Calculation of One-Loop Integrals for Four-Photon Amplitudes by Functional Reduction Method

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Abstract—The method for functional reduction of Feynman integrals, proposed by the author, is used to calculate one-loop integrals corresponding to diagrams with four external lines. The integrals that emerge from amplitudes for the scattering of light by light, the photon splitting in an external field and Delbrück scattering are considered. For master integrals in $d$-dimensions, new analytic results are presented. For $d = 4$, these integrals are given by compact expressions in terms of logarithms and dilogarithms.

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1. INTRODUCTION

In [1] a method for functional reduction of one-loop integrals with arbitrary masses and external momenta was proposed. In the present article I will describe the application of this method for calculating one-loop integrals that arise when computing radiative corrections to important physical processes. As an example, I chose to calculate integrals required for radiative corrections to the amplitudes of scattering of light by light, photon splitting in an external field and Delbrück scattering. First calculations of radiative corrections to these processes were presented in [2–6]. Radiative corrections to the amplitude of the photon splitting in an external field were considered in [7, 8]. Electroweak radiative corrections to the process of photon-photon scattering were investigated in [9, 10].

The study of the scattering of light by light is an important part of the experimental program at the LHC. The main experiment in this study is collisions of lead ions. The first results of these experiments were reported in [11, 12].

The aim of this paper is to calculate integrals that arise when computing amplitudes for the processes with four external photons. Note that integrals of this type can be used to calculate radiative corrections to other processes, as well as to calculate diagrams with five and more external lines that can be reduced to the considered integrals.

2. INTEGRALS AND THE FUNCTIONAL REDUCTION FORMULA

We will consider the calculation of integrals of the following type

$$I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{P_1 P_2 P_3 P_4},$$

where

$$P_j = (k_j - p_j)^2 - m_j^2 + i\epsilon, \quad s_j = (p_j - p_j)^2. \quad (2.2)$$

In what follows, we will omit the small imaginary term $i\epsilon$, implying that all the masses contain it. Figure 1 shows diagrams corresponding to the integrals that we consider in this paper.

To calculate the integral $I_4^{(d)}$, we will need integrals with fewer propagators:

$$I_3^{(d)}(m_1^2, m_2^2, m_3^2; s_{23}, s_{13}, s_{12}) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{P_1 P_2 P_3}, \quad (2.3)$$

$$I_2^{(d)}(m_1^2, m_2^2; s_{12}) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{P_1 P_2}, \quad (2.4)$$

$$I_1^{(d)}(m_1^2) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{P_1}. \quad (2.5)$$

The formula for functional reduction of the integral $I_4^{(d)}$ with arbitrary masses and kinematic variables [1] can be represented as,

$$I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) = \sum_{k=1}^{24} Q_k B_4^{(d)}(R_{1}^{(k)}, R_{2}^{(k)}, R_{3}^{(k)}, R_{4}^{(k)}), \quad (2.6)$$

where $R_{j}^{(k)}$ are the ratios of the modified Cayley determinant to the Gram determinant, $Q_k$ are products of the ratios of polynomials in the squared masses and kinematic variables $s_j$. The function $B_4^{(d)}$ is defined as follows.
The variables are simple combinations of \( R_i \):

\[
z_i = 1 - \frac{R_i}{R_1}, \quad z_2 = \frac{R_2 - R_i}{R_1}, \quad z_3 = \frac{R_3 - R_i}{R_1}. \tag{2.8}
\]

An explicit formula for the functional reduction of the integral \( I_4^{(d)} \) is presented in [1].

### 3. Integral \( I_4^{(d)} \) for the Light by Light Scattering Amplitude

The integral required for computing the light by light scattering amplitude corresponds to the following values of kinematic variables and masses:

\[
s_{12} = s_{23} = s_{34} = s_{14} = 0, \quad m_j^2 = m_j^2, \quad j = 1, \ldots, 4. \tag{3.1}
\]

