Signature of Schwinger’s pair creation rate via radiation generated in graphene by strong electric current

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Electron - hole pairs are copiously created by an applied electric field near the Dirac point in graphene or similar 2D electronic systems. It was shown recently that for sufficiently large electric fields $E$ and ballistic times the I-V characteristics become strongly nonlinear due to Schwinger’s pair creation rate, proportional to $E^{3/2}$. Since there is no energy gap the radiation from the pairs’ annihilation is enhanced. The spectrum of radiation is calculated and exhibits a maximum at $\omega = \sqrt{\epsilon E v_g \gamma}$. The angular and polarization dependence of the emitted photons with respect to the graphene sheet is quite distinctive. For very large currents the recombination rate becomes so large that it leads to the second Ohmic regime due to radiation friction.

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INTRODUCTION

Electronic mobility in graphene, especially one suspended on leads, is extremely large $\sigma$ so that a graphene sheet is one of the purest electronic systems. The relaxation time of charge carriers due to scattering off impurities, phonons, ripplons, etc., in suspended graphene samples of submicron length is so large that the transport is purely ballistic $L/v_g$. The ballistic flight time in these samples can be estimated as $t_{bal} = L/v_g$, where $v_g \simeq 10^6 m/s$ is the graphene velocity characterizing the massless “ultra-relativistic” spectrum of graphene near Dirac points, $\epsilon_k = v_g |k|$, and $L$ is the length of the sample that can exceed several $\mu m \ \ [1, 2, 3]$. The extraordinary physics appears right at the Dirac point at which the density of states vanishes. In particular, at this point graphene exhibits a quasi-Ohmic behaviour, $\mathbf{J} = \sigma \mathbf{E}$, even in the purely ballistic regime.

A physical picture of this “resistivity” without either charge carriers or dissipation is as follows $\mathbf{4}$. The electric field creates electron - hole excitations in the vicinity of the Dirac points similar to the Landau-Zener tunneling effect in narrow gap semiconductors or electron - positron pair creation in Quantum Electrodynamics first studied by Schwinger $\mathbf{7}$ (later referred to as LZS). Importantly, in graphene the energy gap is zero, thus the pair creation is possible at zero temperature and arbitrary small $\mathbf{E}$, even within linear response. Although the absolute value of the quasiparticle velocity $v_g$ cannot be altered by the electric field due to the ”ultra-relativistic“ dispersion relation, the orientation of the velocity can be influenced by the applied field. The electric current, $\mathbf{e} \mathbf{v}$, proportional to the projection of the velocity $\mathbf{v}$ onto the direction of the electric field is increased by the field. These two sources of current, namely creation of moving charges by the electric field (polarization) and their re-orientation (acceleration) are responsible for the creation of a stable current.

Agreement over the qualitative explanation notwithstanding, determination of the value of the minimal DC conductivity at Dirac point in the limit of zero temperature had undergone a period of experimental and theoretical uncertainty. After the value in graphene on substrate was measured to be about $\sigma = 4e^2/h \ \ [4]$, it was shown in experiments on suspended samples $\mathbf{2}$ that the zero temperature limit was not achieved and in fact that these early samples had too many charged ”puddles”, so that they represented an average around the neutrality or the Dirac point. The value in early-on suspended samples $\mathbf{2}$ was half of that and most recently settled at the ”dynamical” $\sigma_2 = \frac{\pi e^2}{h}$ in best samples at $2K$ temperature $\mathbf{4}$. Theoretically several different values appeared. The value $\sigma_1 = \frac{\pi e^2}{2h}$ had been considered as the ”standard” one for several years $\mathbf{9}$ and appeared as a zero disorder limit in many calculations like the self consistent harmonic approximation, although different regularizations within the Kubo formalism resulted in different values $\mathbf{10}$.

The dynamical approach to transport was applied to the tight binding model of graphene $\mathbf{11}$ to resolve this ”regularization ambiguity”. It consists of considering the ballistic evolution of the current density in time after a sudden or gradual switching on of the electric field. The result within linear response is that the current settles very fast, on the microscopic time scale of $t_\gamma = h/\gamma \approx 0.24 \text{ fs}$ (γ being the hopping energy), on the value of $\mathbf{J} = \sigma_2 \mathbf{E}$. The value is identical to the one obtained (at nonzero temperatures) for the AC conductivity $\mathbf{12}$. The two contributions, polarization and attenuation are comparable in strength and combine to produce a constant total current. However a deeper analysis of the ”quasi-Ohmic” graphene system beyond the leading order in...
p perturbation theory in electric field revealed \[13\] that on the time scale

\[ t_{nl} = \sqrt{\frac{\hbar}{eE v_g}}, \]  

