A Grassmannian Étude in NMHV Minors

Dhritiman Nandan, Anastasia Volovich and Congkao Wen

Brown University, Providence, Rhode Island 02912, USA

Abstract

Arkani-Hamed, Cachazo, Cheung and Kaplan have proposed a Grassmannian formulation for the S-matrix of $\mathcal{N} = 4$ Yang-Mills as an integral over link variables. In parallel work, the connected prescription for computing tree amplitudes in Witten’s twistor string theory has also been written in terms of link variables. In this paper we extend the six- and seven-point results of arXiv:0909.0229 and arXiv:0909.0499 by providing a simple analytic proof of the equivalence between the two formulas for all tree-level NMHV superamplitudes. Also we note that a simple deformation of the connected prescription integrand gives directly the ACCK Grassmannian integrand in the limit when the deformation parameters equal zero.
I. INTRODUCTION

The twistor string theory formulation of Yang-Mills scattering amplitudes by Witten [1] has been a great step forward in unearthing a host of properties of scattering amplitudes, hitherto unseen via the standard methods of quantum field theory. A connected prescription formula for computing all tree level superamplitudes in twistor string theory has been written down by Roiban, Spradlin and one of the authors in [2], based on Witten’s proposal that the $N^{k−2}\text{MHV}$ superamplitude should be given by the integral of an open string current algebra correlator over the space of degree $k − 1$ curves in supertwistor space $\mathbb{P}^{3|4}$. As noted in [2] an essential feature of the connected prescription is that the resulting integral for any physical space amplitude completely localizes, allowing it to be expressed as a sum over roots or equivalently as a contour integral (see also [13]). Recently a “linked” version of the formula had been written in [3] and [4] by reformulating the original connected prescription amplitude in terms of the link variables introduced in [5]. A remarkable new contour integral over a Grassmannian of these link variables, which apparently encapsulates information about leading singularities of $\mathcal{N} = 4$ Yang-Mills loop amplitudes in addition to tree-level information, has been written down by Arkani-Hamed, Cachazo, Cheung and Kaplan (ACCK) in [6]. See also [7][8][9][10][11] for related recent developments.

It has been proven for the case of six and seven particles [3][4] that the residues of both the linked-connected formula and the ACCK formula compute BCFW representations [14][15] of tree amplitudes. In this paper we make the connection between the linked-connected prescription formula from twistor string theory and the ACCK proposal more transparent by offering a simple analytic proof between the two formulas for all tree-level NMHV superamplitudes. Also we note that a simple deformation of the connected prescription integrand by non-zero parameters gives directly the Grassmannian integrand in the limit when the deformation parameters equal zero. Specifically, the ACCK Grassmannian integrand arises from the linked-connected formula in a simple limit when the second terms in all sextic polynomials are zero (see formula (18)).

In section II we review some of the recent developments and write down a general formula (15) for $n$-point NMHV amplitudes in terms of minors in a convenient way. In section III we show how to get the BCFW contours from the linked-connected prescription for the six and seven point NMHV amplitudes in a simple way, followed by the general proof for all
n-point NMHV amplitude by using the global residue theorem (GRT). In the appendix we present the ten–point case as a concrete example.

II. REVIEW OF RECENT DEVELOPMENTS

A. Review of Dual S-Matrix Formulation

Recently Arkani-Hamed, Cachazo, Cheung and Kaplan [6] have conjectured a formula for a dual formulation for the S-Matrix of $\mathcal{N} = 4$ SYM. According to their proposal the planar, color stripped, $n$ particle, $N^{k-2}$MHV amplitudes are associated with contour integrals over a Grassmannian

$$\mathcal{L}_{n;k}(W_a) = \frac{1}{\text{Vol}(\text{GL}(k))} \int \frac{d^{k \times n} C_{aa}}{(12 \cdots k)(23 \cdots (k+1)) \cdots (n1 \cdots (k-1))} \prod_{a=1}^{k} \delta^4|4(C_{aa} W_a)| \quad (1)$$

where the $W_a$ are twistor variables obtained by Fourier transforming with respect to the $\lambda_a : W = (W|\bar{\eta}) = (\tilde{\mu}, \tilde{\lambda}|\bar{\eta})$, and

$$(m_1 \cdots m_k) \equiv \epsilon^{\alpha_1 \cdots \alpha_k} C_{\alpha m_1} \cdots C_{\alpha m_k}. \quad (2)$$

Here, $C_{aa}$ is a $k \times n$ matrix and its ‘minor’, $(m_1 \cdots m_k)$ is the determinant of the $k \times k$ sub-matrix made by only keeping the $k$ columns $m_1, \cdots, m_k$. The integrand of this formula has a $\text{GL}(k)$ symmetry under which $C_{aa} \rightarrow L_{\alpha}^{\beta} C_{\beta a}$ for any $k \times k$ matrix $L$, and so one has to gauge fix by dividing by $\text{Vol}(\text{GL}(k))$. This formula has manifest cyclic, parity, superconformal and also dual superconformal symmetry [8].

