Abstract. A survey of goodness-of-fit and symmetry tests based on the characterization properties of distributions is presented. This approach became popular in recent years. In most cases the test statistics are functionals of $U$-empirical processes. The limiting distributions and large deviations of new statistics under the null hypothesis are described. Their local Bahadur efficiency for various parametric alternatives is calculated and compared with each other as well as with diverse previously known tests. We also describe new directions of possible research in this domain.

1. Introduction

This survey is dedicated to the statistical tests based on characterizations. This is a relatively new idea which manifests growing popularity in the context of goodness-of-fit and symmetry testing. The idea to build goodness-of-fit tests using the characterizations of distributions belongs to Yu. V. Linnik \cite{47}. At the end of this wide-ranging paper he wrote: ”... one can raise the issue of the construction of goodness-of-fit tests for testing composite hypotheses based on the equal distribution of the two relevant statistics $g_1(x_1,...,x_r)$ and $g_2(x_1,...,x_r)$, and on the reduction of such question to the homogeneity tests.” This sentence was the guiding star which showed the researchers the right direction in the new and unexplored domain.

Currently, in the world literature there exist hundreds of various characterizations of probability distributions, see, e.g., \cite{37}, \cite{26}, \cite{40}, and \cite{38}. Many characterizations according to Linnik’s idea imply the corresponding statistical tests. Such tests are attractive because they reflect some intrinsic and hidden properties of probability distributions connected with the given characterization, and therefore can be more efficient or more robust than others.

Moreover, one should keep in mind that any hypothesis has to be tested with several possible criteria. The point of the matter is that with absolute confidence we can only reject it, while each new test which fails to reject the null-hypothesis gradually brings the statistician closer to the perception that this hypothesis is true. We find it pertinent to quote here the famous assertion by Einstein \cite{23}: ”No amount of experimentation can ever prove me right; a single experiment can prove me wrong.” Hence, we are interested in building new statistical tests based on novel ideas, specifically using the characterizations.

But the theory of such tests is intricate, and the study of their asymptotic properties including limiting behavior, and especially their asymptotic efficiency began only after 1990. Before that there existed few exceptions like the paper \cite{98}, of which later Mudholkar and Tian \cite{70} wrote: ”Vašicek (1976) was the first to recognize that the characterization results can be logical starting points for developing goodness-of-fit tests.”

2010 Mathematics Subject Classification 60F10, 62F03, 62G20, 62G30.

Key words and phrases. Characterization of distribution, goodness-of-fit, symmetry, $U$-statistics, Bahadur efficiency.
Probably these authors were unfamiliar with the seminal paper by Linnik cited above who was surely the first to propose the idea under discussion. In the abstract of the paper [29] published in 1993, Hashimoto and Shirahata proposed one of the first tests of fit based on characterizations and wrote: "However, since no test statistics based on characterizations are known, our test will be worth considered." This citation shows that in the beginning of 1990-s the tests based on characterizations were unusual and sparse. But since that time the state of affairs changed significantly. Numerous new tests based on characterizations were build, and their study gradually acquired the traits of a theory. We want to trace an outline of this theory and its main achievements within the last 25 years.

We begin by general constructions explaining the structure of tests used in this domain. Next we pass to concrete problems like testing of exponentiality, normality or symmetry, and describe the main developments of last period of time. We are mainly interested in the asymptotic efficiency of our tests though the results of power simulation are also possible and interesting. At the end of the paper we pose some problems and trace new directions of research. In most cases, the proofs are omitted, otherwise this survey would exceed the size of the paper in a journal.

2. \(U\)-statistics and \(U\)-empirical distributions

Let \(X_1, X_2, \ldots, X_n\) be i.i.d. observations with continuous df \(F\). We begin by testing the composite goodness-of-fit problem

\[ H_0 : F \in \mathcal{F}, \]

where \(\mathcal{F}\) is some family of df’s, against the alternative

\[ H_1 : F \notin \mathcal{F}. \]

Typical examples are testing exponentiality, normality or symmetry of a sample.

Next exposition will be based on \(U\)-statistics and their variants. Currently \(U\)-statistics play an important role in Statistics and Probability. They appeared in the middle of 1940-s in problems of unbiased estimation, but after the crucial paper of Hoeffding [32] it became clear that the numerous valuable statistics are just \(U\)-statistics (or von Mises functionals having very similar asymptotic theory.) The most complete exposition of this theory can be found in the monographs [39] and [42].

We consider \(U\)-statistics of the form

\[ U_n = \left( \frac{n}{m} \right)^{-1} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \Psi(X_{i_1}, \ldots, X_{i_m}), \quad n \geq m, \]

where \(X_1, X_2, \ldots\) is a sequence of i.i.d. rv’s with common df \(F\), while the kernel \(\Psi : \mathbb{R}^m \rightarrow \mathbb{R}^1\) is a measurable symmetric function of \(m \geq 1\) variables. The number \(m\) is called the degree of the kernel. We assume that the kernel \(\Psi\) is integrable on \(\mathbb{R}^m\) and denote

\[ \theta(F) := \int \ldots \int_{\mathbb{R}^m} \Psi(x_1, \ldots, x_m) dF(x_1) \ldots dF(x_m). \]

In the sequel we need the notations

\[ \psi(x) := \mathbb{E}_F \{ \Psi(X_1, \ldots, X_m) | X_1 = x \}, \quad \Delta^2 := \mathbb{E}_F \psi^2(X_1) - (\theta(F))^2. \]
The function $\psi$ is called the one-dimensional \textit{projection} of the kernel $\Psi$ and plays an important role in asymptotic theory. If $\Delta^2 > 0$ that specifies the so-called non-degenerate case, the limiting distribution of $U$-statistics is normal as discovered by Hoeffding [32]. He proved that if $\mathbb{E}_F \Psi^2(X_1, \ldots, X_m) < \infty$ and $\Delta^2 > 0$, then as $n \to \infty$ one has convergence in distribution
\[
\frac{\sqrt{n}}{m\Delta} (U_n - \theta(F)) \xrightarrow{d} N(0,1). \tag{1}
\]

Consider, in conformity with Linnik, the characterization of some probability law by the identical distribution of two statistics $g_1(X_1, \ldots, X_r)$ and $g_2(X_1, \ldots, X_s)$. The examples of such characterizations will be given further.

