Stability of the Prékopa-Leindler inequality for log-concave functions

Károly J. Böröczky, Apratim De

August 18, 2020

Abstract

Stability version of the Prékopa-Leindler inequality for log-concave functions on $\mathbb{R}^n$ is established.

MSC Subject Index: 26D15

1 Introduction

For $X \subset \mathbb{R}^n$, we write $\text{conv } X$ to denote the convex hull of $X$, and say that $X$ is homothetic to $Y \subset \mathbb{R}^n$ if $Y = \gamma X + z$ for $\gamma > 0$ and $z \in \mathbb{R}^n$. Writing $|X|$ to denote Lebesgue measure of a measurable subset $X$ of $\mathbb{R}^n$ (with $|\emptyset| = 0$), the Brunn-Minkowski inequality (Schneider [57]) says that if $\alpha, \beta > 0$ and $X, Y, Z$ are bounded measurable subsets of $\mathbb{R}^n$, then

$$|Z|^\frac{1}{n} \geq \alpha |X|^\frac{1}{n} + \beta |Y|^\frac{1}{n} \quad \text{provided} \quad \alpha X + \beta Y \subset Z, \tag{1}$$

and in the case $|X|, |Y| > 0$, equality holds if and only if $\text{conv } X$ and $\text{conv } Y$ are homothetic convex bodies with $|(\text{conv } X) \setminus X| = |(\text{conv } Y) \setminus Y| = 0$ and $\text{conv } Z = \alpha (\text{conv } X) + \beta (\text{conv } Y)$. We note that even if $X$ and $Y$ are Lebesgue measurable, the Minkowski linear combination $\alpha X + \beta Y$ may not be measurable.

*Supported by NKFIH grants KH 129630, K 132002
Because of the homogeneity of the Lebesgue measure, it is an equivalent form of (1) that if \( \lambda \in (0,1) \), then
\[
|Z| \geq |X|^{1-\lambda}|Y|^\lambda \quad \text{provided} \quad (1-\lambda)X + \lambda Y \subset Z.
\]
(2)

In the case \(|X|, |Y| \geq 0\), equality in (2) implies that \( \text{conv} \ X \) and \( \text{conv} \ Y \) are translates.

For convex \( X \) and \( Y \), the first stability forms of the Brunn-Minkowski inequality were due to Minkowski himself (see Groemer [36]). If the distance of the convex \( X \) and \( Y \) is measured in terms of the so-called Hausdorff distance, then Diskant [21] and Groemer [35] provided close to be optimal stability versions (see Groemer [36]). However, the natural distance is in terms volume of the symmetric difference, and the optimal result is due to Figalli, Maggi, Pratelli [27, 28]. To define the “homothetic distance” \( A(K,C) \) of convex bodies \( K \) and \( C \), let \( \alpha = |K|^{\frac{1}{n}} \) and \( \beta = |C|^{\frac{1}{n}} \), and let
\[
A(K,C) = \min \{|\alpha K \Delta (x + \beta C)| : x \in \mathbb{R}^n\}.
\]

In addition, let
\[
\sigma(K,C) = \max \left\{ \frac{|C|}{|K|}, \frac{|K|}{|C|} \right\}.
\]

**THEOREM 1.1 (Figalli, Maggi, Pratelli)** For \( \gamma^* = \left( \frac{2-2\frac{n}{n+1}}{122n} \right)^2 \), and any convex bodies \( K \) and \( C \) in \( \mathbb{R}^n \),
\[
|K + C|^{\frac{1}{n}} \geq (|K|^{\frac{1}{n}} + |C|^{\frac{1}{n}}) \left[ 1 + \frac{\gamma^*}{\sigma(K,C)^{\frac{n}{2}}} \cdot A(K,C)^2 \right].
\]

Here the exponent 2 of \( A(K,C)^2 \) is optimal, see Figalli, Maggi, Pratelli [28]. We note that prior to [28], the only known error term in the Brunn-Minkowski inequality was of order \( A(K,C)^n \) with \( n \geq n \), due to Diskant [21] and Groemer [35] in their work on providing stability result in terms of the Hausdorff distance (see Groemer [36]), and also to a more direct approach by Esposito, Fusco, Trombetti [24]; therefore, the exponent depended significantly on \( n \).

If the \( X, Y \) and \( Z \) in (1) are only assumed to be measurable and may be possibly not convex, then only much weaker estimates are known. Figalli, Jerison [23, 26] managed to prove that if
\[
||X| - 1| + ||Y| - 1| + ||Z| - 1| < \varepsilon \quad \text{and} \quad \frac{1}{2} X + \frac{1}{2} Y \subset Z
\]
for small \( \varepsilon > 0 \), then there exist some convex body (compact convex sets with non-empty interior) \( K \) and \( z \in \mathbb{R}^n \) such that

\[
X \subset K, \ Y + z \subset K \quad \text{and} \quad |K \setminus X| + |K \setminus (Y + z)| < c_0 \varepsilon \eta \quad (3)
\]

where \( c_0, \eta > 0 \) depend on \( n \) and \( \eta < n^{-3n} \). If \( X = Y \), then an essentially optimal version is provided by van Hintum, Spink, Tiba [38]. In one of the sets \( X \) and \( Y \) is convex, then Carlen, Maggi [18] proved a close to be optimal stability version of the Brunn-Minkowski Inequality.

Our main theme is the generalization Prékopa-Leindler inequality of the Brunn-Minkowski inequality. The inequality itself, due to Prékopa [50] and Leindler [42] in dimension one, was generalized in Prékopa [51] and [52], Borell [14] (cf. also Marsiglietti [15]), and in Brascamp, Lieb [16]. Various applications are provided and surveyed in Ball [1], Barthe [6], Fradelizi, Meyer [31] and Gardner [33]. The following multiplicative version from [1] is often more useful and is more convenient for geometric applications.

**THEOREM 1.2 (Prékopa-Leindler)** If \( \lambda \in (0, 1) \) and \( h, f, g \) are non-negative integrable functions on \( \mathbb{R}^n \) satisfying \( h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda}g(y)^\lambda \) for \( x, y \in \mathbb{R}^n \), then

\[
\int_{\mathbb{R}^n} h \geq \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \cdot \left( \int_{\mathbb{R}^n} g \right)^\lambda. \quad (4)
\]

It follows from Theorem 1.2 that the Prékopa-Leindler inequality has a following multifunctional form which resembles Barthe’s Reverse Brascamp-Lieb inequality [5]. If \( \lambda_1, \ldots, \lambda_m > 0 \) satisfy \( \sum_{i=1}^{m} \lambda_i = 1 \) and \( f_1, \ldots, f_m \) are non-negative integrable functions on \( \mathbb{R}^n \), then

\[
\int_{\mathbb{R}^n} \sup_{z=\sum_{i=1}^{m} \lambda_i z_i} \prod_{i=1}^{m} f(x_i)^{\lambda_i} \, dz \geq \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} f_i \right)^{\lambda_i} \quad (5)
\]

where * stands for outer integral in the case the integrand is not measurable.

We say that a function \( f : \mathbb{R}^n \to [0, \infty) \) have positive integral if \( f \) is measurable and \( 0 < \int_{\mathbb{R}^n} f < \infty \). For a convex subset \( \Gamma \subset \mathbb{R}^n \), we say that a function \( f : \Gamma \to [0, \infty) \) is log-concave, if for any \( x, y \in \Gamma \) and \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta = 1 \), we have \( f(\alpha x + \beta y) \geq f(x)^\alpha g(y)^\beta \). The case of equality in Theorem 1.2 has been characterized by Dubuc [22].

3
THEOREM 1.3 (Dubuc) If \( \lambda \in (0, 1) \) and \( h, f, g : \mathbb{R}^n \to [0, \infty) \) have positive integral, satisfy \( h((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda}g(y)^\lambda \) for \( x, y \in \mathbb{R}^n \) and equality holds in (4), then \( f, g, h \) are log-concave up to a set of measure zero, and there exist \( a > 0 \) and \( z \in \mathbb{R}^n \) such that

\[
\begin{align*}
  f(x) &= a^\lambda h(x - \lambda z) \\
  g(x) &= a^{-(1-\lambda)}h(x + (1 - \lambda)z)
\end{align*}
\]

for almost all \( x \).

Our goal is to prove a stability version of the Prékopa-Leindler inequality Theorem 1.2 at least for log-concave functions.

THEOREM 1.4 For some absolute constant \( c > 1 \), if \( \tau \in (0, \frac{1}{2}], \tau \leq \lambda \leq 1 - \tau, h, f, g : \mathbb{R}^n \to [0, \infty) \) are integrable such that \( h((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda}g(y)^\lambda \) for \( x, y \in \mathbb{R}^n \), \( h \) is log-concave and

\[
\int_{\mathbb{R}^n} h \leq (1 + \varepsilon) \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} g \right)^\lambda
\]

for \( \varepsilon \in (0, 1] \), then there exists \( w \in \mathbb{R}^n \) such that for \( a = \frac{\int_{\mathbb{R}^n} f}{\int_{\mathbb{R}^n} g} \), we have

\[
\begin{align*}
  \int_{\mathbb{R}^n} |f(x) - a^\lambda h(x - \lambda w)| dx &\leq c n^n \sqrt{\frac{\varepsilon}{\tau}} \cdot \int_{\mathbb{R}^n} f \\
  \int_{\mathbb{R}^n} |g(x) - a^{-(1-\lambda)}h(x + (1 - \lambda)w)| dx &\leq c n^n \sqrt{\frac{\varepsilon}{\tau}} \cdot \int_{\mathbb{R}^n} g.
\end{align*}
\]

Remark According to Lemma 7.3 (i), if \( f \) and \( g \) are log-concave, then

\[
h(z) = \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda}g(y)^\lambda
\]

is log-concave, as well, and hence Theorem 1.4 applies.

Let us present a version of Theorem 1.4 analogous to Theorem 1.1. If \( f, g \) are non-negative functions on \( \mathbb{R}^n \) with \( 0 < \int_{\mathbb{R}^n} f < \infty \) and \( 0 < \int_{\mathbb{R}^n} g < \infty \), then for the probability densities

\[
\tilde{f} = \frac{f}{\int_{\mathbb{R}^n} f} \quad \text{and} \quad \tilde{g} = \frac{g}{\int_{\mathbb{R}^n} g},
\]

we define

\[
\bar{L}_1(f, g) = \inf_{v \in \mathbb{R}^n} \int_{\mathbb{R}^n} |\tilde{f}(x - v) - \tilde{g}(x)| dx.
\]
COROLLARY 1.5 If \( \tau \in (0, \frac{1}{2}] \), \( \lambda \in [\tau, 1 - \tau] \) and \( f, g \) are log-concave functions with positive integral on \( \mathbb{R}^n \), then

\[
\int_{\mathbb{R}^n} \sup_{z = (1 - \lambda)x + \lambda y} \int_{\mathbb{R}^n} f(x)^{1 - \lambda} g(y)^\lambda \, dz \geq \left( 1 + \gamma \cdot \tau \cdot \tilde{L}_1(f, g)^{19} \right) \left( \int_{\mathbb{R}^n} f \right)^{1 - \lambda} \left( \int_{\mathbb{R}^n} g \right)^\lambda
\]

where \( \gamma = c^n/n^{19n} \) for some absolute constant \( c \in (0, 1) \).

We also deduce a stability version of (5) from Theorem 1.4 (see Corollary [9.4] for the log-concavity of the \( h \) in Theorem [9.4]).

THEOREM 1.6 For some absolute constant \( c > 1 \), if \( \tau \in (0, \frac{1}{m}] \), \( m \geq 2 \), \( \lambda_1, \ldots, \lambda_m \in [\tau, 1 - \tau] \) satisfy \( \sum_{i=1}^{m} \lambda_i = 1 \) and \( f_1, \ldots, f_m \) are log-concave functions with positive integral on \( \mathbb{R}^n \) such that

\[
\int_{\mathbb{R}^n} \sup_{z = \sum_{i=1}^{m} \lambda_i x_i} \prod_{i=1}^{m} f_i(x_i)^{\lambda_i} \, dz \leq (1 + \varepsilon) \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} f_i \right)^{\lambda_i}
\]

for \( \varepsilon \in (0, 1] \), then for the log-concave \( h(z) = \sup_{z = \sum_{i=1}^{m} \lambda_i x_i} \prod_{i=1}^{m} f_i(x_i)^{\lambda_i} \), there exist \( a_1, \ldots, a_m > 0 \) and \( w_1, \ldots, w_m \in \mathbb{R}^n \) such that \( \sum_{i=1}^{m} \lambda_i w_i = 0 \) and for \( i = 1, \ldots, m \), we have

\[
\int_{\mathbb{R}^n} |f_i(x) - a_i h(x + w_i)| \, dx \leq c^n n^5 m^{5/2} \sqrt{\varepsilon} \cdot \int_{\mathbb{R}^n} f_i.
\]

Remark \( a_i = \frac{(f_i)^{1 - \lambda_i}}{\prod_{j \neq i} (f_j)^{\lambda_j}} \) for \( i = 1, \ldots, m \) in Theorem [9.4].

A statement similar to Theorem [1.4] was proved by Ball, Böröczky [4] in the case of even functions.

Recently, various breakthrough stability results about geometric functional inequalities have been obtained. Fusco, Maggi, Pratelli [32] proved an optimal stability version of the isoperimetric inequality (whose result was extended to the Brunn-Minkowski inequality by Figalli, Maggi, Pratelli [27, 28]). Concerning stability versions of the Brunn-Minkowski inequality, see also Eldan, Klartag [23]. Stronger versions of the functional Blaschke-Santaló inequality is provided by Barthe, Böröczky, Fradelizi [7], of the Borell-Brascamp-Lieb inequality is provided by Ghilli, Salani [34] and Rossi, Salani [53], of the Sobolev inequality by Figalli, Zhang [30] (extending Bianchi,
Egnell [8] and Figalli, Neumayer [29]), Nguyen [48] and Wang [58], and of some related inequalities by Caglar, Werner [17].

The next Section 2 reviews the known stability versions of the Prekopa-Leindler inequality for functions on \( \mathbb{R} \), and Section 3 outlines the idea of the proofs of Theorem 1.4, Corollary 1.5 and Theorem 1.6.

2 Known stability versions of the one dimensional Prékopa-Leindler inequality

If \( n = 1 \), then Ball, Böröczky [3] provided the following stability version of the Prekopa-Leindler inequality Theorem 1.2 in the logconcave case.

**THEOREM 2.1** There exists an positive absolute constant \( c \) with the following property: If \( h, f, g \) are non-negative integrable functions with positive integrals on \( \mathbb{R} \) such that \( h \) is log-concave, \( h(\frac{r+s}{2}) \geq \sqrt{f(r)g(s)} \) for \( r, s \in \mathbb{R} \), and

\[
\int_{\mathbb{R}} h \leq (1 + \varepsilon) \sqrt{\int_{\mathbb{R}} f \cdot \int_{\mathbb{R}} g},
\]

for \( \varepsilon > 0 \), then there exist \( a > 0, b \in \mathbb{R} \) such that

\[
\int_{\mathbb{R}} |f(t) - ah(t+b)| \, dt \leq c \cdot \sqrt[3]{\varepsilon} |\ln \varepsilon^{\frac{1}{3}}| \cdot \int_{\mathbb{R}} f(t) \, dt
\]

and

\[
\int_{\mathbb{R}} |g(t) - a^{-1}h(t-b)| \, dt \leq c \cdot \sqrt[3]{\varepsilon} |\ln \varepsilon^{\frac{1}{3}}| \cdot \int_{\mathbb{R}} g(t) \, dt.
\]

**Remark** If \( f \) and \( g \) are log-concave probability distributions then \( a = 1 \) can be assumed, and if in addition \( f \) and \( g \) have the same expectation, then even \( b = 0 \) can be assumed.

