An Empirical Evaluation of the Approximation of Subjective Logic Operators Using Monte Carlo Simulations

Fabio Massimo Zennaro\textsuperscript{a,}\textsuperscript{*}, Magdalena Ivanovska\textsuperscript{a}, Audun Jøsang\textsuperscript{a}

\textsuperscript{a} Department of Informatics, University of Oslo
PO Box 1080 Blindern
0316 Oslo, Norway

Abstract

In this paper we analyze the use of subjective logic as a framework for performing approximate transformations over probability distribution functions. As for any approximation, we evaluate subjective logic in terms of computational efficiency and bias. However, while the computational cost may be easily estimated, the bias of subjective logic operators have not yet been investigated. In order to evaluate this bias, we propose an experimental protocol that exploits Monte Carlo simulations and their properties to assess the distance between the result produced by SL operators and the true result of the corresponding transformation over probability distribution. This protocol allows a modeler to get an estimate of the degree of approximation she must be ready to accept as a trade-off for the computational efficiency and the interpretability of the subjective logic framework. Concretely, we apply our method to the relevant case study of the subjective logic operator for binomial multiplication and we study empirically its approximation.

Keywords: subjective logic, Monte Carlo simulation, binomial product, Beta distributions

1. Introduction

Subjective logic (SL) \cite{Josang2016} defines a framework for expressing uncertain probabilistic statements in the form of subjective opinions. A subjective opinion allows a modeler to state probabilities over a set of alternative events along with a measure of the global uncertainty of such modeling. Subjective opinions thus integrate a form of first-order uncertainty, relative to the distribution of probability mass over events, and a form of second-order uncertainty, due to the incertitude in distributing the probability mass. Subjective opinions provide a simple, clean and interpretable way to encode and manipulate uncertainty: as such, they constitute a useful modelling tool in sensitive scenarios in which statistical models can not be inferred from data, but must be built relying on the domain knowledge or the intuition of experts. In this fashion, SL has been extensively

\*Corresponding author

\textit{Email addresses:} fabiomz@ifi.uio.no (Fabio Massimo Zennaro), magdalei@ifi.uio.no (Magdalena Ivanovska), josang@mn.uio.no (Audun Jøsang)
adopted to model uncertainty in several fields such as trust modeling, biomedical data analysis or forensics analysis [4].

From a purely statistical point of view, subjective opinions can be seen as an alternative representation for standard probability distributions, such as Beta pdfs or Dirichlet pdfs. Indeed, under certain assumptions, it is possible to define a unique mapping between subjective opinions and probability distribution functions [4]. This means that subjective opinions may be interpreted as a re-parametrization of standard distributions from the statistical literature.

SL also defines several operators over subjective opinions. These operators allow to carry out transformations over subjective opinions in a very efficient way. With respect to the underlying probability distributions, SL operators provide an extremely quick approximation of operations over probability distributions that would be otherwise very difficult or impossible to evaluate analytically.

Thus, beyond its original application, SL may also be seen as an effective statistical tool to compute approximate probability distributions generated by the transformations encoded into the SL operators. However, while the efficiency of SL operators may be easily evaluated, estimates about their bias are lacking. This shortcoming may limit the adoption of SL in favor of other better-studied approaches, such as Monte Carlo (MC) simulations. Modern probabilistic programming languages [2] provide a versatile language in which operations over probability distributions may be easily defined and evaluated using pre-coded inference algorithms. While being computationally more expensive, these techniques provide comforting guarantees on the convergence of the algorithms as a function of the number of sampling iterations. These guarantees, contrasting the lack of formal bounds of SL operators, may be a strong argument for many researchers to overlook SL and the related set of operators.

In this paper, we propose a protocol to tackle numerically the problem of characterizing the approximation of SL operators by offering an empirical analysis of their bias with respect to MC simulations. SL operators and MC simulations are taken as two distinct frameworks to approximate operations over pdfs, each one with its strengths and limitations. Our analysis defines a quantitative comparison in which SL operators and MC simulations are contrasted in terms of the trade-off between computational efficiency and bias. More specifically, our approach allows to answer the question: What amount of approximation should we be ready to accept in exchange for the computational efficiency of subjective logic?

To show the usefulness of our protocol, we consider the specific case of binomial multiplication, a simple SL operator that returns the approximation of the product of two Beta pdfs. Computing the product of independent Beta pdfs is a non-trivial problem [1] with relevant applications in fields such as reliability analysis and operations research [8]. Binomial multiplication in SL may then be seen as a simple and effective algorithm to compute an approximate solution to the problem of multiplying together two Beta pdfs. By comparing the approximation obtained using SL to moment-matching approximation and kernel-density approximation produced via MC simulations, we are able to get an understanding of the amount of approximation that we should be ready to accept if we want to work in the framework of SL.

The rest of the paper is organized as follows. Section 2 reviews the basics of subjective logic and Section 3 presents the main aspects of computational statistics relevant to this work. Section 4 describes the computational complexity of SL approximations...
and MC approximations, while Section 4 discusses the bias of the same techniques. Section 5 proposes a grounded framework for evaluating the degree of approximation of SL operators in relation to MC simulations. Section 6 makes this framework concrete by applying it to the case study of the product of Beta pdfs and Section 8 carries out a set of empirical simulations that allow us to estimate the approximation of SL binomial multiplication. Finally, Section 9 summarizes the results and discusses possible directions for future work.

2. Subjective Logic

In this section, we present the fundamentals of SL. We start with a formalization of subjective opinions and we show how they may be mapped to probability distributions.

Subjective opinions. Let $\Omega$ be a discrete collection of $M$ mutually exclusive and exhaustive events. A subjective opinion $\omega$ is a triple:

$$(b, u, a),$$

such that

$$\sum_{i=1}^{M} b_i + u = 1,$$

where $b \in \mathbb{R}^M$, with $b_i \in \mathbb{R}_{\geq 0}$, is the belief vector expressing the probability mass that the modeler places on each event $x_i$ in $\Omega$, $u \in \mathbb{R}_{\geq 0}$ is the uncertainty scalar quantifying the uncertainty of the modeler in its definition of $b$, and $a \in \mathbb{R}^M$ is the prior vector encoding a prior probability distribution over the events in $\Omega$. This subjective opinion is called a multinomial opinion.

Notice that the constraint in Equation 2 limits the degrees of freedom of $b$ and $u$ to $M$ and, consequently, defines an $M$-dimensional simplex on which subjective opinions may be represented.

The limit-case multinomial opinion is the binomial opinion for $M = 2$. In this case $\Omega = \{x, \overline{x}\}$ and the subjective opinion in Equation 1 may be re-written for simplicity as:

$$(b, d, u, a),$$

such that

$$b + d + u = 1,$$

where $b \in \mathbb{R}_{\geq 0}$ is the belief scalar expressing the probability of $x$, $d \in \mathbb{R}_{\geq 0}$ is the disbelief scalar expressing the probability of $\overline{x}$, $u \in \mathbb{R}_{\geq 0}$ is the uncertainty scalar and $a \in \mathbb{R}_{\geq 0}$ is a scalar expressing the prior probability of $x$.

Having only two degrees of freedom, binomial opinions in the form $(b, d, u, a)$ belong to a two-dimensional simplex and may be visualized together with $a$ in a barycentric coordinate system.\footnote{See http://folk.uio.no/josang/sl/BV.html for an illustration.}
Mapping of subjective opinions. In order to ground SL, a mapping has been defined between multinomial opinions and Dirichlet pdfs and between binomial opinions and Beta pdfs.

