Weak type estimates of intrinsic square functions on the weighted Herz-type Hardy spaces

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Abstract

In this paper, by using the atomic decomposition theory of weighted Herz-type Hardy spaces, we will obtain some weighted weak type estimates for intrinsic square functions including the Lusin area function, Littlewood-Paley $g$-function and $g^*_\lambda$-function on these spaces.

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1. Introduction and preliminaries

First, let’s recall some standard definitions and notations. The classical $A_p$ weight theory was first introduced by Muckenhoupt in the study of weighted $L^p$ boundedness of Hardy-Littlewood maximal functions in [15]. A weight $w$ is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere, $B = B(x_0, R)$ denotes the ball with the center $x_0$ and radius $R$. We say that $w \in A_p, 1 < p < \infty$, if

$$\left( \frac{1}{|B|} \int_B w(x) \, dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} \leq C \text{ for every ball } B \subseteq \mathbb{R}^n,$$

where $C$ is a positive constant which is independent of $B$.

For the case $p = 1$, $w \in A_1$, if

$$\frac{1}{|B|} \int_B w(x) \, dx \leq \text{ess inf}_{x \in B} w(x) \text{ for every ball } B \subseteq \mathbb{R}^n.$$

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A weight function $w$ is said to belong to the reverse Hölder class $RH_r$ if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds

$$\left( \frac{1}{|B|} \int_B w(x)^r \, dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B w(x) \, dx \right)$$

for every ball $B \subseteq \mathbb{R}^n$.

It is well known that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. If $w \in A_p$ with $1 \leq p < \infty$, then there exists $r > 1$ such that $w \in RH_r$.

Given a ball $B$ and $\lambda > 0$, $\lambda B$ denotes the ball with the same center as $B$ whose radius is $\lambda$ times that of $B$. For a given weight function $w$, we denote the Lebesgue measure of $B$ by $|B|$ and the weighted measure of $B$ by $w(B)$, where $w(B) = \int_B w(x) \, dx$.

We shall need the following lemmas.

**Lemma A ([4]).** Let $w \in A_p$, $p \geq 1$. Then, for any ball $B$, there exists an absolute constant $C$ such that

$$w(2B) \leq Cw(B).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda B) \leq C\lambda^{np}w(B),$$

where $C$ does not depend on $B$ nor on $\lambda$.

**Lemma B ([4,5]).** Let $w \in A_p \cap RH_r$, $p \geq 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that

$$C_1 \left( \frac{|E|}{|B|} \right)^{p} \leq \frac{w(E)}{w(B)} \leq C_2 \left( \frac{|E|}{|B|} \right)^{(r-1)/r}$$

for any measurable subset $E$ of a ball $B$.

Next we shall give the definitions of the weighted Herz space, weak weighted Herz space and weighted Herz-type Hardy space. In 1964, Beurling [2] first introduced some fundamental form of Herz spaces to study convolution algebras. Later Herz [6] gave versions of the spaces defined below in a slightly different setting. Since then, the theory of Herz spaces has been significantly developed, and these spaces have turned out to be quite useful in harmonic analysis. For instance, they were used by Baernstein and
Sawyer [1] to characterize the multipliers on the classical Hardy spaces, and used by Lu and Yang [12] in the study of partial differential equations.

On the other hand, a theory of Hardy spaces associated with Herz spaces has been developed in [3,10]. These new Hardy spaces can be regarded as the local version at the origin of the classical Hardy spaces $H^p(\mathbb{R}^n)$ and are good substitutes for $H^p(\mathbb{R}^n)$ when we study the boundedness of non-translation invariant operators (see [11]). For the weighted case, in 1995, Lu and Yang introduced the following weighted Herz-type Hardy spaces $H_{\dot{K}}^{\alpha,p}(w_1,w_2)$ and established their atomic decompositions. In 2006, Lee gave the molecular characterizations of these spaces, he also obtained the boundedness of the Hilbert transform and the Riesz transforms on $H_{\dot{K}}^{\alpha,p}(w_1,w_2)$ for $0 < p \leq 1$. For the results mentioned above, we refer the readers to the book [14] and the papers [7,8,9,13] for further details.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{N}$ and $\tilde{\chi}_0 = \chi_{B_0}$, where $\chi_{C_k}$ is the characteristic function of $C_k$. Given a weight function $w$ on $\mathbb{R}^n$, for $1 \leq p < \infty$, we denote by $L^p_w(\mathbb{R}^n)$ the space of all functions satisfying

$$
\|f\|_{L^p_w(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx\right)^{1/p} < \infty.
$$

**Definition 1.** Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$ and $w_1, w_2$ be two weight functions on $\mathbb{R}^n$.

