Determining anisotropic real-analytic metric from boundary electromagnetic information

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Abstract

For a compact, connected, oriented Riemannian 3-manifold $(M, g)$ with smooth boundary $\partial M$, we explicitly give a local representation and a full symbol expression for the electromagnetic Dirichlet-to-Neumann map by factorizing Maxwell’s equations and using an isometric transform. We prove that one can reconstruct a compact, connected, real-analytic Riemannian 3-manifold $M$ with boundary from the set of tangential electric fields and tangential magnetic fields, given on a non-empty open subset $\Gamma$ of the boundary, of all electric and magnetic fields with tangential electric data supported in $\Gamma$. We note that for this result we need no assumption on the topology of the manifold other than compactness and connectedness, nor do we need a priori knowledge of all of $\partial M$. In addition, as a by-product of the explicit symbol expression of $\Lambda_{g, \Gamma}$, we show that for a given smooth Riemannian metric $g$, the electromagnetic Dirichlet-to-Neumann map $\Lambda_{g, \Gamma}$ uniquely determines all order tangential and normal derivatives of electromagnetic parameters $\mu$ and $\sigma$ on $\Gamma$. Therefore, $\mu$ and $\sigma$ are completely determined in $M$ by $\Lambda_{g, \Gamma}$ if these two parameter functions and metric $g$ are all real analytic in $M$ up to $\Gamma$.

1 Introduction

Let $(M, g)$ be a compact, connected, oriented Riemannian manifold with $C^1$-smooth boundary $\partial M$, and assume that $\dim M = 3$. We consider the inverse problem of recovering metric of the medium $(M, g)$ by probing with time-harmonic electromagnetic fields from the measurements made on an open subset $\Gamma \subset \partial M$. The fields in $(M, g)$ are described by $E$ and $H$ (electric and magnetic fields), and the behavior of the fields is governed by the Maxwell equations in $M$,

$$
\begin{cases}
\text{curl} E = i\omega \mu H, \\
\text{curl} H = -i\omega (\varepsilon + i\gamma/\omega)E = -i\omega \sigma E.
\end{cases}
$$

(1.1)

Here the constant $\omega > 0$ is a fixed frequency; $\varepsilon$, $\gamma$ and $\mu$ are electric permittivity, conductivity and magnetic permeability, respectively. We assume that $\varepsilon$, $\gamma$ and $\mu$ are all nonnegative functions in $M$, and $\varepsilon$, $\mu$ are strictly positive on $\bar{M}$. The boundary condition is $\nu \times E|_{\partial M} = f$, the tangential component of the electric field at the boundary, where $\nu$ is the unit outer normal to $\partial M$, and $A \times B$ denotes the vector product of the vectors $A$ and $B$ in the tangential space of Riemannian manifold $(M, g)$ (see section 2). The tangential component of the magnetic field is then $\nu \times H$. A number $\omega$ is said to be a resonant frequency if there exists a non-trivial $(E, H)$ such that

$$
\begin{cases}
\text{curl} E = i\omega \mu H & \text{in } M, \\
\text{curl} H = -i\omega \sigma E & \text{in } M, \\
\nu \times E = 0 & \text{on } \partial M.
\end{cases}
$$

(1.2)

Key Words: Maxwell’s equations; Electromagnetic Dirichlet-to-Neumann map; Pseudodifferential operator; Isometric uniqueness
It is well-known that there is a discrete set \( \mathcal{N} \) of resonant frequencies with accumulation point at infinity such that if \( \omega \) is outside this set, then for any \( f \in TH^2(\partial M) \) (the space of tangential vector fields on \( \partial M \) with components in \( H^2(\partial M) \)), the system \( (1.1) \) has a unique solution \( (E, H) \in (L^2(M))^3 \times (L^2(M))^3 \) (see section 7 of [31] or p. 166 of [27]) satisfying \( v \times E_{|\partial M} = f \). We assume more regularity on \( \Gamma \) and \( g \). In fact we assume \( \Gamma \) is a real analytic piece of boundary, and that the metric tensor \( g \) of \( M \) is real analytic up to \( \Gamma \). Let \( f \in C(\partial M) \cap TH^2(\partial M) \) with \( f \subset \Gamma \). For \( \omega \notin \mathcal{N} \), let \( (E, H) \) be the solution of \( (1.1) \) with \( v \times E = f \) on \( \partial M \). The part boundary map, which we call the electromagnetic Dirichlet-to-Neumann, is

\[
\Lambda_{g,\Gamma} : f \mapsto v \times H_{|\Gamma}.
\]

Similar to one of conjectures for the conductivity equation (see [37]), an open problem for the electromagnetic field is: Can one recover the real-analytic metric \( g \) of \( M \) uniquely up to isometry from the knowledge of the electromagnetic Dirichlet-to-Neumann map \( \Lambda_{g,\Gamma} \)? Or roughly speaking, can one obtain metric information in a real-analytic, inaccessible medium \( M \) when the data consist of electromagnetic field measurements on the part surface of the medium? This open problem also arises naturally, e.g., in medical imaging applications, non-destructive material evaluation and geoelectrics (see [14], p.166 of [27], or [23]; just as wrote by A. Kirsch (see, p.174 of [33]) “A fundamental question for every inverse problem is the question of uniqueness: is the information — at least in principle — sufficient to determine the unknown quantity?”

The root of this problem is in Electrical Impedance Tomography (EIT). The question in EIT is whether one can determine the (anisotropic) electrical conductivity of a medium \( \Omega \) in Euclidean space by making voltage and current measurements at the boundary of the medium. Caldarón proposed this problem [9] motivated by geophysical prospection. The Calderón problem has been studied extensively in past decades, and the problem of uniqueness was solved affirmatively in higher dimensional case (dim \( M \geq 3 \)) for the smooth isotropic conductivity of a body by J. Sylvester and G. Uhlmann [53] and for Lipschitz conductivities on a bounded Lipschitz domain by Caro and Rogers in [11], and in two dimensional case for \( C^2 \)-conductivities by A. Nachman [41] and for Lipschitz conductivities by Brown and Uhlmann in [5]. We also refer the reader to [2, 3, 4, 6, 7, 10, 19, 24, 35, 42, 43, 48, 52, 54].

Let us point out that for many inverse problems determining isotropic quantities, a key method is the construction of complex geometrical optics (CGO) solutions with a large parameter which was introduced by Sylvester and Uhlmann in [52]. Unfortunately, such a very useful method is not valid for any anisotropic inverse problem because one cannot find CGO solutions of the form \( e^{\xi \cdot x} (1 + w_{\xi}) \) with \( \xi \in \mathbb{C}^n \) and \( \xi \cdot \xi = 0 \) for such an anisotropic problem (here the corresponding top-order differential equations have variables coefficients) unless some additional conditions are added. In a celebrated paper [37], by considering the symbol of the Dirichlet-to-Neumann map, Lee and Uhlmann solved a series conjectures and answered the case for the anisotropic conductivity equation \( \sum_{j,k=1}^n \frac{\partial}{\partial x_j} (\gamma^{jk} \frac{\partial u}{\partial x_k}) = 0 \). The same problem was further discussed by Lassas, Taylor and Uhlmann in [36], and the corresponding conjecture for \( n \geq 3 \) is completely solved without any topology assumption on manifold other than connectedness. The reason why the Riemannian manifolds appear naturally in the study of Electrical Impedance Tomography (EIT) (see [LeU]) is that in dimension \( n \geq 3 \), the EIT problem is equivalent to the problem of determining a Riemannian metric \( g \) from Dirichlet-to-Neumann map with

\[
g_{ij} = (\det \gamma^{kl})^{1/(n-2)} (\gamma^{ij})^{-1}.
\]

In addition, in order to reveal the true behavior for the Dirichlet-to-Neumann map on the boundary (even for a bounded domain of the Euclidean space \( \mathbb{R}^n \)), the boundary normal coordinates have to be introduced which leads to Riemannian metric, symbol calculus and differential geometric techniques.

Thus, it is our aim to discuss the more general case of recovering anisotropic metric of the electromagnetic medium by making part boundary measurements, and the equation modeling the problem is the full
system of Maxwell’s equations. As pointed out by V. Isakov in p.166 of [27] “· · · which is not a simple task · · ·. Maxwell’s system is not elliptic, so there are additional difficulties in this case.” For the isotropic inverse problem, the first global uniqueness result for smooth electromagnetic parameters on a smooth domain is [45]; another proof was given in [47]. More recently, in [13] the case of continuously differentiable isotropic electromagnetic parameters on a domain with $C^1$ boundary was examined. The global uniqueness for isotropic Lipschitz electromagnetic parameters in a bounded Lipschitz domain was obtained in a constructive manner in [44] by Pichler. Partial boundary data problems were studied in [12] using the reflection argument introduced in [28], and in [17], extending the ideas from [8] to Maxwell’s equations. The inverse parameter problem on a manifold has been studied in [46], and [31] considered the problem in a non-Euclidean (admissible manifold) setting.

In this paper, we give an affirmative answer to the above open problem by symbol calculus and a method of Green’s function (the later was introduced by Lassas, Taylor and Uhlmann in [36]). Our main results are the following:

**Theorem 1.1.** Let $(M, g)$ be a compact, connected, oriented, smooth Riemannian 3-manifold with $C^1$-smooth boundary $\partial M$, and let $\Gamma \subset \partial M$ be a piece of smooth surface on $\partial M$. Suppose electromagnetic parameters $\mu$, $\sigma$ and metric $g$ are smooth in $M$ up to $\Gamma$, and $\mu$ and $\text{Re}(\sigma)$ are strictly positive on $\bar{M}$. Then the electromagnetic Dirichlet-to-Neumann map $\Lambda_{g, \Gamma}$ has a precise local expression (see Proposition 2.6). Furthermore, the full symbol matrix $\psi(x', \xi')$ of $\Lambda_{g, \Gamma}$

$$\psi(x', \xi') \sim \begin{bmatrix} \psi^{11}(x', \xi') & \psi^{12}(x', \xi') \\ \psi^{21}(x', \xi') & \psi^{22}(x', \xi') \end{bmatrix}$$

can explicitly be given in local boundary normal coordinates (see, the proof of Proposition 3.1), and

$$\Lambda_{g, \Gamma} f(x') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix' \cdot \xi'} \psi(x', \xi') \hat{f}(\xi') \, d\xi'$$

for any $f \in C(\partial M) \cap C^0_0(\Gamma)$, where $\hat{f}(\xi')$ is the Fourier transform of $f$.

If $(M, g)$ and $(\bar{M}, \bar{g})$ are both Riemannian manifolds, a smooth map $\rho : M \to \bar{M}$ is called a (Riemannian) isometry if it is a diffeomorphism that satisfies $\rho^* \bar{g} = g$. If there exists a Riemannian isometry between $(M, g)$ and $(\bar{M}, \bar{g})$, we say that they are isometric as Riemannian manifolds.

**Theorem 1.2.** Let $(M_1, g_1)$ and $(M_2, g_2)$ be compact, connected, oriented, real-analytic Riemannian 3-manifolds with $C^1$-smooth boundaries $\partial M_1$ and $\partial M_2$. Assume that $\partial M_1$ and $\partial M_2$ contain a non-empty open set $\Gamma = \Gamma_1 = \Gamma_2 = \Gamma$, on which each boundary is real analytic, with the metric tensors analytic up to $\Gamma_j$. Furthermore, assume that the electromagnetic parameters $\mu$ and $\sigma$ are real analytic in $M$ up to $\Gamma$, and that $\mu$ and $\text{Re}(\sigma)$ are strictly positive on $\bar{M}$. Finally, assume that the electromagnetic Dirichlet-to-Neumann maps $\Lambda_{g_1, \Gamma}$ and $\Lambda_{g_2, \Gamma}$ coincide. Then there exists a real-analytic diffeomorphism $\rho : M_1 \cup \Gamma \to M_2 \cup \Gamma$ with $\rho|_\Gamma = \text{id}$, such that $g_1 = \rho^* g_2$.

Strictly speaking in the statement above we mean by the set $\Gamma$ the sets $\Gamma_1 \subset \partial M_1$ and $\Gamma_2 \subset \partial M_2$, which are identified by a diffeomorphism.

When $M$ is a bounded domain in $\mathbb{R}^3$ and $g$ is the Euclidean metric, we immediately have the following corollary:

**Corollary 1.3.** Let $M \subset \mathbb{R}^3$ be a bounded domain with $C^1$-smooth boundary, and let $\Gamma$ be a piece of real-analytic surface on $\partial M$. Suppose $\bar{g}$ is a real-analytic metric in $M$ up to $\Gamma$ such that $\Lambda_{\bar{g}, \Gamma} = \Lambda_{g, \Gamma}$, where $g$ is the Euclidean metric. Assume that the electromagnetic parameters $\mu$ and $\sigma$ are real analytic in $M$ up to $\Gamma$, and that $\mu$ and $\text{Re}(\sigma)$ are strictly positive on $\bar{M}$. Then there exists a real-analytic diffeomorphism $\rho : M \cup \Gamma \to \bar{M} \cup \Gamma$ with $\rho|_\Gamma = \text{id}$, such that $g = \rho^* \bar{g}$.
In addition, as a by-product of our new method of full symbol we can show the following:

**Theorem 1.4.** Let \((M, g)\) be a compact, connected, oriented, \(C^\infty\)-smooth Riemannian 3-manifold with \(C^1\)-smooth boundary \(\partial M\). Assume that \(\Gamma\) is a piece of \(C^\infty\)-smooth surface on \(\partial M\). Assume that electromagnetic parameters \(\mu\) and \(\sigma\) are \(C^\infty\)-smooth in \(M\) up to \(\Gamma\), and that \(\mu\text{ and }\text{Re}(\sigma)\) are strictly positive on \(\bar{M}\). Then the electromagnetic Dirichlet-to-Neumann map \(\Lambda_{\mathbb{R}, \Gamma}\) uniquely determines \(\frac{\partial^2 \mu}{\partial x^2}\) and \(\frac{\partial^2 \sigma}{\partial x^2}\) on \(\Gamma\) for all multi-indices \(K = (k_1, k_2, k_3)\) with \(|K| = k_1 + k_2 + k_3 \geq 0\).

Theorem 1.4 generalizes Joshi-McDowall’s result (see [30]) for electromagnetic field (in which they considered the case of a bounded domain with \(C^\infty\)-smooth boundary in Euclidean space \(\mathbb{R}^n\), and \(\Gamma = \partial M\)). This immediately leads to

**Theorem 1.5.** Let \((M, g)\) be a compact, connected, oriented, real analytic Riemannian 3-manifold with \(C^1\)-smooth boundary. Let \(\Gamma \subset \partial M\) be a piece of real-analytic surface. Assume also that \(\omega > 0\) is such that the system (1.1) has a unique solution in \(M\) with the prescribed (compact support) tangential component of the electric field on \(\partial \Omega\), so that \(\Lambda_{\mathbb{R}, \Gamma}\) is well defined. Then the electromagnetic Dirichlet-to-Neumann map \(\Lambda_{\mathbb{R}, \Gamma}\) uniquely determines the function \(\mu\) and \(\sigma\) on \(M \cup \Gamma\).

Let us remark that Theorem 1.5 is not a trivial conclusion. To the best of my knowledge, this is the first novel result on determining the anisotropic parameters for the electromagnetic part boundary measurements.

The main ideas of this paper are as follows. From the Maxwell equations we obtain a second-order partial differential equations

\[
\text{curl curl } E - \left(\text{grad } (\log \mu)\right) \times \text{curl } E - \omega^2 \mu \sigma E = 0 \quad \text{in } M,
\]

which can be rewritten as

\[
(1.4) \quad \left\{ \frac{\partial^2}{\partial x_3^2} I_3 + B \frac{\partial}{\partial x_3} I_3 + C \right\} E = 0
\]

in the boundary normal coordinates. Let us point out that it is a valuable work to obtain explicit expression for \(B\) and \(C\) because some new methods and calculations are needed. The second step is to use the factorization for the left-hand side of (1.4) so that we obtain a pseudodifferential operator \(\Phi\) of order one. This method for the Laplace equation \(\Delta_3 u = 0\) is well known (see, for example, [57], vol. 1, pages 159-161, or [18]) and have been well developed by Lee and Uhlmann in [57]. Since the third component of \(\nabla \times E\) and \(\nabla \times H\) are both vanish in local boundary normal coordinates, we introduce a linear isometric operator \(\varphi: (C^\infty(M))^3 \to (C^\infty(M))^2\) defined by

\[
\varphi \left( \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{for any } (a_1, a_2, 0) \in (C^\infty(M))^3,
\]

then we get an equivalent local representation formula for \(\Lambda_{\mathbb{R}, \Gamma}\) on \(\Gamma \subset \partial M\):

\[
(1.5) \quad \varphi(\nabla \times H) = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \varphi(\nabla \times E) = -\frac{i}{\omega \mu \sqrt{|g|}} \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} -g_{12} \\ -g_{22} \end{bmatrix} \varphi(\nabla \times E),
\]

where \(L^{jk}\) are given by (2.51) (see section 2). This formula transforms a three-dimensional problem into a two-dimensional problem. In order to get (1.5) we also use the equation \(\text{div } (\sigma E) = 0\) in \(M\), which is derived
from Maxwell’s equations (1.1) (In fact, the Maxwell equations (1.1) is equivalent to the following system of equations for electric field $E$ (see section 4):

$$
\begin{align*}
\{ & \text{curl curl } E - (\text{grad } (\log \mu)) \times \text{curl } E - \omega^2 \mu \sigma E = 0 \quad \text{in } M, \\
 \text{div } (\sigma E) = 0 \quad & \text{in } M
\end{align*}
$$

The final step is to determine $g_{\alpha\beta}$ and all their normal derivatives on $\Gamma$ by the full symbol of the electromagnetic Dirichlet-to-Neumann map $\Lambda_{g,\Gamma}$. This step plays a key role in this paper because some subtle new estimates are employed. More precisely, by discussing the principal symbol of pseudodifferential operator $\Lambda^{11}$ we can obtain $g_{\alpha\beta}$ on $\Gamma$ (Of course, by $\Lambda^{22}$ we can also obtain the same conclusion); and by discussing the symbol with homogeneous of degree 0 in $\xi'$ for the operator $L^{11} + L^{22}$, we can get the first-order normal derivatives of $g^{\alpha\beta}$ on $\Gamma$; furthermore, from the symbol with homogeneous of degree $-1$ in $\xi'$ for operator $L^{11} + L^{22}$, we get the second-order normal derivatives of $g^{\alpha\beta}$ on $\Gamma$ (Let us remark that this argument cannot be omitted); generally, by using the induction we can obtain $(m+2)$-order normal derivatives on $\Gamma$ from the symbol with homogeneous of degree $-m-1$ in $\xi'$ for the operator $L^{11} + L^{22}$ for any $m \geq 0$. Combining the expansion of Taylor’s series of $g_{jk}$ at every $x_0 \in \Gamma$ and the analyticity of the Riemannian metric in $M$ up to $\Gamma$, we can prove Theorem 1.2 via a method of Green’s function. The effective method of Green’s function for the Laplace equation $\Delta \mu = 0$ on a Riemannian manifold $M$ was introduced by Lassas, Taylor and Uhlmann in [36]. Note that our method here is slightly different from that of [36]. We attach a collar domain across $\Gamma$ for $M$ instead of a “half-ball” domain of some point $x_0 \in \Gamma$ for $M$, so that we can analytically extend the Riemannian manifolds $M_j$ to the larger Riemannian manifolds $\tilde{M}_j$. The advantage of this new technique is that it can immediately ensure the identity of metric $g$ on $\Gamma$ when $M_1$ and $M_2$ are shown to be isometric. In addition, we will introduce (electric) dyadic Green’s function, which leads to the uniqueness of metric. Finally, by calculating any order (normal and tangent) derivatives of $\mu$ and $\sigma$ on $\Gamma$, we get Theorem 1.4, and hence by using analyticity up to $\Gamma$ we can obtain Theorem 1.5.

This paper is organized as follows: in section 2, we give the local expression of the electromagnetic Dirichlet-to-Neumann map $\Lambda_{g,\Gamma}$. In section 3, we explicitly calculate the full symbol of $\Lambda_{g,\Gamma}$. Therefore, Proposition 2.6 and the proof of Proposition 3.1 have provided the detail proof for Theorem 1.1. Furthermore, we show that the full symbol of $\Lambda_{g,\Gamma}$ uniquely determines $g_{\alpha\beta}$ and their all-order derivatives on $\Gamma$; hence, by using the (electric) dyadic Green’s function we will prove Theorem 1.2 in section 4. Finally, for given metric $g$, Theorem 1.4 and the uniqueness of the anisotropic electromagnetic parameters $\mu$ and $\sigma$ from $\Lambda_{g,\Gamma}$ are proved in section 5.

## 2 A factorization of Maxwell’s equations on Riemannian manifold

Let $M$ be an oriented compact Riemannian 3-manifold endowed with a Riemannian metric $g$. By $TM$ (respectively, $T^*(M)$) we denote the tangent (respectively, cotangent) bundle on $M$, which are the disjoint union of the tangent (respectively, cotangent) spaces at all points of $M$:

$$
TM = \bigoplus_{p \in M} T_p M \quad \text{(respectively, } T^*M = \bigoplus_{p \in M} T_p^* M).
$$

A vector field on $M$ is a section of the map $\pi : TM \to M$. More concretely, a vector field is a smooth map $X : M \to TM$, usually written $p \mapsto X_p$, with the property that

$$
\pi \circ X = \text{Id}_M,
$$

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or equivalently, \(X_p \in T_p M\) for each \(p \in M\). If \((U; x_1, x_2, x_3)\) is any smooth coordinate chart for \(M\), we can write the value of \(X\) at any point \(p \in U\) in terms of the coordinate basis vectors \(\left\{ \frac{\partial}{\partial x_j} \right\}_p\) of \(T_p M\):

\[
X_p = \sum_{j=1}^3 X^j(p) \left. \frac{\partial}{\partial x_j} \right|_p.
\]

This defines three functions \(X^j : U \to \mathbb{R}\), called the component functions of \(X\) in the given chart. It is standard to use the notation \(\mathfrak{X}(M)\) to denote the set of all smooth vector fields on \(M\). Let \(\nabla\) be the associated Levi-Civita connection. For a Riemannian manifold \((M, g)\), the connection coefficients (i.e., Christoffel symbols) associated with the metric \(g\) are given by (see, for example, [55]):

\[
\Gamma^j_{lk} = \frac{1}{2} \sum_{m=1}^3 g^{jm} \left( \frac{\partial g_{km}}{\partial x_l} + \frac{\partial g_{lm}}{\partial x_k} - \frac{\partial g_{kl}}{\partial x_m} \right),
\]

where \([g^{jk}]_{3 \times 3}\) is the inverse of \(g = [g^{jk}]_{3 \times 3}\). It is well-known that in a local coordinate system with the naturally associated frame field on the tangent bundle,

\[
\nabla_{\frac{\partial}{\partial x_k}} X = \sum_{j=1}^n \left( \frac{\partial X^j}{\partial x_k} + \sum_{l=1}^n \Gamma^j_{lk} X^l \right) \frac{\partial}{\partial x_j}
\]

for \(X = \sum_{j=1}^n X^j \frac{\partial}{\partial x_j}\).

