On the Ricci curvature of Kähler-Ricci Flow

Fung Cheuk Yan

Department of Mathematics
Hong Kong University of Science and Technology
cyfungao@connect.ust.hk

Nov 24 2021

Abstract

In this paper, we consider $n$-dimensional compact Kähler manifold with semi-ample canonical line bundle under the long time solution of Kähler Ricci Flow. In particular, if the Kodaira dimension is one, Ricci curvature converge to negative of generalized Kähler Einstein metric $\omega_B$ locally away from singular set in $C^0_{\text{loc}}(\omega(t))$ topology.

Contents

1 Introduction ....................................................... 2
   1.1 Background ............................................... 2
   1.2 Major Result ............................................... 4

2 Gradient estimate ............................................... 8
   2.1 Notation .................................................... 8
   2.2 Preliminary .................................................. 8
   2.3 Preparation for Theorem 2.1 ................................. 10
   2.4 Proof of Theorem 2.1 ....................................... 12

3 Ricci curvature estimate ........................................... 13
   3.1 Preliminary .................................................. 13
   3.2 Preparation for Proof of Theorem 1.2 ....................... 14
   3.3 Proof of Theorem 1.2 ....................................... 15
1 Introduction

1.1 Background

The setup is taken from [11] and [5].

Let \((X, \omega_0)\) be a compact \(n\)-dimensional Kähler manifold with semi-ample canonical line bundle \(K_X\) and \(0 < m := \text{Kod}(X) < n\).

Let \(\omega(t)\) be the long time solution of the normalized Kähler-Ricci flow

\[
\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega) - \omega, \quad \omega|_{t=0} = \omega_0
\]

There exists a holomorphic map:

\[
f : X \to B = X_{\text{can}} \subset \mathbb{CP}^n := \mathbb{C}^H_0(X, K_X^{\otimes l})
\]

for sufficiently large \(l \in \mathbb{Z}^+\) where \(B = X_{\text{can}}\) is the canonical model of \(X\).

We have \(\dim B = m\).

Let \(S'\) be the singular set of \(B\) together with the set of critical values of \(f\), and we define \(S = f^{-1}(S') \subset X\).

Note \(f^*(\mathcal{O}(1)) = K_X^{\otimes l}\). Let \(\chi = \frac{1}{l} \omega_{FS}\). \(f^*\chi\) (also denoted by \(\chi\)) is a smooth semi-positive representative of \(-c_1(X)\) where \(\omega_{FS}\) is the Fubini-Study metric.

Let \(\Omega\) be the smooth volume form on \(X\) with

\[
\sqrt{-1} \partial \bar{\partial} \log(\Omega) = \chi, \quad \int_X \Omega = \binom{n}{m} \int_X \omega_0^{n-m} \wedge \chi^m
\]

Since \(X_y := f^{-1}(y)\) are Calabi-Yau for \(y \in B \setminus S'\), there exists a unique smooth function \(\rho_y\) on \(X_y\) with \(\int_{X_y} \rho_y \omega_0^{n-m} = 0\) and such that

\[
\omega_0|_{X_y} + \sqrt{-1} \partial \bar{\partial} \rho_y =: \omega_y
\]

is the unique Ricci-flat Kähler metric on \(X_y\). Moreover, \(\rho_y\) depends smoothly on \(y\), and so define a global smooth function on \(X \setminus S\). We define a a semi Ricci-flat form \(\omega_{SRF} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \rho\) in which its restriction on fibre \(X_y\) is Ricci-flat.

Define \(F\) on \(X \setminus S\) by

\[
(1.2) \quad F := \frac{\Omega}{\omega_{SRF}^{n-m} \wedge \binom{n}{m} \chi^m}
\]

There exists a unique solution \(v \in PSH(\chi) \cap C^0(B) \cap C^\infty(B \setminus S')\) of

\[
(1.3) \quad (\chi + \sqrt{-1} \partial \bar{\partial} v)^m = F e^v \chi^m
\]

2
Define a Kähler metric on $B$

$$\omega_B := \chi + \sqrt{-1} \partial \bar{\partial} v$$

Note that $\omega_B$ is smooth and satisfies the generalized Einstein equation

$$\text{Ric}(\omega_B) = -\omega_B + \omega_{WP} \text{ on } B \setminus S'$$

where $\omega_{WP}$ is the Weil-Petersson metric induced by the Calabi-Yau fibration $f$.

We also denote $f^*\omega_B$ by $\omega_B$.

In Theorem 1.1 and Theorem 1.2 of [11], it is shown that the metric $\omega(t)$ converges to $\omega_B$ in the $C^0_{\text{loc}}(\omega_0)$ topology on $X \setminus S$.

