Scalar curvature and the moment map in generalized Kähler geometry

Focusing on the cases of compact Lie groups

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Abstract

We introduce a notion of scalar curvature of a twisted generalized Kähler manifold in terms of pure spinors formalism. A moment map framework with a modified action of generalized Hamiltonians on an arbitrary compact generalized Kähler manifold is developed. Then it turns out that a moment map is given by the scalar curvature, which is a generalization of the result of the scalar curvature as a moment map in the ordinary Kähler geometry, due to Fujiki and Donaldson. A noncommutative compact Lie group $G$ does not have any Kähler structure. However, we show that every compact Lie group admits generalized Kähler structures with constant scalar curvature. In particular, generalized Kähler structures with constant scalar curvature on the standard Hopf surface are explicitly given.

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1 Introduction

Let \((M, \omega)\) be a compact manifold of dimension \(2n\) with a symplectic structure \(\omega\). We denote by \(C_\omega\) the set of \(\omega\)-compatible almost complex structures on \((M, \omega)\). Then \(C_\omega\) is an infinite dimensional homogeneous Kähler manifold, which is a set of global sections of a fibre bundle with fibre the Siegel space \(\text{Sp}(2n, \mathbb{R})/U(n)\). Hamiltonian diffeomorphisms act on \(C_\omega\) preserving its Kähler structure. Then Fujiki and Donaldson show that the scalar curvature arises as a moment map of Hamiltonian diffeomorphisms on \(C_\omega\), which is called the moment map framework in Kähler geometry \([9, 8]\).

The main purpose of the paper is to extend the moment map framework to an arbitrary generalized Kähler manifold, that is, \(H\)-twisted generalized Kähler manifolds, where \(H\) denotes a real \(d\)-closed 3-form on \(M\). A generalized Kähler
structure on a manifold is a triple \((g, I, J)\) consisting of a Riemannian metric \(g\) compatible with two complex structures \(I, J\) satisfying the certain conditions, which has an origin in a non-linear sigma model in Mathematical physics. However, a generalized Kähler structure has a natural description using the language of nondegenerate, pure spinors \([21]\), that is, a generalized Kähler structure is defined by a pair \((\phi, \psi)\) of locally defined nondegenerate, pure spinors which induces a generalized Kähler structure \((\mathcal{J}_\phi, \mathcal{J}_\psi)\) \([18]\). We introduce an invariant function \(S(\mathcal{J}_\phi, \mathcal{J}_\psi)\) of a \(H\)-twisted generalized Kähler structure \((\mathcal{J}_\phi, \mathcal{J}_\psi)\) by using pure spinors (see 3.1 in Section 3), which is called the scalar curvature in this paper. Then the scalar curvature admits the following properties,

1. If \((\mathcal{J}_\phi, \mathcal{J}_\psi)\) is coming from an ordinary Kähler structure, then \(S(\mathcal{J}_\phi, \mathcal{J}_\psi)\) is the ordinary scalar curvature. If \((\mathcal{J}_\phi, \mathcal{J}_\psi)\) is a generalized Kähler structure of symplectic type, then \(S(\mathcal{J}_\phi, \mathcal{J}_\psi)\) is the same as in the previous paper \([14]\), \([17]\).

2. The scalar curvature \(S(\mathcal{J}_\phi, \mathcal{J}_\psi)\) is equivalent under the action of an extension \(\widetilde{\text{Diff}}_0(M, H)\) of the group of diffeomorphims of \(M\) by \(b\)-fields.

3. The scalar curvature is invariant under the change of \(\mathcal{J}_\phi\) and \(\mathcal{J}_\psi\), i.e.,

\[
S(\mathcal{J}_\phi, \mathcal{J}_\psi) = S(\mathcal{J}_\psi, \mathcal{J}_\phi).
\]

As in the framework in Kähler geometry, we fix an integrable generalized complex structure \(\mathcal{J}_\psi\) and a volume form \(\text{vol}_M\) which satisfies \(\text{vol}_M = i^{-n}(\psi_{\alpha}, \overline{\psi}_{\alpha})\). We denote by \(\mathcal{B}_{\mathcal{J}_\psi}(M)\) the set of \(\mathcal{J}_\psi\)-compatible almost generalized complex structures, which is also an infinite dimensional homogeneous Kähler manifold, which is the set of global sections of a fibre bundle with fibre \(U(n, n)/U(n) \times U(n)\). The \(\mathcal{J}_\psi\) gives generalized Hamiltonians \(\text{Ham}(M, \mathcal{J}_\psi)\) whose Lie algebra is given by

\[
\mathfrak{ham}(M, \mathcal{J}_\psi) := C^\infty_\psi(M, \mathbb{R}) = \{ f \in C^\infty(M, \mathbb{R}) \mid \pi_T(\zeta_\alpha)f = 0, \int_M f\text{vol}_M = 0 \}.
\]

Then \(\text{Ham}(M, \mathcal{J}_\psi)\) acts on \(\mathcal{B}_{\mathcal{J}_\psi}(M)\) preserving its Kähler structure by the modified action (see Section 5 for more detail).

**Theorem 5.6** Let \(\mathcal{J}_\psi\) be a generalized complex structure on a compact manifold \(M\). Then there exists a moment map

\[
\mu : \mathcal{B}_{\mathcal{J}_\psi}(M) \rightarrow C^\infty_\psi(M, \mathbb{R})^*
\]

for the modified action of the generalized Hamiltonian diffeomorphisms such that \(\mu(\mathcal{J})\) is given by the scalar curvature \(S(\mathcal{J}, \mathcal{J}_\psi)\) for all \(\mathcal{J} \in \mathcal{B}_{\mathcal{J}_\psi}(M)\).
A noncommutative compact Lie group does not have any Kähler structure, however every compact Lie group does have remarkable bihermitian structure twisted by the Cartan 3-form, which gives rise to a twisted generalized Kähler structure [18]. Alekseev, Bursztyn and Meinrenken give an interesting description of Dirac structures on a Lie group $G$ in terms of the double of its Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ equipped with an Adoint invariant metric $B$. We apply their description to construct a family of twisted generalized Kähler structures on a compact Lie group $G$ by using the action of the real Pin group of the double of the real Cartan subalgebra $\mathfrak{h} \oplus \mathfrak{h}$. By using the description of pure spinors on $G$ in terms of pure spinors of $\mathfrak{g} \oplus \mathfrak{g}$, we calculate the scalar curvature of the twisted generalized Kähler structures $(\mathcal{J}_\phi, \mathcal{J}_\psi)$ on $G$. We denote by $\Xi$ the Cartan-Killing 3-form on $\mathfrak{g}$ and by $\hat{B}$ a Hermitian form given by $B$. Then the contraction $\sqrt{-1} \Lambda_B \Xi \in \mathfrak{h}$ is denoted by $P$. We show that the scalar curvature of $(\mathcal{J}_\phi, \mathcal{J}_\psi)$ on $G$ is a constant $2\|P\|^2$, where $\|P\|$ is the norm of $P$ with respect to $\hat{B}$.

This paper is organized as follows. In Section 2, we introduce our notation and preliminary results on generalized complex structures and generalized Kähler structures throughout this paper. In Section 3, we define the scalar curvature of generalized Kähler structures. In Section 4, the notion of generalized Hamiltonians is introduced. In Section 5, the moment map framework is introduced and the main theorem (Theorem 5.6) is proved. In Section 6, we explicitly construct generalized Kähler structures on the standard Hopf surfaces with constant scalar curvature. In Section 7, we give a brief explanation of the results by Alekseev, Bursztyn and Meinrenken. In Section 8, we construct a family of twisted generalized Kähler structures with constant scalar curvature on a compact Lie group $G$.

2 Generalized complex structures and generalized Kähler structures

2.1 Generalized complex structures and nondegenerate, pure spinors

Let $M$ be a differentiable manifold of real dimension $2n$. The bilinear form $\langle \cdot, \cdot \rangle_{T_M \oplus T_M^*}$ on the direct sum $T_M \oplus T_M^*$ over a differentiable manifold $M$ of dim= $2n$ is defined by

$$\langle v + \xi, u + \eta \rangle_{T_M \oplus T_M^*} = \frac{1}{2} (\xi(u) + \eta(v)), \quad u, v \in T_M, \xi, \eta \in T_M^*. $$
Let $\text{SO}(T_M \oplus T_M)$ be the fibre bundle over $M$ with fibre $\text{SO}(2n, 2n)$ which is a subbundle of $\text{End}(T_M \oplus T_M^*)$ preserving the bilinear form $(\; , \; )_{T \oplus T^*}$. An almost generalized complex structure $J$ is a section of $\text{SO}(T_M \oplus T_M^*)$ satisfying $J^2 = -\text{id}$. Then in the case of almost complex structures, an almost generalized complex structure $J$ yields the eigenspace decomposition:

$$(T_M \oplus T_M^*)^C = \mathcal{L}_J \oplus \overline{\mathcal{L}_J},$$

where $\mathcal{L}_J$ is $-\sqrt{-1}$-eigenspace and $\overline{\mathcal{L}_J}$ is the complex conjugate of $\mathcal{L}_J$. Let $H$ be a real $d$-closed 3-form. The Courant bracket of $T_M \oplus T_M^*$ is defined by

$$[u + \xi, v + \eta]_{\omega} = [u, v] + \mathcal{L}_u\eta - \mathcal{L}_v\xi - \frac{1}{2}(d_i u \eta - d_v \xi),$$

where $u, v \in TM$ and $\xi, \eta$ is $T^*M$. The $H$-twisted Courant bracket is given by

$$[u + \xi, v + \eta]_{\omega} = [u, v] + \mathcal{L}_u\eta - \mathcal{L}_v\xi - \frac{1}{2}(d_i u \eta - d_v \xi) - i_v \iota_u H.$$

If $\mathcal{L}_J$ is involutive with respect to the Courant bracket, then $J$ is a generalized complex structure, that is, $[e_1, e_2]_{\omega} \in \Gamma(\mathcal{L}_J)$ for any two elements $e_1 = u + \xi$, $e_2 = v + \eta \in \Gamma(\mathcal{L}_J)$. If $\mathcal{L}_J$ is involutive with respect to the $H$-twisted Courant bracket, the $J$ is a $H$-twisted generalized complex structure. Let $\text{CL}(T_M \oplus T_M^*)$ be the Clifford algebra bundle which is a fibre bundle with fibre the Clifford algebra $\text{CL}(2n, 2n)$ with respect to $(\; , \; )_{T \oplus T^*}$ on $M$. Then a vector $v$ acts on the space of differential forms $\otimes^{2n} \wedge^p T^*M$ by the interior product $i_v$ and a 1-form $\theta$ acts on $\otimes^{2n} \wedge^p T^*M$ by the exterior product $\theta \wedge$, respectively. Then the space of differential forms gives a representation of the Clifford algebra $\text{CL}(T_M \oplus T_M^*)$ which is the spin representation of $\text{CL}(T_M \oplus T_M^*)$. Thus the spin representation of the Clifford algebra arises as the space of differential forms

$$\wedge^* T_M^* = \oplus_p \wedge^p T_M^* = \wedge^{even} T_M^* \oplus \wedge^{odd} T_M^*.$$  

The inner product $(\; , \; )_s$ of the spin representation is given by

$$(\alpha, \beta)_s := (\alpha \wedge \sigma \beta)_{[2n]},$$

where $(\alpha \wedge \sigma \beta)_{[2n]}$ is the component of degree $2n$ of $\alpha \wedge \beta \in \oplus_p \wedge^p T^*M$ and $\sigma$ denotes the Clifford involution which is given by

$$\sigma \beta = \begin{cases} +\beta & \text{deg } \beta \equiv 0, 1 \pmod{4} \\ -\beta & \text{deg } \beta \equiv 2, 3 \pmod{4} \end{cases}$$

We define $\ker \Phi := \{ e \in (T_M \oplus T_M^*)^C \mid e \cdot \Phi = 0 \}$ for a differential form $\Phi \in \wedge^{even/odd} T_M^*$. If $\ker \Phi$ is maximal isotropic, i.e., $\dim_{C} \ker \Phi = 2n$, then $\Phi$ is
called a pure spinor of even/odd type. A pure spinor $\Phi$ is nondegenerate if $\ker \Phi \cap \ker \Phi = \{0\}$, i.e., $(T_M \oplus T^*_M)^C = \ker \Phi \oplus \overline{\ker \Phi}$. Then a nondegenerate, pure spinor $\Phi \in \wedge^* T^*_M$ gives an almost generalized complex structure $J_\Phi$ which satisfies

$$J_\Phi e = \begin{cases} -\sqrt{-1}e, & e \in \ker \Phi \\ +\sqrt{-1}e, & e \in \overline{\ker \Phi} \end{cases}$$

Conversely, an almost generalized complex structure $J$ locally arises as $J_\Phi$ for a nondegenerate, pure spinor $\Phi$ which is unique up to multiplication by non-zero functions. Thus an almost generalized complex structure yields the canonical line bundle $K_J := \mathbb{C}\langle \Phi \rangle$ which is a complex line bundle locally generated by a nondegenerate, pure spinor $\Phi$ satisfying $J = J_\Phi$. Let $d_H$ be a differential operator $d + H$ which acts on differential forms. An almost generalized complex structure $J$ is a $H$-twisted generalized complex structure if and only if $d_H \Phi = \eta \cdot \Phi$ for a section $\eta \in T_M \oplus T^*_M$. The type number of $J = J_\Phi$ is defined as the minimal degree of the differential form $\Phi$. Note that type number Type $J$ is a function on a manifold which is not a constant in general.

**Example 2.1.** Let $J$ be a complex structure on a manifold $M$ and $J^*$ the complex structure on the dual bundle $T^*M$ which is given by $J^* \xi(v) = \xi(Jv)$ for $v \in TM$ and $\xi \in T^*M$. Then a generalized complex structure $J_J$ is given by the following matrix $J_J = \left( \begin{array}{cc} J & 0 \\ 0 & -J^* \end{array} \right)$.

Then the canonical line bundle is the ordinary one which is generated by complex forms of type $(n, 0)$. Thus we have Type $J_J = n$.

**Example 2.2.** Let $\omega$ be a symplectic structure on $M$ and $\hat{\omega}$ the isomorphism from $TM$ to $T^*M$ given by $\hat{\omega}(v) := i_v \omega$. We denote by $\hat{\omega}^{-1}$ the inverse map from $T^*M$ to $TM$. Then a generalized complex structure $J_\phi$ is given by the following matrix $J_\phi = \left( \begin{array}{cc} 0 & -\hat{\omega}^{-1} \\ \hat{\omega} & 0 \end{array} \right)$.

Then the canonical line bundle is given by the differential form $\psi = e^{-\sqrt{-1} \omega}$. Thus Type $J_\phi = 0$.

**Example 2.3 (b-field action).** A real $d$-closed 2-form $b$ acts on a generalized complex structure by the adjoint action of Spin group $e^b$ which provides a generalized complex structure $\text{Ad}_{e^b} J = e^b \circ J \circ e^{-b}$. 

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Example 2.4 (Poisson deformations). Let $\beta$ be a holomorphic Poisson structure on a complex manifold. Then the adjoint action of Spin group $e^{\beta}$ gives deformations of new generalized complex structures by $\mathcal{J}_{\beta t} := \text{Ad}_{e^{\beta t}} \mathcal{J}$. Then Type $\mathcal{J}_{\beta x} = n - 2$ (rank of $\beta_x$) at $x \in M$, which is called the Jumping phenomena of type number.

Let $(M, \mathcal{J})$ be a generalized complex manifold and $\mathcal{T}_{\mathcal{J}}$ the eigenspace of eigenvalue $\sqrt{-1}$. Then we have the Lie algebroid complex $\bigwedge^\bullet \mathcal{T}_{\mathcal{J}}$

$$0 \to \bigwedge^0 \mathcal{T}_{\mathcal{J}} \to \bigwedge^1 \mathcal{T}_{\mathcal{J}} \to \bigwedge^2 \mathcal{T}_{\mathcal{J}} \to \bigwedge^3 \mathcal{T}_{\mathcal{J}} \to \cdots$$

The Lie algebroid complex is the deformation complex of generalized complex structures. In fact, $\varepsilon \in \bigwedge^2 \mathcal{T}_{\mathcal{J}}$ gives deformed isotropic subbundle $E_\varepsilon := \{ e + [\varepsilon, e] \mid e \in \mathcal{L}_{\mathcal{J}} \}$. Then $E_\varepsilon$ yields deformations of generalized complex structures if and only if $\varepsilon$ satisfies Generalized Mauer-Cartan equation

$$\overline{\partial}_{\mathcal{J}} \varepsilon + \frac{1}{2} [\varepsilon, \varepsilon]_{\text{Sch}} = 0,$$

where $[\varepsilon, \varepsilon]_{\text{Sch}}$ denotes the Schouten bracket. The Kuranishi space of generalized complex structures is constructed. Then the second cohomology group $H^2(\bigwedge^\bullet \mathcal{T}_{\mathcal{J}})$ of the Lie algebraic complex gives the infinitesimal deformations of generalized complex structures and the third one $H^3(\bigwedge^\bullet \mathcal{T}_{\mathcal{J}})$ is the obstruction space to deformations of generalized complex structures.

2.2 Generalized Kähler structures

Definition 2.5. A generalized Kähler structure is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ consisting of two commuting generalized complex structures $\mathcal{J}_1, \mathcal{J}_2$ such that $\hat{G} := -\mathcal{J}_1 \circ \mathcal{J}_2 = -\mathcal{J}_2 \circ \mathcal{J}_1$ gives a positive definite symmetric form $G := \langle \hat{G}, \cdot\rangle$ on $T_M \oplus T_M^\ast$.

We call $G$ a generalized metric. A generalized Kähler structure of symplectic type is a generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ such that $\mathcal{J}_2$ is the generalized complex structure $J_\psi$ which is induced from a $d$-closed, nondegenerate, pure spinor $\psi := e^{b - \sqrt{-1} \omega}$.

Each $\mathcal{J}_i$ gives the decomposition $(T_M \oplus T_M^\ast)^C = \mathcal{L}_{\mathcal{J}_i} \oplus \overline{\mathcal{L}}_{\mathcal{J}_i}$, for $i = 1, 2$. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ are commutative, we have the simultaneous eigenspace decomposition

$$(T_M \oplus T_M^\ast)^C = (\mathcal{L}_{\mathcal{J}_1} \cap \mathcal{L}_{\mathcal{J}_2}) \oplus (\overline{\mathcal{L}}_{\mathcal{J}_1} \cap \overline{\mathcal{L}}_{\mathcal{J}_2}) \oplus (\mathcal{L}_{\mathcal{J}_1} \cap \overline{\mathcal{L}}_{\mathcal{J}_2}) \oplus (\overline{\mathcal{L}}_{\mathcal{J}_1} \cap \mathcal{L}_{\mathcal{J}_2}).$$

Since $\hat{G}^2 = +\text{id}$, The generalized metric $\hat{G}$ also gives the eigenspace decomposition: $T_M \oplus T_M^\ast = C_+ \oplus C_-$, where $C_\pm$ denote the eigenspaces of $\hat{G}$ of eigenvalues.
±1. We denote by $L^\pm_{J,1}$ the intersection $L_{J,1} \cap C^\pm_{J,1}$. Then it follows

\[ L_{J,1} \cap L_{J,2} = L_{J,1}, \quad \overline{L}_{J,1} \cap \overline{L}_{J,2} = \overline{L}_{J,1} \]

\[ L_{J,1} \cap L_{J,2} = L_{J,1}, \quad \overline{L}_{J,1} \cap \overline{L}_{J,2} = \overline{L}_{J,1} \]

**Example 2.6.** Let $X = (M, J, \omega)$ be a Kähler manifold. Then the pair $(J, \phi_\psi)$ is a generalized Kähler where $\phi = \exp(-\sqrt{-1} \omega)$.

**Example 2.7.** Let $(J_1, J_2)$ be a generalized Kähler structure. Then the action of $b$-fields gives a generalized Kähler structure $(Ad_e b J_1, Ad_e b J_2)$ for a real $d$-closed 2-form $b$.

**Definition 2.8.** A generalized Kähler structure of symplectic type is a generalized Kähler structure $(J, J_\psi)$, where $J_\psi$ is a generalized complex structure induced from a $d$-closed, nondegenerate, pure spinor $\psi = e^{b-\sqrt{-1} \omega}$ for a $d$-closed 2-form $b$ and a symplectic structure $\omega$.

### 3 Scalar curvature of generalized Kähler manifolds

Let $M$ be a compact, orientable manifold of real dimension $2n$ and $\text{vol}_M$ a volume form on $M$. We denote by $H$ a real $d$-closed 3-form. Let $(J_\phi, J_\psi)$ be an almost $H$-twisted generalized Kähler structure on $M$, where $\phi = \{\phi_\alpha\}$ is local trivializations of the canonical line bundle $K_{J_\phi}$ of generalized complex structure $J_\phi$, that is, each $\phi_\alpha$ is a nondegenerate, pure spinor which induces $J_\phi$ on an open set $U_\alpha$, and $\{U_\alpha\}$ is an open covering of $M$. We also denote by $\psi = \{\psi_\alpha\}$ local trivializations of the canonical line bundle $K_{J_\psi}$ relative to the cover $\{U_\alpha\}$, that is, each $\psi_\alpha$ is a nondegenerate, pure spinor on $U_\alpha$ which induces $J_{J_\psi}$.

From now on, we assume the following normalization for all $\alpha$,

\[ \text{vol}_M := i^{-n} \langle \phi_\alpha, \overline{\phi}_\alpha \rangle_s = i^{-n} \langle \psi_\alpha, \overline{\psi}_\alpha \rangle_s, \quad (3.1) \]

where $\text{vol}_M$ is the fixed volume form on $M$. Note that a generalized Kähler structure gives an orientation of a manifold $M$. Then we can have a normalization as in (3.1), if necessary, by multiplying nonzero real functions on each $\phi_\alpha$ and $\psi_\alpha$. We denote by $d_H$ the differential operator $d + H$ which acts on differential forms on $M$. Then the action of $d_H$ on $\phi_\alpha$ and $\psi_\alpha$ are respectively written as

\[ d_H \phi_\alpha = (\eta_\alpha + N_\phi) \cdot \phi_\alpha, \quad d_H \psi_\alpha = (\zeta_\alpha + N_\psi) \cdot \psi_\alpha. \]
where $\eta_\alpha, \zeta_\alpha \in \sqrt{-1}(T_M \oplus T_M^*)$ are chosen to be pure imaginary and $N_\phi, N_\psi \in \wedge^3(T_M \oplus T_M^*)^\mathbb{R}$ are real. Since $\eta_\alpha, \zeta_\alpha$ are pure imaginary, $\eta_\alpha, \zeta_\alpha$ are uniquely determined. If $N_\phi$ (resp. $N_\psi$) vanishes, then $\mathcal{J}_\phi$ (resp. $\mathcal{J}_\psi$) is integrable, in the sense of generalized complex structure. Thus $N_\phi, N_\psi$ are referred as the Nijenhuis tensors. Note that $N_\phi, N_\psi$ are globally defined real skew-symmetric 3-tensors of $\wedge^3(T_M \oplus T_M^*)^\mathbb{R}$. We denote by $L^H_e$ the $H$-twisted Lie derivative $L^H_e := dHe + edH$ which acts on both differential forms and sections of $T_M \oplus T_M^*$, where $e \in (T_M \oplus T_M^*)^\mathbb{C}$.

