On generalized conditional entropies and information-theoretic Bell inequalities under decoherence

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We consider information-theoretic inequalities of the Bell type in the presence of decoherence. It is natural that too strong coupling with the environment can prevent an observation of quantum correlations. In this regard, the use of various entropic functions may give additional capabilities to reveal desired correlations. It was already shown that the Bell and Leggett–Garg inequalities in terms conditional Tsallis entropies are more sensitive in the cases of detection inefficiencies. In this paper, we study capabilities of generalized conditional entropies of the Tsallis type in analyzing the Bell theorem in the presence of decoherence. Two forms of the conditional Tsallis $q$-entropy are known in the literature. We show that each of them can be used for defining a metric in the probability space of interest. Such metrics can be used in realizing the so-called triangle principle. The triangle principle has recently been proposed as a unifying approach to questions of local realism and non-contextuality. Applying the triangle principle leads to the two families of non-contextuality and local realistic inequalities. Information-theoretic formulations in terms of the $q$-entropic metrics are first discussed for the CHSH scenario in dephasing environment. Then we also revisit $q$-entropic inequalities of the Leggett–Garg type. An environmental influence is modeled by the phase damping channel and by the depolarizing channel.

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I. INTRODUCTION

Non-classical nature of quantum correlations was independently emphasized in the Schrödinger “cat paradox” paper [1] and in the Einstein–Podolsky–Rosen paper [2]. This character is clearly manifested in some experiments such as Bohm’s version of the EPR argument [3]. Quantum correlations are a central subject in quantum information theory. To study different types of correlations, various quantifiers of correlations were considered [4]. Due to the seminal papers by Bell [3] [4] and by Kochen and Specker [7], we know that quantum mechanics is not consistent with the assumptions of local realism and non-contextuality. Hypotheses of such a kind establish that measurable properties of a physical system do not depend on the context, in which they are measured. Bell’s ideas have allowed to recast the problem of hidden variables as an experimentally tested statement [5]. Leggett–Garg macrorealistic inequalities [9] form a direction inspired by the Bell theorem. Such relations are based on the two assumptions known as the macroscopic realism and the noninvasive measurability at the macroscopic level [10]. Since Leggett–Garg inequalities probe correlations of a single system measured at different times, decoherence is one of crucial problems in a practice. Violations of the Leggett–Garg inequalities under decoherence were experimentally studied in Refs. [11, 12]. Quantum non-locality and contextuality properties are related to statistical predictions and probability distributions [13].

Original Bell inequalities were written in terms of mean values [5]. The Greenberger–Horne–Zeilinger approach has given a statement without inequalities [14]. To realize Bell’s theorem in an experimental framework, several scenarios are typically used. The Clauser–Horne–Shimony–Holt (CHSH) scenario [15] is probably the most known setup. Like the Bell paper [3], the CHSH scenario probes entanglement between spacelike separated subsystems. Entropic versions of Bell’s theorem were considered in Refs. [16, 17]. These papers were mainly focused on the CHSH scenario. The Klyachko–Can–Biniçioğlu–Shumovsky (KCBS) scenario [18] is another important setup related to a single spin-1 system. Information-theoretic Bell inequalities for the KCBS scenario were examined in Refs. [19, 20]. For both the CHSH and KCBS scenarios, inequalities in terms of Tsallis $q$-entropies were studied in Ref. [21]. The CHSH and KCBS scenarios can be treated as particular cases of more general $n$-cycle scenario [22, 23]. Inequalities of the Legget–Garg type are related to the case, when cycles are formed by observables taken at different times. In Ref. [24], the triangle principle has been proposed as a new approach to the non-locality and contextuality. Similar ideas were considered in Ref. [25]. Applications of this principle to qutrits with use of the Tsallis-type metrics were
recently examined in Ref. [26].

The aim of the present work is to study information-theoretic Bell inequalities based on generalized conditional entropies. Some advantages of this approach were already examined in Ref. [21, 26, 27]. For instance, variations of the parameter in q-entropic inequalities are shown to be useful in analyzing cases with detection inefficiencies. Formulation of restrictions of the Leggett–Garg type in terms of the Shannon entropies were examined in Ref. [28]. A q-entropic extension of this question has been discussed in Ref. [27]. We also aim to study information-theoretic Bell inequalities in the presence of decoherence. The contribution of the present paper is two-fold. First, we show that each of the two known forms of conditional q-entropy leads to the corresponding metric between random variables. One of the conditional q-entropies obeys the chain rule [29], whereas the triangle inequality for a metric follows for q ≥ 1. However, other conditional q-entropy does not share the chain rule. It is not obvious that a legitimate metric could be obtained in this way. Second, we consider violation of q-metric inequalities of the Bell and Leggett–Garg types under decoherence. It is known that decoherence is one of crucial problems for an observation of quantum correlations in a practice. In particular, dephasing processes can prefer any detection of such correlations. We will argue that q-entropic inequalities could be useful in analysis of data of experiments to test restrictions of such a kind.

The paper is organized as follows. In Section III, we consider those metrics that can be based on the conditional q-entropies. It is shown that the known conditional forms of the Tsallis entropy both lead to a legitimate metric for q ≥ 1. Here, the triangle inequality is most important from the viewpoint of applications of the triangle principle. For one of the cases considered, the triangle inequality directly follows from the chain rule. In the second case, the desired result is obtained due to independent reasons. In Section III, inequalities of the Bell type are written as q-metric inequalities for the CHSH scenario. Each of two particles in the CHSH scenario is assumed to be subjected to the phase damping channel. Section IV is devoted to q-metric Leggett–Garg inequalities under decoherence. We demonstrate advantages of metric inequalities with some parameter that can be varied for maximizing a desired violation. Varying the parameter in q-entropic inequalities, a violation of the restrictions considered may become much more robust to decoherence. As models of quantum noise, the phase damping and depolarizing channels are utilized. In Section V, we conclude the paper with a summary of results.

II. METRICS AND CONDITIONAL TSALLIS ENTROPIES

In this section, we discuss required properties of the q-entropies and their conditional forms. Two kinds of the q-entropic metric will be examined. Let discrete random variable X take values on a finite set ΩX of cardinality #ΩX. The non-extensive entropy of degree q > 0 ̸= 1 is defined by [30]

\[ H_q(X) := \frac{1}{1 - q} \left( \sum_{x \in \Omega_X} p(x)^q - 1 \right). \]

(1)

With the factor \((2^{1-q} - 1)^{-1}\) instead of \((1 - q)^{-1}\), this function was examined by Havrda and Charvát [31] and later by Daróczy [32]. In statistical physics, the entropy (1) is extensively used due to Tsallis [30].

