Inverse problems for Jacobi operators: I. Interior mass–spring perturbations in finite systems

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Abstract
We consider a linear finite spring–mass system which is perturbed by modifying one mass and adding one spring. We study when masses and springs can be recovered from the natural frequencies of the original and the perturbed systems. This is a problem about rank 2 or rank 3 perturbations of finite Jacobi matrices where we are able to describe quite explicitly the associated Green’s functions. We give necessary and sufficient conditions for two given sets of points to be eigenvalues of the original and modified systems, respectively.

1. Introduction
There is a large variety of inverse problems for Jacobi matrices. Their study finds strong motivation both in pure mathematics and physical applications (see, for example, the books [5, 7] and references therein). The earliest known research works on inverse problems for Jacobi matrices in pure mathematics were done by Krein [12, 13] in 1933–34. It seems that his main interest was inverse problems for Sturm–Liouville equations and that discrete problems were studied as approximations to the continuous ones. Borg also worked on inverse problems for Sturm–Liouville operators [4] and noticed the importance of Ambartsumyan’s paper [2]. Borg showed that just one spectrum was not enough to recover the operator. He considered two spectra, corresponding to different boundary conditions and was able to recover the operator from these data. Later this problem was studied by several authors (see, e.g., [14–16] and [17]). Analogs of Borg’s result for Jacobi matrices were first considered by Hochstadt [10, 11] (see also [9, 8, 6] section 5, [18] chapter 7). In this work, we consider a similar problem of ‘Borg type’. We attempt to recover a Jacobi matrix from the information provided by two spectra, that of the original matrix and the spectrum of a special perturbation of it.
The perturbation we consider has a physical interpretation. Finite Jacobi matrices describe the behavior of a mechanical system consisting of \( N \) masses \( m_0, m_1, \ldots, m_{N-1} \) joined by \( N+1 \) springs with stiffness parameters (elastic coefficients) \( k_0, k_1, \ldots, k_N \) and equilibrium lengths \( l_0, l_1, \ldots, l_N \) (see, e.g., \([18, 7]\)).

The first and last springs could either be attached to fixed walls as in figure 1, or be free, in which case the model is the same just setting the elastic coefficient of the spring at the free end equal to zero. The masses are allowed to move in the \( x \)-horizontal direction, with no friction and in the absence of external forces. Using Hooke’s law and Newton’s second law, after normalization the following equation is obtained:

\[
\frac{d^2}{dt^2} \vec{\nu}(t) = -J \vec{\nu}(t),
\]

where the entry \( \nu_i(t) \) of the \( N \)-dimensional vector \( \vec{\nu}(t) \) describes the position of mass \( m_i \) at time \( t \) (see \([7, 19]\)) and \( J \) is the Jacobi matrix

\[
J = \begin{pmatrix}
a_0 & b_0 & 0 & 0 & \cdots \\
b_0 & a_1 & b_1 & 0 & \cdots \\
0 & b_1 & a_2 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & a_{N-1}
\end{pmatrix}
\]

with

\[
a_i = \frac{1}{m_i} (\gamma_{i+1} + \gamma_i), \quad b_i = -\frac{\gamma_{i+1}}{\sqrt{m_im_{i+1}}}, \quad \gamma_i = \frac{k_i}{l_i}
\]

(since we only use the quotients \( \frac{k}{l} \), from now on we consider the elasticity parameters \( \gamma_i = \frac{k_i}{l_i} \) of the springs instead of their length \( l_i \) and Hooke’s coefficients \( k_i \)).

In recent years, several experimental papers \([25, 26]\) were written on possible methods for determining micromasses with the help of oscillating microcantilevers by using the spring–mass system approach. A theoretical framework was suggested by Ram \([24]\) who considered the problem of recovering the Jacobi matrix \((1)\) from its spectrum and the spectrum of the perturbed matrix \( \tilde{J} \) with

\[
\tilde{a}_{N-1} = \frac{m_{N-1}}{m_N} a_{N-1} + \frac{\gamma}{m_N}, \quad \tilde{b}_{N-2} = \sqrt{\frac{m_{N-1}}{m_N}} b_{N-2}, \quad \tilde{m}_{N-1} = 0, \quad \gamma \in \mathbb{R},
\]

the other entries remaining without change. He obtained necessary and sufficient conditions for two point sets to be the spectra of such a pair of matrices \( J, \tilde{J} \) and proposed a method for recovering the matrices from the spectral data. Ram’s results were partially extended by Nylen and Uhlig in \([22, 23]\), who considered the case of an analogous interior perturbation affecting the entries \( a_n, b_n, b_{n-1} \). Namely, they studied the problem of changing the mass \( m_n \) by \( \tilde{m}_n \) for a fixed \( n, 0 \leq n \leq N - 1 \), and attaching to it a new spring of elasticity parameter \( k \), the other end of the spring being fixed: see figure 2.
In this case, the coefficients $a_n, b_{n-1}, b_n$ are modified as follows:

$$\tilde{a}_n = \frac{1}{m_n} (\gamma_n + \gamma_{n+1} + \gamma) = \theta^2 (a_n + M), \quad \tilde{b}_{n-1} = \theta b_{n-1}, \quad \tilde{b}_n = \theta b_n,$$

where

$$\theta = \theta(n) := \sqrt{\frac{m_n}{\tilde{m}_n}}, \quad M := \frac{\gamma}{m_n}, \quad \gamma := \frac{k}{l}.$$  \hspace{1cm} (5)

The coefficient $\tilde{m}_n$ denotes the perturbed mass and $l$ is the length of the new spring. All other entries of $J$ remain unaltered.

The inverse problem for this perturbation is the problem of recovering $J$ and $\tilde{J}$ from their spectra, where $\tilde{J} = \tilde{J}(n)$ is the perturbed matrix with the modified coefficients (4) and all other entries are the same as in $J$.\(^3\)

Nylen and Uhlig obtained necessary conditions for the spectra of the matrices $J$ and $\tilde{J}$ in [22] and offered a method for recovering the possible matrices in the cases where there is a finite number of solutions. However, as will be seen later on, their conditions are not sufficient and the reconstruction does not give all the possible matrices.

The goal of this paper is to develop the direct and inverse spectral theory for the interior perturbations (4) to obtain necessary and sufficient conditions for the spectral data and to give a complete description of the possible Jacobi matrices with such spectral data, providing an explicit algorithm of reconstruction.

The main results are theorem 3 on necessary and sufficient conditions and theorem 4 giving the description of the possible matrices. The algorithm of reconstruction can be found in their proof. In forthcoming papers, we are planning to give a complete solution of this inverse problem for semi-infinite matrices.

The problem considered in this paper may be considered within a class of inverse problems whose purpose is to recover a finite Jacobi matrix by its spectrum and the spectrum of its perturbation. Our mechanical motivation gives a rank 2 or rank 3 perturbation of a special type, while usually the analysis was concentrated on rank 1 perturbations or concerned other types of inverse problems. Our main method is ideologically based on the solution of the rank 1 perturbation problem given in [18] and the $m$-function method for solving the interior inverse problem, proposed by [6]. In contrast to rank 1 perturbations, we do not get uniqueness. Also, degenerated cases are possible for which we have additional manifolds of solutions.

We will use the following notation:

$$\Delta m_n = \tilde{m}_n - m_n, \quad K := \frac{\gamma}{\Delta m_n}$$  \hspace{1cm} (6)

and assume that $\Delta m_n > 0$, i.e. the perturbed mass $\tilde{m}_n$ is greater than the initial mass $m_n$. If $\Delta m_n < 0$, then we just interchange $J$ with $\tilde{J}$, and the analysis remains the same. Taking into account (5) and (6), we immediately have

$$M = (\theta^{-2} - 1)K.$$  \hspace{1cm} (7)

\(^3\) When $n = 0$, we only have the two last equalities in (4), since $b_{n-1}$ is not defined. Analogously, if $n = N - 1$, we have only the first and last equalities in (4), since $b_n$ is not defined.
Thus, we always know one of the three parameters of perturbation $\theta^2$, $M$, $K$ from the two others.

