Gravitational redshift in void-galaxy cross-correlation function in redshift space

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We construct an analytic model for the void-galaxy cross-correlation function that enables theoretical predictions of the dipole signal produced dominantly by the gravitational redshift within voids for the first time. By extending a theoretical formulation for the redshift-space distortion of galaxies to include the second order terms of the galaxy peculiar velocity and the gravitational potential, we formulate the void-galaxy cross-correlation function multipoles in the redshift space, the monopole $\xi_0^{(v)}$, dipole $\xi_1^{(v)}$ and quadrupole $\xi_2^{(v)}$. We find that the dipole $\xi_1^{(v)}$ is dominated by the gravitational redshift, which provides a unique opportunity to detect the gravitational potential of voids. Thus, for the dipole $\xi_1^{(v)}(s)$, the higher order effect is crucial. Although this effect is negligible on the monopole $\xi_0^{(v)}$, it has an influence on the quadrupole $\xi_2^{(v)}$. The effects from the random velocity of galaxies and the definition of the void center on the dipole signal are also discussed. Our model offers a new theoretical probe for detecting gravitational redshift within voids, and further tests on cosmology and gravity.

I. INTRODUCTION

The large-scale structure of the universe is observed in redshift maps of galaxy redshift surveys such as Sloan Digital Sky Survey (SDSS). However, the mapping of galaxies from real space to redshift space produces statistical anisotropies caused by peculiar velocities relevant to gravitational clustering, which is better known as the redshift-space distortion (RSD) [1,2]. Recent galaxy redshift surveys enabled precise measurements of the RSDs in the galaxy map [3,4], which have been used to constrain important cosmological parameters, such as the linear growth rate of structure $f$, useful for distinguishing between various cosmological models (e.g., [10,12]). Hence, the RSD in the large-scale structure of galaxies is beneficial when testing cosmological models, dark matter models, general relativity and its alternative theories such as modified gravity. Recently, general relativistic effects and higher order effects in the redshift map of galaxies have been investigated [13-21]. These works are extensions of precise modeling of galaxy distributions to include higher order effects of redshift-space distortions and other effects. The gravitational redshift and the second order Doppler effect (from the second order peculiar velocity) in the galaxy clusters is one of such topics [22-27]. Measurements of the gravitational redshift with galaxies associated with clusters have been reported [22-24].

The lowest density areas in the large-scale structure larger than $10h^{-1}$Mpc, i.e., void, which is a characteristic structure in the universe, have become a useful tool for testing cosmological models and gravity theories [28-32]. Precise modeling of voids with redshift-space distortions provides us with an approach for testing cosmological models. Accurate models of galaxy distributions of voids in the linear theory of density perturbations have been developed [28,33,34], and a constraint on the linear growth rate has been obtained by considering RSD with voids [35]. The void-galaxy correlation is utilized to compare galaxy distribution in redshift space inside and around voids with observations. The peculiar velocities of the galaxies are essential for void-galaxy correlation in redshift space, however, these models are usually constructed up to the leading order of peculiar velocities of galaxies [34].

In this report, we investigate possible signatures of the gravitational redshift and the higher order effect of peculiar velocities in the galaxy distribution associated with cosmic voids in redshift space, based on the progress of these previous works. We focus our analysis on void-galaxy cross-correlation, which represents the profile of the galaxy distribution of voids. We develop an analytic theoretical formulation for the void-galaxy cross correlation function in redshift space including higher order effects of the second order terms of the galaxy peculiar and the gravitational potential. Our model provides us with theoretical predictions for the multipoles of the void-galaxy cross-correlation function. In particular, we demonstrate that the dipole component reflects the gravitational redshift of a void structure for the first time. This aspect has been mostly neglected in previous works, which rely on the theory of the first order of the peculiar velocity.

This article is organized as follows: In section II, we formulate the redshift space distortion up to the second order of the peculiar velocity and the gravitational potential. Then, we present an expression for the void-galaxy cross-correlation function in redshift space. The multipoles of the void-galaxy cross-correlation function are defined including the effects of the gravitational potential and the second order of the peculiar velocity of the spherical coherent motion of void, as well as random motions. In section III, we adopt a specific model for the density profile of a spherically symmetric void which allows us to demonstrate the multipoles of the void-galaxy cross-correlation function. In section IV, we discuss the results and the physical reasons for the behavior of the multipoles of the
void-galaxy cross-correlation function. We also discuss the non-trivial issue of how to choose the center of a void and its dependence on the results. Finally, we summarize our results and highlight our conclusions in section V. We also briefly discuss prospective applications of our investigation for a comparison with observations. In the appendix, we present derivations of the mathematical formulas in section III.

II. FORMULATION

We will start the formulation for the galaxy distribution in redshift space associated with a void, including the gravitational redshift and the second order of the peculiar velocity. We follow the theoretical formulation developed in Ref. [26] by beginning with a brief review of the formulation. When there is no effect from the gravitational potential and the peculiar velocity, the relation between the comoving distance $\chi$ and the redshift $z$ is given by

$$\chi = \int_0^z \frac{dz'}{H(z')}.$$  

When there is a shift of the redshift from the gravitational potential and the peculiar velocity, $\delta z$, the distance in redshift space can be expressed as

$$S = \int_0^{z+\delta z} \frac{dz'}{H(z')} \simeq \chi + \frac{\delta z}{H(z)} - \frac{H'(z)}{2H^2(z)} \delta z^2,$$

where we evaluated the shift in the comoving distance by $\delta z$ up to the second order.

