NEW PERSPECTIVES OF THE POWER-COMMUTATOR-STRUCTURE: COCLASS TREES OF CF-GROUPS AND RELATED BCF-GROUPS

DANIEL C. MAYER

Abstract. Let $e \geq 2$ be an integer. Among the finite 3-groups $G$ with bicyclic commutator quotient $G/G' \cong C_{3e} \times C_3$, having one non-elementary component with logarithmic exponent $e$, there exists a unique pair of coclass trees with distinguished rank distribution $\varrho \sim (2, 2, 3, 3)$. One tree $T^e(M_1^{(3)})$ consists of CF-groups with coclass $e$, and the other tree $T^{e+1}(M_1^{(3)})$ consists of BCF-groups with coclass $e+1$. It is proved that, due to a chain of periodic bifurcations, the vertices of all pairs $(T^e, T^{e+1})$ with $e \geq 3$ can be constructed as $p$-descendants of the single root $M_1^{(3)}$ of order $3^e$ by means of the $p$-group generation algorithm by Newman and O’Brien.

1. Introduction

We consider finite 3-groups $G$ whose commutator quotient is bicyclic with one non-elementary component, that is, $G/G' \cong C_{3e} \times C_3$ with logarithmic exponent $e \geq 2$. By the Burnside basis theorem, $G = (x, y)$ is two-generated, and we stick to the convention that $w = x^{3^e}$, $w^3 \in G'$ and $y^3 \in G'$. For the generators.

For such groups, we have introduced several invariants [14] in terms of their maximal normal subgroups $H_1, \ldots, H_3; H_4$ of index $(G : H_i) = 3$, where the puncture at the fourth component is motivated by the distinction of the bicyclic quotient $H_4/G' \cong C_{3^{e-1}} \times C_3$, as opposed to the cyclic quotients $H_i/G' \cong C_{3e}$ for $1 \leq i \leq 3$. We have the logarithmic abelian quotient invariants (AQI),

$$\alpha(G) = (H_1/H_1', \ldots, H_3/H_3'; H_4/H_4'),$$

their rank distribution $\varrho(G) = (\text{rank}_3(H_1/H_1'), \ldots, \text{rank}_3(H_3/H_3'); \text{rank}_3(H_4/H_4'))$, and, on the other hand, the punctured transfer kernel type (pTKT),

$$\varkappa(G) = (\ker(T_1), \ldots, \ker(T_3); \ker(T_4)),$$

where $T_i : G/G' \to H_i/H_i'$ denotes the Artin transfer homomorphism from $G$ to $H_i$. AQI and pTKT are combined in the Artin pattern $\text{AP}(G) = (\alpha(G), \varkappa(G))$ of $G$. Since there are only five possibilities for the kernels, the pTKT is abbreviated in the form $\varkappa(G) = (\varkappa_1, \ldots, \varkappa_3; \varkappa_4)$, where

$$\varkappa_i = \begin{cases} 0 & \text{if } \ker(T_i) = \langle w, y \rangle/G' \text{ (complete 3-elementary subgroup)}, \\ j & \text{if } \ker(T_i) = \langle w^{3^{j-1}}, y \rangle/G', 1 \leq j \leq 3, \\ 4 & \text{if } \ker(T_i) = \langle w \rangle/G' \text{ (distinguished third power generator)}. \end{cases}$$

Our special attention is devoted to CF-groups $G$ for which the factors $\gamma_i(G)/\gamma_{i+1}(G)$, $i \geq 2$, of the descending central series $(\gamma_i(G))_{i \geq 1}$ are cyclic of order 3 (cyclic factor groups). Since $\gamma_2(G) = \langle s_2, \gamma_3(G) \rangle$, the second factor is always cyclic, but since $\gamma_3(G) = \langle s_3, t_3, \gamma_4(G) \rangle$, the third factor is usually bicyclic (BCF = bicyclic or cyclic factor groups), and there must exist some relation between $s_3$ and $t_3$ in a CF-group, for instance, either one of the two commutators is trivial or $s_3 = t_3$. (Here, $s_2 = [y, x]$, $s_3 = [s_2, x]$, and $t_3 = [s_2, y]$ denote essential commutators.)
Even more specifically, our focus will lie on coclass trees \[\text{[13] p. 89}\] whose vertices \(G\) share the common rank distribution \(\rho(G) \sim (2, 2, 3, 3)\), that is, trees of CF-groups with mainline of type a.1, \(\pi(G) = (0, 0, 0, 0)\), and trees of BCF-groups with mainline of type d.10, \(\pi(G) \sim (1, 1, 0; 2)\).

After a summary of foundations in \(\S\) 2 we begin with simple laws for all mainlines of CF-coclass trees in \(\S\) 3. Then we extend the investigations to chains of periodic bifurcations in \(\S\) 4, where the complete system of all CF- and BCF-coclass trees with \(e \geq 3\) is shown to arise from a single root.

2. Basic definitions and conventions

The lower exponent \(p\)-central series of a finite \(3\)-group \(G\) will always be denoted by \((P_i(G))_{i \geq 0}\).

**Definition 1.** Let \(D\) be non-trivial finite \(3\)-group with nilpotency class \(c = \text{cl}(D) \geq 1\) and lower exponent \(p\)-class \(c_p = \text{cl}_p(D) \geq 1\), i.e., \(\gamma_c(D) > \gamma_{c+1}(D) = 1\) and \(P_{c_p-1}(D) > P_{c_p}(D) = 1\). Then the quotient \(A = \pi(D) = D/\gamma_c(D)\) is called the parent of \(D\) and the quotient \(A_p = \pi_p(D) = D/P_{c_p-1}(D)\) called the p-parent of \(D\). Conversely, \(D\) is called an immediate descendant of \(A\) and an immediate p-descendant of \(A_p\). By the root path, respectively p-root path, of \(D\) we understand the sequence \((\pi^i(D))_{i \geq 0}\), respectively \((\pi^i_p(D))_{i \geq 0}\), of its iterated parents, respectively p-parents.

**Definition 2.** The propagation from a p-parent \(A\) to a p-descendant \(D\) is called endo-genetic if the commutator quotient remains unchanged, that is, \(D/D' \simeq A/A'\). Otherwise the propagation is called exo-genetic.

The propagation from non-trivial parent \(A\) to non-abelian descendant \(D\) is always endo-genetic, because \(A = D/\gamma_c(D)\) with \(c \geq 2\), and thus \(D/\gamma_2(D) \simeq (D/\gamma_c(D))/(\gamma_2(D)/\gamma_c(D)) \simeq A/\gamma_2(A)\).

**Definition 3.** The descendant tree \(T(R)\), respectively p-descendant tree \(T_p(R)\), with a finite \(3\)-group \(R\) as its root consists of the following vertices and directed edges: the vertices are all isomorphism classes of finite \(3\)-groups \(D\) whose root path, respectively p-root path, contains \(R\), and the directed edges are all pairs \((D, A)\), also denoted by \(D \rightarrow A\), of immediate descendants \(D\) and parents \(A = \pi(D)\), respectively p-parents \(A = \pi_p(D)\), among the vertices of the tree. A descendant tree whose vertices are subject to certain restrictive conditions is called a pruned tree.

**Definition 4.** A pruned tree which contains a unique infinite main line and all of whose vertices share a common coclass \(r\) is called a coclass tree. If the root is \(R\), the tree is denoted by \(T^*(R)\).

The step size of all edges in a coclass tree is necessarily \(s = 1\). Depth-pruned branches of a coclass tree become periodic, beginning with a minimal periodic root on the main line \([13] \text{Thm. 3.1}\).

**Definition 5.** By a tree of type \(X\) we understand a coclass tree whose mainline consists of vertices with (punctured) transfer kernel type \(X\) \([14] \text{Tbl. 1–2, pp. 3–4}\). (For instance \(X = a.1\) or d.10.) We introduce an ostensive terminology in order to illuminate three distinct situations with crucial differences in the construction by means of the p-group generation algorithm \([19, 20, 7, 6]\).

**Definition 6.** A vertex \(V\) on a coclass tree \(T^*\), with \(V/V' \simeq C_{3^r} \times C_3\) and \(r \in \{e, e + 1\}\), lies

- **behind the shock wave**, if \(\text{cl}(V) < r\),
- **on the shock wave**, if \(\text{cl}(V) = r\),
- **ahead of the shock wave**, if \(\text{cl}(V) > r\).

The behavior ahead of the shock wave will turn out to be regular with endo-genetic propagation, dominated by the commutator structure. In contrast, we shall see that the behavior behind the shock wave is irregular with exo-genetic propagation, due to a dominance of the power structure. A singular behavior can be observed on the shock wave, where the propagation is mixed, partially endo-genetic and partially exo-genetic, and periodic bifurcations arise, because both, the commutator structure and the power structure, exert a combined impact.

In order to identify the isomorphism class of a finite \(3\)-group \(G\), several ways are possible. Either the group is characterized by its absolute identifier \(\text{SmallGroup}(o, i)\), or briefly \(\langle o, i \rangle\), in the SmallGroups database \([2]\), where \(o = \#G\) denotes the order of \(G\), bounded by \(o \leq 3^6\), and \(i\) is a positive integer. The short form in angle brackets is returned by the Magma statement \(\text{IdentifyGroup()}\) \([3, 4, 10]\), provided that \(o \leq 3^6\). When the order \(o = 3^e\) or the logarithmic
order \(e\) is given along a scale on the left hand side of a figure illustrating a descendant tree of finite 3-groups, then we omit the order \(o\) in the absolute identifier \(\langle o, i \rangle\) and simply write \(\langle i \rangle\).

Or \(G\) is constructed by means of the Magma statement \(\text{Descendants}(P:\text{StepSizes}:=[s])\) as an immediate step size-\(s\) \(p\)-descendant of a \(p\)-parent group \(P\) and characterized by a relative identifier \(G = P - \#s; j\) with \(1 \leq s \leq n(P)\) and \(1 \leq j \leq N_s(P)\), where \(n(P)\) denotes the nuclear rank of the \(p\)-parent \(P\) and \(N_s(P)\) is the number of immediate step size-\(s\) \(p\)-descendants of \(P\).

Finally, there is always the possibility to give a power commutator (pc-) presentation for \(G\).

### 3. Laws for coclass trees of CF-groups

We separate our main statements into three parts: uniqueness, invariants, and construction.

**Proposition 1.** For each logarithmic exponent \(e \geq 2\), there exists a unique coclass tree \(T^e(M_i^{(e)}) \supset V\) with fixed coclass \(cc(V) = e\), fixed commutator quotient \(V/V' \simeq C_{3^e} \times C_3\), and fixed rank distribution \(q(V) \sim (2, 2, 3; 3)\). Its mainline \((M_i^{(e)})_{i \geq 1}\) is of type \(a.1\), \(\text{cc}(M_i^{(e)}) = (000; 0)\). The tree consists entirely of metabelian CF-groups. The branches are of depth \(3\). (See Figure 2.)

**Proof.** For each commutator quotient \(C_{3^e} \times C_3\) with log exponent \(e \geq 2\), there exists a finite number \(N\) of coclass trees \(T^e(R_i^{(e)})\) with roots \(R_i^{(e)}\), \(1 \leq j \leq N\), and two minimal possible values \(e \leq r \leq e + 1\) for the coclass. Descendant vertices \(V\) of each root share invariants with the root, e.g., the rank distribution \(q(V)\). The roots with \(r = e + 1\) are non-CF groups (called BCF-groups in [18]), i.e., groups with bicyclic or cyclic factors of the lower central series, and the others with \(r = e\) are CF-groups. There are only two trees with rank distribution \(q(V) \sim (2, 2, 3; 3)\), a BCF-tree of type \(d.10\) and a CF-tree of type \(a.1\). The latter is the unique tree with root \(R_1^{(e)} = M_1^{(e)}\), recursively determined by \(M_1^{(3)} = \langle 729, 7 \rangle\) and \(M_1^{(2)} = \langle 243, 17 \rangle\), according to Theorems 1 and 2. Its branches are periodic of length 2 without pre-period, and all of its vertices are metabelian CF-groups, since the vertices of the first two branches are metabelian CF-groups.

