Some properties of block-radial functions and Schrödinger type operators with block-radial potentials

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Abstract

Let $R_{\gamma}B^s_{p,q}(\mathbb{R}^d)$ be a subspace of the Besov space $B^s_{p,q}(\mathbb{R}^d)$ that consists of block-radial functions. We prove that the asymptotic behaviour of the entropy numbers of compact embeddings $id : R_{\gamma}B^s_{p_1,q_1}(\mathbb{R}^d) \to R_{\gamma}B^s_{p_2,q_2}(\mathbb{R}^d)$ depends on the number of blocks of the lowest dimension, the parameters $p_1$ and $p_2$, but is independent of the smoothness parameters $s_1$, $s_2$. We apply the asymptotic behaviour to estimation of powers of a negative spectra of Schrödinger type operators with block-radial potentials. This part essentially relies on the Birman-Schwinger principle.

Keywords: entropy numbers, compact embeddings, Besov spaces, block-radial functions, negative spectrum

1 Introduction

In recent years, some attention has been paid to describing compactness of embeddings of function spaces of Besov and Sobolev type by different quantities, in particular, by corresponding sequences of entropy and approximation numbers. The study was motivated by the program formulated by D. Edmunds and H. Triebel. In [8] they proposed investigation of spectral properties of certain pseudo-differential operators based on the asymptotic behaviour of entropy and approximation numbers, together with Carl’s inequality and the Birman-Schwinger principle. The approach can be used for the pseudo-differential operators that factor over a compact embedding.

Symmetry as well as weights can be used to generate compactness of Sobolev type embeddings on $\mathbb{R}^d$. This was noticed in the case of the first order Sobolev spaces

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of radial functions by W. Strauss in the seventies of the last century, cf. [36]. In the
general framework of Besov and Triebel-Lizorkin spaces a detailed study of radial
distributions has been made in [28], [29] and [30], cf. also [9] for somewhat different
approach. The asymptotic behaviour of entropy numbers of the compact embedding
of radial Besov spaces was described by Th. Kühn, H.-G. Leopold, W. Sickel and
the second named author in [20], corresponding approximation numbers was studied
in [35] and the Gelfand and Kolmogorov numbers by the first named author in [11].

Much less is known about weaker symmetry assumptions, in particular about so
called block-radial symmetry on \( \mathbb{R}^d \). The compactness of the corresponding embed-
dings was noticed by P.L. Lions in [22]. In [34] the second named author extended
this result to Besov and Triebel-Lizorkin spaces. We point out that compactness of
these embeddings is a multidimensional phenomenon, since any block should be at
least of dimension 2. Thus the simplest possible setting is the block-radial symmetry
in four dimensional euclidean space with two 2-dimensional blocks. The application
of block-radial functions to nonlinear elliptic problems can be found in [15, 16].

Now our main aim is to calculate the asymptotic behaviour of entropy numbers
of compact embeddings of Sobolev and Besov spaces of block-radial functions and
present some typical applications of this result to estimate the distribution of eigen-
values of degenerate pseudo-differential operators. Moreover we are interested in
negative spectra of the corresponding Schrödinger type operators with block-radial
potentials. We estimate the number of the negative eigenvalues related to the block
radial eigenfunctions. In particular we show that the Schrödinger type operators
with radial potentials can have block-radial eigenvalues that are not radial.

We recall what we mean by block-radial symmetry. Let \( m \in \{1, \ldots, d\} \) and let
\( \gamma \in \mathbb{N}^m \) be an \( m \)-tuple \( \gamma = (\gamma_1, \ldots, \gamma_m) \), \( \gamma_1 + \ldots + \gamma_m = |\gamma| = d \). The \( m \)-tuple \( \gamma \)
describes the decomposition of \( \mathbb{R}^{|\gamma|} = \mathbb{R}^\gamma_1 \times \cdots \times \mathbb{R}^\gamma_m \) into \( m \) subspaces of dimensions
\( \gamma_1, \ldots, \gamma_m \) respectively. Let
\[
SO(\gamma) = SO(\gamma_1) \times \ldots \times SO(\gamma_m) \subset SO(d)
\]
be a group of isometries on \( \mathbb{R}^{|\gamma|} \). An element \( g = (g_1, \ldots, g_m) \), \( g_i \in SO(\gamma_i) \) acts
on \( x = (\tilde{x}_1, \ldots, \tilde{x}_m) \), \( \tilde{x}_i \in \mathbb{R}^{\gamma_i} \) by \( x \mapsto g(x) = (g_1(\tilde{x}_1), \ldots, g_m(\tilde{x}_m)) \). If \( m = 1 \) then
\( SO(\gamma) = SO(d) \) is the special orthogonal group acting on \( \mathbb{R}^d \). If \( m = d \) then the
group is trivial since then \( \gamma_1 = \ldots = \gamma_m = 1 \) and \( SO(1) = \{ id \} \). We will always
assume that \( \gamma_i \geq 2 \) for any \( i = 1, \ldots, m \).

Let \( B_{p,q}^s(\mathbb{R}^d) \), \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \), be a Besov space and \( R_\gamma B_{p,q}^s(\mathbb{R}^d) \)
be its subspaces consisted of \( SO(\gamma) \)-invariant distributions. It is known that if
\( s_1 - \frac{d}{p_1} > s_2 - \frac{d}{p_2} \) and \( p_1 < p_2 \) then the embedding
\[
R_\gamma B_{p_1,q_1}^{s_1}(\mathbb{R}^d) \hookrightarrow R_\gamma B_{p_2,q_2}^{s_2}(\mathbb{R}^d)
\]
is compact, cf. [34]. Let us assume that \( \gamma_1 \leq \ldots \leq \gamma_m \) and let \( n = \max\{ i : \gamma_i = \gamma_1 \} \). We prove that the entropy numbers \( e_k \) of this embedding have the following
asymptotic behaviour

\[ e_k \sim (k^{-\min_i \gamma_i} (\log k)^{(n-1)(\gamma_1-1)})^{\frac{1}{p_1} - \frac{1}{p_2}}, \]

cf. Theorem 2. If \( m = 1 \) then the space \( R_{\gamma} B^{s_1}_{p_1,q_1} (\mathbb{R}^d) \) consists of radial functions and the above estimates coincide with the estimates proved in [20]. Similarly to the radial case the asymptotic behaviour is independent of the smoothness parameters \( s_1 \) and \( s_2 \). Please note that \( \min_i \gamma_i \leq \frac{d}{m} \leq \max_i \gamma_i \).

In the paper we investigate also the negative spectrum of the self-adjoint operator of the type

\[ H_{s,\beta} = (\text{Id} - \Delta)^{s/2} - \beta V \quad \text{as} \quad \beta \to \infty. \]

We show that if the potential \( V \in L_r(\mathbb{R}^d) \) is \( SO(\gamma) \)-invariant and \( s > \frac{d}{\gamma} \) then the operator has asymptotically at most \( \beta^{r/\gamma_1} (\log \beta)^{(n-1)(\gamma_1-1)/\gamma_1} \) and at least \( \beta^{m/s} \) negative eigenvalues with \( SO(\gamma) \)-invariant eigenfunctions, cf. Theorem 3. If \( V \) is radial and \( \frac{d}{r} < s < [d/2]\frac{d}{r} \) then the operator \( H_{s,\beta,\theta} \) has eigenfunctions that are block-radial but not radial.

**Notation**

Sobolev, Besov and Triebel-Lizorkin spaces are discussed in various places, we refer e.g. to the monographs [37, 38]. We will use only the basic definitions and facts of this theory and will not recall them here. We refer the reader to the quoted literature.

As usual, \( \mathbb{N} \) denotes the natural numbers, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), \( \mathbb{Z} \) denotes the integers and \( \mathbb{R} \) the real numbers. Logarithms are always taken in base 2, \( \log = \log_2 \). If \( X \) and \( Y \) are two Banach spaces, then the symbol \( X \hookrightarrow Y \) indicates that the embedding is continuous. The set of all linear and bounded operators \( T : X \to Y \), denoted by \( \mathcal{L}(X,Y) \), is equipped with the standard norm. As usual, the symbol \( c \) denotes positive constants which depend only on the fixed parameters \( s,p,q \) and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. We will use the symbol \( A \sim B \), where \( A \) and \( B \) can depend on certain parameters. The meaning of \( A \sim B \) is given by: there exist constants \( c_1,c_2 > 0 \) such that inequalities \( c_1 A \leq B \leq c_2 A \) hold for all values of the parameters.

We shall use the following conventions throughout the paper:

- If \( E \) denotes a space of distributions (functions) on \( \mathbb{R}^d \) then by \( R_{\gamma} E \) we mean the subset of \( SO(\gamma) \)-invariant distributions (functions) in \( E \). We endow this subspace with the same norm as the original space. If \( SO(\gamma) = SO(d) \), i.e. if the subspace consists of radial functions, then we will write \( RE \).

Similarly if \( G \) is a finite group of reflections in \( \mathbb{R}^d \) then \( R_G E \) denotes the subspace of those elements of the space \( E \) that are invariant with respect to \( G \).
• If an equivalence class \([f]\) (equivalence with respect to coincidence almost everywhere) contains a continuous representative then we call the class continuous and speak of values of \(f\) at any point (by taking the values of the continuous representative).

• We will use also the following notation related to the action of the group \(SO(\gamma)\) on \(\mathbb{R}^d\),

\[
    r_j = r_j(x) = \left( x_{\gamma_1 + \cdots + \gamma_{j-1} + 1}^2 + \cdots + x_{\gamma_1 + \cdots + \gamma_{j-1} + \gamma_j}^2 \right)^{1/2}.
\]

2 Traces of block-radial functions

Let \(d = |\gamma|\) and \(\gamma_i \geq 2\) for any \(i = 1, \ldots, m\). To simplify the notation we put

\[
    \bar{\gamma}_i = 1 + \sum_{\ell=0}^{i-1} \gamma_\ell \text{ if } i = 1, 2, \ldots, m+1 \text{ with } \gamma_0 = 0.
\]

We define a hyperplane

\[
    H_\gamma = \text{span}\{e_{\bar{\gamma}_1}, e_{\bar{\gamma}_2}, \ldots, e_{\bar{\gamma}_m}\},
\]

where \(e_j, j = 1, \ldots, d\) is a standard orthonormal basis in \(\mathbb{R}^d\). The hyperplane \(H_\gamma\) can be identified with \(\mathbb{R}^m\) in the standard way so we write \((r_1, \ldots, r_m) \in H_\gamma\) if \(r_1 e_{\bar{\gamma}_1} + r_2 e_{\bar{\gamma}_2} + \cdots + r_m e_{\bar{\gamma}_m} \in H_\gamma\). We need also a finite group of reflections \(G(\gamma)\) acting on \(H_\gamma\). The group consists of transformations \(g_{i_1, \ldots, i_m} \in G(\gamma)\) given by

\[
    g_{i_1, \ldots, i_m}(r_1, \ldots, r_m) = (-1)^{i_1} r_1, \ldots, (-1)^{i_m} r_m, \quad (i_1, \ldots, i_m) \in \{1, 2\}^m.
\]

Let \(f : \mathbb{R}^d \to \mathbb{C}\) be a locally integrable \(SO(\gamma)\)-invariant function. By using the Lebesgue point argument its restriction

\[
    f_0(r_1, \ldots, r_m) := f(\tilde{r}_1, \ldots, \tilde{r}_m), \quad \tilde{r}_j = (r_j, 0, \ldots, 0) \in \mathbb{R}^{\gamma_j}, \quad j = 1, \ldots, m
\]

is well-defined a.e. on \(H_\gamma\). However, this restriction need not be locally integrable. A simple example is given by the function

\[
    f(x) := \psi(x) |x|^{-m}, \quad x \in \mathbb{R}^d, \quad \psi \in C_0(\mathbb{R}^d), \quad \psi(0) = 1.
\]

On the other hand, we can start with a measurable \(g : H_\gamma \to \mathbb{C}\), that is invariant with respect to the action of the group \(G(\gamma)\). If \(g\) is locally integrable on all subsets \(\{(r_1, \ldots, r_m) : r_j > 0, j = 1, \ldots, m, \text{ and } a < |(r_1, \ldots, r_m)| < b\}\), \(0 < a < b < \infty\), then (again using the Lebesgue point argument) the function

\[
    f(x) := g(r(x)), \quad x \in \mathbb{R}^d
\]

is well-defined a.e. on \(\mathbb{R}^d\) and is \(SO(\gamma)\)-invariant. In what follows we shall study properties of the associated operators

\[
    \text{tr} : f \mapsto f_0 \quad \text{and} \quad \text{ext} : g \mapsto f.
\]

Both operators are defined pointwise.
2.1 Traces of block-radial $L_p$-spaces.