To include the shift of a photon’s energy caused by the gravitational potential and the Doppler effect of peculiar velocity, we need to consider the second order terms of the peculiar velocity. We work within the Newtonian gauge of cosmological perturbation theory, however, the results will not depend on this choice because the void relevant to our problem is of the sub-horizon scale. Denoting the gravitational potential and the peculiar velocity by $\psi$ and $v$, respectively, we may express $\delta z$ up to the order of $O(v^2)$ [26],

$$\delta z = (1 + z) \left( \gamma \cdot v + \frac{1}{2} v^2 - \psi \right),$$

where $\gamma$ is the unit vector of the line of sight, and $\gamma \cdot v$ in Eq. (4) denotes the usual Doppler effect, while the term $v^2/2$ does the transverse Doppler effect. Inserting Eq. (3) into Eq. (2), we may write $S$ up to the order of $O(v^2)$ as

$$S = \chi + \frac{(1 + z)}{H(z)} \left( \gamma \cdot v + \frac{1}{2} v^2 + (\gamma \cdot v)^2 - \psi \right) - \frac{H'(z)}{2H^2(z)} (1 + z)^2 (\gamma \cdot v)^2.$$

For convenience, we assign our coordinate system by adopting the plane parallel approximation (distant observer approximation). Following this assumption, the coordinates perpendicular to the line of sight direction is the same, and the position of a galaxy in these directions takes the same value between the redshift space and the real space. However, the position parallel to the line of sight direction shifts, which is specified by Eq. (4). We adopt a coordinate system with its origin at the center of a void, and denote the position of a galaxy as $\vec{r}$ and $\vec{s}$, respectively, in the real space and the redshift space. We denote the comoving distance of the center of a void by $\chi_c = \int_0^{z_c} dz'/H(z')$, where $z_c$ is the redshift of the position of the center in the case where there is no effect of the gravitational redshift or the Doppler effect.

As derived in the appendix, when the center of a void and the origin of the coordinate do not change between the real space and the redshift space, $\vec{r}$ and $\vec{s}$ are related by

$$\vec{s} = \vec{r} + \left[ \frac{\gamma \cdot v}{H(z)} + \frac{1}{2} \frac{v^2}{H(z)} + \frac{(\gamma \cdot v)^2}{H(z)} - \psi \frac{H'(z)}{2H^2(z)} (\gamma \cdot v)^2 \right] \gamma,$$

where we introduced $\mathcal{H} = aH$. It might be worth mentioning that the notations $\gamma$ and $\gamma'$ are identical for the convenience of expressing as $\gamma = \gamma'$.  

In the previous equation, we assumed that the center of a void assumes the same position in real space and redshift space. However, as we will discuss later, the definition of the center of a void is not a trivial problem. We may also consider the case that the center of a void shift in the redshift space, taking the gravitational redshift as $\psi_c(0)/H(z_c)$. In this case the center of the void is located at the distance, $S_c = \int_0^{z_c} dz'/H(z') - \psi_c(0)/H(z_c)$, and the redshift space and the real space are related by

$$\vec{s} = \vec{r} + \left[ \frac{\gamma \cdot v}{H(z)} + \frac{1}{2} \frac{v^2}{H(z)} + \frac{(\gamma \cdot v)^2}{H(z)} - \psi \frac{H'(z)}{2H^2(z)} (\gamma \cdot v)^2 + \frac{\psi_c(0)}{H(z_c)} \right] \gamma.'$$
Note that Eq. (5) is reproduced by setting $\psi_c(0) = 0$ in Eq. (6). In the below section, we present the formulation with Eq. (6).

For convenience, we hereby adopt a convention in our derivation that the notation $x$ without a top arrow denotes the magnitude $|\vec{x}|$ for an arbitrary vector $\vec{x}$. For instance, we have $s = |\vec{s}|$ and $r = |\vec{r}|$. To formulate the void-galaxy cross-correlation $\xi^{(s)}(s)$ in redshift space, we need to use the conservation property between redshift space and real space as follows:

$$1 + \xi^{(s)}(s) = (1 + \xi(r))\det \left| \frac{\partial \vec{r}}{\partial \vec{s}} \right|,$$

where the superscript $(s)$ reminds us that $\xi^{(s)}(s)$ is a quantity in redshift space. However, like all the quantities we will introduce in the following sections, this superscript can be omitted if we remember that these quantities as a function of redshift space separation $s$, are defined in redshift space.

To obtain $\xi^{(s)}(s)$ in Eq. (7), we need to express the physical space quantity $\xi(r)$ and the transformation determinant $\det |\partial \vec{r}/\partial \vec{s}|$ using redshift-space quantities related to the separation $s$ within voids. We start by examining the terms in Eq. (9). To calculate the $\gamma \cdot v$ terms in the expression, we assume that the peculiar velocities of the galaxies associated with the void yield to the cosmological continuity equation as

$$v = -\frac{1}{3} f(z) \mathcal{H}(z) \Delta(r) \vec{r},$$

where the structure linear growth rate $f$ and the average density contrast $\Delta(r)$ of the void within the radius $r$ are involved.

Since the peculiar velocity mainly contributes to RSDs along the line-of-sight direction $\vec{\gamma}$, by defining $r_\parallel \equiv \vec{r} \cdot \vec{\gamma}$, we have

$$\frac{\mathbf{v} \cdot \mathbf{\gamma}}{\mathcal{H}(z)} = -\frac{1}{3} f(z) \Delta(r) r_\parallel \simeq -\frac{1}{3} f(z_c) \Delta(r) r_\parallel \equiv \tilde{V}(z_c, r) r_\parallel,$$

where we defined

$$\tilde{V}(z_c, r) \equiv -\frac{1}{3} f(z_c) \Delta(r).$$

Here, we adopt the approximation $z \simeq z_c$ since for a certain galaxy around the void, it is obvious that $z = z(\chi) = z(\chi_c + r_\parallel)$, where $r_\parallel$ is negligible as a tiny quantity compared with the distance $\chi_c$ from distant observers. For further calculations, we need investigations into the relations for quantities $r$ and $s$ between real space and redshift space. To take the anisotropies related to line-of-sight direction into account, we define the dimensionless parameter $\mu = s_\parallel/s$, which is the cosine of the angle between $\vec{s}$ and $\vec{\gamma}$ in redshift space. Then the component that is parallel to $\vec{\gamma}$ and perpendicular to $\vec{\gamma}$ are given as:

$$s_\parallel = s \mu, \quad s_\perp = s \sqrt{1 - \mu^2},$$

respectively. On the other hand, the relation for parallel and perpendicular components between redshift space and real space can be written as:

$$s_\parallel = r_\parallel + \delta r_\parallel, \quad s_\perp = r_\perp,$$

where $\delta r_\parallel$ denotes the shift on $r_\parallel$ in the redshift coordinate along the line-of-sight direction caused by the redshift space distortions, i.e.,

$$\delta r_\parallel = \frac{\mathbf{v} \cdot \mathbf{\gamma}}{\mathcal{H}(z)} + \frac{1}{2} \frac{v^2}{\mathcal{H}(z)} + \frac{(\mathbf{\gamma} \cdot \mathbf{v})^2}{\mathcal{H}(z)} - \psi\frac{\mathcal{H}'(z)}{2\mathcal{H}^2(z)}(\mathbf{\gamma} \cdot \mathbf{v})^2 + \psi_c(0)\frac{1}{\mathcal{H}(z_c)}.$$