We define a real function $S(\mathcal{J}_\phi, \mathcal{J}_\psi)$ by taking the real part

**Definition 3.1.**

$$S(\mathcal{J}_\phi, \mathcal{J}_\psi)\text{vol}_M := \text{Re} \left( i^{-n}\langle \phi_\alpha, L^H_{\eta_\alpha} \overline{\psi}_\alpha \rangle_s + i^{-n}\langle \phi_\alpha, L^H_{\zeta_\alpha} \overline{\phi}_\alpha \rangle_s \right) + 2\langle \zeta_\alpha, \eta_\alpha \rangle_{T\oplus T^*}\text{vol}_M,$$

where $\overline{\phi}_\alpha$ and $\overline{\psi}_\alpha$ denote the complex conjugates of $\phi_\alpha$ and $\psi_\alpha$, respectively. We refer to $S(\mathcal{J}_\phi, \mathcal{J}_\psi)$ as the scalar curvature of $(\mathcal{J}_\phi, \mathcal{J}_\psi)$.

Then we have

**Proposition 3.2.** $S(\mathcal{J}_\phi, \mathcal{J}_\psi)$ does not depend on the choices of local trivializations $\{\phi_\alpha\}, \{\psi_\alpha\}$ of the canonical line bundles $K_{\mathcal{J}_\phi}$ and $K_{\mathcal{J}_\psi}$.

**Proof.** We denote by $S(\phi_\alpha, \psi_\alpha)$ a function as in Definition 3.1

$$S(\phi_\alpha, \psi_\alpha)\text{vol}_M := \text{Re} \left( i^{-n}\langle \phi_\alpha, L^H_{\eta_\alpha} \overline{\psi}_\alpha \rangle_s + i^{-n}\langle \phi_\alpha, L^H_{\zeta_\alpha} \overline{\phi}_\alpha \rangle_s \right) + 2\langle \zeta_\alpha, \eta_\alpha \rangle_{T\oplus T^*}\text{vol}_M,$$

Let $\phi'_\alpha$ and $\psi'_\alpha$ be another local trivializations. Then it follows from the normalization condition 3.1 that one has $\phi'_\alpha = e^{i\eta_\alpha} \phi_\alpha$ and $\psi'_\alpha = e^{iq_\alpha} \psi_\alpha$, where $p_\alpha$ and $q_\alpha$ are real smooth functions on $U_\alpha$. Then it suffices to show $S(\phi_\alpha, \psi_\alpha) = S(\phi'_\alpha, \psi'_\alpha)$. In fact, we define $\eta'_\alpha$ and $\zeta'_\alpha$ by

$$d\phi'_\alpha = \eta'_\alpha \cdot \phi'_\alpha + N_\phi \cdot \phi', \quad d\psi'_\alpha = \zeta'_\alpha \cdot \psi'_\alpha + N_\psi \cdot \psi'$$

Then $\eta'_\alpha = \eta_\alpha + idp_\alpha$ and $\zeta'_\alpha = \zeta_\alpha + idq_\alpha$. Then the Lie derivatives are given by

$$L^H_{\eta_\alpha} \psi'_\alpha = L^H_{\eta_\alpha + idp_\alpha} \psi'_\alpha = L^H_{\eta_\alpha}(e^{i\eta_\alpha} \psi_\alpha) = e^{iq_\alpha} L^H_{\eta_\alpha} \psi_\alpha + (\eta_\alpha(idq_\alpha))e^{iq_\alpha} \psi_\alpha,$$

where $(\eta_\alpha(idq_\alpha))$ denotes the coupling $\eta_\alpha$ and $idq_\alpha$. Taking the complex conjugate, we have

$$L^H_{\eta_\alpha} \overline{\phi}_\alpha = e^{-iq_\alpha} L^H_{\eta_\alpha} \overline{\phi}_\alpha - (\eta_\alpha(idq_\alpha))e^{-iq_\alpha} \overline{\phi}_\alpha$$
Thus
\[ \langle \psi'_\alpha, L^H_{\eta'_\alpha} \bar{\psi}_\alpha \rangle_s = \langle e^{iq^2} \psi, e^{-iq^2} L^H_{\eta} \bar{\psi} \rangle_s - (\eta_\alpha(idq_\alpha)) \langle \psi_\alpha, \bar{\psi}_\alpha \rangle_s \]

One also has
\[ \langle \phi'_\alpha, L^H_{\zeta'_\alpha} \bar{\phi}_\alpha \rangle_s = \langle e^{ip^2} \phi, e^{-ip^2} L^H_{\zeta} \bar{\phi} \rangle_s - (\zeta_\alpha(idp_\alpha)) \langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s \]

Then we have
\[ S(\phi', \psi') vol_M - 2\langle \zeta'_\alpha, \eta'_\alpha \rangle_{T \otimes T^*} vol_M = S(\phi, \psi) vol_M - 2\langle \zeta, \eta \rangle_{T \otimes T^*} vol_M \]

\[ - (\eta_\alpha(idq_\alpha)) i^{-n} \langle \psi_\alpha, \bar{\psi}_\alpha \rangle_s - (\zeta_\alpha(idp_\alpha)) i^{-n} \langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s \]

We also have
\[ 2\langle \zeta'_\alpha, \eta'_\alpha \rangle_{T \otimes T^*} = 2\langle idq_\alpha, \eta_\alpha \rangle_{T \otimes T^*} + 2\langle \zeta_\alpha, idp_\alpha \rangle_{T \otimes T^*} \]

\[ = (\eta_\alpha(idq_\alpha)) + (\zeta_\alpha(idp_\alpha)) \]

From the normalization condition \[3.1\], we obtain
\[ S(\phi', \psi') = S(\phi, \psi). \]

\[ \square \]

**Lemma 3.3.** \[ S(\mathcal{J}_\phi, \mathcal{J}_\psi) \] is also given by
\[ S(\mathcal{J}_\phi, \mathcal{J}_\psi) vol_M := \text{Re} \left( i^{-n} \langle \psi_\alpha, d_H(\eta_\alpha \cdot \bar{\psi}) \rangle_s + i^{-n} \langle \phi_\alpha, d_H(\zeta_\alpha \cdot \bar{\phi}) \rangle_s \right) \]

**Proof.** Since the Lie derivative \( L^H_{\eta_\alpha} \) is given by \( d_H \circ (\eta_\alpha) + (\eta_\alpha) \circ d_H \), it follows
\[ \text{Re} \left( i^{-n} \langle \psi_\alpha, L_H(\eta_\alpha \cdot \bar{\psi}) \rangle_s \right) = \text{Re} \left( i^{-n} \langle \psi_\alpha, d_H(\eta_\alpha \cdot \bar{\psi}) \rangle_s \right) \]
\[ + \text{Re} \left( i^{-n} \langle \psi_\alpha, \eta_\alpha \cdot d_H \bar{\psi} \rangle_s \right) \]

Since \( d_H \bar{\psi} = -\zeta_\alpha \cdot \bar{\psi} \), one has
\[ \text{Re} \left( i^{-n} \langle \psi_\alpha, L_H(\eta_\alpha \cdot \bar{\psi}) \rangle_s \right) = \text{Re} \left( i^{-n} \langle \psi_\alpha, d_H(\eta_\alpha \cdot \bar{\psi}) \rangle_s \right) \]
\[ - \text{Re} \left( i^{-n} \langle \psi_\alpha, \eta_\alpha \cdot \zeta_\alpha \cdot \bar{\psi} \rangle_s \right) \]

One also has
\[ \text{Re} \left( i^{-n} \langle \phi_\alpha, L_H(\zeta_\alpha \cdot \bar{\phi}) \rangle_s \right) = \text{Re} \left( i^{-n} \langle \phi_\alpha, d_H(\zeta_\alpha \cdot \bar{\phi}) \rangle_s \right) \]
\[ - \text{Re} \left( i^{-n} \langle \phi_\alpha, \zeta_\alpha \cdot \eta_\alpha \cdot \bar{\phi} \rangle_s \right) \]
Since $\eta_\alpha$ is pure imaginary, it follows $\eta_\alpha = \eta^{1,0}_\alpha + \eta^{0,1}_\alpha$, where $\eta^{1,0}_\alpha \in \mathcal{L}_{\mathcal{J}_\phi}$ and $\eta^{0,1}_\alpha \in \mathcal{T}_{\mathcal{J}_\phi}$ and $\eta^{1,1}_\alpha = -\eta^{1,0}_\alpha$. Thus $2\text{Re}(\eta^{0,1}_\alpha, \zeta^{1,0}_\alpha)_{/T \oplus T^*} = \text{Re}(\eta_\alpha, \zeta_\alpha)_{/T \oplus T^*}$. Then it follows

$$\text{Re} \left( i^{-n} \langle \psi_\alpha, \eta_\alpha \cdot \zeta_\alpha \cdot \bar{\phi}_\alpha \rangle_s \right) = \text{Re} \left( i^{-n} \langle \psi_\alpha, \eta^{0,1}_\alpha \cdot \zeta^{1,0}_\alpha \cdot \bar{\psi}_\alpha \rangle_s \right)$$

$$= 2\text{Re}(\eta^{0,1}_\alpha, \zeta^{1,0}_\alpha)_{/T \oplus T^*} i^{-n} \langle \psi, \bar{\psi} \rangle_s$$

We also have

$$\text{Re} \left( i^{-n} \langle \phi_\alpha, \zeta_\alpha \cdot \eta_\alpha \cdot \bar{\phi}_\alpha \rangle_s \right) = \langle \zeta_\alpha, \eta_\alpha \rangle_{T \oplus T^*} i^{-n} \langle \phi, \bar{\phi} \rangle_s$$

Thus it follows

$$S(\mathcal{J}_\phi, \mathcal{J}_\psi)_{/\text{vol}_M} = \text{Re} \left( i^{-n} \langle \psi_\alpha, L_H(\eta_\alpha \cdot \bar{\psi}_\alpha) \rangle_s + i^{-n} \langle \phi_\alpha, L_H(\zeta_\alpha \cdot \bar{\phi}_\alpha) \rangle_s \right)$$

$$+ 2\langle \zeta_\alpha, \eta_\alpha \rangle_{T \oplus T^*} i^{-n} \langle \phi_\alpha, d_H(\eta_\alpha \cdot \bar{\psi}_\alpha) \rangle_s + i^{-n} \langle \phi_\alpha, d_H(\zeta_\alpha \cdot \bar{\phi}_\alpha) \rangle_s$$

Hence the result follows. \(\square\)

Let $\text{Diff}_0(M)$ be the identity component of $\text{Diff}(M)$. Define $\text{Diff}_0(M)$ to be the extension of $\text{Diff}_0(M)$ by $2$-forms

$$0 \to \Omega^2(M) \to \text{Diff}_0(M) \to \text{Diff}_0(M) \to 0$$

The group $\text{Diff}_0(M)$ acts on differential forms by pullback of diffeomorphisms together with the exterior product of $e^b$ of the $b$-field. We define a subgroup of $\text{Diff}_0(M)$ by

**Definition 3.4.**

$$\text{Diff}_0^b(M, H) = \{ e^b F \in \text{Diff}_0(M) \mid d_H \circ (e^b F^*) = (e^b F^*) \circ d_H \}$$

Note that $e^b F \in \text{Diff}_0^b(M, H)$ if and only if $db = F^* H - H$. Each $g = e^b F \in \text{Diff}_0^b(M, H)$ acts on $\phi = \{ \phi_\alpha \}$ by $g \cdot \phi = \{ e^b \wedge F^* \phi_\alpha \}$ and also acts on $\psi = \{ \psi_\alpha \}$ by $g \cdot \psi = \{ e^b \wedge F^* \psi_\alpha \}$. Then $(g \cdot \phi, g \cdot \psi)$ gives a generalized Kähler structure $(\mathcal{J}_g \phi, \mathcal{J}_g \psi)$.

**Proposition 3.5.** $S(\mathcal{J}_g \phi, \mathcal{J}_g \psi)$ is equivalent under the action of $\text{Diff}_0^b(M, H)$, in the following sense,

$$S(\mathcal{J}_g \phi, \mathcal{J}_g \psi)_{/\text{vol}_M} = F^* (S(\mathcal{J}_\phi, \mathcal{J}_\psi)_{/\text{vol}_M})$$

for all $g = e^b F \in \text{Diff}_0^b(M, H)$,
PROOF. The result follows from Definition 3.1. In fact, We denote by \( g \cdot \phi \) the action \( g = e^b \mathcal{F} \) on \( \phi = \{ \phi_\alpha \} \). Then

\[
d_H(g \cdot \phi_\alpha) = g d_H \phi_\alpha = g \circ (\eta_\alpha \cdot \phi_\alpha + N_\phi \cdot \phi_\alpha) = (g \circ \eta_\alpha \circ g^{-1}) \cdot g \cdot \phi_\alpha + g \circ N_\phi \circ g^{-1} \circ g \cdot \phi_\alpha,
\]

where \( g \circ \zeta_\alpha \circ g^{-1} \in \sqrt{-1}(T_M \oplus T_M^*) \). We also have

\[
d_H(g \cdot \psi_\alpha) = (g \circ \zeta_\alpha \circ g^{-1}) \cdot g^{-1} \cdot \psi_\alpha + g \circ N_\psi \circ g^{-1} \circ g \cdot \psi_\alpha
\]

Thus one has

\[
S(\mathcal{J}_\phi, \mathcal{J}_\psi) \text{vol}_M = \text{Re} \left( i^{-n} g \cdot \psi_\alpha, \ d_H(g \circ \eta_\alpha \circ g^{-1} \cdot \psi_\alpha) \right) + \text{Re} \left( i^{-n} g \cdot \phi_\alpha, \ d_H(g \circ \zeta_\alpha \circ g^{-1} \cdot \phi_\alpha) \right) + \text{Re} \left( i^{-n} g \cdot \psi_\alpha, \ g \circ d_H(\eta_\alpha \circ \psi_\alpha) \right)
\]

\[
= \text{Re} \left( i^{-n} g \cdot \psi_\alpha, \ g \circ d_H(\eta_\alpha \circ \psi_\alpha) \right) + \text{Re} \left( i^{-n} g \cdot \phi_\alpha, \ g \circ d_H(\zeta_\alpha \circ \phi_\alpha) \right)
\]

\[
= \text{Re} F^*(i^{-n} \langle \psi_\alpha, \ d_H(\eta_\alpha \circ \psi_\alpha) \rangle) + \text{Re} F^*(i^{-n} \langle \phi_\alpha, \ d_H(\zeta_\alpha \circ \phi_\alpha) \rangle)
\]

\[
= F^* S(\mathcal{J}_\phi, \mathcal{J}_\psi) F^* \text{vol}_M
\]

\[
\square
\]

PROPOSITION 3.6. (1) Let \( (\mathcal{J}_1, \mathcal{J}_\omega) \) be a generalized Kähler structure which comes from the ordinary Kähler structure \( (J, \omega) \) as in Example 2.6. Then \( S(\mathcal{J}_\phi, \mathcal{J}_\psi) \) coincides with the ordinary scalar curvature, up to a constant, where \( \text{vol}_M = \frac{\text{vol}_M}{\text{vol}} \).

(2) If \( (\mathcal{J}_\phi, \mathcal{J}_\psi) \) is a GK of symplectic type, then \( S(\mathcal{J}_\phi, \mathcal{J}_\psi) \) is a moment map as in 17, where \( \mathcal{J}_\psi \) is given by an \( e^{b-\sqrt{-1}\omega} \) and \( \text{vol}_M = i^{-n} \langle \psi, \psi \rangle_s \).

(3) \( S(\mathcal{J}_\phi, \mathcal{J}_\psi) = S(\mathcal{J}_\phi, \mathcal{J}_\psi) \) is invariant under the change of \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \).

PROOF. These follow from our definition.

\[
\square
\]

REMARK 3.7. Note that our definition of \( S(\mathcal{J}_\phi, \mathcal{J}_\psi) \) depends on the choice of a volume form on \( M \). Strictly speaking, it might be better to denote the scalar curvature by \( S(\mathcal{J}_\phi, \mathcal{J}_\psi, \text{vol}_M) \). Since a generalized Kähler structure gives the canonical volume form \( \text{vol}_{\text{can}} \), one can define \( S(\mathcal{J}_\phi, \mathcal{J}_\psi, \text{vol}_{\text{can}}) \). However, in order to obtain the moment map framework, we fix \( \mathcal{J}_\psi \) and choose \( \text{vol}_M \) to be \( i^{-n} \langle \psi_\alpha, \overline{\psi_\alpha} \rangle_s \). (See Section 5).

REMARK 3.8. Boulanger also obtained the moment map in the cases of toric generalized Kähler manifolds of symplectic type by using a description of toric
geometry [1]. As though Boulanger’s description of the moment map seems to be different from the one in [17], these should match each other since the moment map is unique modulo constant. In fact, Wang, Yicao actually shows that these are the same by using explicit calculations [24].

**Remark 3.9.** There is a one to one corresponding between bihermitian structures \((I_+, I_-, g, b)\) and generalized Kähler structures \((\mathcal{J}_1, \mathcal{J}_2)\). In Supergravity [9], the notion of generalized scalar curvature is introduced, which is given by depending an arbitrary function \(f\), [6]

\[
GS^f(J) = s_g + 4\Delta_g f - 4|df|^2 - \frac{1}{2}|db|^2
\]

In dimension 4, Boulanger shows that our scalar curvature \(S(J, \mathcal{J}_\omega) = GS^f(J)\) if \(f = -\frac{1}{2}\log(1 - p)\), where \(p = -\frac{3}{2}tr(I_+ I_-)\) is the angle function.

**Remark 3.10.** J. Streets studies problems of generalized Kähler structures by using pluriclosed flows [23]. In the cases of generalized Kähler structures of type \((0, 0)\), (which is also called a degenerate generalized Kähler structure), his definition of generalized Kähler structure with constant scalar curvature is the same as the one in our paper (see also [14], for generalized Kähler structures of type \((0, 0)\)). The Calabi-Yau type problem of generalized Kähler manifolds of type \((0, 0)\) was discussed by Apostolov and Streets in [2]. A generalized Kähler Ricci flow has been explored [7], [3]. It is a remarkable problem to investigate a relation between the scalar curvature as moment map in this paper and the generalized Kähler Ricci flow.

Thus from Proposition 3.4 and Proposition 3.6, \(S(\mathcal{J}_\psi, \mathcal{J}_\psi)\) is regarded as an invariant of generalized Kähler structures.

### 4 Generalized Hamiltonian diffeomorphisms

We shall introduce generalized Hamiltonian diffeomorphisms on a \(H\)-twisted generalized complex manifold. Let \(\mathcal{J}_\psi\) be a \(H\)-twisted generalized complex structure on \(M\) which satisfies

\[
d_H\psi_\alpha = \zeta_\alpha \cdot \psi_\alpha.
\]

Note that in this section we assume that \(\mathcal{J}_\psi\) is integrable. Let \(\{f_1, f_2\}_\psi\) be a function \(L_{\mathcal{J}_\psi} df_1 f_2 - L_{\mathcal{J}_\psi} df_2 f_1\), which is a generalization of the Poisson bracket of symplectic geometry. We denote by \(\pi_T : T_M \oplus T_M \rightarrow T_M\) the projection to the tangent bundle. Then it follows that \(\pi_T(\zeta_\alpha)\) does not depend on the choice of
trivializations $K_{\mathcal{J}_\psi}$ and thus one has a vector field $\pi_T(\zeta_\alpha)$ on $M$, which is globally defined. We consider a set of real smooth normalized functions $C^\infty_\psi(M, \mathbb{R})$ which are invariant under the action of the vector field $\pi_T(\zeta_\alpha)$, that is,

$$C^\infty_\psi(M, \mathbb{R}) := \{ f \in C^\infty(M, \mathbb{R}) \mid \pi_T(\zeta_\alpha)f = 0, \int_M fvol = 0 \}$$

**Lemma 4.1.** We define $\mathfrak{ham}(M, \mathcal{J}_\psi)$ by

$$\mathfrak{ham}(M, \mathcal{J}_\psi) := \{ \mathcal{J}_\psi df \in T_M \oplus T^*_M \mid f \in C^\infty_\psi(M, \mathbb{R}) \}$$

Then $\mathfrak{ham}(M, \mathcal{J}_\psi)$ is closed under the Courant bracket.

**Proof.** We denote by $[,]_H$ the $H$-twisted Courant bracket. Since $\mathcal{J}_\psi$ is integrable, we have

$$[\mathcal{J}_\psi df_1, \mathcal{J}_\psi df_2]_H = [df_1, df_2]_H + \mathcal{J}_\psi[\mathcal{J}_\psi df_1, df_2]_H + \mathcal{J}_\psi df_1, \mathcal{J}_\psi df_2]_H $$

Since $[df_1, df_2]_H = [df_1, df_2]_\text{co} = 0$, we have

$$[\mathcal{J}_\psi df_1, \mathcal{J}_\psi df_2]_H = \mathcal{J}_\psi[\mathcal{J}_\psi df_1, df_2]_\text{co} + \mathcal{J}_\psi[\mathcal{J}_\psi df_1, \mathcal{J}_\psi df_2]_\text{co} \quad (4.1)$$

where $[,]_\text{Dor}$ denotes the Dorman bracket. Since $[\zeta_\alpha, \mathcal{J}_\psi df]_H = [L^H_{\zeta_\alpha}, \mathcal{J}_\psi df] = 0$, one also has

$$\pi_T\zeta_\alpha \{ f_1, f_2 \}_\psi = L^H_{\zeta_\alpha} \{ f_1, f_2 \}_\psi = L^H_{\zeta_\alpha}(L^H_{\mathcal{J}_\psi df_1} df_2 - L^H_{\mathcal{J}_\psi df_2} df_1) = L^H_{\{ \zeta_\alpha, \mathcal{J}_\psi df_1 \}_H} df_2 - L^H_{\{ \zeta_\alpha, \mathcal{J}_\psi df_2 \}_H} df_1 = 0$$

Thus we obtain the result. \hfill $\square$

Since $\mathfrak{ham}(M, \mathcal{J}_\psi)$ is isotropic, it follows that $\mathfrak{ham}(M, \mathcal{J}_\psi)$ is a Lie algebra. In order to obtain a moment map framework, we need to "modify" the action of our hamiltonian in terms of $\zeta_\alpha$.

**Definition 4.2.** Each $f \in \mathfrak{ham}(M, \mathcal{J}_\psi)$ gives rise to the following modified Lie derivative

$$\bar{L}^H_{\mathcal{J}_\psi df} := L^H_{\mathcal{J}_\psi df} - 2\sqrt{-1} f L^H_{\zeta_\alpha} \quad (4.2)$$

Then $f$ acts on both sections of $T_M \oplus T^*_M$ and differential forms on $M$ as in the ordinary Lie derivative.
If $\zeta_\alpha = 0$, it follows $\bar{L}_H^{J_{\psi}} := L^{\bar{J}_{\psi}}_H$ which is the ordinary $H$-twisted Lie derivative. However, if $\zeta_\alpha \neq 0$, we modify the ordinary definition of the action of Hamiltonians. Since $\zeta_\alpha$ is pure imaginary, $\bar{L}_H^{J_{\psi}}$ is defined as a real differential operator. Since $\zeta_\alpha = \zeta + dq_\alpha, \beta$, it follows $L^{J_{\psi}}_\zeta = L^{\bar{J}_{\psi}}_\zeta$. Thus $\bar{L}_H^{J_{\psi}}$ is well-defined on $M$. We need the following lemmas in order to show that the set of differential operators $\{ L^{J_{\psi}}_f | f \in C_\psi^\infty(M, \mathbb{R}) \}$ is closed under the operator bracket.

