The space of arcs of an algebraic variety

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In memory of John Forbes Nash, Jr.

Abstract. The paper surveys several results on the topology of the space of arcs of an algebraic variety and the Nash problem on the arc structure of singularities.

1. Introduction

In 1968, Nash wrote a paper on the arc structure of singularities of complex algebraic varieties [Nas95]. While the paper was only published many years later, its content was promoted by Hironaka and later by Lejeune-Jalabert, and it is thanks to them that the mathematical community came to know about it.

In that paper, the space of arcs of an algebraic variety is regarded for the first time as the subject of investigation in its own right. Other papers which appeared in those years and provide insight to the topological structure of spaces of arcs are [Gre66, Kol73]. Since then, spaces of arcs have become a central object of study by algebraic geometers. They provide the underlying space in motivic integration, where arcs take the role of the $p$-adic integers in $p$-adic integration [Kon95, DL99, Bat99]. The relationship between constructible sets in arc spaces and invariants of singularities in the minimal model program has yielded important applications in birational geometry (e.g., [Mus01, Mus02,EMY03, EM04]).

Nash viewed the space of arcs as a tool to study singularities of complex algebraic varieties, and for this reason he focused on the set of arcs on a variety that originate from the singular points. Nash realized that there is a close connection between the families of arcs through the singularities and certain data associated with resolutions of singularities whose existence had just been established a few years earlier [Hir64]. He gave a precise formulation predicting that such families of arcs should correspond to those exceptional divisors that are “essential” for all resolutions of singularities. Establishing this correspondence became known as the Nash problem.

The Nash problem has remained wide open until recently and is still not completely understood. This note focuses on this problem and the progress made...
around it in recent years. The correspondence proposed by Nash has been shown to hold in dimension two [FdBPP12] and to fail, in general, in all higher dimensions [IK03, dF13, JK13]. However, this should not be viewed as the end of the story, but rather as an indication of the difficulty of the problem. Partial results have been obtained in higher dimensions (e.g., [IK03, dFD16]), and a complete solution of the problem will only be reached once the correct formulation is found.

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2. The space of arcs

We begin this section with a quick overview of the definition of arc space, referring to more in depth references such as [DL99, Voj07, EM09] for further details.

Let $X$ be a scheme of finite type over a field $k$. The space of arcs $X_\infty$ of $X$ is a scheme whose $K$-valued points, for any field extension $K/k$, are formal arcs

$$\alpha: \text{Spec} K[[t]] \to X.$$ 

It is constructed as the inverse limit of the jet schemes $X_m$ of $X$, which parameterize jets $\gamma: \text{Spec} K[t]/(t^{m+1}) \to X$. These schemes are defined as follows.

**Definition 2.1.** For every $m \in \mathbb{N}$, the $m$-th jet scheme $X_m$ of $X$ is the scheme representing the functor that takes a $k$-algebra $A$ to the set of $A$-valued $m$-jets

$$X(A[t]/(t^{m+1})) := \text{Hom}_k(\text{Spec} A[t]/(t^{m+1}), X).$$

For every $p \geq m$ there is a natural projection map $X_p \to X_m$ induced by the truncation homomorphism $A[t]/(t^{p+1}) \to A[t]/(t^{m+1})$. These projections are affine, and hence one can take the inverse limit of the corresponding projective system in the category of schemes.

**Definition 2.2.** The space of arcs (or arc space) of $X$ is the inverse limit

$$X_\infty := \lim_{\leftarrow} X_m.$$ 

The next property is not a formal consequence of the definition, and the proof uses methods of derived algebraic geometry. We do not know if there is a more direct proof.

**Theorem 2.3 (Bhatt [Bha, Corollary 1.2]).** The arc space $X_\infty$ represents the functor that takes a $k$-algebra $A$ to the set of $A$-valued arcs

$$X(A[[t]]) := \text{Hom}_k(\text{Spec} A[[t]], X).$$

The truncations $A[[t]] \to A[[t]/(t^{m+1})]$ induce projection maps $\pi_{X,m}: X_\infty \to X_m$. Taking $m = 0$, we obtain the projection

$$\pi_X: X_\infty \to X$$

which maps an arc $\alpha(t) \in X_\infty(K)$ to the point $\alpha(0) \in X(K)$ where the arc stems from.$^1$

$^1$The notation $\alpha(t)$ refers to the fact that, in local parameters of $X$ at $\alpha(0)$, the arc is given by formal power series in $t$. 

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Remark 2.4. Arc spaces are closely connected to valuation theory (cf. [Reg95, Theorem 1.10]; see also [Pl605]). For any given field extension $K/k$, an arc $\alpha: \text{Spec } K[[t]] \to X$ defines a valuation
\[
\text{val}_\alpha: \mathcal{O}_{X,\alpha(0)} \to \mathbb{N} \cup \{\infty\},
\]
given by $\text{val}_\alpha(h) := \text{ord}_h(\alpha^* h)$. Here $0$ denotes the closed point of $\text{Spec } K[[t]]$. We will denote by $\eta$ the generic point. If $X$ is a variety and $\alpha(\eta)$ is the generic point of $X$, then $\text{val}_\alpha$ extends to a valuation
\[
\text{val}_\alpha: k(X)^* \to \mathbb{Z}.
\]
For every irreducible closed set $C \subset X_\infty$, we denote by $\text{val}_C$ the valuation defined by the generic point of $C$. We will use several times the fact that if an arc $\alpha$ is a specialization of another arc $\beta$, then $\text{val}_\alpha(h) \geq \text{val}_\beta(h)$ for every $h \in \mathcal{O}_{X,\alpha(0)}$, which can be easily seen by observing that, writing $\alpha^* h = \sum a_i t^i$ and $\beta^* h = \sum b_i t^i$, each coefficient $a_i$ is a specialization of the corresponding coefficient $b_i$.

The space of arcs of an affine space $\mathbb{A}^n$ is easy to describe. For every $i \geq 0$, we introduce $n$-ples of variables $x_1^{(i)}, \ldots, x_n^{(i)}$. We identify $x_j = x_j^{(0)}$ and write for short $x_j' = x_j^{(1)}$ and $x_j'' = x_j^{(2)}$. The arc space of $\mathbb{A}^n$ is the infinite dimensional affine space
\[
(\mathbb{A}^n)_\infty = \text{Spec } k[x_j, x_j', x_j'', \ldots]_{1 \leq j \leq n},
\]
where a $K$-valued point $(a_j, a_j', a_j'', \ldots)_{1 \leq j \leq n}$ corresponds to the $K$-valued arc $\alpha(t)$ with components $x_j(t) = a_j + a_j' t + a_j'' t^2 + \ldots$.

If $X$ is an affine scheme, defined by equations $f_i(x_1, \ldots, x_n) = 0$ in an affine space $\mathbb{A}^n$, then $X_\infty$ parameterizes $n$-ples of formal power series $(x_1(t), \ldots, x_n(t))$ subject to the conditions $f_i(x_1(t), \ldots, x_n(t)) = 0$ for all $i$. These conditions describe $X_\infty$ as a subscheme in an infinite dimensional affine space defined by infinitely many equations in infinitely many variables. The equations of $X_\infty$ in $(\mathbb{A}^n)_\infty$ can be generated using Hasse–Schmidt derivations [Voj07]. There is a sequence $(D_0, D_1, D_2, \ldots)$ of $k$-linear maps
\[
D_i: k[x_1, \ldots, x_n] \to k[x_j, x_j', x_j'', \ldots]_{1 \leq j \leq n}
\]
uniquely determined by the conditions
\[
D_i(x_j) = x_j^{(i)}, \quad D_k(f g) = \sum_{i+j=k} D_i(f) D_j(g),
\]
and the ideal of $X_\infty$ in $(\mathbb{A}^n)_\infty$ is generated by all the derivations $D_i(f_j)$, $i \geq 0$, of a set of generators $f_j$ of the ideal of $X$ in $\mathbb{A}^n$.

The arc space of an arbitrary scheme $X$ can be glued together, scheme theoretically, from the arc spaces of its affine charts. The Zariski topology of the arc space agrees with the inverse limit topology. Excluding of course the trivial case where $X$ is zero dimensional, $X_\infty$ is not Noetherian and is not a scheme of finite type. Yet, some finiteness is built into it.

We henceforth assume the following:

$X$ is a variety defined over an algebraically closed field $k$ of characteristic zero.
We will be working in this setting throughout the paper until the last section where varieties over fields of positive characteristics will be considered.

If $X$ is a smooth $n$-dimensional variety, then each jet scheme $X_m$ is smooth and $X_\infty$ is the inverse limit of a system of locally trivial $\mathbb{A}^n$ fibrations $X_{m+1} \to X_m$. This can be seen by reduction to the case of an affine space using Noether normalization, or equivalently by Hensel’s lemma. It follows in this case that $X_\infty$ is an integral scheme and the projections $X_\infty \to X_m$ are surjective. Furthermore, $\pi^{-1}_X(S)$ is irreducible for any irreducible set $S \subset X$.

**Remark 2.5.** The first jet scheme $X_1$ of a smooth variety $X$ is the same as the tangent bundle of $X$. However, for any $m \geq 1$ the fibration $X_{m+1} \to X_m$ does not have a natural structure of vector bundle. For example, the nonlinear change of coordinates $(u, v) = (x + y^2, y)$ on $X = \text{Spec} k[x, y]$ induces the affine change of coordinates $(u'', v'') = (x'' + b^2, y'')$ on the fiber of $X_2 \to X_1$ over a point $(0, 0, a, b) \in X_1 = \text{Spec}[x, y, x', y']$. In general, for every $m$ there is a natural section $X \to X_m$ which takes a point of $X$ to the constant $m$-jet through that point, but there is no natural section $X_m \to X_p$ for $p > m > 0$.

If $X$ is singular, then the jet schemes $X_m$ can have several irreducible components and non-reduced structure, the maps $X_{m+1} \to X_m$ fail to be surjective, there are jumps in their fiber dimensions, and the inverse image $\pi^{-1}_X(S)$ of an irreducible set $S \subset X$ may fail to be irreducible. These pathologies make the study of $X_\infty$ a difficult task.

The systematic study of the space of arcs began in the sixties through the works of Greenberg, Nash, and Kolchin. In this context, Greenberg’s approximation theorem gives the following property.

**Theorem 2.6 (Greenberg [Gre66, Theorem 1]).** For any system of polynomials $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ there are numbers $N, c \geq 1$ and $s \geq 0$ such that for any $m \geq N$ and every $x(t) \in k[[t]]^n$ such that $f_i(x(t)) \equiv 0 \pmod{t^m}$, there exists $y(t) \in k[[t]]^n$ such that $y(t) \equiv x(t) \pmod{t^{m/c-s}}$ and $f_i(y(t)) = 0$.

The image of $X_\infty$ in $X_m$ is the intersection of the images of the jet schemes $X_p$ for $p \geq m$, which form a nested sequence of constructible sets. The content of Greenberg’s theorem is that the sequence stabilizes, which means that the image of $X_\infty$ agrees with the image of $X_p$ for $p \gg m$. It follows in particular that for every $m$ the image of $X_\infty$ in $X_m$ is constructible. This can be viewed as the first structural result on arc spaces.

Following the terminology of [Gro61, Definition (9.1.2)], a constructible set in $X_\infty$ is, by definition, a finite union of finite intersections of retrocompact open sets and their complements, where a subset $Z \subset X_\infty$ is said to be retrocompact if for every quasi-compact open set $U \subset X_\infty$, the intersection $Z \cap U$ is quasi-compact.\(^\dagger\) This means that a subset $C \subset X_\infty$ is constructible if and only if it is the (reduced) inverse image of a constructible set on some finite level $X_m$ [Gre66, Théorème (8.3.11)]. Such sets are nowadays commonly called cylinders. Theorem 2.6 implies the following property.\(^\ddagger\)

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\(^\dagger\) According to this definition, a closed subset of $X_\infty$ needs not be constructible. For instance if $Z \subset X$ is a proper closed subscheme, then $Z_\infty$ is closed in $X_\infty$ but is not constructible.

\(^\ddagger\) The property can also be viewed as a consequence of Pas’ quantifier elimination theorem [Pas89].
Corollary 2.7. The image at any finite level $X_m$ of a constructible subset of $X_\infty$ is constructible.

The second theorem we want to review is Kolchin’s irreducibility theorem. Since the proof goes in the direction of the main focus of this paper, we outline it. The proof given here, which is taken from [EM09], is different from the original proof of Kolchin. Other proofs of this property can be found in [Gil02, IK03, NS10].

Theorem 2.8 (Kolchin [Kol73, Chapter IV, Proposition 10]). The arc space $X_\infty$ of a variety $X$ is irreducible.

