Fusion procedure for the Yang-Baxter equation and Schur-Weyl duality

L. Poulain d’Andecy

Abstract

We first review the fusion procedure for an arbitrary solution of the Yang–Baxter equation and the study of distinguished invariant subspaces for the fused solutions. Then we apply these general results to four particular solutions: the Yang solution, its standard deformation and their generalizations for super vector spaces. For the Yang solution, respectively, its “super” generalization, we explain how, using the fusion formula for the symmetric group together with the (super) Schur–Weyl duality, the fusion procedure allows to construct a family of fused solutions of the Yang–Baxter equation acting on irreducible representations of the general linear Lie algebra, respectively, of the general linear Lie superalgebra. For the deformations of the two previous solutions, we use the fusion formula for the Hecke algebra together with the (super) quantum Schur–Weyl duality to obtain fused solutions acting on irreducible representations of the quantum groups associated to the general linear Lie (super)algebras.

1. Introduction

The Yang–Baxter equation originally appeared in statistical physics and in quantum integrable systems, and its study has then led to the discovery of quantum groups, which have found numerous applications in mathematics and mathematical physics (for literature on the Yang–Baxter equation and related subjects, see, e.g., [1–4] and references therein).

The fusion procedure for the Yang–Baxter equation allows the construction of new solutions of the Yang–Baxter equation starting from a given fundamental solution. We refer to the new solutions as “fused” solutions of the Yang–Baxter equation. The fusion procedure has first been introduced in [5]. Several aspects of the fusion procedure for the Yang–Baxter equation and of its applications to theoretical physics can be found for example in [6–16].

In this paper we start by reviewing the fusion procedure for an arbitrary fundamental solution of the Yang–Baxter equation, together with the identification and the study of distinguished invariant subspaces for the fused solutions. We provide complete proofs as the details are scattered in the literature and as, moreover, often only some particular solutions, depending on the physical model under consideration, were considered.

Let $V$ be a finite-dimensional complex vector space. Starting from an arbitrary solution $R(u)$ of the Yang–Baxter equation on $V \otimes V$ (see Section 2 for more precision), a fused solution is built given two sets of complex numbers $c = (c_1, \ldots, c_n)$ and $c' = (c_1', \ldots, c_n')$ and is denoted by $R_{c,c'}(u)$. The element $R_{c,c'}(u)$ is an endomorphism of the vector space $V^{\otimes n} \otimes V^{\otimes n'}$. Linear

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subspaces $W_c \subset V^\otimes n$ and $W_{\underline{c}} \subset V^\otimes n'$ are constructed, with the property that the fused solution $R_{c,\underline{c}}(u)$ preserves the subspace $W_c \otimes W_{\underline{c}}$ of $V^\otimes n \otimes V^\otimes n'$. Thus, by restriction, the operator $R_{c,\underline{c}}(u)$ induces a solution of the Yang–Baxter equation acting on the space $W_c \otimes W_{\underline{c}}$. A crucial feature of the invariant subspaces is that, for any $c = (c_1, \ldots, c_n)$, the subspace $W_c$ is defined as the image of an endomorphism $F(c)$ of $V^\otimes n$, and moreover the endomorphism $F(c)$ is expressed in terms of the original solution $R(u)$.

The second part of the paper is devoted first to the review of the identification of interesting spaces among the spaces $W_c$ when $R(u)$ is the so-called Yang solution. Then we generalize the obtained results to the case where $R(u)$ is a generalization of the Yang solution for a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V$ (i.e., a “super” vector space), and to the case where $R(u)$ is the standard deformation of the Yang solution, both in the usual and $\mathbb{Z}/2\mathbb{Z}$-graded situations.

For the Yang solution, the main result is the identification of subspaces $W_c$ which are isomorphic to finite-dimensional irreducible representations of the general linear Lie algebra $\mathfrak{gl}_N$, where $N$ is the dimension of $V$. It results in a family of fused solutions of the Yang–Baxter equation acting on finite-dimensional irreducible representations of $\mathfrak{gl}_N$. Let $\lambda$ be a partition of length less than or equal to $N$. A subspace $W_c$ isomorphic to the irreducible representation of highest weight $\lambda$ is obtained when the set of numbers $c$ is a string of classical contents associated to a standard Young tableau of shape $\lambda$. We show moreover that, by replacing the set $c$ by a string of contents associated to another standard Young tableau of the same shape, we obtain an equivalent fused solution up to a change of basis.

As it is suggested in [17], the study of the subspaces $W_c$ when $R(u)$ is the Yang solution is made using the classical Schur–Weyl duality (see, e.g., [18,19]) together with the so-called “fusion formula” for the symmetric group. The fusion formula for the symmetric group $S_n$ originates in [20] and has then been developed in [21], see also [17,22–24]. The fusion formula expresses each element of a complete set of primitive idempotents of the group ring $\mathbb{C}S_n$ as a certain evaluation of a rational function in several variables with coefficients in $\mathbb{C}S_n$ (the rational function is the same for all the idempotents). As the name suggests, the fusion formula is intimately related to the fusion procedure. Indeed the rational function is built from elementary “universal” solutions of the Yang–Baxter equation, called Baxterized elements of $\mathbb{C}S_n$, and it turns out that its expression reproduces the expression for the operator $F(c)$, mentioned above, in terms of $R(u)$.

When $R(u)$ is the generalization of the Yang solution for a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_\uparrow \otimes V_{\uparrow}'$, we still can use the fusion formula for the symmetric group $S_n$. However, in this situation, another action of $S_n$ on $V^\otimes n$ is relevant. This action gives rise to a “super” analogue of the classical Schur–Weyl duality [25,26], which holds between the symmetric group and the general linear Lie superalgebra $\mathfrak{gl}_{N|M}$, where $N := \dim(V_\uparrow)$ and $M := \dim(V_{\uparrow}')$. With these tools, generalizing the preceding results concerning the Yang solution, we obtain a family of fused solutions of the Yang–Baxter equation acting on finite-dimensional irreducible representations of $\mathfrak{gl}_{N|M}$.

For the standard deformation of the Yang solution, we use the fusion formula for the Hecke algebra [27] together with the quantum analogue of the Schur–Weyl duality [28]. This duality holds between the Hecke algebra and the quantum group $U_q(\mathfrak{gl}_N)$ associated to $\mathfrak{gl}_N$. Then we apply the fusion procedure to obtain a family of fused solutions of the Yang–Baxter equation acting on finite-dimensional irreducible representations of $U_q(\mathfrak{gl}_N)$. The parameters $c$ are now related to quantum contents of standard Young tableaux. Here again, replacing a standard Young tableau by another standard Young tableau of the same shape leads to equivalent fused solutions up to a change of basis.

The standard deformation of the Yang solution also admits a generalization for a $\mathbb{Z}/2\mathbb{Z}$-graded
vector space $V$ and we obtain in this case, via the fusion procedure, solutions of the Yang–Baxter equation acting on finite-dimensional irreducible representations of a deformation $U_q(\mathfrak{gl}_{N|M})$ of $U(\mathfrak{gl}_{N|M})$. In this situation, we use the analogue of the Schur-Weyl duality between the Hecke algebra and $U_q(\mathfrak{gl}_{N|M})$ proved in [29].

For an arbitrary fundamental solution $R(u)$, we study the possible equivalence of the fused solutions under a permutation of the parameters $c = (c_1, \ldots, c_n)$. By “equivalence” of two fused solutions, we mean that the two corresponding invariant subspaces have the same dimension, and moreover the restrictions of the fused solutions to these subspaces coincide up to a change of basis. Assuming an unitary property for $R(u)$, we obtain that particular permutations (that we call “admissible” for $c$ and $R(u)$, see Subsection 4.3) of the parameters $c = (c_1, \ldots, c_n)$ lead to equivalent fused solutions. For the four solutions $R(u)$ considered above, the fact that the fused solutions depend on the standard Young tableaux only through their shapes is a consequence of this general property.

We notice that, even for solutions $R(u)$ considered in this paper, the study of (proper) invariant subspaces for the fused solutions is, though quite satisfactory, not fully complete. Namely, it is still an open problem to determine all the sets of parameters $c$ such that the operator $F(c)$ is non-invertible (and thus leads to a proper invariant subspace) and to study the obtained fused solutions. Even the classification of the sets of parameters $c$ such that the operator $F(c)$ is proportional to a projector is not completely understood.

One motivation for a detailed study of the fusion procedure for an arbitrary solution of the Yang–Baxter equation came from recent results generalizing the fusion formula of the symmetric group and of the Hecke algebra to several other structures: the Brauer algebras [30, 31], the Birman–Wenzl–Murakami algebras [32], the complex reflection groups of type $G(d,1,n)$ [33], the Ariki–Koike algebras [34] and the wreath products of finite groups by the symmetric group [35]. These fusion formulas involve as well a rational function (with coefficients in the algebras under consideration) which is built from elementary “universal” solutions of the Yang–Baxter equation, the analogues for these structures of the Baxterized elements of $\mathbb{C}S_n$. However, in these fusion formulas, Baxterized solutions of the reflection equation also appear in the rational functions. It is therefore expected, as it is already mentioned in [32], that these fusion formulas admit an interpretation in the framework of the fusion procedure for the reflection equation. We postpone to another article an analogous study of the fusion procedure for the reflection equation and of invariant subspaces of the fused solutions of the reflection equation.

The paper is organized as follows. In Section 2 we give necessary definitions and notations. In Section 3 starting from an arbitrary solution $R(u)$ of the Yang–Baxter equation acting on $V \otimes V$, we describe the construction of the endomorphisms $R_{c,(u)}$ and prove that the obtained operators $R_{c,(u)}$ are solutions of the Yang–Baxter equation as well. In Section 4 we define endomorphisms $F(c)$ and show that their images provide invariant subspaces for the fused solutions. We also give an alternative formula for $F(c)$ and prove the equivalence of the fused solutions under admissible permutation of the parameters $c$. Both Sections 3 and 4 deal with an arbitrary solution $R(u)$ of the Yang–Baxter equation (only the unitary property is assumed in Subsection 4.3). In Section 5 we apply the whole procedure to the situation where $R(u)$ is the Yang solution, and identify invariant subspaces isomorphic to finite-dimensional irreducible representations of the general linear Lie algebra. The main results are summarized in Corollary 5.5. We also provide explicit examples of fused solutions and discuss open problems concerning “non-standard”
evaluations of the fusion function.

In Sections 6 and 7, we generalize the results of the preceding section to the situation where $R(u)$ is a generalization of the Yang solution for a super vector space and where $R(u)$ is the standard deformation of the Yang solution or of its “super” generalization.

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2. Preliminaries

1. Let $a$ and $b$ be two integers such that $a \leq b$. We will use the following notation for the product of non-commuting quantity $x_i$ depending on $i$:

$$\prod_{i=a,...,b} \rightarrow x_i := x_a x_{a+1} \ldots x_b \quad \text{and} \quad \prod_{i=a,...,b} \leftarrow x_i := x_b \ldots x_{a+1} x_a .$$

Similarly, for non-commuting quantities depending on two indices $i$ and $j$, the arrow “$\rightarrow$” indicates that the factors are ordered lexicographically, while the arrow “$\leftarrow$” indicates that the factors are ordered in the reverse lexicographic order. For example, we have:

$$\prod_{1 \leq i < j \leq 4} \rightarrow x_{ij} := x_{12} x_{13} x_{14} x_{23} x_{24} x_{34} \quad \text{and} \quad \prod_{1 \leq i < j \leq 4} \leftarrow x_{ij} := x_{34} x_{24} x_{23} x_{14} x_{13} x_{12} .$$

2. The product of two (and any number of) permutations is written with the standard notation for the composition of functions, namely, if $\pi, \sigma \in S_n$ are two permutations then $\pi \sigma$ means that $\sigma$ is applied first and then $\pi$; for example, $(1, 2)(2, 3) = (1, 2, 3)$ in the standard cyclic notation.

3. For a vector space $V$ over $\mathbb{C}$, we denote by $\operatorname{End}(V)$ the set of endomorphisms of $V$. We fix a finite-dimensional vector space $V$ over $\mathbb{C}$ and let $\operatorname{Id}$ denote the identity endomorphism of $V$ and $P$ denote the permutation endomorphism of $V \otimes V$ (that is, $P(x \otimes y) := y \otimes x$ for $x, y \in V$). We will denote by $\operatorname{Id}_{V \otimes k}$ the identity operator on $V \otimes k$, for $k \in \mathbb{Z}_{\geq 2}$.

We will use the standard notation for operators in $\operatorname{End}(V \otimes n)$; namely, if $T \in \operatorname{End}(V \otimes n)$ then $T_i$ for $i = 1, \ldots, n$, will denote the operator in $\operatorname{End}(V \otimes n)$ acting as $T$ on the $i$-th copy and trivially anywhere else. For example, for $n = 3$, we have

$$T_1 := T \otimes \operatorname{Id}_{V \otimes 2}, \quad T_2 := \operatorname{Id} \otimes T \otimes \operatorname{Id} \quad \text{and} \quad T_3 := \operatorname{Id}_{V \otimes 2} \otimes T .$$

Similarly, if $R$ is an operator in $\operatorname{End}(V \otimes V)$ then $R_{ij}$ will denote the operator in $\operatorname{End}(V \otimes n)$ acting as $R$ in the $i$-th and $j$-th copies and trivially anywhere else. For example, if $n = 3$:

$$R_{1,2} := R \otimes \operatorname{Id}, \quad R_{2,3} := \operatorname{Id} \otimes R \quad \text{and} \quad R_{1,3} := (\operatorname{Id} \otimes P) R_{1,2} (\operatorname{Id} \otimes P) = P_{2,3} R_{1,2} P_{2,3} .$$

We also have for example $R_{2,1} := P_{1,2} R_{1,2} P_{1,2}$. Note that we slightly abuse by not indicating the integer $n$ in the notation $T_i, R_{i,j}, P_{i,j}$, etc. This should not lead to any confusion as it will be clear on which space the operators act.

We recall some obvious commutation relations that we will often use throughout the text.
without mentioning:
\[
\begin{align*}
T_i T_j &= T_j T_i & \text{if } i \neq j, \\
T_i R_{j,k} &= R_{j,k} T_i & \text{if } i \notin \{j, k\}, \\
R_{i,j} R_{k,l} &= R_{k,l} R_{i,j} & \text{if } \{i, j\} \cap \{k, l\} = \emptyset, \\
P_{i,j} T_k &= T_{s_{i,j}(k)} P_{i,j} & \text{for all } i, j, k = 1, \ldots, n \text{ with } i \neq j, \\
P_{i,j} R_{k,l} &= R_{s_{i,j}(k),s_{i,j}(l)} P_{i,j} & \text{for all } i, j, k, l = 1, \ldots, n \text{ with } i \neq j \text{ and } k \neq l,
\end{align*}
\]
where \( s_{i,j} \) denotes the transposition of \( i \) and \( j \).

4. Yang–Baxter equation. Let \( R \) be a function of one variable \( u \in \mathbb{C} \), taking values in \( \text{End}(V \otimes V) \). The variable \( u \) is called the spectral parameter. The function \( R \) is a solution, on \( V \otimes V \), of the Yang–Baxter equation if the following functional relation is satisfied:
\[
R_{1,2}(u) R_{1,3}(u + v) R_{2,3}(v) = R_{2,3}(v) R_{1,3}(u + v) R_{1,2}(u), \tag{2.1}
\]
where both sides take values in \( \text{End}(V \otimes V) \) and \( R_{1,2}(u) \) means \( R(u)_{1,2} \), etc. (we will always use this standard notation). By abuse of speaking, we will sometimes say that the operator \( R(u) \) itself is a solution of the Yang–Baxter equation.

More generally, let \( E \) be an indexing set, \( \{V_\mu\}_{\mu \in E} \) be a collection of finite-dimensional vector spaces over \( \mathbb{C} \), and let \( \{R_{\mu,\nu}\}_{\mu,\nu \in E} \) be a family of functions of \( u \in \mathbb{C} \) such that \( R_{\mu,\nu} \) takes values in \( \text{End}(V_\mu \otimes V_\nu) \), for any \( \mu, \nu \in E \). We say that the functions \( R_{\mu,\nu} \) form a family of solutions of the Yang–Baxter equation if the following functional relations are satisfied:
\[
R_{\mu,\nu}(u) R_{\mu,\tau}(u + v) R_{\nu,\tau}(v) = R_{\nu,\tau}(v) R_{\mu,\tau}(u + v) R_{\mu,\nu}(u) \quad \text{for any } \mu, \nu, \tau \in E, \tag{2.2}
\]
where both sides take values in \( \text{End}(V_\mu \otimes V_\nu \otimes V_\tau) \) (in (2.2), \( R_{\mu,\nu}(u) \) stands for \( R_{\mu,\nu}(u) \otimes \text{Id}_{V_\tau} \), and similarly for \( R_{\mu,\tau}(u + v) \) and \( R_{\nu,\tau}(v) \)).

Equation (2.1) is the Yang–Baxter equation with “additive” spectral parameters. We will sometimes use the version of the equation with “multiplicative” spectral parameters, namely
\[
R_{1,2}(\alpha) R_{1,3}(\alpha \beta) R_{2,3}(\beta) = R_{2,3}(\beta) R_{1,3}(\alpha \beta) R_{1,2}(\alpha), \tag{2.3}
\]
for a function \( R \) of the variable \( \alpha \in \mathbb{C} \) taking values in \( \text{End}(V \otimes V) \).

