PERIODIC SOLUTIONS TO NONLINEAR WAVE EQUATIONS WITH TIME DEPENDENT PROPAGATION SPEEDS

BOCHAO CHEN, YIXIAN GAO, YONG LI, AND XUE YANG

ABSTRACT. This paper focuses on the existence of families of time-periodic solutions to nonlinear wave equations with time-dependent coefficients. This linear model with variable propagation speed with respect to time describes a change of the quantity of the total energy. For the nonlinear model with general nonlinearity, we give the existence of time-periodic solutions under the periodic boundary conditions in an asymptotically full measure Cantor-like set. The proof relies on a suitable Lyapunov-Schmidt reduction together with a differentiable Nash-Moser iteration scheme.

1. INTRODUCTION

This paper is devoted to the study of time-periodic solutions to nonlinear wave equation subject to the periodic boundary conditions
\[ u_{tt} - a(\omega t)u_{xx} = \epsilon f(\omega t, x, u), \quad x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \]
where \( a(\omega t) > 0 \) with \((2\pi/\omega)\)-periodic in time, \( \epsilon, \omega > 0 \) are parameters, and the nonlinear forcing term \( f(\omega t, x, u) \) is \((2\pi/\omega)\)-periodic in time.

The theory of the model without forcing term may be developed from the following points of view: generalization from the classical wave equation to second order linear strictly hyperbolic equations with variable coefficients, i.e.,
\[ u_{tt} - \sum_{i,j=1}^{n} a_{ij}(t) u_{x_i x_j} = 0 \quad \text{with} \quad \sum_{i,j=1}^{n} a_{ij} \zeta_i \zeta_j \geq a_0 |\zeta|^2, \quad \forall \zeta \in \mathbb{R}^n, \]
where \( a_0 > 0 \). The variable propagation speed with respect to \( t \) describes a change of the quantity of the total energy, hence the effect of time-dependent coefficient is crucial for the asymptotic behavior of the solution. One of the main tasks for such problems is the precise analysis of solutions to (1.2) by virtue of taking into account the properties of the coefficients. There have been many results from the point of view of the Cauchy problem. A pioneer consideration of the local well-posedness on (1.2) was Colombini, De Giorgi and Spagnolo [13], also refers to [15, 16, 18, 23, 24]. Under some non-strict hyperbolicity condition, i.e.,
\[ \sum_{i,j=1}^{n} a_{ij} \zeta_i \zeta_j \geq 0, \quad \forall \zeta \in \mathbb{R}^n, \]
related results are given in [14, 25]. In spite of the wide study for the cauchy problem for the linear model, we more care about the existence of periodic solutions of the nonlinear model (1.1) with \( a(\cdot) \) being \( 2\pi \)-periodic and the forcing term \( f \) depending on \( t, x, u \).

Much attention has been drawn to the problem of finding time-periodic solutions to the classical nonlinear wave equation due to the first pioneering work of Rabinowitz [37–40] dealing with the weakly nonlinear homogeneous string with \( a(t) \equiv 1 \), while the time period \( T \) is required to be a rational multiple of \( \pi \), i.e., the

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frequency \( \omega \) has to be rational. If the forced frequency \( \omega \) is irrational, then it will arise a “small denominators problem”. At the end of the 1980s, a quite different approach which used the Kolmogorov-Arnold-Moser (KAM) theory was developed from the viewpoint of infinite dimensional dynamical systems by Kuksin and Wayne. This method allows one to obtain solutions whose periods are irrational multiples of the length of the spatial interval. In addition, this method is easily extended to construct quasi-periodic solutions, see [20, 22, 35]. Later, in order to overcome some limitations inherent to the usual KAM procedures, the application of Nash-Moser methods to infinite-dimensional dynamical systems has been introduced in the Nineties by Craig, Wayne and Bourgain [11, 12, 17]. Further developments are for example in [8, 10, 35]. The advantage of this approach is to require only the “first order Melnikov” non-resonance conditions, which are essentially the minimal assumptions. For the nonlinear wave equation with \( x \)-dependent coefficients, the existence of periodic solutions with periodic \( T \) is required to be a rational multiple \( \pi \) was studied by Barbu and Pavel [4–6]. Subsequently, Ji and Li obtained a series of results under the general boundary conditions and periodic or anti-periodic boundary conditions, see [26–29], while the kernel space is finite-dimensional.

If the kernel space is of infinite dimension, which was also posed as an open problem by Barbu and Pavel in [6], then the difficulty had been actually overcome by Ji and Li in [30]. In [2], for the forced vibrations of a nonhomogeneous string, under Dirichle boundary condition, Baldi and Berti proved the existence of periodic solutions.

In this paper, we consider the existence of periodic solutions of nonlinear wave equation with time-dependent coefficients. More precisely, under the periodic boundary conditions, when \( \epsilon \) is small and \( (\epsilon, \omega) \) belong to a Cantor set which has positive measure, asymptotically full as \( \epsilon \) tends to zero, we will present that equation (1.1) possesses a family of small amplitude periodic solutions in a suitable Sobolev space. There are two main challenges in looking for periodic solutions of (1.1). The first one is the so-called “small divisors problem” caused by resonances. Such a problem arises in the inversion of the spectrum of the linear operator \( \lambda(t) \partial_{tt} - \partial_{xx} \), whose spectrum is following form

\[
-\omega^2 \lambda_t^\pm + j^2 = -\omega^2 \frac{\lambda_t^2}{\epsilon^2} + j^2 + O(1/j), \quad l \in \mathbb{N}, \; j \in \mathbb{Z}.
\]

Under the assumption \( b \neq 0 \), above spectrum approaches to zero for almost every \( \omega \). Based on this reason, the operator \( \left( \lambda(t) \partial_{tt} - \partial_{xx} \right)^{-1} \) cannot map, in general, a functional space into itself, but only into a large functional space with less regularity. The other one is the influence of the variable coefficients \( a(t) \). In order to overcome the first challenge, we apply a Lyapunov-Schmidt reduction together with a differentiable Nash-Moser method [8, 10] under the “first order Melnikov” non-resonance conditions. In order to overcome the second one, we decompose \( u(t, x) \) with respect to the eigenfunctions \( e^{ijx}, \forall j \in \mathbb{Z} \), also see [11] for the Kirchhoff equations. However the difference with [11] is that the forcing term \( f \) not only depends \( t, x \) but also \( u \), which will leads to \( \omega_j \omega_k \) as a denominator when we verify the invertibility of the linearized operators, where \( \omega_j, \omega_k \) are defined by (4.33). Thus the lower bound of \( \omega_j \omega_k \) has to be given. Moreover we also consider the asymptotic formulae of the eigenvalues to the following Hill’s problem

\[
\begin{cases}
-y_{tt} + \frac{1}{\omega^2} \Pi_v f(t, x, v(\epsilon, \omega, w(t, x))) + w(t, x)y = \lambda a(t)y, \\
y(0) = y(2\pi), \quad y_{tt}(0) = y_{tt}(2\pi),
\end{cases}
\]

which will be helpful to give the invertibility of the linearized operator.

The rest of the paper is organized as follows: we decompose equation (2.1) as the bifurcation equation (Q) and the range equation (P) by a Lyapunov-Schmidt reduction in subsection 2.1. In subsection 2.2 the main results are stated in Theorem 2.2. And the (Q)-equation is solved by the classical implicit function theorem under Hypothesis 1 see Lemma 2.3. Relayed on a differentiable Nash-Moser iteration scheme, section 3 is devoted to solving the (P)-equation under the “first order Melnikov” non-resonance conditions, see the inductive lemma, i.e., Lemma 3.1. The object of section 4 is to check the inversion of the linearized operators, which is the core of the differentiable Nash-Moser iteration. Based on the type of boundary conditions, we first give the asymptotic formulae for the eigenvalues of Hill problem (4.1) in section 4.1. Next, we investigate that the linearized operators are invertible in subsection 4.2. And we show that \( A_\gamma \) defined by (4.51) is a
Cantor-like set in subsection [4.3]. In section [5] we list the the proof of some related results for the sake of completeness. The paper is concluded with some general remarks and directions for future work in section [6].

2. Main results

Rescaling the time \( t \to t/\omega \), we consider the existence of \( 2\pi \)-periodic solutions in time of

\[
\omega^2 u_{tt} - a(t) u_{xx} = \varepsilon f(t, x, u), \quad x \in \mathbb{T}.
\]  

(2.1)

It is obvious that if \( f(t, x, 0) \neq 0 \), then \( u = 0 \) is not the solution of equation (2.1).

For all \( s \geq 0 \), we define the Sobolev spaces \( \mathcal{H}^s \) of real-valued functions

\[
\mathcal{H}^s := \left\{ u(t, x) = \sum_{j \in \mathbb{Z}} u_j(t) e^{ijx} : u_j \in H^1(\mathbb{T}; \mathbb{C}), u_{-j} = \bar{u}_j, \|u\|^2_s := \sum_{j \in \mathbb{Z}} \|u_j\|_{H^1(1+j^{2s})}^2 < +\infty \right\},
\]

(2.2)

where \( \bar{u}_j \) is the complex conjugate of \( u_j \). Moreover define \( C_k \) by

\[
C_k := \left\{ f \in C(\mathbb{T} \times \mathbb{T} \times \mathbb{R} : (x, u) \mapsto f(t, x, u) \in C^k(\mathbb{T} \times \mathbb{R}; H^1(\mathbb{T})) \right\}.
\]

If \( f(t, x, u) = \sum_{j \in \mathbb{Z}} f_j(t, u) e^{ijx} \), then \( u \mapsto f_j(\cdot, u) \in C^k(\mathbb{R}; H^1(\mathbb{T})) \) with \( f_{-j} = \bar{f}_j \). Throughout this paper, our object is to find the solutions in \( \mathcal{H}^s \) with respect to \( (t, x) \in \mathbb{T} \times \mathbb{T} \) and \( f \in C_k \) for \( k \in \mathbb{N} \) large enough.

Remark 2.1. Let \( C(s) \) denote a constant depending on \( s \). The Sobolev space \( \mathcal{H}^s \) with \( s > \frac{1}{2} \) has the following properties:

(i) \( \|uv\|_s \leq C(s) \|u\|_s \|v\|_s, \quad \forall u, v \in \mathcal{H}^s; \)

(ii) \( \|u\|_{L^\infty(\mathbb{T}; H^1(\mathbb{T}))} \leq C(s) \|u\|_s, \quad \forall u \in \mathcal{H}^s. \)

Proof. The proof is given in the Appendix. \( \square \)

2.1. The Lyapunov-Schmidt reduction. For any \( u \in \mathcal{H}^s \), it can be written as the sum of \( u_0(t) + \tilde{u}(t, x) \), where \( \tilde{u}(t, x) = \sum_{j \neq 0} u_j(t) e^{ijx} \). Then we perform the Lyapunov-Schmidt reduction with respect to the following decomposition

\[
\mathcal{H}^s = (V \cap \mathcal{H}^s) \oplus (W \cap \mathcal{H}^s) = V \oplus (W \cap \mathcal{H}^s),
\]

where

\[
\begin{align*}
V := H^1(\mathbb{T}), \quad W := \left\{ w = \sum_{j \in \mathbb{Z}, j \neq 0} w_j(t) e^{ijx} \in \mathcal{H}^0 \right\}.
\end{align*}
\]

Denoting by \( \Pi_V \) and \( \Pi_W \) the projectors onto \( V \) and \( W \) respectively, equation (2.1) is equivalent to the bifurcation equation (Q) and the range equation (P):

\[
\begin{cases}
\omega^2 v'' = \varepsilon (\Pi_V F(v + w), \\
L_\omega w = \varepsilon (\Pi_W F(v + w), \\
\end{cases} \quad (Q),
\]

\[
\begin{cases}
\omega^2 w_{tt} - a(t) w_{xx} = \varepsilon f(t, x, u), \\
\end{cases} \quad (P),
\]

(2.3)

where \( u = v + w \) with \( v \in V, w \in W \), and

\[
L_\omega w := \omega^2 w_{tt} - a(t) w_{xx}, \quad F : u \to f(t, x, u).
\]

(2.4)

Similarly, the nonlinearity \( f \) can be written into

\[
f(t, x, u) = f_0(t, u) + \tilde{f}(t, x, u),
\]

where \( \tilde{f}(t, x, u) = \sum_{j \neq 0} f_j(t, u) e^{ijx} \), which leads to

\[
\Pi_V F(v) = \Pi_V f(t, x, v) = \Pi_V f_0(t, v(t)) + \Pi_V \tilde{f}(t, x, v(t)) = f_0(t, v(t)) \quad \text{if} \quad w = 0.
\]

Hence the (Q)-equation is reduced to the space-independent equation

\[
\omega^2 v'' = \varepsilon f_0(t, v), \quad t \in \mathbb{T} \quad \text{as} \quad w \to 0. \quad (2.5)
\]
Equation (2.5) is also called the infinite-dimensional “zeroth-order bifurcation equation”, see also [17]. We make the following hypothesis.

**Hypothesis 1.** Provided that \( \epsilon/\omega^2 < \tilde{\delta} \) is small enough, the problem

\[
\omega^2 v''(t) = \epsilon f_0(t, v(t)), \quad t \in \mathbb{T}
\]

admits a nondegenerate solution \( v \in H^1(\mathbb{T}) \) with \( \int_{\mathbb{T}} v(t)dt = 0 \), i.e., the linearized equation

\[
\omega^2 h'' = \epsilon f'_0(\hat{v})h
\]

possesses only the trivial solution \( h = 0 \) in \( H^1(\mathbb{T}) \).

Let us explain the rationality of Hypothesis 1. Under \( \int_{\mathbb{T}} h(t)dt = 0 \), linearized equation (2.7) with \( \epsilon = 0 \) possesses only the trivial solution \( h = 0 \). Hence \( \hat{v} = 0 \) is the nondegenerate solution of (2.6) with \( \epsilon = 0 \). It follows from the implicit function theorem that there exists a constant \( \tilde{\delta} > 0 \) small enough such that if \( \epsilon/\omega^2 < \tilde{\delta} \), then linearized equation (2.7) possesses only the trivial solution \( h = 0 \) because of \( \int_{\mathbb{T}} h(t)dt = 0 \).

2.2. **Main results and solution of the bifurcation equation.** To fix ideas, we shall take \( \omega \) inside a fixed sub-interval of \((0, \infty)\), such as \( \omega \in (1, 2) \) (in fact any interval \((a, b) \subset (0, \infty) \) also holds). Let us state our main theorem as follows.

**Theorem 2.2.** Assume that Hypotheses [1] holds for fixed \( \hat{\omega}, \) with \( \hat{\omega} \in (1, 2) \) and \( \frac{\hat{\omega}}{\omega} < \tilde{\delta} \) small enough, and that Hypotheses [2] (see (4.11)) also holds. Fix \( \tau \in (1, 2), \gamma \in (0, 1) \). Let \( a(t) \in H^3(\mathbb{T}) \) with \( a(t) > 0 \), \( f \in \mathcal{C}_k \) with \( k \geq s + \beta + 3 \), where

\[
\beta := \chi(\tau - 1 + \sigma) + \chi(2\tau + 2) + \frac{\chi}{\chi - 1}(\tau - 1 + \sigma)
\]

with \( \chi > 1 \), \( \sigma = (\tau - 1)/(2 - \tau) \). There exist a constant \( K > 0 \) depending on \( a, f, \hat{\omega}, \gamma, \tau, s, \beta, \) a neighborhood \((\epsilon_1, \epsilon_2) \) of \( \hat{\epsilon} \), a neighborhood \((\omega_1, \omega_2) \) of \( \hat{\omega} \), \( 0 < r < 1 \) and a \( C^2 \) map \( v(\epsilon, \omega, w) \) defined on \((\epsilon_1, \epsilon_2) \times (\omega_1, \omega_2) \times \{ w \in W \cap H^s : \|w\|_s < r \} \), with values in \( H^1 \), a map \( w(\epsilon, \omega) \in W \cap H^s \) and a Cantor-like set \( A_\gamma \subseteq (0, \delta^2 \gamma^2) \times (\omega_1, \omega_2) \), where \( A_\gamma \) is defined by (4.51), such that for all \( (\epsilon, \omega) \in A_\gamma \),

\[
u(\epsilon, \omega, w, \omega, \omega) := v(\epsilon, \omega, w(\epsilon, \omega)) + w(\epsilon, \omega) \in H^1(\mathbb{T}) \oplus (W \cap H^s)
\]

with

\[
\int_{\mathbb{T}} u(\epsilon, \omega; \cdot)dt = 0
\]

is a solution of equation (2.1). Moreover, such a solution satisfies

\[
\|w\|_s \leq \frac{K\epsilon}{\gamma}, \quad \|\partial_t w\|_s \leq \frac{K\epsilon}{\gamma^2}, \quad \|v(\epsilon, \omega, w; \cdot) - v(\epsilon, \omega, 0; \cdot)\|_{H^1} \leq \frac{K\epsilon}{\gamma},
\]

\[
\|v(\epsilon, \omega, 0; \cdot) - \tilde{v}(\cdot)\|_{H^1} \leq K(|\epsilon - \tilde{\epsilon}| + |\omega - \tilde{\omega}|).
\]

First of all, we will solve the \((Q)\)-equation relaying on the classical implicit function theorem under Hypothesis 1.

