Regulators and Characteristic Classes of Flat Bundles

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Abstract. In this paper, we prove that on any non-singular algebraic variety, the characteristic classes of Cheeger-Simons and Beilinson agree whenever they can be interpreted as elements of the same group (e.g. for flat bundles). In the universal case, where the base is $BGL(C)^{δ}$, we show that the universal Cheeger-Simons class is half the Borel regulator element. We were unable to prove that the universal Beilinson class and the universal Cheeger-Simons classes agree in this universal case, but conjecture they do agree.

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1. Introduction

For each complex algebraic variety $X$, Deligne and Beilinson [2] have defined cohomology groups

$$H^k_D(X, \mathbb{Z}(p)) \quad k, p \in \mathbb{N}$$

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which map to the usual singular cohomology groups:
\[ H^k_D(X, \mathbb{Z}(p)) \to H^k(X, \mathbb{Z}(p)). \]
(Here, \( \mathbb{Z}(p) \) denotes the subgroup \((2\pi i)^p\mathbb{Z} \) of \( \mathbb{C} \).)\(^{1}\) Beilinson \cite{Beilinson2} and Gillet \cite{Gillet} have constructed Chern classes for algebraic vector bundles \( E \to X \)
\[ c_p^B(E) \in H^{2p}_D(X, \mathbb{Z}(p)) \]
whose images under (1) are the usual Chern classes.

In a different direction, Cheeger and Simons \cite{Cheeger} have defined characteristic classes
\[ \widehat{c}_p(E, \nabla) \in H^{2p-1}(M, \mathbb{C}/\mathbb{Z}(p)) \]
for vector bundles \( E \to M \) endowed with a flat connection \( \nabla \) over a smooth manifold. Flat complex vector bundles over an algebraic manifold \( X \) can be made algebraic; we always want to take, as algebraic structure on the bundle, the unique one with respect to which the connection has regular singularities \cite[p. 98]{Lehmann}.\(^{1}\) We can then associate to such bundles the pair of characteristic classes
\[ \widehat{c}_p(E, \nabla) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)) \text{ and } c_p^B \in H^{2p}_D(X, \mathbb{Z}(p)). \]

There is a natural homomorphism
\[ H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)) \to H^{2p}_D(X, \mathbb{Z}(p)). \]

Our first result is:

**Theorem 1.** If \( E \to X \) is a flat algebraic vector bundle with regular singularities over the algebraic manifold \( X \), then \( c_p^B(E) \) is the image of \( \widehat{c}_p(E, \nabla) \) under (1) for all \( p \geq 1 \).

Previously known cases of this result are due to Bloch \cite{Bloch} (flat bundles with unitary monodromy over a smooth projective base) and Soulé \cite{Soule} (arbitrary flat bundles over a smooth projective base). In \cite{Zu}, there is a proof of the quasi-projective case of Theorem 1 below that is similar to ours (in 4.5) and which invokes our Proposition 6.1.4.

Since both characteristic classes are functorial, the proof of Theorem 1 would be rather straightforward if, given \( E \to X \) as in the theorem, one could find a morphism of \( X \) into some Grassmannian \( f : X \to G(n, \mathbb{C}^m) \) and a connection \( \nabla \) on the universal \( n \)-plane bundle over \( G(n, \mathbb{C}^m) \) that classified both the bundle \( E \) and its flat connection. However, it is easy to see that this is impossible in general; the context of flat vector bundles is just too rigid. For example, if \( X \) were compact and \( E \) had nontrivial monodromy, then \( f \) could not be constant, implying that \( c_1(E) \) is non-zero in \( H^2(X, \mathbb{C}) \), which contradicts the flatness of \( E \). As we shall see, there are ways to evade this problem.

We first state a generalization of Theorem 1 that goes beyond the case of flat vector bundles, providing a sort of Chern-Weil theory for DB-cohomology for algebraic manifolds. This applies to algebraic vector bundles with what we call an \( F^1 \)-connection — see 4.4.1; flat connections with regular singularities at infinity are examples of \( F^1 \)-connections. In order to state the result precisely, one has to first

\(^{1}\) If \( E \) is given a priori as an algebraic bundle on \( X \) with flat connection, it is possible that the two algebraic structures fail to coincide. See the paragraphs following 4.4.2.
write $X = \overline{X} - D$, where $\overline{X}$ is compact and $D$ is a normal crossings divisor. Then one has to define appropriate subgroups

$$\hat{H}^{2p}(X \log D, \mathbb{Z}(p))$$

of $\hat{H}^{2p}(X, \mathbb{Z}(p))$ and show that there is a natural map

$$(3) \quad \hat{H}^{2p}(X \log D, \mathbb{Z}(p)) \to H^{2p}_D(X, \mathbb{Z}(p))$$

and that the Cheeger–Simons Chern classes bundles $E$ over $X$ with an $F^1$-connection lie in this subgroup. This is done in 4.4.

**Theorem 2.** If $E \to X$ is an algebraic vector bundle with $F^1$-connection over the algebraic manifold $X$, then $c^B_p(E)$ is the image of $\hat{c}_p(E, \nabla)$ under (3) for all $p \geq 1$.

We first observe that a necessary condition for Theorem 2 is that the image of $\hat{c}_p(E, \nabla)$ in $H^{2p}_D(X, \mathbb{Z}(p))$ is the same for all $F^1$-connections $\nabla$ on $E$. This is verified a priori—though it appears later in our exposition (as 6.1.4)—and it greatly simplifies the task.

The proof of Theorem 2 in the quasi-projective case, given in 4.5, goes as follows. For any algebraic extension $\overline{E}$ of $E$ to a compactification $\overline{X}$ of $X$, there exists an ample line bundle $L$ on $\overline{X}$ such that both $\overline{E} \otimes L$ and $L$ are generated by global sections, hence, are pullbacks of universal bundles by regular maps. Since the conclusion of Theorem 2 is a tautology for the universal bundles, we get the result by functoriality for $\overline{E} \otimes L$ and $L$; this implies the same for $E$. The reduction from algebraic manifolds to the quasi-projective case is fairly standard; see our 4.6.

We also formulate and prove the analogous result in the Kähler case in 4.7. In order to effect this, one has to make distinctions among the various meromorphic equivalence classes of compactifications, and then of vector bundle extensions. In the case that $X$ is algebraic, we have been tacitly using the obvious choices, viz. the algebraic ones (cf. 4.3.8).

From a different point of view, a trick (see 6.1.5) can be used to construct a model (depending on $E \to X$ and its connection) of the universal $n$-plane bundle $E_n \to BGL_n(\mathbb{C})$ in the category of simplicial varieties, a connection on this bundle, and a morphism of simplicial varieties $X \to BGL_n(\mathbb{C})$ that simultaneously classifies the bundle and the connection. For this reason, it is both natural and tempting to work in the category of simplicial varieties; Cheeger–Simons classes, DB-cohomology and Beilinson Chern classes all extend directly to the simplicial setting (cf. Appendix C). We would then want to view Theorem 2 as a special case of its simplicial analogue. Unfortunately, there remain difficulties in trying to carry out this approach; see Section 6. An abbreviated account of the above is given in 39.

Another reason for working within the simplicial category is that we can work with the “universal case” of a flat bundle. Denote the general linear group of complex $n \times n$ matrices, \textit{endowed with the discrete topology}, by $GL_n(\mathbb{C})^{\delta}$. Its classifying

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\[^2\text{This is, in essence, the argument in 33, where } \overline{X} = X; \text{ see also our crucial 6.1.4. Note that this argument cannot be used in the complex analytic setting, not even for compact manifolds.}]

space $BGL_n(C)\delta$ classifies flat complex vector bundles of rank $n$. The Cheeger-Simons classes of the universal flat bundle

$$\widehat{c}_p \in H^{2p-1}(BGL_n(C)\delta, \mathbb{C}/\mathbb{Z}(p))$$

are the universal Cheeger-Simons classes.

To generalize a classical theorem of Dirichlet, Borel \[4\] defined canonical cohomology classes

$$b_p \in H^{2p-1}(BGL(C)\delta, \mathbb{C}/\mathbb{R}(p))$$

which he used to define “regulators” $r_p : K_{2p-1}(\mathbb{C}) \to \mathbb{C}/\mathbb{R}(p)$ from Quillen’s algebraic $K$-theory of $\mathbb{C}$ into $\mathbb{C}/\mathbb{R}(p) \cong H^1_B(\mathrm{Spec} \mathbb{C}, \mathbb{R}(p))$. Denote by $x_p$ the class in $H^{2p-1}(GL_n(C), \mathbb{Z}(p))$ that transgresses to the universal Chern class $c_p \in H^{2p}(BGL_n(C), \mathbb{Z}(p))$. For each $n$, the cohomology class $b_p$ restricts to the image in

$$H^{2p-1}(GL_n(C), \mathbb{C}/\mathbb{R}(p)) \cong H^{2p-1}(BGL_n(C)\delta, \mathbb{C}/\mathbb{R}(p))$$

of the element $\xi_p$ of $H^{2p-1}(GL_n(C), \mathbb{C}/\mathbb{R}(p))$ that corresponds to the element $x_p$ of $H^{2p-1}(GL_n(C), \mathbb{R}(p))$, under the canonical isomorphism

$$H^{2p-1}(GL_n(C), \mathbb{R}(p)) \xrightarrow{\cong} H^{2p-1}_{\text{clus}}(GL_n(C), \mathbb{C}/\mathbb{R}(p)).$$

This construction and the relevant background is reviewed in Sections 5.1 and 5.2.

Our second main result, proved in Section 5, is:

**Theorem 3.** The image of $\widehat{c}_p$ in $H^{2p-1}(BGL(C)\delta, \mathbb{C}/\mathbb{R}(p))$ is $b_p/2$.

Since $BGL_n(C)\delta$ is a simplicial variety, we also have the Beilinson Chern classes of the universal flat bundle:

$$c_p^B \in H^{2p}(BGL_n(C)\delta, \mathbb{Z}(p)) \cong H^{2p-1}(BGL_n(C)\delta, \mathbb{C}/\mathbb{Z}(p)).$$

The homomorphism $\mathbb{C}/\mathbb{Z}(p) \to \mathbb{C}/\mathbb{R}(p)$ gives rise to a commutative square:

$$\begin{array}{ccc}
H^D_B(BGL(C)\delta, \mathbb{Z}(p)) & \cong & H^{2p-1}(BGL(C)\delta, \mathbb{C}/\mathbb{Z}(p)) \\
\downarrow & & \downarrow \\
H^D_B(BGL(C)\delta, \mathbb{R}(p)) & \cong & H^{2p-1}(BGL(C)\delta, \mathbb{C}/\mathbb{R}(p))
\end{array}$$

We had hoped to prove that

$$c_p^B = \widehat{c}_p \in H^{2p-1}(BGL_n(C)\delta, \mathbb{C}/\mathbb{Z}(p)).$$

as a case of the simplicial version of Theorem 3 which is stated as 5.1.1. Our approach in trying to prove that, and some of the difficulties we encountered in our attempt, are discussed in Section 5. Thus, we are obliged to state the preceding as a conjecture:

**Conjecture 4.** For all $p \geq 1$ and $n \in \mathbb{N} \cup \{\infty\}$,

$$c_p^B = \widehat{c}_p \in H^{2p-1}(BGL_n(C)\delta, \mathbb{C}/\mathbb{Z}(p)).$$

This conjecture, if true, would imply that the universal Beilinson Chern class for flat bundles is represented by half the Borel regulator element. Combining Theorem 3 with the conjecture yields the following conjectural refinement of Beilinson’s result [2, A5.3] (see also the article by Rapoport in [29]) which asserts that there is a non-zero rational constant $\lambda$ such that $b_p \equiv \lambda c_p^B$ modulo the decomposable elements

$$[H^+(GL_n(C)\delta, \mathbb{R}) \cdot H^+(GL_n(C)\delta, \mathbb{R})] \otimes \mathbb{C}/\mathbb{R}(p) \subseteq H^*(GL(C)\delta, \mathbb{C}/\mathbb{R}(p)).$$
He used this to prove that his regulators agreed with those of Borel up to a non-zero rational constant. Theorem 3 yields as a corollary the following strengthening of Beilinson’s result.

**Proposition 5.** If Conjecture 4 is true, then the Borel regulator is two times the Beilinson regulator $K_{2p-1}(\mathbb{C}) \to \mathbb{C}/\mathbb{R}(p)$. Consequently, for all number fields $F$, the Borel regulator and twice the Beilinson regulator

$$K_{2p-1}(F) \to H^1_D(\text{Spec } F, \mathbb{R}(p)) \cong [\mathbb{C}/\mathbb{R}(p)]^{d_p}$$

are equal. Here

$$d_p = \begin{cases} r_1 + r_2 & p \text{ odd}, \\ r_2 & p \text{ even}, \end{cases}$$

where $r_1$ is the number of real embeddings of $F$, and $r_2$ is the number of conjugate pairs of complex embeddings.

We wish to stress that even if we considered only flat bundles in our theorems, it would be necessary to consider Cheeger-Simons classes of bundles with arbitrary connections, as we make essential use of the universal $n$-plane bundle which does not admit a flat connection. By definition, these classes take values in the ring of differential characters. We give a detailed account of Cheeger-Simons classes and differential characters in Section 3, as we could not find in the literature an exposition suitable for our purposes. One of the original approaches to Cheeger-Simons classes uses the construction of a universal connection by Narasimhan-Ramanan [28]. In Section 3 we give an alternative and more canonical approach to universal connections. In Section 4, we review Deligne-Beilinson cohomology, and prove Theorem 2. Next, Theorem 3 is proved in Section 5. Finally, we treat the issues and traps involved in our attempt to prove Conjecture 4 in Section 6.

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**1.1. Conventions.** To make all Chern classes compatible with those used in algebraic geometry, we introduce the algebraic geometers’ Tate twist. Denote by $\mathbb{Z}(p)$ the subgroup of $\mathbb{C}$ generated by $(2\pi i)^p$. For each subgroup $\Lambda$ of $\mathbb{C}$, set

$$\Lambda(p) = \Lambda \otimes_\mathbb{Z} \mathbb{Z}(p).$$

The isomorphism $\mathbb{Z} \to \mathbb{Z}(p)$ that takes 1 to $(2\pi i)^p$ induces a canonical isomorphism

(1.1.1) \hspace{1cm} H^\bullet(X, \mathbb{Z}) \to H^\bullet(X, \mathbb{Z}(p)).

In this paper, the $p$th Chern class of a $GL_n(\mathbb{C})$ (equivalently, a complex $n$-plane) bundle over $X$ is the element of $H^{2p}(X, \mathbb{Z}(p))$ which is the image under (1.1.1) of the usual topological Chern class as defined, for example, in [27].

Let

$$C_k : gl_n(\mathbb{C}) \to \mathbb{C} \quad k = 0, \ldots, n$$
be the \( GL_n(\mathbb{C}) \)-invariant polynomials uniquely determined by
\[
\det(tI - A) = \sum_{k=0}^{n} C_k(A)t^{n-k}.
\]
Explicitly,
\[
C_k(A) = (-1)^k \text{Tr}^\wedge k A.
\]
If \( E \rightarrow M \) is a complex vector bundle with connection \( \nabla \) and curvature \( \Theta \), then it is well known (see \[21\], p. 403) that \( C_k(\Theta) \) is a closed form that represents \( c_k(E) \) in \( H^{2k}(M, \mathbb{C}) \).

All varieties in this paper are defined over the field \( \mathbb{C} \) of complex numbers. We will work in the complex topology unless we explicitly say otherwise. The complex of smooth, \( \mathbb{C} \)-valued forms on a manifold \( X \) will be denoted by \( A^\bullet(X) \).

2. Universal Connections

2.1. Classical formulation. A construction of universal connections was first given in \[28\], where the universal bundle \( U \) on a Grassmannian \( G_C(n) \) (or equivalently, its frame bundle) is endowed with a connection \( \nabla^U \), such that any vector bundle with a unitary connection \( (E, \nabla) \) on the manifold \( M \) is isomorphic to a pull-back of \( (U, \nabla^U) \) via a classifying mapping \( g : M \rightarrow G_C(n) \). More precisely:

**Theorem 2.1.1.** \[28\] I[3]4 Fix positive integers \( m \) and \( n \). Then there is a positive integer \( \ell \) such that any complex \( n \)-plane bundle with a unitary connection \( (E, \nabla) \) on an \( m \)-dimensional manifold \( M \) is the pull-back of the universal one \( (U, \nabla^U) \) on the Grassmannian \( G(n, \mathbb{C}^\ell) \) of \( n \)-planes in \( \mathbb{C}^\ell \). Moreover, the universal connections are compatible with the inclusions \( G(n, \mathbb{C}^\ell) \rightarrow G(n, \mathbb{C}^{\ell+1}) \).

**Remark 2.1.2.** The integer \( \ell \) is given explicitly in \[28\]. Note that it is necessarily larger than would be needed for classifying bundles without connection. Universal connections are constructed for principal bundles with arbitrary connected structure group in \[28\].

2.2. Alternate formulation. For our own purposes, it is better to have an alternate formulation of universal connections. Let \( P \) be a principal bundle, with structure group \( G \) (not assumed to be connected), over the manifold \( Y \). A connection on \( P \) is, by definition, a \( G \)-equivariant lifting of the tangent bundle \( TY \) of \( Y \) to \( TP \) (the tangent bundle of \( P \)). Consider, then, the diagram:

\[
\begin{array}{ccc}
TP & \rightarrow & TP/G \\
\downarrow & & \downarrow p \\
P & \rightarrow & P/G \\
\end{array}
\]

In fact, \( \pi \) is a vector bundle projection, and the left-hand square is cartesian. Put
\[
\tilde{Y} = \{ \alpha \in \text{Hom}(TY, TP/G) : p \circ \alpha = \text{id}_{TY} \}
\]
Let \( q : \tilde{Y} \rightarrow Y \) denote the natural projection. The following is evident:

**Proposition 2.2.2.** The connections on \( P \) are in one-to-one correspondence with the cross-sections of \( q \).
Let $g$ denote the Lie algebra of $G$. Associated to any connection is its connection 1-form:

$$\omega \in \Lambda^1(P, \text{Ad}(g)).$$

In terms of the preceding, $\omega$ can be described as follows. Let $\tilde{\alpha} : TY \to TP$ be the horizontal lift associated to $\alpha$, and let $\xi \in TP$. Then one has the simple formula:

$$\omega(\xi) = \xi - \tilde{\alpha}(p^*\xi). \quad (2.2.3)$$

It is convenient to describe $\tilde{Y}$ in terms of a local trivialization of $P$. Thus, we replace $Y$ by a sufficiently small open subset, which we still call $Y$. Then

$$P \cong Y \times G, \quad (2.2.4)$$

$$TP/G \cong TY \times (TG/G) \cong TY \times g, \quad (2.2.5)$$

so any $\alpha$ in $2.2.1$ is determined by an element $\alpha \in \text{Hom}(TY, g)$. Thus we have:

**Proposition 2.2.6.** $\tilde{Y}$ is an affine-space bundle over $Y$, with fiber $\text{Hom}(T_y Y, g)$ (for any $y \in Y$). In particular, $\tilde{Y}$ is a manifold of the same homotopy type as $Y$.