Given a mapping constant \( W \in \mathbb{R}_{\geq 0} \), it is possible to define a unique mapping from opinions to pdfs. Let \( \omega = (b, u, a) \) be a multinomial opinion with \( u \neq 0 \); \( \omega \) can be mapped to a Dirichlet pdf \( p \) with distribution \( \text{Dir}(\alpha) \), where the vector of parameters \( \alpha \) is defined as:

\[
\alpha = W \left( \frac{b}{u} + a \right).
\]

(5)

For binomial opinions, too, given a mapping constant \( W \in \mathbb{R}_{\geq 0} \), it is possible to define a unique mapping from opinions to pdfs. Let \( \omega = (b, d, u, a) \) be a binomial opinion with \( u \neq 0 \); \( \omega \) can be mapped to a Beta pdf \( p \) with distribution \( \text{Beta}(\alpha, \beta) \), where \( \alpha \) and \( \beta \) are parameters defined as:

\[
\begin{align*}
\alpha &= W \left( \frac{b}{u} + a \right) \\
\beta &= W \left( \frac{d}{u} + (1 - a) \right)
\end{align*}
\]

(6)

Notice that, for reasons of consistency, \( W \) is usually fixed to 2 \([4]\). We then have a mapping \( s \) from opinion \( \omega \) to pdf \( p \):

\[
s : \omega \mapsto p.
\]

Vice versa, given a mapping constant \( W \in \mathbb{R}_{\geq 0} \) and a fixed prior distribution \( a \), it is possible to define a unique mapping from pdfs to opinions. Let \( p \) be a Dirichlet pdf with distribution \( \text{Dir}(\alpha) \) with \( \alpha_i > 1 \); \( p \) can be mapped to a multinomial opinion \( \omega = (b, u, a) \), where the parameters are computed as:

\[
\begin{align*}
b &= \frac{\alpha_i - Wa}{\sum_i (\alpha_i - Wa)} \\
u &= \frac{W}{\sum_i (\alpha_i - Wa)} \\
a &= a
\end{align*}
\]

(7)

Again, for a binomial opinion, given a mapping constant \( W \in \mathbb{R}_{\geq 0} \) and a fixed prior distribution \( a \), it is possible to define a unique mapping from pdfs to opinions. Let \( p \) be a Beta pdf with distribution \( \text{Beta}(\alpha, \beta) \) with \( \alpha, \beta > 1 \); \( p \) can be mapped to a binomial opinion \( \omega = (b, d, u, a) \), where the parameters are computed as:

\[
\begin{align*}
b &= \frac{\alpha - Wa}{\alpha + \beta} \\
d &= \frac{\beta - W(1 - a)}{\alpha + \beta} \\
u &= \frac{W}{\alpha + \beta} \\
a &= a
\end{align*}
\]

(8)

Again, for reasons of consistency, \( W \) is usually fixed to 2 \([4]\). Given a prior distribution \( a \), this generates the mapping \( t \) from pdf \( p \) to opinion \( \omega \):

\[
t : p \mapsto \omega.
\]
Figure 1: If the application of the operator \( \circ_P \) to two pdfs \( p_X \) and \( p_Y \) cannot be solved analytically, we can map \( p_X \) and \( p_Y \) to the opinions \( \omega_X \) and \( \omega_Y \) and apply the SL operator \( \circ_{SL} \) to compute the opinion \( \omega_Z \). The pdf \( p_{SL}Z \) associated with \( \omega_Z \) provides an approximation of \( p_Z \).

Subjective opinion operators. SL defines several operators over subjective opinions, such as addition, product or fusion \(^4\). In general, these operators are computed over the parameters of subjective opinions. Let \( \omega_X = (b_X, u_X, a_X) \) and \( \omega_Y = (b_Y, u_Y, a_Y) \) be two subjective opinions and let \( \circ_{SL} : \mathcal{S} \times \mathcal{S} \to \mathcal{S} \) be a generic operator over the space of subjective opinions \( \mathcal{S} \). Then, \( \omega_Z = (b_Z, u_Z, a_Z) \) resulting from the application of the operator to \( \omega_X \) and \( \omega_Y \) is given as:

\[
\begin{align*}
  b_Z &= f_b(\omega_X, \omega_Y) \\
  u_Z &= f_u(\omega_X, \omega_Y) \\
  a_Z &= f_a(\omega_X, \omega_Y),
\end{align*}
\]

(9)

where \( f_b, f_u, f_a : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{\geq 0} \) are operator-specific functions returning the values of belief, uncertainty and prior for the opinion \( \omega_Z \).

Subjective opinion operators for evaluating operations over pdfs. When properly defined, SL operators can be used to approximate operations over probability distribution functions. Suppose we are given two pdfs, \( p_X \) and \( p_Y \), and we want to compute a generic operation over them, \( \circ_P : \mathcal{P} \times \mathcal{P} \to \mathcal{P} \) over the space of probability distributions \( \mathcal{P} \). Computing this operation over probability distributions may be very complex. However, if we have an SL operator \( \circ_{SL} : \mathcal{S} \times \mathcal{S} \to \mathcal{S} \) that approximates \( \circ_P \), we may find a workaround computing \( p_X \circ_P p_Y \) by projecting the two distribution onto the opinions \( \omega_X \) and \( \omega_Y \), computing the resulting opinion \( \omega_Z = \omega_X \circ_{SL} \omega_Y \), and then mapping the result back onto a probability distribution function \( p_{SL}Z \). In this way, the resulting pdf \( p_{SL}Z \) provides an easy-to-compute approximation of the real pdf \( p_Z \) (see Figure 1).

3. Computational Statistics

In this section, we review some elements of computational statistics that are relevant to our work. We describe how sampling is used in MC simulations; we show how unbiased estimators can be built via MC integration; we discuss how unbiased estimators can be used to build moment-matching approximation; we show how pdfs may be reconstructed through kernel density estimation; and, finally, we bring these parts together to show how MC simulations may be used to compute the product of pdfs via moment-matching or kernel-density estimation.

Monte Carlo sampling. MC simulations are stochastic numerical algorithms designed to find approximate solutions through repeated random sampling. This paradigm has been applied in many areas of research to solve problems whose exact analytical solution is
impossible or too difficult to derive. In statistics, MC simulations are widely used to
evaluate probability distributions whose analytical form cannot be explicitly expressed.
Let $X$ be a random variable with a probability distribution $p_X$ on the support $\Omega$; let
us also assume that the analytical form of $p_X$ is unknown but that we can sample
realizations $x_i$ of the random variable $X$; then, MC simulations allow us to draw a large
number of independent samples $x_i$ and use them to (i) compute useful empirical statistical
descriptors $S_X$ of the probability distribution $p_X$, or, eventually, (ii) reconstruct the
approximate shape of the probability distribution $p_X$.

Monte Carlo integration. In order to compute useful empirical statistical descriptors $S_X$
of the probability distribution $p_X$, MC simulations rely on integration and on the law
of large numbers. Let $X$ be a statistics of the probability distribution $p_X$ that can
be computed from a function $f(\cdot)$ applied to the samples $x_i$. The statistics $S_X$ is then
defined as:

$$ S_X = \int_{\Omega} f(x) p_X(x) dx. $$

(10)

By the law of large numbers, an estimator of $S_X$ can be computed using $N$ samples of $x_i$ as:

$$ \hat{S}_X = \frac{1}{N} \sum_{i=1}^{N} f(x_i). $$

(11)

It is immediate to see that using Equation (11) and choosing an appropriate function $f(\cdot)$
we can directly estimate useful statistics of the distribution $p_X$, such as moments and
quantiles. Thus, through a MC simulation we can sample points from $p_X$ and compute
informative estimator statistics $\hat{S}_X$.

