A Combinatorial Model of Interference in Frequency Hopping Schemes

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January 29, 2022

Abstract

In a frequency hopping (FH) scheme users communicate simultaneously using FH sequences defined on the same set of frequency channels. An FH sequence specifies the frequency channel to be used as communication progresses. Much of the research on the performance of FH schemes is based on either pairwise mutual interference or adversarial interference but not both. In this paper, we evaluate the performance of an FH scheme with respect to both group-wise mutual interference and adversarial interference (jamming), bearing in mind that more than two users may be transmitting simultaneously in the presence of a jammer. We establish a correspondence between a cover-free code and an FH scheme. This gives a lower bound on the transmission capacity. Furthermore, we specify a jammer model and consider what additional properties a cover-free code should have to resist the jammer. We demonstrate that a purely combinatorial approach is inadequate against such a jammer, but that with the use of pseudorandomness, we can have a system that has high throughput as well as security against jamming.

1 Introduction

Frequency hopping is a modulation technique that employs frequency hopping (FH) sequences in spread spectrum transmission. This technology was first introduced to allow multiple users to be co-located within the same spectrum. It also mitigates against interference from unauthorized users, on the assumption that the unauthorised users have no knowledge of the FH sequences being used [19]. Frequency hopping is widely used in signal transmission such as Wi-Fi, Bluetooth and ultrawideband (UWB) communications [12, 7, 21]. FH sequences specify the frequency channels on which a transmitter/receiver sends/receives data as transmission progresses. The main requirement for the transmitter-receiver pair to communicate is that they need to be on the same frequency channel at the same time, that is, a communicating pair of users need to share an FH sequence. When a number of users employ FH sequences which are defined on the

∗Mwawi M. Nyirendda was funded by the Schlumberger foundation, Faculty for the future scholarship.
same set of frequency channels, they form a *frequency hopping multiple access* (FHMA) system. We will consider properties of FH sequences for FHMA systems when used in the presence of adversarial interference.

We will introduce notation and some background as well as related work before describing our contribution.

## 2 Background

### 2.1 Frequency hopping schemes

Let $F = \{f_0, f_1, \ldots, f_{m-1}\}$ be a set of $m$ frequency channels available to an FHMA system. We call $F$ a frequency library.

**Definition 2.1.** A frequency hopping (FH) sequence of length $v$ over $F$ is a sequence $X = (x_t)_{t=0}^{v-1}$, $x_t \in F$, $t = 0, \ldots, v - 1$.

We write $X = (x_t)$ if there is no ambiguity.

**Definition 2.2.** A $(v, k, m)$-frequency hopping scheme ($(v, k, m)$-FHS), is a set $S = \{X_i : 0 \leq i \leq k - 1\}$ of size $k$ where $X_i$ is an FH sequence of length $v$ over a frequency library $F$ of size $m$.

Let $S$ be a $(v, k, m)$-FHS. A transmitter and a receiver share an FH sequence $X = (x_t) \in S$. The channel to be used for transmission/reception at each time slot $t$ is given by $x_t$.

### 2.2 Pairwise mutual interference

The use of the same frequency channel at the same time by two FH sequences (or more) causes *interference*. *Pairwise mutual interference* is where two FH sequences in an FH scheme interfere with each other. It is measured by Hamming correlation. Formally, let $S$ be a $(v, k, m)$-FHS and let $X, Y \in S$, $X = (x_0, x_1, \ldots, x_{v-1})$ and $Y = (y_0, y_1, \ldots, y_{v-1})$. The Hamming correlation $H_{XY}$ at relative time delay $\tau$ between $X$ and $Y$ is

$$H_{XY}(\tau) = \sum_{i=0}^{v-1} h(x_i, y_{i+\tau}), \quad 0 \leq \tau < v, \quad (1)$$

where

$$h(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

The operations on indices are performed modulo $v$. When $X = Y$ we write $H_X(\tau)$ for $H_{XX}(\tau)$ and this is referred to as the *Hamming auto-correlation* of $X$. If $X \neq Y$ we call $H_{XY}(\tau)$ the *Hamming cross-correlation*.
We define the maximum out-of-phase Hamming auto-correlation of an FH sequence \( X \in S \) as
\[
H(X) = \max_{1 \leq \tau < v} \{ H_{XX}(\tau) \},
\]
and the maximum Hamming cross-correlation between any two distinct FH sequences \( X \) and \( Y \) in \( S \) as
\[
H(X, Y) = \max_{0 \leq \tau < v} \{ H_{XY}(\tau) \}.
\]
Further, we define
\[
M(X, Y) = \max \{ H(X), H(Y), H(X, Y) \}.
\]

Lempel and Greenberger [13] developed the following bound on the maximum out-of-phase auto-correlation of a sequence:

**Lemma 2.3.** [13, Lemma 4] For every sequence \( X = (x_t) \) of length \( v \) over \( \mathbb{F} \), \( |\mathbb{F}| = m \),
\[
H(X) \geq \frac{(v-r)(v+r-m)}{m(v-1)}
\]
(2)
where \( r \equiv v \mod m \).

For a \((v, k, m)\)-FHS, \( S \), the maximum periodic Hamming auto-correlation of \( S \), \( H_a(S) \), and the maximum periodic Hamming cross-correlation of \( S \), \( H_c(S) \), are defined as
\[
H_a(S) = \max \{ H(X) | X \in S \},
\]
\[
H_c(S) = \max \{ H(X, Y) | X, Y \in S, X \neq Y \}.
\]
The maximum Hamming correlation of \( S \) is defined as
\[
H_m(S) = \max \{ H_a(S), H_c(S) \}.
\]
(3)
Peng and Fan [17] gave the following bound for the maximum Hamming correlation of a set of sequences:

**Lemma 2.4.** [17, Corollary 1] Let \( S \) be a \((v, k, m)\)-FHS. Let \( I = \lfloor \frac{vk}{m} \rfloor \). Then
\[
H_m(S) \geq \frac{2Iv(k-1)lm}{(vk-1)k}.
\]
(4)

An FH sequence \( X \) is said to be optimal in the Lempel-Greenberger bound if the bound (2) is met, and a \((v, k, m)\)-FHS is said to be an optimal FH scheme in the Peng-Fan bound if the bound (4) is met.