If we denote

\[
X^j_{,k} = \frac{\partial X^j}{\partial x_k} + \sum_{l=1}^n \Gamma^j_{lk} X^l,
\]

then

\[
\nabla Y X = \sum_{j,k=1}^n Y^k X^j_{,k} \frac{\partial}{\partial x_j} \quad \text{for} \quad Y = \sum_{k=1}^n Y^k \frac{\partial}{\partial x_k}.
\]

The associated Riemannian curvature tensor is:

\[
R(X, Y) Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z.
\]

In a local coordinates, the expression for the Riemannian curvature tensor is

\[
R_{klm} = \frac{\partial \Gamma^j_{km}}{\partial x_l} - \frac{\partial \Gamma^j_{kl}}{\partial x_m} + \sum_{s=1}^3 \left( \Gamma^j_{sl} \Gamma^s_{km} - \Gamma^s_{kl} \Gamma^j_{sm} \right),
\]

and the Ricci curvature tensor is defined by

\[
R_{km} = \sum_{j=1}^3 R^j_{kjm}.
\]

It follows from p. 410 of [59] that if \(E\) and \(F\) are two vectors, i.e.,

\[
E = \sum_{j=1}^3 E^j \frac{\partial}{\partial x_j} \quad \text{and} \quad F = \sum_{j=1}^3 F^j \frac{\partial}{\partial x_j},
\]

then the vector product \(E \times F\) of \(E\) and \(F\) is

\[
E \times F = \sqrt{|g|} \left\{ \begin{array}{ccc}
g^{11} & E^2 & E^3 \\
g^{21} & E^1 & E^3 \\
g^{31} & E^1 & E^2 \\
E^1 & F^2 & F^3 \\
E^2 & F^1 & F^3 \\
E^3 & F^1 & F^2 \\
F^1 & F^2 & F^3 \\
F^1 & F^3 & F^2 \\
F^2 & F^3 & F^1 \\
F^3 & F^1 & F^2 \\
\end{array} \right\}.
\]
where $|a|$ denotes the determinant of the matrix
\[
   a = \begin{bmatrix}
      a^{11} & a^{12} & a^{13} \\
      a^{21} & a^{22} & a^{23} \\
      a^{31} & a^{32} & a^{33}
   \end{bmatrix}.
\]

That is,

\[
   \tag{2.10}
   E \times F = \sqrt{|g|} \begin{vmatrix} dx_1 & dx_2 & dx_3 \\ E^1 & E^2 & E^3 \\ F^1 & F^2 & F^3 \end{vmatrix}
   = \sqrt{|g|} \left\{ \left| \begin{array}{ccc}
      E^2 & E^3 & E^1 \\
      F^2 & F^3 & F^1 \\
   \end{array} \right| dx_1 + \left| \begin{array}{ccc}
      E^3 & E^1 & E^2 \\
      F^3 & F^1 & F^2 \\
   \end{array} \right| dx_2 + \left| \begin{array}{ccc}
      E^1 & E^2 & E^3 \\
      F^1 & F^2 & F^3 \\
   \end{array} \right| dx_3 \right\}
\]

with $dx_j = \sum_{k=1}^3 g^{jk} \frac{\partial}{\partial x_k}$, $j = 1, 2, 3$. On the other hand, according to p. 410 of [59] we can also write $E \times F$ as the following form:

\[
   \tag{2.11}
   E \times F = \frac{1}{\sqrt{|g|}} \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \sum_{l=1}^3 g^{1l} E^l & \sum_{l=1}^3 g^{2l} E^l & \sum_{l=1}^3 g^{3l} E^l \\ \sum_{l=1}^3 g^{1l} F^l & \sum_{l=1}^3 g^{2l} F^l & \sum_{l=1}^3 g^{3l} F^l \end{vmatrix}
   = \frac{1}{\sqrt{|g|}} \left\{ \sum_{k=1}^3 g^{1k} \frac{\partial}{\partial x_1} \left( \sum_{l=1}^3 g^{kl} E^l \right) - \sum_{k=1}^3 g^{2k} \frac{\partial}{\partial x_2} \left( \sum_{l=1}^3 g^{kl} E^l \right) + \sum_{k=1}^3 g^{3k} \frac{\partial}{\partial x_3} \left( \sum_{l=1}^3 g^{kl} E^l \right) \right\}
\]

In fact, from (2.10) we have

\[
   \tag{2.12}
   E \times F = \sqrt{|g|} \begin{vmatrix} \sum_{k=1}^3 g^{1k} & \sum_{k=1}^3 g^{2k} & \sum_{k=1}^3 g^{3k} \\ E^1 & E^2 & E^3 \\ F^1 & F^2 & F^3 \end{vmatrix}
   = \sqrt{|g|} \left\{ \sum_{k=1}^3 g^{1k} \frac{\partial}{\partial x_1} \left( \sum_{l=1}^3 g^{kl} E^l \right) - \sum_{k=1}^3 g^{2k} \frac{\partial}{\partial x_2} \left( \sum_{l=1}^3 g^{kl} E^l \right) + \sum_{k=1}^3 g^{3k} \frac{\partial}{\partial x_3} \left( \sum_{l=1}^3 g^{kl} E^l \right) \right\}
\]

so (2.11) is verified.

For any smooth real-valued function $f$ on a Riemannian manifold $(M, g)$, the gradient of $f$ is defined by

\[
   \text{grad } f = (df)^*,
\]

and the divergence operator $\text{div} : \mathcal{X}(M) \to C^0(M)$ is defined by

\[
   \text{div } X = *d *X^*,
\]
where \( * : \wedge^k T^* M \to \wedge^{3-k} T^* M \) is the Hodge star operator, and \( \flat \) and \( \# \) are flat and sharp operators by lowering index and raising index, respectively. In smooth local coordinates, the \( \text{grad} f \) and \( \text{div} X \) have the following expression

\[
\text{grad} f = \sum_{j,k=1}^{3} g^{jk} \frac{\partial f}{\partial x_k} \frac{\partial}{\partial x_j},
\]

(2.13)

\[
\text{div} X = \sum_{j=1}^{3} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} \left( \sqrt{|g|} X^j \right) \quad \text{for } X = \sum_{j=1}^{3} X^j \frac{\partial}{\partial x_j} \in \mathcal{X}(M).
\]

Accordingly, the Laplace-Beltrami operator \( \Delta_g \) is just given by

\[
\Delta_g := \text{div} \text{grad} = \frac{1}{\sqrt{|g|}} \sum_{j,k=1}^{3} \frac{\partial}{\partial x_j} \left( \sqrt{|g|} g^{jk} \frac{\partial}{\partial x_k} \right).
\]

(2.14)

The curl operator, denoted by \( \text{curl} : \mathcal{X}(M) \to \mathcal{X}(M) \), is defined by

\[
\text{curl} X = *(d(X^1))^# \quad \text{for } X \in \mathcal{X}(M).
\]

The curl operator defined only in dimension 3 because it is only in that case that \( \wedge^2 T^* M \) is isomorphic to \( TM \) (via the Hodge star operator). Suppose that \( X = \sum_{j=1}^{3} X^j \frac{\partial}{\partial x_j} \in \mathcal{X}(M) \) in smooth coordinates, then by definition of \( \text{curl} X \), we have (see p.454 of [59])

\[
(2.15) \quad \text{curl} X = \nabla \times X = \frac{1}{\sqrt{|g|}} \left\{ \left( \frac{\partial}{\partial x_2} \left( \sum_{l=1}^{3} g_{3l} X^l \right) - \frac{\partial}{\partial x_3} \left( \sum_{l=1}^{3} g_{2l} X^l \right) \right) \frac{\partial}{\partial x_1} + \left( \frac{\partial}{\partial x_3} \left( \sum_{l=1}^{3} g_{1l} X^l \right) - \frac{\partial}{\partial x_1} \left( \sum_{l=1}^{3} g_{3l} X^l \right) \right) \frac{\partial}{\partial x_2} + \left( \frac{\partial}{\partial x_1} \left( \sum_{l=1}^{3} g_{2l} X^l \right) - \frac{\partial}{\partial x_2} \left( \sum_{l=1}^{3} g_{1l} X^l \right) \right) \frac{\partial}{\partial x_3} \right\} \in \mathcal{X}(M).
\]

The operators \( \text{div} \), \( \text{grad} \) and \( \text{curl} \) on an oriented three-dimensional Riemannian manifold \( M \) are related by the following commutative diagram:

\[
\begin{array}{cccccc}
\wedge^0(M) & \xrightarrow{d} & \wedge^1(M) & \xrightarrow{d} & \wedge^2(M) & \xrightarrow{d} & \wedge^3(M) \\
\text{Id} & \downarrow & \text{grad} & \downarrow & \text{curl} & \downarrow & \text{div} & \downarrow & \text{Id} & \ldots
\end{array}
\]

(2.16)

\[
\text{curl} \text{curl} X = \text{grad} \text{div} X - \sum_{j=1}^{3} \left( \Delta_g X^j + 2 \sum_{k,l,m=1}^{3} g^{ml} \Gamma_{km}^l \frac{\partial X^k}{\partial x_l} + \sum_{k,l,m=1}^{3} g^{ml} \frac{\partial \Gamma_{km}^l}{\partial x_l} X^k \right)
\]

\[
+ \sum_{k,l,m,h=1}^{3} g^{ml} \Gamma_{hl}^k \Gamma_{km}^{lh} X^k - \sum_{k,l,m,h=1}^{3} g^{ml} \Gamma_{hl}^k \Gamma_{km}^{lh} X^k - \sum_{k=1}^{3} R^k X^k \frac{\partial}{\partial x_k},
\]

\[
(2.16) \quad \text{curl} \text{curl} X = \text{grad} \text{div} X - \sum_{j=1}^{3} \left( \Delta_g X^j + 2 \sum_{k,l,m=1}^{3} g^{ml} \Gamma_{km}^l \frac{\partial X^k}{\partial x_l} + \sum_{k,l,m=1}^{3} g^{ml} \frac{\partial \Gamma_{km}^l}{\partial x_l} X^k \right)
\]

\[
+ \sum_{k,l,m,h=1}^{3} g^{ml} \Gamma_{hl}^k \Gamma_{km}^{lh} X^k - \sum_{k,l,m,h=1}^{3} g^{ml} \Gamma_{hl}^k \Gamma_{km}^{lh} X^k - \sum_{k=1}^{3} R^k X^k \frac{\partial}{\partial x_k},
\]

\[
(2.14) \quad \Delta_g := \text{div} \text{grad} = \frac{1}{\sqrt{|g|}} \sum_{j,k=1}^{3} \frac{\partial}{\partial x_j} \left( \sqrt{|g|} g^{jk} \frac{\partial}{\partial x_k} \right).
\]

The identities \( \text{curl} \text{grad} \equiv 0 \) and \( \text{div} \text{curl} \equiv 0 \) follow from \( dd \equiv 0 \) and \( * * \eta = (-1)^{k(n-k)} \eta \) if \( \eta \) is a \( k \)-form on \( n \)-dimensional Riemannian manifolds.

**Lemma 2.1.** Let \( M \) be an oriented compact Riemannian 3-manifold with smooth metric tensor \( g \). If \( X = \sum_{j=1}^{3} X^j \frac{\partial}{\partial x_j} \) is a smooth vector field on \( M \), then in local coordinates,
where $R^j_k = \sum_{m=1}^{3} g^{jm} R_{mk}$.

Proof. In local coordinates, let $X^\flat = \eta$ be the dual 1-form of the vector field $X = \sum_{j=1}^{3} X^j \frac{\partial}{\partial x_j}$, i.e., $\eta = \sum_{j=1}^{3} X^j dx_j$, where $X_j = \sum_{k=1}^{3} g_{jk} X^k$. Then for such a 1-form $\eta$, we have

\[(\ast d)(\ast d)\eta = (\ast d\ast) d\eta,
\]
i.e.,

\[\text{curl curl} X^\flat = \delta d\eta,
\]
where $\delta : \wedge^k(M) \rightarrow \wedge^{k-1}(M)$ is defined by $\delta \zeta = (-1)^{(k+1)+1}(\ast d\ast)\zeta$ for any $k$-form $\zeta$ with $0 \leq k \leq 3 = \dim M$. Recalling that the Hodge Laplacian $\Lambda_\eta = d\delta + \delta d$, we find that

\[\text{(curl curl} X^\flat)^\flat = -d\delta \eta + \Delta_\eta \eta = d(\ast d\ast) \eta + \Delta_\eta \eta = d(\text{div} X) + \Delta_\eta \eta.
\]

It is easy to see that

\[
\Delta_\eta \eta = d\delta \eta + \delta d\eta = -\sum_{j,l,m=1}^{3} X^m_{,j;lm} dx_j + \sum_{j,l,m=1}^{3} (g^{ml} X^m_{,j;l} - g^{ml} X^m_{,j;m}) dx_j
\]

\[= -\sum_{j,l,m=1}^{3} g^{ml} X^m_{,j;lm} dx_j - \sum_{j,l,m=1}^{3} (X^m_{,j;lm} - X^m_{,j;m}) dx_j
\]

\[= -\sum_{j,l,m=1}^{3} g^{ml} X^m_{,j;lm} dx_j + \sum_{j,l,m=1}^{3} \Gamma^m_{ln} X^l dx_j
\]

\[= -\sum_{j,l,m=1}^{3} g^{ml} X^m_{,j;lm} dx_j + \sum_{j,l,m=1}^{3} \Gamma^m_{ln} X^l dx_j,
\]

where $X^m_{,j;lm} = (X^m_{,j;lm})$, $X^m_{,j;lj} = (X^m_{,lj})$, and $X^m_{,j} = \frac{\partial X^m}{\partial x_j} - \sum_{l=1}^{3} \Gamma^m_{lj} X^l$. By using the sharp operator # (i.e., by raising an index) and by noting that $g^{lm} = 0$ for Riemannian metric $g$, we get

\[(\Delta_\eta X^i)^\# = -\sum_{j,l,m=1}^{3} g^{ml} X^j_{,lm} \frac{\partial}{\partial x_j} + \sum_{j,l,m=1}^{3} g^{lm} R_{lmk} X^k \frac{\partial}{\partial x_j}
\]

\[= -\sum_{j,l,m=1}^{3} g^{ml} X^j_{,lm} \frac{\partial}{\partial x_j} + \sum_{j,l,m=1}^{3} \Gamma^i_{lk} \frac{\partial}{\partial x_j}.
\]

In view of

\[X^i_{,jm} = \frac{\partial X^i_{,m}}{\partial x_l} + \sum_{k=1}^{3} X^k_{,m} \Gamma^i_{jk} - \sum_{k=1}^{3} X^j_{,k} \Gamma^i_{ml}
\]

\[= \frac{\partial}{\partial x_l} \left( \frac{\partial X^i}{\partial x_m} + \sum_{k=1}^{3} \Gamma^i_{km} X^k \right) + \sum_{k=1}^{3} \left( \frac{\partial X^i}{\partial x_m} + \sum_{h=1}^{3} \Gamma^i_{hm} X^h \right) \Gamma^j_{kl} - \sum_{k=1}^{3} \left( \frac{\partial X^j}{\partial x_m} + \sum_{h=1}^{3} \Gamma^j_{hm} X^h \right) \Gamma^i_{kl}
\]

\[= \frac{\partial^2 X^i}{\partial x_m \partial x_l} + \sum_{k=1}^{3} \Gamma^i_{km} \frac{\partial X^k}{\partial x_l} + \sum_{k=1}^{3} \Gamma^k_{kl} \frac{\partial X^i}{\partial x_m} + \sum_{k=1}^{3} \Gamma^k_{ml} \frac{\partial X^i}{\partial x_k} + \sum_{k=1}^{3} \left( \frac{\partial X^i}{\partial x_l} + \sum_{h=1}^{3} \Gamma^i_{hl} X^h \right) \Gamma^j_{kl} - \sum_{k=1}^{3} \left( \frac{\partial X^j}{\partial x_l} + \sum_{h=1}^{3} \Gamma^j_{hl} X^h \right) \Gamma^i_{kl} X^k,
\]
we have

\[
\sum_{l,m=1}^{3} g^{ml}X^j_{jm} = \sum_{l,m=1}^{3} \left( g^{ml} \frac{\partial^2 X^j}{\partial x_m \partial x_l} - \sum_{k=1}^{3} g^{ml} \Gamma_{kl}^j \frac{\partial X^j}{\partial x_k} \right) + \sum_{k,l,m=1}^{3} \left( g^{ml} \Gamma_{km}^j \frac{\partial X^k}{\partial x_l} + g^{ml} \Gamma_{lj}^k \frac{\partial X^k}{\partial x_m} \right) + \sum_{k,l,m=1}^{3} \left( g^{ml} \Gamma_{km}^j \frac{\partial X^j}{\partial x_l} + g^{ml} \Gamma_{lj}^k \frac{\partial X^k}{\partial x_m} \right)
\]

+ \sum_{h=1}^{3} \left( g^{ml} \Gamma_{hl}^j \Gamma_{km}^h - \sum_{h=1}^{3} g^{ml} \Gamma_{hl}^j \Gamma_{km}^h \right) X^k

= \Delta g X^j + 2 \sum_{k,l,m=1}^{3} g^{ml} \Gamma_{km}^j \frac{\partial X^k}{\partial x_l} + \sum_{k,l,m=1}^{3} \left( g^{ml} \Gamma_{km}^j \frac{\partial X^j}{\partial x_k} + \sum_{h=1}^{3} g^{ml} \Gamma_{hl}^j \Gamma_{km}^h - \sum_{h=1}^{3} g^{ml} \Gamma_{hl}^j \Gamma_{km}^h \right) X^k

\]

because of \( \sum_{l,m=1}^{3} \left( g^{ml} \frac{\partial^2 X^j}{\partial x_m \partial x_l} - \sum_{k=1}^{3} g^{ml} \Gamma_{kl}^j \frac{\partial X^j}{\partial x_k} \right) = \Delta g X^j \). It follows that

\[
(\Delta_g X^j)^\# = \sum_{j=1}^{3} \left\{ -\Delta_g X^j - 2 \sum_{k,l,m=1}^{3} g^{ml} \Gamma_{km}^j \frac{\partial X^k}{\partial x_l} - \sum_{k,l,m=1}^{3} \left( g^{ml} \Gamma_{km}^j \frac{\partial X^j}{\partial x_k} + \sum_{h=1}^{3} g^{ml} \Gamma_{hl}^j \Gamma_{km}^h - \sum_{h=1}^{3} g^{ml} \Gamma_{hl}^j \Gamma_{km}^h \right) X^k \right\} \frac{\partial}{\partial x_j}.
\]

Noting that \( \text{grad} \text{div} X = (d \text{div} X)^2 \), we find by (2.17) that

\[
\text{curl} \text{curl} X = \text{grad} \text{div} X - \sum_{j=1}^{3} \left( \Delta_g X^j + 2 \sum_{k,l,m=1}^{3} g^{ml} \Gamma_{km}^j \frac{\partial X^k}{\partial x_l} + \sum_{m,k,l=1}^{3} g^{ml} \Gamma_{km}^j \frac{\partial X^j}{\partial x_k} \right)
\]

+ \sum_{k,l,m=1}^{3} g^{ml} \Gamma_{hl}^j \Gamma_{km}^h X^k - \sum_{k,l,m=1}^{3} \left( g^{ml} \Gamma_{hl}^j \Gamma_{km}^h X^k - \sum_{h=1}^{3} g^{ml} \Gamma_{hl}^j \Gamma_{km}^h \right) X^k \frac{\partial}{\partial x_j}.
\]

Remark 2.2. The formula (2.16) can also be obtained by a direct calculation from the definition of curl if we using the relations \( g g^{-1} = \mathbb{I} \), \( \frac{\partial g_{jk}}{\partial x_l} = \sum_{m=1}^{3} \left( g_{jm} \Gamma_{kl}^m + g_{km} \Gamma_{jl}^m \right) \) and \( \frac{\partial g_{jk}}{\partial x_l} = - \sum_{m=1}^{3} \left( g_{jm} \Gamma_{kl}^m + g_{km} \Gamma_{jl}^m \right) \).

Lemma 2.3. Let \((E, H)\) be a solution of the Maxwell equations (1.1). Then

\[
\text{div} (\sigma E) = \text{div} (\mu H) = 0 \quad \text{in M}.
\]

Proof. As pointed out before, \( \text{div} \text{curl} X = 0 \) for any vector field \( X \in \mathbb{X}(M) \). In fact, this also can directly be got as follows. Applying (2.15) and (2.13) we have

\[
\text{div} \text{curl} X = \frac{1}{\sqrt{|g|}} \left\{ \left( \frac{\partial^2 (g_{31} X^1)}{\partial x_1 \partial x_2} - \frac{\partial^2 (g_{21} X^1)}{\partial x_1 \partial x_3} \right) + \left( \frac{\partial^2 (g_{11} X^1)}{\partial x_2 \partial x_3} - \frac{\partial^2 (g_{31} X^1)}{\partial x_2 \partial x_1} \right) + \left( \frac{\partial^2 (g_{21} X^1)}{\partial x_1 \partial x_3} - \frac{\partial^2 (g_{11} X^1)}{\partial x_1 \partial x_2} \right) \right\}
\]

- \frac{1}{2 \sqrt{|g|}} \left\{ \left( \frac{\partial |g|}{\partial x_1} \right) \left( \frac{\partial (g_{31} X^1)}{\partial x_2} - \frac{\partial (g_{21} X^1)}{\partial x_3} \right) + \left( \frac{\partial |g|}{\partial x_2} \right) \left( \frac{\partial (g_{11} X^1)}{\partial x_3} - \frac{\partial (g_{31} X^1)}{\partial x_1} \right) + \left( \frac{\partial |g|}{\partial x_3} \right) \left( \frac{\partial (g_{21} X^1)}{\partial x_1} - \frac{\partial (g_{11} X^1)}{\partial x_2} \right) \right\}
\]

+ \Gamma_{kk} \left( \frac{\partial (g_{31} X^1)}{\partial x_2} - \frac{\partial (g_{21} X^1)}{\partial x_3} \right) + \Gamma_{kk} \left( \frac{\partial (g_{11} X^1)}{\partial x_3} - \frac{\partial (g_{31} X^1)}{\partial x_1} \right) + \Gamma_{kk} \left( \frac{\partial (g_{21} X^1)}{\partial x_1} - \frac{\partial (g_{11} X^1)}{\partial x_2} \right) = 0
\]

because of \( \frac{1}{\sqrt{|g|}} \frac{\partial |g|}{\partial x_k} = \sum_{l=1}^{3} \Gamma_{kl}^j \) for \( k = 1, 2, 3 \). Taking div to Maxwell’s equations and applying the above conclusion, we immediately get the desired result.
By Lemma 2.3, we have
\begin{equation}
0 = \text{div} (\sigma E) = \sigma \text{div} E + \sum_{i=1}^{3} \frac{\partial \sigma}{\partial x_i} E^i, \tag{2.20}
\end{equation}
so that
\[ \text{div} E = -\frac{1}{\sigma} \sum_{i=1}^{3} \frac{\partial \sigma}{\partial x_i} E^i. \]
Thus
\begin{equation}
\text{grad} \text{div} E = - \sum_{j=1}^{3} \left( \sum_{m=1}^{3} g^{jm} \frac{\partial}{\partial x_m} \left( \frac{1}{\sigma} \sum_{i=1}^{3} \frac{\partial \sigma}{\partial x_i} E^i \right) \right) \frac{\partial}{\partial x_j}. \tag{2.21}
\end{equation}

Applying Maxwell’s equations (1.11) again, we obtain
\begin{equation}
\text{curl} \text{curl} E - i\omega \text{curl} (\mu H) = 0 \quad \text{in } M. \tag{2.22}
\end{equation}

Since $H(x) = (H_1(x), H_2(x), H_3(x))$, we get
\begin{equation}
\text{curl} (\mu H) = \text{curl} (\mu H_1(x), \mu H_2(x), \mu H_3(x)) = \frac{1}{\sqrt{|g|}} \left\{ \left( \frac{\partial}{\partial x_2} \left( \sum_{i=1}^{3} g_{3i}(\mu H^i) \right) - \frac{\partial}{\partial x_3} \left( \sum_{i=1}^{3} g_{2i}(\mu H^i) \right) \right) \frac{\partial}{\partial x_1} \\
+ \left( \frac{\partial}{\partial x_3} \left( \sum_{i=1}^{3} g_{1i}(\mu H^i) \right) - \frac{\partial}{\partial x_1} \left( \sum_{i=1}^{3} g_{3i}(\mu H^i) \right) \right) \frac{\partial}{\partial x_2} + \left( \frac{\partial}{\partial x_1} \left( \sum_{i=1}^{3} g_{2i}(\mu H^i) \right) - \frac{\partial}{\partial x_2} \left( \sum_{i=1}^{3} g_{1i}(\mu H^i) \right) \right) \frac{\partial}{\partial x_3} \right\} = \mu \text{curl} H + \left( \text{grad} \mu \right) \times H. \tag{2.23}
\end{equation}