Moment-matching approximation. MC simulation and integration can be used to com-
pute estimators of the $i$-th moment $\hat{M}_i[X]$ of a probability distribution $p_X$. Now, a
probability distribution $p_X$ is completely characterized given the collection of all its mo-
ments; if we know the parametric form of the function $p_X$ from which we are sampling
from, but we ignore the exact value of its parameters, we can compute an estimate $\hat{p}_X$
by setting the moments to the estimated values $\hat{M}_i[X]$.

Kernel density estimation. Beyond computing statistics, it is possible to use samples $x_i$
generated in a MC simulation to reconstruct the actual probability distribution $p_X$. A
standard approach to reconstruct a continuous function $p_X$ from a set of finite points
$x_i$ is kernel density estimation (KDE). Any function may be expressed as a convolution
with a kernel function $\kappa(\cdot)$:

$$ p_X = \int_{\Omega} \kappa(x) dx. $$

(12)

Practically, it is possible to get an empirical approximation using only a finite set of
points $x_i$:

$$ \hat{p}_X(x) = \frac{1}{Nw} \sum_{i=1}^{N} \kappa \left( \frac{x - x_i}{w} \right), $$

(13)

where the kernel $\kappa(\cdot)$ is a symmetric function, like a triangular function or a Gaussian,
and $w$ denotes the width of the kernel; empirical rules are available to select an optimal
value for this parameter in relation to the number of samples available \[9\]. Thus, using the same MC simulation procedure to sample points from \(p_X\) it is possible also to estimate an approximate probability distribution \(\hat{p}_X\).

Monte Carlo simulation for evaluating operations over pdfs. Suppose we are given two probability distributions, \(p_X\) and \(p_Y\), and suppose we want to compute the distribution \(p_Z\) determined by the application of operation \(\circ \): \(\mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}\), that is, \(p_Z = p_X \circ p_Y\). If the pdf \(p_Z\) can not be computed analytically, MC simulations may be used to sample from \(p_Z\) and to estimate a pdf that approximates \(p_Z\). As a first solution, we could rely on the samples \(\{z_1, z_2, \ldots, z_N\}\) obtained by sampling from \(p_X\) and \(p_Y\) to estimate the moments \(\bar{M}_i[Z]\) and then instantiate a moment-matching approximation \(\hat{p}_{\text{MM}}^Z\) (see Figure 2). Alternatively, we could use the same samples \(\{z_1, z_2, \ldots, z_N\}\) from \(p_Z\) to perform a kernel-density estimation and compute the KDE approximation \(\hat{p}_{\text{KDE}}^Z\) (see Figure 3).

Notice that, differently from the SL approximation \(\hat{p}_{\text{SL}}^Z\), we decorate the approximations computed via MC simulations \(\hat{p}_{\text{MM}}^Z\) and \(\hat{p}_{\text{KDE}}^Z\) with a hat to underline that they are empirical statistics.

4. Computational Complexity

In this section, we discuss and compare the computational complexity of SL operators and MC simulations. We will evaluate the computational complexity using the \(\mathcal{O}(\cdot)\) notation as the time complexity of running a given algorithm as a function of its input.

Subjective logic. SL operators are defined to be extremely efficient. Indeed, given two opinions \(\omega_X\) and \(\omega_Y\) and the generic operator \(\circ_{\text{SL}} : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}\), the computation of \(\omega_Z = \omega_X \circ_{\text{SL}} \omega_Y\) usually requires only a limited number of function evaluations, as shown in Equation 9. The number of evaluations is \(2 \cdot M + 1\), where \(M\) is the number of events over which the opinions are defined. Thus, the overall complexity is \(\mathcal{O}(M)\): it depends only on the number of events considered, and it is independent of the actual form of
the mapped distributions. This makes SL operators an attractive choice especially when working in lower dimensions.

Monte Carlo simulation. The MC approach is, by definition, computationally intensive. The computational complexity of a MC simulation scales as a function of the number \( N \) of samples that must be produced. Each iteration requires random sampling and the execution of all the operations necessary to sample from \( p_Z \). Overall, the computational complexity of the MC simulation is \( O(N) \). If we are estimating a moment-matching approximation \( \hat{p}^{M M}_Z \), MC integration allows us to compute statistics from the samples generated during the MC simulation with no additional overall computational complexity. However, if we want to estimate the actual pdf via KDE we have to take into account an increase in the overall computational complexity from the linear order to the quadratic order \( O(N^2) \). Computing the pdf \( \hat{p}^{KDE}_Z \) is then a significantly computationally expensive procedure.

It is evident that, taking into account computational complexity only, SL operators dominate MC simulations, with or without KDE, especially considering that the number \( N \) of samples in a MC simulation is required to grow large in order to return reliable results even in low dimensions.

5. Bias

In this section, we start analyzing the degree of approximation of SL operators and MC simulations. We will evaluate the degree of approximation in terms of bias of the estimator \( \hat{p}_Z \), that is, as the expected value of the difference between the true distribution and the estimated approximation: \( E[p_Z - \hat{p}_Z] \).

Subjective logic. The bias of SL operators is dependent on the definition of the specific operator, and a generic theoretical treatment is not possible. Moreover, an analytic study of the bias is not always available for all possible SL operators. In Section 7 we will consider the case study of the binomial operator for subjective logic and we will analyze more in detail its specific bias.

Monte Carlo simulation. MC simulations are known to provide asymptotically unbiased estimators. If we estimate a statistics \( S_X \) of the pdf \( p_X \) using a MC integration as in Equation [11] then \( \hat{S}_X \) is an asymptotically unbiased estimator, that is, in the limit of infinite samples, it converges to the true quantity it approximates:

\[
\lim_{N \to \infty} \hat{S}_X(N) = S_X,
\]

where we made explicit the dependence of \( \hat{S}_X \) on the number of samples \( N \).

If we use MC integration to estimate the moments \( \hat{M}[Z] \) for a moment-matching approximation \( \hat{p}^{M M}_Z \), the MC simulation provides us with unbiased estimators of the moments; this means that, by increasing the number of samples generated in a MC simulation, we can get arbitrarily close to the true value of the estimated quantity. However, notice that while the estimated moments \( \hat{M}[Z] \) are asymptotically unbiased, the \( \hat{p}^{M M}_Z \) is biased; this bias is due to the limited set of moments \( \hat{M}[Z] \) used to approximate \( p_Z \).
If we use a MC simulation to estimate the true pdf directly via KDE, the empirical pdf $\hat{p}^{KDE}_Z$ is biased. In this case, it is known that the width parameter $w$ of KDE regulates the trade-off between bias and variance. In general, the bias can be shown to be proportional to the width $w$ of the kernel $\kappa(\cdot)$:

$$E_{KDE}[p_Z - \hat{p}^{KDE}_Z] \propto w^2,$$

(15)

under the constraint that $w$ can not be reduced to zero, for statistical and computational reasons [9]. When using a Gaussian kernel, the widely-adopted Silverman rule suggests the adoption of a kernel width of the following size:

$$w = 1.06\hat{\sigma} \frac{1}{\sqrt{N}},$$

(16)

where $\hat{\sigma}$ is the empirical standard deviation computed from the samples:

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \hat{\mu})^2}{N - 1}},$$

(17)

where $\hat{\mu}$ is the empirical mean. It follows, then, that the bias of the KDE approximation is proportional to:

$$E_{KDE}[p_Z - \hat{p}^{KDE}_Z] \propto 1.06^2\hat{\sigma}^2N^{\frac{1}{2}}.$$  

(18)

As said, this bias can never be reduced to zero. However, in specific computational setting, this bias may be bounded by finding an optimal trade-off between the number of samples $N$ and the empirical standard deviation $\hat{\sigma}$. In particular, if the domain of $p_Z$ is a discrete domain, as in the case of multinomial opinions and Dirichlet pdfs which underlie subjective opinions, then the empirical standard deviation $\hat{\sigma}$ may be bounded and it may be possible to estimate the magnitude of the bias as a function of the number of samples $N$.

