Dynamical systems

Combinatorial models for spaces of cubic polynomials

Modèles combinatoires pour les espaces de polynômes cubiques

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Abstract

W. Thurston constructed a combinatorial model of the Mandelbrot set $\mathcal{M}_2$ such that there is a continuous and monotone projection of $\mathcal{M}_2$ to this model. We propose the following related model for the space $\mathcal{MD}_3$ of critically marked cubic polynomials with connected Julia set and all cycles repelling. If $(P, c_1, c_2) \in \mathcal{MD}_3$, then every point $z$ in the Julia set of the polynomial $P$ defines a unique maximal finite set $A_z$ of angles on the circle corresponding to the rays, whose impressions form a continuum containing $z$. Let $G(z)$ denote the convex hull of $A_z$. The convex sets $G(z)$ partition the closed unit disk. For $(P, c_1, c_2) \in \mathcal{MD}_3$ let $c_1^*$ be the co-critical point of $c_1$. We tag the marked dendritic polynomial $(P, c_1, c_2)$ with the set $G(c_1^*) \times G(P(c_2)) \subset \mathbb{D} \times \mathbb{D}$. Tags are pairwise disjoint; denote by $\mathcal{MD}_3^{\text{comb}}$ their collection, equipped with the quotient topology. We show that tagging defines a continuous map from $\mathcal{MD}_3$ to $\mathcal{MD}_3^{\text{comb}}$ so that $\mathcal{MD}_3^{\text{comb}}$ serves as a model for $\mathcal{MD}_3$.

Résumé

W. Thurston a construit un modèle combinatoire de l'ensemble de Mandelbrot $\mathcal{M}_2$ tel qu'il y ait une projection monotone et continue de $\mathcal{M}_2$ sur ce modèle. En relation avec ceci, nous proposons le modèle lié suivant pour l'espace $\mathcal{MD}_3$ des polynômes cubiques à points critiques marqués, avec ensemble de Julia connexe et tous les cycles répulsifs. Si $(P, c_1, c_2) \in \mathcal{MD}_3$, alors chaque point $z$ dans l'ensemble de Julia du polynôme $P$ définit un unique ensemble fini maximal $A_z$ d'angles sur le cercle correspondant aux rayons, dont les impressions forment un continuum contenant $z$. Soit $G(z)$ l'enveloppe convexe de $A_z$. Les ensembles convexes $G(z)$ définissent une partition du disque unité fermé. Pour $(P, c_1, c_2) \in \mathcal{MD}_3$, soit $c_1^*$ le point co-critique de $c_1$. Nous balisons le polynôme dendritique marqué $(P, c_1, c_2)$ avec l'ensemble $G(c_1^*) \times G(P(c_2)) \subset \mathbb{D} \times \mathbb{D}$. Les balises sont deux à deux disjointes ; désignons par $\mathcal{MD}_3^{\text{comb}}$ leur collection, équipée de la topologie quotient. Nous
1. Introduction

Let $\mathbb{D}$ be the open disk $\{z \in \mathbb{C} \mid |z| < 1\}$ in the plane, $\overline{\mathbb{D}}$ be its closure, and $\mathbb{S}$ be its boundary circle. Let $P$ be a polynomial of degree $d$ with connected Julia set $J(P)$. We write $\Phi_P$ for the conformal isomorphism between $\mathbb{C} \setminus \overline{\mathbb{D}}$ and the complement $U$ of the filled Julia set $K(P)$ asymptotic to the identity at infinity. By a theorem of Carathéodory, if $J(P)$ is locally connected, then $\Phi_P$ can be extended to a continuous map $\overline{\Phi}_P : \mathbb{C} \setminus \overline{\mathbb{D}} \to \overline{U}$, under which $\mathbb{S}$ maps onto $J(P)$. Define the \textit{lamination generated by $P$} as the equivalence relation $\sim_P$ on $\mathbb{S}$ identifying points of $\mathbb{S}$ if and only if $\overline{\Phi}_P$ sends them to the same point of $J(P)$.

