Path integrals of a particle in a finite interval and on the half-line

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Abstract

We make use of point transformations to introduce new canonical variables for systems defined on a finite interval and on the half-line so that new position variables should take all real values from \(-\infty\) to \(\infty\). The completeness of eigenvectors of new momentum operators enables us to formulate time sliced path integrals for such systems. Short time kernels thus obtained require extension of the range of variables to the covering space in order to take all reflected paths into account. Upon this extension we determine phase factors attached to the amplitude for paths reflected at boundaries by taking singularities of the potential into account. It will be shown that the phase factor depends on parameters that characterize the potential; and further that the well-know minus sign in the amplitude for odd times reflection of a particle in a box should be understood as the special case for the corresponding value of the parameter of the potential.

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I. INTRODUCTION

Since the early days of quantum mechanics there exists only a limited set of exactly solvable models such as the harmonic oscillator and hydrogen-like potentials, the square well potential, the Morse potential and a fews. The factorization method, introduced by Schrödinger\(^1,2\) and Dirac\(^4\), opened a systematic treatment for dealing with systems of exact solvability. The classification of types in factorization of the second order differential equations was given by Infeld and Hull\(^5\). Today these solvable systems are well studied and understood from the viewpoint of shape invariance\(^6\) or supersymmetric quantum mechanics\(^7\)\(\text{-}10\). When utilized in conjunction with Crum’s theorem\(^11\), shape invariance or the factorization of a Hamiltonian ensures the exact solvability of the model in the Schrödinger picture.

In the framework of the Heisenberg picture, the solvability of a one-dimensional quantum mechanical system is related to the existence of a sinusoidal coordinate. If we can find a sinusoidal coordinate, by introducing creation and annihilation operators as its negative and positive frequency part respectively, we can define the ground state to be destroyed by the annihilation operator; then excited states are generated by repeated multiplication of the creation operator. The existence of a sinusoidal coordinate is connected to the closure relation\(^12\) which explicitly means that the second derivative, when differentiated by time, of a sinusoidal coordinate must be a linear combination of its first derivative as well as itself. Including the harmonic oscillator, there exist several models those can be solved by means of the algebraic method basing upon the creation and annihilation operators mentioned above but not all the systems with shape invariance can be solved by this method.

Considering the solvability of one-dimensional quantum systems from the path integral point of view, it had been restricted for a long time to systems described by quadratic Lagrangians defined over the unrestricted one-dimensional space. The breakthrough was the solvability of the path integral for hydrogen atom by means of the Duru-Kleinert(DK) transformation\(^13\)\(\text{-}14\). Initiated by their brilliant work, there appeared many works as the application of their formalism\(^15\)\(\text{-}29\). Also some attempts were made to elaborate on the meaning of the method and the gauge invariance behind it\(^30\)\(\text{-}32\). In this way, the number of solvable path integrals has been considerably increased with the aid of the DK transformation. In addition to path integrals exactly solvable by means of DK transformation, there have been found path integrals defined on some homogeneous spaces to be evaluated.
exactly by semiclassical method, thus providing examples of the localization formula. The key to understand the solvability of these systems with the localization formula was the existence of the quadratic Hamiltonian with first class constraints. Another kind of exactly solved path integration will be those for a particle in a box or on a circle. Feynman kernels of these geometrically non-trivial systems are characterized by the emergence of one dimensional representation of the fundamental group basing upon the analysis from the viewpoint of homotopy. On the phase factor of a Feynman kernel, we here raise a question: for the Feynman kernel of the symmetric Pöschl-Teller potential, should we apply a phase factor \(-1\) for paths reflected odd times at the boundaries? Since this model involves a free particle in a box as a special case, we may expect it to be true; it does not seem, however, to be a trivial question.

Common to exactly solvable path integrals mentioned above, excepting systems with quadratic Lagrangians on a free one-dimensional space and a particle on a circle, the origin of the difficulty in formulating path integrals would be that these systems are defined on a finite interval or on the half-line upon which we cannot define Hermitian momentum operators. Nevertheless, in many papers, path integrals for these systems are argued in terms of the Lagrangian path integral; and moreover most of them are expressed in the continuum representation. As is well known, Lagrangian path integrals for the harmonic oscillator and a free particle are obtained from the Hamiltonian one after performing the Gaussian integration with respect to the momentum. Furthermore the continuum representation of a path integral can be defined just as a limit from a time sliced one formulated with due care. Absence of the Hermitian momentum operator prevents us from carrying out the procedure of the transition from a Hamiltonian path integral to the corresponding Lagrangian form in most cases excepting the use of radial plane wave for the radial path integral. Regarding the path integral on the half-line, there seems to be an attempt to formulate it from the viewpoint of random walk by modifying the action to incorporate the effect of the boundary condition into as well as the one to define the measure of path integral in a suitable way according to the time paths spend near the boundary. These approaches cannot be, however, generic prescriptions for constructing path integrals of solvable systems. Therefore, including these rather special methods for path integrals on the half-line, Lagrangian path integrals in the continuum representation for solvable models must be validated by testing the consistency with the Schrödinger equation. In view of these facts, there seems to remain a room to
put time sliced path integrals to be examined for solvable systems constructed from the Hamiltonian formulation by keeping rigid connection with the Schrödinger theory. We shall try in this paper to develop a method of conversion from Hamiltonian path integrals into Lagrangian ones for systems on a finite interval and on the half-line. To this aim, we introduce a point transformation\cite{48-51} from the original position variable $x$ to $Q = Q(x)$, which is suitably defined to take all real values from $-\infty$ to $\infty$, for each case. In addition to this new position variable, we define its canonical conjugate $P$ as a new momentum operator.

The completeness of eigenvectors of new momentum operator defined this way will be utilized to formulate Hamiltonian path integrals and convert them into the Lagrangian ones.

This paper is organized as follows: in section 2 we describe the point transformation and see the completeness of the newly defined momentum eigenvectors. Section 3 gives explanation to the formulation of path integrals in terms of the eigenvectors of the new momentum operator. A brief sketch of a technique for evaluating the complicated kinetic term in the time sliced path integral thus formulated will also be given there. We perform explicit calculations of time sliced path integrals of Feynman kernels for symmetric and generalized Pöschl-Teller potential as well as the radial harmonic oscillator in section 4. The final section is devoted to the conclusion. The ambiguity in the form of the potential term in the path integral will be briefly discussed in the appendix.

II. POINT TRANSFORMATION TO NEW CANONICAL VARIABLES

For a system defined on $a < x < b$, we introduce a monotonically increasing function $f(x)$ and define

$$Q(x) \equiv \int_{x_0}^{x} \frac{dx}{f'(x)}, \quad f'(x) \equiv \frac{df(x)}{dx},$$

where $x_0$ should be chosen in a suitable way. We further introduce a differential operator\cite{51} $P$ by

$$P\psi(x) \equiv -\frac{i\hbar}{2} \left\{ f'(x) \frac{d}{dx} + \frac{d}{dx} f'(x) \right\} \psi(x)$$

in addition to $Q(x)$ above. An eigenvalue equation $P\psi_{P'}(x) = P'\psi_{P'}(x)$ can be solved by

$$\psi_{P'}(x) = \frac{1}{\sqrt{2\pi\hbar f'(x)}} e^{iP'\frac{Q(x)}{\hbar}}.$$
If the range of $Q(x)$ covers all real values from $-\infty$ to $\infty$, we find that $\psi_P(x)$ satisfies the normalization

$$\int_a^b \overline{\psi_P^*(x)} \psi_P(x) \, dx = \int_a^b \frac{dx}{2\pi\hbar f'(x)} e^{-i(P_1 - P_2)Q(x)/\hbar}$$

$$= \int_{-\infty}^{\infty} \frac{dQ}{2\pi\hbar} e^{-i(P_1 - P_2)Q/\hbar}$$

$$= \delta(P_1 - P_2)$$

for real values $P_1$ and $P_2$. The completeness of the eigenfunction of $P$ can be easily checked as follows:

$$\int_{-\infty}^{\infty} \psi_P(x) \overline{\psi_P(x')} \, dP = \frac{1}{\sqrt{f'(x)f'(x')}} \delta(Q(x) - Q(x')) = \delta(x - x').$$

We may therefore write the eigenfunction $\psi_P(x)$ as $\psi_P(x) = \langle x | P \rangle$ to write the completeness above as

$$\int_{-\infty}^{\infty} |P\rangle \langle P| \, dP = 1,$$

where 1 is the unity. If we introduce a bra vector $\langle Q |$ by

$$\langle Q | \equiv \sqrt{f'(x)} \langle x |,$$

the definition of the operator $P$ is converted into

$$\langle Q | P = -i\hbar \frac{d}{dQ} \langle Q |.$$

The completeness of $|Q\rangle$ is equivalent to that of $|x\rangle$:

$$\int_{-\infty}^{\infty} |Q\rangle \langle Q| \, dQ = \int_a^b |x\rangle \langle x| \, dx = 1.$$

In this way we see that operators $Q = Q(x)$ and $P$ defined above form a pair of canonical variables which fullfills the canonical commutation relation $[Q, P] = i\hbar$.

