Negative translations not intuitionistically equivalent to the usual ones∗

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Abstract

We refute the conjecture that all negative translations are intuitionistically equivalent by giving two counterexamples. Then we characterise the negative translations intuitionistically equivalent to the usual ones.

1 Introduction

Informally speaking, classical logic $\text{CL}$ is the usual logic in mathematics, and intuitionistic logic $\text{IL}$ is obtained from classical logic by omitting:

- reductio ad absurdum $\neg A;$
- law of excluded middle $A \lor \neg A;$
- law of double negation $\neg\neg A \to A.$

In this sense, $\text{IL}$ is a weakening of $\text{CL}$, that is $\text{IL}$ proves less theorems than $\text{CL}$. At first sight it seems that $\text{IL}$ is just poorer than $\text{CL}$. However, there is a gain in moving from $\text{CL}$ to $\text{IL}$: the theorems of $\text{IL}$ have nicer properties. The main properties gained are

- disjunction property: if $\text{IL} \vdash A \lor B$, then $\text{IL} \vdash A$ or $\text{IL} \vdash B$ (where $A$ and $B$ are sentences);
- existence property: if $\text{IL} \vdash \exists x A(x)$, then $\text{IL} \vdash A(t)$ for some term $t$ (where $\exists x A$ is a sentence).

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Arguably, these two properties are the key criteria to say that a logic is constructive.

On the one hand IL is weaker than CL, on the other hand IL is constructive while CL is not. Given these differences, it is surprising that CL can be faithfully embedded in IL by the so-called negative translations into IL. Negative translations into IL are functions \(N\), mapping a formula \(A\) to a formula \(A^N\), that:

- embed \(CL\) into \(IL\), that is \(CL \vdash A \Rightarrow IL \vdash A^N\);
- are faithful, that is \(CL \vdash A^N \Leftrightarrow A\).

The image of the usual negative translations is (essentially) the negative fragment \(NF\), that is the set of all formulas without \(\lor\) and \(\exists\) and whose atomic formulas are all negated. So \(NF\) is a faithful copy of \(CL\) inside \(IL\). This is pictured in figure 1.

![Figure 1: negative translation \(N\) into \(IL\) embedding \(CL\) in the fragment \(NF\) of \(IL\)](image)

There are four negative translations into \(IL\) usually found in the literature (and recently two new ones were presented [5]). They are introduced in table 1 and defined (by induction on the structure of formulas) in table 2. All these negative translations into \(IL\) are equivalent in \(IL\): given any two of them, say \(M\) and \(N\), we have \(IL \vdash A^M \Leftrightarrow A^N\). This fact leads to the following conjecture that seems to be almost folklore:

if we rigorously define the notion of a negative translation into \(IL\), then we should be able to prove that all negative translations are equivalent in \(IL\).

Curiously, this conjecture apparently has never been studied before. In this article we study it, reaching the following conclusions.

- The conjecture is false and we give two counterexamples.
- The usual negative translations into \(IL\) are characterised by the following two equivalent conditions:
  - to translate into \(NF\) in \(IL\), that is \(A^N\) is equivalent in \(IL\) to a formula in \(NF\);
  - to act as the identity on \(NF\) in \(IL\), that is \(IL \vdash A^N \Leftrightarrow A\) for all \(A \in NF\).
| Year | Name           | Symbol | Note                                                  |
|------|----------------|--------|-------------------------------------------------------|
| 1925 | Kolmogorov     | $Ko$   |                                                       |
| 1933 | Gödel-Gentzen  | $G$    | One variant by Gödel [11] and another one independently by Gentzen [8] |
| 1951 | Kuroda         | $Ku$   |                                                       |
| 1998 | Krivine        | $Kr$   | Maybe better attributed to Streicher and Reus [17]    |

Table 1: the four usual negative translations.

\[
\begin{array}{ll}
PKo & \equiv \neg P \quad (P \neq \bot \text{ atomic}) \\
\bot Ko & \equiv \bot \\
(A \land B)Ko & \equiv \neg (A^{Ko} \land B^{Ko}) \\
(A \lor B)Ko & \equiv \neg (A^{Ko} \lor B^{Ko}) \\
(A \to B)Ko & \equiv \neg (A^{Ko} \to B^{Ko}) \\
(\forall xA)^{Ko} & \equiv \neg (\forall xA^{Ko}) \\
(\exists xA)^{Ko} & \equiv \neg (\exists xA^{Ko}) \\
\end{array}
\]

\[
\begin{array}{ll}
P^G & \equiv \neg P \quad (P \neq \bot \text{ atomic}) \\
\bot^G & \equiv \bot \\
(A \land B)^G & \equiv A^G \land B^G \\
(A \lor B)^G & \equiv (\neg A^G \land \neg B^G) \\
(A \to B)^G & \equiv A^G \to B^G \\
(\forall xA)^G & \equiv \forall xA^G \\
(\exists xA)^G & \equiv \neg \forall x\neg A^G \\
\end{array}
\]

Table 2: definition of the four usual negative translations.
2 Notions

In the rest of this article, \( \text{CL} \) denotes the pure first order classical predicate logic based on \( \bot \), \( \land \), \( \lor \), \( \to \), \( \forall \) and \( \exists \) (where \( \neg A \equiv A \to \bot \), \( A \leftrightarrow B \equiv (A \to B) \land (B \to A) \) and \( \equiv \) denotes syntactical equality) and \( \text{IL} \) and \( \text{ML} \) denote its intuitionistic and minimal counterparts, respectively. All formulas considered belong to the common language of \( \text{CL} \), \( \text{IL} \) and \( \text{ML} \). To save parentheses we adopt the convention that \( \forall \) and \( \exists \) bind stronger than \( \land \) and \( \lor \), which in turn bind stronger than \( \to \).

