TWISTORIAL CONSTRUCTION OF GENERALIZED KÄHLER MANIFOLDS

JOHANN DAVIDOV, OLEG MUSHKAROV

Abstract. The twistor method is applied for obtaining examples of generalized Kähler structures which are not yielded by Kähler structures.

2000 Mathematics Subject Classification 53C15, 53C28.

Key words: generalized Kähler structures, twistor spaces

1. Introduction

The theory of generalized complex structures has been initiated by N. Hitchin [12] and further developed by M. Gualtieri [11]. These structures contain the complex and symplectic structures as special cases and can be considered as a complex analog of the notion of a Dirac structure introduced by T. Courant and A. Weinstein [6, 7] to unify the Poisson and presymplectic geometries. This and the fact that the target spaces of supersymmetric \(\sigma\)-models are generalized complex manifolds motivate the increasing interest to the generalized complex geometry.

The idea of this geometry is to replace the tangent bundle \(TM\) of a smooth manifold \(M\) with the bundle \(TM \oplus T^*M\) endowed with the indefinite metric \(<X + \xi, Y + \eta> = \frac{1}{2}(\xi(Y) + \eta(X)), X, Y \in TM, \xi, \eta \in T^*M\). A generalized Kähler structure is, by definition, a pair \(\{J_1, J_2\}\) of commuting generalized complex structures such that the quadratic form \(<J_1A, J_2A>\) is positive definite on \(TM \oplus T^*M\). According to a result of M. Gualtieri [11] the generalized Kähler structures have an equivalent interpretation in terms of the so-called bi–Hermitian structures.

Any Kähler structure yields a generalized Kähler structure in a natural way. Non-trivial examples of such structures can be found in [2, 3, 5, 13, 14, 15, 16]. The purpose of the present paper is to provide non-trivial examples of generalized Kähler manifolds by means of the R. Penrose [17] twistor construction as developed by M. Atiyah, N. Hitchin and I. Singer [4] in the framework of the Riemannian geometry.

Let \(M\) be a 2-dimensional smooth manifold. Following the general scheme of the twistor construction we consider the bundle \(\mathcal{P}\) over \(M\) whose fibre at a point \(p \in M\) consists of all pairs of commuting generalized complex structures \(\{I, J\}\) on the vector space \(T_pM\) such that the form \(<IA, JA>\) is positive definite on \(T_pM \oplus T^*_pM\). The general fibre of \(\mathcal{P}\) admits two
natural Kähler structures (in the usual sense) and can be identified in a natural way with the disjoint union of two copies of the unit bi-disk. Under this identification, the two structures are defined on the unit bi-disk as \((h \times h, \mathcal{K} \times (\pm \mathcal{K}))\) where \(h\) is the Poincare metric on the unit disk and \(\mathcal{K}\) is its standard complex structure. These two Kähler structures yield a generalized Kähler structure on the fibre of \(\mathcal{P}\) according to the Gualtieri result mentioned above. Moreover, any linear connection \(\nabla\) on \(\mathcal{P}\) gives rise to a splitting of the tangent bundle \(T\mathcal{P}\) into horizontal and vertical parts and this allows one to define two commuting generalized almost complex structures \(\mathcal{I}^\nabla\) and \(\mathcal{J}^\nabla\) on \(\mathcal{P}\) such that the form \(<\mathcal{I}^\nabla, \mathcal{J}^\nabla, \cdot\>\) is positive definite on \(T\mathcal{P} \oplus T^*\mathcal{P}\). The main result of the paper states that if the connection \(\nabla\) is torsion–free, the structures \(\mathcal{I}^\nabla\) and \(\mathcal{J}^\nabla\) are both integrable if and only if \(\nabla\) is flat. Thus any affine structure on \(M\) yields a generalized Kähler structure on the 6-dimensional manifold \(\mathcal{P}\). Note that the only complete affine 2-dimensional manifolds are the plane, a cylinder, a Klein bottle, a torus, or a Mobius band \([10, 9]\).

2. Generalized Kähler structures

Let \(W\) be a \(n\)-dimensional real vector space and \(g\) a metric of signature \((p, q)\) on it, \(p + q = n\). We shall say that a basis \(\{e_1, ..., e_n\}\) of \(W\) is orthonormal if \(||e_1||^2 = ... = ||e_p||^2 = 1, ||e_{p+1}||^2 = ... = ||e_{p+q}||^2 = -1\). If \(n = 2m\) is an even number and \(p = q = m\), the metric \(g\) is usually called neutral. Recall that a complex structure \(J\) on \(W\) is called compatible with the metric \(g\), if the endomorphism \(J\) is \(g\)-skew-symmetric.

Suppose that \(\dim W = 2m\) and \(g\) is of signature \((2p, 2q)\), \(p + q = m\). Denote by \(J(W)\) the set of all complex structures on \(W\) compatible with the metric \(g\). The group \(O(g)\) of orthogonal transformations of \(W\) acts transitively on \(J(W)\) by conjugation and \(J(W)\) can be identified with the homogeneous space \(O(2p, 2q)/U(p, q)\). In particular, \(\dim J(W) = m^2 - m\). The group \(O(2p, 2q)\) has four connected components, while \(U(p, q)\) is connected, therefore \(J(W)\) has four components.

**Example 1** (\([8]\)). The space \(O(2, 2)/U(1, 1)\) is the disjoint union of two copies of the hyperboloid \(x_1^2 - x_2^2 - x_3^2 = 1\).

Consider \(J(W)\) as a (closed) submanifold of the vector space \(so(g)\) of \(g\)-skew-symmetric endomorphisms of \(W\). Then the tangent space of \(J(W)\) at a point \(J\) consists of all endomorphisms \(Q \in so(g)\) anti-commuting with \(J\). Thus we have a natural \(O(g)\) - invariant almost complex structure \(\mathcal{K}\) on \(J(W)\) defined by \(\mathcal{K}Q = J \circ Q\). It is easy to check that this structure is integrable.

Fix an orientation on \(W\) and denote by \(J^{\pm}(W)\) the set of compatible complex structures on \(W\) that induce \(\pm\) the orientation of \(W\). The set \(J^{\pm}(W)\) has the homogeneous representation \(SO(2p, 2q)/U(p, q)\) and, thus, is the union of two components of \(J(W)\).
Suppose that $\dim W = 4$ and $g$ is of split signature $(2,2)$. Let $g(a, b) = -\frac{1}{4} Trace(ab)$ be the standard metric of $so(g)$. The restriction of this metric to the tangent space $T_J$ of $J(W)$ is negative definite and we set $h = -g$ on $T_J$. Then the complex structure $K$ is compatible with the metric $h$ and $(K, h)$ is a Kähler structure on $J(W)$. The space $J^\pm(W)$ can be identified with the hyperboloid $x_1^2 - x_2^2 - x_3^2 = 1$ in $\mathbb{R}^3$ (see e.g. [8, Example 5]) and it is easy to check that, under this identification, the structure $(K, h)$ on $J^\pm(W)$ goes to the standard Kähler structure of the hyperboloid. Thus the Hermitian manifold $(J^\pm(W), K, h)$ is biholomorphically isometric to the disjoint union of two copies of the unit disk endowed with the Poincare-Bergman metric (of curvature $-1$).

Let $\mathfrak{b} : T_J \to T^*_J$ and $\mathfrak{h} = \mathfrak{b}^{-1}$ be the "musical" isomorphisms determined by the metric $h$. Denote by $T^*_J$ the orthogonal complement of $T_J$ in $so(g)$ with respect to the metric $g$; the space $T^*_J$ consists of the skew-symmetric endomorphisms of $W$ commuting with $J$. Consider $T^*_J$ as the space of linear forms on $so(g)$ vanishing on $T^*_J$. Then for every $U \in T_J$ and $\omega \in T^*_J$ we have $U^\flat(A) = -g(U, A)$ and $g(\omega^\flat, A) = -\omega(A)$ for every $A \in so(g)$.

Now let $V$ be a real vector space and $V^*$ its dual space. Then the vector space $V \oplus V^*$ admits a natural neutral metric defined by

$$<X + \xi, Y + \eta> = \frac{1}{2} (\xi(Y) + \eta(X))$$

A generalized complex structure on the vector space $V$ is, by definition, a complex structure on the space $V \oplus V^*$ compatible with its natural neutral metric [12]. If a vector space $V$ admits a generalized complex structure, it is necessarily of even dimension [11]. We refer to [11] for more facts about the generalized complex structures.