Obviously, the entropy (1) is concave for all q > 0. It is convenient rewrite (1) as

\[ H_q(X) = -\sum_{x \in \Omega_X} p(x)^q \ln_q p(x) = -\sum_{x \in \Omega_X} p(x) \ln_q \left( \frac{1}{p(x)} \right). \]

(2)

Here, we used the q-logarithm defined for q > 0 ̸= 1 and ξ > 0 as

\[ \ln_q(\xi) = \frac{\xi^{1-q} - 1}{1 - q}. \]

(3)

In the limit q → 1, we obtain lnq(ξ) → ln ξ and the standard Shannon entropy

\[ H_1(X) = -\sum_{x \in \Omega_X} p(x) \ln p(x). \]

(4)

For the uniform distribution, the entropy (1) reaches the maximal value lnq(#ΩX). Let us introduce the index of coincidence [33]

\[ C(X) := \sum_{x \in \Omega_X} p(x)^2. \]

(5)
In the case \( q = 2 \), we obtain the so-called quadratic entropy \( H_2(X) = 1 - C(X) \). This entropy and related functions are used under several names \[33\]. The Rényi entropies \[34\] form another especially important family of one-parametric extensions of the Shannon entropy. Rényi’s entropies are beyond the scope of the present work. Some properties and applications of such entropies are discussed in the book \[33\].

To define a metric, we will use conditional entropies. For brevity, we will omit symbols such as \( \Omega_x \) in entropic sums. The standard conditional entropy is defined by \[35\]

\[
H_1(X|Y) := \sum_y p(y) H_1(X|y) = -\sum_x \sum_y p(x, y) \ln p(x|y) .
\]

(6)

Here, we use Bayes’ rule \( p(x|y) = p(x, y)/p(y) \) and the particular function

\[
H_1(X|y) = -\sum_x p(x|y) \ln p(x|y) .
\]

(7)

In the literature, two kinds of the conditional \( q \)-entropy were discussed \[29\]. These forms are respectively inspired by the two expressions shown in \(2\). The first form is defined as \[29\]

\[
H_q(X|Y) := \sum_y p(y)^q H_q(X|y) ,
\]

(8)

where

\[
H_q(X|y) := \frac{1}{1-q} \left( \sum_x p(x|y)^q - 1 \right) .
\]

(9)

Similarly to \(2\), the equivalent expressions are written as

\[
H_q(X|y) = -\sum_x p(x|y)^q \ln_q p(x|y)
\]

(10)

\[
= \sum_x p(x|y) \ln_q \left( \frac{1}{p(x|y)} \right) .
\]

(11)

The conditional entropy \[8\] is, up to a factor, the quantity introduced by Daróczy \[32\]. For all \( q > 0 \), we have the chain rule \[29, 32\]

\[
H_q(X, Y) = H_q(Y|X) + H_q(X) = H_q(X|Y) + H_q(Y) .
\]

(12)

Sometimes, this equality is referred to as the stronger chain rule. With \( q = 1 \), we have the chain rule with the standard conditional entropy \[6\]. An immediate extension of \[12\] for more than two random variables can be found in \[29\]. Relations of such a kind play an important role in many information-theoretic derivations. For instance, the Braunstein–Caves derivation \[16\] of entropic Bell inequalities is based on the chain rule for the Shannon entropy.

As was noted in Ref. \[20\], information-theoretic Bell inequalities for the \( n \)-cycle scenario can be represented in terms of the mutual information. Similarly to the standard case, the mutual \( q \)-information can be defined as \[29\]

\[
I_q(X;Y) := H_q(X) - H_q(X|Y) .
\]

(13)

Using the chain rule \[12\], the right-hand side of \[13\] can be rewritten as

\[
H_q(X) + H_q(Y) - H_q(X, Y) = H_q(Y) - H_q(Y|X) .
\]

(14)

Formulation of the Bell theorem in terms of the mutual \( q \)-information has been addressed in Ref. \[24\].

Using the particular functional \[10\], the second form of conditional \( q \)-entropy is introduced as \[29\]

\[
\tilde{H}_q(X|Y) := \sum_y p(y) H_q(X|y) .
\]

(15)

Note that this form of conditional entropy does not share the chain rule of usual kind \[29\]. Hence, it is not directly related to the mutual \( q \)-information. Nevertheless, the entropy \[15\] has found to be useful at least as an auxiliary quantity \[29, 30\]. The conditional entropy \[15\] can also be used for measuring a distance between random variables.

The standard conditional entropy leads to the following metric \[37\]:

\[
\Delta_1(X, Y) := H_1(X|Y) + H_1(Y|X) .
\]

(16)

General properties of information distances are considered in Ref. \[38\]. The author of Ref. \[29\] discussed three forms of an entropic distance based on the Tsallis entropies. First of these distances is defined similarly to \[16\]:

\[
\Delta_q(X, Y) := H_q(X|Y) + H_q(Y|X) .
\]

(17)

Due to \[13\], we can rewrite \[17\] as \(\Delta_q(X, Y) = H_q(X, Y) - I_q(X;Y)\). As was shown in \[29\], the quantity \[17\] is a metric for \( q \geq 1 \). It satisfies the following properties.
the triangle inequality. Another question concerns chain rules for $q$-entropies. For any $q > 0$, the conditional entropy \((15)\) satisfies the chain rule \((12)\). As was already mentioned, the conditional entropy \((15)\) does not share the chain rule \((22)\). Instead, we will use another statement.

\[ H_q(X|Z) \leq H_q(X|Y) + H_q(Y|Z) , \]  

which holds for $q \geq 1$. Other $q$-entropic metrics are defined in terms of correlation coefficients \([29]\). One form of correlation coefficients is introduced as the ratio of the mutual $q$-information to the joint $q$-entropy. Then difference between 1 and this correlation coefficient leads to a metric for $q \geq 1$ \([29]\). It can also be interpreted as the result of division of \((17)\) by the joint entropy $H_q(X,Y)$. Another correlation coefficient is defined as the ratio of the mutual $q$-information to the maximum of the entropies $H_q(X)$ and $H_q(Y)$. Hence, one leads to the third distance considered in Ref. \([29]\). It should be emphasized that the mentioned quantities are metrics only for $q \geq 1$. Further, validity of the triangle inequality for these distances is closely related to the chain rule. In Ref. \([26]\), the mentioned $q$-entropic metrics were used to study Bell inequalities for a pair of entangled qutrits. These metrics can all be represented in terms of the mutual $q$-information together with either $H_q(X,Y)$ or $\max\{H_q(X),H_q(Y)\}$.

We shall now examine a $q$-entropic distance which cannot be expressed in terms of the mutual $q$-information. The conditional $q$-entropy \((15)\) does not share the chain rule. Nevertheless, this conditional form leads to a legitimate metric as well. We shall analyze the question in more details, since it seems to be not addressed in the literature. Similarly to \((17)\), we can introduce another quantity

\[ \tilde{\Delta}_q(X,Y) := \tilde{H}_q(X|Y) + \tilde{H}_q(Y|X) . \]  

It is easy to check that the properties (i)–(iii) remain valid for \((19)\). The only question concerns the triangle inequality. To resolve the question, we will examine some essential properties of the entropy \((15)\).

