Rendezvous of a continuous low-thrust cubesat with a satellite

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Abstract
We give in this paper a new method optimizing continuously the direction of a low continuous thrust to have rendezvous of a cubesat with a given satellite. We study the rendezvous of a cubesat B, with a satellite A, on an elliptic orbit around the Earth. The cubesat B, with 3 kg mass and low-thrust motor, is on a circular orbit around the Earth. The rendezvous is obtained by making the coincidence between the angular momentum vector and the eccentricity vector of B respectively with those of A, then by adjusting the position of B, at the initial instant of thrusting or by convenient choice of this instant. The thrust acceleration is supposed to be constant. The thrust is continuous and its direction is continuously oriented to optimize the time necessary to make the coincidence between the two orbits. A numerical application is given where A is the satellite ‘Almasat-1’.

1. Introduction

The low-thrust spacecraft is very useful for the exploration of the space around the Earth, or around any planet in our solar system. For a given mission, the rendezvous, by continuous low-thrust, is obtained after a long time, but it needs a small quantity of energy. The low-thrust can be produced by a motor which use electrical ionisation. We suppose that the direction of the thrust is controlled at any instant.

Many researches5 use low-thrust to perform rendezvous with asteroids or to make transfers to the Moon or around Mars, or to have optimal Earth-Moon trajectories6. More general information about low-thrust studies can be obtained from [14–19]. In these studies, the thrust is not continuous, but in this paper we use a continuous low-thrust which is continuously oriented to optimize the time necessary to obtain a rendezvous of a cubesat with a given satellite.

We consider a small low-thrust spacecraft, a cubesat B having a mass of 3 kg on a given elliptical orbit around the Earth. The cubesat B is the interceptor. The target is a given non manoeuvrable satellite A having a known position at an instant t1 on an elliptical orbit around the Earth. The motor of B produces a constant continuous thrust whose direction is not constrained. We switch on B’s engine at a given instant t1 in order to make the rendezvous with A, by varying adequately the thrust direction. This amounts to coinciding, at a time t2 > t1, the position and the velocity of B with those of A. We will see how the satellite B can intercept the satellite A. In fact the orbit of the interceptor B is Keplerian when the thrust is stopped, but when the thrust is continuous it is never Keplerian; it is an osculating orbit which is constantly deformed (like a work of potter slowly shaping an amphora) in order to coincide with the orbit of A at the instant t2.

We propose a numerical method to achieve the rendezvous of B with A.

* This paper is dedicated to our colleague Michel Dudeck who died on 2017 April 10th.

5 See [1–8]
6 See [9–13]
2. Equations and notations

We use an inertial direct frame Oxyz whose origin is the centre O of the planet, the Earth for example. An Oz axis passes through the North Pole. An Ox axis in the plane of the equator, points to a fixed inertial direction of space, for example the vernal equinox.

The classical parameters defining the orbit of a satellite in this inertial reference frame are:

- **a**: Semi-major axis of the ellipse,
- **e**: Eccentricity of the orbit,
- **i**: Inclination of the orbit relative to a reference plane, for example the equator.
- **Ω**: Longitude of the ascending node, angle measured in the plane of the equator from Ox axis to the line of intersection with the plane of the orbit.
- **ω**: Argument of pericentre, angle measured in the plane of the orbit from the ascending node to the pericentre.
- **M**: Mean anomaly, angle measured from the pericentre to the radius vector of a fictitious satellite in a circular orbit of same major axis.

This angle helps calculating the true anomaly giving the instantaneous position of the satellite on its orbit.

In the first step, we obtain the coincidence of the B orbit plane with the A orbit plane, by adjusting the angular momentum vector $\mathbf{h}_B$ of B with the angular momentum vector $\mathbf{h}_A$ of A. That is equivalent to make $\Omega_B$ and $i_B$ coincide with $\Omega_A$ and $i_A$, respectively.

In the second step, we obtain the coincidence of the B orbit with the A orbit, by adjusting the eccentricity vector $\mathbf{e}_B$ of B with the eccentricity vector $\mathbf{e}_A$ of A. That is equivalent to make $e_B$ and $\omega_B$ coincide with $e_A$ and $\omega_A$, respectively.

We assume that the thrust acceleration, or thrust per unit mass, $\frac{T(t)}{m(t)}$, has a given constant modulus in advance compatible with the characteristics of the motor on cubesat B. So we pose:

$$\frac{T(t)}{m(t)} = c,$$

$c$ being a given constant such that: $m(t).c < 10^{-4}$ N. We suppose that the mass of the cubesat B is equal to 3 kg. Then $c < \frac{10^{-4}}{3}$ N/kg, $c < \frac{10^{-4}}{3}$ m.s$^{-2}$, or $c < \frac{10^{-2}}{3}$ km.s$^{-2}$. As we will use, in our numerical application, the kilometre as distance unit, we can take

$$c \approx 0.3 \times 10^{-7} \text{ km s}^{-2} \equiv 0.3 \times 10^{-4} \text{ ms}^{-2}.$$

Hence, the constant $c$ has the acceleration dimension. It is the thrust acceleration.

The equations of motion are:

\[
(E): \quad \ddot{r}_B = -\mu_1 \frac{\mathbf{r}_B}{(r_B)^3} + \frac{T}{m},
\]

where $\mathbf{r}_B = \overrightarrow{OB}$, $r_B = |\mathbf{r}_B|$, $\mu_1$ is the geocentric gravitational constant, $\mu = \frac{da}{dt}$ and $\dot{u} = \frac{du}{dt}$ for any $u(t)$. We suppose that B is on an elliptic orbit. At the instant $t_1$, we suppose that B is on the $xy$ plane; but this initial position will be modified after the adjustment of orbits, in order to achieve the rendezvous of B with A.

