Characterization of the Distribution of Twin Primes

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Abstract

We adopt an empirical approach to the characterization of the distribution of twin primes within the set of primes, rather than in the set of all natural numbers. The occurrences of twin primes in any finite sequence of primes are like fixed probability random events. As the sequence of primes grows, the probability decreases as the reciprocal of the count of primes to that point. The manner of the decrease is consistent with the Hardy–Littlewood Conjecture, the Prime Number Theorem, and the Twin Prime Conjecture. Furthermore, our probabilistic model, is simply parameterized. We discuss a simple test which indicates the consistency of the model extrapolated outside of the range in which it was constructed.

Key words: Twin primes
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1 Introduction

Prime numbers [1], with their many wonderful properties, have been an intriguing subject of mathematical investigation since ancient times. The “twin primes,” pairs of prime numbers \{p, p+2\} are a subset of the primes and themselves possess remarkable properties. In particular, we note that the Twin Prime Conjecture, that there exists an infinite number of these prime number pairs which differ by 2, is not yet proven [2, 3].

In recent years much human labor and computational effort have been expended on the subject of twin primes. The general aims of these researches have been three-fold: the task of enumerating the twin primes [4] (i.e., identifying the members of this particular subset of the natural numbers, and its higher-order variants “k-tuples” of primes), the attempt to elucidate how twin primes are distributed among the natural numbers [5, 6, 7, 8] (especially searches for long gaps in the sequence [9, 10, 11]), and finally, the precise estimation of the value of Brun’s Constant [12].

Many authors have observed that the twin primes, along with the primes themselves, generally become more sparse or diffuse as their magnitude increases. In fact, the Prime Number Theorem may be rephrased to state that the number of prime numbers less than or equal to some large (not necessarily prime) number \(N\) is approximately:

\[
\pi_1(N) \sim \int_2^N \frac{1}{\ln(x)} dx.
\]

A similar result is believed to hold for the number of twin primes where each element of the pair is less than or equal to large \(N\):

\[
\pi_2(N) \sim 2c_2 \int_2^N \frac{1}{(\ln(x))^2} dx,
\]

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1In fact, more accurate approximations are known, but the formulae we quote suffice for our purposes.
where the “twin prime” constant $c_2$ has the numerical value $c_2 = 0.661618 \ldots$, and is currently known to many decimal places\footnote{14, 15}. The expression (2) is the first instance of the Hardy–Littlewood conjectures which estimate the multiplicity of $k$-tuples of primes smaller than natural number $N$\footnote{13}.

In our investigation, we sought to account for the effect of the primes themselves becoming more rarefied by examining the distribution of twin primes within the prime numbers. We have done this for the set of prime numbers less than approximately $4 \times 10^9$, and have observed that within this range the occurrence of twin primes may be characterized as slowly-varying-probability random events.

2 Method and Results

To ensure that there is no confusion, let us first make very clear our methodology. We generated prime numbers in sequence, viz, $P_1 = 2, P_2 = 3, P_3 = 5 \ldots$, and within this sequence identified twin primes and their prime separations as illustrated somewhat schematically below.

\[ \cdots P_i (P_i + 1) P_{i+2} P_{i+3} P_{i+4} (P_{i+5} P_{i+6}) P_{i+7} P_{i+8} P_{i+9} P_{i+10} (P_{i+11} P_{i+12}) P_{i+13} \cdots \]

We say that the first pair of twins in the above sequence has a prime separation of 2. There are two non-twin prime numbers, i.e., singletons, which occur between the second prime element, $P_{i+2}$, of the first twin and the first prime element, $P_{i+5}$, of the subsequent twin. Similarly, the second pair of twins has prime separation 4. Note that there are many twins with prime separation equal to zero: for example $(5, 7)(11, 13)$, or $(137, 139)(149, 151)$. Actually, all of the prime 4-tuples $(P, P + 2, P + 6, P + 8)$ are comprised of a pair of twins with zero separation in primes.