Using Eqs. (6) and (9) we can express $\delta r_\parallel$ as a function of $\tilde{V}(z_c, r)$, such that:

$$\delta r_\parallel = \tilde{V}(z_c, r) r_\parallel + \frac{1}{2} \mathcal{H}(z_c) \tilde{V}^2(z_c, r) r_\parallel^2 + \mathcal{H}(z_c) \tilde{V}^2(z_c, r) r_\parallel^2 - \psi(r)\frac{\mathcal{H}'(z_c)}{2\mathcal{H}(z_c)} - \frac{H'(z_c)}{2} \tilde{V}^2(z_c, r) r_\parallel^2 + \psi_c(0)\frac{1}{\mathcal{H}(z_c)}.$$

In the previous expression of $\delta r_\parallel$, except for $\tilde{V}(z_c, r) r_\parallel \sim \mathcal{O}(v)$, all the other terms are of the order $\mathcal{O}(v^2)$, which is supposed to be the same order of $\psi$. This assumption is broadly used in following derivations.
Since the leading term for $\delta r_\parallel$ is $\tilde{V}(z_c, r_\parallel) r_\parallel \sim \mathcal{O}(v)$, it follows that terms up to the order of $\mathcal{O}(\delta r_\parallel^2)$ are sufficient to contain all $\mathcal{O}(v^2)$ terms, thus we express $r$ up to the order of $\mathcal{O}(v^2)$ as:

$$r = \sqrt{r_\parallel^2 + r_\perp^2} = \sqrt{(s_i - \delta s_i)^2 + r_\perp^2} = \sqrt{s^2 - 2s_i \delta r_\parallel + \delta r_\parallel^2} \approx s - \mu \delta r_\parallel + \frac{\delta r_\parallel^2}{2s}(1 - \mu^2),$$

(14)
together with the direct transformation from Eq. (11),

$$r_\perp = s_i - \delta r_\parallel.$$  

(15)

Using Eqs. (14) and (16) and keeping terms up to the order of $\mathcal{O}(v^2)$ equivalent to $\mathcal{O}(\tilde{V}^2)$, we write

$$\tilde{V}(z_c, r_\parallel) r_\parallel = \tilde{V}(z_c, s) s_i - \tilde{V}(z_c, s)^2 s_i - (\tilde{V}(z_c, s) \tilde{V}'(z_c, s)/s)s_i^3,$$

(16)

where we use $V' \equiv \partial V/\partial s$ as a convention. Inserting Eqs. (14) and (16) into the expression for $\delta r_\parallel$ in Eq. (13), up to the order of $\mathcal{O}(v^2)$, we finally have $\delta r_\parallel$ as the function of redshift quantities $s$ and $s_i$ as:

$$\delta r_\parallel = \frac{\psi(0)}{H(z_c)} + \tilde{V}(z_c, s)s_i - \tilde{V}(z_c, s)^2 s_i - (\tilde{V}(z_c, s) \tilde{V}'(z_c, s)/s)s_i^3$$

$$+ \frac{1}{2} H(z_c) \tilde{V}^2(z_c, s)s^2 + H(z_c) \tilde{V}^2(z_c, s)s^2 - \frac{\psi(s)}{H(z_c)} - \frac{H'(z_c)}{2} \tilde{V}^2(z_c, s)s_i^2.$$  

(17)

Thus, Eq. (15) with Eq. (17) give the complete relation between $r_\parallel$ and $s_i$, together with Eq. (10). Replacing $s_\parallel$ with $\mu$ and $s$ using the relation $\partial s/\partial s_\parallel = \mu$, we can finally calculate the coordinate transformation between physical space and redshift space as a function of the redshift-space quantities $\mu$ and $s$ as follows:

$$\det \left| \frac{\partial r_\parallel}{\partial s} \right| = \frac{\partial r_\parallel}{\partial s_i} \frac{\partial s_i}{\partial s} \approx 1 - \tilde{V}'(z_c, s)s\mu^2 - \tilde{V}(z_c, s) + \tilde{V}(z_c, s)^2 s\mu^2 + \tilde{V}^2(z_c, s) + (\tilde{V}(z_c, s) \tilde{V}'(z_c, s)/s)s^3\mu^4$$

$$+ 3(\tilde{V}(z_c, s) \tilde{V}'(z_c, s)/s)s^2\mu^2 - \frac{1}{2} H(z_c) (\tilde{V}^2(z_c, s)s^2)\mu + \frac{\psi(s)}{H(z_c)} \mu$$

$$+ \left\{ -H(z_c) \tilde{V}^2(z_c, s) + \frac{H'(z_c)}{2} \tilde{V}^2(z_c, s) \right\} s^2 \mu^3 + 2 \left\{ -H(z_c) \tilde{V}^2(z_c, s) + \frac{H'(z_c)}{2} \tilde{V}^2(z_c, s) \right\} s \mu.$$  

(18)