As it was observed by C. Borell [14], and later independently by K.M. Ball [1], assigning to any function \( H : [0, \infty] \to [0, \infty] \) the function \( h : \mathbb{R} \to [0, \infty] \) defined by \( h(x) = H(e^x) e^x \), we have the version Theorem 2.2 of the Prékopa-Leindler inequality. We note that if \( H \) is log-concave and decreasing, then \( h \) is log-concave.
THEOREM 2.2 If \( H, F, G : [0, \infty) \to [0, \infty) \) are integrable functions such that \( H(\sqrt{rs}) \geq \sqrt{F(r)G(s)} \) for \( r, s \geq 0 \), then
\[
\int_0^\infty H \geq \sqrt{\int_0^\infty F \cdot \int_0^\infty G}.
\]

Therefore we deduce the following statement by Theorem 2.1:

COROLLARY 2.3 There exists a positive absolute constant \( c_0 > 1 \) such that for \( r, s \geq 0 \), the following property holds: If \( H, F, G : [0, \infty) \to [0, \infty) \) are integrable functions with positive integrals and \( H(\sqrt{rs}) \geq \sqrt{F(r)G(s)} \) for \( r, s \in [0, \infty] \), then
\[
\int_0^\infty H \leq (1 + \varepsilon) \sqrt{\int_0^\infty F \cdot \int_0^\infty G}
\]
for \( \varepsilon \in [0, c_0^{-1}) \), then there exist \( a, b > 0 \) such that
\[
\int_0^\infty |F(t) - a H(bt)| dt \leq c \sqrt{\varepsilon} \ln \varepsilon^{\frac{3}{4}} \cdot \int_0^\infty F(t) dt
\]
and
\[
\int_0^\infty |G(t) - a^{-1} H(b^{-1}t)| dt \leq c \sqrt{\varepsilon} \ln \varepsilon^{\frac{4}{3}} \cdot \int_0^\infty G(t) dt.
\]

Remark If in addition, \( F \) and \( G \) are decreasing log-concave probability distributions then \( a = b \) can be assumed. The condition that \( H \) is log-concave and decreasing can be replaced by the one that \( H(e^t) \) is log-concave.

Unfortunately, no stability version of the Prékopa-Leindler inequality for general measurable functions is known. However, at least the stability of the one-dimensional Brunn-Minkowski inequality has been clarified by Christ \([19]\) (see also Theorem 1.1 in Figalli, Jerison \([26]\)).

THEOREM 2.4 If \( X, Y \subset \mathbb{R} \) are measurable with \( |X|, |Y| > 0 \), and \( |X + Y| \leq |X| + |Y| + \delta \) for some \( \delta \leq \min\{|X|, |Y|\} \), then there exist intervals \( I, J \subset \mathbb{R} \) such that \( X \subset I, Y \subset J, |I \setminus X| \leq \delta \) and \( |J \setminus Y| \leq \delta \).
3 Ideas to verify Theorem 1.4 and its consequences

For Theorem 1.4, the main goal is to prove Theorem 3.1 which is essentially the case $\lambda = \frac{1}{2}$ of Theorem 1.4 for log-concave functions and for small $\varepsilon$, and then the general case is handled in Sections 7 and 8.

**THEOREM 3.1** If $h, f, g : \mathbb{R}^n \to [0, \infty)$ are log-concave, $f, g$ are probability distributions, $h(\frac{x+y}{2}) \geq \sqrt{f(x)g(y)}$ for $x, y \in \mathbb{R}^n$, and
\[
\int_{\mathbb{R}^n} h \leq 1 + \varepsilon
\]
where $0 < \varepsilon < (cn)^{-n}$, then there exists $w \in \mathbb{R}^n$ such that
\[
\int_{\mathbb{R}^n} |f(x) - h(x-w)| \, dx \leq \tilde{c}n^8 \cdot \sqrt{\varepsilon} \cdot |\log \varepsilon|^n
\]
\[
\int_{\mathbb{R}^n} |g(x) - h(x+w)| \, dx \leq \tilde{c}n^8 \cdot \sqrt{\varepsilon} \cdot |\log \varepsilon|^n
\]
where $c, \tilde{c} > 1$ are an absolute constants.

Our proof of the stability version Theorem 3.1 of the Prékopa-Leindler inequality stems from the Ball’s following argument (cf. [1] and Borell [14]) proving the Prékopa-Leindler inequality based on the Brunn-Minkowski inequality.

Let $f, g, h : \mathbb{R}^n \to [0, \infty]$ have positive integrals and satisfy that $h(\frac{x+y}{2}) \geq \sqrt{f(x)g(y)}$ for $x, y \in \mathbb{R}^n$, and for $t > 0$, let
\[
\Phi_t = \{ x \in \mathbb{R}^n : f(x) \geq t \} \quad \text{and} \quad F(t) = |\Phi_t| \\
\Psi_t = \{ x \in \mathbb{R}^n : g(x) \geq t \} \quad \text{and} \quad G(t) = |\Psi_t| \\
\Omega_t = \{ x \in \mathbb{R}^n : h(x) \geq t \} \quad \text{and} \quad H(t) = |\Omega_t|.
\]
As it was observed by Ball [11] and and Borell [14], the condition on $f, g, h$ yields that if $\Phi_r, \Psi_s \neq \emptyset$ for $r, s > 0$, then
\[
\frac{1}{2}(\Phi_r + \Psi_s) \subset \Omega_{\sqrt{rs}}.
\]
Therefore the Brunn-Minkowski inequality implies that
\[
H(\sqrt{rs}) \geq \left( \frac{F(r)^{\frac{1}{n}} + G(s)^{\frac{1}{n}}}{2} \right)^n \geq \sqrt{F(r)} \cdot G(s)
\]
for all $r, s > 0$. In particular we deduce the Prékopa-Leindler inequality by Theorem 2.2 as

$$\int_{\mathbb{R}^n} h = \int_0^\infty H(t) \, dt \geq \sqrt{\int_0^\infty F(t) \, dt \cdot \int_0^\infty G(t) \, dt} = \sqrt{\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} g}.$$ 

Turning to Theorem 3.1, we will need the stability version Corollary 2.3 of the Prekopa-Leindler inequality for function on $\mathbb{R}$, and a stability version of the product form of the Brunn-Minkowski inequality on $\mathbb{R}^n$. Since

$$\frac{1}{2} \left( |K|^{\frac{1}{n}} + |C|^{\frac{1}{n}} \right) = |K|^{\frac{1}{2n}} |C|^{\frac{1}{2n}} \left[ 1 + \frac{1}{2} \left( \sigma(K, C)^{\frac{1}{2n}} - \sigma(K, C)^{\frac{-1}{2n}} \right)^2 \right] \geq |K|^{\frac{1}{2n}} |C|^{\frac{1}{2n}} \left[ 1 + \frac{(\sigma(K, C) - 1)^2}{32n^2 \sigma(K, C)^{\frac{4n-1}{2n}}} \right],$$

using the notation $\sigma = \sigma(K, C) = \max\{\frac{|C|}{|K|}, \frac{|K|}{|C|}\}$, we conclude from the stability version Theorem 1.1 of the Brunn-Minkowski inequality by Figalli, Maggi, Pratelli [28] that

$$\left| \frac{1}{2} (K + C) \right| \geq \sqrt{|K| \cdot |C|} \left[ 1 + \frac{(\sigma - 1)^2}{32n^2 \sigma^2} + \frac{n\gamma^*}{\sigma^2} \cdot A(K, C)^2 \right]. \quad (7)$$

We observe that the volume of the symmetric difference $|K \Delta C|$ of convex bodies $K$ and $C$ is a metric on convex bodies in $\mathbb{R}^n$. We use this fact in the following consequence of Theorem 1.1.

**Lemma 3.2** If $\eta \in (0, \frac{1}{122n^7})$ and $K, C, L$ are convex bodies in $\mathbb{R}^n$ such that $|C| = |K|$, $|L| \leq (1 + \eta)|K|$ and $\frac{1}{2} K + \frac{1}{2} C \subset L$, then there exists $w \in \mathbb{R}^n$ such that

$$|K \Delta (L - w)| \leq 245n^7 \sqrt{\eta} |K| \quad \text{and} \quad |C \Delta (L + w)| \leq 245n^7 \sqrt{\eta} |K|.$$

**Proof:** We may assume that $|C| = |K| = 1$. According to Theorem 1.1 there exists $z \in \mathbb{R}^n$, such that

$$|K \cap (C - z)| \geq 1 - \sqrt{\frac{\eta}{\gamma^*}} > 1 - 122n^7 \sqrt{\eta}.$$
It follows from $z + [K \cap (C - z)] \subset C$ that $M = \frac{1}{2} z + [K \cap (C - z)] \subset L$, and hence $|L| \leq 1 + \eta$ implies $|L \Delta M| < \eta + 122n^7 \sqrt{\eta} < 123n^7 \sqrt{\eta}$. Writing $w = \frac{1}{2} z$, we have

$$|K \Delta (L - w)| \leq |K \Delta (M - w)| + |(M - w) \Delta (L - w)| < 245n^7 \sqrt{\eta},$$

and similar argument yields $|C \Delta (L + w)| < 245n^7 \sqrt{\eta}$. ✷

To prove Theorem 3.1, first we discuss some fundamental estimates for log-concave functions in Section 4, then compare the level sets of $f$, $g$ and $h$ in Theorem 3.1 in Section 5 and finally complete the argument for Theorem 3.1 in Section 6. We verify Theorem 1.4 when $\varepsilon$ is small in Section 7, and prove Theorem 1.4 and Corollary 1.5 in Section 8. Finally, Theorem 1.6 is proved in Section 9.

4 Some properties of log-concave functions

First, we characterize a log-concave function $\varphi$ on $\mathbb{R}^n$ with positive integral; namely, if $0 < \int_{\mathbb{R}^n} \varphi < \infty$. For any measurable function $\varphi$ on $\mathbb{R}^n$, we define

$$M_\varphi = \sup \varphi.$$ 

**Lemma 4.1** Let $\varphi : \mathbb{R}^n \to [0, \infty)$ be log-concave. Then $\varphi$ has positive integral if and only if $\varphi$ is bounded, $M_\varphi > 0$, and for any $t \in (0, M_\varphi)$, the level set $\{ \varphi > t \}$ is bounded and has non-empty interior.

**Proof:** If $\varphi$ has positive integral, then $M_\varphi > 0$, and there exists some $t_0 \in (0, M_\varphi)$ such that the $n$-dimensional measure of $\{ \varphi > t_0 \}$ is positive. As $\{ \varphi > t_0 \}$ is convex, it has non-empty interior. It follows from the log-concavity of $\varphi$ that the level set $\{ \varphi > t \}$ has non-empty interior for any $\varphi \in (0, M_\varphi)$. In turn, we deduce that $\varphi$ is bounded from the log-concavity of $\varphi$ and $\int_{\mathbb{R}^n} \varphi < \infty$.

Next we suppose that that there exists $t \in (0, M_\varphi)$ such that the level set $\{ \varphi > t \}$ is unbounded and seek a contradiction. As $\{ \varphi > t \}$ is convex, there exists a $u \in S^{n-1}$ such that $x + su \in \text{int} \{ \varphi > t \}$ for any $x \in \text{int} \{ \varphi > t \}$ and $s \geq 0$. We conclude that $\int_{\mathbb{R}^n} \varphi = \infty$, contradicting the assumption $\int_{\mathbb{R}^n} \varphi < \infty$. Therefore the level set $\{ \varphi > t \}$ is bounded for any $t \in (0, M_\varphi)$.

Assuming that the conditions of Lemma 4.1 hold, we readily have $\int_{\mathbb{R}^n} \varphi > 0$. To show $\int_{\mathbb{R}^n} \varphi < \infty$, we choose $x_0 \in \mathbb{R}^n$ such that $f(x_0) > 0$, and
let $B$ be an $n$-dimensional ball of centered $x_0$ and radius $\varrho > 0$ containing $\{\varphi > \frac{1}{e} \varphi(x_0)\}$. Let us consider

$$\psi(x) = \varphi(x_0)e^{-\frac{\|x-x_0\|}{\varrho}}.$$  

It follows from the log-concavity of $\varphi$ that $\varphi(x) \leq \psi(x)$ if $\|x-x_0\| \geq \varrho$, and hence

$$\int_{\mathbb{R}^n} \varphi \leq \int_B \varphi + \int_{\mathbb{R}^n\backslash B} \psi < \infty,$$

verifying Lemma 4.1. $lacklozenge$

For a measurable bounded function $\varphi$ on $\mathbb{R}^n$ and for $t \in \mathbb{R}$, let

$$\Xi_{\varphi,t} = \{x \in \mathbb{R}^n : \varphi(x) \geq t\}.$$

If $\varphi$ is log-concave with positive integral, then we consider the symmetric decreasing rearrangement $\varphi^* : \mathbb{R}^n \to \mathbb{R}$ where

$$|\Xi_{\varphi,t}| = |\Xi_{\varphi^*,t}|$$

for any $t > 0$, and if $|\Xi_{\varphi,t}| > 0$, then $\Xi_{\varphi^*,t}$ is a Euclidean ball centered at the origin $o$, and

$$M_\varphi = \max_{x \in \mathbb{R}^n} \varphi(x) = \max_{x \in \mathbb{R}^n} \varphi^*(x) = \varphi^*(o).$$

It follows from Lemma 4.1 that $\varphi^*$ is well-defined. We deduce that $\varphi^*$ is also a log-concave, and

$$\int_{\mathbb{R}^n} \varphi = \int_0^\infty |\Xi_{\varphi,t}| dt = \int_0^\infty |\Xi_{\varphi^*,t}| dt = \int_{\mathbb{R}^n} \varphi^*.$$

We write $B^n$ to denote the unit Euclidean ball in $\mathbb{R}^n$ centered at the origin, and $\kappa_n = |B^n|$, and hence the surface area of $S^{n-1}$ is $n\kappa_n$. For log-concave functions, a useful property of the symmetric decreasing rearrangement is that if $\varrho B^n = \Xi_{\varphi^*,s M_\varphi}$ and $s = e^{-\gamma \varrho}$ for $\gamma, \varrho > 0$, then

$$\varphi^*(x) \geq M_\varphi e^{-\gamma \|x\|} \quad \text{provided} \quad \|x\| \leq \varrho$$

$$\varphi^*(x) \leq M_\varphi e^{-\gamma \|x\|} \quad \text{provided} \quad \|x\| \geq \varrho.$$  \hspace{1cm} (8)

For $s = e^{-\gamma \varrho}$, we have

$$|\Xi_{\varphi,s M_\varphi}| = |\Xi_{\varphi^*,s M_\varphi}| = \kappa_n \varrho^n.$$  \hspace{1cm} (9)
As a related integral, it follows from induction on \( n \) that
\[
\int_0^\infty e^{-\gamma r^{n-1}} \, dr = (n-1)! \cdot \gamma^{-n}.
\] (10)

**Lemma 4.2** If \( \varphi \) is a log-concave probability density on \( \mathbb{R}^n, n \geq 2 \), then
\[
|\Xi_{(1-\tau)M_\varphi}| \geq \frac{1}{n!+1} \tau^n M_\varphi^{-1} \quad \text{for } \tau \in (0, 1).
\] (11)

