Quantum Electrodynamics at Extremely Small Distances

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Abstract

The asymptotics of the Gell-Mann – Low function in QED can be determined exactly, \( \beta(g) = g \) at \( g \to \infty \), where \( g = e^2 \) is the running fine structure constant. It solves the problem of pure QED at small distances \( L \) and gives the behavior \( g \sim L^{-2} \).

According to Landau, Abrikosov, Khalatnikov [1], relation of the bare charge \( e_0 \) with the observable charge \( e \) in quantum electrodynamics (QED) is given by expression

\[
e^2 = \frac{e_0^2}{1 + \beta_2 e_0^2 \ln \Lambda^2/m^2},
\]  

(1)

where \( m \) is the mass of the particle, and \( \Lambda \) is the momentum cut-off. For finite \( e_0 \) and \( \Lambda \to \infty \) the ”zero charge” situation (\( e \to 0 \)) takes place. The proper interpretation of Eq.1 consists in its inverting, so that \( e_0 \) (related to the length scale \( \Lambda^{-1} \)) is chosen to give a correct value of \( e \):

\[
e_0^2 = \frac{e^2}{1 - \beta_2 e^2 \ln \Lambda^2/m^2}.
\]  

(2)

The growth of \( e_0 \) with \( \Lambda \) invalidates Eqs.1,2 in the region \( e_0 \sim 1 \) and existence of ”the Landau pole” in Eq.2 has no physical sense.

The actual behavior of the charge \( e \) as a function of the length scale \( L \) is determined by the Gell-Mann – Low equation [2]

\[
-\frac{dg}{d \ln L^2} = \beta(g) = \beta_2 g^2 + \beta_3 g^3 + \ldots, \quad g = e^2,
\]  

(3)

and depends on appearance of the function \( \beta(g) \). According to classification by Bogolyubov and Shirkov [3], the growth of \( g(L) \) is saturated, if \( \beta(g) \) has a zero for finite \( g \), and continues to infinity, if \( \beta(g) \) is non-alternating and behaves as \( \beta(g) \sim g^\alpha \) with \( \alpha \leq 1 \) for large \( g \); if, however, \( \beta(g) \sim g^\alpha \) with \( \alpha > 1 \), then \( g(L) \) is divergent at finite \( L = L_0 \) (the real Landau pole arises) and the theory is internally inconsistent due to indeterminacy of \( g(L) \) for \( L < L_0 \). Landau and Pomeranchuk [4] tried to justify the latter possibility, arguing that Eq.1 is valid without restrictions; however, it is possible only for the strict equality \( \beta(g) = \beta_2 g^2 \), which is surely invalid due to finiteness of \( \beta_3 \). One can see that solution of the problem of QED at small distances needs calculation of the Gell-Mann – Low function \( \beta(g) \) at arbitrary \( g \), and in particular its asymptotic behavior for \( g \to \infty \).
It was found recently in [5], that strong coupling asymptotic behavior of the renormalization group functions in the actual field theories can be obtained analytically. Attempts to reconstruct the \( \beta \)-function for \( \varphi^4 \) theory, undertaken previously by summation of perturbation series, lead to asymptotics \( \beta(g) = \beta_\infty g^\alpha \) at \( g \to \infty \), where \( \alpha \approx 1 \) for space dimensions \( d = 2, 3, 4 \) [6, 7, 8]. The natural hypothesis arises, that the true asymptotic behavior is \( \beta(g) \sim g \) for all \( d \). Consideration of the "toy" zero-dimensional model confirms the hypothesis and reveals the origin of the linear asymptotics. It is related with unexpected circumstance that the strong coupling limit for the renormalized charge \( g \) is determined not by large values of the bare charge \( g_0 \), but its complex values. More than that, it is sufficient to consider the region \( |g_0| \ll 1 \), where the functional integrals can be evaluated in the saddle-point approximation. If a proper direction in the complex \( g_0 \) plane is chosen, the contribution of the trivial vacuum is comparable with the saddle-point contribution of the main instanton, and a functional integral can turn to zero. The limit \( g \to \infty \) is related with a zero of a certain functional integral and appears to be completely controllable. As a result, it is possible to obtain asymptotic behavior of the \( \beta \)-function and anomalous dimensions: the former indeed appears to be linear.

