Classical Functional Bethe Ansatz for $SL(N)$: Separation of Variables for the Magnetic Chain

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Abstract

The Functional Bethe Ansatz (FBA) proposed by Sklyanin is a method which gives separation variables for systems for which an $R$-matrix is known. Previously the FBA was only known for $SL(2)$ and $SL(3)$ (and associated) $R$-matrices. In this paper I advance Sklyanin’s program by giving the FBA for certain systems with $SL(N)$ $R$-matrices. This is achieved by constructing rational functions $A(u)$ and $B(u)$ of the matrix elements of $T(u)$, so that, in the generic case, the zeros $x_i$ of $B(u)$ are the separation coordinates and the $P_i = A(x_i)$ provide their conjugate momenta. The method is illustrated with the magnetic chain and the Gaudin model, and its wider applicability is discussed.
1 Introduction

1.1 Separation of variables in Integrable Systems

A classical system with a $2n$-dimensional phase space is said to be integrable if there exist $n$ independent functions on the phase space (called integrals of motion) which Poisson commute among themselves and with the Hamiltonian.

Classically a system is said to be separable if there exist variables in which the Hamilton-Jacobi equation separates. Traditionally this means introducing $n$ separation constants so that the Hamilton-Jacobi equation (involving all $n$ pairs of conjugate variables) can be replaced by $n$ equations involving one each (and possibly a further equation constraining the allowed separation constants at each energy).

Liouville’s Theorem [1] says that for integrable systems there exist action-angle variables (at least locally) in which the Hamiltonian is a function of the action variables alone and hence separates. Thus for integrable systems it is known that separation variables exist (at least locally). However the formula for constructing action-angle variables (which involves integrating over invariant tori) is not tractable in general.

What is needed is an explicit method for constructing separation variables for arbitrary integrable systems. Two such methods exist. One uses algebraic geometry and has been applied to loop algebras with linear Poisson brackets [4]. The other, the Functional Bethe Ansatz (FBA) works for systems with $SL(2)$ or $SL(3)$ $R$-matrices of certain forms. In this paper I extend the FBA to $SL(N)$ for $N > 3$. I work with the magnetic chain and Gaudin magnet, but the method and results are much more widely applicable as I will discuss at the end of the paper. Although this extension has equations in common with the algebraic geometric technique no knowledge of algebraic geometry is required to understand this paper.

Some insight may be gained from considering the reverse problem (i.e. how to construct integrals of motion given a separation) which is much better understood. Jacobi’s Theorem [1] can be interpreted as saying that if a system separates then it is integrable, with the integrals of motion corresponding to the separation constants. Thus given a separation of a Hamiltonian there is a systematic method for constructing the integrals of motion, and the separation of the original Hamiltonian induces a separation of these integrals of motion. Thus it is natural in integrable systems to think of separating the system without any reference to a particular integral of motion. Separation of variables may then be (re)defined as seeking variables $x_i$ and $p_i$, with canonical Poisson brackets, which satisfy $n$ separated equations,

$$\Phi_j(x_j, p_j, I_1, \ldots, I_n) = 0 \quad j = 1, \ldots, n$$

where in these equations the dependence on separation constants has been replaced with dependence on the integrals (which are of course constants of the motion). This is the starting point used by Sklyanin [4] for his Functional Bethe Ansatz.
1.2 R-matrices and Lax Pairs

The two methods already proposed and this paper all use the matrix formalism associated with $R$-matrices and Lax Pairs. To understand this paper only the basics of this formalism need be known, further details may be found for example in [4, 5, 6].

In the $R$-matrix approach the operator content of the theory is contained in an $N \times N$ matrix function on the phase space $T(u)$ depending on a spectral parameter $u$, the algebraic structure of the theory is given by the so called $R$-matrix algebra. This can be quadratic,

\[
\{ \frac{1}{1} T(u), \frac{2}{2} T(v) \} = [R(u-v), \frac{1}{1} T(u) \frac{2}{2} T(v)]
\] (2)

or linear,

\[
\{ \frac{1}{1} T(u), \frac{1}{1} T(v) \} = [R(u-v), \frac{1}{1} T(u) + \frac{2}{2} T(v)] .
\] (3)

Here $T$ denotes $T(u) \otimes 1$ etc. and $R$ is an $N^2 \times N^2$ matrix acting in the tensor product.

The $R$-matrix must obey a consistency condition, the Classical Yang-Baxter equation. In the case of Lax Pairs we have a pair of matrices $L(u)$ and $M(u)$ depending on spectral parameter $u$, $L(u)$ is a function on the phase space, while $M(u)$ is usually a constant. $L$ and $M$ give a Lax representation of a system if the equations of motion of the system are equivalent to the following evolution equation for $L$,

\[
\frac{dL(u)}{dt} = [L(u), M(u)] .
\] (4)

From this equation it is clear that the quantities generated by the spectral invariants of $L(u)$ are constants of the motion, but a priori they do not Poisson commute. The necessary and sufficient condition that the eigenvalues Poisson commute is that the Lax matrix obeys an $R$-matrix type Poisson brackets [7], which for antisymmetric $R$-matrices reduces to equation (3).

