Interplay between superconductivity and non-Fermi liquid at a quantum critical point in a metal. II. The $\gamma$-model at a finite $T$ for $0 < \gamma < 1$.

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(Dated: June 5, 2020)
Abstract

Near a quantum-critical point in a metal a strong fermion-fermion interaction, mediated by a soft boson, acts in two different directions: it destroys fermionic coherence and it gives rise to an attraction in one or more pairing channels. The two tendencies compete with each other. We analyze a class of quantum-critical models, in which momentum integration and the selection of a particular pairing symmetry can be done explicitly, and the competition between non-Fermi liquid and pairing can be analyzed within an effective model with dynamical electron-electron interaction \( V(\Omega_m) \propto 1/|\Omega_m|^\gamma \) (the \( \gamma \)-model). In Paper I (arXiv:2004.13220) the two of us analyzed the \( \gamma \)-model at \( T = 0 \) for \( 0 < \gamma < 1 \) and argued that there exist a discrete, infinite set of topologically distinct solutions for the superconducting gap, all with the same spatial symmetry. The gap function \( \Delta_n(\omega_m) \) for the \( n \)-th solution changes sign \( n \) times as the function of Matsubara frequency. In this paper we analyze the linearized gap equation at a finite \( T \). We show that there exist an infinite set of pairing instability temperatures, \( T_{p,n} \), and the eigenfunction \( \Delta_n(\omega_m) \) changes sign \( n \) times as a function of a Matsubara number \( m \). We argue that \( \Delta_n(\omega_m) \) retains its functional form below \( T_{p,n} \) and at \( T = 0 \) coincides with the \( n \)-th solution of the non-linear gap equation. Like in Paper I, we extend the model to the case when the interaction in the pairing channel has an additional factor \( 1/N \) compared to that in the particle-hole channel. We show that \( T_{p,0} \) remains finite at large \( N \) due to special properties of fermions with Matsubara frequencies \( \pm \pi T \), but all other \( T_{p,n} \) terminate at \( N_{cr} = O(1) \). The gap function vanishes at \( T \to 0 \) for \( N > N_{cr} \) and remains finite for \( N < N_{cr} \). This is consistent with the \( T = 0 \) analysis.
I. INTRODUCTION

In this paper we continue our analysis of the competition between non-Fermi liquid (NFL) physics and superconductivity near a quantum-critical point in a metal for a class of models with dynamical four-fermion interaction $V(q, \Omega_m)$, mediated by a critical soft boson. The interaction $V(q, \Omega_m)$ gives rise to strong fermionic self-energy, which destroys Fermi liquid behavior, in most cases in dimensions $D \leq 3$, and also gives rise to an attraction in at least one pairing channel. We consider the class of models in which bosons are slow excitations compared to fermions (like phonons in the case when the Debye frequency is much smaller than the Fermi energy). In this situation, the momentum integration in the expressions for the fermionic self-energy and the pairing vertex in the proper attractive channel can be carried out, and both the self-energy and the pairing vertex are determined by the effective, purely dynamical local interaction $V(\Omega_m) \propto \int dq V(q, \Omega)$, with $q$ connecting points on the Fermi surface. The coupled equations for the self-energy $\Sigma(k_F, \omega_m)$ and the pairing vertex $\Phi(\omega_m)$ have the same structure as Eliashberg equations for a phonon superconductor.

We consider quantum-critical models for which $V(\Omega_m) = \bar{g}^\gamma/|\Omega_m|^\gamma$ (the $\gamma$-model). In the previous paper, Ref.[1], hereafter referred to as Paper I, we listed a number of quantum-critical systems, whose low-energy physics is described by the $\gamma$-model with different $\gamma$. This paper also contains an extensive list of references to earlier publications on this subject.

The interaction $V(\Omega_m)$ is singular, and gives rise to two opposite tendencies: NFL behavior in the normal state, with $\Sigma(\omega_m) \propto \omega_m^{1-\gamma}$, and the pairing. The two tendencies compete with each other as a NFL self-energy reduces the magnitude of the pairing kernel, while the feedback from the pairing reduces fermionic self-energy.

In Paper I we analyzed zero-temperature behavior of the $\gamma$-model for $0 < \gamma < 1$. We found that the system does become unstable towards pairing, i.e., in the ground state the pairing vertex $\Phi(\omega_m)$ and the pairing gap $\Delta(\omega_m) = \Phi(\omega_m)/(1 + \Sigma(\omega_m)/\omega_m)$ are finite. However, in qualitative distinction with BCS/Eliashberg theory of superconductivity, in which there is a single solution of the gap equation, here we found an infinite discrete set of solutions $\Delta_n(\omega_m)$. The solutions have the same spatial gap symmetry, but are topologically distinct as $\Delta_n(\omega_m)$ changes sign $n$ times as a function of Matsubara frequency. The gap functions $\Delta_n(\omega_m)$ tend to finite values at zero frequency, but the magnitude of $\Delta_n(0)$ decreases with $n$ and at large enough $n$ scales as $\Delta_n(0) \propto e^{-An}$, where $A$ is a function of $\gamma$. 


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In this paper we analyze the γ-model for $0 < \gamma < 1$ at finite $T$. We solve the linearized gap equation and show that there is an infinite, discrete set of the onset temperatures for the pairing $T_{p,n}$. The magnitude of $T_{p,n}$ decreases with $n$ and at large $n$, $T_{p,n} \propto e^{-An}$ with the same $A$ as for $\Delta_n(0)$ from the $T = 0$ analysis. The gap function $\Delta_n(\omega_m)$ at $T = T_{p,n} - 0$ changes sign $n$ times as a function of discrete Matsubara frequency $\omega_m = \pi T(2m + 1)$. We argue that $\Delta_n(\omega_m)$, which develops at $T_{p,n}$, retains its functional form with $n$ sign changes also at $T < T_{p,n}$, and at $T \to 0$ approaches the $n$-th solution of the full non-linear gap equation, which we obtained in Paper I.

In Paper I added an extra parameter $N$ to the γ-model to control the ratio of interactions in the particle-particle and the particle-hole channels. Specifically, we added the factor $1/N$ to the interaction in the pairing channel and left the interaction in the particle-hole channel intact. We found that at $T = 0$ there exists a critical $N_{cr} > 1$, which separates the NFL ground state for $N > N_{cr}$ and the state with a non-zero $\Delta$ for $N < N_{cr}$. We found that all $\Delta_n(\omega_m)$ emerge simultaneously at $N = N_{cr} - 0$.

In this paper, we analyze the extended γ-model at a finite $T$. We show that the onset temperature $T_{p,0}$ for sign-preserving gap function $\Delta_0(\omega_m)$ remains finite for all $N$ and at large $N$ scales as $T_{p,0} \propto 1/N^{1/\gamma}$. This $T_{p,0}$ has been studied before, and its existence even for very large $N$ was attributed to special properties of fermions with $\omega_m = \pm \pi T$, for which non-thermal part of the self-energy vanishes. Namely, it was argued that the pairing at $T_{p,0}$ at large $N$ is predominantly between these fermions, and $\Delta_{n=0}(\pm \pi T)$ is much larger than at other $\omega_m$.

Here, we show that there exists an infinite set of other $T_{p,n}$ with $n > 0$, which terminate at $N = N_{cr}$ and are related to the solutions of the non-linear gap equation at $T = 0$, $\Delta_{n>0}(\omega_m)$. We show that the structure of the eigenfunctions $\Delta_{n>0}(\omega_m)$ at $T_{p,n}$ is opposite to that of $\Delta_0(\omega_m)$ in the sense that $\Delta_{n>0}(\omega_m)$ has additional smallness at $\omega_m = \pm \pi T$. We show that for any $N < N_{cr}$, $T_{p,n}$ gradually decrease with $n$, and the set become more dense as $n$ increases. As the consequence, in the limit $T \to 0$, the density of eigenvalues (DoE) remains finite for all $N < N_{cr}$. We show that it is the same at $T = 0$ and at $T \to 0$, i.e., the system evolves smoothly between $T = 0$ and $T > 0$. For completeness, we obtain the DoE away from a QCP, when a pairing boson has a finite mass. We show that in this case the solutions at $T \to 0$ exist only for discrete set of $N$.

The structure of the paper is the following. In Sec. [I] we present Eliashberg equations
for the $\gamma$-model and discuss in some detail the extension to $N \neq 1$ at a finite $T$. In Sec. III we analyze analytically and numerically the linearized gap equation at a finite $T$. We show that there are two possibilities to get a solution at small $T$: one can either set $N \propto 1/T^\gamma$ (the $n = 0$ solution), or keep $N$ finite. We consider these two cases separately in Secs. III A and III B. In Sec. III A we find the analytic solution for $T_{p,0}$, which reproduces earlier results in Refs. 2,3. In Sec. III B we first solve numerically the gap equation at $T \to 0$, obtain the density of eigenvalues, and show that it is non-zero for all $N$ between $N = N_{cr}$ and $N = 0$. This is consistent with $T = 0$ result in Paper I. We then extend calculations to finite $T$ and show that for any $N < N_{cr}$ (and any $\gamma$ from $0 < \gamma < 1$), there exists an infinite, discrete set of onset temperatures for the pairing, $T_{p,n}$. We argue that all $T_{p,n}$ with $n > 0$ terminate at $N = N_{cr}$ and that $T_{p,n}$ at large $n$ should decay exponentially with $n$ as $T_{p,n} \propto e^{-A_n}$. We relate $A$ to the form of $\Delta_n(\omega_m)$ at $T = 0$, express $A$ via $\gamma$ and $N$, and argue that $T_{p,n}$ is proportional to $\Delta_n(0)$ at $T = 0$. In Sec. III C we present numerical results for the gap function $\Delta_n(\omega_m)$ at $T = T_{p,n} - 0$. We show that $\Delta_n(\omega_m)$ changes sign $n$ times as a function of discrete Matsubara frequency. We extend the numerical analysis to a range of $T \leq T_{p,n}$, where $\Delta_n(\omega_m)$ is small but finite, and show that $\Delta_n(\omega_m)$ still changes sign $n$ times, i.e., its topological structure does not change below $T_{p,n}$. We argue that at $T \to 0$ this $\Delta_n(\omega_m)$ approaches the $n$–th solution of the nonlinear gap equation at $T = 0$, which has the same topological structure. In Sec. IV we discuss how the density of eigenvalues at $T \to 0$ and the onset temperatures $T_{p,n}$ evolve away from a QCP, when the pairing boson acquires a finite mass $M$. Finally, in Sec. V we present our conclusions and briefly outline what we will do next.

