Maximal Invariants For Lorentz Wishart Models

Emanuel Ben-David

Department of Statistics, Stanford University

Abstract

In this paper we consider two statistical hypotheses for the families of Wishart type distributions. These distributions are analogs of the Wishart distributions defined and parametrized over a Lorentz cone. We test these hypotheses by means of maximal invariant statistics which are explicitly derived in the paper. The testing problems, respectively, concern the hypothesis that parameters are in a sub-Lorentz-cone, and the the hypothesis that two observations have the same parameter.

Keywords: Lorentz cones, Wishart distributions, maximal invariants, symmetric cones, Jordan algebras

1. Introduction

The primary aim of this paper is a detailed study of maximal invariant statistics for a family of Lorentz type Wishart distributions, or simply a Lorentz Wishart model. A Lorentz type Wishart distribution, similar to the classical Wishart distribution [11], naturally arises as the distribution of the empirical sample covariance of a multivariate normal distribution [8], or directly as the joint spectral density of observable variables [7]. Generally, by employing the theory of Euclidean simple algebras, the Wishart distributions can be defined on each irreducible symmetric cone[5], or more generally, on each homogeneous cone, by employing the theory of Vinberg algebras [2]. According to the classification of irreducible (symmetric) cones [6] there are five types of irreducible cones. The first there types of irreducible cones are, respectively, \( PD_n(\mathbb{R}) \), \( PD_n(\mathbb{C}) \) and \( PD_n(\mathbb{H}) \), i.e., the cone of positive definite matrices over the field of real, complex and quaternion numbers. The forth type of irreducible cone is the Lorentz cone (or Minkowski cone), and the fifth type is the exceptional cone over the Octonion \( \mathbb{O} \). Therefore, correspondingly, the three first types of Wishart distributions are real, complex and quaternion Wishart distributions which have been well studied in statistics and probability literature. In [8] Jensen considered statistical hypotheses for Lorentz Wishart models and obtain a
complete solution to the problems of maximum likelihood inference.

In this paper,\(^1\) we explicitly derive maximal invariant statistics for testing two invariant statistical hypotheses for the family of Lorentz type Wishart distributions. The first hypothesis is studied in [8] too, but our approach is more general and it highlights the role of the maximal invariant statistic. The second hypothesis is an analog of Bartlett's test.

The organization of this paper is as follows. In §2 we give a precise definition of a Lorentz cone \( L \), the description of the automorphism group \( G \) of the Lorentz cone \( L \), and the definition of the Lorentz type Wishart distribution \( W_{n,\sigma}^L \). In §3 we test the hypothesis that the scale parameter \( \sigma \) is in a Lorentz subcone \( L_0 \subset L \). To this end, first we identify a subgroup \( G_0 \subset G \) that acts transitively on \( L_0 \) and derive a maximal invariant statistic associated with the hypothesis under the action of \( G_0 \). In §4 we test the hypothesis that two observed Lorentz type Wishart distributions have the same scale parameter \( \sigma \). For this testing problem we derive a maximal invariant statistic associated with the hypothesis under the action of \( G \).

2. Preliminaries

2.1. The Lorentz cone

Let \( W \neq \{0\} \) be a Euclidean vector space and let
\[
\Psi_W(w, w) = (w, w'), \ w, w' \in W,
\]
denote the inner product on \( W \). Set as usual \( \|w\|^2 := (w, w) \), for each \( w \in W \).

Consider the symmetric form \( \Psi \) on \( \mathbb{R} \times W \) given by \( \Psi((\lambda, w), (\lambda, w')) := \lambda^2 - \|w\|^2 \).

This is a non-singular symmetric form with signature \( (1, \text{Dim}(W)) \). The symmetric form \( \Psi \) partitions according to the decomposition \( \mathbb{R} \times W \) as
\[
\Psi = \begin{pmatrix}
\psi_0 & 0 \\
0 & -\Psi_W
\end{pmatrix}
\]
where \( \psi_0 \in \mathbb{R}^+ \). The set
\[
L := \{ (\lambda, w) \in \mathbb{R} \times W \mid \lambda > 0, \Psi((\lambda, w)) > 0 \}
\]
is a symmetric cone, called the Lorentz cone (generated by \( W \)).

\(^1\)This paper is a revised version of the second part of the author's doctoral thesis [4]. In the first part, using a uniform approach, the maximal invariant statistics over irreducible cones are studied.

\(^2\)we use the notations \( \mathbb{R}^+ \) and \( \mathbb{R}^{++} \), respectively, for the set of non-negative and positive real numbers.
2.2. The automorphism group

Define for any symmetric form $\Psi$ on a Euclidean space $V$ the orthogonal group

$$O(\Psi) := \{ A \in \text{GL}(V) \mid \Psi \circ (A \times A) = \Psi \},$$

the special orthogonal group

$$SO(\Psi) := \{ A \in O(V) \mid \det(A) = 1 \}.$$

The connected component of the identity in $O(\Psi)$, denoted by $SO^\dagger(\Psi)$, is a subgroup of $SO(\Psi)$. If $\Psi$ is positive definite, and hence an inner product on $V$, then $SO^\dagger(\Psi) = SO(\Psi)$.

