THE SZLENK INDEX OF \( L_p(X) \)

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Abstract. We find an optimal upper bound on the values of the weak∗-dentability index \( Dz(X) \) in terms of the Szlenk index \( Sz(X) \) of a Banach space \( X \) with separable dual. Namely, if \( Sz(X) = \omega^\alpha \), for some \( \alpha < \omega_1 \), and \( p \in (1, \infty) \), then

\[
Sz(X) \leq Dz(X) \leq Sz(L_p(X)) \leq \begin{cases} 
\omega^{\alpha+1} & \text{if } \alpha \text{ is a finite ordinal}, \\
\omega^\alpha & \text{if } \alpha \text{ is an infinite ordinal}.
\end{cases}
\]

1. Introduction

Let \( X \) be a Banach space. We say that the dual \( X^* \) is weak∗-dentable if for every nonempty bounded subset \( M \subset X^* \) and for every \( \varepsilon > 0 \) there are \( u \in X \) and \( a \in \mathbb{R} \) such that the slice \( \{ x^* \in M : \langle x^*, u \rangle > a \} \) is nonempty and has diameter less than \( \varepsilon \). We say that \( X^* \) is weak∗-fragmentable if for every nonempty bounded subset \( M \subset X^* \) and for every \( \varepsilon > 0 \) there is a weak∗-open set \( V \subset X^* \) such that the intersection \( M \cap V \) is nonempty and has diameter less than \( \varepsilon \). In [2] Asplund considered the property of \( X \) that every continuous convex function defined on an open set of \( X \) is Fréchet differentiable on a dense \( G_\delta \) set, and we call such a space an Asplund space. The following equivalences between the notions are stated in [11] and gather the results from [2, 19, 21].

Theorem 1.1. [11, Theorem 11.8, p. 486]

Let \( (X, \| \cdot \|) \) be a Banach space. Then the following assertions are equivalent:

(i) \( X^* \) is weak∗-dentable.

(ii) \( X^* \) is weak∗-fragmentable.

(iii) \( X \) is an Asplund space.

(iv) Every separable subspace of \( X \) has a separable dual.

This fundamental result has many ramifications, including for the investigation of the Radon-Nikodým Property and the renorming theory of Banach spaces, see e.g. [7, 11, 15].

Our object of study in this note is the quantitative relationship between weak∗-dentability and weak∗-fragmentability. Our results are expressed in terms of the values of derivation indices, which are naturally associated with the fragmentation properties.

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We begin by defining the Szlenk derivation and the Szlenk index that have been first introduced in [24].

Consider a real Banach space $X$ and a weak$^*$-compact subset $K$ of $X^*$. For $\varepsilon > 0$ we let $V_{(K,\varepsilon)}$ be the set of all relatively weak$^*$-open subsets $V$ of $K$ such that the norm diameter of $V$ is less than $\varepsilon$ and put $s_\varepsilon(K) = K \setminus \bigcup \{ V : V \in V_{(K,\varepsilon)} \}$. Then we define inductively $s_\varepsilon^0(K)$ for any ordinal $\alpha$ by $s_\varepsilon^{\alpha+1}(K) = s_\varepsilon(s_\varepsilon^\alpha K)$ and $s_\varepsilon^\alpha(K) = \cap_{\beta < \alpha} s_\varepsilon^\beta K$, if $\alpha$ is a limit ordinal. We then define $S_{\varepsilon}(X,\varepsilon)$ to be the least ordinal $\alpha$ so that $s_\varepsilon^\alpha(B_{X^*}) = \emptyset$, if such an ordinal exists. Otherwise we write $S_{\varepsilon}(X,\varepsilon) = \infty$. The Szlenk index of $X$ is finally defined to be $S_{\varepsilon}(X) = \sup_{\varepsilon > 0} S_{\varepsilon}(X,\varepsilon)$.

If $K$ is weak$^*$-compact and convex, we call a weak$^*$-slice of $K$ any non empty set of the form $S = \{ x^* \in K, \ x^*(x) > t \}$, where $x \in X$ and $t \in \mathbb{R}$. Then we denote for $\varepsilon > 0$ by $S_{(K,\varepsilon)}$ the set of all weak$^*$-slices of $K$ of norm diameter less than $\varepsilon$ and put $d_\varepsilon(K) = K \setminus \bigcup \{ S : S \in S_{(K,\varepsilon)} \}$. From this derivation, we arrive similarly to the weak$^*$-dentability indices of $X$ that we denote $D_{\varepsilon}(X,\varepsilon)$, for $\varepsilon > 0$, and $D_{\varepsilon}(X) = \sup_{\varepsilon > 0} D_{\varepsilon}(X,\varepsilon)$. Since $S_{(K,\varepsilon)} \subset V_{(K,\varepsilon)}$, for all $\varepsilon > 0$, it follows immediately that $D_{\varepsilon}(X,\varepsilon) \geq S_{\varepsilon}(X,\varepsilon)$, and $D_{\varepsilon}(X) \geq S_{\varepsilon}(X)$. Our problem consists of finding an estimate going in the opposite direction.

In the language of indices Theorem [17] implies that $S_{\varepsilon}(X) \neq \infty$ holds if and only if $D_{\varepsilon}(X) \neq \infty$. Indeed, the respective index is equal to $\infty$ if and only if the dual $X^*$ contains a $w^*$-compact and non empty subset without any $w^*$-open and nonempty subsets (resp. slices) of diameter less than some $\varepsilon > 0$.

