so(4) Plebanski Action and Relativistic Spin Foam Model

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Abstract

In this note we study the correspondence between the “Relativistic Spin Foam” model introduced by Barrett, Crane and Baez and the so(4) Plebanski action. We argue that the so(4) Plebanski model is the continuum analog of the relativistic spin foam model. We prove that the Plebanski action possess four phases, one of which is gravity and outline the discrepancy between this model and the model of Euclidean gravity. We also show that the Plebanski model possess another natural discretisation and can be associated with another, new, spin foam model that appear to be the so(4) counterpart of the spin foam model describing the self dual formulation of gravity.

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\section{I. INTRODUCTION}

The relativistic spin foam model was introduced by Barrett, Crane\textsuperscript{1} and Baez\textsuperscript{2} as a proposal for a state sum formulation of 4-D Euclidean quantum gravity. These authors have analyzed the classical description of a 4-simplex in terms of bivectors, quantized this space and deduced from it a state sum model (or spin foam model in the terminology of Baez). Despite its deep geometrical meaning, this model was not (precisely) interpreted as the partition function associated with a definite classical action.

In this paper we show that the continuum action corresponding to the relativistic spin
foam model \[1,2\], is given by the \(so(4)\) Plebanski \[3\] action. The \(so(4)\) Plebanski model was originally introduced for the study of the possible relation between the various Half-Flat (self-dual) solutions of complexified General Relativity. The action of this model is given by a constrained \(so(4)\) \(BF\) model and its use in the context of classical and quantum gravity has a long history \[1\].

In section \[1\] we recall some of the main results of \[1,2\]. Namely the description of the 4-simplex in terms of bivectors satisfying some constraints, the space of solutions of these constraints and their quantization. We then recall some facts concerning \(BF\) theory, state sum models, and present the relativistic state sum in a form suitable for further analysis. In section \[1\] we present and study the \(so(4)\) Plebanski model. We show that, when the \(B\) field is non degenerate, this model consists of four different sectors, one of which is gravity. Performing a short analysis of the quantization of the \(so(4)\) Plebanski action we show how these sectors interfere and emphasize the resulting discrepancy between this model and Euclidean quantum gravity. In section \[1\] we study the discretization of the Plebanski action and the corresponding state sum models. We show that there exists two possible discretizations of the model which lead to two different state sum models. One state sum model is the Baez-Crane-Barrett model, while the other spin foam model is the \(so(4)\) analog of the Reisenberger spin foam model \[3\] (which corresponds to self-dual formulation of gravity). We inform the reader that after this work was finished we became aware that some of these results have been independently obtained by Reisenberger \[7\].

II. THE BAEZ-BARRETT-CRANE MODEL

In this section we recall the work done by Barrett and Crane \[1\] and further developed by Baez \[2\]. We refer the reader to their works for a deeper understanding of the model. We outline here their construction for reader’s convenience.

We consider, as in \[1,2\], the description of the geometry of a 4-simplex in terms of bivectors \(B\) associated to each 2-simplex (triangle). The properties of the bivectors for a non degenerate 4-simplex are

1. The bivector changes sign if the orientation of the triangle is changed.

2. Each bivector satisfy the simplicity condition i.e.

\[
B \wedge B = 0 \tag{1}
\]

3. If two triangles share a common edge, then the sum of the two bivectors also satisfy the simplicity condition.

4. The sum of the 4 bivectors corresponding to the faces of a tetrahedron is zero. This sum is calculated using the orientation of the triangles given by the boundary of the tetrahedron.

5. The assignment of bivectors to faces is non-degenerate. This means that for six triangles sharing a common vertex, the six bivectors are linearly independent.
Each geometric 4-simplex determines a set of bivectors satisfying these conditions by defining $B = e \wedge e$, where $e$ are vectors associated with the edges of the 4-simplex. Conversely, each set of bivectors satisfying these conditions admits four types of solutions:

$I^\pm : B = \pm e \wedge e,$

$Ii^\pm : B = \pm \star (e \wedge e).$

These bivectors can be identified with elements of the Lie algebra $so(4)$. (Also, it may be profitable to think of them as elements of the dual of this Lie algebra.) The splitting $so(4) \cong su(2) \oplus su(2)$ is then the same as the splitting of the space of bivectors into self-dual and anti-self-dual parts, $\mathbb{R}^4 \wedge \mathbb{R}^4 = \Lambda^2_+ \oplus \Lambda^2_-$. The condition that a bivector $B$ is simple is

$$0 = \langle B, \star B \rangle = \langle B^+, B^+ \rangle - \langle B^-, B^- \rangle,$$

so that the norm of the self-dual and anti-self-dual parts is equal. The area $A$ of the triangle is given by

$$A^2 = \langle B, B \rangle = \langle B^+, B^+ \rangle + \langle B^-, B^- \rangle.$$

The quantization of the model is obtained by labeling the triangles with irreducible representations of $SU(2) \times SU(2)$, and each tetrahedron is labelled with a tensor in the product of the four spaces on its faces. Irreducible representations of $SU(2) \times SU(2)$ are pairs $(j, k)$ of representations of $SU(2)$. The quantum analog of the properties 1–4 of bivectors in a 4-simplex are given by:

Q1 Changing the orientation of a triangle changes the representation to its dual.

Q2 The representations on the triangles are of the form $(j, j)$. These are called simple representations.

Q3 For any pair of faces of a tetrahedron, the pair of representations can be decomposed into its Clebsch-Gordan series for $SU(2) \times SU(2)$. Under this isomorphism, the tensor for the tetrahedron decomposes into summands which are non-zero only for the simple representations of $SU(2) \times SU(2)$.

Q4 The tensor for the tetrahedron is an invariant tensor.

Baez, Barrett and Crane [1] constructed a state sum model incorporating the previous quantum constraints. Let us recall first that there is a state sum formulation of the topological $BF$ theory in 4-dimensions with (or without cosmological constant) given by the Crane-Yetter [8] or Ooguri [9] models. The partition function of the model associated to a triangulation $\Delta$ of a manifold $M$ can be written as:

$$Z_{BF}(\Delta, \Lambda) = \sum_{j_f, \iota_t} \prod_f \text{dim}_q(j_f) \prod_v \phi_q(v, \vec{j}, \vec{i}),$$

(2)

where $q = \exp(i\Lambda)$, $j_f$ denotes a coloring of the faces of $\Delta$ by irreducible representation of $U_q(su(2))$, $\iota_t$ denotes a coloring of the tetrahedra of $\Delta$ by intertwiners and the sum is over all such colorings with $j \leq \frac{2\pi}{\Lambda}$. Moreover $\text{dim}_q(j)$ denotes the quantum dimension of the representation of spin $j$ and $\phi_q(j, \vec{i})$ denote the quantum 15-j symbol associated with the 4-simplex $v$. More precisely, associated to a 4-simplex $v$ we consider a graph $\Gamma_v$ given by
the intersection between the 2-skeleton of the complex which is dual to the triangulation and the boundary of the 4-simplex $v$. $\Gamma_v$ corresponds to the pentagram graph and we color its 10 edges by $\vec{j}$ and its 5 vertices by $\vec{i}$. $\phi_{q,v}(\vec{j},\vec{i})$ corresponds to the Reshetikhin-Turaev evaluation of the colored graph $\Gamma_v$. This state sum does not depend on the triangulation and corresponds to the evaluation of the partition function

$$Z(M, \Lambda) = \int \mathcal{D}A \mathcal{D}B \ e^{i \int_M \text{tr}(B \wedge F(A) - \frac{1}{2} B \wedge B)},$$

(3)

where $F(A)$ is the curvature of the $su(2)$ connection $A$.

The Baez, Barrett and Crane model is a modification of the state sum (3) in the case of the $so(4)$ gauge group in which the conditions corresponding to the quantum 4-simplex conditions (when $\Lambda = 0$) are imposed by hand. Yetter generalized this construction to the case of non vanishing $\Lambda$ [10]. Yetter emphasized that the quantum group that should be considered is $U_q(su(2)) \otimes U_q^{-1}(su(2))$. This state sum model can be written:

$$Z_{BC}(\Delta, \Lambda) = \sum \prod \dim q(j_f) \dim q^{-1}(j'_f) \prod \phi_{q,v}(\vec{j},\vec{i}) \phi_{q^{-1},v}(\vec{j}',\vec{i}') \delta_{\vec{j},\vec{j}'} \delta_{\vec{i},\vec{i}'}.$$  

(4)

So this model corresponds to two copies of $su(2)$ $BF$ theories (or an $so(4)$ $BF$ theory) together with 15 constraints imposed for each 4-simplex. The presence of $\delta_{\vec{j},\vec{j}'}$ corresponds to the constraint $Q2$ while $\delta_{\vec{i},\vec{i}'}$ to the constraint $Q3$. We will give the continuum version of this partition function in equation (28).

