Bounce beyond Horndeski with GR asymptotics and $\gamma$-crossing

S. Mironov, $^{a,c}$ V. Rubakov$^{a,b}$ and V. Volkova$^{a,b}$

$^a$Institute for Nuclear Research of the Russian Academy of Sciences, 60th October Anniversary Prospect, 7a, 117312 Moscow, Russia
$^b$Department of Particle Physics and Cosmology, Physics Faculty, M.V. Lomonosov Moscow State University, Vorobjevy Gory, 119991 Moscow, Russia
$^c$Institute for Theoretical and Experimental Physics, Bolshaya Cheremushkinskaya, 25, 117218 Moscow, Russia
E-mail: sa.mironov.1@physics.msu.ru, rubakov@minus.inr.ac.ru, volkova.viktoriya@physics.msu.ru

Received August 27, 2018
Accepted October 16, 2018
Published October 26, 2018

Abstract. It is known that beyond Horndeski theory admits healthy bouncing cosmological solutions. However, the constructions proposed so far do not reduce to General Relativity (GR) in either infinite past or infinite future or both. The obstacle is so called $\gamma$-crossing, which off hand appears pathological. By working in the unitary gauge, we confirm the recent observation by Ijjas [1] that $\gamma$-crossing is, in fact, healthy. On this basis we construct a spatially flat, stable bouncing Universe solution whose asymptotic past and future are described by GR with conventional massless scalar field.

Keywords: alternatives to inflation, cosmological perturbation theory

ArXiv ePrint: 1807.08361
Contents

1 Introduction and summary  
2 Perturbations in Horndeski theory and beyond  
3 $\gamma$-crossing  
   3.1 Solutions for metric perturbations in the unitary gauge  
   3.2 No-go theorem and $\gamma$-crossing  
4 An example of the bounce with conventional asymptotics  
A Coefficients in quadratic action for perturbations  
B Einstein equations for background  
C Bounce: details of the reconstruction procedure

1 Introduction and summary

Horndeski theories [2–5] are the most general scalar-tensor theories whose equations of motion are second order despite the presence of higher derivatives in the Lagrangian. There is an extension of the general Horndeski theory, which is referred to as “beyond Horndeski” [6, 7]. The difference between Horndeski and beyond Horndeski theories is that in the latter, the equations of motion are third order but still with no Ostrogradsky instabilities arising. Further generalization is dubbed “DHOST” theories [8–10]. Horndeski and beyond Horndeski theories are widely used for constructing various cosmological solutions like cosmological bounce, Genesis, etc. [11–22]. Indeed, (beyond) Horndeski theories are capable of violating the Null Energy Condition without obvious pathologies (for a review see, e.g., ref. [23]). In this paper we concentrate on the classical bouncing solutions. Even though numerous examples of spatially flat bouncing solutions were suggested within the general Horndeski theory, it was shown in refs. [24–26] that bouncing solutions in this class of theories are plagued with gradient instabilities, which arise if one considers the entire evolution. Attempts to evade this no-go theorem within the general Horndeski theory result in either singularity or potential strong coupling. It was found, however, that going beyond Horndeski enables one to satisfy the stability conditions and obtain a complete, healthy, spatially flat bouncing solutions [27–30]. The only drawback left is that the suggested solutions do not have simple asymptotics. Namely, the bouncing solutions designed so far do not reduce to General Relativity (GR) in either the asymptotic past or future or both. This feature appears dissatisfying.

As discussed in detail in ref. [29], the obstacle to having GR in both asymptotic past and future is the so called $\gamma$-crossing. The latter phenomenon has to do with the quadratic action for scalar perturbations in the unitary gauge. The coefficients there involve the denominator,$^1$

\begin{itemize}
\item This denominator is denoted by $\Theta$ in refs. [26, 29], $\gamma$ in refs. [1, 22, 31], and $A_4$ in the current work.
\end{itemize}
and $\gamma$-crossing occurs when this denominator vanishes. Barring fine-tuning, the coefficients in the unitary gauge quadratic action diverge at $\gamma$-crossing.\footnote{It has been noted in ref. [31] that zero denominator corresponds to the interchange of the solution branches for the Hubble parameter in the Friedmann equation.} Until recently, this has been considered unacceptable, and bouncing solutions obtained so far avoided $\gamma$-crossing. The price to pay was the strong modification of gravity at early and/or late times.

The issue of $\gamma$-crossing has been recently discussed in ref. [1] from a new perspective. It has been shown that by choosing the Newtonian gauge one obtains the linearized equations for metric perturbations without any denominator. Hence, there is no problem with $\gamma$-crossing in the Newtonian gauge at all. This apparent discrepancy between the unitary and Newtonian gauges is puzzling. On the one hand, one might suspect that the stability analysis in terms of the unitary gauge set of variables cannot be carried out around $\gamma$-crossing, since the linearized equations become singular at this point. On the other hand, in gauge invariant theories like beyond Horndeski, gauge fixing should not affect the physics.

In this paper we show explicitly that despite the seeming problem with $\gamma$-crossing in the unitary gauge, the solutions for all the perturbation variables in this gauge are regular for any value of the denominator including zero. In other words, the singularities are present only in the linearized equations but not in their solutions. Hence, we safely allow for $\gamma$-crossing. This is in accordance with the general conclusion by Ijjas [1], even though we disagree with her claim that the unitary gauge is ill-defined at $\gamma$-crossing. As we further discuss in this paper, even though $\gamma$-crossing is a healthy phenomenon, it does not enable one to circumvent the above no-go theorem in the Horndeski theory.

Most importantly, healthy $\gamma$-crossing enables us to construct a complete, stable, spatially flat bouncing solution in beyond Horndeski theory whose past and future asymptotics are described by a theory of a conventional massless scalar field and GR. We give an explicit example of such a bouncing solution and check its stability during entire evolution.

This paper is organized as follows. We introduce the beyond Horndeski theory and give the basic formulas of the linearized perturbation theory in section 2. In section 3 we demonstrate that the solutions for linearized perturbations in the unitary gauge are indeed non-singular for any value of the denominator including zero, i.e., that $\gamma$-crossing is not pathological. We also revisit the no-go argument for the general Horndeski theory and stress that healthy $\gamma$-crossing does not help to evade the no-go theorem. In section 4 we present an explicit example of the healthy bouncing solution in beyond Horndeski theory, which connects two asymptotics with a massless scalar field and the conventional Einstein gravity.

2 Perturbations in Horndeski theory and beyond

In what follows we consider both the general Horndeski and beyond Horndeski cases. The Lagrangian of the beyond Horndeski theory has the form (mostly negative metric signature):

$$S = \int d^4 x \sqrt{-g} \left( L_2 + L_3 + L_4 + L_5 + L_{BH} \right),$$

$$L_2 = F(\pi, X),$$

$$L_3 = K(\pi, X) \Box \pi,$$

$$L_4 = -G_4(\pi, X) R + 2G_{4X}(\pi, X) \left[ (\Box \pi)^2 - \pi_{\mu
u} \pi^{\mu
u} \right],$$

$$L_5 = L_{BH}.$$
The scalar field perturbation is denoted by $\delta \pi$. Without loss of generality we partly use the gauge freedom and gauge away the longitudinal component $\partial_i \partial_j E$ from the very beginning. Then, the quadratic action for the scalar perturbations has the form

$$\mathcal{L}_\text{BH} = F_{4}(\pi, X) \epsilon^\mu\nu\sigma\rho\epsilon^\mu\nu\sigma' \pi_\mu\pi_{\nu}\pi_\sigma\pi_\rho' + F_5(\pi, X) \epsilon^\mu\nu\sigma\rho\epsilon^\mu\nu\sigma' \pi_\mu\pi_{\nu}\pi_\sigma\pi_\rho',$$

where $\pi$ is the scalar field (sometimes dubbed generalized Galileon), $X = g^{\mu\nu} \pi_\mu \pi_\nu$, $\pi_\mu = \partial_\mu \pi$, $\pi_{\mu\nu} = \nabla_\mu \nabla_\nu \pi$, $\Box \pi = \nabla^\mu \nabla_\mu \pi$, $G_{4X} = \partial G_4 / \partial X$, etc. The Horndeski theory corresponds to $F_4(\pi, X) = F_5(\pi, X) = 0$.