**Proposition 1** For $q > 0$, the conditional entropy \((15)\) satisfies

\[ \tilde{H}_q(X,Y|Z) \geq \tilde{H}_q(X|Z) . \]  

**Proof.** Since the standard case $q = 1$ is well known, we further assume $q \neq 1$. Let positive numbers $a(y)$ satisfy $\sum_y a(y) = 1$. We then have

\[ \sum_y a(y)^q \geq 1 \quad (0 < q < 1) , \]  

\[ \sum_y a(y)^q \leq 1 \quad (1 < q < \infty) . \]  

Combining these relations with $\sum_y p(x,y|z) = p(x|z)$, we obtain

\[ \sum_y \left( p(x,y|z)^q - p(x,y|z) \right) \geq p(x|z)^q - p(x|z) \quad (0 < q < 1) , \]  

\[ \sum_y \left( p(x,y|z)^q - p(x,y|z) \right) \leq p(x|z)^q - p(x|z) \quad (1 < q < \infty) . \]  

Summarizing with respect to $x$ and taking the sign of the factor $(1 - q)^{-1}$, we have arrived at a conclusion. For all $q > 0 \neq 1$, one gives

\[ H_q(X,Y|z) \geq H_q(X|z) . \]  

Multiplying \((24)\) by $p(z)$ and summing with respect to $z$, we finally obtain \((20)\). \[\Box\]

Note that the facts \((21)\) and \((22)\) are closely related to theorem 19 of the book \([39]\). It is clear that the result \((20)\) can be generalized as follows. For real $q > 0$ and integer $n \geq 1$, we have

\[ \tilde{H}_q(X_1, \ldots, X_n, X_{n+1}|Z) \geq \tilde{H}_q(X_1, \ldots, X_n|Z) . \]  

We refrain from presenting details of the argumentation. The relations \((20)\) and \((26)\) will be used below in deriving the triangle inequality. Another question concerns chain rules for $q$-entropies. For any $q > 0$, the conditional entropy \((15)\) satisfies the chain rule \((12)\). As was already mentioned, the conditional entropy \((15)\) does not share the chain rule \((22)\). Instead, we will use another statement.
Proposition 2 The conditional entropy (15) satisfies the following inequalities:

\[
\tilde{H}_q(X, Y|Z) - \tilde{H}_q(Y|Z) \geq \tilde{H}_q(X|Y, Z) \quad (0 < q < 1),
\]

\[
\tilde{H}_q(X, Y|Z) - \tilde{H}_q(Y|Z) \leq \tilde{H}_q(X|Y, Z) \quad (1 < q < \infty).
\]

Proof. Using \(p(x, y|z)/p(y|z) = p(x|y, z)\) and the definition (2), we merely write

\[
H_q(X, Y|z) - H_q(Y|z) = \frac{1}{1-q} \left( \sum_{x,y} p(x, y|z)^q - \sum_{y} p(y|z)^q \right)
= \sum_{y} p(y|z)^q \frac{1}{1-q} \left( \sum_{x} p(x|y, z)^q - 1 \right) = \sum_{y} p(y|z)^q H_q(X|y, z).
\]

As \(p(y|z) \leq 1\), replacing \(p(y|z)^q\) with \(p(y|z)\) leads to

\[
H_q(X, Y|z) - H_q(Y|z) \geq \sum_{y} p(y|z) H_q(X|y, z) \quad (0 < q < 1),
\]

\[
H_q(X, Y|z) - H_q(Y|z) \leq \sum_{y} p(y|z) H_q(X|y, z) \quad (1 < q < \infty).
\]

Further, we note \(p(z) p(y|z) = p(y, z)\). Multiplying (30) and (31) by \(p(z)\) and summing with respect to \(z\), we complete the proof.  

In principle, the formula (26) can be regarded as a variant of the chain rule. Note that the standard conditional entropy (10) obeys the equality

\[
H_1(X, Y|Z) - H_1(Y|Z) = H_1(X|Y, Z).
\]

We can obtain (32) by taking the limit \(q \to 1\) in both the relations (27) and (28). We are now ready to prove that the conditional entropy (15) of degree \(q \geq 1\) obeys the triangle inequality. This result is formulated as follows.

Proposition 3 For \(q \geq 1\), the conditional entropy (15) satisfies the triangle inequality

\[
\tilde{H}_q(X|Z) \leq \tilde{H}_q(X|Y) + \tilde{H}_q(Y|Z).
\]

Proof. Using the properties (20) and (28) we obtain

\[
\tilde{H}_q(X|Z) \leq \tilde{H}_q(X, Y|Z) \leq \tilde{H}_q(X|Y, Z) + \tilde{H}_q(Y|Z).
\]

The first inequality holds for all \(q > 0\), whereas the second one generally holds for \(q \geq 1\). We further recall the fact that conditioning on more can only reduce the conditional entropy (15). Namely, for all \(q > 0\) we have (40)

\[
\tilde{H}_q(X|Y, Z) \leq \tilde{H}_q(X|Y).
\]

Combining (34) with (35) completes the proof.  

By permutations, we further write \(\tilde{H}_q(Z|X) \leq \tilde{H}_q(Z|Y) + \tilde{H}_q(Y|X)\). Adding the latter to (33), for \(q \geq 1\) we obtain

\[
\tilde{H}_q(X|Y, Z) \leq \tilde{H}_q(X|Y).
\]

In other words, for \(q \geq 1\) the entropic quantity (19) is a legitimate metric. Thus, both the quantities (17) and (19) can be adopted as information distances in realizing the triangle principle. In the next sections, we will consider this question in the context of the Bell theorem and the Leggett–Garg inequalities.

