \mathcal{H}^1(C, \mathcal{G}') \to \mathcal{H}^1(C, \mathcal{G}) \) and consequently

\[
\mathcal{H}_n(C, \mathcal{G}') := \text{Hecke}_n(C/S, \mathcal{G}') \to \mathcal{H}_n(C, \mathcal{G}) := \text{Hecke}_n(C/S, \mathcal{G}), \tag{3.3}
\]
of ind-algebraic stacks. When we further assume that $\mathcal{G}' \to \mathcal{G}$ satisfies a condition similar to 3.1 (i.e. there is a scheme $Y$ affine and of finite type over $X$ with an action $\mathcal{G} \times_X Y \to Y$ of $\mathcal{G}$ and a $\mathcal{G}$-equivariant open immersion $\mathcal{G}/\mathcal{G}' \hookrightarrow Y$), it is quasi-affine and of finite type. Let $\mathcal{E}' := \text{End}(\mathcal{H}^1(C, \mathcal{G}))$ and $\mathcal{E} := \text{End}(\mathcal{H}^1(C, \mathcal{G}'))$. Given morphisms $\varsigma \in \mathcal{E}$ and $\varsigma' \in \mathcal{E}'$, with $\varsigma \circ g_* \cong g_* \circ \varsigma$ then the above morphism 3.3 induces the following morphism of ind-algebraic stacks

$$\Sigma_{\varsigma'}(C, \mathcal{G}') \to \Sigma_{\varsigma}(C, \mathcal{G}).$$

In particular for $\varsigma = \Theta$ and $\varsigma' = \Theta'$ we get a morphism

$$\Sigma_{\Theta'} \to \Sigma_{\Theta}$$

of global affine Grassmannians.

(b) A $\Sigma\mathcal{H}$-datum is a tuple $(\mathcal{G}, Z, \varsigma)$ consisting of a group scheme $\mathcal{G}$ over $C$ together with a boundedness condition $Z$ and an endomorphism $\varsigma$ in $\mathcal{E}(S)$. A morphism $(\mathcal{G}', Z', \varsigma') \to (\mathcal{G}, Z, \varsigma)$ between $\Sigma\mathcal{H}$-data is a morphism $\tilde{\varsigma} : \mathcal{G}' \to \mathcal{G}$ such that $Z' \hookrightarrow \Sigma_{\tilde{\varsigma}} \to \Sigma_{\varphi}$ factors through $Z$, and the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{H}^1(C, \mathcal{G}') & \xrightarrow{\varphi_*} & \mathcal{H}^1(C, \mathcal{G}) \\
\varsigma' \downarrow & & \varsigma \downarrow \\
\mathcal{H}^1(C, \mathcal{G}') & \xrightarrow{\varphi_*} & \mathcal{H}^1(C, \mathcal{G}).
\end{array}
$$

To a morphism $\varphi : (\mathcal{G}', Z', \varsigma') \to (\mathcal{G}, Z, \varsigma)$ as above, one can assign the following 1-morphism of algebraic stacks

$$\varphi_* : \Sigma_{\varsigma'}(C, \mathcal{G}') \to \Sigma_{\varsigma}(C, \mathcal{G}).$$

The following proposition describes the local geometry of the fibers of the family $\Sigma$ over $\mathcal{E}$.

**Proposition 3.20.** Assume that the group $\mathcal{G}$ is smooth over $C$. Then there is the following roof of morphisms

$$
\xymatrix{
\tilde{\Sigma} \ar[dr]^{\pi_{\text{loc}}} & \\
\Sigma \ar[ur]_f & \Sigma_{\Theta} \times \mathcal{E},
}
$$

where
(a) \( \tilde{\Sigma} \) is a \( \mathcal{L}_n^+ \mathfrak{G} \)-torsor over \( \Sigma \) under \( f \),

(b) \( \pi^{loc} \) is formally smooth.

In addition, for a given \( \Sigma \mathcal{H} \)-datum \( (\mathfrak{G}, \mathcal{Z}, \varsigma) \) the above roof induces a roof of morphisms

\[
\xymatrix{
\tilde{\Sigma}_\mathcal{Z} (C, \mathfrak{G}) \ar[dr]_{\pi^{loc}} \ar[rr]^{f} & & \Sigma_\mathcal{Z} (C, \mathfrak{G}) \ar[dl] \\
\mathcal{Z}, & & \text{in a functorial way.}
}
\]

**Proof.** Let \( \tilde{\mathcal{H}} \) denote the stack whose category of \( T \)-valued points parametrizes the tuples \( \tilde{\mathcal{G}} := (\mathcal{G}, \mathcal{G}', \varsigma, \tau : \mathcal{G} \to \mathcal{G}') \) in \( \mathcal{H}(T) \) and \( \varepsilon \) is a trivialization of the restriction \( \mathcal{G}' \) of \( \mathcal{G}' \) to the formal neighborhood of \( \Gamma_{\Sigma} \). Sending \( \tilde{\mathcal{G}}, \varepsilon \) to \( (\mathcal{G}, \mathcal{G}', \varsigma, \varepsilon \circ \tau^{-1}) \) gives a map \( \tilde{\mathcal{H}} \to \Sigma_{\Theta} \). Let \( \Sigma \) (resp. \( f \)) denote the stack (resp. the morphism) defined by the following diagram

\[
\xymatrix{
\Sigma \ar[r]^{f} \ar[d] & \mathcal{H} \ar[d] \\
\mathcal{E} \times \mathcal{Y} \ar[r] & \mathcal{Y} \times \mathcal{Y}.
}
\]

This gives the desired roof of morphisms

\[
\xymatrix{
\tilde{\Sigma} \ar[r]^{f} \ar[d] & \tilde{\mathcal{H}} \ar[r] & \Sigma_{\Theta} \\
\Sigma \ar[r] & \Sigma_{\Theta} \times \mathcal{E},
}
\]

The morphism \( f \) is simply given by forgetting the trivialization \( \varepsilon \), hence it is a torsor under the group \( \mathcal{L}_n^+ \mathfrak{G} \).