The outstanding feature of this formula is that, interpreting the integral as a multidimensional contour integral in momentum space, the residues of the integrand give a basis for obtaining tree level amplitudes as well as all loop leading singularities.

B. NMHV tree amplitude from ACCK

A general formula for determining which residues correspond to tree amplitudes for the $n$ particle NMHV case has been given in [6] which we will now review. Following their notation we denote a residue when $n-5$ minors $(i_1 i_1 + 1 i_1 + 2), \ldots, (i_{n-5} i_{n-5} + 1 i_{n-5} + 2) \rightarrow 0$ as $\{i_1, i_2, \cdots, i_{n-5}\}$, and it is antisymmetric. Then NMHV tree amplitude is given by the sum
of residues

\[ A_{n,\text{BCFW}}^{\text{NMHV}} = (-1)^{n-5} \prod_{i=1}^{n-5} \mathcal{O} \ast \mathcal{E} \ast \mathcal{O} \ast \mathcal{E} \ldots \]

\[(n - 5)\text{ factors} \]

where \( \mathcal{O} \) is the set of odd numbered particles and \( \mathcal{E} \) is the set of even numbered particles

\[ \mathcal{O} = \sum_{k \text{ odd}} \{k\}, \quad \mathcal{E} = \sum_{k \text{ even}} \{k\} \]

and

\[ \{i_1\} \ast \{i_2\} = \begin{cases} \{i_1, i_2\} & \text{if } i_1 < i_2 \\ 0 & \text{otherwise} \end{cases} \]

The above proposal can also be motivated from the geometric picture presented in the recent papers \[12\] and \[9\].

To get \( P(\text{BCFW}) \) (parity-conjugated BCFW terms) from BCFW, one can simply apply the GRT. For example, the BCFW terms of the seven-point NMHV amplitude can be written as

\[ A_7 = \{1, 2\} + \{1, 4\} + \{1, 6\} + \{3, 4\} + \{3, 6\} + \{5, 6\}. \]

C. Review of the Linked-Connected Prescription

Let us begin by reviewing some details of the connected prescription formula \[2\]. The 4|4 component homogeneous coordinates for the \( i \)-th particle in \( \mathbb{P}^{3|4} \) are \( Z_i = (\lambda_i^\alpha, \mu_i^\dot{\alpha}, \eta_i^A) \) with \( \alpha, \dot{\alpha} = 1, 2 \) and \( A = 1, 2, 3, 4 \). The connected formula can be written explicitly in the following form:

\[ A(Z) = \int \frac{d^{4k|4k}A d^n \sigma d^n \xi}{\text{vol } GL(2)} \prod_{i=1}^{n} \delta^{4|4}(Z_i - \xi_i \mathcal{P}(\sigma_i)) \xi_i (\sigma_i - \sigma_{i+1}), \]

where \( \mathcal{P} \) is the degree \( k - 1 \) polynomial given in terms of its \( k \) \( \mathbb{C}^{4|4} \)-valued supercoefficients \( A_d \) by

\[ \mathcal{P}(\sigma) = \sum_{d=0}^{k-1} A_d \sigma^d. \]

As emphasized in \[2\] (see also \[13\]) the integral (7) must be interpreted as a contour integral in a multidimensional complex space. The delta functions specify the contour of integration (specifically they indicate which poles to include in the sum over residues). There is also a \( GL(2) \) invariance, of the integrand and the measure, which needs to be gauged. Taking the
above connected prescription as a starting point and motivated by \[5\] one can express the connected prescription \((7)\) into the form of so-called link representation \([3], [4]\).

One can obtain the physical space amplitude from the link representation

\[
A(\lambda, \bar{\lambda}) = J\delta\left(\sum p_i\right) \int d\tau \ U(c_{ji}(\tau_j)),
\]

where the Jacobian \(J\) generally depends on the parameterization of \(c_{ji}(\tau_j)\). A general form of \(U(c_{ji})\) has been explicitly evaluated by Dolan and Goddard in \([4]\). For an amplitude with helicities \((\epsilon_1, \ldots, \epsilon_n)\) comprising \(p\) strings with \(\epsilon_\alpha = +\) and \(p\) strings with \(\epsilon_\beta = -\), their explicit form is

\[
U(c) = F(c) \prod_{k,t} \frac{1}{S_{kt}}, \quad (10)
\]

where \(S_{kt}\) is the sextic \(S_{IJK:RSkt} = c_{IS}c_{JK:RS}c_{IJ:R}c_{IJK:RS} \) with \(c_{ij:rs} = c_{ir}c_{js} - c_{jr}c_{is}\), and

