Comment on “Solutions of the Schrödinger equation for the time-dependent linear potential”

Jian-Qi Shen 1,2 *

1 Centre for Optical and Electromagnetic Research, State Key Laboratory of Modern Optical Instrumentation, Zhejiang University, Hangzhou SpringJade 310027, P.R. China
2 Zhejiang Institute of Modern Physics and Department of Physics, Zhejiang University, Hangzhou 310027, P.R. China

(December 19, 2021)

We show that the solution obtained by Bekkar et al. in their comment [Phys. Rev. A 68, 016101 (2003)] on Guedes’s work of solving the quantum system with a time-dependent linear potential is still not the general one of the Schrödinger equation. It is concluded that Bekkar et al.’s solution (corresponding to the linear Lewis-Riesenfeld invariant) and our solution (corresponding to the quadratic-form Lewis-Riesenfeld invariant) presented here will constitute together a complete set of solutions (general solutions) of the time-dependent Schrödinger equation of the system under consideration.

PACS number(s): 03.65.Fd, 03.65.Ge

Recently, Guedes used the Lewis-Riesenfeld invariant formulation [1] and solved the one-dimensional Schrödinger equation with a time-dependent linear potential [2], the Hamiltonian of which is $H(t) = \frac{p^2}{2m} + f(t)q$ with $q$ and $p$ being the canonical variables. More recently, Bekkar et al. pointed out that [3] the result obtained by Guedes is merely the particular solution (that corresponds to the null eigenvalue of the linear Lewis-Riesenfeld invariant) rather than a general one. In the comment [3], Bekkar et al. stated that they correctly used the invariant method [1] and gave the general solutions of the time-dependent Schrödinger equation with a time-dependent linear potential [3]. However, in the present comment, we will show that although the solutions of Bekkar et al. is more general than that of Guedes [2], what they finally achieved in their comment [3] may be still not the general solutions, either. On the contrary, I think that their result [3] might also belongs to the particular one. The reason for this may be as follows: according to the Lewis-Riesenfeld invariant method [1], the solutions of the time-dependent Schrödinger equation can be constructed in terms of the eigenstates of the Lewis-Riesenfeld (L-R) invariants. It is known that both the squared of a L-R invariant (denoted by $I_l(t)$) and the product of two L-R invariants are also the invariants, which agree with the Liouville-Von Neumann equation $\frac{d}{dt}I_l(t) + \frac{1}{\hbar} [I_l(t), H(t)] = 0$, and that if $I_a$ and $I_b$ are the two L-R invariants of a certain time-dependent quantum system and $|\psi(t)\rangle$ is the solution of the time-dependent Schrödinger equation (corresponding to one of the invariants, say, $I_a$), then $I_b |\psi(t)\rangle$ is another solution of this quantum system. So, in an attempt to obtain the general solutions of a time-dependent system, one should first analyze the complete set of all L-R invariants of the system under consideration. Historically, in order to obtain the complete set of invariants, Gao et al. suggested the concept of basic invariants which can generate the complete set of invariants [4], as stated in Ref. [4], the basic invariants can be called invariant generators. As far as Bekkar et al.’s result [3] is concerned, the obtained solutions are the ones corresponding only to the linear invariant ($i.e., I_1(t) = A(t)p + B(t)q + C(t)$) that is simply one of the L-R invariants constituting a complete set. It is apparently seen that the quadratic form, $I_q(t) = D(t)p^2 + E(t)(pq + qp) + F(t)q^2 + A'(t)p + B'(t)q + C'(t)$, is also the one that can satisfy the Liouville-Von Neumann equation, since it is readily verified that the generators of $I_q(t)$ form a Lie algebra [5]. However, for the cubic-form invariant, it is easily seen that there exists no such closed Lie algebra. This point holds true also for the algebraic generators in various-power L-R invariants $I^n_l$ ($n > 3$). So, it is concluded that for the driven oscillator, only the linear $I_1(t)$ and quadratic $I_q(t)$ will form a complete set of L-R invariants. Note that here $I_q(t)$ should not be the squared of $I_1(t)$, i.e., $I_q(t) \neq cI_1^2(t)$, where $c$ is an arbitrary c-number. The existence of $I_q(t)$ that cannot be written as the squared of any $I_1$ was demonstrated in Ref. [5]. It is emphasized here that Bekkar et al.’s solution is the one constructed only in terms of the eigenstates of the linear invariant $I_1(t)$. Even though for the linear invariant $I_1(t)$ only, Bekkar et al.’s result [3] can truly be viewed as the complete set of solutions (in a certain sub-Hilbert-space), it still cannot be considered the general one of the Schrödinger equation, since the latter should contain those corresponding to the quadratic invariant $I_q(t)$. In brief, Bekkar et al.’s solution and our solution, which will be found in what follows, together constitute the complete set of solutions of the Schrödinger equation involving a time-dependent linear potential.

In accordance with the L-R theory [1], solving the eigenstates of the quadratic invariant $I_q(t)$ will enable physicists to obtain the solutions of the time-dependent Schrödinger equation. But, unfortunately, it is not easy for us to immediately solve the eigenvalue equation of the time-dependent invariant $I_q(t)$, for $I_q(t)$ involves the time-dependent parameters. So, in the following we will
use the invariant-related unitary transformation formulation [4], under which the time-dependent invariant can be transformed into a time-independent one $I_V$, and if the eigenstates of $I_V$ can be obtained conveniently, the eigenstates of $I_q(t)$ can then be easily achieved.