Proof. Let $f : Y \to X$ be a resolution of singularities. Let $Z \subset X$ be the indeterminacy locus of $f^{-1}$ and $E = f^{-1}(Z)_{\text{red}}$ be the exceptional locus of $f$. Since $Y$ is smooth, its arc space $Y_\infty$ is irreducible. It is therefore sufficient to show that the induced map $f_\infty : Y_\infty \to X_\infty$ is dominant. Since $f$ is an isomorphism in $f^{-1}(X \setminus Z)$, the valuative criterion of properness implies that every arc $\alpha$ on $X$ that is not entirely contained in $Z$ lifts to $Y$. If $Z = \bigcup Z_i$ is the decomposition into irreducible components, then $Z_\infty = \bigcup (Z_i)_\infty$, set theoretically. By induction on dimension, each $(Z_i)_\infty$ is irreducible, and therefore if $U_i \subset Z_i$ is a dense open subset then $(U_i)_\infty$ is dense in $(Z_i)_\infty$. By generic smoothness, we can find a dense open subset $U_i \subset Z_i$ and an open set $V_i \subset E$ such that $f$ restricts to a smooth map $V_i \to U_i$. Every arc on $U_i$ lifts to $V_i$, and hence $(U_i)_\infty \subset f_\infty((V_i)_\infty)$. Therefore each $(Z_i)_\infty$ is in the closure of $f_\infty(Y_\infty)$. This shows that $f_\infty$ is dominant, and the theorem follows. □

The next example shows that $f_\infty$ needs not be surjective.

Example 2.9. Let $X \subset \mathbb{A}^3$ be the Whitney umbrella, defined by the equation $xy^2 = z^2$. Its singular locus $X_{\text{sing}}$ is the $x$-axis, and the normalization $Y \to X$ gives a resolution of singularities. The exceptional divisor $E \subset Y$ maps generically two-to-one over $X_{\text{sing}}$ with ramification at the origin. It follows that for every power series $x(t) \in k[[t]]$ with ord$_t(x(t)) = 1$, the arc $\alpha = (x(t), 0, 0) \in X_\infty$, which is a smooth arc on $X_{\text{sing}}$ passing through the origin, cannot lift to $E$ and hence is not in $f_\infty(Y_\infty)$.

More information on the structure of the arc space of a singular variety $X$ can be obtained by a careful analysis of the truncation maps $X_m \to X_n$, defined for $m > n$, and the maps $f_m : Y_m \to X_m$ induced by a resolution of singularities $f : Y \to X$. Understanding these maps is a delicate but rewarding task. Both sets of maps were studied by Denef and Loeser in connection to motivic integration [DL99], and their description plays a key role in relating the geometry of arc spaces to invariants of singularities in the minimal model program.

A consequence of one of the results of [DL99] is that images of many constructible sets in the arc space $Y_\infty$ of a resolution $Y \to X$ are not far from being constructible in $X_\infty$.

Theorem 2.10. Let $f : Y \to X$ be a resolution of singularities and let $C \subset Y_\infty$ be a constructible set. Assume that none of the irreducible components of $C$ is contained in the arc space of the exceptional locus $\text{Ex}(f)$ of $f$. Then there is a constructible set $D \subset X_\infty$ such that $D \subset f_\infty(C) \subset \overline{D}$.
Proof. Without loss of generality, we can assume that \( \overline{C} \) is irreducible. Then there is a constructible set \( S \subset Y_p \), for some \( p \geq 0 \), such that \( S \) is irreducible and \( C = \pi_{Y,p}^{-1}(S) \), where \( \pi_{Y,p}: Y_\infty \to Y_p \) is the truncation map. Let \( \text{Jac}_f := \text{Fitt}^0(\Omega_{Y/X}) \) denote the Jacobian ideal sheaf of \( f \). Since \( C \not\subset \text{Ex}(f)_\infty \), we have \( e := \text{val}_C(\text{Jac}_f) < \infty \). By replacing \( p \) with a larger integer, we can assume that \( p \geq 2e \) and \( C = \pi_{Y,p}^{-1}(\pi_{Y,p-e}(C)) \).

There is a dense relatively open subset \( S^\circ \subset S \) such that, letting \( C^\circ := \pi_{Y,p}^{-1}(S^\circ) \), we have \( \text{val}_\alpha(\text{Jac}_f) = e \) for all \( \alpha \in C^\circ \). Note that \( C^\circ \) is dense in \( C \) and \( \pi_{Y,m}(C^\circ) \) is constructible in \( Y_m \) for every \( m \).

By [DL99, Lemma 3.4] (see also (a') in the proof), for every \( m \geq p \) the fiber \( F \) of \( f_m: Y_m \to X_m \) through a point of \( \pi_{Y,m}(C^\circ) \) is an affine space of dimension \( e \) which is contained in a fiber of the projection \( Y_m \to Y_{m-e} \). This implies that \( F \) is entirely contained in \( \pi_{Y,m}(C^\circ) \), and therefore we have

\[
f_m^{-1}(f_m(\pi_{Y,m}(C^\circ))) = \pi_{Y,m}(C^\circ).
\]

Using the commutativity of the diagram

\[
\begin{array}{ccc}
Y_\infty & \xrightarrow{f_\infty} & X_\infty \\
\pi_{Y,m} \downarrow & & \downarrow \pi_{X,m} \\
Y_m & \xrightarrow{f_m} & X_m
\end{array}
\]

we see that \( f_\infty(C^\circ) = \pi_{X,m}^{-1}(f_m(\pi_{Y,m}(C^\circ))) \). Note that this is a constructible set in \( X_\infty \) since \( f_m(\pi_{Y,m}(C^\circ)) \) is constructible in \( X_m \). As we have

\[
f_\infty(C^\circ) \subset f_\infty(\overline{C^\circ}) \subset \overline{f_\infty(C^\circ)}
\]

and \( \overline{C^\circ} = C \), we can take \( D := f_\infty(C^\circ) \). \( \square \)

3. Arcs through the singular locus

Let \( X \) be a variety defined over an algebraically closed field \( k \) of characteristic zero.

The main contribution of [Nas95] is the realization that, on \( X \), there are only finitely many maximal families of arcs through the singularities, that is to say that the set \( \pi_X^{-1}(X_{\text{sing}}) \subset X_\infty \) has finitely many irreducible components. Moreover, each such family corresponds to a specific component of the inverse image of \( X_{\text{sing}} \) on a resolution of singularities of \( X \).

This property follows by a variant of the proof of Kolchin’s theorem given in the previous section. We should stress that Nash’s result predates Kolchin’s theorem. The argument goes as follows.

Let \( f: Y \to X \) be a resolution of singularities, and let

\[
f^{-1}(X_{\text{sing}})^\text{red} = \bigcup_{i \in I} E_i
\]

be the decomposition into irreducible components. The set \( I \) is finite because \( Y \) is Noetherian, and each \( \pi_Y^{-1}(E_i) \) is irreducible because \( Y \) is smooth. Arguing as in the proof of Theorem 2.8, one deduces that the map \( f_\infty \) restricts to a dominant
map $\pi_Y^{-1}(f^{-1}(X_{\text{sing}})) \to \pi_X^{-1}(X_{\text{sing}})$, and therefore there is a finite decomposition into irreducible components

$$\pi_X^{-1}(X_{\text{sing}})^{\text{red}} = \bigcup_{i \in I} C_i,$$

where $C_i := f_\infty(\pi_Y^{-1}(E_i)) \subset X_\infty$.

Let $J \subset I$ be the set of indices $j$ for which $C_j$ is a maximal element of $\{C_i\}_{i \in I}$, where maximality is intended with respect to inclusions.

It is convenient at this point to introduce the following notation.

**Definition 3.1.** The maximal divisorial set associated to a prime divisor $E$ on a resolution $Y$ over $X$ is the set

$$C_X(E) := f_\infty(\pi_Y^{-1}(E)) \subset X_\infty.$$  

**Remark 3.2.** The definition of $C_X(E)$ does not require the existence of a resolution. Only assuming that $Y$ is normal one can take $C_X(E) := f_\infty(\pi_Y^{-1}(E \cap Y_{\text{sm}}))$.

In this way, the definition extends to positive characteristics.

We can then state Nash’s result as follows.

**Theorem 3.3 (Nash [Nas95, Propositions 1 and 2]).** The set of arcs through the singular locus $X_{\text{sing}}$ of a variety $X$ has a decomposition into finitely many irreducible components given by

$$\pi_X^{-1}(X_{\text{sing}})^{\text{red}} = \bigcup_{j \in J} C_X(E_j)$$

where each $E_j$ is a prime divisor over $X$.

**Remark 3.4.** To be precise, in [Nas95] arcs are assumed to be defined by converging power series. If $X$ is a complex variety, up to rescaling of the parameter, any such arc is given by a holomorphic map $\alpha : D \to X$, where $D = \{ t \in \mathbb{C} \mid |t| < 1 \}$ is the open disk. It is interesting to compare Nash’s result with the setting considered in [KN15], where holomorphic maps from the closed disk $\overline{D} = \{ t \in \mathbb{C} \mid |t| \leq 1 \}$ are studied instead. In that paper, Kollár and Némethi look at the space of short arcs, which are those holomorphic maps $\phi : \overline{D} \to X$ such that $\text{Supp} \phi^{-1}(X_{\text{sing}}) = \{0\}$. The space of short arcs of a normal surface singularity relates to the link of the singularity, and it satisfies a McKay correspondence property for isolated quotient singularities in all dimensions. In general, the space of short arcs can have infinitely many connected components, thus presenting a quite different behavior from the case of formal arcs.

**Definition 3.5.** An irreducible set $C \subset X_\infty$ is said to be thin if there exists a proper closed subscheme $Z \subset X$ such that $C \subset Z_\infty$. An irreducible set $C \subset X_\infty$ that is not thin is said to be fat.

**Corollary 3.6.** Every irreducible component of $\pi_X^{-1}(X_{\text{sing}})$ is fat in $X_\infty$.

**Proof.** It suffices to observe that the arc corresponding to the generic point of each $C_X(E_i)$ dominates the generic point of $X$. □

Actually, the two cited propositions in [Nas95] deal with arbitrary algebraic sets $W \subset X$; here we are only considering the case $W = X_{\text{sing}}$. It is asserted in [Nas95, Proposition 2] that for an arbitrary algebraic set $W \subset X$, every irreducible component of $\pi_X^{-1}(W)$ corresponds to some component of the inverse image of $W$. 
in the resolution. This property does not seem to hold in such generality, at least in the way we have interpreted its meaning. An example where this property fails is given next.

**Example 3.7.** Let $X = (xy^2 = z^2) \subset \mathbb{A}^3$ be the Whitney umbrella, as in Example 2.9. Denote by $S = (y = z = 0)$ the singular locus of $X$, and let $O \in \mathbb{A}^3$ be the origin in the coordinates $(x, y, z)$.

We claim that $\pi_S^{-1}(O)$ is an irreducible component of $\pi_X^{-1}(O)$. Note that $\pi_S^{-1}(O)$ is irreducible since $S$ is smooth, and it is thin in $X_\infty$ since it is contained in $S_\infty$. In particular, it is not of the form $C_X(E)$ for any prime divisor $E$ over $X$. In fact, as it is explained in Example 2.9, $\pi_S^{-1}(O)$ is not dominated by any set in the space of arcs of any resolution of $X$.

For short, let $F$ and $G$ respectively denote the fibers of $(\mathbb{A}^3)_3 \to \mathbb{A}^3$ and $X_3 \to X$ over $O$. Using the coordinates induced by $x, y, z$ via Hasse–Schmidt derivation, we have $F = \text{Spec } k[x', y', z', x'', y'', z'', x'''', y''', z''']$, and $G$ is defined in $F$ by the equations $(z')^2 = 0$ and $x'(y')^2 - 2z'z'' = 0$. In particular, $G$ has a decomposition into irreducible components $G_{\text{red}} = V(x', z') \cup V(y', z')$. The image of $\pi_S^{-1}(O)$ in $X_3$ is not contained in the component $V(x', z')$ of $G$ since, for instance, it contains the arc $(t, 0, 0)$. On the other hand, every arc $\alpha(t) = (x(t), y(t), z(t))$ in $\pi_X^{-1}(O) \setminus \pi_S^{-1}(O)$ lies over $V(x', z')$. Indeed, since $\alpha$ satisfies $\alpha(0) = 0$ and $(y(t), z(t)) \neq (0, 0)$, the condition $x(t)y(t)^2 = z(t)^2$ implies that the coefficients of $t$ in $x(t)$ and $z(t)$ must be zero. Therefore $\pi_S^{-1}(O)$ is not contained in the closure of $\pi_X^{-1}(O) \setminus \pi_S^{-1}(O)$. This proves our claim.