5. Partitions and Young tableaux. Let \( \lambda \vdash n \) be a partition of a positive integer \( n \), that is, \( \lambda = (\lambda_1, \ldots, \lambda_l) \) is a family of integers such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0 \) and \( \lambda_1 + \cdots + \lambda_l = n \). We say that \( \lambda \) is a partition of size \( n \) and of length \( l \) and set \( |\lambda| := n \) and \( \ell(\lambda) := l \).

The Young diagram of \( \lambda \) is the set of elements \( (x, y) \in \mathbb{Z}^2 \) such that \( x \in \{1, \ldots, l\} \) and \( y \in \{1, \ldots, \lambda_x\} \). A pair \( (x, y) \in \mathbb{Z}^2 \) is usually called a node. The Young diagram of \( \lambda \) will be represented in the plan by a left-justified array of \( l \) rows such that the \( j \)-th row contains \( \lambda_j \) nodes for all \( j = 1, \ldots, l \) (a node will be pictured by an empty box). We number the rows from top to bottom. We identify partitions with their Young diagrams and say that \( (x, y) \) is a node of \( \lambda \), or \( (x, y) \in \lambda \), if \( (x, y) \) is a node of the diagram of \( \lambda \).

For a node \( \theta = (x, y) \), we set \( cc(\theta) := y - x \). The number \( cc(\theta) \) is called the classical content of \( \theta \). For any complex number \( q \), we set \( c^{(q)}(\theta) := q^{2(y-x)} \) and call \( c^{(q)}(\theta) \) the \( q \)-quantum content of \( \theta \), or simply the quantum content of \( \theta \) when \( q \) is fixed.

The hook of a node \( \theta \in \lambda \) is the set of nodes of \( \lambda \) consisting of the node \( \theta \) and the nodes which lie either under \( \theta \) in the same column or to the right of \( \theta \) in the same row; the hook length
Such that, for example, the operator \( R_{1,2}(u) R_{1,3}(u + v) R_{2,3}(v) = R_{2,3}(v) R_{1,3}(u + v) R_{1,2}(u) \),

\begin{equation}
R_{1,2}(u) R_{1,3}(u + v) R_{2,3}(v) = R_{2,3}(v) R_{1,3}(u + v) R_{1,2}(u),
\end{equation}

where both sides operate on \( V \otimes V \otimes V \) (in Sections 3 and 4, we work with the additive convention for the spectral parameters, see paragraph 4 of Section 2; equivalently, we could have chosen the multiplicative version, as indicated in Remark 4.11 below).

**Example 3.1.** One of the simplest example of a solution of Equation (3.1) is the Yang solution:

\[ R(u) = \text{Id}_{V \otimes 2} - \frac{P}{u}. \]

The fact that the function given by (3.2) satisfies the Yang–Baxter equation (3.1) can be checked by a direct calculation.

Let \( n \) and \( n' \) be positive integers. We consider the space \( V^\otimes n \otimes V^\otimes n' \) and label the copies of \( V \) by \( 1, \ldots, n, 1, \ldots, n' \) (from left to right) such that, for example, the operator \( R_{a,b}(u) \) with \( a \in \{1, \ldots, n\} \) and \( b \in \{1, \ldots, n'\} \) stands for the operator \( R_{a,n+b}(u) \) with the notation explained in Section 2 (we prefer the underlined notation to make a clear distinction between the indices corresponding to \( V^\otimes n \) and the indices corresponding to \( V^\otimes n' \)).

Let \( c := (c_1, \ldots, c_n) \) be an \( n \)-tuple of complex parameters and \( \underline{c} := (c_1, \ldots, c_n') \) be an \( n' \)-tuple of complex parameters. We define a function \( R_{c,\underline{c}} \) of one variable \( u \in \mathbb{C} \) taking values in \( \text{End}(V^\otimes n \otimes V^\otimes n') \) by:

\[ R_{c,\underline{c}}(u) := \prod_{i=1,\ldots,n'} R_{n_i}(u + c_n - c_i) \ldots R_{2}(u + c_2 - c_0) R_{1}(u + c_1 - c_0). \]
Moving all the operators with \( n \) as a first index to the left and repeating the process for \( n - 1, n - 2, \ldots, 2 \), we find the following alternative form for \( R_{c,e}(u) \):

\[
R_{c,e}(u) = \prod_{i=1,\ldots,n} R_{i,1}(u + c_i - c_1)R_{i,2}(u + c_i - c_2) \ldots \ldots R_{i,n'}(u + c_i - c_{n'}) . \tag{3.4}
\]

We prove in the following theorem that the set of functions \( \{R_{c,e}\} \), where \( c \in \mathbb{C}^n \), \( e \in \mathbb{C}^{n'} \) and \( n, n' > 0 \), forms a family of solutions of the Yang–Baxter equation. We will call elements of this family “fused solutions” of the Yang–Baxter equation, and operators \( R_{c,e}(u) \) “fused operators”.

**Theorem 3.2.** Let \( n, n' \) and \( n'' \) be positive integers and let \( c := (c_1, \ldots, c_n) \), \( e := (e_1, \ldots, e_{n'}) \) and \( c' := (c_1', \ldots, c_{n''}') \) be, respectively, an \( n \)-tuple, an \( n' \)-tuple and an \( n'' \)-tuple of complex parameters. We have the functional equation

\[
R_{c,e}(u)R_{c,e}(u+v)R_{c,e}(v) = R_{c,e}(v)R_{c,e}(u+v)R_{c,e}(u) , \tag{3.5}
\]

where both sides take values in \( \text{End}(V^\otimes \otimes V'^{\otimes n'} \otimes V''^{\otimes n''}) \) and the copies of \( V \) are labelled by \( 1, \ldots, n, \frac{1}{1}, \ldots, n', \frac{1}{1}, \ldots, n'' \) (from left to right). In Equation (3.5), \( R_{c,e}(u) \) stands for the operator \( R_{c,e}(u) \otimes \text{Id}_{V'^{\otimes n'}} \) and similarly for \( R_{c,e}(u+v) \) and \( R_{c,e}(v) \).

**Proof.** Note first that the equality \((u + c_i - c_j) + (v + c_j - c_k) = u + v + c_i - c_k\), valid for any \( i \in \{1, \ldots, n\}, j \in \{1, \ldots, n'\} \) and \( k \in \{1, \ldots, n''\} \), ensures that, in the product of operators in the left hand side of (3.5), we can always apply the Yang–Baxter equation (3.1) as soon as the indices are properly arranged. In other words, we will never have to worry about the spectral parameters. Therefore, for saving place during the proof, we will drop them out of the notation, namely we set, until the end of the proof,

\[
R_{i,j} := R_{i,j}(u + c_i - c_j) , \quad R_{i,k} := R_{i,k}(u + v + c_i - c_k) \quad \text{and} \quad R_{j,k} := R_{j,k}(v + c_j - c_k) ,
\]

for any \( i \in \{1, \ldots, n\}, j \in \{1, \ldots, n'\} \) and \( k \in \{1, \ldots, n''\} \).

We prove the formula (3.5) by induction on \( n \). If \( n = 1 \) then the left hand side of (3.5) is

\[
R_{1,1}R_{1,2} \ldots R_{1,n'} \cdot R_{1,1}R_{1,2} \ldots R_{1,n''} \cdot \prod_{i=1,\ldots,n'} \left( R_{n',i} \ldots R_{2,i}R_{1,i} \right) \ldots \prod_{i=1,\ldots,n''} \left( R_{1,i} \ldots R_{n'',i} \ldots R_{2,i}R_{1,i} \right) .
\]

We have, for \( i = 1, \ldots, n'' \),

\[
R_{1,1}R_{1,2} \ldots R_{1,n'} \cdot R_{1,i} \cdot R_{n',i} \ldots R_{2,i}R_{1,i} = R_{1,1}R_{1,2} \ldots R_{1,n'-1} \cdot R_{n',i} \cdot R_{1,n'} \cdot R_{n''-1,i} \ldots R_{2,i}R_{1,i} \quad \text{(as} \quad R_{1,n'}R_{1,i} \cdot R_{n',i} = R_{n',i}R_{1,i}R_{1,n'} \quad \text{)}
\]

\[
= R_{n',i} \cdot R_{1,1}R_{1,2} \ldots R_{1,n'-1} \cdot R_{n',i} \cdot R_{n''-1,i} \ldots R_{2,i}R_{1,i} \cdot R_{1,n'} .
\]

Repeating this calculation with \( n' \) replaced by \( n' - 1 \) and so on, we find that

\[
R_{1,1}R_{1,2} \ldots R_{1,n'} \cdot R_{1,i} \cdot R_{n',i} \ldots R_{2,i}R_{1,i} = R_{n',i} \ldots R_{n,1} \cdot R_{1,i} \cdot R_{1,1}R_{1,2} \ldots R_{1,n'} .
\]
So we finally obtain that the left hand side of (3.5) for \( n = 1 \) is equal to

\[
\prod_{i=1,\ldots,n''} \left( R_{n''_1} \cdots R_{2_i} R_{1_i} \cdot R_{1_1} R_{1_2} \cdots R_{1,n'} \right)
= \prod_{i=1,\ldots,n''} \left( R_{n''_1} \cdots R_{2_i} R_{1_i} \cdot R_{1_1} R_{1_2} \cdots R_{1,n''} \cdot R_{1_1} R_{1_2} \cdots R_{1,n'} ,
\right.
\]

which coincides with the right hand side of (3.5) for \( n = 1 \).

Now let \( n > 1 \) and set \( c^{(n-1)} := (c_1, \ldots, c_{n-1}) \). Using (3.4) and commutation relations, we reorganize the left hand side of (3.5) and write it as

\[
R_{n,1} R_{n,2} \cdots R_{n,n'} \cdot R_{n,1} R_{n,2} \cdots R_{n,n''} \cdot R_{c^{(n-1)}} (u) R_{c^{(n-1)}} (u + v) R_{c^{(n-1)}} (v) .
\]

We use the induction hypothesis to transform this expression into

\[
R_{n,1} R_{n,2} \cdots R_{n,n'} \cdot R_{n,1} R_{n,2} \cdots R_{n,n''} \cdot R_{c^{(n)}} (v) R_{c^{(n)}} (u + v) R_{c^{(n)}} (u) .
\]

Then we use the induction basis (with the space labelled here by \( n \) playing the same role as the space labelled by 1 in the calculation for the induction basis) to move \( R_{c^{(n)}} (v) \) to the left and we obtain

\[
R_{c^{(n)}} (v) \cdot R_{n,1} R_{n,2} \cdots R_{n,n'} \cdot R_{n,1} R_{n,2} \cdots R_{n,n''} \cdot R_{c^{(n)}} (u + v) R_{c^{(n)}} (u) .
\]

This is equal, using again commutation relations, to \( R_{c^{(n)}} (v) R_{c^{(n)}} (u + v) R_{c^{(n)}} (u) \), that is, to the right hand side of (3.5). \( \square \)

Remark 3.3. To the operator \( R(u) \) is associated the following figure:

```
       u
      / \
     /   \
    /     \
   /       \
  /         \
/           \
```

Roughly speaking, this represents the interaction of two particles at the intersection of the two lines, and the interaction is governed, in some sense, by the operator \( R(u) \) (see, e.g., [2] for the physical meaning of the graphical formulation of the Yang–Baxter equation). Then, the Yang–Baxter equation (3.1) is formulated graphically as

```
1 2 3
\forward
\forward
\forward
\backward
\backward
\backward
```

The numbers above the lines are the indices of the copy of \( V \) in \( V^\otimes 3 \). By convention, we read the figure from top to bottom and we write the corresponding operators from left to right; for example, the left hand side of the picture above corresponds to \( R_{1,2}(u) R_{1,3}(u + v) R_{2,3}(v) \).

Within this graphical interpretation, the fused solutions, given by Formula (3.3), correspond to interactions of multiplets of particles, namely, \( n \) particles interacting with \( n' \) particles. As an example, for \( n = n' = 2 \), the corresponding interaction is depicted by:
where \( u_{ij} := u + c_i - c_j \) for \( i, j = 1, 2 \), and \( c_1, c_2, c_1', c_2' \in \mathbb{C} \) are the parameters of the fused solutions. We note that there is no ambiguity in the ordering of the factors, because \( R_{1,\perp}(u_{11}) \) and \( R_{2,\perp}(u_{22}) \) commute. The expression obtained from the above picture is equal to the right hand side of the defining formula (3.3) for \( n = n' = 2 \). \( \triangle \)

**Example 3.4.** Let \( n = 2 \) and \( n'' = 1 \). We consider the space \( V^\otimes 2 \otimes V \otimes V \) and the four copies of \( V \) are labelled (from left to right) by \( 1, 2, 1', 1'' \). By definition, we have

\[
R_{c,c}(u) := R_{2,\perp}(u + c_2 - c_1)R_{1,\perp}(u + c_1 - c_2),
\]

\[
R_{c,c}(u) := R_{2,\perp}(u + c_2 - c_1)R_{1,\perp}(u + c_1 - c_2) \quad \text{and} \quad R_{c,c}(u) := R_{1,\perp}(u + c_1 - c_1).
\]

We illustrate on this example the calculation made in the proof of Theorem 3.2 (we use the same abbreviated notation as in the proof):

\[
R_{c,c}(u)R_{c,c}(u + v)R_{c,c}(v) = R_{2,\perp}R_{1,\perp}R_{2,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}
\]

(as \( R_{1,\perp}R_{2,\perp} = R_{2,\perp}R_{1,\perp} \))

\[
= R_{2,\perp}R_{2,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}
\]

(as \( R_{2,\perp}R_{2,\perp}R_{1,\perp}R_{1,\perp} = R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp} \))

\[
= R_{1,\perp}R_{2,\perp}R_{2,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}
\]

(as \( R_{2,\perp}R_{2,\perp}R_{1,\perp}R_{1,\perp} = R_{1,\perp}R_{1,\perp}R_{2,\perp}R_{2,\perp} \))

\[
= R_{1,\perp}R_{2,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{2,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}R_{1,\perp}
\]

(as \( R_{2,\perp}R_{2,\perp}R_{1,\perp}R_{1,\perp} = R_{1,\perp}R_{1,\perp}R_{2,\perp}R_{2,\perp} \))

\[
= R_{c,c}(v)R_{c,c}(u + v)R_{c,c}(u) .
\]

This calculation (and the “higher” ones as well) is maybe more clear within the graphical interpretation explained in Remark 3.3. The calculation above corresponds to the following graphical calculation:

where we omit the spectral parameters at the intersections as they are fixed by the indices of the interacting lines. Moving the line indexed by \( \perp \) in the picture (using the pictorial version of the Yang–Baxter equation of Remark 3.3) corresponds to moving the operator \( R_{1,\perp} \) from right to left in the calculation written above. Here also, we note that there is no ambiguity in the ordering of the factors, due to commutation relations. \( \triangle \)
Example 3.5. In this example, we let $R(u)$ be the Yang solution (3.2). We set $n = 2$, $n' = 1$, $c_1 = c_2 = 0$ and $c_2 = 1$. We have:

$$R_{e,e}(u) = R_{2,1}(u + 1)R_{1,1}(u) = \left(\text{Id}_{V^\otimes 3} - \frac{P_{2,1}}{u + 1}\right) \left(\text{Id}_{V^\otimes 3} - \frac{P_{1,1}}{u}\right). \quad (3.6)$$

We consider the case $\dim(V) = 2$ and let $\{e_1, e_2\}$ be a basis of $V$. We define $e_{11} := e_1 \otimes e_1$, $e_{12} := e_1 \otimes e_2 + e_2 \otimes e_1$, $e_{22} := e_2 \otimes e_2$ and $e_{[2]} := e_1 \otimes e_2 - e_2 \otimes e_1$. We calculate the matrix of the endomorphism (3.6) in the following basis of $V^\otimes 2 \otimes V$:

$$\{e_{11} \otimes e_1, e_{12} \otimes e_1, e_{22} \otimes e_1, e_{11} \otimes e_2, e_{12} \otimes e_2, e_{22} \otimes e_2, e_{[2]} \otimes e_1, e_{[2]} \otimes e_2\}.$$ We obtain (points indicate coefficients equal to 0):

$$\left(\begin{array}{cccccccc}
\frac{u-1}{u} & . & . & . & . & . & . & . \\
. & \frac{u}{u+1} & . & -\frac{1}{u+1} & . & -\frac{1}{u(u+1)} & . & . \\
. & . & 1 & . & -\frac{2}{u+1} & . & . & . \\
. & -\frac{2}{u+1} & . & 1 & . & . & . & . \\
. & . & -\frac{1}{u+1} & . & \frac{u}{u+1} & . & . & . \\
. & . & . & \frac{u-1}{u+1} & . & . & . & . \\
. & . & . & . & \frac{u-2}{u} & . & . & . \\
. & . & . & . & . & \frac{u-2}{u} & . & . \\
\end{array}\right)$$

We remark that, for the particular choice of $c_1, c_2, c_2$ made here, the fused operator $R_{e,e}(u)$ leaves invariant the subspace $S^2V \otimes V$ of $V^\otimes 2 \otimes V$, where $S^2V$ is the symmetric square of $V$. This phenomenon can be explained by the relation:

$$R_{e,e}(u) \cdot R_{1,2}(-1) = R_{2,1}(u + 1)R_{1,1}(u) \cdot R_{1,2}(-1) = R_{1,2}(-1) \cdot R_{1,1}(u)R_{2,1}(u + 1),$$

together with the fact that the space $S^2V$ is the image of the endomorphism $R(-1)$ of $V^\otimes 2$ (since $R(-1) = \text{Id}_{V^\otimes 2} + P$).

