RINGS IN WHICH EVERY UNIT IS A SUM OF A NILPOTENT AND AN IDEMPOTENT

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Dedicated to the memory of Bruno J. Müller

Abstract. A ring $R$ is a UU ring if every unit is unipotent, or equivalently if every unit is a sum of a nilpotent and an idempotent that commute. These rings have been investigated in Călugăreanu [3] and in Danchev and Lam [7]. In this paper, two generalizations of UU rings are discussed. We study rings for which every unit is a sum of a nilpotent and an idempotent, and rings for which every unit is a sum of a nilpotent and two idempotents that commute with one another.

1. Introduction

The motivation of this paper is a recent work of Danchev and Lam [7] on UU rings. Throughout, $R$ is an associative ring with identity. We denote by $J(R)$, $U(R)$, $\text{Nil}(R)$ and $\text{idem}(R)$ the Jacobson radical, the unit group, the set of nilpotent elements and the set of idempotents of $R$, respectively. In [8], Diesl introduced (strongly) nil-clean elements and rings as follows. An element $a$ in a ring $R$ is called (strongly) nil-clean if $a$ is the sum of an idempotent and a nilpotent (that commute with each other), and the ring is called (strongly) nil-clean if each of its elements is (strongly) nil-clean. One of the results in [8] states that a ring $R$ is strongly nil-clean if and only if $R$ is strongly $\pi$-regular with $U(R) = 1 + \text{Nil}(R)$.

This motivated Călugăreanu [3] to introduce and study UU rings (rings whose units are unipotent). Equivalently, a ring is a UU ring if and only if every unit is strongly nil-clean. These rings have been extensively investigated in Danchev and Lam [7], where, among others, it is proved that a ring is strongly nil-clean if and only if it is an exchange (clean) UU ring, and it is asked whether a clean ring $R$ is nil-clean if and only if every unit of $R$ is

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nil-clean. Here we are motivated to study rings whose units are nil-clean. These rings will be called UNC rings.

In section 2, we first prove several basic properties of UNC rings. Especially it is proved that every semilocal UNC ring is nil-clean. This can be seen as a partial answer to the question of Danchev and Lam. We next show that the matrix ring over a commutative ring $R$ is a UNC ring if and only if $R/J(R)$ is Boolean with $J(R)$ nil. As a consequence, the matrix ring over a UNC ring need not be a UNC ring. We also discuss when a group ring is a UNC (UU) ring. In the last part of this section, it is shown that UU rings are exactly those rings whose units are uniquely nil-clean. As the main result in this section, it is proved that a ring $R$ is strongly nil-clean if and only if $R$ is a semipotent UNC ring.

As another natural generalization of UU rings, in section 3 we determine the rings for which every unit is a sum of a nilpotent and two idempotents that commute with one another. We also deal with a special case where every unit of the ring is a sum of two commuting idempotents. These conditions can be compared with the so-called strongly 2-nil-clean rings introduced by Chen and Sheibani [4], and the rings for which every element is a sum of two commuting idempotents, studied by Hirano and Tominaga in [11].

We write $M_n(R)$, $T_n(R)$ and $R[t]$ for the $n \times n$ matrix ring, the $n \times n$ upper triangular matrix ring, and the polynomial ring over $R$, respectively. For an endomorphism $\sigma$ of a ring $R$, let $R[t; \sigma]$ denote the ring of left skew power series over $R$. Thus, elements of $R[t; \sigma]$ are polynomials in $t$ with coefficients in $R$ written on the left, subject to the relation $tr = \sigma(r)t$ for all $r \in R$. The group ring of a group $G$ over a ring $R$ is denoted by $RG$.

## 2. Units being nil-clean

### 2.1. Basic properties

We present various properties of the rings whose units are nil-clean, and prove that every semilocal ring whose units are nil-clean is a nil-clean ring.

**Definition 2.1.** A ring $R$ is called a UNC ring if every unit of $R$ is nil-clean.

Every nil-clean ring is a UNC ring. A ring $R$ is called a UU ring if $U(R) = 1 + \text{Nil}(R)$ (see [3]).
Proposition 2.2. A unit \( u \) of \( R \) is strongly nil-clean if and only if \( u \in 1 + \text{Nil}(R) \). In particular, \( R \) is a UU ring if and only if every unit of \( R \) is strongly nil-clean.

Thus, UU rings can be viewed as the “strong version” of UNC rings.

Lemma 2.3. The class of UNC rings is closed under finite direct sums.

The next lemma was proved for a nil-clean ring in [8] and for a UU ring in [7].

Lemma 2.4. If \( R \) is a UNC ring, then \( J(R) \) is nil and \( 2 \in J(R) \).

Proof. Let \( j \in J(R) \). Then \( 1+j = e+b \) where \( e^2 = e \) and \( b \in \text{Nil}(R) \), and so \( e = (1-b)+j \in U(R) \). It follows that \( e = 1 \), and hence \( j = b \in \text{Nil}(R) \). Write \(-1 = e+b \) where \( e^2 = e \) and \( b \in \text{Nil}(R) \). Then \( e = -1-b \in U(R) \). So \( e = 1 \) and hence \( 2 = -b \in \text{Nil}(R) \). \(\Box\)

The next result is basic for studying the structure of a UNC ring.

Theorem 2.5. Let \( R \) be a ring, and \( I \) a nil ideal of \( R \).

1. \( R \) is a UNC ring if and only if \( J(R) \) is nil and \( R/J(R) \) is a UNC ring.
2. \( R \) is a UNC ring if and only if \( R/I \) is a UNC ring.

Proof. (1) The necessity is clear in view of Lemma 2.4. For the sufficiency, let \( u \in U(R) \). Then \( \bar{u} \in U(R/J(R)) \), and write \( \bar{u} = \bar{e} + \bar{b} \) where \( \bar{e} \in \text{idem}(R/J(R)) \) and \( \bar{b} \in \text{Nil}(R/J(R)) \). As \( J(R) \) is nil, idempotents of \( R/J(R) \) can be lifted to idempotents of \( R \). So we can assume that \( e^2 = e \in R \). Moreover, \( b \in R \) is nilpotent. Thus, for some \( j \in J(R) \), \( u = e+b+j = e+(b+j) \) is nil-clean because \( b+j \in \text{Nil}(R) \).

(2) The proof is similar to (1). \(\Box\)

The following corollary can be quickly verified using Theorem 2.5.

Corollary 2.6. Let \( R, S \) be rings, \( M \) be an \((R,S)\)-bimodule, and \( N \) a bimodule over \( R \).