Going back to the discussion leading to Theorem 3.3, one should remark that while the index set $I$ depends on the choice of resolution, the irreducible decomposition of $\pi_X^{-1}(X_{\text{sing}})$ is intrinsic to $X$. The point is that $J$ may be strictly smaller than $I$, which means that there may be inclusions $C_i \subset C_j$.

Suppose, for instance, that $f$ is an isomorphism over the smooth locus of $X$. Before taking closures, $f_\infty(\pi_Y^{-1}(E_i))$ cannot be a subset of $f_\infty(\pi_Y^{-1}(E_j))$ for $i \neq j$, since $f_\infty$ induces a bijection

$$Y_\infty \setminus (f^{-1}(X_{\text{sing}})_\infty) \xrightarrow{1-1} X_\infty \setminus (X_{\text{sing}})_\infty.$$  

Away from $(f^{-1}(X_{\text{sing}})_\infty)$ and $(X_{\text{sing}})_\infty$, which we can consider as subsets of **measure zero or infinite codimension**, $f_\infty$ is a continuous bijection but not a homeomorphism, and we can regard the two arc spaces as being identified as sets (away from these sets of measure zero), with the left hand side equipped with a stronger topology. This explains why some sets $f_\infty(\pi_Y^{-1}(E_i))$ may lie in the closure of some other sets $f_\infty(\pi_Y^{-1}(E_j))$.

One would like to be able to recognize $J$ in $I$ by only looking at resolution of singularities. Put another way:

Is there a characterization of the irreducible components of $\pi_X^{-1}(X_{\text{sing}})_{\text{red}}$ in terms of resolutions of $X$?

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4These notions can be made precise. Measure zero is intended from the point of view of motivic integration. The codimension of a closed subset of the space of arcs can be defined in two ways, either as the minimal dimension of the local rings at the minimal primes, or as the limit of the codimensions of the projections of the set to the sets of liftable jets. These two notions of codimension may differ, but the property of being finite is equivalent in the two notions.
This question has a natural formulation in the language of valuations. It is elementary to see that the set $C_X(E)$ only depends on the valuation $\text{val}_E$ and 

$$\pi_X(C_X(E)) = c_X(E),$$

the center of $\text{val}_E$ in $X$.\(^5\) The generic point $\alpha$ of $C_X(E)$ is the image of the generic point $\tilde{\alpha}$ of $\pi_Y^{-1}(E)$, and therefore we have

$$\text{val}_\alpha(h) = \text{val}_E(h \circ f) = \text{ord}_E(h \circ f)$$

for any rational function $h \in k(X)^*$. This implies that the valuation associated to $C_X(E)$ is equal to the divisorial valuation defined by $E$.

**Remark 3.8.** The maximal divisorial set associated to a divisorial valuation $\text{val}_E$ captures more information than just the valuation itself. Its codimension computed on the level of jet schemes relates to the order of vanishing along $E$ of $\pi_X^{-1}(X_{\text{sing}})$ and is implicit in the change-of-variable formula in motivic integration. Because of this, maximal divisorial sets provide the essential link between arc spaces and singularities in birational geometry. These sets have been studied from this point of view in [ELM04, Ish08, dFEI08]. The connection between the dimension of the local ring of $X_\infty$ at the generic point of a maximal divisorial set (or of its completion) and other invariants of singularities that are measured by the valuation is more obscure.

Theorem 3.3 yields a natural identification between the irreducible components of $\pi_X^{-1}(X_{\text{sing}})$ and certain divisorial valuations on $X$. Bearing this in mind, we give the following definition.

**Definition 3.9.** A Nash valuation of $X$ is the divisorial valuation $\text{val}_C$ associated to an irreducible component $C$ of $\pi_X^{-1}(X_{\text{sing}})$.

One of the motivations of [Nas95] is to understand Nash valuations from the point of view of resolution of singularities. A precise formulation of this problem, which is discussed in Section 6, has become known as the Nash problem.

To address this problem, given an arbitrary resolution $f: Y \to X$ and a prime divisor $E$ contained in $f^{-1}(X_{\text{sing}})$, one needs to analyze the condition that $C_X(E)$ is strictly contained in some irreducible component of $\pi_X^{-1}(X_{\text{sing}})$.

The idea, which goes back to Lejeune-Jalabert [LJ80], is to detect such proximity by producing a 1-parameter family of arcs which originates in $C_X(E)$ and moves outside of it but still within $\pi_X^{-1}(X_{\text{sing}})$. As intuitive as it may be, the existence of such family of arcs is a delicate fact.

The following curve selection lemma formalizes this idea. It should be clear that the arc $\Phi$ on $X_\infty$ provided by the theorem gives the desired family of arcs on $X$. This result is the key technical tool needed to address the Nash problem.

**Theorem 3.10 (Regueru [Reg06, Corollary 4.8]).** Suppose that $C_X(E) \subset C_X(F)$ for some prime divisors $E$ and $F$ over $X$. Then there is an arc

$$\Phi: \text{Spec } K[[s]] \to X_\infty$$

such that $\Phi(0) \in C_X(E)$ is the generic point of $C_X(E)$ and $\Phi(\eta) \in C_X(F) \setminus C_X(E).$\(^6\)

---

\(^5\) A more natural notation for these sets is $C_X(\text{val}_E)$ and $c_X(\text{val}_E)$.

\(^6\) We are being somewhat sloppy here. To be precise, $\Phi(0)$ dominates the generic point of $C_X(E)$ and $\Phi(\eta)$ dominates a point in $C_X(F) \setminus C_X(E)$. The field $K$ can be chosen to be a finite extension of the residue field of the generic point of $C_X(E)$.
Remark 3.11. The theorem is stated more generally for a larger class of subsets of $X_\infty$ called generically stable sets (we refer to [Reg06] for the precise definition). The fact that $C_X(E)$ and $C_X(F)$ are generically stable sets is a consequence of Theorem 2.10.

In the Noetherian setting, the curve selection lemma essentially follows by cutting down to a curve, normalizing, and completing. The curve selection lemma however fails in general for non Noetherian schemes.

Example 3.12. Let $C = \text{Spec} k[x_1, x_2, x_3, \ldots]/I$ where $I$ is the ideal generated by the polynomials $x_1 - (x_i)^i$ for $i \geq 2$. Let $\Phi : \text{Spec} k[[s]] \to C$ be any morphism such that $\Phi(0) = O$, the origin of $C$. Writing $\Phi(s) = (x_1(s), x_2(s), x_3(s), \ldots)$, we have $\text{ord}_s x_i(s) \geq 1$ for all $i$. From the equations $x_1(s) = x_i(s)^i$, we deduce that $x_i(s) = 0$ for every $i$, and hence $\Phi$ is the constant arc. We note that $C$ can be realized as a closed irreducible set in the space of arcs of any variety.

The main result behind the curve selection lemma is another theorem of Reguera stating that if $\alpha$ is the generic point of $C_X(E)$ (or, more generally, if $\alpha \in X_\infty$ is what is called a stable point), then the completed local ring $\hat{\mathcal{O}}_{X_\infty, \alpha}$ is Noetherian [Reg06, Corollary 4.6]. Once this property is established, the proof of the curve selection lemma follows as a fairly standard application of Cohen’s structure theorem (see [Reg06] for details).

Remark 3.13. The fact that $\hat{\mathcal{O}}_{X_\infty, \alpha}$ is a Noetherian ring is a delicate property. There are examples where, before completion, the local ring $\mathcal{O}_{X_\infty, \alpha}$ is not Noetherian [Reg09, Example 3.16].

Remark 3.14. A related result of Grinberg and Kazhdan [GK00], reproved and extended to all characteristics by Drinfeld [Dri], states that if $\gamma \in X_\infty \setminus (X_{\mathrm{sing}})_\infty$ is a $k$-valued point, then there is an isomorphism

$$\hat{\mathcal{O}}_{X_\infty, \gamma} \cong k[[x_1, x_2, x_3, \ldots]]/I$$

where $I$ is the extension of an ideal in a finite dimensional polynomial ring $k[x_1, \ldots, x_n]$. We will not use this result in this paper.

4. Dimension one

The arc space of a curve is fairly easy to understand. Let $X$ be a curve over an algebraically closed field of characteristic zero, and suppose that $P \in X$ is a singular point. Let $f : Y \to X$ be the normalization, and write $f^{-1}(\{Q_1, \ldots, Q_r\})$. Note that $r$ is the number of analytic branches of $X$ at $P$.

Proposition 4.1. The fiber $\pi_1^{-1}(P)$ has $r$ irreducible components. For every $i = 1, \ldots, r$, the set $f_{\infty}(\pi_1^{-1}(Q_i)_{\text{red}})$ is closed and is one of the irreducible components of $\pi_1^{-1}(P)$.

Proof. For any field extension $K/k$, every constant $K$-valued arc in $X$ through $P$ has $r$ distinct lifts to $Y$, each mapping to a distinct point $Q_i$. By contrast, every non-constant $K$-valued arc through $P$ lifts uniquely to an arc on $Y$ which passes through one of the $Q_i$. This shows two things: for every $i$, the image $f_{\infty}(\pi_1^{-1}(Q_i)_{\text{red}})$ is equal to $C_X(Q_i)$ and hence is closed, and, for every $i \neq j$, the intersection of $f_{\infty}(\pi_1^{-1}(Q_i)_{\text{red}}) \cap f_{\infty}(\pi_1^{-1}(Q_j)_{\text{red}})$ consists only of the trivial arc at $P$. The proposition follows from these two properties. □
Remark 4.2. The sets of jets through the singularity of a curve may of course have more irreducible components than the number of branches. The case of a node \( X = (xy = 0) \subset \mathbb{A}^2 \) provides an elementary example: for every \( m \geq 1 \), the fiber over the origin \( O \in X \) of the truncation map \( \tau_m : X_m \to X \) has a decomposition into \( m \) irreducible components

\[ \tau_m^{-1}(O)_{\text{red}} = \bigcup_{i+j=m+1} C_{i,j} \]

where \( C_{i,j} \) is the closure of the set of \( m \)-jets on \( \mathbb{A}^2 \) with order of contact \( i \geq 1 \) along \((x = 0)\) and \( j \geq 1 \) along \((y = 0)\).

5. Dimension two

Throughout this section, suppose that \( X \) is a surface defined over an algebraically closed field of characteristic zero. There is a natural set of divisorial valuations that one can regard in connection to the Nash valuations, namely, the set of divisorial valuations \( \text{val}_{E_i} \) associated to the exceptional divisors \( E_1, \ldots, E_m \) in the minimal resolution of singularities \( f : Y \to X \).

Since we are not assuming that \( X \) is normal, we should stress that a prime divisor \( E \) on \( Y \) is defined to be exceptional over \( X \) if \( f \) is not an isomorphism at the generic point of \( E \).

In his paper, Nash asked whether there exists a natural one-to-one correspondence between the irreducible components of \( \pi^{-1}(X_{\text{sing}}) \) and the exceptional divisors in the minimal resolution of \( X \) (that is, the irreducible components of \( f^{-1}(X_{\text{sing}}) \)), the correspondence given indeed by identification between the associated valuations. Nash verified the question for \( \mathbb{A}^n \) singularities, where the correspondence is not hard to check.

A particularly simple case which already illustrates in concrete terms the geometry of the correspondence is that of an \( \mathbb{A}^2 \) singularity.

Example 5.1. Let \( X = (xy = z^3) \subset \mathbb{A}^3 \). The blow-up of the origin \( O \in X \) gives the minimal resolution \( f : Y \to X \). Let \( U \subset Y \) be the affine chart with coordinates \((u, v)\) where \( f \) is given by \((x, y, z) = (u^2v, uv^2, uv)\). The two exceptional divisors \( E_1, E_2 \) are given in \( U \) by \( E_1 = (u = 0) \) and \( E_2 = (v = 0) \). Let \( \gamma(t) = (x(t), y(t), z(t)) \) be an arbitrary arc on \( X \) through \( O \). The power series

\[
x(t) = \sum_{i=1}^{\infty} a_{it} t^i, \quad y(t) = \sum_{i=1}^{\infty} b_{it} t^i, \quad z(t) = \sum_{i=1}^{\infty} c_{it} t^i
\]

satisfy the equation \( x(t) y(t) = z(t)^3 \). Expanding, this gives

\[
a_1b_1 t^2 + (a_1b_2 + a_2b_1) t^3 + \cdots + \left( \sum_{i+j=m} a_i b_j \right) t^m + \cdots
\]

\[
= c_1^3 t^3 + \cdots + \left( \sum_{i+j+k=m} c_i c_j c_k \right) t^m + \cdots
\]

Comparing the coefficients of \( t^2 \), we get the equation \( a_1b_1 = 0 \). This leads to two cases.
Suppose $a_1 = 0$. Generically, we have $b_1 \neq 0$, and hence we can solve all remaining equations for $a_i$ ($i \geq 2$) in terms of the $b_j$ and $c_k$, which are free parameters. This gives an irreducible component $C_1$ of $\pi_X^{-1}(O)$ whose generic arc $\alpha(t) = (x(t), y(t), z(t))$ has first entry of order $\text{ord}_t(x(t)) = 2$, and the other two entries have order one. Write

$$\alpha(t) = (t^2 \cdot \overline{x}(t), t \cdot \overline{y}(t), t \cdot \overline{z}(t))$$

where $\overline{x}(t), \overline{y}(t), \overline{z}(t)$ are units. Using the equations $u = x/z$ and $v = y/z$, the lift $\tilde{\alpha}$ of $\alpha$ to $Y$ has entries

$$\tilde{\alpha}(t) = \left( t \cdot \frac{\overline{x}(t)}{\overline{z}(t)}, \frac{\overline{y}(t)}{\overline{z}(t)} \right)$$

in the coordinates $(u, v)$ of $U$. This shows that $\tilde{\alpha}(t)$ has order of contact one with $E_1$ and order of contact zero with $E_2$. In fact, one can argue that $\tilde{\alpha}$ is the generic point of $\pi_Y^{-1}(E_1)$.