III. THE $SO(4)$ PLEBANSKI ACTION

We use the notation given in the appendix. The $so(4)$ Plebanski action [3], which depends on an $so(4)$ connection $\omega = \omega^I_J X_I dx^J$, a two form valued into $so(4)$ $B = B^I_J X^I dx^J$, and a scalar symmetric traceless matrix $\phi_{[IJKL]}$, $(\epsilon^{IJKL} \phi_{IJKL} = 0)$, reads:

$$S[\omega; B; \phi] = \int_M \left[ B^{IJ} \wedge F_{IJ}(\omega) - \frac{\Lambda}{4} \epsilon_{IJKL} B^{IJ} \wedge B^{KL} - \frac{1}{2} \phi(B)_{IJ} \wedge B^{KL} \right],$$  

(5)

where $\phi(B)_{IJ} = \phi_{IJKL} B^{KL}$. The whole set of Euler-Lagrange equations associated to the Plebanski action is given by:

$$\frac{\delta S}{\delta \omega^{(\mu)}_{\omega}} \rightarrow DB = dB + [\omega, B] = 0,$$

(6)

$$\frac{\delta S}{\delta B^{(\omega)}_{\mu\nu}} \rightarrow F_{IJ}[\omega] = \frac{\Lambda}{2} \epsilon^{IJ}_{KLM} B^{KL} + \phi(B)^{IJ},$$  

(7)

$$\frac{\delta S}{\delta \phi^{(\omega)}_{\omega}} \rightarrow B^{IJ} \wedge B^{KL} = e \epsilon^{IJ}_{KLM},$$  

(8)

where

$$e = \frac{1}{4!} \epsilon_{IJKL} B^{IJ} \wedge B^{KL}.$$

(9)
We can rewrite the action using the decomposition of the fields into self-dual and anti-self-dual fields, using the duality in the Lie algebra: \( B = B^{(+)i} + B^{(-)i} \), \( \omega = \omega^{(+)i} + \omega^{(-)i} \), and \( \phi = \phi^{(+)i} + \psi + \phi^{(-)i} + \phi_0 \) (see eq. (A8) in appendix). Here we have decomposed the Lagrange-multiplier field \( \phi_{ijkl} \) into its irreducible components with respect to the symmetry group:

- \( \phi^{(+)ii} \) (2,0) 5 components (left part of the Weyl);
- \( \psi_{ij} \) (1,1) 9 components (traceless part of the Ricci);
- \( \phi^{(-)ij} \) (0,2) 5 components (right part of the Weyl);
- \( \phi_0 \) (0,0) 1 component (scalar curvature).

The action (5) decomposes into three parts.

\[
S = S^+ + S^0 + S^- 
\]

\[
S^\pm = \int_M \left[ \frac{\phi_0 \pm \Lambda}{2} \delta_{ij} B^{(\pm)ij} \wedge F^{(\pm)ij} - \frac{1}{2} \phi^{(\pm)ij} B^{(\pm)ij} \wedge B^{(\pm)ij} \right] 
\]

\[
S^0 = \int_M \left[ -\psi_{ij} B^{(-)ij} \wedge B^{(+)} \right] 
\]

Note that the Ashtekar formulation is derived by considering only the self-dual (left) part of the Plebanski action \( S^+ \) or by imposing \( B^{(+)i} = 0 \).