We consider the reference metric

(1.4) $$\hat{\omega} = e^{-t}\omega_0 + (1 - e^{-t})\chi$$

Some would consider another reference metric alternatively

(1.5) $$\tilde{\omega} = e^{-t}\omega_B + (1 - e^{-t})\omega_{SRF}$$

Writing the metric as

$$\omega(t) = \hat{\omega}(t) + \sqrt{-1} \partial \bar{\partial} \varphi$$

The Kähler-Ricci Flow [11] is equivalent to the Monge-Ampère equation

(1.6) $$\frac{\partial}{\partial t} \varphi = \log \frac{e^{(n-m)t}(\hat{\omega}(t) + \sqrt{-1} \partial \bar{\partial} \varphi(t))^n}{\Omega} - \varphi, \varphi(0) = 0$$

And we consider the function

(1.7) $$u := \frac{\partial \varphi}{\partial t} + \varphi - v$$

The norm $|\cdot|$ in this article, unless specified, mean the norm $\| \cdot \|_{\omega(t)}$. 
In this article, we consider compact sets $K \subset X \setminus S$. Choose some open subset $U'$ such that
\[(1.8) \quad f(K) \subset U' \subset B \setminus S'\]
Define
\[(1.9) \quad U = f^{-1}(U')\]
Then
\[(1.10) \quad K \subset U \subset X \setminus S\]

On any non-singular fibre $X_y$, define
\[g_F(t) := g(t)|_{X_y}\]
Locally, consider holomorphic coordinates as $(z_1, z_2, ..., z_n)$. Since $f$ is a submersion in a neighborhood of point away from singularity, we can write the coordinate of fiber as $\{z_\alpha\}^{n-m}_{\alpha=1}$. At a fixed point $x$, if we can assume $\{z_\alpha\}^{n-m}_{\alpha=1}$ restricted on $X_y$ are normal coordinates and assume $g(t)$ to be identity. And we can write:
\[g_F(t)^{\alpha\beta} = (g_F(t)_{\alpha\beta})^{-1}\]

1.2 Major Result

The uniform Ricci curvature bound on compact set away from singularities has been studied for a long time. When Kod$(X) = 0$ and Kod$(X) = n$, the bound is shown by Cao [1] and Tsuji [12] respectively. When Kod$(X) = n-1$, since general fibre of $X$ over $X_{can}$ are complex tori, the estimate is given by the result of Tosatti-Weinkove-Yang in [11]. The bound when Kod$(X) = 1$ is shown by a recent paper of Jian-Song [7]. In general Kodaira dimension, when regular fibers are biholomorphic to each other, Fong-Lee [2] showed the bound. Jian-Shi [6] further extended the result and showed the local convergence of Ricci curvature in such case:

\[\|\text{Ric} + \omega_B\|_{C^k(K, \omega(t))} \leq C(K)h_{K,k+2}(t)\]

where $h_{K,k+2}(t)$ are positive functions which tends to zero as $t \to \infty$, depending only on $k$ and the domain $K$ away from singularities. And $\tilde{\omega}(t)$ is defined in (1.5). Jian-Shi conjectured that will hold in the general case.

In this paper, inspired by the idea of Jian-Song [7] and the conjecture of Jian-Shi [6], we will show in Corollary [13] that, when Kod$(X) = 1$, the Ricci curvature will converge to negative of the generalized Kähler-Einstein metric $-\omega_B$ in $C^0_{\text{loc}}(\omega(t))$. We will assume Kod$(X) = 1$ in the whole paper.
There are two technical difficulties restricting us to consider the case Kod(X) > 1. The first reason is that, when Kod(X) = 1, it can be shown that Ricci curvature is negative definite in the base × base direction. For detailed explanation, see the paragraph at the end of Corollary 1.4. The second reason comes from a computational obstacle of Laplacian of fibrewise average of the ricci potential u. See the proof of Propositions 2.4 for detailed explanation.

The convergence of $\text{tr}_\omega \omega_B - m$ is one of the most important ingredients of the paper.

**Propositions 1.1.** When Kod(X) = 1, for any compact set $K \subset X \setminus S$. There exists a positive decreasing function:

$$h(t) : [0, \infty) \to [0, \infty)$$

such that the following holds:

$$2|\text{tr}_\omega \omega_B - 1| \leq h(t) \text{ on } K \times [0, \infty)$$

$$h(t) \to 0 \text{ as } t \to \infty$$

$$0 \leq -h'(t) \leq \frac{1}{2} h(t)$$

**Proof.** It is shown by Jian in Proposition 3.2 of [5].

The requirement $0 \leq -h'(t) \leq \frac{1}{2} h(t)$ is equivalent to $(e^{\frac{1}{2}t} h(t))' > 0$. In particular, $h(t)^{-1} \leq e^{t/2}$ and $h(t)^{-1} \leq e^t h(t)$.

The following theorem is an improvement of Proposition 7.1 of [7], where Jian-Song showed that $P := g^{i\bar{\jmath}} g^{\alpha \beta} R_{i\bar{\jmath} \alpha \beta}$ where $i, j = 1, \ldots, n$ include both the base and fibre direction and $\alpha, \beta = 1, \ldots, n-1$ represent the fibre direction, is bounded. We can regard $2P$ as an upper bound of Ricci curvature in the fibre × fibre direction and the fibre × base direction if we consider $g(p, t) = I$ at a point.

The following theorem show $P$ converge to 0 locally away from singularities.