Taking $b_1 = 0$, we get in a similar way the other component $C_2$ of $\pi_X^{-1}(O)$, which corresponds to $E_2$.

The simplicity of this example can be misleading. While arc spaces of $A_n$ singularities are still fairly easy to understand [Nas95], it was only recently that an answer to Nash’s question was given for $D_n$ singularities [Plé08] and for $E_6, E_7, E_8$ [PS12, PP13]. The fact is that, even when dealing with very simple equations like those of rational double points, the complexity of the equations of the arc space can grow very quickly. The case of sandwiched singularities was solved in [LR99], and a general proof for all rational surface singularities was given in [Reg12]. Some families of non-rational surface singularities where Nash’s question has a positive answer were found in [PPP06].

The answer to Nash’s question given in [PP13] for quotient surface singularities uses the reduction to the problem to a topological setting due to [FdB12]. Following the same approach, a complete proof valid for all surfaces was finally found by Fernandez de Bobadilla and Pe Pereira.

**Theorem 5.2** (Fernandez de Bobadilla and Pe Pereira [FdBPP12, Main Theorem]). A valuation on a surface $X$ is a Nash valuation if and only if it is the valuation associated to an exceptional divisor on the minimal resolution of $X$.

We present here a purely algebraic proof of this result that is based on the proof of the main theorem of [dFD16].

**Proof of Theorem 5.2.** Let $f : Y \to X$ be the minimal resolution of singularities. Given what we already discussed about the decomposition of $\pi_X^{-1}(X_{\text{sing}})$ into irreducible components, in order to prove the theorem we only need to show that if $E$ is a prime exceptional divisor on $Y$, then $C_X(E)$ is an irreducible component of $\pi_X^{-1}(X_{\text{sing}})$.

We proceed by way of contradiction and assume that $C_X(E)$ is not an irreducible component of $\pi_X^{-1}(X_{\text{sing}})$. This means that $C_X(E)$ is contained in $C_X(F)$ for some other exceptional divisor $F$.

Let $p \in E$ be a very general closed point.\(^7\) By applying Theorem 3.10 in conjunction with a suitable specialization argument ([LJR12, Proposition 2.9], [dFD16], \footnote{\textit{By very general}, we mean that the point is taken in the complement of countably many proper closed subsets.}
Theorem 7.6], we obtain an arc

\[ \Phi : \text{Spec } k[[s]] \to X_\infty \]

on the space of arcs of \( X \) such that

(a) \( \alpha_0 := \Phi(0) \) is a \( k \)-valued arc on \( X \) whose lift \( \bar{\alpha}_0 \) to \( Y \) is an arc with order of contact one with \( E \) at \( p \) (i.e., \( \bar{\alpha}_0(0) = p \) and \( \text{ord}_{\bar{\alpha}_0}(E) = 1 \)),

(b) \( \alpha_\eta := \Phi(\eta) \) is a \( k((s)) \)-valued point of \( \pi^{-1}_X(X_{\text{sing}}) \setminus C_X(E) \).

By definition, \( \Phi \) is a formal 1-parameter family of arcs giving an infinitesimal deformation of \( \alpha_0 \) in \( X_\infty \). We think of \( \Phi \) as a morphism

\[ \Phi : S = \text{Spec } k[[s,t]] \to X, \quad \Phi(s,t) = \alpha_s(t) \]

from a 2-dimensional regular germ to \( X \). Conditions (a) and (b) imply that the rational map

\[ \tilde{\Phi} := f^{-1} \circ \Phi : S \to Y \]

is not well-defined. Let \( g : Z \to S \) be the minimal sequence of monomial transformations resolving the indeterminacies of \( \tilde{\Phi} \), and let \( g' : Z' \to S \) be the normalized blow-up of the ideal \( \tilde{\Phi}^{-1} a \cdot \mathcal{O}_S \) where \( a \subset \mathcal{O}_X \) is an ideal such that \( Y = \text{Bl}_a X \). We have the commutative diagram

\[
\begin{array}{ccc}
G \cap Z & \phi \to & Y \cap E \\
| & \downarrow \phi' & | \\
Z' & \phi' \to & Y \\
| & \downarrow g & | \\
S & \phi \to & X
\end{array}
\]

where we denote by \( G \) the \( g \)-exceptional divisor intersecting the proper transform \( T \) of the \( t \)-axis \((s = 0) \subset S \). The morphisms \( \phi \) and \( \phi' \) are induced by resolving the indeterminacies of \( \tilde{\Phi} \), and \( h \) is the morphism contracting all \( g \)-exceptional curves that are contracted to a point by \( \phi \). One can check that \( Z \) is regular, \( Z' \) has rational singularities and hence is \( \mathbb{Q} \)-factorial, and \( h \) is the minimal resolution of singularities of \( Z' \) (see [dFD16, Proposition 4.1] for details).

The image of the exceptional locus \( \text{Ex}(g) \) of \( g \) in \( Y \) is contained in the exceptional locus of \( f \), and so is the image of the exceptional locus \( \text{Ex}(g') \) of \( g' \). Recall that none of the irreducible components of \( \text{Ex}(g') \) is contracted by \( \phi' \). Since \( p \) was picked to be a general point of \( E \), and it belongs to \( \phi(G) \), every irreducible component of \( \text{Ex}(g') \) that contains \( h(G) \) must pass through \( p \) and hence dominate \( E \). Note that there is at least one such component of \( \text{Ex}(g') \), since \( g' \) is not an isomorphism. This implies that \( \phi(Z) \) contains the generic point of \( E \). On the other hand, \( \phi(Z) \) is not contained in \( E \), since \( \phi(T) \) is an arc on \( Y \) with finite order of contact with \( E \) and hence not entirely contained in \( E \). We conclude that \( \phi \) is a dominant map.

Let \( K_{Z/Y} \) be the relative canonical divisor of \( Z \) over \( Y \), locally defined by the Jacobian ideal \( \text{Jac}_\phi \subset \mathcal{O}_Z \), and let \( K_{Z'/Y} = h_\ast K_{Z/Y} \), which we think of as the relative canonical divisor of \( Z' \) over \( Y \). Similarly, let \( K_{Z/S} \) be the relative canonical
divisor of $g$ and let $K_{Z'/S} = h_* K_{Z'/S}$.\footnote{The formal definition of relative canonical divisor via sheaves of differentials is not straightforward, as $\Omega_{Z'/k}$ is not the right object in this setting. For a correct formal definition, one needs to replace $\Omega_{Z'/k}$ with the sheaf of special differentials. Once this adjustment is done, the same definitions and properties follow into place as in the usual setting, and therefore we shall omit this discussion here. For details, we refer to [dFD16, Section 4].} We decompose
\[ K_{Z'/Y} = K_{g'}^{g'-\text{exc}} + K_{Z'/Y}^{g'-\text{hor}} \]
where every component of $K_{Z'/Y}^{g'-\text{exc}}$ is $g'$-exceptional and none of the components of $K_{Z'/Y}^{g'-\text{hor}}$ is.

We claim that the following series of inequalities hold:
\[ 1 \leq \ord_G(K_{Z/S}) \leq \ord_G(h^* K_{Z'/S}) \leq \ord_G(h^* K_{Z'/Y}^{g'-\text{exc}}) \leq \ord_G(\phi^* E) = 1. \]
This clearly gives a contradiction, which is what we are after.

The reminder of the proof is devoted to explain these inequalities. We proceed with one inequality at a time.

Inequality (1). The fact that $\ord_G(K_{Z/S}) \geq 1$ holds simply because $S$ is regular and $G$ is $g$-exceptional.

Inequality (2). This inequality follows from the fact that the $\mathbb{Q}$-divisor
\[ K_{Z'/S} = K_{Z'/S} - h^* K_{Z'/S} \]
is $h$-nef and $h$-exceptional since $h$ is the minimal resolution of singularities of $Z'$, and hence is anti-effective by the negative definiteness of the intersection matrix of the $h$-exceptional divisors (see [dFD16, Proposition 4.12]). Here we are using the fact that $Z'$, having rational singularities, is $\mathbb{Q}$-factorial and therefore the pull-back $h^* K_{Z'/S}$ is defined.

Inequality (3). Here is where we use the fact that $f$ is the minimal resolution of singularities of $X$. First, notice that the divisor
\[ K_{Z'/S} - K_{Z'/Y}^{g'-\text{exc}} \]
is $g'$-exceptional. We claim that this divisor is also $g'$-nef. Indeed, we have
\[ K_{Z'/S} - K_{Z'/Y}^{g'-\text{exc}} \sim K_{Z'} - K_{Z'/Y}^{g'-\text{hor}} = (\phi')^* K_Y + K_{Z'/Y}^{g'-\text{hor}}. \]
Since $f$ is the minimal resolution, $K_Y$ is $f$-nef, and hence $(\phi')^* K_Y$ is $g'$-nef. On the other hand, $K_{Z'/Y}^{g'-\text{hor}}$ is clearly $g'$-nef because it is effective and contains no $g'$-exceptional divisors. Therefore $K_{Z'/S} - K_{Z'/Y}^{g'-\text{exc}}$ is $g'$-nef, as claimed. We conclude that this divisor is anti-effective, and this gives the third inequality.

Inequality (4). Let $C_1, \ldots, C_n$ be the irreducible components of $\text{Ex}(g')$ containing $h(G)$. Each $C_i$ dominates $E$, and we have $\ord_{C_i}(K_{Z'/Y}) = \ord_{C_i}((\phi')^* E) - 1$ by a Hurwitz-type computation. This implies that
\[ \ord_G(h^* K_{Z'/Y}^{g'-\text{exc}}) < \ord_G(\phi^* E) \]
(see [dFD16, (5.4)] for more details). Here we are using again that $Z'$ is $\mathbb{Q}$-factorial.

Equality (5). This follows by the way we chose $\Phi$. Recall that $\alpha_0 = \Phi(0)$ lifts to an arc $\tilde{\alpha}_0$ of $\text{Spec} k[[t]] \to Y$ with order of contact one along $E$. This arc is
parametrized by the $t$-axis of $S$. This means that $\tilde{a}_0$ factors through a morphism 
\[ \psi: \text{Spec } k[[t]] \to Z \]
which gives a parameterization of the proper transform $T$ of the $t$-axis. Since
\[ 1 = \text{ord}_{\tilde{a}_0}(E) = \text{ord}_t(\tilde{a}_0 E) \geq \text{ord}_t(\psi^*G) \cdot \text{ord}_G(\psi^*E), \]
we conclude that $\text{ord}_G(\phi^*E) = 1$ (see the discussion leading to [dFD16, (5.5)]).
This proves (5) and hence completes the proof of the theorem. \hfill $\square$

We conclude this section with a brief discussion of the original proof of Theorem 5.2, referring the reader to the original papers [FdB12, FdBPP12] and the survey [PS15] for more rigorous and detailed proofs.

The first step is to reduce to the case where $k = \mathbb{C}$ and $X$ is normal. Once in this situation, let $f: Y \to X$ be the minimal resolution. For simplicity, we assume that the exceptional locus of $f$ is a divisor with simple normal crossings. The proof of the general case is similar but it requires an argument on local deformation to the Milnor fiber which we prefer to omit here.