(1)

the linear response breaks down. For larger times the quasi-Ohmic behavior no longer holds. This is in contrast to dissipative systems, in which the linear response limit can be taken directly at infinite time. This perhaps is the origin of the "regularization" ambiguities in graphene, since large time and small field limits are different. The time scale on which nonlinear effects become dominant is not always very large; for example, in experiments dedicated to breakdown of Quantum Hall effect \[14\] in which \( E = 10^4 V/m \), nonlinearity sets in at \( t_{nl} = 0.3 ps \), that is of order ballistics time for \( L = 0.3 \mu m \). Graphene flakes under larger fields of order \( 2 \cdot 10^6 V/m \) have been studied very recently (at room temperature) in specially designed high current density experiments \[5\]. In this case the nonlinear time is only \( 20 fs \), much lower than the ballistic time \( t_{bal} = 2 ps \) for \( L = 2 \mu m \). Analytic and numerical solutions of the tight binding model \[13\], as well as of the Dirac model describing the physics near the Dirac point demonstrated \[12, 15\] that at \( t_{nl} \) the electron - hole pairs creation becomes dominant and is well described by an adaptation of the well-known (non-analytic in \( E \)) Schweringer electron - positron pair creation rate

\[ \frac{d}{dt} N_p = \frac{3^{3/4}}{29/2v_g^{3/2}} \left( \frac{eE}{\hbar} \right)^{3/2}. \]  

(2)

The difference with the original derivation \[7\] in the context of particle physics is that the fermions are 2+1 dimensional and "massless", thus magnifying the effect. The polarization current is \( J(t) = 2e v_g N(t) \) and therefore Schweringer’s creation rate leads to a linear increase with time \[12, 15\]:

\[ J(t) = \sigma v_g \left( \frac{\sqrt{3}}{2} \right) \left( \frac{e v_g}{\hbar} \right)^{1/2} t. \]  

(3)

The physics of pair creation is highly non-perturbative and non-linear in nature and therefore, instead of the linear response, Schweringer found an exact formula using functional methods. The rate can be intuitively understood using the much simpler instanton approach originally proposed in the context of particle physics \[10\] (extended later to low dimensions \[17\]), but is known in fact in condensed matter physics as the Landau - Zener tunneling probability \[5, 13, 15\]. In particle physics it is extremely difficult to observe Schweringer’s creation rate and it would be interesting to establish experimentally this dynamical phase in low dimensional condensed matter physics featuring the massless Dirac quasiparticle spectrum like graphene or novel materials sharing with it the massless Dirac spectrum like topological insulators or tuned semiconductor heterojunctions \[12\]. Of course transport phenomena at rather large fields always have a background related to possible influence of leads, disorder and thermal effects like local heating, etc.

In this note we draw attention to a direct and unintrusive signature of the dynamical phase of LHS pair creation in a graphene sheet subject to an applied electric field. It is demonstrated that the flux of photons radiated from the surface of the sample is characterized by the creation rate since the photons are emitted via electron - hole pair annihilation and therefore proportional to \( E^{3/2} \), a hallmark of Schwinger’s process. In addition, the frequency, direction and polarization characteristics of the radiation generated by the electric field calculated here all bear footprints of the pair creation dynamics.

**ELECTRON - HOLE RECOMBINATION RATE INTO PHOTONS.**

Amplitude for emission of a single photon

The electrons and their electromagnetic interaction with photons are approximately described near a Dirac point by the Weyl Hamiltonian:

\[ H = \int d^3 r \psi^\dagger \psi \left[ -ie A_0 + i \frac{\hbar}{2} \left( \frac{\partial^2}{2m_e} + i \frac{\hbar}{me} A_z \right)^2 + V_{con_f} (z) \right] \psi. \]  

(4)

Here \( \psi \) is the two component spinor second quantized field and \( (A_0, A_z) \) is the vector potential (bold letters describe vectors in the graphene plane, while \( z \) is the perpendicular direction). Electrons (charge \(-e\)) and holes (charge \(e\)) in the graphene sheet are confined in this model to the \( z = 0 \) plane by a potential \( V_{con_f} \) (small shape changes can be neglected for our purposes). The only requirement from this potential is that it is strong enough to "freeze" the motion along the \( z \) direction. In the single graphene sheet one has two left handed chirality Weyl fermions described by the above Hamiltonian in which \( \sigma \) denotes the in-plane Pauli matrices and two right handed Weyl fermions represented by \( \sigma^\dagger \). To include the topological insulators case \[10\], we first concern ourselves with only one spinor.