It is now clear that a natural quantitative approach to Theorem [17] consists of comparing the values of $S_{\varepsilon}(X)$ and $D_{\varepsilon}(X)$. This problem has received a fair amount of attention in the literature. The first estimates in this direction were purely existential. We recall [24] Lemma 1.6 that if $X^*$ is separable then $S_{\varepsilon}(X) < \omega_1$. In [17] Proposition 2.1 it is shown, using an approach from descriptive set theory due to B. Bossard (see [5] and [6]), that there is a universal function $\psi : \omega_1 \to \omega_1$, such that if $X$ is an Asplund space with $S_{\varepsilon}(X) < \omega_1$, then $D_{\varepsilon}(X) \leq \psi(S_{\varepsilon}(X))$. Using geometrical arguments, Raja [22] Theorem 1.3 has proved that one can use $\psi(\alpha) = \omega^\alpha$ as a growth control function for every ordinal $\alpha$ (i.e., without the restriction $\alpha < \omega_1$). The best value for $\psi(\omega)$, namely $\psi(\omega) = \omega^2$ was obtained in [13] Theorem 4.1.

Our main result, Theorem [17] gives the optimal form of $\psi$, for all $\alpha < \omega_1$. In particular it solves the problem for all separable spaces with separable dual.

**Theorem 1.2.** Let $X$ be an Asplund space and $1 < p < \infty$. If $S_{\varepsilon}(X) = \omega^\alpha$, for some $\alpha < \omega_1$, then

(1) \[ S_{\varepsilon}(X) \leq D_{\varepsilon}(X) \leq S_{\varepsilon}(L_p(X)) \leq \begin{cases} \omega^{\alpha+1} & \text{if } \alpha \text{ is a finite ordinal,} \\ \omega^\alpha & \text{if } \alpha \text{ is an infinite ordinal.} \end{cases} \]

It should be noted [17] Proposition 5.4] that if $S_{\varepsilon}(X) < \omega_1$, then the Szlenk index of $X$ must be of the form $S_{\varepsilon}(X) = \omega^\alpha$, for some ordinal $\alpha$. This was noted independently and also for several other indices in [1 Corollary 3.10]. The same condition holds for the dentability index, i.e., if $D_{\varepsilon}(X) < \omega_1$ then $D_{\varepsilon}(X) = \omega^\alpha$, for some ordinal $\alpha$. So there are no possible intermediate values of indices between $\omega^\alpha$ and $\omega^{\alpha+1}$. Our
result shows that the dentability index is either equal to the Szlenk index, or if \( \alpha \) is finite it may happen that it exceeds Szlenk by just one step. At the end of our note we indicate examples showing that both possibilities may occur in the case that \( \alpha \) is finite.

It should be also noted that both indices \( \text{Sz}(X), \text{Dz}(X) \) have found many applications in the geometry and the structure of Banach spaces, renorming theory and nonlinear theory. This regards also the quantitative estimates of their values, and their relationships. For more details we refer to the survey paper of Lancien [18].

2. Proof of the main result

The proof of the main theorem, which is given at the end of this section, requires several ingredients. We are going to review these ingredients first, together with some necessary technical modifications needed for our proof. The main new idea, contained in Lemma 2.4 and its Corollary, consists of a nonlinear technique for transferring certain trees between pairs of Banach spaces.

Let us denote by \( L_p(X) \) the space of all \( X \)-valued Bochner integrable functions on \([0, 1]\), equipped with the \( L_p \)-norm. By a result of Lancien [18, Lemma 1] if \( p \in (1, \infty) \) then
\[
(2) \quad \text{Dz}(X) \leq \text{Sz}(L_p(X))
\]
for any space \( X \) having a separable dual. The proof in [18] is done for \( p = 2 \), but it can be easily adjusted to any \( p \in (1, \infty) \).

We now recall some standard facts about ordinals and the spaces of continuous function on them. We denote by \( \omega \) the first infinite ordinal and by \( \omega_1 \) the first uncountable ordinal. We always consider sets of ordinals as topological spaces equipped with the order topology.

The isomorphic classification of the spaces \( C([0, \alpha]) \), for \( \alpha < \omega_1 \), is due to C. Bessaga and A. Pełczyński [4, Theorem 1]. They have shown that \( C([0, \omega_\alpha]) \), for \( \alpha < \omega_1 \), are pairwise non-isomorphic spaces, and for every \( \omega_\alpha \leq \beta < \omega_{\alpha+1} \) there is an isomorphism between \( C([0, \beta]) \) and \( C([0, \omega_\alpha]) \). Moreover, every \( C(K) \) space for a countable compact \( K \) is isomorphic to one of these spaces. Samuel [23, Théorème, p.91] computed the precise values of the Szlenk index and showed that
\[
(3) \quad \text{Sz}(C([0, \omega_\alpha])) = \omega^{\alpha+1}, \text{ for all } \alpha < \omega_1,
\]
which implies that the Szlenk index determines the isomorphic classes of the separable \( C([0, \omega_\alpha]) \) spaces. Other proofs of this result were given in [1, 13].

One of the main ingredients of our proof is an alternative description of the Szlenk index introduced in [1], which is based on a derivation and its corresponding index defined for certain trees in the space \( X \). This approach has been further developed e.g. in [8, 12, 20], and we now recall some notion introduced there.

Let \( X \) be a Banach space. We let \( S_X^{\leq \omega} = \bigcup_{n=0}^{\infty} S_X^n \), the set of all finite sequences in \( X \), which includes the sequence of length zero denoted by \( \emptyset \). For \( x \in X \) we shall write \( x \) instead of \((x)\), i.e., we identify \( X \) with sequences of length 1 in \( X \). A tree on \( S_X \) is a non-empty subset \( A \) of \( S_X^{\leq \omega} \) closed under taking initial segments: if \((x_1, \ldots, x_n) \in A\)
and $0 \leq m \leq n$, then $(x_1, \ldots, x_m) \in A$. There is a natural partial order $\leq$ on the elements of the tree $A$, which gives $a \leq b$ if and only if $a$ is an initial segment of $b$.