1. The trivial extension \( R \ltimes N \) is a UNC ring if and only if \( R \) is a UNC ring.
2. For \( n \geq 2 \), \( R[t]/(t^n) \) is UNC ring if and only if \( R \) is a UNC ring.
3. The formal triangular matrix ring \( \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \) is a UNC ring if and only if \( R, S \) are UNC rings.
(4) For \( n \geq 1 \), \( T_n(R) \) is a UNC ring if and only if \( R \) is a UNC ring.

The easiest way to see a UNC ring that is neither UU nor nil-clean is to form the ring direct sum \( R = R_1 \oplus R_2 \), where \( R_1 \) is a UU ring that is not nil-clean and \( R_2 \) is a nil-clean ring that is not UU. For instance, the ring \( \mathbb{Z}_2[t] \oplus M_2(\mathbb{Z}_2) \) is a UNC ring that is neither UU nor nil-clean. The next example gives an indecomposable UNC ring that is neither UU nor nil-clean.

**Example 2.7.** Let \( R = M_n(\mathbb{Z}_2) \) with \( n \geq 2 \), \( S = \mathbb{Z}_2[t] \) and \( M = S^n \). For \( (a_{ij}) \in R \), \( b \in S \) and \( \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in S^n \), define \( (a_{ij})\bar{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{bmatrix} \) and \( \bar{xb} = \begin{bmatrix} x_1b \\ x_2b \\ \vdots \\ x_nb \end{bmatrix} \). Then \( M \) is an \((R, S)\)-bimodule, and the formal triangular matrix ring \( T := \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \) is a UNC ring by Corollary 2.6(3). One can verify that the central idempotents of \( T \) are trivial. So \( T \) is indecomposable. But since \( R \) is not a UU ring and \( S \) is not a nil-clean ring, \( T \) is neither a UU ring nor a nil-clean ring.

**Lemma 2.8.** [19, 20] An element \( a \in R \) is strongly nil-clean if and only if \( a - a^2 \) is nilpotent.

**Corollary 2.9.** Let \( C(R) \) be the center of \( R \).

1. If \( R \) is nil-clean, then \( C(R) \) is strongly nil-clean.
2. If \( R \) is a UNC ring, then \( C(R) \) is a UU ring.

**Proof.** (1) Let \( a \in C(R) \). Then \( a \in R \) is nil-clean and central, so \( a \) is strongly nil-clean in \( R \). Thus \( a^2 - a \) is nilpotent by Lemma 2.8. Hence \( C(R) \) is strongly nil-clean by Lemma 2.8.

(2) The proof is similar to (1). \( \square \)

The proof of the next corollary actually shows that, if \( 2 \in J(R) \), then \( a \in R \) is strongly nil-clean if and only if \( a \) can be written as \( a = e + b \), where \( e^2 = e \in R \), \( b \in \text{Nil}(R) \) and \( eb = ebe \).

**Corollary 2.10.** The following hold for a ring \( R \):

1. \( C(R) \) is strongly nil-clean.
2. If \( R \) is nil-clean, then \( R \) is strongly nil-clean.
3. If \( R \) is a UNC ring, then \( C(R) \) is a UU ring.
(1) R is strongly nil-clean if and only if, for each $a \in R$, $a$ can be written as $a = e + b$, where $e^2 = e \in R$, $b \in \text{Nil}(R)$ and $eb = ebe$.

(2) R is a UU ring if and only if, for each $a \in U(R)$, $a$ can be written as $a = e + b$, where $e^2 = e \in R$, $b \in \text{Nil}(R)$ and $eb = ebe$.

Proof. (1) We show the sufficiency. For $a \in R$, let $a = e + b$ as given as in (1). Then $a^2 = e + eb + be + b^2$, so $a - a^2 = d - x$ where $d = b - b^2$ and $x = eb + be$. We have $ex = eb + ebe = 2eb \in J(R)$ as $2 \in J(R)$ (see [5]). It also follows from $eb = ebe$ that $eb^n = eb^ne$ for any $n \geq 0$, so $eb^n x = eb^nx \in J(R)$. Hence, we deduce that $xod^n = 0$ for all $n \geq 0$. We now show that $a - a^2$ is a nilpotent. As $J(R)$ is nil by [5], it suffices to show that $\bar{a} - \bar{a}^2$ is nilpotent in $R/J(R)$. As $\bar{x}d^n\bar{x} = 0$ in $R/J(R)$ for all $n \geq 0$, we have $(\bar{a} - \bar{a}^2)^{m+1} = (\bar{d} - \bar{x})^{m+1} = \bar{d}^{m+1} - \sum_{i+j=m} \bar{d}^i\bar{x}\bar{d}^j$ for any $m > 0$. So if $\bar{d}^m = 0$, then $(\bar{a} - \bar{a}^2)^{2m+1} = 0$.

(2) The proof is similar to (1). □

By [3], a commutative ring $R$ is a UU ring if and only if so is $R[t]$. Next we present a generalization of this result. The prime radical $N(R)$ of a ring $R$ is defined to be the intersection of the prime ideals of $R$. It is known that $N(R) = \text{Nil}_*(R)$, the lower nilradical of $R$. A ring $R$ is called a 2-primal ring if $N(R)$ coincides with $\text{Nil}(R)$. For an endomorphism $\sigma$ of $R$, $R$ is called $\sigma$-compatible if, for any $a, b \in R$, $ab = 0 \Leftrightarrow a\sigma(b) = 0$ (see [2]), and in this case $\sigma$ is clearly injective.

Theorem 2.11. Let $R$ be a 2-primal ring and $\sigma$ an endomorphism of $R$ such that $R$ is $\sigma$-compatible. The following are equivalent:

(1) $R[t; \sigma]$ is a UNC ring.

(2) $R[t; \sigma]$ is a UU ring.

(3) $R$ is a UU ring.

(4) $R$ is a UNC ring.

(5) $J(R) = \text{Nil}(R)$ and $U(R) = 1 + J(R)$.

Proof. (2) ⇒ (1) and (3) ⇒ (4). The implications are obvious.
(1) \(\Rightarrow\) (4). As \(R[t;\sigma]/(t) \cong R\) and units of \(R[t;\sigma]/(t)\) are lifted to units of \(R[t;\sigma]\), the implication holds.

(2) \(\Rightarrow\) (3). Argue as in proving (1) \(\Rightarrow\) (4).