Our spectral data for the inverse problem of recovering $J$ and $\tilde{J}$ are (a) the spectra of the initial and the perturbed Jacobi matrices $\sigma(J) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, $\sigma(\tilde{J}) = \{\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_N\}$, (b) a given integer number $n \in \{0, \ldots, N - 1\}$ which indicates the place where the mass is modified and (c) the parameter of the perturbation $K$. Note that the multiplying of all the masses $m_i$ and the elasticity parameters $\gamma_i$ in (2) by the same constant does not change the Jacobi matrix (1) and the frequency characteristics of the free oscillations, so we obtain a mechanically ‘equivalent’ system. Thus, we can recover exactly neither the masses nor the elasticity parameters from the frequencies. However, their quotients $\frac{m_i}{m_j}$, $\frac{m_i}{m_n}$, $\frac{2\theta^2}{\gamma_i}$, $\frac{\theta^2}{\gamma_i} = M$ can be found. This is the reason why we will only work with the fractional parameters of perturbation $\theta^2$, $M$, $K$ instead of the masses $m_i$, $\tilde{m}_n$ and the elasticity parameters $\gamma_i$, $\gamma_i$.

2. Auxiliary results

For fundamental aspects of spectral theory for Jacobi matrices, we refer to [27, 3, 1] and the more recent monographs [18] and [28].

The eigenvector $\vec{c}$ of the matrix $J$, which satisfies

$$J\vec{c} = \lambda \vec{c},$$

(8)
can be written after normalization as $\vec{c} = (P_0(\lambda), P_1(\lambda), P_2(\lambda), \ldots, P_{N-1}(\lambda))^T$ with $P_0(\lambda) = 1$

where $P_i(\lambda)$ is a polynomial in $\lambda$ of degree $i \in \{0, 1, \ldots, N-1\}$. From (8) we get the expressions

$$a_0 + b_0 P_1(\lambda) = \lambda,$$

$$\ldots$$

$$b_{i-1} P_{i-1}(\lambda) + a_i P_i(\lambda) + b_i P_{i+1} = \lambda P_i(\lambda),$$

(9)

$$\ldots$$

$$b_{N-2} P_{N-2}(\lambda) + a_{N-1} P_{N-1}(\lambda) = \lambda P_{N-1}(\lambda).$$

Therefore, the polynomials $P_i$ are defined by the conditions

$$P_{i+1} = \frac{1}{b_i} [(\lambda - a_i) P_i - b_{i-1} P_{i-1}], \quad P_{-1} = 0, \quad P_0 = 1,$$

(10)
i = 0, \ldots, N - 2. It follows from the last equation in (9) that if $\lambda$ is an eigenvalue, then $Q_N(\lambda) = 0$, where we define

$$Q_N(\lambda) := \lambda P_{N-1} - a_{N-1} P_{N-1}(\lambda) - b_{N-2} P_{N-2}(\lambda).$$

(11)

and each root of the $N$-degree polynomial $Q_N$ is also an eigenvalue of $J$. The polynomial $Q_N$ is equal to the characteristic polynomial of $J$ times a constant.

Let $P_i$ and $Q_N$ be defined as in (9) and (11) and let $\tilde{P}_i$, $\tilde{Q}_N$ be the corresponding polynomials for the perturbed operator $\tilde{J}$. We now try to get an expression that relates $Q_N$ and $\tilde{Q}_N$.

Fix $n \in \{0, 1, \ldots, N - 1\}$ and define the polynomials $\varphi_i$ for $n \leq i \leq N$ as follows:

$$\varphi_0 = 0, \quad \varphi_{n+1} = \frac{1}{b_n} \varphi_n, \quad \varphi_i = \frac{1}{b_{i-1}} [(\lambda - a_{i-1}) \varphi_{i-1} - b_{i-2} \varphi_{i-2}] \text{ if } n + 2 \leq i,$$

(12)

where we set $b_{N-1} = 1$. Note that the degree of the polynomial $\varphi_i$ is $i - n - 1$. These are the so-called polynomials of the second kind while $P_i$ are called of the first kind; see [1].
Lemma 1. With the definition of $\varphi_i$ given above we have for any $n \in \{0, 1, \ldots, N-1\}$
\[
\tilde{Q}_N = \Gamma(n)(\tilde{Q}_N + A\varphi_N P_n),
\]
where $A = \lambda(\theta^2 - 1) - M$ with $\Gamma(n) = \theta(n)$ (see (5)) if $n \in \{0, N-1\}$ and $\Gamma(n) = 1$ otherwise.

Proof.
(a) Case $0 < n < N - 1$. First we will prove
\[
\tilde{P}_i = \varphi_i AP_n + P_i
\]
for $n + 1 \leq i \leq N - 1$.

Using (4) and the recurrence relations (9) for $P_j$ and $\tilde{P}_j$, we obtain
\[
\tilde{P}_n = \frac{1}{b_n} \left( (\lambda - \tilde{a}_n)\tilde{P}_n - \tilde{b}_{n-1}\tilde{P}_{n-1} \right)
\]
\[
= \frac{1}{\theta b_n} \left( (\lambda - \theta)(a_n + M) - \theta b_{n-1}P_{n-1} \right)
\]
\[
= \frac{1}{b_n} P_n \left( \frac{1}{\theta^2} - 1 \right) - M + P_{n+1}.
\]

To obtain the last equation we added and subtracted $\lambda P_n - \frac{1}{\theta^2} P_n$. Therefore,
\[
\tilde{P}_{n+1} = \frac{1}{b_n} (\lambda - a_{n+1}) P_n + P_{n+1}
\]
\[
= \varphi_{n+1} AP_n + P_{n+1},
\]
so (14) holds with $i = n + 1$.

For $i = n + 2$, we have
\[
\tilde{P}_{n+2} = \frac{1}{b_{n+1}} \left( (\lambda - a_{n+2})\tilde{P}_{n+1} - b_{n+1}P_n \right)
\]
\[
= \varphi_{n+2} AP_n + P_{n+2},
\]
so (14) holds in this case. Assume (14) holds for $i - 2$ and $i - 1$ where $i - 2 \geq n + 1$ and let us prove it holds for $i$. First,
\[
\tilde{P}_i = \frac{1}{b_{i-1}} \left( (\lambda - a_{i-1})\tilde{P}_{i-1} - b_{i-1}P_{i-2} \right),
\]
see (9), where the coefficients $a_{i-1}, b_{i-1}$ and $b_{i-2}$ are unperturbed since $i \geq n + 3$.

Therefore,
\[
\tilde{P}_i = \frac{1}{b_{i-1}} \left( (\lambda - a_i)(\varphi_{i-1} AP_n + P_{i-1}) - b_{i-2}(\varphi_{i-2} AP_i + P_{i-2}) \right)
\]
\[
= \varphi_i AP_n + P_i
\]
and (14) holds. To prove (13) recall that
\[
\tilde{Q}_n = (\lambda - a_{N-1})\tilde{P}_{N-1} - b_{N-2}\tilde{P}_{N-2}
\]
\[
= (\lambda - a_{N-1})(\varphi_{N-1} AP_n + P_{N-1}) - b_{N-2}(\varphi_{N-2} AP_n + P_{N-2})
\]
\[
= \varphi_N AP_n + Q_N
\]
and (13) is proven.
(b) Case $n = 0$. Note that $\tilde{P}_0 = P_0 = 1$. Using the first equality in (9),
$$\tilde{P}_1 = \frac{1}{b_0} (\lambda - \tilde{a}_0) = \frac{1}{\theta b_0} (\lambda - \theta^2 (a_0 + M))$$
$$= \theta (P_1 + \varphi_1 A).$$

Analogously we have
$$\tilde{P}_2 = \theta (P_2 + \varphi_2 A).$$

Using induction as in the case (a) we obtain $\tilde{P}_i = \theta (P_i + \varphi_i A)$ for $i = N - 1, N - 2$ and then $\tilde{Q}_N = \theta (Q_N + A \varphi_N P_0)$.

