Sheet current model for inductances extraction and Josephson junctions devices simulation

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Abstract. Sheet current model for microelectronic superconductor structures, based on Maxwell and London equations, is presented. In this model, magnetic field is three-dimensional but resulting integro-differential equations for current density potential are two-dimensional. These equations are solved using finite element method and matrix of self and mutual inductances is calculated as well as current distribution in superconductors. Then, current distribution can be used for calculation of boundary conditions for Josephson junctions equations. In particular we apply this approach for calculations of bias and control line currents for flux flow oscillator simulation problem. The program and results of calculations are presented.

1. Introduction
We consider the problem of practical 3D electromagnetic modeling of superconductive multilayer, planar, multi-connected microelectronic circuits. These circuits can be digital, SQUIDs, Flux Flow Oscillators (FFO), some modern High-T<sub>c</sub> or microwave devices.

The shape of modern circuits can be very complex [1]. Magnetic field in such layouts has essentially 3D structure [2]. Thus numerical modeling is indispensable for practical design [3]. On practice the values of interest are inductances in form of certain matrix of inductances and current distribution. In the paper, we describe the rigorous numerical approach for extracting inductances and calculating currents from design shapes. The program we evaluated, 3D-MLSI [4], is described, and results of calculations are presented.

2. Problem formulation
2.1. Typical circuits we want to consider
The technique of this paper is applicable to multilayer, planar, multi-connected structures. We want to calculate currents in conducting layers separated by layers of dielectric. The number of conducting layers can be one or more. The circuit can have or don’t have groundplane that carry return current.

The shape of conductors in each layer can be very complicated. These shapes can contain holes.
The key assumption for our technique is that thickness of conducting layers should be less or of order of London penetration depth [1]. In this case, the volume current density can be accurately approximated by sheet current density.

2.2. Simple example
On Fig. 1 simple example of typical structure is presented. It is single layer thin film shape with a hole and two current terminals. Current flow in via $T_1$ terminal and leave circuit via $T_2$ terminal. The hole trap zero or non-zero flux. The inlet and outlet terminal to terminal currents as well as currents circulating around a hole should be accounted in inductance matrix.

Figure 1. Circuit $S$ with 2 terminals $T_1$ and $T_2$ and a hole. $C_1$ and $C_2$ are non-terminal boundaries. Currents can flow around a hole or from terminal to terminal.

3. Integral equations
3.1. Conventional integral equations
Rigorous electromagnetic analysis should start from stationary Maxwell and London equations:

$$\lambda^2 \nabla \times \vec{j} + \vec{H} = 0,$$

$$\nabla \times \vec{H} = \vec{j}. \quad (1)$$

$$\nabla \cdot \vec{j}(r) = 0, \quad \Delta \chi = 0. \quad (4)$$

where $\lambda$ is London penetration depth. Traditionally these equations are reduced to sheet current integral equations using vector potential:

$$\lambda_\perp \vec{J}(r) + \frac{1}{4\pi} \int \int \frac{\vec{J}(r')}{|r-r'|} ds = -\nabla \chi(r), \quad (3)$$

$$\nabla \cdot \vec{J}(r) = 0, \quad \Delta \chi = 0. \quad (4)$$

where where $\lambda_\perp = \lambda^2/t$ is London penetration depth for films, $t$ is the thickness of film.
In (3) $\vec{J}(r)$ is unknown current, $\chi(r)$ is one more unknown function (phase). Equation (3) needs boundary conditions for function $\chi(r)$ and current $\vec{J}(r)$.

Equations similar to (3) are well known for normal conductors [5], [6] where PEEC (Partial Element Equivalent Circuit) method was evaluated. Approaches similar to PEEC for (3) for superconductors are also known [7]. For normal conductor function $\chi(r)$ has sense of voltage potential. Recently fast multipoles technique based program FASTHENRY [8] for (3) was adopted for superconductors [9].

Boundary conditions for (3) are easy for normal conductors where $\chi(r)$ is voltage. For superconductors currents excitation are more natural. Moreover, PEEC-like methods showed to be time and memory consuming in case of superconductors. It was our motivation for developing different numerical approach.

