FREE PICK FUNCTIONS: REPRESENTATIONS, ASYMPTOTIC BEHAVIOR AND MATRIX MONOTONICITY IN SEVERAL NONCOMMUTING VARIABLES

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ABSTRACT. We extend the study of the Pick class, the set of complex analytic functions taking the upper half plane into itself, to the noncommutative setting. R. Nevanlinna showed that elements of the Pick class have certain integral representations which reflect their asymptotic behavior at infinity. Löwner connected the Pick class to matrix monotone functions. We generalize the Nevanlinna representation theorems and Löwner’s theorem on matrix monotone functions to the free Pick class, the collection of functions that map tuples of matrices with positive imaginary part into the matrices with positive imaginary part which obey the free functional calculus.

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1. Introduction

Let $\mathbb{H} \subset \mathbb{C}$ denote the complex upper half plane. That is,

$$\mathbb{H} = \{ z \in \mathbb{C} | \text{Im} \, z > 0 \}.$$

The Pick class is the set of analytic functions $f : \mathbb{H} \rightarrow \mathbb{H}$. The elements of the Pick class are called Pick functions.

Rolf Nevanlinna showed that a subset of the Pick class satisfying an asymptotic condition at infinity is exactly parametrized by positive Borel measures on the real line.

**Theorem 1.1** (R. Nevanlinna [48]). Let $h : \mathbb{H} \rightarrow \mathbb{C}$. There exists a finite Borel positive measure $\mu$ on $\mathbb{R}$ such that

$$h(z) = \int \frac{1}{t - z} \, d\mu(t)$$

if and only if $h$ is in the Pick class and

$$\liminf_{s \rightarrow \infty} s \, |h(is)| < \infty.$$  

Moreover, for any Pick function $h$ satisfying Equation 1.2, the measure $\mu$ in Equation 1.1 is uniquely determined.
The map taking a finite positive Borel measure $\mu$ to a Pick function $h$ via the correspondence in Equation (1.1) is called the Cauchy transform. A naïve interpretation of the asymptotic condition at infinity, Equation (1.2), is that the first residue exists. This was studied rigorously in the guise of a conformally equivalent condition on self maps of the unit disk by Julia [39] and Carathéodory [18].

Nevanlinna applied Theorem 1.1 to the Hamburger moment problem: Given a sequence of real numbers $(\rho_i)_{i=0}^{\infty}$, when does there exist a measure $\mu$ such that for each $i \in \mathbb{N}$, $\rho_i$ is the $i$-th moment of the measure $\mu$, i.e.,

$$\rho_i = \int x^i d\mu.$$

**Theorem 1.2** (R. Nevanlinna [48]). Let $(\rho_i)_{i=1}^{\infty}$ be a sequence of real numbers. The following are equivalent.

1. There is a finite positive Borel measure $\mu$ on $\mathbb{R}$ so that, for each $i \in \mathbb{N}$,

$$\rho_i = \int x^i d\mu.$$

2. There is a Pick function $h$ such that, for every $N \in \mathbb{N}$,

$$h(z) = \sum_{i=0}^{N} \frac{1}{z^{i+1}} \rho_i + O(\frac{1}{|z|^{N+2}}).$$

3. The infinite matrix Hankel matrix

$$A = [\rho_{i+j}]_{0 \leq i,j \leq \infty} = \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots \\ \rho_1 & \rho_2 & \rho_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is positive semidefinite in the sense that, for each in $N \in \mathbb{N}$, the truncated Hankel matrix $[\rho_{i+j}]_{0 \leq i,j \leq N}$ is positive semidefinite.

Pick functions also correspond to matrix monotone functions via Löwner’s theorem. Given a function $f : (a, b) \to \mathbb{R}$, we extend $f$ via the functional calculus to self-adjoint matrices $A$ with spectrum in $(a, b)$ by taking the diagonalization of $A$ by a unitary matrix $U$, that is,

$$A = U^* \begin{bmatrix} \lambda_1 & & \\
& \lambda_2 & \\
& & \ddots \end{bmatrix} U,$$

and defining

$$f(A) = U^* \begin{bmatrix} f(\lambda_1) & & \\
& f(\lambda_2) & \\
& & \ddots \end{bmatrix} U.$$

(1.3)
A function \( f : (a, b) \rightarrow \mathbb{R} \) is called \textit{matrix monotone} if
\[
A \preceq B \Rightarrow f(A) \preceq f(B)
\]
where \( A \preceq B \) means that \( B - A \) is positive semidefinite.

The condition that a function \( f : (a, b) \rightarrow \mathbb{R} \) be matrix monotone is much stronger than that \( f \) should be monotone in the ordinary sense. For example, let the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) be given by the formula
\[
f(x) = x^3.
\]
The function \( f \) is monotone on all of \( \mathbb{R} \). Note that
\[
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \leq \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}
\]
which is not positive semidefinite since \( \det \begin{pmatrix} 9 & 4 \\ 4 & 1 \end{pmatrix} = -5 < 0 \), and so \( f(x) = x^3 \) is not matrix monotone even though it is monotone on all of \( \mathbb{R} \).

In [41], Charles Löwner showed the following theorem.

**Theorem 1.3** (Löwner [41]). Let \( f : (a, b) \rightarrow \mathbb{R} \) be a bounded Borel function. If \( f \) is matrix monotone, then \( f \) analytically continues to the upper half plane as a function in the Pick class.

For a modern treatment of Löwner’s theorem, see e.g. [24,13,14].

Löwner’s theorem can be used to identify whether or not many classically important functions are matrix monotone. For example, \( x^{1/3}, \log x, \) and \( -\frac{1}{x} \) are matrix monotone on the interval \((1,2)\), but \( x^3 \) and \( e^x \) are not.

Interpreting Theorem [13] in the context of Nevanlinna’s solution to the Hamburger moment problem, we obtain the following corollary, which will guide our study of matrix monotone functions in several variables.

**Corollary 1.4.** Let \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) be a power series which converges on a neighborhood of the closed disk \( \overline{D} \). The function \( f \) is matrix monotone if and only if the infinite Hankel matrix \( [a_{i+j+1}]_{0 \leq i,j \leq \infty} \geq 0 \).
Via the connection to moment problems and matrix monotonicity, the theory of Pick functions has deep and well-studied consequences for science and engineering. John von Neumann and Eugene Wigner applied Löwner’s theorem to the theory of quantum collisions [65]. Other applications include quantum data processing [9], wireless communications [38, 16] and engineering [10, 49].

We execute the program above in several noncommuting variables. We now briefly describe the free functional calculus, which gives a meaningful notion of a noncommutative function.

1.1. The free functional calculus. The functional calculus in several variables is less well understood than that given in (1.3) because it is noncommutative. The free functional calculus is modeled on the theory of free polynomials evaluated on tuples of matrices. For this purpose, free polynomials have three important properties which we will now illustrate with an example.

Consider the free polynomial in two variables, $$p(X, Y) = XY + 7XYX.$$ First, given two $n$ by $n$ matrices with entries in $\mathbb{C}$, $X, Y \in \mathcal{M}_n(\mathbb{C})$, the value of $p$ at the point $(X, Y)$, $p(X, Y)$ is again a matrix in $\mathcal{M}_n(\mathbb{C})$. This says that $p$ is a graded function.

Second, for two $n$ by $n$ matrices $X_1, Y_1 \in \mathcal{M}_n(\mathbb{C})$, and two $m$ by $m$ matrices $X_2, Y_2 \in \mathcal{M}_m(\mathbb{C})$, consider the calculation

$$p\left(\left(\begin{array}{c} X_1 \\ X_2 \end{array}\right), \left(\begin{array}{c} Y_1 \\ Y_2 \end{array}\right)\right) = \left(\begin{array}{c} X_1 \\ X_2 \end{array}\right) \left(\begin{array}{c} Y_1 \\ Y_2 \end{array}\right) + 7 \left(\begin{array}{c} X_1 \\ X_2 \end{array}\right) \left(\begin{array}{c} Y_1 \\ Y_2 \end{array}\right) \left(\begin{array}{c} X_1 \\ X_2 \end{array}\right),$$

$$= \left(\begin{array}{c} X_1Y_1 + 7X_1Y_1X_1 \\ X_2Y_2 + 7X_2Y_2X_2 \end{array}\right),$$

$$= \left(\begin{array}{c} p(X_1, Y_1) \\ p(X_2, Y_2) \end{array}\right).$$

The identity

$$p\left(\left(\begin{array}{c} X_1 \\ X_2 \end{array}\right), \left(\begin{array}{c} Y_1 \\ Y_2 \end{array}\right)\right) = \left(\begin{array}{c} p(X_1, Y_1) \\ p(X_2, Y_2) \end{array}\right)$$

says that $p$ respects direct sums.

Third, given two matrices $X, Y \in \mathcal{M}_n(\mathbb{C})$, and an invertible matrix consider the following calculation of the value of $p(S^{-1}XS, S^{-1}YS)$:

$$p(S^{-1}XS, S^{-1}YS) = S^{-1}XSS^{-1}YS + 7S^{-1}XSS^{-1}YSS^{-1}XS,$$

$$= S^{-1}XYS + 7S^{-1}XYS,$$

$$= S^{-1}(XY + 7XYXS),$$

$$= S^{-1}p(X, Y)S.$$
The identity
\[ p(S^{-1}XS, S^{-1}YS) = S^{-1}p(X, Y)S \]
says that \( p \) respects similarity.

These three properties, to be graded, respect direct sums and similarity, constitute the definition of a free function, which we now describe precisely.

Let \( \mathcal{M}^d \) denote the \( d \)-dimensional matrix universe, which is defined by the equation
\[ \mathcal{M}^d = \bigcup_{n=1}^{\infty} \mathcal{M}^d_n(\mathbb{C}). \]

**Definition 1.5.** A set \( D \subset \mathcal{M}^d \) is called a free set if satisfies the following conditions.

1. **\( D \) is closed with respect to direct sums:**
   - If \( X = (X_1, \ldots, X_d) \in D \) and \( Y = (Y_1, \ldots, Y_d) \in D \) if and only if \( (X, Y) = ((X_1, Y_1), \ldots, (X_d, Y_d)) \in D \).
2. **\( D \) is closed with respect to unitary similarity:**
   - If \( X \in D \cap \mathcal{M}^d_n, U \in U_n \), then
     \[ U^*XU = (U^*X_1U, \ldots, U^*X_dU) \in D. \]

Here, \( U_n \) denotes the unitary matrices of size \( n \).

**Definition 1.6.** Let \( D \) be a free set. Let \( f : D \to \mathcal{M}^1 \) be a function. We say that \( f \) is a free function if it satisfies the following conditions.

1. **\( f \) is graded:** If \( X \in D \cap \mathcal{M}^d_n \), then \( f(X) \in \mathcal{M}^1_n \).
2. **\( f \) respects direct sums:** If \( X, Y \in D \) then
   \[ f \left( \begin{pmatrix} X \\ Y \end{pmatrix} \right) = \begin{pmatrix} f(X) \\ f(Y) \end{pmatrix}. \]
3. **\( f \) respects similarity:**
   - If \( S \in GL_n \), and \( X \in D \) such that \( S^{-1}XS \in D \),
   \[ f(S^{-1}XS) = S^{-1}f(X)S. \]

Here, \( GL_n \) denotes the invertible matrices of size \( n \).

It is known that for a free set \( U \), if \( U \cap \mathcal{M}_n \) is open at each level and \( f \) is a locally bounded free function, then for
\[ \hat{U} = \{ z | z \in \sigma(A), A \in U \} \]
there is a unique holomorphic \( \hat{f} : \hat{U} \to \mathbb{C} \) such that \( f(A) = \hat{f}(A) \) for all \( A \in U \). As in the one variable case, if \( U \cap \mathcal{M}_n^d \) is open, then \( f \) is holomorphic on \( U \cap \mathcal{M}_n^d \) as a function of \( dn^2 \) variables.
We note that free polynomials are free functions. Free rational functions also give free functions off of their singular sets. For example,

\[ f(X, Y) = (Y^2 + 5)(1 - X)^{-1}Y \]

is a free function defined on the free set

\[ \{X, Y \in \mathcal{M}^2 | 1 \notin \sigma(X)\} \].

The study of functions in this context is known as free analysis.

1.2. Free Pick functions. We specify an ordering on tuples of matrices where for \( A = (A_1, \ldots, A_d) \) and \( B = (B_1, \ldots, B_d) \), the statement \( A > B \) means \( A_i - B_i \) is strictly positive definite for each \( 1 \leq i \leq d \), and \( A \geq B \) means \( A_i - B_i \) is positive semidefinite for each \( 1 \leq i \leq d \).

We consider the free Pick class \( \mathcal{P}_d \), the set of free functions defined on the domain

\[ \Pi^d := \{X \in \mathcal{M}^d | \text{Im} X_i = \frac{1}{2}(X_i - X_i^*) > 0, i = 1, \ldots, d\} \]

with range in \( \overline{\Pi}^1 \). These functions directly generalize multivariable Pick functions, maps from the polyupperhalfplane \( \Pi_1^d \) to the upper halfplane \( \overline{\Pi}_1^1 \), as

\[ \Pi_1^d = \mathbb{H}^d. \]

Agler, Tully-Doyle, and Young showed the following generalization of Theorem 1.1 [7], presented here in two variables.

**Theorem 1.7** (Type I representation theorem). Let \( h \) be a function defined on \( \Pi^2_1 \). Then there exist a Hilbert space \( \mathcal{H} \), a self-adjoint operator \( A \) on \( \mathcal{H} \), a vector \( v \in \mathcal{H} \), and a positive semidefinite contraction \( Y \) on \( \mathcal{H} \) so that

\[ h(z) = \langle (A - z_1Y - z_2(1 - Y))^{-1}v, v \rangle \]

if and only if \( h \) is a Pick function and

\[ \lim \inf_{s \to \infty} s |h(is, is)| < \infty. \]

We show that the above theorem also holds for free Pick functions. Our Theorem 5.9 contains a precise analogue of Theorem 1.7 among more general representations. That Theorem 1.7 can be extended in this way is an example of noncommutative lifting principle, a guiding principle that states that if a theorem holds by virtue of operator theoretic methods following Agler’s seminal paper of 1990, *On the representation of certain holomorphic functions defined on a polydisc*, then it will hold for free functions. Our proofs of these theorems illustrate the theme of lifting functions from varieties to whole domains which
began with Cartan’s theorems A and B [19] and continued in [22] with bounds and later more precisely by [23, 40], etc.

To give a concrete example of the noncommutative lifting principle, consider a Type I function given by the formula

\[ h(z) = \langle (A - z_1 Y - z_2 (1 - Y))^{-1} \alpha, \alpha \rangle \]

as in Theorem 1.7. We can extend \( h \) to a function \( H \) defined on \( \Pi^2 \). Given a \( Z \in \Pi^2_n \) we obtain \( H(Z) \) via the formula

\[ H(Z) = (\alpha \otimes I_n)^*(A \otimes I_n - Y \otimes Z_1 - (1 - Y) \otimes Z_2)^{-1}(\alpha \otimes I_n). \quad (1.4) \]

This function is well-defined on \( \Pi^2 \) and

\[ H|_{\Pi^2_n} = h. \]

We show that free Pick functions \( H : \Pi^d \to \overline{\Pi}^d \) that restrict to Type I Pick functions on \( \Pi^d_1 \) have representations analogous to that in \((1.4)\).

1.3. Free matrix monotonicity. Löwner’s theorem on matrix monotone functions has been generalized to several commuting functional calculi. Agler, McCarthy and Young proved that matrix monotone functions defined on commuting tuples of matrices extend to functions in the Löwner class [21]. Others have studied matrix monotonicity on an alternate commutative functional calculus involving tensor products [25, 43, 58].