Let us start by motivating our definition of a negative translation.

The main feature of any negative translation \( N \) into \( \text{IL} \) is embedding \( \text{CL} \) into \( \text{IL} \) in the sense of \( \text{CL} \vdash A \Rightarrow \text{IL} \vdash A^N \). We can be even more ambitious and ask for (1) \( \text{CL} + \Gamma \vdash A \Rightarrow \text{IL} + \Gamma^N \vdash A^N \) where \( \Gamma \) is any set of formulas and \( \Gamma^N := \{A^N : A \in \Gamma\} \).

But embedding \( \text{CL} \) into \( \text{IL} \) alone does not seem to capture our intuitive notion of a negative translation. For example, it includes the trivial example \( A^N : \equiv \neg \bot \). The problem with this example is that the meaning of \( A^N \) is unrelated to the meaning of \( A \). So require that a negative translation do not change the meaning of formulas, that is (2) \( A^N \leftrightarrow A \). This equivalence must not be taken in \( \text{IL} \) or \( \text{ML} \), otherwise from (1) and (2) we would get \( \text{CL} = \text{IL} \). So we take the equivalence in \( \text{CL} \), that is \( \text{CL} \vdash A \leftrightarrow A^N \).

**Definition 1.** Let \( N \) be a function mapping each formula \( A \) to a formula \( A^N \).

- The following condition is called **soundness theorem into \( \text{IL} \) (\( \text{ML} \))** of \( N \): for all formulas \( A \) and for all sets \( \Gamma \) of possibly open formulas, we have the implication \( \text{CL} + \Gamma \vdash A \Rightarrow \text{IL} + \Gamma^N \vdash A^N \) (respectively, \( \text{CL} + \Gamma \vdash A \Rightarrow \text{ML} + \Gamma^N \vdash A^N \)).

- The following condition is called **characterisation theorem of \( N \)**: for all formulas \( A \) we have \( \text{CL} \vdash A \leftrightarrow A^N \).

- We say that \( N \) is a **negative translation into \( \text{IL} \) (\( \text{ML} \))** if and only if both the soundness theorem into \( \text{IL} \) (respectively, \( \text{ML} \)) of \( N \) and the characterisation theorem of \( N \) hold.

**Remark 2.** The soundness theorem into \( \text{ML} \) of \( N \) implies the soundness theorem into \( \text{IL} \) of \( N \). So a negative translation into \( \text{ML} \) is in particular a negative translation into \( \text{IL} \).

The conjecture that concerns us mentions equivalence in \( \text{IL} \). For definiteness, we write down exactly what we mean by this.

**Definition 3.** We say that two negative translations \( M \) and \( N \) are **equivalent in \( \text{IL} \) (\( \text{ML} \))** if and only if for all formulas \( A \) we have \( \text{IL} \vdash A^M \leftrightarrow A^N \) (respectively, \( \text{ML} \vdash A^M \leftrightarrow A^N \)).

Later on we will see that what characterises the usual negative translations into \( \text{IL} \) are two properties related to \( \text{NF} \). Again for definiteness we write down the definition of \( \text{NF} \) and of the two properties.

**Definition 4.** The **negative fragment \( \text{NF} \)** is the set of formulas inductively generated by:
• $\bot \in \text{NF}$;
• if $P$ is an atomic formula, then $\neg P \in \text{NF}$;
• if $A, B \in \text{NF}$, then $A \wedge B, A \rightarrow B, \forall x A \in \text{NF}$.

**Definition 5.** Let $N$ be a negative translation into $\text{IL}$.  

• We say that $N$ translates into $\text{NF}$ in $\text{IL}$ ($\text{ML}$) if and only if for all formulas $A$ there exists a $B \in \text{NF}$ such that $\text{IL} \vdash A^N \leftrightarrow B$ (respectively, $\text{ML} \vdash A^N \leftrightarrow B$).

• We say that $N$ acts as the identity on $\text{NF}$ in $\text{IL}$ ($\text{ML}$) if and only if for all $A \in \text{NF}$ we have $\text{IL} \vdash A^N \leftrightarrow A$ (respectively, $\text{ML} \vdash A^N \leftrightarrow A$).

3 Gödel-Gentzen negative translation

We will choose the Gödel-Gentzen negative translation $G$ as a representative of the usual negative translations into $\text{IL}$, so let us take a closer look at it.

We start by motivating the definition of $G$. It is known from proof theory that $\text{CL}$ is conservative over $\text{ML}$ with respect to $\text{NF}$, that is (1) for all $A \in \text{NF}$ we have the implication $\text{CL} \vdash A \Rightarrow \text{ML} \vdash A$. This suggests us that one way of constructing a negative translation into $\text{ML}$ is to rewrite each formula $A$ as a formula $A^N \in \text{NF}$. By rewriting we mean that $A^N$ still has the same meaning as $A$ in the sense of (2) $\text{CL} \vdash A^N \leftrightarrow A$. Then (1) would give us the soundness theorem into $\text{ML}$ of $N$ (almost, because there is no $\Gamma$) and (2) would give us the characterisation theorem of $N$. The natural way of rewriting a formula $A$ as a classically equivalent formula $A^N \in \text{NF}$ (that is having all atomic formulas $P \not\equiv \bot$ negated and using only $\bot$, $\land$, $\rightarrow$ and $\forall$) is:

• rewrite atomic formulas $P \not\equiv \bot$ as $\neg \neg P$;
• rewrite $A \lor B$ as $\neg (\neg A \land \neg B)$;
• rewrite $\exists x A$ as $\neg \forall x \neg A$;
• there’s no need to rewrite $\bot$, $A \land B$, $A \rightarrow B$ and $\forall x A$.

If we formalise these rewritings as a definition of $N$ by induction on the structure of formulas, then we get exactly $G$. As a “tagline” we can say: $A^G$ is the natural rewriting of $A$ into $\text{NF}$.