II. THE $\gamma$-MODEL, ELIASHBERG EQUATIONS

The $\gamma$-model was introduced in Paper I and in earlier papers, and we refer a reader to these works for the justification of the model and its relation to various quantum-critical systems, e.g., fermions at the verge of a nematic of SDW/CDW transition, or SYK-type models of dispersion-less fermions randomly interacting with optical phonons. The model describes low-energy fermions, interacting by exchanging soft bosonic excitations. The effective dynamical 4-fermion interaction, which contributes to both the fermionic self-energy $\Sigma(k_F,\omega_m) = \Sigma(\omega_m)$ and the pairing vertex $\Phi(\omega_m)$, is $V(\Omega_m) = g^\gamma/|\Omega_m|^\gamma$ with the exponent $\gamma$. The effective dynamical 4-fermion interaction, which contributes to both the fermionic self-energy $\Sigma(k_F,\omega_m) = \Sigma(\omega_m)$ and the pairing vertex $\Phi(\omega_m)$, is $V(\Omega_m) = g^\gamma/|\Omega_m|^\gamma$ with the exponent $\gamma$. The effective dynamical 4-fermion interaction, which contributes to both the fermionic self-energy $\Sigma(k_F,\omega_m) = \Sigma(\omega_m)$ and the pairing vertex $\Phi(\omega_m)$, is $V(\Omega_m) = g^\gamma/|\Omega_m|^\gamma$ with the exponent $\gamma$.
\[ \gamma. \] [For \( \Phi(\omega) \), this holds in an attractive channel with a particular spatial symmetry, determined by the underlying model with full momentum dependence of the interaction.] We assume, like in earlier works, that soft bosons are slow compared to dressed fermions. In this situation, the coupled equations for \( \Sigma(\omega_m) \) and \( \Phi(\omega_m) \) are similar to Eliashberg equations for the case when the effective dynamical interaction is mediated by phonons and we will use the term “Eliashberg equations” for our case.

At a finite \( T \) the coupled Eliashberg equations for \( \Phi(\omega_m) \) and \( \Sigma(\omega_m) \) are, in Matsubara formalism

\[
\Phi(\omega_m) = \bar{g}^\gamma \pi T \sum_{m'} \frac{\Phi(\omega_m')}{\sqrt{\bar{\Sigma}^2(\omega_m') + \Phi^2(\omega_m')}} \frac{1}{|\omega_m - \omega_m'|\gamma},
\]

\[
\bar{\Sigma}(\omega_m) = \omega_m + \bar{g}^\gamma \pi T \sum_{m'} \frac{\bar{\Sigma}(\omega_m')}{\sqrt{\bar{\Sigma}^2(\omega_m') + \Phi^2(\omega_m')}} \frac{1}{|\omega_m - \omega_m'|\gamma},
\]

(1)

where \( \bar{\Sigma}(\omega_m) = \omega_m + \Sigma(\omega_m) \). Observe that we define \( \Sigma(\omega_m) \) with the overall plus sign and without the overall factor of \( i \), that is \( G^{-1}(k, \omega_m) = i\Sigma(\omega_m) - \epsilon_k \). In these notations, \( \Sigma(\omega_m) \) is a real function, odd in frequency.

The superconducting gap function \( \Delta(\omega_m) \) is defined as

\[
\Delta(\omega_m) = \frac{\Phi(\omega_m)}{\Sigma(\omega_m)} = \frac{\Phi(\omega_m)}{1 + \Sigma(\omega_m)/\omega_m}
\]

(2)

The equation for \( \Delta(\omega_m) \) is readily obtained from (1):

\[
\Delta(\omega_m) = \bar{g}^\gamma \pi T \sum_{m'} \frac{\Delta(\omega_m') - \Delta(\omega_m)\frac{\omega_m'}{\omega_m}}{\sqrt{(\omega_m')^2 + \Delta^2(\omega_m')}} \frac{1}{|\omega_m - \omega_m'|\gamma}.
\]

(3)

This equation contains a single function \( \Delta(\omega_m) \), but at the cost that \( \Delta(\omega_m) \) appears also in the r.h.s. Both \( \Phi(\omega_m) \) and \( \Delta(\omega_m) \) are defined up to an overall \( U(1) \) phase factor, which we set to zero.

The full set of Eliashberg equations for electron-mediated pairing contains the additional equation, which describes the feedback from the pairing on \( V(\Omega) \). In this study we do not include this feedback into consideration, to avoid additional complications. In general terms, the feedback from the pairing makes bosons less incoherent and can be modeled by taking the exponent \( \gamma \) to be \( T \) dependent and by moving it towards larger values as \( T \) decreases.

Both \( \Phi(\omega_m) \) and \( \Sigma(\omega_m) \) contain divergent contributions from the \( m' = m \) terms in the summation over internal Matsubara frequencies. The divergencies, however, cancel out in
the gap equation (3) by Anderson theorem, because scattering with zero frequency transfer mimics the effect of scattering by non-magnetic impurities. To get rid of divergencies in \( (1) \), one can use the same trick as in the derivation of the Anderson theorem: pull out the terms with \( m' = m \) from the frequency sums, move them to the l.h.s. of the equations, and introduce new variables \( \Phi^*(\omega_m) \) and \( \Sigma^*(\omega_m) \) as

\[
\Phi^*(\omega_m) = \Phi(\omega_m) (1 - Q(\omega_m)), \\
\Sigma^*(\omega_m) = \Sigma(\omega_m) (1 - Q(\omega_m))
\]

where

\[
Q(\omega_m) = \frac{\pi TV(0)}{\sqrt{\Sigma^2(\omega_m) + \Phi^2(\omega_m)}}
\]

The ratio \( \Phi(\omega_m)/\Sigma(\omega_m) = \Phi^*(\omega_m)/\Sigma^*(\omega_m) \), hence \( \Delta(\omega_m) \), defined in (2), is invariant under this transformation. One can easily verify that the equations on \( \Phi^*(\omega_m) \) and \( \Sigma^*(\omega_m) \) are the same as in (1), but the summation over \( m' \) now excludes the divergent term with \( m' = m \).

To simplify the formulas, we re-define \( \Phi^* \) and \( \Sigma^* \) back as \( \Phi \) and \( \Sigma \). We have

\[
\Phi(\omega_m) = \bar{g}^\gamma \pi T \sum_{m' \neq m} \frac{\Phi(\omega_{m'})}{\sqrt{\Sigma^2(\omega_{m'}) + \Phi^2(\omega_{m'})}} \frac{1}{|\omega_m - \omega_{m'}\gamma|}, \\
\Sigma(\omega_m) = \omega_m + \bar{g}^\gamma \pi T \sum_{m' \neq m} \frac{\Sigma(\omega_{m'})}{\sqrt{\Sigma^2(\omega_{m'}) + \Phi^2(\omega_{m'})}} \frac{1}{|\omega_m - \omega_{m'}\gamma|}
\]

In the normal state \( (\Phi \equiv 0) \), the self-energy is \( (m > 0) \)

\[
\Sigma(\omega_m) = \bar{g}^\gamma (2\pi T)^{1-\gamma} \sum_{m' = 1}^{m} \frac{1}{|m'\gamma|} = \bar{g}^\gamma (2\pi T)^{1-\gamma} H_{m,\gamma}
\]

where \( H_{m,\gamma} \) is the Harmonic number. This expression holds for \( \omega_m \neq \pm \pi T \). For the two lowest Matsubara frequencies, \( \Sigma(\pm \pi T) = 0 \). This will be essential for our study. We remind that \( \Sigma(\omega_m) \) in (7) is not the full self-energy, as the summation is only over \( m' \neq 0 \). The full self-energy includes also the term with \( m' = 0 \), and is non-zero for all Matsubara frequencies.

We now extend the model and introduce a parameter \( N \) to control the relative strength of the interactions in the particle-hole and particle-particle channels. We treat \( N \) as a continuous variable (we use the same notations as in earlier works, where \( N \) was originally introduced by extending the original model to a matrix \( SU(N) \) model). With this extension

\[
\Phi(\omega_m) = \frac{\bar{g}^\gamma}{N} \pi T \sum_{m' \neq m} \frac{\Phi(\omega_{m'})}{\sqrt{\Sigma^2(\omega_{m'}) + \Phi^2(\omega_{m'})}} \frac{1}{|\omega_m - \omega_{m'}\gamma|},
\]

\[
\Sigma(\omega_m) = \omega_m + \bar{g}^\gamma \pi T \sum_{m' \neq m} \frac{\Sigma(\omega_{m'})}{\sqrt{\Sigma^2(\omega_{m'}) + \Phi^2(\omega_{m'})}} \frac{1}{|\omega_m - \omega_{m'}\gamma|}.
\]
and
\[
\Delta(\omega_m) = \frac{\bar{g}^\gamma N \pi T}{N} \sum_{m' \neq m} \frac{\Delta(\omega'_m) - N \Delta(\omega_m) \omega'_m}{\omega'_m} \left[ \frac{1}{(\omega'_m)^2 + \Delta^2(\omega'_m)} \left| \frac{\omega_m - \omega'_m}{\omega'_m} \right| \right].
\]

We emphasize that for our study the extension to \( N \neq 1 \) is just a convenient way to better understand the interplay between the tendencies towards NFL and pairing. Our ultimate goal is to understand the physics in the physical case of \( N = 1 \). This is why we first eliminated the divergent terms, which cancel out in the gap equation for \( N = 1 \), and only then extended the model to \( N \neq 1 \). An alternative approach, suggested in Ref.\(^6\), is to extend to \( N \neq 1 \) without first subtracting the \( m' = m \) terms in the equations for \( \Phi(\omega_m) \) and \( \Sigma(\omega_m) \).

In this case, one has to deal with the actual divergencies in the r.h.s of these equations and also in the gap equation.

We will chiefly analyze the linearized equation for the pairing vertex \( \Phi(\omega_m) \)
\[
N \Phi(\omega_m) = \bar{g}^\gamma \pi T \sum_{m' \neq m} \frac{\Phi(\omega'_m)}{|\Sigma(\omega'_m)|} \frac{1}{|\omega_m - \omega'_m|^\gamma},
\]
and the gap function
\[
\Delta(\omega_m) = \frac{\bar{g}^\gamma N \pi T}{N} \sum_{m' \neq m} \frac{\Delta(\omega'_m) - N \Delta(\omega_m) \omega'_m}{\omega'_m} \left[ \frac{1}{|\omega'_m|} \left| \frac{\omega_m - \omega'_m}{\omega'_m} \right| \right].
\]

The two equations are equivalent in the sense that both have a non-trivial solution at the onset temperature for the pairing. We label this temperature as \( T_p \) rather than superconducting \( T_c \) because the latter is smaller due to gap fluctuations. In this paper we focus on the solution of the Eliashberg equations and neglect gap fluctuations. The comparative analysis of \( T_p \) and \( T_c \) will be presented in a separate paper.

To distinguish between finite \( T \) and \( T = 0 \) results, below we label the gap function at a finite \( T \) as \( \Delta_n(m) \), where \( m \) is a discrete Matsubara number, \( \omega_m = \pi T(2m + 1) \), and at \( T = 0 \) as \( \Delta(\omega_m) \), where \( \omega_m \) is a continuous Matsubara frequency. We use the same notations for other quantities.

