ON ONE PROBLEM OF VISCOELASTIC FLUID DYNAMICS WITH MEMORY ON AN INFINITE TIME INTERVAL

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Abstract. In the present paper we establish the existence of weak solutions of one boundary value problem for one model of a viscoelastic fluid with memory along the trajectories of the velocity field on an infinite time interval. We use solvability of related approximating initial-boundary value problems on finite time intervals and responding pass to the limit.

1. Introduction. Let \( Q = (\mathbb{R}^n, T] \times \Omega, \) where \( T > 0, \) be a bounded domain with boundary \( \partial \Omega \in C^2. \) We consider the problem

\[
\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \mu_0 \Delta v - \mu_1 \text{Div} \int_{-\infty}^{t} \exp \left( (s-t)/\lambda \right) \mathcal{E}(v)(s, z(s; t, x)) ds + \nabla p = f(t, x), \quad (t, x) \in Q; \tag{1}
\]

\[
\text{div} v(t, x) = 0, \quad (t, x) \in Q; \tag{2}
\]

\[
z(\tau; t, x) = x + \int_{t}^{\tau} v(s, z(s; t, x)) ds, \quad t, \tau \in (-\infty, T], \quad x \in \Omega; \tag{3}
\]

\[
v(t, x)|_{\Gamma} = 0, \quad (t, x) \in \Gamma = (-\infty, T] \times \partial \Omega. \tag{4}
\]

Here \( v(t, x) = (v_1, \ldots, v_n) \) denotes the velocity, \( p(t, x) \) is the pressure, \( f(t, x) = (f_1, \ldots, f_n) \) denotes the given external body forces, \( \mathcal{E}(v) = \{\mathcal{E}_{ij}(v)\}_{i,j=1}^{n} \) is the strain rate tensor, which is the matrix with components \( \mathcal{E}_{ij}(v) = \frac{1}{2} (\partial v_i/\partial x_j + \partial v_j/\partial x_i), \) \( \mu_0, \lambda > 0, \mu_1 \geq 0 \) are some constants and \( z(\tau; t, x) \) is the solution to the Cauchy problem (in integral form) for the system of ordinary differential equations (3). The sign Div stands for the divergence of a matrix function, i.e. the vector, coordinates of which are divergences of the rows of the matrix.

The problem (1)-(4) describes the motion of a fluid corresponding to the Jeffreys-Oldroyd constitutive law (see [12])

\[
(1 + \lambda D)\sigma = 2\nu(1 + \kappa \nu^{-1} D)\mathcal{E}, \quad \lambda, \kappa, \nu > 0, \tag{5}
\]

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where $D$ means the total derivative with respect to time, $\sigma$ is the deviator of the stress tensor, $E$ is the strain rate tensor, $\mu_0 = 2\kappa/\lambda$ and $\mu_1 = 2(\nu - \kappa)/\lambda \geq 0$. Details can be found in [19].

The presence of the integral term in (1) means that the memory on the stresses is taken into account. Various models with memory was appeared and studied in many papers (see e.g. [1], [3]-[6], [13]-[14], [18]). But as usual, mathematical statements of problems took into account the contribution of memory on a finite time interval but not along trajectories of particles (see e.g. [1], [13]). In this regard, the system (1)-(4) turned out to be without the equation (3) and became much more easier. In the case of a memory along trajectories of particles on a finite time interval problem (1)-(4) is considered in [20], where due to regularization of the velocity field $v$ in (3) its weak solvability is established. A regularization of $v$ provided good properties of the solution $z$ to equation (3) and allowed to prove the solvability of the Cauchy problem (3) in the classical sense. Recent results on the solvability of the Cauchy problem in the class of regular Lagrangian flows (briefly RLF) (see e.g. [2], [7]-[8]) allow to avoid the regularization of (3) and establish the weak solvability of (1)-(4) on a finite time interval (see [21]).

Problems with memory along trajectories on the half-line $(−∞, T]$ occur for many models of viscoelastic fluids motion (see e.g. [3]-[4], [14]), but theorems on the existence of solutions of these problems for infinite intervals $(−∞, T]$ with memory along the trajectories of velocity field are unknown to us.

The goal of the present paper is to establish a weak solvability of problem with a memory along the trajectories (1)-(4) without the requirement of a regularization of the velocity field.

Let us note, that we consider the case of exponential memory kernels, that are typically for Oldroyd type models (see [19], Ch.1). But there are various models with other memory kernels of another types of decay (see e.g. [6]). However, the investigation of such models requires particular analysis.

The paper is organized as follows. In Section 2 we recall some auxiliary results and formulate the main result. Section 3 is devoted to the proof of the main result and it is divided into several Subsections. In Subsection 3.1 we construct a series of approximative problems. In Subsection 3.2 uniform estimates of approximating solutions on the half-line are given. In Subsection 3.3 we find the limit of approximating solutions. In Subsections 3.4 and 3.5 we investigate the properties of the limit function. Finally, Subsection 3.6 completes the proof of the main result.

Constants in inequalities and chains of inequalities which do not depend on significant parameters are denoted by $M$.  

2. Preliminaries and main result. We will need Hilbert spaces $H$ and $V$ (see [16], Chapter I, Section 1.4) of divergence free vector functions. Here $V = \{v \in W^2_2(\Omega)^n : v|\Gamma = 0, \text{div} v = 0\}$ and $H$ is the closure of $V$ with respect to the norm of the space $L^2_2(\Omega)^n$. Let $V^{-1}$ be the space conjugate to $V$.

Norms in the spaces $H$ and $L^2_2(\Omega)^n$ we denote by $|\cdot|_0$, in $V$ by $|\cdot|_1$ and in the space $W^\beta_2(\Omega)$ for $\beta \in R^1$ by $|\cdot|_\beta$, respectively. The norms in $L^2_2(−∞, T; H)$ and $L^2_2(−∞, T; L^2_2(\Omega)^n)$ are defined by the following formulae

$$
\|v\|_{L^2_2(−∞, T; H)} = \left( \int_{−∞}^{T} \|v(t, \cdot)\|^2_0 \, dt \right)^{1/2}, \quad \|v\|_{L^2_2(−∞, T; L^2_2(\Omega)^n)} = \left( \int_{−∞}^{T} |v(t, \cdot)|^2_0 \, dt \right)^{1/2}
$$
and denoted by $\| \cdot \|_0$. The norms
\[
\|v\|_{L^2(-\infty, T; V)} = \left( \int_{-\infty}^{T} |v(t, \cdot)|^2 dt \right)^{1/2}, \quad \|v\|_{L^2(-\infty, T; W^{1}_2(\Omega)^n)} = \left( \int_{-\infty}^{T} |v(t, \cdot)|^2 dt \right)^{1/2}
\]
in $L^2(-\infty, T; V)$ and $L^2(-\infty, T; W^{1}_2(\Omega)^n)$ we denote by $\| \cdot \|_{0,1}$. The norm in the space $L^2(-\infty, T; V^{-1})$ is denoted by $\| \cdot \|_{0,-1}$.

The sign $\langle \cdot, \cdot \rangle$ stands for the scalar product in Hilbert spaces $L^2(\Omega)$, $H$, $L^2(\Omega)^n$, $L^2(\Omega)^{n \times n}$. It is clear from the context in which of them the scalar product is taken. The action of a functional $f \in V^{-1}$ on an element $\varphi \in V$ is denoted by $(f, \varphi)$.

If $v \in L^1(0, T; C^1(\Omega))$ and $v|_{\partial \Omega} = 0$ the problem (3) is nonlocal uniquely solvable in the classical sense (see [13]). However, in the case of only integrable with respect to $x$ derivatives of $v$ the situation is much more complicated, and a more general concept of solution to the problem (3) is required, namely, the concept of regular Lagrangian flows.

Recall some facts about RLF (see [7]-[8]).

**Definition 2.1.** A function $z(\tau; t, x)\colon[0, T] \times [0, T] \times \overline{\Omega} \to \mathbb{R}^n$ is called the regular Lagrangian flow (RLF) associated to $v$ if:

- for a.e. $x$ and any $t \in [0, T]$ the function $\gamma(\tau) = z(\tau; t, x)$ is absolutely continuous and satisfies the equation
  \[
  z(\tau; t, x) = x + \int_{t}^{\tau} v(s, z(s; t, x)) \, ds, \quad t, \tau \in [0, T];
  \]
  (6)

- $m(z(\tau; t, B)) = m(B)$ for any $t, \tau \in [0, T]$ and every Borel set $B \subset \overline{\Omega}$;

- for $t_i \in [0, T]$, $i = 1, 2, 3$ and a.e. $x \in \overline{\Omega}$ the following relation is valid:
  \[
  z(t_3; t_1, x) = z(t_3; t_2, z(t_2; t_1, x)).
  \]

(7)

Here $m$ is the Lebesgue measure in $\mathbb{R}^n$.

Above the definition of RLF is given in the particular case of a bounded domain $\Omega$ and a divergence free field $v$.

Let us mention some results about RLF.

Let $D = [0, T] \times [0, T]$ and $L$ be the set of measurable on $\Omega$ functions considered as a metric space with the metric
\[
d(f, g) = \int_{\Omega} |f(t, x) - g(t, x)|(1 + |f(t, x) - g(t, x)|)^{-1} dt \, dx.
\]

**Theorem 2.2** (see [8]). Let $v \in L^1(0, T; W^{1}_p(\Omega)^n)$, $1 \leq p \leq +\infty$, $\text{div} \, v(t, x) = 0$ and $v|_{\partial \Omega} = 0$. Then there exists a unique RLF $z \in C(D; L^n)$ associated to $v$.

