Fluctuation theorems in the presence of information gain and feedback

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Abstract
In this study, we rederive the fluctuation theorems in the presence of feedback by assuming the known Jarzynski equality and detailed fluctuation theorems. We find that both the classical and quantum systems can be analyzed using a similar treatment in terms of state space trajectories. We first briefly reproduce the already known work theorems for a classical system in order to show its equivalence with the quantum treatment. We then extend the treatment to arrive at new results, namely the generalizations of Seifert’s entropy production theorem and the Hatano–Sasa fluctuation theorem, in the presence of feedback. We have also derived the extended version of the Tasaki–Crooks fluctuation theorem for a quantum particle in the presence of multiple loop feedback. For deriving the extended quantum fluctuation theorems, we have considered open systems. No assumption is made on the nature of environment and the strength of system–bath coupling. However, it is assumed that the measurement process involves classical errors.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In the last couple of decades, active research has been pursued in the field of non-equilibrium statistical mechanics. Until recently, the systems that are far from equilibrium had always eluded exact analytical treatments, in contrast to the well-established theory in the linear response regime. Several new results have been discovered for systems that are far from equilibrium. These relations are generically grouped under the heading fluctuation theorems [1–5]. One of the major breakthroughs has been the Jarzynski equality [3], which states the following. Suppose that the system, consisting of a Brownian particle, is initially prepared in canonical equilibrium with a heat bath at temperature $T$. Now we apply an external protocol
λ(t) that drives the system away from equilibrium. The following equality provided by Jarzynski is valid for this system [3]:

\[ \langle e^{-\beta W} \rangle = e^{-\beta \Delta F} \tag{1} \]

Here, \( W \) is the thermodynamic work defined by \( W \equiv \int_0^\tau (\partial H_\lambda / \partial \lambda) \dot{\lambda} \, dt \), where \( H_\lambda \) is the Hamiltonian with an externally controlled time-dependent protocol \( \lambda(t) \) and \( \tau \) is the time for which the system is driven. \( \Delta F \) is the difference between the equilibrium free energies of the system at the parameter values \( \lambda(0) \) and \( \lambda(\tau) \), which equals the work done in a reversible process. A direct outcome of the above equality is the second law of thermodynamics, which states that the average work done on a system cannot be less than the change in free energy:

\[ \langle W \rangle \geq \Delta F. \]

A further generalization of the above result was provided by Crooks [4]. The Crooks work theorem says that the ratio of the probability of performing work \( W \) on the system, \( P_f(W) \), along the forward process and that of performing work \(-W\) (i.e. extracting work from the system) along the time-reversed process, \( P_r(-W) \), is exponential in the forward work, provided the initial state of either process is at thermal equilibrium:

\[ \frac{P_f(W)}{P_r(-W)} = e^{\beta(W - \Delta F)}. \tag{2} \]

Here, the initial probability density function (pdf) of the time-reversed process is the thermal/Boltzmann pdf corresponding to the final protocol value \( \lambda(\tau) \) of the forward process. It is crucial to note that the fluctuation theorems are in complete conformity with the second law, since the average work always exceeds the free energy, although for individual trajectories this condition may not be meted out [6].

The above theorems are valid for what is known as open-loop feedback, i.e. when the protocol function for the entire process is predetermined. In contrast, in closed-loop feedback, the system state is measured along the forward trajectory, and the protocol is changed depending on the outcomes of these measurements. Such processes are common in today’s experiments because they can enhance the performance characteristics of the system. Examples of classical feedback that help to enhance the performance characteristics of the system are provided in [7–9]. Quantum feedback control or QFC has also become a subject of intense study, primarily because of the experimental progress in mesoscopic systems. Some important applications include the cooling of nanomechanical resonators and atoms [10, 11]. Other applications of the QFC can be found in [12, 13] and in references therein. For such feedback-controlled systems, the fluctuation theorems need modifications so as to account for the information gained through measurement. These extended relations have been derived for both the classical [14–17] and the quantum [18, 19] cases. The classical results have been recently verified experimentally in [20].