The choice of the direction of the thrust vector $\overrightarrow{T}$ will be specified according to the desired result.

Let us define a local reference frame in B, $(\hat{r}, \hat{\theta}, \hat{h})$, such that:

\[
\hat{r} = \frac{\mathbf{r}_B}{r_B}, \quad \hat{h} = (\mathbf{r}_B \times \mathbf{V}_B) / |\mathbf{r}_B \times \mathbf{V}_B| = (\hat{r} \times \mathbf{V}_B) / |\hat{r} \times \mathbf{V}_B|, \quad \hat{\theta} = \hat{h} \times \hat{r},
\]

where $\mathbf{V}_B$ is the velocity vector of B. On this frame, the thrust vector is written: $\overrightarrow{T} = (t_r, t_\theta, t_\theta)$. The angular momentum per unit mass is: $\mathbf{h}_B = \mathbf{r}_B \times \mathbf{V}_B$.

On the local reference frame, the velocity of B can be written:

\[
\mathbf{V}_B = (\mathbf{V}_B \cdot \hat{r}) \hat{r} + |\hat{r} \times \mathbf{V}_B| \hat{\theta} \equiv r_B \hat{r} + r_B \hat{\theta} \hat{\theta},
\]

where $\theta$ is the true anomaly of B. The eccentricity vector is: $\mathbf{e}_B = \frac{1}{\mu} \mathbf{V}_B - \mathbf{r}_B$.

We know that the eccentricity vector is collinear to the major axis and oriented toward pericentre direction. It can be written as $\mathbf{e}_B = (\cos \theta \hat{r} - \sin \theta \hat{\theta})$.

\^ See Appendix A where we reproduce some useful known formulas.
3. Adjustment of angular momentum vectors

To obtain the coincidence of the two angular momentum vectors, in an optimal way, we pose \( \Delta h = h_b - h_A \) and we calculate the derivative of the square of the modulus of this vector with respect to time, where the scalar product of two vectors is denoted by a dot:

\[
S_h \equiv \frac{1}{2} \frac{d}{dt} \{ |\Delta h|^2 \} = \Delta h \cdot \frac{d}{dt}[\Delta h] = \Delta h \cdot \frac{d}{dt}[h_b] = \Delta h \cdot \frac{d}{dt}[\hat{r}_b \times \hat{v}_b]
\]

\[
= \Delta h \cdot \left[ \hat{r}_b \times \frac{\hat{T}}{m} \right] \equiv \frac{\hat{T}}{m} \cdot [\Delta h \times \hat{r}_b] = \left\langle \frac{\hat{T}}{m}, \hat{U} \right\rangle,
\]

with

\[
\hat{U} = [\Delta h \times \hat{r}_b] = - \hat{r}_b \times \Delta h
\]

Consequently, the optimal thrust to make \( \hat{h}_b \) coincide with \( \hat{h}_A \) is to be collinear to vector \( \hat{U} \) and in the opposite direction.

Let \( \hat{u}_h = \hat{U} / |\hat{U}| \). We take \( \hat{T} \) such that \( \frac{\hat{T}}{m} = -c \hat{u}_h \), with \( c = \left| \frac{\hat{T}}{m} \right| \), to make negative and of maximal modulus the derivative of \( |\Delta h|^2 \).

We remark that if \( h_b \) coincide with \( h_A \), then \( \hat{i}_b \) and \( \Omega_b \) coincide respectively with \( \hat{i}_A \) and \( \Omega_A \).

4. Adjustment of eccentricity vectors

Adjusting \( \hat{e}_b \), the eccentricity vector of B, with \( \hat{e}_A \), the eccentricity vector of A, is equivalent to adjust at the same time the eccentricity \( e_b \) with \( e_A \) and the angle \( \omega_A \) with \( \omega_B \), \( \Omega_A \) being (respectively \( \omega_A \)) the angle that \( e_b \) (respectively \( e_A \)) makes with the ascending nodes line of B (respectively of A).

Let \( \Delta e = \hat{e}_b - \hat{e}_A \). Its modulus must be equal to zero when \( e_b \) is coincident with \( e_A \). Let us search for the optimal direction of the thrust \( \hat{T} \) which permits to make \( |\Delta e| \) approach zero.

(a) Let us calculate the derivative of the eccentricity vector for a given thrust.

We have, omitting the index B and writing \( \mu \) for \( \mu_B \):

\[
\hat{h} = \hat{r} \times \hat{v}, \quad \frac{d\hat{h}}{dt} = \hat{r} \times \frac{d\hat{v}}{dt} = \hat{r} \times \left( \frac{-\mu_f}{r^2} + \frac{\hat{T}}{m} \right), \quad \frac{d\hat{h}}{dt} = \hat{r} \times \frac{\hat{T}}{m},
\]

\[
\hat{e} = \frac{1}{\mu} \hat{r} \times \hat{h} - \hat{r}, \quad \frac{d\hat{e}}{dt} = \frac{1}{\mu} \frac{d\hat{v}}{dt} \times \hat{h} - \frac{d\hat{r}}{dt} + \frac{1}{\mu} \hat{v} \times \frac{d\hat{h}}{dt},
\]

\[
\frac{d\hat{e}}{dt} = \frac{d}{dt} \left( \frac{\hat{r}}{r} \right) = \frac{\hat{v}}{r} - \frac{\hat{r}}{r^2},
\]

\[
\frac{d\hat{e}}{dt} = -\frac{1}{\mu} \frac{\hat{r}}{r} \times (\hat{r} \times \hat{v}) + \frac{1}{\mu} \frac{\hat{v}}{m} \times \hat{h} + \frac{1}{\mu} \hat{v} \times \left( \hat{r} \times \frac{\hat{T}}{m} \right) - \frac{\hat{v}}{r} + \frac{\hat{T}}{r^2},
\]

\[
= -\frac{1}{\mu} \left[ (\hat{r} \times \hat{v}) \hat{r} - r^2 \hat{v} \right] + \frac{1}{\mu} \left[ \frac{\hat{v}}{m} \times \hat{h} \right] + \frac{1}{\mu} \left[ \hat{v} \times \left( \hat{r} \times \frac{\hat{T}}{m} \right) \right] - \frac{\hat{v}}{r} + \frac{\hat{T}}{r^2}.
\]