There is an irregularity with our definition of separation for the first few primes: 2 \footnote{3}(3 5)(5 7), where the pairs in fact overlap, yielding a prime separation of $-1$. Fortunately, for very well-known reasons such overlapping twins do not ever recur and we choose to begin our analysis with the twin $(5, 7)$. For instance, in the set of seven twins between 5 and 100,

\[ \{(5, 7)(11, 13)(17, 19)(29, 31)(41, 43)(59, 61)(71, 73)\}, \]

there are 6 separations. Three of these happen to be 0, two are 1, and one is 2, so the relative frequencies for separations $s = 0, 1, 2$ are $\frac{1}{3}, \frac{1}{3},$ and $\frac{1}{6}$, respectively.

From the set of primes, all separations between pairs of twins up to a fixed number $N$ were computed and tabulated. We chose certain values of $N$ in the range 79561 to 4020634603. These particular numbers are the second prime elements of the thousandth and twelve-millionth twins respectively. Many of our $N$’s were chosen such that our analysis started and ended on twins. However the behavior that we observe holds for all $N$, sufficiently large, with the understanding that the singleton primes between the last twin and the upper bound $N$ are ignored. The logarithm of the relative frequency of occurrence of each separation in each of our analyses appears to obey a surprisingly simple relation as illustrated schematically in Figure 1.

Two comments must be made. The first is that this remarkable behavior is perfectly characteristic of a completely random system. We infer that as one approaches each prime member in the sequence of primes following a twin, the likelihood of it being the first member of the next twin prime is constant! By way of analogy, we can consider a radioactive substance. The likelihood of one of its atoms decaying in any short time interval is fixed, with the effect that the probability that the next decay occurs at time $t$ is just $N e^{-\gamma t}$, where $\gamma$ is the decay rate, and $N$ is a constant to ensure appropriate normalization. The measured slope of the line fit to our data provides a decay constant which is particular to the twin primes. Again, recourse to our analogy is warranted. The decay rate of a radioactive element is a defining characteristic of that element. We appear to have the occurrences of twin primes governed by a similar sort of “prime constant.”

We qualify this statement, however, since the curve in Figure 1 is only illustrative because the slope varies inversely with $N$. Figure 2 displays curves (with associated best-fit straight lines) for
Figure 1: Data and Linear Fits for log(frequency) vs. separation, in the case $N = 10^7$.

the twin prime separation data for ranges $[5, 10^6]$, $[5, 10^8]$ and $[5, 10^9]$, illustrating the variation with $N$.

Were it not the case that the magnitude of the slope diminished for larger values of $N$ then the Hardy–Littlewood Conjecture for twins would certainly fail to hold. If the slopes of our lines were indeed the same value for all $N$, meaning that the probability of a given prime being a member of a twin pair is a universal constant, then the number of twin primes $\pi_2(N)$ would just be a fixed fraction of $\pi_1(N)$ in disagreement with Hardy–Littlewood and the empirical data.

The linear fits that we employed were constrained to ensure that the relative frequencies are properly normalized. That is, if the relative frequency with which separation $s$ occurs obeys the exponential relation consistent with our data, then for the sum of frequencies to be normalized to 1, we must have

$$+ \text{(intercept)} = \ln(-\text{(slope)}),$$

so the fit that we performed was constrained by the one-parameter Ansatz

$$f(s) = -ms + \ln(m).$$

Table I gives the best-fit values of this probability-conserving slope ($m$) for various $N$, along with simple statistical estimates of the uncertainty. The quoted error estimates measure only the quality of the estimate of $m$ and and does not account for effects resulting from our arbitrary choices of $N$. It may very well be the case that a more realistic assessment of error would double the values quoted in the table.

Two comments must be made. The first is that all of the separations which appeared in the data received equal (frequency-weighted) consideration in our computation of best-fit slopes. This has a consequence insofar as the large-separation, low-frequency events constituting the tail of the distribution reduce the magnitude of the slope, as is readily seen in Figures 1 and 2. One might well be inclined to truncate the data by excising the tails and fixing the slopes by the (more-strongly-linear) low-separation data for each $N$. We did not do this because it would have entailed a generally systematic discarding of data from pairs appearing near the upper limit of the range,
and thus would nearly correspond to the slope with greater magnitude that one would expect associated with an effective upper limit $N_{\text{eff}} < N$. Viewed from this perspective, it is better to consider all points rather than submit to this degree of uncertainty. The second comment is that we have thus far adhered to the convention of expressing all of our results in terms of the natural number $N$. We shall now pass over to a characterization in terms of $\pi_1$ – itself a function of $N$ – which better suits our viewpoint that the analysis of the distribution of twins is most meaningful when considered in terms of the primes themselves.