Given $f : M \to Y$, the pullback of $P$ is, in terms of $2.2.4$,

$$f^*P \cong M \times G, \quad (2.2.7)$$

The pullback of the connection is represented by $\tilde{\alpha} \circ Tf \in \text{Hom}(TM, g)$.

Consider next the diagram

\[
\begin{array}{ccccccccc}
q^*TP/G & \rightarrow & q^*TP/G & \rightarrow & q^*TP/G \\
\uparrow & & \uparrow & & \downarrow \\
TP & \rightarrow & TP/G & \rightarrow & T\tilde{Y} & \rightarrow & q^*TY & \rightarrow & TY \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{P} & \rightarrow & \tilde{P}/G & \rightarrow & \tilde{Y} & \rightarrow & \tilde{Y} & \rightarrow & q^*Y \\
\end{array}
\]

where $\tilde{P} = q^*P$, the pullback of $P$ along $g$. By construction, $\tilde{P}$ has a tautological connection $\nabla$, given by the pullback of

$$\tilde{\beta} \in \{\beta \in \text{Hom}(q^*TY, q^*TP/G) : \beta \text{ projects to } \text{id}_{q^*TY}\}, \quad (2.2.8)$$

where $\tilde{\beta}$ is defined by: if $\tilde{y} \in q^{-1}(y)$, $\tilde{\beta}(\tilde{y})$ is $\tilde{y} : T_y Y \to (TP/G)_y$. When $P \cong Y \times G$, $2.2.8$ is determined by $\tilde{\beta} \in \text{Hom}(q^*TY, g)$ (recall $2.2.4$ and $2.2.5$).

**Corollary 2.2.9.** For any immersion of manifolds $f : M \to Y$ and connection $\nabla$ on $f^*P$, there is a lifting $\tilde{f} : M \to \tilde{Y}$ such that $\tilde{f}^*\nabla = \nabla$.

**Proof.** By $2.2.4$, the connection $\nabla$ corresponds to a cross-section $\sigma$ of

\[
\begin{array}{ccccccccc}
M & \rightarrow & \{\mu \in \text{Hom}(TM, f^*TP/G) : f^*p \circ \mu = \text{id}_{TM}\} \\
\downarrow & & & & & & & & \\
M & & & & & & & & \\
\end{array}
\]


Since \( f \) is, by hypothesis, an immersion, the bundle mapping \( Tf : TM \to f^*TY \) is an injection. It follows that the natural mapping

\[
\begin{array}{ccc}
\tilde{\nabla} & \longrightarrow & \tilde{Y} \\
\downarrow & \searrow & \downarrow \tilde{f} \\
M & \longrightarrow & Y
\end{array}
\]

is surjective (it is induced by the dual of \( Tf \)). Any splitting of \( Tf \) determines a section \( s \) of \( r \). Pick one, and take \( \tilde{f} = j \circ s \circ \sigma \) (see diagram below).

\[
\begin{array}{ccc}
\tilde{M} & \leftarrow & f^*\tilde{Y} \\
\downarrow & \nearrow & \downarrow \\
M & \longrightarrow & \tilde{Y}
\end{array}
\]

Taking \( Y \) to be \( \tilde{G}_C(n) \), and \( G = GL_n(\mathbb{C}) \), we obtain:

**Proposition 2.2.10.** The universal bundle \( \tilde{U} = q^*U \) on \( \tilde{G}_C(n) \), with its tautological connection \( \tilde{\nabla} \), is universal for connections on \( GL_n(\mathbb{C}) \)-bundles.

**Proof.** Any vector bundle of rank \( n \) on a manifold can be classified by an immersion into \( G_C(n) \). Now apply 2.2.9.

From the above interpretation of universal connections, one gets, for free, the following useful fact.

**Corollary 2.2.11.** If \( \tilde{g}_0, \tilde{g}_1 : M \to \tilde{G}_C(n) \) are two immersions which classify the bundle with connection \((E, \nabla)\). Then there is a piecewise-smooth homotopy \( \tilde{h} : I \times M \to \tilde{G}_C(n) \) from \( \tilde{g}_0 \) to \( \tilde{g}_1 \) such that \( \tilde{h}^*\tilde{\nabla} \) is the “constant” connection \( d_t \oplus \nabla \) on \( p^*E \) (\( p : I \times M \to M \) being the projection).

**Proof.** Let \( g_j : M \to G_C(n) \) be the projection \( q \circ \tilde{g}_j \) of \( \tilde{g}_j \). As \( g_0 \) and \( g_1 \) are necessarily homotopic, pick any homotopy \( \tilde{k} \) from \( g_0 \) to \( g_1 \); we may take \( \tilde{k} : I \times M \to G_C(n) \) to be an immersion. Let \( \tilde{k} : I \times M \to \tilde{G}_C(n) \) be a lifting that classifies \( p^*(E, \nabla) \). This is, of course, a homotopy between its ends, \( \tilde{k}_0 \) and \( \tilde{k}_1 \). By construction,

\[ q \circ \tilde{k}_j = q \circ \tilde{g}_j \quad (j = 0, 1), \]

so \( \tilde{k}_j \) and \( \tilde{g}_j \) can be connected linearly in the affine-space bundle \( \tilde{G}_C(n) \). These homotopies also classify the constant connection. By combining them with \( \tilde{k} \), we obtain \( \tilde{h} \) as desired.

3. The Cheeger-Simons Chern Class

In this section we recall and elaborate on the definition and properties of the Cheeger-Simons invariant, which can be interpreted as a Chern classes in the ring of differential characters, associated to a vector bundle with connection on a manifold \( G \).
3.1. Generalities. Let $M$ be a $C^\infty$ manifold, $S_\bullet(M)$ the complex of $C^\infty$ singular chains on $M$ with integer coefficients, and $Z_\bullet(M)$ the subgroup of cycles. For any abelian group $\Lambda$, one has
\begin{equation}
S^\bullet(M, \Lambda) = \text{Hom}_\mathbb{Z}(S_\bullet(M), \Lambda),
\end{equation}
the corresponding complex of $\Lambda$-valued smooth singular cochains, whose coboundary operator we shall denote by $\delta$.

Let $A^\bullet(M)$ denote the complex of $C^\infty$ differential forms on $M$ with complex coefficients (one could use real coefficients here as well). An element $w \in A^k(M)$ determines a cochain $c_w \in S^k(M, \mathbb{C})$ by the formula
\begin{equation}
c_w(\sigma) = \int_{\Delta^k} \sigma^* w
\end{equation}
whenever $\sigma : \Delta^k \to M$ is a $C^\infty$ singular simplex. This embeds $A^\bullet(M)$ as a subcomplex of $S^\bullet(M, \mathbb{C})$. For any subgroup $\Lambda$ of $\mathbb{C}$, we then get a morphism of complexes.
\begin{equation}
i_\Lambda : A^\bullet(M) \to S^\bullet(M, \mathbb{C}/\Lambda),
\end{equation}
which is an injection whenever $\Lambda$ is totally disconnected.

3.2. Differential characters. Following [9], one makes the following:

**Definition 3.2.1.** The group of mod $\Lambda$ differential characters of degree $k$ on $M$ is the group $\hat{\mathcal{H}}_k(M, \Lambda) = \{ f, \alpha \in \text{Hom}_\mathbb{Z}(Z_{k-1}(M), \mathbb{C}/\Lambda) \oplus A^k(M) : \delta f = i_\Lambda \alpha, \text{ and } d\alpha = 0 \}$.

One sees that $\hat{\mathcal{H}}^\bullet(M, \Lambda)$ is a contravariant functor of $M$; when $\Lambda$ is a ring there is a functorial product [9, (1.11)] that imparts a ring structure to the differential characters.

**Remark 3.2.2.** (i) Though [9] would have us writing $\hat{\mathcal{H}}^k(M, \mathbb{C}/\Lambda)$ in 3.2.1, we have introduced the above change of notation for the sake of consistency with that to be used in §5. We have also taken the liberty of shifting by one the degree of a differential character from that of [9], so that it becomes compatible with other Chern classes. This also makes the ring structure a graded one.

(ii) If $\Lambda$ is totally disconnected, then $\alpha$ above is uniquely determined, 3.1.3, and the second condition of 3.2.1 is a consequence of the first.

It is useful to understand 3.2.1 in terms of conventional homological algebra. We have:

3.2.3. (i) If $\mathcal{F} \in S^{k-1}(M, \mathbb{C}/\Lambda)$ is any $\mathbb{Z}$-linear extension of $f$ to $S_{k-1}(M)$, then $\delta \mathcal{F} = i_\Lambda \alpha$; in particular, if $i(\Lambda)\alpha = 0$, then $\mathcal{F}$ is a cocycle.

(ii) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two extensions as in (i), then $\mathcal{F}_2 - \mathcal{F}_1$ is a $(\mathbb{C}/\Lambda)$-cocycle vanishing on $Z_{k-1}(M)$ which, by a simple argument using the fact that $C_{k-1}(M) = Z_{k-1}(M) \oplus F$, where $F$ is free, implies that the class of $\mathcal{F}_2 - \mathcal{F}_1$ vanishes in $\hat{H}^{k-1}(M, \mathbb{C}/\Lambda)$. We can therefore view $\hat{\mathcal{H}}^k(M, \Lambda)$ as a subgroup of $S^{k-1}(M, \mathbb{C}/\Lambda)/\delta S^{k-2}(M, \mathbb{C}/\Lambda)$ when $\Lambda$ is totally disconnected.
(iii) The form $\alpha$, representing zero in $H^k(M, \mathbb{C}/\Lambda)$, has its periods (on $Z_k(M)$) in $\Lambda$.

From this, one obtains:

**Proposition 3.2.4.** (i) There is a canonical and functorial exact sequence

$$0 \to H^{k-1}(M, \mathbb{C}/\Lambda) \to \hat{H}^k(M, \Lambda) \to A^k_{cl}(M, \Lambda) \to 0,$$

where

$$A^k_{cl}(M, \Lambda) = \{ \varphi \in A^k(M) : d\varphi = 0, \text{ and all periods of } \varphi \text{ on } Z_k(M) \text{ lie in } \Lambda \}. $$

(ii) (cf. [17, §4])

$$\hat{H}^k(M, \Lambda) \cong H^k(\text{cone} \{ A^k(M) \to S^\bullet(M, \mathbb{C}/\Lambda)[−1] \}) = H^{k-1}(M, \text{cone} \{ A^k_M \to S^\bullet_M(\mathbb{C}/\Lambda) \});$$

here, $A^\bullet_M$ and $S^\bullet_M(\mathbb{C}/\Lambda)$ denote the sheaves of $C^\infty$ forms and singular $(\mathbb{C}/\Lambda)$-cochains respectively.

**Corollary 3.2.5.** If $H^{k-1}(M, \mathbb{C}/\Lambda) = 0$, then

$$\hat{H}^k(M, \Lambda) \cong A^k_{cl}(M, \Lambda).$$

**3.3. Cheeger-Simons classes.** Let $(E, \nabla)$ be a $C^\infty$ complex vector bundle with connection on $M$. As in Section 1.1, the image of the Chern class

$$c_p(E) \in H^{2p}(M, \mathbb{Z}(p))$$

in $H^{2p}(M, \mathbb{C})$ is represented in de Rham cohomology by the polynomial $C_p(\Theta)$ in the curvature $\Theta$ of $\nabla$:

$$C_p(\Theta) \in A^{2p}_c(M, \mathbb{Z}(p)).$$

(3.3.1)

As such, 3.3.1 is functorial for pull-backs of bundles with connection. The *Cheeger-Simons invariant* of $(E, \nabla)$ is a functorial lifting of $c_p(E, \nabla)$ to

$$\tilde{c}_p(E, \nabla) \in \hat{H}^{2p}(M, \mathbb{Z}(p))$$

in 3.2.4(i). In terms of 3.2.4(ii), it become a $p^{th}$ Chern class with values in the group of differential characters with $\mathbb{Z}(p)$ coefficients.

**Remark 3.3.2.** If $c_p(E, \nabla) = 0$ (e.g., if $E$ is a flat bundle), then $\tilde{c}_p(E, \nabla)$ is an element of $H^{2p-1}(M, \mathbb{C}/\mathbb{Z}(p))$.

**3.4. Existence and uniqueness of $\tilde{c}_p$.** The existence and uniqueness of the Cheeger-Simon invariants can be deduced from the existence of universal connections (§2). By functoriality, the invariant is completely determined by its value on the universal connection $(\tilde{U}, \tilde{\nabla})$ of 2.2.10. On the other hand, because the odd-dimensional cohomology of a Grassmanian is trivial, there is a unique lifting $\tilde{c}_p(\tilde{U}, \tilde{\nabla})$ of its Chern form to a differential character (see 3.2.4(i)). Thus, at most one Cheeger-Simons invariant can be defined: if $\tilde{g} : M \to G_C(n)$ is a smooth mapping that classifies $(E, \nabla)$, one must take

$$\tilde{c}_p(E, \nabla) = \tilde{g}^*\tilde{c}(\tilde{U}, \tilde{\nabla}).$$

(3.4.1)

One sees that 3.4.1 actually gives a definition of $\tilde{c}_p$ by checking that it is, in fact, independent of the choice of $g$. For this, let $g_0$ and $g_1$ be two classifying mappings,
and apply 2.2.11 to produce a “nice” homotopy \( \tilde{h} \) between \( \tilde{g}_0 \) and \( \tilde{g}_1 \). Then the homotopy formula (see A.0.22 in Appendix A) yields

\[
\tilde{g}_1^* c_p(U, \nabla) - \tilde{g}_0^* c_p(U, \nabla) = \delta(B\tilde{h}^* c_p(U, \nabla)) = B(\tilde{h}^* \delta c_p(U, \nabla)) = 0,
\]
for \( \tilde{h}^* c_p(U, \nabla) \in A^{2p}(I \times M) \) is pulled back from \( M \), so is annihilated by the fiber integration \( B \).

3.5. An intrinsic construction of \( \hat{c}_p \). There is a more intrinsic definition of \( \hat{c}_p(E, \nabla) \), one that uses only functorial constructions on bundles and connections \( \mathfrak{g} \ §4 \). (See also \[1\] §3.) A truly intrinsic construction is given in 6.1.5; that construction uses simplicial methods.

Given integers \( n \geq p \geq 1 \), let \( V^n_p \) denote the Stiefel manifold of linearly independent \((n-p+1)\)-tuples in \( \mathbb{C}^n \). Then \( V^n_p \) has the homotopy type of its submanifold \( \tilde{U}(n) / U(p-1) \) of unitary \((n-p+1)\)-frames. From this, one sees:

**Proposition 3.5.1.** (see \[35\] (25.7))

(i) \[
\pi_1(V^n_p) = \begin{cases} 
0 & \text{for } i < 2p - 1, \\
\mathbb{Z} & \text{for } i = 2p - 1.
\end{cases}
\]

(ii) \( H_{2p-1}(V^n_p, \mathbb{Z}) \cong \mathbb{Z} \), and is generated by the homology class of \( U(p) / U(p-1) \cong S^{2p-1} \).

From a complex vector bundle \( E \) rank \( n \) on \( M \), one can form the associated Stiefel bundle

\[
\pi : V^n(E) \to M,
\]
with fiber \( V^n_p \). (For instance, \( V^1(E) \) is just the frame bundle of \( E \).) By construction, \( \pi^* E \) has \((n-p+1)\) tautological sections, i.e., contains a (canonically) trivial bundle of rank \( n-p+1 \). Therefore, the Chern classes of \( \pi^* E \) are those of any complementary \((p-1)\)-plane bundle \( W \). In particular,

\[
\pi^* c_p(E) = c_p(\pi^* E) = 0 \text{ in } H^{2p}(V^n(E), \mathbb{Z}(p))
\]

Given any connection \( \nabla \) on \( E \), the Chern form \( \pi^* c_p(E, \nabla) \) is thus exact. The first goal is to achieve the exactness in a functorial fashion.

3.5.3. For any \( W \) as above, let \( \nabla^W \) denote the unique connection on \( \pi^* E \) which satisfies both:

(i) \( \nabla^W \) and \( \pi^* \nabla \) agree on \( W \),

(ii) the tautological sections are flat with respect to \( \nabla^W \).

There is a functorial formula expressing the fact that the Chern forms of two connections represent the same cohomology class (see \[10\] p. 48]):

\[
c_p(E, \nabla_1) - c_p(E, \nabla_0) = d\eta_p,
\]
where

\[
\eta_p = p \int_0^1 \mu(\omega, \Theta_1, \ldots, \Theta_1) dt;
\]

here, \( \mu \) is the \( p \)-linear symmetric form with

\[
\mu(\Theta, \ldots, \Theta) = C_p(\Theta),
\]
Given \( 3.5.11 \) \( U \)

In particular, for the universal bundle which can be seen easily by considering the Serre spectral sequence of the fibration.

\[ \eta \]

\[ \text{Observation 3.5.7. Let } \sigma_0, \sigma_1 : \mathcal{M} \to \tilde{\mathcal{M}} \text{ be the sections corresponding to } \nabla_0, \nabla_1 (\text{recall } 2.2.3) \text{; let } h : I \times \mathcal{M} \to \tilde{\mathcal{M}} \text{ classify the connection } \nabla : \tilde{d}_t + (1 - t)\nabla_0 + t\nabla_1. \]

Then formula \( 3.5.4 \) is just what comes out of the homotopy formula \( A.0.19 \) when it is applied to the Chern form of the tautological connection \( \nabla \) on \( \tilde{\mathcal{M}} \), or equivalently, when \( A.0.13 \) is applied to the Chern form of \( \nabla \). To see this, one need only observe that the curvature \( \Theta \) of \( \nabla \) is given by

\[ \Theta = \Theta_t + dt \wedge \omega. \]

For \( \nabla_0 = \nabla^W \) and \( \nabla_1 = \pi^*\nabla \), \( 3.5.2 \) gives

\[ \pi^* c_p(E, \nabla) - c_p(\pi^* E, \nabla^W) = d\eta^W. \]

Since

\[ c_p(\pi^* E, \nabla^W) = 0 \text{ in } A^{2p}(V^p(E)), \]

we can rewrite \( 3.5.8 \) as:

\[ \pi^* c_p(E, \nabla) = d\eta^W. \]

It remains to study the effect of changing \( W \).

\[ \text{Proposition 3.5.10. There is a unique natural choice of } \]

\[ \pi(E, \nabla) \in A^{2p-1}(V^p(E))/dA^{2p-2}(V^p(E)) \]

such that \( d\overline{\eta}(E, \nabla) = \pi^* c_p(E, \nabla) \).

\[ \text{Proof. Let } F \cong V^p_n \text{ denote a fiber of } \pi. \]

From \( 3.5.1 \) and the Leray spectral sequence for \( \pi \), one deduces the exactness of:

\[ 0 \to H^{2p-1}(\mathcal{M}, \mathbb{Z}) \xrightarrow{\delta} H^{2p-1}(V^p(E), \mathbb{Z}) \to H^{2p-1}(F, \mathbb{Z}) \to H^{2p}(\mathcal{M}, \mathbb{Z}), \]

The image of one of the generators of \( H^{2p-1}(F, \mathbb{Z}) \cong \mathbb{Z} \) under \( \delta \) is, in fact, \( c_p(E) \), which can be seen easily by considering the Serre spectral sequence of the fibration.