This paper uses the $R$-matrix formalism. What is important to know for this paper is that in general the spectral invariants of $T(u)$ provide the integrals of motion. So long as they give rise to enough the system is integrable.

1.3 The Method

The search for separation variables can be thought of as happening in two stages, (1) look for variables which give rise to separated equations, then (2) check that these variables have the correct commutation relations.

Crucial to both methods is the idea that if $\zeta_i$ is an eigenvalue of $T(u_i)$ then $u_i$ and $\zeta_i$ automatically obey the following ‘separated’ equation,

\[
\det (\zeta_i - T(u_i)) = 0 .
\] (5)

This is indeed a separated equation in that it only depends on $u_i$, $\zeta_i$ and the spectral invariants of $T(u)$ (i.e. the integrals of motion). Any such $u_i$ and $\zeta_i$ found in this way therefore complete step (1). The problem is to choose them in such a way that they have the correct Poisson brackets and form a basis of the phase space.
1.3.1 Functional Bethe Ansatz

The Functional Bethe Ansatz was proposed by Sklyanin [3, 8, 9, 10] as a method of finding separation variables for integrable systems, this paper is only concerned with the classical version. It is a blending of ideas from the Bethe Ansatz and separation of variables. It is applicable to systems for which an \( R \)-matrix (or indeed Lax) representation of a certain form is known.

The FBA was first proposed for the case of \( 2 \times 2 \) matrices. In this case the basic objects are

\[
T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},
\]

and these obey \( R \)-matrix relations. The Functional Bethe Ansatz says that if the \( \{ x_i, i = 1, \ldots, M \} \) are the zeros of \( B(u) \) then the \( x_i \) provided a set of separation coordinates and the set of \( P_i \) given by \( P_i = A(x_i) \) are their conjugate momenta (or possibly a function of them). The separated equations are then,

\[
\det (P_i - T(x_i)) = 0.
\]

It has been successfully applied in this \( (2 \times 2) \) case to a wide class of systems. The FBA for \( SL(3) \) was studied by Sklyanin in [3] for the \( SL(3) \)-magnetic chain and the Gaudin model. In this paper he conjectures how the method might be extended to \( SL(N) \). For the magnetic chain the conjecture is as follows (with a similar one for the Gaudin magnet but with canonical Poisson brackets),

**Conjecture 1** There exist functions \( A \) and \( B \) on \( GL(N) \) such that the following two assertions are true. Firstly that \( A(T) \) is an algebraic function and \( B(T) \) is a polynomial of degree \( D = MN(N-1)/2 \) in the matrix elements \( T_{mn} \). Secondly that the variables \( x_j, P_j \ (j = 1, \ldots, D) \) defined from the equations,

\[
B(T(x_i)) = 0, \quad P_j = A(T(x_j))
\]

have Poisson brackets

\[
\{x_j, x_k\} = \{P_j, P_k\} = 0, \quad \{P_j, x_k\} = P_j \delta_{jk}
\]

and, besides, are bound to the Hamiltonians (integrals of motion) by the relations

\[
\det (P_i - T(x_i)) = 0.
\]

The scheme proposed by Sklyanin to obtain \( A \) and \( B \) is to put \( T \) into block triangular form by using a similarity transform

\[
T' = K^{-1}TK
\]

where \( K \) is a matrix depending on some parameters \( k_1, \ldots, k_Q \). (I shall show that \( Q \) need only be \( N-2 \).) The \( k_i \) should then be eliminated from the resulting equations to leave a single equation \( B(u) = 0 \). No similarity transformation is required for \( N = 2 \) and the similarity transformation for \( N = 3 \) was performed by Sklyanin in Ref[3], where he proves the commutation relations for the magnetic chain and Gaudin magnet.
1.3.2 Algebraic Geometric Method

M.R. Adams, J.Harnard and J.Hurtubise \[2\] have used algebraic geometric methods to find separation coordinates for systems that have Lax representations with linear $R$-matrices and Lax matrices of the form,

\[ L(u) = uY + u \sum_{i=1}^{M} \frac{L_i}{u - \alpha_i} \quad (12) \]

with $Y \in gl(n)$ and $\alpha_i$ as complex constants (it also works for multiple poles).

In this approach separation variables are constructed as the generically distinct finite solutions of the equation,

\[ \tilde{M}(u, \zeta)V_0 = 0 \quad (13) \]

where $\tilde{M}$ is the classical adjoint of

\[ M(u, \zeta) := T(u) - \zeta I \quad (14) \]

The solutions of this equation give Darboux coordinates in systems with linear Poisson brackets of a certain form. The solutions of this equation are again bound to the integrals of motion by equation (5). As a defining equation for the separation variables equation (13) has the advantage that (because it can be treated in the language of algebraic geometry) degenerate cases can be handled easily. Perhaps, most importantly, the commutation relations can be straightforwardly calculated.

