A geometric proof of the Periodic Averaging Theorem on Riemannian manifolds

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Abstract We present a geometric proof of the averaging theorem for perturbed dynamical systems on a Riemannian manifold, in the case where the flow of the unperturbed vector field is periodic and the $S^1$-action associated to this vector field is not necessarily trivial. We generalize the averaging procedure [2,3] defining a global averaging method based on a free coordinate approach which allow us to formulate our results on any open domain with compact closure.

Keywords Averaging method · Perturbation theory · Periodic flows · Riemannian manifolds · Horizontal lifts · $S^1$-principal bundle

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1 Introduction

The well known averaging method [2,3,15,17,18] is one of the most important methods in perturbation theory and it is based on the idea of splitting the motion of a perturbed system into a slow evolution and rapid oscillations.

Geometrically, the averaging method arises in the context of perturbations of vertical vector fields on a fibered manifold. We consider a smooth fiber bundle $\pi : M \to B$ and a smooth, perturbed vector field $A_\varepsilon = A_0 + \varepsilon A_1$ on
$M$, where $A_0$ is tangent to every fiber. In this situation, each integral curve of $A_0$ is projected by $\pi$ onto a point on the base $B$ and each integral curve of $A_\varepsilon$ is projected into a curve on $B$, whose tangent vector field is of order $\varepsilon$ but, in general, that projected curve is not the integral curve of any vector field on the base $B$. Therefore, a noticeable displacement of the projected curve takes place over time of order $1/\varepsilon$. This situation rises the following question: Is it possible to describe the motion of the projected curve on the base, for a long period of time? The averaging method allow us to describe the motion of this projected curve by means of the integral curve of a certain vector field on the base $B$.

In many applications of the averaging method, the fiber bundles have the following properties: (i) every point of the base has a neighborhood where the fiber bundle is a direct product, and (ii) the fibers are $n$-dimensional tori. However, the only case completely studied is when the fibers are one dimensional tori (circles), the so-called one frequency systems, [3,18].

Consider the product manifold $S^1 \times \mathbb{R}^n$ together with the coordinate system $(\varphi (\text{mod } 2\pi), I)$. The one frequency system is the perturbed dynamical system on $S^1 \times \mathbb{R}^n$ given by

$$\dot{\varphi} = \omega(I) + \varepsilon f(\varphi, I),$$
$$\dot{I} = \varepsilon g(\varphi, I),$$

for $0 \leq \varepsilon \ll 1$, where $f = f(\varphi, I)$ and $g = g(\varphi, I)$ are smooth $2\pi$-periodic in $\varphi$, and $\omega : \mathbb{R}^n \to \mathbb{R}$ is called the frequency function. Notice that for $\varepsilon = 0$, that is, with no perturbation, the system (1), (2) has periodic solutions with frequency $\omega$. If we consider the canonical projection $\pi : S^1 \times \mathbb{R}^n \to \mathbb{R}^n$ over the second factor, we have a trivial fiber bundle with $S^1$ as typical fibers and $\mathbb{R}^n$ as the base. Therefore, for $\varepsilon = 0$, the unperturbed system of (1), (2) defines a vector field which is tangent to the fibers. In order to approximate the projection of the trajectories of the one frequency system (1), (2) over the base $\mathbb{R}^n$, the averaging procedure suggests to replace the slow part (2) by the averaged system

$$\dot{J} = \varepsilon G(J),$$

where

$$G(J) = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi, J) d\varphi,$$

and then compare the trajectories of (2) and (3) at the same initial condition. Remark that equation (3) has the advantage that does not depend on the coordinate $\varphi$.

The classical averaging theorem asserts that if the frequency function satisfies the non degeneracy condition: $\omega(I) > c^{-1} > 0$ for a certain constant $c$, then the solution $J(t)$ of the averaged system (3) remains close enough to the
solution \( I(t) \) of slow part [2], with \( I(0) = J(0) \), for \( \varepsilon \) small enough on the long time scale \( t \sim 1/\varepsilon \), that is, there exist constants \( c_1 > 0 \) and \( \varepsilon_0 > 0 \) such that

\[
\|I(t) - J(t)\| < c_1 \varepsilon \quad \text{if} \quad I(0) = J(0) \quad \text{and} \quad 0 \leq t \leq \frac{1}{\varepsilon},
\]

for all \( \varepsilon < \varepsilon_0 \), [2,3,15,18].

The purpose of this paper is to generalize the classical averaging theorem in the following setting: instead of the frequency system [1], [2] on \( S^1 \times \mathbb{R}^n \), we consider a smooth, perturbed vector field \( X_{\varepsilon} = X_0 + \varepsilon X_1 \) on an arbitrary manifold \( M \), where \( X_0 \) is a vector field with periodic flow, and hence, the vector field \( X_0 \) induces an \( S^1 \)-action on \( M \). Now, assume that this action is free and let \( \mathcal{O} = M/S^1 \) be the orbit space. Thus, \( \mathcal{O} \) is a smooth manifold and the natural projection \( \rho : M \to \mathcal{O} \) is a surjective submersion and hence, we have a fiber bundle \( (M, \rho, \mathcal{O}) \) having \( S^1 \) as typical fiber. Therefore, we are interested in finding some estimates for the projections of the trajectories of the perturbed vector field \( X_{\varepsilon} \) over \( \mathcal{O} \) with respect to the trajectories of a suitable vector field on the base.

If we assume that the fiber bundle \( \rho : M \to \mathcal{O} \) is locally trivial, there exists a local coordinate system where the perturbed vector field \( X_{\varepsilon} \) takes the form [1], [2]. We can try to apply the classical averaging theorem in this setting. However, this approach has a major drawback: it could happen that the integral curves of the perturbed vector field should not be completely contained in the local coordinate system or, else, it could happen that they pass through it only for a short period of time. Since the averaging theorem applies only on the coordinate neighborhood where the perturbed vector field \( X_{\varepsilon} \) takes the form [1], [2], we do not know what occurs outside of this neighborhood. Therefore, we are not able to obtain an approximation of the projected trajectories of \( X_{\varepsilon} \) for a long period of time. This drawback does not occur if the coordinates are well defined on the whole manifold \( M \), but the existence of global action-angle coordinates is a very restrictive situation [6,8,15,16].

In this paper, we prove the averaging theorem on arbitrary manifolds without the assumption of the existence of special coordinate systems like action-angle variables. Actually, we follow a coordinate free approach. We study perturbations of vector fields with periodic flow on arbitrary open domains of a Riemannian manifold \( M \) when the \( S^1 \)-action associated to the unperturbed vector field is not necessarily trivial. Here, we define a global averaging method using the properties of periodic flows which allow us to formulate our results in a global setting.

The proof of the classical averaging theorem [3,18] is based on the following arguments: a near identity transformation whose infinitesimal generator is a solution of a homological type equation, the triangle inequality and some technical estimations (for example, Gronwall type estimation are presented in [18]). In our setting, the proof of the theorem also follows from the same arguments; however, we face with some difficulties which are not present in the classical formulation of the averaging theorem. The main of these difficulties is to get an inequality of Gronwall’s type which help us find an estimation
between the distance of the perturbed trajectory and the averaged trajectory. In the classical setting, this estimation is obtained due to the existence of global minimizing geodesic on $S_1 \times \mathbb{R}^n$. However, this property does not hold in general. To address this problem we use a geometric construction. The idea here is to construct a parameterized surface $\gamma : [0, 1] \times [0, L] \to M$ given by $\gamma(s, t) := \text{Fl}_{t}^{X_{\beta}}(\beta(s))$ where $\beta : [0, 1] \to M$ is a fixed curve and $\text{Fl}_{t}^{X_{\beta}}$ is the flow of a parameterized vector field $X_{\beta}$. Then, for every $t \in [0, L]$ we estimate the distance from $\gamma(0, t)$ and $\gamma(1, t)$ by using Gronwall’s lemma \cite{10, 18} and assuming that the manifold $M$ possesses a suitable Riemannian metric, so we can use such tools like covariant derivatives and horizontal lifts.