We can build two $U$-empirical df’s
\[
L^1_n(t) = \left( \frac{n}{r} \right)^{-1} \sum_{1 \leq i_1 < \ldots < i_r \leq n} 1\{g_1(X_{i_1}, \ldots, X_{i_r}) < t\}, \quad t \in \mathbb{R}^1,
\]
\[
L^2_n(t) = \left( \frac{n}{s} \right)^{-1} \sum_{1 \leq i_1 < \ldots < i_s \leq n} 1\{g_2(X_{i_1}, \ldots, X_{i_s}) < t\}, \quad t \in \mathbb{R}^1.
\]

The theory of $U$-empirical df’s was developed in 80-s, see, e.g. [31], [33], [39] and is similar to the theory of usual empirical df’s. By Glivenko-Cantelli theorem for $U$-empirical df’s we have (wp 1) as $n \to \infty$:
\[
L^1_n(t) \xrightarrow{d} L^1(t) := P(g_1(X_1, \ldots, X_r) < t),
\]
\[
L^2_n(t) \xrightarrow{d} L^2(t) := P(g_2(X_1, \ldots, X_s) < t).
\]

Under $H_0$ for large $n$ we have $L^1_n(t) \approx L^2_n(t)$ so that we can use this closeness for goodness-of-fit testing. Over much of this survey we consider two types of statistics: the integral one
\[
I_n = \int_{\mathbb{R}^1} (L^1_n(t) - L^2_n(t)) \, dF_n(t),
\]
where $F_n(t)$ is the usual empirical df, and of Kolmogorov type, namely
\[
D_n = \sup_{t \in \mathbb{R}^1} |L^1_n(t) - L^2_n(t)|.
\]

Such statistics can have rather different behavior depending on the type of characterization and underlying distribution, accordingly the statistical tests based on them can have distinct limiting properties, power and efficiency.

3. OUTLINE OF BAHADUR THEORY

Suppose that we want compare two sequences of statistics $I_n$ and $D_n$ by their asymptotic efficiency. Among many types of efficiencies, see [74 Ch.1], we select the Bahadur efficiency because, unlike Pitman efficiency, it can be calculated for statistics with non-normal limiting distribution. This is the primary reason to use it in the present context as the Kolmogorov type statistics have non-normal limiting distributions. Hodges-Lehmann efficiency has other drawbacks, in particular, it does not discriminate two-sided tests, see, e.g., [74 Ch.1]. In this section we shortly
describe main points of Bahadur theory, see the complete exposition in [12] and [13].

Let \( s = (X_1, X_2, \ldots) \) be a sequence of i.i.d. rv’s with the distribution \( P_\theta, \theta \in \Theta \), on \((X, A)\). We are testing the null-hypothesis

\[ H_0 : \theta \in \Theta_0 \subset \Theta \subset \mathbb{R}^1 \]

against the alternative

\[ H_1 : \theta \in \Theta_1 \subset \Theta. \]

For this problem we use the sequence of test statistics \( T_n(s) = T_n(X_1, \ldots, X_n) \).

The Bahadur approach prescribes one to fix the power of concurrent tests and to compare the exponential rates of decrease of their sizes for the increasing number of observations and fixed alternative. This exponential rate for a sequence of statistics \( \{T_n\} \) is usually proportional to some non-random function \( c_T(\theta) \) depending on the alternative parameter \( \theta \in \Theta_1 \) which is called the exact slope of the sequence \( \{T_n\} \). The Bahadur asymptotic relative efficiency (ARE) \( e_{V,T}^B(\theta) \) of two sequences of statistics \( \{V_n\} \) and \( \{T_n\} \) is defined by means of the formula

\[ e_{V,T}^B(\theta) = c_V(\theta) / c_T(\theta). \]

The exact slope can be found by the fundamental theorem of Bahadur [12]:

**Theorem 1.** Suppose that the following two conditions hold:

a) \( T_n \overset{P_\theta}{\rightarrow} b(\theta), \quad \theta > 0, \)

where \(-\infty < b(\theta) < \infty\), and \( P_\theta \) denotes convergence in probability under \( P_\theta \).

b) \( \lim_{n \to \infty} n^{-1} \ln P_{H_0}(T_n \geq t) = -h(t) \)

for any \( t \) in an open interval \( I \), on which \( h \) is continuous and \( \{b(\theta), \theta > 0\} \subset I \). Then \( c_T(\theta) = 2 h(b(\theta)) \).

Often the exact Bahadur ARE is uncomputable for any alternative depending on \( \theta \) but it is possible to calculate the local Bahadur ARE as \( \theta \to \Theta_1 \) approaches the null-hypothesis. Then one speaks about the local Bahadur efficiency and local Bahadur exact slopes [74].

Let \( K(\theta, \Theta_0) \equiv K(P_\theta, P_{\Theta_0}) \) be the Kullback-Leibler distance between \( P_\theta \) and \( P_{\Theta_0} \), see, e.g., [13] or [99]. Put for any \( \theta \in \Theta_1 \)

\[ K(\theta, \Theta_0) := \inf \{ K(\theta, \theta_0) : \theta_0 \in \Theta_0 \}. \]

The Bahadur - Raghavachari inequality (the analog of Cramér - Rao inequality in testing), see [12], [74] states that for any \( \theta \in \Theta_1 \) one has

\[ c_T(\theta) \leq 2 K(\theta, \Theta_0). \]

Hence we may define the (absolute) local Bahadur efficiency of the sequence \( \{T_n\} \) by the formula

\[ ef_T = \lim_{\theta \to \partial \Theta_0} c_T(\theta)/2K(\theta, \Theta_0). \]

Only in exceptional cases

\[ c_T(\theta) = 2K(\theta, \Theta_0), \quad \forall \theta \in \Theta_1. \]
Therefore one can be interested in those $F = L(X_1)$ for which

$$f f_T = \lim_{\theta \to \partial \Theta_0} c_T(\theta)/2K(\theta, \Theta_0) = 1.$$ 

We call this property the *local optimality* in Bahadur sense. An interesting question is to describe those alternatives for which the considered tests are locally optimal in Bahadur sense. The idea ascends to Bahadur [13] but was developed by the author, see [75], [74, Ch.6] and subsequent papers, e.g., [80]. However, we leave this direction apart as it requires considerable space to enounce the obtained results.

First condition of Theorem 1 is a variant of the Law of Large Numbers and its verification is easy. On the contrary, the second condition of this theorem describes the rough (logarithmic) large deviation asymptotics of test statistics under the null-hypothesis and is non-trivial. To verify it, we often use the theorem on large deviations of $U$-statistics by Nikitin and Ponikarov [81]:

**Theorem 2.** Let $V_n$ be a sequence of $U$-statistics with centered, bounded and non-degenerate kernel $\Psi$. Then

$$\lim_{n \to \infty} n^{-1} \ln \mathbb{P}\{V_n \geq a\} = -\sum_{j=2}^{\infty} b_j a^j, \quad (2)$$

where the series with numerical coefficients $b_j$ converges for sufficiently small $a > 0$, and $b_2 = (2m^2\Delta^2)^{-1}$, where $\Delta^2$ is the variance of the projection of the kernel $\Psi$.

Large deviations for the supremum of the family of non-degenerate $U$-statistics $\sup_{t \in T} U_n(t)$, where $U_n(t)$ for each $t \in T$ is a $U$-statistic with the non-degenerate kernel $\Xi(X, Y; t)$ which corresponds to Kolmogorov type statistics, were studied in [77]. The result is similar to (2) but slightly more involved.

4. DESU’S CHARACTERIZATION AND CORRESPONDING TESTS OF EXPONENTIALITY

One of most simple characterizations of exponential distribution belongs to Desu, see [21]:

**Theorem 3.** Let $X$ and $Y$ be non-negative i.i.d. rv’s with df differentiable at zero. Then $X \overset{d}{=} 2 \min(X, Y)$ iff $X$ and $Y$ are exponentially distributed.

