Topological contributions in two-dimensional Yang-Mills theory: from group averages to integration over algebras

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Abstract

We show that keeping only the topologically trivial contribution to the average of a class function on $U(N)$ amounts to integrating over its algebra. The goal is reached first by decompactifying an expansion over the instanton basis and then directly, by means of a geometrical procedure.

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I. INTRODUCTION

Yang-Mills theory in two dimensions without dynamical fermions \((YM_2)\) shares important features with topological theories. When considered on the plane and quantized in the light-cone gauge (LCG) it looks indeed trivial, were it not for the very singular nature of correlators at large distances. When infrared singularities are regularized via compactification, namely by putting the theory on a (partially or totally) compact euclidean manifold, dynamics gets hidden in peculiar topological properties.

These features emerge quite naturally if we consider the vacuum average of a regular non self-intersecting Wilson loop. Thanks to the invariance of \(YM_2\) under area-preserving diffeomorphisms, this average turns out to be insensitive to the shape of the contour. Actually one can prove that it can be obtained via invariant integration over the group manifold (in the following we shall limit ourselves to \(U(N)\), the generalization to \(SU(N)\) being straightforward [1]) of a class function (the heat kernel), the outcome exhibiting a pure area dependence [2]. By class function on \(U(N)\) we mean an expression which enjoys the property \(P(U) = P(hUh^\dagger), U \in U(N), \forall h \) in the fundamental representation of \(U(N)\). This implies that \(P\) is a symmetric function of the eigenvalues of \(U\).

The goal of deriving Wilson loops by means of integration over the group can be reached either through a geometrical construction, generalizing techniques which are drawn from the lattice [3–5] to the continuum, or by resumming a perturbative series when the theory is quantized on the light front [6]. Performing a modular inversion [7–10], one can eventually interpret the result as an infinite sum over non-analytic contributions (instantons), tightly related to the group curvature.

If the theory is again quantized in LCG, but at equal times, and the same Wilson loop is computed by resumming the corresponding perturbative series, a quite different behaviour ensues [11]. In particular, in the large-\(N\) limit, confinement is lost. On the other hand, in Ref. [12], it has been shown that perturbation theory, in the equal-time formulation, can only account for the zero-instanton sector (a truly perturbative result).
A further step has been performed in Ref. [13], where the zero-instanton contribution to the average of the class function describing a Wilson loop in two different representations, is obtained by trading integration over the group manifold with integration over its algebra.

The purpose of this note is to elucidate further the geometrical meaning of this procedure as well as to extend it to the vacuum average of a more general set of class-invariant functions.

II. FROM THE GROUP MANIFOLD TO THE GROUP ALGEBRA

As is well known, the partition functions on compact surfaces of genus zero, have a twofold expression, either as an expansion in the characters of the irreducible representations of the gauge group [4], or as a series of terms behaving exponentially with respect to the inverse coupling constant squared $1/g^2$ [1,4,14,8,9,15].

For example, the partition function of $YM_2$ on a cylinder of area $A$ with fixed holonomies at the boundaries $g_1, g_2$ reads

$$K(A; g_2, g_1) = \sum_R \chi^\dagger_R(g_2) \chi_R(g_1) \exp \left[ -\frac{g^2 A}{4} C_2(R) \right]$$

and

$$K(A; g_2, g_1) = N \exp \left[ \frac{g^2 A}{48} N(N^2 - 1) \right] \left( \frac{g^2 A}{2} \right)^{-\frac{N}{2}} \prod_{\{l_i\} = -\infty}^{+\infty} (-1)^{(N-1)\sum l_i}$$

$$\times \sum_{\sigma \in \Pi(1,\ldots,N)} \epsilon(\sigma) \exp \left[ -\frac{1}{g^2 A} \sum_{i=1}^{N} (\theta_i - \phi_{\sigma(i)} - 2\pi l_i)^2 \right].$$