III. METRIC INEQUALITIES FOR THE CHSH SCENARIO IN DEPHASING ENVIRONMENT

In this section, we will formulate \(q\)-metric inequalities for the CHSH scenario. We will also address a possibility to observe violations of the Bell theorem under decoherence. The CHSH scenario is a primary example of the so-called \(n\)-cycle scenarios [22, 23]. It is typically used in studies of conceptual question of quantum theory [41, 42]. The notion of marginal scenarios provides a general way to treat the non-contextuality of probability distributions [20, 42]. Here, we can ask whether a given family of marginal distributions for some set of random variables arises from some joint distribution of these variables [43]. Another general approach is based on the triangle principle [24].
The triangle principle allows to deduce a large class of non-contextuality and locality inequalities. We will formulate quantitative relations in terms of the $q$-metrics \((17)\) and \((19)\) for $q \geq 1$. Let us recall briefly details of the CHSH scenario. In this scenario, we deal with an entanglement of two spacelike separated subsystems $A$ and $B$. Let observables $A$ and $A'$ be used for one subsystem, and let observables $B$ and $B'$ be used for other. No one of the pairs \((A, A')\) and \((B, B')\) is jointly measurable. Each element of \((A, A')\) is compatible with each element of \((B, B')\), since these sets are related to different subsystems. Applying the triangle inequality, we simply obtain two relations

\[
\Delta_q(A, B) \leq \Delta_q(A, B') + \Delta_q(B, A') + \Delta_q(A', B), \quad (37)
\]

\[
\tilde{\Delta}_q(A, B) \leq \tilde{\Delta}_q(A, B') + \tilde{\Delta}_q(B, A') + \tilde{\Delta}_q(A', B). \quad (38)
\]

In the usual CHSH scenario, each of the observables has two possible outcomes rescaled as $\pm 1$. Assumptions of such a kind will determine a concrete form of inequalities with mean values. However, entropic formulations of Bell’s theorem have the same form irrespectively to the number of outcomes or the chosen scale for observables \([20]\). The authors of Ref. \([16]\) derived Bell’s inequality for the CHSH scenario in terms of the Shannon entropies. This inequality is often referred to as the Braunstein–Caves inequality.

Following Ref. \([16]\), we consider a quantum spin-$s$ system. To exemplify violations of the relations \((37)\) and \((38)\), one uses the following setup. Two counter-propagating spin-$s$ particles are emitted by the decay of a system with zero angular momentum. In the simplest case $s = 1/2$, we use the operator $S_z = (\hbar/2)\sigma_z$ with the eigenstates

\[
|0\rangle = \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \quad |1\rangle = \left(\begin{array}{c} 0 \\ 1 \end{array}\right). \quad (39)
\]

The state of two particles with zero total momentum is written as

\[
|\Phi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\right). \quad (40)
\]

We now take the four unit vectors $\vec{a}$, $\vec{a}'$, $\vec{b}$, and $\vec{b}'$. In the quantum-mechanical description, the quantities $A$ and $A'$ are represented as the operators $\vec{a} \cdot \vec{S}$ and $\vec{a}' \cdot \vec{S}$. The quantities $B$ and $B'$ are given in the same way. An example of violations is provided, when the above four vectors are coplanar and \([16]\)

\[
\angle(\vec{a}, \vec{b}) = \angle(\vec{b}', \vec{a}') = \angle(\vec{a}', \vec{b}) = \theta. \quad (41)
\]

Using properties of the standard conditional entropy, Braunstein and Caves formulated an information-theoretic inequality of the Bell type \([14]\). In the considered situation, their result is equivalent to the case $q = 1$ of the formulas \((37)\) and \((38)\). The Braunstein–Caves inequality expresses the fact that there exists some joint probability distribution for the four random variables. In principle, the relation \((17)\) can also be reached on this ground \([21]\). Here, the chain rule \((12)\) is very important. Since the conditional entropy \((15)\) does not share the chain rule, the relation \((35)\) cannot be obtained in such a way.

The following conclusions were found \([16]\). As the value of $s$ increases, the strength of a violation of the Braunstein–Caves inequality also increases. On the other hand, the growth of $s$ leads to decreasing of a range of values $\theta$, for which violations occur. The situation under consideration was also examined within the $q$-entropic approach \([21]\). With the above choice of the vectors, the bell type inequality of \([21]\) is actually tantamount to \((37)\). In this regard, the second $q$-inequality \((38)\) is a novel result. The $q$-entropic approach allow to get some advances \([21]\). First, we can significantly expand a class of probability distributions, for which the non-locality or contextuality are testable in this way. That is, some variations of $q \geq 1$ allows to wide a range of values $\theta$, for which violations occur. Second, the $q$-entropic inequalities are expedient in analyzing cases with detection inefficiencies \([21]\). Two models of detection inefficiencies in combination with the Braunstein–Caves inequality were considered in Ref. \([20]\).

We shall now show that the $q$-entropic approach is very useful in analyzing data of experiments in decohering environment. Here, we will mainly focus on \((38)\), since it was not considered previously. Violations of the Bell inequalities in the presence of decoherence and noise were studied in several papers \([44, 47]\). However, information-theoretic formulations of the Bell theorem were not addressed therein. Experimental studies of the contextual and non-local properties of polarization biphotons under decoherence are reported in \([48]\). Suppose that each of two particles is subjected to dephasing noise during its propagation from the point of emission to the detector. The Kraus operators of the phase damping channel are written as \([49]\)

\[
E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}. \quad (42)
\]
For brevity, we denote \( \lambda(t) = 1 - \exp(-2\gamma t) \). During the time \( t \), the density matrix of a single qubit is mapped as

\[
\rho \mapsto \mathcal{E}(\rho) = E_0 \rho E_0 + E_1 \rho E_1 .
\]

(43)

This channel describes one of fundamental quantum effects. In particular, it is helpful in understanding why a “live-dead” superposition of the Schrödinger cat becomes unlikely. It is also known that the phase damping channel can easily be converted to the phase flip one by a simple recombination [49].

If each of the two qubits is changed as (43), then the initial state \[49] is transformed to

\[
(\mathcal{E} \otimes \mathcal{E})|\Phi\rangle\langle \Phi| = \sum_{j,k=0}^1 (E_j \otimes E_k)|\Phi\rangle\langle \Phi|(E_j \otimes E_k) .
\]

(44)

By \( \delta t_1 \), we further mean the interval between the emission and the first local measurement on \( \mathcal{A} \). The second local measurement on \( \mathcal{B} \) will be performed \( \delta t_2 \) later. By calculations, we then obtain

\[
(\mathcal{E}_1 \otimes \mathcal{E}_1)|\Phi\rangle\langle \Phi| = \exp(-2\gamma \delta t_1)|\Phi\rangle\langle \Phi| + \frac{\lambda(\delta t_1)}{2} \left( |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| \right) .
\]

(45)

Here, the subscript “1” marks that this map is related to the first interval. Let us examine the case, when the plane of the vectors \( \vec{a}, \vec{b}, \vec{c}, \vec{b} \) is orthogonal to the axis \( z \). Dealing with probabilities, we can rescale the considered observables. Suppose that we first measure \( \vec{a} \cdot \vec{\sigma} \) on the qubit \( \mathcal{A} \). With the pre-measurement state \[43\] the outcomes are equiprobable irrespectively to \( \gamma \delta t_1 \). When the outcome \( m \) has been obtained, the post-measurement state is

\[
p(m)^{-1} \left\{ (\mathcal{E}_1 \otimes \mathcal{E}_1)|\Phi\rangle\langle \Phi| \right\} (\Lambda_m(\vec{a} \otimes 1)) = \exp(-2\gamma \delta t_1) \Lambda_m(\vec{a} \otimes \Lambda_{-m}(\vec{a}) + \lambda(\delta t_1) \Lambda_m(\vec{a}) \otimes \rho_* .
\]