We prove a free analogue of Löwner’s theorem in several noncommuting variables. We consider functions on domains contained in the real matrix universe,

\[ \mathcal{R}^d := \{ X \in \mathcal{M}^d | X_i = X_i^*, i = 1 \ldots d \}. \]

A free set \( D \subset \mathcal{R}^d \) is a real free domain if \( D \cap \mathcal{R}^d_n \) is open in the space \( \mathcal{R}^d_n \). We prove Löwner’s theorem for real analytic free functions on convex free sets \( D \), free sets such that \( D_n = D \cap \mathcal{M}^d_n \) is convex for all \( n \). Since a free function is analytic if it is locally bounded [42], a real free function \( f \) is real analytic on \( D \) if for all \( X_0 \in D \) there is a bounded free function defined on a domain containing the set \( \{ X \in \mathcal{M}^d_{nk} | \| X - X_0 \otimes I_k \| < \varepsilon \} \) which agrees with \( f \) on \( D \). We give a formal, intrinsically real definition of real analytic free functions in terms of powers series in Section 2.1.3.

A real analytic free function is matrix monotone if

\[ X \preceq Y \Rightarrow f(X) \preceq f(Y). \]

The following is our generalization of Löwner’s Theorem.
Theorem 1.8. Let \( D \) be a convex free set of \( d \)-tuples of self-adjoint matrices. A real analytic free function \( f : D \to \mathbb{R} \) is matrix monotone if and only if \( f \) analytically continues to \( \Pi^d \) as a function in the free Pick class.

This follows from Theorem 4.3 applied to functions on convex domains.

Convex free sets have a rigid structure because they are simultaneously free sets and convex sets at each level. Convex free sets have been an object of recent work following the trend in semidefinite programming. For example, Helton and McCullough [35] showed that semialgebraic convex sets are LMI domains, a generalization of Lasserre’s result in the commutative case [47]. An LMI domain is the set of tuples of self-adjoint \( X_i \) such that \( \sum A_i \otimes X_i \leq A_0 \) where each \( A_i \) is symmetric.

We remark that the representations which correspond to our free analogue of Theorem 1.7 can be used to manufacture concrete formulas for monotone functions similar to the representations of rational matrix convex functions in work of Helton, McCullough and Vinnikov [36], which show that rational matrix convex functions are all obtained by taking the Schur complement of a monic LMI. In the classical theory, there is a strong connection between convex and monotone functions [13, 45], which seems to be suggested again here.

We now introduce the machinery for understanding free power series. Let \( \mathcal{I} \) denote the set of words in the letters \( x_1, \ldots, x_d \). The set \( \mathcal{I} \) is equipped with an involution \( \ast \) which reverses the letters in a word. For example,

\[
(x_1 x_2)^\ast = x_2 x_1.
\]

For a word \( w \in \mathcal{I} \), we define \( X^w \) recursively. For the empty word \( e \), \( X^e = I \) and \( X^{x_k w} = X_k X^w \). For example, \( (X_1, X_2)^{x_1 x_2 x_1} = X_1 X_2 X_1 \). A free power series is an expression of the form

\[
f(X) = \sum_{I \in \mathcal{I}} c_I X^I.
\]

To prove Löwner’s theorem, we prove an analogue of Corollary 1.4 for free power series.

Theorem 1.9. Let \( f(X) = \sum_{I \in \mathcal{I}} c_I X^I \) be a free power series in \( d \) noncommuting variables which converges for all \( \|X\| \leq d + \epsilon \). The function \( f \) is matrix monotone on the set of \( X \) such that \( \|X\| \leq \frac{1}{\epsilon} \) if and only if for \( 1 \leq k \leq d \), the infinite matrices \([c_I^\ast x_k J]_{I, J \in \mathcal{I}}\) are positive semidefinite.

The preceding is proved as Theorem 4.16.
1.3.1. Löwner’s theorem in one variable. We now sketch the proof (in one variable) of Corollary 1.4 to describe ideas important to the proof in several variables.

For real free analytic functions on convex sets, matrix monotonicity is equivalent to local matrix monotonicity. Let \( Df(X)[H] \) be the Gâteaux derivative at a matrix \( X \) in the direction \( H \). A free function is locally matrix monotone when

\[
Df(X)[H] \geq 0
\]

for all \( X \in \mathcal{R}^d \), whenever \( H \in \mathcal{R}^d \) such that \( H \geq 0 \). Our proof of Theorem 1.8 constructs a formula for the derivative of a locally monotone real analytic free function that can be used to construct an analytic continuation of the function. We call these formulas models.

Given the power series of a monotone function, we construct an explicit model for that function as follows. Consider an analytic function defined by a power series in one variable

\[
f(x) = \sum_{i=0}^{\infty} a_i x^i
\]

defined on a neighborhood of the closed disk. We seek to study the formula,

\[
f'(x) = \left( \frac{1}{x^2} \right)^\ast \left[ \begin{array}{cccc} a_1 & a_2 & a_3 & \ldots \\ a_2 & a_3 & \ldots & \vdots \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{array} \right] \left( \frac{1}{x^2} \right),
\]

in the free case. The derivative of \( f \) evaluated via the functional calculus defined by (1.3) at a matrix \( X \) in the direction \( H \) is given by the formula,

\[
Df(X)[H] = \left( \frac{1}{X^2} \right)^\ast \left[ \begin{array}{cccc} a_{11} & a_{12} & a_{13} & \ldots \\ a_{12} & a_{22} & a_{23} & \ddots \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{array} \right] \left( \frac{1}{X^2} \right). \tag{1.6}
\]

We establish that

\[
\left[ \begin{array}{cccc} a_1 & a_2 & a_3 & \ldots \\ a_2 & a_3 & \ldots & \vdots \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{array} \right] \geq 0. \tag{1.7}
\]

When (1.7) is satisfied, the formula (1.6) for \( f' \) implies that \( f \) can be analytically continued to the upper half plane as a Pick function via the theory of models [3].

In one variable, a Hamburger model for a function defined on a neighborhood of the closed disk is an expression for the derivative of the form

\[
Df(X)[H] = \left( \frac{1}{X^2} \right)^\ast \left[ \begin{array}{cccc} a_{11} & a_{12} & a_{13} & \ldots \\ a_{12} & a_{22} & a_{23} & \ddots \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{array} \right] \left( \frac{1}{X^2} \right). \tag{1.8}
\]
where \((a_{ij})_{ij}\) is positive semidefinite. Note that if (1.7) is satisfied, then (1.6) defines a Hamburger model for the function \(f\). We construct a Hamburger model in Section 4.2, which is unique among expressions of the form (1.8), irrespective of whether \((a_{ij})_{ij}\) is positive semidefinite or not. In Section 4.3, we construct a candidate for the Hamburger model from the power series for a function, which must be equal to the model obtained in Section 4.2 by uniqueness.

To construct the Hamburger model, we establish a sum of squares representation

\[
Df(X)[H] = \sum_{k=0}^{\infty} f_k^* H f_k,
\]

and then reduce it algebraically to the model itself. This method is modeled on the theory of Positivstellensätze. In [34], Helton and McCullough showed the following Positivstellensatz, a generalization of a results of Schmüdgen [57] and Putinar [55] to the free case.

**Theorem 1.10** (Helton, McCullough [34]). Let \(Q\) be a family of free polynomials with the technical assumption that if \(q \in Q\) satisfies \(q(X) \geq 0\) then \(\|X\| \leq M\) for some uniform constant \(M\). If \(f(X) \geq 0\) whenever all \(q \in Q\) satisfy \(q(X) \geq 0\), then

\[
f = \sum g_i^* g_i + \sum h_j^* q_j h_j
\]

for some finite sequences of free polynomials \(g_i, h_j,\) and \(q_j\) where each \(q_j \in Q\).

To establish Equation (1.9), we apply the Choi-Kraus representation theorem [14]. A map \(L : \mathcal{M}_n \rightarrow \mathcal{M}_n\) is called positive if \(H \geq 0 \Rightarrow L(H) \geq 0\). A map \(L : \mathcal{M}_n \rightarrow \mathcal{M}_n\) is said to be completely positive if the extension of \(L\) to \(L^m : \mathcal{M}_n \otimes \mathcal{M}_m \rightarrow \mathcal{M}_n \otimes \mathcal{M}_m\) given on simple tensors by the formula

\[
L^m(A \otimes B) = L(A) \otimes B
\]

is positive for every \(m\).

**Theorem 1.11** (Choi [20], Kraus [46]). A completely positive linear map \(L : \mathcal{M}_n \rightarrow \mathcal{M}_n\) can be written in the form

\[
L(H) = \sum V_i^* H V_i.
\]

In one variable, since \(Df(X)[H]\) is completely positive in \(H\), Theorem 1.11 is used to derive the equation (1.9) locally, by establishing the \(V_i\) are derived from free polynomials \(u_i(X)\). So for each \(X\),

\[
Df(X)[H] = \sum_k u_k(X)^* H u_k(X). \tag{1.11}
\]
The theory of establishing equation (1.11) for locally monotone functions corresponds to Lemma 4.9.

To obtain a global version of (1.11), we introduce the free coefficient Hardy space $H^2_d$, the set of free power series in $d$ variables with coefficients in $l^2$. The free coefficient Hardy space generalizes the classical Hardy space $H^2$, the set of functions on the disk whose power series coefficients are in $l^2$. The classical Hardy space has many important properties, however most important to us are the Szegő kernels, $k_x \in H^2$ such that $\langle f, k_x \rangle = f(x)$. We establish the theory of Szegő kernels for $H^2_d$. For a given $X \in \mathcal{M}^d_n$ we obtain $k_{ij}^X \in H^2_d$ such that $\langle f, k_{ij}^X \rangle = f(X)_{ij}$ for $i, j \leq n$, the Szegő kernels at $X$. The general theory of the free coefficient Hardy space is given in Section 3.1. The theory of free coefficient Hardy space codifies previously studied geometry of a vector of monomials, which was used in the proof that positive free polynomials are sums of squares [29, Section 2] and the proof of the noncommutative Schwarz lemma, as the noncommutative Fock space [32]. For example, in one variable, define $m_X$ to be the list of monomials written as a column vector. That is,

$$m_X = \begin{pmatrix} 1 \\ X \\ X^2 \\ \vdots \end{pmatrix}.$$  

For any $f \in H^2_1$, there is a $u \in l^2$ such that

$$f(X) = u^* m_X.$$  

Via this duality, the theory of $m_X$ is, for our purposes, the theory of the free coefficient Hardy space. For this reason, we formally adopt the view that

$$H^2_1 = l^2(\{0, 1, 2, \ldots\})$$

so that for any $f = (c_i)_{i=0}^\infty \in H^2_1$, and $X$ such that $\|X\| < 1$, the evaluation of $f$ at $X$ is defined via the formula

$$f(X) = \sum_{i=0}^\infty c_i X^i.$$  

The free coefficient Hardy space has appeared in operator theory as the noncommutative Hardy space, where Popescu established the theory of composition operators [34].

Note that each $u_k(X)$ from formula (1.11) can be written in the form of a tensored inner product,

$$u_k(X) = (u_k^* \otimes I) \begin{pmatrix} 1 \\ X \\ X^2 \\ \vdots \end{pmatrix} = (u_{k1}I \ u_{k2}I \ u_{k3}I \ \ldots) \begin{pmatrix} 1 \\ X \\ X^2 \\ \vdots \end{pmatrix}$$

(1.12)
where \( v_k \in H_1^2 \) is the vector of coefficients the polynomial of \( u_k \). The decomposition allows equation (1.11) to be written in the form of equation (1.6),

\[
Df(X)[H] = \left( \begin{array}{c} \frac{1}{X} \\ X^2 \end{array} \right)^* \begin{bmatrix} a_{11}H & a_{12}H & a_{13}H & \cdots \\ a_{21}H & a_{22}H & \cdots \\ a_{31}H & \cdots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} \left( \begin{array}{c} \frac{1}{X} \\ X^2 \end{array} \right), \tag{1.13}
\]

where the matrix \((a_{ij})_{ij}\) is positive semidefinite since \((a_{ij})_{ij} = \sum v_k v_k^*\).

The matrix \((a_{ij})_{ij}\) in formula (1.13) is unique when restricted the space spanned by the Szegö kernels at \( X \). As the spectrum of \( X \) grows, the Szegö kernels exhaust the space and we obtain one matrix \((a_{ij})_{ij}\) that satisfies equation (1.8) for all \( X \) that must agree with equation (1.6) by uniqueness. Since \((a_{ij})_{ij}\) is positive, equation (1.7) is satisfied, which allows us to conclude that \( f \) extends to the upper half plane as a Pick function. The full construction is given in Section 4.2.

1.4. The structure of the paper. The paper is structured as follows. In Section 2 we describe free analysis in detail and how it relates to the classical functional calculus. In Section 3, we develop the foundations of our paper. The begins with a discussion of the free coefficient Hardy space. Then, we establish the lurking isometry argument, a tool to represent functions. In Section 4, we prove Löwner’s theorem, using the method described above. In Section 5, we prove our Nevanlinna representations, and characterize them using asymptotic behavior.

2. Background

We fix \( \mathcal{H} \) to be a infinite-dimensional separable Hilbert space.

2.1. Free analysis.

2.1.1. Matrix universe. Let \( \mathcal{M}_n(\mathbb{C}) \) denote the \( n \times n \) matrices over \( \mathbb{C} \). We define the matrix universe to be

\[
\mathcal{M} = \bigcup \mathcal{M}_n(\mathbb{C}),
\]

and the \( d \)-dimensional matrix universe by

\[
\mathcal{M}^d = \bigcup \mathcal{M}_n(\mathbb{C})^d.
\]

Most function theory is proven for maps from \( \mathcal{M}^d \) to \( \mathcal{M}^{d'} \). However, we require a slight generalization of this calculus. Let \( \mathcal{V} \) be a vector space over \( \mathbb{C} \). We define the \( \mathcal{V} \) matrix universe as

\[
\mathcal{V} \otimes \mathcal{M} = \bigcup \mathcal{V} \otimes \mathcal{M}_n(\mathbb{C}).
\]
Similarly, we define the real matrix universe to be
\[ \mathcal{R} = \{ X \in M | X = X^* \}, \]
and the real d-dimensional matrix universe by
\[ \mathcal{R}^d = \bigcup \mathcal{R}^d_n. \]
Let \( \mathcal{V} \) be a vector space over \( \mathbb{R} \). We define the real \( \mathcal{V} \) matrix universe
\[ \mathcal{V} \otimes \mathcal{R} = \bigcup \mathcal{V} \otimes \mathcal{R}^n. \]

The vertical tensor decomposition of data reflects the structure of our arguments. We take the opinion that the first slot contains extrinsic data, data related to the evaluation of functions, and that the second slot contains intrinsic data, information about the input and output of functions. Data that has been decomposed into its intrinsic and extrinsic parts will be presented visually as

Extrinsic data

\[ \otimes \]

Intrinsic data

for organizational purposes. We have denoted our tensor products vertically to make formulas more clear. For example, we desire
\[ \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} AC \\ BD \end{pmatrix}. \]

Moreover, we will often encounter expressions of the form of Equation 1.6, which can be rewritten in vertical tensor notation as
\[ \begin{pmatrix} \frac{1}{X} \\ X^2 \end{pmatrix} \otimes \begin{pmatrix} a_1 & a_2 & a_3 & \ldots \\ a_2 & a_3 & \ldots \\ a_3 & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{X} \\ X^2 \end{pmatrix} \]
which is symmetric. We believe adding the visual symmetry induced by adopting vertical tensor notation will make our arguments more clear.