The Euler-Lagrange equations (8) can be rewritten as:

\[
\frac{\delta S}{\delta \phi^{(+)ij}} \propto B^{(+)ij} \wedge B^{(+)ij} - \frac{1}{3} \delta_{ij} \delta_{kl} B^{(+)kl} \wedge B^{(+)kl} = 0 
\]

\[
\frac{\delta S}{\delta \phi^{(-)ij}} \propto B^{(-)ij} \wedge B^{(-)ij} - \frac{1}{3} \delta_{ij} \delta_{kl} B^{(-)kl} \wedge B^{(-)kl} = 0 
\]

\[
\frac{\delta S}{\delta \psi_{ij}} \propto B^{(-)ij} \wedge B^{(+)ij} = 0 
\]

\[
\frac{\delta S}{\delta \phi_0} \propto \delta_{ij} \left[ B^{(+)ij} \wedge B^{(+)ij} + B^{(-)ij} \wedge B^{(-)ij} \right] = 0 
\]

The set of equations (13,14,15,16) is of course equivalent to equation (8), with

\[
\frac{1}{3} \delta_{kl} B^{(+)kl} \wedge B^{(+)kl} = -\frac{1}{3} \delta_{kl} B^{(-)kl} \wedge B^{(-)kl} = 2\epsilon. 
\]

We can now state the following:

**Theorem 1** If

\[
\tilde{\epsilon} = \frac{1}{4!} \epsilon_{ijkl} \epsilon^{\mu \nu \rho \sigma} B^{IJ}_{\mu \nu} B^{KL}_{\rho \sigma} \neq 0 \ , 
\]

then equation (8) is equivalent to equation

\[
\epsilon_{ijkl} B^{IJ}_{\mu \nu} B^{KL}_{\rho \sigma} = \epsilon_{\mu \nu \rho \sigma} \tilde{\epsilon} \ . 
\]

Moreover, in this case, (8) and (I8) are fulfilled iff there exists a real tetrad field \( e^I = e^I_\mu dx^\mu \) such that one among the following equalities is satisfied:
\[
\begin{align*}
I^+ & \quad B^{IJ} = +e^I \wedge e^J \\
I^- & \quad B^{IJ} = -e^I \wedge e^J \\
II^+ & \quad B^{IJ} = -\frac{1}{2} e^{IJ} K L e^I \wedge e^J \\
II^- & \quad B^{IJ} = -\frac{1}{2} e^{IJ} K L e^I \wedge e^J 
\end{align*}
\]

**Proof**

The proof of the equivalence of conditions (8) and (18) when \( \tilde{e} \neq 0 \) is quite simple. In fact, we can define:

\[
\Sigma_{IJ}^{\mu \nu} = \frac{1}{\tilde{e}} \epsilon^{\mu \nu \rho \sigma} \epsilon_{IJKL} B^{KL}_{\rho \sigma},
\]

and in terms of \( B \) and \( \Sigma \) conditions (8) and (18) respectively read:

\[
\begin{align*}
\Sigma_{IJ}^{\mu \nu} B_{\rho \sigma}^{IJ} & = \delta_{\rho \sigma}^{\mu \nu} \quad (20) \\
\Sigma_{IJ}^{\mu \nu} B_{\mu \nu}^{KL} & = \delta_{IJ}^{KL}. \quad (21)
\end{align*}
\]

It is easy to verify the if part of the theorem, and to see that equation (8) is invariant under the change \( B \to -B \) or \( B \to \ast B \). The only if part can be proved as follows. From Reisenberger [5], we know that equation (13) (resp eq. (14)) implies that there exists a real tetrad field \( a^I_f \) (resp. \( b^I_f \)) such that \( B^{(+)i}_{IJ} = \pm T^{(+)i}_{IJ} a^I \wedge a^J \) (resp. \( B^{(-)i}_{IJ} = \pm T^{(-)i}_{IJ} b^I \wedge b^J \)). The condition \( \tilde{e} \neq 0 \) implies that \( a^I \) is a basis of one-forms, so there exists a \( 4 \times 4 \) real matrix \( A \) such that:

\[
b^I = A^I_J a^J. \quad (22)
\]