Inserting these values in the final formula for functional reduction of the integral \( I_4^{(d)} \), given in [1], we get:

\[
I_4^{(d)}(m^2, m_j^2, m_j^2, m_j^2; 0, 0, 0, 0, s_{24}, s_{13}) = \frac{\tilde{s}_{13}}{\tilde{s}_{24}} I_4^{(d)}(\tilde{s}_{13}, M_z^2, m_j^2) \tag{3.2} + \frac{\tilde{s}_{24}}{\tilde{s}_{24} + \tilde{s}_{13}} L^{(d)}(\tilde{s}_{24}, M_z^2, m_j^2),
\]

where

\[
L^{(d)}(\tilde{s}_{ij}, M_z^2, m_j^2) = I_4^{(d)}(m_j^2 - M_z^2, m_j^2, m_j^2 - \tilde{s}_{ij}, m_j^2; M_z^2, -\tilde{s}_{ij}, -\tilde{s}_{ij}, 0, M_z^2 - \tilde{s}_{ij}); \tag{3.3}
\]

\[
M_z^2 = \frac{\tilde{s}_{24} \tilde{s}_{13}}{\tilde{s}_{24} + \tilde{s}_{13}}, \quad \tilde{s}_{ij} = \frac{s_{ij}}{4}. \tag{3.4}
\]

The resulting formula (3.4) represents the integral \( I_4^{(d)} \) depending on three variables in terms of integrals also depending on three variables. However, calculating the integral \( L^{(d)} \) is simpler than calculating the original integral. We will consider different methods of calculating \( L^{(d)} \).

It turns out that the recurrence equation with respect to \( d \) provides an easy way to derive a compact expression for the function \( L^{(d)} \). Such an equation for the integral \( I_3^{(d)}(\tilde{s}_{ij}, M_z^2, m_j^2) \) has the form [1]:

\[
(d - 3) I_3^{(d+2)}(\tilde{s}_{ij}, M_z^2, m_j^2) = -2m_0^2 I_3^{(d)}(\tilde{s}_{ij}, M_z^2, m_j^2) - I_3^{(d)}(m_j^2, \tilde{s}_{ij}), \tag{3.5}
\]

where

\[
I_3^{(d)}(m_j^2, \tilde{s}_{ij}) = I_3^{(d)}(m_j^2, m_j^2 - \tilde{s}_{ij}, m_j^2; 0, -\tilde{s}_{ij}), \tag{3.6}
\]

\[
m_0^2 = m_j^2 - M_z^2. \tag{3.7}
\]

The solution of Eq. (3.5) can be represented as:

\[
I_3^{(d)}(\tilde{s}_{ij}, M_z^2, m_j^2) = \frac{(m_0^2)^d c_4^{(d)}(\tilde{s}_{ij}, M_z^2, m_j^2)}{\Gamma\left(d - \frac{3}{2}\right) \sin \frac{\pi d}{2}} \tag{3.8}
\]

\[
- \frac{1}{2m_0^2} \sum_{r=0}^{\infty} \left( \frac{d - 3}{2} \right) \rho_3^{(d+2r)}(m_j^2, \tilde{s}_{ij}),
\]

where \( c_4^{(d+2)}(\tilde{s}_{ij}, M_z^2, m_j^2) = c_4^{(d)}(\tilde{s}_{ij}, M_z^2, m_j^2) \) is an arbitrary periodic function of the parameter \( d \). Using the method of [13], we obtained for the integral \( I_3^{(d)}(m_j^2; \tilde{s}_{ij}) \) a simple functional relation

\[
I_3^{(d)}(m_j^2; \tilde{s}_{ij}) = I_3^{(d)}(m_j^2, m_j^2, m_j^2; 4 \tilde{s}_{ij}, 0, 0), \tag{3.9}
\]

which reduces this integral to the well-known result [14]

\[
I_3^{(d)}(m_j^2; \tilde{s}_{ij}) = \frac{1}{\Gamma\left(3 - \frac{d}{2}\right)} \int_{F_3} \left[ 1, 1, 3 - \frac{d}{2}, \frac{\tilde{s}_{ij}}{2}, 2 \frac{3}{2}, \frac{\tilde{s}_{ij}}{4m_j^2} \right] . \tag{3.10}
\]

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Fig. 1. Diagrams corresponding to scattering of light by light, photon splitting and Delbrück scattering. The thick lines correspond to the off-shell photon.
We find the function $c_2^{(d)}(\vec{s}_y, M_s^2, m^2)$ as a solution of a differential equation that can be obtained, for example, from a differential equation for the integral $L^{(d)}(\vec{s}_y, M_s^2, m^2)$
\[
\frac{\partial c^{(d)}_2(\vec{s}_y, M_s^2, m^2)}{\partial \vec{s}_y} + c^{(d)}_4(\vec{s}_y, M_s^2, m^2) = 0.
\] (3.12)