Before we turn to the description of the trace classes of block-radial Besov and Sobolev spaces with $1 \leq p \leq \infty$ we start with almost trivial results for $L_p$-functions. We need a further notation. By $L_p(\mathbb{R}^m, w)$ we denote the weighted Lebesgue space equipped with the norm

$$
\| f \|_{L_p(\mathbb{R}^m, w)} := \left( \int_{\mathbb{R}^m} |f(x)|^p w(x) \, dx \right)^{1/p}
$$

with the usual modification if $p = \infty$. We will use a weight

$$
w_\gamma(r_1, \ldots, r_m) = \prod_{i=1}^m |r_i|^\gamma_i - 1.
$$

(1)

Direct calculations show that $w_\gamma$ is a Muckenhoupt weight. More precisely $w_\gamma \in A_\rho$ for any $\rho > \max \gamma_i$. We recall the definition of the $A_\rho$ classes in Appendix B.

**Lemma 1** We assume that $d \geq 2$.

(i) Let $0 < p < \infty$. Then $\text{tr} : R_\gamma L_p(\mathbb{R}^d) \to R_\gamma L_p(H_\gamma, w_\gamma)$ is a linear isomorphism with inverse $\text{ext}$.

(ii) Let $p = \infty$. Then $\text{tr} : R_\gamma L_\infty(\mathbb{R}^d) \to R_\gamma L_\infty(H_\gamma)$ is a linear isomorphism with inverse $\text{ext}$.

**Proof.** Introducing the radial coordinates on each block we get

$$
\int_{\mathbb{R}^d} |f(x)|^p \, dx =
$$

(2)

$$
\frac{2^{d/2}}{\Gamma(\gamma_1/2) \ldots \Gamma(\gamma_m/2)} \int_0^\infty \ldots \int_0^\infty |f_0(r_1, \ldots, r_m)|^p r_1^{\gamma_1 - 1} \ldots r_m^{\gamma_m - 1} \, dr_1 \ldots dr_m.
$$

On the other hand the formula

$$
\int_0^\infty \ldots \int_0^\infty |f_0(r_1, \ldots, r_m)|^p r_1^{\gamma_1 - 1} \ldots r_m^{\gamma_m - 1} \, dr_1 \ldots dr_m =
$$

$$
= \lim_{\varepsilon \downarrow 0} \int_0^\varepsilon \ldots \int_0^\varepsilon |f_0(r_1, \ldots, r_m)|^p r_1^{\gamma_1 - 1} \ldots r_m^{\gamma_m - 1} \, dr_1 \ldots dr_m,
$$

implies that test functions supported in the interior of $[0, \infty)^m$ are dense in $L_p([0, \infty)^m, w_\gamma)$. In consequence the formula (2) can be read from the other side i.e.

$$
\int_{\mathbb{R}^d} |\text{ext} \, g(x)|^p \, dx =
$$

$$
= \frac{2^{d/2}}{\Gamma(\gamma_1/2) \ldots \Gamma(\gamma_m/2)} \int_0^\infty \ldots \int_0^\infty |g(r_1, \ldots, r_m)|^p r_1^{\gamma_1 - 1} \ldots r_m^{\gamma_m - 1} \, dr_1 \ldots dr_m,
$$

for all $g \in L_p([0, \infty)^m, w_\gamma)$. This proves (i). Part (ii) is obvious.  

\[ \blacksquare \]
Lemma 1 means that whenever the Besov $B^s_{p,q}(\mathbb{R}^d)$ or Sobolev space $H^s_p(\mathbb{R}^d)$ is contained in $L_1(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$, then tr is well-defined on its block-radial subspace. It is well known that

$$B^s_{p,q}(\mathbb{R}^d), H^s_p(\mathbb{R}^d) \rightarrow L_1(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$$

if $s > d \max(0, \frac{1}{p} - 1)$, see e.g. [32].

This is in some contrast to the general theory of traces on these spaces. Generally to guarantee that $B^s_{p,q}(\mathbb{R}^d)$ or $H^s_p(\mathbb{R}^d)$ has a trace on $\mathbb{R}^m$ one has to assume that

$$s > \frac{d - m}{p} + m \max\left(0, \frac{1}{p} - 1\right),$$

cf. e.g. [10], [38, Rem. 2.7.2/4]. This condition is stronger.

### 2.2 Traces of block-radial Sobolev and Besov spaces

Let $n \in \mathbb{N}_0$. Then $W^n_p(\mathbb{R}^d)$ denotes the collection of all functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$ such that all weak derivatives $D^\alpha f$ of order $|\alpha| \leq n$ exist and belong to $L_p(\mathbb{R}^d)$. The norm in $W^n_p(\mathbb{R}^d)$ is defined by

$$\|f|W^n_p(\mathbb{R}^d)\| := \sum_{|\alpha| \leq n} \|D^\alpha f|L_p(\mathbb{R}^d)\|.$$

**Proposition 1** Let $d \geq 2$. For $1 < p < \infty$ and $n = \max(0, \frac{\max n}{p} - 1), n \in \mathbb{N}$ the mapping tr is a linear isomorphism of $R_G W^n_p(\mathbb{R}^d)$ onto $R_G W^n_p(H_\gamma, w_\gamma)$ with the inverse ext.

**Proof** Step 1. (The trace operator.) The operator tr is well-defined on $R_G W^n_p(\mathbb{R}^d)$ and $\text{tr} (R_G W^n_p(\mathbb{R}^d)) \subset R_G L_p(H_\gamma, w_\gamma)$ since $R_G W^n_p(\mathbb{R}^d) \subset R_G L_p(\mathbb{R}^d)$, cf. Lemma 1. One can easily see that if $f \in R_G C^\infty_0(\mathbb{R}^d)$ then $f_0 = \text{tr} f \in R_G C^\infty_0(H_\gamma)$ since

$$\frac{\partial^{|\alpha|} f_0}{\partial x_\gamma_1 \cdots \partial x_\gamma_n}(r(x)) = \frac{\partial^{|\alpha|} f}{\partial x_\gamma_1 \cdots \partial x_\gamma_n}(x), \quad |\alpha| \leq n.$$ 

In consequence introducing the block spherical coordinates we get

$$\|f_0|W^n_p(\mathbb{R}^m, w_\gamma)\| \leq C \|f|W^n_p(\mathbb{R}^d)\|.$$

It was proved in [34] that the space $R_G W^n_p(\mathbb{R}^d)$ is a complemented subspace of $W^n_p(\mathbb{R}^d)$ and that the corresponding projection maps $C^\infty_0(\mathbb{R}^d)$ onto $R_G C^\infty_0(\mathbb{R}^d)$. Thus the space $R_G C^\infty_0(\mathbb{R}^d)$ is dense in $R_G W^n_p(\mathbb{R}^d)$. This proves that

$$\text{tr} : R_G W^n_p(\mathbb{R}^d) \rightarrow R_G W^n_p(H_\gamma, w_\gamma).$$
Step 2. (The extension operator) Since \( 1 < p < \infty \) one can define an equivalent norm in the Sobolev spaces by

\[
\| f \|_{W^m_p(\mathbb{R}^d)} := \| f \|_{L^p(\mathbb{R}^d)} + \sum_{i=1}^{d} \left\| \frac{\partial^n f}{\partial x_i^n} \right\|_{L^p(\mathbb{R}^d)}. \tag{3}
\]

Let \( f_0 \in R_G C_0^0(H_\gamma) \) and \( f = \text{ext} f_0 \). First we fix \((x_{\gamma_1+1}, \ldots, x_d) \in \mathbb{R}^{d-\gamma_1}\). The function

\[
\tilde{f}(x_1, \ldots, x_{\gamma_1}) := f((x_1, \ldots, x_d)) = f_0(r_1(x), \ldots, r_m(x))
\]

is radial on \( \mathbb{R}^{\gamma_1} \). Moreover \( \tilde{f} \) is the radial extension of an even function \( \tilde{f}_0(r) = f_0(r, r_2(x), \ldots, r_m(x)), r \in \mathbb{R} \). By the trace result for radial functions, Theorem 3, Theorem 8 and Theorem 9 in [30], we deduce that there is a constant \( C > 0 \) independent of \( f_0 \) such that the inequality

\[
\int_{\mathbb{R}^{\gamma_1}} \left| \partial^\alpha f(x_1, \ldots, x_d) \right|^p dx_1 \ldots dx_{\gamma_1} \leq C \| \tilde{f}_0 \|_{W^m_p(\mathbb{R}, |r|^{\gamma_1-1})}^p = (4)
\]

holds for any \( \alpha \in \{1, \ldots, \gamma_1\}^{\gamma_1}, |\alpha| \leq n \). The function

\[
\mathbb{R}^d \ni x \mapsto f_0(r, r_2(x), \ldots, r_m(x))
\]

is invariant with respect of any isometry belonging to \( \{\text{id}\} \times SO(\gamma_2) \times \ldots \times SO(\gamma_m) \), therefore integrating the inequality (4) with respect to the variables \((x_{\gamma_1+1}, \ldots, x_d)\) we obtain

\[
\int_{\mathbb{R}^d} \left| \partial^\alpha f(x_1, \ldots, x_d) \right|^p dx_1 \ldots dx_d \leq C \sum_{i=0}^{n} \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{d-\gamma_1}} \left| \frac{d^i}{dr^i} f_0(r, r_2(x), \ldots, r_m(x)) \right|^p dx_{\gamma_1+1} \ldots dx_d \right) |r|^{\gamma_1-1} dr = \\
= C \sum_{i=0}^{n} \int_{\mathbb{R}^m} \left| \frac{d^i}{dr^i} f_0(r_1, r_2, \ldots, r_m) \right|^p |r_1|^{\gamma_1-1} \ldots |r_m|^{\gamma_m-1} dr_1 \ldots dr_m \leq C \| f_0 \|_{W^m_p(\mathbb{R}^m, w_\gamma)}^p.
\]

Similar argument works for any \( \alpha \in \{\bar{\gamma}_i, \ldots, \bar{\gamma}_{i+1} - 1\}^{\gamma_i}, i = 2, \ldots, m - 1 \), \( |\alpha| \leq n \). So for any \( j = 1, \ldots, d \) we have

\[
\int_{\mathbb{R}^d} \left| \frac{\partial^n}{\partial x_j^n} f(x_1, \ldots, x_d) \right|^p dx_1 \ldots dx_d \leq C \| f_0 \|_{W^m_p(\mathbb{R}^m, w_\gamma)}^p.
\]

Now using the norm (3) we conclude

\[
\| \text{ext} f_0 \|_{W^m_p(\mathbb{R}^d)} = \| f \|_{W^m_p(\mathbb{R}^d)} \leq C \| f_0 \|_{W^m_p(\mathbb{R}^m, w_\gamma)}.
\]
Smooth compactly supported functions are dense in $W^n_p(R^m, w)$, cf. [2]. The space $R_G W^n_p(H, w)$ is a complemented subspace of $W^n_p(H, w)$ since

$$P : W^n_p(R^m, w) \ni h \mapsto \frac{1}{|G(\gamma)|} \sum_{g \in G(\gamma)} h \circ g$$

is a continuous projection onto $R_G W^n_p(H, w)$. Moreover $P$ maps $C^\infty_0(R^m)$ onto $R_G C^\infty_0(H)$. Thus $R_G C^\infty_0(H)$ is a dense subspace of $R_G W^n_p(H, w)$. Thus we can extend the operator $\text{ext}$ to a continuous operator defined on $R_G W^n_p(H, w)$. This proves the proposition.

**Corollary 1** Let $d \geq 2$. Let $1 < p < \infty$, $s > \left[\max(0, \frac{\max \gamma}{p} - 1)\right] + 1$, and $0 < q \leq \infty$.

(a) The mapping $\text{tr}$ is a linear isomorphism of $R_G H^s_p(R^d)$ onto $R_G H^s_p(H, w)$ with the inverse $\text{ext}$.

(b) The mapping $\text{tr}$ is a linear isomorphism of $R_G B^s_{p,q}(R^d)$ onto $R_G B^s_{p,q}(H, w)$ with the inverse $\text{ext}$.