The paper is organized as follows. In Sec. 2, we introduce the averaging and integrating operator associated to a vector field with periodic flow. In Sec. 3, we state our hypotheses and main result. Then, in Sec. 4 we show how to construct a near identity transformation putting the perturbed vector field $X_\varepsilon = X_0 + \varepsilon X_1$ into its $S^1$-invariant normal form relative to the $S^1$-action induced by $X_0$. To achieve this goal, a kind of homological equation must be solved. In Sec. 5, we state a Gronwall’s type inequality on Riemannian manifolds. In Sec. 6, we define an $S^1$-invariant horizontal distribution on $TM$ using the fact that in the $S^1$ bundle $(S^1, \rho, O)$ the map $\rho$ is Riemannian submersion. We also show the basic properties of the horizontal lifts of curves and vector fields. Sec. 7 is devoted to the proof of the main theorem. Finally, in Sec. 8 we make use of the averaging theorem (Theorem \ref{thm:averaging}) in order to construct adiabatic invariants for perturbed vector fields.

## 2 Averaging operators associated to periodic flows

Let $M$ be a smooth manifold and let $X_0$ be a complete vector field on $M$ with periodic flow $\text{Fl}_{t}^{X_0}$ and frequency function $\omega : M \to \mathbb{R}, \omega > 0$, that is, for any $p \in M$

$$\text{Fl}_{t}^{X_0}(p) = \text{Fl}_{t}^{X_0}(p), \quad \forall t \in \mathbb{R}, \tag{4}$$

where $T(p) := 2\pi/\omega(p)$ is the period of the orbit of $X_0$ passing through $p$. The vector field $X_0$ induces an $S^1$-action on $M$ given by $(t, p) \mapsto \text{Fl}_{tX_0}^{X_0}(p)$, with coordinate $t \mod 2\pi$. We denote the infinitesimal generator of the $S^1$-action by $\Upsilon$, which can be computed in terms of the vector field $X_0$, and is given by

$$\Upsilon = \frac{1}{\omega}X_0. \tag{5}$$

Now, in order to settle the main result of the paper, let us recall some useful facts. A tensor field $R \in \Gamma T^s_p(M)$ is said to be $S^1$-invariant if and only if $(\text{Fl}_{t\Upsilon}^{X_0})^* R = R$. Equivalently, $\mathcal{L}_\Upsilon R = 0$, where $\mathcal{L}$ denotes the usual Lie derivative. In most cases, we will be using this definition for smooth functions (tensor fields in $\Gamma T_0^s(M)$) and smooth vector fields (tensor fields in $\Gamma T^s_0(M)$).

For any tensor field $R \in \Gamma T^s_p(M)$, we associate, with the $S^1$-action on $M$, defined by $X_0$, the following operators acting on $T^s_p(M)$:
1. The **averaging operator** which is the tensor field defined by

\[
\langle R \rangle := \frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}^t_\Upsilon)^* R.
\] (6)

Notice that \(\langle R \rangle\) is a tensor field of the same type as \(R\).

2. The **integrating operator**, which is the tensor field defined by

\[
S(R) := \frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(\text{Fl}^t_\Upsilon)^* R.
\] (7)

It is clear that \(S(R)\) is a tensor field of the same type as \(R\).

For every \(R \in \Gamma T^*_s(M)\), the operators defined in (6) and (7) have the following properties [4,13]:

1. \(\mathcal{L}_\Upsilon \langle R \rangle = 0\).
2. \(R\) is \(S^1\)-invariant if and only if \(\langle R \rangle = R\).
3. \(\langle \mathcal{L}_\Upsilon R \rangle = \mathcal{L}_\Upsilon \langle R \rangle\).
4. For an \(S^1\)-invariant function \(g \in C^\infty(M)\) we have \(\langle gR \rangle = g\langle R \rangle\) and \(\mathcal{S}(gR) = g\mathcal{S}(R)\).
5. \(\mathcal{S}(\langle R \rangle) = \langle \mathcal{S}(R) \rangle = 0\)
6. \(\mathcal{L}_\Upsilon \circ \mathcal{S}(R) = R - \langle R \rangle\).

A key property relating the averaging operator and the integrating operator is given by the following result.

**Proposition 1** For every \(R \in \mathfrak{X}(M)\), the vector field \(Z = \frac{1}{\omega}S(R) + \frac{1}{\omega^2}\mathcal{S}^2(\mathcal{L}_R\omega)X_0\) satisfies the homological type equation

\[
\mathcal{L}_{X_0}Z = R - \langle R \rangle.
\] (8)

The proof of this proposition follows from properties 1-6, above (see [4]).

3 Main result: The periodic averaging theorem

Let \(M\) be a connected manifold and let \(X_0\) be a vector field on \(M\). We assume that the vector field \(X_0\) satisfies the following symmetry hypothesis:

\((SH)\) \(X_0\) is a vector field on \(M\) with periodic flow and the action of the circle \(S^1 = \mathbb{R}/2\pi \mathbb{Z}\) on \(M\) associated with the infinitesimal generator \(\Upsilon = \frac{1}{\omega}X_0\) is free.

Let \(\mathcal{O} = M/S^1\) be the orbit space of the \(S^1\)-action and denote by \(\rho : M \to \mathcal{O}\) the natural projection. It follows from well-known properties of free actions of compact Lie groups [11,12] that there exists a unique manifold structure on \(\mathcal{O}\) such that \(\rho\) is a smooth surjective submersion (a fiber bundle). Moreover, \(\rho\) is a principal \(S^1\)-bundle over \(\mathcal{O}\). For each \(S^1\)-invariant vector field \(Y\) on \(M\)
there exists a unique vector field $Y_O$ on the orbit space $O$ which is $\rho$-related with $Y$, that is,

$$\rho_* Y = Y_O.$$ 

It is clear that $Y_O \equiv 0$.

Now, let us choose an $S^1$-invariant metric $g$ on $M$. Such a Riemannian metric always exists and can be obtained from an arbitrary Riemannian metric on $M$ by applying the averaging procedure, [4]. Since the $S^1$-action is free and proper, there exists a unique Riemannian metric on $O$, denoted by $g^O$, such that the projection $\rho : M \to O$ is a Riemannian submersion [9,12]. We also denote by $\text{dist}^O : O \times O \to \mathbb{R}$ the corresponding distance function.

In order to formulate the main result of this paper, we start with a perturbed vector field on $M$, which is close to the smooth vector field $X_0$ in the following sense:

$$X_\varepsilon = X_0 + \varepsilon X_1, \quad \varepsilon \ll 1.$$ (9)

As usual, $X_1$ is also a smooth vector fields on $M$, known as the perturbed part of $X_\varepsilon$. The averaging theorem establishes that the projection over the orbit space $O$ of a trajectory of $X_\varepsilon$ (9) can be approximated by a trajectory of the vector field $\langle X_1 \rangle^O$, where $\langle X_1 \rangle^O = \rho_* \langle X_1 \rangle$.

**Theorem 1 (Periodic Averaging Theorem on Manifolds)** Let $M$ be a connected manifold. Assume that the vector field $X_0$ satisfies the symmetry hypothesis (SH) above. Fix $m^0 \in M$ and suppose that there exists a constant $L > 0$ such that the trajectory of $\langle X_1 \rangle^O$ through $z^0 = \rho(m^0) \in O$ is defined for all $t \in [0, L]$ and remains in an open domain $D_0$ with compact closure. Then, there exist constants $\varepsilon_0 > 0$, $L \geq L_0 > 0$ and $c > 0$ such that

$$\text{dist}^O \left( \rho \circ \text{Fl}^t_{X_\varepsilon}(m^0), \text{Fl}^t_{\langle X_1 \rangle^O}(z^0) \right) \leq c \varepsilon,$$ (10)

for all $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, L_0/\varepsilon]$.

If $\varepsilon \to 0$, we have $\rho \circ \text{Fl}^t_{X_\varepsilon}(m^0) \to \rho(m^0)$ and $\text{Fl}^t_{\langle X_1 \rangle^O}(z^0) = \text{Fl}^t_{\langle X_1 \rangle^O}(z^0) \to z^0$. Therefore, the left hand side of (10) is identically zero and the conclusion of the Theorem remains true for $\varepsilon = 0$ which corresponds to the unperturbed case.