A disadvantage as compared with Sklyanin’s scheme is that the defining equation involves two parameters (the spectral parameter and an eigenvalue parameter).

1.4 Overview of Paper

In this paper I advance Sklyanin’s program by giving the FBA for $SL(N)$. This is done by constructing polynomials $A$ and $B$ in the matrix elements, so that generically the zeros $x_i$ of $B(u)$ give the separation coordinates and the $P_i = A(x_i)$ provide their conjugate momenta in the cases of the magnetic chain and Gaudin model. This paper deals primarily with the magnetic chain which is reviewed in section (2) (the Gaudin model being dealt with at the end (section (5))).

In section (3) I obtain candidates for $A$ and $B$ for the case of $N \times N$ matrices, in the sense that produce variables that give rise to separated equations, without however any consideration of whether they have the correct commutation relations at this stage. Sklyanin’s program of similarity transforms is used.

In section (4) it is shown that generically $A$ and $B$ give the same separation coordinates as obtained from equation (13), and then this equivalence is used to prove the commutation relations for the magnetic chain. In doing this the crucial calculation in Ref. 2 is extended to show that equation (13) gives separation variables in the case of quadratic Poisson brackets (given by the permutation $R$-matrix, $\frac{P}{u-v}$ see equation (20)).

In section (5) it is shown how the results carry over to the (linear $R$-matrix) case of the Gaudin Model, and to systems with more general $R$-matrices.
\section{The Magnetic Chain}

The variables of the non-homogeneous classical $SL(N)$ magnetic chain are $s^{(m)}_{\alpha \beta}, \alpha, \beta = 1, \ldots, N; m = 1, \ldots, M$ (where $M$ is the length of the spin chain). The variables are not completely independent but a related by $\sum_{\alpha=1}^{N} s^{(m)}_{\alpha \alpha} = 0$. They obey the following Poisson brackets

$$\{s^{(m)}_{\alpha_1 \beta_1}, s^{(m)}_{\alpha_2 \beta_2}\} = \left(s^{(m)}_{\alpha_1 \beta_2} \delta_{\alpha_2 \beta_1} - s^{(m)}_{\alpha_2 \beta_1} \delta_{\alpha_1 \beta_2}\right) \delta_{mn}$$

which define the Kirillov-Kostant Poissonian structure on the direct product of $M$ coadjoint orbits of $SL(N)$. The center of the algebra is generated by the eigenvalues $\lambda^m_\alpha$ of the matrices $s^{(m)}$

$$\det(s^{(m)} - \lambda) = \prod_{\alpha=1}^{N} (\lambda^m_\alpha - \lambda) \cdot \sum_{\alpha=1}^{N} \lambda^m_\alpha = 0$$

I shall fix the orbit, by taking $\lambda^m_\alpha$ to be fixed numbers. Furthermore I shall assume that I have a generic orbit by requiring the eigenvalues of $s^{(m)}$ are distinct. The manifold defined by equations (16) and having dimension $MN(N-1)$, is then equipped with a non-degenerate Poisson bracket (15).

The monodromy matrix may be defined as,

$$T(u) = Z(u - \delta_M + S^{(M)}) \cdots (u - \delta_2 + S^{(2)})(u - \delta_1 + S^{(1)})$$

where $Z$ is an $N \times N$ number matrix with distinct eigenvalues and $\delta_m$ are some fixed numbers, and $u$ is the spectral parameter.

The matrix elements of $T$ are polynomial in $u$ of degree $M$ (length of the magnetic chain). The $T(u)$ obey the following quadratic R-matrix relations,

$$\{T_{\alpha_1 \beta_1}(u), T_{\alpha_2 \beta_2}(v)\} = \frac{1}{u-v}(T_{\alpha_2 \beta_1}(u)T_{\alpha_1 \beta_2}(v) - T_{\alpha_1 \beta_2}(u)T_{\alpha_2 \beta_1}(v))$$

or in the formalism used earlier,

$$\{T(u), T(v)\} = \left[R(u-v), \frac{1}{u} T(u) \frac{1}{v} T(v)\right]$$

$$R(u) = \frac{\mathcal{P}}{u}$$

where $\mathcal{P}$ is the permutation matrix in the tensor product i.e.,

$$R(u) = \frac{1}{u} \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji}$$

where the $e_{ij}$ form the usual basis for $N \times N$ matrices with a 1 in the $ij$th position and zeros elsewhere. The spectral invariants $t_\nu(u)$ of the matrix $T(u)$ may be defined as the elementary symmetric polynomials of its eigenvalues, $t_\nu(u) \equiv \text{tr} \wedge^\nu T(u), \nu = 1, \ldots, N$. $t_N(u) = \det(T(u))$ contains the central elements and is therefore taken to be a constant function of $u$. The non-leading coefficients of the remaining invariants provide a commuting family of $MN(N-1)/2$ independent Hamiltonians (see e.g. [3]), since this is half the dimension of the phase space the system is integrable.
3 $\mathcal{A}, \mathcal{B}$ in the $SL(N)$ case