Incidentally, Gödel’s and Gentzen’s negative translations differ only in the way they translate $A \rightarrow B$: Gödel translates to $\neg (A^G \land \neg B^G)$ while Gentzen translates to $A^G \rightarrow B^G$. By the above discussion, we find Gentzen’s variant more natural and so we adopt it.

Now we turn to the main properties of $G$. We can prove that $G$:

• is a negative translation into $\text{ML}$;
• translates into $\text{NF}$ in $\text{ML}$;
acts as the identity on \( \text{NF} \) in \( \text{ML} \).

We can even prove strengthenings of the second and third properties above:

- for all formulas \( A \) we have \( A^G \in \text{NF} \);
- for all formulas \( A \in \text{NF} \) we have \( A^G \equiv A \) (modulo identifying \( \neg
\neg
\neg P \) with \( \neg P \) for atomic formulas \( P \)).

These two strengthenings are specific of \( G \): they do not hold for \( \text{Ko}, \text{Ku} \) and \( \text{Kr} \).

To finish this section we discuss \( G \) as a representative of the usual negative translations into \( \text{IL} \). We can prove that \( \text{Ko}, G, \text{Ku} \) and \( \text{Kr} \) are equivalent in \( \text{IL} \). So any of them can be taken as a representative of the usual negative translations into \( \text{IL} \). We choose to take \( G \) as a representative due to nice syntactical properties of \( G \) like \( (A \leftrightarrow B)^G \equiv A^G \leftrightarrow B^G \) and the two strengthenings above. These properties allow us to work many times with syntactical equalities instead of equivalences, thus avoiding the question of where (\( \text{CL}, \text{IL} \) or \( \text{ML} \)) the equivalences are provable.

### 4 Two negative translations not intuitionistically equivalent to the usual ones

Before we present our two counterexamples to the conjecture, let us draw a scale to roughly measure how provable or refutable a formula \( F \) is. This scale will be useful to picture our main theorem about the counterexamples. We draw the scale following this set of instructions.

- We plot along an axis all possible pairs of combinations of
  
  \( \text{CL} \vdash F \), \( \text{CL} \nvdash F \) and \( \text{CL} \nvdash \neg F \), \( \text{CL} \vdash \neg F \)

  with

  \( \text{IL} \vdash F \), \( \text{IL} \nvdash F \) and \( \text{IL} \nvdash \neg F \), \( \text{IL} \vdash \neg F \).

- Actually, we do not plot impossible pairs (for example, “\( \text{CL} \vdash F \) and \( \text{IL} \vdash \neg F \)”)
  and redundant entries in pairs (for example, the entry “\( \text{CL} \vdash F \)” in the pair “\( \text{CL} \vdash F \) and \( \text{IL} \vdash F \)”).

- The plotting is ordered from provability of \( F \) on the left to refutability of \( F \)
  on the right (for example, “\( \text{IL} \vdash F \)” stands on the left of “\( \text{CL} \vdash F \) and \( \text{IL} \nvdash F \)”
  because “\( \text{IL} \vdash F \)” states a stronger form of provability).

The resulting scale is pictured in figure 2.

Now we present our two counterexamples.

- The first counterexample \( N_1 \) is a weakening of \( G \) obtained by weakening \( A^G \)
  to \( A^G \lor F \) (for suitable formulas \( F \)).

- The second counterexample \( N_2 \) is a variant of \( G \) obtained by making \( \perp \) in \( A^G \)
  “less false” in the sense of replacing \( \perp \) by \( F \) in \( A^G \), that is \( A^G[F/\perp] \) (again,
  for suitable \( F \)).
Definition 6. Fix a formula $F$. We define two functions $N_1$ and $N_2$, mapping formulas to formulas, by

- $A^{N_1} \equiv A \lor F$;
- $A^{N_2} \equiv A[F/\bot]$.

Since $N_1$ and $N_2$ depend on the chosen $F$, in rigour we should write something like $A^{N_1(F)}$ and $A^{N_2(F)}$, but we avoid this cumbersome notation.

We found $N_2$ in an article by Ishihara [13] and in a book chapter by Coquand [2, section 2.3]. Maybe Ishihara drew inspiration from an article by Flagg and Friedman [6] where a similar translation appears. It is even possible that $N_2$ is folklore.

For our two counterexamples to work, we need the formula $F$ to be classically refutable but intuitionistically acceptable. In the next lemma we prove that there are such formulas $F$.

Lemma 7.

1. There exists a formula $F$ such that $\text{CL} \vdash \neg F$ but $\text{IL} \not\vdash \neg F$.

2. Any such formula $F$ is not equivalent in $\text{IL}$ to a formula in $\text{NF}$.

Proof.

1. Let $P$ be an unary predicate symbol. We are going to prove that $F \equiv \neg \forall x P(x) \land \forall x \neg \neg P(x)$ is such that $\text{CL} \vdash \neg F$ but $\text{IL} \not\vdash \neg F$. Since $\text{CL} \vdash \neg F$ is obvious, we move on to prove $\text{IL} \not\vdash \neg F$ by showing that the Kripke model $K$ from figure 3 forces $F$.

- $K$ forces $\neg \forall x P(x)$ because no node forces $\forall x P(x)$.
- $K$ forces $\forall x \neg \neg P(x)$ because every node $k$ forces $\neg \neg P(d)$ for all $d$ in its domain $\{0, \ldots, k\}$ since the node $k+1$ forces $P(d)$.

2. If $F$ were equivalent in $\text{IL}$ to a formula in $\text{NF}$, then $
eg F$ would also be equivalent in $\text{IL}$ to a formula in $\text{NF}$, so from $\text{CL} \vdash \neg F$ and the fact that $\text{CL}$ is conservative over $\text{IL}$ with respect to $\text{NF}$ we would get $\text{IL} \vdash \neg F$, contradicting point 1.