III. CONVERSION OF THE HAMILTONIAN AND THE CONSTRUCTION OF THE PATH INTEGRAL

On an interval $a < x < b$, we may consider a quantum system described by the Schrödinger equation

$$\left\{-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)\right\} \psi(x) = E\psi(x)$$

(III.1)
with boundary conditions \( \psi(a) = \psi(b) = 0 \). Since wave functions must vanish at the endpoints, the formally defined momentum operator \( p \) given by
\[
\langle x | p = -i\hbar \frac{d}{dx} \langle x |
\]
(III.2)
cannot be self-adjoint when \( a \) and/or \( b \) being finite. For such cases, we cannot utilize the completeness of the eigenvector of the momentum operator. To avoid this inconvenient situation, we rewrite the Hamiltonian
\[
H = \frac{1}{2m} p^2 + V(x)
\]
(III.3)
in terms of \( P \) introduced in the previous section. By a simple and straightforward manipulation, we obtain
\[
H = \frac{1}{2m} \frac{1}{f'(x)} \frac{1}{f'(x)} P^2 f'(x) + \frac{\hbar^2}{8m} \left[ \left\{ \frac{f''(x)}{f'(x)} \right\}^2 - \frac{2f'''(x)}{f'(x)} \right] + V(x).
\]
(III.4)
It will be useful here to notice that a commutation relation of a function \( G(x) \) and the operator \( P \) is given by
\[
\left[ G(x), P \right] = i\hbar f'(x) G'(x).
\]
(III.5)
We now make use of the completeness of the eigenvector of \( P \) to find a short time kernel
\[
K(x, x'; \epsilon) = \langle x | \left( 1 - \frac{\epsilon}{\hbar} H \right) | x' \rangle
\]
(III.6)
with the imaginary time \( \epsilon = \beta/N(\beta > 0) \) for \( N \rightarrow \infty \). For the term proportional to \( P^2 \), we evaluate it as
\[
\langle x | \frac{1}{f'(x)} \frac{1}{f'(x)} P^2 f'(x) f'(x') \int_{-\infty}^{\infty} P^2 \langle x | P \rangle \langle P | x' \rangle dP \]
(III.7)
and we may assume some ordering prescription in the evaluation of potential terms to find
\[
\langle x | \left\{ \frac{\hbar^2}{8m} \left[ \left\{ \frac{f''(x)}{f'(x)} \right\}^2 - \frac{2f'''(x)}{f'(x)} \right] + V(x) \right\} | x' \rangle = V_{\text{eff}}(x, x') \int_{-\infty}^{\infty} \langle x | P \rangle \langle P | x' \rangle dP.
\]
(III.8)
The precise form of \( V_{\text{eff}}(x, x') \), the dependences on \( x \) and \( x' \) in particular, will depend on systems we are dealing with. Keeping this in mind, we exponentiate the \( \epsilon \)-number Hamiltonian to obtain a Gaussian integral
\[
K(x, x'; \epsilon) = \int_{-\infty}^{\infty} \frac{dP}{2\pi \hbar \sqrt{f'(x)f'(x')}} \times \exp \left[ \frac{i}{\hbar} P \left\{ Q(x) - Q(x') \right\} - \frac{\epsilon}{\hbar} \left\{ \frac{P^2}{2mf'(x)f'(x')} + V_{\text{eff}}(x, x') \right\} \right]
\]
(III.9)
which results in
\[ K(x, x'; \epsilon) = \sqrt{\frac{m}{2\pi\hbar\epsilon}} \exp \left[ -\frac{m}{2\hbar\epsilon} f'(x)f'(x') \{Q(x) - Q(x')\}^2 - \frac{\epsilon}{\hbar} V_{\text{eff}}(x, x') \right]. \]

We thus obtain a time sliced path integral
\[ \langle x_F | e^{-\beta H/\hbar} | x_I \rangle = \lim_{N \to \infty} \left( \frac{m}{2\pi\hbar\epsilon} \right)^{N/2} \int_a^b \prod_{i=1}^{N-1} dx_i \times \exp \left[ -\sum_{j=1}^{N} \left\{ \frac{m}{2\hbar\epsilon} f'(x_j)f'(x_{j-1}) \{Q(x_j) - Q(x_{j-1})\}^2 + \frac{\epsilon}{\hbar} V_{\text{eff}}(x_j, x_{j-1}) \right\} \right], \]

where we have set \( x_F = x_N \) and \( x_I = x_0 \).

We will observe in the following, if \( x \) ranges over the whole real axis so that the equation \( Q(x) - Q(x') = 0 \) has only a single solution \( x - x' = 0 \), that the time sliced path integral given by (III.11) can be converted into
\[ \langle x_F | e^{-\beta H/\hbar} | x_I \rangle = \lim_{N \to \infty} \left( \frac{m}{2\pi\hbar\epsilon} \right)^{N/2} \int_{-\infty}^{\infty} \prod_{i=1}^{N-1} dx_i \exp \left[ -\sum_{j=1}^{N} \left\{ \frac{m}{2\hbar\epsilon} (\Delta x_j)^2 + \frac{\epsilon}{\hbar} V(x_j, x_{j-1}) \right\} \right], \]

where \( \Delta x_j \equiv x_j - x_{j-1} \) and \( V(x_j, x_{j-1}) = \langle x_j | V(x) | x_{j-1} \rangle \). Obviously this is nothing but a time sliced path integral we usually obtain by making use of the completeness of the eigenvector of the momentum operator \( p \). It is, therefore, trivial result for systems defined on the whole real line. The assumption that the path integral is dominated by the contribution from the saddle point at \( \Delta x_j = 0 \) will not be, however, fulfilled when both \( a \) and \( b \) are, or at least one of them is, finite. For such cases we have to extend the domain of the integration form \( a < x < b \) to the whole real line in order to take contributions from multiple saddle points into account. This will be explained in the next section through solving examples explicitly and in an exact manner.

Let us check the validity of the short time kernel (III.10) and observe that (III.11) is equivalent to (III.12) which possesses an Euclidean Lagrangian
\[ L_E = \frac{m}{2m} \dot{x}^2 + V(X) \]

in its exponent. To this aim we first consider
\[ \psi(x, \beta + \epsilon) = \int_a^b K(x, x'; \epsilon)\psi(x', \beta) \, dx' \]
for infinitesimally small \( \epsilon \). Note that the range of the integration above should not be extended to outside the physical domain \( a < x < b \) because the wave function \( \psi(x, \beta) \) is defined on this domain. Therefore, even for systems on the half-line or on a finite interval, the evaluation of the integration given below remains to be valid. If we assume that exists no other critical points in the exponent of the short time kernel other than the one at \( x' = x \), we may expect that, in the limit \( \epsilon \to 0 \), the short time kernel tends to the delta function \( \delta(x - x') \). Therefore the integration with respect to \( x' \) will be dominated by the contribution from the vicinity of \( x \). By setting \( x' = x + \eta \), we find

\[
f'(x) f'(x') \{ Q(x) - Q(x') \}^2 = \eta^2 + \left\{ f'(x) Q''(x) + \frac{f''(x)}{f'(x)} \right\} \eta^3 + \frac{1}{4} \left\{ f'(x) Q''(x) \right\}^2 + \frac{1}{3} f'(x) Q'''(x) + f''(x) Q''(x) + \frac{f'''(x)}{2 f'(x)} \right] \eta^4 + O(\eta^5) \tag{III.15}
\]

where use has been made of Taylor expansions for \( f'(x') \) and \( Q(x') \). If we recall the definition of \( Q(x) \), we observe that coefficients in the right hand side above are given by

\[
f'(x) Q''(x) + \frac{f''(x)}{f'(x)} = 0 \tag{III.16}
\]

and

\[
\frac{1}{4} \left\{ f'(x) Q''(x) \right\}^2 + \frac{1}{3} f'(x) Q'''(x) + f''(x) Q''(x) + \frac{f'''(x)}{2 f'(x)} = -\frac{1}{12} \left[ \left\{ \frac{f''(x)}{f'(x)} \right\}^2 - \frac{2 f'''(x)}{f'(x)} \right]. \tag{III.17}
\]

In the same way, we expand \( \psi(x', \beta) \) into the Taylor series to obtain

\[
\psi(x', \beta) = \psi(x, \beta) + \frac{\partial \psi(x, \beta)}{\partial x} \eta + \frac{1}{2} \frac{\partial^2 \psi(x, \beta)}{\partial x^2} \eta^2 + O(\eta^3). \tag{III.18}
\]

The integration in the right hand side of (III.13) is now rewritten as

\[
\sqrt{\frac{m}{2\pi \hbar \epsilon}} \int_{a-x}^{b-x} \exp \left( -\frac{m}{2 \hbar \epsilon} \eta^2 \right) \left\{ 1 + \frac{m}{2 \hbar \epsilon} \left[ \left\{ \frac{f''(x)}{f'(x)} \right\}^2 - \frac{2 f'''(x)}{f'(x)} \right] \right\} \eta^4 \times \left\{ 1 - \frac{\epsilon}{\hbar} V_{\text{eff}}(x, x) \right\} \left\{ \psi(x, \beta) + \frac{\partial \psi(x, \beta)}{\partial x} \eta + \frac{1}{2} \frac{\partial^2 \psi(x, \beta)}{\partial x^2} \eta^2 \right\} d\eta \tag{III.19}
\]

by discarding irrelevant terms in the limit \( \epsilon \to 0 \). We here make a change of variables given by

\[
\xi \equiv \sqrt{\frac{m}{2 \hbar \epsilon}} \eta, \tag{III.20}
\]