(i) The homogeneous weighted Herz space $\dot{K}^{\alpha,p}_{\dot{w}}(w_1,w_2)$ is defined by

$$
\dot{K}^{\alpha,p}_{\dot{w}}(w_1,w_2) = \{f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}, w_2) : \|f\|_{\dot{K}^{\alpha,p}_{\dot{w}}(w_1,w_2)} < \infty\},
$$

where

$$
\|f\|_{\dot{K}^{\alpha,p}_{\dot{w}}(w_1,w_2)} = \left(\sum_{k \in \mathbb{Z}} (w_1(B_k))^{\alpha p/n} \|f\chi_k\|_{L^p_{w_2}}^p\right)^{1/p}.
$$

(ii) The non-homogeneous weighted Herz space $K^{\alpha,p}_{\dot{w}}(w_1,w_2)$ is defined by

$$
K^{\alpha,p}_{\dot{w}}(w_1,w_2) = \{f \in L^q_{\text{loc}}(\mathbb{R}^n, w_2) : \|f\|_{K^{\alpha,p}_{\dot{w}}(w_1,w_2)} < \infty\},
$$

where

$$
\|f\|_{K^{\alpha,p}_{\dot{w}}(w_1,w_2)} = \left(\sum_{k=0}^{\infty} (w_1(B_k))^{\alpha p/n} \|f\tilde{\chi}_k\|_{L^p_{w_2}}^p\right)^{1/p}.
$$

For $k \in \mathbb{Z}$ and $\lambda > 0$, we set $E_k(\lambda, f) = |\{x \in C_k : |f(x)| > \lambda\}|$. Let $\tilde{E}_k(\lambda, f) = E_k(\lambda, f)$ for $k \in \mathbb{N}$ and $\tilde{E}_0(\lambda, f) = |\{x \in B(0,1) : |f(x)| > \lambda\}|$.
Definition 2. Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$ and $w_1, w_2$ be two weight functions on $\mathbb{R}^n$.

(i) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the homogeneous weak weighted Herz spaces $W\hat{K}_q^{\alpha,p}(w_1, w_2)$ if

$$
\|f\|_{W\hat{K}_q^{\alpha,p}(w_1, w_2)} = \sup_{\lambda > 0} \left( \sum_{k \in \mathbb{Z}} \frac{w_1(B_k)^{\alpha p/n} w_2(E_k(\lambda, f))^{p/q}}{\lambda^{p/q}} \right)^{1/p}.
$$

(ii) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the non-homogeneous weak weighted Herz spaces $W\hat{K}_q^{\alpha,p}(w_1, w_2)$ if

$$
\|f\|_{W\hat{K}_q^{\alpha,p}(w_1, w_2)} = \sup_{\lambda > 0} \left( \sum_{k \in \mathbb{Z}} \frac{w_1(B_k)^{\alpha p/n} w_2(\mathcal{E}_k(\lambda, f))^{p/q}}{\lambda^{p/q}} \right)^{1/p}.
$$

Let $\mathcal{S}(\mathbb{R}^n)$ be the class of Schwartz functions and let $\mathcal{S}'(\mathbb{R}^n)$ be its dual space. For $f \in \mathcal{S}'(\mathbb{R}^n)$, the grand maximal function of $f$ is defined by

$$
G(f)(x) = \sup_{\varphi \in \mathcal{S}_N |y-x| < t} \sup_{\varphi(x/t)} \|\varphi \ast f(y)\|,
$$

where $N = n + 1$, $\mathcal{S}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \leq N} |x^\alpha D^\beta \varphi(x)| \leq 1\}$ and $\varphi_t(x) = t^{-n} \varphi(x/t)$.

Definition 3. Let $0 < \alpha < \infty$, $0 < p < \infty$, $1 < q < \infty$ and $w_1, w_2$ be two weight functions on $\mathbb{R}^n$.

(i) The homogeneous weighted Herz-type Hardy space $H\hat{K}_q^{\alpha,p}(w_1, w_2)$ associated with the space $\hat{K}_q^{\alpha,p}(w_1, w_2)$ is defined by

$$
H\hat{K}_q^{\alpha,p}(w_1, w_2) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G(f) \in \hat{K}_q^{\alpha,p}(w_1, w_2)\}
$$

and we define $\|f\|_{H\hat{K}_q^{\alpha,p}(w_1, w_2)} = \|G(f)\|_{\hat{K}_q^{\alpha,p}(w_1, w_2)}$.