There are many FH sequence constructions that are optimal in the Lempel-Greenberger bound or the Peng-Fan bound. We list a few here that use techniques from algebra, combinatorial designs as well as codes. Lempel and Greenberger [13] construct optimal FH sequences from algebraic transforms of m-sequences. Fuji-Hara et al. [8] provide constructions of optimal sequences using affine geometries, cyclic designs and difference families. Using cyclotomy as well as quadratic residues, Chung and Yang [4] obtain optimal FH...
sequences with new parameters. Ding et al. [5] employ Reed-Solomon codes to obtain sets of FH sequences optimal in the Peng-Fan bound. Using cyclotomy and the Chinese remainder theorem, Ren et al. [20] also obtain sets of FH sequences meeting the Peng-Fan bound. Ren’s constructions are a generalisation of the constructions in [5] and [25] which also use cyclotomy over finite fields. Ding et al. [5] also use the trace function to construct sets of FH sequences that are optimal in the Peng-Fan bound. The FH sequences constructions given above are all optimal in the sense of meeting either the Lempel-Greenberger bound or the Peng-Fan bound. However, both bounds are based on pairwise Hamming correlation. We will explain in Section 2.4 the insufficiency of using pairwise correlation and propose using Hamming group correlation to measure group-wise mutual interference.

2.3 Adversarial interference

Interference originating from unauthorised entities where signals are deliberately transmitted to interfere with legitimate transmission is called adversarial interference or jamming. We discuss this in more detail in Section 3.2.

Now we describe the work of Bag et al. [2], Nyirenda et al. [15] and Emek and Wattenhofer [6] who focused on adversarial interference rather than mutual interference. What is common across these three constructions is the use of pseudorandomness. However, the capabilities of the jammers differ. We will elaborate a little on this.

Bag et al. [2] use mutually orthogonal Latin squares to obtain sets of FH sequences. These FH sequences achieve maximum transmission rate of 100% without adversarial interference. A transmitter and a receiver share a pair of secret pseudorandom numbers before the start of communication and uses them for the entire session. The jammer is assumed to be able to jam at most a certain number of the frequency channels. However, it was shown in [15] that a jammer only needs to eavesdrop on a single time slot to obtain the pair of secret shared pseudorandom numbers. This allows the jammer to derive the FH sequences and thus interfere with any FH sequence of its choice. This weakness was amended in [15], where it was proposed that there should be a new secret pseudorandom number for each time slot. We will review one of the schemes of [15] in Section 5 in light of our model described in Section 3.

Emek and Wattenhofer [6] construct FH sequences as a random walk on an expander graph. The authors consider a single pairwise communication where subsequent channels for transmission are included in the data transmitted. A jammer can eavesdrop and jam a certain fraction of the available frequency channels. In this paper two adversarial models were considered. In the first model, a jammer can only acquire information about the channel (but not the content) that was used in previous time slots after a certain number of time slots have lapsed, while in the second model it has knowledge of both channel and content. Knowing the transmitted messages is important since the content specifies the next channels. At any time slot, the FH sequence is guaranteed successful transmission with some minimum probability. However, it is not clear what happens if more than one pair (transmitter/receiver) of communication occurs simultaneously.
2.4 Our contributions

In this paper, we evaluate the performance of an FH scheme with respect to both group-wise mutual interference and adversarial interference. This framework was introduced in [15], motivated by the fact that more than two pairs of users may be transmitting simultaneously in the presence of a jammer. This means that measuring pairwise mutual interference, while giving some idea of the throughput of the scheme, is not adequate.

An overview of our contributions is as follows. We refine the system and jammer model which was introduced in [15]. We establish a correspondence between a cover-free code and an FH scheme. We show that when a cover-free code is considered as an FH scheme, a user can successfully transmit in at least a specified fraction of time in the presence of a given number of interfering FH sequences. We specify a jammer model for an FH scheme. Considering the resources and knowledge of a jammer, we look at how an FH scheme can mitigate against the jammer. We examine necessary and desirable additional properties of cover-free codes such that they can be used in the presence of adversarial interference: a cover-free code allows us to determine the throughput of an FH sequence in the presence of other interfering FH sequences but not in the presence of adversarial interference. Finally we discuss the limitations of cover-free codes against a jammer, and demonstrate that one effective way to improve resistance against a jammer is to use pseudorandomness.

The rest of the paper is organised as follows. In Section 3 we introduce the system and jammer model and the necessary notation. In Section 4 we introduce cover-free codes and show their equivalence to FH schemes. Section 4 also examines the additional properties of cover-free codes to defend against jamming. In Section 5 in particular we discuss how pseudorandomness can be used to strengthen cover-free codes to withstand a jammer so that the FH sequences can be used for longer periods of time. We conclude in Section 6.

3 System and jammer model

3.1 System model

Let \( S \) be a \((v, k, m)\)-FHS. A single FH sequence is used by a single transmitter/receiver pair to communicate in the presence of both mutual and adversarial interference. We first consider the case of mutual interference.