**Lemma 2.3.** Let Hypothesis 1 hold for some \( \hat{\epsilon}, \hat{\omega}, \) with \( \hat{\omega} \in (1, 2) \) and \( \frac{\hat{\omega}}{\omega} < \tilde{\delta} \). There exists a neighborhood \((\epsilon_1, \epsilon_2) \) of \( \hat{\epsilon} \), \((\omega_1, \omega_2) \) of \( \hat{\omega} \), \( 0 < r < 1 \), and a \( C^2 \) map

\[v : (\epsilon_1, \epsilon_2) \times (\omega_1, \omega_2) \times \{ w \in W \cap H^s : \|w\|_s < r \} \to H^1(\mathbb{T}),
\]

\[(\epsilon, \omega, w) \mapsto v(\epsilon, \omega, w; \cdot)
\]

with \( \int_{\mathbb{T}} v(\epsilon, \omega, w; t)dt = 0 \) and

\[
\|v(\epsilon, \omega, w; \cdot) - v(\epsilon, \omega, 0; \cdot)\|_{H^1} \leq C\|w\|_s, \quad \|v(\epsilon, \omega, 0; \cdot) - \tilde{v}(\cdot)\|_{H^1} \leq C(|\epsilon - \tilde{\epsilon}| + |\omega - \tilde{\omega}|),
\]

(2.10)

such that \( v(\epsilon, \omega, w; \cdot) \) solves the \((Q)\)-equation in (2.3).
Moreover we also have properties (Fréchet derivative of $F$)
\[
P \frac{\partial F}{\partial P} = C \frac{\partial F}{\partial H}
\]
is invertible on $H^1(\mathbb{T})$. Since $f \in C_k$, by Lemma 5.4, it holds that the following map
\[
(\epsilon, \omega, w, v) \mapsto \omega^2 v'' - \epsilon \Pi_V F(v + w)
\]
is $C^2(\{\epsilon_1, \epsilon_2\} \times \{\omega_1, \omega_2\} \times (W \cap H^s) \times H^1(\mathbb{T}) ; H^1(\mathbb{T}))$. Therefore, by the implicit function theorem, there is a $C^2$-path
\[
(\epsilon, \omega, w) \mapsto v(\epsilon, \omega, w; \cdot)
\]
with $\int_0^t v(\epsilon, \omega, w; t)dt = 0$ and $v(\epsilon, \omega, w; t) = \hat{v}(t)$ such that $v(\epsilon, \omega, w; t)$ is a solution of the $(Q)$-equation in (2.3) and satisfies (2.10). \hfill \Box

3. Solution of the $(P)$-Equation

In this section, our goal is to solve the $(P)$-equation in (2.3), i.e.,
\[
L_\omega w = \epsilon \Pi_W F(\epsilon, \omega, w),
\]
where $F(\epsilon, \omega, w) := F(\epsilon, \omega, w) + w$. Denote by the symbol $\lfloor \cdot \rfloor$ the integer part. Consider the orthogonal splitting $W = W_{N_0} \oplus W_{N_0}^\perp$, where
\[
W_{N_0} := \left\{ w \in W : w = \sum_{1 \leq |j| \leq N_0} w_j(t)e^{ijx} \right\}, \quad W_{N_0}^\perp := \left\{ w \in W : w = \sum_{|j| > N_0} w_j(t)e^{ijx} \right\},
\]
with
\[
N_0 := \lfloor e^{\chi N_0} \rfloor, \quad q = \ln N_0, \quad \chi > 1.
\]
Let $P_{N_0}$ and $P_{N_0}^\perp$ denote the orthogonal projectors onto $W_{N_0}$ and $W_{N_0}^\perp$, respectively, i.e.,
\[
P_{N_0} : W \to W_{N_0}, \quad P_{N_0}^\perp : W \to W_{N_0}^\perp.
\]
If $f \in C_k$ with $k \geq s^f + 3$, then the composition operator $F$ has the following standard properties:
- $(P1)$ (Regularity) $F \in C^2(\mathcal{H}^s; \mathcal{H}^s)$ and $F, D_\omega F, D_{\omega^2} F$ are bounded on $\{ ||w||_s \leq 1 \}$, where $D_\omega F$ is the Fréchet derivative of $F$ with respect to $w$.
- $(P2)$ (Tame) $\forall s \leq s' \leq k - 3, \forall w \in \mathcal{H}^{s'}$ with $||w||_s \leq 1$,
\[
||F(\epsilon, \omega, w)||_{s'} \leq C(s')(1 + ||w||_{s'}),
\]
\[
||D_\omega F(\epsilon, \omega, w)||_{s'} \leq C(s')(||w||_{s'}||h||_s + ||h||_{s'}),
\]
\[
||D_{\omega^2} F(\epsilon, \omega, w)||_{s'} \leq C(s')(||w||_{s'}||h||_s||h||_s + ||h||_{s'}||h||_s + ||h||_s||h||_{s'}).\]
- $(P3)$ (Taylor Tame) $\forall s \leq s' \leq k - 3, \forall w \in \mathcal{H}^{s'}, \forall h \in \mathcal{H}^s$ and $||w||_s \leq 1, ||h||_s \leq 1$,
\[
||F(\epsilon, \omega, w + h) - F(\epsilon, \omega, w) - D_\omega F(\epsilon, \omega, w)[h]||_s \leq C||h||_s^2,
\]
\[
||F(\epsilon, \omega, w + h) - F(\epsilon, \omega, w) - D_\omega F(\epsilon, \omega, w)[h]||_{s'} \leq C(s')(||w||_{s'}||h||_s^2 + ||h||_s||h||_{s'}).
\]
Moreover we also have properties $(P4)$–$(P5)$,
- $(P4)$ (Smoothing) For all $N \in \mathbb{N}\setminus\{0\}$, one has
\[
||P_{N^+} u||_s \leq N^r ||u||_s, \quad \forall u \in \mathcal{H}^s,
\]
\[
||P_{N^+}^1 u||_s \leq N^{-r} ||u||_{s+r}, \quad \forall u \in \mathcal{H}^{s+r}.
\]
- $(P5)$ (Invertibility of $\mathcal{L}_N$) Define the linearized operator as
\[
\mathcal{L}_N(\epsilon, \omega, w)[h] := -L_\omega h + \epsilon P_N \Pi_W D_\omega F(\epsilon, \omega, w)[h], \quad \forall h = \sum_{1 \leq |j| \leq N} h_j(t)e^{ijx} \in W_N,
\]
where $L_\omega$ is given by (2.4). Denote by $\lambda^\pm_l (\epsilon, \omega, w), l \geq 0$ the eigenvalues of Hill’s problem (4.1), where, abusing notation, we set $\lambda^\pm_l (\epsilon, \omega, w)$ for $\lambda_0 (\epsilon, \omega, w)$. Let $(\epsilon, \omega) \in \Delta^\gamma_{N} (w)$ for fixed $\gamma \in (0, 1), \tau \in (1, 2)$, where

$$\Delta^\gamma_{N} (w) := \left\{ (\epsilon, \omega) \in (\epsilon_1, \epsilon_2) \times (\omega_1, \omega_2) : \epsilon \leq \delta_5 \gamma^2, \left| \omega \sqrt{\lambda^\pm_l (\epsilon, \omega, w)} - j \right| > \frac{\gamma}{j^\tau}, \right. \left. \right\}$$

and $|\omega - j| > \frac{\gamma}{j^\tau}, \forall j = 1, 2, \cdots, N, \forall l \geq 0$,

with $\epsilon_i, \omega_i, i = 1, 2$ being given by Lemma 2.3 $c$ being defined by (4.4) and $\delta_5$ being given in Lemma 4.8. There exist $K, K(s') > 0$ such that if

$$\|w\|_{s+\sigma} \leq 1 \quad \text{with} \quad \sigma := \tau(\tau - 1)/(2 - \tau),$$

then $L_N (\epsilon, \omega, w)$ is invertible with

$$\|L_N^{-1} (\epsilon, \omega, w) h\|_s \leq K \gamma^{-1} N^{\tau-1} \|h\|_s, \forall s > 1/2$$

$$\|L_N^{-1} (\epsilon, \omega, w) h\|_{s'} \leq K(s') \gamma^{-1} N^{\tau-1} (\|h\|_{s'} + \|w\|_{s'+\sigma} \|h\|_s), \forall s' \geq s > 1/2. $$

Denote by $A_0$ the open set

$$A_0 := \left\{ (\epsilon, \omega) \in (\epsilon_1, \epsilon_2) \times (\omega_1, \omega_2) : \omega \sqrt{\lambda^\pm_l (\epsilon, \omega, w)} - j \right\}$$

and, abusing notations, we set $\lambda^\pm_l$ for $\lambda_0$ and $\lambda^\pm_l, l \geq 0$ are the eigenvalues of the Sturm-Liouville problem

$$-y''(t) = \lambda \alpha(t)y(t),$$

$$y(0) = y(2\pi) = 0, \quad y'(0) = y'(2\pi). \quad (3.12)$$

Furthermore let $B(0, \rho)$ stand for the open ball of center 0 and radius $\rho$.

**Lemma 3.1.** (inductive scheme) For all $n \in \mathbb{N}$, there exists a sequence of subsets $(\epsilon, \omega) \in A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1 \subseteq A_0$, where

$$A_n := \left\{ (\epsilon, \omega) \in A_{n-1} : (\epsilon, \omega) \in \Delta^\gamma_{N_n} (w_{n-1}) \right\},$$

and a sequence $w_n (\epsilon, \omega) \in W_{N_n}$ with

$$\|w_n\|_{s+\sigma} \leq 1, \quad (3.13)$$

and

$$\|w_0\| \leq \frac{K_1 \epsilon}{\gamma} N_0^{\tau-1}, \|w_k - w_{k-1}\| \leq \frac{K_2 \epsilon}{\gamma} N_0^{-\sigma-1}, \forall 1 \leq k \leq n, \quad (3.14)$$

$$\|\partial_{\alpha} w_0\| \leq \frac{K'_1 \epsilon}{\gamma^2 \omega^2} N_0^{\tau+1}, \|\partial_{\alpha} (w_k - w_{k-1})\| \leq \frac{K'_2 \epsilon}{\gamma^2 \omega^2} N_0^{-1}, \forall 1 \leq k \leq n, \quad (3.15)$$

where $K_i, K'_i > 0, i = 1, 2$ (depending on $a, f, \tilde{\delta}, \tilde{\omega}, \gamma, \tau, s, \beta$ at most), $\epsilon \gamma^{-1} K_1 N_0^{\tau} < r$ and $\epsilon \gamma^{-1} K_2 < r$, such that if $(\epsilon, \omega) \in A_n$ and $\epsilon \gamma^{-1} \leq \delta_4$ is small enough ($\delta_4$ is seen in Lemma 3.4), then $w_n (\epsilon, \omega)$ is a solution of

$$L_\omega w - \epsilon P_{N_n} W\mathcal{F}(\epsilon, \omega, w) = 0. \quad (P_{N_n})$$

**Proof:** The proof of the lemma is given by induction. At the first step of iteration, since the eigenvalues of $1/\alpha L_\omega$ on $W_{N_0}$ are

$$- \omega^2 \lambda^\pm_l + j^2, \quad \forall 1 \leq |j| \leq N_0, \forall l \geq 0 \quad (3.16)$$

with $\lambda^\pm_l$ are the eigenvalues of (3.12), by the definition of $A_0$, solving equation ($P_{N_0}$) is reduced to look for the fixed point problem:

$$w = U_0 (w),$$
where

\[ U_0 : \ W_{N_0} \to W_{N_0}, \ w \mapsto \epsilon (\frac{1}{a} L_\omega)^{-1} \frac{1}{a} P_{N_0} \Pi W \mathcal{F}(\epsilon, \omega, w). \]  

(3.17)

Let us show that \( U_0 \) is a contraction map, i.e., the following lemma.

**Lemma 3.2.** If \( \epsilon \gamma^{-1} \leq \delta_1 N_0^{-1 - \tau} \leq \frac{\delta(s + \beta)}{L(s + \beta)} \) is small enough, then the map \( U_0 \) is a contraction in

\[ B(0, \rho_0) := \{ w \in W_{N_0} : \| w \|_s \leq \rho_0 \} \quad \text{with} \quad \rho_0 := \epsilon \gamma^{-1} K_1 N_0^{-1}, \]

where \( \epsilon \gamma^{-1} K_1 N_0^{-1} < r \) (recall Lemma 2.3). Note that the condition \( \delta_1 N_0^{-1 - \tau} \leq \delta(s + \beta)/L(s + \beta) \) is to guarantee that Lemma 3.7 holds.

**Proof.** By the definition of \( A_0 \) and property (P1), one has

\[ \| U_0(w) \|_s \leq \epsilon \gamma^{-1} N_0^{-\tau - 1} \| 1/\alpha \|_{H^1} \| P_{N_0} \Pi W \mathcal{F}(\epsilon, \omega, w) \|_s \leq \epsilon \gamma^{-1} K_1 N_0^{-1} \]

for \( \epsilon \gamma^{-1} N_0^{-1} \leq \delta_1 \) small enough.

Moreover proceeding the proof as above yields

\[ \| D_u U_0(w) \|_s = \| (\epsilon \frac{1}{a} L_\omega)^{-1} \frac{1}{a} P_{N_0} \Pi W D_u \mathcal{F}(\epsilon, \omega, w) \|_s \leq \epsilon \gamma^{-1} K_1 N_0^{-1} \leq 1/2 \]

(3.18)

for \( \epsilon \gamma^{-1} N_0^{-1} \leq \delta_1 \) small enough. Thus the map \( U_0 \) is a contraction in \( B(0, \rho_0) \).

Let us continue to the proof of the first step iteration. Denote by \( w_0 \) the unique solution of equation \( (P_{N_0}) \) in \( B(0, \rho_0) \). It is obvious that

\[ \omega^2 \partial_{tt} w_0 = a(t) \partial_{xx} w_0 + \epsilon P_{N_0} \Pi W \mathcal{F}(\epsilon, \omega, w_0), \]

which establishes

\[ \| \partial_{tt} w_0 \|_s \leq K \frac{\epsilon \gamma}{\gamma^2} N_0^{\tau + 1} \]

due to (3.8), Lemma 3.2 and \( \gamma \in (0, 1) \).

Moreover it follows from formula (3.8) and Lemma 3.2 that

\[ \| w_0 \|_{s + \beta} = \| (\epsilon \frac{1}{a} L_\omega)^{-1} \frac{1}{a} P_{N_0} \Pi W \mathcal{F}(\epsilon, \omega, w_0) \|_{s + \beta} \leq K \]

(3.19)

for \( \epsilon \gamma^{-1} N_0^{-1} \leq \delta_1 \) small enough.

At the second step of iteration, assume that we have obtained a solution \( w_n \in W_{N_n} \) of \( (P_{N_n}) \) satisfying conditions (3.13)–(3.15). Our goal is to find a solution \( w_{n+1} \in W_{N_{n+1}} \) of \( (P_{N_{n+1}}) \) with conditions (3.13)–(3.15) at \((n + 1)\)-th step. Denote by

\[ w_{n+1} = w_n + h \quad \text{with} \quad h \in W_{N_{n+1}} \]

a solution of

\[ L_\omega w - \epsilon P_{N_{n+1}} \Pi W \mathcal{F}(\epsilon, \omega, w) = 0. \]

(\(P_{N_{n+1}}\))

It follows from \( L_\omega w_n - \epsilon P_{N_n} \Pi W \mathcal{F}(\epsilon, \omega, w_n) = 0 \) that

\[ L_\omega (w_n + h) - \epsilon P_{N_{n+1}} \Pi W \mathcal{F}(\epsilon, \omega, w_n + h) = L_\omega h + L_\omega w_n - \epsilon P_{N_{n+1}} \Pi W \mathcal{F}(\epsilon, \omega, w_n + h) \]

\[ = - L_{N_{n+1}} (\epsilon, \omega, w_n) h + R_n(h) + r_n, \]

where

\[ R_n(h) := - \epsilon P_{N_{n+1}} (\Pi W \mathcal{F}(\epsilon, \omega, w_n + h) - \Pi W \mathcal{F}(\epsilon, \omega, w_n) - \Pi W D_u \mathcal{F}(\epsilon, \omega, w_n)[h]), \]

(3.20)

\[ r_n := \epsilon (P_{N_n} \Pi W \mathcal{F}(\epsilon, \omega, w_n) - P_{N_{n+1}} \Pi W \mathcal{F}(\epsilon, \omega, w_n)) = - \epsilon P_{N_n}^\perp P_{N_{n+1}} \Pi W \mathcal{F}(\epsilon, \omega, w_n). \]  

(3.21)
By means of (3.13) and $(\epsilon, \omega) \in A_{n+1} \subseteq A_n$, property (P5) shows that the linearized operator $L_{N_{n+1}}(\epsilon, \omega, w_n)$ (recall (3.10)) is invertible with
\[
\|L_{N_{n+1}}^{-1}(\epsilon, \omega, w_n)h\|_s \leq K N^{-1}_{n+1}\|h\|_s, \quad \forall s > 1/2, \tag{3.22}
\]
\[
\|L_{N_{n+1}}^{-1}(\epsilon, \omega, w_n)h\|_{s'} \leq K(s')^{-1}N^{-1}_{n+1}(\|h\|_{s'} + \|w\|_{s'+\delta}\|h\|_s), \quad \forall s' \geq s > 1/2. \tag{3.23}
\]

Then we reduce solving $(P_{N_{n+1}})$ to find the fixed point of $h = U_{n+1}(h)$, where
\[
U_{n+1}: W_{N_{n+1}} \rightarrow W_{N_{n+1}}, \quad h \mapsto L_{N_{n+1}}(\epsilon, \omega, w_n)^{-1}(R_n(h) + r_n). \tag{3.24}
\]

Moreover set
\[
S_n = 1 + \|w_n\|_{\beta}. \tag{3.25}
\]

Similar to Lemma 3.2, we prove the map $U_{n+1}$ is a contraction, i.e., the following lemma.

**Lemma 3.3.** For $(\epsilon, \omega) \in A_{n+1}$ and $\epsilon^{-1} \leq \delta_2 \leq \delta_1 N^{-1}_0$ small enough, there exists $K_2 > 0$ such that the map $U_{n+1}$ is a contraction in
\[
B(0, \rho_{n+1}) := \{ h \in W_{N_{n+1}} : \|h\|_s \leq \rho_{n+1} \} \quad \text{with} \quad \rho_{n+1} := \epsilon^{-1} K_2 N^{-\sigma}_{n+1}, \tag{3.26}
\]
where $\epsilon^{-1} K_2 < r$ ($r$ is given by Lemma 2.3). Moreover the unique fixed point $h_{n+1}(\epsilon, \omega)$ of $U_{n+1}$ satisfies
\[
\|h_{n+1}\|_s \leq \epsilon^{-1} K_2 N^{-1}_{n+1} N^{-\beta}_{\gamma} S_n. \tag{3.27}
\]

**Proof.** It follows from (3.20)–(3.21) that
\[
\|R_n(h)\|_s \leq \epsilon C \|h\|_s^2, \quad \|r_n\|_s \leq \epsilon C(\beta) N^{-\beta}_{\gamma} S_n. \tag{3.28}
\]

Let us claim that, for $\epsilon^{-1} \leq \delta_2$ small enough, the following holds
\[
(F1): \quad \nu_n \leq \tilde{C} S_n N^{-\tau}_{n+1}(n + \sigma) \tag{3.29}
\]
for some constant $\tilde{C} := \tilde{C}(\chi, q, \tau, \sigma) > 0$. The proof of (F1) will be given in Lemma 3.5.

For $\epsilon^{-1} \leq \delta_2$ small enough, combining (3.24) with (3.22), (3.28), (3.26) yields
\[
\|U_{n+1}(h)\|_s \leq \epsilon K' \gamma N^{-1}_{n+1}\|h\|_s^2 + \epsilon K' \gamma N^{-1}_{n+1} N^{-\beta}_{\gamma} S_n \tag{3.30}
\]
\[
\leq \epsilon K' \gamma N^{-1}_{n+1} \rho_{n+1} + \epsilon K' \gamma N^{-1}_{n+1} N^{-\beta}_{\gamma} S_n.
\]

Using the definition of $\rho_{n+1}$, (3.29), (3.8) and (3.19), we derive
\[
\frac{\epsilon K' \gamma N^{-1}_{n+1} \rho_{n+1}}{2} \leq \frac{\epsilon K' \gamma N^{-1}_{n+1} N^{-\beta}_{\gamma} S_n}{2}, \tag{3.31}
\]
which leads to $\|U_{n+1}(h)\|_s \leq \rho_{n+1}$. Moreover taking the derivative of $U_{n+1}$ with respect to $h$ yields
\[
D_h U_{n+1}(h)[w] = -\epsilon L_{N_{n+1}}^{-1}(\epsilon, \omega, w_n) P_{N_{n+1}}(\Pi_W D_w F(\epsilon, \omega, w_n + h) - \Pi_W D_w F(\epsilon, \omega, w_n)) w. \tag{3.32}
\]

For $\epsilon^{-1} \leq \delta_2$ small enough, it follows from (3.31)–(3.32) that
\[
\|D_h U_{n+1}(h)[w]\|_s \leq \frac{\epsilon K' \gamma N^{-1}_{n+1} \rho_{n+1}}{2} \|w\|_s \|w\|_s \leq \frac{\epsilon K' \gamma N^{-1}_{n+1} \rho_{n+1}}{2} \|w\|_s \leq \frac{1}{2} \|w\|_s. \tag{3.33}
\]

Hence $U_{n+1}$ is a contraction in $B(0, \rho_{n+1})$.