It remains to justify that \( \pi^{loc} : \tilde{\Sigma} \to \Sigma_{\Theta} \times \mathcal{E} \) is formally smooth. Consider the following commutative diagram
Where $i : \mathcal{T} \to T$ is defined by a nilpotent sheaf of ideals. We need to show that the map $\alpha$ which fits in the above commutative diagram exists. This question reduces to the fact that a trivialization of a $\mathfrak{g}$-bundle $\tilde{\mathcal{G}}$ over infinitesimal neighborhood of $\Gamma_{x}$ lifts over $\Gamma_{x}$. This holds due to smoothness of $\mathfrak{g}$ and the infinitesimal criterion for smoothness. Compare also proof of [Ara, Theorem 3.12]. Part b) follows from part a) and definition of boundedness condition, see Definition-Remark and Definition-Remark 3.13 and Definition-Remark 3.19.

\[ \square \]

3.3.2 Moduli of Higgs bundles

Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$. The stack $\mathcal{H}^{1}(C, \mathfrak{z})$ operates on $\mathcal{H}^{1}(C, \mathfrak{g})$ via twisting torsors

$$\mathcal{H}^{1}(C, \mathfrak{z}) \times \mathcal{H}^{1}(C, \mathfrak{g}) \to \mathcal{H}^{1}(C, \mathfrak{g}).$$

It is defined by sending $(\mathcal{T}, \mathcal{G})$ to $\mathcal{T} \times^3 \mathcal{G}$. This induces the following morphism of stacks

$$\mathcal{H}^{1}(C, \mathfrak{z}) \to \mathcal{E}, \mathcal{T} \mapsto \varsigma_{\mathcal{T}} \quad (3.4)$$

(resp. $\mathcal{H}^{1}(C, \mathfrak{z}) \to \text{Hom} := \text{Hom} (\mathcal{H}^{1}(C, \mathfrak{g}), \mathcal{E})$ obtained by composing with natural morphism $\mathcal{E} \to \text{Hom}$). The pull back of the family $\Sigma^{\mathfrak{z}} \to \mathcal{E}$ (resp. $\Sigma^{\mathfrak{z}, \mathfrak{a}^{\mathfrak{z}}(s)} \to \text{Hom}$ under the above map yields a family $(\Sigma^{\mathfrak{z}})^{\dagger} \to \mathcal{H}^{1}(C, \mathfrak{z})$

$$\stackrel{(\Sigma^{\mathfrak{z}})^{\dagger}}{\longrightarrow} \Sigma^{\mathfrak{z}} \quad \downarrow \quad \downarrow \quad \mathcal{H}^{1}(C, \mathfrak{z}) \quad \longrightarrow \quad \mathcal{E}.$$
Proposition 3.21. Let $\mathfrak{G}$ be a constant reductive group, i.e. $\mathfrak{G} = G \times_S C$ for reductive group $G$ over $S$. The morphism $\left( \Sigma^{z,a,-(s)s} \right)^\dagger \rightarrow \mathcal{H}^1(C, \mathfrak{Z})$ is proper, for relevant type $(x,P)$ and stability condition $a$.

Proof. This statement follows from [AraHar2, Proposition 3.12], Proposition 3.12 and Theorem 3.4.

Let $S = \text{Spec} R$, for a complete dvr $(R, m_R)$, with special point $s$ and generic point $\eta$. Let $\kappa(s) := R/m_R$ denote the residue field and set $\overline{S} := \text{Spec} \kappa(s)$. Let $Z$ be a boundedness condition and let $Z := Z \times_S S$. We set $Y' := \mathcal{H}^1(C, \mathfrak{Z})$. Let $\overline{Y} := Y' \times_S \overline{S}$, $\overline{H} := H \times_R \overline{S}$ and set $pr^\leftarrow := pr^\leftarrow \times_S \overline{S}$ (resp. $pr^\rightarrow := pr^\rightarrow \times_S \overline{S}$). Let $\Sigma$ denote the corresponding family over $\mathcal{E} := \text{End}(\overline{Y})$, see Definition 2.1. We similarly use the notation $\Sigma$ for the corresponding bounded family and $\Sigma^{z,a,-(s)s}$ for the corresponding family with stability conditions of the given type $(x,P)$.

Corollary 3.22. Keep the above notation. Let $T_0$ be a $\mathfrak{Z}$-bundle in $Y'(S)$. Twisting by $T_0$ defines a morphism $\varsigma_{T_0}$ in $\mathcal{E}$, and consequently in $\text{Hom}$. Consider the family $\Sigma^Z \rightarrow \mathcal{E}$ (resp. $\Sigma^{z,a,-(s)s}$ $\rightarrow \text{Hom}$) and let $\overline{X} := \Sigma^Z_{T_0}$ (resp. $\overline{X}' := \left( \Sigma^{z,a,-(s)s}_{(x,P)} \right)_{T_0}$) denote the fiber above $\varsigma_{T_0}$. Then

a) there is a deformation $\hat{X}$ (resp. $\hat{X}'$) of $\overline{X}$ (resp. $\overline{X}'$) over $S$,

b) Let $\nu$ be a $n$-tuple of distinct sections $\nu_i : S \rightarrow C$. Assume that $Z_{\nu_i} := Z \times_{C, \nu_i} S$ is smooth for every $i$, and let $\hat{X}_{\nu_i}'$ denote the corresponding fiber above $\nu : S \rightarrow C^n$ of the projection map $\hat{X}' \rightarrow C^n$. Then

$$H^m((\hat{X}_{\nu_i}')_{T_0}, \mathbb{Q}_\ell) \cong H^m((\hat{X}_{\nu_i}')_{T_0}, \mathbb{Q}_\ell).$$

Proof. The $\mathfrak{Z}$-bundle $T_0$ defines a point $\overline{S} \rightarrow \mathcal{H}^1(C, \mathfrak{Z}) \rightarrow \text{Hom}$. The statement follows from Proposition 3.21, Proposition 3.20 and Lemma 2.8; see [Wan, Proposition 6.0.18] for smoothness of $\mathcal{H}^1(C, \mathfrak{Z})$. Note that by Proposition 3.20 we observe that $R\Psi(\mathbb{Q}_\ell)$ is constant.

Bellow we address the formally properness of the induced families over $\text{Pic}(C/S)$. For the notion of formally proper morphism see [HLP].

Corollary 3.23. Suppose $C \rightarrow S$ admits a section. For a cocharacter $\lambda$ of $\mathfrak{Z}$, the family $(\Sigma^Z)^\dagger$ yields a family $(\Sigma^Z)^\dagger_\lambda$ over $\text{Pic}(C/S)$. After restricting to $\mathcal{U}_\alpha$ this gives a formally proper family $(\Sigma^Z)^\dagger_\lambda \rightarrow \text{Pic}(C/S)$. 

Proof. The cocharacter \( \lambda \) induces a closed immersion

\[ \mathcal{H}^1(C, \mathbb{G}_m) \to \mathcal{H}^1(C, \mathfrak{g}). \]

Note that the compositum \( \lambda \) of this morphism followed by \( \mathbf{3.4} \) is given by sending a \( \mathbb{G}_m \)-torsor \( \mathcal{L} \) to the morphism \( \underline{\mathcal{L}} \), defined by sending \( \mathcal{G} \) to its twist \( \mathcal{G} \times_{\mathbb{G}_m} \mathcal{L} \) by \( \mathcal{L} \). We may restrict the family \( (\Sigma^Z)^\dagger_\lambda \to \mathcal{H}^1(C, \mathfrak{g}) \) to obtain a family \( (\Sigma^Z)^\dagger_\lambda \) on \( \mathcal{H}^1(C, \mathbb{G}_m) \). Since \( C \to S \) admits a section, there is an isomorphism \( \mathcal{H}^1(C, \mathbb{G}_m) \cong \text{Pic}(C/S) \times B \mathbb{G}_m \). This gives a morphism \( (\Sigma^Z)^\dagger_\lambda \to \text{Pic}(C/S) \). The formal properness follows from proposition \( \mathbf{3.21} \) and \cite[Example 4.3.1]{HL}.