In this paper, we rederive the results for the classical systems, assuming the known fluctuation theorems in their integral as well as detailed form (section 3). The same treatment goes through for deriving the generalized Hatano–Sasa identity, which provides equalities for a driven system from one steady state to another (section 4). We also extend the same treatment to the quantum case (section 5) and show that no matter how many intermediate projective measurements and subsequent feedback are performed, the extended form of the Tasaki–Crooks fluctuation theorem [21] remains unaffected. The efficacy parameters for classical and quantum systems are also obtained. This is a generalization of the study carried out in [19].

2. The system

We have a Brownian particle that is initially prepared in canonical equilibrium with a heat bath at temperature \( T \). Now, we apply an external drive \( \lambda_0(t) \) from time \( t_0 = 0 \) up to \( t = t_1 \). At
Figure 1. A typical phase space trajectory $x(t)$ versus $t$. The actual and the measured states at time $t_k$ are $x_k$ and $m_k$, respectively.

$t_1$, we measure the state of the system and find it to be $m_1$ (see figure 1). Then, we modify our protocol from $\lambda_0(t)$ to $\lambda_{m_1}(t)$ and evolve the system up to time $t_2$, where we perform a second measurement with outcome $m_2$. Subsequently, the protocol is changed to $\lambda_{m_2}(t)$, and so on up to the final form of the protocol $\lambda_{m_N}(t)$, which ends at $t = \tau$ (total time of observation). However, the time instants at which such measurements are taken need not to be equispaced. We assume that there can be a measurement error with the probability $p(m_k|x_k)$, where $m_k$ is the measurement outcome at time $t_k$ when the system’s actual state is $x_k$. Obviously, the value of $\Delta F$ will be different for different protocols $\lambda_{m_k}(t)$.

3. Rederivation of previous results

3.1. Extended Jarzynski equality

The results of this section have already been derived in [17]. The result for a single measurement has been derived in [14]. We briefly outline the derivation in a simpler way, assuming the already known fluctuation theorems in their integral as well as detailed forms. This would be helpful in bringing out the similarity between the classical and quantum approaches. For a given set of observed values $\{m_k\} \equiv \{m_1, m_2, \ldots, m_N\}$, we have a fixed protocol $\{\lambda_{m_k}(t)\}$ which depends on all the measured values $\{m_k\}$, as explained above. For such a given protocol trajectory, the Jarzynski equality must be satisfied. Equation (1) can be rewritten as

$$\int \mathcal{D}[x(t)] p_{eq}(x_0) P_{\{\lambda_{m_k}\}}[x(t)|x_0] \exp[-\beta W(x(t), \{m_k\}) + \beta \Delta F(\lambda_{m_N}(\tau))] = 1,$$

(3)

where $p_{eq}(x_0)$ is the equilibrium distribution of the system at the beginning of the protocol, $P_{\{\lambda_{m_k}\}}[x(t)|x_0]$ is the path probability (from given initial point $x_0$) for this fixed protocol, and $W$ is a function of the path. Now, we average over all possible protocols:

$$\int \{dm_k\} P[\{m_k\}] \int \mathcal{D}[x(t)] p_{eq}(x_0) P_{\{\lambda_{m_k}\}}[x(t)|x_0] \exp[-\beta W(x(t), \{m_k\}) + \beta \Delta F(\lambda_{m_N}(\tau))] = 1.$$  

(4)

Here, $\{dm_k\} \equiv dm_1 dm_2 \cdot \ldots dm_N$ and $P[\{m_k\}]$ is the joint probability of a set of measured values $\{m_k\}$. The mutual information is defined as [17]

$$I = \ln \left[ \frac{p(m_1|x_1)p(m_2|x_2) \cdots p(m_N|x_N)}{P[\{m_k\}]} \right].$$

(5)
The path probability $P_{\{m_{k}\}|x(t)|x_{0}}$ for a fixed protocol and fixed $x_{0}$ is given by

$$P_{\{m_{k}\}|x(t)|x_{0}} = P_{m_{0}}[x_{0} \rightarrow x_{1}]P_{m_{1}}[x_{1} \rightarrow x_{2}] \cdots P_{m_{N}}[x_{N} \rightarrow x_{T}]. \quad (6)$$

Using (5) and (6) in (1), we obtain

$$\int [dm_{k}] \int D[x(t)]p_{eq}(x_{0})P_{m_{0}}[x_{0} \rightarrow x_{1}]p(m_{1}|x_{1})P_{m_{1}}[x_{1} \rightarrow x_{2}] \cdots p(m_{N}|x_{N})P_{m_{N}}[x_{N} \rightarrow x_{T}] \exp \{-\beta W + \beta \Delta F - I\} = 1. \quad (7)$$