\[
F(c) = (c_{IJ:RS})^{N_R-p+2} c_{1R}^{p-3} c_{1S}^{p-3} c_{JR}^{p-3} c_{JS}^{p-3} \prod_{i \in P'} c_{l-3}^{l-3} c_{Jt}^{l-3} \prod_{k \in N'} c_{kR}^{m-3} c_{kS}^{m-3} \prod_{k \in \mathcal{N}} \prod_{\alpha=1}^{N_R} \frac{1}{d_{\alpha,\alpha+1}} \quad (11)
\]

where

\[
d_{ir} = c_{ir}, \quad d_{ri} = c_{ir}, \quad d_{ij} = \frac{c_{iR}c_{jS}c_{jR}c_{iS}}{c_{iR}c_{jS} - c_{jR}c_{iS}}, \quad d_{rs} = \frac{c_{Ir}c_{Js}c_{Is}c_{Jr}}{c_{Ir}c_{Js} - c_{Is}c_{Jr}}, \quad i, j \in \mathcal{N}, \ r, s \in P.
\]

We denote \(P\) as the set of positive helicity particles and \(\mathcal{N}\) as the set of negative helicity particles, and \(N_R\) is the number of independent sextics, \(l\) is the number of the negative helicity particles, \(m\) the number of the positive helicity particles and \(n = m + p\) is the total number of particles.\(^1\)

**D. NMHV tree amplitude from the connected prescription**

In order to make the connection between the linked-connected and ACCK formulas more transparent, in this section we will express the linked-connected formula in terms of minors as in the ACCK approach.\(^2\)

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\(^1\) Here we exchange the helicities \(+ \leftrightarrow -\), at the same time \(c_{ij} \rightarrow c_{ji}\) with respect to \([4]\).

\(^2\) We are grateful to Freddy Cachazo for encouraging us to rewrite everything in terms of minors. There are many different ways to write the formulas, but we will pick the one which makes the proof simpler and has many other nice properties as we will discuss later.
Let us start with helicity \((- + - + + + \cdots + +\)), and take \(I = 1, J = 3, R = 2, S = 4\), then formula \((10)\) becomes

\[
U(c) = (c_{52}c_{54}c_{13:24})^{n-6}(c_{12}c_{32}c_{34}c_{54}c_{56}c_{1n}) \prod_{\alpha = 6}^{n-1} \frac{c_{1\alpha}c_{1\alpha+1}c_{3\alpha}c_{3\alpha+1}}{c_{13,\alpha,\alpha+1}} \prod_{k \in \mathbb{P}, t \in \mathbb{N}} \frac{1}{n} \prod_{i=6}^{n} \frac{1}{S_{135:24i}}.
\]

Using the identity

\[
\delta(S_{ijk:rst})\delta(S_{ijk:rst'}) = \delta(S_{ijk:rst})\delta(S_{ijk:rst''}) (c_{it}c_{jk}c_{is}c_{jk}),
\]

\((13)\)

we can transform the sextics \(S_{135:24i}\) in \((12)\) to \(S_{135:246}, S_{135:2,n-1,n}, \) and \(S_{135:i-1,i,i+1}\) to arrive at

\[
U'(c) = \frac{c_{35:26}c_{12}c_{13:n-1,n}c_{5,n-1} \prod_{\alpha = 8}^{n} c_{5\alpha} \prod_{\beta = 7}^{n-1} c_{3\beta} \prod_{\gamma = 6}^{n-2} c_{1\gamma}}{c_{52}c_{14}c_{13:67}c_{35:n-1,n}} \frac{1}{S_1S_2 \ldots S_{n-5}}.
\]

\((14)\)

We then translate it into minors, the result is\(^3\)

\[
A_n = \frac{N}{(123)(345)(567)(n-1)n1} \frac{1}{S_1S_2 \ldots S_{n-5}},
\]

\((15)\)

where the numerator is given as

\[
N = (135)(612)(235)(5n-1n)(13n-1) \prod_{\alpha = 8}^{n} (13\alpha) \prod_{\beta = 7}^{n-1} (15\beta) \prod_{\gamma = 6}^{n-2} (35\gamma).
\]

\((16)\)

The sextics can be written as

\[
S_1 = (234)(456)(612)(135) - (123)(345)(561)(246),
\]

\[
S_2 = (n12)(13n-1)(235)(5n-1n) - (123)(35n-1)(5n2)(n-1n1),
\]

\[
S_{i-3} = (i+i+1+i+2)(13i+2)(15i+1)(35i) - (135)(3i+i+2)(5i+1)(i+1i+21),
\]

\((17)\)

where \(6 \leq i \leq n-2\).

Several comments about this formula are in order.