Using this characterization we will show how to build and analyze the corresponding tests of exponentiality.

Let $X_1, \ldots, X_n$ be i.i.d. observations with non-degenerate df $F$, and let $F_n$ be the corresponding empirical df. We are testing the composite hypothesis

$$H_0 : \text{the df of exponential law with the density } f(x) = \lambda e^{-\lambda x}, x \geq 0,$$

where $\lambda > 0$ is some unknown parameter, against the alternative $H_1$ under which the hypothesis $H_0$ is wrong.

In this case we need the $U$-statistical empirical df $H_n$ which is defined as

$$H_n(t) = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} 1\{2 \min(X_i, X_j) < t\}, \ t \geq 0.$$
We will study two statistics

\[ I_n = \int_0^\infty (F_n(t) - H_n(t))dF_n(t), \]

and

\[ D_n = \sup_{t \geq 0} |F_n(t) - H_n(t)|. \]

Clearly their distribution under the null-hypothesis does not depend on \( \lambda. \)

The statistic \( I_n \) is asymptotically equivalent to the \( U \)-statistic of degree 3 with the centered kernel

\[ \Psi(X, Y, Z) = \frac{1}{2} - \frac{1}{3} \{2 \min(X, Y) < Z\} - \{2 \min(Y, Z) < X\} - \{2 \min(X, Z) < Y\}. \]

The projection of this kernel is

\[ E[\Psi(X, Y, Z)|Z = t] := \psi(s) = \frac{1}{3} e^{-s} - \frac{1}{18} - \frac{4}{9} e^{-3s}, \]

and the variance of the projection is \( \Delta^2 := E\psi^2(Z) = \frac{11}{3780} \approx 0.003. \)

By Hoeffding’s theorem, see [32] we get

**Theorem 4.** Under the hypothesis \( H_0 \) one has convergence in distribution

\[ \sqrt{n}I_n \xrightarrow{d} N(0, 9\Delta^2), \quad \text{as} \quad n \to \infty. \]

As to the large deviations, in our case we get for \( a > 0 \)

\[ \lim_{n \to \infty} n^{-1} \ln P(I_n > a) = -f_I(a), \]

where the function \( f_I \) is continuous for sufficiently small \( a > 0 \), and, moreover,

\[ f_I(a) = \frac{210}{11} a^2 (1 + o(1)), \quad \text{as} \quad a \to 0. \]

By way of an example let calculate the local Bahadur efficiency of \( I_n \) for the Weibull alternative. This means that the alternative df of observations is

\[ F(x, \theta) = 1 - \exp(-x^{1+\theta}), x \geq 0, \theta \geq 0. \]

We find after some simple calculations that as \( \theta \to 0 \)

\[ c_I(\theta) \sim b_I(\theta)^2/(9\Delta^2) \sim 1.147\theta^2. \]

The Kullback-Leibler distance \( K(\theta) \) between \( H_0 \) and \( H_1 \) satisfies

\[ K(\theta) \sim \pi^2 \theta^2/12, \quad \theta \to 0. \]

The local Bahadur ARE of our test is consequently equal to

\[ eff(I) := \lim_{\theta \to 0} \frac{c_I(\theta)}{2K(\theta)} \approx 0.697. \]

Consider now the Kolmogorov-type statistic \( D_n \). The difference \( F_n(t) - H_n(t) \) is a family of \( U \)-statistics with the kernels depending on \( t \geq 0 : \)

\[ \Xi(X, Y; t) = \frac{1}{2} \{X < t\} + \{Y < t\} - \{2 \min(X, Y) < t\}. \]
The limiting distribution of the sequence $D_n$ is unknown. Critical values for statistics $D_n$ can be found via simulation.

In our case the family $\{\Xi(X, Y; t), t \geq 0\}$ from [3] is centered, bounded, non-degenerate and hence satisfies all conditions of Theorem 2.4 from [77] on large deviations of $U$-empirical Kolmogorov statistics. Therefore as $a > 0$ by [77]

$$\lim_{n \to \infty} n^{-1} \ln P(D_n > a) = -f_D(a),$$

where

$$f_D(a) = 2a^2(1 + o(1)), \text{ as } a \to 0.$$

Consider again the Weibull alternative. Arguments similar to the case of integral statistic, see [77], show that the local Bahadur efficiency of the sequence $D_n$ is equal to 0.158. We see that this efficiency is low and considerably smaller than in the integral case. It is a rule that Kolmogorov-Smirnov type statistics are less efficient than integral ones. There exist some exceptions but they are rare.

5. Tests of exponentiality based on characterizations

There are numerous characterizations of exponential law, probably more than of any other probability law, see, e.g., [5], [10], [14] and [26]. We consider only few typical examples where the tests of fit are build and studied.

5.1. Lack of memory property and corresponding tests. First, we mention the celebrated ”lack of memory” property which consists in that only the exponential distribution satisfies the functional equation in df’s

$$1 - F(x + y) - (1 - F(x))(1 - F(y)) = 0 \quad \forall x, y \geq 0.$$

Replacing $F$ by empirical df $F_n$, one obtains some empirical field, and the functionals of it can be used as test statistics for exponentiality, see as examples of many papers in this direction [1], [11] and [30].

The ”lack of memory” property can be simplified. Denote, following Angus [7], the class $D_1$ of right-continuous df’s $F$ with $F(0-) = 0$ and

$$\lim_{h \to 0} \frac{F(h) - F(0)}{h} = l \in [0, \infty].$$

Let $\bar{F}(x) = 1 - F(x)$. Angus used the following statement that belongs to Arnold and Gupta: the functional equation

$$\bar{F}(2x) = \bar{F}^2(x) \quad \forall x \geq 0$$

characterizes the exponential distribution in the class of such distributions in $D_1$ which are not concentrated at 0. He introduced a Kolmogorov type test based on this characterization and studied its properties in [7]. Later its local Bahadur efficiency against standard alternatives was calculated in [70] and [77]. It turned out to be rather low.
5.2. Characterizations based on order statistics. Another example is given by Riedel-Rossberg characterization in terms of order statistics, see [91]. Denote, as usually, $X_{k,n}$ the $k$-th order statistic in the sample of size $n$, $1 \leq k \leq n$. Then the following characterization holds.

**Theorem 5.** Two statistics $X_{2,3} - X_{1,3}$ and $\min(X_1, X_2)$ are identically distributed iff the sample $X_1, X_2, X_3$ consists of exponential rv's.

The construction of tests based on this characterization and their asymptotic analysis is performed similarly to the case of Desu characterization, see [100].

Next consider the Ahsanullah’s characterization. Suppose that the df $F$ belongs to the class of df’s $F_1$, where the failure rate function $f(t)/(1 - F(t))$ is monotone for $t \geq 0$. Ahsanullah [2] proved some characterizations of exponentiality within the class $F_1$. We consider here only one of his characterizations.