**Proof:** To prove (11) based on (8), let \( \gamma, \kappa > 0 \) be such that \( \kappa B^n = \Xi_{\varphi^*,(1-\tau)M_\varphi} \) and \( 1 - \tau = e^{-\gamma \varphi} \). Here \( e^{-\gamma \varphi} > 1 - \gamma \varphi \) yields \( \gamma \varphi \geq \tau \), thus it follows from (9) and (10) that
\[
1 = \int_{\mathbb{R}^n} \varphi^*(x) \, dx \leq |\varphi B^n| \cdot M_\varphi + \int_{\mathbb{R}^n \setminus \varphi B^n} \varphi^*(x) \, dx
\]
\[
\leq |\Xi_{\varphi,(1-\tau)M_\varphi}| \cdot M_\varphi + \int_{\mathbb{R}^n} M_\varphi e^{-\gamma \|x\|} \, dx
\]
\[
= M_\varphi \cdot |\Xi_{\varphi,(1-\tau)M_\varphi}| + M_\varphi n \kappa_n \int_0^\infty e^{-\gamma r^{n-1}} \, dr
\]
\[
= M_\varphi \cdot |\Xi_{\varphi,(1-\tau)M_\varphi}| + M_\varphi n \kappa_n \cdot \gamma^{-n}
\]
\[
\leq M_\varphi \cdot |\Xi_{\varphi,(1-\tau)M_\varphi}| + M_\varphi n \kappa_n \cdot \frac{\varphi^n}{\tau^n}
\]
\[
= M_\varphi \cdot |\Xi_{\varphi,(1-\tau)M_\varphi}| \left(1 + \frac{n!}{\tau^n}\right),
\]
proving (11). \( \square \)

We note that the estimate in (11) is close to be optimal because if the probability density is of the form \( \varphi(x) = Mf e^{-\gamma \|x\|} \) for suitable \( \gamma > 0 \), then
\[
|\Xi_{\varphi,(1-\tau)M_\varphi}| = \frac{|\ln(1-\tau)|^n}{n!M_\varphi} < \frac{\varphi^n}{n!M_\varphi} \quad \text{if } \tau \in (0, \frac{1}{n}).
\]

For a log-concave probability density \( \varphi \), let \( \mu_\varphi \) be the probability measure associated to \( \varphi \); namely, \( d\mu_\varphi(x) = \varphi(x) \, dx \). According to Lemma 5.16 in Lovász, Vempala [43], if \( s \in (0, e^{-4(n-1)}) \), then
\[
\mu_\varphi(\varphi < sM_\varphi) \leq \frac{e^{n-1}}{(n-1)^{n-1}} \cdot s \cdot |\ln s|^{n-1} \leq s \cdot |\ln s|^n.
\] (12)

But what we really need is the following estimate.
LEMMA 4.3 If $s \in (0, e^{-4(n-1)})$ and $\varphi$ is a log-concave probability density on $\mathbb{R}^n$, $n \geq 2$, then

$$|\Xi_{\varphi,s,M_\varphi}| < \frac{2|\ln s|^n}{n!M_\varphi},$$

(13)

$$\int_0^{sM_\varphi} |\Xi_{\varphi,t}| \, dt < \left(1 + \frac{1}{M_\varphi}\right)s \cdot |\ln s|^n.$$  
(14)

Proof: To prove (14) based on (8), let $\gamma, \theta > 0$ be such that $\theta B^n = \Xi_{\varphi^*,s,M_\varphi}$ and $s = e^{-\gamma \theta}$. Since $s \in (0, e^{-4(n-1)})$, we deduce that

$$\gamma \theta > 4(n-1).$$
(15)

It follows from (10) and integration by parts that

$$\int_\theta^{\infty} e^{-\gamma r} r^{n-1} \, dr = e^{-\gamma \theta} \int_0^{\infty} e^{-\gamma r} r^{n-1} \, dr \cdot \sum_{k=0}^{n-1} \frac{(\gamma \theta)^k}{k!}.$$  
(16)

Here the well-known estimate $(n-1)! > \frac{(n-1)^{n-1}}{e^{n-1}}$ implies that if $k = 1, \ldots, n-1$, then

$$(n-1) \cdot \ldots \cdot (n-k) > \frac{(n-1)^k}{e^k}.$$  
(17)

In addition, $1 + e \cdot s < e^{\frac{3}{4}s}$ holds for $s \geq 4$. Combining this with (17) yields that if $t > 4(n-1)$, then

$$\sum_{k=0}^{n-1} \frac{t^k}{k!} < \sum_{k=0}^{n-1} \left(\frac{n-1}{k}\right) \left(\frac{et}{n-1}\right)^k = \left(1 + \frac{et}{n-1}\right)^{n-1} < e^{\frac{3}{4}t}.$$  
(18)

Therefore, we deduce from (15), (16) and (18) that

$$\int_\theta^{\infty} e^{-\gamma r} r^{n-1} \, dr < e^{-\gamma \theta} \int_0^{\infty} e^{-\gamma r} r^{n-1} \, dr \cdot e^{\frac{3}{4}\gamma \theta} = e^{-\gamma \theta} \int_0^{\infty} e^{-\gamma r} r^{n-1} \, dr \cdot e^{\frac{3}{4}\gamma \theta} < \frac{1}{2} \int_0^{\infty} e^{-\gamma r} r^{n-1} \, dr.$$  
(19)
In particular, using (8), (19) and later (9), we have
\[ 1 \geq \int_{\phi B^n} \varphi^*(x) \, dx \geq \int_{\phi B^n} M \varphi e^{-\gamma \|x\|} \, dx \]
\[ = n \kappa_n M \varphi \int_0^\theta e^{-\gamma r} r^{n-1} \, dr \]
\[ \geq n \kappa_n M \varphi \cdot \frac{1}{2} \int_0^\infty e^{-\gamma r} r^{n-1} \, dr = \frac{n! M \varphi \kappa_n g^n}{2\gamma^n g^n} \]
\[ = \frac{n! M \varphi}{2} \cdot \frac{|\Xi_{\varphi,s}|}{\ln s^n}. \]

We conclude that if \( s \in (0, e^{-4(n-1)}) \), then
\[ |\Xi_{\varphi,s}| \leq \frac{2|\ln s|^n}{n! M \varphi}. \]

Combining the last inequality with (12), we deduce that if \( s \in (0, e^{-4(n-1)}) \), then
\[ \int_0^{sM \varphi} |\Xi_{\varphi,t}| \, dt = |\Xi_{\varphi,s}| \cdot s + \mu \varphi(\varphi < sM \varphi) < e \left( 1 + \frac{1}{M \varphi} \right) s \cdot |\ln s|^n, \]
proving (14). \( \square \)

We note that the estimate in (13) is close to be optimal because if again the probability density is of the form \( \varphi(x) = M_f e^{-\gamma \|x\|} \) for suitable \( \gamma > 0 \), then \( |\Xi_{\varphi,sM \varphi}| = \frac{|\ln s|^n}{n! M \varphi} \).

5 The area of the level sets in Theorem 3.1

Let \( f, g, h \) as in Theorem 3.1. We may assume that
\[ f(o) = \max\{f(x) : x \in \mathbb{R}^n\} \text{ and } g(o) = \max\{g(x) : x \in \mathbb{R}^n\}. \quad (20) \]

According to Lemma 4.1 for \( t > 0 \), we may consider the bounded convex sets
\[ \Phi_t = \{ x \in \mathbb{R}^n : f(x) \geq t \} \text{ and } F(t) = |\Phi_t| \]
\[ \Psi_t = \{ x \in \mathbb{R}^n : g(x) \geq t \} \text{ and } G(t) = |\Psi_t| \]
\[ \Omega_t = \{ x \in \mathbb{R}^n : h(x) \geq t \} \text{ and } H(t) = |\Omega_t| \]
where (20) yields
\[ o \in \Phi_t \cap \Psi_t, \]  
and we have
\[ \int_0^\infty F = \int_0^{M_t} F = \int_{\mathbb{R}^n} f = 1 \quad \text{and} \quad \int_0^\infty G = \int_0^{M_0} G = \int_{\mathbb{R}^n} g = 1. \]  

As it was observed in K.M. Ball [1], the condition on \( f, g, h \) yields that if \( \Phi_r, \Psi_s \neq \emptyset \) for \( r, s > 0 \), then
\[ \frac{1}{2}(\Phi_r + \Psi_s) \subset \Omega_{\sqrt{rs}}. \]  

Therefore the Brunn-Minkowski inequality yields that
\[ H(\sqrt{rs}) \geq \left( \frac{F(r)^{\frac{1}{n}} + G(s)^{\frac{1}{n}}}{2} \right)^n \geq \sqrt{F(r) \cdot G(s)} \]  
for all \( r, s > 0 \).

Let \( c_0 > 1 \) be the absolute constant of Corollary 2.3 and if \( 0 < \varepsilon < 1/c_0 \), then let
\[ \omega(\varepsilon) = c_0 \cdot \sqrt[n]{\ln \varepsilon} \]  
be the error estimate in Corollary 2.3.

The main goal of this section is to prove

**Lemma 5.1** If \( 0 < \varepsilon < \frac{1}{c_n^4} \) for suitable absolute constant \( c > 1 \), then
\[ \int_0^\infty ||\Phi_t| - |\Omega_t|| \, dt \leq 49\sqrt{n} \sqrt{\omega(\varepsilon)} \]  
and
\[ \int_0^\infty ||\Psi_t| - |\Omega_t|| \, dt \leq 49\sqrt{n} \sqrt{\omega(\varepsilon)}. \]  

**Proof:** The absolute constant \( c > 1 \) is defined in a way such that
\[ \omega(\varepsilon) < \frac{1}{512n} \]  
(cf 25).

We observe that \( \Phi_t, \Psi_t, \Omega_t \) are convex bodies, and \( F(t), G(t), H(t) \) are decreasing and log-concave, and \( F, G \) are probability distributions on \([0, \infty)\)
by (22). Since \( \int_0^\infty H = \int_{\mathbb{R}^n} h \leq 1 + \varepsilon \), it follows from Corollary 2.3 that there exists some \( b > 0 \) such that
\[
\begin{align*}
\int_0^\infty |bF(bt) - H(t)| dt & \leq \omega(\varepsilon) \\
\int_0^\infty |b^{-1}G(b^{-1}t) - H(t)| dt & \leq \omega(\varepsilon).
\end{align*}
\] (28)

We may assume that \( b \geq 1 \).

For \( t > 0 \), let
\[
\begin{align*}
\tilde{\Phi}_t &= b^{\frac{1}{n}} \Phi_{bt} \quad \text{if} \quad \tilde{\Phi}_t \neq \emptyset \\
\tilde{\Psi}_t &= \tilde{\Psi}_t = b^{-\frac{1}{n}} \Psi_{b^{-1}t} \quad \text{if} \quad \tilde{\Psi}_t \neq \emptyset.
\end{align*}
\]

These sets satisfy \( |\tilde{\Phi}_t| = bF(bt) \), \( |\tilde{\Psi}_t| = b^{-1}G(b^{-1}t) \) and
\[
\begin{align*}
\int_0^\infty |\tilde{\Phi}_t| - H(t) dt & \leq \omega(\varepsilon) \\
\int_0^\infty |\tilde{\Psi}_t| - H(t) dt & \leq \omega(\varepsilon).
\end{align*}
\] (29) (30)

In addition, (23) yields that if \( \tilde{\Phi}_t \neq \emptyset \) and \( \tilde{\Psi}_t \neq \emptyset \), then
\[
\frac{1}{2} (b^{\frac{1}{n}} \tilde{\Phi}_t + b^{-\frac{1}{n}} \tilde{\Psi}_t) \subset \Omega_t.
\] (31)

We dissect \( [0, \infty) \) into \( I \) and \( J \), where \( t \in I \), if \( \frac{3}{4} H(t) < |\tilde{\Phi}_t| < \frac{5}{4} H(t) \) and \( \frac{3}{4} H(t) < |\tilde{\Psi}_t| < \frac{5}{4} H(t) \), and \( t \in J \) otherwise. For \( J \), since \( \varepsilon < \frac{1}{c n} \), we choose \( c > 1 \) in a way such that (27) holds, (29) and (30) yield that
\[
\int_J H(t) dt \leq 4 \int_J \left( |\tilde{\Phi}_t| - H(t) + |\tilde{\Psi}_t| - H(t) \right) dt \leq 8 \omega(\varepsilon) < \frac{1}{2}. \] (32)

Turning to \( I \), it follows from the Prékopa-Leindler inequality and (32) that
\[
\int_I H(t) dt \geq 1 - \int_J H(t) dt > \frac{1}{2}. \] (33)

For \( t \in I \), we define \( \alpha(t) = |\tilde{\Phi}_t|/H(t) \) and \( \beta(t) = |\tilde{\Psi}_t|/H(t) \), and hence \( \frac{3}{4} < \alpha(t), \beta(t) < \frac{5}{4} \), and (29) and (30) imply
\[
\int_0^\infty H(t) \cdot (|\alpha(t) - 1| + |\beta(t) - 1|) dt \leq 2 \omega(\varepsilon). \] (34)
We set
\[ \eta(t) = \left( \frac{b \pi |\Phi_t|^{\frac{1}{m}}}{|\Psi_t|^{\frac{1}{m}}} - \frac{b \pi |\Psi_t|^{\frac{1}{m}}}{|\Phi_t|^{\frac{1}{m}}} \right)^2. \]

As for \( s \geq 1 \), we have
\[ s^{\frac{1}{m}} - s^{-\frac{1}{m}} = s^{\frac{1}{m}}(s^{\frac{1}{m}} - 1) \geq s^{\frac{1}{m}} \cdot \frac{s^{\frac{1}{m} - 1}}{2n}(s - 1) \geq \frac{1}{2n} \left( s - \frac{1}{s} \right), \]

it follows from the definition of \( \alpha(t) \) and \( \beta(t) \) that
\[ \eta(t) \geq \frac{1}{4n^2} \left( \frac{b^2 \beta(t)}{\alpha(t)} - \frac{\alpha(t)}{b^2 \beta(t)} \right)^2. \] (35)

We deduce from (24) that
\[ H(t) \geq \left( \frac{|\Phi_t|^{\frac{1}{m}} + |\Psi_{b-1,t}|^{\frac{1}{m}}}{2} \right)^n = \left( \frac{b \pi |\Phi_t|^{\frac{1}{m}} + b \pi |\Psi_t|^{\frac{1}{m}}}{2} \right)^n \]
\[ = \sqrt{|\Phi_t| \cdot |\Psi_t|} \left( 1 + \frac{1}{2} \left( \frac{b \pi |\Phi_t|^{\frac{1}{m}}}{|\Psi_t|^{\frac{1}{m}}} - \frac{b \pi |\Psi_t|^{\frac{1}{m}}}{|\Phi_t|^{\frac{1}{m}}} \right)^2 \right)^n \]
\[ \geq H(t) \cdot \sqrt{\alpha(t) \cdot \beta(t)} \left( 1 + \frac{n}{2} \cdot \eta(t) \right). \] (36)

We claim that if \( t \in I, \) then
\[ \sqrt{\alpha(t) \cdot \beta(t)} \left( 1 + \frac{n}{2} \cdot \eta(t) \right) \geq 1 - 2|\alpha(t) - 1| - 2|\beta(t) - 1| + \frac{(b - 1)^2}{16n \cdot b^2}. \] (37)

Since \( \sqrt{\alpha(t) \cdot \beta(t)} \geq 1 - |\alpha(t) - 1| - |\beta(t) - 1|, \) (37) readily holds if \( |\alpha(t) - 1| - |\beta(t) - 1| \geq \frac{(b - 1)^2}{16n \cdot b^2}. \) Therefore we may assume that
\[ |\alpha(t) - 1| + |\beta(t) - 1| \leq \frac{(b - 1)^2}{16n \cdot b^2} < \frac{1}{2}, \] (38)
which condition in turn yields that
\[ \frac{b^2 \beta(t)}{\alpha(t)} \geq \frac{b^2 \left( 1 - \frac{(b - 1)^2}{16n \cdot b^2} \right)}{1 + \frac{(b - 1)^2}{16n \cdot b^2}} \geq b^2 \left( 1 - 2 \cdot \frac{(b - 1)^2}{16n \cdot b^2} \right) \geq b^2 \left( 1 - \frac{b - 1}{b} \right) = b. \] (39)
We deduce first applying (38), then (35) and finally (39) that
\[
\sqrt{\alpha(t) \cdot \beta(t)} \left( 1 + \frac{n}{2} \cdot \eta(t) \right) \geq \left( 1 - |\alpha(t) - 1| - |\beta(t) - 1| \right) \left( 1 + \frac{n}{2} \cdot \eta(t) \right) \\
\geq 1 - |\alpha(t) - 1| - |\beta(t) - 1| + \frac{n}{4} \cdot \eta(t) \\
\geq 1 - |\alpha(t) - 1| - |\beta(t) - 1| + \frac{1}{16n} \left( \frac{b^2 \beta(t)}{\alpha(t)} - \frac{\alpha(t)}{b^2 \beta(t)} \right)^2 \\
\geq 1 - |\alpha(t) - 1| - |\beta(t) - 1| + \frac{1}{16n} \left( b - \frac{1}{b} \right)^2,
\]
proving (37) also under the assumption (38), as well.