**Remark 3.24** (Hitchin morphism for \( \mathfrak{g} = \text{GL}_r \)). Let \( \mathfrak{g} := \text{GL}_r \) and let \( \lambda \) be the embedding \( \mathbb{G}_m \cong \mathfrak{g} \to \mathfrak{g} \) of the center. Fix an isomorphism \( \mathcal{V}(-) : \mathcal{H}^1(C, \mathfrak{g}) \cong \text{Vect}_C \), where \( \text{Vect}_C \) denotes the stack of vector bundles of rank \( r \) over \( C \). One can construct a morphism from \( (\Sigma^Z)^\dagger_\lambda \) to an affine bundle over \( \mathcal{H}^1(C, \mathfrak{g}) \times C^m \) in the following way. Let us take the following variant of the boundedness conditions. For a point \((\mathcal{G}, \mathcal{G}', \mathcal{S}, \tau) \) in \( \text{Hecke}_n(C, \mathfrak{g})(T) \) we require that \( \tau^{-1} \mathcal{V}(\mathcal{G}) \subseteq \mathcal{V}(\mathcal{G}') \otimes \mathcal{O}_{C_T}(D_N(\mathcal{S})) \), where \( D_N(\mathcal{S}) \) denotes the relative divisor \( \sum_i N_i \cdot \Gamma_x \). Let \( \text{Hecke}^N_n(C, \mathfrak{g}) \) (resp. \( \Sigma^N \)) be the moduli stack obtained by imposing this boundedness condition. Then \( T \)-points of \( (\Sigma^N)^\dagger_\lambda \) can be described as tuples \((\mathcal{V}, \mathcal{S}, \tau, \mathcal{L}) \), where \( \tau \) is a morphism of vector bundles with \( \tau^{-1} \mathcal{V} \subseteq \mathcal{V} \otimes \mathcal{L}(D_N(\mathcal{S})) \), and \( \mathcal{L} \) is a line bundle over \( C \times T \). Here \( \mathcal{L}(D_N(\mathcal{S})) := \mathcal{L} \otimes_C \mathcal{O}(D_N(\mathcal{S})) \). Consider the affine bundle \( \mathcal{A}(-, \mathcal{N}) \) over \( \text{Pic}(C) \), defined by the following functor of points

\[ T \mapsto \{(\mathcal{L}, \mathcal{S} := (s_i)_i, \mathcal{L} := (t_i)_i) ; \mathcal{L} \in \text{Pic}(C)(T), s_i \in C(T), t_i \in H^0(C_T, \mathcal{L}^\otimes i \cdot D_N(\mathcal{S})) \} \]

over \( C^m \).

A point \( \mathcal{V} := (\mathcal{V}, \mathcal{S}, \tau, \mathcal{L}) \in (\Sigma^N)^\dagger_\lambda \), determines a global section \( \text{Tr} \tau^{-1} \) of \( \mathcal{L}(D_N(\mathcal{S})) \) via the composition \( \mathcal{O}_C \to \mathcal{V} \otimes \mathcal{V} \to \mathcal{L}(D_N(\mathcal{S})) \), where \( \mathcal{V} \otimes \mathcal{V} \to \mathcal{L}(D_N) \) is induced by \( \tau^{-1} \) and the first map takes 1 to \( \text{id}_\mathcal{V} \). Similarly we define \( \text{Tr}(\wedge^i \tau^{-1}) \). This yields the following map

\[ (\Sigma^N)^\dagger_\lambda \to \mathcal{A}(-, \mathcal{N}). \quad (3.5) \]

**Remark 3.25** (The fiber above \( \text{id} \) and periodic dynamics). Let \( \zeta = \text{id} \in \mathcal{E} \) (or equivalently consider the trivial bundle \( \mathcal{L} = \mathcal{O}_C \)), let \( \Sigma^Z_{\text{id}} \) be the fiber above the identity. Fix a representation \( \rho : \mathfrak{g} \to \text{GL}(\mathcal{V}_0) \), where \( \mathcal{V}_0 \) is a vector bundle over \( C \). Assume that the induced morphism \( \Sigma^Z_{\text{id}}(C, \mathfrak{g}) \hookrightarrow \text{Hecke}_n(C, \text{GL}_r) \) factors through \( \text{Hecke}^N_n(C, \text{GL}_r) \) for some \( \mathcal{N} \). Then composing \( \Sigma^Z_{\text{id}}(C, \mathfrak{g}) \to \Sigma^Z_{\text{id}}(C, \text{GL}_r) \) with \( \mathbf{3.5} \) for \( \mathcal{L} = \mathcal{O}_C \), gives

\[ \Sigma^Z_{\text{id}}(C, \mathfrak{g}) \to \mathcal{A}(\mathcal{N}). \quad (3.6) \]
Here \( \mathcal{A}(\mathcal{O}_C, N) := \mathcal{A}(\mathcal{O}_C, N) \) denote the fiber of \( \mathcal{A}(\mathcal{O}_C, N) \to \text{Pic}(C) \) over the trivial bundle \( \mathcal{O}_C \). Assume that \( \zeta^m = \text{id} \) and consider the map \( \alpha^m : \Sigma\rightarrow \Sigma\circ m \), defined by sending \( (\mathcal{G}, \zeta\mathcal{G}, \mathcal{G}, \tau : \mathcal{G} \rightarrow \mathcal{G}) \) to \( (\mathcal{G}, \zeta^m\mathcal{G}, \mathcal{G}, \tau \circ \ldots \circ \zeta^m\tau) \). This gives \( \alpha : \Sigma\rightarrow \Sigma\circ \text{id} \). After imposing the boundedness conditions, and composing with the map \( \Sigma\rightarrow \mathcal{A}(N) \).

3.3.3 Moduli Of \( \mathfrak{G} \)-Shtukas

To construct the moduli stack of global \( \mathfrak{G} \)-shtukas, one needs Frobenius symmetry on \( \mathcal{Y} \). To provide this, we have to pass to the formal completions at a fixed prime \( p \in \mathbb{Z} \).