As usual, one assumes by contradiction that there are two exceptional divisors $E$ and $F$ on $Y$ such that $C_X(E) \subset C_X(F)$. Like in the algebraic proof we gave above, the curve selection lemma yields a map $\Phi: S = \text{Spec } \mathbb{C}[[s,t]] \to X$ with the properties listed in the proof. Such a map is called a formal wedge. By the results of [FdB12] which rely on Popescu’s approximation theorem, one can replace $\Phi$ with a convergent wedge, and hence assume without loss of generality that $S \subset \mathbb{C}^2$ is a small open neighborhood of the origin.\footnote{A rigorous discussion of what follows requires working with Milnor representatives of $X$ and the wedge.}

Fix a sufficiently small $\epsilon > 0$, and let $\mathbb{D}_\epsilon = \{t \in \mathbb{C} \mid |t| < \epsilon\}$. For every $s \in \mathbb{C}$ with $|s| < \epsilon$, we have a holomorphic map
\[ \alpha_s: \mathbb{D}_\epsilon \to X, \quad \alpha_s(t) := \Phi(s, t). \]
The image of this map is not contained in $X_{reg}$ and lifts uniquely to a holomorphic map $\tilde{\alpha}_s: \mathbb{D}_\epsilon \to Y$. Let $D_s \subset Y$ denote the image of $\tilde{\alpha}_s$. Since $D_0$ is the support of a small curve whose germ at the point of contact with $E$ is a smooth arc, it is homeomorphic to an open disk. One deduces from this that if $s$ is sufficiently small then $D_s$ is homeomorphic to an open disk.

As $s$ approaches 0, $D_s$ degenerates to a cycle
\[ D_0 + \sum a_i E_i \]
supported within the union of $D_0$ and the exceptional divisor $\text{Ex}(f) = \sum E_i$. Let $\Gamma = \text{Supp}(D_0 + \sum a_i E_i)$. Let $I$ be the union of $\{0\}$ with the index set of the components $E_i$ appearing in $\Gamma$, and let $J$ be the index set for the singular points $p_j$ of $\Gamma$. Note that $E = E_i$ for some $i \in I$, say for $i = 1$. Suppose that $0 \in J$ is the index such that $p_0$ is the point of intersection of $D_0$ with $E_1$.

For every $j \in J$, let $B_j \subset Y$ be a small ball around $p_j$, and for every $i \in I$, let $T_i$ be a small tubular neighborhood of $D_0$ if $i = 0$, and of $E_i$ if $i \neq 0$. We assume that the sectional radius of $T_i$ is chosen sufficiently small with respect to the radius of the balls $B_j$ so that the boundary of $T_i$ intersects the boundary of $B_j$ transversally and all such intersections are disjoint. Let $T_i^o$, $D_0^o$, and $E_i^o$ denote the restrictions of $T_i$, $D_0$, and $E_i$ to the complement of $\bigcup B_j$. Fix $s$ with $0 < |s| < \epsilon$ so that $D_s$...
is contained in \((\bigcup T_i) \cup (\bigcup B_j)\). We assume that \(D_s\) intersects transversally the boundary of each \(B_j\). We have
\[
\chi(D_s) = \sum \chi(D_s \cap T_i^0) + \sum \chi(D_s \cap B_j),
\]
where \(\chi\) is the Euler–Poincaré characteristic.

For \(i = 0\), we have \(\chi(D_s \cap T_i^0) = \chi(D_i^0) = 0\), and for \(i \neq 0\) we have
\[
\chi(D_s \cap T_i^0) \leq a_i \chi(E_i^0)
\]
by Hurwitz formula.\(^{10}\) To bound \(\chi(D_s \cap B_j)\), we observe that \(D_s \cap B_j\) is a union of disjoint orientable surfaces with boundary. Those homeomorphic to the disk are the only components contributing positively to the characteristic, and each such component must intersect \(\Gamma\) at some point in \(B_j\). It follows that
\[
\chi(D_s \cap B_j) \leq \sum_{p \in B_j} i_p(D_s, \Gamma),
\]
where we denote by \(i_p\) the intersection multiplicity at a point \(p\) (see [FdBPP12, Lemma 7] for more details). For \(j = 0\), this estimate can be improved. Indeed, \(D_s \cap B_0\) must have at least one connected component whose boundary is the union of at least two circles, one contained in \(T_0 \cap B_0\) and the other contained in \(T_1 \cap B_0\). Such component intersects both branches of \(\Gamma \cap B_j\) and does not contribute positively to the characteristic. This implies that
\[
\chi(D_s \cap B_0) \leq -2 + \sum_{p \in B_0} i_p(D_s, \Gamma)
\]
(we refer to the discussion leading to [FdBPP12, (12)] for more details). Putting everything together and suitably rearranging the terms, one gets
\[
\chi(D_s) = \sum \chi(D_s \cap T_i^0) + \sum \chi(D_s \cap B_j) \leq \sum a_i (2 - 2g(E_i) + E_i^2).
\]

By the adjunction formula, the right-hand side is equal to \(-K_Y \cdot \sum a_i E_i\). As \(K_Y\) is nef over \(X\) (\(f\) being the minimal resolution) and \(\sum a_i E_i\) is \(f\)-exceptional, this number is \(\leq 0\). Since, on the other hand, \(D_s\) is homeomorphic to the unit disk and hence \(\chi(D_s) = 1\), we get a contradiction.

6. Higher dimensions

Moving on to higher dimensional singularities, it becomes less clear which exceptional divisors should correspond to Nash valuations. The reason is that in dimension \(\geq 3\) there is no minimal resolution available to determine a natural set of candidates. In fact, some varieties may have small resolutions, which extract no divisors at all. With this in mind, Nash proposed to consider the following set of valuations.

Throughout this section, let \(X\) be a variety of positive dimension defined over an algebraically closed field of characteristic zero.

**Definition 6.1.** An **essential valuation** of \(X\) is a divisorial valuation whose center on every resolution of singularities \(f: Y \to X\) is an irreducible component of \(f^{-1}(X_{\text{sing}})\).

\(^{10}\)Here we are implicitly using that the boundaries of \(D_s \cap T_i^0\) and \(D_i^0\) are unions of circles, and hence they can be added in without altering the computation.
Proposition 6.2 (Nash [Nas95, Corollary]). Every Nash valuation of $X$ is essential.

Proof. Let $\text{val}_C$ be the Nash valuation associated to an irreducible component of $\pi^{-1}_X(X_{\text{sing}})$. We already know that $\text{val}_C$ is a divisorial valuation. Let $f: Y \rightarrow X$ be an arbitrary resolution of singularities. As we argued in the proof of Theorem 3.3, there is an irreducible component $E$ of $f^{-1}(X_{\text{sing}})$ such that

$$C = \text{fq}_{\pi^{-1}(E)}.$$

In other words, the generic point of $C$ is the image of the generic point of $\pi^{-1}(E)$. This implies that the center of $\text{val}_C$ in $Y$ is $E$. Since $f$ is an arbitrary resolution, we conclude that $\text{val}_C$ is an essential valuation. □

Definition 6.3. After identifying the irreducible components of $\pi^{-1}_X(X_{\text{sing}})$ with the valuations they define, the inclusion of the set of Nash valuations into the set of essential valuations is known as the Nash map.

Nash asked whether the Nash map is surjective, that is, whether the property of being essential characterizes Nash valuations. This question became known as the Nash problem.

Theorem 5.2 states that this is the case in dimension two. However, after years of speculation, this turned out to be false in general: counter-examples where first found in dimensions $\geq 4$ [IK03], and later in dimension 3 as well [dF13]. A larger class of counter-examples showing that this phenomenon is actually quite common and not limited to few sporadic examples was finally produced in [JK13].

In [JK13], Nash valuations of a $cA_1$-type singularity $X = (xy = f(z_1, \ldots, z_n)) \subset \mathbb{A}^{n+2}$ (where $\text{mult}(f) \geq 2$) are completely determined, and essential valuations are characterized when $\text{mult}(f) = 2$. A special case of their result, stated next, shows that the Nash map is not surjective about half of the times for 3-dimensional $cA_1$ singularities.

Theorem 6.4 (Johnson and Kollár [JK13, Theorem 1 and Proposition 9]). For $m \geq 3$, the singular threefold $X = (xy = z^2 - w^m) \subset \mathbb{A}^4$ has one Nash valuation, and the number of essential valuations of $X$ is one if $m$ is even or $m = 3$, and two if $m$ is odd $\geq 5$.

We extract from this result the case $m = 5$, which gives the simplest counterexample to the Nash problem. We review the proof of this case. The proof of Lemma 6.6, which gives the count of Nash valuations for this example, formalizes the type of discussion given in Example 5.1 based on localization and elimination of variables, and is inspired by some computations we learned from Ana Reguera.

Corollary 6.5. The Nash map is not surjective for $X = (xy = z^2 - w^5) \subset \mathbb{A}^4$.

Proof. A resolution of $X$ can be obtained by taking two blow-ups. The blow-up $f: Y \rightarrow X$ of the origin $O$ produces a model with an isolated singularity $P \in Y$ whose tangent cone is the affine cone over a (singular) quadric surface. Blowing up the point $P$ gives a resolution $g: Z \rightarrow Y$ of $Y$, and hence of $X$. Let $F \subset Y$ be the exceptional divisor of $f$ and $G \subset Z$ the exceptional divisor of $g$.

Since $Z \rightarrow X$ is a resolution which only extracts two divisors, it follows that there are at most two essential valuations. Moreover, we have

$$\pi^{-1}_X(O_{\text{red}}) = C_X(F) \cup C_X(G),$$
and hence there are at most two Nash valuations. The precise count of Nash valuations and essential valuations is given in the next two lemmas which, combined, yield the corollary.

Lemma 6.6. With the above notation, we have $\pi_{X}^{-1}(O)_{\text{red}} = C_{X}(F)$, and therefore $\text{val}_{F}$ is the only Nash valuation of $X$.

Proof. First note that $C_{X}(F) \not\subset C_{X}(G)$ because every arc $\gamma \in C_{X}(G)$ has $\text{ord}_{\gamma}(x) \geq 2$, whereas if $\alpha \in C_{X}(F)$ is the generic point then $\text{ord}_{\alpha}(x) = 1$. Therefore the statement is equivalent to showing that $\pi_{X}^{-1}(O)$ is irreducible. This set is defined by the vanishing of the derivations of the polynomial

$$h(x, y, z, w) = xy - z^2 + w^5$$

and the pull-back of the maximal ideal. Explicitly, let $(x_i, y_i, z_i, w_i)_{i \geq 0}$ denote the coordinates of $(\mathbb{A}^{4})_{\infty}$ defined by setting $x_i = D_i(x)$, $y_i = D_i(y)$, $z_i = D_i(z)$, and $w_i = D_i(w)$, where $D_i$ are the Hasse–Schmidt derivations. The fiber $\pi_{X}^{-1}(O)$ is defined in $(\mathbb{A}^{4})_{\infty}$ by the equations

$$x_0 = y_0 = z_0 = w_0 = 0 \quad \text{and} \quad h_i := D^i(h(x_0, y_0, z_0, w_0)) = 0 \quad \text{for} \quad i \geq 0.$$ 

Let $\overline{h}_i$ denote the polynomial $h_i$ once we set $x_0 = y_0 = z_0 = w_0 = 0$. Note that $\overline{h}_0$ and $\overline{h}_1$ vanish identically.

Let $U \subset \pi_{X}^{-1}(O)$ be the open set obtained by inverting $w_1$ and $x_2$. Using the equations $\overline{h}_i = 0$, for $i \geq 6$, to eliminate the variables $w_j$ for $j \geq 2$, we see that $U = \text{Spec} \, R$ where

$$R = (k[w_1][x_i, y_i, z_i]_{i \geq 1}/(\overline{h}_2, \ldots, \overline{h}_5))_{w_1 x_2}.$$ 

We claim that $R$ is a domain. Since the polynomials $\overline{h}_2, \ldots, \overline{h}_5$ do not depend on the variables $x_i, y_i, z_i$ for $i \geq 5$, it suffices to show that

$$S = (k[w_1][x_i, y_i, z_i]_{1 \leq i \leq 4}/(\overline{h}_2, \ldots, \overline{h}_5))_{w_1 x_2}$$

is a domain. It can be checked that $S_{x_1}$ is a domain of dimension 9 and $S/(x_1)$ is a domain of dimension 8. From this it follows that $S$, and hence $R$, are domains, and therefore $U$ is irreducible. Since $U$ has nonempty intersection with both $C_{X}(F)$ and $C_{X}(G)$, we conclude that $\pi_{X}^{-1}(O)$ is irreducible.

Lemma 6.7. With the above notation, both $\text{val}_{F}$ and $\text{val}_{G}$ are essential valuations of $X$, and therefore $X$ has two essential valuations.