Given $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ in $X^{<\omega}$, we write $(x, y)$ for the concatenation of $x$ and $y$:

$$(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n).$$

Given $A \subset S_X^{<\omega}$ and $x \in S_X^{<\omega}$, we let

$$A(x) = \{ y \in S_X^{<\omega} : (x, y) \in A \}.$$

Let $S$ be a set consisting of sequences in $S_X$. In our case $S$ will be the set of normalized weakly null sequences in $X$. For a tree $A$ on $X$ the $S$-derivative $A'_S$ of $A$ consists of all finite sequences of two kinds:

1. first kind: $x \in X^{<\omega}$, for which there is a sequence $(y_i)_{i=1}^\infty \in S$ with $(x, y_i) \in A$ for all $i \in \mathbb{N}$,
2. second kind: initial segments $(x_1, \ldots, x_m)$, $m \leq n$, where $(x_1, \ldots, x_n)$ is a sequence of the first kind.

Note that $A'_S \subseteq A$ and that $A'_S$ is also a tree unless it is empty.

We define higher order derivatives $A^{(\alpha)}_S$ for ordinals $\alpha < \omega_1$ by recursion as follows.

$A^{(0)}_S = A$, $A^{(\alpha+1)}_S = (A^{(\alpha)}_S)'_S$, for $\alpha < \omega_1$, and $A^{(\lambda)}_S = \bigcap_{\alpha < \lambda} A^{(\alpha)}_S$ for limit ordinals $\lambda < \omega_1$.

It is clear that $A^{(\alpha)}_S \supset A^{(\beta)}_S$, whenever $\alpha \leq \beta$, and that $A^{(\alpha)}_S$ is a tree or empty, for all $\alpha$. An easy induction also shows that

$$(A(x))^{(\alpha)}_S = (A^{(\alpha)}_S)(x)$$

for all $x \in S_X^{<\omega}$ and all ordinals $\alpha$.

Our proof will rely on the use of trees with the next additional heredity property. We will say that $A$ is a hereditary tree (H-tree, for short) if for every sequence $x \in A$, every subsequence of $x$ is also in $A$. Note that in this case all elements of the second kind are also of the first kind and that $A'$ consists therefore of all sequences in $A$ which are of the first kind. Taking the $S$-derivative of an H-tree therefore amounts to removing all elements which are not of the first kind. It is clear that the property of being an H-tree is preserved under taking $S$-derivatives of any ordinal order.

We now define the $S$-index $I_S(A)$ of $A$ by

$$I_S(A) = \min \{ \alpha < \omega_1 : A^{(\alpha)}_S = \emptyset \}$$

if there exists $\alpha < \omega_1$ with $A^{(\alpha)}_S = \emptyset$, and $I_S(A) = \infty$ otherwise.

Note that if $I_S(A) \neq \infty$, it will always be a successor ordinal. Indeed, if $\lambda$ is a limit ordinal and $I_S(A^{(\alpha)}) > 0$ for all $\alpha < \lambda$, then, since $\emptyset \in \bigcap_{\alpha < \lambda} A^{(\alpha)} = A^{(\lambda)}$ we get $I_S(A) > \lambda$.

If $A$ is a tree on $S_X$ we call a subset $B \subset A$ a subtree if it is also a tree on $S_X$. Let $Y$ be another Banach space and let $A \subset S_Y^{<\omega}$ and $B \subset S_Y^{<\omega}$ be trees on $S_X$ and $S_Y$, respectively. We say that $A$ order isomorphically embeds into $B$ if there is a injective map $\Psi : A \to B$, with the property that $\Psi(x) \prec \Psi(z)$ if and only if $x \prec z$. In that case $\Psi$ is called an order isomorphism from $A$ to $B$. 
In this paper we will only consider the case that $S$ consists of the normalized weakly null sequences and will therefore write $\mathcal{A}'$ and $\mathcal{A}^{(\alpha)}$, for a tree $\mathcal{A} \subset S_X^{<\omega}$, instead of $\mathcal{A}'_S$ and $\mathcal{A}^{(\alpha)}_S$, respectively, and we put $I_w(\mathcal{A}) = I_S(\mathcal{A})$, which we call the weak index of $\mathcal{A}$.

The following Proposition describes a sufficient condition for $I_w(\mathcal{A}) \leq I_w(\mathcal{B})$, if $\mathcal{A}$, $\mathcal{B}$ are two trees on the sphere of two Banach spaces $X$ and $Y$.

**Proposition 2.1.** Assume that $X$ and $Y$ are two Banach spaces, and $\mathcal{A} \subset S_X^{<\omega}$ and $\mathcal{B} \subset S_Y^{<\omega}$ are trees on $S_X$ and $S_Y$, respectively, and assume that there is an order isomorphism $\Psi$ from $\mathcal{A}$ to $\mathcal{B}$, with the following property:

1. If $x \in \mathcal{A}$ and if $(x_k) \subset S_X$ is weakly null, with $(x, x_k) \in \mathcal{A}$, for $k \in \mathbb{N}$, then there is a weakly null sequence $(y_k) \subset S_Y$, so that $\Psi(x, x_k) = (\Psi(x), y_k)$.

Then $I_w(\mathcal{A}) \leq I_w(\mathcal{B})$.

