Knot invariants and higher representation theory II: 
the categorification of quantum knot invariants

Ben Webster
Department of Mathematics
Northeastern University
Boston, MA
Email: b.webster@neu.edu

Abstract. We construct knot invariants categorifying the quantum knot vari-
ants for all representations of quantum groups. We show that these invariants 
coincide with previous invariants defined by Khovanov for $sl_2$ and $sl_3$ and by 
Mazorchuk-Stroppel and Sussan for $sl_q$.

Our technique uses categorifications of the tensor product representations 
of Kac-Moody algebras and quantum groups, constructed in part I of this 
paper. These categories are based on the pictorial approach of Khovanov and 
Lauda. In this paper, we show that these categories are related by functors cor-
responding to the braiding and (co)evaluation maps between representations 
of quantum groups. Exactly as these maps can be used to define quantum 
invariants attached to any tangle, their categorifications can be used to define 
knot homologies.

We also investigate the structure of the categorifications of tensor prod-
ucts more thoroughly; in particular, we show that the projectives categorify 
Lusztig’s canonical tensor of tensor products when the algebra is of symmetric 
type (the corresponding result for non-symmetric type is false).

Much of the theory of quantum topology rests on the structure of monoidal cat-
egories and their use in a variety of topological constructions. In this paper, we 
define a categorification of one of these: the R-matrix construction of quantum knot 
invariants, following Reshetikhin and Turaev [Tur88, RT90].

They construct polynomial invariants of framed knots by assigning natural maps 
between tensor products of representations of a quantized universal enveloping 
algebra $U_q(\mathfrak{g})$ to each ribbon tangle labeled with representations. These maps are 
natural with respect to tangle composition; thus they can be reconstructed from a 
small number of constituents, most notably the maps associated to a single ribbon 
twist, single crossing, single cup and single cap. The map associated to a link whose 
components are labeled with a representation of $\mathfrak{g}$ (or the corresponding highest 
weight) is thus simply a Laurent polynomial.

Particular cases of these include:

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- the **Jones polynomial** when $\mathfrak{g} = \mathfrak{sl}_2$ and all strands are labeled with the defining representation.
- the **colored Jones polynomials** for other representations of $\mathfrak{g} = \mathfrak{sl}_2$.
- specializations of the **HOMFLYPT polynomial** for the defining representation of $\mathfrak{g} = \mathfrak{sl}_n$.
- the **Kauffman polynomial** (not to be confused with the Kauffman bracket, a variant of the Jones polynomial) for the defining representation of $\mathfrak{so}_n$.

These special cases have been categorified to knot homologies from a number of perspectives by Khovanov and Khovanov-Rozansky [Kho00, Kho02, Kho04, Kho07, KR08a, KR07, KR08b], Stroppel and Mazorchuk-Stroppel [Str05, MSa], Sussan [Sus07], Seidel-Smith [SS06], Manolescu [Man07], Cautis-Kamnitzer [CK08a, CK08b], Mackaay, Stošić and Vaz [MSV09, MSV] and the author and Williamson [WW]. However all of these have only considered minuscule representations (of which there are only finitely many in each type).

There has been some progress on other representations of $\mathfrak{sl}_2$. In a paper still in preparation, Stroppel and Sussan also consider the case of the colored Jones polynomial [SS] (building on previous work with Frenkel [FS]); it seems likely their construction is equivalent to ours via the constructions of Section 4. Similarly, Cooper, Hogancamp and Krushkal have given a categorification of the 2-colored Jones polynomial in Bar-Natan’s cobordism formalism for Khovanov homology [?].

On the other hand, the work of physicists suggests that categorifications for all representations exist; one schema for defining them is given by Witten [Wit]. The relationship between these invariants arising from gauge theories and those presented in this paper is completely unknown (at least to the author) and presents a very interesting question for consideration in the future.

However, the vast majority of representations previously had no homology theory attached to them. In this paper, we will construct such a theory for any labels; that is,

**Theorem A** For each simple complex Lie algebra $\mathfrak{g}$, there is a homology theory $\mathcal{K}(L, \{\lambda_i\})$ for links $L$ whose components are labeled by finite dimensional representations of $\mathfrak{g}$ (here indicated by their highest weights $\lambda_i$), which associates to such a link a bigraded vector space whose graded Euler characteristic is the quantum invariant of this labeled link.

This theory coincides up to grading shift with Khovanov’s homologies for $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$ when the link is labeled with the defining representation of these algebras, and the Mazorchuk-Stroppel-Sussan homology for the defining representation of $\mathfrak{sl}_n$.

Conjecturally, the Mazorchuk-Stroppel-Sussan homology is canonically isomorphic to Khovanov-Rozansky homology (see [MSa §7]); they both categorify the same knot invariants.
At the moment, we have not proven that this theory is functorial, but we do have a proposal for the map associated to a cobordism when the weights $\lambda_i$ are all minuscule. As usual in knot homology, this proposed functoriality map is constructed by picking a Morse function on the cobordism, and associating simple maps to the addition of handles. At the moment, we have no proof that this definition is independent of Morse function and we anticipate that proving this will be quite difficult.

Our method for this construction is to categorify every structure on the ribbon category of $U_q(g)$-representations used in the original definition: its braiding, ribbon structure, and rigid structure (the functor of taking duals). This approach was pioneered by Stroppel for the defining rep of $sl_2$ [Str] and was extended to $sl_n$ by Sussan [Sus07] and Mazorchuk-Stroppel [MSa]. But for our approach, we must use much less familiar categories than the variations of category $\mathcal{O}$ used by those authors. These categories are introduced by the author in [Web], and our primary task in this paper to construct and check relations between functors analogous to the translation and twisting functors that appear in the $sl_n$ case (which our construction will specialize to).

The principal result of [Web] is that for each ordered $\ell$-tuple $\Lambda = (\lambda_1, \ldots, \lambda_\ell)$ of dominant weights of $g$, there is a graded finite dimensional algebra $T^{\Lambda}$ whose representations are a module category for the categorification of $U_q(g)$ in the sense of Rouquier and Khovanov-Lauda and whose graded Grothendieck group $K_0(T^{\Lambda})$ is an integral form of the $U_q(g)$-representation $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}$.

In this paper, we strengthen the case for viewing $\mathcal{V}^{\Lambda}$, the category of finite dimensional $T^{\Lambda}$-modules and its derived category $\mathcal{V}^{\Lambda} = D(\mathcal{V}^{\Lambda})$ as categorifications of tensor products of $U_q(g)$-modules:

**Theorem B** The derived category $\mathcal{V}^{\Lambda}$ carries functors categorifying all the structure maps of the ribbon category of $U_q(g)$-modules:

(i) If $\sigma$ is a braid, then we have an exact functor $B_\sigma: \mathcal{V}^{\Lambda} \to \mathcal{V}^{\tau(\Lambda)}$ such that the induced map $K_0(T^{\Lambda}) \to K_0(T^{\tau(\Lambda)})$ is the action of the appropriate composition of $R$-matrices and flips. Furthermore, these functors induce a weak action of the braid groupoid on the categories associated to permutations of the set $\Lambda$.

(ii) If two consecutive elements of $\Lambda$ label dual representations and $\Lambda^{-}$ denotes the sequence with these removed, then there are functors $T, E: \mathcal{V}^{\Lambda} \to \mathcal{V}^{\Lambda^{-}}$ which induces the quantum trace and evaluation on the Grothendieck group, and similarly functors $K, C: \mathcal{V}^{\Lambda} \to \mathcal{V}^{\Lambda}$ for the coevaluation map and quantum cotrace maps.

(iii) When $g = sl_n$, the structure functors above can be described in terms of twisting and Enright-Shelton functors on $\mathcal{O}$. 

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The indecomposable projective objects in $\mathcal{B}^\lambda$ define a basis of $K^0(T^\lambda)$, which we call the \textbf{orthodox} basis; when the Cartan matrix is symmetric, this basis coincides with Lusztig’s canonical basis for tensor products [Lus92]. This basis always has the property that totally non-negative elements of a group exponentiating $g$ act with non-negative real matrix coefficients.

As mentioned earlier, these functors have a topological interpretation: the algebra $T^\lambda$ is defined using red strands labeled with weights; we imagine placing them in $\mathbb{R}^3$ and thickening them to ribbons (so that we keep track of twists in them). Then our functors correspond to the following operations on ribbons:

- Crossing two ribbons: the corresponding operator in representations of the quantum group is called the \textbf{braiding} or \textbf{R-matrix}.\footnote{As usual, the R-matrix is a map between tensor products of representations $V \otimes W \to V \otimes W$ intertwining the usual and opposite coproducts; we use the term braiding to refer to the composition of this with the usual flip map, which is thus a homomorphism of representations $V \otimes W \to W \otimes V$.}
- Creating a cup, or closing a cap: the corresponding operators in representations of the quantum group are called the \textbf{coevaluation} and \textbf{quantum trace}.
- Adding a full twist to one of the ribbons: the corresponding operator in the quantum group is called the \textbf{ribbon element}.

Since all ribbon knots can be built using these operations, the quantum knot invariants are given by a composition of the decategorifications of the functors constructed in Theorem B, as described in [CP95, §4]; combining the functors themselves in the same pattern gives the knot homology of Theorem A.

Let us now summarize the structure of the paper.

- In Section 1, we prove Theorem B(i). That is, we construct the functor lifting the braiding of the monoidal category of $\mathcal{U}_q(\mathfrak{g})$-representations. This functor is derived tensor product with a natural bimodule. A particularly interesting and important special case is the functors corresponding to the half-twist braid, which sends projective modules to tiltings and the full twist braid, which we show gives the right Serre functor of $\mathcal{V}^\lambda$. In this section, we study the orthodox basis mentioned above and show it coincides with Lusztig’s canonical basis in the case of symmetric Cartan matrix.
- In Section 2, we prove Theorem B(ii). The most important element of this is to identify a special simple module in the category for a pair of dual fundamental weights, which categorifies an invariant vector. Interestingly, we are essentially forced to choose a non-standard ribbon element in order to obtain a ribbon functor which fits the same compatibilities. This means we will categorify the knot invariants for a slightly unusual ribbon structure on

\footnote{The word “orthodox” comes from the Greek ὀρθός “correct” + δόξα “belief”; it is a basis we can believe in.}
the category of \( U_q(\mathfrak{g}) \) modules, but this will only have the effect of multiplying the quantum invariants by an easily determined sign (see Proposition 3.3).

- In Section 3, we prove Theorem A using the functors constructed in Theorem B and a small number of explicit computations. We also suggest a map for the functoriality along a cobordism between links. However, this map is defined by choosing a handle decomposition of the cobordism, and at the moment, we have no proof that the induced map is independent of this choice.

- In Section 4, we consider the special case where \( \mathfrak{g} \cong \mathfrak{sl}_N \); in our previous paper [Weba], we showed that the categories in this case are related to category \( \mathcal{O} \) for \( \mathfrak{gl}_N \). Now we relate the functors appearing Theorem B to previously defined functors on category \( \mathcal{O} \). This allows us to show the portions of Theorem A regarding comparisons to Khovanov homology and Mazorchuk-Stroppel-Sussan homology.

**Notation.** We let \( \mathfrak{g} \) be a finite-dimensional simple complex Lie algebra, which we will assume is fixed for the remainder of the paper. In future work, we will investigate tensor products of highest and lowest weight modules for arbitrary symmetrizable Kac-Moody algebras, hopefully allowing us to extend the contents of Sections 1, 2 and 3 to this case.

We fix from now on an order on the simple roots of \( \mathfrak{g} \), which we will simply denote with \( i < j \) for two nodes \( i, j \). This choice is purely auxiliary, but will be useful for breaking symmetries.

Consider the weight lattice \( Y(\mathfrak{g}) \) and root lattice \( X(\mathfrak{g}) \), and the simple roots \( \alpha_i \) and coroots \( \alpha_i^\vee \). Let \( c_{ij} = \alpha_j^\vee(\alpha_i) \) be the entries of the Cartan matrix. Let \( D \) be the determinant of the Cartan matrix. For technical reasons, it will often be convenient for us to adjoint a \( D \)th root of \( q \), which we denote \( q^{1/D} \).