(4) \(\Rightarrow\) (5). As \(R\) is 2-primal, \(\text{Nil}(R) \subseteq J(R)\), so \(J(R) = \text{Nil}(R)\) by Lemma 2.4. Hence \(R/J(R)\) is a reduced ring that is a UNC ring. It follows that every unit of \(R/J(R)\) is an idempotent, so \(U(R/J(R)) = \{\overline{1}\}\). That is, \(U(R) = 1 + J(R)\).

(5) \(\Rightarrow\) (2). As \(R\) is a 2-primal ring, we deduce from (5) that \(J(R) = \text{Nil}_*(R) = \text{Nil}(R)\).

By [7], a ring \(R\) is strongly nil-clean if and only if \(R\) is a clean, UU ring. It is asked in [7] whether a clean, UNC ring is nil-clean (the converse holds clearly). We show that every semilocal UNC ring is nil-clean. In particular, a semiperfect, UNC ring is nil-clean. The following lemma is implicit in the proof of [12, Theorem 3].

**Lemma 2.13.** [12] Let \(D\) be a division ring. If \(|D| \geq 3\) and \(a \in D \setminus \{0, 1\}\), then \(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_2(D)\) is not nil-clean.

The equivalence (2) \(\Leftrightarrow\) (3) of the next theorem was proved in [12].

**Theorem 2.14.** The following are equivalent for a semilocal ring \(R\):

1. \(R\) is a UNC ring.
(2) $R$ is a nil-clean ring.

(3) $J(R)$ is nil and $R/J(R)$ is a finite direct sum of matrix rings over $F_2$.

Proof. (2) $\iff$ (3). The equivalence is [12, Corollary 5]. The implication (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (3). By Lemma 2.4, $J(R)$ is nil. As $R$ is semilocal, $R/J(R) = R_1 \oplus R_2 \oplus \cdots \oplus R_n$ where each $R_i$ is a matrix ring over a division ring $D_i$. If $|D_i| > 2$, let $a \in D_i \setminus \{0, 1\}$. Then, for any $n \geq 1$, $A := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_n(D_i)$ is not nil-clean by Lemma 2.13 and this implies that $I_n - A$ is not nil-clean in $M_n(D_i)$. But $I_n - A$ is a unit of $M_n(D_i)$, we deduce that $M_n(D_i)$ is not a UNC ring. Hence $R_i$ is not a UNC ring, so $R$ is not a UNC ring by Lemma 2.3 and Theorem 2.5. This contradiction shows that $|D_i| = 2$. □

2.2. UNC matrix rings. Any proper matrix ring cannot be a strongly nil-clean ring by [8] (indeed, cannot be a UU ring by [3]). It is still unknown whether the matrix ring over a nil-clean ring is again nil-clean (see [8, Question 3]). Next we determine when the matrix ring over a commutative ring is a UNC ring. As a consequence, the matrix ring over a UNC ring need not be a UNC ring.

The $(i, j)$-cofactor of an $n \times n$ matrix $A$ over a commutative ring, denoted by $A_{ij}$, is $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ submatrix obtained from $A$ by deleting row $i$ and column $j$. Let $E_{ij}$ be the square matrix with $(i, j)$-entry 1 and all other entries 0.

Lemma 2.15. [13] Let $R$ be a commutative ring and let $n \geq 1$. If $A \in M_n(R)$ and $x \in R$, then \( \det(xE_{ij} + A) = xA_{ij} + \det(A) \).

Theorem 2.16. Let $R$ be a commutative ring and $n \geq 2$. Then $M_n(R)$ is a UNC ring if and only if $J(R)$ is nil and $R/J(R)$ is Boolean.

Proof. ($\Leftarrow$). This is by [14, Corollary 6.2].

($\Rightarrow$). By Lemma 2.4, $M_n(J(R))$ is nil, so $J(R)$ is nil. Thus, it suffices to show that $R/J(R)$ is Boolean. As $M_n(R/J(R)) \cong M_n(R)/J(M_n(R))$ is a UNC ring (by Theorem 2.5) and $R/J(R)$ is reduced, we can assume without loss of generality that $R$ is reduced. We next show that $R$ is Boolean. As every commutative reduced ring is a subdirect product.
of integral domains, there exist a family of ideals \( \{ I_\alpha \} \) of \( R \) such that \( \cap I_\alpha = 0 \) and \( R/I_\alpha \) is an integral domain for each \( \alpha \). To show that \( R \) is Boolean, it suffices to show that each \( R/I_\alpha \cong \mathbb{Z}_2 \).

Firstly, we show that \( R/I_\alpha \) is a field. For \( a \in R \), write \( \overline{a} = a + I_\alpha \in R/I_\alpha \). For \( A = (a_{ij}) \in M_n(R) \), write \( \overline{A} = (\overline{a}_{ij}) \in M_n(R/J(R)) \). Assume that \( R/I_\alpha \) is not a field. Then there exists \( \overline{0} \neq \overline{x} \in R/I_\alpha \) such that \( \overline{x} \notin U(R/I_\alpha) \). The matrix

\[
U := I_n + xE_{12} + xE_{21} + x^2 E_{22}
\]

is a unit in \( S := M_n(R) \), so \( U \) is nil-clean in \( S \). Hence \( \overline{U} \) is nil-clean in \( \overline{S} = M_n(R/I_\alpha) \). Write \( \overline{U} = \epsilon + \beta \) where \( \epsilon^2 = \epsilon \in \overline{S} \) and \( \beta \in \text{Nil}(\overline{S}) \). One easily sees that \( \epsilon \neq I_n \). So \( \det(\epsilon) \neq 1 \), and hence \( \det(\epsilon) = 0 \). Thus, \( 0 = \det(\epsilon) = \det(U - \beta) = \det(xE_{12} + xE_{21} + x^2 E_{22} + (I_n - \beta)) \).

By Lemma 2.13 there exist \( a, b, c \in R/I_\alpha \) such that

\[
\det(xE_{12} + xE_{21} + x^2 E_{22} + (I_n - \beta)) = \overline{x}a + \det(xE_{21} + x^2 E_{22} + (I_n - \beta))
\]

\[
= \overline{x}a + \overline{a}b + \det(x^2 E_{22} + (I_n - \beta))
\]

\[
= \overline{x}a + \overline{a}b + \overline{x}^2 c + \det(I_n - \beta).
\]

It follows that \( \overline{x}(a + b + \overline{x}c) = -\det(I_n - \beta) \). As \( \beta \) is nilpotent, \( I_n - \beta \) is invertible, \( \det(I_n - \beta) \) is a unit in \( R/I_\alpha \). Hence, we deduce that \( \overline{x} \notin U(R/I_\alpha) \), a contradiction. Thus, we have proved that \( R/I_\alpha \) is a field.