Now we prove our main theorem giving two counterexamples to the conjecture: $N_1$ and $N_2$ are negative translations into $\text{IL}$ (even into $\text{ML}$) not equivalent in $\text{IL}$ to the usual negative translations into $\text{IL}$ (for suitable formulas $F$). The claims of this theorem are summarised in figure 4.
Theorem 8. The functions $N_1$ and $N_2$:

1. have a soundness theorem into $\mathsf{ML}$ for all formulas $F$;
2. have a characterisation theorem if and only if $\mathsf{CL} \vdash \neg F$;
3. are equivalent in $\mathsf{IL}$ to $G$ if and only if $\mathsf{IL} \vdash \neg F$.

So, if $\mathsf{CL} \vdash \neg F$ but $\mathsf{IL} \not\vdash \neg F$, then $N_1$ and $N_2$ are negative translations into $\mathsf{ML}$ not equivalent in $\mathsf{IL}$ to $G$.

\begin{figure}[h]
\centering
\begin{tikzpicture}[scale=0.8]
\node (0) at (0,0) {$0$};
\node (1) at (1,0) {$1$};
\node (2) at (2,0) {$2$};
\node (3) at (3,0) {$3$};
\node (00) at (0,-1) {$\{0\}$};
\node (01) at (1,-1) {$\{0,1\}$};
\node (012) at (2,-1) {$\{0,1,2\}$};
\node (0123) at (3,-1) {$\{0,1,2,3\}$};
\node (P0) at (0,1) {$P(0)$};
\node (P1) at (1,1) {$P(1)$};
\node (P2) at (2,1) {$P(2)$};
\draw (0) edge (00) (0) edge (01) (0) edge (012) (0) edge (0123) (1) edge (01) (1) edge (012) (2) edge (012) (2) edge (0123) (3) edge (0123) (P0) edge (0) (P1) edge (1) (P2) edge (2);
\end{tikzpicture}
\caption{Figure 3: a Kripke model $\mathcal{K}$ forcing $\neg \forall x P(x) \land \forall x \neg P(x)$.}
\end{figure}

\begin{figure}[h]
\centering
\begin{tabular}{ccccccc}
\hline
& strongly & & provable & & & strongly & \\
& provable & & (but not strongly) & & & refutable & \\
& undecidable & & (but not strongly) & & & refutable & \\
\hline
$\mathsf{IL} \vdash \neg F$ & $\mathsf{CL} \vdash \neg F$ & $\mathsf{CL} \not\vdash \neg F$ & $\mathsf{CL} \vdash \neg F$ & $\mathsf{IL} \vdash \neg F$ & $\mathsf{IL} \not\vdash \neg F$ & $\mathsf{IL} \vdash \neg F$
\hline
$\mathsf{CL} \vdash A_{N_1} \leftrightarrow \vdash A^G$ & $\mathsf{CL} \vdash A^G$ & $\mathsf{CL} \vdash \neg A^G$ & $\mathsf{CL} \vdash \neg A_{N_1} \leftrightarrow \vdash A^G$ & $\mathsf{CL} \vdash \neg A^G$ & $\mathsf{CL} \vdash \neg A_{N_2} \leftrightarrow \vdash A^G$ & $\mathsf{CL} \vdash \neg A_{N_2} \leftrightarrow \vdash A^G$
\hline
\end{tabular}
\caption{Figure 4: theorem 8 on the scale of provability-refutability.}
\end{figure}

\textit{Proof.}

1. Consider an arbitrary formula $F$.

First let us consider the case of $N_1$. By direct proof, consider an arbitrary set of formulas $\Gamma$ and an arbitrary formula $A$, assume $\mathsf{CL} + \Gamma \vdash A$ and let us prove $\mathsf{ML} + \Gamma^{N_1} \vdash A^{N_1}$. Since a proof in $\mathsf{CL}$ of $A$ uses only finitely many formulas $A_1, \ldots, A_n$ from $\Gamma$, then $\mathsf{CL} + A_1 + \cdots + A_n \vdash A$. By the soundness theorem into $\mathsf{ML}$ of $G$ we get $\mathsf{ML} + A_{i}^{G} + \cdots + A_{n}^{G} \vdash A^G$ (where $A^G_i$ abbreviates $(A_i)^G$), that is (1) $\mathsf{ML} \vdash A_{i}^{G} \land \cdots \land A_{n}^{G} \rightarrow A^G$ by the deduction theorem of $\mathsf{ML}$.

Let us show (2) $\mathsf{ML} + A_{i}^{G} \lor F + \cdots + A_{n}^{G} \lor F \vdash A^G \lor F$. We argue inside $\mathsf{ML}$. Assume $A_{i}^{G} \lor F, \ldots, A_{n}^{G} \lor F$. Each $A_{i}^{G} \lor F$ gives us two cases: the case of $A_{i}^{G}$ and the case of $F$. 

\begin{figure}[h]
- If for some $A_i^G \lor F$ we have the case $F$, then trivially $A^G \lor F$.
- Otherwise in all $A_i^G \lor F$ we have the case of $A_i^G$, so we have $A_1^G \land \cdots \land A_n^G$, thus $A^G$ by (1), therefore trivially $A^G \lor F$.

So we have (2) as we wanted. This argument is illustrated for $n = 2$ in figure 5.

![Figure 5: argument of $\text{ML} + A_2^G \lor F + \cdots + A_n^G \lor F \vdash A^G \lor F$ for $n = 2$.](image)

But (2) is $\text{ML} + A^{N_1} + \cdots + A^{N_1} \vdash A^{N_1}$, so we get $\text{ML} + \Gamma^{N_1} \vdash A^{N_1}$, as we wanted.