**Example 2** ([11] [12] [13]). Every complex structure $K$ and every symplectic form $\omega$ on $V$ (i.e. a non-degenerate 2-form) induce generalized complex structures on $V$ in a natural way. If we denote these structures by $J$ and $S$, respectively, the structure $J$ is defined by $J = K$ on $V$ and $J = -K^*$ on $V^*$, where $(K^* \xi)(X) = \xi(KX)$ for $\xi \in V^*$ and $X \in V$.

The map $X \to \iota_X \omega$ (the interior product) is an isomorphism of $V$ onto $V^*$. Denote this isomorphism also by $\omega$. Then the structure $S$ is defined by $S = \omega$ on $V$ and $S = -\omega^{-1}$ on $V^*$.

**Example 3** ([11] [12] [13]). Any 2-form $B \in \Lambda^2 V^*$ acts on $V \oplus V^*$ via the inclusion $\Lambda^2 C V^* \subset \Lambda^2 (V \oplus V^*) \cong so(V \oplus V^*)$; in fact this is the action $X + \xi \to \iota_X B; X \in V, \xi \in V^*$. Denote the latter map again by $B$. Then the invertible map $e^B$ is given by $X + \xi \to X + \xi + \iota_X B$ and is an orthogonal transformation of $V \oplus V^*$. Thus, given a generalized complex structure $J$ on $V$, the map $e^B J e^{-B}$ is also a generalized complex structure on $V$, called the $B$-transform of $J$.

Similarly, any 2-vector $\beta \in \Lambda^2 V$ acts on $V \oplus V^*$. If we identify $V$ with $(V^*)^*$, so $\Lambda^2 V \cong \Lambda^2 (V^*)^*$, the action is given by $X + \xi \to \iota_X \beta \in V$. Denote
This shows that the restriction of the generalized complex structure $I$ with $J$ a symplectic form on $V$ generalized complex structure $J$. Such a structure is said to be if its $+$ structures.

Let $\{e_i\}$ be an arbitrary basis of $V$ and $\{\eta_i\}$ its dual basis, $i = 1, ..., 2n$. Then the orientation of the space $V \oplus V^*$ determined by the basis $\{e_i, \eta_i\}$ does not depend on the choice of the basis $\{e_i\}$. Further on, we shall always consider $V \oplus V^*$ with this canonical orientation. The sets $J^\pm (V \oplus V^*)$ of generalized complex structures on $V$ inducing $\pm$ the canonical orientation of $V \oplus V^*$ will be denoted by $G^\pm (V)$.

Example 4. A generalized complex structure on $V$ induced by a complex structure (see Example 2) always yields the canonical orientation of $V \oplus V^*$. A generalized complex structure on $V$ induced by a symplectic form yields the canonical orientation of $V \oplus V^*$ if and only if $n = \frac{1}{2} \dim V$ is an even number. The $B$- or $\beta$-transform of a generalized complex structure $J$ on $V$ yields the canonical orientation of $V \oplus V^*$ if and only if $J$ does so.

Example 5. Let $V$ be a 2-dimensional real vector space. Take a basis $\{e_1, e_2\}$ of $V$ and let $\{\eta_1, \eta_2\}$ be its dual basis. Then $\{Q_1 = e_1 + \eta_1, Q_2 = e_2 + \eta_2, Q_3 = e_1 - \eta_1, Q_4 = e_2 - \eta_2\}$ is an orthonormal basis of $V \oplus V^*$ with respect to the natural neutral metric and is positively oriented with respect to the canonical orientation of $V \oplus V^*$. Put $\varepsilon_k = ||Q_k||^2$, $k = 1, ..., 4$, and define skew-symmetric endomorphisms of $V \oplus V^*$ setting $S_{ij}Q_k = \varepsilon_k (\delta_{ik}Q_j - \delta_{kj}Q_i)$, $1 \leq i, j \leq 4$. Then the endomorphisms

\[
I_1 = S_{12} - S_{34}, \quad J_1 = S_{12} + S_{34}, \\
I_2 = S_{13} - S_{24}, \quad J_2 = S_{13} + S_{24}, \\
I_3 = S_{14} + S_{23}, \quad J_3 = S_{14} - S_{23}
\]

constitute a basis of the space of skew-symmetric endomorphisms of $V \oplus V^*$. Let $I \in G^+(V)$ and $J \in G^-(V)$. Then $I = \sum_i x_i I_i$ with $x_i^2 - x_i^2 - x_i^2 = 1$ and $J = \sum_s y_s J_s$ with $y_s^2 - y_s^2 - y_s^2 = 1$. It follows that

\[
I e_1 = x_2 e_1 + (x_1 + x_3)e_2, \quad J e_1 = y_2 e_1 + (y_1 - y_3)\eta_2, \\
I e_2 = -(x_1 - x_3)e_1 - x_2 e_2, \quad J e_2 = y_2 e_2 - (y_1 - y_3)\eta_1, \\
I \eta_1 = -x_2 \eta_1 + (x_1 - x_3)\eta_2, \quad J \eta_1 = (y_1 + y_3)e_2 - y_2 \eta_1, \\
I \eta_2 = -(x_1 + x_3)\eta_1 + x_2 \eta_2, \quad J \eta_2 = -(y_1 + y_3)e_1 - y_2 \eta_2.
\]

This shows that the restriction of $I$ to $V$ is a complex structure on $V$ inducing the generalized complex structure $I$ (as in Example 2). In contrast, the generalized complex structure $J$ is not induced by a complex structure or a symplectic form on $V$. Moreover $J$ is not a $B$- or $\beta$-transform of such structures.

A general almost complex structure on an even-dimensional smooth manifold $M$ is, by definition, an endomorphism $J$ of the bundle $TM \oplus T^*M$ with $J^2 = -Id$ which preserves the natural neutral metric of $TM \oplus T^*M$. Such a structure is said to be integrable or a generalized complex structure if its $+i$-eigensubbundle of $(TM \oplus T^*M) \otimes \mathbb{C}$ is closed under the Courant
Recall that if $X, Y$ are vector fields on $M$ and $\xi, \eta$ are 1-forms, the Courant bracket \cite{6} is defined by the formula
\[
[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi),
\]
where $[X, Y]$ on the right hand-side is the Lie bracket and $\mathcal{L}$ means the Lie derivative. As in the case of almost complex structures, the integrability condition for a generalized almost complex structure $J$ is equivalent to the vanishing of its Nijenhuis tensor $N$, the latter being defined by means of the Courant bracket:
\[
N(A, B) = -[A, B] - J[A, JB] - J[JA, B] + [JA, JB], \quad A, B \in TM \oplus T^* M.
\]

**Example 6** (\cite{11}). A generalized complex structure $K$ induced by an almost complex structure $K$ on $M$ (see Example 2) is integrable if and only the structure $K$ is integrable. A generalized complex structure yielded by a non-degenerate 2-form $\omega$ on $M$ is integrable if and only if the form $\omega$ is closed.

**Example 7** (\cite{11}). Let $J$ be a generalized almost complex structure and $B$ a closed 2-form on $M$. Then the $B$-transform of $J, e^BJe^{-B}, \text{ (see Example 3) is integrable if and only if the structure } J \text{ is integrable.}$

Let us note that the notion of $B$-transform plays an important role in the local description of the generalized complex structures given by M. Gualtieri \cite{11} and M. Abouzaid - M. Boyarchenko \cite{11}.

The existence of a generalized almost complex structure on a 2n-dimensional manifold $M$ is equivalent to the existence of a reduction of the structure group of the bundle $TM \oplus T^* M$ to the group $U(n, n)$. Further, to reduce the structure group to the subgroup $U(n) \times U(n)$ of $U(n, n)$ is equivalent to choosing two commuting generalized almost complex structures $\{J_1, J_2\}$ such that the quadratic form $<J_1A, J_2A>$ on $TM \oplus T^* M$ is positive definite \cite{11}. A pair $\{J_1, J_2\}$ of generalized complex structures with these properties is called an almost generalized Kähler structure. It is said to be a generalized Kähler structure if $J_1$ and $J_2$ are both integrable \cite{11}.

**Example 8** (\cite{11}). Let $(J, g)$ be a Kähler structure on a manifold $M$ and $\omega$ its Kähler form, $\omega(X, Y) = g(JX, Y)$. Let $J_1$ and $J_2$ be the generalized complex structures on $M$ induced by $J$ and $\omega$. Then the pair $\{J_1, J_2\}$ is a generalized Kähler structure.