In summary, from the point of view of approximation, MC simulations represents a safer choice than SL operators, as they are grounded in solid theory and they allow us to quantify and to control the bias. The lack of any bound for SL operators may be seen as an obstacle in adopting them when working in critical domains where precise approximations are required. In the next section, we will introduce our protocol to solve this problem and estimate the degree of approximation of SL operators.

6. Computational Evaluation of the Degree of Approximation of Subjective Logic Operators

In this section we present a framework to evaluate the bias of an SL operator. We start by discussing how MC approximations may be related to SL approximation using a distance measure; then, we define what precise distance measure we will use and how it relates to bias.
In the previous sections we illustrated two methodologies for finding an approximation of the pdf $p_Z$, one based on SL operators ($p_{SL}^Z$) and one relying on MC simulations ($\hat{p}_{KDE}^Z, \hat{p}_{MM}^Z$). Figure 4 merges the graphs in Figure 1, 2 and 3 to illustrate the alternative computational paths that are offered to compute an approximation of $p_Z$; starting from the distributions $(p_X, p_Y)$, the upper path represents the SL approach to finding an approximation of $p_Z = p_X \circ_{p_Y} p_Z$, while the lower paths represent MC approaches to finding an approximation of the same quantity $p_Z$.

Now, approximate methods trade off precision in the results for simplicity in computation. In order to make a grounded decision on which approximation path in Figure 4 to use, it is necessary to quantify the trade-off between computational complexity and bias. As discussed in Section 4 and 5, in the case of KDE approximation via MC simulations, both complexity and bias are known. However, in the case of SL operators, we may easily derive their computational complexity, but we have no simple way of evaluating their bias. Exploiting the properties of MC integration and the idea of distance between pdfs, it is possible to assess the degree of approximation of SL operators in a computational fashion by relating them to MC simulations.

A simple way to evaluate how well a pdf $p$ approximates another pdf $q$ is to estimate the distance between them, $D[p, q]$, where $D[\cdot, \cdot]$ is a measure of distance or divergence between pdfs [6]. The degree of approximation of $p_{SL}^Z$ could then be obtained by measuring the distance from the true pdf $p_Z$:

$$D[p_Z, p_{SL}^Z].$$

(19)

However, since the true pdf $p_Z$ is taken to be unknown or hard to compute, it is challenging to get a direct estimate of these quantities. Since we cannot rely directly on $p_Z$, we can instead exploit MC simulations and its properties.

From Equation 15 in Section 5 we know that the KDE estimation $p_{KDE}^Z$ is biased and we know how to evaluate it. Moreover, from Equation 18 in Section 5 we see that this bias depends on the number of samples $N$ and the standard deviation $\hat{\sigma}$. Now, if the domain of $p_Z$ is a discrete domain, as in the case of multinomial opinions and Dirichlet
pdfs, then the empirical standard deviation \( \hat{\sigma} \) may be bounded and it may be possible to estimate the magnitude of the bias as a function of the number of samples \( N \). It may be possible to select a number of samples \( N \) that shrinks the bias to a negligible quantity; in such case, we can then accept the KDE estimation \( \hat{p}_{KDE}^Z \) as a close approximation of the true pdf \( p_Z \):

\[
D \left[ p_Z, \hat{p}_{KDE}^Z(N) \right] \approx 0. \tag{20}
\]

We underline that this approximation holds only under the assumption that, for an increasing number of samples \( N \), the bias of the estimate \( \hat{p}_{KDE}^Z(N) \) tends, if not to zero, to a quantity whose order of magnitude is negligible with respect to further analysis; in other words, the validity of the approximation in Equation 20 is conditional on the pdf \( p_Z \) we are considering, the analysis we will be carrying out, and the number of samples we can produce (for an example of an evaluation of these conditions, see the application to the case study of the product of Beta pdfs in Section 7 and Section 8).

The approximation in Equation 20 is extremely useful because it means that while we can not evaluate absolute distances with respect to the true distribution \( p_Z \), we can still evaluate the relative distance between the KDE approximation and the SL approximation, and use it as a proxy for the distance between the SL approximation \( p_{SL}^Z \) and the true distribution \( p_Z \):

\[
D \left[ \hat{p}_{KDE}^Z(N), p_{SL}^Z \right] \approx D \left[ p_Z, p_{SL}^Z \right]. \tag{21}
\]

Thus, given only a finite set of samples \( N \) we can obtain an empirical statistic of the distance as:

\[
\hat{D} \left[ p_Z, p_{SL}^Z \right] \doteq D \left[ \hat{p}_{KDE}^Z(N), p_{SL}^Z \right]. \tag{22}
\]

If the condition in Equation 20 holds, we expect the distance \( D \left[ \hat{p}_{KDE}^Z(N), p_{SL}^Z \right] \) to be orders of magnitudes greater than \( D \left[ p_Z, \hat{p}_{KDE}^Z(N) \right] \); this would indeed confirm that the bias of \( \hat{p}_{KDE}^Z(N) \) is negligible and that the computation of \( \hat{D} \left[ p_Z, p_{SL}^Z \right] \) with a finite number of samples provides a good estimate of the degree of approximation offered by the SL approximation.

Relating distance measure to bias. So far, we have discussed distance measures in abstract terms. The quantity \( \hat{D} \left[ p_Z, p_{SL}^Z \right] \) may indeed be computed using different pdf distance, such as \( \phi \)-divergences or integral probability metrics \([11]\).

In this paper, we will rely on computing a simple integral distance, defined as:

\[
D_I [p, q] = \int_{-\infty}^{+\infty} |p(x) - q(x)| \, dx. \tag{23}
\]

This distance \( D_I [p, q] \) is the same as the total variation distance except for the scaling constant:

\[
D_{TV} [p, q] = \frac{1}{2} \int_{-\infty}^{+\infty} |p(x) - q(x)| \, dx. \tag{24}
\]

The constant \( \frac{1}{2} \) rescales the distance on the interval \([0, 1] \). However, in order to get an absolute evaluation of how the mass of the two distributions \( p \) and \( q \) overlaps, we drop the scaling constant.
The choice of an integral distance $D_I [\cdot, \cdot]$ is justified for three reasons. First, from a conceptual point of view, an integral distance allows us to get a complete picture of the difference between two pdfs. While measures based on the evaluation of a limited set of synthetic statistics such as moments would provide us with a rough evaluation of the difference between two distributions, an integral distance provides a more precise way to assess the distribution of the mass of probability, taking into account, for instance, the potential presence of multiple modes or how mass subtly distributes on the tails.

Second, from a computational point of view, the integral distance $D_I [\cdot, \cdot]$ allows us, once again, to exploit MC integration. Recall that we want to get an estimation of $\hat{D}_{p_{\text{Z}} Z, p_{\text{SL}} Z}$ via KDE, we can estimate the integral distance via MC integration over the domain $\Omega$ of the events as:

$$
\int_{\Omega} |\hat{p}_{\text{KDE}} Z(z) - p_{\text{SL}} Z(z)| dz \approx \frac{1}{N} \sum_{i=1}^{N} |\hat{p}_{\text{KDE}} Z(z_i) - p_{\text{SL}} Z(z_i)|.
$$

(25)

Third, from a theoretical point of view, the integral in Equation 25 is related to the bias:

$$
\int_{\Omega} |\hat{p}_{\text{KDE}} Z(z) - p_{\text{SL}} Z(z)| dz \approx E \left[ |\hat{p}_{\text{KDE}} Z(Z) - p_{\text{SL}} Z(Z)| \right] \geq E \left[ |\hat{p}_{\text{KDE}} Z(Z) - p_{\text{SL}} Z(Z)| \right].
$$

(26)

Thus, using the integral distance $D_I [\hat{p}_{\text{KDE}} Z, p_{\text{SL}} Z]$ we can obtain an estimation of the distance $\hat{D}_{p_{\text{Z}} Z, p_{\text{SL}} Z}$ as well as an upper bound on the bias of $p_{\text{SL}} Z$. Notice that the absolute value in the integral distance provides a more honest evaluation of the absolute difference between pdfs, avoiding an averaging effect in absence of the absolute value operator.