In this and next sections we concentrate mostly on the scalar sector of perturbations about the spatially flat FLRW background. In this case the ADM decompositon of the linearized metric has the following form:

$$ds^2 = (1 + 2\alpha) dt^2 - \partial_i \partial_j E dx^i dx^j - a^2(1 + 2\zeta \delta_{ij} + 2 \partial_i \partial_j E) dx^i dx^j.$$  

The scalar field perturbation is denoted by $\delta \pi = \chi$. Without loss of generality we partly use the gauge freedom and gauge away the longitudinal component $\partial_i \partial_j E$ from the very beginning. Then, the quadratic action for the scalar perturbations has the form

$$S^{(2)} = \int dt d^3 x a^3 \left( A_1 \dot{\chi}^2 + A_2 \left( \frac{\nabla \dot{\chi}}{a^2} \right)^2 + A_3 \alpha^2 + A_4 \frac{\nabla^2 \beta}{a^2} + A_5 \chi \frac{\nabla^2 \beta}{a^2} + A_6 \alpha \ddot{\chi} 
+ A_7 \alpha \frac{\nabla^2 \xi}{a^2} + A_8 \alpha \frac{\nabla \chi}{a^2} + A_9 \frac{\nabla^2 \chi}{a^2} + A_{10} \chi \ddot{\chi} + A_{11} \alpha \chi + A_{12} \frac{\nabla^2 \chi}{a^2} + A_{13} \frac{\nabla^2 \xi}{a^2} 
+ A_{14} \chi^2 + A_{15} \left( \frac{\nabla \chi}{a^2} \right)^2 + B_{16} \frac{\nabla^2 \chi}{a^2} + A_{17} \alpha \chi + A_{18} \dot{\chi} + A_{19} \chi + A_{20} \chi^2 \right),$$

where an overdot stands for derivative with respect to cosmic time $t$, and coefficients $A_i$ are expressed in terms of the Lagrangian functions and their derivatives. Their explicit expressions are collected in appendix A. Note that the terms $\alpha \zeta$ and $\zeta \chi$ have vanishing coefficients thanks to the background equations. The correspondence between our coefficients $A_i$ and those in refs. [26, 29] is

$$A_1 = -3 \dot{\mathcal{G}}_T, \ A_2 = \mathcal{F}_T, \ A_3 = \Sigma, \ A_4 = -2 \Theta, \ A_5 = 2 \dot{\mathcal{G}}_T, \ A_6 = 6 \Theta, \ A_7 = -2 \mathcal{G}_T.$$  

Also, the coefficient $A_4$ is denoted in refs. [1, 22, 31] by

$$A_4 = 2 \gamma.$$  

It is important for what follows that the coefficients $A_5$ and $(-A_7)$ differ only by beyond Horndeski terms. Explicitly, see appendix A,

$$A_5 + A_7 = -B_{10} \ddot{\pi} = 4F_4 \ddot{\pi}^4 + 12HF_5 \ddot{\pi}^5.$$  

The quadratic action (2.3) is invariant under the residual gauge transformations:

$$\alpha \to \alpha + \xi_0, \ \beta \to \beta - \xi_0, \ \chi \to \chi + \xi_0 \ddot{\pi}, \ \zeta \to \zeta + \xi_0 H,$$

where $H$ is the Hubble parameter and $\xi_0$ is the gauge function.
Lapse ($\alpha$) and shift ($\beta$) variables are non-dynamical, and variation of the action (2.3) with respect to them leads to the following constraints:

$$\alpha = -\frac{1}{A_4} \left( A_5 \dot{\zeta} + A_9 \dot{\chi} + A_{12} \chi \right),$$

$$\frac{\nabla^2 \beta}{a^2} = -\frac{1}{A_4} \left( A_7 \frac{\nabla^2 \zeta}{a^2} + A_8 \frac{\nabla^2 \chi}{a^2} + A_6 \dot{\zeta} + A_{11} \dot{\chi} + A_{17} \chi \right) \tag{2.8b}$$

By utilizing the constraints (2.8), we integrate out $\alpha$ and $\beta$. We write the resulting action in terms of gauge invariant combination of the curvature and scalar field perturbations:

$$S^{(2)} = \int dt \, d^3 x \, a^3 \left( A \left( \frac{d}{dt} [\zeta \dot{\pi} - \chi H] \right)^2 + B \cdot (\zeta \dot{\pi} - \chi H)^2 - C \left( \frac{\nabla^2 \zeta}{a} \dot{\pi} - \frac{\nabla^2 \chi}{a} H \right)^2 \right), \tag{2.9}$$

where

$$A = \frac{1}{\pi^2} \left( A_1 + \frac{A_3 \cdot A_5}{A_4^2} - \frac{A_5 \cdot A_6}{A_4} \right) = \frac{1}{\pi^2} \left( \frac{4 A_1^2 \cdot A_3}{9 A_4^2} - A_1 \right), \tag{2.10a}$$

$$B = A \dot{\pi} + \dot{A} \pi + 3 A H \dot{\pi}, \tag{2.10b}$$

$$C = \frac{1}{\pi^2} \left( \frac{1}{a} \frac{d}{dt} \left[ a \dot{A} \cdot A_7 \right] - A_2 \right). \tag{2.10c}$$

The potentially problematic situation occurs if the coefficient $A_4$ crosses zero. Following refs. [1, 22, 31] we call it $\gamma$-crossing, see eq. (2.5). Indeed, according to constraints (2.8), both $\alpha$ and $\beta$ appear singular when $A_4 = 0$. Moreover, the coefficients $A$, $B$ and $C$ in the quadratic action (2.9) hit singularity, making the stability analysis tricky.

However, it has been shown in ref. [1], that in the Newtonian gauge, the solutions for the variables $\alpha$, $\zeta$ and $\chi$ are regular for all values of $A_4$, including zero. This implies that the solutions for all scalar perturbations are everywhere regular in any other gauge. To see explicitly that this is indeed the case, we carry out in section 3 calculations analogous to ref. [1] but in the unitary gauge and show that the solutions for all variables in the unitary gauge, namely, $\zeta$, $\alpha$ and $\beta$, are in fact regular at $\gamma$-crossing.

3 \textbf{$\gamma$-crossing}

3.1 \textbf{Solutions for metric perturbations in the unitary gauge}

In this section we obtain the solutions for $\zeta$, $\alpha$ and $\beta$ in the unitary gauge and show that these are regular despite the seeming pathology of eqs. (2.8) and action (2.9) at $\gamma$-crossing.

As the first step, let us assume that $\alpha$ and $\beta$ are finite and can be found from eqs. (2.8) for any value of $A_4$ including zero (below we explicitly show that this assumption does hold). This enables one to legitimately obtain the quadratic action (2.9) in a standard manner. Upon imposing the unitary gauge $\chi = 0$ in the action (2.9), one obtains the linearized equation for $\zeta$:

$$A \dot{\pi}^2 \cdot \dot{\zeta} + \left( A \dot{\pi}^2 + 2 A \dot{\pi} \dot{\pi} + 3 A H \dot{\pi}^2 \right) \cdot \dot{\zeta} - C \pi^2 \cdot \frac{\nabla^2 \zeta}{a^2} = 0, \tag{3.1}$$
In what follows, we keep track of the coefficient $A_4$ and its time derivatives only. Making use of the definitions (2.10) and performing Fourier transformation, we write eq. (3.1) in the following form:

$$(1 + c_1 \cdot A_4^2) \cdot \ddot{\zeta} + \left( c_2 + c_3 \cdot A_4^2 - 2 \cdot \frac{\dot{A}_4}{A_4} \right) \dot{\zeta} + \frac{k^2}{a^2} \left( c_4 \cdot \dot{A}_4 + c_5 \cdot A_4 + c_6 \cdot A_4^2 \right) \cdot \zeta = 0,$$  \hspace{1cm} (3.2)

where $c_i$ are combinations of the coefficients $A_i$, $i \neq 4$. These combinations are non-singular at $\gamma$-crossing. Since for homogeneous background the coefficients $A_i$ are functions of time only, so are the coefficients $c_i$ in eq. (3.2).

To study the behavior of metric perturbations at $\gamma$-crossing, we choose the origin of time in such a way that $\gamma$-crossing occurs at $t = 0$ and write

$$A_4 = C \cdot t + \ldots,$$  \hspace{1cm} (3.3)

where $C$ is a constant and dots denote terms of higher order in $t$.