It follows first from (33), after that from (36) and (37) and finally from (34) that
\[
\frac{(b - 1)^2}{32n \cdot b^2} \leq \int_I H(t) \cdot \frac{(b - 1)^2}{16n \cdot b^2} \, dt \leq \int_I H(t) \cdot (2|\alpha(t) - 1| + 2|\beta(t) - 1|) \, dt \leq 4\omega(\varepsilon).
\]
Since \( \omega(\varepsilon) < \frac{1}{512n} \) (cf. (27)) and \( \frac{(b - 1)^2}{b^4} \) is monotone increasing for \( b \geq 1 \), we deduce that \( b < 2 \); therefore,
\[
b \leq 1 + \sqrt{512n \omega(\varepsilon)} < 24\sqrt{n\sqrt{\omega(\varepsilon)}}. \tag{40}
\]

Next we claim that
\[
\int_0^\infty |\Phi_t| - |\bar{\Phi}_t| \, dt \leq 48\sqrt{n\sqrt{\omega(\varepsilon)}} \tag{41}
\]
\[
\int_0^\infty |\Psi_t| - |\bar{\Psi}_t| \, dt \leq 48\sqrt{n\sqrt{\omega(\varepsilon)}}.
\]
Since \( |\Phi_{bt}| \leq |\Phi_t| \), we have
\[
\int_0^\infty |\Phi_t| - |\bar{\Phi}_t| \, dt = \int_0^\infty ||\Phi_t| - b|\Phi_{bt}|| \, dt \\
\leq \int_0^\infty ||\Phi_t| - b|\Phi_t|| \, dt + b \int_0^\infty ||\Phi_t| - |\Phi_{bt}|| \, dt \\
= (b - 1) + b \int_0^\infty |\Phi_t| - |\Phi_{bt}| \, dt \\
= 2(b - 1) \leq 48\sqrt{n\sqrt{\omega(\varepsilon)}}.
\]

18
Similarly, $|\Psi_t| \leq |\Psi_{b^{-1}t}|$, and hence

\[
\int_0^\infty |\Psi_t| - |\tilde{\Psi}_t| \, dt = \int_0^\infty |\Psi_t| - b^{-1}|\Psi_{b^{-1}t}| \, dt \\
\leq \int_0^\infty |\Psi_t| - b^{-1}|\Psi_t| \, dt + b^{-1}\int_0^\infty |\Psi_t| - |\Psi_{b^{-1}t}| \, dt \\
= (1 - b^{-1}) + b^{-1}\int_0^\infty |\Psi_{b^{-1}t}| - |\Psi_t| \, dt \\
= 2(1 - b^{-1}) \leq 48\sqrt{n} \sqrt{\omega(\varepsilon)},
\]

proving (41).

We conclude (26) from combining (29), (30) and (41). \qed

As a first consequence of (26), we verify the following.

**COROLLARY 5.2** There exists an absolute constant $c > 1$ such that if $0 < \varepsilon < (cn)^{-n}$, then $\frac{1}{2} < M_f/M_g < 2$ and $\frac{1}{2} < M_f/M_h < 2$.

**Proof:** First we prove the estimate about $M_g$ using that (26) yields

\[
\int_0^\infty |\Phi_t| - |\Psi_t| \, dt \leq 98\sqrt{\omega(\varepsilon)}.
\]

We may assume that $1 = M_f \geq M_g$. Since $|\Psi_t| = 0$ if $t > M_g$, we deduce first from (42), and then from (11) and $k! < (\frac{1}{e})^k \sqrt{2\pi(k + 1)}$ that

\[
98\sqrt{\omega(\varepsilon)} \geq \int_{M_g}^1 |\Phi_t| \, dt \geq \frac{1}{2 \cdot n!} \int_{M_g}^1 (1 - t)^n \, dt \\
= \frac{1}{2 \cdot n!} \frac{(1 - M_g)^{n+1}}{n + 1} > \frac{e^{n+1}}{2(n + 1)^{n+1} \sqrt{2\pi(n+2)}} \cdot (1 - M_g)^{n+1},
\]

and hence

\[
1 - M_g < c_1 n \omega(\varepsilon)^{\frac{1}{2(n+1)}}
\]

for an absolute constant $c_1 > 0$. In particular, we deduce from (25) that for some absolute constant $c > 1$, if $0 < \varepsilon < (cn)^{-n}$, then $M_g > \frac{1}{2}$.

The proof of $\frac{1}{2} < M_f/M_h < 2$ is analogous based directly on (26). \qed
6 Proof of Theorem 3.1

We use the notation set up in Section 5 with the additional assumption
\( f(o) = 1 \), and hence
\( f(o) = M_f = 1 \) and \( g(o) = M_g \). (43)

First we assume that
\( \varepsilon < c^{-n}n^{-n} \) (44)

for suitably large absolute constant \( c > 1 \). According to (43), (44) and Corollary 5.2, we have
\[
\frac{1}{2} < g(o) = M_g < 2 \\
\frac{1}{2} < M_h < 2. 
\] (45)

We write \( u_0 \) to denote the \((n+1)\)th basis vector in \( \mathbb{R}^{n+1} \) orthogonal to \( \mathbb{R}^n \). We set
\[
\xi = \frac{\sqrt[6]{\omega(\varepsilon)}}{\ln \omega(\varepsilon) |^2} 
\] (46)

where (44) ensures that
\[
\xi < \frac{e^{-4(n-1)}}{2} \quad \text{and} \quad 6e\xi \cdot |\ln \xi|^n < \frac{1}{2}. 
\] (47)

We observe that \( \Phi_t = \{ x \in \mathbb{R}^n : x + w \ln t \in K \} \) for \( \xi \leq t \leq f(x) \), etc. In particular, using the substitution \( s = \ln t \), it follows from \( M_f = 1 \) (see (43)), \( \frac{1}{2} < M_g, M_h < 2 \) (see (45) and (11)) and (47) that
\[
\int_\xi^1 |\Phi_t| dt = \int_\xi^{M_f} |\Phi_t| dt > 1 - 2e \cdot \xi \cdot |\ln \xi|^n > \frac{1}{2} 
\] (48)
\[
\int_\xi^2 |\Psi_t| dt = \int_\xi^{M_g} |\Psi_t| dt > 1 - 3e \cdot \frac{\xi}{M_g} \cdot |\ln \frac{\xi}{M_g}|^n \\
> 1 - 6e\xi \cdot |\ln \xi|^n > \frac{1}{2} 
\] (49)
\[
\int_\xi^2 |\Omega_t| dt = \int_\xi^{M_h} |\Omega_t| dt > 1 - 6e\xi \cdot |\ln \xi|^n > \frac{1}{2} 
\] (50)
We consider the following convex bodies in $\mathbb{R}^{n+1}$:

$$K = \{ x + u_0 \ln t : x \in \Phi_\xi \text{ and } \xi \leq t \leq f(x) \}$$

$$C = \{ x + u_0 \ln t : x \in \Psi_\xi \text{ and } \xi \leq t \leq g(x) \}$$

$$L = \{ x + u_0 \ln t : x \in \Omega_\xi \text{ and } \xi \leq t \leq h(x) \}.$$

We write $V(\cdot)$ to denote volume ($(n+1)$-dimensional Lebesgue measure) in $\mathbb{R}^{n+1}$. It follows from (48) and (49) that

$$V(K) = \int_{\ln \xi}^{0} |\Phi_{e^s}| \cdot \frac{1}{t} \, dt \geq \int_{\ln \xi}^{1} |\Phi_{t}| \cdot \frac{1}{t} \, dt > \frac{1}{2},$$

$$V(C) = \int_{\ln \xi}^{\ln 2} |\Psi_{e^s}| \cdot \frac{1}{t} \, dt \geq \int_{\ln \xi}^{2} |\Psi_{t}| \cdot \frac{1}{t} \, dt > \frac{1}{4}.$$  

Since $K$ is contained in a right cylinder whose base is a translate of $\Phi_\xi$ and height is $|\ln \xi|$, and $C$ is contained in a right cylinder whose base is a translate of $\Psi_\xi$ and height is $|\ln \xi| + \ln 2 < 2|\ln \xi|$, we deduce from (13) that

$$V(K) \leq \frac{2}{n!} \cdot |\ln \xi|^{n+1},$$

$$V(C) \leq \frac{4}{n!} \cdot |\ln \xi|^{n+1}. $$

It follows from (26), $f(o) = 1$, (45) and using the substitution $s = \ln t$ that

$$|V(K) - V(L)| = \left| \int_{\ln \xi}^{\ln 2} (|\Phi_{e^s}| - |\Omega_{e^s}|) \, ds \right| = \left| \int_{\xi}^{2} (|\Phi_{t}| - |\Omega_{t}|) \cdot \frac{1}{t} \, dt \right|$$

$$\leq \frac{1}{\xi} \int_{\xi}^{2} ||\Phi_{t}| - |\Omega_{t}|| \cdot dt \leq 49 \sqrt{n} \frac{\sqrt{\omega(\varepsilon)}}{\xi},$$

and similarly

$$|V(C) - V(L)| \leq 49 \sqrt{n} \frac{\sqrt{\omega(\varepsilon)}}{\xi}. $$

Combining (56) and (57) leads to

$$|V(C) - V(K)| \leq 98 \sqrt{n} \frac{\sqrt{\omega(\varepsilon)}}{\xi}. $$

The condition $h(\frac{x+y}{2}) \geq \sqrt{f(x)g(y)}$ for $x, y \in \mathbb{R}^n$ in Theorem 3.1 implies that

$$\frac{1}{2} K + \frac{1}{2} C \subset L.$$
LEMMA 6.1 Assuming the condition $\varepsilon < c^{-n}n^{-n}$ as in (11), there exist $w \in \mathbb{R}^n$ and absolute constant $\gamma > 1$ such that

$$V(K \Delta (L - w)) \leq \gamma n^8 \cdot \sqrt[3]{\frac{\omega(\varepsilon)}{\sqrt{\xi}}} \cdot |\ln \xi|^{n+1}$$

$$V(C \Delta (L + w)) \leq \gamma n^8 \cdot \sqrt[3]{\frac{\omega(\varepsilon)}{\sqrt{\xi}}} \cdot |\ln \xi|^{n+1}.$$  

Proof: First we verify a slightly weaker statement; namely, we allow to choose the translation vectors from $\mathbb{R}^{n+1}$, not only from $\mathbb{R}^n$. More precisely, we claim that there exist $\tilde{w} \in \mathbb{R}^{n+1}$ and absolute constant $\gamma > 1$ such that

$$V(K \Delta (L - \tilde{w})) \leq \gamma n^8 \cdot \sqrt[3]{\frac{\omega(\varepsilon)}{\sqrt{\xi}}} \cdot |\ln \xi|^{n+1} \quad (60)$$

$$V(C \Delta (L + \tilde{w})) \leq \gamma n^8 \cdot \sqrt[3]{\frac{\omega(\varepsilon)}{\sqrt{\xi}}} \cdot |\ln \xi|^{n+1}. \quad (61)$$

To prove (60) and (61), let us consider a homothetic copy $K_0 \subset K$ of $K$ and homothetic copy $C_0 \subset C$ of $C$ with $V(K_0) = V(C_0) = \min\{V(K), V(C)\}$, and hence either $K = K_0$ or $C = C_0$, and

$$\frac{1}{2} K_0 + \frac{1}{2} C_0 \subset L. \quad (62)$$

It also follows from (56) and (57), and from the fact that either $K = K_0$ or $C = C_0$, that

$$V(L) - V(K_0) \leq 49 \sqrt{n} \cdot \sqrt[6]{\frac{\omega(\varepsilon)}{\xi V(K_0)}} \cdot V(K_0) \quad (63)$$

where (14) (provided $\tilde{c}$ is large enough), (54) and $\xi = \sqrt[6]{\omega(\varepsilon)/|\ln \omega(\varepsilon)|^{1/2}}$ yield that $\frac{\sqrt[6]{\omega(\varepsilon)}}{\xi V(K_0)} < 2 \sqrt{n} \cdot \frac{\sqrt[6]{\omega(\varepsilon)/|\ln \omega(\varepsilon)|^{1/2}}}{2} \sqrt[6]{V(K_0)}$ is small enough to apply Lemma 3.2 to $K_0$, $C_0$ and $L$. In particular, we deduce from (62), (63) and Lemma 3.2 that there exist $\tilde{w} \in \mathbb{R}^{n+1}$ and an absolute constant $\gamma_0 > 1$ such that

$$V(K_0 \Delta (L - \tilde{w})) \leq \gamma_0 n^8 \cdot \sqrt[6]{\frac{\omega(\varepsilon)}{\sqrt[6]{\xi V(K_0)}}} \cdot V(K_0) = \gamma_0 n^8 \sqrt[6]{\frac{\omega(\varepsilon)}{\sqrt[6]{\xi}}} \cdot \sqrt{V(K_0)}$$

$$V(C_0 \Delta (L + \tilde{w})) \leq \gamma_0 n^8 \cdot \sqrt[6]{\frac{\omega(\varepsilon)}{\sqrt[6]{\xi}}} \cdot \sqrt{V(K_0)}.$$
In turn, it follows from (54), (58) and the properties of $K_0$ and $C_0$ that

$$V(K \Delta (L - \tilde{w})) \leq (\gamma_0 + 1)n^8 \cdot \sqrt[4]{\frac{\omega(\varepsilon)}{\xi}} \cdot \sqrt{V(K)}$$

$$V(C \Delta (L + \tilde{w})) \leq (\gamma_0 + 1)n^8 \cdot \sqrt[4]{\frac{\omega(\varepsilon)}{\xi}} \cdot \sqrt{V(K)}.$$

Therefore, (55) implies (60) and (61).

Next we verify that if $w \in \mathbb{R}^n$ satisfies that $\tilde{w} = w + pu_0$ for $p \in \mathbb{R}$, then, then we have

$$V(K \Delta (L - w)) \leq 3V(K \Delta (L - \tilde{w})) \quad (64)$$

$$V(C \Delta (L + w)) \leq 3V(C \Delta (L + \tilde{w})). \quad (65)$$

For (64), we may assume that $p \neq 0$, and we distinguish two cases.