**Lemma 4.3.** The operator bracket of $L_{e_1}$ and $L_{e_2}$ is denoted by $[L_{e_1}, L_{e_2}] := L_{e_1}L_{e_2} - L_{e_2}L_{e_1}$, for $e_1, e_2 \in T_M \oplus T_M^*$. Then one has

$$[L_{e_1}, L_{e_2}] = L_{[e_1, e_2]}$$

**Proof.** The operator bracket is given in terms of the Dorfman bracket of $e_1, e_2$,

$$[L_{e_1}, L_{e_2}] = L_{[e_1, e_2]}_{Dor}$$

Since the difference between the Dorfman bracket and the Courant bracket is given by $[e_1, e_2]_{Dor} - [e_1, e_2]_{co} = d(e_1, e_2)_{T \oplus T^*}$, we have $L_{[e_1, e_2]}_{Dor} = L_{[e_1, e_2]_{co}}$. Thus we have $[L_{e_1}, L_{e_2}] = L_{[e_1, e_2]_{co}}$. □

**Lemma 4.4.** $[L^{H}_{e_1}, L^{H}_{e_2}] = L^{H}_{[e_1, e_2]}$

**Proof.** For $e_1 = v_1 + \theta_1, e_2 = v_2 + \theta_2 \in T_M \oplus T_M^*$, one has

$$[L^{H}_{e_1}, L^{H}_{e_2}] = L_{[e_1, e_2]_{co}} + d_{v_1}i_{v_2}H + i_{[v_1, v_2]}H$$

$$= L_{[e_1, e_2]_{H}} + i_{[v_1, v_2]}H$$

$$= L^{H}_{[e_1, e_2]}$$

□

**Lemma 4.5.** $L^{H}_{\zeta_\alpha}J_\psi = 0$

**Proof.** It suffices to show that $L^{H}_{\zeta_\alpha} \psi_\alpha = f \psi_\alpha$ for a function $f$. Since $d_H \psi_\alpha = \zeta_\alpha \cdot \psi_\alpha$ and $d_H \circ d_H = 0$, we have

$$L^{H}_{\zeta_\alpha} \psi_\alpha = d_H (\zeta_\alpha \cdot \psi_\alpha) + \zeta_\alpha \cdot d_H \psi_\alpha$$

$$= d_H d_H \psi + \zeta_\alpha \cdot \zeta_\alpha \cdot \phi_\alpha$$

$$= (\zeta_\alpha, \zeta_\alpha)_{T \oplus T^*} \psi_\alpha$$

Since $(\zeta_\alpha, \zeta_\alpha)_{T \oplus T^*}$ is a function, it implies that $L^{H}_{\zeta_\alpha} \psi_\alpha \in K_{J_\psi}$. Thus $L^{H}_{\zeta_\alpha}$ preserves $J_\psi$. Then we have $L^{H}_{\zeta_\alpha} J_\psi = 0$. □
Hence we obtain

Thus we have

We also have

Then it follows

\begin{align*}
[\tilde{L}_J^H, \tilde{L}_J^H] &= \tilde{L}_J^H (f_1, f_2)_\nu.
\end{align*}

**Proof.** From Lemma 4.4 and (4.1), we have

\begin{align*}
[\tilde{L}_J^H, \tilde{L}_J^H] &= \tilde{L}_J^H (f_1, f_2)_\nu = L_J^H (f_1, f_2)_\nu.
\end{align*}

Since \( f_1, f_2 \in C^\infty_\psi (M, \mathbb{R}) \), we have

\begin{align*}
[L_J^H, f_2 L_J^H] &= (L_J^H, f_2) L_J^H + f_2 [L_J^H, L_J^H] \\
&= \pi_T (J^\psi (f_1)) f_2 L_J^H + f_2 L_J^H [J^\psi (f_1), \zeta\nu]_H.
\end{align*}

Since \( L_{[J^\psi (f_1), \zeta\nu]} = -L_J^H (J^\psi (f_1), \zeta\nu) \), we have

\begin{align*}
f_2 L_J^H [J^\psi (f_1), \zeta\nu]_H &= -f_2 L_J^H [J^\psi (f_1)]_H \\
&= -f_2 L_J^H [J^\psi (f_1)]_H + f_2 L_J^H [\pi_T (J^\psi (f_1), \zeta\nu) H].
\end{align*}

Then from Lemma 4.5, \( L_J^H = L_{\zeta\nu} + \pi_T (\zeta\nu) \cdot H \) and \( L_{\zeta\nu} f_1 = 0 \), we have

\begin{align*}
[J^\psi (f_1), \zeta\nu]_H &= L_{\zeta\nu} (J^\psi (f_1)) = L_{\zeta\nu} (J^\psi (f_1) + \pi_T (J^\psi (f_1) \cdot \pi_T (\zeta\nu) H) \\
&= J^\psi (dL_{\zeta\nu} f_1) + \pi_T (J^\psi (f_1) \cdot \pi_T (\zeta\nu) H) \\
&= \pi_T (J^\psi (f_1) \cdot \pi_T (\zeta\nu) H).
\end{align*}

Thus we have

\begin{align*}
f_2 L_J^H [J^\psi (f_1), \zeta\nu]_H &= 0 \quad (4.3)
\end{align*}

Hence we obtain

\begin{align*}
[L_J^H, f_2 L_J^H] &= (L_J^H, f_2) L_J^H \quad (4.4)
\end{align*}

We also have

\begin{align*}
[f_1 L_J^H, L_J^H] &= -(L_J^H, f_2) L_J^H \quad (4.5)
\end{align*}

Then it follows

\begin{align*}
[L_J^H, f_2 L_J^H] + [f_1 L_J^H, L_J^H] &= \{ f_1, f_2 \}_\nu L_J^H.
\end{align*}
Since $\tilde{L}^H_{\zeta_\alpha} f_1 = L^H_{\zeta_\alpha} f_2 = 0$, we have

$$\left[ \tilde{L}^H_{f_1}, \tilde{L}^H_{f_2} \right] = \left[ L^H_{f_1}, L^H_{f_2} \right] - 2\sqrt{-1} [L^H_{f_1}, f_2 L^H_{\zeta_\alpha}] - 2\sqrt{-1} [f_1 L^H_{\zeta_\alpha}, L^H_{f_2}] - 4 [f_1 L^H_{\zeta_\alpha}, f_2 L^H_{\zeta_\alpha}]$$

$$= L^H_{f_1 d(f_1, f_2) \psi} - 2\sqrt{-1} [f_1, f_2] \psi L^H_{\zeta_\alpha}$$

$$= \tilde{L}^H_{f_1 d(f_1, f_2) \psi}$$

\[ \blacksquare \]

**Proposition 4.7.** We also have

$$L^H_{\tilde{f}} J \psi = 0$$

**Proof.** Since $d_H \psi_\alpha = \zeta_\alpha \cdot \psi_\alpha$, we have

$$L^H_{\tilde{f}} \psi_\alpha = d_H (J \psi_\alpha) \cdot \psi_\alpha + (J \psi_\alpha) d_H \psi_\alpha = d_H (\sqrt{-1} df) \cdot \psi_\alpha + (J \psi_\alpha) d_H \psi_\alpha$$

$$= - (\sqrt{-1} df) \cdot d_H \psi_\alpha + (J \psi_\alpha) d_H \psi_\alpha$$

$$= - \sqrt{-1} df + J \psi_\alpha \cdot \zeta_\alpha \cdot \psi_\alpha$$

Since $(-\sqrt{-1} df + J \psi_\alpha) \in \mathcal{L}_\psi$, we have $(-\sqrt{-1} df + J \psi_\alpha) \cdot \zeta_\alpha \cdot \psi_\alpha \in K_{\tilde{J}_\psi}$. Thus $L^H_{\tilde{f}} \psi_\alpha \in K_{\tilde{J}_\psi}$. Hence we have $L^H_{\tilde{f}} J \psi = 0$.

\[ \blacksquare \]

**Lemma 4.8.**

$$\tilde{L}^H_{J \psi} J \psi = 0$$

**Proof.** The result follows from Proposition [4.7] and Lemma [4.3] \[ \blacksquare \]

We denote by Ham($M, J_\psi$) the Lie group associated with the Lie algebra $\mathfrak{ham}(M, J_\psi)$. From Proposition [4.6] the modified action $f \mapsto \tilde{L}^H_{f \psi} J_\psi$ is a Lie algebra homomorphism. Then one has the action of the Lie group Ham($M, J_\psi$) on the set of almost generalized complex structures which preserves $J_\psi$.

## 5 Scalar curvature as moment map

### 5.1 Moment map framework

Let $B(M)$ be the set of almost generalized complex structures on a differentiable compact manifold $M$ of dimension $2n$, that is,

$$B(M) := \{ \mathcal{J} : \text{almost generalized complex structure on } M \}.$$
We also define $B_{\text{int}}^0(M)$ as the set of $H$-twisted generalized complex structures on $M$, i.e., integral ones

$$B_{\text{int}}^0(M) := \{ \mathcal{J} : H$-twisted generalized complex structure on $M \}.$$ 

We fix a $H$-twisted generalized complex structure $\mathcal{J}_\psi$ which is defined by a set of nondegenerate, pure spinor $\psi := \{ \psi_\alpha \}$ relative to a cover $\{ U_\alpha \}$ of $M$. Then we have

$$dH\psi_\alpha = \zeta_\alpha \cdot \psi_\alpha, \quad (5.1)$$

where $\zeta_\alpha \in \sqrt{-1}(T_M \oplus T^*_M)$. We can take $\{ \psi_\alpha \}$ which satisfies $\langle \psi_\alpha, \bar{\psi}_\alpha \rangle_s = \langle \psi_\beta, \bar{\psi}_\beta \rangle_s$ if $U_\alpha \cap U_\beta \neq \emptyset$. The we define a volume form $\text{vol}_M$ by

$$\text{vol}_M := (\sqrt{-1})^{-n}\langle \psi_\alpha, \bar{\psi}_\alpha \rangle_s.$$

for each $\alpha$ which is globally defined. An almost generalized complex structure $\mathcal{J}$ is $\mathcal{J}_\psi$-compatible if and only if the pair $(\mathcal{J}, \mathcal{J}_\psi)$ is an almost generalized Kähler structure. Let $B_{\mathcal{J}_\psi}(M)$ be the set of $\mathcal{J}_\psi$-compatible almost generalized complex structure,

$$B_{\mathcal{J}_\psi}(M) := \{ \mathcal{J} \in B(M) : (\mathcal{J}, \mathcal{J}_\psi) \text{ is an almost generalized Kähler structure} \}.$$ 

We also define $B_{\mathcal{J}_\psi}^0(M)$ to be the set of $\psi$-compatible generalized complex structures. For each point $x \in M$, we define $B_{\mathcal{J}_\psi}(M)_x$ to be the set of $\psi_x$-compatible almost generalized complex structures on $T_xM \oplus T^*_xM$,

$$B_{\mathcal{J}_\psi}(M)_x := \{ \mathcal{J}_x \mid (\mathcal{J}_x, \mathcal{J}_\psi)_x : \text{almost generalized Kähler structure at } x \}.$$ 

Then it follows that $B_{\mathcal{J}_\psi}(M)_x$ is given by the Riemannian Symmetric space of type AIII

$$U(n, n)/U(n) \times U(n)$$

which is biholomorphic to the complex bounded domain $\{ h \in M_n(\mathbb{C}) \mid 1_n - h^*h > 0 \}$, where $M_n(\mathbb{C})$ denotes the set of complex matrices of $n \times n$.

**Remark 5.1.** In Kähler geometry, the set of almost complex structures compatible with a symplectic structure $\omega$ is given by the Riemannian symmetric space $\text{Sp}(2n)/U(n)$ which is biholomorphic to the Siegel upper half plane

$$\{ h \in \text{GL}_n(\mathbb{C}) \mid 1_n - h^*h > 0, h^t = h \}$$

Let $P_{\mathcal{J}_\psi}$ be the fibre bundle over $M$ with fibre $B_{\mathcal{J}_\psi}(M)_x$, that is,

$$P_{\psi} := \bigcup_{x \in M} B_{\mathcal{J}_\psi}(M)_x \to M,$$
Then $\mathcal{B}_{\mathcal{J}_\psi}(M)$ is given by smooth sections $\Gamma(M, P_{\mathcal{J}_\psi})$ which contains the integral ones $\mathcal{B}_{\mathcal{J}_\psi}^\text{int}(M)$. We can introduce a Sobolev norm on $\mathcal{B}_{\mathcal{J}_\psi}(M)$ such that $\mathcal{B}_{\mathcal{J}_\psi}(M)$ becomes a Banach manifold in the standard method. The tangent bundle of $\mathcal{B}_{\mathcal{J}_\psi}(M)$ at $\mathcal{J}$ is given by

$$T_{\mathcal{J}}\mathcal{B}_{\mathcal{J}_\psi}(M) = \{ \dot{\mathcal{J}} \in \text{so}(T_M) : \dot{\mathcal{J}}\mathcal{J} + \mathcal{J}\dot{\mathcal{J}} = 0, \mathcal{J}\mathcal{J}_\psi = \mathcal{J}_\psi \dot{\mathcal{J}} \},$$

where $\text{so}(T_M)$ denotes the set of sections of Lie algebra bundle of $\text{SO}(T_M)$. Then it follows that there exists an almost complex structure $J_B$ on $\mathcal{B}_{\mathcal{J}_\psi}(M)$ which is given by

$$J_B(\dot{\mathcal{J}}) := J\dot{\mathcal{J}}, \quad (\dot{\mathcal{J}} \in T_{\mathcal{J}}\mathcal{B}_{\mathcal{J}_\psi}(M))$$

We also have a Riemannian metric $g_B$ and a 2-form $\Omega_B$ on $\mathcal{B}_{\mathcal{J}_\psi}(M)$ by

$$g_B(\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2) := \int_M \text{tr}(\dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2) \text{vol}_M \quad (5.2)$$

$$\Omega_B(\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2) := \int_M \text{tr}(\mathcal{J}\dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2) \text{vol}_M \quad (5.3)$$

for $\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2 \in T_{\mathcal{J}}\mathcal{B}_{\mathcal{J}_\psi}(M)$.

**Proposition 5.2.** $J_B$ is integrable almost complex structure on $\mathcal{B}_{\mathcal{J}_\psi}(M)$ and $\Omega_B$ is a Kähler form on $\mathcal{B}_{\mathcal{J}_\psi}(M)$.

**Proof.** Let $\mathcal{J}_V$ be an almost generalized complex structure on a real vector space $V$ of dimension $2n$. We denote by $X_n$ the Riemannian symmetric space $U(n,n)/U(n) \times U(n)$ which is identified with the set of almost generalized complex structures compatible with $\mathcal{J}_V$. We already see that $\mathcal{B}_{\mathcal{J}_\psi}(M)$ is the set of global sections of the fibre bundle $P_{\mathcal{J}_\psi}$ over a manifold $M$ with fibre $X_n$ which is biholomorphic to the bounded domain $\{ h \in M_n(\mathbb{C}) \mid 1_n - h^*h > 0 \}$. If $\mathcal{B}_{\mathcal{J}_\psi}(M)$ is not empty, we have a global section $\mathcal{J}_0$. Then the fibre bundle is identified with the space of maps from $M$ to the complex bounded domain $\{ h \in M_n(\mathbb{C}) \mid 1_n - h^*h > 0 \}$ which is open set in the complex vector space $M_n(\mathbb{C})$. Since the almost complex structure $J_B$ is induced from the complex structure of the complex bounded domain, $\mathcal{B}_{\mathcal{J}_\psi}(M)$ admits complex coordinates and $J_B$ is integrable. We denote by $g_{X_n}$ the Riemannian metric on $X_n$ and by $\omega_{X_n}$ the Kähler form which are respectively given by

$$g_{X_n}(\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2) = \text{tr}(\dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2)$$

$$\omega_{X_n}(\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2) = -\text{tr}(\mathcal{J}\dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2),$$
where $\hat{J}_1, \hat{J}_2 \in T_J X_n$. The complex bounded domain $\{ h \in \text{GL}_n(\mathbb{C}) \mid 1_n - h^* h > 0 \}$ admits a Kähler structure which is given by

$$4\sqrt{-1} \partial \bar{\partial} \log \det(1_n - h^* h).$$

Then under the identification $X_n \cong \{ h \in M_n(\mathbb{C}) \mid 1_n - h^* h > 0 \}$ by using $\mathcal{J}_\psi$, we have $\omega_{X_n} = 4\sqrt{-1} \partial \bar{\partial} \log \det(1_n - h^* h)$. Then the space of maps $B_{\mathcal{J}_\psi}(M)$ inherits a Riemannian metric $g_B$ and a symplectic structure $\Omega_B$ which are given by

$$g_B(\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2) := \int_M \text{tr}(\dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2) \text{vol}_M \quad (5.4)$$

$$\Omega_B(\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2) := \int_X \text{tr}(\mathcal{J} \dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2) \text{vol}_M = -4\sqrt{-1} \partial \bar{\partial} \int_M \log \det(1_n - h^* h) \text{vol}_M \quad (5.5)$$

Hence $\omega_B$ is closed. Thus $(B_{\mathcal{J}_\psi}(M), \mathcal{J}_B, \Omega_B)$ is a Kähler manifold.

Let $\widehat{\text{Diff}}(M)$ be an extension of diffeomorphisms of $M$ by 2-forms which is defined as

$$\widehat{\text{Diff}}(M) := \{ e^b F : F \in \text{Diff}(M), b : 2\text{-form} \}.$$  

Note that the product of $\widehat{\text{Diff}}(M)$ is given by

$$(e^{b_1} F_1)(e^{b_2} F_2) := e^{b_1 + F_1^*(b_2)} F_1 \circ F_2,$$

where $F_1, F_2 \in \text{Diff}(M)$ and $b_1, b_2$ are real 2-forms. The action of $\widehat{\text{Diff}}(M)$ on $G_C(M)$ by

$$e^b F_\# \circ \mathcal{J} \circ F_\#^{-1} e^{-b} = e^b F_\# \circ \mathcal{J} \circ F_\#^{-1} e^{-b} = \mathcal{J}_\psi \quad (5.7)$$

where $F \in \text{Diff}(M)$ acts on $\mathcal{J}$ by $F_\# \circ \mathcal{J} \circ F_\#^{-1}$ and and $e^b$ is regarded as an element of $SO(T_M \oplus T_M^*)$ and $F_\#$ denotes the bundle map of $T_M \oplus T_M^*$ which is the lift of $F$. For a (integral) generalized complex structure $\mathcal{J}_\psi$, We define $\widehat{\text{Diff}}_{\mathcal{J}_\psi}(M)$ to be a subgroup consists of elements of $\widehat{\text{Diff}}(M)$ which preserves $\mathcal{J}_\psi$,

$$\widehat{\text{Diff}}_{\mathcal{J}_\psi}(M) = \{ e^b F \in \widehat{\text{Diff}}(M) : e^b F_\# \circ \mathcal{J}_\psi \circ F_\#^{-1} e^{-b} = \mathcal{J}_\psi \}.$$  

Then from (5.2), we have the following,

**Proposition 5.3.** The symplectic structure $\Omega_B$ is invariant under the action of $\psi$-preserving group $\widehat{\text{Diff}}_{\mathcal{J}_\psi}(M)$.

**Proof.** The result follows from (5.5) and (5.7) since $\text{vol}_M$ is invariant under the action of $\text{Diff}_{\mathcal{J}_\psi}(M)$.

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Proposition 5.4. Consider the infinitesimal action of \( \text{ham}(M, \mathcal{J}_\psi) \) on \( \Omega_B \) which is given by \( L^H_{\mathcal{J}_\psi df} \). Let \( \text{Ham}(M, \mathcal{J}_\psi) \) be the generalized Hamiltonian diffeomorphisms associated with the Lie algebra is \( \text{ham}(M, \mathcal{J}_\psi) \). Then the action of \( \text{Ham}(M, \mathcal{J}_\psi) \) also preserves \( \Omega_B \).

Proof. The Lie algebra of \( \widetilde{\text{Diff}}(M) \) is given by \( TM \oplus \Omega^2(M) \), where \( \Omega^2(M) \) denotes the 2-forms on \( M \). Then the Lie bracket of \( TM \oplus \Omega^2(M) \) is given by

\[
[L_{v_1 + b_1}, L_{v_2 + b_2}] = L_{[v_1, v_2]} + L_{v_1} b_2 - L_{v_2} b_1
\]

Let \( \iota \) be the map \( T_M \oplus T^*_M \rightarrow TM \oplus \Omega^2(M) \) which is given by

\[
\iota(v, \theta) := (v, i_v H + d\theta),
\]

where \( v \in T_M \) and \( \theta \in T^*_M \). Then as shown in Lemma 4.4, one has

\[
L^H_{[v_1 + \theta_1, v_2 + \theta_2]} = [L^H_{v_1 + \theta_1}, L^H_{v_2 + \theta_2}],
\]

for \( v_1 + \theta_1, v_2 + \theta_2 \in T_M \oplus T^*_M \).

Since \( \text{ham}(M, \mathcal{J}_\psi) = \{ \mathcal{J}_\psi df, | f \in C^\infty(M, \mathbb{R}) \} \subset T_M \oplus T^*_M \), then one see that the map \( \iota \) restricted to \( \text{ham}(M, \mathcal{J}_\psi) \) gives the inclusion from \( \text{ham}(M, \mathcal{J}_\psi) \) to the Lie algebra of \( \text{Diff}_{\mathcal{J}_\psi}(M) \) which is a homomorphism between Lie algebras. Thus \( \text{Ham}_{\mathcal{J}_\psi}(M) \) is a Lie subgroup of \( \text{Diff}_{\mathcal{J}_\psi}(M) \). Then the result follows form Proposition 5.4.

The modified action defined in Definition 4.2 also gives an infinitesimal action of \( \text{ham}(M, \mathcal{J}_\psi) \) on \( \Omega_B \). Then we also have the modified action of \( \text{Ham}(M, \mathcal{J}_\psi) \) on \( \Omega_B \). Then one has

Proposition 5.5. The modified action of \( \text{Ham}(M, \mathcal{J}_\psi) \) as in Definition 4.2 preserves \( \Omega_B \).

Proof. The infinitesimal action is given by

\[
\tilde{L}^H_{\mathcal{J}_\psi df} := L^H_{\mathcal{J}_\psi df} - 2\sqrt{-1} f L^H_{\zeta_\alpha} \quad (5.8)
\]

\[
= L^H_{\mathcal{J}_\psi df} - L^H_{2\sqrt{-1} f \zeta_\alpha} + 2\sqrt{-1} df \cdot \zeta_\alpha \quad (5.9)
\]

As in the proof of Proposition 5.4, \( L^H_{\mathcal{J}_\psi df} \) and \( L^H_{2\sqrt{-1} f \zeta_\alpha} \) preserves \( \Omega_B \) infinitesimally. The Clifford action \( df \cdot \zeta_\alpha \) is also preserves \( \Omega_B \). Then the result follows.

Recall that the Lie algebra \( \text{ham}_{\mathcal{J}_\psi}(M) \) is given by \( C^\infty_{\psi}(M, \mathbb{R}) \), where \( C^\infty_{\psi}(M, \mathbb{R}) = \{ f \in C^\infty(M, \mathbb{R}) | \pi_T(\zeta_\alpha f) = 0, \int_M f \text{vol}_M = 0 \} \). Then \( e := \mathcal{J}_\psi(df) \in T_M \oplus T^*_M \).
is referred as referred a generalized Hamiltonian element. Note that we have $e \cdot \psi_\alpha = -\sqrt{-1} df \cdot \bar{\psi}_\alpha$.

Then we obtain

**Theorem 5.6.** Let $J_\psi$ be a $H$-twisted generalized complex structure on a compact manifold $M$. Then there exists a moment map

$$
\mu : B_{J_\psi}(M) \to C^\infty_\psi(M, \mathbb{R})^*
$$
on $B_{J_\psi}$ for the modified action of the generalized Hamiltonian diffeomorphisms $\text{Ham}(M, J_\psi)$ such that $\mu(J)$ is given by the scalar curvature $S(J, J_\psi)$ for all $J \in B_{J_\psi}(M)$.