The matrix universe \( \mathcal{M} \) has a direct sum, and this is inherited by the tensor product as follows on simple tensors
\[ \begin{array}{c} A \\ B \end{array} \otimes \begin{array}{c} C \\ D \end{array} = \begin{array}{c} A \\ B \otimes 0 \end{array} + \begin{array}{c} 0 \otimes 0 \\ C \left\otimes D \right\otimes \begin{array}{c} 0 \otimes 0 \end{array} \end{array} \]
and is extended by linearity on the entire tensor product.

We will sometimes use flat tensors of two pieces of intrinsic or extrinsic data. The flat tensor \( \otimes \) is implemented so that it is right distributive over \( \oplus \),
\[ A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)). \]
2.1.2. Free maps. A free set $D \subseteq \mathcal{V}_\mathcal{M}$ satisfies the following axioms

(1) $A, B \in D \iff A \oplus B \in D$,
(2) $\forall A \in D \cap \mathcal{V}_{\mathcal{M}_n(\mathbb{C})}, \forall U \in \mathcal{U}_n(\mathbb{C}), \ I_U \otimes A \otimes U \in D$.

A free domain $D \subset \mathcal{M}_d$ is a free set in the $d$-dimensional matrix universe that is open in the disjoint union topology. Given two free sets $D$ and $D'$, a free map $f : D \to D'$ is a map so that the graph of $f$ is a free set and if for some invertible matrix $S$, $A \in D$, and $I_{S^{-1}} \otimes S^{-1} A \otimes S \in D$, then

$$f(I_{S^{-1}} \otimes S^{-1} A \otimes S) = I_{S^{-1}} \otimes S^{-1} f(A) \otimes S.$$ 

2.1.3. Real free maps. A real free set $D \subseteq \mathcal{V}_\mathcal{R}$ satisfies the following axioms

(1) $A, B \in D \iff A \oplus B \in D$
(2) $\forall A \in D \cap \mathcal{V}_{\mathcal{R}_n}, U \in \mathcal{U}_n(\mathbb{C}), \ I_U^* \otimes A \otimes U \in D$.

A real free domain is a real free set in the $d$-dimensional matrix universe, $D \subset \mathcal{R}_d$, that is open in the disjoint union topology restricted to self-adjoint matrices. A real free map $f : D \to D'$ is a map so that the graph of $f$ is a real free set and if for some unitary $U$, $A \in D$, and $I_U^* \otimes A \otimes U \in D$, then

$$f(I_U^* \otimes A \otimes U) = I_U^* \otimes f(A) \otimes U.$$

We define the complexified tangent bundle of a domain as follows, noting that we need the vector component of the tangent bundle to have the same dimension as the point over which it lies. Let $D \subset \mathcal{R}_d$ be a real free domain (resp. free domain). The complexified tangent bundle (resp. tangent bundle) is the set $T(D)$ given by the formula

$$T(D) = \bigcup_n D_n \times \mathcal{M}_n^d,$$

where $D_n = D \cap \mathcal{M}_n$.

We now begin the discussion of power series and real analyticity. We adopt the convention that if $w$ is a word in the letters $x_1, \ldots, x_k$, and $X = (X_1, \ldots, X_k)$, then $X^w = w(X)$. For example $X^{x_1 x_2 x_1} = X_1 X_2 X_1$. Furthermore, we define an involution $*$ on words which reverses their letters. For example $(x_1 x_2)^* = x_2 x_1$. We use $|w|$ to denote the length of a word.

To discuss analyticity, we first define a meaningful way to talk about local coordinates.
Definition 2.1. A \( n \)-frame is a basis \( F = (F_i)_{i=1}^{dn^2} \) of \( \mathcal{R}_n^{dn^2} \) such that \((Fi)_j\) is positive semidefinite over all \( i, j \), where \((Fi)_j\) is the matrix in the \( j \)th slot of the \( i \)th basis element.

We denote the local coordinate function with respect to \( F \) as

\[
X_F = \sum_{i=1}^{dn^2} F_i \otimes X_i.
\]

A real free map is analytic if for each point \( A_0 \in D \cap \mathcal{R}_n^{dn^2} \), the function of \( dn^2 \) noncommuting variables has a power series for each of its noncommuting entries.

Definition 2.2. Let \( f : D \rightarrow \mathcal{R} \) be a real free function. We say \( f \) is real analytic if for any \( A_0 \in D \cap \mathcal{R}_n^{dn^2}, n \)-frame \( F \), and vectors \( u, v \in \mathbb{C}^n \), there are \( c_I \in \mathbb{C} \) such that

\[
u^* \otimes I f(A_0 \otimes I + X_F)v \otimes I = \sum_I c_I X^I
\]

for all \( X \in \mathcal{R}^{dn^2} \) such that \( \|X\| < \epsilon \).

We note that the definition of a real free analytic map \( f \) implies that for each \( A_0 \in D \), there is a free domain \( D' \), the domain of convergence of the power series for \( f \) at \( A_0 \), containing \( A_0 \) such that \( f \) analytically continues to \( D' \cap D \) as a function of the entries of the input as tuples of matrices. However, \( a \ priori \), it is not clear that the continuation of \( f \) is a free function. This issue is resolved by the following lemma combined with observation that the derivative is a linear map.

Lemma 2.3. Let \( D \subset \mathcal{M}_n^{dn^2} \). Let \( f : D \rightarrow \mathcal{M}_n \) be a differentiable function. Let \([,\cdot] \) denote the commutator. (That is, \([X,Y] = XY - YX\).)

\begin{enumerate}
\item \( f \) respects unitary similarity if and only if for all \( A \in \mathcal{R}_n \)
\[
Df(X)[[iA,X]] = [iA,f(X)].
\]
\item \( f \) respects similarity if and only if for all \( T \in \mathcal{M}_n \)
\[
Df(X)[[T,X]] = [T,f(X)].
\]
\end{enumerate}

Proof. Suppose \( f \) respects unitary similarity. Let \( A \in \mathcal{R}_n \) Since \( f \) respects unitary similarity,

\[
e^{-itA}f(X)e^{itA} = f(e^{-itA}Xe^{itA}).
\]

Differentiating this equation at 0 via the chain rule gives

\[
Df(X)[[iA,X]] = [iA,f(X)].
\]
The fundamental theorem of calculus proves the converse. That is, since
\[
\frac{d}{dt} f(e^{-itA}X e^{itA}) = Df(e^{-itA}X e^{itA})[[iA, e^{-itA}X e^{itA}]]
\]
and
\[
\frac{d}{dt} e^{-itA} f(X) e^{itA} = [iA, e^{-itA} f(X) e^{itA}],
\]
then
\[
Df(e^{-itA}X e^{itA})[[iA, e^{-itA}X e^{itA}]] = [iA, e^{-itA} f(X) e^{itA}]
\]
implies
\[
f(e^{-itA}X e^{-itA}) = e^{-itA} f(X) e^{itA} + C
\]
and \(C\) must be 0 since evaluating at 0 gives \(f(X) = f(X) + C\). Since, every unitary is of the form \(e^{iA}\) for some self-adjoint \(A\), (see [28, Chapter 2]) we are done.

The proof of 2 is similar and is left to the reader. \( \square \)

Since the derivative is complex linear, Lemma 2.3 immediately implies the following.

Corollary 2.4. A complex analytic real free function is a free function.

2.1.4. The real free derivative identity. In the work of Helton, Klep, and McCullough [31], Voiculescu [62, 63] and Kaliuzhnyi-Verbovetskyi and Vinnikov [42], a number of identities have been proven to compute derivatives of free functions. We will need the following two identities for real free functions. The following identity is given for commutative matrix functions in the proof of the commuting multivariable Löwner theorem of Agler, McCarthy and Young [6] and proven in [15].

Proposition 2.5. Let \( f : D \to \mathcal{M} \) be a differentiable real free function.
\[
Df (X, Y) \left[ X^{-Y, X^{-Y}} \right] = \left( f(X) - f(Y) \right)^{f(X) - f(Y)}
\]

Proof. Let \( A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \). By Lemma 2.3
\[
Df (X, Y) \left[ [iA, (X, Y)] \right] = \left[ iA, \left( f(X), 0 \right) f(Y) \right] .
\]
Substituting \( A \) and simplifying obtains the desired result.
\[
Df (X, Y) \left[ X^{-Y, X^{-Y}} \right] = \left( f(X) - f(Y) \right)^{f(X) - f(Y)} .
\] \( \square \)
2.1.5. Dominating points. Given a finite set of ordered pairs 
\[(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2,\]
such that all \(x_i\) are distinct, there is a real analytic function \(f: \mathbb{R} \to \mathbb{R}\) such that \(f(x_i) = y_i\).

In the free setting this assertion is no longer true. For example, the value of a function \(f\) at 
\[X = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\]
completely determines the value at the point 
\[Y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\]
since 
\[f \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} f(1) & f(2) \end{pmatrix} \begin{pmatrix} 3 \end{pmatrix} .\]

We now draw on the notion of dominating points which was used in Helton and McCullough [35].

**Definition 2.6.** Let \(X, Y \in \mathcal{R}^d\). We say \(X\) dominates \(Y\) if \(f(X) = 0\) implies that \(f(Y) = 0\).

Given an \(X\), we desire to find an algebraically well-conditioned point which dominates \(X\).

**Definition 2.7.** Let \(X \in \mathcal{R}^d\). We say that \(X\) is reduced if the algebra generated by \(X_1, \ldots, X_d\) is equal to 
\[\bigoplus_i M_{n_i}(\mathbb{C})\]
for some finite sequence of integers \(n_i\). Elementary techniques from the theory of finite dimensional C*-algebras provide a reduced dominating point for a given \(X\).

**Lemma 2.8.** Let \(X \in \mathcal{R}^d\). There exists an \(X^0 \in \mathcal{R}^d\) such that \(X^0\) dominates \(X\), \(X\) dominates \(X^0\), and \(X^0\) is reduced.

**Proof.** Let \(\mathcal{A}\) denote the algebra generated by the components of \(X\). Note that \(\mathcal{A}\) is a *-algebra since the generators are self-adjoint.

By an Artin-Wedderburn type theorem for finite dimensional C*-algebras [23 Theorem III.1.1], \(\mathcal{A} \cong \bigoplus_i M_{n_i}(\mathbb{C})\) for some finite sequence of integers \(n_i\). So let \(\pi : \bigoplus_i M_{n_i}(\mathbb{C}) \to \mathcal{A}\) be a *-isomorphism. By a structure theorem in [23 Theorem III.1.2], there is a unitary \(U\) and integers \(m_i\) such that for every \(\bigoplus_i Y_i \in \bigoplus_i M_{n_i}(\mathbb{C})\), we can write the homomorphism via the formula 
\[\pi(\bigoplus_i Y_i) = U^* \bigoplus_i (Y_i \otimes I_{m_i}) U.\]
Let $X^0 = \pi^{-1}(X)$. Since real free functions respect direct sums and unitary similarity, $X^0$ dominates $X$ and $X$ dominates $X^0$. Since the coordinates of $X^0$ generate $\bigoplus_i M_{n_i}(\mathbb{C})$, $X^0$ is reduced. \hfill \Box

We now will show that if $X$ dominates $Y$ then the derivative of a function $f$ at $X$ determines the derivative of $f$ at $Y$.

**Lemma 2.9.** Let $X \in \mathcal{R}^d_n, Y \in \mathcal{R}^d_m$ be points such that $X$ dominates $Y$. Let $L : \mathcal{R}^d_n \to \mathcal{R}^d_n$ be a linear map. There is a unique map $K : \mathcal{R}^d_m \to \mathcal{R}^d_m$ such that if $f$ is a real analytic free function, and $Df(X) = L$ as a linear map, then $Df(Y) = K$.

The proof of the lemma follows immediately from the following free identity.

**Proposition 2.10.** Let $f : D \to M$ be a differentiable real free function. Let $A \in \mathcal{R}^d$ and $B \in \mathcal{R}$. 

$$Df(X \otimes I_n)[A \otimes B] = Df(X)[A] \otimes B.$$ 

*Proof.* Let $U \in U_n$ be unitary such that 

$$B = U^* DU$$

where $D$ is some real diagonal matrix.

So, 

$$Df(X \otimes I_n)[A \otimes B] = (I \otimes U^*)Df(X \otimes I_n)[A \otimes D](I \otimes U)$$

$$= (I \otimes U^*)Df\left(\bigoplus_{i=1}^n X, \bigoplus_{i=1}^n d_i A\right)(I \otimes U)$$

$$= (I \otimes U^*) \bigoplus_{i=1}^n Df(X)[d_i A](I \otimes U)$$

$$= (I \otimes U^*) \bigoplus_{i=1}^n d_i Df(X)[A](I \otimes U)$$

$$= Df(X)[A] \otimes D(I \otimes U)$$

$$= Df(X)[A] \otimes B.$$ 

\hfill \Box

*Proof of Lemma 2.4.* Let $X^0$ be the reduction of $X$ given in the proof of Theorem 2.3. By Proposition 2.10, the derivative at $X^0$ is determined by $L$, since there is a $k$, a unitary $U$ and a matrix tuple $W$ such that $X_0 \otimes I_k = U^* X \oplus WU$. That is, there is an $L_0$ such that for any $f$ such that $Df(X) = L$, $Df(X^0) = L_0$. It can be shown that there is a homomorphism $\pi$ taking the algebra generated by the components of
$X^0$ to the algebra generated by the components of $Y$ so that $\pi(X^0_i) = Y_i$. Furthermore, for some unitary $V$,

$$\pi(\bigoplus_l Z_l) = V^* \bigoplus_l (Z_l \otimes I) V.$$ 

Thus, by Proposition 2.10, the derivative at $X^0$ determines the derivative at $Y$. □

3. Foundations

3.1. The free coefficient Hardy space. Introduced in [27], and [26], the classical Hardy space $H^2$ is typically defined as the set of analytic functions on the disk so that

$$\lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})| d\theta \leq \infty.$$ 

The space $H^2$ is endowed with an inner product given by the formula

$$\langle f, g \rangle = \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta})\overline{g(re^{i\theta})} d\theta$$

and is indeed a Hilbert space. In an alternative characterization, the functions $z^n$ form an orthonormal basis for $H^2$. Thus, $H^2$ is also the set of functions on the disk such that their coefficients in a power series at the origin are sequences in $l^2$.

In the classical Hardy space $H^2$, there exists a function, the Szegő kernel [59, 44], $k_\alpha$ such that if $f \in H^2$, then

$$f(\alpha) = \langle f, k_\alpha \rangle.$$ 

We generalize the second interpretation of the Hardy space as the free coefficient Hardy space.

**Definition 3.1.** Let $\mathcal{I}$ be the set of monomial indices in the free algebra with $d$ variables. The free coefficient Hardy space $H^2_d$ is $l^2(\mathcal{I})$ where for $f \in H^2_d$ such that $f = (a_I)_{I \in \mathcal{I}}$ the value of $f$ at $X$ is defined on tuples of matrices $X$ of norm less than $\frac{1}{d}$ by the formula

$$f(X) = \sum_{I \in \mathcal{I}} a_I X^I.$$ 

Importantly, this space has a Szegő kernel itself.

**Definition 3.2.** Let $\mathcal{I}$ be the set of monomial indices in the free algebra with $d$ variables. Let $X \in \mathcal{M}^n$. Define the monomial basis vector $m_X = (X^I)_{I \in \mathcal{I}}$. The Szegő kernel is given by $k_X^{ij} = (m_X^i m_X^j)^{\mathcal{I}}$. That is, it is the sequence $(i, j)$-th entries of each monomial $I$ evaluated at $X$. 