In fact, we can calculate constant $c$ in (10) and its value depends only on the Riemannian structure of $M$, the $S^1$-action induced by $X_0$ and the choice of the open domain $D_0$ (see Corollary 1 and Remark 1).

**Example 1 (One-frequency systems)** Let us consider $M = S^1 \times \mathbb{R}^k$ with the usual angular coordinate $\varphi \mod 2\pi$ and $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$. Let $X_\varepsilon = X_0 + \varepsilon X_1$ be a perturbed vector field where

$$X_0 = \omega(x) \frac{\partial}{\partial \varphi}, \quad X_1 = f(x, \varphi) \frac{\partial}{\partial \varphi} + g_i(x, \varphi) \frac{\partial}{\partial x_i}.$$
with frequency function $\omega > 0$ and $f = f(x, \varphi)$, $g_i = f(x, \varphi)$, 2\pi-periodic functions in $\varphi$. The vector field $\partial/\partial\varphi$ is the infinitesimal generator of the $S^1$-action induced by $X_0$. The orbit space $\mathcal{O}$ can be identified with $\mathbb{R}^k$, $\mathcal{O} \cong \mathbb{R}^k$.

The average of $X_1$ is given by

$$\langle X_1 \rangle = \langle f \rangle(x) \frac{\partial}{\partial\varphi} + \langle g_i \rangle(x) \frac{\partial}{\partial x_i},$$

and its reduced vector field is $\langle X_1 \rangle_{\mathcal{O}} = \langle g_i \rangle(x) \frac{\partial}{\partial x_i}$. If a trajectory of $\langle X_1 \rangle_{\mathcal{O}}$ remains, over time $t = T/\varepsilon$, on an open domain $A \subset \mathbb{R}^k$ having compact closure, then Theorem 1 reduces to the classical result of averaging for a single-frequency system $[2,3,15,18]$.

4 $S^1$-invariant normalization of perturbed vector fields

In the proof of the classical averaging theorem $[2,3,15]$, an important role is played by coordinate changes taking the original perturbed vector field $X_\varepsilon$ into its $S^1$-invariant normal form of first order, (see Definition 1). More precisely, such a change of coordinates allow us to replace the vector field $X_1$ by the $S^1$-invariant vector field $\langle X_1 \rangle$ together with a small perturbation (of order $\varepsilon^2$). The change of coordinates that needs to be performed belongs to the class of near identity transformations. A precise definition and a procedure to construct such a kind of transformations is given in what follows.

For a nonempty open domain $N \subset M$ and a constant $\delta > 0$ a near identity transformation is a smooth mapping $\Phi : (-\delta, \delta) \times N \to M$ such that for every $\varepsilon \in (-\delta, \delta)$, the map defined by $\Phi_\varepsilon(x) := \Phi(\varepsilon, x)$ is a diffeomorphism onto its image and $\Phi_0 \equiv \text{id}$.

Near identity transformations have the following important property: for a perturbed vector field $X_\varepsilon = X_0 + \varepsilon X_1$, the pullback $\Phi_\varepsilon^* X_\varepsilon$ is again a perturbed vector field whose unperturbed part is $X_0$,

$$\Phi_\varepsilon^* X_\varepsilon \bigg|_{\varepsilon=0} = X_0.$$

By the Flow Box Theorem $[11]$, for any open domain $N \subset M$ with compact closure and for an arbitrary smooth vector field $Z$ on $M$ there exists $\delta > 0$ such that the mapping $\Phi : (-\delta, \delta) \times N \to M$ given by

$$\Phi_\varepsilon := \text{Fl}_Z^t \bigg|_{t=\varepsilon}$$

is a near identity transformation.

Now we deal with the problem of $S^1$-invariant normal forms of perturbed vector fields.

Definition 1 Let $X_\varepsilon = X_0 + \varepsilon X_1 + O(\varepsilon^2)$ be a perturbed vector field on $M$. It is said that $X_\varepsilon$ is in $S^1$-invariant normal form of first order if $X_1$ is $S^1$-invariant. Moreover, $X_\varepsilon$ admits a global $S^1$-normalization of first order if for any open domain $N$, with compact closure, there exists $\delta > 0$ and a near
identity transformation \( \Phi_\varepsilon : (-\delta, \delta) \times N \to M \) which brings \( X_\varepsilon \) to a \( S^1 \)-invariant normal form of first order. In this case, \( \Phi_\varepsilon \) is called a normalization transformation.

If we take the vector field
\[
Z = \frac{1}{\omega}S(X_1) + \frac{1}{\omega^3}S^2(\mathcal{L}X_1, \omega)X_0,
\]
and consider the near identity transformation (11), we get the following \( S^1 \)-invariant normal form result.

**Proposition 2** Let \( X_0 \) and \( X_1 \) be vector fields on \( M \). If \( X_0 \) has periodic flow, then for any open domain \( N \subset M \), having compact closure, there exists a constant \( \delta > 0 \) such that the near identity transformation (11) given by the flow of vector field \( Z \) in (12), brings \( X_\varepsilon = X_0 + \varepsilon X_1 \) into the \( S^1 \)-invariant normal form
\[
\Phi_\varepsilon^* X_\varepsilon = X_0 + \varepsilon \langle X_1 \rangle + \varepsilon^2 R_\varepsilon,
\]
where \( R = R_\varepsilon \) is a vector field on \( M \), smoothly depending on \( \varepsilon \).

**Proof** By using the non-canonical Lie transform method and Deprit’s diagram [7], we can compute the expansion of \( (\Phi_\varepsilon)^* X_\varepsilon \) up to any order in \( \varepsilon \). In particular, we have
\[
(\Phi_\varepsilon)^* X_\varepsilon = X_0 + \varepsilon (X_1 - \mathcal{L}X_0, Z) + O(\varepsilon^2),
\]
where \( Z \) is the infinitesimal generator of the mapping \( \Phi_\varepsilon \), as in (11). It follows that \( \Phi_\varepsilon \) is a normalization transformation if and only if there exist vector fields \( W \) and \( Z \) satisfying the homological type equation
\[
\mathcal{L}X_0, Z = X_1 - W, \tag{14}
\]
where \( W \) is \( S_1 \)-invariant. Since \( X_0 \) has periodic flow, Proposition 1 implies that vector fields \( W = \langle X_1 \rangle \) and \( Z = \frac{1}{\omega}S(X_1) + \frac{1}{\omega^3}S^2(\mathcal{L}X_1, \omega)X_0 \) are solutions of equation (14).

5 Gronwall’s type estimations on Riemannian manifolds

Suppose that \( \varphi : I \subset \mathbb{R} \to \mathbb{R} \) is a continuous function such that for \( t_0 \leq t \leq t_0 + L \), we have
\[
\varphi(t) \leq \delta_2 (t - t_0) + \delta_1 \int_{t_0}^{t} \varphi(\tau) d\tau + \delta_3,
\]
with constants \( \delta_1 > 0 \) and \( \delta_2 \geq 0, \delta_3 \geq 0 \). The well known Gronwall’s lemma asserts that
\[
\varphi(t) \leq \left( \frac{\delta_2}{\delta_1} + \delta_3 \right) e^{\delta_1 (t - t_0)} - \frac{\delta_2}{\delta_1}, \tag{15}
\]
holds for \( t_0 \leq t \leq t_0 + L \). By using this fundamental inequality, it is possible to get some estimates for the time evolution of the distance between points of trajectories of two vector fields on a general Riemannian manifold.

Let \((M, g)\) be a connected Riemannian manifold. Denote by \(\text{dist} : M \times M \to \mathbb{R}\) the distance function induced by the Riemannian metric \(g\). For a submanifold \(N \subset M\), \(\text{dist}_N\) will denote the restriction of the distance function to \(N\).

Let \(\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)\) be the Levi-Civita connection associated to \((M, g)\). From the basic properties of the Levi-Civita connection, \((\nabla_YY)_{(m)}\) depends only on the value of \(Y\) at \(m\). Therefore, for each \(X \in \mathfrak{X}(M)\) and \(m \in M\) we have a linear map \((\nabla_X)_{(m)} : (T_m M, g_m) \to (T_m M, g_m)\), \(Y(m) \mapsto \nabla_{Y(m)} X\).