Let us first obtain the solutions to eq. (3.2) to the leading order. Keeping only the dominant terms in the vicinity of $t = 0$, we find that eq. (3.2) is reduced to

$$\ddot{\zeta} - \frac{2}{t} \dot{\zeta} = 0,$$  \hspace{1cm} (3.4)

and the two solutions to this equation are

$$\zeta = \lambda = \text{const},$$  \hspace{1cm} (3.5)

and

$$\zeta = \delta \cdot t^3, \; \delta = \text{const}.$$  \hspace{1cm} (3.6)

Importantly, the corrections to the solution (3.5) start with $t^2$:

$$\zeta = \lambda \left( 1 + \frac{C}{2} \frac{k^2}{a^2} c_4 \cdot t^2 + \ldots \right),$$  \hspace{1cm} (3.7)

where the coefficient $c_4$ is, explicitly,

$$c_4 = \frac{3}{4} \cdot \frac{A_7}{A_1 \cdot A_3}.$$  \hspace{1cm} (3.8)

Thus, even though the linearized equation for $\zeta$ is singular at $t = 0$, the solutions (3.5) and (3.7) are regular.

Let us see that the lapse and shift are regular as well. Making use of the relations $A_5 = -\frac{2}{3} A_1$ and $A_6 = -3 A_4$ (see eqs. (A.5) and (A.6) in appendix A) we write eqs. (2.8) in the unitary gauge:

$$\alpha = -\frac{A_5}{A_4} \cdot \dot{\zeta},$$  \hspace{1cm} (3.9a)

$$-\frac{k^2}{a^2} \beta = \frac{A_7}{A_4 a^2} k^2 \cdot \zeta + \left( 3 - \frac{4}{3} \frac{A_1 A_3}{A_4^2} \right) \cdot \dot{\zeta},$$  \hspace{1cm} (3.9b)
Because of eq. (3.7), the shift perturbation (3.9a) is obviously regular. Off hand, the right hand side of eq. (3.9b) is of order $t^{-1}$. However, the terms of order $t^{-1}$ cancel out: in view of (3.3), (3.7) and (3.8), we have
\[
\frac{A_7 k^2}{A_4 a^2} \cdot \zeta - \left(3 - \frac{4 A_1 A_3}{3 A_4^2}\right) \cdot \dot{\zeta} = \frac{1}{A_4} \left[A_7 \frac{k^2}{a^2} \cdot \lambda - \frac{4 A_1 A_3}{3 A_4} \left(\frac{k^2}{a^2} \chi_4 \cdot C t\right) \lambda + O(t)\right] = O(1).
\]
Therefore, the shift perturbation $\beta$ is regular as well. Moreover, it is now evident that the unitary and Newtonian gauges are related by a non-singular transformation. Indeed, moving from the unitary to Newtonian gauge amounts to gauging away $\beta$ and introducing $\chi$ back. Since $\beta$ is regular, the corresponding gauge function $\xi_0$ in (2.7) is regular as well.

Hence, we have explicitly shown that there is nothing wrong with letting $A_4$ to cross zero. It is still possible to analyse the stability in terms of the unitary gauge set of variables around this point.

As discussed in refs. [1, 29], $\gamma$-crossing is essential for constructing spatially flat bouncing solutions, which connect two asymptotic states of the Universe, in which the field $\pi$ is a conventional scalar field and gravity is described by conventional GR. Indeed, in that case $G_4 \to 1/2$, $K, G_5, F_4, F_5 \to 0$ as $|t| \to \infty$, and, according to the explicit expression (A.4) given in appendix A, $A_4 \to -2H$. Bouncing solution implies that the Hubble parameter, and hence $(-A_4)$, are negative at early times and positive at late times, so $A_4$ crosses zero somewhere. Since in many previous works $\gamma$-crossing was believed to be troublesome, the scenario with restored Einstein gravity long before and after the bounce was not considered. In the section 3 we construct a specific example of this type of bouncing solution.

### 3.2 No-go theorem and $\gamma$-crossing

Let us now briefly revisit the no-go theorem [26] for the general Horndeski theory and emphasise that $\gamma$-crossing does not help to evade this theorem. To this end, let us recall the form of the quadratic action for tensor perturbations valid both in Horndeski and beyond Horndeski theories:

\[
S_{\text{tensor}}^{(2)} = \int dt d^3x a^3 \frac{A_5}{2} \left(h_{ik}^T\right)^2 - A_2 \left(\nabla^2 h_{ik}^T\right)^2, \tag{3.11}
\]

where $h_{ik}^T$ denotes transverse traceless tensor perturbation. The quadratic action in the scalar sector has the form (2.9). To avoid ghost and gradient instabilities one requires $A_5 > 0$, $A > 0$ and $A_2 > 0, \mathcal{C} > 0$. The main no-go argument is based on the requirement of the absence of gradient instabilities in the scalar sector, i.e. $\mathcal{C} > 0$. Taking into account the positivity of both $A_2$ and $\mathcal{C}$ and using the definition of $\mathcal{C}$ in (2.10c), we write this requirement in the following form:\footnote{In addition to (2.4), we note that our notations are related to those in refs. [26, 29] as $A = G \phi$, $\mathcal{C} = F \phi$.}

\[
\frac{d}{dt} \left[\frac{a A_5 \cdot A_7}{2 A_4}\right] = a \cdot (C \dot{\pi}^2 + A_2) > 0, \tag{3.12}
\]

The point is that

\[
\xi = \frac{a A_5 \cdot A_7}{2 A_4}, \tag{3.13}
\]

is, therefore, a monotonously growing function. If the theory at $t \to \pm \infty$ is more or less conventional, the coefficients $A_2$ and $\mathcal{C} \dot{\pi}^2$ in (3.11) and (2.9) tend to positive constants as
Figure 1. $\xi(t)$ with $\gamma$-crossing (without loss of generality $\gamma$-crossing takes place at $t = 0$). Dashed line shows $\xi(t)$ which asymptotically tends to zero and faces strong coupling problem at both infinities: $\dot{\xi} \to 0$. Solid line represents a healthy behavior of $\xi$: $\dot{\xi} > \text{const} > 0$ and $\xi$ crosses zero twice.

$t \to \pm \infty$ (or, more generally, are bounded from below by positive constants); then $\dot{\xi} > \text{const} > 0$ for all times and, thus, $\xi$ cannot asymptotically tend to a constant value or zero. Hence, $\xi$ necessarily crosses zero somewhere during the evolution. Importantly, this occurs both in Horndeski and beyond Horndeski theories and irrespectively of the $\gamma$-crossing, at which $|\xi| = \infty$. This situation is shown in figure 1 by solid lines.

In the case of the general Horndeski theory, one has

$$A_5 = -A_7,$$

(3.14)

see (2.6), and $A_5$ has to be positive to avoid ghost instabilities in the tensor sector. So, in Horndeski theory with conventional asymptotics $\xi$ necessarily crosses zero, and this occurs when $A_4 \to \infty$. This scenario of infinite $A_4$ implies the singularity in the classical solution; we get back to the no-go theorem of refs. [24, 26] which, we emphasize, holds even in the presence of $\gamma$-crossing.

There are in principle two ways out in Horndeski theory. Sticking to conventional asymptotics, and hence $\xi$ crossing zero, one can have $A_4 = 0$ and $A_5 = -A_7 = 0$ at this point [22, 26] (we give concrete example below in figure 6). This case is not only fine-tuned, but also faces strong coupling problem in the tensor sector, see eq. (3.11). The second possibility is to give up the conventional asymptotics of the action for quadratic perturbations and consider models with $A_2, C \to 0$ as $t \to -\infty$ and/or $t \to +\infty$. This scenario is shown by dashed lines in figure 1; it faces the danger of strong coupling regime in either distant past or distant future, or both.