In the same way, we expand \( \psi(x', \beta) \) into the Taylor series to obtain

\[
\psi(x', \beta) = \psi(x, \beta) + \frac{\partial \psi(x, \beta)}{\partial x} \eta + \frac{1}{2} \frac{\partial^2 \psi(x, \beta)}{\partial x^2} \eta^2 + O(\eta^3). \tag{III.18}
\]

The integration in the right hand side of (III.13) is now rewritten as

\[
\sqrt{\frac{m}{2\pi \hbar \epsilon}} \int_{a-x}^{b-x} \exp \left( -\frac{m}{2 \hbar \epsilon} \eta^2 \right) \left\{ 1 + \frac{m}{2 \hbar \epsilon} \left[ \left\{ \frac{f''(x)}{f'(x)} \right\}^2 - \frac{2 f'''(x)}{f'(x)} \right] \right\} \eta^4 \times \left\{ 1 - \frac{\epsilon}{\hbar} V_{\text{eff}}(x, x) \right\} \left\{ \psi(x, \beta) + \frac{\partial \psi(x, \beta)}{\partial x} \eta + \frac{1}{2} \frac{\partial^2 \psi(x, \beta)}{\partial x^2} \eta^2 \right\} d\eta \tag{III.19}
\]

by discarding irrelevant terms in the limit \( \epsilon \to 0 \). We here make a change of variables given by

\[
\xi \equiv \sqrt{\frac{m}{2 \hbar \epsilon}} \eta, \tag{III.20}
\]
to regard the integration with respect to $\xi$ as a Gaussian integration from $-\infty$ to $\infty$ for small $\epsilon$. We then obtain

$$
\psi(x, \beta+\epsilon) = \psi(x, \beta) - \frac{\epsilon}{\hbar} \left\{ - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_{\text{eff}}(x, x) - \frac{\hbar^2}{8m} \left\{ \frac{f''(x)}{f'(x)} \right\}^2 - \frac{2f'''(x)}{f'(x)} \right\} \psi(x, \beta).
$$

(III.21)

Since $V_{\text{eff}}(x, x)$ is given by

$$
V_{\text{eff}}(x, x) = \frac{\hbar^2}{8m} \left\{ \frac{f''(x)}{f'(x)} \right\}^2 - \frac{2f'''(x)}{f'(x)} + V(x),
$$

(III.22)

it is evident that the first term in $V_{\text{eff}}(x, x)$ is precisely canceled by the last term in the right hand side of (III.21). We hence observe that the result obtained above is equivalent to

$$
-\frac{\hbar}{\beta} \frac{\partial}{\partial \beta} \psi(x, \beta) = \left\{ - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right\} \psi(x, \beta).
$$

(III.23)

This is precisely the imaginary time version of the Schrödinger equation. We have thus confirmed the validity of the short time kernel given by (III.10). This fact suggests that we can convert the time sliced path integral (III.11) into (III.12) at least for a system on the whole real axis.

To achieve the aim we rewrite $f'(x)$ and $f'(x')$ by expanding them around $x^{(\alpha)} = \bar{x} - \alpha \Delta x (\bar{x} \equiv (x + x')/2, \Delta x \equiv x - x')$ as

$$
f'(x) = f'(x^{(\alpha)}) + f''(x^{(\alpha)}) \left( \frac{1}{2} + \alpha \right) \Delta x + \frac{1}{2} f'''(x^{(\alpha)}) \left( \frac{1}{2} + \alpha \right)^2 (\Delta x)^2 + \cdots
$$

(III.24)

and

$$
f'(x') = f'(x^{(\alpha)}) + f''(x^{(\alpha)}) \left( \frac{1}{2} - \alpha \right) \Delta x + \frac{1}{2} f'''(x^{(\alpha)}) \left( \frac{1}{2} - \alpha \right)^2 (\Delta x)^2 + \cdots
$$

(III.25)

to find

$$
\begin{align*}
&f'(x)f'(x') = \left\{ f'(x^{(\alpha)}) \right\}^2 \times \left[ 1 + \alpha \frac{f''(x^{(\alpha)})}{f'(x^{(\alpha)})} \Delta x + \frac{1}{2} \left\{ (\alpha^2 + 1/4) \frac{f'''(x^{(\alpha)})}{f'(x^{(\alpha)})} - \frac{1}{4} \left( \frac{f''(x^{(\alpha)})}{f'(x^{(\alpha)})} \right)^2 \right\} (\Delta x)^2 + \cdots \right]^2.
\end{align*}
$$

(III.26)

Then, in the same way, we expand $Q(x) - Q(x')$ into a series as

$$
Q(x) - Q(x') = -\frac{\Delta x}{f'(x^{(\alpha)})} \left[ 1 - \alpha \frac{f''(x^{(\alpha)})}{f'(x^{(\alpha)})} \Delta x + \left\{ \left( \frac{f''(x^{(\alpha)})}{f'(x^{(\alpha)})} \right)^2 - 2f'''(x^{(\alpha)}) \right\} \frac{1}{2} (\alpha^2 + 1/12) (\Delta x)^2 + \cdots \right],
$$

(III.27)
so that we obtain
\[ f'(x)f'(x')\{Q(x) - Q(x')\}^2 = \{R(x^{(a)}, \Delta x)\Delta x\}^2 \] (III.28)
in which \(R(x^{(a)}, \Delta x)\) being given by
\[ R(x^{(a)}, \Delta x) = 1 - \frac{1}{24} \left\{ \left( \frac{f''(x^{(a)})}{f'(x^{(a)})} \right)^2 - \frac{2f'''(x^{(a)})}{f'(x^{(a)})} \right\} (\Delta x)^2 + \cdots \] (III.29)

According to ref. 32, a time sliced path integral with a complicated kinetic term given by
\[ \frac{1}{2\lambda} \{\Delta x_j R(x^{(a)}_j, \Delta x_j)\}^2, \quad R(x^{(a)}_j, \Delta x_j) = 1 + a_2(x^{(a)}_j)\Delta x_j + a_3(x^{(a)}_j)(\Delta x_j)^2 + \cdots, \] (III.30)

can be converted into its equivalent with the standard kinetic term:
\[
\left( \frac{1}{2\pi \lambda} \right)^{N/2} \int_{-\infty}^{\infty} \prod_{i=1}^{N-1} dx_i \exp \left[ -\sum_{j=1}^{N} \frac{1}{2\lambda} \{\Delta x_j R(x^{(a)}_j, \Delta x_j)\}^2 \right] = \left( \frac{1}{2\pi \lambda} \right)^{N/2} \int_{-\infty}^{\infty} \prod_{i=1}^{N-1} dx_i \exp \left[ -\sum_{j=1}^{N} \left\{ \frac{(\Delta x_j)^2}{2\lambda} + \lambda U_{\text{add}}(x^{(a)}_j) \right\} \right]
\] (III.31)

where \(\lambda = \hbar \epsilon/m\) and the additional potential being given by
\[ U_{\text{add}}(x) = 3a_3(x) - 3(\alpha - 1/2)a_2'(x) - 2\{a_2(x)\}^2 \] (III.32)

which is equivalent to
\[ V_{\text{add}}(x) = \frac{\hbar^2}{m} \left\{ 3a_3(x) - 3(\alpha - 1/2)a_2'(x) - 2\{a_2(x)\}^2 \right\}. \] (III.33)

Here we have written \(\lambda U_{\text{add}}(x)\) as
\[ \lambda U_{\text{add}}(x) = \frac{\epsilon}{\hbar} V_{\text{add}}(x). \] (III.34)

In view of the series expansion (III.29) above, we immediately obtain an additional potential
\[ V_{\text{add}}(x) = -\frac{\hbar^2}{8m} \left\{ \left( \frac{f''(x)}{f'(x)} \right)^2 - \frac{2f'''(x)}{f'(x)} \right\} \] (III.35)
as contribution to the path integral through rewriting (III.10) to have the kinetic term in the standard form. Since the effective potential in the exponent of (III.11) can be taken to be
\[ V_{\text{eff}}(x_j, x_{j-1}) = \frac{\hbar^2}{8m} \left\{ \left( \frac{f''(x^{(a)}_j)}{f'(x^{(a)}_j)} \right)^2 - \frac{2f'''(x^{(a)}_j)}{f'(x^{(a)}_j)} \right\} + V(x^{(a)}_j), \] (III.36)
the cancellation of the term, that emerged by rewriting the Hamiltonian in terms of $P$, by $V_{\text{add}}(x_j^{(\alpha)})$ is evident. We therefore obtain the time sliced path integral (III.12) as an equivalent one for (III.11). As can be easily seen, the exact cancellation above holds true at each saddle point even when there exist multiple saddle points if the system possesses suitable periodicity or a symmetry. Here, a comment is in need; we have employed the $\alpha$-ordering for evaluating the potential term of the path integral at $x^{(\alpha)}$. This is useful to check the ordering independence of the path integral for the case $x$ takes its value from $-\infty$ to $\infty$. For systems defined on the half-line or on a finite interval, there will be other natural and suitable scheme for evaluating potential term. For example, as the midpoint prescription for a system on the half-line, we may write $V(x,x')$ in the path integral as $V(\sqrt{x x'})$ because the geometric mean will be more suitable than taking the arithmetic average as $V(\bar{x})$.