Let $h_{n+1}(\epsilon, \omega)$ denote the unique fixed point of $U_{n+1}$. With the help of (3.24)–(3.26), (3.30)–(3.31), one arrives at
\[
\|h_{n+1}\|_s \leq \frac{1}{2} \|h_{n+1}\|_s + \epsilon K' \gamma N^{-1}_{n+1} N^{-\beta}_{\gamma} S_n,
\]
which leads to (3.27).
Finally, let us complete the proof of Lemma 3.1. Since $w_n, w_{n+1}$, with $w_{n+1} = w_n + h_{n+1}$, are solutions of equations $(P_{N_0})$, $(P_{N_{n+1}})$ respectively, one has

$$
\omega^2 \partial_{tt} h_{n+1} = a(t) \partial_x h_{n+1} + \epsilon P_{N_{n+1}} (\Pi W F(\epsilon, \omega, w_n + h_{n+1}) - \Pi W F(\epsilon, \omega, w_n))
\leq
+ \epsilon P_{N_n} P_{N_{n+1}} \Pi W F(\epsilon, \omega, w_n).
$$

(3.34)

It follows from (3.34), (3.27) and $\gamma \in (0, 1)$ that

$$
\|\partial_{tt} h_{n+1}\|_s \leq C'(s) e^{\omega^2} \left(\|h_{n+1}\|_{s+2} + \|h_{n+1}\|_s + N_{n-\beta} S_n\right) \leq \frac{K_4 e}{\gamma \omega^2} N_{n+1}^{\tau-1} N_{n-\beta} S_n,
$$

(3.35)

which leads to

$$
\|\partial_{tt} h_{n+1}\|_s \leq \frac{K_4 e}{\gamma \omega^2} N_{n+1}^{\tau-1}
$$

by (3.8), (3.29) and (3.19). Moreover if $\gamma^{-1} \leq \delta_3$ with $\delta_3 \leq \delta_2$ is small enough, according to Lemmata 3, 4 then the following holds:

$$
\|w_{n+1}\|_{s+\sigma} \leq \sum_{k=0}^{n+1} \|h_k\|_{s+\sigma} \leq \sum_{k=0}^{n+1} N_k^\sigma \|h_k\|_s \leq \sum_{k=1}^{n+1} N_k^\sigma \frac{K_2 e N_k^{\tau-1}}{\gamma} + N_0^\sigma \frac{K_1 N_0^{\tau-1}}{\gamma} \leq 1,
$$

which verifies that condition (3.13) holds at $(n+1)$-th step.

Let us show the estimates on the derivatives of $h_k$ with respect to $\omega$.

**Lemma 3.4.** For $(\epsilon, \omega) \in A_k$, if $\gamma^{-1} \leq \delta_4 \leq \delta_3$ is small enough, then the map $h_k = w_k - w_{k-1} \in C^1(A_k; W_{N_k})$ with

$$
\|\partial_{\omega} w_0(\epsilon, \cdot)\|_s \leq \frac{K_3 e}{\gamma \omega^2}, \quad \|\partial_{\omega} h_k(\epsilon, \cdot)\|_s \leq \frac{K_4 e}{\gamma \omega^2} N_k^{\tau-1}, \quad \forall 1 \leq k \leq n
$$

(3.36)

for some constants $K_3, K_4 > 0$.

**Proof.** The proof is divided into three parts.

Step1: Let us define

$$
\mathcal{U}_0(\epsilon, \omega, w) := w - \epsilon \left(\frac{1}{a} L_\omega \right)^{-1} \frac{1}{a} P_{N_0} \Pi W F(\epsilon, \omega, w).
$$

(3.37)

It follows (3.17), Lemma 3.2 that $\mathcal{U}_0(\epsilon, \omega, w_0) = 0$.

For $\gamma^{-1} N_0^{\tau-1} \leq \delta_1$ small enough, formulae (3.37) and (3.17)–(3.18) imply that

$$
D_{\omega} \mathcal{U}_0(\epsilon, \omega, w_0) = I_d - \epsilon \left(\frac{1}{a} L_\omega \right)^{-1} \frac{1}{a} P_{N_0} \Pi W D_{\omega} F(\epsilon, \omega, w_0)
$$

is invertible. Obviously, we can get $w_0 \in C^1(A_0; W_{N_0})$ by the implicit function theorem. Moreover taking the derivative of the identity $(\frac{1}{a} L_\omega)(\frac{1}{a} L_\omega)^{-1} w = w$ with respect to $\omega$ yields

$$
\partial_{\omega}(\frac{1}{a} L_\omega)^{-1} w = -\frac{1}{a} (\partial_{tt}(\frac{1}{a} L_\omega))^{-1} w,
$$

which then gives that $\partial_{\omega} w_0$ is equal to

$$
-(I_d - D_{\omega} U_0(w_0))^{-1} \left(\frac{1}{a} (L_\omega)^{-1} (\frac{2\omega}{\alpha} \partial_x w_0) + \epsilon \left(\frac{1}{a} L_\omega \right)^{-1} \frac{1}{a} P_{N_0} \Pi W \partial_{\omega} L_\omega f(\epsilon, \omega, v(\epsilon, \omega, w_0) + w_0) \partial_{\omega} v\right)
$$

by (3.17) and taking the derivative of $w_0 = \epsilon (\frac{1}{a} L_\omega)^{-1} \frac{1}{a} P_{N_0} \Pi W F(\epsilon, \omega, w_0)$ with respect to $\omega$. Then applying (3.15)–(3.16), the definition of $A_0$ and Lemma 2.3, it yields

$$
\|\partial_{\omega} w_0\|_s \leq \frac{K_3 e}{\gamma \omega^2}.
$$

(3.38)

Combining this with (3.8) shows that for $\gamma^{-1} \leq \delta_4$ small enough,

$$
\|\partial_{\omega} w_0\|_{s+\beta} \leq \hat{K}(\gamma \omega)^{-1}.
$$

(3.39)
Step 2: Assume that we have obtained a solution $h_k(\epsilon, \cdot) \in C^1(A_k; W_{N_k})$, $1 \leq k \leq n$ satisfying (3.36). Hence it is straightforward that
\[
\|\partial_\omega w_n\|_s \leq \tilde{K}\epsilon (\gamma^2\omega)^{-1}.
\] (3.39)

Step 3: Define
\[
\mathcal{U}_{n+1}(\epsilon, \omega, h) := -L_\omega(w_n + h) + \epsilon P_{N_{n+1}} \Pi W \mathcal{F}(\epsilon, \omega, w_n + h).
\] (3.40)
Lemma 3.3 shows that $h_{n+1}(\epsilon, \omega)$ is a solution of (3.40), i.e., $\mathcal{U}_{n+1}(\epsilon, \omega, h_{n+1}) = 0$, which gives rise to
\[
D_h \mathcal{U}_{n+1}(\epsilon, \omega, h_{n+1}) = \mathcal{L}_{n+1}(\epsilon, \omega, w_n)(\text{Id} - D_h U_{n+1}(h_{n+1})).
\] (3.41)
It follows from (3.33) that $\mathcal{L}_{n+1}(\epsilon, \omega, w_{n+1})$ is invertible with
\[
\|\mathcal{L}_{n+1}^{-1}(\epsilon, \omega, w_{n+1})w\|_s \leq 2K\gamma^{-1}N_{n+1}^{-1}\|w\|_s.
\] (3.42)
Then using the implicit function theorem, we can investigate
\[
h_{n+1}(\epsilon, \cdot) \in C^1(A_{n+1}; W_{n+1}).
\]
Hence one has
\[
\partial_\omega \mathcal{U}_{n+1}(\epsilon, \omega, h_{n+1}) + D_h \mathcal{U}_{n+1}(\epsilon, \omega, h_{n+1})\partial_\omega h_{n+1} = 0,
\] carrying out
\[
\partial_\omega h_{n+1} = -\mathcal{L}_{n+1}^{-1}(\epsilon, \omega, w_{n+1})\partial_\omega \mathcal{U}_{n+1}(\epsilon, \omega, h_{n+1})
\] (3.43)
with
\[
\partial_\omega \mathcal{U}_{n+1}(\epsilon, \omega, h_{n+1}) = -2\omega(h_{n+1})_{tt} + \epsilon P_{N_{n+1}} \Pi W d_\omega \mathcal{F}(\epsilon, \omega, w_n) + \epsilon P_{N_{n+1}}(\Pi W d_\omega \mathcal{F}(\epsilon, \omega, w_n + h_{n+1}) - \Pi W d_\omega \mathcal{F}(\epsilon, \omega, w_n)).
\] (3.44)
For $\epsilon\gamma^{-1} \leq \delta_4$ small enough, let us claim that there exists some constant $\tilde{C}_1 := \tilde{C}_1(\chi, q, \tau, \sigma)$ such that
\[
(F2) : \quad S'_n \leq \tilde{C}_1 S_0 N_{n+1}^{2r+\sigma+\frac{1}{\chi-1}(\tau-1+\sigma)} + \frac{\tilde{C}_1}{\gamma \omega} S_0 N_{n+1}^{2r+\sigma+\frac{1}{\chi-1}(\tau-1+\sigma)},
\] (3.45)
where
\[
S'_n := 1 + \|\partial_\omega w_n\|_{s+\beta}.
\] (3.46)
The proof of (F2) will be given in Lemma 3.7
Moreover we have
\[
\|\Pi W d_\omega \mathcal{F}(\epsilon, \omega, w)\|_{s+\beta} \leq C(\beta)\|w\|_{s+\beta}(1 + \|\partial_\omega w_n\|_s) + C(\beta)(1 + \|\partial_\omega w_n\|_{s+\beta}),
\] (3.47)
\[
\|\Pi W d_\omega \mathcal{F}(\epsilon, \omega, w_{n+1}) - \Pi W d_\omega \mathcal{F}(\epsilon, \omega, w_n)\|_s \leq C(1 + \|\partial_\omega w_n\|_s)\|h_{n+1}\|_s,
\] (3.48)
\[
\|\Pi W d_\omega \mathcal{F}(\epsilon, \omega, w_{n+1}) - \Pi W d_\omega \mathcal{F}(\epsilon, \omega, w_{n+1})\|_{s+\beta} \leq C(\beta)(\|w_n\|_{s+\beta} + \|h_{n+1}\|_{s+\beta})(1 + \|\partial_\omega w_n\|_s)
\] + $C(\beta)(1 + \|\partial_\omega w_n\|_{s+\beta})$.
(3.49)
with the help of Lemma 3.5 Lemma 2.3 (3.26) and 5.2. For $\epsilon\gamma^{-1} \leq \delta_4$ small enough, by (3.27), (3.35), (3.39) and (3.47)–(3.48), we obtain
\[
\|\partial_\omega \mathcal{U}_{n+1}(\epsilon, \omega, h_{n+1})\|_s \leq \frac{2\omega}{\gamma \omega} (h_{n+1})_{tt} + \epsilon C'(\beta)\|\partial_\omega w_n\|_s\|h_{n+1}\|_s
\] + $\epsilon C(\beta)N^{-\beta}_n S_n + \epsilon C_2 N^{-\beta}_n S'_n$,
(3.50)
which leads to
\[
\|\partial_\omega h_{n+1}\|_s \leq \frac{\epsilon K'}{\gamma^2 \omega} N_{n+1}^{2r} N^{-\beta}_n S_n + \frac{\epsilon K'}{\gamma} N_{n+1}^{-1} N^{-\beta}_n S'_n,
\] (3.51)
Furthermore it follows from (3.2) that

\[ N_{n+1} \leq e^{qN_{n+1}} < N_{n+1} + 1 < 2N_{n+1}. \] (3.51)

Thus using (3.29), (3.45) and (2.8), we derive

\[ ||\partial_\omega h_{n+1}||_s \leq \frac{K_4e^{N_{n+1}}}{2\gamma \omega}, \]

which completes the proof. \( \square \)

Let us give the proof of (F1) (recall (3.29)).

**Lemma 3.5.** If \( \epsilon \gamma^{-1} \leq \delta_2 \) with \( \delta_2 \leq \delta_1 \) is small enough, then there exists \( \tilde{C} := \tilde{C}(\chi, q, \tau, \sigma) > 0 \) such that

\[ S_n \leq \tilde{C}S_0N_{n+1}^{\chi-1}(\tau-1+\sigma), \]

where \( S_n \) is given in (3.25).

**Proof.** First of all, for \( \epsilon \gamma^{-1} \leq \delta_2 \) small enough, we claim

\[ S_n \leq (1 + N_{n+1}^{r_1+\sigma})S_{n-1}. \] (3.52)

In fact, from (3.25) we have

\[ S_n \leq 1 + ||w_{n-1}||_{s+\beta} + ||h_n||_{s+\beta} = S_{n-1} + ||h_n||_{s+\beta}, \] (3.53)

which implies that we only give the upper bound of \( ||h_n||_{s+\beta} \). It follows from Lemma 3.3, (3.20)–(3.21), (P1), (3.26), (3.3) and (3.6)–(3.7) that

\[ ||r_{n-1}||_s \leq C, \quad ||R_{n-1}(h_n)||_s \leq C\rho_n, \quad ||r_{n-1}||_{s+\beta} \leq \epsilon C(\beta)S_{n-1}, \]

\[ ||R_{n-1}(h_n)||_{s+\beta} \leq C(\beta)(\rho_n^2S_{n-1} + \rho_n||h_n||_{s+\beta}). \]

Hence combining this with the equality \( h_n = L_{N_n}^{-1}(\epsilon, \omega, w_{n-1})(R_{n-1}(h_n) + r_{n-1}) \) yields

\[ ||h_n||_{s+\beta} \leq \epsilon \gamma^{-1}K'(\beta)N_{n}^{\tau-1}(\rho_n^2S_{n-1} + S_{n-1} + \rho_n||h_n||_{s+\beta} + N_{n-1}^{\sigma}\rho_n^2S_{n-1} + N_{n-1}^{\sigma}S_{n-1}) \leq \epsilon \gamma^{-1}K''(\beta)N_{n}^{\tau-1+\sigma}S_{n-1} + \epsilon \gamma^{-1}K''(\beta)||h_n||_{s+\beta}, \]

where we use \( \sigma > \tau - 1 \) according to \( \tau \in (1, 2) \) and (3.11). Hence there exists \( \delta_2 > 0 \) with \( \epsilon \gamma^{-1} \leq \delta_2 \) small enough such that \( \epsilon \gamma^{-1}K''(\beta) \leq \frac{1}{2} \), which implies

\[ ||h_n||_{s+\beta} \leq 2\epsilon \gamma^{-1}K''(\beta)N_{n}^{\tau-1+\sigma}S_{n-1} \leq N_{n}^{\tau-1+\sigma}S_{n-1}. \] (3.54)

Substituting (3.54) into (3.53) gives rise to (3.52). Hence by (3.4) and (3.31)–(3.32), it yields

\[ S_n \leq S_0 \prod_{k=1}^{n}(1 + N_{k}^{\tau-1+\sigma}) \leq S_0 \prod_{k=1}^{n}(1 + e^{qN_{k}^{(r-1+\sigma)}}) \]

\[ = S_0 \prod_{k=1}^{n}e^{qN_{k}^{(r-1+\sigma)}} \prod_{k=1}^{n}(1 + e^{-qN_{k}^{(r-1+\sigma)}}) \]

\[ \leq \tilde{C}S_0N_{n+1}^{\chi-1}(\tau-1+\sigma), \]

where \( \tilde{C} = 2L_{N_{n+1}}^{-1}(\epsilon, \omega, w_{n+1})||w||_{s+\beta} \). \( \square \)

To prove (F2), we have to estimate the upper bound of \( L_{N_{n+1}}^{-1}(\epsilon, \omega, w_{n+1}) ||w||_{s+\beta} \) on \((s + \beta)\)-norm for all \( \omega \in W_{N_{n+1}} \), where \( L_{N_{n+1}}^{-1}(\epsilon, \omega, w_{n+1}) \) is given by (3.4).
Lemma 3.6. For \((\epsilon, \omega) \in A_{n+1}\) and \(\epsilon \gamma^{-1} \leq \delta_4 \leq \delta_3\) small enough, one has
\[
\|L_{N_{n+1}}^{-1}(\epsilon, \omega, w_{n+1})w\|_{s+\beta} \leq K_5 \gamma^{-1} N_{n+1}^{\tau-1} \|w\|_{s+\beta} + K_5 \gamma^{-1} N_{n+1}^{2(\tau-1)} (\|w_{n+1}\|_{s+\beta+\sigma} + \|h_{n+1}\|_{s+\beta})\|w\|_s
\]
for some constant \(K_5 > 0\), where \(\beta, \sigma\) are given by (2.8), (3.11) respectively.

Proof. Let \(L'(h_{n+1}) := (\text{Id} - D_h U_{n+1}(h_{n+1}))^{-1}w\). It is obvious that
\[
L'(h_{n+1}) = w + D_h U_{n+1}(h_{n+1}) L'(h_{n+1}).
\]
Taking \(s' = s + \beta\) and \(s'' = s\) in (3.4) and applying (3.32) and (3.23), we derive
\[
\|D_h U_{n+1}(h_{n+1})\|_{s+\beta} \leq \epsilon \gamma^{-1} K'(\beta) N_{n+1}^{\tau-1} (\|w_n\|_{s+\beta}\|h_{n+1}\|_s + \|h_{n+1}\|_{s+\beta} + \|w_n\|_{s+\beta+\sigma}\|h_{n+1}\|_s) \\
\leq \epsilon \gamma^{-1} K''(\beta) N_{n+1}^{\tau-1} (\|w_n\|_{s+\beta+\sigma}\|h_{n+1}\|_s + \|h_{n+1}\|_{s+\beta}).
\]
Hence combining this with (3.2), (3.33), (3.55) and (3.26) yields
\[
\|L'(h_{n+1})\|_{s+\beta} \leq \epsilon \|w\|_{s+\beta} + \epsilon \gamma^{-1} K''(\beta) N_{n+1}^{\tau-1} (\|w_n\|_{s+\beta+\sigma}\|h_{n+1}\|_s + \|h_{n+1}\|_{s+\beta})\|w\|_s \\
+ \epsilon \gamma^{-1} K''(\beta) N_{n+1}^{\tau-1} (\|w_n\|_{s+\beta+\sigma}\|h_{n+1}\|_s + \|h_{n+1}\|_{s+\beta})|L'(h_{n+1})|_{s+\beta}.
\]
By the fact \(\tau \in (1, 2)\), it is obvious that \(N_{n+1}^{\tau-1} \rho_n < 1\). Using (3.11) and (3.26), for \(\epsilon \gamma^{-1} \leq \delta_4\) small enough, we can deduce
\[
\|L'(h_{n+1})\|_{s+\beta} \leq 2\|w\|_{s+\beta} + 2\epsilon \gamma^{-1} K'''(\beta) N_{n+1}^{\tau-1} (\|w_n\|_{s+\beta+\sigma}\|h_{n+1}\|_s + \|h_{n+1}\|_{s+\beta})\|w\|_s.
\]
Hence, for \(\epsilon \gamma^{-1} \leq \delta_4\) small enough, it follows from (3.22)–(3.23) that the conclusion of this lemma holds.

Now let us verify (F2) (recall (3.45)) holds.

Lemma 3.7. There exists \(\tilde{C}_1 := \tilde{C}_1(\chi, q, \tau, \sigma)\) such that
\[
S'_n \leq \tilde{C}_1 S_0 N_{n+1}^{2r+\sigma+\frac{1}{\chi-1}(\tau-1+\sigma)} + \frac{\tilde{C}_1}{\gamma \omega} S_0 N_{n+1}^{2r+\sigma+\frac{1}{\chi-1}(\tau-1+\sigma)}
\]
for \(\epsilon \gamma^{-1} \leq \delta_4 \leq \delta_3\) small enough, where \(S'_n\) is defined in (3.46).

Proof. First of all, for \(\epsilon \gamma^{-1} \leq \delta_4\) small enough, let us check that there exists some constant \(C_1 > 0\) such that
\[
S'_n \leq (1 + N_{n+1}^{\tau-1}) S_{n-1}' + C_1 (\gamma \omega)^{-1} N_{n+1}^{2r+\sigma} S_{n-1}.
\]
Note that by (3.46), it has \(S'_n \leq 1 + \|\partial_{\omega} w_{n-1}\|_{s+\beta} + \|\partial_{\omega} h_{n}\|_{s+\beta} = S'_{n-1} + \|\partial_{\omega} h_{n}\|_{s+\beta}\). Thus we verify the upper bound of \(\partial_{\omega} h_{n}\) on \((s + \beta)\)-norm.