$C_2(R)$ is the quadratic Casimir invariant of the irreducible representation $R$, $\theta_i$ and $\phi_i$ are the invariant angles corresponding to $g_1$ and $g_2$, respectively, $N$ is a normalization factor and $J(\theta) \equiv J(\theta_1, \ldots, \theta_N) = \prod_{i<j} 2 \sin \left( \frac{\theta_i - \theta_j}{2} \right)$. In the equation above, $\sigma \in \Pi$ denotes a permutation and $\epsilon(\sigma)$ its signature. Both the expressions (1), (2) are solutions of the heat-kernel equation

$$\frac{4}{g^2} \frac{\partial}{\partial A} K(A; g_2, g_1) = \Delta_\theta K(A; g_2, g_1),$$

(3)
\( \Delta_\theta \) being the laplacian over the group manifold, with the boundary condition (which fixes the normalization constant \( N \))

\[
\lim_{g^2 A \to 0} K(A; g_2, g_1) = \delta(g_2 - g_1),
\]

\( \delta(g_2 - g_1) \) being the class-invariant \( \delta \)-distribution.

Equations (1), (2) are linked by what is called a modular inversion [7–10]. Actually they represent two different, unitarily equivalent, harmonic analyses of the class function \( K \). In turn the kernel \( K \) is the basic quantity for writing the partition function on the sphere of area \( A \)

\[
Z(A) = K(A; 1, 1)
\]
as well as, more generally, the expectation value of a non self-intersecting Wilson loop in the \( R \) representation, enclosing an area \( A_1 \) and winding \( n \) times [4,5,16]

\[
Z(A) W_n(A_1, A) = \int DU K(A_1; \mathbf{1}, U) K(A - A_1; U, \mathbf{1}) \text{Tr} U^n_R.
\]

Since our goal will be to single out the zero-instanton contribution to any class function in \( L^2(U) \), as a warm-up we begin by considering the simple case of the Wilson loop in Eq. (3). It is immediately realized that Eq. (2) provides the natural starting point, being a series of exponentials of \( \frac{1}{g^2} \). However the limit \( g_2 \to 1 \) is to be performed carefully; the result is

\[
K(A; \mathbf{1}, g_1) = \tilde{N}(g^2 A) \exp \left[ \frac{g^2 A}{48} N(N^2 - 1) \right] \times \sum_{\{ l_i \} = -\infty}^{+\infty} \sum_{\{ m_i \} = -\infty} \frac{\sum_{i<j} [\theta_i - \theta_j + 2\pi (l_i - l_j)]}{J(\theta + 2\pi l)} \exp \left[ -\frac{1}{g^2 A} \sum_{i=1}^N (\theta_i + 2\pi l_i)^2 \right].
\]

Hereafter Eq. (3), in the case of the fundamental representation, becomes

\[
W_n(A_1, A) Z(A) = \sum_{\{ m \}} \sum_{\{ l \}} \int_{0}^{2\pi} d\theta_1 \cdots d\theta_N |\Delta(e^{i\theta})|^2 \frac{\Delta(\theta + 2\pi I)}{J(\theta + 2\pi I)} \frac{\Delta(\theta + 2\pi m)}{J(\theta + 2\pi m)}
\]

\[
\times \exp \left[ -\frac{1}{g^2 (A - A_1)} \sum_{i=1}^N (\theta_i + 2\pi l_i)^2 - \frac{1}{g^2 A_1} \sum_{i=1}^N (\theta_i + 2\pi m_i)^2 \right] \times \exp \left[ \sum_{k=1}^N \frac{1}{g} \sum_{i=1}^N (\theta_k + 2\pi n_i)^2 \right].
\]
where $\Delta(a) = \prod_{i<j}(a_i - a_j)$ is the Vandermonde determinant and $|\Delta(e^{i\theta})|^2 d\theta_1 \ldots d\theta_N$ is the group invariant measure. Notice that, in Eq. (7), the product $J(\theta + 2\pi m)J(\theta + 2\pi l)$ simplifies with $|\Delta(e^{i\theta})|^2$ leaving the phase factor $\exp[i\pi(N-1)\sum_i(l_i - m_i)]$, and that an area dependent normalization factor in $\mathcal{W}_n$ was dropped since it can be absorbed in $\mathcal{Z}$ by suitably rescaling the integration variables. In the sequel it will be understood that the expression of $\mathcal{Z}(A)$ is recovered by performing the average of the identity. It is now easy to get rid of the sums over $l_i$ by enlarging the range of integration over $\theta_i$; exploiting the symmetry over the angles, one has