If $p < 0$, then let

$$K_{(p)} = \{x \in K : \ln \xi \leq \langle x, u_0 \rangle < |p| + \ln \xi \} \subset K \setminus (L - \tilde{w}).$$

Using the fact that $\Phi_t$ is decreasing as a set as $t > 0$ increases and the Fubini theorem, we have

$$V(K \Delta (K + |p|u_0)) = 2V(K_{(p)}) \leq 2V(K \Delta (L - \tilde{w}));$$

therefore, the triangle inequality for the symmetric difference metric implies

$$V(K \Delta (L - w)) = V((K + |p|u_0) \Delta (L - \tilde{w}))$$

$$\leq V((K + |p|u_0) \Delta K) + V(K \Delta (L - \tilde{w}))$$

$$\leq 3V(K \Delta (L - \tilde{w})).$$

Similarly, if $p > 0$, then we consider

$$L_{(p)} = \{x \in L : \ln \xi \leq \langle x, u_0 \rangle < p + \ln \xi \}$$

satisfying

$$L_{(p)} + \tilde{w} \subset (L + \tilde{w}) \setminus K.$$ Using that $\Omega_t$ is decreasing as a set as $t > 0$ increases, we deduce

$$V((L + \tilde{w}) \Delta (L + w)) = 2V(L_{(p)}) \leq 2V(K \Delta (L + \tilde{w}));$$

23
therefore, the triangle inequality for the symmetric difference metric implies
\[ V(K \Delta (L + w)) \leq V(K \Delta (L + \tilde{w})) + V((L + \tilde{w}) \Delta (L + w)) \leq 3V(K \Delta (L + \tilde{w})), \]
completing the proof of (64). The argument for (65) is similar.

In turn, combining (60), (61), (64) and (65) yields Lemma 6.1.

\[ \square \]

**Proof of Theorem 3.1** We may assume that \( f \) and \( g \) are log-concave probability distributions satisfying
\[ f(o) = M_f = 1 \text{ and } g(o) = M_g. \]

In particular, (45) implies \( \frac{1}{2} < M_f, M_h < 2 \). We consider the convex bodies 
\( K, C, L \subset \mathbb{R}^{n+1} \) defined by (51), (52) and (53), and let \( w \in \mathbb{R}^n \) be provided by Lemma 6.1.

For (14), we have \( \frac{1}{2} < M_f, M_h < 2 \) and hence \( \frac{\xi}{M_f}, \frac{\xi}{M_h} < 2\xi \) and both \( 1 + \frac{\xi}{M_f} \) and \( 1 + \frac{\xi}{M_h} \) are at most 3. For the functions \( f, g, h \), it follows from \( n \geq 2 \), (14) (compare the condition (47)), Lemma 6.1, \( \xi = \sqrt[\frac{1}{4}]{\omega(\varepsilon)/|\ln \omega(\varepsilon)|^{\frac{1}{4}}} \) and the \( \gamma \) of Lemma 6.1 that
\[
\int_{\mathbb{R}^n} |f(x) - h(x - w)| \, dx \leq 2 \gamma n^8 \cdot \sqrt[\frac{1}{2}]{\omega(\varepsilon)} \cdot |\ln \omega(\varepsilon)|^{\frac{n-1}{4}} + 2 \cdot 3 \cdot (2\xi) \cdot |\ln(2\xi)|^n
\]
\[
\leq 2\gamma n^8 \cdot \sqrt[\frac{1}{4}]{\omega(\varepsilon)} \cdot |\ln \omega(\varepsilon)|^{n-\frac{4}{4}}.
\]
Similarly, we have
\[
\int_{\mathbb{R}^n} |g(x) - h(x + w)| \, dx \leq 2\gamma n^8 \cdot \sqrt[\frac{1}{4}]{\omega(\varepsilon)} \cdot |\ln \omega(\varepsilon)|^{n-\frac{4}{4}}.
\]
Since \( \omega(\varepsilon) = c_0 \varepsilon |\ln \varepsilon|^\frac{4}{3} \) for an absolute constant \( c_0 > 1 \) (cf. \((25)\)), we conclude that
\[
\int_{\mathbb{R}^n} |f(x) - h(x - w)| \, dx \leq \gamma_0 n^8 \cdot \sqrt[3]{\varepsilon} \cdot |\log \varepsilon|^n
\]
\[
\int_{\mathbb{R}^n} |g(x) - h(x + w)| \, dx \leq \gamma_0 n^8 \cdot \sqrt[3]{\varepsilon} \cdot |\log \varepsilon|^n
\]
for an absolute constant \( \gamma_0 > 1 \), proving Theorem 3.1. \( \square \)

7 A version of Theorem 1.4 when \( \varepsilon \) is small

The goal of this section is to prove the following version of Theorem 1.4.

**THEOREM 7.1** For some absolute constant \( c > 1 \), if \( \tau \in (0, \frac{1}{2}] \), \( \tau \leq \lambda \leq 1 - \tau \), \( h, f, g : \mathbb{R}^n \rightarrow [0, \infty) \) are integrable such that \( h((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda \) for \( x, y \in \mathbb{R}^n \), \( h \) is log-concave and
\[
\int_{\mathbb{R}^n} h \leq (1 + \varepsilon) \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} g \right)^\lambda
\]
for \( \varepsilon \in (0, \tau) \) for \( \varepsilon_0 = c^{-n}n^{-n} \), then there exists \( w \in \mathbb{R}^n \) such that for \( a = \int_{\mathbb{R}^n} f / \int_{\mathbb{R}^n} g \), we have
\[
\int_{\mathbb{R}^n} |f(x) - a^\lambda h(x - \lambda w)| \, dx \leq cn^8 \frac{\sqrt[3]{\varepsilon}}{\tau} \cdot |\log \frac{\varepsilon}{\tau}| \int_{\mathbb{R}^n} f
\]
\[
\int_{\mathbb{R}^n} |g(x) - a^{-(1-\lambda)} h(x + (1 - \lambda)w)| \, dx \leq cn^8 \frac{\sqrt[3]{\varepsilon}}{\tau} \cdot |\log \frac{\varepsilon}{\tau}| \int_{\mathbb{R}^n} g.
\]

For a bounded measurable function \( f : \mathbb{R}^n \rightarrow [0, \infty) \), the log-concave hull \( \tilde{f} : \mathbb{R}^n \rightarrow [0, \infty) \) of is
\[
\tilde{f}(z) = \sup_{\varepsilon = \sum_{i=1}^k \alpha_i z_i, \sum_{i=1}^k \alpha_i = 1, \gamma_{\alpha_i} \geq 0} \prod_{i=1}^k f(x_i)^{\alpha_i}.
\]
To show that \( \tilde{f} \) is log-concave, it is equivalent to prove that if \( \varepsilon, \alpha, \beta \in (0, 1) \) and \( x, y \in \mathbb{R}^n \), then
\[
\tilde{f}(\varepsilon x + \beta y) \geq (1 - \varepsilon)\tilde{f}(x)^\alpha \tilde{f}(y)^\beta. \quad (66)
\]
We observe that there exist \( x_1, \ldots, x_k, y_1, \ldots, y_m \in \mathbb{R}^n \) and \( \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m \geq 0 \) with \( \sum_{i=1}^k \alpha_i = 1, \sum_{j=1}^m \beta_j = 1, x = \sum_{i=1}^k \alpha_i x_i \) and \( y = \sum_{i=1}^k \beta_j y_j \) such that
\[
(1 - \varepsilon)\tilde{f}(x) \geq \prod_{i=1}^k f(x_i)^{\alpha_i} \quad \text{and} \quad (1 - \varepsilon)\tilde{f}(y) \geq \prod_{j=1}^m f(y_j)^{\beta_j}.
\]

Since
\[
\alpha x + \beta y = \sum_{i=1}^k \sum_{j=1}^m (\alpha_i \beta_j \alpha x_i + \alpha_i \beta_j \beta y_j) \quad \text{where} \quad \sum_{i=1}^k \sum_{j=1}^m (\alpha_i \beta_j \alpha + \alpha_i \beta_j \beta) = 1,
\]
we deduce that
\[
\tilde{f}(\alpha x + \beta y) \geq \prod_{i=1}^k \prod_{j=1}^m f(x_i)^{\alpha_i \beta_j \alpha} f(y_j)^{\alpha_i \beta_j \beta}
\]
\[
= \left( \prod_{i=1}^k f(x_i)^{\alpha_i} \right)^{\alpha} \left( \prod_{j=1}^m f(y_j)^{\beta_j} \right)^{\beta}
\]
\[
\geq (1 - \varepsilon)^\alpha \tilde{f}(x)^\alpha (1 - \varepsilon)^\beta \tilde{f}(y)^\beta = (1 - \varepsilon)^\alpha \tilde{f}(x)^\alpha \tilde{f}(y)^\beta,
\]
proving that \( \tilde{f} \) is log-concave via (66).

We note that if \( a_0 > 0 \) and \( z_0 \in \mathbb{R}^n \) and \( f_0(z) = a_0 f(z - z_0) \), then
\[
\tilde{f}_0(z) = a_0 \tilde{f}(z - z_0).
\]

We prepare the proof of Theorem 7.1 by the three technical statements Lemma 7.2, Lemma 7.3 and Lemma 7.4 about log-concave functions.

**Lemma 7.2** If \( \lambda \in (0, 1) \), \( h \) is a log-concave function on \( \mathbb{R}^n \) with positive integral, and \( f, g : \mathbb{R}^n \to [0, \infty) \) are measurable satisfying \( \int_{\mathbb{R}^n} f > 0, \int_{\mathbb{R}^n} g > 0 \) and \( h(1 - \lambda)x + \lambda y) \geq f(x)^{1 - \lambda} g(y)^{\lambda} \) for \( x, y \in \mathbb{R}^n \), then \( f \) and \( g \) are bounded, and their log-concave hulls \( \tilde{f} \) and \( \tilde{g} \) satisfy that \( h((1 - \lambda)x + \lambda y) \geq \tilde{f}(x)^{1 - \lambda} \tilde{g}(y)^{\lambda} \) for \( x, y \in \mathbb{R}^n \).

**Remark** The Prékopa-Leindler inequality yields \( \int_{\mathbb{R}^n} \tilde{f} < \infty \) and \( \int_{\mathbb{R}^n} \tilde{g} < \infty \).

**Proof:** To show that \( f \) is bounded, we choose \( y_0 \in \mathbb{R}^n \) with \( g(y_0) > 0 \). For any \( x \in \mathbb{R}^n \), we have \( h((1 - \lambda)x + \lambda y_0) \geq f(x)^{1 - \lambda} g(y_0)^\lambda \); therefore,
\[
f(x) \leq \frac{h((1 - \lambda)x + \lambda y_0)^{1 - \lambda}}{g(y_0)^{1 - \lambda}} \leq \frac{M_{\tilde{h}}}{M_{\tilde{g}}} g(y_0)^{1 - \lambda}.
\]

26
Similar argument yields that \( g \) is bounded. For \( x, y \in \mathbb{R}^n \), it is sufficient to prove that for any \( \varepsilon \in (0, 1) \), we have

\[
h((1 - \lambda)x + \lambda y) \geq (1 - \varepsilon)\tilde{f}(x)^{1-\lambda}\tilde{g}(y)^{\lambda}.
\]

(69)

We choose \( x_1, \ldots, x_k, y_1, \ldots, y_m \in \mathbb{R}^n \) and \( \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m \geq 0 \) with \( \sum_{i=1}^k \alpha_i = 1, \sum_{j=1}^m \beta_j = 1, \) \( x = \sum_{i=1}^k \alpha_i x_i \) and \( y = \sum_{i=1}^k \beta_j y_j \) such that

\[
(1 - \varepsilon)\tilde{f}(x) \geq \prod_{i=1}^k f(x_i)^{\alpha_i} \quad \text{and} \quad (1 - \varepsilon)\tilde{f}(y) \geq \prod_{j=1}^m f(y_j)^{\beta_j}.
\]

It follows from (67) and the log-concavity of \( h \) that

\[
h((1 - \lambda)x + \lambda y) = h\left( \sum_{i=1}^k \sum_{j=1}^m \alpha_i \beta_j ((1 - \lambda)x_i + \lambda y_j) \right)
\geq \prod_{i=1}^k \prod_{j=1}^m h((1 - \lambda)x_i + \lambda y_j)^{\alpha_i \beta_j} \geq \prod_{i=1}^k \prod_{j=1}^m f(x_i)^{(1-\lambda)\alpha_i \beta_j} g(y_j)^{\lambda \alpha_i \beta_j}
\]

\[
= \left( \prod_{i=1}^k f(x_i)^{\alpha_i} \right)^{1-\lambda} \left( \prod_{j=1}^m f(y_j)^{\beta_j} \right)^{\lambda}
\]

\[
\geq (1 - \varepsilon)^{1-\lambda}\tilde{f}(x)^{1-\lambda}(1 - \varepsilon)^{\lambda}\tilde{g}(y)^{\lambda} = (1 - \varepsilon)\tilde{f}(x)^{1-\lambda}\tilde{g}(y)^{\lambda},
\]

proving (69). \( \square \)

**Lemma 7.3** Let \( f, g : \mathbb{R}^n \to [0, \infty) \) be log-concave and have positive integral.

(i) For \( \lambda \in [0, 1] \), the function \( h_\lambda : \mathbb{R}^n \to [0, \infty) \) defined by

\[
h_\lambda(z) = \sup_{z = (1-\lambda)x + \lambda y} f(x)^{1-\lambda} g(y)^{\lambda}
\]

is log-concave, has positive integral, and satisfies \( h_0 = f \) and \( h_1 = g \).

(ii) The function \( \lambda \mapsto \int_{\mathbb{R}^n} h_\lambda \) is log-concave for \( \lambda \in [0, 1] \).
Proof: For (i), readily $h_0 = f$ and $h_1 = g$. Next let $\lambda \in (0, 1)$. To show the log-concavity of $h_\lambda$, it is sufficient to prove that if $z_1, z_2 \in \mathbb{R}_n$, $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and $\varepsilon \in (0, 1)$, then

$$h_\lambda(\alpha z_1 + \beta z_2) \geq (1 - \varepsilon)h_\lambda(z_1)^\alpha h_\lambda(z_2)^\beta. \quad (70)$$

We choose $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ such that $z_1 = (1 - \lambda)x_1 + \lambda y_1$, $z_2 = (1 - \lambda)x_2 + \lambda y_2$ and

$$f(x_1)^{1-\lambda}g(y_1)^\lambda \geq (1 - \varepsilon)h_\lambda(z_1) \quad \text{and} \quad f(x_2)^{1-\lambda}g(y_2)^\lambda \geq (1 - \varepsilon)h_\lambda(z_2).$$

It follows that $\alpha z_1 + \beta z_2 = (1 - \lambda)(\alpha x_1 + \beta x_2) + \lambda(\alpha y_1 + \beta y_2)$, and the log-concavity of $f$ and $g$ yields

$$h_\lambda(\alpha z_1 + \beta z_2) = h_\lambda((1 - \lambda)(\alpha x_1 + \beta x_2) + \lambda(\alpha y_1 + \beta y_2)) \geq f(\alpha x_1 + \beta x_2)^{1-\lambda}g(\alpha y_1 + \beta y_2)^\lambda \geq f(x_1)^{\alpha(1-\lambda)}f(x_2)^{\beta(1-\lambda)}g(y_1)^{\alpha(\lambda)}g(y_2)^{\beta(\lambda)} = (f(x_1)^{1-\lambda}g(y_1)^\lambda)^\alpha (f(x_2)^{1-\lambda}g(y_2)^\lambda)^\beta \geq (1 - \varepsilon)^\alpha h_\lambda(z_1)^\alpha (1 - \varepsilon)^\beta h_\lambda(z_2)^\beta = (1 - \varepsilon)^\alpha h_\lambda(z_1)^\alpha h_\lambda(z_2)^\beta,$

proving (70), and in turn the log-concavity of $h_\lambda$.