In addition, $z(s; t, \cdot) \in W^{1}_1(\Omega)$, $z(\tau; t, \overline{\Omega}) \subset \overline{\Omega}$ (except on a null set) and
\[
dz(\tau; t, x)/d\tau = v(\tau, z(\tau; t, x)), \quad t, \tau \in [0, T], \quad \text{a.e.} \ x \in \Omega.
\]

**Theorem 2.3.** Let $v, v^m \in L^1(0, T; W^{1}_p(\Omega)^n)$, $m = 1, 2, \ldots$ for some $p > 1$. Let $\text{div} \, v = 0$, $\text{div} \, v^m = 0$, $v^m|_{\partial \Omega} = 0$, $v|_{\partial \Omega} = 0$. Let inequalities
\[
\|v\|_{L^1(0, T; L^p(\Omega)^n)} + \|v\|_{L^1(0, T; L^1(\Omega)^n)} \leq M,
\]
\[
\|v^m\|_{L^1(0, T; L^p(\Omega)^n)} + \|v^m\|_{L^1(0, T; L^1(\Omega)^n)} \leq M
\]
be fulfilled. Let the sequence $v^m$ converges to $v$ in $L_1(Q_T)^n$ as $m \to +\infty$. Let $z^m(\tau; t, x)$ and $z(\tau; t, x)$ be RLF associated to $v^m$ and $v$, respectively. Then the sequence of $z^m$ converges to $z$ in Lebesgue measure in $[0, T] \times \Omega$ for $t \in [0, T]$.

Here $v_x$ is the Jacobian matrix of a vector-function $v$.

In a more general form this result is given in [7], Corollaries 3.6, 3.7 and 3.9. Same results hold if we replace $[0, T]$ by any interval $[r, T] \subset (0, \infty)$.

We introduce the following functional spaces

$W_1 = \{ v : v \in L_2(-\infty, T; V) \cap L_\infty(-\infty, T; H), \; dv/dt \in L_2(-\infty, T; V^{-1}) \}, \; n = 2,$

$W_2 = \{ v : v \in L_2(-\infty, T; V) \cap L_\infty(-\infty, T; H), \; dv/dt \in L_{4,1,loc}(\infty, T; V^{-1}) \}, \; n = 3.$

Let $W = W_1$ for $n = 2$ and $W = W_2$ for $n = 3$.

**Definition 2.4.** Let $f \in L_2(-\infty, T; V^{-1})$. A weak solution of the problem (1)–(4) is a function $v \in W$, satisfying the identity

$$
\frac{d}{dt}(v, \varphi) - \sum_{i=1}^{n} (v_i v, \partial \varphi/\partial x_i) + \mu_0(E(v), E(\varphi))
+ \mu_1 \int_{-\infty}^{t} \exp((s-t)/\lambda) E(v)(s, z(s; t, x)) ds, E(\varphi) = \langle f, \varphi \rangle
$$

(8)

for any $\varphi \in V$ and a.e. $t \in (0, T)$. Here $z$ is the RLF associated to $v$.

The scalar products in the third and fourth terms of (8) are taken in $L_2(\Omega)^{n \times n}$.

Let us formulate the main result.

**Theorem 2.5.** Given $f \in L_2(\infty, T; V^{-1})$, the problem (1)–(4) has at least one weak solution.

3. **Proof of theorem 2.5.** The proof of Theorem 2.5 we will carry out in several steps.

3.1. **Approximative problems.** Consider the family of regularized initial-boundary value problems

$$
\partial v^m/\partial t + K_m(v^m) - \mu_0 \Delta v^m - \mu_1 \text{Div} \int_{-m}^{t} \exp((s-t)/\lambda) E(v^m)(s, z^m(s; t, x)) ds
+ \nabla p^m = f^m, \quad (t, x) \in Q_m = [-m, T] \times \Omega; \quad \text{div } v^m = 0, \quad (t, x) \in Q_m; \quad (9)
$$

$$
z^m(\tau; t, x) = x + \int_{t}^{\tau} \dot{v}^m(s, z^m(s; t, x)) ds, x \in \Omega, \quad t, \tau \in [-m, T]; \quad (10)
$$

$$
v^m(-m, x) = 0, x \in \Omega; \quad v^m|_{\Gamma_m} = 0, \Gamma_m = [-m, T] \times \partial \Omega. \quad (11)
$$

Here $m = 1, 2, \ldots$, $f^m$ is the restriction of $f$ on $[-m, T] \times \Omega$,$

K_m(v) = \sum_{i=1}^{n} v_i(1 + m^{-1}|v|^2)^{-1}\partial v/\partial x_i.$

The motivation of introduction of the regularization operator $K_m$ in (9) is caused by the reason that $K_m(v) \in L_2(0, T; V^{-1})$ while $K(v) = \sum_{i=1}^{n} v_i \partial v/\partial x_i$ belongs, generally speaking, only to $L_1(0, T; V^{-1})$ for $v \in W$. The change of $K(v)$ by $K_m(v)$
in (1) provides the existence of solution to problem (9)-(11) in the class $W$, together with the a regularization (smoothing) $\tilde{v}^m$ of the field $v^m$ in (10).

The regularization of the velocity field in (10) is caused by the fact that the study of Cauchy problem (10) for $v^m \in W$ runs into difficulty because in this case the velocity field, generally speaking, doesn’t determine the trajectory of fluid particles. One possible way to avoid this situation (see [19], section 7.1) is regularization of the field $v^m$.

We will use the regularization operator $S_{1/m} : H \to C^1(\overline{\Omega})$ from [19], Section 7.7. Recall some properties of $S_{1/m}$.

Lemma 3.1. For operator $S_{1/m}$ the following properties hold:

$$\|S_{1/m}\|_{H^2} \leq M,$$

$$\|S_{1/m}\|_{C^1(\overline{\Omega})} \leq M \quad m = 1, 2, \ldots;$$

$$|S_{1/m}v - v|_0 \to 0 \text{ as } m \to 0 \text{ for any } v \in H. \quad (13)$$

Introduce the functional space

$$W(m) = L_2(\overline{\Omega}, T; V) \cap L_\infty(\overline{\Omega}, T; H) \cap W^2_m(\overline{\Omega}, T; V^{-1}).$$

Definition 3.2. Let $f^m \in L_2(\overline{\Omega}, T; V^{-1})$. We say that $v^m \in W(m)$ is a weak solution to problem (9)-(11) if the identity

$$\frac{d}{dt}(v^m, \varphi) - \sum_{i=1}^n (v_i^m(1 + m^{-1}|v^m|^2)^{-1}v^m, \partial\varphi/\partial x_i) + \mu_0(\mathcal{E}(v^m), \mathcal{E}(\varphi)) + \mu_1 \int_{-m}^t \exp((s-t)/\lambda)\mathcal{E}(v^m)(s, z^m(s; t, x)) \, ds, \mathcal{E}(\varphi) = \langle f^m, \varphi \rangle$$

holds for all $\varphi \in V$ and a.e. $t \in [-m, T]$ and the initial condition in (11) holds. Here $z^m$ is RLF associated to $\tilde{v}^m$.

Theorem 3.3 (see [20]). Given $f^m \in L_2(\overline{\Omega}, T; V^{-1})$, the problem (9)-(11) has at least one weak solution $v^m \in W(m)$ and inequalities hold:

$$\sup_{-m \leq t \leq T} |v^m(t, \cdot)|_0 + \|v^m\|_{L_2(\overline{\Omega}, T; V)} \leq M(m) \|f^m\|_{L_2(\overline{\Omega}, T; V^{-1})}, \quad (15)$$

$$\|v^m\|_{W^1_2(\overline{\Omega}, T; V^{-1})} \leq M(m)(1 + \|f^m\|_{L_2(\overline{\Omega}, T; V^{-1})} + \|f^m\|_{L_2(\overline{\Omega}, T; V^{-1})}^2). \quad (16)$$

Remark 1. Note that in [20] the problem (9)-(11) has been considered on the interval $[0, T]$. It is obvious, however, that the change of variable $s = T(m + t)/(m + T)^{-1}$, $t \in [-m, T]$, reduces problem (9)-(11) on the interval $[-m, T]$ to the problem on $[0, T]$. As to the constants $M(m)$ in inequalities (15)-(16), below we prove necessary estimates of the approximative solutions with independent of $m$ constant.

Note also that the solution $z^m(\tau; t, x)$ to the Cauchy problem (10) belongs to the space $C([-m, T] \times [-m, T], C^1(\overline{\Omega}))$. By virtue of Theorem 2.2 the solution $z^m(\tau; t, x)$ defines RLF associated to $\tilde{v}^m$.

Remark 2. Estimates (15)-(16) imply the embedding $W(m) \subset C([-m, T], V^{-1})$ and hence $v^m \in C([-m, T], V^{-1})$.

Setting $v^m(t, x) = 0$ and $f^m(t, x) = 0$ for $t \leq -m$, respectively we extend functions $v^m$, $\tilde{v}^m$ and $f^m$ on $(-\infty, T] \times \Omega$. Setting $z^m(\tau; t, x) \equiv x$ for $\tau, t \leq -m$ we extend $z^m(\tau; t, x)$ on $(-\infty, T] \times (-\infty, T] \times \overline{\Omega}$. Let us save for extended functions $f^m$, $z^m$ and $v^m$ the same notations. Observe that extended $z^m$ are RLFs associated
to extended $\tilde{v}^m$. It is clear that the extension of function $v^m$ satisfies the identity (3.2) by a.e. $t \in (-\infty, -T]$.