Next we show that \( R/I_\alpha \cong \mathbb{Z}_2 \). Let \( \overline{y} \in R/I_\alpha \). Then the matrix

\[
V = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & y & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\]

is a unit in \( S \), so \( V \) is nil-clean in \( S \). Hence \( \overline{V} \) is nil-clean in \( \overline{S} \). Write \( \overline{V} = \beta + \epsilon \) where \( \epsilon^2 = \epsilon \in \overline{S} \) and \( \beta \in \text{Nil}(\overline{S}) \). By [21], \( \epsilon \) is similar to a diagonal matrix, so it is similar to \( \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \) for some \( 0 < k < n \). Moreover, as \( \beta \) is nilpotent, it is similar to a strictly upper triangular matrix. Thus, as the trace is similarity-invariant, we obtain that \( \text{trace}(\beta) = 0 \)
and \( \text{trace}(\epsilon) = \bar{k} \). As \( \bar{2} = \bar{0} \) in \( \mathbb{S} \), for any \( m \in \mathbb{Z}, \bar{m} = \bar{0} \) or \( \bar{m} = \bar{1} \). So we see that \( \text{trace}(\bar{V}) = \bar{y} + m - \bar{2} \), which is equal to \( \bar{y} \) or \( \bar{y} + \bar{1} \), and that \( \text{trace}(\epsilon) \) is equal to \( \bar{0} \) or \( \bar{1} \). Therefore, from \( \text{trace}(\bar{V}) = \text{trace}(\beta) + \text{trace}(\epsilon) \), we deduce that \( \bar{y} = \bar{0} \) or \( \bar{y} = \bar{1} \). Hence \( R/I_\alpha \cong \mathbb{Z}_2 \).

The matrix ring over a UNC ring need not be a UNC ring.

**Example 2.17.** For any \( n \geq 2 \), \( M_n(\mathbb{Z}_2[\bar{t}]) \) is not a UNC ring, while \( \mathbb{Z}_2[\bar{t}] \) is a UNC ring.

In [7, Theorem 2.6], it is proved that any unital subring of a UU ring is again a UU ring. But a subring of a UNC ring may not be a UNC ring.

**Example 2.18.** Take \( u = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in M_3(\mathbb{Z}_2) \). Then \( u^3 = 1 \). Let \( R \) be the unital subring of \( M_3(\mathbb{Z}_2) \) generated by \( u \). That is,

\[
R = \{a1 + bu + cu^2 : a, b, c \in \mathbb{Z}\}
= \{0, 1, u, u^2, 1 + u, 1 + u^2, u + u^2, 1 + u + u^2\}.
\]

One easily sees that \( R \) is reduced. As \( u^2 \neq u \), \( u \) is not nil-clean. So \( R \) is not a UNC ring, but \( M_3(\mathbb{Z}_2) \) is a UNC ring.

### 2.3. UNC group rings

(Strongly) nil-clean group rings have been discussed in [13, 17, 19].

Here we consider when a group ring is a UNC or UU ring following the idea in [19]. A group \( G \) is called locally finite if every finitely generated subgroup of \( G \) is finite. Let \( p \) be a prime number. A group \( G \) is called a \( p \)-group if the order of each element of \( G \) is a power of \( p \). The center of a group \( G \) is denoted by \( Z(G) \).

For the group ring \( RG \) of a group \( G \) over a ring \( R \), the ring homomorphism \( \omega : RG \to R, \Sigma r_g g \mapsto \Sigma r_g \) is called the augmentation map, and the kernel \( \ker(\omega) \) is called the augmentation ideal of the group ring \( RG \) and is denoted by \( \Delta(RG) \). Note that \( \Delta(RG) \) is an ideal of \( RG \) generated by the set \( \{1 - g : g \in G\} \).

**Proposition 2.19.** If \( R \) is a UNC ring and \( G \) is a locally finite \( 2 \)-group, then \( RG \) is a UNC ring.
Proof. As $G$ is locally finite, to show that $RG$ is a UNC ring it suffices to show that $GH$ is a UNC ring for any finite subgroup $H$ of $G$. So, without loss of generality, one can assume that $G$ is a finite 2-group. As $R$ is a UNC ring, $2 \in J(R)$ is nilpotent by Lemma 2.4. So, by [9, Theorem 9], $\triangle(RG)$ is nilpotent. As $RG/\triangle(RG) \cong R$, it follows from Theorem 2.5 that $RG$ is a UNC ring.

The hypercenter of a group $G$, denoted by $H(G)$, is defined to be the union of the (transfinite) upper central series of the group $G$.

**Theorem 2.20.** Let $R$ be a ring and $G$ be a group. If $RG$ is a UNC ring, then $R$ is a UNC ring and $H(G)$ is a 2-group.

**Proof.** Let $1 = Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) \subseteq \ldots \subseteq Z_\alpha(G) = H(G)$ be the upper central series of length $\alpha$ for $G$. Clearly, $Z_0(G)$ is a 2-group. Assume that, for some $\beta \leq \alpha$, $Z_\sigma(G)$ is a 2-group for all $\sigma < \beta$. We next verify that $Z_\beta(G)$ is a 2-group. This is certainly true if $\beta$ is a limit ordinal. If $\beta$ is not a limit ordinal, then $Z_{\beta - 1}(G)$ is a 2-group, and $Z_\beta(G)/Z_{\beta - 1}(G) = Z(G)$, where $G = G/Z_{\beta - 1}(G)$. Let $\overline{R} = R/2R$. As $\overline{RG}$ is an image of $RG$, it is a UNC ring. So, for $g \in Z(G)$, $g$ is nil-clean in $\overline{RG}$. As $g$ is central, it is strongly nil-clean. Hence, by Lemma 2.8, $g(1 - g) = g - g^2$ is nilpotent. It follows that $1 - g \in \overline{RG}$ is nilpotent. So, for some $n > 0$, $(1 - g)^{2^n} = 0$. That is, $g^{2^n} = 1$. Hence, $Z_\beta(G)/Z_{\beta - 1}(G) = Z(G)$ is a 2-group. As $Z_{\beta - 1}(G)$ is a 2-group, it follows that $Z_\beta(G)$ is a 2-group. Therefore, by the Transfinite Induction, $H(G) = Z_\alpha(G)$ is a 2-group.