**Theorem 6.** Let $X$ and $Y$ be non-negative i.i.d. rv’s from class $F_1$. Then $|X - Y| \overset{d}{=} 2 \min(X, Y)$ iff $X$ and $Y$ are exponentially distributed.

Corresponding tests were build and analyzed by Nikitin and Volkova in [85].

5.3. Characterization of Arnold and Villaseñor. Recently Arnold and Villaseñor [9] expressed in the form of hypothesis the following characterization of exponentiality:

Let $X_1, X_2, \ldots$ be non-negative i.i.d. rv’s with the density $f$ having derivatives of all orders around zero. Then for any $k \geq 2$

$$\max(X_1, X_2, \ldots, X_k) \overset{d}{=} \frac{1}{k} \sum_{i=1}^{k} X_i$$

iff $f$ is exponential.

Arnold and Villaseñor were able to prove this hypothesis only for $k = 2$. Later Yanev and Chakraborty [105] proved it is true for $k = 3$ and later in [106] proved it for arbitrary $k$, see also [59]. Tests of exponentiality based on these characterizations and their efficiencies were studied in [35] and in [103].

Other tests based on characterizations of exponential distribution in terms of order statistics were build and studied in [54] and [58]. One can mention also the characterization of exponential law by the same distribution of $X$ and $|X - Y|$ where $X, Y$ are i.i.d. rv’s having absolute continuous distribution, see [89]. Some steps toward using it for testing were made in [77].

5.4. Table of efficiencies. Now we present a table of local Bahadur efficiencies of the majority of tests of exponentiality described above. We will compare them with well-known classic scale-free tests of exponentiality based on Greenwood statistic $R_n$, Moran statistic $M_n$ and Gini statistic $G_n$. We recall that

$$R_n = 2 - \frac{1}{n} \sum_{i=1}^{n} \left( \frac{X_i}{X} \right)^2, \quad M_n = \frac{1}{n} \sum_{i=1}^{n} \ln \left( \frac{X_i}{X} \right) + C, \quad G_n = \frac{\sum_{i,j=1}^{n} |X_i - X_j|}{2n(n-1)X},$$
where \( C \) denotes the Euler constant. We consider also the famous Lilliefors statistic \([43]\) which has the form
\[
L_i n = \sup_{x \geq 0} |1 - F_n(x) - e^{-x/X}|,
\]
and belongs to Kolmogorov type statistics with estimated parameters. On efficiencies of these statistics see \([39, 82, 95]\).

We consider the following standard alternatives against exponentiality:

i) Weibull alternative with the density
\[
(1 + \theta)x^\theta \exp(-x^{1+\theta}), \theta \geq 0, x \geq 0;
\]

ii) Makeham alternative with the density
\[
(1 + \theta(1 - e^{-x})) \exp(-x - \theta(e^{-x} - 1 + x)), \theta \geq 0, x \geq 0;
\]

iii) linear failure rate alternative with the density
\[
(1 + \theta x)e^{-x-\frac{\theta}{2}x^2}, \theta \geq 0, x \geq 0;
\]

Now let compare the values of local Bahadur efficiency for various statistics. All them are collected in Table 1 below and were calculated according to the approach developed above for the tests based on Desu characterization. The superscripts Ross and Ahs denote the statistics based on Riedel-Rossberg’s or Ahsanullah’s characterization.

**Table 1.** Local efficiencies of tests for exponentiality.

| Statistic | Alternative Weibull | Alternative Makeham | Alternative linear failure rate |
|-----------|---------------------|---------------------|-------------------------------|
| \( F_{n}^{Ross} \) | 0.650 | 0.450 | 0.119 |
| \( F_{n}^{Ahs} \) | 0.795 | 0.692 | 0.257 |
| Gini | 0.876 | 1 | 0.750 |
| Moran | 0.943 | 0.694 | 0.388 |
| Greenwood | 0.608 | 0.750 | 1 |
| \( D_{n}^{Ross} \) | 0.320 | 0.207 | 0.047 |
| \( D_{n}^{Ahs} \) | 0.450 | 0.470 | 0.187 |
| Angus | 0.158 | 0.187 | 0.073 |
| Lilliefors | 0.538 | 0.607 | 0.356 |

We see that our tests based on characterizations are competitive with respect to other tests of exponentiality, the more that the alternatives were taken almost at random. However, the Gini test reaffirms its high reputation.
6. Tests of normality

Characterizations of normality are also numerous and mathematically content-rich. They are described in [4], [37], [53], and [19], apart from many articles. We discuss here only few papers based on selected characterizations.

6.1. Polya characterization. One of first characterizations in the history of Statistics belongs to Polya [88].

Theorem 7. Let $X$ and $Y$ be i.i.d. centered rv’s. Then $X \overset{d}{=} (X + Y)/\sqrt{2}$ iff $X$ and $Y$ have the normal distribution with some positive variance.

The integral test of normality based on this property was proposed by Muliere and Nikitin, see [72]. Their statistic is asymptotically normal with the variance $9\delta^2$, where

$$
\delta^2 = \frac{13}{108} - \frac{4}{9\pi} \left( \arctan \sqrt{\frac{3}{5}} + \frac{1}{2} \arctan \frac{1}{\sqrt{7}} \right) \approx 1.571236 \cdot 10^{-3} > 0.
$$

The expression for the variance shows the non-trivial character of the calculations. The efficiency of this test is very high and equals 0.967 for shift and skew (see [11]) alternatives.

We can generalize these findings considering more general characterization, which is the particular case of [37, Theor.13.7.2]:

Theorem 8. Let $X$ and $Y$ are centered i.i.d. rv’s, and $a$ and $b$ are such constants that $0 < a, b < 1, a^2 + b^2 = 1$. Then $X \overset{d}{=} aX + bY$ iff $X, Y \in \mathcal{N}(0, \sigma^2)$.

We can rebuild our statistics using Theorem 8, and the result should depend on $a$. The theory of integral statistic in this generalized setting is developed in [50]. In particular, the local efficiency of integral test for shift alternative equals to

$$
eff^*(a) = \left( a - 1 + \sqrt{1-a^2} \right)^2 / \Omega(a),$$

where

$$
\Omega(a) = \left( \frac{7}{3} \pi - 4 \arctan \sqrt{\frac{1+a^2}{3-a^2}} - 4 \arctan \sqrt{\frac{2-a^2}{2+a^2}} - 4 \arctan \sqrt{\frac{1-a^2}{3+a^2}} - 4 \arctan \sqrt{\frac{a^2(1-a^2)}{4-a^2}} \right).
$$

The maximum of $\eff^*(a)$ is 1 but is attained for $a = 0$ and $a = 1$, where the test is inconsistent. The worst case (quite unexpectedly) is just the Polya case for $a = \sqrt{\frac{7}{2}}$ with the efficiency 0.966. We recommend $a = \frac{24}{25}$, and $b = \frac{7}{25}$. Then we have $a^2 + b^2 = 1$, and the efficiency is 0.990, this is a very high value.

The Kolmogorov type test based on this characterization was studied in [51]. The results are similar but the efficiencies are considerably lower.