Finally, we get:

\[
\frac{d\hat{e}}{dt} = \frac{1}{\mu m} \left[ \hat{T} \times \hat{h} + \hat{v} \times (\hat{r} \times \hat{T}) \right]
\]

(5)

(b) Let us calculate the derivative of \( |\Delta e|^2 \)

We have: \( \Delta e = \hat{e}_b - \hat{e}_A = [(\Delta e) \cdot \hat{r} + (\Delta e \cdot \hat{h}) \hat{h} + (\Delta e \cdot \hat{T}) \hat{T}] \), so, we have, if we use the mixed product notation \( (\hat{A}, \hat{B}, \hat{C}) \equiv \hat{A}(\hat{B} \times \hat{C}) \):

\[
\frac{1}{2} \frac{d}{dt} \{ |\Delta e|^2 \} = \frac{d}{dt} \cdot |\Delta e| = \Delta e \cdot \frac{d\hat{e}}{dt} = \frac{1}{\mu m} \Delta e \cdot [\hat{T} \times \hat{h} + \hat{v} \times (\hat{r} \times \hat{T})]
\]

\[
= \frac{1}{\mu m} (\Delta e, \hat{T}, \hat{h}) + \frac{1}{\mu m} (\Delta e, \hat{v}, \hat{r} \times \hat{T}) = \frac{1}{\mu m} (\hat{T}, \hat{h}, \Delta e) + \frac{1}{\mu m} (\hat{r} \times \hat{T}, \Delta e, \hat{v})
\]
\[
\frac{1}{\mu m} (\vec{T}, \vec{h}, \vec{\Delta}_h) + \frac{1}{\mu m} (\vec{r} \times \vec{T}) \cdot (\vec{\Delta} \times \vec{V}) = \frac{1}{\mu m} (\vec{T}, \vec{h}, \vec{\Delta}_h) + \frac{1}{\mu m} (\vec{\Delta} \times \vec{V}, \vec{r}, \vec{T})
\]
\[
= \frac{1}{\mu m} \langle \vec{T}, \vec{U} \rangle, \text{ scalar product, up to a positive factor,} \quad \frac{1}{\mu m} \text{ of the vectors} \quad \vec{T} \quad \text{and} \quad \vec{U}, \text{ with}
\]
\[
\vec{U}^T = [\vec{h} \times \vec{\Delta}_h + (\vec{\Delta} \times \vec{V}) \times \vec{r}]
\]
Consequently, for a thrust \(\vec{T}\) of given modulus, the maximum decrease of \(|\vec{\Delta}_h|^2\) is obtained when the thrust is collinear to the vector \(\vec{U}^T\) and in opposite direction of it.

For the adjustment of \(\vec{e}_h\) with \(\vec{e}_a\), the optimal choice of \(\vec{T}\) is such that:
\[
\frac{\vec{r}}{m} = -c \quad \hat{u}_e, \quad \text{where} \quad c = \left| \frac{\vec{r}}{m} \right| \quad \text{and} \quad \hat{u}_e = \vec{U}^T/|\vec{U}^T|.
\]

5. Adjustment of the two orbits

It is possible to choose an optimal direction of thrust to make \(\vec{e}_h\) and \(\vec{h}_a\) coincide respectively with \(\vec{e}_a\) and \(\vec{h}_a\). For that, we will consider the sum:
\[
S_p = \frac{1}{2} \left( |\vec{\Delta}_h|^2 + \rho \mu |\vec{\Delta}_a|^2 \right)
\]
We introduce the positive factor \(\rho\) (expressed in km) such that \(\rho \mu |\vec{\Delta}_a|^2\) is, at the instant \(t_1\), near the expression of \(|\vec{\Delta}_h|^2\). That is \(\mu |\vec{\Delta}_a|^2\) is, in general, very smaller than \(|\vec{\Delta}_h|^2\). Hence we can take \(\rho\) near the value of \(|\vec{\Delta}_h|^2 / (\mu |\vec{\Delta}_a|^2)\) at the instant \(t_1\).

The expression of \(S_p\) tends to zero if and only if \(|\vec{\Delta}_h|\) and \(|\vec{\Delta}_a|\) tend to zero. Let us calculate the derivative of \(S_p\) with respect to time:
\[
\frac{d}{dt} S_p = \frac{1}{2} \frac{d}{dt} \left( |\vec{\Delta}_h|^2 + \rho \mu |\vec{\Delta}_a|^2 \right) = \vec{\Delta}_h \cdot \frac{d}{dt}[\vec{\Delta}_h] + \rho \mu \vec{\Delta}_a \cdot \frac{d}{dt}[\vec{\Delta}_a]
\]
\[
= \left\{ \frac{\vec{r}}{m} \cdot \vec{U} \right\} + \left\{ \frac{\vec{r}}{m} \cdot \rho \vec{U}^T \right\} = \left\{ \frac{\vec{r}}{m} \cdot \vec{U} + \rho \vec{U}^T \right\} = \left\{ \frac{\vec{r}}{m} \cdot \vec{U}^T \right\},
\]
with \(\vec{U}^T = \vec{U} + \rho \vec{U}^T\), where \(\vec{U}\) and \(\vec{U}^T\) are given by (4) and (6).