![Diagram](image)
Fig. 1. Feynman diagrams representing the major electromagnetic processes in graphene. a. The one photon emission.

![Feynman diagram](image)

b. The two photon emission

We consider the emission of a photon with wave vector \((\mathbf{k}, k_z)\) and frequency \(\omega = c\sqrt{k^2 + k_z^2}\), described by a linearly polarized plane wave,

\[
A_{\text{ph}} = \frac{2E_0}{\omega} e^{i(\lambda)} \sin (k \cdot r + k_z z - \omega t),
\]

whereas the DC applied field is \(A_{\text{ext}} = (0, -cEt)\). For regularization we make use of a finite box \(L \times L \times L_z\), so that momenta are discrete and the single photon’s electric field is \(E_0^\prime = \hbar \omega / (L^2 L_z)\). The unit vectors

\[
e^{(1)} = (- \sin \varphi, \cos \varphi); \quad n_z^{(1)} = 0;
\]
\[
e^{(2)} = - \cos \theta (\cos \varphi, \sin \varphi); \quad n_z^{(2)} = \sin \theta,
\]

describe polarizations that are conveniently chosen similarly to a recent calculation of electromagnetic emission due to thermal fluctuations [21]. The vectors \(e^{(1)}\) and \(e^{(2)}\) represent the “in plane” and the “out of plane” polarizations, respectively. The electron and the hole wave functions are \(\frac{1}{\sqrt{2L}} e^{i\mathbf{p} \cdot \mathbf{r}} u(\mathbf{p}) \psi_n(z)\) and \(\frac{1}{\sqrt{2L}} e^{i\mathbf{p}^\prime \cdot \mathbf{r}} v(\mathbf{p}^\prime) \psi_n(z)\), with spinors defined by

\[
u(\mathbf{p}) = \left( \begin{array}{c} 1 \\ -ie^{i\phi} \end{array} \right), \quad \nu^\prime(\mathbf{p}^\prime) = \left( \begin{array}{c} 1 \\ ie^{i\phi^\prime} \end{array} \right),
\]

with \(\mathbf{p} + \frac{e}{\hbar} A_{\text{ext}} = \mathbf{p} (\cos \phi, \sin \phi)\). \(\psi_n(z)\) are wave functions of the confinement. The interaction with a photon at time \(t\) happens when the momentum is minimally shifted due to the DC field. The Golden rule photon emission rate (for an “initial” electron with momentum \(\mathbf{p}\) and a “final” hole \(\mathbf{p}^\prime\) and a photon of polarization \(\lambda\) and momentum \((\mathbf{k}, k_z)\)) is

\[
W_{\text{nn}}^{(\lambda)}(\mathbf{p}, \mathbf{p}^\prime, k_z, t) = \frac{2\pi}{\hbar} \left| F_{\text{nn}}^{(\lambda)} \right|^2 N_p(t) N_{-p^\prime}(t) \delta(h\nu_g (p + p^\prime) - \hbar \omega).
\]

In terms of Feynman diagrams of Quantum Electrodynamics [20] it corresponds to the diagram in Fig. 1a. Here \(N_p(t)\) is the density of electrons in a certain momentum range produced by the electric field \(E\), and \(N_{-p^\prime}(t)\) the density of holes (equal to that of the electrons at the opposite momentum due to particle - hole symmetry). The density calculated using the simple Landau - Zener creation rate expression for one of the flavours is [13, 15, 16]:

\[
N_p(t) = \Theta(p_y) \Theta\left(\frac{\epsilon}{\hbar}Et - p_y\right) \exp \left(-\frac{\pi \hbar v_y p_y^2}{eE^2 t_z^2}\right),
\]

where \(\Theta\) are the Heaviside functions. The transition amplitude is given by

\[
F_{\text{nn}}^{(\lambda)} = \frac{E_0}{\omega} \frac{e}{2L^2} e^{-i(v_y (p + p^\prime) - \omega)} F_{\text{nn}}^{(\lambda)}(\mathbf{p}, \mathbf{p}^\prime) \int dz e^{ik_z z} \psi_n^* (z) \psi_n (z) \delta (\mathbf{p} + \mathbf{p}^\prime - \mathbf{k}),
\]

where matrix elements \(F_{\text{nn}}^{(\lambda)}(\mathbf{p}, \mathbf{p}^\prime) \equiv v^\dagger (-\mathbf{p}') \sigma \cdot e^{(\lambda)} u (\mathbf{p})\) are

\[
\left| F_{\text{nn}}^{(1)}(\mathbf{p}, \mathbf{p}^\prime) \right|^2 = 2 \left[ 1 - \cos \left( 2\varphi - \phi - \phi^\prime \right) \right];
\]
\[
\left| F_{\text{nn}}^{(2)}(\mathbf{p}, \mathbf{p}^\prime) \right|^2 = 2 \cos^2 \theta \left[ 1 + \cos \left( 2\varphi - \phi - \phi^\prime \right) \right].
\]