At the present paper we show that the same idea can be applied to QED. Attempted reconstruction of the \( \beta \)-function in this theory [9] gives for it non-alternating behavior (Fig.1) with the asymptotics \( \beta(g) = \beta_\infty g^\alpha \), where

\[
\alpha = 1.0 \pm 0.1, \quad \beta_\infty = 1.0 \pm 0.3.
\]
Within uncertainty, the obtained $\beta$-function satisfies inequality

$$0 \leq \beta(g) < g,$$  \hfill (5)

established in [10, 11] from the spectral representations, while the asymptotics (4) corresponds to the upper bound of (5). Such coincidence does not look incident and indicates that asymptotics $\beta(g) = g$ is an exact result. We show below that it is so indeed.

The general functional integral of QED contains $M$ photon and $2N$ fermionic fields in the pre-exponential,

$$I_{M,2N} = \int D\bar{A}D\psi D\bar{\psi} A_{\mu_1}(x_1)\ldots A_{\mu_M}(x_M) \psi(y_1)\bar{\psi}(z_1)\ldots \psi(y_N)\bar{\psi}(z_N) \exp \left(-S\{A, \psi, \bar{\psi}\}\right),$$  \hfill (6)

where $S\{A, \psi, \bar{\psi}\}$ is the Euclidean action,

$$S\{A, \psi, \bar{\psi}\} = \int d^4x \left[ \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \bar{\psi}(i\partial - m_0 + e_0 \lambda)\psi \right],$$  \hfill (7)

while $e_0$ and $m_0$ are the bare charge and mass. Fourier transforms of the integrals $I_{M,N}$ with excluded $\delta$-functions of the momentum conservation will be referred as $\tilde{K}_{MN}(q_i, p_i)$ after extraction of the usual factors depending on tensor indices $\tilde{q}_i$ and $p_i$ are momenta of photons and electrons. Introducing Green’s functions $G^{(M,N)} = \tilde{K}_{MN}/\tilde{K}_{00}$, we can define amputated vertices $\Gamma^{(M,N)}$ with $M$ photon and $N$ electron external legs:

$$\Gamma^{(0,2)}(p) = 1/G^{(0,2)}(p) \equiv 1/G(p), \quad \Gamma^{(2,0)}(q) = 1/G^{(2,0)}(q) \equiv 1/D(q),$$

$$G^{(1,2)}(q, p, p') = D(q)G(p)G(p')\Gamma^{(1,2)}(q, p, p'),$$  \hfill (8)

etc., where $G(p)$ and $D(q)$ are the exact electron and photon propagators.

Multiplicative renormalizability of the vertex $\Gamma^{(M,N)}$ means that

$$\Gamma^{(M,N)}(q_i, p_i; e_0, m_0, \Lambda) = Z_3^{-M/2}Z_2^{-N/2} \Gamma_R^{(M,N)}(q_i, p_i; e, m),$$  \hfill (9)

i.e. its divergency at $\Lambda \to \infty$ disappears after extracting the proper $Z$-factors and transferring to the renormalized charge $e$ and mass $m$. Renormalization conditions at zero momenta are accepted

$$\Gamma^{(0,2)}_R(p)\big|_{p=0} = p^2 - m,$$

$$\Gamma^{(2,0)}_R(q)\big|_{q=0} = q^2,$$

$$\Gamma^{(1,2)}_R(q, p, p')\big|_{q,p,p'=0} = e,$$  \hfill (10)

where the usual pole structure of the electron and photon propagators is taken into account. Substitution of (10) into (9) determines $e, m, Z_2, Z_3$ in terms of the bare quantities

$$Z_2 = \left( \frac{\partial}{\partial p} \Gamma^{(0,2)}(p; e_0, m_0, \Lambda)\big|_{p=0} \right)^{-1}$$

\hfill 1 A specific form of the bare factors is inessential, since the results are independent on the absolute normalization of $e$ and $m$.  