3.2. Better integral equations

Our approach is based on potential representation $\psi(r)$ of two-dimensional sheet current $\vec{J}$ which is called a stream function:

$$J_x(r) = \frac{\partial \psi(r)}{\partial y}, \quad J_y(r) = -\frac{\partial \psi(r)}{\partial x}$$  \hspace{1cm} (5)

Then using stationary London and Maxwell equations (2) it is possible to obtain the following expression in terms of stream functions $\psi(r)$:

$$-\lambda_\perp \Delta \psi(r_0) + \frac{1}{4\pi} \int_S (\nabla \psi(r), \nabla_{xy} G(r, r_0)) ds = -H_{z,ext}(r_0).$$  \hspace{1cm} (6)

Here $H_{z,ext}(r_0)$ is external or exciting magnetic field. For very thin conductors $G(r, r_0)$ can be taken as

$$G(r, r_0) = \frac{1}{|r - r_0|}.$$  \hspace{1cm} (7)

In [10] simple expressions for $G(r, r_0)$ are evaluated that take into account finite thickness of conducting objects.

Equation (6) should be completed by boundary conditions. These boundary conditions are simple first kind boundary conditions when values of stream function on the boundary are given:

$$\psi(r) = I_{h,k}, \quad r \in \partial S_{h,k},$$  \hspace{1cm} (8)

$$\psi(r) = F(r), \quad r \in \partial S_{ext}.$$  \hspace{1cm} (9)

Here $I_{h,k}$ is the full current circulating around hole $k$ with boundary $S_{h,k}$. On the external boundary function $F(r)$ can be easily evaluated [10].

Mathematically problem (6), (9) is very similar to boundary problem for Poisson equation [10, 16].

We prefer to solve equation (6) instead of (3) because (6) accounts important physical features of the problem and because of numerical effectivity considerations:

- Many superconductivity problems are based solely on currents and magnetic field. In these cases it is difficult to define boundary conditions for $\chi(r)$.
- Holes in $S$ is a problem for (3). But it is an easy task for (6). Given currents circulating around holes are accounted in boundary conditions in function $I_{h,k}$ (9). Non-decaying currents circulating around holes are typical for problems in superconductivity.
- FEM for (6) has better numerical approximation then PEEC and thus can give smaller system of linear equations.
FEM for (6) can be adopted for solution of very large problems when matrix of equations is so large then can’t be allocated in computer memory.

After function $\psi(r)$ is calculated, energy functional also can be calculated as well as inductance matrix [10], [11].

4. Numerical technique

4.1. Finite Elements Method

Equation (6) together with boundary conditions (9) are ready to solution using Finite Element Method (FEM) [15]. Finite elements method is based on bilinear form. The bilinear form $a(u, v)$ for "weak" formulation of the problem (6) is:

$$
a(u, v) = \lambda \int_{S} (\nabla u(r), \nabla v(r)) ds + \frac{1}{4\pi} \int_{S} \int_{S} (\nabla u(r), \nabla v(r)) G(r, r_0) ds.
$$

Here the principal value integral in (6) was integrated by parts. Form $a(u, v)$ is symmetric and for thin enough conductors positively definite because for $u = v$ it gives the expression for full energy.

Then we apply conventional FEM based on triangular meshing of conductors and linear elements. There are no limitations on shape and number of conductors. More details can be found in [10, 16].

4.2. Large problems

The FEM matrix is the sum of sparse matrix resulted from Laplace operator and dense part resulted from integral operator. Dense part of the matrix can be a problem for numerical solution. If number of nodes in triangulation of $S$ is moderate then finite element equations can be solved using Cholesky factorization. The problem arise when number of nodes is several thousands or tens thousands of nodes. It is not possible to store this matrix in dense format or to factor it. To overcome this limitation we evaluated a sparsification procedure integral equation matrix.

Consider finite element matrix elements for integral operator in (6) $b_{ij}$. For non-intersecting finite elements supports $S_i \cap S_j = \emptyset$ we have

$$
b_{ij} = \int_{S_i} ds \int_{S_j} ds' \frac{(\nabla u_j^h(r'), \nabla u_i^h(r))}{|r - r'|} ds' = - \int_{S_i} ds \int_{S_j} ds' \frac{u_j^h(r') u_i^h(r)}{|r - r'|^3} ds'.
$$

Here $u_j^h$ and $u_i^h$ are simplectic finite element shapes, $|u_j^h| \leq 1$. It is seen that $b_{ij}$ is small if distance between $S_i$ and $S_j$ is large because the denominator in (11) is large and quickly grow up with the distance.