Spectral emittance

For tight confinement to the \(z = 0\) plane one should consider only the ground state \(n = n' = 0\). Note that the perpendicular component of the wave vector \(k_z\) is “free” from conservation that prohibits the process in fully relativistic QED [20]. The phase space for annihilation is very limited due to \(v_y < c\), see Appendix A and leads to important simplifications.

![Graphene intensity](image)
where the integration over graphene flake, of the radiation in the direction perpendicular to the momentum. We first study the frequency dependence of the delta function expressing the conservation of momentum, summing over the polarizations
\[ \omega_{\text{t}<<f} \]
frequencies as function of time, increases linearly for \( t \to \infty \), then rises sharply approaching a maximum at \( t = \omega t_{nl} \). For each ballistic time the curve has two parts. The first follows the universal dependence given by Eq. (13). Therefore the frequency for observation of the Schwinger effect, not surprisingly, should exceed \( \omega_{\text{min}} = \sqrt{eEeV/h} \), that amounts to 3.6THz for \( E = 10^4V/m \), and 50THz for \( E = 2 \cdot 10^9V/m \). At a higher frequency \( \omega_{\text{max}} = 2t/t_{nl} \) the emittance sharply drops. Therefore the frequency does not exceed 2\( t_{bal}/t_{nl} \).

Let us define the spectral emittance per volume of the \( k \)-space (and area of the graphene flake) as

\[
\mathcal{M}^{(\lambda)}(k, z, t) = \frac{4\pi\hbar}{L^2} \sum_{p, p'} \frac{d}{dk_z dk} W^{(\lambda)}(p, p', k, z, t)
\]

\[
= e^2 v_g^2 \left( \frac{2\pi}{\lambda} \right)^3 \int \frac{d^2 p}{N_p N_{p-k}} \delta \left( v_g (p + |p - k|) - \omega \right),
\]

where the integration over \( p' \) was performed using the delta function expressing the conservation of momentum. We first study the frequency dependence of the radiation in the direction perpendicular to the graphene flake, \( k = 0 \). Multiplying with 4 for the spin and valley degeneracy, summing over the polarizations
\[ \sum_{\lambda} |F_{p, p}^{(\lambda)}|^2 = 4 \], and integrating over \( p \) one obtains, using \( \omega = ck_z \), the spectral emittance

\[
\mathcal{M}(k = 0, \omega, t) = e^2 v_g^2 t_{nl}^2 \frac{\omega}{\pi^4} \int_{-t/t_{nl}}^{0} d\bar{p} \Theta \left( t_{nl} \omega / 2 + \bar{p} \right)
\]

\[
\times \exp \left[ -2\pi \frac{(t_{nl}^2 \omega^2 / 4 - \bar{p}^2)}{\omega (t_{nl}^2 \omega^2 / 4 - \bar{p}^2)^{1/2}} \right].
\]

The spectral emittance, presented in Fig.2 for various frequencies as function of time, increases linearly for \( t << \omega^{-1} \), \( \mathcal{M}(k = 0, \omega, t) = e^2 t_{nl}^2 \omega / \pi \), then rises sharply approaching a maximum at \( t = \omega t_{nl} / 2 \) and stabilizes at

\[
\mathcal{M}(k = 0, \omega, t >> t_{nl}) = e^2 \frac{\omega}{\pi^3} \exp \left( -\pi t_{nl}^2 \omega^2 / 4 \right) I_0 \left( \frac{\pi t_{nl}^2 \omega^2}{4} \right),
\]

where \( I_0 \) is the modified Bessel function. The asymptotic value rises linearly with frequency, \( \pi^{-3} \omega \), in the infrared, reaches its maximum at \( \omega = t_{nl}^{-1} \) and falls slightly to \( \sqrt{2\pi} t_{nl}^2 \) in the ultraviolet. In Fig.3 the emittance at various ballistic times is given as function of frequency.