By Thurston [7], the map $P$ restricted to its locally connected Julia set $J(P)$ is topologically conjugate to a self-mapping $f_{-P}$ of the quotient space $\mathbb{S}/\sim_P = J_{-P}$ induced by $z^d \mid_{\mathbb{S}} = \sigma_d z$; denote this conjugacy by $\Psi_P : J(P) \to J_{-P}$. The mapping $f_{-P}$ is called a \textit{topological polynomial}. The quotient map of $\mathbb{S}$ onto $\mathbb{S}/\sim_P$ is denoted by $\pi_{-P}$. Given a point $z \in J(P)$, we let $G_P(z) = G(z)$ denote the convex hull of the set $\pi_{-P}^{-1}(\Psi_P(z))$. In other words, we represent $z$ by the point $\Psi_P(z)$ of the model topological Julia set $J_{-P}$ and then take all angles associated with $\Psi_P(z)$ in the sense of the lamination $\sim_P$. By [7], for two points $z$ and $w$, the sets $G(z)$ and $G(w)$ either coincide or are disjoint.

The \textit{geolamination} (from geodesic or geometric lamination) of $P$ is the collection of chords, each of which is an edge of the convex hull of a $\sim_P$-class. Geolaminations geometrically interpret and “topologize” laminations, reflecting limit transitions among them. Both laminations and their geolaminations can be defined intrinsically (without polynomials). Then some geolaminations will not directly correspond to an equivalence relation on $\mathbb{S}$, but the family of all geolaminations will be closed. This allows one to work with limits of geolaminations and limits of polynomials (which might have non-locally connected Julia sets).

Thurston [7] models polynomials by their geolaminations, and families of quadratic polynomials by families of quadratic geolaminations. He “tags” quadratic geolaminations with their \textit{minors} which form the \textit{quadratic minor geolamination} QML and generate the corresponding lamination $\sim_{QML}$. The quotient space $\mathbb{S}/\sim_{QML}$ models the boundary of the Mandelbrot set $\mathcal{M}_2$ (this is the set of all parameters $c$ such that polynomials $z^2 + c$ have connected Julia set; it is also called the \textit{quadratic connected locus}). The induced quotient space of $\overline{\mathbb{D}}$ serves as a model for $\mathcal{M}_2$. Conjecturally, it is homeomorphic to $\mathcal{M}_2$.

Call a polynomial with connected Julia set \textit{dendritic} if all its periodic points are repelling. By [5], for any dendritic polynomial $P$, even if $J(P)$ is not locally connected, there is a lamination $\sim_P$ such that there exists a \textit{monotone semi-conjugacy} $\Psi_P$ between $P|_{J(P)}$ and the topological polynomial $f_{-P}$. Thus the sets $G_P(z) = \pi_{-P}^{-1}(\Psi_P(z))$ are well defined for every dendritic polynomial $P$ and every point $z \in J(P)$. As we will see, these nice properties of \textit{individual} dendritic polynomials result in nice properties of \textit{families} of cubic dendritic polynomials.

Let $D_2 \subset \mathcal{M}_2$ be the set of all parameters $c \in \mathcal{M}_2$ such that the polynomial $P_c(z) = z^2 + c$ is dendritic. Set $H_c = G_{P_c}(c)$, and let $\mathcal{H}$ stand for the collection of all sets $H_c, c \in D_2$. We denote the union $\bigcup_{c \in D_2} H_c$ by $\mathcal{H}^+$ (in what follows, for any collection $\mathcal{A}$ of sets, we write $\mathcal{A}^+$ for the union of all sets in $\mathcal{A}$). By a part of a major result of [7], for two parameter values $c, c' \in D_2$, the sets $H_c$ and $H_{c'}$ are either disjoint or equal. Moreover, the mapping $c \mapsto H_c$ from $D_2$ to $\mathcal{H}$ is upper semi-continuous (if a sequence of dendritic parameters $c_n$ converges to a dendritic parameter $c$, then $\limsup_{n \to \infty} G_{c_n} \subset G_c$). The set $D_2$ (or, equivalently, the set of all dendritic quadratic polynomials defined up to a Moebius change of coordinates) projects continuously onto the quotient space of $\mathcal{H}^+$ defined by the partition of $\mathcal{H}^+$ into sets $H_c$ with $c \in D_2$.