Readily, $\int_{\mathbb{R}^n} h_\lambda > 0$. If follows from Lemma 4.4 that $0 < M_f, M_g < \infty$, and hence

$$M = M_{h_\lambda} = M_f^{1-\lambda}M_g^\lambda.$$ 

If $t \in (0, M)$ and $h_\lambda(z) > t$, then there exist $x, y \in \mathbb{R}^n$ such that

$$z = (1 - \lambda)x + \lambda y \quad (71)$$

and $f(x)^{1-\lambda}g(y)^\lambda > t$. It follows that

$$f(x) > \left(\frac{t}{M_g^\lambda}\right)^{\frac{1}{1-\lambda}} \quad \text{and} \quad g(y) > \left(\frac{t}{M_f^{1-\lambda}}\right)^{\frac{1}{\lambda}}. \quad (72)$$

We conclude from (71), (72) and Lemma 4.4 that $h_\lambda(z) > t$ is bounded; therefore, $h_\lambda$ has positive integral by Lemma 4.4.

Finally, to verify that the function $\lambda \mapsto \int_{\mathbb{R}^n} h_\lambda$ is log-concave for $\lambda \in [0, 1]$, it is enough to prove that if $\lambda_1, \lambda_2 \in [0, 1]$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$, then for $\lambda = \alpha \lambda_1 + \beta \lambda_2$, we have

$$\int_{\mathbb{R}^n} h_\lambda \geq \left(\int_{\mathbb{R}^n} h_{\lambda_1}\right)^\alpha \left(\int_{\mathbb{R}^n} h_{\lambda_2}\right)^\beta. \quad (73)$$
According to the Prékopa-Leindler inequality Theorem 1.2, it is sufficient to show that if \( z = \alpha z_1 + \beta z_2 \), \( z_1, z_2 \in \mathbb{R}^n \), then
\[
h_\lambda(z) \geq h_{\lambda_1}(z_1)^\alpha h_{\lambda_2}(z_2)^\beta.
\]
In turn, (73) is a consequence of the claim that if \( z = \alpha z_1 + \beta z_2 \) for \( z_1, z_2 \in \mathbb{R}^n \) and \( \varepsilon \in (0, 1) \), then
\[
h_\lambda(z) \geq (1 - \varepsilon) h_{\lambda_1}(z_1)^\alpha h_{\lambda_2}(z_2)^\beta.
\]
For \( i = 1, 2 \), there exist \( x_i, y_i \in \mathbb{R}^n \) such that
\[
f(x_i)^{1-\lambda_i} g(y_i)^{\lambda_i} \geq (1 - \varepsilon) h_\lambda(z_i).
\]
As \( \lambda = \alpha \lambda_1 + \beta \lambda_2 \) and \( z = \alpha z_1 + \beta z_2 \), we observe that \( 1 - \lambda = \alpha (1 - \lambda_1) + \beta (1 - \lambda_2) \) and
\[
z = \alpha z_1 + \beta z_2 = \alpha [ (1 - \lambda_1) x_1 + \lambda_1 y_1 ] + \beta [ (1 - \lambda_2) x_2 + \lambda_2 y_2 ]
\]
\[
\quad \quad = (1 - \lambda) \cdot \left( \frac{\alpha (1 - \lambda_1)}{1 - \lambda} \cdot x_1 + \frac{\beta (1 - \lambda_2)}{1 - \lambda} \cdot x_2 \right) + \lambda \cdot \left( \frac{\alpha \lambda_1}{\lambda} \cdot y_1 + \frac{\beta \lambda_2}{\lambda} \cdot y_2 \right).
\]
It follows from the log-concavity of \( f \) and \( g \) and later from (76) that
\[
h_\lambda(z) \geq f \left( \frac{\alpha (1 - \lambda_1)}{1 - \lambda} \cdot x_1 + \frac{\beta (1 - \lambda_2)}{1 - \lambda} \cdot x_2 \right)^{1-\lambda} g \left( \frac{\alpha \lambda_1}{\lambda} \cdot y_1 + \frac{\beta \lambda_2}{\lambda} \cdot y_2 \right)^\lambda
\]
\[
\quad \quad \geq f(x_1)^{\alpha (1-\lambda_1)} f(x_2)^{\beta (1-\lambda_2)} g(y_1)^{\alpha \lambda_1} g(y_2)^{\beta \lambda_2}
\]
\[
\quad \quad = (f(x_1)^{1-\lambda_1} g(y_1)^{\lambda_1})^\alpha (f(x_2)^{1-\lambda_2} g(y_2)^{\lambda_2})^\beta
\]
\[
\quad \quad \geq (1 - \varepsilon)^\alpha h_{\lambda_1}(z_1)^\alpha (1 - \varepsilon)^\beta h_{\lambda_2}(z_2)^\beta = (1 - \varepsilon) h_{\lambda_1}(z_1)^\alpha h_{\lambda_2}(z_2)^\beta;
\]
proving (74), and in turn (73). \( \square \)

**Lemma 7.4** If \( \lambda \in (0, 1) \), \( \eta \in (0, 2 \cdot \min\{1 - \lambda, \lambda\}) \) and \( \varphi \) is a log-concave function on \([0, 1]\) satisfying \( \varphi(\lambda) \leq (1 + \eta) \varphi(0)^{1-\lambda} \varphi(1)^\lambda \), then
\[
\varphi \left( \frac{1}{2} \right) \leq \left( 1 + \frac{\eta}{\min\{1 - \lambda, \lambda\}} \right) \sqrt{\varphi(0) \varphi(1)}
\]

29
Proof: We may assume that \(0 < \lambda < \frac{1}{2}\), and hence \(\lambda = (1 - 2\lambda) \cdot 0 + 2\lambda \cdot \frac{1}{2}\), \(\varphi(\lambda) \leq (1 + \eta)\varphi(0)^{1-\lambda}\varphi(1)^{\lambda}\) and the log-concavity of \(\varphi\) yield

\[
(1 + \eta)\varphi(0)^{1-\lambda}\varphi(1)^{\lambda} \geq \varphi(\lambda) \geq \varphi(0)^{1-2\lambda}\varphi(\frac{1}{2})^{2\lambda}.
\]

Thus \((1 + \eta)\frac{1}{\lambda} \leq e^{\frac{1}{\lambda}} \leq 1 + \frac{\eta}{\lambda}\) implies

\[
\varphi(\frac{1}{2}) \leq (1 + \eta)\frac{1}{\lambda} \leq \varphi(0)\varphi(1) \leq (1 + \frac{\eta}{\lambda}) \sqrt{\varphi(0)\varphi(1)}.
\]

Proof of Theorem 7.1: For the \(\lambda\) in Theorem 7.1, we may assume that \(0 < \lambda \leq \frac{1}{2}\), and hence \(\min\{1 - \lambda, \lambda\} = \lambda\).

For suitable \(d, e > 0\) and \(w \in \mathbb{R}^n\), we may replace \(f(z)\) by \(d \cdot f(z - w)\), \(g(z)\) by \(e \cdot g(z + w)\) and \(h(z)\) by \(d^{1-\lambda}e^{\lambda}h(z + (2\lambda - 1)w)\) where \(e\) and \(d\) will be defined by (78) below, and \(w\) will be defined by (81) and (82).

Let \(\tilde{f}\) and \(\tilde{g}\) be the log-concave hulls of \(f\) and \(g\); therefore, Lemma 7.2 yields

\[
h((1 - \lambda)x + \lambda y) \geq \tilde{f}(x)^{1-\lambda}\tilde{g}(y)^{\lambda}
\]

for \(x, y \in \mathbb{R}^n\). (77)

We may assume by (68) that

\[
\int_{\mathbb{R}^n} \tilde{f} = \int_{\mathbb{R}^n} \tilde{g} = 1.
\]

(78)

It follows from Lemma 7.3 that

\[
h_t(z) = \sup_{x=(1-t)x+ty} \tilde{f}(x)^{1-t}\tilde{g}(y)^t
\]

satisfies that

\[
\varphi(t) = \int_{\mathbb{R}^n} h_t
\]

is log-concave on \([0, 1]\). In particular, (78) implies that

\[
\varphi(0) = \varphi(1) = 1.
\]

(79)

It follows from (77), (78), the Prékopa-Leindler inequality Theorem 1.2 and the conditions in Theorem 7.1 that

\[
1 = \left(\int_{\mathbb{R}^n} \tilde{f}\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} \tilde{g}\right)^{\lambda} \leq \int_{\mathbb{R}^n} h_\lambda \leq \int_{\mathbb{R}^n} h \leq (1 + \varepsilon) \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g\right)^{\lambda} \leq (1 + \varepsilon) \left(\int_{\mathbb{R}^n} \tilde{f}\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} \tilde{g}\right)^{\lambda} = 1 + \varepsilon.
\]

(80)
On the other hand, (80), Lemma 7.4 and finally (79) yield that
\[ \int_{\mathbb{R}^n} h_{1/2} = \varphi \left( \frac{1}{2} \right) \leq \left( 1 + \frac{\varepsilon}{\lambda} \right) \sqrt{\varphi(0)\varphi(1)} = 1 + \frac{\varepsilon}{\lambda}. \]

In turn, we deduce from Theorem 3.1 that there exists \( w \in \mathbb{R}^n \) such that
\[ \int_{\mathbb{R}^n} |\tilde{f}(z) - h_{1/2}(z + w)| \, dz \leq \tilde{c} n \varepsilon^{18} \sqrt{\frac{\varepsilon}{\lambda}} \cdot \log \frac{\varepsilon}{\lambda} \cdot \sqrt{\varphi(0) \varphi(1)} = 1 + \varepsilon \lambda. \]

Replacing \( f(z) \) by \( f(z - w) \) and \( g(z) \) by \( g(z + w) \) (cf. (68)), the function \( h_{1/2} \) does not change, and we have
\[ \int_{\mathbb{R}^n} |\tilde{f} - h_{1/2}| \leq \tilde{c} n \varepsilon^{18} \sqrt{\frac{\varepsilon}{\lambda}} \cdot \log \frac{\varepsilon}{\lambda}. \]

To replace \( h_{1/2} \) by \( h \) in (83) and (84), we claim that
\[ \int_{\mathbb{R}^n} |h - h_{1/2}| \leq 5\tilde{c} n \varepsilon^{18} \sqrt{\frac{\varepsilon}{\lambda}} \cdot \log \frac{\varepsilon}{\lambda}. \]

To prove (85), we consider
\[ X_- = \{ x \in \mathbb{R}^n : h(x) \leq h_{1/2}(x) \} \]
\[ X_+ = \{ x \in \mathbb{R}^n : h(x) > h_{1/2}(x) \}. \]

It follows from (77) that for any \( x \in X_- \), we have
\[ h(x) \geq \tilde{f}(x)^{1-\lambda} \tilde{g}(x)^{\lambda} \geq \min\{ \tilde{f}(x), \tilde{g}(x) \}, \]
or in other words, if \( x \in X_- \), then
\[ 0 \leq h_{1/2}(x) - h(x) \leq |h_{1/2}(x) - \tilde{f}(x)| + |h_{1/2}(x) - \tilde{g}(x)|. \]

In particular, (83) and (84) imply
\[ \int_{X_-} |h - h_{1/2}| = \int_{X_-} (h_{1/2} - h) \leq \int_{X_-} (|h_{1/2} - \tilde{f}| + |h_{1/2} - \tilde{g}|) \]
\[ \leq 2\tilde{c} n \varepsilon^{18} \sqrt{\frac{\varepsilon}{\lambda}} \cdot \log \frac{\varepsilon}{\lambda} \cdot \sqrt{\varphi(0) \varphi(1)} = 1 + \varepsilon \lambda. \]
On the other hand, $\int_{\mathbb{R}^n} h < 1 + \varepsilon$ and $\int_{\mathbb{R}^n} h_{1/2} \geq 1$ by (80), thus (86) implies

$$\int_{X_+} |h - h_{1/2}| = \int_{X_+} (h - h_{1/2}) = \int_{\mathbb{R}^n} h - \int_{\mathbb{R}^n} h_{1/2} + \int_{X_-} (h_{1/2} - h) \leq \varepsilon + \int_{X_-} (h_{1/2} - h) \leq 3\tilde{c}n^8 \frac{1}{\lambda} \sqrt{\varepsilon} \cdot |\log \frac{\varepsilon}{\lambda}|. \quad (87)$$

We conclude (85) by (86) and (87).

To replace $\tilde{f}$ and $\tilde{g}$ by $f$ and $g$ in (83) and (84), we claim that

$$\int_{\mathbb{R}^n} |f - \tilde{f}| \leq \varepsilon \quad \text{and} \quad \int_{\mathbb{R}^n} |g - \tilde{g}| \leq \varepsilon. \quad (88)$$

Readily $\tilde{f} \geq f$ and $\tilde{g} \geq g$. It follows from (80) and $\int_{\mathbb{R}^n} g \leq \int_{\mathbb{R}^n} \tilde{g} = 1$ that

$$\int_{\mathbb{R}^n} |f - \tilde{f}| = \int_{\mathbb{R}^n} \tilde{f} - \int_{\mathbb{R}^n} f \leq 1 - \frac{1}{1 + \varepsilon} < \varepsilon,$$

and similar argument for $g$ and $\tilde{g}$ completes the proof of (88).

We conclude from (83), (84), (85) and (88) that

$$\int_{\mathbb{R}^n} |f - h| \, dx \leq 7\tilde{c}n^8 \frac{1}{\lambda} \sqrt{\varepsilon} \cdot |\log \frac{\varepsilon}{\lambda}|,$$

$$\int_{\mathbb{R}^n} |g - h| \, dx \leq 7\tilde{c}n^8 \frac{1}{\lambda} \sqrt{\varepsilon} \cdot |\log \frac{\varepsilon}{\lambda}|,$$

proving Theorem 7.1. \(\square\)

8 Proof of Theorem 1.4 and Corollary 1.5

First we verify a simple estimate.

**Lemma 8.1** If $\varrho > 0$, $t > 1$ and $n \geq 2$, then

$$(\log t)^n \leq \left( \frac{n \varrho}{e} \right)^n t^{\frac{1}{t}}.$$
Proof: We observe that for $s = \log t$, differentiating the function $s \mapsto n \log s - \frac{s}{e}$ yields

$$\log \left( \frac{(\log t)^n}{t^n} \right) = n \log s - \frac{s}{e} \leq n(\log(ne) - 1) = n \log \frac{ne}{e}.$$ 

In turn, we conclude Lemma 8.1. \endproof

Proof of Theorem 1.4 and Corollary 1.5: We may assume that $f$ and $g$ are probability densities.

For Theorem 1.4, we deduce from Theorem 7.1 and Lemma 8.1 the following statement: For some absolute constants $c_1, c_2 > 1$, if $\epsilon < c_1^{-n\cdot n^{-n}} \cdot \tau$, then there exists $w \in \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x) - h(x - \lambda w)| \, dx \leq c_2^n n^{-n^{19} \sqrt{\epsilon \tau}}, \quad \quad (89)$$

$$\int_{\mathbb{R}^n} |g(x) - h(x + (1 - \lambda)w)| \, dx \leq c_2^n n^{-n^{19} \sqrt{\epsilon \tau}}, \quad \quad (90)$$

settling Theorem 1.4 if $\epsilon < c_1^{-n\cdot n^{-n}} \cdot \tau$.

On the other hand, if $\epsilon \geq c_1^{-n\cdot n^{-n}} \cdot \tau$, then we use that the left hand sides of (89) and (90) are at most $2 + \epsilon \leq 3$; therefore both (89) and (90) readily hold for suitable absolute constant $c_2 > 1$. This proves Theorem 1.4.