Here, the second term of the last equality follows from the definition of vector product (see (2.11)). Combining (2.22), (2.23) and Maxwell’s equations, we get
\begin{equation}
\text{curl} \text{curl} E - \omega^2 \mu \sigma E - \left( \text{grad} \left( \log \mu \right) \right) \times \text{curl} E = 0 \quad \text{in } M. \tag{2.24}
\end{equation}

Because
\begin{align*}
H &= \frac{1}{i\omega \mu} \text{curl} E = \frac{1}{i\omega \mu \sqrt{|g|}} \left\{ \left( \frac{\partial}{\partial x_2} \left( \sum_{s=1}^{3} g_{3s}E^s \right) - \frac{\partial}{\partial x_3} \left( \sum_{s=1}^{3} g_{2s}E^s \right) \right) \frac{\partial}{\partial x_1} + \left( \frac{\partial}{\partial x_3} \left( \sum_{s=1}^{3} g_{1s}E^s \right) \right) \frac{\partial}{\partial x_2} \right\} \\
&\quad - \frac{\partial}{\partial x_1} \left( \sum_{s=1}^{3} g_{3s}E^s \right) \frac{\partial}{\partial x_2} + \left( \frac{\partial}{\partial x_1} \left( \sum_{s=1}^{3} g_{2s}E^s \right) - \frac{\partial}{\partial x_2} \left( \sum_{s=1}^{3} g_{1s}E^s \right) \right) \frac{\partial}{\partial x_3},
\end{align*}
we have
\begin{align*}
&\frac{1}{\sqrt{|g|}} \left\{ \left( \frac{\partial}{\partial x_2} \sum_{i=1}^{3} g_{3i}H^i - \frac{\partial}{\partial x_3} \sum_{i=1}^{3} g_{2i}H^i \right) \frac{\partial}{\partial x_1} + \left( \frac{\partial}{\partial x_3} \sum_{i=1}^{3} g_{1i}H^i - \frac{\partial}{\partial x_1} \sum_{i=1}^{3} g_{3i}H^i \right) \frac{\partial}{\partial x_2} \\
&+ \left( \frac{\partial}{\partial x_1} \sum_{i=1}^{3} g_{2i}H^i - \frac{\partial}{\partial x_2} \sum_{i=1}^{3} g_{1i}H^i \right) \frac{\partial}{\partial x_3} \right\}.
\end{align*}
\[\begin{align*}
&= \frac{1}{i\alpha \mu |g|} \left\{ \left( \frac{\partial}{\partial x_1} g_{31} \left( \frac{\partial}{\partial x_2} \sum_{s=1}^{3} g_{3s} E^s \right) - \frac{\partial}{\partial x_3} \left( \sum_{s=1}^{3} g_{2s} E^s \right) \right) + g_{32} \left( \frac{\partial}{\partial x_2} \left( \sum_{s=1}^{3} g_{1s} E^s \right) - \frac{\partial}{\partial x_1} \left( \sum_{s=1}^{3} g_{3s} E^s \right) \right) \\
&+ g_{33} \left( \frac{\partial}{\partial x_1} \left( \sum_{s=1}^{3} g_{2s} E^s \right) - \frac{\partial}{\partial x_3} \left( \sum_{s=1}^{3} g_{1s} E^s \right) \right) \right\} \frac{\partial}{\partial x_1}
\end{align*}\]
Combining Lemma 2.1, Lemma 2.3, (2.21) and (2.22), we obtain

\[
(2.25) \quad - \sum_{j=1}^{3} \left( \Delta x E^{j} + 2 \sum_{k,l,m=1}^{3} g^{ml} \Gamma^{j}_{kn} \frac{\partial E^{k}}{\partial x_{j}} + \sum_{k,l,m=1}^{3} g^{ml} \frac{\partial \Gamma^{j}_{kn}}{\partial x_{j}} E^{k} + \sum_{k,l,m,h=1}^{3} g^{ml} \Gamma^{j}_{kn} \Gamma^{h}_{kn} E^{k} - \sum_{k,l,m,h=1}^{3} g^{ml} \Gamma^{j}_{kh} \Gamma^{h}_{ml} E^{k} - R^{j}_{k} E^{k} + \omega^{2} \mu \sigma E^{j} \right) \frac{\partial}{\partial x_{j}} - \frac{1}{\mu |g|} \left( \left\{ \frac{\partial \mu}{\partial x_{2}} g_{31} - \frac{\partial \mu}{\partial x_{3}} g_{21} \right\} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{3s}}{\partial x_{2}} - g_{2s} \frac{\partial g_{3s}}{\partial x_{3}} + \frac{\partial g_{3s}}{\partial x_{1}} \right) \right) \frac{\partial}{\partial x_{1}} \\
+ \left\{ \frac{\partial \mu}{\partial x_{2}} g_{32} - \frac{\partial \mu}{\partial x_{3}} g_{22} \right\} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{3s}}{\partial x_{2}} - g_{2s} \frac{\partial g_{3s}}{\partial x_{3}} + \frac{\partial g_{3s}}{\partial x_{1}} \right) \frac{\partial}{\partial x_{1}} \\
+ \left\{ \frac{\partial \mu}{\partial x_{3}} g_{33} - \frac{\partial \mu}{\partial x_{1}} g_{31} \right\} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{3s}}{\partial x_{3}} - g_{2s} \frac{\partial g_{3s}}{\partial x_{1}} + \frac{\partial g_{3s}}{\partial x_{2}} \right) \frac{\partial}{\partial x_{1}} \\
+ \left\{ \frac{\partial \mu}{\partial x_{1}} g_{12} - \frac{\partial \mu}{\partial x_{2}} g_{11} \right\} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{1s}}{\partial x_{2}} - g_{2s} \frac{\partial g_{1s}}{\partial x_{1}} + \frac{\partial g_{1s}}{\partial x_{3}} \right) \frac{\partial}{\partial x_{1}} \\
+ \left\{ \frac{\partial \mu}{\partial x_{1}} g_{21} - \frac{\partial \mu}{\partial x_{2}} g_{11} \right\} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{2s}}{\partial x_{2}} - g_{2s} \frac{\partial g_{2s}}{\partial x_{1}} - \frac{\partial g_{2s}}{\partial x_{3}} \right) \frac{\partial}{\partial x_{1}} \\
+ \left\{ \frac{\partial \mu}{\partial x_{1}} g_{22} - \frac{\partial \mu}{\partial x_{2}} g_{22} \right\} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{1s}}{\partial x_{2}} - g_{2s} \frac{\partial g_{1s}}{\partial x_{1}} + \frac{\partial g_{1s}}{\partial x_{3}} \right) \frac{\partial}{\partial x_{1}} \\
+ \left\{ \frac{\partial \mu}{\partial x_{1}} g_{23} - \frac{\partial \mu}{\partial x_{2}} g_{23} \right\} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{1s}}{\partial x_{2}} - g_{2s} \frac{\partial g_{1s}}{\partial x_{1}} + \frac{\partial g_{1s}}{\partial x_{2}} \right) \right) = 0.
\]

Write the above equation as the form of components relative to coordinates:

\[
(2.26) \quad \mathcal{M}_{g} E = \begin{cases} 
\Delta x E^{j} + & \left[ \sum_{l,m=1}^{3} g^{ml} \Gamma^{j}_{kn} \frac{\partial}{\partial x_{j}} \right] \frac{\partial}{\partial x_{j}} \\
+ \frac{1}{\mu |g|} \left\{ \frac{\partial \mu}{\partial x_{2}} g_{31} - \frac{\partial \mu}{\partial x_{3}} g_{21} \right\} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{3s}}{\partial x_{2}} - g_{2s} \frac{\partial g_{3s}}{\partial x_{3}} + \frac{\partial g_{3s}}{\partial x_{1}} \right) \frac{\partial}{\partial x_{1}} \\
+ \frac{\partial \mu}{\partial x_{2}} g_{32} - \frac{\partial \mu}{\partial x_{3}} g_{22} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{3s}}{\partial x_{2}} - g_{2s} \frac{\partial g_{3s}}{\partial x_{3}} + \frac{\partial g_{3s}}{\partial x_{1}} \right) \frac{\partial}{\partial x_{1}} \\
+ \frac{\partial \mu}{\partial x_{3}} g_{33} - \frac{\partial \mu}{\partial x_{1}} g_{31} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{3s}}{\partial x_{3}} - g_{2s} \frac{\partial g_{3s}}{\partial x_{1}} + \frac{\partial g_{3s}}{\partial x_{2}} \right) \frac{\partial}{\partial x_{1}} \\
+ \frac{\partial \mu}{\partial x_{1}} g_{12} - \frac{\partial \mu}{\partial x_{2}} g_{11} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{1s}}{\partial x_{2}} - g_{2s} \frac{\partial g_{1s}}{\partial x_{1}} + \frac{\partial g_{1s}}{\partial x_{3}} \right) \frac{\partial}{\partial x_{1}} \\
+ \frac{\partial \mu}{\partial x_{1}} g_{21} - \frac{\partial \mu}{\partial x_{2}} g_{11} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{2s}}{\partial x_{2}} - g_{2s} \frac{\partial g_{2s}}{\partial x_{1}} - \frac{\partial g_{2s}}{\partial x_{3}} \right) \frac{\partial}{\partial x_{1}} \\
+ \frac{\partial \mu}{\partial x_{1}} g_{22} - \frac{\partial \mu}{\partial x_{2}} g_{22} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{1s}}{\partial x_{2}} - g_{2s} \frac{\partial g_{1s}}{\partial x_{1}} + \frac{\partial g_{1s}}{\partial x_{2}} \right) \frac{\partial}{\partial x_{1}} \\
+ \frac{\partial \mu}{\partial x_{1}} g_{23} - \frac{\partial \mu}{\partial x_{2}} g_{23} \sum_{s=1}^{3} \left( g_{1s} \frac{\partial g_{1s}}{\partial x_{2}} - g_{2s} \frac{\partial g_{1s}}{\partial x_{1}} + \frac{\partial g_{1s}}{\partial x_{2}} \right) \right) \frac{\partial}{\partial x_{1}},
\end{cases}
\]
has the form (see p. 532 of [55])

Under this normal coordinates, we take the outward unit norm of vector
\[ A \]

\[
\begin{align*}
\sum_{m=1}^{3} g^{1m} \left( \frac{\partial}{\partial x_m} \left( \frac{1}{\sigma} \frac{\partial \sigma}{\partial x_1} \right) + \frac{1}{\sigma} \frac{\partial \sigma}{\partial x_1} \frac{\partial}{\partial x_m} \right) + \\
\sum_{m=1}^{3} g^{2m} \left( \frac{\partial}{\partial x_m} \left( \frac{1}{\sigma} \frac{\partial \sigma}{\partial x_2} \right) + \frac{1}{\sigma} \frac{\partial \sigma}{\partial x_2} \frac{\partial}{\partial x_m} \right) + \\
\sum_{m=1}^{3} g^{3m} \left( \frac{\partial}{\partial x_m} \left( \frac{1}{\sigma} \frac{\partial \sigma}{\partial x_3} \right) + \frac{1}{\sigma} \frac{\partial \sigma}{\partial x_3} \frac{\partial}{\partial x_m} \right)
\end{align*}
\]

\[
\left[ \begin{array}{c}
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial x_3}
\end{array} \right] = \begin{bmatrix}
g_{11}(x',x_3) & g_{12}(x',x_3) & 0 \\
g_{21}(x',x_3) & g_{22}(x',x_3) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

By (2.28) we immediately see that the inverse of metric tensor \( g \) in the boundary normal coordinates has form:

\[
g^{-1}(x',x_3) = \begin{bmatrix}
g^{11}(x',x_3) & g^{12}(x',x_3) & 0 \\
g^{21}(x',x_3) & g^{22}(x',x_3) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Under this normal coordinates, we take the outward unit normal vector \( \nu(x) = [0, 0, -1]^T \), where \( A' \) denotes the transpose of a vector \( A \).

In what follows, we will let Greek indices run from 1 to 2 and Roman indices from 1 to 3. Let \( I_3 \) denote the 3 \times 3 identity matrix. In boundary normal coordinates, we have that \( g^{k3} = g^{3k} = 0 \) for any \( 1 \leq k \leq 2 \), and

\[
\begin{align*}
\Gamma^3_{k3} &= \frac{1}{2} \sum_{m=1}^{3} g^{km} \left( \frac{\partial g_{km}}{\partial x_3} + \frac{\partial g_{km}}{\partial x_3} - \frac{\partial g_{km}}{\partial x_m} \right) = \frac{1}{2} \left( \frac{\partial g_{31}}{\partial x_3} + \frac{\partial g_{32}}{\partial x_3} - \frac{\partial g_{33}}{\partial x_3} \right) = 0, \\
\Gamma^k_{33} &= \frac{1}{2} \sum_{m=1}^{3} g^{km} \left( \frac{\partial g_{km}}{\partial x_3} + \frac{\partial g_{km}}{\partial x_3} - \frac{\partial g_{km}}{\partial x_m} \right) = 0
\end{align*}
\]

Thus, in local boundary normal coordinates, we can rewrite (2.26) as

\[
\mathcal{M}_g E = \left\{ \left( \frac{\partial^2}{\partial x_3^2} + \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3} \frac{\partial}{\partial x_3} + \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial^2}{\partial x_3 \partial x_\alpha} + \sum_{\alpha, \beta} \left( g^{\alpha \beta} \sum_{\gamma} \Gamma^\gamma_{\alpha \beta} + \frac{\partial g_{\alpha \beta}}{\partial x_\alpha} \right) \frac{\partial}{\partial x_\beta} \right) I_3 \right\}
\]
Here, for the sake of simplicity, we have used the relationships that

\[(2.30) \quad \frac{1}{g} \frac{\partial |g|}{\partial x_k} = \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_k} = \sum_{\gamma} \Gamma_{\gamma k}^r \]

Consequently, we have

\[(2.31) \quad \mathcal{M}_E = \left\{ \left( \frac{\partial^2}{\partial x_3^2} I \right) + B \left( \frac{\partial}{\partial x_3} I \right) + C \right\} \begin{bmatrix} E^1 \\ E^2 \\ E^3 \end{bmatrix} = 0, \]

where

\[B := \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3} I_3 + \begin{bmatrix} 2 \Gamma_{13}^1 & 2 \Gamma_{23}^1 & 0 \\ 2 \Gamma_{13}^2 & 2 \Gamma_{23}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{\mu} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{\partial \mu}{\partial x_3} & 0 \\ 0 & 0 & -\frac{\partial \mu}{\partial x_3} \end{bmatrix}, \]

\[C := \sum_{\alpha, \beta} \left( g^{\alpha \beta} \frac{\partial^2}{\partial x_3 \partial x_\beta} + \left( g^{\alpha \beta} \sum_{\gamma} \Gamma_{a \gamma}^r + \frac{\partial g_{\alpha \beta}}{\partial x_3} \right) \frac{\partial}{\partial x_\beta} \right) I_3 + \begin{bmatrix} 2 \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{1a}^1 \frac{\partial}{\partial x_3} & 2 \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{2a}^1 \frac{\partial}{\partial x_3} & 2 \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{3a}^1 \frac{\partial}{\partial x_3} \\ 2 \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{2a}^2 \frac{\partial}{\partial x_3} & 2 \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{2a}^2 \frac{\partial}{\partial x_3} & 2 \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{3a}^2 \frac{\partial}{\partial x_3} \\ 2 \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{3a}^3 \frac{\partial}{\partial x_3} & 2 \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{3a}^3 \frac{\partial}{\partial x_3} & 2 \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{3a}^3 \frac{\partial}{\partial x_3} \end{bmatrix} \]
There exists a pseudodifferential operator modulo a smoothing operator, where
\[
(D_{x'})^{2.32} \text{ and } D_{x'} = \frac{\partial}{\partial x_j}.
\]

Let
\[
(a_{11} + \omega^2 \mu \sigma - R_1^1, a_{12} - R_2^1, a_{13} - R_3^1, a_{21} - R^2, a_{22} + \omega^2 \mu \sigma - R^2, a_{23} - R^3, a_{31} - R_1^3, a_{32} - R_2^3, a_{33} + \omega^2 \mu \sigma - R^3)
\]

be the full symbols of \(\Phi(x, D_{x'})\) and \(B(x, D_{x'})\) respectively. Clearly, \(\phi(x, \xi') \sim\)
It follows that
\[ b_0(x, \xi') = \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3} I_3 + \left[ \begin{array}{ccc} 2 \Gamma_{13}^1 & 2 \Gamma_{12}^1 & 0 \\ 2 \Gamma_{23}^2 & 2 \Gamma_{23}^3 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\sigma} \frac{\partial \sigma}{\partial x_1} & \frac{1}{\sigma} \frac{\partial \sigma}{\partial x_2} & \frac{1}{\sigma} \frac{\partial \sigma}{\partial x_3} \end{array} \right] + \frac{1}{\mu} \left[ \begin{array}{ccc} \frac{\partial \mu}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial \mu}{\partial x_1} & \frac{\partial \mu}{\partial x_2} \end{array} \right], \]

\[ c_2(x, \xi') = -\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta I_3, \]

\[ c_1(x, \xi') = i \sum_{\alpha, \beta} \left( g^{\alpha \beta} \sum_\gamma \Gamma_{\gamma \alpha \beta} + \frac{\partial g^{\alpha \beta}}{\partial x_\alpha} \right) \xi_\beta I_3 + \left[ 2i \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{1a}^1 \xi_\beta \right] + \left[ 2i \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{2a}^1 \xi_\beta \right] + \left[ 2i \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{3a}^1 \xi_\beta \right], \]

\[ \sum_{j \geq 0} \phi_{1-j}(x, \xi'), \quad b(x, \xi') = b_0(x, \xi') \quad \text{and} \quad c(x, \xi') = c_2(x, \xi') + c_1(x, \xi') + c_0(x, \xi'), \quad \text{where} \]

\[ (2.34) \]

\[ \frac{1}{\mu} \left[ \begin{array}{ccc} \frac{\partial g_{21}}{\partial x_3} & \frac{\partial g_{21}}{\partial x_1} & \frac{\partial g_{21}}{\partial x_2} \\ \frac{\partial g_{12}}{\partial x_3} & \frac{\partial g_{12}}{\partial x_1} & \frac{\partial g_{12}}{\partial x_2} \end{array} \right], \]

\[ \frac{1}{\mu} \left[ \begin{array}{ccc} \frac{\partial g_{21}}{\partial x_3} & \frac{\partial g_{21}}{\partial x_1} & \frac{\partial g_{21}}{\partial x_2} \\ \frac{\partial g_{12}}{\partial x_3} & \frac{\partial g_{12}}{\partial x_1} & \frac{\partial g_{12}}{\partial x_2} \end{array} \right]. \]

Note that for any smooth vector-valued function \( v \),

\[ \left( \phi \left( \frac{\partial}{\partial x_3} I_3 \right) - \left( \frac{\partial}{\partial x_3} I_3 \right) \phi \right) v = \phi \left( \frac{\partial}{\partial x_3} I_3 \right) v - \left( \frac{\partial}{\partial x_3} I_3 \right) \left( \phi v \right) = \phi \left( \frac{\partial}{\partial x_3} I_3 \right) v - \phi \left( \frac{\partial}{\partial x_3} I_3 \right) v = - \left( \frac{\partial}{\partial x_3} \phi \right) v. \]

It follows that

\[ (2.35) \]

\[ \left( \phi \left( \frac{\partial}{\partial x_3} I_3 \right) - \left( \frac{\partial}{\partial x_3} I_3 \right) \phi \right) = - \frac{\partial}{\partial x_3} \phi. \]
Combining this, the right-hand side of (2.33) and symbol formula for product of two pseudodifferential operators (see, for example, p.37 of [57], p.13 of [55], Theorem 18.1.8 of [26] or [34]) we get the full symbol equation for (2.33):

\[
\sum_{|\sigma| \geq 0} \frac{(-i)^{|\sigma|}}{\vartheta!} \left( \partial_\vartheta^\vartheta \phi \right) \left( \partial_\vartheta^\vartheta \phi \right) - b_0 \phi - \frac{\partial \phi}{\partial x_3} + c = 0,
\]

where \( \partial_\vartheta^\vartheta = \frac{\partial}{\partial x_1^\vartheta}, \partial_\vartheta^\vartheta = \frac{\partial}{\partial x_2^\vartheta}, \) and \( \vartheta = (\vartheta_1, \vartheta_2) \) is a 2-tuple of nonnegative integers.

Group the homogeneous terms of degree two in (2.36) we get

\[
\sum_{i_1, i_2} g^{i_1 i_2} \xi_{i_1} \xi_{i_2} I_3.
\]

Because \( v \) is the outward normal vector of \( \partial M \), we may take

\[
\phi_1(x, \xi') = \sqrt{\sum_{i, j} g^{i j} \xi_i \xi_j} I_3.
\]

The terms of degree one in (2.36) give

\[
\phi_1 \phi_0 + \phi_0 \phi_1 - i \sum_{m=1}^2 \frac{\partial \phi_1}{\partial x_m} \frac{\partial \phi_1}{\partial x_m} - b_0 \phi_1 - \frac{\partial \phi_1}{\partial x_3} + c_1 = 0.
\]

By (2.37) we see that \( \phi_1 \) has commutative property with any \( 3 \times 3 \) matrix, and

\[
\phi_1^{-1}(x, \xi') = \frac{1}{\sqrt{\sum_{i, j} g^{i j} \xi_i \xi_j}} I_3.
\]

From this, we immediately get

\[
\phi_0 = \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta \left\{ i \sum_{m=1}^2 \frac{\partial \phi_1}{\partial x_m} \frac{\partial \phi_1}{\partial x_m} + \frac{1}{2} \sum_{i, j} g^{i j} \frac{\partial g_{i j}}{\partial x_3} I_3 + \begin{pmatrix} 2 \Gamma_{1 3}^3 & 2 \Gamma_{2 3}^3 & 0 \\ 2 \Gamma_{1 3}^3 & 2 \Gamma_{2 3}^3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \phi_1 + \frac{\partial \phi_1}{\partial x_3}
\]

\[
- \frac{i}{\alpha, \beta} \left( \sum_{\gamma} g^{\alpha \beta} \Gamma_{1 \gamma}^{\alpha \gamma} + \frac{\partial g^{\alpha \beta}}{\partial x_\alpha} \right) \xi_\beta I_3 - \begin{pmatrix} 2i \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{1 1}^{\alpha 1} & 2i \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{1 2}^{\alpha 2} & 2i \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{1 3}^{\alpha 3} \\ 2i \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{2 1}^{\alpha 1} & 2i \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{2 2}^{\alpha 2} & 2i \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{2 3}^{\alpha 3} \\ 2i \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{3 1}^{\alpha 1} & 2i \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{3 2}^{\alpha 2} & 2i \sum_{\alpha, \beta} g^{\alpha \beta} \Gamma_{3 3}^{\alpha 3} \end{pmatrix}
\]

\[
- \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
0
\]

\[
0
\]

\[
0
\]
The terms of degree zero are

\[ \phi_1 \phi_{-1} + \phi_{-1} \phi_1 + \phi_0^2 - i \sum_{m=1}^2 \left( \frac{\partial \phi_1}{\partial x_m} \frac{\partial \phi_0}{\partial x_m} + \frac{\partial \phi_0}{\partial x_m} \frac{\partial \phi_1}{\partial x_m} - \frac{1}{2} \sum_{m,l=1}^2 \frac{\partial^2 \phi_1}{\partial x_m \partial x_l} \frac{\partial^2 \phi_1}{\partial x_m \partial x_l} - b_0 \phi_0 - \frac{\partial \phi_0}{\partial x_3} + c_0 = 0. \]

It follows from (2.37) and (2.38) that

\[ (2.40) \]

The proof is completed.
We have obtained the full symbol $\phi(x, \xi') \sim \sum_{\ell=1} \phi_{\ell}(x, \xi')$ of the pseudodifferential operator $\Phi$ from above Proposition 2.4. This implies that modulo a smoothing operator, the pseudodifferential operator $\Phi$ has been determined on $\partial M$.