#### 3.1. Vertices on the mainline (with depth 0).

**Proposition 2.** For \(e \geq 3\), invariants of vertices on the mainline \((M_i^{(e)})_{i \geq 1}\) of the coclass tree \(T^e(M_1^{(e)})\) are given as follows:

\[
\text{logarithmic order } \text{lo}(M_i^{(e)}) = e + i + 2, \quad \text{nilpotency class } \text{cl}(M_i^{(e)}) = i + 2, \quad \text{for } i \geq 1,
\]

\[
p\text{-class } \text{cl}_p(M_i^{(e)}) = \begin{cases} 
    i + 2 & \text{if } i > e - 2, \\
    e & \text{if } i \leq e - 2,
\end{cases}
\]

\[
p\text{-coclass } cc_p(M_i^{(e)}) = \begin{cases} 
    e & \text{if } i > e - 2, \\
    i + 2 & \text{if } i \leq e - 2.
\end{cases}
\]

**Proof.** Proposition 2 remains true when the mainline vertex \(M_i^{(e)}\) is replaced by any proper descendant vertex \(V_i^{(e)}\) with \(i \geq 2\). All coclass trees under investigation start at a root of class \(\text{cl}(M_1^{(e)}) = 3 = 1 + 2\), for each \(e \geq 2\). Thus, proper descendants possess nilpotency class \(\text{cl}(V_i^{(e)}) = i + 2 \geq 4\). By definition, all vertices \(V\) of the coclass tree \(T^e(M_1^{(e)})\) share the common coclass \(cc(V) = e\). Consequently, the logarithmic order is the sum \(\text{lo}(V_i^{(e)}) = \text{cl}(V_i^{(e)}) + cc(V_i^{(e)}) = i + 2 + e\). Finally, the power structure of all finite 3-groups \(G\) with commutator quotient \(G/G' \simeq C_{3^e} \times C_3\) is responsible for the constant \(p\)-class \(\text{cl}_p(V_i^{(e)}) = e\), independently of the class \(\text{cl}(V_i^{(e)}) = i + 2 \leq e\), in the finite region on and behind the shock wave.

**Theorem 1.** Vertices on the mainline \((M_i^{(e)})_{i \geq 1}\) of the coclass tree \(T^e(M_1^{(e)})\) can be constructed recursively, according to three laws in dependence on the nilpotency class,

- **by irregular exo-genetic propagation (behind the shock wave)**
  \[
  M_i^{(e)} = M_i^{(e-1)} - \#1; 1, \quad \text{for } e \geq 4, \quad i \leq e - 3, \quad \text{i.e. } \text{cl}(M_i^{(e)}) < e,
  \]

- **by singular exo-genetic propagation (bifurcation on the shock wave)**
  \[
  M_i^{(e)} = M_i^{(e-1)} - \#2; 1, \quad \text{for } e \geq 4, \quad i = e - 2, \quad \text{i.e. } \text{cl}(M_i^{(e)}) = e,
  \]
• by regular endo-genetic propagation (ahead of the shock wave)

\[ M_i^{(e)} = M_{i-1}^{(e)} - \#1;1, \text{ for } e \geq 3, \text{ } i \geq e - 1, \text{ i.e. } \text{cl}(M_i^{(e)}) > e. \]

**Remark 1.** Formula (3.4) is the well-known old law for the construction of the mainline of coclass trees with elementary commutator quotient \( C_3 \times C_3 \). Formulas (3.3) and (3.2) constitute the new deterministic laws in the finite region on and behind the shock wave, in the case of non-elementary commutator quotients \( C_{3e} \times C_3, e \geq 4 \). The statements are illuminated graphically in Figure 1.

![Figure 1. Mainlines of CF-coclass trees and their various mechanisms of propagation](image)

In Figure 1 the nilpotency class is selected as the unifying invariant on the left hand scale, since all coclass trees start at a root of class \( \text{cl}(M_1^{(e)}) = 3 \), for \( e \geq 2 \). The trees are drawn for \( e \leq 8 \).

The mainline of the leftmost coclass tree \( T^3(M_1^{(2)}) \) is actually not involved in the propagation, since it is completely regular and endo-genetic: see Figure 2.

Exceptionally, the arrows of directed edges are drawn in reverse orientation, in order to point out the ostensive direction of bifurcation and propagation (irregular exo-genetic propagation in horizontal direction, singular exo-genetic propagation in diagonal direction, and regular endo-genetic propagation in vertical direction).

Figure 1 impressively shows that the root \( M_1^{(3)} \) of the coclass tree \( T^3(M_1^{(3)}) \) is a common ancestor of all mainline vertices of all CF-coclass trees \( T^e(M_1^{(3)}), e \geq 3 \), under investigation. It is clear that \( M_1^{(3)} \) is infinitely capable (root of a coclass tree), but the aforementioned fact emphasizes that \( M_1^{(3)} = \langle 729, 7 \rangle \) has the remarkable property of being infinitely capable of higher order.

**Proof.** (Theorem 1) Let \( G = \langle x, y \rangle \) be a two-generated finite 3-group. Then we denote the main commutator by \( s_2 = [y, x] \) and higher commutators by \( s_{j+1} = s_j = [s_{j-1}, x], t_j = [s_{j-1}, y] \). If the commutator quotient is bicyclic \( G/G' \simeq C_{3e} \times C_3 \) with one non-elementary component having logarithmic exponent \( e \geq 2 \), then we assume \( w^3 \in G' \) for \( w = x^{3^{e-1}} \) and \( y^3 \in G' \).
All mainline vertices involved in Theorem 1 possess a parametrized pc-presentation $M_i^{(e)} = $

\[
\langle x, y \mid x^{3^c-1} = w, \ w^3 = 1, \ y^3 = 1, \ \forall j = 2 \ s_j = s_j^2 s_{j+3}, \ s^3_{c-2} = s^2_c, \ s^3_{c-1} = s^3_c = 1, \\
\forall j = 3 \ s_j = t_j, \ s_{c+1} = t_{c+1} = 1 \rangle
\]

with two parameters, logarithmic exponent $e \geq 2$, and nilpotency class $c = \text{cl}(M_i^{(e)}) = i + 2 \geq 3$.

Recall that only the \textit{commutator structure} enters the recursive definition of the descending central series $\gamma_i(G) = G$, and $\gamma_i(G) = [\gamma_{i-1}(G), G]$, for $i \geq 2$, but also the \textit{power structure} is included in the lower exponent $p$-central series $P_0(G) = G$, and $P_i(G) = P_{i-1}(G)^3 \cdot [P_{i-1}(G), G]$, for $i \geq 1$.

Generally, for a descendant $D$, the parent is $A = \pi(D) = D/\gamma_c(D)$, and $A_p = \pi_p(D) = D/P_{c_p-1}(D)$ is the $p$-parent, where $c = \text{cl}(D)$ is the class, and $c_p = \text{cl}_p(D)$ is the $p$-class. Now we put $D := M_i^{(e)}$ and consider three situations.

1. \textit{Behind} the shock wave: If $c < e$, i.e. $i + 2 < e$ resp. $i \leq e - 3$, then $\gamma_c(D) = \langle s_c \rangle$ and $P_{c_p-1}(D) = \langle w \rangle$. Consequently, we obtain $s_c = 1$ in $A = \pi(D)$ but $w$ persists, that is $A = M_i^{(e-1)}$, if $i \geq 2$. However, in $A_p = \pi_p(D)$, we get $w = 1$ but $s_c$ persists, that is $A_p = M_i^{(e-1)}$, provided that $e \geq 4$ (for $e \leq 3$, the condition $3 \leq c < e$ cannot occur).

2. \textit{On} the shock wave: If $c = e$, i.e. $i + 2 = e$ resp. $i = e - 2$, then $\gamma_c(D) = \langle s_c \rangle$ and $P_{c_p-1}(D) = \langle s_c, w \rangle$ is bicyclic. Thus, we have $s_c = 1$ but $w$ persists in $A$, that is $A = M_i^{(e-1)}$, if $e \geq 4$. However, in $A_p$ both, $s_c = 1$ and $w = 1$, become trivial, whence $A_p = M_i^{(e-1)}$ with step size $s = 2$ reveals a bifurcation, provided that $e \geq 4$ and thus $i = e - 2 \geq 2$.

3. \textit{Ahead of} the shock wave: If $c > e$, i.e. $i + 2 > e$ resp. $i \geq e - 1$, then $c_p = e$ and $\gamma_c(D) = P_{c_p-1}(D) = \langle s_c \rangle$. So we get $s_c = 1$ in $A = A_p$ but $w$ persists, that is $A = A_p = M_i^{(e-1)}$, since $i \geq 3 - 1 = 2$ for each $e \geq 3$.

Strictly speaking, the preceding considerations only prove that $M_i^{(e)} = M_i^{(e-1)} - \#1; k$, resp. $M_i^{(e)} = M_i^{(e-1)} - \#2; \ell$, resp. $M_i^{(e)} = M_i^{(e-1)} - \#1; m$, with positive integers $k, \ell, m$, but actual computations with Magma [10] show that $k = \ell = m = 1$ for vertices on the mainline. \hfill $\square$

3.2. \textbf{Vertices remote from the mainline with depth} 1. Concerning vertices $V$ on the coclass tree $T^v(M_i^{(e)})$, $e \geq 3$, which are remote from the mainline, we restrict ourselves to those with depth $\text{dp}(V) = 1$ and omit the investigation of others with depth $2 \leq \text{dp}(V) \leq 3$. Let $V_i^{(e)}$ with $i \geq 2$ be an \textit{offside} immediate descendant of a mainline vertex $M_i^{(e)}$, and let $\zeta$ be its centre.

\textbf{Theorem 2.} The vertices $V_i^{(e)}$ remote from the mainline of the coclass tree $T^v(M_i^{(e)})$ can be constructed recursively, according to four laws in dependence on nilpotency class and centre $\zeta$,

- by \textit{irregular exo-genetic} propagation (behind the shock wave, \underline{with stable type})

\[
V_i^{(e)} = V_i^{(e-1)} - \#1; 1, \text{ for } \zeta \text{ bicyclic, } e \geq 5, \ 2 \leq i \leq e - 3, \ \text{i.e. } \text{cl}(V_i^{(e)}) < e,
\]

- by \textit{singular exo-genetic} propagation (bifurcation on the shock wave)

\[
V_i^{(e)} = M_i^{(e-1)} - \#2; \ell, \text{ for } \zeta \text{ bicyclic, } e \geq 4, \ i = e - 2, \ \text{i.e. } \text{cl}(V_i^{(e)}) = e,
\]

- by \textit{regular endo-genetic} propagation (ahead of the shock wave)

\[
V_i^{(e)} = M_i^{(e)} - \#1; m, \text{ for } \zeta \text{ bicyclic, } e \geq 3, \ i \geq e - 1, \ \text{i.e. } \text{cl}(V_i^{(e)}) > e,
\]

- by \textit{permanent regular endo-genetic} propagation (independent of the shock wave)

\[
V_i^{(e)} = M_i^{(e)} - \#1; q, \text{ for } \zeta \text{ cyclic, } e \geq 3, \ i \geq 2.
\]

\textbf{Proof.} For each \textit{periodic sequence} (also called \textit{coclass family}), the vertices $V$ have a parametrized pc-presentation with two parameters $e$ and $c$. By the \textbf{mainline principle}, the generating commutator of the last non-trivial lower central $\gamma_c(V) = \langle s_c \rangle$ does not enter the relations for the mainline, but enters at least one typical relation, in \textbf{boldface} font, for each vertex off mainline.
The branches of the coclass trees under investigation are periodic with length 2. On every branch, there is a unique mainline vertex \( M \) of type a.1, \( \zeta(M) = (000; 0) \). We recall its pc-presentation:

\[
\langle x, y \mid x^{3} = w, \ y^{3} = 1, \ \forall j = 2 \ s_{j} = s_{j}^{2} s_{j+3}, \ s_{c}^{3} = s_{c}^{2}, \ s_{c-1}^{3} = s_{c}^{2} = 1, \ \forall j = 3 \ s_{j} = t_{j}, \ s_{c+1} = t_{c+1} = 1 \rangle.
\]

(3.10)

Furthermore, there is a unique leaf \( V \) of type b.16, \( \zeta(V) = (004; 0) \):

\[
\langle x, y \mid x^{3} = w, \ y^{3} = 1, \ y^{3} = s_{c}, \ \forall j = 2 \ s_{j} = s_{j}^{2} s_{j+3}, \ s_{c-2}^{3} = s_{c}^{2}, \ s_{c-1}^{3} = s_{c}^{2}, \ s_{c}^{2} = 1, \ \forall j = 3 \ s_{j} = t_{j}, \ s_{c+1} = t_{c+1} = 1 \rangle.
\]

(3.11)