Theorem 3.3 implies following estimates for extended functions $v^m$:

$$\sup_{-\infty < t \leq T} |v^m(t, \cdot)|_0 + \|v^m\|_{0,1} \leq M(m)\|f\|_0, \quad (17)$$

$$\|v^m\|_{W^{1,2}(-\infty, T; V^{-1})} \leq M(m)(1 + \|f\|_{L_2(-m, T; V^{-1})} + \|f\|_{L_2(-m, T; V^{-1})}^2). \quad (18)$$

Below we will show that $v^m$ (with up to a subsequence) converges to $v$ weakly in $L_2(-\infty, T; V)$, *-weakly in $L_\infty(-\infty, T; H)$ and strongly in $L_2(-\infty, T; H)$. Using these facts we prove that the limit function $v$ is a weak solution of the problem (1)-(4).

Uniform on $m$ estimates similar to (17)-(18) play an important role for our proof.

3.2. Uniform on $m$ estimates of approximating solutions on the half-line.

**Lemma 3.4.** Let $f \in L_2(-\infty, T; V^{-1})$. Then for extended solutions $v^m$ of problems (9)-(11) the following uniform on $m$ estimate holds:

$$\sup_{-\infty < t \leq T} |v^m(t, \cdot)|_0 + \|v^m\|_{0,1} \leq M\|f\|_0. \quad (19)$$

**Proof of Lemma 3.4.** In the same way as in [20] we obtain from (9)

$$\frac{1}{2} \frac{d}{dt} |v^m(t, \cdot)|^2_0 + \mu_0 \mathcal{E}(v^m)(t, \cdot)^2_0$$

$$+ \mu_1 \left( \int_{-m}^t \exp \left( (s-t)/\lambda \right) \mathcal{E}(v^m)(s, z^m(s; t, x)) ds, \mathcal{E}(v^m)(t, x) \right)$$

$$= \frac{1}{2} \frac{d}{dt} |v^m(t, \cdot)|^2_0 + \mu_0 \mathcal{E}(v^m)(t, \cdot)^2_0$$

$$+ \mu_1 \left( \int_{-\infty}^t \exp \left( (s-t)/\lambda \right) \mathcal{E}(v^m)(s, z^m(s; t, x)) ds, \mathcal{E}(v^m)(t, x) \right)$$

$$= \langle f^m(t, x), v^m(t, x) \rangle. \quad (20)$$

Here we have in mind that $v^m = 0$ for $t \leq -m$.

Integrating (20) with respect to time over $(-\infty, t]$, we get

$$\frac{1}{2} |v^m(t, \cdot)|^2_0 + \mu_0 \int_{-\infty}^t \mathcal{E}(v^m)(s, \cdot)^2_0 ds$$

$$+ \mu_1 \int_{-\infty}^t \left( \int_{-\infty}^\tau \exp \left( (s-\tau)/\lambda \right) \mathcal{E}(v^m)(s, z^m(s; \tau, x)) ds, \mathcal{E}(v^m)(\tau, x) \right) d\tau$$

$$= \int_{-\infty}^t \langle f^m(s, x), v^m(s, x) \rangle ds. \quad (21)$$

For the right hand side of (21) we have for arbitrary $\varepsilon > 0$

$$\left| \int_{-\infty}^t \langle f^m(s, x), v^m(s, x) \rangle ds \right| \leq \int_{-\infty}^t |f^m(s, \cdot)|_{-1} v^m(s, \cdot) ds.$$
Denote the third term in the left hand side of (21) by

\[ \leq M_1(\varepsilon) \int_{-\infty}^{t} |f^m(s, \cdot)|^2 ds \]

\[ + \varepsilon \int_{-\infty}^{t} |v^m(s, \cdot)|^2 ds \leq M_1(\varepsilon) \int_{-\infty}^{t} |f(s, \cdot)|^2 ds + \varepsilon \int_{-\infty}^{t} |v^m(s, \cdot)|^2 ds. \]  \hspace{1cm} (22)

Let us transform \( K_3(t) \). Using the change of variables \( x = z^m(\tau; T, y) \) we have

\[ K_3(t) = \mu_1 \int_{-\infty}^{t} \left( \int_{-\infty}^{\tau} \exp \left( \frac{(s - \tau)}{\lambda} \right) \mathcal{E}(v^m)(s, z^m(s; \tau, x)) ds, \mathcal{E}(v^m)(\tau, x) \right) d\tau. \]  \hspace{1cm} (23)

Here \( A : B \) for matrices \( A = \{a_{ij}\} \) and \( B = \{b_{ij}\} \) means \( A : B = \sum_{i,j=1}^{n} a_{ij}b_{ij} \).

Note that in (24) it was used the fact that the Jacobian \( J(\tau; T, y) = \det z^m_y(\tau; T, y) \) of the change \( x = z^m(\tau; T, y) \) is equal to one due to divergence free \( v^m \). In fact, it is well known that \( \partial_\tau J(\tau; T, y) = (\text{div } v^m)(\tau, z^m(\tau; T, y))J(\tau; T, y) \). Consequently, \( J(\tau; T, y) = \exp((\text{div } v^m)(\tau, z^m(\tau; T, y)))J(T; T, y) \). Since \( \text{div } v^m = 0 \) and \( J(T; T, y) = \det z^m_y(T; T, y) = 1 \) we obtain the identity \( J(\tau; T, y) = 1 \).

It follows from (23) and (24) that

\[ K_3(t) = \mu_1 \int_{-\infty}^{t} \exp \left( \frac{(s - \tau)}{\lambda} \right) \mathcal{E}(v^m)(s, z^m(s; T, y)) ds \]  \hspace{1cm} (25)

Using (22), (25) and Korn’s inequality (see e.g. [11]) we get from (21)
Lemma 3.5. Let \( k > m, f \in L_2(-\infty, T; V^{-1}) \). Then for functions \( v^m \) the estimate

\[
\|dv^m/dt\|_{L^1(-k,T;V^{-1})} \leq C(k)(1 + \|f\|_{L^1(-k,T;V^{-1})} + \|f\|_{L^1(-k,T;V^{-1})}^2)
\]

holds with independent on \( m \) but dependent on \( k \) constant \( C(k) \).

Proof of Lemma 3.5. Since \( u(t, x) = v^m(t, x) \) is continuous with respect to \( t \) as function with values in \( V^{-1} \), it can be considered on \([-k, T]\) as a weak solution to the initial-boundary value problem

\[
\partial u/\partial t + K_m(u) - \mu_0 \Delta u - \mu_1 \text{Div} \int_{-m}^t \exp\left((s-t)/\lambda\right) \mathcal{E}(u)(s, z(s; t, x))\, ds
\]

\[
+ \nabla p = f^m; \quad \text{div} u = 0, \quad (t, x) \in Q_k = [-k, T] \times \Omega;
\]

\[
z(\tau; t, x) = x + \int_0^\tau \mathcal{u}(s, z(s; t, x))\, ds, x \in \Omega, \quad t, \tau \in [-k, T];
\]

\[
u(-k, x) = v^m(-k, x), \quad x \in \Omega; \quad u|\Gamma_k = 0, \quad \Gamma_k = [-k, T] \times \partial \Omega.
\]

From [20] there follows an estimate for \( u(t, x) \)

\[
\|du/dt\|_{L^1(-k,T;V^{-1})} \leq C_1(k)(1 + \|f^m\|_{L^1(-k,T;V^{-1})} + \|f^m\|_{L^1(-k,T;V^{-1})}^2 + \|v^m(-k, \cdot)\|_0)
\]

with depending on \( k \) constant \( C_1(k) \).

The assertion of Lemma 3.5 follows from this estimate, equality \( u(t, x) = v^m(t, x) \) and (19).

3.3. Passage to the limit. Estimate (19) means that the sequence \( v^m \) is bounded in \( L_2(-\infty, T; V) \) and \( L_\infty(-\infty, T; H) \). This implies the existence of function \( v \in L_2(-\infty, T; V) \cap L_\infty(-\infty, T; H) \) such that \( v^m \) (with up to a subsequence) converges to \( v \) weakly in \( L_2(-\infty, T; V) \) and *-weakly in \( L_\infty(-\infty, T; H) \). Using properties of lower limits of weakly and *-weakly converging sequences (see e.g. [17], Theorems 1 and 9, Chapter V, section 1) we get from estimates (19) and (27) the following inequality

\[
\sup_t |v(t, \cdot)|_0 + \|v\|_{0,1} \leq M \|f\|_0.
\]

In addition, estimates (19) and (27) entail (see e.g. [16], Chapter III, proof of Theorem 3.2) the convergence of \( v^m \) to \( v \) (with up to a subsequence) a.e. on \([-k, T] \times \Omega \) for any \( k > 0 \), and, therefore, on \((-\infty, T] \times \Omega \).

Let us show that the limit function \( v \) is a weak solution to problem (1)-(4).