In what follows, \(\nabla_X\) will denote the covariant derivative of the vector field \(X\) and \(\|\nabla_X\|_m\) the operator norm defined by

\[
\|\nabla_X\|_m := \sup\{ \|\nabla_{Y(m)} X\|_m : Y \in \mathfrak{X}(M) \text{ and } \|Y(m)\|_m = 1 \}
\]

where \(\|Y\|_m := g_m(Y, Y)\). Given a diffeomorphism \(\varphi : M \to M\), the push-forward \(\varphi_* \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)\) of the covariant derivative \(\nabla\) is defined by

\[
(\varphi_* \nabla)_Y \cdot \varphi_* X = \varphi_* (\nabla_Y X)
\]

for all \(X, Y \in \mathfrak{X}(M)\). Hence, if \(\varphi\) is an isometry, then \(\varphi\) preserves the connection \(\nabla\), that is, \(\varphi_* \nabla = \nabla\).

We prove the following technical fact.

**Lemma 1** Let \(\varphi\) be an isometry on the Riemannian manifold \((M, g)\) and let \(X \in \mathfrak{X}(M)\) be a vector field. Then, \(\|\varphi_* X\|_m = \|X\|_{\varphi^{-1}(m)}\) and for the vector bundle morphisms \(\nabla(\varphi_* X)\) and \(\nabla X\) we have

\[
\|\nabla(\varphi_* X)\|_m = \|\nabla X\|_{\varphi^{-1}(m)}.
\]

**Proof** Since \(\varphi\) is an isometry, we have that \(\nabla_{\varphi_* Y} \varphi_* X = \varphi_* (\nabla_Y X)\) and hence

\[
\nabla_v(\varphi_* X) = (d_{\varphi^{-1}(m)} \varphi)(\nabla_{d_m \varphi^{-1}(v)} X),
\]

for every \(v \in T_{m} M\). It follows that

\[
\|\nabla(\varphi_* X)(v)\|_m = \|\nabla X(d_m \varphi^{-1}(v))\|_{\varphi^{-1}(m)}.
\]

This last equality together with \(\|d_m \varphi^{-1}(v)\|_{\varphi^{-1}(m)} = \|v\|_m\) implies \(16\).

Now, we will construct a parameterized surface \((t, s) \mapsto \gamma(t, s)\) on a manifold \(M\), which is generated by trajectories of a parameter dependent family of vector fields, as follows. Let \(X_s\) be a one-parameter vector field on \(M\), smoothly depending on the parameter \(s \in [0, 1]\). \(\text{Fl}_{X_s}\) denotes the flow of \(X_s\) for each \(s\). Let \(\beta : [0, 1] \to M\) be a parameterized smooth curve on \(M\). Then,
one can fix $L > 0$ such that for each $s \in [0, 1]$, the trajectory $t \mapsto \text{Fl}_{X_s}(\beta(s))$ is defined for all $t \in [0, L]$. The resulting parameterized surface is given by

$$\gamma : [0, L] \times [0, 1] \to M$$

$$\gamma(t, s) := \text{Fl}_{X_s}(\beta(s)).$$

(17)

Notice that $\text{Fl}_{X_s}(\beta(s)) = \beta(s)$, thus, for each $t$, the $s$-curve $s \mapsto \gamma_t(s) = \gamma(t, s)$ can be viewed as the time evolution of the “initial” curve $\beta(s)$ under the flow of $X_s$.

**Proposition 3** The length $L(t)$ of the $s$-curve $s \mapsto \gamma_t(s)$ on the parameterized surface $\gamma(t)$ satisfies the Gronwall’s type estimate

$$L(t) \leq \left(\frac{C_2}{C_1} + L(0)\right) e^{C_1 t} = \frac{C_2}{C_1},$$

(18)

for all $t \in [0, L]$. Here

$$C_1 = \sup_{m \in \gamma([0, T) \times [0, 1])} \|\nabla X_m\|, \quad \text{and} \quad C_2 = \sup_{t \in [0, T], s \in [0, 1]} \left\|\frac{d}{ds} X_s\right\|_{\gamma_t(s)}.$$

**Proof** It follows directly from the inequality $\left\|\frac{d}{dt}\partial \gamma/\partial \theta\right\| \leq \left\|\nabla_{\frac{d}{dt}} (\partial \gamma/\partial \theta)\right\|$ that

$$\left\|\frac{d}{ds} \gamma(t, s)\right\|_{\gamma_t(s)} \leq \left\|\frac{d}{ds} \gamma(0, s)\right\|_{\gamma_0(s)} + \int_0^t \left\|\nabla_{\frac{d}{dt}} (\partial \gamma/\partial \theta(t', s))\right\|_{\gamma_t(s)} dt'.$

Taking into account that the Levi-Civita connection is torsion free and $L(t) = \int_0^1 \left\|\frac{d}{ds} \gamma(t, s)\right\|_{\gamma_t(s)} ds$, integration in $s$ gives

$$L(t) \leq L(0) + \int_0^t \int_0^1 \left\|\nabla_{\frac{d}{dt}} \frac{d}{ds} \gamma(t, s)\right\|_{\gamma_t(s)} ds dt'.$$

(19)

Furthermore, for every $t \in [0, L]$ the vector field $\frac{d}{dt} \gamma(t, s)$ satisfies the relation

$$\nabla_{\frac{d}{dt}} (\partial \gamma/\partial \theta) = \left(\frac{d}{ds} X_s\right) (\gamma(t, s)) + (\nabla X_s)_{\gamma_t(s)} (\partial \gamma/\partial \theta),$$

and thus, we get

$$\left\|\nabla_{\frac{d}{dt}} (\partial \gamma/\partial \theta)\right\|_{\gamma_t(s)} \leq \left\|\frac{d}{ds} X_s\right\|_{\gamma_t(s)} + \left\|(\nabla X_s)_{\gamma_t(s)}\right\| \left\|\frac{\partial \gamma}{\partial \theta}\right\|_{\gamma_t(s)}.$$

Putting this inequality into (19), we get

$$L(t) \leq L(0) + C_1 \int_0^t L(t') dt' + C_2 t.$$

Now, applying the usual Gronwall’s lemma to last inequality leads to (18).
6 Riemannian submersions on the $S^1$-principal bundle $(M, \rho, \mathcal{O})$

Let $(M, g)$ be a connected Riemannian manifold and let $X_0$ be a vector field with periodic flow. Assume that the Riemannian metric $g$ is invariant with respect to the $S^1$-action induced by $X_0$, and the flow $\text{Fl}_t^Y$ is an isometry on $(M, g)$, where $Y$ is the infinitesimal generator of the $S^1$-action induced by $X_0$. If the $S^1$-action on $M$ induced by $X_0$ is free, then the triple $(M, \rho, \mathcal{O})$ is an $S^1$-principal bundle. The vertical subbundle $V := \ker d\rho$ coincides with the one dimensional distribution given by $D := \{D_m \subset T_m M | D_m = \text{Span}\{Y(m)\}\}$. In this case, we choose the horizontal subbundle as the orthogonal complement to $V$, $H = V^\perp$.

Since the Riemannian metric is $S^1$-invariant, the horizontal subbundle is also invariant with respect to the $S^1$-action, 

$$(d_m \text{Fl}_t^Y)(H_m) = H_{\text{Fl}_t^Y(m)}, \quad \forall m \in M.$$ 

Thus, we have the $S^1$-invariant orthogonal splitting $TM = H \oplus V$, and every vector field $Y$ on $M$ decomposes into its horizontal and vertical parts, as $Y = Y^\text{hor} + Y^\text{vert}$.

It is clear that the restriction of the differential $d_m \rho : T_m M \to T_{\rho(m)} \mathcal{O}$ to $H_m$ is an isomorphism. Hence, for every vector field $v \in X(M)$ there exists a unique vector field $\text{hor}(v) \in X(M)$, called the horizontal lift of $v$, which is tangent to $H$ and $d\rho \circ \text{hor}(v) = v \circ \rho$.