In this section I use the similarity transformation equations to show that the following are candidates for $\mathcal{A}$ and $\mathcal{B}$, i.e. that the $x_i$ and $P_i$ derived from these equations allow separation (the proof that these variables obey the correct commutation relations is postponed to the next section.)

$$\mathcal{A}(T(u)) = \epsilon_{i_1 \cdots i_{N-1}} T_{i_1 N} T_{i_2 N}^2 \cdots T_{i_{N-2} N}^{N-2} \frac{T_{i_{N-1} N}^{N-1}}{\det(M)}$$  \hspace{1cm} (21)

$$\mathcal{B}(T(u)) = \epsilon_{i_1 \cdots i_{N-1}} T_{i_1 N} T_{i_2 N}^2 \cdots T_{i_{N-2} N}^{N-2} T_{i_{N-1} N}^{(N-1)}$$  \hspace{1cm} (22)

In the case the case of the magnetic chain, where $T(u)$ is a polynomial in $u$ of degree $M$, $\mathcal{B}(u)$ is generically degree $MN(N-1)/2$ insuring that it defines the correct number of separation variables.

In this section indices labeled $i, j, k$ range from 1 to $N-1$, indices labelled $m, n$ range from 2 to $N-2$ and repeated indices are summed over their appropriate ranges.

Let

$$K = \begin{pmatrix}
1 & k_2 & k_3 & \cdots & k_{N-1} & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \cdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1 & 0
\end{pmatrix}.$$  \hspace{1cm} (23)

The requirement that,

$$KTK^{-1} = \begin{pmatrix}
\mathcal{A}(u) & 0 & \cdots & 0 \\
# & \cdots & \cdots & # \\
\vdots & \ddots & \ddots & \vdots \\
# & \cdots & \cdots & #
\end{pmatrix}$$  \hspace{1cm} (24)

where # denotes arbitrary matrix elements, implies that,

$$\mathcal{A}(T(u)) = k_i T_{i1}$$  \hspace{1cm} (25)

and that the $k_i$’s must satisfy the following set of $N-1$ equations,

$$k_i T_{im} - k_m k_i T_{i1} = 0 \quad (m = 2, \ldots, N-1)$$  \hspace{1cm} (26)

$$k_i T_{iN} = 0$$  \hspace{1cm} (27)

where $i$ is summed from 1 to $N-1$. These are $N-1$ equations in $N-2$ unknowns our aim is to eliminate the unknowns and obtain a single consistency equation $\mathcal{B}(T(u))=0$. The first $N-2$ equations are quadratic and only the last is linear, however an equivalent set of linear equations can be obtained from these. Multiplying the equations (26) by $T_{mN}$ and summing from 2 to $N-1$, one obtains,

$$k_j T_{jm} T_{mN} - k_m T_{mN} k_i T_{i1} = 0$$  \hspace{1cm} (28)

and the last linear equation may be used to replace the last term by $k_i T_{i1} T_{1N}$, thus

$$k_j T_{jk} T_{kN} = 0.$$  \hspace{1cm} (29)
Now we have two linear equations, further equations can now be obtained by iterating this procedure (e.g. to obtain the next equation in the series multiply each of the original quadratic equations by $T_{mi}T_{iN}$ and sum them using (29) to eliminate the quadratic terms). Introducing the notation

$$\Sigma_{ik}^{(a)} = T_{ij}\Sigma_{jk}^{(a-1)}, \quad \Sigma_{ik}^{(1)} = T_{ik}$$

the linear equations read,

$$k_i\Sigma_{iN}^{(a)} = 0. \quad (31)$$

The first $N-2$ of these equations are sufficient to determine the $k_i$’s, and may be written in matrix form as,

$$M_{mn}k_n = -\Sigma_{1N}^{(m)} \quad (32)$$

where $M$ is the $(N-2) \times (N-2)$ matrix,

$$M_{mn} = \Sigma_{nN}^{(m-1)}. \quad (33)$$

Hence,

$$\det(M)k_m = N_{mn}\Sigma_{1N}^{(n)} \quad (34)$$

where $N$ is the transpose of the matrix of cofactors. Multiplying the L.H.S. of equation (31) (with $a = N - 1$) by $\det(M)$ and substituting,

$$\det(M)k_i\Sigma_{iN}^{(a)} = \det(M)\Sigma_{1N}^{(N-1)} + N_{mn}\Sigma_{1N}^{(m)}\Sigma_{nN}^{(N-1)}$$

$$= \epsilon_{i1i2\cdots i_{N-1}}\Sigma_{i1N}^{(1)}\Sigma_{i2N}^{(2)}\cdots\Sigma_{i_{N-1}N}^{(N-1)} \quad (35)$$

this is chosen to be $B$. Likewise $A$ is obtained by eliminating the $k_i$’s, giving,

$$A(T(u)) = \epsilon_{i1i2\cdots i_{N-1}}\Sigma_{i1N}^{(1)}\Sigma_{i2N}^{(2)}\cdots\Sigma_{i_{N-1}N}^{(N-2)}\frac{T_{iN}^{(N-1)}}{\det(M)}$$

$$B(T(u)) = \epsilon_{i1i2\cdots i_{N-1}}\Sigma_{i1N}^{(1)}\Sigma_{i2N}^{(2)}\cdots\Sigma_{i_{N-1}N}^{(N-2)}\Sigma_{i_{N-1}N}^{(N-1)} \quad (36)$$

