ALGEBRAIC PROPERTIES OF THE 1 + 1 DIMENSIONAL HEISENBERG SPIN FIELD MODEL

E. Alfinito, M. Leo, R. A. Leo, M. Palese and G. Soliani

Dipartimento di Fisica dell’Università and Sezione INFN, 73100 Lecce, Italy

Abstract. The Estabrook-Wahlquist prolongation method is applied to the (compact and noncompact) continuous isotropic Heisenberg model in $1 + 1$ dimensions. Using a special realization (an algebra of the Kac-Moody type) of the arising incomplete prolongation Lie algebra, a whole family of nonlinear field equations containing the original Heisenberg system is generated.
1. In the study of nonlinear field equations (NLF), the Estabrook-Wahlquist (EW) prolongation method [1] constitutes a systematic analytical procedure which enables one, in principle, to associate a linear problem with the equation under consideration. Within the EW method, one has that nonlinear prolongation algebras are related to integrable NLF equations which can be expressed by means of closed differential ideals. Such algebras arise via the introduction of an arbitrary number of prolongation forms containing new dependent variables (called pseudopotentials), and by requiring the algebraic equivalence between the generators of the prolonged ideals and their exterior differentials.

It turns out that the integrability property of NLF equations is closely connected with the existence of incomplete prolongation Lie algebras (in the sense that not all of the commutators are known). We recall that so far the EW method has been applied mostly to look for the prolongation algebra of a given integrable NLF equation ("direct" prolongation method) [2]. Conversely, a minor attention has been payed to the "inverse" procedure. This consists in starting from a certain incomplete Lie algebra to obtain the class of NLF equations whose prolongation structure it is. The inverse (prolongation) problem has been handled by a few authors who have investigated some particular cases in 1+1 dimensions [3,4,5]. We point out that the prolongation studies suggest the existence of a one-to-one correspondence (eventually up to simple transformations) between a given NLF equation and the associated incomplete Lie algebra. However, in order to achieve a better understanding of this correspondence, we need to deal with new case studies. Following this idea, in this Letter we show that the above equivalence holds for the (1 + 1)-dimensional continuous isotropic Heisenberg model in both the compact and the noncompact version.

On the other hand, we prove that using a special realization of the incomplete prolongation Lie algebra of the model (i.e., an infinite dimensional Lie algebra of the Kac-Moody type with a loop structure), we obtain a whole family of NLF equations containing the original Heisenberg system.

2. Let us consider the continuous isotropic Heisenberg model in 1 + 1 dimensions

\[(\Sigma \vec{S})_t = \vec{S} \times \vec{S}_{xx}, \tag{2.1}\]

where \(\vec{S} = \vec{S}(x,t)\) is a classical spin field vector, \(\Sigma\) is a 3×3 diagonal matrix

diag \((1,1,\kappa^2)\), \(\kappa^2 = \pm 1\), subscripts denote partial derivatives, and the symbol \(\times\) stands for the usual vector product. The spin field components \(S_j\) are assumed to obey the constraint

\[(\Sigma \vec{S}) \cdot \vec{S} = \kappa^2, \tag{2.2}\]

where \(\kappa^2 = \pm 1\) refers to the compact and the noncompact case, respectively. In other words, for \(\kappa^2 = 1\) the quantities \(S_j\) belong to the unitary sphere \(SU(2)/U(1)\), while for \(\kappa^2 = -1\), the \(S_j\)'s range over a shield of the two-fold hyperboloid \(SU(1,1)/U(1)\).

In order to apply the Estabrook-Wahlquist prolongation method, let us introduce the differential ideal defined by the two vector 2-forms

\[\bar{\alpha}_1 = d\vec{S} \wedge dt - \vec{S}_x dx \wedge dt, \tag{2.3a}\]

\[\bar{\alpha}_2 = d(\Sigma \vec{S}) \wedge dx + \vec{S} \times d\vec{S}_x \wedge dt, \tag{2.3b}\]
and by the scalar 2-form

$$
\beta = d(\Sigma \vec{S}) \cdot \vec{S}_x \wedge dt + (\Sigma \vec{S}_x) \cdot d\vec{S} \wedge dt,
$$

(2.3c)

where $\wedge$ means the wedge product. One can verify directly that the ideal (2.3) is closed and equivalent to Eq. (2.1) together with the condition (2.2).

