Physical Resurgent Extrapolation

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Expansions of physical functions are controlled by their singularities, which have special structure because they themselves are physical, corresponding to instantons, caustics or saddle configurations. Resurgent asymptotics formalizes this idea mathematically, and leads to significantly more powerful extrapolation methods to extract physical information from a finite number of terms of an expansion, including precise decoding of non-perturbative effects.

I. INTRODUCTION

An important problem in physics is the following: given a physical quantity (free energy, correlator, scattering amplitude, \ldots) expanded in a parameter (temperature, distance, coupling, \ldots) to a finite number of terms in some parametric limit, we wish to extract as much physical information as possible about the function in other parametric regimes \cite{1-19}. For e.g., an extrapolation between weak and strong coupling, real and complex fugacity, or Euclidean and Minkowski space. The original expansion may be convergent, but in many practical cases it is the start of an asymptotic series. If computing further terms is not possible, such an extrapolation appears to be a prohibitively difficult task. However, the series expansions of physical functions are not completely generic; they have further structure which we can exploit. This extra structure arises because saddle points and critical points have physical meaning, and tend to interact in specific ways. Mathematically, this extra structure follows from recent work in resurgent asymptotics \cite{20-22} which shows that functions arising as solutions to systems of equations (differential, difference, integral, \ldots), generally have special orderly structure in the Borel plane. Some ingredients of our analysis are familiar: Borel summation, Padé approximation, conformal mapping, asymptotics of orthogonal polynomials, capacity theory, but we combine these in new ways. This leads to new quantitative measures of the precision of different extrapolations, and novel strategies for decoding non-perturbative physics from limited perturbative information. This motivates the use of resurgence as a discovery tool, an approach with a steadily growing body of evidence in a wide variety of branches of physics \cite{23-39}.

A broad class of physical problems involves analyzing a finite number of terms of an expansion of a function in a physical variable $x$, computed in the limit $x \to +\infty$:

$$F_{2N}(x) = \sum_{n=0}^{2N} \frac{a_n}{x^{n+1}}, \quad x \to +\infty \quad (1)$$

Often this is an asymptotic expansion, with factorial leading large order behavior \cite{40-43}:

$$a_n \sim (-1)^n \Gamma(n - \alpha) \frac{S^n}{n!}, \quad n \to \infty \quad (2)$$

We illustrate our results with this divergent structure because of its physical relevance, but the general results extend to all resurgent functions \cite{44}. The parameters $S$ and $\alpha$ have physical meaning: $S$ is related to the action of a dominant saddle configuration, and $\alpha$ to the power of $x$ in the prefactor from fluctuations about this configuration. We have deliberately chosen the coefficients $a_n$ to be alternating in sign, in order to be as far as possible from a Stokes line, since one of our goals is to probe a non-perturbative Stokes transition by extrapolating from a distant perturbative regime.

There are (at least) 5 natural methods for extrapolating the truncated asymptotic expansion \cite{4}: (i) $F_{2N}(x)$ itself; (ii) Padé in the physical $x$ plane; (iii) Borel-Padé: Padé in the Borel $p$ plane; (iv) Taylor-Conformal-Borel: truncated
extrapolation method with a concrete example that captures the Bender-Wu-Lipatov asymptotics in [2] (we scale x to set S = 1):  

\[ F(x; \alpha) = \frac{e^x \Gamma(1 + \alpha, x)}{x^{1+\alpha}} \sim \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n - \alpha)}{\Gamma(-\alpha)} x^{n+1} \quad (3) \]

\( \Gamma(\beta, x) \) is the incomplete gamma function. \( F(x; \alpha) \) has a branch cut (with parameter \( \alpha \)) along the negative x axis, far from our perturbative region, with a non-perturbative Stokes jump across the cut:

\[ F(e^{i\pi} x; \alpha) - F(e^{-i\pi} x; \alpha) = -\frac{2\pi i}{\Gamma(-\alpha)} e^{-x} \quad (4) \]

We probe: (i) extrapolation from \( x = +\infty \) down to \( x = 0 \); (ii) extrapolation into the complex plane, rotating from the positive to negative real x axis. Case (i) is an analog of a high to low temperature extrapolation, and (ii) is an analog of a non-perturbative Stokes transition, like [1].

