Clifford Algebra Approach to Superenergy Tensors

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Senovilla has recently defined an algebraic construction of a superenergy tensor $T\{A\}$ from any arbitrary tensor $A$, by structuring it as an $r$-fold form. This superenergy tensor satisfies automatically the dominant superenergy property. We present a more compact definition using the $r$-direct product Clifford algebra $\mathcal{C}_{p,q}$. This form for the superenergy tensors allows to obtain an easy proof of the dominant superenergy property valid for any dimension.

1 Introduction

The name of superenergy was first applied to the Bel-Robinson (BR) and Bel tensors defined from the conformal Weyl tensor and the Riemann tensor respectively. The motivation for this name is that they share many properties with energy-momentum tensors. The prefix super appears because they are rank-4 tensors instead of rank-2. Senovilla has recently defined an algebraic construction of superenergy tensors $T\{A\}$ from arbitrary seed tensors $A$, which unifies in a single procedure the BR and Bel tensors and many energy-momentum tensors from different physical fields. Given any tensor $A$, its superenergy tensor $T\{A\}$ satisfies the Dominant Superenergy Property (DSEP). These superenergy tensors have been used to study the interchange of superenergy between different fields and to provide a simple geometric criterion for the causal propagation of physical fields. Bergqvist has found an expression with 2-spinor formalism, valid for dimension 4, which leads to an elegant proof of the DSEP. In this article we introduce an alternative formulation for superenergy tensors using real Clifford algebra. Our expression is simpler and more compact than both, standard-tensorial and spinorial definitions, and it allows a proof of the DSEP, as elegant as the 2-spinors one and valid for any dimension.

2 Multivectors

Let us consider the tangent vector space $T(\mathcal{M})$ of some real manifold $\mathcal{M}$ of dimension $n$ and signature $\{p, q\}$, with $n = p + q$. A multivector is an element of the Grassmann or exterior algebra $\Lambda \equiv \Lambda(T(\mathcal{M}))$, i.e. it is a linear combination of scalar, vector, bivector, etc. It is over this tangent multivector space $\Lambda$ that Clifford algebra $\mathcal{C}_{p,q}$ is built, by endowing $\Lambda$ with the Clifford geometric product. The geometric product is an associative product which, in contrast to the exterior product, is specific to the metric.

To write a multivector $A \in \mathcal{C}_{p,q}$ in a complete basis of the linear space $\mathcal{C}_{p,q} = \Lambda$ we will use a multiindex, denoted in a latin capital letter,

$$A = A^I e_I$$
A basic operation on multivectors is the degree projection \( \langle A \rangle_s \), which selects from the multivector \( A \) its \( s \)-vector part (0=scalar, 1=vector, 2=bivector \ldots). There are two natural antiinvolutions, which are independent of the metric but are fixed by the graded structure of the multivectors space \( \Lambda \): Reversion, denoted with a tilde \( \widetilde{\cdot} \), and Clifford conjugation, with an overline \( \bar{\cdot} \). They both reverse the order of products \( AB = \bar{B} \bar{A} \), but reversion keeps vectors unchanged \( \widetilde{A} = A \), while Clifford conjugation changes the sign of vectors \( \bar{a} = -a \). Taking, for instance, a factorizable trivector

\[
(a \land b \land c)^\sim = c \land b \land a = -a \land b \land c,
\bar{a} \land b \land c = (-c) \land (-b) \land (-a) = a \land b \land c
\]

### 3 r-fold multivectors

An \( r \)-fold multivector \( A \) is an element of the \( r \)-direct product of the Clifford algebra \( \mathcal{C}_{p,q} \).

\[
A \in \mathcal{C}_{p,q}^r = \mathcal{C}_{p,q} \otimes \mathcal{C}_{p,q} \otimes \cdots \otimes \mathcal{C}_{p,q} = \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_r \equiv \Lambda^r
\]