Equation (16) is equivalent to the requirement that \( \det(A) = \pm 1 \) so that \( \pm A \in SL(4) \). Equation (15) is equivalent to the requirement that \( \pm A \in SO(4) \). Using the decomposition \( SO(4) = (SU(2) \times SU(2))/Z_2 \) we can decompose \( A \) into a commuting product of self-dual and anti-self dual rotations, \( \pm A = O^{\pm 1} O^- \). If we define \( e^I = O_{+j}^I b^j = \pm O_{-j}^I a^j \), then

\[
B^{(+)}_{ij} = \pm T^{(+)}_{IJ} e^I \wedge e^J, \quad B^{(-)}_{ij} = \pm T^{(-)}_{IJ} e^I \wedge e^J. \quad (23)
\]

\[\square\]

It is interesting to note that given an \( so(4) \) two-form field \( B^{IJ} \), it is possible to construct two Urbanke \( su(2) \) metrics

\[
g_{\mu \nu}^{(+)} = -\frac{2}{3 \tilde{e}} \epsilon_{ijk} \epsilon^{\alpha \beta \gamma \delta} \left[ B^{(+)}_{I \alpha} B^{(+)}_{J \beta} B^{(-)}_{K \delta} \right] \\
g_{\mu \nu}^{(-)} = -\frac{2}{3 \tilde{e}} \epsilon_{ijk} \epsilon^{\alpha \beta \gamma \delta} \left[ B^{(-)}_{I \alpha} B^{(-)}_{J \beta} B^{(-)}_{K \delta} \right]. \quad (24, 25)
\]

For a general field \( B \) these two metrics are unrelated, but when equation (8) is satisfied they are equal up to a sign:
\[
\begin{align*}
I^+ & \quad g^{(+)}_{\mu\nu} = -g^{(-)}_{\mu\nu} = \eta_{IJ}e^I_\mu e^J_\nu \\
I^- & \quad g^{(+)}_{\mu\nu} = -g^{(-)}_{\mu\nu} = -\eta_{IJ}e^I_\mu e^J_\nu \\
II^+ & \quad g^{(+)}_{\mu\nu} = g^{(-)}_{\mu\nu} = \eta_{IJ}e^I_\mu e^J_\nu \\
II^- & \quad g^{(+)}_{\mu\nu} = g^{(-)}_{\mu\nu} = -\eta_{IJ}e^I_\mu e^J_\nu
\end{align*}
\]

Case I corresponds to the sector where the action becomes:

\[
S_{I\pm} = \int \left[ \pm e^I \wedge e^J \wedge F_{IJ} + \frac{\Lambda}{4} \epsilon_{IJRS} e^I \wedge e^J \wedge e^R \wedge e^S \right],
\] (26)

and case II to the gravitational sector:

\[
S_{II\pm} = \int \left[ \pm \epsilon_{IJRS} e^I \wedge e^J \wedge F_{RS} + \frac{\Lambda}{4} \epsilon_{IJRS} e^I \wedge e^J \wedge e^R \wedge e^S \right].
\] (27)

After integration over the field \( \phi \), the partition function of the Plebanski model reads:

\[
Z_{Pl}(M, \Lambda) = \int DA^+ DA^- DB^+ DB^- \delta(f(B)) \times \\
\quad e^{i \int_M tr(B^+ \wedge F(A^+)) - \frac{\Lambda}{2} B^+ \wedge B^+} \times \\
\quad e^{i \int_M tr(B^- \wedge F(A^-)) + \frac{\Lambda}{2} B^- \wedge B^-},
\] (28)

where \( \delta(f(B)) \) is a 20 dimensional delta function corresponding to the set of constraints (9). This expression clearly appears to be the continuous analogues of (4) (we will prove this relation in the next section). The fact that the \((+ \) and \((-\) parts have opposite cosmological constant (which is the only possibility due to the constraint (16)) is the field-theoretic motivation for the Yetter [10] choice of the quantum group \( U_q(su(2)) \otimes U_{q^{-1}}(su(2)) \).