Example 3.6. We still take for $R(u)$ the Yang solution (3.2) and we set again $n = 2$, $n' = 1$ and $c_1 = c_2 = 0$. But now we set $c_2 = -1$. We then have

$$R_{e,e}(u) = R_{2,1}(u - 1)R_{1,1}(u) = \left(\text{Id}_{V^\otimes 3} - \frac{P_{2,1}}{u - 1}\right) \left(\text{Id}_{V^\otimes 3} - \frac{P_{1,1}}{u}\right). \quad (3.7)$$

We consider again the case $\dim(V) = 2$ and calculate the matrix of the endomorphism (3.7) in the same basis of $V^\otimes 2 \otimes V$ as in Example 3.5. We obtain (points indicate coefficients equal to 0):

$$\left(\begin{array}{cccccccc}
\frac{u-2}{u} & . & . & . & . & . & . & . \\
. & \frac{u-1}{u} & . & -\frac{1}{u} & . & -\frac{2}{u} & . & . \\
. & . & 1 & . & -\frac{2}{u} & . & . & . \\
. & -\frac{2}{u} & . & 1 & . & . & . & . \\
. & . & -\frac{1}{u} & . & \frac{u-1}{u} & . & . & . \\
. & . & . & \frac{u-2}{u} & . & . & . & . \\
. & . & . & . & \frac{u-2}{u} & . & . & . \\
. & . & . & . & . & \frac{u-2}{u} & . & . \\
\end{array}\right)$$

We remark now that, for the particular choice of $c_1, c_2, c_2$ made here, the fused operator $R_{e,e}(u)$ leaves invariant the subspace $\Lambda^2V \otimes V$ of $V^\otimes 2 \otimes V$, where $\Lambda^2V$ is the alternating square of $V$. Similarly to the preceding example, this phenomenon can be explained by the relation:

$$R_{e,e}(u) \cdot R_{1,2}(1) = R_{2,1}(u - 1)R_{1,1}(u) \cdot R_{1,2}(1) = R_{1,2}(1) \cdot R_{1,1}(u)R_{2,1}(u - 1),$$
together with the fact that the space $A^2V$ is the image of the endomorphism $R(1)$ of $V^\otimes 2$ (since $R(1) = \text{Id}_{V^\otimes 2} - P$).

\[ \triangle \]

## 4. Invariant subspaces for fused solutions

In both Examples 3.5 and 3.6 we have noted that, for some choices of the parameters $c$ and $c'$, the fused operator $R_{c,c'}(u)$ leaves invariant a subspace of $V^\otimes n \otimes V^\otimes n'$; moreover this subspace can be seen as the image of an endomorphism of $V^\otimes n \otimes V^\otimes n'$ written in terms of the original solution $R(u)$ of the Yang–Baxter equation. The goal of this Section is to show that Examples 3.5 and 3.6 are actually (simplest) examples of a general phenomenon.

### 4.1 Invariant subspaces as images of certain operators

Let $n$ be a positive integer such that $n \geq 2$ and $c := (c_1, \ldots, c_n)$ be an $n$-tuple of complex parameters. Recall that $R(u)$ is an arbitrary solution of the Yang–Baxter equation (3.1) on $V \otimes V$. We define the following endomorphism of $V^\otimes n$:

\[ F(c) := \prod_{1 \leq i < j \leq n} R_{i,j}(c_i - c_j). \quad (4.1) \]

It will be useful to consider that $c_1, \ldots, c_n$ are complex variables and to see $F$ as a function of $c$ with values in $\text{End}(V^\otimes n)$. Depending on the function $R$, the function $F$ can be singular for some values of $c$ (see, for example, (3.2)).

We will need a preliminary Lemma concerning the operator $F(c)$.

**Lemma 4.1.** We have

\[ F(c) = \prod_{1 \leq i < j \leq n} R_{i,j}(c_i - c_j). \quad (4.2) \]

**Proof.** As in the proof of the Theorem 3.2 we do not need to pay attention to the spectral parameters since $(c_i - c_j) + (c_j - c_k) = c_i - c_k$. Thus, until the end of the proof, we use the abbreviated notation $R_{i,j} := R_{i,j}(c_i - c_j)$ for any $1 \leq i < j \leq n$.

We prove the formula (4.2) by induction on $n$. For $n = 2$ there is nothing to prove, so we let $n > 2$ and write

\[ F(c) = \prod_{1 \leq i < j \leq n} R_{i,j} = \left( \prod_{1 \leq i < j \leq n-1} R_{i,j} \right) \cdot R_{1,n}R_{2,n} \cdots R_{n-1,n}, \]

where we have moved to the right the operators $R_{i,n}$ using commutation relations. Now we use the induction hypothesis and we have

\[ F(c) = \left( \prod_{1 \leq i < j \leq n-1} R_{i,j} \right) \cdot R_{1,n}R_{2,n} \cdots R_{n-1,n} = \left( \prod_{2 \leq i < j \leq n-1} R_{i,j} \right) \cdot R_{1,n-1} \cdots R_{1,2} \cdot R_{1,n}R_{2,n} \cdots R_{n-1,n}. \]

Note that

\[ R_{1,n-1} \cdots R_{1,3} \cdot R_{1,2}R_{1,n}R_{2,n} \cdot R_{3,n} \cdots R_{n-1,n} = R_{1,n-1} \cdots R_{1,3} \cdot R_{2,n}R_{1,n}R_{1,2} \cdot R_{3,n} \cdots R_{n-1,n} = R_{2,n} \cdot R_{1,n-1} \cdots R_{1,3}R_{1,n}R_{3,n} \cdots R_{n-1,n} \cdot R_{1,2}, \]
and thus, repeating a similar calculation the necessary number of times, we arrive at

\[ F(c) = \left( \prod_{2 \leq i < j \leq n-1} R_{i,j} \right) R_{2,n} \ldots R_{n-1,n} \cdot R_{1,n} R_{1,n-1} \ldots R_{1,2}. \]

We use again the induction hypothesis and commutation relations to conclude as follows:

\[
F(c) = \left( \prod_{2 \leq i < j \leq n} R_{i,j} \right) R_{2,n} \ldots R_{n-1,n} \cdot R_{1,n} R_{1,n-1} \ldots R_{1,2} \\
= \left( \prod_{2 \leq i < j \leq n} R_{i,j} \right) R_{1,n} R_{1,n-1} \ldots R_{1,2} \\
= \left( \prod_{2 \leq i < j \leq n} R_{i,j} \right) R_{1,n} R_{1,n-1} \ldots R_{1,2} = \prod_{1 \leq i < j \leq n} R_{i,j}.
\]

By Theorem 4.2, the fused operator \( R_{c,c}(u) \) preserves the subspace \( W_c \otimes W_c \) of \( V^\otimes n \otimes V^\otimes n' \). 

**Proof.** Let \( v \) be a complex variable. We first prove the following Lemma:

**Lemma 4.3.** (i) We consider the space \( V \otimes V^\otimes n' \) with the copies of \( V \) labelled by \( 0, 1, \ldots, n' \) (from left to right). The endomorphism \( R_{0,1}(v-c_1)R_{0,2}(v-c_2) \ldots R_{0,n'}(v-c_n') \) preserves the subspace \( V \otimes W_c \subset V \otimes V^\otimes n' \).

(ii) We consider the space \( V^\otimes n \otimes V \) with the copies of \( V \) labelled by \( 1, \ldots, n, n+1 \) (from left to right). The endomorphism \( R_{n,n+1}(c_n-v) \ldots R_{2,n+1}(c_2-v)R_{1,n+1}(c_1-v) \) preserves the subspace \( W_c \otimes V \subset V^\otimes n \otimes V \).

**Proof of the lemma.** (i) Set \( c_0 := v \). The image of the endomorphism \( \text{Id} \otimes F(c) \) of \( V \otimes V^\otimes n' \) is \( V \otimes W_c \). We have, using Lemma 4.1,

\[
R_{0,1}(v-c_1)R_{0,2}(v-c_2) \ldots R_{0,n'}(v-c_n') \cdot (\text{Id} \otimes F(c)) \\
= \prod_{0 \leq i < j \leq n'} R_{i,j}(c_i - c_j) = \prod_{0 \leq i < j \leq n'} R_{j,i}(c_i - c_j) \\
= \left( \prod_{1 \leq i < j \leq n'} R_{i,j}(c_i - c_j) \right) \cdot R_{0,n'}(v-c_n') \ldots R_{0,2}(v-c_2)R_{0,1}(v-c_1) \\
= (\text{Id} \otimes F(c)) \cdot R_{0,n'}(v-c_n') \ldots R_{0,2}(v-c_2)R_{0,1}(v-c_1).
\]

Thus the operator \( R_{0,1}(v-c_1)R_{0,2}(v-c_2) \ldots R_{0,n'}(v-c_n') \) restricted on \( V \otimes W_c \) has its image contained in \( V \otimes W_c \).
(ii) Now set $c_{n+1} := v$. The image of the endomorphism $F(c) \otimes \text{Id}$ of $V^\otimes n \otimes V$ is $W_c \otimes V$. We have

$$R_{n,n+1}(c_n - v) \ldots R_{2,n+1}(c_2 - v)R_{1,n+1}(c_1 - v) \cdot (F(c) \otimes \text{Id})$$

$$= R_{n,n+1}(c_n - v) \ldots R_{2,n+1}(c_2 - v)R_{1,n+1}(c_1 - v) \cdot \left( \prod_{1 \leq i < j \leq n} R_{i,j}(c_i - c_j) \right)$$

$$= \left( \prod_{1 \leq i < j \leq n+1} R_{i,j}(c_i - c_j) \right)$$

$$= \left( \prod_{1 \leq i < j \leq n} R_{i,j}(c_i - c_j) \right)$$

$$= \left( \prod_{1 \leq i < j \leq n} R_{i,j}(c_i - c_j) \right) \cdot R_{1,n+1}(c_1 - v)R_{2,n+1}(c_2 - v) \ldots R_{n,n+1}(c_n - v)$$

$$= (F(c) \otimes \text{Id}) \cdot R_{1,n+1}(c_1 - v)R_{2,n+1}(c_2 - v) \ldots R_{n,n+1}(c_n - v).$$

We used Lemma 4.1 in the first and third equalities, and commutation relations in the second and fourth equalities. We conclude that the image of the restriction on $W_c \otimes V$ of the operator $R_{n,n+1}(c_n - v) \ldots R_{2,n+1}(c_2 - v)R_{1,n+1}(c_1 - v)$ is contained in $W_c \otimes V$. 

We return to the proof of Theorem 4.2. Recall that the operator $R_{c,\underline{\ell}}(u)$ is given by Formula (3.3), namely

$$R_{c,\underline{\ell}}(u) := \prod_{i=1,\ldots,n} R_{n,\underline{\ell}}(u + c_n - c_{\underline{\ell}}) \ldots R_{2,\underline{\ell}}(u + c_2 - c_{\underline{\ell}})R_{1,\underline{\ell}}(u + c_1 - c_{\underline{\ell}}).$$

For any $i \in \{1, \ldots, n\}$, applying Lemma 4.3(ii) with $v = c_{\underline{\ell}} - u$ and the label $n + 1$ replaced by $i$, we obtain that the operator $R_{n,\underline{\ell}}(u + c_n - c_{\underline{\ell}}) \ldots R_{2,\underline{\ell}}(u + c_2 - c_{\underline{\ell}})R_{1,\underline{\ell}}(u + c_1 - c_{\underline{\ell}})$ preserves the subspace $W_c \otimes V^\otimes n$, and so in turn that the operator $R_{c,\underline{\ell}}(u)$ preserves $W_c \otimes V^\otimes n$.

Now recall the alternative formula (3.4) for the operator $R_{c,\underline{\ell}}(u)$:

$$R_{c,\underline{\ell}}(u) = \prod_{i=1,\ldots,n} R_{i,\underline{\ell}}(u + c_i - c_{\underline{\ell}})R_{i,2}(u + c_i - c_2) \ldots R_{i,n'}(u + c_i - c_{n'}).$$

For any $i \in \{1, \ldots, n\}$, applying Lemma 4.3(i) with $v = u + c_i$ and the label $\underline{\ell}$ replaced by $i$, we obtain that the operator $R_{i,1}(u + c_i - c_{\underline{\ell}})R_{i,2}(u + c_i - c_2) \ldots R_{i,n'}(u + c_i - c_{n'})$ preserves the subspace $V^\otimes n \otimes W_{\underline{\ell}}$, and so in turn that the operator $R_{c,\underline{\ell}}(u)$ preserves $V^\otimes n \otimes W_{\underline{\ell}}$.

We conclude that the operator $R_{c,\underline{\ell}}(u)$ preserves the subspace $W_c \otimes W_{\underline{\ell}}$. 

Remark 4.4. For $n = n' = 2$, part of Theorem 4.2 is illustrated as follows, using the pictorial version of the Yang–Baxter equation explained in Remark 3.5:

![Diagram](image-url)
Remark 4.5. The content of Theorem 4.2 is empty if the endomorphisms $F(c)$ and $F(c)$ are both invertible (as in this situation $W_c \otimes W_c = V^\otimes n \otimes V^\otimes n'$). This is the goal of Sections 5–7 to explain that, for standard examples of $R(u)$, there are some values of $c$ such that the images of the operators $F(c)$ are interesting proper subspaces of $V^\otimes n$. △

4.2 Alternative formula for the operator $F(c)$

We will give an alternative formula for $F(c)$ which will be useful later. To do this, we define a function $\hat{R}$ with values in $\text{End}(V \otimes V)$ by:

$$\hat{R}(u) := R(u)P,$$

where we recall that $P$ is the permutation operator on $V \otimes V$.

For any $\pi$ in the symmetric group $S_n$ on $n$ letters, we define $P_\pi \in \text{End}(V^\otimes n)$ by:

$$P_\pi(x_1 \otimes x_2 \otimes \cdots \otimes x_n) := x_{\pi(1)} \otimes x_{\pi(2)} \otimes \cdots \otimes x_{\pi(n)} \quad \text{for } x_1, \ldots, x_n \in V. \quad (4.5)$$

Note that if $\pi$ is written as a product of transposition, say $\pi = (i_1, j_1)(i_2, j_2) \cdots (i_k, j_k) \in S_n$, then $P_\pi = P_{i_1,j_1}P_{i_2,j_2} \cdots P_{i_k,j_k}$.

Let $w_n$ be the longest element of the symmetric group $S_n$. We recall the following property of the element $w_n$:

$$w_n = (1, 2)(2, 3) \cdots (n-1, n) \cdot w_{n-1} \quad \text{and} \quad w_n(i) = n - i + 1 \quad \text{for all } i = 1, \ldots, n, \quad (4.6)$$

where $w_{n-1}$ is the longest element of $S_{n-1}$, seen as an element of $S_n$ acting only on the letters $1, \ldots, n-1$. Note that $w_n$ is an involution.

Lemma 4.6. We have

$$F(c) = \left( \prod_{i=1,\ldots,n-1} \hat{R}_{i,i+1}(c_i - c_{i+1}) \cdots \hat{R}_{2,3}(c_1 - c_2) \hat{R}_{1,2}(c_1 - c_{i+1}) \right) \cdot P_{w_n}. \quad (4.7)$$

Proof. We prove Formula (4.7) by induction on $n$. For $n = 2$ there is nothing to prove. Let $n > 2$ and write

$$F(c) = \prod_{1 \leq i < j \leq n} R_{i,j}(c_i - c_j) = \left( \prod_{1 \leq i < j \leq n-1} R_{i,j}(c_i - c_j) \right) \cdot R_{1,n}(c_1 - c_n)R_{2,n}(c_2 - c_n) \cdots R_{n-1,n}(c_{n-1} - c_n).$$

The induction hypothesis allows to replace $\prod_{1 \leq i < j \leq n-1} R_{i,j}(c_i - c_j)$ by

$$\left( \prod_{i=1,\ldots,n-2} \hat{R}_{i,i+1}(c_i - c_{i+1}) \cdots \hat{R}_{2,3}(c_1 - c_2) \hat{R}_{1,2}(c_1 - c_{i+1}) \right) \cdot P_{w_{n-1}}.$$

So it remains to prove that

$$P_{w_{n-1}}R_{1,n}(c_1 - c_n)R_{2,n}(c_2 - c_n) \cdots R_{n-1,n}(c_{n-1} - c_n) = \hat{R}_{n-1,n}(c_1 - c_n) \cdots \hat{R}_{2,3}(c_{n-2} - c_n) \hat{R}_{1,2}(c_{n-1} - c_n)P_{w_n}. \quad (4.8)$$
4.3 Admissible permutation of the parameters $(c_1, \ldots, c_n)$

Let $n, n'$ be positive integers such that $n, n' \geq 2$, let $c := (c_1, \ldots, c_n)$ be an $n$-tuple of complex numbers and $c' := (c_1', \ldots, c_{n'})$ be an $n'$-tuple of complex numbers.

For any permutation $\pi$ in the symmetric group $S_n$, we define $c^{(\pi)} := (c_{\pi(1)}, \ldots, c_{\pi(n)})$. We denote by $s_k$ the transposition $(k, k+1) \in S_n$, for $k = 1, \ldots, n-1$; then we have, for $k = 1, \ldots, n-1$, $c^{(s_k)} = (c_1, \ldots, c_{k+1}, c_k, \ldots, c_n)$.

We recall that the copies of $V$ in $V^{\otimes n} \otimes V^{\otimes n'}$ are labelled by $1, \ldots, n, 1', \ldots, n'$ (from left to right).