**Definition 3.1.** The Hamming group correlation \( G(X, \mathcal{U}) \) between an FH sequence \( X \in S \) and the FH sequences in \( \mathcal{U} \subseteq S \setminus \{X\}, |\mathcal{U}| = w, 1 \leq w < k \), is defined as the number of time slots in \( X \) that use the same frequency channels as the corresponding time slots of some FH sequence in \( \mathcal{U} \):

\[
G(X, \mathcal{U}) = |\{x_t \exists Y \in \mathcal{U} \text{ s.t. } x_t = y_t, t = 0, \ldots, v-1\}|. 
\] (5)

Note that the Hamming group correlation is a generalisation of Hamming correlation. When \(|\mathcal{U}| = 1\), say \( \mathcal{U} = \{Y\} \), then \( G(X, \mathcal{U}) = H_{XY}(0) \). Note also that in the following,
while the definitions and results work for \( w = 0 \), we will assume that \( w > 0 \) for the idea of group correlation to be meaningful.

**Lemma 3.2.** Let \( S, U, X \) be as above. Let \( H_m(S) \) be the maximum Hamming correlation of \( S \). Then \( G(X, U) \leq wH_m(S) \).

**Proof.** Let \( U = \{ Y_1, \ldots, Y_w \} \). Then \( G(X, U) \leq H_{XY_1}(0) + \cdots + H_{XY_w}(0) \). Since \( H_{XY_i}(0) \leq H_m(S) \) for \( 1 \leq i \leq w \), we have \( G(X, U) \leq wH_m(S) \). \(\square\)

The notion of Hamming group correlation is the complement of the group distance defined in \([10]\) when a \((v, k, m)\)-FHS is considered as a set of \( k \) codewords of length \( v \) over \( \mathcal{F} \). Hamming group correlation \( G(X, U) \) gives the number of time slots of an FH sequence \( X \) that are blocked by the FH sequences in the \( w \)-subset \( U \) of \( S \).

We define a *session* as being made up of \( v \) time slots, that is one full length of an FH sequence. We now define the throughput of an FH sequence.

**Definition 3.3.** Let \( S \) be a \((v, m, k)\)-FHS. Let \( X \in S \) and let \( U \subseteq S \setminus \{X\} \), \( |U| = w \), \( 1 \leq w < k \). Then the *\( w \)-throughput of an FH sequence* \( X \) with respect to \( U \) is the rate of successful transmission of \( X \) in a session in the presence of FH sequences in \( U \):

\[
\rho_w(X, U) = 1 - \frac{G(X, U)}{v}.
\] (6)

It is desirable that \( \rho_w(X, U) \) be large, which means an FH sequence transmits successfully in many time slots. Here we focus on the worst-case \( w \)-throughput of a \((v, k, m)\)-FHS, which gives the lowest possible throughput over a session in a communication system. We first look at the case without a jammer. The worst-case \( w \)-throughput in the presence of a jammer will be dealt with in Section 3.2.

**Definition 3.4.** Given a \((v, k, m)\)-FHS, \( S \), the *worst-case \( w \)-throughput of an FH sequence* \( X \in S \) is

\[
\hat{\rho}_w(X, S) = \min_{U \subseteq S \setminus \{X\}} \{ \rho_w(X, U) \}.
\] (7)

So, given an FH sequence \( X \in S \), (7) gives the minimum number of time slots in which the FH sequence can transmit data if some other \( w \) FH sequences in \( S \) are also in use.

Next we consider the worst-case \( w \)-throughput of a particular subset of a \((v, k, m)\)-FHS.

**Definition 3.5.** Let \( S \) be a \((v, k, m)\)-FHS and let \( V \subseteq S \), \( |V| = w + 1 \). The *worst-case \( w \)-throughput of* \( V \) is the minimum number of time slots in which any \( X \in V \) can transmit information if the other \( w \) FH sequences in \( V \setminus \{X\} \) are in use:

\[
\hat{\rho}_w(V) = \min_{X \in V} \{ \rho_w(X, V \setminus \{X\}) \}.
\] (8)

We conclude with the worst-case \( w \)-throughput of an FHS.
Definition 3.6. The worst-case $w$-throughput of a $(v, k, m)$-FHS, $S$, is the minimum of the values $\hat{\rho}_w(V)$ for each $(w + 1)$-set $V \subseteq S$:

$$\hat{\rho}_w(S) = \min_{V \subseteq S, |V| = w + 1} \{\hat{\rho}_w(V)\}.$$  (9)

We write $(v, k, m; \hat{\rho}_w(S))$-FHS for a $(v, k, m)$-FHS, $S$, with worst-case $w$-throughput $\hat{\rho}_w(S)$. For any $X \in S$ we can estimate the worst-case $w$-throughput of $X$:

Lemma 3.7. Let $S$ be a $(v, k, m)$-FHS and let $X \in S$. Then $\hat{\rho}_w(X, S) \geq \hat{\rho}_w(S)$.

Proof. Suppose $\hat{\rho}_w(X, S) < \hat{\rho}_w(S)$, that is, there is some $(w + 1)$-subset $V \subseteq S$ containing $X$ with $\rho_w(X, V \setminus \{X\}) < \hat{\rho}_w(S)$. This contradicts the definition of $\hat{\rho}_w(S)$. \qed

Note however, that $\hat{\rho}_w(V)$ and $\hat{\rho}_w(X, S)$ are not necessarily comparable. For example, suppose a $(w + 1)$-subset $V$ containing $X$ satisfies $\hat{\rho}_w(X, S) = \rho_w(X, V \setminus \{X\})$. This does not rule out the existence of an FH sequence $Y \in V$ such that $\rho_w(X, V \setminus \{Y\}) < \rho_w(X, V \setminus \{X\})$. Similarly, suppose an FH sequence $X$ satisfies $\hat{\rho}_w(V) = \rho_w(X, V \setminus \{X\})$. This does not rule out the existence of another $(w + 1)$-subset $V' \subseteq S$ with $\hat{\rho}_w(X, S) = \rho_w(X, V' \setminus \{X\})$.