Using (3.43) and Lemma 3.6, we can derive
\[
\|\partial_{\omega} h_{n}\|_{s+\beta} \leq K_5 \gamma^{-1} N_{n+1}^{\tau-1} \|\partial_{\omega} \mathcal{U}_n(\epsilon, \omega, h_{n})\|_{s+\beta} \\
+ K_5 \gamma^{-1} N_{n+1}^{2(\tau-1)} (\|w_{n-1}\|_{s+\beta+\sigma} + \|h_{n}\|_{s+\beta})\|\partial_{\omega} \mathcal{U}_n(\epsilon, \omega, h_{n})\|_s.
\]
Let us show the upper bound of \(\|\partial_{\omega} \mathcal{U}_n(\epsilon, \omega, h_{n})\|_{s+\beta}\). For \(\epsilon \gamma^{-1} \leq \delta_4\) small enough, it follows from (3.34), (3.54) and \(\gamma \in (0, 1)\) that
\[
\|(h_{n})_{tt}\|_{s+\beta} \leq C' \omega^{-2} (N_{n}^{2r} \|h_{n}\|_{s+\beta+\sigma} + \|w_{n-1}\|_{s+\beta} \|h_{n}\|_s + \epsilon(1 + \|w_{n-1}\|_{s+\beta}))
\]
\[
\leq C'' \omega^{-2} N_{n+1}^{2r+\sigma} S_{n-1}.
\]
Then, for \(\epsilon \gamma^{-1} \leq \delta_4\) small enough, applying (3.44), (3.46)–(3.47), (3.49), (3.29), (3.39) and (3.54) yields
\[
\|\partial_{\omega} \mathcal{U}_n(\epsilon, \omega, h_{n})\|_{s+\beta} \leq C_2 \omega^{-1} N_{n+1}^{2r+\sigma} S_{n-1} + \epsilon C_2 S'_{n-1}.
\]
Since \( \| \partial_\omega \mathcal{W}_{n+1}(\epsilon, \omega, h_{n+1}) \|_s \leq \frac{\epsilon \gamma}{\gamma^2} \) according to (3.50), (2.8), (3.29) and (3.45), by means of (3.25) and (3.54), we have

\[
\| \partial_\omega h_n \|_{s+\beta} \leq C_1(\gamma^\omega)^{-1} N_{n+\beta} \leq K_5(\gamma^\omega)^{-1} N_{n-1} S_{n-1} + N_{n-1} S_{n-1}^{\prime}
\]

for \( \epsilon \gamma^{-1} \leq \delta_4 \) small enough. This gives rise to (3.56).

Letting \( \alpha_1 := \tau - 1, \alpha_2 := 2 + \pi, \alpha_3 := \tau - 1 + \pi \), by virtue of (3.56), a simple calculation gives

\[
S_n^{\prime} \leq S_1 + S_2 \quad \text{with} \quad S_1 = S_0 \prod_{k=1}^n (1 + N_k), \quad S_2 = \frac{C_1}{\gamma^\omega} \sum_{k=1}^n \prod_{j=1}^k (1 + N_{n-j}) N_{n-j}^{\alpha_1} \leq \epsilon \gamma^{-1} \delta_4 \leq (3.56).
\]

Since the upper bound on \( S_1 \) is proved in the same way as shown in Lemma 3.5, the detail is omitted. Thus we arrive at

\[
S_1 \leq C_1 S_0 N^{\frac{1}{n+1}}.
\]

We write \( S_2 = \sum_{k=1}^n S_{2,k} \), where

\[
S_{2,1} = \frac{C_1}{\gamma^\omega} N_{n+1} S_{n-1}, \quad S_{2,k} = \frac{C_1}{\gamma^\omega} \prod_{j=2}^k (1 + N_{n-j}^{\alpha_1}) N_{n-j}^{\alpha_2} S_{n-k} \quad \forall 2 \leq k \leq n.
\]

On the one hand, it follows from (3.2) and (3.29) that

\[
S_{2,1} \leq \frac{C_1 S_0}{\gamma^\omega} e^{\omega q^n} e^{\alpha_3 \chi^n_{n-1}} = \frac{C_1 S_0}{\gamma^\omega} e^{\omega q^n (\alpha_2 + \frac{1}{\chi^n_{n-1}} - \alpha_3)}.
\]

And on the other hand, a simple computation yields

\[
\sum_{k=2}^n S_{2,k} \leq \frac{C_1 S_0}{\gamma^\omega} e^{\omega q^n} \sum_{k=2}^n \left( \frac{\chi^n_{n+1}}{\chi^n_{n-1}} e^{\omega q^n \chi^n_{n+2-k}} e^{\alpha_3 q^n \chi^n_{n-1}} \right)
\]

\[
\leq \frac{C_1 S_0}{\gamma^\omega} e^{\omega q^n} \sum_{k=2}^n \left( \frac{\chi^n_{n+1}}{\chi^n_{n-1}} \right)
\]

\[
\leq \frac{C_1 S_0}{\gamma^\omega} N^{\alpha_2 + \frac{1}{\chi^n_{n-1} - \alpha_3}}.
\]

Hence using (3.2), (3.57)-(3.59) implies the conclusion in the lemma.

\[ \square \]

4. INVERTIBILITY OF THE LINEARIZED OPERATORS

The invertibility of the linearized operators, i.e., property (P5) is the core of any Nash-Moser iteration. Let us show the proof.

4.1. The perodic boundary value problem. We first propose the asymptotic formulae of the eigenvalues to the following Hill’s problem

\[
\begin{aligned}
-\gamma y'' + \omega^2 y &= \lambda a(t) y, \\
y(0) &= y(2\pi), \quad y'(0) = y'(2\pi),
\end{aligned}
\]

where

\[
\omega(t) := \frac{\epsilon}{\omega^2} \Pi_V f(t, x, v(\epsilon, \omega, w(t, x)) + w(t, x)) \in L^2(0, 2\pi).
\]

Since \( a(\cdot) \) is 2\( \pi \)-periodic, making the Liouville substitution

\[
t = \psi(\xi) \Leftrightarrow \xi = g(t) \quad \text{with} \quad \xi \in [0, 2\pi], \quad g(t) := \frac{1}{\epsilon} \int_0^t \sqrt{a(s)} ds,
\]

\[ (4.1) \]

\[ (4.2) \]
where
\[ c := \frac{1}{2\pi} \int_0^{2\pi} \sqrt{a(s)} \, ds, \quad \text{(4.4)} \]

yields that \( \lambda \) and \( y(\psi(\xi)) \) satisfy
\[
\begin{align*}
- y\xi - c + c^2 \frac{a}{2} y &= c^2 \lambda y, \\
y(0) &= y(2\pi), \quad y(\xi) = y(2\pi),
\end{align*}
\quad \text{(4.5)}
\]

Moreover we make the Liouville change
\[ y = z/\tau \quad \text{with} \quad \tau(t) = (a(t))^{1/2}. \quad \text{(4.6)} \]
The system (4.5) can be reduced into
\[
\begin{align*}
- z\xi + \varrho(\xi)z(\xi) &= \mu z(\xi), \\
z(0) &= z(2\pi), \quad z(\xi) = z(2\pi),
\end{align*}
\quad \text{(4.7)}
\]

where
\[
\begin{align*}
\varrho(\xi) &= q(\xi) + \vartheta(\xi), \\
q(\xi) &= c^2 Q(\psi(\xi)) \quad \text{with} \quad Q(t) = \frac{a_{tt}(t)}{4a^2(t)} - \frac{11a^2(t)}{16a^3(t)}, \quad \text{(4.9)}
\end{align*}
\[
\vartheta(\xi) = c^2 \frac{q(\psi(\xi))}{a(\psi(\xi))), \quad \mu = c^2 \lambda. \quad \text{(4.10)}
\]

**Hypothesis 2.** We have to make an additional hypothesis:
\[
\varrho(\xi) > 0, \quad \forall \xi \in [0, 2\pi]. \quad \text{(4.11)}
\]

It is known [33, 34] that the eigenvalues of problem (4.7) are arranged as an increasing unbounded sequence \( \mu_0 < \mu_1^- \leq \mu_1^+ < \cdots < \mu_l^- \leq \mu_l^+ < \cdots \) so that if the equality sign is present, then the corresponding eigenvalue is double, and the zeros of an eigenfunction on the segment \( [0, 2\pi] \) are equal to \( 2l \), where \( l \) is the number of the corresponding eigenvalues. Now we consider
\[
\begin{align*}
\varphi_l''(\xi) + (\mu_l - \varrho(\xi))\varphi_l(\xi) &= 0, \\
\varphi_l(0) &= \varphi_l(2\pi), \quad \varphi_l'(0) = \varphi_l'(2\pi),
\end{align*}
\quad \text{(4.12)}
\]

where \( \varphi_l'(\xi) = \frac{d}{d\xi} \varphi_l(\xi), \quad \forall l \in \mathbb{N}. \) Formulae (4.7)–(4.8) show that \( \mu_l \) depend on \( \varrho(\cdot) \). However, for brevity, we do not write \( \varrho(\cdot) \). Let us verify the following properties on the eigenvalues of (4.12)–(4.13).

**Lemma 4.1.** Under Hypothesis 2 the eigenvalues of (4.12)–(4.13) satisfy \( \mu_l \geq \varrho_0 \) for all \( l \in \mathbb{N} \), where
\[
\varrho_0 := \inf_{\xi \in [0, 2\pi]} \varrho(\xi) > 0. \quad \text{(4.14)}
\]

**Proof.** Multiplying both sides of (4.12) by \( \varphi_l(\xi) \) and integrating over \([0, 2\pi]\) yield
\[
\begin{align*}
\mu_l \int_0^{2\pi} \varphi_l''(\xi) d\xi &= \int_0^{2\pi} \varrho(\xi)\varphi_l''(\xi) d\xi + \int_0^{2\pi} (\varphi_l'(\xi))^2 d\xi - (\varphi_l(2\pi)\varphi_l'(2\pi) - \varphi_l(0)\varphi_l'(0)) \\
&\geq \varrho_0 \int_0^{2\pi} \varphi_l^2(\xi) d\xi,
\end{align*}
\quad \text{(4.13)}
\]

which establishes \( \mu_l \geq \varrho_0 \) for all \( l \in \mathbb{N} \).
\[ \square \]
Lemma 4.2 (Asymptotic formulae). Denote by \( \mu_0 < \mu_1^- \leq \mu_1^+ \leq \cdots < \mu_i^- \leq \mu_i^+ < \cdots \) and \( \varphi_0, \varphi_1^+, \varphi_1^- , \cdots \), respectively, the eigenvalues and the corresponding real orthonormal eigenfunctions of (4.12) – (4.13). Then the following asymptotic formulae hold:

\[
\sqrt{\mu_i^\pm} = l + \eta_i^\pm \quad \text{with} \quad \frac{\sqrt{1 + \varrho_0} - 1}{l} \leq \eta_i^\pm \leq \frac{\varrho_1}{l}, \quad l \in \mathbb{N}^+,
\]

and \( \eta_i^\pm \to 0 \) as \( l \to +\infty \),

where \( \varrho_0 \) is defined in (4.14) and \( \varrho_1 := \frac{1}{\pi} \int_0^{2\pi} \varrho(\xi) d\xi \).

Proof. Before proving the conclusion, we first claim

\[
l^2 + \varrho_0 \leq \mu_l^\pm \leq l^2 + \varrho_1, \quad \forall l \in \mathbb{N}^+.
\]

By (4.16), it is clear that

\[
0 < \frac{\varrho_0}{\sqrt{l^2 + \varrho_0} + l} = \sqrt{l^2 + \varrho_0} - l \leq \sqrt{\mu_l^\pm} - l \leq \sqrt{l^2 + \varrho_1} - l = \frac{\varrho_1}{\sqrt{l^2 + \varrho_1} + l} < \frac{\varrho_1}{l}.
\]

Combining this with

\[
\frac{\varrho_0}{\sqrt{l^2 + \varrho_0} + l} = \frac{\varrho_0}{l \sqrt{1 + \frac{\varrho_0}{l^2}} + 1} \geq \frac{\varrho_0}{l \sqrt{1 + \varrho_0} + 1} = \frac{\sqrt{1 + \varrho_0} - 1}{l} > 0
\]

yields that (4.15) holds.

Now let us check (4.16). In fact, Lemma 4.1 shows that \( \mu_l^\pm > \varrho_0 \) for all \( l \in \mathbb{N}^+ \). On the one hand, in equality (4.12), using the Prüfer transformation

\[
\varphi_l^\pm = r \sin \theta, \quad \varphi_l^{\pm'} = \left( \sqrt{\mu_l^\pm - \varrho_0} \right) r \cos \theta, \quad \forall l \in \mathbb{N}^+,
\]

where \( r(\xi) > 0 \), we derive

\[
\begin{cases}
    r' \sin \theta + r \theta' \cos \theta = \left( \sqrt{\mu_l^\pm - \varrho_0} \right) r \cos \theta, \\
    r' \cos \theta - r \theta' \sin \theta = -\frac{\mu_l^\pm - \varrho(\xi)}{\sqrt{\mu_l^\pm - \varrho_0}} r \sin \theta,
\end{cases}
\]

which may carry out

\[
\frac{d\theta}{d\xi} = \left( \sqrt{\mu_l^\pm - \varrho_0} \right) \cos^2 \theta + \frac{\mu_l^\pm - \varrho(\xi)}{\sqrt{\mu_l^\pm - \varrho_0}} \sin^2 \theta
\]

\[
= \left( \sqrt{\mu_l^\pm - \varrho_0} \right) \cos^2 \theta + \frac{\mu_l^\pm - \varrho_0 + \varrho_0 - \varrho(\xi)}{\sqrt{\mu_l^\pm - \varrho_0}} \sin^2 \theta
\]

\[
\leq \sqrt{\mu_l^\pm - \varrho_0}.
\]

Let \( \kappa_1 < \kappa_2 < \cdots < \kappa_{2l} \) denote 2l zeros in \([0, 2\pi]\) of \( \varphi_l \). Correspondingly, we set \( \theta_i = \theta(\kappa_i), i = 1, \cdots, 2l \). Hence we may take \( \theta_i = i \pi, i = 1, 2, \cdots, 2l \). The definition of \( \kappa_1 \) implies either \( \kappa_1 = 0 \) or \( \kappa_1 > 0 \).

By the first term in (4.13), in the first case \( \theta(2\pi) = (2l + 1)\pi \) and in the latter \( \theta(2\pi) = 2l\pi + \theta_0 \) with \( \theta_0 := \theta(0) \in (0, \pi) \). This shows \( \theta(2\pi) - \theta(0) = 2l\pi \). Integrating (4.17) over \([0, 2\pi]\) yields

\[
2l\pi \leq 2\pi \sqrt{\mu_l^\pm - \varrho_0} \Rightarrow \mu_l^\pm \geq l^2 + \varrho_0, \quad \forall l \in \mathbb{N}^+.
\]

And on the other hand, in equality (4.12), we introduce the Prüfer transformation

\[
\varphi_l = r \sin \theta, \quad \varphi_l' = \sqrt{\mu_l^\pm} r \cos \theta, \quad l \in \mathbb{N}^+,
\]
where \( r(\xi) > 0 \), which leads to
\[
\frac{d\theta}{d\xi} = \sqrt{\mu_i^\pm - \frac{\varrho(\xi)}{\sqrt{\mu_i^\pm}} \sin^2 \theta} \geq \sqrt{\mu_i^\pm - \frac{\varrho(\xi)}{\sqrt{\mu_i^\pm}}}. \tag{4.18}
\]

The similar argument as above gives \( \theta(2\pi) - \theta(0) = 2l\pi \). Integrating (4.18) over \([0, 2\pi]\) yields
\[
2l\pi \geq 2\pi \sqrt{\mu_i^\pm - \int_0^{2\pi} \varrho(\xi) d\xi},
\]
which is equivalent to \((\sqrt{X})^2 - B\sqrt{X} - C \leq 0\), where \( X = \mu_i^+, \) \( B = l, \) \( C = \frac{1}{2\pi} \int_0^{2\pi} \varrho(\xi) d\xi \). It follows from the quadratic formula and the elementary inequality \( \sqrt{1 + x} \leq 1 + \frac{x}{2} \) that
\[
\sqrt{X} \leq (B + \sqrt{B^2 + 4C})/2 \Rightarrow X \leq (2B^2 + 4C + 2B\sqrt{B^2 + 4C})/4 \Rightarrow X \leq (2B^2 + 4C + 2B^2\sqrt{1 + 4C/B^2})/4 \Rightarrow X \leq (2B^2 + 4C + 2B^2(1 + 2C/B^2))/4 = B^2 + 2C,
\]
i.e., \( \mu_i^\pm \leq l^2 + g_1 \). This completes the proof. \(\square\)

For brevity, abusing notations, let \( \psi_i^+ (\epsilon, \omega, w) := \psi_0 (\epsilon, \omega, w) \) and \( \hat{\psi}_0^+ (\cdot) := \hat{\psi}_0 \).

**Lemma 4.3.** Denote by \( \lambda_i^\pm (\epsilon, \omega, w) \) and \( \psi_i^\pm (\epsilon, \omega, w) \), \( l \in \mathbb{N} \) the eigenvalues and the eigenfunctions of the Hill’s problem (4.1) respectively. Given Hypothesis 2, one has
\[
0 < m_0 \leq \lambda_0 (\epsilon, \omega, w) < \lambda_1 (\epsilon, \omega, w) \leq \cdots \leq \lambda_\ell (\epsilon, \omega, w) \leq \cdots \leq \lambda_i^+ (\epsilon, \omega, w) < \cdots \tag{4.19}
\]
with \( \lambda_i^+ (\epsilon, \omega, w) \to +\infty \) as \( l \to +\infty \), and for all \( \epsilon \in (\epsilon_1, \epsilon_2), \omega \in (\omega_1, \omega_2), w \in \{ W \cap H^s : \| w \|_s < r \}, \)
\[
\sqrt{\lambda_i^+ (\epsilon, \omega, w)} = \frac{l}{c} + \tilde{\eta}_i^\pm (\epsilon, \omega, w), \quad \forall l \in \mathbb{N}^+
\tag{4.20}
\]
with \( \tilde{\eta}_i^\pm (\epsilon, \omega, w) \to 0 \) as \( l \to +\infty \) and
\[
0 < \inf_{t \in (0, 2\pi), \epsilon \in (\epsilon_1, \epsilon_2), \omega \in (\omega_1, \omega_2), w \in \{ W \cap H^s : \| w \|_s < r \}} \frac{\sqrt{\frac{h}{l} + m(t, \epsilon, \omega, w) - \frac{1}{l}}}{l} \leq \eta_i^\pm (\epsilon, \omega, w) \leq \frac{1}{\pi} \int_0^{2\pi} m(t, \epsilon, \omega, w)a(t) dt, \tag{4.21}
\]
where \( c \) is defined in (4.4) and
\[
m_0 := \inf_{t \in (0, 2\pi), \epsilon \in (\epsilon_1, \epsilon_2), \omega \in (\omega_1, \omega_2), w \in \{ W \cap H^s : \| w \|_s < r \}} m(t, \epsilon, \omega, w),
\]
\[
m(t, \epsilon, \omega, w) = Q(t) + \frac{\epsilon}{\omega^2 a(t)} \Pi_V f^t(t, x, v(\epsilon, \omega, w(t, x)) + w(t, x))
\]
with \( Q \) is given by (4.9). And \( \psi_i^\pm (\epsilon, \omega, w) \) form an orthogonal basis of \( L^2(0, 2\pi) \) with the scalar product
\[
(y, z)_{L^2} := c^{-1} \int_0^{2\pi} ay z dt.
\]
Moreover define an equivalent scalar product \( (\cdot, \cdot)_{\epsilon, \omega, w} \) on \( H^1(0, 2\pi) \) by
\[
(y, z)_{\epsilon, \omega, w} := \frac{1}{c} \int_0^{2\pi} y' z' + \frac{\epsilon}{\omega^2} \Pi_V f^t(v(\epsilon, \omega, w) + w)) y z + ay z dt
\]
with
\[
L_1 \| y \|_{H^1} \leq \| y \|_{\epsilon, \omega, w} \leq L_2 \| y \|_{H^1}, \quad \forall y \in H^1(0, 2\pi)
\tag{4.22}
\]
for some constants $L_1$, $L_2 > 0$. The eigenfunctions $\psi_{1}^{\pm}(\epsilon, \omega, w)$ are also an orthogonal basis of $H^1(0, 2\pi)$ with respect to the scalar product $(\cdot, \cdot)_{L_2}$ and one has that, for all $y = \sum_{l \geq 0} \hat{y}_l^\pm \psi_l^\pm(\epsilon, \omega, w)$,

$$
\|y\|_{L_2}^2 = \sum_{l \geq 0} (\hat{y}_l^\pm)^2, \quad \|y\|_{L_2}^2 = \sum_{l \geq 0} (\lambda_l^\pm(\epsilon, \omega, w) + 1)(\hat{y}_l^\pm)^2.
$$

(4.23)

**Proof.** Clearly, $\mu_l^\pm(\hat{\vartheta}) = e^2 \lambda_l^\pm(\vartheta)$ by virtue of (4.10). Then it follows from Hypothesis 2 and Lemma 4.1 that (4.19) holds for all $\epsilon \in (\epsilon_1, \epsilon_2)$, $\omega \in (\omega_1, \omega_2)$ and $w \in \{W \cap H^s : \|w\|_s < \tau\}$. Moreover using Lemma 4.2 and the inverse Liouville substitution of (4.3), yields that eigenvalues $\lambda_l^\pm(\vartheta)$, $l \in \mathbb{N}_1$ of (4.4) have the asymptotic formulae (4.20) satisfying (4.21) and that the eigenfunctions $\varphi_l^\pm$ form an orthonormal basis for $L^2$.