$$
\mathcal{W}_n \mathcal{Z} = \sum_{\{m\}} e^{i\pi(N-1)\sum_i m_i} \int_{-\infty}^{+\infty} d\theta_1 \ldots d\theta_N \Delta(\theta) \Delta(\theta + 2\pi m) \times \exp \left[ -\frac{1}{2g^2(A - A_1)} \sum_{i=1}^{N} \theta_i^2 - \frac{1}{2g^2A_1} \sum_{i=1}^{N} (\theta_i + 2\pi m_i)^2 \right] e^{i\theta_1 n}.
$$

(8)

After performing suitable shifts and rescalings over the integration variables, we obtain

$$
\mathcal{W}_n \mathcal{Z} = \exp \left[ -\frac{n^2g^2A_1A_2}{4A} \right] \sum_{\{m\}} \exp \left[ i\pi(N-1)\sum_i m_i \right] \times \exp \left[ -\frac{4\pi^2}{g^2A} \sum_{i=1}^{N} m_i^2 - 2\pi i \frac{A_2}{A} n m_1 \right] \int_{-\infty}^{+\infty} d\theta_1 \ldots d\theta_N \exp \left[ -\frac{1}{g^2A} \sum_{i=1}^{N} \theta_i^2 \right] \times \Delta \left( \theta_1 - 2\pi \sqrt{\frac{A_2}{A_1}} m_1 + \frac{img^2}{2} \sqrt{A_1A_2}, \theta_2 - 2\pi \sqrt{\frac{A_2}{A_1}} m_2, \ldots, \theta_N - 2\pi \sqrt{\frac{A_2}{A_1}} m_N \right) \\
\times \Delta \left( \theta_1 + 2\pi \sqrt{\frac{A_1}{A_2}} m_1 + \frac{img^2}{2} \sqrt{A_1A_2}, \theta_2 + 2\pi \sqrt{\frac{A_1}{A_2}} m_2, \ldots, \theta_N + 2\pi \sqrt{\frac{A_1}{A_2}} m_N \right)
$$

(9)

where $A_2 = A - A_1$. Using now the identity

$$
\int_{-\infty}^{+\infty} dz_1 \ldots dz_N \ e^{-\frac{1}{h^2} \sum z_i^2} \ \Delta(z_i + a_i/h) \ \Delta(z_i + b_i h) \\
= \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \ e^{-\frac{1}{h^2} \sum z_i^2} \ \Delta(z_i + a_i) \ \Delta(z_i + b_i)
$$

(10)

with $a_i, b_i$ and $h$ complex quantities, which can be proven for instance by expanding the Vandermonde determinants in terms of Hermite polynomials, Eq. (8) becomes

$$
\mathcal{W}_n \mathcal{Z} = \sum_{\{m\}} e^{i\pi(N-1)\sum_i m_i} \int_{-\infty}^{+\infty} d\theta_1 \ldots d\theta_N \Delta(\theta_1 - 2\pi m_1 - img^2 A_2 A_1 / 4, \ldots, \theta_N - 2\pi m_N)
$$
\( \times \Delta(\theta_1 + 2\pi m_1 + i g^2 \frac{A_2 - A_1}{4}, \ldots, \theta_N + 2\pi m_N) \exp\left(-\frac{1}{g^2 A} \sum_{i=1}^{N} \theta_i^2 + i n \theta_1 \right) \)

\( \times \exp\left[\frac{n^2 g^2 (A_2 - A_1)^2}{16 A}\right] \exp\left[-\frac{4\pi^2}{g^2 A} \sum_{i=1}^{N} m_i^2 - 2\pi i \frac{A_2}{A} n m_1\right]. \) (11)

Eq. (11) coincides with the result of Ref. [12], derived from the representation in terms of group characters (our Eq. (1)) via a Poisson transformation. Nevertheless we found it instructive to derive it from the representation (2) since the same procedure can be easily transferred to more general situations.