We let \( \langle \cdot, \cdot \rangle \) denote the symmetrized inner product on \( Y(\mathfrak{g}) \), fixed by the fact that the shortest root has length \( \sqrt{2} \) and

\[
2 \frac{\langle \alpha_i, \lambda \rangle}{\langle \alpha_i, \alpha_i \rangle} = \alpha_i^\vee(\lambda).
\]

As usual, we let \( 2d_i = \langle \alpha_i, \alpha_i \rangle \), and for \( \lambda \in Y(\mathfrak{g}) \), we let

\[
\lambda^i = \alpha_i^\vee(\lambda) = \langle \alpha_i, \lambda \rangle / d_i.
\]

We let \( \rho \) be the unique weight such that \( \alpha_i^\vee(\rho) = 1 \) for all \( i \) and \( \rho^\vee \) the unique coweight such that \( \rho^\vee(\alpha_i) = 1 \) for all \( i \). We note that since \( \rho \in 1/2X \) and \( \rho^\vee \in 1/2Y^* \), for any weight \( \lambda \), the numbers \( \langle \lambda, \rho \rangle \) and \( \rho^\vee(\lambda) \) are not necessarily integers, but \( 2\langle \lambda, \rho \rangle \) and \( 2\rho^\vee(\lambda) \) are (not necessarily even) integers.

Throughout the paper, we will use \( \Lambda = (\lambda_1, \ldots, \lambda_\ell) \) to denote an ordered \( \ell \)-tuple of dominant weights, and always use the notation \( \lambda = \sum \lambda_i \).
We let \( U_q(g) \) denote the deformed universal enveloping algebra of \( g \); that is, the associative \( \mathbb{C}(q^{1/p}) \)-algebra given by generators \( E_i, F_i, K_\mu \) for \( i \) and \( \mu \in \mathfrak{Y}(g) \), subject to the relations:

i) \( K_0 = 1, K_\mu K_{\mu'} = K_{\mu + \mu'} \) for all \( \mu, \mu' \in \mathfrak{Y}(g) \),
ii) \( K_\mu E_i = q^{\alpha_i(\mu)} E_i K_\mu \) for all \( \mu \in \mathfrak{Y}(g) \),
iii) \( K_\mu F_i = q^{-\alpha_i(\mu)} F_i K_\mu \) for all \( \mu \in \mathfrak{Y}(g) \),
iv) \( E_i F_j - F_j E_i = \delta_{ij} \frac{\tilde{K}}{q_{i-j} - q^{-i-j}} \), where \( \tilde{K}_{\pm i} = K_{\pm d_i \mu} \).

v) For all \( i \neq j \)

\[
\sum_{a+b=-c_{ij}+1} (-1)^a E_i^{(a)} E_j^{(b)} = 0 \quad \text{and} \quad \sum_{a+b=-c_{ij}+1} (-1)^a F_i^{(a)} F_j^{(b)} = 0.
\]

This is a Hopf algebra with coproduct on Chevalley generators given by

\[
\Delta(E_i) = E_i \otimes 1 + \tilde{K}_i \otimes E_i \quad \Delta(F_i) = F_i \otimes \tilde{K}_i + 1 \otimes F_i
\]

and antipode on these generators defined by \( S(E_i) = -\tilde{K}_i E_i, S(F_i) = -F_i \tilde{K}_i \).

We should note that this choice of coproduct coincides with that of Lusztig [Lus93], but is opposite to the choice in some of our other references, such as [CP95, ST]. In particular, we should not use the formula for the R-matrix given in these references, but that arising from Lusztig’s quasi-R-matrix. There is a unique element \( \Theta \in U_q(g) \otimes U_q^\ast(g) \) such that \( \Delta(u) \Theta = \Theta \Delta(u) \), where

\[
\Delta(E_i) = E_i \otimes 1 + \tilde{K}_i \otimes E_i \quad \Delta(F_i) = F_i \otimes \tilde{K}_i + 1 \otimes F_i.
\]

If we let \( A \) be the operator which acts on weight vectors by \( A(v \otimes w) = q^{\langle \text{wt}(v), \text{wt}(w) \rangle} v \otimes w \), then as noted by Tingley [Tin 2.10], \( R = A \Theta^{-1} \) is a universal R-matrix for the coproduct \( \Delta \) (which Tingley denotes \( \Delta^p \)). This is the opposite of the R-matrix of [CP95] (for example).

We let \( U_{q^2}(g) \) denote the Lusztig (divided powers) integral form generated over 
\( \mathbb{Z}[q^{1/D}, q^{-1/D}] \) by \( \frac{E_i}{\frac{1}{d_i}} \otimes \frac{F_i}{\frac{1}{d_i}} \) for all integers \( n \) of this quantum group. The integral form of the representation of highest weight \( \lambda \) over this quantum group will be denoted by \( V_\lambda^{\mathbb{Z}} \), and \( V_\lambda^{\mathbb{Z}} = V_\lambda^{\mathbb{Z}_1} \otimes \mathbb{Z}[q^{1/D}, q^{-1/D}] \) \( \cdots \otimes \mathbb{Z}[q^{1/D}, q^{-1/D}] \) \( V_\lambda^{\mathbb{Z}} \). We let \( V_\Lambda = V_\Lambda^{\mathbb{Z}} \otimes \mathbb{Z}[q^{1/D}, q^{-1/D}] \mathbb{Z}(q^{1/D}) \) be the tensor product with the ring of integer valued Laurent series in \( q^{1/D} \); this is the completion of \( V_\Lambda^{\mathbb{Z}} \) in the \( q \)-adic topology.

We let \( T^\mathbb{Z} \) be the algebra of red and black strands defined in [Weba, §2] and let \( \mathfrak{B}_{\mathbb{Z}} = T^\mathbb{Z} \mod \) the category of graded finite dimensional representations of \( T^\mathbb{Z} \) graded by \( 1/D \mathbb{Z} \). This is a minor conventional difference with [Weba], where \( \mathbb{Z} \)-graded modules were used, but this is such a minor difference we felt it did not merit a notational change.

We let \( \mathcal{V} = D^1(\mathfrak{B}_{\mathbb{Z}}) \) be the derived category of complexes of projective objects in \( \mathfrak{B}_{\mathbb{Z}} \) which are 0 in homological degree \( j \) and internal degree \( i \) if \( i + j \ll 0 \) or \( j \gg 0 \).
(here we take the convention that the differential increases homological degree). This notation agrees with that of [BGS96 §2.12].

As before, the ring $\mathbb{Z}[q^{1/q}, q^{-1/q}]$ acts on $K_0(\mathcal{T}_\Lambda)$ by $q^A[M] = [M(A)]$ for any $A \in \mathbb{Q}\mathbb{Z}$. We note that $K_0(\mathcal{V}_\Lambda)$ is the completion of $K_0(\mathcal{T}_\Lambda)$ in the $q$-adic topology; thus corresponding to the isomorphism of [Weba, Theorem 3.6], we also have an isomorphism $K_0(\mathcal{V}_\Lambda) \cong V_\Lambda$ as $\mathbb{Z}((q^{1/q}))$-modules.

We will freely use other notation from the companion paper [Weba], but as a courtesy to the reader, we include a list of the most important such notations:

1. **Braiding functors**

1.1. **Braiding.** Recall that the category of integrable $U_q(g)$ modules (of type I) is a braided category; that is, for every pair of representations $V, W$, there is a natural isomorphism $\sigma_{VW}: V \otimes W \to W \otimes V$ satisfying various commutative diagrams (see, for example, [CP95 5.2B], where the name “quasi-tensor category” is used instead). This braiding is described in terms of an $R$-matrix $R \in U(g) \otimes U(g)$, where we complete the tensor square with respect to the kernels of finite dimensional representations, as usual.

As we mentioned earlier, we were left at times with difficult decisions in terms of reconciling the different conventions which have appeared in previous work. One which we seem to be forced into is to use the opposite $R$-matrix from that usually chosen (for example in [CP95]), which would usually be denoted $R_{21}$. Thus, we must be quite careful about matching formulas with references such as [CP95].

Our first task is to describe the braiding in terms of an explicit bimodule $\mathcal{B}_\sigma$ attached to each braid. Let us describe the bimodule $\mathcal{B}_{\sigma_k}$ attached to a single positive crossing of the $k$th and $k + 1$st strands.

Like the algebra $\mathcal{T}_\Lambda$, the bimodule $\mathcal{B}_{\sigma_k}$ is spanned by pictures. In fact, it is spanned by pictures which are identical to those used in the definition of $\mathcal{T}_\Lambda$, except that we must have a single crossing between the $k$th and $k + 1$st red strands. These pictures are acted upon on the left by $\mathcal{T}_\Lambda$ and on the right by $T_{\sigma_k}\Lambda$ in the obvious way. More
generally, we can view the sum of these over all \( \lambda \) as a bimodule over the universal algebra \( T = \bigoplus \lambda T^\lambda \). The module \( \mathcal{B}_\sigma \) is homogeneous, where a diagram is assigned a grading as in \([\text{Weba}, \S 2.1]\), but with the red crossing given degree \(-\langle \lambda_k, \lambda_{k+1} \rangle\).

\[
\begin{array}{cccc}
\lambda_1 & \cdots & \lambda_k & \lambda_{k+1} \\
\vdots & & \vdots & \vdots \\
\lambda_1 & \cdots & \lambda_k & \lambda_{k+1}
\end{array}
\]

**Figure 1.** An example of an element of \( \mathcal{B}_\sigma \).

As before, we need to mod out by relations:

- We impose all local relations from \([\text{Weba}, \S 2]\), including planar isotopy.
- Furthermore, we have to add the relations (along with their mirror images)

\[
\begin{align*}
\begin{array}{ccc}
\lambda_k & \lambda_{k-1} & \\
\lambda_{k-1} & \lambda_k & \\
\lambda_k & \lambda_{k-1} & \\
\lambda_{k-1} & \lambda_k &
\end{array} & = & \begin{array}{ccc}
\lambda_k & \lambda_{k+1} & \\
\lambda_{k+1} & \lambda_k & \\
\lambda_k & \lambda_{k+1} & \\
\lambda_{k+1} & \lambda_k &
\end{array}
\end{align*}
\]

Following our convention in \([\text{Weba}\]), we use \( \tilde{\mathcal{B}}_\sigma \) to denote the corresponding \( \tilde{T}^A - \tilde{T}^\sigma \cdot \tilde{A} \) bimodule where the relation that any diagram with a violating strand is 0 is *not* imposed.

Recall that for any permutation \( w \), there is a unique positive braid \( \sigma_w \) which induces that permutation on the ends of the strands of the same length of the permutation, constructed by a picking a reduced expression \( w = s_{i_1} \cdots s_{i_m} \), and taking the product \( \sigma_w = \sigma_{i_1} \cdots \sigma_{i_m} \). We call this the permutation’s **minimal lift**. If \( \sigma = \sigma_{i_1} \cdots \sigma_{i_m} \) is a positive braid, we let

\[
\begin{align*}
\mathcal{B}_\sigma &= \mathcal{B}_{\sigma_{i_1}} \otimes_T \cdots \otimes_T \mathcal{B}_{\sigma_{i_m}} \\
\tilde{\mathcal{B}}_\sigma &= \tilde{\mathcal{B}}_{\sigma_{i_1}} \otimes_T \cdots \otimes_T \tilde{\mathcal{B}}_{\sigma_{i_m}}
\end{align*}
\]
Recall from [Weba, §2.3] that for any reduced word in $S_{n+\ell}$ which permutes the red strands according to $\sigma$, we obtain an element $\psi_w \in \tilde{\mathfrak{B}}_\sigma$. Fix a choice of reduced word $w$ for each such permutation.

**Proposition 1.1** If $\sigma$ is a minimal lift, then the bimodule $\tilde{\mathfrak{B}}_\sigma$ has a basis given by diagrams $\psi_w$ times an arbitrary monomial in the dots on black strands. These elements span $\mathfrak{B}_\sigma$.

**Proof.** The proof is essentially identical to that of [Weba, Theorem 2.4]; the argument that these elements span is literally the same.

Linear independence is slightly more complex. We note that we have a natural map $\tilde{\mathfrak{B}}_\sigma \otimes \tilde{\mathfrak{B}}_{\sigma'} \to \tilde{\mathfrak{B}}_{\sigma\sigma'}$ given by stacking which makes the sum over all positive braids $\sigma$ into a ring. This ring has a polynomial representation, just like that defined by $\tilde{T}_\lambda$ in the proof of [Weba, Theorem 2.4]. This shows that the map $R \to \tilde{\mathfrak{B}}_\sigma$ given by horizontal composition is injective (since the image acts by Khovanov and Lauda’s polynomial representation). We can reduce to this case by taking any other relation, and composing at the top and bottom with elements pulling all strands to the right. Thus, a non-trivial relation between our claimed basis vectors would give a nontrivial relation between Khovanov and Lauda’s basis for $R$, which is thus a contradiction. □

**Definition 1.2** Let $\mathfrak{B}_{\sigma_k}$ be the functor $- \otimes \mathfrak{B}_{\sigma_k} : D^-(\mathfrak{A}) \to D^-(\mathfrak{A}_{\sigma_k})$.

Here, $D^-(\mathfrak{A})$ refers to the bounded above derived category of $\mathfrak{A}$; a priori, the functor $\mathfrak{B}_{\sigma_k}$ does not obviously preserve the subcategory $\mathfrak{A} \subset D^-(\mathfrak{A})$. In order to show this, and certain other important properties of this functor, we require some technical results.