Let \( S = \text{Spec} \, R \) (resp. \( \hat{S} = \text{Spf} \, R \)) for a complete dvr \((R, m_R)\) with special point \( s \) and generic point \( \eta \). Let \( \kappa(s) \) denote the residue field \( R/m_R \) and set \( \overline{S} := \text{Spec} \, \kappa(s) \). Let \( \mathfrak{G} \) be a smooth affine group scheme over \( S \), sending \( (\mathcal{G}, \zeta\mathcal{G}, \mathcal{G}, \tau : \mathcal{G} \rightarrow \mathcal{G}) \) to \( (\mathcal{G}, \zeta\mathcal{G}, \mathcal{G}, \tau \circ \ldots \circ \zeta\tau) \). This gives \( \alpha : \Sigma\rightarrow \Sigma\circ \text{id} \). After imposing the boundedness conditions, and composing with the map \( \Sigma\rightarrow \mathcal{A}(N) \).

\[ \Sigma^Z (C, \mathfrak{G}) \rightarrow \mathcal{A}(N). \]

Theorem 3.26. Let \( \mathcal{W} \) be a smooth stack and let \( f : \mathcal{W} \rightarrow \mathcal{Y} \) be a morphism of algebraic stacks over \( S \). Assume that \( \overline{\mathcal{Y}} := \mathcal{Y} \times_{\mathcal{Y}} \mathcal{W} \) lifts to a morphism \( \overline{\mathcal{W}} = \mathcal{W} \times \text{Spf} \, R \rightarrow \mathcal{W} \). We have the following statements

(a) There is a formal algebraic stack \( \Sigma_{D, \mathcal{A}}^Z \) over \( \text{Spf} \, R \) whose special fiber \( \Sigma_{D, \mathcal{A}}^Z \) is Deligne-Mumford and coincides \( \nabla_n^Z(D, \mathcal{Y}, \mathcal{W}) \times_{\mathcal{Y}} \mathcal{W} \). Here \( \nabla_n^Z(D, \mathcal{Y}, \mathcal{W}) \) denotes the moduli of global \( \mathfrak{G} \)-shtukas.

(b) There is a natural morphism \( \Sigma_{D, \mathcal{A}}^Z \rightarrow \Sigma_{D, \mathcal{A}}^Z \) for \( D \subseteq D' \), which is an étale morphism of formal algebraic stacks.

(c) Assume that \( \mathcal{W} \) is proper. Then the restriction \( \Sigma_{D, \mathcal{A}, \mathcal{X}}^Z \) of \( \Sigma_{D, \mathcal{A}}^Z \) to \( \mathcal{X} \times_{\mathcal{Y}} \mathcal{W} \) is algebraizable for large enough \( D \), i.e. there is \( \Sigma_{D, \mathcal{A}, \mathcal{X}}^Z \rightarrow \text{Spec} \, R \) with \( \Sigma_{D, \mathcal{A}, \mathcal{X}}^Z = \Sigma_{D, \mathcal{A}, \mathcal{X}}^Z \times \text{Spf} \, R \).

Proof. (a) \( \mathcal{Y}_D \rightarrow \mathcal{Y} \) is a \( \mathfrak{G}_D \)-torsor, where \( \mathfrak{G}_D \) denote the Weil restriction of scalars \( R_D/S \mathfrak{G} \). Let \( \widehat{\mathcal{Y}} := \mathcal{Y} \times_{\mathcal{Y}} \mathcal{W} \) and \( \widehat{\mathcal{Y}}_D := \mathcal{Y}_D \times_{\mathcal{Y}} \mathcal{W} \). Let us set \( \widehat{\mathcal{W}}_D := \widehat{\mathcal{W}} \times \widehat{\mathcal{Y}}_D \). The map \( \overline{\mathcal{W}} \) lifts to a morphism \( \overline{\mathcal{W}}_D := \widehat{\mathcal{W}} \times \widehat{\mathcal{Y}}_D \). Consider \( \mathcal{H}^Z := \mathcal{H} \times \mathcal{Y} \) as a family over \( \mathcal{Y}_D \times \mathcal{Y} \) and let \( H := \mathcal{H}^Z \times \mathcal{Y} \). Define \( \widehat{\Sigma}_{D, \mathcal{A}}^Z := \Sigma(H)_{\mathcal{A}} \).
Here \( \hat{\mathcal{H}} \) denote the \( HR \)-tuple \( \hat{\mathcal{H}}, \text{char} : \hat{\mathcal{H}} \to \hat{\mathcal{C}}^n := C^n \times \hat{\mathcal{S}}, \text{pr}^{-}, \text{pr}^{-} : \hat{\mathcal{H}} \cong \hat{\mathcal{W}}_D \).

By construction the special fiber \( \Sigma^Z \) coincides \( \nabla^Z_n \mathcal{M}_1(C, \overline{\mathcal{S}}) \times_y \hat{\mathcal{W}}; \) see AraHar2 and AraHab. In particular it is Deligne-Mumford; see AraHab Theorem 3.1.7.

(b) Flatness of this morphism can be checked over the special fiber according to EGA, Lemme 11.3.10.1; see also Definition-Remark 3.13 b). Now the statement follows from AraHar2 Theorem 3.15 and BoLu, Lemma 1.2.

(c) Since \( Z \) is proper, see Definition-Remark 3.13 b), by Proposition 3.12 we observe that \( \mathcal{H}^Z \to \mathcal{Y}^Z \) is proper and \( \mathcal{H}^Z \to \mathcal{Y}^Z \times \mathcal{Y}^Z \) are proper. Consequently \( \hat{\Sigma}^Z_{\alpha, \mathcal{W}} \) is proper over \( \hat{\mathcal{W}}_{\alpha} := \hat{\mathcal{W}} \times_y \mathcal{Y}^Z. \) Notice that \( \hat{\Sigma}^Z_{\alpha, \mathcal{W}} \) is equipped with an ample line bundle (i.e. a system of line bundles \( L_n \) on \( \hat{\Sigma}^Z_{\alpha, \mathcal{W}} \)) which is inherited from an ample line bundle on \( \mathcal{H}^Z_{\alpha, \mathcal{W}} \) for large enough \( D \). The existence of ample line bundle on \( \mathcal{H}^Z_{\alpha, \mathcal{W}} \) is a consequence of Proposition 3.12 and Wan, Thorem 5.0.14. Now the statement follows from Grothendieck’s algebraization theorem; EGA, III, Thm. 5.4.5]

\[ \square \]

Proposition 3.27. Assume that \( \mathcal{W} \to \mathcal{Y} \) is a smooth affine chart, then there is a lift \( \hat{\Sigma}^Z_{\alpha, \mathcal{W}} \) of \( \nabla^Z_n \mathcal{M}_1(C, \overline{\mathcal{S}}) \times_y \mathcal{W} \) as in Theorem 3.26 (a).

Proof. As \( \mathcal{S} \) is smooth we see that \( \mathcal{Y} \) and thus \( \mathcal{W} \), are smooth. Since \( \mathcal{W} \) is in addition affine, we see by infinitesimal criterion for smoothness that the Frobenius \( \sigma_\mathcal{W} \) lifts, and gives \( \hat{\sigma} : \hat{\mathcal{W}} \to \hat{\mathcal{W}}. \)

\[ \square \]

Fix a \( n \)-tuple \( \nu := (\nu_i) \) of disjoint characteristic sections on \( C \). That is \( \nu : \text{Spf} \ R \to C^n \).

Let \( Z_\nu := (Z_{\nu_i}) \) denote the associated tuple of local bounds corresponding to \( Z \) at the places \( \nu_i \), see AraHab, Subsection 4.3.