III. THE LINEARIZED GAP EQUATION, ANALYTIC CONSIDERATION

We re-write the self-energy \([7]\) as
\[
\Sigma(m) = \pi TKA(m) \text{sgn}(2m + 1)
\]
where

\[ A(m) = 2 \sum_{1}^{m} \frac{1}{n^{\gamma}}, \quad m > 0 \]
\[ A(m) = A(-m - 1), \quad m < -1 \]
\[ A(0) = A(-1) = 0 \quad (13) \]

and

\[ K = \left( \frac{\bar{g}}{2\pi T} \right)^{\gamma} \quad (14) \]

Using these notations and the fact that \( \Phi(m) \) is an even under \( m \to -m - 1 \), we re-write Eq. (10) for \( \Phi(m) \) as the set of two coupled equations, by singling out \( \Phi(0) = \Phi(-1) \) (Ref. 2):

\[ (N - K)\Phi(0) = \sum_{n=1}^{\infty} \frac{\Phi(n)}{A(n) + 2n + 1} \left( \frac{1}{n^{\gamma}} + \frac{1}{(n + 1)^{\gamma}} \right) \quad (15) \]
\[ \Phi(m > 0) = \frac{1}{N} \sum_{n=1, n \neq m}^{\infty} \frac{\Phi(n)}{A(n) + 2n + 1} \left( \frac{1}{n - m^{\gamma}} + \frac{1}{(n + m + 1)^{\gamma}} \right) + \frac{\Phi(0)K}{N} \left( \frac{1}{n^{\gamma}} + \frac{1}{(n + 1)^{\gamma}} \right) \quad (16) \]

Eliminating \( \Phi(0) \) from this set we obtain

\[ N\Phi(m > 0) = \sum_{n=1, n \neq m}^{\infty} \frac{\Phi(n)}{A(n) + 2n + 1} \left( \frac{1}{n - m^{\gamma}} + \frac{1}{(n + m + 1)^{\gamma}} \right) + \sum_{n=1}^{\infty} \frac{\Phi(n)}{A(n) + 2n + 1} \left( \frac{1}{n^{\gamma}} + \frac{1}{(n + 1)^{\gamma}} \right) + \frac{K}{N - K} \sum_{n=1}^{\infty} \frac{\Phi(n)}{A(n) + 2n + 1} \left( \frac{1}{n^{\gamma}} + \frac{1}{(n + 1)^{\gamma}} \right) \left( \frac{1}{m^{\gamma}} + \frac{1}{(m + 1)^{\gamma}} \right) \quad (17) \]

A quick inspection of Eq. (17) shows that at low \( T \), when \( K \gg 1 \), there are two regions of \( N \), in which the solution with \( \Phi(m) \neq 0 \) may appear: a) \( N \approx K \), such that \( N - K = \text{const} \), and b) \( N = O(1) \), such that \( K \gg N \). We consider these two regions separately.

A. The region of large \( N, N \approx K \)

We express \( N \) as \( N = K + b_{\gamma} \), substitute into (17), and take the limit \( K \to \infty \). The divergent \( K \) cancels out, and we obtain

\[ b_{\gamma}\Phi(m > 0) = \sum_{n=1}^{\infty} \frac{\Phi(n)}{A(n)} \left( \frac{1}{n^{\gamma}} + \frac{1}{(n + 1)^{\gamma}} \right) \left( \frac{1}{m^{\gamma}} + \frac{1}{(m + 1)^{\gamma}} \right) \quad (18) \]

The form of the kernel implies that the solution should be in the form

\[ \Phi(m > 0) = C \left( \frac{1}{m^{\gamma}} + \frac{1}{(m + 1)^{\gamma}} \right) \quad (19) \]

Substituting this form into (18) we obtain that \( b_{\gamma} \) is given by

\[ b_{\gamma} = \sum_{n=1}^{\infty} \frac{1}{A(n)} \left( \frac{1}{n^{\gamma}} + \frac{1}{(n + 1)^{\gamma}} \right)^{2} \quad (20) \]
At large $n$, $A(n) \approx 2n^{1-\gamma}/(1-\gamma)$. The sum in the r.h.s. of (20) then converges for all $0 < \gamma < 1$, i.e., $b_\gamma$ is indeed of order one. We plot $b_\gamma$ in Fig. 1. At small $\gamma$, $b_\gamma \propto 1/\gamma$. At $\gamma = 1$, $b_1 = 1.63303$.

Using $K = N - b_\gamma$, we obtain the onset temperature for the pairing

$$T_p = \left( \frac{\bar{g}}{2\pi} \right) \frac{1}{N^{1/\gamma}} \left( 1 + \frac{b_\gamma}{N\gamma} \right)$$

We emphasize that $T_p$ remains finite for arbitrary large $N$, i.e., for arbitrary weak pairing interaction. Eq. (21) has been earlier obtained in Ref.\cite{2} using somewhat different computational procedure.

Using (15), (19), and (20), we find $\Phi(0) = C$. Comparing with (19), we see that the magnitude of the pairing vertex at Matsubara frequencies $\omega_m = \pm \pi T$ is of the same order as at other frequencies (a more detailed analysis shows that $\Phi(m)$ is the largest at $m = 2$, see Fig. 8). At the same time, the pairing gap $\Delta(m)$ is strongly peaked at $m = 0, -1$: $\Delta(0) = \Delta(-1) = C$, while $\Delta(m > 0) \sim C/K$, i.e., is much smaller. In fact, the leading term in the expression for $T_p$ in Eq. (21) can be obtained from the gap equation (9) if we keep there only the terms with $m = 0, -1$ and use the fact that $\sum'_m \text{sign}(2m' + 1)/|m'|^{\gamma} = 0$. In this case, the gap equation simplifies to

$$\Delta(0) = \frac{\bar{g}^\gamma \Delta(-1)}{N (2\pi T)^\gamma}$$

$$\Delta(-1) = \frac{\bar{g}^\gamma \Delta(0)}{N (2\pi T)^\gamma}$$

Solving this set we obtain $T_p = \bar{g}/(2\pi N^{1/\gamma})$. The outcome is that at large $N$, the pairing predominantly involves fermions with Matsubara frequencies $\omega_m = \pm \pi T$ (Matsubara num-
FIG. 2. The gap function and the density of states below $T_{p,0}$, for $N > N_{cr}$. We set $\gamma = 0.9$, in which case $N_{cr} = 1.340$. Panel (a): $\Delta(m = 0) \equiv \Delta(\pi T)$ as a function of $T$. Panel (b): the density of states $\nu(\omega)$ in real frequency, normalized to its value in the normal state. The density of states shows a gap-filling behavior: at $T$ increases towards $T_{p,0}$, states at low frequencies fill in, and the position of the maximum shifts to higher $\omega$.

Below we classify the solutions for the pairing vertex function by an integer $n$, which is the number of times $\Phi(m)$ changes sign as a function of Matsubara number $m$. In this classification, the sign-preserving pairing vertex $\Phi(m)$ in (19) corresponds to $n = 0$. Accordingly, $T_p$ in (21) is $T_{p,0}$ and $\Phi(m) = \Phi_0(m), \Delta(m) = \Delta_0(m)$.

At a first glance, the result that $T_{p,0}$ is non-zero at large $N$ contradicts the $T = 0$ analysis in Paper I and earlier studies that at $T = 0$ there exists critical $N = N_{cr}$, which separates the state with a finite $\Delta(\omega_m)$ for $N < N_{cr}$ and the NFL normal state for $N > N_{cr}$. We, however, showed in Refs.\textsuperscript{3} that $\Phi(m)$ and $\Delta(m)$ evolve non-monotonically below $T_{p,0}$ and for $N > N_{cr}$ vanish at $T \to 0$. We show the behavior of $\Delta(m = 0) \equiv \Delta(\pi T)$ vs temperature in Fig. 2a. Such a state has a number of other unusual properties, e.g., the gap-filling behavior of the density of states and the spectral function (see Fig. 2b). We refer a reader to Refs.\textsuperscript{3} for the detailed analysis of the Eliashberg equations below $T_{p,0}$ and of the role of gap fluctuations. In Appendix A we compute the free energy $F_{0,p}$ of a state with a finite $\Delta_0(m)$ for $N > N_{cr}$ and analyze $\delta F = F_{p,0} - F_n$, where $F_n$ is the free energy at $\Delta_0(m) = 0$. We use $\delta F = \delta E - T\delta S$ and study the two terms separately. We find that at $T \leq T_{p,0}$, negative $\delta F$ comes primarily from negative $\delta E$, while $-T\delta S$ is positive. However, below a certain $T$, close to where $\Delta(m = 0)$ changes its behavior in Fig. 2, $-T\delta S$ becomes...
FIG. 3. Two potential phase diagrams of the $\gamma$-model. (a) There exists only one onset temperature $T_{p,0}$ for any given $N$, like in BCS theory. In this case, $N_{cr}$ is an isolated $T = 0$ QCP, with no transition line attached to it. (b) There exists an infinite set of $T_{p,n}$, which all terminate at $N = N_{cr}$. In this case, a QCP at $N = N_{cr}$ is critical point of an infinite order. We show that the correct phase diagram is the one in the panel (b).

negative, and $\delta E$ almost vanishes. In this $T$ range, a negative $\delta F$ comes primarily from the entropy. This is consistent with the vanishing of $\Delta_0 (m)$ at $T = 0$.

Now, if $T_{p,0}$ was the only solution of the linearized gap equation, the phase diagram within the Eliashberg theory would be as in Fig. 3(a), i.e., there would be an isolated quantum-critical point (QCP) at $T = 0$ and $N = N_{cr}$, with no transition line coming out of it. We show below that this is not the case, and the actual phase diagram is the one in Fig. 3(b), with infinite number of lines terminating at $N_{cr}$. For this to hold, the linearized gap equation must have an infinite set of onset temperatures $T_{p,n}$ for any $N < N_{cr}$. This is what we analyze next.

**B. The region $N = O(1)$**

Our reasoning to search for additional solutions of the gap equation in the region $N = O(1)$ comes from the $T = 0$ analysis in Paper I. The key result of that Paper I is that for any $N < N_{cr}$ (which is $O(1)$ for a generic $\gamma < 1$), there exists an infinite, discrete set of solutions of the non-linear gap equation, $\Delta_n (\omega_m)$. If system properties evolve smoothly
between \( T = 0 \) and \( T > 0 \), each gap function evolves with \( T \) and has to vanish at some finite \( T_{p,n} \). By this logic, there must be an infinite set of \( T_{p,n} \) for any given \( N < N_{cr} \). Because all \( \Delta_n(\omega_m) \) at \( T = 0 \) vanish at \( N = N_{cr} \), all \( T_{p,n}(N) \) with finite \( n \) must approach 0 at \( N = N_{cr} \), as in Fig.3(b).

We first verify the very assumption that the system behavior evolves smoothly between \( T = 0 \) and \( T > 0 \). This is not a priori guaranteed, as at \( T = 0 \) the infinite set of solutions for all \( N < N_{cr} \) emerges due to a fine balance between the tendencies towards NFL and pairing. A finite \( T \) perturbs this balance, and it is possible in principle that an infinite set of solutions exists only at \( T = 0 \), while at a finite \( T \) only a single solution with a finite \( \Delta \) survives.