**Lemma 4.8.** (i) For $k = 1, \ldots, n-1$, we have, on $V^{\otimes n} \otimes V^{\otimes n'}$,$$P_{k,k+1}R_{k+1,k}(c_{k+1} - c_k) \cdot R_{c_L}(u) = R_{c^{(s_k)L}}(u) \cdot P_{k,k+1}R_{k+1,k}(c_{k+1} - c_k) .$$ (4.10)

(ii) For $k = 1, \ldots, n-1$, we have, on $V^{\otimes n}$,$$P_{k,k+1}R_{k+1,k}(c_{k+1} - c_k) \cdot F(c) = F(c^{(s_k)}) \cdot P_{k,k+1}R_{k,k+1}(c_k - c_{k+1}) .$$ (4.11)

**Proof.** (i) We have, for any $i \in \{1, \ldots, n\}$,$$P_{k,k+1}R_{k+1,k}(c_{k+1} - c_k) \cdot \prod_{j=1,\ldots,n} R_{j,i}(u + c_j - c_i)$$

$$= P_{k,k+1} \left( \prod_{j=1,\ldots,n} R_{s_k(j),i}(u + c_{s_k(j)} - c_i) \right) \cdot R_{k+1,k}(c_{k+1} - c_k)$$

$$= \left( \prod_{j=1,\ldots,n} R_{j,i}(u + c_{s_k(j)} - c_i) \right) \cdot P_{k,k+1}R_{k+1,k}(c_{k+1} - c_k) ;$$

we use in the second equality commutation relations, together with the Yang–Baxter relation on the copies labelled by $k+1$, $k$ and $i$. Due to Formula (3.3), this proves the item (i).

(ii) Formula (4.11) is equivalent to

$$R_{k+1,k}(c_{k+1} - c_k) \cdot F(c) = \left( \prod_{1 \leq i < j \leq n} R_{s_k(i),s_k(j)}(c_{s_k(i)} - c_{s_k(j)}) \right) \cdot R_{k,k+1}(c_k - c_{k+1}) .$$ (4.12)
We prove Formula (4.12) by induction on \( n \). If \( n = 2 \), Formula (4.12) is trivial.

Let \( n > 2 \). We deal first with the case \( k = 1 \). We have, using commutation relations,

\[
F(c) = \left( \prod_{1 \leq i < j \leq n-1} R_{i,j}(c_i - c_j) \right) \cdot R_{1,n}(c_1 - c_n)R_{2,n}(c_2 - c_n) \cdots R_{n-1,n}(c_{n-1} - c_n) .
\]

The induction hypothesis gives

\[
R_{2,1}(c_2 - c_1) \cdot \left( \prod_{1 \leq i < j \leq n-1} R_{i,j}(c_i - c_j) \right) = \left( \prod_{1 \leq i < j \leq n-1} R_{s_{1}(i),s_{1}(j)}(c_{s_{1}(i)} - c_{s_{1}(j)}) \right) \cdot R_{1,2}(c_1 - c_2) ,
\]

and besides we have

\[
R_{1,2}(c_1 - c_2) \cdot R_{1,n}(c_1 - c_n)R_{2,n}(c_2 - c_n) \cdots R_{n-1,n}(c_{n-1} - c_n) = R_{2,n}(c_2 - c_n)R_{1,n}(c_1 - c_n) \cdots R_{n-1,n}(c_{n-1} - c_n) \cdot R_{1,2}(c_1 - c_2) ,
\]

where we used the Yang–Baxter relation on the copies labelled by 1, 2 and \( n \), and commutation relations. Formula (4.12) for \( k = 1 \) follows.

Then we assume that \( k > 1 \), and we write

\[
F(c) = R_{1,2}(c_1 - c_2)R_{1,3}(c_1 - c_3) \cdots R_{1,n}(c_1 - c_n) \cdot \left( \prod_{2 \leq i < j \leq n} R_{i,j}(c_i - c_j) \right).
\]

Similarly to above, we first have, using commutation relations and the Yang–Baxter equation on the copies labelled by 1, \( k + 1 \) and \( k \), that

\[
R_{k+1,k}(c_{k+1} - c_k) \cdot R_{1,2}(c_1 - c_2) \cdots R_{1,k}(c_1 - c_k)R_{1,k+1}(c_1 - c_{k+1}) \cdots R_{1,n}(c_1 - c_n) = R_{1,2}(c_1 - c_2) \cdots R_{1,k+1}(c_1 - c_{k+1})R_{1,k}(c_1 - c_k) \cdots R_{1,n}(c_1 - c_n) \cdot R_{k+1,k}(c_{k+1} - c_k) ,
\]

and then we use the induction hypothesis, namely (since \( k > 1 \))

\[
R_{k+1,k}(c_{k+1} - c_k) \cdot \left( \prod_{2 \leq i < j \leq n} R_{i,j}(c_i - c_j) \right) = \left( \prod_{2 \leq i < j \leq n} R_{s_{k}(i),s_{k}(j)}(c_{s_{k}(i)} - c_{s_{k}(j)}) \right) \cdot R_{k,k+1}(c_{k+1} - c_k) .
\]

This yields the desired result.

For the remaining of this section, we assume that

\[
R(u)R_{2,1}(-u) = \gamma(u) \cdot \text{Id}_{V^{\otimes 2}} \quad \text{for a complex-valued function } \gamma \text{ of } u ,
\]

where \( R_{2,1}(u) := PR(u)P \). In the physical literature, this condition is called the “unitarity” condition. We note that \( \gamma(-u) = \gamma(u) \).

**Definition 4.9.** For \( k \in \{1, \ldots, n-1 \} \), we say that the transposition \( s_k = (k, k+1) \) is admissible for \( R(u) \) and for the set of parameters \( c = (c_1, \ldots, c_n) \) if \( \gamma(c_k - c_{k+1}) \neq 0 \).

Let \( k \in \{1, \ldots, n-1 \} \) such that the transposition \( s_k \) is admissible for \( R(u) \) and for the set of parameters \( c = (c_1, \ldots, c_n) \). We define an endomorphism \( A_k \) of \( V^{\otimes n} \) by

\[
A_k := P_{k,k+1}R_{k,k+1}(c_{k+1} - c_k) .
\]

The admissibility of \( s_k \) implies that the operator \( A_k \) is invertible with, from (4.13),

\[
A_k^{-1} = \gamma_k^{-1}R_{k,k+1}(c_k - c_{k+1})P_{k,k+1} , \quad \text{where } \gamma_k := \gamma(c_k - c_{k+1}) .
\]
In this situation, the assertions of Lemma 4.8 can be rewritten, with \( \tilde{A}_k := P_{k,k+1} A_k P_{k,k+1} \),
\[
R_{c^{(s_k)},c}(u) = (A_k \otimes \text{Id}_{V^\otimes n'}) R_{c,c}(u) (A_k^{-1} \otimes \text{Id}_{V^\otimes n'}) \quad \text{and} \quad A_k F(c) = \gamma_k F(c^{(s_k)}) \tilde{A}_k^{-1}. \tag{4.15}
\]

We sum up the consequences of these two formulas in the following proposition. We assume that \( W_c \) (and thus \( W_{c^{(s_k)}} \) as well) are defined.

**Proposition 4.10.** Let \( k \in \{1, \ldots, n-1\} \) such that the transposition \( s_k \) is admissible for \( R(u) \) and for the set of parameters \( c = (c_1, \ldots, c_n) \). Let \( e_1, \ldots, e_l \in V^\otimes n \) such that \( B_c := \{ F(c)e_1, \ldots, F(c)e_l \} \) is a basis of the subspace \( W_c \).

(i) Then \( B_{c^{(s_k)}} := \{ A_k F(c)e_1, \ldots, A_k F(c)e_l \} \) is a basis of \( W_{c^{(s_k)}} \). In particular, \( W_c \) and \( W_{c^{(s_k)}} \) have the same dimension.

(ii) Moreover, for any basis \( B_c \) of \( W_c \), the matrix of the endomorphism \( R_{c,c}(u) \big|_{W_c \otimes W_c} \) in the basis \( B_c \otimes B_c \) coincides with the matrix of the endomorphism \( R_{c^{(s_k)},c}(u) \big|_{W_{c^{(s_k)}} \otimes W_{c^{(s_k)}}} \) in the basis \( B_{c^{(s_k)}} \otimes B_{c^{(s_k)}} \) (we use the notation \( B_c \otimes B_c := \{ a \otimes b \mid a \in B_c, b \in B_c \} \) and similarly for \( B_{c^{(s_k)}} \otimes B_{c^{(s_k)}} \).

**Proof.**

(i) As the operator \( A_k \) is invertible, the linear independence of the vectors in \( B_{c^{(s_k)}} \) is immediate. Moreover, due to the second relation in (4.15), we have that the dimensions of \( W_c \) and \( W_{c^{(s_k)}} \) coincide and also that the vectors in \( B_{c^{(s_k)}} \) belong to \( W_{c^{(s_k)}} \). This proves the item (i).

(ii) The item (ii) is a direct consequence of the first formula in (4.15). \( \square \)

**Remark 4.11.** In Sections 3 and 4 we worked with a solution of the Yang–Baxter equation with “additive” spectral parameters (see paragraph 4 in Section 2). The whole construction in Sections 3 and 4 has an equivalent “multiplicative” version, where one replaces addition for the spectral parameters by multiplication. As examples, the fused solution (3.3) is given by
\[
R_{c,c}(\alpha) := \prod_{i=1,\ldots,n'} R_{n,i} \left( \frac{\alpha c_i}{c_i} \right) \ldots \frac{\alpha c_2}{c_2} R_{1,2} \left( \frac{\alpha c_1}{c_1} \right), \tag{4.16}
\]
the invariant subspace for this fused solution is the image in \( V^\otimes n \) of the operator
\[
F(c) := \prod_{1 \leq i < j \leq n} R_{i,j} \left( \frac{c_i}{c_j} \right), \tag{4.17}
\]
and the unitarity condition (4.13) becomes \( R(\alpha) R_{2,1} \left( \frac{1}{\alpha} \right) = \gamma(\alpha) \cdot \text{Id}_{V^\otimes 2} \), with \( \gamma(\alpha) = \gamma \left( \frac{1}{\alpha} \right) \in \mathbb{C} \).

All proofs and calculations made in Sections 3 and 4 are completely similar for the multiplicative version. In the sequel we will use both versions. \( \square \)

5. Invariant subspaces and representations of \( U(\mathfrak{gl}_N) \)

Starting from an arbitrary solution \( R(u) \) of the Yang–Baxter equation on \( V \otimes V \), we obtained in the preceding sections, via the fusion procedure, a set \( \{ R_{c,c} \} \) of functions of \( u \in \mathbb{C} \), where \( c \in \mathbb{C}^n \), \( c \in \mathbb{C}^{n'} \) and \( n, n' > 0 \), which forms a family of solutions of the Yang–Baxter equation, and where \( R_{c,c}(u) \in \text{End}(V^\otimes n \otimes V^\otimes n') \). Moreover, for generic \( c \) and \( c' \), we identified a subspace \( W_c \otimes W_{c'} \subset V^\otimes n \otimes V^\otimes n' \) which is preserved by \( R_{c,c}(u) \). Therefore, by restriction, the operators \( R_{c,c}(u) \) induce a family of solutions of the Yang–Baxter equation acting on the spaces \( W_c \otimes W_{c'} \).
In this Section, we will further study the subspaces $W_c$, when we start from the Yang solution, $\text{(5.1)}$ below, of the Yang–Baxter equation on $V \otimes V$. This will lead to a family of fused solutions acting on irreducible representations of the general linear Lie algebra $\mathfrak{gl}_N$.

We fix in this Section:

$$ R(u) := \text{Id}_{V \otimes z} - \frac{P_u}{u} . \quad (5.1) $$

Let $n > 1$ and recall that $F$ is the following function of $c := (c_1, \ldots, c_n) \in \mathbb{C}^n$ with values in $\text{End}(V^{\otimes n})$:

$$ F(c) := \prod_{1 \leq i < j \leq n} R_{i,j}(c_i - c_j) . \quad (5.2) $$

Moreover, recall that the subspace we are interested in, denoted by $W_c$, is the image of the operator $F(c)$, for points $c = (c_1, \ldots, c_n)$ where the function $F$ is non-singular. For the solution $\text{(5.1)}$, the expression for the operator $F(c)$ reads:

$$ F(c) := \prod_{1 \leq i < j \leq n} (\text{Id}_{V^{\otimes n}} - \frac{P_{i,j}}{c_i - c_j}) . \quad (5.3) $$

### 5.1 Schur–Weyl duality

Set $N := \dim(V)$ and denote by $\mathfrak{gl}_N$ the Lie algebra of endomorphism of $V$ and by $U(\mathfrak{gl}_N)$ its universal enveloping algebra. The Lie algebra $\mathfrak{gl}_N$ acts on the tensor product $V^{\otimes n}$ and we denote by $\chi$ this representation, given by:

$$ \chi(g) := g \otimes \text{Id} \otimes \cdots \otimes \text{Id} + \text{Id} \otimes g \otimes \text{Id} \otimes \cdots + \cdots + \text{Id} \otimes \cdots \otimes \text{Id} \otimes g , \quad g \in \mathfrak{gl}_N. $$

The representation $\chi$ extends to a representation, which we still denote by $\chi$, of $U(\mathfrak{gl}_N)$.

The symmetric group $S_n$ acts on $V^{\otimes n}$ by permuting the copies and we denote this representation of $S_n$ by $\rho$. Explicitly, the representation $\rho$ is given by:

$$ \rho(\pi) = P_\pi \quad \text{for all } \pi \in S_n, \quad (5.4) $$

where $P_\pi$ is defined by $\text{(4.5)}$. The representation $\rho$ extends to a representation, which we still denote by $\rho$, of the group algebra $\mathbb{C}S_n$.

The irreducible complex representations of $\mathbb{C}S_n$ are parametrized by the partitions of $n$. For any partition $\lambda$ of $n$, we denote by $M_{\lambda}^{\mathbb{C}S_n}$ the corresponding irreducible $\mathbb{C}S_n$-module (with the convention that $M_{(n)}^{\mathbb{C}S_n}$ is the trivial representation).

For any partition $\lambda$ such that $\ell(\lambda) \leq N$, we denote by $M_\lambda^{U(\mathfrak{gl}_N)}$ the irreducible highest-weight $U(\mathfrak{gl}_N)$-module of highest weight $\lambda$.

The Schur–Weyl duality between the symmetric group $S_n$ and the algebra $U(\mathfrak{gl}_N)$ can be expressed by the following assertions (see e.g. $\text{[18][19]}$).

**Theorem 5.1.**

(i) The subalgebra $\rho(\mathbb{C}S_n)$ of $\text{End}(V^{\otimes n})$ is the centraliser of $\chi(U(\mathfrak{gl}_N))$.

(ii) The subalgebra $\chi(U(\mathfrak{gl}_N))$ of $\text{End}(V^{\otimes n})$ is the centraliser of $\rho(\mathbb{C}S_n)$.

(iii) As an $(U(\mathfrak{gl}_N) \otimes \mathbb{C}S_n)$-module defined by $\chi$ and $\rho$, the space $V^{\otimes n}$ decomposes as:

$$ V^{\otimes n} \cong \bigoplus_\lambda M_\lambda^{U(\mathfrak{gl}_N)} \otimes M_\lambda^{\mathbb{C}S_n} , \quad (5.5) $$

where $\lambda$ runs over the set of partitions such that $|\lambda| = n$ and $\ell(\lambda) \leq N$. 

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5.2 Fusion formula for idempotents of the symmetric group

According to Formulas (5.3) and (5.4), we are interested in the following rational function in $u_1, \ldots, u_n$ with values in the group algebra $\mathbb{C}S_n$:

$$
\Phi(u_1, \ldots, u_n) := \prod_{1 \leq i < j \leq n} \left(1 - \frac{(i, j)}{u_i - u_j}\right). \quad (5.6)
$$

We refer to the rational function $\Phi$ as the “fusion function” of the symmetric group. It is easy to check that, for distinct $i, j, k \in \{1, \ldots, n\}$,

$$
(1 - \frac{(i, j)}{u})(1 - \frac{(i, k)}{u + v})(1 - \frac{(j, k)}{v}) = (1 - \frac{(j, k)}{v})(1 - \frac{(i, k)}{u + v})(1 - \frac{(i, j)}{u}) . \quad (5.7)
$$

The functions $\Phi$ with values in $\mathbb{C}S_n$ are called Baxterized elements. We say that they are “universal” solutions of the Yang–Baxter equation (associated to the symmetric group) as they provide solutions of the Yang–Baxter equation in the representations of $S_n$.

Let $\lambda$ be a partition of $n$ and let $T$ be a standard Young tableau of shape $\lambda$. For brevity, set $cc_i := cc(T| i)$ for $i = 1, \ldots, n$. The following result traces back to the work of Jucys [20], see also [17, 21–24] (we use a formulation as in [17]).

**Theorem 5.2.** The element obtained by the following consecutive evaluations

$$
f(\lambda)\Phi(u_1, \ldots, u_n) \bigg|_{u_1 = cc_1, u_2 = cc_2, \ldots, u_n = cc_n}
$$

is a primitive idempotent of $\mathbb{C}S_n$ which generates a minimal left ideal isomorphic, as an $\mathbb{C}S_n$-module, to the irreducible module $M^\lambda_{\mathbb{C}S_n}$.

Recall that $f(\lambda)$ is the non-zero complex number defined in (2.4).