We can now single out the zero-instanton contribution from the above formula by retaining only the term with \( m_i = 0, \ \forall i. \)

In Ref. [13] it is shown how this zero-instanton contribution can be written as an integral over the group algebra for a Wilson loop in the fundamental and in the adjoint representation, namely

\[ W_0^n(A_1, A) Z_0^n(A) = \int \mathcal{D}F \exp\left(-\frac{1}{2} \text{Tr} \ F^2\right) \text{Tr} \exp\left[igF \sqrt{\frac{A_1(A - A_1)}{2A}}\right] \] (12)

\[ = \int \mathcal{D}F e^{-\frac{1}{2} \text{Tr} \ (F^2)} \chi_{\text{fund}}(e^{ig\mathcal{E}}) \]

and

\[ W_0^n(A_1, A) Z_0^n(A) = \int \mathcal{D}F \left(\left| \text{Tr} \exp\left[igF \sqrt{\frac{A_1(A - A_1)}{2A}}\right]\right|^2 - 1 \right) \] (13)

\[ \times \exp\left(-\frac{1}{2} \text{Tr} \ F^2\right) = \int \mathcal{D}F e^{-\frac{1}{2} \text{Tr} \ (F^2)} \chi_{\text{adj}}(e^{ig\mathcal{E}}), \]

respectively, with \( \mathcal{E} = \sqrt{\frac{A_1(A - A_1)}{2A}}, \) and \( F \) a Hermitian \( N \times N \) matrix.

We want to extend this property to a generic class function \( S(U) \equiv \mathcal{P}(e^{i\theta_j}), \) \( \mathcal{P} \) being a \( L^2 \)-summable symmetric function of the eigenvalues, which can be expanded in the group characters

\[ \mathcal{P} = \sum_R b_R \chi_R(e^{i\theta_j}) \] (14)
The set of group characters \( \{ \chi_R \} \) represents an orthogonal basis in the space of class functions. Alternatively, recalling that each character is a symmetric polynomial in \( e^{\pm i \theta} \), one can write

\[
P = \sum_{\{q\}} p_{\{q\}} S_{\{q\}},
\]

where \( S_{\{q\}} = \sum_{\sigma \in \Pi(1, \ldots, N)} (e^{i \theta_{\sigma(1)}})^{q_1} \ldots (e^{i \theta_{\sigma(N)}})^{q_N} \) and \( q_1, \ldots q_N \) are integers. Generalizing Eq. (7) we get

\[
SZ = \sum_{\{m\}} \sum_{\{l\}} (-1)^{(N-1)\sum_i (m_i - l_i)} \int_{-\infty}^{2\pi} d\theta_1 \ldots d\theta_N \Delta(\theta + 2\pi l) \Delta(\theta + 2\pi m)
\]

\[
\times \exp \left[ -\frac{1}{g^2 A_1} \sum_{i=1}^{N} (\theta_i + 2\pi l_i)^2 - \frac{1}{g^2 A_2} \sum_{i=1}^{N} (\theta_i + 2\pi m_i)^2 \right] P(e^{i \theta_j}).
\]

We can now repeat the previous procedure to obtain

\[
SZ = \sum_{\{q\}} \sum_{\sigma \in \Pi(1, \ldots, N) \{m\}} (-1)^{(N-1)\sum_i m_i} \int_{-\infty}^{+\infty} d\theta_1 \ldots d\theta_N \exp \left[ -\frac{1}{g^2 A} \sum_{i=1}^{N} \theta_i^2 \right]
\]

\[
\times \Delta \left( \theta_1 - 2\pi m_1 - i \alpha q_{\sigma(1)}, \ldots, \theta_N - 2\pi m_N - i \alpha q_{\sigma(N)} \right)
\]

\[
\times \Delta \left( \theta_1 + 2\pi m_1 + i \alpha q_{\sigma(1)}, \ldots, \theta_N + 2\pi m_N + i \alpha q_{\sigma(N)} \right) e^\frac{i}{2} \sum_k \theta_k q_{\sigma(k)}
\]