4 An example of the bounce with conventional asymptotics

Let us now take advantage of the safety of $\gamma$-crossing and construct a bouncing solution in beyond Horndeski theory, where the driving field $\pi$ reduces to a conventional massless scalar field and gravity tends to GR in both distant past and future.
Without loss of generality we choose the following form of the scalar field

$$\pi(t) = t,$$  \hspace{1cm} (4.1)

so that $X = 1$. Indeed, assuming that the scalar field monotonously increases, one can always obtain (4.1) by field redefinition. Then the asymptotics of the Lagrangian functions as $t \to \pm \infty$ are (we set $M_{Pl}^2/(8\pi) = 1$)

$$F(\pi, X) = \frac{X}{3\pi^2} = \frac{1}{3t^2},$$ \hspace{1cm} (4.2a)

$$G_4(\pi, X) = \frac{1}{2},$$ \hspace{1cm} (4.2b)

$$G_5(\pi, X) = F_4(\pi, X) = F_5(\pi, X) = 0.$$ \hspace{1cm} (4.2c)

Equations (4.2b) and (4.2c) ensure that gravity is described by GR, while the choice (4.2a) indeed implies that $\varphi = \sqrt{\frac{2}{3}} \log(t)$ is a conventional massless scalar field. Its equation of state is $p = \rho$, and hence

$$H = \frac{1}{3t}, \hspace{1cm} t \to \pm \infty.$$ \hspace{1cm} (4.3)

Note that the field equation $\ddot{\varphi} + 3H\dot{\varphi} = 0$ is satisfied for $\varphi = \sqrt{\frac{2}{3}} \log(t)$. In this section we choose a specific form of $H$ and reconstruct the Lagrangian functions of the beyond Horndeski theory which yield the chosen solution. This approach is by now standard [22, 24, 29, 30]. Our main concern is the stability of the solution during the entire evolution.

Let us choose the following form of the Hubble parameter,

$$H(t) = \frac{t}{3(\tau^2 + t^2)},$$ \hspace{1cm} (4.4)

so that

$$a(t) = (\tau^2 + t^2)^{\frac{1}{2}},$$ \hspace{1cm} (4.5)

and the bounce occurs at $t = 0$. The parameter $\tau$ in (4.4) determines the duration of the bouncing stage; we take $\tau \gg 1$, so that the time scale inherent in the solution greatly exceeds the Planck time. To reconstruct the theory which admits the solution (4.4) we use the following Ansatz for the Lagrangian functions

$$F(\pi, X) = f_0(\pi) + f_1(\pi) \cdot X + f_2(\pi) \cdot X^2,$$ \hspace{1cm} (4.6a)

$$G_4(\pi, X) = \frac{1}{2} + g_{40}(\pi) + g_{41}(\pi) \cdot X,$$ \hspace{1cm} (4.6b)

$$F_4(\pi, X) = f_{40}(\pi) + f_{41}(\pi) \cdot X,$$ \hspace{1cm} (4.6c)

while $K(\pi, X) = 0, G_5(\pi, X) = 0, F_5(\pi, X) = 0$. Let us note that in full analogy with ref. [29] there is no need to employ both beyond Horndeski functions $F_4(\pi, X)$ and $F_5(\pi, X)$: one of these functions, $F_4(\pi, X)$ in our case, is sufficient to get around the no-go theorem and satisfy the stability conditions.

Our tactics is to choose $F_4(\pi, X)$, $G_4(\pi, X)$ and also $f_2(\pi)$ in (4.6) in such a way that the stability conditions are satisfied, and find $f_0(\pi)$ and $f_1(\pi)$ from the background equations of motion. Indeed, there are two independent field equations which can be chosen, e.g. as $(00)$- and $(ij)$-components of the generalized Einstein equations (see appendix B for their
explicit forms). These equations can be used to find $f_0(t)$ and $f_1(t)$ in terms of other functions in (4.6). Once our $G_4(\pi, X)$ and $F_4(\pi, X)$ have the asymptotics (4.2b), (4.2c) and the Hubble parameter asymptotes (4.3), the function $F(\pi, X)$ automatically has the asymptotics (4.2a).

To clarify the reasons behind further choice of functions in (4.6), let us give an explicit form of the quadratic action in beyond Horndeski theory, which includes both tensor and scalar dynamical degrees of freedom (with unitary gauge imposed):

$$S^{(2)} = \int dt d^3 x a^3 \left[ \frac{A_5}{2} \left( \tilde{h}_{ik} \right)^2 - A_2 \left( \frac{\nabla \tilde{h}_{ik}}{a^2} \right)^2 + A \cdot \zeta^2 - C \cdot \left( \frac{\nabla \zeta}{a^2} \right)^2 \right],$$

(4.7)

where $A$ and $C$ are defined in eq. (2.10). We present the detailed reconstruction of the Lagrangian functions in appendix C, and here we give the results only. The functions $f_0(t)$, $f_1(t)$, $f_2(t)$, $g_{40}(t)$, $g_{41}(t)$, $f_{40}(t)$ and $f_{41}(t)$ entering (4.6) are shown in figure 2 (the analytical expressions are gathered in appendix C). Their asymptotic behavior as $t \to \pm \infty$ is as follows:

$$f_1(t) = \frac{1}{3t^2}, \quad f_0(t) = f_2(t) \propto \frac{1}{t^4}, \quad g_{40}(t) = f_{40}(t) = f_{41}(t) \propto e^{-2t/\tau}, \quad g_{41}(t) \propto t \cdot e^{-2t/\tau}. \tag{4.8}$$

As promised, the Lagrangian functions $F(\pi, X)$, $G_4(\pi, X)$ and $F_4(\pi, X)$ have the asymptotics (4.2). The coefficients $A$ and $C$ are shown in figure 3; note that they are positive everywhere and infinite at some point ($\gamma$-crossing). Their ratio $c_s^2 = C/A$ (sound speed

---

**Figure 2.** The Lagrangian functions $f_0(t)$, $f_1(t)$, $f_2(t)$, $g_{40}(t)$, $g_{41}(t)$, $f_{40}(t)$ and $f_{41}(t)$, with the following choice of the parameters involved in the analytical expressions (see appendix C): $u = 1/10$, $w = 1$ and $\tau = 10$. This choice guarantees that the bouncing solution is not fine-tuned and its duration safely exceeds the Planck time. Note that the functions $f_0(t)$ and $f_2(t)$ almost coincide for the chosen values of parameters.
Figure 3. The coefficients $A$ and $C$; the parameters $u$, $w$ and $\tau$ are the same as in figure 2.

Figure 4. Sound speed squared for the scalar perturbations is non-negative for all times and asymptotically tends to 1 in both infinite past and future. Right panel shows the vicinity of the bounce. The parameters $u$, $w$, $\tau$ are the same as in figure 2.

squared) is given in figure 4, which shows that the propagation is subluminal in the scalar sector. We choose the functions $g_{40}(\pi)$, $g_{41}(\pi)$, $f_{40}(\pi)$ and $f_{41}(\pi)$ in (4.6b) and (4.6c) in such a way that

$$A_2 = 1, \quad A_5 = 2,$$

hence, the tensor perturbations are stable and strictly luminal.

Thus, the stability requirements in both tensor and scalar sectors are satisfied and our bouncing solution indeed has conventional asymptotics in both distant past and future.

Finally, let us compare bouncing models with and without $\gamma$-crossing. The inequality (3.12) must be satisfied in beyond Horndeski theory, so the function $\xi(t)$ defined in (3.13) must grow monotonously. The difference, as compared to the Horndeski theory, is that eq. (3.14) does not hold any more, so $A_7$ is no longer constrained. In a model without $\gamma$-crossing, $A_4$ is always positive, so $\xi(t)$ crosses zero due to the zero of $A_7(t)$. This situation is
Figure 5. The coefficients $A_4$, $A_7$ and $\xi$ for two scenarios: without $\gamma$-crossing (top panel) and with $\gamma$-crossing (bottom panel). The former case was studied in ref. [29].

shown in figure 5, top panel. The fact that $(-A_7)$ is negative at early times while $A_5$ is always positive (see (3.11)) reiterates that the beyond Horndeski term is relevant at early times. In a model with $\gamma$-crossing, $\xi$ diverges at $\gamma$-crossing, so $\xi$, and hence $A_7$, crosses zero twice. This enables one to have $-A_7 = A_5 = 1$ both at early and late times, which corresponds to GR asymptotics. This case is shown in figure 5, lower panel.

Acknowledgments

The authors are grateful to Eugeny Babichev and Alexander Vikman for useful comments and fruitful discussions. This work has been supported by Russian Science Foundation grant 14-22-00161.