The solution of this equation is
\[
c^{(d)}_4(\vec{s}_y, M_s^2, m^2) = \frac{c^{(d)}_3(M_s^2, m^2)}{\vec{s}_y},
\] (3.13)

where $c^{(d)}_3(M_s^2, m^2)$ is the integration constant of the differential Eq. (3.12). Using the boundary value of the integral $L^{(d)}$ at $\vec{s}_y = 0$, we get
\[
c^{(d)}_0(\vec{s}_y, M_s^2, m^2) = 0.
\] (3.14)

For the hypergeometric function from Eq. (3.10), we used the following representation [15]
\[
\begin{aligned}
3F_2 \left[ \begin{array}{c} 1, 1, 3 - d, \frac{d}{2} \\
2, \frac{3}{2}
\end{array} ; x \right] &= \frac{-\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(3 - \frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)} x^{\frac{d}{2}} \\
&\times \int_0^1 dz z^{\frac{d-3}{2}} (1 - z)^{\frac{d-5}{2}} \ln(1 - xz).
\end{aligned}
\] (3.15)

Employing this representation and taking into account Eq. (3.8), the solution (3.14) can be written as
\[
L^{(d)}(\vec{s}_y, M_s^2, m^2) = -\frac{1}{8\vec{s}_y} \frac{\pi^2(m^2)^{d-3}}{\Gamma\left(\frac{d}{2}\right)}
\times \int_0^1 dz z^{\frac{d-3}{2}} (1 - z)^{\frac{d-5}{2}} \ln\left(1 - \frac{\vec{s}_y}{M_s^2}ight).
\] (3.16)

Substituting this expression into Eq. (3.2), we get
\[
I^{(d)}_4(m^2, m^2, m^2, m^2; 0, 0, 0, 0, s_{24}, s_{13})
= -\frac{\pi^2(m^2)^{d-3}}{2(s_{24} + s_{13})\Gamma\left(\frac{d}{2}\right)}
\times \frac{1}{1 - M_s^2/m^2} \ln \left(1 - \frac{s_{13}}{4M_s^2} \right) + \ln \left(1 - \frac{s_{24}}{4M_s^2} \right).
\] (3.17)

This is the simplest representation known so far for this integral. Note that for $d = 4$ in [16], this integral was obtained in terms of the hypergeometric Appell function $F_5$, which was expressed as a two-fold integral.

The representation (3.16) is convenient for a series expansion in $\varepsilon = (4 - d)/2$. For $d = 4$, we have
\[
L^{(d)}(\vec{s}_y, M_s^2, m^2) = -\frac{1}{8\vec{s}_y} \frac{\pi^2(M_s^2)^{d-3}}{\Gamma\left(\frac{d}{2}\right)}
\times \int_0^1 dz z^{d-3} (1 - z)^{d-5} \left[ \ln \left(1 - \frac{s_{13}}{4M_s^2} \right) + \ln \left(1 - \frac{s_{24}}{4M_s^2} \right) \right].
\] (3.18)

where
\[
\beta_y = \sqrt{1 - \frac{m^2}{\vec{s}_y}} \quad (ij = 13, 24) \quad \beta_2 = \sqrt{1 - \frac{m^2}{M_s^2}}
\] (3.19)

Using (3.18), from the equation (3.2), we get
\[
I^{(d)}_4(m^2, m^2, m^2, m^2; 0, 0, 0, 0, s_{24}, s_{13}) = \frac{1}{8\vec{s}_y s_{24} \beta_3}
\times \left[ \begin{array}{c}
2 \ln^2 \left(\beta_2 + \beta_3\right) + \ln \left(\beta_2 - \beta_3\right) \ln \left(\beta_2 - \beta_{24}\right) \\
\frac{\pi^2}{2} + \sum_{i=1324} \left[ \begin{array}{c}
2\ln^2 \left(\beta_2 - \beta_3\right) + \ln \left(\beta_2 - \beta_{24}\right) \ln \left(\beta_2 - \beta_{24}\right) \\
- \frac{\pi^2}{2} + \ln \left(\beta_2 - \beta_3\right) + \ln \left(\beta_2 - \beta_{24}\right)
\end{array} \right]
\end{array} \right].
\] (3.20)

In the sum of two $L^{(d)}$ the third terms with $\ln \beta_2$ from (3.18) cancel. The expression (3.20) agrees with the result obtained in [16].