Defined on \((-\infty, T]\) function \( v^m \) satisfies, obviously, the identity

\[
-\sum_{i=1}^n \int_{-\infty}^T (v_i^m(t, x)(1 + m^{-1}|v|^2)^{-1}v^m(t, x), \partial \varphi(x)/\partial x_i) \psi(t)\, dt
\]
Lemma 3.7. The sequence \( k,T \) have that

\[
\begin{align*}
&+ \mu_1 \int_{-\infty}^T \left( \int_{-\infty}^t \exp \left( (s - t)/\lambda \right) \mathcal{E}(v^m)(s, z^m(s; t, x)) \, ds, \mathcal{E}(\varphi)(x) \right) \psi(t) \, dt \\
&+ \int_{-\infty}^T (v^m(t, x), \varphi) \psi'(t) \, dt = \int_{-\infty}^T (f^m(t, x), \varphi(x)) \psi(t) \, dt
\end{align*}
\]

(29)

for any \( \varphi \in V \) and \( \psi \in C_0^\infty(-\infty, T) \).

Let \( \text{supp} \, \psi \subset [-k, T] \), where \( k > 0 \), so the outer integrals are in fact performed on interval \([-k, T] \).

It is easy to see that the sequence \( f^m \) converges to \( f \) in \( L_2(-\infty, T; H) \). From estimates (18) and (19) it follows (see e.g. [20]) that the sequence \( v^m \) converges (with up to subsequence) to \( v \) weakly in \( L_2(-\infty, T; V) \), \( * \)-weakly in \( L_\infty(-\infty, T; H) \), strongly in \( L_2(Q)^n \), a.e. on \( Q = (-\infty, T] \times \Omega \), and the sequence of derivatives \( dv^m/dt \) is bounded in the norm of the space \( L_1(-k, T; V^{-1}) \) and converges to \( dv/dt \) in the sense of distributions on \([k, T]\) for any \(-\infty < k < T\).

Next, we will need several Lemmas.

Lemma 3.6. The sequence \( \tilde{v}^m \) converges to \( v \) in \( L_2(-\infty, T; H) \).

Proof of Lemma 3.6. It is easy to see that \( \tilde{v}^m - v = I_1(m) + I_2(m) \), where

\[
I_1(m) = S_{1/m}(v^m - v), \quad I_2(m) = S_{1/m}v - v.
\]

From the first estimate (12) it follows that \( \| S_{1/m} \|_{L_2(-\infty, T; H)} \to L_2(-\infty, T; H) \leq M \). Then

\[
\| I_1(m) \|_0 \leq M \| v - v^m \|_0 \to 0 \quad \text{as} \quad m \to +\infty.
\]

For the proof that \( \| I_2(m) \|_0 \to 0 \) as \( m \to +\infty \) it suffices to show that

\[
\int_{-\infty}^T |S_{1/m}v(t, \cdot) - v(t, \cdot)|^2_0 \, dt \to 0 \quad \text{as} \quad m \to +\infty.
\]

(30)

But in virtue of (13) we have for integrand in (30)

\[
g(t) \equiv |\tilde{v}^m(t, \cdot) - v(t, \cdot)|_0 = |S_{1/m}v(t, \cdot) - v(t, \cdot)|_0 \to 0 \quad \text{as} \quad m \to +\infty.
\]

In addition, on the strength of the first inequality in (12) function \( g^2(t) \) satisfies the inequality \( g^2(t) \leq M|v(t, \cdot)|^2_0 \).

Hence, in virtue of the Lebesgue Theorem \( \| I_2(m) \|_0 \to 0 \) as \( m \to +\infty \).

The assertion of Lemma 3.6 follows from the fact that \( \| I_k(m) \|_0 \to 0 \) as \( m \to +\infty \) for \( k = 1, 2 \).

Consider the Cauchy problem (3) for the limit function \( v \). Since \( v \in W \), we have that \( v \) satisfies the conditions of Theorem 2.2 on any bounded interval \([-k, T] \), \(-\infty < -k \leq T \). Then from Theorem 2.2 and Remark 4.3 from [2] there follows the existence of associated to \( v \) RLF \( z(\tau; t, x) \) in \((\infty, T] \times (\infty, T] \times \overline{\Omega}) \).

Lemma 3.7. The sequence \( z^m(\tau; t, x) \) converges in \((\tau, x)\) Lebesgue measure on \([k, T] \times \Omega \) to \( z(\tau; t, x) \) for \( t \in [-k, T] \) and for any \( k \geq 0 \).
Proof of Lemma 3.7. By the second estimate from (12) we have the uniform inequality
\[ \|\tilde{v}_m\|_{L^1(\Omega)} \leq M|\tilde{v}_m|_0 \leq M|S_{1/m}v^m(t,\cdot)|_1 \leq M|v(t,\cdot)|_1. \]
Hence, from estimates (19) there follows a uniform on \(m\) estimate of norms \(\|v^m\|_{0,1}\) and, moreover, of norms \(\|v^m\|_{L^1(-k,T;L^1(\Omega))}\) and \(\|v^m\|_{L^1(\cdot;L^1(\Omega))}\).

Using now Theorem 2.3 we get the assertion of Lemma 3.7. \( \square \)

By virtue of Lemma VI.5.1, [10] from the statement of Lemma 3.7 it follows that the sequence \(z^m(\tau;t,x)\) converges (up to subsequence) to \(z(\tau;t,x)\) a.e. on \(Q(k,T) = \{(\tau,x) \in [-k,T] \times \Omega\} \) for any \(t \in [-k,T]\) where \(-k \in (-\infty,T)\).

3.4. Proof of identity (8) for the limit function \(v\). Let us prove an auxiliary integral identity for \(v\).

Lemma 3.8. The limit function \(v(t,x)\) satisfies the identity
\[
\int_{-\infty}^{T} (v(t,x),\varphi(x))\psi'(t) dt - \sum_{i=1}^{n} \int_{-\infty}^{T} (v_i(t,x)v(t,x),\partial\varphi(x)/\partial x_i)\psi(t) dt \\
+ \mu_0 \int_{-\infty}^{T} (E(v)(t,x),E(\varphi)(x))\psi(t) dt \\
+ \mu_1 \int_{-\infty}^{T} \int_{-\infty}^{t} \exp\left((s-t)/\lambda\right)E(v)(s,z(s;t,x)) ds, E(\varphi)(x))\psi(t) dt \\
= \int_{-\infty}^{T} (f(t,x),\varphi(x))\psi(t) dt \tag{31}
\]
for any \(\varphi \in V\) and \(\psi \in C_0^\infty(-\infty,T)\).

Proof of Lemma 3.8. First, let \(\varphi \in V\) be smooth. Introduce the notation for terms in the left hand side of (29):
\[
J_1^m = \int_{-\infty}^{T} (v^m(t,x),\varphi(x))\psi'(t) dt, \\
J_2^m = \sum_{i=1}^{n} \int_{-\infty}^{T} (v_i^m(t,x)(1 + m^{-1}|v^m|^2)^{-1}v^m(t,x),\partial\varphi(x)/\partial x_i)\psi(t) dt, \\
J_3^m = \mu_0 \int_{-\infty}^{T} (E(v^m)(t,x),E(\varphi)(x))\psi(t) dt, \\
J_4^m = \mu_1 \int_{-\infty}^{T} \int_{-\infty}^{t} \exp\left((s-t)/\lambda\right)E(v^m)(s,z^m(s;t,x)) ds, E(\varphi)(x))\psi(t) dt.
\]
Let the corresponding terms in the left hand side of (31) be
\[ J_1 = \int_{-\infty}^{T} (v(t,x), \varphi(x)) \psi'(t) \, dt \quad J_2 = \sum_{i=1}^{n} \int_{-\infty}^{T} (v_i(t,x) \varphi(x), \partial \varphi / \partial x_i) \psi(t) \, dt, \]
\[ J_3 = \mu_0 \int_{-\infty}^{T} (\mathcal{E}(v)(t,x), \mathcal{E}(\varphi)(x)) \psi(t) \, dt, \]
\[ J_4 = \mu_1 \int_{-\infty}^{T} \left( \int_{-\infty}^{t} \exp \left( (s-t)/\lambda \right) \mathcal{E}(v)(s,z(s;t,x)) \psi(s) \, ds, \mathcal{E}(\varphi)(x) \right) \psi(t) \, dt. \]

Here \( z \) is the RLF associated to \( v \).

**Proposition 1.** \( J_i^m \) converges to \( J_i \) for \( i = 1, 2, 3 \).

**Proof of Proposition 1.** The weak convergence of \( v^m \) to \( v \) in \( L_2(-\infty, T; V) \) implies that \( J_i^m \) converges to \( J_i \) for \( i = 1, 3 \).

Next, the sequence of functions \( v^m_i(t,x) v^m(t,x)(1 + m^{-1}v^m)^{-1} \) weakly converges to \( v_i(t,x) v(t,x) \) as \( m \to +\infty \) in \( L_2([t_1, t_2] \times \Omega) \) on a bounded interval \([t_1, t_2]\) (see [9]). Since the integration in \( J_2^m \) and \( J_2 \) is performed over the finite interval \([k,T] \supset \text{supp} \psi\), then in virtue of Lemma 2.2 from [9] there follows the convergence of \( J_2^m \) to \( J_2 \) as \( m \to +\infty \).

Proposition 1 is proved. \( \square \)

Consider the term \( J_4^m \). Obviously,
\[ J_4^m = \mu_1 \int_{-\infty}^{T} \left( \int_{-\infty}^{t} \exp \left( (s-t)/\lambda \right) \mathcal{E}(v^m)(s,z^m(s;t,x)) : \mathcal{E}(\varphi)(x) \, dx \, ds \right) \psi(t) \, dt, \]
\[ J_4 = \mu_1 \int_{-\infty}^{T} \left( \int_{-\infty}^{t} \exp \left( (s-t)/\lambda \right) \mathcal{E}(v)(s,z(s;t,x)) : \mathcal{E}(\varphi)(x) \, dx \, ds \right) \psi(t) \, dt. \]  

We will show that \( \lim_{m \to \infty} J_4^m = J_4 \).