According to the Liouville substitution (4.6), we obtain

$$
\varphi_l^\pm = \psi_l^\pm(\epsilon, \omega, w) \tau \quad \text{with } \tau(t) = (a(t))^{\frac{1}{2}},
$$

which leads to

$$
\int_{0}^{2\pi} \varphi_l^\pm(\xi) \varphi_l^{\pm}(\xi) d\xi = \int_{0}^{2\pi} \psi_l^\pm(\epsilon, \omega, w)(t)\tau(t)\psi_l^\pm(\epsilon, \omega, w)(t)\tau(t) \frac{1}{c} \sqrt{a(t)} dt
$$

$$
= \frac{1}{c} \int_{0}^{2\pi} \psi_l^\pm(\epsilon, \omega, w)(t)\psi_l^\pm(\epsilon, \omega, w)(t) a(t) dt
$$

by the inverse Liouville substitution of (4.3). Hence the eigenfunctions $\psi_{1}^{\pm}(\epsilon, w)$ of (4.1) form an orthogonal basis for $L^2$ with respect to the scalar product $(\cdot, \cdot)_{L_2}$. A simple calculation gives (4.22). Furthermore it is obvious that

$$
-(\psi_{1}^{\pm})''(\epsilon, \omega, w) + \frac{\epsilon}{\omega^2} \Pi \psi(t, v(\epsilon, \omega, w) + w) \psi_{1}^{\pm}(\epsilon, \omega, w) = \lambda_{1}^{\pm}(\epsilon, \omega, w) a \psi_{1}^{\pm}(\epsilon, \omega, w).
$$

Multiplying above equality by $\psi_{1}^{\pm}(\epsilon, \omega, w)$ and integrating by parts yield

$$
(\psi_{1}^{\pm}, \psi_{1}^{\pm})_{\epsilon, \omega, w} = \delta_{1, t}(\lambda_{1}^{\pm}(\epsilon, \omega, w) + 1),
$$

which implies (4.23).

By Lemma 4.3 we present that the $s$-norm (recall (2.2)) restricted to $W \cap \mathcal{H}^s$ is equivalent to the norm

$$
\left( \sum_{j \in \mathbb{Z}_1 \setminus \{0\}, l \in \mathbb{N}} (\lambda_j^{\pm}(\epsilon, \omega, w) + 1)(\hat{w}_j^{\pm})^2(1 + j^{2s}) \right)^{1/2},
$$

i.e., for all $w = \sum_{j \in \mathbb{Z}_1 \setminus \{0\}} w_j(t) e^{ijx} = \sum_{j \in \mathbb{Z}_1 \setminus \{0\}, l \in \mathbb{N}} \hat{w}_j^{\pm}(\epsilon, \omega, w)e^{ijx}$,

$$
L_s^2 \|w\|_s^2 \leq \sum_{j \in \mathbb{Z}_1 \setminus \{0\}, l \in \mathbb{N}} ((\lambda^{\pm}_{j}(\epsilon, \omega, w) + 1)(\hat{w}_{j}^{\pm})^2(1 + j^{2s}) \leq L_s^2 \|w\|_s^2,
$$

(4.24)

where $L_i$, $i = 1, 2$ are seen in (4.22).

4.2 **Invertibility of the linearized operator.** The linearized operator $\mathcal{L}_{N}(\epsilon, \omega, w)$ (recall (3.10)) may be written as

$$
\mathcal{L}_{N}(\epsilon, \omega, w)[h] := -L_\omega h + \epsilon P_N \Pi W D_w F(v(\epsilon, \omega, w) + w)[h]
$$

$$
= \mathcal{L}_1(\epsilon, \omega, w)[h] + \mathcal{L}_2(\epsilon, \omega, w)[h]
$$

for all $h \in W_N$, where $L_\omega$ is defined by (2.4) and

$$
\mathcal{L}_1(\epsilon, \omega, w)[h] := -L_\omega h + \epsilon P_N \Pi W f'(t, x, v(\epsilon, \omega, w) + w)h,
$$

$$
\mathcal{L}_2(\epsilon, \omega, w)[h] := \epsilon P_N \Pi W f'(t, x, v(\epsilon, \omega, w) + w)D_w v(\epsilon, \omega, w)[h].
$$
Let \( b(t, x) := f'(t, x, v(\epsilon, \omega, w(t, x)) + w(t, x)) \). Using (5.13), \( \|w\|_{s+\sigma} \leq 1 \) and Lemma 2.3, we derive
\[
\|b\|_{s} \leq \|b\|_{s+\sigma} \leq C, \quad \forall s > 1/2, \tag{4.25}
\]
\[
\|b\|_{s'} \leq C(s')(1 + \|w\|_{s'}), \quad \forall s' > 1/2. \tag{4.26}
\]
Moreover, with the help of decomposing \( b(t, x) = \sum b_k(t)e^{ikx}, h(t, x) = \sum_{1 \leq |j| \leq N} h_j(t)e^{ijx} \), the operator \( \mathcal{L}_1(\epsilon, \omega, w) \) can be written as
\[
\mathcal{L}_1(\epsilon, \omega, w)[h] = \sum_{1 \leq |j| \leq N} \left( -\omega^2 \partial_t h_j - a(t) j^2 h_j \epsilon^{ijx} + \epsilon P_N \Pi_w \left( \sum_{k,N \leq |j| \leq N} b_{k-j} h_j e^{ikx} \right) \right) + \epsilon \sum_{1 \leq |j| \leq N} b_{k-j} h_j e^{ikx}
\]
where \( b_0 = \Pi_v f'(t, x, v(\epsilon, \omega, w) + w) \) and
\[
\mathcal{L}_{1,D} h := \sum_{1 \leq |j| \leq N} \left( -\frac{1}{a} \omega^2 \partial_t h_j - j^2 h_j + \epsilon b_0 \right) h_j e^{ijx},
\]
\[
\mathcal{L}_{1,ND} h := -\frac{\epsilon}{a} \sum_{1 \leq |j|, |k| \leq N, j \neq k} b_{k-j} h_j e^{ikx}. \tag{4.27}
\]
Let us check the invertibility of the linearized operator \( \mathcal{L}_N(\epsilon, \omega, w) \). It follows from Lemma 4.3 that
\[
\frac{1}{\alpha} \omega^2 h_j'' - j^2 h_j + \epsilon b_0 h_j = \sum_{l=0}^{+\infty} (\omega^2 \lambda_l^\pm(\epsilon, \omega, w) - j^2) \hat{h}_{j,l}^\pm \psi_l^\pm(\epsilon, \omega, w), \quad \forall h_j = \sum_{l=0}^{+\infty} \hat{h}_{j,l}^\pm \psi_l^\pm(\epsilon, \omega, w),
\]
where, abusing notation, we set \( \hat{h}_{j,0}^\pm \) for \( \hat{h}_{j,0} \). Then \( \mathcal{L}_{1,D} \) is a diagonal operator on \( W_N \). Define \( |\mathcal{L}_{1,D}|^{1/2} \) by
\[
|\mathcal{L}_{1,D}|^{1/2} h = \sum_{1 \leq |j| \leq N} \sum_{l=0}^{+\infty} \sqrt{|\omega^2 \lambda_l^\pm(\epsilon, \omega, w) - j^2|} \hat{h}_{j,l}^\pm \psi_l^\pm(\epsilon, \omega, w)e^{ijx}, \quad \forall h \in W_N.
\]
If \( \omega^2 \lambda_l(\epsilon, \omega, w) - j^2 \neq 0, \forall 1 \leq |j| \leq N, \forall l \geq 0 \), then its invertibility is
\[
|\mathcal{L}_{1,D}|^{-1/2} h = \sum_{1 \leq |j| \leq N} \sum_{l=0}^{+\infty} \frac{\hat{h}_{j,l}^\pm}{\sqrt{|\omega^2 \lambda_l^\pm(\epsilon, \omega, w) - j^2|}} \psi_l^\pm(\epsilon, \omega, w)e^{ijx}.
\]
Hence, the operator \( \mathcal{L}_N(\epsilon, \omega, w) \) is reduced to
\[
\mathcal{L}_N(\epsilon, \omega, w) = a|\mathcal{L}_{1,D}|^{1/2} (|\mathcal{L}_{1,D}|^{-1/2} \mathcal{L}_1 D |\mathcal{L}_{1,D}|^{-1/2} - R_1 - R_2)|\mathcal{L}_{1,D}|^{1/2},
\]
where
\[
R_1 = |\mathcal{L}_{1,D}|^{-1/2} \mathcal{L}_{1,ND} |\mathcal{L}_{1,D}|^{-1/2}, \quad R_2 = -|\mathcal{L}_{1,D}|^{-1/2} (1/a \mathcal{L}_2) |\mathcal{L}_{1,D}|^{-1/2} \tag{4.28}
\]
Moreover the definitions of \( \mathcal{L}_{1,D}, |\mathcal{L}_{1,D}|^{-1/2} \) read
\[
(|\mathcal{L}_{1,D}|^{-1/2} \mathcal{L}_1 D |\mathcal{L}_{1,D}|^{-1/2}) h = \sum_{1 \leq |j| \leq N} \sum_{l=0}^{+\infty} \text{sgn}(\omega^2 \lambda_l^\pm(\epsilon, \omega, w) - j^2) \hat{h}_{j,l}^\pm \psi_l(e, \omega, w)e^{ijx}, \quad \forall h \in W_N,
\]
where \( \text{sgn}(\omega^2 \lambda_l^\pm(\epsilon, \omega, w) - j^2) \in \{ \pm 1 \} \). Combining this with (4.24) implies that it is invertible with
\[
\|(|\mathcal{L}_{1,D}|^{-1/2} \mathcal{L}_1 D |\mathcal{L}_{1,D}|^{-1/2})^{-1} h\|_s \leq \frac{1}{L_1} \sum_{1 \leq |j| \leq N} (\lambda_j^\pm(\epsilon, \omega, w) + 1)(\hat{h}_{j,l}^\pm)^2 (1 + j^2) \leq \frac{L_2^2}{L_1} \|h\|_s^2 \tag{4.29}
\]
for all \( h \in W_N \) and \( s \geq 0 \). Therefore \( \mathcal{L}_N(\epsilon, \omega, w) \) can be written as

\[
\mathcal{L}_N(\epsilon, \omega, w) = a|\hat{\mathcal{L}}_{1,D}|^{-\frac{1}{2}} ((\hat{\mathcal{L}}_{1,D})^{-\frac{1}{2}} \mathcal{L}_1 \mathcal{D} \mathcal{D}^{-\frac{1}{2}} (\text{Id} - \mathcal{R}))|\hat{\mathcal{L}}_{1,D}|^{\frac{1}{2}},
\]

where \( \mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 \) with

\[
\mathcal{R}_1 = (|\hat{\mathcal{L}}_{1,D}|^{-\frac{1}{2}} \mathcal{L}_1 \mathcal{D} \mathcal{D}^{-\frac{1}{2}})^{-1} \mathcal{R}_1, \quad \mathcal{R}_2 = (|\hat{\mathcal{L}}_{1,D}|^{-\frac{1}{2}} \mathcal{L}_1 \mathcal{D} \mathcal{D}^{-\frac{1}{2}})^{-1} \mathcal{R}_2.
\]

To verify the invertibility of the operator \( \text{Id} - \mathcal{R} \) in (4.30), we have to suppose some non-resonance conditions.

For \( \tau \in (1, 2) \), assume the following “Melinkov’s” non-resonance conditions:

\[
\left| \omega \sqrt{\lambda^+_{l_j}(\epsilon, \omega, w)} - j \right| \geq \frac{\gamma}{j^\tau}, \quad \forall 1 \leq j \leq N, \forall l \geq 0,
\]

which leads to

\[
|\omega^2 \lambda^+_{l_j}(\epsilon, \omega, w) - j^2| = |\omega \sqrt{\lambda^+_{l_j}(\epsilon, \omega, w)} - j||\omega \sqrt{\lambda^+_{l_j}(\epsilon, \omega, w)} + j| > \frac{\gamma}{j^{\tau-1}}.
\]

Furthermore denote

\[
\omega_j := \min_{j \geq 0} |\omega^2 \lambda^+_{l_j}(\epsilon, \omega, w) - j^2| = |\omega^2 \lambda^+_{l_j}(\epsilon, \omega, w) - j^2|, \quad \forall 1 \leq |j| \leq N.
\]

It is clear that \( \omega_j = \omega_{-j} \) for all \( 1 \leq j \leq N \).

**Lemma 4.4.** Under assumption (4.31), for all \( s \geq 0 \), the operator \( |\hat{\mathcal{L}}_{1,D}|^{-\frac{1}{2}} \) satisfies

\[
\| |\hat{\mathcal{L}}_{1,D}|^{-\frac{1}{2}} h \|_s \leq \frac{\sqrt{L_2}}{\sqrt{L_1}} \|h\|_{s + \frac{1}{2}}, \quad \forall h \in W_N,
\]

\[
\| |\hat{\mathcal{L}}_{1,D}|^{-\frac{1}{2}} h \|_s \leq \frac{\sqrt{L_2}}{\sqrt{L_1}} N^{\frac{s}{2}} \|h\|_s, \quad \forall h \in W_N.
\]

**Proof.** Since \( |j|^{\tau-1} (1 + j^{2s}) < 2(1 + |j|^{2s+\tau-1}) \) for all \( |j| \geq 1 \), by (4.24), (4.32)–(4.33), we get

\[
\| |\hat{\mathcal{L}}_{1,D}|^{-\frac{1}{2}} h \|_s^2 \leq \frac{1}{L_1^2} \sum_{1 \leq |j| \leq N, l \in \mathbb{N}} (\lambda^+_{l_j}(\epsilon, \omega, w) + 1) \left( \frac{\hat{h}^+_{l_j}}{\sqrt{\omega^2 \lambda^+_{l_j}(\epsilon, \omega, w) - j^2}} \right)^2 (1 + j^{2s})
\]

\[
\leq \frac{1}{L_1^2} \sum_{1 \leq |j| \leq N, l \in \mathbb{N}} \frac{\lambda^+_{l_j}(\epsilon, \omega, w) + 1}{\omega_j} (\hat{h}^+_{l_j})^2 (1 + j^{2s})
\]

\[
\leq \frac{1}{\gamma L_1^2} \sum_{1 \leq |j| \leq N, l \in \mathbb{N}} |j|^{\tau-1} (\lambda^+_{l_j}(\epsilon, \omega, w) + 1) (\hat{h}^+_{l_j})^2 (1 + j^{2s})
\]

\[
\leq \frac{2L_2}{\gamma L_1^2} \|h\|_s^2 \leq \frac{2L_2^2 \gamma^{\tau-1}}{\gamma L_1^2} \|h\|_s^2,
\]

which completes the proof. \( \square \)

The next step is to verify the upper bounds of \( \|\mathcal{R}_i h\|_{s'} \), \( i = 1, 2 \).

For \( \tau \in (1, 2) \), we also assume that “Melinkov’s” non-resonance conditions holds:

\[
\left| \omega \frac{l}{c} - j \right| > \frac{\gamma}{j^\tau}, \quad \forall 1 \leq j \leq N, \forall l \geq 0,
\]

where \( c \) is defined by (4.4). Note that condition (4.36) will be applied in the proof of (F3).

**Lemma 4.5.** Under non-resonance conditions (4.31) and (4.36), if \( \|w\|_{s+\sigma} \leq 1 \), then there exists some constant \( L(s') > 0 \) such that

\[
\|\mathcal{R}_1 h\|_{s'} \leq \frac{\epsilon L(s')}{2\gamma^2} (\|h\|_{s'} + \|w\|_{s'+\sigma} \|h\|_s), \quad \forall h \in W_N, \forall s' \geq s > 1/2.
\]
Proof. Let us first claim the following: (F3): Supposed that (4.31) and (4.36) hold, for all \( \omega \in (1, 2) \), \( \tau \in (1, 2) \) and all \( |j|, |k| \in \{1, \cdots, N\} \) with \( j \neq k \), one has
\[
\sqrt{\omega_j \omega_k} \geq \gamma^2 \tilde{L}|j-k|^{-\sigma}
\] for some constant \( \tilde{L} > 0 \), where \( \omega_j, \omega_k \) are defined by (4.33), \( \sigma \) is given in (3.11).

It follows from formulae (4.27)–(4.28) and the definition of \( |\mathcal{Q}_{1,D}|^{-\frac{1}{2}} \) that
\[
R_1 h = |\mathcal{Q}_{1,D}|^{-\frac{1}{2}} \mathcal{Q}_{1,ND} |\mathcal{Q}_{1,D}|^{-\frac{1}{2}} h
\]
\[
= |\mathcal{Q}_{1,D}|^{-\frac{1}{2}} \mathcal{Q}_{1,ND} \left( \sum_{1 \leq |j| \leq N, j \neq k, l \geq 0} \frac{\hat{h}_{j,l}^\pm}{\sqrt{[\omega^2 \lambda_i^\pm(\epsilon, \omega, w) - j^2]}} \psi_i^\pm(\epsilon, \omega, w)e^{ijx} \right)
\]
\[
= -\epsilon |\mathcal{Q}_{1,D}|^{-\frac{1}{2}} \left( \sum_{1 \leq |j| \leq N, j \neq k, l \geq 0} \frac{\hat{h}_{j,l}^\pm}{\sqrt{[\omega^2 \lambda_i^\pm(\epsilon, \omega, w) - k^2]}} \frac{b_{k-j}}{a} \psi_i^\pm(\epsilon, \omega, w)e^{ikx} \right)
\]
which then gives
\[
(R_1 h)_k = -\epsilon \sum_{1 \leq |j| \leq N, j \neq k, l \geq 0} \frac{\hat{h}_{j,l}^\pm}{\sqrt{[\omega^2 \lambda_i^\pm(\epsilon, \omega, w) - k^2]}} \frac{b_{k-j}}{a} \psi_i^\pm(\epsilon, \omega, w).
\]

Combining this with (4.22)–(4.23), (4.33) yields
\[
||(R_1 h)_k||_{H^1} \leq \epsilon \frac{L_2}{L_1} \left( \sum_{1 \leq |j| \leq N, j \neq k} \frac{1}{\omega_k \omega_j} ||b_{k-j}/a||_{H^1} ||h_j||_{H^1} \right)
\]
\[
\leq \frac{\epsilon L_2}{\gamma^2 LL_1} \sum_{1 \leq |j| \leq N, j \neq k} ||b_{k-j}/a||_{H^1} |k-j|^{\sigma} ||h_j||_{H^1}.
\]

Let us define
\[
g(x) := \sum_{1 \leq |j| \leq N, j \neq k} ||b_{k-j}/a||_{H^1} |k-j|^{\sigma} ||h_j||_{H^1} e^{ix} \quad \text{with } b_0 = 0,
\]
\[
p(x) := \sum_{|j| \in \mathbb{Z}} ||b_j/a||_{H^1} |j|^{\sigma} e^{ix}, \quad q(x) := \sum_{1 \leq |j| \leq N} ||h_j||_{H^1} e^{ix}.
\]

It is straightforward that \( g = P_N(pq) \). Moreover one has
\[
||p||_{s'} \leq \sqrt{2} ||1/a||_{H^1} ||b||_{s'+\sigma} \leq C'(s')(1+||w||_{s'+\sigma}) \quad \text{and} \quad ||q||_{s'} = ||h||_{s'}, \quad \forall s' \geq s > \frac{1}{2}.
\]

Hence if \( ||w||_{s+\sigma} \leq 1 \), then we can deduce
\[
||R_1 h||_{s'} \leq \frac{\epsilon L_2}{\gamma^2 LL_1} C'(s')(||p||_{s'} ||q||_s + ||p||_s ||q||_{s'}) \leq \frac{\epsilon C''(s')}{2\gamma^2} (||w||_{s'+\sigma} ||h||_s + ||h||_{s'}).\]

Let \( L(s') = \frac{\epsilon}{L_1} C''(s') \). Combining above inequality with (4.29) completes the proof of the lemma. \( \square \)
Lemma 4.6. Given non-resonance conditions \((4.31)\) and \(\|w\|_{s+\sigma} \leq 1\), one has
\[
\|R_2 h\|_{s'} \leq \frac{L(s')}{2\gamma} (\|h\|_{s'} + \|w\|_{s'+\sigma}\|h\|_s), \quad \forall h \in W_N, \forall s' \geq s > \frac{1}{2}
\] (4.39)
for some constant \(L(s') > 0\).