Using the result of Theorem 1, this partition function can be written (when \( \Lambda = 0 \))

\[
Z_{Pl}(M, \Lambda) = \int DADE \left[ \cos(\int_M e^I \wedge e^J \wedge F_{IJ}) + \cos(\int_M \epsilon_{IJRS} e^I \wedge e^J \wedge F_{RS}) \right].
\] (29)

In this expression we have neglected contributions from degenerate \( B \)'s \( (e = 0) \) (which is not justified at this point). The expression (29) clearly shows the difference between the Plebanski model and pure gravity. In the Plebanski model, even if we consider only non degenerate \( B \), we still integrate over configurations of the \( B \) fields that can globally belong to different sectors, which results in interference between different sectors. We have seen previously that the \( so(4) \) Plebanski model possess a \( Z_2 \times Z_2 \) symmetry \( B \rightarrow -B, B \rightarrow *B \), this discrete symmetry exchange the different sectors and is responsible for the existence of interference. The interference kill for instance the imaginary part of the amplitude. Gravity is obtained by restricting the \( B \) fields to be always in the \( II^+ \) sector. For example, the partition function of Euclidian pure gravity without cosmological constant is:

\[
Z_{GR} = \int DADE e^{i \int_M \epsilon_{IJRS} e^I \wedge e^J \wedge F_{RS}}
\] (30)
If we allow in the partition function degenerate configurations, the situation is even worse because we should then integrate between configurations of the $B$ fields which can belong to different sectors at different points.

This means that in all cases, there is a discrepancy between the Barret-Crane model (so(4) Plebanski model) and gravity due to interference between different sector. It is important to note that this discrepancy between $BF$ formulation of gravity and Einstein theory already appears in the case of 3-dimensional Euclidian gravity \cite{12} and also in the self-dual formulation of 4-dimensional Euclidian gravity.

### IV. THE DISCRETIZATION OF THE CONSTRAINTS

In order to link the partition function \eqref{28} to the Baez-Barrett-Crane model we have to discretize the set of constraints \eqref{8,18} along 4-simplices. Using the $B$ field, we can associate an element of so(4) to each 2-dimensional surface $S$ embedded in $M^4$ as follows:

$$B^{IJ}[S] = \int_S B^I_{\mu\nu} dx^\mu \wedge dx^\nu.$$ \hfill (31)

Given a triangulation of our manifold $M^4$ by 4-simplices, we take the $B$ field to be constant inside each 4-simplex, so that $dB^{IJ} = 0$. Using \eqref{31}, we can associate a bivector to each face (2-simplex) of the 4-simplex. Then, by Stokes’ theorem we see that the sum of bivectors over all faces of a given tetrahedron belonging to the 4-simplex is 0: this is the closure constraint $Q4$.

We have seen, in theorem \cite{4}, that in the continuum case the constraints on the $B$ field can be written, when $B$ is non degenerate, in two different but equivalent forms, \eqref{8} or \eqref{18}. Namely

$$\epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} = \tilde{\epsilon} \epsilon_{\mu\nu\rho\sigma}, \hfill (32)$$

or

$$\epsilon^{\mu\nu\rho\sigma} B^{IJ}_{\mu\nu} B_{KL}^{KL} = \tilde{\epsilon} \epsilon^{IJKL}. \hfill (33)$$

It is important to understand that these two equivalent form of the constraints lead to two different discretizations and then to two, a priori different, state sum models. As we will see, one state sum model (or spin foam model in the terminology of Baez), corresponding to \eqref{32} is the Baez-Crane-Barrett model, while the other spin foam model, corresponding to \eqref{33}, is the so(4) analog of the Reisenberger spin foam model \cite{6} (which corresponds to self-dual formulation of gravity).

Let us consider first the discretization of \eqref{32}. The free indices of \eqref{32} are space-time indices so that we can saturate them by integrating in all possible ways over faces of the 4-simplex. A straightforward computation show that \eqref{32} implies:

$$V(S, \tilde{S}) = \epsilon_{IJKL} B^{IJ}[S] B^{KL}[\tilde{S}] = \delta_{ij} \left[ B^{(i)j}[S] B^{(i)j}[\tilde{S}] - B^{(i)j}[S] B^{(j)j}[\tilde{S}] \right],$$

where $V(S, \tilde{S}) = \int_{x \in S, y \in \tilde{S}} \tilde{\epsilon} \epsilon_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dy^\rho \wedge dy^\sigma$ is the 4-volume spanned by $S$ and $\tilde{S}$. In particular, this means
\[ \delta_{ij} \left[ B^{(+)}i[S]B^{(+)}j[S] - B^{(-)}i[S]B^{(-)}j[S] \right] = 0, \]  
\[ \delta_{ij} \left[ B^{(+)}i[\tilde{S}]B^{(+)}j[S] - B^{(-)}i[\tilde{S}]B^{(-)}j[S] \right] = 0, \]