Now let us consider the case of $N_2$. By direct proof, consider an arbitrary set of formulas $\Gamma$ and an arbitrary formula $A$, assume $\text{CL} + \Gamma \vdash A$ and let us prove $\text{ML} + \Gamma^{N_2} \vdash A^{N_2}$. By the soundness theorem into $\text{ML}$ of $G$ we get $\text{ML} + A_1^G + \cdots + A_n^G \vdash A^G$. Since $\bot$ is treated as an arbitrary propositional letter in $\text{ML}$, we can replace $\bot$ by $F$ getting $\text{ML} + A_1^G[F/\bot] + \cdots + A_n^G[F/\bot] \vdash A^G[F/\bot]$, that is $\text{ML} + A_1^{N_2} + \cdots + A_n^{N_2} \vdash A^{N_2}$, as we wanted.

2. First let us consider the case of $N_1$.

$(\Rightarrow)$ By direct proof, assume that $N_1$ has a characterisation theorem and let us prove $\text{CL} \vdash \neg F$. By the characterisation theorem of $N_1$ we have $\text{CL} \vdash \bot^{N_1} \leftrightarrow \bot$ where $\bot^{N_1} \equiv \bot \lor F$, so $\text{CL} \vdash \neg F$, as we wanted.

$(\Leftarrow)$ By direct proof, assume $\text{CL} \vdash \neg F$, consider an arbitrary formula $A$ and let us prove $\text{CL} \vdash A^{N_1} \leftrightarrow A$. By the characterisation theorem of $G$ we have $\text{CL} \vdash A^G \leftrightarrow A$. Since $\text{CL} \vdash \neg F$ by assumption, it makes no difference in $\text{CL}$ to replace $A^G$ by $A^G \lor F$. So $\text{CL} \vdash A^G \lor F \leftrightarrow A$, that is $\text{CL} \vdash A^{N_1} \leftrightarrow A$, as we wanted.

Now let us consider the case of $N_2$.

$(\Rightarrow)$ Analogous to the case of $N_1$.

$(\Leftarrow)$ By direct proof, assume $\text{CL} \vdash \neg F$, consider an arbitrary formula $A$ and let us prove $\text{CL} \vdash A^{N_2} \leftrightarrow A$. By the characterisation theorem of $G$ we have $\text{CL} \vdash A^G \leftrightarrow A$. Since $\text{CL} \vdash \neg F$ by assumption, it makes no difference in $\text{CL}$ to replace $\bot$ by $F$. So $\text{CL} \vdash A^G[F/\bot] \leftrightarrow A$, that is $\text{CL} \vdash A^{N_2} \leftrightarrow A$, as we wanted.
3. First let us consider the case of \( N_1 \).

(\( \Rightarrow \)) By direct proof, assume that \( N_1 \) and \( G \) are equivalent in \( \mathbb{IL} \) and let us prove \( \mathbb{IL} \vdash \neg F \). By the assumption we have \( \mathbb{IL} \vdash \perp \iff \perp^G \) where \( \perp^N_1 \equiv \perp \lor F \) and \( \perp^G \equiv \perp \). So \( \mathbb{IL} \vdash \neg F \), as we wanted.

(\( \Leftarrow \)) By direct proof, assume \( \mathbb{IL} \vdash \neg F \), take an arbitrary formula \( A \) and let us prove \( \mathbb{IL} \vdash A^N_1 \iff A^G \). By the assumption it makes no difference in \( \mathbb{IL} \) to replace \( \perp \) by \( F \). So \( \mathbb{IL} \vdash A^G[F/\perp] \iff A^G \), that is \( \mathbb{IL} \vdash A^N_1 \iff A^G \), as we wanted.

Now let us consider the case of \( N_2 \).

(\( \Rightarrow \)) Analogously to the case of \( N_1 \).

(\( \Leftarrow \)) By direct proof, assume \( \mathbb{IL} \vdash \neg F \), take an arbitrary formula \( A \) and let us prove \( \mathbb{IL} \vdash A^N_2 \iff A^G \). By the assumption it makes no difference in \( \mathbb{IL} \) to replace \( \perp \) by \( F \). So \( \mathbb{IL} \vdash A^G[F/\perp] \iff A^G \), that is \( \mathbb{IL} \vdash A^N_2 \iff A^G \), as we wanted.

We saw in theorem \( \S \) that \( N_1 \) and \( N_2 \) are two counterexamples to the conjecture (for suitable \( F \)). Now in proposition \( 9 \) we clarify that these two counterexamples are different (for the same suitable \( F \)).

**Proposition 9.** If \( \mathbb{CL} \vdash \neg F \) but \( \mathbb{IL} \not\vdash \neg F \), then \( N_1 \) and \( N_2 \) are not equivalent in \( \mathbb{IL} \).

**Proof.** By direct proof, assume that \( \mathbb{CL} \vdash \neg F \) but \( \mathbb{IL} \not\vdash \neg F \) and let us prove \( \mathbb{IL} \not\vdash A^N_1 \iff A^N_2 \). We start by making two observations about Kripke models.

1. There exists a Kripke model \( K \), with a bottom node, that forces \( \neg F \).

   Let us prove this claim. Since \( \mathbb{CL} \vdash \neg F \) by assumption, any classical model forces \( \neg F \). Regarding a classical model as a Kripke model with only one node, we have a Kripke model, with a bottom node, forcing \( \neg F \), as we wanted.

   For example, for the \( F \equiv \neg \forall x P(x) \land \forall x \neg \neg P(x) \) used in the proof of lemma \( 7 \) we can take \( K \) to be the Kripke model of figure \( 6 \):

   \[
   \{0\} \bullet P(0)
   \]

   Figure 6: a Kripke model \( K \) forcing \( \neg F \) where \( F \equiv \neg \forall x P(x) \land \forall x \neg \neg P(x) \).