**Proof.** The point (a) can be proved by complex interpolation and the retraction-coretraction method. We known that

$$[W^{n_1}_p(R^m, w), W^{n_2}_p(R^m, w)]_{\theta} = H^s_p(R^m, w) \quad \text{if} \quad (1 - \theta)n_1 + \theta n_2 = s,$$

with $w$ being the Muckenhoupt weight, cf. [31]. Moreover it was proved in [34] that the space $R_G W^n_p(R^d)$ is a complemented subspace of $W^n_p(R^d)$. Analogously $R_G W^n_p(H, w)$ is a complemented subspace of $W^n_p(H, w)$. So using the retraction-coretraction method we can prove that

$$[R_G W^{n_1}_p(R^d), R_G W^{n_2}_p(R^d)]_{\theta} = R_G H^s_p(R^d) \quad \text{if} \quad (1 - \theta)n_1 + \theta n_2 = s,$$

$$[R_G W^{n_1}_p(H, w), R_G W^{n_2}_p(H, w)]_{\theta} = R_G H^s_p(H, w) \quad \text{if} \quad (1 - \theta)n_1 + \theta n_2 = s.$$

The point (b) can be proved similarly, but know we use the real interpolation. More precisely one should use the formula

$$[W^{n_1}_p(R^m, w), W^{n_2}_p(R^m, w)]_{\theta,q} = B^s_{p,q}(R^m, w) \quad \text{if} \quad \theta k = s, \ k > s, \ 0 < q \leq \infty,$$

proved by Bui in [2].

Now we consider the case $p = \infty$.

**Proposition 2** Let $d \geq 2$, $0 < s < 1$, $p = \infty$ and $0 < q \leq \infty$. The mapping $\text{tr}$ is a linear isomorphism of $R_G B^s_{p,q}(R^d)$ onto $R_G B^s_{p,q}(H, w)$ with the inverse $\text{ext}$.
Proof. It is obvious that \( \text{tr} \) is a linear isomorphism of \( R_\gamma C(\mathbb{R}^d) \) onto \( R_\gamma C(H_\gamma) \) with the inverse \( \text{ext} \). We prove that it is also a linear isomorphism of \( R_\gamma C^1(\mathbb{R}^d) \) onto \( R_\gamma C^1(H_\gamma) \) with the same inverse. Here \( C^1(\mathbb{R}^d) \) denotes the collection of all functions \( f : \mathbb{R}^d \to \mathbb{C} \) such that all their derivatives of the first order exist, are uniformly continuous and bounded. The spaces \( C^1(\mathbb{R}^d) \) is equipped with the norm
\[
\| f | C^1(\mathbb{R}^d) \| := \sum_{|\alpha| \leq 1} \| D^\alpha f | L_\infty(\mathbb{R}^d) \|.
\]

It should be clear that if \( f \) is \( SO(\gamma) \)-invariant on \( \mathbb{R}^d \) then \( \text{tr} f \) is a continuous \( G \)-invariant function on \( H_\gamma \). Vice versa, if \( f_0 \in C^1(H_\gamma) \) is \( G \)-invariant on \( H_\gamma \) then \( \text{ext} f_0 \) is a continuous \( SO(\gamma) \)-invariant on \( \mathbb{R}^d \).

Let \( f \in R_\gamma C^1(\mathbb{R}^d) \). For \( r = (r_1, \ldots, r_m) \in H_\gamma \) we put \( \tilde{r} = (x_1, \ldots, x_d) \) with \( x_\ell = r_\ell \) if \( \ell = \bar{\gamma}_j \) and \( x_\ell = 0 \) otherwise. We obviously have
\[
\frac{\partial f_0}{\partial r_j}(r_1, \ldots, r_m) = \frac{\partial f}{\partial x_{\gamma_j}}(\tilde{r}) ,
\]
which proves the estimate
\[
\| \text{tr} f | C^1(H_\gamma) \| \leq \| f | C^1(\mathbb{R}^d) \|
\]
and at the same time the continuity of the function \( \text{tr} f = f_0 \) and its derivative.

Now, we assume that \( f_0 \in R_\gamma C^1(H_\gamma) \). Let \( f := \text{ext} f_0 \). If \( r_j(x) \neq 0 \) then we have
\[
\frac{\partial f}{\partial x_\ell}(x) = \frac{\partial f_0}{\partial r_j}(r(x)) \frac{x_\ell}{r_j(x)} , \quad \bar{\gamma}_j \leq \ell < \bar{\gamma}_{j+1} . \tag{5}
\]

Let \( r_j(x) = 0 \). The function \( f_0 \) is \( G \)-invariant, so the function \( r_i \mapsto f_0(r_1, \ldots, r_m) \) is even for any \( i = 1, \ldots, m \). In consequence derivatives of the continuously differentiable function \( f_0 \) satisfies
\[
\frac{\partial f_0}{\partial r_j}(r_1, \ldots, r_{j-1}, 0, r_{j+1}, \ldots, r_m) = 0 . \tag{6}
\]

So, if \( \bar{\gamma}_j \leq \ell < \bar{\gamma}_{j+1} \) then (6) implies
\[
\frac{\partial f}{\partial x_\ell}(x) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_\ell + h, \ldots, x_d) - f(x_1, \ldots, x_d)}{h} = \lim_{h \to 0} \frac{f_0(r_1, \ldots, r_{j-1}, |h|, r_{j+1}, \ldots, r_m) - f_0(r_1, \ldots, r_{j-1}, 0, r_{j+1}, \ldots, r_m)}{h} = 0 . \tag{7}
\]

Now (5) and (7) give us
\[
\sup_x \left| \frac{\partial f}{\partial x_\ell}(x) \right| \leq \sum_{j=1}^m \sup_r \left| \frac{\partial f_0}{\partial r_j}(r) \right| .
\]

It remains to deal with the continuity of the derivatives \( \frac{\partial f}{\partial x_\ell} \). Let \( \bar{\gamma}_j \leq \ell < \bar{\gamma}_{j+1} \). If \( r_j(x) \neq 0 \) then the continuity follows immediately from (5) and the continuity of derivatives of function \( f_0 \).
Let \( r_j(x) = 0 \) and let \( x^{(k)} \to x \) in \( \mathbb{R}^d \) as \( k \to \infty \), \( r_j(x^{(k)}) \neq 0 \). Then \( r_j(x^{(k)}) \to 0 \) so (5), (6) and the continuity of the partial derivatives of \( f_0 \) imply the continuity of the partial derivative of \( f \) at \( x \). This proves the claim.

It remains to extend the statement to Besov spaces \( B_{\infty,q}^s \). Once more this can be done via interpolation since

\[
(C(\mathbb{R}^d), C^1(\mathbb{R}^d))_{s,q} = B_{\infty,q}^s(\mathbb{R}^d), \quad 0 < s < 1 \quad \text{and} \quad 0 < q \leq \infty,
\]

cf. Theorem 2.7.2 and Theorem 1.10.2 in [37].

\[\square\]

**Corollary 2** Let \( d \geq 2 \), \( s \in \mathbb{R} \), \( 1 < p \leq \infty \) and \( 0 < q \leq \infty \). Then the space \( R_\gamma B_{p,q}^s(\mathbb{R}^d) \) is isomorphic to \( R_\gamma B_{p,q}^s(H_\gamma, w_\gamma) \). Analogously the space \( R_\gamma H_p^s(\mathbb{R}^d) \) is isomorphic to \( R_\gamma H_p^s(H_\gamma, w_\gamma) \).

**Proof.** If \( 1 < p < \infty \) and \( s \) is sufficiently large then the corollary follows immediately from Corollary 1. For other value of \( s \) the statement follows by the lift property. The statement can be extented to smaller values of \( s \) by the lifting property for (weighted) Besov and Sobolev spaces, cf. [2].

The same argument works for \( p = \infty \). Now we should use Proposition 2 instead of Corollary 1.

\[\square\]

### 3 Entropy numbers of embeddings of spaces of block-radial functions

In this section we estimate entropy numbers of compact Sobolev embeddings of block-radial functions. It was proved in [34] that the embedding

\[
id : R_\gamma B_{p_1,q_1}^{s_1}(\mathbb{R}^d) \to R_\gamma B_{p_2,q_2}^{s_2}(\mathbb{R}^d)
\]

is compact if and only if

\[
p_1 < p_2, \quad \delta = s_1 - \frac{d}{p_1} - s_2 + \frac{d}{p_2} > 0 \quad \text{and} \quad \min_i \gamma_i \geq 2.
\]

For convenience of the reader we recall the basic definitions. Let \( X \) and \( Y \) be Banach spaces and \( T \in \mathcal{L}(X,Y) \). The \( k \)-th entropy number of \( T \), \( k \in \mathbb{N} \), is defined in the following way

\[
e_k(T) := \inf \{ \epsilon > 0 : T(B_X) \text{ can be covered by } 2^{k-1} \text{ balls of radius } \epsilon \in Y \},
\]

where \( B_X \) denotes the closed unit ball in \( X \). The sequence of entropy numbers can be viewed as quantification of the notion of compactness since the operator is compact if and only if \( e_k(T) \to 0 \), as \( k \to \infty \).
The entropy numbers have properties of multiplicativity and additivity, i.e.
\[ e_{n+k-1}(T \circ S) \leq e_n(T) \cdot e_k(S) \quad \text{and} \quad e_{n+k-1}(T + S) \leq e_n(T) + e_k(S). \]

For further their properties we refer to [24], [14], [5] or [8].

To estimate entropy numbers we use the technique of quasi-normed operator ideals. In the context of entropy numbers the approach goes back to B. Carl’s paper [3]. For Sobolev embeddings it was used in [17] and [20] for the first time. Let \( \omega = (\omega_n) \) be an increasing sequence of positive real numbers satisfying the regularity condition \( \omega_{2k} \sim \omega_k \). The properties of entropy numbers imply that for operators \( T \in \mathcal{L}(X,Y) \) between Banach spaces the formula
\[
L^{(e)}_{\omega}(T) := \sup_k \omega_k e_k(T)
\]
defines a quasi-norm in the vector space
\[
\mathcal{L}^{(e)}_{\omega}(X,Y) = \{ T \in \mathcal{L}(X,Y) : L^{(e)}_{\omega}(T) < \infty \}.
\]

In fact, \( (\mathcal{L}^{(e)}_{\omega}, L^{(e)}_{\omega}) \) is a quasi-normed operator ideal in the sense of Pietsch [23, Definition 6.1.1].

By Corollary 2 we can reduce investigation of an asymptotic behaviour of entropy numbers of the embeddings (8) to estimation of embeddings of the corresponding weighted spaces with the Muckenhoupt weight \( w_\gamma \), cf. (1), i.e.
\[
e_k(id : R_\gamma B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \to R_\gamma B^{s_2}_{p_2,q_2}(\mathbb{R}^d)) \sim
\sim e_k(id : R_G B^{s_1}_{p_1,q_1}(\mathbb{R}^m, w_\gamma) \to R_G B^{s_2}_{p_2,q_2}(\mathbb{R}^m, w_\gamma)).
\]

Furthermore using the wavelet characterization of Besov spaces with \( A_\infty \) weights we can use the technique of discretization i.e., we can reduce the problem to the corresponding problem for suitable sequence spaces, cf. [12, Theorem 1.13]. However, the resulting sequence spaces are still complicated, therefore a further reduction is necessary. We will use the following result concerning the entropy numbers of general diagonal operators proved by Th. Kühn, cf. [18, 19].