**Angular and polarization distribution**

Next we consider the angular and polarization dependence of the radiated power per unit area defined as the
spectral intensity, Eq.(12), integrated over frequencies:

\[ \mathcal{L}^{(\lambda)}(\theta, \varphi, t) \equiv \int_0^\infty dw \frac{\omega^2}{c^2} M^{(\lambda)}(k, \omega, t). \]  

Performing integrations and simplifying, utilizing the small parameter \( v \equiv v_g/c \simeq 1/300 << 1 \), see Appendix B for details, one obtains:

\[ \mathcal{L}^{(1)}(\varphi, t) = \frac{e^4 v^4 E^2}{2^{5/2} \pi^4 c^4 \hbar^2} \left( \frac{t}{4 \pi \ell_{nl}} \cos^2 \varphi + \frac{t^3}{3 \ell_{nl}^3} \sin^2 \varphi \right); \]

\[ \mathcal{L}^{(2)}(\theta, \varphi, t) = \frac{e^4 v^4 E^2}{2^{5/2} \pi^4 c^4 \hbar^2} \cos^2 \theta \times \left( \frac{t}{4 \pi \ell_{nl}} \sin^2 \varphi + \frac{t^3}{3 \ell_{nl}^3} \cos^2 \varphi \right). \]  

The radiant flux from a flake of a \( \mu m \times \mu m \) size is \( 4.7 \cdot 10^{-21} \text{W} \), for \( E = 10^4 \text{V/m} \) corresponding to the emission rate of just 10 photons per second, yet for the high current samples, \( \ell_{nl} \) with \( E = 2 \cdot 10^6 \text{V/m} \) of the same area one gets a more significant output: the radiant flux is \( 1.3 \cdot 10^{-17} \text{W} \), corresponding to the emission rate of \( 3 \cdot 10^4 \) photons per second.

**DISCUSSION**

Now we elaborate on a number of related issues and comment on the experimental feasibility of exploring the Schwinger phase physics. We start with a qualitative discussion of the rather unusual physics arising at strong applied fields, when the pair recombination becomes an important relaxation channel.

**Coulomb interaction and the formation of the neutral electron-hole plasma**

Even if the ballistic time and the relaxation time are very large, Schwinger’s dynamical pair creation phase cannot persist for a long time at large field since density of charges (or both signs) becomes large. When the density of quasiparticles reaches the order of \( \rho_p = 10^{11} \text{cm}^{-2} \) a neutral electron - hole plasma is created (like in some semiconductor systems under irradiation). In this state electrostatic interactions (despite being screened at large distances) become dominant, as was discussed extensively in connection with electron - positron pairs creation in Quantum Electrodynamics\[^{20}\]. When electrons and holes are close enough they strongly attract each other effectively facilitating the recombination process. The rate therefore far exceeds the one calculated within perturbation theory in Section III. Let us first estimate when this state is achieved at experimentally accessible situations.

Assuming the Schwinger pair creation rate, Eq.(2), the density will approach \( \rho_p \) at times of order \( t_p \propto \rho_p / E^{3/2} \). With a moderate field value of \( E = 10^4 \text{V/m} \)\[^{14}\], the “plasma time” \( t_p \approx 140 \text{ps} \simeq 400 \tau_{rd} \) exceeds the ballistic time of the \( L = 1 \mu m \) long sample (and probably also the relaxation time in current experiments on graphene). Yet with higher achievable fields \( E = 2 \cdot 10^6 \text{V/m} \)\[^{2}\], the plasma time is reduced to \( t_p \approx 40 \text{fs} \simeq 2 \tau_{rd} \ll t_{rd} \). Therefore one expects that the “radiation friction” dissipation channel opens up: electron - hole pairs annihilate emitting photons, which take energy out of the graphene sheet and thus a new Ohmic behavior is reached. This
is roughly the ballistic time range for which the emission was calculated in Section III. Of course, due to the Coulomb attraction enhancement of the recombination the intensity becomes grossly underestimated in the plasma regime. The pair density will have to be recalculated via Boltzmann equations; this will be done in a separate publication. One however might expect qualitatively that this conductivity in the "second" Ohmic regime certainly exceeds $\sigma_2$ and is likely to reach several times $\sigma_3$. Data presented in ref. [5] for clean samples, see Figs. 3 and 12 therein, is consistent with the linear (Ohmic) I-V curve at such conductivity value. However the experimental situation in the transport experiment is rather complex as discussed next.