**Proof.** We verify by transfinite induction that for all ordinals $\alpha$

$$\Psi(\mathcal{A}^{(\alpha)}) \subset \mathcal{B}^{(\alpha)}.$$  

If $\alpha = 0$ this is just our assumption. If (4) holds for some ordinal $\alpha$ and if $x \in \mathcal{A}^{(\alpha+1)}$, then there is a weakly null sequence $(x_k) \subset S_X$ so that $(x, x_k) \in \mathcal{A}^{(\alpha)}$, for all $k \in \mathbb{N}$, and, by assumption (4), we can choose a weakly null sequence $(y_k) \subset S_Y$ so that $\Psi(x, x_k) = (\Psi(x), y_k)$. By the induction hypothesis $\Psi(x, x_k) \in \mathcal{B}^{(\alpha)}$ for all $k \in \mathbb{N}$. Now, since $(y_k)$ is weakly null, this implies that $\Psi(x) \in \mathcal{B}^{(\alpha+1)}$.

If $\lambda$ is a limit ordinal and (5) holds for all $\alpha < \lambda$, then

$$\Psi(\mathcal{A}^{(\lambda)}) = \bigcap_{\alpha < \lambda} \Psi(\mathcal{A}^{(\alpha)}) \subset \bigcap_{\alpha < \lambda} \mathcal{B}^{(\alpha)} = \mathcal{B}^{(\lambda)}.$$

$\square$

The following characterization of the Szlenk index was proven in [1].

**Theorem 2.2.** [1, Theorem 4.2] If $X$ is a separable Banach space not containing $\ell_1$ then

$$\text{Sz}(X) = \sup_{\rho > 0} I_w(\mathcal{F}_\rho),$$

where for $\rho > 0$, we let

$$\mathcal{F}_\rho = \mathcal{F}_\rho^X = \left\{ (x_1, x_2, \ldots, x_n) \in S_X^{<\omega} : \left\| \sum_{i=1}^n a_i x_i \right\| \geq \rho \sum_{i=1}^n a_i, \text{ for all } (a_i)_{i=1}^n \subset [0, \infty) \right\}.$$

It is important to note, and it will be used repeatedly in what follows, that $\mathcal{F}_\rho$ is in fact an H-tree, and, thus, that all its derivatives are H-trees.

**Remark.** In [1, Definition 3.6] the set $\mathcal{F}_\rho$ was actually defined differently, namely

$$\tilde{\mathcal{F}}_\rho = \left\{ (x_1, x_2, \ldots, x_n) \in S_X^{<\omega} : \left\| \sum_{i=1}^n a_i x_i \right\| \geq \rho \sum_{i=1}^n a_i, \text{ for all } (a_i)_{i=1}^n \subset [0, \infty) \text{ and } (x_1, x_2, \ldots, x_n) \text{ is } \frac{1}{\rho} \text{-basic} \right\}.$$
This was necessary in [1] because in that paper the $S$-derivatives for several other sets $S$ of sequences were considered.

In the case that one only considers derivatives with respect to the weakly null sequences the restriction to $1$-$basic$ sequences is superfluous, as the next proposition shows.

**Proposition 2.3.** Let $A \subset S^<\omega_X$ be an $H$-tree and $c > 1$. Then

$$I_w(A) = I_w\left(A \cap \{(x_1, x_2, \ldots, x_n) \in S^<\omega : (x_1, x_2, \ldots, x_n) \text{ is } c\text{-basic}\}\right).$$

**Proof.** For $c > 1$ and a finite dimensional subspace $F$ of $X$ we put

$$A_{(F,c)} = \left\{(x_1, x_2, \ldots, x_n) \in A : \left\|a_0y_0 + \sum_{i=1}^{m} a_ix_i\right\| \leq c\left\|a_0y_0 + \sum_{i=1}^{n} a_ix_i\right\|, \text{ for all } y_0 \in F, (a_i)_{i=0}^{n} \subset \mathbb{R}, \text{ and } 0 \leq m \leq n \right\}.$$

By transfinite induction we will show that for all $\alpha < \omega_1$, if $A^{(\alpha)} \neq \emptyset$, then $A_{(F,c)}^{(\alpha)} \neq \emptyset$, for all $c > 1$ and all finite dimensional subspaces $F \subset X$. Then our claim follows simply by letting $F = \{0\}$.

If $A = A^{(0)} \neq \emptyset$, then $0 \in A$ and thus $0 \in A_{(F,c)}$, for all $c > 1$ and all finite dimensional subspaces $F \subset X$.

Assume that our claim is true for some ordinal $\alpha$ and assume that $A^{(\alpha+1)} \neq \emptyset$. Let $F \subset X$ be finite dimensional and $c > 1$. Choose $c' = \sqrt{c}$. Since $0 \in A^{(\alpha+1)}$, there exists a weakly null sequence $(y_j) \subset A^{(\alpha)}$ and, thus, $(A(y_j))^{(\alpha)} = A^{(\alpha)}(y_j) \neq \emptyset$, for all $j \in \mathbb{N}$. Put $F_j = \text{span}(F \cup \{y_j\})$, for $j \in \mathbb{N}$. From the induction hypothesis we deduce that $(A(y_j))^{(\alpha)}_{(F_j,c')} \neq \emptyset$, for all $j \in \mathbb{N}$. But now we note that $(x_1, \ldots, x_n) \in (A(y_j))^{(\alpha)}_{(F_j,c')}$, for some $j \in \mathbb{N}$, means that $(x_1, x_2, \ldots, x_n) \in (A(y_j))^{(\alpha)}$ and $\left\|a_0y + \sum_{i=1}^{m} x_i\right\| \leq c'\left\|a_0y + \sum_{i=1}^{n} x_i\right\|$, for all $y \in F_j, (a_i)_{i=0}^{n} \subset \mathbb{R}$, and $m \leq n$. The first condition means that $(y_j, x_1, \ldots, x_n) \in A^{(\alpha)}$. Since $(y_j)$ is weakly null the second condition implies for large enough $j_0 \in \mathbb{N}$ and $j \geq j_0$ that