Let $G_W = G$ denote the connected component of the automorphism group $\text{Aut}(\mathcal{L})$. Then for the symmetric form $\Psi$ defined in §2.1 we have

$$G = \mathbb{R}^+ \times SO^\dagger(\Psi),$$

where

$$SO^\dagger(\Psi) = \left\{ A = \begin{pmatrix} a_0 & a_{0W} \\ a_{W0} & A_W \end{pmatrix} \in SO(\Psi) \mid a_0 > 0 \right\}$$

(1)

with $a_{0W} : W \to \mathbb{R}$, $a_{W0} : \mathbb{R} \to W$ and $A_W : W \to W$ being linear mappings. Note that $\mathbb{R}^+ \times SO^\dagger$ acts transitively on $\mathcal{L}$ and for $(a, A) \in \mathbb{R}^+ \times SO^\dagger(\mathbb{R} \times W)$ we have

$$(a, A)(\lambda, w) = aA \left( \begin{pmatrix} \lambda \\ w \end{pmatrix} \right).$$

2.3. The Lorentz type Wishart distributions

Fix an element $e \in W$. Let $W_1$ be the orthogonal complement of $Re$, i.e., $Re^\perp$. Thus $W = Re \oplus W_1$ can be identified with $\mathbb{R} \times W_1$. Under this identification, the Lorentz cone $\mathcal{L}$ is isomorphic to the homogenous cone

$$\mathcal{P}_2(W) := \{ S = \begin{pmatrix} \lambda_1 & w_1 \\ w_1 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{R}^+, w_1 \in W_1, \det(S) := \lambda_1\lambda_2 - \|w_1\|^2 > 0 \},$$

studied in [2]. The isomorphism is given by

$$\phi : \mathcal{P}_2(W) \to \mathcal{L}$$

$$\begin{pmatrix} \lambda_1 & w_1 \\ w_1 & \lambda_2 \end{pmatrix} \mapsto \left( \frac{\lambda_1 + \lambda_2}{2}, \frac{\lambda_1 - \lambda_2}{2}, w_1 \right),$$

(2)
where the inverse mapping is given by

$$\phi^{-1}: \mathcal{L} \rightarrow \mathcal{P}_2(W)$$

$$(\lambda_1, (\lambda_2, s)) \mapsto \left( \begin{array}{cc} \lambda_1 + \lambda_2 & w_1 \\ w_1 & \lambda_1 - \lambda_2 \end{array} \right).$$

(3)

By using their general theory of the Wishart distributions for homogeneous cones, in [2] Andersson et al. derive the Wishart distribution on $\mathcal{P}_2(V)$ as

$$dW_{\mathcal{P}_2(V)}(S|\eta, \Sigma) = \eta^{m\eta} \det(S)\eta^{-m+1} \exp\{-\eta \text{tr}(\Sigma^{-1}S)\} 1_{\mathcal{P}_2(V)}(S)dS,$$

where $\eta > m/2$ is shape parameter, $m = \dim_{\mathbb{R}}(W)$, and $\Sigma \in \mathcal{P}_2(W)$ is the expectation vector. Therefore, one natural way to define the Wishart distribution on $\mathcal{L}$ is via the image of $W_{\mathcal{P}_2(V)}(S|\eta, \Sigma)$ under the mapping $\phi$.

**Definition 2.1.** The Lorentz type Wishart distribution, denoted by $W^\mathcal{L}_{\eta,\sigma}$, is the image of the Wishart distribution $W_{\mathcal{P}_2(V)}(S|\eta, \phi^{-1}(\sigma))$ under the mapping $\phi$ in Eq. (2). One can check that the Wishart distribution on $\mathcal{L}$ with shape parameter $\eta > \frac{m-1}{2}$ and expectation $\sigma = (\lambda, w) \in \mathcal{L}$ is given by

$$dW^\mathcal{L}_{\eta,\sigma}(y, z) = \frac{\eta^{m\eta} \det(S)\eta^{-m+1}}{\pi^{m\eta} \Gamma(\eta)\Gamma(\eta - \frac{m-1}{2}) \det(\Sigma)^\eta} \exp\{-2\eta(\frac{\lambda y - w \cdot z}{\lambda^2 - \|w\|^2})\} 1_{\mathcal{L}}(y, z)d\lambda d\Sigma,$$

(4)

where $(y, z) \in \mathbb{R} \times W$ and $k(m, \eta) = 2\pi^{m\eta} \Gamma(\eta)\Gamma(\eta - \frac{m-1}{2})\eta^{-2\eta}$.

**Remark 2.1.** First note that the density given in Eq. (4) is the same as formula (29) in [8]. Moreover, for every irreducible symmetric cone $\Omega$, the Wishart distribution on $\Omega$, denoted by $W^\Omega_{\eta,\sigma}$, is well defined and given by the density

$$dW^\Omega_{\eta,\sigma}(x) = \frac{1}{2^{\frac{m\eta}{2}} \Gamma_{\Omega}(\frac{\eta}{2}) \det(\sigma)^{\frac{m\eta}{2}}} \exp\{-\frac{1}{2} \text{tr}(\sigma^{-1}x)\} \det(x)^{\frac{\eta}{2} - \frac{m-1}{2}} 1_{\Omega}(x),$$