6.2. Characterization by Shepp property. In 1964 Shepp [92] proved that if $X$ and $Y$ are i.i.d., $X, Y \in \mathcal{N}(0, \tau^2)$, then the rv

$$
k(X, Y) := 2XY/\sqrt{X^2 + Y^2} \in \mathcal{N}(0, \tau^2) \quad \text{again}.
$$

This statement is usually called the Shepp property.
Later Galambos and Simonelli [25] proved that the Shepp property characterizes the normal law in some class $\mathcal{F}_0$ which consists of such df’s $F$ which satisfy $0 < F(0) < 1$ and for which $F(x) - F(-x)$ is changing regularly in zero with the exponent 1. They proved the following result

**Theorem 9.** Let $X$ and $Y$ be i.i.d. rv’s with common df $F$ from the class $\mathcal{F}_0$. Then the equality in distribution $X \overset{d}{=} k(X,Y)$ takes place iff $X \in \mathcal{N}(0, \tau^2)$ for some variance $\tau^2 > 0$.

Nikitin and Volkova in [101] constructed tests of normality based on this characterization and found the efficiencies of corresponding tests. It turned out that for shift and skew alternatives the efficiencies coincide and are equal in case of integral and supremum tests to

$$
eff_I = \frac{3}{\pi} = 0.955, \quad \neff_D = \frac{2}{\pi} = 0.637.$$

7. **Tests of fit for other distributions**

The reader has probably noticed that the majority of characterizations used above for testing exponentiality and normality was formulated in terms of equal distribution of some simple statistics. There arises the question if such characterizations exists for other probability laws and if it is possible to build goodness-of-fit tests based on them. The answer is positive, but the set of corresponding characterizations is more sparse, the calculations are more involved and therefore the whole subject is underdeveloped.

7.1. **Puri-Rubin characterization.** We begin by the characterization of the power law. We are testing the composite hypothesis

$$H_0 : F \text{ is the df of the power law so that } F(x) = x^\mu, x \in [0, 1], \mu > 0,$$

against general alternatives. We use the characterization which is given in the paper by Puri and Rubin [89].

**Theorem 10.** Let $X$ and $Y$ be i.i.d. non-negative rv’s with df $F$. Then the equality

$$X \overset{d}{=} \min\left(\frac{X}{Y}, \frac{Y}{X}\right)$$

takes place iff $X$ and $Y$ have the power distribution.

The tests for the power law based on this characterization were build and studied by Nikitin and Volkova [104]. The efficiencies of integral test are between 0.71 and 0.97, the efficiencies of the Kolmogorov test are between 0.47 and 0.63 depending on the alternative under consideration.

7.2. **Some other laws.** The power law is closely related to the Pareto law, so Obradovic, Jovanovic and Miloševic, see [87], were able to use almost the same characterization (by replacing min by max) when testing for Pareto law. Volkova [102] introduced and studied some tests of fit for the Pareto distribution based on another characterization.
Goodness-of-fit test for the Cauchy law was built and studied by Litvinova [49]. She used the characterization of Ramachandran and Rao [90]. Its simplified variant is as follows:

**Theorem 11.** Let $X$ and $Y$ be i.i.d. rv’s. Then $X$ and $\frac{1}{3}X - \frac{2}{3}Y$ are identically distributed iff $X$ and $Y$ have the Cauchy df with arbitrary scale factor.

Litvinova in [49] explored the integral test, its local efficiency under the shift alternative turned out to be 0.665.

Some tests of uniformity based on characterizations were developed in [22], [29], [64]. In [57] there are interesting efficiency calculations for such tests.

We finish this section by briefly mentioning numerous results on testing goodness-of-fit based on characteristic properties of entropy and Kullback-Leibler information, see [8], [98], [22], [70], [28], [86], etc. However, there is almost nothing known on efficiencies of new tests, these tests are mainly compared on the basis of simulated power.

8. **Testing of symmetry**

Testing of *symmetry* based on characterizations has been much less explored than goodness-of-fit testing. Consider the classical hypothesis

$$H_0 : 1 - F(x) - F(-x) = 0, \quad \forall x \in \mathbb{R}^1,$$

against the alternative $H_1$ under which the equality (4) is violated at least in one point. The first step in construction of such tests was done in the crucial paper by Baringhaus and Henze [16].

Suppose that $X$ and $Y$ are i.i.d. rv’s with continuous df $F$. Baringhaus and Henze proved that the equidistribution of rv’s $|X|$ and $|\max(X,Y)|$ is valid iff $F$ is symmetric with respect to zero, that is (1) holds. They also proposed suitable Kolmogorov-type and omega-square type tests of symmetry. Some efficiency calculations for Kolmogorov type test were later performed in [78], see also [77]. Integral test of symmetry was next proposed and studied by Litvinova [48].

Another characterization of symmetry with respect to 0 belongs to Ahsanullah and was published in [3].

**Theorem 12.** Suppose that $X_1, ..., X_k$, $k \geq 2$, are i.i.d. rv’s with absolutely continuous df $F(x)$. Denote $X_{1,k} = \min(X_1, ..., X_k)$ and $X_{k,k} = \max(X_1, ..., X_k)$. Then

$$|X_{1,k}| \overset{d}{=} |X_{k,k}|$$

iff $F$ is symmetric about zero, i.e.

$$1 - F(x) - F(-x) = 0 \quad \forall x \in \mathbb{R}^1.$$

Subsequently we refer to this result as *Ahsanullah’s characterization of order k*.

Nikitin and Ahsanullah [29] published a paper on tests of symmetry based on these characterizations and their efficiencies. It was found that corresponding tests of symmetry for $k = 2$ and $k = 3$ are asymptotically equivalent to the test of Litvinova and to the Kolmogorov-type test of Baringhaus and Henze. In case of location alternative they are competitive and manifest rather high local Bahadur efficiency in comparison to many other tests of symmetry. At the same time, higher
values of \( k, k > 3 \), lead us to different tests with lower values of efficiencies in case of common alternatives. It would be interesting to calculate the efficiencies of such tests for more realistic alternatives, for instance for skew alternatives, see [11]. First steps in this direction were undertaken in the recent paper [18].

Similar research based on a certain modification of Ahsanullah’s characterization was done recently by Obradovic and Miloševic [60]. The authors of [60] were able to build corresponding integral test and the test of Kolmogorov type based on their theorem and studied its efficiency.

9. Directions of further research and perspectives

9.1. Tests based on characterizations of stable laws. Only three stable laws have explicit densities: normal, Cauchy and Lévy one-sided density given by the formula

\[
l(x) = \frac{1}{\sqrt{2\pi x^3}} \exp \left( -\frac{1}{2x} \right), \quad x \geq 0.
\]

The tests for normal and Cauchy law based on characterizations were described above. One of simplest characterizations of the Lévy law obtained by Ahsanullah and Nevzorov [6] looks as follows:

**Theorem 13.** Let \( X, Y \) and \( Z \) be i.i.d. rv’s. Then the equality in distribution

\[
X \overset{d}{=} \frac{Y + Z}{4}
\]

takes place iff \( X, Y \) and \( Z \) have the one-sided Lévy distribution with arbitrary scale factor.