Let $g^\mathcal{O}$ be the unique Riemannian metric on the orbit space $\mathcal{O}$ such that the projection $\rho$ is a Riemannian submersion (see [9,12]), 

$$g_m(u_1, u_2) = g^\mathcal{O}_{\rho(m)}((d_m \rho)u_1, (d_m \rho)u_2)$$ 

for any $m \in M$ and $u_1, u_2 \in H_m$. Denote by $\text{dist}^\mathcal{O} : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$, the distance function associated to the Riemannian metric $g^\mathcal{O}$ on $\mathcal{O}$, and by $\nabla$ and $\nabla^\mathcal{O}$ the Levi-Civita connections on the Riemannian manifolds $(M, g)$ and $(\mathcal{O}, g^\mathcal{O})$, respectively.

Lemma 2 Let $\gamma : [0, 1] \to M$, $s \mapsto \gamma(s)$, be a smooth curve on $M$ and let $\alpha := \rho \circ \gamma : [0, 1] \to \mathcal{O}$, $s \mapsto \rho(\gamma(s))$, be its projection onto the orbit space. Let $(\frac{d\gamma}{ds})^\text{hor} \in H_{\gamma(s)}$ and $(\frac{d\gamma}{ds})^\text{vert} \in V_{\gamma(s)}$ be, respectively, the horizontal and vertical components in the orthogonal decomposition

$$\frac{d\gamma}{ds} = \left(\frac{d\gamma}{ds}\right)^\text{hor} + \left(\frac{d\gamma}{ds}\right)^\text{vert}.$$ 

Then,

(a) The arc lengths $L(\gamma)$ and $L(\alpha)$ of the curves $\gamma$ and $\alpha$, respectively, satisfy the inequalities

$$L(\alpha) \leq L(\gamma),$$

(22)
\[
L(\gamma) \leq L(\alpha) + \int_0^1 \left\| \left( \frac{d\gamma}{ds} \right)^{\text{vert}} \right\|^2 ds \leq \sqrt{2} L(\gamma). \tag{23}
\]

The equality \( L(\alpha) = L(\gamma) \) holds if and only if the curve \( \gamma \) is horizontal, that is, \( \left( \frac{d\gamma}{ds} \right)^{\text{vert}} = 0 \).

(b) For any \( p, q \in M \), we have
\[
\text{dist}^O(\rho(p), \rho(q)) \leq \text{dist}(p, q). \tag{24}
\]

Proof Part (a) is evident and follows from the relation \( \frac{d\alpha}{ds} = (d\gamma(x)\rho)(\frac{d\gamma}{ds})^{\text{hor}} \), the orthogonal decomposition \( 21 \) and the equality
\[
\left\| \frac{d\alpha}{ds} \right\|^O = \left\| \left( \frac{d\gamma}{ds} \right)^{\text{hor}} \right\|,
\tag{25}
\]
which is a consequence of the property that \( \rho \) is a Riemannian submersion. In order to prove part (b), for arbitrary \( p, q \in M \), let us choose a curve \( \gamma \) on \( M \) joining points \( p \) and \( q \) and such that \( \text{dist}(p, q) + \Delta \geq L(\gamma) \), for some \( \Delta > 0 \). Then, by \( 22 \) we get
\[
\text{dist}^O(\rho(p), \rho(q)) \leq L(\rho \circ \gamma) \leq L(\gamma) \leq \text{dist}(p, q) + \Delta.
\]

Since \( \Delta > 0 \) is arbitrary, inequality \( 24 \) is satisfied.

Now, let \( \beta : [a, b] \to \mathcal{O} \) be a smooth curve on \( \mathcal{O} \) passing trough the point \( \beta(a) = x \in \mathcal{O} \). Let \( m \in \rho^{-1}(x) \) be a point in the fiber over \( x \). A lifting of \( \beta \) trough \( m \) is a smooth curve \( \tilde{\beta} : [a, b] \to M \) such that

(i) \( m = \tilde{\beta}(a) \), and

(ii) \( \rho \circ \tilde{\beta} = \beta \).

In this case, \( \tilde{\beta} \) is called the lift of curve \( \beta \) or lifted curve. A lifted curve \( \tilde{\beta} \) is called horizontal if, in addition, it satisfies the following property:
\[
\frac{d}{dt} \tilde{\beta}(t) \in H_{\tilde{\beta}(t)}, \quad \forall t \in [a, b]. \tag{26}
\]

Since \( S^1 \) in compact, the fibers are compact. Hence, it follows that for any smooth curve \( \beta : [a, b] \to \mathcal{O} \), the horizontal lift of \( \alpha \) trough \( m \) always exists \( 12 \). The following statement gives us a key property for the horizontal lift.

Proposition 4 Let \( X \in X(M) \) be a vector field and \( \gamma : [0, T(m^0)] \to M \) the trajectory of \( X \) through \( m^0 \in M, \gamma(t) = F^t_X(m^0) \). Consider the projection \( \alpha = \rho \circ \gamma \) and its horizontal lift \( \tilde{\alpha} : [0, T] \to M, t \mapsto \tilde{\alpha}(t) \) through \( m^0 \), \( \tilde{\alpha}(0) = m^0 \). Then, there exists a smooth function \( \tau : [0, T(m^0)] \to \mathbb{R} \) such that \( \tau(0) = 0 \) and
\[
\tilde{\alpha}(t) = \phi^t(\gamma(t)), \tag{27}
\]
where
\[
\phi^t = F^t_X. \tag{28}
\]
Moreover, the curve \( t \mapsto \tilde{\alpha}(t) \in M \) is the trajectory through \( m^0 \) of the horizontal \( t \)-dependent vector field
\[
\tilde{X}_t = (\varphi^t)_* X^\text{hor},
\]
that is,
\[
\frac{d\tilde{\alpha}(t)}{dt} = \tilde{X}_t(\tilde{\alpha}(t)),
\]
and the following properties hold:
\[
\|\tilde{X}_t\|_{\tilde{\alpha}(t)} = \|X^\text{hor}\|_{\gamma(t)},
\]
\[
\|\nabla_v \tilde{X}_t\|_{\tilde{\alpha}(t)} \leq \|\nabla X^\text{hor}\|_{\gamma(t)} \cdot \|v\|_{\tilde{\alpha}(t)},
\]
for every \( v \in T\tilde{\alpha}(t)M \).

**Proof** By definition, for each \( t \), the points \( \tilde{\alpha}(t) \) and \( \gamma(t) \) belong to the same fiber \( \rho^{-1}(\alpha(t)) \) and thus they can be joined by a segment of the periodic trajectory of \( \Upsilon \), for time \( \tau = \tau(t) \). Differentiating both sides of (24) with respect to \( t \) and using decomposition (21), we get
\[
\frac{d}{dt} \tilde{\alpha}(t) = (d(\gamma(t))\varphi^t)\frac{d\gamma(t)}{dt} + \tau'(t)\Upsilon(\gamma(t))
\]
\[
= (d(\gamma(t))\varphi^t)X^\text{hor}(\gamma(t)) + (d(\gamma(t))\varphi^t)X^\text{vert}(\gamma(t)) + \tau'(t)\Upsilon(\gamma(t)).
\]
Remark that the flow of \( \Upsilon \) is an isometry which preserves the splitting of \( TM \) into horizontal and vertical subbundles. Hence, the diffeomorphisms \( \varphi^t \) have the same properties. From here and the fact that the velocity \( \frac{d\tilde{\alpha}(t)}{dt} \) is a horizontal vector field, we deduce, from (24), the relations
\[
\frac{d\tilde{\alpha}(t)}{dt} = (d(\gamma(t))\varphi^t)X^\text{hor}(\gamma(t)),
\]
and
\[
\tau'(t)\Upsilon(\gamma(t)) = - (d(\gamma(t))\varphi^t)X^\text{vert}(\gamma(t)).
\]
Notice that formula (34) defines the function \( \tau = \tau(t) \). Putting \( \gamma(t) = (\varphi^t)^{-1}(\tilde{\alpha}(t)) \) into (33) leads to the relation
\[
\frac{d\tilde{\alpha}(t)}{dt} = (d(\varphi^t)^{-1}(\tilde{\alpha}(t))\varphi^t)X^\text{hor}((\varphi^t)^{-1}(\tilde{\alpha}(t)))
\]
\[
= (\varphi^t)_* X^\text{hor}(\tilde{\alpha}(t)),
\]
which says that \( \tilde{\alpha}(t) \) is the trajectory through \( m^0 \) of the vector field \( \tilde{X}_t \) in (24). Equality (30) follows from the property that the differential of \( \varphi^t \) is a linear isometry and the representation \( \tilde{X}_t(m) = (d_m\varphi^t)X((\varphi^t)^{-1}m) \). Finally, applying Lemma 1, we get
\[
\|\nabla_v \tilde{X}_t\|_{\tilde{\alpha}(t)} = \|\nabla(d_{\tilde{\alpha}(t)}\varphi^t)^{-1}v X^\text{hor}\|_{\gamma(t)} \leq \|\nabla X^\text{hor}\|_{\gamma(t)} \cdot \|v\|_{\tilde{\alpha}(t)}
\]
7 Proof of main result