At this stage, let us consider the prolongation 1-forms

$$
\omega^k = - dy^k + F^k(\vec{S}, \vec{S}_x; y) dx + G^k(\vec{S}, \vec{S}_x; y) dt,
$$

(2.4)

where $y = \{ y^m \}$, $k, m = 1, 2, ..., N$ (N arbitrary), and $F^k, G^k$ are, respectively, the pseudopotential and functions to be determined.

Now, by requiring that $d\omega^k \in I(\vec{\alpha}^i, \beta, \omega^k)$, $I$ being the ideal generated by $\vec{\alpha}^i, \beta$ and $\omega^k$, we obtain the relations

$$
F^k_{\vec{S}_x} = 0,
$$

(2.5a)

$$
G^k_{\vec{S}_x} - (\Sigma F^k_j) \times \vec{S} = 0,
$$

(2.5b)

$$
G^k_{\vec{S}} \cdot \vec{S}_x - [F, G]^k = 0,
$$

(2.5c)

where

$$
F^k_{\vec{S}} = \text{grad}_{\vec{S}} F^k \equiv (F^k_{S_1}, F^k_{S_2}, F^k_{S_3}),
$$

$$
[F, G]^k = F^j G^k_j - G^j F^k_j,
$$

and $F^k = \partial F^k/\partial S_1$, and so on. For simplicity, in the following we shall drop the index $k$.

From (2.5) we get

$$
F = \vec{X}(y) \cdot \vec{S} + Y(y),
$$

(2.6a)

$$
G = (\Sigma \vec{X} \times \vec{S}) \cdot \vec{S}_x - S_1 [X_2, X_3] + S_2 [X_1, X_3] - \kappa^2 S_3 [X_1, X_2] + Z(y),
$$

(2.6b)

and the commutation relations:

$$
[X_1, X_2] = X_4, [X_1, X_3] = X_5, [X_2, X_3] = X_6, [X_3, X_4] = - \kappa^2 [X_3, X_5],
$$

$$
[X_1, X_3] = - \kappa^2 [X_2, X_3], [X_1, X_6] = [X_3, X_4] = - [X_2, X_3],
$$

$$
[Y, Z] = - \kappa^2 [X_2, X_3], Y = [X_1, X_6] = [X_3, X_4] = - [X_2, X_3],
$$

3
\[ [Y, X_1] = [Y, X_2] = [Y, X_3] = [Z, X_1] = [Z, X_2] = [Z, X_3] = 0. \] (2.7)

Here \( \vec{X} \equiv (X_1, X_2, X_3) \), where \( X_j, Y \) and \( Z \) are arbitrary functions depending on the pseudopotential \( y \) only.

We notice that one can determine a homomorphism between the algebra defined by (2.7) and the \( sl(2, c) \) algebra

\[ [X_1, X_2] = 2i\lambda \kappa^2 X_3, [X_1, X_3] = -2i\lambda X_2, [X_2, X_3] = 2i\lambda X_1, \] (2.8)

where \( \lambda \) is a free parameter. This can be seen assuming that \( X_1, X_2, \) and \( X_3 \) are independent and \( X_4, X_5, X_6, Y \) and \( Z \) are a linear combination of the preceding operators. In such a way one finds \( X_4 = 2i\lambda \kappa^2 X_3, \quad X_5 = -2i\lambda X_2, \quad X_6 = 2i\lambda X_1, \) and \( Y = Z = 0. \)

Exploiting (2.7) and (2.8), Eqs.(2.6) yield the spectral problem associated with the model (2.1), both in the compact \( (\kappa^2 = 1) \) and in the noncompact \( (\kappa^2 = -1) \) case.