A nilpotent group is a group $G$ such that $G = Z_n(G)$ for a finite number $n$.

**Theorem 2.21.** Let $R$ be a ring and $G$ be a nilpotent group. Then $RG$ is a UNC ring if and only if $R$ is a UNC ring and $G$ is a 2-group.

**Proof.** The claim follows from Proposition 2.19 and Theorem 2.20.

**Theorem 2.22.** If $RG$ is a UU ring, then $R$ is a UU ring and $G$ is a 2-group. The converse holds if $G$ is locally finite.
Proof. If \( RG \) is a UU ring, then as an image of \( RG \), \( R \) is certainly a UU ring. Moreover, \( G \) is a 2-group by the same argument as in the proof of Theorem 2.20 or by [7]. Conversely, as \( R \) is a UU ring, \( 2 \in J(R) \) is nilpotent by Lemma 2.4 or [7]. As \( G \) is a locally finite 2-group, \( \triangle(RG) \) is locally nilpotent by [6, Corollary, p.682]. Let \( x \in U(RG) \). Then \( \omega(x) \in U(R) \) is strongly nil-clean, so \( \omega(x) - \omega(x)^2 \in R \) is nilpotent. Hence, for some \( n > 0 \), \( \omega((x - x^2)^n) = (\omega(x - x^2))^n = 0 \). So \( (x - x^2)^n \in \triangle(RG) \). It follows that \( x - x^2 \) is nilpotent. So \( x \) is strongly nil-clean by Lemma 2.8. \( \square \)

2.4. Semipotent UU rings. An element \( a \) in a ring \( R \) is called uniquely nil-clean if there exists a unique idempotent \( e \) in \( R \) such that \( a - e \) is nilpotent, and the ring \( R \) is called uniquely nil-clean if each element of \( R \) is uniquely nil-clean. Uniquely nil-clean rings were characterized in Diesl [8]. Here, using a recent result of Šter in [20], we first show that UU rings are exactly those rings whose units are uniquely nil-clean.

**Theorem 2.23.** A ring \( R \) is a UU ring if and only if every unit of \( R \) is uniquely nil-clean.

**Proof.** (⇒). Let \( u \in U(R) \). By the hypothesis, \( u = 1 + b \) with \( b \in \mathrm{Nil}(R) \). Assume that \( u = e + x \) where \( e^2 = e \) and \( x \in \mathrm{Nil}(R) \). Then \( 1 - e = x - b \). By Šter [20, Corollary 2.13], \( \mathrm{Nil}(R) \) is closed under addition. So \( x - b \in \mathrm{Nil}(R) \), and hence \( 1 - e \) is nilpotent. It follows that \( e = 1 \) and \( x = b \). Hence \( u \) is uniquely nil-clean.

(⇐). Let \( u \in U(R) \). Then there exist \( e^2 = e \in R \) and \( x \in \mathrm{Nil}(R) \) such that \( u = e + x \). Thus, \( u = u^{-1}eu + u^{-1}xu \) is another nil-clean decomposition in \( R \). So it follows that \( e = u^{-1}eu \). This gives \( eu = ue \). So \( u \) is strongly nil-clean. \( \square \)

In contrast to Theorem 2.23, a unipotent unit need not be uniquely nil-clean, even in a nil-clean ring.

**Example 2.24.** Let \( R = M_3(\mathbb{Z}_2) \). Then \( R \) is a nil-clean ring. As observed in [9], if \( A \) is the strictly upper triangular matrix in \( R \) whose all entries above the diagonal are equal to 1 and let \( B \) be the transpose of \( A \), then \( A, B \) are nilpotent and \( E := A + B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \) is a nonzero idempotent. Thus, \( U := I_3 - A = (I_3 - E) + B \) is a unipotent unit that is not uniquely nil-clean.
A ring $R$ is strongly $\pi$-regular if, for each $a \in R$, $a^n \in a^{n+1}R \cap Ra^{n+1}$ for some $n \geq 1$. A ring is clean if every element is a sum of a unit and an idempotent, and a ring $R$ is an exchange ring if, for each $a \in R$, $a - e \in (a - a^2)R$ for some $e^2 = e \in R$ (see [18]). A ring $R$ is semipotent if every right ideal not contained in $J(R)$ contains a non-zero idempotent.

We have the implications: strongly $\pi$-regular $\Rightarrow$ clean $\Rightarrow$ exchange $\Rightarrow$ semipotent; and none of the arrows is reversible.

Diesl [8, Corollary 3.11] proved that a ring $R$ is strongly nil-clean if and only if $R$ is a strongly $\pi$-regular, UU ring. Danchev and Lam [7, Theorem 4.3] generalized the direction “$\Leftarrow$” by showing that a ring $R$ is strongly nil-clean if and only if $R$ is an exchange (or clean), UU ring. We now generalize the direction “$\Leftarrow$” further by showing the following

**Theorem 2.25.** A ring $R$ is a strongly nil-clean ring if and only if it is a semipotent, UU ring.

**Proof.** ($\Rightarrow$). The implication is clear.

($\Leftarrow$). Let $R$ be a semipotent, UU ring. So $R/J(R)$ is a semipotent, UU ring (by [7, Theorem 2.4(2)]), and moreover, $J(R)$ is nil. So $R$ is strongly nil-clean if and only if so is $R/J(R)$ by [14, Theorem 2.7]. Hence, to show the necessity, we can assume that $J(R) = 0$.

We now show that $R$ is a reduced ring. Assume $a^2 = 0$ for some $0 \neq a \in R$. As $R$ is a semipotent ring with $J(R) = 0$, there exists $e^2 = e \in R$ such that $eRe$ is isomorphic to a $2 \times 2$ matrix ring over a non trivial ring by Levitzki [16, Theorem 2.1]. But, as $eRe$ is again a UU ring by [3, Proposition 2.5], this gives a contradiction to [3, Corollary 3.3]. Hence $R$ is reduced. It follows that $U(R) = \{1\}$.

We next show that $R$ is a Boolean ring. Assume on the contrary that $a^2 \neq a$ for some $a \in R$. As $R$ is semipotent with $J(R) = 0$, $(a - a^2)R$ contains a nonzero idempotent, say $e$. Write $e = (a - a^2)b$ with $b \in R$. Then $e = e(a - a^2)b = ea \cdot e(1 - a)b = e(1 - a) \cdot eab$. As $eRe$ is reduced, $ea$ and $e(1 - a)$ are units of $eRe$. As $U(R) = \{1\}$, we have $U(eRe) = \{e\}$. Hence $ea = e$ and $e(1 - a) = e$. It follows that $e = 0$, a contradiction. □
In [10, Theorem 2.9], Henriksen proved that a von Neumann regular ring $R$ with $U(R) = \{1\}$ is Boolean. Danchev and Lam [7, Corollary 4.2] proved that a ring $R$ is an exchange ring with $U(R) = \{1\}$ if and only if $R$ is Boolean.