The tests based on this characterization are unknown. Nothing is known about testing for general stable distributions using similar characterizations.

9.2. Tests based on characterizations by independence. The characterization of distributions can be formulated not only in terms of the equidistribution of statistics as in majority of examples given above but also in terms of their independence. Consider as an example the well-known classical result obtained independently by Kac [36] and Bernstein [17] long ago.

**Theorem 14.** If \( X \) and \( Y \) are independent rv’s, then \( X + Y \) and \( X - Y \) are independent iff \( X \) and \( Y \) are normal.

As far as we know, this approach is unexplored. Further development of the plot led finally to the famous Skitovich-Darmois theorem [19], [37] which is also suitable for the construction of tests. We can construct corresponding \( U \)-empirical distributions and test statistics which are more difficult for analysis. Nobody has studied corresponding goodness-of-fit tests.

Another option consists in the well-known result that the independence of \( \bar{x} \) and \( s^2 \) implies normality which was first proved by Geary in [27]. The same is true for higher central moments. This characterizations were used in a number of papers, see [52], [40], [71], and a preprint by Thulin [96] with power study via simulations.
However, there are no calculations of efficiency and analytic comparison with other tests of normality.

First steps in the calculations of efficiency for tests based on characterizations by independence were done recently by Miloševic and Obradovic [62]. For instance, they used the following characterization of the exponential law from [24]:

**Theorem 15.** If $X$ and $Y$ are independent i.i.d. random variables with an absolutely continuous distribution and if $\min\{X,Y\}$ and $|X-Y|$ are independent, then both $X$ and $Y$ have exponential distribution with distribution function $F(x) = 1 - e^{-\lambda x}$, $x > 0$, $\lambda > 0$.

In [62] there are also related results concerning other distributions.

9.3. **Use of empirical integral transforms.** For certain characterizations one can build the test statistics based not on $U$-empirical distributions but on empirical transforms, e.g. on empirical characteristic functions or empirical Laplace transforms.

Let $f_n(t) = n^{-1}\sum_{k=1}^n \exp\{itX_k\}$ be the empirical characteristic function of the sample $X_1,\ldots,X_n$. Then it is clear that using the Polya characterization ($X \sim (X + Y)/\sqrt{2}$) we have

$$f_n(t) - f_n^2(t/\sqrt{2}) \approx 0.$$

Hence the statistics for testing normality of the sample can be

$$Z_n = \sup_t |f_n(t) - f_n^2(t/\sqrt{2})|,$$

or

$$W_n = \int_{-\infty}^{\infty} |f_n(t) - f_n^2(t/\sqrt{2})|^2 Q(t)\,dt,$$

where $Q$ is some appropriate weight function.

Their asymptotic properties and efficiencies are unknown. However, the technique of asymptotic analysis of similar statistics was substantially developed in recent years, see, e.g., the papers by Meintanis and Jimenez-Gamero, see [55], [54], [34], [56], etc.

The use of empirical Laplace transform with interesting calculation of efficiencies for testing of exponentiality is presented in [61].

9.4. **Characterizations based on records.** There are many characterizations of distributions based on record statistics, see, e.g., [74], [20], [15], [93], and many others. Only few of them have been used for construction of goodness-of-fit tests, mainly in the works of Morris and Szynal, see, e.g., [68], [69], [66], [94] where they essentially used the characterizations based on moments of record values. However, nothing is known about the efficiencies of such tests.

9.5. **Characterizations based on moments.** Some characterizations of distributions are based on their moments or on moments of corresponding order statistics, see, e.g., [44], [45], [63], [65], [77]. They can be used for the construction of goodness-of-fit tests, but their efficiencies are unexplored.
9.6. **Multivariate generalizations.** It seems that little or nothing is known about multivariate goodness-of-fit tests and multivariate symmetry tests. One of few exceptions is the recent paper [62].

10. **Acknowledgements**

This research was supported by grant RFBR No. 16-01-00258 and by grant SPbSU-DFG No. 6.65.37.2017. The author is thankful to the referee for careful reading of the paper and many useful remarks.

**References**

[1] I. Ahmad, I. Alwasel, *A goodness-of-fit test for exponentiality based on the memoryless property*. J. Roy. Statist. Soc. 61(3)(1999), 681–689.

[2] M. Ahsanullah, *On a characterization of the exponential distribution by spacings*, Ann. Inst. Statist. Math. A30(1978), 163–166.

[3] M. Ahsanullah, *On some characteristic property of symmetric distributions*. Pakist. J. Statist. 8(1992), 19–22.

[4] M. Ahsanullah, B. M. G. Kibria, M. Shakil, *Normal and Student’s t-distributions and their applications*, Atlantis Press, Paris, 2014.

[5] M. Ahsanullah, G. G. Hamedani, *Exponential distribution*, Nova Science Publ., NY, 2010.

[6] M. Ahsanullah, V. B. Nevzorov, *Some inferences on the Lévy distribution*. J. Statist. Theor. Appl., 13(2014), 205–211.

[7] J. E. Angus, *Goodness-of-fit tests for exponentiality based on a loss-of-memory type functional equation*, J. Stat. Plann. Infer., 6(1982), 241–251.

[8] I. Arizono, H. Ohta, *A test for normality based on Kullback-Leibler information*, The Amer. Statistician, 43(1)(1989), 20–22.

[9] B. C. Arnold, J. A. Villaseñor, *Exponential characterizations motivated by the structure of order statistics in samples of size two*, Stat. Probab. Lett., 83(2013), 596–601.

[10] T. A. Azlarov, N. A. Volodin, *Characterization problems associated with the exponential distribution*, Springer-Verlag, NY, 1986.

[11] A. Azzalini, *The Skew-Normal and Related Families*, Cambridge University Press, 2014.

[12] R. R. Bahadur, *Some limit theorems in statistics*, Philadelphia, SIAM, 1971.

[13] R. R. Bahadur, *Rates of convergence of estimates and test statistics*, Ann. Math. Statist., 38(1967), 303–324.

[14] N. Balakrishnan, A. P. Basu, *Exponential distribution: theory, methods and applications*, CRC Press, 1996.

[15] N. Balakrishnan, A. Stepanov, *Two characterizations based on order statistics and records*, J. Statist. Plann. Infer., 124(2004), 273–287.

[16] L. Baringhaus, N. Henze, *A characterization of and new consistent tests for symmetry*, Commun. Statist.- Theor. Meth., 21(1992), 1555–1566.

[17] S. Bernstein, *Sur une propriété caractéristique de la loi de Gauss*, Trans. Leningrad Polytech. Inst. 3(1941), 21–22 (in Russian).

[18] G. T. Bookiya, Ya. Yu. Nikitin, *Asymptotic efficiency of new nonparametric symmetry tests for generalized skew alternatives*, Zapiski Nauchn. Semin. POMI, 454(2016), 82–101.

[19] W. Bryc, *Normal distribution: characterization with applications*, Lect. Notes Stat. 100(1995), Berlin, Springer Science.