Another possible realization of (2.7) is given by an infinite dimensional Lie algebra of the Kac-Moody type. In fact, let us suppose that \( Y = Z = 0 \) and, in opposition to what happens for the previous case, \( X_4, X_5 \) and \( X_6 \) are independent from \( X_1, X_2 \) and \( X_3 \). Therefore, now the commutators \( [X_2, X_6], [X_3, X_5], [X_3, X_6], [X_4, X_5], [X_4, X_6] \) and \( [X_5, X_6] \) are unknown. Hence, a realization of the incomplete Lie algebra (2.7) is

\[ X_1 = \kappa T_1^{(1)}, X_2 = \kappa T_2^{(1)}, X_3 = T_3^{(1)}, \]
\[ X_4 = i\kappa^2 T_3^{(2)}, X_5 = -i\kappa T_2^{(2)}, X_6 = i\kappa T_1^{(2)}, \] (2.9)

where the vector fields \( T_i^{(m)} \) \( (i = 1, 2, 3; m \in Z) \) obey the commutation relations

\[ [T_i^{(m)}, T_j^{(n)}] = i\epsilon_{ijk} T_k^{(m+n)}, \] (2.10)

\( \epsilon_{ijk} \) being the Ricci tensor. A representation of (2.10) in terms of the prolongation variables
is

\[
T_1^{(m)} = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \left[ y_2^{(m+n)} \partial / \partial y_1^{(n)} + y_1^{(m+n)} \partial / \partial y_2^{(n)} \right],
\]

(2.11a)

\[
T_2^{(m)} = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \left[ y_2^{(m+n)} \partial / \partial y_1^{(n)} - y_1^{(m+n)} \partial / \partial y_2^{(n)} \right],
\]

(2.11b)

\[
T_3^{(m)} = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \left[ y_1^{(m+n)} \partial / \partial y_1^{(n)} - y_2^{(m+n)} \partial / \partial y_2^{(n)} \right],
\]

(2.11c)

where the pseudopotential \( y \) is expressed in terms of the infinite-dimensional vectors \( y_1 \) and \( y_2 \) [\( y_1^{(n)} \) and \( y_2^{(n)} \) denote the \( n \)-th components of \( y_1 \) and \( y_2 \), respectively].

Now we point out that, keeping in mind (2.9), (2.11) and the condition \( Y = Z = 0 \), from (2.6) we obtain the spectral problem for Eq. (2.1). We have

\[
\begin{pmatrix}
  y_1^{(i)} \\
  y_2^{(i)}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
  S_3 & \kappa S_+ \\
  \kappa S_+^* & -S_3
\end{pmatrix} \begin{pmatrix}
  y_1^{(1+i)} \\
  y_2^{(1+i)}
\end{pmatrix},
\]

(2.12a)

where \( S_+ = S_1 + iS_2 \).

Furthermore, by introducing the 2x2 matrix \( M \) whose elements are

\[
M_{11} = \kappa^2 (S_1 S_{2x} - S_2 S_{1x}),
\]

\[
M_{12} = i\kappa (S_3 S_{1x} - S_1 S_{3x}) + \kappa (S_2 S_{3x} - S_3 S_{2x}),
\]

\[
M_{21} = -i\kappa (S_3 S_{1x} - S_1 S_{3x}) + \kappa (S_2 S_{3x} - S_3 S_{2x}),
\]

\[
M_{22} = -\kappa^2 (S_1 S_{2x} - S_2 S_{1x}),
\]

we find

\[
\begin{pmatrix}
  y_1^{(i)} \\
  y_2^{(i)}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
  S_3 & \kappa S_+ \\
  \kappa S_+^* & -S_3
\end{pmatrix} \begin{pmatrix}
  y_1^{(1+i)} \\
  y_2^{(1+i)}
\end{pmatrix}.
\]

(2.12b)

On the other hand, by resorting to the formal expansion

\[
\psi(\epsilon) = \sum_{n=-\infty}^{+\infty} \epsilon^n y^{(n)},
\]

(2.13)

where \( \epsilon \) is a constant, \( \psi(\epsilon) = \begin{pmatrix} \psi_1(\epsilon) \\ \psi_2(\epsilon) \end{pmatrix} \) and \( y^{(n)} = \begin{pmatrix} y_1^{(n)} \\ y_2^{(n)} \end{pmatrix} \), Eqs. (2.12a) and (2.12b) can be written as

\[
\psi_1(\epsilon) = \frac{1}{2\kappa} \begin{pmatrix}
  S_3 & \kappa S_+ \\
  \kappa S_+^* & -S_3
\end{pmatrix} \psi(\epsilon),
\]

(2.14a)

\[
\psi_2(\epsilon) = \frac{1}{2\kappa} \begin{pmatrix}
  S_3 & \kappa S_+ \\
  \kappa S_+^* & -S_3
\end{pmatrix} \psi(\epsilon) - \frac{i}{2\kappa^2} M \psi(\epsilon).
\]

(2.14b)

These equations reproduce just the spectral problem (with spectral parameter \( \lambda = \frac{1}{\kappa} \)) related to Eq. (2.1).