The function $L^{(d)}$ can be obtained as a solution to Eq. (3.11). For $d = 4$, the solution of this equation has the form
\[
L^{(d)}(\vec{s}_y, M_s^2, m^2) = -\frac{1}{8\vec{s}_y M_s^2 \beta_2}
\times \int_0^1 dz z^{d-3} (1 - z)^{d-5} \left[ F(y_1, y_2) - 2\ln \left(1 - y_2\right) \right].
\] (3.21)
5. INTEGRALS $I^{(d)}_4$ FOR THE DELBRÜCK SCATTERING AMPLITUDE

We now turn to the more complicated integrals corresponding to the diagrams (c) and (d) in Fig. 1.

Inserting the values of masses and kinematic variables for the integral represented by the diagram (c),

$$s_{23} = s_{14} = 0, \quad m_k^2 = m^2, \quad k = 1, \ldots, 4.$$  \hspace{1cm} (5.1)

in the final formula for the functional reduction of $I^{(d)}_4$ from [1], we get

$$I^{(d)}_4(m^2, m^2, m^2, m^2; s_{12}, 0, s_{24}, 0, s_{24}, 0, s_{24}, 0, s_{24}, 0, s_{24}, s_{13}) = s_{12}L^{(d)}(s_{12}, M^2_4, m^2) - s_{13}L^{(d)}(s_{13}, M^2_4, m^2) \tag{5.2}$$

or

$$I^{(d)}_4(m^2, m^2, m^2, m^2; s_{12}, 0, s_{24}, 0, s_{24}, 0, s_{24}, s_{13}) = F(y_{13}, y_4) + F(y_{24}, y_4) - F(y_{12}, y_4) - F(y_{13}, y_2) - F(y_{12}, y_3). \tag{5.4}$$

where

$$y_4 = \frac{\beta_4 - 1}{\beta_4 + 1}, \quad \beta_4 = \sqrt{1 - \frac{m^2}{M^2_4}}. \tag{5.5}$$

Another integral contributing to Delbrück scattering cross section is represented by the diagram (d) in Fig. 1. The kinematics corresponding to this diagram reads

$$s_{24} = s_{14} = 0, \quad m_k^2 = m^2, \quad k = 1, \ldots, 4.$$  \hspace{1cm} (5.6)

Substituting these values into the final formula for functional reduction [1], we obtain

$$I^{(d)}_4(m^2, m^2, m^2, m^2; s_{12}, s_{23}, s_{24}, s_{13}, s_{24}, s_{13}) = \frac{1}{2} d_s s_{24} I^{(d)}_4(m_0^2, m_0^2, m^2 - \tilde{s}_{24}, m^2) + \frac{1}{2} d_s s_{24} I^{(d)}_4(m_0^2, m_0^2, m^2 - \tilde{s}_{24}, m^2) \tag{5.7}$$

and

$$I^{(d)}_4(m^2, m^2, m^2, m^2; s_{12}, s_{23}, s_{24}, s_{13}, s_{24}, s_{13}) = F(y_{13}, y_3) + F(y_{24}, y_3) - F(y_{12}, y_3) - 2L_2(1 - y_3). \tag{4.4}$$

where

$$y_3 = \frac{\beta_3 - 1}{\beta_3 + 1}, \quad \beta_3 = \sqrt{1 - \frac{m^2}{M^2_5}}. \tag{4.5}$$

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where

$$F(y, y_k) = \text{Li}_2(1 - y y_k)$$

$$+ \text{Li}_2\left(1 - \frac{y_k}{y}\right) + \frac{1}{2} m^2 y_y, \quad \text{for} \quad y_y \rightarrow 1/y_y$$

Substituting (3.21) into the formula (3.2), we get

$$I^{(d)}_4(m^2, m^2, m^2, m^2; 0, 0, 0, 0, s_{24}, s_{13}) = \frac{1}{2s_{13}s_{24} \beta_2} [F(y_{13}, y_2) + F(y_{24}, y_2) - 4L_2(1 - y_2)]. \tag{3.24}$$

Series representations of $L^{(d)}(\tilde{s}_{13}, M^2_3, m^2)$ can be obtained from the integral representations (2.7).