Recall that if $\omega$ is not a resonant frequency, then for $f \in TH^1(\partial M) \cap C(\partial M)$ with $\text{supp} f \subset \Gamma$, there exists a unique solution $(E, H) \in (\mathcal{D}'(M))^3 \times (\mathcal{D}'(M))^3$ of Maxwell’s equations

(2.42)
\[
\begin{align*}
\text{curl} E &= i\omega \mu H & \text{in } M, \\
\text{curl} H &= -i\omega \sigma E & \text{in } M, \\
\nu \times E &= f & \text{on } \partial M.
\end{align*}
\]

**Proposition 2.5.** If $E$ solves (2.42), then

(2.43)
\[
\left( \frac{\partial}{\partial x_3} I_3 \right) E = \Phi E |_{\partial M}
\]

modulo a smoothing operator.

**Proof.** Clearly, $\mathcal{M}(x, D)E = 0$. It is well-known (see [57], Ch. III, Remark 1.2 & 4.1) that the solution $(E, H)$ of Maxwell’s equations (2.42) is smooth in normal variable, i.e., in boundary normal coordinates $(x', x_3)$ with $x_3 \in [0, T]$, $E \in (C^\infty([0, T]; \mathcal{D}'(\mathbb{R}^2)))^3$ locally. Actually, $E$ is smooth in the interior of $M$ by interior regularity for elliptic equation system $\mathcal{M}_g$ (see, for example, [11] or [21]). From Proposition 2.4, we see that (2.42) is locally equivalent to the following system of equations for $E, U \in (C^\infty([0, T]; \mathcal{D}'(\mathbb{R}^2)))^3$:

\[
\begin{align*}
\left( \frac{\partial}{\partial x_3} I_3 + \Phi \right) E &= U, \\
\nu \times E |_{x_3=0} &= f, \\
\left( \frac{\partial}{\partial x_3} I_3 + B - \Phi \right) U &= W \in (C^\infty([0, T] \times \mathbb{R}^2))^3.
\end{align*}
\]

Making the substitution $t = T - x_3$ for the second equation mentioned above (as done in [37]), we get a backwards generalized heat equation system:

\[
\left( \frac{\partial}{\partial t} I_3 \right) U - (-\Phi + B)U = -W.
\]

In view of the principal symbol $\phi_1(x, \xi')$ of $\Phi$ is strictly positive for any $\xi' \neq 0$, we get that the solution operator for this heat equation system is smooth for $t > 0$ (see p.134 of [57] or [20]). This implies that $U$ is smooth in the interior of $M$, and hence $U |_{x_3=T}$ is smooth. Therefore,

\[
\left( \frac{\partial}{\partial x_3} I_3 \right) E + \Phi E \in (C^\infty([0, T] \times \mathbb{R}^2))^3
\]

locally. Setting $Rf = U |_{\partial M}$, we immediately see that $R$ is a smoothing operator and

\[
\left. \left( \frac{\partial}{\partial x_3} I_3 \right) E \right|_{\partial M} = -\Phi E |_{\partial M} + Rf.
\]

The desired result is obtained. \qed

Recall that if $(E, H)$ solves (2.42) then the electromagnetic Dirichlet-to-Neumann map $\Lambda_{g, \Gamma}$ is the map $\Lambda_{g, \Gamma} : \nu \times E |_{\Gamma} \rightarrow \nu \times H |_{\Gamma}$, where $\nu = (0, 0, -1)$ is the outward unit normal to the boundary $\Gamma \subset \partial M$ in the
boundary normal coordinates. Then we have

\[
\mathbf{v} \times E = \sqrt{|g|} \left\{ \begin{array}{l}
g_{11} \partial_{x_1} + g_{12} \partial_{x_2} + g_{13} \partial_{x_3} \\
g_{21} \partial_{x_1} + g_{22} \partial_{x_2} + g_{23} \partial_{x_3} \\
g_{31} \partial_{x_1} + g_{32} \partial_{x_2} + g_{33} \partial_{x_3} \end{array} \right. 
\]

which can be rewritten as the components of vector field with respect to coordinates basis \( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \):

\[
\mathbf{v} \times E = \left[ \begin{array}{c}
\sqrt{|g|}(-g^{21}E_1 + g^{11}E_2) \\
\sqrt{|g|}(-g^{22}E_1 + g^{12}E_2) \\
\sqrt{|g|}(-g^{23}E_1 + g^{13}E_2) \\
\end{array} \right] 
\]

because of \( g^{13} = g^{23} = 0 \) in boundary normal coordinates. From (2.15) and (2.42), we have

\[
\mathbf{v} \times H = \frac{1}{i \omega \mu} \mathbf{v} \times (\text{curl} \ E) 
\]

Thus, by rewriting the components of vector fields with respect to the coordinates basis \( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \), the map \( \Lambda : \mathbf{v} \times E \to \mathbf{v} \times H \) becomes

\[
\Lambda : \left[ \begin{array}{c}
\sqrt{|g|}(-g^{21}E_1 + g^{11}E_2) \\
\sqrt{|g|}(-g^{22}E_1 + g^{12}E_2) \\
0 \\
\end{array} \right] \mapsto \frac{1}{i \omega \mu} \left[ \begin{array}{c}
g^{11} \sum_{l=1}^{3} \left( \frac{\partial(g_{11}E_l^l)}{\partial x_1} - \frac{\partial(g_{31}E_l^l)}{\partial x_3} \right) \\
g^{12} \sum_{l=1}^{3} \left( \frac{\partial(g_{12}E_l^l)}{\partial x_1} - \frac{\partial(g_{32}E_l^l)}{\partial x_3} \right) \\
g^{13} \sum_{l=1}^{3} \left( \frac{\partial(g_{13}E_l^l)}{\partial x_1} - \frac{\partial(g_{33}E_l^l)}{\partial x_3} \right) \\
0 \\
\end{array} \right]. 
\]

Note that

(2.44)

\[
\begin{align*}
g^{11}g_{11} + g^{12}g_{21} &= 1, & g^{11}g_{12} + g^{12}g_{22} &= 0, \\
g^{21}g_{11} + g^{22}g_{21} &= 0, & g^{21}g_{12} + g^{22}g_{22} &= 1. 
\end{align*}
\]
Therefore, in boundary normal coordinates, we find by (2.44) that

\[ \mathbf{v} \times \mathbf{H} = \frac{1}{i \omega \mu} \begin{bmatrix} g^{11} \left( \sum_{\alpha} \frac{\partial (g_{1\alpha} E^\alpha)}{\partial x_3} - \frac{\partial E^3}{\partial x_3} \right) + g^{21} \left( \sum_{\alpha} \frac{\partial (g_{2\alpha} E^\alpha)}{\partial x_3} - \frac{\partial E^3}{\partial x_3} \right) \\
0 \end{bmatrix} \]

\[ = \frac{1}{i \omega \mu} \begin{bmatrix} \sum_{\alpha} \left( g^{11} g_{1\alpha} \frac{\partial E^\alpha}{\partial x_3} + g^{11} \frac{\partial g_{1\alpha}}{\partial x_3} E^\alpha + g^{21} g_{2\alpha} \frac{\partial E^\alpha}{\partial x_3} + g^{21} \frac{\partial g_{2\alpha}}{\partial x_3} E^\alpha \right) - g^{11} \frac{\partial E^3}{\partial x_3} - g^{21} \frac{\partial E^3}{\partial x_3} \\
0 \end{bmatrix} \]

\[ = \frac{1}{i \omega \mu} \begin{bmatrix} \sum_{\alpha} \left( g^{12} g_{1\alpha} \frac{\partial E^\alpha}{\partial x_3} + g^{12} \frac{\partial g_{1\alpha}}{\partial x_3} E^\alpha + g^{22} g_{2\alpha} \frac{\partial E^\alpha}{\partial x_3} + g^{22} \frac{\partial g_{2\alpha}}{\partial x_3} E^\alpha \right) - g^{12} \frac{\partial E^3}{\partial x_3} - g^{22} \frac{\partial E^3}{\partial x_3} \\
0 \end{bmatrix} \]

\[ = \frac{1}{i \omega \mu} \begin{bmatrix} \frac{\partial E^1}{\partial x_3} + \sum_{\alpha, \beta} g^{\beta 1} \frac{\partial g_{\beta \alpha}}{\partial x_3} E^\alpha - \sum_{\alpha} g^{\alpha 1} \frac{\partial E^3}{\partial x_3} \\
0 \end{bmatrix} \]

According to (2.20), we have

\[ 0 = \sigma \text{div} E + \sum_{k=1}^{3} \frac{\partial \sigma}{\partial x_k} E^k \]

\[ = \sigma \left( \sum_{k=1}^{3} \frac{\partial E^k}{\partial x_k} + \sum_{k,l=1}^{3} \Gamma^l_{kl} E^k \right) + \sum_{k=1}^{3} \frac{\partial \sigma}{\partial x_k} E^k. \]

This implies

\[ \sigma \left( \frac{\partial E^3}{\partial x_3} + \sum_{l=1}^{3} \Gamma^l_{3l} E^3 \right) + \frac{\partial \sigma}{\partial x_3} E^3 = -\sigma \left( \sum_{k=1}^{3} \frac{\partial E^k}{\partial x_k} + \sum_{k=1}^{3} \sum_{l=1}^{3} \Gamma^l_{kl} E^k \right) - \sum_{k=1}^{3} \frac{\partial \sigma}{\partial x_k} E^k, \]

i.e.,

\[ \sigma \left( \frac{\partial E^3}{\partial x_3} + \sum_{l=1}^{3} \Gamma^l_{3l} E^3 \right) + \frac{\partial \sigma}{\partial x_3} E^3 = -\sigma \left( \sum_{\beta} \frac{\partial E^\beta}{\partial x_\beta} + \sum_{\beta} \sum_{\gamma} \Gamma^\gamma_{\beta \gamma} E^\beta \right) - \sum_{\beta} \frac{\partial \sigma}{\partial x_\beta} E^\beta. \]

It follows from Proposition 2.5 that

\[ \left( \frac{\partial}{\partial x_3} l_3 \right) E = \Phi E, \]

or equivalently,

\[
\begin{bmatrix}
\frac{\partial}{\partial x_3} & 0 & 0 \\
0 & \frac{\partial}{\partial x_3} & 0 \\
0 & 0 & \frac{\partial}{\partial x_3}
\end{bmatrix}
\begin{bmatrix}
E^1 \\
E^2 \\
E^3
\end{bmatrix}
=
\begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} \\
\Phi_{21} & \Phi_{22} & \Phi_{23} \\
\Phi_{31} & \Phi_{32} & \Phi_{33}
\end{bmatrix}
\begin{bmatrix}
E^1 \\
E^2 \\
E^3
\end{bmatrix}
\]
This can be written as
\[
(2.47) \quad \frac{\partial E^k}{\partial x_3} = \Phi^{k1}E^1 + \Phi^{k2}E^2 + \Phi^{k3}E^3 \quad \text{for } k = 1, 2, 3.
\]

By taking \( k = 3 \) in (2.47) and inserting the result into (2.46) we get
\[
\sigma(\Phi^{31}E^1 + \Phi^{32}E^2 + \Phi^{33}E^3 + \sum_\beta \Gamma^3_\beta E^3) + \frac{\partial \sigma}{\partial x_3}E^3 = -\sigma \left( \sum_\beta \frac{\partial E^\beta}{\partial x_3} + \sum_\gamma (\sum_\beta \Gamma^\gamma_\beta E^\beta) \right) \frac{\partial \sigma}{\partial x_3}E^\beta,
\]
\[\text{i.e.,}\]
\[
(2.48) \quad \left( \sigma(\Phi^{33} + \sum_\beta \Gamma^3_\beta) + \frac{\partial \sigma}{\partial x_3} \right)E^3 = -\sigma \left( \Phi^{31}E^1 + \Phi^{32}E^2 + \sum_\beta \frac{\partial E^\beta}{\partial x_3} + \sum_\gamma (\sum_\beta \Gamma^\gamma_\beta E^\beta) \right) \frac{\partial \sigma}{\partial x_3}E^\beta.
\]

Let \( Q(x', D_x) \) be a pseudodifferential operator of order \(-1\) in \( x' \) such that
\[
(2.49) \quad Q \left( \sigma(\Phi^{33} + \sum_\beta \Gamma^3_\beta) + \frac{\partial \sigma}{\partial x_3} \right) = I
\]
modulo a smoothing operator (We will determine \( Q \) by calculating the full symbol of \( Q \) in section 3). Then
\[
E^3 = -Q \left[ \sigma \left( \Phi^{31}E^1 + \Phi^{32}E^2 + \sum_\beta \frac{\partial E^\beta}{\partial x_3} + \sum_\gamma (\sum_\beta \Gamma^\gamma_\beta E^\beta) \right) + \sum_\beta \frac{\partial \sigma}{\partial x_3}E^\beta \right]
\]
\[
- Q \left[ \sigma \left( \sum_\beta \Phi^{3\beta}E^\beta + \sum_\beta \frac{\partial E^\beta}{\partial x_3} + \sum_\gamma (\sum_\beta \Gamma^\gamma_\beta E^\beta) + \sum_\beta \frac{\partial \sigma}{\partial x_3}E^\beta \right) \right]
\]
\[
= -Q \left[ \sum_\beta \left( \sigma \left( \Phi^{3\beta} + \frac{\partial}{\partial x_3} + \sum_\gamma \Gamma^\gamma_\beta \right) + \frac{\partial \sigma}{\partial x_3} \right)E^\beta \right].
\]

Inserting this into the right-hand side of (2.45), we find by (2.47) and (2.50) that
\[
\nu \times H = \frac{1}{i \omega \mu} \left[ \begin{array}{c} \Phi^{11}E^1 + \Phi^{12}E^2 + \Phi^{13}E^3 + \sum_{\alpha, \beta} g^{\alpha 1} \frac{\partial g_{\alpha \beta}}{\partial x_3} E^\alpha - \sum_{\alpha} g^{\alpha 1} \frac{\partial g_{\alpha 3}}{\partial x_3} E^\alpha \\ \Phi^{21}E^1 + \Phi^{22}E^2 + \Phi^{23}E^3 + \sum_{\alpha, \beta} g^{\alpha 2} \frac{\partial g_{\alpha \beta}}{\partial x_3} E^\alpha - \sum_{\alpha} g^{\alpha 2} \frac{\partial g_{\alpha 3}}{\partial x_3} E^\alpha \\ 0 \end{array} \right]
\]
\[
= \frac{1}{i \omega \mu} \left[ \begin{array}{c} \sum_\beta \Phi^{1\beta}E^\beta + \frac{\partial}{\partial x_3} \left( \Phi^{13} + \sum_{\alpha} g^{\alpha 1} \frac{\partial}{\partial x_3} \right) E^3 + \sum_{\alpha, \beta} g^{\alpha 1} \frac{\partial g_{\alpha \beta}}{\partial x_3} E^\beta \\ \sum_\beta \Phi^{2\beta}E^\beta + \frac{\partial}{\partial x_3} \left( \Phi^{23} + \sum_{\alpha} g^{\alpha 2} \frac{\partial}{\partial x_3} \right) E^3 + \sum_{\alpha, \beta} g^{\alpha 2} \frac{\partial g_{\alpha \beta}}{\partial x_3} E^\beta \\ 0 \end{array} \right]
\]
\[
= \frac{1}{i \omega \mu} \left[ \begin{array}{c} \sum_\beta \Phi^{1\beta}E^\beta + \left( \Phi^{13} + \sum_{\alpha} g^{\alpha 1} \frac{\partial}{\partial x_3} \right) \left\{ Q \left[ \sigma \left( \sum_{\alpha, \beta} \frac{\partial}{\partial x_3} \Phi^{\beta 3} + \sum_{\gamma} \Gamma^\gamma_\beta E^\beta \right) + \sum_\beta \frac{\partial \sigma}{\partial x_3} E^\beta \right] \right\} + \sum_{\alpha, \beta} g^{\alpha 1} \frac{\partial g_{\alpha \beta}}{\partial x_3} E^\beta \\ \sum_\beta \Phi^{2\beta}E^\beta + \left( \Phi^{23} + \sum_{\alpha} g^{\alpha 2} \frac{\partial}{\partial x_3} \right) \left\{ Q \left[ \sigma \left( \sum_{\alpha, \beta} \frac{\partial}{\partial x_3} \Phi^{\beta 3} + \sum_{\gamma} \Gamma^\gamma_\beta E^\beta \right) + \sum_\beta \frac{\partial \sigma}{\partial x_3} E^\beta \right] \right\} + \sum_{\alpha, \beta} g^{\alpha 2} \frac{\partial g_{\alpha \beta}}{\partial x_3} E^\beta \\ 0 \end{array} \right]
\]
\[
= \frac{1}{i \omega \mu} \left[ \sum_\beta \Phi^{1\beta}E^\beta + \left( \Phi^{13} + \sum_{\alpha} g^{\alpha 1} \frac{\partial}{\partial x_3} \right) Q \left[ \sigma \left( \sum_{\beta} \frac{\partial}{\partial x_3} \Phi^{\beta 3} + \sum_{\gamma} \Gamma^\gamma_\beta E^\beta \right) + \sum_\beta \frac{\partial \sigma}{\partial x_3} E^\beta \right] + \sum_{\alpha, \beta} g^{\alpha 1} \frac{\partial g_{\alpha \beta}}{\partial x_3} E^\beta \\ \sum_\beta \Phi^{2\beta}E^\beta + \left( \Phi^{23} + \sum_{\alpha} g^{\alpha 2} \frac{\partial}{\partial x_3} \right) Q \left[ \sigma \left( \sum_{\beta} \frac{\partial}{\partial x_3} \Phi^{\beta 3} + \sum_{\gamma} \Gamma^\gamma_\beta E^\beta \right) + \sum_\beta \frac{\partial \sigma}{\partial x_3} E^\beta \right] + \sum_{\alpha, \beta} g^{\alpha 2} \frac{\partial g_{\alpha \beta}}{\partial x_3} E^\beta \\ 0 \end{array} \right]
\]
\[
= \frac{1}{i \omega \mu} \left[ \sum_\beta \Phi^{1\beta}E^\beta + \left( \Phi^{13} + \sum_{\alpha} g^{\alpha 1} \frac{\partial}{\partial x_3} \right) Q \left[ \sigma \left( \sum_{\beta} \frac{\partial}{\partial x_3} \Phi^{\beta 3} + \sum_{\gamma} \Gamma^\gamma_\beta E^\beta \right) + \sum_\beta \frac{\partial \sigma}{\partial x_3} E^\beta \right] + \sum_{\alpha, \beta} g^{\alpha 1} \frac{\partial g_{\alpha \beta}}{\partial x_3} E^\beta \\ \sum_\beta \Phi^{2\beta}E^\beta + \left( \Phi^{23} + \sum_{\alpha} g^{\alpha 2} \frac{\partial}{\partial x_3} \right) Q \left[ \sigma \left( \sum_{\beta} \frac{\partial}{\partial x_3} \Phi^{\beta 3} + \sum_{\gamma} \Gamma^\gamma_\beta E^\beta \right) + \sum_\beta \frac{\partial \sigma}{\partial x_3} E^\beta \right] + \sum_{\alpha, \beta} g^{\alpha 2} \frac{\partial g_{\alpha \beta}}{\partial x_3} E^\beta \\ 0 \end{array} \right].
\]
Then
\[
\begin{pmatrix}
L^{11} & L^{12} \\
L^{21} & L^{22}
\end{pmatrix}
\begin{bmatrix}
E^1 \\
E^2
\end{bmatrix},
\]
where
\[
L_{jk} = \Phi_{jk} - \Phi_{j3} Q \left[ \sigma \left( \frac{\partial}{\partial x_k} + \Phi_{3k} + \sum_{\gamma} \Gamma_{\gamma}^k \right) \right] + \sum_{a} \left\{ g^{\alpha j} \frac{\partial}{\partial x_a} \left( Q \left[ \sigma \left( \frac{\partial}{\partial x_k} + \Phi_{3k} + \sum_{\gamma} \Gamma_{\gamma}^k \right) \right] \right) \right\} + g^{\alpha j} \left[ \sigma \left( \frac{\partial}{\partial x_k} + \Phi_{3k} + \sum_{\gamma} \Gamma_{\gamma}^k \right) \right],
\]
and
\[
\partial (Pu) = \left( \frac{\partial P}{\partial x_\alpha} + P \frac{\partial}{\partial x_\alpha} \right) u,
\]
where the operators \( \frac{\partial P}{\partial x_\alpha} \) and \( P \frac{\partial}{\partial x_\alpha} \) have the full symbols \( \frac{\partial p(x, \xi)}{\partial x_\alpha} \) and \( p(x, \xi)(i\xi_\alpha) \), respectively. Indeed, according to the definition of the symbol in a Riemannian manifold (for example, §10 of Chapter 7 in [55]) in every local chart we have
\[
Pu(x) = \int_{\mathbb{R}^n} e^{i(x, \xi)} p(x, \xi) \hat{u}(\xi) d\xi,
\]
so that
\[
\frac{\partial (Pu(x))}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha} \int_{\mathbb{R}^n} e^{i(x, \xi)} p(x, \xi) \hat{u}(\xi) d\xi = \int_{\mathbb{R}^n} e^{i(x, \xi)} \left( \frac{\partial p(x, \xi)}{\partial x_\alpha} + p(x, \xi)(i\xi_\alpha) \right) \hat{u}(\xi) d\xi,
\]
which implies that the operator \( u \mapsto \frac{\partial (Pu(x))}{\partial x_\alpha} \) has the full symbol \( \frac{\partial p(x, \xi)}{\partial x_\alpha} + p(x, \xi)(i\xi_\alpha) \). Or equivalently, \( \frac{\partial (Pu)}{\partial x_\alpha} \) can be written as (2.52).