**Remark 5.3.** For some standard Young tableau $T$, it may happen that $cc(T| j) = cc(T| k)$ for $j \neq k$ (for example, if the standard Young tableau $T$ has the shape (2, 2)). In this situation, Theorem 5.2 asserts in particular that the rational function $\Phi(u_1, \ldots, u_n)$ is non-singular for the evaluations as in (5.8), even if the factor $\left(1 - \frac{(j, k)}{u_j - u_k}\right)$ is singular at $u_j = cc(T| j)$ and $u_k = cc(T| k)$.

\[\triangle\]

5.3 Fused solutions on $M^U(\mathfrak{g}_N) \otimes M^U(\mathfrak{g}_N)$

Let $T$ be a standard Young tableau such that $sh_T = \lambda$ with $|\lambda| = n$ and $\ell(\lambda) \leq N$. We let

$$
F(T) := F(c) \bigg|_{c_1 = cc(T|1), c_2 = cc(T|2), \ldots, c_n = cc(T|n)}, \quad (5.9)
$$

where $c = (c_1, \ldots, c_n)$ is seen here as an $n$-tuple of variables.

Now, in view of Theorem 5.2, we define

$$
E_T := f(\lambda)\Phi(u_1, \ldots, u_n) \bigg|_{u_1 = cc(T|1), u_2 = cc(T|2), \ldots, u_n = cc(T|n)}. \quad (5.10)
$$

Due to Formulas (5.3) and (5.4), we have that

$$
F(T) = \rho \left(f(\lambda)^{-1}E_T\right). \quad (5.11)
$$

The assertions of Theorem 5.2 imply that $F$ is non-singular for the consecutive evaluations of the variables as in (5.9). We denote by $W_T$ the image of the operator $F(T)$, which is well-defined.
and which coincides, due to \( (5.11) \), with the image in \( V^\otimes n \) of the operator \( \rho(E_T) \). Moreover, the element \( E_T \) acts on the irreducible \( \mathbb{C}S_n \)-module \( M^\mathbb{C}S_n \) as a projector on a one-dimensional subspace, and annihilates any irreducible \( \mathbb{C}S_n \)-module \( M^\mathbb{C}S_n' \) with \( \lambda' \neq \lambda \).

The preceding discussion, which follows from Theorem 5.2 together with the Schur–Weyl duality, namely Formula (5.5) in Theorem 5.1, leads to the identification of the space \( V \) with the symmetric square of \( \mathbb{C}U \).

**Theorem 5.4.** The subspace \( W_T \) of \( V^\otimes \mathbb{C} \) is an irreducible \( U(\mathfrak{gl}_N) \)-module isomorphic to \( M^U(\mathfrak{gl}_N) \).

Let \( \mathcal{T}, \mathcal{T}' \) be two standard Young tableaux such that \( \text{sh} \mathcal{T} = \lambda \) and \( \text{sh} \mathcal{T}' = \lambda' \), with \( |\lambda| = n \), \( |\lambda'| = n' \) and \( \ell(\lambda), \ell(\lambda') \leq N \). We set \( c^\mathcal{T} := (\text{cc}(\mathcal{T}|1), \ldots, \text{cc}(\mathcal{T}|n)) \), \( c^T := (\text{cc}(\mathcal{T}'|1), \ldots, \text{cc}(\mathcal{T}'|n')) \) and we define

\[
R^\text{res}_{\mathcal{T}, \mathcal{T}'}(u) := R^\mathcal{T, \mathcal{T}'}(u)|_{W_{\mathcal{T}} \otimes W_{\mathcal{T}'},}
\]

the restriction of the operator \( R^\mathcal{T, \mathcal{T}'}(u) \) to the subspace \( W_{\mathcal{T}} \otimes W_{\mathcal{T}'} \) of \( V^\otimes n \otimes V^\otimes n' \).

Let \( k \in \{1, \ldots, n-1\} \) and denote by \( \mathcal{T}'(s_k) \) the Young tableau of shape \( \lambda \) obtained from \( \mathcal{T} \) by exchanging the nodes with numbers \( k \) and \( k+1 \). We assume that \( \mathcal{T}'(s_k) \) is also standard. This is equivalent to say that \( \text{cc}(\mathcal{T}|k+1) \neq \text{cc}(\mathcal{T}|k) \pm 1 \).

Moreover, the Yang solution (5.1) satisfies the following unitarity condition:

\[
R(u)R_{2,1}(-u) = \frac{u^2 - 1}{u^2} \cdot \text{Id}_{V^\otimes 2}.
\]  (5.12)

Therefore, according to Definition 4.9, the condition for the tableau \( T'(s_k) \) to be standard is equivalent to the condition for \( s_k = (k, k+1) \) to be an admissible transposition for the Yang solution \( R(u) \) and the set of parameters \( c^\mathcal{T} \). Theorem 5.4 implies that \( W_{\mathcal{T}} \otimes W_{\mathcal{T}'} \) and \( W_{\mathcal{T}'(s_k)} \otimes W_{\mathcal{T}'} \) are both isomorphic to \( M^U(\mathfrak{gl}_N) \otimes M^U(\mathfrak{gl}_N) \) and, moreover, Proposition 4.10 applied here asserts that the endomorphisms \( R^\text{res}_{\mathcal{T}, \mathcal{T}'}(u) \) and \( R^\text{res}_{\mathcal{T}'(s_k), \mathcal{T}'}(u) \) coincide up to a change of basis.

Besides, it is well-known that, for two standard Young tableaux \( \mathcal{T} \) and \( \tilde{\mathcal{T}} \) of the same shape \( \lambda \), there is a sequence of admissible transpositions \( \pi = s_{i_1} \ldots s_{i_l} \in S_n \) transforming \( \mathcal{T} \) into \( \tilde{\mathcal{T}} \).

Finally, we sum up the results obtained in this section for the Yang solution. The item (i) below follows from Theorems 3.2, 4.2 and 5.4. The item (ii) is a consequence of the above discussion together with the Proposition 4.10.

**Corollary 5.5.** (i) The set of functions \( \{R^\text{res}_{\mathcal{T}, \mathcal{T}'}\} \), where \( \mathcal{T}, \mathcal{T}' \) are standard Young tableaux such that \( \ell(\text{sh} \mathcal{T}), \ell(\text{sh} \mathcal{T}') \leq N \), forms a family of solutions of the Yang–Baxter equation, where \( R^\text{res}_{\mathcal{T}, \mathcal{T}'}(u) \) is an endomorphism of a space isomorphic to \( M^U(\mathfrak{gl}_N) \otimes M^U(\mathfrak{gl}_N) \).

(ii) For four standard Young tableaux \( \mathcal{T}, \tilde{\mathcal{T}}, \mathcal{T}' \) and \( \tilde{\mathcal{T}}' \) as above such that \( \text{sh} \mathcal{T} = \text{sh} \tilde{\mathcal{T}} \) and \( \text{sh} \mathcal{T}' = \text{sh} \tilde{\mathcal{T}}' \), the endomorphisms \( R^\text{res}_{\mathcal{T}, \tilde{\mathcal{T}}'}(u) \) and \( R^\text{res}_{\mathcal{T}', \tilde{\mathcal{T}}'}(u) \) coincide up to a change of basis.

**Example 5.6.** Let \( \mathcal{T} = \mathcal{T}' = \begin{array}{c} 1 \\ 2 \end{array} \). The expression for the operator \( R_{\mathcal{T}, \mathcal{T}'}(u) \) is

\[
R_{\mathcal{T}, \mathcal{T}'}(u) = (\text{Id}_{V^\otimes 4} - \frac{P_{2,1}}{u+1})(\text{Id}_{V^\otimes 4} - \frac{P_{1,1}}{u})(\text{Id}_{V^\otimes 4} - \frac{P_{2,2}}{u})(\text{Id}_{V^\otimes 4} - \frac{P_{1,2}}{u-1}).
\]

The results of this section, applied to this example, give that the operator \( R_{\mathcal{T}, \mathcal{T}'}(u) \) preserves the subspace \( W_\boxplus \otimes W_\boxplus \subset V^\otimes 2 \otimes V^\otimes 2 \), where \( W_\boxplus \) is the space of the irreducible representation of \( U(\mathfrak{gl}_N) \) corresponding to the partition \( \lambda = (2) \). In fact the subspace \( W_\boxplus \subset V^\otimes 2 \) coincides with the symmetric square of \( V \), since it is the image of the operator \( R(-1) = \text{Id}_{V^\otimes 2} + P \).
Fusion procedure and Schur-Weyl duality

Let \( N = 2 \) and fix a basis \( \{e_1, e_2\} \) of \( V \). We set \( \tilde{e}_1 := e_1 \otimes e_1, \tilde{e}_2 := e_1 \otimes e_2 + e_2 \otimes e_1 \) and \( \tilde{e}_3 := e_2 \otimes e_2 \). We give the matrix of the endomorphism \( R_{\tilde{T}, \tilde{T}'}^{\text{res}}(u) \) of \( W_\infty \otimes W_\infty \) written in the basis \( \{\tilde{e}_i \otimes \tilde{e}_j\}_{i,j=1,2,3} \) ordered lexicographically (that is, \( \tilde{e}_1 \otimes \tilde{e}_1, \tilde{e}_1 \otimes \tilde{e}_2, \tilde{e}_1 \otimes \tilde{e}_3, \tilde{e}_2 \otimes \tilde{e}_1, \ldots \)):

\[
\begin{pmatrix}
\frac{(u-2)(u-1)}{u(u+1)} & \frac{u-1}{u+1} & \frac{-2(u-1)}{u(u+1)} & \cdots & \cdots & \cdots \\
\frac{u-1}{u+1} & 1 & \frac{-4}{u+1} & \cdots & \cdots & \cdots \\
\frac{-2(u-1)}{u(u+1)} & \frac{u-1}{u+1} & \frac{2}{u+1} & \cdots & \cdots & \cdots \\
\frac{-2(u-1)}{u(u+1)} & \frac{u-1}{u+1} & \frac{2}{u+1} & \cdots & \cdots & \cdots \\
\frac{2}{u(u+1)} & \frac{-4}{u+1} & \frac{-2(u-1)}{u(u+1)} & 1 & \cdots & \cdots \\
\frac{2}{u(u+1)} & \frac{-4}{u+1} & \frac{-2(u-1)}{u(u+1)} & \frac{u-1}{u+1} & \cdots & \cdots \\
\frac{(u-2)(u-1)}{u(u+1)} & \frac{u-1}{u+1} & \frac{-2(u-1)}{u(u+1)} & \frac{u-1}{u+1} & \cdots & \cdots \\
\end{pmatrix}, \quad (5.13)
\]

where the points indicate the coefficients equal to 0.

Example 5.7. Let \( \tilde{T} = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}, \quad \tilde{T}' = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} \) and \( T' = \begin{pmatrix} 1 \end{pmatrix} \). The expression for the operator \( R_{\tilde{T}, \tilde{T}'}(u) \) is:

\[
R_{\tilde{T}, \tilde{T}'}(u) = (\text{Id}_{V \otimes^4} - \frac{P_{3,1}}{u+1})(\text{Id}_{V \otimes^4} - \frac{P_{2,1}}{u-1})(\text{Id}_{V \otimes^4} - \frac{P_{1,1}}{u}).
\]

Here \( W_{\tilde{T}'} = V \), whereas the subspace \( W_{\tilde{T}} \) is the image in \( V^{\otimes^3} \) of the operator

\[
F := (\text{Id}_{V \otimes^3} - P_{1,2})(\text{Id}_{V \otimes^3} + P_{1,3})(\text{Id}_{V \otimes^3} + \frac{P_{2,3}}{2}).
\]

Let \( N = 2 \) and fix a basis \( \{e_1, e_2\} \) of \( V \). One can directly verify that \( \{e_1^\top, e_2^\top\} \) is a basis of the subspace \( W_{\tilde{T}} \) where

\[
e_1^\top := e_1 \otimes e_2 \otimes e_1 - e_2 \otimes e_1 \otimes e_1 \quad \text{and} \quad e_2^\top := e_1 \otimes e_2 \otimes e_2 - e_2 \otimes e_1 \otimes e_2;
\]

(we indicate that \( e_1^\top = \frac{3}{2} F(e_1 \otimes e_2 \otimes e_1) \) and \( e_2^\top = -\frac{3}{2} F(e_2 \otimes e_1 \otimes e_2) \)). We give the matrix of the endomorphism \( R_{\tilde{T}, T'}^{\text{res}}(u) \) in the basis \( \{e_1^\top \otimes e_1, e_1^\top \otimes e_2, e_2^\top \otimes e_1, e_2^\top \otimes e_2\} \) of \( W_{\tilde{T}} \otimes W_{T'} \) (points replace coefficients equal to 0):

\[
\begin{pmatrix}
u & -1 & -\frac{1}{u+1} \\
-1 & \frac{1}{u+1} & 1 \\
-\frac{1}{u+1} & 1 & \cdots \\
\end{pmatrix}, \quad (5.14)
\]

On the other hand, the expression for the operator \( R_{\tilde{T}, T'}(u) \) is:

\[
R_{\tilde{T}, T'}(u) = (\text{Id}_{V \otimes^4} - \frac{P_{3,1}}{u+1})(\text{Id}_{V \otimes^4} - \frac{P_{2,1}}{u+1})(\text{Id}_{V \otimes^4} - \frac{P_{1,1}}{u}).
\]

Here, the subspace \( W_{\tilde{T}} \) is the image in \( V^{\otimes^3} \) of the operator

\[
\bar{F} := (\text{Id}_{V \otimes^3} + P_{1,2})(\text{Id}_{V \otimes^3} - P_{1,3})(\text{Id}_{V \otimes^3} - \frac{P_{2,3}}{2}).
\]

As \( \bar{T} = T^{(e_2)} \) and \( \text{cc}(T, 3) \neq \pm 1 \) \( \text{cc}(T, 2) \), Proposition[10] or item (ii) in Corollary[5,3] asserts that the matrix of the endomorphism \( R_{\tilde{T}, T'}^{\text{res}}(u) \) in a certain basis of \( W_{\tilde{T}} \otimes W_{T'} \) coincides with.
Moreover, Proposition 4.10 provides explicitly this basis. Indeed, define, according to Formula (4.14), the following operator on $V^\otimes 3$:

$$A := P_{2,3}(\mathrm{Id}_{V^\otimes 3} - \frac{P_{2,3}}{2}) = (P_{2,3} - \frac{\mathrm{Id}_{V^\otimes 3}}{2}) .$$

Then the basis of $W_\sim \otimes W_\sim'$ in which the matrix of the endomorphism $R_{\sim,T'}^\mathrm{res}(u)$ coincides with (5.14) is $\{ A(e_1^T) \otimes e_1, A(e_1^T) \otimes e_2, A(e_2^T) \otimes e_1, A(e_2^T) \otimes e_2 \}$.

In this simple example, the assertions of Proposition 4.10 can be directly checked since, using that

$$A(e_1^T) = e_1 \otimes e_1 \otimes e_2 - \frac{1}{2}(e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1) , \quad A(e_2^T) = -e_2 \otimes e_2 \otimes e_1 + \frac{1}{2}(e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2) ,$$

we easily verify that $\tilde{F}(e_1 \otimes e_1 \otimes e_2) = 2A(e_1^T)$ and $\tilde{F}(e_2 \otimes e_2 \otimes e_1) = -2A(e_2^T)$, and in turn that $\{ A(e_1^T), A(e_2^T) \}$ is a basis of $W_\sim$. The calculation of the matrix of the endomorphism $R_{\sim,T'}^\mathrm{res}(u)$ in the basis $\{ A(e_1^T) \otimes e_1, A(e_1^T) \otimes e_2, A(e_2^T) \otimes e_1, A(e_2^T) \otimes e_2 \}$ is straightforward and leads, as expected, to the matrix (5.14).

5.4 “Non-standard” evaluations of the fusion function

In the framework of the fusion procedure for the Yang solution, we are interested in any set of complex parameters $c = (c_1, \ldots, c_n)$ such that the fusion function $\Phi$, defined by

$$\Phi(u_1, \ldots, u_n) := \prod_{1 \leq i < j \leq n} \left( 1 - \frac{(i,j)}{u_i - u_j} \right) , \quad (5.15)$$

is non-singular at $c$, and moreover such that the resulting element of $C S_n$ is non-invertible. Indeed, such a set of parameters $c$ will lead, via (5.4), to a non-invertible endomorphism $F(c)$, and in turn to a proper subspace $W_c \subset V^\otimes n$ such that $W_c \otimes V^\otimes n'$ is invariant for the fused operators of the form $R_{c,c'}(u)$. With the help of Theorem 5.2, we obtained in the preceding subsection such a set of parameters for any standard Young tableau (actually, in these situations, the evaluations of the fusion function are proportional to primitive idempotents of $C S_n$).

One may easily see that there are other sets of parameters $c = (c_1, \ldots, c_n)$ such that the fusion function is non-singular and gives a non-invertible element of $C S_n$. The classification of such sets of parameters $c$ and the study of the associated subspaces $W_c$ and the associated fused solutions is an open question. We note also that the less general problem of finding all sets of parameters $c = (c_1, \ldots, c_n)$ such that $\Phi$ is non-singular at $c$ and gives an element proportional to an idempotent is not solved. We will give in this subsection some examples for small $n$.