4. INTEGRAL $I^{(d)}_4$ FOR THE PHOTON SPLITTING AMPLITUDE

In this section, we turn to the integral with one external line off-shell, which is associated with the diagram (b) in Fig. 1. This integral corresponds to the kinematics

$$s_{23} = s_{14} = 0, \quad m_k^2 = m^2.$$  \hspace{1cm} (4.1)

Using the final formula for functional reduction of $I^{(d)}_4$ from [1], we get

$$(s_{12} - s_{24} - s_{13})I^{(d)}_4(m^2, m^2, m^2, m^2, s_{12}, 0, 0, 0, s_{24}, s_{13}) = s_{12}L^{(d)}(s_{12}, M^2_3, m^2) - s_{13}L^{(d)}(s_{13}, M^2_3, m^2) \tag{4.2}$$

$$- s_{24}L^{(d)}(s_{24}, M^2_3, m^2).$$

where

$$M^2_3 = \frac{s_{24}s_{13}}{4(s_{13} + s_{24} - s_{12})}. \tag{4.3}$$

For $d = 4$, we have

$$I^{(d)}_4(m^2, m^2, m^2, m^2, s_{12}, 0, 0, 0, s_{24}, s_{13}) = F(y_{13}, y_3) + F(y_{24}, y_3) - F(y_{12}, y_3) - 2L_2(1 - y_3). \tag{4.4}$$

where

$$y_3 = \frac{\beta_3 - 1}{\beta_3 + 1}, \quad \beta_3 = \sqrt{1 - \frac{m^2}{M^2_5}}. \tag{4.5}$$
where
\[
m_0^2 = m^2 - \tilde{s}_1, \quad \tilde{m}_0^2 = \tilde{m}^2 - \tilde{s}_2, \quad \tilde{s}_1 = \frac{s_{12}^2 + s_{13}^2 + s_{23}^2}{4d_1}, \\
\tilde{s}_2 = -\frac{s_{12}^2 s_{13}^2}{d_1},
\]
(5.8)
\[
n_i = 2s_{12}s_{23} - s_{12}s_{24} + s_{13}s_{24} - s_{23}s_{24}, \\
d_i = s_{12}^2 + s_{13}^2 + s_{23}^2 - 2s_{12}s_{13} - 2s_{12}s_{23} - 2s_{13}s_{23}, \\
d_2 = s_{12}s_{23} - s_{12}s_{24} + s_{13}s_{24} - s_{23}s_{24} + s_{24}^2.
\]
(5.9)

The first four integrals in Eq. (5.7) are actually expressed in terms of the function \( L^{(d)} \). The three remaining integrals \( I_4^{(d)}(m_0^2, \tilde{m}_0^2, m^2 - \tilde{s}_{ij}, m^2) \), can be calculated using the formula from [1]
\[
I_4^{(d)}(r_{1234}, r_{234}, r_{34}, r_4) = \int_0^1 \int_0^1 \int_0^1 \frac{\Gamma(4 - \frac{d}{2})}{\left[ a - bx_1^2 - cx_1^2 x_2 - ex_1^2 x_2 x_3 \right]^\frac{d}{2}} \prod_{i=1}^{d-1} dx_i dx_2 dx_3
\]
(5.10)
where
\[
a = r_{1234}, \quad b = r_{234} - r_{234}, \\
c = r_{34} - r_{34}, \quad e = r_4 - r_4.
\]
(5.11)

For \( d = 4 \), the integral (5.10) can be calculated as a combination of functions \( L^{(d)} \) with various arguments. The detailed derivation of the result will be described in an expanded version of this article.

6. CONCLUSIONS

Our results clearly indicate that the application of the functional reduction method makes it possible to reduce complicated integrals to simpler integrals. Even if the number of variables in the integrals resulting from applying the functional reduction is the same as in the original integral, these integrals are simpler than the original integral. In general, integrals depending on more than four variables are reduced to a combination of integrals with four or fewer variables, which are also simpler than the original integral.

The fact that the results for the integrals discussed in this paper are expressed in terms of the same function \( L^{(d)} \) may be useful for improving the accuracy and efficiency of calculating radiative corrections. The integral representations (3.16), (5.10) can significantly simplify the calculation of higher order terms in the \( \varepsilon \) expansion of integrals considered in the article.

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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