IV. SYSTEMS WITH NON-TRIVIAL GEOMETRY

In this section, we consider path integrals for systems confined within a finite interval and for systems on the half-line as examples for the time sliced path integral (III.11). For a system defined on $0 < x < L$, we may expect that $f'(x)$ behaves like $f'(x) \propto x$ near the origin and like $f'(x) \propto L-x$ when $x$ approaches to $L$ so that the new coordinate $Q(x)$ takes its value from $-\infty$ to $\infty$. We may further assume that the Hamiltonian should have a term proportional to $1/\{f'(x)\}^2$ as a potential for the physical requirement that wave functions must vanish at boundaries. In the same way, we may assume $f'(x) \propto x$ near the origin and a potential proportional to $1/x^2$ for a system on the half-line. If we introduce potential terms in this way, we can expect wave functions to behave like $\{f'(x)\}^\alpha$ where $\alpha$ being determined by the coefficient of the potential term. If $\alpha$ is not an integer, this behavior makes wave functions be multi-valued when considered on the extended domain which is needed to take effects of reflections at boundaries into account. Since $f'(x)$ vanishes at the boundary, the sign of this function will change if we put $x$ outside the original domain. It is, therefore, important to take the phase factor of the short time kernel into account when we extend the domain to evaluate contributions from multiple saddle points.
A. Systems on a finite interval

We consider here a system described by the Hamiltonian

\[ H = \frac{1}{2m} p^2 + \mathcal{R} \frac{\nu(\nu - 1)}{\sin^2 \theta}, \quad \mathcal{R} \equiv \frac{\hbar^2}{2ma^2}, \quad \theta \equiv \frac{x}{a}, \quad a \equiv \frac{L}{\pi}, \] (IV.1)
on a finite interval \(0 < x < L\). A suitable choice for the function \(f(x)\) will be given by

\[ f(x) = -a \cos \theta \] (IV.2)
to yield

\[ Q(x) = a \log \{\tan(\theta/2)\} \] (IV.3)
and

\[ P = -\frac{i\hbar}{2a} \left( \sin \theta \frac{d}{d\theta} + \frac{d}{d\theta} \sin \theta \right). \] (IV.4)
The eigenfunction of this operator is given by

\[ \psi_P(x) = \frac{1}{\sqrt{2\pi \hbar \sin \theta}} \exp \left[ \frac{i}{\hbar} a P' \log \{\tan(\theta/2)\} \right] \] (IV.5)
whose completeness can be seen as, \(\theta'\) corresponding to \(x'\) like \(\theta\) for \(x\) above,

\[ \int_{-\infty}^{\infty} \psi_P(x) \psi_P(x') \, dP = \frac{1}{a \sqrt{\sin \theta \sin \theta'}} \delta(\log \{\tan(\theta/2)\} - \log \{\tan(\theta'/2)\}) = \delta(x - x') \] (IV.6)
for \(x\) and \(x'\) belonging to the interval \((0, L)\). We can utilize this completeness relation to find the short time kernel

\[ \mathcal{K}(\theta, \theta'; \epsilon) = \frac{1}{\sqrt{2\pi \lambda}} \exp \left[ -\frac{1}{2\lambda} \sin \theta \sin \theta' \left\{ \log \frac{\tan(\theta/2)}{\tan(\theta'/2)} \right\}^2 - \lambda U_{\text{eff}}(\theta, \theta') \right], \quad \lambda \equiv \frac{\hbar \epsilon}{ma^2} \] (IV.7)
where the effective potential being given by

\[ U_{\text{eff}}(\theta, \theta') = \frac{1}{8} \left( 1 + \frac{1}{\sin \theta \sin \theta'} \right) + \frac{1}{2} \frac{\nu(\nu - 1)}{\sin \theta \sin \theta'} \] (IV.8)
for the Hamiltonian given by (IV.1). Note that the kernel (IV.7) is normalized to fit the integration with respect to \(\theta\) instead of \(x\). This allows us to make expressions below considerably simple. If we divide it by \(a\), we will obtain the one corresponding to the integration with respect to \(x\). It will be clear that we have employed the geometric mean as the ordering prescription for \(U_{\text{eff}}(\theta, \theta')\).
In addition to the saddle point located at $\theta - \theta' = 0$, there exist infinitely many saddle points $\theta - \theta' = \pm 2n\pi(n = 1, 2, 3, \ldots)$ for this system. Therefore we have to take contributions from all these saddle points into account. We thus collect contributions from paths reflected even times at boundaries to obtain

$$K^{(e)}(\theta, \theta'; \epsilon) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2\lambda} (\theta - \theta' - 2n\pi)^2 - \frac{\lambda}{2} \frac{\nu(\nu-1)}{\sin \theta \sin \theta'} \right]. \quad \text{(IV.9)}$$

Remembering the periodicity and its Taylor expansion of $1 - \cos \Delta \theta$, we can sum up the series above to obtain

$$K^{(e)}(\theta, \theta'; \epsilon) = \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{\lambda} \cos(\theta - \theta') - \frac{\lambda}{8} - \frac{\lambda}{2} \frac{\nu(\nu-1)}{\sin \theta \sin \theta'} \right]. \quad \text{(IV.10)}$$

Note that $2(1 - \cos \Delta \theta)$ is expressed for small $\Delta \theta$ as

$$2(1 - \cos \Delta \theta) = \left\{ \Delta \theta \left( 1 - \frac{1}{24} (\Delta \theta)^2 + \cdots \right) \right\}^2, \quad \text{(IV.11)}$$

the kinetic term $(1 - \cos \Delta \theta)/\lambda$ generates a constant additional potential $-\lambda/8$ when converted into the standard form.

In order to obtain the component of the kernel composed of contributions from paths reflected odd times at boundaries, we need to replace $\sin \theta \sin \theta'$, $\tan(\theta/2)$ and $\tan(\theta'/2)$ in (IV.7) with $|\sin \theta \sin \theta'|$, $|\tan(\theta/2)|$ and $|\tan(\theta'/2)|$, respectively. In addition to this change, we have to determine the phase factor which the kernel acquires when $\theta$ or $\theta'$ goes outside the original domain. This can be achieved by considering solutions of the stationary Schrödinger equation for the Hamiltonian (IV.1). In view of singularities of the potential at boundaries, we see that eigenfunctions must be proportional to $(\sin \theta)^\nu$. Let us keep $\theta'$ remaining in the original domain and consider $\theta$ goes outside the domain. On the original domain, we set $\arg(\sin \theta) = 0$. We then define a contour for extending the domain to the covering space as shown in Fig.1. As can be seen in Fig.1 we let $\theta$ go along an infinitesimally small semicircle below $(2n + 1)\pi$ and above $2n\pi(n = -\infty, \ldots, \infty)$. Let us define $D^{(e)} = \{(2n\pi, (2n+1)\pi) \mid n = 0, \pm 1, \pm 2, \ldots \}$ as well as $D^{(o)} = \{((2n+1)\pi, (2n+2)\pi) \mid n = 0, \pm 1, \pm 2, \ldots \}$. It is easy to see that $\arg(\sin \theta) = 0$ if $\theta$ belongs to $D^{(e)}$ and $\arg(\sin \theta) = \pi$ if $\theta$ belongs to $D^{(o)}$. Taking $\arg(\sin \theta) = 0$ for $D^{(e)}$ is consistent with definition of $K^{(e)}(\theta, \theta'; \epsilon)$ above and the kernel must be proportional to $e^{\nu\pi i}$ if $\theta$ belongs to $D^{(o)}$. 

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FIG. 1. The contour for extending the domain to the covering space.

We thus obtain

\[ K^{(o)}(\theta, \theta'; \epsilon) = e^{\nu \pi i} \frac{1}{\sqrt{2\pi \lambda}} \exp \left( - \frac{1 - \cos \theta \cos \theta' - \lambda}{\lambda} - \frac{\nu}{8} \right) \times \exp \left\{ - \frac{\sin \theta \sin \theta'}{\lambda} + \frac{\lambda}{2} \frac{\nu(\nu - 1)}{\sin \theta \sin \theta'} \right\} \]  \hspace{1cm} (IV.12)

as the sum of contributions from paths reflected odd times at boundaries. This can be seen explicitly if we rewrite it as

\[ K^{(o)}(\theta, \theta'; \epsilon) = e^{\nu \pi i} \sqrt{\frac{2}{\sqrt{2\pi \lambda}}} \exp \left[ - \frac{1 - \cos(\theta + \theta')}{\lambda} - \frac{\lambda}{8} + \frac{\lambda}{2} \frac{\nu(\nu - 1)}{\sin \theta \sin \theta'} \right] = e^{\nu \pi i} \sum_{n=-\infty}^{\infty} \sqrt{\frac{2}{\sqrt{2\pi \lambda}}} \exp \left[ - \frac{1}{2\lambda} (\theta + \theta' + 2n\pi)^2 + \frac{\lambda}{2} \frac{\nu(\nu - 1)}{\sin \theta \sin \theta'} \right]. \]  \hspace{1cm} (IV.13)