**Proposition 1.3** The functors $\mathfrak{B}_{\sigma_k}$ commute with all 1-morphisms in $\mathcal{U}$.

**Proof.** Of course, we only need to check this for $\mathfrak{B}_i$ and $\mathfrak{C}_i$. In both cases, there is an obvious map $u \circ \mathfrak{B}_{\sigma_k} \to \mathfrak{B}_{\sigma_k} \circ u$, which is an isomorphism on the $^\sim$-level, by the basis given in Proposition 1.1. The preimage of any element with a violating strand under this map also has a violating strand, so it gives the desired isomorphism. □

**Proposition 1.4** $\mathfrak{B}_j\left(\mathfrak{S}(P_{i_1,\ldots,i_{j+1}})\right) \cong \mathfrak{S}(P_{i_1,\ldots,i_{j+1}})\left((\lambda_j - \alpha(j), \lambda_{j+1})\right)$

**Proof.** The higher terms of the tensor product vanish, since we can reduce to question to one where the crossing is of the only two strands, in which case, $\mathfrak{S}(P_{i,j})$ is projective. Thus $\mathfrak{B}_j\left(\mathfrak{S}(P_{i_1,\ldots,i_{j+1}})\right)$ is the naive tensor product and is generated by the single diagram shown in Figure 2. □

**Corollary 1.5** The action of $\mathfrak{B}_{\sigma}$ categorifies the action of the braiding.
Proof. By Proposition 1.3, the induced action on $V_\lambda$, which we denote by $R_\sigma$, commutes with the action of $U_q^-(g)$. Thus we need only calculate the action of $R_\sigma$ on a pure tensor of a weight vectors with a highest weight vector $v_h$ in the $j + 1$st place, since these generate $V_\lambda$ as a $U_q^-(g)$-representation.

The space of such vectors is spanned by the classes of the form $S_\lambda(P_{i_1} \ldots i_j; \emptyset; \ldots)$. Thus, Proposition 1.4 implies that

$$R_\sigma(v_1 \otimes \cdots \otimes v_j \otimes v_h \otimes \cdots \otimes v_{\ell}) = q^{\langle \wt(v_j), \lambda_{j+1} \rangle} v_1 \otimes \cdots \otimes v_h \otimes v_j \otimes \cdots \otimes v_{\ell}$$

which is exactly what the braiding does to vectors of this form as we noted in the Notation section. Since vectors of this form generate the representation, there is a unique endomorphism with this behavior, and $R_\sigma$ is the braiding. \qed

Lemma 1.6 If $\sigma = \sigma_{i_1} \cdots \sigma_{i_m}$ is a positive braid, then the functor $B_\sigma = B_{\sigma_{i_1}} \cdots B_{\sigma_{i_m}}$ is independent of the choice of word in the generators (up to canonical isomorphism).

If $\sigma$ is a minimal lift of a permutation, then for any projective $P_\kappa^\epsilon$, the module $B_\sigma(P_\kappa^\epsilon)$ has a standard filtration and $B_\sigma(S_\kappa^\epsilon)$ is a module (that is, $\Tor_{i}^{B_\sigma}(S_\kappa^\epsilon, \mathcal{B}_\sigma) = 0$).

In particular, $B_\sigma$ sends $V_\lambda$ to $V_{\sigma \cdot \lambda}$.

Proof. First, we note that assuming the first two paragraphs of the theorem, we find that $\mathcal{B}_{\sigma_{i_1}}$ considered as a left module (which is the same as $\mathcal{B}_{\sigma_{j}}$) has a finite length free resolution. So any projective module $M$ is sent to a finite length complex; since there are only finitely many indecomposable projectives, the amount which this can decrease the lowest degree is bounded below. Thus, a complex of projectives in $C^+(\mathcal{B}_\sigma^\epsilon)$ is sent to another collection of projectives in $C^+(\mathcal{B}_{\sigma \cdot \lambda}^\epsilon)$.

Secondly, note that the independence of choice of word for all positive braids is equivalent to that for minimal lifts, since the braid relations only involve minimal lifts.

We now turn to the other statements of the theorem, and prove these by induction on the length of $w$. This induction is slightly subtle, so rather than attempt each step
in one go, we break the theorem into 3 statements, and induct around a triangle. Consider the three statements (for each positive integer $n$):

\begin{align*}
  p_n &: \text{For all } \sigma \text{ with } \ell(\sigma) = n, \mathcal{B}_\sigma \text{ sends projectives to modules.} \\
  f_n &: \text{For all } \sigma \text{ with } \ell(\sigma) = n, \mathcal{B}_\sigma \text{ sends projectives to objects with standard filtrations, and is independent of reduced expression.} \\
  s_n &: \text{For all } \sigma \text{ with } \ell(\sigma) = n, \mathcal{B}_\sigma \text{ sends standards to modules.}
\end{align*}

Our induction proceeds by showing

$$
\cdots \rightarrow p_n \rightarrow f_n \rightarrow s_n \rightarrow p_{n+1} \rightarrow \cdots
$$

These are all obviously true for $\sigma = 1$, so this covers the base of our induction.

$f_n \rightarrow s_n$: Consider $\text{Tor}^i(S^\kappa_i, S^\kappa'_i)$. By symmetry, we may assume that $(\kappa, i) \not< (\kappa', i')$ in which case $S^\kappa_i$ has a projective resolution where all higher terms are killed by tensor product with $S^\kappa'_i$, since they are projective covers of simples which do not appear as composition factors in $S^\kappa'_i$. Thus, we have $\text{Tor}^i(S^\kappa_i, S^\kappa'_i) = 0$.

Let $\bar{\sigma}$ be a reduced positive braid for the inverse of $\sigma$. Then if we let $\mathcal{B}_\sigma'$ be $\mathcal{B}_\sigma$ with the left and right actions reversed by the dot-anti-automorphism, then $\mathcal{B}_\sigma' \cong \mathcal{B}_\sigma$.

By $f_n$, the bimodule $\mathcal{B}_\sigma$ has a standard filtration as a right module, so $\mathcal{B}_\sigma'$ has a standard filtration as a left module. Thus, we have $\text{Tor}^i(S^\kappa_i, S^\kappa'_i)$ for $i > 0$ and the same holds for any module with a standard filtration.

$s_n \rightarrow p_{n+1}$: We can write $\mathcal{B}_\sigma = \mathcal{B}_{\sigma'} \mathcal{B}_{\sigma''}$ where $\sigma', \sigma''$ are of length $< n + 1$. Thus, by assumption, $\mathcal{B}_{\sigma'}$ sends projectives to standard filtered modules, and $\mathcal{B}_{\sigma''}$ sends standards to modules. The result follows.

$p_n \rightarrow f_n$: Since $\mathcal{B}_\sigma$ sends projectives to modules, the bimodule $\mathcal{B}_\sigma$ is the naive tensor product of those corresponding to individual crossings. The commutation of crossings with no common strands is clear. In order to do a Reidemeister III move, note that any bunch of 3 red strands which does a full twist can have its “triangle” entirely cleared if black strands (since any black strand passing through the triangle must touch two of the sides, and thus can be slid through the place where they cross), by Proposition [11]. The isomorphism is given by simply doing Reidemeister III on the red strands, which interferes with no black ones.

Now, we construct the standard filtration on $D = \mathcal{B}_\sigma P^\kappa_i$. Let $\Phi$ be the parameter set of the standard filtration on the projective as defined in [Weba, §3.4]. We compose each of these permutations with the permutation of the blocks of black strands between two consecutive red strands according to the action of $\sigma$ on the red strands at the left of each block. As before, we can place a partial order on these by considering the preorder on the labeling of the tops of the strands, and then within each labeling using the Bruhat order. The element $y_{\phi}$ which we attach to $\phi \in \Phi$ is again the diagram which permutes the red and black strands according to a reduced word of the permutation.
We construct a filtration $D_{\leq \phi}, D_{< \phi}$ out of these elements and partial order; while the element $y_{\phi}$ involves a choice of reduced word, this filtration is independent of it. Multiplication by $y_{\phi}$ gives a surjection $d : S_{i_{\phi}}^{\kappa_{\phi}} \rightarrow D_{\leq \phi}/D_{< \phi}$, which we aim to show is an isomorphism.

Since $B_{\sigma}$ categorifies the braiding, when $q$ is specialized to 1, it categorifies the permutation map $V_{A} \rightarrow V_{\sigma A}$ and is thus an isometry for $\langle - , - \rangle_1$. In particular,

$$\dim B_{\sigma} = \langle [T^{\sigma A}], \sigma \cdot [T^{A}] \rangle_1 = \sum_{\phi \in \Phi} \langle [T^{\sigma A}], [S_{i_{\phi}}^{\kappa_{\phi}}] \rangle_1 = \sum_{\phi \in \Phi} \dim S_{i_{\phi}}^{\kappa_{\phi}},$$

which shows that all the maps $S_{i_{\phi}}^{\kappa_{\phi}} \rightarrow D_{\leq \phi}/D_{< \phi}$ must be isomorphisms. \[\square\]

Let $\tau$ be a positive lift of the longest element. This is essentially a half twist, but with the blackboard framing, not the one with ribbon half-twists as well.

Recall that a module $M$ over a standardly stratified algebra is called tilting if $M$ has a standard filtration, and $M^\star$ has a filtration by standardizations (which is weaker than a filtration by standards, since those are standardizations of projectives).

**Theorem 1.7** The modules $B_{\tau} P_{i}^{\kappa}$ are tilting, and every indecomposable tilting module is a summand of these tiltings.

**Proof.** We show first that $B_{\tau} P_{i}^{\kappa}$ is self-dual. The pairing that achieves this duality is a simple variant on that described in [Weba, §1.3], where as before, we form a closed diagram and evaluate its constant term. This pairing is pictorially represented in Figure 3.

The non-degeneracy of this pairing follows from that on $P_{i}^{0}$. In [Weba, Lemma 3.19], we have shown that $P_{i}^{\kappa}$ has an embedding into $P_{i}^{c}$ into $P_{i}^{0}$ consistent with the standard filtration, given by left multiplication by the element $\theta_{k}$. By the Tor-vanishing of $B_{\tau}$ paired with any module with a standard filtration, this map induces an inclusion $B_{\tau} P_{i}^{c} \rightarrow P_{i}^{0}$.
By Proposition 1.1, any non-zero diagram in $\mathcal{B}_\tau P_\kappa^i$ can be drawn with a section in the middle where all black strands are right of all red strands. Thus, the map $P_\kappa^0 \to P_\kappa^i$ given by multiplication by $\theta_\kappa$ is not surjective, but the induced map $P_\kappa^0 \to \mathcal{B}_\tau P_\kappa^i$ is.

The pairing of Figure 3 is that induced by these maps. This shows immediately that the perpendicular to the image of the inclusion contains the kernels of the surjection. Since these have the same dimension, they coincide and the pairing is non-degenerate. Thus, $\mathcal{B}_\tau P_\kappa^i$ is self-dual.

By Lemma 1.6, $\mathcal{B}_\tau P_\kappa^i$ has a filtration by standards for any indecomposable projective; the element $\tau$ reverses the pre-order on standards, every standard which appears is below $(\kappa', i')$ is the sequence obtained from reversing the blocks of $(\kappa, i)$, so if $(\kappa, i)$ (and thus $(\kappa', i)$) is stringy, the tilting whose highest composition factor is the head of $S_{\kappa'}^\lambda$. Thus, any tilting is a summand of $\mathcal{B}_\tau$ applied to a projective. □

**Theorem 1.8** The functor $\mathcal{B}_\sigma$ is an equivalence and thus gives a weak braid groupoid action on our categories.

**Proof.** We have already checked the braid relations for positive braids, so the only difficulty is in showing that $\mathcal{B}_\sigma$ is a derived equivalence. We will first show this for $\mathcal{B}_\tau$. The higher Ext’s between tilting modules always vanish so we always have that $\text{Ext}^{>0}(\mathcal{B}_\tau P_\kappa^i, \mathcal{B}_\tau P_{\kappa'}^{i'}) = 0$; thus we need only show that induced map between endomorphisms of these modules is an isomorphism.

It follows from Corollary 1.5 that

$$\dim \text{Hom}(\mathcal{B}_\tau P_\kappa^i, \mathcal{B}_\tau P_{\kappa'}^{i'}) = \langle [\mathcal{B}_\tau P_\kappa^i], [\mathcal{B}_\tau P_{\kappa'}^{i'}] \rangle_1 = \langle [P_\kappa^i], [P_{\kappa'}^{i'}] \rangle_1 = \dim \text{Hom}(P_\kappa^i, P_{\kappa'}^{i'}).$$

The functor $\mathcal{B}_\tau$ induces a map

$$\text{Hom}(P_\kappa^i, P_{\kappa'}^{i'}) \to \text{Hom}(\mathcal{B}_\tau P_\kappa^i, \mathcal{B}_\tau P_{\kappa'}^{i'}).$$

This is injective, since no element of the image kills the element which pulls all black strands to the right of all red strands below all crossings, by [Weba, Lemma 3.19]. Thus, it is surjective by the dimension calculation above.