We propose a related model for the space $\mathcal{M}_3$ of \textit{marked dendritic cubic polynomials} $(P, c_1, c_2)$ with connected Julia set $(c_1, c_2)$ are the critical points of $P$). Define the \textit{co-critical} point associated with a critical point $\tau$ of $P$ as the only point $\tau^*$ such that $P(\tau^*) = P(\tau)$, $\tau^* \neq \tau$ unless $P$ has a unique critical point, in which case $\tau = \tau^*$. Then, with every marked dendritic cubic polynomial $(P, c_1, c_2)$, we associate the corresponding \textit{mixed tag} $\text{Tag}(P, c_1, c_2) = (G_{c_1^+} \times G_{\overline{P}(c_2)}) \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. This defines the mixed tag $\text{Tag}(P, c_1, c_2)$ for all \textit{marked dendritic cubic polynomials}. Our choice of tags is based on the following two requirements. Firstly, the tag of $\text{Tag}(P, c_1, c_2)$ must determine $\sim_P$. Secondly, different tags must be disjoint. It is easy to see that the post-critical tag $G(P(c_1)) \times G(P(c_2))$ does not determine $G(c_1)$ and $G(c_2)$. Hence it does not determine $\sim_P$ either. Co-critical tags $G(c_1^+) \times G(c_2^+)$ do not satisfy our requirements either since these tags may intersect without being the same (this happens, e.g., for unicritical polynomials). For this reason, we use mixed tags.

\textbf{Theorem 1.1.} Mixed tags of elements in $\mathcal{M}_3$ are disjoint or coincide so that sets $\text{Tag}(P, c_1, c_2)$ form a partition of the set $\text{Tag}(\mathcal{M}_3)^+ \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$ and generate the corresponding quotient space of $\text{Tag}(\mathcal{M}_3)^+$ denoted by $\mathcal{M}_3^{\text{comb}}$. Then $\mathcal{M}_3^{\text{comb}}$ is a separable metric space and the map $\text{Tag} : \mathcal{M}_3 \to \mathcal{M}_3^{\text{comb}}$ is continuous.
There are few papers studying the parameterization of geolaminations of higher degree. One of them is due to D. Schleicher [6], who extended Thurston’s results to geolaminations of any degree with one critical set. We have also heard of an old preprint of D. Ahmadi and M. Rees, in which they study cubic laminations.

The results of this paper are based upon [2], which in fact applies to laminations of any degree. An extended version of the present paper can be found in [3].

2. Main ideas of the proof

Let us begin with the notions and tools developed for polynomials of any degree. If $G$ is the convex hull of some set $G' \subset \mathbb{S}$, then we write $\sigma_d(G)$ for the convex hull of the set $\sigma_d(G')$.

**Definition 2.1 (Geolaminations).** Two distinct chords of $\mathbb{D}$ are said to be linked if they intersect in $\mathbb{D}$. A geolamination $L$ is a collection of pairwise unlinked chords in $\mathbb{D}$, called leaves of $L$, such that the union $L^+ \cup \mathbb{D}^-$ of all leaves is compact, and every point in $\mathbb{S}$ is a degenerate leaf of $L$. A gap of $L$ is the closure of a component of $\mathbb{D} \setminus L^+$. A chord of $L$ is a chord of $\mathbb{D}$ that is either a leaf of $L$ or is disjoint from $L^+$ in $\mathbb{D}$.