We are now ready to extend the definition of the short time kernel (IV.7) to the covering space given by the sum of \( D^{(e)} \) and \( D^{(o)} \). The short time kernel (IV.7) defined on the original domain should be evaluated on the covering space to be written as

\[ K(\theta, \theta'; \epsilon) = K^{(e)}(\theta, \theta'; \epsilon) + K^{(o)}(\theta, \theta'; \epsilon). \]  \hspace{1cm} (IV.14)

In view of (IV.12) and its counterpart for \( K^{(e)}(\theta, \theta'; \epsilon) \), it is clear that \( K^{(e)}(\theta, \theta'; \epsilon) \) and \( K^{(o)}(\theta, \theta'; \epsilon) \) are corresponding to the first and second terms respectively in the asymptotic form of the modified Bessel function

\[ I_{\nu-1/2} \left( \frac{\sin \theta \sin \theta'}{\lambda} \right) \sim \sqrt{\frac{\lambda}{2\pi \sin \theta \sin \theta'}} \times \exp \left\{ - \frac{\sin \theta \sin \theta'}{\lambda} - \frac{\lambda}{2} \frac{\nu(\nu - 1)}{\sin \theta \sin \theta'} \right\} + e^{\nu \pi i} \exp \left\{ - \frac{\sin \theta \sin \theta'}{\lambda} + \frac{\lambda}{2} \frac{\nu(\nu - 1)}{\sin \theta \sin \theta'} \right\} \]  \hspace{1cm} (IV.15)

in the limit \( \lambda \to 0 \). We therefore observe that the short time kernel (IV.7) can be rewritten as

\[ K(\theta, \theta'; \epsilon) = \exp \left( - \frac{1 - \cos \theta \cos \theta'}{\lambda} - \frac{\lambda}{8} \right) \frac{(\sin \theta \sin \theta')^{1/2}}{\lambda} I_{\nu-1/2} \left( \frac{\sin \theta \sin \theta'}{\lambda} \right). \]  \hspace{1cm} (IV.16)

for infinitesimally small \( \lambda \).
In order to deduce eigenfunctions of the Hamiltonian from the short time kernel (IV.16), we here make use of a formula (see e.g. Ch. 11.5 of ref. 52 or Ch. 8.8 of ref. 28)

\[
\frac{(\sin \theta \sin \theta')^{1/2}}{\lambda} \exp \left( \frac{\cos \theta \cos \theta'}{\lambda} \right) I_{\nu-1/2} \left( \frac{\sin \theta \sin \theta'}{\lambda} \right) = \frac{2^{2\nu} \{\Gamma(\nu)\}^2}{\sqrt{2\pi} \lambda} (\sin \theta \sin \theta')^\nu \sum_{n=0}^{\infty} \frac{n! (\nu + n)}{\Gamma(2\nu + n)} I_{\nu+n} \left( \frac{1}{\lambda} \right) C_n^{\nu}(\cos \theta) C_n^{\nu}(\cos \theta').
\]  

(IV.17)

Comparing the left hand side above with the right hand side of (IV.16), we find

\[
K(\theta, \theta'; \epsilon) = \sum_{n=0}^{\infty} \left\{ \sqrt{\frac{2\pi}{\lambda}} I_{\nu+n} \left( \frac{1}{\lambda} \right) \exp \left( -\frac{1}{\lambda} - \frac{\lambda}{8} \right) \right\} \phi_n^{(\nu)}(\theta) \phi_n^{(\nu)}(\theta'),
\]  

(IV.18)

where the eigenfunction \( \phi_n^{(\nu)}(\theta) \) being given by

\[
\phi_n^{(\nu)}(\theta) = 2^{\nu} \Gamma(\nu) \sqrt{\frac{n! (\nu + n)}{2\pi \Gamma(2\nu + n)}} (\sin \theta)^\nu C_n^{\nu}(\cos \theta)
\]  

(IV.19)

in terms the Gegenbauer polynomial \( C_n^{\nu}(\cos \theta) \). If we make use of the orthogonality

\[
\int_0^\pi \phi_n^{(\nu)}(\theta) \phi_{n'}^{(\nu)}(\theta) d\theta = \delta_{n,n'},
\]  

(IV.20)

we can easily check that there holds

\[
\int_0^\pi K(\theta, \theta'; \epsilon) K(\theta', \theta''; \epsilon) d\theta' = \sum_{n=0}^{\infty} \left\{ \sqrt{\frac{2\pi}{\lambda}} I_{\nu+n} \left( \frac{1}{\lambda} \right) \exp \left( -\frac{1}{\lambda} - \frac{\lambda}{8} \right) \right\} ^2 \phi_n^{(\nu)}(\theta) \phi_n^{(\nu)}(\theta').
\]  

(IV.21)

This can be repeated \( N - 1 \) times to result in

\[
\langle \theta | e^{-\beta H/\hbar} | \theta' \rangle = \lim_{N \to \infty} \sum_{n=0}^{\infty} \left\{ \sqrt{\frac{2\pi}{\lambda}} I_{\nu+n} \left( \frac{1}{\lambda} \right) \exp \left( -\frac{1}{\lambda} - \frac{\lambda}{8} \right) \right\} ^N \phi_n^{(\nu)}(\theta) \phi_n^{(\nu)}(\theta')
\]  

(IV.22)

in which we have written \( |\theta\rangle = |x\rangle \sqrt{a} \) and \( E_n^{(\nu)} = (n + \nu)^2 \mathcal{R}(n = 0, 1, 2, \ldots) \) for energy eigenvalues. This is the eigenfunction expansion of the Feynman kernel for a finite imaginary time \( \beta \). Thus we have obtained eigenvalues and corresponding eigenfunctions for the Hamiltonian (IV.1) solely by means of path integral method.

We now generalize the consideration above to a system described by the Hamiltonian

\[
H = \frac{1}{2m} p^2 + \frac{\mathcal{R}}{4} \left\{ \frac{\mu (\mu - 1)}{\cos^2(\theta/2)} + \frac{\nu (\nu - 1)}{\sin^2(\theta/2)} \right\}
\]  

(IV.23)
on the same domain $0 < x < L$. The short time kernel for this system will be same as \((\text{IV.7})\) if we replace the effective potential to the one given by
\[
U_{\text{eff}}(\theta, \theta') = \frac{1}{8} \left(1 + \frac{1}{\sin \theta \sin \theta'}\right) + \frac{1}{8} \left\{ \frac{\mu(\mu - 1)}{\cos(\theta/2) \cos(\theta'/2)} + \frac{\nu(\nu - 1)}{\sin(\theta/2) \sin(\theta'/2)} \right\}.
\]
(IV.24)

Since this potential is periodic with a period $4\pi$, the first saddle point at $\theta - \theta' = 0$ is copied to the ones at $\theta - \theta' = 4n\pi (n = \pm 1, \pm 2, \pm 3, \ldots)$. We can sum up contributions from these saddle points to find a partial kernel
\[
K^{(e,e)}(\theta, \theta'; \epsilon) = \frac{1}{\sqrt{2\pi \lambda}} \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{1}{2\lambda} (\theta - \theta' - 4n\pi)^2 - \frac{\lambda}{8} \left\{ \frac{\mu(\mu - 1)}{\cos(\theta/2) \cos(\theta')} + \frac{\nu(\nu - 1)}{\sin(\theta/2) \sin(\theta')} \right\} \right]
\]
\[
= \frac{1}{\sqrt{2\pi \lambda}} \exp \left[ -\frac{4}{\lambda} \left\{ 1 - \cos \left(\frac{\theta - \theta'}{2}\right) \right\} \right] - \frac{\lambda}{32} \frac{\mu(\mu - 1)}{\cos(\theta/2) \cos(\theta'/2)} + \frac{\nu(\nu - 1)}{\sin(\theta/2) \sin(\theta'/2)} \right].
\]
(IV.25)

Here the symbol $(e,e)$ above designates that this kernel is the sum of contributions from paths which are reflected even times at both boundaries. If we write $(e,o)$, this means those from paths reflected even times by the one at $x = 0$ and odd times by the one at $x = L$.