On the other hand, using the relation (14), we can expand $\xi(r)$ around $s$ for a small $\delta s$ still keeping terms up to $\mathcal{O}(v^2)$ as:

$$\xi(r) \approx \xi(s) - \xi'(s)\mu \left( \frac{\psi(0)}{H(z_c)} + \tilde{V}(z_c, s)s\mu - \tilde{V}(z_c, s)^2 s\mu - (\tilde{V}(z_c, s) \tilde{V}'(z_c, s))s^2\mu^3 + \frac{1}{2} H(z_c) \tilde{V}^2(z_c, s)s^2 \right.$$

$$+ H(z_c) \tilde{V}^2(z_c, s)s^2\mu^2 - \frac{\psi(s)}{H(z_c)} - \frac{H'(z_c)}{2} \tilde{V}^2(z_c, s)s^2\mu^2 \left. + \xi'(s) \right\} \frac{1}{2} (1 - \mu^2) s \mu (\tilde{V}(z_c, s)^2 + \frac{1}{2} \xi''(s)s \mu^2 (\tilde{V}(z_c, s)s\mu)^2.$$  

(19)

Having calculated $\xi(r)$ with respect to $s$ and $\mu$ in Eq. (19) and the determinant for coordinate transformation in Eq. (18), we eventually determine the redshift-space correlation function $\xi(s, \mu)$ through the conservation property Eq. (7), as:

$$\xi(s, \mu) = -1 + (1 + \xi(s)) \left\{ 1 - \tilde{V} + \tilde{V}^2 + \left[ (H' - 3\mathcal{H}) \tilde{V} + \mathcal{H} \tilde{V}'s \right] s\mu + (5\tilde{V} \tilde{V}' - \tilde{V})s\mu^2 \right.$$

$$(H' - 2\mathcal{H}) \tilde{V} \tilde{V}'s^3 + (-\tilde{V}' + \tilde{V}'r + \tilde{V}'s) s^2 \mu^2 + \tilde{V} \tilde{V}'s^2 s\mu^3 \right.$$  

$$+ \frac{1}{2} (5\tilde{V} - 2) \tilde{V} s^2 \mu^2 + \frac{1}{2} (H' - 2\mathcal{H}) \tilde{V} \tilde{V}'s^2 s^2 \mu^3 + (-\frac{1}{2} \tilde{V} + 2\tilde{V}'s) s^2 \mu^4 \right.$$  

$$+ \frac{1}{2} \tilde{V} s^2 \mu^4 \xi''(s).$$  

(20)

For more details of the derivation process from Eq. (4) to Eq. (20), please refer to Appendix A.
where we defined $\psi$ that only the dipole is influenced by the gravitational potential among the multipoles up to account for $\xi$. For simplicity, we adopt the following expression to take the random motions of galaxies with velocity dispersion into $\xi$ representing contribution from the velocity component also influenced by the second order terms of the velocity, hence it is reasonable to decompose the dipole into terms.

Furthermore, for convenience, we divide $\xi_{1\ell}^{(s)}$ into two terms as

$$\xi_{1\ell}^{(s)}(s) = \xi_{1\ell\psi}^{(s)}(s) + \xi_{1\ell v}^{(s)}(s),$$

where we defined

$$\xi_{1\ell\psi}^{(s)}(s) = \frac{\psi'(s)}{3H} (1 + \xi(s)) + \frac{\psi(s) - \psi_c(0)}{3H} \xi'(s),$$

$$\xi_{1\ell v}^{(s)}(s) = (1 + \xi(s)) \left( \frac{H' - 3H}{3} \bar{V}^2 s + \frac{3H' - 11H}{15} \bar{V}' \bar{V}'s^2 \right) + \frac{3H' - 11H}{30} \bar{V}'^2 s^2 \xi'(s).$$

Furthermore, for convenience, we divide $\xi_{1\ell\psi}^{(s)}$ into two terms as

$$\xi_{1\ell\psi}^{(s)}(s) = \xi_{1\ell\psi 0}^{(s)}(s) + \xi_{1\ell\psi 1}^{(s)}(s),$$

with

$$\xi_{1\ell\psi 0}^{(s)}(s) = \frac{\psi(s) - \psi_c(0)}{3H} \xi'(s),$$

$$\xi_{1\ell\psi 1}^{(s)}(s) = \frac{\psi'(s)}{3H} (1 + \xi(s)).$$

In the previous derivation, the velocity field is supposed to be a coherent field which follows from the linear theory of density perturbations. Random motions from the nonlinear effects, which causes the Fingers of God (FOG) effect could also be important. We may include the effect from the random velocities by following the Gaussian streaming model. This model re-maps the galaxy distribution through a Gaussian random process with velocity dispersion $\sigma_v$. For simplicity, we adopt the following expression to take the random motions of galaxies with velocity dispersion into account for $\xi^{(s)}(s, \mu)$ of Eq. (24)

$$1 + \xi^\sigma(s, \mu) = \int \frac{1 + \xi^{(s)}(s', \mu')}{\sqrt{2\pi}\sigma_v} \exp \left( -\frac{(v_\parallel)^2}{2\sigma_v^2} \right) dv_\parallel,$$
| Parameter | Value | Remark                   |
|-----------|-------|--------------------------|
| $z$       | 0.5   | overall redshift         |
| $b$       | 2     | galaxy linear bias       |
| $\gamma$  | 0.55  | growth index for growth rate |
| $\Omega_m$| 0.3   | matter density parameter |
| $\Delta_c$| −0.4  | unbiased void central density |
| $\alpha$  | 3     | void shape parameter     |

**TABLE I:** Values for the parameters in our demonstration. These values were used unless otherwise expressly stated.

with

$$s^\sigma = \sqrt{s_\perp^2 + (s_\parallel - \frac{v_\parallel}{aH})^2},$$

$$\mu^\sigma = \frac{s_\parallel - \frac{v_\parallel}{aH}}{s^\sigma} \sqrt{s_\perp^2 + (s_\parallel - \frac{v_\parallel}{aH})^2},$$

(32)

where the upper index $\sigma$ takes quantities with a random velocity $v_\parallel$ into account, and the transformed quantity $\xi^\sigma(s, \mu)$ naturally contains the effect from the random velocity with the velocity dispersion $\sigma_\parallel^2$.