**Case $n=2$.** In this situation, the fusion function, with values in $C S_2$, is given by

$$\Phi(u_1, u_2) = 1 - \frac{(1,2)}{u_1 - u_2} .$$

It is immediate to see that the only pairs of complex numbers $(c_1, c_2)$ such that the evaluation $\Phi(c_1, c_2)$ is non-singular and is a non-invertible element of $C S_2$ are the pairs such that $c_2 = c_1 \pm 1$ (this can be seen from (5.12)). As the rational function $\Phi(u_1, u_2)$ only depends on $u_1 - u_2$, there is no loss of generality to consider that $c_1 = 0$. As a conclusion, for $n = 2$, the only pairs $(c_1, c_2)$ leading to a proper subspace $W_{(c_1,c_2)} \subset V^\otimes 2$ are the pairs associated to the standard Young tableaux $\begin{array}{|c|}
|-----|
\hline
1 \\
\hline
2 \\
\hline
\end{array}$ and $\begin{array}{|c|}
|-----|
\hline
2 \\
\hline
1 \\
\hline
\end{array}$.
**Fusion Procedure and Schur-Weyl Duality**

**Case n=3.** The fusion function with values in $\mathbb{C}S_3$ is given by

$$
\Phi(u_1, u_2, u_3) = (1 - \frac{(1,2)}{u_1 - u_2})(1 - \frac{(1,3)}{u_1 - u_3})(1 - \frac{(2,3)}{u_2 - u_3}).
$$

(5.16)

As above, we can fix the freedom of translating all variables $u_1, u_2, u_3$ by the same constant with the assumption that $u_1$ is evaluated to 0.

**Proposition 5.8.** The evaluation of the function $\Phi(0, u_2, u_3)$ at $u_2 = c_2$ and $u_3 = c_3$ is proportional to an idempotent of $\mathbb{C}S_3$ if and only if:

$$(c_2, c_3) \in \{ (1,2), (1,-1), (-1,1), (-1,-2), (2,1), (-2,-1) \}.$$  

*Proof.* To prove the Proposition, we solve by a direct analysis the system of 6 equations in $x, c_2, c_3$, obtained by equating to 0 the coefficient in front of each element of $S_3$ in the expression $(\Phi(0, c_2, c_3))^2 - x\Phi(0, c_2, c_3)$ (we skip the details). □

The first four evaluations found in the preceding proposition correspond to standard tableaux, namely, with the same notation as in (5.10),

$$
E_{\{1,2,3\}} = \frac{1}{6}\Phi(0, 1, 2), \quad \begin{aligned}
E_{\{1,2\}} &= \frac{1}{3}\Phi(0, 1, -1), \\
E_{\{1,3\}} &= \frac{1}{3}\Phi(0, -1, 1), \\
E_{\{2,3\}} &= \frac{1}{6}\Phi(0, -1, -2).
\end{aligned}
$$

For brevity, let $E_1 := E_{\{1,2,3\}}$ and $E_4 := E_{\{1,2\}}$. For the two “non-standard” evaluations of the fusion function obtained in the above proposition, one can verify that the following elements

$$
E_2 := \frac{1}{3}\Phi(0, 2, 1) \quad \text{and} \quad E_3 := \frac{1}{3}\Phi(0, -2, -1)
$$

(5.17)

are idempotents of $\mathbb{C}S_3$ such that

$$
E_i E_j = E_j E_i = \delta_{i,j} E_i \quad \text{for} \; i, j = 1, 2, 3, 4, \quad \text{and} \quad E_1 + E_2 + E_3 + E_4 = 1_{\mathbb{C}S_3},
$$

where $1_{\mathbb{C}S_3}$ is the unit element of $\mathbb{C}S_3$. Thus, $\{E_1, E_2, E_3, E_4\}$ is a complete system of pairwise orthogonal primitive idempotents of $\mathbb{C}S_3$, and $E_2$ (respectively, $E_3$) generates a minimal left ideal isomorphic, as an $\mathbb{C}S_3$-module, to the irreducible module corresponding to the partition $\lambda = (2,1)$.

By the same arguments as in the preceding subsection, these two non-standard evaluations (namely, with $(c_2, c_3) = (2,1)$ or $(-2,-1)$) lead to fused solutions of the Yang-Baxter equation acting on spaces $M_{U(g\lambda)}^{U(g\lambda')}$, where $\lambda'$ is any partition, alternative to the ones obtained in Corollary 5.5. It is an open question to determine if these “non-standard” fused solutions are related in some sense to the standard ones. In general (for any $n$), we think that “non-standard” fused solutions deserve a better study.

**Remark 5.9.** It is interesting to note that the two non-standard evaluations found in Proposition 5.8 correspond actually to classical contents of non-standard Young tableaux, namely the Young tableaux $\begin{ytableau} 1 & 3 & 2 \end{ytableau}$ and $\begin{ytableau} 2 & 1 \end{ytableau}$, Further, translating the three variables by 1 or $-1$, which do not affect the fusion function, the non-standard evaluations also correspond to the non-standard Young tableaux $\begin{ytableau} 3 & 2 \end{ytableau}$ and $\begin{ytableau} 1 & 3 \end{ytableau}$ (we note that these Young tableaux are maybe more natural in view of the coefficients $\frac{1}{3}$ in (5.17)).
Finally, we indicate that, for \( n = 4 \), there are some evaluations of the fusion function leading to an element of \( \mathbb{C}S_4 \) proportional to an idempotent and which do not correspond to any Young tableaux, standard or not (such an evaluation is, for example, \((u_1, u_2, u_3, u_4) = (0, 1, 5, 2)\), using a notation as in \((5.16)\) extended for \( n = 4 \)). \( \triangle \)

**Remark 5.10.** As we have already explained above, not only the evaluations of the fusion function providing elements proportional to idempotents are of interest for the fusion procedure; the evaluations of the fusion function giving non-invertible elements of \( \mathbb{C}S_n \) furnish proper invariant subspaces as well. For \( n = 3 \), with the notation \((5.16)\) and \( u_1 \) evaluated to 0, obvious examples of such evaluations (which include the ones studied previously) are

\[
\{ u_2 = \pm 1, u_3 \neq 0, u_2 \}, \quad \{ u_3 = \pm 1, u_2 \neq 0, u_3 \}, \quad \{ u_3 = u_2 \pm 1, u_2, u_3 \neq 0 \}.
\]

Further, one can verify that the two elements obtained by the following consecutive evaluations

\[
\Phi(0, u_2, u_3)|_{u_2 = \pm 1, u_3 = 0}
\]

are non-singular and non-invertible (we also mention that the evaluations described in this remark exhaust the evaluations of \( \Phi(0, u_2, u_3) \) providing non-invertible elements of \( \mathbb{C}S_3 \)). \( \triangle \)

6. Invariant subspaces and representations of \( U(\mathfrak{gl}_{N|M}) \)

We explain how, with the help of a “super” analogue of the Schur–Weyl duality \([25,26]\), the fusion formula for the symmetric group can also be used in the context of the fusion procedure for a generalization of the Yang solution. We obtain a family of fused solutions acting on irreducible representations of the general linear Lie superalgebra \( \mathfrak{gl}_{N|M} \).

**Generalization of the Yang solution for a \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space.** We fix a \( \mathbb{Z}/2\mathbb{Z} \)-decomposition of the vector space \( V \) as \( V = V_{\mathbb{Z}_0} \oplus V_{\mathbb{Z}_1} \) and we set \( N := \dim(V_{\mathbb{Z}_0}) \) and \( M := \dim(V_{\mathbb{Z}_1}) \). A non-zero vector \( x \in V \) is called homogeneous if \( x \in V_{\mathbb{Z}_0} \) or \( x \in V_{\mathbb{Z}_1} \). For a homogeneous vector \( x \in V_{\mathbb{Z}_i}, i = 0, 1 \), we set \( |x| := i \) and let \( \tilde{I}_V \) be the endomorphism of \( V \) defined by \( \tilde{I}_V(x) := (-1)^{|x|}x \) for any homogeneous vector \( x \in V \).

We define an operator \( \tilde{I}_{V \otimes 2} \in \text{End}(V \otimes V) \) by:

\[
\tilde{I}_{V \otimes 2}(x \otimes y) := (-1)^{|x||y|}x \otimes y \quad \text{for any homogeneous vectors } x, y \in V,
\]

and we let \( R \) be the following function of \( u \in \mathbb{C} \) with values in \( \text{End}(V \otimes V) \):

\[
R(u) := \tilde{I}_{V \otimes 2} - \frac{P}{u},
\]

where \( P \) is the permutation operator on \( V \otimes V \). It is straightforward to check that the function \( R \) is a solution of the Yang–Baxter equation on \( V \otimes V \):

\[
R_{1,2}(u)R_{1,3}(u + v)R_{2,3}(v) = R_{2,3}(v)R_{1,3}(u + v)R_{1,2}(u).
\]

Moreover, this solution satisfies the same unitarity condition \((5.12)\) as the Yang solution, namely

\[
R(u)R_{2,1}(-u) = \frac{u^2 - 1}{u^2} \cdot \text{Id}_{V \otimes 2}.
\]

**Schur–Weyl duality in the \( \mathbb{Z}/2\mathbb{Z} \)-graded setting.** Let \( n > 1 \). We recall that the symmetric group \( S_n \) on \( n \) letters is isomorphic to the group generated by elements \( s_1, \ldots, s_{n-1} \) subject to
the defining relations:
\[
\begin{align*}
    s_i^2 &= 1 & \text{for } i = 1, \ldots, n - 1, \\
    s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} & \text{for } i = 1, \ldots, n - 2, \\
    s_is_j &= s_js_i & \text{for } i, j = 1, \ldots, n - 1 \text{ such that } |i - j| > 1,
\end{align*}
\]
(6.5)

the isomorphism being given by \( s_i \mapsto (i, i + 1) \), \( i = 1, \ldots, n - 1 \).

We define an operator \( \tilde{P} \in \text{End}(V \otimes V) \) by \( \tilde{P} := \tilde{I}_V \otimes \tilde{P} \), that is, we have:
\[
\tilde{P}(x \otimes y) := (-1)^{|x||y|} y \otimes x \quad \text{for any homogeneous vectors } x, y \in V.
\]
(6.6)

The following map from the set of generators \( \{s_1, \ldots, s_n\} \) to the set \( \text{End}(V^{\otimes n}) \),
\[
s_i \mapsto \tilde{P}_{i,i+1} \quad \text{for } i = 1, \ldots, n - 1,
\]
(6.7)

extends to an algebra homomorphism from \( \mathbb{C}S_n \) to \( \text{End}(V^{\otimes n}) \) (one easily checks that the relations (6.5) are satisfied by the images of the generators). We denote by \( \tilde{\rho} \) this representation of \( \mathbb{C}S_n \).

Let \( \mathfrak{g}l_{N|M} \) be the Lie superalgebra of endomorphisms of \( V \), defined by the following \( \mathbb{Z}/2\mathbb{Z} \)-grading: a non-zero element \( g \in \mathfrak{g}l_{N|M} \) is homogeneous of degree 0 if \( g(V_T^1) \subset V_T, \) \( i = 0, 1 \), and is homogeneous of degree 1 if \( g(V_T^1) \subset V_T^1 \) and \( g(V_T^2) \subset V_T^2 \). We let \( |g| \) denote the degree of an homogeneous element \( g \in \mathfrak{g}l_{N|M} \).

Let \( U(\mathfrak{g}l_{N|M}) \) be the universal enveloping algebra of the Lie superalgebra \( \mathfrak{g}l_{N|M} \). The map, with values in \( \text{End}(V^{\otimes n}) \), defined on homogeneous element \( g \in \mathfrak{g}l_{N|M} \) by
\[
g \mapsto \begin{cases} 
g \otimes \text{Id} \otimes \cdots \otimes \text{Id} + \text{Id} \otimes g \otimes \text{Id} \otimes \cdots \otimes \text{Id} + \cdots + \text{Id} \otimes \cdots \otimes \text{Id} \otimes g, & \text{if } |g| = 0, \\
g \otimes \text{Id} \otimes \cdots \otimes \text{Id} + \tilde{I}_V \otimes g \otimes \text{Id} \otimes \cdots \otimes \text{Id} + \cdots + \tilde{I}_V \otimes \cdots \otimes \tilde{I}_V \otimes g, & \text{if } |g| = 1,
\end{cases}
\]
extends to a representation of the algebra \( U(\mathfrak{g}l_{N|M}) \) on \( V^{\otimes n} \). We denote by \( \tilde{\chi} \) this representation.

We recall the generalization of the Schur–Weyl duality, which holds between the symmetric group \( S_n \) and the algebra \( U(\mathfrak{g}l_{N|M}) \) [25][26].

**Theorem 6.1.**

(i) The subalgebra \( \tilde{\rho}(\mathbb{C}S_n) \) of \( \text{End}(V^{\otimes n}) \) is the centraliser of \( \tilde{\chi}(U(\mathfrak{g}l_{N|M})) \).

(ii) The subalgebra \( \tilde{\chi}(U(\mathfrak{g}l_{N|M})) \) of \( \text{End}(V^{\otimes n}) \) is the centraliser of \( \tilde{\rho}(\mathbb{C}S_n) \).

(iii) As an \( (U(\mathfrak{g}l_{N|M}) \otimes \mathbb{C}S_n) \)-module defined by \( \tilde{\chi} \) and \( \tilde{\rho} \), the space \( V^{\otimes n} \) decomposes as:
\[
V^{\otimes n} \cong \bigoplus_\lambda M_\lambda^{U(\mathfrak{g}l_{N|M})} \otimes M_\lambda^{\mathbb{C}S_n},
\]
(6.8)

where \( \lambda = (\lambda_1, \ldots, \lambda_t) \) runs over the set of partitions such that \( |\lambda| = n \) and \( \lambda_j \leq M \) if \( j > N \), and where the modules \( M_\lambda^{U(\mathfrak{g}l_{N|M})} \) are irreducible \( U(\mathfrak{g}l_{N|M}) \)-modules.

**Fused solutions on** \( M_\lambda^{U(\mathfrak{g}l_{N|M})} \otimes M_\lambda^{U(\mathfrak{g}l_{N|M})} \). As a consequence of Formulas (6.2) and (6.7), we have:
\[
\tilde{\rho}(s_i - \frac{1}{u}) = R_{i,i+1}(u)P = \tilde{R}_{i,i+1}(u) \quad \text{for } i = 1, \ldots, n - 1.
\]

Therefore, defining the following rational function in the complex variables \( u_1, \ldots, u_n \) with values in \( \mathbb{C}S_n \):
\[
\tilde{\Phi}(u_1, \ldots, u_n) := \prod_{i=1}^{n-1} \left( s_i - \frac{1}{c_1 - c_{i+1}} \right) \cdots \left( s_2 - \frac{1}{c_{i-1} - c_{i+1}} \right) \left( s_1 - \frac{1}{c_1 - c_{i+1}} \right),
\]

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we have, recalling the definition \((4.9)\),
\[
\tilde{\rho} \left( \Phi(u_1, \ldots, u_n) \right) = \tilde{F}(u_1, \ldots, u_n).
\]

Moreover, the following Lemma relates the function \(\tilde{\Phi}\) with the fusion function \(\Phi\), defined by \((5.6)\), of the symmetric group.

**Lemma 6.2.** We have (where \(w_n\) is the longest element of \(S_n\)):
\[
\Phi(u_1, \ldots, u_n) = \Phi(u_1, \ldots, u_n) w_n.
\]

**Proof.** Recall that the relation \((5.7)\) holds for the functions \((1 - (i,j)/u)\) with values in \(CS_n\). Moreover, the following Lemma relates the function \(\tilde{\Phi}\) with the fusion function \(\Phi\), defined by \((5.6)\), of the symmetric group.

Thus, as \(w_n\) is invertible, the fusion formula for the symmetric group in Theorem \((5.2)\) can be used, as in Subsection \((5.3)\) to analyze the image of \(\tilde{\rho} \left( \Phi(u_1, \ldots, u_n) \right) \) in \(V^{\otimes n}\) when \(u_1, \ldots, u_n\) are evaluated to classical contents of standard Young tableaux as in \((5.8)\).

Using Corollary \((4.7)\) Formula \((5.9)\) and the Schur–Weyl duality stated in Theorem \((6.1)\) we reproduce the same reasoning as in Subsection \((5.3)\) to obtain the generalization of the Corollary \((5.3)\) for the solution \((6.2)\). Namely, let \(\mathcal{T}, \mathcal{T}'\) are two standard Young tableaux such that \(\text{sh}_{\mathcal{T}} = (\lambda_1, \ldots, \lambda_l)\) and \(\text{sh}_{\mathcal{T}'} = (\lambda_1', \ldots, \lambda_l')\), with \(|\text{sh}_{\mathcal{T}}| = n\), \(|\text{sh}_{\mathcal{T}'}| = n'\) and \(\lambda_j, \lambda_j' \leq M\) if \(j > N\). We obtain, by restriction of the fused operators to their invariant subspaces, a function \(R_{\mathcal{T},\mathcal{T}'}^{\text{res}}(u)\) with the following properties.

**Corollary 6.3.** (i) The set of functions \(\{R_{\mathcal{T},\mathcal{T}'}^{\text{res}}\}\), where \(\mathcal{T}, \mathcal{T}'\) are standard Young tableaux as above, forms a family of solutions of the Yang–Baxter equation, and \(R_{\mathcal{T},\mathcal{T}'}^{\text{res}}(u)\) is an endomorphism of a space isomorphic to \(M^{U(\text{gl}_N|\text{M})}_{\text{sh}_{\mathcal{T}}'} \otimes M^{U(\text{gl}_N|\text{M})}_{\text{sh}_{\mathcal{T}}'}\).

(ii) For four standard Young tableaux \(\mathcal{T}, \bar{\mathcal{T}}, \mathcal{T}'\) and \(\bar{\mathcal{T}}'\) as above such that \(\text{sh}_{\mathcal{T}} = \text{sh}_{\bar{\mathcal{T}}}\) and \(\text{sh}_{\mathcal{T}'} = \text{sh}_{\bar{\mathcal{T}}'}\), the endomorphisms \(R_{\mathcal{T},\mathcal{T}'}^{\text{res}}(u)\) and \(R_{\mathcal{T}',\bar{\mathcal{T}}'}^{\text{res}}(u)\) coincide up to a change of basis.