**Example 9** (\cite{11}). If $\{J_1, J_2\}$ is a generalized Kähler structure and $B$ is a closed 2-form, then its $B$-transform $\{e^B J_1 e^{-B}, e^B J_2 e^{-B}\}$ is also a generalized Kähler structure.

It has been observed by Gualtieri \cite{11} that an almost generalized Kähler structure $\{J_1, J_2\}$ on a manifold $M$ determines the following data on $M$: 1) a Riemannian metric $g$; 2) two almost complex structures $J_2$ compatible with $g$; 3) a 2-form $b$. Conversely, the almost generalized Kähler structure
\{J_1, J_2\} can be reconstructed from the data \((g, J_+, J_-, b)\). In fact, Gualtieri [11] has given an explicit formula for \(J_1\) and \(J_2\) in terms of this data.

**Example 10.** Let \(V\) be a 2-dimensional real vector spaces and \(G^{\pm}(V)\) the space of generalized complex structures on \(V\) yielding \(\pm\) the canonical orientation of \(V \oplus V^*\). Let \((h, \mathcal{K})\) be the Kähler structure on \(G^{\pm}(V)\) defined above. Consider the manifold \(G^+(V) \times G^-(V)\) with the product metric \(g = h \times h\) and the complex structures \(J_+ = K \times K\) and \(J_- = K \times (-K)\). According to [11, formula (6.3)] the generalized Kähler structure \(\{I, J\}\) on \(G^+(V) \times G^-(V)\) determined by \(g, J_+, J_-\) and \(b = 0\) is given by

\[
\begin{align*}
I(U, V) &= I \circ U - V^\flat \circ J, \\
J(U, V) &= J \circ V - U^\flat \circ I \\
I(\varphi, \psi) &= -\varphi \circ I + J \circ \psi^\flat, \\
J(\varphi, \psi) &= -\psi \circ J + I \circ \varphi^\flat
\end{align*}
\]

for \(U \in T_I G^+(V)\), \(V \in T_J G^-(V)\) and \(\varphi \in T^*_I G^+(V)\), \(\psi \in T^*_J G^-(V)\).

Gualtieri [11] has also proved that the integrability condition for \(\{J_1, J_2\}\) can be expressed in terms of the data \((g, J_+, J_-, b)\) in a nice way. In particular, in the case when \(b = 0\), the structures \(\{J_1, J_2\}\) are integrable if and only if the almost-Hermitian structures \((g, J_\pm)\) are Kählerian.

**Example 11.** According to the Gualtieri's result the structure \(\{I, J\}\) defined by (2) is a generalized Kähler structure. Of course, the integrability of \(I\) and \(J\) can be directly proved.

Let \(V\) be an even-dimensional real vector space. The group \(GL(V)\) acts on \(V \oplus V^*\) by letting \(GL(V)\) act on \(V^*\) in the standard way. This action preserves the neutral metric \(II\) and the canonical orientation of \(V \oplus V^*\). Thus, we have an embedding of \(GL(V)\) into the group \(SO(<, >)\) and, via this embedding, \(GL(V)\) acts on the manifold \(G^{\pm}(V)\) in a natural manner. Denote by \(P(V)\) the open subset of \(G^{+}(V) \times G^-(V)\) consisting of those \((I, J)\) for which the quadratic form \(<IA, JA>\) is positive definite on \(V \oplus V^*\). It is clear that the natural action of \(GL(V)\) on \(G^{+}(V) \times G^-(V)\) leaves \(P(V)\) invariant. Suppose that \(dim V = 2\). Let \(I \in G^{+}(V)\) and \(J \in G^-(V)\). Then it is easy to see that, under the notations in Example 5, the quadratic form \(<IA, JA>\) is positive definite if and only if either \(x_1 + x_3 > 0, y_1 + y_3 > 0\) or \(x_1 + x_3 < 0, y_1 + y_3 < 0\). This is equivalent to the condition that either \(x_1 > 0, y_1 > 0\) or \(x_1 < 0, y_1 < 0\). Thus \(P(V)\) is the disjoint union of two products of one-sheeted hyperboloids. Therefore \(P(V)\) endowed with the complex structure \(K \times K\) and the metric \(h \times h\) is biholomorphically isometric to the disjoint union of two copies of the unit bi-disk endowed with the Bergman metric. Note also that, when \(dim V = 2\), every \(I \in G^{+}(V)\) commutes with every \(J \in G^-(V)\) (see Example 5). Thus, in this case, every pair \((I, J) \in P(V)\) is a generalized Kähler structures on the manifold \(V\).

### 3. The twistor space of generalized Kähler structures

Let \(M\) be a smooth manifold of dimension 2. Denote by \(\pi : G^{\pm} \to M\) the bundle over \(M\) whose fibre at a point \(p \in M\) consists of all generalized...
complex structures on $T_p M$ that induce $\pm$ the canonical orientation of $T_p M \oplus T^*_p M$. This is the associated bundle

$$GL(M) \times_{GL(2,\mathbb{R})} G^{\pm}(\mathbb{R}^2),$$

where $GL(M)$ denotes the principal bundle of linear frames on $M$. Consider the product bundle $\pi : G^+ \times G^- \to M$ and denote by $\mathcal{P}$ its open subset consisting of those pairs $K = (I, J)$ for which the quadratic form $\langle IA, JA \rangle$ on $T_p M \oplus T^*_p M$, $p = \pi(K)$, is positive definite. Clearly $\mathcal{P}$ is the associated bundle

$$\mathcal{P} = GL(M) \times_{GL(2,\mathbb{R})} P(\mathbb{R}^2).$$

The projection maps of the bundles $G^\pm$ and $\mathcal{P}$ to the base space $M$ will be denoted by $\pi$.

Let $\nabla$ be a linear connection on $M$. Following the standard twistor construction we can define two commuting almost generalized complex structures $\mathcal{I}^\nabla$ and $\mathcal{J}^\nabla$ on $\mathcal{P}$ as follows: The connection $\nabla$ gives rise to a splitting $\mathcal{V} \oplus \mathcal{H}$ of the tangent bundle of any bundle associated to $GL(M)$ into vertical and horizontal parts. The vertical space $\mathcal{V}_K$ of $\mathcal{P}$ at a point $K = (I,J)$ is the direct sum $\mathcal{V}_K = \mathcal{V}_I G^+ \oplus \mathcal{V}_J G^-$ of vertical spaces and we define $\mathcal{I}^\nabla$ and $\mathcal{J}^\nabla$ on $\mathcal{V}_K$ by means of [2] where the "musical" isomorphisms are determined by the metric $h$ on $\mathcal{V}_I G^+$ and $\mathcal{V}_J G^-$. The horizontal space $\mathcal{H}_K$ is isomorphic via the differential $\pi_* K$ to the tangent space $T_p M, p = \pi(K)$. Denoting $\pi_* K|\mathcal{H}$ by $\pi_\mathcal{H}$, we define $\mathcal{I}^\nabla$ and $\mathcal{J}^\nabla$ on $\mathcal{H}_K \oplus \mathcal{H}_K^*$ as the lift of the endomorphisms $I$ and $J$ by the map $\pi_\mathcal{H} \oplus (\pi_\mathcal{H}^{-1})^*$.

**Remark.** Neither of the generalized almost complex structures $\mathcal{I}^\nabla$ and $\mathcal{J}^\nabla$ is induced by an almost complex or symplectic structure on $\mathcal{P}$. Moreover they are not $B$- or $\beta$-transforms of such structures.

Further on, the generalized almost complex structures $\mathcal{I}^\nabla$ and $\mathcal{J}^\nabla$ will be simply denoted by $\mathcal{I}$ and $\mathcal{J}$ when the connection $\nabla$ is understood. The image of every $A \in T_p M \oplus T^*_p M$ under the map $\pi_\mathcal{H}^{-1} \oplus \pi_\mathcal{H}^*$ will be denoted by $A^h$. The elements of $\mathcal{H}_J^*$, resp. $\mathcal{V}_J^*$ will be considered as 1-forms on $T_j \mathcal{G}$ vanishing on $\mathcal{V}_J$, resp. $\mathcal{H}_J$.