It follows that $\mathcal{B}_\tau$ is an equivalence. Since it factors through any $\mathcal{B}_{\sigma_k}$ on the left and right, $\mathcal{B}_{\sigma_k}$ is an equivalence as well. □

Recall that the Ringel dual of a standardly stratified category is the category of modules over the endomorphism ring of a tilting generator, that is, the opposite category to the heart of the $t$-structure in which the tiltings are projective.

**Corollary 1.9** The Ringel dual of $\mathcal{B}_\lambda^\Lambda$ is equivalent to $\mathcal{B}_\lambda^{\Lambda^\perp}$.

If $C_i$ and $C'_i$ are semi-orthogonal decompositions indexed by $i \in [1, n]$ then $C'_i$ is the mutation of $C_i$ by a permutation $\sigma$ if the category generated by $C_i$ for $i \leq j$ is the same as that generated by $C'_{\sigma(i)}$ for $i \leq j$. 

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Proposition 1.10 For any braid $\sigma$, $B_\sigma$ sends the semi-orthogonal decomposition of [Weba Proposition 3.20] to its mutation by $\sigma$.

Proof. First, note that we need only show this for $\sigma_k$. Of course, an equivalence sends one semi-orthogonal decomposition to another. Thus, the only point that remains to show is that $B_\sigma \sigma_k$ for $a \leq \beta$ generates the same subcategory as $S'_a$ for $\sigma_k^{-1}(a) \leq \beta$, where $S'_a$ denotes the appropriate standard module in $B_\sigma$. This follows from the fact that $B_\sigma S'_a = B_\sigma P'_a = S''_a$ modulo smaller $S''_a$ where $\kappa'$ and $i'$ are arrived at by moving the $i$th red strand and all black strands between that and the $(i + 1)$-st rightward to the immediate left of the $(i + 2)$-nd. □

1.2. Serre functors. It is a well-supported principle (see, for example, Beilinson, Bezrukavnikov and Mirković [BBM04] or Mazorchuk and Stroppel [MSb]) that for any suitable braid group action on a category, the Serre functor will be given by the full twist. Here the same is true, up to grading shift. Let $\mathcal{R} = B_\sigma^2$ be the functor given by a full positive twist of the red strands. Let $\Xi'$ be the functor sends $M \in V_\lambda$ to $M(-\langle \lambda, \lambda \rangle + \sum_{i=1}^\ell \langle \lambda_i, \lambda_i \rangle)$. Let $V_\lambda^{\text{per}}$ be the full subcategory of $V_\lambda$ given by bounded perfect complexes, that is, objects which have finite projective dimension. We note that in general, this subcategory does not contain many of the important objects in $V_\lambda$; for example, it will contain all simple modules if and only if all $\lambda$ are minuscule.

Proposition 1.11 The right Serre functor of $V_\lambda^{\text{per}}$ is given by $\Xi = \mathcal{R} \Xi'$.

Proof. First consider the action of $\Xi$ on projective-injectives: the twists of red strands are irrelevant to black strands that begin to the right of all of them, so

$$\mathcal{R} \cong \text{Id} \left( \langle \lambda, \lambda \rangle - \sum_{i=1}^\ell \langle \lambda_i, \lambda_i \rangle \right)$$

as functors on the projective-injective category. We let $I^\kappa_i$ be the injective hull of the cosocle of $P^\kappa_i$. Since $I^\kappa_i \cong P^\kappa_i(\langle \lambda, \lambda \rangle - \langle \alpha, \alpha \rangle)$ on this subcategory $\Xi P^\kappa_i \cong P^\kappa_i(\langle \lambda, \lambda \rangle - \langle \alpha, \alpha \rangle) \cong I^\kappa_i$ and so $\Xi$ is the graded Serre functor.

Since they both have costandard and standard filtrations and the same class in the Grothendieck group, we have that $B_\tau^{-1} I^\kappa_i$ and $B_\tau P^\kappa_i$ are the same self-dual tilting module (ignoring grading for the moment). Thus, $\mathcal{R} P^\kappa_i \cong I^\kappa_i$ (again, ignoring the grading). In particular, $\mathcal{R}$ sends projectives to injectives, and is an equivalence by Theorem 1.8. By [MSb Theorem 3.4], the result follows. □

1.3. Canonical bases. For each simple module $L$ (considered without fixed grading), there is a unique stringy sequence (as defined in [Weba §3.3]) $(\kappa, i)$ corresponding to $L$. 14
**Definition 1.12** Let \( P_L \) be the summand of \( P^\kappa_i \) which is a projective cover of \( L \).

This definition is important in that it fixes a preferred grading for every indecomposable projective. We let \( T^\Delta \cong \text{End}(\oplus_i P_L) \); this is a ring which is basic (all simple representations have dimension 1) and graded Morita equivalent to \( T^\Delta \).

**Definition 1.13** The **orthodox basis** is the basis of \( V^Z_\Delta \) defined by the classes of \( [P_L] \).

This is very appealing basis for a number of reasons, the most important of which is its positivity properties:

**Theorem 1.14** Let \( u \in U(g) \) be the class of any 1-morphism in \( U \) (particular, we could take \( u = E_i^{(n)} \) or \( F_i^{(n)} \)); the matrix coefficients of \( u \) acting on the orthodox basis are non-negative integers.

In particular, if \( g \in G \) (an algebraic group with Lie algebra \( g \)) is totally non-negative in the sense of Lusztig \[Lus94\], the matrix coefficients of \( g \) acting on the orthodox basis are positive.

**Question 1.15** Is there a purely algebraic characterization of the orthodox basis?

As we noted in \[Weba, \S 4.3\], if the Cartan matrix is symmetric and \( \kappa \) is characteristic 0, the ring \( T^\Delta \) is positively graded; for a non-symmetric Cartan matrix, it seems that this ring is very rarely positively graded. The arguments below show that this ring being positively graded implies that the orthodox and canonical bases coincide, and in particular implies that the canonical basis having positive structure coefficients, which is known to be false. For example, counterexamples appear in \( G_2, C_3 \) and \( B_4 \); we thank Shunsuke Tsuchioka for pointing us to these counterexamples \[Tsu\].

**Proposition 1.16** For all \( L \), the simple quotient of \( P_L \) is self-dual, and

\[
\text{Hom}_{T^\Delta}(P_L, T^\Delta)^* \cong (\Xi(P_L))^* \cong P_L.
\]

**Proof.** We note that if \((i, \kappa)\) is a stringy sequence for \( L \), then \((P_L/\text{rad } P_L)e_{i,\kappa}\) is 1-dimensional and concentrated in degree 0, so it is graded self-dual. By definition \( \Xi(P_L) \) is the injective hull of the cosocle of \( P_L \), so \((\Xi(P_L))^* \) is the projective cover of the dual of the socle of \( P_L \); the result follows by uniqueness of projective covers. \( \square \)

Recall that in \[Lus93, \text{Ch. } 27.3\], Lusztig defines a \( \mathbb{Z}[q, q^{-1}] \)-antilinear map \( \Psi : V^Z_\Delta \to V^Z_\Delta \) by

\[
\Psi(v_1 \otimes \cdots \otimes v_\ell) = \Theta^{(\ell)}(\bar{v}_1 \otimes \cdots \otimes \bar{v}_\ell)
\]
where $\Theta^{(\ell)}$ is the $\ell$-fold quasi-R-matrix (discussed in [Weba, §3.2]) and $\overline{\cdot}$ is the unique involution satisfying $\overline{\vartheta} = \vartheta$ and $\overline{\theta \cdot \vartheta} = \overline{\theta} \overline{\vartheta}$ for the bar involution on $U_q(\mathfrak{g})$ given by

$$E_i = E_i \quad F_i = F_i \quad K_i = K_i^{-1} \quad \overline{q} = q^{-1}.$$

**Proposition 1.17** The self-map $V^Z_A \to V^Z_A$ induced by $\star \circ \Xi$ is $\Psi$.

**Proof.** The set of $\Psi$-invariant vectors is preserved by the maps $F_i \cdot -$ and $- \otimes \vartheta$. Thus, for any $P^\kappa_i$ we have that $\Psi([P^\kappa_i]) = [P^\kappa_i]$; since these vectors span $V^Z_A$, the map $\Psi$ is uniquely characterized by this property.

On the other hand, for any finite dimensional algebra, the Serre functor on perfect complexes of $A$-modules sends $A$ as a left-module to $A^*$ (with the left-module structure induced by right multiplication). Since $\hat{e}_i, \kappa = \hat{e}_i, \kappa$, we have that $\Xi P^\kappa_i = (P^\kappa_i)^*$ and we are done. $\square$

**Proposition 1.18** If the Cartan matrix is symmetric and $\mathbb{k}$ is characteristic 0, the successive quotients of the standard filtration on $P_L$ consist of one copy of $S_L$ in its natural grading and copies of other standards in strictly positive shifts from their natural gradings.

**Proof.** Any other standard $S_{L'}$ appearing as a subquotient of $P_L$ is induced by a map $P_{L'} \to P_L$ which we already know is positively shifted from the natural grading. $\square$

Lusztig has defined a canonical basis of the tensor product in [Lus92] and [Lus93, Ch. 27.3] when $\mathfrak{g}$ is finite type. For infinite symmetric type, we can use Proposition 1.17 as a definition of the bar involution on a tensor product, and the argument below shows that the orthodox basis has an algebraic characterization exactly like that used by Lusztig in finite type using our bar involution.

**Theorem 1.19** If the Cartan matrix is symmetric, $\mathbb{k}$ is characteristic 0 and either (1) $\mathfrak{g}$ is finite-dimensional or (2) $\mathcal{A} = (\lambda)$, then the orthodox and canonical bases of the tensor product coincide.

**Proof.** First, we prove this result when $\mathcal{A} = (\lambda)$. In this case, we can simply apply the work of Vasserot and Varagnolo [VV, Theorem 4.5]; they show that the indecomposable objects in $\mathcal{U}^-$ correspond to powers of $q$ times Lusztig’s canonical basis under the isomorphism $K_0(\mathcal{U}^-) \cong U(n)$.

Acting on $P_\emptyset$ with such an object in $\mathcal{U}^-$ gives an indecomposable projective (since thought of as a module of the quiver Hecke algebra by pull-back, this is the quotient of an indecomposable projective, and thus indecomposable). Thus the actual canonical basis is the classes of the indecomposable projectives fixed by $\star \circ \Xi$, which we have already described in terms of stringy sequences.
The \( \mathbb{Z}[q^{-1}] \)-lattice Lusztig denotes \( L \) is the \( \mathbb{Z} \)-span of the classes \([S_L(-a)]\) for \( a \in \mathbb{Z}_{\geq 0} \); by Proposition 1.18, this is the same as the \( \mathbb{Z} \)-span of the classes \([P_L(-a)]\) for \( a \in \mathbb{Z}_{\geq 0} \).

The classes of \([S_L(-a)]\) are the elements of the form \( b \otimes b' \) in Lusztig’s notation; by Proposition 1.18, this is the same as the \( \mathbb{Z} \)-span of the classes \([P_L(-a)]\) for \( a \in \mathbb{Z}_{\geq 0} \).

Thus, they are the elements \( b \odot b' \) considered in [Lus93, 27.3.2]. □

2. R rigidity structures

2.1. Coevaluation and evaluation for a pair of representations. Now, we must consider the cups and caps in our theory. The most basic case of this is \( \lambda = (\lambda, \lambda^*) \), where we use \( \lambda^* = -w_0 \lambda \) to denote the highest weight of the dual representation to \( V_\lambda \). It is important to note that \( V_\lambda \cong V_{\lambda^*} \), but this isomorphism is not canonical.

In fact, the representation \( K_0(T^\lambda) \) comes with more structure, since it is an integral form \( V_\Lambda^\mathbb{Z} \). In particular, it comes with a distinguished highest weight vector \( v_h \), the class of the unique simple in \( V_\lambda \) which is 1-dimensional and concentrated in degree 0. Thus, in order to fix the isomorphism above, we need only fix a lowest weight vector \( v_l \) of \( V_{\lambda^*} \), and take the unique invariant pairing such that \( \langle v_h, v_l \rangle = 1 \).