To address this issue, we recall that the origin for the appearance of the set of \( \Delta_n(\omega_m) \) at \( T = 0 \) is the existence of the solution of the linearized gap equation for all \( N < N_{cr} \) rather than only for \( N = N_{cr} \). In Paper I we obtained the exact solution of the linearized gap equation for all \( N < N_{cr} \) and \( 0 < \gamma < 1 \). At small \( \omega_m \ll \bar{g} \), the exact solution reduces to

\[
\Delta(\omega_m) = E|\omega_m|^{\gamma/2} \cos \left( \beta_N \log \frac{|\omega_m|}{\bar{g}} + \phi_N \right) \tag{23}
\]

where \( E \) is an infinitesimally small overall magnitude, \( \phi_N = O(1) \) is the phase factor, and \( \beta_N \) is the real solution of \( \epsilon_{\beta_N} = N \), where

\[
\epsilon_{\beta} = \frac{1 - \gamma}{2} \frac{|\Gamma(\gamma/2(1 + 2i\beta))|}{\Gamma(\gamma)} \left( 1 + \frac{\cosh(\pi\gamma\beta)}{\cos(\pi\gamma/2)} \right) \tag{24}
\]

A solution of this equation with a real \( \beta = \pm \beta_N \) exists for \( N < N_{cr} \), where

\[
N_{cr} = \frac{1 - \gamma}{2} \frac{\Gamma^2(\gamma/2)}{\Gamma(\gamma)} \left( 1 + \frac{1}{\cos(\pi\gamma/2)} \right) \tag{25}
\]

One can verify that \( N_{cr} > 1 \) for \( 0 < \gamma < 1 \). At \( \gamma \ll 1 \), \( N_{cr} \approx 4/\gamma \). At \( \gamma \to 1 \), \( N_{cr} \to 1 \).

It is instructive to interpret \( \epsilon_{\beta_N} = N \) as the dispersion relation and identify \( \beta_N \) with the effective momentum and \( N \) with the effective energy. Then one can define the DoE as

\[
\nu(N) \propto \left. \frac{d\beta}{d\epsilon_{\beta}} \right|_{\beta = \beta_N} \tag{26}
\]

We plot this function in Fig.4. As expected, it is non-zero for all \( N < N_{cr} \). It is singular near \( N = 0 \), where \( \nu(N) \propto (1/N)^{(2-\gamma)/(1-\gamma)} \) and near \( N_{cr} \), where \( \nu(N) \propto 1/\sqrt{N_{cr} - N} \). This last singularity, however, affects \( \nu(N) \) only in the immediate vicinity of \( N = N_{cr} \), as one can
FIG. 4. The DoE $\nu(N)$ (in arbitrary units) at $T = 0$, for $\gamma = 0.5$. The DoE is non-zero for all $N < N_{cr}$. It has a strong singularity near $N = 0$ and a weaker singularity at $N = N_{cr}$.

see from Fig. 4. The DoE in (26) is defined up to an overall factor. In Fig. 4, we plot $\nu(N)$ without normalizing it.

Now, for a smooth evolution of system properties between $T = 0$ and $T > 0$ the same $\nu(N)$ must emerge if we solve the linearized gap equation by approaching the $T = 0$ limit from a finite $T$. To verify whether this is the case, we keep $N = O(1)$ and set $K \propto 1/T^{\gamma}$ to infinity in (17).

1. The limit $T \to 0$

Keeping $N$ finite and setting $K$ to infinity, we obtain from (17), after symmetrization and rescaling

$$\tilde{\Phi}_{m>0} = \sum_{n=1, n \neq m}^{\infty} \tilde{\Phi}_n \frac{K_{n,m}}{(B_mB_n)^{1/2}},$$

(27)

where $\tilde{\Phi}_m = \Phi_m (S_m/A_m)^{1/2}$, $B_m = S_mA_m$, and $S_m$ and $K_{n,m}$ are given by

$$S_m = N - \frac{1}{A_m} \left( \frac{1}{(2m+1)^\gamma} - \left( \frac{1}{m^\gamma} + \frac{1}{(m+1)^\gamma} \right)^2 \right),$$

(28)

$$K_{n,m} = \frac{1}{|n - m|^\gamma} + \frac{1}{(n+m+1)^\gamma} - \left( \frac{1}{m^\gamma} + \frac{1}{(m+1)^\gamma} \right) \left( \frac{1}{n^\gamma} + \frac{1}{(n+1)^\gamma} \right)$$

(29)

In explicit form,

$$B_m = 2N \sum_{n=1}^{\infty} \frac{1}{n^\gamma} - \frac{1}{(2m+1)^\gamma} + \left( \frac{1}{m^\gamma} + \frac{1}{(m+1)^\gamma} \right)^2$$

(30)
At large $n, m \gg 1$, $B_m \approx 2N m^{1-\gamma}/(1-\gamma)$, $K_{n,m} \approx \frac{1}{|n-m|^\gamma} + \frac{1}{(n+m)^\gamma}$, $\Phi_m \approx \Phi_m$, and the equation for the pairing vertex can be approximated by an integral equation

$$\Phi_m = \frac{1 - \gamma}{2N} \int_0^\infty dn \frac{\Phi_n}{n^{1-\gamma}} \left( \frac{1}{|n-m|^\gamma} + \frac{1}{(n+m)^\gamma} \right)$$  \tag{31}$$

This equation has been analyzed in Paper I and in earlier works. The solution is

$$\Phi_{m>0} = \frac{E}{m^{\gamma/2}} \cos (\beta_N \log m + \phi)$$  \tag{32}$$

where $\beta_N$ is the same as in (23) and $E$ is an infinitesimally small overall factor. For these $m$, $\Sigma_m \propto m^{1-\gamma}$, hence

$$\Delta_{m>0} = E m^{\gamma/2} \cos (\beta_N \log m + \phi)$$  \tag{33}$$

The functional form of $\Delta_m$ is the same as for the $T=0$ solution, but at this stage $\phi$ is a free parameter. To be the solution of the gap equation for all $m$, this $\Delta_m$ (or $\Phi_m$) has to match with the solution at small $m$, when the discreteness of Matsubara numbers becomes relevant. If this can be achieved by fixing the value of $\phi$, then $\nu(N)$ at $T \to 0$ is the same as at $T = 0$.

This is similar to the $T = 0$ case, where the oscillating $\Delta(\omega_m)$ from (23) has to match with the non-oscillating solution $\Delta(\omega_m) \propto 1/|\omega_m|^\gamma$ for $|\omega_m| \gg \bar{g}$. For $T = 0$, the exact solution and the approximate, but highly accurate, analytical solution show that this is the case. Namely, for some particular $\phi = \phi_N$, the exact solution has the form of Eq. (23) for $|\omega_m| \ll \bar{g}$ and decays as $1/|\omega_m|^\gamma$ for $|\omega_m| \gg \bar{g}$. For $T \to 0$, we solve the gap equation numerically. We present the results for the DoE in Fig. 5. While for any finite number $M$ of Matsubara points in numerical calculations the DoE consists of a discrete set of points, we see from the histogram of the eigenvalues in panel (a) that at larger $M$, more eigenvalues move to larger $N$ and the number of eigenvalues in any fixed interval of $N$ increases. The “smoothened” $\nu(N)$ in panel (b) weakly depends on $M$ and is quite similar to the DoE at $T = 0$ in Fig. 4. This strongly indicates that Eq. (27) has a solution for any $N < N_{cr}$, like at $T = 0$, i.e. the system evolves continuously between $T = 0$ and $T \to 0$.

2. The case of small but finite $T$

We next consider the case when $T$ is small, but finite. Like we said at the beginning of Sec. III B at $T = 0$ and $N < N_{cr}$, there exists an infinite, discrete set of solutions of the
non-linear gap equation, $\Delta_n(\omega_m)$. It is natural to expect that each $\Delta_n$ smoothly evolves with $T$ and ends at a finite $T_{p,n}$. We then expect that Eq. (17) should have an infinite set of solutions at large but finite $K$. The equation for the pairing vertex in this case has the same form as in Eq. (27), but one should keep the bare $\omega$ term along with the self-energy, i.e., replace $A_m$ by

$$ A_m + \frac{2m + 1}{K} \tag{34} $$

The implication here is for large enough $m \sim K^{1/\gamma}$, the second term becomes comparable to the first one, and this will modify the functional form of $\Delta_m$ compared to that in Eq. (32). The other dependence on $K$, from $K/(N - K) \approx -1 + N/K$, is irrelevant as $1/K$ term never becomes comparable to other terms in the r.h.s. of (17).

A qualitative argument for the existence of the infinite set of $T_{p,n}$ is the following:

- at $m \gg 1$, the difference between summation over Matsubara numbers $n$ and integration is negligible. Hence, the solution of the gap equation, expressed in terms of $\omega_m \approx 2\pi m T$, should be the same as the exact solution at $T = 0$, Eq. (23). In terms
of $m$ we have

$$\Delta_m = Em^{\gamma/2} \cos \left( \beta_N \log m - \beta_N \log K^{1/\gamma} + \phi_N \right) \quad (35)$$

The phase $\phi_N$ is fixed by the requirement that $\Delta_m \propto 1/|m|^\gamma$ for $m > K^{1/\gamma}$, but there is another phase factor $-\beta_N \log K^{1/\gamma}$, which depends on $K$.

- From the analysis at $T \to 0$ we know that the gap equation has solutions for $N < N_{cr}$. The form of a solution is the same as in (35), $\Delta_m = Em^{\gamma/2} \cos \left( \beta_N \log m + \bar{\phi}_N \right)$, with some particular $\bar{\phi}_N$ which depends on $N$ (for a fixed $\gamma$).

- We can match the two forms of $\Delta_m$ by relating the phases:

$$\bar{\phi}_N = \phi_N - \beta_N \log K^{1/\gamma} + \pi n, \quad (36)$$

where $n$ is integer. Solving (36), we obtain the set of $K_n$, for which this identity holds. Note that the additional factor is $\pi n$, not $2\pi n$, because we can independently flip the sign of $\Delta_m$ at $T \to 0$ keeping $\Delta_m$ at $T = 0$ intact.

Solving Eq. (36) for the critical temperature, we obtain

$$T_{p,n} \sim \bar{g} e^{-An}, \quad A = \frac{\pi}{\beta_N \gamma} \quad (37)$$

This is consistent with the result in Paper I that at $T = 0$, $\Delta_n(\omega_m = 0) \sim \bar{g} e^{-An}$. As $N$ increases towards $N_{cr}$, $\beta_N \propto (N_{cr} - N)^{1/2}$ gets smaller, and all $T_{p,n}$ with $n > 0$ become exponentially small in $N_{cr} - N$:

$$T_{p,n} \sim \bar{g} e^{-bn/(N_{cr} - N)^{1/2}}, \quad (38)$$

where $b = O(1)$. Eq. (38) shows that all $T_{p,n}(N)$ terminate simultaneously at $N = N_{cr}$.