**Experimental evidence of the pair creation in samples of mesoscopic size**

In this subsection we use the above "radiation friction" scenario to discuss whether there is a clear and unambiguous signature of the Schwinger’s pair creation phase in transport experiments. In a series of remarkable experiments the nonlinear I-V were measured at high electric fields of the order $2 \cdot 10^3 V/m$ at room temperature [3]. The samples were treated in such a way that, despite being non-suspended, the typical charge asymmetry did not appear and the Dirac point was accessed convincingly at zero gate voltage (this demonstrates high quality and is in variance with most samples on substrate). The I-V curves were studied in various high and low mobility samples (up to $\mu = 7000 cm^2/(V s))$ and effects of disorder were partially controlled by irradiating the samples. The samples were $L = 1 - 2 \mu m$ long and rather narrow ($W = 0.5 \mu m$) and the four-probe technique was applied. Although a nonlinear I-V dependence $I \propto V^n$ with exponent $\alpha = 1.3 - 1.5$ was observed at Dirac point, surprisingly the nonlinearity disappeared in the highest mobility samples. In these experiments the ballistic time $t_{bat} = 2 ps$ is much larger than $t_{nl} = 20 fs$ at the highest applied voltage of 4V. Unfortunately the crossover voltages, namely when $t_{nl}(V) = t_{bat}$ or

$$V_{nl} = \frac{h \nu_g}{eL} = 0.32 mV \quad (18)$$

were not probed since at room temperature $k_B T = 25 m eV$. As argued in the previous subsection the radiation friction causes a second Ohmic regime and the I-V curve in the clean samples is expected to be linear. It is disorder that might have caused the observed non-linearity in irradiated samples. This requires an additional theoretical study that includes the effect of pair recombination. As argued above it becomes as important as the Landau-Zener-Schwinger pair creation process at such currents.

**CONCLUSIONS**

To summarize, electron - hole pairs are copiously created via Landau-Zener-Schwinger mechanism near the Dirac points in graphene or similar 2D electronic systems by an applied electric field, provided the available ballistic time exceeds $t_{nl}$, Eq. (1). The recombination into photons produces a characteristic signal proportional to $E^{3/2}$ at frequencies of order $t_{nl}^{-1}$ that enables unintrusive and unambiguous experimental observation of the Schwinger phenomenon. The angular and polarization dependence of the emitted photons with respect to the graphene sheet was calculated. At very high currents and sufficiently long ballistic times the recombination process becomes greatly enhanced by the electron - hole attraction and the radiation becomes an effective channel of dissipation, the radiation friction.

The calculation can be trivially extended to any system with a Dirac point - like spectrum as double layer graphene and the recently synthesized family of materials called "topological insulators" [19] in which surface excitations are similar to those in graphene with the notable exception of chirality. Schwinger’s mechanism is also expected in these materials since the mechanism does not involve chirality (left and right movers contribute equally to the emission rate of graphene). These materials have an advantage of not being strictly two dimensional, although ballistic times might be shorter at present. The driving current should not necessarily be DC, a sufficient condition is $\omega_{ext} << t_{nl}^{-1}$. Detectors of light (photon counters) in the microwave-infrared which are sensitive enough have recently been developed [22]. Hopefully Schwinger’s pair creation rate formula can be directly tested using novel condensed matter materials endowed with relativistic fermion spectra.

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**APPENDIX A. PHASE SPACE OF THE ONE-PHTON PROCESS**

Planar electrons and holes are described by their momenta $p$ and $p'$ in the $x - y$ plane, (see Fig.1a and subsection IIA for notations), while the momentum of the photon $k$ is three dimensional. The conservation of the in-plane momentum and the conservation of energy read:
\[ \mathbf{p} + \mathbf{p'} = \mathbf{k}; \quad v_g (p + p') = ck. \]  

(19)

The momentum in the z direction is not conserved; it is balanced by the elasticity of the graphene flake. Since 

\[ k = \sqrt{k_x^2 + |\mathbf{k}|^2} \geq |\mathbf{k}|, \]  

in terms of \( \mathbf{p} \) and \( \mathbf{p'} \) one has the inequality

\[ v^2 (p + p')^2 - |\mathbf{p} + \mathbf{p'}|^2 \geq 0, \]

(20)
or, in the polar coordinates,

\[ (1 - v^2) (p^2 + p'^2) + 2 [\cos (\phi - \phi') - v^2] pp' \leq 0. \]

(21)

With the above constraint the condition that \( pp' \) has real solutions leads to

\[ \cos (\phi - \phi') \leq -1 + 2v^2. \]

(22)

As \( v \ll 1 \), it is obvious that \( \phi' - \phi \) is very close to \( \pi \). By defining \( \Delta \phi = \pi - (\phi' - \phi) \), one see the above condition simplifies to

\[ -2v \leq \Delta \phi \leq 2v. \]

(23)

Substituting the above result back into Eq.(21) it can be seen that \( pp' \) is very close to \( p \). By introducing \( \Delta r = 1 - p'/p \), the condition becomes

\[ \Delta r^2 + \Delta \phi^2 \leq 4v^2. \]

(24)

Therefore the allowed region is a disk of radius \( 2v \).