Our first step is to better understand the lowest weight category \( V_\lambda \omega_0 \lambda \); consider a reduced expression \( s \) in the Weyl group \( W \) of \( g \), and let \( s_j \) be the product of the first \( j \) reflections in this word. Consider the sequence

\[
\begin{align*}
&\ i_s^\lambda = (i_1^\lambda, i_2^\lambda, \ldots, i_k^\lambda) \\
&\ i_s^\lambda = (i_1^{(s_1 \lambda)^2}, i_2^{(s_2 \lambda)^2}, \ldots, i_k^{(s_k \lambda)^2})
\end{align*}
\]

Proposition 2.1 The projective \( P_0^0_{i_k^s} \) over \( T^\lambda \) is irreducible, and only depends on the product \( s_k \in W \).

Proof. Let us show this induction. The base case is when the expression is length 1, which is the case of \( \mathfrak{sl}_2 \), which was shown by Lauda [Lau] (this corresponds to the fact that the Grassmannian of \( k \)-planes in \( k \)-space is a point).

In general, it is clear from 1-dimensionality of extremal weight spaces that the category \( \mathbb{B}_{s_k \lambda}^\lambda \) has a unique indecomposable projective and a unique simple, so we need only show that \( \text{Hom}(P_0^0_{i_k^s}, P_0^0_{i_k^s}) = 1 \). Thus, we need only consider diagrams beginning and ending with our preferred idempotent. We claim that such diagrams can be written as a sum of diagrams where no lines of different colors cross. This reduces our proposition to the \( \mathfrak{sl}_2 \) case.

Now consider an arbitrary diagram, and consider the left-most block of strands of a single color whose members cross strands of other colors. If no strands start in this block at the bottom and end up in a different block at the top, then we can simply “pull straight” and have a diagram where the first “bad block” is further right.

If a strand does leave this block traveling upward, it must be matched by one which leaves it traveling downward, and the strands must cross. Using RIII moves,
one can move this crossing left (with correction terms that have fewer such strands, since the correction terms smooth crossings), so that all differently colored strands pass to its left. But then at this crossing, we have reordered the strands so that we get $i^\ast_\lambda$ for some truncation of our word, and then a repetition of the last element. This is a composition of induction functors corresponding to an empty weight space, so is 0. Thus, by induction, we are done.

Fix an expression $s_0$ for the longest element $w_0$ and consider this construction for $i^\lambda = i^\lambda_{s_0}$. We fix $v_l = [P^0_{i^\lambda}]$, and use this to fix an isomorphism $V_{w_0} \cong V_{\lambda}^\ast$ which we use freely throughout the rest of the paper.

We can now consider $P^0_{i^\lambda}$ standardized in two different ways, obtaining two standard modules: $S_{i^\lambda}^{0,2\rho^\vee(\lambda)} = P_{i^\lambda}^{0,2\rho^\vee(\lambda)}$ and $S^0_{i^\lambda}$. Proposition 2.1 shows that the first has simple cosocle and the second is itself simple. We denote the cosocles of these representations by $L$ and $M$.

Recall that the coevaluation $\mathcal{Z}((q)) \to V_{w_0,w_0}$ is the map sending 1 to the canonical element of the pairing we have fixed, and evaluation is the map induced by the pairing $V_{w_0,w_0} \to \mathcal{Z}((q))$.

**Definition 2.2** Let

\[ K^0_{\lambda,\lambda} : D^\dagger(\text{gVect}) \to \mathcal{V}^{\lambda,\lambda} \text{ be the functor } \text{RHom}_k(\hat{L}_\lambda, -)(2(\lambda, \rho))[-2\rho^\vee(\lambda)] \]

and

\[ \mathcal{E}^0_{\lambda,\lambda} : \mathcal{V}^{\lambda,\lambda} \to D^\dagger(\text{gVect}) \text{ be the functor } \otimes_{\text{F}} L^\dagger. \]

These functors preserve the appropriate categories since by [Hua, Theorem 3.15], $L$ has a projective resolution in $D^\dagger(\text{gVect})$.

**Proposition 2.3** The functor $K^0_{\lambda,\lambda}$ categorifies the coevaluation, and $\mathcal{E}^0_{\lambda,\lambda}$ the evaluation.

**Proof.** Since $L$ is self-dual, we must first check that $[L]$ is invariant. Of course, the invariants are the space of vectors of weight 0 such that $[v]|E_i[v] = 0$ for any $i$. Since $P^0_{i^\lambda}$ has no positive degree endomorphisms, any diagram in which a strand passes over the second red strand is in a proper submodule of $P^0_{i^\lambda}$, and so $E_i L = 0$ for all $i$. Thus $[L]$ is invariant. In fact, $L$ is the only such representation, since the $-\lambda^\ast$-weight space of $V_{w_0}$ is 1 dimensional.

Now, we need just check the normalization is correct. Of course, $[L]$’s projection to $(V_{w_0})_{\text{low}} \otimes (V_{w_0})_{\text{high}}$ is

\[ [P^0_{i^\lambda}] \otimes [P^0_{\emptyset}] = F_{i^\lambda} v_h \otimes v_{l^\ast}. \]

Thus, by invariance, the projection to $(V_{w_0})_{\text{high}} \otimes (V_{w_0})_{\text{low}}$ is

\[ v_h \otimes S(F_{i^\lambda}) v_{l^\ast} = (-1)^{2\rho^\vee(\lambda)} q^{-2(\lambda, \rho)} v_h \otimes v_{l^\ast}. \]
On the other hand, one can easily check that $- \otimes_{\mathcal{L}} L_{\Lambda}$ kills all modules of the form $\mathcal{G}_J M$, so it gives an invariant map, whose normalization we, again, just need to check on one element. For example, $P_i^{(0, 0)}(0, 2) \otimes L_{\Lambda} \cong k$, so we get 1 on $v_l \otimes v_h$, which is the correct normalization for the evaluation. □

We represent these functors as leftward oriented cups as in the usual diagrammatic approach to quantum groups, as shown in Figure 4.

In order to analyze the structure of $L_{\Lambda}$ and $M_{\Lambda}$, we must understand some projective resolutions of standards. This can be done with surprising precision in the case where $\ell = 2$.

Define a map $\kappa_j : [1, 2] \to [0, n]$ by $\kappa_j(2) = j$ and $\kappa_j(1) = 0$. Given a subset $T \subset [j + 1, n]$, we let $i_T$ be the sequence given by $i_1, \ldots, i_j$ followed by $T$ in reversed sequence, and then $[j + 1, n] \setminus T$ in sequence and let $\kappa_T(2) = j + \#T$. Let

$$\chi_T = \sum_{k \in T} \left( \alpha_{i_k} - \lambda_2 + \sum_{j < m < k} \alpha_{i_m} \right).$$

**Proposition 2.4** The standard $S_{i_j}^{K_j}$ has a projective resolution of the form

$$\cdots \to \bigoplus_{|T|=n} P_{i_T}^{K_T}(\chi_T) \to \cdots \to P_{i_j}^{K_j} \to S_{i_j}^{K_j} \to 0.$$

**Proof.** We induct on $n - j$. If $j = n$, then $S_{i_j}^{K_j}$ is itself projective, so we may take the trivial resolution. Let $i'$ be $i$ with its last entry removed, and $i''$ be $i$ with its last entry moved to the $j + 1$st position. As we showed in the proof of [Weba, Theorem 3.7], we have an exact sequence

$$0 \to S_{i''}^{K_{i''}}(\langle \alpha_{i''} - \lambda_2 + \sum_{j < \ell < n} \alpha_{i''} \rangle) \to \mathcal{G}_{i''} S_{i''}^{K_{i''}} \to S_{i''}^{K_j} \to 0.$$

Applying the inductive hypothesis, we obtain projective resolutions of the left two factors. Furthermore, we can lift the leftmost map to a map between projective resolutions. The cone of this map is the desired projective resolution of $S_{i_j}^{K_j}$. □
The same principle can be used for any value of $\ell$ to construct an explicit description of a projective resolution for any standard, but carefully writing this down is a bit more subtle and difficult than the $\ell = 2$ case, so we will not do so here. This provides a resolution of $M_\lambda$, since it is itself standard. In particular, it shows that

**Corollary 2.5** $\text{Ext}^i(M_\lambda, L_\lambda) = \begin{cases} 0 & \text{if } i \not= 2\rho^\vee(\lambda) \\ \mathbb{k}(2\langle \lambda, \rho \rangle) & \text{if } i = 2\rho^\vee(\lambda) \end{cases}$.

**Proof.** All of the projectives which appear in the resolution of $M_\lambda$ has no maps to $L_\lambda$ except the last term. We can break up the grading shift of this term into the pieces corresponding to simple reflections in a reduced expression for a longest word of $W$, which are in turn in canonical bijection is with the set of positive roots $R^+$. Thus, we have

$$\sum_{i=1}^n \left( \alpha_{i, r} - \lambda^* + \sum_{m<k} \alpha_{i, m} \right) = \sum_{\alpha \in R^+} \langle \alpha, -\lambda^* \rangle = -2\langle \lambda^*, \rho \rangle = -2\langle \lambda, \rho \rangle$$

which is $P_{i_1}^{2\rho^\vee(\lambda)}(-2\langle \lambda, \rho \rangle)$. Thus we have

$$\text{Ext}^i(M_\lambda, L_\lambda) \cong \text{Ext}^{i-2\rho^\vee(\lambda)}(P_{i_1}^{2\rho^\vee(\lambda)}(-2\langle \lambda, \rho \rangle), L_\lambda)$$

and the result follows. $\square$

It also shows more indirectly that $L_\lambda$ has a beautiful, if more complicated resolution.

**Proposition 2.6** There is a resolution

$$\cdots \rightarrow M_j \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow L_\lambda \rightarrow 0$$

of $L_\lambda$ with the property that

- $M_{2\rho^\vee(\lambda)-j}$ lies in the subcategory generated by $S_i^{k_j}$ for all different choices of $i$.
  
  In particular, if $j > 2\rho^\vee(\lambda)$, then $M_j = 0$.

- $M_{2\rho^\vee(\lambda)} \cong M_\lambda(-2\langle \lambda, \rho \rangle)$.

**Proof.** Since we have

$$\text{Ext}^i(S_i^{k_j}, (S_i^{k_i})^*) = 0 \quad \text{if } j \not= k \text{ or } i > 0,$$

the first property is equivalent to showing that

$$\text{Ext}^m(L_\lambda, (S_i^{k_j})^*) = 0 \text{ if } m \not= j.$$

This follows immediately from replacing $S_i^{k_j}$ by its projective resolution defined in Proposition 2.4.

For the second, we must more carefully analyze this Ext group. By our projective resolution, we have

$$\text{Hom}(M_{2\rho^\vee(\lambda)}, (S_i^{k_j})^*) \cong \text{Ext}^{2\rho^\vee(\lambda)}(L_\lambda, (S_i^{k_j})^*) \cong \mathbb{k}(-2\langle \lambda, \rho \rangle).$$
Thus, we must have $M_{2\rho^\vee(\lambda)} \cong M_{\lambda}(-2\langle \lambda, \rho \rangle)$.

**Corollary 2.7** \( \text{Ext}^i(L_\lambda, M_\lambda) = \begin{cases} 0 & i \neq 2\rho^\vee(\lambda) \\ \mathbb{K}(2\langle \lambda, \rho \rangle) & i = 2\rho^\vee(\lambda) \end{cases} \).

**Corollary 2.8** \( \text{Tor}^i(M_\lambda, \hat{L}_\lambda) = \begin{cases} 0 & i \neq 2\rho^\vee(\lambda) \\ \mathbb{K}(-2\langle \lambda, \rho \rangle) & i = 2\rho^\vee(\lambda) \end{cases} \).

### 2.2. Ribbon structure

This calculation is also important for showing how $L_\lambda$ behaves under braiding.

**Proposition 2.9** \( B_{\sigma^1} L_\lambda \cong L_\lambda[-2\rho^\vee(\lambda)][-2\langle \lambda, \rho \rangle - \langle \lambda, \lambda \rangle] \).

**Proof.** Unless $i$ is a sequence corresponding to weight 0 and $j = \langle \lambda, \rho \rangle$, we have that $B \otimes \hat{P}_i^j$ is of the form $\mathbb{K}(B \otimes \hat{P}_i^j)$ for a shorter sequence $i'$. Thus, $B \otimes \hat{P}_i^j$ has a projective resolution in which $P_{i'}^{k(\lambda,\nu)}$ never appears, and

\[
\mathbb{B}L_\lambda e(i, \kappa) \cong L_\lambda \otimes B \otimes \hat{P}_i^j \cong 0.
\]

Thus, we have an isomorphism of vector spaces

\[
\mathbb{B}L_\lambda e(i, \kappa) \cong L_\lambda \otimes B \otimes \hat{P}_i^j \cong L_\lambda \otimes M_{\lambda}(-\langle \lambda, \lambda \rangle) \cong \mathbb{K}[-2\rho^\vee(\lambda)](-2\langle \lambda, \rho \rangle - \langle \lambda, \lambda \rangle).
\]

As a $T^{X,\lambda}$ representation, $\mathbb{B}L_\lambda$ must be simple, and thus

\[
\mathbb{B}L_\lambda \cong L_\lambda[-2\rho^\vee(\lambda)](-\langle \lambda, \lambda \rangle - 2\langle \lambda, \rho \rangle).
\]

Now, in order to define quantum knot invariants, we must also have have quantum trace and cotrace maps, which can only be defined after one has chosen a ribbon structure. The Hopf algebra $U_q(g)$ does not have a unique ribbon structure; in fact topological ribbon elements form a torsor over the characters $Y/X \rightarrow \{ \pm 1 \}$. Essentially, this action is by multiplying quantum dimension by the value of the character.