In particular, for the universal bundle \( U \) on \( G_{C(n)} \)

\[ H^{2p-1}(V^p(U), \mathbb{Z}) \cong H^{2p-1}(G_{C(n)}, \mathbb{Z}) = 0. \]

Given \( \varphi \), the condition \( d\eta = \varphi \) determines \( \eta \) up to a closed form. From \( 3.5.11 \), a closed \((2p - 1)\)-form on \( V^p(U) \) is exact. It follows that \( \overline{\eta}(U, \nabla) \) is uniquely determined. Therefore, there is at most one natural construction of \( \overline{\eta}(E, \nabla) \): it must be \( g^* \overline{\eta}(U, \nabla) \) whenever \( g : \mathcal{M} \to G_{C(n)} \) classifies \( E \). The proof that this is well-defined goes as in \( 3.4.2 \) \( \Box \)

As the preceding is contrary to the spirit with which we began subsection \( 3.5 \), we will show directly, after all, that \( \eta^W \) (from \( 3.5.4 \)) modulo exact forms, independent of \( W \). Let \( W_0 \) and \( W_1 \) be two such. Consider the following connection on the pullback of \( E \) to \( I \times I \times \mathcal{M} \):

\[ d_s + d_t + (1 - t)[(1 - s)\nabla^{W_0} + s\nabla^{W_1}] + t(\pi^* \nabla); \]

it corresponds to a mapping

\[ I \times I \times \mathcal{M} \to \tilde{\mathcal{M}}. \]
Let
\[ (3.5.13) \quad B : A^{2p}(I \times I \times M) \to A^{2p-2}(M) \]
denote the fiber integration, which is just the iteration of \(A.0.12\) in Appendix A; as such, we write \(B = B_2 \circ B_1\). From \(A.0.18\) it follows that
\[ (3.5.14) \quad d(B \varphi) = (B_2 \varphi_{1,t} - B_2 \varphi_{0,t}) - (B_1 \varphi_{1,s} - B_1 \varphi_{0,s}) \]
where \(\varphi_{0,t}\) is the restriction of \(\varphi\) to \(\{0\} \times I \times M\), \(\varphi_{1,s}\) to \(I \times \{1\} \times M\), etc. Let \(\varphi\) be the \(p\)th Chern form of \(3.5.12\). Then, by \(3.5.7\)
\[ B_1 \varphi_{1,s} = 0, \quad B_2 \varphi_{1,t} = \eta^W, \quad B_2 \varphi_{0,t} = \eta^W \]
This gives
\[ d(B \varphi) = \eta^W - \eta^W + B_1 \varphi_{0,s}. \]
The desired conclusion follows from the realization that \(B_1 \varphi_{0,s}\) is the zero form. The reason for this is that when \(t = 0\), \(3.5.12\) gives the linear interpolation between \(\nabla^W\) and \(\nabla^W\), so \(\nabla^W - \nabla^W\) is an endomorphism of \(\pi^* E\) that is zero on the tautological sub-bundle (so is of rank at most \(p - 1\)). The same holds for the curvature \(\Theta_s\). Since \(\mu\) in \(3.5.5\) is induced by polarization from trace \(\wedge^p\), it follows that \(B_1 \varphi_{0,s} = 0\).

We can now complete the direct construction of \(\hat{c}_p(E, \nabla)\).

**Proposition 3.5.15.** There is a unique element \(\hat{\eta} \in \hat{H}^{2p}(M, \mathbb{Z}(p))\) with \(\pi^* \hat{\eta} = [\eta^W, \pi^* c_p(E, \nabla)]\) (and \(\tilde{\delta} \hat{\eta} = t_{2p} c_p(E, \nabla)\)).

**Proof.** The uniqueness of \(\hat{\eta}\) is easy, for it follows from the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & H^{2p-1}(F, \mathbb{C}/\mathbb{Z}(p)) \\
\uparrow & & \uparrow \\
0 & \rightarrow & H^{2p-1}(V^p(E), \mathbb{C}/\mathbb{Z}(p)) \\
\uparrow & & \uparrow \pi^* \\
0 & \rightarrow & H^{2p-1}(M, \mathbb{C}/\mathbb{Z}(p)) \\
\uparrow & & \uparrow \\
0 & & 0 \\
\end{array}
\]

that \(\pi^*\) is injective on differential characters. Likewise, we argue that \([\eta^W]\) is in the image of \(\pi^*\) by showing that the restriction of \(\eta^W\) to the fiber \(F\) is the zero differential character, or equivalently,
\[
\int_{S^{2p-1}} \eta^W \in \mathbb{Z}(p).
\]
(Recall that \(d\eta^W = \pi^* c_p(\nabla, E)\) vanishes on \(F\).) A little bizarrely, we pass over what might have been a direct calculation on the space \(V^p\), the partial frames in the trivial \(n\)-bundle over a point, and check it in the universal situation. There, \(S^{2p-1}\) bounds in \(V^p(U)\), so we write \(S^{2p-1} = \partial \Gamma\). Then \(Z = \pi_s \Gamma\) is a cycle on the Grassmannian, with
\[
\int_{S^{2p-1}} \eta^W = \int_{\Gamma} \pi^* c_p(\nabla^U) = \int_Z c_p(\nabla^U) \in \mathbb{Z}(p)
\]
This completes the proof of \(3.5.15\). \(\square\)

With the preceding accomplished, we now make the:
Definition 3.5.16. The differential character \( \hat{\eta} \) above is the Cheeger-Simons Chern class of \((E, \nabla)\). We write \( \hat{\eta} = \hat{c}_p(E, \nabla) \).

Remark 3.5.17. By 3.2.3(ii), the Cheeger-Simons class \( \hat{c}_p \) is the choice of an element
\[
y \in S^{2p-1}(M, \mathbb{C}/\mathbb{Z}(p))/\delta S^{2p-2}(M, \mathbb{C}/\mathbb{Z}(p))
\]
with \( \delta y = \iota_{Z(p)} c_p(E, \nabla) \). In terms of the cone on \( \iota_{Z(p)} \) (recall 3.2.4(ii)), it is represented by
\[
(c_p(E, \nabla), -y) \in A^{2p}(M) \oplus S^{2p-1}(M, \mathbb{C}/\mathbb{Z}(p))/\delta S^{2p-2}(M, \mathbb{C}/\mathbb{Z}(p)),
\]
which is a cocycle by virtue of the calculation
\[
D(c_p(E, \nabla), -y) = (-dc_p(E, \nabla), -\delta y + \iota_{Z(p)} c_p(E, \nabla)) = (0, 0).
\]

4. Deligne-Beilinson Cohomology

In this section we recall the definition of Deligne-Beilinson cohomology, and establish the precise relation with differential characters. References on Deligne-Beilinson cohomology include \cite{2} and \cite{18}.

4.1. Deligne-Beilinson (DB-) cohomology. Suppose that \( Y \) is a smooth algebraic variety. By resolution of singularities \cite{24}, we can find a smooth completion \( \overline{Y} \) of \( Y \) such that \( \overline{Y} - Y \) is a normal crossings divisor \( D \). Denote by \( I_p \) the composite
\[
F^p A^\bullet(\overline{Y} \log D) \to A^\bullet Y \xrightarrow{I} S^\bullet(Y),
\]
where \( F \) denotes the Hodge filtration of the de Rham complex.

Definition 4.1.1. Suppose that \( \Lambda \) is a subring of \( \mathbb{R} \) and that \( p \in \mathbb{N} \). The Deligne-Beilinson (or DB-) cohomology \( H_D^p(Y, \Lambda(p)) \) of the variety \( Y \) is the cohomology of the complex
\[
D^\bullet(Y, \overline{Y}; \Lambda(p)) = \text{cone}(F^p A^\bullet(\overline{Y} \log D) \xrightarrow{I} S^\bullet(Y, \mathbb{C}/\Lambda(p)))[-1].
\]

As the notation suggests, the DB-cohomology of a variety \( Y \) is independent of the compactification \( \overline{Y} \) chosen. This follows from standard arguments (cf. \cite{13} (8.3.2)). As DB-cohomology is constructed from a cone, we have:

Proposition 4.1.2. For each smooth algebraic variety \( Y \), there is natural long exact sequence
\[
\ldots \to H^{k-1}(Y, \mathbb{C}/\Lambda(p)) \to H^k_D(Y, \Lambda(p)) \to F^p H^k(Y, \mathbb{C}) \to H^k(Y, \mathbb{C}/\Lambda(p)) \to \ldots.
\]

Corollary 4.1.4. For each variety
(i) For each variety \( Y \), there is a natural homomorphism
\[
H^{k-1}(Y, \mathbb{C}/\Lambda(p)) \to H^k_D(Y, \Lambda(p)).
\]
(ii) If \( H^{2p-1}(Y, \mathbb{C}/\Lambda(p)) = 0 \), there is a natural isomorphism
\[
H_{D}^{2p}(Y, \Lambda(p)) \cong H^{2p}(Y) \cap \text{image}(H^{2p}(Y, \Lambda(p)) \to H^{2p}(Y, \mathbb{C})].
\]
There are products defined in Deligne cohomology:

\[(4.1.5) \quad H^k_D(Y, \Lambda(p)) \times H^l_D(Y, \Lambda(q)) \rightarrow H^{k+l}_D(Y, \Lambda(p+q))\]

(see [2, 1.5.1]). In particular, \(\bigoplus_p H^{2p}_D(Y, \Lambda(p))\) is a ring.

Bellinon [2] and Gillet [20] have defined Chern classes

\[c^B_p(E) \in H^{2p}_D(Y, \mathbb{Z}(p))\]

for vector bundles \(E \rightarrow Y\) over a variety. The construction in [2, 1.7], is in the manner of [22]: since the splitting principle holds for DB-cohomology, the existence of Chern classes in Deligne cohomology reduces to the existence of first Chern classes

\[c_1(L) \in H^2_D(Y, \mathbb{Z}(1))\]

for line bundles \(L \rightarrow Y\).

4.2. Differential characters and DB-cohomology. Recall from [3, 2] that the mod \(\Lambda\) differential characters of degree \(k\) on a \(C\) manifold \(M\), \(\hat{H}^k(M, \Lambda)\), are represented by the cocycles of degree \(k-1\) in cone\(\{A^\geq k(M) \xrightarrow{\iota} S^\bullet(M, \mathbb{C}/\Lambda)\}\).

When \(X\) is a complex manifold, of the form \(X = \overline{X} - D\), with \(D\) a divisor with normal crossings \(\overline{X}\), we can incorporate the Hodge filtration and growth conditions on \(A^\bullet(X)\) to define subgroups of \(\hat{H}^k(X, \Lambda)\). Specifically, take

\[(4.2.1) \quad F^p\hat{H}^k(\overline{X} \log D, \Lambda) = H^{k-1}(\text{cone}\{F^p A^\geq k(\overline{X} \log D) \xrightarrow{\iota} S^\bullet(\overline{X}, \mathbb{C}/\Lambda)\})\]

and

\[\hat{H}^k(\overline{X} \log D, \Lambda) = F^0\hat{H}^k(\overline{X} \log D, \Lambda).\]

It is a subgroup of \(\hat{H}^k(X, \Lambda)\). When \(\overline{X}\) is compact, there is an exact sequence:

\[(4.2.2) \quad 0 \rightarrow H^k(X, \mathbb{C}/\Lambda) \rightarrow \hat{H}^k(\overline{X} \log D, \Lambda) \rightarrow F^p A^k(\overline{X} \log D, \Lambda) \rightarrow 0\]

where \(A^k(\overline{X} \log D, \Lambda)\) is the set of closed elements of \(A^\bullet(\overline{X} \log D)\) whose periods lie in \(\Lambda\).

The reason for introducing (4.2.1) comes from its similarity to (4.1.2). The cone in (4.2.1) is a subset of the cone in (4.1.1); for \(k = 2p\), one obtains, for \(X\) algebraic and for any \(\Lambda\), a functorial exact sequence:

\[(4.2.3) \quad F^p A^{2p-1}(\overline{X} \log D) \rightarrow F^p\hat{H}^{2p}(\overline{X} \log D, \Lambda) \rightarrow H^{2p}_D(X, \Lambda) \rightarrow 0.\]

4.3. Meromorphic equivalence. Though it may seem premature, it is helpful to leave the setting of algebraic varieties and algebraic vector bundle for awhile.

Let \(X\) be a compactifiable complex manifold, in the sense that \(X\) admits a compactification \(\overline{X}\) that is a compact manifold, and for which \(D = \overline{X} - X\) is an analytic subvariety. We may then modify \(\overline{X}\) by blow-ups with smooth center [25] to make \(D\) a divisor with normal crossings.

It becomes necessary to divide the compactifications of \(X\) into meromorphic equivalence classes.

**Definition 4.3.1.** Two compactifications \(\overline{X}_1\) and \(\overline{X}_2\) of \(X\) are said to be **meromorphically equivalent** if there exists a compactification \(\overline{X}_3\) of \(X\) and morphisms of compactifications, i.e., extensions of the identity map of \(X\), \(\overline{X}_3 \rightarrow \overline{X}_1\) and \(\overline{X}_3 \rightarrow \overline{X}_2\).
The above is easily seen to be an equivalence relation.

We point out that (the underlying complex manifold of) a smooth algebraic variety can admit compactifications that are not meromorphically equivalent to the algebraic ones. A simple example of this is provided by \( (\mathbb{C}^p+q \times C) \):

**Example 4.3.2.** Let \( C \) be an elliptic curve. Then \( X = \mathbb{C}^p+q \times C \) admits non-algebraic compactifications \( \overline{X}_{p,q} \) \((p > 0, 0 \leq q \leq p)\) with the following properties:

(i) There is a principal \( C \)-fibration \( \pi : \overline{X}_{p,q} \to \mathbb{P}^p(\mathbb{C}) \times \mathbb{P}^q(\mathbb{C}) \).

(ii) Every meromorphic function on \( \overline{X}_{p,q} \) is a pullback from \( \mathbb{P}^p(\mathbb{C}) \times \mathbb{P}^q(\mathbb{C}) \); the algebraic dimension of \( \overline{X}_{p,q} \) is less than its complex dimension, so it is not algebraic.

(iii) \( \overline{X}_{p,q} \) is diffeomorphic to the product of spheres \( S^{2p+1} \times S^{2q+1} \). It follows that \( H^2(\overline{X}_{p,q}, \mathbb{C}) = 0 \), and thus \( \overline{X}_{p,q} \) is not Kähler.

Furthermore, it is possible for a complex manifold to admit more than one algebraic structure; it can have inequivalent algebraic compactifications:

**Example 4.3.3.** Let \( C \) again be an elliptic curve. The universal vector extension of \( C \), as a complex manifold, can be given Hodge-theoretically as follows. Put \( H_Z = H^1(C, \mathbb{Z}) \), \( H = H^1(C, \mathbb{C}) \), and \( F = F^1 H^1(C, \mathbb{C}) \). There is a short exact sequence of abelian groups:

\[
0 \to F \to H/H_Z \to H/(H_Z + F) \to 0,
\]

which is isomorphic to

\[
0 \to \mathbb{C} \to (\mathbb{C}^*)^2 \to C \to 0.
\]

The associated \( \mathbb{P}^1 \)-bundle:

\[
\begin{array}{ccc}
\mathbb{P}^1 & \longrightarrow & \overline{X} \\
& & \downarrow \\
& & C
\end{array}
\]

then gives an algebraic variety \( \overline{X} \) that is a compactification of \( X = (\mathbb{C}^*)^2 \). However, \( \overline{X} \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \) are not meromorphically (birationally) equivalent, for they have non-isomorphic function fields; only the latter one is an algebraic completion of the algebraic variety \( X \).

We also recall the notion of a meromorphic mapping, from complex geometry.

**Definition 4.3.4.** A map \( f : X \to Y \) is said to be *meromorphic* with respect to \( \overline{X} \) and \( \overline{Y} \) (and one writes \( f : \overline{X} \to \overline{Y} \)) if the closure of the graph of \( f \) in \( \overline{X} \times \overline{Y} \) is a subvariety of \( \overline{X} \times \overline{Y} \).

Because of the existence of resolutions of singularities for complex analytic varieties (\( \mathbb{C} \)), two compactifications of \( X \) are meromorphically equivalent if and only if the identity map of \( X \) defines a meromorphic map between them.

There is an obvious notion of the *extendability* of a holomorphic vector bundle \( E \) on \( X \) to \( \overline{X} \). We next recall the notion of meromorphic equivalence of bundle extensions; it is presented in a somewhat different, though equivalent, manner in [12, p. 65ff].
**Definition 4.3.5.** Let $\overline{X}$ be a complex manifold, $D$ a divisor on $\overline{X}$, and put $X = \overline{X} - D$. Let $E$ be a vector bundle on $X$. Two extensions $\overline{E}$ and $\overline{E}'$ of $E$ to a vector bundle on $\overline{X}$ are said to be meromorphically equivalent if, whenever $U$ is a nice open neighborhood of a point of $D$ in $\overline{X}$, on which both bundles are trivial, the change-of-frame matrix from one to the other (a priori holomorphic on $U \cap X$) is meromorphic (i.e., does not have essential singularities) along $U \cap D$.

**Remark 4.3.6.** If $M$ is a meromorphic equivalence class of compactifications of $X$, there are corresponding notions of the $M$-extendability of $E$ and $M$-meromorphic equivalence classes of extensions of $E$.

One can always regard a vector bundle as a complex manifold, forgetting the linear structure. The following is almost a tautology:

**Lemma 4.3.7.** With notation as above,

(i) Two bundle extensions $\overline{E}$ and $\overline{E}'$ are meromorphically equivalent if and only if $\overline{E}$ and $\overline{E}'$ are equivalent partial compactifications of $E$.

(ii) If $\overline{E}$ and $\overline{E}'$ are meromorphically equivalent, then the meromorphic map $\overline{E} \rightarrow \overline{E}'$ induces a meromorphic map $\mathbb{P}(\overline{E}) \rightarrow \mathbb{P}(\overline{E}')$.

**Remark 4.3.8.** When $X$ is an algebraic manifold, there is a canonical equivalence class of compactifications of $X$, namely the smooth algebraic completions. Algebraic vector bundles admit algebraic extensions to suitable completions, and this class of extensions is likewise canonical. We understand algebraic extensions of algebraic vector bundles on an algebraic variety if it is not specified otherwise.

### 4.4. $F^1$-connections

Throughout, we let $X \subseteq \overline{X}$ be as in 4.2, and $E$ a vector bundle on $X$.

**Definition 4.4.1.** An $F^1$-connection on $E$ (relative to $\overline{X}$) is a connection for which there exists a vector bundle $\overline{E}$ over $\overline{X}$ such that

(i) the restriction of $\overline{E}$ to $X$ is $E$;

(ii) the local connection forms (with respect to holomorphic frames of $\overline{E}$), and therefore also the curvature, lie in $F^1A^\bullet(\overline{X} \log D, \text{End}(E))$.