So, we have:
\[
\vec{U}^T = [\vec{\Delta}_h \times \vec{r}_h] + \rho [\vec{h} \times \vec{\Delta}_h + (\vec{\Delta} \times \vec{V}) \times \vec{r}]
\]
Consequently, we take \(\vec{T}\) such that \(\frac{\vec{r}}{m} = -c \quad \hat{u}_p\), where \(\hat{u}_p = \vec{U}^T/|\vec{U}^T|\) and \(c = \left| \frac{\vec{r}}{m} \right|\); hence we make \(\frac{d}{dt} S_p\) negative with maximum modulus, at each point of the orbit. We expect that, in the beginning of the convergence process, \(|\vec{\Delta}_h|\) will decrease up to be near \(|\vec{\Delta}_a|\), then \(S_p\) will tend rapidly to zero.

We make the calculation in Fortran (FT95) to test the convergence speed, for the adjustment of both orbits, for different values of \(\rho\). We notice that the number \(t_1\) of days necessary to obtain this adjustment is a function of \(\rho\) which has a unique minimum for some \(\rho = \rho_1\). Hence, the optimal direction of the thrust is obtained for \(\rho = \rho_1\).

Finally, for \(\rho = \rho_1\) the direction of \(\vec{T}\) becomes optimal. Then, the integration of differential equations (E), insures, in a minimal time, the convergence of \(S_p = \rho \mu |\vec{\Delta}_a|^2 + |\vec{\Delta}_a|^2\) to zero and the coincidence of B orbit with A orbit.

6. Numerical results

We recall that the mass of the Earth is equal to \(M_T = 5.9736 \times 10^{24}\) kg, the constant of the gravitation is equal to \(G = 6.67 \times 10^{-11}\) N⋅m²⋅kg⁻², the geocentric gravitational constant is equal to \(\mu_T = G \cdot M_T = 398 600.4418 \text{ (km)}^3 \text{ s}^{-2}\).

The radius of the Earth is equal to 6378 km.

We suppose that B is, at the instant \(t_1\), on the xy plane; but this position will be modified after the adjustment of orbits, in order to achieve the rendezvous of B with A.

We take the example where B has a circular orbit: \(e_h = 0, r_h = 6982\text{ km}, i_h = 69°\) and \(\Omega_h = 237°\). We suppose that B is, at the instant \(t_1\) (calculated below), in the xy plane where \(\vec{r}_h\) makes with Ox axis an angle equal to \(\Omega_h = 237°\). The coordinates of B, for \(t \leq t_1\), can be written before the instant \(t_1\) of the thrust beginning:
where $n_B = \sqrt{\frac{\mu_B}{r_B^3}}$.

The method that we propose, in this paper, is valid for any choice of the satellite A. We will carry out the numerical applications by taking as satellite A, the satellite ALMASAT-1 whose orbit parameters are:

- $i_A = 69.4857$°, $\Omega_A = 238.7662$°, $e_A = 0.0792733$,
- $\omega_A = 45.0462$°, $M_A = 191.9858$°.

where $i_A$ is the inclination of the orbit of A, $\Omega_A$ is the angle made by the line of nodes with the x-axis of the Galilean frame centred on the Earth, $\omega_A$ is the argument of the latitude of the pericentre counted from the line of the ascending nodes and $M_A$ is the mean anomaly which is written:

$M_A = n_A(t - t_p)$, with $n_A = 2\pi / T_A = \sqrt{\frac{\mu_A}{a_A^3}},$

where $t_p$ is the instant passage of ALMASAT-1 to pericentre and $a_A$ is the semi-major axis of the orbit of A.

We know that the ALMASAT-1 satellite makes 14.056 604 35 revolutions a day around the Earth, giving the period:

$T_A = 24 \times 3600/14.056 604 35 = 6146.577 \ s = 102.4429 \ m.$

We deduce: $n_A = 2\pi / T_A = 0.001 022 22$, which gives:

$t_1 - t_p = M_A / n_A = M_A T_A / (2\pi) = 3.350784328 \times 6146.577 / (2\pi) = 3277.931913 \ s.$

We take, in the following, $t_p = 0$, so we have: $t_1 \approx 3277.932 \ s.$

Moreover, having $\mu_A$ and $n_A$, we obtain the semi-major axis $a_A$:

$a_A = \left(\frac{\mu_A}{n_A^2}\right)^{1/3} = \left[\frac{\mu_A}{(T_A)^2 / (4\pi^2)}\right]^{1/3} \approx 7252.394 \ km.$

We know that the eccentric anomaly $E$ is related to the mean anomaly by:

$E - e \sin (E) = M$

Knowing $M_A$ and $e_A$, we deduce by numerical computation that the eccentric anomaly $E_A$ at the instant $t_1$ is:

$E_A = 3.335 508 165$°. Then using the formula: $\tan (\theta/2) = \sqrt{\frac{1 + e}{1 - e}} \tan (E/2)$, which links the true anomaly $\theta$ to the eccentric anomaly $E$ (see figure 1),

![Figure 1](image.png)

**Figure 1.** Parameters of the elliptic trajectory of a particle S a semi-major axis, $\theta$: true anomaly, $E$: eccentric anomaly.
we obtain:

\[
\theta_{A1} = 2 \tan \left( \frac{\tan (\theta_{A1}/2)}{1 + \tan (\theta_{A1}/2)} \right) = 2(-1.481201720) = -2.962403336 \text{ rad},
\]

then we obtain, at the instant \( t_1 \), the value \( \alpha_{A1} \) of the angle \( \alpha = \omega_A + \theta \):

\[
\alpha_{A1} = \theta_{A1} + \omega_A = -2.962403336 + 0.7862045057 = -2.176198830 \text{ rad},
\]

We deduce also the value \( r_{A1} \) of \( r_A \):

\[
r_{A1} = a_A(1 - e_A^2) / [1 + e_A \cos (\theta_{A1})] \equiv a_A[1 - e_A \cos (\theta_{A1})]
\]

\[
= 7816.537634 \text{ km}.
\]

### 6.1. Calculation program

We specify that the calculation program performs the following operations:

\( \alpha \) It integrates the differential equations by the Runge-Kutta method of the fourth order, every second, from time \( t_1 \), over the interval \([t_1 + i - 1, t_1 + i], i = 1, 2, ... \)

\( \beta \) It calculates the position and the velocity of the cubesat B, at time \( t_1 \), over the orbit of B coincides with the orbit of A.