**Remark 5.7.** In the previous paper [14], [17], the existence of the moment map was shown in the rather restricted cases of generalized Kähler manifolds of symplectic type or the case of $\zeta_\alpha = 0$. Theorem 5.6 shows the existence of a moment map for all twisted generalized Kähler structures. It is remarkable that the scalar curvature $S(J, J_\psi)$ is invariant under the change of $J_\phi$ and $J_\psi$. In fact, if we fix $J_\phi$ instead of $J_\psi$ and consider the infinite dimensional Kähler manifold $B_{J_\phi}(M)$ on which $\text{Ham}(M, J_\phi)$ acts, then a moment map is also given by the scalar curvature $S(J_\phi, J_\psi)$.

### 5.2 Preliminary results

In order to give a proof of Theorem 5.6, we show several Lemmas, in particular on the Nijenhuis tensor $N$. Let $J \in B_{J_\psi}(M)$ be an almost generalized complex structure which is induced from a set of nondegenerate, pure spinors $\phi = \{\phi_\alpha\}$. We normalize $\{\phi_\alpha\}$ such that $\langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s = \text{vol}_M$ for each $\alpha$. Then $d\phi_\alpha$ is given by

$$
d\phi_\alpha = (\eta_\alpha + N_\alpha) \cdot \phi_\alpha, \tag{5.10}
$$

where $\eta_\alpha \in \sqrt{-1}(T_M \oplus T^*_M)$ and $N_\alpha \in (\wedge^3 L_J \oplus \wedge^3 L_J)^\mathbb{R}$.

**Remark 5.8.** Note that $N_\alpha$ is a real element. $N_\alpha = N_\beta$ for all $\alpha, \beta$. Then $N_\alpha$ defines a global element $N$, which is called Nijenhuis tensor.

**Lemma 5.9.** $N \cdot \psi_\alpha = 0$

**Proof.** Since $N$ is uniquely defines by (5.10), for $e_1, e_2, e_3 \in L_J$, we have

$$
N(e_1, e_2, e_3) \langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s = \langle d\phi_\alpha, e_1 \cdot e_2 \cdot e_3 \cdot \bar{\phi}_\alpha \rangle_s = -\langle e_1 \cdot e_2 \cdot d\phi_\alpha, e_3 \cdot \bar{\phi}_\alpha \rangle_s
$$

$$
= -\langle [e_1, e_2]_{\omega_{J_\phi}}, \phi_\alpha, e_3 \cdot \bar{\phi}_\alpha \rangle_s
$$

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Thus we have
\[ N(e_1, e_2, e_3) = 2 \langle [e_1, e_2]_\omega, e_3 \rangle_{T \oplus T^*} \quad (5.11) \]
This implies that \( N = 0 \) if and only if \( \mathcal{J} \) is integrable. By using \( \mathcal{J}_\psi \), we have the decomposition \( \mathcal{L}_J = \mathcal{L}_J^+ \oplus \mathcal{L}_J^- \) and \( \mathcal{T}_J = \mathcal{T}_J^+ \oplus \mathcal{T}_J^- \). Since \( \ker \psi = \mathcal{L}_J^+ \oplus \mathcal{L}_J^- \) and \( N \in (\wedge^3 \mathcal{L}_J \oplus \wedge^3 \mathcal{T}_J)^{Re} \), we have \( N \cdot \psi = (N^+ + N^-) \cdot \psi \), where \( N^+ \in \wedge^3 \mathcal{T}_J^+ \) and \( N^- \in \wedge^3 \mathcal{L}_J^- \). From \( (5.11) \), we see

\[ N(e_1^-, e_2^-, e_3^-) = \langle [e_1^-, e_2^-]_\omega, e_3^- \rangle_{T \oplus T^*} \]

Since \( \mathcal{J}_\psi \) is integrable, it follows that \( [e_1^-, e_2^-]_\omega \in \mathcal{L}_J \). Since \( e_3^- \in \mathcal{T}_J \), we have \( N(e_1^-, e_2^-, e_3^-) = 0 \). Thus \( N^- = 0 \). Since \( N \) is real, we have \( N^- = 0 \). Hence \( N \cdot \psi = 0 \).

**Remark 5.10.** In the cases of the ordinary Kähler manifolds, \( \psi \) is given by the exponential \( e^{\sqrt{-1} \omega} \), where \( \omega \) denotes a symplectic form and \( \sqrt{-1} \omega = \sum \theta^i \wedge \theta^i \), where \( \{\theta^i\}_{i=1}^n \) denotes the local unitary frame of \( \wedge^{1,0} \). We also denote by \( \{v_i\}_{i=1}^n \) the dual basis of \( T^{1,0}_M \). Then the Nijenhuis tensor \( N \) is given by \( N = \sum N_{ijk} \theta^i \wedge \theta^j \wedge \theta^k \), where \( N_{ijk} \theta^i \wedge \theta^j \wedge \theta^k \) is the \((0,2)\)-component of \( d\theta \). The Nijenhuis tensor \( N \) acts on \( \omega \) by the interior and exterior product, that is, the action of Clifford algebra. Since \( d\omega = 0 \) implies \( N \cdot \omega = 0 \). Thus one may have \( N \cdot \psi = 0 \).

**Lemma 5.11.** We have

\[ \langle e \cdot N \cdot \phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s = 0 \]

**Proof.** Since \( h = h^{2,0} + h^{0,2} \in \wedge^2 \mathcal{L}_J \oplus \wedge^2 \mathcal{T}_J \) and \( N = N^{3,0} + N^{0,3} \in \wedge^3 \mathcal{L}_J \oplus \wedge^3 \mathcal{T}_J \), we have

\[ \langle e \cdot N \cdot \phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s = - \langle N \cdot \phi_\alpha, e \cdot h \cdot \overline{\phi}_\alpha \rangle_s \]

\[ = - \langle N^{0,3} \cdot \phi_\alpha, e^{1,0} \cdot h^{2,0} \cdot \overline{\phi}_\alpha \rangle_s \]

Since \( e^{1,0} \cdot h^{2,0} = h^{2,0} \cdot e^{1,0} \), we have

\[ \langle e \cdot N \cdot \phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s = - \langle N^{0,3} \cdot \phi_\alpha, h^{2,0} \cdot e^{1,0} \cdot \overline{\phi}_\alpha \rangle_s \]

\[ = \langle h^{2,0} \cdot N^{0,3} \cdot \phi_\alpha, e^{1,0} \cdot \overline{\phi}_\alpha \rangle_s \]
We denote by \([h^{2,0}, N^{0,3}]^{0,1}_s \in \mathcal{L}_T\) the component of \([h^{2,0}, N^{0,3}]\). Then we obtain

\[
\langle h^{2,0}, N^{0,3} \cdot \phi_\alpha, \ e^{1,0} \cdot \overline{\phi_\alpha} \rangle_s = \langle [h^{2,0}, N^{0,3}]^{0,1}_s \cdot \phi_\alpha, \ e^{1,0} \cdot \overline{\phi_\alpha} \rangle_s
\]

\[
= 2 \langle [h^{2,0}, N^{0,3}]^{0,1}_s, \ e^{1,0} \rangle_{T \otimes T} \cdot \langle \phi_\alpha, \overline{\phi_\alpha} \rangle_s
\]

\[
= 2 \langle [h^{2,0}, N^{0,3}]^{0,1}_s, \ e^{1,0} \rangle_{T \otimes T} \cdot \langle \psi_\alpha, \overline{\psi_\alpha} \rangle_s
\]

\[
= \langle [h^{2,0}, N^{0,3}]^{0,1}_s \cdot \psi_\alpha, \ e^{1,0} \rangle_{\overline{\phi_\alpha}} \rangle_s
\]

From Lemma 5.9 and \(h \cdot \psi = 0\), we have \([h, N] \cdot \psi = 0\). Thus we have \([h^{2,0}, N^{0,3}]^{0,1}_s \cdot \overline{\psi} = 0\) and \([h^{2,0}, N^{0,3}]^{0,1}_s \cdot \psi = 0\). Hence we obtain \(\langle e \cdot N \cdot \phi_\alpha, \ h \cdot \overline{\phi_\alpha} \rangle_s = 0\). 

**LEMMA 5.12.** We also have

\[\langle N \cdot e \cdot \phi_\alpha, \ h \cdot \overline{\phi_\alpha} \rangle_s = 0\]

**PROOF.** The result follows as in the proof of Lemma 5.11.

Let \(J_t\) be deformations of \(J\) such that \((J_t, J_\psi)\) is an almost generalized Kähler structures. Then we have a family \(\{\phi_\alpha(t)\}\) which gives rise to \(J_t\) depending smoothly on parameter \(t\). Then \(d\phi_\alpha(t) = (\eta_\alpha(t) + N(t)) \cdot \phi_\alpha(t)\), where \(\eta_\alpha(t) \in \sqrt{-1}(T_M \oplus T^*_M)\) and \(N \in (\wedge^3 L_{J_t} \oplus \wedge^3 L_{J_t})(Re)\).

**LEMMA 5.13.** Let \(\dot{N} = \frac{d}{dt}N(t)|_{t=0}\). Then we have

\[\dot{N} \cdot \psi_\alpha = 0\]

**PROOF.** From Lemma 5.9 we have \(N(t) \cdot \psi_\alpha = 0\) for all \(t\). Since \(\psi_\alpha\) is fixed, then we have the result.

**LEMMA 5.14.** \(\langle e \cdot \phi_\alpha, \ \dot{N} \cdot \overline{\phi_\alpha} \rangle_s = 0\).

**PROOF.** The space \(\wedge^4(T_M \oplus T^*_M)\) is decomposed into \(\wedge^4 T_M \oplus (\wedge^3 T_M \oplus T^*_M) \oplus (\wedge^2 T_M \oplus \wedge^2 T^*_M) \oplus (T_M \oplus \wedge^3 T_M) \oplus \wedge^4 T_M\). We denote by \(\mathrm{Cont}^{2,2}\) the contraction of the component \(\wedge^2 T_M \oplus \wedge^2 T^*_M\) which yields a map from \(\wedge^4(T_M \oplus T^*_M)\) to \(C^\infty(M)\). Then it follows

\[
\langle e \cdot \phi_\alpha, \ \dot{N} \cdot \overline{\phi_\alpha} \rangle_s = - \langle \phi_\alpha, e \cdot \dot{N} \cdot \overline{\phi_\alpha} \rangle_s
\]

\[
= - \mathrm{Cont}^{2,2}(e \cdot \dot{N} \langle \phi_\alpha, \ \overline{\phi_\alpha} \rangle_s
\]

\[\text{(5.12)}\]

\[\text{(5.13)}\]
Since $(\phi_{\alpha}, \bar{\phi}_{\alpha})_s = (\psi_{\alpha}, \bar{\psi}_{\alpha})_s$, we have

\[
\langle e \cdot \phi_{\alpha}, \dot{N} \cdot \bar{\phi}_{\alpha} \rangle_s = -\text{Cont}^2(e \cdot \dot{N})\langle \psi_{\alpha}, \bar{\psi}_{\alpha} \rangle_s
\]

\[= \langle e \cdot \psi_{\alpha}, \dot{N} \cdot \bar{\psi}_{\alpha} \rangle_s \tag{5.14}\]

Since $\dot{N}$ is real, it follows from Lemma \[5.13\] that $\dot{N} \cdot \bar{\psi}_{\alpha} = 0$. Hence we have

\[
\langle e \cdot \phi_{\alpha}, \dot{N} \cdot \bar{\phi}_{\alpha} \rangle_s = 0. \tag{5.15}\]

\section{Proof of the scalar curvature as a moment map}

This subsection is devoted to a proof of our main theorem, i.e., Theorem \[5.6\].

\textbf{Proof of Theorem \[5.6\]} Let $\mathcal{J}_\phi$ be an almost generalized complex structure which is induced from $\phi = \{\phi_{\alpha}\}$ as before. We denote by $\mathcal{J}_t$ small deformations of $\mathcal{J}_\phi$. Any infinitesimal deformations $\dot{\mathcal{J}} := \frac{d}{dt}\mathcal{J}_t|_{t=0}$ of $\mathcal{J}$ is given by the adjoint action of $h$

\[
\dot{\mathcal{J}}_h := [h, \dot{\mathcal{J}}]
\]

where $h$ denotes a real $C^\infty$ global section of $(\wedge^2 L_{\mathcal{J}_\phi} \oplus \wedge^2 L_{\mathcal{J}_\phi})^R$. Then the corresponding infinitesimal deformation of $\phi$ is given by the Clifford action of $h$ on each $\phi_{\alpha}$,

\[
\dot{\phi}_{\alpha} = h \cdot \phi_{\alpha}
\]

Note that we can assume $\dot{\phi}_{\alpha}$ is transversal to the canonical line bundle $K_{\mathcal{J}_\phi}$ in the following sense,

\[
\langle \phi_{\alpha}, \bar{\phi}_{\alpha} \rangle_s = 0.
\]

An Hamiltonian element $e = \mathcal{J}_\phi df$ gives the infinitesimal deformation $L_e^H \mathcal{J}$ of $\mathcal{J}$. Since $d_H \phi_{\alpha} = \eta_{\alpha} \cdot \phi_{\alpha} + N \cdot \phi_{\alpha}$, then the corresponding infinitesimal deformations of $\phi_{\alpha}$ is given by $L_e^H \phi_{\alpha}$. Then it follows

\[
L_e^H \phi_{\alpha} = d_H e \cdot \phi_{\alpha} + e \cdot d_H \phi_{\alpha} = d_H e \cdot \phi_{\alpha} + e \cdot (\eta_{\alpha} + N) \cdot \phi_{\alpha},
\]

where $d_H$ denotes $d + H$. Recall our modified action defined in Definition \[4.2\] is

\[
\tilde{L}_e^H := L_e^H - 2\sqrt{-1}fL_{\phi_{\alpha}}^H
\]

Then in order to show the existence of a moment map, we shall calculate $\Omega_B(\tilde{L}_e^H \mathcal{J}, \dot{\mathcal{J}}_h)$. Recall the formula (see Section 7 in \[14\]).

\[
\Omega_B(\dot{\mathcal{J}}_{h_1}, \dot{\mathcal{J}}_{h_2}) = \text{Im} \left( i^{-n} \int_M \langle h_1 \cdot \phi_{\alpha}, h_2 \cdot \bar{\phi}_{\alpha} \rangle_s \right), \tag{5.16}\]
Applying (5.16), we obtain
\[ \Omega_B(L^H_e \mathcal{J}, \mathcal{J}_h) = \text{Im} \left( i^{-n} \int_M (\bar{L}^H_e \phi_\alpha, h \cdot \bar{\phi}_\alpha)_s \right) \] (5.17)

\[ = \text{Im} \left( i^{-n} \int_M (d_H e \cdot \phi_\alpha + e \cdot (\eta_\alpha + N) \cdot \phi_\alpha, h \cdot \bar{\phi}_\alpha)_s \right) \] (5.18)

\[ - \text{Im} \left( i^{-n} \int_M (2\sqrt{-1}fL^H_e \phi_\alpha, h \cdot \bar{\phi}_\alpha)_s \right) \] (5.19)

First we shall calculate the term \( \Omega_B(L^H_e \mathcal{J}, \mathcal{J}_h) \) in (5.18). Since \( h \in (\wedge^2 \mathcal{L}_V \oplus \wedge^2 \mathcal{L}_H)^R \), we have \( \langle \phi_\alpha, h \cdot \bar{\phi}_\alpha \rangle = 0 \). Since \( e \cdot \eta_\alpha + \eta_\alpha \cdot e = 2(e \cdot \eta_\alpha)_{T \oplus T^*} \), we have

\[ \langle e \cdot \eta_\alpha \cdot \phi_\alpha, h \cdot \bar{\phi}_\alpha \rangle_s = -\langle \eta_\alpha \cdot e \cdot \phi_\alpha, h \cdot \bar{\phi}_\alpha \rangle_s \]

Applying Lemma 5.11 we have
\[ \Omega_B(L^H_e \mathcal{J}, \mathcal{J}_h) = \text{Im} \left( i^{-n} \int_M ((d_H - \eta_\alpha)e \cdot \phi_\alpha, h \cdot \bar{\phi}_\alpha)_s \right) \] (5.20)

Since \( d\langle (e \cdot \phi_\alpha), h \cdot \bar{\phi}_\alpha \rangle_{2n-1} = \langle d_H (e \cdot \phi_\alpha), h \cdot \bar{\phi}_\alpha \rangle_s - \langle (e \cdot \phi_\alpha), d_H (h \cdot \bar{\phi}_\alpha) \rangle_s \) and \( \langle (e \cdot \phi_\alpha), h \cdot \bar{\phi}_\alpha \rangle_{2n-1} \) gives a globally defined \((2n - 1)\)-form on \( M \), then applying the Stokes Theorem, we have

\[ \int_M \langle d_H (e \cdot \phi_\alpha), h \cdot \bar{\phi}_\alpha \rangle_s = \int_M \langle e \cdot \phi_\alpha, d_H (h \cdot \bar{\phi}_\alpha) \rangle_s, \]

where we are also applying \( \langle H \cdot (e \cdot \phi_\alpha), h \cdot \bar{\phi}_\alpha \rangle_s = \langle e \cdot \phi_\alpha, H \cdot (h \cdot \bar{\phi}_\alpha) \rangle_s \). Since \( \eta_\alpha \) is in \( \sqrt{-1}(T_M \oplus T^*_M) \), it follows \( \bar{\eta}_\alpha = -\eta_\alpha \). Then we have

\[ \langle \eta_\alpha \cdot e \cdot \phi_\alpha, h \cdot \bar{\phi}_\alpha \rangle_s = -\langle e \cdot \phi_\alpha, \eta_\alpha \cdot h \cdot \bar{\phi}_\alpha \rangle_s. \]

Substituting them into (5.20), we obtain
\[ \Omega_B(L^H_e \mathcal{J}, \mathcal{J}_h) = \text{Im} \left( i^{-n} \int_M (e \cdot \phi_\alpha, (d_H + \eta_\alpha) \cdot (h \cdot \bar{\phi}_\alpha))_s \right) \]

Let \( \phi(t) = \{ \phi_\alpha(t) \} \) be a one parameter family of nondegenerate, pure spinors which gives

\[ \frac{d}{dt} \phi_\alpha(t)|_{t=0} = h \cdot \phi_\alpha. \]

We assume that \( \phi_\alpha(t) \) depends on a parameter \( t \) smoothly. Then \( d\phi_\alpha(t) \) is given by

\[ d\phi_\alpha(t) = (\eta_\alpha(t) + N(t)) \cdot \phi_\alpha(t). \]

Taking the differential of both sides at \( t = 0 \), we have

\[ d(h \cdot \phi_\alpha) = (\dot{\eta}_\alpha + \dot{N}) \cdot \phi_\alpha + (\eta_\alpha + N) \cdot (h \cdot \phi_\alpha), \]

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where \( \dot{\eta}_\alpha = \frac{d}{dt} \eta_\alpha |_{t=0} \) and \( \dot{N} = \frac{d}{dt} N(t) |_{t=0} \). Since \( \eta_\alpha(t) \) is pure imaginary and \( N(t) \) is real, we have

\[
d(h \cdot \overline{\phi_\alpha}) = (-\dot{\eta}_\alpha + \dot{N}) \cdot \overline{\phi_\alpha} + (-\eta_\alpha + N) \cdot (h \cdot \overline{\phi_\alpha})
\]

Thus we obtain

\[
\langle e \cdot \phi_\alpha, (d + \eta_\alpha - N) \cdot (h \cdot \overline{\phi_\alpha}) \rangle_s = -\langle e \cdot \phi_\alpha, (\dot{\eta}_\alpha - \dot{N}) \cdot \overline{\phi_\alpha} \rangle_s
\]

From Lemma 5.14 we have \( \langle e \cdot \phi_\alpha, \dot{N} \cdot \overline{\phi_\alpha} \rangle_s = 0 \). Applying Lemma 5.12 we obtain

\[
\langle e \cdot \phi_\alpha, (d + \eta_\alpha) \cdot (h \cdot \overline{\phi_\alpha}) \rangle_s = -\langle e \cdot \phi_\alpha, \dot{\eta}_\alpha \cdot \overline{\phi_\alpha} \rangle_s
\]

We decompose \( e \) as \( e^{1,0} + e^{0,1} \), where \( e^{1,0} \in \mathcal{L}_J \) and \( e^{0,1} \in \overline{\mathcal{L}_J} \). We also decompose \( \dot{\eta}_\alpha = \dot{\eta}_\alpha^{1,0} + \dot{\eta}_\alpha^{0,1} \), where \( \dot{\eta}_\alpha^{1,0} \in \mathcal{L}_J \) and \( \dot{\eta}_\alpha^{0,1} \in \overline{\mathcal{L}_J} \).