Note that by a straightforward calculation,
\[ f(X)_{ij} = \langle f, k_{ij}^X \rangle. \]
Thus, the behavior of \( f \) at \( X \) is determined by the projection of \( f \) onto the space spanned by \( k_{ij}^X \).

**Definition 3.3.** Define
\[ \mathcal{V}_X = \text{span}_{ij}\{k_{ij}^X\} \subset H_2^d. \]
Furthermore, define \( P_X : H_2^d \to \mathcal{V}_X \) to be the projection onto \( \mathcal{V}_X \).

Thus, given some pair \((X, Y)\) such that there is a free polynomial with \( p(X) = Y \), we can correspond to \( p \) a unique \( f \) of minimum norm in \( H_2^d \) such that \( f(X) = Y \) by assigning \( f = P_X p \).

The following is essentially the statement that the values of a function determine the function.

**Proposition 3.4.**
\[ \bigcup_{\|X\| \leq \frac{1}{d}} \mathcal{V}_X = H_2^d. \]

**Proof.** Suppose \( \bigcup_{\|X\| \leq \frac{1}{d}} \mathcal{V}_X \neq H_2^d \). Then there exists a nonzero function \( f \in \bigcup_{\|X\| \leq \frac{1}{d}} \mathcal{V}_X^\perp \). So, \( 0 = \langle f, k_{ij}^X \rangle \) for all \( i, j \) and \( X \) which implies that \( f \) is zero. This is a contradiction. \( \square \)

In general we will often forget coordinate systems, so we use an alternative characterization of \( \mathcal{V}_X \).

**Proposition 3.5.** The space \( \mathcal{V}_X \) is the unique vector space such that
\[ \mathcal{V}_X \cong H_2^d. \]

To prove this we need a decomposition theorem, a *detensoring lemma* that will allow us to show that spaces like the one in the above lemma are well-defined.

**Lemma 3.6 (Detensoring lemma).** For a Hilbert space \( \mathcal{H} \), let \( \mathcal{V} \) be a subspace of \( \mathcal{H} \otimes \mathbb{C}^n \) such that if \( U \in \mathcal{U}_n \) then
\[ \mathcal{V} \otimes U = \mathcal{V}. \]

Then there exists \( \mathcal{V}' \), a subspace of \( \mathcal{H} \), so that \( \mathcal{V} = \mathcal{V}' \otimes \mathbb{C}^n \).
Proof. Let $e_1, \ldots, e_n$ be an orthonormal basis for $\mathbb{C}^n$.

Let $V'$ be the subspace $\{v| e_i \otimes v \in V\}$.

We first show $V \subseteq V' \otimes \mathbb{C}^n$. Let $v \in V$. Then

$$v = \sum_i v_i \otimes e_i.$$ 

Let $U_i$ be the unitary that fixes $e_i$ and sends $e_j$ to $-e_j$ if $i \neq j$. So,

$$I \otimes U_i v = \sum_i (-1)^{i\neq j} v_i \otimes e_i \in V$$

by assumption. Therefore

$$\frac{1}{2} \left( I \otimes U_i v + v \right) = \sum_i v_i \otimes e_i \in V.$$ 

Let $W_i$ be a unitary taking $e_i$ to $e_1$. So

$$I \otimes W_i v_i \otimes e_i = v_i \otimes e_1.$$ 

Thus, each $v_i \in V'$ and so $v \in V' \otimes \mathbb{C}^n$.

We now show $V \supseteq V' \otimes \mathbb{C}^n$. Suppose $v \in V' \otimes \mathbb{C}^n$, so that,

$$v = \sum_i v_i \otimes e_i$$

where each $v_i \in V$. Let $W_i$ be a unitary taking $e_1$ to $e_i$. Thus,

$$v = \sum_i I \otimes W_i v_i \otimes e_1 \in V$$

since each $I \otimes W_i v_i \otimes e_1 \in V$ by definition. \hfill \Box

We now prove Proposition 3.5.

Proof of Proposition 3.5. By the detensoring lemma, there is a unique vector space $V$ such that

$$V = \mathbb{C}^n = \text{span}\{I \otimes U_m X c| c \in \mathbb{C}^n, U \in U_n\}.$$ 

Let $U_1$ be the unitary sending $e_i$ to $-e_i$. Note,

$$k_{ij} X e_i = \frac{1}{2} \left[ I \otimes m_X e_j - I \otimes U_i m_X e_j \right].$$

Thus, $V$ contains $V_X$ as the $k_{ij}$ form a basis for $V_X$.

Let $E_{ij}$ designate the matrix in $M_n$ with 1 in the $ij$th position and 0 elsewhere. Note, $m_X = \sum E_{ij} k_{ij} X$. So, since the unitary matrices in $M_n$ span $M_n$ itself, it can be derived that

$$\text{span}\{I \otimes m_X W|U, W \in U_n\} = \text{span}\{k_{ij} X A|A \in M_n\}.$$
Now, if \( f \in \mathcal{V} \), there is some \( l \in \mathbb{C}^n \) such that
\[
\frac{f}{l} = \sum_{ij} k_{ij}^X \otimes_A c_{ij}.
\]
Note that \( Ac_{ij} = l \), since \( k_{ij}^X \). So \( \frac{f}{l} = \sum_{ij} k_{ij}^X \otimes l \).

The value of \( f \in H_2^2 \) at \( X \in D \) is defined to be:
\[
f(X) = \frac{f}{l} \otimes \varphi(X).
\]

3.1.1. *The coefficient Hardy space of a general free vector-valued function.* The free coefficient Hardy space is useful for interpolation problems to obtain existence and uniqueness results. If we relax the choice of basis of functions, we still obtain uniqueness results. For our constructions, this is often enough.

**Definition 3.7.** Let \( \varphi \) be a free function on a domain \( D \subset \mathcal{M}^d \) and taking values in \( \mathcal{M} \). The **free coefficient Hardy space** for \( \varphi \), denoted by \( H_2^2 \), is given by
\[
H_2^2 = \{ u \in \mathcal{H} | \forall X \in D, \frac{u}{l} \otimes \varphi(X) = 0 \}^\perp.
\]
The value of \( f \in H_2^2 \) at \( X \in D \) is defined to be:
\[
f(X) = \frac{f}{l} \otimes \varphi(X).
\]

We define the vector spaces from Definition \[3.3\] in the second abstract characterization which is easier to state in this context. However, a choice of basis will yield Szegö kernels as in the original definition.

**Definition 3.8.** Let \( \varphi \) be a free function on a domain \( D \subset \mathcal{M}^d \) and taking values in \( \mathcal{M} \), and let \( X \in \mathcal{M}_n^d \). The **space** \( \mathcal{V}_X^\varphi \) is the unique vector space such that
\[
\mathcal{V}_X^\varphi = \operatorname{span}\{ \frac{l}{U} \otimes \varphi(X)c | c \in \mathbb{C}^n, U \in U_n \}.
\]

Define the projection \( P_X^{\varphi} : H_2^2 \to \mathcal{V}_X^\varphi \) to be the projection onto \( \mathcal{V}_X^\varphi \).

We note again that for \( f, g \in H_2^2 \), \( f(X) = g(X) \) if and only if \( P_X^{\varphi} f = P_X^{\varphi} g \). Furthermore, the spaces \( \mathcal{V}_X^\varphi \) exhaust \( H_2^2 \).

**Proposition 3.9.**
\[
\bigcup_{X \in D} \mathcal{V}_X^\varphi = H_2^2.
\]

**Proof.** Suppose \( f \in (\bigcup_{X \in D} \mathcal{V}_X^\varphi)^\perp \). So, for every \( X \), \( P_X f = 0 = P_X 0 \). Thus, \( f(X) \equiv 0 \). \( \square \)
3.2. Models. As in the commutative case, model formulas are a powerful tool for investigating various classes of holomorphic functions on different domains. For example, for scalar valued functions in the Schur class in two variables, we have the following theorem:

**Theorem 3.10** (Agler [1]). Let $U \subseteq \mathbb{D}^2$. Let $\varphi : U \rightarrow \mathbb{D}$. There is an analytic continuation of $\varphi$ to $\mathbb{D}^2$, $\varphi : \mathbb{D}^2 \rightarrow \mathbb{D}$ if and only if there exists a separable Hilbert space $\mathcal{H}$, an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and a holomorphic function $u : U \rightarrow \mathcal{H}$ so that
\[
1 - \varphi(\mu)\varphi(\lambda) = \langle (1 - \mu^*\lambda)u_{\lambda}, u_{\mu} \rangle,
\]
where $\lambda$ is viewed as the operator $\lambda = \lambda_1 P_{\mathcal{H}_1} + \lambda_2 P_{\mathcal{H}_2}$.

To use models for analytic continuation techniques in the free setting, we use the following interpolation theorem. Let $B^d$ be the free set of $d$-tuples of contractions
\[
B^d = \{ X = (X_1, \ldots, X_d) \in \mathcal{M}^d | \|X_i\| < 1 \text{ for } i = 1, \ldots, d \}. \tag{3.1}
\]

**Theorem 3.11** (Agler, M\textsuperscript{c}Carthy [2]). Let $D \subset B^d$ be a free set. Let $\varphi : D \rightarrow B$ be a free function. There is an extension of $\varphi$ to $B^d$ as a free function $\varphi : B^d \rightarrow B$ if and only if there are functions $u_1, \ldots, u_d : D \rightarrow B \otimes_{\mathcal{M}} (B(\mathbb{C}, \mathcal{H}) \otimes \mathcal{H})$ so that for any $X, Y \in D$,
\[
I - \varphi(Y)^*\varphi(X) = \sum_i u_i(Y)^* \frac{I}{I - Y^*_i X_i} u_i(X). \tag{3.2}
\]

Transformed to the upper half plane via a Möbius transform, the model theorem is as follows.

**Theorem 3.12.** Let $D \subset \Pi^d$ be a free set. Let $\varphi : D \rightarrow \Pi$ be a free function. There is an extension of $\varphi$ to $\Pi^d$ as a free function $\varphi : \Pi^d \rightarrow \Pi$ if and only if there are functions $u_1, \ldots, u_d : D \rightarrow B(\mathbb{C}, \mathcal{H}) \otimes_{\mathcal{M}}$ so that for any $X, Y \in D$,
\[
\varphi(X) - \varphi(Y)^* = \sum_i u_i(Y)^* \frac{I}{X_i - Y_i^*} u_i(X). \tag{3.3}
\]

3.3. The lurking isometry argument for linear forms. The proof of Theorem 3.11 relies on the existence of a free version of the standard lurking isometry argument. The additional complications inherent in the free algebraic structure of the models warrant an argument establishing the existence of these isometries in the free case. We prove a lurking isometry argument for linear forms, which makes slightly different assumptions, but can also be used to make model theoretic arguments.
Proposition 3.13. Suppose that for free functions \( \theta \) and \( \varphi \) taking a set \( D \subset \mathcal{M} \) into \( \frac{\mathcal{H}}{\mathcal{M}} \), we have the relation
\[
\theta(X)^* \otimes_H \theta(X) = \varphi(X)^* \otimes_H \varphi(X)
\]
for all \((X, H) \in T(D)\). Let \( \mathcal{V}_X^\theta \) and \( \mathcal{V}_X^\varphi \) be the vector spaces given in Definition 3.8. Then for each \( X \in D \cap \mathcal{M}_n^d \), there exists a unique unitary operator \( U_X : \mathcal{V}_X^\theta \rightarrow \mathcal{V}_X^\varphi \) such that for any unitary \( U \in \mathcal{U}_n \) and any \( c \in \mathbb{C}^n \),
\[
U_X \otimes_U \theta(X)c = \otimes_U \varphi(X)c.
\]
Proof. Let \( c_1, c_2 \in \mathbb{C}^n \) and \( U_1, U_2 \in \mathcal{U}_n \). Then
\[
c_2 \theta(X)^* \otimes_{U_2^* U_1} \theta(X)c_1 = \left( \otimes_{U_2^*} \varphi(Y)c_2 \right)^* \left( \otimes_{U_1} \varphi(X)c_1 \right).
\]
Since these inner products agree, there is a uniquely determined partial isometry \( \tilde{U}_X : \frac{\mathcal{V}_X^\theta}{\mathbb{C}^n} \rightarrow \frac{\mathcal{V}_X^\varphi}{\mathbb{C}^n} \) so that
\[
\tilde{U}_X \otimes_U \theta(X)c = \otimes_U \varphi(X)c
\]
for all \( U \) and \( c \). Now, note that for \( U_1, U_2 \in \mathcal{U}_n \),
\[
\tilde{U}_X \otimes_{U_1^*} \otimes_{U_2} \theta(X)c = \otimes_{U_1^*} \otimes_{U_2} \varphi(X)c.
\]
Rearranging this equation gives
\[
\tilde{U}_X \otimes_{U_1^*} \otimes_{U_2} \theta(X)c = \otimes_{U_2^*} \varphi(X)c.
\]
Note that the uniqueness implies
\[
\tilde{U}_X \otimes_{U_1^*} \otimes_{U_2} \theta(X)c = \otimes_{U_2^*} \varphi(X)c.
\]
and thus \( \tilde{U}_X = \frac{U_X}{U} \).

The uniqueness of \( U_X \) gives the following.

Proposition 3.14. Suppose that for free functions \( \theta \) and \( \varphi \) taking a set \( D \subset \mathcal{M} \) into \( \frac{\mathcal{H}}{\mathcal{M}} \), we have the relation
\[
\theta(X)^* \otimes_H \theta(X) = \varphi(X)^* \otimes_H \varphi(X)
\]
for all \((X, H) \in T(D)\). Let \( X, Y \in D \) such that \( Y \) dominates \( X \). For \( U_X \) as defined in Proposition 3.13,
\[
P_X^\varphi U_Y P_X^\theta = U_X.
\]
Proof. Note \( P_X^\varphi U_Y P_X^\theta \) is unitary and thus by uniqueness, \( P_X^\varphi U_Y P_X^\theta = U_X \). □
Theorem 3.15 (Lurking isometry argument for linear forms). Let $\theta$ and $\varphi$ be free functions on a set $D \subset \mathcal{M}^d$ taking values in $\frac{H}{\mathcal{M}}$. If for all $(X, H) \in T(D)$,

$$\theta(X)^* \otimes_H \theta(X) = \varphi(X)^* \otimes_H \varphi(X),$$

then there exists an isometry $U : H^2_{\theta} \rightarrow H^2_{\varphi}$ such that for all $X \in D$,

$$U \otimes_I \theta(X) = \varphi(X).$$

Proof. The direct limit of the $U_X$ with respect to inclusion of domains

$$\hat{U} : \bigcup_{X \in D} \mathcal{V}_X^\theta \rightarrow \bigcup_{X \in D} \mathcal{V}_X^\varphi$$

is well defined by 3.14. Furthermore, $\hat{U}$ extends as an isometry $U : H^2_{\theta} \rightarrow H^2_{\varphi}$ since the domain of $\hat{U}$ is dense in $H^2_{\theta}$ by Proposition 3.9.

Since $P_X^\varphi U P_X^\theta = U_X$, $U$ satisfies the required properties. □

4. Löwner’s theorem

Let $f : (a, b) \rightarrow \mathbb{R}$. We say $f$ is matrix monotone if

$$A \leq B \Rightarrow f(A) \leq f(B).$$

In 1934, Löwner [41] showed that if $f$ is matrix monotone, then $f$ analytically continues to $\Pi_1 \cup (a, b)$ so that $f : \Pi_1 \cup (a, b) \rightarrow \mathbb{R}_+^+.$

In general, we define locally monotone functions as follows. This definition agrees with classical monotonicity on convex sets since

$$f(X) - f(Y) = \int_0^1 Df(X + t(X - Y))[X - Y] \, dt$$

by the fundamental theorem of calculus.