$$\left\|b_0y\right\| \leq c'\left\|b_0y + b_1y_j\right\| \leq c'c\left\|b_0y + b_1y_j + \sum_{i=1}^{n} a_ix_i\right\|,$$

for all $y \in F$ and $b_0, b_1, a_1, a_2, \ldots a_n \in \mathbb{R}$. Thus $(y_j, x_1, \ldots, x_n) \in A^{(\alpha)}_{(F,c)}$ for all $j \geq j_0$.

We deduce that $A^{(\alpha+1)}_{(F,c)} \neq \emptyset$, which finishes the induction step for successor ordinals.

If $\lambda$ is a limit ordinal and $A^{(\lambda)} \neq \emptyset$ it follows that $0 \in A^{(\alpha)}$, for all $\alpha < \lambda$, and thus, by the induction hypothesis $0 \in A^{(\alpha)}_{(F,c)}$ for any $c > 1$ and finite dimensional subspace $F \subset X$, which implies that $0 \in \bigcap_{\alpha < \lambda} A^{(\alpha)}_{(F,c)} = A^{(\lambda)}_{(F,c)}$. This finishes the induction step, and the proof of our claim.

The following Lemma compares the weak index of trees which are in a certain sense close to each other.

**Lemma 2.4.** Let $X$ and $Y$ be subspaces of a Banach space $Z$, with $X^*$ and $Y^*$ being separable, and let $\varepsilon > 0$. Assume that $\text{dist}(x,Y) < \varepsilon \|x\|$, for each $x \in X$. 

Then it follows for any H-tree $A$ on $S_X$, with $I_w(A) < \infty$, that $I_w(A) \leq I_w(B)$, where

$$B = \{ (y_1, y_2, \ldots, y_n) \in S_Y^< \omega : \exists (x_1, x_2, \ldots, x_n) \in A \; \| x_j - y_j \| \leq 4\varepsilon, \text{ for } j = 1, 2 \ldots n \}.$$  

Proof. We first prove the following

**Claim 1.** For every weakly null sequence $(x_j) \subset S_X$ there is a subsequence $(x'_k)$ of $(x_j)$ and a weakly null sequence $(y_k)$ in $S_Y$ so that $\| x'_k - y_k \| \leq 4\varepsilon$.

For a Banach space $U$ we denote the weak topology on $U$ by $\sigma(U, U^*)$ and the weak* topology on $U^*$ by $\sigma(U^*, U)$. By assumption we can find $\tilde{x}_j \in Y$, for every $j \in \mathbb{N}$, with $\| \tilde{x}_j - x_j \| < \varepsilon$. We choose an element

$$z^{**} = \left( \bigcap_{n=1}^{\infty} \{ \tilde{x}_j - x_j : j \geq n \} \right)^{\sigma(Z^{**}, Z^*)} \subset \varepsilon B_{Z^{**}}$$

(considering $Z$ as a subspace of $Z^{**}$ via the canonical map). We let $I = \mathbb{N} \times U$, where $U$ is a neighborhood basis of 0 in $\sigma(Z^{**}, Z^*)$, and consider the order on $I$ defined by $(n, U) \leq (n', U')$ if and only if $n \leq n'$ and $U \supset U'$. We pick for every $i = (n, U) \in I$ an element $\tilde{x}_i - x_i \in \{ \tilde{x}_j - x_j : j \geq n \} \cap (z^{**} + U)$ and note that $(\tilde{x}_i - x_i : i \in I)$ is a net which $\sigma(Z^{**}, Z^*)$-converges to $z^{**}$. Since $(x_j)$ is $\sigma(X, X^*)$-null, it follows that $\sigma(Z^{**}, Z^*) - \lim_{i \in I} x_i = 0$, and thus, since $Y^{**}$ is $\sigma(Z^{**}, Z^*)$-closed in $Z^{**}$,

$$z^{**} = \sigma(Z^{**}, Z^*) - \lim_{i \in I} \tilde{x}_i - x_i = \sigma(Z^{**}, Z^*) - \lim_{i \in I} \tilde{x}_i \in Y^{**}.$$  

Since $Y^*$ is separable the $\sigma(Y^{**}, Y^*)$-topology is metrizable on $B_{Y^{**}}$, and we can find by Goldstine’s Theorem a sequence $(u_n) \subset \varepsilon B_Y$ which $\sigma(Y^{**}, Y^*)$-converges to $z^{**}$. This implies that $0 \in \bigcap_{n \in \mathbb{N}} \{ \tilde{x}_j - u_k : j, k \geq n \}$, and using again the separability of $Y^*$ we can find strictly increasing sequences $(m(k))$ and $(n(k))$ such that $(\tilde{x}_{m(k)} - u_{n(k)})_{k \in \mathbb{N}}$ converges in $\sigma(Y, Y^*)$ to 0. We deduce now our claim by letting $x'_k = x_{m(k)}$, and $y_k = (\tilde{x}_{m(k)} - u_{n(k)})/\| \tilde{x}_{m(k)} - u_{n(k)} \|$, and noting that

$$\| x_{m(k)} - \frac{\tilde{x}_{m(k)} - u_{n(k)}}{\| \tilde{x}_{m(k)} - u_{n(k)} \|} \| \leq \| x_{m(k)} - \tilde{x}_{m(k)} \| + \| u_{n(k)} \| + \| \tilde{x}_{m(k)} - u_{n(k)} \| - 1 \| \leq 4\varepsilon.$$  

Next we prove the following claim by transfinite induction for all ordinals $\alpha$, which will yield, together with Proposition [2.1], the assertion of our lemma.