A remark is in order here. Eq. (37) and the $T = 0$ result for $\Delta_n$ are based on the assumption that the solution of the linearized gap equation oscillates up to $\omega_m \sim \bar{g}$ and decays as $1/|\omega_m|^\gamma$ at larger $\omega_m$. This holds for most of $\gamma < 1$, but for very small $\gamma$, oscillations extend to larger scale. In this case, Eq. (36) has to be modified. We discuss this in Appendix B.

In Fig. 6 we show the numerical results for $T_{p,n}$ for $n = 1 - 4$, along with the result for $T_{p,0}$ from Eq. (21). We set representative $\gamma$ to be $\gamma = 0.3$ and $\gamma = 0.5$. We verified that $T_{p,0}$ behaves as $1/N^{1/\gamma}$, as expected. At other $T_{p,n}$ lines exponentially approach zero as $N$
The numerical solution of the gap equation for small but finite $T$ for $\gamma = 0.3$ and $\gamma = 0.5$. The temperatures $T_{p,n}$ are the onset temperatures for the pairing in different topological sectors (the corresponding eigenfunctions change sign $n$ times as functions of discrete Matsubara frequency). The highest $T_{p,0} \propto 1/N^{1/\gamma}$ terminates at $N = \infty$. Analytical reasoning shows that all other $T_{p,n}$ vanish at $N = N_{cr}$ (big red dot). Numerical results show that $T_{p,n}$ with finite $n > 0$ indeed approach zero at $N = N_{cr}$.

Tends to $N_{cr}$, and the slope becomes larger as $n$ increases. To obtain this behavior we used a “hybrid frequency scale” method, which allowed us to numerically cover an exponentially large frequency range and reach very low $T$, while keeping track of the Matsubara summation at the lowest frequencies. This is achieved by adopting a frequency mesh that overlaps with Matsubara frequencies $\omega_m = \pi T, 3\pi T, \cdots$ at small values and crosses over to a logarithmical spacing beyond a certain scale, above which the discreteness of the Matsubara sum becomes unimportant. We discuss this method in Appendix C. It was also used in Ref. 6 in addressing the interplay between the first-Matsubara physics and thermal fluctuations.

In Fig. 7 we plot $T_{p,n}$ with $n$ up to 17 for particular $N = 1$. We clearly see that $T_{p,n}$ scale as $e^{-A_n}$, as in Eq. (37). We extracted $\beta_{N=1}$ from the fit to the exponential form and obtained $\beta_{N=1} = 1.62$ for $\gamma = 0.3$ $\beta_{N=1} = 1.12$ for $\gamma = 0.5$. These values are quite close to the exact values, extracted from Eq. (24): $\beta_{N=1} = 1.71$ for $\gamma = 0.3$ $\beta_{N=1} = 1.27$ for $\gamma = 0.5$. The small difference comes from the numerical error of the hybrid frequency scale, which effectively shifts $N$ up by roughly 0.07 for $\gamma = 0.3$ and 0.13 for $\gamma = 0.5$. 

FIG. 6. The numerical solution of the gap equation for small but finite $T$ for $\gamma = 0.3$ and $\gamma = 0.5$. The temperatures $T_{p,n}$ are the onset temperatures for the pairing in different topological sectors (the corresponding eigenfunctions change sign $n$ times as functions of discrete Matsubara frequency). The highest $T_{p,0} \propto 1/N^{1/\gamma}$ terminates at $N = \infty$. Analytical reasoning shows that all other $T_{p,n}$ vanish at $N = N_{cr}$ (big red dot). Numerical results show that $T_{p,n}$ with finite $n > 0$ indeed approach zero at $N = N_{cr}$.
C. The structure of the gap function, $\Delta_n(m)$

Another result of the $T = 0$ analysis is that the solutions of the non-linear gap equation with different $n$ are topologically distinct — the gap function $\Delta_n(\omega_m)$ changes sign $n$ times as a function of Matsubara frequency. Because we expect that $\Delta_n(m)$, which develops below $T_{p,n}$, becomes $\Delta_n(\omega_m)$ at $T = 0$, it should change sign $n$ times as a function of Matsubara number $m$. The same should be true for the pairing vertex $\Phi_n(m)$. In Fig. 8 we show $\Phi_n(m)$ for a few smallest $n$ and for $n = 16, 17$. We see that $\Phi_n(m)$ indeed changes sign $n$ times. At large $n$, $\Phi_n(m)$ oscillates at large $m$ as a function of $\log m$, with the amplitude proportional to $1/|m|^\gamma/2$. This is exactly the same behavior as in Eq. (35), given that $\Delta_n(m) \sim \Phi_n(m)|m|^{\gamma}$. For comparison, in the last panel of Fig. 8 we plot the exact $\Phi(\omega_m)$ at $T = 0$. We see that at $T = T_{p,n}$, the form of $\Phi_n(m)$ for $m \gg 1$ is quite similar to that at $T = 0$.

That $\Phi_n(m)$ has to change the sign at least once follows from the relation between $\Phi(0)$ and $\Phi(m > 0)$, Eq. (15). For $K \gg N$:

$$\Phi(0) \approx -\frac{1}{K} \sum_{m=1}^{\infty} \frac{\Phi(m)}{A(m) + \frac{2m+1}{K}} \left( \frac{1}{m^{\gamma}} + \frac{1}{(m+1)^{\gamma}} \right). \quad (39)$$

This relation shows that even if $\Phi(m)$ has the same sign for all $m > 0$, $\Phi(0)$ would still be of the opposite sign. This is consistent with Fig. 8 which shows that $\Phi_1(m)$ changes sign at
FIG. 8. Panels (a)-(e) – The pairing vertex, $\Phi_n(m)$ at $T = T_{p,n}$, as a function of the Matsubara frequency $\omega_m = \pi T(2m + 1)$ for representative parameters $\gamma = 0.5$ and $N = 1$. We show $\Phi_n(m)$ for $n = 0, 1, 2, 16, 17$. The corresponding $T_{p,n}$ are shown in the figures. For $n = 0$, $\Phi_0(m) \propto (1|m|^{\gamma} + 1/|m + 1|^{\gamma})$ does not change sign. Other $\Phi_n(m)$ change sign $n$ times, and $T_{p,n} \propto e^{-An}$. The results for $n = 16, 17$ show that $\Phi_n(m)$ oscillates at $m \gg 1$ as a function of $\log m$, with the amplitude proportional to $1/|m|^{\gamma/2}$. Panel(f) - $\Phi_0(\omega_m)$ from the exact solution of the linearized equation for the pairing vertex at $T = 0$. The positions of zeros of $\Phi_n(m)$ are marked by crosses. The smallest frequency $\omega_0 = \pi T_{p,n}$ (different for different $n$) are shown by arrows.
FIG. 9. The solution of the non-linear equation for the pairing vertex for $T = 0.9T_{p,n}$, along with the solution at $T = T_{p,n} - 0$. The three panels are the solutions for $n = 0, 1, 2$. The number of sign changes remains the same at $T_{p,n}$ and $0.9T_{p,n}$, as indicated by the blue arrows, and the frequencies, at which $\Phi_n(m)$ changes sign, do not shift with $T$.

$m = O(1)$ and keeps the same sign at larger $m$. The same holds for larger $n$ - the first sign change occurs at $m = O(1)$, i.e., at $\omega_m \sim T_{p,n}$. In Fig. 9 we show the results for $\Phi_n(m)$ for $n = 0, 1, 2$ obtained by solving the non-linear equation for the pairing vertex for $T \leq T_{p,n}$. We expanded to order $\Phi^3_m$ and used the solution at $T = T_{p,n}$ as the source. We see that the number of sign changes remains the same, and the frequencies, at which the sign of $\Phi_n(m)$ changes, remain essentially independent on $T$. This is consistent with the result in Paper I that at $T = 0$, $\Delta_n(\omega_m)$ changes sign $n$ times at finite $\omega_m$.

Finally, in Fig. 10 we follow $\Delta_n(m)$ along the line $T_{p,n}(N)$ and show the evolution of the frequency $\omega_{max}$, at which $\Delta_n(m)$ changes sign last time (i.e., all sign changes are at $\omega_m < \omega_{max}$). Because all $T_{p,n}$ terminate at $N = N_{cr}$, $\omega_{max}$ must shrink, as $T_{p,n}$ decreases, and vanish at $T_{p,n} \to 0$, because right at $N = N_{cr}$ and $T = 0$, $\Delta(\omega_m)$ is sign-preserving (see Paper I). The data in Fig. 10 show that $\omega_{max}$ indeed decreases with decreasing $T_{p,n}$ for all values of $n$, as long as $n$ remains finite.
FIG. 10. The highest frequency $\omega_{\text{max}}$, at which $\Delta_n(m)$ changes sign, plotted vs $T_{p,n}$ for various $n$. The arrows indicate the direction towards smaller $T_{p,n}$. We see that $\omega_{\text{max}}$ decreases together with $T_{p,n}$, i.e., $n$ sign changes of $\Delta_n(m)$ occur at progressively smaller Matsubara frequencies.

IV. AWAY FROM A QCP

We now analyze how $T_{p,n}$ and the DoE at $T \to 0$ change away from a QCP, when the pairing boson acquires a finite mass, $M_b$. We argue that a finite $M_b$ introduces qualitative changes in the system behavior, i.e., there is a qualitative difference between the structure of the DoE at a finite $M_b$ and at $M_b = 0$. Specifically, we argue that a finite $M_b$ (i) makes the number of $T_{p,n}$ for a given $N$ to be finite, and (ii) splits $T_{p,n}$ at the smallest $T$ such that different $T_{p,n}$ terminate at different $N_{cr,n}$, all are smaller than $N_{cr}$. The temperature $T_{p,0}$ is still non-zero for any $N$, but at a finite $M_b$ it acquires a conventional form $T_{p,0} \sim M_b e^{-1/\lambda}$, where $\lambda \propto 1/N$ (see Eq. (43) below).