Then we have

\[
-\text{Im}(i^{-n}(e \cdot \phi_\alpha, \dot{\eta}_\alpha \cdot \overline{\phi_\alpha})) = -\text{Im}(i^{-n}(e^{0,1} \cdot \phi_\alpha, \dot{\eta}_\alpha^{0,1} \cdot \overline{\phi_\alpha})) = \text{Im}(i^{-n}(\dot{\eta}_\alpha^{0,1} \cdot \phi_\alpha, \overline{\phi_\alpha})) = \text{Im}(i^{-n}2(e^{0,1} \cdot \dot{\eta}_\alpha^{1,0} \cdot \overline{\phi_\alpha}))
\]

Since \( e \) is real and \( \eta_\alpha \) is pure imaginary, we have

\[
\langle \dot{\eta}_\alpha, e \rangle_{T \oplus T^*} = \langle \eta_\alpha^{1,0}, e^{1,0} \rangle_{T \oplus T^*} + \langle \eta_\alpha^{0,1}, e^{0,1} \rangle_{T \oplus T^*}
\]

\[
= \langle \eta_\alpha^{1,0}, e^{1,0} \rangle_{T \oplus T^*} + \langle -\eta_\alpha^{1,0}, e^{1,0} \rangle_{T \oplus T^*}
\]

\[
= \langle \eta_\alpha^{1,0}, e^{1,0} \rangle_{T \oplus T^*} - \langle \eta_\alpha^{1,0}, e^{1,0} \rangle_{T \oplus T^*}
\]

\[
= 2\sqrt{-1} \text{Im}(\eta_\alpha^{1,0}, e^{1,0})_{T \oplus T^*}
\]

Since \( i^{-n}(\phi_\alpha, \overline{\phi_\alpha}) = i^{-n}(\psi, \overline{\psi}) = \text{vol}_M \), we have

\[
-\text{Im}(i^{-n}(e \cdot \phi_\alpha, \dot{\eta}_\alpha \cdot \overline{\phi_\alpha})) = \text{Im}(\dot{\eta}_\alpha, e)_{T \oplus T^*} \text{vol}_M
\]

Since \( i^{-n}(\psi_\alpha, \overline{\psi_\alpha}) = \text{vol}_M \), we have

\[
\text{Im}(\dot{\eta}_\alpha, e)_{T \oplus T^*} \text{vol}_M = \text{Im}(\dot{\eta}_\alpha, e)_{T \oplus T^*} i^{-n}(\psi_\alpha, \overline{\psi_\alpha})
\]

Then as in before, we obtain

\[
\text{Im}(\dot{\eta}_\alpha, e)_{T \oplus T^*} \text{vol}_M = \text{Im}(\dot{\eta}_\alpha, e)_{T \oplus T^*} i^{-n}(\psi_\alpha, \overline{\psi_\alpha}) = \text{Im}(i^{-n}(e \cdot \psi_\alpha, \dot{\eta}_\alpha \cdot \overline{\psi_\alpha}))
\]
Since $e$ is a generalized hamiltonian element, we have $e \cdot \phi_\alpha = -\sqrt{-1} df \cdot \psi_\alpha$. Applying $d_H \psi_\alpha = \zeta_\alpha \cdot \psi_\alpha$, we have
\[
e \cdot \psi_\alpha = \frac{-\sqrt{-1}}{2} (d_H \psi_\alpha - f \zeta_\alpha \cdot \psi_\alpha) = -\sqrt{-1}(d_H - \zeta_\alpha)(f \psi_\alpha)
\]
Then we obtain
\[
\Omega_{B}(L^H_e \mathcal{J}, \mathcal{J}_h) = - \text{Im} \left( i^{-n} \int_M \langle e \cdot \psi_\alpha, \eta_\alpha \cdot \overline{\psi}_\alpha \rangle_s \right) \tag{5.21}
\]
\[
= \text{Im} \left( \int_M i^{-n+1} (d_H - \zeta_\alpha)(f \psi_\alpha, \eta_\alpha \cdot \overline{\psi}_\alpha)_s \right) \tag{5.22}
\]
Since infinitesimal deformations are given by the action of a global section $h$, it follows that $\eta_\alpha = \eta_\beta$. Thus $\langle \psi_\alpha, \eta_\alpha \cdot \overline{\psi}_\alpha \rangle_{2n-1}$ defines a global $(2n-1)$-form on $M$. We have
\[
d(f \psi_\alpha, \eta_\alpha \cdot \overline{\psi}_\alpha)_{2n-1} = (d_H(f \psi_\alpha), \eta_\alpha \cdot \overline{\psi}_\alpha)_s + (f \psi_\alpha, d_H(\eta_\alpha \cdot \overline{\psi}_\alpha))_s
\]
Applying the Stokes theorem again, The first term $\Omega_{B}(L^H_e \mathcal{J}, \mathcal{J}_h)$ is given by
\[
\Omega_{B}(L^H_e \mathcal{J}, \mathcal{J}_h) = \text{Im} \left( i^{-n+1} \int_M (f \psi_\alpha, (d_H + \zeta_\alpha) \cdot (\eta_\alpha \cdot \overline{\psi}_\alpha)_s) \right) \tag{5.23}
\]
Then $\Omega_{B}(L^H_e \mathcal{J}, \mathcal{J}_h)$ is written as
\[
\Omega_{B}(L^H_e \mathcal{J}, \mathcal{J}_h) = \text{Im} \int_M i^{-n} \langle f \psi_\alpha, (d_H + \zeta_\alpha)(\eta_\alpha \cdot \overline{\psi}_\alpha) \rangle_s
\]
\[
- \text{Im} \int_M i^{-n} \langle 2\sqrt{-1} f L^H_{\zeta_\alpha} \phi_\alpha, \overline{\phi_\alpha} \rangle_s
\]
\[
= \text{Re} \int_M i^{-n} \langle (f \psi_\alpha), (d_H + \zeta_\alpha)(\eta_\alpha \cdot \overline{\psi}_\alpha) \rangle_s
\]
\[
- 2 \text{Re} \int_M i^{-n} \langle f L^H_{\zeta_\alpha} \phi_\alpha, \overline{\phi_\alpha} \rangle_s,
\]
where we are changing from the imaginary part to the real part.

Since one has
\[
2\text{Re} i^{-n} \langle L^H_{\zeta_\alpha} \phi_\alpha, \overline{\phi_\alpha} \rangle_s = i^{-n} \langle L^H_{\zeta_\alpha} \phi_\alpha, \overline{\phi_\alpha} \rangle_s + i^{-n} \langle L^H_{\zeta_\alpha} \phi_\alpha, \overline{\phi_\alpha} \rangle_s
\]
\[
= i^{-n} \langle L^H_{\zeta_\alpha} \phi_\alpha, \overline{\phi_\alpha} \rangle_s - (-i)^{-n} \langle L^H_{\zeta_\alpha} \phi_\alpha, \phi_\alpha \rangle_s
\]
\[
= i^{-n} \langle L^H_{\zeta_\alpha} \phi_\alpha, \overline{\phi_\alpha} \rangle_s - i^{-n} \langle \phi_\alpha, L^H_{\zeta_\alpha} \overline{\phi_\alpha} \rangle_s,
\]
then we have
\[
\left. 2 \frac{d}{dt} \right|_{t=0} \text{Re} i^{-n} \langle L^H_{\zeta_\alpha} \phi_\alpha(t), \overline{\phi_\alpha(t)} \rangle_s = i^{-n} \langle L^H_{\zeta_\alpha} \phi_\alpha, \overline{\phi_\alpha} \rangle_s - i^{-n} \langle \phi_\alpha, L^H_{\zeta_\alpha} \overline{\phi_\alpha} \rangle_s
\]
\[
+ i^{-n} \langle L^H_{\zeta_\alpha} \phi_\alpha, \overline{\phi_\alpha} \rangle_s - i^{-n} \langle \phi_\alpha, L^H_{\zeta_\alpha} \overline{\phi_\alpha} \rangle_s
\]
Since \( \langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s = 0 \), one has
\[
0 = L^H_{\zeta_\alpha} \langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s = \langle L^H_{\zeta_\alpha} \phi_\alpha, \bar{\phi}_\alpha \rangle_s + \langle \phi_\alpha, L^H_{\zeta_\alpha} \bar{\phi}_\alpha \rangle_s
\]
Taking the complex conjugate, we have also
\[
\langle L^H_{\zeta_\alpha} \bar{\phi}_\alpha, \phi_\alpha \rangle_s + \langle \bar{\phi}_\alpha, L^H_{\zeta_\alpha} \phi_\alpha \rangle_s = 0
\]
Thus we have
\[
\frac{d}{dt} \bigg|_{t=0} \text{Re } i^{-n} \langle L^H_{\zeta_\alpha} \phi_\alpha(t), \bar{\phi}_\alpha(t) \rangle_s = -i^{-n} \langle \phi_\alpha, L^H_{\zeta_\alpha} \bar{\phi}_\alpha \rangle_s + i^{-n} \langle L^H_{\zeta_\alpha} \phi_\alpha, \bar{\phi}_\alpha \rangle_s
\]
\[
= 2 \text{Re } i^{-n} \langle L^H_{\zeta_\alpha} \phi_\alpha, \bar{\phi}_\alpha \rangle_s
\]
Since our infinitesimal deformations of \( J_\phi \) are fixing \( J_\psi \), both \( \psi \) and \( \zeta \) do not change. Then we obtain
\[
\frac{d}{dt} \bigg|_{t=0} \int_M \text{Re } i^{-n} \langle f \psi_\alpha, (d_H + \zeta_\alpha)(\eta_\alpha \cdot \bar{\psi}_\alpha) \rangle_s - i^{-n} \langle f L^H_{\zeta_\alpha} \phi_\alpha, \bar{\phi}_\alpha \rangle_s
\]
\[
= \int_M \text{Re } \left( i^{-n} \langle f \psi_\alpha, (d_H + \zeta_\alpha)(\eta_\alpha \cdot \bar{\psi}_\alpha) \rangle_s - 2i^{-n} \langle f L^H_{\zeta_\alpha} \phi_\alpha, \bar{\phi}_\alpha \rangle_s \right)
\]
\[
= \Omega_B(\bar{L}^H_{J_\phi}, \dot{J}_h)
\]
Then we have
\[
\text{Re } i^{-n} \langle f \psi_\alpha, (d_H + \zeta_\alpha)(\eta_\alpha \cdot \bar{\psi}_\alpha) \rangle_s - \text{Re } i^{-n} \langle f L^H_{\zeta_\alpha} \phi_\alpha, \bar{\phi}_\alpha \rangle_s
\]
\[
= \text{Re } i^{-n} f \langle \psi_\alpha, L^H_{\zeta_\alpha} \bar{\psi}_\alpha \rangle_s + \text{Re } i^{-n} f \langle \phi_\alpha, L^H_{\zeta_\alpha} \bar{\phi}_\alpha \rangle_s + 2 \langle \zeta_\alpha, \eta_\alpha \rangle_{\mathfrak{t} \oplus \mathfrak{t}} f \text{vol}_M
\]
\[
= f S(J_\phi, J_\psi) \text{vol}_M
\]
Thus we obtain
\[
\Omega_B(\bar{L}^H_{J_\phi}, J_\psi) = \frac{d}{dt} \bigg|_{t=0} \int_M f S(J_\phi(t), J_\psi) \text{vol}_M
\]
(5.24)
Therefore, there exists a moment map \( \mu : \mathcal{B}_{J_\psi}(M) \to C^\infty_v(M)^* \) which is given by the scalar curvature
\[
\mu(J) = S(J, J_\psi),
\]
where \( C^\infty_v(M)^* \) is identified with \( C^\infty_v(M) \) by using the \( L^2 \)-metric.
Finally we show that \( \mu : \mathcal{B}_{J_\psi}(M) \to C^\infty_v(M)^* \) satisfies the co-equivalent condition of moment map under the modified action of \( \text{Ham}(M, J_\psi) \). Recall that the bracket of the Lie algebra \( \mathfrak{ham}(M, J_\psi) \) of \( \text{Ham}(M, J_\psi) \) is given by \{ \, , \}_\psi and the infinitesimal action of \( \mathfrak{ham}(M, J_\psi) \) is given by \( \bar{L}^H_{J_\phi}df \) on \( \mathcal{B}_{J_\psi}(M) \). Thus it suffices to check the following infinitesimal condition at every \( J_\phi \in \mathcal{B}_{J_\psi}(M) \),
\[
d\langle \mu, \{ f, f_1 \}_\psi \rangle = \langle \bar{L}^H_{J_\phi}df, J_\phi \rangle
\]
(5.25)
for all $f, f_1 \in C^\infty(M)$. As shown in \((5.24)\)

$$\Omega_B(\tilde{L}_J^H \psi \eta, \tilde{J}) = d(\mu, f)(\tilde{J})$$

Thus it follows

$$d(\mu, f)(\tilde{L}_J^H \psi \eta, J) = \Omega_B(\tilde{L}_J^H \psi \eta, \tilde{J}) = -\Omega_B(\tilde{L}_J^H \psi \eta, \tilde{J})$$

Since $\tilde{L}_J^H \psi \eta f_1 = L_J^H \psi \eta f - 2\sqrt{-1} f_1 L_J^H \psi \eta$, the left hand side of \((5.25)\) is given by

$$d(\mu, f)(L_J^H \psi \eta f_1 J) - 2\sqrt{-1} f_1 d(\mu, f)(L_J^H \psi \eta) \quad (5.26)$$

From Proposition 3.5, the map $\mu$ is co-equivalent under the action of $\tilde{\text{Diff}}_0(M, H)$. Thus the first term $d(\mu, f)(L_J^H \psi \eta f_1 J)$ of \((5.26)\) is given by

$$d(\mu, f)(L_J^H \psi \eta f_1 J) = -\langle \mu, L_J^H \psi \eta f_1 \rangle = \langle \mu, \{f, f_1\}_\psi \rangle \quad (5.27)$$

Applying the co-equivalency of the map $\mu$ under the action of $\tilde{\text{Diff}}_0(M, H)$ again, it follows from $\pi_T(\zeta_\alpha) f = 0$ that one has

$$d(\mu, f)(L_J^H \psi \eta J) = -\langle \mu, L_J^H \psi \eta f \rangle = 0. \quad (5.28)$$

Thus the second term of \((5.26)\) vanishes. Hence we have \((5.25)\)

$$d(\mu, f)(\tilde{L}_J^H \psi \eta J) = \langle \mu, \{f, f_1\}_\psi \rangle$$

\[\square\]

### 6 Scalar curvature of generalized Kähler structures on the standard Hopf surfaces

Two kinds of generalized Kähler structures are known on the standard Hopf surfaces $\mathbb{H}$. The one is of odd type, which is a generalized Kähler structure consisting of a pair of generalized complex structures of odd type. The other is of even type. Our explicit descriptions of nondegenerate, pure spinors of generalized Kähler structures enable us to calculate their scalar curvature. Further an action of the Pin group converts the odd one to the even one. First we start to discuss the generalized Kähler structure of odd type.
6.1 Generalized Kähler structures of odd type on the standard Hopf surfaces

Let \((z_1, z_2)\) be complex coordinates of \(\mathbb{C}^2\). We define two differential forms on \(\mathbb{C}^2 \setminus \{0\}\) by

\[
\phi = dz_1 \wedge e^{-\sqrt{-1} \frac{r^2}{r}}, \quad \psi = dz_2 \wedge e^{-\sqrt{-1} \frac{r^2}{r}},
\]

where

\[
\omega = \sqrt{-1} (dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2), \quad r^2 = |z_1|^2 + |z_2|^2,
\]

Then \(\phi\) and \(\psi\) are nondegenerate, pure spinors which give rise to almost generalized complex structures \(J_\phi, J_\psi\), respectively. In fact, \(J_\phi\) comes from the ordinary complex structure along \(z_1\)-direction and \(J_\phi\) is induced from the real symplectic structure \(\exp(\frac{dz_2}{r} \wedge d\overline{z}_2)\) along \(z_2\)-direction. Thus one has

\[
J_\phi(r \frac{\partial}{\partial z_2}) = \sqrt{-1} \frac{dz_2}{r}, \quad J_\phi(\frac{dz_2}{r}) = \sqrt{-1} r \frac{\partial}{\partial z_2}
\]

On the other hand, \(J_\psi\) comes from the ordinary complex structure along \(z_2\)-direction and \(J_\psi\) is induced from the real symplectic structure \(\exp(\frac{dz_1}{r} \wedge d\overline{z}_1)\) along \(z_1\)-direction. Thus one also has

\[
J_\psi(r \frac{\partial}{\partial z_1}) = \sqrt{-1} \frac{dz_1}{r}, \quad J_\psi(\frac{dz_1}{r}) = \sqrt{-1} r \frac{\partial}{\partial z_1}
\]

We denote by \(\text{vol}\) the volume form \(-\frac{1}{2!} \omega \wedge \omega\). The radiation vector field \(r \frac{\partial}{\partial r}\) and the logarithmic 1-form \(\frac{dr}{r}\) are given by

\[
\frac{dr}{r} = \frac{dr^2}{2r^2} = \frac{1}{2r^2} (z_1 d\overline{z}_1 + \overline{z}_1 dz_1 + z_2 d\overline{z}_2 + \overline{z}_2 dz_2)
\]

\[
r \frac{\partial}{\partial r} = \frac{z_1}{r} \frac{\partial}{\partial z_1} + \frac{z_2}{r} \frac{\partial}{\partial z_2} + \frac{\overline{z}_1}{r} \frac{\partial}{\partial \overline{z}_1} + \frac{\overline{z}_2}{r} \frac{\partial}{\partial \overline{z}_2}
\]

We define a real 3-form \(H\) by

\[
H := -i \frac{\partial}{\partial r} (\text{vol}) = -\frac{1}{r^4} (z_1 d\overline{z}_1 \wedge dz_2 + d\overline{z}_2 - \overline{z}_1 dz_1 \wedge dz_2 + d\overline{z}_2 - z_2 d\overline{z}_1 \wedge dz_2 + d\overline{z}_2 - \overline{z}_1 dz_1 \wedge dz_2 + d\overline{z}_2)
\]

Then one has

\[
d\phi + H \wedge \phi = 0, \quad d\psi + H \wedge \psi = 0.
\]

Then it follows that the pair \((J_\phi, J_\psi)\) gives a \(H\)-twisted generalized Kähler structure on \(\mathbb{C}^2 \setminus \{0\}\). We see that \((J_\phi, J_\psi)\) gives the bihermitian structure \((g, I^+, I^-)\), where \(g\) is \(\frac{1}{r^4} g_{eu}\) and \(I^+\) is the standard complex structure on \(\mathbb{C}^2\) and \(I^-\) has holomorphic coordinates \((z_1, \overline{z}_2)\), where \(g_{eu}\) denotes the Euclid metric.
on \( \mathbb{C}^2 \cong \mathbb{R}^4 \). We consider an action of \( \mathbb{Z} \) on \( \mathbb{C}^2 \) which is given by the diagonal multiplication of \( \alpha \), \(|\alpha| \neq 1\), that is, \((z_1, z_2) \mapsto (\alpha z_1, \alpha z_2)\). Then the quotient \( X := \mathbb{C}^2 \setminus \{0\}/\mathbb{Z} \) is called the standard Hopf surface. Since \( \phi, \psi \) are equivalent under the action of \( \alpha \), it follows that the generalized Kähler structure descends to the underlying differential manifold of the standard Hopf surface \( X \).

In order to normalize \( \phi \) and \( \psi \), we replace \((\phi, \psi)\) with 
\[
(\tilde{\phi}, \tilde{\psi}) := \left( \frac{1}{\sqrt{2}} \frac{\phi}{r}, \frac{1}{\sqrt{2}} \frac{\psi}{r} \right).
\]

Then we see
\[
i^{-2} \langle \frac{1}{\sqrt{2}} \frac{\phi}{r}, \frac{1}{\sqrt{2}} \frac{\phi}{r} \rangle_s = \frac{1}{r^4} (dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2) = \text{vol}
\]
Since \( \sqrt{-1} \mathcal{J}_\phi \frac{dr}{r} + \frac{dr}{r} \in \mathcal{L}_{\mathcal{J}_\phi} = \text{ker} \phi \), one has
\[
d_H(\frac{\phi}{r}) = -\frac{dr}{r} \frac{\phi}{r} + (d_H \phi) \frac{1}{r} = (\sqrt{-1} \mathcal{J}_\phi \frac{dr}{r}) \frac{\phi}{r}.
\]
We also have
\[
d_H(\frac{\psi}{r}) = (\sqrt{-1} \mathcal{J}_\psi \frac{dr}{r}) \frac{\psi}{r}.
\]
Since \( \eta \) and \( \zeta \) should be pure imaginary as in Definition of the scalar curvature, we obtain
\[
\eta = (\sqrt{-1} \mathcal{J}_\phi \frac{dr}{r}), \quad \zeta = (\sqrt{-1} \mathcal{J}_\psi \frac{dr}{r}). \quad (6.1)
\]
The following lemma is a key to calculate the scalar curvature of \((\mathcal{J}_\phi, \mathcal{J}_\psi)\),

**Lemma 6.1.**
\[-\mathcal{J}_\psi \mathcal{J}_\phi \frac{dr}{r} = r \frac{\partial}{\partial r} \]

**Proof.** In fact, we have
\[
-\mathcal{J}_\psi \mathcal{J}_\phi \frac{dr}{r} = -\mathcal{J}_\psi \left( \sqrt{-1} \left( \frac{1}{r^2} (z_1 d\bar{z}_1 - \bar{z}_1 dz_1) + \sqrt{-1} \frac{1}{r} (z_2 r \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2}) \right) \right)
= z_1 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + z_2 \frac{\partial}{\partial z_2} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2}
= r \frac{\partial}{\partial r}
\]
Then the result follows. \( \square \)

**Lemma 6.2.** One has
\[
\eta \cdot \overline{\psi} = - (\mathcal{J}_\phi \mathcal{J}_\psi \frac{dr}{r}) \psi = r \frac{\partial}{\partial r} \psi \quad (6.2)
\]
Proof. Since \((\eta - \sqrt{-1} J_\phi \eta) \in \mathcal{L}_\psi\), it follows \((\eta - \sqrt{-1} J_\psi \eta) \cdot \bar{\psi} = 0\). From (6.1) and Lemma 6.1, we have \(\eta \cdot \bar{\psi} = \sqrt{-1} J_\psi \eta \cdot \bar{\psi} = - J_\psi J_\phi \frac{dr}{r}\). Then the result follows. 

\[ \square \]

Proposition 6.3. The scalar curvature of \((J_\phi, J_\psi)\) is a constant, that is, 

\[ S(J_\phi, J_\psi) = 1 \]

Proof. The scalar curvature \(S(J_\phi, J_\psi)\) is given by the real part of the following:

\[ S(J_\phi, J_\psi) \text{vol}_M = \text{Re} \left( i^{-n} \langle \tilde{\phi}, H(\zeta \cdot \tilde{\phi}) \rangle_s + i^{-n} \langle \tilde{\psi}, H(\eta \cdot \tilde{\psi}) \rangle_s \right) \tag{6.3} \]

Since \(r \frac{\partial}{\partial r}\) is the vector field defined from the dilation action, one has

\[ L_r \frac{\partial}{\partial r} \psi = 0. \]

One also has \(i_r \frac{\partial}{\partial r} H = -i_r \frac{\partial}{\partial r} i_r \frac{\partial}{\partial r} \text{vol} = 0\). Then it follows

\[ d_H(r \frac{\partial}{\partial r}) = L_r H \frac{\partial}{\partial r} + r \frac{\partial}{\partial r} \cdot d_H = L_r \frac{\partial}{\partial r} - r \frac{\partial}{\partial r} \cdot d_H. \]

From Lemma 6.2, one has

\[ d_H(\eta \cdot \bar{\psi}) = d_H(r \frac{\partial}{\partial r} \bar{\psi}) = -(r \frac{\partial}{\partial r}) \cdot d_H \bar{\psi} = (r \frac{\partial}{\partial r}) \cdot (\frac{dr}{r} \bar{\psi}) \]

Thus one has

\[ i^{-2} \langle \frac{\psi}{r}, d_H(\eta \cdot \bar{\psi}) \rangle_s = i^{-2} \langle \frac{\psi}{r}, (r \frac{\partial}{\partial r}) \cdot (\frac{dr}{r} \bar{\psi}) \rangle_s \]

Taking the real part, we have the coupling \((r \frac{\partial}{\partial r})\) and \((\frac{dr}{r})\),

\[ \text{Re} \left( i^{-2} \langle \frac{\psi}{r}, d_H(\eta \cdot \bar{\psi}) \rangle_s \right) = \frac{1}{2} \text{Re} \left( i^{-2} \langle \frac{\psi}{r}, \bar{\psi} \rangle \right)_s \]

Thus we have

\[ \text{Re} \left( i^n \langle \tilde{\psi}, H(\eta \cdot \bar{\psi}) \rangle_s \right) = \frac{1}{2} \text{vol}_M \]

We also have

\[ \text{Re} \left( i^n \langle \tilde{\phi}, H(\zeta \cdot \tilde{\phi}) \rangle_s \right) = \frac{1}{2} \text{vol}_M \]

Thus we obtain \(S(J_\phi, J_\psi) = 1\). 

\[ \square \]
6.2 Generalized Kähler structures of even type on the standard Hopf surfaces

We use the same notation as in Subsection 6.1. We define
\[ E_\pm \] by
\[ E_\pm := r \frac{\partial}{\partial r} \pm \frac{dr}{r}. \]
Since \( E_\pm \cdot E_\pm = \pm 1 \), real \( E_\pm \) are elements of the real Pin group. Since the real Pin group acts on the set of almost generalized Kähler structures, the pair \((E_\pm \cdot \phi, E_\pm \cdot \psi)\) gives an almost generalized Kähler structure. We shall show that \((E_\pm \cdot \phi, E_\pm \cdot \psi)\) is a generalized Kähler structure.