(ii) The non-homogeneous weighted Herz-type Hardy space $HK_q^{\alpha,p}(w_1, w_2)$ associated with the space $K_q^{\alpha,p}(w_1, w_2)$ is defined by

$$
HK_q^{\alpha,p}(w_1, w_2) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G(f) \in K_q^{\alpha,p}(w_1, w_2)\}
$$

and we define $\|f\|_{HK_q^{\alpha,p}(w_1, w_2)} = \|G(f)\|_{K_q^{\alpha,p}(w_1, w_2)}$.

2. The atomic decomposition

In this article, we will use the atomic decomposition theory for weighted Herz-type Hardy spaces in [8,9]. We characterize these spaces in terms of atoms in the following way.
Definition 4. Let $1 < q < \infty$, $n(1 - 1/q) \leq \alpha < \infty$ and $s \geq [\alpha + n(1/q - 1)]$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha, q, s)$-atom with respect to $(w_1, w_2)$ (or a central $(\alpha, q, s; w_1, w_2)$-atom), if it satisfies

(a) $\text{supp} a \subseteq B(0, R) = \{ x \in \mathbb{R}^n : |x| < R \}$,

(b) $\|a\|_{L^q_{w_2}} \leq w_1(B(0, R))^{-\alpha/n}$,

(c) $\int_{\mathbb{R}^n} a(x)x^\beta \, dx = 0$ for every multi-index $\beta$ with $|\beta| \leq s$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha, q, s)$-atom of restricted type with respect to $(w_1, w_2)$ (or a central $(\alpha, q, s; w_1, w_2)$-atom of restricted type), if it satisfies the conditions (b), (c) above and

(a') $\text{supp} a \subseteq B(0, R)$ for some $R > 1$.

Theorem C. Let $w_1, w_2 \in A_1$, $0 < p < \infty$, $1 < q < \infty$ and $n(1 - 1/q) \leq \alpha < \infty$. Then we have

(i) $f \in H^{\alpha, p}_{K_q}(w_1, w_2)$ if and only if

$$f(x) = \sum_{k \in \mathbb{Z}} \lambda_k a_k(x), \quad \text{in the sense of } \mathcal{F}'(\mathbb{R}^n),$$

where $\sum_{k \in \mathbb{Z}} |\lambda_k|^p < \infty$, each $a_k$ is a central $(\alpha, q, s; w_1, w_2)$-atom. Moreover,

$$\|f\|_{H^{\alpha, p}_{K_q}(w_1, w_2)} \approx \inf \left( \sum_{k \in \mathbb{Z}} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all the above decompositions of $f$.

(ii) $f \in H^{\alpha, p}_{K_q}(w_1, w_2)$ if and only if

$$f(x) = \sum_{k=0}^{\infty} \lambda_k a_k(x), \quad \text{in the sense of } \mathcal{F}'(\mathbb{R}^n),$$

where $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$, each $a_k$ is a central $(\alpha, q, s; w_1, w_2)$-atom of restricted type. Moreover,

$$\|f\|_{H^{\alpha, p}_{K_q}(w_1, w_2)} \approx \inf \left( \sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all the above decompositions of $f$.

3. The intrinsic square functions and our main results

The intrinsic square functions were first defined by Wilson in [17] and [18]. For $0 < \beta \leq 1$, let $C_\beta$ be the family of functions $\varphi$ defined on $\mathbb{R}^n$, such
that $\varphi$ has support containing in $\{x : |x| \leq 1\}$, $\int_{\mathbb{R}^n} \varphi(x) \, dx = 0$ and for all $x, x' \in \mathbb{R}^n$,
$$|\varphi(x) - \varphi(x')| \leq |x - x'|^\beta.$$ 
For $(y, t) \in \mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$ and $f \in L^1_{loc}(\mathbb{R}^n)$, we set
$$A_\beta(f)(y, t) = \sup_{\varphi \in C_\beta} |f * \varphi_t(y)|.$$ 
Then we define the intrinsic square function of $f$ (of order $\beta$) by the formula
$$S_\beta(f)(x) = \left( \iint_{\Gamma(x)} \left( A_\beta(f)(y, t) \right)^2 \frac{dydt}{tn+1} \right)^{1/2},$$
where $\Gamma(x)$ denotes the usual cone of aperture one:
$$\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}.$$ 
We can also define varying-aperture version of $S_\beta(f)$ by the formula
$$S_{\beta,\gamma}(f)(x) = \left( \iint_{\Gamma_\gamma(x)} \left( A_\beta(f)(y, t) \right)^2 \frac{dydt}{\gamma n+1} \right)^{1/2},$$
where $\Gamma_\gamma(x)$ is the usual cone of aperture $\gamma > 0$:
$$\Gamma_\gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < \gamma t\}.$$ 
The intrinsic Littlewood-Paley $g$-function (could be viewed as “zero-aperture” version of $S_\beta(f)$) and the intrinsic $g^*_\Lambda$-function (could be viewed as “infinite aperture” version of $S_\beta(f)$) will be defined respectively by
$$g_\beta(f)(x) = \left( \int_0^\infty \left( A_\beta(f)(x, t) \right)^2 \frac{dt}{t} \right)^{1/2}$$
and
$$g^*_\Lambda,\beta(f)(x) = \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t + |x - y|} \right)^\Lambda \left( A_\beta(f)(y, t) \right)^2 \frac{dydt}{\gamma n+1} \right)^{1/2}.$$ 
In [18], Wilson showed the following weighted $L^p$ boundedness of the intrinsic square functions.
Theorem D. Let $w \in A_p$, $1 < p < \infty$ and $0 < \beta \leq 1$. Then there exists a constant $C > 0$ such that