for each face (2-simplex) \( S \) of the 4-simplex (conditions Q2), and

for each couple of faces \( S, \tilde{S} \) which share one edge (that, combined with (34), impose condition Q3). Therefore the bivectors associated via (31) to the face \( s \) of the tetrahedra satisfy the Baez-Barrett-Crane constraints of section II. We can thus conclude that the state sum model (4) corresponds to a natural discretization of the main sector of the \( so(4) \) Plebanski model.

We now consider the continuum constraint in the form (33). In this form of the constraint, the space-time indices are saturated by the \( \epsilon \) tensor while the internal indices are free. A general procedure is presented in [13] that describes the construction of spin foam models from classical actions of certain \( BF \) type. The form (33) of the constraints allows us to use the results of [13] from which we can deduce the discretization of (33) and the corresponding spin foam model. For the reader’s convenience we nevertheless give a self contained justification. The discretization of the \( B \) field inside a 4-simplex amounts to decompose the 2-form \( B \), inside the 4-simplex, into a sum of singular 2-forms associated with the faces of the 4-simplex:

\[ B^{IJ}(x) = \sum_{S} B^{IJ}_{S}(x), \]  

where the sum is over all faces of the 4-simplex and \( B^{IJ}_{S} \) is a two form such that:

\[ \int_{S} B^{IJ}_{S} \wedge J = \int_{S^*} B^{IJ}[S] J, \]

where \( J \) is any 2-form, and \( S^* \) denote the dual face of \( S \), i.e., the 2-cell dual to \( S \) that belongs to the dual complex of the 4-simplex. With this definition it is clear that:

\[ \int_{S} B^{IJ}_{S} = \delta_{S,\tilde{S}} B^{IJ}[\tilde{S}], \]  
\[ \int_{M^4} B^{IJ}_{S} \wedge B^{KL}_{S} = B^{IJ}[S] B^{KL}[\tilde{S}] \epsilon(S, \tilde{S}), \]

where \( \epsilon(S, \tilde{S}) \) is equal to the sign of the oriented volume \( V(S, \tilde{S}) \) spanned by the 2 faces \( S \) and \( \tilde{S} \). So \( \epsilon(S, \tilde{S}) = \pm 1 \) if \( S, \tilde{S} \) are 2 faces of the 4-simplex which do not share an edge and \( \epsilon(S, \tilde{S}) = 0 \) if \( S, \tilde{S} \) do share an edge. Using these definitions and properties it is straightforward to see that the constraint (33), after integration over the 4-simplex, leads to:

\[ \Omega^{IJKL} - \epsilon^{IJKL} \frac{1}{4!} \epsilon_{ABCD} \Omega^{ABCD} = 0, \]  

where
\[
\Omega^{IJKL} = \sum_{S, \bar{S}} \epsilon(S, \bar{S}) B^{IJ}[\bar{S}] B^{KL}[S]. 
\]

This form of the constraint is analogous to the Reisenberger constraint appearing as a discretization of the self-dual formulation of gravity \([14]\). And we can construct (as in \([6]\)) the corresponding spin foam model (see also \([13]\)).

If we decompose \(\Omega\) into its self-dual and antiself-dual components the constraints \((40)\) are equivalent to:

\[
\begin{align*}
\Omega^{ij}_{++} &= \delta^{ij} \frac{1}{3} tr(\Omega_{++}) \\
\Omega^{ij}_{--} &= \delta^{ij} \frac{1}{3} tr(\Omega_{--}) \\
\Omega^{ij}_{+-} &= 0 \\
tr(\Omega_{++}) + tr(\Omega_{--}) &= 0,
\end{align*}
\]

where \(\Omega^{ij}_{+-} = T_{IJ}^{(ij)} T_{KL}^{(ij)} \Omega^{IJKL}\). These constraints are clearly the discrete analogous of constraints \((13, 14, 15, 16)\). Moreover, \((42)\) is precisely the Reisenberger constraint that appears in the spin foam model corresponding to self-dual formulation of gravity \([14, 6]\).