By considering singularities of the potential in the stationary Schrödinger equation, we see that the Feynman kernel should behave like $\{\sin(\theta/2)\}^\nu$ near $x = 0$ and like $\{\cos(\theta/2)\}^\mu$ around $x = L$. Taking into account of these behavior, we extend the domain to the covering space which will be composed of copies of intervals $-\pi < \theta + \theta' < \pi$, $-\pi < \theta - \theta' < \pi$, $\pi < \theta + \theta' < 3\pi$ and $\pi < \theta - \theta' < 3\pi$ corresponding to partial kernels $K^{(o,e)}(\theta, \theta'; \epsilon)$, $K^{(e,e)}(\theta, \theta'; \epsilon)$, $K^{(e,o)}(\theta, \theta'; \epsilon)$ and $K^{(o,o)}(\theta, \theta'; \epsilon)$ in this order, respectively.

\[\theta \in \mathbb{R}\]

**FIG. 2.** The contour for extending the domain to the covering space for the generalized Pöschl-Teller potential.

A suitable choice of the contour to move $\theta$ when we make $\theta'$ stay within the original domain will be given by the one shown in Fig.2. In the original domain, we choose such that $\arg(\sin(\theta/2)) = \arg(\cos(\theta/2)) = 0$. Then, in the interval $\pi < \theta < 2\pi$, $\arg(\cos(\theta/2))$ changes
to $\pi$ while $\text{arg}(\sin(\theta/2))$ remains to be 0. If $\theta$ exceeds $2\pi$, $\text{arg}(\sin(\theta/2))$ decreases to $-\pi$ without changing $\text{arg}(\cos(\theta/2)) = \pi$. For $3\pi < \theta < 4\pi$, $\text{arg}(\sin(\theta/2))$ remains to be $-\pi$ while $\text{arg}(\cos(\theta/2))$ decreases to 0. On going along the semicircle below $4\pi$, $\text{arg}(\sin(\theta/2))$ increases to 0 to recover original values of these arguments in $4\pi < \theta < 5\pi$ and to complete the cycle. This cycle will be repeated infinitely many times to give us, in addition to $K^{(e,o)}(\theta, \theta'; \epsilon)$, the following partial kernels

$$K^{(o,o)}(\theta, \theta'; \epsilon)$$

$$= \frac{e^{\mu\pi i}}{\sqrt{2\pi \lambda}} \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{1}{2\lambda} \{\theta + \theta' + (4n + 2)\pi\}^2 - \frac{\lambda}{8} \left\{ -\frac{\mu(\mu - 1)}{\cos(\theta/2) \cos(\theta')} + \frac{\nu(\nu - 1)}{\sin(\theta/2) \sin(\theta')} \right\} \right]$$

$$= \frac{e^{\mu\pi i}}{\sqrt{2\pi \lambda}} \exp \left[ -\frac{4}{\lambda} \left\{ 1 + \cos \left( \frac{\theta + \theta'}{2} \right) \right\} - \frac{\lambda}{32} \right]$$

$$\left\{ -\frac{\mu(\mu - 1)}{\cos(\theta/2) \cos(\theta')} + \frac{\nu(\nu - 1)}{\sin(\theta/2) \sin(\theta')} \right\} \right] \right], \quad \text{(IV.26)}$$

and

$$K^{(o,e)}(\theta, \theta'; \epsilon)$$

$$= \frac{e^{-\nu\pi i}}{\sqrt{2\pi \lambda}} \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{1}{2\lambda} \{\theta + \theta' + 4n\pi\}^2 - \frac{\lambda}{8} \left\{ -\frac{\mu(\mu - 1)}{\cos(\theta/2) \cos(\theta')} - \frac{\nu(\nu - 1)}{\sin(\theta/2) \sin(\theta')} \right\} \right]$$

$$= \frac{e^{-\nu\pi i}}{\sqrt{2\pi \lambda}} \exp \left[ -\frac{4}{\lambda} \left\{ 1 - \cos \left( \frac{\theta + \theta'}{2} \right) \right\} - \frac{\lambda}{32} \right]$$

$$\left\{ -\frac{\mu(\mu - 1)}{\cos(\theta/2) \cos(\theta')} - \frac{\nu(\nu - 1)}{\sin(\theta/2) \sin(\theta')} \right\} \right]. \quad \text{(IV.28)}$$

By making use of the addition theorem of the trigonometric function, we rewrite
\[ \mathcal{K}^{(e,e)}(\theta, \theta'; \epsilon) \] as
\[
\mathcal{K}^{(e,e)}(\theta, \theta'; \epsilon) = \frac{1}{\sqrt{2\pi \lambda}} \exp \left( -\frac{4}{\lambda} - \frac{\lambda}{32} \right) \times \exp \left\{ \frac{4}{\lambda} \cos \frac{\theta}{2} \cos \frac{\theta'}{2} - \frac{\lambda}{8} \cos(\theta/2) \cos(\theta'/2) \right\} \times \exp \left\{ \frac{4}{\lambda} \sin \frac{\theta}{2} \sin \frac{\theta'}{2} - \frac{\lambda}{8} \sin(\theta/2) \sin(\theta'/2) \right\}
\]

(IV.29)

to find that \( \mathcal{K}^{(e,e)}(\theta, \theta'; \epsilon) \) is proportional to the leading term in the asymptotic form of the product of modified Bessel functions \( I_{\mu-1/2}(4 \cos(\theta/2) \cos(\theta'/2) / \lambda) I_{\nu-1/2}(4 \sin(\theta/2) \sin(\theta'/2) / \lambda) \) when \( \cos(\theta/2) \cos(\theta'/2) > 0 \) and \( \sin(\theta/2) \sin(\theta'/2) > 0 \). In the same way, we observe that each of the four kernels above corresponds to a term generated by the product of asymptotic expansions of modified Bessel functions as follows
\[
2\pi(z\zeta)^{1/2} I_{\mu-1/2}(z) I_{\nu-1/2}(\zeta) \\
\sim \left\{ e^{-\mu(\mu-1)/(2z)} + e^{\nu\pi i} e^{-z+\mu(\mu-1)/(2z)} \right\} \left\{ e^{\nu(\nu-1)/(2\zeta)} + e^{-\nu\pi i} e^{-\zeta+\nu(\nu-1)/(2\zeta)} \right\}
\]

(IV.30)

where \( z = 4 \cos(\theta/2) \cos(\theta'/2) / \lambda \) and \( \zeta = 4 \sin(\theta/2) \sin(\theta'/2) / \lambda \). We thus find that the short time kernel \( \mathcal{K}(\theta, \theta'; \epsilon) \), given by the sum \( \mathcal{K}^{(e,e)}(\theta, \theta'; \epsilon) + \mathcal{K}^{(e,o)}(\theta, \theta'; \epsilon) + \mathcal{K}^{(o,e)}(\theta, \theta'; \epsilon) \) for the Hamiltonian (IV.23), results in
\[
\mathcal{K}(\theta, \theta'; \epsilon) = \frac{1}{\sqrt{2\pi \lambda}} \exp \left( -\frac{4}{\lambda} - \frac{\lambda}{32} \right) \frac{8\pi}{\lambda} \left( \cos \frac{\theta}{2} \cos \frac{\theta'}{2} \right)^{1/2} \left( \sin \frac{\theta}{2} \sin \frac{\theta'}{2} \right)^{1/2} \\
\times I_{\mu-1/2} \left( \frac{4}{\lambda} \cos \frac{\theta}{2} \cos \frac{\theta'}{2} \right) I_{\nu-1/2} \left( \frac{4}{\lambda} \sin \frac{\theta}{2} \sin \frac{\theta'}{2} \right)
\]

(IV.31)

for infinitesimally small \( \lambda \).

Deduction of eigenfunctions and eigenvalues of the Hamiltonian (IV.23) is achieved if we make use of a formula, which is equivalent to Bateman’s expansion (see e.g. Ch. 11.6 in ref. 52 or Ch. 8.8 in ref. 28), given by
\[
\frac{2}{\lambda} \left( \cos \frac{\theta}{2} \cos \frac{\theta'}{2} \right)^{1/2} \left( \sin \frac{\theta}{2} \sin \frac{\theta'}{2} \right)^{1/2} I_{\mu-1/2} \left( \frac{4}{\lambda} \cos \frac{\theta}{2} \cos \frac{\theta'}{2} \right) I_{\nu-1/2} \left( \frac{4}{\lambda} \sin \frac{\theta}{2} \sin \frac{\theta'}{2} \right)
\]

(IV.32)

\[
= \sum_{n=0}^{\infty} I_{\mu+\nu+2n+1} \left( \frac{4}{\lambda} \right) \frac{(\mu + \nu + 2n)! \Gamma(\mu + \nu + n)}{\Gamma(\mu + n + 1/2) \Gamma(\nu + n + 1/2)} \\
\times P_n^{(\nu-1/2,\mu-1/2)}(\cos \theta) P_n^{(\nu-1/2,\mu-1/2)}(\cos \theta') \left( \cos \frac{\theta}{2} \cos \frac{\theta'}{2} \right)^{\mu} \left( \sin \frac{\theta}{2} \sin \frac{\theta'}{2} \right)^{\nu}
\]
where \( P^{(\nu-1/2, \mu-1/2)}_n(\cos \theta) = (\Gamma(\nu+n+1/2)/(n!\Gamma(\nu+1/2)))_2 F_1(-n, \mu+\nu+n; \nu+1/2; \sin^2 \theta) \)

is the Jacobi polynomial. By comparing the left hand side above with (IV.31), we find that the short time kernel can be rewritten as

\[
\mathcal{K}(\theta, \theta'; \epsilon) = \sum_{n=0}^{\infty} \left\{ \sqrt{\frac{8\pi}{\lambda}} I_{\mu+\nu+2n+1} \left( \frac{4}{\lambda} \right) \exp \left( -\frac{4}{\lambda} - \frac{\lambda}{32} \right) \right\} \phi_n^{(\mu, \nu)}(\theta) \phi_n^{(\mu, \nu)}(\theta'), \quad (IV.33)
\]

where the eigenfunction belonging to the eigenvalue \( E_n^{(\mu, \nu)} = \{n + (\mu + \nu)/2\}^2 R \) is given by

\[
\phi_n^{(\mu, \nu)}(\theta) = \sqrt{\frac{(\mu + \nu + 2n)! \Gamma(\mu + \nu + n)}{\Gamma(\mu + n + 1/2) \Gamma(\nu + n + 1/2)}} P_n^{(\nu-1/2, \mu-1/2)}(\cos \theta) \{\cos(\theta/2)\}^\mu \{\sin(\theta/2)\}^\nu
\]

for the Hamiltonian (IV.23). A similar relation like (IV.21) holds again. Hence we obtain

\[
\langle \theta | e^{-\beta H/\hbar} | \theta' \rangle = \lim_{N \to \infty} \sum_{n=0}^{\infty} \left\{ \sqrt{\frac{8\pi}{\lambda}} I_{\nu+n} \left( \frac{4}{\lambda} \right) \exp \left( -\frac{4}{\lambda} - \frac{\lambda}{32} \right) \right\} N \phi_n^{(\mu, \nu)}(\theta) \phi_n^{(\mu, \nu)}(\theta')
\]

\[
= \sum_{n=0}^{\infty} e^{-\beta E_n^{(\mu, \nu)}/\hbar} \phi_n^{(\mu, \nu)}(\theta) \phi_n^{(\mu, \nu)}(\theta')
\]