**Theorem 1 (cf. [19])** Let \( 0 < p_1, p_2 \leq \infty \), and let \( \sigma = (\sigma_k) \) be a non-increasing sequence satisfying the doubling condition \( \sigma_k \sim \sigma_{2k} \) and, in addition,
\[
\sup_{n \geq k} \frac{\sigma_n}{\sigma_k} \cdot \left( \frac{n}{k} \right)^\alpha < \infty \quad \text{for some} \quad \alpha > \max(1/p_2 - 1/p_1, 0).
\]

Then
\[
e_k(D_\sigma : \ell_{p_1} \to \ell_{p_2}) \sim k^{\frac{1}{p_2} - \frac{1}{p_1} \sigma_k}.
\]

Now we recall the definition of the sequence spaces. Let \( Q_{\nu,n} \) denote a dyadic cube in \( \mathbb{R}^m \), centred at \( 2^\nu n, n \in \mathbb{Z}^m, \nu \in \mathbb{N}_0 \), and with the side length \( 2^{-\nu} \). For
where \( \Omega \subset Q \) function of the cube \( \sigma \) where

\[
\text{If } \sigma = 0 \text{ we write } b_{p,q}(w) \text{ instead of } b^\sigma_{p,q}(w); \text{ moreover, if } w \equiv 1 \text{ we write } b^\sigma_{p,q} \text{ instead of } b^\sigma_{p,q}(w). \]

Using the same arguments as in [12] we can prove that

\[
e_k(id : B_{p_1,q_1}^\sigma (\mathbb{R}^m, w_\gamma) \to B_{p_2,q_2}^\sigma (\mathbb{R}^m, w_\gamma)) \sim e_k(id : b_{p_1,q_1}^\sigma (w_\gamma) \to b_{p_2,q_2}^\sigma (w_\gamma)),
\]

where \( \sigma_i = s_i + \frac{m}{2} - \frac{m}{p_i}, i = 1, 2. \)

For later use we introduce an abbreviation

\[
w_\gamma(\Omega) = \int_\Omega w_\gamma(x) dx,
\]

where \( \Omega \subset \mathbb{R}^m \) is some bounded, measurable set. For any dyadic cube \( Q_{\nu,n} \) we have

\[
w_\gamma(Q_{\nu,n}) \sim \prod_{i=1}^m \int_{2^{-\nu n_i} r_i}^{2^{-\nu(n_i+1)}} |r_i|^{-1}dr_i \sim \prod_{i=1}^m \int_0^{2^{-\nu r_i}} (r_i + 2^{-\nu n_i})^{-1}dr_i \sim 2^{-\nu(\gamma_1 + \ldots + \gamma_m)} \prod_{i=1}^m \max\{1, |n_i|\}^{-1} \sim 2^{-\nu d} w_\gamma(Q_{0,n}). \quad (11)
\]

Moreover, one can easily verify that the expression

\[
\left( \sum_{\nu=0}^\infty 2^{\nu q} \left( \sum_{n \in \mathbb{Z}^m} |\lambda_{\nu,n}|^p 2^{\nu n} w_\gamma(Q_{\nu,n}) \right)^{q/p} \right)^{1/q}
\]

is an equivalent (quasi)-norm in \( b^\sigma_{p,q}(w) \). We will used this norm in the sequel. Please note that the conditions (3) are equivalent to the necessary and sufficient conditions for compactness of the embedding \( id : B_{p_1,q_1}^\sigma (\mathbb{R}^m, w_\gamma) \to B_{p_2,q_2}^\sigma (\mathbb{R}^m, w_\gamma) \) given in [12] Proposition 3.1.

**Lemma 2** Let \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{N}^m \) be a multi-index such that \( \gamma_i \geq 2 \) for any \( i = 1, \ldots, m. \) We assume that

\[
\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_m \quad (12)
\]
and put
\[ n = \max\{i : \gamma_i = \gamma_1\}. \] (13)

Let \( \tau \) denote a bijection of \( \mathbb{Z}^m \) onto \( \mathbb{N}_0 \) such that \( \tau(k) < \tau(\ell) \) if \( w_\gamma(Q_{0,k}) < w_\gamma(Q_{0,\ell}) \). Then there are positive constants \( c_1 \) and \( c_2 \) such that for sufficiently large \( L \in \mathbb{N} \) the inequalities
\[ c_1 2^{L(d-m)} \leq w_\gamma(Q_{0,k}) \leq c_2 2^{L(d-m)} \] if and only if
\[ c_1 2^{\frac{L(d-m)}{\alpha - 1}} L^{n-1} \leq \tau(k) \leq c_2 2^{\frac{L(d-m)}{\alpha - 1}} L^{n-1}. \] (15)

**Proof.** Step 1. To simplify our notation we put \( \alpha_i = \gamma_i - 1 \). By assumptions \( 0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m < 1 \). We consider two sets
\[ W_L = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : 2^L \leq \prod_{i=1}^m \max(1, |x_i|)^{\alpha_i} < 2^{L+1}\}, \]
and
\[ \widetilde{W}_L = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : |x_i| \geq 1 \text{ and } 2^L \leq \prod_{i=1}^m |x_i|^{{\alpha_i}} < 2^{L+1}\}. \]

If \((k_1, \ldots, k_m) \in \mathbb{N}_0^m \) belongs to \( W_L \) then \((\max(1, k_1), \ldots, \max(1, k_m)) \in \widetilde{W}_L \), in consequence
\[ \#\{k \in \mathbb{N}_0^m : k \in \widetilde{W}_L\} \leq \#\{k \in \mathbb{N}_0^m : k \in W_L\} \leq 2^m \#\{k \in \mathbb{N}_0^m : k \in \widetilde{W}_L\}. \]

We prove that
\[ \#\{k \in \mathbb{N}_0^m : k \in \widetilde{W}_L\} \sim 2^{L/\alpha_1} L^n. \] (16)

First let us note that
\[ \#\{k \in \mathbb{N}_0^m : k \in \widetilde{W}_L\} \sim \text{vol}_m(\widetilde{W}_L) = \text{vol}_m(V_{2L+1}^m) - \text{vol}_m(V_{2L}^m), \] (17)
where \( V_{2L}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : |x_i| \geq 1 \text{ and } \prod_{i=1}^m |x_i|^{{\alpha_i}} \leq 2^L\}. \)

It is sufficient to restrict our attention to the first octant \( \{(x_1, \ldots, x_m) : x_i \geq 0, i = 1, \ldots, m\} \). A bit more generally we consider the sets
\[ V_R^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 1 \text{ and } \prod_{i=1}^n x_i^{{\alpha_i}} \leq R, R > 1, n = 1, \ldots, m, \]
and show that
\[ \text{vol}_m(V_R^m) \sim R^{1/\alpha_1} (\log R)^{n-1}, \] (18)
for sufficiently large \( R \).
Step 2. First we consider the special case $n = m$ i.e. $\alpha_1 = \ldots = \alpha_m$. The volume estimate in this case has been already calculated by Th. Kühn, W. Sickel and T. Ullrich in [21], cf. Lemma 3.2 ibidem. Let

$$\mathcal{H}_\ell(r) := \{x \in [1, \infty)^\ell : \prod_{i=1}^\ell x_i \leq r\}, \quad f_\ell(r) := r^{(\ln r)^{\ell-1}/(\ell - 1)!}, \quad r > 1, \quad \ell \in \mathbb{N}.$$ 

It was proved that

$$\begin{align*}
\text{vol}_\ell(\mathcal{H}_\ell(r)) &\leq f_\ell(r) \quad \text{for all}, \quad r \geq 2^\ell \quad \text{and} \quad \ell \geq 1; \\
\text{vol}_\ell(\mathcal{H}_\ell(r)) &\geq f_\ell(r) - f_{\ell-1}(r) \quad \text{for all}, \quad r \geq 2^\ell \quad \text{and} \quad \ell \geq 2.
\end{align*}$$

(19) (20)

One can easily observe that $f_\ell(r) - f_{\ell-1}(r) \geq f_\ell(r)/2$ if $r \geq e^{2(\ell-1)}$, so (20) gives

$$\text{vol}_\ell(\mathcal{H}_\ell(r)) \geq \frac{1}{2} f_\ell(r) \quad \text{for all} \quad r \geq e^{2(\ell-1)}. \quad (21)$$

The last inequality holds also for $\ell = 1$ and $r \geq 2$. and that $V^m_R = \mathcal{H}_m(R^{1/\alpha_1})$. Let

$$c_\ell = \alpha_1^{1-\ell} \frac{(\ln 2)^{\ell-1}}{(\ell - 1)!}, \quad \ell = 1, \ldots, m.$$ 

The above estimates give us

$$\frac{1}{2} c_m R^{1/\alpha_1} (\log R)^{m-1} \leq \text{vol}_m(V^m_R) \leq c_m R^{1/\alpha_1} (\log R)^{m-1}, \quad R \geq e^{2\alpha_1(m-1)}, \quad (22)$$

since $\ell = n = m \geq 2$.

Step 3. Now we prove upper and lower estimates in the case $n < m$. Let $\alpha_1 = \ldots = \alpha_n < \alpha_{n+1} \leq \ldots \leq \alpha_m$. We use the following relation between $\text{vol}_{\ell+1}(V^{\ell+1}_R)$ and $\text{vol}_{\ell}(V^{\ell}_R)$

$$\text{vol}_{\ell+1}(V^{\ell+1}_R) = \int_1^{R^{1/\alpha_{\ell+1}}} \text{vol}_{\ell}(V^{\ell}_R) dx_{\ell+1}. \quad (23)$$

First we take $\ell = n$. The inequality (19) implies

$$\text{vol}_{n+1}(V^{n+1}_R) \leq \int_1^{R^{-1}} \text{vol}_n(\mathcal{H}_n(R^{1/\alpha_1}/x_{n+1}^{\alpha_{n+1}/\alpha_1})) dx_{n+1} \leq c_n R^{1/\alpha_1} (\log R)^{n-1} \int_1^\infty x^{-\alpha_{n+1}/\alpha_1} dx \leq c_n \frac{\alpha_1}{\alpha_{n+1} - \alpha_1} R^{1/\alpha_1} (\log R)^{n-1},$$

Iterating this argument $m - n$ times we get the estimate

$$\text{vol}_m(V^m_R) \leq c_n \prod_{\ell = n+1}^m \frac{\alpha_1}{\alpha_{\ell+1} - \alpha_1} R^{1/\alpha_1} (\log R)^{n-1}, \quad R \geq 2^{\alpha_1 n}. \quad (24)$$

Using (21) instead of (19) we can prove the lower estimates in the form

$$c_n \prod_{\ell = n+1}^m \frac{\alpha_1}{\alpha_{\ell+1} - \alpha_1} \leq \text{vol}_m(V^m_R), \quad R \geq \max\{2^{\alpha_1}, e^{2\alpha_1(n-1)}\}. \quad (25)$$
This proves (18).

Step 4. It remains to prove the estimates (16). By (17), (24) and (25) we get
\[
\#\{k \in \mathbb{N}_0^m : k \in \widetilde{W}_L\} \leq 2^m c_n \prod_{\ell=n+1}^m \frac{\alpha_1}{\alpha_{\ell+1} - \alpha_1} 2^{L/\alpha_1} L^{n-1},
\]
and
\[
\#\{k \in \mathbb{N}_0^m : k \in \widetilde{W}_L\} \geq 2^m c_n \prod_{\ell=n+1}^m \frac{\alpha_1}{\alpha_{\ell+1} - \alpha_1} (2^{\frac{1}{\alpha_1} - 1}) 2^{L/\alpha_1} L^{n-1},
\]
for sufficiently large $L$. Please note that $2^{\frac{1}{\alpha_1} - 1} > 0$ since $\alpha_1 < 1$.

Proposition 3 Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{N}_0^m$, $m \in \mathbb{N}$, be a multi-index such that $2 \leq \gamma_1 \leq \ldots \leq \gamma_m$, $d = \gamma_1 + \ldots + \gamma_m$, and let $n = \max\{i : \gamma_i = \gamma_1\}$. Let $1 \leq p_1 < p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$ and $s_1, s_2 \in \mathbb{R}$. If $\delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2}) > 0$
then
\[
e_k(id : b_{p_1,q_1}(w_\gamma) \to b_{p_2,q_2}(w_\gamma)) \sim (k^{-\gamma_1} (\log k)^{(n-1)(\gamma_1 - 1)})^{\frac{m}{p_1} - \frac{m}{p_2}},
\]
where $\sigma_i = s_i + \frac{m}{2} - \frac{m}{p_i}$, $i = 1, 2$.

Proof. Step 1. It is convenient to change slightly the notation. Let $0 < p, q \leq \infty$, $\sigma \in \mathbb{R}$ and let $\mathcal{X}$ denote $\mathbb{Z}^m$ or $\mathbb{N}_0$. We introduce the following sequence space,
\[
\ell_q(2^{\nu_1} \ell_p(\mathcal{X}, w)) := \{ \lambda = \{\lambda_{\nu,\ell} : \lambda_{\nu,\ell} \in \mathbb{C} \mid ||\lambda||_{\ell(q,2^{\nu_1} \ell_p(\mathcal{X}, w))} = \left( \sum_{\nu=0}^{\infty} 2^{\nu q} \left( \sum_{\ell \in \mathcal{X}} |\lambda_{\nu,\ell}|^p w_{\nu,\ell} \right)^{q/p} \right)^{1/q} \leq \infty \},
\]
where $w = (w_{\nu,\ell})_{\nu,\ell}$, $w_{\nu,\ell} > 0$. As usual we write $\ell_q(\ell_p(\mathcal{X}, w))$ if $\sigma = 0$ and $\ell_q(2^{\nu_1} \ell_p(\mathcal{X}))$ if $w \equiv 1$.