On odd branches, we have a single leaf, on even branches, we have two leaves, \( V \) of type b.3, \( \zeta(V) = (001; 0) \), with cyclic centre \( \zeta \) and exponent \( n = 1 \), resp. \( 1 \leq n \leq 2 \):

\[
\langle x, y \mid x^{3} = w, \ y^{3} = 1, \ \forall j = 2 \ s_{j} = s_{j}^{2} s_{j+3}, \ s_{c-2}^{3} = s_{c}^{2}, \ s_{c-1}^{3} = s_{c}^{2}, \ s_{c}^{2} = 1, \ \forall j = 3 \ s_{j} = t_{j}, \ s_{c+1} = t_{c+1} = 1 \rangle.
\]

(3.12)

On every branch, there is a unique root \( V \) of type a.1, \( \zeta(V) = (000; 0) \), of a twig which goes down to depth 3 (we devote our attention to the root alone and abstain from its descendants):

\[
\langle x, y \mid x^{3} = w, \ y^{3} = 1, \ y^{3} = s_{c}, \ s_{c} = 1, \ \forall j = 2 \ s_{j} = s_{j}^{2} s_{j+3}, \ s_{c-2}^{3} = s_{c}^{2}, \ s_{c-1}^{3} = s_{c}^{2}, \ t_{3} = s_{3} s_{c}, \ \forall j = 4 \ s_{j} = t_{j}, \ s_{c+1} = t_{c+1} = 1 \rangle.
\]

(3.13)

On every branch, there are two leaves \( V \) of type a.1, \( \zeta(V) = (000; 0) \), with bicyclic centre \( \zeta \) and exponent \( 1 \leq n \leq 2 \):

\[
\langle x, y \mid x^{3} = w, \ y^{3} = 1, \ y^{3} = s_{c}, \ s_{c} = 1, \ \forall j = 2 \ s_{j} = s_{j}^{2} s_{j+3}, \ s_{c-2}^{3} = s_{c}^{2}, \ s_{c-1}^{3} = s_{c}^{2}, \ t_{3} = s_{3} s_{c}, \ \forall j = 4 \ s_{j} = t_{j}, \ s_{c+1} = t_{c+1} = 1 \rangle.
\]

(3.14)

On odd branches, we have a single leaf, on even branches, we have two leaves, \( V \) of type a.1, \( \zeta(V) = (000; 0) \), with cyclic centre \( \zeta \) and exponent \( n = 1 \), resp. \( 1 \leq n \leq 2 \):

\[
\langle x, y \mid x^{3} = w, \ y^{3} = 1, \ y^{3} = s_{c}, \ s_{c} = 1, \ \forall j = 2 \ s_{j} = s_{j}^{2} s_{j+3}, \ s_{c-2}^{3} = s_{c}^{2}, \ s_{c-1}^{3} = s_{c}^{2}, \ t_{3} = s_{3} s_{c}, \ \forall j = 4 \ s_{j} = t_{j}, \ s_{c+1} = t_{c+1} = 1 \rangle.
\]

(3.15)

Similarly as in the proof of Theorem 11 for a descendant \( D \), the parent is \( \pi(D) = D/\gamma_{c}(D) \), and the p-parent is \( A_{p} = \pi_{p}(D) = D/\gamma_{c-1}(D) \), where \( c = cl(D) \) is the class, and \( c_{p} = cl_{p}(D) \) is the p-class. Now we put \( D := V_{i}^{(e)}(e) \) and consider four situations. For the first three items, let \( D \) be a vertex with bicyclic centre, and thus with one of the presentations (3.11), (3.13) or (3.14).

1. **Behind** the shock wave: If \( c < e \), i.e. \( i + 2 < e \) resp. \( i \leq e - 3 \), then \( \gamma_{c}(D) = \langle s_{c} \rangle \) and \( P_{c_{p}}(D) = \langle w \rangle \). Consequently, we obtain \( s_{c} = 1 \) in \( A = \pi(D) = \pi_{c}(D) \), but \( w \) persists, that is \( A = M_{i}^{(e)} \), if \( i \geq 2 \). However, in \( A_{p} = \pi_{p}(D) \), we get \( w = 1 \) but \( s_{c} \) (and the distinguished relation) persists, that is \( A_{p} = V_{i}^{(e-1)} \), same type, provided that \( e \geq 5 \) (for \( e \leq 4 \), condition \( 4 \leq e < c \) cannot occur).

2. **On the shock wave**: If \( c = e \), i.e. \( i + 2 = e \) resp. \( i = e - 2 \), then \( \gamma_{c}(D) = \langle s_{c} \rangle \) and \( P_{c_{p}}(D) = \langle s_{c}, w \rangle \) is bicyclic. Thus, we have \( s_{c} = 1 \) but \( w \) persists in \( A \), that is \( A = M_{i}^{(e)} \), if \( e \geq 4 \). However, in \( A_{p} \) both, \( s_{c} = 1 \) and \( w = 1 \), become trivial, whence \( A_{p} = M_{i}^{(e-1)} \) with step size \( s = 2 \) reveals a bifurcation, provided that \( e \geq 4 \) and thus \( i + 2 = e - 2 \geq 2 \).

3. **Ahead of** the shock wave: If \( c > e \), i.e. \( i + 2 > e \) resp. \( i \geq e - 1 \), then \( c_{p} = c \) and \( \gamma_{c}(D) = P_{c_{p}}(D) = \langle s_{c} \rangle \). So we get \( s_{c} = 1 \) (the distinguished relation degenerates) in \( A = A_{p} \), but \( w \) persists, i.e. we get a mainline vertex \( A = A_{p} = M_{i}^{(e)} \), since \( i \geq 3 - 1 = 2 \) for each \( e \geq 3 \).

4. Let \( D \) be a vertex with cyclic centre, and thus with one of the presentations (3.12) or (3.15).

Although the p-class \( c_{p} \) may be bigger than the class \( c \) of \( D \), nevertheless, the last non-trivial lower centre and lower p-centre coincide \( \gamma_{c}(D) = P_{c_{p}}(D) = \langle s_{c} \rangle \), due to the exceptional relation \( w^{3} = s_{c} \). So we get \( s_{c} = 1 \) in \( A = A_{p} \) and \( w = 1 \) becomes trivial, that is, the propagation is always regular and endo-genetic with coinciding parent and p-parent \( A = A_{p} = M_{i}^{(e)} \) a mainline vertex, where \( i + 2 = e \geq 4 \), i.e. \( i - 1 = c - 3 \geq 1 \).
Strictly speaking, the preceding considerations only prove that \( V_{1}^{(e)} = V^{(e-1)} - \#1; k \), resp. \( V_{1}^{(e)} = M_{i-1}^{(e-1)} - \#1; \ell \), resp. \( V_{1}^{(e)} = M_{i-1}^{(e)} - \#1; m \), resp. \( V_{1}^{(e)} = M_{i-1}^{(e)} - \#1; q \), with positive integers \( k, \ell, m, q \), but actual computations with Magma [13] show that \( k = 1 \), and \( \ell, m, q \geq 2 \). □

**Remark 2.** It should be pointed out that the descendant vertices \( V \) of each root \( M_{i}^{(e)} \) share further invariants with the root, aside from the rank distribution \( r(V) \). They have closely related transfer kernel types \( x(V) \) with three identical components (the stabilization) and a single varying component (the polarization). For all CF trees in this paper, the polarization is located at the third component, and thus distinct from the puncture, which is the fourth component, by convention.

4. **Laws for coclass trees of BCF-groups**

**Proposition 3.** For each log exponent \( e \geq 2 \), there exists a unique coclass tree \( T^{e+1}(M_{i}^{(e+1)}) \) \( \ni V \) with fixed coclass \( cc(V) = e + 1 \), fixed commutator quotient \( V/V' \simeq C_{3^r} \times C_{3^s} \), and fixed rank distribution \( r(V) \sim (2, 2, 3; 3) \). Its mainline \( (M_{i}^{(e+1)})_{i \geq 1} \) is of type \( d.10 \), \( r(M_{i}^{(e+1)}) \sim (110; 2) \). The tree contains metabelian and non-metabelian BCF-groups. The branches are of depth 4.

(See Figures 5 – 7 for the depth-pruned metabelian skeleton, when \( 2 \leq e \leq 4 \).)

**Proof.** According to the proof of Proposition 1 there are only two coclass trees \( T^{r}(R_{j}^{(e)}) \) with rank distribution \( r(V) \sim (2, 2, 3; 3) \), for each \( e \geq 2 \), \( e \leq r \leq e + 1 \), a BCF-tree of type \( d.10 \) and a CF-tree of type \( a.1 \). The former is the unique tree with root \( R_{j}^{(e+1)} = M_{i}^{(e+1)}, \) recursively determined by the BCF-group \( M_{1}^{(3)} = (729, 7) \) and the BCF-group \( M_{1}^{(3)} = (729, 13) \). Its depth-pruned metabelian branches are periodic of length 2 without pre-period, and all of its vertices are BCF-groups, since the vertices of the first two branches are BCF-groups.

**Proposition 4.** For \( e \geq 2 \), invariants of vertices on the mainline \( (M_{i}^{(e+1)})_{i \geq 1} \) of the coclass tree \( T^{e+1}(M_{i}^{(e+1)}) \) are given as follows:

\[
\begin{align*}
\log \text{order} \text{ } & \log(M_{i}^{(e+1)}) = e + i + 3, \text{ } \text{nilpotency class} \text{ } & cl(M_{i}^{(e+1)}) = i + 2, \text{ } \text{for } i \geq 1, \\
\text{p-class} \text{ } & cl_{p}(M_{i}^{(e+1)}) = \begin{cases} 
i + 2 & \text{if } i > e - 1, \\
i + 1 & \text{if } i \leq e - 1, \end{cases} \\
\text{p-coclass} \text{ } & cc_{p}(M_{i}^{(e+1)}) = \begin{cases} e + 1 & \text{if } i > e - 1, \\
i + 2 & \text{if } i \leq e - 1. \end{cases}
\end{align*}
\]

**Proof.** Proposition 1 remains true when the mainline vertex \( M_{i}^{(e+1)} \) is replaced by any proper descendant vertex \( V_{i}^{(e+1)} \) with \( i \geq 2 \). All coclass trees under investigation start at a root of class \( cl(M_{i}^{(e+1)}) = 3 = 1 + 2 \), for each \( e \geq 2 \). Thus, proper descendants possess nilpotency class \( cl(V_{i}^{(e+1)}) = i + 2 \geq 4 \). By definition, all vertices \( V \) of the coclass tree \( T^{e+1}(M_{i}^{(e+1)}) \) share the common coclass \( cc(V) = e + 1 \). Consequently, the logarithmic order is the sum \( \log(V_{i}^{(e+1)}) = cl(V_{i}^{(e+1)}) + cc(V_{i}^{(e+1)}) = i + 2 + e + 1 \). Finally, the power structure of all finite 3-groups \( G \) with commutator quotient \( G/G' \simeq C_{3^r} \times C_{3^s} \) is responsible for the constant \( p \)-class \( cl_{p}(V_{i}^{(e+1)}) = e + 1 \), independently of the class \( cl(V_{i}^{(e+1)}) = i + 2 \leq e + 1 \), in the finite region on and behind the shock wave.

Concerning vertices \( V \) on the coclass trees \( T^{e+1}(M_{i}^{(e+1)}) \), \( e \geq 2 \), which are remote from the mainline, we restrict ourselves to the metabelian with depth \( dp(V) = 1 \), and we omit the investigation of others with depth \( 2 \leq dp(V) \leq 4 \).

Let \( V_{i}^{(e+1)} \) with \( i \geq 1 \) be any metabelian vertex on or remote from the mainline \( (M_{i}^{(e+1)})_{i \geq 1} \).