Let $K = (I,J) \in \mathcal{P}, A \in T_{\pi(K)} M \oplus T^*_{\pi(K)} M, W = (U,V) \in \mathcal{V}_K$ and $\Theta = (\varphi, \psi) \in \mathcal{V}_K^*$. Then we have

$$\langle \mathcal{I}(A^h + W + \Theta), \mathcal{J}(A^h + W + \Theta) \rangle = \langle IA, JA \rangle + ||U||^2 + ||V||^2 + ||\varphi||^2 + ||\psi||^2.$$

Therefore the quadratic form $\langle \mathcal{I}, \mathcal{J} \rangle$ is positive definite. Thus the pair $(\mathcal{I}, \mathcal{J})$ is an almost generalized Kähler structure.

We shall show that for a torsion-free connection $\nabla$ the integrability condition for $\mathcal{I}$ and $\mathcal{J}$ can be expressed in terms of the curvature of $\nabla$ (as is usual in the twistor theory).

Let $A(M)$ be the bundle of the endomorphisms of $TM \oplus T^*M$ which are skew-symmetric with respect to its natural neutral metric $\langle , \rangle$; the fibre
of this bundle at a point \( p \in M \) will be denoted by \( A_p(M) \). The connection \( \nabla \) on \( TM \) induces a connection on \( A(M) \), thus a connection on the bundle \( A(M) \oplus A(M) \), both denoted again by \( \nabla \).

Consider the bundle \( \mathcal{P} \) as a subbundle of the bundle \( \pi : A(M) \oplus A(M) \to M \). Then the inclusion \( \mathcal{P} \) is fibre-preserving and the horizontal space of \( \mathcal{P} \) at a point \( K \) coincides with the horizontal space of \( A(M) \oplus A(M) \) at that point since the inclusion \( P(\mathbb{R}^2) \subset so(2,2) \times so(2,2) \) is \( SO(2,2) \)-equivariant.

Let \( (U,x_1,x_2) \) be a local coordinate system of \( M \) and \( \{Q_1,\ldots,Q_4\} \) an orthonormal frame of \( TM \oplus T^*M \) on \( U \). Set \( \varepsilon_k = ||Q_k||^2 \), \( k = 1,\ldots,4 \), and define sections \( S_{ij} \), \( 1 \leq i,j \leq 4 \), of \( A(M) \) by the formula

\[
S_{ij}Q_k = \varepsilon_k(\delta_{ik}Q_j - \delta_{kj}Q_i).
\]

Then \( S_{ij} \), \( i < j \), form an orthogonal frame of \( A(M) \) with respect to the metric \( <a,b> = -\frac{1}{2}Trace(a \circ b) \); \( a,b \in A(M) \); moreover \( ||S_{ij}||^2 = \varepsilon_i\varepsilon_j \) for \( i \neq j \). For \( c = (a,b) \in A(M) \oplus A(M) \), we set

\[
\bar{s}_m(c) = x_m \circ \pi(c), \quad y_{ij}(c) = \varepsilon_i\varepsilon_j < a, S_{ij} >, \quad z_{ij}(c) = \varepsilon_i\varepsilon_j < b, S_{ij} >.
\]

Then \( (\bar{s}_m, y_{ij}, z_{kl}) \), \( m = 1,2,1 \leq i < j \leq 4,1 \leq k < l \leq 4 \), is a local coordinate system on the total space of the bundle \( A(M) \oplus A(M) \). Note that \( (\bar{s}_m, y_{ij}) \) and \( (\bar{s}_m, z_{kl}) \) are local coordinate systems of the manifold \( A(M) \).

Let

\[
U = \sum_{i<j} u_{ij} \frac{\partial}{\partial y_{ij}}(I), \quad V = \sum_{i<j} v_{ij} \frac{\partial}{\partial z_{ij}}(J)
\]

be vertical vectors of \( \mathcal{G}^+ \) and \( \mathcal{G}^- \) at some points \( I \) and \( J \) with \( \pi(I) = \pi(J) \).

It is convenient to set \( u_{ij} = -u_{ji}, \quad v_{ij} = -v_{ji} \) for \( i \geq j \), \( 1 \leq i,j \leq 4 \). Then the endomorphism \( U \) of \( T_pM \oplus T_p^*M \), \( p = \pi(I) \), is determined by \( UQ_i = \sum_{j=1}^4 \varepsilon_i u_{ij} Q_j \); similarly for the endomorphism \( V \) of \( T_pM \oplus T_p^*M \). Moreover

\[
K^+_I U^h = -(IU)^h = \sum_{i<j} \varepsilon_i \varepsilon_j \sum_{r=1}^4 u_{ir}y_{rj}(I)\varepsilon_r(dy_{ij})I.
\]

Similar formula holds for \( K^+_I V^h \). Thus we have

\[
\mathcal{I}(U,V) = \sum_{i<j} \sum_r u_{ir}y_{rj}(I)\varepsilon_r \frac{\partial}{\partial y_{ij}}(I) - \sum_{k<l} \varepsilon_k \varepsilon_l \sum_s v_{ks}z_{sl}(J)\varepsilon_s(dz_{kl})J
\]

and

\[
\mathcal{J}(U,V) = \sum_{k<l} \sum_s v_{ks}z_{sl}(J)\varepsilon_s \frac{\partial}{\partial z_{kl}}(J) - \sum_{i<j} \varepsilon_i \varepsilon_j \sum_r u_{ir}y_{rj}(I)\varepsilon_r(dy_{ij})I.
\]

Note also that, for every \( A \in T_pM \oplus T_p^*M \), we have

\[
A^h = \sum_{i=1}^{4n} (< A, Q_i > \circ \pi) \varepsilon_i Q_i^h
\]
and
\[(7) \ \mathcal{I}A^h = \sum_{i,j=1}^{4} (\langle A, Q_i \rangle \circ \pi) y_{ij} Q_j^h, \quad \mathcal{J}A^h = \sum_{k,l=1}^{4} (\langle A, Q_k \rangle \circ \pi) z_{kl} Q_l^h.\]

For each vector field
\[X = \sum_{i=1}^{2} X_i \frac{\partial}{\partial x_i}\]
on \(U\), the horizontal lift \(X^h\) on \(\pi^{-1}(U)\) is given by
\[(8) \ X^h = \sum_m (X^m \circ \pi) \frac{\partial}{\partial \tilde{x}_m} - \sum_{i<j} \sum_{a<b} y_{ab} (\langle \nabla_X S_{ab}, S_{ij} \rangle \circ \pi) \varepsilon_i \varepsilon_j \frac{\partial}{\partial y_{ij}} - \sum_{k<l} \sum_{c<d} z_{cd} (\langle \nabla_X S_{cd}, S_{kl} \rangle \circ \pi) \varepsilon_k \varepsilon_l \frac{\partial}{\partial z_{kl}}.\]

Let \(c = (a, b) \in A(M) \oplus A(M)\) and \(p = \pi(c)\). Then \(\mathcal{I}\) implies that, under the standard identification of \(T_c(A_p(M) \oplus A_p(M))\) with the vector space \(A_p(M) \oplus A_p(M)\), we have
\[(9) \ [X^h, Y^h]_c = [X, Y]^h_c + R(X, Y)c,\]
where \(R(X, Y)c = (R(X, Y)a, R(X, Y)b)\) is the curvature of the connection \(\nabla\) on \(A(M) \oplus A(M)\) (for the curvature tensor we adopt the following definition: \(R(X, Y) = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y]\)).

**Notation.** Let \(K = (I, J) \in \mathcal{P}\) and \(p = \pi(K)\). There exists an oriented orthonormal basis \(\{a_1, ..., a_4\}\) of \(T_p M \oplus T^*_p M\) such that \(a_2 = Ia_1, a_4 = Ia_3\) and \(Ja_1 = \varepsilon a_2, Ja_3 = -\varepsilon a_4\), where \(\varepsilon = +1\) or \(-1\). Let \(\{Q_i\}, i = 1, ..., 4,\) be an oriented orthonormal frame of \(TM \oplus T^*M\) near the point \(p\) such that
\[Q_i(p) = a_i \text{ and } \nabla Q_i|_p = 0, \ i = 1, ..., 4.\]

Define sections \(S\) and \(T\) of \(A(M)\) by setting
\[SQ_1 = Q_2, \quad SQ_2 = -Q_1, \quad SQ_3 = Q_4, \quad SQ_4 = -Q_3\]
\[TQ_1 = \varepsilon Q_2, \quad JQ_2 = -\varepsilon Q_1, \quad TQ_3 = -\varepsilon Q_4, \quad TQ_4 = \varepsilon Q_3.\]

Then \(\nu = (S, T)\) is a section of \(\mathcal{P}\) such that
\[\nu(p) = K, \quad \nabla \nu|_p = 0\]
(considering \(\nu\) as a section of \(A(M) \oplus A(M)\)). Thus \(X^h_K = \nu_* X\) for every \(X \in T_p M\).