(46)

By \( \Lambda_m(\vec{a}) \), we denote the orthogonal projector on the eigenstate of \( \vec{a} \cdot \vec{\sigma} \) corresponding to the eigenvalue \( m \). By \( \rho_* = 1/2 \), we mean the completely mixed state of a qubit. The post-first-measurement state is further mapped by the phase damping channel during the interval \( \delta t_2 \). Since the state \[46\] is separable, it is merely mapped to

\[
\exp(-2\gamma \delta t_1) \mathcal{E}_2(\Lambda_m(\vec{a} \otimes 1)) \otimes \mathcal{E}_2(\Lambda_{-m}(\vec{a} \otimes 1)) + \lambda(\delta t_1) \mathcal{E}_2(\Lambda_m(\vec{a} \otimes 1)) \otimes \rho_* .
\]

(47)

Here, the subscript “2” marks that this map is related to the second interval. We also used that the completely mixed state is a fixed point the phase damping channel. The pre-measurement state of the qubit \( \mathcal{B} \) is obtained by the partial trace operation:

\[
\exp(-2\gamma \delta t_1) \mathcal{E}_2(\Lambda_{-m}(\vec{a} \otimes 1)) + \lambda(\delta t_1) \rho_* = \frac{1}{2} (1 + \vec{v} \cdot \vec{\sigma}) .
\]

(48)

That is, the pre-measurement density matrix is represented by its Bloch vector \( \vec{v} = (v_x,v_y,v_z) \). In terms of the Bloch vector, one gets

\[
\vec{v} = -m \exp(-2\gamma \delta t_1) \exp(-\gamma \delta t_2) \vec{a}.
\]

(49)

The action of \( \mathcal{E}_2 \) merely shrinks horizontal components of the initial Bloch vector by the factor \( \sqrt{1 - \lambda(\delta t_2)} = \exp(-\gamma \delta t_2) \). Measuring the observable \( \vec{b} \cdot \vec{\sigma} \), the outcome \( m' \) occurs with the probability

\[
\frac{1}{2} \left( 1 + m' \vec{b} \cdot \vec{v} \right) = \frac{1 - m'm \exp(-\gamma \delta t) \cos \theta}{2} = p(B = m'|A = m) .
\]

(50)

Here, the angle between \( \vec{b} \) and \( \vec{a} \) is denoted as \( \theta \), and \( \delta t = 2\delta t_1 + \delta t_2 \). Thus, we obtain the conditional probability \( p(B = m'|A = m) \). It turns out that the final expression \[50\] is symmetric in the labels \( m \) and \( m' \). For other pairs of jointly measurable observables of interest, conditional probabilities are obtained by replacing \( \theta \) with \( \theta/3 \) in the formula \[50\].

We shall now show that the probability distributions calculated above sometimes violate the locality conditions \[87\] and \[88\]. To characterize a violation of the restriction \[88\], we introduce a characteristic quantity

\[
\mathcal{C}_q := \bar{\Delta}_q(A,B) - \bar{\Delta}_q(A,B') - \bar{\Delta}_q(B',A') - \bar{\Delta}_q(A',B) .
\]

(51)

Strictly positive values of \[51\] will reveal violations of the \( q \)-metric inequality \[88\] in the presence of decoherence. A possibility to detect such violations essentially depends on the entropic parameter \( q \). Note also that a strength of
violations depends on values of the parameter $\gamma \delta t$. This parameter characterizes the influence of an environment in the model considered. To study the question, we also put the ratio

$$\kappa := \frac{\gamma \delta t}{\theta/3}. \quad (52)$$

This quantity linearly increases with $\gamma$ as well as with $\delta t$. The characteristic quantity (51) is some function $C_q(\theta, \kappa)$ of two variables. We would like to see a trade-off between $\kappa$ and $q$. To do so, we define the quantity

$$S_q(\kappa) := \sup_{\theta} C_q(\theta, \kappa) \quad (53)$$

Such an approach is meaningful, since in real experiments we want to maximize a possible violation to be tested. It is very useful that the range of positivity of (53) essentially depends on $q$. To be more precise, we further introduce the bound

$$\kappa_s(q) = \sup\{\kappa : \kappa \geq 0, \; S_q(\kappa) > 0\} \quad (54).$$

The first fact is that the strength of violations is essentially depends on the parameter $q \geq 1$. To consider a behavior of the range of violations, we focus on $\kappa_s(q)$. It turns out that $\kappa_s(q)$ increases with $q$. The dependence of $S_q(\kappa)$ on $\kappa$ for several values of $q$ is shown on Fig. 1. For the first time, both the strength and the range of positivity are increased with $q$. Indeed, the curves with $q > 1$ all lie over the curve for $q = 1$. For sufficiently large $q$, however, the strength of positivity becomes reducing. Nevertheless, the least point $\kappa_s(q)$ still slowly increases with growing $q$. Note that similar conclusions were found in Ref. [26]. The authors of this paper modeled a noise by adding a completely mixed term to the noise-free density matrix of qutrit pair. We developed more detailed approach, in which the environmental influence is taken into account through the phase damping channel applied to each of the particles. Thus, the $q$-entropic approach can be used in analyzing data of Bell-type experiments in dephasing environments.

IV. ON METRIC LEGGETT–GARG INEQUALITIES UNDER DECOHERENCE

In this section, we will deal with information-theoretic inequalities of the Leggett–Garg type. These inequalities are based on the following two concepts [10]. First, we assume that physical properties of a macroscopic object preexist irrespectively to the act of observation. Second, measurements are non-invasive in the sense that the measurement of an observable at any instant of time does not alert its subsequent evolution. Using the $q$-metrics (17) and (19), we
will apply the triangle principle to the described situation. Following the ideas of Ref. [28], we consider the two-level system with the self Hamiltonian

\[ H = -\omega S_z = -\frac{\hbar \omega}{2} \sigma_z. \]  

(55)

Its eigenstates are the ground state \(|0\rangle\) with energy \(-\hbar \omega/2\) and the excited state \(|1\rangle\) with energy \(+\hbar \omega/2\). These states are explicitly written as \([39]\). In studies of restrictions of the Leggett–Garg type, the Heisenberg picture is more convenient. The operator of unitary evolution is

\[ U(t) = \exp(-i\hbar^{-1}tH) = \exp\left(+i(\omega t/2)\sigma_z\right). \]

(56)