We define a sort of “universal $F^1$-connection” in Appendix B. Note that 4.4.1 coincides with 3.0.26 if and only if $X = \overline{X}$.

**Remark 4.4.2.** This notion occurs in a well-known fact about Hermitian geometry (see [21, p.73]): If $E$ is an Hermitian vector bundle over the compact complex manifold $X$, there exists a unique $F^1$-connection on $E$ with respect to which the Hermitian metric on $E$ is horizontal. In particular, an extendable vector bundle always admits an $F^1$-connection (without singularities).

The following is evident:

**Proposition 4.4.3.** If $\nabla$ is an $F^1$-connection on $E$ (relative to $\overline{X}$), then the Cheeger-Simons class $\hat{c}_p(E, \nabla)$ is in the subgroup $F^p\hat{H}^{2p}(\overline{X} \log D, \Lambda)$ of $\hat{H}^{2p}(X, \Lambda)$.

To be able to talk sensibly about $F^1$-connections, one needs to know the following:
Proposition 4.4.4. Let $\bar{X}$ be a compact complex manifold, $D$ a divisor with normal crossings on $\bar{X}$, and put $X = \bar{X} - D$. Let $(E, \nabla)$ be a vector bundle with connection on $X$. Suppose that $E$ and $E'$ are vector bundles on $\bar{X}$ with respect to which $\nabla$ is an $F^1$-connection. Then $E$ and $E'$ are meromorphically equivalent.

Proof. The issue is fairly elementary. Let $U$ be open in $\bar{X}$. If $\omega$ is the connection form with respect to a frame of $E$ on $U \subset \bar{X}$, and $\omega'$ is the connection form with respect to a frame of $E'$ on $U$, suppose that $\omega$ and $\omega'$ have first order-poles on $U \cap D$. Let $A$ be the matrix expressing the frame for $E'$ in terms of the one for $E$ (thus $\omega' = A\omega A^{-1} + (dA)A^{-1}$). We wish to conclude that the entries of $A$ must be meromorphic along $U \cap D$.

Like holomorphy, meromorphy can be detected by a curve test, i.e., a function of several variables is meromorphic if and only if its restriction to sufficient curves is—e.g., if it is meromorphic in each variable separately—(see [32]; cf. [12, II, 4.1.1]). Since connections are functorial, we may assume that $U$ is the unit disc in $\mathbb{C}$, and $D = \{0\}$. In this case, the assertion is proved in [12, II, 1.19].

Remark 4.4.5. Note that Prop. 4.4.4 does not say that there exists $E$ for which the connection form is logarithmic; nor does it say that if such $E$ exists, the connection form has logarithmic poles for all $E'$ meromorphically equivalent to $E$ (counterexamples abound). Also, there is no contradiction between Remark 4.4.2 and Prop. 4.4.4.

According to [12, II, (5.2)], given any compactification $\bar{X}$ of $X$, a flat bundle $E$ on $X$ admits a vector bundle extension $\bar{E}$ to $\bar{X}$ with respect to which the flat connection can be seen to be an $F^1$-connection. In fact, according to 4.4.4 any two such $\bar{E}$ are meromorphically equivalent. We point out that $\bar{E}$ can be determined from constructions that are local on $\bar{X}$ along $D$. Local connection forms are then to be computed on $\bar{X}$ in terms of local frames of $\bar{E}$. In particular, there is no analogue of 4.4.4 for equivalence classes of compactifications.

On the other hand, if one starts with an $F^1$-connection on an algebraic vector bundle on $X$, relative to an algebraic compactification $\bar{X}$, it does not follow that the meromorphic equivalence class of extensions of $E$ distinguished by 4.4.4 are the algebraic vector bundles on $\bar{X}$. However, the two are known to coincide in an important class of examples, namely the flat bundles “coming from algebraic geometry.” By that, one means that the fibers of $E$ are cohomology groups for a family of algebraic varieties over $X$, and $\nabla$ is the Gauss-Manin connection. This result is commonly called the regularity theorem in algebraic geometry. (See [26] and its generalizations, e.g., [34, §5].)

The following should shed some light on the issue.

Example 4.4.6. Let $j : X \hookrightarrow \bar{X}$ denote the inclusion. Given the holomorphic vector bundle $E$, the specification of $\bar{E}$ is equivalent to selecting a locally-free subsheaf $(\mathcal{O}_{\bar{X}}(E))$ of $j_*\mathcal{O}_X(E)$ on $\bar{X}$.

Suppose that $\dim X = 1$ (so $\bar{X}$ is automatically algebraic). Let $\Delta \cong U \subset \bar{X}$ be a disc, with coordinate $t$, for which the restriction $j|U$ is $\Delta^* \hookrightarrow \Delta$. Also, take $E$ to be a line bundle. Restricting to $U$, we consider the two inequivalent extensions

3Since we will be reducing to curves in $U$, there is actually no need to assume that the poles are logarithmic, for the two notions coincide on a curve.
of $E$, $\overline{E} = O_\Delta$ and $\overline{E} = O_\Delta \cdot e^{t - 1}$. Let $\nabla$ be $\frac{d}{dt}$ (connection matrix $\omega = 0$) and $\nabla' = \nabla + \omega'$, where $\omega' = -t^{-2}dt$. Clearly, $\nabla$ is $F^1$ with respect to $\overline{E}$ (indeed with respect to $\ell^k\overline{E}$ for any $k \in \mathbb{Z}$); whereas $\nabla'$, not $F^1$ with respect to $\overline{E}$, is $F^1$ with respect to $\overline{E}'$. (We leave it to the reader to squelch the possible misconception that the preceding contradicts GAGA.) Note that the preceding discussion is global when $X = C^*$.

Fixing a meromorphic equivalence class $\mathcal{M}$ of compactifications of $X$, we use (4.2.1) to define

$$ F^p \hat{H}^{2p}(X, \Lambda)_{\mathcal{M}, \log} = \lim F^p \hat{H}^{2p}(\overline{X} \log D, \Lambda), $$

where the limit is taken over $\overline{X} \in \mathcal{M}$; and $F^p A^{2p-1}(X, \Lambda)_{\mathcal{M}, \log}$ is defined analogously.

The following variant of 4.4.1 is inevitable.

**Definition 4.4.8.** An $F^1$-connection on $E$ (relative to $\mathcal{M}$) is a connection that is $F^1$ relative to some member of $\mathcal{M}$.

Then 4.4.3 yields immediately:

**Proposition 4.4.9.** If $\nabla$ is an $F^1$-connection on $E$ relative to $\mathcal{M}$, then the Cheeger-Simons class $\hat{c}_p(E, \nabla)$ lies in the subgroup $F^p \hat{H}^{2p}(X, \Lambda)_{\mathcal{M}, \log}$ of $\hat{H}^{2p}(X, \Lambda)$.

When $X$ is an algebraic manifold and $\mathcal{M}$ is the class generated by algebraic completions, we drop the subscript "$\mathcal{M}$" in (4.4.7)–(4.4.9); cf. 4.3.8.

**4.5. Proof of Theorem 2 (quasi-projective case).** We return to algebraic manifolds. First let $X$ be a smooth projective variety (a fortiori compact), and $E$ an algebraic vector bundle on $X$. There exists an ample line bundle $L$ on $X$ such that both $L$ and $E' = E \otimes L$ are generated by global sections, so that both vector bundles are pullbacks of universal bundles on Grassmannians under holomorphic maps. We get a pullback diagram:

$$
\begin{array}{ccc}
E & \rightarrow & p_1^* U_n \otimes p_2^* U_1^{-1} \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & G_C(n) \times G_C(1)
\end{array}
$$

in which $n$ denotes the rank of $E$, $p_1 \circ g$ classifies $E'$, and $p_2 \circ g$ classifies $L$.

Also, choose any $F^1$-connection $\nabla^L$ on $L$. The latter, together with any given $F^1$-connection $\nabla^{E'}$ on $E'$, induces the tensor product $F^1$-connection

$$\nabla^{E'}(e \otimes \ell) = \nabla^E e \otimes \ell + e \otimes \nabla^L \ell.$$  

Likewise, the tensor product of $\nabla^{E'}$ and the dual of $\nabla^L$ is just $\nabla^E$ again (under the isomorphism $E \cong E' \otimes L^{-1}$).

As the image in Deligne cohomology is independent of the $F^1$-connection (see 1.1.4), we may assume that the connections are pullbacks of $F^1$-connections on $G_C(n)$ and $G_C(1)$, which exist by 1.4.2. Since $G_C(n) \times G_C(1)$ satisfies the hypothesis of 3.2.7 and 3.1.4, (ii), the assertion of Theorem 2 holds for $p_1^* U_n \otimes p_2^* U_1^{-1}$, so by functoriality for $E$. This completes the proof when $X$ is compact.

\footnote{4 in terms of 3.0.28, \ the tensor product construction corresponds to $\text{Hom}(T^{1.0} Y, g) \times \text{Hom}(T^{1.0} Y, \mathbb{C}) \rightarrow \text{Hom}(T^{1.0} Y, g)$.}
We also use functoriality to get the assertion for non-compact \( X \). Specifically, let \( \overline{X} \) be an algebraic completion of the sort considered in \( \text{[1.2]} \) for which \( E \) extends to an algebraic vector bundle \( E \) on \( \overline{X} \). Then there is a commutative diagram:

\[
\begin{array}{c c c c c c}
F^p \tilde{H}^2(p)(\overline{X}, \mathbb{Z}(p)) & \longrightarrow & H^2_p(\overline{X}, \mathbb{Z}(p)) \\
\downarrow & & & & \\
F^p \tilde{H}^2(p)(\overline{X} \log D, \mathbb{Z}(p)) & \longrightarrow & H^2_p(\overline{X}, \mathbb{Z}(p))
\end{array}
\]

(4.5.1)

Let \( \nabla' \) be any \( F^1 \)-connection on \( E \). By functoriality, \( \tilde{c}_p(E, \nabla') \) restricts to \( \tilde{c}_p(E, \nabla) \) and \( c^p_\ell(E) \) maps to \( c^p_\ell(E) \). By \( \text{[4.6.3]} \) again, \( \tilde{c}_p(E, \nabla) \) has the same image in \( H^2_p(X, \mathbb{Z}(p)) \) as \( \tilde{c}_p(E, \nabla) \), and we are done.

### 4.6. General algebraic manifolds.

We wish to generalize Theorem 2 to the context of general algebraic manifolds, i.e., to ones that are not quasi-projective, so need not even be Kähler. This is done by reducing to the quasi-projective case as we next describe.

**Proposition 4.6.1.** Suppose that \( f : Y \to X \) is a morphism of algebraic varieties satisfying:

(i) \( f^* : H^{2p-1}(X, \mathbb{Z})/\text{tors} \to H^{2p-1}(Y, \mathbb{Z})/\text{tors} \) is injective with image a direct summand in the sense of mixed Hodge structures over \( \mathbb{Z} \);

(ii) \( f^* : H^{2p}(X, \mathbb{Z})_{\text{tors}} \to H^{2p}(Y, \mathbb{Z})_{\text{tors}} \) is injective.

Then \( f^* : H^2_p(X, \mathbb{Z}(p)) \to H^2_p(Y, \mathbb{Z}(p)) \) is injective.

**Proof.** The exact sequence \( \text{[4.1.3]} \) gives a commutative diagram with exact rows:

\[
\begin{array}{c c c c c c}
0 & \longrightarrow & J^p(X) & \longrightarrow & H^2_D(X, \mathbb{Z}(p)) & \longrightarrow & F^p H^2(X, \mathbb{C}) \\
\downarrow & & & & & & \\
0 & \longrightarrow & J^p(Y) & \longrightarrow & H^2_D(Y, \mathbb{Z}(p)) & \longrightarrow & F^p H^2(Y, \mathbb{C})
\end{array}
\]

(4.6.2)

The following is standard:

**Lemma 4.6.3.** Given the commutative diagram with exact rows:

\[
\begin{array}{c c c c c c}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\
& & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C'
\end{array}
\]

A sufficient condition for the injectivity of \( \beta \) is that \( \alpha \) and \( \gamma \) be injective.

We apply the lemma twice. The hypotheses imply that the rightmost vertical arrow in \( \text{[1.6.3]} \) is an injection. Thus, to reach the conclusion, it suffices by \( \text{[4.6.3]} \) to show that leftmost arrow is injective. Recall that \( J^p \) in the above is an extension by the intermediate Jacobian \( J^p \), having the functorial exact sequence

\[
0 \to J^p(X) \to J^p(X) \to H^2(p)(X; \mathbb{Z}(p))_{\text{tors}} \to 0.
\]

The hypothesis implies the injectivity of \( J^p(X) \to J^p(Y) \) and of \( H^2(p)(X; \mathbb{Z}(p))_{\text{tors}} \to H^2(p)(Y; \mathbb{Z}(p))_{\text{tors}} \). By \( \text{[4.6.3]} \), we are done.

**Corollary 4.6.4.** Under the hypotheses of Proposition \( \text{[4.6.1]} \), if the conclusion of Theorem 2 holds for \( Y \), then it also holds for \( X \).
Proof. Consider the diagram that one gets by functoriality:

\[
\begin{array}{ccc}
F^p \tilde{H}^{2p}(X) & \longrightarrow & H^2_B(X, \mathbb{Z}(p)) \\
\downarrow & & \downarrow \\
F^p \tilde{H}^{2p}(Y) & \longrightarrow & H^2_B(Y, \mathbb{Z}(p))
\end{array}
\]

(4.6.5)

Then \(\hat{c}_p(f^*E, \nabla) \in F^p \tilde{H}^{2p}(X)\) maps to \(c^B_p(f^*E) = f^*(c^B_p(E))\). Since the right vertical map is injective by 4.6.4, \(\hat{c}_p(E, \nabla)\) can only map to \(c^B_p(E)\) in \(H^2_B(X, \mathbb{Z}(p))\).

We next show that such \(Y\) exist:

**Proposition 4.6.6.**

(i) Given a smooth algebraic variety \(X\), there is a smooth quasi-projective variety \(Y\) with surjective birational map \(Y \to X\).

(ii) Under the conditions of (i), there is a direct sum decomposition in the derived category of \(X\):

\[
Rf_*Z_Y \cong Z_X \oplus \ker \text{Tr}(f).
\]

(iii) In particular, \(H^*(Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z}) \oplus \ker f_*\), and the corresponding mixed Hodge structures are isomorphic over \(\mathbb{Z}\).

**Proof.** The argument for (i) was used already in [13, II:(3.2)] for extending Hodge theory to general algebraic varieties, and we sketch it here. By a theorem of Nagata and the Chow Lemma, \(X\) fits into a cartesian diagram

\[
\begin{array}{ccc}
Y & \rightarrow & \overline{Y} \\
\downarrow f & & \downarrow \overline{f} \\
X & \rightarrow & \overline{X}
\end{array}
\]

(4.6.7)

with \(\overline{X}\) a completion of \(X\), \(\overline{Y}\) projective, and \(\overline{f}\) birational. (By resolution of singularities [24], we may assume that \(\overline{X}\) and \(\overline{Y}\) are smooth, and \(\overline{X} - X\) and \(\overline{Y} - Y\) are divisors with normal crossings.)

Statement (ii) is a topological assertion that follows from the identity \(\text{Tr}(f) \circ f^* = 1_X\). Then (iii) follows immediately.

Combining Propositions 4.6.1 and 4.6.6, and keeping in mind our convention 4.3.8, we obtain:

**Theorem 4.6.8.** If \(X\) is a non-singular complex algebraic variety, \(E\) an algebraic vector bundle on \(X\) that extends to an algebraic vector bundle on some completion of \(X\), and \(\nabla\) an \(F^1\)-connection on \(E\) (relative to said extension of \(E\)), then \(c^B_p(E)\) is the image of \(\hat{c}_p(E, \nabla)\) for all \(p \geq 1\).

**Corollary 4.6.9.** The conclusion of Theorem 4 holds for all non-singular algebraic varieties.

**4.7. Chern classes in \(\mathfrak{M}\)-DB-cohomology.** We wish to define the analogue of \(c^B_p(E)\) in the complex analytic setting. Fix the meromorphic equivalence class \(\mathfrak{M}\) of compactifications \(\overline{X}\) of \(X\). We can use the formula in Def. 4.1.1 to define \(H^*_B(X, \overline{X}; \Lambda(p))\). When \(\overline{X}\) is Kähler, this is seen to be independent of \(\overline{X} \in \mathfrak{M}\), for the same reason it was true in 4.1 (see 4.7.9), so we rename it:
**Definition 4.7.1.** The \(\mathcal{M}\)-DB-cohomology \(H^*_D(X, \Lambda(p))_{\mathcal{M}}\) of \(X\) with coefficients in \(\Lambda(p)\) is the common value of \(H^*_D(X, X; \Lambda(p))\) for \(X \in \mathcal{M}\).

Then (4.2.3) gives rise to the exact sequence:

\[
F^p A^{2p-1}(X, \Lambda)_{\mathcal{M}, \text{log}} \to F^p \widehat{H}^{2p}(X, \Lambda)_{\mathcal{M}, \text{log}} \to H^{2p}_D(X, \Lambda)_{\mathcal{M}} \to 0.
\]

Next, fix a meromorphic equivalence class \(\mathcal{E}\) of extensions of \(E\) to the members of \(\mathcal{M}\) (note that \(\mathcal{E}\) subsumes \(\mathcal{M}\)). We call these the \(\mathcal{E}\)-extensions of the vector bundle \(E\) on \(X\). We will construct Chern classes

\[c^B_p(E; \mathcal{E}) \in H^*_D(X, Z(p))_{\mathcal{M}}\]

in more or less the same way as in the algebraic setting. Choose \(X \in \mathcal{M}\) to which \(E\) extends to \(\overline{E} \in \mathcal{E}\), and let \(j: X \hookrightarrow \overline{X}\) denote the inclusion.

As usual, we consider first the case where \(E\) is a line bundle. Then there is a short exact sequence of complexes of sheaves on \(\overline{X}\):

\[
0 \to \text{cone}(\mathcal{Z}_\overline{X}(1) \to \mathcal{O}_\overline{X})[-1] \to \text{cone}(Rj_* \mathcal{Z}_X(1) \to \mathcal{O}_\overline{X})[-1] \to \text{cone}(\mathcal{Z}_\overline{X}(1) \to Rj_* \mathcal{Z}_X(1)) \to 0.
\]

From this, we extract the following commutative diagram:

\[
\begin{array}{c}
H^2_D(\overline{X}, \mathcal{Z}(1)) \longrightarrow H^2_D(X, \mathcal{Z}(1))_{\mathcal{M}} \\
\uparrow d \log \quad \uparrow c^B_p(\mathcal{E}) \\
H^1(\overline{X}, \mathcal{O}_\overline{X}^\vee) \longrightarrow H^1(X, \mathcal{O}_X^\vee)_{\mathcal{E}} \hookrightarrow H^1(X, \mathcal{O}_X^\vee),
\end{array}
\]

where \(H^1(X, \mathcal{O}_X^\vee)_{\mathcal{E}}\) denotes the \(\mathcal{E}\)-extendable line bundles on \(X\). Since any two extensions in \(\mathcal{E}\) differ by a divisor class supported on \(D\), we see that the Chern class \(c^B_p(E; \mathcal{E}) \in H^2_D(X, Z(1))_{\mathcal{M}}\) of an \(\mathcal{E}\)-extendable line bundle \(E\) is well-defined.