In the dynamical context, we use Definition 2.2, slightly different from Thurston’s [7].

**Definition 2.2 (Invariant geolaminations [1]).** A chord (leaf) of a geolamination $L$ is critical if its two distinct endpoints are mapped to the same point under $\sigma_d$. A geolamination $L$ is $\sigma_d$-invariant if for any $\ell \in L$ there exists $\ell^* \in L$ such that $\sigma_d(\ell^*) = \ell$, and, if $L$ is non-critical and non-degenerate, then $\sigma_d(\ell) \in L$ and there exist $d$ pairwise disjoint leaves $\ell_1, \ell_2, \ldots, \ell_d$ in $L$ with $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all $i = 1, \ldots, d$.

In the sequel, by invariant laminations we always understand the above-defined notion. Below we use the positive (counter-clockwise) circular order on $\mathbb{S}$.

**Definition 2.3 (Critical quadrilaterals).** A critical quadrilateral is a circularly ordered quadruple of points $a, b, c, d$ in $\mathbb{S}$, not necessarily different, such that $a \neq c$ are mapped to the same point under $\sigma_d$, and the same is true for the pair $b \neq d$. We refer to chords and polygons just by listing their vertices. Also, we identify quadrilaterals $abcd, bcda, cdab$ and $dabc$, and call the chords $ac$ and $bd$ diagonals of the critical quadrilateral $abcd$.

A critical chord $xy$ can be viewed as a critical quadrilateral $xxyy$. A triangle $abc$ with critical edges can be viewed as a critical quadrilateral $abbc$, or $bcca$, or $caab$.

**Definition 2.4 (Strong linkage).** Let $A$ and $B$ be two quadrilaterals. Say that $A$ and $B$ are strongly linked if the vertices of $A$ and $B$ can be numbered so that

$$a_0 \leq b_0 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq a_3 \leq b_3 \leq a_0$$

with respect to the circular order of points on $\mathbb{S}$, where $a_i, 0 \leq i \leq 3$, are vertices of $A$ and $b_i, 0 \leq i \leq 3$, are vertices of $B$.

We now consider geolaminations with sufficiently many critical quadrilaterals.

**Definition 2.5 (Qc-portraits).** An ordered $(d-1)$-tuple QCP of critical quadrilaterals $Q_1, \ldots, Q_{d-1}$ is called a quadratically critical portrait (qc-portrait) if there is a geolamination $L$ such that every $Q_i$ is a gap or a leaf of $L$ and any collection $\ell_i \subset Q_i$, $1 \leq i \leq d-1$ of diagonals of $Q_i$’s contains no loops (call such collections full). The pair $(L, \text{QCP})$ is then called a geolamination with qc-portrait. The space of all qc-portraits is denoted by $\text{QCP}^d$; the space of all geolaminations with qc-portraits is denoted by $L_{\text{QCP}}^d$. Here both spaces are equipped with the topology induced by the Hausdorff metric on sets $\text{QCP}^+$ (in the case of $QCP_d$) or $(L^+, \text{QCP}^+)$ (in the case of $L_{\text{QCP}}^d$).

If an invariant geolamination $L$ has a gap whose boundary maps forward $k$-to-$1$ with $\infty > k > 1$ as a covering, then there is no qc-portrait for $L$. In the term “quadratically critical portrait”, the word “quadratic” refers to the analogy between a critical quadrilateral and a simple (quadratic) critical point. Quadrilaterals $Q_i$ and $Q_j$ from a qc-portrait for $L$ cannot share a diagonal, as otherwise the two coinciding diagonals form a degenerate loop. The “no-loop” condition in Definition 2.5 guarantees that a qc-portrait captures all critical objects of $L$: if, say, $L$ is a degree-$4$ laminuation with a triangle $\Delta$ formed by critical leaves and another critical leaf $\tau$, then any qc-portrait of $L$ must contain $\tau$; the collection of all edges of $\Delta$ is not a qc-portrait exactly because the edges form a loop. A condition equivalent to the “no-loop” condition can be stated as follows: for each component $E$ of the closed disk minus the union of sets in the qc-portrait, the map $\sigma_d$ maps the boundary of $E$ forward in the one-to-one fashion, except for (possibly existing) critical edges of the boundary of $E$.