By standard arguments we get
\[
e_k(id : b_{p_1,q_1}^{\sigma_1}(w_\gamma) \to b_{p_2,q_2}^{\sigma_2}(w_\gamma)) \\
\sim e_k(id : \ell_{q_1}(\ell_{p_1}(\mathbb{Z}^m)) \to \ell_{q_2}(2^{\nu_1} \ell_{p_2}(\mathbb{Z}^m, w(\gamma)))).
\]
where $\sigma = s_2 - s_1$ and $w(\gamma) = (w_{\nu,\gamma})$ with $w_{\nu,\gamma} = w_{\gamma}(Q_{\nu,\gamma})^{1 - \frac{m}{p_1}}$.

Step 2. Now we prove that
\[
e_k(id : \ell_{q_1}(\ell_{p_1}(\mathbb{Z}^m)) \to \ell_{q_2}(2^{\nu_1} \ell_{p_2}(\mathbb{Z}^m, w(\gamma)))) \\
\sim e_k(id : \ell_{q_1}(\ell_{p_1}(\mathbb{N}_0)) \to \ell_{q_2}(2^{\nu_1} \ell_{p_2}(\mathbb{N}_0, w(\gamma)))),
\]

(26)
where \( \tilde{w}_\ell^{(\gamma)} = \max(1, \ell \log^{1-n} \ell)^{(\gamma-1)(1-\frac{p_2}{p_1})} \). The estimate of the weight on the cubes \((\text{II})\) and the definition of the bijection \(\tau\), cf. Lemma \(2\) give us
\[
\left\| \lambda \ell_{q_2}(2^{\nu_2} \ell_{p_2}(\Z^m, w^{(\gamma)})) \right\| = \\
= \left( \sum_{\nu=0}^{\infty} 2^{\nu q_2} \left( \sum_{n \in \Z^m} |\lambda_{\nu,n}|^{p_2} w_\gamma(Q_{\nu,n})^{1-\frac{p_2}{p_1}} \right)^{q_2/p_2} \right)^{1/q_2} \\
\sim \left( \sum_{\nu=0}^{\infty} 2^{\nu q_2} \left( \sum_{n \in \Z^m} |\lambda_{\nu,n}|^{p_2} w_\gamma(Q_{\nu,n})^{1-\frac{p_2}{p_1}} \right)^{q_2/p_2} \right)^{1/q_2} \\
\sim \left( \sum_{\nu=0}^{\infty} 2^{-\nu \delta q_2} \left( \sum_{\ell=0}^{\infty} |\lambda_{\nu,\ell-1}|^{p_2} \max(1, \ell \log^{1-n} \ell)^{(\gamma-1)(1-\frac{p_2}{p_1})} \right)^{q_2/p_2} \right)^{1/q_2},
\]
where \( \delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2}) \).

If \( L, \ell \in \N_0 \) and \( 2^{\frac{L-1}{n-1}} L^{n-1} \leq \ell < 2^{\frac{(L+1)(d-m)}{\gamma-1}} (L+1)^{n-1} \) then
\[
w_\gamma(Q_{0,\ell-1}) \sim 2^{L(d-m)} \sim (\ell \log^{1-n} \ell)^{(\gamma-1)}, \tag{27}
\]
cf. \((\text{II})\). By what we have already proved
\[
\left\| \lambda \ell_{q_2}(2^{\nu_2} \ell_{p_2}(\Z^m, w^{(\gamma)})) \right\| \sim \\
\sim \left( \sum_{\nu=0}^{\infty} 2^{-\nu \delta q_2} \left( \sum_{\ell=0}^{\infty} |\lambda_{\nu,\ell-1}|^{p_2} \max(1, \ell \log^{1-n} \ell)^{(\gamma-1)(1-\frac{p_2}{p_1})} \right)^{q_2/p_2} \right)^{1/q_2}.
\]
This justifies the equivalence \((\text{II})\) since the estimate
\[
\left\| \lambda \ell_{q_1}(\ell_{p_1}(\Z^m)) \right\| \sim \left( \sum_{\nu=0}^{\infty} \left( \sum_{\ell=0}^{\infty} |\lambda_{\nu,\ell-1}|^{p_1} \right)^{q_1/p_1} \right)^{1/q_1}
\]
is obvious.

**Step 3.** We prove the upper estimate of the entropy numbers
\[
\epsilon_k \left( \text{id : } \ell_{q_1}(\ell_{p_1}(\N_0)) \rightarrow \ell_{q_2}(2^{-\nu \delta} \ell_{p_2}(\N_0, \tilde{w}^{(\gamma)})) \right) \leq \left( \frac{k^{\gamma_1}(\log k)^{\gamma_1(n-1)\gamma_1^{-1}}}{1^{\gamma_1} - \frac{1}{p_2}} \right) \frac{1}{p_1} - \frac{1}{p_2}.
\]
Let us consider the projection \( P_\nu : \ell_{q_1}(\ell_{p_1}(\N_0)) \rightarrow \ell_{p_1}(\N_0) \) onto the \( \nu \)-th vector-coordinate, and the embedding operator \( E_\nu : \ell_{p_2}(\N_0) \rightarrow \ell_{q_2}(2^{-\nu \delta} \ell_{p_2}(\N_0)) \),
\[
(E_\nu(y))_{\mu,\ell} = \begin{cases} 
  y_\ell & \text{if } \mu = \nu \\
  0 & \text{otherwise, } y \in \ell_{p_2}(\N_0),
\end{cases}
\]
It is obvious that
\[
\|P_\nu\| = 1 \quad \text{and} \quad \|E_\nu\| = 2^{-\nu \delta}.
\]
Let \( D_\gamma \) denote the diagonal operator \( D_\gamma : \ell_{p_1}(\N_0) \rightarrow \ell_{p_2}(\N_0) \) generated by the sequence \( \sigma_\ell = (\tilde{w}_\ell^{(\gamma)})^{\frac{1}{p_2}} \) i.e.
\[
(D_\gamma(\lambda))_\ell = (\tilde{w}_\ell^{(\gamma)})^{\frac{1}{p_2}} \lambda_\ell.
\]
\[16\]
Then
\[ \text{id} = \sum_{\nu=0}^{\infty} \text{id}_\nu, \quad \text{where} \quad \text{id}_\nu = E_\nu D_\gamma P_\nu. \]

The multiplicativity of the entropy numbers yields
\[ e_k(\text{id}_\nu) \leq c 2^{-\nu\delta} e_k(D_\gamma). \]

Now using Theorem 1 we get
\[ e_k(D_\gamma) \sim k^{-(1/p_1-1/p_2)}(k \log^{1-n} k)^{-(\gamma_1-1)(1/p_1-1/p_2)}. \]

So
\[ e_k(\text{id}_\nu) \leq c 2^{-\nu\delta}(k^{\nu_1}(\log k)^{(n-1)(\gamma_1-1)})^{1/p_1-1/p_2} \]
with a constant \( c \) independent of \( \nu \) and \( k \). Now taking \( \omega_k = (k^{\nu_1}(\log k)^{(n-1)(\gamma_1-1)})^{1/p_1-1/p_2} \) we have
\[ L_\omega^{(e)}(\text{id}_\nu) = \sup_k \omega_k e_k(\text{id}_\nu) \leq c 2^{-\nu\delta}. \]

Since quasi-norm \( L_\omega^{(e)} \) is equivalent to an \( r \)-norm for some \( r, 0 < r \leq 1 \), we arrive at
\[ L_\omega^{(e)}(\text{id})^r \leq c \sum_{\nu=0}^{\infty} L_\omega^{(e)}(\text{id}_\nu)^r \leq c \sum_{\nu=0}^{\infty} 2^{-\nu\delta r} < \infty, \]
which proves the upper estimate.

Step 4. In this step we estimate the entropy numbers from below. For any given \( k \in \mathbb{N} \) we consider \( k \)-dimensional vector spaces \( \ell^k_{p_1} \) and the following commutative diagram
\[
\begin{array}{ccc}
\ell^k_{p_1} & \xrightarrow{T} & \ell_{q_1}(\ell_{p_1}(\mathbb{N}_0)) \\
\text{id} & \downarrow & \downarrow \text{id} \\
\ell^k_{p_2} & \xleftarrow{S} & \ell_{q_2}(2^{-\nu\delta} \ell_{p_2}(\mathbb{N}_0, \tilde{w}(\gamma))).
\end{array}
\]

Here the operators \( S \) and \( T \) are defined by
\[
(T(\xi_1, \ldots, \xi_k))_{\nu,\ell} = \begin{cases} 
\xi_{\ell+1-k} & \text{if } \nu = 0 \text{ and } k \leq l \leq 2k - 1, \\
0 & \text{otherwise}
\end{cases}
\]
and
\[
S((\lambda_{\nu,\ell})_{\nu,\ell}) = (\lambda_{0,k}, \ldots, \lambda_{0,2k-1}).
\]

The norms of the above operators have the obvious estimates
\[
\|T\| \leq 1 \quad \text{and} \quad \|S\| \leq (\tilde{w}_k^{(\gamma)})^{-\frac{1}{p_2}}.
\]
Using Schütz’s description of asymptotic behaviour of entropy numbers for embeddings between the finite dimensional spaces $\ell^k_p$ - see [26], and again the multiplicativity of the entropy numbers we get,

$$ck^{-\left(\frac{1}{n^2}+\frac{1}{n}+\frac{1}{2}\right)} \leq e_k(id : \ell^k_{p_1} \rightarrow \ell^k_{p_2})$$

$$\leq \|S\| e_k(id : \ell_{q_1}(\ell_{p_1}(N_0)) \rightarrow \ell_{q_2}(2^{-\nu\delta}\ell_{p_2}(N_0, \tilde{w}(\gamma)))) \|T\| \leq (k \log^{-1} n)^{\left(1\right)} \left(\frac{1}{n^2}+\frac{1}{n}+\frac{1}{2}\right) e_k(id : \ell_{q_1}(\ell_{p_1}(N_0)) \rightarrow \ell_{q_2}(2^{-\nu\delta}\ell_{p_2}(N_0, \tilde{w}(\gamma))))$$

with some constant $c$ independent of $k$. This proves the proposition.

**Proposition 4** Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{N}^m$, $m \in \mathbb{N}$, $2 \leq \gamma_1 \leq \ldots \leq \gamma_m$, $d = \gamma_1 + \ldots + \gamma_m$, and let $n = \max\{i : \gamma_i = \gamma_1\}$. Let $1 \leq p_1 < p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$ and $s_1, s_2 \in \mathbb{R}$. If $s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0$ then

$$e_k\left(id : R_G B^s_{p_1, q_1}(\mathbb{R}^m, w_\gamma) \rightarrow R_G B^s_{p_2, q_2}(\mathbb{R}^m, w_\gamma)\right) \sim (k^{-\gamma_1}(\log k)^{(n-1)(\gamma_1-1)})^{\frac{1}{p_1} - \frac{1}{p_2}}.$$

**Proof.** Proposition 3 and the wavelet decomposition of the spaces give us

$$e_k\left(id : B^s_{p_1, q_1}(\mathbb{R}^m, w_\gamma) \rightarrow B^s_{p_2, q_2}(\mathbb{R}^m, w_\gamma)\right) \sim (k^{-\gamma_1}(\log k)^{(n-1)(\gamma_1-1)})^{\frac{1}{p_1} - \frac{1}{p_2}} \quad (28)$$

So it remains to show that we have the same estimate for the $G(\gamma)$-invariant subspaces. The operator

$$P f(x) = \frac{1}{|G(\gamma)|} \sum_{g \in G(\gamma)} f(g(x))$$

is a bounded projection of $B^s_{p,q}(\mathbb{R}^m, w_\gamma)$ onto $R_G B^s_{p,q}(\mathbb{R}^m, w_\gamma)$ since $G(\gamma)$ is a finite group of linear isometries. So the inequality

$$e_k\left(id : R_G B^s_{p_1, q_1}(\mathbb{R}^m, w_\gamma) \rightarrow R_G B^s_{p_2, q_2}(\mathbb{R}^m, w_\gamma)\right) \leq C(k^{-\gamma_1}(\log k)^{(n-1)(\gamma_1-1)})^{\frac{1}{p_1} - \frac{1}{p_2}}$$

follows from (28) and the following commutative diagram

$$\begin{array}{ccc}
R_G B^s_{p_1, q_1}(\mathbb{R}^m, w_\gamma) & \xrightarrow{id} & R_G B^s_{p_2, q_2}(\mathbb{R}^m, w_\gamma) \\
\downarrow{id} & & \downarrow{id} \\
B^s_{p_1, q_1}(\mathbb{R}^m, w_\gamma) & \xrightarrow{id} & B^s_{p_2, q_2}(\mathbb{R}^m, w_\gamma).
\end{array}$$