**Remark 6.4.** Assume that \(M > 0\). Let \(n = 2\) and define
\[
\tilde{R}(u) := \tilde{\rho}(1 - (1,2)/u) = \text{Id}_{V^{\otimes 2}} - \frac{\bar{P}}{u} \in \text{End}(V^{\otimes 2}).
\]

Note that, contrary to the situation \(M = 0\), the function \(\tilde{R}\) is not a solution of the Yang–Baxter equation. This does not contradict the relation \((5.7)\) because if \(n > 2\) and \(|i - j| > 1\), we have \(\tilde{R}_{i,j}(u) \neq \tilde{\rho}(1 - (i,j)/u)\). Actually, we have, if \(|i - j| > 1\),
\[
\tilde{\rho}(1 - (i,j)/u) = \bar{P}_{-i,j} \bar{P}_{-j-2} \cdots \bar{P}_{i+1,i+2,\ldots,\bar{R}_{i,i+1}(u) \bar{R}_{i+1,i+2} \cdots \bar{R}_{j-2,j-1}} \bar{P}_{j-1,ij}.
\]

It follows from \((6.7)\) and \((5.7)\) that the function \(\tilde{R}(u)\) satisfies, instead of \((6.3)\), a “braided” Yang–Baxter equation:
\[
\tilde{R}_{1,2}(u)\tilde{R}_{1,3}(u + v)\tilde{R}_{2,3}(v) = \tilde{R}_{2,3}(v)\tilde{R}_{1,3}(u + v)\tilde{R}_{1,2}(u),
\]
with the braiding defined by the operator \(\bar{P}\), namely with \(\tilde{R}_{1,3}(u) := \bar{P}_{2,3}(u \tilde{R}_{1,2}(u) \bar{P}_{2,3})\) and \(\tilde{R}_{2,3}(u) := \bar{P}_{1,2}(\tilde{R}_{1,3}(u) \bar{P}_{1,2})\). \(\triangle\)
Fusion procedure and Schur-Weyl duality

7. Invariant subspaces and representations of $U_q(\mathfrak{gl}_N)$

In this Section, we will consider another example of solution of the Yang–Baxter equation, which is the standard deformation of the Yang solution (5.1) considered in Section 5. We present the generalization of the construction in Section 5, which leads here to a family of fused solutions of the Yang–Baxter equation acting on irreducible representations of the quantum group $U_q(\mathfrak{gl}_N)$ (the standard deformation of $U(\mathfrak{gl}_N)$).

7.1 Deformation of the Yang solution

We fix a basis $\{e_i\}_{i=1,\ldots,N}$ of the vector space $V$ where $N := \dim(V)$. Let $q$ be a non-zero complex number and define an endomorphism $\hat{R}$ of $V \otimes V$ by, for $i, j = 1, \ldots, N$,

$$\hat{R}(e_i \otimes e_j) := \begin{cases} q e_i \otimes e_j & \text{if } i = j, \\ e_j \otimes e_i & \text{if } i < j, \\ e_j \otimes e_i + (q - q^{-1}) e_i \otimes e_j & \text{if } i > j. \end{cases}$$

(7.1)

It is well-known, and it can be directly checked, that $\hat{R}$ satisfies the quadratic relation

$$\hat{R}^2 - (q - q^{-1}) \hat{R} - \text{Id} = 0,$$

(7.2)

and verifies as well, on the space $V \otimes V \otimes V$,

$$\hat{R}_{1,2} \hat{R}_{2,3} \hat{R}_{1,2} = \hat{R}_{2,3} \hat{R}_{1,2} \hat{R}_{2,3}. \quad (7.3)$$

Relations (7.2) and (7.3) imply by a direct calculation that the function of $\alpha \in \mathbb{C}$, with values in $\text{End}(V \otimes V)$, given by:

$$\hat{R}(\alpha) := \hat{R} + (q - q^{-1}) \frac{\text{Id}_{V \otimes V}}{\alpha^{-1} - 1},$$

(7.4)

satisfies the equation (operators act on $V \otimes V \otimes V$)

$$\hat{R}_{1,2}(\alpha) \hat{R}_{2,3}(\alpha \beta) \hat{R}_{1,2}(\beta) = \hat{R}_{2,3}(\beta) \hat{R}_{1,2}(\alpha \beta) \hat{R}_{2,3}(\alpha).$$

As a direct consequence, the function $R$ given by:

$$R(\alpha) := \hat{R}(\alpha) \text{P}, \quad (7.5)$$

is a solution, on $V \otimes V$, of the Yang–Baxter equation with multiplicative spectral parameters:

$$R_{1,2}(\alpha) R_{1,3}(\alpha \beta) R_{2,3}(\beta) = R_{2,3}(\beta) R_{1,3}(\alpha \beta) R_{1,2}(\alpha).$$

Remark 7.1. The solution $R(\alpha)$ is a deformation of the Yang solution (5.1) in the following sense. Consider $q$ as a variable in $\mathbb{C}\{0\}$ and set $\alpha = q^{2u}$. Since $\hat{R} \big|_{q=1} = \text{P}$, we obtain

$$\hat{R}(q^{2u}) \big|_{q=1} = \text{P} - \frac{\text{Id}_{V \otimes V}}{u},$$

where we have used that $1 - q^{-2u} = (q - q^{-1})(q^{-1} + q^{-3} + \cdots + q^{3-2u} + q^{1-2u})$. Thus

$$R(q^{2u}) \big|_{q=1} = \text{Id}_{V \otimes V} - \frac{\text{P}}{u}. \quad \Box$$

Example 7.2. In this example, let $N = 2$. We write the matrix of the endomorphism $R(\alpha)$ in...
the basis \( \{ e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2 \} \) of \( V \otimes V \) (points indicate coefficients equal to 0):

\[
\begin{pmatrix}
\frac{q \alpha^{-1} - q^{-1}}{\alpha^{-1} - 1} & \cdot & \cdot & \cdot \\
\cdot & \frac{q - q^{-1}}{\alpha^{-1} - 1} & \cdot & \cdot \\
\cdot & \cdot & \frac{(q - q^{-1}) \alpha^{-1}}{\alpha^{-1} - 1} & \cdot \\
\cdot & \cdot & \cdot & \frac{q \alpha^{-1} - q^{-1}}{\alpha^{-1} - 1}
\end{pmatrix}
\] (7.6)

\[ \triangle \]

7.2 Jimbo–Schur–Weyl duality

From now on, we assume that \( q \in \mathbb{C} \setminus \{0\} \) is not a root of unity.

**Hecke algebra.** The Hecke algebra (of type A) is the associative algebra \( H_n(q) \) over \( \mathbb{C} \) generated by \( \sigma_1, \ldots, \sigma_{n-1} \) with the defining relations:

\[
\begin{align*}
\sigma_i^2 &= (q - q^{-1}) \sigma_i + 1 \quad \text{for} \ i = 1, \ldots, n - 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for} \ i = 1, \ldots, n - 2, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{for} \ i, j = 1, \ldots, n - 1 \text{ such that } |i - j| > 1.
\end{align*}
\] (7.7)

The following map from the set of generators of \( H_n(q) \) to \( \text{End}(V^{\otimes n}) \):

\[
\sigma_i \mapsto \tilde{R}_{i,i+1} \quad \text{for} \ i = 1, \ldots, n - 1,
\] (7.8)

extends to an algebra homomorphism. This follows from Relations (7.2) and (7.3), together with the obvious commutation relation \( \tilde{R}_{i,i+1} \tilde{R}_{j,j+1} = \tilde{R}_{j,j+1} \tilde{R}_{i,i+1} \) if \( |i - j| > 1 \). We denote by \( \rho \) this representation of \( H_n(q) \) on the space \( V^{\otimes n} \).

**Quantum algebra** \( U_q(\mathfrak{gl}_N) \). The standard deformation of the universal enveloping algebra of \( \mathfrak{gl}_N \) is the associative algebra \( U_q(\mathfrak{gl}_N) \) over \( \mathbb{C} \) defined by generators and relations as follows. The generators are \( K_i^{\pm 1}, i = 1, \ldots, N, \) and \( E_j, F_j, j = 1, \ldots, N - 1, \) and the defining relations are:

\[
\begin{align*}
K_i K_j^{-1} &= K_j K_i^{-1} = 1, \\
K_i E_j K_i^{-1} &= K_j E_j, \\
K_i F_j K_i^{-1} &= K_j^{-1} F_j, \\
E_i F_j - F_j E_i &= \delta_{i,j} \frac{K_i K_{i+1}^{-1} - K_{i+1}^{-1} K_i}{q - q^{-1}} \quad \text{for} \ i, j = 1, \ldots, N - 1,
\end{align*}
\]

where \( \delta_{i,j} \) is the Kronecker delta, \( a_{i,i} = -a_{i+1,i} = 1, i = 1, \ldots, N - 1, \) and \( a_{i,j} = 0 \) otherwise, together with

\[
\begin{align*}
E_i E_i^2 - (q + q^{-1}) E_i E_i + E_i^2 &= 0 \quad \text{for} \ i, j = 1, \ldots, N - 1 \text{ such that } |i - j| = 1, \\
F_j F_j^2 - (q + q^{-1}) F_j F_j + F_j^2 &= 0 \quad \text{for} \ i, j = 1, \ldots, N - 1 \text{ such that } |i - j| = 1, \\
E_i E_j &= E_j E_i, \\
F_i F_j &= F_j F_i \quad \text{for} \ i, j = 1, \ldots, N - 1 \text{ such that } |i - j| > 1.
\end{align*}
\]

The algebra \( U_q(\mathfrak{gl}_N) \) admits a Hopf algebra structure such that, in particular, the coproduct is the algebra homomorphism \( \Delta \) from \( U_q(\mathfrak{gl}_N) \) to \( U_q(\mathfrak{gl}_N) \otimes U_q(\mathfrak{gl}_N) \) defined on the generators by:

\[
\begin{align*}
\Delta(K_i^{\pm 1}) &= K_i^{\pm 1} \otimes K_i^{\pm 1} \quad \text{for} \ i = 1, \ldots, N, \\
\Delta(E_j) &= E_j \otimes 1 + K_j K_{j+1}^{-1} \otimes E_j, \\
\Delta(F_j) &= F_j \otimes K_{j+1} K_j + 1 \otimes F_j \quad \text{for} \ j = 1, \ldots, N - 1.
\end{align*}
\]
The vector representation $\eta$ of the algebra $U_q(\mathfrak{gl}_N)$ on $V$ is given, on the basis $\{e_k\}_{k=1,\ldots,N}$, by:
\[
\eta(K_i^{\pm 1})(e_k) = \delta_{i,k}q^{\pm 1}e_k \quad \text{for } i = 1, \ldots, N, \\
\eta(E_j)(e_k) = \delta_{j+1,k}e_{k-1}, \quad \eta(F_j)(e_k) = \delta_{j-1,k}e_{k+1} \quad \text{for } k = 1, \ldots, N \text{ and } j = 1, \ldots, N-1.
\]

Via the coproduct $\Delta$, the representation $\eta$ induces a representation $\chi$ of the algebra $U_q(\mathfrak{gl}_N)$ on the space $V^{\otimes n}$. More precisely, let $\Delta^{(2)} := \Delta$ and define inductively $\Delta^{(k)} := (\Delta^{(k-1)} \otimes 1) \circ \Delta$ for $k = 3, \ldots, n$, where $1$ is the identity homomorphism of $U_q(\mathfrak{gl}_N)$. Thus, for all $k = 2, \ldots, n$, $\Delta^{(k)}$ is an algebra homomorphism from $U_q(\mathfrak{gl}_N)$ to $U_q(\mathfrak{gl}_N)^{\otimes k}$. Explicitly, we have:
\[
\Delta^{(k)}(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1} \otimes \cdots \otimes K_i^{\pm 1} \quad \text{for } i = 1, \ldots, N, \\
\Delta^{(k)}(E_j) = \sum_{p=0,\ldots,k-1} (K_jK_{j+1}^{-1})^{\otimes p} \otimes E_j \otimes 1^{\otimes k-1-p} \quad \text{for } j = 1, \ldots, N-1, \\
\Delta^{(k)}(F_j) = \sum_{p=0,\ldots,k-1} 1^{\otimes p} \otimes F_j \otimes (K_j^{-1}K_{j+1})^{\otimes k-1-p} \quad \text{for } j = 1, \ldots, N-1.
\]

Then the representation $\chi$ of the algebra $U_q(\mathfrak{gl}_N)$ on the space $V^\otimes n$ is given by:
\[
\chi(X) := (\eta \otimes \eta \otimes \cdots \otimes \eta) \circ \Delta^{(n)}(X) \quad \text{for any } X \in U_q(\mathfrak{gl}_N).
\]

**Jimbo–Schur–Weyl duality.** To state the analogue of the Schur–Weyl duality (Theorem 5.1) between the Hecke algebra $H_n(q)$ and the quantum algebra $U_q(\mathfrak{gl}_N)$, we recall that the irreducible representations of $H_n(q)$ are parametrized by the partitions of $n$. For any partition $\lambda$ of $n$, we denote by $M^H_\lambda$ the corresponding irreducible $H_n(q)$-module (with the convention that $M^H_\lambda$ is the one-dimensional $H_n(q)$-module on which the generators $s_1, \ldots, s_{n-1}$ act by multiplication by $q$).

For any partition $\lambda$ such that $\ell(\lambda) \leq N$, we denote by $M^{U_q(\mathfrak{gl}_N)}_\lambda$ the irreducible highest weight $U_q(\mathfrak{gl}_N)$-module of highest weight $\lambda$ (where the eigenvalues of the generators $K_1, \ldots, K_N$ are of the form $q^m$, $m \in \mathbb{Z}$).

Then the Jimbo–Schur–Weyl duality consists in the following assertions [2N], analogous to the assertions of Theorem 5.1.

**Theorem 7.3.** (i) The subalgebra $\rho(H_n(q))$ of $\text{End}(V^\otimes n)$ is the centraliser of $\chi(U_q(\mathfrak{gl}_N))$.

(ii) The subalgebra $\chi(U_q(\mathfrak{gl}_N))$ of $\text{End}(V^\otimes n)$ is the centraliser of $\rho(H_n(q))$.

(iii) As an $(U_q(\mathfrak{gl}_N) \otimes H_n(q))$-module defined by $\chi$ and $\rho$, the space $V^\otimes n$ decomposes as:
\[
V^\otimes n \cong \bigoplus_\lambda M^{U_q(\mathfrak{gl}_N)}_\lambda \otimes M^H_\lambda,
\]
where $\lambda$ runs over the set of partitions such that $|\lambda| = n$ and $\ell(\lambda) \leq N$.

### 7.3 Fusion formula for idempotents of the Hecke algebra

Let $n > 1$ and recall the definition (4.9) of the function $\widehat{F}$ of $\mathbf{c} := (c_1, \ldots, c_n) \in \mathbb{C}^n$ (with the multiplicative version for the spectral parameters, see Remark 4.11) with values in $\text{End}(V^\otimes n)$:
\[
\widehat{F}(\mathbf{c}) := \prod_{i=1,\ldots,n-1} \widehat{R}_{i, i+1}(c_{i+1} / c_i) \cdots \widehat{R}_{2, 3}(c_3 / c_2) \widehat{R}_{1, 2}(c_2 / c_1).
\]

Due to Corollary 4.7, when the function $\widehat{F}$ is non-singular at a particular value $\mathbf{c} := (c_1, \ldots, c_n)$, the subspace $W_\mathbf{c} \subset V^\otimes n$ is the image of the operator $\widehat{F}(\mathbf{c})$. 

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Following Formulas (7.10), (7.8) and (7.4), we define the following rational function in variables $\alpha_1, \ldots, \alpha_n$ with values in the algebra $H_n(q)$:

$$
\Psi(\alpha_1, \ldots, \alpha_n) := \prod_{i=1,\ldots,n-1} \sigma_i(\frac{\alpha_1}{\alpha_i+1}) \ldots \sigma_2(\frac{\alpha_{i-1}}{\alpha_{i+1}}) \sigma_1(\frac{\alpha_i}{\alpha_{i+1}}), \quad (7.11)
$$

where $\sigma_i(\alpha) := \sigma_i + \frac{q - q^{-1}}{\alpha - 1}$ for all $i = 1, \ldots, n - 1$. The elements $\sigma_i(\alpha)$ with values in $H_n(q)$ are the Baxterized elements of the Hecke algebra. We also set

$$
T_{wn} := \sigma_1(\sigma_2\sigma_1) \ldots (\sigma_{n-2} \ldots \sigma_2\sigma_1)(\sigma_{n-1} \ldots \sigma_2\sigma_1) = \prod_{i=1,\ldots,n-1} \sigma_i \ldots \sigma_2\sigma_1.
$$

The main result concerning the function $\Psi(u_1, \ldots, u_n)$ is the following (we follow [27]). Let $\lambda$ be a partition of $n$ and let $T$ be a standard Young tableau of shape $\lambda$. For brevity, set $c_i(q) := c_i(q)(T|i)$ for $i = 1, \ldots, n$.

**Theorem 7.4.** The element obtained by the following consecutive evaluations

$$
f(q)(\lambda)\Psi(\alpha_1, \ldots, \alpha_n)T_{wn}^{-1}|_{\alpha_1=c_1(q)}|_{\alpha_2=c_2(q)} \cdots |_{\alpha_n=c_n(q)} \quad (7.12)
$$

is a primitive idempotent of $H_n(q)$ which generates a minimal left ideal isomorphic, as an $H_n(q)$-module, to the irreducible module $M^\lambda_{\text{Id}}(q)$.