We can also relate the worst-case $w$-throughput of an FH scheme with the maximum Hamming correlation of an FH scheme. Given a $(v, m, k)$-FHS, $S$ with maximum Hamming correlation $H_m(S)$, we can obtain a lower bound on the worst-case $w$-throughput of $S$:

Lemma 3.8.

$$\hat{\rho}_w(S) \geq 1 - \frac{w \cdot H_m(S)}{v}.$$  (10)

Proof. By definition, $\hat{\rho}_w(S) = \rho_w(X, V \setminus \{X\})$ for some $(w + 1)$-subset $V \subseteq S$ containing $X$. The inequality follows from Definition 3.1 and Lemma 3.2. \qed

It also follows from Lemma 3.7 that $\hat{\rho}(X, S) \geq 1 - \frac{w \cdot H_m(S)}{v}$.

3.2 Jammer model

We consider the presence of a jammer who sends noisy signals on frequency channels to block the signal transmissions of legitimate users. The jammer knows $F$, the $(v, k, m)$-FHS, $S$, and the number of FH sequences used in a session, $w + 1$ ($0 \leq w < k$). However, the jammer has no knowledge of the actual FH sequences to be used. Its strategy is to eavesdrop and jam. At each time slot it has enough resources to eavesdrop on $\theta_1 m$ channels, $0 \leq \theta_1 \leq 1$, and jam on $\theta_2 m$ channels, $0 \leq \theta_2 < 1$. (We assume that it cannot jam all the frequency channels at each time slot.) It can use the information it acquires while eavesdropping to make choices of which channels to jam. This jammer is described as a $(\theta_1, \theta_2)$-adaptive jammer. When a signal is jammed, legitimate users hear noise and acknowledge failure of transmission. So we treat a jamming signal as an erasure. The goal of the jammer is to reduce the worst-case $w$-throughput of $S$.  


We note here that we have allowed the possibility of \( w = 0 \) here, that is, only one FH sequence is used. While it is not meaningful to talk about correlation when there is only one sequence, it is still perfectly reasonable for a jammer to want to identify that one sequence.

We model a jammer’s channel selection for jamming as a set of FH sequences \( J = \{Y_i | i = 0, \ldots, \theta_2m - 1\} \), where \( Y_i \) is an FH sequence of length \( v \) over \( F \). We will call \( J \) the jamming sequences.

**Definition 3.9.** Let \( S \) be a \((v, k, m)\)-FHS over \( F \). Let \( X \in S \) and let \( U \subseteq S \setminus \{X\} \), \( |U| = w \), \( 0 \leq w < k \). Suppose \( J \) is a set of \( \theta_2m \) jamming sequences of length \( v \) over \( F \). Then the \((w, J)\)-throughput of \( X \) in the presence of both legitimate FH sequences in \( U \) and jamming sequences \( J \) is:

\[
\rho_{w,J}(X, U) = 1 - \frac{G(X, U \cup J)}{v}.
\]

The rest of the measures introduced in Section 3.1 can be easily modified to measure the worst case throughput of a \((v, k, m)\)-FHS in the presence of both mutual interference and jamming. These are summarised in Table 1.

| Worst-case \((w, J)\)-throughput of \( X \in S \) | \( \rho_{w,J}(X, S) = \min_{u \subseteq S \setminus \{X\}, |U| = w} \{\rho_{w,J}(X, U)\} \) |
|-----------------------------------------------|-------------------------------------------------------------------------|
| Worst-case \((w, J)\)-throughput of \( V \subseteq S \), \( |V| = w + 1 \) | \( \hat{\rho}_{w,J}(V) = \min_{X \in V} \{\rho_{w,J}(X, V \setminus \{X\})\} \) |
| Worst-case \((w, J)\)-throughput of \( S \) | \( \hat{\rho}_{w,J}(S) = \min_{V \subseteq S, |V| = w + 1} \{\hat{\rho}_{w,J}(V)\} \) |

Table 1: Performance measures of FH scheme in the presence of mutual interference and jamming.

Suppose \( X \in S \) is an FH sequence in use. Clearly if \( X \in J \) then the jammer would have succeeded in reducing the worst case throughput to 0. In Section 4.1 we will discuss the strategy the jammer might adopt to discover \( X \).

In the literature, jammers are classified according to their capabilities (broadband or narrowband) and their behaviour (constant, random or reactive) [16, 18, 24]. Our \((\theta_1, \theta_2)\)-adaptive jammer includes these jammers. For example, a broadband jammer means the jammer jams on contiguous set of channels and so we can consider this as \( \theta_2m > 1 \) and the jamming sequences \( Y_1, Y_2, \ldots \) have neighbouring channels in each time slot. If we consider a constant jammer who always jams on the same channel(s) \( f_{i_1}, \ldots, f_{i_{m\theta_2}} \), then we have a \((\theta_1, \theta_2)\)-adaptive jammer with \( J = \{Y_j = (f_{i_1}, f_{i_2}, \ldots, f_{i_j}) | j = 1, \ldots, m\theta_2\} \).

### 4 Cover-free codes and frequency hopping schemes

We now model a \((v, k, m)\)-FHS as a code and consider the desirable correlation properties in this light.
Recall that $\mathcal{F} = \{f_0, f_1, \ldots, f_{m-1}\}$ is the alphabet over which we defined the FH sequences. Let $S$ be a $(v, k, m)$-FHS defined over $\mathcal{F}$. We may treat each sequence of $S$ as a $v$-tuple in $\mathcal{F}^v$ and therefore treat $S$ as a code of length $v$ and size $k$ over $\mathcal{F}$. The Hamming distance $d_H(X, Y)$ between two codewords $X, Y$, is the number of places where the two codewords differ. The following is easy to verify:

**Lemma 4.1.** Let $S$ be a $(v, k, m)$-FHS defined over $\mathcal{F}$ considered as a code, and let $X, Y \in S$. Then $d_H(X, Y) = v - H_{XY}(0)$. The minimum distance $d_S$ of $S$ as a code satisfies

$$d_S = \min_{X, Y \in S} \{d_H(X, Y)\} \geq v - H(X, Y).$$

We will call $S$ a $(v, k, m; \hat{\rho}_w(S))$-FHS or a $(v, k, m; d_S)$-code (omitting $\hat{\rho}_w(S)$and $d_S$ if they are not known) depending on context. Clearly a $(v, k, m)$-FHS can always be treated as a $(v, k, m)$-code and vice versa.