To prove the opposite inequality we use the wavelet decomposition, cf. Appendix B. The group $G(\gamma)$ divides $\mathbb{R}^m$ into the finite sum of cones, that have pairwise disjoint interiors. Let us choose one of those cones and denote it by $\tilde{C}$. Moreover let $\eta$ be a function belonging to $C_0^\ell(\mathbb{R}^d)$, $\ell > s_1$, such that $\text{supp } \eta \subset \tilde{C}$ and $\eta(x) = 1$ if $x \in C = \{x \in \tilde{C} : \text{dist}(x, \partial \tilde{C}) > \varepsilon\}$, for some fixed sufficiently small $\varepsilon > 0$. We consider the family $\mathcal{K} = \{k \in \mathbb{Z}^m : \text{supp } \phi_{0,k} \subset C\}$. The group $G(\gamma)$ is finite.
therefore for any $0 < c_1 < c_2$ we can find $L_0$ such that for any $L \geq L_0$, $L \in \mathbb{N}$, we have

$$\# \{ k \in \mathcal{K} : c_1 2^{L(d-m)} \leq w(Q_{0,k}) \leq c_2 2^{L(d-m)} \} \sim \# \{ k \in \mathbb{Z}^d : c_1 2^{L(d-m)} \leq w(Q_{0,k}) \leq c_2 2^{L(d-m)} \}. \quad (29)$$

Please note that $L$ large means that the cubes $Q_{0,k}$ are located far from the origin, cf. (11). The above considerations and Lemma 2 yield the existence of the bijection $\sigma : \mathcal{K} \to \mathbb{N}_0$ such that

$$w_\gamma(Q_{0,k}) \sim 2^{L(d-m)} \quad \iff \quad \sigma(k) \sim 2^{\frac{L(d-m)}{\gamma_1-1}} L^{n-1}.$$

Let $v_\gamma(\ell) = w_\gamma(Q_{0,\sigma^{-1}(\ell)})$. Then (27) and (29) imply

$$v_\gamma(\ell) \sim (\ell \log^{1-n} \ell)^{\gamma_1-1}. \quad (30)$$

For further arguments we need three linear bounded operators: $T : \ell_{p_1}(\mathbb{N}_0, v_\gamma) \to \ell_q(\ell_{p_1}(\mathbb{Z}^m, w_\gamma))$, $M_q : B^{s_z}_{p_2,q_2}(\mathbb{R}^m, w_\gamma) \to B^{s_z}_{p_2,q_2}(\mathbb{R}^m, w_\gamma)$ and $S : \ell_{q_2}(\ell_{p_2}(\mathbb{Z}^m, w_\gamma)) \to \ell_{p_2}(\mathbb{N}_0, v_\gamma)$. The operators are defined in the following way:

$$(T\lambda)_{j,k} = \begin{cases} 
\lambda_{\sigma(k)} & \text{if } k \in \mathcal{K} \text{ and } j = 0, \\
0 & \text{otherwise},
\end{cases} \quad \lambda \in \ell_{p_1}(\mathbb{N}_0, v_\gamma);$$

$$M_q(f) = |G(\gamma)| \eta \cdot f, \quad f \in B^{s_z}_{p_2,q_2}(\mathbb{R}^m, w_\gamma);$$

$$S(\lambda) = \lambda_{0,\sigma^{-1}(\ell)} \quad \text{it } \lambda \in \ell_{q_2}(\ell_{p_2}(\mathbb{Z}^m, w_\gamma)).$$

Using these operators we can construct the following commutative diagram

$$
\begin{array}{ccccccc}
\ell_{p_1}(\mathbb{N}_0, v_\gamma) & \xrightarrow{T} & \ell_q(\ell_{p_1}(\mathbb{Z}^m, w_\gamma)) & \xrightarrow{W^{-1}} & B^{s_z}_{p_1,q_1}(\mathbb{R}^m, w_\gamma) & \xrightarrow{P} & R_G B^{s_z}_{p_1,q_1}(\mathbb{R}^m, w_\gamma) \\
\text{Id} & & \downarrow & & \downarrow \text{id} & & \\
\ell_{p_2}(\mathbb{N}_0, v_\gamma) & \xleftarrow{S} & \ell_{q_2}(\ell_{p_2}(\mathbb{Z}^m, w_\gamma)) & \xleftarrow{W} & B^{s_z}_{p_2,q_2}(\mathbb{R}^m, w_\gamma) & \xleftarrow{M} & R_G B^{s_z}_{p_2,q_2}(\mathbb{R}^m, w_\gamma).
\end{array}
$$

Here $W$ is the isomorphism defined by the wavelet basis, cf. Appendix B. It follows from the above diagram that

$$e_k(\text{Id} : \ell_{p_1}(\mathbb{N}_0, v_\gamma) \to \ell_{p_2}(\mathbb{N}_0, v_\gamma)) \leq C e_k(\text{id}). \quad (31)$$

But

$$e_k(\text{Id} : \ell_{p_1}(\mathbb{N}_0, v_\gamma) \to \ell_{p_2}(\mathbb{N}_0, v_\gamma)) \sim e_k(D_\gamma : \ell_{p_1}(\mathbb{N}_0) \to \ell_{p_2}(\mathbb{N}_0)), \quad (32)$$

where $D_\gamma$ denote the diagonal operator generated by the sequence $\sigma_\ell = (\ell \log^{1-n} \ell)^{(\gamma_1-1)/(1/p_2-1/p_1)}$ i.e.

$$(D_\gamma(\lambda))_\ell = (\ell \log^{1-n} \ell)^{(\gamma_1-1)/(1/p_2-1/p_1)} \lambda_\ell.$$
Using once more Kühn’s results from [18] or [19] and (30)-(32) we have

\[ k^{1/p_1-1/p_2}(k \log^{1-n} k)^{(\gamma_1-1)(1/p_1-1/p_2)} \sim e_k(D_{\gamma_1}) \leq e_k(id). \]

This proves the proposition.

Summarizing, we have the following theorem.

**Theorem 2** Let \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{N}^m, m \in \mathbb{N}, \) be a multi-index such that \( 2 \leq \gamma_1 \leq \ldots \leq \gamma_m, d = \gamma_1 + \ldots + \gamma_m, \) and let \( n = \max\{i: \gamma_i = \gamma_1\}. \)

Let \( 1 < p_1 < p_2 \leq \infty, 0 < q_1, q_2 \leq \infty \) and \( s_1, s_2 \in \mathbb{R}. \) If \( \delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2}) > 0 \) then

\[ e_k\left(id : R_\gamma B_{p_1,q_1}(\mathbb{R}^d) \to R_\gamma B_{p_2,q_2}(\mathbb{R}^d)\right) \sim \left(k^{-\gamma_1} (\log k)^{(n-1)(\gamma_1-1)}\right)^{\frac{1}{p_1} - \frac{1}{p_2}} \quad (33) \]

and

\[ e_k\left(id : R_\gamma H_{p_1}^{s_1}(\mathbb{R}^d) \to R_\gamma H_{p_2}^{s_2}(\mathbb{R}^d)\right) \sim \left(k^{-\gamma_1} (\log k)^{(n-1)(\gamma_1-1)}\right)^{\frac{1}{p_1} - \frac{1}{p_2}}. \quad (34) \]

**Remark 1** If \( m = 1 \) then the spaces \( R_\gamma B_{p,q}^s(\mathbb{R}^d) \) and \( R_\gamma H_p^s(\mathbb{R}^d) \) consists of radial distributions and the estimates (33)-(34) coincides with the estimates for radial functions proved in [20]. If \( \gamma_i \neq \gamma_j \) for any \( i \neq j \) then

\[ e_k(id) \sim k^{-\gamma_1}\left(\frac{1}{p_1} - \frac{1}{p_2}\right). \]

So the asymptotic behaviour is the same as for the corresponding radial subspaces defined on the block with the lowest dimension. On the other hand if \( \gamma_1 = \ldots = \gamma_m \) then

\[ e_k(id) \sim \left(k^{-\gamma_1} (\log k)^{(m-1)(\gamma_1-1)}\right)^{\frac{1}{p_1} - \frac{1}{p_2}}. \]

In that case the sequence of the entropy numbers for spaces of block-radial functions goes asymptotically to zero slower than the entropy numbers for radial function defined on any block.

## 4 Block-radial bounded states of Schrödinger type operators

The interest in studying the ‘negative’ spectrum (bound states) comes from quantum mechanics, generalizing the classical hydrogen operator,

\[ H = -\Delta - \frac{c}{|x|}, \quad c > 0, \]

in \( L_2(\mathbb{R}^3). \) Thus ‘potentials’ \( V(x) \) with \( V(x) \sim |x|^{-a}, a > 0, \) are of peculiar interest. These potentials have not only local singularities and some decay properties at
infinity but also they are radial. Here, more generally, we want to consider the ‘potentials’ which have block-radial symmetry. We want to estimate the number of negative eigenvalues that corresponds to block-radial eigenvectors, but first we briefly describe the general setting.

4.1 The Birman-Schwinger principle and the Carl inequality

We adapt the Birman-Schwinger principle as described in [25] and [33] to our concrete situation. Let $A$ be a self-adjoint positive-definite operator and let $B$ be a symmetric relatively compact operator in the Hilbert space $\mathcal{H}$. Let $\sigma_p$ denote the point spectrum and $\sigma_e$ denote the essential spectrum of a self-adjoint operator. Then the eigenvalues $\{\mu_k\}_k$ of $BA^{-1}$ are real, and $(BA^{-1})^* = A^{-1}B$ is the adjoint operator after extension by continuity from $\text{dom}(B)$ to $\mathcal{H}$. Furthermore, the operator $A + B$ with $\text{dom}(A + B) = \text{dom}(A)$, is self-adjoint, with $\sigma_e(A + B) = \sigma_e(A)$, and

$$\# \{\sigma_p(A + B) \cap (-\infty, 0]\} = \# \{\sigma(A + B) \cap (-\infty, 0]\}$$

$$= \# \{k \in \mathbb{N} : \mu_k(BA^{-1}) \leq -1\} < \infty. \quad (35)$$

This is usually called the Birman-Schwinger principle. It goes back to [1, 27], proofs may be found in [33, Chapter 7] and [25, Chapter 8, §5]. A short description has also been given in [8, Section 5.2.1, p. 186]. Our formulation is different and adapted to our later needs.

Our approach is based upon the relation between the eigenvalues of a compact operator and its entropy numbers described by the Carl inequality. If $(\lambda_k(T))_{k \in \mathbb{N}}$ is a decreasing sequence of all non-zero eigenvalues of a compact operator $T$, repeated according to their algebraic multiplicities then the following inequality

$$|\lambda_k(T)| \leq \sqrt{2} e_k(T) \quad (36)$$

holds, cf. [4] [6], [8, Theorem 1.3.4]. Using (36) with $T = BA^{-1}$ one obtains by (35)

$$\# \{\sigma_p(A + B) \cap (-\infty, 0]\} \leq \# \left\{k \in \mathbb{N} : \sqrt{2} e_k \left( BA^{-1} \right) \geq 1 \right\}.$$

This entropy version of the Birman-Schwinger principle appeared first in [13, Theorem 2.4], cf. also [8, Corollary, p. 186].

We shall concentrate on the special case when $B = -V$ is a multiplication operator where (in a slight abuse of notation) $V$ is a nonnegative measurable and $SO(\gamma)$-invariant function, finite a.e., typically belonging to some space $L_r(\mathbb{R}^d)$.