The crudest approach is to use the truncated series [1], but the principle of least-term truncation [15] implies one can typically only extrapolate from \( x \rightarrow +\infty \) down to \( x_{\text{min}} \sim N \). Padé approximation in x yields a significant improvement. Padé is a simple algorithmic reprocessing of the input coefficients \( a_n \) [16, 17]. For \( F(x; \alpha) \) in [3], Padé can be written in closed-form in terms of Laguerre polynomials, using the fundamental connection between Padé and orthogonal polynomials (App. A). Large N asymptotics of Laguerre polynomials leads to a uniform estimate for the fractional error, implying that a desired level of precision can be achieved down to a minimum x that scales with the truncation order as \( x_{\text{min}} \sim 1/N \). See Fig. 1.

Borel methods directly yield a further \( \frac{1}{N} \) factor improvement. See Fig. 1. The truncated Borel transform, \( B_{2N}(p) = \sum_{n=0}^{2N} \frac{a_n}{n!} p^n \), regenerates the original truncated series by a Laplace transform:

\[ F_{2N}(x) = \int_0^\infty dp e^{-px} B_{2N}(p) \]

Borel extrapolation is achieved by analytic continuation of the truncated Borel transform \( B_{2N}(p) \). The quality of this continuation in the Borel plane determines the quality of the extrapolation for \( F_{2N}(x) \) in the physical \( x \) plane. For \( F(x; \alpha) \) in [3], the exact Borel transform is

\[ B(p; \alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n n! (-\alpha)}{(\alpha)!} n! p^n = (1 + p)^\alpha \quad (5) \]

with a branch cut on the negative p axis: \( p \in (-\infty, -1) \). The closed-form expression for the diagonal Padé approximation of \( B_{2N}(p) \) is:

\[ \text{PB}_{[N,N]}(p; \alpha) = \frac{P_N^{(\alpha,-\alpha)}(1 + \frac{1}{p})}{P_N^{(-\alpha,\alpha)}(1 + \frac{1}{p})} \quad (6) \]

\( P_n^{(\alpha,\beta)} \) is the \( N^{\text{th}} \) Jacobi polynomial. This Padé-Borel approximation is a ratio of polynomials, with only pole singularities. Padé attempts to represent a cut with an interlacing set of zeros and poles [11, 18, 19]. We see this clearly here because Jacobi polynomial zeros lie on the real axis in the interval \(-1,1\), so the zeros of the denominator in [6] lie along the Borel plane cut, \( p \in (-\infty, -1) \), accumulating to \( p = -1 \).

Away from the cut, the Padé-Borel transform \( \text{PB}_{[N,N]}(p; \alpha) \) is remarkably accurate. Uniform large N asymptotics of the Jacobi polynomials

\[
\begin{array}{|c|c|}
\hline
\text{extrapolation} & x_{\text{min}} \text{ scaling} \\
\hline
\text{truncated series} & x_{\text{min}} \sim N \\
x \text{ Padé} & x_{\text{min}} \sim N^{-1} \\
\text{Padé-Borel} & x_{\text{min}} \sim N^{-2} \\
\text{Taylor-Conformal-Borel} & x_{\text{min}} \sim N^{-2} \\
\text{Padé-Conformal-Borel} & x_{\text{min}} \sim N^{-4} \\
\hline
\end{array}
\]

TABLE I. The scaling with truncation order parameter \( N \) of the minimum real \( x \) value at which a chosen precision can be obtained, for each of the five extrapolation methods discussed here.
quantifies this statement:

$$\text{PB}_{N,N}(p; \alpha) \sim \frac{I_\alpha \left( (N + \frac{1}{2}) \ln \left[ \frac{\sqrt{1 + p^2} + 1}{\sqrt{1 + p^2} - 1} \right] \right)}{I_{-\alpha} \left( (N + \frac{1}{2}) \ln \left[ \frac{\sqrt{1 + p^2} + 1}{\sqrt{1 + p^2} - 1} \right] \right)},$$  \hfill (7)

$I_\alpha$ is the modified Bessel function. For Borel extrapolation, small $z$ behavior is controlled by large $p$ behavior of the Borel transform. Eq. (7) implies $\text{PB}_{N,N}(p; \alpha) \sim p^{\alpha} \left( \frac{N}{p} \right)^{\alpha}$ as $p \to +\infty$.