[20] P. Deheuvels, P., *The characterization of distributions by order statistics and record values a unified approach*. J. Appl. Prob., 21(1984), 326–334.

[21] M. M. Desu, *A characterization of the exponential distribution by order statistics*, Ann. Math. Stat., 42(1971), 837–838.

[22] E. J. Dudewicz, E. C. van der Meulen, *Entropy-based test for uniformity*, J. Amer. Statist. Assoc. 76(1981), 967–974.
[23] A. Einstein, *Collected Papers. The Berlin Years: Writings, 1918-1921*, 7, Document 28, Princeton University Press, 2002.

[24] M. Fisz, *Characterization of some probability distributions*, Skand. Aktuarietidskr., 41(1958), 65–67.

[25] J. Galambos, I. Simonelli, Comments on a recent limit theorem of Quine, Stat. Probab. Lett., 63(2003), 89–95.

[26] J. Galambos, S. Kotz, *Characterizations of probability distributions*, Lect. Notes in Math. 675(1978), Springer, NY.

[27] R.C. Geary, *The Distribution of "Student’s" ratio for non-normal samples*, Suppl. J. Roy. Stat. Soc., 3(1936), 178–184.

[28] D.V. Gokhale *On entropy-based goodness-of-fit tests*, Comput. Stat. Data Anal. 1(1983), 157–165.

[29] T. Hashimoto, S. Shirahata, *A goodness-of-fit test based on a characterization of uniform distribution*, J. Japan Stat. Soc., 23(1993), 123–130.

[30] J. Haywood, E. Khmaladze, *On distribution-free goodness-of-fit testing of exponentiality*, J. Econometr., 143(2008), 5–18.

[31] R. Helmers, P. Janssen, R. Serfling, *Glivenko-Cantelli properties of some generalized empirical DF’s and strong convergence of generalized L-statistics*, Probab. Theor. Relat. Fields, 79(1988), 75–93.

[32] W. Hoeffding, *A class of statistics with asymptotically normal distribution*, Ann. Math. Stat. 19(1948), 293–395.

[33] P.L. Janssen, *Generalized empirical distribution functions with statistical applications*, Limburgs Universitair Centrum, Diepenbeek, 1988.

[34] M.D. Jimenez-Gamero et al., *Goodness-of-fit tests based on empirical characteristic functions*, Comput. Stat. Data Anal., 53(2009), 3957–3971.

[35] M. Jovanovic, B. Milosevic, Y. Y. Nikitin, M. Obradovic, K. Volkova, *Tests of exponentiality based on Arnold-Villasenor characterization and their efficiencies*, Comput. Stat. Data Anal., 90(2015), 100–113.

[36] M. Kac, *On a characterization of the normal distribution*, Amer. Journ. Mathem., 61(1939), 726–728.

[37] A.M. Kagan, Y.V. Linnik, C.R. Rao, *Characterization Problems in Mathematical Statistics*, Wiley, New York, 1973.

[38] A.V. Kakosyan, L.B. Klebanov, J.A. Melamed, *Characterization of distributions by the method of intensively operators*, Lect. Notes in Math., 1088(1984), New York, Springer.

[39] S. Kotz, *Characterizations of statistical distributions: a supplement to recent surveys*, Intern. Stat. Rev., 42(1974), 39–65.

[40] N.L. Kou, *A test for new better than used*, Commun. Stat. Theor. Meth., 6(1977), 563–573.

[41] A.J. Lee, *U-statistics: Theory and Practice*, Dekker, NY, 1990.

[42] H. Lilliefors, *On the Kolmogorov-Smirnov test for the exponential distribution with mean unknown*, J. Amer. Stat. Ass., 64(1969), 387–389.

[43] G.D. Lin, *Characterizations of continuous distributions via expected values of two functions of order statistics*, Sankhyā, A52(1990), 84–90.

[44] G.D. Lin, C.Y. Hu, *On characterizations of the logistic distribution*, J. Stat. Plann. Infer. 138(2008), 1147–1156.

[45] C.C. Lin, G.S. Mudholkar, *A simple test for normality against asymmetric alternatives*, Biometrika, 67(1980), 455–461.

[46] Yu.V. Linnik, *Linear forms and statistical criteria I, II*, Ukrain. Mathem. J., 5(1953), 207–243; 247–290 (in Russian). Engl. transl. in Selected Transl. in Mathem. Stat. and Probab., 3(1963), Amer. Math. Soc., Providence, RI, 1–90.

[47] V.V. Litvinova, *New nonparametric test for symmetry and its asymptotic efficiency*, Vestnik of Saint-Petersburg Univ. Mathematics, 34(2001), 12–14.

[48] V.V. Litvinova, *Two criteria of goodness-of-fit for Cauchy distributions based on characterizations*, J. Math. Sci., 127(2005), 1752–1756.
[50] V. V. Litvinova, Y. Y. Nikitin, *Two families of normality tests based on Polya-type characterization and their efficiencies*, J. Math. Sci.  **139**(3)(2006), 6582–6588.

[51] V. V. Litvinova, Ya. Yu. Nikitin, *Kolmogorov tests of normality based on some variants of Polya characterization*, J. Math. Sci. **219** (2016), 782–788.

[52] K. McDonald, S. K. Katti, *Test for normality using a characterization*, A Modern Course on Statistical Distributions in Scientific Work, Springer, Netherlands, 1975, 91–104.

[53] A. M. Mathai, G. Pederzoli, *Characterizations of the normal probability law*, Wiley, NY, 1977.

[54] S. G. Meintanis, *A Kolmogorov - Smirnov type test for skew normal distributions based on the empirical moment generating function*, J. Stat. Plann. Infer., **137**(2007), 2681–2688.

[55] S. G. Meintanis, M. D. Jimenez-Gamero, V. Alba-Fernandez, *A class of goodness-of-fit tests based on transformation*, Commun. Stat. -Theor. Meth., 43(2014), 1708–1735.

[56] S. G. Meintanis, J. Swanepoel, J. Allison, *The probability weighted characteristic function and goodness-of-fit testing*, J. Stat. Plann. Infer. **146**(2014), 122–132.

[57] B. Milošević B. *Asymptotic efficiency of goodness-of-fit tests based on Tu-Lo characterizaton*, arXiv:1508.05514, 2015.

[58] B. Milošević, *Asymptotic efficiency of new exponentiality tests based on a characterization*, Metrika, **79**(2016), 221–236.

[59] B. Milošević, M. Obradovic, *Some characterizations of exponential distribution based on order statistics*, Appl. Anal. Discrete Math., **10**(2016), 394–407.

[60] B. Milošević, M. Obradovic, *Characterization based symmetry tests and their asymptotic efficiencies*, Stat. Prob. Lett., **119**(2016), 155–162.

[61] B. Milošević, M. Obradovic, *New class of exponentiality tests based on U-empirical Laplace transform*, Stat. Papers., **57**(2016), 977–990.

[62] B. Milošević, M. Obradovic, *Two-dimensional Kolmogorov-type goodness-of-fit tests based on caracterisations and their asymptotic efficiencies*, Journ. Nonparam. Stat., **28**(2016), 413–427.