Definition 4.1. A real analytic free function $f : D \rightarrow \mathbb{R}$ is locally monotone if

$$H \geq 0 \Rightarrow Df(X)[H] \geq 0$$

for all $(X, H) \in T(D)$.

The following definition codifies the extension property in the free setting.

Definition 4.2. A real analytic free function $f : D \rightarrow \mathbb{R}$ has a Löwner extension if there is a continuous free function $F : \Pi^n \cup D \rightarrow \mathbb{R}$ such that $F|_D = f$.

Our goal is to give a version of Löwner’s theorem for the noncommutative functional calculus.
**Theorem 4.3.** A real analytic free function $f : D \to \mathbb{R}$ is locally monotone if and only if $f$ has a Löwner extension.

4.1. The Hamburger model. Our first kind of model, the Hamburger model gives information about the derivative.

**Definition 4.4.** Let $f : D \to \mathcal{R}$. A Hamburger model for $f$ is a list of $d$ real free functions $u_i : D \to \mathcal{H} \otimes \mathcal{R}$ such that for all $(X, H) \in T(D)$,

$$Df(X)[H] = \sum_i u_i(X)^* \frac{I}{H} \otimes u_i(X).$$

Our second kind of model gives nonlocal data. This has been analyzed in the commutative case on polydisks by Ball and Bolotnikov [12].

**Definition 4.5.** Let $f : D \to \mathcal{R}$. A boundary Nevanlinna model for $f$ is a list of $d$ real free functions $u_i : D \to \mathcal{H} \otimes \mathcal{R}$ such that for all $X, Y$ of the same dimension,

$$f(X) - f(Y)^* = \sum_i u_i(Y)^* \frac{I}{X_i - Y_i} \otimes u_i(X)$$

and for all $(X, H) \in T(D)$,

$$Df(X)[H] = \sum_i u_i(X)^* \frac{I}{H} \otimes u_i(X).$$

The Hamburger model and the boundary Nevanlinna model are related in these sense that if we have one, we can obtain the other via the relations of free analysis. These can be explicitly computed and are essentially equivalent up to isometry by the lurking isometry for linear forms. We discuss these computations in Section 4.3.

The following expands Theorem 4.3 to give the actual strategy for proof.

**Theorem 4.6.** Let $f : D \to \mathbb{R}$ be a real analytic free function. The following are equivalent:

1. $f$ is locally monotone,
2. $f$ has a Hamburger model,
3. $f$ has a boundary Nevanlinna model,
4. $f$ has a Löwner extension.

We regard the implication (1 $\Rightarrow$ 2) to be the novel part of the proof. The implication (2 $\Leftarrow$ 1) holds a fortiori because of the form of the Hamburger model. We devote the rest of this section to proving the
simpler parts, and will prove \((1 \Rightarrow 2)\) as Section 4.2. The implication \((1 \Rightarrow 4)\) in Theorem 4.16 is proven as Theorem 4.19.

The following lemma proves \((2 \Leftrightarrow 3)\) in Theorem 4.6.

**Lemma 4.7.** A real free analytic function \(f : D \rightarrow \mathbb{R}\) has a Hamburger model if and only if \(f\) has a boundary Nevanlinna model.

**Proof.** The reverse implication holds by definition.

Suppose \(f\) has a Hamburger model \(u\). That is, 
\[
Df(X)[H] = \sum_i u_i(X)^* \frac{I}{H} u_i(X).
\]

So, 
\[
Df(X)\frac{X-Y}{X-\mathbb{Y}} = \sum_i u_i(X)^* \frac{I}{X_i-Y_i} u_i(X).
\]

Via the formula in Proposition 2.5,
\[
\left[f(X) - f(Y)\right] = \sum_i \left[u_i(X)^* \frac{I}{X_i-Y_i} u_i(Y)\right].
\]

Multiplying on the second slot,
\[
\left[f(X) - f(Y)\right] = \sum_i \left[u_i(Y)^* \frac{I}{X_i-Y_i} u_i(X)\right].
\]

This implies
\[
f(X) - f(Y) = \sum_i u_i(Y)^* \frac{I}{X_i-Y_i} u_i(Y).
\]

Thus,
\[
f(X) - f(Y)^* = \sum_i u_i(Y)^* \frac{I}{X_i-Y_i} u_i(Y).
\]

\(\square\)

The following lemma proves \((4 \Rightarrow 1)\) in Theorem 4.6.

**Lemma 4.8.** If a real free analytic function \(f : D \rightarrow \mathbb{R}\) has a Löwner extension, then \(f\) is locally monotone.

**Proof.** Suppose \(f\) has a Löwner extension and \(f\) is not monotone. Then, there is a point \(X\) and a positive semidefinite \(H\) such that \(D(X)[H]\) is not positive semidefinite. Since
\[
f(X + itH) = f(X) + itD(X)[H] + O(t^2),
\]
\[
\text{Im } f(X + itH) = tD(X)[H] + O(t^2)
\]
which for small $t \geq 0$ is not positive semidefinite. This is a contradiction. □

4.2. The Hamburger model construction. We now begin the construction of a Hamburger model. The following is a reduction of the Choi-Kraus theorem which gives the raw data used in the construction locally.

**Lemma 4.9.** Suppose $f : \mathcal{D} \to \mathcal{R}$ is locally monotone on $D$. For any point $X \in D_n$ there are real free functions $u_{ij}$ such that for all $H \in M_n^d$,

$$Df(X)[H] = \sum_i \sum_j u_{ij}(X)^* H_i u_{ij}(X).$$

**Proof.** Suppose $f$ is locally monotone. Let $X \in \mathcal{D} \cap M_n^d$. Without loss of generality, we let $X = \bigoplus X_i$ where $X$ generates the algebra $\bigoplus M_n(C)$ where $\sum n_i = n$. (This follows from Lemmas 2.9 and 2.8.) Note $Df(X) : M_n^d \to M_n$ is completely positive in each coordinate since the extension of $Df(X)$ to $M_n^d \otimes M_k$ via Formula (1.10) is given by $Df(X \otimes I_n)$ by Proposition 2.10 which is positive by the assumption of local monotonicity. By the Choi-Kraus theorem [13],

$$Df(X)[H] = \sum_i \sum_j V_{ij}^* H_i V_{ij}.$$ 

We now show that the $V_{ij}$ are in the algebra generated by $X$. That is, they are free polynomial functions of $X$. Let $P^l$ be the projection onto the $l$-th component of $X$. Let $P^l_i$ be a tuple that equals $P^l$ on the $i$-th coordinate and 0 elsewhere. Consider $Df(X)[P^l_i]$.

$$Df(X)[P^l_i] = \sum_j V_{ij}^* P^l V_{ij}.$$ 

Block decompose

$$V_{ij} = \sum_{l,m} P^l V_{ij} P^m.$$ 

So,

$$Df(X)[P^l_i] = \sum_j \sum_m \sum_n P^m V_{ij}^* P^l V_{ij} P^n.$$ 

So, in the block decomposition of $Df(X)[P^l_i]$ the $(m, m)$ entry is

$$\sum_j P^m V_{ij}^* P^l V_{ij} P^m.$$ 

However, since $Df(X)[H]$ is a free function,

$$Df(X)[P^l_i] = \sum_j P^l V_{ij}^* P^l V_{ij} P^l.$$
So, if \( l \neq m \) the \((m, m)\) entry is 0. That is, \( P^m V_{ij}^* P^l V_{ij} P^m = 0 \). Thus, if \( l \neq m \) \( P^l V_{ij} P^m = 0 \). This implies that

\[
V_{ij} = \sum_l P^l V_{ij} P^l.
\]

Since \( X \) generates \( \bigoplus M_m(\mathbb{C}) \), \( V_{ij} \) is in the algebra generated by \( X \). Thus, each \( V_{ij} \) is in the algebra generated by \( X \) so there are free polynomials \( u_{ij} \) such that \( u_{ij}(X) = V_{ij} \).

**Definition 4.10.** A global monomial basis vector for \( D \) is a free function \( m \) on \( D \) given by the formula \( m_X = (c_I X^I) \), such that \( \|m_X\| \) is locally bounded on \( D \) and each \( c_I > 0 \).

We now use free coefficient Hardy space methods to establish local uniqueness of the Hamburger model. We refine the the raw data obtained in Lemma 4.9 into a canonical object.

**Lemma 4.11.** Let \( m \) be a global monomial basis vector. If \( f : D \to \mathcal{R} \) is locally monotone on \( D \), for any matrix tuple \( X \in D_n \) there are unique finite rank operators \( A^*_X \in \mathcal{B}(H^2_{m_X}) \geq 0 \) such that if at a tuple of operators \( B_i \in \mathcal{B}(H^2_{m_X}) \)

\[
Df(X)[H] = \sum_i m_X^* \cdot_{B_i} m_X.
\]

for all \( H \in \mathcal{M}_n^d \), then \( P^m X B_{ij} P^m X = A^*_X \).

**Proof.** Note by Lemma 4.9 there are polynomials \( u_{ij} \) such that

\[
Df(X)[H] = \sum_i \sum_j u_{ij}(X)^* H_i u_{ij}(X).
\]

Define \( u_i \) to be the function given by \((u_{ij}(X))_j\) as a column vector. So,

\[
Df(X)[H] = \sum_i u_i(X)^* \cdot_{H_i} u_i(X).
\]

Note, since the entries of \( u_i \) are polynomials, there are bounded finite rank operators \( K_i \) so that \( u_i(X) = \frac{K_i}{H_i} m_X \). So

\[
Df(X)[H] = \sum_i m_X^* \cdot_{H_i} m_X.
\]

Define \( A^*_X = P^m X K_i^* K_i P^m X \).

\[
Df(X)[H] = \sum_i m_X^* \cdot_{H_i} m_X.
\]
Suppose, \( B_i \) satisfies

\[
Df(X)[H] = \sum_i m_X^* \frac{B_i}{H_i} m_X
\]

and \( P_X^m B_i P_X^m = B_i \). So if \( U_1, U_2 \in U \)

\[
0 = m_X^* \frac{A_X^i - B_i}{U_1^i U_2} m_X = m_X^* \frac{I}{U_1^i} \frac{I}{U_2} m_X
\]

So, \( \langle (A_X^i - B_i) u, v \rangle = 0 \) for any \( u, v \in V_X^m \) and is thus 0. \( \square \)

By patching these together, we obtain the Hamburger model.

**Lemma 4.12.** If \( f : D \rightarrow \mathcal{R} \) is locally monotone on \( D \), then \( f \) has a Hamburger model.

**Proof.** By Lemma 4.11, for each \( X \) there is a unique \( A_X^i \) such that

\[
Df(X)[H] = \sum_i m_X^* \frac{A_X^i}{H_i} m_X
\]

and \( P_X^m A_X^i P_X^m = A_X^i \).

Let \( q^i(u, v) \) be the direct limit of the semidefinite sesquilinear forms

\[
q^i_X(u, v) = \langle A_X^i u, v \rangle \text{ defined on } \mathcal{V} = \bigcup_{X \in D} V_X^m.
\]

Let \( K_i \) be the Hilbert space formed by completing the quotient \( \mathcal{V}/\ker q^i(\cdot, \cdot) \). Let \( T_i : \mathcal{V} \rightarrow K_i \) be the inclusion map. Define \( u_i = \frac{T_i}{I} \otimes m_X \). Now \( u_i \) form a Hamburger model for \( f \). \( \square \)

### 4.3. The localizing matrix construction of the Hamburger model.

Let \( f(X) = \sum_I c_I X^I \) be a power series. The \( x_k \)-localizing matrix of coefficients is the infinite matrix with rows and columns indexed by monomials \( (c_I x_k^I)_{I,J} \). In the section, we will show that if \( f \) is monotone, then the \( x_k \)-localizing matrix of coefficients must be positive semi-definite. This application mirrors the use of classical Hankel matrices in the study of the Hamburger moment problem [43, 51]. Localizing matrices have been used to study multivariate moment problems [22], and more recently to study noncommutative convex hulls [30].

The following gives a condition for a free power series to be uniformly and absolutely convergent.

**Lemma 4.13.** Let \( \epsilon > 0 \). Suppose a series in \( d \) noncommuting variables \( \sum_I c_I X^I \) is convergent for all \( \|X\| < d + \epsilon \). Then \( \sum_I c_I X^I \) is absolutely and uniformly convergent for all \( \|X\| < 1 \). Furthermore, there is an \( N \) such that if \( |I| \geq N \),

\[
|c_I| \leq \left( d + \frac{\epsilon}{2} \right)^{-|I|}.
\]

**Proof.** Note that

\[
\lim_{n \to \infty} \max_{|I|=n} \|c_I X^I\| = 0
\]
for all \( \|X\| < d + \epsilon \). Substituting in the tuple \((d + \epsilon/2, \ldots, d + \epsilon/2)\) gives that

\[
\lim_{n \to \infty} \max_{|I| = n} |c_I|(d + \epsilon/2)^{|I|} = 0
\]

which implies that for large \( n \), \( |c_I|(d + \epsilon/2)^{|I|} \leq 1 \) which implies the claim. \( \square \)

We will now establish that for power series that converge on large enough sets, the \( x_k \)-localizing matrices are compact, which will be useful in establishing formulas for the derivative of a real free power series.

**Lemma 4.14.** If \( f(X) = \sum_I c_I X^I \) is a real analytic locally monotone free function for \( \|X\| < d + \epsilon \), then for each \( k \), the \( x_k \)-localizing matrix of coefficients \( (c_{I^*x_k})_{I,J} \) is compact.

**Proof.** The compactness of each \( (c_{I^*x_k})_{I,J} \) follows from the decay of the entries given in Lemma 4.13. That is, if \( E_{I,J} \) is the infinite matrix with entry 1 in the \((I, J)\)-th slot and zero elsewhere, then

\[
(c_{I^*x_k})_{I,J} = \sum_{n=1}^{\infty} \sum_{|I_k, J| = n} c_{I^*x_k} E_{I,J}
\]

is a convergent formula in the norm topology since it is Cauchy via the estimate (relying on the combinatorial observation that the total number of words of length \( n \) in \( d \) letters is \( d^n \) and the estimate in Lemma 4.13)

\[
\left\| \sum_{n=M}^{N} \sum_{|I_k, J| = n} c_{I^*x_k} E_{I,J} \right\| \leq \sum_{n=M}^{N} \sum_{|I_k, J| = n} |c_{I^*x_k}|
\]

\[
= \sum_{n=M}^{N} \sum_{|I_k, J| = n} (d + \epsilon/2)^{|I_k, J|}
\]

\[
\leq \sum_{n=M}^{N} (d + \epsilon/2)^{-n}
\]

\[
\leq \sum_{n=M}^{N} d^n (d + \epsilon/2)^{-n}
\]

\[
= \sum_{n=M}^{N} (1 + \frac{\epsilon}{2d})^{-n}
\]

\[
\leq \sum_{n=M}^{\infty} \left(1 + \frac{\epsilon}{2d}\right)^{-n}
\]
Thus, since \((c_{I^*x_kJ})_{I,J}\) is well-approximated by finite rank operators, \((c_{I^*x_kJ})_{I,J}\) is compact \([21\text{, Theorem } 4.4]\). \(\square\)

We will need the following elementary fact about the derivative.