$$\|S_\beta(f)\|_{L^p_w} \leq C\|f\|_{L^p_w}.$$ 

In [16], the author obtained some boundedness properties of intrinsic square functions on the homogeneous and non-homogeneous weighted Herz-type Hardy spaces. As a continuation of [16], the aim of this paper is to discuss their weak type estimates. Our main results are stated as follows.

Theorem 1. Let $w_1, w_2 \in A_1$, $0 < p \leq 1$, $1 < q < \infty$, $0 < \beta < 1$ and $\alpha = n(1 - 1/q) + \beta$. Then there exists a constant $C$ independent of $f$ such that

$$\|g_\beta(f)\|_{WK_q^{\alpha,p}(w_1,w_2)} \leq C\|f\|_{HK_q^{\alpha,p}(w_1,w_2)}$$

$$\|S_\beta(f)\|_{WK_q^{\alpha,p}(w_1,w_2)} \leq C\|f\|_{HK_q^{\alpha,p}(w_1,w_2)}.$$ 

Theorem 2. Let $w_1, w_2 \in A_1$, $0 < p \leq 1$, $1 < q < \infty$, $0 < \beta < 1$ and $\alpha = n(1 - 1/q) + \beta$. Then there exists a constant $C$ independent of $f$ such that

$$\|S_\beta(f)\|_{WK_q^{\alpha,p}(w_1,w_2)} \leq C\|f\|_{HK_q^{\alpha,p}(w_1,w_2)}$$

$$\|S_\beta(f)\|_{WK_q^{\alpha,p}(w_1,w_2)} \leq C\|f\|_{HK_q^{\alpha,p}(w_1,w_2)}.$$ 

Theorem 3. Let $w_1, w_2 \in A_1$, $0 < p \leq 1$, $1 < q < \infty$, $0 < \beta < 1$ and $\alpha = n(1 - 1/q) + \beta$. If \( \lambda > 3 + (2\beta)/n \), then there exists a constant $C$ independent of $f$ such that

$$\|g_{\lambda,\beta}(f)\|_{WK_q^{\alpha,p}(w_1,w_2)} \leq C\|f\|_{HK_q^{\alpha,p}(w_1,w_2)}$$

$$\|g_{\lambda,\beta}(f)\|_{WK_q^{\alpha,p}(w_1,w_2)} \leq C\|f\|_{HK_q^{\alpha,p}(w_1,w_2)}.$$ 

Throughout this article, we will use $C$ to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence.

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. First we note that our assumption $\alpha = n(1 - 1/q) + \beta$ implies that $s = [\alpha + n(1/q - 1)] = [\beta] = 0$. For every $f \in HK_q^{\alpha,p}(w_1,w_2)$, then by Theorem C, we have the decomposition $f = \sum_{j \in \mathbb{Z}} \lambda_j a_j$, where $\sum_{j \in \mathbb{Z}} |\lambda_j|^p < \infty$ and each $a_j$ is a central $(\alpha, q, 0; w_1, w_2)$-atom. Without loss of generality, we may assume that $\text{supp} \ a_j \subseteq B(0, R_j)$ and $R_j = 2^j$. For any given $\sigma > 0$, we write
\[
\sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2(\{ x \in C_k : |g_\beta(f)(x)| > \sigma \})^{p/q}
\]

\[
\leq \sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2(\{ x \in C_k : \sum_{j=k-1}^{\infty} |\lambda_j||g_\beta(a_j)(x)| > \sigma/2 \})^{p/q}
\]

\[
+ \sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2(\{ x \in C_k : \sum_{j=-\infty}^{k-2} |\lambda_j||g_\beta(a_j)(x)| > \sigma/2 \})^{p/q}
\]

\[= I_1 + I_2.\]