With the aforementioned transformation, we can define the RSD multipoles with velocity dispersion similar to Eq. (21) as

$$\xi^\sigma_\ell(s) = \frac{1}{2} \int_{-1}^{+1} \xi^\sigma(s, \mu) P_\ell(\mu) d\mu.$$  (33)

For simplicity, the superscript $(s)$ in Eq. (31) and Eq. (33) is omitted since it is well-understood that our model for void RSD is constructed in redshift space.

**III. ANALYSIS ON SIMPLE MODEL FOR VOID**

In Sec. II, we have established the theoretical formulation for the void-galaxy cross-correlation function in redshift space and its multipoles, including the second order terms of the peculiar velocity as well as the gravitational potential. In this section, we demonstrate the behavior of the multipoles by adopting a specific model for the void density profile. To this end, we adopt the simplest model of a spherical void density profile proposed in Ref. [32], which assumes the integrated density contrast of matter in the form

$$\Delta(r) = \Delta_c e^{-(r/r_v)\alpha},$$

(34)

where $\Delta$, $r_v$, and $\alpha$ are the parameters, $\Delta_c$ specifies the amplitude of the matter density contrast, $r_v$ is the characteristic radius, and $\alpha$ characterizes the steepness of the void wall. To include the linear galaxy bias in our model, specifically $b = 2$, we adopt $\Delta_c = -0.4$ and $\alpha = 3$ (see Table I). Here $\Delta(r)$ is related to the matter density contrast $\delta(r)$ and the gravitational potential $\psi$ by relation:

$$\Delta(r) = \frac{3}{r^3} \int_0^r dr' r'^2 \delta(r'),$$

(35)

$$\Delta\psi(r) = 4\pi G \bar{\rho}_m(a) \alpha \delta(r),$$

(36)

where $\bar{\rho}_m(a)$ is the background matter density. Assuming a spatially flat cosmology with a cosmological constant, we may write $\bar{\rho}_m(a) = 3H_0^2\Omega_m/(8\pi G a^3)$, where $\Omega_m$ is the matter density parameter and $H_0$ is the Hubble parameter at the present epoch. Then, the density contrast and the gravitational potential for the void are given by:

$$\delta(r) = \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^3 \Delta(r)}{3} \right) = \Delta_c \left( 1 - \frac{\alpha}{3} \left( \frac{r}{r_v} \right)^\alpha \right) e^{-(r/r_v)\alpha},$$

(37)

$$\psi(r) = -\frac{3\Omega_m}{2a} H_0^2 \int_r^\infty dr' r' \frac{\Delta(r')}{3} = -\frac{H_0^2 r_v^2}{2} \frac{\Omega_m \Delta_c}{\alpha a} \Gamma(2/\alpha, (r/r_v)\alpha),$$

(38)
FIG. 1: Monopole $\xi_0(=\xi_{0}^{(s)})$ and $\xi_0^\sigma$ as a function of $s/r_v$. The solid curve plots $\xi_0$, Eq. (24), while the symbols show $\xi_0^\sigma$, Eq. (33), with the different values of the velocity dispersion $\sigma_v$, whose values are noted in the figure. The dashed curve, which nearly overlaps with the solid curve, is $\xi_0$ including only the term up to $O(v)$.

where $\Gamma(z,a)$ is the incomplete Gamma function.

By solving the continuity equation, the peculiar velocity of the radial direction can be written (see, e.g., Ref. [32]) as

$$v(r) = -Hr\Delta(r)\frac{f(a)}{3},$$

(39)

where $f(a) = d\ln D_1(a)/d\ln a$ is the linear growth rate defined by the logarithmic differentiation with respect to the cosmological scale factor $a$, which is approximately written as $f(a) = [\Omega_m(a)]^{\gamma}$ with $\Omega_m(a) = a^{-3}\Omega_m/(a^{-3}\Omega_m + 1 - \Omega_m)$ and $\gamma = 0.55$. We here assume that the coherent peculiar velocity of the galaxies follows Eq. (39).

Based on the modeling and parameterization above, we are now able to evaluate the void-galaxy cross-correlation function multipoles. Figs. 1 and 2 plot the monopole component and the quadrupole component as functions of the dimensionless characteristic separation $s/r_v$, in correspondence to Eq. (22) and Eq. (23). In each figure, the solid
curve shows the result including the contribution from the terms up to $O(v^2)$ and the gravitational potential, while the dashed curve shows the result for the terms up to $O(v)$. These two curves overlap in Fig. 4 while the two curves are deviated in Fig. 2. This means that the calculation up to $O(v)$ is sufficient to predict the monopole component and that the higher order terms slightly affect the prediction of the quadrupole component. The symbols in Figs. 4 and 2 represent a plot of the case which includes the random velocity, Eq. (31). They demonstrate that the effect of the random motion is an important factor for the quadrupole component. The monopole and the dipole are irrelevant to gravitational potential terms in our model, thus we may concentrate mainly on the analysis for the dipole signal.

Fig. 3 plots the dipole component as a function of $s/r_v$. The solid curve is the prediction without the random velocity, Eq. (24), while the symbols show the case including the contribution from the random velocity Eq. (31). For the dipole component, there is no contribution from the order of $O(v)$. Figure 4 compares the details of the dipole component. $\xi_{1\psi}^{(s)}$ is the contribution from the second order terms of the velocity, and $\psi_{1\psi}$ and $\xi_{1\psi}^{(s)}$ are the contributions from the gravitational potential and the radial differentiation respectively, which are defined with Eqs. (28) and (25).

This figure shows that $\xi_{1\psi}^{(s)}$ is a minor contribution to the dipole and that the dipole is dominated by the contribution from the gravitational potential. Furthermore, this figure shows that $\xi_{1\psi}$ arises from the terms $\psi(s)$ and $\psi'(s)$, which make contributions to the total result of the dipole at the same level.