2. There exists a Kripke model \( L \), with a bottom node, that forces \( F \).

   Let us prove this claim. Since \( \mathbb{IL} \not\vdash \neg F \) by assumption, there exists a Kripke model \( L' \) that does not force \( \neg F \), that is some node \( n' \) of \( L' \) does not force \( \neg F \). Then there exists a node \( n \) above or equal to \( n' \) that forces \( F \). By restricting \( L' \) to all the nodes above or equal to \( n \) we get a Kripke model \( L \), with bottom node \( n \), that forces \( F \), as we wanted.

   For example, for the \( F \equiv \neg \forall x P(x) \land \forall x \neg \neg P(x) \) used in the proof of lemma \( 7 \) we can take \( L \) to be the Kripke model of figure \( 3 \).
Now let us return to our goal: $\mathbf{IL} \not\vDash A^{N_1} \leftrightarrow A^{N_2}$. Consider a fresh nullary predicate $Q \not\equiv \bot$. Since $Q$ is fresh and $Q \not\equiv \bot$,

- $\mathcal{L}$ forces $\neg Q$;
- we can force $Q$ in $\mathcal{K}$;
- forcing $Q$ in $\mathcal{K}$ will not collide with $\mathcal{K}$ forcing $\neg F$.

We will show $\mathbf{IL} \not\vDash Q^{N_2} \rightarrow Q^{N_1}$, where $Q^{N_2} \equiv (Q \rightarrow F) \rightarrow F$ and $Q^{N_1} \equiv \neg Q \lor F$.

The base nodes of $\mathcal{K}$ and $\mathcal{L}$ have (by definition of Kripke model) non empty domains. We can assume (renaming elements if necessary) that those domains share a common element $d$. Consider the Kripke model $\mathcal{M}$ from figure 7 obtained by:

- connecting a fresh bottom node 0, with domain $\{d\}$, to the bottom nodes of $\mathcal{K}$ and $\mathcal{L}$;
- for every node $n$ of $\mathcal{M}$, forcing $Q$ in $n$ if and only if $n$ forces $\neg F$; or equivalently, forcing $Q$ in $\mathcal{K}$ but not in $\mathcal{L}$ and 0.

$$Q, \neg F \bullet \mathcal{K} \quad \mathcal{L} \bullet F, \neg Q$$

$$\{d\} \bullet 0$$

Figure 7: a Kripke model $\mathcal{M}$ not forcing $((Q \rightarrow F) \rightarrow F) \rightarrow \neg Q \lor F$.

Note that $\mathcal{M}$ is well-defined because:

- the domains of $\mathcal{M}$ are monotone since $\{d\}$ is contained in the domains of $\mathcal{K}$ and $\mathcal{L}$;
- the forcing relation in $\mathcal{M}$ is monotone since $Q$ is forced only in the entire $\mathcal{K}$.

Now we argue that $\mathcal{M}$ does not force $(\ast)$.

- The node 0 does not force $\neg \neg Q$ because $\mathcal{L}$ forces $\neg Q$.
- The node 0 does not force $F$ because $\mathcal{K}$ forces $\neg F$.
- Let us show that the node 0 forces $(Q \rightarrow F) \rightarrow F$, that is any node $n$ does not force $Q \rightarrow F$ or forces $F$. We consider the following three cases.
  
  - If $n$ is in $\mathcal{K}$, then $n$ does not force $Q \rightarrow F$ because $\mathcal{K}$ forces $Q$ (by construction of $\mathcal{M}$) and $\neg F$.
  - If $n$ is in $\mathcal{L}$, then $n$ forces $F$ because $\mathcal{L}$ forces $F$.
  - If $n$ is 0, then $n$ does not force $Q \rightarrow F$, otherwise $\mathcal{K}$ would force $Q \rightarrow F$ and we already saw that this is false.

We conclude that the node 0 does not force $(\ast)$, as we wanted. \qed
As a curiosity, let us see that we have the factorisations $N_2 = FD \circ G$ and $N_2 = FD' \circ G$ of $N_2$ in terms of Friedman-Dragalin translation $FD$ \cite{Friedman-Dragalin} (better known as Friedman’s $A$-translation), its refinement $FD'$ \cite{Friedman-Dragalin-refine} and $G$. The translation $FD$ was used by Friedman and Dragalin to prove that certain intuitionistic theories $\mathsf{IT}$ are closed under Markov rule in the sense of $\mathsf{IT} \vdash \neg \neg \exists x P(x) \Rightarrow \mathsf{IT} \vdash \exists x P(x)$ where $P(x)$ is an atomic formula.

**Definition 10.** Fix a formula $F$.

- **Friedman-Dragalin translation** $FD$ maps each formula $A$ to the formula $A^{FD}$ obtained from $A$ by simultaneously replacing in $A$:
  - $\bot$ by $F$;
  - all atomic subformulas $P \not\equiv \bot$ by $P \lor F$.

- The **refined Friedman-Dragalin translation** $FD'$ maps each formula $A$ to the formula $A^{FD'} := A[F/\bot]$.

Naming $FD'$ a refinement of $FD$ is a little bit misleading, as we explain now. On the one hand, $FD'$ simplifies $FD$ by dropping the replacement of atomic subformulas $P \not\equiv \bot$ by $P \lor F$. On the other hand,

- $FD$ is sound in the sense of $\mathsf{IL} \vdash A \Rightarrow \mathsf{ML} \vdash A^{FD}$;
- in general $FD'$ is sound only in the weaker sense of $\mathsf{ML} \vdash A \Rightarrow \mathsf{ML} \vdash A^{FD'}$.