We turn to study the behaviour of the part of negative spectrum of the self-adjoint unbounded operator

$$H_{s, \theta, \beta} = (\theta \text{Id} - \Delta)^{s/2} - \beta V \quad \text{as} \quad \beta \to \infty; \quad 0 < \theta \leq 1, \quad (37)$$
corresponding to the $SO(\gamma)$-invariant eigenfunctions. We assume that $s > 0$, $\beta > 0$ and $V \geq 0$ is an $SO(\gamma)$-invariant potential. The operator $H_{s,\theta,\beta}$ is a bounded below, self-adjoint operator in $L_2(\mathbb{R}^d)$ with the domain $D(H_{s,\theta,\beta}) = H_2^s(\mathbb{R}^d)$. Let $\sigma_p^\gamma$ denote the part of the point spectrum of the operator (37) that corresponds to the $SO(\gamma)$-invariant eigenfunctions. By the Birman-Schwinger principle with $\mathcal{H} = R_\gamma L_2$ as the basic space we get

$$\#\{\sigma_p^\gamma(H_{s,\theta,\beta}) \cap (-\infty, 0]\} \leq \# \left\{ k \in \mathbb{N} : \sqrt{2} e_k \left( V^\frac{1}{2}(\theta \text{Id} - \nabla)^{-s/2} V^\frac{1}{2} \right) \geq \lambda^{-1} \right\}.$$ 

Thus we should consider the compactness and asymptotic behaviour of entropy numbers of the operators $V_2 \Delta^{-s/2}_\theta V_1$ where $V_1, V_2$ are positive block-radial functions and $\Delta_\theta := \theta \text{Id} - \nabla$.

**Lemma 3** Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{N}^m$, $m \in \mathbb{N}$, be a multi-index such that $2 \leq \gamma_1 \leq \cdots \leq \gamma_m$, $d = \gamma_1 + \cdots + \gamma_m$, and let $n = \max\{i : \gamma_i = \gamma_1\}$. Let $1 \leq p \leq \infty$, $s > 0$, $V_i \in R_\gamma L_{r_i}(\mathbb{R}^d)$, $V_2 \in R_\gamma L_{r_2}(\mathbb{R}^d)$ and $\frac{\lambda}{d} > \frac{1}{r_1} + \frac{1}{r_2} > 0$. If $p' < r_1 \leq \infty$ and $p \leq r_2 \leq \infty$ then the operator

$$V_2 \Delta^{-s/2}_\theta V_1 : R_\gamma L_p(\mathbb{R}^d) \to R_\gamma L_p(\mathbb{R}^d)$$

is compact. Moreover its eigenvalues and entropy numbers satisfy the following estimate

$$\lambda_k(V_2 \Delta^{-s/2}_\theta V_1) \leq \sqrt{2} e_k(V_2 \Delta^{-s/2}_\theta V_1) \leq C[k^{-\gamma_1}(\log k)^{(n-1)(\gamma_1-1)/(\gamma_1+1)}] ||V_1||_{L_{r_1}(\mathbb{R}^d)} ||V_2||_{L_{r_2}(\mathbb{R}^d)}.$$  

**Proof.** The reasoning goes by standard factorization

$$R_\gamma L_p(\mathbb{R}^d) \xrightarrow{V_2 \Delta^{-s/2}_\theta V_1} R_\gamma L_p(\mathbb{R}^d) \xleftarrow{V_2} R_\gamma B^{s}_{r,\infty}(\mathbb{R}^d),$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{r_1}$, $\frac{1}{t} = \frac{1}{p} - \frac{1}{r_2}$ and $V_i$ denotes the operator of multiplication by function $V_i$, $i = 1, 2$. The operator $(\Delta_\theta)^{-s/2} : R_\gamma B^s_{r,\infty}(\mathbb{R}^d) \to R_\gamma B^s_{r,\infty}(\mathbb{R}^d)$ is an isomorphism and the Sobolev embedding $\text{Id} : R_\gamma B^s_{r,\infty}(\mathbb{R}^d) \to R_\gamma B^0_{t,1}(\mathbb{R}^d)$ is compact since $s > \frac{d}{r_1} + \frac{d}{r_2} = \frac{d}{r} - \frac{d}{t} > 0$. Moreover, $\frac{1}{p} = \frac{1}{r} - \frac{1}{r_1} = \frac{1}{t} + \frac{1}{r_2}$ therefore by Hölder inequality and elementary embeddings $B^0_{p,1}(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d) \hookrightarrow B^0_{p,\infty}$ the operators $V_1 : R_\gamma L_p(\mathbb{R}^d) \to R_\gamma B^0_{r,\infty}(\mathbb{R}^d)$, $V_2 : R_\gamma B^0_{r,\infty}(\mathbb{R}^d) \to R_\gamma L_p(\mathbb{R}^d)$ are bounded. The lemma follows from Theorem 2 and the Carl inequality (36). 

**Remark 2** The constant $C$ in (38) depends on $s, p, r_1, r_2$ and $\theta$. It follows from the proof that $C = c\|(\Delta_\theta)^{-s/2}\|$, where $c$ is independent of $\theta$.

In the similar way one can estimate eigenvalues and entropy numbers of the operators $V \Delta^{-s/2}_\theta$ and $\Delta^{-s/2}_\theta V$. 

22
4.2 The negative spectrum of Schrödinger type operators

We are interested in a number of negative eigenvalues of $H_{s,\beta,\theta}$ with $SO(\gamma)$-invariant eigenfunctions. We put

$$N_{\gamma,\beta} = \# \{ \lambda : \lambda \leq 0, \quad H_{s,\theta,\beta} f = \lambda f, \quad f \in R_{\gamma}L_2(\mathbb{R}^d), \quad f \neq 0 \}.$$ 

The main theorem of this section reads as follows

**Theorem 3** Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{N}^m$, $m \in \mathbb{N}$, be a multi-index such that $2 \leq \gamma_1 \leq \ldots \leq \gamma_m$, $d = \gamma_1 + \ldots + \gamma_m$, and let $n = \max \{ i : \gamma_i = \gamma_1 \}$. Let $V$ be a nonnegative block-radial function invariant with respect to $SO(\gamma)$ such that $\|V\|_{L^r(\mathbb{R}^d)} = 1$, $1 < r < \infty$. Let $0 < \theta < 1$, $s > 0$ and $\beta > 0$. We assume that $s d > 1/r$. If $H_{s,\theta,\beta}$ is the operator defined by (37) with domain $H_{s}^2(\mathbb{R}^d)$ then

$$N_{\gamma,\beta} \leq c \beta^{\gamma_1} (\log \beta)^{(n-1)\gamma_1-1}.$$ 

Moreover, if there exist $\delta > 0$, $\varepsilon > 0$ and $1 < \rho_1, \ldots, \rho_m$ such that

$$V(x) \geq \varepsilon \quad \text{if} \quad x \in A = \{ x \in \mathbb{R}^d : \rho_i \leq r_i(x) \leq \rho_i + \delta, \quad i = 1, \ldots, m \}$$

then

$$c \beta^{\frac{\delta}{r}} \leq N_{\gamma,\beta}.$$ 

**Proof.** **Step 1.** The number $\lambda$ is a negative eigenvalue of $H_{s,\theta,\beta}$ with block-radial eigenfunction if and only if $\lambda$ belongs to the spectrum of $H_{s,\theta,\beta}$ regarded as an operator in $R_{\gamma}L_2(\mathbb{R}^d)$ with domain $R_{\gamma}H_{s}^2(\mathbb{R}^d)$. Thus it is sufficient to consider the last operator. The operator $\beta \sqrt{\Delta^s/2} \sqrt{V}$ is compact in $R_{\gamma}L_2(\mathbb{R}^d)$. This follows from the factorization (39) with $p = 2$ and $r_1 = r_2 = 2r$. By the Birman-Schwinger principle the operator $H_{s,\theta,\beta}$ is self-adjoint with the same domain as $\Delta^s/2$, cf. [8, Proposition 5.4.1]. Moreover

$$N_{\gamma,\beta} \leq \# \{ k \in \mathbb{N} : \sqrt{2}e_k(\beta \sqrt{\Delta^s/2} \sqrt{V}) \geq 1 \},$$

So the upper estimate follows from Lemma 3 since

$$e_k(\beta \sqrt{\Delta^s/2} \sqrt{V}) \leq C (k^{-\gamma} (\log k)^{(n-1)(\gamma_1-1)})^{\frac{1}{r}} \beta,$$

cf. (38).

**Step 2.** Now we prove the estimate from below. We use the Max-Min principle and the method of atomic decompositions, cf. Appendix A. We have

$$H_{s,\theta,\beta} f, f) \sim \| f \|_{H_{s}^2(\mathbb{R}^d)}^2 - \beta (Vf, f)_{L_2(\mathbb{R}^d)}$$
$$\sim \| f \|_{B_{r,2}^s(\mathbb{R}^d)}^2 - \beta \| f \sqrt{V} \|_{L_2(\mathbb{R}^d)}^2, \quad f \in H_{s}^2(\mathbb{R}^d).$$
Substep 2.1. Let $\eta \in C_0^\infty (\mathbb{R})$ be a smooth function such that supp $\eta \subset [0, \delta]$, $0 \leq \eta(t) \leq 1$ and $\eta(t) = 1$ if $t \in [\frac{\delta}{4}, \frac{3\delta}{4}]$. Let $\eta_j(t) = \eta(2^j t)$, $j = 0, 1, 2, \ldots$. We put

$$r_{j,\nu}^{(i)} = \rho_i + \nu 2^{-j} \delta,$$

$$\tilde{r}_{j,\nu}^{(i)} = r_{j,\nu}^{(i)} + 2^{-j-2}\delta$$

and

$$\tilde{r}_{j,\nu}^{(i)} = r_{j,\nu}^{(i)} + 3 \cdot 2^{-j-2}\delta,$$

where $\nu = 0, 1, 2, \ldots 2^j - 1$.

We choose the following functions

$$\psi_{j,\tilde{\nu}}(x) = 2^{-j\frac{s-d}{2}} \prod_{i=1}^m r_{j,\nu_i}^{(i)}(r_i(x)), ~ x \in \mathbb{R}^d$$

where $\eta_{j,\nu_i}^{(i)}(t) = \eta_j(t - r_{j,\nu_i}^{(i)})$, and

$$j \in \mathbb{N}, \tilde{\nu} = (\nu_1, \ldots, \nu_m), ~ \nu_i = 0, \ldots, 2^j - 1, ~ i = 1, \ldots, m.$$

Any function $\psi_{j,\tilde{\nu}}$ is smooth and $SO(\gamma)$-invariant. Such a function is supported in the set

$$\{ x \in \mathbb{R}^d : r_{j,\nu_i}^{(i)} \leq r_i(x) \leq r_{j,\nu_i+1}^{(i)}, ~ i = 1, \ldots, m \} \subset A$$

and it takes value $2^{-j\frac{s-d}{2}}$ on the set $A_{j,\tilde{\nu}} = \{ x \in \mathbb{R}^d : \tilde{r}_{j,\nu_i}^{(i)} \leq r_i(x) \leq \tilde{r}_{j,\nu_i}^{(i)}, ~ i = 1, \ldots, m \}$. Moreover there is a positive constant $C$ such that

$$\left| \partial^\alpha \psi_{j,\tilde{\nu}}(x) \right| \leq C \cdot 2^{-j|\frac{s}{2} - |\alpha| + \delta|}, \quad |\alpha| \leq s + 1. \quad (40)$$

Let $\{ x_{k,\tilde{\nu}}^{(i)} \}$ be $(2^{-j}\delta)$-discretization described in the appendix. Then the balls $\{ B(x_{k,\tilde{\nu}}^{(i)}, 2\delta 2^{-j}) \}_{k,\tilde{\nu}}$ form a uniformly locally finite covering of $\mathbb{R}^d$, cf. Remark 4 ibidem. So there exists a resolution of unity $(\varphi_{j,\tilde{k},\tilde{\ell}})_{k,\tilde{\ell}}$ related to this covering such that

$$\left| \partial^\alpha \varphi_{j,\tilde{k},\tilde{\ell}}(x) \right| \leq C \cdot 2^{||\alpha||}, \quad |\alpha| \leq s + 1. \quad (41)$$

It follows from (40) and (41) that functions

$$a_{j,\tilde{k},\tilde{\ell}}(x) = \varphi_{j,\tilde{k},\tilde{\ell}}(x) \psi_{j,\tilde{\nu}}(x)$$

are $(\frac{s}{2}, 2)$-atoms and

$$\psi_{j,\tilde{\nu}} = \sum_k \sum_{\tilde{\ell}} a_{j,\tilde{k},\tilde{\ell}}$$

is the atomic decomposition of $\psi_{j,\tilde{\nu}}$.