Thus Padé-Borel is good up to $p \sim N^2$, translating to an $x$ space extrapolation extending down to $x_{\text{min}} \sim 1/N^2$. (See Fig. 1 & App. A). This explains why Padé in the Borel plane is generally more precise than Padé in the physical plane, an old empirical observation in [50].

The most interesting thing about our uniform Padé-Borel approximation (7) is the appearance of the conformally mapped variable $z$:

$$z = \frac{\sqrt{1 + p} - 1}{\sqrt{1 + p} + 1} \quad \leftrightarrow \quad p = \frac{4z}{(1 - z)^2},$$  \hfill (8)

which maps the cut Borel $p$ plane to the interior of the unit disc, $|z| < 1$. Conformal maps are well-known tools for physical resummation problems [5] [17], but the result (7) now explains why and how it works so well: the conformal variable is the natural variable of large order Padé asymptotics. This is a general property of Padé approximations [44] [48] [10], not just for the function $F(x; \alpha)$ in (3).

Another common physical extrapolation, Taylor-Conformal-Borel, does not use Padé, but conformally maps the truncated Borel function to the unit disc in $z$, re-expands and maps back to the Borel $p$ plane [8] [8]. Our methods show that this procedure is comparable to Padé-Borel, with $x_{\text{min}}$ also scaling as $1/N^2$, but sub-leading terms tend to make it slightly better.

A significantly better Borel extrapolation [5] [12] [51] [53] combines the conformal map with a Padé approximation in the conformal $z$ variable, before mapping back to the Borel $p$ plane. We show that this simple extra Padé step yields a further factor of $1/N^2$ improvement in the extrapolation down towards $x = 0$. The closed-form diagonal Padé approximant is now:

$$\text{PCB}_{N,N}(p; \alpha) = \frac{P_N^{(2\alpha,-2\alpha)}(\sqrt{1 + p^2} + 1)}{P_{-2\alpha,2\alpha}^{(2\alpha)}(\sqrt{1 + p^2} - 1)} \quad \hfill (9)$$

$P_{N}^{(\alpha,\beta)}$ is again the $N$th Jacobi polynomial. Uniform large $N$ asymptotics yields (App. A):

$$\text{PCB}_{N,N}(p; \alpha) \sim \frac{I_{2\alpha} \left( (N + \frac{1}{2}) \ln |h(p)| \right)}{I_{-2\alpha} \left( (N + \frac{1}{2}) \ln |h(p)| \right)},$$  \hfill (10)

where the argument now involves the function $h(p) = \left( \frac{\sqrt{1 + p^2} + 1}{\sqrt{1 + p^2} - 1} \right)$, and the Bessel index is $2\alpha$. Contrast (10) with the Padé-Borel result (7). The small $x$ behavior is controlled by the large $p$ behavior of the Borel transform. As $p \to +\infty$, we find $\text{PCB}_{N,N}(p; \alpha) \sim p^{\alpha} \left( \frac{N}{p} \right)^{\alpha}$. Thus $\text{PCB}_{N,N}(p; \alpha)$ extends out to large $p$ scaling like $N^4$, corresponding to extrapolation in $x$ down to $x_{\text{min}}$ scaling as $1/N^4$. See Fig. 1. Since the large $N$ asymptotics (6) [10] are uniform in $p$, we can probe the quality of the extrapolations throughout the complex $x$ plane. The most dramatic superiority of the Padé-Conformal-Borel extrapolation is seen in the non-perturbative region near the negative $x$ axis, which is "as far
as possible” from the starting perturbative expansion region $x \to +\infty$. This region is governed by the Borel transform near the Borel plane cut: $p \in (-\infty, -1]$. Both Padé-Borel and Taylor-Conformal-Borel have unphysical oscillations near the cut, while the Padé-Conformal-Borel transform is extremely accurate. See Fig. 2. This is because the argument of the Jacobi polynomials in (9) is $\frac{1}{z}$, the inverse of the conformal variable $z$ in (8). The Jacobi zeros lie in the interval $(-1, 1)$, so $z$ lies outside the conformal unit disc. Therefore the Padé singularities are on the next Riemann sheet when mapped back to the Borel plane. In other words, the Padé-Conformal-Borel transform has no poles or singularities along the cut. See Fig. 2. It is therefore far better representing non-perturbative Stokes phenomena: see Fig. 3. With just 10 perturbative input coefficients the Padé-Conformal-Borel extrapolation encodes the exact Stokes jump (4), even at small $|x|$. The Padé-Borel extrapolation fails at small $|x|$, due to unphysical poles on the Borel cut. The $x$ space Padé extrapolation is much worse, due to unphysical poles on the $x$ space cut.