(5)

where $r$ is the rank of $\Omega$, and $\Gamma_{\Omega}(\cdot)$ is the gamma function associated with $\Omega$ (see [10] or [3] for detail). Since $\mathcal{L}$ is an irreducible symmetric cone of rank 2, Eq. (5) directly defines the Wishart distribution on $\mathcal{L}$.
3. Testing for scale parameter in a Lorentz subcone

Suppose that shape parameter $\eta$ is known and consider the Lorentz Wishart model

$$\mathcal{M} \equiv \{ W_{\eta,\sigma}^\mathcal{L} : \sigma \in \mathcal{L} \}.$$  \hfill (6)

The standard theory of exponential families implies that the ML estimator $\hat{\sigma}_{mle}$ of $\sigma \in \mathcal{L} \in \mathcal{M}$ exists for any observation $(y, z) \in \mathcal{L}$ and is given by

$$\hat{\sigma}_{mle}((y, z)) = (y, z).$$  \hfill (7)

Now suppose that $W_0 \neq \{0\}$ is a subspace of $W$. Let $\mathcal{L}_0$ denote the Lorentz cone generated by $W_0$. Under the inclusion mapping $\iota : \mathcal{L}_0 \rightarrow \mathcal{L}$ the Lorentz Wishart model

$$\mathcal{M}_0 \equiv \{ W_{\eta,\iota(\sigma_1)}^\mathcal{L} : \sigma_1 \in \mathcal{L}_0 \}$$  \hfill (8)

is a submodel of $\mathcal{M}$. Let us consider the hypothesis

$$H_0 : \sigma \in \mathcal{L}_0 \text{ vs. } H : \sigma \in \mathcal{L}.$$  \hfill (T1)

We will test the hypothesis (T1) by a maximal invariant statistic we shall derive in §3.2. First we identify a subgroup $G_0 \subset G$ such that the hypothesis is invariant under it. We proceed as follows.

3.1. The subgroup of $G$ with invariant action on $\mathcal{L}_0$

Recall the definition of the symmetric form $\Psi$ and the inner product $\Psi_W$ in §2.1. Let $W_0^\perp$ be the orthogonal complement of $W_0 \subset W$, with respect to $\Psi_W$. We set

$$\Psi_{W_0} := \Psi_W|_{W_0 \times W_0}, \quad \Psi_{W_0^\perp} := \Psi_W|_{W_0^\perp \times W_0^\perp}, \quad \text{and } \Psi_0 := \Psi|_{(\mathbb{R} \times W_0) \times (\mathbb{R} \times W_0)}.$$

By these conventions $\Psi$ partitions according to the decomposition $\mathbb{R} \times W_0 \times W_0^\perp$ as

$$\Psi = \begin{pmatrix} \psi_0 & 0 & 0 \\ 0 & -\Psi_{W_0} & 0 \\ 0 & 0 & -\Psi_{W_0^\perp} \end{pmatrix}.$$

Note that $\Psi_0$ and $\Psi_W$ partition with respect to the decompositions $\mathbb{R} \times W_0$ and $W = W_0 \times W_0^\perp$ accordingly as

$$\Psi_0 = \begin{pmatrix} \psi_0 & 0 \\ 0 & -\Psi_{W_0} \end{pmatrix}, \quad \text{and } \Psi_W = \begin{pmatrix} \Psi_{W_0} & 0 \\ 0 & \Psi_{W_0^\perp} \end{pmatrix}.$$
Since \((\mu, \mathbf{w}_0) \in \mathcal{L}_0\) implies that \(\mu^2 > \psi_{\mathbf{w}_0}(\mathbf{w}) = \psi_{\mathbf{w}}(\mathbf{w})\), clearly, \(\mathcal{L}_0 \subseteq \mathbb{R} \times \mathbf{w}_0 \subset \mathbb{R} \times \mathbf{W}\) is a subcone of \(\mathcal{L}\), i.e., \(\mathcal{L}_0 \subseteq \mathcal{L}\).

Next we determine the group \(G_0 := \{ g \in G | g(\mathcal{L}_0) \subseteq \mathcal{L}_0 \}\). Note that

\[
\mathbb{R}^{++} \subset \mathbb{R}^{++} \times SO^1(\Psi)
\]

is obviously contained in \(G_0\).

**Proposition 3.1.** Let \(A \in SO^1(\Psi)\). Then \(A(\mathcal{L}_0) \subseteq \mathcal{L}_0\) if and only if \(A\) partition as

\[
\begin{pmatrix}
A_0 & 0 \\
0 & A_{W_0^\perp}
\end{pmatrix}
\]

with

(i) \(A_0 \in O(\Psi_0)\), i.e., \(\Psi_0 \circ (A_0 \times A_0) = \Psi_0\),

(ii) \(A_{W_0^\perp} \in O(\Psi_{W_0^\perp})\), i.e., \(\Psi_{W_0^\perp} \circ (A_{W_0^\perp} \times A_{W_0^\perp}) = \Psi_{W_0^\perp}\),

(iii) \(a_0 > 0\), and

(iv) \(\det(A_{W_0^\perp}) \cdot \det(A_{W_0^\perp}) > 0\).