To keep the notation simple, the arguments of the joint probability for a phase space trajectory $x(t)$ and measured values $\{m_{k}\}$ in the forward process (for any initial point $x_{0}$) is precisely the factor appearing before $e^{-\beta(W-\Delta F-I)}$ in the integrand in (7). Thus, we arrive at the following generalized Jarzynski equality [14–17]:

$$e^{-\beta(W-\Delta F-I)} = 1. \quad (8)$$

Jensen’s inequality leads to the second law of thermodynamics which is generalized due to the information gain, namely $\langle W \rangle \geq \langle \Delta F \rangle - k_{B}T(I)$. Since $\langle I \rangle \geq 0$ (being a relative entropy [22]), the work performed on a thermodynamic system can be lowered by feedback control.

### 3.2. Detailed fluctuation theorem

To generate a time-reversed trajectory, we first select one of the measurement trajectories $\{m_{k}\}$ (with probability $P(\{m_{k}\})$). We then begin with the system being at canonical equilibrium at the final value of the protocol $\lambda_{m_{N}}(\tau_{f})$, and blindly run the full forward protocol in reverse, i.e. $\{\lambda_{k}(t)\} \rightarrow \{\lambda_{k}^{\dagger}(t)\} \equiv \{\lambda_{k}(\tau - t)\}$. We stress that no feedback is performed during the reverse process in order to respect causality [17]. We denote a point in phase space by $x \equiv (x, p)$ and the corresponding point along the time-reversed trajectory by $x^{\dagger} \equiv (x, -p)$. Here, $x$ and $p$ denote the coordinates and momenta of the particle, respectively. In this case, the probability of a reverse trajectory becomes

$$P_{\{\lambda_{k}^{\dagger}\}|x^{\dagger}(t)|x_{T}} = \frac{P(\{m_{k}\})P_{\lambda_{m_{0}}}[x_{0} \leftarrow x_{1}^{\dagger}]P_{\lambda_{m_{1}}}[x_{1}^{\dagger} \leftarrow x_{2}^{\dagger}] \cdots P_{\lambda_{m_{N}}}[x_{N}^{\dagger} \leftarrow x_{T}^{\dagger}]}{\int [dm_{k}] \int D[x(t)]p_{eq}(x_{0})P_{m_{0}}[x_{0} \rightarrow x_{1}]p(m_{1}|x_{1})P_{m_{1}}[x_{1} \rightarrow x_{2}] \cdots p(m_{N}|x_{N})P_{m_{N}}[x_{N} \rightarrow x_{T}] \exp \{-\beta W + \beta \Delta F - I\}}. \quad (9)$$

$P_{\{\lambda_{k}^{\dagger}\}|x^{\dagger}(t)|x_{T}}$ should not be confused with the joint probability $P_{\{\lambda_{k}\}|x(t),\{m_{k}\}}$. The former represents the probability of the reverse path with fixed values $\{m_{k}\}$ multiplied by the probability $P(\{m_{k}\})$ of choosing the set $\{m_{k}\}$, while the latter represents the probability of a trajectory $x(t)$ along with the measured outcomes $\{m_{k}\}$, if measurements are performed along the reverse process. Now, we take the ratio of the forward and reverse trajectories, and using the definition of mutual information (equation (5)) and the Crooks work theorem for a predetermined protocol [4, 5]

$$\frac{P_{\{\lambda_{k}\}|x(t)}{P_{\{\lambda_{k}^{\dagger}\}|x^{\dagger}(t)|x_{T}}} = e^{\beta(W-\Delta F)} \quad \text{(which in turn depends on the principle of local detailed balance or microscopic reversibility)}, \quad \text{we obtain} \quad [16, 17]$$

$$\frac{P_{\{\lambda_{k}\}|x(t),\{m_{k}\}|x_{T}}}{P_{\{\lambda_{k}^{\dagger}\}|x^{\dagger}(t)|x_{T},\{m_{k}\}}} = e^{\beta(W-\Delta F)+I}. \quad (10)$$