In light of the above comments, we sketch in Figure 3 a plot of the slopes (computed in the manner described) versus $\log(\pi_1)$. The trend seen on the graph may be well-described by the function (remember that the error bars are understated)

$$ -m(x) \approx (1.321 \pm 0.010)/x, \quad \text{for } x = \log(\pi_1(N)). $$

There are two amazing features of this functional form for the dependence of the slope on $\pi_1$. The first is that the factor which appears looks suspiciously like $-2c_2$, the twin primes constant! This result will be confirmed in the next section. The second feature is that

$$ \lim_{x \to \infty} -m(x) = 0^- , $$

i.e., as one progresses through the infinite set of primes, the slope which governs the distribution of twins does not crash through zero. This is consistent with the Hardy–Littlewood conjecture insofar as the twins become progressively more sparse within the set of primes. In addition, it is consistent with the Twin Prime Conjecture in that the reciprocal of the slope admits the interpretation of being the “expected number of primes interspersed between a given twin and the next twin in order.”

$$ \bar{s} = \frac{1}{m}, \quad \text{for random fixed probability events.} $$

Since $\bar{s}$ remains finite for all $\pi_1$, then we can conjecture that wherever one happens to be in the infinite set of primes it is possible to characterize the expected number of primes that will be
Table 1: Values of slope, statistical error, \( \pi_1(N) \), and \( \pi_2(N) \) for certain \( N \) from 79561 to 4020634603.

| \( \pi_2(N) \) | slope  | stat. error (±) | \( \pi_1(N) \) | \( N \)    |
|----------------|--------|-----------------|----------------|-----------|
| \( 1 \times 10^3 \) | 0.141667 | 0.00599         | 7793           | 79561     |
| \( 5 \times 10^3 \) | 0.122415 | 0.00315         | 45886          | 557521    |
| \( 1 \times 10^4 \) | 0.114097 | 0.00325         | 97255          | 1260991   |
| \( 5 \times 10^4 \) | 0.104126 | 0.00105         | 556396         | 8264959   |
| \( 1 \times 10^5 \) | 0.096421 | 0.00095         | 1175775        | 18409201  |
| \( 5 \times 10^5 \) | 0.086700 | 0.00056         | 6596231        | 115438669 |
| \( 1 \times 10^6 \) | 0.081143 | 0.00041         | 13804822       | 252427603 |
| \( 3 \times 10^6 \) | 0.075491 | 0.00035         | 44214960       | 863029303 |
| \( 5 \times 10^6 \) | 0.073150 | 0.00031         | 75860671       | 1523975911|
| \( 8 \times 10^6 \) | 0.070965 | 0.00032         | 124538861      | 2566997821|
| \( 1 \times 10^7 \) | 0.070154 | 0.00029         | 157523559      | 3285916171|
| \( 1.2 \times 10^7 \) | 0.069814 | 0.00024         | 190894477      | 4020634603|

encountered on the way to the next twin. This is also consistent with the Twin Prime Conjecture, although unfortunately it is empirical and does not constitute a proof.

3 Interpretive Framework

Recall that, up to this point, all of our results have been purely empirical. Now, we will argue for their essential truth and consistency beyond the range of our data.