**Proof.** Suppose \(A \in SO^1(\Psi)\) acts invariantly on \(\mathcal{L}_0\), i.e., \(A(\mathcal{L}_0) \subseteq \mathcal{L}_0\). Let

\[
A = \begin{pmatrix}
\begin{bmatrix}
a_0 & \mathbf{a}_{0W_0} \\
\mathbf{a}_{W_00} & A_{W_0}
\end{bmatrix} & \begin{bmatrix}
\mathbf{a}_{0W_0^\perp} \\
\mathbf{a}_{W_0W_0^\perp}
\end{bmatrix}
\end{pmatrix} = \begin{pmatrix}
A_0 & \mathbf{a}_{0W_0^\perp} \\
A_{W_00W_0} & A_{W_0^\perp}
\end{pmatrix}
\]

and

\[
\Psi = \begin{pmatrix}
\Psi_0 & 0 \\
0 & -\Psi_{W_0^\perp}
\end{pmatrix}
\]

be the partitions of \(A\), \(A\), and \(\Psi\) with respect to the decompositions \(\mathbb{R} \times \mathbf{W}_0 \times \mathbf{W}_0^\perp\), \((\mathbb{R} \times \mathbf{W}_0) \times \mathbf{W}_0^\perp\), and \((\mathbb{R} \times \mathbf{W}_0) \times \mathbf{W}_0^\perp\), respectively. Note that

\[
A_{W_0^\perp0\mathbf{w}_0} = (\mathbf{a}_{W_0^\perp0\mathbf{w}_0}, A_{W_0^\perp0\mathbf{w}_0}) : \mathbb{R} \times \mathbf{W}_0 \to \mathbf{W}_0^\perp,
\]

and \(A(\mathcal{L}_0) \subseteq \mathcal{L}_0\) implies that \(A_{W_0^\perp0\mathbf{w}_0}(\lambda, \mathbf{w}_0) = 0\) for all \((\lambda, \mathbf{w}_0) \in \mathbb{R} \times \mathbf{W}_0\). Therefore \(A_{W_0^\perp0\mathbf{w}_0} = 0\). Since \(A \in SO^1(\Psi)\) we also have \(\Psi \circ (A \times A) = \Psi_\ast\), i.e.,

\[
\begin{pmatrix}
\Psi_0 & 0 \\
0 & -\Psi_{W_0^\perp}
\end{pmatrix} \circ \begin{pmatrix}
\begin{bmatrix}
A_0 & \mathbf{a}_{0W_0^\perp} \\
\mathbf{a}_{W_00} & A_{W_0}
\end{bmatrix} \times \begin{bmatrix}
A_0 & \mathbf{a}_{0W_0^\perp} \\
\mathbf{a}_{W_0W_0^\perp}
\end{bmatrix}
\end{pmatrix} = \begin{pmatrix}
\Psi_0 & 0 \\
0 & -\Psi_{W_0^\perp}
\end{pmatrix}.
\]
This is equivalent to the followings.

(i) \( \Psi_0 \circ (A_0 \times A_0) = \Psi_0 \).

(ii) \( \Psi_0 \circ (A_0 \times A_{W_0^\perp}) = 0 \).

(iii) \( \Psi_{W_0^\perp} \circ (A_{W_0^\perp} \times A_{W_0^\perp}) = \Psi_{W_0^\perp} \).

\[ \square \]

3.2. A maximal invariant statistic for testing the hypothesis

Now consider the action of \( G_0 \) on \( \mathcal{L} \), i.e., the restriction of the (transitive) action of \( G \) on \( \mathcal{L} \) to \( G_0 \). We define the statistic

\[ m : \mathcal{L} : \rightarrow \mathbb{R}_{++} \]

\[ (\lambda, \mathbf{w}_0, \mathbf{w}_0^\perp) \mapsto \frac{\Psi_{W_0^\perp}(\mathbf{w}_0^\perp, \mathbf{w}_0^\perp)}{\Psi_0((\lambda, \mathbf{w}_0), (\lambda, \mathbf{w}_0))}. \tag{9} \]

where \((\lambda, \mathbf{w}_0, \mathbf{w}_0^\perp)\) represents a typical element of \( \mathbb{R} \times W_0 \bigoplus W_0^\perp \).

**Proposition 3.2.** The mapping \( m \) in Eq. (9) is a faithful representation of the orbit projection \( \pi : \mathcal{L} \rightarrow \mathcal{L}/G_0 \) and therefore a maximal invariant statistic.

**Proof.** First for notational convenience set \( \|\mathbf{w}\|^2 = \Psi_W(\mathbf{w}, \mathbf{w}) \), for \( \mathbf{w} \in W \). Thus

\[ m((\lambda, \mathbf{w}_0, \mathbf{w}_0^\perp)) = \frac{\|\mathbf{w}_0^\perp\|^2}{\lambda^2 - \|\mathbf{w}_0\|^2}, \quad \forall (\lambda, \mathbf{w}_0, \mathbf{w}_0^\perp) \in \mathcal{L}. \]

For each \((a, A) \in G_0 \) by partitioning \( A \) as in Proposition 3.1 we obtain

\[ m((a, A)(\lambda, \mathbf{w}_0, \mathbf{w}_0^\perp)) = \frac{\|aA_{W_0^\perp}\mathbf{w}_0^\perp\|^2}{(a\lambda)^2 - \|aA_{W_0}\mathbf{w}_0\|^2} = \frac{\|A_{W_0^\perp}\mathbf{w}_0^\perp\|^2}{\lambda^2 - \|A_{W_0}\mathbf{w}_0\|^2} = \frac{\|\mathbf{w}_0^\perp\|^2}{\lambda^2 - \|\mathbf{w}_0\|^2}. \]