To obtain the same for higher-rank \(E\), one invokes the splitting principle. In the formulation of [22, p. 140, A1], one must know 4.7.6 below. Because we will invoke a little Hodge theory in the argument, we must impose a condition on \(X\).

**Definition 4.7.5.** A complex manifold \(X\) is said to be quasi-c-Kähler if it admits a compactification \(\overline{X}\) that is a compact Kähler manifold, and for which \(D = \overline{X} - X\) is an analytic subvariety.

When \(X\) is quasi-c-Kähler, one can then modify such \(\overline{X}\) by a sequence of blow-ups with smooth center, a process that preserves the Kähler condition, to make \(D\) a divisor with normal crossings \([25]\).

Let \(\psi: \mathcal{P}(E) \to X\) be the projectivization of \(E\), \(L\) the tautological line sub-bundle \(\mathcal{O}(1)\) of \(\psi^* (E)\), and \(\mathcal{M}\) the meromorphic equivalence class of \(\{\mathcal{P}(\overline{E}) : \overline{E} \in \mathcal{E}\}\) an \(\mathcal{E}\)-extension of \(E\).

**Proposition 4.7.6.** Let \(X\) be quasi-c-Kähler, and \(\mathcal{M}\) a meromorphic equivalence class of Kähler compactifications. Let \(\xi = c_1(L; \mathcal{M})\). Then there is an isomorphism of additive groups

\[
H^*_D(\mathcal{P}(E), Z(p))_{\mathcal{M}} \cong \bigoplus_{0 \leq j < r} H^*_D(X, Z(p))_{\mathcal{M}} \cdot \xi^j,
\]

where \(r\) is the rank of \(E\); in other words, there is an injective map

\[
H^*_D(X, Z(p))_{\mathcal{M}} \to H^*_D(\mathcal{P}(E), Z(p))_{\mathcal{M}},
\]
and the right-hand side of the above is freely generated, as a module over the left-hand side, by \( \{ \xi^j : 0 \leq j < r \} \).

**Proof.** We want to reduce the assertion to more elementary cohomology for which one already knows the splitting principle. As in 4.1.2, there is a long exact sequence:

\[
\cdots \to H^{2p-1}(F^p A^\bullet(X \log D)) \to H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)) \\
\to H^{2p}_M(X, \mathbb{Z}(p)) \to H^{2p}(F^p A^\bullet(X \log D)) \to \cdots
\]

Next, understanding \( \xi \) to denote generically the Chern class of \( L \) in any cohomology group, multiply \( 4.7.7 \) by \( \xi^j \) \((0 \leq j < r)\) and take the direct sum over \( j \) yields

\[
\oplus H^{2p-1-2j}(X, \mathbb{C}/\mathbb{Z}(p-j)) \cdot \xi^j \to \oplus H^{2p-2j}(X, \mathbb{Z}(p-j)) \cdot \xi^j \to \oplus H^{2p-2j}(F^p_i A^\bullet(X \log D)) \cdot \xi^j \downarrow \\
H^{2p-1}(P(E), \mathbb{C}/\mathbb{Z}(p)) \to H^{2p}_M(P(E), \mathbb{Z}(p)) \to H^{2p}(F^p A^\bullet(P(E) \log D))
\]

Now, we know the splitting principle for \( \mathbb{Z} \)-coefficients, hence also for \( \mathbb{Z}(p) \)- and \( \mathbb{C} \)-coefficients, and then also for \( (\mathbb{C}/\mathbb{Z}(p)) \)-coefficients. In other words, the left vertical arrow in the display \( 4.7.8 \) is an isomorphism. The easiest way to deal with the one on the right is to use Hodge theory: there is, in general, a canonical surjection

\[
H^{2k}(F^k A^\bullet(X \log D)) \to F^k H^{2k}(X, \mathbb{C}),
\]

and this is an isomorphism when \( X \) is Kähler. Thus, the right vertical arrow is the Hodge filtered version of the one for \( \mathbb{C} \)-coefficients, so it is likewise an isomorphism. We conclude by the five-lemma that our assertion holds.

The above determines 4.7.3. We recall that \( c_p(E; \mathcal{E}) \) is, up to a sign, the coefficient in \( \mathbb{H}^{2p}_M(X, \mathbb{Z}(p)) \) of \( \xi^r-p \) in the formula for \( \xi^r \) in terms of the additive decomposition given in 4.7.6.

Because of (6.1.4), we obtain the generalization of Theorem 3 to the Kähler case:

**Theorem 4.7.10.** Let \( X \) be a quasi-c-Kähler manifold, \( \mathcal{M} \) an equivalence class of Kähler compactifications of \( X \), \( E \) an \( \mathcal{M} \)-extendable vector bundle on \( X \), and \( \nabla \) an \( F^1 \)-connection on \( E \). Then \( c_p^E(E; \mathcal{E}) \) is the image of \( \hat{c}_p(E, \nabla) \) for all \( p \geq 1 \). Here, \( \mathcal{E} \) is the unique equivalence class of \( \mathcal{M} \)-extensions of \( E \) implied by 4.4.4.

**Corollary 4.7.11.** The conclusion of Theorem 1 holds when \( X \) is a quasi-c-Kähler manifolds for the \( \mathcal{M} \)-DB Chern classes of flat vector bundles that are regular with respect to \( \mathcal{E} \).

5. Proof of Theorem 3

Throughout this section, the complexification \( \mathfrak{g} \otimes \mathbb{C} \) of a real Lie algebra \( \mathfrak{g} \) will be denoted \( \mathfrak{g}_{\mathbb{C}} \). In addition, we make use of simplicial methods; all relevant definitions can be found in Appendix C.
5.1. Continuous cohomology. The continuous cohomology $H_{cts}^\bullet(G, V)$ of a topological group $G$ with coefficients in a real topological vector space $V$, on which $G$ acts, is defined to be the cohomology of the complex $C_{cts}^\bullet(G, V)$, whose elements in degree $m$ are $G$-equivariant continuous maps $f : G^{m+1} \to V$ (with the usual coboundary map).

Now let $G$ be the real points of a reductive algebraic group defined over $\mathbb{R}$, and $K$ a maximal compact subgroup. We recall that the space $G/K$ is contractible, and thus $G$ and $K$ are of the same homotopy type. Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra of $G$. By van Est’s Theorem \[19\], whenever $V$ is finite-dimensional with trivial $G$-action, there is a canonical isomorphism

\[
H_{cts}^\bullet(G, V) \cong H^\bullet(\mathfrak{g}, K) \otimes V. 
\]

Here, the right-hand side is the relative Lie algebra cohomology, and is canonically isomorphic to $(\wedge^\bullet \mathfrak{p}^*)^K \otimes V$, the complex of $G$-invariant $V$-valued differentials forms on $G/K$. The isomorphism can be realized as follows. Fix a point $e$ of $G/K$, and endow the latter with a $G$-invariant Riemannian metric. For $(g_0, \ldots, g_m) \in G^{m+1}$, define $\Delta_e(g_0, \ldots, g_m)$ to be the “geodesic simplex” in $G/K$ with vertices $g_0 e, \ldots, g_m e$; it is constructed inductively as the cone swept out by all geodesics from $g_0 e$ to $\Delta_e(g_1, \ldots, g_m)$. (Note that, in general, the order of the vertices makes a difference, since the curvature may not be constant.) Define a homomorphism

\[
\wedge^\bullet \mathfrak{p}^* \to C_{cts}^\bullet(G, \mathbb{R})
\]

by taking an $m$-form $\omega$ to the function

\[
(g_0, \ldots, g_m) \mapsto \int_{\Delta_e(g_0, \ldots, g_m)} \omega.
\]

This induces \eqref{5.1.1}

We now recall the standard trick for computing the continuous cohomology of a reductive group. The compact real form of $\mathfrak{g} \otimes \mathbb{C}$ is the Lie algebra $\mathfrak{u} = \mathfrak{t} \oplus i \mathfrak{p} \subseteq \mathfrak{g}_C$. Let $U$ be the corresponding subgroup of $G(\mathbb{C})$. Since $U$ is compact, there are canonical isomorphisms

\[
H^\bullet(U/K, V) \cong H^\bullet(\mathfrak{u}, K; V) \cong (\wedge^\bullet (i \mathfrak{p})^*)^K \otimes V.
\]

Composing these isomorphisms with the isomorphism

\[
\wedge^\bullet (i \mathfrak{p})^* \to \wedge^\bullet \mathfrak{p}^*
\]

induced by multiplication by $i$, we obtain an isomorphism

\[
H^\bullet(U/K, \mathbb{C}) \cong H_{cts}^\bullet(G, \mathbb{C}),
\]

which carries $H^m(U/K, \mathbb{R}(p))$ onto the subspace $i^m H_{cts}^m(G, \mathbb{R}(p))$ of $H_{cts}^m(G, \mathbb{C})$.

5.2. The Borel regulator elements. We now take $G$ to be $GL_n(\mathbb{C})$, viewed as a real group, and we take $K$ to be $U(n)$. In this case, one identifies the complexification of $G$ as $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$; the natural map $G \to G_\mathbb{C}$ is the homomorphism $\iota: GL_n(\mathbb{C}) \to GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$,

\[
\iota(g) = (g, \overline{g}).
\]

The corresponding map on Lie algebras

\[
\gll_n(\mathbb{C}) \to \gll_n(\mathbb{C}) \oplus \gll_n(\mathbb{C})
\]

takes $X$ to $(X, \overline{X})$; indeed, it is not hard to see that the identification

\[
\gll_n(\mathbb{C}) \otimes \mathbb{C} \cong \gll_n(\mathbb{C}) \oplus \gll_n(\mathbb{C})
\]
is given by: for \( X, Y \in \mathfrak{gl}_n(\mathbb{C}) \),
\[
5.2.3 \quad X \otimes 1 + Y \otimes i \mapsto (X + iY, \overline{X} + i\overline{Y}).
\]
It follows that the Lie algebra \( \mathfrak{u}_n \) of \( U(n) \) appears in \( 5.2.1 \) as the "anti-diagonal"
\[
\{(X, -iX) \mid X \in \mathfrak{u}_n\}.
\]

It is clear that the compact real form \( U \) of \( G_\mathbb{C} \) is \( U(n) \times U(n) \). The quotient space \( U/K \) is therefore \( (U(n) \times U(n))/U(n) \), which we identify with \( U(n) \) via the map induced by inclusion as the first factor of the product. Since \( GL_n(\mathbb{C}) \) and \( U(n) \) are homotopy-equivalent, we obtain from \( 5.1.3 \) an isomorphism
\[
H^\bullet(GL_n(\mathbb{C}), \mathbb{C}) \cong H^\bullet_{cts}(GL_n(\mathbb{C}), \mathbb{C})
\]
which takes \( H^m(GL_n(\mathbb{C}), \mathbb{R}(p)) \) onto \( i^mH^m_{cts}(GL_n(\mathbb{C}), \mathbb{R}(p)) \). This yields a canonical isomorphism
\[
5.2.4 \quad H^{2p-1}(GL_n(\mathbb{C}), \mathbb{R}(p)) \xrightarrow{\sim} H^p_{cts}(GL_n(\mathbb{C}), \mathbb{C}/\mathbb{R}(p)).
\]

The real cohomology of \( GL_n(\mathbb{C}) \) is an exterior algebra on generators \( x_1, \ldots, x_n \), where \( x_p \) is in degree \( 2p - 1 \). We can choose \( x_p \) canonically by insisting that it be the unique (necessarily primitive) class which transgresses, via
\[
d_{2p} : H^0(BGL_n(\mathbb{C})) \rightarrow H^{2p-1}(GL_n(\mathbb{C})),
\]
to the universal Chern class \( c_p \in H^{2p}(BGL_n(\mathbb{C})) \) in the Leray spectral sequence associated to the universal \( GL_n(\mathbb{C}) \)-bundle; it lies in
\[
W_0H^{2p-1}(GL_n(\mathbb{C}), \mathbb{Z}(p)),
\]
as \( \mathbb{Z}(p) \) is of weight \(-2p\). Define \( b_p \in H^{2p-1}(BGL_n(\mathbb{C}), \mathbb{C}/\mathbb{R}(p)) \) to be the image of \( x_p \) under the composite of \( 5.2.4 \) and the natural map
\[
H^p_{cts}(GL_n(\mathbb{C}), \mathbb{C}/\mathbb{R}(p)) \rightarrow H^p_{cts}(GL_n(\mathbb{C}), \mathbb{C}/\mathbb{R}(p)) \cong H^p(BGL_n(\mathbb{C}), \mathbb{C}/\mathbb{R}(p)).
\]
This is called the \( p \)-th Borel regulator element.

### 5.3. The Weil algebra

Let \( G \) be a real reductive group and \( K \) any compact subgroup. Denote the corresponding Lie algebras by \( \mathfrak{g} \) and \( \mathfrak{k} \). Consider the (real) Weil algebra \( \mathfrak{w}(\mathfrak{g}) \)
\[
5.3.1 \quad \mathfrak{w}(\mathfrak{g}) = \wedge^\bullet \mathfrak{g}^* \otimes S^\bullet(\mathfrak{g}^*),
\]
a d.g.a. that is a \( \mathfrak{g} \)-module via the coadjoint action (here \( S \) denotes the symmetric algebra over \( \mathbb{R} \)). Denote the set of invariant polynomials \( S^\bullet(\mathfrak{g}^*)^G \), which is a subalgebra of \( \mathfrak{w}(\mathfrak{g}) \), by \( I^\bullet(G) \).

Let \( \omega \in A^1(P, \mathfrak{g}) \) be a connection on a principal bundle \( P \rightarrow M \) over a (simplicial) manifold, which will be viewed as a map \( \mathfrak{g}^* \rightarrow A^1(P) \). Then \( \omega \) extends uniquely to a d.g.a. homomorphism
\[
5.3.2 \quad k_P(\omega) : \mathfrak{w}(\mathfrak{g}) \rightarrow A^\bullet(P),
\]
for \( F \) a number field or \( \mathbb{R} \) or \( \mathbb{C} \).

---

\(^5\) Some prefer to define the Borel regulator element to be the class in \( H^{2p-1}(GL_n(\mathbb{C}), \mathbb{C}/\mathbb{R}(p)) \) that corresponds to the element \( y_p \) of
\[
W_0H^{2p-1}(GL_n(\mathbb{C}), \mathbb{Q}(p)) \cong \mathbb{Q} \oplus H^{2p-1}(GL_n(\mathbb{C}), \mathbb{Q}(p)) \cong H^{2p-1}(C_p - \{0\}, \mathbb{Q}(p)) \cong \mathbb{Q}
\]
that takes the value 1 on the generator of \( \pi_{2p-1}(GL_n(\mathbb{C})) \). Since \( x_p = \pm(p-1)!y_p \) \([24.5.2]\), it follows that \( y_p \) corresponds, up to a sign, to twice the degree \( p \) part of the Chern character
\[
ch_p : K_{2p-1}(F) \rightarrow H^2_{2p}(\text{Spec} F, \mathbb{R}(p)).
\]
such that for $\Phi \in S^p(g^\ast)$, 
\[ k_P(\omega)(\Phi) = \Phi(\Theta^p), \]
where $\Theta$ denotes the curvature of the connection.

The \textit{relative Weil algebra} $W(g, K)$ is the subspace of \textit{“$K$-basic”} elements \[(5.3.3) \quad [\wedge^\bullet (g/t)^\ast \otimes S^\bullet(g^\ast)]^K \]
of $W(g)$. One recovers \[5.3.1\] when $K$ is trivial; $W(g, K)$ is a contravariant functor of pairs $(g, K)$, and it contains $I^\bullet (G)$ for all $K$. A connection $\omega \in A^1(P, g)$ on a principal bundle $P \to M$ induces a d.g.a. homomorphism \textit{(Chern-Weil homomorphism)} \[(5.3.4) \quad k_P(\omega): W(g, K) \to A^\bullet (P/K) \]
by restriction of \[5.3.2\] The natural map $W(g, K) \to I^\bullet (K)$ (which factors through $I^\bullet (G)$) is a quasi-isomorphism \[8\], hence
\[ H^\bullet (W(g, K)) \cong H^\bullet (K, \mathbb{R}). \]

A consequence of this is that if $\Phi \in I^\bullet (G)$ has the property that its image in $I^\bullet (K)$ is zero, then there exists $T \in W(g, K)$ such that $dT = \Phi$. Note that when $K$ is \textit{maximal} compact, $K \hookrightarrow G$ is a homotopy equivalence, so $P/K \to P/G = M$ is a homotopy equivalence, and \[5.3.4\] induces a map \[ H^\bullet (G, \mathbb{R}) \to H^\bullet (M, \mathbb{R}). \]

For a real vector space $V$, denote the Weil algebra $W(V) \otimes V$ with coefficients in $V$ by $W_V(V)$, and the corresponding Weil algebra in the \textit{relative case} by $W_V(g, K)$. In particular, we have the Weil algebras $W_C(g)$ (\textit{quasi}-isomorphic to $A^\bullet (EG) \cong \mathbb{C}$) and $W_C(g, K)$, as well as $W_{\mathbb{R}(p)}(g, K)$. When $g$ and $t$ are \textit{complex} Lie algebras, one can also define the \textit{complex} Weil algebra $\mathfrak{W}(g, K)$ by doing the linear algebra in \[5.3.3\] over $\mathbb{C}$ instead of $\mathbb{R}$. We observe that when $g$ is a real Lie algebra, $W_C(g, K) \cong \mathfrak{W}(g_C, K_C)$ as $\mathbb{C}$-algebras.

\textbf{5.4. The class $\tau_p$ and Cheeger-Simons classes.} In this section, $g$ is $\mathfrak{gl}_n(\mathbb{C})$ viewed as a real Lie algebra, and $t = u_n$. Let 
\[ C_p(X) = (-1)^p \, \text{Tr}(\wedge^p X), \]
the invariant polynomial (of degree $p$) on $\mathfrak{gl}_n(\mathbb{C})$ which determines the $p$th Chern class. We can express it as 
\[ C_p(X) = P_p(X) + Q_p(X), \]
where 
\[ P_p(X) = \frac{1}{2} \left[ C_p(X) + (-1)^p C_p(\overline{X}) \right] \quad \text{and} \quad Q_p(X) = \frac{1}{2} \left[ C_p(X) - (-1)^p C_p(\overline{X}) \right] \]
are elements of $S^p(g^\ast)$. Note that $P_p$ is $\mathbb{R}(p)$-valued, and $Q_p$ is $\mathbb{R}(p - 1)$-valued. If $X \in u_n$, then $\overline{X} = -t X$, so $Q_p(X) = 0$. It follows from the discussion in the previous section that there exists $T_p \in W_{\mathbb{R}(p)}(\mathfrak{gl}_n(\mathbb{C}), U(n))$ such that $dT_p = Q_p$.