Each multivector space \( \Lambda \) will be called a block. Its expression in a basis, using multiindices, is

\[
A = A^{I_1} I_2 \cdots I_r e_{I_1} \otimes e_{I_2} \otimes \cdots \otimes e_{I_r}.
\]

The natural associative product defined between two \( r \)-fold multivectors is the \( r \)-direct Clifford product, which involves an independent Clifford product in each block.

\[
AB = \left( A^{I_1} \cdots I_r e_{I_1} \otimes \cdots \otimes e_{I_r} \right) \left( B^{J_1} \cdots J_r e_{J_1} \otimes \cdots \otimes e_{J_r} \right)
\]

\[
= A^{I_1} \cdots I_r B^{J_1} \cdots J_r (e_{I_1} e_{J_1}) \otimes \cdots \otimes (e_{I_r} e_{J_r})
\]

To shorten expressions we introduce multi-fold multiindices. These collect a list of multiindices and are denoted with an underline, \( \underline{I} = \{ I_1, I_2, \cdots, I_s \} \). Using them the expression (1) simplifies to \( A = \Lambda^2 e_{\underline{I}} \). Multi-fold multiindices with \( s < r \) will be used to make a single block explicit.

\[
A = A^{L} \otimes e_{L_1} \otimes e_{I_1} \otimes e_{L_2} \equiv e_{L} \otimes e_{I_1} \otimes e_{L_2}
\]

where \( \underline{C} = \{ I_1, \cdots, I_{s-1} \} \) and \( \underline{D} = \{ I_{s+1}, \cdots, I_r \} \). Note that \( A^{L} \otimes e_{I_1} \otimes e_{L_2} \) is not just a scalar component but a multivector.

The basic operations of degree projection, reversion and Clifford conjugation, acting on multivectors, can be extended to \( r \)-fold multivectors. Thus, we define the \( r \)-fold degree projection, the \( r \)-fold reversion and the \( r \)-fold Clifford conjugation as the result of applying the corresponding operation independently on every block, and will use the same notation as with simple multivectors.

\[
\langle A \rangle_s = A^{I_1} I_2 \cdots I_r \langle e_{I_1} \rangle_s \otimes \langle e_{I_2} \rangle_s \otimes \cdots \otimes \langle e_{I_r} \rangle_s,
\]

\[
\widetilde{A} = A^{I_1} I_2 \cdots I_r \widetilde{e}_{I_1} \otimes \widetilde{e}_{I_2} \otimes \cdots \otimes \widetilde{e}_{I_r}
\]

and

\[
\bar{A} = A^{I_1} I_2 \cdots I_r \bar{e}_{I_1} \otimes \bar{e}_{I_2} \otimes \cdots \otimes \bar{e}_{I_r}
\]
4 Superenergy tensors

Given an arbitrary tensor
\[ \hat{A} = \hat{A}^{\mu_1 \cdots \mu_s} e_{\mu_1} \otimes \cdots \otimes e_{\mu_s}, \]
the procedure defined by Senovilla gives an associated \textit{basic superenergy tensor} \( T\{A\} \). The seed tensor \( \hat{A} \) is treated as an \( r \)-fold form, by reordering and grouping antisymmetric indices in separated blocks. Thus, the reordered tensor \( A \) is an \( r \)-fold \((n_1, n_2, \ldots, n_r)\)-form.

\[ A \in \Lambda^{n_1} \otimes \Lambda^{n_2} \otimes \cdots \otimes \Lambda^{n_r} \subset \,^r\mathcal{C}_p,q \]

We say that \( A \) is a \textit{degree defined} \( r \)-fold multivector.

An energy condition has sense only for Lorentzian manifolds, that is, for signatures \( \{1, p\} \) or \( \{p, 1\} \). Here we will use the signature \( \{p, 1\} \), although everything can be done in the reversal signature without any trouble.

