Note on Rényi vertex contributions and twist operator weights for free scalar fields

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I give an analytical proof of the conjecture of Bueno, Myers and Witczak-Krempa regarding the relation between the universal corner contributions and twist operator conformal weights for Renyi entropy in the case of free scalar fields.
1. Introduction.

Entanglement and Rényi entropies have become useful notions in the study of systems in condensed matter physics, quantum gravity and related topics. A considerable amount of work has been done to find renormalisation independent (universal) constants that appear during the calculation of these quantities. One such constant, $\sigma_n$, is associated with the existence of a corner, or vertex, singularity in the entangling surface. The subscript $n \in \mathbb{N}$ is the order of the replica covering used to compute the entropies. For the entanglement entropy the limit $n \to 1$ is required but the general case is relevant for Rényi entropies.

Casini and Huerta [1] gave an integral form for $\sigma_n$ for free scalar fields and, recently, Elvang and Hadjiantonis, [2], have, remarkably, been able to reduce the rather complicated integral to a finite sum which can be evaluated.

More recently Bueno, Myers and Witczak-Krempa, [3], have used these results to make the conjecture that $\sigma_n$ is related to the conformal weight, $h_n$, of twist operators by

$$\sigma_n = \frac{1}{\pi} \frac{h_n}{n(n - 1)}$$

for any three-dimensional conformal field theory. For free fields, they checked this numerically for a large range of $n$, using the finite sum for $\sigma_n$ and an integral form for $h_n$, derived earlier, which they also transformed into a finite trigonometric sum for integral $n$.

In this short note, which is really an addendum to a previous work, [4], I wish to show analytically that this conjecture is true, at least for free scalars.

2. The calculation

I begin by writing down Elvang and Hadjiantonis’s expression for $\sigma_n$,

$$\sigma_n = \frac{1}{24\pi n^3(n - 1)} \sum_{k=1}^{n-1} k(n - k)(n - 2k) \tan \frac{k\pi}{k},$$

obtained after considerable manipulation.

In [4] a similar form was found for $h_n$ and I now show that they are identical, taking (1) into account. The method in [4] utilised a conformal transformation to a (Euclidean) flat manifold but one with a conical singularity having a flux $\delta$, passing through the axis, This can be interpreted as a chemical potential but I will not use this language here. The angle of the cone is $2\pi/q$. 

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In the absence of a conical singularity \((q = 1)\), from the definition of \(h_n\) in terms of the energy density, it is shown in [5] that, in three dimensions,
\[
h_1(\delta) = \frac{1}{6} \left( \frac{1}{4} - \delta^2 \right) \delta \cot \pi \delta,
\]
where \(\delta = \delta - 1/2\). This expression has to be periodised for \(\delta\) outside the range \(0 < \delta \leq 1\) using \(\delta + 1 \equiv \delta\). \(\delta = 1\) gives zero flux.

Next an image construction, which follows from the periodicity of the Green function, [5], allows the weight for replica order \(n\), (and no flux), \(i.e.\ h_n\), to be written as the finite sum,
\[
h_n = \sum_{s=1}^{n} h_1 \left( \frac{s}{n} + \frac{1}{2} - \frac{1}{2} \right).
\]

All that remains is to substitute (3) into (4) and we are almost done, except for a small technical point.

First, direct substitution gives,
\[
h_n = -\frac{1}{6} \sum_{s=1}^{n} \left( \frac{1}{4} - \left( \frac{s}{n} + \frac{1}{2} \right)^2 \right) \left( \frac{s}{n} + \frac{1}{2} \right) \tan \frac{\pi (s + 1)}{n},
\]
with the periodisation still to be allowed for. This is done by noting that the only term for which \(\delta > 1/2\), \(i.e.\ \delta > 1\) is \(s = n\), when \(\delta = 1/2 + 1/n\). Subtracting 1 gives the equivalent \(\delta = 1/n - 1/2\) which corresponds to \(s = 0\) so that periodisation is effected simply by a change in the summation limits, and, actually,
\[
h_n = -\frac{1}{6} \sum_{s=0}^{n-1} \left( \frac{1}{4} - \left( \frac{s+1}{n} - \frac{1}{2} \right)^2 \right) \left( \frac{s+1}{n} - \frac{1}{2} \right) \tan \frac{\pi (s+1)}{n}
\]
\[
= -\frac{1}{6} \sum_{k=1}^{n} \left( \frac{1}{4} - \left( \frac{k}{n} - \frac{1}{2} \right)^2 \right) \left( \frac{k}{n} - \frac{1}{2} \right) \tan \frac{\pi k}{n}
\]
\[
= \frac{1}{12n^3} \sum_{k=1}^{n-1} k(n-k)(n-2k) \tan \frac{\pi k}{n},
\]
which is the same as (2) with (1) apart from a factor of 2 because I have to use complex fields.

This result can be taken as an analytical proof of the conjecture, (1), of Bueno, Myers and Witczak-Krempa, [3] as promised.
3. Higher dimensions

It is easy to find the higher dimensional weights, \( h_n \), but the corresponding vertex constants, \( \sigma_n \), are not available.

A conjecture, similar to (1), has been made by Bueno and Myers, [6]. For example, in five dimensions,
\[
\sigma_n^{(5)} = \frac{16}{9} \frac{h_n}{n-1},
\]
which, using the results in [4], yields,
\[
\sigma_n^{(5)} = \frac{1}{360\pi (n-1) n^5} \sum_{k=1}^{n-1} k(n-k)(n-2k)(n+2k)(3n-2k) \tan \frac{\pi k}{n},
\]
as my prediction for the corner contribution in 5 dimensions. Other dimensions are easily calculated.

4. Conclusion

This proof of the conjecture of [3] shows the utility, in some circumstances, of conformally transforming to the flat conical space rather than to the hyperbolic cylinder.

The considerable simplification effected by Elvang and Hadjianthonis of the integral to a sum which has another interpretation would suggest that there is a way of obtaining this form more directly.
References.

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