The notion of cover-free codes has been used in [10, 11, 22] for blacklisting and traitor tracing schemes. In this paper we use the definition of Staddon et al [22].

**Definition 4.2.** [Cover-free codes][22] Let $S$ be a $(v, k, m)$-code. For any subset $S' \subseteq S$ and any $X \in \mathcal{F}^v$, define:

$$I(X, S') = \{i : x_i = y_i \text{ for some } Y \in S'\}. \tag{12}$$

Let $w \geq 1$ be an integer and let $0 \leq \alpha < 1$. Then $S$ is called $(w, \alpha)$-cover-free code, denoted $(w, \alpha)$-CFC, if $|I(Z, S')| < (1 - \alpha)v$ for any $S' \subseteq S$, $|S'| = w$ and any $Z \in S \setminus S'$.

We see that $|I(X, S')|$ gives us the number of “incidents” between $X$ and the codewords in $S'$. If we treat $S$ as a $(v, k, m)$-FHS, then for $S' \subseteq S$, $X \in S \setminus S'$, $|I(X, S')|$ is precisely $G(X, S')$ from Definition 3.1. Hence we have a direct correspondence between an FH scheme with a given Hamming group correlation and a cover-free code.

**Theorem 4.3.** Let $S$ be a $(v, k, m)$-code over $\mathcal{F}$, $|\mathcal{F}| = m$. Then $S$ is a $(w, \alpha)$-CFC if and only if $S$ is a $(v, k, m)$-FHS with worst-case $w$-throughput $\hat{\rho}_w(S)$ greater than $\alpha$.

**Proof.** Suppose $S$ is a $(w, \alpha)$-CFC. Then, $|I(X, S')| < (1 - \alpha)v$ for all $S' \subseteq S$, $|S'| = w$, $X \notin S'$. Since $G(X, S') = |I(X, S')|$, we have $\rho(X, S') = 1 - G(X, S')/v > \alpha$ for all $S' \subseteq S$, $|S| = w$, $X \notin S'$. Since $\hat{\rho}_w(S) = \rho(X, S')$ for some $X$, $S'$ by definition, we have $\hat{\rho}_w(S) > \alpha$.

Conversely, let $\hat{\rho}_w(S) > \alpha$. Suppose $S$ is not a $(w, \alpha)$-CFC, that is, there exist some $Z, S', S' \subseteq S$, $|S| = w$, $Z \notin S'$, such that $|I(X, S')| \geq (1 - \alpha)v$. This implies that $\rho(Z, S') = 1 - G(Z, S')/v \leq \alpha$. However, by assumption, $\hat{\rho}_w(S) > \alpha \geq \rho(Z, S')$, which contradicts Lemma 3.7.

Hence the problem of designing frequency hopping schemes with high throughput is equivalent to finding cover-free codes with low “incidents”. Note that a $(v, k, m; d)$-code is a $(1, d/v)$-CFC. We are interested in the cases where $w > 1$. It was shown in [22] that codes with large minimum distance are cover-free codes.
Theorem 4.4. [22] Theorem 4.3] Suppose that $\mathcal{S}$ is a $(v, k, m; d)$-code such that $d > v(1 - \frac{1}{w})$. Then $\mathcal{S}$ is a $(w, 1 - \frac{1}{w})$-CFC.

We have the following as a corollary of Theorems 4.3 and 4.4.

**Corollary 4.5.** A $(v, k, m; d)$-code with $d > v(1 - \frac{1}{w})$ gives a $(v, k, m)$-FHS with worst-case $w$-throughput greater than $1 - 1/w$.

**Example 4.6.** Let $v$ and $w$ be integers where $v \geq 2$ and $w \geq 2$. Let $m$ be a prime power such that $m \geq v$. Let $\mathcal{F}$ be the finite field of cardinality $m$ and let $\alpha_1, \alpha_2, \ldots, \alpha_v \in \mathcal{F}$ be distinct. Define a length $v$ Reed-Solomon code $\mathcal{S}$ over $\mathcal{F}$ by:

$$\mathcal{S} = \{ (f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_v)) : f \in \mathcal{F}[X] \text{ and } \deg f < \left\lceil \frac{v}{w} \right\rceil \}.$$  

Then $\mathcal{S}$ is a $(w, 1 - 1/w)$-CFC code, which is a $(v, m\left\lceil \frac{1}{w} \right\rceil, m)$-FHS with worst case $w$-throughput greater than $1 - 1/w$.

### 4.1 Jamming resistance properties for cover-free codes

Theorem 4.3 shows that cover-free codes give FHSs with a guaranteed minimum throughput without adversarial presence. We now consider how a jammer influences the throughput of such FHSs. We delve into further properties that cover-free codes should have to mitigate a $(\theta_1, \theta_2)$-adaptive jammer. For simplicity, we assume $\theta_1 m = \theta_2 m = 1$.

We introduce some terms and notation first.