**Proof.** Since \(\sigma = \tau(\tau - 1)/(2 - \tau) > \tau - 1\) by the fact \(1 < \tau < 2\), using \((4.28)\), \((4.34)\) and Lemma 2.3 yields
\[
\|R_2 h\|_{s'} \leq \frac{\sqrt{2}L_2}{\sqrt{\gamma}L_1} \|1/a\|_{H^1} \|b(t, x)D_w v(e, w)[\|\mathcal{L}_{1,D}]^{-\frac{1}{2}} h\|_{s'} + \|b\|_{s}\|D_w v[\|\mathcal{L}_{1,D}]^{-\frac{1}{2}} h\|_{H^1})
\]
\[
\leq \frac{\sqrt{2}L_2}{\sqrt{\gamma}L_1} \|1/a\|_{H^1} C(s') (\|b\|_{s'+\sigma}\|\mathcal{L}_{1,D}]^{-\frac{1}{2}} h\|_{s} + \|b\|_{s+\sigma}\|\mathcal{L}_{1,D}]^{-\frac{1}{2}} h\|_{s'-1})
\]
\[
\leq \frac{\sqrt{2}L_2}{\sqrt{\gamma}L_1} \|1/a\|_{H^1} C(s') (\|b\|_{s'+\sigma}\|\mathcal{L}_{1,D}]^{-\frac{1}{2}} h\|_{s} + \|b\|_{s+\sigma}\|\mathcal{L}_{1,D}]^{-\frac{1}{2}} h\|_{s'-1})
\] (4.38)
\[
\leq \frac{\sqrt{2}L_2}{\sqrt{\gamma}L_1} \|1/a\|_{H^1} C(s') (\|b\|_{s'+\sigma}\|\mathcal{L}_{1,D}]^{-\frac{1}{2}} h\|_{s} + \|b\|_{s+\sigma}\|\mathcal{L}_{1,D}]^{-\frac{1}{2}} h\|_{s'-1})
\] (4.39)
Combining this with \((4.29)\) yields that if \(L(s') = \frac{L_2}{L_1}C(s')\), then \((4.39)\) holds. \(\square\)

Lemma 4.7. Provided \(\|w\|_{s+\sigma} \leq 1\), for \(\varepsilon \gamma^{-2} L(s') \leq \delta(s')\) small enough, one has that the operator \(\text{Id} - R\) is invertible with
\[
\|\text{(Id} - R)^{-1} h\|_{s'} \leq 2(\|h\|_{s'} + \|w\|_{s'+\sigma}\|h\|_s), \quad \forall h \in W_N, \forall s' \geq s > \frac{1}{2}
\] (4.40)

**Proof.** It follows from Lemmata 4.5, 4.6 that the operator \(\text{Id} - R\) is invertible for \(\varepsilon \gamma^{-2} L(s') \leq c(s')\) small enough. Furthermore let us claim the following:

**F4:** If \(\|w\|_{s+\sigma} \leq 1\), then
\[
\|R^p h\|_{s'} \leq (\varepsilon \gamma^{-2} L(s'))^p (\|h\|_{s'} + \|w\|_{s'+\sigma}\|h\|_s), \quad \forall h \in W_N, \forall p \in \mathbb{N}^+.
\] (4.41)

Hence, for \(\varepsilon \gamma^{-2} L(s') \leq c(s')\) small enough, inequality \((4.41)\) reads
\[
\|\text{(Id} - R)^{-1} h\|_{s'} = \|\text{(Id} + \sum_{p \in \mathbb{N}^+} \mathcal{R}^p) h\|_{s'} \leq \|h\|_{s'} + \sum_{p \in \mathbb{N}^+} \mathcal{R}^p h\|_{s'}
\]
\[
\leq \|h\|_{s'} + \sum_{p \in \mathbb{N}^+} (\varepsilon \gamma^{-2} L(s'))^p (\|h\|_{s'} + \|w\|_{s'+\sigma}\|h\|_s)
\]
\[
\leq 2\|h\|_{s'} + 2\|w\|_{s'+\sigma}\|h\|_s
\]
for all \(h \in W_N\) and \(s' \geq s > \frac{1}{2}\).

Let us prove \((F4)\) by induction. For \(p = 1\), \((4.37)\) and \((4.39)\) establish
\[
\|R h\|_{s'} \leq \varepsilon \gamma^{-2} L(s') (\|h\|_{s'} + \|w\|_{s'+\sigma}\|h\|_s).
\] (4.42)

In particular, because of formula \((4.42)\) and inequality \(\|w\|_{s+\sigma} \leq 1\), we have
\[
\|R h\|_s \leq \varepsilon \gamma^{-2} L\|h\|_s \quad \text{with} \quad L \leq L(s').
\] (4.43)
Suppose that \((4.41)\) holds for \(p = 1\) with \(I \in \{1 \in \mathbb{N}^+ : I \geq 2\}\). Let us show that \((4.41)\) holds for \(p = I + 1\). Based on the assumption for \(p = 1\) and \((4.42)-(4.43)\), we can obtain
\[
\|R^{I+1} h\|_{s'} = \|R^I(R h)\|_{s'} \leq (\varepsilon \gamma^{-2} L(s'))^I \|R h\|_{s'} + I\|w\|_{s'+\sigma}\|R h\|_s
\]
\[
\leq (\varepsilon \gamma^{-2} L(s'))^I (\varepsilon \gamma^{-2} L(s')\|h\|_{s'} + (I\varepsilon \gamma^{-2} L + \varepsilon \gamma^{-2} L(s'))\|w\|_{s'+\sigma}\|h\|_s)
\]
\[
\leq (\varepsilon \gamma^{-2} L(s'))^{I+1} (\|h\|_{s'} + (I+1)\|w\|_{s'+\sigma}\|h\|_s),
\]
which completes the proof of \((F4)\). \(\square\)
The proof of (P5). It follows from (4.29), (4.35) and (4.40) that
\[
\|L_N^{-1}(\epsilon, \omega, w)h\|_{s'} \leq \frac{\sqrt{2}L_2}{\sqrt{L_1}} \|1/a\|_{H_1} N^{\frac{s-1}{2}} \|1/(\mathcal{R} - \mathcal{R}^{-1})(|L_{1,D}|^{-\frac{1}{2}} L_{1,D}|L_{1,D}|^{-\frac{1}{2}})^{-1} L_{1,D}^{-\frac{1}{2}} h\|_{s'}
\]
\[
\leq K(s') \gamma^{-1} N^{\tau-1} \|(h)_{s'} + \|w\|_{s'+\sigma} \|h\|_s),
\]
where \(K(s') := \frac{8L_2^3}{L_1^2} \|1/a\|_{H_1}\). In particular, by means of the fact \(\|w\|_{s'+\sigma} \leq 1\), we get
\[
\|L_N^{-1}(\epsilon, \omega, w)h\|_s \leq K\gamma^{-1} N^{\tau-1} \|h\|_s.
\]
Thus the property (P5) holds. \(\square\)

To prove (F3), some additional properties on \(\lambda^\pm_t(\epsilon, \omega, w)\) are given. Using (4.20)–(4.21) yields that there exists some constant \(m > 0\) such that
\[
\sqrt{\lambda^\pm_t(\epsilon, \omega, w) - l/c} \leq \frac{m}{t}, \quad \forall t \in \mathbb{N}^+.
\]
Moreover if \(j^2 - \omega^2 \lambda^\pm_t(\epsilon, \omega, w) > 0\), then \(l^* \geq 1\), where \(l^*\) is seen in (4.33). Hence there exists \(\hat{L} > 0, \epsilon > 0\) such that for every \(j > \omega \hat{L}\),
\[
\omega_j = \min_{i \geq 0} |\omega^2 \lambda^\pm_{k}^t(\epsilon, \omega, w) - j^2| = |\omega^2 \lambda^\pm_{k}^t(\epsilon, \omega, w) - j^2| \Rightarrow l^* \geq \frac{\epsilon}{\omega}(j - \omega \hat{L})
\]
(4.45)

Now let us show the proof of (F3) (recall (4.38)).

The proof of (F3). The fact \(1 < \tau < 2\) shows \(\varsigma = (2 - \tau)/\tau \in (0, 1)\). Denote
\[
\omega_j := |\omega^2 \lambda^\pm_{k}(\epsilon, \omega, w) - j^2|, \quad \omega_k := |\omega^2 \lambda^\pm_{k}(\epsilon, \omega, w) - k^2|, \quad l^*, i^* \geq 0.
\]

Case 1: \(2|k - j| > (\max \{k, j\})^\varsigma\). Condition (4.32) shows
\[
\omega_j \omega_k \geq \frac{\gamma^2}{(jk)^{\tau-1}} \geq \frac{\gamma^2}{(\max \{k, j\})^{2(\tau-1)}} > \frac{\gamma^2}{2^{2(\tau-1)/\varsigma}|k - j|^{2(\tau-1)/\varsigma}}.
\]

Case 2: \(0 < 2|k - j| \leq (\max \{k, j\})^\varsigma\). Since \(\varsigma \in (0, 1)\), if \(k > j\), then
\[
2(k - j) \leq k^\varsigma \Rightarrow 2j \geq 2k - k^\varsigma > k.
\]

In the same way, if \(j > k\), then \(2k > j\). As a result
\[
k/2 < j < 2k.
\]

(i) It follows from \(\varsigma = (2 - \tau)/\tau\) and \(\tau \in (1, 2)\) that \(\varsigma < 1\). We first consider the case \(\max \{k, j\} = j > j_*\) with \(j_* := \max \left\{2\omega \hat{L}, \left(\frac{6}{\gamma c}\omega^2 + 2\omega \hat{L}\right)^{\frac{1}{1-\tau}}\right\}\). The definition of \(j_*\) shows that \(l^*, i^* \geq 1\). Using (4.36), (4.44–4.45), \(j < 2k\) and \(\tau \in (1, 2)\), we derive
\[
\bigg|\left(\omega \sqrt{\lambda^\pm_{l}(\epsilon, \omega, w) - j} - \omega \sqrt{\lambda^\pm_{i}(\epsilon, \omega, w) - k}\right)
\]
\[
= \omega \left(\sqrt{\lambda^\pm_{l}(\epsilon, \omega, w) - \frac{i^*}{c}} - \omega \left(\sqrt{\lambda^\pm_{i}(\epsilon, \omega, w) - \frac{i^*}{c}}\right) + \omega \left(\frac{l^* - i^*}{c} - (j - k)\right)\right)
\]
\[
\geq \frac{\gamma}{(j - k)^{\tau}} - \omega \frac{m}{l^*} - \omega \frac{m}{i^*} \geq \frac{\gamma}{(j - k)^{\tau}} - \omega^2 \frac{m}{\epsilon(j - \omega \hat{L})} - \omega^2 \frac{m}{\epsilon(k - \omega \hat{L})} \geq \frac{2^\tau \gamma}{j^{\tau}} - \frac{3m\omega^2}{\epsilon(j - 2\omega \hat{L})},
\]
\[
> \frac{\gamma}{2j^{\tau}} + \frac{\gamma}{j^{\tau}} + \frac{\gamma}{2j^{\tau}} - \frac{3m\omega^2}{\epsilon(j - 2\omega \hat{L})}.
\]
Hence, by means of the fact $j < 2k$, we obtain
\[
\left| \omega \sqrt{\lambda_n^\pm(\epsilon, \omega, w)} - j \right| + \left| \omega \sqrt{\lambda_n^\pm(\epsilon, \omega, w)} - k \right| > \frac{1}{2} \left( \frac{\gamma}{j^{\xi}} + \frac{\gamma}{k^{\xi}} \right).
\] (4.47)

The same conclusion is reached if $\max \{j, k\} = k > j_*$. In addition from formula (4.47) implies that

\[
\left| \omega \sqrt{\lambda_n^\pm(\epsilon, \omega, w)} - j \right| > \frac{\gamma}{2j^{\xi}} \text{ or } \left| \omega \sqrt{\lambda_n^\pm(\epsilon, \omega, w)} - k \right| > \frac{\gamma}{2k^{\xi}}
\]
holds. Without loss of generality, we suppose $|\omega \sqrt{\lambda_n^\pm(\epsilon, \omega, w)} - j| > \frac{\gamma}{2j^{\xi}}$. Then
\[
\omega_j = |\omega^2 \lambda_n^\pm(\epsilon, \omega, w) - j^2| = \left| \left( \omega \sqrt{\lambda_n^\pm(\epsilon, \omega, w)} + j \right) \left( \omega \sqrt{\lambda_n^\pm(\epsilon, \omega, w)} - j \right) \right| > \frac{\gamma}{2} j^{1-\xi},
\]
which leads to
\[
\omega_j \omega_k > \frac{\gamma^2}{(jk)^{\tau-1}} > \frac{\gamma^2}{(j_*)^{2(\tau-1)}} = \frac{\gamma^2}{(2\omega_L)^{2(\tau-1)}} = \frac{\gamma^2}{(4\hat{L})^{2(\tau-1)}} \quad (4.49)
\]
for all $\omega \in (1, 2)$. On the other hand, if $j_* = 2\omega\hat{L}$, then
\[
\omega_j \omega_k > \frac{\gamma^2}{(jk)^{\tau-1}} > \frac{\gamma^2}{(j_*)^{2(\tau-1)}} = \frac{\gamma^2}{(6m^2\omega^2 + 2\omega\hat{L})} = \frac{\gamma^4}{16 \left( \frac{6m^2}{	au} \right)^2} \quad (4.50)
\]
for all $\omega \in (1, 2)$. Formula (4.38) is reached if we take the minimums of lower bounds in (4.46), (4.48)-(4.50). Since $\omega_j = \omega_{-j}, \omega_k = \omega_{-k}$, the remainder of the lemma may be proved in the similar way as shown above when $l \geq 1, k \leq -1$, or $l \leq -1, k \geq 1$, or $l, k \leq -1$. Thus we complete the proof of (F3). □

4.3. Measure estimate. Let us show the measure estimate on $A_\gamma$ defined by (4.51).

Lemma 4.8. Let $I = (\tilde{\omega}_1, \tilde{\omega}_2)$ with $\omega_1 < \tilde{\omega}_1 < \tilde{\omega}_2 < \omega_2$. There exists some constant $\delta_5 \in (0, \delta_4]$ such that, for all $\gamma \in (0, 1)$, the Lebesgue measures of the Cantor set
\[
A_\gamma := \{(\epsilon, \omega) \in A_0, \forall n \in \mathbb{N}, \epsilon < \delta_5 \gamma \}
\] (4.51)
and its sections $A_\gamma(\epsilon) := \{\omega : (\epsilon, \omega) \in A_\gamma\}$ have the following property: there exists a constant $\hat{C} = \hat{C}(I)$, independent on $\gamma$ and $\epsilon$, such that
\[
\frac{\text{meas}(I \cap A_\gamma(\epsilon))}{\text{meas}(I)} > 1 - \hat{C} \gamma, \quad \frac{\text{meas}(B_\gamma \cap A_\gamma)}{\text{meas}(B_\gamma)} > 1 - \hat{C} \gamma, \quad \forall \epsilon < \delta_5 \gamma, \quad (4.52)
\]
where $B_\gamma$ is the rectangular region $B_\gamma = (0, \delta_5 \gamma) \times I$.

To give the measure estimates on $A_\gamma$ defined by (4.51), we have to introduce the following “perturbation of self-adjoint operators” result developed by T. Kato [31]. Denote by $\mathcal{H}$ and $\mathcal{B}(\mathcal{H})$ a Hilbert space and the space of bounded operators from $\mathcal{H}$ to $\mathcal{H}$ respectively.

Theorem 4.9. [31] Theorem 4.10] Define $T_1 = T_2 + S$ with $T_2$ self-adjoint in $\mathcal{H}$ and $S \in \mathcal{B}(\mathcal{H})$ symmetric. Then $T_1$ is a self-adjoint operator with dist$(\Sigma(T_1), \Sigma(T_2)) \leq \|S\|$, namely
\[
\sup_{\zeta \in \Sigma(T_1)} \text{dist}(\zeta, \Sigma(T_2)) \leq \|S\|, \quad \sup_{\zeta \in \Sigma(T_2)} \text{dist}(\zeta, \Sigma(T_1)) \leq \|S\|,
\]
where $\Sigma(T_1)$ and $\Sigma(T_2)$ are spectrums of $T_1$ and $T_2$ respectively.
This implies the following lemma.

**Lemma 4.10.** Fixing $\epsilon \in (\epsilon_1, \epsilon_2)$, for all $\omega_1 < \omega'_1 < \omega'_2 < \omega_2$, the eigenvalues of (4.1) satisfy

$$|\lambda_{j_1}^{\pm}(\epsilon, \omega_1, w(\rho, \omega_1)) - \lambda_{j_2}^{\pm}(\epsilon, \omega_2, w(\rho, \omega_2))| \leq \frac{\epsilon \kappa}{\omega_1^3} |\omega_1' - \omega_2'| + \frac{\epsilon \kappa}{\omega_1^2} \|w(\rho, \omega_1) - w(\rho, \omega_2)\|_s, \quad \forall l \geq 0$$

for some constant $\kappa > 0$.