### APPENDIX B. DERIVATION OF THE AMPLITUDE AND SPECTRAL EMITTANCE

The Golden rule photon emission rate (for an "initial" electron with momentum \( \mathbf{p} \), a "final" hole \( \mathbf{p'} \) and a photon of polarization \( \lambda \) and momentum \( (\mathbf{k}, k_z) \)) is given by Eq.(5). Correspondingly the rate defined in Eq.(12) is

\[ \mathcal{M}^{(\lambda)} (\theta, \varphi, \frac{\omega}{c}, t) = \frac{e^2 v_g^2}{(2\pi)^4} \int d\mathbf{p} \left| F_{p,k-p}^{(\lambda)} \right|^2 N_{p} N_{k-p} \delta (v_g (p + |\mathbf{k} - \mathbf{p}|) - \omega) \]

\[ = \frac{e^2 v_g^2}{(2\pi)^3} \int p(\phi) d\phi \left| F_{p,k-p}^{(\lambda)} \right|^2 N_{p} N_{k-p} \cdot J(\phi). \]

(25)

Because of the delta function, the integration over \( \mathbf{p} \) imposes the condition of energy conservation. As a result, the squares of the matrix elements, Eq.(11) simplify:

\[ \left| F_{p,p'}^{(1)} \right|^2 = \frac{4 \left[ \cos (\phi - \varphi) - v \sin \theta \right]^2}{(1 - 2v \sin \theta \cos (\phi - \varphi) + v^2 \sin^2 \theta)}. \]

\[ \left| F_{p,p'}^{(2)} \right|^2 = \frac{4 \cos^2 \theta \sin^2 (\phi - \varphi)}{(1 - 2v \sin \theta \cos (\phi - \varphi) + v^2 \sin^2 \theta)}. \]

(26)

Moreover, there is an additional factor from the delta function:

\[ J(\phi) = \frac{(1 - 2v \sin \theta \cos (\phi - \varphi) + v^2 \sin^2 \theta)}{2v_g (1 - 2v \sin \theta \cos (\phi - \varphi))^2}. \]

(27)

The Jacobian of the transition to polar coordinates is

\[ p(\phi) = \frac{\omega (1 - v^2 \sin^2 \theta)}{2v_g (1 - 2v \sin \theta \cos (\phi - \varphi))}. \]

(28)

In view of the step functions for the LLS density, Eq.(4), the following possibilities occur:

a. **into the forward direction (positive projection on the electric field)** \( 0 < \varphi < \pi \):

The conditions imposed by the step functions are \( 0 < -p_y \) and \( 0 < \frac{eE}{\hbar} t + p_y - k_y \). In terms of the polar coordinates in the momentum space

(i) for \( 0 < \frac{2\pi}{\omega v_g} \sin \theta < 1 + \sin \theta \sin \varphi \), the allowed regions are \( \Delta_+ - \phi_0 < \phi < 0 \) or \( -\pi < \phi < -\pi + \phi_0 - \Delta_+ \), with

\[ \phi_0 = \arcsin \left( \frac{2t}{\omega v_g} \right), \]

\[ \Delta_+ = v \sin \theta \tan \phi_0 (\cos (\varphi + \phi_0) + 2 \sin \varphi \csc \phi_0) \]

(29)

(ii) for \( 1 + \sin \theta \sin \varphi < \frac{2\pi}{\omega v_g} \), the allowed region for \( \phi < -\pi < \phi < 0 \).

b. **into the backward direction (negative projection on the electric field)** \( -\pi < \varphi < 0 \):

The conditions imposed by the step functions now become \( 0 < -p_y + k_y \) and \( 0 < \frac{eE}{\hbar} t + p_y \). In terms of the polar coordinates in the momentum space

(i) for \( -2v \sin \theta \sin \varphi < \frac{2\pi}{\omega v_g} \sin \varphi < 1 - v \sin \theta \sin \varphi \), the allowed regions are \( -\left( \phi_0 - \Delta_- \right) < \phi < 2v \sin \theta \sin \varphi \) or \( -\pi - 2v \sin \theta \sin \varphi < \phi < -\pi + \phi_0 - \Delta_- \), where

\[ \Delta_- = v \sin \theta \tan \phi_0 \cos (\varphi + \phi_0). \]

(30)

(ii) for \( 1 - v \sin \theta \sin \varphi < \frac{2\pi}{\omega v_g} \), the allowed region for \( \phi \) is \( -\pi - 2v \sin \theta \sin \varphi < \phi < -2v \sin \theta \sin \varphi \).