Since \( w_2 \in A_1 \), then \( w_2 \in A_q \) for any \( 1 < q < \infty \). Note that \( 0 < p \leq 1 \). Applying Chebyshev's inequality and Theorem D, we have

\[
I_1 \leq \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} \left( \sum_{j=k-1}^{\infty} |\lambda_j||g_\beta(a_j)\chi_k|_{L^p_{w_2}}^n \right)^p
\]

\[
\leq \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} \left( \sum_{j=k-1}^{\infty} |\lambda_j|^p|g_\beta(a_j)|_{L^p_{w_2}}^p \right)^p
\]

\[
\leq C \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} \left( \sum_{j=k-1}^{\infty} |\lambda_j|^p|a_j|_{L^p_{w_2}}^p \right)^p.
\]

Changing the order of summation yields

\[
I_1 \leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^p \left( \sum_{k=-\infty}^{j+1} w_1(B_k)^{\alpha p/n} w_1(B_j)^{-\alpha p/n} \right).
\]

Since \( w_1 \in A_1 \), then we know \( w \in RH_r \) for some \( r > 1 \). When \( k \leq j + 1 \), then \( B_k \subseteq B_{j+1} \). It follows from Lemma B that

\[
w_1(B_k) \leq C w_1(B_{j+1}) |B_k|^\delta |B_{j+1}|^{-\delta},
\]

where \( \delta = (r - 1)/r > 0 \).

By using Lemma A and the inequality (1), we can get

\[
\sum_{k=-\infty}^{j+1} w_1(B_k)^{\alpha p/n} w_1(B_j)^{-\alpha p/n}
\]

\[
\leq C \sum_{k=-\infty}^{j+1} \left( \frac{w_1(B_{j+1})}{w_1(B_j)} \right)^{\alpha p/n} \left( \frac{|B_k|}{|B_{j+1}|} \right)^{\alpha \delta p/n}
\]

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\[ I_1 \leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^p \leq C \|f\|^p_{L^p(K^n\alpha,q)}(w_1,w_2). \]  

(2)

We now turn to estimate \( I_2 \). We first claim that for any \((x, t) \in \mathbb{R}^{n+1}\) and \(j \in \mathbb{Z}\), the following inequality holds

\[ A_{\beta}(a_j)(x, t) \leq C \cdot 2^{jn}w_1(B_j)^{-\alpha/n}w_2(B_j)^{-1/q}. \]  

(3)

Actually, this result was already given in [16] in a general form. Here, we give its proof for completeness. For any \(\varphi \in C^\infty_{\beta}\), by the vanishing moment condition of central atom \(a_j\), we can get

\[ |a_j * \varphi_t(x)| = \left| \int_{B_j} (\varphi_t(x - y) - \varphi_t(x)) a_j(y) \, dy \right| \leq \int_{B_j} |y|^\beta |a_j(y)| \, dy \leq 2^{jn} \int_{B_j} |a_j(y)| \, dy. \]  

(4)

Denote the conjugate exponent of \(q > 1\) by \(q' = q/(q-1)\). Hölder’s inequality and the \(A_q\) condition imply

\[ \int_{B_j} |a_j(y)| \, dy \leq \left( \int_{B_j} |a_j(y)|^q w_2(y) \, dy \right)^{1/q} \left( \int_{B_j} (w_2^{-1/q} y^{q'}) \, dy \right)^{1/q'} \leq C \cdot 2^{jn}w_1(B_j)^{-\alpha/n}w_2(B_j)^{-1/q}. \]  

(5)

Substituting the above inequality (5) into (4) and taking the supremum over all functions \(\varphi \in C^\infty_{\beta}\), we obtain the estimate (3).

Note that when \(j \leq k - 2\), then for any \(y \in B_j\) and \(x \in C_k = B_k \setminus B_{k-1}\), we have \(|x| \geq 2|y|\). We also note that \(\text{supp } \varphi \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\}\), then we can get \(t \geq |x - y| \geq \frac{1}{2}|x|\). Hence by the inequality (3), we deduce
\[(g_\beta(a_j)(x))^2 \leq C \cdot \left(2^{j(n+\beta)}w_1(B_j)^{-\alpha/n}w_2(B_j)^{-1/q}\right)^2 \int_{|x|^{2n+2\beta+1}}^\infty \frac{dt}{t^{2n+2\beta+1}} \]  
\[\leq C \cdot \left(2^{j(n+\beta)}w_1(B_j)^{-\alpha/n}w_2(B_j)^{-1/q}\right)^2 \frac{1}{|x|^{2n+2\beta}}. \]  
(6)

Since \(B_j \subseteq B_{k-2}\), then by using Lemma B, we obtain

\[w_i(B_j) \geq Cw_i(B_{k-2})|B_{j-2}|^{-1} \text{ for } i = 1 \text{ or } 2.\]

From our assumption \(\alpha = n(1 - 1/q) + \beta\) and (6), it follows immediately that

\[g_\beta(a_j)(x) \leq C \left(\frac{2^j}{2^{k-2}}\right)^{n+\beta-\alpha-n/q}w_1(B_{k-2})^{-\alpha/n}w_2(B_{k-2})^{-1/q} \]  
\[\leq Cw_1(B_{k-2})^{-\alpha/n}w_2(B_{k-2})^{-1/q}. \]  
(7)

Set \(A_k = w_1(B_{k-2})^{-\alpha/n}w_2(B_{k-2})^{-1/q}\).