Firstly, one can deform the sextics by any non-zero parameters \(a_j\), namely

\[
S_j \rightarrow S'_j = (klm)(mnp)(pqk)(qln) - a_j(qkl)(lmm)(npq)(kmp).
\]

\((18)\)

As we will prove in next section, interestingly, the final amplitude does not depend on \(a_j\) at all. Taking the limit \(a_j \rightarrow 0\) one gets ACCK formula directly. This appears to be a

\(^3\) When \(n = 6\) or \(n = 7\) the minor \((567)\) does not appear in the denominator. And we put the minor \((135)\) in the numerator by hand to make the scale right, since \((135) = 1\) for the helity we started.
general fact, not specific to just NMHV amplitudes: the ACCK Grassmanian integrand arises from the linked-connected formula in a simple limit when the second terms in all sextic polynomials are zero.

Secondly, the formula has $GL(3)$ symmetry for the Grassmanian, even though we had started with the link representation for a particular helicity configuration. We should point out that for some particular gauge fixings, we do not always get the form of each sextic as a polynomial of degree 6 in the $c'_J$'s. But, nevertheless, one can numerically check that we do indeed get the tree amplitudes for the connected prescription, namely, the residues at the locus where all the sextics simultaneously vanish.

Thirdly, writing sextics in terms of minors has a simple geometrical interpretation. The minor $(i j k) = 0$ in twistor space means the points $i, j, k$ lie on a line. For NMHV, the sextics $S_{ijklmn} = 0$ means that these six points $i, j, k, l, m, n$ lie a conic curve, which is consistent with the origin of the connected prescription—integrating out degree two curves in twistor space as in formula (7).

III. FROM THE CONNECTED TO ACCK USING GRT

In this section we will use the multidimensional Global Residue Theorem (GRT) to analytically derive the BCFW contour of ACCK as in (3) from the connected prescription formula (15).

A. $n=6$ and $n=7$

We begin with $n = 6$ and $n = 7$ cases, which were previously done in [3], [4].

• For the six-point amplitude, the connected formula gives

$$A_6 = \frac{(135)}{(123)(345)(561)} S,$$

where

$$S = (234)(456)(612)(135) - (123)(345)(561)(246).$$

This was emphasised to us by Freddy Cachazo.
Cauchy’s theorem states that the sum of residues in this expression is zero, so

$$\{S\} = -\{1\} - \{3\} - \{5\}, \quad (21)$$

which is ACCK formula (3) for \(n = 6\).

• For the seven-point amplitude,

$$A_7 = \frac{(135)(235)(612)(136)}{(123)(345)(671)} \frac{1}{S_1 S_2}, \quad (22)$$

where

$$S_1 = (234)(456)(612)(135) - (123)(345)(561)(246),$$

$$S_2 = (567)(712)(235)(136) - (123)(356)(572)(671). \quad (23)$$

By applying GRT, we get

$$\{S_1, S_2\} = \{1, S_1\} + \{3, S_1\} + \{6, S_1\}. \quad (24)$$

On the poles \((123) = 0\) and \((345) = 0\), the second term of \(S_1\) vanishes and we get

$$\{1, S_1\} = \{1, 2\} + \{1, 4\}, \quad \{3, S_1\} = \{3, 2\} + \{3, 4\}. \quad (25)$$

Note that the terms with non-adjacent minors do not contribute because they would be cancelled by the numerator of \(A_7\). Moreover, the condition of the residue \(\{3, 2\}\) implies that the points 2, 3, 4, 5 lie on a line and hence \((235) = 0\), which is a term in the numerator of \(A_7\). To simplify the residue \(\{6, S_1\}\) we use GRT again

$$\{6, S_1\} = - \{6, S_2\} + \{6, 1\} + \{6, 3\} \quad (26)$$

$$= - \{6, 5\} + \{6, 7\} + \{6, 1\} + \{6, 3\}. \quad (27)$$

Again, \((671) = 0\) makes the second term of \(S_2\) vanish, hence \(\{6, S_2\} = \{6, 5\} + \{6, 7\}\). But the condition of \(\{6, 7\}\) implies that \((612) = 0\), which is a term in the numerator of \(A_7\). So finally, collecting all the residues we get

$$\{S_1, S_2\} = \{1, 2\} + \{1, 4\} + \{1, 6\} + \{3, 4\} + \{3, 6\} + \{5, 6\}. \quad (28)$$

These are exactly the BCFW contours of the ACCK formula (3).

Let us conclude this section by saying that there are two useful properties which play an important role in making the above proof simple. First, the second terms of the sextics vanish for some particular contours. Second, the residue vanishes if one of the non-adjacent minors in the first term of the sextic vanishes. We will use these two simple facts in the general proof, which follows in the next section.
B. All $n$ proof

Let us first note that one can easily check that the second terms of the sextics vanish for any BCFW contours. It means that whenever we get a BCFW contour (3) by applying GRT, we are sure that our NMHV formula for the amplitude is exactly of the same form as in ACCK amplitude, namely all the non-adjacent minors cancel out.