**Proof.** Let us define

$$T_1 u := \frac{d^2}{d\xi^2} u + \vartheta_1(\xi) u, \quad T_2 u := \frac{d^2}{d\xi^2} u + \vartheta_2(\xi) u,$$

where $\vartheta_i(\xi) = \frac{e^2}{\omega(\xi)} (\xi)$ (see (4.10)), $i = 1, 2$. It follows from formula (4.2) and Lemma 5.3 that

$$\|\vartheta_i\|_{L^2} \leq \|\vartheta_i\|_{L^\infty} \|u\|_{L^2} \leq \|e^2 \psi(\psi)/a(\psi)\|_{L^1} \|u\|_{L^2} \leq \mathcal{C} \|u\|_{L^2},$$

which leads to $\vartheta_i \in \mathcal{L}(L^2, L^2)$. It is obvious that $T_1, T_2$ are self-adjoint using Theorem 4.9. By means of Theorem 4.9, Lemma 5.5, and the inverse Liouville substitution of (4.3), one has

$$|\lambda_{j_1}^{\pm}(\omega_1) - \lambda_{j_2}^{\pm}(\omega_2)| \leq \frac{1}{\epsilon^2} |\lambda_{j_1}^{\pm}(\vartheta_2) - \lambda_{j_1}^{\pm}(\vartheta_1)| \leq \frac{1}{\epsilon^2} \|\vartheta_2 - \vartheta_1\|_{L^1} \leq \frac{1}{\epsilon^2} \|\vartheta_2 - \vartheta_1\|_{L^2} \leq \frac{\epsilon \kappa}{\omega_1^3} |\omega_1' - \omega_2'| + \frac{\epsilon \kappa}{\omega_1^2} \|w(\rho, \omega_1) - w(\rho, \omega_2)\|_s.$$

\[\square\]

**The proof of Lemma 4.8.** Fix $\epsilon \in (\epsilon_1, \epsilon_2)$, and set

$$\mathcal{R}_n = A_n(\epsilon) \setminus A_{n+1}(\epsilon), \quad \forall n \in \mathbb{N},$$

where $A_n(\epsilon) := \{\omega: (\epsilon, \omega) \in A_n\}$. Let us show that $\cup_{n \in \mathbb{N}} \mathcal{R}_n$ has small measure. Define

$$\Omega^{n,1}_{j,l} := \left\{\omega \in (\omega_1, \omega_2) : \left|\omega \sqrt{\lambda_{j_1}^{\pm}(\epsilon, \omega, w)} - j \right| \leq \frac{\gamma}{j^{\tau}}\right\}, \quad \Omega^{n,2}_{j,l} := \left\{\omega \in (\omega_1, \omega_2) : \left|\omega - \frac{l}{\epsilon} - j \right| \leq \frac{\gamma}{j^{\tau}}\right\}.$$

Obviously, since $\Omega^{n,2}_{j,0} = \emptyset$ due to $\gamma < 1$, we have

$$\mathcal{R}_n \subseteq \left( \bigcup_{1 \leq j \leq N_n+1, l \geq 0} \Omega^{n,1}_{j,l} \cap \tilde{A}_n(\epsilon) \cap (\tilde{\omega}_1, \tilde{\omega}_2) \right) \cup \left( \bigcup_{N_n < j \leq N_{n+1+1}, l \geq 1} \Omega^{n,2}_{j,l} \cap (\tilde{\omega}_1, \tilde{\omega}_2) \right),$$

where $\tilde{A}_0 := A_0$ and

$$\tilde{A}_n := \left\{(\epsilon, \omega) \in \tilde{A}_{n-1} : \left|\omega \sqrt{\lambda_{j_1}^{\pm}(\epsilon, \omega, w)} - j \right| > \frac{\gamma}{j^{\tau}}, \forall j = 1, \ldots, N_n, \forall l \geq 0\right\}, \quad \forall n \in \mathbb{N}^+.$$

Moreover it follows from the definition of $\Omega^{n,1}_{j,l}$, (4.20)–(4.21) and the fact $\gamma < 1$ that

$$C_{1,j} \omega \sqrt{\lambda_{j_1}(\epsilon, \omega, w)} < C_{2,j}, \quad C_{1,j} \omega \omega \omega_{j_1} < C_{2,j}, \quad \forall \omega \in \Omega^{n,1}_{j,l}, l \in \mathbb{N}^+$$

for some constants $C_1, C_2 > 0$.

Since $\lambda_{j_1}(\epsilon, \omega, w) \geq m_0 > 0$ (recall (4.19)), by Lemma 4.10, one has

$$\left|\sqrt{\lambda_{j_1}^{\pm}(\epsilon, \omega_2, w(\rho, \omega_2))} - \sqrt{\lambda_{j_1}^{\pm}(\epsilon, \omega_1, w(\rho, \omega_1))}\right| \leq \frac{|\lambda_{j_1}(\epsilon, \omega_1, w(\rho, \omega_1)) - \lambda_{j_1}(\epsilon, \omega_2, w(\rho, \omega_2))|}{\sqrt{\lambda_{j_1}^{\pm}(\epsilon, \omega_2, w(\rho, \omega_2))} + \sqrt{\lambda_{j_1}^{\pm}(\epsilon, \omega_1, w(\rho, \omega_1))}}$$

$$\leq \frac{\epsilon \kappa}{\omega_1^3 \sqrt{m_0}} |\omega_1' - \omega_2'| + \frac{\epsilon \kappa}{\omega_1^2} \|w(\rho, \omega_1) - w(\rho, \omega_2)\|_s$$

(4.54)
for all $\omega'_1, \omega'_2 \in A_n(\epsilon)$ with $\omega'_1 < \omega'_2$. Hence if $e\gamma^{-1} \leq \delta_5$ is small enough, then it follows from (4.20)–(4.21) and (4.35) that

$$|\omega'_2 \sqrt[2]{\lambda \gamma^2(e, \omega, w_1)} - \omega'_1 \sqrt[2]{\lambda \gamma^2(e, \omega, w_1)}| \leq |\omega'_2 - \omega'_1| \frac{l}{c} - \frac{ek}{\sqrt{m_0}}|\omega'_1 - \omega'_2| - \frac{ek}{\sqrt{m_0} \gamma^2}|\omega'_1 - \omega'_2| > |\omega'_2 - \omega'_1| \frac{l}{4c}, \quad \forall l \in \mathbb{N}^+.$$  \hspace{1cm} (4.55)

In particular, one has

$$|\omega'_2 \sqrt[2]{\lambda_0(e, \omega, w_1)} - \omega'_1 \sqrt[2]{\lambda_0(e, \omega, w_1)}| > |\omega'_2 - \omega'_1| \frac{\sqrt{m_0}}{4} \quad \text{for } e\gamma^{-1} \leq \delta_5 \text{ small enough.}$$  \hspace{1cm} (4.56)

Consequently, if $\Omega^{n,1}_{j,l} \cap \tilde{A}_n(e) \cap (\tilde{\omega}_1, \tilde{\omega}_2)$ is nonempty, then using (4.55)–(4.56) and (4.53) yields

$$\text{meas}(\Omega^{n,1}_{j,l}) \leq \frac{8\gamma}{\sqrt{m_0}j^2}, \quad \text{meas}(\Omega^{n,1}_{j,l}) \leq \frac{8\gamma}{lj^2} < C\tilde{\omega}_2 \gamma_j, \quad \forall l \in \left( \frac{C_1}{\tilde{\omega}_2}, \frac{C_1}{\tilde{\omega}_1} \right) =: \mathcal{I}(j), j \geq 1.$$  \hspace{1cm} (4.57)

Since

$$\bigcup_{1 \leq j \leq N, l \geq 0} \Omega^{n,1}_{j,l} \cap \tilde{A}_0(e) \cap (\tilde{\omega}_1, \tilde{\omega}_2) \subseteq \bigcup_{1 \leq j \leq N} \Omega^{n,1}_{j,l} \cap (\tilde{\omega}_1, \tilde{\omega}_2),$$

in view of (4.57), we deduce

$$\text{meas} \left( \bigcup_{1 \leq j \leq N, l \geq 0} \Omega^{n,1}_{j,l} \cap (\tilde{\omega}_1, \tilde{\omega}_2) \right) \leq C_1 \gamma \sum_{1 \leq j \leq N} \frac{1}{j^2}.$$ \hspace{1cm} (4.58)

Let us estimate the measure of $\bigcup_{1 \leq j \leq N, l \geq 0} \Omega^{n,1}_{j,l} \cap \tilde{A}_n(e) \cap (\tilde{\omega}_1, \tilde{\omega}_2), \forall n \in \mathbb{N}^+$. We consider either $N_n < j \leq N_{n+1}$ or $1 \leq j \leq N_n$. In the first case

$$\text{meas} \left( \bigcup_{N_n < j \leq N_{n+1}, l \geq 0} \Omega^{n,1}_{j,l} \cap \tilde{A}_n(e) \cap (\tilde{\omega}_1, \tilde{\omega}_2) \right) \leq C_1 \gamma \sum_{N_n < j \leq N_{n+1}} \frac{1}{j^2} \quad \text{in the latter}$$ \hspace{1cm} (4.59)

and in the latter

$$\left| \omega \sqrt[2]{\lambda \gamma^2(e, \omega, w_{n-1})} - j \right| \leq \left| \omega \sqrt[2]{\lambda \gamma^2(e, \omega, w_n)} - j \right| + \left| \omega \sqrt[2]{\lambda \gamma^2(e, \omega, w_{n-1})} - \omega \sqrt[2]{\lambda \gamma^2(e, \omega, w_n)} \right| \leq \frac{\gamma}{j^2} + \frac{eC}{\tilde{\omega}_1 \gamma} \left| w_n - w_{n-1} \right| \leq \frac{\gamma}{j^2} + eC N_{n-1} N_{n-1}^{-\beta} S_{n-1}.$$ \hspace{1cm} (4.54)

This also shows that for $1 \leq j \leq N_n$,

$$\Omega^{n,1}_{j,l} \cap \tilde{A}_n(e) \subseteq \left\{ \omega : \frac{\gamma}{j^2} < \left| \omega \sqrt[2]{\lambda \gamma^2(e, \omega, w_{n-1})} - j \right| \leq \frac{\gamma}{j^2} + \frac{eC}{\tilde{\omega}_1 \gamma} N_{n-1} N_{n-1}^{-\beta} S_{n-1} \right\}$$

which leads to

$$\text{meas} \left( \Omega^{n,1}_{j,l} \cap \tilde{A}_n(e) \cap (\tilde{\omega}_1, \tilde{\omega}_2) \right) \leq \frac{eC'}{\tilde{\omega}_1 \gamma} \left| w_n - w_{n-1} \right| \leq C_1 \gamma N_{n-1}^{-\beta} S_{n-1}.$$ \hspace{1cm} (4.20)

Hence one has that for $1 \leq j \leq N_n$,

$$\text{meas} \left( \bigcup_{1 \leq j \leq N, l \geq 0} \Omega^{n,1}_{j,l} \cap \tilde{A}_n(e) \cap (\tilde{\omega}_1, \tilde{\omega}_2) \right) \leq \sum_{1 \leq j \leq N_n} \frac{eC'}{\tilde{\omega}_1 \gamma} N_{n-1} N_{n-1}^{-\beta} S_{n-1} \leq C_1 \gamma N_{n-1}^{-1}.$$ \hspace{1cm} (4.60)
Moreover we also have
\[ C_1 j < \omega l < C_2 j, \quad \forall \omega \in \Omega_{j,l}^{n,2}, \forall l \in \mathbb{N}^+. \]
Applying the same technique as above yields \( \text{meas}(\Omega_{j,l}^{n,2}) < \frac{C_2 \gamma}{j^{\tau+1}}, \forall l \in I(j) \), which then gives
\[
\text{meas} \left( \bigcup_{N_n < j \leq N_{n+1},l \geq 1} \Omega_{j,l}^{n,2} \cap (\bar{\omega}_1, \bar{\omega}_2) \right) \leq C_1 \gamma \sum_{N_n < j \leq N_{n+1}} \frac{1}{j^\tau}, \quad \forall n \in \mathbb{N}.
\] (4.61)
It follows from (4.58)–(4.61) and the fact \( \tau > 1 \) that
\[
\text{meas} \left( \bigcup_{n \in \mathbb{N}} R_n \cap (\bar{\omega}_1, \bar{\omega}_2) \right) \leq C \gamma \left( \sum_{n \in \mathbb{N}^+} N_{n-1} + \sum_{j \geq 1} \frac{1}{j^\tau} \right) = O(\gamma).
\]
The second estimate in (4.52) holds because
\[
\text{meas}(B_\gamma \cap A_\gamma) = \int_0^{\delta_\gamma} \text{meas}(A_\gamma(\epsilon) \cap (\bar{\omega}_1, \bar{\omega}_2)) \, d\epsilon.
\] □

Theorem 2.2 follows from Lemmata 2.3, 3.1 and 4.8. Let us complete the proof of Theorem 2.2.

The proof of Theorem 2.2. Lemmata 3.1 and 4.8 present that \( w(\epsilon, \omega) \in C^1(A_\gamma; W \cap \mathcal{H}^s) \) solves the range equation \( (P) \) in (2.3) for all \( (\epsilon, \omega) \in A_\gamma \subseteq (0, \delta_\gamma^{-1}) \times (\omega_1, \omega_2) \), where \( A_\gamma \) is defined by (4.51). And it follows from Lemma 3.1 that
\[
\|w\|_s \leq \sum_{k=0}^{+\infty} \|h_k\|_s \leq \sum_{k=1}^{+\infty} K_2 e^{-\gamma^{-1}N_k^{\sigma-1}} + K_1 e^{-\gamma^{-1}N_k^{\tau-1}} < r.
\]
Then using Lemma 2.3 yields that \( v(\epsilon, \omega, w) \) solves the bifurcation equation \( (Q) \) in (2.3). Hence
\[
u(\epsilon, \omega, w) := v(\epsilon, \omega, w(\epsilon, \omega)) + w(\epsilon, \omega) \in H^1(\mathbb{T}) \oplus (W \cap \mathcal{H}^s)
\] with \( \int_{\mathbb{T}} u(\epsilon, \omega; \cdot, \cdot) \, dt \, dx = 0 \) is a solution of equation (2.1). And estimate (2.9) follows by (3.14)–(3.15) and (2.10).

Moreover, since \( u \) solves \( \omega^2 u_{tt} = \epsilon f(t, x, u) + a(t)u_{xx} \), we obtain
\[
 u_{tt} \in H^1(\mathbb{T}), \quad \forall x \in \mathbb{T},
\]
implying \( u(t, x) \in H^3(\mathbb{T}) \subset C^2(\mathbb{T}) \) for all \( x \in \mathbb{T} \). □

5. APPENDIX

The proof of Remark 2.1. (i) Decomposing \( uv \) as
\[
 uv = \sum_{j \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} u_{j-k}v_k \right) e^{ijx}, \quad \forall u, v \in \mathcal{H}^s
\] and using the Cauchy inequality can yield
\[
\|uv\|^2_s = \sum_j \left( 1 + j^{2s} \right) \| \sum_k u_{j-k}v_k \|^2_{H^1} \leq \sum_j \left( \sum_k \left( 1 + j^{2s} \right) \frac{1}{c_{jk}} \| u_{j-k}v_k \|_{H^1} \right)^2 \frac{1}{c_{jk}} \leq \sum_j \left( \sum_k \frac{1}{c_{jk}^2} \right) \left( \sum_k \| u_{j-k} \|^2_{H^1} (1 + (j - k)^2s) \| v_k \|^2_{H^1} (1 + k^{2s}) \right),
\]
where
\[
c_{jk} := \sqrt{\frac{(1 + k^{2s})(1 + (j - k)^2s)}{1 + j^{2s}}}.
\]
A simple calculation yields
\[ 1 + j^{2s} \leq 1 + (k + j - k)^{2s} \leq 1 + 2^{2s-1}(k^{2s} + (j-k)^{2s}) \]
\[ < 2^{2s-1}(1 + k^{2s}) + 2^{2s-1}(1 + (j-k)^{2s}), \]
which leads to
\[ \sum_{k \in \mathbb{Z}} \frac{1}{2^k} < 2^{2s-1}\left( \sum_{k \in \mathbb{Z}} \frac{1}{1 + k^{2s}} + \sum_{k \in \mathbb{Z}} \frac{1}{1 + (j-k)^{2s}} \right) = 2^{2s-1}\sum_{k \in \mathbb{Z}} \frac{1}{1 + k^{2s}} =: C(s)^2. \]
Hence \(\|uv\|_s\) may be bounded from above by \(C(s)\|u\|_s\|v\|_s\).
(ii) It also follows from the Cauchy inequality that
\[ \sum_{j \in \mathbb{Z}} \|u_j\|_{H^1} \leq \left( \sum_{j \in \mathbb{Z}} \|u_j\|_{H^2}^2 \right)^{1/2} \left( \sum_{j \in \mathbb{Z}} \frac{1}{1 + j^{2s}} \right)^{1/2} \leq C(s)\|u\|_s. \]
\[ \square \]
By the definition of \(\mathcal{H}^s\), for completeness, we list Lemmata 5.1–5.3 and the proof can be found in [8].

**Lemma 5.1** (Moser-Nirenberg). For all \(u_1, u_2 \in \mathcal{H}^{s'} \cap \mathcal{H}^s\) with \(s' \geq 0\) and \(s > \frac{1}{2}\), we have
\[ \|u_1u_2\|_{s'} \leq C(s')\left( \|u_1\|_{L^\infty(T;H^1)} \|u_2\|_{s'} + \|u_1\|_{s'} \|u_2\|_{L^\infty(T;H^1)} \|u_1\|_{s'} \right) \]
\[ \leq C(s')\left( \|u_1\|_{s'} \|u_2\|_{s'} + \|u_1\|_{s'} \|u_2\|_{s'} \right). \]

**Lemma 5.2** (Logarithmic convexity). Let \(0 \leq a' \leq a \leq b \leq b'\) satisfy \(a + b = a' + b'\). Taking \(p := \frac{b' - a}{b - a'}\), one has
\[ \|u_1\|_a \|u_2\|_b \leq p\|u_1\|_{a'} \|u_2\|_{b'} + (1-p)\|u_2\|_{a'} \|u_1\|_{b'}, \quad \forall u_1, u_2 \in \mathcal{H}^{b'}. \]
In particular
\[ \|u\|_a \|u\|_b \leq \|u\|_{a'} \|u\|_{b'}, \quad \forall u \in \mathcal{H}^{b'}. \]

Let \(\mathcal{C}_k\) denote the following space composed by the space-independent functions:
\[ \mathcal{C}_k := \left\{ f \in C(T \times \mathbb{R}; \mathbb{R}) : u \mapsto f(\cdot, u) \text{ belongs to } C^k(\mathbb{R}; H^1(T)) \right\}. \]

**Lemma 5.3.** Let \(f \in \mathcal{C}_1, \mathcal{H} := \|u\|_{L^\infty(T)}\). Then the composition operator \(u(t) \mapsto f(t, u(t))\) belongs to \(C(H^1(T); H^1(T))\) with
\[ \|f(t, u(t))\|_{H^1} \leq C\left( \max_{u \in [-\mathcal{H}, \mathcal{H}]} \|f(\cdot, u)\|_{H^1} + \max_{u \in [-\mathcal{H}, \mathcal{H}]} \|\partial_u f(\cdot, u)\|_{H^1} \|u\|_{H^1} \right). \]
In particular, one has \(\|f(t, 0)\|_{H^1} \leq C\).

With the help of Lemmata 5.1–5.3, the following lemma can be obtained.