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\begin{equation}
Z_3 = \left( \frac{\partial}{\partial q^2} \Gamma^{(2,0)}(q; e_0, m_0, \Lambda) \bigg|_{q=0} \right)^{-1}
\end{equation}

\begin{equation}
m = -Z_2 \Gamma^{(0,2)}(p; g_0, m_0, \Lambda) \bigg|_{p=0}
\end{equation}

\begin{equation}
e = Z_2 Z_3^{1/2} \Gamma^{(1,2)}(q, p, p'; e_0, m_0, \Lambda) \bigg|_{q,p,p'=0}
\end{equation}

The Gell-Mann – Low function is defined in this scheme as

\begin{equation}
\beta(g) = \left. \frac{dg}{d \ln m^2} \right|_{e_0, \Lambda=const},
\end{equation}

where \( g = e^2 \) is the running fine structure constant. Using Eq.8 and definition of \( G^{(M,N)} \),

\begin{equation}
\Gamma^{(0,2)}(p) = \frac{K_{00}}{K_{02}(p)}, \quad \Gamma^{(2,0)}(q) = \frac{K_{00}}{K_{20}(q)}, \quad \Gamma^{(1,2)} = \frac{K_{12}^2 K_{00}}{K_{02}^2 K_{20}},
\end{equation}

where zero momenta are implied in the last relation. Setting for small momenta

\begin{equation}
K_{02}(p) = K_{02} + \tilde{K}_{02} p, \quad K_{20}(q) = K_{20} + \tilde{K}_{20} q^2,
\end{equation}

and taking (11) into account, we have

\begin{equation}
Z_2 = -\frac{K_{02}^2}{K_{00} K_{02}}, \quad Z_3 = -\frac{K_{20}^2}{K_{00} K_{20}}, \quad m = \frac{K_{02}}{K_{02}}, \quad g = -\frac{K_{12}^2 K_{00}}{K_{02}^2 K_{20}},
\end{equation}

Denoting differentiation over \( m_0 \) by prime, one has from (15)

\begin{equation}
\frac{dm}{dm_0} = \left( \frac{K_{02}}{K_{02}} \right)' = \frac{K_{02}' \tilde{K}_{02} - K_{02} \tilde{K}_{02}'}{K_{02}^2}
\end{equation}

Since differentiation in (12) occurs at \( e_0, \Lambda = const \), the latter parameters are considered to be fixed throughout all calculations: then \( m \) is a function of only \( m_0 \) and Eq.16 defines also the derivative \( dm_0/dm \). Using definition of the \( \beta \)-function (12),

\begin{equation}
\beta(g) = m \left( \frac{K_{12}^2 K_{00}}{K_{02}^2 K_{20}} \right)' \frac{dm_0}{dm},
\end{equation}

and making simple transformations, we end with equations

\begin{equation}
g = -\frac{K_{12}^2 K_{00}}{K_{02}^2 K_{20}},
\end{equation}

\begin{equation}
\beta(g) = \frac{1}{2} \frac{K_{02} \tilde{K}_{02} - K_{02}' \tilde{K}_{02}'}{K_{02}^2 K_{20}} \left\{ \frac{2 K_{12}'}{K_{12}} + \frac{K_{00}'}{K_{00}} - 2 \frac{\tilde{K}_{02}'}{\tilde{K}_{02}} - \frac{\tilde{K}_{20}'}{\tilde{K}_{20}} \right\}
\end{equation}
Equations (18),(19) define the dependence \( \beta(g) \) in the parametric form. Their right hand sides depend on \( m_0, g_0, \Lambda \) with two latter parameters being fixed. Solving Eq.18 for \( m_0 \) and substituting into (19), one obtains \( \beta \) as a function of \( g, g_0 \) and \( \Lambda \); in fact, the dependence on the latter two parameters is absent due to the general theorems [12, 13].