**Corollary 2.26.** A ring $R$ is semipotent with $U(R) = \{1\}$ if and only if $R$ is Boolean.

The question of Danchev and Lam whether a clean, UNC ring is nil-clean is still open. In view of Theorem 2.25 [8, Corollary 3.11] and [8, Question 4], the following questions arise.

**Questions 2.27.** Is a semipotent, UNC ring a nil-clean ring? What can be said about strongly $\pi$-regular UNC rings?

3. **Units being sums of a nilpotent and two idempotents**

As a generalization of a strongly nil-clean ring, a strongly 2-nil-clean ring was introduced by Chen and Sheibani [4] to be the ring for which every element is a sum of a nilpotent and two idempotents that commute with one another. The structure of these rings is obtained in [4]. In this section, we consider the “unit” version of strongly 2-nil-clean rings. That is, the rings for which every unit is a sum of a nilpotent and two idempotents that commute with one another. These rings extend UU rings, and will be completely characterized here. A special situation of a strongly 2-nil-clean ring is the property that every element of a ring is a sum of two commuting idempotents, first considered by Hirano and Tominaga [11]. Here the “unit” version of this property is also discussed.

3.1. **Units being sums of a nilpotent and two idempotents.**

**Definition 3.1.** A ring $R$ is called a UNII-ring if every unit of $R$ is a sum of a nilpotent and two idempotents. The ring $R$ is called a strong UNII-ring if for each $u \in U(R)$, $u = b + e + f$ where $b \in \text{Nil}(R)$, $e, f \in \text{idem}(R)$ such that $b, e, f$ commute.

**Lemma 3.2.** A ring $R$ is strong UNII-ring if and only if $R = A \oplus B$ where $A, B$ are strong UNII-rings, $2 \in J(A)$ is nilpotent and $3 \in J(B)$ is nilpotent.

**Proof.** ($\Leftarrow$). The implication is clear.
Write $-1 = b + e + f$ where $b \in \text{Nil}(R)$, $e, f \in \text{idem}(R)$ and $b, e, f$ all commute. Then
\[
1 + 2b + b^2 = (-1 - b)^2 = (e + f)^2 = e + f + 2ef = (-1 - b) + 2e(-1 - b - c)
\]
\[
= -1 - b - 4e - 2eb,
\]
which gives $2 + 4e = -3b - b^2 - 2eb$. So $6e = (2 + 4e)e = (-3b - b^2 - 2eb)e = -(5e + be)b$ is nilpotent. Similarly, $6f$ is nilpotent. Thus $0 = 6^n(e + f) = 6^n(-1 - b)$. As $-1 - b \in U(R)$, $6^n = 0$, i.e., $2^nR \cap 3^nR = 0$. So $R = A \oplus B$ where $A \cong R/2^nR$ and $B \cong R/3^nR$, and $A, B$ are strong UNII-rings. □

**Lemma 3.3.** The following are equivalent for a ring $R$:

(1) A ring $R$ is a strong UNII-ring with 2 nilpotent.

(2) Each unit of $R$ is a sum of a nilpotent and a tripotent that commute and 2 is nilpotent.

(3) $R$ is a UU ring.

**Proof.** (3) ⇒ (1). The implication is clear.

(1) ⇒ (2). Let $u \in U(R)$ and write $u = b + e + f$ where $b \in \text{Nil}(R)$, $e, f \in \text{idem}(R)$ and $b, e, f$ all commute. Then $g := e + f - 2ef$ is an idempotent and $c := u - g = b + 2ef$ is nilpotent. Moreover, $g, c$ commute. So $u = c + g$ is strongly nil-clean. Hence (2) holds.

(2) ⇒ (3). Let $u \in U(R)$ and write $u = b + t$ where $b \in \text{Nil}(R)$, $t^3 = t$ and $bt = tb$. Then $(t - t^2)^2 = 2(t^2 - t) \in \text{Nil}(R)$, so $b + (t - t^2)$ is nilpotent. Thus, $u = (b + t - t^2) + t^2$ is strongly clean. So $R$ is a UU ring. □

**Lemma 3.4.** Let $R$ be a ring with $3 \in \text{Nil}(R)$. The following are equivalent:

(1) $R$ is a strong UNII-ring.

(2) Each unit of $R$ is a sum of a nilpotent and a tripotent that commute.

(3) $1 - u^2$ is nilpotent for every $u \in U(R)$.

**Proof.** (1) ⇒ (2). Let $u \in U(R)$ and write $u = b + e + f$ where $b \in \text{Nil}(R)$, $e, f \in \text{idem}(R)$ and $b, e, f$ all commute. Then $g := e + f - 3ef$ is a tripotent and $b + 3ef$ is nilpotent. Moreover, $g, b + 3ef$ commute. So $u = (b + 3ef) + g$ is a sum of a nilpotent and a tripotent that commute. Hence (2) holds.
(2) ⇒ (3). Let \( u \in U(R) \) and write \( u = b + f \) where \( b \in \text{Nil}(R) \), \( f \) is a tripotent and \( b, f \) all commute. Then \( u^3 \equiv b^3 + f \pmod{3R} \), so \( u - u^3 \equiv b - b^3 \pmod{3R} \). As \( 3, b \in \text{Nil}(R) \), \( u - u^3 \in \text{Nil}(R) \), so \( 1 - u^2 \) is nilpotent.

(3) ⇒ (1). Let \( u \in U(R) \) and write \( u = b + x \) where \( b \in \text{Nil}(R) \), \( x^3 = x \) and \( xb = bx \). Then \( u = b + x^2 + (x - x^2) \), and \( (x - x^2) - (x - x^2)^2 = 3(x - x^2) \in J(R) \) is nilpotent. By [22, Lemma 3.5], there exists a polynomial \( \theta(t) \in \mathbb{Z}[t] \) such that \( \theta(x)^2 = \theta(x) \) and \( j := (x - x^2) - \theta(x) \in J(R) \). Thus, \( u = (b + j) + x^2 + \theta(x) \), where \( x^2, \theta(x) \) are idempotents, and \( b + j \) is nilpotent and they commute with each other. □

**Theorem 3.5.** The following are equivalent for a ring \( R \):

1. \( R \) is a strong UNII-ring.
2. \( 1 - u^2 \) is nilpotent for every \( u \in U(R) \) and \( 6 \) is nilpotent.
3. \( R \) is one of the following types:
   a. \( R \) is a UU ring.
   b. \( 1 - u^2 \) is nilpotent for every \( u \in U(R) \) and \( 3 \) is nilpotent.
   c. \( R = A \oplus B \), where \( A \) is a UU ring, \( 1 - u^2 \) is nilpotent for every \( u \in U(B) \) and \( 3 \in B \) is nilpotent.