If all the sets $Q_i$ of a qc-portrait are gaps or leaves of a geolamination $L$, then a pair of $L$ can be recovered uniquely by taking pullbacks of $Q_i$’s that are disjoint from $Q_j$’s. Thus, parameterizing geolaminations is closely related to parameterizing qc-portraits.
Lemma 2.6. The spaces $QCP_d$ and $\mathbb{L}QCP_d$ are compact.

The following lemma explains the importance of full collections of diagonals.

Lemma 2.7. If $C$ is a full collection of diagonals and $W$ is a complementary component of $\overline{D} \setminus C^+$, then the circle arcs from the boundary of $W$ add up to the total length $\frac{1}{2}$ and the restriction of $\sigma_d$ to the union of these circle arcs is an orientation-preserving homeomorphism, except for the endpoints. Thus, a pair of linked chords disjoint from $C^+$ is mapped to a pair of linked chords preserving orientation on their endpoints.

If $L$ has a simple loop of critical leaves, then it does not matter how to choose critical quadrilaterals in the polygon bounded by this loop. This motivates the following definition.

Definition 2.8. A critical cluster of $L$ is a convex subset of $\mathbb{D}$ whose boundary is a union of critical leaves (e.g., a critical leaf is itself a critical cluster).

To avoid confusion, we will use the notation $L^q$ (with subscripts) for geolaminations with qc-portraits.

Definition 2.9 (Linked geolaminations). Let $L^q_1, L^q_2$ be geolaminations with qc-portraits $QCP_1 = (Q^1_i)_{i=1}^{d-1}$, $QCP_2 = (Q^2_i)_{i=1}^{d-1}$ (the sets $Q^1_i, Q^2_i, 1 \leq i \leq d - 1$ are called associated). Let $k, 0 \leq k \leq d - 1$ be such that:

1. for every $i, 1 \leq i \leq k$, the quadrilaterals $Q^1_i$ and $Q^2_i$ are strongly linked;
2. for each $j > k$ the sets $Q^1_j$ and $Q^2_j$ are contained in a common critical cluster of $L_1$ and $L_2$ (in what follows these common clusters will be called special clusters).

Then qc-portraits $QCP_1, QCP_2$, and geolaminations with qc-portraits $(L^q_1, QCP_1)$ and $(L^q_2, QCP_2)$, are called linked.

In what follows, we fix linked geolaminations with qc-portraits $(L^q_1, QCP_1)$ and $(L^q_2, QCP_2)$.

Definition 2.10 (Accordions). If a leaf $\ell_1 \in L^q_1$ is not contained in a special cluster, then the union $A_{L^q_2}(\ell_1)$ of $\ell_1$ and all leaves of $L^q_2$ linked with $\ell_1$ is called an accordion. The union $A_{L^q_2}(\ell_1)$ of $\ell_1$ and all leaves from the orbit of a leaf $\ell_2 \in L^q_2$ that are linked with $\ell_1$ is also called an accordion (see Fig. 1).

Lemma 2.11 is used in studying accordions of linked geolaminations with qc-portraits.

Lemma 2.11 (Smart criticality). If $\ell_1 \in L^q_1$ is not contained in a special cluster, then every critical set of QCP$_2$ has a diagonal unlinked with $\ell_1$ or coinciding with $\ell_1$. Denote this full collection of diagonals by $\mathcal{E}$. Then $A = A_{L^q_2}(\ell_1)$ is contained in the closure of a component of $D \setminus \mathcal{E}^+$ and $\sigma_d|_{\mathcal{A}\mathcal{E}}$ is (non-strictly) monotone.