Senovilla’s definition for the basic superenergy tensor has the following form
\[ T\{A\} = \frac{1}{2} \sum_P A_P \times A_P \quad (2) \]
where \( A_P \) denote the \( r \)-fold form \( A \) transformed by a combination of duals: that is, \( P \) codifies taking the Hodge dual on some blocks and keeping the rest of blocks unchanged. Thus, the summation runs through the \( 2^r \) possible combinations. The cross product \( A \times A \) is defined here as the contraction in every block of all the indices except one in each factor. Therefore, \( T\{A\} \) has \( r \) pairs of indices:

\[ (A \times A)_{\mu_1 \nu_1 \cdots \mu_r \nu_r} = \left( \prod_{i=1}^{r} \frac{1}{(n_i - 1)!} \right) A_{\mu_1 \lambda_1 \cdots \lambda_{n_1} \cdots \mu_r \lambda_{n_2} \cdots \lambda_{n_r}} A_{\nu_1 \lambda_1 \cdots \lambda_{n_1} \cdots \nu_r \lambda_{n_2} \cdots \lambda_{n_r}} \quad (3) \]

The use of the \( r \)-fold Clifford algebra \( \,^r\mathcal{C}_p,1 \) allows us to introduce an alternative and much more compact definition of the basic superenergy tensor. Our expression, inspired by the Clifford geometric algebra formulation of the electromagnetic stress-energy, defines the tensor directly applied to \( r \) pairs of vectors.

\[ T\{A\}(u_1 \otimes u_2 \otimes \cdots \otimes u_r)(v_1 \otimes v_2 \otimes \cdots \otimes v_r) = (-1)^r \frac{1}{2} \left( A (u_1 \otimes \cdots \otimes u_r) \overline{A} (v_1 \otimes \cdots \otimes v_r) \right) \quad (4) \]

If we are interested in its components in a basis \( \{e_\mu\} \), we simply apply the tensor to the corresponding basis elements.

\[ T\{A\}_{\mu_1 \nu_1 \cdots \mu_r \nu_r} = (-1)^r \frac{1}{2} \left( A (e_{\mu_1} \otimes \cdots \otimes e_{\mu_r}) \overline{A} (e_{\nu_1} \otimes \cdots \otimes e_{\nu_r}) \right) \quad (5) \]

In the remaining part of the section we will show that both expressions correspond to the same object. To this purpose let us start with the Clifford algebra definition (4) and expand it to obtain the standard-tensorial definition (2). Let us concentrate on a single arbitrary block. Using multi-fold multiindices, the components (5) can be written as

\[ T\{A\}_{\mu_1 \nu_1 \cdots \mu_r \nu_r} = \]
\[ (-1)^r \frac{1}{2} \left\langle e_{e(\mu_1,\ldots,\mu_{r-1})} e_{e(v_{1},\ldots,v_{r-1})} \right\rangle_0 \left\langle \overline{Ae\cdot e_{\mu_1}} \overline{Ae\cdot e_{\nu_1}} \right\rangle_0 \left\langle e_{e(\mu_{r+1},\ldots,\mu_r)} e_{e(v_{r+1},\ldots,v_r)} \right\rangle_0 \]

To reexpress the result in that ith block we take into account two essential facts. The first is that the Clifford product of any multivector with a vector can be splitted into inner and exterior products.

\[ A^{\mathbb{C}D} e_{\mu_i} = A^{\mathbb{C}D} \cdot e_{\mu_i} + A^{\mathbb{C}D} \wedge e_{\mu_i} \]

With this expansion the block \( \left\langle (A^{\mathbb{C}D} e_{\mu_i}) (\overline{Ae\cdot e_{\nu_i}}) \right\rangle_0 \) splits also into 2 terms, since the cross terms vanish.