**Theorem 3.** The vertices \( V_{i}^{(e+1)} \) on and remote from the mainline of the coclass tree \( T^{e+1}(M_{i}^{(e+1)}) \) can be constructed recursively, according to three laws in dependence on the nilpotency class,

- by irregular endo-genetic propagation (behind the shock wave, with type change)

\[
V_{i}^{(e+1)} = V_{i}^{(e)} - \#1; k, \text{ for } e \geq 3, \text{ } 1 \leq i \leq e - 2, \text{ i.e. } cl(V_{i}^{(e+1)}) < e + 1,
\]

where \( d.10, B.2, D.10, C.4, D.5 \) of \( V_{i}^{(e+1)} \) correspond to \( a.1, a.1, b.16, a.1, a.1 \) of \( V_{i}^{(e)} \),...
\begin{itemize}
  \item by singular exo-genetic propagation (bifurcation on the shock wave)
  \begin{equation}
  \nu^{(e+1)}_i = M^{(e)}_{i-1} - \#2; i, \text{ for } e \geq 3, \quad \text{if } e = 1, \quad i = e - 1, \quad \text{i.e. } \mathrm{cl}(\nu^{(e+1)}_i) = e + 1,
  \end{equation}

  \item by regular endo-genetic propagation (ahead of the shock wave)
  \begin{equation}
  \nu^{(e+1)}_i = M^{(e+1)}_{i-1} - \#1; m, \text{ for } e \geq 2, \quad \text{if } e \geq e, \quad \text{i.e. } \mathrm{cl}(\nu^{(e+1)}_i) > e + 1,
  \end{equation}
\end{itemize}

\textbf{Proof.} For each periodic sequence (or coclass family), the vertices $V$ have a parametrized presentation with two parameters $e$ and $c$. According to the \textbf{mainline principle}, the generating commutator of the last non-trivial lower central $\gamma_c(V) = \langle s_c \rangle$ does not enter the relations for the mainline, but enters at least one typical relation, in \textbf{boldface} font, for each vertex off mainline. Branches of coclass trees under investigation are periodic with length $2$. On every branch, there is a unique mainline vertex $M$ of type $d.10$, $\kappa(M) \sim (110; 2)$. Its pc-presentation is given by:

\begin{equation}
\langle x, y \mid x^3 = w, \quad w^3 = 1, \quad y^3 = 1, \quad \nu^{(e-3)}_{j=2} s_j = s_{j+2}^3 s_{j+3}, \quad s_{c-2}^3 = s_{c-1}^3 = s_c^3 = 1, \quad t_3 = s_3 w, \quad \forall_{j=4}^c s_j = t_j, \quad s_{c+1} = t_{c+1} = 1 \rangle.
\end{equation}

On odd branches, we have a single vertex, on even branches, we have two vertices, $V$ of type $D.10$, $\kappa(V) \sim (114; 2)$, with exponent $n = 1$, resp. $1 \leq n \leq 2$:

\begin{equation}
\langle x, y \mid x^3 = w, \quad w^3 = 1, \quad y^3 = 1, \quad \nu^{(e-3)}_{j=2} s_j = s_{j+2}^3 s_{j+3}, \quad s_{c+2}^3 = s_c^3 = 1, \quad t_3 = s_3 w, \quad \forall_{j=4}^c s_j = t_j, \quad s_{c+1} = t_{c+1} = 1 \rangle.
\end{equation}

On odd branches, we have a single root, on even branches, we have two roots, $V$ of type $B.2$, $\kappa(V) \sim (111; 2)$, of a twig, with exponent $n = 1$, resp. $1 \leq n \leq 2$:

\begin{equation}
\langle x, y \mid x^3 = w, \quad w^3 = 1, \quad y^3 = 1, \quad \nu^{(e-3)}_{j=2} s_j = s_{j+2}^3 s_{j+3}, \quad s_{c+2}^3 = s_c^3 = 1, \quad t_3 = s_3 w, \quad \forall_{j=4}^c s_j = t_j, \quad s_{c+1} = t_{c+1} = 1 \rangle.
\end{equation}

On odd branches, we have a single vertex, on even branches, we have two vertices, $V$ of type $C.4$, $\kappa(V) \sim (112; 2)$, with exponent $n = 1$, resp. $1 \leq n \leq 2$:

\begin{equation}
\langle x, y \mid x^3 = w, \quad w^3 = 1, \quad y^3 = 1, \quad \nu^{(e-3)}_{j=2} s_j = s_{j+2}^3 s_{j+3}, \quad s_{c+2}^3 = s_c^3 = 1, \quad t_3 = s_3 s_c^3 w, \quad \forall_{j=4}^c s_j = t_j, \quad s_{c+1} = t_{c+1} = 1 \rangle.
\end{equation}

On odd branches, we have a single vertex, on even branches, we have two vertices, $V$ of type $D.5$, $\kappa(V) \sim (113; 2)$, with exponent $n = 1$, resp. $1 \leq n \leq 2$:

\begin{equation}
\langle x, y \mid x^3 = w, \quad w^3 = 1, \quad y^3 = 1, \quad \nu^{(e-3)}_{j=2} s_j = s_{j+2}^3 s_{j+3}, \quad s_{c+2}^3 = s_c^3 = 1, \quad t_3 = s_3 s_c^{3-n} w, \quad \forall_{j=4}^c s_j = t_j, \quad s_{c+1} = t_{c+1} = 1 \rangle.
\end{equation}

Similarly as in the proof of Theorem 2, for a descendant $D$, the parent is $A = \pi(D) = D/P_{c-1}(D)$ is the $p$-parent, where $c = \mathrm{cl}(D)$ is the class, and $c_p = \mathrm{cl}_p(D)$ is the $p$-class. Now we put $D := V_i^{(e+1)}$ and consider three situations.

1. \textit{Behind the shock wave:} If $c < e + 1$, i.e. $i + 2 < e + 1$ resp. $i \leq e - 2$, then $\gamma_c(D) = \langle s_c, t_c \rangle$, if $i = 1$, $\gamma_c(D) = \langle s_c \rangle$, if $i \geq 2$, and $P_{c-1}(D) = \langle w \rangle$. Consequently, if $i \geq 2$, we obtain $s_c = 1$ in $A = \pi(D)$ but $w$ persists, all Formulas (4.10), (4.11), (4.12), (4.13), (4.14) degenerate to (4.15), that is $A = M_{i-1}^{(e+1)}$, if $i \geq 2$. However, in $A_p = \pi_p(D)$, we get $w = 1$ but $s_c$ (and the distinguished relation persists), Formula (4.15), resp. (4.16), (4.17), (4.18), (4.19), becomes Formula (4.10), resp. (4.11), (4.13), (4.14), (4.15) i.e. $A_p = V_i^{(e)}$, provided $e \geq 4$ (for $e \leq 3$, condition $4 \leq c < e + 1$ cannot occur) or $e = 3$ and $c = 3$.

2. \textit{On the shock wave:} If $c = e + 1$, i.e. $i + 2 = e + 1$ resp. $i = e - 1$, then $\gamma_c(D) = \langle s_c \rangle$ and $P_{c-1}(D) = \langle s_c, w \rangle$ is bicyclic. Thus, we have $s_c = 1$ but $w$ persists in $A$, that is $A = M_{i-1}^{(e+1)}$, if $e \geq 3$. However, in $A_p$ both, $s_c = 1$ and $w = 1$, become trivial, all Formulas (4.10), (4.11), (4.12), (4.13), (4.14), (4.15) degenerate to (4.10), whence $A_p = M_{i-1}^{(e)}$ with step size $s = 2$ reveals a bifurcation, provided that $e \geq 3$ and thus $i = e - 1 \geq 2$.

3. \textit{Ahead of the shock wave:} If $c > e + 1$, i.e. $i + 2 > e + 1$ resp. $i \geq e$, then $c_p = c$ and $\gamma_c(D) = P_{c-1}(D) = \langle s_c \rangle$. So we get $s_c = 1$ in $A = A_p$ but $w$ persists, all Formulas (4.10), (4.11),
\( (1.17), (4.8), (4.9) \) degenerate to \((4.5)\), that is a mainline vertex \( A_i = A_{p} = M_{i-1}^{(e+1)} \), since \( i \geq 2 \) for each \( e \geq 2 \).

In the preceding, we have proved that \( V_i^{(e+1)} = V_i^{(e)} - \#1; k \), resp. \( V_i^{(e+1)} = M_{i-1}^{(e)} - \#2; \ell \), resp. \( V_i^{(e+1)} = M_{i-1}^{(e+1)} - \#1; m \), with positive integers \( k, \ell, m \).

**Figure 2.** Coclass-2 tree of type a.1, rooted in original Ascione, for AQI (21)

In Figure 2, the coclass tree with Ascione’s CF-group \( A \) as its root \((243, 17) = (81, 3) - \#1; 6\) is drawn up to order \(3^{11}\). The branches are periodic with length 2 and naturally bounded depth 3, without artificial pruning. The infinite main line is of type a.1 and consists of \( \sigma \)-groups with GI action by \( C_2 \). The other vertices are \( \sigma \)-groups for even branches and non-\( \sigma \) groups for odd branches. Vertices with positive depth are of type b.16 or b.3 or a.1. All vertices of type b.3 and some of type a.1 have a cyclic centre \( \zeta \approx C_9 \). Vertices of depth \( dp \in \{2, 3\} \) are exclusively of type a.1 with cyclic centre \( \zeta \approx C_9 \). They are drawn for the first and second branch only. On the first branch, \( \langle 373 \rangle \) gives rise to \((2208) \ldots (2212)\), and \( \langle 378 \rangle \) gives rise to \((2213) \ldots (2216)\). On the second branch, \( \langle 2200 \rangle \) gives rise to 5 non-\( \sigma \)-descendants, and \( \langle 2205 \rangle \) gives rise to 5 descendants with generator inverting (GI) and relator inverting (RI) action by \( C_2 \).
Remark 3. The purely graph theoretic structure of the coclass tree in Figure 2 was indicated in Tbl. 6, p. 272 with much less details and only up to order $3^3$. The root $A = (243, 17)$ was described ten years earlier by James [8] as a member of the first branch $\Phi_p^{(1)}$ of Hall’s isoclinism class $\Phi_p$. See the pc-presentation for $\Phi_p(2111) \simeq A$ in [9, p. 620]. Another pc-presentation for $A$ can be extracted from the microfiches in [1, p. 320, folio I02].

Figure 3. Coclass-3 tree of type a.1, rooted in generalized Ascione A, for AQI $\Phi_3(31)$

Similarly as in Figure 2, the propagation in Figure 3 is also completely regular, and purely endo-
genetic with respect to the coclass tree $T^3(M_1^{(3)})$. However, the generalized CF-group $A$ of Ascione, $M_1^{(3)} \simeq (729, 7)$, is located on the shock wave, and consequently also possesses additional exo-
genetic $p$-descendants with step size $s = 1$, namely $M_i^{(3)} - \#1; i$ with $i \in \{1, 2\}$. This leads to exceptional relative identifiers for the vertices of the first branch $B(M_1^{(3)})$, beginning with $M_2^{(3)} = M_1^{(3)} - \#1; 3 \simeq (2187, 113)$. (The mainline vertex $M_1^{(3)}$ of all other branches with $i \geq 3$ has relative identifier $M_{i-1}^{(3)} - \#1; 1$.)
As opposed to the Figures 2 and 3, the entire first branch is exceptional in Figure 4. The root $M_1^{(4)} \simeq \langle 2187, 111 \rangle$, that is the generalized CF-group $\text{A}$ of Ascione, arises by irregular exo-genetic propagation as $p$-descendant $M_1^{(3)} - \#1; 1$ with step size $s = 1$ of the periodic root $M_1^{(3)} \simeq \langle 729, 7 \rangle$ of bifurcations, according to Formula (3.2). The vertices $V_2^{(4)} = M_1^{(3)} - \#2; i$ with $1 \leq i \leq 5$ and bicyclic centre arise by singular exo-genetic propagation as $p$-descendants with step size $s = 2$ of $M_1^{(3)}$, according to Formulas (3.3) and (3.7). The vertices $V_2^{(4)} = M_1^{(4)} - \#1; i$ with $i \in \{3, 4\}$ and cyclic centre, however, arise by permanent regular endo-genetic propagation as $p$-descendants with step size $s = 1$ of the root $M_1^{(4)}$ of a periodic chain with step size $s = 1$, according to Formulas (3.4). See also the Exhaustion Theorem 6.
In Figure 5, the depth-pruned metabelian skeleton of the coclass tree with root $\langle 729, 13 \rangle = \langle 81, 3 \rangle - \#2; 10$ is drawn up to order $3^{12}$. Without artificial pruning, the branches have naturally bounded depth 4 (not drawn). The depth-pruned metabelian branches are periodic with length 2. The infinite main line is of type d.10 and consists of $\sigma$-groups with GI action by $C_2$, every other even with RI action. The offside vertices are $\sigma$-groups for even branches and non-$\sigma$ groups for odd branches. Vertices with positive depth are of type D.10, D.5, C.4, B.2 or d.10. Metabelian vertices of depth $dp \geq 2$ are exclusively of type B.2, for instance, $\langle 2187, 170 \rangle$ has 105 descendants with step size $s = 1$, whereas $\langle 6561, 1682 \rangle$ and $\langle 6561, 1685 \rangle$ have 58 descendants each.