Our sparsification procedure is simple. Matrix elements below some tolerance are dropped. It makes matrix of integral equation sparse. Then, sparse Cholesky factorization can be applied. As result, large size circuits with detailed mesh can be calculated with some small accuracy trade-off.

4.3. The program

Our program, 3D-MLSI, is written on C++ with Windows GUI. Non-GUI part can be compiled for Unix. The program can be found here [4].
5. Some results of calculations

5.1. Inductances calculations for HTC transformer

As an example of a real device, a two-layer YBCO circuit [11] was calculated, see Fig. 2.

Both superconducting layers are 0.2 μm thick. The first layer is a ground plane and contains 12 × 4 μm hole to increase the mutual inductance. The second layer contains superconducting wiring. The narrow part of wires is 3 μm wide. The thickness of the insulator between the layers is 0.19 μm. London penetration depth \( \lambda = 0.18 \) μm. The circuit is symmetric (see Fig. 2).

We assume that the ground plane carries all return currents. It is possible to calculate full 3×3 inductance matrix: for two symmetrical current loops via top layer wires and the current around the loop in the ground plane. For the circuit analysis this matrix is unsuitable because the hole circulating currents are not excited independently on the terminal currents. It is known that for any currents flowing in the circuit the hole keeps constant trapped flux, which can be taken equal to zero, as it, in any case, does not affect the reduced inductance matrix [1]. The program calculates 2×2 inductance matrix with self inductances \( L \) and mutual inductances \( -M \).

For basic configuration with rectangular hole \( w \times d = 12 \times 4 \) μm the self-inductance is calculated to be \( L = 13.61 \) pH, \( M = 1.651 \) pH (transformation ratio 0.12). For the same structure without a hole \( L = 10.65 \) pH, \( M = 0.04 \) pH (transformation ratio 0.0039). For these calculations, a typical PC was used. The whole set of calculations took several hours in batch mode. The exact time depends on CPU but can’t be large even for low performance computer.

5.2. Bias and control line currents calculations for FFO

FFO (Flux Flow Oscillator) is microwave Josephson junction device [2], [12]. Typically these devices are two-layer structures where layers of superconductor partly overlap. Part of overlapping region occupy long Josephson junction. The device is driven by two currents, bias current \( I_b(t) \) that pass from one sheet to another across JJ and controlling current \( I_{cl}(t) \) that flow on top or bottom sheet. Schematically FFO is presented on Fig. 4.

Simplest mathematical model of FFO is perturbed sine-Gordon equation with constant coefficients subject to Neumann boundary conditions [2]:

\[
\ddot{\varphi} + \alpha \dot{\varphi} - \varphi_{xx} = \eta(x) - \sin \varphi \\
\frac{\partial \varphi(0,t)}{\partial x} = \frac{\partial \varphi(L,t)}{\partial x} = \Gamma.
\]  

(12)

In this equation function \( \eta(x) \) and value of \( \Gamma \) should be defined using given full control line and bias currents as it is shown on Fig. 4.
Using so called “locking principle” for superconductors, we adopted our program 3D-MLSI for calculations of right part and coefficients in (12). 3D-MLSI calculate inlet currents on the boundary of overlapping region on Fig. (4). Then this data was passed to program FFOSIM [14] where final evaluation of right part in (12) was performed. It was found that practical distribution of control line an bias currents is very different for from simple profiles used in
numerical calculations of (12). For one of devices [12, 13], the graphics of control line and bias components of $\eta(x)$ is presented on Fig. 5. For results of simulation of FFO, see [14].

![Figure 5.](image)

6. Conclusions
We evaluated modeling technique and software 3D-MLSI for currents calculation and inductances extraction. The program can solve problems of arbitrary geometry of the circuits. The program have good user interface and visualization of current density and flow lines. Also, the program can calculate magnetic field in given set of points in space. The program can solve large problems when number of mesh points is too large to be stored in matrix in computer memory.

Our plans are to implement fluxes calculations for given contours and evaluate better triangulation of conductor shapes with external mesh generation program.

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