Let \( \Phi: (C^\infty(M))^3 \to (C^\infty(M))^2 \) be a linear isometric operator defined by
\[
\Phi \left( \begin{bmatrix} A_1 \\ A_2 \\ 0 \end{bmatrix} \right) \mapsto \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.
\]
Then
\[
\Phi(\nu \times E) = \Phi \left[ \begin{bmatrix} \sqrt{|g|}(-g^{21}E^1 + g^{11}E^2) \\ 0 \end{bmatrix} \right] = \sqrt{|g|} \left[ \begin{bmatrix} -g^{21}E^1 + g^{11}E^2 \\ -g^{22}E^1 + g^{12}E^2 \end{bmatrix} \right] = \sqrt{|g|} \left[ \begin{bmatrix} g^{21} & g^{11} \\ g^{22} & g^{12} \end{bmatrix} \right] \begin{bmatrix} E^1 \\ E^2 \end{bmatrix}
\]
and
\[
\Phi(\nu \times H) = \frac{1}{i\omega \mu} \begin{pmatrix} L^{11} & L^{12} \\
L^{21} & L^{22}\end{pmatrix} \begin{bmatrix} E^1 \\ E^2 \end{bmatrix}.
\]
From (2.53) we have
\[
\begin{bmatrix}
E^1 \\
E^2
\end{bmatrix} = \frac{1}{\sqrt{|g|}} \begin{bmatrix}
-g^{21} & g^{11}
\end{bmatrix}^{-1} (\varphi(v \times E)) = \frac{1}{\sqrt{|g|}} \begin{bmatrix}
-g_{12} & -g_{22} \\
g_{11} & g_{12}
\end{bmatrix} \varphi(v \times E),
\]
so that
\[
(2.55) \quad \varphi(v \times H) = \frac{1}{|i \omega \mu \sqrt{|g|}|} \begin{bmatrix}
L^{11} & L^{12} \\
L^{21} & L^{22}
\end{bmatrix} \begin{bmatrix}
-g_{12} & -g_{22} \\
g_{11} & g_{12}
\end{bmatrix} \varphi(v \times E),\]

Thus we have obtained the following proposition:

**Proposition 2.6.** In boundary normal coordinates, the electromagnetic Dirichlet-to-Neumann map \(\Lambda_{g, \Gamma}\) is equivalent to the following operator which maps \(\varphi(v \times E)\) to \(\varphi(v \times H)\) defined by
\[
(2.56) \quad \varphi(v \times H) = \begin{bmatrix}
\Lambda_{11} & \Lambda_{12} \\
\Lambda_{21} & \Lambda_{22}
\end{bmatrix} \varphi(v \times E) \quad \text{on} \ \Gamma,
\]

where
\[
(2.57) \quad \Lambda_{11} = \frac{1}{|i \omega \mu \sqrt{|g|}|} (-g_{12}L^{11} + g_{11}L^{12}), \quad \Lambda_{12} = \frac{1}{|i \omega \mu \sqrt{|g|}|} (-g_{22}L^{11} + g_{12}L^{12}),
\]
\[
\Lambda_{21} = \frac{1}{|i \omega \mu \sqrt{|g|}|} (-g_{12}L^{21} + g_{11}L^{22}), \quad \Lambda_{22} = \frac{1}{|i \omega \mu \sqrt{|g|}|} (-g_{22}L^{21} + g_{12}L^{22}),
\]
and \(L^{11}, L^{12}, L^{21}, L^{22}\) are given by (2.51).

We will still denote the above equivalent operator by \(\Lambda_{g, \Gamma}\).

## 3 Determining metric of manifold from the electromagnetic Dirichlet-to-Neumann map

We first calculate the full symbol of \(Q\), which was introduced in section 2. Recall that (see (2.49) and (2.30))
\[
Q \{ \sigma (\Phi^{33} + \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x^3}) + \frac{\partial \sigma}{\partial x^3} \} = I.
\]

We wish to define \(Q\) so that
\[
(3.58) \quad \imath \left( Q \left( \sigma (\Phi^{33} + \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x^3}) + \frac{\partial \sigma}{\partial x^3} \right) \right) \sim 1,
\]
where \(\imath(P)\) denotes the full symbol of pseudodifferential operator \(P\). Let \(q_l(x, \xi') \sim \sum_{l \leq -1} q_l(x, \xi')\) and \(\phi \sim \sum_{l \leq 1} \phi_l(x, \xi')\) be the full symbols of pseudodifferential operators \(Q\) and \(\Phi\), respectively. Here, the definition of \(\Phi\) is as in Proposition 2.4, and
\[
\phi_l = \begin{bmatrix}
\phi_{11}^l & \phi_{12}^l & \phi_{13}^l \\
\phi_{21}^l & \phi_{22}^l & \phi_{23}^l \\
\phi_{31}^l & \phi_{32}^l & \phi_{33}^l
\end{bmatrix}
\]
is the symbol matrix of \(\Phi\) with homogenous of degree \(l\) in \(\xi'\), \(l = 1, 0, -1, -2, \ldots\).
Then (3.58) leads to the following full symbol equation:

\[
q_{-1}(\sigma \phi_1^{33}) = 1,
\]

so that

\[
q_{-1} = \frac{1}{\sigma \phi_1^{33}} = \frac{1}{\sigma \sqrt{\sum_{\alpha, \beta} g_{\alpha \beta} \zeta_{\alpha} \zeta_{\beta}}}.
\]

The terms of degree \(-1\) in (3.59) are

\[
q_{-1}\left(\sigma \left(\phi_0^{33} + \frac{1}{2} \sum_{\alpha, \beta} g_{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3}\right) + \frac{\partial \sigma}{\partial x_3}\right) + q_{-2}(\sigma \phi_1^{33}) - i \sum_{m=1}^{2} \frac{\partial q_{-1}}{\partial x_m} \frac{\partial (\sigma \phi_1^{33})}{\partial x_m} = 0,
\]

which implies

\[
q_{-2} = -\frac{1}{\sigma \phi_1^{33}} \left\{ q_{-1}\left(\sigma \left(\phi_0^{33} + \frac{1}{2} \sum_{\alpha, \beta} g_{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3}\right) + \frac{\partial \sigma}{\partial x_3}\right) + q_{-2}(\sigma \phi_1^{33}) - i \sum_{m=1}^{2} \frac{\partial q_{-1}}{\partial x_m} \frac{\partial (\sigma \phi_1^{33})}{\partial x_m} \right\}.
\]

The terms of degree \(-2\) in (3.59) are

\[
q_{-1}(\sigma \phi_1^{33}) + q_{-2}(\sigma \left(\phi_0^{33} + \frac{1}{2} \sum_{\alpha, \beta} g_{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3}\right) + \frac{\partial \sigma}{\partial x_3}\right) + q_{-3}(\sigma \phi_1^{33}) - i \sum_{m=1}^{2} \frac{\partial q_{-1}}{\partial x_m} \frac{\partial (\sigma \phi_1^{33})}{\partial x_m} - \frac{1}{2} \sum_{m,k=1}^{2} \frac{\partial^2 q_{-1}}{\partial x_m \partial x_k} \frac{\partial^2 (\sigma \phi_1^{33})}{\partial x_m \partial x_k} = 0,
\]

so

\[
q_{-3} = -\frac{1}{\sigma \phi_1^{33}} \left\{ q_{-1}(\sigma \phi_1^{33}) + q_{-2}\left(\sigma \left(\phi_0^{33} + \frac{1}{2} \sum_{\alpha, \beta} g_{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3}\right) + \frac{\partial \sigma}{\partial x_3}\right) - i \sum_{m=1}^{2} \frac{\partial q_{-1}}{\partial x_m} \frac{\partial (\sigma \phi_1^{33})}{\partial x_m} - \frac{1}{2} \sum_{m,k=1}^{2} \frac{\partial^2 q_{-1}}{\partial x_m \partial x_k} \frac{\partial^2 (\sigma \phi_1^{33})}{\partial x_m \partial x_k} \right\}.
\]

Proceeding recursively, the terms of degree \(-m, (m > 1)\), are

\[
q_{-m-1}(\sigma \phi_1^{33}) + \sum_{k,l,\phi} \frac{(-1)^{|\phi|}}{\phi!} \left(\frac{\partial}{\partial \phi} q_{-k-1}\right) \left(\frac{\partial}{\partial \phi} \phi_1^{33}\right) = 0,
\]

where

\[
\phi_1^{33} = \begin{cases} 
\sigma \phi_1^{33} & \text{if } l \neq 0, \\
\sigma(\phi_0^{33} + \frac{1}{2} \sum_{\alpha, \beta} g_{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3}) + \frac{\partial \sigma}{\partial x_3} & \text{if } l = 0,
\end{cases}
\]
and hence

\[
q_{m-1} = -\frac{1}{\sigma \Phi_l^{3s}} \left( \sum_{k,l,\theta} \frac{(-i)^{|\theta|}}{\theta!} \left( \partial_\theta^q q_{-k} \right) \left( \partial_\theta \Phi_l^{3s} \right) \right), \quad m > 1.
\]

The following calculation will be needed late. It follows from the symbol formula of product of two pseudodifferential operators (see p.37 of [57] or p.13 of [55]) that the full symbol of operator \(Q \left( \sigma \frac{\partial}{\partial x_i} + \Phi^{3s} + \frac{1}{2} \sum_{\alpha,\beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_i} + \frac{\partial \sigma}{\partial x_i} \right)\) for each \(s = 1, 2\) is

\[
\sum_{|\theta| \geq 0} \frac{(-i)^{|\theta|}}{\theta!} \left( \partial^\theta \left( q_{-1} + q_{-2} + q_{-3} + \cdots \right) \right) \left( \partial^\theta \left( \sigma (i\xi_s + \Phi_l^{3s}) + \frac{1}{2} \sum_{\alpha,\beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_i} + \Phi_l^{3s} + \cdots \right) \right).
\]

The corresponding terms of degree zero, degree \(-1\) and degree \(-2\) are

\[
q_{-1} \left( \sigma (i\xi_s + \Phi_l^{3s}) \right),
\]

\[
q_{-1} \left( \sigma (i\xi_s + \Phi_l^{3s}) + \frac{1}{2} \sum_{\alpha,\beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_i} + \frac{\partial \sigma}{\partial x_i} \right) + q_{-2} \left( \sigma (i\xi_s + \Phi_l^{3s}) \right) - i \sum_{m=1}^{2} \frac{\partial q_{-m}}{\partial x_m} \frac{\partial (\sigma (i\xi_s + \Phi_l^{3s}))}{\partial x_m},
\]

\[
q_{-1} (\sigma \Phi_l^{3s}) + q_{-2} \left( \sigma (i\xi_s + \Phi_l^{3s}) + \frac{1}{2} \sum_{\alpha,\beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_i} + \frac{\partial \sigma}{\partial x_i} \right) + q_{-3} \left( \sigma (i\xi_s + \Phi_l^{3s}) \right) - i \sum_{m=1}^{2} \frac{\partial q_{-m}}{\partial x_m} \frac{\partial (\sigma (i\xi_s + \Phi_l^{3s}))}{\partial x_m} - i \sum_{m, l=1}^{2} \frac{\partial^2 q_{-1}}{\partial x_m \partial x_l} \frac{\partial^2 (\sigma (i\xi_s + \Phi_l^{3s}))}{\partial x_m \partial x_l},
\]

respectively. Generally, the terms of degree \(-m\) of the symbol of \(Q \left( \sigma \frac{\partial}{\partial x_i} + \Phi^{3s} + \frac{1}{2} \sum_{\alpha,\beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_i} + \frac{\partial \sigma}{\partial x_i} \right)\) for each \(s = 1, 2\) is

\[
\sum_{k,l,\theta} \frac{(-i)^{|\theta|}}{\theta!} \left( \partial_\theta^q q_{-k} \right) \left( \partial_\theta \Phi_l^{3s} \right),
\]

where

\[
\Phi_l^{3s} = \begin{cases} 
\sigma \Phi_l^{3s} & \text{if } l \leq -1, \\
\sigma (i\xi_s + \Phi_l^{3s}) & \text{if } l = 0, \\
\sigma (i\xi_s + \Phi_l^{3s}) & \text{if } l = 1.
\end{cases}
\]

**Proposition 3.1.** Suppose \(\text{dim} M = 3\), and assume the real parts of electromagnetic parameters \(\mu\) and \(\sigma\) are positive functions in \(\bar{M}\). Let \((x_1, x_2)\) be any local coordinates for an open set \(W \subset \Gamma \subset \partial M\), and let \(\psi \sim \sum_{j \geq 1} \psi_j\) denote the full symbol of \(\Lambda_{\psi, \Gamma}\) in these coordinates. Then, \(g_{jk}\) and their partial derivatives up to order \(m\) are determined by \(\psi_1, \psi_0, \cdots, \psi_{m+1}\) on \(\Gamma\) for any \(m \geq 0\). Furthermore, for any \(x_0 \in W\), the full Taylor series of \(g\) at \(x_0\) in boundary normal coordinates is given by explicit formula in terms of the matrix-valued functions \(\{\psi_j\}_{j \leq 1}\) and their tangential derivatives at \(x_0\).

**Proof:** Denote by \((x_1, x_2, x_3)\) the boundary normal coordinates associated with \(x' = (x_1, x_2)\) as in section 2. According to the form of metric (2.28) which we have chosen, we immediately see that it suffices to show...
that the matrix-valued functions \( \{ \psi_j \} \) determine the metric \( [g_{\alpha \beta}]_{2 \times 2} \) and all its normal derivatives along \( \Gamma \subset \partial M \). Noticing that \( \frac{\partial g_{\alpha \beta}}{\partial x_3} = - \sum_{\rho, \gamma} g_{\alpha \rho} \frac{\partial g_{\rho \gamma}}{\partial x_3} g_{\gamma \beta} \), it is also enough to determine the inverse matrix \( [g^{\alpha \beta}]_{2 \times 2} \) and all its normal derivatives.

First, according to (2.57) and (2.51) we have

\[
\Lambda^{11} = \frac{1}{i \omega \mu \sqrt{|g|}} \left( -g_{12} \Lambda^{12} + g_{11} L^{12} \right)
\]

\[
= \frac{1}{i \omega \mu \sqrt{|g|}} \left\{ -g_{12} \left[ \Phi^{11} - \Phi^{13} Q \left( \sigma \left( \frac{\partial}{\partial x_1} + \Phi^{31} + \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_1} \right) + \frac{\partial \sigma}{\partial x_1} \right) + \sum_{\alpha} g^{\alpha 1} \frac{\partial}{\partial x_\alpha} \left( Q \left( \sigma \left( \frac{\partial}{\partial x_1} + \Phi^{31} + \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_1} \right) + \frac{\partial \sigma}{\partial x_1} \right) \right] 
\right. \\
+ \sum_{\beta, \gamma} g^{\beta \gamma} \left( g^{\beta \gamma} \frac{\partial g_{\beta \gamma}}{\partial x_1} + \frac{\partial \sigma}{\partial x_1} \right) + \sum_{\alpha} g^{\alpha 1} \frac{\partial}{\partial x_\alpha} \left( g^{\alpha 1} \frac{\partial g_{\alpha 1}}{\partial x_3} \right) \right\} + g_{11} \left[ \Phi^{12} - \Phi^{13} Q \left( \sigma \left( \frac{\partial}{\partial x_2} + \Phi^{32} + \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_2} \right) + \frac{\partial \sigma}{\partial x_2} \right) + \sum_{\alpha} g^{\alpha 1} \frac{\partial}{\partial x_\alpha} \left( Q \left( \sigma \left( \frac{\partial}{\partial x_2} + \Phi^{32} + \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_2} \right) + \frac{\partial \sigma}{\partial x_2} \right) \right] 
\left. + \sum_{\beta, \gamma} g^{\beta \gamma} \left( g^{\beta \gamma} \frac{\partial g_{\beta \gamma}}{\partial x_2} + \frac{\partial \sigma}{\partial x_2} \right) + \sum_{\alpha} g^{\alpha 1} \frac{\partial}{\partial x_\alpha} \left( g^{\alpha 1} \frac{\partial g_{\alpha 2}}{\partial x_3} \right) \right\}.
\]

Let \( \delta_{jk} \) be the Kronecker symbol defined by

\[
\delta_{jk} = \begin{cases} 
1 & \text{for } j = k, \\
0 & \text{for } j \neq k.
\end{cases}
\]

Since the principal symbols of \( Q \) and \( \Phi^{jk}, (j, k = 1, 2, 3), \) are

\[
q_{-1} = \frac{1}{\sigma} \sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}
\]

and

\[
\phi_{-1}^{jk} = \sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} \delta_{jk},
\]

respectively, we see that the symbol of the operator \( \Lambda^{11} \) with homogeneous of degree one is

(3.66)

\[
\psi_{11} = \frac{1}{i \omega \mu \sqrt{|g|}} \left\{ -g_{12} \left( \phi_{11} + \sum_{\alpha} g^{\alpha 1} q_{-1} \left( \sigma i \xi_1 \right) \left( i \xi_\alpha \right) \right) + g_{11} \left( \sum_{\alpha} g^{\alpha 1} q_{-1} \left( \sigma i \xi_2 \right) \left( i \xi_\alpha \right) \right) \right\}
\]

\[
= \frac{1}{i \omega \mu \sqrt{|g|}} \left( -g_{12} \left( \sum_{\alpha} g^{\alpha \beta} \xi_\alpha \xi_\beta + g_{12} \sum_{\alpha} g^{\alpha 1} \frac{\sigma \xi_1 \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} - g_{11} \sum_{\alpha} g^{\alpha 1} \frac{\sigma \xi_2 \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} \right) \right) - g_{12} \left( \sum_{\alpha} g^{\alpha \beta} \xi_\alpha \xi_\beta + g_{12} \sum_{\alpha} g^{\alpha 1} \frac{\xi_1 \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} + g_{11} \sum_{\alpha} g^{\alpha 2} \frac{\xi_2 \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} \right)
\]

\[
= \frac{1}{i \omega \mu \sqrt{|g|}} \left( -g_{12} \left( \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta + g_{12} \sum_{\alpha} g^{\alpha 1} \frac{\sigma \xi_1 \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} - g_{11} \sum_{\alpha} g^{\alpha 1} \frac{\sigma \xi_2 \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} \right) \right) - g_{12} \left( \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta + g_{12} \sum_{\alpha} g^{\alpha 1} \frac{\xi_1 \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} + g_{11} \sum_{\alpha} g^{\alpha 2} \frac{\xi_2 \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} \right).
\]
Here the third equality follows from the relation $g_{11}g^{11} + g_{12}g^{12} = 1$ and $g_{11}g^{21} + g_{12}g^{22} = 0$ (see (2.44)). Since $\Lambda^{11}$ is uniquely determined by $\Lambda_{s, \Gamma}$, so the principal symbol $-\frac{\xi_1 \xi_2}{i\omega \mu \sqrt{\sum_{\alpha, \beta} |g| g^{\alpha\beta} \xi_\alpha \xi_\beta}}$ of $\Lambda^{11}$ is uniquely determined by $\Lambda_{s, \Gamma}$ at each boundary point $x_0$. This implies that $|g| g^{\alpha\beta}$ are determined by $\Lambda_{s, \Gamma}$ at each boundary point $x_0$ for all $1 \leq \alpha, \beta \leq 2$. However, $\det(|g| g^{\alpha\beta})$ determine those of $|g|$, and we can thus recover the values of $g^{\alpha\beta}$ itself along $\Gamma$.

Next, recall that

$$
\varrho(v \times H) = \frac{1}{i\omega \mu \sqrt{|g|}} \begin{bmatrix}
L^{11} & L^{12} \\
L^{21} & L^{22}
\end{bmatrix} \begin{bmatrix}
-g_{12} & -g_{22} \\
g_{11} & g_{12}
\end{bmatrix} \varrho(v \times E).
$$

Since $g_{\alpha\beta}$ have been determined by the principal symbol $\psi^1_{11}$ of component $\Lambda^{11}$, it follows that $\Lambda$ determines the operator

$$
\begin{bmatrix}
L^{11} & L^{12} \\
L^{21} & L^{22}
\end{bmatrix}.
$$

From (2.51) and (3.64), we see that the terms of degree zero in $\xi'$ of the symbol of $L^{jj}$, $(j = 1, 2)$, are

$$
(3.67)
$$

where and throughout the proof, each $T_0^{(s)}(g_{\alpha\beta})$ is an expressions involving only the boundary values of $g_{\alpha\beta}$, $g^{\alpha\beta}$, and their tangential derivatives. The last two equalities follow from the fact that, for $1 \leq \gamma \leq 2$,

$$
(3.68)
$$
It follows from (3.61) and (2.39) that $q_{-2}$ and $\phi^{jk}_0$ can be written as

\begin{equation}
q_{-2} = -\frac{1}{\sigma} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3} \left( \phi^{11}_0 + \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3} \right) + T^{(4)}_0 (g_{\alpha \beta}),
\end{equation}

and

\begin{equation}
\phi^{jk}_0 = \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3} \left( \left( \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3} \right) \phi^{11}_0 + 2 \Gamma^{(j)}_{kl} \phi^{11}_0 + \frac{\partial \phi^{11}_0}{\partial x_3} - 2 \Gamma^{(j)}_{kl} \phi^{11}_0 + \frac{\partial \phi^{11}_0}{\partial x_3} \right) + T^{(5)}_0 (g_{\alpha \beta}),
\end{equation}

respectively. Noting that $\phi^{jk}_1 = 0$ if $j \neq k$, we find by (3.67), (3.69) and (3.70) that

\[ l^{11}_0 + l^{22}_0 = \phi^{11}_0 + \phi^{22}_0 - \sum_{\alpha, \beta, \gamma} g^{\alpha \gamma} g^3 \frac{\partial g_{\alpha \gamma}}{\partial x_3} \frac{\partial g_{\alpha \beta}}{\partial x_3} + \sum_{\alpha, \beta, \gamma} g^{\alpha \gamma} g^3 \frac{\partial g_{\alpha \gamma}}{\partial x_3} \frac{\partial g_{\alpha \beta}}{\partial x_3} - q_{-2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3} + T^{(6)}_0 (g_{\alpha \beta}). \]

Since

\[ \Gamma^{\eta}_{\alpha \eta} = \frac{1}{2} \sum_{\rho} g^{\eta \rho} \frac{\partial g_{\rho \alpha}}{\partial x_3} + \frac{\partial g_{\rho \alpha}}{\partial x_3} - \frac{\partial g_{3 \alpha}}{\partial x_3} = \frac{1}{2} \sum_{\rho} g^{\eta \rho} \frac{\partial g_{\rho \alpha}}{\partial x_3} + T^{(7)}_0 (g_{\alpha \beta}), \]

\[ \Gamma^{3}_{\gamma \eta} = \frac{1}{2} \sum_{\gamma} g^{3 \gamma} \frac{\partial g_{\gamma \eta}}{\partial x_3} + \frac{\partial g_{\gamma \eta}}{\partial x_3} - \frac{\partial g_{3 \eta}}{\partial x_3} = \frac{1}{2} \sum_{\gamma} g^{3 \gamma} \frac{\partial g_{\gamma \eta}}{\partial x_3} + T^{(8)}_0 (g_{\alpha \beta}), \]

and

\[ \sum_{\gamma, \eta} g^{\alpha \gamma} \frac{\partial g_{\gamma \eta}}{\partial x_3} = -\frac{\partial g_{\alpha \beta}}{\partial x_3}, \]

we see by applying (3.69) that

\[ l^{11}_0 + l^{22}_0 = \phi^{11}_0 + \phi^{22}_0 - \sum_{\alpha, \beta, \gamma, \rho} g^{\alpha \gamma} g^3 \frac{\partial g_{\rho \alpha}}{\partial x_3} \frac{\partial g_{\rho \beta}}{\partial x_3} + \sum_{\alpha, \beta, \gamma, \rho} g^{\alpha \gamma} g^3 \frac{\partial g_{\rho \alpha}}{\partial x_3} \frac{\partial g_{\rho \beta}}{\partial x_3} - q_{-2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3} + T^{(9)}_0 (g_{\alpha \beta}) \]

\[ = \phi^{11}_0 + \phi^{22}_0 + \phi^{33}_0 - \sum_{\alpha, \beta, \gamma, \rho} g^{\alpha \gamma} g^3 \frac{\partial g_{\rho \alpha}}{\partial x_3} \frac{\partial g_{\rho \beta}}{\partial x_3} + \sum_{\alpha, \beta, \gamma, \rho} g^{\alpha \gamma} g^3 \frac{\partial g_{\rho \alpha}}{\partial x_3} \frac{\partial g_{\rho \beta}}{\partial x_3} - q_{-2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3} + T^{(9)}_0 (g_{\alpha \beta}) \]

\[ = \phi^{11}_0 + \phi^{22}_0 + \phi^{33}_0 - \sum_{\alpha, \beta, \gamma, \rho} g^{\alpha \gamma} g^3 \frac{\partial g_{\rho \alpha}}{\partial x_3} \frac{\partial g_{\rho \beta}}{\partial x_3} + \sum_{\alpha, \beta, \gamma, \rho} g^{\alpha \gamma} g^3 \frac{\partial g_{\rho \alpha}}{\partial x_3} \frac{\partial g_{\rho \beta}}{\partial x_3} - q_{-2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3} + T^{(9)}_0 (g_{\alpha \beta}) \]

But, it follows from (3.70) and $\phi^{kk}_1 = \sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3}}$ for $k = 1, 2, 3$ that

\[ \phi^{11}_0 + \phi^{22}_0 + \phi^{33}_0 = \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_3} \left( \phi^{kk}_1 + \frac{\partial \phi^{kk}_1}{\partial x_3} \right) + T^{(10)}_0 (g_{\alpha \beta}). \]
determine the quadratic form along \( \mathcal{L}_3 \).