This shows that \( m \) is invariant under the action of \( G_0 \) on \( \mathcal{L} \). Now suppose that

\[ m((\lambda, \mathbf{w}_0, \mathbf{w}_0^\perp)) = m((\mu, \mathbf{u}_0, \mathbf{u}_0^\perp)), \]

i.e.,

\[ \frac{\|\mathbf{w}_0^\perp\|^2}{\lambda^2 - \|\mathbf{w}_0\|^2} = \frac{\|\mathbf{u}_0^\perp\|^2}{\mu^2 - \|\mathbf{u}_0\|^2}. \]
If $\mathbf{u}_0^\perp = 0$, then $\mathbf{w}_0^\perp = 0$ and, since $G_0$ acts transitively on $\mathcal{L}_0$, we can find $(a, A) \in G_0$ such that

$$(a, A)(\lambda, \mathbf{w}_0, 0) = (\mu, \mathbf{u}_0, 0),$$

which shows that $(\mu, \mathbf{u}_0, \mathbf{u}_0^\perp)$ and $(\mu, \mathbf{w}_0, \mathbf{w}_0^\perp)$ are in the same $G_0$-orbit.

Now assume $\mathbf{w}_0^\perp \neq 0$, and therefore $\mathbf{u}_0^\perp \neq 0$. We set

$$a := \frac{\|\mathbf{u}_0^\perp\|}{\|\mathbf{w}_0^\perp\|} = \sqrt{\frac{\mu^2 - \|\mathbf{u}_0\|^2}{\lambda^2 - \|\mathbf{w}_0\|^2}}. $$

Choose $A_0 \in SO^+(\Psi_0)$ and $A_{W_0^\perp} \in SO(\Psi_{W_0^\perp})$ such that

$$A_0\left(\frac{\lambda, \mathbf{w}_0}{\sqrt{\lambda^2 - \|\mathbf{w}_0\|^2}}\right) = \frac{(\mu, \mathbf{u}_0)}{\sqrt{\mu^2 - \|\mathbf{u}_0\|^2}},$$

and

$$A_{W_0^\perp}\left(\frac{\mathbf{w}_0^\perp}{\|\mathbf{w}_0^\perp\|}\right) = \frac{\mathbf{u}_0^\perp}{\|\mathbf{u}_0^\perp\|}.$$

For $A := \begin{pmatrix} A_0 & 0 \\ 0 & A_{W_0^\perp} \end{pmatrix}$ we have $(a, A) \in G_0$ and

$$(a, A)(\lambda, \mathbf{w}_0, \mathbf{w}_0^\perp) = a(A_0(\lambda, \mathbf{w}_0), A_{W_0^\perp}(\mathbf{w}_0^\perp))$$

$$= a\left(\sqrt{\frac{\lambda^2 - \|\mathbf{w}_0\|^2}{\mu^2 - \|\mathbf{u}_0\|^2}}, \frac{\|\mathbf{w}_0\|^2}{\|\mathbf{w}_0\|^2} \frac{\mathbf{w}_0^\perp}{\|\mathbf{u}_0^\perp\|}\right)$$

$$= a(a^{-1}(\mu, \mathbf{u}_0), a^{-1}\mathbf{u}_0^\perp)$$

$$= (\mu, \mathbf{u}_0, \mathbf{u}_0^\perp).$$

\[\square\]

**Remark 3.1.** From statistical point of view, it is more useful to write the maximal invariant statistics $\mathbf{m}$ in Eq. (9) as

$$\mathbf{m}(\lambda, \mathbf{w}) = \frac{\|\mathbf{w} - \mathbf{p}(\mathbf{w})\|^2}{\lambda^2 - \|\mathbf{p}(\mathbf{w})\|^2} \quad \forall (\lambda, \mathbf{w}) \in \mathcal{L},$$

where $\mathbf{p} : W \to W_0$ is the orthogonal projection of $W$ onto $W_0$. 

8
3.3. Testing the hypothesis (T1)
Finally we are in the position to give a test statistic for testing the hypothesis (T1). First note that the hypothesis (T1) is invariant under $G_0$.

**Theorem 3.1.** Consider $W_{\eta,\nu(\sigma)}^\mathcal{L}$, the Wishart distribution on $\mathcal{L}$, where $\sigma \in \mathcal{L}_0$ and $\nu(\sigma) \in \mathcal{L}$ is the embedding of $\sigma$ into $\mathcal{L}$. Let $t : \mathcal{L} \to \mathcal{L}_0$ be the mapping $(y, z) \mapsto (y, p(z))$, where $(y, z) \in \mathcal{L}$. Then for the mapping

$$(t, m) : \mathcal{L} \to \mathcal{L}_0 \times \mathbb{R}^+$$

$$(y, z) \mapsto ((y, p(z)), m(y, z))$$

we have $(t, m)(W_{\eta,\nu(\sigma)}^\mathcal{L}) = W_{\eta,\nu(\sigma)}^\mathcal{L} \otimes m(W_{\eta,\nu(\sigma)}^\mathcal{L})$.