**Theorem 1.2.** Let $(X, \omega_0)$ be a compact n-dimensional Kähler manifold with semi-ample canonical line bundle $K_X$ such that Kod(X) = 1. Let $\omega(t)$ satisfy the Kähler-Ricci Flow equation (1.1) on $X$ and $h(t)$ be defined as in Propositions 1.1. Then for any compact set $K \subset X \setminus S$, there exists a constant $C = C(K)$ such that

$$P := g^{i\bar{\jmath}} g^{\alpha \beta} R_{i\bar{\jmath} \alpha \beta} \leq C(K) h(t)^{\frac{1}{2}}$$
The breakthrough comes from a sharper estimate of $|\nabla F u|^2$ where it is shown in Proposition 6.1 of [7] that $|\nabla F u|^2 \leq C(K)e^{-t}$ locally away from singularities. In Theorem 2.1 we will show a sharper estimate. The main breakthrough comes from computational techniques which consider as well the rate of convergence of $|\text{tr}_\omega \omega_B - 1|$.

For the reason why we have to consider $\text{Kod}(X) = 1$. See the proof of Propositions 2.4 for detailed explanation.

The proof of Theorem 1.2 will be shown in Section 3.

The idea of the following corollaries comes from the proof of Theorem 7.1 of [7].

**Corollary 1.3.** When $\text{Kod}(X) = 1$, for any $K \subset \subset X \setminus S$, there exists a positive decreasing function $h_1(t)$ such that on $K \times [0, \infty)$, we have

$$\|\nabla |\rho|_\omega(t) - 1\| \leq h_1(t)$$

$h_1(t) \to 0$ as $t \to \infty$

**Proof.** By Theorem 1.1 of [5] and Theorem 1.2, there exists a positive decreasing function $A(t)$ that tends to 0 as $t \to \infty$ such that on $K \times [0, \infty)$, we have:

$$P + |R + 1| \leq A(t)$$

Fix a point $(p, t)$. Consider a local coordinate where $\{z_\alpha\}_{\alpha=1}^{n-1}$ is the coordinate of fibre and $\{z_n\}$ is the coordinate of base. And at $(p, t)$, $g(p, t)$ is identity.

In such coordinate, we have:

$$P = \sum_{i=1}^{n} \sum_{a=1}^{n-1} |R_{ia}|^2$$

$$\|\nabla |\rho|_\omega(t) - 1\| = \|R_{nn}|^2 + \sum_{(i,j) \neq (n,n)} |R_{ij}|^2 - 1|$$

$$\leq \|R - \sum_{i \neq n} |R_{ii}|^2 - 1| + 2P$$

$$\leq |R^2 - 1| + 2|R \sum_{i \neq n} R_{ii}| + \sum_{i \neq n} |R_{ii}|^2 + 2P$$

$$\leq C|R + 1| + CP^2 + CP$$

$$\leq CA(t)^{\frac{3}{2}}$$
Corollary 1.4. When $\text{Kod}(X) = 1$, for any $K \subset X \setminus S$, there exists a positive decreasing function $h_2(t)$ such that on $K \times [0, \infty)$, we have

$$\|\text{Ric} + \omega_B\|_{\omega(t)}^2 \leq h_2(t)$$

$$h_2(t) \to 0 \text{ as } t \to \infty$$

Proof. By Theorem 1.1 of [5], Proposition 3.2 in [5] and Theorem 1.2, there exists a positive decreasing function $A(t)$ that tends to 0 as $t \to \infty$ such that on $K \times [0, \infty)$, we have:

$$P + |R + 1| + ||\omega_B||_{\omega(t)}^2 - 1 \leq A(t)$$

Choose coordinate as in Corollary 1.3 and let $n$ to be the base direction, we have:

$$\|\text{Ric} + \omega_B\|_{\omega(t)}^2 = (R_{\bar{n}n} + \omega_{Bn\bar{n}})^2 + \sum_{(i,j) \neq (n,n)} |R_{ij}|^2$$

$$\leq (R_{\bar{n}n} + 1 + \omega_{Bn\bar{n}} - 1)^2 + 2P$$

$$\leq 2(R_{\bar{n}n} + 1)^2 + 2(\omega_{Bn\bar{n}} - 1)^2 + 2P$$

$$\leq C|R + 1| \sum_{i \neq \bar{n}} |R_{ii}|^2 + 2(\omega_{Bn\bar{n}} - 1)^2 (\omega_{Bn\bar{n}} + 1)^2 + 2P$$

$$\leq C|R + 1|^2 + 2C(|R + 1| \sum_{i \neq \bar{n}} |R_{ii}| + C|\sum_{i \neq \bar{n}} |R_{ii}||^2$$

$$+ 2(||\omega_B||_{\omega(t)}^2 - 1)^2 + 2P$$

$$\leq A(t)$$

Notice that even if Theorem 1.2 is generalized to other $\text{Kod}(X)$ i.e. Ricci curvature along $\text{Fibre} \times \text{Fibre}$ and $\text{Fibre} \times \text{Base}$ direction tends to 0 in general $\text{Kod}(X)$, it is not sufficient to generalize Corollary 1.4. The reason is that, when $\text{Kod}(X) = 1$, we make use of the convergence of scalar curvature to -1 to conclude Ricci curvature is negative definite in the base $\times$ base direction. And this argument does not hold for $\text{Kod}(X) > 1$.