As every, one refers to measured results of some spin component as \(p\)-values, we have \(0 \leq p \leq 1\). Let us study information-theoretic inequalities of the Leggett–Garg type for the \(x\)-component of the spin. Assuming a validity of the macrorealistic approach, we will deal with the quantity \(S_x(t)\). Our description of \(S_x(t)\) should be carried out in line with the macroscopic realism per se and the non-invasive measurability. Let \(\tau, \tau', \tau''\) be three instants of the time. Denoting \(X = S_x(\tau), X' = S_x(\tau'), X'' = S_x(\tau'')\), for \(q \geq 1\) we write the conditions

\[ \Delta_q(X, X'') \leq \Delta_q(X, X') + \Delta_q(X', X''), \]

\[ \tilde{\Delta}_q(X, X'') \leq \tilde{\Delta}_q(X, X') + \tilde{\Delta}_q(X', X''). \]

(57)

(58)

These formulas give a formulation of the Leggett–Garg inequalities in terms of the metrics \([17]\) and \([19]\), respectively. The inequalities \([57]\) and \([58]\) are sometimes violated by probability distributions calculated in quantum-mechanical way. We further assume that the initial state is chosen to be completely mixed \([28]\), i.e., \(\rho_0 = \rho_s\). In the basis \(\{|0\rangle, |1\rangle\}\), eigenstates of the operator \(S_x = (\hbar/2)\sigma_x\) are written as

\[ |x\pm\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ \pm 1 \end{array} \right). \]

(59)

Using the Heisenberg picture, we describe the evolution of the \(x\)-component of the spin by the operator

\[ U(t) \dagger S_x U(t). \]

(60)

The aim is to obtain the corresponding conditional probabilities. They will show a violation of the inequalities \([57]\) and \([58]\) with a concrete example of spin-1/2 particle.

If the system is not altered by the environment, then its initial state \(\rho_0 = \rho_s\) remains unchanged up to measurement. When we measure the \(x\)-component at the moment \(t = \tau\), the outcome \(m = \pm 1\) occurs with probability

\[ p(m) = \text{Tr}(\Pi_m(\tau)\rho_0), \]

(61)

where the corresponding projector

\[ \Pi_m(\tau) = U(\tau) \dagger |x_m\rangle\langle x_m| U(\tau). \]

(62)

The post-measurement state is described by the density matrix \(p(m)^{-1}\Pi_m(\tau) \rho_0 \Pi_m(\tau)\). For the initial state \(\rho_0 = \mathbb{1}/2\), we have \(p(m) = 1/2\) and the post-measurement state \(\Pi_m(\tau)\). Then the conditional probability of obtaining the outcome \(m'\) at the next time \(t = \tau'\) is equal to

\[ p(m'|m) = \text{Tr}(\Pi_{m'}(\tau')\Pi_m(\tau)) = \langle x_{m'} | U(\tau') \dagger U(\tau) | x_m \rangle^2. \]

(63)

The right-hand side of \([63]\) is immediately connected with elements of the corresponding rotation matrix. Such matrices are well studied \([50]\). In the case considered, we obtain the probabilities

\[ p(m'|m) = \frac{1 + m'm \cos \omega(\tau' - \tau)}{2}, \]

(64)

where \(m, m' = \pm 1\). As was shown in Refs. \([27, 28]\), conditional probabilities of the form \([64]\) can violate entropic inequalities of the Leggett–Garg type. In real experiments, however, quantum systems are inevitably exposed to noise. We shall theoretically study possibilities to test a violation of the macrorealistic picture in the presence of decoherence.
Let us consider the situation, in which our qubit is exposed to the influence of environment. In this case, the transformation (60) implies the use of the interaction picture instead of the Heisenberg one. For all operators of interest, we will therefore consider a transformation of the form

$$X \mapsto U(t) \dagger X U(t).$$  

(65)

The density matrix denoted by $$\rho_i$$ is now related to the interaction picture and changed during the evolution. These changes fully relate to the environmental influence. Like the analysis of the previous section, we can assume that the density matrix is mapped by some quantum channel. In terms of the Bloch vector, we represent changes fully relate to the environmental influence. Like the analysis of the previous section, we can assume that the density matrix denoted by $$\rho$$ is now related to the interaction picture and changed during the evolution. These changes fully relate to the environmental influence.

As above, the dynamical variable is represented by the operator (60). When the system evolution is described by the Kossakowski–Lindblad form (52, 53), the transformation (65) will imply replacing the Schrödinger picture by the interaction picture. We will consider two important models of decohering processes described by the phase damping and depolarizing channels.

Let $$\delta \tau$$ be the time interval between two successive measurements of the $$x$$-component of the spin. We first suppose that the Bloch vector is mapped as

$$(w_x, w_y, w_z) \mapsto (\sqrt{1 - \lambda w_x}, \sqrt{1 - \lambda w_y}, w_z),$$  

(68)

where $$\lambda(\delta \tau) = 1 - \exp(-2\gamma \delta \tau)$$. This discrete transformation of the Bloch vector corresponds to the phase damping channel. To put the transformation (68) into the form (67), we introduce the operator

$$N := |1\rangle\langle 1|.$$  

(69)

It shows the number of excitations, since $$N|0\rangle = 0|0\rangle$$ and $$N|1\rangle = 1|1\rangle$$. The qubit in dephasing environment can be described by the quantum master equation

$$\frac{d}{dt} \rho_S(t) = -\frac{i}{\hbar}[H, \rho_S] + \gamma (2N\rho_S N - N\rho_S - \rho_S N).$$  

(70)

Here, we take into account that the operator (69) is Hermitian and projective, i.e., $$N^2 = N$$. We also note that the operator (60) commutes with $$\sigma_z$$ and is herewith invariant under the transformation (65). For a convenience in consequent expressions, the parameter $$\gamma$$ is rescaled as well. In the interaction picture, the corresponding density matrix

$$\rho_i(t) = U(t) \dagger \rho_S(t) U(t)$$  

(71)

is changed according to the equation

$$\frac{d}{dt} \rho_i(t) = \gamma (2N\rho_i N - N\rho_i - \rho_i N).$$  

(72)

Substituting (66) to (72) finally results in $$\dot{w}_x = -\gamma w_x$$, $$\dot{w}_y = -\gamma w_y$$, and $$\dot{w}_z = 0$$. So, after the time $$t$$ the initial horizontal components are multiplied by $$\exp(-\gamma t)$$; the $$z$$-component remains constant. In other words, off-diagonal elements of the density matrix (66) are exponentially decayed. By the phase damping channel, the Bloch ball is turned into an ellipsoid touching the Bloch sphere at the north and south poles.