A Coefficients in quadratic action for perturbations

In this appendix we collect the expressions for coefficients $A_i$ entering the quadratic action (2.3):

$$A_1 = 3 \left[ -2G_4 + 4G_{4X} \dot{\pi}^2 - G_{5\pi} \dot{\pi}^2 + 2HG_{5X} \ddot{\pi}^3 + 2F_4 \dot{\pi}^4 + 6HF_5 \dot{\pi}^5 \right],$$  \hspace{1cm} (A.1)

$$A_2 = 2G_4 - 2G_{5X} \dot{\pi}^2 \ddot{\pi} - G_{5\pi} \dot{\pi}^2,$$  \hspace{1cm} (A.2)
\begin{align}
A_3 &= F_X \dot{\pi}^2 + 2F_{XX} \dot{\pi}^4 + 12HK_X \dot{\pi}^5 - K_\pi \dot{\pi}^2 - K_\pi \dot{\pi}^4 \\
&
- 6H^2G_4 + 42H^2G_{4XX} \dot{\pi}^2 + 96H^2G_{4XXX} \dot{\pi}^4 + 24H^2G_{4XXXX} \dot{\pi}^6 \\
&
- 6HG_{4\pi} \dot{\pi} - 30HG_{4\pi X} \dot{\pi}^3 - 12HG_{4\pi XX} \dot{\pi}^5 + 30H^3G_{5\pi \pi} \dot{\pi}^3 \\
&
+ 2H^2G_{5\pi XX} \dot{\pi}^3 + 4H^2G_{5\pi XXX} \dot{\pi}^7 - 18H^2G_{5\pi \pi} \dot{\pi}^2 - 27H^2G_{5\pi X} \dot{\pi}^4 \\
&
- 6H^2G_{5\pi XX} \dot{\pi}^6 + 90H^2F_4 \dot{\pi}^4 + 78H^2F_{4X} \dot{\pi}^6 + 12H^2F_{4XX} \dot{\pi}^8 \\
&
+ 168H^3F_5 \dot{\pi}^5 + 102H^3F_5X \dot{\pi}^7 + 12H^3F_{5XX} \dot{\pi}^8,
\end{align}

\begin{align}
A_4 &= 2\left[K_X \dot{\pi}^3 - 2G_4H + 8HG_{4X} \dot{\pi}^2 + 8HG_{4XX} \dot{\pi}^4 - 2G_{4\pi} \dot{\pi}^2 - 2G_{4\pi X} \dot{\pi}^4 \\
&
+ 3H^2G_{5\pi} \dot{\pi}^3 + 2H^2G_{5\pi X} \dot{\pi}^5 - 3H^2G_{5\pi XX} \dot{\pi}^6 - 2HG_{5\pi X} \dot{\pi}^4 \\
&
+ 10H^2F_4 \dot{\pi}^5 + 4HF_{4X} \dot{\pi}^6 + 21H^2F_5 \dot{\pi}^5 + 6H^2F_{5X} \dot{\pi}^7\right],
\end{align}

\begin{align}
A_5 &= -\frac{2}{3}A_1, \\
A_6 &= -3A_1, \\
A_7 &= -A_5 - B_{10} \dot{\pi},
\end{align}

\begin{align}
A_8 &= 2\left[K_X \dot{\pi}^3 - 2G_4H + 8HG_{4X} \dot{\pi}^2 + 8HG_{4XX} \dot{\pi}^4 - 2G_{4\pi} \dot{\pi}^2 - 2G_{4\pi X} \dot{\pi}^4 \\
&
- 2HG_{5\pi X} \dot{\pi}^3 + 3H^2G_{5\pi X} \dot{\pi}^5 + 2H^2G_{5\pi XX} \dot{\pi}^4 + 10HF_{4\pi} \dot{\pi}^3 + 4HF_{4X} \dot{\pi}^5 \\
&
+ 21H^2F_5 \dot{\pi}^4 + 6H^2F_{5X} \dot{\pi}^6\right],
A_9 &= -(A_6 - B_{16} H), \\
A_{10} &= -3(A_8 - B_{16} H).
\end{align}