The integrand in the formula (33) contains the superposition of functions \( \mathcal{E}(v) = 2^{-1}(v_x + v_y^*) \) and \( z \). Here \( * \) means the transpose of a matrix. Let us show the summability of this superposition. First, let us establish

**Proposition 2.** Let \( h \in L_1(\Omega) \). Then \( h(z(s;t,\cdot)) \in L_1(\Omega) \) for \( s,t \in (-\infty, T] \) and
\[ \|h(z(s;t,\cdot))\|_{L_1(\Omega)} = \|h(\cdot)\|_{L_1(\Omega)}. \]  

**Proof of Proposition 2.** Let \( h_n(x), n = 1, 2, \ldots \) be a sequence of smooth functions on \( \Omega \) such that \( \|h_n(x) - h(x)\|_{L_1(\Omega)} \to 0 \) as \( n \to +\infty \) and \( h_n(x) \to h(x) \) on a dense subset \( \Omega_0 \subset \Omega \). The function \( h_n(z(s;t,x)) \) is measurable. Using the change of variable \( x = z(t; s, y) \) \( (y = z(s; t, x) \) is the reverse change) we obtain the equality
\[ \int_{\Omega} |h_n(z(s;t,x))| \, dx = \int_{\Omega} |h_n(y)| \, dy. \]

We will show that \( h_n(z(s;t,x)) \to h(z(s;t,x)) \) as \( n \to +\infty \) a.e. on \( \Omega \). Let \( \Omega_1 = \{ x : x = z(t; s, y), y \in \Omega_0 \} \). The definition of RLF yields \( \overline{\Omega}_1 = \Omega \). By (7) we have
for $x \in \Omega_1$

$$h_n(z(s; t, x)) = h_n(z(s; t, z(s; t, y))) = h_n(y) \text{ for } y \in \Omega_0.$$ 

It follows that $h_n(z(s; t, x)) = h_n(y) \rightarrow h(y) = h(z(s; t, x))$ on the dense set $\Omega_1$.

Thus, $h(z(s; t, x))$ is measurable. From (35) and the convergence of $h_n(x)$ to $h(x)$ in $L_1(\Omega)$ it follows that the sequence $\|h_n(z(s; t, x))\|_{L_1(\Omega)}$ is uniformly bounded. Using the Fatou theorem for $|h_n(z(s; t, x))|$ we get the summability of $h(z(s; t, x))$.

The relation (34) for a bounded function $h(x)$ follows from (35). In the case of arbitrary $h(x)$ it suffices to take a sequence of truncation functions $[h(z(s; t, x))]_k$ for $h(z(s; t, x))$ and using the equality $[h(z(s; t, x))]_k = [h]_k(z(s; t, x))$, to obtain (34) by means of the passage to the limit as $k \rightarrow +\infty$. Recall that $[h(x)]_k = h(x)$ for $|h(x)| \leq k$, $[h(x)]_k = k$ if $h(x) > k$ and $[h(x)]_k = -k$ if $h(x) < -k$.

Proposition 2 is proved.

The proof of the summability of $v_x(s, z(s; t, x))$ as a function of three variables is similar to the proof of Proposition 2 but it consists of some tedious arguments and we skip it.

**Proposition 3.** $J_4^4$ converges to $J_4$.

**Proof of Proposition 3.** It is easy to see that

$$J_4^m - J_4 = Z_1^m + Z_2^m,$$

where

$$Z_1^m = \mu_1 \int_{-k}^{T} \left( \int_{-\infty}^{t} \exp \left( (s - t)/\lambda \right) \right. \times \left( \mathcal{E}(v^m)(s, z^m(s; t, x)) - \mathcal{E}(v)(s, z^m(s; t, x)) \right) : \mathcal{E}(\varphi)(x) \, dx \, ds \psi(t) \, dt;$$

$$Z_2^m = \mu_1 \int_{-k}^{T} \left( \int_{-\infty}^{t} \exp \left( (s - t)/\lambda \right) \right. \times \left. \int_{\Omega} \left[ \mathcal{E}(v)(s, z^m(s; t, x)) - \mathcal{E}(v)(s, z(s; t, x)) \right] : \mathcal{E}(\varphi)(x) \, dx \, ds \psi(t) \, dt. \quad (37)$$

We will show that $\lim_{m \rightarrow \infty} Z_1^m = 0$.

Denote the integral over $\Omega$ in $Z_1^m$ by

$$I = \int_{\Omega} \left( \mathcal{E}(v^m)(s, z^m(s; t, x)) - \mathcal{E}(v)(s, z^m(s; t, x)) \right) : \mathcal{E}(\varphi)(x) \, dx.$$

Let’s make in $I$ the change of variable

$$x = z^m(t; s, y) \quad \text{(the reverse change is } y = z^m(s; t, x)). \quad (38)$$

Then $I = \int_{\Omega} \left[ \mathcal{E}(v^m)(s, y) - \mathcal{E}(v)(s, y) \right] : \mathcal{E}(\varphi)(z^m(t; s, y)) \, dy$. By changing the integration order this relation implies
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\[ Z_m^1 = \int_{-k}^{T} \int_{-\infty}^{t} \exp((s - t)/\lambda) \]
\[ \times \int_{\Omega} (\mathcal{E}(v^m)(s, y) - \mathcal{E}(v)(s, y)) : \mathcal{E}(\varphi)(z^m(t; s, y)) dy dtds \]
\[ = \int_{-k}^{T} \int_{-\infty}^{t} \exp((s - t)/\lambda) \]
\[ \times \int_{\Omega} (\mathcal{E}(v^m)(s, y) - \mathcal{E}(v)(s, y)) : \mathcal{E}(\varphi)(z^m(t; s, y)) dy dt ds \]
\[ + \int_{-k}^{T} \int_{s}^{T} \exp((s - t)/\lambda) \]
\[ \times \int_{\Omega} (\mathcal{E}(v^m)(s, y) - \mathcal{E}(v)(s, y)) : \mathcal{E}(\varphi)(z^m(t; s, y)) dy dt ds = Z_{11}^m + Z_{12}^m. \quad (39) \]

Note, that in what follows index \( j \) means that \( Z_{ij} \) is the \( j \)-th summand in \( Z_i \), index \( k \) means that \( Z_{ijk} \) is the \( k \)-th summand in \( Z_{ij} \), etc. It is obvious that

\[ Z_{12}^m = \int_{-k}^{T} \int_{s}^{T} \exp((s - t)/\lambda) \]
\[ \times \int_{\Omega} (\mathcal{E}(v^m)(s, y) - \mathcal{E}(v)(s, y)) : (\mathcal{E}(\varphi)(z^m(t; s, y)) - \mathcal{E}(\varphi)(z(t; s, y))) dy dt ds \]
\[ + \int_{-k}^{T} \int_{-k}^{s} \exp((s - t)/\lambda) \int_{\Omega} (\mathcal{E}(v^m)(s, y) - \mathcal{E}(v)(s, y)) : \mathcal{E}(\varphi)(z(t; s, y)) dy dt ds \]
\[ = Z_{121}^m + Z_{122}^m. \quad (40) \]

Using the boundedness of functions \( \psi \) and \( \exp((s - t)/\lambda) \) and applying the Cauchy-Schwartz and Hölder inequalities we get

\[ |Z_{121}^m|^2 \leq M \left( \int_{-k}^{T} \int_{s}^{T} \exp((s - t)/\lambda) \right. \]
\[ \left. \times |v^m(s, \cdot) - v(s, \cdot)|_1 |\varphi_x(z^m(t; s, \cdot)) - \varphi_x(z(t; s, \cdot))|_0 \psi(t) dt ds \right)^2 \]
\[ \leq M \left( \int_{-k}^{T} |v^m(s, \cdot) - v(s, \cdot)|_1 \int_{s}^{T} |\varphi_x(z^m(t; s, \cdot)) - \varphi_x(z(t; s, \cdot))|_0 dt ds \right)^2 \]
\[ \leq M \int_{-k}^{T} |v^m(s, \cdot) - v(s, \cdot)|_1^2 ds \int_{s}^{T} \left( \int_{-k}^{T} |\varphi_x(z^m(t; s, \cdot)) - \varphi_x(z(t; s, \cdot))|_0 dt \right)^2 ds \]
\[ \leq M ||v^m - v||_{L^2(\Omega)}^2 \int_{-\kappa}^{T} \int_{s}^{T} |\varphi_x(z^m(t; s, \cdot) - \varphi_x(z(t; s, \cdot)))|_0 dt^2 ds. \]  

(41)

Let us show that \( Z^m_{121} \to 0 \) as \( m \to +\infty \). Denote the last factor in (41) by

\[ \Psi_m(s) = \int_{-\kappa}^{T} \int_{s}^{T} |\varphi_x(z^m(t; s, \cdot) - \varphi_x(z(t; s, \cdot)))|_0 dt^2 ds. \]  

(42)

Represent it in the form

\[ \Psi_m(s) = \int_{-\kappa}^{T} g_m(s) ds, \quad g_m(s) = \int_{s}^{T} |\varphi_x(z^m(t; s, y) - \varphi_x(z(t; s, y)))|_0 dt^2. \]

Let us establish the convergence \( g_m(s) \to 0 \) as \( m \to +\infty \) for all \( s \in [-\kappa, T] \).