(c) Case $n = N - 1$. From (9) we know that
$$P_{N-1} = \frac{1}{b_{N-2}} (\lambda - a_{N-2}) P_{N-2} - b_{N-3} P_{N-3}$$

and
$$\tilde{P}_{N-1} = \frac{1}{b_{N-2}} (\lambda - a_{N-2}) P_{N-2} - b_{N-3} P_{N-3} = \frac{b_{N-2}}{b_{N-2}} P_{N-1} = \frac{1}{\theta} P_{N-1}.$$ 

Therefore,
$$\tilde{P}_{N-1} = \frac{1}{\theta} P_{N-1}. \quad (17)$$

Analogously and considering (11), we find that
$$\tilde{Q}_N (\lambda) = (\lambda - a_{N-1}) \tilde{P}_{N-1} (\lambda) - \tilde{b}_{N-2} P_{N-2} (\lambda)$$
$$= (\lambda - \theta^2 (a_{N-1} + M)) \frac{1}{\theta} P_{N-1} - \theta b_{N-2} P_{N-2}.$$ 

Adding and subtracting $\theta \lambda P_{N-1} - \frac{1}{\theta} P_{N-1}$, we get
$$= \theta Q_N (\lambda) + \theta \left( \lambda \left( \frac{1}{\theta^2} - 1 \right) - M \right) P_{N-1} (\lambda).$$

Therefore,
$$\tilde{Q}_N = \theta (Q_N + A \varphi_N P_{N-1}).$$

Observe that $\varphi_N = 1$ if $n = N - 1$. □

Green’s function is defined by
$$G(\lambda, j, j) := \langle \delta_j, (J - \lambda)^{-1} \delta_j \rangle,$$
where $\langle, \rangle$ denotes the scalar product and $\delta_j (i) = 1$ if $i = j$ and 0 if $i \neq j$, $i, j \in \{0, 1, \ldots, N-1\}$.

We occasionally allude to $G(\lambda, n, n)$ as Green’s function at the point $nn$ or the $nn$ Green’s function.

**Lemma 2.** For $0 \leq n \leq N - 1$, $\lambda \in \mathbb{C}$,
$$G(\lambda, n, n) = -\varphi_n (\lambda) P_n (\lambda) / Q_N (\lambda).$$
**Proof.** From the definition of the polynomials $P_n$, see (9) or (10), we know that, for $n \geq 1$, $P_n(\lambda) = 0$ if and only if $\lambda$ is an eigenvalue of $J_{(0,n-1)}$, the $n \times n$ upper-left corner of $J$, that is,

$$J_{(0,n-1)} = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 \\ b_0 & a_1 & b_1 & \cdots & 0 \\ 0 & b_1 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \end{pmatrix}.$$ 

Inductively it follows from the definition that

$$P_n(\lambda) = \frac{1}{b_0b_1 \cdots b_{n-1}} \lambda^n + \text{lower degree in } \lambda.$$ 

Since $b_0b_1 \cdots b_{n-1}P_n(\lambda)$ and $\det(\lambda - J_{(0,n-1)})$ are monic polynomials of degree $n$ and with the same zeros, they are equal, i.e.

$$P_n(\lambda) = \frac{1}{b_0b_1 \cdots b_{n-1}} \det(\lambda - J_{(0,n-1)}). \quad (18)$$

Analogously from (12) it follows inductively that

$$\varphi_n(\lambda) = \frac{1}{b_{n-2} \cdots b_n} \lambda^{N-n-1} + \text{lower degree in } \lambda$$

(recall $b_{N-1} := 1$). If we define the matrix, $n \leq N - 2$,

$$J_{(n+1,N-1)} = \begin{pmatrix} a_{n+1} & b_{n+1} & 0 & \cdots & 0 \\ b_{n+1} & a_{n+2} & b_{n+2} & \cdots & 0 \\ 0 & b_{n+2} & a_{n+3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-1} \end{pmatrix},$$

then $\varphi_n$ is related to the matrix $J_{(n+1,N-1)}$ in the same way as $P_n$, defined by (9), is related to the matrix $J$. In fact, the $\varphi_i$ are the $P_i$ for the matrix $J_{(n+1,N-1)}$, multiplied by $\frac{1}{b_n}$.

We get similarly as (18)

$$\varphi_n(\lambda) = \frac{1}{b_n \cdots b_{n-2}} \det(\lambda - J_{(n+1,N-1)}) \quad (19)$$

and

$$Q_n(\lambda) = \frac{1}{b_0b_1 \cdots b_{N-2}} \det(\lambda - J). \quad (20)$$

Therefore, if $1 \leq n \leq N - 2$,

$$\frac{\varphi_n(\lambda)P_n(\lambda)}{Q_n(\lambda)} = \frac{\det(\lambda - J_{(0,n-1)}) \det(\lambda - J_{(n+1,N-1)})}{\det(\lambda - J)}. \quad (21)$$

When $n = 0$,

$$\frac{\varphi_n(\lambda)P_n(\lambda)}{Q_n(\lambda)} = \frac{\det(\lambda - J_{(0,N-1)})}{\det(\lambda - J)}. \quad (22)$$

and for $n = N - 1$

$$\frac{\varphi_n(\lambda)P_n(\lambda)}{Q_n(\lambda)} = \frac{\det(\lambda - J_{(0,N-2)})}{\det(\lambda - J)}. \quad (23)$$

Recall $\varphi_N(\lambda) = 1$ if $n = N - 1$.

According to Cramer’s rule, the solution of $(\lambda - J)u = \delta_n$ is the vector $u = (u(0), u(1), \ldots, u(N-1))^T$ with

$$u(j) = \frac{\det(\lambda - J)}{\det(\lambda - J)} \frac{\det(\lambda - J)_j}{\det(\lambda - J)}, \quad j = 0, \ldots, N - 1.$$
where \((\lambda - J)\) is the matrix \(\lambda - J\) with the \(j\)-column substituted by \(\delta_n\). Since \(u(n) = \langle \delta_n, (\lambda - J)^{-1}\delta_n \rangle = -G(\lambda, n, n)\), we get\[ -G(\lambda, n, n) = \frac{\det(\lambda - J)_n}{\det(\lambda - J)}. \] (24)

Now observe \(\det(\lambda - J)_n = \det(\lambda - J_{[0,n-1]}) \det(\lambda - J_{[n+1,N-1]})\) when \(1 \leq n \leq N - 2\). This can be checked expanding the determinant on the left side by cofactors with respect to the \(n\)th column. When \(n = 0, N - 1\) we get \(\det(\lambda - J)_0 = \det(\lambda - J_{[1,N-1]})\) and \(\det(\lambda - J)_{N-1} = \det(\lambda - J_{[0,N-2]})\), respectively. Therefore, from formulas (21)–(24) the theorem follows. \(\square\)

With the help of the two lemmas proved above, we give an explicit expression for Green’s function at the point \(mn\).