So we can say that $FD'$ only really refines $FD$ on $\mathsf{ML}$, not on $\mathsf{IL}$. This limitation of $FD'$ is a problem if we want to apply a Friedman-Dragalin-like translation in $\mathsf{IL}$. But it is not problem if we only want to apply a Friedman-Dragalin-like translation after a negative translation into $\mathsf{ML}$ (not just into $\mathsf{IL}$).

**Proposition 11** (factorisations $N_2 = FD \circ G$ and $N_2 = FD' \circ G$).

1. For all formulas $A$ we have $\mathsf{ML} \vdash A^{N_2} \leftrightarrow (A^G)^{FD}$.
2. For all formulas $A$ we have $A^{N_2} \equiv (A^G)^{FD'}$.

**Proof.**

1. Let us abbreviate $(A^G)^{FD}$ by $A^{GFD}$. First we recall the definition of $G$ writing all negations $\neg A$ in the form $A \to \bot$:

$$P^G := (P \to \bot) \to \bot \ (P \neq \bot \text{ atomic}),$$

$$\bot^G := \bot,$$

$$(A \land B)^G := A^G \land B^G,$$

$$(A \lor B)^G := (A^G \to \bot) \land (B^G \to \bot) \to \bot,$$

$$(A \to B)^G := A^G \to B^G,$$

$$(\forall x A)^G := \forall x A^G,$$

$$(\exists x A)^G := \exists x (A^G \to \bot) \to \bot.$$
Using this we unfold $N_2$ and $GFD$ by induction on the structure of formulas:

\[ P^{N_2} \equiv (P \to F) \to F \ (P \neq \bot \text{ atomic}), \]
\[ \bot^{N_2} \equiv F, \]
\[ (A \land B)^{N_2} \equiv A^{N_2} \land B^{N_2}, \]
\[ (A \lor B)^{N_2} \equiv (A^{N_2} \to F) \land (B^{N_2} \to F) \to F, \]
\[ (A \to B)^{N_2} \equiv A^{N_2} \to B^{N_2}, \]
\[ (\forall x A)^{N_2} \equiv \forall x A^{N_2}, \]
\[ (\exists x A)^{N_2} \equiv \exists x (A^{N_2} \to F) \to F, \]
\[ P^{GFD} \equiv (P \lor F \to F) \to F \ (P \neq \bot \text{ atomic}), \]
\[ \bot^{GFD} \equiv F, \]
\[ (A \land B)^{GFD} \equiv A^{GFD} \land B^{GFD}, \]
\[ (A \lor B)^{GFD} \equiv (A^{GFD} \to F) \land (B^{GFD} \to F) \to F, \]
\[ (A \to B)^{GFD} \equiv A^{GFD} \to B^{GFD}, \]
\[ (\forall x A)^{GFD} \equiv \forall x A^{GFD}, \]
\[ (\exists x A)^{GFD} \equiv \exists x (A^{GFD} \to F) \to F. \]

Now we prove $ML \vdash A^{N_2} \leftrightarrow A^{GFD}$ by induction on the structure of formulas. The only non-trivial case is the one of atomic formulas $P \neq \bot$. In this case we argue $ML \vdash P^{N_2} \leftrightarrow P^{GFD}$ using $ML \vdash (P \to F) \leftrightarrow (P \lor F \to F)$.

2. Just note that $A^{N_2}$ and $(A^{G})^{FD'}$ are both syntactically equal to $A^G[F/\bot]$: we have we have $A^{N_2} \equiv A^G[F/\bot]$ by definition of $N_2$ and we have $(A^{G})^{FD'} \equiv A^G[F/\bot]$ by definition of $FD'$.

\[ \square \]

5 Characterisation of the negative translations intuitionistically equivalent to the usual ones

There are two properties relative to $NF$ that the usual negative translations share:

- to translate into $NF$ in $IL$;
- to act as the identity on $NF$ in $IL$.

We show that these two properties are not shared by $N_1$ and $N_2$.

**Proposition 12.** If $CL \vdash \neg F$ but $IL \nvdash \neg F$, then $N_1$ and $N_2$:

1. do not translate into $NF$ in $IL$;
2. do not act as the identity on $NF$ in $IL$.

**Proof.** We do the proof only for $N_1$ since the case of $N_2$ is analogous. By direct proof, assume $CL \vdash \neg F$ but $IL \nvdash \neg F$ and let us prove points 1 and 2.
1. If $N_1$ would translate into $\mathbf{NF}$ in $\mathbf{IL}$, then $\bot^{N_1} \equiv \bot \lor F$, which is equivalent in $\mathbf{IL}$ to $F$, would be equivalent in $\mathbf{IL}$ to a formula in $\mathbf{NF}$, contradicting point 2 of lemma 7.

2. If $N_1$ would act as the identity on $\mathbf{NF}$ in $\mathbf{IL}$, then $\mathbf{IL} \vdash \bot \leftrightarrow \bot$ (since $\bot \in \mathbf{NF}$) where $\bot^{N_1} \equiv \bot \lor F$, so $\mathbf{IL} \not\vdash \neg F$, contradicting the assumption $\mathbf{IL} \not\vdash \neg F$. \qed

Proposition 12 suggests that the two properties relative to $\mathbf{NF}$ may tell the difference between the usual negative translations into $\mathbf{IL}$ and other negative translations into $\mathbf{IL}$. Indeed, now we prove that they characterise the usual negative translations into $\mathbf{IL}$.

**Theorem 13.** Let $N$ be a negative translation into $\mathbf{IL}$ (ML). The following properties are equivalent.

1. $N$ is equivalent in $\mathbf{IL}$ (respectively, ML) to $G$.
2. $N$ translates into $\mathbf{NF}$ in $\mathbf{IL}$ (respectively, ML).
3. $N$ acts as the identity on $\mathbf{NF}$ in $\mathbf{IL}$ (respectively, ML).