Figure 5 summarizes our overall framework to evaluate the degree of approximation of the SL approximation as the integral distance $\int |\hat{p}_{\text{SL}} Z - \hat{p}_{\text{KDE}} Z|$, under the assumption that $D_{p_{\text{Z}}, \hat{p}_{\text{KDE}} Z (N)} \approx 0$. This approach is generic and it is not tied to the SL approximation. If the condition in Equation 20 can be guaranteed, the same approach may be used to get an estimation of the distance between the true pdf $p_{\text{Z}}$ and other potential approximation. For instance, Figure 5 shows our methodology applied also to the problem of estimating the distance from the true pdf of the moment-matching approximation $D_{p_{\text{Z}}, \hat{p}_{\text{MM}} Z (N)}$ by computing the distance $\int |\hat{p}_{\text{MM}} Z - \hat{p}_{\text{KDE}} Z|$.

7. Case Study: Product of Beta Distributions

In this section, we show how our framework may be applied to the problem of computing the product of Beta distributions. We first recall the definition of a Beta distribution and the definition of the product of Beta distributions; we then introduce the SL operator for binomial multiplication and we discuss how it can be used for approximating the product of Beta distributions; we work out the computational complexity of binomial multiplication and show the lack of generic estimations of its degree of approximation;
\[
(\omega_X, \omega_Y) \xrightarrow{\text{MCI}} \omega_S \xrightarrow{\text{MCS}} p_Z \xrightarrow{\text{KDE}} \hat{p}_Z \xrightarrow{\text{MCI}} \hat{D}_I [\hat{p}_Z, p_Z]
\]

\[
(p_X, p_Y) \xrightarrow{\text{SL}} p_S \xrightarrow{\text{KDE}} \hat{p}_Z \xrightarrow{\text{MCI}} \hat{D}_I [\hat{p}_Z, p_Z]
\]

\[
\{z_1, z_2 \ldots z_N\} \xrightarrow{\text{MCS}} \hat{M}_I [Z] \xrightarrow{\text{MCI}} \hat{p}_Z \xrightarrow{\text{KDE}} \hat{p}_Z \xrightarrow{\text{MCI}} \hat{D}_I [\hat{p}_Z, p_Z]
\]

Figure 5: Evaluations of the distances between \(p_Z\) and its approximations based on the assumption that \(D[p_Z, \hat{p}_Z^{\text{KDE}(N)}] \approx 0\). MCS stands for MC sampling, MCI stands for MC integration, KDE stands for kernel-density estimation.

finally, we apply our framework to get an evaluation of the degree of approximation of binomial multiplication.

**Beta pdf.** Let \(X\) be a random variable on the support \([0, 1]\); we say that \(X\) follows a Beta distribution \(X \sim \text{Beta}(\alpha, \beta)\) with parameters \(\alpha \in \mathbb{R}_{\geq 0}\) and \(\beta \in \mathbb{R}_{\geq 0}\) when its probability density function \(p_X\) has the following form:

\[
p_X(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, \tag{29}
\]

where \(B(\alpha, \beta)\) is the Beta function.

**Product of Beta pdfs.** Let \(X \sim \text{Beta}(\alpha_X, \beta_X)\) and \(Y \sim \text{Beta}(\alpha_Y, \beta_Y)\) be two Beta random variables with associated pdfs \(p_X\) and \(p_Y\). Let us define a third random variable \(Z\) as the product of the two Beta random variables \(Z = X \cdot Y\). The probability density function \(p_Z\) of \(Z\) does not follow a Beta distribution anymore, and its precise analytical form can not be easily expressed using elementary functions \([7]\).

An analytical solution to the evaluation of the pdf of the product of two Beta distributions has been offered in \([8]\):

\[
p_Z(z; \alpha_X, \alpha_Y, \beta_X, \beta_Y) = \frac{B(\beta_X, \beta_Y) \cdot z^{-\beta_X} \cdot (1-z)^{\beta_X+\beta_Y-1} \cdot z^{\alpha_Y}}{F_D^{(3)}(\beta_X,1-\alpha_X,1-\alpha_Y,\alpha_X+\beta_X-1,\beta_X+\beta_Y,0,\frac{1}{\beta},\frac{1}{\alpha})}, \tag{30}
\]

where \(F_D^{(3)}\) is the Lauricella D hyper-geometric series. While this formula provides an elegant solution to the problem of finding the pdf of the product of two Beta pdfs, its

---

2This paper actually presents the more generic solution to the problem of multiplying two general Beta distribution, which subsume the multiplication of two simple Beta distributions as defined above.
straightforward evaluation is challenging as the Lauricella function requires the computation of factorial products and series.

Other analytical approaches to evaluate the product of two or more Beta distributions include relying on high-order functions, such as the Meijer G-function or Fox’s H function, or modeling the pdf of the product using an infinite mixture of simpler distributions \cite{12,1}. These approaches also present computational challenges, despite more efficient solutions have been investigated \cite{12,1}.

Finally, common approaches rely on MC simulations to sample points from the probability distribution of \(Z\) and to compute statistics of the pdf by matching the moments or the quantiles of \(Z\) \cite{7}, as we reviewed in Section 3.

Subjective logic binomial multiplication. An alternative solution to compute the product of Beta distributions is based on the use of the SL operator for binomial multiplication. Given two binomial opinions \(\omega_X\) and \(\omega_Y\) defined on the same domain \(\Omega\), the binomial opinion \(\omega_Z\) resulting from the multiplication \(\omega_X \cdot \omega_Y\) is computed as \cite{4}:

\[
\begin{align*}
\omega_Z &= \begin{cases} 
  b_Z &= b_X b_Y + \frac{(1-a_X-a_Y) b_X b_Y + a_X (1-a_Y) b_X b_Y}{1-a_X a_Y} \\
  d_Z &= d_X + d_Y - d_X d_Y \\
  u_Z &= u_X u_Y + \frac{(1-a_Y) u_X u_Y + (1-a_X) u_X b_Y}{1-a_X a_Y} \\
  a_Z &= a_X a_Y.
\end{cases}
\end{align*}
\]

In the domain of probability distributions, the multiplication of opinions \(\omega_Z = \omega_X \cdot \omega_Y\) translates into the multiplication of the mapped pdfs \(p_Z = p_X \cdot p_Y\).

Approximating the product of Beta pdfs. Now, assume we are interested in computing the product \(Z = X \cdot Y\), where \(X\) and \(Y\) are two Beta random variables. Since an analytic solution is hard to compute, we may decide to rely either on the SL approximation or on a MC approximation.

Concerning moment-matching approximations we may consider a Gaussian pdf and a Beta pdf. Using a Gaussian pdf \(\hat{p}_Z^{GAUSS}\) is a choice motivated by the simplicity and the ubiquity of this distribution; however, this is clearly a naive choice, as a Gaussian pdf has an unbounded support, is symmetrical and it assumes all the moments greater than two being zero. Using a Beta distribution \(\hat{p}_Z^{BETA}\) is a more prudent choice: even if it is known that the product of two Betas is not, in general, a Beta distribution, a Beta pdf still fits the right support and it may have several moments different from zero. After using MC integration to estimate the mean \(\hat{\mu}\) and the variance \(\hat{\sigma}^2\) of \(p_Z\), the Gaussian approximation \(\hat{p}_Z^{GAUSS}\) is instantiated as \(N(\hat{\mu}, \hat{\sigma}^2)\), while the Beta approximation \(\hat{p}_Z^{BETA}\) is defined as \(\text{Beta}\left(\frac{\hat{\mu} (\hat{\sigma}^2 + \hat{\sigma}^2 - \hat{\mu})}{\hat{\sigma}^2}, \frac{(\hat{\mu} - 1) (\hat{\sigma}^2 + \hat{\mu}^2 - \hat{\mu})}{\hat{\sigma}^2}\right)\), thus guaranteeing that \(p_Z, \hat{p}_Z^{GAUSS}\) and \(\hat{p}_Z^{BETA}\) have the same mean and variance.