We begin with the analytical analysis. We model the bosonic propagator away from a QCP by

$$V(\Omega_m) = \frac{\tilde{g}^\gamma}{(\Omega_m^2 + M_b^2)^{\gamma/2}}$$

(40)

The linearized gap equation becomes

$$\Delta(\omega_m) = \frac{\tilde{g}^\gamma}{N} \pi T \sum_{m' \neq m} \frac{\Delta(\omega_{m'}) - N \Delta(\omega_m) \frac{\omega_{m'}}{\omega_m}}{|\omega_{m'}|} \frac{1}{(|\omega_m - \omega_{m'}|^2 + M_b^2)^{\gamma/2}}.$$  

(41)

Consider first the limit $T \to 0$. Replacing the summation over frequency by integration with $T$ as the lower limit, we immediately find that at a finite $M_b$ there is a simple difference
between sign-preserving and sign-changing solutions for $\Delta(\omega_m)$. For the sign-preserving solution, $\Delta(0)$ is finite. Taking properly the limit $\omega_m \to 0$, we obtain $T_{p,0}$ from self-consistent equation on $\Delta(0)$:

$$
(1 + \left( \frac{\bar{g}}{M_b} \right)^\gamma) = \frac{1}{N} \left( \frac{\bar{g}}{M_b} \right)^\gamma \log \frac{M_b}{T_{p,0}}.
$$

(42)

The second term in the l.h.s. is the contribution from the self-energy, which away from a QCP has a Fermi liquid form at frequencies below $M_b$. Solving for $T_{p,0}$, we find

$$
T_{p,0} \sim M_b e^{-N(1+\lambda)/\lambda}, \quad \lambda = 1 + \left( \frac{\bar{g}}{M_b} \right)^\gamma
$$

(43)

We see that $T_{p,0}$ is still non-zero for any $N$, however its dependence on $N$ is exponential. This is similar to the case of a BCS superconductor, where $T_c$ is finite for arbitrary weak coupling, albeit exponentially small. For $N = 1$, Eq. (43) has the same structure as McMillan formula $T_c \sim \omega_D e^{-(1+\lambda)/\lambda}$ (Ref.9). In qualitative distinction to the behavior at a QCP, now the existence of a non-zero $T_{p,0}$ for arbitrary large $N$ is due to ordinary Cooper logarithm in a Fermi liquid rather than to special properties of fermions with $\omega_m = \pm \pi T$ in a NFL regime. As a consequence, $\Delta_0(\omega_m)$, emerging below $T_{p,0}$, does not vanish at $T = 0$, i.e., at any $N$ the ground state is a superconductor. In this respect, there is no critical $N_{cr}$, separating normal and superconducting states at $T = 0$.

For solutions with $n > 0$, $\Delta_n(\omega_m)$ must vanish at $\omega = 0$ because there is just a single solution with a finite $\Delta(0)$. This sets the condition

$$
\Delta(0) \left( 1 + \left( \frac{\bar{g}}{M_b} \right)^\gamma \right) = \frac{\bar{g}^\gamma}{N} \int_0^\infty \frac{\Delta_n(\omega'_m)}{\omega'_m} \frac{d\omega'_m}{((\omega'_m)^2 + M_b^2)^{\gamma/2}} = 0
$$

(44)

We show below that $\Delta_n(\omega'_m)$ scale as $(\omega'_m)^2$ at small $\omega'_m$, hence the integral in (44) is infrared convergent. Eq. (44) then implies that $\Delta_n(\omega_m)$ must change sign $n$ times at some finite $\omega'_m$. This is qualitatively different from the situation at a QCP. There, all $T_{p,n}$ with finite $n$ terminate at $T = 0$ at $N = N_{cr}$. The gap function at $N = N_{cr}$ vanishes at $\omega_m = 0$, yet it remains sign-preserving (see Paper I for details). This holds at a QCP because $\Delta(\omega'_m) \propto |\omega'_m|^\gamma/2$, in which case $\Delta(\omega'_m)/\omega'_m$ is singular at $\omega_m \to 0$, and one cannot just set $\omega_m = 0$ in the gap equation, as it is done in Eq. (44). At small $M_b$, $T_{p,n}$ approaches zero at $N$ only slightly below $N_{cr}$, and $\Delta_n(\omega'_m)$ must recover the gap function at a QCP at $N = N_{cr}$ at frequencies above some scale, which vanishes when $M_b \to 0$. Because $\Delta(\omega_m)$ at $N = N_{cr}$ is sign-preserving, the $n$ sign changes of $\Delta_n(\omega_m)$ have to occur below this scale.
FIG. 11. The solutions of the linearized gap equation for a finite boson mass $M_b$. Different panels are for different $M_b/\bar{g}$, shown in the figures. We set $\gamma = 0.5$. The critical temperatures $T_{p,n}$ now terminate at different $N_{cr,n}$. This is qualitatively different from the behavior at a QCP, where all $T_{p,n}$ with $n > 0$ terminate at the same $N = N_{cr}$.

We now expand in $\omega_m$ in the r.h.s. of the gap equation (41) for $\Delta_n(\omega_m)$. Expanding and using (44) to cancel out the leading term, we obtain $\Delta_n(\omega_m) = A_n(\omega_n/M_b)^2$, where $A_n$ is given by

$$A_n = \left( \frac{\bar{g}}{M_b} \right)^\gamma \frac{\gamma}{N} \int_0^\infty dx \frac{\Delta_n(x) x^2 (1 + \gamma) - 1}{x (x^2 + 1)^{2+\gamma/2}}$$  \hspace{1cm} (45)

where $x = \omega_m/M_b$. The integral in (45) converges at $x = O(1)$, hence by order of magnitude $A_n \sim \bar{g}^\gamma / (M_b^{2+\gamma} N)$ with $n-$dependent prefactor. For an estimate, we assume that at large $n$ the integral is determined by $x_n \sim 1/n$, before oscillations begin, and that $\Delta_n(x < x_n) \approx A_n x^2$. Substituting into (45), we obtain that the solution is possible only for a given $N \sim 1/n^2$

$$N \gamma = -\frac{x_n^2}{2} \left( \frac{\bar{g}}{M_b} \right)^\gamma \int_0^{x_n} dx x^2 (1 + \gamma) - 1 \left( x^2 + 1 \right)^{2+\gamma/2}$$  \hspace{1cm} (46)

The outcome of this analysis is that at a non-zero $M_b$ the solutions of the linearized gap equation for different $n$ exist at different $N$, i.e., each $T_{p,n}$ terminates at its own $N_{cr,n}$.

In Fig. 11 we show the results of the numerical solution of the gap equation for a finite
\( \gamma = 0.5, M_b/\bar{g} = 10^{-3} \)

FIG. 12. The histogram of the DoE for a finite boson mass \( M_b/\bar{g} = 10^{-3} \) and \( \gamma = 0.5 \). We see that the histogram is heavily shifted towards \( N = 0 \) because now there is a finite number of points in a given interval around a particular \( N \) even when the total number of sampling Matsubara points \( M \) tends to infinity. This is qualitatively different from the case \( M_b = 0 \) in Fig. 5, where the number of points in a given interval around any \( N < N_{cr} \) scales with \( M \).

We see that \( T_{p,n} \) indeed terminate at different \( N \). We verified that \( T_{p,0} \) is exponential in \( N \), like in Eq. 43. The number of solutions, for which \( T_{p,n} \) crosses \( N = 1 \), is finite for any non-zero \( M_b \). It decreases one by one with increasing \( M_b \) and vanishes once \( M_b \) exceeds some critical value. At larger \( M_b \), there is only one onset temperature \( T_{p,0} \) for the physical case \( N = 1 \), and the behavior below \( T_{p,0} \) is qualitatively the same as in BCS theory.

The same behavior shows up in the analysis of the DoE at \( T \to 0 \). Because there is only a finite number of termination points of \( T_{p,n} \) in any finite interval of \( N \), the normalized DoE \( \nu(N) \) vanishes for all \( N \neq 0 \) in the formal limit \( M \to \infty \), where \( M \) is the number of Matsubara points, probed in a numerical calculation. Because termination lines cluster around \( N = 0 \), and the total \( \int dN \nu(N) = 1 \), the DoE becomes \( \nu(N) = \delta(N) \) in this limit. In practice, this implies that the DoE will decrease for any finite \( N \) when \( M \) increases, and the histogram of \( \nu(N) \) will shift with increasing \( M \) towards \( N = 0 \). In Fig... we show the numerical results for \( \nu(N) \). We see precisely this behavior. Note also that the largest \( N_{cr,1} \) shifts down from \( N_{cr} \) as \( M_b \) increases.
V. CONCLUSIONS

In this paper, the second in a series, we continued our analysis of the interplay between pairing and NFL in the model of fermions with singular dynamical interaction $V(\Omega_m) \propto 1/|\Omega_m|^\gamma$ (the $\gamma$-model). We introduced a knob (the parameter $N$) to vary the relative strength of the dynamical interaction in the particle-hole and particle-particle channels. In Paper I we studied the $\gamma$-model at $T=0$ and $0<\gamma<1$. We found that superconducting order exists for $N<N_{cr}(\gamma)$ and that for any such $N$ there is an infinite number of solutions of the non-linear gap equation, $\Delta_n(\omega_m)$, specified by the number of times ($n$) the gap function changes sign as a function of $\omega_m$. At large $n$, $\Delta_n(0) = e^{-an}$ is exponentially small in $n$.

In this paper, we analyzed the same model at a finite $T$. We showed the DoE remains continuous at $T \to 0$ and coincides with the one at $T=0$, i.e., the existence of an infinite set of solutions of the gap equation is not a special property of $T=0$. We further showed that each $\Delta_n(\omega_m)$ vanishes at a critical temperature $T_{p,n} \sim \Delta_n(0)$. Viewed as functions of $N$, all $T_{p,n}(N)$ with $n>0$ terminate at the same $N=N_{cr}$, which turns out to be a multicritical point (see Fig.3(b) and Fig.6). The eigen-function $\Delta_n(m)$ at $T = T_{p,n} - 0$ changes sign $n$ times as a function of Matsubara number $m$, and retains this property down to $T=0$, where it becomes a continuous function $\Delta_n(\omega_m)$. The temperature $T_{p,0}$ behaves differently – it remains finite for all $N$ and scales at large $N$ as $T_{p,0} \sim 1/N^{1/\gamma}$. In this limit, the pairing at $T_{p,0}$ predominantly involves fermions with the two lowest Matsubara frequencies $\pm \pi T$, because for these fermions non-thermal self-energy vanishes, which implies that pairing does not compete with NFL. At $N > N_{cr}$, $\Delta_0(m)$ initially grows below $T_{p,0}$, but then changes trend and vanishes at $T \to 0$.

This behavior holds only at a QCP. Away from a QCP, $T_{p,n}$ with different $n$ terminate at different $N_{cr,n}$, and for any $N > 0$, the number of $T_{p,n}$ is finite. It drops one by one as the bosonic mass $M_b$ gets larger, and disappears above a certain $M_b$. The onset temperature $T_{p,0}$ is still finite for any $N$, but the gap $\Delta_0(m)$ does not vanish at $T \to 0$, i.e., the ground state is a superconductor for any value of $N$.

The next paper will be devoted to analysis of how the physics of the $\gamma$-model at a QCP changes between $\gamma < 1$ and $\gamma > 1$.
ACKNOWLEDGMENTS

We thank I. Aleiner, B. Altshuler, E. Berg, D. Chowdhury, L. Classen, R. Combescot, R. Fernandes, A. Finkelstein, E. Fradkin, A. Georges, S. Hartnol, S. Karchu, S. Kivelson, I. Klebanov, A. Klein, R. Laughlin, G. Lonzarich, D. Maslov, M. Metlitski, W. Metzner, A. Millis, D. Mozyrsky, V. Pokrovsky, N. Prokofiev, S. Raghu, S. Sachdev, T. Senthil, D. Scalapino, Y. Schattner, J. Schmalian, D. Son, G. Tarnopolsky, A-M Tremblay, A. Tsvelik, G. Torroba, E. Yuzbashyan, and J. Zaanen for useful discussions. The work by AVC was supported by the NSF DMR-1834856.