Proposition 3.28. Keep the notation in theorem 3.26. Furthermore assume that \( \mathcal{W} \) is proper and \( Z_{\nu_i} \) is smooth (e.g. it comes from minuscule coweights) for every \( i \), then there is an isomorphism

\[
H^q((\hat{\Sigma}^Z_{\alpha, \mathcal{W}})_{\mathcal{Y}}, Q_\ell) \cong H^q((\nabla^Z_n \mathcal{M}_1)(C, \overline{\mathcal{S}}) \times_y \mathcal{W}, Q_\ell).
\]

In particular the cohomology of the generic fiber \( (\hat{\Sigma}^Z_{\alpha, \mathcal{W}})_{\mathcal{Y}} \) is independent of the choice of the lift \( \sigma \).

Proof. We may take \( D \) enough large such that \( \hat{\Sigma}^Z_{\alpha, \mathcal{W}} \) becomes algebraizable. The projection morphism \( \mathcal{H}^Z \to \mathcal{Y} \) is proper, see proof of theorem 3.26 (c). Since \( \mathcal{W} \) is proper, we observe that \( \mathcal{H}^Z \times_y \mathcal{Y} \to \mathcal{Y} \times \mathcal{W} \) is proper and hence \( \hat{\Sigma}^Z_{\alpha, \mathcal{W}} \) is proper. Now the isomorphism follows from Lemma 2.8 and Proposition 3.20 which implies that \( R\Psi(Q_\ell) \) is constant; see also Lemma 3.31 below.

\[ \square \]
Remark 3.29. For the case \( \ell = p \), and for the tuple of smooth bounds \( Z_\Sigma \), it can be seen that the cohomology \( H^q((\hat{X}_\Sigma)_\eta, Q_\ell) \) of \( \hat{X}_\Sigma = (\hat{\Sigma}_{D,\hat{\sigma},\nu})_\eta \) is also independent of the choice of the lift \( \hat{\sigma} \). This follows from the spectral sequence

\[
H^p(X, i^* R^q j_* Q_\ell) \Rightarrow H^{p+q}(\hat{X}_\Sigma, Q_\ell),
\]

see for example [BK], corresponding to the following diagram of formal stacks

\[
\begin{array}{ccc}
\hat{X}_\Sigma & \longrightarrow & X \\
\downarrow & & \downarrow \\
\eta & \longrightarrow & \bar{S}
\end{array}
\]

The bar in the above notation indicates corresponding algebraic or integral closures.

Question 3.30. Fix a global boundedness condition \( Z \). Consider the stack \( \mathcal{Y}_{\underline{x}, \underline{P}}^{\mathrm{a-ss}} := \mathcal{H}^1(C, \mathcal{G}, \underline{x}, \underline{P})_{\underline{a}^{\mathrm{a-ss}}} \) of \( \underline{a} \)-semistable flagged principal \( \mathcal{G} \)-bundles of type \((\underline{x}, \underline{P})\).

It is natural to ask

- whether one can take \( \mathcal{W} \) to be \( \mathcal{Y}_{\underline{x}, \underline{P}}^{\mathrm{a-ss}} \) (i.e. if the Frobenius lifts over \( \hat{\mathcal{W}} = \mathcal{Y}_{\underline{x}, \underline{P}}^{\mathrm{a-ss}} \times \text{Spf } R \))?

and second,

- does the cohomology remain independent of the choice of the lift \( \hat{\sigma} \) in the global situation, namely for the stack \( \hat{\Sigma}_Z \)? Note that according to the lemma 3.31, the moduli stack \( \hat{\Sigma}_Z \) is flat over \( C^n \), provided that \( Z \) is flat over \( C^n \). Furthermore, as \( \pi : \hat{X}_\Sigma := (\hat{\Sigma}_Z)^{\underline{a}^{\mathrm{a-ss}}} \rightarrow C^n \) is proper, the higher direct image \( R^i \pi_* \mathcal{F} \) is constructible and we have the following convergence of Leray spectral sequence

\[
H^i(C^n, R^i \pi_* \mathcal{F}) \Rightarrow H^{i+j}(\hat{X}_\Sigma, \mathcal{F}).
\]

Moreover we have

\[
(R^i \pi_* \mathcal{F})_y \cong H^i((\hat{X}_\Sigma)_y, \mathcal{F}|_y)
\]

e.g. see [Mil] Ch. VI Cor. 2.7].

Lemma 3.31. Keep the notations in theorem 3.26 and assume that \( \mathcal{W} \rightarrow \mathcal{Y} \) is étale (resp. smooth). Let \( Z \) be a boundedness condition in the sense of 3.12 b). For a point \( y \) in \( \hat{\Sigma}_{D,\hat{\sigma}} \) there exist an étale neighborhood \( U_y \) of \( y \) and a roof

\[
\begin{array}{ccc}
\hat{\Sigma}_{D,\hat{\sigma}} & \longrightarrow & \hat{Z} \\
\downarrow & & \downarrow \\
U_y & \longrightarrow & \hat{\Sigma}_{D,\hat{\sigma}}
\end{array}
\]
of étale morphisms. Here \( \hat{\mathcal{Z}} := \mathcal{Z} \times_R \text{Spf } R \).

Proof. By theorem 3.26 (b) we can forget \( D \)-level structure. Let \( y' \) be the image of \( y \) in \( C^n \times \mathcal{Y} \) under the projection sending \((\mathcal{G}, \mathcal{G}', \hat{s}, \tau)\) to \((\hat{s}, \mathcal{G}')\). According to Proposition 3.12 we may take an étale neighborhood \( U \to C^n \times_S \mathcal{Y} \) of \( y' \), such that the restriction \( U' \) of \( \mathcal{H} \) to \( U \) and the restriction \( U'' \) of \( \Sigma_\Theta \times \mathcal{Y} \) to \( U \) become isomorphic. Now, set \( \tilde{U} := U' \times_\mathcal{H} \hat{\mathcal{Z}} \).

We deduce the following roof of morphisms

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\varphi} & \hat{\mathcal{Z}} \\
\leftarrow & & \downarrow \\
\hat{\Sigma}_\hat{\mathcal{Z}} & \rightarrow & \hat{\mathcal{Z}}
\end{array}
\]

It remains to check that \( \varphi \) is an étale morphism (resp. smooth). To see this, consider the morphism \( \overline{\varphi} : \overline{U} := U \times_R \mathbb{R} \mathbb{R}/m \mathbb{R} \mathbb{R} \to \overline{\mathcal{Z}} := \hat{\mathcal{Z}} \times_R \mathbb{R}/m \mathbb{R} \mathbb{R} \). The morphism \( \overline{\varphi} \) is étale according to [AraHab, Theorem 3.2.1]. Now as in the proof of 3.26 (b) we may argue by [EGA] IV, Lemme 11.3.10.1 and [BoLo] Lemma 1.2.

\[ \square \]

4 The Hecke stack over the moduli of G-Shtukas

In this subsection we discuss another sample of the construction we described in section 2. Namely, we consider the Hecke stack over the moduli stack \( \nabla_n \mathcal{H}(C, \mathfrak{G}) \) (see theorem 3.26 for the notation) of \( \mathfrak{G} \)-shtukas. Throughout this subsection we let \( S := \text{Spec } \mathbb{F}_q \).