3. The prolongation structure of NLF equation can be interpreted as a Cartan-Ehresman connection, so that an incomplete Lie algebra of vector fields can be associated with a differential ideal. Conversely, starting from a prolongation algebra, one can determine the differential ideal related to a certain NLF equations specifying the form of the connection.
This inverse prolongation method will be applied below to the incomplete Lie algebra $X_i (i = 1, 2, \ldots, 6)$ (see (2.7)) of the Heisenberg model (2.1). In doing so, let us suppose that the connection
\[
\omega^k = -dy^k + X^k_j \vartheta^j
\]  
exists, such that
\[
d\omega^k = X^k_j d\vartheta^j - \frac{1}{2} [X_i, X_j]^k \vartheta^i \wedge \vartheta^j \quad (mod \omega^k),
\]  
where $j = 1, 2, \ldots, 6$, $k = 1, 2, \ldots, N$, and $\vartheta^j$ are 1-forms. [mod $\omega^k$ means that all the exterior products between $\omega^k$ and 1-forms of the Grassmann algebra have not to be considered].

With the help of (2.7), Eq. (3.2) provides the relations
\[
\begin{align*}
\vartheta^1 &= \vartheta^2 = \vartheta^3 = 0, \\
\vartheta^4 - \vartheta^1 \wedge \vartheta^2 &= 0, \\
\vartheta^5 - \vartheta^1 \wedge \vartheta^3 &= 0, \\
\vartheta^6 - \vartheta^1 \wedge \vartheta^2 - \vartheta^2 \wedge \vartheta^3 &= 0,
\end{align*}
\]  
which yield
\[
\vartheta^1 = u_1 \eta, \quad \vartheta^5 = u_2 \eta, \quad \vartheta^6 = u_3 \eta,
\]  
where $\eta$ is an exact 1-form such that $\eta = dt$ and $u_j (j = 1, 2, 3)$ are 0-forms satisfying the condition
\[
\kappa^2 u_1^2 + u_2^2 + u_3^2 = \kappa^2.
\]  
From (3.3) we have also
\[
\vartheta^1 = u_3 \alpha + \Lambda_1 dt, \quad \vartheta^2 = -u_1 \alpha + \Lambda_2 dt,
\]
\[
\vartheta^3 = \kappa^2 u_1 \alpha + \Lambda_3 dt,
\]  
where $\alpha$ and $\Lambda_j (j = 1, 2, 3)$ are a 1-form and 0-forms, respectively.

After some manipulations, from (3.3) we obtain

\[
d[\kappa^2 \alpha + (u_3 \Lambda_1 - u_2 \Lambda_2 + u_1 \Lambda_3) dt] = 0,
\]  
from which
\[
\alpha = \kappa^2 [dx - (u_3 \Lambda_1 - u_2 \Lambda_2 + u_1 \Lambda_3)] dt.
\]  
At this stage, by virtue of (3.6), (3.8) and the conditions $d\vartheta^j = 0 (j = 1, 2, 3)$ (see (3.3)), we arrive at the system of NLF equations
\[
u_{1t} = u_3 \nu_{2x} - u_2 \nu_{3x},
\]  
(3.9a)
Here we shall tackle the inverse (prolongation) problem, which consists in finding the class of NLF equations whose prolongation structure is assumed to be given by the 1-forms

\[ T_i \]  

and

\[ T_j \]