The standard convention is to choose the ribbon element so that all quantum dimensions are Laurent polynomials in $q$ with positive coefficients; however, the calculation above shows that this choice is not compatible with our categorification! By Proposition 2.9, we have

\[
\mathbb{B}^2 L_\lambda = L_\lambda[-4\rho^\vee(\lambda)](-4\langle \lambda, \rho \rangle - 2\langle \lambda, \lambda \rangle).
\]

Thus, if we wish to define a ribbon functor $\mathbb{R}$ to satisfy the equations

\[
\mathbb{B}^2 L_\lambda \cong \mathbb{R}^{-2} L_\lambda = \mathbb{R}^{-2} L_\lambda = \mathbb{R}_1^{-1} \mathbb{R}_2^{-1} L_\lambda,
\]

which are necessary for topological invariance (as we depict in Figure 5).

**Definition 2.10** *The ribbon functor* $\mathbb{R}_i$ *is defined by*

\[
\mathbb{R}_i M = M[2\rho^\vee(i_i)][2\langle i_i, \rho \rangle + \langle i_i, i_i \rangle).
\]

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Taking Grothendieck group, we see that we obtain the ribbon element in $U_q(\mathfrak{g})$ uniquely determined by the fact that it acts on the simple representation of highest weight $\lambda$ by $(-1)^{2\rho^\vee(\lambda)}q^{(\lambda,\lambda)+2(\lambda,\rho)}$. This is the inverse of the ribbon element constructed by Snyder and Tingley in [ST]; we must take inverse because Snyder and Tingley use the opposite choice of coproduct from ours. See Theorem 4.6 of that paper for a proof that this is a ribbon element. From now on, we will term this the ST ribbon element. It may seem strange that this element seems more natural from the perspective of categorification than the standard ribbon element, but it is perhaps not so surprising; the ST ribbon element is closely connected to the braid group action on the quantum group, which also played an important role in Chuang and Rouquier’s early investigations on categorifying $\mathfrak{sl}_2$ in [CR08]. It is not surprising at all that we are forced into a choice, since ribbon structures depend on the ambiguity of taking a square root; while numbers always have 2 or 0 square roots in any given field (of characteristic $\neq 2$), a functor will often only have one.

Due to the extra trouble of drawing ribbons, we will draw all pictures in the blackboard framing.

This different choice of ribbon element will not seriously affect our topological invariants; we simply multiply the invariants from the standard ribbon structure by a sign depending on the framing of our link and the Frobenius-Schur indicator of the label, as we describe precisely in Proposition 3.8.

**Proposition 2.11** The quantum trace and cotrace for the ST ribbon structure are categorified by the functors

$$C_{\lambda}^{\Lambda} : D^+(\mathfrak{g}\text{-Vect}) \to \mathcal{V}$$

given by $\text{RHom}(\hat{L}_{\lambda'}^\Lambda,-)[2(\lambda,\rho)]$
and

\[ T^0_{\lambda,\lambda'} : \mathcal{V}^{\lambda,\lambda'} \to D^1(\text{gVect}) \text{ given by } - \otimes T_\perp \hat{L}. \]

Proof. As the picture Figure 6 suggests, by definition the quantum trace is given by applying a negative ribbon twist of one strand, and then applying a positive braiding, followed by the evaluation; that is, it is categorified by

\[ (\text{BR}_1 -) \otimes \hat{L}_\lambda \cong - \otimes (\text{BR}_1 \hat{L}_\lambda) \cong - \otimes \hat{L}_\lambda. \]

The result thus immediately follows from Proposition 2.9 and our definition of \( R \). The same relation between evaluation and quantum trace follows from adjunction. \( \square \)

2.3. Coevaluation and quantum trace in general. More generally, whenever we are presented with a sequence \( \underline{\lambda} \) and a dominant weight \( \mu \), we wish to have a functor relating the categories \( \underline{\lambda} \) and \( \underline{\lambda}^+ = (\lambda_1, \ldots, \lambda_{j-1}, \mu, \mu^*, \lambda_j, \ldots, \lambda_\ell) \). This will be given by left tensor product with a particular bimodule.

The coevaluation bimodule \( \mathcal{R}^{\lambda'}_{\underline{\lambda}} \) is generated by the diagrams of the form
where $v$ is an element of $L_\lambda$ and diagrams only involving the strands between $\mu$ and $\mu^*$ act in the obvious way, modulo the relation (and its mirror image).

One can think of the relation above as categorifying the equality $(F_i v) \otimes K = F_i(v \otimes K)$ for any invariant element $K$.

Let $\mathcal{F}_i^\kappa$ denote composition of functors where one reads the corresponding idempotent from left to right, and applies $\mathcal{F}_i$ when passing a black strand labeled $i$, and $\mathcal{F}_\lambda$ when passing a red strands labeled $\lambda$. This has the useful property that $\mathcal{F}_i^\kappa P_\emptyset = P_i^\kappa$.

We can write $\lambda = \lambda' \lambda''$ and $i = i' i''$ as the union of the red/black strands that come before and after the point where $\mu, \mu^*$ are inserted, with $\kappa', \kappa''$ be the corresponding $\kappa$-functions. Then, we can give an alternate definition of this bimodule by the formula.

$$P_i^\kappa \otimes \mathcal{R}_\kappa^{\kappa''} \cong \mathcal{F}_\kappa^{\kappa''} \left( T^{\lambda'}(\mu, \mu^*) (P_i^\kappa \otimes L_{\mu}) \right).$$

**Definition 2.12** The coevaluation functor is

$$\mathbb{K}_{\lambda}^{\lambda'} = \text{RHom}_{\mathcal{A}}(\mathcal{R}_{\lambda}^{\lambda'}, -) \left(2(\lambda, \rho)[-2\rho^\vee(\lambda)]\right) : \mathcal{V}_{\lambda} \to \mathcal{V}_{\lambda'}.$$  

Similarly, the quantum trace functor is the left adjoint to this given by

$$\mathbb{T}_{\lambda}^{\lambda'} = - \mathcal{L}_{\lambda}^{\lambda'} \mathcal{R}_\lambda^{\lambda'} : \mathcal{V}_{\lambda'} \to \mathcal{V}_{\lambda}.$$  

The evaluation and quantum cotrace are defined similarly.

Since $\mathcal{R}_{\lambda}^{\lambda'}$ is projective as a right module, Hom with it gives an exact functor. The quantum trace functor, however, is very far from being exact.

**Proposition 2.13** $\mathbb{K}_{\lambda}^{\lambda'}$ categorifies the coevaluation and $\mathbb{T}_{\lambda}^{\lambda'}$ the quantum trace.

**Proof.** We need only prove the former, since the latter follows by adjunction. Furthermore, we may reduce to the case where $\mu$ is added at the end of the sequence, since all other cases are obtained from this by the action of $\mathcal{U}$.

In this case, consider $\mathbb{K}_{\lambda}^{\lambda'} (S_i^\kappa)$. The resulting module is isomorphic to the standardization

$$\mathcal{S}_{\lambda, \mu}^{\mu, \mu'}(S_i^\kappa \otimes L_{\mu}) (2(\lambda, \rho)[-2\rho^\vee(\lambda)])$$

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Figure 8. The “S-move”

since any diagram with a left crossing involving the red lines from $\lambda_m$’s is trivial since we are considering a standardization and any with a left crossing on the strand labeled $\mu$ is killed since it is positive degree.

This reduces to the case where $\underline{\lambda} = \emptyset$, which we have covered in Propositions 2.3 and 2.11.

The most important property of these functors is that they satisfy the obvious isotopy; there are two functors

\[ S_1 = T_{\underline{\lambda}_1}^{\underline{\lambda}_1;\underline{\lambda}_1;\underline{\lambda}_2} K_{\underline{\lambda}_2;\underline{\lambda}_1;\underline{\lambda}_1} \quad S_2 = T_{\underline{\lambda}_2}^{\underline{\lambda}_1;\underline{\lambda}_1;\underline{\lambda}_1} K_{\underline{\lambda}_1;\underline{\lambda}_1;\underline{\lambda}_2} \]

which come from adding a pair of the representations are added on the left of an entry $\lambda$, and removing them on the right of $\lambda$ or vice versa.

**Proposition 2.14** The functors $S_1$ and $S_2$ are isomorphic to the identity functor.

**Proof.** As in Proposition 2.13, we can easily reduce to the case where $\underline{\lambda}_1 = \underline{\lambda}_2 = \emptyset$. Furthermore, these functors commute with the action of $U$, and so it suffices to check this equality on $P_\emptyset$. To prove the result for $S_2$, we must check that

\[ S_{\lambda;\lambda;\lambda}(P_\emptyset \otimes L_{\lambda}) \otimes_{T_{\lambda}} S_{\lambda;\lambda;\lambda}(L_{\lambda} \otimes P_\emptyset)(2\langle \lambda, \rho \rangle) \equiv \mathbb{k} \]

Applying the dot involution to switch left/right, the symmetry of tensor product shows that $S_1$ reduces to the same calculation.

We can use Lemma 2.6 to expand $L_{\lambda}$ into a complex, and then use the spectral sequence attached to tensoring these complexes. The $E^2$-page of this spectral sequence has entries

\[ E^2_{k,m} = \bigoplus_{i+j=m} \text{Tor}^k \left( S_{\lambda;\lambda;\lambda}(P_\emptyset \otimes M_i), S_{\lambda;\lambda;\lambda}(M_j \otimes P_\emptyset)(2\langle \lambda, \rho \rangle) \right) \]

By the Tor-vanishing discussed in the proof of 1.6, this will be 0 unless the two factors lie in the same piece of the semi-orthogonal decomposition, that is, if $i = 0, j = 2\rho^\vee(\lambda)$ and $k = 0$. This term is exactly

\[ S_{\lambda;\lambda;\lambda}(P_\emptyset \otimes P_{1i} \otimes P_\emptyset)(2\langle \lambda, \rho \rangle) \equiv \mathbb{k}[-2\rho^\vee(\lambda)] \]

with the homological shift canceling the fact that $j = 2\rho^\vee(\lambda)$. Thus, the result follows. □
This move is depicted in more usual topological form in Figure 8; it is extremely tempting to conclude that this proposition shows that the functors $\mathcal{K}$ and $\mathcal{T}$ are biadjoint; in fact, they are not always, though the adjunction on one side is clear from the definition. Rather, this is reflecting some sort of biadjunction between the 2-functors of “tensor with $\mathcal{V}^\lambda$” and “tensor with $\mathcal{V}^{\lambda\ast}$” on the 2-category of representations of $\mathcal{U}$. While there is not a unified construction of a tensor product of two $\mathcal{U}$ categories, one can easily generalize the definition of $\mathcal{V}^\lambda$ to describe auto-2-functors of $\mathcal{U}$ representations given by adding one red line; we will discuss this construction in more detail in forthcoming work [Webb].

3. Knot invariants

3.1. Constructing knot and tangle invariants. Now, we will use the functors from the previous section to construct tangle invariants. Using these as building blocks, we can associate a functor $\Phi(T) : \mathcal{V}_\lambda \to \mathcal{V}_\mu$ to any diagram of an oriented labeled ribbon tangle $T$ with the bottom ends given by $\lambda = \{\lambda_1, \ldots, \lambda_\ell\}$ and the top ends labeled with $\mu = \{\mu_1, \ldots, \mu_m\}$.

As usual, we choose a projection of our tangle such that at any height (fixed value of the $x$-coordinate) there is at most a single crossing, single cup or single cap. This allows us to write our tangle as a composition of these elementary tangles.

For a crossing, we ignore the orientation of the knot, and separate crossings into positive (right-handed) and negative (left-handed) according to the upward orientation we have chosen on $\mathbb{R}^2$.

- To a positive crossing of the $i$ and $i + 1$st strands, we associate the braiding functor $\mathcal{B}_{\sigma_i}$.
- To a negative crossing, we associate its adjoint $\mathcal{B}_{\sigma_i^{-1}}$ (the left and right adjoints are isomorphic, since $\mathcal{B}$ is an equivalence).