Proof. The fact that $\text{val}_{F}$ is essential follows by Lemma 6.6 and Proposition 6.2. Suppose that $\text{val}_{G}$ is not essential. Then there is a resolution $\mu: W \to X$ such that the center $C = c_{W}(G)$ is not an irreducible component of $\mu^{-1}(O)$. Let $T$ be an irreducible component of $\mu^{-1}(O)$ containing $C$. We have the commutative diagram

$$\begin{array}{ccc}
G & \xrightarrow{g} & Z \\
\downarrow \psi & & \downarrow \\
P & \xrightarrow{\phi} & W \xrightarrow{\mu} T \supset C \\
\downarrow f & & \downarrow \\
O & \xrightarrow{\mu} & X
\end{array}$$
For short, for any prime divisor $D$ over a $\mathbb{Q}$-Gorenstein variety $V$ we denote by $k_D(V)$ the coefficient of $D$ in the relative canonical divisor $K_{V'/V}$ on some (smooth or normal) model $V'$ over $V$ on which $D$ is a divisor, and call it the \textit{discrepancy} of $D$ over $V$.\footnote{Here we are a bit sloppy and identify divisors across different models when they define the same valuation.}

A direct computation shows that $k_F(X) = 1$ and $k_G(X) = 2$. In particular $X$ is terminal and $F$ is the only exceptional divisor over $O \subset X$ with discrepancy one.

Since $X$ is factorial, the exceptional locus $\text{Ex}(\mu)$ of $\mu$ is a divisor, as otherwise the push-forward of a general hyperplane section of $W$ would be a Weil divisor in $X$ that is not Cartier. Note that $K_{W/X} \geq \text{Ex}(\mu)$ because $X$ has terminal singularities. This implies that $k_G(W) \leq k_G(X) - 1 = 1$. Since, on the other hand, $k_G(W) \geq \text{codim}_W(C) - 1 \geq 1$ because $W$ is smooth, we conclude that $k_G(W) = 1$. $C$ is a $1$-dimensional set contained in a unique $\mu$-exceptional divisor $E$, and $k_E(X) = 1$.

Since $C \subset T \subset E$, we must have $T = E$, and hence $E$ is a divisor with center $O$ in $X$. Since there is only one exceptional divisor over $X$ with discrepancy one and center $O$, we deduce that $E$ is the proper transform of $F$. Taking into account that $\text{val}_E(m_{X,O}) = \text{val}_E(m_{X,O}) = 1$, we see that $m_{X,O} \cdot \mathcal{O}_W$ is equal to $\mathcal{O}_W(-E)$ in a neighborhood of the generic point of $C$, and hence it is locally principal there. As $f$ is the blow-up of $m_{X,O}$, this means that the map $\phi: W \dashrightarrow Y$ is well-defined at the generic point of $C$. Replacing $W$ with a higher model without blowing-up near the generic point of $C$, we can assume that $\phi$ is everywhere well defined and projective. Note that $\phi$ contracts $C$ to the point $P$, since $g(G) = P$. Then $\phi$ is a resolution of $Y$ whose exceptional locus has a component of codimension 2. This contradicts the fact that $Y$ is factorial. \qed

In the surface case, it is clear that a divisorial valuation is essential if and only if it is defined by an exceptional divisor on the minimal resolution. In the terminology introduced in this section, Theorem 5.2 simply states that the Nash map is surjective in dimension two.

Even though the Nash map is not always surjective in higher dimensions, the result on surfaces still admits a natural generalization to all dimensions. This is possible by interpreting the minimal resolution of a surface from the point of view of the minimal model program. With this in mind, we give the following definition.

\textbf{Definition 6.8.} A \textit{terminal valuation} of $X$ is a valuation defined by an exceptional divisor on a minimal model $f: Y \rightarrow X$ over $X$.

A minimal model $f: Y \rightarrow X$ over $X$ is, by definition, the outcome of a minimal model program over $X$ started from any resolution of singularities of $X$. It is characterized by two properties: $Y$ has terminal singularities and $K_Y$ is relatively nef over $X$. The minimal resolution of a surface is the unique minimal model over it, and hence a valuation on a surface is terminal if and only if it is essential. In higher dimensions, a variety $X$ can admit several relative minimal models over itself, but all of them are isomorphic in codimension one, and therefore the set of terminal valuations of $X$ is determined by the set of the exceptional divisors of any one of them.

The following theorem is the natural generalization of Theorem 5.2 to higher dimensions. The proof is similar to the proof of Theorem 5.2 given in this paper.
(with some technical adjustments needed to take into account the dimension of \( X \)), so we omit it.

**Theorem 6.9** (de Fernex and Docampo \[dFD16, Theorem 1.1\]). *Every terminal valuation of \( X \) is a Nash valuation.*

One way of thinking about this result is to contrast it to Proposition 6.2. While the proposition gives a necessary condition to be a Nash valuation, the theorem provides a sufficient condition, thus squeezing the set of Nash valuations from the other side.

**Remark 6.10.** There are no terminal valuations on a variety with terminal singularities, nor over a variety which admits a small resolution, simply because in both case a minimal model over the variety does not extract any divisor. The above result sheds no light on Nash valuations over varieties with such singularities.

There is another (more elementary) sufficient condition to be a Nash valuation. We define a partial order among divisorial valuations on \( X \) as follows. Given two divisorial valuations \( v \) and \( v' \) on \( X \) we write \( v \leq v' \) if \( c_X(v) \supset c_X(v') \) and \( v(h) \leq v'(h) \) for every \( h \in \mathcal{O}_{X, c_X(v)} \). If moreover \( v \neq v' \), then we write \( v < v' \).

**Definition 6.11.** A divisorial valuation \( v \) centered in the singular locus of \( X \) is said to be a minimal valuation if it is minimal (with respect to the above partial order) among all divisorial valuations centered in \( X_{\text{sing}} \).

**Proposition 6.12.** *Every minimal valuation of \( X \) is a Nash valuation.*

**Proof.** Let \( \text{val}_E \) be a minimal valuation of \( X \), and let \( C_X(E) \subset X_{\infty} \) the associated maximal divisorial set. As \( \text{val}_E \) is centered in the singular locus of \( X \), we have \( C_X(E) \subset \pi_X^{-1}(X_{\text{sing}}) \). Let \( C \) be an irreducible component of \( \pi_X^{-1}(X_{\text{sing}}) \) containing \( C_X(E) \). Let \( \alpha \in C_X(E) \) and \( \beta \in C \) be the respective generic points, so that \( \text{val}_\alpha = \text{val}_E \) and \( \text{val}_\beta = \text{val}_C \). Since \( \alpha \) is a specialization of \( \beta \), we have \( \text{val}_\beta \leq \text{val}_\alpha \). The hypothesis that \( \text{val}_E \) is minimal implies that \( \text{val}_E = \text{val}_C \). Therefore \( \text{val}_E \) is a Nash valuation. \( \Box \)

Toric varieties provide another important class of varieties where the Nash map is surjective.

**Theorem 6.13** (Ishii and Kollár \[IK03, Theorem 3.16\]). *For a divisorial valuation \( v \) centered in the singular locus of a toric variety \( X \) the following properties are equivalent:

(a) \( v \) is a minimal valuation,
(b) \( v \) is a Nash valuation,
(c) \( v \) is an essential valuation.

In particular, the Nash map is surjective for every toric variety.*

**Proof.** We already know that (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) by Propositions 6.12 and 6.2. We are left to prove that (c) \( \Rightarrow \) (a).

Without loss of generality, we may assume that \( X = X(\sigma) \) is the affine toric variety associated to a cone \( \sigma \subset N \otimes \mathbb{R} \), where \( N \) is the lattice dual to the character lattice \( M \) of the torus. The elements of \( \sigma \cap N \) are in bijection with the torus-invariant valuations on \( X \). Let \( \sigma_{\text{sing}} := \bigcup_\tau \tau^\circ \), where \( \tau \) ranges over all singular
faces of $\sigma$ and $\tau^o$ denotes the relative interior of $\tau$.

The elements in $\sigma_{\text{sing}} \cap N$ are in bijection with the torus-invariant valuations on $X$ centered in $X_{\text{sing}}$. Given two vectors $v, v' \in \sigma \cap N$, we write $v \leq_{\sigma} v'$ if $v' \in v + \sigma$. If moreover $v \neq v'$, then we write $v <_{\sigma} v'$. This defines a partial order on $\sigma \cap N$, and hence on $\sigma_{\text{sing}} \cap N$.

Since $X$ admits a torus-invariant resolution of singularities $f: Y \to X$ such that $f^{-1}(X_{\text{sing}})$ is the union of torus-invariant divisors, every essential valuation of $X$ is torus-invariant and, as such, it corresponds to an element in $\sigma_{\text{sing}} \cap N$.

Let $v$ be a toric valuation centered in $X_{\text{sing}}$, and assume that $v$ is not a minimal valuation of $X$. Then there is a divisorial valuation $\text{val}_F$ on $X$ with center $c_X(F)$ contained in $X_{\text{sing}}$ and containing the center of $v$, such that $\text{val}_F < v$. Since $c_X(F) \subset X_{\text{sing}}$, we have $C_X(F) \subset \pi_X^{-1}(X_{\text{sing}})$. Let $C'$ be an irreducible component of $\pi_X^{-1}(X_{\text{sing}})$ containing $C_X(F)$, and let $v' = \text{val}_{C'}$. Note that $v'$ is torus-invariant, since, being a Nash valuation, it is essential. We identify $v$ and $v'$ with the corresponding elements in $\sigma_{\text{sing}} \cap N$. We have $v' \leq \text{val}_F$ by the inclusion $C_X(F) \subset C'$. It follows that $v' < v$, and hence $v' <_{\sigma} v$.

We have $v = v' + v''$ for some $v'' \in \sigma \cap N \setminus \{0\}$. Let $\tau$ be the 2-dimensional cone spanned by $v'$ and $v''$, and let $\Gamma$ be the fan spanned by all elements in $\tau \cap N$ that are minimal with respect to the partial order $\leq_{\tau}$. Geometrically, $X(\tau)$ is a surface singularity and $X(\Gamma)$ is its minimal resolution. Since $v$ is not minimal in $\tau \cap N$ with respect to $\leq_{\tau}$, it must belong to the interior of a 2-dimensional face of $\Gamma$.

For a suitable choice of vectors $v'$ and $v''$ adding up to $v$, the subdivision $\Gamma$ of $\tau$ can be extended to a subdivision $\Delta$ of $\sigma$ so that the faces of $\Gamma$ are faces in $\Delta$ and $g: X(\Delta) \to X(\sigma)$ is a resolution of singularities such that $g^{-1}(X_{\text{sing}})$ is a union of divisors and $g$ is an isomorphism over the nonsingular locus of $X(\sigma)$ (we refer to the proof of [IK03, Lemma 3.15] for more details). By construction, $v$ does not belong to any ray in $\Delta$. This means that the center of $v$ in $X(\Delta)$ is not a divisor, and therefore it cannot be an irreducible component of $g^{-1}(X_{\text{sing}})$. We conclude that $v$ is not an essential valuation.

There are other examples where the Nash map is known to be surjective. Theorem 6.13 is extended to non-normal pretoric varieties in [Ish05], and locally analytically pretoric singularities in [Ish06], which implies in particular that the Nash map is surjective for quasi-ordinary singularities since any such singularity decomposes, locally analytically, into analytically irreducible quasi-ordinary singularities, which are analytically pretoric. This result was further extended in [GP07] to cover the case of reduced germs of quasi-ordinary hypersurface singularities. More examples were discovered in [PPP08, LJR12, LA16].

The following recent result provides yet another class of examples.

Theorem 6.14 (Docampo and Nigro [DN, Proposition 11.2]). The Nash map is surjective for Schubert varieties in Grassmannians.

Reviewing the proof of this theorem would require setting up some notation about Schubert varieties which we prefer to avoid here, so we limit ourselves to outline the argument. If $X$ is a Schubert variety and $X_{\text{sing}} = \bigcup Z_i$ is the decomposition of the singular locus of $X$ into irreducible components, then there is a

12 A rational polyhedral cone $\tau \subset N \otimes \mathbb{R}$ is regular if the primitive elements in the rays of $\tau$ form a part of a basis of $N$, and is singular otherwise.
resolution $f: Y \to X$ such that $f^{-1}(Z_i)$ is irreducible for every $i$, so that

$$f^{-1}(X_{\text{sing}})_{\text{red}} = \bigcup f^{-1}(Z_i)$$

is the decomposition into irreducible components of $f^{-1}(X_{\text{sing}})$. The construction of the resolution is well-known to the experts; it appears for example in [Zel83]. This fact immediately implies that

$$\pi_X^{-1}(X_{\text{sing}})_{\text{red}} = \bigcup \pi_X^{-1}(Z_i)$$

is the decomposition into irreducible components of $\pi_X^{-1}(X_{\text{sing}})$. Therefore, if $E_i$ is the divisor dominating $Z_i$ in the blow-up of $Y$ along $Z_i$, then $\text{val}_{E_i}$ is a Nash valuation. Note that $\text{val}_{E_i}$ is also both minimal and essential. In particular, the Nash map is surjective and the sets of minimal valuations, Nash valuations, and essential valuations are all the same.