### 3.3. The generalized Jarzynski equality and the efficacy parameter

The Jarzynski equality can also be extended to a different form in the presence of information in terms of the efficacy parameter $\gamma$ [14, 15]. The efficacy parameter $\gamma$ defines how efficient
our feedback control is. Following similar mathematical treatment as in the derivation of the extended Jarzynski equality, we have

\[
\gamma = \langle e^{-\beta(W-\Delta F)} \rangle = \int \mathcal{D}[x(t)]\mathcal{P}P_{\lambda_m}[x(t), \{m_k\}]e^{-\beta(W-\Delta F)}
\]

\[
= \int \mathcal{D}[x(t)]\mathcal{P}P_{\lambda_m}[x^+(t); \{m_k\}]e^\gamma.
\]  

In the last step, we have used relation (10). Using definition (9) for the reverse trajectory, equation (5) for the mutual information and the assumption of time-reversal symmetry of the measurements [14], i.e. \( p(m_k^x|x^+_t) = p(m_k|x_t) \), we obtain

\[
\gamma = \int \mathcal{D}[x(t)]\mathcal{P}P_{\lambda_m}[x^+(t)]p(m_1^x|x_1^x)p(m_2^x|x_2^x)\cdots p(m_N^x|x_N^x),
\]

\[
= \int \mathcal{D}[x(t)]\mathcal{P}P_{\lambda_m}[x^+(t), \{m_k^x\}],
\]

\[
= \int \{dm_k\} \mathcal{P}P_{\lambda_m}[\{m_k^x\}],
\]  

(12)

Here, \( P_{\lambda_m}[x^+(t), \{m_k^x\}] \) is the joint probability of obtaining both the trajectory \( x(t) \) and the set of measured outcomes \( \{m_k^x\} \) in the reverse process under the protocol \( \{x_m^+,\} \), while \( P_{\lambda_m}[\{m_k^x\}] \) is the probability of observing the measurement trajectory \( \{m_k^x\} \) for the reverse process. Physically, this means that \( \gamma \) also describes the sum of the probabilities of observing time-reversed outcomes in the time-reversed protocols, for all sets of possible protocols.

4. Modification in Seifert’s and Hatano–Sasa identities

Now, we derive other new identities which are straightforward generalizations of their earlier counterparts in the presence of information. The mathematics involved is the same as in sections 3.1 and 3.2. For a predetermined protocol, if the pdf of the initial states for the forward path (denoted by \( p_0(x_0) \)) is arbitrary rather than being the Boltzmann distribution, and that of the reverse path is the final distribution of states (denoted by \( p_r(x_r) \)) attained in the forward path, we obtain Seifert’s theorem in lieu of the Jarzynski equality [23, 24]:

\[
\langle e^{-\Delta s_{tot}} \rangle \equiv 1.
\]  

(13)

Here, \( \Delta s_{tot} = \Delta s_{m} + \Delta s \) is the change in the total entropy of bath and system (in units of \( k_B \)). The path-dependent medium entropy change is given by \( \Delta s_{m} = Q/T \), where \( Q \) is the heat dissipated into the bath. \( \Delta s \) is the change in the system entropy given by \( \Delta s = \ln[p_0(x_0)/p_r(x_r)] \). Then, equation (13) can be written as

\[
\int \mathcal{D}[x(t)]p_0(x_0)P_{\lambda_m}[x(t)|x_0] \exp \{-\Delta s_{tot}[x(t)]\} = 1.
\]  

(14)

Averaging over different sets of protocols determined by the different sets of \( \{m_k\} \) values, we obtain

\[
\int \{dm_k\} \mathcal{P}[\{m_k\}] \int \mathcal{D}[x(t)]p_0(x_0)P_{\lambda_m}[x(t)|x_0] \exp \{-\Delta s_{tot}[x(t)]\} = 1.
\]  

(15)

Proceeding exactly in the same way as in section 3.1 (equations (1)–(8)), we readily obtain

\[
\langle e^{-\Delta s_{tot}-I} \rangle \equiv 1.
\]  

(16)

Equation (16) is the generalization of the entropy production theorem and it gives the second law in the form

\[
\langle \Delta s_{tot} \rangle \geq -\langle I \rangle.
\]  