(2.10) is furnished, in correspondence of any set of independent functions \( \Psi \). The compatibility condition \( \Psi \) vanish implies

\[ \omega = -dy + \sum_{i=1}^{3} S_i X_i dx + \sum_{i=1}^{6} \Psi_i (S_j, S_{jx}) X_i dt, \quad (4.1) \]

where \( j = 1, 2, 3 \),

\[ X_i = t_i T_i^{(n)} \quad (i = 1, 2, 3; \quad n_i \in \mathbb{Z}), \quad (4.2a) \]

and \( t_i, t_i' \) are arbitrary constants. The functions \( \Psi_i \) \( (i = 1, 2, \ldots, 6) \) have to be determined in such a way that the operators \( T_i \) are the generators of the Kac - Moody algebra (2.10). Of course, once a representation of the algebra (2.10) is furnished, in correspondence of any set of independent functions \( \Psi_i \), an evolution equation of the form \( S_{it} = F (S_j, S_{jx}, S_{jxx}) \quad (j = 1, 2, 3) \) exists which can be obtained simply by equating the 1-forms (4.1) to zero. To be precise, let us consider as a representation of the algebra (2.10) the expressions (2.11). Then, the requirement that the 1-forms (4.1) vanish implies

\[ \begin{align*}
\chi_x^{(i)} & = \frac{1}{2} t_1 S_1 \sigma_1 \chi^{(n_1+i)} + \frac{1}{2} t_2 S_2 \sigma_2 \chi^{(k_1+i)} + \frac{1}{2} t_3 S_3 \sigma_3 \chi^{(m_1+i)}, \\
\chi_t^{(i)} & = \frac{1}{2} t_1' \chi^{(n_1+i)} + \frac{1}{2} t_2' \chi^{(k_1+i)} + \frac{1}{2} t_3' \chi^{(m_1+i)} + \\
&\quad \frac{1}{2} t_4' \chi^{(m_2+i)} + \frac{1}{2} t_5' \chi^{(k_2+i)} + \frac{1}{2} t_6' \chi^{(n_2+i)},
\end{align*} \]

where \( \sigma_1, \sigma_2, \sigma_3 \) are the Pauli matrices acting on the vectors \( \chi^{(i)} = \left( \chi_x^{(i)} \chi_t^{(i)} \right) \).

The compatibility condition \( \chi_x^{(i)} = \chi_t^{(i)} \) for the system (4.3) yields

\[ \begin{align*}
t_1 (S_{11} - \psi_1 S_{jx} S_{jxx}) \chi^{(n_1+i)} & - t_1' (\psi_6 S_{jx} + \psi_6 S_{jxx}) \chi^{(n_2+i)} \\
+ it_2 t_3 (S_2 \psi_3 - S_3 \psi_2) \chi^{(k_1+m_1+i)} & + it_2' t_3' S_2 \psi_2 \chi^{(m_2+k_1+i)} - it_3 t_2' S_3 \psi_3 \chi^{(m_1+k_2+i)} = 0,
\end{align*} \]

(4.4a)
where the property of linear independence of the $\sigma_j$’s has been used.

Since the quantities $\chi^{(i)}(i \in Z)$ are linearly independent, assuming that $n_1 = k_1 = m_1 \equiv n$, and $n_2 = k_2 = m_2 \equiv 2n$ ($n \neq 0$), Eqs.(4.4) provide

\[ t_1(S_{1t} - \psi_{1S_x}S_{jx} - \psi_{1S_x}S_{jxx}) = 0, \]

\[ t_1'(\psi_{6S_x}S_{jx} + \psi_{6S_x}S_{jxx}) = it_2t_3(S_{2}\psi_3 - S_3\psi_2), \quad (4.5a) \]

\[ t_2t_3'S_2\psi_4 = t_3't_2'S_3\psi_5, \]

\[ t_2(S_{2t} - \psi_{2S_x}S_{jx} - \psi_{2S_x}S_{jxx}) = 0, \]

\[ t_2'(\psi_{5S_x}S_{jx} + \psi_{5S_x}S_{jxx}) = -it_1t_3(S_1\psi_3 - S_3\psi_1), \quad (4.5b) \]

\[ t_1't_3'S_1\psi_4 = t_3't_1'S_3\psi_5, \]

\[ t_3(S_{3t} - \psi_{3S_x}S_{jx} - \psi_{3S_x}S_{jxx}) = 0, \]

\[ t_3'(\psi_{4S_x}S_{jx} + \psi_{4S_x}S_{jxx}) = it_1t_2(S_1\psi_2 - S_2\psi_1), \quad (4.5c) \]

\[ t_1t_2'S_1\psi_5 = t_2't_1'S_2\psi_6. \]