(IV.35)

solely by means of path integral method. We should add here a comment: the model under consideration reduces to the previous one by setting \( \mu = \nu \). The energy spectrum and eigenfunctions clearly exhibits this relation. We can easily see that \( \mathcal{K}^{(e, e)}(\theta, \theta'; \epsilon) + \mathcal{K}^{(o, o)}(\theta, \theta'; \epsilon) \) tends to the partial kernel \( \mathcal{K}^{(e)}(\theta, \theta'; \epsilon) \) of the previous model. On the other hand, the behavior of \( \mathcal{K}^{(e, o)}(\theta, \theta'; \epsilon) + \mathcal{K}^{(o, e)}(\theta, \theta'; \epsilon) \) needs to be examined. At first sight, \( \mathcal{K}^{(o, o)}(\theta, \theta'; \epsilon) \) does not seem to transform properly since it has opposite sign in its phase factor \( e^{-\nu \pi i} \) while \( \mathcal{K}^{(e, o)}(\theta, \theta'; \epsilon) \) generates a half of \( \mathcal{K}^{(o)}(\theta, \theta'; \epsilon) \) by setting \( \mu = \nu \). The factor \( e^{-\nu \pi i} \) in \( \mathcal{K}^{(o, o)}(\theta, \theta'; \epsilon) \) can be changed to \( e^{\nu \pi i} \), however, if we recall the fact that \( \text{arg}(\sin \theta) \) for this partial kernel differs from that of \( \mathcal{K}^{(e, o)}(\theta, \theta'; \epsilon) \) by the amount of \(-2\pi\).

Since we are dealing with the kernel with the help of asymptotic forms of the modified Bessel functions, this phase factor can be converted to \( e^{\nu \pi i} \) by making use of the relation

\[
I_\mu(e^{2\pi i} z) = e^{2\mu \pi i} I_\mu(z)
\]

through analytic continuation. It will be clear that the origin of this discrepancy in appearance is nothing but the difference between contours we have employed in the extension of the domain to the covering space. The change in the phase mentioned above precisely corresponds to the change of the contour for the extension. We now observe that \( \mathcal{K}^{(e, o)}(\theta, \theta'; \epsilon) + \mathcal{K}^{(o, e)}(\theta, \theta'; \epsilon) \) properly transforms into \( \mathcal{K}^{(o)}(\theta, \theta'; \epsilon) \) of the previous model.

We have thus succeeded in formulation and evaluation of path integrals for generalized Pöschl-Teller potential as well as the symmetric one. We should also emphasize here that
the phase factor $e^{\nu \pi i}$ in the partial kernel $K^{(o)}(\theta, \theta'; \epsilon)$ for the symmetric case tends to $-1$ if we set $\nu = 1$, that is, our result reproduces the well-known minus sign in the odd times reflected component of the Feynman kernel for a free particle in a box because for $\nu = 1$ the potential disappears. Interestingly, if we set $\nu$ to be an even integer, we observe that the phase factor becomes unity so that paths reflected odd times also contribute the kernel in an additive way. In this regard, the factor $-1$ for the amplitude of a particle in a box is usually understood from the viewpoint of boundary condition at the endpoints. Our derivation of this factor is based on the consideration of the behavior of wave functions around boundaries not the requirement for kernel to vanish at boundaries. However, we are convinced, from the eigenfunction expansion, that our prescription yields correct kernel that fulfills the boundary conditions.

B. Systems on the half-line

We proceed now to consider path integrals for systems on the half-line. Typical example is given by the radial Schrödinger equation. If we define the inner product of wave functions by $\int_0^\infty \psi^* (x) \phi(x) \, dx$, the Hamiltonian for such a system may have the form given by (III.3) in which the potential will be written as

$$V(x) = \frac{\hbar^2 \nu(\nu - 1)}{2m x^2} + U(x) \quad (0 < x < \infty),$$

where $\nu = l + (D - 1)/2$ being specified by the angular momentum $l(l + D - 2)$ for the radial Hamiltonian in $D$-dimensional space.

It is natural to choose $f(x)$ to be given by $f(x) = x^2/(2a)$ which yields

$$Q(x) = a \log \frac{x}{a},$$

where $a$ is positive constant carrying the dimension of length. For the radial oscillator, whose potential being specified by $U(x) = m\omega^2 x^2/2$, $a = \sqrt{\hbar/m\omega}$ will be a natural choice and $a = a_B$, $a_B = h^2/(km)$ for the radial Coulomb system, specified by $U(x) = -k/x(k > 0)$, will be suitable. The corresponding momentum operator $P$ reads

$$P = -\frac{i\hbar}{2a} \left(x \frac{d}{dx} + \frac{d}{dx} x\right)$$

whose eigenfunction being given by

$$\psi_P(x) = \frac{1}{\sqrt{2\pi \hbar x/a}} \exp \left(\frac{i}{\hbar} a P \log \frac{x}{a}\right).$$
The completeness of the eigenvectors can be seen by calculating

\[ \int_{-\infty}^{\infty} \psi_P(x) \psi_P(x') \, dP = \frac{1}{\sqrt{xx'}} \delta(\log x - \log x') = \delta(x - x') \]  

(IV.40)

for positive \( x \) and \( x' \).

Since we are aiming here to show the usefulness of (III.10), let us restrict ourselves to the case of radial oscillator so that we can find its exact solution. Introducing a dimensionless variable \( u = x/a \), \( a = \sqrt{\hbar/m\omega} \), we find the short time kernel (III.10) for this system should be given by

\[ K(u, u'; \epsilon) = \frac{1}{\sqrt{2\pi \lambda}} \exp \left[ -\frac{1}{2\lambda} uu' \left( \log \frac{u}{u'} \right)^2 - \lambda U_{\text{eff}}(u, u') \right], \quad \lambda \equiv \frac{\hbar \epsilon}{ma^2} = \omega \epsilon, \]  

(IV.41)

in which the effective potential being given by

\[ U_{\text{eff}}(u, u') = \frac{1}{8uu'} + \frac{\nu(\nu - 1)}{2uu'} + \frac{1}{4} (u^2 + u'^2). \]  

(IV.42)

The first term of \( U_{\text{eff}}(u, u') \) has appeared through rewriting the Hamiltonian in terms of \( P \).

Again, the kernel above is normalized to fit integration with respect to \( u \) instead of \( x \). We may replace the last term in the effective potential by \( uu'/2 \) as the result of the geometric mean. It does not, however, affect the argument below. We therefore use \( (u^2 + u'^2)/4 \) as the harmonic potential in the short time kernel for convenience. See appendix on the detail of this ambiguity in the form of the potential term.

We first examine the contribution from the saddle point at \( u - u' = 0 \) to obtain

\[ K^{(e)}(u, u'; \epsilon) = \frac{1}{\sqrt{2\pi \lambda}} \exp \left[ -\frac{1}{2\lambda} (u - u')^2 - \frac{\lambda}{2} \frac{\nu(\nu - 1)}{uu'} - \frac{\lambda}{4} (u^2 + u'^2) \right] \]  

(IV.43)

whose exponent can be arranged as

\[ \frac{1}{2\lambda} (u - u')^2 + \frac{\lambda}{2} \frac{\nu(\nu - 1)}{uu'} + \frac{\lambda}{4} (u^2 + u'^2) = \frac{1}{2\lambda} \left( 1 + \frac{\lambda^2}{2} \right) (u^2 + u'^2) - \frac{uu'}{\lambda} + \frac{\lambda}{2} \frac{\nu(\nu - 1)}{uu'}. \]  

(IV.44)

We can always modify the exponent of a time sliced path integral by adding higher order terms of \( \lambda \). Therefore we can rewrite (IV.43) as

\[ K^{(e)}(u, u'; \epsilon) = \frac{1}{\sqrt{2\pi \sinh \lambda}} \exp \left[ -\frac{\cosh \lambda}{2} (u^2 + u'^2) + \frac{uu'}{\sinh \lambda} - \frac{\sinh \lambda}{2} \frac{\nu(\nu - 1)}{uu'} \right] \]  

(IV.45)
without changing the result of the time sliced path integral. Since $K^{(e)}(u, u'; \epsilon)$ can now be identified with the asymptotic form of

$$\frac{\sqrt{uu'}}{\sinh \lambda} \exp \left[ -\frac{\coth \lambda}{2} (u^2 + u'^2) \right] I_{\nu - 1/2} \left( \frac{uu'}{\sinh \lambda} \right)$$

(IV.46)

for $uu' / \sinh \lambda \to \infty$, we can utilize this expression to consider the contribution from the reflected paths by the analytic continuation of the modified Bessel function to find

$$K^{(o)}(u, u'; \epsilon) = e^{\nu \pi i} \frac{1}{\sqrt{2\pi \sinh \lambda}} \exp \left[ -\frac{1}{2\lambda} (u + u')^2 + \frac{\lambda}{2} \frac{\nu(\nu - 1)}{uu'} - \frac{\lambda}{4} (u^2 + u'^2) \right].$$

(IV.47)

We can rewrite it as

$$K^{(o)}(u, u'; \epsilon) = e^{\nu \pi i} \frac{1}{\sqrt{2\pi \lambda}} \exp \left[ -\frac{1}{2\lambda} (u + u')^2 + \frac{\lambda}{2} \frac{\nu(\nu - 1)}{uu'} - \frac{\lambda}{4} (u^2 + u'^2) \right].$$

(IV.48)

to make it clear that $K^{(o)}(u, u'; \epsilon)$ can be viewed as the contribution from the saddle point at $u + u' = 0$.