The polarization of all CF-descendants of $\langle 243, 17 \rangle = \langle 81, 3 \rangle - \#1; 6$ in Figure 2 was in the third component. In Figure 5, we see that, astonishingly, the polarization of all BCF-descendants of $\langle 729, 13 \rangle = \langle 81, 3 \rangle - \#2; 10$ is in the first component, although both roots are constructed as immediate descendants of the same parent. However, this discrepancy does not really matter, since the coclass trees $T^2(\langle 243, 17 \rangle)$ and $T^3(\langle 729, 13 \rangle)$ are actually completely independent, in contrast to the following coclass trees in Figure 6 and 7.
In Figure 6, the root \(\langle 2187, 112 \rangle = \langle 729, 7 \rangle - \#1; 2\) is \(p\)-terminal, and all its immediate descendants on the first branch are step size-2 \(p\)-descendants of \(\langle 729, 7 \rangle\). This is the prototype of an application of Formula (5.13), \(m_1^{(4)} = M_1^{(3)} - \#1; 2\), and (5.13), \(\psi_{2,t-5}^{(4)} = M_1^{(3)} - \#2; t\), both for \(e = e_0 = 3\), the latter for \(t \in \{6, 7, 8, 9, 10\}\). The second branch is regular with root \(\langle 6561, 98 \rangle = \langle 729, 7 \rangle - \#2; 6\), but its relative identifiers are exceptional, since \(\langle 6561, 98 \rangle - \#1\); \(i\) with \(1 \leq i \leq 2\) are non-metabelian and thus remain hidden in the metabelian skeleton. Beginning with the third branch, all odd branches are regular with regular relative identifiers, according to Formula (5.2). Beginning with the fourth branch, all even branches are regular with regular relative identifiers, according to Formula (5.4). Less explicitly, Formula (5.14) can be expressed by Formula (4.2), and Formula (5.15) by Formula (4.3). The propagation in regular branches is covered by Formula (4.4).
In Figure 7, the root and the initial branches 1 and 2 of the BCF-tree $\mathcal{T}^5$ are constituted by immediate descendants of CF-groups. The root $\langle 6561, 953 \rangle = \langle 2187, 111 \rangle - \#1; 2$ propagates endo-genetically from the root of the CF-tree $\mathcal{T}^4$ with (logarithmic) commutator quotient (41), according to Formula (4.2). The first branch comes from distinct mainline and offside vertices, $\langle 6561, i \rangle - \#1; 2$ with $93 \leq i \leq 97$, on the first branch of $\mathcal{T}^4$, according to the same Formula (4.2) with type change. The entire second branch uniformly propagates from the second mainline vertex of $\mathcal{T}^4$, according to Formula (4.3). Although branch 3 is regular, its relative identifiers are exceptional, since $\langle 6561, 93 \rangle - \#2; 6 - \#1; 1$ is non-metabelian and thus does not show up in the metabelian skeleton. Regular branches (third, etc.) are constructed according to Formula (4.4).
5. Periodic bifurcations and periodic chains

The statements in this section exhibit several new kinds of periodicities in \( p \)-descendant trees. The notations are based on both preceding sections, §3 on CF-groups, and §4 on BCF-groups.

Generally, it is convenient to view a coclass tree \( T^e \) as union of a finite pre-period \( V \) and an infinite disjoint union \( T^e = V \cup (\bigcup_{k \geq 0} P_k) \) of copies \( P_k = \bigcup_{i=0}^{k-1} B_{p+kt+i} \) of a collection of finitely many branches \( \bigcup_{i=0}^{\ell} B_{p+i} \), the period with length \( \ell \geq 1 \) and starting subscript \( p \geq 1 \), such that the branches \( (V^l_{k=0}) \cup (V^l_{i=0}) \cup B_{p+kt+i} \) are isomorphic as finite graphs.

In the present article, all coclass trees are depth-1 pruned metabelian skeletons without pre-period, \( V = \emptyset \), minimal starting subscript \( p = 1 \), and period length \( \ell = 2 \), that is, we have \( T^e = \bigcup_{k \geq 0} P_k \) with \( P_k = B_{1+2k} \cup B_{2+2k} \).

**Definition 7.** For each integer \( i \geq 1 \), the finite subtree \( B(M_i^{(e)}) = T^e(M_i^{(e)}) \setminus T^e(M_{i+1}^{(e)}) \) of the depth-pruned CF-coclass tree \( T^e(M_i^{(e)}) \) is called \( i \)-th depth-1 pruned branch, and the finite subtree \( B(M_i^{(e+1)}) = T^{e+1}(M_i^{(e+1)}) \setminus T^{e+1}(M_{i+1}^{(e+1)}) \) of the depth-pruned metabelian BCF-coclass tree \( T^{e+1}(M_i^{(e+1)}) \) is called \( i \)-th depth-1 pruned metabelian branch.

From now on, we omit the phrase “depth-1 pruned metabelian”. The precise constitution of the branches in Definition 7 by CF-vertices, respectively BCF-vertices, is given experimentally:

**Proposition 5. (Odd branches)** Let \( e \geq 2 \) be a logarithmic integer exponent.

For each odd integer \( i \geq 1 \), the \( i \)-th CF-branch \( B(M_i^{(e)}) = \{M_i^{(e)}\} \cup \{(V_{i+1,j}^{(e)})_{j=2}\} \) consists of the mainline vertex \( M_i^{(e)} \) (branch root) and its immediate step size-1 offside descendants

\[
\begin{align*}
V_{i+1,2}^{(e)} & \text{ of type b.16, } \sim (004; 0), \\
V_{i+1,3}^{(e)} & \text{ of type a.1, } \sim (000; 0), \text{ capable twig root,} \\
V_{i+1,4}^{(e)} & \text{ of type a.1, } \sim (000; 0), \text{ bicyclic centre (e - 1, 1),} \\
V_{i+1,5}^{(e)} & \text{ of type a.1, } \sim (000; 0), \text{ bicyclic centre (e - 1, 1),} \\
V_{i+1,6}^{(e)} & \text{ of type b.3, } \sim (001; 0), \text{ cyclic centre (e),} \\
V_{i+1,7}^{(e)} & \text{ of type a.1, } \sim (000; 0), \text{ cyclic centre (e).}
\end{align*}
\]

\( (5.1) \)

For each odd integer \( i \geq 1 \), the \( i \)-th BCF-branch \( B(M_i^{(e+1)}) = \{M_i^{(e+1)}\} \cup \{(V_{i+1,j}^{(e+1)})_{j=2}\} \) consists of the mainline vertex \( M_i^{(e+1)} \) (branch root) and its immediate step size-1 offside descendants

\[
\begin{align*}
V_{i+1,2}^{(e+1)} & \text{ of type D.10, } \sim (114; 2), \\
V_{i+1,3}^{(e+1)} & \text{ of type B.2, } \sim (111; 2), \text{ capable twig root,} \\
V_{i+1,4}^{(e+1)} & \text{ of type C.4, } \sim (112; 2), \\
V_{i+1,5}^{(e+1)} & \text{ of type D.5, } \sim (113; 2).
\end{align*}
\]

\( (5.2) \)

**Remark 4.** 1. In order to be able to include the mainline vertex, we always assume \( V_{i+1,1}^{(e)} = M_i^{(e+1)} \) of type a.1 for CF-groups and \( V_{i+1,1}^{(e+1)} = M_i^{(e+1)} \) of type d.10 for BCF-groups (Propositions 4 – 6) where the ordering of the offside vertices usually coincides with the ordering in Figures 3 – 4.

2. The branches for even subscripts \( i \geq 2 \) have bigger cardinality (9 for CF, 9 for BCF in Proposition 5) than those for odd subscripts \( i \geq 1 \) (7 for CF, 5 for BCF in Proposition 5).

3. It must be emphasized very clearly that the (abstract) descendants in Propositions 4 – 6 are not \( p \)-descendants in the region behind and on the shock wave.
Proposition 6. (Even branches) Let $e \geq 2$ be a logarithmic integer exponent. For each even integer $i \geq 2$, the $i$-th CF-branch $B(M_i^{(e)}) = \{M_i^{(e)}\} \cup \{(V_i^{(e)})^9\}_{j=2}$ consists of the mainline vertex $M_i^{(e)}$ (branch root) and its immediate step size-1 offside descendants

$$
\begin{align*}
V_{i+1.2}^{(e)} & \text{ of type b.16, } \sim (004; 0), \\
V_{i+1.3}^{(e)} & \text{ of type a.1, } \sim (000; 0), \text{ capable twig root}, \\
V_{i+1.4}^{(e)} & \text{ of type a.1, } \sim (000; 0), \text{ bicyclic centre (e - 1, 1)}, \\
V_{i+1.5}^{(e)} & \text{ of type a.1, } \sim (000; 0), \text{ bicyclic centre (e - 1, 1)}, \\
V_{i+1.6}^{(e)} & \text{ of type b.3, } \sim (001; 0), \text{ cyclic centre (e)}, \\
V_{i+1.7}^{(e)} & \text{ of type a.1, } \sim (000; 0), \text{ cyclic centre (e)}, \\
V_{i+1.8}^{(e)} & \text{ of type b.3, } \sim (001; 0), \text{ cyclic centre (e)}, \\
V_{i+1.9}^{(e)} & \text{ of type a.1, } \sim (000; 0), \text{ cyclic centre (e)}.
\end{align*}
$$

(5.3)

For each even integer $i \geq 2$, the $i$-th BCF-branch $B(M_i^{(e+1)}) = \{M_i^{(e+1)}\} \cup \{(V_i^{(e+1)})^9\}_{j=2}$ consists of the mainline vertex $M_i^{(e+1)}$ (branch root) and its immediate step size-1 offside descendants

$$
\begin{align*}
V_{i+1.2}^{(e+1)} & \text{ of type D.10, } \sim (114; 2), \\
V_{i+1.3}^{(e+1)} & \text{ of type D.10, } \sim (114; 2), \\
V_{i+1.4}^{(e+1)} & \text{ of type B.2, } \sim (111; 2), \text{ capable twig root}, \\
V_{i+1.5}^{(e+1)} & \text{ of type C.4, } \sim (112; 2), \\
V_{i+1.6}^{(e+1)} & \text{ of type D.5, } \sim (113; 2), \\
V_{i+1.7}^{(e+1)} & \text{ of type B.2, } \sim (111; 2), \text{ capable twig root}, \\
V_{i+1.8}^{(e+1)} & \text{ of type D.5, } \sim (113; 2), \\
V_{i+1.9}^{(e+1)} & \text{ of type C.4, } \sim (112; 2).
\end{align*}
$$

(5.4)

We are now in the position to clarify in depth the structure of periodic bifurcations, that is, periodic chains with constant step size $s = 2$. This phenomenon concerns only the links between CF-coclass trees.

Proposition 7. (Periodic size-2 chain) For each commutator quotient $C_{3r} \times C_3$ with $e \geq 3$, the mainline of the unique CF-coclass tree $\mathcal{T}^e(M_1^{(e)})$, which starts at the root $M_1^{(e)}$ with cc = 3, $lo = 3 + e$, contains a unique vertex $M_{e-2}^{(e)}$ with bifurcation due to nuclear rank $n = 2$, and with cc = e, $lo = 2e$. The complete periodic size-2 chain is given by $\{M_{e-2}^{(e)}\}_{e \geq 3}$.