In these expressions each index is summed from 1 to $N-1$ however $A$ and $B$ remain unchanged if the sums are extends to sums from 1 to $N$, thus

$$A(T(u)) = \epsilon_{i1i2\cdots i_{N-1}}T_{i1N}T_{i2N}^2\cdots T_{i_{N-2}N}^{N-2}\frac{T_{iN}^{N-1}}{\det(M)} \quad (37)$$

$$B(T(u)) = \epsilon_{i1i2\cdots i_{N-1}}T_{i1N}T_{i2N}^2\cdots T_{i_{N-2}N}^{N-2}T_{i_{N-1}N}^{(N-1)} \quad (38)$$

To see this notice that

$$\Sigma_{iN}^{(r)} = (T^r)_{iN} + (T^{r-1})_{iN}F^{(1)} + \cdots + T_{iN}F^{(r-1)}$$

and that in $A$ and $B$ these appear within antisymmetric sums so only the first terms contribute.
4 Proof of commutation relations

Summation convention is NOT used in this section.

In this section I prove that the $x_i$ and $P_i$ given by $B(x_i) = 0$ and $P_i = A(x_i)$ have the following commutation relations,

$$\{x_i, x_j\} = \{P_i, P_j\} = 0$$  \hspace{1cm} (42)
$$\{x_i, P_j\} = P_i \delta_{ij} \hspace{1cm} (43)$$

The proof works in two stages first I show that the $(x_i, P_i)$ can be equivalently defined as the solutions of another equation. Then this equation is used to calculate the Poisson brackets.

4.1 Equivalent defining equation

In this subsection I show that (if the $B$ has the correct number of zeros then) then $(x_i, P_i)$ defined by $A$ and $B$ can be equivalently defined as the generically distinct finite solutions of the equation,

$$\tilde{M}(u, \zeta) V_0 = 0$$  \hspace{1cm} (44)

where $\tilde{M}$ is the classical adjoint of

$$M(u, \zeta) := T(u) - \zeta I$$  \hspace{1cm} (45)

It is very natural to think of the separation variables as being defined by these equations since they do give eigenvalues and moreover define the variables in terms of things for which we can calculate Poisson brackets. The solutions of this equation were shown to give Darboux coordinates in systems with linear Poisson brackets [2].

To see this consider,

$$M'_i(\zeta) = K(u_i) M(u_i, \zeta) K^{-1}(u_i) = \begin{pmatrix}
P_i - \zeta & 0 & 0 & \cdots & 0 & 0 \\
\# & \cdots & \cdots & \cdots & \cdots & \# \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\# & \cdots & \cdots & \cdots & \cdots & \# 
\end{pmatrix}$$  \hspace{1cm} (46)

where $\#$ denotes arbitrary matrix elements. Factors of $(P_i - \zeta)$ may be pulled out of all but one of the non-zero cofactors,

$$\tilde{M}'_i(\zeta) = \begin{pmatrix}
\# & (P_i - \zeta)c_i \\
0 & (P_i - \zeta)B_{ij}
\end{pmatrix}$$  \hspace{1cm} (47)

where $c_i$ and $B_{ij}$ are the relevant (determinant) factors that multiply $(P_i - \zeta)$ in each cofactor.

Thus for generic $V_0$ (i.e. one for which $V_0^1 = 0$), $z_i = P_i$ solves the equation,

$$\tilde{M}'(u_i, z_i) V_0 = 0$$  \hspace{1cm} (48)

and hence $\lambda = u_i$ and $\zeta = P_i$ are generic solutions of the equivalent equation

$$\tilde{M}(\lambda, \zeta) V_0 = 0$$  \hspace{1cm} (49)

In the case that $B$ does not give as many solutions as (44) this proof breaks down. Indeed if $B$ does not give the same number of zeros as half the dimension of the phase space then the method can not be applied in its present form. For example the $R$-matrices of Kuznetsov [11] (for the reducible systems of Kalnins et al. [12, 13]) cannot be used to define separation variables in this manner. It would be very satisfying to have a FBA type method that could handle such degenerate $R$-matrices.
4.2 Proof of commutation relations

Since the \( x_i, P_i \) are given as the generic solutions of equation (14) we may choose \( (V_0)_i = \delta_{iN_0} \) where \( N_0 \) is fixed but arbitrary (and could be chosen to be 2, 3, ..., \( N-1 \) or \( N \)). Then \( (x_i, P_i) \) are given by the conditions,

\[
\tilde{M}_{kN_0}(\lambda, \zeta) = 0
\]