\( \gamma \) It displays every day (86 400 s) the position, the velocity of B and the parameters of the orbit of B, namely:

- \( t_1 \): the number of days after the instant \( t_1 \);
- \( t_B \): the period of B orbit in seconds;
- \( e_B \): B orbit eccentricity;
- \( a_B \): B orbit semi-major axis;
- \( i_B \): the argument of the latitude of pericentre of B, expressed in degrees;
- \( i_B \): the argument of the latitude of B pericentre.

We saw, above, that the optimal direction of the thrust is such that:

\[
f = c \ u_{op}, \text{ with } c = -\frac{T}{m} \text{ and } \hat{u}_{op} = \frac{\hat{U}^o}{\lVert U^o \rVert}, \text{ where } \hat{U}^o \text{ is giving by (8)}.
\]

\( \theta_B \) the true anomaly of B;

\( \theta_{A1} \): the true anomaly of \( A \);

\( r_{AB} \): the distance AB;

\( \alpha_B = \omega_B + \theta_B \); where \( \omega_B \) is the argument of the latitude of B pericentre.

\( \alpha_A = \omega_A + \theta_A \), where \( \omega_A \) is the argument of the latitude of A pericentre.

### 6.2. Adjustment of the two orbits

We saw, above, that the optimal direction of the thrust is such that:

\[
f = c \ u_{op}, \text{ with } c = -\frac{T}{m} \text{ and } \hat{u}_{op} = \frac{\hat{U}^o}{\lVert U^o \rVert}, \text{ where } \hat{U}^o \text{ is giving by (8)}.
\]

This choice of the thrust direction at any point of B trajectory permits to optimize the adjustment of angular momentum vectors and eccentricity vectors, at the same time. So, at the end of the convergence process, \( \hat{h}_B \) coincide with \( \hat{h}_A \) and \( \hat{\theta}_B \) with \( \hat{\theta}_A \). Then the parameters \( a, e, i, \Omega \) of B coincide respectively with those of \( A \), namely the orbit of B coincides with the orbit of A.

For a given \( \rho \), we will make the numerical calculation from the instant \( t_1 \), with this choice of \( \hat{T} \) which leads to the adjustment of the two orbits. So, we take, at the initial instant \( t_1 \):

\[
[f_B = 0, r_B = 6982 \text{ km}, i_B = 69^\circ, z_B = 0, \Omega_B = 237^\circ].
\]

At the instant \( t_1 \), the parameters of A are:

\[
[a_A = 7252.394 \text{ km}, e_A = 0.079 \text{ km/s}, i_A = 69^\circ, \Omega_A = 238^\circ, \omega_A = 45^\circ].
\]

In this numerical example, we have at the instant \( t_1 \):

\[
\frac{\lVert \Delta \theta \rVert^2}{\lVert \mu \lVert^2} \approx 1302.35 \equiv \rho_0
\]

\( \alpha \) The calculation is stopped when the following conditions are verified:

\[
\frac{\lVert \Delta \theta \rVert}{\lVert \mu \rVert} < 0.001 \text{ km/s and } \frac{\lVert \Delta \rVert^2}{\mu^2} < 0.00001
\]

Hence, for this example, we can take \( \rho = \rho_0 \) to optimize the orientation of the thrust. But we notice that the time \( t_F \), necessary to obtain the adjustment of the two orbits, is a decreasing function of \( \rho \) for \( \rho \in [1, 1150] \). We can verify that \( t_F \) has a unique minimum equal approximately to 208 days 2 h 36 min, reached near \( \rho = 1150 \). For
\( \rho > 1150 \), \( t_j \) is an increasing function of \( \rho \). The time \( t_j \) increases slowly up to nearly 223 days near \( \rho = \rho_0 \) and up to 284 days near \( \rho = 50,000 \). These results are summarized in the following Figure 2.

Hence, for this example, the orientation of the trust is optimal near \( \rho_1 = 1150 \), if we stop the calculation when conditions (9) are verified.

The numerical calculation for \( \rho = \rho_1 \), shows that, at the instant \( t_2 = 17,984,215 \) s, namely 208 days 2 h 19 m 11 s after the instant \( t_1 \), of the beginning of thrust, the orbit of B coincides with the orbit of A. The curves of \( \delta t \) as a function of \( \rho \), in the appendix B, show that \( a_B, e_B, i_B, \Omega_B \) and \( \omega_B \); these parameters of B coincide with the parameters of A, for \( t = t_2 \).

\( \beta \) The calculation is stopped when the following conditions are verified:

\[
|\Delta_n| < 10^{-4} \text{ km}^2/\text{s} \quad \text{and} \quad |\Delta_e| < 10^{-5}
\]

Then we notice that the time \( t_j \), necessary to obtain the adjustment of the two orbits, has, as a function of \( \rho \), a unique minimum near \( \rho_2 = 1000 \). For this value of \( \rho \), the numerical calculation shows that the orbit of B coincide with the orbit of A at the instant \( t_2 = 18,144,389 \) s = 210 days 0 h 6 min 29 s.

### 6.3. Achievement of the rendezvous

\( \alpha \) We recall that, for \( \rho = \rho_1 = 1150 \), we make the calculation using a FORTRAN 95 program, with initial conditions at the instant \( t_1 \), \( M_{A1} = 191^\circ.986 \) and \( M_{B1} = 0 \). We stop the calculation when the two conditions (9) are verified.