The outline of paper is as follows. In Section 2 we will show Theorem 2.1 which is required to show Theorem 1.2. In Section 3, the proof of Theorem 1.2 will be shown.

Acknowledgement: The author would like to thank his advisor Frederick Fong Tsz-Ho for his guide and support. The author would also like to thank Wangjian Jian for his useful discussion.
2 Gradient estimate

In this section, we will improve the gradient estimate shown in Propositions 6.1 of [7], where Jian-Song showed that $|\nabla^F u|^2 \leq C(K)e^{-t}$ locally.

**Theorem 2.1.** Assume $\text{Kod}(X) = 1$, for any $K \subset \subset X \setminus S$, let $h(t)$ be defined in Propositions [17]. There exists a constant $C(K)$ depending compact set $K$ away from singularities such that on $K \times [0, \infty)$, we have:

$$|\nabla^F u|^2 \leq C(K)e^{-t}h(t)$$

Before proving the theorem, we have to introduce certain notation and compute certain quantity first.

2.1 Notation

Recall in [17], $u$ is defined as

$$u := \frac{\partial \varphi}{\partial t} + \varphi - v$$

Please note that there are slight difference in definition of ”$u$” in this paper and [7]. The ”$u$” defined in [7], denoted by $f$ here is defined as below:

$$f := \frac{\partial \varphi}{\partial t} + \varphi$$

Define $\bar{u}$ to be fibrewise average of $u$:

$$\bar{u} := \frac{\int_{X_y} u \omega^{n-1}}{\int_{X_y} \omega^{n-1}}$$

Define

$$a(t) := \frac{1}{\int_{X_y} \omega^{n-1}}$$

Define the fibrewise gradient:

$$\nabla^F u := (\nabla|_{X_y}) (u|_{X_y})$$

The norm of fibrewise gradient of $u$ is given by the following:

$$|\nabla^F u|^2 = (g_F)^{\alpha\beta} u_\alpha u_\beta$$

We also use the following notation:

$$|\nabla \nabla^F u|^2 := g^{ij}g_F^{\alpha\beta}(\nabla_i \nabla_\alpha u)(\nabla_j \nabla_\beta u)$$

$$|\nabla \nabla^F u|^2 := g^{ij}g_F^{\alpha\beta}(\nabla_j \nabla_\alpha u)(\nabla_i \nabla_\beta u)$$

$$\langle \nabla^F \text{tr}_\omega \omega_B \cdot \nabla^F u \rangle = g_F^{\alpha\beta} \nabla_\alpha \text{tr}_\omega \omega_B \cdot \nabla_\beta u$$
2.2 Preliminary

Propositions 2.2. For any $K \subset X \setminus S$, on $K \times [0, \infty)$, we have:

$$\left( \frac{\partial}{\partial t} - \Delta \right) u = \text{tr}_\omega \omega_B - 1$$

$$\left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_\omega \omega_B \leq C(K)$$

Proof. They are shown in Lemma 3.2 and Lemma 3.3 in \cite{11} by Tosatti-Weinkove-Yang. The original proof comes from Schwarz Lemma computation from Theorem 4.3 of \cite{9} by Song-Tian.

Propositions 2.3. For any $K \subset X \setminus S$, on $K \times [0, \infty)$, we have:

$$|u - \bar{u}| + |\nabla^F u|^2 \leq C(K)e^{-t}$$

Proof. Let $\bar{f}$ denote fiberwise average of $f$ as in the definition of $\bar{u}$ in (2.2). By Proposition 6.1 of \cite{7} by Jian-Song, we have:

$$|f - \bar{f}| + |\nabla^F f|^2 \leq C(K)e^{-t}$$

Note that $f - u = v$ and $v$ defined in (1.3) is a function depends on bases only. Note $|\nabla^F v| = 0$ and the fiberwise average $\bar{v}$ of $v$ is equal to $v$. So we have

$$|u - \bar{u}| + |\nabla^F u|^2 \leq C(K)e^{-t}$$

Propositions 2.4. For any $K \subset X \setminus S$, on $K \times [0, \infty)$, we have:

$$\left( \frac{\partial}{\partial t} - \Delta \right) \bar{u} = a(t) \int_{X_y} (\text{tr}_\omega \omega_B - 1) \omega^{n-1} + a(t) \int |\nabla^F u|^2 \omega^{n-1}$$

Proof. It is an analogue of Lemma 6.2 in \cite{7} by Jian-Song. Note that we require Kod$(X) = 1$ here. We have to make use of Lemma 6.1 in \cite{7} which assumed Kod$(X) = 1$ to show

$$\Delta \bar{u} = \frac{\int_{X_y} \Delta u \omega^{n-1}}{\int_{X_y} \omega^{n-1}}$$

If this obstacle can be overcome, Theorem 1.2 can be generalized to other Kod$(X)$.
Propositions 2.5. For any $K \subset \subset X \setminus S$, on $K \times [0, \infty)$, we have:

$$\left( \frac{\partial}{\partial t} - \Delta \right) |\nabla^F u|^2 = |\nabla^F u|^2 - |\nabla\nabla^F u|^2 - |\nabla F u|^2 + 2 \text{Re}(\nabla^F \tromega B \cdot \nabla^F u)$$

Proof. It is an analogue of Lemma 6.4 from [7] by Jian-Song.