Further, given a smooth manifold \(N\), the natural projections of \(TN \oplus T^*N\) onto \(TN\) and \(T^*N\) will be denoted by \(\pi_1\) and \(\pi_2\), respectively.

We shall use the above notations throughout this section.
The next three technical lemmas can be easily proved by means of \([\mathcal{L}], \mathfrak{N}\) and \([J]\).

**Lemma 1.** If \(A\) and \(B\) are sections of the bundle \(TM \oplus T^*M\) near \(p\), then:

(i) \([\pi_1(A^h), \pi_1(TB^h)]_K = [\pi_1(A), \pi_1(SB)]_K^h + R(\pi_1(A), \pi_1(IB))K\).

(ii) \([\pi_1(TA^h), \pi_1(TB^h)]_K = [\pi_1(SA), \pi_1(SB)]_K^h + R(\pi_1(IA), \pi_1(IB))K\).

**Lemma 2.** Let \(A\) and \(B\) be sections of the bundle \(TM \oplus T^*M\) near \(p\), and let \(Z \in T_p M\), \(W = (U, V) \in \mathcal{V}_K = \mathcal{V}_1G^+ \oplus \mathcal{V}_1G^-\). Then:

(i) \((\mathcal{L}_{\pi_1(A^h)}\pi_2(B^h))_K = (\mathcal{L}_{\pi_1(A)}\pi_2(B))_K^h\).

(ii) \((\mathcal{L}_{\pi_1(A^h)}\pi_2(TB^h))_K = (\mathcal{L}_{\pi_1(A)}\pi_2(SB))_K^h\).

(iii) \((\mathcal{L}_{\pi_1(A)}\pi_2(TB^h))_K(Z^h + W) = (\mathcal{L}_{\pi_1(SA)}\pi_2(B))_K^h(Z^h) + (\pi_2(B))_p(\pi_1(UA))\).

(iv) \((\mathcal{L}_{\pi_1(A)}\pi_2(SB))_K(Z^h) + (\pi_2(IB))_p(\pi_1(UA))\).

**Lemma 3.** Let \(A\) and \(B\) are sections of the bundle \(TM \oplus T^*M\) near \(p\). Let \(Z \in T_p M\) and \(W = (U, V) \in \mathcal{V}_K = \mathcal{V}_1G^+ \oplus \mathcal{V}_1G^-\). Then:

(i) \((d i_{\pi_1(A^h)}\pi_2(B^h))_K = (d i_{\pi_1(A)}\pi_2(B))_K^h\).

(ii) \((d i_{\pi_1(A^h)}(TB^h))_K(Z^h + W) = (d i_{\pi_1(A)}\pi_2(SB))_K^h(Z^h) + (\pi_2(UB))_p(\pi_1(A))\).

(iii) \((d i_{\pi_1(TA^h)}\pi_2(B^h))_K(Z^h + W) = (d i_{\pi_1(SA)}\pi_2(B))_K^h(Z^h) + (\pi_2(B))_p(\pi_1(UA))\).

(iv) \((d i_{\pi_1(TA^h)}\pi_2(TB^h))_K(Z^h + W) = (d i_{\pi_1(SA)}\pi_2(SB))_K^h(Z^h) + (\pi_2(IB))_p(\pi_1(UA))\).

**Proposition 1.** Suppose that the connection \(\nabla\) is torsion-free and let \(K = (I, J) \in \mathcal{P}\). Then

(i) \(N^2(A^h, B^h) = 0\) for every \(A, B \in T_{\pi(K)} M \oplus T_{\pi(K)}^* M\).
(ii) $N^J(A^h, B^h) = 0$ for every $A, B \in T_{\pi(K)}^* M \oplus T^*_{\pi(K)} M$ if and only if $R(X, Y)J = 0$ for every $X, Y \in T_{\pi(K)} M$.

**Proof.** First we shall show that

$$N^J(A^h, B^h)_K = -R(\pi_1(A), \pi_1(B))I - I \circ R(\pi_1(A), \pi_1(IB))I$$

\begin{equation}
- I \circ R(\pi_1(IA), \pi_1(B))I + R(\pi_1(IA), \pi_1(IB))I
- R(\pi_1(A), \pi_1(B))J + R(\pi_1(IA), \pi_1(IB))J
+ K^*_J(R(\pi_1(A), \pi_1(IB))J) + K^*_J(R(\pi_1(IA), \pi_1(IB))J).
\end{equation}

Similar formula holds for the Nijenhuis tensor $N^J$ with interchanged roles of $I$ and $J$ in the right-hand side of (10).

Set $p = \pi(K)$ and extend $A$ and $B$ to (local) sections of $TM \oplus T^* M$, denoted again by $A, B$, in such a way that $\nabla A|_p = \nabla B|_p = 0$.

Let $\nu = (S, T)$ be the section of $\mathcal{P}$ defined above with the property that $\nu(p) = K$ and $\nabla \nu|_p = 0$ ($\nu$ being considered as a section of $A(M) \oplus A(M)$).

According to Lemmas 1, 2 and 3, the part of $N^J(A^h, B^h)_K$ lying in $\mathcal{H}_K \oplus \mathcal{H}^*_K$ is given by

$$\mathcal{(H \oplus H^*)N^J(A^h, B^h)_K =}
\begin{equation}
- [A, B] - S[A, SB] - S[SA, B] + [SA, SB] b_K.
\end{equation}

Note that we have $\nabla \pi_1(A)|_p = \pi_1(\nabla A)|_p = 0$ and $\nabla \pi_1(SA)|_p = \pi_1((\nabla S)|_p(A) + S(\nabla A)|_p) = 0$. Similarly, $\nabla \pi_2(A)|_p = 0$ and $\nabla \pi_2(SA)|_p = 0$. We also have $\nabla \pi_1(B)|_p = 0$, $\nabla \pi_1(SB)|_p = 0$ and $\nabla \pi_2(B)|_p = 0$, $\nabla \pi_2(SB)|_p = 0$. Now, since $\nabla$ is torsion-free, we can easily see that every bracket in (11) vanishes by means of the following simple observation: Let $Z$ be a vector field and $\omega$ a 1-form on $M$ such that $\nabla Z|_p = 0$ and $\nabla \omega|_p = 0$. Then for every $T \in T_p M$

$$\mathcal{(L_Z} \omega)(T)_p = (\nabla Z \omega)(T)_p = 0 \quad \text{and} \quad (d_iZ \omega)(T)_p = (\nabla_T \omega)(Z)_p = 0.$$

By Lemmas 1, 2, 3 the part of $N^J(A^h, B^h)_K$ lying in $\mathcal{V}_K$ is

$$- R(\pi_1(A), \pi_1(B))I - I \circ R(\pi_1(A), \pi_1(IB))I
- I \circ R(\pi_1(IA), \pi_1(B))I + R(\pi_1(IA), \pi_1(IB))I
- R(\pi_1(A), \pi_1(B))J + R(\pi_1(IA), \pi_1(IB))J.$$

Finally, the part of $N^J(A^h, B^h)_K$ lying in $\mathcal{V}^*_K$ is the vertical form whose value at every vertical vector $W = (U, V) \in \mathcal{V}_K$ is equal to

$$\mathcal{1/2}\left\{- \pi_2(UB)(\pi_1(A)) - \pi_2(A)(\pi_1(UB))
+ \pi_2(UA)(\pi_1(B)) + \pi_2(B)(\pi_1(UA))
+ \pi_2(IB)(\pi_1(UA)) + \pi_2(UA)(\pi_1(IB))
- \pi_2(IA)(\pi_1(UB)) - \pi_2(UB)(\pi_1(IA))\right\}
+ K^*_J(R(\pi_1(A), \pi_1(IB))J) + K^*_J(R(\pi_1(IA), \pi_1(IB))J)^b.$$. 
The endomorphism $U$ of $T_pM \oplus T^*_pM$ is skew-symmetric with respect to the metric $\langle , \rangle$ and anti-commutes with $I$. Thus we have
\[ \langle IU A, B \rangle = \langle IA, UB \rangle. \]

This identity reads as
\[ \pi_2(IUA)(\pi_1(B)) + \pi_2(B)(\pi_1(IUA)) = \pi_2(IA)(\pi_1(UB)) + \pi_2(UB)(\pi_1(IA)). \]

Therefore the part of $N^T(A^h, B^h)_K$ lying in $V^*_R$ is
\[ K^j_\bullet(R(\pi_1(A), \pi_1(IB))J)^j + K^j_\bullet(R(\pi_1(IB), \pi_1(B))J)^j. \]

This proves formula (10).