\[
\times \exp \left[ \frac{g^2 (A_1 - A_2)^2}{16 A} \sum_{i=1}^{N} q_i^2 \right] \exp \left[ -\frac{4\pi^2}{g^2 A} \sum_{i=1}^{N} m_i^2 - 2\pi i A_2 \sum_k q_{\sigma(k)} m_k \right]
\]

with \( \alpha = \frac{g^2}{4} (A - 2A_1) \). By retaining only the zero-instanton contribution \( (m_i = 0, \forall i) \), we find

\[
S^0 Z^0 = \sum_{\{q\}} \sum_{\sigma \in \Pi(1, \ldots, N) \{m\}} \int_{-\infty}^{+\infty} d\theta_1 \ldots d\theta_N \Delta(\theta)^2 \exp \left[ -\frac{1}{2} \sum_{i=1}^{N} \theta_i^2 \right] e^{igF \sum_k \theta_k q_{\sigma(k)}}
\]

\[
= \int_{-\infty}^{+\infty} d\theta_1 \ldots d\theta_N \Delta(\theta)^2 \exp \left[ -\frac{1}{2} \sum_{i=1}^{N} \theta_i^2 \right] \sum_{\{q\}} p_{\{q\}} \sum_{\sigma \in \Pi(1, \ldots, N)} \left( e^{igF \theta_1} g_{\sigma(1)} \right) \ldots \left( e^{igF \theta_N} g_{\sigma(N)} \right)
\]

\[
= \int_{-\infty}^{+\infty} \mathcal{D}F e^{-\frac{1}{4} Tr[F^2]} P(e^{igF}).
\]

To get such a result, we have used the relation [13]

\[
\int_{-\infty}^{+\infty} d\theta_k \text{He}_{r_k-1}(\theta_k - i\alpha q_{\sigma(k)}) \sqrt{\frac{2g^2}{A}} \text{He}_{s_k-1}(\theta_k + i\alpha q_{\sigma(k)}) \sqrt{\frac{2g^2}{A}}
\]

\[
= \sqrt{\frac{2g^2}{A}} \text{He}_{r_k-1}(\theta_k) \sqrt{\frac{2g^2}{A}} \text{He}_{s_k-1}(\theta_k)
\]

(19)
\[
\begin{align*}
&\times \exp \left( -\frac{1}{2} \theta_k^2 + \frac{i}{2} \sqrt{\frac{g^2 A}{2}} q_{\sigma(k)} \theta_k \right) = \exp \left[ -\frac{g^2 (A_2 - A_1)^2}{16 A} q_{\sigma(k)}^2 \right] (A - A_1)^{r_k - s_k} A_1^{s_k - r_k} \\
&\times \int_{-\infty}^{+\infty} d\theta_k \exp \left( -\frac{1}{2} \theta_k^2 + ig \sqrt{\frac{A_1 (A - A_1)}{2A}} q_{\sigma(k)} \theta_k \right) \ He_{r_k-1}(\theta_k) \ He_{s_k-1}(\theta_k)
\end{align*}
\]

and taken into account that \( \sum_k r_k = \sum_k s_k \).

Thus far we have shown that retaining only the zero-instanton contribution to the average of a class function amounts to integrate over the group algebra. One may wonder how this happens.

We see from Eq. (17) that the zero-instanton contribution dominates in the limit \( g^2 A \to 0 \). Intuitively, in this limit both heat-kernel solutions become the class-invariant \( \delta \)-distribution, and integration over the group manifold is turned into integration over its tangent space, namely its algebra.