The following allows us to consider cutoff function.

Lemma 2.6. Let $K \subset \subset X \setminus S$ be a compact set. Choose some open subset $U'$ such that $f(K) \subset \subset U' \subset \subset B \setminus S'$. Let $U = f^{-1}(U')$. Then there exists a smooth cutoff function $\rho$ such that $\rho$ is compactly supported on $U$, $0 \leq \rho \leq 1$ and for each Kähler-Ricci flow $\omega(t)$ there exists $1 \leq C$ depending on the initial metric $\omega_0$ and $U$ with

$$|\partial \rho|^2 \omega(t) + |\Delta \omega(t) \rho| \leq C \text{ on } U \times [0, \infty)$$

Proof. It is taken from Lemma 4.2 in [5] by Jian.

2.3 Preparation for Theorem 2.1

Propositions 2.7. For any $K \subset \subset X \setminus S$, let $h(t)$ be defined as in Proposition 1.1, on $K \times [0, \infty)$, we have:

$$|\left( \frac{\partial}{\partial t} - \Delta \right) (u - \bar{u})| \leq C(K)h(t)$$

Proof. Using formulas for evolution of $u$ and $\bar{u}$ in Propositions 2.2 and Propositions 2.8, we have:

$$|\left( \frac{\partial}{\partial t} - \Delta \right) (u - \bar{u})| = |\tromega B - 1 - a(t) \int_{X_y} (\tromega B - 1) \omega^{n-1} - a(t) \int |\nabla^F u|^2 \omega^{n-1}|$$

$$\leq h(t) + h(t)a(t) \int_{X_y} \omega^{n-1} + Ce^{-t}a(t) \int_{X_y} \omega^{n-1}$$

$$\leq Ch(t)$$

In the second last line, we used estimate of $|\nabla^F u|^2$ in Propositions 2.4.

Propositions 2.8. For any $K \subset \subset X \setminus S$, let $h(t)$ be defined as in Proposition 1.1, on $K \times [0, \infty)$, we have:

$$\left( \frac{\partial}{\partial t} - \Delta \right) (e^t h(t)^{-1}|\nabla^F u|^2) \leq Ce^{2t}|\nabla^F u|^2 + h(t)^{-2}|\nabla^F \tromega B|^2$$
Proof. Using formula of evolution of $|\nabla^F u|^2$ from Propositions 2.5, we have:

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \left( e^t h(t)^{-1} |\nabla^F u|^2 \right) \\
= (e^t h(t)^{-1} + e^t \left( \frac{h'(t)}{h(t)} \right) h(t)^{-1}) |\nabla^F u|^2 \\
+ e^t h(t)^{-1} \left( |\nabla^F u|^2 - |\nabla^F u|^2 - |\nabla^F u|^2 + 2 \text{Re}(\nabla^F \omega_B \cdot \nabla^F u) \right) \\
\leq C e^t h(t)^{-1} |\nabla^F u|^2 \\
+ e^t h(t)^{-1} \left( |\nabla^F u|^2 - 0 - 0 \right) + 2 e^t h(t)^{-1} \text{Re}(\nabla^F \omega_B \cdot \nabla^F u) \\
\leq C e^t h(t)^{-1} |\nabla^F u|^2 + 2 h(t)^{-1} |\nabla^F \omega_B| e^t |\nabla^F u| \\
\leq C e^{2t} |\nabla^F u|^2 + h(t)^{-2} |\nabla^F \omega_B|^2 \\
\]

In the last step, we use Cauchy inequality and $e^{2t} |\nabla^F u|^2$ absorb the term $C e^t h(t)^{-1} |\nabla^F u|^2$.

Proposition 2.9. For any $K \subset \subset X \setminus S$, let $h(t)$ be defined as in Proposition 1.1. Choosing large enough $C(K)$ such that $|u - \bar{u}| \leq \frac{1}{2} C(K) e^{-t}$ by Propositions 2.5 and applying Propositions 2.7 for estimate of $u - \bar{u}$, on $K \times [0, \infty)$, we have:

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \log \left( \frac{C e^{-t}}{C e^{-t} - (u - \bar{u})} \right) \leq e^t h(t) - C e^{2t} |\nabla^F u|^2 + C \\
\]

Proof.