**Theorem 1.**

\[
G(\lambda, n, n) = \frac{1}{1 - b^2} \left\{ \frac{\theta^2}{\lambda - K} - \frac{1}{\lambda - K} \left( \prod_{j=1}^{N} (\lambda - \tilde{\lambda}_j) \right) \right\}. \tag{25}
\]

**Proof.** From lemmas 1 and 2, we get

\[
\frac{\Gamma(n)\hat{Q}_N - \hat{Q}_N}{\Gamma(n)\hat{Q}_N(\lambda(\theta^2 - 1) - M)} = G(\lambda, n, n). \tag{26}
\]

Similar to (20), we have

\[
\hat{Q}_N(\lambda) = \frac{1}{b_0 \ldots b_{N-2}} \det(\lambda - \check{J}). \tag{27}
\]

Using (4) and writing the determinant as a product involving the eigenvalues of \(\check{J}\), we get

\[
\hat{Q}_N = \frac{\Gamma(n)}{\theta^2 b_0 \ldots b_{N-2}} \prod_{j=1}^{N} (\lambda - \tilde{\lambda}_j). \tag{28}
\]

(\(\Gamma(n)\) defined in lemma 1), and from (20)

\[
\hat{Q}_N = \frac{1}{b_0 \ldots b_{N-2}} \prod_{j=1}^{N} (\lambda - \lambda_j). \tag{29}
\]

Using (28) and (29) in (26) we get for \(0 \leq n \leq N - 1\)

\[
\frac{m_n}{\lambda(\Delta m_n) - \gamma} = \frac{\hat{m}_n}{\lambda(\Delta m_n) - \gamma} \left( \prod_{j=1}^{N} (\lambda - \tilde{\lambda}_j) \right) = G(\lambda, n, n), \tag{30}
\]

where we made use of (5). Multiplying both sides of the last equation by \(\frac{\Delta m_n}{\hat{m}_n} = \theta^2 - 1\) and taking into account (6), we get (25). \(\square\)

**Remark 1.** Multiplying both sides of (30) by \(\lambda(\Delta m_n) - \gamma\) we obtain, if \(\frac{\gamma}{\Delta m_n}\) is not a pole of \(G(\lambda, n, n)\),

\[
m_n = \frac{\hat{m}_n}{\prod_{j=1}^{N} (\lambda - \tilde{\lambda}_j)} \iff \lambda = \frac{\gamma}{\Delta m_n} \text{ or } G(\lambda, n, n) = 0. \tag{31}
\]

From lemma 2 and equations (18) and (19), we know that the roots of \(G(\lambda, n, n)\) are the eigenvalues of \(J_{[0,n-1]}\) and \(J_{[n+1,N-1]}\). It will be seen in lemma 5 that common eigenvalues of \(J\) and \(\check{J}\) are also roots of \(G(\lambda, n, n)\). So, if we know one of these points, or the value of \(\frac{\gamma}{\Delta m_n}\) if not a pole (for example not an eigenvalue of \(J\), plus \(m_n\) and the eigenvalues of \(J\) and \(\check{J}\), then \(\hat{m}_n\) can be determined from (31). A particular case of (31) appears in [24], formula 2.17.
3. Direct problem

Let \( \psi_1, \psi_2, \ldots, \psi_N \) be an orthonormal system of eigenvectors for \( J \) with the corresponding eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_N \). For example, one way to get a system of orthonormal eigenvectors of \( J \) is to consider the polynomials \( P_n \) defined by (10) and normalize the eigenvectors \( (1, P_1(\lambda_k), P_2(\lambda_k), \ldots, P_{N-1}(\lambda_k)) \) corresponding to the eigenvalue \( \lambda_k \). We obtain in this case \( |\psi_k(i)|^2 = \frac{|P(\lambda_k)|^2}{\sum_{i=0}^{N}|P(\lambda_k)|^2} \).

Lemma 3. \( G(z, j) := \langle \delta_j, (J - z)^{-1}\delta_j \rangle = \sum_{k=1}^{N} |\psi_k(j)|^2 / \lambda_k - z \).

Proof. Let \( \delta_j = \sum_{k=1}^{N} \alpha_k \psi_k \); then, \( \alpha_k = \langle \delta_j, \psi_k \rangle = \psi_k(j) \) and

\[
\left\langle \sum_{k=1}^{N} \alpha_k \psi_k, (J - z)^{-1} \sum_{l=1}^{N} \alpha_l \psi_l \right\rangle = \sum_{k=1}^{N} \sum_{l=1}^{N} \alpha_k \alpha_l \psi_k \frac{1}{\lambda_k - z} \psi_l = \sum_{k=1}^{N} \frac{|\psi_k(j)|^2}{\lambda_k - z}.
\]

The following lemma can be proven using lemma 3. The spectrum of the operator \( T \) will be denoted by \( \sigma(T) \).

Lemma 4. Let \( \beta \in \sigma(J) = [\lambda_1, \lambda_2, \ldots, \lambda_N] \)

(a) If \( 0 < n < N - 1 \), then \( G(\beta, n) = \infty \) or \( G(\beta, n, n) = 0 \)

(b) If \( n = 0 \) or \( n = N - 1 \), then \( G(\beta, n, n) = \infty \).

Proof.

(a) From lemma 3, we know

\[
G(z, n, n) = \sum_{k=1}^{N} |\psi_k(n)|^2 / \lambda_k - z.
\]

Assume \( \beta = \lambda_{k_0} \in \sigma(J) \). There are two possibilities: either \( \psi_{k_0}(n) \neq 0 \) or \( \psi_{k_0}(n) = 0 \). If the first holds, then \( \lambda_{k_0} \) is a pole of \( G(z, n, n) \). In fact, taking the left and right limits along the real axis, we get \( G(\lambda_{k_0} - n, n) = +\infty \) and \( G(\lambda_{k_0} + n, n) = -\infty \), respectively. If the second possibility holds, then the eigenvector \( \psi_{k_0} \), which corresponds to the eigenvalue of \( \beta = \lambda_{k_0} \) of \( J \), vanishes at \( n \). This implies that \( \lambda_{k_0} \) is also the eigenvalue of \( J_{[0,n-1]} \) and \( J_{[n+1,N-1]} \). Therefore, \( \varphi_N(\lambda_{k_0}) = P_n(\lambda_{k_0}) = 0 \) and it follows from lemma 2 that \( G(\beta, n, n) = 0 \).

(b) Observe that \( P_0 = 1 \) and similarly \( P_{N-1} \neq 0 \), since otherwise \( Q_N(\lambda_{k_0}) = P_{N-1}(\lambda_{k_0}) = 0 \) and this would imply \( P_i = 0 \) for all \( i \). Therefore, using \( |\psi_k(r)|^2 = \frac{|P_k(\lambda_{k_0})|^2}{\sum_{i=0}^{N}|P_i(\lambda_{k_0})|^2} \), \( \psi_{k_0}(r) \neq 0 \) if \( r = 1 \) or \( r = N - 1 \), and then \( G(\beta, n, n) = \infty \) follows.

Using lemma 3 and formula (25), we obtain the next formula, which will be used in the forthcoming results:

\[
\frac{\theta^2}{1 - \theta^2} - \frac{1}{1 - \theta^2} N(\lambda) = (\lambda - K) \sum_{l=1}^{N} \frac{|\psi_l(n)|^2}{\lambda_l - \lambda},
\]  

(32)
where

\[ N(\lambda) := \prod_{j=1}^{N}(\lambda - \hat{\lambda}_j) \prod_{j=1}^{N}(\lambda - \lambda_j) \quad (33) \]

and \( \psi_1, \psi_2, \ldots, \psi_N \) is an orthonormal system of eigenvectors of \( J \) with corresponding eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_N \).

We call an eigenvalue \( \lambda_r \in \sigma(J) \) unmovable if it does not change after the perturbation, i.e. if \( \lambda_r \in \sigma(J) \cap \sigma(\tilde{J}) \).

**Lemma 5.** (Properties for the unmovable eigenvalues and \( \theta \))

\[
\sigma(J) \cap \sigma(\tilde{J}) = \sigma(J) \cap (\{ \lambda : G(\lambda, n, n) = 0 \} \cup \{ K \}),
\]

(34)

\[
\#(\sigma(J) \cap \sigma(\tilde{J}) \cap \{ \lambda : G(\lambda, n, n) = 0 \}) < \min(n, N - n - 1),
\]

(35)

\[
\lambda_r \in \sigma(J) \cap \{ \lambda : G(\lambda, n, n) = 0 \} \implies N(\lambda_r) = \theta^2,
\]

(36)

\[
\lambda_r = K \in \sigma(J) \implies N(\lambda_r) = \theta^2 + (1 - \theta^2)|\psi_r(n)|^2 \geq \theta^2,
\]

(37)

\[
K \notin \sigma(J) \implies N(K) = \theta^2,
\]

(38)

\[
K \in \sigma(J) \cup \sigma(\tilde{J}) \implies K \in \sigma(J) \cap \sigma(\tilde{J}).
\]

(39)

If \( \lambda_N \neq K \), then \( \tilde{\lambda}_N \neq \lambda_N \), and if \( \lambda_1 \neq K \), then \( \tilde{\lambda}_1 \neq \lambda_1 \).