**III. A GEOMETRICAL APPROACH**

This issue can be made mathematically more precise according to the following argument. The exact heat-kernel solution lives on the topologically non-trivial \( U(N) \) manifold. Thus, neglecting instantons amounts to map the manifold \( U(N) \) into a topologically trivial one. Of course this mapping cannot be a global diffeomorphism; nevertheless we may require it to be local, in order to preserve the original differential structure. We can consider the map \( U \to -i \log U \), which is a multivalued immersion of the group into its algebra, infer the image \( \Delta_X \) of the laplacian operator \( \Delta_\theta \) and seek for solutions of the heat-kernel equation

\[
\Delta_X \mathcal{H}(A; \theta) = \frac{4}{g^2} \frac{\partial}{\partial A} \mathcal{H}(A; \theta)
\]

in the algebra manifold, obeying

\[
\lim_{g^2 A \to 0} \mathcal{H}(A; \theta) = \delta(\theta).
\]

The group algebra manifold is \( R^{N^2} \) with canonical coordinates \( (\omega_1, \ldots, \omega_{N^2}) \). Consider the exponential map
\((\omega_1, \ldots, \omega_{N^2}) \mapsto e^{i\omega_a T_a}\). \quad (22)

This is a locally invertible differentiable map, and therefore defines a local diffeomorphism. The metric on the group manifold is given by \([1,17]\)

\[
d s^2 = \text{Tr} (U^{-1} dU (U^{-1} dU)^\dagger) = \sum_{k=1}^{N} d\theta_k^2 + \sum_{j \neq k=1}^{N} |e^{i\theta_j} - e^{i\theta_k}|^2 |(V^t dV)_{jk}|^2, \quad (23)
\]

where \(V\) is a unitary matrix such that \(U = V \text{diag}(e^{i\theta_k}) V^\dagger\), whereas on the tangent space it reads

\[
d s^2 = \text{Tr} (dF dF^\dagger) = \sum_{k=1}^{N} d\theta_k^2 + \sum_{j \neq k=1}^{N} |\theta_j - \theta_k|^2 |(V^t dV)_{jk}|^2. \quad (24)
\]

The laplacian operator acting on class functions over the group

\[
\Delta_\theta = \frac{1}{J(\theta)} \sum_{k=1}^{N} \left( \frac{\partial}{\partial \theta_k} \right)^2 J(\theta) + \frac{1}{12} N(N^2 - 1) \quad (25)
\]

becomes

\[
\Delta_X = \frac{1}{\Delta(\theta_k)} \sum_{k=1}^{N} \partial^2_{\theta_k} \Delta(\theta_k). \quad (26)
\]

Then the solution of the heat-kernel equation on the algebra manifold is

\[
\mathcal{H}(A; \theta) = (\pi g^2 A)^{-\frac{N^2}{2}} \exp \left( -\frac{1}{g^2 A} \sum_{k=1}^{N} \theta_k^2 \right) \quad (27)
\]

and consequently, generalizing Eq. (27), we check that

\[
\mathcal{S}^0 Z^0 = \int \mathcal{D}F \mathcal{H}(A_1; \theta) \mathcal{H}(A_2; \theta) \mathcal{P}(e^{iF})
\]

\[
= \int \mathcal{D}F e^{-\frac{1}{g^2 A_1} \text{Tr} F^2} e^{-\frac{1}{g^2 A_2} \text{Tr} F^2} \mathcal{P}(e^{iF})
\]

\[
= \int \mathcal{D}F e^{-\frac{1}{g^2 f_2} \text{Tr} F^2} \mathcal{P}(e^{iF})
\]

is equivalent to Eq. (18), as expected.
IV. CONCLUSIONS

In this letter we showed that the topologically trivial contribution to the average over $U(N)$ of a class function belonging to $L^2(U)$ corresponds to its average over the group algebra and thereby to a matrix model.

This was first derived generalizing the harmonic analysis of the Wilson loop average in the fundamental and in the adjoint representations performed in Ref. [13]. We started from an expansion in terms of instantons and retained only the trivial sector. This procedure makes clear that instantons are indeed related to the group curvature.

Then we presented a direct approach relying on purely geometrical considerations: we mapped the heat equation from the group manifold to its tangent space and exploited the basic heat-kernel sewing property.

In our procedure, when singling out the trivial topological sector, the entire group $U(N)$ underwent decompactification. It might be worth examining the consequences on the instanton patterns ensuing from partial decompactifications, namely understanding to what extent different instanton sectors turn out to be intertwined.

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