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \log \left( \frac{C e^{-t}}{C e^{-t} - (u - \bar{u})} \right) \\
= \left( \frac{\partial}{\partial t} - \Delta \right) \left( C e^{-t} - (u - \bar{u}) \right) \frac{1}{(C e^{-t} - (u - \bar{u}))} - \frac{|\nabla(u - \bar{u})|^2}{(C e^{-t} - (u - \bar{u}))^2} - 1 \\
= \left( -C e^{-t} + (\frac{\partial}{\partial t} - \Delta) (u - \bar{u}) \right) \frac{1}{C e^{-t} - (u - \bar{u})} - \frac{|\nabla(u - \bar{u})|^2}{(C e^{-t} - (u - \bar{u}))^2} - 1 \\
\leq \frac{e^{-t} + h(t)}{\frac{1}{2} e^{-t}} - \frac{|\nabla^F u|^2}{4 C e^{-2t}} + 0 \\
\leq e^t h(t) - C e^{2t} |\nabla^F u|^2 + C \\
\]

Note that, by our choice of $C$, we have

\[
\log \frac{1}{2} \leq \log \left( \frac{C e^{-t}}{C e^{-t} - (u - \bar{u})} \right) \leq \log 2
\]

11
Propositions 2.10. For any $K \subset X \setminus S$, let $h(t)$ be defined as in Proposition 1.4. Using upper bound of evolution of $\text{tr}_\omega \omega_B$ in Propositions 2.8, on $K \times [0, \infty)$, we have:

$$\left( \frac{\partial}{\partial t} - \Delta \right) (h(t)^{-2}(\text{tr}_\omega \omega_B - 1 + h(t))^2) \leq C + Ch(t)^{-1} - 2h(t)^{-2}|\nabla F \text{tr}_\omega \omega_B|^2$$

Proof. $$\left( \frac{\partial}{\partial t} - \Delta \right) (h(t)^{-2}(\text{tr}_\omega \omega_B - 1 + h(t))^2)$$

$$= 2h(t)^{-2}\left( -\frac{h(t)}{h(t)}(\text{tr}_\omega \omega_B - 1 + h(t))^2 \right)$$

$$+ h(t)^{-2}\left( 2(\text{tr}_\omega \omega_B - 1 + h(t))\frac{\partial}{\partial t} - \Delta)(\text{tr}_\omega \omega_B + h(t)) - 2|\nabla \text{tr}_\omega \omega_B|^2 \right)$$

$$\leq Ch(t)^{-2}(Ch(t))^2$$

$$+ h(t)^{-2}(2(Ch(t))(C) - 2|\nabla \text{tr}_\omega \omega_B|^2)$$

$$= C + Ch(t)^{-1} - 2h(t)^{-2}|\nabla \text{tr}_\omega \omega_B|^2$$

$$\leq C + Ch(t)^{-1} - 2h(t)^{-2}|\nabla F \text{tr}_\omega \omega_B|^2$$

Note that we only have an upper bound of evolution of $\text{tr}_\omega \omega_B$ and we do not have a lower bound for it. Note that in the first inequality, $\text{tr}_\omega \omega_B - 1 + h(t)$ is a non-negative function and we can apply upper bound of evolution of $\text{tr}_\omega \omega_B$. \qed

Note also, by Propositions 1.1, $h(t)^{-2}(\text{tr}_\omega \omega_B - 1 + h(t))^2$ is bounded.

2.4 Proof of Theorem 2.1

Now we are ready to prove Theorem 2.1

Proof. Let $K \subset X \setminus S$. Define

$$Q := e^t h(t)^{-1}|\nabla F u|^2 + C_2 \log \left( \frac{C e^{-t}}{C e^{-t} - (u - \bar{u})} \right) + C_1 h(t)^{-2}(\text{tr}_\omega \omega_B - 1 + h(t))^2$$

By Proposition 2.8, Proposition 2.9 and Proposition 2.10, and choosing large $C_1$ and $C_2$,

$$\left( \frac{\partial}{\partial t} - \Delta \right) Q \leq -e^t|\nabla F u|^2 + C + h(t)^{-1} + e^t h(t)$$

$$\leq -e^t|\nabla F u|^2 + e^t h(t)$$

In the last step, $e^t h(t)$ absorb $h(t)^{-1}$ as explained at the end of Propositions 1.1.
By Lemma 2.6, let $\rho$ be cutoff function s.t. $|\nabla \rho| \leq C$, $|\Delta \rho| \leq C$ and $\rho \equiv 1$ on $K$, $\text{supp } \rho \subset \subset U$.

\[
(\frac{\partial}{\partial t} - \Delta) (\rho^4 Q) = \rho^4 (\frac{\partial}{\partial t} - \Delta) Q - Q \Delta \rho^4 - 2 \text{Re}(\nabla Q \cdot \bar{\nabla} \rho^4) \\
\leq \rho^4 (-e^{2t}|\nabla F u|^2 + e^t h(t)) - Q \Delta \rho^4 - 2 \text{Re}(\nabla Q \cdot \bar{\nabla} \rho^4)
\]

\[-Q \Delta \rho^4 \leq C \rho^2 Q \leq Ch(t)^{-1} \leq Ce^t h(t)\]

\[-2 \text{Re}(\nabla Q \cdot \bar{\nabla} \rho^4) = -8\rho^{-1} \text{Re}(\nabla (\rho^4 Q) \cdot \bar{\nabla} \rho) + 32\rho^2 Q|\nabla \rho| \leq -8\rho^{-1} \text{Re}(\nabla (\rho^4 Q) \cdot \bar{\nabla} \rho) + Ce^t h(t)\]