Now let \( \{Q_1, Q_2 = IQ_1, Q_3, Q_4 = IQ_3\} \) be an orthonormal basis of $T_pM \oplus T^*_pM$. To prove that $N^T(A^h, B^h)_K = 0$ it is enough to show that $N^T(Q^h_1, Q^h_2)_K = 0$ since $N^T(IE, F) = N^T(E, IF) = -T^N^T(E, F)$ for every $E, F \in TP$.

Let $\pi_1(Q_i) = e_i$, $i = 1, \ldots, 4$. Then, according to (10)
\[ N^T(Q^h_1, Q^h_3) = [-R(e_1, e_3)I + R(e_2, e_4)I] - I \circ [R(e_1, e_4)I + R(e_2, e_3)I] \]
\[ + K^j_\bullet(R(e_1, e_4)J + R(e_2, e_3)J)^j. \]

Since $I$ yields the canonical orientation of $T_pM \oplus T^*_pM$, the latter expression vanishes in view of the following simple algebraic fact proved in [3]:

**Lemma 4.** Let $V$ be a 2-dimensional real vector space and let $\{Q_i = e_i + \eta_i\}$, $1 \leq i \leq 4$, be an orthonormal basis of the space $V \oplus V^*$ endowed with its natural neutral metric (11). Then $\{e_1, e_2\}$ is a bases of $V$ and
\[
\begin{align*}
    e_3 &= a_{11}e_1 + a_{12}e_2 \\
    e_4 &= a_{21}e_1 + a_{22}e_2
\end{align*}
\]
where $A = [a_{kl}]$ is an orthogonal matrix. If $\det A = 1$, the basis $\{Q_i\}$ yields the canonical orientation of $V \oplus V^*$ and if $\det A = -1$ it yields the opposite one.

To prove statement (ii), take an orthonormal basis $\{Q_1, Q_2 = JQ_1, Q_3, Q_4 = JQ_3\}$ and set $\pi_1(Q_i) = e_i$, $i = 1, \ldots, 4$. Suppose that $N^J(Q^h_1, Q^h_3) = 0$. Then, according to the analog of (10) for $N^J(A^h, B^h)_K$, we have
\[ -R(e_1, e_3)J + R(e_2, e_4)J - J \circ [R(e_1, e_4)J + R(e_2, e_3)J] = 0. \]

Since $J$ yields the orientation of $T_pM \oplus T^*_pM$ opposite to the canonical one, then, by Lemma (11) $e_3 = \cos t e_1 + \sin t e_2$, $e_4 = \sin t e_1 - \cos t e_2$ for some $t \in \mathbb{R}$. Thus
\[ -\sin t \cdot R(e_1, e_2)J + \cos t \cdot J \circ R(e_1, e_2)J = 0, \]
which implies
\[ \cos t \cdot R(e_1, e_2)J + \sin t \cdot J \circ R(e_1, e_2)J = 0. \]

Therefore $R(e_1, e_2)J = 0$, so $R(X, Y)J = 0$ for every $X, Y \in T_pM$.

Conversely, if the latter identity holds, the analog of (10) shows that $N^J(A^h, B^h)_K = 0$. \(\square\)
Proposition 2. Suppose that the connection $\nabla$ is torsion-free and let $K = (I, J) \in \mathcal{P}$. Then

(i) $N^J(A^h, W) = 0$ for every $A \in T_{\pi(K)}M \oplus T^*_{\pi(K)}M$ and $W \in \mathcal{V}_K$ if and only if $R(X, Y)J = 0$ for every $X, Y \in T_{\pi(K)}M$.

(ii) $N^J(A^h, W) = 0$ for every $A \in T_{\pi(K)}M \oplus T^*_{\pi(K)}M$ and $W \in \mathcal{V}_K$ if and only if $R(X, Y)I = 0$ for every $X, Y \in T_{\pi(K)}M$.

Proof. Set $p = \pi(K)$ and $W = (U, V)$. Extend $A$ to a section of $TM \oplus T^*M$ denoted again by $A$. Take sections $a$ and $b$ of $A(M)$ such that

$$a(p) = U, \quad b(p) = V, \quad \nabla a|_p = \nabla b|_p = 0.$$ 

Define vertical vector fields $\tilde{a}$ and $\tilde{b}$ on $\mathcal{G}^+$ and $\mathcal{G}^-$, respectively, setting

$$\tilde{a}_{i'} = a_{\pi(i')} + I \circ a_{\pi(i')} \circ I', \quad \tilde{b}_{j'} = b_{\pi(j')} + J \circ b_{\pi(j')} \circ J', \quad \tilde{a}_{i'}, \tilde{b}_{j'} \in \mathcal{G}^+, \quad \tilde{a}_{i'}, \tilde{b}_{j'} \in \mathcal{G}^-.$$ 

Then

$$\tilde{W}_{i', j'} = (\tilde{a}_{i'}, \tilde{b}_{j'}), \quad (I', J') \in \mathcal{P},$$

is a vertical vector field on $\mathcal{P}$ with $\tilde{W}_K = 2W$.

Let $a(Q_i) = \sum_j \varepsilon_i a_{ij} Q_j$, $\quad b(Q_i) = \sum_j \varepsilon_i b_{ij} Q_j$. Then, in the local coordinates introduced above,

$$\tilde{W} = \sum_{i<j} (\tilde{a}_{ij} \frac{\partial}{\partial y_{ij}} + \tilde{b}_{ij} \frac{\partial}{\partial z_{ij}}),$$

where

$$\tilde{a}_{ij} = a_{ij} \circ \pi + \sum_{k,l} y_{ik}(a_{kl} \circ \pi)y_{lj} \varepsilon_k \varepsilon_l, \quad \tilde{b}_{ij} = b_{ij} \circ \pi + \sum_{k,l} z_{ik}(b_{kl} \circ \pi)z_{lj} \varepsilon_k \varepsilon_l.$$ 

In view of \ref{12}, for any vector field $X$ on $M$ near the point $p$, we have

$$X_K^h = \sum_m X^m(p) \frac{\partial}{\partial x_m}(K), \quad [X_K^h, \frac{\partial}{\partial y_{ij}}]_K = [X_K^h, \frac{\partial}{\partial z_{ij}}]_K = 0,$$

and

$$0 = (\nabla_{X_p} a)(Q_i) = \sum_j \varepsilon_i X_p(a_{ij}) Q_j, \quad 0 = (\nabla_{X_p} b)(Q_i) = \sum_j \varepsilon_i X_p(b_{ij}) Q_j$$

since $\nabla Q_i|_p = 0$ and $\nabla S_{ij}|_p = 0$. In particular, $X_p(a_{ij}) = X_p(b_{ij}) = 0$,

hence

$$X_K^h(\tilde{a}_{ij}) = X_K^h(\tilde{b}_{ij}) = 0.$$ 

Now simple calculations making use of \ref{14}, \ref{12} and \ref{13} give

$$[X_K^h, \tilde{W}]_K = 0.$$ 

Let $\omega$ be a 1-form on $M$. It is easy to see that for every vertical vector field $W'$ on $\mathcal{P}$

$$[\omega^h, W'] = 0.$$
Therefore, by (14) and (17), we have

\[(A^h, \tilde{W})_K = 0.\]  

Next, in view of (17), (4), (14) and (15), we have

\[[A^h, I\tilde{W}]_K = [\pi_1(A^h), I\tilde{W}]_K = (\mathcal{L}_{\pi_1(A^h)}\pi_2(I\tilde{W}))_K.\]

Let \(W' = (U', V') \in \mathcal{V}_K.\) Take sections \(a', b'\) of \(A(M)\) such that \(a'(p) = U',\) \(b'(p) = V',\) \(\nabla a'|_p = \nabla b'|_p = 0.\) Define vertical vector fields \(\tilde{a}'\) and \(\tilde{b}'\) on \(G^+\) and \(G^-\) by means of (12) and set \(\tilde{W}' = (\tilde{a}', \tilde{b}')\) on \(\mathcal{P}.\) Then \([X^h, \tilde{W}']_K = 0\) for every vector field \(X\) near the point \(p\) and an easy computation making use of (12), (14) and (15) gives

\[(\mathcal{L}_{\pi_1(A^h)}\pi_2(I\tilde{W}))_K(W') = \frac{1}{2}(\mathcal{L}_{\pi_1(A^h)}\pi_2(I\tilde{W}))_K(\tilde{W}') = 0.\]