**Lemma 6.4.** \( E_\pm \cdot (J_\phi \frac{dr}{r}) + (J_\phi \frac{dr}{r}) \cdot E_\pm = 0 \)

**Proof.** It suffices to show that \( \langle E_\pm, J_\phi \frac{dr}{r} \rangle_{T \oplus T^*} = 0 \). Since one has \( \langle \frac{dr}{r}, J_\phi \frac{dr}{r} \rangle_{T \oplus T^*} = 0 \), it follows \( \langle E_\pm, J_\phi \frac{dr}{r} \rangle_{T \oplus T^*} = \langle r \frac{\partial}{\partial r}, J_\phi \frac{dr}{r} \rangle_{T \oplus T^*} \). From Lemma 6.1, one has \( -J_\psi J_\phi \frac{dr}{r} = r \frac{\partial}{\partial r} \).

Thus one has
\[
\langle r \frac{\partial}{\partial r}, J_\phi \frac{dr}{r} \rangle_{T \oplus T^*} = \langle -J_\psi J_\phi \frac{dr}{r}, \frac{dr}{r} \rangle_{T \oplus T^*} = 0
\]
Hence one has \( \langle E_\pm, J_\phi \frac{dr}{r} \rangle_{T \oplus T^*} = 0 \).

**Proposition 6.5.** \((E_\pm \cdot \phi, E_\pm \cdot \psi)\) are \( H \)-twisted generalized Kähler structures.

**Proof.** We need to calculate \( d_H(E_\pm \cdot \phi) \). Since \( d_H \circ E_\pm = L^H_{E_\pm} - E_\pm \circ d_H \) and \( L^H_{E_\pm} (\frac{\phi}{r}) = L_{E_\pm} (\frac{\phi}{r}) + (i(r \frac{\partial}{\partial r})H) \wedge (\frac{\phi}{r}) = 0 \), we have
\[
d_H(E_\pm \cdot \phi) = -E_\pm d_H(\frac{\phi}{r}) = E_\pm \cdot \frac{dr}{r} \cdot (\frac{\phi}{r})
\]
Applying Lemma 6.4 we have
\[
d_H(E_\pm \cdot \phi) = \sqrt{-1}(J_\phi \frac{dr}{r}) \cdot E_\pm \cdot (\frac{\phi}{r}) \tag{6.4}
\]
Hence both \((E_\pm \cdot \phi)\) give integrable generalized complex structures. It also follows that \((E_\pm \cdot \psi)\) give integrable generalized complex structures. Hence \((E_\pm \cdot \phi, E_\pm \cdot \psi)\) are generalized Kähler structures.
Next we shall calculate the scalar curvature of the generalized Kähler structures \((\mathcal{J}_{E^\pm}, \mathcal{J}_{E^\pm})\) which are given by \((E^\pm \cdot \phi, E^\pm \cdot \psi)\). We define a volume form \(\text{vol}_{E^\pm}\) by \(\mp \text{vol}\). Then we have the normalization condition
\[
i^{-2} \langle E^\pm \phi, E^\pm \phi \rangle_s = i^{-2} \langle E^\pm \cdot \psi, E^\pm \cdot \psi \rangle_s = \text{vol}_{E^\pm}
\]
We already have
\[
d_H(E^\pm \cdot \phi) = \sqrt{-1}(\mathcal{J}_{\phi} \frac{dr}{r}) \cdot E^\pm \cdot (\phi_r)
\]
\[
d_H(E^\pm \cdot \psi) = \sqrt{-1}(\mathcal{J}_{\psi} \frac{dr}{r}) \cdot E^\pm \cdot (\psi_r)
\]
Thus we have \(\eta = \sqrt{-1}(\mathcal{J}_{\phi} \frac{dr}{r})\), \(\zeta = \sqrt{-1}(\mathcal{J}_{\psi} \frac{dr}{r})\). Applying Lemma 6.1 and Lemma 6.4, again, we have
\[
d_H(\eta \cdot E^\pm \cdot \psi_r) = \pm \sqrt{-1} d_H(\mathcal{J}_{\phi} \frac{dr}{r}) \cdot E^\pm \cdot \psi_r = - \sqrt{-1} d_H E^\pm \cdot (\mathcal{J}_{\phi} \frac{dr}{r}) \cdot \psi_r
\]
\[
= - d_H E^\pm \cdot (\mathcal{J}_{\phi} \frac{dr}{r}) \cdot \psi_r = d_H E^\pm \cdot (r \frac{\partial}{\partial r}) \cdot \psi_r
\]
Since \(L^H_{E^\pm} ((r \frac{\partial}{\partial r}) \cdot \psi) = L_{r \frac{\partial}{\partial r}} ((r \frac{\partial}{\partial r}) \cdot \psi) = ((r \frac{\partial}{\partial r}) \cdot L_{r \frac{\partial}{\partial r}} \psi) = 0\), one has
\[
d_H(\eta \cdot E^\pm \cdot \psi_r) = - E^\pm \cdot d_H (r \frac{\partial}{\partial r}) \cdot \psi_r = E^\pm \cdot (r \frac{\partial}{\partial r}) \cdot d_H \psi_r
\]
\[
= - E^\pm \cdot (r \frac{\partial}{\partial r}) \cdot \frac{dr \psi_r}{r}
\]
Taking the complex conjugate \(\overline{\psi}\), we have
\[
i^{-2} \langle E^\pm \cdot \psi_r, \overline{d_H(\eta \cdot E^\pm \cdot \overline{\psi})}_s \rangle = i^{-2} \langle E^\pm \cdot \psi_r, E^\pm \cdot (r \frac{\partial}{\partial r}) \cdot \frac{dr \overline{\psi}_r}{r} \rangle_s
\]
\[
= \mp \frac{1}{2} i^{-2} \langle \frac{\psi_r}{r}, \frac{\overline{\psi}_r}{r} \rangle_s
\]
Hence we obtain
\[
\text{Re} \ i^{-n} \left( \langle E^\pm \cdot \psi_r, \overline{d_H(\eta \cdot E^\pm \cdot \overline{\psi})}_s \rangle \right) = \mp \frac{1}{2} i^{-2} \langle \frac{\psi_r}{r}, \frac{\overline{\psi}_r}{r} \rangle_s
\]
We also have
\[
\text{Re} \ i^{-n} \left( \langle E^\pm \cdot \phi_r, \overline{d_H(\zeta \cdot E^\pm \cdot \overline{\phi})}_s \rangle \right) = \mp \frac{1}{2} i^{-2} \langle \frac{\phi_r}{r}, \frac{\overline{\phi}_r}{r} \rangle_s
\]
Hence we have
\[
\text{Re} \left( i^{-n} \langle E^\pm \cdot \psi_r, \overline{d_H(\eta \cdot E^\pm \cdot \overline{\psi})}_s \rangle + i^{-n} \langle E^\pm \cdot \phi_r, \overline{d_H(\zeta \cdot E^\pm \cdot \overline{\phi})}_s \rangle \right)
\]
\[
= \text{vol}_{E^\pm}
\]
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Proposition 6.6. The scalar curvature of \((J_{E^+}, J_{E^-})\) is 1, where the volume form is given by \(\text{vol}_{E^\pm}\).

Proof. The result follows from (6.5). \(\square\)

7 Review of results of Lie groups, due to Alekseev, Bursztyn and Meinrenken

We give a review of results of the paper [1] on pure spinors and Dirac structures of Lie groups, which we will need to obtain generalized Kähler structures on compact Lie groups. The results in [1] are also effective to calculate the scalar curvature of the generalized Kähler structures on compact Lie groups.

7.1 The isomorphism \(T_M \oplus T_M^* \cong G \times (\mathfrak{g} \oplus \overline{\mathfrak{g}})\)

Let \(G\) be a Lie group, and let \(\mathfrak{g}\) be its Lie algebra. We denote by \(\xi^R, \xi^L \in \mathfrak{X}(G)\) the left-, right-invariant vector field on \(G\) which are equal to \(\xi \in \mathfrak{g} \cong T_e G\) at the group unit. Let \(\theta^L, \theta^R \in \Omega(G) \otimes \mathfrak{g}\) be the left-, right-Maurer-Cartan forms, i.e., \(\iota(\xi^L) \theta^L = \iota(\xi^R) \theta^R = \xi\). They are related by \(\theta^R_g = \text{Ad}_g(\theta^L_g)\), for all \(g \in G\).

The corresponding infinitesimal action is given by the vector fields

\[ A_{\text{ad}}(\xi) = \xi^L - \xi^R \]

Suppose that the Lie algebra \(\mathfrak{g}\) of \(G\) carries an invariant inner product. By this we mean an Ad-invariant, non-degenerate symmetric bilinear form \(B\). Equivalently, \(B\) defines a bi-invariant pseudo Riemannian metric on \(G\). Given \(B\), we can define the Ad-invariant 3-form \(H \in \Omega^3(G)\),

\[ H := \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) \]

Since \(H\) is bi-invariant, it is closed. and it defines an \(H\)-twisted Courant bracket \([,]_H\) on \(G\). The conjugation action \(A_{\text{ad}}\) extends to an action of "the double group" \(D = G \times G\) on \(G\), by

\[ A : D \to \text{Diff}(G), \quad A(a, a') = l_a \circ r_{(a')^{-1}}, \]

where \(l_a(g) = ag\) and \(r_a(g) = ga\). The corresponding infinitesimal action

\[ A : \mathfrak{d} \to \mathfrak{X}(G), \quad A(\xi, \xi') = \xi^L - (\xi')^R \]

lifts to a map

\[ s : \mathfrak{d} \to \Gamma(T_M \oplus T_M^*), \quad s(\xi, \xi') = s^L(\xi) - s^R(\xi'), \]

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where
\[ s^L(\xi) = \xi^L + B(\theta^L, \xi), \quad s^R(\xi') = -(\xi')^R + B(\theta^R, \xi) \]

Let us equip \( \mathfrak{d} \) with the bilinear form \( B_0 \) given by \( +B \) on the first \( g \)-summand and \( -B \) on the second summand. Thus \( \mathfrak{d} = g \oplus \mathfrak{g} \) is an example of a Lie algebra with invariant split bilinear form.

**Proposition 7.1.** \([1]\) The map \( s : \mathfrak{d} \to \Gamma(TG \oplus T^*G) \) is \( D \)-equivalent, and satisfies

1. \( (s(\zeta_1), s(\zeta_2))_{\mathfrak{T} \oplus \mathfrak{T}^*} = B_0(\zeta_1, \zeta_2) \)
2. \( [s(\zeta_1), s(\zeta_2)]_H = s([\zeta_1, \zeta_2])_0 \)

for all \( \zeta_1, \zeta_2 \in \mathfrak{d} \).

**Proof.** For completeness of our paper, we will give a proof.

(1) It suffices to show that
\[ (s(\zeta), s(\zeta))_{\mathfrak{T} \oplus \mathfrak{T}^*} = B_0(\zeta, \zeta) \]
for all \( \zeta = (\xi, \xi') \). Since \( s(\zeta) \) is given by
\[ s(\zeta) = \xi^L - \xi^R + B(\theta^L, \xi) + B(\theta^R, \xi') \]
one has
\[ (s(\zeta), s(\zeta))_{\mathfrak{T} \oplus \mathfrak{T}^*} = B(\theta^L, \xi)(\xi^L) + B(\theta^R, \xi')(\xi^L) \]
\[ = B(\theta^L, \xi)(\xi^L) - B(\theta^R, \xi)(\xi'^R) \]

From the definition of the Maurer-Cartan form, one has \( B(\theta^L, \xi)(\xi^L) = B(\xi, \xi) \) and \( B(\theta^R, \xi')(\xi'^R) = B(\xi', \xi') \). Since \( \xi^L = (\text{Ad}_g \xi)^R \) and \( \xi^R = (\text{Ad}_{g^{-1}} \xi)^L \) at \( g \in G \), we have
\[ B(\theta^L, \xi)(\xi^L) = B(\theta^R, \xi')(\text{Ad}_g \xi)^R = B(\text{Ad}_g \xi, \xi') \]
\[ B(\theta^L, \xi)(\xi^L) = B(\theta^R, \xi)(\text{Ad}_{g^{-1}} \xi')^L = B(\text{Ad}_{g^{-1}} \xi', \xi) \]

Since \( B \) is \( \text{Ad} \)-invariant, it follows \( B(\text{Ad}_g \xi, \xi') = B(\text{Ad}_{g^{-1}} \xi', \xi) \). Thus we have
\[ (s(\zeta), s(\zeta))_{\mathfrak{T} \oplus \mathfrak{T}^*} = B(\xi, \xi) - B(\xi', \xi') \]

(2) We shall show that \( [s(\zeta_1), s(\zeta_2)]_H = s([\zeta_1, \zeta_2]) \) for all \( \zeta_1 = (\xi_1, \xi'_1), \zeta_2 = (\xi_2, \xi'_2) \). Let \( s^L \) be the following
\[ s^L(\xi_1) = \xi^L_1 + \theta^L_1, \quad s^L(\xi_2) = \xi^L_2 + \theta^L_2, \]

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where \( \theta_1^l = B(\theta^L, \xi_1) \) and \( \theta_2^L = B(\theta^L, \xi_2) \). We also denote by \( s^R \) the following
\[
s^R(\xi'_1) = -\xi'^R_1 + \theta^R_1, \quad s^R(\xi'_2) = -\xi'^R_2 + \theta^R_2,
\]
where \( \theta^R_1 = B(\theta^R, \xi'_1) \) and \( \theta^R_2 = B(\theta^R, \xi'_2) \). Then one see
\[
s(\xi_1) = s^L(\xi_1) + s^R(\xi'_1), \quad s(\xi_2) = s^L(\xi_2) + s^R(\xi'_2)
\]
From the definition of \( H \)-twisted bracket, one has
\[
[s^L(\xi_1), s^L(\xi_2)]_H = [\xi_1^L, \xi_2^L] + \frac{1}{2} \left( L_{\xi_1^l} \theta_1^l - L_{\xi_2^l} \theta_1^l \right)
- \frac{1}{2} \left( i_{\xi_1^l} d\theta_1^l - i_{\xi_2^l} d\theta_2^l \right) - i_{\xi_1^l} i_{\xi_2^l} H
\]
Since \( i_{\xi_1^l} \theta_2^l = B(\xi_1, \xi_2) \) and \( di_{\xi_1^l} \theta_2^l = 0 \), then we have
\[
[s^L(\xi_1), s^L(\xi_2)]_H = [\xi_1^L, \xi_2^L] + 2B(\theta^L, [\xi_1, \xi_2]_g) - i_{\xi_1^l} i_{\xi_2^l} H
\]
Since \( i_{\xi_1^l} \theta_1^l = -B(\theta^L, [\xi_1, \xi_2]_g) \), we have
\[
[s^L(\xi_1), s^L(\xi_2)]_H = [\xi_1^L, \xi_2^L] + 2B(\theta^L, [\xi_1, \xi_2]_g) - i_{\xi_1^l} i_{\xi_2^l} H
\]
Since \( i_{\xi_1^l} i_{\xi_2^l} i_{\xi_1^l} H = -B([\xi_1, \xi_2], \xi_3) \), we have
\[
[s^L(\xi_1), s^L(\xi_2)]_H = [\xi_1^L, \xi_2^L] + B(\theta^L, [\xi_1, \xi_2]_g)
\]
\[
[s^R(\xi'_1), s^R(\xi'_2)]_H \text{ is calculated by using } [\xi'^R_1, \xi'^R_2] = [-\xi'^l_1, \xi'^l_2] \text{ and } i_{\xi'^l_1} d\theta_1^l = B(\theta^R, [\xi_1, \xi_2]_g)_g,
\]
\[
[s^R(\xi_1), s^R(\xi_2)]_H = [-\xi'^l_1, \xi'^l_2] + 2B(\theta^R, [\xi'^l_1, \xi'^l_2]_g) + i_{\xi'^l_1} i_{\xi'^l_2} H
- [-\xi'^l_1, \xi'^l_2]_g + B(\theta^R, [\xi'^l_1, \xi'^l_2]_g)
\]
Then we have
\[
[s^L(\xi_1), s^L(\xi_2)]_H = s^L([\xi_1, \xi_2]_g), \quad [s^R(\xi'_1), s^R(\xi'_2)]_H = s^R([\xi'^l_1, \xi'^l_2]_g) \quad (7.1)
\]
We also have
\[
[s^L(\xi_1), s^R(\xi'_2)]_H = 0, \quad [s^R(\xi'_1), s^R(\xi_2)]_H = 0
\]
Thus we obtain the result. \( \square \)
7.2 The isomorphism $\wedge T^* G \cong G \times Cl(\mathfrak{g})$

Let us now assume that the adjoint action $\text{Ad} : G \to O(\mathfrak{g})$ lifts to a group homomorphism

$$\tau : G \to \text{Pin}(\mathfrak{g}) \subset Cl(\mathfrak{g}) \quad (7.2)$$

If $G$ is connected and $\pi_1(G)$ is torsion free, then this is automatic. Note that $(7.2)$ is consistent with our previous notation $\tau(\xi) = q(\lambda(\xi))$, since

$$\tau(\xi) = \left. \frac{d}{dt} \right|_{t=0} \tau(\exp(t\xi))$$

We will write $N(g) = N(\tau(g)) = \pm 1$ for the image under the norm homomorphism, and $|g| = |\tau(g)|$ for the parity of $\tau(g)$. Since $\tau(g)$ lifts $\text{Ad}_g$, one has $(-1)^{|g|} = \text{det} \text{Ad}_g$. The definition of the Pin group implies that conjugation by $\tau(g)$ is the twisted adjoint action,

$$\tau(g)x\tau(g^{-1}) = \tilde{\text{Ad}}_g(x) := (-1)^{|g|} \text{Ad}_g(x)$$

This twisted adjoint action extends to an action of the group $D$ on $Cl(\mathfrak{g})$,

$$A^{cl}(a, a')(x) = \tau(a)x\tau((a')^{-1})$$

Let us now fix a generator $\mu \in \det(\mathfrak{g})$, and consider the corresponding star operator $\star : \mathfrak{g}^* \to \wedge \mathfrak{g}$. The star operator satisfies

$$\text{Ad}_g \circ \star = (-1)^{|g|} \star \circ \text{Ad}_{g^{-1}}$$

We use the left-invariant forms to identify $\wedge T^* G \cong G \times \wedge \mathfrak{g}^*$. Applying $\star$ point-wise, we obtain an isomorphism $q \circ \star : \wedge T^*_g G \to Cl(\mathfrak{g})$ for each $g \in G$. Let us define the linear map

$$R : Cl(\mathfrak{g}) \to \Omega(G), \quad R(x) = (q \circ \star)^{-1}(x\tau(g)) \quad (7.3)$$

We denote by $\mu^*$ the dual generator defined by $\iota((\mu^*)^T)\mu = 1$, and let $\mu_G$ be the left-invariant volume form on $G$ defined by $\mu^*$. Let $q : \wedge \mathfrak{g} \to Cl(\mathfrak{g})$ be the natural inclusion from the skewsymmetric algebra to the Clifford algebra. We define $\Xi \in \wedge^3 \mathfrak{g}$ by

$$i_{\xi_3}i_{\xi_2}i_{\xi_1}\Xi = B(\xi_1, [\xi_2, \xi_3])$$

We denote by $q(\Xi) \in Cl(\mathfrak{g})$ the image of the 3-form $\Xi$ by the map $q$. We define the Clifford differential $d^{cl} : Cl(\mathfrak{g}) \to Cl(\mathfrak{g})$ by the action of $\Xi$

$$d^{cl}(x) = -[q(\Xi), x]_{Cl}.$$

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Proposition 7.2. The map \( \mathcal{R} \) has the following properties:

(a) \( \mathcal{R} \) intertwines the Clifford actions, in the sense that
\[
\mathcal{R}(\rho^{cl}(\zeta)x) = \rho(s(\zeta))\mathcal{R}(x), \quad \forall x \in Cl(\mathfrak{g}), \ \zeta \in \mathfrak{d}
\]
Up to a scalar function, \( \mathcal{R} \) is uniquely characterized by this property.

(b) \( \mathcal{R} \) intertwines differentials:
\[
\mathcal{R}(d^{cl}(x)) = (d + H)\mathcal{R}(x)
\]

(c) \( \mathcal{R} \) satisfies the following \( D \)-equivalent property: For any \( h = (a, a') \in D \), and at any given point \( g \in G \), and for all \( x, x' \in Cl(\mathfrak{g}) \),
\[
\mathcal{A}(h^{-1})^*\mathcal{R}(x) = (-1)^{|a||g|+|x|}\mathcal{R}(\mathcal{A}^{cl}(h)x)
\]

(d) \( \mathcal{R} \) relates the bilinear pairings on the Clifford modules \( Cl(\mathfrak{g}) \) and \( \Omega(G) \) as follows: At any given point \( g \in G \), and for all \( x, x' \in Cl(\mathfrak{g}) \),
\[
\langle \mathcal{R}(x), \mathcal{R}(x') \rangle_s = (-1)^{|g|(\dim G + 1)}N(g)(x, x')_{Cl(\mathfrak{g})}\mu_G
\]
Here the pairing \( \langle ., . \rangle_{Cl(\mathfrak{g})} \) is viewed as scalar-valued, using the trivialization of \( \det(\mathfrak{g}) \) defined by \( \mu_G \).

Notice that the signs in part (c), (d) disappear if \( G \) is connected.

8 Scalar curvature of generalized Kähler structures on compact Lie groups

Typical generalized Kähler structures on compact Lie groups are known [18]. We shall give a different construction of generalized complex structures on compact Lie groups by using Proposition 7.1.

8.1 Construction of generalized Kähler structures on compact Lie groups

Let \( G \) be a compact Lie group of even dimension, and let \( \mathfrak{g} \) be its Lie algebra. Suppose that \( \mathfrak{g} \) carries a bi-invariant metric \( B \) which is positive definite in this section. Let \( \mathfrak{g}^\mathbb{C} \) be the complexification \( \mathfrak{g} \otimes \mathbb{C} \) of \( \mathfrak{g} \). The metric \( B \) is linearly
extended to the bilinear form on $\mathfrak{g}^\mathbb{C}$ over $\mathbb{C}$. We denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}^\mathbb{C}$ with $\dim \mathfrak{h} = 2l$. Then we have the root decomposition:

$$\mathfrak{g}^\mathbb{C} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$$

Choosing a positive root system $\{e_\alpha \in \mathfrak{g}_\alpha | \alpha > 0\}$, we have the triangular decomposition

$$\mathfrak{g}^\mathbb{C} = \mathfrak{h} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha$$

Let $I_\mathfrak{h}$ be an almost complex structure on the real part $\mathfrak{h}^\mathbb{R} = \mathfrak{g} \cap \mathfrak{h}$ of $\mathfrak{h}$ which is compatible with the bi-invariant metric $B$. Then $I_\mathfrak{h}$ gives the decomposition of $\mathfrak{h}$ into the following $(1, 0)$ and $(0, 1)$-components:

$$\mathfrak{h} = \mathfrak{h}^{1,0} \oplus \mathfrak{h}^{0,1}.$$ 

We define an almost complex structure on $I_\mathfrak{g}$ on $\mathfrak{g}$ by taking the following $(1, 0)$ and $(0, 1)$-components

$$\mathfrak{g}^\mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1},$$

where $\mathfrak{g}^{1,0} = \mathfrak{h}^{1,0} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$ and $\mathfrak{g}^{0,1} = \mathfrak{h}^{0,1} \oplus \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha$. The complex conjugation is given by $\tau_\alpha = e^{-\alpha}$.

**Proposition 8.1.** $\mathfrak{g}^{1,0}$ is closed under the bracket of the Lie algebra $\mathfrak{g}$. The metric $B$ is a symmetric form of type $(1,1)$.