**Proposition 4.15.** Let \(\epsilon > 0\). Suppose a series in \(d\) noncommuting variables \(\sum_I c_I X^I\) is convergent for all \(\|X\| < d + \epsilon\). The derivative of \(f\) at \(X \in D_n\) in the direction \(H \in M_n^d\) is, for \(\|X\| \leq \frac{1}{d}\), given by the formula

\[
Df(X)[H] = \sum_k m_X^* (c_{I^*x_kJ})_{I,J} \otimes_{H_k} m_X.
\]

**Proof.** By Lemma 4.14 \((c_{I^*x_kJ})_{I,J}\) is compact and thus it will be sufficient to show that, for the functions \(g_K(X) = X^K\),

\[
Dg_K(X) = \sum_k m_X^* (\chi(I^*x_kJ = K))_{I,J} \otimes_{H_k} m_X.
\]

where \(\chi\) is the indicator function. Since

\[
\sum_k m_X^* (\chi(I^*x_kJ = K))_{I,J} \otimes_{H_k} m_X = \sum_k X^J \chi(I^*x_kJ = K) H_k X^I
\]

and the right hand side of the preceding equation is the derivative by the product rule, we are done. \(\square\)

The following gives a characterization of monotone functions in terms of their power series, similarly to Nevanlinna’s solution to the Hamburger moment problem\([48]\).

**Theorem 4.16.** If \(f(X) = \sum_I c_I X^I\) is a real analytic locally monotone free function for \(\|X\| < d + \epsilon\), then for each \(k\), the \(x_k\)-localizing matrix of coefficients \((c_{I^*x_kJ})_{I,J}\) is positive semidefinite and compact.

**Proof.** Let \(m_X = (X^I)_I\). Note, for \(\|X\| < \frac{1}{d}\), \(m_X\) is bounded and

\[
Df(X)[H] = \sum_k m_X^* (c_{I^*x_kJ})_{I,J} \otimes_{H_k} m_X
\]

via Lemma 4.15.

By Lemma 4.14

\[
P_X(c_{I^*x_kJ})_{I,J} P_X
\]

is positive semidefinite, and thus since \(\mathcal{V} = \cup_X \mathcal{V}_X\) is dense in \(H_d^2\) by Proposition 3.4, \((c_{I^*x_kJ})_{I,J}\) is positive semidefinite. \(\square\)
We remark that expressions of the form (4.1) have been used to analyze the noncommutative Hessian to obtain results on convex functions.

Reinterpreting Theorem 4.16 we immediately obtain a model for a locally monotone function.

**Corollary 4.17.** Let $f(X) = \sum c_i X^i$ be a real analytic locally monotone free function for $\|X\| < d + \epsilon$. The Hamburger model for $f$ on $\|X\| < 1/d$ is given by the formula

$$Df(X)[H] = \sum_k m_X^* \left( \frac{c_{i^* k^*}^{i^* j^*}}{I} \right)_{i^* j^*}^{1/2} \otimes I \otimes H_k \left( \frac{c_{i^* k^*}^{i^* j^*}}{I} \right)_{i^* j^*}^{1/2} m_X.$$ 

**Lemma 4.18.** Let $f : D \to \mathbb{R}$ be a real analytic locally monotone free function. For any $X \in D$, the Hamburger model for $f$ analytically continues to a free domain containing $X$.

**Proof.** Let $f$ have the Hamburger model

$$Df(X)[H] = \sum_i u_i(X)^* \otimes I \otimes u_i(X).$$

Let $X_0 \in D \cap \mathbb{R}^d$ and $F_1, \ldots, F_d$ be tuples of positive semidefinite matrices which span $\mathbb{R}^d$ large enough so that the function of $dn^2$ variables

$$h_i(X) = (e_i \otimes I)^* f(X_0 \otimes I + X_F)(e_i \otimes I)$$

is analytic for all $\|X\| \leq dn^2 + \epsilon$. Note that each $h_i$ is a monotone function. Thus for each $i$

$$Dh_i(X)[H] = \sum_k m_X^* \left( \frac{c_{i^* k^*}^{i^* j^*}}{I} \right)_{i^* j^*}^{1/2} \otimes I \otimes H_k \left( \frac{c_{i^* k^*}^{i^* j^*}}{I} \right)_{i^* j^*}^{1/2} m_X.$$ 

Note that

$$(e_j \otimes I)^* Df(X_0 \otimes I + X_F)[H_F](e_j \otimes I) = Dh_j(X)[H].$$

So,

$$\sum_j \sum_i (e_j \otimes I)^* u_i(X_0 \otimes I + X_F)^* \otimes I \otimes u_i(X_0 \otimes I + X_F)(e_j \otimes I)$$

$$= \sum_j \sum_k m_X^* \left( \frac{c_{i^* k^*}^{i^* j^*}}{I} \right)_{i^* j^*}^{1/2} \otimes I \otimes H_k \left( \frac{c_{i^* k^*}^{i^* j^*}}{I} \right)_{i^* j^*}^{1/2} m_X.$$
Expanding out the frame gives
\[
\sum_k \sum_j \sum_i (e_j \otimes I)^* u_i (X_0 \otimes I + X_F)^* \frac{I}{(F_k \otimes H)} u_i (X_0 \otimes I + X_F) (e_j \otimes I)
\]
\[
= \sum_j \sum_k m_X^* \frac{(c_{j+k,j})^2}{I} \frac{I}{H_k} \frac{(c_{j+k,j})^2}{j} m_X.
\]

Thus, by the lurking isometry argument for linear forms, there is a unitary \(U\) such that \(\otimes_I U\) takes
\[
\theta(X) = \bigoplus_{i,j,k} \frac{I}{(F_k \otimes I)} \frac{1}{2} u_i (X_0 \otimes I + X_F) (e_j \otimes I)
\]
to
\[
\varphi(X) = \bigoplus_{j,k} \frac{(c_{j+k,j})^2}{I} m_X.
\]

Since \(\varphi(X)\) analytically continues to a neighborhood of 0 via its formula, so does \(\theta(X)\).

The above implies each \(u_i\) itself must analytically continue, since the value of \(\theta(X)\) determines the values of each \(u_i(X)\). The continuation of \(u_i\) a free function by Corollary 2.4. Thus, the analytic continuation of the Hamburger model is given by the formula
\[
D f(Z) | H = \sum_i u_i(Z^*) \frac{I}{H_i} u_i(Z)
\]

since this agrees with the Hamburger model on a neighborhood of \(X_0\).

Thus, we obtain the desired Löwner extension. Thus, we obtain Löwner’s theorem by a rearrangement argument.

**Theorem 4.19.** Let \(D\) be a free domain. If \(f : D \to R\) is a real analytic locally monotone free function, then \(f\) has a Löwner extension.

**Proof.** Let \(X_0 \in D\).

Note that formula
\[
D f(Z) | H = \sum_i u_i(Z^*) \frac{I}{H_i} u_i(Z)
\]
can be used to derive a Nevanlinna model
\[
f(X) - f(Y)^* = \sum_i u_i(Y)^* \frac{I}{X_i - Y_i} u_i(X).
\]
by Proposition 2.5. Thus, there is some $D^X$ containing $X_0$ such that $f|_{D^X}$ has an analytic continuation to $\Pi^d$ by Lemma 4.18 via Theorem 3.12.

Note that for $X_0, Y_0 \in D$ there is a Löwner extension at $X_0 \oplus Y_0$ which induces the Löwner extension at $X_0$ and $Y_0$ and thus must be the same. □

5. The Nevanlinna Representations

Rolf Nevanlinna characterized the class of Pick functions in terms of three parameters: a real number, a nonnegative real number and a finite positive Borel measure on the real line.

**Theorem 5.1** (R. Nevanlinna [48]). Let $h : \mathbb{H} \to \mathbb{C}$. The function $h$ is a Pick function if and only if there exist $a \in \mathbb{R}, b \geq 0$, and a finite positive Borel measure $\mu$ on $\mathbb{R}$ such that

$$h(z) = a + bz + \int \frac{1 + tz}{t - z} d\mu(t)$$

for all $z \in \mathbb{H}$. Moreover, for any Pick function $h$, the numbers $a \in \mathbb{R}, b \geq 0$ and the measure $\mu \geq 0$ are uniquely determined.

This representation parametrizes all Pick functions, generalizing Theorem 1.1 which represents Pick functions satisfying the growth condition (1.2).

The two representations given in Theorem 5.1 and Theorem 1.1 which we refer to as Nevanlinna representations, were recently generalized to several variables in the work of Agler, McCarthy, and Young [6], and later in Agler, Tully-Doyle, and Young for multivariable Pick functions that are also in the so-called Löwner class [8, 7].

The $d$-variable Löwner class $\mathcal{L}_d$ is the set of analytic functions that lift via the functional calculus to act on $d$-tuples of commuting operators with the property that they possess analytic extensions that take $\Pi^d \to \mathbb{T}$. In the case of one-variable Pick functions, the Löwner class is the entire Pick class by von Neumann’s inequality [64, 60]. For two variables, the classes coincide by Andô’s Theorem [11]. In three or more variables, the Löwner class is a proper subset of the Pick class [50, 61]. The Löwner class is conformally equivalent to the well-studied Schur-Agler class via a Möbius transform taking the upper half plane $\mathbb{H}$ to the disk to obtain a map $\mathbb{D}^d \to \mathbb{D}$.

The multivariable Nevanlinna representations discussed in [7] partitions the Löwner class $\mathcal{L}_d$ into four types depending on asymptotic behavior at infinity. Note that these not only give representations of
Pick functions, but also give formulas for constructing new Pick functions.

**Definition 5.2.** A *positive decomposition* is a collection of positive operators \( Y_1, \ldots, Y_d \) on \( H \) so that \( \sum_i Y_i = I_H \), and the operator \( z_Y : \mathbb{C}^d \rightarrow \mathcal{H} \) is given by

\[
z_Y = \sum_{i=1}^d z_i Y_i.
\]

An *orthogonal decomposition* is a positive decomposition where each \( Y_i \) is a projection.

By the following theorem, every \( h \) in \( L_d \) has a representation of at least one type. Recall that \( \mathbb{H}^d = \Pi_1^d \).

**Theorem 5.3** (Agler, Tully-Doyle, Young [7]). Let \( h \) be a function defined on \( \Pi_1^d \).

**Type 4:** The function \( h \in L_d \) if and only if there exist an orthogonally decomposed Hilbert space \( \mathcal{H} \) and a vector \( v \in \mathcal{H} \) such that

\[
h(z) = \langle M(z)v, v \rangle,
\]

where \( M(z) \) is the matricial resolvent as defined in Proposition 3.1 in [7].

**Type 3:** There exist \( a \in \mathbb{R} \), a Hilbert space \( \mathcal{H} \), a self-adjoint operator \( A \) on \( \mathcal{H} \), a vector \( v \in \mathcal{H} \), and a positive decomposition \( Y \) of \( H \) so that

\[
h(z) = a + \langle (1 - iA)(A - z_Y)^{-1}(1 + z_Y A)(1 - iA)^{-1}v, v \rangle
\]

if and only if \( h \in L_d \) and

\[
\liminf_{s \to \infty} \frac{1}{s} \text{Im} \ h(is, \ldots, is) = 0.
\]

**Type 2:** There exist \( a \in \mathbb{R} \), a Hilbert space \( \mathcal{H} \), a self-adjoint operator \( A \) on \( \mathcal{H} \), a vector \( v \in \mathcal{H} \), and a positive decomposition \( Y \) of \( H \) so that

\[
h(z) = a + \langle (A - z_Y)^{-1}v, v \rangle
\]

if and only if \( h \in L_d \) and

\[
\liminf_{s \to \infty} s \text{Im} \ h(is, \ldots, is) < \infty.
\]

**Type 1:** There exist a Hilbert space \( \mathcal{H} \), a self-adjoint operator \( A \) on \( \mathcal{H} \), a vector \( v \in \mathcal{H} \), and a positive decomposition \( Y \) of \( H \) so that

\[
h(z) = \langle (A - z_Y)^{-1}v, v \rangle
\]
if and only if $h \in \mathcal{L}_d$ and
\[
\liminf_{s \to \infty} s |h(is, \ldots, is)| < \infty.
\]

In the following sections, we will extend this theorem to characterize free Pick functions.

5.1. **The free Herglotz representation formula.** A Herglotz function is a holomorphic function $h$ on $\mathbb{D}$ with $\text{Re} \, h \geq 0$. Herglotz functions were characterized by C. Carathéodory in [17] and given an integral representation by Gustav Herglotz in [37]. J. Agler generalized this representation for functions in the $d$-variable strong Herglotz class, a conformal equivalent of the Schur-Agler class, in [1].

To define free Herglotz functions, we need an analogue of the right halfplane. Denote by $\Psi \subset \mathcal{M}$ the right matrix polyhalfplane
\[
\Psi = \{ X \in \mathcal{M} | \text{Re} X = \frac{1}{2} (X + X^*) > 0 \}.
\]
Recall that $\mathcal{B}^d$ is the set of $d$-tuples of strict contractions. Say that $h$ is a free Herglotz function if $h$ is a free holomorphic function from $\mathcal{B}^d \to \Psi$.

The representation of a classical Herglotz function $h$ with $h(0) = 1$ is given by a probability measure $\mu$ on the unit circle so that
\[
h(z) = \int_0^{2\pi} \frac{1 + e^{-i\theta}}{1 - e^{-i\theta}} d\mu(\theta).
\]

The following theorem gives an analogous formula in the free case. The noncommutative Herglotz representation was originally proved by G. Popescu in [53, Theorem 3.1]. We express the representation in terms of the geometry from the commutative case in Agler, Tully-Doyle and Young’s proof of the Nevanlinna representations for Lowner functions in several variables [7].

**Theorem 5.4** (Popescu [53]). Let $h$ be a free holomorphic function with $h(0) = 1$. The function $h$ is a free Herglotz function, that is $h : \mathcal{B}^d \to \Psi$, if and only if there exist a Hilbert space $\mathcal{H}_d = \bigoplus_{i=1}^d \mathcal{H}$, a unitary operator $U$ on $\mathcal{H}_d$ and an isometry $V : \mathbb{C} \to \mathcal{H}_d$ so that
\[
h(X) = V^* \left( I + \frac{U}{I} \delta(X) \right) \left( I - \frac{U}{I} \delta(X) \right)^{-1} \frac{V}{I}, \tag{5.1}
\]
where
\[
\delta(X) : \mathcal{M}^d \to \mathcal{H}_d \otimes \mathcal{M} = \bigoplus_{i=1}^d \mathcal{H} \otimes X^i.
\]
Corollary 5.5. If \( h \) is a free Herglotz function on \( \mathcal{B}^d \), then there exists a Hilbert space \( \mathcal{H}_d = \bigoplus_i \mathcal{H}_i \), a real constant \( a \), a unitary operator \( L \in \mathcal{B}(\mathcal{H}_d) \), and a vector \( v \in \mathcal{H}_d \) such that for all \( X \in \mathcal{B}^d \),

\[
h(X) = -i \otimes I + v^* \left( \frac{L}{I} - \delta(X) \right)^{-1} \left( \frac{L}{I} + \delta(X) \right) \otimes I,
\]

(5.2)

where \( \delta(X) = \bigoplus_{i=1}^d \frac{I}{X_i} \).

Conversely, any \( h \) defined by an equation of the form (5.2) is a free Herglotz function.

Connections between Pick functions, Herglotz functions, and Schur functions are given by the Cayley transform. The particular Cayley transform given by

\[
\lambda = \frac{z - 1}{z + 1}, \quad z = \frac{1 + \lambda}{1 - \lambda}
\]

is a conformal mapping between the right half-plane and the disk. G. Popescu showed that the noncommutative Cayley transform is a well-defined bijection between free Herglotz functions and free Schur functions in [52, Section 1]. Via the Cayley transform, we will use representation of free Herglotz functions to derive our Nevanlinna representations.

5.2. The structured resolvents. The expression \( f(z) = (t - z)^{-1} \) in Theorem 1.1 suggests the resolvent operator \( R(Z) = (A - z)^{-1} \). We will present, in the following section, four representations based on the following structured resolvents (properties of structured resolvents are discussed at length in [7]).