We can further check that there are no ‘spurious’ solutions, having non-vanishing contribution, from the connected contour. Spurious solutions are those where the sextics vanish because individual minors in the expressions for the sextics vanish (non-spurious solutions are those where the two terms in every sextic are separately non-zero). We should exclude these solutions simply because the vanishing of any individual minor of the sextics means that the conic curve is not smooth anymore$^5$.

The way to get BCFW contours from connected prescription is simply to get rid of all the sextics in the connected contour by applying GRT repeatedly. Let us remind you that the poles in formula (15) are

$$(123)(345)(567)(n-1\ n\ 1)S_1S_2\ldots S_{n-5}.$$  \hfill (29)

Use GRT we have

$$\{S_2S_1\ldots S_{n-5}\} = -(\{1S_1S_3\ldots S_{n-5}\} + \{3S_1S_3\ldots S_{n-5}\}$$
$$+ \{5S_4S_3\ldots S_{n-5}\} + \{(n-1)S_1S_3\ldots S_{n-5}\})$$
$$= -(\{12S_3\ldots S_{n-5}\} + \{14S_3\ldots S_{n-5}\} + \{34S_3\ldots S_{n-5}\}$$
$$+ \{32S_3\ldots S_{n-5}\} + \{(n-1)S_16\ldots (n-2)\})$$,  \hfill (30)

where $\{1S_1S_3\ldots S_{n-5}\}$ is the residue of (123) = $S_1 = S_3 = \ldots = S_{n-5} = 0$, and etc.

In order to explain why $\{5S_1S_3\ldots S_{n-5}\} = 0$ first notice that $\{5S_1S_3S_4\ldots S_{n-5}\} = \{5S_1678\ldots(n-2)\}$. This is true because on (567) = 0 the second term of $S_3$ vanishes and hence $\{5S_1S_3\ldots S_{n-5}\} = \{5S_16S_4\ldots S_{n-5}\}$. Now in addition to (567) = 0, we also have (678) = 0 which implies that the points 5, 6, 7, 8 lie on a line and hence (578) = 0, resulting in the second term of $S_4$ vanishing. So, we get $\{5S_16S_4\ldots S_{n-5}\} = \{5S_167S_5\ldots S_{n-5}\}$. We can again apply similar arguments on $S_4$ and reduce it to (789), and this goes on until the last sextic of the residue, which is $S_{n-5}$. Now, $\{5S_1678\ldots(n-2)\}$ means that the

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$^5$ The same reasoning holds for the validity of identity (13).
points 5, 6, ..., n lie on a straight line, so \((5n - 1)/n\) in the numerator vanishes, and hence 
\[5S_1S_3 \ldots S_{n-5} = 0.\]

The equality \(\{(n-1)S_1S_3 \ldots S_{n-5}\} = \{(n-1)S_16 \ldots (n-2)\}\) in (30) can also be explained along the same lines, but starting from the fact that, due to \((n-1)/n = 0\), \(S_{n-5}\) is replaced by \((n-2)/n\). Finally \(\{32S_3 \ldots S_{n-5}\} = 0\) simply because \((345) = (234) = 0\) implies \((235) = 0\), which is a term in the numerator.

In the following, we will study each term from (30) individually. In the process, we will ignore all the vanishing terms without explanation, since the reasons are very similar.

1. \(\{(n-1)S_16 \ldots (n-2)\}\) term

By applying GRT again, with the poles
\[
(123)(345)(567)(n-1)/n)S_1(n12)(678)(789) \ldots (n-2)/n-1\)
we get the following non-vanishing residues
\[
-\{(n-1)S_16 \ldots (n-2)\} = \{(n-1)16 \ldots (n-2)\} + \{(n-1)36 \ldots (n-2)\} + \{(n-1)56 \ldots (n-2)\}.
\]
Actually these three terms are all the contours of the form \{i6\ldots\} and i can be 1, 3 or 5, and they have the correct signs.