**Proof.** First write $W_{\eta,\nu(\sigma)}^\mathcal{L}$ as a density with respect to the invariant measure

$$d\nu(y, z) = (y^2 - \|z\|^2)^{\nu - \frac{m+1}{2}} dydz,$$

and rewrite the density in terms of $t(y, z)$ and $m(y, z)$. Then apply [1, Lemma 3] to $\nu$ and state the transformation result. \hfill \Box

**Corollary 3.1.** Let $m_0 := \dim_\mathcal{L}(W_0)$ and $m_1 = m - m_0$. Then the transformed measure $m(W_{\eta,\nu(\sigma)}^\mathcal{L})$ is the beta distribution $\beta(\eta - \frac{m-1}{2}, \frac{m+1}{2})$.

**Proof.** We start with writing $(t, m) : \mathcal{L} \to \mathcal{L}_0 \times \mathbb{R}^+$ as the composition of the mappings

$$\mathcal{L} \to \mathbb{R}^+ \times W_0 \oplus W_0^+ \to \mathbb{R}^+ \times W_0 \times \mathbb{R}^+$$

$$(y, z) \mapsto (y, z_0, z_1) := \left(y, p(z), \frac{z - p(z)}{\sqrt{y^2 - \|p(z)\|^2}}\right)$$

$$\mapsto (y, z_0, u) := (y, z_0, \|z_1\|^2).$$

Using these compositions we transfer the probability density of the Wishart distribution $W_{\eta,\nu(\sigma)}^\mathcal{L}$ as follows:

$$dW_{\eta,\nu(\sigma)}^\mathcal{L}(y, z) \rightarrow \frac{k(m_0, \eta)}{k(m, \eta)} \left(1 - \|z_1\|^2\right)^{\eta - \frac{m+1}{2}} 1_{\{\|z_1\| < 1\}}(z_1)dW_{\eta,\nu(\sigma)}(y, z_0)dz_1$$

$$\rightarrow \frac{k(m_0, \eta)\pi^{\frac{mn}{2}}}{k(m, \eta)\Gamma\left(\frac{m_1}{2}\right)} (1 - u)^{\eta - \frac{m+1}{2}} u^{\frac{m_1}{2} - 1} [0, 1](u)dW_{\eta,\nu(\sigma)}(y, z_0)du$$

$$= \frac{\Gamma(\eta - \frac{m-1}{2})}{\Gamma(\eta - \frac{m-1}{2})\Gamma\left(\frac{m_1}{2}\right)} (1 - u)^{\eta - \frac{m+1}{2}} u^{\frac{m_1}{2} - 1} [0, 1](u)dW_{\eta,\nu(\sigma)}(y, z_0)du$$

$$= dW_{\eta,\nu(\sigma)}(y, z_0)d\beta(\eta - \frac{m-1}{2}, \frac{m_1}{2})(u).$$
Thus $m(y, z) \sim \beta(\eta - \frac{m_1}{2}, \frac{m_1}{2})$. 

**Proposition 3.3.** The likelihood ratio LR statistic $Q$ for hypothesis (T1) is given by

$$Q = \left( \frac{y^2 - \|z\|^2}{y^2 - \|p(z)\|^2} \right)^\eta = m(y, z)^\eta. \quad (10)$$

Moreover, $Q$ is independent $W_{L_0, \sigma}$ and $Q^\frac{1}{\eta} \sim \beta(\eta - \frac{m_1 - 1}{2}, \frac{m_1}{2} - m_0^2)$.

**Proof.** First note that for the submodel (8) and the observation $(y, z)$, the ML estimator of $\sigma \in L_0$ becomes

$$\hat{\sigma}_1((y, z)) = (y, p(z)),$$

where $p$ is the orthogonal projection on the subspace $W_0$. Thus LR is given by Eq. (10). The rest of the proof are direct consequences of Proposition 3.1 and Corollary 3.1. 

Note that the Proposition 3.3 reduces to Theorem 5 in [8] for choice of $\eta = N/4$.

**4. The Bartlett’s test for Lorentz Wishart Models**

As before, suppose $\eta$ is known, and consider the statistical model

$$\mathcal{W}_{\eta, \sigma_1} \otimes \mathcal{W}_{\eta, \sigma_2} : (\sigma_1, \sigma_2) \in L \times L$$

and its submodel

$$\mathcal{W}_{\eta, \sigma} \otimes \mathcal{W}_{\eta, \sigma} : \sigma \in L.$$

Consider the hypothesis

$$H_0 : \sigma_1 = \sigma_2 = \sigma \hspace{1em} \text{vs.} \hspace{1em} H : \sigma_1 \neq \sigma_2. \quad (T2)$$

4.1. A maximal invariant statistic associated with the hypothesis (T2)

Note that $L$ can be considered a subcone of $L \times L$, via diagonal embedding. Also the action of $G$ on $L$ can be, canonically, extended to an action on $L \times L$ given by

$$g(\sigma_1, \sigma_2) = (g\sigma_1, g\sigma_2) \quad \forall g \in G, \forall (\sigma_1, \sigma_2) \in L \times L. \quad (11)$$

Under this consideration the hypothesis (T2) is invariant under $G$, and therefore a maximal invariant statistic is desired. To obtain a maximal invariant we proceed with the following lemma.
Lemma 4.1. For every $\sigma_1$ and $\sigma_2 \in \mathcal{L}$

$$\Psi(\sigma_1, \sigma_2) \geq \sqrt{\Psi(\sigma_1, \sigma_1)\Psi(\sigma_2, \sigma_2)},$$

(12)

and the equality holds if and only if $g\sigma_1 = \sigma_2$ for some $g \in G$.