**Claim 2.** For any H-tree $A$ on $S_X$, with $I_w(A) = \alpha + 1$, there exist a subtree $\tilde{A}$ of $A$, and a length preserving order isomorphism $\Psi : \tilde{A} \rightarrow B$, so that

(6) \quad $I_w(\tilde{A}) = I_w(A) = \alpha + 1,$ and

(7) \quad $\Psi$ satisfies condition (4) of Proposition [2.1].

If $\alpha = 0$ and $I_w(A) = 1$, we simply can take $\tilde{A} = \{ \emptyset \}$ and put $\Psi(\emptyset) = \emptyset$. Assume now that our claim is true for $\alpha$ and that $A$ is an H-tree with $I_w(A) = \alpha + 2$. We deduce, that $\emptyset \in A^{(\alpha+1)}$ and that there is a weakly null sequence $(x_k)_{k \in \mathbb{N}} \subset S_X$, so that $x_k = (0, x_k) \in A^{(\alpha)}$, which means that $I_w(\mathcal{A}(x_k)) \geq \alpha + 1$, for $k \in \mathbb{N}$.

After passing to a subsequence of $(x_k)$ we can, using Claim 1, assume that there is a weakly null sequence $(y_k) \subset S_X$ so that $\| x_k - y_k \| \leq 4\varepsilon$, for all $k \in \mathbb{N}$. After
passing to a cofinite subsequence of \((x_k)\) we can assume that \(I_w(\mathcal{A}(x_k)) = \alpha + 1\) for all \(k \in \mathbb{N}\). Indeed, otherwise we could pass to a subsequence \((x'_k)\) of \((x_k)\), so that \(I_w(\mathcal{A}(x'_k)) \geq \alpha + 2\) for all \(k \in \mathbb{N}\), which would imply that \(x'_k \in \mathcal{A}^{(\alpha+1)}\) for all \(k \in \mathbb{N}\), and thus \(\emptyset \in \mathcal{A}^{(\alpha+2)}\), which would mean that \(I_w(\mathcal{A}) \geq \alpha + 3\), a contradiction.

Applying the inductive hypothesis we find for every \(k \in \mathbb{N}\) a subtree \(\tilde{\mathcal{A}}_k\) of \(\mathcal{A}(x_k)\), with \(I_w(\tilde{\mathcal{A}}_k) = I_w(\mathcal{A}_k) = \alpha + 1\), and a length preserving isomorphism \(\Psi_k : \tilde{\mathcal{A}}_k \to \mathcal{B}\), which satisfies \([7]\).

We glue these trees, and isomorphisms together by letting
\[
\tilde{\mathcal{A}} = \big\{ (x_k, x^{(k)}) : k \in \mathbb{N} \text{ and } x^{(k)} \in \tilde{\mathcal{A}}_k \big\} \cup \{ \emptyset \}
\]
and
\[
\Psi : \tilde{\mathcal{A}} \to \mathcal{B}, \quad x \mapsto \begin{cases} 
\emptyset & \text{if } x = \emptyset, \\
(y_k, \Psi(x^{(k)})) & \text{if } x = (x_k, x^{(k)}), \text{ for some } k \in \mathbb{N} \text{ and } x^{(k)} \in \tilde{\mathcal{A}}_k.
\end{cases}
\]
It is now routine to verify that \(\tilde{\mathcal{A}}, \mathcal{B}\) and \(\Psi\) satisfy conditions \([6]\) and \([7]\).

In the case that \(\alpha\) is a limit ordinal and we assume that our claim holds for all \(\alpha' < \alpha\) we proceed as follows. Assume that \(I_w(\mathcal{A}) = \alpha + 1\). Let \((\alpha_n)\) be a sequence in \([0, \alpha)\) which increases to \(\alpha\). For each \(n \in \mathbb{N}\), we can pick a weakly null sequence \((u(n,j))_{j \in \mathbb{N}} \subset S_X\), so that \(u(n,j) \in \mathcal{A}(\alpha_n)\), for all \(n, j \in \mathbb{N}\). Since \(X^*\) is separable, the weak topology on \(B_X\) is metrizable, and we can find a diagonal sequence \((x_n) = (u(n,j_n))\) which is also weakly null. It follows that \(I_w(\mathcal{A}(x_n)) \geq \alpha_n\), for all \(n \in \mathbb{N}\). After passing to a subsequence of \((x_n)\) we can assume, again using Claim 1, that there is a weakly null sequence \((y_n) \subset S_Y\), so that \(\|x_n - y_n\| \leq 4\varepsilon\), for all \(n \in \mathbb{N}\). After passing to a cofinite subsequence of \((x_n)\) we can assume that \(I_w(\mathcal{A}(x_n)) < \alpha\), for all \(n \in \mathbb{N}\). Indeed, otherwise there is an infinite subsequence \((x'_n)\) of \((x_n)\), so that \(I_w(\mathcal{A}(x'_n)) \geq \alpha + 1\) (recall that \(I_w(\cdot)\) takes only values among the successor ordinals), and thus \(x'_n \in \mathcal{A}(\alpha)\), for all \(n \in \mathbb{N}\), which implies that \(I_w(\mathcal{A}) \geq \alpha + 2\), a contradiction.