**Proof.** \( \supseteq \) for (34). If \( \lambda \in \sigma(J) \) but \( \lambda \notin \sigma(\tilde{J}) \), then \( \lambda \) is a pole of \( N \), by the definition of \( N \).

If \( G(\lambda, n, n) = 0 \), then from (25), (32), we get

\[
\theta^2 = N(\lambda)
\]

(40)

and \( \lambda \) is not a pole of \( N \). Therefore, if one eigenvalue \( \lambda_{j_1} \) of \( J \) coincides with one root of \( G(\lambda, n, n) \), then an eigenvalue of \( \tilde{J} \) has to coincide with \( \lambda_{j_1} \). Since the spectra of \( J \) and \( \tilde{J} \) are simple, no more eigenvalues coincide at that point. In case the eigenvalue \( \lambda_{j_1} \) of \( J \) coincides with \( K \), then from (32) we get

\[
\frac{\theta^2}{1 - \theta^2} - \frac{1}{1 - \theta^2}N(\lambda) \rightarrow -|\psi_{j_1}|^2 \text{ when } \lambda \rightarrow K
\]

(41)

and \( \lambda_{j_1} = K \) is not a pole of \( N \). Then, an eigenvalue of \( \tilde{J} \) has to coincide with \( \lambda_{j_1} \).

\( \subseteq \) for (34). If \( \lambda \notin \sigma(J) \cap \sigma(\tilde{J}) \), then \( \lambda \) is a pole of \( N \) by the definition of \( N \). It is enough to consider the case \( \lambda \neq K \). Using lemma 4 we know that \( \lambda \) is either a pole or a zero of \( G(\cdot, n, n) \). From (32), poles of \( G \) that are not \( K \), are poles of \( N \). Therefore, \( \lambda \) is a root of \( G \) if \( \lambda \in \sigma(J) \cap \sigma(\tilde{J}) \) and \( \lambda \neq K \).

Further, as mentioned in lemma 4 (a), if an eigenvalue of \( J \) is a zero of \( G(\lambda, n, n) \), then it is a common eigenvalue of \( J_{[0,n-1]} \) and \( J_{[n+1,N-1]} \). Since, according to [6], there are at most \( \min(n, N - 1 - n) \) of them, we conclude that the common eigenvalues of \( J \) and \( \tilde{J} \) are at most \( \min(n, N - 1 - n) \) plus possibly the point \( K \), so (35) is proven. Note that by (b) of lemma 4, \( \sigma(J) \cap \{ \lambda : G(\lambda, n, n) = 0 \} = \emptyset \) when \( n = 0 \) or \( N - 1 \), so the only possible common eigenvalue of \( J \) and \( \tilde{J} \) in this case is \( K \).

Implications (36)–(39) are obtained using the proved part of the lemma and formula (32).

Let us now pass to the last assertion. If the matrix \( A_{[0,N-2]} \) is constructed from a Jacobi matrix \( A_{[0,N-1]} \) by deleting the last column and row (similarly first column and row), then \( \sigma(A_{[0,N-2]}) \subset (\beta_1, \beta_2) \) where \( \beta_1, \beta_2 \) are, respectively, the smallest and largest...
eigenvalues of $A_{[0,N-1]}$ (see corollary 2.5 in [6], for example). Using this fact with matrices $J = J_{[0,N-1]}$, $J_{[0,N-2]}$, and so on, we get that the eigenvalues $\lambda_1$ and $\lambda_N$ of $J$ cannot be eigenvalues of the sub-matrices $J_{[0,n-1]}$, $J_{[r+1,N-1]}$ defined in theorem 2, and therefore cannot be roots of $G(\cdot, n, n)$. From lemma 5 follows $\lambda_r \neq \tilde{\lambda}_r$, $r = 1, N$, unless equal to $K$. \hfill $\square$

**Theorem 2.** (Interlacing properties for two spectra) Let $\sigma(J) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, $\sigma(\tilde{J}) = \{\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_N\}$. Take $p \in \{0, 1, \ldots, N\}$ such that $\lambda_p < K \leq \lambda_{p+1}$, where we define $\lambda_0 = -\infty$ and $\lambda_{N+1} = \infty$ and let $K$ be the parameter of the perturbation (4), (6). Then, there is exactly one eigenvalue of $J$ in each of the following intervals:

\[ [\lambda_j, \lambda_{j+1}], \quad j = 1, \ldots, p - 1, \]

\[ [\lambda_j, \lambda_{j+1}], \quad j = p + 1, \ldots, N - 1, \]

and

\[ [\lambda_p, K], \quad (K, \lambda_{p+1}] \text{ if this last interval is not empty.} \]

**Proof.** Let us find if there is an eigenvalue of $\tilde{J}$, that is, a root of $N(\lambda)$, in the interval $[\lambda_p, K]$. Using (32) and (37), we get $N(K) \geq \frac{m}{\nu_n} > 0$. From lemma 4, we have two possibilities at $\lambda_p$: either $G(\lambda_p, n, n) = 0$ or $\infty$. If the first happens, then $\lambda_p$ coincides with an eigenvalue of $\tilde{J}$ by lemma 5 (and $N(\lambda_p) = \theta$ by (36)). If the second possibility holds, then from (32) $N(\lambda_p+) = -\infty$. Since $N$ is continuous in $(\lambda_p, K)$ there is at least a zero of $N$ in this interval. Therefore, there is at least one eigenvalue of $\tilde{J}$ in the interval $[\lambda_p, K]$.

Now consider the interval $[\lambda_{p-1}, \lambda_p)$. The two options mentioned above for $\lambda_p$ give us either $N(\lambda_p) = \theta$ or $N(\lambda_p-) = \infty$. The two options for $\lambda_{p-1}$ are either $N(\lambda_{p-1}) = \theta$, and in this case $\lambda_{p-1}$ is the eigenvalue of $\tilde{J}$, or $N(\lambda_{p-1}+) = -\infty$. If the second possibility happens, since $N(\lambda) > 0$ for $\lambda$ near $\lambda_p$, again from the continuity of $N(\lambda)$ in $(\lambda_{p-1}, \lambda_p)$ it follows that $N$ has at least a zero in this interval. Therefore, there is at least one eigenvalue of $\tilde{J}$ in $[\lambda_{p-1}, \lambda_p)$. Continuing in this way, we get one eigenvalue of $\tilde{J}$ in each interval $[\lambda_j, \lambda_{j+1})$ with $j = 1, \ldots, p - 1$. Therefore, if $K > \lambda_p$, we get at least one eigenvalue of $\tilde{J}$ in each $[\lambda_j, \lambda_{j+1})$ with $j = 1, \ldots, p - 1$ and one in $[\lambda_p, K)$, a total of at least $p$ eigenvalues of $\tilde{J}$ in the interval $[\lambda_1, K)$.

Now consider the case $K < \lambda_{p+1}$ and let us see whether there is an eigenvalue of $\tilde{J}$ in $(K, \lambda_{p+1}]$. From (32) and (37), $N(K) \geq \theta$. Now, from lemma 4, we have two possibilities at $\lambda_{p+1}$: either $G(\lambda_{p+1}, n, n) = 0$ or $\infty$. If the first happens, then $\lambda_{p+1}$ coincides with an eigenvalue of $\tilde{J}$ by (34) (and $N(\lambda_{p+1}) = \theta$ by (36)). If the second possibility holds, then from (32), $N(\lambda_{p+1}-) = -\infty$. Since $N$ is continuous in $(K, \lambda_{p+1})$ there is at least a zero of $N$ in this interval. Therefore, there is at least one eigenvalue of $\tilde{J}$ in the interval $(K, \lambda_{p+1}]$. Now consider the interval $(\lambda_{p+1}, \lambda_{p+2}]$. The two options mentioned above for $\lambda_{p+1}$ give us either $N(\lambda_{p+1}) = \theta$ or $N(\lambda_{p+2}+) = \infty$. The two options for $\lambda_{p+2}$ are $N(\lambda_{p+2}) = \theta$ and in this case $\lambda_{p+2}$ is the eigenvalue of $\tilde{J}$, or $N(\lambda_{p+2}-) = -\infty$. If the second possibility happens, since $N(\lambda) > 0$ for $\lambda$ near $\lambda_{p+1}$, again from the continuity of $N(\lambda)$ in $(\lambda_{p+1}, \lambda_{p+2})$ it follows that $N$ has at least a zero in this interval. Therefore, there is at least one eigenvalue of $\tilde{J}$ in $(\lambda_{p+1}, \lambda_{p+2}]$. Continuing in this way, we get one eigenvalue of $\tilde{J}$ in each interval $(\lambda_j, \lambda_{j+1})$ with $j = p + 1, \ldots, N - 1$.