Consider our approximation (7) for \( \pi_1(N) \). Making the additional draconian approximation that the integrand is constant at its minimum value, and discarding a small term, we get the oft-quoted estimate

\[
\tilde{\pi}_1(N) \sim \frac{N}{\log(N)}.
\]

In precisely the same manner we get

\[
\tilde{\pi}_2(N) \sim 2c_2 \frac{N}{(\log(N))^2}.
\]

The entire set of prime numbers less than \( N \) consists of the \( 2 \times \pi_2(N) \) elements which occur together in twins and \( \pi_1(N) - 2 \times \pi_2(N) \) singletons. Now, let us suppose that the set of twins is randomly interspersed among the singletons. This would imply that between each twin pair there will appear, on average, \( s_0(N) \) singletons, where

\[
s_0(N) = \frac{\pi_1 - 2\pi_2}{\pi_2}.
\]

Note that there is an essential distinction between \( \bar{s} \) which arises from the actual distribution of separations and \( s_0 \) which, in effect, assumes that the twins are evenly spaced. That is, from a value of \( \bar{s}(\pi_1(N)) \) one can infer \( m \) and thus the probability distribution of twin prime separations characteristic of the set of primes less than \( N \). On the other hand, \( s_0(N) \) is an average value in which no account is taken of the details of the distribution and thus no more information is contained in it.

In the approximation scheme developed in (7) and (8),

\[
\bar{s}_0 = \frac{\log(N) - 4c_2}{2c_2} \simeq \frac{\log(N)}{2c_2}.
\]
Furthermore, to very lowest-order
\[ \log(N) \approx \log(\tilde{\pi}_1) + \log(\log(\tilde{\pi}_1)) , \]
and hence
\[ \tilde{s}_0 \approx \frac{\log(\tilde{\pi}_1)}{2c_2} . \]
Finally, taking this as the expected number of singleton primes occurring between twin prime pairs for numbers less than or equal to \( N \) we see immediately that
\[ \tilde{m} = \frac{1}{\tilde{s}_0} \approx \frac{2c_2}{\log(\tilde{\pi}_1)} \]
is completely consistent with the Prime Number Theorem, the Hardy–Littlewood Conjecture, and with our empirical results.

As an aside, one might consider the effect of attempting to improve upon the draconian approximation. It turns out that any reasonable improvement merely results in the addition of (small) constant terms which may be neglected in the limit of large \( N \).

We are quite surprised that our empirical results yield the large \( N \) limit with such accuracy.

Another test of the general consistency of our model is by comparison with \( m_0 \), where
\[ m_0(N) = \frac{1}{s_0(N)} = \frac{\pi_2(N)}{\pi_1(N) - 2\pi_2(N)} . \]
Bearing in mind that we are modeling more accurately the actual distribution of prime separations with \( \tilde{s} \) than with \( s_0 \), we do not expect perfect agreement, but rather that the general trend exposed by \( s_0 \) will be followed by \( \tilde{s} \) if investigated beyond the range thus far examined. We sketch below a plot of \( m_0 \) vs. \( \log \pi_1 \) using precise values for \( \pi_1 \) and \( \pi_2 \) computed by T.R. Nicely. Note that we have made the small adjustments of decrementing the published \( \pi_2 \)'s by one and decrementing the \( \pi_1 \) by two to take into account our skipping the anomalous prime 2 and twin (3 5). We are quite encouraged by the correspondence of the data on graph, and believe that the distributional model does extend itself well beyond the range of our present data.
Figure 4: $m_0$ using Nicely Data and our empirical fit $\frac{1.321}{x}$ vs. $\log(\pi_1)$.

4 Conclusion

We believe that we have constructed a novel characterization of the distribution of twin primes. The most essential feature of our approach is that we consider the spacings of twins among the primes themselves, rather than among the natural numbers. Secondly, we modeled the distribution empirically — without preconceptions — and argued that for any given $N$ (larger than $10^4$, say) the twin primes appear amongst the sequence of primes in a manner characteristic of a completely random, fixed probability system. Again working empirically, we noted that the “fixed” probability varied with $N$, in a manner consistent with Theorems and with Conjectures that are believed to hold. We have parameterized the variation of the “separation constant” in terms of $\pi_1$, as suggested by our outlook, and have discovered that it has a particularly simple functional form and is also consistent with the established Theorems and Conjectures.

With this model for the distribution now in hand and assumed viable, we are beginning to investigate other consequences. These will be reported upon in a forthcoming paper [17].

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