We apply the inductive hypothesis for each \(n \in \mathbb{N}\) to obtain a subtree \(\tilde{\mathcal{A}}_n\) of \(\mathcal{A}(x_n)\), with \(I_w(\tilde{\mathcal{A}}_n) = I_w(\mathcal{A}(x_n))\), a tree \(\mathcal{B}_n\) on \(S_Y\), and an order isomorphism from \(\tilde{\mathcal{A}}_n\) onto \(\mathcal{B}_n\), so that the conditions \([6]\) and \([7]\) are satisfied. We now can define \(\tilde{\mathcal{A}}, \mathcal{B}\) and \(\Psi\) as before to verify our claim in the case that \(\alpha\) is a limit ordinal.

**Corollary 2.5.** Let \(X\) and \(Z\) be Banach spaces and \(Y\) be a subspace of \(Z\). Assume that \(Y^*\) and \(X^*\) are separable and assume for some \(\rho \in (0, 1)\) and \(\varepsilon \in (0, \rho/6)\) there is an embedding \(i : X \to Z\), with \(\|i\| \cdot \|i^{-1}\| \leq 1 + \varepsilon\), so that \(\text{dist}(i(x), y) \leq \varepsilon\|x\|\), for all \(x \in X\). Then
\[
I_w(\mathcal{F}^X_\rho) \leq I_w(\mathcal{F}^Y_{\rho-6\varepsilon}).
\]

**Proof.** If \((x_1, x_2, \ldots, x_n) \in \mathcal{F}^X_\rho\), and we let \(z_j = i(x_j)/\|i(x_j)\|\) for \(j = 1, 2, \ldots, n\), we deduce that for \((a_j)_{j=1}^n \subset [0, \infty)\) that
\[
\left\| \sum_{j=1}^n a_j z_j \right\| \geq \left\| \sum_{j=1}^n a_j i(x_j) \right\| - \sum_{j=1}^n a_j \|i(x_j)\| - 1 \geq \left( \frac{\rho}{1+\varepsilon} - \varepsilon \right) \sum_{j=1}^n a_j \geq (\rho-2\varepsilon) \sum_{j=1}^n a_j.
\]
It follows therefore that \(I_w(\mathcal{F}^X_\rho) \leq I_w(\mathcal{F}^Y_{\rho-2\varepsilon})\). Replacing \(X\) by \(i(X)\), and \(\rho\) by \(\rho - 2\varepsilon\), we can assume that \(X\) is a subspace of \(Z\) and need to show that \(I_w(\mathcal{F}^X_\rho) \leq I_w(\mathcal{F}^Y_{\rho-4\varepsilon})\).
We apply Lemma 2.4 to $A = \mathcal{F}_\rho^X$ (recall that $\mathcal{F}_\rho^X$ is an H-tree) and note that the tree $\mathcal{B}$ on $S_Y$, as defined in Lemma 2.4, is a subtree of $\mathcal{F}_\rho^Y$. Indeed, if $(y_1, y_2, \ldots, y_n) \in \mathcal{B}$, and if $(x_1, x_2, \ldots, x_n) \in \mathcal{F}_\rho^X$, is such that $\|x_j - y_j\| \leq 4\varepsilon$, then for all $(a_j)_{j=1}^n \subset [0, \infty)$,

$$\left\| \sum_{j=1}^n a_j y_j \right\| \geq \left\| \sum_{j=1}^n a_j x_j \right\| - 4\varepsilon \sum_{j=1}^n a_j \geq \rho \sum_{j=1}^n a_j - 4\varepsilon \sum_{j=1}^n a_j = (\rho - 4\varepsilon) \sum_{j=1}^n a_j.$$ 

\[\Box\]

The conditions described by our previous Lemma 2.4 and Corollary 2.5 are fulfilled in the situation described by the next theorem, which is essentially due to Zippin [25, Theorem 1.2]. Our formulation is explicitly due to Benyamini [3, page 27].

**Theorem 2.6.** Let $X$ be a space with separable dual and $0 < \varepsilon < \frac{1}{2}$, and let $K$ be a $w^*$-closed and totally disconnected subset of $B_{X^*}$, which is $(1 - \varepsilon)$-norming $X$.

Then there exist $\beta < \omega^{\text{Sz}(X, \varepsilon^{-1}) + 1}$ and a subspace $Y$ of $C(K)$, isometric to $C([0, \beta])$, so that

$$\text{dist}(i(x), Y) \leq 2\varepsilon \|x\|, \text{ for } x \in X,$$

where $i : X \to C(K)$ is the embedding defined by $i(x)(x^*) = x^*(x)$, for $x^* \in K$ and $x \in X$.

**Remark.** The proof of [25, Theorem 1.2] shows that for any Banach space $X$ with a separable dual, and $\varepsilon > 0$, we can find a $w^*$-closed totally disconnected $(1 - \varepsilon)$-norming subset of $B_{X^*}$. An explicit construction of such a set $K \subset B_{X^*}$ can also be found in [10, Lemmas 1.3 and 1.4].

Let us also note that in [25] and [3] another index $\eta(X, \varepsilon)$ was used, replacing in the statement of Theorem 2.6 our index $\text{Sz}(X, \varepsilon)$. But since it was shown for $\eta(\varepsilon, X)$ in [11, page 22] (note that in [11] $\eta(\varepsilon, \cdot)$ was called $\eta'(\cdot, \cdot)$, while $\text{Sz}(\cdot, \cdot)$ was named $\eta(\cdot, \cdot)$) that $\eta(X, \varepsilon) \leq \text{Sz}(X, \varepsilon) \leq \eta(X, \varepsilon/2)$ for all $\varepsilon > 0$, our statement of Theorem 2.6 follows from the statement in [3].