**Lemma 5.4.** Let \(f \in \mathcal{C}_k\) with \(k \geq 1\). Then, for all \(s > \frac{1}{2}, 0 \leq s' \leq k - 1\), the composition operator \(u(t, x) \mapsto f(t, x, u(t, x))\) is in \(C(\mathcal{H}^s \cap \mathcal{H}^{s'}; \mathcal{H}^{s'})\) with
\[ \|f(t, x, u)\|_{s'} \leq C(s', \|u\|_s)(1 + \|u\|_{s'}). \]

**Proof.** If \(s' = p\) is an integer, for all \(p \in \mathbb{N}\) with \(p \leq k - 1\), \(u \in \mathcal{H}^s \cap \mathcal{H}^p\), we have to prove
\[ \|f(t, x, u)\|_{p} \leq C(p, \|u\|_s)(1 + \|u\|_{p}) \]
and that
\[ f(t, x, u_n) \rightarrow f(t, x, u) \quad \text{as } u_n \rightarrow u \text{ in } \mathcal{H}^s \cap \mathcal{H}^p. \]
Let us verify (5.5)–(5.6) by a recursive argument. For all \(g \in \mathcal{C}_1\), it is clear that
\[ \|g(t, u(t))\|_{H^1} \leq C(1 + \|u\|_{H^1}) \]
(5.7)
by Lemma 5.3. First, for \( p = 0 \), using (5.7) and Remark 2.1 (ii) yields
\[
\| f(t, x, u) \|_0 \leq C \max_{x \in \mathbb{T}} \| f(\cdot, x, u(\cdot, x)) \|_{H^1(\mathbb{T})} \leq C'(1 + \max_{x \in \mathbb{T}} \| u(\cdot, x) \|_{H^1(\mathbb{T})})
\]
\[
\leq C''(1 + \| u \|_s) =: C(\| u \|_s), \quad \forall x \in \mathbb{T}.
\] (5.8)

If \( k \geq 2 \), then a similar argument as above can yield
\[
\| \partial_x f(t, x, u) \|_0 \leq C(\| u \|_s), \quad \max_{x \in \mathbb{T}} \| \partial_u f(\cdot, x, u(\cdot, x)) \|_{H^1(\mathbb{T})} \leq C(\| u \|_s).
\] (5.9)

Moreover Remark 2.1 (ii) shows that
\[
\max_{x \in \mathbb{T}} \| u_n(\cdot, x) - u(\cdot, x) \|_{H^1(\mathbb{T})} \rightarrow 0 \quad \text{as} \quad u_n \rightarrow u \quad \text{in} \quad \mathcal{H}^s \cap \mathcal{H}^0.
\]

Then it follows from the continuity property in Lemma 5.3 and the compactness of \( \mathbb{T} \) that
\[
\| f(t, x, u_n) - f(t, x, u) \|_0 \leq C \max_{x \in \mathbb{T}} \| f(\cdot, x, u_n(\cdot, x)) - f(\cdot, x, u(\cdot, x)) \|_{H^1(\mathbb{T})} \rightarrow 0
\]
as \( u_n \rightarrow u \) in \( \mathcal{H}^s \cap \mathcal{H}^0 \).

Assume that (5.5) holds for \( p = n \) with \( n \in \mathbb{N}^+ \), then we have to verify that it holds for \( p = n + 1 \) with \( n + 1 \leq k - 1 \).

Since \( \partial_x f, \partial_u f \in C_{k-1} \), by the above assumption for \( p = n \), we have that for all \( u \in \mathcal{H}^s \cap \mathcal{H}^{n+1} \),
\[
\| \partial_x f(t, x, u) \|_n \leq C(n, \| u \|_s)(1 + \| u \|_n), \quad \| \partial_u f(t, x, u) \|_n \leq C(n, \| u \|_s)(1 + \| u \|_n).
\] (5.10)

Letting \( \bar{f}(t, x) := f(t, x, u(t, x)) \), we write \( \bar{f} \) as the form \( \bar{f}(t, x) = \sum_{j \in \mathbb{Z}} \bar{f}_j(t) e^{ijx} \). It is obvious that \( \partial_x \bar{f}(t, x) = \sum_{j \in \mathbb{Z}} j \bar{f}_j(t) e^{ijx} \). By the definition of \( s \)-norm (recall (2.2)), one has
\[
\| \bar{f} \|_{n+1}^2 = \sum_{j \in \mathbb{Z}} (1 + j^2(2n+1)) |\bar{f}_j|^2_{L^2} = \sum_{j \in \mathbb{Z}} |\bar{f}_j|_{L^2}^2 + \sum_{j \in \mathbb{Z}} j^2 |\bar{f}_j|_{L^2}^2 \leq \| \bar{f}(t, x) \|_n^2 + \| \partial_x \bar{f}(t, x) \|_n^2
\]
\[
= \left( \| \bar{f}(t, x) \|_n + \| \partial_x \bar{f}(t, x) \|_n \right)^2,
\]
which then leads to
\[
\| f(t, x, u) \|_{n+1} \leq \| f(t, x, u) \|_0 + \| \partial_x f(t, x, u) \|_n + \| \partial_u f(t, x, u) \|_n.
\] (5.11)

For \( n = 0 \) (\( p = 1 \)), formulae (5.9)–(5.11) carry out
\[
\| f(t, x, u) \|_1 \leq \| f(t, x, u) \|_0 + \| \partial_x f(t, x, u) \|_0 + C \max_{x \in \mathbb{T}} \| \partial_u f(\cdot, x, u(\cdot, x)) \|_{H^1(\mathbb{T})} \| \partial_x u \|_0
\]
\[
\leq 2C(\| u \|_s) + C'(\| u \|_s) \| u \|_1 \leq C(1, \| u \|_s)(1 + \| u \|_1),
\]
where \( C(1, \| u \|_s) := \max\{2C(\| u \|_s), C'(\| u \|_s)\} \).

Letting \( s \in (1/2, \min(1, s)) \), it is straightforward to establish that
\[
\begin{cases}
\frac{1}{s} < n < s + 1 < n + 1, & n = 1, \\
\frac{1}{s} < s + 1 < n < n + 1, & \forall n \geq 2.
\end{cases}
\]

which then gives rise to
\[
\| u \|_{n} \| u \|_{s+1} \leq \| u \|_{n+1} \| u \|_{s} \leq \| u \|_{n+1} \| u \|_{s}.
\] (5.12)
As a consequence, by (5.1), (5.8)–(5.12), Remark 2.1 (ii) and the above assumption for \( p = n \), we obtain
\[
\|f(t, x, u)\|_{n+1} \leq C(\|u\|_s) + C(n, \|u\|_s)(1 + \|u\|_n) + C(n)\|\partial_u f(t, x, u)\|_{L^\infty(T, H^1(T))} \\
+ C(n)\|\partial_u f(t, x, u)\|_{L^\infty(T, H^1(T))} \|u\|_{n+1} \\
\leq C(\|u\|_s) + C(n, \|u\|_s)(1 + \|u\|_n) + C(n)C(n, \|u\|_s)(1 + \|u\|_n) \|u\|_{n+1} \\
\leq C(n + 1, \|u\|_s)(1 + \|u\|_{n+1}),
\]
where \( C(n + 1, \|u\|_s) := 4 \max\{C(\|u\|_s), C(n, \|u\|_s), C(n)C(n, \|u\|_s)(1 + \|u\|_n), C(n)\}(\|u\|_s)\).

This implies that (5.5) holds for \( p = n + 1 \).

Finally, we assume that (5.6) holds for \( p = n \). Using inequality (5.11), we may get the continuity property \( f \) with respect to \( u \) for \( \ell = n + 1 \) with \( n + 1 \leq k - 1 \).

When \( s' \) is not an integer, the result is verified by the Fourier dyadic decomposition. The argument is similar to the proof of Lemma A.1 in [19].

**Lemma 5.5.** Letting \( f \in C_k \) with \( k \geq 3 \), for all \( 0 \leq s' \leq k - 3 \), a map \( F \) is defined as
\[
F: \mathcal{H}^s \cap \mathcal{H}^{s'} \to \mathcal{H}^{s'}, \\
u \mapsto f(t, x, u).
\]

Then \( F \) is a \( C^2 \) map with respect to \( u \) and
\[
\|\partial_u f(t, x, u)\|_{s'} \leq C(s', \|u\|_s)(1 + \|u\|_s'), \quad \|\partial_u^2 f(t, x, u)\|_{s'} \leq C(s', \|u\|_s)(1 + \|u\|_s'). \tag{5.13}
\]

**Proof.** It is straightforward that \( \partial_u f \in C_{k-1}, \partial_u^2 f \in C_{k-2} \). Then it follows from Lemma 5.4 that the maps \( u \mapsto \partial_u f(t, x, u), u \mapsto \partial_u^2 f(t, x, u) \) are continuous and that (5.13) holds.

Let us investigate that \( F \) is \( C^2 \) with respect to \( u \). Using the continuity property of \( u \mapsto \partial_u f(t, x, u) \), we deduce
\[
\|f(t, x, u + h) - f(t, x, u) - \partial_u f(t, x, u)h\|_{s'} = \int_0^1 (\partial_u f(t, x, u + vh) - \partial_u f(t, x, u)) \, dw \|_{s'} \\
\leq C(s')\|h\|_{\mathcal{H}^{\max\{s, s'\}}} \max_{w \in [0, 1]} \|\partial_u f(t, x, u + vh) - \partial_u f(t, x, u)\|_{\mathcal{H}^{\max\{s, s'\}}} \\
= o(\|h\|_{\mathcal{H}^{\max\{s, s'\}}}),
\]
which leads to
\[
\partial_u f(t, x, u + h) - \partial_u f(t, x, u)h - \partial_u^2 f(t, x, u)h^2 = h^2 \int_0^1 (\partial_u^2 f(t, x, u + vh) - \partial_u^2 f(t, x, u)) \, dw.
\]

The same discussion as above yields that \( F \) is twice differentiable with respect to \( u \) and that \( u \mapsto \partial_u^2 F(u) \) is continuous.

**The proof of properties (P1)–(P4).** By (2.4), the definition of \( F \) and Lemma 5.5, we obtain
\[
F(\epsilon, \omega, w) = F(v(\epsilon, \omega, w) + w) = f(t, x, v(\epsilon, \omega, w) + w), \\
D_wF(\epsilon, \omega, w) = D_aF(v(\epsilon, \omega, w) + w)(1 + D_w v(\epsilon, \omega, w)) = \partial_a f(t, x, v(\epsilon, \omega, w) + w)(1 + D_w v(\epsilon, \omega, w)), \\
D_w^2F(\epsilon, \omega, w) = D_a^2F(v(\epsilon, \omega, w) + w)(1 + D_w v(\epsilon, \omega, w))^2 + D_aF(v(\epsilon, \omega, w) + w)D_w^2 v(\epsilon, \omega, w) \\
= \partial_a^2 f(t, x, v(\epsilon, \omega, w) + w)(1 + D_w v(\epsilon, \omega, w))^2 + \partial_a f(t, x, v(\epsilon, \omega, w) + w)D_w^2 v(\epsilon, \omega, w).
\]
For all \( s' \geq s > 1/2 \), it follows from Lemmata 5.4, 5.5, 2.3 and \( \|w\|_s \leq 1 \) that

\[
\|F(\epsilon, \omega, w)\|_{s'} \leq C(s', \|v + w\|_s)(1 + \|v + w\|_{s'}) \leq C(s')(1 + \|w\|_{s'}), \tag{5.14}
\]

\[
\|D_w F(\epsilon, \omega, w)\|_{s'} \leq (1 + \|D_w v(\epsilon, \omega, w)\|_{H^1})\|\partial_x f(t, x, v(\epsilon, \omega, w) + w)\|_{s'} \\
\leq C(s')(1 + \|w\|_{s'}), \tag{5.15}
\]

\[
\|D_w^2 F(\epsilon, \omega, w)\|_{s'} \leq (1 + \|D_w v(\epsilon, \omega, w)\|_{H^1})^2\|\partial_x^2 f(t, x, v(\epsilon, \omega, w) + w)\|_{s'} \\
+ \|D_w v(\epsilon, \omega, w)\|_{H^1}\|\partial_x f(t, x, v(\epsilon, \omega, w) + w)\|_{s'} \leq C(s')(1 + \|w\|_{s'}). \tag{5.16}
\]

Clearly, we can obtain \( F \in C^2(\mathcal{H}^s; \mathcal{H}^s) \) using Lemmata 5.5 and 2.3. Hence formulae (5.14)–(5.16) with \( s' = s \) and \( \|w\|_s \leq 1 \) yield that property \((P_1)\) holds. Moreover, by means of (5.14), one has that (3.3) holds. Because of (5.2) and (5.13)–(5.16), we also have

\[
\|D_w F(\epsilon, \omega, w)h\|_{s'} \leq \frac{C(s')}{\|w\|_{s'}}\|h\|_s + \|h\|_{s'},
\]

\[
\|D_w^2 F(\epsilon, \omega, w)[h, h]\|_{s'} \leq \frac{C(s')}{\|w\|_{s'}}\|h\|_s\|h\|_s + \|h\|_{s'}\|h\|_s + \|h\|_{s'}\|h\|_{s'},
\]

which lead to formulae (3.2)–(3.5). Since

\[
F(\epsilon, \omega, w + h) - F(\epsilon, \omega, w) - D_w F(\epsilon, \omega, w)[h] = \int_0^1 D_w^2 F(\epsilon, \omega, w + \psi h)[h, h]d\psi,
\]

combining (3.5) with \( \|w\|_s \leq 1 \) and \( \|h\|_s \leq 1 \) yields property \((P_3)\). Obviously, owing to (3.1) and the definition of \( s \)-norm (recall (2.2)), we can obtain property \((P_4)\). \( \square \)

6. Concluding Remarks

We have discussed the existence of periodic solutions to nonlinear wave equations with time-dependent coefficients. This model with time dependent propagation speeds in the linear case describes a change of the quantity of the total energy. For the nonlinear model under periodic boundary conditions, the periodic solutions have been constructed by a Lyapunov-Schmidt reduction together with a Nash-Moser iteration scheme. We will investigate the nonlinear Schrödinger equations with time-dependent coefficients, which arises as an envelope equation for electromagnetic wave propagation in optical fibers. The results will be reported elsewhere.

References

[1] P. Baldi. Periodic solutions of forced Kirchhoff equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5), 8(1):117–141, 2009.
[2] P. Baldi and M. Berti. Forced vibrations of a nonhomogeneous string. *SIAM J. Math. Anal.*, 40(1):382–412, 2008.
[3] P. Baldi, M. Berti, and R. Montalto. KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation. *Math. Ann.*, 359(1-2):471–536, 2014.
[4] V. Barbu and N. H. Pavel. Periodic solutions to one-dimensional wave equation with piece-wise constant coefficients. *J. Differential Equations*, 132(2):319–337, 1996.
[5] V. Barbu and N. H. Pavel. Determining the acoustic impedance in the 1-d wave equation via an optimal control problem. *SIAM J. Control Optim.*, 35(5):2035–2048, 1997.
[6] V. Barbu and N. H. Pavel. Periodic solutions to nonlinear one-dimensional wave equation with \( x \)-dependent coefficients. *Trans. Amer. Math. Soc.*, 349(5):2035–2048, 1997.
[7] M. Berti and P. Bolle. Cantor families of periodic solutions for completely resonant nonlinear wave equations. *Duke Math. J.*, 134(2):359–419, 2006.
[8] M. Berti and P. Bolle. Cantor families of periodic solutions of wave equations with \( C^1 \) nonlinearities. *NoDEA Nonlinear differ. eqn. appl.*, 15(1-2):247–276, 2008.
[9] M. Berti and P. Bolle. Sobolev periodic solutions of nonlinear wave equations in higher spatial dimensions. *Arch. Ration. Mech. Anal.*, 195(2):609–642, 2010.
[10] M. Berti, L. Corsi, and M. Procesi. An abstract Nash-Moser theorem and quasi-periodic solutions for NLW and NLS on compact Lie groups and homogeneous manifolds. *Comm. Math. Phys.*, 334(3):1413–1454, 2015.
[11] J. Bourgain. Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE. *Internat. Math. Res. Notices*, (11):475ff., approx. 21 pp. (electronic), 1994.
[12] J. Bourgain. Construction of periodic solutions of nonlinear wave equations in higher dimension. Geom. Funct. Anal., 5(4):629–639, 1995.
[13] F. Colombini, E. De Giorgi, and S. Spagnolo. Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 6(3):511–559, 1979.
[14] F. Colombini, E. Jannelli, and S. Spagnolo. Well-posedness in the gevrey classes of the cauchy problem for a nonstrictly hyperbolic equation with coefficients depending on time. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 10(2):291–312, 1983.
[15] F. Colombini, E. Jannelli, and S. Spagnolo. Hyperbolic equations and classes of infinitely differentiable functions. Ann. Mat. Pura Appl. (4), 143:187–195, 1986.
[16] F. Colombini and G. Métivier. Counterexamples to the well posedness of the Cauchy problem for hyperbolic systems. Anal. PDE, 8(2):499–511, 2015.
[17] W. Craig and C. E. Wayne. Newton’s method and periodic solutions of nonlinear wave equations. Comm. Pure Appl. Math., 46(11):1409–1498, 1993.
[18] M. D’Abbicco and M. Reissig. Long time asymptotics for 2 by 2 hyperbolic systems. J. Differential Equations, 250(2):752–781, 2011.
[19] J.-M. Delort. Periodic solutions of nonlinear Schrödinger equations: a paradifferential approach. Anal. PDE, 4(5):639–676, 2011.
[20] L. H. Eliasson, B. Grébert, and S. B. Kuksin. KAM for the nonlinear beam equation. Geom. Funct. Anal., 26(6):1588–1715, 2016.
[21] Y. Gao, Y. Li, and J. Zhang. Invariant tori of nonlinear Schrödinger equation. J. Differential Equations, 246(8):3296–3331, 2009.
[22] E. Haus and M. Procesi. KAM for beating solutions of the quintic NLS. Comm. Math. Phys., 354(3):1101–1132, 2017.
[23] F. Hirosawa. On the asymptotic behavior of the energy for the wave equations with time depending coefficients. Math. Ann., 339(4):819–838, 2007.
[24] F. Hirosawa. Energy estimates for wave equations with time dependent propagation speeds in the gevrey class. J. Differential Equations, 248(12):2972–2993, 2010.
[25] F. Hirosawa and H. Ishida. On second order weakly hyperbolic equations and the ultradifferentiable classes. J. Differential Equations, 255(7):1437–1468, 2013.
[26] S. Ji. Time periodic solutions to a nonlinear wave equation with x-dependent coefficients. Calc. Var. Partial Differential Equations, 32(2):137–153, 2008.
[27] S. Ji. Time-periodic solutions to a nonlinear wave equation with periodic or anti-periodic boundary conditions. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 465(2103):895–913, 2009.
[28] S. Ji and Y. Li. Periodic solutions to one-dimensional wave equation with x-dependent coefficients. J. Differential Equations, 229(2):466–493, 2006.
[29] S. Ji and Y. Li. Time-periodic solutions to the one-dimensional wave equation with periodic or anti-periodic boundary conditions. Proc. Roy. Soc. Edinburgh Sect. A, 137(2):349–371, 2007.
[30] S. Ji and Y. Li. Time periodic solutions to the one-dimensional nonlinear wave equation. Arch. Ration. Mech. Anal., 199(2):435–451, 2011.
[31] T. Kato. Perturbation theory for linear operators. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
[32] S. B. Kuksin. Hamiltonian perturbations of infinite-dimensional linear systems with imaginary spectrum. Funktsional. Anal. i Prilozhen., 21(3):22–37, 95, 1987.
[33] B. M. Levitan and I. S. Sargsjan. Sturm-Liouville and Dirac operators, volume 59 of Mathematics and its Applications (Soviet Series). Kluwer Academic Publishers Group, Dordrecht, 1991. Translated from the Russian.
[34] V. Marchenko. Sturm-liouville operators and their applications. Kiev Izdatel Naukova Dumka, 1977.
[35] R. Montalto. Quasi-periodic solutions of forced Kirchhoff equation. NoDEA Nonlinear Differential Equations Appl., 24(1):Art. 9, 71 pp, 2017.
[36] C. Procesi and M. Procesi. A KAM algorithm for the resonant non-linear Schrödinger equation. Adv. Math., 272:399–470, 2015.
[37] P. H. Rabinowitz. Periodic solutions of nonlinear hyperbolic partial differential equations. Comm. Pure Appl. Math., 20:145–205, 1967.
[38] P. H. Rabinowitz. Periodic solutions of nonlinear hyperbolic partial differential equations. II. Comm. Pure Appl. Math., 22:15–39, 1968.
[39] P. H. Rabinowitz. Time periodic solutions of nonlinear wave equations. Manuscripta Math., 5:165–194, 1971.
[40] P. H. Rabinowitz. Free vibrations for a semilinear wave equation. Comm. Pure Appl. Math., 31(1):31–68, 1978.
[41] C. E. Wayne. Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. Comm. Math. Phys., 127(3):479–528, 1990.
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