Let $\mathcal{S}$ be a $(v, k, m; \rho_w(\mathcal{S}))$-FHS over $\mathcal{F} = \{ f_0, \ldots, f_{m-1} \}$. In any session of $v$ time slots, there are $w + 1$ FH sequences that are in use by legitimate users. We call these active sequences. Let $\mathcal{S}' \subseteq \mathcal{S}$ be the set of active FH sequences. At any time slot $t$, $0 \leq t \leq v - 1$, there are at most $w + 1$ frequency channels in use, which we call active channels. At any time slot $t$, let the multiset $\mathcal{F}_t = \{ x_0^t, \ldots, x_{w-1}^t \}$ denote all the channels that appear in all the FH sequences in $\mathcal{S}$ at that time. The $m$-tuple $M_t = (a_0, \ldots, a_{m-1})$ denotes the multiplicity of channels $f_0, \ldots, f_{m-1}$ at time slot $t$, where $a_i = |\{ j : x_j^t = f_i \}|$.

The aim of a jammer is to reduce the worst-case $w$-throughput of the FHS. To this end a jammer would aim to identify an active FH sequence as quickly as possible. It can then reduce the worst-case $w$-throughput to 0, or close to 0. We call the jammer’s search space the set of FH sequences $\mathcal{S}_0' \subseteq \mathcal{S}$ which the jammer needs to look at to identify an active FH sequence at time slot $t$. At the beginning of a session, $t = 0$, the search space is the whole FH scheme, $\mathcal{S}_0' = \mathcal{S}$. Let the number of time slots it takes a jammer to determine an active FH sequence be denoted $\gamma v$, $0 \leq \gamma \leq 1$. It is desirable that $\gamma$ be large. The aim of a $(v, k, m; \rho_w(\mathcal{S}))$-FHS is to make the jammer’s advantage not much better than a random guess.

We now explore the behaviour of a jammer given $\mathcal{S}$ and consider what properties $\mathcal{S}$ should have as a defence. A jammer can trivially reduce the worst-case $w$-throughput of $\mathcal{S}$ to 0 if $\mathcal{S}'' = \mathcal{S}$, since all sequences are active and the jammer can choose any $X \in \mathcal{S}$ to jam the entire session. As a mitigation strategy we have:
M1 Use only a fraction of $S$, that is $S' \subset S$.

If the jammer does not know which sequences or channels are being used then it will have to guess which frequency channel to eavesdrop on. At time $t = 0$, there are $k$ FH sequences assumed to be equally likely over $m$ frequency channels, and for each frequency channel $x_i$ there are $a_i$ FH sequences of that frequency channel. The probability that frequency channel $x_i$ is active is:

$$
Prob(x_i \text{ is active}) = 1 - \frac{(k - a_i)}{w + 1}/\left(\frac{k}{w + 1}\right).
$$

The probability in (13) is maximum for a frequency channel $x_i$ such that $a_i \geq a_j$ for all $i \neq j$. Therefore, if there exists such an $x_i$, then a jammer will choose it and will have a higher chance of jamming an active sequence.

Hence we propose that:

M2 A $(v, k, m)$-FHS should have the property that all frequency channels used at any time slot $t$ are uniformly distributed, that is we should have $a_0 = a_1 = \ldots = a_{m-1}$.

Recall that for a $(\frac{1}{m}, \frac{1}{m})$-adaptive jammer, what happens at time $t$ informs its next action at time $t + 1$, therefore we also propose that:

M3 For all FH sequences with frequency channel $x_i$ at time slot $t$, all frequency channels on the next time slot $t + 1$ should be uniformly distributed. This forces a jammer to guess randomly at any time slot.

An FHS satisfying properties M2 and M3 would mean that a jammer has no better chance of identifying an active channel at any time slot than randomly picking a channel. Hence we would like our FHS to possess these properties. Properties M2 and M3 describe an orthogonal array:

**Definition 4.7.** [9] A $k \times v$ array $A$ with entries from $\mathcal{F}$, $|\mathcal{F}| = m$, is said to be an orthogonal array with $m$ levels, strength $t'$, $1 \leq t' \leq v$, and index $\lambda$ if every $k \times t'$ subarray of $A$ contains each $t'$-tuple over $\mathcal{F}$ exactly $\lambda$ times as a row. We denoted this as $OA_{\lambda}(k, v, m, t')$.

Clearly an $OA_{\lambda}(k, v, m, t')$ is also an $OA_{\lambda m}(k, v, m, t' - 1)$ and $k = \lambda m^{t'}$.

Suppose we treat an $OA_{\lambda}(k, v, m, t')$ as a $(v, k, m)$-FHS, $S$. We want to know how long such an FHS can resist a jammer, that is, we want to know how many time slots a jammer would need to identify an active FH sequence.

Consider first the situation where only one sequence is active in $S$ and consider a $(\frac{1}{m}, \frac{1}{m})$-jammer in our FHS. At $t = 0$ (or indeed, at any time slot $t$), every channel in $\mathcal{F}$ appears $\lambda m^{t'-1}$ times, so the jammer has no better chance than randomly guessing a channel $x$ to eavesdrop on. If $x$ is active we will say that the jammer is lucky, otherwise we say that the jammer is unlucky.
Now, if the jammer is lucky, that means that the active sequence must be one of the $\lambda m^{t'-1}$ sequences with $x$ at $t = 0$. Thus the jammer would be able to reduce the search space for $t = 1$ to $\lambda m^{t'-1}$, by a factor of $m$. If the jammer is unlucky, then it will be able to remove from its search space the sequences with $x$ at time $t = 0$, and on $t = 1$ continues its search on the remaining sequences in $S$. The search space would be reduced from $\lambda m^{t'}$ to $\lambda m^{t'} - \lambda m^{t'-1} = \lambda m^{t'}(m - 1)/m$, a factor of $(m - 1)/m$. In summary,

**Lemma 4.8.** Let $S$ be a $(v, k, m)$-FHS which is also an OA$_{\lambda}(k, v, m, t')$. Suppose only one sequence is active in $S$. Then for a $(\frac{1}{m}, \frac{1}{m})$-jammer, the size of the search space $S_t^*$ at time $t$, $0 \leq t \leq t'$, is given by $|S_t^*| = \lambda(m - 1)^B m^{t'-t}$ where $B, 0 \leq B \leq t \leq t'$, is the number of time slots in which a jammer has been unlucky.