Finally, we highlight the importance of the choice of the center of a void in this section. The details will be discussed later and indicate that the choice for the center of a void is not trivial for the dipole signal. In Figs. 3 and 4, we showed the case where the gravitational redshift of the center of a void $\psi_c(0)$ is non-zero. This is the case when the shift of the center of a void through the gravitational redshift is included. However, it depends on the strategy employed to define the void center. For example, when the center of a void is determined using galaxies far from the central region, the position of the center is determined without the information $\psi_c(0)$, irrespective of the gravitational redshift of $\psi_c(0)$. In such a case, we should assume $\psi_c(0) = 0$. Figure 5 compares the case with/without the term $\psi_c(0)/H(z_c)$. Thus, the definition of the center of a void shifts the center, and the dipole signal depends on the choice as well as the strategy of defining the center, although the monopole and the quadrupole do not depend on this choice.

IV. DISCUSSIONS

In the previous section, we demonstrated how the higher order terms of the peculiar velocity and the gravitational potential influences the void-galaxy correlation functions through redshift space distortions. The higher order effect is not very significant for the monopole and the quadrupole components. However, the most interesting finding from the higher order effects is that the dipole component in the void-galaxy correlation functions, dominantly reflects the gravitational potential influences the void-galaxy correlation functions through redshift space distortions. The higher order effect is the origin of the dipole signal in the void-galaxy correlation function. Most importantly, the dipole signal is therefore determined by the gravitational potential profile of the void. However, the dipole signal is not simply dominated by the gravitational potential $\psi(r)$. The contribution of the gravitational redshift to the dipole, Eq. (22), is given by the combination of the two terms,

$$\xi_{1\psi}^{(s)}(s) \approx \xi_{1\psi}^{(s)}(s) = \xi_{1\psi0}(s) + \xi_{1\psi1}(s) = \frac{\psi'(s)}{3H(1 + \psi(s))} + \frac{\psi(s) - \psi_c(0)}{3H} \xi'(s),$$

(40)

Namely the terms from the gravitational potential $\psi(s)$ and its gradient $\psi'(s)$ contribute equally to the dipole signal, as is demonstrated in Fig. 4.

Measurements of the gravitational potential $\psi(s)$ provides a unique chance to test general relativity and other gravity theories. Theories kept in the linear regime for low density regions associated with voids will make such tests less difficult compared to those that require nonlinear clustering of the galaxies. To be a useful tool for testing gravity, the robustness of theoretical predictions is necessary. This will be discussed in the latter part of this section, although we restrict ourselves to the predictions based on general relativity, and discuss aspects of the robustness of our theoretical predictions.

We first discuss the choice of the center of a void in detail. The influences of $\psi_c(0)$ on the dipole component is shown in Fig. 5 which demonstrates how the definition of the center of a void changes the dipole signal. This causes a difficulty in the comparison of our results with observations. The choice of the center of a void depends on an algorithm used in data analysis. In a realistic analysis of void-finding algorithms, there are two algorithms best suited to define the center of a void. One is the lowest-density center, which is determined as the circumcenter of the lowest density galaxy in a void and its three most adjacent neighboring galaxies. This algorithm is equivalent to finding the largest empty sphere that can be found within the void. Another is the volume-weighted barycenter which is mainly used for irregular-shaped voids (Ref. 38). Roughly, the first definition of the center by the minimum density center
1. The scope of the present paper.

2. For finding the center of a void with mock catalogs of galaxies from numerical simulations, although this is beyond the outskirt region of a void. We need to investigate the behavior of the dipole signal by adopting the realistic algorithm.

3. Figure 3 shows the case with the non-zero value of $\xi$. The robustness of our prediction of the results against the dependence on the model parameters. In Fig. 6, the dipole signal corresponds to the case including $\phi_c(0)$, while the second volume-weighted barycenter might correspond to the case without $\phi_c(0)$. This is because $\phi_c(0)$ should be included when the center of a void is determined by the information of the central region, while $\phi_c(0)$ should not be included when the center is determined by the information of the outskirt region of a void. We need to investigate the behavior of the dipole signal by adopting the realistic algorithm for finding the center of a void with mock catalogs of galaxies from numerical simulations, although this is beyond the scope of the present paper.

In the previous section, we fixed the model parameters of a void (see Table I). It will therefore be useful to check the robustness of our prediction of the results against the dependence on the model parameters. In Fig. 6, the dipole signal $\xi_1^\phi$ is plotted by fixing the velocity dispersion $\sigma_v = 450$ km/s, by varying the set of parameters as, $(\alpha = 3, z = 0.5)$, $(\alpha = 2, z = 0.5)$, $(\alpha = 4, z = 0.5)$ and $(\alpha = 3, z = 1)$. We see that the dipole signal changes with the different values of $\alpha$, which characterizes the steepness of a void potential wall, and also the value of the redshift $z$, which corresponds to different cosmological epoch. As the void potential becomes steeper, the gravitational potential takes a larger amplitude inside a void in the general $\psi_c(0) \neq 0$ case. In the cold dark matter model with a cosmological constant, the gravitational potential decreases in proportion to $D_1(a)/a$. These properties explain the behavior in Fig. 6. However, our conclusions are not qualitatively altered by the choice of these parameters.
In this paper, we have presented an analytic model for the void-galaxy correlation function in redshift space including the higher order terms of the peculiar velocity and the gravitational potential through redshift space distortions. By adopting a simple specific model for a void density profile, we have quantitatively demonstrated the influence of the higher order terms on the multipole components of the void-galaxy correlation. In particular, we have found that the dipole signal dominantly reflects the gravitational potential through the gravitational redshift. Our conclusion is qualitatively robust against the change of the model parameters. However, we have also discussed the possible dependence of the dipole signal on the algorithm for determining the center of a void. This dependence should be investigated with the use of numerical simulations with mock catalogs by adopting a practical algorithm to determine the center of a void, including other systematic errors. This is left as a future investigation. However, in principle, our finding presents the possibility of a new approach to direct measurements of the gravitational potential of voids.

In ref. [28], the monopole and the quadrupole multipoles of the void-galaxy cross-correlation function were measured with the SDSS III LOWZ sample and the CMASS sample. The error bars of the multipoles are quite small and we may therefore detect the dipole component in future analysis. Such an analysis would help in the understanding of voids and provide a test for general relativity and cosmological models in combination with observations using other

FIG. 5: Same as Fig. 3 but with the added cases under the condition \( \psi_v(0) = 0 \), for comparison.