We shall now recalculate the conditional probabilities (64) in the presence of dephasing environment. During the time interval from the first to the second measurements, components of the Bloch vector of any post-first-measurement state is changed as follows. First, a unitary transformation with the generator $$H$$ rotates the Bloch vector around the
z-axis. Second, the phase damping rescales horizontal components by the factor $\exp(-\gamma \delta \tau)$, where $\delta \tau = \tau' - \tau$. More precisely, in the interaction picture we write

$$p(m'|m) = \text{Tr}(\Pi_m(\tau') \rho_{lm}(\tau')) .$$

(73)

Here, the density matrix $\rho_{lm}(\tau')$ is obtained from the post-first-measurement state $\Pi_m(\tau)$ according to the phase damping with the factor $\exp(-\gamma \delta \tau)$. It follows from (62) that

$$\rho_{lm}(\tau') = \frac{1}{2} \left( \mathbb{1} + \exp(-\gamma \delta \tau) m U(\tau) \sigma_z U(\tau) \right) .$$

(74)

Similarly to (64), we then get the final expression

$$p(m'|m) = \frac{1 + m'm \exp(-\gamma \delta \tau) \cos \omega \delta \tau}{2} .$$

(75)

Thus, the influence of dephasing environment merely results in exponential decay of non-trivial terms of conditional probabilities.

As above, we assume that the initial qubit state is completely mixed. The completely mixed state is a fixed point for a unitary evolution as well as for the phase damping channel. Indeed, the phase damping channel is unital. The role of unitarity against unitarity in the context of quantum fluctuation theorems was recently revealed [53–57]. Unitary channels with controlled amount of noise were used in experimental studies of the non-local and contextual properties of biphotons [48]. Following Refs. [27, 28], we consider three measurements in equidistant time intervals. Measuring channels with controlled amount of noise were used in experimental studies of the non-local and contextual properties of biphotons [48]. Following Refs. [27, 28], we consider three measurements in equidistant time intervals. Measuring

$\tilde{C}_q := \tilde{\Delta}_q(X, X'') - \tilde{\Delta}_q(X, X') - \tilde{\Delta}_q(X', X'') .$$

(76)

In (76), the distance $\tilde{\Delta}_q(X, X')$ is found from the probabilities $p(m, m')$. The distances $\tilde{\Delta}_q(X', X'')$ and $\tilde{\Delta}_q(X, X'')$ are obtained in the same manner. Here, the conditional probabilities $p(m, m')$ of the outcomes $m$ at $t = \tau$ and $m'$ at $t = \tau'$. We also note that the expression (75) is symmetric with respect to the labels $m$ and $m'$. These points allow to evaluate the corresponding distance. Instead of (51), the characteristic quantity is now expressed as

$$C_q := \gamma \omega .$$

(77)

In the notation of (77), the parameter $\gamma$ is taken to be a half of the relaxation rate. Of course, the quantity (76) also depends on the entropic parameter $q$. It is convenient to put an auxiliary variable $\theta = \omega \delta \tau$. This substitution will allow us to exploit a similarity between the conditional probabilities (51) and (76). The characteristic quantity (76) then becomes some function $C_q(\theta, \kappa)$ of two variables. In the previous section, the angle $\theta$ was a characteristic of the geometry of experiment. In this section, however, the treatment of $\theta$ is purely temporal. For the given frequency $\omega$, values of the variable $\theta$ can be controlled by choosing $\delta \tau$.

Focusing on a behavior with respect to $\kappa$, we will again take the optimization of $C_q(\theta, \kappa)$ over $\theta$ for the fixed $\kappa$ and $q$. Formally, the function $S_q(\kappa)$ is again defined by (52). The only change is that the term $C_q$ is defined by (76) instead of (51). Strictly positive values of the quantity (53) will reveal a violation of the Leggett–Garg restrictions in the presence of dephasing environment. Additional ways of analysis of experimental data are provided by a possibility to vary the entropic parameter $q$. As was shown, the $q$-entropic approach can allow to reduce an amount of required detection efficiency [21, 27]. We shall now motivate that a possibility to vary $q$ is also significant from the viewpoint of analyzing data of experiments in the presence of decoherence.

It is instructive to discuss the quantity $\kappa_\epsilon(q)$ introduced by (51) as well. It is important that the range $[0; \kappa_\epsilon(q)]$ essentially depends on $q \geq 1$. On Fig. 2 we have shown $S_q(\kappa)$ versus $\kappa$ for several values of $q$, including the standard case $q = 1$. With growing $q$, both the strength and the range of positivity are firstly increased. In effect, the curve with $q > 1$ all go over the curve for $q = 1$. With growing $q > 1$, the point $\kappa_\epsilon(q)$ also increases. For sufficiently large $q$, however, the strength of positivity becomes reducing. Nevertheless, the least point $\kappa_\epsilon(q)$ is still slowly increasing with growth of $q$. In principle, we can actually restrict a consideration to values of $q$ around the point $q = 2.0$. 

$兹\text{ briefly, we will again have outcomes } m = \pm 1 \text{ with the probability } p(m) = 1/2. \text{ The latter is conditioned by the choice of the initial state. Taking } p(m) \text{ and } p(m'|m), \text{ we obtain the joint probabilities } p(m, m') \text{ of the outcomes } m \text{ at } t = \tau \text{ and } m' \text{ at } t = \tau'. \text{ We also note that the expression (75) is symmetric with respect to the labels } m \text{ and } m'. \text{ These points allow to evaluate the corresponding distance. Instead of (51), the characteristic quantity is now expressed as }$

$$C_q := \tilde{\Delta}_q(X, X'') - \tilde{\Delta}_q(X, X') - \tilde{\Delta}_q(X', X'') .$$

(76)

In (76), the distance $\tilde{\Delta}_q(X, X')$ is found from the probabilities $p(m, m')$. The distances $\tilde{\Delta}_q(X', X'')$ and $\tilde{\Delta}_q(X, X'')$ are obtained in the same manner. Here, the conditional probabilities $p(m, m')$ are expressed like (75), but with the interval $2 \delta \tau$ instead of $\delta \tau$.

It is natural that an influence of the phase damping process is dependent on the ratio of its rate to the excitation frequency. To study this question, we introduce an analog of (52) written as

$$\kappa := \frac{\gamma}{\omega} .$$

(77)

In the notation of (77), the parameter $\gamma$ is taken to be a half of the relaxation rate. Of course, the quantity (76) also depends on the entropic parameter $q$. It is convenient to put an auxiliary variable $\theta = \omega \delta \tau$. This substitution will allow us to exploit a similarity between the conditional probabilities (51) and (76). The characteristic quantity (76) then becomes some function $C_q(\theta, \kappa)$ of two variables. In the previous section, the angle $\theta$ was a characteristic of the geometry of experiment. In this section, however, the treatment of $\theta$ is purely temporal. For the given frequency $\omega$, values of the variable $\theta$ can be controlled by choosing $\delta \tau$.

Focusing on a behavior with respect to $\kappa$, we will again take the optimization of $C_q(\theta, \kappa)$ over $\theta$ for the fixed $\kappa$ and $q$. Formally, the function $S_q(\kappa)$ is again defined by (52). The only change is that the term $C_q$ is defined by (76) instead of (51). Strictly positive values of the quantity (53) will reveal a violation of the Leggett–Garg restrictions in the presence of dephasing environment. Additional ways of analysis of experimental data are provided by a possibility to vary the entropic parameter $q$. As was shown, the $q$-entropic approach can allow to reduce an amount of required detection efficiency [21, 27]. We shall now motivate that a possibility to vary $q$ is also significant from the viewpoint of analyzing data of experiments in the presence of decoherence.