Moreover, for every vector field \(Z\) on \(M\) near the point \(p\) we have

\[(\mathcal{L}_{\pi_1(A^h)}\pi_2(I\tilde{W}))_K(Z^h) = -\pi_2(I\tilde{W})((\pi_1(A^h), Z^h)_K) = 2V^h(J(R(\pi_1(A), Z), J) = 2 < JV, R(\pi_1(A), Z) >.)\]

by (2) and (2). It is convenient to define a 1-form \(\gamma_A\) on \(T_p M\) setting

\[\gamma_A(Z) = < JV, R(\pi_1(A), Z) >, \quad Z \in T_p M.\]

Then

\[[A^h, I\tilde{W}]_K = 2\gamma_A^h.\]

Computations in local coordinates involving (7), (11), (14) and (15) show that

\[[I A^h, \tilde{W}]_K = -2(U(A))_K^h.\]

and

\[[I A^h, I\tilde{W}]_K = -2((I U)(A))_K^h + 2\gamma_A^h.\]

It follows that

\[N^h(A^h, W) = \frac{1}{2}N^h(A^h, \tilde{W})_K = -J\gamma_A^h + \gamma_A^h.\]

Let \(\{e_1, e_2\}\) be a bases of \(T_p M\) and denote by \(\{\eta_1, \eta_2\}\) its dual bases. Then \(Q_1 = e_1 + \eta_1, Q_2 = e_2 + \eta_2, Q_3 = e_1 - \eta_1, Q_4 = e_2 - \eta_2\) constitute an orthonormal bases of \(T_p M \oplus T_p M^*\) yielding its canonical orientation. According to Example 5, every generalized complex structure \(J \in G^-(T_p M)\) is given by

\[
\begin{align*}
Q_1 & \to y_1 Q_2 + y_2 Q_3 + y_3 Q_4, & Q_2 & \to -y_1 Q_1 + y_2 Q_4 - y_3 Q_3 \\
Q_3 & \to -y_1 Q_4 + y_2 Q_1 - y_3 Q_2, & Q_4 & \to y_1 Q_3 + y_2 Q_2 + y_3 Q_1,
\end{align*}
\]

where \(y_1^2 - y_2^2 - y_3^2 = 1, y_1, y_2, y_3 \in \mathbb{R}.\) Then

\[
J\gamma_A^h = \gamma_A(e_1)(J\eta_1)^h + \gamma_A(e_2)(J\eta_2)^h =
\]

\[-(y_1 + y_3)\gamma_A(e_2)e_1^h + (y_1 + y_3)\gamma_A(e_1)e_2^h - y_2\gamma_A(e_1)\eta_1^h - y_2\gamma_A(e_2)\eta_2^h.\]
Therefore the identity $N^2(A^h, W) = 0$ implies $\gamma_A(e_1) = \gamma_A(e_2) = 0$, i.e. $\gamma_A = 0$. This proves statement (i). The proof of (ii) is similar. \hfill \Box

Now suppose that $R(X, Y)I = 0$ for every generalized complex structure $I \in G^+(T_pM)$, $X, Y \in T_pM$ being fixed. Take a basis $\{e_1, e_2\}$ of $T_pM$, denote by $\{\eta_1, \eta_2\}$ its dual bases and set $Q_1 = e_1 + \eta_1$, $Q_2 = e_2 + \eta_2$, $Q_3 = e_1 - \eta_1$, $Q_4 = e_2 - \eta_2$. Then every $I$ is given by (see Example 5)

$$
\begin{align*}
Q_1 &\rightarrow x_1Q_2 + x_2Q_3 + x_3Q_4, & Q_2 &\rightarrow -x_1Q_1 - x_2Q_4 + x_3Q_3 \\
Q_3 &\rightarrow -x_1Q_4 + x_2Q_1 + x_3Q_2, & Q_4 &\rightarrow -x_1Q_3 - x_2Q_2 + x_3Q_1,
\end{align*}
$$

where $x_1^2 - x_2^2 - x_3^2 = 1$, $x_1, x_2, x_3 \in \mathbb{R}$. The identity $R(X, Y)I = 0$ implies

$$
<R(X, Y)e_1, \eta_k> + <R(X, Y)e_1, \eta_k> = 0, \quad k = 1, 2,
$$

which is equivalent to

$$
(x_1 + x_3)\eta_1(R(X, Y)e_2) + (x_1 - x_3)\eta_2(R(X, Y)e_1) = 0,
$$

$$
2x_2\eta_2(R(X, Y)e_2) - (x_1 + x_3)\eta_1(R(X, Y)e_1) + (x_1 + x_3)\eta_2(R(X, Y)e_2) = 0.
$$

It follows that $R(X, Y)I = 0$ for every $I \in G^+(T_pM)$ if and only if $\eta_1(R(X, Y)e_1) + \eta_2(R(X, Y)e_2) = 0$.

Thus if the structures $\mathcal{I}$ and $\mathcal{J}$ are both integrable, then the connection $\nabla$ is flat. The converse is also true as the following result shows.

**Theorem 1.** Let $M$ be a 2-dimensional manifold and $\nabla$ a torsion-free connection on $M$. Then the generalized almost complex structures $\mathcal{I}$ and $\mathcal{J}$ induced by $\nabla$ on the twistor space $\mathcal{P}$ are both integrable if and only if the connection $\nabla$ is flat.

**Proof.** Since the structures $\mathcal{I}$ and $\mathcal{J}$ on $\mathcal{V} \oplus \mathcal{V}^*$ are induced by complex structures on the fibres of $\mathcal{P}$ the Nijenhuis tensors of $\mathcal{I}$ and $\mathcal{J}$ vanish on $\mathcal{V} \oplus \mathcal{V}^*$. Thus, in view of Propositions 1 and 2 we have to consider these tensors only on $\mathcal{H} \times \mathcal{V}^*$.

Suppose that the connection $\nabla$ is flat. Let $K = (I, J) \in \mathcal{P}$. Fix bases $\{U_1, U_2 = \mathcal{K}^+U_1\}$ of $\mathcal{V}_p^+\mathcal{G}^+$ and $\{V_1, V_2 = \mathcal{K}^-V_1\}$ of $\mathcal{V}_p\mathcal{G}^-$. Take sections $a_1$ and $b_1$ of $A(M)$ near the point $p = \pi(K)$ such that $a_1(p) = U_1$, $b_1(p) = V_1$ and $\nabla a_1|_p = \nabla b_1|_p = 0$. Define vertical vector fields $\tilde{a}_1$ and $\tilde{b}_1$ on $\mathcal{G}^+$ and $\mathcal{G}^-$ by means of (12). Set $\tilde{a}_2 = \mathcal{K}^+\tilde{a}_1$, $\tilde{b}_2 = \mathcal{K}^-\tilde{b}_1$. Then $\{\tilde{a}_1, \tilde{a}_2\}$ and $\{\tilde{b}_1, \tilde{b}_2\}$ are frames of the vertical bundles $\mathcal{V}\mathcal{G}^+$ and $\mathcal{V}\mathcal{G}^-$ near the points $I$ and $J$, respectively. Denote by $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ the dual frames of $\{\tilde{a}_1, \tilde{a}_2\}$ and $\{\tilde{b}_1, \tilde{b}_2\}$. Set $\tilde{W}_i = (\tilde{a}_i, 0)$, $\tilde{\gamma}_i = (\alpha_i, 0)$ and $\tilde{W}_{i+2} = (0, \tilde{b}_i)$, $\gamma_{i+2} = (0, \beta_i)$ for $i = 1, 2$. Then $\{\tilde{W}_1, \tilde{W}_2, \tilde{W}_3, \tilde{W}_4\}$ is a frame of the vertical bundle $\mathcal{V}$ near the point $K$ and $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ is its dual frame. We have $\gamma_2 = \mathcal{I}_{\gamma_1}, \gamma_3 = \mathcal{I}_{\gamma_2} = -\beta_2^0, \gamma_4 = -\beta_2^1$. If $A \in T_pM \oplus T_p^*M$, then $\mathcal{I}N^2(A^h, \gamma_3) = -N^2(A^h, \mathcal{I}_{\gamma_3}) = -N^2(A^h, \beta_2^2) = 0$ by Proposition 2. Hence $N^2(A^h, \gamma_3) = 0$. Similarly, $N^2(A^h, \gamma_4) = 0$.