Appendix A: Free energy below $T_{p,0}$ at large $N$

In this Appendix we address one issue about the system behavior at large $N$, below $T_{p,0}$. As we said in the text and in earlier papers, $T_{p,0}$ remains finite for arbitrary large $N$ because the pairing in the large $N$ limit predominantly involves fermions with Matsubara frequencies $\omega_m = \pm \pi T$ (Matsubara numbers $m = 0, -1$), for which in Eliashberg theory the non-thermal part of the self-energy vanishes. Because only non-thermal part of the self-energy appears in the gap equation, these fermions can be viewed as free quasiparticles for the purpose of the pairing. For all other fermions, non-thermal self-energy is strong and acts against pairing. We argue that the pairing at $T_{p,0}$ involves fermions with $\omega_m = \pm \pi T$, which develop $\Delta_0(m = 0) = \Delta_0(-1)$ below $T_{p,0}$. A much smaller gap $\Delta_0(m)$ at other Matsubara frequencies is then induced by proximity. The gap $\Delta_0(0)$ initially increases as $T$ decreases, but never exceeds $T$. At the smallest $T$, $\Delta_0(0)$ decreases proportional to $T$ and vanishes at $T = 0$, where a Matsubara frequency becomes a continuous function, and the special role of frequencies $\pm \pi T$ is lost.

The very existence of the solution of the Eliashberg equations with a non-zero $\Delta_0(m)$ implies that the free energy of such state, $F_{p,0}$, is smaller than that of the normal state (see Paper I and Refs.10-12). In explicit form, in this case

$$\delta F = F_{p,0} - F_n = -\frac{g^2 \nu_0}{2N^{1+2/\gamma}} t^{2-\gamma} (t^{\gamma} - 1)^2$$

(A1)

where $t = T/T_{p,0}$ and $\nu_0$ is the density of states at the Fermi level in the normal state.

The issue we address here is whether a negative $\delta F = \delta E - T \delta S$ comes from the change of the energy, like in BCS superconductor, or from the change of the entropy. In our case,
we can compute both terms explicitly at large $N$. We obtain
\begin{align}
\delta S &= S_{p,0} - S_n = \frac{\bar{g} \nu_0 \pi}{N^{1+1/\gamma}} t^{1-\gamma} (t^\gamma - 1) (t^\gamma (\gamma + 2) + \gamma - 2) \\
\delta E &= E_{p,0} - E_n = \frac{\bar{g}^2 \nu_0}{2 N^{1+2/\gamma}} t^{2-\gamma} (t^\gamma - 1) (t^\gamma (\gamma + 1) + \gamma - 1)
\end{align}

(A2)

We plot $\delta F, \delta S$, and $\delta E$ in Fig. 13 for a particular case of $\gamma = 0.5$ and $N = 5 > N_{cr}(\gamma = 0.5) = 4.476 < 5$. We clearly see that all three quantities vanish at $T = 0$, as expected. We also see that there is a clear change of behavior at larger $t$, when $\Delta_0(m)$ increases with decreasing $t$, and at smaller $t$, where $\Delta_0(m)$ scales with $t$. In the first case, $\delta E$ is negative and $\delta S$ is positive, i.e., negative $\delta F$ is due to negative $\delta E$, as in a BCS superconductor. However, at smaller $t$, $\delta E$ is reduced and actually becomes positive at the smallest $t$. In this $t$ range, a negative $\delta F$ is entirely due to positive $T\delta S$, i.e smaller free energy for the state with a finite $\Delta_0(m)$ is an entropic effect. This is also consistent with the fact that the gap function vanishes at $T = 0$.

**Appendix B: Onset temperatures $T_{p,n}$ at small $\gamma$**

The limit $\gamma \to 0$ of the $\gamma$ model attracted a lot of attention from various sub-communities in physics and has been analyzed in both Eliashberg-type and renormalization group methods.
approaches. In this Appendix, we present two expressions for $T_{p,n}$ at small $\gamma$ and $N = O(1)$, which is much smaller than $N_{cr} = 4/\gamma$. One expression is for the pure $\gamma$ model with $V(\Omega) = (\bar{g}/|\Omega|)^\gamma$. Another is for the case when we introduce an upper cutoff for $V(\Omega_m)$ at $\Omega_m = \Lambda$, i.e., modify the interaction to

$$V(\Omega_m) = \left(\frac{\bar{g}}{|\Omega_m|}\right)^\gamma \left[1 - \left(\frac{|\Omega_m|}{\Lambda}\right)^\gamma\right]$$

(B1)

for $|\Omega_m| < \Lambda$, and $V(\Omega_m) = 0$ for $|\Omega_m| > \Lambda$. The distinction is important for small $\gamma$ because at $\gamma \to 0$ $V(\Omega) \propto |\Omega|^{-\gamma}$ becomes frequency independent, and the pairing problem within the $\gamma$-model becomes equivalent to BCS, but without the upper cutoff for the interaction. In this situation, the cutoff at $\Lambda$ is necessary to avoid ultra-violet singularities. At the same time, at a finite $\gamma$, the gap equation is free from ultra-violet singularities already without $\Lambda$, and if $T_{p,n}$ are smaller than $\Lambda$, the cutoff can be safely neglected.

We consider the cases when $\Lambda$ is set to infinity (the pure $\gamma$ model) and when $\Lambda$ is finite separately, and then obtain an interpolation formula linking the two cases.

1. $T_{p,n}$ in the pure $\gamma$-model

The point of departure for the calculation of $T_{p,n}$ at small $\gamma$ is the observation in Paper I and in earlier works that for $N = O(1)$, $T_{p,0}$ is determined by fermions with frequencies much larger than $\bar{g}$. For these fermions, the normal state self-energy at $T = 0$, $\Sigma(\omega_m) \approx |\omega_m|^{1-\gamma}\bar{g} \text{sgn} \omega_m$ is smaller than the bare $\omega_m$ and can be safely neglected. Without the self-energy, $\Phi(m) = \Delta(m)$ and the gap equation reduces to

$$\Delta(m) = \frac{K}{N} \sum_{m' \neq m} \frac{\Delta(m')}{|2m'+1|} \frac{1}{|m-m'|^{\gamma}}.$$  

(B2)

where, we remind, $K = (\bar{g}/(2\pi T))^{\gamma}$. For $m, m' \gg 1$, the summation over $m'$ can be replaced by the integration. Also, for small $\gamma$, $1/|m-m'|^{\gamma}$ is reasonably well approximated by $1/|m|^{\gamma}$ for $|m| > |m'|$ and by $1/|m'|$ for $|m'| > |m|$. As a result, the gap equation for $m \gg 1$ can be approximately re-expressed as

$$\Delta(m) = \frac{K}{N} \left(\frac{1}{|m|^{\gamma}} \int_{0}^{m} \frac{dm'}{m'} \Delta(m')^{1+\gamma}\right)$$

(B3)

Introducing $z = m^\gamma$ and differentiating twice over $z$, we obtain differential equation

$$(z\Delta(z))'' = -\frac{K}{N\gamma} \frac{\Delta(z)}{z^2}$$

(B4)
The solution of this equation must satisfy two boundary conditions. One follows from Eq. \( \Delta(z \to \infty) \propto \frac{1}{z} \). Another comes from the boundary at \( m = O(1) \), i.e., at \( z \approx 1 \), where the discretization in Eq. \( B2 \) becomes relevant. Sign-preserving \( \Delta_0(m) \) is flat at \( m = O(1) \) and deviates from \( \Delta_0(0) \) only by \( O(\gamma) \). Then, to leading order in \( \gamma \), the second boundary condition is \( \Delta'(z = 1) = 0 \). For sign-changing \( \Delta_n(m) \), we can use the fact that the gap function changes sign at \( m = O(1) \) and use \( \Delta(z = 1) = 0 \).

The generic solution of (B4) is

\[
\Delta(z) = \frac{1}{\sqrt{z}} \left[ A_1 J_1 \left( 2 \sqrt{\frac{K}{N \gamma z}} \right) + A_2 Y_1 \left( 2 \sqrt{\frac{K}{N \gamma z}} \right) \right],
\]

where \( J_1 \) and \( Y_1 \) are Bessel and Neumann functions.

At large \( z \), \( J_1 \propto 1/\sqrt{z} \) and \( Y_1 \propto \sqrt{z} \). To satisfy the boundary condition at \( z \to \infty \) we must set \( A_2 = 0 \). Then

\[
\Delta(z) = \frac{A_1}{\sqrt{z}} J_1 \left( 2 \sqrt{\frac{K}{N \gamma z}} \right).
\]

A similar result has been obtained in Ref.\(^7\). Using the second boundary condition for sign-preserving solution (\( n = 0 \) in our nomenclature) and using \((xJ_1(x))' = xJ_0(x)\), we obtain

\[
J_1 \left( 2 \sqrt{\frac{K}{N \gamma}} \right) = 0,
\]

where, we remind, \( K = (\bar{g}/(2\pi T))^{\gamma} \). The highest \( T \) at which this equation has the solution is \( T = T_{p,0} \), where

\[
T_{p,0} = c\bar{g}(1.4458N\gamma)^{-1/\gamma}
\]

where \( c = O(1) \). For \( N = 1 \) this agrees with earlier results\(^12\,14\,15\). We see that \( T_{p,0} \) is indeed much larger than \( \bar{g} \) and tends to infinity when \( \gamma \to 0 \). As we said, this is the consequence of the fact that at \( \gamma \to 0 \) the \( \gamma \) model reduces to the BCS model without the cutoff on the interaction.

The prefactor \( c \) has been computed in Ref.\(^13\), and the result is \( c = 0.502 \). The gap at \( T = 0 \), \( \Delta_0(\omega_m = 0) \) has been obtained in Paper I and in Ref.\(^13\). \( \Delta_0(0) = 0.885\bar{g}(1.4458N\gamma)^{-1/\gamma} \).

The ratio \( 2\Delta/T_c = 3.53 \), as in BCS theory.

For sign-changing boundary conditions, the boundary condition \( J_1(2\sqrt{K/N\gamma}) = 0 \) yields

\[
T_{p,n} = c_n\bar{g}(a_nN\gamma)^{-1/\gamma}
\]
where \( c = O(1) \) and \( a_1 = 3.6705, a_2 = 12.3047 \), etc. This formula is valid up to some \( n \), for which \( T_{p,n} = O(\bar{g}) \). For larger \( n \), \( T_{p,n} \propto e^{-A_n} \), like in Eq. (37). The ratio \( 2\Delta_n(0)/T_{p,n} \) is of order of one for \( n = O(1) \), but we didn’t compute it explicitly.

Note that although \( T_{p,n} \) with \( n = O(1) \) are much larger than \( \bar{g} \), the ratio \( T_{p,1}/T_{p,0} \sim (0.394)^1/\gamma \) is parametrically small and vanishes when \( \gamma \to 0 \).