Therefore, if $K < \lambda_{p+1}$, we get at least one eigenvalue of $\tilde{J}$ in each $(\lambda_j, \lambda_{j+1})$ with $j = p + 1, \ldots, N - 1$ and one in $(K, \lambda_{p+1}]$, a total of at least $N - p$ eigenvalues of $\tilde{J}$ in the interval $(K, \lambda_N]$. Since the $p$ eigenvalues of $\tilde{J}$ in $[\lambda_1, K]$ plus the $N - p$ eigenvalues of $\tilde{J}$ in
\((K, \lambda_N)\) give all the eigenvalues of \(\tilde{J}\), we conclude that at most there is one eigenvalue of the perturbed operator in each one of the intervals considered.

In the case \(K = \lambda_{p+2}\), we analyze first the interval \((K, \lambda_{p+2})\) exactly as above, and find at least one eigenvalue of \(\tilde{J}\) in it. Continuing with the other intervals as before, we conclude that there is at least one eigenvalue of \(\tilde{J}\) in \((\lambda_j, \lambda_{j+1})\) with \(j = p + 2, \ldots, N - 1\). Therefore, we get \(N - p - 1\) eigenvalues of \(\tilde{J}\) in \((K, \lambda_N)\). These plus \(\lambda_{p+1}\) and the \(p\) eigenvalues in \([\lambda_1, K)\) give all the \(N\) eigenvalues of the perturbed operator. Therefore, there is at most one eigenvalue of \(\tilde{J}\) in each of the intervals considered.

\[\square\]

**Lemma 6.** If \(K \in \sigma(J) \cap \sigma(\tilde{J})\), then the following alternative holds. Either

(a) \(N(K) = \theta^2\), and then \(N'(K) = 0\), \(G(K, n, n) = 0\) and there are at most \(\min(n - 1,\allowbreak\allowbreak N - n - 2)\) other common points of \(\sigma(J)\) and \(\sigma(\tilde{J})\)

or

(b) \(N(K) > \theta^2\), and then \(G(K, n, n) = \infty\) and there may be \(\min(n, N - n - 1)\) other common points of \(\sigma(J)\) and \(\sigma(\tilde{J})\).

**Proof.** According to lemma 4, \(G(K, n, n) = 0\) or \(G(K, n, n) = \infty\). In the first case, by formula (25), it means that the function

\[
\frac{1}{\lambda - K} \frac{\theta^2 - N(\lambda)}{1 - \theta^2}
\]

has a zero at \(K\); thus,

\[
N(K) = \prod_{\lambda_j \neq \tilde{\lambda}_j} (K - \tilde{\lambda}_j) \prod_{\lambda_j \neq \tilde{\lambda}_j} (K - \lambda_j) = \theta^2
\]

and, moreover,

\[
N'(K) = 0.
\]

Since \(G(\lambda, n, n)\) may vanish only at \(\min(n, N - n - 1)\) points of the spectrum of \(J\), in the first case there may be at most \(\min(n - 1, N - n - 2)\) other points of \(\sigma(J) \cap \sigma(\tilde{J})\).

In the second case, the function \(\theta^2 - N(\lambda)\) may not have a zero at \(K\) of order greater than 1, because otherwise \(G(\lambda, n, n)\) would not have a pole at this point. We also know that \(G(\lambda, n, n)\) has a negative residue at \(K\). Thus, \(N(K) > \theta^2\). In the second case, there may be \(\min(n, N - n - 1)\) more points of \(\sigma(J) \cap \sigma(\tilde{J})\) where \(G(\lambda, n, n)\) vanish.

\[\square\]

4. Inverse problem

It turns out that the properties of the spectral data, described in the previous section, are sufficient. We assume the following data are given.

(i) \(\sigma = \{\lambda_1, \ldots, \lambda_N\}\) and \(\tilde{\sigma} = \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_N\}\) which are two finite subsets of \(\mathbb{R}\), \(\lambda_i < \lambda_{i+1}\) and \(\tilde{\lambda}_i < \tilde{\lambda}_{i+1}\) for \(i = 1, \ldots, N - 1\).

(ii) \(K \in \mathbb{R}\).

(iii) An integer number \(n \in \{0, 1, \ldots, N - 1\}\).
We introduce the following notation:
\[ \{\mu_1, \ldots, \mu_q\} =: \sigma \cap \hat{\sigma} - \{K\}. \]  
If \( \sigma \cap \hat{\sigma} - \{K\} = \emptyset \), then \( q := 0 \);  
\[ \hat{N}(\lambda) := \frac{\prod_{j=1}^{N}(\lambda - \hat{\lambda}_j)}{\prod_{j=1}^{N}(\lambda - \hat{\lambda}_j)}; \]  
\[ \theta^2 := \hat{N}(\mu_1) \] if \( \sigma \cap \hat{\sigma} - \{K\} = \emptyset \), then \( \theta^2 \) is any fixed point in \( (0, \hat{N}(K)) \);  
\[ \check{n} := \min(n, N - n - 1). \]

**Theorem 3.** (Necessary and sufficient conditions)

The conditions

(I) \( \hat{\sigma} \) and \( K \) interlace as in theorem 2;

(II) \( \theta^2 = \hat{N}(\mu_1) = \hat{N}(\mu_2) = \cdots = \hat{N}(\mu_q) \in (0, 1) \);

(III) if \( K \notin \sigma \cup \hat{\sigma} \), then \( q \leq \check{n} \) and \( \hat{N}(K) = \theta^2 \);

(IV) if \( K \in \sigma \cup \hat{\sigma} \), then \( K \in \sigma \cap \hat{\sigma} \) and either

(a) \( q \leq \check{n} \) and \( \hat{N}(K) > \theta^2 \) or

(b) \( q < \check{n} \) and \( \hat{N}(K) = \theta^2 \), \( \hat{N}'(K) = 0 \) (** denotes derivative)

are necessary and sufficient for the existence of \( N \times N \) Jacobi matrices \( J \) and \( \hat{J} \), where \( \hat{J} \) is obtained by perturbing \( J \) at the \( n \)th place as described in (4), that is, \( \hat{a}_n = \theta^2(a_n + M), \ \hat{b}_{n-1} = \theta b_{n-1} \); \( \hat{b}_n = \theta b_n \) with \( M = (\theta^2 - 1)K \), such that

\[ \sigma = \text{spectra of } J, \quad \hat{\sigma} = \text{spectra of } \hat{J}. \]

**Remark 2.** Observe that condition (I) implies \( \hat{N}(K) \in (0, 1) \).

**Theorem 4.** Assume that the conditions of the previous theorem hold. If \( K \notin \sigma \cup \hat{\sigma} \) or \( K \in \sigma \cup \hat{\sigma} \) and option (IV) (a) happens, then when \( q \neq 0 \), there are infinitely many pairs \( J, \hat{J} \) of \( N \times N \) Jacobi matrices. Indeed this inverse spectral family is a collection of

\[ \binom{N - 2q - 1}{n - q} \]

disjoint manifolds of dimension \( q \) and diffeomorphic to a \( q \)-dimensional open ball. If \( q = 0 \), then there is only the finite

\[ \binom{N - 1}{n} \]

number of pairs \( J, \hat{J} \).