We recall that $f(q)(\lambda)$ is the non-zero complex number defined by (2.5).

**Remark 7.5.** In the same spirit as in Remark 7.1, we verify that

$$
\sigma_i(q^{2u})|_{q=1} = (i, i+1) - \frac{1}{u} \quad \text{for } i = 1, \ldots, n - 1,
$$

where we recall that $H_n(1) \cong \mathbb{C}S_n$ with the isomorphism given by $\sigma_i|_{q=1} \mapsto (i, i+1)$, $i = 1, \ldots, n - 1$. Thus, setting $\alpha_j = q^{2u_j}$, for $j = 1, \ldots, n$, we obtain:

$$
\Psi(\alpha_1, \ldots, \alpha_n)|_{q=1} = \prod_{i=1,\ldots,n-1} \left( (i, i+1) - \frac{1}{u_1 - u_{i+1}} \right) \cdots (2, 3) - \frac{1}{u_{i-1} - u_{i+1}} \left( (1, 2) - \frac{1}{u_i - u_{i+1}} \right).
$$

Using Lemma 6.2 we conclude that

$$
\Psi(\alpha_1, \ldots, \alpha_n)T_{wn}^{-1}|_{q=1} = \Phi(u_1, \ldots, u_n),
$$

where $\Phi$ is the fusion function of the symmetric group defined by (5.6). In this sense, Theorem 5.2 is the classical limit ($q \rightarrow 1$) of Theorem 7.4. \(\triangle\)

### 7.4 Fused solutions on $M^U_{\mathfrak{gl}_N}(\mathfrak{gl}_N) \otimes M^U_{\mathfrak{gl}_N}(\mathfrak{gl}_N)$

Let $T$ be a standard Young tableau such that $\text{sh}_T = \lambda$ with $|\lambda| = n$ and $\ell(\lambda) \leq N$. We let

$$
\tilde{F}(T) := \tilde{F}(c)|_{c_1=c(q)(T|1)}|_{c_2=c(q)(T|2)} \cdots |_{c_n=c(q)(T|n)} \quad (7.13)
$$

where $c = (c_1, \ldots, c_n)$ is seen as an $n$-tuple of variables.

In view of Theorem 7.4 we define

$$
E_T^{(q)} := f(q)(\lambda)\Psi(\alpha_1, \ldots, \alpha_n)T_{wn}^{-1}|_{\alpha_1=c_1(q)}|_{\alpha_2=c_2(q)} \cdots |_{\alpha_n=c_n(q)} \quad (7.14)
$$
Fusion procedure and Schur-Weyl duality

According to Formulas (7.10) and (7.11), we have that

\[ \hat{F}(\mathcal{T}) = \rho \left( (f^{(q)}(\lambda))^{-1} E^{(q)}_\mathcal{T} T_{\alpha_n} \right) \]

The assertions of Theorem 7.4 imply that \( \hat{F} \) is non-singular for the consecutive evaluations of the variables as in (7.13). Thus the image \( W_\mathcal{T} \subset V^{\otimes n} \) of the operator \( \hat{F}(\mathcal{T}) \) is well-defined and coincides, since \( T_{\alpha_n} \) is invertible, with the image in \( V^{\otimes n} \) of the operator \( \rho \left( E^{(q)}_\mathcal{T} \right) \).

With a similar reasoning as in Subsection 5.3 before Theorem 5.4, we conclude that Theorem 7.4 together with Formula (7.9) in Theorem 7.3 implies:

**Theorem 7.6.** The subspace \( W_\mathcal{T} \) of \( V^{\otimes n} \) is an irreducible \( U_q(\mathfrak{gl}_N) \)-module isomorphic to \( M_{\lambda}^{U_q(\mathfrak{gl}_N)} \).

Let \( k \in \{1, \ldots, n-1\} \) and denote by \( \mathcal{T}^{(s_k)} \) the Young tableau of shape \( \lambda \) obtained from \( \mathcal{T} \) by exchanging the nodes with numbers \( k \) and \( k+1 \). We assume that \( \mathcal{T}^{(s_k)} \) is also standard, which is equivalent to the fact that \( c^{(q)}(\mathcal{T}|k+1) \neq c^{(q)}(\mathcal{T}|k)q^2 \). The solution \( (7.5) \) satisfies the following unitarity condition, implied by (7.2),

\[ R(\alpha) R_{2,1} \left( \frac{1}{\alpha} \right) = \frac{(\alpha - q^2)(\alpha + q^{-2})}{(\alpha - 1)^2} \cdot \text{Id}_{V^{\otimes 2}}. \]  

(7.15)

Therefore, according to Definition 4.9 the condition for the tableau \( \mathcal{T}^{(s_k)} \) to be standard is equivalent to the condition for \( s_k = (k, k+1) \) to be an admissible transposition for \( R(\alpha) \) and the set of parameters \( c^\mathcal{T} \).

As in Subsection 5.3, Corollary 5.5 we sum up the results obtained for the standard deformation of the Yang solution. For two standard Young tableaux \( \mathcal{T}, \mathcal{T}' \) such that \( \text{sh}_\mathcal{T} = \lambda \) and \( \text{sh}_{\mathcal{T}'} = \lambda' \), with \( |\lambda| = n, |\lambda'| = n' \) and \( \ell(\lambda), \ell(\lambda') \leq N \), we set \( c^\mathcal{T} : = (c^{(q)}(\mathcal{T}|1), \ldots, c^{(q)}(\mathcal{T}|n)), c^{\mathcal{T}'} : = (c^{(q)}(\mathcal{T}'|1), \ldots, c^{(q)}(\mathcal{T}'|n')) \) and define \( R_{\mathcal{T},\mathcal{T}'}^\text{res}(u) \) to be the restriction of the operator \( R_{c^\mathcal{T},c^{\mathcal{T}'}}(u) \) to the subspace \( W_\mathcal{T} \otimes W_{\mathcal{T}'} \subset V^{\otimes n} \otimes V^{\otimes n'} \):

\[ R_{\mathcal{T},\mathcal{T}'}^\text{res}(u) := R_{c^\mathcal{T},c^{\mathcal{T}'}}(u) \bigg|_{W_\mathcal{T} \otimes W_{\mathcal{T}'}}. \]

**Corollary 7.7.** (i) The set of functions \( \{ R_{\mathcal{T},\mathcal{T}'}^\text{res} \} \), where \( \mathcal{T}, \mathcal{T}' \) are standard Young tableaux such that \( \ell(\text{sh}_\mathcal{T}), \ell(\text{sh}_{\mathcal{T}'}) \leq N \), forms a family of solutions of the Yang–Baxter equation, where \( R_{\mathcal{T},\mathcal{T}'}^\text{res}(\alpha) \) is an endomorphism of a space isomorphic to \( M_{\lambda}^{U_q(\mathfrak{gl}_N)} \otimes M_{\lambda'}^{U_q(\mathfrak{gl}_N)} \).

(ii) For four standard Young tableaux \( \mathcal{T}, \mathcal{T}', \mathcal{T}'' \) and \( \mathcal{T}''' \) as above such that \( \text{sh}_{\mathcal{T}} = \text{sh}_{\mathcal{T}'} \) and \( \text{sh}_{\mathcal{T}''} = \text{sh}_{\mathcal{T}'''} \), the endomorphisms \( R_{\mathcal{T},\mathcal{T}''}^\text{res}(\alpha) \) and \( R_{\mathcal{T}',\mathcal{T}'''}^\text{res}(\alpha) \) coincide up to a change of basis.

**Example 7.8.** This example is the (deformed) analogue of Example 5.6. Namely, we consider the fused operator \( R_{c,\mathcal{E}}(\alpha) \), with \( n = n' = 2, c_1 = c_1 = 1 \) and \( c_2 = c_2 = q^2 \), obtained from the solution \( (7.5) \). The expression for \( R_{c,\mathcal{E}}(\alpha) \) is

\[ R_{c,\mathcal{E}}(\alpha) = R_{2,1}(\alpha q^2) R_{1,1}(\alpha) R_{2,2}(\alpha) R_{1,2}(\alpha q^{-2}). \]

The results of this section, applied to this example, give that the operator \( R_{c,\mathcal{E}}(\alpha) \) preserves the subspace \( W^{(q)}_{\mathcal{E}} \otimes W^{(q)}_{\mathcal{E}} \subset V^{\otimes 2} \otimes V^{\otimes 2} \), where \( W^{(q)}_{\mathcal{E}} \) is the space of the irreducible representation of \( U_q(\mathfrak{gl}_N) \) corresponding to the partition \( \lambda = (2) \). The subspace \( W^{(q)}_{\mathcal{E}} \subset V^{\otimes 2} \) is the image of the operator \( R(q^{-2}) \) and is a (quantum) analogue of the symmetric square of \( V \).
Let \( N = 2 \) and fix a basis \( \{ e_1, e_2 \} \) of \( V \). From the matrix in Example 7.2 with \( \alpha = q^{-2} \), we find that the following vectors form a basis of \( W^{(q)}_{(2)} \):

\[
e_1 := e_1 \otimes e_1, \quad e_2 := e_1 \otimes e_2 + q e_2 \otimes e_1 \quad \text{and} \quad e_3 := e_2 \otimes e_2.
\]

We give the matrix of the endomorphism \( R_{\mathbb{C} e_2}(\alpha) \) on the space \( W^{(q)}_{(2)} \otimes W^{(q)}_{(2)} \) written in the basis \( \{ e_i \otimes e_j \}_{i,j=1,...,3} \) ordered lexicographically (the points indicate the coefficients equal to 0):

\[
\begin{pmatrix}
[-2]_{\alpha} & [-1]_{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\
[0]_{\alpha} & [1]_{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\
[0]_{\alpha} & [0]_{\alpha} & [1]_{\alpha} & 0 & 0 & 0 & 0 & 0 \\
[0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [1]_{\alpha} & 0 & 0 & 0 & 0 \\
[0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [1]_{\alpha} & 0 & 0 & 0 \\
[0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [1]_{\alpha} & 0 & 0 \\
[0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [1]_{\alpha} & 0 \\
[0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [0]_{\alpha} & [1]_{\alpha} \\
\end{pmatrix}
\]

where we have set \(( n \in \mathbb{Z}) \)

\[
[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_\alpha := \frac{q^n \alpha - q^{-n}}{q - q^{-1}} \quad \text{and} \quad P(\alpha) := \frac{q^{-1} \alpha^2 + (q^3 - 2q - 2q^{-1} + q^{-3})\alpha + q}{(q - q^{-1})^2}.
\]

Taking \( \alpha = q^{2n} \) and performing the “classical” limit \(( q \to 1 \) on the coefficients of the above matrix, we obtain the matrix \((5.13) \) displayed in Example 5.6 (as it should be in view of Remark 7.1). To perform the classical limit, we use that:

\[
[n]_{q=1} = n, \quad [n]_{q^2u} = u + n \quad \text{and} \quad P(q^{2n})_{q=1} = u^2 - u + 2;
\]

this can be obtained, for example, by setting \( q = e^h \), expanding in powers of \( h \) (up to the order 2) and letting \( h \) tend to 0. We remark the following facts appearing in the above deformed version of the matrix \((5.13) \):

- While the factors 2 in \((5.13) \) are deformed in \([2] \), the factors 4 become \([2]_2 \) (and not \([4] \)).
- The factors \((u + n) \) are deformed into \([n]_\alpha \), while the factor \( u^2 - u + 2 \) is deformed into \( P(\alpha) \).
- In addition to the “deformation rules” described in the preceding items, powers of \( q \) and of \( \alpha \), which do not affect the classical limit, appear in some coefficients of the matrix.  \( \triangle \)

**Remark 7.9.** As in Subsection 5.4, other evaluations of the fusion function \((7.11) \) leading to non-invertible elements of the Hecke algebra \( H_n(q) \) are of interest in the framework of the fusion procedure for the solution \((7.3) \). We only indicate here the analogues of the “non-standard” idempotents found in Proposition 5.8 (see \((7.17) \)):

\[
E_2^{(q)} := \frac{1}{q^2 + 1 + q^{-2}} \Psi(1, q^4, q^2) T_{w_3}^{-1} \quad \text{and} \quad E_3^{(q)} := \frac{1}{q^2 + 1 + q^{-2}} \Psi(1, q^{-4}, q^{-2}) T_{w_3}^{-1},
\]

which are pairwise orthogonal primitive idempotents of \( H_3(q) \) generating minimal left ideals isomorphic to the irreducible module \( M^{H_3(q)}_{(2,1)} \).
7.5 Invariant subspaces and representations of $U_q(\mathfrak{gl}_N|\mathfrak{M})$

The construction of fused solutions acting on irreducible representations of $U(\mathfrak{gl}_N|\mathfrak{M})$ in Section 6 also admits, for any $M$, a generalization to the “quantum” setting. It relies on the Schur–Weyl duality between the Hecke algebra $H_n(q)$ and a quantum analogue of $U(\mathfrak{gl}_N|\mathfrak{M})$ proved in [29]. We only indicate the relevant generalization (for any $M$) of the solution $R(\alpha)$, see (7.5), and the corresponding action of $H_n(q)$ on $V^\otimes n$.

**Generalization of the solution $R(\alpha)$ for a $\mathbb{Z}/2\mathbb{Z}$-graded vector space.** We fix a $\mathbb{Z}/2\mathbb{Z}$-decomposition of the vector space $V$ as $V = V_0^\oplus V_1$ and we set $N := \dim(V_0)$ and $M := \dim(V_1)$. We use the notations introduced in Section 6.

We fix a basis $\{e_i\}_{i=1, \ldots, N+M}$ of the vector space $V$, such that $\{e_1, \ldots, e_N\}$ is a basis of the subspace $V_0$ and $\{e_{N+1}, \ldots, e_{N+M}\}$ is a basis of the subspace $V_1$. Let $\hat{R} \in \text{End}(V \otimes V)$ be defined by, for $i, j = 1, \ldots, N+M$,

$$\hat{R}(e_i \otimes e_j) := \begin{cases} \frac{(-1)^{|e_i|}(q + q^{-1}) + (q - q^{-1})}{2} e_i \otimes e_j & \text{if } i = j, \\ (-1)^{|e_i||e_j|} e_j \otimes e_i & \text{if } i < j, \\ (-1)^{|e_i||e_j|} e_j \otimes e_i + (q - q^{-1}) e_j \otimes e_i & \text{if } i > j. \end{cases} \quad (7.16)$$

Note that, for $M = 0$, this operator coincides with the operator defined by (7.1), and that, if we take $q = 1$, we obtain the operator $\hat{P}$ defined by (6.6). As in Subsection 7.1, one directly checks that:

$$\hat{R}^2 - (q - q^{-1}) \hat{R} - \text{Id} = 0, \quad (7.17)$$

and thus that the following function of $\alpha \in \mathbb{C}$ with values in $\text{End}(V \otimes V)$:

$$R(\alpha) := \hat{R}P + (q - q^{-1}) \frac{P}{\alpha^{-1} - 1}, \quad (7.18)$$

is a solution, on $V \otimes V$, of the Yang–Baxter equation with multiplicative spectral parameters:

$$R_{1,2}(\alpha)R_{1,3}(\alpha\beta)R_{2,3}(\beta) = R_{2,3}(\beta)R_{1,3}(\alpha\beta)R_{1,2}(\alpha).$$

**Example 7.10.** Let $N = M = 1$. We write the matrix of the endomorphism $R(\alpha)$ in the basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ of $V \otimes V$ (points indicate coefficients equal to 0):

$$R(\alpha) = \begin{pmatrix} \frac{q\alpha^{-1} - q^{-1}}{\alpha^{-1} - 1} & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{q - q^{-1}}{\alpha^{-1} - 1} & \cdot \\ \cdot & \cdot & \frac{(q - q^{-1})\alpha^{-1}}{\alpha^{-1} - 1} & 1 \\ \cdot & \cdot & \cdot & \frac{q - q^{-1}\alpha^{-1}}{\alpha^{-1} - 1} \end{pmatrix}. \quad (7.19)$$

Due to Relations (7.17), the map

$$\sigma_i \mapsto \hat{R}_{i,i+1} \quad \text{for } i = 1, \ldots, n - 1, \quad (7.20)$$

from the set of generators of $H_n(q)$ to $\text{End}(V^{\otimes n})$ extends to an algebra homomorphism.

For this action of $H_n(q)$, an analogue of the Jimbo–Schur–Weyl duality, Theorem 7.3 is proved in [29], and involves a quantum deformation of $U(\mathfrak{gl}_N|\mathfrak{M})$ instead of $U_q(\mathfrak{gl}_N)$ (we refer
to \cite{29} for the precise definition of the deformation of $U(gl_{N|M})$. Alternatively, the result in \cite{29} can be seen as a quantum analogue of Theorem \cite{6.1} with $U(gl_{N|M})$ replaced by its quantum deformation and the symmetric group $S_n$ replaced by the Hecke algebra $H_n(q)$. We note that the direct sum in (6.8) remains, in the quantum setting, over the same set of partitions.

Now, the exact same reasoning as in Subsection 7.4 can be reproduced and leads to the construction of fused solutions acting on spaces isomorphic to irreducible representations of the quantum deformation of $U(gl_{N|M})$. Namely, we obtain a $\mathbb{Z}/2\mathbb{Z}$-graded analogue of Corollary 7.7. or in other words, a quantum analogue of Corollary 6.3. We omit the details as they are repetitions of the preceding sections.

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L. Poulain d’Andecy loic.poulan-d-andecy@uvsq.fr
Mathematics Laboratory of Versailles, CNRS UMR 8100 Versailles Saint-Quentin University, 45 avenue des Etas-Unis, 78035 Versailles Cedex, France