At this point it is convenient to adopt a vector notation. Precisely, let us introduce the vectors

\[ \vec{A} \equiv (t_1S_1, t_2S_2, t_3S_3), \]

\[ \vec{\Phi} \equiv (t_1\psi_1, t_2\psi_2, t_3\psi_3), \]

and
\[ \vec{x} \equiv (t'_{1}\psi_{6}, t'_{2}\psi_{5}, t'_{3}\psi_{4}). \]

Hence, Eqs. (4.5) can be cast into the forms

\[ \vec{\Phi} = (\vec{A} \cdot \vec{B})(\vec{A}_x \times \vec{A}) + \gamma \vec{A}, \quad (4.6) \]

\[ \vec{\chi} = -i(\vec{A} \cdot \vec{B})\vec{A}, \quad (4.7) \]

\[ \vec{A}_t = \vec{\Phi}_x = (\vec{A} \cdot \vec{B})_x(\vec{A}_x \times \vec{A}) + (\vec{A} \cdot \vec{B})(\vec{A}_{xx} \times \vec{A}) + \gamma \vec{A}_x, \quad (4.8) \]

where \( \vec{A}^2 = 1, \gamma \) is an arbitrary constant and \( \vec{B} \) is an arbitrary vector functionally dependent on \( \vec{A} \) and \( \vec{A}_x \) only. By setting \( t' = t \) and \( x' = x + \gamma t \), Eq. (4.8) takes the form of a conservation law, namely

\[ \vec{A}_{t'} = \partial_{x'}[(\vec{A} \cdot \vec{B})(\vec{A}_{x'} \times \vec{A})], \quad (4.9) \]

We remark that for \( \vec{B} = -\vec{A} \), Eq. (4.9) reproduces just the Heisenberg system (2.1).

5. The calculations carried out in this Letter shows the great versatility of the Estabrook - Wahlquist prolongation method in the study of integrable nonlinear field equations. The method can be applied with benefit both in the "direct" and "inverse" direction. The investigation made here by dealing with the 1+1 dimensional continuous isotropic Heisenberg model confirms that a close connection exists between the incomplete prolongation Lie algebra (2.7) associated with the model and its integrability property. This feature is common to other well-known integrable NLF equations [2].

We notice that an infinite dimensional Lie algebra of the Kac-Moody type is found as a realization of the prolongation algebra (2.7). Moreover, the incomplete Lie algebra enables one to obtain the linear spectral problem related to the system under consideration. This goal is achieved exploiting a (finite) quotient algebra of the original incomplete Lie algebra (2.7).

On the other hand, within the inverse procedure the incomplete Lie algebra (2.7) is used to generate the field equations whose prolongation structure it is. In doing so, we arrive at the family of spin field models (4.9), which reproduces the Heisenberg system (2.1) for a special choice of the vector \( \vec{B} \). In general, we have that the correspondence between incomplete Lie algebras and integrable NLF equations is not unique, because the resulting equations depend on what we take as independent variables.

It should be interesting to extend the prolongation technique to the case of higher dimension NLF equations. In this context, although so far some attempts have been done within the direct framework [6,7,8], at the best of our knowledge the inverse method has been never explored.
References

1. Estabrook, F. B. and Wahlquist, H. D., Prolongation structures of nonlinear evolution equations. II J. Math. Phys. 17, 1293 (1976).

2. See, for example, the references quoted in Rogers, C. and Shadwick, W. F., Bäcklund Transformations and Their Applications, Academic Press, New York, 1982.

3. Estabrook, F. B., Moving frames and prolongation algebras, J.Math. Phys. 23, 2071 (1982).

4. Hoenselaers, C., More Prolongation Structures, Prog. Theor. Phys. 75, 1014 (1986).

5. Leo, R. A. and Soliani, G., Incomplete algebras generating integrable nonlinear field equations, Phys. Lett. B 222, 415 (1989).

6. Morris, H. C., Prolongation structures and nonlinear evolution equations in two spatial dimensions, J. Math. Phys. 17, 1870 (1976).

7. Morris, H. C., Inverse scattering problem in higher dimensions: Yang-Mills fields and the supersymmetric sine-Gordon equation, J. Math. Phys. 21, 327 (1980).

8. Tondo, G. S., The eigenvalue problem for the three-wave resonant interaction in (2+1) dimensions via the prolongation structure, Lett. Nuovo Cimento 44, 297 (1985).