Proof. Let $\sigma_1 \cdot \sigma_2 := \Psi(\sigma_1, \sigma_2)$, where $\sigma_1 = (\lambda, w)$ and $\sigma_2 = (\mu, u)$. Then

$$\begin{align*}
(\sigma_1 \cdot \sigma_2)^2 - \|\sigma_1\|^2\|\sigma_2\|^2 &= (\lambda \mu - w \cdot u)^2 - (\lambda^2 - \|w\|^2)(\mu^2 - \|u\|^2) \\
&\geq \lambda^2\|u\|^2 - 2\lambda \mu \|u\|\|w\| + \mu^2\|w\|^2 \\
&= (\lambda\|u\| - \mu\|w\|)^2 \geq 0.
\end{align*}$$

The equality holds if and only if $\lambda\|u\| = \mu\|w\|$ which is equivalent to $\|\sigma_2\| = \frac{\mu}{\lambda}\|\sigma_1\|$.

Notation 1. In the remainder of this paper for brevity we use the notation $\sigma_1 \cdot \sigma_2$ for the quadratic product $\Psi(\sigma_1, \sigma_2)$.

Proposition 4.1. The mapping $\pi : \mathcal{L} \times \mathcal{L} \to \mathbb{R}^2_{++}$ with $$(\sigma_1, \sigma_2) \mapsto (\xi_1, \xi_2),$$

$$\begin{align*}
\xi_1 &= \frac{\sigma_1 \cdot \sigma_2 + \sqrt{(\sigma_1 \cdot \sigma_2)^2 - \|\sigma_1\|^2\|\sigma_2\|^2}}{\|\sigma_1\|^2} \\
\xi_2 &= \frac{\sigma_1 \cdot \sigma_2 - \sqrt{(\sigma_1 \cdot \sigma_2)^2 - \|\sigma_1\|^2\|\sigma_2\|^2}}{\|\sigma_1\|^2},
\end{align*}$$

is maximal invariant under the action of $G$, defined in Eq. (11), and $\xi_1$, $\xi_2$ are eigenvalues of $\sigma_2$ with respect to $\sigma_1$.

Proof. By Lemma 4.1 the mapping $\pi$ is well-defined. Next we show that $\xi_1 \geq \xi_2$ are eigenvalues of $\sigma_1$ with respect to $\sigma_2$. Note that the characteristic polynomial of $\sigma_2$ with respect to $\sigma_1$ is

$$p(\ell) := \det(\sigma_2 - \ell \sigma_1) = \|\sigma_1\|^2\ell^2 - 2(\sigma_1 \cdot \sigma_2)\ell - \|\sigma_2\|^2.$$ (13)

One then can easily check that $\xi_1$ and $\xi_2$ are indeed the roots of $p(\ell)$. Next we show that $\pi$ is, moreover, onto. Let the real numbers $r_1 \geq r_2 \geq 0$ be given. Set $\lambda := \frac{r_1 + r_2}{2}$. Choose a vector $w \in W$ such that $\|w\| = \frac{r_1 - r_2}{2}$. Let $\sigma = (\lambda, w)$ and $e = (1, 0)$. Then $\pi((\sigma, e)) = (r_1, r_2)$ as desired. It is clear that $\pi$ is invariant under the action of $G$. Suppose $\pi(\sigma_1, \sigma_2) = \pi(\sigma'_1, \sigma'_2)$. Since $G$ acts transitively on $\mathcal{L}$ and
\( \pi \) is invariant under the action of \( G \), without loss of generality, we may assume that \( \sigma_2 = \sigma'_2 = e \). We have
\[
\pi(\sigma_1, e) = \pi(\sigma'_1, e).
\]

Eq. (13) implies that the characteristic polynomials \( p_1(\ell) := \|\sigma_1\|_2^2 - 2(\sigma_1, e)\ell - 1 \) and \( p_2(\ell) := \|\sigma'_1\|_2^2 - 2(\sigma'_1, e)\ell - 1 \) are identical. Therefore, if \( \sigma_1 = (\lambda, w) \) and \( \sigma'_1 = (\lambda', u) \), then
\[
\lambda = \sigma_1 \cdot e = \sigma'_1 \cdot e = \lambda',
\]
\[
\lambda^2 - \|w\|^2 = \|\sigma_1\|^2 = \|\sigma'_1\|^2 = \lambda'^2 - \|u\|^2.
\]

Thus \( (\sigma_1, e) \) and \( (\sigma'_1, e) \) are in the same \( G \)-orbit. \( \Box \)

### 4.2. Testing the hypothesis (T2)

Now by using the maximal invariant obtained in Proposition 4.1 we can test the hypothesis (T2) as follows.