Combining above, we have:

\[
(\frac{\partial}{\partial t} - \Delta) (\rho^4 Q) \leq -e^{2t}\rho^4 |\nabla F u|^2 + e^t h(t) - 8\rho^{-1} \text{Re}(\nabla (\rho^4 Q) \cdot \bar{\nabla} \rho)
\]

At max point, we have:

\[
\rho^4 |\nabla F u|^2 \leq Ce^t h(t)
\]

As explained at the end of Proposition 2.9 and Proposition 2.10 the two terms below is bounded:

\[
C_2 \log \left( \frac{Ce^{-t}}{Ce^{-t} - (u - \bar{u})} \right) + C_1 h(t)^{-2}(\text{tr} \omega \omega_B - 1 + h(t))^2
\]

we have finished the proof. \(\square\)

## 3 Ricci curvature estimate

### 3.1 Preliminary

**Propositions 3.1.** For any $K \subset X \setminus S$, let $P$ be defined as in Theorem 1.2 on $K \times [0, \infty)$, we have:

\[
P \leq C(K)
\]

**Proof.** It is taken from Proposition 7.1 of [7] by Jian-Song. \(\square\)
Proposition 3.2. For any $K \subset \subset X \setminus S$, let $P$ be defined as in Theorem 1.2 on $K \times [0, \infty)$, we have:
\[
\left( \frac{\partial}{\partial t} - \Delta \right) P \leq C(K)e^t
\]

Proof. By Lemma 7.2 of \cite{7} by Jian-Song,
\[
\left( \frac{\partial}{\partial t} - \Delta \right) P \leq C(K)e^t + C(K)e^t P
\]

By Propositions 3.1 we have show the Propositions. \hfill \square

3.2 Preparation for Proof of Theorem 1.2

Proposition 3.3. For any $K \subset \subset X \setminus S$, let $h(t)$ be defined as in Proposition 1.1 and $P$ be defined as in Theorem 1.2 on $K \times [0, \infty)$, we have:
\[
\left( \frac{\partial}{\partial t} - \Delta \right) (h(t)^{-1} P^2) \leq Ch(t)^{-1}e^t P + Ch(t)^{-1}
\]

Proof.
\[
\left( \frac{\partial}{\partial t} - \Delta \right) (h(t)^{-1} P^2) = \left( -\frac{h'(t)}{h(t)} \right) h(t)^{-1} P^2 + h(t)^{-1} \left( \frac{\partial}{\partial t} - \Delta \right) P^2 \\
\leq Ch(t)^{-1} + h(t)^{-1}(2P \left( \frac{\partial}{\partial t} - \Delta \right) P - 2|\nabla P|^2) \\
\leq Ch(t)^{-1} + Ch(t)^{-1}e^t P
\]

In the first inequality, we have used property of $h(t)$ from its definition in Propositions 1.1. In the last step, we have used Propositions 3.2 for the estimate of evolution of $P$ and Propositions 3.1 for an upper bound for $P$. \hfill \square

The idea of considering $P^2$ comes from Proof of Theorem 1.6 in \cite{6} by Jian-Shi.

Propositions 3.4. For any $K \subset \subset X \setminus S$, let $h(t)$ be defined as in Proposition 1.1 and $P$ be defined as in Theorem 1.2 on $K \times [0, \infty)$, we have:
\[
\left( \frac{\partial}{\partial t} - \Delta \right) (h(t)^{-1} e^t |\nabla^F u|^2) \leq -Ch(t)^{-1}e^t P + C e^t h(t) + h(t)^{-2} |F_{\omega B}|^2
\]
Proof. Using formula of evolution of $|\nabla F u|^2$ from Propositions 2.5, we have:

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \left( h(t)^{-1} e^t |\nabla F u|^2 \right)
\]
\[
= \left( h(t)^{-1} e^t + e^t \left( \frac{h'(t)}{h(t)} \right) h(t)^{-1} \right) |\nabla F u|^2
\]
\[
+ h(t)^{-1} e^t \left( |\nabla F u|^2 - |\nabla \nabla F u|^2 - |\nabla F u|^2 + 2 \text{Re}(\nabla F \text{tr}_\omega \omega_B \cdot \nabla F u) \right)
\]
\[
\leq Ch(t)^{-1} e^t |\nabla F u|^2
\]
\[
+ h(t)^{-1} e^t \left( |\nabla F u|^2 - 0 - |\nabla \nabla F u|^2 \right) + 2(h(t)^{-1} |\nabla F \text{tr}_\omega \omega_B|)(e^t |\nabla F u|)
\]
\[
\leq -Ch(t)^{-1} e^t |\nabla F u|^2 + Ce^t |\nabla F u|^2 + h(t)^{-2} |\nabla F \text{tr}_\omega \omega_B|^2
\]
\[
\leq -Ch(t)^{-1} e^t |\nabla F u|^2 + Ce^t h(t) + h(t)^{-2} |\nabla F \text{tr}_\omega \omega_B|^2
\]
\[
= -Ch(t)^{-1} e^t P + Ce^t h(t) + h(t)^{-2} |\nabla F \text{tr}_\omega \omega_B|^2
\]

In the second last line, we applied Theorem 2.1 for the estimate of $|\nabla F u|^2$.