It is easy to see that

\[ g_m(s) = \left( \int_{s}^{T} |\varphi_x(z^m(t; s, \cdot) - \varphi_x(z(t; s, \cdot)))|_0 dt^2 \right)^{1/2}. \]  

(43)

Let \( \varepsilon > 0 \) be a small number. The continuity of function \( \varphi_x \) in \( \overline{\Omega} \) implies the existence of \( \delta_1(\varepsilon) > 0 \) such that for \( |x'' - x'| \leq \delta_1(\varepsilon) \) then

\[ |\varphi_x(x'') - \varphi_x(x')| \leq \varepsilon. \]  

(44)

Since the sequence \( z^m(t; s, y) \) converges to \( z(t; s, y) \) in \( (t, y) \) Lebesgue measure on \( [s, T] \times \Omega \) for \( s \in [-\kappa, T] \), then for \( \delta_1(\varepsilon) \) there exists such \( N = N(\delta_1(\varepsilon)) \) that as \( m \geq N \) the following inequality holds

\[ m(\{(t, y) : |z(t, s, y) - z^m(t, s, y)| \geq \delta_1(\varepsilon)\}) \leq \varepsilon. \]  

(45)

Let us denote

\[ Q(> \delta_1(\varepsilon)) = \{(t, y) \in Q : |z(t, s, y) - z^m(t, s, y)| > \delta_1(\varepsilon)\}, \]

\[ Q(\leq \delta_1(\varepsilon)) = \{(t, y) \in Q : |z(t, s, y) - z^m(t, s, y)| \leq \delta_1(\varepsilon)\}. \]

Then from (43) it follows that

\[ g_m(s) \leq M_1(G_3 + G_4), \]  

(46)

where

\[ G_3 = \int_{Q(> \delta_1(\varepsilon))} |\varphi_x(z(t, s, y)) - \varphi_x(z^m(t, s, y)))|_0^2 dy dt, \]

\[ G_4 = \int_{Q(\leq \delta_1(\varepsilon))} |\varphi_x(z(t, s, y)) - \varphi_x(z^m(t, s, y)))|_0^2 dy dt. \]

For \( G_4 \) by (44) we have \( |z(t, s, y) - z^m(t, s, y)| \leq \delta_1(\varepsilon) \), and, therefore,

\[ G_4 \leq \int_{Q(\leq \delta_1(\varepsilon))} \varepsilon^2 dy dt \leq M_2 \varepsilon^2. \]  

(47)
In virtue of (45) we have \( m(\Phi(>\delta_1(\varepsilon))) \leq \varepsilon \). Then
\[
G_3 \leq 2\|\varphi_\lambda\|_{C(\Omega)} \int_{\Phi(>\delta_1(\varepsilon))} dy dt \leq 2\|\varphi_\lambda\|_{C(\Omega)} \varepsilon. \tag{48}
\]
From estimates (46), (47) and (48) it follows that for \( m \),
\[
\text{the convergence } g_m(s) \to M_3 \varepsilon^{1/2}. \tag{49}
\]
The convergence \( g_m(s) \to 0 \) as \( m \to +\infty \) for all \( s \in [-k, T] \) is established. Moreover, \( g_m(s) \) is bounded because \( \varphi_\lambda(s) \) is smooth. Therefore, \( \Psi_m \to 0 \).

The first factor in the right hand side of (41) is bounded w.r.t. \( m \) due to the boundedness of \( \|v^m\|_{0,1} \) and for the second factor the convergence \( \Psi_m \to 0 \) as \( m \to +\infty \) holds.

Thus, (41) and (42) imply \( Z_{121}^m \to 0 \) as \( m \to +\infty \).

The convergence \( Z_{122}^m \to 0 \) as \( m \to +\infty \) follows from the weak convergence of \( v^m \to v \) in \( L_3(-\infty, T; V) \). From \( Z_{121}^m \to 0 \) and \( Z_{122}^m \to 0 \) as \( m \to +\infty \) and (40) we get \( Z_{12}^m \to 0 \) as \( m \to +\infty \).

Consider \( Z_{11}^m \). It is easy to see that for arbitrary \(-\infty < R < -k\)
\[
Z_{11}^m = \int_{-\infty}^{R} \int_{-k}^{T} \exp((s-t)/\lambda) \times \int_{\Omega} (E(v^m)(s,y) - E(v)(s,y)) : E(\varphi)(z^m(t,s,y)) dy \psi(t) dt ds
\]
\[+ \int_{R}^{T} \int_{-k}^{T} \exp((s-t)/\lambda) \times \int_{\Omega} (E(v^m)(s,y) - E(v)(s,y)) : E(\varphi)(z^m(t,s,y)) dy \psi(t) dt ds = Z_{111}^m + Z_{112}^m. \tag{50}
\]

Using the boundedness of \( \varphi_\lambda \) and \( \psi \), we get
\[
|Z_{111}^m| \leq M \int_{-\infty}^{R} \int_{-k}^{T} \exp(-t/\lambda) \exp(s/\lambda)
\times |v^m(s,\cdot) - v(s,y)|_1 |\varphi(z^m(t,s,\cdot))|_1 dt ds
\]
\[\leq M \int_{-k}^{T} \exp(-t/\lambda) dt \int_{-\infty}^{R} \exp(s/\lambda) |v^m(s,\cdot) - v(s,y)|_1 ds
\]
\[\leq M \int_{-k}^{T} \exp(-t/\lambda) dt \int_{-\infty}^{R} \exp(2/\lambda) ds^{1/2} \int_{-\infty}^{R} |v^m(s,\cdot) - v(s,\cdot)|^2 ds^{1/2}
\]
\[\leq M \exp(R/\lambda) \|v^m - v\|_{0,1}. \tag{51}
\]

It is clear that for a sufficiently large \( |R| \) we can make \( |Z_{111}^m| \) small enough. The relation \( |Z_{112}^m| \to 0 \) as \( m \to +\infty \) is established similarly to the case of \( |Z_{12}^m| \).

Thus, we have proved that
\[
|Z_{1}^m| \to 0 \text{ as } m \to +\infty. \tag{52}
\]
Show that if \( m \to +\infty \) then
\[
|Z^m_2| \to 0. \tag{53}
\]

Let us estimate \( |Z^m_2| \). Let us approximate \( v(t, x) \) by some smooth and finite on \((-\infty, T) \times \Omega\) function \( \tilde{v} \), so that \( \|v - \tilde{v}\|_{0,1} \leq \varepsilon_2 \) and \( \tilde{v}(t, x) \equiv 0 \) for \( t < k_1 \), where \( k_1 < T \), and \( \varepsilon_2 > 0 \) is an arbitrarily small number.

Then
\[
|Z^m_2| \leq M(Z^m_{21} + Z^m_{22} + Z^m_{23}), \tag{54}
\]

where
\[
Z^m_{21} = \int_{-k}^T \int_{-\infty}^t \exp ((s-t)/\lambda) |v(s, z^m(s; t, x)) - \tilde{v}(s, z^m(s; t, x))|_1 \, ds \, dt,
\]
\[
Z^m_{22} = \int_{-k}^T \int_t^T \exp ((s-t)/\lambda) |\tilde{v}(s, z(s; t, x)) - \tilde{v}(s, z(s; t, x))|_1 \, ds \, dt,
\]
\[
Z^m_{23} = \int_{-k}^T \int_{-\infty}^t \exp ((s-t)/\lambda) |\tilde{v}(s, z(s; t, x)) - v(s, z(s; t, x))|_1 \, ds \, dt.
\]

Making the change of variable (38) in the integral that defines the norm \( |\cdot|_1 \) in \( Z^m_{21} \), we get
\[
|v(s, z^m(s; t, x)) - \tilde{v}(s, z^m(s; t, x))|_1 = |v(s, y) - \tilde{v}(s, y)|_1. \tag{55}
\]

Similarly, using the change of variable \( y = z(s; t, x) \) we get for \( Z^m_{23} \)
\[
|\tilde{v}(s, z(s; t, x)) - v(s, z(s; t, x))|_1 = |\tilde{v}(s, y) - v(s, y)|_1. \tag{56}
\]

From inequalities (55) and (56) it follows that
\[
Z^m_{21} + Z^m_{23} \leq M T \int_{-k}^T \int_{-\infty}^t \exp ((s-t)/\lambda) |v(s, \cdot) - \tilde{v}(s, \cdot)|_1 \, ds \, dt
\]
\[
\leq M \|v - \tilde{v}\|_{0,1} \leq M \varepsilon_2. \tag{57}
\]

Since \( \tilde{v} \) is compactly supported, then
\[
Z^m_{22} \leq M \int_{-k}^T \left( \int_{k_1}^t \exp ((s-t)/\lambda) \int_{\Omega} |\tilde{v}_x(s, z^m(s; t, x)) - \tilde{v}_x(s, z^m(s; t, x))|_1 \, dx \, ds \right) dt.
\]

Since \( z^m(s; t, x) \) converges a.e. to \( z^m(s; t, x) \) on \([k_1, 0] \times \Omega\) uniformly with respect to \( t \) and the function \( \tilde{v}_x(t, x) \) is smooth and bounded, we get the convergence (53) as \( m \to +\infty \) by the Lebesgue theorem. The assertion of Proposition 3 follows from (52) and (53).