**Theorem 4.1.** For the observation \((\tau_1, \tau_2) \in \mathcal{L} \times \mathcal{L}\), the **ML** estimator of \( \sigma \) under \( H_0 \) is
\[
\hat{\sigma}(\tau_1, \tau_2) := \frac{\tau_1 + \tau_2}{2},
\]
and the **LR** statistic for testing hypothesis (T2) is
\[
\left( 16 \prod_{j=1}^2 \frac{\xi_j}{(1 + \xi_j)^2} \right)^{\eta},
\]
where \( \xi_1 > \xi_2 \) are eigenvalues of \( \tau_2 \) with respect to \( \tau_1 \). Furthermore, under the hypothesis \( H_0 \) the statistics \( \hat{\sigma}(\tau_1, \tau_2) \) and \( \pi(\tau_1, \tau_2) = (\xi_1, \xi_2) \) are independently distributed, \( \hat{\sigma}(\tau_1, \tau_2) \sim \mathcal{W}_{\eta, \sigma}^\mathcal{L} \) and the density of \( \pi(\tau_1, \tau_2) = (\xi_1, \xi_2) \) is given by
\[
\frac{(2\pi)^{n-2}}{B_{\mathcal{L}}(\eta, \eta)\Gamma_{\mathcal{L}}(n-2)} (\xi_1 - \xi_2)^2 \left( \prod_{j=1}^2 \xi_j \right)^{\eta - \frac{n}{2}} \left( \prod_{j=1}^2 (1 + \xi_j) \right)^{2\eta}.
\]

**Proof.** This follows from Proposition 4.1 and Theorem 6.1 in [3] when the irreducible cone \( \Omega \) is the Lorentz cone \( \mathcal{L} \). \( \Box \)

We should mention that the proof of Theorem 6.1 [3], which simultaneously applies to all five types of irreducible cones, heavily rests on the analysis of simple Euclidean Jordan algebras.
Remark 4.1. Recall that in the classical multivariate statistics, the Bartlett’s test is testing

\[ H_0 : \Sigma = \sigma^2 I_n \quad \text{vs.} \quad H : \Sigma \neq \sigma^2 I_n, \quad (14) \]

for a Gaussian model, where the sample space is \( \mathbb{R}^n \), the distributions are multivariate normal distribution \( \mathcal{N}_n(0, \Sigma) \) and the parameter space is \( \mathcal{P}_D_n(\mathbb{R}) \). Therefore the Bartlett’s test is testing whether \( n \) univariate Gaussian distributions are independent and have the same variance \( \sigma \).

5. Closing Remarks

In this paper we have shown how maximal invariant statistics can be derived and used for testing two specific invariant statistical hypotheses for Lorentz Wishart models. Analogs of such hypotheses have been already studied for real, complex and quaternion type Wishart models [1]. An interesting topic of future research, which its analog has been studied in [1], is testing the hypothesis that the scale parameter \( \Sigma \) of the real Wishart distribution has a Lorentz structure. To clarify what is meant by a Lorentz type we note that every Lorenz cone \( \mathcal{L} \) is isomorphic to a subcone of \( \mathcal{P}D_n(\mathbb{R}) \) for a suitable \( n \) (see [6], [8] for detail), which means that there exist a linear injection \( \rho : \mathbb{R} \times W \to \mathbb{R}^{n \times n} \) such that \( \rho(\lambda, w) \) is positive definite, for each \( (\lambda, w) \in \mathcal{L} \). Therefore testing whether \( \Sigma \) has a Lorentz structure requires to show that \( \Sigma = \rho(\lambda, w) \) for some \( (\lambda, w) \in \mathcal{L} \).

References

[1] S. Andersson, Hans K. Brons, and Soren T. Jensen, Distribution of eigenvalues in multivariate statistical analysis, Ann. Statist., 11 (1983), pp. 392–415.

[2] S. A. Andersson and G. G. Wojnar, Wishart distributions on homogeneous cones, J. Theor. Probab., 17 (2004), pp: 781–818.

[3] E. Ben-David, Some hypothesis tests for Wishart models on symmetric cones, Contemp. Math., 516 (2010), pp. 327345.

[4] E. Ben-David, Some hypothesis tests for Wishart models on symmetric cones, Thesis (Ph.D.)–Indiana University (2008).

[5] M. Casalis and G. Letac, The Lukacs-Olkin-Rubin characterization of Wishart distributions on symmetric cones, Ann. Statist. 24 (1996), pp. 763–786.
[6] Jacques Faraut and Adam Korany, *Analysis on symmetric cones*, Oxford University Press, (1994).

[7] P. Feinsilver, J. Kocik, and M. Giering, *Canonical variables and analysis on so(n, 2)*, J. Phys. A: Math. Gen., 34 (2001), pp. 2367–2376.

[8] Soren T. Jensen, *Covariance hypothesis which are linear in both the covariance and the inverse covariance*, Ann. Statist., 6 (1988), pp. 302–322.

[9] Yoshihiko Konno, *Estimation of normal covariance matrices parametrized by irreducible symmetric cones under steins loss*, J. Multivar. Anal., 98 (2007), pp. 295–316.

[10] G. Letac and H. Massam, *All invariant moments of the Wishart distribution*, Scand. J. Stat., 31 (2004), pp. 295–318.

[11] Robb J. Muirhead, *Aspects Of Multivariate Statistical Theory*, Wiley, New York, (1982).