As in the proof of Proposition 2 it is not hard to see that $[\pi_1(A^h), \tilde{W}_r]|_K = 0$, $r = 1, \ldots, 4$, $[\pi_1(\mathcal{I}A^h), \tilde{W}_i]|_K = -(\pi_1(\mathcal{I}\tilde{a}_i(A)))|_K$ and $[\pi_1(\mathcal{I}A^h), \tilde{W}_{i+2}]|_K = 0$,
\[ i = 1, 2. \] In particular \( \pi_1(A^h), \tilde{W}_r \) and \( \pi_1(ZA^h), \tilde{W}_r \) are horizontal vectors for every \( r = 1, \ldots, 4 \). It follows, in view of (9) and Lemma (12), that for every \( Z \in T_p M, r = 1, \ldots, 4 \) and \( s = 1, 2 \)

\[
(\mathcal{L}_{\pi_1(A^h)\gamma_s})_K(Z^h + \tilde{W}_r) = -\alpha_s(R(\pi_1(A), Z)I) = 0,
\]

\[
(\mathcal{L}_{\pi_1(ZA^h)\gamma_s})_K(Z^h + \tilde{W}_r) = -\alpha_s(R(\pi_1(IA), Z)I) = 0
\]

since the connection \( \nabla \) is flat. This implies \( N^2(A^h, \gamma_s)_K = 0 \) for \( s = 1, 2 \).

It follows that \( N^2(A^h, \Theta)_K = 0 \) for every \( \Theta \in \mathcal{V}_K \). Similarly, \( N^2(A^h, \Theta)_K = 0 \).

Denote by \( (g, J_+, J_-, b) \) the data on \( \mathcal{P} \) determined by the almost generalized Kähler structure \( \{ T^\nabla, J^\nabla \} \) as described in (11). It is not hard to see that the metric \( g \), the almost complex structures \( J_\pm \) and the 2-form \( b \) are given as follows. Let \( K = (I, J) \in \mathcal{P}, X, Y \in T_{\pi(K)} M, W = (U, V) \in \mathcal{V}_K \).

Let \( \{e_1, e_2\} \) be a local frame of \( TM \) near the point \( \pi(K) \) and denote by \( \{\eta_1, \eta_2\} \) its dual co-frame. Define endomorphisms \( I_r, J_s, r, s = 1, 2, 3 \), by means of \( e_1, e_2, \eta_1, \eta_2 \) as in Example 5. Then \( I = \sum_r x_r I_r, J = \sum_s y_s J_s \) with \( x_1^2 - x_2^2 - x_3^2 = 1, y_1^2 - y_2^2 - y_3^2 = 1 \). Let \( X = X_1 e_1 + X_2 e_2, Y = Y_1 e_1 + Y_2 e_2 \). Then

\[
g(X^h, Y^h)_K = \frac{1}{y_1 + y_3}[(x_1 + x_3)X_1 Y_1 - x_2 (X_1 Y_2 + X_2 Y_1) + (x_1 - x_3)X_2 Y_2],
\]

\[
g(X^h, W)_K = 0, \quad g(\mathcal{V}_K \times \mathcal{V}_K) = h.
\]

\[
J^h X^h_K = (IX)^h, \quad J^h X^h_K = (IX)^h,
\]

\[
J^h(U, V) = (I \circ U, J \circ V), \quad J^h(U, V) = (I \circ U, -J \circ V).
\]

\[
b(X^h, Y^h)_K = \frac{y_2}{y_1 + y_3}(X_1 Y_2 - X_2 Y_1),
\]

\[
b(X^h, W)_K = 0, \quad b(\mathcal{V}_K \times \mathcal{V}_K) = 0.
\]

In particular, the almost complex structures \( J_+ \) and \( J_- \) commutes and \( J_+ \neq \pm J_- \).

Computations similar to that above show that the almost complex structures \( J_\pm \) are both integrable for any torsion-free connection \( \nabla \). Denote by \( \omega_\pm \) the Kähler form of the Hermitian structure \( (g, J_\pm) \) on \( \mathcal{P} \). Then

\[
\omega_\pm(X^h, Y^h)_K = (y_1 + y_3)^{-1}(X_1 Y_2 - X_2 Y_1), \quad \omega_\pm(X^h, W)_K = 0,
\]

\[
\omega_\pm(W, W') = h(I \circ U, U') \pm h(J \circ V, V'), \quad \text{where } W' = (U', V') \in \mathcal{V}_K.
\]

Set \( V = \sum_s v_s J_s \). Then we easily obtain that

\[
3d\omega_\pm(X^h, Y^h, W)_K = -(v_1 + v_3)(y_1 + y_3)^{-2}((X_1 Y_2 - X_2 Y_1)
\]

\[
+ h(R(X, Y)I, I \circ U) \pm h(R(X, Y)J, J \circ U) \]
in view of (2) and the fact that $[X^h, W]_K$ and $[Y^h, W]_K$ are vertical vectors. Moreover
\[
h(R(X, Y), J, J \circ V) = - <R(X, Y), J \circ V> = 2(y_1 + y_3)(y_2(v_1 - v_3) + v_2(y_1 - y_3))(\eta_1(R(X, Y)e_1) + \eta_2(R(X, Y)e_2))\]
Thus putting $y_1 = 2, y_2 = 0, y_3 = \sqrt{3}, U = 0, v_1 = \sqrt{3}, v_2 = 0, v_3 = 2$ we see that $d\omega_\pm(X^h, Y^h, W) \neq 0$. Therefore the structure $(g, J_\pm)$ is not Kählerian.

Acknowledgement: We would like to thank W. Goldman for sending us a proof of the classification of complete affine 2-dimensional manifolds.

References

[1] M.Abouzaid, M.Boyarchenko, Local structure of generalized complex manifolds, J.Sympl.Geom. 4(2006), 43-62, arxiv:math.DG/0412084.
[2] V.Apostolov, P.Gauduchon, G.Grantcharov, Bihermitian structures on complex surfaces, Proc.London Math.Soc. (3) 79 (1999), 414-428. Corrigendum, 92 (2006), 200-2002.
[3] V.Apostolov, M.Gualtieri, Generalized Kähler manifolds with split tangent bundle, arxiv:math.DG/0605342.
[4] M.F.Atiyah, N.J.Hitchin, I.M.Singer, Self-duality in four-dimensional Riemannian geometry, Proc.R.Soc.London, Ser.A 362 (1978), 425-461.
[5] H.Bursztyn, G.Cavalcanti, M.Gualtieri, Reduction of Courant algebroids and generalized complex structures, arxiv:math.DG/0509640.
[6] T.Courant, Dirac manifolds, Trans.Amer.Math.Soc. 319 (1990), 631-661.
[7] T.Courant, A.Weinstein, Beyond Poisson structures, in Action Hamiltoniennes de Groupes. Troisième Théorème de Lie (Lyon, 1986), in: Travaux en Cours, vol.27, Hermann, Paris, 1988, pp. 39-49.
[8] J.Davidov, O.Muskarov, Twistor spaces of generalized complex structures, J.Geom.Phys. 56 (2006), 1623-1636, arxiv:math.DG/0501396.
[9] D.Fried, W.Goldman, Three-dimensional affine crystallographic groups, Adv.Math. 47 (1983), 1-49.
[10] W.Goldman, Private communication.
[11] M.Gualtieri, Generalized complex geometry, Ph.D. Thesis, St John’s College, University of Oxford, 2003, arxiv:math.DG/0401221.
[12] N.Hitchin, Generalized Calabi-Yau manifolds, Q.J.Math. 54 (2004), 281-308, arxiv:math.DG/0209099.
[13] N.Hitchin, Instantons, Poisson structures and generalized Kähler geometry, Comm.Math.Phys. 265 (2006), 131-164, arxiv:math.DG/0503432.
[14] N.Hitchin, Bihermitian metrics on Del Pezzo surface, arxiv:math.DG/0608213.
[15] P.Kobak, Explicit doubly-Hermitian metrics, Diff.Geom.Appl. 10 (1999), 179-185.
[16] Y.Lin, S.Tolman, Symmetries in generalized Kähler geometry, arxiv:math.DG/0509069.
[17] R.Penrose, Twistor theory, its aims and achievements, in: C.J.Isham, R.Penrose, D.W.Sciama (Eds.), Quantum gravity, an Oxford Symposium, Clarendon Press, Oxford, 1975, 268-407.