The jammer continues this action until either an active codeword is identified or the session ends.

One well-known class of orthogonal arrays is the maximum distance separable (MDS) codes: An OA$_1(m^{t'}, v, m, t')$ is a $(v, m^{t'}, m; v - t' + 1)$-MDS code. Example 4.6 gives an example of Reed-Solomon codes which are MDS codes that are both orthogonal arrays and cover-free codes. In this case we may be more specific:

**Corollary 4.9.** Let $S$ be a $(v, k, m)$-FHS which is also an OA$_1(k, v, m, t')$. Suppose there is only one active sequence in $S$. Then a $(\frac{1}{m}, \frac{1}{m})$-jammer which is lucky all the time will be able to identify the active sequence in $t'$ times slots. Otherwise it will take more than $t'$ times slots to identify an active sequence.

However, the analysis above does not apply when there are more than one active sequence. If the jammer eavesdrop on channel $x$, say, at time slot $t = 0$ and is unlucky, it can rule out all sequences with $x$ at $t = 0$, hence reducing the search space for the next time slot. If it is lucky, all it could conclude is that one of the active sequences is one of the $\lambda m^{t'}$ sequences that has $x$ at $t = 0$. It cannot rule out the possibility that there are other active sequences that do not have $x$ at $t = 0$. Therefore it cannot reduce the search space. However, this is related to the notion of descendants in fingerprinting codes. We will explore this briefly in the next section.

### 4.2 Another model of FHS

Let $S$ be a $(v, k, m)$-FHS and suppose $w + 1$ of the $k$ sequences are in use. Now, the jammer wishes to identify one of these active sequences. A possible strategy for the jammer is:

1. Pick the first channel $e_0$ which has the highest number of occurrences in $S$.

2. For $t \geq 1$, Record $E = (e_0, \ldots, e_{t-1})$. Suppose there are $t_1$ active channels at time $i_0, \ldots, i_{t_1-1}$, and $t_2$ inactive channels at time $j_0, \ldots, j_{t_2-1}$, $t_1 + t_2 = t$.

3. At time $t$, pick $e_t$ to eavesdrop on.
(a) If $e_t$ is active, compile a collection of subsets of $S$ capable of giving rise to the active parts of $(e_0, \ldots, e_{t-1}, e_t)$ and attempt to identify an active sequence.

(b) If $e_t$ is inactive, compile a subset of $S$, getting rid of sequences that cannot possibly be active given $(e_0, \ldots, e_{t-1}, e_t)$.

Our aim is to design $S$ so that any $(w+1)$-subset is able to withstand such an attack for as long as possible, that is, it takes the jammer as long as possible to identify an active sequence. One possible approach to this is to view $S$ as a code and $E$ as a “descendant” of subsets of $S$. An active sequence should belong to some parent set. We will introduce some notation and terminology here to better discuss this (see [3] for an introduction to fingerprinting codes).

Let $S = \{X^i = (x^i_0, \ldots, x^i_{v-1}) \mid x^i_j \in \mathcal{F}, i = 1, \ldots, k\}$. Treat $S$ as a code and $X^i$ as codewords over $\mathcal{F}$. Let $D = (d_0, \ldots, d_{v-1})$. We say $D$ is a descendant of a subset $\{X^0, \ldots, X^w\}$ if $d_i \in \{x^0_i, \ldots, x^w_i\}$ for all $i = 0, \ldots, v-1$. We call $\{X^0, \ldots, X^w\}$ a parent set of $D$. These terms are used to define traceability codes and codes with the identifiable parent property (IPP) - given a word $D$ we can identify at least one codeword in $S$ that gives rise to $D$. This finds application in traitor tracing ([3]).

Here we would like the opposite - given a word $E$ obtained by the jammer we would like the jammer NOT to be able to identify a parent. In addition we would need to capture the partial and sequential nature of how $E$ is obtained.

**Definition 4.10.** [Partial descendants and parent sets.] Let $i_0, i_1, \ldots, i_{t_1-1} \in \{0, 1, \ldots, v-1\}$, $t_1 \leq v$. We say that $D = (d_0, \ldots, d_{v-1})$ is an $(i_0, \ldots, i_{t_1-1})$-partial descendant of a set $\{X^0, \ldots, X^w\}$ if $d_{i_j} \in \{x^0_{i_j}, \ldots, x^w_{i_j}\}$ for $j \in \{0, \ldots, t_1 - 1\}$, ignoring all other positions.

We call $\{X^0, \ldots, X^w\}$ a $t_1$-partial parent set.

We consider the properties we would like for $S$ that would defend against the jammer strategy. Considering Step 3(a), we would like the following property:

**M4** The intersection of all the $t_1$-partial parent set should be the empty set for as large a $t_1$ as possible (otherwise an active sequence is identified). This should hold for all positions $(i_0, \ldots, i_{t_1-1})$.

Considering Step 3(b), we have:

**M5** For every inactive channel that the jammer eavesdrop on, if the jammer can discard some codewords at this step then property **M4** should be preserved after expurgation.