Since \( v \equiv v_g/c \approx 1/300 < 1 \), one can neglect higher order correction in \( v \). The conditions for case (i) simplify into \( 0 < \frac{2\pi}{\omega v_g} < 1 \) and \( -\phi_0 < \phi < 0 \) or \( -\pi < \phi < \)
Similarly, the conditions for case (ii) simplify:
\[ 0 \quad \text{and} \quad -g \]
gives rise to the following S-matrix element above formula one can first integrate over \( \omega \) as the spectral intensity emittance Eq.(13) and Eq.(14).

\[ M^{(1)} \left( \theta, \varphi, \frac{\omega}{c}, t \right) = \frac{e^2 \omega}{8\pi^4} \int_{-\infty}^{0} d\phi \left( \cos^2 \phi \cos^2 \varphi + \sin^2 \phi \sin^2 \varphi \right) \exp \left[ -\frac{\pi}{2} \omega^2 t_{nl}^2 \cos^2 \varphi \right] \]
\[ M^{(2)} \left( \theta, \varphi, \frac{\omega}{c}, t \right) = \frac{e^2 \cos^2 \theta}{8\pi^4} \int_{0}^{\infty} d\phi \left( \cos^2 \phi \sin^2 \varphi + \sin^2 \phi \cos^2 \varphi \right) \exp \left[ -\frac{\pi}{2} \omega^2 t_{nl}^2 \cos^2 \varphi \right] \]

Similarly, the conditions for case (ii) simplify: \( \frac{2e}{\hbar c} > 1 \) and \( -\pi < \phi < 0 \). Thus

\[ M^{(1)} \left( \theta, \varphi, \frac{\omega}{c}, t \right) = \frac{e^2}{16\pi^2} \omega \int_{0}^{\pi} d\phi \cos^2 (\phi - \varphi) \exp \left[ -\frac{\pi}{2} t_{nl}^2 \omega^2 \cos^2 \phi \right] \]
\[ = \frac{e^2}{32\pi^3} \omega e^{-\pi t_{nl}^2 \omega^2/4} \left[ I_0 \left( \frac{\pi t_{nl}^2 \omega^2}{4} \right) - \cos 2\varphi I_1 \left( \frac{\pi t_{nl}^2 \omega^2}{4} \right) \right] \]
\[ M^{(2)} \left( \theta, \varphi, \frac{\omega}{c}, t \right) = \frac{e^2}{16\pi^4} \omega \cos^2 \theta \int_{-\infty}^{0} d\phi \sin^2 (\phi - \varphi) \exp \left[ -\frac{\pi}{2} t_{nl}^2 \omega^2 \cos^2 \phi \right] \]
\[ = \frac{e^2}{32\pi^3} \omega e^{-\pi t_{nl}^2 \omega^2/4} \cos^2 \theta \left[ I_0 \left( \frac{\pi t_{nl}^2 \omega^2}{4} \right) + \cos 2\varphi I_1 \left( \frac{\pi t_{nl}^2 \omega^2}{4} \right) \right] \]

These expressions lead to the final results for the spectral emittance Eq.(13) and Eq.(14).

Next we consider the angular and polarization dependence of the radiated power per unit area defined as the spectral intensity \( \mathcal{L}(\lambda) (\theta, \varphi, t) \), Eq.(13). In the above formulae one can first integrate over \( \omega \) in the range \( 0 < \omega < \frac{2e}{\hbar c} \csc \phi \), and then integrate over \( \phi \) in the region \( [0, \frac{\pi}{2}] \). This leads to Eqs.(14) and (17) for the luminosity integrated over frequencies.

**APPENDIX C. PHASE SPACE OF THE TWO-PHOTON PROCESS**

As in QED [29], the two-photon diagram, Fig. 1b, gives rise to the following S-matrix element

\[ F_2^{(\lambda, \lambda')}(p, p', k, k') = \frac{ie^2}{4\sqrt{\omega \epsilon e^2}} \mathcal{F}_2^{(\lambda, \lambda')}(2\pi)^4 \delta \left( v_g (p + p') - c(k + k') \right), \]

where

\[ \mathcal{F}_2^{(\lambda, \lambda')} = v^l \left( -p' \right) \Xi^{(\lambda, \lambda')} u(p) \]
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