**Remark 6.15.** It was noticed in [Zel83] that all Schubert varieties (in Grassmannians) admit small resolutions. This implies that they do not have any terminal valuations.

Much of the work on the Nash problem for surfaces that preceded [FdBPP12] has been based on explicit analysis of the singularities, thus resulting in more or less explicit descriptions of spaces of arcs. Arc spaces have also been studied on varieties which possess additional structure such as a group action or a combinatorial nature, not just in relation to the Nash problem but also for its own sake.

An example is the paper [DN] we just mentioned, whose main purpose is certainly not to solve the Nash problem for Schubert varieties which mainly relies upon the existence of such nice resolutions, but rather to describe the space of arcs of the Grassmannian which is done by means of a decomposition of the arc space that resembles the Schubert cell decomposition of the Grassmannian itself. As it is explained at the end of [DN], these general results already give a way of understanding directly the decomposition of $\pi_X^{-1}(X_{\text{sing}})$ into irreducible components for any Schubert subvariety $X$ of the Grassmannian that does not make use of any explicit resolution of singularities. Arc spaces were previously studied for toric varieties in [Ish04] and for determinantal varieties in [Doc13].

The next corollary summarizes the general results on the Nash problem stated in this section.

**Corollary 6.16.** For any variety $X$, there are inclusions

$$\{ \text{ minimal val's} \} \cup \{ \text{ terminal val's} \} \subset \{ \text{ Nash val's} \} \subset \{ \text{ essential val's} \}.$$  

**Proof.** The first inclusion follows by Theorem 6.9 and Proposition 6.12. The second inclusion is the Nash map, which is defined by Proposition 6.2. \qed

Surfaces, toric varieties, and Schubert varieties (in Grassmannians) form three classes of varieties for which the Nash map is known to be a bijection (Theorems 5.2, 6.13 and 6.14). However, the surjectivity of the Nash map in these cases appears to hold for different reasons.

For surfaces, the surjectivity follows by the fact that every essential valuation is a terminal valuation. This means that, for surfaces, we have

$$\{ \text{ minimal val's} \} \subset \{ \text{ terminal val's} \} = \{ \text{ Nash val's} \} = \{ \text{ essential val's} \},$$
and the first inclusion may be strict.

By contrast, for toric varieties and Schubert varieties, the surjectivity follows by the fact that every essential valuation is a minimal valuation, which gives

\[ \{ \text{terminal val's} \} \subset \{ \text{minimal val's} \} = \{ \text{Nash val's} \} = \{ \text{essential val's} \}, \]

and the first inclusion may be strict.

We do not know any example where the first inclusion in Corollary 6.16 is a strict inclusion. If the inclusion turned out to be always an equality, this would provide a complete solution to the Nash problem. It is however quite possible that the inclusion is strict in general.

Remark 6.17. It would be interesting to see whether there are other classes of varieties, like those listed above, for which all Nash valuations are either minimal or terminal, and to find examples where this property holds but neither the set of minimal valuations nor the set of terminal valuations suffices, alone, to exhaust all Nash valuations.

Approaching the set of Nash valuations from the other side, one could try to fix the Nash problem by modifying the definition of essential valuation. The definition given in [Nas95] requires testing a certain condition on the centers of the valuations on all resolutions, but such definition is not restrictive enough to characterize Nash valuations. By enlarging the class of models where the condition is tested, one may hope to get a more restrictive version of essential valuations that agrees with Nash valuations.

This idea is explored in [JK13] for valuations on a normal threefold \( X \). The focus is on valuations centered at closed points in \( X \).

Definition 6.18. An isolated threefold singularity \( Q \in Y \) is \textit{arc-wise Nash-trivial} if for every general arc \( \alpha: \text{Spec} \ k[[t]] \to Y \) passing through a singular point \( Q \in Y \) there is a morphism \( \Phi: \text{Spec} \ k[[s,t]] \to Y \) such that \( \alpha(t) = \Phi(0,t) \) and \( \Phi^{-1}(Q) \) is zero-dimensional.

Definition 6.19. A divisorial valuation \( v \) centered at a closed point \( P \) of a normal threefold \( X \) is a \textit{very essential valuation} for \( (P \in X) \) if for every proper birational model \( f: Y \to X \) where \( Y \) has only isolated, \( Q \)-factorial, arc-wise Nash-trivial singularities, either the center \( C_Y(v) \) is an irreducible component of \( f^{-1}(P) \), or the maximal divisorial set \( C_Y(v) \) is an irreducible component of \( \pi_Y^{-1}(Q) \) for some singular point \( Q \in Y \) such that \( \dim_Q(f^{-1}(P)) \leq 1. \)

The set of valuations that are very essential for a normal threefold singularity \( (P \in X) \) clearly contains the set of valuations defined by the irreducible components of \( \pi_X^{-1}(P) \). The converse is not known.

Question 6.20 (Johnson–Kollár [JK13, Problem 38]). Let \( P \) be a closed point of a normal threefold \( X \). Is every very essential valuation for \( (P \in X) \) defined by an irreducible component of \( \pi_X^{-1}(P) \)?

One can also consider the following alternative approach which is perhaps too optimistic but it is easier to state in all dimensions. Let \( X \) be any variety.

\footnote{It is suggested in [JK13] that, in this definition, one may need to allow \( Y \) to be an algebraic space.}
Definition 6.21. A proper birational morphism $f: Y \to X$ is an arc-wise semi-resolution if for every irreducible component $E$ of $f^{-1}(X_{\text{sing}})$, the set $\pi_Y^{-1}(E)$ is irreducible.

Definition 6.22. A divisorial valuation on $X$ is strongly essential if its center on any arc-wise semi-resolution $f: Y \to X$ is an irreducible component of $f^{-1}(X_{\text{sing}})$.

Clearly, strongly essential valuations are essential, and the same argument proving that Nash valuations are essential shows that they are strongly essential.

Question 6.23. Is a valuation on a variety $X$ a Nash valuation if and only if it is a strongly essential valuation?

7. The Nash problem in the analytic topology

The space of arcs can be defined for any complex analytic variety $V$. If $V$ is defined by the vanishing of finitely many holomorphic functions $f_i(x) = 0$ in an analytic domain $U \subset \mathbb{C}^n$, then $V_{\infty}$ is defined as the set of $n$-ples of power series $x(t) \in \mathbb{C}[[t]]^n$ such that $f_i(x(t)) = 0$ for all $i$. The jet spaces $V_m$ are defined similarly, and $V_{\infty}$ is their inverse limit. As such, it inherits the inverse limit analytic topology.

Suppose that $V = X^{an}$ is the analytification of some complex algebraic variety $X$. Then the points of $V_m$ are in bijection with $X_m(\mathbb{C})$ and the points of $V_{\infty}$ with $X_\infty(\mathbb{C})$. By Theorem 2.6, the image of $\pi_V^{-1}(V_{\text{sing}})$ in any finite level $V_m$ is a constructible set and the decomposition of its closure into irreducible analytic subvarieties of $V_m$ stabilizes for $m \gg 1$. It follows that the truncation maps $V_{m+1} \to V_m$ establish a one-to-one correspondence between such decompositions. By passing to the limit as $m \to \infty$, one obtains a decomposition of $\pi_V^{-1}(V_{\text{sing}})$ into finitely many families of arcs which agrees with the decomposition into irreducible components of $\pi_X^{-1}(X_{\text{sing}})$ given in Theorem 3.3.

We can define essential valuations over $V$ analogously to the algebraic setting, by looking at the centers of the valuation on all analytic resolutions $W \to V$. It turns out that the notion of essential valuation depends on the category.

Theorem 7.1 (de Fernex [dF13, Theorem 5.1]). There is a divisorial valuation over a complex threefold $X$ that is essential in the category of schemes but not in the analytic category.

The example is a threefold $X \subset \mathbb{C}^4$ with an isolated singularity $O$. In this example, $\pi_X^{-1}(O)$ is irreducible. Blowing up the singular point produces a variety $Y$ with an ordinary double point $P$, and the exceptional divisor $F$ defines the only Nash valuation on $X$. A resolution of $X$ is obtained by blowing up $P \in Y$. The divisor $G$ extracted by this second blow-up defines a valuation $\text{val}_G$ over $X$ which is essential in the category of schemes. This valuation is however not essential in the analytic category. This is due to the fact that $Y^{an}$ admits a small analytic resolution $W \to Y^{an}$, which is not defined in the category of schemes, where the center of $\text{val}_G$ is not an irreducible component of the inverse image of $(X^{an})_{\text{sing}}$. The difference in the notion of essential valuation is reflected in this example in the fact that $X$ is locally $\mathbb{Q}$-factorial in the Zariski topology but not in the analytic topology.
One can also formulate the Nash problem in the category of algebraic spaces. Just like in the analytic setting, there are more resolutions in the category of algebraic spaces than the category of schemes, and this can affect the notion of essential valuation.

8. The Nash problem in positive characteristics

Two ingredients play a pivotal role in the treatment of the subject in characteristic zero: resolution of singularities and generic smoothness. Both are used in the proofs of Theorems 2.8 and 3.3 given here.

Kolchin’s original proof of Theorems 2.8 does not use resolution of singularities and works over any field of characteristic zero. A geometric proof in this generality is given in [NS10, Theorem 3.6]. Irreducibility fails however in positive characteristics. The following example was suggested by János Kollár.

**Example 8.1.** Consider the $p$-fold Withney umbrella $X = (xy^p = z^p) \subset \mathbb{A}^3$ over a field $k$ of characteristic $p > 0$. The singular locus is the $x$-axis $X_{\text{sing}} = (y = z = 0)$. The ideal in $k[x, y, z, x', y', z']$ of the first jet scheme $X_1 \subset (\mathbb{A}^3)_1$ is generated by $xy^p - z^p$ and $x'y^p$, and its primary decomposition is $(xy^p - z^p, x') \cap (y^p, z^p)$. Therefore $X_1$ has two irreducible components: $V(xy^p - z^p, x')$ and $V(y, z)$. The first component is the closure of the image of $(X \setminus X_{\text{sing}})_\infty$. On the other hand, the arc $\alpha = (t, 0, 0) \in (X_{\text{sing}})_\infty$ maps to a 1-jet that does not belong to such component. It follows that $(X_{\text{sing}})_\infty$ is not contained in the closure of $(X \setminus X_{\text{sing}})_\infty$, and hence $X_\infty$ is not irreducible.

The next theorem tells us that this is the worst that can happen, for arc spaces of varieties over perfect fields.

**Theorem 8.2 (Reguera [Reg09, Theorem 2.9]).** The arc space $X_\infty$ of a variety $X$ defined over a perfect field $k$ has a finite number of irreducible components only one of which is not contained in $(X_{\text{sing}})_\infty$.

An example where irreducibility fails for a regular variety defined over a non-perfect field can be found in [NS10, Theorem 3.19].

The Nash problem was discussed for varieties in positive characteristics in [IK03]. For the reminder of the section, we shall assume that $X$ is a variety over an algebraically closed field $k$ of positive characteristic.

We write the decomposition of $\pi_X^{-1}(X_{\text{sing}})_{\text{red}}$ into irreducible components as follows:

$$\pi_X^{-1}(X_{\text{sing}})_{\text{red}} = \left( \bigcup_{i \in I} C_i \right) \cup \left( \bigcup_{j \in J} D_j \right),$$

where $C_i \not\subset (X_{\text{sing}})_\infty$ and $D_j \subset (X_{\text{sing}})_\infty$.

**Definition 8.3.** The components $C_i$ are called the good components of $\pi_X^{-1}(X_{\text{sing}})$.

In order to gain some control on the decomposition, one needs to assume something about resolution of singularities. Since we cannot rely on generic smoothness, we consider resolutions that are isomorphisms over the smooth locus.

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14This example is discussed in [IK03, Example 2.13] in regard to the decomposition of $\pi_X^{-1}(X_{\text{sing}})$ (cf. Example 8.7).
DEFINITION 8.4. A divisorial valuation $v$ on $X$ is essential if for any resolution $f: Y \to X$ inducing an isomorphism over $X_{\text{sm}}$, the center of $v$ on $Y$ is an irreducible component of $f^{-1}(X_{\text{sing}})$.