Evaluating this on unit vectors \( \mathbf{h} \), we have

\[
\left( 3 \sum_{\alpha,\beta} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_3} + \frac{3 \sum_{\alpha,\beta} \frac{\partial^2 g_{\alpha\beta}}{\partial x_3^2} \xi_{\alpha} \xi_{\beta} + T_0^{(10)}(g_{\alpha\beta})}{2 \sqrt{\sum_{\alpha,\beta} g^{\alpha\beta} \xi_{\alpha} \xi_{\beta}}} \right)
\]

where, in the second equality we have used \( \sum_{k=1}^{l} \Gamma_{k}^{k} = \frac{1}{4} \sum_{\alpha,\beta} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_3} \). Hence we have

\[
l_0^{11} + l_0^{22} = \frac{11}{4} \sum_{\alpha,\beta} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_3} + \frac{7 \sum_{\alpha,\beta} \frac{\partial g_{\alpha\beta}}{\partial x_3} \xi_{\alpha} \xi_{\beta} + T_0^{(11)}(g_{\alpha\beta})}{4 \sqrt{\sum_{\alpha,\beta} g^{\alpha\beta} \xi_{\alpha} \xi_{\beta}}}.
\]

In view of

\[
(3.71)
\]

\[
\sum_{\alpha,\beta} g^{\alpha\beta} g_{\alpha\beta} = 2,
\]

we have

\[
\sum_{\alpha,\beta} \left( \frac{\partial g^{\alpha\beta}}{\partial x_3} g_{\alpha\beta} + g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_3} \right) = 0,
\]

i.e.,

\[
(3.72)
\sum_{\alpha,\beta} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_3} = -\sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial g^{\alpha\beta}}{\partial x_3}.
\]

Therefore,

\[
l_0^{11} + l_0^{22} = \frac{11}{4} \sum_{\alpha,\beta} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_3} + \frac{7 \sum_{\alpha,\beta} \frac{\partial g_{\alpha\beta}}{\partial x_3} \xi_{\alpha} \xi_{\beta} + T_0^{(11)}(g_{\alpha\beta})}{4 \sqrt{\sum_{\alpha,\beta} g^{\alpha\beta} \xi_{\alpha} \xi_{\beta}}}.
\]

If we set \( h_{1}^{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial x_3} \) and \( h_{1} = \sum_{\alpha,\beta} g_{\alpha\beta} h_{1}^{\alpha\beta} \), then

\[
(3.73)
l_0^{11} + l_0^{22} = \frac{1}{4} \sum_{\alpha,\beta} g^{\alpha\beta} \xi_{\alpha} \xi_{\beta} \left( 7h_{1}^{\alpha\beta} - 11h_{1} g^{\alpha\beta} \right) \xi_{\alpha} \xi_{\beta} + T_0^{(11)}(g_{\alpha\beta}).
\]

Evaluating this on unit vectors \( \xi \in T^{s}(\Gamma) \) implies that \( l_0^{11} + l_0^{22} \) and the values of \( g_{\alpha\beta} \) along \( \partial M \) completely determine the quadratic form

\[
(3.74)
k_{1}^{\alpha\beta} = 7h_{1}^{\alpha\beta} - 11h_{1} g^{\alpha\beta}
\]

along \( \Gamma \). From (3.74), we have

\[
\sum_{\alpha,\beta} g_{\alpha\beta} k_{1}^{\alpha\beta} = 7 \sum_{\alpha,\beta} g_{\alpha\beta} h_{1}^{\alpha\beta} - 11h_{1} \sum_{\alpha,\beta} g_{\alpha\beta} g^{\alpha\beta} = 7h_{1} - 22h_{1} = -15h_{1},
\]

so

\[
(3.75)
h_{1} = -\frac{1}{15} \sum_{\alpha,\beta} g_{\alpha\beta} k_{1}^{\alpha\beta},
\]
and hence

\[
(3.76) \quad \frac{\partial g^{\alpha\beta}}{\partial x_3} = h_1^{\alpha\beta} = \frac{1}{7} \left( k_1^{\alpha\beta} - \frac{11}{15} \sum_{\gamma,\nu} g_{\gamma\nu} k_1^{\gamma\nu} g^{\alpha\beta} \right).
\]

This shows that \( \frac{\partial g^{\alpha\beta}}{\partial x_3} \) on \( \partial M \) are uniquely determined by the symbol of degree zero of \( L^{11} + L^{22} \) as well as the symbol of order 1 of \( \Lambda^{11} \) along \( \Gamma \), and hence \( \frac{\partial g^{\alpha\beta}}{\partial x_3} \) on \( \Gamma \) are uniquely determined by \( \Lambda_{g,\Gamma} \).

Now, we calculate the terms of degree \(-1\) in \( \xi^j \) of the symbol of \( L^{11} + L^{22} \). It follows from \((2.51), (3.64)\) and the symbol formula for the product of two pseudodifferential operators that

\[
(3.77) \quad l_{-1}^{jj} = \phi_{-1}^{jj} - \left\{ \phi_{1}^{j3} \left( q_{-1}(\sigma \phi_{-1}^{3j}) + q_{-2}(\sigma(\phi_{0}^{3j} + \frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta} \partial g^{\alpha\beta}_{\partial x_3}) + \frac{\partial \sigma}{\partial x_3} \right) + q_{-3}\sigma(i\xi_j + \phi_{1}^{3j}) \right. \\
\left. - i \sum_{m=1}^{2} m q_{-1} \frac{\partial}{\partial \xi_m} \left( q_{-1}(\sigma(\phi_{0}^{3j} + \frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta} \partial g^{\alpha\beta}_{\partial x_3}) + \frac{\partial \sigma}{\partial x_3} \right) + q_{-2}(\sigma(i\xi_j + \phi_{1}^{3j})) \right. \\
\left. - i \sum_{m=1}^{2} m q_{-1} \frac{\partial}{\partial \xi_m} \left( q_{-1}(\sigma(\phi_{0}^{3j} + \frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta} \partial g^{\alpha\beta}_{\partial x_3}) + \frac{\partial \sigma}{\partial x_3} \right) + q_{-2}(\sigma(i\xi_j + \phi_{1}^{3j})) \right) \\
\left. + \sum_{\alpha} g^{\alpha j} \frac{\partial}{\partial x_{\alpha}} \left( q_{-1}(\sigma(\phi_{0}^{3j} + \frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta} \partial g^{\alpha\beta}_{\partial x_3}) + \frac{\partial \sigma}{\partial x_3} \right) + q_{-2}(\sigma(i\xi_j + \phi_{1}^{3j})) \right) \right\}
\]

Clearly, there is not other terms containing \( \frac{\partial^2 g^{\alpha\beta}}{\partial x_3^2} \) except for \( \phi_{-1}^{jj}, \phi_{-1}^{3j}, \phi_{-1}^{j3} \) and \( q_{-3} \) on the right-hand side of the above equality. Thus

\[
l_{-1}^{jj} = \phi_{-1}^{jj} - \phi_{-1}^{3j} q_{-1} \sigma i\xi_j + \sum_{\alpha} g^{\alpha j} q_{-1}(\sigma \phi_{-1}^{3j} + q_{-3}(\sigma i\xi_j)) i\xi_{\alpha} + T_{-1}^{(1)}(g^{\alpha\beta}) \quad j = 1, 2,
\]

where each \( T_{-1}^{(s)}(g^{\alpha\beta}) \) only involves the boundary values of \( g^{\alpha\beta}, g_{\alpha\beta} \), and their normal derivatives of order at most one. By \((3.62)\) we have

\[
q_{-3} = -\frac{1}{\sigma_{1}^{3j}} q_{-1}\phi_{-1}^{3j} + T_{-1}^{(2)}(g^{\alpha\beta}),
\]
so that

\[ l_{ij}^{(j)} = \phi_{-1}^{ij} - \phi_{-3}^{j} q_{-1} \sigma i \xi_{j} + \sum_{\alpha} g^{\alpha j} q_{-1} \sigma \phi_{-3}^{j} i \xi_{\alpha} + \sum_{\alpha} g^{\alpha j} \left( \frac{\xi_{j} \xi_{\alpha}}{\phi_{1}^{3 \alpha}} \right) q_{-1} \sigma \phi_{-1}^{3 \alpha} + T_{-1}^{(3)} (g_{\alpha \beta}), \quad j = 1, 2. \]

It follows that

\[ (3.78) \]

\[ l_{11}^{(1)} + l_{22}^{(2)} = \phi_{-1}^{11} + \phi_{-1}^{22} - \sum_{\beta} \phi_{-1}^{\beta 3} q_{-1} \sigma i \xi_{\beta} + \sum_{\alpha} g^{\alpha \beta} q_{-1} \sigma \phi_{-1}^{\beta 3} i \xi_{\alpha} + \sum_{\alpha} g^{\alpha \beta} \left( \frac{\xi_{\beta} \xi_{\alpha}}{\phi_{1}^{3 \alpha}} \right) q_{-1} \sigma \phi_{-1}^{3 \alpha} + T_{-1}^{(4)} (g_{\alpha \beta}) \]

\[ = \phi_{-1}^{11} + \phi_{-1}^{22} + \phi_{-1}^{33} - \frac{1}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_{\alpha} \xi_{\beta}}} \left( \sum_{\beta} \phi_{-1}^{\beta 3} i \xi_{\beta} + \sum_{\alpha, \beta} g^{\alpha \beta} \phi_{-1}^{3 \beta} i \xi_{\alpha} + T_{-1}^{(4)} (g_{\alpha \beta}) \right). \]

From (2.40) and (3.70) we have

\[ \phi_{-1}^{kk} = \frac{1}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_{\alpha} \xi_{\beta}}} \left( - (\phi_{1}^{k})^{2} + \frac{1}{2} \sum_{l=1}^{3} \sum_{l=1}^{3} \left( \frac{\partial \phi_{l}^{k}}{\partial \xi_{l}} \frac{\partial \phi_{l}^{k}}{\partial \xi_{l}} + \frac{\partial \phi_{l}^{k}}{\partial x_{l}} \frac{\partial \phi_{l}^{k}}{\partial x_{l}} \right) + \frac{1}{2} \sum_{l, m=1}^{3} \frac{\partial^{2} \phi_{l}^{k}}{\partial \xi_{l} \partial \xi_{m}} \frac{\partial^{2} \phi_{l}^{k}}{\partial x_{m} \partial x_{l}} \right) \]

\[ + \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g^{\alpha \beta}}{\partial x_{3}} \phi_{k}^{kk} + 2 \sum_{l=1}^{3} \Gamma_{l3}^{k} \phi_{0}^{l} + \phi_{0}^{k} \frac{\partial \phi_{0}^{k}}{\partial x_{3}} - a_{kk} + R_{k} + \alpha^{2} \mu \sigma \right) + T_{-1}^{(5)} (g_{\alpha \beta}) \]

\[ = \frac{1}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_{\alpha} \xi_{\beta}}} \left( \frac{\partial \phi_{0}^{k}}{\partial x_{3}} - a_{kk} + R_{k} \right) + T_{-1}^{(6)} (g_{\alpha \beta}), \quad k = 1, 2, 3. \]

According to the definitions of \( a_{kk} \) and \( R_{k} \), we have that for \( k = 1, 2, 3 \) (see (2.27) and (2.8)),

\[ (3.79) \]

\[ a_{kk} = \sum_{l, m=1}^{3} g^{ml} \left( \frac{\partial \Gamma_{km}^{l}}{\partial x_{l}} + \sum_{h} \Gamma_{hl}^{k} \Gamma_{km}^{h} - \sum_{h} \Gamma_{hm}^{k} \Gamma_{kl}^{h} \right) \]

\[ = \sum_{l, m=1}^{3} g^{ml} \frac{\partial \Gamma_{km}^{l}}{\partial x_{l}} + T_{-1}^{(7)} (g_{\alpha \beta}) \]

\[ = \frac{\partial \Gamma_{k}}{\partial x_{3}} + T_{-1}^{(7)} (g_{\alpha \beta}) \]

\[ = \frac{\partial}{\partial x_{3}} \left( \frac{1}{2} \sum_{s=1}^{3} g^{ks} \left( \frac{\partial g_{ks}}{\partial x_{3}} + \frac{\partial g_{ks}}{\partial x_{k}} - \frac{\partial g_{ks}}{\partial x_{3}} \right) + T_{-1}^{(7)} (g_{\alpha \beta}) \right) \]

\[ = \begin{cases} \frac{1}{2} \sum_{\beta} g^{\beta \xi_{\beta}} \frac{\partial^{2} g^{\alpha \beta}}{\partial x_{3}} + T_{-1}^{(8)} (g_{\alpha \beta}) & \text{when } k = 1, 2, \\ 0 & \text{when } k = 3, \end{cases} \]

\[ (3.80) \]

\[ R_{k}^{2} = \sum_{m=1}^{3} g^{kn} R_{mk} \]

\[ = \sum_{l, m=1}^{3} g^{km} \left( \frac{\partial \Gamma_{mk}^{l}}{\partial x_{l}} - \frac{\partial \Gamma_{ml}^{k}}{\partial x_{k}} \right) + T_{-1}^{(9)} (g_{\alpha \beta}) \]

\[ = \begin{cases} - \frac{1}{2} \sum_{\beta} g^{\beta \xi_{\beta}} \frac{\partial^{2} g^{\alpha \beta}}{\partial x_{3}} + T_{-1}^{(10)} (g_{\alpha \beta}) & \text{when } k = 1, 2, \\ - \frac{1}{2} g^{\alpha \beta} \frac{\partial^{2} g^{\alpha \beta}}{\partial x_{3}} + T_{-1}^{(11)} (g_{\alpha \beta}) & \text{when } k = 3. \end{cases} \]
Also,

\[
\begin{align*}
\alpha_{3\beta} &= 0 + T_{-1}^{(12)}(g_{\alpha\beta}), \\
\alpha_{33} &= 0 + T_{-1}^{(13)}(g_{\alpha\beta}), \\
R_{\beta}^3 &= 0 + T_{-1}^{(14)}(g_{\alpha\beta}), \\
R_{3\beta} &= 0 + T_{-1}^{(15)}(g_{\alpha\beta}).
\end{align*}
\]

(3.81) and (3.80) imply that

\[
\sum_{k=1}^{3}(R_{k}^{k} - a_{kk}) = \frac{3}{2} \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial^2 g_{\alpha\beta}}{\partial x_{3}^2} + T_{-1}^{(16)}(g_{\alpha\beta}).
\]

It follows from (2.39) that

\[
\frac{\partial \phi_{0}^{kk}}{\partial x_{3}} = \frac{\partial}{\partial x_{3}} \left\{ \frac{1}{2} \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_{3}} + 2 R_{k}^{k} \right\} \sqrt{\sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_{3}}} + \frac{\partial}{\partial x_{3}} \left( \sum_{\alpha,\beta} g_{\alpha\beta} \xi_{\alpha} \xi_{\beta} \right) + T_{-1}^{(17)}(g_{\alpha\beta})
\]

for \( k = 1, 2, 3 \).

Therefore

\[
\phi_{-1}^{11} + \phi_{-1}^{22} + \phi_{-1}^{33} = \frac{1}{2} \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_{3}} + \frac{3}{4} \sum_{\alpha,\beta} \frac{\partial g_{\alpha\beta}}{\partial x_{3}} + \sum_{k=1}^{3} R_{k}^{k} \right\} + T_{-1}^{(18)}(g_{\alpha\beta})
\]

\[
+ \frac{3}{2} \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial^2 g_{\alpha\beta}}{\partial x_{3}^2} + T_{-1}^{(18)}(g_{\alpha\beta})
\]

\[
= \frac{1}{2} \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_{3}} + \frac{3}{4} \sum_{\alpha,\beta} \frac{\partial g_{\alpha\beta}}{\partial x_{3}} + \sum_{k=1}^{3} R_{k}^{k} \right\} + T_{-1}^{(19)}(g_{\alpha\beta})
\]

\[
+ \frac{3}{2} \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial^2 g_{\alpha\beta}}{\partial x_{3}^2} + T_{-1}^{(19)}(g_{\alpha\beta})
\]

\[
= \frac{1}{2} \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_{3}} + \frac{3}{4} \sum_{\alpha,\beta} \frac{\partial g_{\alpha\beta}}{\partial x_{3}} + \sum_{k=1}^{3} R_{k}^{k} \right\} + T_{-1}^{(20)}(g_{\alpha\beta}).
\]

The last equality follows from the relation \( \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial^2 g_{\alpha\beta}}{\partial x_{3}^2} = \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial^2 g_{\alpha\beta}}{\partial x_{3}^2} + T_{-1}^{(21)}(g_{\alpha\beta}) \) which can be derived by the equality \( \frac{\partial}{\partial x_{3}} \left( \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_{3}} \right) = - \frac{\partial}{\partial x_{3}} \left( \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial^2 g_{\alpha\beta}}{\partial x_{3}^2} \right). \) In addition, from (2.39), (2.40) and (3.81) we have

\[
\phi_{-1}^{11} = \frac{1}{2} \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_{3}} + \sum_{k=1}^{3} R_{k}^{k} \right\} + T_{-1}^{(22)}(g_{\alpha\beta})
\]

\[
= \frac{1}{2} \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_{3}} + T_{-1}^{(23)}(g_{\alpha\beta})
\]
\[
\sum_{i=1}^{15} \frac{1}{\sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}} \left\{ \sum_{\alpha, \beta} g^\alpha \left[ \frac{\partial}{\partial x_3} \frac{2\sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}}{2 \sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}} \xi_i' \right] \right\} + T_{-1}^{(24)}(g_{\alpha\beta})
\]

\[
\sum_{i=1}^{15} \frac{1}{\sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}} \left\{ \sum_{\alpha, \beta} g^\alpha \left[ \frac{\partial}{\partial x_3} \frac{2\sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}}{2 \sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}} \xi_i' \right] \right\} + T_{-1}^{(25)}(g_{\alpha\beta})
\]

\[
\sum_{i=1}^{15} \frac{1}{\sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}} \left\{ \sum_{\alpha, \beta} g^\alpha \left[ \frac{\partial}{\partial x_3} \frac{2\sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}}{2 \sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}} \xi_i' \right] \right\} + T_{-1}^{(25)}(g_{\alpha\beta})
\]

and

\[
\phi_i^{(3)} - \sum_{\alpha, \beta} g^\alpha \xi_\alpha' \frac{\partial \phi_i^{(3)}}{\partial x_3} (\phi_i^{(3)} - \alpha_3') + T_{-1}^{(27)}(g_{\alpha\beta})
\]

so that

\[
(3.83) \quad - \sum_{\beta} \phi_\beta' \phi_\beta^{(3)} \xi_\beta' \xi_\beta = \frac{-1}{4(\sum_{\beta, \gamma} g^{\beta\gamma} \xi_\beta' \xi_\gamma')^{3/2}} \left( \sum_{\beta, \gamma} \frac{\partial^2 g^{\beta\gamma}}{\partial x_3^2} (i\xi_\gamma' (i\xi_\beta')) + T_{-1}^{(30)}(g_{\alpha\beta}) \right)
\]

and

\[
(3.84) \quad \sum_{\alpha, \beta} g^\alpha \xi_\alpha' \phi_i^{(3)} \xi_\beta' \xi_\alpha' \xi_\beta' = \frac{1}{2 \sum_{\alpha, \beta} g^\alpha \xi_\alpha' \xi_\beta'} \left\{ \sum_{\alpha, \beta} g^\alpha \left[ \frac{\partial}{\partial x_3} \frac{2\sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}}{2 \sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}} \xi_i' \right] \right\} + T_{-1}^{(31)}(g_{\alpha\beta})
\]

\[
= \frac{1}{2 \sum_{\alpha, \beta} g^\alpha \xi_\alpha' \xi_\beta'} \left\{ \sum_{\alpha, \beta} g^\alpha \left[ \frac{\partial}{\partial x_3} \frac{2\sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}}{2 \sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}} \xi_i' \right] \right\} + T_{-1}^{(32)}(g_{\alpha\beta})
\]

\[
= \frac{1}{2 \sum_{\alpha, \beta} g^\alpha \xi_\alpha' \xi_\beta'} \left\{ \sum_{\alpha, \beta} g^\alpha \left[ \frac{\partial}{\partial x_3} \frac{2\sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}}{2 \sqrt{\sum_{\alpha, \beta} g^\alpha \xi_\alpha'}} \xi_i' \right] \right\} + T_{-1}^{(32)}(g_{\alpha\beta})
\]
Therefore (3.86) imply that
\[ g \] is uniquely determined by \( l \). Setting \( l \) is determined by \( \xi \) all we thus have
\[
\frac{\sum_{\alpha \beta} \partial^2 g_{\alpha \beta} \xi_{\alpha} \xi_{\beta}}{4 \left( \sum_{\alpha \beta} g_{\alpha \beta} \xi_{\alpha} \xi_{\beta} \right)^{3/2}} + T_{-1}^{(33)}(g_{\alpha \beta}).
\]

Combining (3.78), (3.82), (3.83) and (3.84) we have
\[
l_{11} + l_{22} = -\frac{1}{8} \left( \sum_{\alpha \beta} g_{\alpha \beta} \xi_{\alpha} \xi_{\beta} \right)^{3/2} \left[ \sum_{\alpha \beta} \left( h_{2} g_{\alpha \beta} + 7 \frac{\partial^2 g_{\alpha \beta}}{\partial x_3} \right) \xi_{\alpha} \xi_{\beta} \right] + T_{-1}^{(34)}(g_{\alpha \beta}).
\]

Setting
\[ (3.85) \]
\[ h_2 := \sum_{\alpha \beta} g_{\alpha \beta} \frac{\partial^2 g_{\alpha \beta}}{\partial x_3}, \]
we thus have
\[
l_{11} + l_{22} = -\frac{1}{8} \left( \sum_{\alpha \beta} g_{\alpha \beta} \xi_{\alpha} \xi_{\beta} \right)^{3/2} \left[ \sum_{\alpha \beta} \left( h_{2} g_{\alpha \beta} + 7 \frac{\partial^2 g_{\alpha \beta}}{\partial x_3} \right) \xi_{\alpha} \xi_{\beta} \right] + T_{-1}^{(34)}(g_{\alpha \beta}).
\]

Note that \( g_{\alpha \beta} \) have been known by \( \Lambda^{11}_{\xi} \) according to the discussion before. Evaluating the above result for all \( \xi' \in T^*(\Gamma) \) shows that the symbol of degree \(-1\) of \( L_{11} + L_{22} \) completely determines the quadratic form
\[ (3.86) \]
\[ k_{2}^{\alpha \beta} = h_2 g_{\alpha \beta} + 7 \frac{\partial^2 g_{\alpha \beta}}{\partial x_3} \]
along \( \partial M \). From this quadratic form, (3.71) and (3.85) we find that
\[
\sum_{\alpha \beta} g_{\alpha \beta} k_{2}^{\alpha \beta} = h_2 \sum_{\alpha \beta} g_{\alpha \beta} g_{\alpha \beta} + 7 \sum_{\alpha \beta} g_{\alpha \beta} \frac{\partial^2 g_{\alpha \beta}}{\partial x_3} = 2h_2 + 7h_2 = 9h_2.
\]
Therefore (3.86) imply that
\[
\frac{\partial^2 g_{\alpha \beta}}{\partial x_3} = \frac{1}{7} \left( k_{2}^{\alpha \beta} - \frac{1}{9} \left( \sum_{\gamma \rho} g_{\gamma \rho} k_{2}^{\gamma \rho} \right) g_{\alpha \beta} \right)
\]
is uniquely determined by \( l_{11} + l_{22} \) and \( \psi^{11}_{-1} \), and hence \( \frac{\partial^2 g_{\alpha \beta}}{\partial x_3} \) are uniquely determined on \( \Gamma \) by \( \Lambda_{\xi, \Gamma} \).