To prove Lemma 2.11, observe that by the assumption, critical chords from special clusters are unlinked with $\ell_1$. Otherwise, take a pair of associated critical quadrilaterals $A \in L_1, B \in L_2$ with non-strictly alternating on $S$ vertices $a_0 \leq b_0 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq a_3 \leq b_3 \leq a_0$, and observe that $\ell_1$ is contained, say, in the circle arc $[a_0, a_1]$, and hence is unlinked with the diagonal $b_1b_3$ of $B$.  

Fig. 1. This picture shows an accordion $\ell_1 \cup \ell_2$ for two linked cubic geolaminations. The first geolamination is sketched in light gray lines, while the second one in dotted boldface lines.
To treat sets $X$ formed by linked leaves of two linked geolaminations with qc-portraits, we vary our choice of the full collection of diagonals for successive images of $X$ on each step, so that the orbit of $X$ avoids that particular full collection of diagonals on that particular step (thus smart criticality). Therefore, similarly to the case of one geolamination, any power of the map is order preserving on $X$ (see Lemma 2.7). This serves as the basis for Theorem 2.12. Suppose that the orbit of a quadrilateral $Q$ is the union of $k \geq 1$ components permuted by $\sigma_d$. Suppose that either all components are single images of $Q$ or all components are unions of $m > 1$ images of $Q$ such that $\sigma_d^{(i)}(Q) \cap \sigma_d^{j+k}(Q) \neq 0$, the $\sigma_d^j$-images of the first diagonal of $Q$ form a convex $m$-gon, the images of the second diagonal of $Q$ form a convex $m$-gon, and these two polygons have vertices alternating on $S$. Then we say that $Q$ gives rise to a periodic cluster.

**Theorem 2.12 (Dynamics of accordions).** Let $\ell_1, \ell_2$ be linked leaves of $L^i_1, L^i_2$. The set $B = CH(\ell_1, \ell_2)$ is either wandering or, for some $k$, the sets $\sigma_d^{(i)}(B), 0 < i < k$ are pairwise disjoint and $\sigma_d^{(i)}(B)$ gives rise to a periodic cluster unless, for some $t$, there are two chains of diagonals of $QCP_1$ and of $QCP_2$ connecting two adjacent on the circle endpoints of $\sigma_d^{(i)}(\ell_1) \cup \sigma_d^{j}(\ell_2)$.

Theorem 2.12 implies Corollary 2.13.

**Corollary 2.13.** The set of all leaves of $L^i_2$ non-disjoint from a leaf $\ell_1$ of $L^i_1$ is at most countable. Thus, if $\ell_2$ is an accumulation set of uncountably many leaves of $L^i_2$, then $\ell_2$ is unlinked with any leaf of $L^i_1$.

To apply Corollary 2.13, we need the following definition.

**Definition 2.14 (Perfect sublamination).** For a geolamination $L$, the maximal sublamination $L^c \subset L$ of $L$ without isolated leaves is called the perfect sublamination of $L$. If $L^c = L$, then $L$ is called perfect.

Note that for any $\ell \in L^c$ and any neighborhood $U$ of $\ell$, there are uncountably many leaves of $L^c$ in $U$.

**Theorem 2.15.** We have $(L^i_1)^c = (L^i_2)^c$. Moreover, suppose that $L_1, L_2$ are geolaminations with finite critical sets and there are linked geolaminations with qc-portraits $(L^i_1, QCP_1), (L^i_2, QCP_2)$ such that $L^i_1 \supset L_1$ and $L^i_2 \supset L_2$. Then $L^i_1 = L^i_2$.