Our quantitative extrapolation analysis for the physically motivated model function $F(x; \alpha)$ in (3) generalizes to all resurgent functions, which are universal in physical applications [44]. Resurgent functions have isolated algebraic or logarithmic Borel branch cuts. Even for simple structures with multiple singularities, Padé-Borel fails because it places unphysical poles on artificial arcs along or crossing the Borel integration axis [44, 48, 49], while Padé-Conformal-Borel does not.

A further advantage is that, generically for non-linear problems, each Borel singularity $p_k$ is repeated at integer multiples along the direction $\arg(p_k)$: a physical “multi-instanton” expansion or renormalon structure. Here Padé-Borel fails because it places unphysical poles along this direction (Fig. 2), thereby obscuring the further resurgent Borel singularities. On the other hand, Padé-Conformal-Borel can accurately represent this line of cuts, resolving higher resurgent singularities. This has been demonstrated to high precision for the Painlevé I equation [53], which describes the double-scaling limit of matrix models for 2d quantum gravity [54]. Another physical example is the cusp anomalous dimension, denoted $\Gamma(g)$, in maximally supersymmetric Yang-Mills theory in 4 spacetime dimensions. This quantity satisfies a system of non-linear integral equations, the Beisert-Eden-Staudacher (BES) equations [55]. It is convergent at weak coupling, but
divergent at strong coupling \cite{56}. Its resurgent properties have been studied in \cite{57,58}: \( \Gamma(g) \) has a trans-series structure, as a sum over an infinite tower of saddles, and the fluctuation about each saddle is an asymptotic series. Padé-Borel analysis of the fluctuations about the first and second saddles, suggests an asymmetric Borel plane structure, with leading singularities at \( p = 1 \) and \( p = -4 \), while for the fluctuations about the third saddle the leading singularities are at \( p = \pm 1 \) \cite{57,58}. Padé-Borel methods are not sufficiently powerful to probe beyond these leading singularities, but Padé-Conformal-Borel transforms reveal an intricate structure of repeated higher singularities.

In \cite{44} we prove that for any resurgent function \( f \) the optimal reconstruction accuracy is obtained from the truncated Taylor series of \( f \circ \psi^{-1} \), where \( \psi \) is a uniformization map from the Riemann surface of \( f \) onto the unit disk, with \( \psi(0) = 0 \). Our resurgent analysis also leads to new approximation procedures. Singularity elimination allows one to probe the vicinity of any given Borel singularity with extreme sensitivity. (This can be applied not just in the Borel plane, but also to analyze branch cut singularities in the physical plane, e.g. in the study of phase transitions and critical exponents \cite{2,4,9,10}). A chosen singularity can be modified by Laplace convolution, implementable directly on the series coefficients \cite{44,59}, and a suitable conformal map eliminates it completely \cite{44}. With this method, an initial estimate for the singularity location and exponent can be iteratively refined with extraordinary precision. This is implementable locally on any isolated singularity. The capacity theory interpretation of Padé in terms of a minimal capacitor \cite{48,49}, by which poles are placed as charges on a graph of minimal capacitance, leads to new physically motivated methods to move poles out of the way, to break unphysical pole arcs, and to zoom in on a chosen singularity, leading to dramatic increases in precision \cite{44}. We anticipate that efficient numerical conformal mapping algorithms \cite{60} will be useful for analysis of realistic physical models. Further physical applications will be described elsewhere.