\begin{align}
A_{11} &= 2\left[-F_X \dot{\pi} - 2F_{XX} \dot{\pi}^3 + K_\pi \dot{\pi} - 6HK_X \dot{\pi}^4 - 9HK_X \dot{\pi}^2 + K_\pi \dot{\pi}^3 \\
&
+ 3H^2G_{4\pi} + 2HG_{4\pi X} \dot{\pi}^2 + 12HG_{4\pi XX} \dot{\pi}^4 - 18H^2G_{4X} \dot{\pi} - 72H^2G_{4XX} \dot{\pi}^3 \\
&
- 24H^2G_{4XXX} \dot{\pi}^5 + 9H^2G_{5\pi} \dot{\pi} + 21H^2G_{5\pi X} \dot{\pi}^3 + 6H^2G_{5\pi XX} \dot{\pi}^5 \\
&
- 15H^3G_{5\pi} \dot{\pi}^2 - 20H^3G_{5\pi X} \dot{\pi}^4 - 4H^3G_{5\pi XX} \dot{\pi}^6 - 60H^2F_4 \dot{\pi}^3 - 66H^2F_{4X} \dot{\pi}^5 \\
&
- 12H^2F_{4XX} \dot{\pi}^7 - 105H^3F_5 \dot{\pi}^4 - 84H^3F_{5X} \dot{\pi}^6 - 12H^3F_{5XX} \dot{\pi}^8\right],
A_{12} &= 2\left[F_X \dot{\pi} - K_\pi \dot{\pi} + 3HK_X \dot{\pi}^2 - HG_{4\pi} + G_{4\pi X} \dot{\pi} - 10HG_{4\pi X} \dot{\pi}^2 + 6H^2G_{4X} \dot{\pi} \\
&
+ 12H^2G_{4XX} \dot{\pi}^3 - 3H^2G_{5\pi} \dot{\pi} + HG_{5\pi X} \dot{\pi}^2 - 4H^2G_{5\pi X} \dot{\pi}^3 + 3H^3G_{5\pi X} \dot{\pi}^2 \\
&
+ 3H^2G_{5\pi XX} \dot{\pi}^4 + 12H^2F_5 \dot{\pi}^3 + 6H^2F_{5X} \dot{\pi}^5 - 2HF_{4\pi} \dot{\pi}^4 + 15H^3F_5 \dot{\pi}^4 + 6H^3F_{5X} \dot{\pi}^6 \\
&
- 3H^2F_{5XX} \dot{\pi}^5\right],
A_{13} &= 2\left[4HG_{4X} \dot{\pi} + 4G_{4\pi X} \dot{\pi} + 8G_{4\pi XX} \dot{\pi}^2 \dot{\pi} - 2G_{4\pi} + 4G_{4\pi X} \dot{\pi}^2 + 2H^2G_{5\pi X} \dot{\pi}^2 \\
&
+ 2HG_{5\pi} \dot{\pi}^2 + 4HG_{5\pi X} \dot{\pi}^3 \dot{\pi} + 4HG_{5\pi XX} \dot{\pi}^3 \dot{\pi} - 2HG_{5\pi} \dot{\pi}^2 - 2G_{5\pi X} \dot{\pi} - 2HG_{5\pi X} \dot{\pi}^3 \\
&
- 2G_{5\pi XX} \dot{\pi}^3 \dot{\pi} - G_{5\pi X} \dot{\pi}^2 + 2HF_{4\pi} \dot{\pi}^3 + 4F_{4\pi} \dot{\pi}^2 + 4F_{4\pi} \dot{\pi}^4 + 2HF_{5\pi} \dot{\pi}^3 \\
&
+ 6H^2F_5 \dot{\pi}^4 + 12H^2F_{5X} \dot{\pi}^5 + 6H^2F_{5XX} \dot{\pi}^6\right],
A_{14} &= F_X + 2F_{XX} \dot{\pi}^2 - K_\pi + 6HK_X \dot{\pi} - K_\pi \dot{\pi}^2 + 6HK_{XX} \dot{\pi}^3 + 6H^2G_{4X} \\
&
- 18HG_{4\pi X} \dot{\pi} + 48H^2G_{4XX} \dot{\pi}^2 - 12HG_{4\pi XX} \dot{\pi}^3 + 24H^2G_{4XXX} \dot{\pi}^4 + 6H^3G_{5\pi X} \dot{\pi} \\
&
- 3H^2G_{5\pi} - 15H^2G_{5\pi X} \dot{\pi}^2 + 14H^2G_{5\pi XX} \dot{\pi}^3 + 4H^2G_{5\pi XXX} \dot{\pi}^5 - 6H^2G_{5\pi X} \dot{\pi}^4 \\
&
+ 36H^2F_4 \dot{\pi}^5 + 54H^2F_{4X} \dot{\pi}^4 + 12H^2F_{4XX} \dot{\pi}^6 + 6H^3F_5 \dot{\pi}^3 + 6H^3F_{5X} \dot{\pi}^5 \\
&
+ 12H^3F_{5XX} \dot{\pi}^7,
\end{align}
\[ A_{15} = - F_{X} - 4H K_{X} \pi - 2K_{X} \pi^{2} - 2K_{XX} \dot{\pi}^{2} - 6H^{2} G_{4} X \]
\[ - 4H G_{4X} - 20H^{2} G_{4XX} \pi^{2} - 8H G_{4XX} \dot{\pi}^{2} - 24H G_{4XX} \ddot{\pi} + 12 H G_{4X} \dot{\pi} \]
\[ + 6G_{4} \pi \dot{\pi} - 16H G_{4XX} \dot{\pi}^{2} + 8H G_{4X} \ddot{\pi} + 4G_{4XX} \ddot{\pi} - 2G_{4X} \dot{\pi}^{2} \]
\[ - 4H^{3} G_{5X} \pi - 4H H G_{5X} \dot{\pi} - 2H^{2} G_{5X} \pi + 3H^{2} G_{5} + 2H G_{5} + 5H^{2} G_{5X} \ddot{\pi} \]
\[ + 2H G_{5X} \dot{\pi} + 8H G_{5X} \dot{\pi}^{2} - 4H^{3} G_{5XX} \pi^{3} - 4H H G_{5XX} \dot{\pi}^{3} - 10H^{2} G_{5XX} \ddot{\pi} \]
\[ - 4H^{2} G_{5XX} \ddot{\pi} + 2H^{2} G_{5X} \dot{\pi}^{2} + 4H G_{5XX} \ddot{\pi} + 2H G_{5X} \ddot{\pi}^{2} - 20 \dot{F}_{4X} H^{2} \pi^{2} \]
\[ - 10 \dot{F}_{4X} \pi + 24H \dot{F}_{4X} \ddot{\pi} - 4H \dot{F}_{4X} \dddot{\pi} - 36 H \dot{F}_{4X} \dddot{\pi}^{3} - 36 \dot{F}_{4X} H^{2} \dddot{\pi} + 12 \dot{F}_{5X} \dot{\pi} \]
\[ - 12 H \dot{F}_{5X} \dot{\pi}^{2} - 6H^{2} F_{5X} \dot{\pi}^{2} + 12 H^{2} F_{5X} \ddot{\pi}^{2} + 6H^{2} F_{5X} \ddot{\pi}^{2} + 12 \dot{F}_{5X} \dot{\pi}^{2} \]
\[ (A.15) \]
\[ B_{16} = 4F_{4} \dot{\pi}^{3} + 12H F_{5} \dot{\pi}^{4} \]
\[ A_{17} = F_{4} - 2F_{4X} \dot{\pi}^{2} + K_{X} \pi^{2} - 6H K_{X} \dot{\pi}^{2} + 6G_{4X} H^{2} + 6G_{4X} \dot{\pi} - 24G_{4X} H^{2} \pi^{2} \]
\[ + 108G_{4X} \pi^{3} - 24G_{4X} \dot{\pi}^{2} + 72G_{4X} \ddot{\pi}^{2} + 216G_{4X} \dot{\pi}^{2} \]
\[ - 24G_{4} \dddot{\pi} + 36G_{4X} \dot{\pi}^{2} + 12G_{4X} \dddot{\pi} - 96G_{4X} \dddot{\pi} - 96G_{4X} \dot{\pi} \]
\[ - 4G_{5X} \dot{\pi}^{2} + 24G_{5X} \ddot{\pi} + 4G_{5X} \dddot{\pi} + 12G_{5X} \dddot{\pi}^{2} + 36G_{5X} \dddot{\pi}^{2} + 12G_{5X} \dddot{\pi}^{2} \]
\[ + 36G_{5X} \dddot{\pi}^{2} - 84G_{5X} \dddot{\pi}^{2} - 48G_{5X} \dddot{\pi}^{2} - 24G_{5X} \dddot{\pi}^{2} \]
\[ (A.16) \]
\[ A_{18} = - 6F_{X} \dot{\pi} + 6K_{X} \dot{\pi} - 36H K_{X} \dot{\pi}^{2} - 2K_{X} \dot{\pi}^{2} - 12K_{XX} \dot{\pi}^{2} \]
\[ - 12K_{XX} \dot{\pi}^{2} - 12K_{XX} \dot{\pi}^{2} - 12K_{XX} \dot{\pi}^{2} - 12K_{XX} \dot{\pi}^{2} + 24G_{4} H \]
\[ - 24G_{4X} H \pi^{2} - 24G_{4X} \dot{\pi} + 72G_{4X} \ddot{\pi}^{2} + 216G_{4X} \dddot{\pi}^{2} \]
\[ - 48G_{4X} \dddot{\pi} + 36G_{4X} \dot{\pi}^{2} + 12G_{4X} \dddot{\pi} - 96G_{4X} \dddot{\pi} - 96G_{4X} \dot{\pi} \]
\[ - 4G_{5X} \dot{\pi}^{2} + 24G_{5X} \ddot{\pi} + 4G_{5X} \dddot{\pi} + 12G_{5X} \dddot{\pi}^{2} + 36G_{5X} \dddot{\pi}^{2} + 12G_{5X} \dddot{\pi}^{2} \]
\[ + 36G_{5X} \dddot{\pi}^{2} - 84G_{5X} \dddot{\pi}^{2} - 48G_{5X} \dddot{\pi}^{2} - 24G_{5X} \dddot{\pi}^{2} \]
\[ (A.17) \]
\[ A_{19} = 3F_{X} - 18F_{X} \dot{\pi}^{2} - 6F_{XX} \dot{\pi}^{2} + 12F_{XX} \dot{\pi}^{2} + 18H K_{X} \pi + 6K_{X} \dot{\pi}^{2} - 54H^{2} K_{X} \pi^{2} \]
\[ - 36H K_{X} \dot{\pi}^{2} - 36H K_{XX} \dot{\pi}^{2} + 3K_{XX} \dot{\pi}^{2} + 18H K_{XX} \dot{\pi}^{2} + 3K_{XX} \dot{\pi}^{2} \]
\[ + 36G_{4X} \dot{\pi}^{2} + 18G_{4X} \dot{\pi} - 108G_{4X} H \dot{\pi}^{2} - 72G_{4X} H \dot{\pi} + 36G_{4X} H \ddot{\pi}^{2} + 108G_{4X} H \dddot{\pi}^{2} \]
\[ + 36G_{4X} H \dddot{\pi} + 288G_{4X} \dot{\pi}^{2} - 216G_{4X} \dot{\pi}^{2} + 144G_{4X} H \dddot{\pi}^{2} \]
\[ - 72G_{4X} H \dddot{\pi}^{2} + 36G_{4X} \dot{\pi}^{2} + 72G_{4X} \dddot{\pi}^{2} + 144G_{4X} H \dddot{\pi}^{2} \]
\[ - 54G_{5X} \dddot{\pi}^{2} + 36G_{5X} \dddot{\pi}^{2} + 36G_{5X} \dddot{\pi}^{2} + 36G_{5X} \dddot{\pi}^{2} \]
\[ (A.18) \]
Note that $B_{16} = 0$ in the general Horndeski theory.

B Einstein equations for background

This appendix gives the (00)- and $(ij)$- components of the generalized Einstein equations for spatially flat FLRW background in beyond Horndeski theory (2.1):

$$
\delta g^{00} : F - 2F_XX - 6HKX \pi + K_\pi X + 6H^2G_4 + 6HG_{4\pi} \pi \\
- 24H^2X(G_{4\pi} + G_{4XX}X) + 12HG_{4\pi X}X \pi \\
- 2H^3X(5G_{5\pi X} + 2G_{5XX}X) + 3H^2X(3G_{5\pi} + 2G_{5\pi X}X) \\
- 6H^2X(5F_4 + 2F_{4X}X) - 6H^3X^2(7F_5 + 2F_{5X}X) = 0,
$$

$$
\delta g^{ij} : F - X(2KX \pi + K_\pi) + 2(3H^2 + 2H)G_4 - 12H^2G_{4XX}X \\
- 8HG_{4XX}X - 8HG_{4\pi X}X \pi - 16HG_{4XX}X \pi \pi - 2(\pi + 2H \pi)G_{4\pi} \\
+ 4XG_{4\pi}(\pi - 2H \pi) + 2XG_{4\pi}X - 2XG_{5X}(2H^3 \pi + 2H \pi + 3H^2 \pi) \\
- 4H^2G_{5XX}X^2 \pi + G_{5n}(3H^2X + 2H \pi + 4H \pi \pi) \\
+ 2H(G_{5\pi X}X(2\pi \pi - HX) + 2HG_{5\pi X}X \pi \\
- 2F_4X(3H^2X + 2H \pi + 5H \pi) - 8HF_{4\pi X}X^2 \pi \pi - 4HF_{4\pi}X^2 \pi \\
- 6HF_{5XX}X^2(2H^2 \pi + 2H \pi + 5H \pi) - 12H^2F_{5XX}X^3 \pi - 6H^2F_{5\pi}X^3 = 0,
$$

where $X = \pi^2$.