Indeed, otherwise choose a leaf $\ell^i_1 \in (L^i_1)^c \setminus L^i_2$. By Corollary 2.13, the leaf $\ell^i_1$ (except for its endpoints) is contained in the interior of a gap $G$ of $L^i_2$ (if not, a leaf of $L^i_2$ linked with $\ell^i_1$ would have an uncountable accordion). Since $(L^i_2)^c$ is perfect, from at least one side all one-sided neighborhoods of $\ell^i_1$ contain uncountably many leaves of $(L^i_1)^c$. Hence $G$ is uncountable with uncountably many leaves of $(L^i_1)^c$ connecting points of $G \cap S$. Interiors of images of $G$ are disjoint from the critical sets of $L^i_2$, since these critical sets are finite. Hence eventually $G$ maps to a Siegel gap, i.e. a gap on which the appropriate iterate of $\sigma_d$ is semi-conjugate to an irrational rotation. This forces images of leaves of $(L^i_1)^c$ inside $G$ to intersect, a contradiction. The second part of Theorem 2.15 follows easily.

From now on we consider only the cubic case (i.e. $d = 3$). Call a geolamination dendritic if all its gaps are finite and pairwise disjoint. It is known that dendritic geolaminations are perfect. If $L$ is a cubic dendritic geolamination, then it has either two disjoint critical sets of degree two each, or one critical set of degree three. For a critical set $Q$ of $L$ its co-critical set $Q^+$ is defined as follows: if $Q$ is of degree three, set $Q^+ = Q$, otherwise $Q^+$ is the convex hull of all points in $S \setminus Q$ that map to $\sigma_d(Q)$ under $\sigma_d$. By a marked cubic dendritic geolamination we mean a triple $(L, Q_1, Q_2)$ where $Q_1$ and $Q_2$ are critical sets of $L$ and $Q_1 \neq Q_2$ if possible; the family of all of them is denoted by $L_{MD_2}$. We now introduce a labeling of such pairs of sets $(Q_1, Q_2)$.

**Definition 2.16 (Mixed tags).** The mixed tag of $(L, Q_1, Q_2)$ is the set $\text{Tag}(L, Q_1, Q_2) = Q_1^c \times \sigma_d(Q_2)$.

**Lemma 2.17** is based on simple geometric considerations and Theorem 2.15.

**Lemma 2.17.** The mixed tags of two distinct elements of $L_{MD_2}$ are non-disjoint if and only if these elements of $L_{MD_2}$ coincide (see Fig. 2).

To prove Lemma 2.17, assume that $(L, C_1, C_2) \in L_{MD_2}$ and $(T, D_1, D_2) \in L_{MD_2}$ have non-disjoint mixed tags. Then $C_1^+$ and $D_1^+$ have either a common vertex or two linked edges. The definition of the co-critical set implies then that either $C_1$ and $D_1$ have a common critical diagonal, or they contain strongly linked critical quadrilaterals with opposite edges being edges of $C_1, D_1$. Using this and analyzing the fact that $\sigma_d(C_2)$ and $\sigma_d(D_2)$ are non-disjoint, one can see that the sets $C_2, D_2$ also either share a critical diagonal, or contain strongly linked quadrilaterals with opposite edges being edges of $C_2, D_2$. If we insert the just-found shared critical diagonals or strongly linked quadrilaterals in critical sets of our geolaminations and pull the inserted objects back, we will construct two cubic geolaminations with qc-portraits that are linked. By Theorem 2.15, this implies that $(L, C_1, C_2) = (T, D_1, D_2)$. 


Fig. 2. This picture illustrates Lemma 2.17. The leaf $a_1b_1$ is a co-critical set of a critical quadrilateral $a_2b_2a_3b_3$. The leaf $x_1y_1$ is a co-critical set of a critical quadrilateral $x_2y_2x_3y_3$. Again, the two geolaminations are sketched in light gray lines and dotted boldface lines.

Standard topological arguments, based on properties of dendritic geolaminations and results of [4], imply now Theorem 1.1.

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