**Proof.** Since $[e_\alpha, e_\beta] \subset \mathfrak{g}_{\alpha + \beta}$, for $e_\alpha \in \mathfrak{g}_\alpha$ and $e_\beta \in \mathfrak{g}_\alpha$, the space of positive roots $\bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$ is closed under the bracket of the Lie algebra $\mathfrak{g}$. Since $[h, e_\alpha] = \alpha(h)e_\alpha$ and $\mathfrak{h}$ is commutative, $\mathfrak{g}^{1,0}$ is closed under the bracket of the Lie algebra $\mathfrak{g}$. The metric $B$ is ad-invariant, it follows that $B([h, e_\alpha], e_\beta) + B(e_\alpha, [h, e_\beta]) = 0$ for $h \in \mathfrak{h}$ and $e_\alpha \in \mathfrak{g}_\alpha, e_\beta \in \mathfrak{g}_\alpha$. Then one has $\alpha(h)B(e_\alpha, e_\beta) + \beta(h)B(e_\alpha, e_\beta) = 0$. If $\alpha, \beta$ are positive, $\alpha + \beta \neq 0$. Thus it follows $B(e_\alpha, e_\beta) = 0$ for all positive roots $\alpha, \beta$. Since $B$ is real, it follows $B(e_{-\alpha}, e_{-\beta}) = 0$. Thus $B$ is of type $(1,1)$.

We choose a unitary basis $\{u_i\}$ of $\mathfrak{g}^{1,0}$ with respect to $B$ and denote by $\{\overline{u}_i\}$ the conjugate basis of $\mathfrak{g}^{0,1}$. We recall the Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$ which is the direct sum of $\mathfrak{g}$ and $\overline{\mathfrak{g}}$. Let $\mathfrak{d}$ equipped with the bilinear form $B_\mathfrak{d}$ given by $+B$ on the first summand and $-B$ on the second summand.

**Definition 8.2.** We define $\mathcal{L}_{\phi}$ by the direct sum $\mathfrak{g}^{1,0} \oplus \mathfrak{g}^{1,0}$ which is a subspace of $\mathfrak{d}$. We also define $\mathcal{L}_{\psi}$ by the direct sum $\mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$.

Then one has
Proposition 8.3. The subspace $\mathcal{L}_\phi$ is maximal isotropic, which gives a decomposition $\mathfrak{d} \otimes \mathbb{C} = \mathcal{L}_\phi \oplus \overline{\mathcal{L}_\phi}$. The subspace $\mathcal{L}_\psi$ is also maximal isotropic, which gives a decomposition $\mathfrak{d} \otimes \mathbb{C} = \mathcal{L}_\psi \oplus \overline{\mathcal{L}_\psi}$.

Proof. Since $B$ is of type $(1,1)$, one see that

$$B_\phi((u_i, u_j), (u_i, u_j)) = B(u_i, u_i) - B(u_j, u_j) = 0,$$

for all $(u_i, u_j) \in \mathcal{L}_\phi$. Thus $\mathcal{L}_\phi$ is maximal, isotropic. One also has

$$B_\psi((u_i, \overline{u}_j), (u_i, \overline{u}_j)) = B(u_i, u_i) - B(\overline{u}_j, \overline{u}_j) = 0,$$

Thus $\mathcal{L}_\psi$ is also maximal isotropic. \qed

Proposition 8.4. Both $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ are closed under the bracket of the Lie algebra $\mathfrak{d}$.

Proof. Since $\mathfrak{g}^{1,0}$ is closed under the bracket, thus $\mathcal{L}_\phi = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{1,0}$ is closed. Since $\mathfrak{g}^{0,1}$ is also closed under the bracket, thus $\mathcal{L}_\psi = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ is closed also. \qed

We define $\mathcal{L}_\phi^+$ and $\mathcal{L}_\phi^-$ by

$$\mathcal{L}_\phi^+ = \mathcal{L}_\phi \cap \mathcal{L}_\psi, \quad \mathcal{L}_\phi^- = \mathcal{L}_\phi \cap \overline{\mathcal{L}_\psi}$$

We also denote by $\overline{\mathcal{L}_\phi^\pm}$ the conjugate of $\mathcal{L}_\phi^\pm$, respectively. Then one see that

$$\mathcal{L}_\phi^+ = \mathfrak{g}^{1,0} \oplus 0, \quad \mathcal{L}_\phi^- = 0 \oplus \mathfrak{g}^{1,0}$$

$$\overline{\mathcal{L}_\phi^+} = \mathfrak{g}^{0,1} \oplus 0, \quad \overline{\mathcal{L}_\phi^-} = 0 \oplus \mathfrak{g}^{0,1}$$

We define real subspaces $C_\pm$ by

$$C_\pm = (\mathcal{L}_\phi^\pm \oplus \overline{\mathcal{L}_\phi^\pm}) \cap \mathfrak{g}$$

Then we see that

Lemma 8.5. $C_+ = \mathfrak{g} \oplus 0$ and $C_- = 0 \oplus \mathfrak{g}$.

Proof. The result follows from our definition of $C_\pm$. \qed

Thus we have

Proposition 8.6. The pair of isotropic subspaces $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ gives a generalized Kähler structure on the vector space $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$. 

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Proof. The result follows from Lemma 8.5.

Proposition 8.7. The pair \((\mathcal{L}_\phi, \mathcal{L}_\psi)\) gives rise to a generalized Kähler structure \((\mathcal{J}_\phi, \mathcal{J}_\psi)\) on a Lie group \(G\).

Proof. Applying Proposition 7.1 and Proposition 8.4, we have \(\mathcal{J}_\phi\) and \(\mathcal{J}_\psi\) give integrable generalized complex structures on \(G\). Then the result follows from Proposition 8.6.

8.2 Calculation of the scalar curvature of generalized Kähler structures on compact Lie groups

Our observation of calculations of the scalar curvature of generalized Kähler structures on the Hopf surfaces can be extended to the cases of compact Lie groups. We shall show that the scalar curvature of \((G, \mathcal{J}_\phi, \mathcal{J}_\psi)\) is a constant.

Let \((\mathcal{J}_\phi, \mathcal{J}_\psi)\) be a generalized Kähler structure on \(G\) as in Proposition 8.7. Then we shall show the following:

Proposition 8.8. The scalar curvature \(S(\mathcal{J}_\phi, \mathcal{J}_\psi)\) is a constant.

We need several lemmas to show Proposition 8.8. The proof will be given at the end of this section. In order to show that \(S(\mathcal{J}_\phi, \mathcal{J}_\psi)\) is a constant, we determine nondegenerate, pure spinors corresponding to \(\mathcal{J}_\phi, \mathcal{J}_\psi\), which are given by nondegenerate, pure spinors of \(Cl(g)\). As in (7.3), the map

\[ R : Cl(g) \to \Omega(G) \]

gives a map from nondegenerate, pure spinors of \(Cl(g)\) to nondegenerate, pure spinors of \(\Omega(G)\). Note that there is no loss of generality, we can assume nondegenerate, pure spinors \(R(\phi), R(\psi)\) are globally defined on \(G\). We shall give an explicit description of \((\phi, \psi)\) which gives \((\mathcal{J}_\phi, \mathcal{J}_\psi)\) on \(G\). Let \(\{t_i\}\) be a unitary basis of \(\mathfrak{h}^{1,0}\) and \(\{\bar{t}_i\}\) the conjugate basis of \(\mathfrak{h}^{0,1}\). We denote by \(\{\theta_\alpha\}_{\alpha > 0}\) a unitary basis of \(\oplus_{\alpha > 0} \mathfrak{g}_\alpha\) and by \(\{\bar{\theta}_\alpha\}_{\alpha > 0}\) the conjugate basis of \(\oplus_{\alpha < 0} \mathfrak{g}_\alpha\), where note that \(\bar{\theta}_\alpha = \theta_{-\alpha}\). Then we have

Lemma 8.9. \(\phi\) and \(\psi\) are given by

\[ \phi = \left( \prod_{i=1}^l t_i \cdot \prod_{\alpha > 0} \theta_\alpha \right) \]
\[ \psi = \left( \prod_{i=1}^l t_i \cdot \bar{t}_i \right) \cdot \left( \prod_{\alpha > 0} \theta_\alpha \cdot \bar{\theta}_\alpha \right) \]
Proof. Since \( t_i \cdot t_j = 0 \) for all \( i, j = 1, \cdots, l \) and \( \theta_\alpha \cdot \theta_\beta = 0 \) for all positive roots \( \alpha, \beta \), it follows \( t_i \cdot \phi = \phi \cdot t_i = 0 \) and \( \theta_\alpha \cdot \phi = \phi \cdot \theta_\alpha = 0 \). Thus \( \ker \phi = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1} \). Then it also follows \( \ker \psi = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1} \). □

We define \( \text{vol}_g \) by

\[
\text{vol}_g = i^{-n} \left( \prod_{i=1}^{l} t_i \cdot \vec{t}_i \right) \cdot \left( \prod_{\alpha > 0} \theta_\alpha \cdot \vec{\theta}_\alpha \right).
\]

Then we have

\[
\text{vol}_g = i^{-n} \langle \phi, \vec{\phi} \rangle_s = i^{-n} \langle \psi, \vec{\psi} \rangle_s.
\]

The volume form on \( G \) is given by

\[
\text{vol}_G := i^{-n} \langle \mathcal{R}(\phi), \mathcal{R}(\vec{\phi}) \rangle_s = i^{-n} \langle \mathcal{R}(\psi), \mathcal{R}(\vec{\psi}) \rangle_s
\]

As in \( [\text{24}] \), \( \mathcal{R} \) intertwines differentials: \( \mathcal{R}(d^4(x)) = (d + H)\mathcal{R}(x) \). By using this, we shall calculate \( d^4 \phi \) and \( d^4 \psi \). The Clifford differential \( d^4 \) is defined by \( d^4 := -[q(\Xi), \cdot]_{\mathcal{G}} \) as before, where \( \Xi \) is the Cartan 3-form of \( \mathfrak{g} \) and the map \( q : \wedge^2 \mathfrak{g}^C \rightarrow \text{Cl}(\mathfrak{g}^C) \) gives \( q(\Xi) \in \text{Cl}(\mathfrak{g}^C) \).

Lemma 8.10. The 3-form \( \Xi \) is given by

\[
\Xi = \sum_{\alpha > 0} 2P_\alpha \wedge \theta_\alpha \wedge \vec{\theta}_\gamma + \sum_{\alpha + \beta = \gamma} C_{\alpha,\beta,\gamma} \theta_\alpha \wedge \theta_\beta \wedge \vec{\theta}_\gamma + \sum_{\alpha + \beta = \gamma} C_{\alpha,\beta,\gamma} \vec{\theta}_\alpha \wedge \vec{\theta}_\beta \wedge \theta_\gamma,
\]

where \( P_\alpha = C_{i,\alpha,\alpha} t_i + C_{i,\alpha,\alpha} \bar{t}_i \in \mathfrak{h} \) is pure imaginary and \( C_{i,\alpha,\alpha}, C_{\alpha,\beta,\gamma} \) and \( C_{\alpha,\beta,\gamma} \) are structure constants of \( \mathfrak{g}^C \) and the summation runs over all triples of positive roots \( (\alpha, \beta, \gamma) \) satisfying \( \alpha + \beta = \gamma \).

Proof. Recall the real 3-form \( \Xi \) is given by

\[
\Xi(x, y, z) = B([x, y], z), \quad x, y, z \in \mathfrak{g}.
\]

Since \([e_\alpha, e_\beta] \in \mathfrak{g}_{\alpha + \beta} \) and \( B \) is of type \((1, 1)\), it follows \( \Xi(\theta_\alpha, \theta_\beta, \theta_\gamma) = 0 \) for all positive roots \( \alpha, \beta, \gamma \). Since \( \Xi \) is real, one has \( \Xi(\bar{\theta}_\alpha, \bar{\theta}_\beta, \bar{\theta}_\gamma) = 0 \) for all positive roots \( \alpha, \beta, \gamma \). If \([e_\alpha, e_\beta] \in \mathfrak{h} \), then \( \alpha + \beta = 0 \). Thus \( \Xi \) is of type \((1, 2)\) and of type \((2, 1)\). Both \( \Xi(t_i, e_\alpha, e_\beta) \) and \( \Xi(\bar{t}_i, e_\alpha, e_\beta) \) give the first term and \( \Xi(e_\alpha, e_\beta, \bar{e}_\gamma) \) and \( \Xi(\bar{e}_\alpha, \bar{e}_\beta, e_\gamma) \) give the second and third term. Since \( \mathfrak{h} \) is commutative, \( \Xi(t_i, t_j, e_\alpha) = 0 \). Then we have the result. □

Let \( P_\alpha = C_{i,\alpha,\alpha} t_i + C_{i,\alpha,\alpha} \bar{t}_i \in \mathfrak{h} \) as before. We denote by \( P \) the sum \( \sum_{\alpha > 0} P_\alpha \) and by \( e(P) \) the diagonal element \((P, P)\) of \( \mathfrak{g} \oplus \mathfrak{h} \) which acts on \( \text{Cl}(\mathfrak{g}^C) \) by the Clifford multiplication \( \rho^4 \), that is, \( \rho^4(e(P)) = P \cdot x - (-1)^{|x|} x \cdot P \). We also denote by \( a(P) \) the anti-diagonal element \((P, -P)\) which acts on \( x \in \text{Cl}(\mathfrak{g}^C) \) by \( \rho^4(a(P)) = p \cdot x + (-1)^{|x|} x \cdot P \)
Lemma 8.11. Then the Clifford differential of $\phi$ and $\psi$ are given by

$$d^c \phi = -\rho^c(\alpha(P))\phi, \quad d^c \psi = -\rho^c(e(P))\psi,$$

where note that $e(P), \alpha(P)$ are pure imaginary elements of $\mathfrak{g}^C \oplus \mathfrak{g}^{\gamma}$. 

Proof. First we consider the Clifford differential $d^c \phi$. Since $\{\theta_\alpha\}_{\alpha > 0}$ is a unitary basis of the space of positive roots and $\tau_\alpha = e_{-\alpha}$, one has

$$\theta_\alpha \cdot \theta_\beta + \theta_\beta \cdot \theta_\alpha = \begin{cases} 1 & \alpha = -\beta, \\ 0 & \alpha \neq -\beta \end{cases}$$

In particular, $\theta_\alpha \cdot \theta_\alpha = 0$ for every positive root $\alpha$. Recall that $q(\theta_1 \wedge \theta_2 \wedge \theta_3)$ is given by

$$\theta_1 \wedge \theta_2 \wedge \theta_3 = \frac{1}{6} \sum_{\sigma \in S_3} (-1)^{|\sigma|} (\theta_{\sigma(1)} \cdot \theta_{\sigma(2)} \cdot \theta_{\sigma(3)}),$$

where $\sigma$ runs over all permutation of 1, 2, 3 and $|\sigma|$ denotes the signature of $\sigma$. Thus both right and left multiplications of $q(\theta_\alpha \wedge \theta_\beta \wedge 2\theta_\gamma)$ on $(\prod_{\alpha > 0} \theta_\alpha)$ vanish since $\alpha, \beta \neq \gamma$. It also follows that both right and left multiplications of $q(\theta_\alpha \wedge \theta_\beta \wedge \theta_\gamma)$ on $(\prod_{\alpha > 0} \theta_\alpha)$ vanish since $\alpha, \beta \neq \gamma$. We also have $q(\theta_\alpha \wedge \theta_\beta) = 1/(2(\theta_\alpha \cdot \theta_\alpha - \theta_\alpha \cdot \theta_\alpha))$. Then it follows that

$$q(\theta_\alpha \wedge \theta_\alpha) \cdot \theta_\alpha = \frac{1}{2}(\theta_\alpha \cdot \theta_\alpha - \theta_\alpha \cdot \theta_\alpha) \cdot \theta_\alpha = \frac{1}{2} \theta_\alpha \quad (8.3)$$

$$\theta_\alpha \cdot q(\theta_\alpha \wedge \theta_\alpha) = \frac{1}{2} \theta_\alpha \cdot (\theta_\alpha \cdot \theta_\alpha - \theta_\alpha \cdot \theta_\alpha) = -\frac{1}{2} \theta_\alpha \quad (8.4)$$

Then it follows that the right multiplications of $2q(\theta_\alpha \wedge \theta_\alpha)$ on $(\prod_{\alpha > 0} \theta_\alpha)$ is trivial and the left one is $(-1)$-times. Thus one has

$$[q(2\alpha \wedge \theta_\alpha \wedge \theta_\beta), \phi]_{C^1} = P_\alpha \cdot \phi + (-1)^{|\phi|} \phi \cdot P_\alpha = \rho^c(\alpha(P))\phi$$

As in Lemma 8.10 $\Xi$ is given by

$$\Xi = \sum_{\alpha > 0} 2P_\alpha \wedge \theta_\alpha \wedge \theta_\beta + \sum_{\alpha, \beta, \gamma > 0} C_{\alpha, \beta, \gamma} \theta_\alpha \wedge \theta_\beta \wedge \theta_\gamma + \sum_{\alpha, \beta, \gamma > 0} C_{\alpha, \beta, \gamma} \theta_\alpha \wedge \theta_\beta \wedge \theta_\gamma,$$

Thus one has $-d^c \phi = \rho^c(e(\Xi))\phi = \sum_\alpha \rho^c(\alpha(P_\alpha))\phi = \rho^c(\alpha(P))\phi$. We also have

$$q(\theta_\alpha \wedge \theta_\alpha) \cdot (\theta_\alpha \cdot \theta_\alpha) = \frac{1}{2} (\theta_\alpha \cdot \theta_\alpha - \theta_\alpha \cdot \theta_\alpha) \cdot (\theta_\alpha \cdot \theta_\alpha) = \frac{1}{2} (\theta_\alpha \cdot \theta_\alpha) \quad (8.5)$$

$$(\theta_\alpha \cdot \theta_\alpha) \cdot q(\theta_\alpha \wedge \theta_\alpha) = \frac{1}{2} (\theta_\alpha \cdot \theta_\alpha) \cdot (\theta_\alpha \cdot \theta_\alpha - \theta_\alpha \cdot \theta_\alpha) = \frac{1}{2} (\theta_\alpha \cdot \theta_\alpha) \quad (8.6)$$

(Note that the sign of (8.6) is different from the case of (8.3).) Thus one has

$$-d^c \psi = \rho^c(e(\Xi))\psi = \sum_\alpha [P_\alpha, \psi]_{C^1} = \rho^c(e(P))\psi \quad \square$$
Next we shall calculate $d^4\rho^c(a(P))\psi$ and $d^4\rho^c(e(P))\phi$. First we shall show the following,

**Lemma 8.12.**

\[
\begin{align*}
\rho^c(e(\Xi))\rho^c(a(P)) + \rho^c(a(P))\rho^c(e(\Xi)) &= \rho^c(a([\Xi, P]_{Cl})) \quad (8.7) \\
\rho^c(e(\Xi))\rho^c(e(P)) + \rho^c(e(P))\rho^c(e(\Xi)) &= \rho^c(e([\Xi, P]_{Cl})) \quad (8.8)
\end{align*}
\]

**Proof.** We shall show the Lemma in a general form. Let $E = (u_1, u_2)$ be a real element of $\mathfrak{h} \oplus \mathfrak{g}$ which satisfies $B(E, E)_\mathfrak{g} = B(u_1, u_1) - B(u_2, u_2) \neq 0$. Then $E$ is an element of Pin group Pin(\mathfrak{h}). We denote by $e(\Xi)$ the diagonal $(\Xi, \Xi) \in Cl(\mathfrak{g} \oplus \mathfrak{g})$. $Cl(\mathfrak{g})$ is a $Cl(\mathfrak{g} \oplus \mathfrak{g})$-module and we denote by $\rho^c$ the action of $Cl(\mathfrak{g} \oplus \mathfrak{g})$ on $Cl(\mathfrak{g})$. Then for $x \in \mathfrak{g}$

\[
\rho^c(e(\Xi))\rho^c(E)x = \rho^c(e(\Xi)) \left( u_1 \cdot x - (-1)^{|x|} u_1 \cdot x \cdot \Xi \right.
\]
\[
\left. + (-1)^{|x|} x \cdot u_2 + (-1)^{|x| + 1} x \cdot u_2 \cdot \Xi \right)
\]

\[
\rho^c(E)\rho^c(e(\Xi))x = \rho^c(E) \left( \Xi \cdot x - (-1)^{|x|} x \cdot \Xi \right.
\]
\[
\left. + u_1 \cdot \Xi \cdot x - (-1)^{|x| + 1} u_1 \cdot x \cdot \Xi \right.
\]
\[
\left. - (-1)^{|x| + 1} \Xi \cdot x \cdot u_2 + (-1)^{|x| + 1} x \cdot u_2 \cdot \Xi \right)
\]

Since $[e(\Xi), E]_{Cl} = ([\Xi, u_1]_{Cl}, [\Xi, u_2]_{Cl}) = (\Xi \cdot u_1 + u_1 \cdot \Xi, \Xi \cdot u_2 + u_2 \cdot \Xi)$ is even, one has

\[
\rho^c([e(\Xi), E]_{Cl}) = \rho^c(E)\rho^c(e(\Xi)) + \rho^c(e(\Xi))\rho^c(E) \quad (8.9)
\]

Applying $E = e(P)$ or $a(P)$, we obtain the result.

