Adaptive ensemble Kalman filtering of nonlinear systems

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We consider a system of the form:

\[ x_{k+1} = f(x_k) + \omega_{k+1} \quad \text{with} \quad \omega \approx \mathcal{N}(0, Q) \]

\[ y_{k+1} = h(x_{k+1}) + \nu_{k+1} \quad \text{with} \quad \nu \approx \mathcal{N}(0, R) \]

We initially assume Gaussian system and observation noise.

Our goal is to estimate the covariance matrices \( Q \) and \( R \) as part of the filter procedure.

Later we consider \( Q \) to be an additive inflation which attempts to compensate for model error.
Nonlinear Kalman-type Filter: Influence of $Q$ and $R$

- Simple example with full observation and diagonal noise covariances
- Red indicates RMSE of unfiltered observations
- Black is RMSE of ‘optimal’ filter (true covariances known)
Nonlinear Kalman-type Filter: Influence of $Q$ and $R$

Standard Kalman Update:

$$
\begin{align*}
P^f_k &= F_{k-1} P^a_{k-1} F_{k-1}^T + Q_{k-1} \\
P^y_k &= H_k P^f_k H_k^T + R_{k-1} \\
K_k &= P^f_k H_k^T (P^y_k)^{-1} \\
P^a_k &= (I - K_k H_k) P^f_k \\
\epsilon_k &= y_k - y^f_k = y_k - H_k x^f_k \\
x^a_k &= x^f_k + K_k \epsilon_k
\end{align*}
$$
Nonlinear Kalman-type Filter: Influence of $Q$ and $R$

- Covariances $Q$ and $R$ effect filter performance
- Seems better to underestimate observation noise
- Seems better to overestimate ‘model error’
- Can we estimate these parameters from the data?
Adaptive Filter: Estimating $Q$ and $R$

- Innovations contain information about $Q$ and $R$
  \[ \epsilon_k = y_k - y_k^f \]
  \[ = h(x_k) + \nu_k - h(x_k^f) \]
  \[ = h(f(x_{k-1}) + \omega_k) - h(f(x_{k-1}^a)) + \nu_k \]
  \[ \approx H_k F_{k-1}(x_{k-1} - x_{k-1}^a) + H_k \omega_k + \nu_k \]

- IDEA: Use innovations to produce samples of $Q$ and $R$:
  \[ \mathbb{E}[\epsilon_k \epsilon_k^T] \approx H P^f H^T + R \]
  \[ \mathbb{E}[\epsilon_{k+1} \epsilon_k^T] \approx H F P^e H^T - H F K \mathbb{E}[\epsilon_k \epsilon_k^T] \]
  \[ P^e \approx F P^a F^T + Q \]

- In the linear case this is rigorous and was first solved by Mehra in 1970
Adaptive Filter: Estimating $Q$ and $R$

- To find $Q$ and $R$ we estimate $H_k$ and $F_{k-1}$ from the ensemble and invert the equations:

$$
\mathbb{E}[\epsilon_k \epsilon_k^T] \approx H P_f H^T + R
$$
$$
\mathbb{E}[\epsilon_{k+1} \epsilon_k^T] \approx H F P_e H^T - H F K \mathbb{E}[\epsilon_k \epsilon_k^T]
$$

- This gives the following empirical estimates of $Q_k$ and $R_k$:

$$
P_k^e = (H_{k+1} F_k)^{-1} (\epsilon_{k+1} \epsilon_k^T + H_{k+1} F_k K_k \epsilon_k \epsilon_k^T) H_k^{-T}
$$
$$
Q_k^e = P_k^e - F_{k-1} P_{k-1}^a F_{k-1}^T
$$
$$
R_k^e = \epsilon_k \epsilon_k^T - H_k P_k^f H_k^T
$$

- Note: $P_k^e$ is an empirical estimate of the background covariance
An Adaptive Kalman-Type Filter for Nonlinear Problems