Definition 5.6. Let \( \mathcal{H} \) be a Hilbert space. A positive decomposition \( Y \) of \( \mathcal{H} \) is a collection of positive operators \( Y_1, \ldots, Y_d \) summing to \( I \). An orthogonal decomposition \( P \) of \( \mathcal{H} \) is a collection of pairwise orthogonal projections \( P_1, \ldots, P_d \) on \( \mathcal{H} \) summing to \( I \).

Definition 5.7 (structured resolvents).

Type 2: Let \( A \) be a closed densely defined self-adjoint operator on a Hilbert space \( \mathcal{H} \) and let \( Y \) be a positive decomposition of \( \mathcal{H} \). The structured resolvent of type 2 corresponding to \( Y \) is the function from \( \Pi^d \rightarrow \mathcal{B}(\mathcal{H}) \otimes_{\mathcal{M}} \) given by

\[
M_2(Z) = \left( \frac{A}{I} - \delta_Y(Z) \right)^{-1},
\]

(5.3)

where

\[
\delta_Y(Z) = \sum_{i=1}^d \frac{Y_i}{Z_i}.
\]
**Type 3:** Let $A$ be a closed, densely defined, self-adjoint operator on a Hilbert space $\mathcal{H}$, and let $Y$ be positive decomposition of $\mathcal{H}$. The *structured resolvent of type 3* corresponding to $Y$ is the function from $\Pi^d \to B(\mathcal{H}) \otimes \mathcal{M}$ given by

$$M_3(Z) = \left( I - iA \right) - \delta_Y(Z) \left( I + \delta_Y(Z) \right)^{-1},$$

where

$$\delta_Y(Z) = \sum_i \frac{Y_i}{Z_i}.$$

**Type 4:** Let $\mathcal{H}$ be a Hilbert space decomposed orthogonally as $\mathcal{H} = \mathcal{N} \oplus \mathcal{K}$. Let $A$ be a densely defined, self-adjoint operator on $\mathcal{K}$ with domain $D(A)$, and let $P$ be an orthogonal decomposition of $\mathcal{H}$. The structured resolvent of type 4 resolvent is a function from $\Pi^d \to B(\mathcal{H}) \otimes \mathcal{M}$ of the form

$$M_4(Z) = \left[ \begin{array}{cc} -i & 0 \\ 0 & I \end{array} \right] - \delta_P(Z) \left[ \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right]^{-1} \times \left[ \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right] + \delta_P(Z) \left[ \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right]^{-1},$$

where

$$\delta_P(Z) = \sum_i \frac{P_i}{Z_i}.$$

For each $2 \leq i \leq 4$ and each $Z \in \Pi^d$, the expression $M_i(Z)$ is a bounded operator and has positive imaginary part, which follows directly from proofs given in [7].

With the structured resolvents (5.7), (5.7), and (5.5), we now present representations for free Pick functions that generalize the classical Nevanlinna representations given in Theorems 1.1 and 5.1.

**Recall** that the free Pick class $\mathcal{P}_d$ consists of analytic functions $h : \Pi^d \to \Pi^1$.

**Definition 5.8.** The following are the representations for functions in $\mathcal{P}_d$.

**Type 4:** A Nevanlinna representation of type 4 of a function $h$ in the free Pick class on $\mathcal{M}^d$ is

$$h(z) = \sum_i \frac{v_i^*}{I} + \sum_i \frac{v_i^*}{I} M_4(Z) \frac{v_i}{I},$$
where $M_4(Z)$ is a type 4 structured resolvent as given in Definition 5.7 with associated Hilbert space $\mathcal{H}$, $v$ is a vector in $\mathcal{H}$, and $a \in \mathbb{R}$.

**Type 3:** A Nevanlinna representation of type 3 of a function $h$ in the free Pick class on $\mathcal{M}^d$ is

$$h(z) = \frac{a}{I} + \frac{v^*}{I} M_3(Z) \frac{v}{I},$$

where $M_3(Z)$ is a type 3 structured resolvent as given in Definition 5.7 with associated Hilbert space $\mathcal{H}$, $v$ is a vector in $\mathcal{H}$, and $a \in \mathbb{R}$.

**Type 2:** A Nevanlinna representation of type 2 of a function $h$ in the free Pick class on $\mathcal{M}^d$ is

$$h(Z) = \frac{a}{I} + \frac{v^*}{I} M_2(Z) \frac{v}{I},$$

where $M_2(Z)$ is a type 2 structured resolvent as given in Definition 5.7 with associated Hilbert space $\mathcal{H}$, $v$ is a vector in $\mathcal{H}$, and $a \in \mathbb{R}$.

**Type 1:** A Nevanlinna representation of type 1 of a function $h$ in the free Pick class on $\mathcal{M}^d$ is

$$h(Z) = \frac{v^*}{I} M_2(Z) \frac{v}{I},$$

where $M_2(Z)$ is a type 2 structured resolvent as given in Definition 5.7 with associated vector space $\mathcal{H}$, and $v$ is a vector in $\mathcal{H}$.

In [7, Section 6], one of the authors with Agler and Young discussed the connections between asymptotic behavior of a function $h$ in $\mathcal{L}_n$ along the upper imaginary polyaxis and the existence of representations of a given type for $h$. It turns out that those results lift to results about functions in the free Pick class $\mathcal{P}_d$. In fact, the structure of free Pick functions is determined by their behavior on the first level of $\Pi^d$, that is, $d$-tuples of complex numbers. In the following discussion, we will denote by

$$\chi = (1, \ldots, 1)$$

the element of $\mathcal{M}_1^d$ consisting of all ones. When evaluating a function $h$ on the ray

$$is\chi = (is, is, \ldots, is) \in \mathcal{M}_1^d,$$

we will make the identification $\mathbb{C} \otimes \mathcal{M}_1^d \cong \mathbb{C}$. Every $h \in \mathcal{P}_d$ has a representation of type 4. We show the following analogue of Theorem 5.3.
Theorem 5.9. The following criteria characterize representations for functions in the free Pick class \( \mathcal{P}_d \). Let \( h \) be a function defined on \( \mathbb{P}^d \).

**Type 4:** The following are equivalent:
1. The function \( h \) has a representation of type 4.
2. The function \( h \) is a free Pick function.

**Type 3:** The following are equivalent:
1. The function \( h \) has a Nevanlinna representation of type 3;
2. The function \( h \) is a Pick function such that
   \[
   \liminf_{s \to \infty} \frac{1}{s} \text{Im} h(is\chi) = 0; \tag{5.6}
   \]
3. The function \( h \) is a Pick function such that
   \[
   \lim_{s \to \infty} \frac{1}{s} \text{Im} h(is\chi) = 0. \tag{5.7}
   \]

**Type 2:** The following are equivalent:
1. The function \( h \) has a Nevanlinna representation of type 2;
2. The function \( h \) is a Pick function such that
   \[
   \liminf_{s \to \infty} s \text{Im} h(is\chi) \leq \infty. \tag{5.8}
   \]
3. The function \( h \) is a Pick function such that
   \[
   \lim_{s \to \infty} s \text{Im} h(is\chi) \tag{5.9}
   \]
   exists and
   \[
   \lim_{s \to \infty} s \text{Im} h(is\chi) \leq \infty. \tag{5.10}
   \]

**Type 1:** The following are equivalent:
1. The function \( h \) has a Nevanlinna representation of type 1;
2. The function \( h \) is a Pick function such that
   \[
   \liminf_{s \to \infty} |sh(is\chi)| \leq \infty. \tag{5.11}
   \]
3. The function \( h \) is a Pick function such that
   \[
   \lim_{s \to \infty} |sh(is\chi)| \tag{5.12}
   \]
   exists and
   \[
   \lim_{s \to \infty} |sh(is\chi)| \leq \infty. \tag{5.13}
   \]

The representations have some geometric relationships. Type 3 representations are the restrictions of type 4 representations onto subspaces. Type 1 representations are a special case of type 2. We can complete the hierarchical description by considering the connection between representations of type 3 and type 2. Proofs of the following
propositions use the same arguments as in Propositions 5.3 and 5.5 in [7].

**Proposition 5.10.** If a free Pick function $h$ has a type 2 representation, then it has type 3 representation. Conversely, if a free Pick function $h$ has a type 3 representation and in addition the vector $v \in \mathcal{D}(A)$, then $h$ has a type 2 representation.

We now begin the proof of Theorem 5.9. We essentially follow [7].

5.3. **Proof of Theorem 5.9.**

5.3.1. **Type 4.**

**Proof of 5.9: Type 4.** We use a Cayley transform to connect the free Pick class to the free Herglotz class. The Cayley transform between the disc and the right halfplane is given by

$$z = \frac{i + \lambda}{1 - \lambda}, \quad \lambda = \frac{z - i}{z + i},$$

for $z \in \Pi$, $\lambda \in \mathbb{D}$. For a given Pick function $f$, define a Herglotz function $h$ for $X \in \mathcal{B}^d$ by

$$h(X) = -if(Z),$$

where $Z$ is the coordinate-wise Cayley transform of $X$, i.e.

$$Z_i = i(I - X_i)^{-1}(I + X_i).$$

Let $f$ be a free Pick function, and let $h$ be the associated Herglotz function. Then by Corollary 5.5

$$f(Z) = ih(X)$$

$$= \bigotimes_i a_i + i \bigotimes_i v^* \left( \bigotimes_i \frac{L}{I} - \delta_P(X) \right)^{-1} \left( \bigotimes_i \frac{L}{I} + \delta_P(X) \right) \bigotimes_i v$$

$$= \bigotimes_i a_i + i \bigotimes_i v^* \left( \bigotimes_i \frac{L}{I} - \delta_P((Z + i)^{-1}(Z - i)) \right)^{-1} \times$$

$$\left( \bigotimes_i \frac{L}{I} + \delta_P((Z + i)^{-1}(Z - i)) \right) \bigotimes_i v$$

$$= \bigotimes_i a_i + i \bigotimes_i v^* \left( \bigotimes_i \frac{L}{I} - \delta_P((Z + i)^{-1})\delta_P((Z - i)) \right)^{-1} \times$$

$$\left( \bigotimes_i \frac{L}{I} + \delta_P((Z + i)^{-1})\delta_P((Z - i)) \right) \bigotimes_i v,$$

where

$$\delta_P(Z) = \sum_i \frac{P_i}{Z_i}.$$
Now let $M(Z)$ be the expression

$$M(Z) = i \left( \frac{L}{I} - \delta_P((Z+i)^{-1})\delta_P((Z-i)) \right)^{-1} \times \left( \frac{L}{I} + \delta_P((Z+i)^{-1})\delta_P((Z-i)) \right).$$

With the notation $P_i = P_{h_i}$, the operator $M(Z)$ can be written

$$M(Z) = i \left[ \delta_P((Z+i)^{-1}) \left( \delta_P(Z+i) \frac{L}{I} - \delta_P((Z-i)) \right) \right]^{-1} \times \delta_P((Z+i)^{-1}) \left( \delta_P(T+i) \frac{L}{I} + \delta_P((Z-i)) \right)$$

$$= i \left( \delta_P(Z+i) \frac{L}{I} - \delta_P((Z-i)) \right)^{-1} \left( \delta_P(Z+i) \frac{L}{I} + \delta_P((Z-i)) \right)$$

$$= \left( \sum_i \frac{P_i}{Z_i+i} - \sum_i \frac{P_i}{Z_i-i} \right) \left( \sum_i \frac{P_i}{Z_i+i} + \sum_i \frac{P_i}{Z_i-i} \right)$$

$$= i \left( \sum_i \frac{P_i(L-I)}{Z_i} - \sum_i \frac{P_i(L+I)}{Z_i} \right)^{-1} \left( \sum_i \frac{P_i(L+I)}{Z_i} + \sum_i \frac{P_i(L-I)}{Z_i} \right)$$

$$= i \left( \delta_P(Z) \frac{L-I}{I} - i \frac{L+I}{I} \right)^{-1} \left( \delta_P(Z) \frac{L+I}{I} + i \frac{L-I}{I} \right).$$

Let $\mathcal{N} = \ker(I - L)$. Decompose $L$ according to $\mathcal{H} = \mathcal{N} \oplus \mathcal{K}$, where $\mathcal{K} = \mathcal{N}^\perp$, so that

$$L = \begin{bmatrix} I & 0 \\ 0 & L_0 \end{bmatrix} \begin{array}{c} \mathcal{N} \\ \mathcal{K} \end{array}$$

where $L_0$ is unitary and $\ker(I - L_0) = \{0\}$. 
Then

\[
M(Z) = i \left( \delta_P(Z) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + i \begin{bmatrix} 2 & 0 \\ 0 & L_0 + I \end{bmatrix} \right)^{-1} \times \\
\left( \delta_P(Z) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + i \begin{bmatrix} 0 & 0 \\ 0 & L_0 + I \end{bmatrix} \right)
\]

\[
= \left( -\delta_P(Z) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + i \begin{bmatrix} 2i & 0 \\ 0 & L_0 \end{bmatrix} \right)^{-1} \times \\
\left( \delta_P(Z) \begin{bmatrix} 2i & 0 \\ 0 & i(I + L_0) \end{bmatrix} + i \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right) .
\] (5.14)

We would like to continue by writing

\[
M(Z) = \begin{bmatrix} -\frac{i}{2} & 0 \\ 0 & (I - L_0)^{-1} \end{bmatrix} \left( -\delta_P(Z) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + i \begin{bmatrix} 0 & 0 \\ 0 & I + L_0 \end{bmatrix} \right)^{-1} \times \\
\left( \delta_P(Z) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + i \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} 2i & 0 \\ 0 & I - L_0 \end{bmatrix} ,
\] (5.15)

but to do so, we need to show that the unbounded, partially defined operator above makes sense. Let

\[
A = \frac{I + L_0}{I - L_0} .
\]

\(A\) is self-adjoint and densely defined on \(\mathcal{K}\) as \(L_0\) is unitary on \(\mathcal{K}\) and \(\ker(I - L_0) = \{0\}\) [56, Section 22]. Let \(\mathcal{D}(A)\) be the domain of \(A\). Then \(\mathcal{D}(A)\) is the dense subspace \(\text{ran}(I - L_0)\) of \(\mathcal{K}\). Then by the definition of \(A\),

\[
(I - L_0)^{-1} = \frac{1}{2}(I - iA),
\]

which is an equation between bijective operators from \(\mathcal{D}(A) \to \mathcal{M}\). Likewise, the equation

\[
I + L_0 = -2iA(I - iA)^{-1}
\]

relates bounded operators from \(\mathcal{D}(A) \to \mathcal{M}\).