If \(\{x \in C_k : \sum_{j=-\infty}^{k-2} |\lambda_j||g_\beta(a_j)(x)| > \sigma/2\} = \emptyset\), then the inequality

\[I_2 \leq C\|f\|_{H^{\alpha,p}_{\lambda_1}(w_1,w_2)}^p\]

holds trivially.

If \(\{x \in C_k : \sum_{j=-\infty}^{k-2} |\lambda_j||g_\beta(a_j)(x)| > \sigma/2\} \neq \emptyset\), then by the inequality (7), we have

\[\sigma < C \cdot A_k \left(\sum_{j \in \mathbb{Z}} |\lambda_j|\right) \]  
\[\leq C \cdot A_k \left(\sum_{j \in \mathbb{Z}} |\lambda_j|^p\right)^{1/p} \]  
\[\leq C \cdot A_k \|f\|_{H^{\alpha,p}_{\lambda_1}(w_1,w_2)}. \]

It is easy to verify that \(\lim_{k \to \infty} A_k = 0\). Then for any fixed \(\sigma > 0\), we are able to find a maximal positive integer \(k_\sigma\) such that

\[\sigma < C \cdot A_{k_\sigma} \|f\|_{H^{\alpha,p}_{\lambda_1}(w_1,w_2)}. \]

From the above discussion, we have that \(B_{k-2} \subseteq B_{k_\sigma-2}\), then by using Lemma B again, we obtain

\[\frac{w_i(B_{k-2})}{w_i(B_{k_\sigma-2})} \leq C \left(\frac{|B_{k-2}|}{|B_{k_\sigma-2}|}\right)^{\delta} \text{ for } i = 1 \text{ or } 2. \]
Furthermore, it follows from Lemma A that
\[
\frac{w_i(B_k)}{w_i(B_{k-2})} \leq C \left( \frac{|B_{k-2}|}{|B_k|} \right) \delta \quad \text{for } i = 1 \text{ or } 2.
\]
Therefore
\[
I_2 \leq \sigma^p \cdot \sum_{k=\infty}^{k_a} w_1(B_k)^{\alpha p/n} w_2(B_k)^{p/q}
\]
\[
\leq C \|f\|_{H^{\alpha, p}_{1, q}(w_1, w_2)}^p \sum_{k=\infty}^{k_a} \left( \frac{w_1(B_k)}{w_1(B_{ka})} \right)^{\alpha p/n} \left( \frac{w_2(B_k)}{w_2(B_{ka})} \right)^{p/q}
\]
\[
\leq C \|f\|_{H^{\alpha, p}_{1, q}(w_1, w_2)}^p \sum_{k=\infty}^{k_a} \frac{1}{2((k_a-k)\delta)}
\]
\[
\leq C \|f\|_{H^{\alpha, p}_{1, q}(w_1, w_2)}^p.
\]
Combining the above estimates (2) and (8) and taking the supremum over all \(\sigma > 0\), we complete the proof of Theorem 1.

**Proof of Theorem 2.** The proof is similar. We only point out the main differences. Write
\[
\sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2(\{x \in C_k : |S_\beta(f)(x)| > \sigma\})^{p/q}
\]
\[
\leq \sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2(\{x \in C_k : \sum_{j=k-1}^{\infty} |\lambda_j||S_\beta(a_j)(x)| > \sigma/2\})^{p/q}
\]
\[
+ \sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2(\{x \in C_k : \sum_{j=-\infty}^{k-2} |\lambda_j||S_\beta(a_j)(x)| > \sigma/2\})^{p/q}
\]
\[
= J_1 + J_2.
\]
Using the same arguments as in the proof of Theorem 1, we can prove
\[
J_1 \leq C \|f\|_{H^{\alpha, p}_{1, q}(w_1, w_2)}^p.
\]
To estimate \(J_2\), we note that if \(j \leq k-2\), then for any \(x \in C_k = B_k \setminus B_{k-1}\) and \(z \in B_j\), we have \(|z| \leq \frac{1}{2}|x|\). Furthermore, when \(|x-y| < t\) and \(|y-z| < t\), then we deduce
\[
2t > |x-z| \geq |x| - |z| \geq \frac{1}{2}|x|.
\]
By using the inequality (3), we thus obtain
\[
(S_\beta(a_j)(x))^2 \leq C \cdot \left( 2^{j(n+\beta)} w_1(B_j)^{-\alpha/n} w_2(B_j)^{-1/q} \right)^2 \cdot \int_\mathbb{R} \int_{|y-x|<t} \frac{dydt}{t^{2n+2\beta+n+1}} \leq C \cdot \left( 2^{j(n+\beta)} w_1(B_j)^{-\alpha/n} w_2(B_j)^{-1/q} \right)^2 \frac{1}{|x|^{2n+2\beta}},
\]
which is equivalent to
\[
S_\beta(a_j)(x) \leq C \cdot 2^{j(n+\beta)} w_1(B_j)^{-\alpha/n} w_2(B_j)^{-1/q} \frac{1}{|x|^{n+\beta}}.
\]