As in before, $d^4$ is given by the action $-\rho^c(e(\Xi))$. From Lemma 8.12 and $-d^4\psi = \rho^c(e(\Xi))\psi = \rho^c(e(P))\psi$, we have

\[
-d^4\rho^c(a(P))\psi = \rho^c(e(\Xi))\rho^c(a(P))\psi \quad (8.10)
\]

\[
= - \rho^c(a(P))\rho^c(e(\Xi))\psi + \rho^c(a([\Xi, P]_{Cl}))\psi \quad (8.11)
\]

\[
= - \rho^c(a(P))\rho^c(e(P))\psi + \rho^c(a([\Xi, P]_{Cl}))\psi \quad (8.12)
\]

We also have

\[
-d^4\rho^c(e(P))\phi = \rho^c(e(\Xi))\rho^c(e(P))\phi \quad (8.13)
\]

\[
= - \rho^c(e(P))\rho^c(e(\Xi))\phi + \rho^c(e([\Xi, P]_{Cl}))\phi \quad (8.14)
\]

\[
= - \rho^c(e(P))\rho^c(a(P))\phi + \rho^c(e([\Xi, P]_{Cl}))\phi \quad (8.15)
\]

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Lemma 8.13.

\[ \text{Re } i^{-n} \langle \psi, \rho^\ell (a(P)) \rho^\ell (e(P)) \overline{\psi} \rangle_s = B(P, P)i^{-n} \langle \psi, \overline{\psi} \rangle_s \]

\[ \text{Re } i^{-n} \langle \phi, \rho^\ell (e(P)) \rho^\ell (a(P)) \overline{\phi} \rangle_s = B(P, P)i^{-n} \langle \phi, \overline{\phi} \rangle_s, \]

Proof. Since \( P \) is pure imaginary, the result follows. \( \square \)

Lemma 8.14.

\[ \rho^\ell (e(\Xi, P)_{CI}) \phi = 2B(P, P)\phi \quad (8.16) \]

\[ \rho^\ell (a(\Xi, P)_{CI}) \psi = 2B(P, P)\psi \quad (8.17) \]

Proof. We have

\[ \Xi, P]_{CI} = \sum_\alpha [P_\alpha \cdot (\theta_\alpha \cdot \overline{\theta}_\alpha - \overline{\theta}_\alpha \cdot \theta_\alpha), P]_{CI} \]

\[ = \sum_\alpha B(P_\alpha, P) (\theta_\alpha \cdot \overline{\theta}_\alpha - \overline{\theta}_\alpha \cdot \theta_\alpha) \]

Then we have

\[ \rho^\ell (e(\Xi, P)_{CI}) \phi = \sum_\alpha B(P_\alpha, P) (\theta_\alpha \cdot \overline{\theta}_\alpha - \overline{\theta}_\alpha \cdot \theta_\alpha) \cdot \phi \]

\[ - B(P_\alpha, P) \phi \cdot (\theta_\alpha \cdot \overline{\theta}_\alpha - \overline{\theta}_\alpha \cdot \theta_\alpha) \]

\[ = \sum_\alpha B(P_\alpha, P) \phi + B(P_\alpha, P) \phi \]

\[ = 2B(P, P) \phi \]

We also have

\[ \rho^\ell (a(\Xi, P)_{CI}) \psi = \sum_\alpha B(P_\alpha, P) (\theta_\alpha \cdot \overline{\theta}_\alpha - \overline{\theta}_\alpha \cdot \theta_\alpha) \cdot \psi \]

\[ + B(P_\alpha, P) \psi \cdot (\theta_\alpha \cdot \overline{\theta}_\alpha - \overline{\theta}_\alpha \cdot \theta_\alpha) \]

\[ = \sum_\alpha B(P_\alpha, P) \psi + B(P_\alpha, P) \psi \]

\[ = 2B(P, P) \psi \]

\( \square \)

Lemma 8.15.

\[ - i^{-n} \langle \psi, d^\ell \rho^\ell (a(P)) \overline{\psi} \rangle_s = 2\|P\|^2 i^{-n} \langle \phi, \overline{\phi} \rangle_s \quad (8.18) \]

\[ - i^{-n} \langle \phi, d^\ell \rho^\ell (a(P)) \overline{\phi} \rangle_s = 2\|P\|^2 i^{-n} \langle \psi, \overline{\psi} \rangle_s, \quad (8.19) \]

where \( \|P\|^2 = -B(P, P) \geq 0 \). Note that \( P \) is pure imaginary.
PROOF. Taking the complex conjugate of (8.10) and (8.13), we have
\[-d^l \rho^{cl}(a(P))\overline{\psi} = \rho^{cl}(a(P))\rho^{cl}(e(P))\overline{\psi} + \rho^{cl}(a([\Xi, P]_{Cl})\overline{\psi}\]
\[-d^l \rho^{cl}(e(P))\overline{\phi} = \rho^{cl}(e(P))\rho^{cl}(a(P))\overline{\phi} + \rho^{cl}(e([\Xi, P]_{Cl})\overline{\phi}\]

From Lemma 8.13 and Lemma 8.14 we have
\[-i^{-n}\langle \psi, d^l \rho^{cl}(a(P))\overline{\psi} \rangle_s = i^{-n}\langle \psi, \rho^{cl}(a(P))\rho^{cl}(e(P))\overline{\psi} \rangle_s + i^{-n}\langle \psi, \rho^{cl}(a([\Xi, P]_{Cl})\overline{\psi} \rangle_s + i^{-n}\langle \psi, \rho^{cl}(a([\Xi, P]_{Cl})\overline{\psi} \rangle_s = ||P||^2 i^{-n}\langle \psi, \overline{\psi} \rangle_s\]

One also has
\[-i^{-n}\langle \phi, d^l \rho^{cl}(e(P))\overline{\phi} \rangle_s = i^{-n}\langle \phi, \rho^{cl}(e(P))\rho^{cl}(e(P))\overline{\phi} \rangle_s + i^{-n}\langle \phi, \rho^{cl}(a([\Xi, P]_{Cl})\overline{\phi} \rangle_s + i^{-n}\langle \phi, \rho^{cl}(a([\Xi, P]_{Cl})\overline{\phi} \rangle_s = ||P||^2 i^{-n}\langle \phi, \overline{\phi} \rangle_s\]

Proof of Proposition 8.8 The scalar curvature \(S(\mathcal{J}_\phi, \mathcal{J}_\psi)\) is given by
\[S(\mathcal{J}_\phi, \mathcal{J}_\psi)vol_G = \text{Re} \ i^{-n}\langle \mathcal{R}(\psi), d_H(\eta \cdot \overline{\mathcal{R}(\psi)}) \rangle_s + \text{Re} \ i^{-n}\langle \mathcal{R}(\phi), d_H(\zeta \cdot \overline{\mathcal{R}(\phi)}) \rangle_s,\]

Then substituting \(\eta = -s(a(P))\) and \(\zeta = -s(e(P))\), we have
\[\eta \cdot \overline{\mathcal{R}(\psi)} = -s(a(P)) \cdot \overline{\mathcal{R}(\psi)} = -\mathcal{R}(\rho^{cl}(a(P))\overline{\psi})
\zl{\zeta \cdot \overline{\mathcal{R}(\phi)} = -s(e(P)) \cdot \overline{\mathcal{R}(\phi)} = -\mathcal{R}(\rho^{cl}(e(P))\overline{\phi})\]

Since \(d_H \mathcal{R}(x) = \mathcal{R}(d^l x)\), we have
\[d_H(\eta \cdot \overline{\mathcal{R}(\psi)}) = -\mathcal{R}(d^l \rho^{cl}(a(P))\overline{\psi})
\[d_H(\zeta \cdot \overline{\mathcal{R}(\phi)}) = -\mathcal{R}(d^l \rho^{cl}(e(P))\overline{\phi})\]

Applying (7.5), one has, for each \(g \in G\),
\[\langle \mathcal{R}(\psi), d_H(\eta \cdot \overline{\mathcal{R}(\psi)}) \rangle_s = -\langle \psi, d^l \rho^{cl}(a(P))\overline{\psi} \rangle_s \cdot (1)_{g([\text{dim}G+1]} N(g)_{\mu_G}
\[\langle \mathcal{R}(\phi), d_H(\zeta \cdot \overline{\mathcal{R}(\phi)}) \rangle_s = -\langle \phi, d^l \rho^{cl}(e(P))\overline{\phi} \rangle_s \cdot (1)_{g([\text{dim}G+1]} N(g)_{\mu_G}\]

Then from Lemma 8.15 we have
\[i^{-n}\langle \mathcal{R}(\psi), d_H(\eta \cdot \overline{\mathcal{R}(\psi)}) \rangle_s = ||P||^2 (-1)_{g([\text{dim}G+1]} N(g)_{i^{-n}\langle \psi, \overline{\psi} \rangle_s \mu_G}
\[i^{-n}\langle \mathcal{R}(\phi), d_H(\zeta \cdot \overline{\mathcal{R}(\phi)}) \rangle_s = ||P||^2 (-1)_{g([\text{dim}G+1]} N(g)_{i^{-n}\langle \phi, \overline{\phi} \rangle_s \mu_G}\]

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Recall the volume form is normalized by
\[
\text{vol}_G = i^{-n} \langle \mathcal{R}(\phi), \mathcal{R}(\phi) \rangle_s = i^{-n} \langle \mathcal{R}(\psi), \mathcal{R}(\psi) \rangle_s.
\]
Thus the form \( \text{vol}_G \) on \( G \) is given by
\[
\text{vol}_G = (-1)^{|g|(\dim G + 1)} N(g) i^{-n} \langle \phi, \phi \rangle_s \mu_G = (-1)^{|g|(\dim G + 1)} N(g) i^{-n} \langle \psi, \psi \rangle_s \mu_G
\]
Then one has
\[
\text{Re } i^{-n} \langle \mathcal{R}(\psi), d_H (\eta \cdot \mathcal{R}(\psi)) \rangle_s = \| P \|^2 \text{vol}_G
\]
Then the scalar curvature is given by
\[
S(\mathcal{J}_\phi, \mathcal{J}_\psi) = 2 \| P \|^2.
\]
\[\square\]

8.3 Pin groups action of generalized Kähler structures on compact Lie groups

As in the cases of the standard Hopf surfaces, the action of the real Pin group gives deformations of generalized Kähler structures on compact Lie groups. Recall that the Cartan 3-form \( \Xi \) is given by (see Lemma 8.10)
\[
\Xi = \sum_{\alpha > 0} 2 P_\alpha \wedge \theta_\alpha \wedge \overline{\theta}_\alpha + \sum_{\alpha, \beta \neq 0} C_{\alpha, \beta, \gamma} \theta_\alpha \wedge \theta_\beta \wedge \theta_\gamma + \sum_{\alpha, \beta \neq 0} C_{\alpha, \beta, \gamma} \overline{\theta}_\alpha \wedge \overline{\theta}_\beta \wedge \theta_\gamma,
\]
Let \( u \) be a real element of \( \mathfrak{h} \). Then one has
\[
[\Xi, u]_{\text{Cl}} = \Xi \cdot u + u \cdot \Xi = \sum_{\alpha} (P_\alpha \cdot u + u \cdot P_\alpha) (\theta_\alpha \cdot \overline{\theta}_\alpha - \overline{\theta}_\alpha \cdot \theta_\alpha)
= \sum_{\alpha} B(P_\alpha, u) (\theta_\alpha \cdot \overline{\theta}_\alpha - \overline{\theta}_\alpha \cdot \theta_\alpha)
\]
Let \( \phi \in \text{Cl}(\mathfrak{g}^C) \) be a nondegenerate, pure spinor as in Lemma 8.9 which satisfies \(-d^A \phi = \rho^A(e(\Xi)) \phi = \rho^A(a(P)) \phi \) where \( a(P) \in \sqrt{-1} \mathfrak{g} \). We denote by \( \psi \in \text{Cl}(\mathfrak{g}^C) \) a nondegenerate, pure spinor as in Lemma 8.9 which satisfies \(-d^A \psi = \rho^A(e(\Xi)) \psi = \rho^A(e(P)) \psi \) where \( e(P) \in \sqrt{-1} \mathfrak{g} \).
Let \( E = (u_1, u_2) \) be an element of \( \mathfrak{h}^R \oplus \overline{\mathfrak{h}}^R \) satisfying \( \langle E, E \rangle_0 = B(u_1, u_1) - B(u_2, u_2) = \pm 1 \). Then \( E \) is an element of real Pin group \( \text{Pin}(\mathfrak{g} \oplus \overline{\mathfrak{g}}) \). Since the
set of almost generalized Kähler structures forms an orbit of the real Pin group, \((\rho^{cl}(E), \phi, \rho^{cl}(E)\psi)\) is an almost generalized Kähler structure. Then one has

\[
d^{cl} \rho^{cl}(E)\phi = -\rho^{cl}(e(\Xi))\rho^{cl}(E)\phi
\]
\[
= \rho^{cl}(E)\rho^{cl}(e(\Xi))\phi - \rho^{cl}([e(\Xi), E]_{C1})\phi
\]
\[
= \rho^{cl}(E)\rho^{cl}(a(P))\phi - \rho^{cl}([e(\Xi), E]_{C1})\phi
\]

Applying \(\rho^{cl}(E)\rho^{cl}(a(P)) + \rho^{cl}(a(P))\rho^{cl}(E) = \rho^{cl}([a(P), E]_{C1})\), we have

\[
d^{cl} \rho^{cl}(E)\phi = -\rho^{cl}(a(P))\rho^{cl}(E)\phi - \rho^{cl}([e(\Xi), E]_{C1})\phi + \rho^{cl}([a(P), E]_{C1})\phi
\]

\[(8.20)\]

Then one has

\[
[e(\Xi), E]_{C1} = \sum_{\alpha} ([P_{\alpha}, u_1]_{C1}(\theta_{\alpha} \cdot \bar{\theta}_{\alpha} - \bar{\theta}_{\alpha} \cdot \theta_{\alpha}), [P_{\alpha}, u_2]_{C1}(\theta_{\alpha} \cdot \bar{\theta}_{\alpha} - \bar{\theta}_{\alpha} \cdot \theta_{\alpha}))
\]

Since \([P, u_i]_{C1} = P \cdot u_i + u_i \cdot P = B(P, u_i), (i = 1, 2)\), we have

\[
\rho^{cl}([e(\Xi), E]_{C1})\phi = [P, u_1]_{C1} \cdot \phi + \phi \cdot [P, u_2]_{C1}
\]
\[
= B(P, u_1 + u_2)\phi
\]

On the other hand, one has

\[
\rho^{cl}([a(P), E]_{C1})\phi = [P, u_1]_{C1} \cdot \phi + \phi \cdot [P, u_2]_{C1}
\]
\[
= B(P, u_1 + u_2)\phi
\]

From (8.20), we have

\[
d^{cl} \rho^{cl}(E)\phi = -\rho^{cl}(a(P))\rho^{cl}(E)\phi
\]

\[(8.21)\]

We also have

\[
d^{cl} \rho^{cl}(E)\psi = -\rho^{cl}(e(\Xi))\rho^{cl}(E)\psi
\]
\[
= \rho^{cl}(E)\rho^{cl}(e(\Xi))\psi - \rho^{cl}([e(\Xi), E]_{C1})\psi
\]
\[
= \rho^{cl}(E)\rho^{cl}(e(P))\psi - \rho^{cl}([e(\Xi), E]_{C1})\psi
\]

Applying \(\rho^{cl}(E)\rho^{cl}(e(P)) + \rho^{cl}(e(P))\rho^{cl}(E) = \rho^{cl}([e(P), E]_{C1})\), we have

\[
d^{cl} \rho^{cl}(E)\psi = -\rho^{cl}(e(P))\rho^{cl}(E)\psi - \rho^{cl}([e(\Xi), E]_{C1})\psi + \rho^{cl}([e(P), E]_{C1})\psi
\]

\[(8.22)\]
Then one has
\[ \rho^{cl}([e(\Xi), E]_{Cl})\psi = [P, u_1]_{cl} \cdot \psi - \psi \cdot [P, u_2]_{cl} = B(P, u_1 - u_2)\psi \]
\[ \rho^{cl}([e(P), E]_{Cl})\psi = [P, u_1]_{cl} \cdot \psi - \psi \cdot [P, u_2]_{cl} = B(P, u_1 - u_2)\psi \]

Then we also have
\[ d^{cl}\rho^{cl}(E)\psi = -\rho^{cl}(e(P))\rho^{cl}(E)\psi \]

Thus we obtain

**Proposition 8.16.** Let \((\phi, \psi)\) be a generalized Kähler structure on \(G\) as in Proposition 8.7. We denote by \(E = (u_1, u_2)\) a real element of \(\mathfrak{h} \oplus \mathfrak{h}\) satisfying \(B_3(E, E) \neq 0\). Then \((\mathcal{R}(\rho^{cl}(E)\phi), \mathcal{R}(\rho^{cl}(E)\psi))\) gives a generalized Kähler structure on \(G\).

**Proof.** We already show
\[ d^{cl}\rho^{cl}(E)\phi = -\rho^{cl}(a(P))\rho^{cl}(E)\phi, \]
\[ d^{cl}\rho^{cl}(E)\psi = -\rho^{cl}(e(P))\rho^{cl}(E)\psi. \]

Thus \(\rho^{cl}(E)\phi\) and \(\rho^{cl}(E)\psi\) satisfy the integrability condition. Hence the result follows.

Since every element \(g\) of the real Pin group \(\text{Pin}(\mathfrak{h})\) is given by a simple product \(E_m \cdots E_2 \cdot E_1\), where each \(E_i\) satisfies \(B_3(E_i, E_i) = \pm 1\). Then we have

**Proposition 8.17.** Let \((\phi, \psi)\) be a generalized Kähler structure on \(G\) as in Proposition 8.7 and denote by \(g\) an arbitrary element of the real Pin group \(\text{Pin}(\mathfrak{h} \oplus \mathfrak{h})\). Then \((\mathcal{R}(\rho^{cl}(g)\phi), \mathcal{R}(\rho^{cl}(g)\psi))\) gives a generalized Kähler structure on \(G\).

**Proof.** As in the proof of Proposition 8.16 we have
\[ d^{cl}\rho^{cl}(g)\phi = -\rho^{cl}(e(\Xi))\rho^{cl}(g)\phi \]
\[ = -(-1)^m\rho^{cl}(g)\rho^{cl}(e(\Xi))\phi - \rho^{cl}([e(\Xi), g]_{Cl})\phi \]
\[ = -(-1)^m\rho^{cl}(g)\rho^{cl}(a(P))\phi - \rho^{cl}([e(\Xi), g]_{Cl})\phi \]

Applying \(\rho^{cl}(a(P))\rho^{cl}(g) - (-1)^m\rho^{cl}(g)\rho^{cl}(a(P)) = \rho^{cl}([a(P), g]_{Cl})\), we have
\[ d^{cl}\rho^{cl}(g)\phi = -\rho^{cl}(a(P))\rho^{cl}(g)\phi - \rho^{cl}([e(\Xi), g]_{Cl})\phi + \rho^{cl}([a(P), g]_{Cl})\phi. \]

(8.24)
We denote by \( Q_\alpha \) an element \( P_\alpha \cdot (\theta_\alpha \cdot \theta_\alpha - \bar{\theta}_\alpha \cdot \bar{\theta}_\alpha) \). Since \( g \) is an element of \( \text{Pin}(\mathfrak{h} \oplus \mathfrak{n}) \), one has

\[
[e(\Xi), g]_{Cl} = \sum_{\alpha} [e(Q_\alpha), g]_{Cl}.
\]

Then we have

\[
\rho^{cl}([e(Q_\alpha), g]_{Cl})\phi = \rho^{cl}(e(Q_\alpha))\rho^{cl}(g)\phi - (-1)^m \rho^{cl}(g)\rho^{cl}(e(Q_\alpha))\phi
\]

Since \( (\theta_\alpha \cdot \theta_\alpha - \bar{\theta}_\alpha \cdot \bar{\theta}_\alpha) \cdot \phi = \phi \) and \( \phi \cdot (\theta_\alpha \cdot \theta_\alpha - \bar{\theta}_\alpha \cdot \bar{\theta}_\alpha) = -\phi \), one has

\[
\rho^{cl}(e(Q_\alpha))\phi = (P_\alpha \cdot (\theta_\alpha \cdot \theta_\alpha - \bar{\theta}_\alpha \cdot \bar{\theta}_\alpha)) \cdot \phi
- (-1)^{|\phi|} (P_\alpha \cdot (\theta_\alpha \cdot \theta_\alpha - \bar{\theta}_\alpha \cdot \bar{\theta}_\alpha))
= P_\alpha \cdot \phi + (-1)^{|\phi|} P_\alpha
= \rho^{cl}(a(P_\alpha))\phi
\]

Since \( g \) and \( (\theta_\alpha \cdot \theta_\alpha - \bar{\theta}_\alpha \cdot \bar{\theta}_\alpha) \) commute, one has

\[
\rho^{cl}(e(Q_\alpha))\rho^{cl}(g)\phi = (P_\alpha \cdot (\theta_\alpha \cdot \theta_\alpha - \bar{\theta}_\alpha \cdot \bar{\theta}_\alpha)) \cdot \rho^{cl}(g)\phi
- (-1)^{|\rho^{cl}(g)|} (P_\alpha \cdot (\theta_\alpha \cdot \theta_\alpha - \bar{\theta}_\alpha \cdot \bar{\theta}_\alpha))
= P_\alpha \cdot \rho^{cl}(g)\phi + (-1)^{|\rho^{cl}(g)|} P_\alpha \rho^{cl}(g)\phi
= \rho^{cl}(a(P_\alpha))\rho^{cl}(g)\phi
\]

Then we have

\[
\rho^{cl}([e(Q_\alpha), g]_{Cl})\phi = \rho^{cl}(e(Q_\alpha))\rho^{cl}(g)\phi - (-1)^m \rho^{cl}(g)\rho^{cl}(e(Q_\alpha))\phi
= \rho^{cl}(a(P_\alpha))\rho^{cl}(g)\phi - (-1)^m \rho^{cl}(g)\rho^{cl}(a(P_\alpha))\phi
= \rho^{cl}([a(P_\alpha), g]_{Cl})\phi
\]

Thus we obtain

\[
\rho^{cl}([e(\Xi), g]_{Cl})\phi = \rho^{cl}([a(P), g]_{Cl})\phi. \tag{8.25}
\]

From \([8,24]\), we have

\[
\delta^c \rho^{cl}(g)\phi = -\rho^{cl}(a(P))\rho^{cl}(g)\phi \tag{8.26}
\]

We also have

\[
\delta^c \rho^{cl}(g)\psi = -\rho^{cl}(e(\Xi))\rho^{cl}(g)\psi
= -(-1)^m \rho^{cl}(g)\rho^{cl}(e(\Xi))\phi - \rho^{cl}([e(\Xi), g]_{Cl})\phi
= -(-1)^m \rho^{cl}(g)\rho^{cl}(e(P))\phi - \rho^{cl}([e(\Xi), g]_{Cl})\phi
\]

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Applying $\rho^cl(e(P))\rho^cl(g) - (-1)^m \rho^cl(g)\rho^cl(e(P)) = \rho^cl([e(P), g]_{Cl})$, we have
\begin{equation}
d^cl \rho^cl(g)\psi = -\rho^cl(e(P))\rho^cl(g)\psi - \rho^cl([e(\Xi), g]_{Cl})\psi + \rho^cl([e(P), g]_{Cl})\psi \tag{8.27}
\end{equation}
Applying
\[
(\theta_\alpha \cdot \overline{\theta_\alpha} - \overline{\theta_\alpha} \cdot \theta_\alpha) \cdot \psi = \psi
\]
\[
\psi \cdot (\theta_\alpha \cdot \overline{\theta_\alpha} - \overline{\theta_\alpha} \cdot \theta_\alpha) = \psi,
\]
one has
\[
\rho^cl([e(\Xi), g]_{Cl})\psi = \rho^cl([e(P), g]_{Cl})\psi
\]
Then we also have
\begin{equation}
d^cl \rho^cl(g)\psi = -\rho^cl(e(P))\rho^cl(g)\psi \tag{8.28}
\end{equation}
Thus we obtain the result.

**Proposition 8.18.** Let $(J_\phi, J_\psi)$ be the generalized Kähler structure on a compact Lie group $G$ as in Proposition 8.17 by an action of $g \in \text{Pin}(h^R \oplus \overline{h^R})$. Then the scalar curvature of $(J_\phi, J_\psi)$ is a constant.

**Proof.** We denote by $\phi_g$ the nondegenerate, pure spinor $g \cdot \phi$ and let $\psi_g$ be $g \cdot \psi$. From (8.26) and (8.28), one has $\eta = -a(P)$ and $\zeta = -e(P)$. Then we have
\begin{align*}
-i^n \langle \psi_g, d^cl (a(P))\overline{\psi_g} \rangle_s &= i^n \langle \psi_g, \rho^cl(a(P))\rho^cl(e(P))\overline{\psi_g} \rangle_s \\
&\quad + i^n \langle \psi_g, \rho^cl(a([\Xi, P]_{Cl})\psi_g) \rangle_s \\
&= \|P\|^2 i^n \langle \psi_g, \overline{\psi_g} \rangle_s
\end{align*}
\begin{align*}
-i^n \langle \phi_g, d^cl (e(P))\overline{\phi_g} \rangle_s &= i^n \langle \phi_g, \rho^cl(e(P))\rho^cl(e(P))\overline{\phi_g} \rangle_s \\
&\quad + i^n \langle \phi_g, \rho^cl([e(P), P]_{Cl})\overline{\phi_g} \rangle_s \\
&= \|P\|^2 i^n \langle \phi_g, \overline{\phi_g} \rangle_s
\end{align*}
For each $g \in \text{Pin}(h^R \oplus \overline{h^R})$, the volume form $\text{vol}_{G,g}$ is given by
\[
\text{vol}_{G,g} := i^{-n}i^{-n} \langle \mathcal{R}(\psi_g), \mathcal{R}(\overline{\psi_g}) \rangle_s = i^{-n} \langle \mathcal{R}(\phi_g), \mathcal{R}(\overline{\phi_g}) \rangle_s
\]
Then the scalar curvature is a constant $2\|P\|^2$. 

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