Finally, we will consider the symbol \( L_{11} + L_{22} \) of degree \(-m - 1\) for \( L_{11} + L_{22} \) for the general integer \( m \geq 1 \). From (2.71), we see that
\[
l_{m-1}^{1j} = \phi^{1j}_{-m-1} - \phi^{3j}_{-m-1} q_{-1} i_{-j} + \sum_{\alpha} g_{\alpha j} \left( q_{-1} \sigma_{-m-1} + q_{-m-3} \sigma_{-j} i_{-j} \right) i_{\xi_{\alpha}} + T_{-m-1}^{(1)}(g_{\alpha \beta}), \quad \text{for } j = 1, 2,
\]
where $T^{(s)}_{m-1}(g_{\alpha \beta})$, ($s = 1, 2, \cdots$), are expressions involving only the boundary values of $g^{\alpha \beta}$, $g_{\alpha \beta}$, and their normal derivatives of order at most $m + 1$. From (3.63) we have

$$q_{m-3} = -\frac{1}{\sigma \phi_{3}^{3}} q_{1} \sigma \phi_{3}^{3} + T^{(2)}_{m-1}(g_{\alpha \beta}).$$

It follows that

$$t^{l_{m-1}} + t^{l_{m-1}} = \phi_{m-1}^{l_{m}} + \phi_{m-1}^{22} + \phi_{m-1}^{33} - q_{m-1} \sigma i \xi_{j} + \sum_{j=1}^{2} \phi_{m-1}^{3j} \sigma i \xi_{j} + T^{(3)}_{m-1}(g_{\alpha \beta}).$$

From (2.41) we have

$$\phi^{j}_{m-1} = \frac{1}{2\sqrt{\sum_{\alpha, \beta} \alpha \beta \xi \alpha \xi \beta}} \frac{\partial \phi^{j}_{m-1}}{\partial x_{3}} + T^{(4)}_{m-1}(g_{\alpha \beta}) \quad \text{for} \quad j, k = 1, 2, 3.$$

We will give the corresponding estimates for $\sum_{k=1}^{3} \phi_{m-1}^{kk}$, $\frac{\sum_{\alpha, \beta} \alpha \beta \xi \alpha \xi \beta}{\sqrt{\sum_{\alpha, \beta} \alpha \beta \xi \alpha \xi \beta}} g^{\alpha \beta}$, and $\frac{\sum_{\alpha, \beta} \alpha \beta \xi \alpha \xi \beta}{\sqrt{\sum_{\alpha, \beta} \alpha \beta \xi \alpha \xi \beta}}$ by induction.

We first show that for all $m \geq 1$,

$$\sum_{k=1}^{3} \phi_{m-1}^{kk} = \left(2 \sqrt{\sum_{\alpha, \beta} \alpha \beta \xi \alpha \xi \beta} \right)^{-m} \left(\frac{1}{4} \sum_{k=1}^{3} g^{\alpha \beta} \frac{\partial g^{m+1}_{\alpha \beta}}{\partial x_{3}^{m+1}} + \frac{3}{4} \sum_{\alpha, \beta} \frac{\partial g^{m+1}_{\alpha \beta}}{\partial x_{3}^{m+1}} \right) + T^{(5)}_{m-1}(g_{\alpha \beta}).$$

Suppose we have shown that for $1 \leq r \leq m$,

$$\sum_{k=1}^{3} \phi_{r-1}^{kk} = \left(2 \sqrt{\sum_{\alpha, \beta} \alpha \beta \xi \alpha \xi \beta} \right)^{-r} \left(\frac{1}{4} \sum_{k=1}^{3} g^{\alpha \beta} \frac{\partial g^{r+1}_{\alpha \beta}}{\partial x_{3}^{r+1}} + \frac{3}{4} \sum_{\alpha, \beta} \frac{\partial g^{r+1}_{\alpha \beta}}{\partial x_{3}^{r+1}} \right) + T^{(6)}_{r-1}(g_{\alpha \beta}).$$

Clearly, this estimates actually holds when $r = 1$ by (3.82). Then, from (3.88) and (3.90), we have

$$\sum_{k=1}^{3} \phi_{m-1}^{kk} = \left(2 \sqrt{\sum_{\alpha, \beta} \alpha \beta \xi \alpha \xi \beta} \right)^{-m} \left(\frac{1}{4} \sum_{k=1}^{3} g^{\alpha \beta} \frac{\partial g^{m+1}_{\alpha \beta}}{\partial x_{3}^{m+1}} + \frac{3}{4} \sum_{\alpha, \beta} \frac{\partial g^{m+1}_{\alpha \beta}}{\partial x_{3}^{m+1}} \right) + T^{(7)}_{m-1}(g_{\alpha \beta}).$$

$$= \left(2 \sqrt{\sum_{\alpha, \beta} \alpha \beta \xi \alpha \xi \beta} \right)^{-m} \frac{\partial}{\partial x_{3}^{m+1}} \left[\left(\frac{1}{4} \sum_{k=1}^{3} g^{\alpha \beta} \frac{\partial g^{m+1}_{\alpha \beta}}{\partial x_{3}^{m+1}} + \frac{3}{4} \sum_{\alpha, \beta} \frac{\partial g^{m+1}_{\alpha \beta}}{\partial x_{3}^{m+1}} \right) \right] + T^{(8)}_{m-1}(g_{\alpha \beta}).$$

$$= \left(2 \sqrt{\sum_{\alpha, \beta} \alpha \beta \xi \alpha \xi \beta} \right)^{-m} \frac{\partial}{\partial x_{3}^{m+1}} \left[\left(\frac{1}{4} \sum_{k=1}^{3} g^{\alpha \beta} \frac{\partial g^{m+2}_{\alpha \beta}}{\partial x_{3}^{m+2}} + \frac{3}{4} \sum_{\alpha, \beta} \frac{\partial g^{m+2}_{\alpha \beta}}{\partial x_{3}^{m+2}} \right) \right] + T^{(9)}_{m-1}(g_{\alpha \beta}).$$

$$= \left(2 \sqrt{\sum_{\alpha, \beta} \alpha \beta \xi \alpha \xi \beta} \right)^{-m} \frac{\partial}{\partial x_{3}^{m+2}} \left[\left(\frac{1}{4} \sum_{k=1}^{3} g^{\alpha \beta} \frac{\partial g^{m+3}_{\alpha \beta}}{\partial x_{3}^{m+3}} + \frac{3}{4} \sum_{\alpha, \beta} \frac{\partial g^{m+3}_{\alpha \beta}}{\partial x_{3}^{m+3}} \right) \right] + T^{(10)}_{m-1}(g_{\alpha \beta}).$$
Thus, by induction we get that (3.89) holds for all \( m \geq 1 \). Next, we will prove by induction that

\[
\frac{\sum_{\alpha, \beta} g^{\alpha \beta} \phi_{m-1}^{3 \beta} i \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} = \frac{1}{2} \left( 2 \left( \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta \right)^{-m-1} \left( \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial^{m+2} \phi^{3 \beta} \phi_{m-1}^{3 \beta} \xi_\alpha \xi_\beta}{\partial x^3_\alpha \partial x^3_\beta} \right) + T_{m-1}^{(11)}(g_{\alpha \beta}) \right).
\]

Suppose we have also shown that, when \( 1 \leq r \leq m \),

\[
\frac{\sum_{\alpha, \beta} g^{\alpha \beta} \phi_{m-1}^{3 \beta} i \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} = \frac{1}{2} \left( 2 \left( \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta \right)^{-r} \left( \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial^{r+1} \phi^{3 \beta} \phi_{m-1}^{3 \beta} \xi_\alpha \xi_\beta}{\partial x^3_\alpha \partial x^3_\beta} \right) + T_{m-1}^{(12)}(g_{\alpha \beta}) \right).
\]

Again, this actually is true when \( r = 1 \) by (3.84). By (3.88) we find that

\[
\phi_{m-1}^{3 \beta} = \frac{1}{2 \sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} \frac{\partial \phi_{m-1}^{3 \beta}}{\partial x_3} + T_{m-1}^{(13)}(g_{\alpha \beta}) \quad \text{for } j, k = 1, 2, 3.
\]

It follows from this and (3.93) that

\[
\frac{\sum_{\alpha, \beta} g^{\alpha \beta} \phi_{m-1}^{3 \beta} i \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} = \frac{1}{2 \sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} \frac{\partial}{\partial x_3} \left( \frac{\sum_{\alpha, \beta} g^{\alpha \beta} \phi_{m-1}^{3 \beta} \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} \right) + T_{m-1}^{(14)}(g_{\alpha \beta})
\]

\[
= \frac{1}{2 \sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} \frac{\partial}{\partial x_3} \left[ \frac{1}{2} \left( \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta \right)^{-m} \left( \sum_{\alpha, \beta} g^{\alpha \beta} \phi_{m-1}^{3 \beta} \xi_\alpha \xi_\beta \right) \right] + T_{m-1}^{(15)}(g_{\alpha \beta})
\]

\[
= \frac{1}{2} \left( 2 \left( \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta \right)^{-m} \left( \sum_{\alpha, \beta} g^{\alpha \beta} \phi_{m-1}^{3 \beta} \xi_\alpha \xi_\beta \right) \right) + T_{m-1}^{(16)}(g_{\alpha \beta}).
\]

Thus (3.92) holds for all \( m \geq 1 \) by induction. Similarly, by applying induction we can easily prove that for all \( m \geq 1 \),

\[
- \frac{\sum_{\alpha, \beta} g^{\alpha \beta} \phi_{m-1}^{3 \beta} i \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} = \frac{1}{2} \left( 2 \left( \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta \right)^{-m-1} \left( \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial^{m+2} \phi^{3 \beta} \phi_{m-1}^{3 \beta} \xi_\alpha \xi_\beta}{\partial x^3_\alpha \partial x^3_\beta} \right) + T_{m-1}^{(18)}(g_{\alpha \beta}) \right).
\]

Hence

\[
\frac{\sum_{\alpha, \beta} g^{\alpha \beta} \phi_{m-1}^{3 \beta} i \xi_\alpha}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} = \frac{1}{2} \left( 2 \left( \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta \right)^{-m} \left( \sum_{\alpha, \beta} g^{\alpha \beta} \phi_{m-1}^{3 \beta} \xi_\alpha \xi_\beta \right) \right) + T_{m-1}^{(19)}(g_{\alpha \beta})
\]

\[
= \frac{1}{2} \left( 2 \left( \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta \right)^{-m-1} \left( \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial^{m+2} \phi^{3 \beta} \phi_{m-1}^{3 \beta} \xi_\alpha \xi_\beta}{\partial x^3_\alpha \partial x^3_\beta} \right) + T_{m-1}^{(20)}(g_{\alpha \beta}) \right).
\]
Consequently, the operator \( \Xi_U \Gamma \in \partial \) Let \( \rho \) map \( x \Gamma, \) and the proof of Proposition 3.1 is completed.

Putting \( \bar{x} \), we see that \( \bar{x} \) and satisfies \( \psi_1^{(3.98)} \) (respectively, \( x \) and \( x \)). Assume also that the electromagnetic parameters \( \mu \) and \( \sigma \) are strictly positive on \( \mathbb{M} \). Since

\[
\sum_{\alpha, \beta} g_{\alpha \beta} \kappa_{m+2} = h_{m+2} \sum_{\alpha, \beta} g_{\alpha \beta} g^{\alpha \beta} + 7 \sum_{\alpha, \beta} g_{\alpha \beta} \frac{\partial^{m+2} g^{\alpha \beta}}{\partial x_3^{m+2}} = 9h_{m+2},
\]

we have

\[
h_{m+2} = \frac{1}{9} \sum_{\alpha, \beta} g_{\alpha \beta} \kappa_{m+2}.
\]

so that

\[
\frac{\partial^{m+2} g^{\alpha \beta}}{\partial x_3^{m+2}} = \frac{1}{7} \left( \kappa_{m+2} - \frac{1}{9} \left( \sum_{\gamma, \rho} g_{\gamma \rho} \kappa_{m+2} \right) g^{\alpha \beta} \right).
\]

This implies that \( \frac{\partial^{m+2} g^{\alpha \beta}}{\partial x_3^{m+2}} \) is uniquely determined by \( \psi_1^{(3.99)}, l_{11}^{(3.99)} + l_{22}^{(3.99)} \) and \( \psi_1^{(3.99)} \). Consequently, the operator \( \Lambda_{x, \Gamma} \) uniquely determines \( g^{\alpha \beta} \) and all order normal derivatives of \( g^{\alpha \beta} \) along on \( \Gamma \); and the proof of Proposition 3.1 is completed.

Lemma 3.2. Let \( \tilde{\mathcal{M}} \) be a three-dimensional Riemannian manifold having compact closure and C^1-smooth boundary \( \partial \mathcal{M} \), and let \( \Gamma \) be a real analytic piece of boundary \( \partial \mathcal{M} \). Assume that \( \mathbb{M} \) and \( \tilde{\mathbb{M}} \) are real analytic in \( \mathbb{M} \) up to \( \Gamma \). Assume also that the electromagnetic parameters \( \mu \) and \( \sigma \) are real analytic in \( \mathbb{M} \) up to \( \Gamma \), and that \( \mu \) and \( \Re(\sigma) \) are strictly positive on \( \mathbb{M} \). If \( \Lambda_{x, \Gamma} = \Lambda_{x, \Gamma} \), then there exists a neighborhood \( \mathcal{U} \) of \( \Gamma \) in \( \mathcal{M} \) and a real-analytic map \( \rho_0 : \mathcal{U} \rightarrow \mathcal{M} \) such that \( \rho_0|_{\Gamma} = \text{identity} \) and \( g = \rho_0^* \tilde{g} \).

Proof. Let \( \mathcal{U} \) be some connected open set in the half-space \( \{ x_3 \geq 0 \} \subset \mathbb{R}^3 \) containing the origin. For any \( x_0 \in \Gamma \) we define a real-analytic diffeomorphism \( \tilde{\xi}_{x_0} : \mathcal{U} \rightarrow \Lambda_{x_0} \) (respectively, \( \tilde{\xi}_{x_0} : \mathcal{U} \rightarrow \Lambda_{x_0} \)) where \( (x_1, x_2, x_3) \) (respectively, \( (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \)) define the corresponding boundary normal coordinates for \( g \) (respectively, \( \tilde{g} \)) defined on a connected neighborhood \( \Lambda_{x_0} \) (respectively, \( \tilde{\Lambda}_{x_0} \)) of \( x_0 \) (see (3.7)). It follows from Proposition 3.1 that the metric \( \tilde{\xi}_{x_0}^* g \) and \( \tilde{\xi}_{x_0}^* \tilde{g} \) are real-analytic metric on \( \mathcal{U} \), where Taylor series at origin by explicit formulas involving the symbol of \( \tilde{\mathcal{E}}_g \) in \( \{ x_3 \} \) coordinates. Clearly, these two metric must be identical on \( \mathcal{U} \). Set \( \rho_{x_0} = \tilde{\xi}_{x_0} \circ \tilde{\xi}_{x_0}^{-1} : \mathcal{U}_{x_0} \rightarrow \Lambda_{x_0} \). Then, \( \rho_{x_0} \) is a real-analytic diffeomorphism which fixes the portion of \( \Gamma \) lying in \( \Lambda_{x_0} \) and satisfies \( \rho_{x_0}^* \tilde{g} = g \). Therefore, repeating this construction for each \( x_0 \in \Gamma \), we obtain a real analytic map \( \rho_0 : \mathcal{U} \rightarrow \tilde{\mathcal{M}} \) such that \( \rho_0|_{\Gamma} = \text{identity} \) and \( \rho_0^* \tilde{g} = g \).
4 Determining metric tensor from electromagnetic Dirichlet-to-Neumann map on a piece of analytic surface

Recall that $\Gamma$ is a real analytic piece of boundary, and that the metric tensor of $M$ is real analytic up to $\Gamma$. Let $f \in C(\partial M) \cap TH^2(\partial M)$ with $\text{supp } f \subset \Gamma$, and let $(E, H) \in (L^2(M))^3 \times (L^2(M))^3$ be the unique solution of the Maxwell’s equations:

$$
\begin{align*}
\text{curl } E &= i \omega \mu H \quad \text{in } M, \\
\text{curl } H &= -i \omega \sigma E \quad \text{in } M, \\
\nu \times E &= f \quad \text{on } \partial M.
\end{align*}
$$

(4.99)

We assume that we know the all boundary data $(\mu \times E, \nu \times H)$ on $\Gamma$ of all possible solutions of (4.99), for $f$ supported on $\Gamma$. Equivalently, we assume that we know the electromagnetic Dirichlet-to-Neumann map

$$
\Lambda_{g,\Gamma} : f \rightarrow \nu \times H \big|_\Gamma
$$

(4.100)

Let $\mathcal{O}$ be a boundary normal coordinate collar neighborhood of $\Gamma$ (i.e., $\mathcal{O} = \Gamma \times (-r, r)$ in the boundary normal coordinates). Set $U = \mathcal{O} \setminus \bar{M}$. Then, we define a manifold $\tilde{M}$ by gluing to $M \cup \Gamma$ the boundary normal coordinate collar $U$ (i.e., $U = \partial M \times (-r, 0)$) with metric described as follows. In section 3, we have shown that for a connected Riemannian manifold with compact closure, the $\Lambda_{g,\Gamma}$ determines $g_{jk}$ and all order normal derivatives $\frac{\partial^k g_{jk}}{\partial x^k}(x)$, $k \geq 0$ of the metric tensor $g$ on $\Gamma$ (This result is based on the local fact that when $\Lambda_{g,\Gamma}$ is considered as a pseudodifferential operator, its full symbol determines the all order derivatives of the metric in boundary normal coordinates). By Taylor’s series we can continue the metric so that the new metric is real-analytic in $\tilde{M}$ when $r$ is small enough (In other words, the real analytic metric $g_{jk}$ is uniquely determined by $\Lambda_{g,\Gamma}$ in $\tilde{M}$, see the proof of Lemma 3.2). We denote the new metric of $\tilde{M}$ also by $g_{jk}$.

\[\text{Figure 1: The manifold } M, \text{ the set } \Gamma \subset \partial M \text{ and the extension of the manifold over } \Gamma.\]

From (2.24) and (2.19), we see that if $(E, H)$ is a solution of Maxwell’s equations (4.99) then $E$ satisfies

$$
\begin{align*}
\text{curl curl } E - (\text{grad } (\log \mu)) \times \text{curl } E - \omega^2 \mu \sigma E &= 0 \quad \text{in } M, \\
\text{div } (\sigma E) &= 0 \quad \text{in } M, \\
\nu \times E &= f \quad \text{on } \partial M.
\end{align*}
$$

(4.101)

Conversely, let $E$ be a solution of (4.101) then $E$ and $H := \frac{1}{i \omega \mu} \text{curl } E$ satisfies Maxwell’s equations (4.99). In the local (or local boundary normal) coordinates, (4.101) can be written as

$$
\begin{align*}
\mathcal{M}_g E &= 0, \quad \text{in } M, \\
\text{div } (\sigma E) &= 0 \quad \text{in } M, \\
\nu \times E &= f \quad \text{on } \partial M.
\end{align*}
$$

(4.102)
Example 4.1

**dyadic Green function for Maxwell’s equations:**

It is well-known that the (electric) dyadic Green function \(G^e(x,y)\) is a real-analytic function of \(x\) when \(x \notin \{y\} \cup \partial M\). Moreover, when \(x\) is near to a given \(y\) it has the asymptotics (see \([14]\))

\[
G^e(x,y) \to \infty \quad \text{as } x \to y.
\]

The (electric) dyadic Green function \(G^e(x,y)\) is a generalization of the following classical (electric) dyadic Green function for Maxwell’s equations:

**Example 4.1 (see p.31 of \([14]\))**

For the classical Maxwell’s equations (where \(g_{jk} = \delta_{jk}\), and \(\mu\) and \(\sigma\) are constants), the classical dyadic Green’s function (for electric field) satisfies the equation

\[
\text{curl} \text{ curl } G^e(x,y) - \omega^2 \mu \sigma G^e(x,y) = \delta(x-y)I.
\]

Then, after post-multiplying (4.105) by \(G^e(x,y)\), pre-multiplying (4.105) by \(E(x)\), subtracting the resultant equations and integrating the difference over \(M\), we have

\[
E(y) = \int_M \left[ E(x) \cdot (\text{curl} \text{ curl } G^e(x,y)) + (\text{curl} \text{ curl } E(x)) \cdot G^e(x,y) \right] dV_x.
\]

Next, using the vector identity that

\[
-\nabla \cdot \left[ E(x) \cdot (\text{curl} \text{ curl } G^e(x,y)) + (\text{curl} \text{ curl } E(x)) \cdot G^e(x,y) \right] = E(x) \cdot (\text{curl} \text{ curl } G^e(x,y)) - \text{curl} \text{ curl } E(x) \cdot G^e(x,y),
\]

and applying Gauss’ divergence theorem, one has

\[
E(y) = -\int_{\partial M} \left[ \nabla \times E(x) \cdot \text{curl} G^e(x,y) + i\omega \mu \nabla \times H(x) \cdot G^e(x,y) \right] dS.
\]

Again, notice that (4.106) is derived via the use of (4.105), but no boundary condition has yet been imposed on \(G^e(x,y)\) on \(\partial M\). Now, if we require that \(\nabla \times G^e(x,y) = 0\) for \(x \in \partial M\), then (4.106) becomes

\[
E(y) = -\int_{\partial M} \left[ \nabla \times E(x) \cdot \text{curl} G^e(x,y) \right] dS.
\]

Note that if no boundary condition is imposed on \(G^e(x,y)\), then

\[
[G^e(x,y)]^T = G^e(y,x) = \left[ I + \frac{\nabla_y \nabla_y}{\omega^2 \mu \sigma} \right] g(y-x), \quad g(y-x) = \frac{e^{i\omega \sqrt{\mu \sigma}|y-x|}}{4\pi |y-x|}.
\]

Let us next consider two manifolds \(M_1\) and \(M_2\) for which we have identified \(\Gamma_1 = \Gamma_2 = \Gamma\) and \(\Lambda_{x_1,\Gamma} = \Lambda_{x_2,\Gamma}\). Using the previous construction of the set \(U\) and the metric tensor on \(U\), which is the same for both manifolds, we can attach this set and the metric on it to both manifolds, i.e.,

\[
\tilde{M}_1 = M_1 \cup U, \quad \tilde{M}_2 = M_2 \cup U.
\]
Then the corresponding (electric) dyadic Green functions of \( \tilde{M}_j \), \( j = 1, 2 \), satisfying
\[
\begin{align*}
\mathcal{M}_g G_j^r(x, y) &= \delta_y \quad \text{in } \tilde{M}_j, \\
\text{div } G_j^r(x, y) &= 0 \quad \text{in } \tilde{M}_j \setminus \{y\}, \\
G_j^r(x, y) \big|_{\partial \tilde{M}_j} &= 0.
\end{align*}
\]
(4.108)

**Proof of Theorem 1.2.** First, we will show that the (electric) dyadic Green’s function \( G_j^r(x, y) \) satisfy
\[
G_j^r(x) = G_j^r(x, y), \quad (x, y) \in \Gamma', \quad V_0(x) = 0, \quad x \in \partial M_2 \setminus \partial'.
\]
(4.109)

Pick \( y \in U \), and define \( V_0 \in C(\partial M_2) \) by
\[
V_0(x) = G_j^r(x, y), \quad x \in \Gamma \cap \partial'; \quad V_0(x) = 0, \quad x \in \partial M_2 \setminus \partial'.
\]
(4.110)

Let \( V \) be the solution on \( M_2 \) to
\[
\begin{align*}
\mathcal{M}_g V &= 0 \quad \text{in } M_2, \\
\text{div } (\sigma E) &= 0 \quad \text{in } M_2, \\
V &= V_0 \quad \text{on } \partial M_2.
\end{align*}
\]

From the hypothesis \( \Lambda_{\Gamma, g_1} = \Lambda_{\Gamma, g_2} \), we have
\[
\nu \times V(x) = \nu \times G_j^r(x, y) \quad \text{and} \quad \nu \times \left( \frac{1}{i \omega \nu} \text{curl } E(x) \right) = \nu \times \left( \frac{1}{i \omega \nu} \text{curl } G_j^r(x, y) \right)
\]
for \( x \in \Gamma' \),

so that, by Holmgren’s theorem for Maxwell’s equations (see Theorem 6.5 on p.194 of [15]) and unique continuation of real analytic function (see, for example, p.65 in [29]), \( V \) continues analytically to \( \tilde{V} \in C^\infty(\tilde{M}_2 \setminus \{y\}) \), with \( \tilde{V}(x) = G_j^r(x, y) \) for \( x \in U \setminus \{y\} \). This satisfies
\[
\begin{align*}
\mathcal{M}_g \tilde{V} &= \delta_y \quad \text{in } \tilde{M}_2, \\
\text{div } \tilde{V}(x) &= 0 \quad \text{in } \tilde{M}_2 \setminus \{y\}, \\
\tilde{V} \big|_{\partial \tilde{M}_2} &= 0.
\end{align*}
\]

Clearly \( \tilde{V}(x) = G_j^r(x, y) \) for \( x \in \tilde{M}_2 \setminus \{y\} \) since \( \tilde{M}_2 \) has compact closure. Thus, (4.109) holds.