(50)

Generically these points are specified by just two of these, choose

\[
\tilde{M}_{1N_0} = \tilde{M}_{2N_0} = 0
\]

(51)

and the matrix

\[
F_{\mu} := \begin{pmatrix}
\frac{\partial \tilde{M}_{1N_0}}{\partial u} & \frac{\partial \tilde{M}_{1N_0}}{\partial \zeta} \\
\frac{\partial \tilde{M}_{2N_0}}{\partial u} & \frac{\partial \tilde{M}_{2N_0}}{\partial \zeta}
\end{pmatrix} (u_\nu, \zeta_\nu)
\]

(52)

is invertible. Hence

\[
\begin{pmatrix}
u_{\mu} u_\mu \\ \zeta_{\nu} u_\mu
\end{pmatrix} \frac{\partial \tilde{M}_{1N_0}}{\partial \zeta_\nu} \begin{pmatrix}
u_{\mu} u_\mu \\ \zeta_{\nu} u_\mu
\end{pmatrix}^{-1}
\]

\[
\begin{pmatrix}
u_{\mu} u_\mu \\ \zeta_{\nu} u_\mu
\end{pmatrix} \begin{pmatrix} F_{\mu}^{-1} \end{pmatrix} \begin{pmatrix}
u_{\mu} u_\mu \\ \zeta_{\nu} u_\mu
\end{pmatrix}
\]

(53)

The Poisson brackets of the adjoint can be calculated from the Poisson brackets of \( M \) by using the derivation property,

\[
\{\tilde{M}_{ij}(u, \zeta), \tilde{M}_{kl}(v, \eta)\} = \sum_{pqrs} \frac{\partial \tilde{M}_{ij}(u, \zeta)}{\partial M_{pq}(u, \zeta)} \frac{\partial \tilde{M}_{kl}(v, \eta)}{\partial M_{rs}(v, \eta)} \{M_{pq}(u, \zeta), M_{rs}(v, \eta)\}
\]

(54)

Now

\[
\{M_{pq}(u, \zeta), M_{rs}(v, \eta)\} = \frac{1}{u-v} (T_{pq}(u)T_{rs}(v) - T_{ps}(u)T_{qr}(v))
\]

(55)

and it follows from \( \tilde{M}(u, \zeta)M(u, \zeta) = \text{det}(M(u, \zeta)) I \) that

\[
\frac{\partial \tilde{M}_{ij}(u, \zeta)}{\partial M_{pq}(u, \zeta)} = \frac{\tilde{M}_{ij}(u, \zeta) \tilde{M}_{pq}(u, \zeta) - \tilde{M}_{ip}(u, \zeta) \tilde{M}_{jq}(u, \zeta)}{\text{det} M(u, \zeta)}
\]

(56)

Substituting these into (54) one obtains,

\[
\{\tilde{M}_{1N_0}, \tilde{M}_{kN_0}\} = \sum_{pqrs} (\delta_{qs} \tilde{M}_{1N_0}(u) - \delta_{is} \tilde{M}_{qN_0}(u)) \left( \delta_{sq} \tilde{M}_{kN_0}(v) - \delta_{kq} \tilde{M}_{sN_0}(v) \right)
\]

\[
- \left( \delta_{tp} \tilde{M}_{iN_0}(u) - \delta_{rN_0} \tilde{M}_{ip}(u) \right) \left( \delta_{ps} \tilde{M}_{kN_0}(v) - \delta_{pq} \tilde{M}_{kN_0}(v) \right) - \delta_{rN_0} \tilde{M}_{ip}(u) \right)
\]

\[
+ \lambda \frac{\partial \tilde{M}_{kN_0}(u)}{\partial M_{pq}(u)} \left( \delta_{ps} \tilde{M}_{kN_0}(v) - \delta_{pq} \tilde{M}_{kN_0}(v) \right) - \delta_{rN_0} \tilde{M}_{ip}(u) \right)
\]

(57)

Thus by taking the appropriate limit with \( \nu \neq \mu \) the left hand side vanishes. (After noticing that \( \tilde{M}_{aN_0}(u_i, \zeta_i) = 0 \) the two remaining terms vanish since \( \frac{\partial \tilde{M}_{aN_0}}{\partial M_{ai}} = 0 \).)

\[
\{\tilde{M}_{1N_0}(u_\nu, \zeta_\nu), \tilde{M}_{1N_0}(v_\mu, \eta_\mu)\} = \{\tilde{M}_{1N_0}(u_\nu, \zeta_\nu), \tilde{M}_{2N_0}(v_\mu, \eta_\mu)\}
\]

\[
= \{\tilde{M}_{2N_0}(u_\nu, \zeta_\nu), \tilde{M}_{2N_0}(v_\mu, \eta_\mu)\} = 0
\]