The rest of the proof is exactly the same as that of Theorem 1, we can get
\[
J_2 \leq C \|f\|_{H^{\alpha,p}(w_1,w_2)}^p.
\]
This completes the proof of Theorem 2.

5. Proof of Theorems 3

In [16], the author have already established the following propositions.

Proposition 5.1. Let \(w \in A_1, 0 < \beta < 1\). Then for any \(i \in \mathbb{Z}_+\), we have
\[
\|S_{\beta,2^i}(a)\|_{L^2_w} \leq C \cdot 2^{in/2}\|S_\beta(a)\|_{L^2_w}.
\]

Proposition 5.2. Let \(w \in A_1, 0 < \beta < 1\) and \(2 < q < \infty\). Then for any \(i \in \mathbb{Z}_+\), we have
\[
\|S_{\beta,2^i}(a)\|_{L^q_w} \leq C \cdot 2^{in/2}\|S_\beta(a)\|_{L^q_w}.
\]

Proposition 5.3. Let \(w \in A_1, 0 < \beta < 1\) and \(1 < q < 2\). Then for any \(i \in \mathbb{Z}_+\), we have
\[
\|S_{\beta,2^i}(a)\|_{L^q_w} \leq C \cdot 2^{in/q}\|S_\beta(a)\|_{L^q_w}.
\]

Using the above three estimates, we can prove the following result.

Proposition 5.4. Let \(w \in A_1, 0 < \beta < 1\) and \(1 < q < \infty\). Then for every \(\lambda > 2\), we have
\[
\|g_{\lambda,\beta}(a)\|_{L^q_w} \leq C \|a\|_{L^q_w}.
\]
Proof. From the definition, we readily see that
\[
g^{*}_{\lambda, \beta}(a)(x)^2 = \int_{\mathbb{R}^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} (A_{\beta}(a)(y, t))^2 \frac{dy dt}{(t + |x - y|)^{n+1}}
\]
\[
= \int_{0}^{\infty} \int_{|x - y| < t} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} (A_{\beta}(a)(y, t))^2 \frac{dy dt}{(t + |x - y|)^{n+1}}
\]
\[
+ \sum_{i=1}^{\infty} \int_{i}^{\infty} \int_{2^{i-1}t \leq |x - y| < 2^i t} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} (A_{\beta}(a)(y, t))^2 \frac{dy dt}{(t + |x - y|)^{n+1}}
\]
\[
\leq C \left[ S_{\beta}(a)(x)^2 + \sum_{i=1}^{\infty} 2^{-i\lambda n} S_{\beta, 2^i}(a)(x)^2 \right]. \quad (9)
\]

Applying Proposition 5.1–5.3 and Theorem D, we have
\[
\|g^{*}_{\lambda, \beta}(a)\|_{L^p_w} \leq C \left( \|S_{\beta}(a)\|_{L^p_w} + \sum_{i=1}^{\infty} 2^{-i\lambda n} \|S_{\beta, 2^i}(a)\|_{L^p_w} \right)
\]
\[
\leq C \left( \|S_{\beta}(a)\|_{L^p_w} + \sum_{i=1}^{\infty} 2^{-i\lambda n} \cdot 2^i \|S_{\beta}(a)\|_{L^p_w} \right)
\]
\[
\leq C \|a\|_{L^p_w} \left( 1 + \sum_{i=1}^{\infty} 2^{-i\lambda n} \cdot 2^i \right)
\]
\[
\leq C \|a\|_{L^p_w}.
\]

We are done. \qed

We are now in a position to give the proof of Theorem 3.