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Appendix A: Appendix

In this Appendix we present some further details of the analytic comparisons between the five different extrapolation methods described in this paper. Table I summarizes at a glance how the minimal \( x \) at which a chosen level of precision can be achieved scales with the truncation order parameter \( N \). Fig. 4 displays the logarithm of the fractional error, as a function of the truncation order parameter \( N \), in the extrapolation from \( x = \infty \) down to a fixed reference value \( x = 1 \). The truncated series at fixed \( x \) gets dramatically worse for larger \( N \), while all other extrapolations improve in precision with increasing \( N \).

![Graph showing logarithm of fractional error](attachment:fig_4.png)

**FIG. 4.** Logarithm of the fractional error in the extrapolation of \( F(x = 1; -\frac{1}{4}) \), as a function of the input truncation order parameter \( N \), extrapolated from a perturbative expansion at \( x = \infty \) down to a fixed reference value \( x = 1 \). The truncated series at fixed \( x \) gets dramatically worse for larger \( N \), while all other extrapolations improve in precision with increasing \( N \).

1. **Truncated series:** For a truncated asymptotic series with coefficients growing like \( n! \), the optimal truncation order is at \( N \sim x \), so if \( N \) is
fixed we can achieve a reasonable precision only for \( x \) extrapolated from \( x = +\infty \) down to some \( x_{\text{min}} \) that scales with \( N \) as \( x_{\text{min}} \sim N \).

2. \( x \)-Padé: An improved extrapolation is achieved by computing a Padé approximant of the truncated asymptotic series in the physical \( 1/x \) variable. For our physical test function \([3]\), with Bender-Wu-Lipatov asymptotics, this Padé approximant can be computed in closed form, which leads to precise asymptotic precision estimates. We find the closed-form:

\[
P_{[N-1,N]}(F(x;\alpha)) = \frac{R_{N-1}(x;\alpha)}{S_N(x;\alpha)} \quad (A1)
\]

where the polynomials \( R_{N-1}(x;\alpha) \) and \( S_N(x;\alpha) \) are in terms of Laguerre polynomials:

\[
S_N(x;\alpha) = N! L_N^{(-1-\alpha)}(-x) \quad (A2)
\]

\[
R_{N-1}(x;\alpha) = \sum_{j=0}^{\left[\frac{x-1}{2}\right]} \frac{\Gamma(N-j)\Gamma(1+\alpha)}{\Gamma(1+\alpha-j)\Gamma(1-j)} L_N^{(2j+1-\alpha)}(-x) \quad (A3)
\]

A general feature of Padé is that the difference between successive near-diagonal approximants can be expressed in terms of successive denominator factors \([46, 47]\). Here this reads:

\[
P_{[N,N+1]}(F(x;\alpha)) - P_{[N-1,N]}(F(x;\alpha)) = \frac{\Gamma(N-\alpha)}{\Gamma(-\alpha)(N+1)! L_N^{(-1-\alpha)}(-x) L_N^{(-1-\alpha)}(-x)} \quad (A4)
\]

The large \( N \) uniform asymptotics of Laguerre polynomials therefore leads to a uniform estimate for the fractional error:

\[
F(x;\alpha) - P_{[N-1,N]}(F(x;\alpha)) \sim e^{-\sqrt{8N\pi}} \quad (A5)
\]

Thus, for a chosen level of precision, one can extrapolate from \( x = +\infty \) down to \( x_{\text{min}} \) which scales with the truncation order as \( x_{\text{min}} \sim \frac{1}{N} \). This is a significant improvement over the naive truncated series \([1]\). Padé is a nonlinear operation, so it does not commute with the Borel transform step. It had been observed empirically in the analysis of the spectrum of the quantum anharmonic oscillator \([50]\), that a Padé approximation in the Borel plane produced more precise results than a Padé approximation in the coupling plane. See also \([61]\). Here we explain why this is the case, and furthermore we quantify the degree of improvement.