\[ \left\langle A^{\mathbb{C}D} e_{\mu_i} \overline{Ae\cdot e_{\nu_i}} \right\rangle_0 = \left\langle (A^{\mathbb{C}D} \cdot e_{\mu_i}) (\overline{Ae\cdot e_{\nu_i}}) \right\rangle_0 + \left\langle (A^{\mathbb{C}D} \wedge e_{\mu_i}) (\overline{Ae\cdot e_{\nu_i}}) \right\rangle_0 \]

The second fact is that an exterior product can be written, with the help of Hodge duality, as an inner product. Applying it, the second term has the same structure as the first term, but where in the first we have the original \( n \)-form, \( A^{\mathbb{C}D} \), in the second we have the dual \( (n-n_i) \)-form, \( *A^{\mathbb{C}D} \).

\[ \left\langle A^{\mathbb{C}D} e_{\mu_i} \overline{Ae\cdot e_{\nu_i}} \right\rangle_0 = \left\langle (A^{\mathbb{C}D} \cdot e_{\mu_i}) (\overline{Ae\cdot e_{\nu_i}}) \right\rangle_0 + \left\langle ([*A^{\mathbb{C}D}] \cdot e_{\mu_i}) ([*Ae\cdot e_{\nu_i}] \cdot e_{\mu_i}) \right\rangle_0 \]

Repeating this expansion for every block we obtain \( 2^r \) terms, corresponding to all possible combinations that take the dual in some blocks and leave unchanged the rest.

\[ T \{ A \}_{\mu_1\nu_1\ldots\mu_r\nu_r} = \frac{1}{2} \sum_\mathcal{P} (-1)^r \left\langle (A_{\mathcal{P}} \cdot (e_{\mu_1} \circ \cdots \circ e_{\mu_r})) (\overline{A_{\mathcal{P}} \cdot (e_{\nu_1} \circ \cdots \circ e_{\nu_r})}) \right\rangle_0 \tag{6} \]

where the dot denotes the inner product in every block, and we have used \( \mathcal{P} \) again to indicate each combination of Hodge duals. Finally, comparing this last expression with (3), we only have to check that the terms of this summation (8) coincide with the Senovilla’s cross product (3).

\[ (-1)^r \left\langle (A \cdot e_{\mu_1,\ldots,\mu_r}) (\overline{A} \cdot e_{\nu_1,\ldots,\nu_r}) \right\rangle_0 = (A \times A)_{\mu_1\nu_1\ldots\mu_r\nu_r} \]

This can be seen by realizing that the dot products in (3) fix one index in each factor for each block. The scalar projection of the product selects the terms that correspond to the contraction of the rest of the indices. It is easy to check that the signs coincide. Then, the proof is complete.

\[ (-1)^r \left\langle (A \cdot e_{\mu})(\overline{A} \cdot e_{\nu}) \right\rangle_0 = \left\langle (A \cdot e_{\mu})(e_{\nu} \cdot \overline{A}) \right\rangle_0 = \left\langle (A \cdot e_{\mu})(\widetilde{e_{\nu}} \cdot A) \right\rangle_0 \]

5 Dominant superenergy property (DSEP)

In this section we present a simple proof of the DSEP for the superenergy tensor \( T \{ A \} \), using its expression in the \( r \)-fold Clifford algebra \( \mathcal{D}_{p,1} \). A superenergy tensor satisfies the DSEP if for all collection \( \{ u_i, v_i \} \) of causal and future-pointing (f-p) vectors

\[ T \{ A \}(u_1 \circ \cdots \circ u_r)(v_1 \circ \cdots \circ v_r) = (-1)^r \frac{1}{2} \left\langle A (u_1 \circ \cdots \circ u_r) \overline{A} (v_1 \circ \cdots \circ v_r) \right\rangle_0 \geq 0 \]
Let us recall, first, that a time-like f-p vector $u$ can always be expressed as the result of applying a local Lorentz transformation and a dilation to a chosen unitary time-like f-p vector $e_0$. This transformation is performed by means of an even multivector inside the same Clifford algebra.