The connection on the universal flat bundle $P \to BGL_n(\mathbb{C})$ induces a Chern-Weil homomorphism 
\[ k: W_C(\mathfrak{gl}_n(\mathbb{C}), U(n)) \to A^\bullet (P/U(n)). \]
Denote $k(T_p) \in A^{2p-1}(P/U(n))$ by $\tau_p$; the curvature $\Theta$ of $P$ is zero. We have 
\[ d\tau_p = Q_p(\Theta) = 0, \]
and thus $\tau_p$ defines a class in 
$$H^{2p-1}(P/U(n), \mathbb{C}) \cong H^{2p-1}(GL_n(\mathbb{C}), \mathbb{C}).$$

The first task is to show:

**Proposition 5.4.1.** The class of $\tau_p$ in $H^{2p-1}(GL_n(\mathbb{C}), \mathbb{C}/\mathbb{R}(p))$ is the negative of the universal real Cheeger-Simons class $\hat{c}_p$.

**Proof.** Let $E \to B$ be the model of the universal $GL_n(\mathbb{C})$-bundle constructed from $P \to BGL_n(\mathbb{C})$ (over the standard model of $BGL_n(\mathbb{C})$), as in §1.3. Recall that $E$ has a canonical connection, induced by that of $P$, and there is a tautological map $BGL_n(\mathbb{C}) \to B$ which classifies both the universal flat bundle and its connection. Consider the Chern-Weil homomorphism 
$$k_E : W_C(g\mathfrak{l}_n(\mathbb{C}), U(n)) \to A^\bullet(E/U(n))$$
associated to $E$ and its connection, writing $\tau_p^E$ for $k_E(T_p)$. Since the polynomial $P_p$ is $\mathbb{R}(p)$-valued,
$$C_p(\Theta_E) \equiv Q_p(\Theta_E) \mod \mathbb{R}(p);$$
and since $dT = Q_p$, it follows that $d\tau_p^E = Q_p(\Theta_E)$, so $(Q_p(\Theta_E), -\tau_p^E)$ represents the “universal Cheeger-Simons class”
$$\hat{c}_p \in \hat{H}^{2p}(E/U(n), \mathbb{R}(p)),$$
i.e., that of the pullback of $E \to B$ to $E/U(n) \simeq B$. The naturality of Chern-Weil construction implies that the diagram
$$W_C(g\mathfrak{l}_n(\mathbb{C}), U(n)) \xrightarrow{k} A^\bullet(E/U(n)) \xrightarrow{k} A^\bullet(P/U(n))$$
commutes. It follows that the Cheeger-Simons class of the pullback of $P \to BGL_n(\mathbb{C})$ to $P/U(n)$ is represented by $(Q_p(\Theta), -\tau_p) = (0, -\tau_p)$, and this gives the desired assertion. \(\square\)

**5.5. $\tau_p$ and $T_p$.** The next step is to obtain a formula for the cocycle given by $\tau_p$ in terms of $T_p$. For this we will need an explicit section of the bundle $P/U(n) \to BGL_n(\mathbb{C})$. It is more convenient to write down a section of $EGL_n(\mathbb{C})/U(n) \to BGL_n(\mathbb{C})$ and pull it back to the universal flat bundle. Actually, we will write down a section of the bundle $\pi : EG/K \to BG$, where $K$ is a maximal compact subgroup of an arbitrary real reductive group $G$, following 14.

The $m$-simplex $G^{m+1}$ of the usual model of $EG$ (see §1.3, and take $X_*$ to be a point) is identified with $G^{m+1}$ in the inhomogeneous, or reduced, model via the map 
$$(g_0, g_1, \ldots, g_m) \mapsto (g_0, g_1g_0^{-1}, \ldots, g_mg_{m-1}^{-1}).$$

This map is $G$-equivariant with respect to the diagonal right $G$-action on the left-hand side, and the right action of $G$ on the first factor of the right-hand side. It follows that in the reduced model, the space of $m$-simplices of $EG/K$ is $G/K \times G^m$. In particular, when $G = K$, we have that the space of $m$-simplices of the reduced model of $BG$ is $G^m$.

Suppose now that $G$ is reductive and that $K$ is a maximal compact subgroup. For each $m$, define a map (which is a homotopy equivalence)
$$s_m : G^m \times \Delta^m \to G/K \times G^m \times \Delta^m$$
(5.5.1)
by
\[
((g_1, \ldots, g_m), (t_0, \ldots, t_m)) \mapsto (\tilde{\Delta}_e((g_1 \cdot g_2 \cdots g_m g_{m-1}^{-1} \cdot g_1) \cdot (t_0, \ldots, t_m)), (g_1, \ldots, g_{m-1}, t_0, \ldots, t_m))
\]
where \(\tilde{\Delta}_e : G^m \times \Delta^m \to G/K\) parametrizes the geodesic simplices defined in 5.1.2.

**Proposition 5.5.2.** [14, p. 241] The maps 5.5.1 are compatible with the face maps and therefore induce a section \(s : BG \to (EG)/K\) of \(\pi\).

We now show that the cohomology class on \(GL_n(\mathbb{C})^d\) defined by the continuous cohomology class corresponding to \(T_p \in \wedge^{2p-1} p^*\) equals the cohomology class defined by \(s^* \tau_p\). With respect to the inhomogeneous model (see 5.5.1), the connection form of the universal flat bundle is given locally by the Maurer-Cartan form in the vertical direction, and is zero in the horizontal directions. Consequently, with respect to the inhomogeneous model, the restriction of \(\tau_p\) to \(GL_n(\mathbb{C})/U(n) \times G^m \times \Delta^m\) satisfies
\[
\tau_p \in A^{2p-1}(GL_n(\mathbb{C})/U(n)) \subseteq A^{2p-1}(GL_n(\mathbb{C})/U(n) \times G^m \times \Delta^m),
\]
and that it is the image of \(T_p\) under the natural inclusion
\[
\wedge^{2p-1} p^* \hookrightarrow A^{2p-1}(GL_n(\mathbb{C})/U(n)).
\]
It follows immediately that the value of the cocycle
\[
s^* \tau_p \in A^{2p-1}(BGL_n(\mathbb{C})^d)
\]
on the simplex of \(BGL_n(\mathbb{C})\) with vertices \((g_0, \ldots, g_m)\) is given by the integral
\[
\int_{\tilde{\Delta}_e(g_0, \ldots, g_m)} T_p.
\]

**5.6. \(T_p\) and the Borel regulator element.** In the final step, the idea is to use the complexification to show that \(T_p\) can be interpreted as a class that transgresses to the \(p\)th Chern class of \(E \to B\).

An invariant polynomial on \(\mathfrak{gl}_n(\mathbb{C})\) defines, in an obvious way, an invariant function on \(\mathfrak{g}_C\). Using 5.2.2 to write the elements of \(\mathfrak{g}_C\) as pairs \((X, Y)\), with \(X, Y \in \mathfrak{g}_l_n(\mathbb{C})\), we have for \(Q_p\) of (7.4):
\[
Q_p(X, Y) = \frac{1}{2} [C_p(X) - (-1)^p C_p(Y)].
\]
From this, we see that for all \(X \in \mathfrak{g}_l_n(\mathbb{C})\),
\[
C_p(X) = 2 Q_p(X, 0).
\]
Likewise, we can view \(T_p\) (also from 5.3) as an element of the complex Chern-Weil algebra \(\mathfrak{W}(\mathfrak{g}_C, U(n)_{\mathbb{C}})\), which then determines elements of \(W_C(u_n \oplus u_n, U(n))\) and \(W_C(u_n \oplus \{0\}) \cong W_C(u_n)\), which we continue to denote \(T_p\).

From 5.6.2, it follows that, in \(W_C(u_n)\), the relation \(2dT_p = C_p\) holds. In other words, the invariant form on \(U(n)\) defined by \(2T_p\) represents a class which transgresses to the \(p\)th Chern class. The corresponding class in
\[
\wedge^* (u_n)^* \overset{pr^*}{\to} \wedge^* ((u_n \oplus u_n)^*/u_n)^* \cong \wedge^* (ip^*)
\]
is the image of \(2T_p\) in \(\wedge^{2p-1}(ip^*)\). By the definition given in 5.2, the Borel element is represented by the differential form
\[
t^{2p-1} (2T_p|_{\wedge^{2p-1} ip}) = (-1)^{2p-1} (2T_p|_{\wedge^{2p-1} ip}) = -2T_p \in \wedge^{2p-1} p^*.
\]
Combining the above with 5.4 and 5.5, we have shown that the Borel element is twice the Cheeger-Simons class, and Theorem 3 is proved.

6. Towards a Proof of Conjecture 4

In this section we make use of simplicial methods. Again, see Appendix C for definitions and constructions.

6.1. The goal. We would have preferred to have Theorem 2 and Conjecture 4 as special cases of the simplicial analogue of Theorem 2, which we have not been able to prove:

Statement 6.1.1. Suppose that \( \pi : \mathcal{P}_* \to \mathcal{X}_* \) is a morphism in the category of smooth simplicial varieties, with \( \mathcal{X}_* \) complete, and that \( D_* \) and \( Q_* = \pi^{-1}(D_*) \) are divisors with normal crossings in \( \mathcal{X}_* \) and \( \mathcal{P}_* \), respectively. Let \( P_* = \mathcal{P}_* - Q_* \) and \( X_* = \mathcal{X}_* - D_* \). If the restriction \( P_* \to X_* \) of \( \pi \) is a principal \( GL_n(\mathbb{C}) \)-bundle with \( F^1 \)-connection — i.e., the connection form \( \omega \) satisfies

\[
\omega \in F^1 E^1(\mathcal{P}_* \log Q_* \oplus \text{End}(E_*)),
\]

then the image of its Cheeger-Simons class \( \hat{c}_p \) in \( H^2_{\mathcal{D}}(X_*, \mathbb{Z}(p)) \) under the natural homomorphism

\[
F^p \widehat{H}^2_{\mathcal{D}}(X_* \log D, \mathbb{Z}(p)) \to H^2_{\mathcal{D}}(X_*, \mathbb{Z}(p))
\]

is the Beilinson Chern class \( c^B_p \).

Remark 6.1.2. Note that we are not assuming in the above that \( P_* \) is proper over \( \mathcal{X}_* \), but only that it is so “in the horizontal direction”. In practice, \( P_* \) would be the frame bundle of a vector bundle \( E_* \) on \( \mathcal{X}_* \), and \( P_* \) the frame bundle of an extension \( E_* \) of \( E_* \) to \( \mathcal{X}_* \).

The following lemma, whose proof is quite direct, provides good evidence for the conjecture. (We retain the previous notation.)

Lemma 6.1.3. Suppose that one has two connections \( \nabla_0 \) and \( \nabla_1 \) on \( E_* \), and denote by \( \Theta_0 \) and \( \Theta_1 \) the respective curvature forms. Write \( \omega = \nabla_1 - \nabla_0 \) and \( \eta_p \) for the solution of

\[
d\eta_p = c_p(E_*, \nabla_1) - c_p(E_*, \nabla_0)
\]

given by 3.5.5. If

\[
\Theta_0, \Theta_1 \in F^1 A^2(\mathcal{X}_* \log D_*, \text{End}(E_*)), \quad \omega \in F^1(A^1(\mathcal{X}_* \log D_*, \text{End}(E_*))),
\]

then

(i) \( \eta_p \in F^p A^{2p-1}(\mathcal{X}_* \log D_*)) \); 
(ii) \( c_p(E_*, \nabla_0) \) and \( c_p(E_*, \nabla_1) \) represent the same cohomology class in

\[
H^{2p}(F^p A^* (\mathcal{X}_* \log D_*)) = F^p H^{2p}(\mathcal{X}_*, \mathbb{C}).
\]

Proof. This is a direct consequence of 3.5.4 and 3.5.5, as the assumptions imply that \( \Theta_1 \) is in \( F^1 \).

The following “refinement” of 6.1.3, which we will soon prove, is a necessary condition for 6.1.1.

Proposition 6.1.4. Under the conditions of 6.1.3, these two Cheeger-Simons classes have the same image in \( H^2_{\mathcal{D}}(X_*, \mathbb{Z}(p)) \).
The basic theme in any proof of 6.1.1 is to reduce to the universal case (where the assertion is a tautology; see 3.2.4 and 6.1.10 below), invoking functoriality. We wanted to make use of the following device:

**Proposition 6.1.5.** If $X_\bullet$ is a simplicial manifold and $P_\bullet \to X_\bullet$ is a principal $GL_n(\mathbb{C})$-bundle with connection given by $\omega \in A^1(|P_\bullet|, \text{Ad}(gl_n(\mathbb{C})))$,

then there exists a smooth bisimplicial variety $B_{\bullet\bullet}$ which has the homotopy type of $BGL_n(\mathbb{C})$ and a $GL_n(\mathbb{C})$-principal bundle $U_{\bullet\bullet} \to B_{\bullet\bullet}$ with connection $\omega_U \in A^1(|U_{\bullet\bullet}|, \text{Ad}(gl_n(\mathbb{C})))$ and a morphism of $GL_n(\mathbb{C})$-bundles

$$
P_\bullet \xrightarrow{G} U_{\bullet\bullet}$$

$$\downarrow \quad \downarrow$$

$$X_\bullet \xrightarrow{g} B_{\bullet\bullet}$$

such that $G^* \omega_U = \omega$. Moreover, if $X_\bullet$ is a smooth simplicial variety such that $\omega \in F^1 A^1(|P_\bullet|, \text{Ad}(gl_n(\mathbb{C})))$,

then we can choose $\omega_U$ to satisfy

$$\omega_U \in A^{1,0}(|U_{\bullet\bullet}|, \text{Ad}(gl_n(\mathbb{C}))).$$

**Proof.** Consider the bisimplicial variety $U_{\bullet\bullet}$ whose simplicial variety of $m$ simplices is the simplicial variety $(P_\bullet)^{m+1}$ with face maps

$$d_i : (P_\bullet)^{m+1} \to (P_\bullet)^m \quad 0 \leq i \leq m$$

$$(u_0, \ldots, u_m) \mapsto (u_0, \ldots, \hat{u}_i, \ldots, u_m).$$

The geometric realization of this simplicial variety is contractible as it is the coskeleton of the trivial covering (cf. [1] or [31, p. 107]). The free $GL_n(\mathbb{C})$ action on $P_\bullet$ induces a free $GL_n(\mathbb{C})$ action on $U_{\bullet\bullet}$. Let $B_{\bullet\bullet}$ be the quotient of $U_{\bullet\bullet}$. The resulting principal bundle $U_{\bullet\bullet} \to B_{\bullet\bullet}$ is a model of the universal $GL_n(\mathbb{C})$-bundle.

Define a connection $\omega_U \in A^1(|U_{\bullet\bullet}|, \text{gl}_n(\mathbb{C}))$ on the bundle $|U_{\bullet\bullet}| \to |B_{\bullet\bullet}|$ by the compatible family of 1-forms

$$\sum_{j=0}^m t_j \pi_j^* \omega \in A^1(|P_\bullet|^{m+1} \times \Delta^m),$$

where $(t_0, \ldots, t_m)$ are the barycentric coordinates of $\Delta^m$, and $\pi_j : |P_\bullet|^{m+1} \to |P_\bullet|$ denote the canonical projections, $0 \leq j \leq m$.

The canonical isomorphism $P_\bullet \to U_{\bullet0}$ induces a $GL_n(\mathbb{C})$-equivariant map $G : P_\bullet \to U_{\bullet\bullet}$, and therefore a map $g : X_\bullet \to B_{\bullet\bullet}$ such that the pair of maps $(g, G)$ classifies the bundle and the connection. The remaining assertion is easy to verify.

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6This may be proved directly as follows. First show that the geometric realization of the simplicial space is simply connected. This is not difficult, as the fundamental group depends only on the 2-skeleton. One then shows that this space has trivial integral homology, and is therefore contractible, by standard arguments.
Remark 6.1.6. In other words, given a principal $GL_n(\mathbb{C})$-bundle with connection over a complete smooth simplicial variety, we can build a model of the universal $GL_n(\mathbb{C})$-bundle, and put a connection on this bundle, such that a fixed mapping of our variety into this model of $BGL_n(\mathbb{C})$ simultaneously classifies the bundle and its connection. Unfortunately the above does not yield a proof of 6.1.1, for the Maurer-Cartan forms of $GL_n(\mathbb{C})$, and hence the connection constructed in 6.1.7 are not logarithmic at infinity for $n > 1$. This was our fundamental obstacle.

However, 6.1.5 does yield at once the following useful observation:

Proposition 6.1.7. Suppose that $\nabla_1$ is a second connection on $E_*$ and that $\widehat{\gamma}_p(E_*, \nabla_0)$ is represented by $(c_p(E_*, \nabla_0), -\gamma)$, as in 7.5.17. Let $\eta_p$ be as in 3.5.5. Then $\widehat{\gamma}_p(E_*, \nabla_1)$ is represented by

$$(c_p(E_*, \nabla_1), -(y + i\zeta(p)(\eta_p))) = (c_p(E_*, \nabla_0), -y) + (d\eta_p, -i\zeta(p)(\eta_p)).$$

Proof. By the functoriality of $c_p$, $\eta_p$ and $\widehat{\gamma}_p$, it suffices to check this “universally”, on the model of $BGL_n(\mathbb{C})$ from 3.1.7. But there, the Cheeger-Simons class is completely determined by the Chern form (see 3.3), so it comes down to the calculation:

$$\delta(y + i\zeta(p)(\eta_p)) = i\zeta(p)(c_p(E_*, \nabla_0)) + i\zeta(p)(d\eta_p) = i\zeta(p)(c_p(E_*, \nabla_1)).$$

Using 6.1.7, we can now give:

Proof of 6.1.4. In the DB-complex of $X_*$, we have that:

$$(c_p(E, \nabla_0), -y) - (c_p(E, \nabla_1), -(y + i\zeta(p)(\eta_p))) = (-d\eta_p, i\zeta(p)(\eta_p)) = D(\eta_p, 0),$$

so 6.1.4 follows.

The simplicial version of 4.2.1 is

$$\text{Proposition 6.1.10. If } H^{2p-1}(|X_*|, \mathbb{C}/\mathbb{Z}(p)) = 0, \text{ then the assertion in 6.1.4 holds for every holomorphic vector bundle on } |X_*|.$$
always be done when $X_\bullet$ is just a single algebraic manifold $X = \overline{X} - D$. Thus, the proof of Theorem 2 (the non-simplicial version of 6.1.1) reduced to the compact case (see 4.5).

Remark 6.1.11. It should be apparent that the preceding development goes through verbatim to the setting of simplicial complex manifolds $X_\bullet$ that are of the form $\overline{X_\bullet} - D_\bullet$, with $\overline{X_\bullet}$ compact Kähler and $D_\bullet$ a divisor with normal crossings in $\overline{X_\bullet}$. In fact, even the condition that $\overline{X_\bullet}$ be Kähler can be dispensed with; one simply makes the distinction between the two sides (isomorphic when $\overline{X_\bullet}$ is Kähler) of

$$F^p H^\bullet(\overline{X_\bullet}; \mathbb{C}) \leftarrow H^\bullet(F^p A^\bullet([\overline{X_\bullet} \log D_\bullet]));$$

and

$$H^\bullet(\overline{X_\bullet}; \mathbb{C})/F^p H^\bullet(\overline{X_\bullet}; \mathbb{C}) \hookrightarrow H^\bullet(A^\bullet([\overline{X_\bullet} \log D_\bullet])/F^p),$$

and takes the right-hand member wherever we have written the left.