Proof. By definition of the coclass tree $\mathcal{T}^e(M_1^{(e)})$, all of its vertices share the common coclass $cc = e$. Each tree arises from a root $M_1^{(e)}$ with class cc = 3 = 1 + 2, whence generally each mainline vertex $M_i^{(e)}$ is of class cc = i + 2, for $i \geq 1$. In particular, the distinguished vertex $M_{e-2}^{(e)}$ has class $cc = (e - 2) + 2 = e$. Evidence of its elevated nuclear rank $n = 2$ will be provided by the following Theorem 1. The logarithmic order is always the sum $lo = cl + cc$ of class and coclass. □

Each of the step sizes, $s = 1$ and $s = 2$, of the bifurcations generates both, exo-genetic and endo-genetic $p$-descendants. Whereas Propositions 5 - 6 are not constructive, the following Theorems 4 and 5 can be viewed as deterministic laws for the construction of vertices with the aid of the $p$-group generation algorithm [19, 20, 7, 6]. Both theorems are experimental.
Theorem 4. (Structure of bifurcations) For each integer \( e \geq 3 \), the distinguished CF-mainline vertex \( B := M^{(e)}_{e-2} \), which possesses bifurcation, gives rise to

- 1 exo-, and 8, respectively 10, endo-genetic propagations with step size \( s = 1 \),
  \[ M^{(e+1)}_{e-2} = B - \#1; 1 \text{ (exo-genetic)}, \]
  \[ M^{(e+1)}_{e-2} = B - \#1; 2, \]
  \[ M^{(e)}_{e-1} = B - \#1; 3, \]
  \[ V^{(e)}_{e-1, i-2} = B - \#1; i \text{ with } 4 \leq i \leq 9 \text{ (e odd), respectively } \leq 11 \text{ (e even);} \]
- 5 exo-, and 5, respectively 9, endo-genetic propagations with step size \( s = 2 \),
  \[ M^{(e+1)}_{e-1} = B - \#2; 1 \text{ (exo-genetic recursion),} \]
  \[ V^{(e+1)}_{e-1, i} = B - \#2; i \text{ with } 2 \leq i \leq 5 \text{ (exo-genetic),} \]
  \[ M^{(e+1)}_{e-1} = B - \#2; 6, \]
  \[ V^{(e+1)}_{e-1, i-5} = B - \#2; i \text{ with } 7 \leq i \leq 10 \text{ (e odd), respectively } \leq 14 \text{ (e even).} \]

In particular, by endo-genetic propagations, \( B \) generates the complete branch \( B(M^{(e)}_{e-2}) \) of CF-groups (with \( s = 1 \)), and the complete branch \( B(M^{(e+1)}_{e-2}) \) of BCF-groups (with \( s = 2 \)), with exception of the branch roots, \( M^{(e)}_{e-2} \) and \( M^{(e+1)}_{e-2} \) (i.e., only vertices with depth \( dp = 1 \)).

In the following Theorem 5, we abstain from vertices with brushwood type B.2 of high complexity, and we restrict our attention to vertices with types d.10, D.10, C.4, and D.5. Again, each member of the chains generates both, exo-genetic and endo-genetic \( p \)-descendants.

Theorem 5. (Periodic size-1 chains) For each integer \( e \geq 3 \),

- \( M := M^{(e+1)}_{e-2} = B - \#1; 1 \text{ of type a.1 gives rise to} \)
  \[ M^{(e+2)}_{e-2} = M - \#1; 1 \text{ of type a.1 (exo-genetic recursion),} \]
  \[ M^{(e+2)}_{e-2} = M - \#1; 2 \text{ of type d.10,} \]
  \[ V^{(e+1)}_{e-1, i+3} = M - \#1; i \text{ of types b.3, a.1, with } 3 \leq i \leq 4, \]
  \[ V^{(e+1)}_{e-1, i+3} = M - \#1; i \text{ of types b.3, a.1, with } 5 \leq i \leq 6, \text{ only for even } e. \]
- \( V := V^{(e+1)}_{e-1, 2} = B - \#2; 2 \text{ of type b.16 gives rise to} \)
  \[ V^{(e+2)}_{e-1, 2} = V - \#1; 1 \text{ of type b.16 (exo-genetic recursion),} \]
  \[ V^{(e+2)}_{e-1, 2} = V - \#1; 2 \text{ of type D.10,} \]
  \[ V^{(e+2)}_{e-1, 3} = V - \#1; 3 \text{ of type D.10, only for even } e. \]
- \( V := V^{(e+1)}_{e-1, 4} = B - \#2; 4 \text{ of type a.1 gives rise to} \)
  \[ V^{(e+2)}_{e-1, 4} = V - \#1; 1 \text{ of type a.1 (exo-genetic recursion),} \]
  \[ V^{(e+2)}_{e-1, 4} = V - \#1; 2 \text{ of type C.4, with } i = 4 \text{ (e odd) or } i = 5 \text{ (e even),} \]
  \[ V^{(e+2)}_{e-1, 9} = V - \#1; 3 \text{ of type C.4, only for even } e. \]
- \( V := V^{(e+1)}_{e-1, 5} = B - \#2; 5 \text{ of type a.1 gives rise to} \)
  \[ V^{(e+2)}_{e-1, 5} = V - \#1; 1 \text{ of type a.1 (exo-genetic recursion),} \]
  \[ V^{(e+2)}_{e-1, 5} = V - \#1; 2 \text{ of type D.5, with } i = 5 \text{ (e odd) or } i = 6 \text{ (e even),} \]
  \[ V^{(e+2)}_{e-1, 8} = V - \#1; 3 \text{ of type D.5, only for even } e. \]
Figures 8 and 9 provide a graphical illumination of the statements in Theorem 4 and 5. The periodicity of length two is indicated by roman numerals (I) and (II) at identification points.

**Figure 8.** Details of exo- and endo-genetic propagation at a bifurcation \((e \geq 3\) odd)

**Figure 9.** Details of exo- and endo-genetic propagation at a bifurcation \((e \geq 4\) even)
Eventually, we state the main theorem as the coronation of the present article.

**Theorem 6. (Exhaustion theorem)** Due to an infinite chain of periodic bifurcations, the \( p \)-descendant tree \( T_p(R) \) of the metabelian root \( R = \langle 729, 7 \rangle \) with abelianization \( R/R' \cong C_{27} \times C_3 \) includes as subsets, for every commutator quotient \( C_{3^e} \times C_3 \) with logarithmic exponent \( e \geq 3 \), all depth-pruned coclass trees \( T^e(M_1^{(e)}) \) of CF-groups with rank distribution \( \varphi \sim (223; 3) \) and all metabelian skeletons of depth-pruned coclass trees \( T^{e+1}(M_1^{(e+1)}) \) of BCF-groups with rank distribution \( \varphi \sim (223; 3) \). The former are of type a.1, \( \kappa = (000; 0) \), the latter of type d.10, \( \kappa = (110; 2) \). The depth-pruning process eliminates all vertices with depth \( dp \geq 2 \).

We point out that we cannot speak about subtrees, because the coclass trees are completely disconnected as subgraphs of \( p \)-descendants in the finite region behind the shock wave. The coclass trees are not subtrees of \( T_p(R) \) (the problem are the different edges, not the vertices).

**Proof.** Let \( e \geq 3 \) be the logarithmic exponent of an assigned non-elementary bicyclic commutator quotient \( C_{3^e} \times C_3 \).

First, we show that all vertices of the CF coclass tree \( T^e(M_1^{(e)}) \) are \( p \)-descendants of the root \( M_1^{(3)} \cong \langle 729, 7 \rangle \).

- Vertices ahead of the shock wave, with class \( c > e \), are constructed as regular descendants with endo-genetic propagation by iteration of Formula (3.4) along the main line, and a single application of Formula (3.8) for vertices off main line.
- For \( e \geq 4 \), vertices on the shock wave, with class \( c = e \), are constructed as singular \( p \)-descendants with exo-genetic propagation by a single application of Formula (3.3), if they are main line, and Formula (3.7), if they are offside with bicyclic centre. If they are offside with cyclic centre, they are constructed as regular \( p \)-descendants with endo-genetic propagation by Formula (3.9).
- For \( e \geq 4 \), all roots of CF coclass trees, with class \( c = 3 \), are constructed as irregular \( p \)-descendants with exo-genetic propagation by a single application of Formula (3.2).
- In the case \( e \geq 5 \), vertices behind the shock wave, with class \( 3 < c < e \), are constructed as irregular \( p \)-descendants with exo-genetic propagation by iteration of Formula (3.2), if they are main line, and Formula (3.6), if they are offside with bicyclic centre. If they are offside with cyclic centre, they are constructed as regular \( p \)-descendants with endo-genetic propagation by Formula (3.9).

Second, we show that all vertices of the BCF coclass tree \( T^{e+1}(M_1^{(e+1)}) \) are also \( p \)-descendants of the same root \( M_1^{(3)} \cong \langle 729, 7 \rangle \).

- Vertices ahead of the shock wave, with class \( c > e + 1 \), are constructed as regular descendants with endo-genetic propagation by iteration of Formula (4.4) along the main line, followed by a single application of the same Formula (4.4) for vertices off main line.
- Vertices on the shock wave, with class \( c = e + 1 \), are constructed as singular \( p \)-descendants with exo-genetic propagation by a single application of Formula (4.3).
- All roots of BCF coclass trees, with class \( c = 3 \), are constructed as irregular \( p \)-descendants with exo-genetic propagation by a single application of Formula (4.2).
- In the case \( e \geq 4 \), vertices behind the shock wave, with class \( 3 < c < e + 1 \), are constructed as irregular \( p \)-descendants with exo-genetic propagation by a single application of Formula (4.2).

By the preceding distinction of cases, all claimed metabelian depth-pruned vertices are exhausted.
The **Exhaustion Theorem** can be viewed from another perspective: instead of recursion formulas, completely explicit instructions are given for the construction of vertices on coclass trees of CF-groups and BCF-groups. Assume \( e_0 \) is a starting exponent and \( e \geq e_0 \) is a variable exponent.

For \( e_0 \geq 3 \) odd,

CF-groups are constructed as vertices on mainlines of type a.1,

\[
M^{(e)}_{e_0 - 2} = M^{(e_0)}_{e_0 - 2} [-\#1; 1]^{e - e_0},
\]

offside vertices of types b.3 and a.1 with cyclic centre,

\[
V_{e_0 - 1,j + 3}^{(e)} = M^{(e_0)}_{e_0 - 2} [-\#1; 1]^{e - e_0} - \#1; j, \ e \geq e_0 + 1, \ j \in \{3, 4\},
\]

and offside vertices of types b.16, a.1 twig, and two a.1 with bicyclic centre,

\[
V_{e_0 - 1,t}^{(e)} = M^{(e_0)}_{e_0 - 2} - \#2; t[-\#1; 1]^{e - (e_0 + 1)} - \#1; 2, \ \ e \geq e_0 + 1, \ t \in \{2, 3, 4, 5\};
\]

BCF-groups are constructed as vertices on mainlines of type d.10,

\[
M^{(e + 1)}_{e_0 - 2} = M^{(e_0)}_{e_0 - 2} [-\#1; 1]^{e - e_0} - \#1; 2,
\]

and offside vertices of types D.10, B.2, C.4, D.5,

\[
V_{e_0 - 1,t}^{(e + 1)} = M^{(e_0)}_{e_0 - 2} - \#2; t[-\#1; 1]^{e - (e_0 + 1)} - \#1; 2, \ \ e \geq e_0 + 1, \ t \in \{2, 3, 4, 5\}.
\]

For \( e_0 \geq 4 \) even,

CF-groups are constructed as vertices on mainlines of type a.1,

\[
M^{(e)}_{e_0 - 2} = M^{(e_0)}_{e_0 - 2} [-\#1; 1]^{e - e_0},
\]

offside vertices of types b.3, a.1, b.3, and a.1 with cyclic centre,

\[
V_{e_0 - 1,j + 3}^{(e)} = M^{(e_0)}_{e_0 - 2} [-\#1; 1]^{e - e_0} - \#1; j, \ e \geq e_0 + 1, \ j \in \{3, 4, 5, 6\},
\]

and offside vertices of types b.16, a.1 twig, and two a.1 with bicyclic centre,

\[
V_{e_0 - 1,t}^{(e)} = M^{(e_0)}_{e_0 - 2} - \#2; t[-\#1; 1]^{e - (e_0 + 1)} - \#1; 2, \ \ e \geq e_0 + 1, \ t \in \{2, 3, 4, 5\};
\]

BCF-groups are constructed as vertices on mainlines of type d.10,

\[
M^{(e + 1)}_{e_0 - 2} = M^{(e_0)}_{e_0 - 2} [-\#1; 1]^{e - e_0} - \#1; 2,
\]

and offside vertices of types D.10, B.2, C.4, D.5,

\[
V_{e_0 - 1,t}^{(e + 1)} = M^{(e_0)}_{e_0 - 2} - \#2; t[-\#1; 1]^{e - (e_0 + 1)} - \#1; 2, \ \ e \geq e_0 + 1, \ j \in \{3, 4\}, \ t \in \{2, 3, 4, 5\},
\]

where

\[
k(j) = \begin{cases} 
  j & \text{if } t = 2, \\
  4 & \text{if } t = 3, \ j = 2, \\
  7 & \text{if } t = 3, \ j = 3, \\
  5 & \text{if } t = 4, \ j = 2, \\
  9 & \text{if } t = 4, \ j = 3, \\
  6 & \text{if } t = 5, \ j = 2, \\
  8 & \text{if } t = 5, \ j = 3.
\end{cases}
\]
6. Extension and unification of excited states

The results concerning periodic non-metabelian Schur $\sigma$-groups $G$ with moderate rank distribution $\varrho(G)$ and types D.10, C.4, D.5 in [14,16,17] can be restated, extended, and unified in the terminology and notation of the present article. Periodic chains of both step sizes $s \in \{1,2\}$ must be employed, bifurcations with step size $s = 2$ for the selection of excited states $n \geq 0$, and chains with step size $s = 1$ for growing commutator quotients with logarithmic exponents $e \geq 2$.