For the cups and caps, it is necessary to consider the orientation, following the pictures of Figures 4 and 7.

- To a clockwise oriented cup, we associate the coevaluation.
- To a clockwise oriented cap, we associate the quantum trace.
- To a counter-clockwise cup, we associate the quantum cotrace.
- To a counter-clockwise cap, we associate the evaluation.

Proposition 3.1 The map induced by $\Phi(T) : \mathcal{V}_\lambda \to \mathcal{V}_\mu$ on the Grothendieck groups $V_\lambda \to V_\mu$ is that assigned to a ribbon tangle by the structure maps of the category of $U_q(\mathfrak{g})$ with the ST ribbon structure.

In particular, the graded Euler characteristic of the complex $\Phi(T)(\mathbb{C})$ for a closed link is the quantum knot invariant for the ST ribbon element.
Proof. We need only check this for each elementary tangle, which was done in Corollary 1.5, Section 2.2 and Proposition 2.13. □

Theorem 3.2 The cohomology of \( \Phi(T)(\mathbb{k}) \) is finite-dimensional in each homological degree, and each graded degree is a complex with finite dimensional total cohomology. In particular the bigraded Poincaré series

\[
\varphi(T)(q,t) = \sum_i (-t)^i \dim_q H^i(\Phi(T)(\mathbb{k}))
\]

is a well-defined element of \( \mathbb{Z}[q^{1/\beta}, q^{-1/\beta}]((t)) \).

Proof. We note that the category \( V^0 \) is the category of complexes of graded finite dimensional vector spaces

\[
\cdots \leftarrow M^{i+1} \leftarrow M^i \leftarrow M^{i-1} \leftarrow \cdots
\]

such that \( M^i = 0 \) for \( i \gg 0 \) and for some \( k \), the vector space \( M^i \) is concentrated in degrees above \( k-i \). Thus, \( \Phi(T)(\mathbb{k}) \) lies in this category. In particular, each homological degree and each graded degree of \( \Phi(T)(\mathbb{k}) \) is finite-dimensional. □

The only case where the invariant is known to be finite dimensional is when the representations \( \underline{\lambda} \) are minuscule; recall that a weight \( \mu \) is called minuscule if every weight with a non-zero weight space in \( V^\mu \) is in the Weyl group orbit of \( \mu \).

Proposition 3.3 If all \( \lambda_i \) are minuscule, then the cohomology of \( \Phi(T)(\mathbb{k}) \) is finite-dimensional.

Proof. If all \( \lambda_i \) are minuscule, then the standard modules form a full exceptional collection. Any category with a finite full exceptional collection where each element has a finite projective resolution has finite projective dimension. Thus, in this case, the functor given by \( R\text{Hom} \) or \( L \otimes \) with a finite dimensional module preserves being quasi-isomorphic to a finite length complex. □

3.2. The unknot for \( \mathfrak{g} = \mathfrak{sl}_2 \). Unfortunately, the cohomology of the complex \( \Phi(T)(\mathbb{k}) \) is not always finite-dimensional. This can be seen in examples as simple as the unknot \( U \) for \( \mathfrak{g} = \mathfrak{sl}_2 \) and label 2.

In this case, the module \( L_2 \) with has a standard resolution of the form

\[
0 \to S_{1,2}^{(0,0)}(-2) \to S_{1,1}^{(0,1)}/(y_1 + y_2)(-1) \to S_{1,2}^{(0,2)} \to L_{\underline{\lambda}} \to 0.
\]

We let \( A = \text{End}_{\mathfrak{sl}_2}(S_{1,1}^{(0,1)}, S_{1,1}^{(0,1)}) \cong \mathbb{k}[y_1, y_2]/(y_1^2, y_2^2); \) the middle piece of the semi-orthogonal decomposition is equivalent to representations of this algebra.
Taking $\otimes$ of this resolution to its dual, we observe that all Tor’s vanish between terms that do not lie in the same piece of the semi-orthogonal decomposition, so

$$\text{Tor}^*(L_\lambda, L_\lambda) = \text{Tor}^*(S^{(0,2)}_{12}, (S^{(0,2)}_{12})^*)$$

$$\oplus \text{Tor}^*(S^{(0,1)}_{1,1}/(y_1 + y_2), (S^{(0,1)}_{1,1}/(y_1 + y_2))^*)[2](-2) \oplus \text{Tor}^*(S^{(0,2)}_{12}, (S^{(0,2)}_{12})^*)[4](-4)$$

$$\cong \mathbb{k} \oplus \text{Tor}^*(A/(y_1 + y_2)A, A/(y_1 + y_2)A)[2](-2) \oplus \mathbb{k}[4](-4)$$

The module $A/(y_1 + y_2)A$ has a minimal projective resolution given by

$$\cdots \longrightarrow A(-4) \xrightarrow{y_1+y_2} A(-2) \xrightarrow{y_1-y_2} A \longrightarrow A/(y_1 + y_2)A \longrightarrow 0.$$

which after taking $\otimes$ becomes

$$\cdots \xrightarrow{y_1+y_2} A/(y_1 + y_2)A(-4) \xrightarrow{y_1-y_2} A/(y_1 + y_2)A(-2) \xrightarrow{y_1+y_2} A/(y_1 + y_2)A \longrightarrow 0.$$

Thus, we have that

$$\text{Tor}^i_A(A/(y_1 + y_2)A, A/(y_1 + y_2)A) \cong \begin{cases} A/(y_1 + y_2)A & i = 0 \\ \mathbb{k}(-2i) & i > 0, \text{ odd} \\ \mathbb{k}(-2i - 2) & i > 0, \text{ even} \end{cases}$$

Thus, we have that

**Proposition 3.4** \(\varphi(U) = q^{-2}t^2 + 1 + q^2t^{-2} + \frac{q^{-2} - q^{-2}t}{1 - t^2q^{-4}}.\)

It is easy to see that the Euler characteristic is \(q^{-2} + 1 + q^2 = [3]_q,\) the quantum dimension of \(V_2.\) As this example shows, infinite-dimensionality of invariants is extremely typical behavior, and quite subtle. This same phenomenon of infinite dimensional vector spaces categorifying integers has also appeared in the work of Frenkel, Sussan and Stroppel [FSS], and in fact, their work could be translated into the language of this paper using the equivalences of [Weba, §5]; it would be quite interesting to work out this correspondence in detail.

**Conjecture 3.5** The invariant \(\Phi(L)\) for a link \(L\) is only finite-dimensional if all components of \(L\) are labeled with minuscule representations.

3.3. Independence of projection. While Theorem 3.1 shows the action on the Grothendieck group is independent of the presentation of the tangle, it doesn’t establish this for the functor \(\Phi(T)\) itself.

**Theorem 3.6** The functor \(\Phi(T)\) does not depend (up to isomorphism) on the projection of \(T.\)
Proof. We have already proved the ribbon Reidemeister moves in at least one position: RI in Proposition [2.9] and RII and RIII as part of Theorem [1.8] and also the “S-move” shown in Figure [8] in Proposition [2.14]. There is only one move of importance left for us to establish: the pitchfork move, shown in Figure [10].

Once we have established this move, we can easily show the others which are necessary. The illustrative example of the “χ-move” is given in Figure [9]. The other moves in the list of Ohtsuki [Oht02, Theorem 3.3] follow in the same way.

So, let us turn to the pitchfork. We may assume that the pictured red strands are the only ones. We must prove that this move holds for all reflections and orientations. The vertical reflection of the version shown follows from that illustrated by adjunction. We may assume that the cup is clockwise oriented, since the counter clockwise move can be derived from that one using Reidemeister moves II and III. The orientation of the “middle tine” is irrelevant, so we will ignore it.

For the orientation shown in Figure [11] we need only show this move holds for $P_0$ again, since we again have commutation with Hecke functors.

$$\Pi_1 = B_{\mu_1^{-1}} \circ S_{\mu,\lambda+\lambda'} (P_0 \boxtimes -) \quad \Pi_2 = B_{\mu_2} \circ S_{\lambda+\lambda',\mu} (- \boxtimes P_0)$$

Lemma 3.7 The functors $\Pi_1$ and $\Pi_2$ coincide.

Proof. First, we multiply both sides by $B_{\mu_2}$, so we must show that we have isomorphisms of functors

$$S_{\mu,\lambda+\lambda'} (P_0 \boxtimes -) \cong B_{\mu_1} \circ S_{\lambda+\lambda',\mu} (- \boxtimes P_0).$$

Since they generate the category, we need only show this isomorphism can be exhibited on the level of projectives.
Knot invariants and higher representation theory II

The isomorphism is given by Figure 11, and is essentially the same as that of Proposition 1.4. We note that this element has degree zero because we are assuming that the roots on the black strands add to $\lambda + \lambda^*$. Any diagram in the module $B_{\sigma_1}B_{\sigma_2}S^{1+1,\mu}(P_{\rho} \boxtimes P_{\theta})$ can be prefixed by this element, so the map is surjective. Any element which is sent to 0 by adjoining this diagram is easily seen to be 0, since the standardly violating strand can be slid downward to become a violating strand, so the map is also injective.

Figure 11. The isomorphism of Lemma 3.7

The pitchfork move shown in Figure 10 follows from this lemma, since two sides of the depicted move are

$$- \otimes_{T} \Pi_1 L_\lambda(2(\lambda, \rho))[2\rho^\vee(\lambda)] \quad \text{and} \quad - \otimes_{T} \Pi_2 L_\lambda(2(\lambda, \rho))[2\rho^\vee(\lambda)].$$

The only variation remaining to check is the case where the move is reflected through the page (i.e. with the signs of the crossings given reversed), but this follows from the lemma as well since the two sides are

$$- \otimes_{T} (\Pi_1 L_\lambda)^* (2(\lambda, \rho))[2\rho^\vee(\lambda)] \quad \text{and} \quad - \otimes_{T} (\Pi_2 L_\lambda)^* (2(\lambda, \rho))[2\rho^\vee(\lambda)].$$

Some care must be exercised with the normalization of these invariants, since as we noted in Section 2.2, they are the Reshetikhin-Turaev invariants for a slightly different ribbon element from the usual choice. However, the difference is easily understood. Let $L$ be a link drawn in the blackboard framing, and let $L_i$ be its components, with $L_i$ labeled with $\lambda_i$. Recall that the writh $wr(K)$ of an oriented ribbon knot is the linking number of the two edges of the ribbon; this can be calculated by drawing the link the blackboard framing and taking the difference between the number of positive and negative crossings. Here we give a slight extension of the proposition of Snyder and Tingley relating the invariants for different framings [ST, Theorem 5.21]:

**Proposition 3.8** The invariants attached to $L$ by the standard and Snyder-Tingley ribbon elements differ by the scalar $\prod_i (-1)^{2\rho^\vee(\lambda_i) (wr(L_i) - 1)}$.

**Proof.** The proof is essentially the same as that of [ST, Theorem 5.21] with a bit more attention paid to the case where the components have different labels. The proof is an induction on the crossing number of the link. The formula is correct for any framing of an unlink, which gives the base case of our induction.
Now note that the ratio between the knot invariants only depends on the number of rightward oriented cups and caps, so both the ratio between the invariants for the usual and ST ribbon structures and the formula given are insensitive to Reidemeister II and III as well as crossing change (which changes the writhe, but by an even number). Since these operations can be used to reduce any link to an unlink, we are done. □

Since one of the main reasons for interest in these quantum invariants of knots is their connection to Chern-Simons theory and invariants of 3-manifolds, it is natural to ask:

**Question 3.9** Can these invariants glue into a categorification of the Witten-Reshetikhin-Turaev invariants of 3-manifolds?

**Remark 3.10** The most naive ansatz for categorifying Chern-Simons theory, following the development of Reshetikhin and Turaev [RT91] would associate

- a category $C(\Sigma)$ to each surface $\Sigma$, and
- an object in $C(\Sigma)$ to each isomorphism of $\Sigma$ with the boundary of a 3-manifold such that

  - the invariants $\mathcal{K}$ we have given are the Ext-spaces of this object for a knot complement with fixed generating set of $C(T^2)$ labeled by the representations of $g$, and
  - the categorification of the WRT invariant of a Dehn filling is the Ext space of this object with another associated to the torus filling.

While some hints of this structure appear in the constructions of this paper, it’s far from clear how they will combine.

3.4. **Functoriality.** One of the most remarkable properties of Khovanov homology is its functoriality with respect to cobordisms between knots [Jac04]. This property is not only theoretically satisfying but also played an important role in Rasmussen’s proof of theunknotting number of torus knots [Ras]. Thus, we certainly hope to find a similar property for our knot homologies. While we cannot present a complete picture at the moment, there are promising signs, which we explain in this section. We must restrict ourselves to the case where the weights $\lambda_i$ are minuscule, since even the basic results we prove here do not hold in general. We will assume this hypothesis throughout this subsection.