FIG. 6: Dependence of the dipole \( \xi_1 \) on different model parameters with fixed velocity dispersion \( \sigma_v = 450 \text{ km/s} \), and different set of model parameters \( (\alpha = 3, z = 0.5), (\alpha = 2, z = 0.5), (\alpha = 4, z = 0.5) \) and \( (\alpha = 3, z = 0.1) \). Both cases \( \psi_v(0) \neq 0 \) and \( \psi_v(0) = 0 \) are shown. All the curves treat univariate parameter changes compared to the original parameter set in Table I.

V. SUMMARY AND CONCLUSION
methods, e.g., measurement of the thermal Sunyaev-Zel’dovich effect around voids and the weak lensing measurements of voids [39].

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Appendix A: Complementary Derivations for Eq. 20

We will present some details for derivations in Sec. II. Since we adopted the plane parallel approximation, we will assign the three dimensional coordinates $\vec{S}$ and $\vec{R}$ in redshift space and real space, respectively, as

$$\vec{S} = S \vec{\gamma} + \vec{x}_\perp,$$

$$\vec{R} = \chi \vec{\gamma} + \vec{x}_\perp,$$

(A.1)
where $\vec{r}_\perp$ is the coordinate perpendicular to the line of sight direction. Denoting the position of the center of a void as $\vec{R}_c = R_c \vec{\gamma}$ and $\vec{S}_c = S_c \vec{\gamma}$ in real space and the redshift space, respectively we have

$$\vec{s} + \vec{S}_c = \vec{r} + \vec{R}_c + \left[ \frac{(1+z)}{H(z)} \left( \begin{array}{c} \gamma \cdot \vec{v} + \frac{1}{2} v^2 + (\gamma \cdot \vec{v})^2 - \frac{H'(z)}{2H(z)^2} (1+z)^2 (\gamma \cdot \vec{v})^2 \end{array} \right) \right] \vec{\gamma},$$

(A.2)

from Eq. (3), where we adopted the coordinate system with its origin at the center of a void $\vec{r}$ and $\vec{s}$ in the real space and the redshift space, respectively.

When we choose $S_c = R_c = \chi_c = \int_0^{z_c} dz'/H(z')$, we have

$$\vec{s} = \vec{r} + \left[ \frac{(1+z)}{H(z)} \left( \gamma \cdot \vec{v} + \frac{1}{2} v^2 + (\gamma \cdot \vec{v})^2 - \frac{H'(z)}{2H(z)^2} (1+z)^2 (\gamma \cdot \vec{v})^2 \right) \right] \vec{\gamma}.$$  

(A.3)

Introducing the conformal Hubble parameter $H = aH$ and using the relation $a = 1/(1+z)$ will slightly simplify the verbose expression as: Eq. (5) that can be expressed as:

$$\vec{s} = \vec{r} + \left[ \gamma \cdot \vec{v} + \frac{1}{2} v^2 + (\gamma \cdot \vec{v})^2 - \frac{H'(z)}{2H(z)^2} (\gamma \cdot \vec{v})^2 \right] \vec{\gamma}. $$

(A.4)

This is for the case of the center of a void and the origin of the coordinate does not change between the real space and the redshift space.

When we choose $S_c + \psi_c(0)/H(z_c) = R_c = \chi_c = \int_0^{z_c} dz'/H(z')$, we have

$$\vec{s} = \vec{r} + \left[ \frac{\gamma \cdot \vec{v}}{H(z)} + \frac{1}{2} v^2 + \frac{\gamma \cdot \vec{v}^2}{H(z)} - \frac{\psi}{H(z)} - \frac{H'(z)}{2H(z)^2} (\gamma \cdot \vec{v})^2 + \frac{\psi(0)}{H(z_c)} \right] \vec{\gamma},$$

(A.5)

This is for the case when the center of the void and the origin of the coordinate shift from the real space to the redshift space.

Note that Eq. (A.4) is reproduced by setting $\psi_c(0) = 0$ in Eq. (A.5). Then we present the formulation with Eq. (A.5) in the following section.

Then the component parallel to $\vec{\gamma}$ in Eq. (3) is in fact

$$s_\parallel = r_\parallel + \left[ \gamma \cdot \vec{v} \frac{1}{H(z)} + \frac{1}{2} \frac{v^2}{H(z)} + \frac{\gamma \cdot \vec{v}^2}{H(z)} - \frac{\psi}{H(z)} - \frac{H'(z)}{2H(z)^2} (\gamma \cdot \vec{v})^2 + \frac{\psi(0)}{H(z_c)} \right].$$

(A.6)

Comparing this with the first line of Eq. (11), where $s_\parallel = r_\parallel + \delta r_\parallel$, we get Eq. (12) for $\delta r_\parallel$.

Substituting the dimensionless quantity we defined in Eq. (9) into Eq. (12), we write Eq. (13) as:

$$\delta r_\parallel = \tilde{V}(z_c, r) r_\parallel + \frac{1}{2} H(z_c) \tilde{V}^2(z_c, r) r_\parallel + \tilde{H}(z_c) \tilde{V}^2(z_c, r) r_\parallel^2 - \frac{H'(z)}{2} \tilde{V}^2(z_c, r) r_\parallel^2 + \frac{\psi(0)}{H(z_c)} - \frac{\psi(r)}{H(z_c)}.$$  

In $\delta r_\parallel$ in the previous derivation, all terms are $O(v^2)$ terms except $\tilde{V}(z_c, r) r_\parallel \sim O(v)$, thus we can write

$$\delta r_\parallel = \tilde{V}(z_c, r) r_\parallel + O(v^2),$$

(A.7)

which will be convenient in later derivations.

As an example of how we keep terms up to the order of $O(v^2)$, we now consider the $\tilde{V}(z_c, r) r_\parallel$ term which frequently appears in the expression for $\delta r_\parallel$ as Eq. (13). Remembering Eq. (A.7) will be very helpful in the process of keeping terms up to the order of $O(v^2) \sim O(V^2)$ in the following section.