In this section, we present the proof of the periodic averaging theorem (Theorem 1), which is done in several steps. Here, as in the previous sections, we denote by $g$ an $S^1$-invariant metric on the manifold $M$ and $\text{dist} : M \times M \to \mathbb{R}$, the corresponding distance function. Also, let $g^O$ be the unique Riemannian metric on $O$ such that the projection $\rho : M \to O$ is a Riemannian submersion and $\text{dist}^O : O \times O \to \mathbb{R}$ its distance function.

Now, take an open domain $D$ in $O$ having compact closure and such that $D_0 \subset D$. Thus, $N_0 = \rho^{-1}(D_0)$ and $N = \rho^{-1}(D)$ are open domains in $M$, with compact closure. Since for every $m \in N$ the fiber through $m$ is contained in $N$, the sets $N_0$ and $N$ are invariant with respect to the $S^1$-action. We also assume that $N$ and $N_0$ are connected.

**Step 1 (Normalization of the perturbed vector field.)** By Proposition 2 there exists $\delta > 0$ such that the flow of vector field

$$Z = \frac{1}{\omega} S(X_1) + \frac{1}{\omega^3} S^2(\mathcal{L}(X_1)\omega) X_0$$  \hspace{1cm} (35)

is a near identity transformation, $\Phi : (-\delta, \delta) \times N \to M$, which takes the vector field $X_\varepsilon$ into the $S^1$-invariant normal form,

$$\Phi^*_\varepsilon X_\varepsilon = X_0 + \varepsilon(X_1) + \varepsilon^2 R_\varepsilon.$$

Now, for each $s \in [0, 1]$, define the $(\varepsilon, s)$-dependent vector field

$$\tilde{X}_{\varepsilon,s} = X_0 + \varepsilon(X_1) + s\varepsilon^2 R_\varepsilon.$$  \hspace{1cm} (36)

**Step 2 (Triangle inequality.)** It is easy to see that $\tilde{X}_{\varepsilon,1}$ is the $S^1$-invariant normal form of first order of $X_\varepsilon$ and $\tilde{X}_{\varepsilon,0} = X_0 + \varepsilon(X_1)$ is an $S^1$-invariant vector field $\rho$-related with $\varepsilon(X_1)$. By the triangle inequality, we get

$$\text{dist}^O \left( \rho \circ \text{Fl}^t_{\tilde{X}_{\varepsilon,s}}(m^0), \rho \circ \text{Fl}^t_{\tilde{X}_{\varepsilon,0}}(m^0) \right) \leq \text{dist}^O \left( \rho \circ \text{Fl}^t_{\tilde{X}_{\varepsilon,s}}(m^0), \rho \circ \text{Fl}^t_{\tilde{X}_{\varepsilon,1}}(m^0) \right)$$

$$+ \text{dist}^O \left( \rho \circ \text{Fl}^t_{\tilde{X}_{\varepsilon,1}}(m^0), \rho \circ \text{Fl}^t_{\tilde{X}_{\varepsilon,0}}(m^0) \right),$$  \hspace{1cm} (37)

where $\varepsilon^0 = \rho(m^0)$ and $m_{\varepsilon} = \Phi^{-1}_\varepsilon(m^0)$ can be though as a parameterized curve depending smoothly on $\varepsilon$.

Since $\Phi_\varepsilon$ is a near identity transformation, there exists a constant $\delta_0 \in (0, \delta]$ such that $m_{\varepsilon} \in N_0$ for all $\varepsilon \in [0, \delta_0]$.

**Lemma 3** Let $[0, \delta_0] \to D_0$ be a parameterized curve on the orbit space $O$ given by $\varepsilon \mapsto \rho(m_\varepsilon)$. Then, the inequality

$$\text{dist}^O(\rho(m^0), \rho(m_{\varepsilon})) \leq \text{dist}(m^0, m_\varepsilon) \leq \varepsilon_0 \varepsilon,$$

holds for $\varepsilon \in [0, \delta_0]$, with

$$\varepsilon_0 = \sup_{m \in N_0} \|Z(m)\|_m,$$  \hspace{1cm} (38)

where $Z$ is the vector field given by $35$. 

Proof Let $L_\varepsilon$ be the arc length of the parameterized curve $m_\varepsilon = \Phi^{-1}_\varepsilon(m^0)$. Taking into account that $\frac{d}{dx} \frac{dm_\varepsilon}{dx} = -Z(m_\varepsilon)$, we get

$$L_\varepsilon = \int_0^\varepsilon \left\| \frac{d}{d\varepsilon'} \left( \frac{d}{dx} \frac{dm_\varepsilon}{dx} \right) \right\| d\varepsilon' = \int_0^\varepsilon \|Z(m_\varepsilon)\| d\varepsilon' \leq \varepsilon \sup_{m \in N_0} \|Z(m)\|_m.$$ 

Here we use the fact that $\rho$ is a Riemannian submersion and the properties of the distance function.

Lemma 4 Consider the $(\varepsilon, s)$-dependent vector field $\tilde{X}_{\varepsilon,s}$ given in (48). Then, there exist $L_0 > 0$ and $\varepsilon_0 \in (0, \delta_0]$ such that every trajectory of $\tilde{X}_{\varepsilon,s}$ through $m_{\varepsilon,s}$,

$t \mapsto \gamma_{\varepsilon,s}(t) := \text{Fl}^t_{\tilde{X}_{\varepsilon,s}}(m_{\varepsilon,s}) \in N,$

is defined for $t \in [0, L_0/\varepsilon]$ if $\varepsilon \in (0, \varepsilon_0]$ and $s \in [0, 1]$.

Proof By direct computations, we have

$$\frac{d}{dt} \left. \left( \text{Fl}^t_{\tilde{X}_{\varepsilon,s}} \circ \text{Fl}^t_{\tilde{P}_s}(m) \right) \right|_{t=0} = \tilde{X}_{\varepsilon,s}(m), \quad \forall \, m \in M,$$

where

$$P_t = (P_{\varepsilon,s})_t = \langle X_1 \rangle - t(\mathcal{L}_{\langle X_1 \rangle})\omega + \varepsilon s \langle \text{Fl}^t_{\tilde{X}_{\varepsilon,s}} \rangle R_{\varepsilon}$$

is a time-dependent vector field parameterically depending on $(\varepsilon, s)$ in a smooth way. Hence, we have the following identity

$$\text{Fl}^t_{\tilde{X}_{\varepsilon,s}} = \text{Fl}^t_{\tilde{X}_{\varepsilon,s}} \circ \text{Fl}^t_{\tilde{P}_s}. \quad (40)$$

Since the flow of $X_0$ is periodic, it is enough to show that for small enough $\varepsilon$, there exists a fixed interval $[0, L_0]$ which belongs to the interval of definition of the trajectory of $P_t$ through $m_{\varepsilon,s}$. The vector field $(P_{\varepsilon,s})_t = \langle X_1 \rangle - t(\mathcal{L}_{\langle X_1 \rangle})\omega$ is $S^1$-invariant and $\rho$-related with $\langle X_1 \rangle\circ$. Hence, the trajectory of this field through $m^0$ is defined for $t \in [0, L_0]$, and there exists $\varepsilon_0 \in (0, \delta_0]$ such that for every $\varepsilon \in [0, \varepsilon_0]$ and $s \in [0, 1]$, the trajectory of $(P_{\varepsilon,s})_t = (P_{\varepsilon,s})_t + \varepsilon s \langle \text{Fl}^t_{\tilde{X}_{\varepsilon,s}} \rangle R_{\varepsilon}$ through $m_{\varepsilon,s}$ is also defined for all $t \in [0, L_0]$. Here we use the following well-known property (see [1], page 222): If $[0, L_0]$ is contained in the domain of definition of the trajectory through $m^0$, then there exists a neighborhood $U$ of $m^0$ such that any $m \in U$ has a trajectory existing for time $t \in [0, L_0]$.