For the physical model function \( F(x;\alpha) \) in \([3]\), the closed-form Padé-Borel transform is expressed as a ratio of Jacobi polynomials in \([6]\), and the uniform large \( N \) limit is presented in Eq. \((7)\) as a ratio of modified Bessel functions. This large \( N \) limit is remarkably precise even for small values of \( N \). At small \( p \), which governs the large \( x \) behavior of the extrapolated function in the physical \( x \) plane, we find a fractional error:

\[
\frac{(1+p)^\alpha - \mathcal{PB}_{[N,N]}(p;\alpha)}{(1+p)^\alpha} \sim 2\sin(\pi\alpha) \left(\frac{\sqrt{1+p} - 1}{\sqrt{1+p} + 1}\right)^{2N+1} \quad (A6)
\]

Note the appearance of the conformal variable \( z \) from \([8]\) in this limit. This is general \([44, 48]\). In the opposite limit, as \( p \to +\infty \), which governs the small \( x \) behavior of the extrapolated function in the physical \( x \) plane, we have:

\[
\mathcal{PB}_{[N,N]}(p;\alpha) \sim \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \frac{\Gamma(N+1+\alpha)}{\Gamma(N+1-\alpha)} \times \left(1 + \frac{2\alpha N(N+1)}{(a^2-1)p} + \ldots\right) \quad (A7)
\]

In other words, while the true Borel transform has large \( p \) behavior \( B(p;\alpha) \sim p^\alpha \), the Padé-Borel approximation behaves as \( \mathcal{PB}(p;\alpha) \sim N^{2\alpha} \), implying that in a uniform large \( N \) and large \( p \) limit, the Borel variable \( p \) scales with \( N^2 \). Thus, there is good agreement between \( \mathcal{PB}(p;\alpha) \) and the true Borel transform up to a \( p \) value that scales as \( N^2 \) with the truncation
order. A large $N$ analysis of the Laplace integral, $F_{2N}(x) = \int_0^\infty dp e^{-px} B_{2N}(p)$, using the uniform asymptotics in (7) leads to the fractional error of the Padé-Borel extrapolation in the physical $x$ plane:

$$\text{fractional error}_{\text{PB}}(x; N; \alpha) \sim 2 \sqrt{\frac{\pi}{3}} \frac{\sin(\pi \alpha)}{\sqrt{x}} \left( \frac{2N}{x} \right)^{(2\alpha+1)/3} \times \exp\left[-3 \left(4N^2 x^{1/3} + \frac{x^2}{3} + \ldots\right)\right] \quad (A8)$$

This shows that the leading behavior has $x_{\text{min}}$ scaling with $\frac{1}{\sqrt{x}}$, also specifying the subleading corrections, and agrees well with the numerical results shown in Fig. 4. Note that the dependence on the cut exponent $\alpha$ is subleading.

4. Taylor-Conformal-Borel: Another approximation method, of precision comparable with the Padé-Borel method, does not use a Padé approximation, but instead makes a conformal map in the Borel plane $[6, 8, 11, 13]$. This Taylor-Conformal-Borel extrapolation consists of re-expanding the Borel transform function in the conformal variable to the same order as the original truncated series. This can then be mapped back to the original Borel plane to perform the integral, or equivalently the integral can be done inside the unit disc of the conformal $z$ plane. For our model function $F(x; \alpha)$ in (3), the mapped Taylor-Conformal-Borel transform has the explicit closed-form expression

$$\text{TCB}_{2N}(p; \alpha) = \sum_{l=0}^{2N} \frac{(2\alpha)}{l!} \left( \frac{\sqrt{1+p} - 1}{\sqrt{1+p} + 1} \right)^l \quad (A9)$$

enabling rigorous estimates of the precision [41]. The resulting precision is comparable with, but due to subleading terms is generally slightly better than, the Padé-Borel extrapolation described above. See Fig. 4 and Table I.

5. Padé-Conformal-Borel: A far better extrapolation, which combines the advantages of the Padé-Borel method with those of conformal mapping, is obtained by adding a simple extra step of Padé approximation in the conformally mapped $z$ plane before mapping back to the Borel plane $[9, 12, 61, 63]$. This straightforward extra Padé step leads to a dramatic further improvement in the resulting extrapolation. See Figs. 1 and 4 and Table I.