2. \( T_{p,n} \) in the \( \gamma \) model with the infra-red cutoff

We now consider the opposite limit when \( \gamma \to 0 \), but the upper cutoff for \( V(\Omega_m) \) is a finite \( \Lambda \). In this case, we use Eq. (B1) for the interaction.

The derivation of the differential equation proceeds along the same lines as before, and the result is the same Eq. (B4) as before. However, now the solution, \( \Delta(z) \), has to satisfy the boundary condition \( \Delta(K^*) = 0 \), where \( K^* = (\Lambda/2\pi T)^\gamma \). Such a solution is

\[
\Delta(z) \propto \frac{1}{\sqrt{z}} \times \left[ J_1 \left( 2\sqrt{\frac{K}{N\gamma z}} \right) Y_1 \left( 2\sqrt{\frac{K}{N\gamma K^*}} \right) - Y_1 \left( 2\sqrt{\frac{K}{N\gamma z}} \right) J_1 \left( 2\sqrt{\frac{K}{N\gamma K^*}} \right) \right]. \tag{B10}
\]

In the limit \( K/(N\gamma K^*) = (\bar{g}/\Lambda)^\gamma/(N\gamma) \gg 1 \), valid when \( \gamma \to 0 \) at a finite \( \Lambda \), we use the asymptotic forms of Bessel and Neumann functions at large values of the argument:

\[
J_1(x) \approx \sqrt{\frac{2}{\pi x}} \cos (x - 3\pi/4),
\]
\[
Y_1(x) \approx \sqrt{\frac{2}{\pi x}} \sin (x - 3\pi/4), \tag{B11}
\]

and obtain

\[
\Delta(z) \propto \frac{1}{z^{1/4}} \sin \left( 2\sqrt{\frac{K}{N\gamma K^*}} - 2\sqrt{\frac{K}{N\gamma z}} \right). \tag{B12}
\]

To the leading order in \( \gamma \), this reduces to

\[
\Delta(m) \propto \sin \left( \frac{\gamma}{N} \log \frac{2\pi T|m|}{\Lambda} \right). \tag{B13}
\]

Using the second boundary condition for sign-preserving solution, we find

\[
T_{p,0} \sim \Lambda e^{-\pi/2\sqrt{\lambda\gamma}}, \quad \lambda_\gamma = \frac{\gamma}{N} \tag{B14}
\]

This agrees with Refs. 15,17. For \( n > 0 \), using \( \Delta(z = 1) = 0 \), we obtain

\[
T_{p,n} \sim \Lambda e^{-n\pi/\sqrt{\lambda\gamma}}, \quad n = 1, 2... \tag{B15}
\]
Again, $T_{p,1}$ is parametrically smaller than $T_{p,0}$, and the ratio $T_{p,1}/T_{p,0}$ vanishes at $\gamma \to 0$, $T_{p,n}$ with larger $n$ are even smaller. Note that $T_{p,n}$ is independent on $\gamma$.

The zero-temperature gap $\Delta_n(\omega_m = 0)$ in the same limit has been computed in Paper I. Comparing it with $T_{p,n}$, Eqs. (B14) and (B15), we find that $T_{p,n}$ and $\Delta_n(0)$ are of the same order. A higher accuracy of calculations is required to compute $2\Delta/T_p$ ratios.

3. **Intermediate regime**

We now argue that Eq. (B10), describes the gap functions in both, the $\gamma$ model with the cutoff at $\Lambda$, and the pure $\gamma$ model. Indeed, in the opposite limit of large $\Lambda$ and finite $\gamma$, $K/(N\gamma K^*) = (\bar{g}/\Lambda)^\gamma/(N\gamma) \ll 1$. In this case $Y_1(2\sqrt{K/(N\gamma K^*)}) \gg J_1(2\sqrt{K/(N\gamma K^*)})$. Then only $Y_1(2\sqrt{K/(N\gamma K^*)})$ should be kept in (B10), and we recover Eq. (B6). In this limit, the functional form of $\Delta(z)$ does not depend on $\Lambda$ and we recover the result for $T_{p,n}$ for the pure $\gamma$ model. The crossover between the two regimes occurs at $(\bar{g}/\Lambda)^\gamma/(N\gamma) = O(1)$.

**Appendix C: The hybrid frequency scale**

Within our computation capacity, the maximum size of frequency mesh we can take, is approximately $10^4$. If we directly sum over Matsubara frequencies $\omega_m = (2m+1)\pi T$, the maximum frequency we can reach is around $10^5 T$. Because we are interested in frequencies $\omega_m < \bar{g}$, the lowest temperature one can reach is $T \sim 10^{-5}\bar{g}$. To access exponentially lower temperatures, one can in principle use a logarithmical frequency mesh in which $\log(\omega_m/\pi T) \propto m$, but in this scheme one loses the special role of the first Matsubara frequency and, more generally, of Matsubara frequencies with $m = O(1)$, which, e.g., set the phase in the expression for $\Delta(\omega_m)$ at $T \to 0$, Eq. (33). To overcome this difficulty we adopt a hybrid frequency scaling. Namely, we set

$$\omega_m = \begin{cases} (2m+1)\pi T, & m < m_L \\ (2m+1)\pi T + e^{km-b} \pi T, & m \geq m_L \end{cases} \tag{C1}$$

Here $n_L$ is some number, below which we adopt the original formula for a Matsubara frequency, and beyond which we add an exponentially growing term. In practice, we have taken $m_L \sim 0.1 M$, where $M$ is the total number of frequency points. One should also
properly choose the parameters $k$ and $b$ such that when $m \to m_L$, this exponential term can be neglected compared to $\pi T$ and when $m \to M$, the exponential term dominates over the linear term and can reach our upper limit of frequency. The change of the frequency form also induces a corresponding change to the measure of summation through a Jacobian, i.e. when the hybrid frequency is used ($m \geq m_L$), the following adjustment should be applied,

$$
\pi T \sum \ldots \to \pi T \sum \left( 1 + \frac{k}{2} e^{km-b} \right) \ldots
$$

On the other hand, the Matsubara summation for the self-energy is well approximated using the Euler-Maclaurin formula. In the normal state and at zero bosonic mass $M_b = 0$, $\Sigma(\omega_m) = \bar{g}^\gamma (2\pi T)^{1-\gamma} H_{m,\gamma}$, Eq. (7), where $H_{m,\gamma}$ is a generalized harmonic number. Although $H_{m,\gamma}$ is a tabulated function in some computation libraries, we note that it is well approximated by

$$
H_{m,\gamma} = \sum_{p=1}^{m} \frac{1}{p^{\gamma}} \approx \frac{1}{1-\gamma} \left( m^{1-\gamma} - 1 \right) + \frac{1}{2} \left( 1 + \frac{1}{m^{\gamma}} \right)
+ \frac{\gamma}{12} \left( 1 - \frac{1}{m^{\gamma+1}} \right) - \frac{\gamma(\gamma + 1)(\gamma + 2)}{720} \left( 1 - \frac{1}{m^{\gamma+3}} \right)
+ \frac{\gamma(\gamma + 1)(\gamma + 2)(\gamma + 3)(\gamma + 4)}{30240} \left( 1 - \frac{1}{m^{\gamma+5}} \right)
$$

In Fig. 14 we compare our numerical $\Sigma(\omega_m)$ at about the smallest $T$ that we used, to the zero temperature expression $\Sigma(\omega_m) = \bar{g}^\gamma (2\pi T)^{1-\gamma} \omega_m^{-\gamma}$. We see that the numerical and the $T = 0$ expressions expectedly coincide at Matsubara numbers $m \gg 1$, but differ at $m = O(1)$, and the difference increases as $\gamma$ gets larger.

In the case when the bosonic mass $M_b$ is finite, the self-energy in the normal state is given by $\Sigma(\omega_m) = \bar{g}^\gamma (2\pi T)^{1-\gamma} H_{m,\gamma}(\delta)$, where $\delta = M_b/(2\pi T)$ and $H_{m,\gamma}(\delta) = \sum_{\nu=1}^{m} \frac{1}{(\nu^2 + \delta^2)^{\gamma/2}}$. This $H_{m,\gamma}(\delta)$ is not a tabulated function, but based on what we know about the case $M_b = 0$, we
FIG. 14. Comparison between self energy obtained though finite temperature \((T/g = 10^{-50})\) analysis and zero temperature analysis. In all subplots the red curves represent results obtained via hybrid scaling frequency technique and though the Euler-Maclaurin formula, while the dark blue curves represent \(T = 0\) analysis \(\Sigma(\omega) = \frac{g^2}{1-\gamma} \omega^{1-\gamma}\).

We expect the Euler-Maclaurin formula to work well. Using this formula, we obtain

\[
H_{m,\gamma}(\delta) = \sum_{p=1}^{m} \frac{1}{(p^2 + \delta^2)^{\gamma/2}} \\
\approx \left(\frac{1}{\delta}\right)^\gamma \left[m_2 F_1[\frac{3}{2}, \frac{3}{2}, 3 - \frac{m^2}{\delta^2}] - \frac{m^2}{\delta^2} F_1[\frac{1}{2}, \frac{3}{2}, 3 - \frac{1}{\delta^2}]\right] \\
+ \frac{1}{2} \left[\frac{1}{(1 + \delta^2)^{\gamma/2}} + \frac{1}{(m^2 + \delta^2)^{\gamma/2}}\right] \\
+ \frac{\gamma}{12} \left[\frac{-m}{(m^2 + \delta^2)^{\gamma/2+1}} + \frac{1}{(1 + \delta^2)^{\gamma/2+1}}\right] \\
- \frac{1}{720} \gamma(\gamma + 2) \left[\frac{-m^3(\gamma + 4)}{(m^2 + \delta^2)^{\gamma/2+3}} + \frac{3m}{(m^2 + \delta^2)^{\gamma/2+2}} + \frac{\gamma + 4}{(1 + \delta^2)^{\gamma/2+3}} - \frac{3}{(1 + \delta^2)^{\gamma/2+2}}\right] \\
+ \frac{1}{30240} \gamma(\gamma + 2)(\gamma + 4) \left[\frac{-m^3(\gamma + 6)(\gamma + 8)}{(m^2 + \delta^2)^{\gamma/2+5}} + \frac{10m^3(\gamma + 6)}{(m^2 + \delta^2)^{\gamma/2+4}} - \frac{15m}{(m^2 + \delta^2)^{\gamma/2+3}}\right] \\
+ \gamma(\gamma + 6)(\gamma + 8) \left[\frac{-m^3(\gamma + 8)}{(1 + \delta^2)^{\gamma/2+5}} + \frac{10(\gamma + 6)}{(1 + \delta^2)^{\gamma/2+4}} + \frac{15}{(1 + \delta^2)^{\gamma/2+3}}\right].
\]

We used this form to compute \(\Sigma(\omega_m)\) for a finite \(M_6\).
All numerical results, reported in this paper, are obtained using Matlab2017a with the default double precision and Wolfram Mathematica 11.1.

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