Figure 6 provides a concrete instantiation of the diagram in Figure 4, in which the generic operators \(\circ_P\) and \(\circ_{SL}\) have been substituted with multiplication and the generic moment-matching approximation \(\hat{p}_Z^{MM}\) has been replaced by the Gaussian approximation \(\hat{p}_Z^{GAUSS}\) and the Beta approximation \(\hat{p}_Z^{BETA}\).

As discussed earlier, choosing which path to take, whether to follow the SL approximation path in upper part of the graph or opt for one of the MC approximations in the lower part, requires evaluating the trade-off between computational complexity and
degree of approximation of the different approaches. As these parameters are known in the case of MC simulations, we will review here the computational complexity and the approximation of the binomial multiplication.

**Computational complexity of the binomial operator.** Binomial multiplication is extremely efficient. Given two binomial opinions \( \omega_X \) and \( \omega_Y \) it is possible to compute their product \( \omega_Z \) through a fixed and finite number of arithmetic operations. Independently from the actual form of the mapped distributions, the product is always computed in the same amount of time. As such, the computational complexity of these SL operators is constant \( \mathcal{O}(1) \).

**Approximation of the binomial operator.** The original paper that introduced the SL operator for binomial multiplication \cite{5} proposed a first qualitative analysis of the degree of approximation of this operator. In particular, it considered the specific instance of the multiplication of two Beta pdfs of the form \( X, Y \sim \text{Beta}(1, 1) \); the pdf of \( X \) and \( Y \) reduces to a uniform distribution over \([0, 1]\), which is taken to be a worst-case scenario with maximal variance and entropy. The analytical solution \( Z = X \cdot Y \) to this particular case was then computed and graphically compared to the pdf associated with product \( \omega_Z = \omega_X \cdot \omega_Y \). This study provided a clear visual appraisal of the difference between the exact pdf and the SL-approximated pdf, but no quantitative estimation were provided for more general cases.

**Relating the binomial multiplication and Monte Carlo approximations.** In order to compute a numerical estimation of the degree of approximation of the SL operator for binomial multiplication we want to rely on the framework described in Section 6.

The basic condition expressed in Equation \(20\) requires the bias of \( \hat{p}_{KDE}^Z \) to be bounded and negligible. Recall that this bias, using a Gaussian kernel with width computed using the Silverman rule, is:

\[
E_{KDE} [p_Z - \hat{p}_Z^{KDE}] \propto 1.06^2 \sigma^2 N^{\frac{1}{5}},
\]  

(32)
where
\[ \hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \hat{\mu})^2}{N - 1}}. \] (33)

Now, notice that on our bounded support \([0, 1]\) we can expect the difference \((x_i - \hat{\mu})\) to be at most, in the order of \(10^{-1}\). This implies that, in the worst case, the order of magnitude of \(\hat{\sigma}\) may be estimated as:
\[ \hat{\sigma} < \sqrt{\frac{N (10^{-1})^2}{N - 1}} \] (34)
\[ \hat{\sigma} < 10^{-1}. \] (35)

Consequently, relying on Silverman rule in Equation 16, the order of magnitude of largest kernel width \(w\) may be bounded as:
\[ w < 1.06 \cdot 10^{-1} \cdot \frac{1}{\sqrt{N}} \] (36)
\[ w < 10^{-1} N^{-\frac{1}{4}}. \] (37)

As such, from Equation 18 the bias will be proportional to this upper bound:
\[ E_{KDE} [p_Z - \hat{p}_Z^{KDE}] \propto \left(10^{-1} N^{-\frac{1}{4}}\right)^2. \] (38)

Thus, for instance, if we were to run our MC simulation sampling \(N = 10^5\) samples, then we can expect the bias of \(\hat{p}_Z^{KDE}\) to be in the order of \(10^{-4}\). This upper bound on the bias allows us to consider the bias negligible if we are comparing it with quantities, such as \(D[p_Z^{KDE}(N), p_Z^{SL}]\), order of magnitude greater than \(10^{-4}\). If this condition is met, then we can estimate the degree of approximation of binomial multiplication adopting the framework illustrated in Figure 5 and instantiated for this specific SL operator as in Figure 7.

8. Case Study: Empirical Evaluation

In this section we describe our experimental simulations for the evaluation of the degree of approximation of the binomial product. We will first offer a qualitative analysis of the SL approximation \(p_Z^{SL}\) and the approximation generated via MC simulation \(p_Z^{MC}\); then, we will provide a quantitative statistical assessment of the distance \(D[p_Z, p_Z^{SL}]\); next, we will analyze a specific study case concerning the worst-case scenario of the product of two degenerate Beta random variables; finally, we will assess quantitatively the degree of approximation in the product of multiple opinions.

In all our simulations, opinions \(\omega = (b, d, u, a)\) are sampled randomly. The parameters \(b, d\) and \(u\) must be sampled from a simplex defined by the constraint \(b + d + u = 1\); therefore we sample them from a Dirichlet distribution \(\text{Dir}(\boldsymbol{\alpha})\) with \(\boldsymbol{\alpha} = [1, 1, 1, 1]\), which guarantees a uniform sampling over the simplex. The parameter \(a\), instead, is sampled from a uniform distribution \(\text{Unif}(0, 1)\). When sampling Beta pdfs \(\text{Beta}(\alpha, \beta)\), we will draw the parameters \(\alpha\) and \(\beta\) from a uniform pdf on a bounded domain, \(\text{Unif}(0, 10)\).
8.1. Qualitative simulations

In the qualitative simulations we aim at getting a first intuitive feeling about the approximation of \( p_{\text{SL}}^Z \).

Protocol. In order to compare the SL approximation \( p_{\text{SL}}^Z \) and the MC approximation \( \hat{p}_{\text{MC}}^Z \), we adopt the following protocol: we sample two random opinions \( \omega_X \) and \( \omega_Y \); we compute their product \( \omega_Z \) and project the result onto the distribution \( p_{\text{SL}}^Z \). In parallel, we project the opinions \( \omega_X \) and \( \omega_Y \) onto the Beta distributions \( p_X \) and \( p_Y \) and we recreate \( \hat{p}_{\text{MC}}^Z \) using MC simulation to draw \( N \) samples \( \{z_1, z_2, \ldots, z_N\} \) from \( p_Z \). Finally, we plot \( p_{\text{SL}}^Z \) against \( \hat{p}_{\text{MC}}^Z \) numerically, without any smoothing or interpolation. In addition, we use the samples \( \{z_1, z_2, \ldots, z_N\} \) to estimate moments via MC integration and then instantiate the moment-matching approximations \( \hat{p}_{\text{GAUSS}}^Z \) and \( \hat{p}_{\text{BETA}}^Z \).

Results. Figures 8 and 9 illustrates the difference between the SL binomial multiplication \( p_{\text{SL}}^Z \) and the approximation of the true pdf \( p_Z \) plotted via MC. In some instances, \( p_{\text{SL}}^Z \) seems to provide a very good approximation of \( p_Z \), as shown in Figure 8. In other instances, as shown in Figure 9, this approximation is more coarse, especially when it comes to values of the support near the extremes.