Proposition 3 is proved.

Thus, by virtue of Propositions 1 and 3 it is possible to pass to the limit in each term in (29), that yields the identity (31) for any smooth \( \varphi \).

Let us establish identity (31) for arbitrary \( \varphi \in V \) and \( \psi \in C^\infty(-\infty, T) \). We rewrite (31) for a smooth \( \varphi \) in the form
\[
[G_1, \varphi] - [G_2, \varphi] = 0, \tag{58}
\]
where

\[
[G_1, \varphi] = \int_{-\infty}^{T} (v(t, x), \varphi') dt - \sum_{i=1}^{n} \int_{-\infty}^{T} (v_i(t, x)v(t, x), \partial \varphi(x)/\partial x_i) dt \\
+ \mu_0 \int_{-\infty}^{T} (\mathcal{E}(v)(t, x), \mathcal{E}(\varphi)(x)) \psi(t) dt \\
+ \mu_1 \int_{-\infty}^{T} (\int_{-\infty}^{t} \exp((s-t)/\lambda)\mathcal{E}(v)(s, z(s; t, x)) ds, \mathcal{E}(\varphi)(x)) \psi(t) dt,
\]

(59)

\[
[G_2, \varphi] = \int_{-\infty}^{T} (f(t, x), \varphi(x)) \psi(t) dt.
\]

(60)

Let us obtain estimates of $[G_i, \varphi], i = 1, 2$, for arbitrary $\varphi \in V$.

**Proposition 4.** Let $\varphi$ be smooth. Then

\[
|G_1, \varphi| \leq M|\varphi|_1, \quad |G_2, \varphi| \leq M|\varphi|_1.
\]

(61)

**Proof of Proposition 4.** We denote

\[
g(t) = -(f(t, \cdot), \varphi(\cdot)) + \sum_{i=1}^{n} (v_i(t, \cdot)v(t, \cdot), \partial \varphi(\cdot)/\partial x_i) + \mu_0 (\mathcal{E}(v)(t, \cdot), \mathcal{E}(\varphi)(\cdot)) \\
+ \mu_1 \left( \int_{-\infty}^{t} \mathcal{E}(v)(s, z(s; t, \cdot)) ds, \mathcal{E}(\varphi)(\cdot) \right) = \sum_{i=1}^{4} R_i.
\]

(62)

Here

\[
R_1 = -(f(t, \cdot), \varphi(\cdot)), \quad R_2 = \sum_{i=1}^{n} (v_i(t, \cdot)v(t, \cdot), \partial \varphi(\cdot)/\partial x_i),
\]

\[
R_3 = \mu_0 (\mathcal{E}(v)(t, \cdot), \mathcal{E}(\varphi)(\cdot)), \quad R_4 = \mu_1 \left( \int_{-\infty}^{t} \mathcal{E}(v)(s, z(s; t, \cdot)) ds, \mathcal{E}(\varphi)(\cdot) \right).
\]

Let us estimate $R_i$. It is easy to see that for any $-\infty < m_1 < m_2 \leq T$ the following inequality holds

\[
\int_{m_1}^{m_2} |R_1(t)| dt \leq \int_{m_1}^{m_2} |f(t, \cdot)| dt \leq M(m_1, m_2) \int_{m_1}^{m_2} |f(t, \cdot)| dt.
\]

(63)

The continuous embedding $W_2^1(\Omega) \subset L_4(\Omega)$ for $n = 2, 3$, inequalities

\[
\|u\|_{L_4(\Omega)} \leq 2^{1/4}|u_{01}|^{1/2}|u_{12}|^{1/2}, \quad n = 2,
\]

\[
\|u\|_{L_4(\Omega)} \leq M|u_{02}|^{1/4}|u_{12}|^{3/4}, \quad n = 3
\]
The elementary calculations yield
\[ \int_{m_1}^{m_2} |R_2(t)| \, dt \leq \sum_{i=1}^{n} \int_{m_1}^{m_2} |v_i(t, \cdot) v(t, \cdot)|_0 |\partial \varphi(\cdot)/\partial x|_0 \, dt \]
\[ \leq M \int_{m_1}^{m_2} |v(t, \cdot)|_1^{3/2} |v(t, \cdot)|_0^{1/2} \, dt |\varphi|_1. \]

The elementary calculations yield
\[ \int_{m_1}^{m_2} |R_2(t)| \, dt \leq M \int_{m_1}^{m_2} (|v(t, \cdot)|_2^2 + |v(t, \cdot)|_0^2) \, dt |\varphi|_1 \]
\[ \leq M(m_1, m_2)(\|v\|_{0,1}^2 + \sup_t |v(t, \cdot)|_0^2) |\varphi|_1 \leq M(m_1, m_2)\|f\|_0^2 |\varphi|_1. \] (64)

Estimate (28) implies for \( R_3(t) \)
\[ \int_{m_1}^{m_2} |R_3(t)| \, dt \leq M \int_{m_1}^{m_2} |v(t, \cdot)|_0^2 \, dt |\varphi|_1 \leq M \|f\|_0^2 |\varphi|_1. \] (65)

Finally, by the preceding arguments we have for \( R_4(t) \)
\[ \int_{m_1}^{m_2} |R_4(t)| \, dt \leq M \int_{m_1}^{m_2} \left| \int_{-\infty}^{t} \exp \left( (s-t)/\lambda \right) \mathcal{E}(v)(s, z(s; t, x)) \, ds \right|_0 |\varphi(\cdot)|_0 \, dt \]
\[ \leq M \int_{m_1}^{m_2} \left( \int_{-\infty}^{t} \exp \left( (s-t)/\lambda \right) \mathcal{E}(v)(s, z(s; t, x)) \, ds \right)_{0,1} \, dt |\varphi|_1 \]
\[ \leq M \int_{m_1}^{m_2} \left( \int_{-\infty}^{t} \exp \left( 2(s-t)/\lambda \right) \, ds \right)^{1/2} \left( \int_{-\infty}^{t} |v(s, y)|_2^2 \, ds \right)^{1/2} \, dt |\varphi|_1 \]
\[ \leq M(m_1, m_2)\|f\|_0 |\varphi|_1. \] (66)

Assuming \( \text{supp } \psi \subset [-k, T] \) and taking into account smoothness of \( \psi(t) \), relation (62) and estimates (63)-(66) we get (61).

Proposition 4 is proved. \( \square \)

Since the set of smooth functions is dense in \( V \), for \( \varphi \in V \) there exists a sequence of smooth functions \( \varphi^l \in V \) such that \( |\varphi^l - \varphi|_1 \to 0 \) as \( l \to +\infty \). By (58) for \( \varphi = \varphi^l \) we have
\[ [G_1, \varphi] - [G_2, \varphi] = [G_1, \varphi - \varphi'] - [G_2, \varphi - \varphi'] + [G_1, \varphi'] - [G_2, \varphi'] \]
\[ = [G_1, \varphi - \varphi'] - [G_2, \varphi - \varphi']. \]

This equality and (61) entail
\[ |[G_1, \varphi] - [G_2, \varphi]| \leq M |\varphi - \varphi'|_1. \] (67)
Taking into account inequality (67) and passing to the limit when \( l \to +\infty \) in (31) for \( \varphi = \varphi^l \) we get (58), or that is the same, the identity (31) for any \( \varphi \in V \).

In order to prove that \( v \) is a weak solution to problem (1)-(4) it is now enough to prove that \( v \in W \) and satisfies the identity (8).

First, let us prove the second assertion.

**Lemma 3.9.** The limit function \( v \) satisfies the identity (8).

The proof of Lemma 3.9 consists in the deduction of (8) from (31) and is a rather standard application of the following fact, formulated in Lemma 1.1, Chapter III from [16].

**Proposition 5.** Let \( X \) be a Banach space, \( u, g \in L_1(a, b; X) \), \( -\infty \leq a < b \leq +\infty \). Then the following statements are equivalent:

1) for any \( \psi \in C_0^\infty(a, b) \) the identity holds

\[
\int_a^b u(t)\psi'(t) \, dt = -\int_a^b g(t)\psi(t) \, dt \quad (\psi' = d\psi/dt);
\]

2) for each \( \eta \in X' \) (the space \( X' \) is adjoint to \( X \)) \( d\langle u, \eta \rangle/dt = \langle g, \eta \rangle \) takes place in the sense of distributions on \((a, b)\);

3) \( u(t) \) is differentiable by a.e. \( t \) as a function with values in \( X \) and \( du(t)/dt = g(t) \).

To complete the proof of Theorem 2.5 it remains to show that \( v \in W \).

**Lemma 3.10.** The limit function \( v \) belongs to the space \( W \).

3.5. **Proof of Lemma 3.10.** It is convenient to rewrite problem (1)-(4) in an operator form.