For Corollary 1.5, the functions $f$ and $g$ are log-concave probability densities on $\mathbb{R}^n$. In this case, we define

$$h(z) = \sup_{z=(1-\lambda)x + \lambda y} f(x)^{1-\lambda} g(y)^\lambda,$$

which is log-concave on $\mathbb{R}^n$ according to Lemma 7.3 (i). For the $w$ in (89) and (90), we deduce that

$$\bar{L}_1(f, g) \leq \int_{\mathbb{R}^n} |f(x + w) - g(x)| \, dx \leq 2c_2^n n^{19} \sqrt{\epsilon \tau},$$

yielding Corollary 1.5. \endproof
9 Proof of Theorem 1.6

We deduce from Lemma 7.3 (i) and induction on $m$ the following corollary.

**COROLLARY 9.1** If $\lambda_1, \ldots, \lambda_m > 0$ satisfy $\sum_{i=1}^m \lambda_i = 1$ and $f_1, \ldots, f_m$ are log-concave functions with positive integral on $\mathbb{R}^n$, then

$$h(z) = \sup_{z = \sum_{i=1}^m \lambda_i x_i} \prod_{i=1}^m f(x_i)^{\lambda_i}$$

is log-concave and has positive integral.

The first main step towards proving Theorem 9.4 is the case when each $\lambda_i$ in Theorem 9.4 is $1/m$.

**THEOREM 9.2** Let $c > 1$ be the absolute constant in Theorem 7.1, let $\gamma_0 = cn^8$ and $\epsilon_0 = c/n - n$. If $f_1, \ldots, f_m$, $m \geq 2$ are log-concave probability densities on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \sup_{mz = \sum_{i=1}^m x_i} \prod_{i=1}^m f(x_i)^{\frac{1}{m}} \, dz \leq 1 + \epsilon$$

for $0 < \epsilon < \epsilon_0/m^4$, then for the log-concave $h(z) = \sup_{mz = \sum_{i=1}^m x_i} \prod_{i=1}^m f(x_i)^{\frac{1}{m}}$, there exist $w_1, \ldots, w_m \in \mathbb{R}^n$ such that $\sum_{i=1}^m w_i = 0$ and

$$\int_{\mathbb{R}^n} |f_i(x) - h(x + w_i)| \, dx \leq m^4 \cdot \gamma_0 \cdot |\log \epsilon|^n.$$ 

**Proof:** Instead of Theorem 9.2 we prove that if $0 < \epsilon < \epsilon_0/4^{[\log_2 m]}$, then there exist $w_1, \ldots, w_m \in \mathbb{R}^n$ such that $\sum_{i=1}^m w_i = 0$ and

$$\int_{\mathbb{R}^n} |f_i(x) - h(x + w_i)| \, dx \leq 4^{[\log_2 m]} \cdot \gamma_0 \cdot |\log \epsilon|^n. \quad (91)$$

Since $4^{[\log_2 m]} < 4^{\log_2 m} = m^4$, (91) yields Theorem 9.2.

We prove (91) by induction on $[\log_2 m] \geq 1$. If $[\log_2 m] = 1$, and hence $m = 2$, then

$$h(z) = \sup_{z = \lambda_1 x_1 + \lambda_2 x_2} f_1(x_1)^{\lambda_1} f_2(x_2)^{\lambda_2}$$

34
is log-concave by Lemma 7.3 (i). Therefore, the case $m = 2$ of (91) follows from Theorem 7.1.

Next, we assume that $[\log_2 m] > 1$, and let $k = \lfloor m/2 \rfloor$, and hence $[\log_2 (m - k)] \leq \lfloor \log_2 k \rfloor = \lfloor \log_2 m \rfloor - 1$. We consider the coefficient

$$\lambda = \frac{m - k}{m}$$

satisfying $\frac{1}{3} \leq \lambda \leq \frac{1}{2}$.

and the functions

$$h(z) = \sup_{mz = \sum_{i=1}^m x_i} \prod_{i=1}^m f_i(x_i)^{\frac{1}{m}}$$

$$f(z) = \sup_{kz = \sum_{i=1}^k x_i} \prod_{i=1}^k f_i(x_i)^{\frac{1}{k}}$$

$$g(z) = \sup_{(m-k)z = \sum_{i=k+1}^m x_i} \prod_{i=k+1}^m f_i(x_i)^{\frac{1}{m-k}}$$

which are log-concave by Corollary 9.1. In particular, we have

$$h(z) = \sup_{z = \lambda x + (1-\lambda)y} f(x)^{1-\lambda} g(y)^{\lambda}.$$

It follows from the Prékopa-Leindler inequality that

$$\int_{\mathbb{R}^n} f \geq 1$$

$$\int_{\mathbb{R}^n} g \geq 1$$

$$\int_{\mathbb{R}^n} h \geq \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} g \right)^{\lambda} \geq 1.$$  

(97)

Since $\int_{\mathbb{R}^n} h < 1 + \varepsilon$, we deduce on the one hand, that

$$\int_{\mathbb{R}^n} f \leq \left( 1 + \varepsilon \right)^{\frac{1}{1-\lambda}} \leq (1 + \varepsilon)^3 \leq 1 + 4\varepsilon$$

$$\int_{\mathbb{R}^n} g \leq 1 + 4\varepsilon,$$

(99) (100)
and on the other hand, Theorem 7.1 yields that for \( a = \int_{\mathbb{R}^n} g f \), there exists \( w \in \mathbb{R}^n \) such that

\[
\int_{\mathbb{R}^n} |f(x) - a^\lambda h(x - \lambda w)| \, dx \leq \gamma_0 \frac{\varepsilon}{\sqrt{1/3}} \cdot |\log \varepsilon|^n \int_{\mathbb{R}^n} f
\]

\[
\int_{\mathbb{R}^n} |g(x) - a^{-(1-\lambda)} h(x + (1 - \lambda)w)| \, dx \leq \gamma_0 \frac{\varepsilon}{\sqrt{1/3}} \cdot |\log \varepsilon|^n \int_{\mathbb{R}^n} g
\]

(101)

We deduce from (95), (96), (99), (100) that \( 1 + 4 \varepsilon > a, a^{-1} > \frac{1}{1+4\varepsilon} > 1 - 4\varepsilon \); therefore, \( \frac{1}{3} \leq \lambda \leq \frac{2}{3}, \int_{\mathbb{R}^n} h < 1 + \varepsilon, (101) \) and (102) yield

\[
\int_{\mathbb{R}^n} |f(x) - h(x - \lambda w)| \, dx \leq 4\gamma_0 \frac{\varepsilon}{\sqrt{4\varepsilon}} \cdot |\log \varepsilon|^n
\]

(103)

\[
\int_{\mathbb{R}^n} |g(x) - h(x + (1 - \lambda)w)| \, dx \leq 4\gamma_0 \frac{\varepsilon}{\sqrt{4\varepsilon}} \cdot |\log \varepsilon|^n.
\]

(104)

Since \( \lfloor \log_2 (m - k) \rfloor \leq \lfloor \log_2 k \rfloor = \lfloor \log_2 m \rfloor - 1 \), induction and (91), (93) and (92) yield that there exist \( \tilde{w}_1, \ldots, \tilde{w}_m \in \mathbb{R}^n \) such that

\[
\sum_{i=1}^k \tilde{w}_i = \sum_{j=k+1}^m \tilde{w}_j = 0,
\]

(105)

and if \( i = 1, \ldots, k \) and \( j = k + 1, \ldots, m \), then

\[
\int_{\mathbb{R}^n} |f_i(x) - f(x + \tilde{w}_i)| \, dx \leq 4^{\lfloor \log_2 k \rfloor} \gamma_0 \frac{\varepsilon}{\sqrt{4\varepsilon}} \cdot |\log \varepsilon|^n,
\]

(106)

\[
\int_{\mathbb{R}^n} |f_j(x) - g(x + \tilde{w}_j)| \, dx \leq 4^{\lfloor \log_2 (m-k) \rfloor} \gamma_0 \frac{\varepsilon}{\sqrt{4\varepsilon}} \cdot |\log \varepsilon|^n.
\]

(107)

Combining (103), (104), (106) and (107) shows the existence of \( w_1, \ldots, w_m \in \mathbb{R}^n \) such that if \( i = 1, \ldots, m \), then

\[
\int_{\mathbb{R}^n} |f_i(x) - h(x + w_i)| \, dx \leq 4 \cdot 4^{\lfloor \log_2 m \rfloor - 1} \gamma_0 \frac{\varepsilon}{\sqrt{4\varepsilon}} \cdot |\log \varepsilon|^n = 4^{\lfloor \log_2 m \rfloor} \gamma_0 \frac{\varepsilon}{\sqrt{4\varepsilon}} \cdot |\log \varepsilon|^n;
\]

namely,

\[
w_i = -\lambda w - \tilde{w}_i \text{ for } i = 1, \ldots, k,
\]

\[
w_j = (1 - \lambda)w - \tilde{w}_j \text{ for } j = k + 1, \ldots, m.
\]
Since \( \lambda = \frac{m-k}{m} \) and \( 1 - \lambda = \frac{k}{m} \), we have

\[
\sum_{i=1}^{m} w_i = -k \cdot \lambda w - \left( \sum_{i=1}^{k} \bar{w}_i \right) + (m-k)(1-\lambda)w - \left( \sum_{j=k+1}^{m} \bar{w}_j \right) = o,
\]
proving (91). \( \square \)

For \( m \geq 2 \), we consider the \((m-1)\)-simplex

\[
\Delta^{m-1} = \{ p = (p_1, \ldots, p_m) \in \mathbb{R}^m : p_1 + \ldots + p_m = 1 \}.
\]

The proof of Lemma 7.3 readily extends to verify the following statement.

**Lemma 9.3** Let \( f_1, \ldots, f_m, m \geq 2 \) be log-concave probability densities with positive integral on \( \mathbb{R}^n \), \( n \geq 2 \). For \( p = (p_1, \ldots, p_m) \in \Delta^{m-1} \), the function

\[
h_p(z) = \sup_{z = \sum_{i=1}^{m} p_i x_i} \prod_{i=1}^{m} f_i(x_i)^{p_i}
\]

is log-concave on \( \mathbb{R}^n \), and the function \( p \mapsto \int_{\mathbb{R}^n} h_p \) is log-concave on \( \Delta^{m-1} \).

**Theorem 9.4** For some absolute constant \( \tilde{\gamma} > 1 \), if \( \tau \in (0, \frac{1}{m}] \), \( m \geq 2 \), \( \lambda_1, \ldots, \lambda_m \in [\tau, 1-\tau] \) satisfy \( \sum_{i=1}^{m} \lambda_i = 1 \) and \( f_1, \ldots, f_m \) are log-concave functions with positive integral on \( \mathbb{R}^n \) such that

\[
\int_{\mathbb{R}^n} \sup_{z = \sum_{i=1}^{m} \lambda_i x_i} \prod_{i=1}^{m} f_i(x_i)^{\lambda_i} \, dz \leq (1 + \varepsilon) \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} f_i \right)^{\lambda_i}
\]

for \( 0 < \varepsilon < \tau \cdot \tilde{\gamma}^{-n} n^{-n} / m^4 \), then for the log-concave \( h(z) = \sup_{z = \sum_{i=1}^{m} \lambda_i x_i} \prod_{i=1}^{m} f_i(x_i)^{\lambda_i} \), there exist \( a_1, \ldots, a_m > 0 \) and \( w_1, \ldots, w_m \in \mathbb{R}^n \) such that \( \sum_{i=1}^{m} \lambda_i w_i = o \) and for \( i = 1, \ldots, m \), we have

\[
\int_{\mathbb{R}^n} |f_i(x) - a_i h(x + w_i)| \, dx \leq \tilde{\gamma} m^5 n^8 \sqrt{\varepsilon} \cdot \left| \log \frac{\varepsilon}{m \tau} \right| \int_{\mathbb{R}^n} f_i.
\]

**Remark** \( a_i = \frac{(f_0 \circ f_i)^{1-\lambda_i}}{\prod_{j \neq i} (f_0 \circ f_j)^{\lambda_j}} \) for \( i = 1, \ldots, m \) in Theorem 9.4

**Proof:** Let \( \tau \in (0, \frac{1}{m}] \), let \( \lambda_1, \ldots, \lambda_m \in [\tau, 1-\tau] \) with \( \lambda_1 + \ldots + \lambda_m = 1 \) and let \( f_1, \ldots, f_m \) be log-concave with positive integral as in Theorem 9.4.
In particular, \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \Delta^{m-1} \). We may assume that \( f_1, \ldots, f_m \) are probability densities, \( \lambda_1 \geq \ldots \geq \lambda_m \) and \( \lambda_m < \frac{1}{m} \) (as if \( \lambda_m = \frac{1}{m} \), then Theorem 9.2 implies Theorem 9.4).

Let \( \tilde{p} = \left( \frac{1}{m}, \ldots, \frac{1}{m} \right) \), and for \( i = 1, \ldots, m \), let \( v(i) \in \mathbb{R}^m \) be the vector whose \( i \)th coordinate is 1, and the rest is 0, and hence \( v(1), \ldots, v(m) \) are the vertices of \( \Delta^{m-1} \), and \( \lambda = \sum_{i=1}^{m} \lambda_i v(i) \). For \( p = (p_1, \ldots, p_m) \in \Delta^{m-1} \), we write

\[
h_p(z) = \sup_{z = \sum_{i=1}^{m} p_i x_i} \prod_{i=1}^{m} f_i(x_i)^{p_i},
\]

and hence \( h_{v(i)} = f_i \). According the conditions in Theorem 9.4,

\[
\int_{\mathbb{R}^n} h_{\lambda} < 1 + \varepsilon. \tag{108}
\]

Since \( \lambda_i - \lambda_m \geq 0 \) for \( i = 1, \ldots, m - 1 \) and \( m \lambda_m < 1 \), it follows that

\[
q = \sum_{i=1}^{m-1} \frac{\lambda_i - \lambda_m}{1 - m \lambda_m}, \quad v(i) \in \Delta^{m-1},
\]

and Corollary 9.3 and \( \int_{\mathbb{R}^n} h_{v(i)} = \int_{\mathbb{R}^n} f_i = 1 \) yield that \( \int_{\mathbb{R}^n} h_q \geq 1 \). Since

\[
\lambda = m \lambda_m \tilde{p} + \sum_{i=1}^{m-1} (\lambda_i - \lambda_m) v(i) = m \lambda_m \tilde{p} + (1 - m \lambda_m) q,
\]

(108) and Corollary 9.3 imply that

\[
1 + \varepsilon > \int_{\mathbb{R}^n} h_{\lambda} \geq \left( \int_{\mathbb{R}^n} h_q \right)^{1 - m \lambda_m} \left( \int_{\mathbb{R}^n} h_{\tilde{p}} \right)^{m \lambda_m} \geq \left( \int_{\mathbb{R}^n} h_{\tilde{p}} \right)^{m \lambda_m};
\]

therefore, \( \varepsilon \leq m \tau \leq m \lambda_m \) yields

\[
\int_{\mathbb{R}^n} h_{\tilde{p}} < (1 + \varepsilon)^{\frac{1}{m \lambda_m}} < e^{\frac{\varepsilon}{m \lambda_m}} < 1 + \frac{2 \varepsilon}{m \lambda_m}.
\]

According to Theorem 9.2 there exist \( w_1, \ldots, w_m \in \mathbb{R}^n \) such that \( \sum_{i=1}^{m} w_i = 0 \) and

\[
\int_{\mathbb{R}^n} |f_i(x + w_i) - h_{\tilde{p}}(x)| \, dx \leq m^4 \cdot \gamma_0 \sqrt{\frac{2 \varepsilon}{m \lambda_m} \cdot \log \frac{2 \varepsilon}{m \lambda_m}}.
\]

38
for $i = 1, \ldots, m$. Replacing $f_i(x)$ by $f_i(x + w_i)$ for $i = 1, \ldots, m$ does not change $h_{\tilde{\varphi}}$ by the condition $\sum_{i=1}^{m} w_i = 0$; therefore, we may assume that

$$\int_{\mathbb{R}^n} |f_i(x) - h_{\tilde{\varphi}}(x)| \, dx \leq m^4 \gamma_0^{1/8} \sqrt{\frac{2\varepsilon}{m \lambda m}} \cdot \left| \log \frac{2\varepsilon}{m \lambda m} \right|^n \quad (109)$$

for $i = 1, \ldots, m$.