For this aim, we will employ two time-dependent unitary transformation operators

$$V_1(t) = \exp[q(t)q + \beta(t)p], \quad V_2(t) = \exp[\alpha(t)p^2 + \rho(t)q^2]$$

(1)

to get a time-independent $I_V$. The time-dependent parameters $\eta, \beta, \alpha$ and $\rho$ in (1) are purely imaginary functions, which will be determined in what follows [5]. Since the canonical variables (operators) $q$ and $p$ form a non-semisimple Lie algebra, here the first step is to transform $I_q(t)$ into $I_1(t)$, i.e., $I_1(t) = V_1^\dagger(t)I_q(t)V_1(t)$, which no longer involves the canonical variables $q$ and $p$, and the retained Lie algebraic generators in $I_1(t)$ are only $p^2, pq + qp, q^2$. Here the time-independent parameters in $V_1(t)$ are chosen [5]

$$\eta = \frac{EB^\prime - FA^\prime}{2i(E^2 - DF)}, \quad \beta = \frac{DB^\prime - EA^\prime}{2i(E^2 - DF)}.$$  

(2)

Note that the three generators $(p^2, pq + qp, q^2)$ in $I_1(t)$ also form a Lie algebra [5]. The second step is to obtain the time-independent $I_V$, which will be gained via the calculation of $I_V = V_2^\dagger(t)I_1(t)V_2(t)$. In this step, the obtained $I_V$ has no other generators (and time-dependent c-numbers) than $p^2$ and $q^2$, namely, $I_V$ may be written in the form $I_V = \frac{1}{2}\varsigma(p^2 + q^2)$ with $\varsigma$ being a certain parameter independent of time [5]. For the detailed and complicated derivation of the functions $\alpha$ and $\rho$, readers may be referred to Ref. [5]. It is well known that the eigenvalue equation of $I_V$ is of the form $I_V|n, q\rangle = (n + \frac{1}{2})\varsigma|n, q\rangle$, where $|n, q\rangle$ stands for the familiar stationary harmonic-oscillator wavefunction. Hence, the eigenstates of the time-dependent L-R invariant $I_q(t)$ can be achieved and the final result is $V_1(t)V_2(t)|n, q\rangle$ with the eigenvalue being $(n + \frac{1}{2})\varsigma$.

According to the L-R invariant theory [1], the particular solution $|n; q, t\rangle_S$ of the time-dependent Schrödinger equation is different from the eigenfunction of the invariant $I_q(t)$ only by a phase factor $\exp\left[\frac{1}{i}\phi_n(t)\right]$, the time-dependent phase of which is written as (in the unit $\hbar = 1$)

$$\phi_n(t) = \int_0^t\langle n, q|V^\dagger(t')\left[H(t') - i\frac{\partial}{\partial t'}\right]V(t')|n, q\rangle dt'$$

(3)

with $V(t) = V_1(t)V_2(t)$. This phase $\phi_n(t)$ can be calculated with the help of the Glauber formula and the Baker-Campbell-Hausdorff formula [6,7].

The particular solution $|n; q, t\rangle_S$ of the time-dependent Schrödinger equation corresponding to the invariant eigenvalue $(n + \frac{1}{2})\varsigma$ is thus of the form

$$|n; q, t\rangle_S = \exp\left[\frac{1}{i}\phi_n(t)\right]V_1(t)V_2(t)|n, q\rangle.$$  

(4)

Hence the general solution of the Schrödinger equation (corresponding to $I_q(t)$) can be written in the form

$$|\Psi(q, t\rangle_S = \sum_n c_n |n; q, t\rangle_S,$$  

(5)

where the time-independent c-number $c_n$’s are determined by the initial conditions, i.e., $c_n = \langle n, q|V^\dagger(t = 0)|n, q\rangle_S$.

Thus we found the general solutions of the Schrödinger equation for the time-dependent linear potential, which corresponds only to the quadratic-form invariant. As stated above, Bekkar et al.’s solution is not the general one of the Schrödinger equation. Likewise, the solution obtained here still does not form a complete set of solutions of this time-dependent Schrödinger equation, either. We conclude that Bekkar et al.’s solution and our solution presented here will together constitute such a complete set of solutions of the Schrödinger equation.

In addition, Guedes recently stated in his reply to Bekkar et al. that in order to obtain the general solutions of the time-dependent Schrödinger equation one must follow the L-R invariant theory step by step [8]. I think that “step by step” may not be the essence of getting the general solutions of Schrödinger equation. Instead, the key point for the present subject is that one should first find the complete set of all L-R invariants of the time-dependent quantum systems under consideration.

[1] H.R. Lewis, Jr. and W.B. Riesenfeld, J. Math. Phys.(N.Y) 10, 1458 (1969).
[2] I. Guedes, Phys. Rev. A 63, 034102 (2001).
[3] H. Bekkar, F. Benamira, and M. Maamache, Phys. Rev. A 68, 016101 (2003).
[4] X.C. Gao, J.B. Xu, and T.Z. Qian, Phys. Rev. A 44, 7016 (1991).
[5] J.Q. Shen, arXiv: quant-ph/0310179 (2003).
[6] J. Wei and E. Norman, J. Math. Phys.(N.Y) 4, 575 (1963).
[7] J.Q. Shen, H.Y. Zhu, and P. Chen, Euro. Phys. J. D 23, 305 (2003).
[8] I. Guedes, Phys. Rev. A 68, 016102 (2003).