Let $k_{i,\tilde{\nu}}$ be an integer such that $2^{-j}(k_{i,\tilde{\nu}} - 1) < r_{j,\nu_i}^{(i)} \leq 2^{-j}k_{i,\tilde{\nu}}$. The atomic decomposition theorem and (45) give us

$$\| \psi_{j,\tilde{\nu}} | H_2^{s/2}(\mathbb{R}^d) \| \leq C_1 \left( \prod_{\tilde{\nu}} k_{i,\tilde{\nu}}^{-1} \right)^{1/2}. \quad (42)$$
On the other hand direct calculations show that the measure of the set $A_{j,\tilde{\nu}}$ is equivalent to $2^{-jd} \prod_{i=1}^{m} k_{i,\tilde{\nu}}^{\gamma_{i}-1}$. So

$$\left\| \psi_{j,\tilde{\nu}} \sqrt{V} L_2(\mathbb{R}^d) \right\| > 2^{-j s/2} 2^{jd} \left( \int_{A_{j,\tilde{\nu}}} V(x) dx \right)^{1/2} \geq C_{s} 2^{-j s/2} |A_{j,\tilde{\nu}}|^{1/2} \geq C_{2} 2^{-j s/2} \left( \prod_{i=1}^{m} k_{i,\tilde{\nu}}^{\gamma_{i}-1} \right)^{1/2}. \quad (43)$$

**Substep 2.2.** We choose $j = \left\lfloor s^{-1} \log_{2} (C_{1}^{-1} C_{2}^{2} \beta) \right\rfloor$. Inequalities (42) and (43) imply

$$\left( \Delta_{s/2} \psi_{j,\tilde{\nu}}, \psi_{j,\tilde{\nu}} \right) \leq C_{1} \left( \prod_{i=1}^{m} k_{i,\tilde{\nu}}^{\gamma_{i}-1} \right) \leq C_{1} C_{2}^{-2} 2^{sj} \left\| \psi_{j,\tilde{\nu}} \sqrt{V} L_2(\mathbb{R}^d) \right\|^{2} < \beta \left( V \psi_{j,\tilde{\nu}}, \psi_{j,\tilde{\nu}} \right).$$

The subspace $M = \text{span}\{ \psi_{j,\tilde{\nu}} \}$ has the dimension $\dim M = 2^{jm} \sim \beta^{m/s}$. The functions $\psi_{j,\tilde{\nu}}$ are pairwise orthogonal therefore for any $\psi \in M$ we have

$$\left( H_{s,\beta,\theta} \psi, \psi \right) < 0.$$

For any subspace $N \subset R_{\gamma} L_2(\mathbb{R}^d)$ of dimension $\dim M - 1$ one can find a function $\psi \in M$ such that $\|\psi\| = 1$ and $\psi \perp N$. In consequence

$$\sup_{N} \inf_{\psi \in D(\text{H}_{s,\beta,\theta}) \|\psi\|=1, \psi \perp N} \left( H_{s,\beta,\theta} \psi, \psi \right) < 0,$$

where the supremum is taken over all $M - 1$ dimensional subspaces of $R_{\gamma} L_2(\mathbb{R}^d)$. So the Max-Min principle implies that $H_{s,\beta,\theta}$ has at least $\dim M \sim \beta^{m/s}$ negative eigenvalues, cf. eg. [7, p.489].

**Remark 3**

1) Since $s > \frac{d}{r}$ and $\frac{m}{s} < \frac{rm}{d} \leq \frac{1}{\gamma_{1}}$ we have always the gap between the upper and lower estimates. The estimate is more precise for small values of $s$.

2) Apart of the radial case we have no block-radial functions satisfying the assumption of Theorem 2 and Theorem 3 in dimensions $d = 2$ and $d = 3$. So the smallest possible dimension for which the assumption of Theorem 3 are satisfied is $d = 4$ with $\gamma_{1} = \gamma_{2} = 2$.

3) If $d \geq 4$ and $V \in L^{r}(\mathbb{R}^d)$ is the radial potential satisfying the assumption of Theorem 3 then the operator $H_{s,\beta,\theta}$ has asymptotically at most $\beta^{r/d}$ eigenvalues with radial eigenfunctions. On the other hand such potential is $SO(\gamma)$-invariant for any $\gamma$. So choosing $\gamma$ such that $\min_{i} \gamma_{i} \geq 2$ we have asymptotically at least $\beta^{m/s}$ eigenvalues with $SO(\gamma)$-invariant eigenfunctions. If $\frac{d}{r} < s < \lceil d/2 \rceil \frac{d}{r}$ then the operator $H_{s,\beta,\theta}$ has eigenfunctions that are block-radial but not radial.
A Atomic decomposition for subspaces of invariant functions

We recall the main idea of the method of atomic decomposition, and we refer the reader to [34] where the atomic decompositions related to the action of compact groups of isometries are described in details. We assume that \( \gamma_i \geq 2 \) for any \( i = 1, \ldots, m \).

We start with the following notions of separations and discretizations.

**Definition 1** Let \( \varepsilon > 0 \) be a positive number, \( \alpha = 1, 2, \ldots \) be a positive integer and \( X \) a nonempty subset of \( \mathbb{R}^d \).

(a) A subset \( H \) of \( X \) is said to be an \( \varepsilon \)-separation of \( X \), if the distance between any two distinct points of \( H \) is greater than or equal to \( \varepsilon \).

(b) A subset \( H \) of \( X \) is called an \((\varepsilon, \delta)\)-discretization of \( X \) if it is an \( \varepsilon \)-separation of \( X \) and \( X \subset \bigcup_{x \in H} B(x, \delta \varepsilon) \).

**Remark 4** Let \( n \) be a positive integer. If \( H \) is an \((\varepsilon, \delta)\)-discretization of \( \mathbb{R}^d \) and \( n \geq \delta \), then the family \( \{B(x, n\varepsilon)\}_{x \in H} \) is an uniformly locally finite covering of \( \mathbb{R}^d \) with multiplicity that can be estimated from above by a constant depending on \( d \) and \( n \), but independent of \( \varepsilon \).

Now we describe discretizations related to the group \( SO(\gamma) \). In this case we can proceed in the following way. Let \( \{x_{k,\ell}^{(j,i)}\} \), \( k \in \mathbb{N}_0 \), and \( \ell = 0, \ldots, k^{\gamma_i-1} \) be a \((2^{-j}, \delta_i)\)-discretization in \( \mathbb{R}^{\gamma_i} \) related to the action of special orthogonal group \( SO(\gamma_i) \) on \( \mathbb{R}^{\gamma_i} \). We refer to [28] for the construction of this type of discretization. In particular we have

\[
|x_{k,\ell}^{(j,i)}| \sim k 2^{-j} \quad \text{and} \quad x_{k,\ell}^{(j,i)} \in SO(\gamma_i) \cdot x_{k,0}^{(j,i)}, \quad \ell = 0, \ldots, k^{\gamma_i-1}.
\]

We put

\[
H_j = \{x_{k,\ell}^{(j)} = (x_{k,\ell_1}^{(j,1)}, \ldots, x_{k,\ell_m}^{(j,m)}): \; \tilde{k} = (k_1, \ldots, k_m), \; \tilde{\ell} = (k_1, \ldots, k_m)\},
\]

and

\[
\delta = \sqrt{d} \max_i \{\delta_i\}.
\]

The set \( H_j \) is a \((2^{-j}, \delta)\)-discretization of \( \mathbb{R}^d \). Please note that \( \tilde{\ell} = 0 \) if \( \tilde{k} = 0 \) and that \( x_{0,0}^{(j)} \) is the origin. If \( \tilde{k} \neq 0 \) then

\[
SO(\gamma)(x_{k,\ell}^{(j,i)}) = \prod_{i,k_i \neq 0} SO(\gamma_i)(x_{k,\ell_i}^{(j,i)})
\]
We assume that $s > 0$ and $1 \leq p \leq \infty$. The function $a_{j,k,i}$ is called an $(s,p)$-atom centred at the point $x_{j,k,i} \in \mathcal{H}_j$ if:

$$\supp a_{j,k,i} \subset B(x_{j,k,i}, 2\delta 2^{-j}),$$

$$\sup y \in \mathbb{R}^d |\partial^\alpha a_{j,k,i}(y)| \leq 2^{-j(s-|\alpha|\frac{d}{p})}, \quad |\alpha| \leq s + 1.$$

Let $f \in R_\gamma B^s_{p,q}(\mathbb{R}^d)$. The atomic decomposition theorem asserts that any function $f \in B^s_{p,q}(\mathbb{R}^d)$ can be decomposed in the following way

$$f = \sum_{j=0}^{\infty} \sum_{\tilde{k} \in \mathbb{N}^m_0} \sum_{\tilde{\ell} \in \mathbb{N}^m_0} s_{j,k} a_{j,k,i}, \quad \text{(convergence in} \quad \mathcal{S}') \quad (46)$$

with

$$\left( \sum_{j=0}^{\infty} \left( \sum_{\tilde{k} \in \mathbb{N}^m_0} C(j,\tilde{k}) |s_{j,k}|^p \right)^{q/p} \right)^{1/q} < \infty, \quad s_{j,k} \in \mathbb{C} \quad (47)$$

(usual change if $q = \infty$). On the other hand any distribution represented by (46) with (47) belongs to $R_\gamma B^s_{p,q}(\mathbb{R}^d)$. Moreover the infimum over all possible representations of the expressions (47) give us an equivalent norm in $R_\gamma B^s_{p,q}(\mathbb{R}^d)$. For the proof we refer to [34, Theorem 2], cf. also Remark 7 ibidem.

### B Wavelet characterizations of Besov spaces with $A_\infty$ weights

A locally integrable function $w : \mathbb{R}^d \to \mathbb{R}_+$ belongs to the class $A_\rho$, $1 < \rho < \infty$ if it satisfies the inequality

$$\frac{1}{|Q|} \int_Q w(x)dx \left( \frac{1}{|Q|} \int_Q w(x)^{-\rho'/\rho} dx \right)^{\rho'/\rho} \leq A < \infty$$

for all cubes $Q$ in $\mathbb{R}^d$. The class $A_\infty$ is the union of all the $A_\rho$ classes. We recall briefly the wavelet characterization of the weighted Besov spaces proved in [12]. Further information and references concerning the wavelet theory can be found there.

Let $\tilde{\phi}$ be an orthogonal scaling function on $\mathbb{R}$ with compact support and of sufficiently high regularity. Let $\tilde{\psi}$ be an associated wavelet. Then the tensor-product ansatz yields a scaling function $\phi$ and associated wavelets $\psi_1, \ldots, \psi_{2d-1}$, all defined now on $\mathbb{R}^d$. We suppose

$$\tilde{\phi} \in C^{N_1}(\mathbb{R}) \quad \text{and} \quad \text{supp} \ \tilde{\phi} \subset [-N_2, N_2]$$
for certain natural numbers $N_1$ and $N_2$. This implies
\[ \phi, \psi_i \in C^{N_1}(\mathbb{R}) \quad \text{and} \quad \text{supp } \phi, \text{supp } \psi_i \subset [-N_3, N_3]^d, \quad i = 1, \ldots, 2^n - 1. \quad (48) \]

We shall use the standard abbreviations
\[ \phi_{\nu,k}(x) = 2^{\nu d/2} \phi(2^\nu x - k) \quad \text{and} \quad \psi_{i,\nu,k}(x) = 2^{\nu d/2} \psi_i(2^\nu x - k). \quad (49) \]

**Theorem 4** Let $0 < p, q \leq \infty$ and let $s \in \mathbb{R}$. Let $\phi$ be a scaling function and let $\psi_i, i = 1, \ldots, 2^d - 1$, be the corresponding wavelets satisfying (48). We assume that $|s| < N_1$. Then a distribution $f \in S'(\mathbb{R}^d)$ belongs to $B^{s}_{p,q}(\mathbb{R}^d, w)$, if, and only if,
\[
\| f | B^{s}_{p,q}(\mathbb{R}^d, w) \|_* = \| \{ \langle f, \phi_{0,k} \rangle \}_{k \in \mathbb{Z}^d} | \ell^p(w) \|
+ \sum_{i=1}^{2^d-1} \| \{ \langle f, \psi_{i,\nu,k} \rangle \}_{\nu \in N_0, k \in \mathbb{Z}^n} | b^{\sigma}_{p,q}(w) \| < \infty, 
\]
where $\sigma = s + \frac{d}{2} - \frac{d}{p}$. Furthermore, $\| f | B^{s}_{p,q}(\mathbb{R}^d, w) \|_*$ may be used as an equivalent (quasi-) norm in $B^{s}_{p,q}(\mathbb{R}^d, w)$ and the map
\[ \mathcal{W} : B^{s}_{p,q}(\mathbb{R}^d, w) \ni f \mapsto \left( \langle f, \phi_{0,k} \rangle, \{ \langle f, \psi_{i,\nu,k} \rangle \}_{(i,\nu,k)} \right) \in \ell^p(w) \oplus b^{\sigma}_{p,q}(w) \]
is a topological linear isomorphism.

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