THEOREM 8.5 (Ishii–Kollár [IK03, Theorem 2.15]). Assume that $X$ has a resolution $f: Y \to X$ that is an isomorphism over $X_{\text{sm}}$. Then for any good component $C_i$ of $\pi_X^{-1}(X_{\text{sing}})$ there is a divisor $E_i$ over $X$ such that $C_X(E_i) = C_i$. Moreover, the center of $\text{val}_{E_i}$ on any such resolution $f$ is an irreducible component of $f^{-1}(X_{\text{sing}})$. In particular, $\pi_X^{-1}(X_{\text{sing}})$ has only finitely many good components, and the valuation $\text{val}_{C_i}$ associated to any such component is equal to $\text{val}_{E_i}$ and hence is essential.

DEFINITION 8.6. We say that a divisorial valuation $v$ on $X$ is a Nash valuation if $v = \text{val}_{C_i}$ for some good component $C_i$ of $\pi_X^{-1}(X_{\text{sing}})$.

The theorem implies that every Nash valuation is essential. Just like in characteristic zero, one can formulate the Nash problem by asking for which varieties Nash valuations are the same as essential valuations.

In the terminology introduced in Definition 3.5, Theorem 8.5 implies that the $C_i$ are the fat components of $\pi_X^{-1}(X_{\text{sing}})$. The $D_j$, instead, are the thin components because they are contained in $(X_{\text{sing}})_{\infty}$.

The argument in the proof of Theorem 3.3 showing that $f_{\infty}$ induces a dominant map from $\pi_Y^{-1}(f^{-1}(X_{\text{sing}}))$ to $\pi_X^{-1}(X_{\text{sing}})$ breaks down in positive characteristics, and thin components can actually occur in the decomposition of $\pi_X^{-1}(X_{\text{sing}})$.

EXAMPLE 8.7. Let $X = (xy^p = z^q) \subset \mathbb{A}^3$ defined over a field $k$ of characteristic $p$ as in Example 8.1. Since $(X_{\text{sing}})_{\infty}$ is an irreducible component of $X_{\infty}$, it is also an irreducible component of $\pi_X^{-1}(X_{\text{sing}})$. In particular, it is a thin component of this set. For a different argument which looks at the normalization of $X$, see [IK03, Example 2.13].

COROLLARY 8.8. If we assume the existence of a resolution of singularities of $X$, then $\pi_X^{-1}(X_{\text{sing}})$ has finitely many irreducible components.

PROOF. We already know by Theorem 8.5 that there are finitely many good components $C_i$, and the question is whether the number of thin components $D_j$ is finite. Since $D_j \subset (X_{\text{sing}})_{\infty} \subset \pi_X^{-1}(X_{\text{sing}})$, each $D_j$ is an irreducible component of $(X_{\text{sing}})_{\infty}$. Therefore it suffices to check that $(X_{\text{sing}})_{\infty}$ has only finitely many irreducible components. Since $(X_{\text{sing}})_{\infty}$ is, set-theoretically, the union of the arc spaces of the irreducible components of $X_{\text{sing}}$, this property follows from Theorem 8.2.  

A possible way that may get rid of the thin components in the decomposition of $\pi_X^{-1}(X_{\text{sing}})$ is to restrict the attention to the main component $X_{\infty}^{\text{main}}$ of $X_{\infty}$, namely, the irreducible component that dominates $X$. Let

$$\pi_X^{\text{main}}: X_{\infty}^{\text{main}} \to X$$

denote the restriction of $\pi_X$.

QUESTION 8.9. Is every irreducible component of $(\pi_X^{\text{main}})^{-1}(X_{\text{sing}})$ a good component of $\pi_X^{-1}(X_{\text{sing}})$?

Resolutions which induce isomorphisms over the smooth locus are known to exist in positive characteristics for surfaces and toric varieties. It is therefore natural to consider the Nash problem for these classes of varieties.
Regarding toric varieties, the proof of Theorem 6.13 is characteristic free, and therefore the statement holds in all characteristics.

As for the case of surfaces, both proofs of Theorem 5.2 (the original one from [FdBPP12] and the one based on [dFD16]) use characteristic zero in an essential way. This is clear for the original proof where the problem is translated into a topological problem. Most of the other proof (which is the one given here) is characteristic free, but inequality (4) relies on a computation which fails if the map $\phi'$ is wildly ramified at the generic point of some of the divisors $C_i$.

This leaves the following question open.

**Question 8.10.** For a surface defined over an algebraically closed field of positive characteristic, is every valuation associated to an exceptional divisor on the minimal resolution a Nash valuation?

Some cases are known, for instance the case of sandwiches singularities [LJR99].

References

- [Bha] Bhargav Bhatt, *Algebraization and Tannaka duality*. Preprint, arXiv:1404.7483.
- [Bat99] Victor V. Batyrev, *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, J. Eur. Math. Soc. (JEMS) 1 (1999), no. 1, 5–33.
- [dF13] Tommaso de Fernex, *Three-dimensional counter-examples to the Nash problem*, Compos. Math. 149 (2013), no. 9, 1519–1534.
- [dFD16] Tommaso de Fernex and Roi Docampo, *Terminal valuations and the Nash problem*, Invent. Math. 203 (2016), no. 1, 303–331.
- [dFEI08] Tommaso de Fernex, Lawrence Ein, and Shihoko Ishii, *Divisorial valuations via arcs*, Publ. Res. Inst. Math. Sci. 44 (2008), no. 2, 425–448.
- [DL99] Jan Denef and François Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. 135 (1999), no. 1, 201–232.
- [Doc13] Roi Docampo, *Arcs on determinantal varieties*, Trans. Amer. Math. Soc. 365 (2013), no. 5, 2241–2269.
- [DN] Roi Docampo and Antonio Nigro, *The arc space of the Grassmannian*. Preprint, arXiv: 1510.08833.
- [Dri] Vladimir Drinfeld, *On the Grinberg–Kazhdan formal arc theorem*. Preprint, arXiv: math.AG/ 0203263.
- [ELM04] Lawrence Ein, Robert Lazarsfeld, and Mircea Mustaţă, *Contact loci in arc spaces*, Compos. Math. 140 (2004), no. 5, 1229–1244.
- [EM04] Lawrence Ein and Mircea Mustaţă, *Inversion of adjunction for local complete intersection varieties*, Amer. J. Math. 126 (2004), no. 6, 1355–1365.
- [EM09] ————, *Jet schemes and singularities*, Algebraic geometry—Seattle 2005. Part 2, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 505–546.
- [EMY03] Lawrence Ein, Mircea Mustaţă, and Takehiko Yasuda, *Jet schemes, log discrepancies and inversion of adjunction*, Invent. Math. 153 (2003), no. 3, 519–535.
- [FdB12] Javier Fernández de Bobadilla, *Nash problem for surface singularities is a topological problem*, Adv. Math. 230 (2012), no. 1, 131–176.
- [FdBPP12] Javier Fernández de Bobadilla and María Pe Pereira, *The Nash problem for surfaces*, Ann. of Math. (2) 176 (2012), no. 3, 2003–2029.
- [Gil02] Henri Gillet, *Differential algebra—a scheme theory approach*, Differential algebra and related topics (Newark, NJ, 2000), World Sci. Publ., River Edge, NJ, 2002, pp. 95–123.
- [GP07] P. D. González Pérez, *Bijectiveness of the Nash map for quasi-ordinary hypersurface singularities*, Int. Math. Res. Not. IMRN 19 (2007).
- [Gre66] Marvin J. Greenberg, *Rational points in Henselian discrete valuation rings*, Inst. Hautes Études Sci. Publ. Math. 31 (1966), 59–64.
- [GK00] M. Grinberg and D. Kazhdan, *Versal deformations of formal arcs*, Geom. Funct. Anal. 10 (2000), no. 3, 543–555.
28 TOMMASO DE FERNEX

[Gro61] A. Grothendieck, Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I, Inst. Hautes Études Sci. Publ. Math. 11 (1961), 167 (French).

[Gro66] ———, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III, Inst. Hautes Études Sci. Publ. Math. 28 (1966), 255.

[Hir64] Heisuke Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, I, II, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) 79 (1964), 205–326.

[Ish04] Shihoko Ishii, The arc space of a toric variety, J. Algebra 278 (2004), no. 2, 666–683.

[Ish05] ———, Arcs, valuations and the Nash map, J. Reine Angew. Math. 588 (2005), 71–92.

[Ish06] ———, The local Nash problem on arc families of singularities, Ann. Inst. Fourier (Grenoble) 56 (2006), no. 4, 1207–1224 (English, with English and French summaries).

[Ish08] ———, Maximal divisorial sets in arc spaces, Algebraic geometry in East Asia—Hanoi 2005, Adv. Stud. Pure Math., vol. 50, Math. Soc. Japan, Tokyo, 2008, pp. 237–249.

[IK03] Shihoko Ishii and János Kollár, The Nash problem on arc families of singularities, Duke Math. J. 120 (2003), no. 3, 601–620.

[JK13] Jennifer M. Johnson and János Kollár, Arc spaces of cA-type singularities, J. Singul. 7 (2013), 238–252.

[Kol73] E. R. Kolchin, Differential algebra and algebraic groups, Academic Press, New York, 1973. Pure and Applied Mathematics, Vol. 54.

[KN15] János Kollár and András Némethi, Holomorphic arcs on singularities, Invent. Math. 200 (2015), no. 1, 97–147.

[LJ80] Monique Lejeune-Jalabert, Arcs analytiques et résolution minimale des surfaces quasihomogènes, Séminaire sur les Singularités des Surfaces (Palaiseau, France, 1976/1977), Lecture Notes in Math., vol. 777, Springer, Berlin, 1980, pp. 303-332 (French).

[LJR99] Monique Lejeune-Jalabert and Ana J. Reguera, Arcs and wedges on sandwiched surface singularities, Amer. J. Math. 121 (1999), no. 6, 1191–1213.

[LJR12] ———, Exceptional divisors that are not uniruled belong to the image of the Nash map, J. Inst. Math. Jussieu 11 (2012), no. 2, 273–287.

[LA10] Maximiliano Leyton-Alvarez, Familles d’espaces de m-jets et d’espaces d’arcs, J. Pure Appl. Algebra 220 (2016), no. 1, 1–33 (French, with English and French summaries).

[Mus01] Mircea Mustață, Jet schemes of locally complete intersection canonical singularities, Invent. Math. 145 (2001), no. 3, 397–424. With an appendix by David Eisenbud and Edward Frenkel.

[Mus02] ———, Singularities of pairs via jet schemes, J. Amer. Math. Soc. 15 (2002), no. 3, 599-615 (electronic).

[Nas95] John F. Nash Jr., Arc structure of singularities, Duke Math. J. 81 (1995), no. 1, 31–38 (1996). A celebration of John F. Nash, Jr.

[NS10] Johannes Nicaise and Julien Sebag, Greenberg approximation and the geometry of arc spaces, Comm. Algebra 38 (2010), no. 11, 4077–4096.

[Pas89] Johan Pas, Uniform p-adic cell decomposition and local zeta functions, J. Reine Angew. Math. 399 (1989), 137–172.

[PP13] Maria Pe Pereira, Nash problem for quotient surface singularities, J. Lond. Math. Soc. (2) 87 (2013), no. 1, 177–203.

[Plé05] Camille Plénat, À propos du problème des arcs de Nash, Ann. Inst. Fourier (Grenoble) 55 (2005), no. 3, 805–823 (French, with English and French summaries).

[Plé08] ———, The Nash problem of arcs and the rational double points Da, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 7, 2249–2278 (English, with English and French summaries).

[PPP06] Camille Plénat and Patrick Popescu-Pampu, A class of non-rational surface singularities with bijective Nash map, Bull. Soc. Math. France 134 (2006), no. 3, 383–394 (English, with English and French summaries).

[PPP08] ———, Families of higher dimensional germs with bijective Nash map, Kodai Math. J. 31 (2008), no. 2, 199–218.
Camille Plénat and Mark Spivakovsky, *The Nash problem of arcs and the rational double point $E_6$*, Kodai Math. J. **35** (2012), no. 1, 173–213.

______, *The Nash problem and its solution: a survey*, J. Singul. **13** (2015), 229–244.

Ana J. Reguera, *Families of arcs on rational surface singularities*, Manuscripta Math. **88** (1995), no. 3, 321–333.

______, *A curve selection lemma in spaces of arcs and the image of the Nash map*, Compos. Math. **142** (2006), no. 1, 119–130.

______, *Towards the singular locus of the space of arcs*, Amer. J. Math. **131** (2009), no. 2, 313–350.

______, *Arcs and wedges on rational surface singularities*, J. Algebra **366** (2012), 126–164.

Paul Vojta, *Jets via Hasse-Schmidt derivations*, Diophantine geometry, CRM Series, vol. 4, Ed. Norm., Pisa, 2007, pp. 335–361.

A. V. Zelevinski˘ı, *Small resolutions of singularities of Schubert varieties*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 2, 75–77 (Russian).

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