For the physical model function $F(x; \alpha)$ in (3), the closed-form Padé-Borel transform is expressed as a ratio of Jacobi polynomials in (6), and the uniform large $N$ limit is presented in Eq. (7) as a ratio of modified Bessel functions. This large $N$ limit is remarkably precise even for small values of $N$. At small $p$, which governs the large $x$ behavior of the extrapolated function in the physical $x$ plane, we have a fractional error:

$$\frac{(1+p)^\alpha - \text{PCB}_{[N,N]}(p; \alpha)}{(1+p)^\alpha} \sim 2 \sin(2\pi \alpha) \left( \frac{\sqrt{1+p} - 1}{1 + (1+p)^{1/4} 2\pi} \right)^{2N+1} \quad (A10)$$

There are two important differences compared to the corresponding result for the Padé-Borel transform in (A6). First, the branch cut exponent $\alpha$ appears as $\sin(2\pi \alpha)$ instead of $\sin(\pi \alpha)$, reflecting the fact that for a square root branch cut the conformally mapped function is already rational, so the Padé step is in fact exact. The other difference is the different function of $p$ in (A10). This leads to a further gain of a factor of $1/4\pi$ in the precision at small $p$, and hence a similar gain in precision at large $x$.

In the opposite limit, as $p \to +\infty$, which governs the small $x$ behavior of the extrapolated function in the physical $x$ plane, we find:

$$\text{PCB}_{[N,N]}(p; \alpha) \sim \Gamma(1-2\alpha) \Gamma(N+1+2\alpha) \Gamma(N+1-2\alpha) \times \left(1 + \frac{4N(N+1)}{(4\alpha^2 - 1)} \frac{1}{\sqrt{p}} + \ldots\right) \sim \frac{\Gamma(1-2\alpha)}{\Gamma(1+2\alpha)} p^\alpha \left( \frac{N^4}{p} \right)^\alpha \quad (A11)$$

Thus PCB$_{[N,N]}(p; \alpha)$ extends accurately out to large $p$ scaling like $N^4$, which translates to a high quality extrapolation in $x$ down to $x_{\text{min}}$. 


scaling like $1/N^4$. See Figs. 1 and 4 and Table 1. A large $N$ analysis of the Borel integral back to the physical $x$ plane, using the uniform asymptotics in (10), leads to the fractional error of the Padé-Conformal-Borel extrapolation in the physical $x$ plane:

$$\text{fractional error}_{\text{PCB}}(x,N;\alpha) \sim 2\sqrt{\frac{\pi}{2}} \frac{\sin(2\pi\alpha)}{\sqrt{x}} \left( N^4 x \right)^{2(2\alpha+1)/5} \exp \left[ -5 \left( N^4 x \right)^{1/5} - \frac{4}{5} \frac{N^2}{x} \left( N^4 x \right)^{3/5} + 3x \right]$$

This shows that the leading behavior has $x_{\text{min}}$ scaling with $1/N^4$, also specifying the subleading corrections, and agrees well with the numerical results shown in Fig. 1.

Another instructive visualization of the relative quality of the extrapolation methods based on Borel transforms involves comparing the accuracy with which the method approximates the true Borel function, especially near the Borel plane cut. This is shown in Fig. 5 for our physical model function (6), where we see that the Padé-Conformal-Borel transform is extremely precise near the cut, while the Padé-Borel transform introduces unphysical poles along the cut, and the Taylor-Conformal-Borel transform has unphysical oscillations along the cut.

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FIG. 5. Plots of the imaginary part of the Borel transform in the complex $p$ plane, with parameters chosen to be $\alpha = -\frac{1}{3}$, and truncation order parameter $N = 5$. The first plot is the exact Borel function, $B(p) = (1 + p)^{-1/3}$, with a cut along $p \in (-\infty, -1]$. The next shows the Padé-Borel approximation, with its spurious poles placed along the cut. The third is the Taylor-Conformal-Borel transform, which also has unphysical oscillations along the cut, but of smaller magnitude. The last plot is the Padé-Conformal-Borel transform, which is extremely precise near the cut, with no unphysical singularities on the cut.