Definition 4.1. Fix integers \( m, n \) and \( n' \). Set \( \nabla_n \mathcal{H} := \nabla_n \mathcal{H}^1(C, \mathfrak{G}) \) and \( \nabla_{n'} \mathcal{H} := \nabla_{n'} \mathcal{H}^1(C, \mathfrak{G}) \).

(a) Define the algebraic stack \( \tilde{\mathcal{H}}_{n,m,n'} \) as the stack whose points over a scheme \( T \) over \( S \) consists of tuples \((\mathcal{G}, \mathcal{G}', \mathcal{L}, \varphi)\), consisting of the following data

(i) \( \mathcal{G} := (G, s, \tau) \) in \( \nabla_n \mathcal{H}(T) \) and \( \mathcal{G}' := (G', s', \tau') \) in \( \nabla_{n'} \mathcal{H}(T) \),

(ii) \( m \)-tuple \( \mathcal{L} := (c_i) \) in \( C^m(T) \),

(iii) a commutative diagram

\[
\begin{array}{ccc}
\sigma^* \mathcal{G}' & \xrightarrow{\tau'} & \mathcal{G}' \\
\sigma^* \varphi \downarrow & & \downarrow \varphi \\
\sigma^* \mathcal{G} & \xrightarrow{\tau} & \mathcal{G},
\end{array}
\]

where \( \varphi \) is defined over \( C_T \setminus \Gamma_{\mathcal{L}} \) and the diagram is defined after restricting to \( C_T \setminus \Gamma_{\mathcal{L}} \cup \Gamma_{\mathcal{L}'} \cup \Gamma_{\mathcal{L}} \).
(b) The moduli stack \( \tilde{\mathcal{H}}_{n,m,n'} \) is fibered over \( \nabla_n \mathcal{H} \) (resp. \( \nabla_{n'} \mathcal{H} \)) through projections to the first (resp. second) factor. We have the following map

\[
(pr^\leftarrow, pr^\rightarrow, \text{char} \circ pr^\leftarrow, \text{char} \circ pr^\rightarrow, \text{char}) : \tilde{\mathcal{H}}_{n,m,n'} \to \nabla_n \mathcal{H} \times \nabla_{n'} \mathcal{H} \times C^n \times C^{n'} \times C^m,
\]

(4.7)
given by sending the \( T \)-point \( (G, s, \tau) \), \( G' := (G', s', \tau') \), in \( \tilde{\mathcal{H}}_{n,m,n'}(T) \) to \( (G, G', s, s', \tau) \).

(c) There is an obvious projection

\[ \mathcal{H}_{n,m,n'} \to \text{Hecke}_m(C, \mathcal{G}), \]

defined by sending \( (G, G', \varphi) \) to \( (G, G', \varphi) \). Set

\[ \mathcal{H}_{n,m,n'}^{\mathcal{Z}, \mathcal{Z}'} := \mathcal{H}_{n,m,n'} \times_{\nabla_n \mathcal{H} \times \nabla_{n'} \mathcal{H} \times \text{Hecke}_m(C, \mathcal{G})} \left( \nabla_n^\mathcal{Z} \mathcal{H} \times \nabla_{n'}^\mathcal{Z'} \mathcal{H} \times \text{Hecke}_{\mathcal{Z}}(C, \mathcal{G}) \right). \]

Here \( \mathcal{Z} \) (resp. \( \mathcal{Z}' \)) is a bound in \( GR_n \) (resp. \( GR_{n'} \)).

(d) Let \( \text{Hom}_{n,n'} \) denote the stack \( \text{Hom}(\nabla_n \mathcal{H}, \nabla_{n'} \mathcal{H}) \) and let

\[ \mathcal{H}_{n,m,n'} := (\mathcal{H}_{n,m,n'}, \text{char}, pr^\leftarrow, pr^\rightarrow). \]

Let \( \Sigma_{n,m,n'}/\nabla \mathcal{H} := \Sigma(\mathcal{H}_{n,m,n'}) \) denote the corresponding family over \( \text{Hom}_{n,n'} \). For \( n = n' \), set \( \mathcal{E}_n = \text{Hom}_{n,n'} \).

(e) Let \( D \subseteq C \) be a closed subscheme. One can equip the Hecke stack \( \mathcal{H}_{n,m,n'} \) (resp. \( \mathcal{H}_{n,m,n'}^{\mathcal{Z}, \mathcal{Z}'} \)) with a \( D \)-level structure in an obvious sense. We denote the resulting moduli stack by \( \mathcal{H}_{n,m,n'}^D \) (resp. \( \mathcal{H}_{n,m,n'}^{\mathcal{Z}, \mathcal{Z}'}^D \)).

**Remark 4.2.** Let \( I \) (resp. \( J \), resp. \( K \)) be an index set with cardinality \( n \) (resp. \( n' \), resp. \( n + n' \)) and fix a bijection \( \iota : K \to I \sqcup J \). To this, one assigns the following closed immersion of stacks

\[ C^{n'} \times \nabla_n \mathcal{H}^1(C, \mathcal{G}) \to \nabla_{n+n'} \mathcal{H}^1(C, \mathcal{G}), \]

which is defined by sending \( (s_j)_{j \in I} \times (G, (s_i)_{i \in I}, \tau) \) to \( (G, (s_k)_{k \in K}, \tau) \), where \( s_k := s_{\iota k} \).

**Proposition 4.3.** Fix a \( n \)-tuple \( \nu := (\nu_i) \) of closed points \( \nu_i \) in \( C \). Let \( A_{\nu} \) be the completion of the stalk of \( \mathcal{O}_{C^n} \) at the point \( \nu \). Set \( \mathcal{H}_m := \mathcal{H}_{n,m,n} \), and let \( \mathcal{H}_{m,\nu} := \mathcal{H}_m \times C^n \text{Spf} A_{\nu} \) (resp. \( \nabla_n \mathcal{H}_m := \nabla_n \mathcal{H} \times \text{Spf} A_{\nu} \)). The projection map \( \mathcal{H}_m \to \nabla_n \mathcal{H}_m \times \nabla_n \mathcal{H}_m \times C^m \) is formally unramified.
Proof. This follows from rigidity of quasi-isogenies between $𝔖$-shtukas, see [AraHar1, Proposition 5.9].

Remark 4.4. For an ind-scheme $X^\bullet = \lim_{i \in I} X^i$, one defines the Chow group $Ch_d(X^\bullet)$ as the direct limit of $Ch_d(X^i)$ under $t_{i,j}^*: X^i \to X^j$ denotes the closed immersion $X^i \to X^j$ for $i < j$. For construction of the Chow groups of algebraic spaces, flat pull back, proper push forward and further properties see [Stacks, Chapter 80, Tag 0EDQ]. Concerning the construction of Chow groups for Deligne-Mumford stacks, e.g. see [Vis], one similarly defines the (homological) Chow group $\text{Ch}$. For an ind-scheme $X^\bullet = \lim_{i \in I} X^i$, one defines the Chow group $Ch_d(X^\bullet)$, for ind-Deligne-Mumford stack $X^\bullet = \lim_{i \in I} X^i$. Recall that there is an ind-Deligne-Mumford structure $\nabla_n \mathcal{H}_D = \lim_{i \in I} \nabla_n^i \mathcal{H}_D^j(C, \mathfrak{G})$, where $\mathfrak{G}$ lies in a countable set of cocharacters of $SL(V)$ for some vector bundle $V$; see [AraHar2, Theorem 3.15].