V. CONCLUSION

In this note, we have analyzed the connection between the Barrett-Crane model and so(4) Plebanski’s action for Euclidean General Relativity. In particular, we have shown that the Barrett-Crane conditions on the allowed so(4) representation are the quantum transcription of the Plebanski constraints \((18)\). This result implies that the Barrett-Crane state sum model is associated to the partition function \((29)\). This model is related to, but different from, pure gravity due to the presence of interference terms between different sectors. We also showed that the so(4) Plebanski model admits another possible discretisation that is analogous to the one already used for the self-dual formulation of gravity. Thus, it seems important to understand the link between the Barrett-Crane constraint and the constraint \((40)\). One would expect that, at the quantum level, the space of solutions of the two constraints are in one-to-one correspondence or that the two corresponding state sum models converge to each other when the triangulation becomes finer (if degenerate solutions are not relevant). We leave these questions for future analysis.

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APPENDIX A: NOTATION FOR THE SO(4) AND SO(3,1) GROUPS.

We use capital Latin letter for internal vector index $I, J = 0, 1, 2, 3$ and the internal metric $\eta_{IJ} = \text{diag}(1, 1, 1, 1)$. We denote with $(\alpha)$ the group indices and use as fundamental basis

$$T_{(\alpha)}^{IJ} = -\epsilon^{0iIJ} \quad T_{(\alpha)}^{IJ} = \eta^{Ii} \eta^{0j} - \eta^{0i} \eta^{jI}$$  \hspace{1cm} (A1)

In these two groups the duality transformations is given by:

$$(^* T_{(\alpha)})^{IJ} = \frac{1}{2} \epsilon^{IJK} T_{(\alpha)}^{KL} \quad ^* T_{(\alpha)} = T_{(\alpha)+3} \quad ^* T_{(\alpha)+3} = T_{(\alpha)}.$$ \hspace{1cm} (A2)

It is profitable to consider the duality basis

$$T_{(\pm)ij}^{IJ} = \frac{T_{ij}^{IJ} + T_{ij+3}^{IJ}}{2} \quad T_{(\pm)ij}^{IJ} = \frac{T_{ij}^{IJ} - T_{ij+3}^{IJ}}{2}$$ \hspace{1cm} (A3)

where we have

$$^* T_{(\pm)ij} = T_{(\pm)ji} \quad ^* T_{(\pm)ji} = -T_{(\pm)ji}$$ \hspace{1cm} (A4)

In this bases we have the relations

$$T_{(\pm)IJ}^{IK} T_{(\pm)JK}^{J} = \frac{1}{2} \epsilon^{ijk} T_{(\pm)IJ}^{JK} - \frac{1}{4} \delta^{ij} \eta^{IK}$$ \hspace{1cm} (A5)

$$\sum_{i} T_{(\pm)ji}^{IJ} T_{(\pm)KJI} = \frac{1}{2} \delta_{[K} \delta_{I]}^{J} \pm \frac{1}{4} \epsilon^{IJ}$$ \hspace{1cm} (A6)

and the duality projection of bivector is defined by:

$$B^{IJ} = T_{(\pm)ij}^{IJ} B_{(\pm)ij} + T_{(\pm)ij}^{IJ} B_{(\pm)ij}.$$ \hspace{1cm} (A7)

Finally, the decomposition of the Lagrange multiplier field $\phi_{IJKL}$ is given by:

$$\phi_{IJKL} = \phi_{ij}^{(0)} T_{ij}^{IJ} T_{jK}^{KL} + \phi_{ij}^{(-)} T_{ij}^{IJ} T_{jK}^{KL}$$
$$+ \psi_{ij} (T_{ij}^{(0)} T_{jK}^{(0)} + T_{ij}^{(0)} T_{jK}^{(0)})$$
$$+ \phi_0 \delta_{ij} (T_{ij}^{(0)} T_{jK}^{(0)} + T_{ij}^{(0)} T_{jK}^{(0)})$$ \hspace{1cm} (A8)

where $\phi_{ij}^{(0)}$ and $\phi_{ij}^{(-)}$ are symmetric and traceless.
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