Since $X_\varepsilon = (\Phi_\varepsilon)_\ast \tilde{X}_{\varepsilon,1}$, we get $\text{Fl}^t_{\tilde{X}_{\varepsilon,1}}(m_\varepsilon) = \Phi^{-1}_\varepsilon \circ \text{Fl}^t_{X_\varepsilon}(m^0)$. Taking into account that $\rho$ is a Riemannian submersion, it follows from Lemma 5 that

$$\text{dist}^C(\rho \circ \text{Fl}^t_{X_\varepsilon}(m^0), \rho \circ \text{Fl}^t_{\tilde{X}_{\varepsilon,1}}(m_\varepsilon)) \leq \text{dist}^C(\text{Fl}^t_{X_\varepsilon}(m^0), \text{Fl}^t_{\tilde{X}_{\varepsilon,1}}(m_\varepsilon)) \leq \text{dist}^C(\text{Fl}^t_{X_\varepsilon}(m^0), \Phi^{-1}_\varepsilon \circ \text{Fl}^t_{X_\varepsilon}(m^0)) \leq \varepsilon_0 \varepsilon. \quad (41)$$
By Lemma 4, estimation holds for $t \in [0, L_0/\varepsilon]$, whenever $\varepsilon \in [0, \varepsilon_0]$ and $s \in [0, 1]$.

**Step 3 (Gronwall’s estimations.)** In order to get an estimation of first order in $\varepsilon$ for the second term in (47), we proceed as follows: for each fixed $\varepsilon$ we define the trajectory $\gamma_\varepsilon : [0, L_0/\varepsilon] \to N$ of the vector field $\tilde{X}_{\varepsilon, 1}$ through $m_\varepsilon$, that is, $\gamma_\varepsilon = \text{Fl}^t_{\tilde{X}_{\varepsilon, 1}}(m_\varepsilon)$.

Now, let $\alpha_\varepsilon = \rho \circ \gamma_\varepsilon$ be the projection of $\gamma_\varepsilon$ on the orbit space and $\tilde{\alpha}_\varepsilon$ the horizontal lift of $\alpha_\varepsilon$ through $m_\varepsilon$. Then, by Proposition 4, for every $t$, there exists a fiberwise diffeomorphism $\varrho^t$ on $N$ defined by (28), such that $\varrho^0 = \text{id}$ and

$$
\tilde{\alpha}_\varepsilon(t) = \varrho^t(\gamma_\varepsilon(t)).
$$

Moreover, $\tilde{\alpha}_\varepsilon(t)$ is the trajectory of the time dependent vector field $(\varrho^t)_* \tilde{X}_{1, \varepsilon}^\text{hor}$, where

$$
\tilde{X}_{1, \varepsilon}^\text{hor} = \varepsilon (X_1)^\text{hor} + \varepsilon^2 R^\text{hor}.
$$

Since $\varrho^t$ is defined as the reparameterized flow of the infinitesimal generator of the $S^1$-action, we have that $(\varrho^t)_* (X_1)^\text{hor} = (X_1)^\text{hor}$, and hence,

$$
(\varrho^t)_* \tilde{X}_{1, \varepsilon}^\text{hor} = \varepsilon (X_1)^\text{hor} + \varepsilon^2 (\varrho^t)_* R^\text{hor}.
$$

For every $\varepsilon \in [0, \varepsilon_0]$ and $s \in [0, 1]$, consider the following horizontal time dependent vector field on $N$:

$$
Y_{\varepsilon}(s, t) = \varepsilon (X_1)^\text{hor} + s \varrho^t(\gamma_\varepsilon(t))^\text{hor}.
$$

We define the following parameterized surface in $N$,

$$
\Sigma_{\varepsilon} : [0, L_0/\varepsilon] \times [0, 1] \ni (t, s) \mapsto \Sigma_{\varepsilon}(t, s) := \text{Fl}^t_{Y_{\varepsilon}}(m_{s\varepsilon}).
$$

It is clear that

$$
\Sigma_{\varepsilon}(t, 0) = \text{Fl}^t_{(X_1)^\text{hor}}(m_0).
$$

Since $Y_{\varepsilon}(t, 1)$ coincides with $(\varrho^t)_* \tilde{X}_{1, \varepsilon}^\text{hor}$ we have that $\tilde{\alpha}_\varepsilon(t) = \Sigma_{\varepsilon}(t, 1)$. Thus, $\alpha_\varepsilon(t) = \rho \circ \Sigma_{\varepsilon}(t, 1)$ and $\rho \circ \text{Fl}^t_{\tilde{X}_{1, \varepsilon}}(m_0) = \rho \circ \Sigma_{\varepsilon}(t, 0)$. By construction, we have

$$
\text{dist}^\mathcal{O}
\left(\rho \circ \text{Fl}^t_{\tilde{X}_{1, \varepsilon}}(m_\varepsilon), \rho \circ \text{Fl}^t_{\tilde{X}_{1, \varepsilon}}(m_0)\right) = \text{dist}^\mathcal{O}
\left(\rho \circ \Sigma_{\varepsilon}(t, 1), \rho \circ \Sigma_{\varepsilon}(t, 0)\right).
$$

By part (b) of Lemma 4, we have the estimation

$$
\text{dist}^\mathcal{O}
\left(\rho \circ \Sigma_{\varepsilon}(t, 1), \rho \circ \Sigma_{\varepsilon}(t, 0)\right) \leq \text{dist}^\mathcal{O}
\left(\Sigma_{\varepsilon}(t, 1), \Sigma_{\varepsilon}(t, 0)\right).
$$

Combining (43) and (44) we can get an estimation for the second term in (47) by studying the lengths of the $s$-curves in the surface $\Sigma_{\varepsilon}$.

Now, for a fixed $t$, consider the horizontal $s$-curve $s \mapsto \Sigma_{\varepsilon}(s) := \Sigma_{\varepsilon}(t, s)$ and its arc length

$$
L_{\varepsilon}(t) := \int_0^1 \left\| \frac{d}{ds} \Sigma_{\varepsilon}(s) \right\| ds.
$$
Lemma 5 For all $\varepsilon \in [0, \varepsilon_0]$ and $t \in [0, L_0/\varepsilon]$, the following estimate holds:

$$L_\varepsilon(t) \leq \left[ \left( \frac{x_2}{x_1} + x_0 \right) e^{\varepsilon x_1 t} - \frac{x_2}{x_1} \right] \varepsilon,$$

(45)

where $x_0$ is given by \ref{eq:5} and

$$x_1 = \sup_{m \in N} \| \nabla \langle X_1 \rangle_{\text{hor}} \|_m + \varepsilon \| R^\text{hor}_\varepsilon \|_m,$$

(46)

$$x_2 = \sup_{m \in N} \| R^\text{hor}_\varepsilon \|_m.$$

(47)

Proof Applying the basic inequality \ref{eq:15}, we have

$$L_\varepsilon(t) \leq L_\varepsilon(0) + \int_0^t \int_0^1 \left\| \nabla \frac{\partial \Sigma_\varepsilon}{\partial t'} \right\|_{\Sigma_\varepsilon} \, ds \, dt'.$$

(48)