Theorem 4.5. Fix a boundedness condition $\mathcal{Z}$, and an $\alpha$ as in Remark 3.2. Let $\nabla^Z_n \mathcal{H}_{D,\alpha} := \nabla_n^Z \mathcal{H}_D \times_{\nabla_n^Z \mathcal{H}_{\alpha}} \mathcal{U}_\alpha$. Let $\mathcal{C}_{n}\mathfrak{G}(GR_m)$ denote the free $\mathbb{Z}$-module generated by $\mathcal{C}_{n}\mathfrak{G}$-equivariant closed equidimensional subschemes of global affine Grassmannian $GR_m$, which are flat over $C^m$ and of dimension $\tilde{d}$. An element $\tilde{\mathcal{Z}}$ in $\mathcal{C}_{\tilde{d}}\mathfrak{G}(GR_m)$ induces a morphism

$$M^c(\nabla^Z_n \mathcal{H}_{D,\alpha})(\tilde{d})[2(\tilde{d})] \to M^c(\nabla^Z_{n+\alpha} \mathcal{H}_D) \otimes M(C)^{\otimes m},$$

for sufficiently large boundedness condition $\mathcal{Z}'$. Similarly it induces a morphism in

$$\text{Hom}_{DM_{gm}}(M^c(X_{D,n,\alpha})(\tilde{d})[2\tilde{d}], M^c(X_{D,n+\alpha}')(\tilde{d})[2\tilde{d}]) \otimes M(C \otimes D)^{\otimes m},$$

here $X_{D,n,\alpha}$ (resp. $X_{D,n+\alpha}'$) denotes the coarse moduli space for $\nabla^Z_n \mathcal{H}_{D,\alpha}$ (resp. $\nabla^Z_{n+\alpha} \mathcal{H}_D$).

Proof. As the statement with $D$-level structure follows similarly, we just explain the situation where there is no level structure. Note first that for any boundedness condition $\mathcal{Z}$, the moduli stack $\mathcal{X}^{\mathcal{Z}}_n := \nabla^Z_n \mathcal{H}$ (resp. $\mathcal{X}^{\mathcal{Z}}_{n,\alpha} := \nabla^Z_n \mathcal{H}_{\alpha}$) is Deligne-Mumford. It is separated and its inertia is finite over $\nabla^Z_n \mathcal{H}$ (resp. $\nabla^Z_n \mathcal{H}_{\alpha}$) see [AraHar2, theorem 3.15], [AraHar1, theorem 3.1.7] and [AraHar2, Corollary 3.16]. Therefore by Keel-Mori’s theorem, [Co], it admits a coarse moduli space $X^{\mathcal{Z}}_n$ (resp. $X^{\mathcal{Z}}_{n,\alpha}$).

Let $\tilde{\mathcal{Z}}$ be an irreducible cycle in $\mathcal{C}_{\tilde{d}}\mathfrak{G}(GR_m)$. Let $\mathcal{H}^{-\tilde{\mathcal{Z}},-}_n$ denote the substack of $\mathcal{H}_{n,m,n+m}$ defined by restricting $\mathcal{H}_{n,m,n+m}$ to $\mathcal{U}_\alpha$, and then imposing the boundedness condition $\tilde{\mathcal{Z}}$ to the universal isomorphism $\varphi_u$ of the universal tuple $(\mathcal{G}_u, \mathcal{G}_u', \mathcal{L}_u, \phi_u : \mathcal{G}_{u} \to \mathcal{G}_{u'})$. The family $\mathcal{H}^{-\tilde{\mathcal{Z}},-}_n \to \mathcal{H}_n$ is locally on the base isomorphic to the family $\mathcal{H}_m \to \mathcal{H}_m$. In particular it is flat of relative dimension $\tilde{d}$; see Proposition 3.12.
By [AraHar2] Theorem 3.15 the morphism \( \text{pr} : \mathcal{H}_{(n,m,n+m),\alpha} \to \nabla_{n+m}\mathcal{H} \) factors through \( \mathcal{X}_{n+m} := \nabla_{n+m}^{Z'}\mathcal{H} \) for some sufficiently large \( Z' \). Therefore we get the following roof

\[
\begin{array}{ccc}
\mathcal{H}_{(n,m,n+m),\alpha}^{Z,\tilde{Z}} & \xrightarrow{\quad} & \mathcal{X}_{n+m}^{Z} \\
\downarrow & & \downarrow \\
\mathcal{X}_{n+m}^{Z'} & \xrightarrow{\quad} & S \times C^n.
\end{array}
\]

which induces a morphism

\[
M^{c}(X_{n,a}^Z)(\tilde{d})[2\tilde{d}] = M^{c}(\mathcal{X}^Z)(\tilde{d})[2\tilde{d}] \to M^{c}(\mathcal{X}^{Z'}) = M^{c}(X^{Z'}).
\]

\textbf{Remark 4.6.} Note in particular that when \( \nabla_{n}\mathcal{H}_D \) is proper we have

\[
\text{Hom}_{\text{DM}_{gm}}(M^{c}(X_{n,a}^Z)(\tilde{d})[2\tilde{d}], M^{c}(X_{n+m}^Z)) = Ch_{d+m}(X_{n}^Z \times S X^{Z'}).
\]

This follows from the above discussion and [VSF] Chapter 5, Proposition 4.2.3]. Note further that by Zariski’s main theorem we observe that \( \mathcal{X}^Z \) is proper and thus \( M^{c}(\mathcal{X}^Z) = M(\mathcal{X}^Z) \). For examples of such cases see [Lau].

\textbf{Corollary 4.7.} Keep the notation in 4.3. For \( D \) sufficiently large, the element \( \tilde{Z} \) in \( C_{d}^{\mathcal{X}^Z}(GR_m) \) induces a cycle \( \Phi^{\tilde{Z}} \) in

\[
A_{d,0}(\nabla_{n}^{Z}\mathcal{H}_{D,\alpha}, \nabla_{n+m}^{Z'}\mathcal{H}_{D,\alpha} \times S (C \setminus D)^m).
\]

Furthermore when \( \tilde{Z} \) comes from minuscule coweights \( \mu := (\mu_i)_i \), it induces an element in \( Ch_{d+m}(\nabla_{n}^{Z}\mathcal{H}_{D}^{\mu} \times S \nabla_{n+m}^{Z'}\mathcal{H}_{D} \times S (C \setminus D)^m) \). Here \( \nabla_{n}^{Z}\mathcal{H}_{D}^{\mu} := \nabla_{n}^{Z}\mathcal{H}_{D} \times_{C^n} (C^n \setminus \Delta_{\text{big}}) \), where \( \Delta_{\text{big}} \) denotes the big diagonal in \( C^n \), and \( d = \dim Z \).