If \( K \in \sigma \cup \hat{\sigma} \) and option (IV) (b) happens, there are infinitely many pairs. The inverse spectral family is a collection of

\[ \binom{N - 2q - 3}{n - q - 1} \]

disjoint manifolds of dimension \( q + 1 \) and diffeomorphic to a \( q + 1 \)-dimensional open ball.

**Remark 3.** Note that when \( K \notin \sigma \cup \hat{\sigma} \) and in case (IVb), the parameter \( \theta^2 \) is uniquely determined by the spectral parameters (i–iii), and in case (IVA) \( \theta^2 \) is arbitrary in \( (0, \hat{N}(K)) \). In particular, in theorem 4 this means that if \( \sigma \cap \hat{\sigma} = \{K\} \) \( \theta^2 = 0 \) and \( \hat{N}(K) = 0 \), then there are a finite number \( \binom{N}{n} \) of solutions for each \( \theta^2 \in (0, \hat{N}(K)) \) and a collection of \( \binom{N}{n-1} \) disjoint one-dimensional manifolds of solutions for \( \theta^2 = \hat{N}(K) \).
Remark 4. In theorem 5.2 of [6] and theorem 3 of chapter 7 of [19], the problem of obtaining information about the Jacobi matrix from two spectra is studied. The perturbations considered just affect one entry (the first one in [6] and the last one in [19]) whereas we consider perturbations, multiplicative and additive, which may happen at other entries of the matrix.

Proof. Here we prove simultaneously theorems 3 and 4.

The necessity of conditions (I)–(IV) is already proved in the previous section: theorem 2 proves the necessity of condition (I). Assertions (34) and (36) of lemma 5 prove the necessity of condition (II). For condition (III), use assertions (38) and (35) of lemma 5. The first part of condition (IV) is (39). Lemma 6 (a) implies condition (IV) b. Lemma 6 (b) and (37) imply condition (IV) a. Now we prove the sufficiency part of theorem 3 and theorem 4 by finding all pairs of Jacobi matrices that have the given spectral data.

Consider the function
\[ \hat{G}(\lambda) = 1 - \frac{\theta^2}{\lambda - K} - \frac{\hat{N}(\lambda)}{\lambda - K} \]
(compare to formula (25)). Let us now prove that this is Green’s function \( G(\lambda, n, n) \) of a Jacobi matrix. We consider the following two cases.

Case A. \( K \not\in \sigma \cup \tilde{\sigma} \)

Expanding \( \hat{N}(\lambda) \lambda - K \) in partial fractions, we get
\[ \hat{G}(\lambda) = 1 - \frac{\theta^2 - \beta_0}{\lambda - K} - \sum_{j=1}^{N} \frac{\beta_j}{\lambda - \lambda_j} \]
where
\[ \beta_j = \lim_{\lambda \to \lambda_j} \hat{N}(\lambda) \frac{\lambda - \lambda_j}{\lambda - K} = \frac{\prod_{i=1}^{N} (\lambda_j - \hat{\lambda}_i)}{\prod_{i \neq j} (\lambda_j - \lambda_i) (\lambda_j - K)} \quad \text{if} \quad j \neq 0 \]
and
\[ \beta_0 = \hat{N}(K). \]

From condition (III) and (49), we obtain
\[ \hat{G}(\lambda) = 1 - \frac{1}{1 - \theta^2} \sum_{j=1}^{N} \frac{\beta_j}{\lambda_j - \lambda}. \]

Now, from (46) and (50) we find that
\[ \lim_{\lambda \to \infty} \lambda \hat{G}(\lambda) = -1 \quad \text{and} \quad \lim_{\lambda \to \infty} \lambda \hat{G}(\lambda) = -\frac{1}{1 - \theta^2} \sum_{i=1}^{N} \beta_i, \]
respectively. Therefore,
\[ \frac{1}{1 - \theta^2} \sum_{i=1}^{N} \beta_i = 1 \]
and
\[ \hat{G}(\lambda) = \sum_{j=1}^{N} \frac{\alpha_j}{\lambda_j - \lambda} \quad \text{with} \quad \sum_{j=1}^{N} \alpha_j = 1, \]
where \( \alpha_j := \frac{\beta_j}{1 - \theta^2}. \)
From (48) we know that \( \beta_j = 0 \) if and only if \( \lambda_j \in \sigma \cap \hat{\sigma} \). From condition (III), it follows that the sum in (53) has \( k := N - q \) terms and \( N - \tilde{n} \leq k \leq N \). Using expression (48) and the interlacing condition (I), it follows that if \( \alpha_i \neq 0 \), then \( \alpha_i > 0 \). Therefore,

\[
\hat{G}(\lambda) = \sum_{i=1}^{k} \frac{\alpha_i}{{\lambda}_i - \lambda} \quad \text{with} \quad \sum_{i=1}^{k} \alpha_i = 1 \quad \text{and} \quad \alpha_i > 0. \tag{54}
\]

According to theorem 6.2 of [6], (54) implies that \( \hat{G}(\lambda) \) has the form of an \( nn \) Green’s function for at least one Jacobi matrix \( J \). Now we will describe the family of matrices which have \( \hat{G}(\lambda) \) as its \( nn \) Green’s function and moreover have a spectrum equal to the given set \( \sigma \).

All finite Jacobi operators with an \( nn \) Green’s function given by \( \hat{G}(\lambda) \) in (54) have the same eigenvalues \( \lambda_i, l = 1, \ldots, k \), but the other \( N - k \) eigenvalues may change. To study the family of operators which correspond to a given Green’s function, we will use the theory of interior inverse problems for finite Jacobi matrices, developed in [6], theorems 6.1–4. The key formula of this method is (2.18) in [6]:

\[
-\langle \delta_n, (J - z)^{-1} \delta_n \rangle^{-1} = \lambda - a_n + b_n^2 m_+(\lambda, n) + b_{n-1}^2 m_-(\lambda, n), \tag{55}
\]

where

\[
m_+(\lambda, n) := \langle \delta_{n+1}, (J_{[n+1,N-1]} - z)^{-1} \delta_{n+1} \rangle,
\]

\[
m_-(\lambda, n) := \langle \delta_{n-1}, (J_{[0,n-1]} - z)^{-1} \delta_{n-1} \rangle
\]

are the so-called Weyl m-functions. The matrices \( J_{[n+1,N-1]} \) and \( J_{[0,n-1]} \) were defined in lemma 2. It happens that \( m_+(\lambda, n) \) determines \( J_{[n+1,N-1]} \) uniquely (see remark 5), and has the form

\[
m_+(\lambda, n) = \sum_{i=1}^{N-n-1} \frac{\gamma_i}{\lambda - \lambda_i}, \quad \gamma_i > 0, \tag{56}
\]

where \( \sum_{i=1}^{N-n-1} \gamma_i = 1 \) and the \( \lambda_i \) are the eigenvalues of \( J_{[n+1,N-1]} \). Any sum of this form is legal for \( m_+(\lambda, n) \). Similarly \( m_-(\lambda, n) \) determines uniquely \( J_{[0,n-1]} \) and has the form

\[
m_-(\lambda, n) = \sum_{i=1}^{n} \frac{\kappa_i}{\lambda - \lambda_i}, \quad \kappa_i > 0, \tag{57}
\]

where \( \sum_{i=1}^{n} \kappa_i = 1 \) and \( \lambda_i \) are the eigenvalues of \( J_{[0,n-1]} \). Any such sum is allowed for \( m_-(\lambda, n) \).

The reconstruction procedure is as follows.