The discrepancy shown in Figure 9 may be theoretically imputed to a poor approximation of the MC simulation due to a limited number of samples. In order to confute this hypothesis, another identical simulation with a number of samples one order of magnitude larger was run. Figure 10 shows that this simulation returned the same qualitative result. This suggests that the gap between \( p_{\text{MC}}^Z \) and \( p_{\text{SL}}^Z \) may not be imputed to a poor MC approximation.
Figure 8: Qualitative simulation. The first opinion $\omega_X$ has parameters $b = 0.61, d = 0.30, u = 0.09$ and $a = 0.79$, the second opinion $\omega_Y$ has parameters $b = 0.28, d = 0.66, u = 0.06$ and $a = 0.46$. The number of samples is $N = 10^5$. 
Figure 9: Qualitative simulation. The first opinion $\omega_X$ has parameters $b = 0.16$, $d = 0.55$, $u = 0.29$ and $a = 0.001$, the second opinion $\omega_Y$ has parameters $b = 0.43$, $d = 0.16$, $u = 0.41$ and $a = 0.39$. The number of samples is $N = 10^5$.

Figure 10: Qualitative simulation. Same settings as in Figure 9, except for the number of samples $N = 10^6$. 
Figure 11: Qualitative simulation. The first opinion $\omega_X$ has parameters $b = 0.35$, $d = 0.23$, $u = 0.42$ and $a = 0.83$, the second opinion $\omega_Y$ has parameters $b = 0.14$, $d = 0.26$, $u = 0.60$ and $a = 0.77$. The number of samples is $N = 10^5$.

Figure 11 offers a visual comparison of the approximations offered by $\hat{p}^{MC}_Z$ and $\hat{p}^{SL}_Z$ contrasted now with the Gaussian $\hat{p}^{GAUSS}_Z$ and the Beta $\hat{p}^{BETA}_Z$ approximations. Evidently the Gaussian approximation is far from providing a good model for $p_Z$, while the Beta approximation follows very closely $\hat{p}^{SL}_Z$ and $\hat{p}^{MC}_Z$.

Discussion. This analysis suggests that the SL approximation $p^{SL}_Z$ may provide a good and useful estimation of the true pdf $p_Z$; indeed, $p^{SL}_Z$ follows very closely the shape $p^{MC}_Z$ which, in turn, is close to $p_Z$. Given that $p^{SL}_Z$ consists of a smooth Beta distribution, it is not surprising that the approximation suffers the worst near the boundaries where the MC estimate $p^{MC}_Z$ diverges from $p^{SL}_Z$, as shown in Figure 9. The results also rule out the naive possibility of using a Gaussian approximation, since $p^{GAUSS}_Z$ clearly mismodels the true pdf $p_Z$. Instead, a Beta approximation $\hat{p}^{BETA}_Z$ using the mean and the variance computed through MC integration, provides an approximation qualitatively very close to $p^{SL}_Z$ and $\hat{p}^{MC}_Z$; however, notice that this Beta approximation requires the same amount of computation of the MC simulation but it does not offer similar theoretical guarantees; as such, it does not provide the computational simplicity of $p^{SL}_Z$ nor the statistical guarantees of $\hat{p}^{MC}_Z$.

8.2. Quantitative simulations

Quantifying the gap between $p_Z$ and $p^{SL}_Z$ that we observed in the qualitative study above is the aim of the quantitative simulations.
Protocol. The first part of our quantitative protocol is the same as the qualitative protocol: we sample two random opinions \( \omega_X \) and \( \omega_Y \), we compute their product \( \omega_Z \), and we project the result onto the distribution \( p_Z^{SL} \); at the same time, we project the opinions \( \omega_X \) and \( \omega_Y \) onto the Beta distributions \( p_X \) and \( p_Y \), and we use MC simulation to draw \( N \) samples \( \{x_1, x_2, \ldots, x_N\} \) from \( p_Z \). At this point, instead of plotting our results, we use a KDE to explicitly estimate \( p_Z^{KDE} \). This allows us to compute via MC integration the area determined by the integral \( \int_0^1 [p_Z^{SL}(z) - p_Z^{KDE}(z)] \, dz \). In order to get significant statistical result, we repeat this procedure 100 times and we compute the mean and the standard deviation of the distance \( D_I[p_Z, p_Z^{SL}] \).

For completeness, we also run this simulation starting in the domain of pdfs: we start sampling two random Beta pdfs \( p_X \) and \( p_Y \); we run a MC simulation to draw \( N \) samples \( \{x_1, x_2, \ldots, x_N\} \) from \( p_Z = p_X \cdot p_Y \). We then project the pdfs onto the opinions \( \omega_X \) and \( \omega_Y \), we compute their product \( \omega_Z \), and we project back the result onto the distribution \( p_Z^{SL} \). We then apply KDE and compute the integral distance, as before. This simulation is also repeated 100 times in order to get statistically significant results.

Notice that, since the pdf \( p_Z^{KDE} \) that we are trying to estimate is defined on a bounded interval, using a Gaussian kernel is a sub-optimal choice. The Gaussian kernel distributes the mass of probability over the entire real line, and thus we would inevitably spill part of the probability mass beyond the domain [0,1]. To solve this problem we adopt the logit trick [10]: instead of applying a Gaussian KDE to estimate \( p_Z^{KDE} \) directly from the samples \( \{z_1, z_2, \ldots, z_N\} \), we use a logit transform \( \logit(x) = \log \frac{x}{1-x} \) to project the sample \( \{z_1, z_2, \ldots, z_N\} \) onto the entire real line; we then apply a Gaussian KDE to the projected samples and rescale back the learned pdf to \( p_Z^{KDE} \).

Again, in order to have a comparative assessment, we also evaluate the distance between the true distribution \( p_Z \) and a Gaussian \( p_Z^{GAUSS} \) and a Beta \( p_Z^{BETA} \) approximation; these two approximations are generated following the same protocol used in the qualitative analysis in Section 8.1: we first compute mean and variance of \( p_Z \) via MC integration, and then we initialize a Gaussian and a Beta distribution with the same parameters.

Refer to Figure 12 for the diagram of the experimental protocol for the quantitative simulations.

Results. Figure 12 shows the variation in the distances \( D[p_Z, \cdot] \) estimated as \( D[p_Z^{KDE}(N), \cdot] \) as a function of the number samples \( N \) generated in the MC simulation. All the statistics are computed from 100 repetitions and using \( 10^3 \) uniformly sampled points on the support [0,1] to perform MC integration.

The stable trend of all the distances \( D[p_Z^{KDE}(N), \cdot] \) suggests that the MC simulations sampled enough points, for all the values of \( N \) that we considered, to return a good approximation.

More importantly, recall that our whole analysis holds only if Equation [21] is satisfied. Using \( N = 10^5 \), we know from Equation [38] that the bias in evaluating \( p_Z^{KDE}(N) \) is in the order of \( 10^{-4} \). Thus compared to the scale of the mean and variance error in our results, which are in the scale of \( 10^{-1} \), we can confirm that the bias is negligible. We can then state that \( D[p_Z^{KDE}(N), \cdot] \) does indeed provide a good estimate of \( D[p_Z, \cdot] \).

Consistently with the previous experiments, the Gaussian approximation provides by far the worst approximation. Indeed, with a distance \( D[p_Z^{KDE}(N), p_Z^{GAUSS}] \) averaging around 1.5, we can expect only one quarter of the probability mass of the Gaussian approximation \( p_Z^{GAUSS} \) to overlap with the true distribution \( p_Z \).
The Beta approximation $\hat{p}^{\text{BETA}}_Z$ clearly offers a better solution. Even if the product of two Beta distributions $p_Z$ is not a Beta distribution, it is clear from these results that the shape of $p_Z$ is in general very close to a Beta pdf. Indeed, the expected value of $D[\hat{p}^{\text{KDE}}_Z(N), \hat{p}^{\text{BETA}}_Z]$ points out that 95% of the mass of $\hat{p}^{\text{BETA}}_Z$ and $p_Z$ overlap with very limited variance.