(17)
Thus, with the help of information (feedback), the lower limit of change in the average entropy can be made less than zero, by an amount given by the average mutual information gained. We can similarly show that, in steady states, the detailed fluctuation theorem for the total entropy becomes

$$\frac{P(\Delta s_{\text{tot}})}{P(-\Delta s_{\text{tot}})} = e^{\Delta s_{\text{tot}} + I}.$$  (18)

The Hatano–Sasa identity [25] for overdamped systems can also be similarly generalized:

$$\langle \exp \left[ -\int_0^\tau d\lambda \frac{\partial \phi(x, \lambda)}{\partial \lambda} - I \right] \rangle = 1,$$  (19)

where $\phi(x, \lambda) \equiv -\ln \rho_{ss}(x, \lambda)$, the negative logarithm of the non-equilibrium steady state pdf corresponding to the parameter value $\lambda$. The derivations of (18) and (19) are simple and similar to the earlier derivations (see sections 3.1 and 3.2), so they are not reproduced here.

In terms of the excess heat $Q_{\text{ex}}$, which is the heat dissipated when the system moves from one steady state to another, the above equality (equation (19)) can be rewritten in the following form (for details see [25]):

$$\langle \exp[-\beta Q_{\text{ex}} - \Delta \phi - I] \rangle = 1.$$  (20)

Using Jensen’s inequality, the generalized second law for transitions between non-equilibrium steady states follows, namely

$$T\langle \Delta s \rangle \geq -\langle Q_{\text{ex}} \rangle - k_BT \langle I \rangle,$$  (21)

where $\Delta s$ is the change in the system entropy defined by $\Delta s \equiv -\ln \frac{\rho_{ss}(x, \lambda_0)}{\rho_{ss}(x, \lambda)} = \Delta \phi$.

5. Quantum case

Now, we extend the above treatment to the case of an open quantum system. Hanggi et al have shown [26] that for a closed quantum system, the fluctuation theorems remain unaffected even if projective measurements are performed in-between. This happens in spite of the fact that the probabilities of the forward and backward paths (by ‘path’ we mean here a collection of the successive eigenstates to which the system collapses each time a projective measurement is performed) do change in general. Taking clue from this result, we proceed as follows. The supersystem consisting of the bath and the system evolves under the total Hamiltonian

$$H(t) = H_S(t) + H_{SB} + H_B.$$  (22)

The bath Hamiltonian $H_B$ and the interaction Hamiltonian $H_{SB}$ have been assumed to be time independent, whereas the system Hamiltonian $H_S(t)$ depends explicitly on time through a time-dependent external drive $\lambda(t)$. We first prepare the supersystem at canonical equilibrium at temperature $T$. At initial time $t_0 = 0$, we measure the state of the total system and the collapsed eigenstate is $|i_0\rangle$. The notation means that the total system has collapsed to the $i_0$th eigenstate (of the corresponding measured operator, which is the total Hamiltonian $H(0)$ at $t = 0$) when measured at time $t = 0$. The supersystem consisting of the bath and the system is described by

$$\rho(0) \equiv \frac{e^{-\beta H(0)}}{Y(0)} \Rightarrow \rho_{\text{bath}} = \frac{e^{-\beta E_0}}{Y(0)}.$$  (23)

In the above relation, $\rho_{\text{bath}}$ are the diagonal elements of the initial density matrix of the supersystem and $Y(0)$ is the partition function for the entire supersystem:

$$Y(0) \equiv \text{Tr} \ e^{-\beta H(0)}.$$  (24)

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We then evolve the system up to time $t_1$ under a predetermined protocol $\lambda_0(t)$, and at $t_1$ we once again measure some observable $M$ of the system. Let the outcome be $m_1$, whereas the actual collapsed state is $|i_1\rangle$ corresponding to eigenvalue $M_{i_1}$. This outcome is obtained with the probability $p(m_1|i_1)$, which is an assumed classical error involved in the measurement. Now, we introduce the feedback by modifying the original protocol to $\lambda_{m_1}(t)$, and then continue up to $t_2$, where we perform the measurement to get the outcome $m_2$, with the actual state being $|i_2\rangle$; subsequently, our protocol becomes $\lambda_{m_2}(t)$ and so on. Thus, the probability of getting the set of eigenstates $\{i_k\} = \{|i_0\rangle, |i_1\rangle, \ldots, |i_N\rangle\}$ with the measurement trajectory $\{m_k\}$ is given by

$$P_{\{i_k\}, \{m_k\}} = \rho_{i_0|m_0}|\langle i_0|U_{m_0}(t_1, 0)|i_0\rangle|^2 p(m_1|i_1)|\langle i_1|U_{m_1}(t_2, t_1)|i_1\rangle|^2 \times \cdots \times p(m_N|i_N)|\langle i_N|U_{m_N}(\tau, t_N)|i_N\rangle|^2. \tag{25}$$