By definition, the $t$-curves in $\Sigma_\varepsilon$ are horizontal and

$$\frac{\partial}{\partial t} \Sigma_\varepsilon(t, s) = (\varepsilon \langle X_1 \rangle_{\text{hor}} + s \varepsilon^2 \langle g' \rangle_{\text{hor}} R^\text{hor}_\varepsilon) \circ \Sigma_\varepsilon.$$

It follows that

$$\left\| \nabla \frac{\partial \Sigma_\varepsilon}{\partial t'} \right\|_{\Sigma_\varepsilon} \leq \varepsilon \left\| \nabla \langle X_1 \rangle_{\text{hor}} \right\|_{\Sigma_\varepsilon} \cdot \left\| \frac{\partial}{\partial s} \Sigma_\varepsilon \right\|_{\Sigma_\varepsilon} + s \varepsilon^2 \left\| \nabla \frac{\partial}{\partial s} ((g')_{\text{hor}} R^\text{hor}_\varepsilon) \right\|_{\Sigma_\varepsilon} + \varepsilon^2 \left\| (g')_{\text{hor}} R^\text{hor}_\varepsilon \right\|_{\Sigma_\varepsilon}.$$n

By Lemma \ref{lem:1} we deduce

$$\left\| \nabla \frac{\partial}{\partial s} ((g')_{\text{hor}} R^\text{hor}_\varepsilon) \right\|_{\Sigma_\varepsilon} \leq \left\| \nabla ((g')_{\text{hor}} R^\text{hor}_\varepsilon) \right\|_{\Sigma_\varepsilon} \cdot \left\| \frac{\partial}{\partial s} \Sigma_\varepsilon \right\|_{\Sigma_\varepsilon}$$

and

$$\| (g')_{\text{hor}} R^\text{hor}_\varepsilon \|_{\Sigma_\varepsilon} = \| R^\text{hor}_\varepsilon \|_{\vartheta^{-1} \Sigma_\varepsilon}.$$n

Putting these relations into \ref{eq:48} we arrive at the inequality

$$L_\varepsilon(t) \leq \varepsilon x_0 + \varepsilon x_1 \int_0^t L_\varepsilon(t') \, dt' + \varepsilon^2 x_2 t,$$

and, by applying the specific Gronwall’s lemma, we get \ref{eq:45}.
Finally, we need to prove that estimation (10) holds for all \( \varepsilon \in (0, \varepsilon_0] \) and \( t \in [0, L_0/\varepsilon] \). This fact concludes the proof of Theorem 1. From estimation (41), we have

\[
\text{dist}^O \left( \rho \circ \text{Fl}^{\xi}_X(m^0), \rho \circ \text{Fl}^{\xi}_{X_{\varepsilon, i}}(m_{\varepsilon}) \right) \leq \kappa_0 \varepsilon.
\]

(49)

By equations (43), (44), the inequality \( \text{dist} \left( \Sigma_{\varepsilon}(t, 1), \Sigma_{\varepsilon}(t, 0) \right) \leq L_\varepsilon(t) \) and Lemma 5, we have

\[
\text{dist}^O \left( \rho \circ \text{Fl}^{\xi}_{X_{\varepsilon, i}}(m^0) \right) \leq \left[ \left( \frac{\kappa_2}{\kappa_1} + \frac{\kappa_0}{\kappa_1} \right) e^{\kappa_1 T_0} - \frac{\kappa_2}{\kappa_1} \right] \varepsilon.
\]

(50)

Therefore, the desired result follows from triangle inequality (37) and inequalities (49) and (50).

**Corollary 1** The \( \varepsilon \)-independent constant in (10) can be chosen as follows:

\[
c = \kappa_0 + \left( \frac{\kappa_2}{\kappa_1} + \frac{\kappa_0}{\kappa_1} \right) e^{\kappa_1 T_0} - \frac{\kappa_2}{\kappa_1},
\]

(51)

where the constants \( \kappa_0, \kappa_1, \) and \( \kappa_2 \) are given by (38), (46) and (47), respectively.

**Remark 1** Taking into account that \( \kappa_0 = \sup_{m \in N_0} \| Z(m) \|_m \) (Lemma 3) where \( Z \) is the vector field given by

\[
Z = \frac{1}{\omega} X_1 + \frac{1}{\omega^2} S^2(\mathcal{L}(X_1) \omega) X_0,
\]

\( \kappa_0 \) is the unique constant of equation (51) that can be expressed only in terms of vector fields \( X_0 \) and \( X_1 \).

8 Application of the averaging theorem to adiabatic invariants

Here, we present an application of Theorem 1 in the context of adiabatic invariants which appear in many important problems of mathematical-physics [15].

**Adiabatic invariants.** An adiabatic invariant of a perturbed vector field is a function which changes very little along the trajectories of the vector field over a long period of time. More precisely, let \( M \) be a smooth manifold and let \( X_0 \) be a complete vector field on \( M \). A function \( I \in C^\infty(M) \) is called an adiabatic invariant of the perturbed vector field \( X_\varepsilon = X_0 + \varepsilon X_1 \), if there exists a constant \( c \) such that for every \( x \in M \) and \( \varepsilon > 0 \), the following inequality holds,

\[
|I \circ \text{Fl}^{\xi}_{X_\varepsilon}(x) - I(x)| \leq c \varepsilon, \quad \text{for } 0 \leq t \leq \frac{1}{\varepsilon}.
\]
Now we prove a result that states the conditions for the existence of an adiabatic invariant of a perturbed vector field $X_\varepsilon = X_0 + \varepsilon X_1$ which is $\varepsilon$-close to a vector field with periodic flow. Since this proposition relies on Theorem 1, the $S^1$-action induced by $X_0$ must be free and $M$ a connected manifold.

**Proposition 5** Assume also that the reduced averaged vector field $\langle X_1 \rangle_\mathcal{O}$ on the orbit space $\mathcal{O} = M / S^1$ satisfies the hypothesis of Theorem 1 and admits a smooth first integral $J_\mathcal{O} : \mathcal{O} \to \mathbb{R}$,

$$\mathcal{L}_{\langle X_1 \rangle_\mathcal{O}} J_\mathcal{O} = 0.$$  \hfill (52)

Then, the function $J := J_\mathcal{O} \circ \rho$ is an adiabatic invariant for $X_\varepsilon$:

$$|J \circ \text{Fl}^{\varepsilon t}_{X_\varepsilon}(m^0) - J(m^0)| = O(\varepsilon),$$

for $m^0 \in \mathcal{D}_0$ , $\varepsilon$ small enough and $t \in [0, T_0/\varepsilon]$.

**Proof** Since the closure of the open domain $\mathcal{D}$ is compact, the function $J_\mathcal{O}$ has the Lipschitz property on $\overline{\mathcal{D}}$ (see, for example, [1,18]),

$$|J_\mathcal{O}(z) - J_\mathcal{O}(y)| \leq \lambda_J \|z - y\|_\mathcal{O}.$$

Then, by condition (52) and Theorem 1 we have

$$|J \circ \text{Fl}^{\varepsilon t}_{X_\varepsilon}(m^0) - J(m^0)|$$

$$= |J_\mathcal{O}(\rho \circ \text{Fl}^{\varepsilon t}_{X_\varepsilon}(m^0)) - J_\mathcal{O}(\text{Fl}^{\varepsilon t}_{\langle X_1 \rangle_\mathcal{O}}(\rho(m^0)))|$$

$$\leq \lambda_J \|\rho \circ \text{Fl}^{\varepsilon t}_{X_\varepsilon}(m^0) - \text{Fl}^{\varepsilon t}_{\langle X_1 \rangle_\mathcal{O}}(\rho(m^0))\|_\mathcal{O}$$

$$\leq \lambda_J \varepsilon,$$

where the constant $c$ is given by (51).

This result is well known for perturbed Hamiltonian vector fields, where the unperturbed part is a one degree of freedom Hamiltonian system, see [3,15], and the proof in this case relies on the classical averaging theorem and the existence of action-angle variables, (coordinate approach). A study of the existence of adiabatic invariant for perturbed vector fields with a free coordinate approach can be found in [5].

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