We will now justify factoring the expression in \((5.14)\). Since \(\ker(I - L_0) = \{0\}\),

\[
\begin{bmatrix} 2i & 0 \\ 0 & I - L_0 \end{bmatrix}
\]

is a bijection between \(\mathcal{H}\) and \(\mathcal{N} \oplus \mathcal{D}(A)\). Now decompose the \(P_i\) with respect to \(\mathcal{H} = \mathcal{N} \oplus \mathcal{K}\), so that for each \(i = 1, \ldots, d\),

\[
P_i = \begin{bmatrix} X_i & B_i \\ B_i^* & Y_i \end{bmatrix} .
\]
which allows us to write $\delta_P(Z)$ as

$$
\delta_P(Z) = \sum_{i=1}^d P_i \otimes Z_i = \sum_i \left[ \begin{array}{c|c}
X_i & B_i \\
\hline
Z_i & Y_i
\end{array} \right].
$$

Then

$$
\left( -\delta_P(Z) \begin{bmatrix} [0 & 0] \\ I \end{bmatrix} + [I & 0] \right)^{-1} = \left( -\left( \sum_i P_i \otimes Z_i \right) \begin{bmatrix} [0 & 0] \\ I \end{bmatrix} + [I & 0] \right)^{-1}
$$

$$
= \left( -\left( \sum_i \left[ \begin{array}{c|c}
X_i & B_i \\
\hline
Z_i & Y_i
\end{array} \right] \right) \begin{bmatrix} [0 & 0] \\ I \end{bmatrix} + [I & 0] \right)^{-1}
$$

$$
= \left( -\left( \sum_i \left[ \begin{array}{c}
0 \\
A \otimes -\delta_Y(Z)
\end{array} \right] \right) \begin{bmatrix} [0 & 0] \\ I \end{bmatrix} + [I & 0] \right)^{-1}
$$

$$
= \left[ \begin{array}{c|c}
I - \sum_i B_i \\ Z_i & 0
\end{array} \right]^{-1}
$$

$$
= \left[ \begin{array}{c|c}
A & -\delta_Y(Z) \\
\hline
I & 0
\end{array} \right]^{-1}.
$$

(5.16)

That $g(Z) = \left( \frac{A}{I} - \delta_Y(Z) \right)^{-1}$ is a well-defined function bounded on $D(A)$ follows by an argument similar to that in [7]. Thus

$$
\left( -\delta_P(Z) \begin{bmatrix} [0 & 0] \\ I \end{bmatrix} + [I & 0] \right)^{-1}
$$

$$
= \left[ \begin{array}{c}
-\frac{i}{2} \\
\hline
0
\end{array} \right] \left( -\delta_P(Z) \begin{bmatrix} [0 & 0] \\ I \end{bmatrix} + [I & 0] \right)^{-1}
$$

$$
= \left[ \begin{array}{c}
-\frac{i}{2} \\
\hline
0
\end{array} \right] \left( \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \otimes I - \delta_P(Z) \begin{bmatrix} [0 & 0] \\ I \end{bmatrix} \right)^{-1}
$$

is also a bijection from $N \oplus D(A) \to H$, and so we can apply inverses to the left factor of the right-hand side of (5.14). Similar reasoning
allows us to conclude that
\[
\left( \delta_P(Z) \left[ \begin{array}{c c} 2i & 0 \\ 0 & \frac{1}{2}i \end{array} \right] \right) + \left[ \begin{array}{c c} 0 & 0 \\ 0 & \frac{1}{2}i \end{array} \right] \left[ \begin{array}{c c} 0 & 0 \\ 0 & \frac{1}{2}i \end{array} \right] = \left( \delta(Z) \left[ \begin{array}{c c} 1 & 0 \\ 0 & \frac{1}{2}i \end{array} \right] \right) + \left[ \begin{array}{c c} 0 & 0 \\ 0 & \frac{1}{2}i \end{array} \right] \left[ \begin{array}{c c} 0 & 0 \\ 0 & \frac{1}{2}i \end{array} \right] \left[ \begin{array}{c c} -\frac{1}{2}i & 0 \\ 0 & 1 \end{array} \right]^{-1} \left[ \begin{array}{c c} 0 & 0 \\ 0 & -\frac{1}{2}i \end{array} \right]^{-1} \left[ \begin{array}{c c} 0 & 0 \\ 0 & -\frac{1}{2}i \end{array} \right]^{-1}
\]
(5.18)
as operators on \( \mathcal{H} \). Thus, upon combining (5.14), (5.17), and (5.18) and pre- and post-multiplying by \( \frac{2}{1} \) and \( \frac{1}{2} \), we have established
\[
M(Z) = \left[ \begin{array}{c c} -i & 0 \\ 0 & iA \end{array} \right] \left( \left[ \begin{array}{c c} [1] & [0] \\ [0] & i \end{array} \right] - \delta_P(Z) \left[ \begin{array}{c c} 0 & 0 \\ 0 & i \end{array} \right] \right) \left[ \begin{array}{c c} -i & 0 \\ 0 & iA \end{array} \right]^{-1} \left[ \begin{array}{c c} 0 & 0 \\ 0 & -i \end{array} \right]^{-1} \left[ \begin{array}{c c} 0 & 0 \\ 0 & -i \end{array} \right]^{-1} \left[ \begin{array}{c c} 0 & 0 \\ 0 & -i \end{array} \right]^{-1}
\]
(5.19)
Thus, we have shown that \( M \) is a type 4 resolvent, and therefore that \( h \) has a Nevanlinna representation of type 4, i.e.
\[
h(Z) = \frac{a}{Z} + v^* \odot M(Z) \odot \frac{v}{Z}.
\]
The converse follows from the fact that the imaginary part of \( M_4 \) is positive as was remarked in Definition 5.7. \( \square \)

5.3.2. Type 3.

Proof of Theorem 5.13: Type 3. (1) \( \Rightarrow \) (3): Follows from Theorem 5.3.
(3) \( \Rightarrow \) (2) is trivial.
(2) \( \Rightarrow \) (1): Suppose that condition (2) holds, that is \( h \) is a free Pick function with
\[
\lim_{s \to \infty} \text{Im} \frac{1}{s} h(is\chi) = 0.
\]
As a Pick function, \( h \) has a type 4 Nevanlinna representation, that is there exist \( a \in \mathbb{R}, \mathcal{H}, \mathcal{N} \subset \mathcal{H} \), operators \( A, Y \) on \( \mathcal{N}^\perp \), an orthogonal decomposition \( \mathcal{P} \) of \( \mathcal{H} \) and a vector \( v \in \mathcal{H} \) such that
\[
\frac{a}{Z} + v^* \odot M(Z) \odot \frac{v}{Z},
\]
where
\[
M(Z) = \left[ \begin{array}{c c} -i & 0 \\ 0 & 1-iA \end{array} \right] \left( \left[ \begin{array}{c c} [1] & [0] \\ [0] & i \end{array} \right] - \delta_P(Z) \left[ \begin{array}{c c} 0 & 0 \\ 0 & i \end{array} \right] \right) \left[ \begin{array}{c c} -i & 0 \\ 0 & 1-iA \end{array} \right]^{-1} \left[ \begin{array}{c c} 0 & 0 \\ 0 & -i \end{array} \right]^{-1} \left[ \begin{array}{c c} 0 & 0 \\ 0 & -i \end{array} \right]^{-1} \left[ \begin{array}{c c} 0 & 0 \\ 0 & -i \end{array} \right]^{-1},
\]
For $Z = is\chi$, 

$$\delta_P(Z) = \frac{is}{1_c},$$

so for $s > 0$, $M(is\chi)$ becomes

$$M(is\chi) = \left[ \begin{array}{cc} -i & 0 \\ 0 & 1 \end{array} \right] \left( \begin{array}{cc} 1 & 0 \\ 0 & A \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right) \right)^{-1} \times \left( \begin{array}{cc} \frac{is}{1} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right)^{-1}

= \left[ \begin{array}{cc} -i & 0 \\ 0 & 1 \end{array} \right] \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right)^{-1} 

= \left[ \begin{array}{cc} -i & 0 \\ 0 & 0 \end{array} \right] \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right)^{-1}

= \left[ \begin{array}{cc} -i & 0 \\ 0 & 0 \end{array} \right] \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right)^{-1} 

= \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right)^{-1} 

= \left[ \begin{array}{cc} -i & 0 \\ 0 & 0 \end{array} \right] \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right)^{-1} \right] .

Now let $v_1 = P_Nv$ and $v_2 = P_{N^\perp}v$. Then

$$h(is\chi) = a + v_2^* v_1 + v_2^* (1 - iA)(A - is)^{-1}(1 + isA)(1 - iA)^{-1} v_2$$

$$= a + is \|v_1\|^2 + \langle (1 - iA)(A - is)^{-1}(1 + isA)(1 - iA)^{-1} v_2, v_2 \rangle_{N^\perp} .$$

To compute $1/s \text{Im} h(is\chi)$, we find

$$\frac{1}{s} \text{Im} (a + is \|v_1\|^2 + \langle (1 - iA)(A - is)^{-1}(1 + isA)(1 - iA)^{-1} v_2, v_2 \rangle)$$

$$= \|v_1\|^2 + \frac{1}{s} \text{Im} \langle (1 - iA)(A - is)^{-1}(1 + isA)(1 - iA)^{-1} v_2, v_2 \rangle$$

$$\geq \|v_1\|^2$$

by Corollary 2.7 in [7]. By hypothesis,

$$0 = \liminf_{s \to \infty} \frac{1}{s} h(is\chi)$$

$$= \liminf_{s \to \infty} \|v_1\|^2 + \frac{1}{s} \text{Im} \langle (1 - iA)(A - is)^{-1}(1 + isA)(1 - iA)^{-1} v_2, v_2 \rangle$$

$$\geq \|v_1\|^2 ,$$

and so $v_1 = 0$. We claim that with the Hilbert space $N^\perp$, the positive decomposition $Y$ of $N^\perp$ given by the compression of the orthogonal decomposition $P$ to $N^\perp$, the operator $A$ on $N^\perp$, the vector $v_2 \in N^\perp$, and the real number $a$, we get that the compression of the type 4 representation to $\otimes_{M^\perp}$ is a type 3 representation for $h$. Recall that
Proof of Theorem 5.9: Type 2.

5.3.3. that is, $h$ from the multiplication to $v$

Since $v_1 = 0$, we can compress the rather unwieldy operator resulting from the multiplication to $N_{\mathcal{M}}^e$, which gives

$h(Z) = \frac{a}{I} + v^* \otimes I M(Z) \otimes I$

\[
= \frac{a}{I} + v^* \otimes I \left[ \begin{array}{c}
\delta_p(Z) \left( \begin{array}{c}
[1 \ 0]_l 
\end{array} \right) - \delta_p(Z) \left( \begin{array}{c}
[0 \ 0]_l 
\end{array} \right) \end{array} \right]^{-1} \times \left( \begin{array}{c}
\delta_p(Z) \left( \begin{array}{c}
[1 \ 0]_l 
\end{array} \right) + \left( \begin{array}{c}
[0 \ 0]_l 
\end{array} \right) \end{array} \right) \times \left( \begin{array}{c}
\sum \left[ \begin{array}{c}
0 \ 0 
\end{array} \right] 
\end{array} \right) \otimes I
\]

\[
= \frac{a}{I} + v^* \otimes I \left[ \begin{array}{c}
\delta_B(Z) \left( \begin{array}{c}
[1 \ 0]_l 
\end{array} \right) - \delta_B(Z) \left( \begin{array}{c}
[0 \ 0]_l 
\end{array} \right) \end{array} \right]^{-1} \times \left( \begin{array}{c}
\delta_B(Z) \left( \begin{array}{c}
[1 \ 0]_l 
\end{array} \right) + \left( \begin{array}{c}
[0 \ 0]_l 
\end{array} \right) \end{array} \right) \times \left( \begin{array}{c}
\sum \left[ \begin{array}{c}
0 \ 0 
\end{array} \right] 
\end{array} \right) \otimes I
\]

\[
= \frac{a}{I} + v^* \otimes I \left[ \begin{array}{c}
\delta_B(Z) \left( \begin{array}{c}
[0 \ 0]_l 
\end{array} \right) \end{array} \right]^{-1} \times \left( \begin{array}{c}
\delta_B(Z) \left( \begin{array}{c}
[1 \ 0]_l 
\end{array} \right) + \left( \begin{array}{c}
[0 \ 0]_l 
\end{array} \right) \end{array} \right) \times \left( \begin{array}{c}
\sum \left[ \begin{array}{c}
0 \ 0 
\end{array} \right] 
\end{array} \right) \otimes I
\]

Since $v_1 = 0$, we can compress the rather unwieldy operator resulting from the multiplication to $N_{\mathcal{M}}^e$, which gives

$h(Z) = \frac{a}{I} + v_2(1-iA) \otimes I \left( \begin{array}{c}
\delta_B(Z) \left( \begin{array}{c}
[0 \ 0]_l 
\end{array} \right) \end{array} \right) \times \left( \begin{array}{c}
\delta_B(Z) \left( \begin{array}{c}
[1 \ 0]_l 
\end{array} \right) + \left( \begin{array}{c}
[0 \ 0]_l 
\end{array} \right) \end{array} \right) \times \left( \begin{array}{c}
\sum \left[ \begin{array}{c}
0 \ 0 
\end{array} \right] 
\end{array} \right) \otimes I
\]

that is, $h$ has a type 3 Nevanlinna representation. 

5.3.3. Type 2.

Proof of Theorem 5.9. Type 2. (1) $\Rightarrow$ (2): Follows from Theorem 5.3.

(3) $\Rightarrow$ (2) is trivial.
(2) ⇒ (1): Suppose that for \( h \in \mathcal{P}_d \),
\[
\liminf_{s \to \infty} s \text{ Im } h(is\chi) < \infty. \tag{5.20}
\]
This obviously implies that
\[
\liminf_{s \to \infty} \frac{1}{s} \text{ Im } h(is\chi) = 0,
\]
and so by Theorem \( 5.9 \) Type 3, there exist \( a, \mathcal{H}, Y, A, \) and \( v \) so that \( h \) has a type 3 representation. As
\[
\delta_Y(is\chi) = \frac{is}{ic},
\]
\[
h(is\chi) = a + \alpha^*(1 - iA)(A - is)^{-1}(1 + isA)(1 - iA)^{-1}\alpha
\]
\[
= a + \langle (1 - iA)(A - is)^{-1}(1 + isA)(1 - iA)^{-1}v, v \rangle_{\mathcal{H}}.
\]
Let \( \nu_{v,v} = \nu \) be the scalar spectral measure for \( A \). Then for \( s > 0 \),
\[
s \text{ Im } h(is\chi) = s \text{ Im } \int \frac{1 + ist}{t - si} \, d\nu(t)
\]
\[
= \int s^2 \frac{(1 + t^2)}{t - is} \, d\nu(t).
\]
As \( s \to \infty \), the integrand increases monotonically to \( 1 + t^2 \). Then by \( 5.20 \)
\[
\int (1 + t^2) \, d\nu(t) < \infty,
\]
and so by the Spectral Theorem
\[
\langle (1 + A^2)v, v \rangle < \infty,
\]
which gives \( v \in D(A) \). Therefore, by Theorem \( 5.10 \) \( h \) has a type 2 representation. \( \square \)

5.3.4. Type 1.

Proof of \( 5.9 \) Type 1. (1) ⇒ (3) follows from \( 5.3 \)

(3) ⇒ (2) is trivial.

(2) ⇒ (1): Suppose that
\[
\liminf_{s \to \infty} s |h(is\chi)| < \infty. \tag{5.21}
\]
As
\[
\liminf_{s \to \infty} s \text{ Im } h(is\chi) \leq \liminf_{s \to \infty} s |h(is\chi)|,
\]
by Theorem \( 5.9 \) Type 2, \( h \) has a Nevanlinna representation of type 2, that is there exist \( \mathcal{H}, Y, A, \) and \( \alpha \in D(A) \) such that
\[
h(Z) = \frac{a}{i} + \alpha^* \left( \frac{A}{i} - \delta_Y(Z) \right)^{-1} \frac{\alpha}{i}.
\]
It remains to show that $a = 0$. For $Z = is\chi$,

$$\delta_Y(is\chi) = \frac{is}{iC}.$$

Then

$$h(is\chi) = a + \langle (A - is)^{-1}\alpha, \alpha \rangle_H.$$

By (5.21), there must exist a sequence $s_n \to \infty$ such that $h(is_n\chi) \to 0$. But on this sequence,

$$\text{Re } h(is_n\chi) = a + \langle A(A^2 + s_n^2)^{-1}\alpha, \alpha \rangle \to a,$$

and therefore it must be the case that $a = 0$. Thus $h$ has a type 1 representation.

□

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