2. \(\{34S_3 \ldots S_{n-5}\}\) term

Now, in this case the poles are
\[
(123)(345)(567)(n-1)/n1)(234)(456)(n12)S_3S_4 \ldots S_{n-5}.
\]
Again using GRT we get
\[
-\{34S_3 \ldots S_{n-5}\} = \underbrace{345S_4 \ldots S_{n-5}}_{A_1} + \{34(n-1)7 \ldots (n-2)\}.
\]
The second term in the previous equation is a BCFW term and we use GRT again on the term \(A_1\) to generate another BCFW term in the next step
\[
\{345S_4 \ldots S_{n-5}\} = -\underbrace{3456S_5 \ldots S_{n-5}}_{A_2} + \{345(n-1)8 \ldots (n-2)\}.
\]
Similarly, we can keep on using GRT repeatedly on one of the two terms, generated at each step by using GRT in the previous step. In the final step of this iteration, by applying GRT we get two terms, \{34567 \ldots (n-4)(n-1)\} and \{34567 \ldots (n-3)\}. So in this way, we generate \{347 \ldots (n-1)\} + \{3458 \ldots (n-1)\} + \{34569 \ldots (n-1)\} + \cdots + \{34567 \ldots (n-3)\}, which are all the BCFW contours of the form \{34 \ldots \}.

3. \{14S_3 \ldots S_{n-5}\} term

Now, let us consider the contours of the form \{14 \ldots \}. Here the poles are given as

\[(123)(345)(567)(n-1 \ldots n)\](345)(456)(n-1)S_3S_4 \ldots S_{n-5}.

Using GRT we get the following

\[-\{14S_3 \ldots S_{n-5}\} = \{14(n-1)7 \ldots (n-2)\} + \underbrace{142S_4 \ldots S_{n-5}}_{x_1} + \underbrace{145S_4 \ldots S_{n-5}}_{B_1}. \tag{35}\]

Apart from the BCFW term \{147 \ldots (n-1)\} we also have other non-BCFW terms. Out of these, we will see that the terms like \(X_1\) generated at each step will cancel out later from the same terms generated by \{12S_3 \ldots S_{n-5}\} in the next subsection. We can again apply GRT on \(B_1\). Now, we can see the pattern of BCFW terms generated from the \(B_i\) terms, and here we will not write the non-BCFW terms explicitly at each step

\[
\{14S_3 \ldots S_{n-5}\} \Rightarrow \{147 \ldots (n-1)\}, \\
\{145S_4 \ldots S_{n-5}\} \Rightarrow \{1458 \ldots (n-1)\}, \\
\{1456S_5 \ldots S_{n-5}\} \Rightarrow \{14569 \ldots (n-1)\}, \\
\ldots...
\]

In the final step of this series, by applying GRT, we have two terms, \{14567 \ldots (n-4)(n-1)\} and \{145678 \ldots (n-3)\}. So by using GRT repeatedly, we get all the BCFW contours of the type \{14 \ldots \}, namely \{147 \ldots (n-1)\} + \{1458 \ldots (n-1)\} + \{14569 \ldots (n-1)\} + \cdots + \{145678 \ldots (n-3)\}.

4. \{12S_3 \ldots S_{n-5}\} term

Finally, we look at the remaining contours \{12S_3 \ldots S_{n-5}\} in equation (30).
Let us apply $\mathrm{GRT}$ and we get
\begin{align*}
-\{12S_3 \ldots S_{n-5}\} &= \{126 \ldots (n-3)3\} + \{126 \ldots (n-3)5\} \\
&\quad + \underbrace{\{12S_3 \ldots S_{n-6}(n-1)\}}_{C_1} + \{12S_3 \ldots S_{n-6}4\}. \\
&\quad + \underbrace{\{12S_3 \ldots S_{n-7}(n-2)(n-1)\}}_{D_1} + \{12S_3 \ldots S_{n-7}4(n-1)\}. \\
&\quad + \underbrace{\{12S_3 \ldots S_{n-7}(n-5)(n-2)(n-1)\}}_{E_1} + \{12S_3 \ldots S_{n-7}569(n-1)\}.
\end{align*}

(37)

We can apply GRT on the term $C_1$ in (37) again, and we will deal with the term $D_1$ later.

From $C_1$ we get
\begin{align*}
\{12S_3 \ldots S_{n-6}(n-1)\} &= -(\{126 \ldots (n-4)3(n-1)\} + \{126 \ldots (n-4)5(n-1)\} \\
&\quad + \underbrace{\{12S_3 \ldots S_{n-7}(n-2)(n-1)\}}_{C_2} + \{12S_3 \ldots S_{n-7}4(n-1)\}) \\
&\quad + \underbrace{\{12S_3 \ldots S_{n-7}(n-5)(n-2)(n-1)\}}_{E_2} + \{12S_3 \ldots S_{n-7}569(n-1)\}.
\end{align*}

(38)