Secondly, as in [36], we introduce the maps
\[
\mathcal{G}_j : \tilde{M}_j \to (H^s(U))^3, \quad (\text{any } s < -\frac{1}{2}),
\]

defined by
\[
\mathcal{G}_j(x)(y) = G_j(x, y), \quad x \in \tilde{M}_j, \quad y \in U.
\]

Since \( \delta_x \in (H^{s-2}(U))^3 \) depends continuously on \( x \), we see that \( \mathcal{G}_j(x) \in (H^s(U))^3 \) depends continuously on \( x \) for \( s < -\frac{1}{2} \), and the maps \( \mathcal{G}_j, j = 1, 2 \) are \( C^1 \). Because \( G_j(x, y) \) are real-analytic functions of \( x \) in \( \tilde{M}_j \setminus \{y\} \), we conclude that maps \( \mathcal{G}_j \) are real analytic on \( M_j \). Noting that the derivative of \( \mathcal{G}_j \)
\[
D\mathcal{G}_j(x) : T_x \tilde{M}_j \to (H^s(U))^3
\]
is defined by
\[
D\mathcal{G}_j(x)v = vG_j(x, \cdot) = \sum_{k=1}^3 \nu_k \text{div } G_j(x, \cdot) \bigg|_{x},
\]

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where \( v = \psi^k \frac{\partial}{\partial x_k} \in T_x \tilde{M}, \) we immediately see that the map \( D \mathcal{G}_j(x) \) is injective for each \( x \in \tilde{M}_j. \) Furthermore, it can be shown that the map \( \mathcal{G}_j : \tilde{M}_j \rightarrow (H^s(U))^3 \) is an embedding. In fact, it remains to show that \( x_1 \neq x_2 \) in \( \tilde{M}_j \) implies that \( \mathcal{G}_j(x_1) \neq \mathcal{G}_j(x_2). \) Suppose this is not the case, then

\[
G_j^*(x_1, y) = G_j^*(x_2, y)
\]

for all \( y \in U, \) therefore, by analyticity, (4.111) holds for all \( y \in \tilde{M}_j \setminus \{x_1, x_2\}. \) However, \( G_j^*(x_1, \cdot) \) is singular only at \( y = x_1 \) and \( G_j^*(x_2, \cdot) \) is singular only at \( y = x_2, \) which implies that \( x_1 = x_2. \)

Finally, it is clear that \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) coincide in the set \( U \) according to (4.109). Completely similar to proof of Theorem 3.3 of [36] we can show that the sets \( \mathcal{G}_1(\tilde{M}_1) \) and \( \mathcal{G}_2(\tilde{M}_2) \) are identical subsets of \((H^s(U))^3, \) and the map \( J := \mathcal{G}_2 \circ \mathcal{G}_1^{-1} : \tilde{M}_1 \rightarrow \tilde{M}_2 \) is an isometry. Combining the arguments in section 3 (see also the proof of Lemma 3.2) and the definition of \( J, \) we immediately see that \( J \) is an identity on \( \Gamma. \) Therefore, the desired result is immediately obtained.

\[ \square \]

**Remark 4.2.** In [38], among other things, the author of this paper also proved that the elastic Dirichlet-to-Neumann map determines the metric \( g \) uniquely up to isometry for a strong convex or extendable real-analytic manifold \( M \) with boundary. Actually, we can show the isometric result for the elastic problem without topology assumption other than compactness and connectedness by completely similar to the proof of Theorem 1.2.

## 5 Determining electromagnetic parameters from the electromagnetic Dirichlet-to-Neumann map

**Proof of Theorem 1.4.** Let \((x_1, x_2, x_3)\) be the local boundary normal coordinates associated with \((x_1, x_2)\) for \( \Gamma \subset \partial \Omega \) as in section 2. According to (3.66), the principal symbol \( \psi_{11}^i \) of the operator \( \Lambda_{11} \) is

\[
\psi_{11}^i = \frac{1}{i \alpha \mu \sqrt{|g|}} \left\{ -g_{12} (\phi_{11} + \sum_{\alpha} g^{\alpha j} a_{j q_1 - 1} (\sigma i \xi^j_1 \xi^q_1 \alpha) \right\} + g_{11} (\sum_{\alpha} g^{\alpha j} a_{j q_1 - 1} (\sigma i \xi^j_2 \xi^q_2 \alpha) \right\}
\]

Since \( g = (g_{jk}) \) are the known Riemannian metric on \( M, \) we see that the parameter \( \mu \) is uniquely determined by \( \psi_{11}^1(x', \xi') \) on \( \Gamma. \) Furthermore, all their tangential derivatives along \( \Gamma \) are determined by \( \psi_{11}^1(x', \xi'). \) By a completely similar discussion of impedance map \( \Lambda_{g,1}^{-1}, \) we can show that the highest terms of homogeneity in the expansion of the symbol for \( \Lambda_{g,1}^{-1} \) uniquely determines the \( \sigma \) on \( \Gamma. \)

Next, by (2.55) we see that \( \Lambda_{g,1}^{-1} \) uniquely determines \( L^n, \) \((1 \leq j, k \leq 2). \) It follows from (3.67) that

\[
l_{0j} = \phi_{0j} - \phi_{0j}^{33} q_2 (\sigma i \xi_j) + \sum_{\alpha} g^{\alpha j} \frac{\partial}{\partial x_{\alpha}} (q_1 - (\sigma i \xi_j)) + \sum_{\alpha} g^{\alpha j} \left\{ q_1 (\sigma \phi_{0j}^{3j}) + \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\partial g_{\alpha \beta}}{\partial x_j} + \frac{\partial \sigma}{\partial x_j} (\sigma i \xi_j) + \frac{1}{2} \sum_{m=1}^2 \frac{\partial q_2}{\partial x_m} \frac{\partial (\sigma i \xi_j)}{\partial x_m} \right\}
\]

Observe that there is not other terms containing \( \frac{\partial \sigma}{\partial x_5} \) or \( \frac{\partial \sigma}{\partial x_3} \) except for \( \phi_{0j}^{3j}, \phi_{0j}^{33} \) and \( q_2 \) on the right-hand
side of the above equality. In view of (3.61) we have
\[ q_{-2} = -\frac{1}{\sigma \phi_3^{33}} \left\{ q_{-1} \left( \sigma \phi_0^{33} + \frac{\partial \sigma}{\partial \xi} \right) \right\} + S_0^{(1)}(\mu, \sigma), \]

here and throughout the proof, each \( S_0^{(j)}(\mu, \sigma) \) is an expression involving only the \( \mu, \sigma \) and their tangential derivatives. Noting that \( \phi_0^{3j} = 0 + S_0^{(3)}(\mu, \sigma) \) for \( j = 1, 2 \) (see (3.68)), we get

(5.113)

\[
l_{0}^{jj} = \phi_0^{jj} - \phi_0^{j3} q_{-1} \sigma i \xi_j + \sum_{\alpha} g^{\alpha j} \left( q_{-1} \sigma \phi_0^{3j} + q_{-2} \sigma i \xi_j \right) + S_0^{(2)}(\mu, \sigma)
\]

\[
= \frac{1}{2} \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta \left\{ \frac{1}{\mu |g|} \left( \frac{\partial \mu}{\partial \xi_j} \right) \right\} + \sum_{\alpha} g^{\alpha j} \left( q_{-1} \sigma \phi_0^{3j} + \frac{\partial \sigma}{\partial \xi_j} \right) i \xi_j + \frac{1}{\nu} \sum_{\alpha} g^{\alpha j} \left( q_{-1} \sigma \phi_0^{3j} + \frac{\partial \sigma}{\partial \xi_j} \right) i \xi_j + S_0^{(3)}(\mu, \sigma)
\]

For each \( x \in \Gamma \), since the determinant

\[
\begin{vmatrix}
\frac{1}{2 \mu} - \frac{g_{21} \xi_1 - g_{22} \xi_2}{2 \mu |g| \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} & \frac{\sum_{\alpha} g^{\alpha j} \xi_\alpha \xi_j}{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} \\
\frac{1}{2 \mu} + \frac{g_{21} \xi_1 - g_{22} \xi_2}{2 \mu |g| \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} & \frac{\sum_{\alpha} g^{\alpha j} \xi_\alpha \xi_j}{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}
\end{vmatrix} = \frac{g^{11} \xi_1 - g^{22} \xi_2}{2 \mu \sigma \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta},
\]

we get the above determinant does not vanish when \( (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \mathcal{B} \) because the set \( \mathcal{B} := \{ (\xi_1, \xi_2) | g^{11}(x) \xi_1^2 - g^{22}(x) \xi_2^2 = 0 \} \) is at most two one-dimensional curves in \( \mathbb{R}^2 \). For each \( x \in \Gamma \), we may choose \( (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \mathcal{B} \). By solving the linear equation system

\[
\begin{align*}
\frac{\partial \mu}{\partial \xi_3} \left( \frac{1}{2 \mu} - \frac{g_{21} \xi_1 - g_{22} \xi_2}{2 \mu |g| \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} \right) + \frac{\partial \sigma}{\partial \xi_3} \left( \frac{\sum_{\alpha} g^{\alpha j} \xi_\alpha \xi_j}{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} \right) &= l_{10}^{j1}, \\
\frac{\partial \mu}{\partial \xi_3} \left( \frac{1}{2 \mu} + \frac{g_{21} \xi_1 - g_{22} \xi_2}{2 \mu |g| \sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} \right) + \frac{\partial \sigma}{\partial \xi_3} \left( \frac{\sum_{\alpha} g^{\alpha j} \xi_\alpha \xi_j}{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} \right) &= l_{20}^{j2},
\end{align*}
\]

we see that \( \frac{\partial \mu}{\partial \xi_3} \) and \( \frac{\partial \sigma}{\partial \xi_3} \) are uniquely determined on \( \Gamma \) by \( l_{10}^{j1} \) and \( l_{20}^{j2} \) (and hence by the operator \( \Lambda_{8, \Gamma} \)).

Because there are not \( \frac{\partial^2 \mu}{\partial \xi_3^2} \) and \( \frac{\partial^2 \sigma}{\partial \xi_3^2} \) in the expression of \( l_{0}^{jj} \) except for \( \phi_1^{jj}, \phi_{-1}^{33}, \phi_{-1}^{3j}, \) and \( q_{-3} \), it follows from (3.77) that

\[ j_{-1}^{jj} = \phi_{-1}^{jj} - \phi_{-1}^{j3} q_{-1} (\sigma i \xi_j) + \sum_{\alpha} g^{\alpha j} \left( q_{-1} \sigma \phi_0^{3j} + q_{-3} \sigma i \xi_j \right) i \xi_\alpha + S_{-1}^{(1)}(\mu, \sigma), \quad j = 1, 2, \]
where each \( S^{(m)}_{-1}(\mu, \sigma) \) denotes an expression involving only \( \mu, \sigma, \frac{\partial \mu}{\partial x_3} \) and \( \frac{\partial \sigma}{\partial x_3} \). According to (3.62), we see that

\[
q_{-3} = -\frac{1}{\sigma \Phi_1^l} \left( q_{-3} \phi_{3j}^j \right) + S^{(2)}_{-1}(\mu, \sigma)
\]

\[
= -\frac{1}{\sigma} \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\xi_\alpha \xi_\beta}{\xi_\alpha \xi_\beta} \left( \phi_{3j}^j + \frac{1}{\sigma} \frac{1}{\Phi_1^l} \frac{\partial \phi_{3j}^j}{\partial x_3} \right) + S^{(3)}_{-1}(\mu, \sigma)
\]

\[
= -\frac{1}{2\sigma} \left( \sum_{\alpha, \beta} g^{\alpha \beta} \frac{\xi_\alpha \xi_\beta}{\xi_\alpha \xi_\beta} \right)^{3/2} \left( \frac{\partial}{\partial x_3} \left( \frac{\partial \phi_{3j}^j}{\partial x_3} \right) \right) + \frac{\partial^2 \sigma}{\partial x_3} + S^{(4)}_{-1}(\mu, \sigma)
\]

so that for \( j = 1, 2 \),

(5.114)

\[
l_{-1}^{ij} = \phi_{-1}^{ij} - \phi_{-1}^{j3} \frac{i \xi_j}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} + \sum_{\alpha} g^{\alpha j} \left( \frac{1}{\sigma} \phi_{3j}^j + \frac{1}{\sigma} \frac{1}{\Phi_1^l} \frac{\partial^2 \sigma}{\partial x_3} i \xi_j \right) i \xi_\alpha + S^{(6)}_{-1}(\mu, \sigma)
\]

\[
= \phi_{-1}^{ij} - \phi_{-1}^{j3} \frac{i \xi_j}{\sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}} + \sum_{\alpha} g^{\alpha j} \frac{\partial \phi_{3j}^j}{\partial x_3} + \frac{1}{2} \sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} \left( \frac{\partial^2 \sigma}{\partial x_3} i \xi_j + S^{(8)}_{-1}(\mu, \sigma) \right)
\]

\[
= \frac{1}{2} \sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} \frac{\partial}{\partial x_3} \left( \frac{1}{\sigma} \phi_{3j}^j \right) - \frac{1}{2} \sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} \left( \frac{\partial \phi_{3j}^j}{\partial x_3} \right) + \sum_{\alpha} g^{\alpha j} \frac{1}{2} \frac{\partial^2 \sigma}{\partial x_3} i \xi_j + S^{(10)}_{-1}(\mu, \sigma)
\]

By taking \( j = 1, 2 \) and solving the above linear equation system, we immediately get that the \( l_{-1}^{11} \) and \( l_{-1}^{22} \) uniquely determines \( \frac{\partial \mu}{\partial x_3} \) and \( \frac{\partial \sigma}{\partial x_3} \) on \( \Gamma \).
Finally, we will consider the symbol $l_{-m-1}^{11} + l_{-m-1}^{22}$ with homogeneous of degree $-m - 1$, $(m \geq 1)$, for $L_{11} + L_{22}$. Obviously, there is not other terms containing $\frac{\partial^{m+2}u}{\partial x_{5}^{m+2}}$ and $\frac{\partial^{m+2}v}{\partial x_{5}^{m+2}}$ except for $\phi_{j}^{ij}$, $\phi_{-1}^{3j}$, $\phi_{-1}^{-j}$ and $q_{-3}$ on the right-hand side of (2.51), so that

$$l_{-m-1}^{ij} = \phi_{-m-1}^{-j} - \phi_{-m-1}^{3j} q_{-1}^{i} \xi_{j} + \sum_{\alpha} g_{j}^{a\alpha j} (q_{-1}^{\alpha} \phi_{-m-1}^{3j} + q_{-m-3}^{\alpha} \xi_{j}) \phi_{-m-1}^{j} + S_{-m-1}^{(1)}(\mu, \sigma), \quad j = 1, 2,$$

where each $S_{-m-1}^{(s)}$ is an expression involving only the boundary values of $\mu, \sigma$, and their normal derivatives of order at most $m + 1$. From (3.63) we have

$$q_{-m-3} = -\frac{1}{\sigma_{3}} q_{-1}^{i} \phi_{-m-1}^{33} + S_{-m-1}^{(2)}(\mu, \sigma).$$

It follows that

$$l_{-m-1}^{ij} = \phi_{-m-1}^{-j} - \frac{i \xi_{j}}{\sqrt{\sum_{\alpha, \beta} g_{\alpha\beta}^{x_{3}\xi_{\alpha}} \xi_{\beta}}} \phi_{-m-1}^{3j} + \sum_{\alpha} g_{j}^{a\alpha j} \left(\frac{1}{\sqrt{\sum_{\alpha, \beta} g_{\alpha\beta}^{x_{3}\xi_{\alpha}} \xi_{\beta}}} \phi_{-m-1}^{3j} - \frac{1}{\phi_{33}} \frac{1}{\sqrt{\sum_{\alpha, \beta} g_{\alpha\beta}^{x_{3}\xi_{\alpha}} \xi_{\beta}}} \phi_{-m-1}^{3j} i \xi_{j} \right) \phi_{-m-1}^{j} + S_{-m-1}^{(3)}(\mu, \sigma)$$

$$= \phi_{-m-1}^{-j} - \frac{i \xi_{j}}{\sqrt{\sum_{\alpha, \beta} g_{\alpha\beta}^{x_{3}\xi_{\alpha}} \xi_{\beta}}} \phi_{-m-1}^{3j} + \frac{1}{\phi_{33}} \frac{1}{\sqrt{\sum_{\alpha, \beta} g_{\alpha\beta}^{x_{3}\xi_{\alpha}} \xi_{\beta}}} \sum_{\alpha} g_{j}^{a\alpha j} i \xi_{\alpha} \phi_{-m-1}^{j} + S_{-m-1}^{(3)}(\mu, \sigma).$$

From (2.41) we have

$$l_{-m-1}^{ij} = \frac{1}{2 \sqrt{\sum_{\alpha, \beta} g_{\alpha\beta}^{x_{3}} x_{3}}^{2} x_{3}} \frac{\partial \phi_{-m}^{j}}{\partial x_{3}} + S_{-m-1}^{(4)}(\mu, \sigma) \quad \text{for } j, k = 1, 2, 3.$$

We will end our proof by induction. Suppose we have shown that for $1 \leq r \leq m$,

$$l_{-r}^{ij} = \left(2 \sqrt{\sum_{\alpha, \beta} g_{\alpha\beta}^{x_{3}} x_{3}}\right)^{-r} \left\{ \frac{\partial^{r+1} \mu}{\partial x_{3}^{r+1}} \left( \frac{1}{2 \mu} + \frac{(-1)^{j-1} (g_{3-j, 2} \xi_{3-j} - g_{3-j, 1} \xi_{3}) \xi_{j}}{2 \mu \xi_{1}} \left( \sum_{\alpha, \beta} g_{\alpha\beta}^{x_{3}} x_{3} \right) \right) \right\} + S_{-r}^{(5)}(\mu, \sigma).$$

Clearly, this estimates actually holds when $r = 1$ by (5.114). It follows from (5.115) and (2.41) that

$$l_{-m}^{ij} = \frac{1}{2 \sqrt{\sum_{\alpha, \beta} g_{\alpha\beta}^{x_{3}} x_{3}}^{2} x_{3}} \frac{\partial \phi_{-m}^{j}}{\partial x_{3}} \left\{ \phi_{-m}^{-j} - \frac{i \xi_{j}}{\sqrt{\sum_{\alpha, \beta} g_{\alpha\beta}^{x_{3}} x_{3}}^{2} x_{3}} \phi_{-m}^{3j} + \frac{1}{\phi_{33}} \frac{1}{\sqrt{\sum_{\alpha, \beta} g_{\alpha\beta}^{x_{3}} x_{3}}^{2} x_{3}} \sum_{\alpha} g_{j}^{a\alpha j} i \xi_{\alpha} \phi_{-m}^{j} \right\} + S_{-m-1}^{(7)}(\mu, \sigma)$$

$$= \frac{1}{2 \sqrt{\sum_{\alpha, \beta} g_{\alpha\beta}^{x_{3}} x_{3}}^{2} x_{3}} \frac{\partial l_{-m}^{ij}}{\partial x_{3}} + S_{-m-1}^{(8)}(\mu, \sigma).$$
Combing this and (5.117) we have

\[
(5.118) \quad l_{m-1}^{jj} = \frac{1}{2} \sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} \frac{\partial}{\partial x_3} \left\{ \left( 2 \sqrt{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta} \right)^{-m} \left[ \frac{\partial^{m+1} \mu}{\partial x_3^{m+1}} \left( -\frac{1}{2 \mu} + (-1)^{j-1} (g_{3-j,2 \xi_1}^1 - g_{3-j,1 \xi_2}^2) \xi_j \right) + \frac{\sum_{\alpha, \beta} g^{\alpha \beta} \xi_\alpha \xi_\beta}{2 \sigma} \frac{\partial^{m+2} \sigma}{\partial x_3^{m+2}} \right] + S_{m-1}^{(11)}(\mu, \sigma) \right\}
\]

Solving the above linear equation system, we get that \( \frac{\partial^{m+2} \mu}{\partial x_3^{m+2}} \) and \( \frac{\partial^{m+2} \sigma}{\partial x_3^{m+2}} \) can uniquely be determined on \( \Gamma \) by \( l_{m-1}^{11} \) and \( l_{m-1}^{1j} \). More precisely, we can solve for the first \((m+2)\)-order normal derivatives of \( \mu \) and \( \sigma \) in terms of \( l_{m-1}^{jj} \) for \( k = 1, 2, \cdots, m+1 \) and \( j = 1, 2 \).

**Proof of Theorem 1.5.** Let \((x_1, x_2)\) be any local coordinates for an open set \( W \subset \Gamma \), and let \( \{ \psi_j \}_{j \leq 1} \) denote the full symbol of \( \Lambda_{\partial \Sigma} \) in these coordinates. Then for any \( x_0 \in W \), \( \frac{\partial^{K} \mu}{\partial x^{K}} \) and \( \frac{\partial^{K} \sigma}{\partial x^{K}} \) for all multi-indices \( K = (k_1, k_2, k_3) \) with \( |K| \geq 0 \) at \( x_0 \) in boundary normal coordinates is given by explicit formula in terms of the matrix-valued functions \( \{ \psi_j \}_{j \leq 1} \) and their tangent derivatives at \( x_0 \). This implies that we can determines the functions \( \mu \) and \( \sigma \) at a small neighborhood of \( x_0 \) by the real-analyticity of \( \mu \) and \( \sigma \) on \( M \cup \Sigma \). Hence, by unique continuation of real analytic function (see, for example, p.65 in [29]), we can uniquely determine \( \mu \) and \( \sigma \) on real-analytic manifold \((M, g)\).

**Remark 5.1.** It is easy to verify that Theorem 1.5 still holds for piece-wise real-analytic manifold \((M, g)\).

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