Using Eq. (14) for $r$ and Eq. (15) for $r_\parallel$ together with Eq. (A.7) repeatedly, we write

$$\tilde{V}(z_c, r) r_\parallel = \tilde{V}(z_c, s - \mu \delta r_\parallel)(s_\parallel - \delta r_\parallel)$$

$$\simeq \left( \tilde{V}(z_c, s) - \mu \delta r_\parallel \tilde{V}'(z_c, s) \right) (s_\parallel - \tilde{V}(z_c, r) r_\parallel + O(v^2))$$

$$\simeq \tilde{V}(z_c, s) s_\parallel - \tilde{V}(z_c, s) \tilde{V}(z_c, r) r_\parallel - \mu s_\parallel \tilde{V}'(z_c, s) \delta r_\parallel$$

$$\simeq \tilde{V}(z_c, s) s_\parallel - \tilde{V}(z_c, s) \tilde{V}(z_c, s - \mu \delta r_\parallel)(s_\parallel - \delta r_\parallel) - \mu s_\parallel \tilde{V}'(z_c, s) \tilde{V}(z_c, r) r_\parallel.$$

(A.8)
In the above expression, noticing that Eq. (A.7) again, we further write for the second term in the expression above as
\[ \tilde{V}(z_c, s)\tilde{V}(z_c, s - \mu \delta r_i) (s_i - \delta r_i) \simeq \tilde{V}(z_c, s)\tilde{V}(z_c, s - \mu \delta r_i) (s_i - \mu \tilde{V}(z_c, r)_i) \]
\[ \simeq \tilde{V}(z_c, s) (\tilde{V}(z_c, s) - \mu \delta r_i \tilde{V}'(z_c, s)) s_i + \mathcal{O}(v^3) \]
\[ \simeq \tilde{V}(z_c, s)^2 s_i. \]

Also, for the third term in Eq. (A.8), using the definition for \( \mu \) in Eq. (10), transforming \( \tilde{V}(z_c, r) r_i \) similarly to Eq. (A.8) will lead to:
\[ \mu s \tilde{V}'(z_c, s)\tilde{V}(z_c, s) r_i \simeq \frac{s^2}{s} \tilde{V}'(z_c, s)\tilde{V}(z_c, s) s_i + \mathcal{O}(v^3) \]
\[ \simeq \frac{s^3}{s} \tilde{V}'(z_c, s)\tilde{V}(z_c, s). \] (A.9)

Inserting Eqs. (A.9) and (A.10) back into Eq. (A.8) will lead us to Eq. (16).

On the other hand, relation give as:
\[ r = s - \mu \left( \frac{\psi_c(0)}{\mathcal{H}(z_c)} + \tilde{V}(z_c, s) s_i - \tilde{V}^2(z_c, s) s_i - \frac{\tilde{V}(z_c, s)\tilde{V}'(z_c, s) s_i}{s} \right) \]
\[ + \frac{1}{2} \mathcal{H}(z_c)\tilde{V}^2(z_c, s) s_i^2 + \mathcal{H}(z_c)\tilde{V}^2(z_c, s) s_i^2 \]
\[ - \frac{\psi(s)}{\mathcal{H}(z_c)} - \frac{H'(z_c)}{2} \tilde{V}^2(z_c, s) s_i^2 \]
\[ + \frac{\tilde{V}^2(z_c, s) s_i^2}{2 s} \]
\[ + \frac{1}{2} \mathcal{H}(z_c)\tilde{V}^2(z_c, s) s_i^2 \]
\[ - \frac{\psi(s)}{\mathcal{H}(z_c)} - \frac{H'(z_c)}{2} \tilde{V}^2(z_c, s) s_i^2 \]
\[ + \frac{\tilde{V}^2(z_c, s) s_i^2}{2 s} \]
\[ (1 - \mu^2) \] (A.13)

It is also obvious that
\[ s_i = \mu s, \]
\[ \frac{\partial s_i}{\partial s_i} = \frac{1}{2 s} \cdot 2 s_i = s_i = \mu. \] (A.12)

Combining Eqs. (A.11) and (A.12) we obtain Eq. (18).

On the other hand, relation give as:
\[ r = s - \mu \left( \frac{\psi_c(0)}{\mathcal{H}(z_c)} + \tilde{V}(z_c, s) s_i - \tilde{V}^2(z_c, s) s_i - \frac{\tilde{V}(z_c, s)\tilde{V}'(z_c, s) s_i}{s} \right) \]
\[ + \frac{1}{2} \mathcal{H}(z_c)\tilde{V}^2(z_c, s) s_i^2 \]
\[ - \frac{\psi(s)}{\mathcal{H}(z_c)} - \frac{H'(z_c)}{2} \tilde{V}^2(z_c, s) s_i^2 \]
\[ + \frac{\tilde{V}^2(z_c, s) s_i^2}{2 s} \]
\[ (1 - \mu^2) \]
\[ \text{simply leads to Eq. (19).} \]

Appendix B: Potentially Useful Quantities

From Eq. (20), it might also be noted that for \( \mu = 0 \), which stands for the direction perpendicular to the line-of-sight direction, we have:
\[ \xi(s) = -\tilde{V} + \tilde{V}^2 + (1 - \tilde{V} + \tilde{V}^2)\xi(s) \]
\[ = (1 + \xi(s))(1 - \tilde{V} + \tilde{V}^2) - 1. \] (B.1)

The cosine angle between the galaxy displacement vector \( \tilde{r} \) from the void center and the line-of-sight \( \tilde{r}' \) in real space is
\[ \mu_r \equiv \frac{\tilde{r}_i}{\tilde{r}} = \frac{\mu s - \delta r_i}{s - \mu \delta r_i + \delta r_i^2 (1 - \mu^2)/2s} \]
\[ \simeq \mu - \frac{\delta r_i}{s} (1 - \mu^2) + \frac{3}{2} \left( \frac{\delta r_i}{s} \right)^2 (1 - \mu^2). \] (B.2)