C Bounce: details of the reconstruction procedure

In this appendix we describe in detail the reconstruction procedure for the Lagrangian functions shown in figure 2.

The procedure is based, in particular, on the form of the quadratic action (4.7) for tensor and scalar perturbations in beyond Horndeski theory. As before, the coefficients $A$
and \( C \) are given by (2.10). Making use of expansions (4.6), we cast \( A_2, A_3, A_4, A_5 \) and \( A_7 \) (eqs. (A.2), (A.3), (A.4), (A.5) and (A.7) in appendix A) in the following form:

\[
A_2 = 2 \left( \frac{1}{2} + g_{40}(t) + g_{41}(t) \right),
\]

\[
A_3 = f_1(t) + 6 f_2(t) - 3H^2 \cdot [1 + 30 f_{40}(t) + 56 f_{41}(t) + 2g_{40}(t) - 12g_{41}(t)]
\]

\[
- 6H \cdot [g_{40}(t) + 6g_{41}(t)],
\]

\[
A_4 = -2 \cdot H(t) \cdot [1 + 2g_{40}(t) - 6g_{41}(t) + 10f_{40}(t) + 14f_{41}(t)] - 2\dot{g}_{40}(t) - 6\dot{g}_{41}(t),
\]

\[
A_5 = 4 \left( \frac{1}{2} + g_{40}(t) + g_{41}(t) \right) - 8g_{41}(t) + 4 (f_{40}(t) + f_{41}(t)),
\]

\[
A_7 = -4 \left( \frac{1}{2} + g_{40}(t) + g_{41}(t) \right) + 8g_{41}(t).
\]

We heavily rely on these expressions when choosing the functions \( g_{40}(t), g_{41}(t), f_{40}(t), f_{41}(t) \) and \( f_2(t) \).

First, we require that ghost and gradient instabilities are absent in the tensor sector, i.e. \( A_5 > 0 \) and \( A_2 > 0 \). According to the requirement of asymptotically vanishing \( F_4(\pi, X) \) in (4.2c), one possible choice for \( f_{40}(t) \) in (C.1d) is the following:

\[
f_{40}(t) = -f_{41}(t) + w \cdot \sech^2 \left( \frac{t}{\tau} + u \right),
\]

where \( u \) and \( w \) are constants, which are introduced to avoid fine-tuning. We give a detailed discussion of the fine-tuning issue below.

In order to avoid superluminal propagation of the tensor modes, let us choose

\[
A_2 = 1, \quad A_5 = 2,
\]

and express \( g_{41}(t) \) and \( g_{40}(t) \) using eqs. (C.1d) and (C.1a), respectively,

\[
g_{40}(t) = -g_{41}(t) = -\frac{w}{2} \sech^2 \left( \frac{t}{\tau} + u \right).
\]

Thus, we have completely defined \( G_4(\pi, X) \) in (4.6b).

Let us now use the stability conditions for the scalar sector of action (4.7). To have no gradient instabilities one requires \( C > 0 \). So according to eqs. (2.10c)

\[
\frac{1}{a} \frac{d}{dt} \xi = \frac{1}{a} \frac{d}{dt} \left[ \frac{a A_5 \cdot A_7}{2 A_4} \right] > A_2
\]

where the expression for \( \xi \) given by (3.13) is used. Since \( A_2 > 0 \), \( \xi \) must be a monotonously growing function of time, so it crosses zero at some point(s). We have already made a choice (C.3), so we are left with \( A_7 \) and \( A_4 \) in (C.5). Note that, unlike in the case of the general Horndeski theory, \( A_7 \) is not constrained by any stability conditions. In fact, \( A_7 \) is completely determined by eqs. (C.1e) and (C.4):

\[
A_7 = 4w \cdot \sech^2 \left( \frac{t}{\tau} + u \right) - 2.
\]

Now, having defined \( A_7 \) in (C.6), there is \( A_4 \) in (C.5) to be determined. According to the explicit form of \( A_4 \) in (C.1c), the only yet unknown function there is \( f_{41}(t) \). Recall that
the main requirement is that $\pi$ becomes a conventional scalar field and General Relativity is restored in both distant past and future. Thus, eq. (C.1c) shows that in both asymptotic past and future $A_4 = -2H$. We require that $(-A_4/2)$ is reasonably close to the Hubble parameter (4.4) at all times. We achieve this by choosing

$$f_{41}(t) = \frac{3w \cdot \text{sech}^2\left(\frac{t}{\tau} + u\right)}{2t\tau} \cdot \left[ t^2 \cdot \tanh\left(\frac{t}{\tau} + u\right) + \tau^2 \cdot \tanh\left(\frac{t}{\tau}\right) - t \cdot \tau \right].$$ \hspace{1cm} (C.7)

This completes our definition of $F_4(\pi, X)$ in (4.6c).

A comment on the fine-tuning issue is in order. Under fine-tuning we mean the situation when the coefficient $A_4$ crosses zero at some moment of time $t^*$, while $A_7$ touches zero at $t = t^*$ and remains non-negative [22, 28]. This situation is shown in figure 6. In the case of fine-tuning eq. (C.5) is satisfied, and both $C \neq \infty$ and $A \neq \infty$. In contrast, we aim at avoiding fine-tuning by introducing the constants $u$ and $w$ in eq. (C.2). We choose these constants in such a way that $A_4$ and $A_7$ cross zero at different times, see the bottom panel of figure 5.

The only function still to be found is $F(\pi, X)$. We make use of (00)- and (ij)-components of equations of motion (see appendix B) to relate $f_0(t), f_1(t)$ and $f_2(t)$:

\begin{align*}
 f_0(t) - f_1(t) - 3f_2(t) + \frac{t}{3\tau \cdot (\tau^2 + t^2)^2} \left[ t \cdot \tau + 6w \cdot \text{sech}^2\left(\frac{t}{\tau} + u\right) \right] & = 0, \tag{C.8a} \\
 3f_0(t) + 3f_1(t) + 3f_2(t) + \frac{2\tau^2 - t^2}{(\tau^2 + t^2)^2} & = 0, \tag{C.8b}
\end{align*}

where we make use of eqs. (C.4), (C.2) and (C.7). From eqs. (C.8) one expresses $f_0(t)$ and

Figure 6. A fine-tuned solution.
\[ f_0(t) = \frac{1}{3t(t^2 + \tau^2)^2} \left[ -t^3 + 3\tau(t^2 + \tau^2)^2 \cdot f_2(t) - 3t\tau^2w \cdot \text{sech}^2\left(\frac{t}{\tau} + u\right) \tanh\left(\frac{t}{\tau}\right) \right] \tag{C.9a} \]

\[ + 3t^2w \cdot \text{sech}^2\left(\frac{t}{\tau} + u\right) \tanh\left(\frac{t}{\tau} + u\right) + 6t\tau^2w \cdot \text{sech}^2\left(\frac{t}{\tau} + u\right) \tanh\left(\frac{t}{\tau} + u\right) \right], \]

\[ f_1(t) = -\frac{1}{3t(t^2 + \tau^2)^2} \left[ \frac{3}{t^2 + \tau^2} \cdot \left[ -2t^3 + 3\tau(t^2 + \tau^2)^2 \cdot f_2(t) - 3t\tau^2w \cdot \text{sech}^2\left(\frac{t}{\tau} + u\right) \tanh\left(\frac{t}{\tau}\right) \right] \right] \tag{C.9b} \]

\[ + 3t^2w \cdot \text{sech}^2\left(\frac{t}{\tau} + u\right) \tanh\left(\frac{t}{\tau} + u\right) + 6t\tau^2w \cdot \text{sech}^2\left(\frac{t}{\tau} + u\right) \tanh\left(\frac{t}{\tau} + u\right) - t^2 \right]. \]