It is not clear at this stage how properties **M4** and **M5** relates to properties of other types of fingerprinting codes. We can, however, make the following statement about the Reed-Solomon code of Example 4.6:
Theorem 4.11. Let $m$ be a prime power such that $m \geq v$. Let $\delta$ be an integer, $0 < \delta \leq v$. Let $\mathcal{F}$ be the finite field of cardinality $m$ and let $\alpha_1, \alpha_2, \ldots, \alpha_v \in \mathcal{F}$ be distinct. Define a length $v$ Reed-Solomon code $\mathcal{S}$ over $\mathcal{F}$ by:

$$\mathcal{S} = \{(f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_v)) : f \in \mathcal{F}[X] \text{ and } \deg f < \delta\}.$$ 

Treat $\mathcal{S}$ as a $(v, m^\delta, m^\delta)$-FHS. Suppose there are $w + 1$ active sequence. If the jammer eavesdrop at $\delta(w + 1) + 1$ time slots and is lucky in all of them, then the jammer can determine an active sequence.

Proof. For ease of notation we assume that the jammer was lucky at time slots $t = 0, \ldots, \delta(w + 1)$ and the active channels are $E = (e_0, \ldots, e_{\delta(w+1)})$.

Every $\delta$ positions determines a unique sequence, so by taking combinations of $\delta$ of the $e_i$s we can determine at least

$$\binom{\delta(w+1)}{\delta} \binom{\delta w}{\delta} \cdots \binom{2\delta}{\delta}$$

parent sets, each containing $w + 1$ sequences.

Since there are $w + 1$ active sequences, at least one active sequence would have contributed $\delta$ of the $e_i$s, it will appear in at least one of the parent sets.

Suppose all active sequences contributed the same number $\delta$ of $e_i$, then all the active sequences would appear somewhere in the parent sets, and $e_{\delta(w+1)}$ would identify one of them.

Suppose there is one active sequence that contributes more than $\delta$ of the $e_i$ then this sequence would appear multiple times in some parent set and thus would be identified. \qed

On the other hand, if we have a scheme that has properties $M4$, $M5$, that means that too many codewords can give rise to $E$. This implies that too many codewords have the same symbol at a position, which contradicts the requirement for low correlation. Hence there is a trade-off between throughput and jamming resistance. This is a subject that warrants further research. In the next section we consider FHSs that achieve both high throughput and jammer-resistance, at the expense of computational costs.

5 A secure and efficient FHS

Section 4.1 demonstrates the limits of how secure an FH scheme based on codes can be. While MDS codes give a guarantee of throughput, they may not resist a jammer for very long. One possible solution would be to restart the FH scheme every $\gamma v$ time slots. We therefore propose that in order to withstand the attack of an adaptive jammer, some form of pseudorandomness must be introduced. Indeed, the schemes in [15] suggests that pseudorandomness is a necessary component of an FH scheme secure against an adaptive jammer. We include the description of one of the schemes here to illustrate this. This scheme is able to withstand an adaptive jammer for the entire session, at the expense
of additional computational burden, and on the assumption of a secure pseudorandom number generator.

The “strongly resilient Latin square (sR-LS) scheme” \cite{15} is constructed using a Latin square:

Let $F = \mathbb{Z}_v$, the set of integers modulo $v$. A Latin square of order $v$ defined over $F$ is a $v \times v$ array $L$ such that no element of $F$ appears more than once in a row or in any column of $L$. Suppose $x \in F$. Let $L + x = [\beta_{ij}]_{v \times v}$ where $\beta_{ij} = \alpha_{ij} + x \mod v$. Then $L + x$ is also a Latin square.

The sR-LS scheme is constructed from a Latin square $L = [\alpha_{ij}]_{v \times v}$ of order $v$ over $F$ as follows.

Let $K$ be a long term key shared by all legitimate users. Let $g$ be a pseudorandom function that takes as input $K$, the session number $s$, and the current time slot $t$, and outputs an element of $F$. A slot key $x_t$ is generated at each time slot as $x_t = g(K, s, t)$. The FH sequences are given as:

$$X_i = (\beta_{ij}),$$

where $\beta_{ij} = \alpha_{ij} + x_t \mod v$. Note that the sR-LS scheme has $v$ FH sequences used in a session, and a worst-case $(v - 1)$-throughput of 1. It can be viewed in two ways,

- A collection of $v$ $(v, v, 1)$-FHS which are MDS codes of minimum distance $v$, where each FH scheme is used only once.
- An MDS code with minimum distance 1, that is, a $(v, v, 1)$-FHS. Only $v$ sequences are used and these are determined by the pseudorandom number generator.

It can be seen that in this scheme the $v$ frequency channels at each time slot are unique. Therefore we have the maximum achievable $w$-throughput of 1 for any $w, 1 \leq w < v$. Now, consider a $(\frac{1}{m}, \frac{1}{m})$-adaptive jammer. The jammer has no knowledge of $K$, as it is shared by only the legitimate users. Further, a fresh pseudorandom number $x_t$ is generated at each time slot. Therefore a $(\frac{1}{m}, \frac{1}{m})$-adaptive jammer cannot identify an active FH sequence being used at a time slot. So we have an FH scheme that achieves maximum $w$-throughput of 1 and can withstand a $(\frac{1}{m}, \frac{1}{m})$-adaptive jammer for the entire session, that is $\gamma v = v$.

6 Conclusion

In this paper we have discussed FH schemes in the presence of both legitimate sequences of the system as well as jamming sequences of an adaptive jammer. We have provided a system model and jammer model where the performance of the FH schemes can be analysed in the presence of both mutual interfering FH sequences as well as jamming sequences.

So, it is desirable to know the throughput of an FH scheme under these circumstances. We explored using cover-free codes as FH schemes as they give a lower bound on the worst-case throughput. Further we considered mitigating strategies for cover-free codes.
to be used in the presence of jamming. However, we showed that in the presence of our
adaptive jammer, the FH schemes based on cover-free codes do not withstand the attack
for long. With this analysis, and with the example of the strongly resilient Latin square
(sR-LS) scheme proposed in [15] we conclude that pseudorandomness is a necessity in
providing jamming resistance of FH sequences.

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