6.1. Ground state. Pairs of periodic Schur $\sigma$-groups for the ground state, $n = 0$, were discovered in [14] § 9, Thm. 12, Eqn. (9.1)–(9.3)]. For each of the types D.10, C.4, D.5, determined by the fixed parameter $t \in \{2,4,5\}$, they were given by the sequence of doublets $G = G(e,i) = (3^8,93) - #2; t[-#1;1]^{e-5} - #1; i - #1; 1$ with running parameter $e \geq 5$ and selector $i \in \{2,3\}$.

The constitution by an infinite main trunk and finite twigs was illuminated more closely in [17] § 4, Thm. 3–6, Eqn. (11)–(22)]. For each $t \in \{2,4,5\}$, a periodic chain of CF-groups $T_e = (3^8,93) - #2; t[-#1;1]^{e-5}$ with $e \geq 5$ forms the trunk of type b.16 for $t = 2$, and of type a.1 for $t \in \{4,5\}$. Each of these vertices $T_e$ gives rise to a finite double twig of depth two, consisting of BCF-groups, the metabelianizations $M_{e,i} = T_e - #1; i \simeq G_{e,i}/G_{e,i}'$ with $i \in \{2,3\}$ in depth one, and the Schur $\sigma$-groups $G_{e,i} = M_{e,i} - #1; 1$ in depth two. The type is D.10 for $t = 2$, C.4 for $t = 4$, and D.5 for $t = 5$.

In both previous papers [14,17], a connection between the ground state and branches of coclass trees is missing. The completely explicit notation of the present article admits the following restatement of all facts concerning the ground state.

**Theorem 7.** The metabelianizations of the ground state of Schur $\sigma$-groups with type D.10 for $t = 2$, C.4 for $t = 4$, and D.5 for $t = 5$ are given by

\[
V^{(e+1)}_{3,k(j)} = V^{(e)}_{3,t} - #1; j \quad \text{with} \quad j \in \{2,3\}, \quad V^{(e)}_{3,t} = (3^8,93) - #2; t[-#1;1]^{e-5},
\]

for each $e \geq 5$. The subscript $k(j)$ is given by Formula (5.21). $V^{(e)}_{3,t}$ belongs to the second branch $B(M^{(e)}_{2})$ of the CF-coclass tree $T^{e}(M^{(e)}_{1})$, and $V^{(e+1)}_{3,k(j)}$ belongs to the second branch $B(M^{(e+1)}_{2})$ of the BCF-coclass tree $T^{e+1}(M^{(e+1)}_{1})$. The Schur $\sigma$-group $V^{(e+1)}_{3,k(j)} - #1; 1$ has soluble length three.

**Remark 5.** So the new insight in comparison to [14,17] is that the endo-genetic propagation behind the shock wave establishes a branchwise mapping $V^{e}_{3,t} \mapsto (V^{(e+1)}_{3,k(2)}, V^{(e+1)}_{3,k(3)})$ from the CF-coclass tree $T^{e}(M^{(e)}_{1})$ to the BCF-coclass tree $T^{e+1}(M^{(e+1)}_{1})$, for each $e \geq 5$.

**Proof.** With respect to Schur $\sigma$-groups as possible descendants, only distinguished CF-mainline vertices $M^{(e)}_{e,2}$ with even coclass $e \geq 4$ are relevant. For the ground state, we need the smallest even bifurcation $M^{(4)}_{2}$ with $e = 4$ and exo-genetic offside $p$-descendants $V^{(5)}_{3,t} = M^{(4)}_{2} - #2; t$ with types b.16 for $t = 2$, and a.1 for $t \in \{4,5\}$, each of them root of a periodic chain with step size $s = 1$, namely $V^{(e)}_{3,t} = M^{(4)}_{2} - #2; t[-#1;1]^{e-5}$ for $e \geq 5$, according to Formula (5.18). These CF-groups give rise to pairs of BCF-groups as endo-genetic $p$-descendants $V^{(6)}_{3,k(j)} = V^{(5)}_{3,t} - #1; j$, and more generally, for $e \geq 5$, $V^{(e+1)}_{3,k(j)} = V^{(e)}_{3,t} - #1; j$ with $j \in \{2,3\}$, according to Formula (5.24). In the SmallGroups library [2], $M^{(4)}_{2}$ has the absolute identifier $(3^8,93)$, which completes the proof. □
6.2. First excited state. Pairs of periodic Schur $\sigma$-groups for the first excited state, $n = 1$, were discovered in [10] § 2, Thm. 2, Eqn. (2)–(4). For each of the types D.10, C.4, D.5, determined by the fixed parameter $\ell \in \{2,4,5\}$, they were given by the sequence of doublets $G = G(e,i) \simeq M(e,i)\langle -\#1;1\rangle^2$ with metabelianization $M = G(e,i)/G(e,i)^\prime = \simeq W_t\langle -\#1;1\rangle^e - \#1;1,\ell$, where $e \geq 7$, $i \in \{2,3\}$, and $W_t = (3^8,93)[-\#2;1]^2 - \#2;\ell$.

The constitution by an infinite main trunk and finite twigs was illuminated more closely in [17] § 5, Thm. 8–10, Eqn. (27)–(38)]. For each $t \in \{2,4,5\}$, a periodic chain of CF-groups $T_e = (3^8,93)[-\#2;1]^2 - \#2;2t[-\#1;1]e^{-7} - \#1;1,\ell$ with $e \geq 7$ forms the trunk of type b.16 for $t = 2$, and of type a.1 for $t \in \{4,5\}$. Each of these vertices $T_e$ gives rise to a finite double twig of depth three, consisting of BCF-groups, the metabelianizations $M_{e,i} = T_e - \#1;i \simeq G_{e,i}/G_{e,i}^{\prime\prime}$ with $i \in \{2,3\}$ in depth one, and the Schur $\sigma$-groups $G_{e,i} = M_{e,i}[\#1;1]^2$ in depth three. The type is D.10 for $t = 2$, C.4 for $t = 4$, and D.5 for $t = 5$.

As before, in both previous papers [10] [17], a connection between the first excited state and branches of coclass trees is missing. Again, the completely explicit notation of the present article admits the following restatement of all facts concerning the first excited state.

**Theorem 8.** The metabelianizations of the first excited state of Schur $\sigma$-groups with type D.10 for $t = 2$, C.4 for $t = 4$, and D.5 for $t = 5$ are given by

&
(6.2) \ \forall_{5,k(j)}(e) = V_{5,t}^{(e)} - \#1; j \ \text{with} \ j \in \{2,3\}, \ V_{5,t}^{(e)} = (3^8,93)[-\#2;1]^2 - \#2;2t[-\#1;1]e^{-7},

for each $e \geq 7$. The subscript $k(j)$ is given by Formula (5.21). $V_{5,t}^{(e)}$ belongs to the fourth branch $B(M_4^{(e)})$ of the CF-coclass tree $T^e(M_1^{(e)})$, and $V_{5,k(j)}^{(e+1)}$ belongs to the fourth branch $B(M_4^{(e+1)})$ of the BCF-coclass tree $T^{e+1}(M_1^{(e+1)})$. The Schur $\sigma$-group $V_{5,k(j)}^{(e+1)}[\#1;1]^2$ has soluble length three.

**Remark 6.** Again, the new insight in comparison to [10] [17] is that the endo-genetic propagation behind the shock wave establishes a branchwise mapping $V_{5,t}^{(e)} \to (V_{5,k(2)}^{(e+1)}, V_{5,k(3)}^{(e+1)})$ from the CF-coclass tree $T^e(M_1^{(e)})$ to the BCF-coclass tree $T^{e+1}(M_1^{(e+1)})$, for each $e \geq 7$.

**Proof.** For the first excited state, we need the next event bifurcation $M_4^{(e)}$ with $e = 6$ and endo-genetic offside $p$-descendants $V_{5,t}^{(e)} = M_4^{(e)} - \#2;2t$ with types b.16 for $t = 2$, and a.1 for $t \in \{4,5\}$, according to Formula (5.18). These CF-groups give rise to pairs of BCF-groups as endo-genetic $p$-descendants $V_{5,k(j)}^{(e)} = V_{5,t}^{(e)} - \#1; j$ with $j \in \{2,3\}$, according to Formula (5.21). In order to start within the SmallGroups database [2], we observe that $M_4^{(6)} = M_2^{(4)}[-\#2;1]^2$, where $M_2^{(4)} \simeq (3^8,93)$. \hfill $\square$

6.3. nth excited state. Now we can easily extend the previous results by generalization to the $n$-th excited state for $n \geq 2$. For the sake of completeness, we include $n = 0$ and $n = 1$.

**Theorem 9.** The metabelianizations of the $n$th excited state of Schur $\sigma$-groups with type D.10 for $t = 2$, C.4 for $t = 4$, and D.5 for $t = 5$ are given by

&
(6.3) \ \forall_{3+2n,k(j)}^{(e+1)} = V_{3+2n,t}^{(e)} - \#1; j \ \text{with} \ j \in \{2,3\}, \ V_{3+2n,t}^{(e)} = (3^8,93)[-\#2;1]^{2n} - \#2;2t[-\#1;1]^e(-5+2n),

for each $e \geq 5+2n$. The subscript $k(j)$ is given by Formula (5.21). $V_{3+2n,t}^{(e)}$ belongs to the $(2n+1)$-th branch $B(M_2^{(e)})$ of the CF-coclass tree $T^e(M_1^{(e)})$, and $V_{3+2n,k(j)}^{(e+1)}$ belongs to the $(2n+1)$-th branch $B(M_2^{(e+1)})$ of the BCF-coclass tree $T^{e+1}(M_1^{(e+1)})$. The Schur $\sigma$-group $V_{3+2n,k(j)}^{(e+1)}[\#1;1]^{n+1}$ has soluble length three.

**Proof.** By induction with respect to the excited state $n \geq 2$, using Theorem 7 for $n = 0$ and Theorem 8 for $n = 1$ as induction hypothesis. \hfill $\square$
In the proofs of Theorem 1 and 3, we had to exclude the investigation of parents $A = \pi(D) = D/\gamma_e(D)$ of the roots $D = M_1^{(e)}$ respectively $D = M_1^{(e+1)}$ of coclass trees for $e \geq 2$. In Lemma 1, we construct a periodic chain with step size $s = 1$, which consists precisely of these parents. Since the distinction between CF- and BCF-groups begins with class three, the parents $\pi(D)$ are neither CF nor BCF but simply class two.

**Lemma 1. (Unbounded extensible 3-groups of class 2)**

For each logarithmic exponent $e \geq 2$, the unique infinitely capable 3-group $B_e$ of class 2 and type a.1, $\varpi = (000; 0)$, with commutator quotient $C_3 \times C_3$ is given as the following member of a periodic chain with step size $s = 1$. It is parent of both, $M_1^{(e)}$ and $M_1^{(e+1)}$.

\[ (7.1) \quad B_e := B[−#1; 1]^{e−2}, \quad \pi(M_1^{(e)}) = B_e, \quad \pi(M_1^{(e+1)}) = B_e, \]

where $B = B_2 \cong \text{SmallGroup}(81, 3)$ denotes the root of the chain.

**Example 1.** Aside from the root $B_2 \cong (81, 3)$, the SmallGroups database [2] also contains $B_3 \cong (243, 12)$, $B_4 \cong (729, 61)$, $B_5 \cong (2187, 315)$, and $B_6 \cong (6561, 2063)$. As the elementary analogue, we can view the extra special group $B_1 \cong (27, 3)$ with commutator quotient $C_3 \times C_3$.