The final step is to choose \( f_2(t) \) in such a way that \( A > 0 \) (no ghosts in the scalar sector of (4.7)) and \( A > C \) (scalar perturbations propagate at subluminal speed). The only unconstrained coefficient left in the definition of \( A \) in (2.10a) is \( A_3 \), which, according to (C.1b), involves the yet undetermined function \( f_2(t) \). To satisfy both \( A > 0 \) and \( A > C \) we choose \( f_2(t) \) in such a way that the term involving \( A_3 \) is always dominating in (2.10a). Since we chose \( A_2 = 2 \) and have \( A_1 = -\frac{3}{2}A_5 = -3 \) (see eq. (A.5)), it is sufficient to choose \( A_3 \) as follows:

\[ A_3 = \left[ 1 + \left(\frac{t}{\tau}\right)^2 \right]^{-2}, \tag{C.10} \]

which gives

\[ f_2(t) = \frac{1}{12t(t^2 + \tau^2)^2} \left[ \left( 3\tau^3 + 3\tau^5 + 4t\tau w \cdot \text{sech}^2\left(\frac{t}{\tau} + u\right) \left[ -4t\tau + 9\tau^2 \tanh\left(\frac{t}{\tau}\right) \right] \right) \right] \]

\[ + 3(t^2 - 2\tau^2) \tanh\left(\frac{t}{\tau} + u\right) \right]. \]

This completes the reconstruction of the Lagrangian functions (4.6).

References

[1] A. Ijjas, *Space-time slicing in Horndeski theories and its implications for non-singular bouncing solutions*, JCAP **02** (2018) 007 [arXiv:1710.05990] [SPIRE].

[2] G.W. Horndeski, *Second-order scalar-tensor field equations in a four-dimensional space*, Int. J. Theor. Phys. **10** (1974) 363 [SPIRE].

[3] D.B. Fairlie, J. Govaerts and A. Morozov, *Universal field equations with covariant solutions*, Nucl. Phys. B **373** (1992) 214 [hep-th/9110022] [SPIRE].

[4] A. Nicolis, R. Rattazzi and E. Trincherini, *The Galileon as a local modification of gravity*, Phys. Rev. D **79** (2009) 064036 [arXiv:0811.2197] [SPIRE].

[5] C. Deffayet, G. Esposito-Farese and A. Vikman, *Covariant Galileon*, Phys. Rev. D **79** (2009) 084003 [arXiv:0901.1314] [SPIRE].

[6] M. Zumalacárregui and J. García-Bellido, *Transforming gravity: from derivative couplings to matter to second-order scalar-tensor theories beyond the Horndeski Lagrangian*, Phys. Rev. D **89** (2014) 064046 [arXiv:1308.4685] [SPIRE].

[7] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, *Healthy theories beyond Horndeski*, Phys. Rev. Lett. **114** (2015) 211101 [arXiv:1404.6495] [SPIRE].


[8] D. Langlois and K. Noui, *Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability*, JCAP **02** (2016) 034 [arXiv:1510.06930] [SPIRE].

[9] J. Ben Achiou, M. Crisostomi, K. Koyama, D. Langlois, K. Noui and G. Tasinato, *Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order*, JHEP **12** (2016) 100 [arXiv:1608.08135] [SPIRE].

[10] D. Langlois, M. Mancarella, K. Noui and F. Vernizzi, *Effective Description of Higher-Order Scalar-Tensor Theories*, JCAP **05** (2017) 033 [arXiv:1703.03797] [SPIRE].

[11] P. Creminelli, M.A. Luty, A. Nicolis and L. Senatore, *Starting the Universe: Stable Violation of the Null Energy Condition and Non-standard Cosmologies*, JHEP **12** (2006) 080 [hep-th/0606090] [SPIRE].

[12] T. Kobayashi, M. Yamaguchi and J. Yokoyama, *G-inflation: Inflation driven by the Galileon field*, Phys. Rev. Lett. **105** (2010) 231302 [arXiv:1105.5723] [SPIRE].

[13] T. Kobayashi, M. Yamaguchi and J. Yokoyama, *Generalized G-inflation: Inflation with the most general second-order field equations*, Prog. Theor. Phys. **126** (2011) 511 [arXiv:1008.0603] [SPIRE].

[14] D.A. Easson, I. Sawicki and A. Vikman, *G-Bounce*, JCAP **11** (2011) 021 [arXiv:1109.1047] [SPIRE].

[15] Y.-F. Cai, D.A. Easson and R. Brandenberger, *Towards a Nonsingular Bouncing Cosmology*, JCAP **08** (2012) 020 [arXiv:1206.2382] [SPIRE].

[16] M. Koehn, J.-L. Lehners and B. Ovrut, *Cosmological super-bounce*, Phys. Rev. D **90** (2014) 025005 [arXiv:1310.7577] [SPIRE].

[17] D. Pirtskhalava, L. Santoni, E. Trincherini and P. Utinymyar, *Inflation from Minkowski Space*, JHEP **12** (2014) 151 [arXiv:1410.0882] [SPIRE].

[18] T. Qiu and Y.-T. Wang, *G-Bounce Inflation: Towards Nonsingular Inflation Cosmology with Galileon Field*, JHEP **04** (2015) 130 [arXiv:1501.05710] [SPIRE].

[19] T. Kobayashi, M. Yamaguchi and J. Yokoyama, *Galilean Creation of the Inflationary Universe*, JCAP **07** (2015) 017 [arXiv:1504.05710] [SPIRE].

[20] Y. Wan, T. Qiu, F.P. Huang, Y.-F. Cai, H. Li and X. Zhang, *Bounce Inflation Cosmology with Standard Model Higgs Boson*, JCAP **12** (2015) 019 [arXiv:1509.08772] [SPIRE].

[21] M. Koehn, J.-L. Lehners and B. Ovrut, *Nonsingular bouncing cosmology: Consistency of the effective description*, Phys. Rev. D **93** (2016) 103501 [arXiv:1512.03807] [SPIRE].

[22] A. Ijjas and P.J. Steinhardt, *Classically stable nonsingular cosmological bounces*, Phys. Rev. Lett. **117** (2016) 121304 [arXiv:1606.08880] [SPIRE].

[23] V.A. Rubakov, *The Null Energy Condition and its violation*, Phys. Usp. **57** (2014) 128 [arXiv:1401.4024] [SPIRE].

[24] M. Libanov, S. Mironov and V. Rubakov, *Generalized Galileons: instabilities of bouncing and Genesis cosmologies and modified Genesis*, JCAP **08** (2016) 037 [arXiv:1605.05992] [SPIRE].

[25] R. Kolevatov and S. Mironov, *Cosmological bounces and Lorentzian wormholes in Galileon theories with an extra scalar field*, Phys. Rev. D **94** (2016) 123516 [arXiv:1607.04099] [SPIRE].

[26] T. Kobayashi, *Generic instabilities of nonsingular cosmologies in Horndeski theory: A no-go theorem*, Phys. Rev. D **94** (2016) 043511 [arXiv:1606.05831] [SPIRE].

[27] Y. Cai, Y. Wan, H.-G. Li, T. Qiu and Y.-S. Piao, *The Effective Field Theory of nonsingular cosmology*, JHEP **01** (2017) 090 [arXiv:1610.03400] [SPIRE].
[28] P. Creminelli, D. Pirtskhalava, L. Santoni and E. Trincherini, *Stability of Geodesically Complete Cosmologies*, *JCAP* **11** (2016) 047 [arXiv:1610.04207] [inSPIRE].

[29] R. Kolevatov, S. Mironov, N. Sukhov and V. Volkova, *Cosmological bounce and Genesis beyond Horndeski*, *JCAP* **08** (2017) 038 [arXiv:1705.06626] [inSPIRE].

[30] Y. Cai and Y.-S. Piao, *A covariant Lagrangian for stable nonsingular bounce*, *JHEP* **09** (2017) 027 [arXiv:1705.03401] [inSPIRE].

[31] D.A. Dobre, A.V. Frolov, J.T.G. Gherci, S. Ramazanov and A. Vikman, *Unbraiding the Bounce: Superluminality around the Corner*, *JCAP* **03** (2018) 020 [arXiv:1712.10272] [inSPIRE].