Under these conditions, the B orbit coincide with the A orbit, for \( t_2 = 17,982,828 \) s = 208 days 3 h 13 m 48 s, namely 208 days 2 h 19 m 10 s after the instant \( t_1 \). But at the instant \( t_2 \), we obtain \( M_{B2} = 184^\circ.816 \), whereas the mean anomaly of A is: \( M_{A2} = 239^\circ.690 \).

Thus a final step is necessary to achieve the rendezvous: we must choose a convenient position of B on his initial circular orbit, to make B coincide with A at the final time, \( t_2 \).

The satellite B is behind the satellite A when B arrives at the instant \( t_2 \) in the orbit of A, with the following difference between mean anomalies:

\[
M_{A2} - M_{B2} = 54^\circ.874423 = 0.957739 \text{ rad}
\]

This difference corresponds in time to \( \delta t = (M_{A2} - M_{B2})/n_A = 936,916,284 \) s. Then \( \delta t \) gives: \( \delta M_B \equiv \delta t \cdot n_B = 1.013,910 \text{ rad} = 58^\circ.092,798 \).

Consequently, at the instant \( t_1 \), the mean anomaly of B must be: \( M_{B1} = 58^\circ.092,798 \). We expect, with this new initial condition that the cubeSat B will coincide with A at the instant \( t_2 \). But the calculation, made with this new initial condition for the mean anomaly of B, shows a drift for \( t_2 \) and for \( M_B \); at the instant \( t_2 = 17,981,273 \) s, we obtain \( M_{A2} = 148^\circ.614,920 \) and \( M_{B2} = 154^\circ.806,118 \). Then a correction is needed for \( M_{B1} \), in the following way.

We have \( M_{A2} - M_{B2} = -6^\circ.191,198 \), \( \delta M_B = (M_{A2} - M_{B2})/n_B = -6^\circ.554,310 \)

Then we take the new value for \( M_{B1} = 58^\circ.092,77 - 6^\circ.554,310 = 51^\circ.538,488 \)

We remake a similar calculation, as above, using the initial conditions at the instant \( t_1 \), \( M_{A1} = 191^\circ.986 \) and \( M_{B1} = 51^\circ.538,488 \).
We stop the calculation when the conditions (9) are verified. By repeating this calculation, we verify that $|\delta M_b|$ decreases rapidly and tends to zero. Finally, when we take $M_B = 52^\circ.181 \, 928$, the rendezvous is obtained, at $t_2 = 17 \, 981 \, 421 \, s \equiv 208$ days $2 \, h \, 50 \, m \, 21 \, s$, where $M_{A2} - M_{B2} = 0^\circ.000 \, 05$, with a distance between $A$ and $B$ approximately equal to $7252 \times (0.000 \, 05 \times \pi/180) \, km \approx 6.32 \, m$.

We summarize this result in the following Table 1, where the convergence of this process is obtained after 9 steps:

| $M_{A1}$ | $M_{B1}$ | $M_{A2}$ | $M_{B2}$ | $\delta M_B$ | $t_2$ |
|---------|---------|---------|---------|-------------|------|
| $191^\circ.986$ | 0 | $239^\circ.690 \, 003$ | $184^\circ.815 \, 580$ | $58^\circ.092 \, 798$ | $17 \, 982 \, 828 \, s$ |
| $191^\circ.986$ | $58^\circ.092 \, 798$ | $148^\circ.614 \, 920$ | $154^\circ.806 \, 118$ | $-6^\circ.534 \, 310$ | $17 \, 981 \, 273 \, s$ |
| $191^\circ.986$ | $51^\circ.538 \, 488$ | $158^\circ.220 \, 267$ | $157^\circ.545 \, 778$ | $0^\circ.714 \, 048$ | $17 \, 981 \, 437 \, s$ |
| $191^\circ.986$ | $52^\circ.252 \, 536$ | $157^\circ.166 \, 021$ | $157^\circ.239 \, 976$ | $-0^\circ.402 \, 030$ | $17 \, 981 \, 419 \, s$ |
| $191^\circ.986$ | $51^\circ.850 \, 506$ | $157^\circ.810 \, 282$ | $157^\circ.462 \, 854$ | $0^\circ.367 \, 8046$ | $17 \, 981 \, 430 \, s$ |
| $191^\circ.986$ | $52^\circ.218 \, 311$ | $157^\circ.224 \, 591$ | $157^\circ.262 \, 673$ | $-0^\circ.040 \, 316$ | $17 \, 981 \, 420 \, s$ |
| $191^\circ.986$ | $52^\circ.177 \, 995$ | $157^\circ.283 \, 160$ | $157^\circ.278 \, 986$ | $0^\circ.004 \, 419$ | $17 \, 981 \, 421 \, s$ |
| $191^\circ.986$ | $52^\circ.182 \, 414$ | $157^\circ.283 \, 160$ | $157^\circ.283 \, 619$ | $-0.000 \, 486$ | $17 \, 981 \, 421 \, s$ |
| $191^\circ.986$ | $52^\circ.181 \, 928$ | $157^\circ.283 \, 160$ | $157^\circ.283 \, 108$ | $0^\circ.000 \, 050$ | $17 \, 981 \, 421 \, s$ |

(1) Let us use the same algorithm when the calculation is stopped under the conditions (10). We take $\rho = \rho_2 = 1000$, and the initial conditions: $M_{A1} = 191^\circ.986$ and $M_{B1} = 0$, at the instant $t_1$. Then the orbit of $B$ coincide with the orbit of $A$, for $t_2 = 18 \, 144 \, 389 \, s \equiv 210$ days $0 \, h \, 6 \, min \, 29 \, s$, namely 209 days $23 \, h \, 11 \, min \, 51 \, s$ after the instant $t_1$. We obtain at the instant $t_2$:

$M_{A2} = 342^\circ.186073$, $M_{B2} = 289^\circ.130545$,

$M_{A2} - M_{B2} = 53^\circ.055528$, new $M_{B1} = 56^\circ.167226$

We repeat this calculation until $\delta M_{B1}$ tends to zero. The convergence of this process is obtained after 8 steps, as we can see in the following table 2:

| $M_{A1}$ | $M_{B1}$ | $M_{A2}$ | $M_{B2}$ | $\delta M_B$ | $t_2$ |
|---------|---------|---------|---------|-------------|------|
| $191^\circ.986$ | 0 | $342^\circ.186 \, 073$ | $289^\circ.130 \, 545$ | $56^\circ.167 \, 226$ | $18 \, 144 \, 389 \, s$ |
| $191^\circ.986$ | $56^\circ.167 \, 226$ | $216^\circ.463 \, 976$ | $216^\circ.058 \, 809$ | $-6^\circ.5543$ | $18 \, 142 \, 140 \, s$ |
| $191^\circ.986$ | $50^\circ.744 \, 257$ | $223^\circ.349 \, 197$ | $222^\circ.780 \, 097$ | $-0^\circ.602 \, 478$ | $18 \, 142 \, 360 \, s$ |
| $191^\circ.986$ | $50^\circ.846 \, 735$ | $222^\circ.060 \, 675$ | $222^\circ.118 \, 877$ | $-0^\circ.061 \, 615$ | $18 \, 142 \, 338 \, s$ |
| $191^\circ.986$ | $50^\circ.785 \, 119$ | $222^\circ.177 \, 813$ | $222^\circ.171 \, 866$ | $0^\circ.006 \, 295$ | $18 \, 142 \, 340 \, s$ |
| $191^\circ.986$ | $50^\circ.791 \, 415$ | $222^\circ.177 \, 813$ | $222^\circ.178 \, 422$ | $-0^\circ.000 \, 644$ | $18 \, 142 \, 340 \, s$ |
| $191^\circ.986$ | $50^\circ.790 \, 771$ | $222^\circ.177 \, 813$ | $222^\circ.177 \, 751$ | $0^\circ.000 \, 066$ | $18 \, 142 \, 340 \, s$ |
| $191^\circ.986$ | $50^\circ.790 \, 836$ | $222^\circ.177 \, 813$ | $222^\circ.177 \, 819$ | $-0^\circ.000 \, 006$ | $18 \, 142 \, 340 \, s$ |

Finally, for $M_{B1} = 50^\circ.790 \, 836$, we obtain the adjustment of orbits at $t_2 = 18 \, 142 \, 340 \, s \equiv 209$ days $23 \, h \, 32 \, m \, 20 \, s$, with $M_{A2} = 222^\circ.177 \, 813$, $M_{B2} = 222.177 \, 819$, $M_{A2} - M_{B2} = -0^\circ.000 \, 006$

Hence, the rendezvous is obtained at $t_2 = 18 \, 142 \, 340 \, s \equiv 209$ days $23 \, h \, 32 \, m \, 20 \, s$, with a distance between $A$ and $B$ approximately equal to $7252 \times (0.000 \, 006 \times \pi/180) \, km \approx 0.76 \, m$.

The curves of figures C1–C6, in appendix C, show clearly the achievement of the rendezvous, since $\Omega_{B}, e_{B}, \omega_{B}$ and $\mu_{B}$ tend respectively to $a_{A}, e_{A}, i_{A}, \Omega_{A}$, $\omega_{A}$ and $M_{A}$; these parameters of $B$ coincide with the parameters of $A$ for $t = t_2$.

**Remark:** To achieve the rendezvous, after the adjustment of orbits, an alternative to this procedure is to use the method of local approach exposed in our previous work.8

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8 See [20] and [21].
7. Conclusion

We present in this paper a new method which permits to a cubesat, equipped with a low-thrust propulsion system which covers an orbit around the Earth, to make rendezvous with a given satellite on an elliptic orbit around the Earth. The thrust is very low but continuous; its acceleration is constant and its direction is adjusted continuously to obtain optimal convergence of the angular momentum vector and the eccentricity vector of the cubesat respectively to the angular vector and the eccentricity vector of the satellite. We gave a numerical application of this method taking ALMASAT-1 as a satellite, where the cubesat starts on circular orbit. Evidently this method can be applied for any satellite around the Earth or another planet. It can be used to remove old satellites or debris around the Earth by deorbiting them, in the aim to burn the debris in the dense atmosphere.

Appendix A. Some formulas

The eccentricity vector can be written, on the local frame as follows, where $\theta$ is the true anomaly:

$$e_B = \left( \frac{\mu}{r^2} V^2 - 1 \right) \hat{\tau} - \frac{r}{\mu} \hat{\tau} \times |V| \hat{\theta} = \left( \frac{r^3 \dot{\theta}^2}{\mu} - 1 \right) \hat{\tau} - \frac{r^2 \dot{\theta}}{\mu} \hat{\theta} = \left[ a \left( 1 - e^2 \right) - 1 \right] \hat{\tau} - \hat{\tau} \sqrt{a \left( 1 - e^2 \right) / \mu} \hat{\theta} = e \left( \cos \theta \hat{\tau} - \sin \theta \hat{\theta} \right)$$