We present in Appendix D a couple of plausible techniques we tried in our unsuccessful attempt to prove 6.1.1.

Appendix A. Fiber integration and the homotopy formula

In this appendix, we present some basic facts needed in Section 3. Let $p : I \times M \to M$ denote the canonical projection of the product of the unit interval with an $m$-manifold $M$ onto $M$. For all $i$, there is a linear mapping

\[ B : A^{i+1}(I \times M) \to A^i(M) \]  

characterized by the formula

\[ T(B\varphi) = ([I] \times T) \]  

for all compactly supported $i$-currents $T$ on $M$. Whenever $T$ is the current $T_\psi$ defined by

\[ T_\psi(\varphi) = \int_M \varphi \wedge \psi \quad (\psi \in A^{m-i}_c(M)), \]

for which

\[ \partial T_\psi = (-1)^{i+1} T_{d\psi} \text{ and } [I] \times T_\psi = T_{p^*\psi}, \]

becomes $\int_M B\varphi \wedge \psi = \int_{I \times M} \varphi \wedge p^*\psi$. In fact, in general, $B\varphi$ is just the integral over $I$ of $(\partial_1|_I)\varphi$. The mapping $B$ is functorial, in the sense that if $g : N \to M$ is smooth, inducing $\tilde{g} : I \times N \to I \times M$, then

\[ B(\tilde{g}^*\varphi) = g^* B\varphi. \]

The identity

\[ \partial([I] \times T) = \partial[I] \times T - [I] \times \partial T \]

yields at once

\[ B(d\varphi) + d(B\varphi) = \varphi_1 - \varphi_0, \]

where $\varphi_1 = \varphi|_{\{t\} \times M}$. Applied to $\varphi = h^*\omega$, where $h : M \times I \to N$ is a smooth homotopy, and $\omega \in A^i(N)$, \( A.0.18 \) gives the homotopy formula:

\[ (B \circ h^*)d\omega + d(B \circ h^*)\omega = h^1_1 \omega - h^0_0 \omega. \]
The dual version for currents is
\begin{equation}
\partial(h_*(I \times T)) + h_*(I \times \partial T) = (h)_* T - (h_0)_* T.
\end{equation}

There is a parallel treatment for singular cochains with any coefficients, as follows. A smooth singular simplex \( \sigma : \Delta^i \to M \) defines a current:
\begin{equation}
[\sigma] = \sigma^* [\Delta^i],
\end{equation}
such that \([\partial \sigma] = \partial [\sigma]\). Using the standard triangulation of \( I \times \Delta^i \), a smooth mapping \( \tau : I \times \Delta^i \to N \) defines a singular chain \([\tau]\) on \( N \). Taking \( \tau \) of the form \( h^* (1 \times \sigma) \) : \( I \times \Delta^i \to T \), we have
\begin{equation}
\partial h^* (1 \times \sigma) + \partial h^* (1 \times \partial \sigma) = (h)_1 \sigma - (h_0)_* \sigma,
\end{equation}
which under (A.0.21) recovers (A.0.20). Define, for a smooth singular cochain \( f \) on \( I \times M \),
\begin{equation}
(B_f)(\sigma) = f(1 \times \sigma)
\end{equation}
(cf. (A.0.13); putting \( f = h^* \tau \), we get from (A.0.22)
\begin{equation}
B(h^* \partial \sigma) + \delta (Bh^* \tau) = h^*_1 \tau - h^*_0 \tau.
\end{equation}
When \( \tau \) is a differential form (see 3.1), one recovers (A.0.19), i.e.:
\begin{proposition}
The homotopy formula for smooth singular cochains, when restricted to differential forms, coincides with the homotopy formula for differential forms.
\end{proposition}

Appendix B. Universal weak \( F^1 \)-connections

We show that if one ceases to impose conditions at infinity, it is possible to construct a universal “\( F^1 \)-connection.”

We start with a definition:

\begin{definition}
A weak \( F^1 \)-connection on \( E \) is a connection for which the local connection forms and therefore also the curvature lie in \( F^1 A^*(X, \text{End}(E)) \).
\end{definition}

Imposing the growth condition along \( D \), as in (4.2.1) in (B.0.20) gives (4.4).

We introduce the notion of a universal weak \( F^1 \)-connection. The discussion proceeds as in that of 2.2.3. In the setting of 2.2.1, suppose that \( Y \) is a complex manifold, and take \( G = GL_n(\mathbb{C}) \). Let \( P \to Y \) be a holomorphic principal \( GL_n(\mathbb{C}) \)-bundle, and
\begin{equation}
p : T^{1,0}P \to T^{1,0}Y
\end{equation}
be the induced map of holomorphic tangent bundles. We put
\begin{equation}
\tilde{Y}^{(1)} = \{ \hat{\alpha} \in \text{Hom}(T^{1,0}Y, T^{1,0}P/G) : \hat{\rho} \circ \hat{\alpha} = \text{id}_{T^{1,0}Y} \},
\end{equation}
where \( \hat{\rho} : T^{1,0}P/G \to T^{1,0}Y \) is induced by \( p \). Then \( \tilde{Y}^{(1)} \) can be identified with a submanifold of the space \( \tilde{Y} \) in (2.2.1) in the following way. One makes use of the canonical splitting \( (TY)_C = T^{1,0}Y \oplus T^{0,1}Y \). The projection \( q^{(1)} : \tilde{Y}^{(1)} \to Y \) then
becomes a complex affine space sub-bundle of \( q : \tilde{Y} \rightarrow Y \) (via \( \alpha = \hat{\alpha} \oplus 0 \)); a local holomorphic trivialization of \( P \) gives rise to the local description (cf. 2.2.6):

\[
\tilde{Y}^{(1)} \cong \text{Hom}(T^{1,0}Y, g).
\]

(B.0.28)

There is a tautological holomorphic connection on \( q^{(1)*}P \).

**Proposition B.0.29.** The weak \( F^1 \)-connections on \( P \) are in one-to-one correspondence with the \( C^\infty \) sections of \( q^{(1)} \); the holomorphic sections of \( q^{(1)} \) correspond to the holomorphic connections on \( P \).

One must remember that the notion of weak \( F^1 \)-connection is respected under pullback only by holomorphic maps. For a holomorphic map \( f : X \rightarrow Y \), the pullback connection on \( f^*P \) is induced by

\[
\text{Hom}(T f_{(x)}^1 Y, g) \rightarrow \text{Hom}(T x^1 X, g).
\]

Taking \( Y = G_C(n) \), we deduce (cf. 2.2.9, 2.2.10):

**Corollary B.0.30.** If a holomorphic vector bundle \( E \) is classified by a holomorphic immersion \( f : X \rightarrow G_C(n) \), then every weak \( F^1 \)-connection on \( E \) is the pullback of the tautological weak \( F^1 \)-connection on \( \tilde{G}_C(n)^{(1)} \) by a \( C^\infty \) lifting \( \tilde{f} : X \rightarrow \tilde{G}_C(n)^{(1)} \) of \( f \).

**Appendix C. Simplicial de Rham Theory**

In this appendix we give a resume of simplicial manifolds, tailored to our present needs. The principal reference is [15].

**C.1. Simplicial manifolds.** Let \( \Delta \) be the category whose objects are the finite ordinals

\[
[n] = \{0 < 1 < \ldots < n\} \quad n \in \mathbb{N}
\]

and whose morphisms are order-preserving monomorphisms. Of special importance are the face maps \( d_j : [n - 1] \rightarrow [n] \), the unique morphism that omits the value \( j \).

View \([n]\) as the set of the set of vertices of the standard \( n \)-simplex \( \Delta^n \). Then, each morphism \( f : [n] \rightarrow [m] \) of \( \Delta \) induces a simplicial map

\[
|f| : \Delta^n \rightarrow \Delta^m.
\]

**Definition C.1.1.** A (strict) simplicial (resp. cosimplicial) object of a category \( C \) is a contravariant (resp. covariant) functor \( \Delta \rightarrow C \). Morphisms between strict (co)simplicial objects in \( C \) are natural transformations of functors.

Denote the category of smooth manifolds, each of whose connected components are second countable, and smooth maps by \( \mathcal{M} \).

**Definition C.1.2.** A simplicial manifold \( M_* \) is a strict simplicial object of \( \mathcal{M} \). The object of \( \mathcal{M} \) corresponding to the ordinal \([n]\) will be denoted by \( M_n \). The smooth map induced by the morphism \( f : [n] \rightarrow [m] \) will be denoted by

\[
f_M : M_m \rightarrow M_n.
\]

The geometric realization \(|M_*|\) of the simplicial manifold \( M_* \) is defined to be the topological space

\[
\prod_{n \geq 0} (M_n \times \Delta^n) / \sim
\]
endowed with the quotient topology, where $\sim$ is the equivalence relation which identifies the points

$$(x, |f|\xi) \in M_m \times \Delta^m \quad \text{and} \quad (f_M(x), \xi) \in M_n \times \Delta^n$$

whenever $f : [n] \to [m]$ is a morphism.

Observe that simplicial sets can be viewed as simplicial manifolds. In particular, the standard simplicial model of the classifying space of $GL_n(\mathbb{C})^\delta$, the general linear group with the discrete topology, is a simplicial manifold.

There is a fully faithful imbedding of $\mathcal{M}$ into $\mathcal{M}\Delta$, the category of simplicial manifolds: The manifold $M$ is taken to the simplicial manifold $M_\bullet$ with

$$M_n = \begin{cases} M & n = 0 \\ \emptyset & n > 0. \end{cases}$$

**C.2. The de Rham functor.** Suppose that $C^\bullet$ is a strict cosimplicial cochain complex of $\mathbb{C}$-vector spaces. Denote the cochain complex corresponding to $[n]$ by $C^\bullet[n]$.

**Definition C.2.1.** [23 (5.3)] The de Rham complex $DC^\bullet$ of $C^\bullet$ is the total complex of the double complex $D^{s,t}C^\bullet$, where $D^{s,t}C^\bullet$ consists of those elements $(w_n)$ of

$$\prod_{n \geq 0} A^s(\Delta^n) \otimes C^t[n]$$

that satisfy the compatibility condition:

$$(|f|^* \otimes \text{id})w_n = (\text{id} \otimes f_*)w_m \in A^s(\Delta^m) \otimes C^t[n],$$

whenever $f : [m] \to [n]$ is a morphism. The differentials in the double complex arise from exterior differentiation in the simplicial direction $s$ and from the differential of $C^\bullet$ in the $t$ direction.

Now suppose that $M_\bullet$ is a simplicial manifold. Applying the usual de Rham complex functor to $M_\bullet$ we obtain a cosimplicial cochain complex $A^\bullet(M_\bullet)$ whose value on the ordinal $[n]$ is $A^\bullet(M_n)$.

**Definition C.2.2.** The de Rham complex $A^\bullet|M_\bullet|$ of the simplicial manifold $M_\bullet$ is the de Rham complex $DA^\bullet(M_\bullet)$ of $A^\bullet(M_\bullet)$. It is naturally bigraded:

$$A^k|M_\bullet| = \bigoplus_{s+t=k} A^{s,t}|M_\bullet|,$$

where

$$A^{s,t}|M_\bullet| = D^sA^t(M_\bullet).$$

If $M_\bullet$ is a simplicial set, then $A^\bullet|M_\bullet|$ is just the Thom-Whitney de Rham complex of $\mathbb{C}$-valued forms on $M_\bullet$ (cf. [37], [38]).

**Proposition C.2.3.** The association of $A^\bullet|M_\bullet|$ to the simplicial manifold $M_\bullet$ defines a contravariant functor from the category of simplicial manifolds into the category of commutative differential graded algebras. \qed
C.3. The de Rham Theorem. In this section we sketch the de Rham theorem for simplicial manifolds. It is first convenient to introduce the functor \( C^\bullet \) from the category of (strict) cosimplicial abelian groups into the category of cochain complexes: The cosimplicial abelian group \( A \) is taken to the cochain complex \( C^\bullet A \), where
\[
C^n A = A[n]
\]
and the differential \( \delta : C^n A \to C^{n+1} A \) is the alternating sum
\[
\delta = \sum_{j=0}^{n+1} (-1)^j d^j
\]
of the coface maps \( d^i : A[n] \to A[n+1] \). When \( A^\bullet \) is a strict cosimplicial cochain complex, \( C^\bullet A^\bullet \) is a double complex.

The classical de Rham theorem states that if \( M \) is a smooth manifold, the integration map
\[
\int : A^\bullet(M) \to S^\bullet(M, \mathbb{C})
\]
is a quasi-isomorphism which induces an algebra isomorphism on cohomology.

Now, to each simplicial manifold \( M_\bullet \), we can associate the strict cosimplicial cochain complex \( S^\bullet(M_\bullet) \) whose value on the ordinal \([n]\) is \( S^\bullet(M_n) \). Applying the cochain functor \( C \) above, we obtain a double complex \( C^\bullet S^\bullet(M_\bullet) \). Denote the associated single complex (total complex) by \( S^\bullet(M_\bullet) \). Note that if \( M_\bullet \) is a strict simplicial set, then \( S^\bullet(M_\bullet) \) denotes the usual complex of simplicial cochains associated to \( M_\bullet \).

For each simplicial manifold \( M_\bullet \), define a \( C^\bullet \)-linear function
\[
A^{n,t}(M_\bullet) \to S^t(M_\bullet, \mathbb{C})
\]
by taking the compatible family
\[
(w_n) \in \prod_{n \geq 0} A^s(\Delta^n) \otimes A^t(M_n)
\]
to the \( t \)-cochain
\[
\sigma \mapsto \int_{\Delta^s \times \Delta^t} w_s, \quad \sigma : \Delta^t \to M_s.
\]

The following result is a direct consequence of Stokes’ Theorem and the compatibility conditions.

**Proposition C.3.2.** The mapping \( \boxed{C.3.1} \) is a morphism of double complexes.

**Corollary C.3.3.** The mapping \( \boxed{C.3.1} \) induces a morphism of complexes
\[
\int : A^\bullet(M_\bullet) \to S^\bullet(M_\bullet, \mathbb{C}).
\]

The following de Rham theorem is a modest generalization of those of Thom (see [37]) and Sullivan [36]. Proofs of statements similar to the following appear in [14, 15, 23, (5.4)].
Theorem C.3.4. For each simplicial manifold $M_\bullet$, the integration mapping
\[ \int : A^\bullet |M_\bullet| \to S^\bullet(M_\bullet, \mathbb{C}) \]
is a quasi-isomorphism that induces an algebra isomorphism
\[ H^\bullet(A^\bullet |M_\bullet|) \to H^\bullet(S^\bullet(M_\bullet, \mathbb{C}) \cong H^\bullet(|M_\bullet|, \mathbb{C}). \]

\[ \square \]

C.4. Bundles over simplicial manifolds. As is well-known, we may talk equivalently about vector bundles or about principal bundles, and here we choose the latter. Let $G$ be a Lie group. A principal $G$-bundle over the simplicial manifold $M_\bullet$ is a simplicial manifold $P_\bullet$ and a morphism $\pi : P_\bullet \to M_\bullet$ of simplicial manifolds satisfying:

(i) for each $n, \pi : P_n \to M_n$ is a principal $G$-bundle;
(ii) for each morphism $f : [m] \to [n]$ of $\Delta$, $f_P$ is a morphism of $G$-bundles:

\[
\begin{array}{ccc}
P_n & \xrightarrow{f_P} & P_m \\
\downarrow & & \downarrow \\
M_n & \xrightarrow{f_M} & M_m
\end{array}
\]

The second condition guarantees that the principal $G$-action on $P_\bullet$ passes to its geometric realization:

Proposition C.4.1. If $\pi : P_\bullet \to M_\bullet$ is a principal $G$-bundle over a simplicial manifold, then $|\pi| : |P_\bullet| \to |M_\bullet|$ is a principal $G$-bundle, with $G$-action induced by

\[
\Delta^n \times P_n \times G \to \Delta^n \times P_n
\]
\[(t, x, g) \mapsto (t, xg).\]

\[ \square \]

One can define vector bundles over simplicial manifolds similarly.

Definition C.4.2. A connection $\nabla$ on the principal $G$-bundle $P_\bullet \to M_\bullet$ over a simplicial manifold $M_\bullet$ is a $G$-invariant 1-form

\[ \omega \in A^1(|P_\bullet|, \text{Ad} g) \]
taking values in $g$ (cf. [2,2]), the Lie algebra of $G$, on which $G$ acts via the adjoint representation.

As in the classical case, the $g$-valued 2-form

\[ d\omega + \frac{1}{2}[\omega, \omega] \in A^2(|P_\bullet|, \text{Ad} g) \]
descends to a $g$-valued 2-form $\Theta \in A^2(|M_\bullet|, \text{Ad} g)$. The 2-form $\Theta$ is necessarily unique and is called the curvature of $\nabla$.

We conclude this subsection with a review of the extension of Chern-Weil theory to simplicial manifolds (cf. [14,15]). We restrict our attention to Chern classes.

Definition C.4.3. The $k$th Chern form $c_k(\nabla)$ of a connection $\nabla$ on the principal $GL_n(\mathbb{C})$-bundle $P_\bullet \to M_\bullet$ over a simplicial manifold is:

\[ c_k(\nabla) := C_k(\Theta) \in A^{2k}|M_\bullet|, \]
where Θ is the curvature of ∇ and C_k is the invariant polynomial defined in Section 1.1.

The standard arguments can be used to prove the following result (see [15]).

**Theorem C.4.4.** If c_0(∇), . . . , c_n(∇) are the Chern forms of a connection on a principal GL_n(C) bundle over a simplicial manifold, then

(i) each c_k(∇) is closed;
(ii) the cohomology class of c_k(∇) is the image under the canonical map

\[ H^{2k}(|M\|, \mathbb{Z}(k)) \to H^{2k}(|M\|, \mathbb{C}) \]

of the kth Chern class of |P\| \to |M\|.

**C.5. Cheeger-Simons classes.** Suppose that \( M \) is a simplicial manifold and \( \Lambda \) an abelian group. Denote by \( C^\bullet(M, \Lambda) \) the cochain complex of compatible cochains on \(|M\|\). This is, \( C_k(M, \Lambda) \) consists of those elements \((c_n)\) of

\[ \prod_{n \geq 0} S^k(\Delta^n \times M_n) \]

that satisfy the compatibility condition

\[ (|f| \times \text{id})^*c_n = (\text{id} \times f_M)^*c_m \in S^k(\Delta^m \times M_n) \]

for each morphism \( f : [m] \to [n] \) of \( \Delta \).

The following fact is standard:

**Proposition C.5.1.** The natural inclusion \( C^\bullet(M, \Lambda) \hookrightarrow S^\bullet(|M\|, \Lambda) \) is a quasi-isomorphism.