The SL approximation $p^{\text{SL}}_Z$ also offers a good solution. The result of the simulation in which we started from opinions and the one in which we started from Beta pdfs are extremely close. This offers a confirmation of the robustness of the transformations between the domain of opinions and the domain of pdfs. Overall, the expected value of $D[\hat{p}^{\text{KDE}}_Z(N), p^{\text{SL}}_Z]$ suggests that the typical overlap between the mass of $p^{\text{SL}}_Z$ and $p_Z$ settles around 90%, slightly worse than $\hat{p}^{\text{BETA}}_Z$. The high variance points to a strong case-by-case variability: in certain scenario $p^{\text{SL}}_Z$ may provide a model as good or better than $\hat{p}^{\text{BETA}}_Z$, but on other instances its quality may degrade further.

Discussion. The results of our quantitative analysis agree with the qualitative study. A Gaussian approximation $\hat{p}^{\text{GAUSS}}_Z$ was shown to be an unsuitable option for modelling the product of two Beta distributions (and, for this reason, we will drop this approximation from the next simulations). Instead, the SL approximation $p^{\text{SL}}_Z$ and the Beta approximation $\hat{p}^{\text{BETA}}_Z$ are both good approximations, assuming that we can accept a difference between the true pdf and the approximation up to 5% – 10% of the probability density.

If we were to have the resources to run a MC simulation and perform a MC integration, $\hat{p}^{\text{BETA}}_Z$ seems to be, on average, the best bet.
8.3. Limit-case Study

In this limit-case study, we consider the worst-case scenario considered in [5]. This study provides a way to enrich the previous study and reconnect this paper to it.

Protocol. We quantitatively analyze the case in which both opinions $\omega_X$ and $\omega_Y$ are degenerate Beta pdfs of the form $\text{Beta}(1, 1)$ with $a = \frac{1}{2}$. To provide a quantitative analysis we follow the same paradigm used in Section 8.2: first, we derive the MC approximation $\hat{p}^\text{MC}_Z$, the SL approximation $\hat{p}^\text{SL}_Z$, and the Beta approximation $\hat{p}^\text{BETA}_Z$; then, compute the distance between the aforementioned distributions and the true pdf $p_Z$, whose exact form, $-\log(z)$, is given in [5].

Results. Table 13 shows the evaluation of the distance $\hat{D}[p_Z, \cdot]$ with respect to the true pdf $p_Z = -\log(z)$, when performing MC simulations with $N = 10^6$ points and using $10^3$ uniformly sampled points on the support $[0, 1]$ to perform MC integration.

The results show that the difference between the three approximations is about one order of magnitude from each other. The MC approximation is, as expected, very close to the true pdf $p_Z$, with a distance averaging around $9 \cdot 10^{-3}$. This is higher than the theoretically computed value, likely due to the fact that we are evaluating a limit case; however, this difference is still small enough to allow us a comparison with the other approximations. The Beta approximation, which relies on MC approximation, has a slightly higher distance around $3 \cdot 10^{-2}$. Finally, the SL approximation has the highest distance at around $2 \cdot 10^{-1}$, meaning that the probability mass of $p^\text{SL}_Z$ and $p_Z$ overlap for about 90%. All the results also show a high variance, which is caused by the difficulty in numerically approximating values near 0, where the true pdf $p_Z = -\log(z)$ diverges.

Discussion. The results are consistent with our previous results obtained in the quantitative analysis in Section 8.2 and they confirm that the scenario considered in [5] with two degenerate Beta pdfs of the form $\text{Beta}(1, 1)$ and $a = \frac{1}{2}$ is indeed a hard case for SL approximation. The MC approximation performs better in modeling the true form of the pdf $-\log(z)$ and, compared to it, the SL approximation is two orders of magnitude less precise in terms of integral distance. This simulation thus clearly highlights the cost in terms of accuracy that the computational simplicity of SL implies.

8.4. Multiple Products

In this last experimental section we consider the product of multiple opinions and we examine how approximation spread.
Protocol. We quantitatively evaluate the product of multiple opinions $\omega_{X_1}, \omega_{X_2} \ldots \omega_{X_L}$ by randomly sampling $L$ opinions and then defining $\omega_Z = (\omega_{X_1} \cdot \omega_{X_2} \cdot \omega_{X_3} \ldots \omega_{X_L})$ and $p_Z = p_{X_1} \cdot p_{X_2} \cdot \ldots \cdot p_{X_L}$. The following analysis adopts the same protocol used in the quantitative simulations in Section 8.2 in order to evaluate the integral distance $\hat{D} [p_Z^{\text{KDE}}, p_Z^{\text{SL}}]$ where now the final pdf over $Z$ is given by the product of multiple opinions. Notice that, while the convergence properties of the MC simulation remains the same, we may expect the precision of the SL approximation to degrade over multiple products as successive approximations cumulate.

Results. Figure 14 shows the variation of the distance $\hat{D} [p_Z^{\text{KDE}}, p_Z^{\text{SL}}]$ as a function of the number $L$ of opinions that are multiplied together to determine $Z$. A slight increase in the degree of approximation may be observed as the number of factors increases from 2 to 5.

Discussion. The hypothesis that the degree of approximation of the SL operator degrades over multiple products because of the accumulation of approximation appears to be correct. As more factors are taken into consideration, $p_Z^{\text{SL}}$ slowly diverges from the true pdf $p_Z$. A modeler should be aware of these dynamics in case she were to use SL to approximate the product of multiple Beta random variables.

9. Conclusion and Future Work

In this study we studied the use of subjective logic as a framework for approximating operations over probability distributions. As in the case of any approximation, we considered SL operators from the perspective of the trade-off between the computational simplicity they guarantee and the precision they sacrifice. We proposed a protocol based on MC simulations to evaluate quantitatively this trade-off, estimating the distance between the SL approximation and a KDE estimation, under the assumption of a negligible bias between the KDE reconstruction and the true probability distribution.
We applied our protocol to the case study of the product of two Beta distributions. This scenario is relevant to fields like reliability analysis, and SL provides a well-defined operator to approximate this operation. In general, the SL binomial operator guarantees the preservation of the first moment, but does not strictly preserve higher moments or quantiles. To quantify the degree of approximation of the SL approximation, we compared it with other standard approximations, such as moment-matching with a Gaussian pdf, moment-matching with a Beta pdf, and KDE via MC.

Our simulations showed that, at the cost of accepting a difference between the SL approximation and the true pdf of the product of two Beta distributions up to 10% of the probability mass, the SL approximation offers a computationally efficient approximation. Indeed the KDE approximation and the Beta approximation provided better estimation, but at a computational cost that is quadratic in the number of samples (for KDE), or linear (for moment-matching).

In summary, it is possible to enjoy the computational efficiency and the interpretability of SL if the modeling scenario allows room for approximation up to the amount estimated using our protocol. The recommendation is that, were SL operators to be used to model critical systems (as in the case of reliability analysis or when higher-order moments are critical), this divergence between the true pdf and the SL approximation that we highlighted should be factored in the analysis.

Further work will be developed for better characterizing the difference between true pdfs and SL approximations; in particular, understanding how the mass is differently allocated with respect to the overall shape of the pdf, whether, for instance, these differences are more accentuated near the mode (assuming one exists) or around the tail. According to the way in which probability mass is misplaced in SL approximations different forms of correction may be then considered.

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