**Proof.** We do the proof only for negative translations into $\mathbf{IL}$ since the case of negative translations into $\mathbf{ML}$ is analogous.

(1 $\Rightarrow$ 2) By direct proof, if $N$ is equivalent in $\mathbf{IL}$ to $G$, then $N$ translates into $\mathbf{NF}$ in $\mathbf{IL}$ because $G$ does so, as we wanted.

(2 $\Rightarrow$ 3) By direct proof, assume that $N$ translates into $\mathbf{NF}$ in $\mathbf{IL}$, consider an arbitrary formula $A \in \mathbf{NF}$ and let us prove $\mathbf{IL} \vdash A^N \leftrightarrow A$. By assumption the formula $A^N$ is equivalent in $\mathbf{IL}$ to a formula in $\mathbf{NF}$, and we have $A \in \mathbf{NF}$, so the formula $A^N \leftrightarrow A$ is equivalent in $\mathbf{IL}$ to a formula in $\mathbf{NF}$. Since $\mathbf{CL} \vdash A^N \leftrightarrow A$ by the characterisation theorem of $N$, and since $\mathbf{CL}$ is conservative over $\mathbf{NF}$ with respect to $\mathbf{NF}$, we have $\mathbf{IL} \vdash A^N \leftrightarrow A$, as we wanted.

(3 $\Rightarrow$ 1) By direct proof, assume that $N$ acts as the identity on $\mathbf{NF}$ in $\mathbf{IL}$, consider an arbitrary formula $A$ and let us prove $\mathbf{IL} \vdash A^N \leftrightarrow A^G$. By the characterisation theorem of $G$ we have $\mathbf{CL} + A \vdash A^G$ and $\mathbf{CL} + A^G \vdash A$. So by the soundness theorem into $\mathbf{IL}$ of $N$ we get $\mathbf{IL} + A^N \vdash A^{GN}$ and $\mathbf{IL} + A^{GN} \vdash A^N$ (where $A^{GN}$ abbreviates $(A^G)^N$). Therefore by the deduction theorem of $\mathbf{IL}$ we have (1) $\mathbf{IL} \vdash A^N \leftrightarrow A^{GN}$. Since $A^G \in \mathbf{NF}$ by a property of $G$, by the assumption we have (2) $\mathbf{IL} \vdash A^{GN} \leftrightarrow A^G$. From (1) and (2) we get $\mathbf{IL} \vdash A^N \leftrightarrow A^G$, as we wanted. \qed

Another property shared by the usual negative translations into $\mathbf{IL}$ is idempotence in $\mathbf{IL}$, that is $N \circ N = N$ in the sense of: $\mathbf{IL} \vdash (A^N)^N \leftrightarrow A^N$ for all formulas $A$. Idempotence in $\mathbf{IL}$ is sometimes proved using the properties relative to $\mathbf{NF}$. The proof roughly proceeds like this: if $N$ is a negative translation into $\mathbf{IL}$ that (1) translates into $\mathbf{NF}$ in $\mathbf{IL}$ and (2) acts as the identity on $\mathbf{NF}$ in $\mathbf{IL}$, then $A^N \in \mathbf{NF}$ by (1), so $\mathbf{IL} \vdash (A^N)^N \leftrightarrow A^N$ by (2). (This argument is not rigorous since from (1) we only get that $A^N$ is equivalent in $\mathbf{IL}$ to a formula in $\mathbf{NF}$, not that $A^N \in \mathbf{NF}$.) This relation
of idempotence in \( \text{IL} \) with the properties relative to \( \text{NF} \) can make us suspect that idempotence in \( \text{IL} \) also characterises the usual negative translations into \( \text{IL} \). But this is not so because, as we will show now, all negative translations into \( \text{IL} \) are idempotent in \( \text{IL} \) (but not equivalent in \( \text{IL} \), as we already saw).

**Definition 14.** Let \( N \) be a negative translation into \( \text{IL} \). We say that \( N \) is **idempotent in \( \text{IL} \) (ML)** if and only if for all formulas \( A \) we have \( \text{IL} \models (A^N)^N \leftrightarrow A^N \) (respectively, \( \text{ML} \models (A^N)^N \leftrightarrow A^N \)).

**Proposition 15.** All negative translations into \( \text{IL} \) (ML) are idempotent in \( \text{IL} \) (respectively, \( \text{ML} \)).

**Proof.** We do the proof only for negative translations into \( \text{IL} \) since the case of negative translations into \( \text{ML} \) is analogous.

Consider an arbitrary negative translation \( N \) into \( \text{IL} \), an arbitrary formula \( A \) and let us prove \( \text{IL} \models (A^N)^N \leftrightarrow A^N \). By the characterisation theorem of \( N \) we have \( \text{CL} + A^N \models A \) and \( \text{CL} + A \models A^N \). So by the soundness theorem into \( \text{IL} \) of \( N \) we get \( \text{IL} + (A^N)^N \models A^N \) and \( \text{IL} + A^N \models (A^N)^N \). Then by the deduction theorem of \( \text{IL} \) we have \( \text{IL} \models (A^N)^N \leftrightarrow A^N \), as we wanted. \( \square \)

### 6 Conclusion

The main three points of this article are the following.

- **Conjecture** The fact that the usual negative translations into \( \text{IL} \) are equivalent in \( \text{IL} \) leads to the conjecture: if we rigorously define the notion of a negative translation into \( \text{IL} \), then we should be able to prove that all negative translations are equivalent in \( \text{IL} \).

- **Refutation** We refuted the conjecture by presenting two counterexamples.

- **Characterisation** We characterised the usual negative translations into \( \text{IL} \) as being the ones that translate into \( \text{NF} \) in \( \text{IL} \), or equivalently, that act as the identity on \( \text{NF} \) in \( \text{IL} \).

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