**Proof.** With our usual convention $s_2 = [y, x]$ for the main commutator of a finite two-generated 3-group $G = \langle x, y \rangle$, a parametrized pc-presentation of all members of the chain is given by

\[ (7.2) \quad B_e = \langle x, y \mid x^{3^{e−1}} = w, \quad w^3 = 1, \quad y^3 = 1, \quad s_2^e = 1 \rangle. \]

Whereas the nilpotency class of all members is constant $\text{cl}(B_e) = 2$, the $p$-class $c_p = \text{cl}(B_e) = e$ depends on the logarithmic exponent $e$. Since the last non-trivial lower exponent-$p$ central is $P_{e−1}(B_e) = \langle w \rangle$, it follows that $\pi_p(B_e) = B_e/P_{e−1}(B_e) \cong B_{e−1}$ for $e \geq 3$. Actual computation with Magma [10] shows that $B_e = B_{e−1} − #1; 1$ for $e \geq 3$, and thus by induction $B_e = B_2(−#1; 1)^{e−2}$ for $e \geq 2$.

Now we come to the justification of the parent relations. First observe that Formula (7.2) degenerates to

\[ (7.3) \quad M_1^{(e)} = \langle x, y \mid x^{3^{e−1}} = w, \quad w^3 = 1, \quad y^3 = 1, \quad s_2^e = s_3^e = 1, \quad s_4 = t_3, \quad s_4 = t_4 = 1 \rangle. \]

in the special case of the root with class $c = 3$, for each $e \geq 2$. We put $D = M_1^{(e)}$. For $e \geq 4$, we are in the irregular region behind the shock wave, and we have $c = 3$, $c_p = e$, $\gamma_3(D) = \langle s_3 \rangle$ and $P_{e−1}(D) = \langle w \rangle$, whence $A = \pi(D) = D/\gamma_3(D) \cong B_3$, as claimed, and $A_p = \pi_p(D) = D/P_{e−1}(D) \cong M_1^{(e−1)}$, as known from Formula (7.2). For $e = 3$, the behavior on the shock wave is singular, i.e. $c = c_p = 3$, but $\gamma_3(D) = \langle s_3 \rangle$ as opposed to $P_2(D) = \langle s_3, w \rangle$. Thus $A = \pi(D) = D/\gamma_3(D) \cong B_3$, but $A_p = \pi_p(D) = D/P_2(D) \cong B_2$, due to bifurcation. For $e = 2$, the situation is regular (ahead of the shock wave), i.e. $c = c_p = 3$, $\gamma_3(D) = P_2(D) = \langle s_3 \rangle$ and $A_p = \pi_p(D) = A = \pi(D) = D/\gamma_3(D) \cong B_2$.

On the other hand, note that Formula (7.2) degenerates to

\[ (7.4) \quad M_1^{(e+1)} = \langle x, y \mid x^{3^e} = w, \quad w^3 = 1, \quad y^3 = 1, \quad s_2^e = s_3^e = 1, \quad t_3 = s_3 w, \quad s_4 = t_4 = 1 \rangle. \]

in the special case of the root with class $c = 3$, for each $e \geq 2$. We put $D = M_1^{(e+1)}$. Then we have $c = 3$, $c_p = e+1$, $\gamma_3(D) = \langle s_3, t_3 \rangle = \langle s_3, w \rangle$ and $P_e(D) = \langle w \rangle$, whence $A = \pi(D) = D/\gamma_3(D) \cong B_e$, as claimed, and $A_p = \pi_p(D) = D/P_e(D) \cong M_1^{(e)}$, as known from Formula (7.2).
8. Conclusion

In a series of preceding papers \[14, 15, 16, 17\], we have developed a new theory of finite 3-groups \(G\) with bicyclic commutator quotient \(G/G' \simeq C_3^\times C_3\) having one non-elementary component with logarithmic exponent \(e \geq 2\). Theoretical foundations were based on two invariants of \(G\) with respect to its four maximal subgroups \(H_1, \ldots, H_3; H_4\) (with distinguished \(H_4\), the abelian quotient invariants \((\text{AQI})\) \(\alpha(G) = (H_i/H_i')_{i=1}^4\) and the punctured transfer kernel type \((\text{pTKT})\) \(\varphi(G) = (\ker(T_i))^4_{i=1}\), combined in the Artin pattern \(\text{AP}(G) = (\alpha(G), \varphi(G))\)).

The primary motivation for these works was the application to possible automorphism groups \(\text{Gal}(\mathbb{F}_3^\omega(K)/K)\) of 3-class field towers over imaginary quadratic number fields \(K = \mathbb{Q}(\sqrt{d})\), \(d < 0\), which must be \textit{Schur \(\sigma\)-groups} (with balanced presentation and generator inverting \((\text{GI})\) automorphism). In the justification of newly discovered periodicities among such groups, two strange phenomena attracted our vigilance and attention:

- cumbersome difficulties in the construction of groups with small nilpotency class \(\text{cl}(G) \leq e\),
- unexpected connections and relationships between \(\text{CF}\)-groups \([1]\) and \(\text{BCF}\)-groups \([13]\).

In the present article, we abandoned all motivations by algebraic number theory and class field theory, we removed the focus on Schur groups and even on \(\sigma\)-groups (except in \(\S\ 6\)), and we solved the above mentioned two problems completely for two infinite families of coclass trees \([11, 12]\), one, \(T_\epsilon(M_{1\epsilon}^{(3)})\), consisting of \(\text{CF}\)-groups and mainline of type a.1, the other, \(T_{\epsilon+1}(M_{1\epsilon+1}^{(3)})\), consisting of \(\text{BCF}\)-groups and mainline of type d.10, and unbounded \(e \geq 3\) in both situations.

The first difficulty is explained by shedding new light on the \textit{commutator structure} and \textit{power structure} and their impact on the descending central series, the lower exponent-\(p\) central series, and the \(p\)-group generation algorithm \([19, 20, 7, 6]\) (also called \textit{extension algorithm} in \([1]\)).

The second phenomenon is due to closely related \textit{power-commutator-presentations} for certain \(\text{CF}\)-groups and \(\text{BCF}\)-groups, the \textit{mainline principle} for the generator of the last non-trivial lower central \(\gamma_c(G) = \langle s_c \rangle\), and peculiarities of the last non-trivial lower \(p\)-central \(P_{e-1}(G) = \langle w \rangle\).

The marvellous and astonishing statement of Theorems 1 and 2 is the constructibility of all vertices \(V_i^{(3)}\), \(i \geq 2\), on infinitely many \(\text{CF}\) coclass trees \(T_\epsilon(M_{1\epsilon}^{(3)})\), \(e \geq 3\), of type a.1, \(\kappa = (0, 0, 0; 0)\), with rank distribution \(\varrho \sim (2, 2, 3; 3)\), as descendants of a single root \(M_{1}^{(3)} = \langle 729, 7 \rangle\), which is the analogue of \textit{Ascione’s CF-group A} for the commutator quotient \((27, 3)\). The highlight of this work, completely unexpected up to now, asserts the constructibility of all vertices \(V_i^{(e+1)}\), \(i \geq 2\), on infinitely many \(\text{BCF}\) coclass trees \(T_{\epsilon+1}(M_{1\epsilon+1}^{(3)})\), \(e \geq 3\), of type d.10, \(\kappa = (1, 1, 0; 2)\), also with rank distribution \(\varrho \sim (2, 2, 3; 3)\), as descendants of the same CF-root \(M_{1}^{(3)} = \langle 729, 7 \rangle\), according to Theorem 3.

9. Outlook

In view of future research, it should be pointed out that three similar theorems can be proved for the root \(M_{1}^{(3)} = \langle 729, 6 \rangle\), the analogue of \textit{Ascione’s CF-group G} for the commutator quotient \((27, 3)\), which gives rise to infinitely many \(\text{CF}\) coclass trees \(T_\epsilon(M_{1\epsilon}^{(3)})\), \(e \geq 3\), of the same type a.1, \(\kappa = (0, 0, 0; 0)\), and to infinitely many \textit{pairs of BCF coclass trees} \(T_{\epsilon+1}(M_{1\epsilon+1}^{(3)})\) and \(T_{\epsilon+1}(M_{1\epsilon+1}^{(3)})\), \(e \geq 3\), of type e.14, \(\kappa = (1, 2, 3; 0)\), all three with a distinct rank distribution \(\varrho \sim (2, 2, 2; 3)\). As opposed to the trees in the present article, the \textit{polarization} for all these trees coincides with the \textit{puncture} at the fourth component.

Since the main line of a coclass tree, \(T_{\epsilon}(M_{1\epsilon}^{(3)})\) respectively \(T_{\epsilon+1}(M_{1\epsilon+1}^{(3)})\), gives rise to an infinite projective limit of the same coclass, \(M_{\infty}^{(3)} = \lim_{\epsilon} M_{1\epsilon}^{(3)}\) respectively \(M_{\infty}^{(e+1)} = \lim_{\epsilon} M_{1\epsilon+1}^{(e+1)}\), it would be interesting to investigate whether \(M_{\infty}^{(3)}\) “generates” all limit groups \(M_{\infty}^{(e)}\) with \(e \geq 4\) and \(M_{\infty}^{(e+1)}\) with \(e \geq 3\), in some sense.
References

[1] J. A. Ascione, G. Havas, and C. R. Leedham-Green, *A computer aided classification of certain groups of prime power order*, Bull. Austral. Math. Soc. **17** (1977), 257–274, microfiche supplement p. 320.

[2] H. U. Besche, B. Eick, and E. A. O’Brien, *The SmallGroups Library — a Library of Groups of Small Order*, 2005, an accepted and refereed GAP package, available also in MAGMA.

[3] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[4] W. Bosma, J. J. Cannon, C. Fieker, A. Steels (eds.), *Handbook of Magma functions*, Ed. 2.26, Sydney, 2021.

[5] B. Eick, C. R. Leedham-Green, M. F. Newman, and E. A. O’Brien, *On the classification of groups of prime-power order by coclass: The 3-groups of coclass 2*, Int. J. Algebra Comput. **23** (2013), 1243–1288.

[6] G. Gamble, W. Nickel, and E. A. O’Brien, *ANU p-Quotient — p-Quotient and p-Group Generation Algorithms*, 2006, an accepted GAP package, available also in MAGMA.

[7] D. F. Holt, B. Eick, and E. A. O’Brien, *Handbook of computational group theory*, Discrete mathematics and its applications, Chapman and Hall/CRC Press, Boca Raton, 2005.

[8] R. James, *The groups of order $p^6$ ($p \geq 3$)*, Ph. D. Thesis, Univ. of Sydney, 1968.

[9] R. James, *The groups of order $p^6$ ($p$ an odd prime)*, Math. Comp. **34**, no. 150, 613–637.

[10] MAGMA Developer Group, *MAGMA Computational Algebra System*, Version 2.26-10, Univ. Sydney, 2021, [http://magma.maths.usyd.edu.au](http://magma.maths.usyd.edu.au).

[11] D. C. Mayer, *Periodic bifurcations in descendant trees of finite $p$-groups*, Adv. Pure Math. **5** (2015), No. 1, 162–195, DOI 10.4236/apm.2015.54020, Special Issue on Group Theory, March 2015.

[12] D. C. Mayer, *Artin transfer patterns on descendant trees of finite $p$-groups*, Adv. Pure Math. **6** (2016), No. 2, 66–104, DOI 10.4236/apm.2016.62008, Special Issue on Group Theory Research, January 2016.

[13] D. C. Mayer, *Modeling rooted $m$-trees by finite $p$-groups*, Chapter 5, pp. 85–113, in the Open Access Book *Graph Theory — Advanced Algorithms and Applications*, Ed. B. Sirmacek, InTech d.o.o., Rijeka, January 2018, DOI 10.5772/intechopen.68703.

[14] D. C. Mayer, *Bicyclic commutator quotients with one non-elementary component*, [arXiv:2108.10754](http://arxiv.org/abs/2108.10754).

[15] D. C. Mayer, *BCF-groups with elevated rank distribution*, [arXiv:2110.03558](http://arxiv.org/abs/2110.03558).

[16] D. C. Mayer, *First excited state with moderate rank distribution*, [arXiv:2110.06511](http://arxiv.org/abs/2110.06511).

[17] D. C. Mayer, *Periodic Schur $\sigma$-groups of non-elementary bicyclic type*, [arXiv:2110.13880](http://arxiv.org/abs/2110.13880).

[18] B. Nebelung, *Klassifikation metabelscher 3-Gruppen mit Faktorkommutatorgruppe vom Typ (3, 3) und Anwendung auf das Kapitulationsproblem*, Inauguraldissertation, Universität zu Köln, 1989.

[19] M. F. Newman, *Determination of groups of prime-power order*, pp. 73–84 in: Group Theory, Canberra, 1975, Lecture Notes in Math., Vol. **573** (1977), Springer, Berlin.

[20] E. A. O’Brien, *The $p$-group generation algorithm*, J. Symbolic Comput. **9** (1990), 677–698.