We notice that one of the non-BCFW terms, $C_2$, is a similar kind of term to $C_1$. Terms which are similar to $E_1$ and generated at each step, will combine with other terms generated from the subsequent steps of applying GRT. The general trend of BCFW contours generated from the $C_i$ terms are
\begin{align*}
\{12S_3 \ldots S_{n-5}\} &\Rightarrow \{1236 \ldots (n-3)\} + \{1256 \ldots (n-3)\}, \\
\{12S_3 \ldots S_{n-6}(n-1)\} &\Rightarrow \{1236 \ldots (n-4)(n-1)\} + \{1256 \ldots (n-4)(n-1)\}, \\
\{12S_3 \ldots S_{n-7}(n-2)(n-1)\} &\Rightarrow \{1236 \ldots (n-5)(n-2)(n-1)\} \\
&\quad + \{1256 \ldots (n-5)(n-2)(n-1)\}, \\
\end{align*}

\begin{align*}
\ldots &\ldots
\end{align*}

(39)

Note that at each step of the iteration we also generate some non-BCFW terms(not explicitly written down in the above pattern) which need to be dealt with as before. The final step in the above series generates the BCFW terms $\{1238 \ldots (n-1)\}, \{1258 \ldots (n-1)\}$ and $\{1278 \ldots (n-1)\}$.

By similar methods we can generate the other BCFW contours of the form $\{12 \ldots \}$ by using non-BCFW terms generated in previous steps. Since all the steps are similar, here we only give some examples of generating BCFW terms, without showing the details
\begin{align*}
\{12S_3 \ldots S_{n-6}4\} &\Rightarrow \{12347 \ldots (n-3)\}, \\
\{125S_4 \ldots S_{n-6}4\} &\Rightarrow \{123458 \ldots (n-3)\}, \\
\{1256S_5 \ldots S_{n-6}4\} &\Rightarrow \{1234569 \ldots (n-3)\}, \\
\ldots &\ldots
\end{align*}

(40)
Again the last step of this iterative process is special, the BCFW term generated is \(\{1234\ldots(n-5)\}\). We will give a few examples of how non-BCFW terms combine to generate BCFW terms and we choose these particular examples as they give residues related to the ones in (40). Firstly
\[
\{12(n-1)S_4\ldots S_{n-6}4\} + \{12S_3\ldots S_{n-7}4(n-1)\} \Rightarrow \{12347\ldots(n-4)(n-1)\},
\]
\[
\{12(n-1)S_4\ldots S_{n-7}(n-2)4\} + \{12S_3\ldots S_{n-s}4(n-2)(n-1)\}
\Rightarrow \{12347\ldots(n-5)(n-2)(n-1)\},
\]
\[\ldots\]

The BCFW term generated from the last step of the above series is \(\{12349\ldots(n-1)\}\). Next example is
\[
\{125(n-1)S_5\ldots S_{n-6}4\} + \{12(n-1)S_4\ldots S_{n-7}54\} \Rightarrow \{123458\ldots(n-4)(n-1)\},
\]
\[
\{125(n-1)S_5\ldots S_{n-7}(n-2)4\} + \{12(n-1)S_4\ldots S_{n-s}5(n-2)4\}
\Rightarrow \{123458\ldots(n-5)(n-2)(n-1)\},
\]
\[\ldots\]

The last step generates BCFW term \(\{1234510\ldots(n-1)\}\). And one more example will be
\[
\{125(n-1)S_5\ldots S_{n-764}\} + \{1256(n-1)S_6\ldots S_{n-6}4\} \Rightarrow \{1234569\ldots(n-4)(n-1)\},
\]
\[
\{125(n-1)S_5\ldots S_{n-s}6(n-2)4\} + \{1256(n-1)S_6\ldots S_{n-s}7(n-2)4\}
\Rightarrow \{1234569\ldots(n-5)(n-2)(n-1)\},
\]
\[\ldots\]

The BCFW term generated in the last step is \(\{12345611\ldots(n-1)\}\).

From the above mentioned examples, we can see the general pattern: the first term in (40), \(\{1234\ldots(n-3)\}\), combining with all the terms from (41) generates all the contours of the form \(\{12347\ldots\}\); similarly, the second term in (40), \(\{123458\ldots(n-3)\}\), and all the terms in (42) give us all the contours of the form \(\{123458\ldots\}\); the third term in (40), \(\{1234569\ldots(n-3)\}\), together with all the terms of (43) give us all the contours of the form \(\{1234569\ldots\}\). It is not hard to see that all the other BCFW terms of the form \(\{1234\ldots\}\) can be generated in a similar way. So we have generated all the contours of the form \(\{12\ldots\}\) and we notice that they can be grouped into contours of the form, \(\{1236\ldots\}, \{1256\ldots\}, \{1238\ldots\}, \{1258\ldots\}, \{1278\ldots\}\) and \(\{1234\ldots\}\).
As we had seen so far, each GRT step also generates terms which have no contribution to BCFW contours. These terms, typically, look like \( \{124 \ldots i, S_i, \ldots, S_{n-5}\} \), but they just cancel out at each step. At each of the final steps, we also generate terms like \( \{124 \ldots i, i + 4, \ldots, (n - 1)\} \) and \( \{124 \ldots (n - 4)\} \), and they also cancel out. In the Appendix we can see all these cancelations explicitly in the 10-point example.