(58)
Hence
\[ \{ u_\nu, u_\mu \} = \{ \zeta_\nu, \zeta_\mu \} = \{ u_\alpha, \zeta_\mu \} = 0 \quad \nu \neq \mu \] (59)

It remains to calculate the Poisson bracket \( \{ u_\nu, \zeta_\nu \} \). One may use,
\[
\{ M_{pq}(u, \zeta), M_{rs}(u, \zeta) \} = \frac{dT_{rq}}{du} T_{ps} - \frac{dT_{ps}}{du} T_{rq}
\] (60)
Thus
\[
\{ M_{1N_0}(u, \zeta), M_{2N_0}(u, \zeta) \} = \sum_{pqr} \left( \frac{\partial \tilde{M}_{1N_0}}{\partial M_{pq}} \frac{\partial \tilde{M}_{2N_0}}{\partial M_{rs}} - \frac{\partial \tilde{M}_{2N_0}}{\partial M_{pq}} \frac{\partial \tilde{M}_{1N_0}}{\partial M_{rs}} \right) \frac{dM_{ps}}{du} (M_{rq} + \zeta \delta_{rq})
\] (61)
and from equation (56) we see,
\[
\left( \frac{\partial \tilde{M}_{1N_0}}{\partial M_{pq}} \frac{\partial \tilde{M}_{2N_0}}{\partial M_{rs}} - \frac{\partial \tilde{M}_{2N_0}}{\partial M_{pq}} \frac{\partial \tilde{M}_{1N_0}}{\partial M_{rs}} \right) = \frac{1}{(\det M)^2} \left( - \tilde{M}_{1N_0} \tilde{M}_{2N_0} (\tilde{M}_{1N_0} \tilde{M}_{2r} - \tilde{M}_{2N_0} \tilde{M}_{1r}) + \tilde{M}_{sr} \tilde{M}_{qN_0} (\tilde{M}_{1N_0} \tilde{M}_{2p} - \tilde{M}_{2N_0} \tilde{M}_{1p}) + \tilde{M}_{qN_0} \tilde{M}_{sN_0} (\tilde{M}_{1p} \tilde{M}_{2r} - \tilde{M}_{2p} \tilde{M}_{1r}) \right)
\] (62)
The last term in brackets contains three zeros and never gives a contribution. The other terms also vanish when they are multiplied by \( \frac{dM_{ps}}{du} M_{rq} \) and summed over (\( M_{rq} \) may always be multiplied by an \( \tilde{M} \) that is not of the form \( \tilde{M}_{\#N_0} \) thereby cancelling a \( \det M \) in the denominator but leaving two zeros in the numerator). Thus the only contribution comes from the product of the first two terms of (62) and \( \zeta \frac{dM_{ps}}{du} \delta_{rq} \). However this is equal to \( \zeta \det F_\nu \). To see this recall that,
\[
det F_\nu = \sum_{pqr} \left( \frac{\partial \tilde{M}_{1N_0}}{\partial M_{pq}} \frac{\partial \tilde{M}_{2N_0}}{\partial M_{rs}} - \frac{\partial \tilde{M}_{2N_0}}{\partial M_{pq}} \frac{\partial \tilde{M}_{1N_0}}{\partial M_{rs}} \right) \frac{dM_{ps}}{du} \delta_{qs}
\] (63)
Thus explicit calculation shows that,
\[
\left( \frac{\partial \tilde{M}_{1N_0}}{\partial M_{pq}} \frac{\partial \tilde{M}_{2N_0}}{\partial M_{rs}} - \frac{\partial \tilde{M}_{2N_0}}{\partial M_{pq}} \frac{\partial \tilde{M}_{1N_0}}{\partial M_{rs}} \right) \left( \frac{dM_{ps}}{du} \delta_{rq} - \frac{dM_{pr}}{du} \delta_{qs} \right) = \sum_{pqr} \frac{1}{(\det M)^2} \frac{dM_{ps}}{du} \left( \tilde{M}_{1N_0} \tilde{M}_{rN_0} (\tilde{M}_{sr} \tilde{M}_{2p} - \tilde{M}_{sp} \tilde{M}_{2r}) + \tilde{M}_{2N_0} \tilde{M}_{rN_0} (\tilde{M}_{sp} \tilde{M}_{1r} - \tilde{M}_{sr} \tilde{M}_{1p}) + \tilde{M}_{1N_0} \tilde{M}_{sN_0} (\tilde{M}_{rr} \tilde{M}_{2p} - \tilde{M}_{rp} \tilde{M}_{2r}) + \tilde{M}_{2N_0} \tilde{M}_{sN_0} (\tilde{M}_{rp} \tilde{M}_{1r} - \tilde{M}_{1p} \tilde{M}_{rr}) \right) = 0
\] (64)
where the terms in brackets give the required third zeros because they are the subdeterminants of a matrix of rank 1. Hence
\[
\{ M_{1N_0}(u, \zeta), M_{2N_0}(u, \zeta) \} = \zeta \det F_\nu
\] (65)
thus substituting in equation (53) we find
\[
\{ u_\nu, \zeta_\nu \} = \zeta_\nu
\] (66)
as required.
5 The Gaudin Model and other systems