Here, $U_{m_1}(t_{i+1}, t_i)$ is the unitary evolution operator from time $t_i$ to time $t_{i+1}$. The reverse process is generated by starting with the supersystem in the canonical equilibrium with the protocol value $\lambda_{m_1}(\tau)$, and blindly reversing one of the chosen protocols of the forward process. Now, we need to perform measurements along the reverse process as well, simply to ensure that the state does collapse to specific eigenstates and we do obtain an unambiguous reverse trajectory in each experiment. However, in order to respect causality [17], we do not perform feedback during the reverse process. The probability for a trajectory that starts from the initial collapsed energy eigenstate $|i_1\rangle$ and follows the exact sequence of collapsed states as the forward process is given by

$$P_{\{i_k\}, \{m_k\}} = |\langle i_0|\Theta^* U_{\lambda_{m_1}^*}^\dagger (\tilde{t}_1, \tilde{t}_1)\Theta |i_1\rangle|^2 |\langle i_1|\Theta^* U_{\lambda_{m_1}^*}^\dagger (\tilde{t}_2, \tilde{t}_1)\Theta |i_2\rangle|^2 \times \cdots \times |\langle i_N|\Theta^* U_{\lambda_{m_1}^*}^\dagger (\tilde{t}_N, \tilde{t}_N)\Theta |i_N\rangle|^2 \rho_{i_{1N}}P\{m_k\}. \tag{26}$$

Here, $\Theta$ is a time-reversal operator [26] and $|\langle i_0|\Theta^* U_{\lambda_{m_1}^*}^\dagger (\tilde{t}_1, \tilde{t}_1+1)\Theta |i_1\rangle|^2$ is the probability of transition from the time-reversed state $\Theta|\tilde{t}_{k+1}\rangle$ to $\Theta|\tilde{t}_k\rangle$ under the unitary evolution with the reverse protocol: $U_{\lambda_{m_1}^*}^\dagger (\tilde{t}_k, \tilde{t}_{k+1}) = U_{\lambda_{m_1}^*}^\dagger (\tau - t_k, \tau - t_{k+1})$. Here, the tilde symbol implies time calculated along the reverse trajectory: $\tilde{t} \equiv \tau - t$. $\rho_{i_{1N}}$ is the diagonal element of the density matrix when the system is at canonical equilibrium at the beginning of the reverse process:

$$\rho_{i_{1N}} = \frac{e^{-\beta E_{i_{1N}}}}{\cal Z(\tau)}. \tag{27}$$

Now, we have

$$\Theta^* U_{\lambda_{m_1}^*}^\dagger (\tilde{t}_k, \tilde{t}_{k+1})\Theta = \Theta^* T \exp \left[ -\frac{i}{\hbar} \int_{\tilde{t}_{k+1}}^{\tilde{t}_k} H_{\lambda_{m_1}^*}(t) \, dt \right] \Theta$$

$$= T \exp \left[ +\frac{i}{\hbar} \int_{\tilde{t}_{k+1}}^{\tilde{t}_k} H_{\lambda_{m_1}^*}(t) \, dt \right],$$

where $T$ implies time-ordering. Changing the variable $t \rightarrow \tau - t$, we obtain

$$\Theta^* U_{\lambda_{m_1}^*}^\dagger (\tilde{t}_k, \tilde{t}_{k+1})\Theta = T \exp \left[ -\frac{i}{\hbar} \int_{\tilde{t}_{k+1}}^{\tilde{t}_k} H_{\lambda_{m_1}^*}(\tau - t) \, dt \right],$$

$$= T \exp \left[ +\frac{i}{\hbar} \int_{\tilde{t}_{k+1}}^{\tilde{t}_k} H_{\lambda_{m_1}^*}(t) \, dt \right],$$

$$= U_{\lambda_{m_1}^*}(t_k, t_{k+1}) = U_{\lambda_{m_1}^*}^\dagger (t_{k+1}, t_k). \tag{28}$$