The weakest form of functoriality is putting a Frobenius structure on the vector space associated to a circle. This vector space, as we recall, is

$$A_{\lambda} = \text{Ext}^\bullet(L_{\lambda_1}, L_{\lambda_2})[2 \rho^\vee(\lambda)](2(\lambda, \rho)).$$
This algebra is naturally bigraded by the homological and internal gradings. The algebra structure on it is that induced by the Yoneda product. Recall that $\mathcal{S}$ denotes the right Serre functor of $\mathcal{V}_\lambda$, discussed in Section 1.2.

**Theorem 3.11** For minuscule weights $\lambda$, we have a canonical isomorphism

$$\mathcal{S}L_\lambda \cong L_\lambda(-4\langle\lambda, \rho\rangle)[-4\rho^\vee(\lambda)].$$

Thus, the functors $\mathcal{K}$ and $\mathcal{T}$ are biaadjoint up to shift.

In particular, $\text{Ext}^{4\langle\lambda, \rho\rangle}(L_\lambda, L_\lambda) \cong \text{Hom}(L_\lambda, L_\lambda)^*$, and the dual of the unit $\iota^* : \text{Ext}^{4\langle\lambda, \rho\rangle}(L_\lambda, L_\lambda) \to \mathbb{C}$ is a symmetric Frobenius trace on $A_\lambda$ of degree $-4\langle\lambda, \rho\rangle$.

One should consider this as an analogue of Poincaré duality, and thus is a piece of evidence for $A_\lambda$’s relationship to cohomology rings.

**Proof.** As we noted in the proof of 3.3, $T_\lambda$ has finite global dimension if the weights $\lambda$ are minuscule. The result then follows immediately from Proposition 1.11. □

It would be enough to show that this algebra is commutative to establish the functoriality for flat tangles; we simply use the usual translation between 1+1 dimensional TQFTs and commutative Frobenius algebras (for more details, see the book by Kock [Koc04]). At the moment, not even this very weak form of functoriality is known.

**Question 3.12** Is there another interpretation of the algebra $A_\lambda$? Is it the cohomology of a space?

One natural guess, based on the work of Mirković-Vilonen [MV07] and the symplectic duality conjecture of the author and collaborators [BLPW], is that $A_\lambda$ is the cohomology of the corresponding Schubert variety $\text{Gr}_\lambda$ in the Langlands dual affine Grassmannian.

Another candidate algebra is the multiplication induced on $V_\lambda$ by the quantized “shift of function algebra” $A_f$ for a regular nilpotent element $f$ studied by Feigin, Frenkel, and Rybnikov [FFR].

We can use the biaadjunction to give a rather simple prescription for functoriality: for each embedded cobordism in $I \times S^3$ between knots in $S^3$, we can isotope so that the height function is a Morse function, and thus decompose the cobordism into handles. Furthermore, we can choose this so that the projection goes through these handle attachments at times separate from the times it goes through Reidemeister moves. We construct the functoriality map by assigning

- to each Reidemeister move, we associate a fixed isomorphism of the associated functors.
• to the birth of a circle (the attachment of a 2-handle), we associate the unit of the adjunction \((K, T)\) or \((C, E)\), depending on the orientation.
• to the death of a circle (the attachment of a 0-handle), we associate the counits of the opposite adjunctions \((T, K)\) or \((E, C)\) (i.e., the Frobenius trace).
• to a saddle cobordism (the attachment of a 1-handle), we associate (depending on orientation) the unit of the second adjunction above, or the counit of the first.

**Conjecture 3.13** This assignment of a map to a cobordism is independent of the choice of Morse function, i.e. this makes the knot homology theory \(\mathcal{K}(\cdot)\) functorial.

4. Comparison to other knot homologies

A great number of other knot homologies have appeared on the scene in the last decade, and obviously, we would like to compare them to ours. While several of these comparisons are out of reach at the moment, in this section we check the one which seems most straightforward based on the similarity of constructions: we describe an isomorphism to the invariants constructed by Mazorchuk-Stroppel and Sussan for the fundamental representations of \(\mathfrak{sl}_n\).

To do this, we will use the functor \(\Xi : \mathfrak{B}^\lambda \to \hat{\mathcal{O}}^p\) constructed in [Weba, §5] (as before, we will freely use notation from this preceding paper). Here we use \(\lambda\) to construct a Young pyramid \(\pi\) whose column lengths are the indices of the fundamental weights appearing in the expansion of \(\lambda_j\), and let \(p\) be a parabolic subalgebra of \(\mathfrak{gl}_N\) which precisely preserves a flag of type corresponding to the pyramid \(\pi\). Given this data, we let \(\hat{\mathcal{O}}^p\) be a graded lift of a block of \(p\)-parabolic category \(\mathcal{O}\).

In order to compare knot homologies, we must compare the functors we have described on our categories \(\mathcal{V}^\lambda\) and those on \(\hat{\mathcal{O}}^p\). For simplicity, in this section we will assume that \(\lambda\) is a sequence of fundamental weights. In this paper, we are only concerned about commuting up to isomorphism of functors; thus when we say a diagram of functors “commutes” we mean that the functors for any two paths between the same points are isomorphic.

First, let us consider the braiding functors. Associated to each permutation of \(N\) letters, we have a derived twisting functor \(T_w : D^\dagger(\hat{\mathcal{O}}) \to D^\dagger(\hat{\mathcal{O}})\) (see [AS03] for more details and the definition).
Proposition 4.1 When $\underline{\lambda} = (\omega_1, \ldots, \omega_1)$, then $v = b$ and we have a commutative diagram

\[
\begin{array}{ccc}
D^\dagger(\tilde{O}_n) & \xrightarrow{T_v} & D^\dagger(\tilde{O}_n) \\
\uparrow \Xi & & \uparrow \Xi \\
V^\perp & \xrightarrow{\Xi \circ B_v} & V^\perp \\
\end{array}
\]

Proof. We note that both functors $T_v$ and $B_v$ commute with translation functors by [AS03, Lemma 2.1(5)]. The same holds for $\Xi \circ B_v \circ \Xi$ by [Weba, Proposition 5.9] and Proposition [1.3].

So as usual, we need only compute their behavior on parabolic Verma modules on the level of objects in order to check isomorphisms of functors. Furthermore, both send parabolic Verma modules to their mutations by a particular change of order. For $T_v$ the mutation is that associated to the action of $v$ on tableaux, and for $\Xi \circ B_v \circ \Xi$, it is given by using $v$ to reorder the root function $\alpha$ given by the sum of the roots that appear between the red lines. These coincide, so the functors are the same. □

Finally, we turn to describing the functors associated to cups and caps. If $\pi$ has a column of height $n$, then any block of category $\tilde{O}_n^\pi$ is equivalent to the block of category $\tilde{O}_n^{\pi'}$ associated to $\pi'$, the diagram $\pi$ with that column of height $n$ removed. The content of the tableaux in the new block is that of the original block with the multiplicity of each number in $[1, n]$ reduced by 1. The effect of this functor on the simples, projectives and Verma is simply removing that column of height $n$ (which by column strictness must be the numbers $[1, n]$ in order). The functor that realizes this equivalence $\zeta : \tilde{O}_n^\pi \to \tilde{O}_n^{\pi'}$ is the Enright-Shelton equivalence (based on the paper [ES87] chapter 11), but developed more fully by Bernstein, Frenkel and Khovanov in [BFK99]; also, note that we use the Koszul dual to their equivalence.

We will also use also have Zuckerman functors, which are the derived functors of sending a module in $\tilde{O}$ to its largest quotient which is locally finite for $\mathfrak{p}$. These are left adjoint to the forgetful functor $D^b(\tilde{O}) \to D^b(\tilde{O})$.

Begin with a pyramid $\pi$, and assume $\pi'$ is obtained from $\pi$ by replacing a pair of consecutive columns whose lengths add up to $n$ (a pair of consecutive dual representations in the sequence $\underline{\lambda}$), with one of length $n$, and $\pi''$ is obtained by deleting them altogether.

Definition 4.2 The ES-cup functor $K : \tilde{O}_n^{\pi''} \to \tilde{O}_n^{\pi'}$ is the composition of the inverse of the Enright-Shelton equivalence for $\pi''$ and $\pi'$ with the forgetful functor from $\tilde{O}_n^{\pi'}$ to $\tilde{O}_n^\pi$ (which corresponds to an inclusion of parabolic subgroups).
The **ES-cap functor** \( T : \tilde{\mathcal{O}}^\pi \to \tilde{\mathcal{O}}^{\pi''} \) is the composition of the Zuckerman functor from \( \tilde{\mathcal{O}}^\pi \) to \( \tilde{\mathcal{O}}^{\pi'} \) with the Enright-Shelton functor \( \zeta : \tilde{\mathcal{O}}^{\pi'} \to \tilde{\mathcal{O}}^{\pi''} \).

**Proposition 4.3** Both squares in the diagram below commute.

![Diagram](attachment:diagram.png)

**Proof.** We need only check this for \( K \), since in both cases, the functors above are in adjoint pairs.

In the case where \( \pi \) has 2 columns and \( N = n \) (so \( \pi'' = \emptyset \)), then this is clear, since \( K \) sends \( \kappa \) to the simple for the tableau which places the integers \([\pi_2 + 1, n]\) in the first column, and \([1, \pi_2]\) in the second. This is sent under \( \Xi \) to the simple \( L_\lambda \). All other cases follow from this one, using the compatibility results for functors proved in [Weba] Propositions 5.9 & 5.10.

These propositions show that our work matches with that of Sussan [Sus07] and Mazorchuk-Stroppel [MSa], though the latter paper is “Koszul dual” to our approach above. Recall that each block of \( \tilde{\mathcal{O}}_n \) has a Koszul dual, which is also a block of parabolic category \( \mathcal{O} \) for \( \mathfrak{gl}_N \) (see [Bac99]). In particular, we have a Koszul duality equivalence

\[ \mathcal{S}_\lambda : D^!(\tilde{\mathcal{O}}^p_n) \to D^!(\tilde{\mathcal{O}}^p) \]

where \( \tilde{\mathcal{O}} \) is the direct sum over all \( n \) part compositions \( \mu \) (where we allow parts of size 0) of a block of \( p_\mu \)-parabolic category \( \tilde{\mathcal{O}} \) for \( \mathfrak{gl}_N \) with a particular central character depending on \( p \).

Now, let \( T \) be an oriented tangle labeled with \( \underline{\lambda} \) at the bottom and \( \underline{\lambda}' \) at top, with all appearing labels being fundamental. Then, as before, associated to \( \underline{\lambda} \) and \( \underline{\lambda}' \) we have parabolics \( p \) and \( p' \).
**Proposition 4.4** Assume $\Lambda$ only uses the fundamental weights $\omega_1$ and $\omega_{n-1}$. Then we have a commutative diagram

$$
\begin{array}{ccc}
D^\dagger(\tilde{O}_n) & \xrightarrow{\mathcal{F}(T)} & D^\dagger(\bar{O}_n) \\
\downarrow \Phi(T) & & \downarrow \Phi(T) \\
D^\dagger(\tilde{O}_n^p) & \xrightarrow{\mathcal{F}(T)} & D^\dagger(\bar{O}_n^p) \\
\uparrow \Psi & & \uparrow \Psi \\
\mathcal{V}^\Lambda & \xrightarrow{\mathcal{F}(T)} & \mathcal{V}^{\Lambda'}
\end{array}
$$

where $\mathcal{F}(T)$ is the functor for a tangle defined by Sussan in [Sus07] and $\mathcal{F}(T)$ is the functor defined by Mazorchuk and Stroppel in [MSa].

Our invariant $\mathcal{K}$ thus coincides with the knot invariants of both the above papers all components are labeled with the defining representation, and thus coincides with Khovanov homology when $\mathfrak{g} = \mathfrak{sl}_2$ and Khovanov-Rozansky homology when $\mathfrak{g} = \mathfrak{sl}_3$.

**Proof.** This follows immediately from [Weba, Proposition 5.8], Propositions 4.1 and 4.3 and the definitions given in the papers referred to above of $\mathcal{F}(T)$ and $\mathcal{F}(T)$.

We believe strongly that this homology agrees with that of Khovanov-Rozansky when one uses the defining representation for all $n$ (this is conjectured in [MSa]), but actually proving this requires an improvement in the state of understanding of the relationship between the foam model of Mackaay, Stošić and Vaz [MSV09] and the model we have presented. It would also be desirable to compare our results to those of Cautis-Kamnitzer for minuscule representations, and Khovanov-Rozansky for the Kauffman polynomial, but this will require some new ideas, beyond the scope of this paper.

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