Let us conclude with our main result
\[
\oint_C \frac{\mathcal{N}}{(123)(345)(567)(n - 1 \ n \ 1)} S_1 S_2 \ldots S_{n-5} = \oint_B \frac{1}{(123)(234) \ldots (n12)},
\]
where contour \( C \) is the connected contour, and \( B \) is the BCFW contour. One can apply GRT again and show that the same equality is true for the \( P(BCFW) \) contour.

Since for any BCFW contour the second terms of sextics vanish, so as a byproduct, we also proved the statement we made before that deforming sextics by some non-zero parameters still gives us the correct tree amplitude.

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**APPENDIX A: LINKED-CONNECTED TO BCFW–10 POINT AMPLITUDE**

Let us consider one higher-point non-trivial case, the 10-point amplitude, here we will ignore all vanishing terms. For \( n = 10 \), the poles are \((123)(345)(567)(9 \ 10 \ 1)S_1 S_2 S_3 S_4 S_5\). Using GRT, we have
\[
\{S_2 S_1 S_3 S_4 S_5\} = -\left\{1S_1 S_3 S_4 S_5\right\} + \left\{3S_1 S_3 S_4 S_5\right\} + \left\{5S_1 S_3 S_4 S_5\right\} + 9S_1 S_3 S_4 S_5
\]
\[
\begin{align*}
&= -\left\{12S_3 S_4 S_5\right\} + \left\{14S_3 S_4 S_5\right\} + \left\{34S_3 S_4 S_5\right\} + \left\{9S_1 678\right\}. \\
\end{align*}
\]
As in the general case, we can apply GRT again
\[
-\{9S_1 678\} = \{91678\} + \{93678\} + \{95678\},
\]

\( A1 \)
and
\[-\{34S_3S_4S_5\} = \{345S_4S_5\} + \{34978\} \]
\[= - (\{3456S_5\} + \{34598\}) + \{34978\} \]
\[= \{34569\} + \{34567\} + \{34589\} + \{34789\}. \tag{A3}\]
Now let us consider the contours of the form \(\{12\ldots\}\) and \(\{14\ldots\}\).
\[-\{12S_3S_4S_5\} = \{12S_3S_43\} + \{12S_3S_45\} + \{12S_3S_49\} + \{12S_3S_44\} \]
\[= \{12673\} + \{12675\} + \{12S_3S_49\} + \{12S_3S_44\}. \tag{A4}\]
Applying GRT again we get
\[\{12S_3S_49\} = - (\{12639\} + \{12659\} + \{12S_349\} + \{12S_389\}) , \tag{A5} \]
\[\{12S_3S_44\} = - \left( \{12374\} + \{125S_44\} + \{129S_44\} + \underbrace{\{12S_5S_44\}}_1 \right) , \tag{A5} \]
and
\[-\{14S_3S_4S_5\} = \{145S_4S_5\} + \{14978\} + \{142S_4S_5\} \]
\[= - (\{14598\} + \{1452S_5\} + \{1456S_5\}) + \{14978\} + \{142S_4S_5\} \]
\[= \{14589\} + \{14978\} + \underbrace{\{142S_4S_5\}}_1 - \underbrace{\{1452S_5\}}_2 + \{14569\} + \{14562\} + \{14567\} , \tag{A6} \]
and also
\[-\{12S_389\} = \{12389\} + \{12589\} + \{12489\} + \{12789\} , \tag{A7} \]
\[-\{125S_44\} = \{12534\} + \{12594\} + \{12564\} + \{125S_34\} . \tag{A7} \]
Now we take \(\{12S_349\}\) and \(\{129S_44\}\) from equation \(\text{(A5)}\) and applying GRT we get
\[\{129S_44\} + \{12S_349\} = \{12934\} + \underbrace{\{12954\} + \{12984\}}_4 . \tag{A8} \]
Note the all the underbraced terms cancel out \(^6\), and it is a general feature even for general \(n\)-point cases as we pointed out before. Now we collect all the non-vanishing term together
\[\{S_2S_3S_4S_5\} = \{91678\} + \{93678\} + \{95678\} + \{34569\} + \{34567\} + \{34589\} + \{34789\} \]
\[+ \{12673\} + \{12675\} - \{12639\} - \{12659\} - \{12374\} + \{14589\} + \{14978\} \]
\[+ \{14569\} + \{14567\} + \{12389\} + \{12589\} + \{12789\} + \{12534\} + \{12934\} , \tag{A9} \]
\(^6\) The numbering marks out which terms cancel.
which are exactly the BCFW contours for 10 points as predicted by the ACCK formula, and have the right signs.

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