In the last step, by direct computation, we have

\[
\partial \bar{\partial} u = -\text{Ric} - \omega_B
\]

Also, note $\omega_B$ vanish on fibre $\times$ fibre and fibre $\times$ base direction. We have:

\[
|\nabla \nabla F u|^2 = P
\]

Note by Theorem 2.1 $h(t)^{-1} e^t |\nabla F u|^2$ is bounded.

3.3 Proof of Theorem 1.2

Proof. Let $K \subset \subset X \setminus S$. Define:

\[
Q := h(t)^{-1} P^2 + C_1 h(t)^{-1} e^t |\nabla F u|^2 + C_2 h(t)^{-2} (\text{tr}_\omega \omega_B - 1 + h(t))^2
\]

By Proposition 2.10, Proposition 3.3, Proposition 3.4 and choosing large enough $C_1$ and $C_2$, we have:

\[
\left( \frac{\partial}{\partial t} - \Delta \right) Q \leq -h(t)^{-1} e^t P + Ce^t h(t) + h(t)^{-1}
\]
\[
\leq -h(t)^{-1} e^t P + Ce^t h(t)
\]
By Lemma 2.6 let \( \rho \) be cutoff function s.t. \( |\nabla \rho| \leq C, |\Delta \rho| \leq C \) and \( \rho \equiv 1 \) on \( K \), \( \text{supp} \rho \subset \subset U. \)

\[
(\frac{\partial}{\partial t} - \Delta) (\rho^4 Q) = \rho^4 (\frac{\partial}{\partial t} - \Delta) Q - Q \Delta \rho^4 - 2 \Re(\nabla Q \cdot \bar{\nabla} \rho^4) \\
\leq \rho^4 (-h(t)^{-1} e^t P + e^t h(t)) \\
- Q \Delta \rho^4 - 2 \Re(\nabla Q \cdot \bar{\nabla} \rho^4)
\]

\[-Q \Delta \rho^4 \leq C \rho^2 Q \leq C h(t)^{-1} \leq C e^t h(t)\]

\[-2 \Re(\nabla Q \cdot \bar{\nabla} \rho^4) = -8 \rho^{-1} \Re(\nabla (\rho^4 Q) \cdot \bar{\nabla} \rho) + 32 \rho^2 Q |\nabla \rho|^2 \\
\leq -8 \rho^{-1} \Re(\nabla (\rho^4 Q) \cdot \bar{\nabla} \rho) + C e^t h(t)\]

Combining above, we have:

\[
(\frac{\partial}{\partial t} - \Delta) (\rho^4 Q) \leq -\rho^4 h(t)^{-1} e^t P + e^t h(t) - 8 \rho^{-1} \Re(\nabla (\rho^4 Q) \cdot \bar{\nabla} \rho)
\]

At max point, we have

\[
\rho^3 P \leq h(t)^2
\]

In particular,

\[
\rho^4 P^2 \leq C \rho^4 P \leq C h(t)^2 \leq h(t)
\]

Since other terms in \( Q \) are bounded (explained in Proposition 2.10 and Proposition 3.4), we have proved the theorem.

\[\square\]

**References**

[1] Cao, H.-D. *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. 81 (1985), no. 2, 359–372

[2] Fong, F.T.-H., Lee, M.C. *Higher-order estimates of long-time solutions to the Kähler-Ricci flow*, J. Funct. Anal. 281 (2021), no. 11, Paper No. 109235

[3] Fong, F.T.-H., Zhang, Y. *Local curvature estimates of long-time solutions to the Kähler-Ricci flow*, Adv. Math. 375. (2020) 107416. MR 4170232

[4] Gill, M. *Collapsing of products along the Kähler-Ricci flow*, Trans. Amer. Math. Soc. 366 (2014), no. 7, 3907–3924.
[5] Jian, W. *Convergence of scalar curvature of Kähler-Ricci flow on manifolds of positive Kodaira dimension*, Adv. Math. 371. (2020) 107253. MR 4108223

[6] Jian, W., Shi, Y. *A "boundedness implies convergence" principle and its applications to collapsing estimates in Kähler geometry* arXiv.1909.05521, accepted by Nonlinear Analysis

[7] Jian, W., Song J. *Diameter and ricci curvature estimates for long-time solutions of the Kähler-ricci flow*. arxiv.2101.04277

[8] Song, J., Tian, G. *Canonical measures and Kähler-Ricci flow*, J. Amer. Math. Soc. 25 (2012), no. 2, 303–353.

[9] Song, J., Tian, G. *The Kähler-Ricci flow on surfaces of positive Kodaira dimension*, Invent. Math. 170 (2007), no. 3, 609–653.

[10] Tosatti, V. *Adiabatic limits of Ricci-flat Kähler metrics* J. Differential Geom. 84 (2010), no.2, 427–453.

[11] Tosatti, V., Weinkove, B. and Yang, X. *The Kähler-Ricci flow, Ricci-flat metrics and collapsing limit*, Amer. J. Math. 140 (2018), no. 3, 653–698

[12] Tsuji, H. *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. 281 (1988), 123–133