Given \( G(\lambda) \) as in (54),

\[
-\hat{G}(\lambda)^{-1} = z - a + \sum_{l=1}^{k-1} \frac{\beta_l}{\eta_l - \lambda}, \tag{58}
\]

where \( \eta_1 < \eta_2 < \cdots < \eta_{k-1} \) are the zeros of \( \hat{G}(\lambda) \). The numbers \( \eta_l, a, \) and \( \beta_l \) > 0 are determined by \( \alpha_i \) and \( \lambda_i \) in expression (54). Now we have to write the right-hand side of equality (58) in the form of the right-hand side of equality (55) for some \( a_n, b_n^2, b_{n-1}^2 \) and \( m_+(\lambda, n), m_-(\lambda, n) \) of the form described in (56) and (57). If we do this, then we have according to (55) that \( \hat{G}(\lambda) \) is an \( nn \) Green’s function for a matrix \( J \) with corresponding entries \( a_n, b_n, b_{n-1} \) and sub-matrices \( J_{[n+1,N-1]} \) and \( J_{[0,n-1]} \) determined by the Weyl m-functions. From condition (II), we know that the \( q \) points of \( \sigma \cap \hat{\sigma} \) are among zeros of \( \hat{G}(\lambda) \). We will construct \( m_+(\lambda, n) \) and \( m_-(\lambda, n) \) in such a way that these \( q \) points are their common poles. The other \( k - 1 - q \) zeros of \( \hat{G}(\lambda) \) will be poles of just one of the Weyl m-functions. Since \( m_-(\lambda, n) \) has \( n \) poles, there are

\[
\binom{k - 1 - q}{n - q} = \binom{N - 2q - 1}{n - q}. \tag{59}
\]
possibilities of distributing non-common poles. For each of the $q$ common poles $\mu_i$, we pick a decomposition $\beta_j = \beta_j^{(1)} + \beta_j^{(2)}$ such that the addends $\beta_j^{(i)}$, $i = 1, 2$, appear each in one of the sums (56), (57). So we have $q$ parameters which generate a manifold for each one of the possible choices (59). That these manifolds are diffeomorphic to a sphere follows from theorem 3.6 of [6]. Since we have constructed the Weyl $m$-functions, the matrices $J_{(n+1,n-1)}$ and $J_{[0,n-1]}$ are determined. We can fix $a = a_n$ and

$$b_n^2 = \sum_{i=1}^{l} \beta_i + \sum_{i=1}^{l} \beta_i^{(1)}$$

$$b_n^2 = \sum_{i=1}^{l} \beta_i + \sum_{i=1}^{l} \beta_i^{(2)},$$

where $f_i$ and $g_i$ are defined in (56) and (57). So given $\hat{G}(\lambda)$, we have constructed a family of Jacobi matrices so that each of its members has $\hat{G}(\lambda)$ as its $mn$ Green’s function and its eigenvalues are exactly the points of $\sigma$. We only have to prove that if we perturb one of these matrices $J$, then the perturbation $\hat{J}$ has a spectrum exactly $\hat{\sigma}$. Consider then the Jacobi matrix $\hat{J}$, obtained from $J$ by formulas (4) with $M := (\theta^2 - 1)K$, where $K$ and $\theta^2$ are as defined above in theorems 3. Then, by formulas (25) and (46),

$$\frac{1}{1 - \theta^2} \left\{ \frac{\theta^2}{\lambda - K} - \hat{N}(\lambda) \right\} = \frac{1}{1 - \theta^2} \left\{ \frac{\theta^2}{\lambda - K} - N(\lambda) \right\}$$

which implies $\hat{N}(\lambda) = N(\lambda)$ and $\prod(\lambda - \hat{\lambda}_j) = \prod(\lambda - \lambda_j)$; thus,

$$\hat{\sigma} = [\hat{\lambda}_1, \ldots, \hat{\lambda}_N] = [\lambda_1, \ldots, \lambda_N] = \sigma(J).$$

This proves all assertions of theorems 3 and 4 for case (A) $K \notin \sigma \cup \sigma$. Case B. $K \in \sigma \cup \sigma$

In this case from condition (IV) we know that there exist $j_0 \in \{1, \ldots, N\}$ such that

$$K = \lambda_{j_0} = \hat{\lambda}_{j_0}.$$ 

Then, (47) takes the form

$$\hat{G}(\lambda) = \frac{1}{1 - \theta^2} \left\{ \frac{\theta^2 - \beta_{j_0}}{\lambda - \lambda_{j_0}} - \sum_{j \neq j_0} \frac{\beta_j}{\lambda - \lambda_j} \right\},$$

where

$$\beta_j = \frac{\prod_{j \neq j_0} (\lambda_j - \hat{\lambda}_{j_0})}{\prod_{j \neq j_0} (\lambda_j - \lambda_{j_0})} \quad \text{if} \quad j \neq j_0$$

and

$$\beta_{j_0} = \hat{N}(K).$$

Analogously to what was done in (51), (52) and (53), we get

$$\frac{1}{1 - \theta^2} \sum_{i=1}^{N} \beta_i = \frac{\theta^2}{1 - \theta^2} = 1$$

(63) and

$$\hat{G}(\lambda) = \sum_{i=1}^{N} \frac{\alpha_i}{\lambda_i - \lambda} \quad \text{with} \quad \sum_{i=1}^{N} \alpha_i = 1$$

(64)

where $\alpha_i := \frac{\beta_i}{1 - \theta^2}$ if $i \neq j_0$, $\alpha_{j_0} = \beta_{j_0}. \theta^2$. For $i \neq j_0$ we get using (61) that $\alpha_i = 0$ if and only if $\lambda_i = \lambda_{j_0}$, that is, exactly $q$ times according to (42). Using the interlacing condition (I) and (61), it follows that $\alpha_i > 0$ if $\alpha_i \neq 0$. Therefore,

$$\hat{G}(\lambda) = \sum_{i=1}^{k-1} \frac{\alpha_i}{\lambda_i - \lambda} + \frac{\alpha_{j_0}}{\lambda_{j_0} - \lambda} \quad \text{with} \quad \sum_{i=1}^{k-1} \alpha_i + \alpha_{j_0} = 1, \quad \alpha_i > 0,$$

(65)

where $k := N - q$. Now we have two options.
(i) If situation (IV)a happens, then we get exactly (54) since \( \alpha_{j_0} > 0 \), \( K \) is a pole of \( \hat{G}(\lambda) \) and the analysis is completely analogous to case A.

(ii) If (IV)b holds, then \( \alpha_{j_0} = 0 \), and then

\[
\hat{G}(\lambda) = \sum_{l=1}^{k-1} \frac{\alpha_l}{\lambda_{j_0} - \lambda} \quad \text{with} \quad \alpha_l > 0
\]

and \( K \) is a zero of \( \hat{G}(\lambda) \). So we have \( q + 1 \) fixed zeros. An analysis similar to the one for case A gives

\[
\begin{pmatrix}
 k - 2 - (q + 1) \\
 n - (q + 1)
\end{pmatrix} = \begin{pmatrix}
 N - 2q - 3 \\
 n - q - 1
\end{pmatrix}
\]

possible choices and then \( q + 1 \) parameters. \( \square \)

Remark 5. To find the entries of the matrices \( J_{[0,n-1]} \) and \( J_{[n+1,N-1]} \) one could use the continuous fraction expansions of the Weyl functions [19]:

\[
-m_-(\lambda, n)^{-1} = \lambda - a_{n-1} - \frac{b_{n-2}^2}{\lambda - a_{n-2} - \frac{b_{n-3}^2}{\lambda - a_{n-3} - \cdots \frac{b_0^2}{\lambda - a_0}}} \quad (67)
\]

and

\[
-m_+(\lambda, n)^{-1} = \lambda - a_{n+1} - \frac{b_{n+1}^2}{\lambda - a_{n+2} - \frac{b_{n+2}^2}{\lambda - a_{n+3} - \cdots \frac{b_{N-2}^2}{\lambda - a_{N-1}}}} \quad (68)
\]

These expansions are unique and can be obtained using Euclid’s algorithm. We have to choose the negative square root of \( b_i^2 \) since the off-diagonal terms of our Jacobi matrices are negative; see formula (2).

Remark 6. There are practical applications in which the position \( n \) of the perturbation is not given a priori; see [20] and [21]. Our method does not allow us to find this number directly. All we know is that \( n + 1 \) and \( N - n \) are greater or equal than the number of common points of \( \sigma(J) \) and \( \sigma(\tilde{J}) \). One way out would be to recover the system for all \( n = 0, \ldots, N - 1 \) and then choose the most plausible solution. We thank one of the referees for pointing out the above references to us.

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