We combine the estimates of $Q$ and $R$ with a moving average

**Original Kalman Eqs.**

\[
\begin{align*}
P^f_k &= F_{k-1} P^a_{k-1} F_{k-1}^T + Q_{k-1} \\
P^y_k &= H_k P^f_k H_k^T + R_{k-1} \\
K_k &= P^f_k H_k^T (P^y_k)^{-1} \\
P^a_k &= (I - K_k H_k) P^f_k
\end{align*}
\]

\[
\begin{align*}
\epsilon_k &= y_k - y^f_k \\
x^a_k &= x^f_k + K_k \epsilon_k
\end{align*}
\]

**Our Additional Update**

\[
\begin{align*}
P_{k-1}^e &= F_{k-1}^{-1} H_k^{-1} \epsilon_k \epsilon_k^T H_k^{-T} \\
K_{k-1} &= K_{k-1} \epsilon_k \epsilon_k^T H_k^{-T} \\
Q_{k-1}^e &= P_{k-1}^e - F_{k-2} P^a_{k-2} F_{k-2}^T \\
R_{k-1}^e &= \epsilon_k \epsilon_k^T H_k^{-1} P^f_{k-1} H_k^{-T} \\
Q_k &= Q_{k-1} + (Q_{k-1}^e - Q_{k-1})/\tau \\
R_k &= R_{k-1} + (R_{k-1}^e - R_{k-1})/\tau
\end{align*}
\]
How does this compare to inflation?

- We extend Kalman’s equations to estimate $Q$ and $R$
- Estimates converge for linear models with Gaussian noise
- When applied to nonlinear, non-Gaussian problems
  - We interpret $Q$ as an additive inflation
  - $Q$ can have complex structure, possibly more effective than multiplicative inflation?
  - Downside: many more parameters than multiplicative inflation
- Somewhat less ad hoc than other inflation techniques?
We will apply the adaptive EnKF to the 40-dimensional Lorenz96 model integrated over a time step $\Delta t = 0.05$

$$\frac{dx^i}{dt} = -x^{i-2}x^{i-1} + x^{i-1}x^{i+1} - x^i + F$$

We augment the model with Gaussian white noise

$$x_k = f(x_{k-1}) + \omega_k \quad \omega_k = \mathcal{N}(0, Q)$$
$$y_k = h(x_k) + \nu_k \quad \nu_k = \mathcal{N}(0, R)$$

We will consider full and sparse observations

The Adaptive EnKF uses $F = 8$

We will consider model error where the true $F^i = \mathcal{N}(8, 16)$
Recovering $Q$ and $R$, Full Observability

RMSE shown for the initial guess covariances (red) the true $Q$ and $R$ (black) and the adaptive filter (blue)
Recovering $Q$ and $R$, Sparse Observability

Observing 10 sites results in divergence with the true $Q$ and $R$

\[ \begin{array}{cccc}
\text{True Covariance} & \text{Initial Guess} & \text{Final Estimate} & \text{Difference} \\
Q & & & \\
R & & & \\
\end{array} \]

RMSE shown for the initial guess covariances (red) the true $Q$ and $R$ (black) and the adaptive filter (blue)
Compensating for Model Error

The adaptive filter compensates for errors in the forcing $F^i$

RMSE shown for the initial guess covariances (red) an Oracle EnKF (black) and the adaptive filter (blue)
Integration with the LETKF

Simply find a local $Q$ and $R$ for each region

| Site Number | Relative Variance |
|-------------|-------------------|
| 0           | 0                 |
| 10          | 0.2               |
| 20          | 0.4               |
| 30          | 0.6               |
| 40          | 0.8               |

| Filter Steps | RMSE |
|--------------|------|
| 0            | 0    |
| 0.5          | 0.2  |
| 1            | 0.4  |
| 1.5          | 0.6  |
| 2            | 0.8  |

RMSE shown for the initial guess covariances (red) the true $Q$ and $R$ (black) and the adaptive filter (blue)
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