According to [5], the strong coupling regime for renormalized interaction is related with a zero of a certain functional integral. It is clear from (18) that the limit \( g \to \infty \) can be realized by two ways: tending to zero either \( \tilde{K}_{02} \), or \( \tilde{K}_{20} \). For \( \tilde{K}_{02} \to 0 \), equations (18,19) are simplified,

\[
g = -\frac{K_{12}^2 K_{00}}{\tilde{K}_{02} \tilde{K}_{20}}, \quad \beta(g) = -\frac{K_{12}^2 K_{00}}{\tilde{K}_{02} \tilde{K}_{20}},
\]

and the parametric representation is resolved in the form

\[
\beta(g) = g, \quad g \to \infty.
\]

For \( \tilde{K}_{20} \to 0 \), one has

\[
g \propto \frac{1}{\tilde{K}_{20}}, \quad \beta(g) \propto \frac{1}{\tilde{K}_{20}^2},
\]

and hence

\[
\beta(g) \propto g^2, \quad g \to \infty.
\]

Consequently, there are two possibilities for the asymptotics of \( \beta(g) \), either (21), or (23). The second possibility is in conflict with inequality (5), while the first possibility is in excellent agreement with results (4) obtained by summation of perturbation series. In our opinion, it is sufficient reason to consider Eq.21 as an exact result for the asymptotics of the \( \beta \)-function. It means that the fine structure constant in pure QED behaves as \( g \propto L^{-2} \) at small distances \( L \).

Above we have in mind that the mechanism determining the asymptotics of \( \beta(g) \) is the same as in \( \varphi^4 \) theory. Strictly speaking, one cannot exclude possibility that the strong coupling regime in QED is determined by the different mechanism, e.g. by the large value of the integral \( K_{12} \). However, such possibility looks improbable: if one make rough estimate of pre-exponential in (6), supposing that all fields are localized at the unit length scale,

\[
K_{12} \sim \langle A \rangle \langle \psi \bar{\psi} \rangle K_{00}, \quad \tilde{K}_{02} \sim K_{02} \sim \langle \psi \bar{\psi} \rangle K_{00}, \quad \tilde{K}_{20} \sim K_{20} \sim \langle A \rangle^2 K_{00},
\]

then substitution into (18) gives \( g \sim 1 \). The change of the general length scale does not affect the quantity \( g \) due to its dimensionless character. Consequently, it is impossible to achieve large values of \( g \) by a simple change of the amplitude of fields \( A, \psi, \bar{\psi} \), or the scale of their spatial localization. It is necessary to suggest that the average value of \( \langle A \rangle \) or \( \langle \psi \bar{\psi} \rangle \) is anomalously small for one of the integrals, but it returns us to already considered possibilities.

In the analogy with [5], the zeroes of functional integrals can be obtained for the complex \( g_0 \) with \( |g_0| \ll 1 \) by compensation of the saddle-point contribution of trivial vacuum with the saddle-point contribution of the instanton configuration with minimum action. The
latter contribution is well-known from the studies of large-order behavior of perturbation series \[14, 15, 16, 9\] and has the form

\[
[K_{M,N}(q_i, p_i)]^{inst} = ic(q_i, p_i) \left( \frac{S_0}{g_0^2} \right)^b e^{-S_0/g_0^2}
\]

(25)

where \(S_0\) is the instanton action, \(b = (M + r)/2\) and \(r\) is the number of zero modes. Setting \(t^2 = -S_0/g_0^2\), we come to expressions of the same kind, as were analyzed in \[5\]. It is easy to be convinced that zeroes of different integrals \(K_{M,N}\) and their derivatives lie in different points.

In conclusion, we have determined the exact asymptotics of the \(\beta\)-function in QED, which determines the behavior of the effective interaction at extremely small distances.

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