A direct consequence of the compatibility condition of \[\text{C.2.1}\] is that integration induces a chain map

\[ I : A^\bullet(|M\|, \Lambda) \to C^\bullet(M, \mathbb{C}). \]

This is easily seen to be a quasi-isomorphism.

**Definition C.5.2.** Suppose that \( \Lambda \) is a subgroup of \( \mathbb{C} \). The group of mod \( \Lambda \) differential characters of degree \( k \) of a simplicial manifold \( M \) is defined by

\[ \hat{H}^k(M, \Lambda) = H^{k-1}(\text{cone}(A^\geq k|M\| \xrightarrow{\iota} C^\bullet(M, \mathbb{C}/\Lambda))) \],

where \( \iota \Lambda \) is the composite of \( I \) with the natural map \( C^\bullet(M, \mathbb{C}) \to C^\bullet(M, \mathbb{C}/\Lambda). \)

Since the differential characters are constructed from a cone, we have the short exact sequence

(C.5.3) \[ 0 \to H^{k-1}(|M\|, \mathbb{C}/\Lambda) \to \hat{H}^k(M, \Lambda) \to A^k_{\text{cl}}(|M\|, \Lambda) \to 0, \]

where

\[ A^k_{\text{cl}}(|M\|, \Lambda) = \{ \varphi \in A^k|M\| : d\varphi = 0 \text{ and the periods of } \varphi \text{ on } H_k(|M\|) \text{ lie in } \Lambda \} \].

The construction of the Cheeger-Simons invariant \( \hat{c}_p \) given in \[\text{C.3}\] works equally well in the simplicial setting. We use it to define Cheeger-Simons classes

\[ \hat{c}_p(E\bullet, \nabla) \in \hat{H}^{2p}(M\bullet, \mathbb{Z}(p)) \]

for complex vector bundles \( E\bullet \to M\bullet \) with connection over simplicial manifolds.
C.6. Simplicial varieties. Let $\text{Alg}$ be the category whose objects are arbitrary disjoint unions of complex algebraic manifolds and whose morphisms are morphisms of varieties on each component.

**Definition C.6.1.** A smooth simplicial variety is a strict simplicial object of $\text{Alg}$. The category of simplicial varieties will be denoted by $\text{Alg}^\Delta$.

For later use we record the following result whose proof follows from standard results in algebraic geometry.

**Proposition C.6.2.**

(a) If $Y_\bullet$ is a simplicial variety, then there is a simplicial variety $\overline{Y}_\bullet$ and an open immersion $Y_\bullet \to \overline{Y}_\bullet$ such that each $\overline{Y}_n$ is complete and such that $\overline{Y}_n - Y_n$ is a normal crossing divisor in $\overline{Y}_n$. Moreover, if $f : Y_\bullet \to Z_\bullet$ is a morphism of simplicial varieties, then there exist smooth completions $\overline{Y}_\bullet$ of $Y_\bullet$ and $\overline{Z}_\bullet$ of $Z_\bullet$ as above and a morphism $\overline{f} : \overline{Y}_\bullet \to \overline{Z}_\bullet$ whose restriction to $Y_\bullet$ is $f$.

(b) If $Y'_\bullet$ and $Y''_\bullet$ are two smooth completions as in (a) of a simplicial variety $Y_\bullet$, then there exists a third completion $\overline{Y}_\bullet$ of $Y_\bullet$ and morphisms $\overline{Y}_\bullet \to Y'_\bullet$, $Y''_\bullet \to \overline{Y}_\bullet$ such that the diagram

$$
\begin{array}{ccc}
Y'_\bullet & \xleftarrow{\sim} & \overline{Y}_\bullet \\
\downarrow & & \downarrow \\
Y''_\bullet & \xrightarrow{\sim} & \overline{Y}_\bullet
\end{array}
$$

commutes.

C.7. Complements. Because the cohomology of $BGL_n(\mathbb{C})^\delta$ and other simplicial varieties that occur naturally in the proof of Theorems 1 and 2 are not of finite type, we need to say a few words about generalizations of Hodge theory to that case.

We will say that a simplicial manifold is of finite type if the cohomology of its $n$-simplices is finite dimensional for each $n$. Every simplicial algebraic variety can be written as the direct limit of simplicial varieties which are of finite type. It follows that the cohomology, with characteristic zero field coefficients, of every simplicial variety is the inverse limit of an inverse system of mixed Hodge structures. We view the cohomology of a simplicial variety as a topological vector space, where the neighbourhoods of 0 are the kernels of restriction maps to the cohomology of simplicial varieties of finite type. It is complete in this topology. We extend the notion of mixed Hodge structures to such topological spaces.

Suppose that $\Lambda$ is a subring of $\mathbb{R}$ with quotient field $\mathbb{L}$. Denote the category of $\Lambda$-mixed Hodge structures by $\mathcal{H}_\Lambda$. Consider the category of completed $\Lambda$-mixed Hodge structures $\hat{\mathcal{H}}_\Lambda$ whose objects consist of triples

$$(H_{\Lambda}, (H_L, W_\bullet), (H_C, F^\bullet)),$$

where $H_{\Lambda}$ is a $\Lambda$-module; $H_L = H_{\Lambda} \otimes_\Lambda \mathbb{L}$ a complete topological $\mathbb{L}$-vectorspace with an increasing filtration $W_\bullet$ by closed subspaces; $H_C$ the complexification of $H_L$ with the corresponding topology and $F^\bullet$ a decreasing filtration by closed subspaces. In addition we suppose that there is a base of neighborhoods $(N_\alpha)$ of 0 in $H_L$ such that for each $\alpha$;

$$(H_L/(H_L \cap N_\alpha), (H_L/N_\alpha, W_\bullet), (H_C/N_\alpha \otimes \mathbb{C}, W_\bullet \otimes \mathbb{C}, F^\bullet))$$

7The modifier strict means that this is the category whose objects come without degeneracy maps.
is a $\Lambda$-mixed Hodge structure. A morphism
\[(H_{\Lambda}, (H_L, W_*), (H_C, F^*)) \to (H'_{\Lambda}, (H'_L, W_*), (H'_C, F'^*))\]
consists of a $\Lambda$-module homomorphism $H_{\Lambda} \to H'_{\Lambda}$ that induces continuous, filtration
preserving homomorphisms $H_L \to H'_L$ and $H_C \to H'_C$.

It is straightforward to show that the cohomology of every simplicial variety
inherits a functorial mixed Hodge structure, in the above sense, from its de Rham
complex (cf. \[13, (8.1.19)]).

\[C.8. \text{Deligne-Beilinson (DB) cohomology.} \]
Suppose that $Y_*$ is a smooth
simplicial variety. By \[C.6.2\] we can find a smooth completion \(\overline{Y}_*\) of $Y_*$ such that
each $Y_n - Y_n$ is a normal crossings divisor $D_n$. Denote by $I_p$ the composite
\[F^p A^*([\overline{Y}_*] \log D_*) \to A^*|Y_*| \xrightarrow{I} S^*(Y_*)\],
where $F$ denotes the Hodge filtration of the de Rham complex.

**Definition C.8.1.** Suppose that $\Lambda$ is a subring of $\mathbb{R}$ and that $p \in \mathbb{N}$. The
Deligne-Beilinson (or DB) cohomology $H^*_D(Y_*, \Lambda(p))$ of the simplicial variety $Y_*$ is
the cohomology of the complex
\[D^*(Y_*, \overline{Y}_*; \Lambda(p)) = \text{cone}\{F^p A^*([\overline{Y}_*] \log D_*) \xrightarrow{I} S^*(Y_*, C/\Lambda(p))\}[-1].\]

As DB-cohomology is constructed from a cone, we have:

**Proposition C.8.2.** For each simplicial variety $Y_*$, there is natural long exact
sequence
\[\ldots \to H^{k-1}(|Y_*|, C/\Lambda(p)) \xrightarrow{\partial} H^k_D(Y_*, \Lambda(p)) \to \] 
\[F^p H^k(|Y_*|, C) \to H^k(|Y_*|, C/\Lambda(p)) \to \ldots.\]

**Corollary C.8.4.** For each simplicial variety $Y_*$, there is a natural homo-
morphism
\[H^{k-1}(|Y_*|, C/\Lambda(p)) \to H^k_D(Y_*, \Lambda(p)).\]

Since a point is a smooth projective variety of dimension 0, every simplicial set
can be regarded as a simplicial variety. Since the Hodge filtration of the de Rham
complex of such a simplicial variety satisfies $F^p = 0$ when $p > 0$ we have:

**Corollary C.8.5.** If $K_*$ is a simplicial set and $p > 0$, then the natural homo-
morphism
\[H^{k-1}(|K_*|, C/\Lambda(p)) \to H^k_D(K_*, \Lambda(p))\]
is an isomorphism.

**Corollary C.8.6.** There is a natural isomorphism
\[H^{2p}_D(BGL_n(C)^{\delta}, \Lambda(p)) \cong H^{2p-1}(\delta BGL_n(C)^{\delta}, C/\Lambda(p)),\]
where $BGL_n(C)^{\delta}$ denotes the classifying space of the general linear group endowed
with the discrete topology.
C.9. Chern classes in DB cohomology. A vector bundle of rank \( r \) over a simplicial variety \( Y_\bullet \) is a morphism of simplicial varieties \( E_\bullet \to Y_\bullet \) where, for each \( n \), \( E_n \to Y_n \) is an algebraic vector bundle of rank \( r \). Beilinson [2] and Gillet [20] have defined Chern classes

\[
c^R_p(E_\bullet) \in H^p_D(Y_\bullet, \mathbb{Z}(p))
\]

for vector bundles \( E_\bullet \to Y_\bullet \) over a simplicial varieties: Since the splitting principle holds for DB-cohomology, the existence of Chern classes in Deligne cohomology reduces to the existence of first Chern classes

\[
c_1(L_\bullet) \in H^1_D(Y_\bullet, \mathbb{Z}(1))
\]

for line bundles \( L_\bullet \to Y_\bullet \). These are constructed in [2, (1.7)], in the manner of [22].

By a connection on a vector bundle \( E_\bullet \to Y_\bullet \) we mean a connection on the associated \( GL_n(\mathbb{C}) \) bundle \( P(E_\bullet) \to Y_\bullet \) over the underlying simplicial manifold.

**Proposition C.9.1.** Suppose that \( Y_\bullet \) is a smooth simplicial variety and that

\[
H^{2p-1}(|Y_\bullet|, \mathbb{C}/\mathbb{Z}(p)) = 0.
\]

If \( \nabla \) is a connection on the vector bundle \( E_\bullet \to Y_\bullet \) whose \( p \)th Chern form satisfies

\[
c_p(\nabla) \in F^p A^{2p}(|Y_\bullet|, \log D_\bullet),
\]

for some smooth completion \( Y_\bullet \) of \( Y_\bullet \) where \( Y_\bullet - Y_\bullet \) is a divisor with normal crossings, then

\[
(c_p(\nabla), -\eta) \in D^{2p}(Y_\bullet, Y_\bullet; \mathbb{Z}(p))
\]

represents \( c^B_p(E_\bullet) \), where \( \eta \) is any element of \( S^{2p-1}(Y_\bullet, \mathbb{C}/\mathbb{Z}(p)) \) such that \( \delta \eta = c_p(\nabla) \).

**Proof.** The result follows from [C.8.1] and the vanishing of \( H^{2p-1}(|Y_\bullet|, \mathbb{C}/\mathbb{Z}(p)) \) which imply that

\[
H^p_D(Y_\bullet, \mathbb{Z}(p)) \to F^p H^{2p}(|Y_\bullet|, \mathbb{C})
\]

is injective.

C.10. Multisimplicial manifolds and varieties. In some constructions that one would like to do, such as 6.1.5, one needs to work with **multisimplicial varieties**. Define the category of 0-simplicial manifolds to be \( \mathcal{M} \), the category of manifolds. For each \( n \geq 1 \), we define inductively an \( n \)-simplicial manifold to be a strict simplicial object in the category of \( (n-1) \)-simplicial manifolds; morphisms between two \( n \)-simplicial manifolds are natural transformations. The category of \( n \)-simplicial manifolds and morphisms will be denoted by \( \mathcal{M}^{\Delta^n} \). Just as every manifold can be regarded as a simplicial manifold, every \( n \)-simplicial manifold can be regarded as an \( (n+1) \) simplicial manifold by placing it in degree 0 of the \( (n+1) \)-simplicial manifold, and the empty set in positive degrees. Objects of one of the categories \( \mathcal{M}^{\Delta^n} \) are called a **multisimplicial manifolds**.

The geometric realization \( |M_\bullet| \) of an \( n \)-simplicial space \( M_\bullet \) is defined inductively to be the geometric realization of the simplicial topological space \( |M|_\bullet \), whose space of \( k \)-simplices is the geometric realization \( |M_k| \) of the \( (n-1) \)-simplicial manifold \( M_k \).

One can extend the definition of de Rham complex, the de Rham theorem, Chern-Weil theory and Cheeger-Simons Chern classes to multisimplicial manifolds, and the definition of Deligne-Beilinson cohomology to multisimplicial varieties, all in a straightforward way.
Appendix D. Additional techniques

D.1. Killing the unipotent directions. Let $B$ be a Borel subgroup of $G = GL_n(C)$. Then $G$ is a principal $B$-bundle over the Grassmannian $G/B$. If one decomposes $B$ as $T \times U$, where $T$ is the maximal torus and $U$ is the unipotent radical, one sees that the trouble in 6.1.6 is that the Maurer-Cartan forms have second-order poles in the direction of $U$. But $U$ is contractible. One might try to replace $G$ by $G/U$, an algebraic variety of the same homotopy type. We didn’t see that this was helpful.

D.2. The $Q$-filtration. Because of 6.1.6, we were led to introduce another filtration on the de Rham complex that is coarser than the Hodge filtration, yet induces the same filtration on cohomology.

We begin by letting $X$ be the disc $\Delta$, and $D = \{0\}$. The logarithmic (holomorphic) de Rham complex $\Omega^\bullet_X(\log D) = \{O_X \rightarrow \Omega^1 X(\log D)\}$ comes with its usual Hodge filtration $F$. Let $\Omega^\bullet_X(D)$ denote the larger complex of forms that are meromorphic along $D$. We define a decreasing filtration $Q$ on $\Omega^\bullet_X(D)$ by:

$$Q^0\Omega^\bullet_X(D) = \Omega^\bullet_X(D), \quad Q^1\Omega^\bullet_X(D) = F^1\Omega^\bullet_X(\log D) = \Omega^1 X(\log D)[-1],$$

and $Q^2\Omega^\bullet_X(D) = 0$. In other words, we “discount” a 1-form that has worse than logarithmic singularities. It is standard that the inclusion $\Omega^\bullet_X(\log D) \subset \Omega^\bullet_X(D)$ is a quasi-isomorphism, and it respects the filtrations.

**Proposition D.2.2.** The inclusion of filtered complexes

$$(\Omega^\bullet_X(\log D), F) \hookrightarrow (\Omega^\bullet_X(D), Q)$$

is a filtered quasi-isomorphism.

**Proof.** The real content of the above assertion is that

$$Gr^Q_0\Omega^\bullet_X(\log D) \rightarrow Gr^0_0\Omega^\bullet_X(D)$$

is a quasi-isomorphism. This is equivalent to the exactness of

$$0 \rightarrow O_X \rightarrow O_X(D) \xrightarrow{d} \Omega^1_X(D)/\Omega^1_X(\log D) \rightarrow 0,$$

which is elementary.

Next, we consider products of the preceding. For $X = \Delta^k$, and $D$ the union of the coordinate axes, so that $X - D = (\Delta^*)^k$, we use the tensor product of the $Q$-filtrations of the factors to define the $Q$-filtration on $\Omega^\bullet_X(D)$; and for the product of the this pair with a polydisc $\Delta^l$, we take the tensor product of the preceding with the Hodge filtration $F$ of $\Delta^l$. It is an easy exercise to check:

**Lemma D.2.3.** Let $X$ and $D$ be as above, so $X - D \cong (\Delta^*)^k \times \Delta^l$. Then the filtration $Q$ is independent of the choice of coordinates. The inclusion

$$(\Omega^\bullet_X(\log D), F) \hookrightarrow (\Omega^\bullet_X(D), Q)$$

is a filtered quasi-isomorphism.

This yields at once the following globalization:
Proposition D.2.4. Let $X$ be a complex manifold, and $D$ a divisor with normal crossings on $X$. Then there is a uniquely-defined filtration $Q$ of $\Omega^\bullet_X(*D)$, given by the above in any system of local coordinates adapted to $D$. Moreover, the inclusion
\[(\Omega^\bullet_X(\log D), F) \hookrightarrow (\Omega^\bullet_X(*D), Q)\]
is a filtered quasi-isomorphism.

Remark D.2.5. (i) In contrast to $F$, the filtration $Q$ is not functorial: e.g., in two variables $dz/w^2 \in Q^1$, whereas its restriction to the diagonal is $dz/z^2 \notin Q^1$. However, if $X$ and $D$ are as in the proposition, and $f : X' \rightarrow X$ is a morphism of complex manifolds, such that $f^*(D)$ is a reduced divisor with normal crossings, then pulling back by $f$ respects the $Q$-filtrations.

(ii) $Q$ is not multiplicative: both $dw$ and $dz/w^2$ are in $Q^1$, but $dw \wedge dz/w^3 \notin Q^2$. However, it does induce a functorial, multiplicative filtration on $H^\bullet(X - D, \mathbb{C})$, via the quasi-isomorphism of Proposition D.2.4.

Finally, let $X_\bullet$ be a smooth simplicial variety with smooth completion $\overline{X}_\bullet$, such that $D_\bullet = \overline{X}_\bullet - X_\bullet$ has normal crossings, and where the face maps preserve the $Q$ filtration. One defines in the obvious way a subcomplex $A^\bullet(\overline{X}_\bullet)_{\log D_\bullet}$ of the simplicial de Rham complex $A^\bullet(|\overline{X}_\bullet| \log D_\bullet)$, that contains $A^\bullet(|\overline{X}_\bullet|) \log D_\bullet$. One places the $Q$-filtration on $A^\bullet(\overline{X}_\bullet)_{\log D_\bullet}$ by taking the $Q$-filtration on each $A^\bullet(|\overline{X}_n| + D_n)$. Since the Hodge filtration $F$ of $A^\bullet(|\overline{X}_\bullet| \log D_\bullet)$ is defined similarly, it follows from Proposition D.2.4 that

Proposition D.2.6. Let $\overline{X}_\bullet$ and $D_\bullet$ be as above. Then the inclusion
\[(A^\bullet(|\overline{X}_\bullet| \log D_\bullet), F) \hookrightarrow (A^\bullet(|\overline{X}_\bullet| + D_\bullet, Q)\]
is a filtered quasi-isomorphism.

One can determine, at least, that the Maurer-Cartan connection forms, and then the Chern forms for the connection constructed in 6.1.5, are in $Q^1$ along the zero and polar loci of the determinant function (recall 6.1.6). However, for this fact to be useful, one needs a serviceable compactification of the simplicial model of $BG$.

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