We have in the Galilean frame, where the index B is omitted to simplify the notations:

$$\hat{r} = [\cos \alpha \cos \Omega - \sin \alpha \sin \Omega \cos i, \cos \alpha \cdot \sin \Omega + \sin \alpha \cos \Omega \cos i, \sin \alpha \sin i]$$

$$\hat{\theta} = [\sin \alpha \cos \Omega + \cos \alpha \sin \Omega \cos i, \sin \theta \sin \Omega - \cos \alpha \cos \Omega \cos i, -\cos \alpha \sin i],$$

$$\hat{h} = [\sin \Omega \sin i, -\cos \Omega \sin i, \cos \Omega].$$

We have other useful formulas for an elliptic orbit with $\omega = \alpha + \theta, \dot{\alpha} \equiv \dot{\theta}$:

$$(V, \hat{r}) = \hat{r}, \hat{r} \times V = r \hat{\alpha} [\sin i \sin \Omega, -\sin i \cos \Omega, \cos i] = r \alpha \hat{\alpha},$$

$$r = a(1 - e^2)/(1 + e \cos \theta), \quad h = \sqrt{\mu} a \left( 1 - e^2 \right) = r^2 \dot{\theta}, \quad V^2 = \mu \left( 2/r - 1/a \right)$$

$$(V^2 + \mu/a) / 2 = \mu/r, \quad V = \sqrt{\mu \left( 1 + e^2 + 2 \cdot e \cos \theta \right)} a \left( 1 - e^2 \right)^{-1}, \quad \hat{r} \left( \mu \mu \right) \frac{\mu}{a \left( 1 - e^2 \right)} e \sin \theta$$

$$\dot{\theta} = \sqrt{\frac{\mu}{a^3 \left( 1 - e^2 \right)^3}} (1 + e \cos \theta)^2,$$

On the other hand, we have the position and the velocity of B calculated on the Galilean frame (see figure A1):

$$x = r [\cos \alpha \cos \Omega \cos i - \sin \alpha \sin \Omega \cos i], \quad y = r [\cos \alpha \sin \Omega + \sin \alpha \cos \Omega \cos i],$$

$$z = r \sin \alpha \sin i,$$

$$\dot{x} = \dot{r} [\cos \alpha \cos \Omega - \sin \alpha \sin \Omega \cos i] - r [\sin \alpha \cos \Omega + \cos \alpha \sin \Omega \cos i] \alpha,$$

$$\dot{y} = \dot{r} [\cos \alpha \sin \Omega + \sin \alpha \cos \Omega \cos i] + r [\cos \alpha \cos \Omega \cos i + \sin \alpha \sin \Omega - \cos \alpha \sin \Omega \cos i] \alpha,$$

$$\dot{z} = \dot{r} \sin \alpha \sin i + r \cos \alpha \sin i \alpha + r \sin \alpha \cos i.$$
Appendix B. Adjustment of the orbit of B with the orbit of A

We take $\rho = 1150$ and we stop the calculation when conditions (9) are verified. Then the orbit of B coincides with the orbit of A at the instant $t_2 = 17,981,421$ s, namely 208 days 2 h 50 m 43 s after the instant $t_1$ of the beginning of the thrust. The following curves show that $a_B, e_B, i_B, \omega_B$ tend respectively to $a_A, e_A, i_A, \omega_A$ and they coincide for $t = t_2$.

1) Semi-major axis $a_B$:

![Figure B1. Adjustment of orbits. The semi-major axis $a_B$ as a function of $t$, for $\rho = 1150$.](image)

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9 See section 6.2 on page 6 and section ($\alpha$) on page 6.
2) Eccentricity $e_B$:

![Figure B2](image2.png)

*Figure B2. Adjustment of orbits. The eccentricity $e_B$ as a function of $t$, for $\rho = 1150$.*

3) Inclination $i_B$:

![Figure B3](image3.png)

*Figure B3. Adjustment of orbits. The inclination $i_B$ as a function of $t$, for $\rho = 1150$.*

4) Longitude of ascending node $\Omega_B$:

![Figure B4](image4.png)

*Figure B4. Adjustment of orbits. The Longitude of ascending node $\Omega_B$ as a function of $t$, for $\rho = 1150$.*

5) Argument of pericentre latitude $\omega_B$:

![Figure B5](image5.png)

*Figure B5. Adjustment of orbits. Argument of pericentre latitude $\omega_B$ as a function of $t$, for $\rho = 1150$.*
Appendix C. The rendezvous of B with A

We take $\rho = 1000$. The calculation is stopped when the conditions (10) are verified\textsuperscript{10}. The rendezvous is obtained at $t_2 = 18 \, 142 \, 340$ s, or 209 days 22 h 37 m 42 s after the instant $t_1$. The following curves show that $a_B$, $e_B$, $i_B$, $\Omega_B$, $\omega_B$, and $M_A$ tend respectively to $a_A$, $e_A$, $i_A$, $\Omega_A$, $\omega_A$, and $M_A$; they coincide for $t = t_2$.

1) The semi-major axis $a_B$ of B:

![Figure C1](image1.png)

Figure C1. The achievement of the rendezvous. The semi-major axis $a_B$ as a function of $t$, for $\rho = 1000$.

2) The eccentricity of B:

![Figure C2](image2.png)

Figure C2. The achievement of the rendezvous. The eccentricity $e_B$ as a function of $t$, for $\rho = 1000$.

3) The inclination $i_B$ of B:

![Figure C3](image3.png)

Figure C3. The achievement of the rendezvous. The inclination $i_B$ as a function of $t$, for $\rho = 1000$.

\textsuperscript{10} See section (\beta) on page 8.
4) The ascending node longitude $\Omega_B$:

![Figure C4. The achievement of the rendezvous. The ascending node longitude $\Omega_B$ as a function of $t$, for $\rho = 1000$.](image)

5) The latitude argument of pericentre $\omega_B$:

![Figure C5. The achievement of the rendezvous. The latitude argument of pericentre $\omega_B$ as a function of $t$, for $\rho = 1000$.](image)

6) The mean anomaly of B:

![Figure C6. The achievement of the rendezvous. The mean anomaly $M_B$ of B as a function of $t$, for $\rho = 1000$.](image)

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