[63] K. W. Morris, D. Szynal, *Goodness-of-fit tests based on characterizations of continuous distributions*, Applic. Mathem., **27**(2000), 475–488.

[64] K. W. Morris, D. Szynal, *A goodness-of-fit test for the uniform distribution based on a characterization*, J. Math. Sci., **106**(2001), 2719–2724.

[65] K. W. Morris, D. Szynal, *Goodness-of-fit tests using dual versions of characterizations via moments of order statistics*, J. Math. Sci., **122**(2004), 3365–3383.

[66] K. Morris, D. Szynal, *Goodness-of-fit tests via characterizations*, Intern. J. Pure Appl. Math., **23**(2005), 491–554.

[67] K. Morris, D. Szynal, *Goodness-of-fit tests based on characterizations involving moments of order statistics*, Intern. J. Pure Appl. Mathem., **38**(2007), 83–121.

[68] K. Morris, D. Szynal, *Some U-statistics in goodness-of-fit tests derived from characterizations via record values*, Intern. J. Pure Appl. Mathem., **46**(2008), 339–414.

[69] K. Morris, D. Szynal, *Goodness-of-fit tests via characterizations. General approach*, J. Math. Sci., **191**(2013), 538–549.

[70] G. S. Mudholkar, L. Tian, *An entropy characterization of the inverse Gaussian distribution and related goodness-of-fit test*, J. Stat. Plann. Infer., **102**(2002), 211–221.

[71] G. S. Mudholkar, C. E. Marchetti, C. T. Lin, *Independence characterizations and testing normality against restricted skewness-kurtosis alternatives*, J. Stat. Plann. Infer., **104**(2004), 485–501.

[72] P. Muliere, Y. Nikitin, *Scale-invariant test of normality based on Polya’s characterization*, Metron, **60**(1-2)(2002), 21–33.

[73] H. N. Nagaraja, V. B. Nevzorov, *On characterizations based on record values and order statistics*, J. Stat. Plann. Infer., **63**(1997), 271–284.

[74] Ya. Yu. Nikitin, *Asymptotic Efficiency of Nonparametric Tests*, Cambridge University Press, New York, 1995.

[75] Y. Y. Nikitin, *Local asymptotic Bahadur optimality and characterization problems*, Theor. Probab. Appl., **29**(1)(1985), 79–92.
[76] Ya. Yu. Nikitin, Bahadur efficiency of a test of exponentiality based on a loss-of-memory type functional equation, J. Nonparam. Stat., 6(1996), 13–26.

[77] Ya. Yu. Nikitin, Large deviations of U-empirical Kolmogorov-Smirnov tests, and their efficiency, J. Nonparam. Stat., 22(5)(2010), 649–668.

[78] Ya. Yu. Nikitin, On Baringhaus-Henze test for symmetry: Bahadur efficiency and local optimality for shift alternatives, Math. Methods Stat., 5(1996), 214–226.

[79] Y. Y. Nikitin, M. Ahsanullah, New U-empirical tests of symmetry based on extremal order statistics, and their efficiencies, in: Mathematical Statistics and Limit Theorems, Springer International Publishing, 2015, pp. 231–248.

[80] Y. Y. Nikitin, I. Peaucelle, Efficiency and local optimality of nonparametric tests based on U- and V-statistics, Metron - Intern. J. Stat., 62(2004), 185–200.

[81] Ya. Yu. Nikitin, E. V. Ponikarov, Rough large deviation asymptotics of Chernoff type for von Mises functionals and U-statistics, Proc. Saint-Petersburg Mathem. Soc., 7(1999), 124–167; Engl. transl. in AMS Transl., ser. 2, 203(2001), 107–146.

[82] Ya. Yu. Nikitin, A. V. Tchirina, Bahadur efficiency and local optimality of a test for the exponential distribution based on the Gini statistic, J. Math. Sci., 128(2005), 2640–2655.

[83] Ya. Yu. Nikitin, K. Yu. Volkova, Asymptotic efficiency of exponentiality tests based on order statistics characterization, Georgian Math. J., 17(2010), 749–763.

[84] Ya. Yu. Nikitin, K. Yu. Volkova, Exponentiality tests Based on Ahsanullahs characterization and their efficiency, J. Math. Sci., 204(2015), 42–54.

[85] H. A. Noughabi, N. R. Arghami, General treatment of goodness-of-fit tests based on Kullback-Leibler information, J. Stat. Comput. Simul., 83(2013), 1556–1569.

[86] M. Obradovic, M. Jovanovic, B. Miloševic, Goodness-of-fit tests for Pareto distribution based on a characterization and their asymptotics, Statistics, 49(2015), 1026–1041.

[87] G. Polya, Herleitung des Gauss’schen Fehlergesetzes aus einer Funktionalsgleichung, Math. Zeitschr., 18(1923), 96–108.

[88] P. S. Puri, H. A. Rubin, A characterization based on the absolute difference of two i.i.d. random variables, Ann. Math. Stat., 41(1970), 2113–2122.

[89] A. V. Tchirina, Large deviations for a class of scale-free statistics under the gamma distribution, J. Math. Sci., 128(2005), 2640–2655.

[90] M. Thulin, On two simple tests for normality with high power, arXiv:1008.5319, 2010.

[91] A. W. Van der Vaart, Asymptotic statistics, Cambridge University Press, 2000.

[92] K. Volkova, On asymptotic efficiency of exponentiality tests based on Rosbjerg’s characterization, J. Math. Sci., 167(4)(2010), 486–494.

[93] K. Volkova, Y. Y. Nikitin, On the asymptotic efficiency of normality tests based on the Shepp property, Vestnik St. Petersburg Univ. Mathematics, 42(2009), 256–261.

[94] R. Volkova, Goodness-of-fit tests for the Pareto distribution based on its characterization, Stat. Meth. Applic., 25(2015), 1–23.
[103] K. Volkova, *Goodness-of-fit tests for expomatility based on Yanev-Chakraborty characterization and their efficiencies*, Proc. 19th Europ. Young Statisticians Meet., Nagy S., ed., Prague, 2015, 156–159.

[104] K. Y. Volkova, Y. Y. Nikitin, *Goodness-of-fit tests for the power function distribution based on the Puri-Rubin characterization and their efficiencies*, J. Math. Sci., **199**(2)(2014), 130–138.

[105] G. P. Yanev, S. Chakraborty, *Characterizations of exponential distribution based on sample of size three*, Pliska Studia Math. Bulg., **23**(2013), 237–244.

[106] G. P. Yanev, S. Chakraborty, *A characterization of exponential distribution and the Sukhatme - Rényi decomposition of exponential maxima*, Stat. Probab. Lett., **110**(2016), 94–102.

Department of Mathematics and Mechanics,
Saint-Petersburg State University, Universitetskaia nab. 7/9,
Saint-Petersburg, 199034, Russia

National Research University - Higher School of Economics,
Souza Pechatnikov, 16, St.Petersburg 190008, Russia

e-mail y.nikitin@spbu.ru