To replace $h_{\tilde{\varphi}}$ by $h_{\lambda}$ in (109), we claim that

$$\int_{\mathbb{R}^n} |h_{\lambda} - h_{\tilde{\varphi}}| \leq 3m^5 \gamma_0^{1/8} \sqrt{\frac{2\varepsilon}{m \lambda m}} \cdot \left| \log \frac{2\varepsilon}{m \lambda m} \right|^n. \quad (110)$$

To prove (110), we consider

$$X_- = \{ x \in \mathbb{R}^n : h_{\lambda}(x) \leq h_{\tilde{\varphi}}(x) \} \quad X_+ = \{ x \in \mathbb{R}^n : h_{\lambda}(x) > h_{\tilde{\varphi}}(x) \}. \quad \text{(111)}$$

It follows from the definition of $h_{\lambda}$ that for any $x \in X_-$, we have

$$h_{\lambda}(x) \geq \prod_{i=1}^{m} f_i(x)^{\lambda_i} \geq \min\{f_1(x), \ldots, f_m(x)\},$$

or in other words, if $x \in X_-$, then

$$0 \leq h_{\tilde{\varphi}}(x) - h_{\lambda}(x) \leq \sum_{i=1}^{m} |f_i(x) - h_{\tilde{\varphi}}(x)|.$$ 

In particular, (109) implies

$$\int_{X_-} |h_{\lambda} - h_{\tilde{\varphi}}| = \int_{X_-} (h_{\lambda} - h_{\tilde{\varphi}}) \leq \sum_{i=1}^{m} \int_{X_-} |f_i(x) - h_{\tilde{\varphi}}(x)| \leq m^5 \cdot \gamma_0^{1/8} \sqrt{\frac{2\varepsilon}{m \lambda m}} \cdot \left| \log \frac{2\varepsilon}{m \lambda m} \right|^n. \quad (111)$$

On the other hand, $\int_{\mathbb{R}^n} h_{\lambda} < 1 + \varepsilon$ and the Prékopa-Leindler inequality yields $\int_{\mathbb{R}^n} h_{\tilde{\varphi}} \geq 1$, thus (111) implies

$$\int_{X_+} |h_{\lambda} - h_{\tilde{\varphi}}| = \int_{X_+} (h_{\lambda} - h_{\tilde{\varphi}}) = \int_{\mathbb{R}^n} h_{\lambda} - \int_{\mathbb{R}^n} h_{\tilde{\varphi}} + \int_{X_-} (h_{\tilde{\varphi}} - h_{\lambda}) \leq \varepsilon + \int_{X_-} (h_{\tilde{\varphi}} - h_{\lambda}) \leq 2m^5 \cdot \gamma_0^{1/8} \sqrt{\frac{2\varepsilon}{m \lambda m}} \cdot \left| \log \frac{2\varepsilon}{m \lambda m} \right|^n \quad (112)$$
We conclude (110) by (111) and (112).

Finally, combining (109) and (110) prove Theorem 9.4. □

Proof of Theorem 1.6 We may assume that $\int_{\mathbb{R}^n} f_i = 1$ for $i = 1, \ldots, m$ in Theorem 1.6 for the log-concave functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$.

Let $\tau \in (0, \frac{1}{m}]$ for $m \geq 2$, and let $\lambda_1, \ldots, \lambda_m \in [\tau, 1-\tau]$ satisfy $\sum_{i=1}^m \lambda_i = 1$ such that

$$\int_{\mathbb{R}^n} \sup_{z=\sum_{i=1}^m \lambda_i x_i} \prod_{i=1}^m f_i(x_i)^{\lambda_i} \, dz \leq 1 + \varepsilon$$

for $\varepsilon \in (0, 1]$.

For the absolute constant $\tilde{\gamma} > 1$ of Theorem 9.4 if

$$0 < \varepsilon < \tau \cdot \tilde{\gamma}^{-n} n^{-n} / m^4,$$  \hfill (113)

then for the log-concave $h(z) = \sup_{z=\sum_{i=1}^m \lambda_i x_i} \prod_{i=1}^m f_i(x_i)^{\lambda_i}$, there exist $w_1, \ldots, w_m \in \mathbb{R}^n$ such that $\sum_{i=1}^m \lambda_i w_i = o$ and for $i = 1, \ldots, m$, we have

$$\int_{\mathbb{R}^n} |f_i(x) - h(x + w_i)| \, dx \leq \tilde{\gamma} m^5 n^8 \sqrt{\varepsilon / m \tau} \cdot \left|\log \frac{\varepsilon}{m \tau}\right|^n.$$

We deduce from Lemma 8.1 that

$$\int_{\mathbb{R}^n} |f_i(x) - h(x + w_i)| \, dx \leq \tilde{\gamma}_0 n^n m^{5} \sqrt{\varepsilon / m \tau} \quad \hfill (114)$$

for $i = 1, \ldots, m$ and some absolute constant $\tilde{\gamma}_0 \geq \max\{\tilde{\gamma}, 3\}$, proving Theorem 1.6 if (113) holds. Finally if $\varepsilon \geq \tau \cdot \tilde{\gamma}^{-n} n^{-n} / m^4$, then (114) readily holds as the left hand side is at most $2 + \varepsilon \leq 3$. □

Acknowledgement We are grateful for useful discussions with Alessio Figalli.

References

[1] K.M. Ball: PhD thesis, University of Cambridge, 1988.

[2] K.M. Ball: An elementary introduction to modern convex geometry. In: Flavors of geometry (Silvio Levy, ed.), Cambridge University Press, 1997, 1-58.
[3] K.M. Ball, K.J. Böröczky: Stability of the Prékopa-Leindler inequality. Mathematika 56 (2010), 339-356.

[4] K.M. Ball, K.J. Böröczky: Stability of some versions of the Prkopa-Leindler inequality. Monatsh. Math., 163 (2011), 1-14.

[5] F. Barthe: On a reverse form of the Brascamp-Lieb inequality. Invent. Math., 134 (1998), 335-361.

[6] F. Barthe: Autour de l’inégalité de Brunn-Minkowski. Mémoire d’Habilitation, 2008.

[7] F. Barthe, K.J. Böröczky, M. Fradelizi: Stability of the functional forms of the Blaschke-Santal inequality. Monatsh. Math. 173 (2014), no. 2, 135-159.

[8] G. Bianchi, H. Egnell: A note on the Sobolev inequality. J. Funct. Anal. 100 (1991), 18-24.

[9] W. Blaschke: Über affine Geometrie I. Isoperimetrische Eigenschaften von Ellipse und Ellipsoid. Leipz. Ber., 68 (1916), 217-239.

[10] W. Blaschke: Über affine Geometrie VII. Neue Extremeigenschaften von Ellipse und Ellipsoid. Leipz. Ber., 69 (1917), 306-318.

[11] W. Blaschke: Über affine Geometrie XXXVII. Eine Verschärfung von Minkowskis Ungleichheit für den gemischten Flächeninhalt. Hamb. Abh., 1 (1922), 206-209.

[12] B. Bollobás, I. Leader: Products of unconditional bodies. Geometric aspects of functional analysis (Israel, 19921994), Oper. Theory Adv. Appl., 77, Birkhauser, Basel, (1995), 13-24.

[13] T. Bonnesen, W. Fenchel: Theory of convex bodies. BCS Associates, 1987.

[14] C. Borell: Convex set functions in d-space. Period. Math. Hungar. 6 (1975), 111-136.

[15] K.J. Böröczky: Stability of the Blaschke-Santaló and the affine isoperimetric inequality. Advances in Mathematics, 225 (2010), 1914-1928.
[16] H.J. Brascamp, E.H. Lieb: On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. J. Functional Analysis 22 (1976), 366-389.

[17] U. Caglar, E.M. Werner: Stability results for some geometric inequalities and their functional versions. Convexity and concentration, IMA Vol. Math. Appl., 161, Springer, New York, 2017, 541-564.

[18] E. Carlen, F. Maggi: Stability for the Brunn-Minkowski and Riesz rearrangement inequalities, with applications to Gaussian concentration and finite range non-local isoperimetry. Canad. J. Math., 69 (2017), 1036-1063.

[19] M. Christ: An approximate inverse RieszSobolev inequality, preprint, available online at http://arxiv.org/abs/1112.3715, 2012.

[20] D. Cordero-Erausquin, M. Fradelizi, B. Maurey: The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems. J. Funct. Anal., 214 (2004), 410-427.

[21] V.I. Diskant: Stability of the solution of a Minkowski equation. (Russian) Sibirsk. Mat. Ž. 14 (1973), 669–673. [Eng. transl.: Siberian Math. J., 14 (1974), 466–473.]

[22] S. Dubuc: Critères de convexité et inégalités intégrales. Ann. Inst. Fourier Grenoble 27 (1) (1977), 135-165.

[23] R. Eldan, B. Klartag, Dimensionality and the stability of the Brunn-Minkowski inequality. Annali della Scuola Normale Superiore di Pisa, Classe di Scienze (5) 13 (2014), 975–1007.

[24] L. Esposito, N. Fusco, C. Trombetti: A quantitative version of the isoperimetric inequality: the anisotropic case. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4(4), (2005), 619-651.

[25] A. Figalli, D. Jerison: Quantitative stability for sumsets in \( \mathbb{R}^n \). J. Eur. Math. Soc. (JEMS) 17 (2015), 1079-1106.

[26] A. Figalli, D. Jerison: Quantitative stability for the Brunn-Minkowski inequality. Adv. Math. 314 (2017), 1-47.
[27] A. Figalli, F. Maggi, A. Pratelli: A refined Brunn-Minkowski inequality for convex sets. Annales de IHP, 26 (2009), 2511-2519.

[28] A. Figalli, F. Maggi, A. Pratelli: A mass transportation approach to quantitative isoperimetric inequalities. Invent. Math., 182 (2010), 167-211.

[29] A. Figalli, R. Neumayer: Gradient stability for the Sobolev inequality: the case \( p \geq 2 \). J. Eur. Math. Soc. (JEMS), 21 (2019), 319-354.

[30] A. Figalli, Yi Ru-Ya Zhang: Sharp gradient stability for the Sobolev inequality. [arXiv:2003.04037]

[31] M. Fradelizi, M. Meyer: Some functional forms of Blaschke-Santaló inequality. Math. Z., 256 (2007), 379–395.

[32] N. Fusco, F. Maggi, A. Pratelli: The sharp quantitative isoperimetric inequality. Ann. of Math. 168 (2008), no. 3, 941–980.

[33] R.J. Gardner: The Brunn-Minkowski inequality. Bull. AMS, 29 (2002), 335-405.

[34] D. Ghilli, P. Salani: Quantitative Borell-Brascamp-Lieb inequalities for power concave functions. J. Convex Anal. 24 (2017), 857–888.

[35] H. Groemer: On the Brunn-Minkowski theorem. Geom. Dedicata, 27 (1988), 357–371.

[36] H. Groemer: Stability of geometric inequalities. In: Handbook of convex geometry (P.M. Gruber, J.M. Wills, eds), North-Holland, Amsterdam, 1993, 125–150.

[37] P.M. Gruber: Convex and discrete geometry. Springer, Berlin, 2007.

[38] P. van Hintum, H. Spink, M. Tiba: Sharp Stability of Brunn-Minkowski for Homothetic Regions. Journal EMS, accepted. [arXiv:1907.13011]

[39] D. Hug: Contributions to affine surface area. Manuscripta Math., 91 (1996), 283–301.

[40] F. John: Polar correspondence with respect to a convex region. Duke Math. J., 3 (1937), 355–369.
[41] K. Leichtweiß: Affine geometry of convex bodies. Johann Ambrosius Barth Verlag, Heidelberg, 1998.

[42] L. Leindler: On a certain converse of Hölder’s inequality. II. Acta Sci. Math. (Szeged) 33 (1972), 217-223.

[43] L. Lovász, S. Vempala: The geometry of logconcave functions and sampling algorithms. Random Structures Algorithms 30 (2007), 307-358.

[44] E. Lutwak: Selected affine isoperimetric inequalities. In: Handbook of convex geometry, North-Holland, Amsterdam, 1993, 151–176.

[45] A. Marsiglietti: Borell’s generalized Prkopa-Leindler inequality: a simple proof. J. Convex Anal. 24 (2017), 807-817.

[46] M. Meyer, A. Pajor: On the Blaschke-Santaló inequality. Arch. Math. (Basel) 55 (1990), 82–93.

[47] M. Meyer, S. Reisner: Shadow systems and volumes of polar convex bodies. Mathematika, 53 (2006), 129–148.

[48] V.H. Nguyen: New approach to the affine Polya-Szego principle and the stability version of the affine Sobolev inequality. Adv. Math. 302 (2016), 1080–1110.

[49] C.M. Petty: Affine isoperimetric problems. Discrete geometry and convexity (New York, 1982), 113–127, Ann. New York Acad. Sci., 440, New York Acad. Sci., New York, 1985.

[50] A. Prékopa: Logarithmic concave measures with application to stochastic programming. Acta Sci. Math. (Szeged) 32 (1971), 301-316.

[51] A. Prékopa: On logarithmic concave measures and functions. Acta Sci. Math. (Szeged) 34 (1973), 335-343.

[52] A. Prékopa: New proof for the basic theorem of logconcave measures. (Hungarian) Alkalmaz. Mat. Lapok 1 (1975), 385-389.

[53] A. Rossi, P. Salani: Stability for Borell-Brascamp-Lieb inequalities. Geometric aspects of functional analysis, Lecture Notes in Math., 2169, Springer, Cham, (2017), 339–363.
[54] J. Saint-Raymond: Sur le volume des corps convexes symétriques. Initiation Seminar on Analysis: G. Choquet-M. Rogalski-J. Saint-Raymond, 20th Year: 1980/1981, Exp. No. 11, 25 pp., Publ. Math. Univ. Pierre et Marie Curie, 46, Univ. Paris VI, Paris, 1981.

[55] L.A. Santaló: An affine invariant for convex bodies of $n$-dimensional space. (Spanish) Portugaliae Math., 8 (1949), 155-161.

[56] C. Saroglou: Remarks on the conjectured log-Brunn-Minkowski inequality. Geom. Dedicata 177 (2015), 353-365.

[57] R. Schneider: Convex Bodies: The Brunn-Minkowski Theory. Cambridge University Press, 2014.

[58] T. Wang, The affine Polya-Szego principle: Equality cases and stability, J. Funct. Anal., 265 (2013), 1728–1748.

Károly J. Böröczky
Alfréd Rényi Institute of Mathematics, Reáltanoda u. 13-15, H-1053 Budapest, Hungary, and
Department of Mathematics, Central European University, Nádor u. 9, H-1051, Budapest, Hungary
boroczky.karolyj@renyi.hu

Apratim De
Department of Mathematics, Central European University, Nádor u. 9, H-1051, Budapest, Hungary
de.apratim91@gmail.com