5.1 The Gaudin Model

In the Gaudin model (which may be considered a degenerate case of the magnetic chain) $T(u)$ has the following form,

$$T(u) \equiv Z + \sum_{m=1}^{M} \frac{S^{(m)}}{u - \delta_m}$$

where $S^{(m)}$ obey equation (13) as before. $T(u)$ obeys the following linear $R$-matrix relation,

$$\{T_{\alpha_1\beta_1}(u), T_{\alpha_2\beta_2}(v)\} = \frac{1}{u - v}((T_{\alpha_2\beta_1}(u) - T_{\alpha_2\beta_1}(v))\delta_{\alpha_1\beta_2} + (T_{\alpha_1\beta_2}(v) - T_{\alpha_1\beta_2}(u))\delta_{\alpha_2\beta_1})$$

Once again the spectral invariants contain the integrals of motion. Sklyanin’s conjecture for the Gaudin model is,

Conjecture 2 Let $A$ and $B$ be the same functions on $GL(N)$ as above. Then the variables $x_j$ and $p_j$ defined by the equations,

$$B(T(x_i)) = 0, \quad p_j = A(T(x_j))$$

have canonical Poisson brackets and, besides, are bound to the Hamiltonians by the relation

$$\det (p_i - T(x_i)) = 0$$

This is true generically. To prove this one only needs to show that $x_i$ and $p_j$ have the correct commutation relations. The proof works the same way as the quadratic case and will not be repeated (it can be found in [2]).

5.2 Other systems

In this paper the procedure is illustrated only in the simplest cases of the magnetic chain and the Gaudin model, however the method is much more widely applicable. Since the equation $\hat{M}(u, \zeta)V_0 = 0$ and the calculation used to obtain $A$ and $B$ is independent of the particular $R$-matrix algebra, the separation variables thus obtained satisfy condition (1) of section (1.3) (i.e. give separated equations) automatically.

However it must be checked that they have the correct commutation relations and at present I do not have a general proof of this. Nevertheless it can be proved (by direct calculation) that one obtains the correct commutation relations for systems with $R$-matrices of the following form

$$R = \sum f_{ij}(u)e_{ij} \otimes e_{ji}$$

provided that the $f_{ij}(u) = \frac{1}{u} + O(1)$ as $u \to 0$. With the $e_{ij}$ as in equation (20) (which is clearly of this form). This allows the result to be extended from rational to trigonometric and elliptic $R$-matrices. Applications in these systems and systems with dynamical $R$-matrices will be discussed in a future publication.
6 Conclusion

In this paper I have shown that generically the following polynomials in the $T$ matrix coefficients maybe used to obtain separation variables as illustrated in the case of the magnetic chain and Gaudin model,

$$A(T(u)) = \epsilon_{i_1i_2...i_{N-1}} T_{i_1i_N} T_{i_2i_N}^2 \cdots T_{i_{N-2}i_N}^{N-2} \frac{T_{i_{N-1}i_1}}{\det(M)}$$ (72)

$$B(T(u)) = \epsilon_{i_1i_2...i_{N-1}} T_{i_1i_N} T_{i_2i_N}^2 \cdots T_{i_{N-2}i_N}^{N-2} T_{i_{N-1}i_N}^{(N-1)}$$ (73)

in the sense that separation variables, with canonical Poisson brackets, are given by $B(x_i) = 0$, $e^{p_i} = P_i = A(x_i)$ for the magnetic chain and by $B(x_i) = 0$, $p_i = A(x_i)$ for the Gaudin model.

With $A$ and $B$ in this form it is clear to see that $B$ has an $SL(N-1)$ symmetry corresponding to similarity transform which leave the last row and column fixed, and that $A$ has an $SL(N-2)$ corresponding to those transformations which leave the first and last rows and columns fixed. Clearly in the construction and proof it does not matter which column we choose to call the first and last, and these associated $A$’s and $B$’s also give rise to separation variables. The meaning of the symmetry and non-uniqueness is not yet clear.

As remarked, $B$ only gives the correct number of separation coordinates in the generic case. For non-generic cases the equation

$$\tilde{M}(u, \zeta)V_0 = 0$$ (74)

must be used, it is desirable to extend the FBA to these cases. One would need to provide a method that told one how to decouple equation (14) into an equation(s) for the $x_i$ and an equation(s) for the $p_i$s (in terms of the $x_i$s). The reducible systems of Kalnins et al. [11, 13] provide examples of such problems and the author conjectures that the notion of irreducibility in these systems is related to this problem of non-degeneracy.

The classical Functional Bethe Ansatz in its $2 \times 2$ version has been successfully applied to systems with a wide variety of $R$-matrices including dynamical $R$-matrices, as has its quantum counterpart. It is hoped that the $SL(N)$ FBA proposed here and its quantum counterpart might prove equally useful.

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