Proof of Theorem 3. As in the proof of Theorem 1, we write
\[
\sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2 \left\{ x \in C_k : |g^{*}_{\lambda, \beta}(f)(x)| > \sigma \right\}^{p/q}
\]
\[
\leq \sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2 \left\{ x \in C_k : \sum_{j=k-1}^{\infty} |\lambda_j| |g^{*}_{\lambda, \beta}(a_j)(x)| > \sigma/2 \right\}^{p/q}
\]
\[
+ \sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2 \left\{ x \in C_k : \sum_{j=-\infty}^{k-2} |\lambda_j| |g^{*}_{\lambda, \beta}(a_j)(x)| > \sigma/2 \right\}^{p/q}
\]
\[
= K_1 + K_2.
\]

Note that $0 < p \leq 1$ and $\lambda > 3 + (2\beta)/n > 2$. Applying Chebyshev’s inequality and Proposition 5.4, we have
\[ K_1 \leq \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha/p} \left( \sum_{j=k-1}^{\infty} |\lambda_j| \|g_{\lambda,\beta}(a_j)\chi_k\|_{L^p_{w_2}} \right)^p \]

\[ \leq \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha/p} \left( \sum_{j=k-1}^{\infty} |\lambda_j|^p \|g_{\lambda,\beta}(a_j)\|_{L^p_{w_2}} \right)^p \]

\[ \leq C \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha/p} \left( \sum_{j=k-1}^{\infty} |\lambda_j|^p \|a_j\|_{L^p_{w_2}} \right)^p \]

Changing the order of summation gives

\[ K_1 \leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^p \left( \sum_{k=-\infty}^{j+1} w_1(B_k)^{\alpha/p} w_1(B_j)^{-\alpha/p} \right). \]

Following the same lines as that of Theorem 1, we can show

\[ K_1 \leq C \|f\|_{H_{K^\alpha,p}(w_1,w_2)}^p. \]

We now turn to deal with \( K_2. \) In the proof of Theorem 2, for any fixed \( j \) with \( j \leq k - 2 \), we have already showed

\[ (S_{\beta}(a_j)(x))^2 \leq C \cdot \left( 2^{(n+\beta)\alpha/n} w_1(B_j)^{-\alpha/n} w_2(B_j)^{-1/q} \right)^2 |x|^{-2n-2\beta}. \quad (10) \]

We are going to estimate \( (S_{\beta,2\alpha}(a_j)(x))^2 \) for \( i = 1, 2, \ldots \). For any given \( (y, t) \in \Gamma_2(x), x \in B_k \setminus B_{k-1} \), then a simple calculation shows that \( t \geq \frac{1}{2n+2|x|}. \) Hence, by the inequality (3), we can get

\[ (S_{\beta,2\alpha}(a_j)(x))^2 \leq C \cdot \left( 2^{(n+\beta)\alpha/n} w_1(B_j)^{-\alpha/n} w_2(B_j)^{-1/q} \right)^2 \int_{|x|/2+2}^{\infty} \int_{|y-x|<2t} dydt \int_{|x|/2+2}^{\infty} dt \leq C \cdot \left( 2^{(n+\beta)\alpha/n} w_1(B_j)^{-\alpha/n} w_2(B_j)^{-1/q} \right)^2 2^{(3n+2\beta)|x|^{-2n-2\beta}}. \quad (11) \]

It follows immediately from the inequalities (9), (10) and (11) that

\[ \left( g_{\lambda,\beta}(a_j)(x) \right)^2 \leq C \cdot \left( 2^{(n+\beta)\alpha/n} w_1(B_j)^{-\alpha/n} w_2(B_j)^{-1/q} \right)^2 |x|^{-2n-2\beta} \left( 1 + \sum_{i=1}^{\infty} 2^{-i(\lambda n - 3n - 2\beta)} \right) \]

\[ \leq C \cdot \left( 2^{(n+\beta)\alpha/n} w_1(B_j)^{-\alpha/n} w_2(B_j)^{-1/q} \right)^2 |x|^{-2n-2\beta}, \]
where the last series is convergent since $\lambda > 3 + (2\beta)/n$. As a consequence,

$$g_{\lambda,\beta}^*(a_j)(x) \leq C \cdot 2^{j(n+\beta)}w_1(B_j)^{-\alpha/n}w_2(B_j)^{-1/q} \frac{1}{|x|^{n+\beta}}.$$ 

Again, the rest of the proof is exactly the same as that of Theorem 1, we can obtain

$$K_2 \leq C \|f\|_{H_k^{\alpha, p}(w_1, w_2)}^p.$$ 

Therefore, we conclude the proof of Theorem 3.

\[\square\]

**Remark.** The corresponding results for non-homogeneous weighted Herz-type Hardy spaces can also be proved by atomic decomposition theory. The arguments are similar, so the details are omitted here.

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