**Proof.** (1) ⇔ (3). The equivalence follows from Lemmas 3.2–3.4.

(3) ⇒ (2). The implication is clear.

(2) ⇒ (3). Let \( 6^n = 0 \) with \( n \geq 1 \). Then \( R = A \oplus B \) where \( A \cong R/2^nR \) and \( B \cong R/3^nR \). The hypothesis on \( R \) shows that \( 1 - u^2 \) is nilpotent if \( u \in U(A) \) or \( u \in U(B) \). For \( u \in U(A) \), \( (1 - u)^2 = (1 - u^2) + 2(u^2 - u) \) is nilpotent, so \( 1 - u \) is nilpotent. This shows that \( A \) is a UU ring. □

The condition that \( 6 \) is nilpotent can not be removed in Theorem 3.5(2): As \( U(\mathbb{Z}) = \{-1, 1\} \), \( 1 - u^2 = 0 \) for all \( u \in U(\mathbb{Z}) \), but \( 6 \in \mathbb{Z} \) is not nilpotent. Theorem 3.6 below can be viewed as an extension of Theorem 2.25.

**Theorem 3.6.** A ring is a semipotent, strong UNII-ring if and only if \( R = A \oplus B \), where \( A \) is zero or \( A/J(A) \) is Boolean with \( J(A) \) nil and \( B \) is zero or \( B/J(B) \) is a subdirect product of \( \mathbb{Z}_3 \)'s with \( J(B) \) nil.
Proof. (⇒). As $A$ is strongly nil-clean, it is a strong UNII-ring. For any $u \in U(B)$, $u^3 - u$ is nilpotent. So $(u^2)^3 - u^2 = u^3(u^3 - u)$ is nilpotent and $(u - u^2)^2 - (u - u^2) = (u - 2)(u^3 - u) + 3(u^2 - u)$ is nilpotent. By [22] lemma 3.5, there exist $\theta_1(t), \theta_2(t) \in \mathbb{Z}[t]$ such that $\theta_1(u)^2 = \theta_1(u), \theta_2(u)^2 = \theta_2(u)$, and $u^2 - \theta_1(u), (u - u^2) - \theta_2(u)$ are nilpotent. So $u = \theta(u) + \theta_1(u) + \theta_2(u)$, where $\theta(u) = (u^2 - \theta_1(u)) + [(u - u^2) - \theta_2(u)]$ is nilpotent. So $B$ is a strong UNII-ring. Hence $R$ is a strong UNII-ring.

(⇒). By Theorem [3, 5], $R = A \oplus B$, where $A$ is zero or $A$ is a UU ring, and $B$ is zero or $1 - u^2 \in \text{Nil}(B)$ for every $u \in U(B)$ with $3 \in \text{Nil}(B)$. We can assume that $A \neq 0$ and $B \neq 0$. As $R$ is semipotent, $A, B$ are semipotent. So $A$ is strongly nil-clean by Theorem [22, 25] and hence $A/J(A)$ is Boolean with $J(A)$ nil. For $j \in J(B), (1 + j)^2 - 1 = j(2 + j) \in \text{Nil}(B)$. As $2 + j \in U(B), j \in \text{Nil}(B)$. So $J(B)$ is nil. We next show that $\overline{B} := B/J(B)$ is reduced. Assume $x^2 = 0$ for some $0 \neq x \in \overline{B}$. As $\overline{B}$ is semiprimitive semipotent, there exists $f^2 = f \in \overline{B}$ such that $f\overline{B}f \cong M_2(S)$ for a non trivial ring $S$ by Levitzki [16] Theorem 2.1. In $M_2(S)$, \[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}^2 - \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\] is not nilpotent. So, there exists a unit $v$ of $f\overline{B}f$ such that $f - v^2$ is not nilpotent. Thus $w := v + 1 - f$ is a unit of $\overline{B}$ and $1 - u^2 = f - v^2$ is not nilpotent. This is a contradiction. Hence $\overline{B}$ is reduced. It follows that $w^2 = 1$ for any $w \in U(\overline{B})$. Next we show that $y^3 = y$ in $\overline{B}$ for every $y \in \overline{B}$. Assume that $y - y^3 \neq 0$ for some $y \in \overline{B}$. As $\overline{B}$ is semipotent, there exists $0 \neq e^2 = e \in (y - y^3)\overline{B}$. Write $e = (y - y^3)z$ with $z \in \overline{B}$. Then $e = ey \cdot e(1 - y^3)z = e(1 - y) \cdot ey(1 + y)z = e(1 + y) \cdot ey(1 - y)z$. As $e\overline{Be}$ is reduced, $ey, e(1 - y)$ and $e(1 + y)$ are units of $e\overline{Be}$. As $u^2 = 1$ for each $u \in \overline{B}, v^2 = e$ for all $v \in e\overline{Be}$. Hence, $e = (ey)^2 = (e(1 + y))^2 = (e(1 - y))^2$. It follows that $2e = 0$. As $2$ is a unit of $e\overline{Be}$, $e = 0$. This contradiction shows that $y^3 = y$ in $\overline{B}$ for every $y \in \overline{B}$. As $3 = 0$ in $\overline{B}$, $\overline{B}$ is a subdirect product of $\mathbb{Z}_3$’s (see [15] Ex.12.11)).

Corollary 3.7. A ring is a strongly 2-nil-clean ring if and only if it is a semipotent, strong UNII-ring.

Proof. This is by Theorem [3, 6] and [3] Theorem 4.2.

Corollary 3.8. If $R$ is a semipotent strong UNII-ring, then every element of $R$ is a sum of a nilpotent and two idempotents that commute with one another.
Proof. This is by Theorem 3.6 and [4, Theorem 4.5].

3.2. Units being sums of two idempotents.

Hirano and Tominaga [11] have characterized the rings in which every element is the sum of two commuting idempotents.