It will be now evident that $K^{(e)}(u, u'; \epsilon)$ and $K^{(o)}(u, u'; \epsilon)$ are corresponding to the terms in the asymptotic from of the modified Bessel function

$$I_{\nu - 1/2} \left( \frac{uu'}{\sinh \lambda} \right) \sim \sqrt{\frac{\sinh \lambda}{2\pi uu'}} \times \left[ \exp \left\{ -\frac{uu'}{\sinh \lambda} - \frac{\sinh \lambda}{2} \frac{\nu(\nu - 1)}{uu'} \right\} + e^{\nu \pi i} \exp \left\{ -\frac{uu'}{\sinh \lambda} + \frac{\sinh \lambda}{2} \frac{\nu(\nu - 1)}{uu'} \right\} \right].$$

(IV.49)

Therefore the decomposition of the kernel $K(u, u'; \epsilon)$ into the sum of $K^{(e)}(u, u'; \epsilon)$ and $K^{(o)}(u, u'; \epsilon)$ explains the extension to the covering space from the original domain. We thus obtain

$$K(u, u'; \epsilon) = \frac{\sqrt{uu'}}{\sinh \lambda} \exp \left[ -\frac{\coth \lambda}{2} (u^2 + u'^2) \right] I_{\nu - 1/2} \left( \frac{uu'}{\sinh \lambda} \right)$$

(IV.50)

as the sum of $K^{(e)}(u, u'; \epsilon)$ and $K^{(o)}(u, u'; \epsilon)$. If we make use of the formula

$$\int_0^\infty e^{-ax^2} x I_\nu(px) I_\nu(qx) \, dx = \frac{1}{2a} e^{(p^2 + q^2)/4a} I_\nu \left( \frac{pq}{2a} \right)$$

(IV.51)

which is valid for $a > 0$, we can verify that there holds

$$\int_0^\infty K(u, u', \epsilon)K(u', u'', \epsilon) \, du' = K(u, u'', 2\epsilon).$$

(IV.52)
This proves that the form of the short time kernel given by (IV.50) is already exact and hence $\epsilon$ can be finite. We then obtain $K(u, u'; \beta)$ in the same form as the one in (IV.50) by substituting $\omega\beta$ to $\lambda$ for finite $\beta$.

We may make use of a formula (see e.g. Ch. 4.17 in ref. 53)

$$\sum_{n=0}^{\infty} n! L_n^\alpha(x)L_n^\alpha(y)t^n = \frac{e^{-(x+y)t/(1-t)}}{1-t}(xyt)^{-\alpha/2}I_{\alpha}\left(\frac{2(xyt)^{1/2}}{1-t}\right)$$

(IV.53)

to find that the Euclidean kernel $K(u, u'; \beta)$ has an expansion

$$K(u, u'; \beta) = \sum_{n=0}^{\infty} e^{-(2n+\nu+1/2)\lambda} \phi_n^{(\nu)}(u)\phi_n^{(\nu)}(u')$$

(IV.54)

in terms of the eigenfunction of the Hamiltonian. The explicit form of the eigenfunction $\phi_n^{(\nu)}(u)$ for this case is given by

$$\phi_n^{(\nu)}(u) \equiv \sqrt{\frac{2n!}{\Gamma(n+\nu+1/2)}} e^{-u^2/2} u^{\nu/2} L_n^{\nu-1/2}(u^2)$$

(IV.55)

in terms of the Laguerre polynomial $L_n^{\nu-1/2}(u^2)$. The eigenvalue of the Hamiltonian is also obtained immediately from (IV.54) to be $E_n^{(\nu)} = (2n + \nu + 1/2)\hbar\omega$. In this way, we have successfully formulated and solved path integral for the radial oscillator in an exact manner.

V. CONCLUSION

We have developed a method to obtain Lagrangian path integrals for systems defined on a finite interval as well as for systems on the half-line. The first step of our method is finding a suitable function of the position variable $x$ so that we are able to define a point transformation from $x$ to $Q(x)$ which covers all real values. This point transformation allows us to introduce the canonical momentum $P$ associated with $Q(x)$. It is the completeness of the eigenvectors of this new momentum operator that plays the role in our method to convert a Hamiltonian path integral into the corresponding Lagrangian one via the Gaussian integration. For systems on a finite interval or on the half-line, the kinetic term of the Lagrangian path integral thus formulated possesses multiple saddle points to give contributions to the path integral. We have taken them into account by extending the original domain to the covering space in obtaining suitable expressions for the short time kernel. In solving the path integral of a radial oscillator, we have succeeded formulating and carrying out the
path integral entirely within the framework of path integral method. In this regard, we have to content ourselves to rely on some identities to convert short time kernels into the corresponding eigenfunction expansions. It will be beautiful and useful if we could obtain short time kernels for these systems in a closed form by which we can validate the reproducing property of kernels in an exact manner. In this regard our formulae, given by (IV.21) for the expression of the short time kernel (IV.18) and the similar one for (IV.33), are almost exact. If we make use of the asymptotic form of the modified Bessel functions in (IV.18) and (IV.33), both the formula (IV.21) and the corresponding one for (IV.33) become exactly the reproducing property of the kernel but they are not expressed in the closed form. The phase factor \( e^{\nu \pi i} \) appeared in the partial kernel for the symmetric Pöschl-Teller potential generalizes the factor \(-1\) for the odd times reflected amplitude of a particle in a box. Our result clearly shows that the phase factor Feynman kernel acquires upon reflections at boundaries depends on the dynamics of the system and therefore cannot be determined solely by the consideration on the geometry of the system.

On the point transformation we have employed to formulate path integrals for systems with non-trivial geometry, we may fully utilize the completeness of eigenvectors of \( Q = Q(x) \) as well as \( P \) leaving from the viewpoint of path integrals for Feynman kernels in terms of the original variable \( x \). This new pair of canonical variables is quite suitable for making consideration on the DK transformed path integral in the time sliced representation. Relations among solvable models, such as DK equivalence of the radial Coulomb path integral to that of the radial oscillator\(^{32}\), can be deduced in a rigorous manner by formulating time sliced path integrals in terms of these new canonical variables.

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Appendix A: Ambiguity in the form of the potential term of a path integral

In this appendix, we consider the possibility of the different choice of the form of potential term in the short time kernel (IV.41). If we evaluate the matrix element \( \langle x | x^2 | x' \rangle \) as \( xx' \langle x | x' \rangle \),

24
the exponent of the kernel becomes

\[- \frac{1}{2\lambda} (u - u')^2 - \frac{\lambda}{2} \frac{\nu(\nu - 1)}{uu'} - \frac{\lambda}{2} uu' \]  

(A.1)

which can be arranged to yield

\[- \frac{1}{2\lambda} (u^2 + u'^2) - \frac{\lambda}{2} \frac{\nu(\nu - 1)}{uu'} + \left(1 - \frac{\lambda^2}{2}\right) \frac{uu'}{\lambda} \]  

(A.2)

We may set \( v = (1 - \lambda^2/2)^{1/2}u \) and write \( u \) for \( v \) again to find that the expression above is equivalent to

\[- \frac{1}{2\lambda} \left(1 + \frac{\lambda^2}{2}\right) (u^2 + u'^2) - \frac{\lambda}{2} \frac{\nu(\nu - 1)}{uu'} + \frac{uu'}{\lambda} \]  

(A.3)

by discarding the irrelevant terms. Since the Jacobian of the change of variables from \( u \) to \( v \) converges to unity in the continuum limit, we can regard it as unity from very beginning. It is, therefore, clear that the different choice for the matrix element \( \langle x|x^2|x' \rangle \) to be \( xx' \langle x|x' \rangle \) does not affect the final form of the Euclidean kernel. A more generalized scheme for this matrix element will be given by \( x^{1+\alpha} x'^{1-\alpha} \langle x|x' \rangle \). This is, however, expressed as

\[ x^{1+\alpha} x'^{1-\alpha} = xx' \left(\frac{x}{x'}\right)^{\alpha} \]  

(A.4)

to exhibit the fact that \( \alpha \)-dependent factor generates irrelevant terms in addition to unity in the exponent of a path integral. Therefore, it is equivalent to the symmetric one considered above.
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