We introduce:

1) the functional on \( V \) and, hence, the operator

\[
A : V \to V^{-1}, \quad \langle A(u), h \rangle = \mu_0(\mathcal{E}(u), \mathcal{E}(h)) |_{L_2(\Omega)^n \times n}, \quad u, h \in V;
\]

2) the operator

\[
K : V \to V^{-1}, \quad \langle K(v), h \rangle = \sum_{i=1}^n (v_i v, \partial h_i/\partial x_j) |_{L_2(\Omega)}, \quad u, h \in V;
\]

3) for \( v \in L_2(-\infty, T; V) \), the associated to \( v \) RLF \( z \) and for every fixed \( t \in (-\infty, T) \) the functional on \( V \)

\[
\langle C(v, z)(t), h \rangle = \mu_1(\int_{-\infty}^t \exp((s - t)\lambda) \mathcal{E}(v)(s, z(s; t, x)) ds, \mathcal{E}(h)) |_{L_2(\Omega)^n \times n}.
\]

Let us establish properties of introduced maps.
Proposition 6. Let \( v \in L_2(-\infty, T; V) \cap L_{-\infty}(-\infty, T; H) \). Then

\[
\|A(v)\|_{0,-1} \leq M\|v\|_{0,1};
\]

(68)

\[
\|C(v, z)(t)\|_{0,-1} \leq M\|v\|_{0,1};
\]

(69)

\[
\|K(v)\|_{L_2(-\infty, T; V^{-1})} \leq M \sup_t |v(t, \cdot)|_0 \left( \int_{-\infty}^T |v(t, \cdot)|^2_1 dt \right)^{1/2} \quad \text{(by } n = 2); \]

(70)

\[
\|K(v)\|_{L_4(\Omega; V^{-1})} \leq M \sup_t |v(t, \cdot)|^{1/2}_0 \left( \int_{-\infty}^T |v(t, \cdot)|^3_1 dt \right)^{3/4} \quad \text{(by } n = 3). \]

(71)

Proof of Proposition 6. It is easy to see that for fixed \( u \in V \) and for every \( h \in V \)

\[
\langle (A(u), h) \rangle \leq M\|\langle \mathcal{E}(u), \mathcal{E}(h) \rangle \|_{L_2(\Omega) \times \Omega} \leq M\|u\|_1\|h\|_1.
\]

Therefore, \( |A(u)|_1 \leq M\|u\|_1 \). This easily yields (68).

By the standart arguments we get for \( C \)

\[
|C(v, z)(t)|_1 \leq M \int_{-\infty}^t \exp((s - t)\lambda)\mathcal{E}(v)(s, z(s; t, x)) \, ds|_0
\]

\[
\leq M \int_{-\infty}^t \exp((s - t)\lambda)|v_x(s, z(s; t, x))|_0 \, ds. \tag{72}
\]

Using the change of variable \( y = z(s; t, x) \), we have

\[
I = |v_x(s, z(s; t, x))|^2_0 = \int_\Omega |v_x(s, z(s; t, x))|^2 \, dx = \int_\Omega |v_x(s, y)|^2 \, dy = |v_x(s, \cdot)|^2_0.
\]

Hence from (72) using the change of variable \( \tau = s - t \) we obtain

\[
|C(v, z)(t)|_1 \leq M \int_{-\infty}^t \exp((s - t)\lambda)|v_x(s, \cdot)|_0 \, ds
\]

\[
\leq M \int_{-\infty}^t \exp((s - t)\lambda)|v(s, \cdot)|_1 \, ds = M \int_{-\infty}^0 \exp(\lambda\tau)|v(t + \tau, \cdot)|_1 \, d\tau. \tag{73}
\]

Then, we deduce from (73) by virtue of the Minkowski integral inequality

\[
\|C(v, z)(t)\|_{0,-1} \leq M \| \int_{-\infty}^0 \exp(\lambda\tau)|v(t + \tau, \cdot)|_1 \, d\tau \|_{L_2(-\infty, T]}
\]

\[
\leq M \int_{-\infty}^0 \exp(\lambda\tau)\|\|v(t + \tau, \cdot)\|_1 \|_{L_2(-\infty, T]} \, d\tau \leq M\|v\|_{0,1}.
\]

Inequality (69) is proved.

Consider \( K \). Let \( n = 2 \). Using inequality \( \|u\|_{L_4(\Omega)} \leq M\|u\|_0^{1/2}\|u\|_1^{1/2} \) for \( u \in V \) we get

\[
|K(v)|_1 \leq M \sum_{i=1}^n v_i v_i |0| \leq M\|v\|_{L_4(\Omega)}^2 \leq M\|v\|_0\|v\|_1, \quad v \in V. \tag{74}
\]
From inequality (74) it follows that
\[
\|K(v)\|_{L_2(-\infty, T; V^{-1})} = \left( \int_{-\infty}^{T} |K(v)|_{-1}^2 \, dt \right)^{1/2} \left( \int_{-\infty}^{T} |v(t, \cdot)|_0^2 \, dt \right)^{1/2} \\
\leq M \sup_t |v(t, \cdot)|_0 \left( \int_{-\infty}^{T} |v(t, \cdot)|_1^2 \, dt \right)^{1/2} \quad \text{if } n = 2.
\]

Inequality (70) is proved. Now let \( n = 3 \). Then, since \( \|v\|_{L_4(\Omega)} \leq M |v|_{1/4}^{1/4} |v|_1^{3/4} \) for \( n = 3 \), one has
\[
|K(v)|_{-1} \leq M \sum_{i=1}^n v_i |v|_0 \leq M \|v\|_{L_4(\Omega)}^2 \leq M |v|_{1/2}^{1/2} |v|_1^{3/2}. \tag{75}
\]

From inequality (75) it follows that
\[
\|K(v)\|_{L_{4/3}(-\infty, T; V^{-1})} = \left( \int_{-\infty}^{T} |K(v)|_{-1}^{4/3} \, dt \right)^{3/4} \leq \left( \int_{-\infty}^{T} |v(t, \cdot)|_0^{2/3} |v(t, \cdot)|_1^{2/3} \, dt \right)^{3/4} \\
\leq M \sup_t |v(t, \cdot)|_0^{1/2} \left( \int_{-\infty}^{T} |v(t, \cdot)|_1^2 \, dt \right)^{3/4} \quad \text{if } n = 3.
\]

Inequality (71) is proved. This completes the proof of Proposition 6. \( \square \)

**Proposition 7.** Function \( v \) satisfies the equation
\[
dv/dt + A(v) - K(v) + C(v, z) = f \quad \text{for a.e. } t \in (-\infty, T]. \tag{76}
\]
in the Hilbert space \( V^{-1} \).

**Proof of Proposition 7.** Use the statement of Proposition 5. From Lemma 3.9 it follows that \( v \) satisfies by a.e. \( t \) the identity (8). We rewrite it in the form
\[
d(v, \varphi)/dt = \langle \hat{g}, \varphi \rangle, \tag{77}
\]
where
\[
\hat{g} = f - A(v) + K(v) - C(v, z). \tag{78}
\]
Find out summability properties of the summands in the right hand side of (78).

From inequality (68) and estimate (28) it follows that
\[
\|A(v)\|_{0, -1} \leq M \|f\|_{0, -1}. \tag{79}
\]

From inequality (73) and estimate (28) it follows that
\[
\|C(v, z)\|_{0, -1} \leq M \int_{-\infty}^{t} \exp \left( (-t)/\lambda \right) |v(s, \cdot)|_1 \, ds \|_0 \leq M \|f\|_{0, -1}. \tag{80}
\]

From estimates (28), (70) (71) it follows that
\[
\|K(v)\|_{L_2(-\infty, T; V^{-1})} \leq M \|f\|_{0, -1}^{2} \quad \text{for } n = 2. \tag{81}
\]
\[
\|K(v)\|_{L_{4/3}(-\infty, T; V^{-1})} \leq M \|f\|_{0, -1}^{2} \quad \text{for } n = 3. \tag{82}
\]
From (78) and inequalities (79), (80), (81) it follows that \( \hat{g} \in L_2(-\infty,T;V^{-1}) \) for \( n = 2 \) and the following inequality

\[
\|\hat{g}\|_{0,-1} \leq M(\|f\|_{0,-1}^3 + \|f\|_{0,-1})
\]

holds.

Let \( n = 3 \). From (78) and inequalities (79), (80), (82) for an arbitrary natural number \( k \) there follows the validity of inequality

\[
\|\hat{g}\|_{L_{4/3}(k,T;V^{-1})} \leq \|f\|_{L_{4/3}(k,T;V^{-1})} + \|K(v)\|_{L_{4/3}(k,T;V^{-1})} + \|C(v,z)\|_{L_{4/3}(k,T;V^{-1})} + \|A(v)\|_{L_{4/3}(k,T;V^{-1})} + \|K(v)\|_{L_{4/3}(k,T;V^{-1})} \leq M(k)(\|f\|_{L_{2}(k,T;V^{-1})}^2 + \|f\|_{0,-1}).
\]

By the Proposition 5, from the summability of function \( \hat{g} \) there follows the existence of \( dv/dt \) as a function with values in \( V^{-1} \), the equality \( dv/dt = \hat{g} \) for a.e. \( t \in (-\infty,T] \) and, in virtue of (77) and (78), validity of equation (76).

Proposition 7 is established.

From equation (76) and estimates of the terms of the function \( \hat{g} \) it follows the validity of inequalities

\[
\|dv/dt\|_{L_2(-\infty,T;V^{-1})} \leq M(\|f\|_{0,-1}^2 + \|f\|_{0,-1}), \quad \text{for } n = 2,
\]

\[
\|dv/dt\|_{L_{4/3}(k,T;V^{-1})} \leq M(k)(\|f\|_{0,-1}^2 + \|f\|_{0,-1}), \quad k \in (-\infty,T), \quad \text{for } n = 3.
\]

The last inequalities and (28) means that \( v \in W \).

Lemma 3.10 is proved.

3.6. Conclusion of the proof of Theorem 2.5. Lemmas 3.9 and 3.10 imply the assertion of Theorem 2.5.

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