It is instructive to discuss the quantity $\kappa_\epsilon(q)$ introduced by (51) as well. It is important that the range $[0; \kappa_\epsilon(q)]$ essentially depends on $q \geq 1$. On Fig. 2 we have shown $S_q(\kappa)$ versus $\kappa$ for several values of $q$, including the standard case $q = 1$. With growing $q$, both the strength and the range of positivity are firstly increased. In effect, the curve with $q > 1$ all go over the curve for $q = 1$. With growing $q > 1$, the point $\kappa_\epsilon(q)$ also increases. For sufficiently large $q$, however, the strength of positivity becomes reducing. Nevertheless, the least point $\kappa_\epsilon(q)$ is still slowly increasing with growth of $q$. In principle, we can actually restrict a consideration to values of $q$ around the point $q = 2.0$. 

$兹\text{ briefly, we will again have outcomes } m = \pm 1 \text{ with the probability } p(m) = 1/2. \text{ The latter is conditioned by the choice of the initial state. Taking } p(m) \text{ and } p(m'|m), \text{ we obtain the joint probabilities } p(m, m') \text{ of the outcomes } m \text{ at } t = \tau \text{ and } m' \text{ at } t = \tau'. \text{ We also note that the expression (75) is symmetric with respect to the labels } m \text{ and } m'. \text{ These points allow to evaluate the corresponding distance. Instead of (51), the characteristic quantity is now expressed as }$
Let us define two unit vectors in three dimensions, namely \( \vec{n} \) and \( \vec{r} \). The operators \( \sigma_j \) are mutually orthogonal in the sense of the Hilbert–Schmidt product. Writing \( \sigma_j \rho \sigma_j - \rho \) for \( j = x, y, z \), we finally get

\[
U(t) \hat{\sigma}_x U(t) = \vec{n} \cdot \vec{\sigma}, \quad U(t) \hat{\sigma}_y U(t) = \vec{r} \cdot \vec{\sigma}.
\]

In the interaction picture, we then obtain

\[
\frac{d}{dt} \rho(t) = \gamma \left\{ \hat{\sigma}_x \rho \hat{\sigma}_x + \hat{\sigma}_y \rho \hat{\sigma}_y + \hat{\sigma}_z \rho \hat{\sigma}_z - 3 \rho \right\}.
\]

The operators \( \vec{n} \cdot \vec{\sigma} \), \( \vec{r} \cdot \vec{\sigma} \), and \( \sigma_z \) are mutually orthogonal in the sense of the Hilbert–Schmidt product. Writing the Bloch vector of \( \rho \) in the orthonormal basis \( \{ \vec{n}, \vec{r}, \vec{e}_z \} \), we finally get

\[
\dot{w}_n = -4\gamma w_n, \quad \dot{w}_r = -4\gamma w_r, \quad \dot{w}_z = -4\gamma w_z,
\]

where the horizontal components \( w_n = \vec{n} \cdot \vec{w} \) and \( w_r = \vec{r} \cdot \vec{w} \). Thus, the Bloch vector of \( \rho \) is merely multiplied by the factor \( \exp(-4\gamma t) \). In other words, the Bloch vector is transformed as

\[
(w_x, w_y, w_z) \mapsto \left( (1 - 4\mu/3)w_x, (1 - 4\mu/3)w_y, (1 - 4\mu/3)w_z \right),
\]

where \( \mu(t) = (3/4)(1 - \exp(-4\gamma t)) \). The transformation \( [52] \) corresponds to the depolarizing channel with the four Kraus operators \( \sqrt{1 - \mu} \mathbb{1} \) and \( \sqrt{\mu/3} \hat{\sigma}_j \) for \( j = x, y, z \). This channel merely shrinks the Bloch ball \([33]\).

Let us recalculate the conditional probabilities \( [64] \) in the presence of depolarizing environment. Instead of shrinking of horizontal components of the Bloch vector, we now deal with shrinking of the vector as a whole. As was discussed right before \( [75] \), conditional probabilities of interest are determined by changes of horizontal components of the Bloch vector. It is clear, therefore, that the expression \( [75] \) is simply replaced with

\[
p(m'|m) = \frac{1 + m' m \exp(-4\gamma \Delta \tau) \cos \omega \delta \tau}{2}.
\]

The only distinction is that decaying of non-trivial terms in conditional probabilities is much faster than in \( [76] \). Thus, the above conclusions can all be applied to the case of depolarizing environment. We should only rescale the ratio \( [77] \) appropriately. We hope that the presented results could be useful in analysis of real experiments to test the Leggett–Garg inequalities.
V. CONCLUSION

We examined capabilities of conditional entropies of the Tsallis type in studying restrictions of the Bell type in the presence of decoherence. One of unifying approaches to problems of local realism and non-contextuality is provided by the so-called triangle principle. The mentioned questions are related to statistical predictions and, therefore, deal with probability distributions. Any use of the triangle principle is based on some metric in the probability space of interest. There are several realizations of the triangle principle with the use of conditional $q$-entropies of the Tsallis type. It turned out that both the known forms of the conditional $q$-entropy leads to a legitimate metric. Such metrics are naturally treated as an information-theoretic distance between random variables related to any pair of measurement. It seems that the metric based on the second conditional form was previously not considered in the literature. Using the defined metrics, we obtained the two families of non-contextuality and local realistic inequalities, which depends on one entropic parameter. We further considered information-theoretic inequalities of the Bell and Leggett–Garg types under decoherence. This question is studied in detail for the system of spin-1/2 particle in the presence of dephasing or depolarizing environment. We considered a situation, when each of two particles in the CHSH scenario is independently subjected to the phase damping channel. As calculations showed, a violation of the corresponding $q$-entropic inequality become much more robust to decoherence by adopting the parameter value. Similar conclusions were made in the taken example of $q$-entropic restrictions of the Leggett–Garg type. A dynamics of this system is modeled by a master equation written in the Kossakowski–Lindblad form. In the interaction picture, a relaxation of the corresponding density matrix is described by the phase damping or depolarizing channels. It is natural that too fast relaxation will actually prefer tests of the Leggett–Garg inequalities in real experiments. At the same time, both the strength and range of violations can be increased by adopting suitable values of the entropic parameter. For the given ratio of relaxation rate to the excitation frequency, the violation may still be testable with certain values of $q$. Of coarse, a rate of relaxation process should be low enough. Thus, the $q$-entropic formulation could be useful in analysis of recent experiments to test the contextual and non-local properties.

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