Accordingly,

$$|\langle i_k|\Theta^* U_{\lambda_{m_1}^*}^\dagger (\tilde{t}_k, \tilde{t}_{k+1})\Theta |i_{k+1}\rangle| = |\langle i_k|U_{\lambda_{m_1}^*}^\dagger (t_{k+1}, t_k)\tilde{i}_{k+1}\rangle| = |\langle i_{k+1}|U_{\lambda_{m_1}^*} (t_{k+1}, t_k)\tilde{i}_k\rangle|^\dagger. \tag{29}$$
Thus, while dividing (25) by (26), all the modulus squared terms cancel, and we have
\[
\frac{P_{[\lambda_{m_{1}}]}([i_{k}], \{m_{k}\})}{P_{[\lambda_{m_{1}}]}([i_{k}]; \{m_{k}\})} = \frac{p(m_{1}|i_{1}) \cdots p(m_{N}|i_{N})}{P[m_{k}]} = \frac{Y(\tau)}{Y(0)} e^{\beta W(I)},
\]
where \(W \equiv E_{t} - E_{0}\) is the work done by the external drive on the system. This follows from the fact that the external forces act only on the system \(S\). Now, we follow [27] and define the equilibrium free energy of the system, \(F_{S}(t)\), as the thermodynamic free energy of the open system, which is the difference between the total free energy \(F(t)\) of the supersystem and the bare bath free energy \(F_{B}\):
\[
F_{S}(t) = F(t) - F_{B}.
\]
Here, \(t\) specifies the values of the external parameters in the course of the protocol at time \(t\). From the above equation, the partition function for the open system is given by [27]
\[
Z_{S}(t) = \frac{Y(t)}{Z_{B}} = \frac{\text{Tr}_{S,B} e^{-\beta H(t)}}{\text{Tr}_{B} e^{-\beta H_{B}}},
\]
where \(S\) and \(B\) represent the system and bath variables, respectively. Since \(Z_{B}\) is independent of time, using (32) in (30), we have
\[
\frac{P_{[\lambda_{m_{1}}]}([i_{k}], \{m_{k}\})}{P_{[\lambda_{m_{1}}]}([i_{k}]; \{m_{k}\})} = \frac{Z_{S}(\tau)}{Z_{S}(0)} e^{\beta(W(I))}.
\]

The above relation is the extended form of the Tasaki–Crooks detailed fluctuation theorem for open quantum systems in the presence of feedback where the measurement process involves classical errors. From (33), the quantum Jarzynski equality follows:
\[
\sum_{\{i_{1}\}} \int [dm_{k}] P_{[\lambda_{m_{1}}]}([i_{k}], \{m_{k}\}) e^{-\beta(W(I))} = \sum_{\{i_{1}\}} \int [dm_{k}] P_{[\lambda_{m_{1}}]}([i_{k}]; \{m_{k}\}] = 1,
\]
i.e.
\[
\langle e^{\beta(W(I)) - 1} \rangle = 1.
\]
This is valid for the open quantum system and is independent of the coupling strength and the nature of the bath.

The quantum efficacy parameter is defined as \(\langle e^{\beta(W(I)) - 1} \rangle \equiv \gamma\), and the calculation of \(\gamma\) is exactly in the spirit of section 3.3, except that \(\int D[x(t)]\) is replaced by \(\sum_{\{i_{1}\}}\), i.e. summation over all possible eigenstates. Finally, we obtain the same result, namely
\[
\gamma = \int [dm_{k}] P_{[\lambda_{m_{1}}]}([m_{k}]).
\]

6. Discussion and conclusions

In conclusion, we have rederived several extended fluctuation theorems in the presence of feedback. To this end, we have used several equalities given by the already known fluctuation theorems. We have extended the quantum fluctuation theorems for open systems, following the earlier treatment [26, 27]. No assumption is made on the strength of the system–bath coupling and the nature of the environment. However, we have assumed that the measurement process leading to the information gain involves classical errors. The fluctuation–dissipation theorem is violated in non-equilibrium systems. The effects of feedback on this violation are being studied [28]. This is the subject of our ongoing work.
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