The final key ingredient of our proof is the actual computation of the dentability index of $C([0, \alpha])$, $\alpha < \omega_1$, which was done in [14, Proposition 12].

\[9\] \[Dz(C([0, \omega^\alpha])) = \text{Sz}(L_\rho(C([0, \omega^\alpha]))) = \begin{cases} \omega^{\alpha+2} & \text{if } \alpha \text{ is finite}, \\ \omega^{\alpha+1} & \text{if } \alpha \text{ is an infinite ordinal}. \end{cases}\]

We are now ready for the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Lancien showed in [17, Propositions 3.1 and 3.2] that $\text{Sz}(X)$ and $\text{Dz}(X)$ are separably determined, provided they are countable, so we may assume without loss of generality that $X$ is separable.

We will show that for any $\rho \in (0, 1)$ and $1 < p < \infty$ we have

\[10\] \[I_w(\mathcal{F}_\rho^{L_p(X)}) < \begin{cases} \omega^\alpha & \text{if } \alpha \text{ is infinite}, \\ \omega^{\alpha+1} & \text{if } \alpha \text{ is finite}. \end{cases}\]

Then our claim follows from Theorem 2.2.
Let $\varepsilon \in (0, \rho/12)$ and apply Theorem 2.6 which provides us with a $w^*$-closed, totally disconnected and $(1 - \varepsilon)$-norming $X$ subset $K$ of $B_{X^*}$, an ordinal $\beta < \omega^{\operatorname{Sz}(X, \varepsilon/8) + 1}$ and a subspace $Y \subset C(K)$, isometric to $C([0, \beta])$, so that $\operatorname{dist}(i(x), Y) \leq 2\varepsilon \|x\|$ for all $x \in X$, where $i : X \to C(K)$, is defined by $i(x)(x^*) = x^*(x)$, for $x^* \in K$ and $x \in X$. Since $\beta < \omega^{\operatorname{Sz}(X, \varepsilon/8)} < \omega^\alpha$ equation (3) yields that $\operatorname{Sz}(Y) \leq \omega^\alpha$. Indeed, if $\alpha = \gamma + 1$ for some $\gamma < \omega_1$, then $\beta < \omega^{\omega^\gamma - k}$ for some $k \in \mathbb{N}$ and we deduce from (3) that $\operatorname{Sz}(Y) \leq \operatorname{Sz}(C([0, \omega^{\omega^\gamma - k}])) = \operatorname{Sz}(C([0, \omega^\gamma])) = \omega^{\gamma + 1} = \omega^\alpha$. On the other hand, if $\alpha$ is a limit ordinal we deduce that $\beta < \omega^\gamma$, for some $\gamma < \alpha$, and we derive our claim the same way.

We define

$$I : \operatorname{Lp}(X) \to \operatorname{Lp}(C(K)), \quad f \mapsto i \circ f,$$

and note that $\|f\|(1 - \varepsilon) \leq \|I(f)\| \leq \|f\|$ for $f \in \operatorname{Lp}(X)$. Observe that $\operatorname{Lp}(Y)$ embeds naturally and isometrically into $\operatorname{Lp}(C(K))$ and that $\operatorname{dist}(I(f), \operatorname{Lp}(Y)) \leq 2\varepsilon \|f\|$ for all $f \in \operatorname{Lp}(X)$. The last observation follows easily for step functions and from the fact that $[0, 1]$ with the Lebesgue measure is a probability space, and for general elements of $\operatorname{Lp}(X)$ by approximation.

This means that the spaces $\tilde{X} = \operatorname{Lp}(X)$, $\tilde{Y} = \operatorname{Lp}(Y)$ and $\tilde{Z} = \operatorname{Lp}(C(K))$, and the isomorphic embedding $I$ satisfy the conditions of Corollary 2.5 for $2\varepsilon$ instead of $\varepsilon$. We conclude therefore from Corollary 2.5 and (3) that

$$I_w(\mathcal{E}^{L_p(X)}_{\rho}) \leq I_w(\mathcal{E}^{L_p(Y)}_{\rho - 12\varepsilon}) < \operatorname{Sz}(\operatorname{Lp}(Y)) \leq \begin{cases} \omega^{\alpha + 1} & \text{if } \alpha \text{ is finite}, \\ \omega^\alpha & \text{if } \alpha \text{ is an infinite ordinal}. \end{cases}$$

which proves our claim and finishes the proof of our theorem.

We remark that if $\alpha$ is finite and $\operatorname{Sz}(X) = \omega^\alpha$ then the precise value of $\operatorname{Dz}(X)$ depends on the geometry of $X^*$. Indeed, since $\operatorname{Lp}(\operatorname{Lp}(X))$ and $\operatorname{Lp}(X)$ are isomorphic for any Banach spaces $X$, it follows that $\operatorname{Sz}(\operatorname{Lp}(\operatorname{Lp}(X))) = \operatorname{Lp}(X)$. But for $X = C([0, \omega^\alpha])$, where $\alpha$ is finite, it was shown in [14, Theorem 2] that $\operatorname{Dz}(X) > \operatorname{Sz}(X)$ and thus by Theorem 1.2 $\operatorname{Dz}(\operatorname{Lp}(X)) = \operatorname{Sz}(\operatorname{Lp}(X)) = \omega^{\alpha + 2}$.

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