\textbf{Proof.} First we can take \( D \) sufficiently large that \( \nabla_{n}^{Z}\mathcal{H}_{D,\alpha} \times_S \nabla_{n+m}^{Z'}\mathcal{H}_{D,\alpha} \) becomes representable by a quasi-projective variety. According to 4.5, \( \tilde{Z} \) induces a morphism

\[
M^{c}(\nabla_{n}^{Z}\mathcal{H}_{D,\alpha})(\tilde{d})[2\tilde{d}] \to M^{c}(\nabla_{n+m}^{Z'}\mathcal{H}_{D,\alpha} \times (C \setminus D)^m).
\]

After composing the canonical morphism \( M(\nabla_{n}^{Z}\mathcal{H}_{D,\alpha}) \to M^{c}(\nabla_{n}^{Z}\mathcal{H}_{D,\alpha}) \), e.g. see [MVW] Example 16.2, with the above morphism we obtain

\[
M(\nabla_{n}^{Z}\mathcal{H}_{D,\alpha})(\tilde{d})[2\tilde{d}] \to M^{c}(\nabla_{n+m}^{Z'}\mathcal{H}_{D,\alpha} \times (C \setminus D)^m).
\]
Note further that
\[ \operatorname{Hom}(M(\nabla_n^Z \mathcal{H}_{D,\alpha})[\tilde{d}][2\tilde{d}], M^c(\nabla_{n+m}^Z \mathcal{H}_{D,\alpha} \times (C \setminus D)^m)) \cong A_{d,0}(\nabla_n^Z \mathcal{H}_{D,\alpha}, \nabla_{n+m}^Z \mathcal{H}_{D,\alpha} \times (C \setminus D)^m); \]
see [CD] Theorem 8.4. Finally when \( Z \) comes from minuscul coweights, the restriction \( \mathcal{Z} \times_{C^n} (C^n \setminus \Delta^{bg}) \) is smooth, and therefore \( \nabla_n^Z \mathcal{H}_{D,\alpha} \) is smooth and of dimension \( d = n + \sum_i \mu_i \); see [AraHab] Theorem 3.2.1. Hence by duality, see [MVW] Theorem 16.24 and see also [Kel] Theorem 5.4.20, we deduce
\[ A_{d,0}(\nabla_n^Z \mathcal{H}_{D,\alpha}, \nabla_{n+m}^Z \mathcal{H}_{D,\alpha}) = A_{d,0}(S, \nabla_n^Z \mathcal{H}_{D,\alpha} \times_S \nabla_{n+m}^Z \mathcal{H}_{D,\alpha}) \]
\[ = \mathcal{C}_{h_{d+q}}(\nabla_n^Z \mathcal{H}_{D,\alpha} \times_S \nabla_{n+m}^Z \mathcal{H}_{D,\alpha}) \]
\[ \Box \]

**Remark 4.8** (global Rapoport-Zink spaces). Let \( Z \) be a boundedness condition in \( \mathcal{G}_n \). Let \( \mathcal{H}_m^Z \mathcal{H} := \mathcal{H}_m^Z \mathcal{H} \times_{\mathcal{H}, \mathcal{H}, \mathcal{H}} \mathcal{H}_m^Z \mathcal{H} \). Let \( \Sigma^Z_{m/\mathcal{H}} := \Sigma(\mathcal{H}_m^Z \mathcal{H}, \mathcal{H}, \mathcal{H}) \) and view it as a family over \( \operatorname{Hom}(\nabla_n^Z \mathcal{H}, \nabla \mathcal{H}) \), according to the construction described in Section 2.1. Fix a global \( \mathcal{G}_0 \) and let \( \mathcal{G}_{\mathcal{G}_0} \) denote the corresponding constant morphism in \( \mathfrak{E} \). The fiber \( \Sigma^Z_{m/\mathcal{H}} \mathcal{G}_{\mathcal{G}_0} \) above \( \mathcal{G}_{\mathcal{G}_0} \) is called **global Rapoport-Zink space** corresponding to \( Z \) and \( \mathcal{G}_0 \). Note that there is a natural projection morphism
\[ \Sigma^Z_{m/\mathcal{H}} \mathcal{G}_{\mathcal{G}_0} \to \mathcal{G}_{\mathcal{G}_0} \]
defined by sending \( (\mathcal{G} := (\mathcal{G}, \tau_0), \mathcal{Z}, \varphi) \) to \( (\mathcal{G}, \mathcal{Z}, \varphi : \mathcal{G}|_{C_T \setminus C_Z} \to \mathcal{G}_0|_{C_T \setminus C_Z}) \); see definition 3.3.9

**Remark 4.9.** Consider the morphism \( \nabla \mathcal{H} \to \nabla \mathcal{H} \) induced by Frobenius \( \sigma : \mathcal{H}^1(C, \mathfrak{G}) \to \mathcal{H}^1(C, \mathfrak{G}) \). We again denote this morphism by \( \sigma \) and we let \( \Sigma_{m/\nabla \mathcal{H}, \sigma} \) denote the fiber above \( \sigma \). There is a morphism
\[ \Psi_0 : \nabla \mathcal{H} \to \Sigma_{m/\nabla \mathcal{H}, \sigma}, \]
which is defined by sending \( \mathcal{G} := (\mathcal{G}, s, \tau) \) to \( (\mathcal{G}, s^* \mathcal{G}, s, \tau : \sigma^* \mathcal{G} \to \mathcal{G}) \).

On the other hand there is a morphism
\[ \Psi_\ell : \Sigma_{m/\nabla \mathcal{H}, \sigma} \to \Sigma_{m/\nabla \mathcal{H}, \sigma^\ell}, \]
defined by sending \( (\mathcal{G}, \sigma^* \mathcal{G}, \mathcal{Z}, \varphi) \) to \( (\sigma^\ell)^* \mathcal{G}, \mathcal{Z} \mathcal{G} \sigma^\ell \mathcal{Z} \sigma^\ell \mathcal{Z} \sigma^\ell \mathcal{Z} \sigma^\ell \varphi \cdots \sigma^\ell \varphi \). In particular for a finite field extension \( k/\mathbb{F}_q \) of degree \( \ell \), composing \( \Psi_\ell \) with \( \Psi_0 \) gives
\[ \nabla \mathcal{H}(k) \to \Sigma_{m/\nabla \mathcal{H}, \sigma^\ell} = \Sigma_{m/\nabla \mathcal{H}, \mathcal{I}}(k). \]

The above map sends \( \mathcal{G} \) to the Frobenius isogeny \( \Phi_{\mathcal{G}} \in \operatorname{QEnd}(\mathcal{G}) \). Note that, when \( \mathfrak{G} = \operatorname{GL}_n \), the coefficients of the minimal polynomial of \( \Phi_{\mathcal{G}} \in \operatorname{QEnd}(\mathcal{G}) \) determines the quasi-isogeny class of \( \mathcal{G} \); see [AraHar3] Section 5.
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