$$u = R_u e_0 \overline{R_u}, \quad R_u \in \mathcal{A}_{p,1}^+$$

The same expression applies for a null vector $u$ with $R_u$ a singular transformation. For the tensor product of $r$ f-p vectors $u \equiv u_1 \otimes u_2 \otimes \cdots \otimes u_r \in \mathcal{A}_{p,1}^r$, the operator which implements the transformation also belongs to $\mathcal{A}_{p,1}^r$

$$R_u \equiv R_{u_1} \otimes R_{u_2} \otimes \cdots \otimes R_{u_r} \in \mathcal{A}_{p,1}^r, \quad e_0 \equiv e_0 \otimes \cdots \otimes e_0, \quad \text{so} \quad u = R_u e_0 \overline{R_u}$$

Our proof of the DSEP proceeds in two steps. First, using the operators $R_u \in \mathcal{A}_{p,1}^r$, we express the result of applying $T\{A\}$ to any set of $2r$ f-p vectors $u, v$ as the $\{0, \ldots, 0\}$ component of another superenergy tensor $T\{A'\}$.

$$2T\{A\}(u)(v) = (-1)^r \left\langle A \ u \overline{A} \ v \right\rangle_0 = (-1)^r \left\langle A \left( R_u e_0 \overline{R_u} \right) \overline{A} \left( R_u e_0 \overline{R_u} \right) \right\rangle_0$$

$$= (-1)^r \left\langle \left( R_u A R_u \right) e_0 \overline{\left( R_u A R_u \right) e_0} \right\rangle_0 = 2T\{A'\}(e_0)(e_0)$$

where $A' \equiv R_u A R_u \in \mathcal{A}_{p,1}^r$ is also an $r$-fold multivector.

The second step proves that this component is non negative for any $A' \in \mathcal{A}_{p,1}^r$.

$$T\{A'\}(e_0)(e_0) = \frac{1}{2} (-1)^r \left\langle A' e_0 \overline{A'} e_0 \right\rangle_0 \geq 0 \quad \forall A' \in \mathcal{A}_{p,1}^r$$

To see this we realize a splitting of $A'$ into parts orthogonal and parallel to the direction $e_0$. This splitting corresponds to the isomorphism of linear spaces, though not as algebras,

$$\mathcal{A}_{p,1}^r \cong \mathcal{A}_{p,0}^r \otimes \mathcal{A}_{0,1}^r$$

where $\mathcal{A}_{0,1}$ is the space generated by vector $e_0$ and $\mathcal{A}_{p,0}$ is the space generated by the Euclidean space orthogonal to $e_0$. A basis for $\mathcal{A}_{0,1}$ has the $2^r$ elements:

$$\{e_P\} = \{1 \otimes \cdots \otimes 1 \otimes e_0, \ 1 \otimes \cdots \otimes e_0 \otimes 1, \ \ldots, \ e_0 \otimes \cdots \otimes e_0 \otimes e_0\}$$

Now, expanding $A'$ in the basis $\{e_P\}$ of $\mathcal{A}_{0,1}$ with components in $\mathcal{A}_{p,0}$

$$A' = A' e_P, \quad A' e_P \in \mathcal{A}_{p,0}, \quad e_P \in \mathcal{A}_{0,1}$$

we finally complete the proof

$$(-1)^r \left\langle A' e_0 \overline{A'} e_0 \right\rangle_0 = \left\langle A' e_P \overline{A'} e_P \right\rangle_0 (e_0 \overline{A e_0}) e_0^{-1} = \left\langle A' e_P e_P \overline{A e_0} e_0^{-1} \right\rangle_0$$

$$= \sum_P \left\langle A' e_P e_P \right\rangle_0 \geq 0$$

The summation is always positive since, $\forall B \in \mathcal{A}_{p,0}$, which is an algebra generated by an Euclidian metric, $\left\langle B \overline{B} \right\rangle_0$ is a positive defined norm.
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