Definition 3.9. A ring $R$ is called a UII-ring if every unit of $R$ is a sum of two idempotents, and $R$ is called a strong UII-ring if every unit of $R$ is a sum of two commuting idempotents.

Lemma 3.10. A finite direct sum $R_1 \oplus R_2 \oplus \cdots \oplus R_n$ of rings is a (strong) UII-ring if and only if each $R_i$ is a (strong) UII-ring.

Lemma 3.11. A ring $R$ is a UII-ring if and only if $R = A \oplus B$ where $A, B$ are UII-rings, $2 = 0$ in $A$ and $3 = 0$ in $B$.

Proof. ($\Rightarrow$). The implication is clear.

($\Leftarrow$). Write $-1 = e + f$ where $e, f$ are idempotents of $R$. Then $-1 - e = f = f^2 = (-1 - e)^2 = 1 + 3e$, so $4e + 2 = 0$. Thus, $0 = (4e + 2)e = 6e$. Similarly, $6f = 0$. Hence $6 = -6(e + f) = -6e - 6f = 0$, so $2R \cap 3R = 0$. It follows that $R = A \oplus B$ where $A \cong R/2R$ and $B \cong R/3R$. Moreover, $A, B$ are clearly UII-rings.

Lemma 3.12. A ring $R$ is strong UII with $ch(R) = 2$ if and only if $U(R) = \{1\}$.

Proof. If $U(R) = \{1\}$, then $R$ is clearly strong UII, and $-1 = 1$; so $2 = 0$ in $R$. If $R$ is strong UII with $2R = 0$, then, for any $u \in U(R)$, $u = e + f$ where $e, f$ are commuting idempotents of $R$. Thus $u + e = f = f^2 = (u + e)^2 = u^2 + 2ue + e = u^2 + e$. It follows that $u^2 = u$, i.e., $u = 1$.

Lemma 3.13. A ring $R$ is a strong UII-ring with $ch(R) = 3$ if and only if $U(R) = 1 + idem(R)$.

Proof. ($\Leftarrow$). Suppose $U(R) = 1 + idem(R)$. Then $2 = 1 + 1 \in U(R)$. As $-1 = 1 + (-2)$, we infer that $(-2)^2 = -2$, so $-2 = 1$. That is, $3 = 0$ in $R$. Moreover, $R$ is clearly a strong UII-ring.
For $e^2 = e \in R$, $1 + e = 1 - 2e \in U(R)$. Let $u \in U(R)$. Write $u = e + f$ where $e, f$ are commuting idempotents. Then $u - e = f = f^2 = (u - e)^2 = u^2 - 2ue + e$, so
\[ e = eu - u + u^2. \]

Multiplying the equality by $e$ from the left gives $e = eu^2$. It follows that $eu^2 = eu - u + u^2$, and hence $eu = e - 1 + u$, i.e., $e = 1 - u + eu$. We deduce that $1 - u + eu = eu - u + u^2$, so $u^2 = 1$. This gives that $1 = u^2 = (e + f)^2 = e + f + 2ef = u + 2ef$. That is, $u = 1 + g$ with $g = ef$. □

**Theorem 3.14.** A ring $R$ is a strong UII-ring if and only if $R$ is one of the following types:

1. $U(R) = \{1\}$.
2. $U(R) = 1 + \text{idem}(R)$.
3. $R = A \oplus B$ where $U(A) = \{1\}$ and $U(B) = 1 + \text{idem}(B)$.

**Proof.** It follows from Lemmas 3.10-3.13. □

**Corollary 3.15.** If $R$ is a strong UII-ring, then $J(R) = 0$ and $\text{Nil}(R) = 0$.

**Proof.** One easily sees that $U(R) = \{1\}$ or $U(R) = 1 + \text{idem}(R)$ implies that $J(R) = 0$ and $\text{Nil}(R) = 0$. □

In view of Corollary 3.15, we see that, for any ring $R$ and $n \geq 2$, the rings $M_n(R)$, $T_n(R)$ and $R[[t]]$ are not strong UII-rings.

**Proposition 3.16.** Let $\sigma$ be an endomorphism of a ring $R$. Then $R[t; \sigma]$ is a strong UII-ring if and only if $R$ is a strong UII-ring and $\sigma(e) = e$ for all $e \in \text{idem}(R)$.

**Proof.** ($\Rightarrow$). For $u \in U(R)$, $u \in U(R[t; \sigma])$, so $u = \sum_{i \geq 0} a_i t^i + \sum_{i \geq 0} b_i t^i$ is a sum of two commuting idempotents in $R[t; \sigma]$. Thus $u = a_0 + b_0$ is a sum of two commuting idempotents in $R$. So $R$ is a strong UII-ring. Since $R[t; \sigma]$ is a reduced ring, each $e \in \text{idem}(R)$ is central in $R[t; \sigma]$. So $et = te = \sigma(e)t$, showing that $\sigma(e) = e$.

($\Leftarrow$). As $R$ is a strong UII-ring, $R$ is a reduced ring. So idempotents of $R$ are central, and hence units of $R$ are central. Moreover, the assumption that $\sigma(e) = e$ for all $e^2 = e \in R$
implies that \( \sigma(u) = u \) for all \( u \in U(R) \). Now one can easily show that \( \text{idem}(R[t; \sigma]) = \text{idem}(R) \) and \( U(R[t; \sigma]) = U(R) \). It follows that \( R[t; \sigma] \) is a strong UII-ring. \( \square \)

**Proposition 3.17.** Let \( R \) be a ring and \( G \) a nontrivial group. Then \( RG \) is a strong UII-ring if and only if \( U(R) = 1 + \text{idem}(R) \) and \( G \) is a group of exponent 2.

**Proof.** (\( \Rightarrow \)). Let \( u \in U(R) \). Then \( u = \alpha + \beta \), where \( \alpha, \beta \) are commuting idempotents in \( RG \). So \( u = \omega(\alpha) + \omega(\beta) \) is a sum of two commuting idempotents in \( R \). So \( R \) is a strong UII-ring. By Theorem 3.14 \( R = A \oplus B \) where \( A, B \) are strong UII-rings, \( 2 = 0 \) in \( A \) and \( 3 = 0 \) in \( B \). Thus \( RG \cong AG \oplus BG \), so \( AG \) is a strong UII-ring. As \( 2 = 0 \) in \( AG \), \( U(AG) = \{1\} \) by Lemma 3.12. Because \( G \) is a nontrivial group, it must be that \( A = 0 \). So \( R = B \), and hence \( U(R) = 1 + \text{idem}(R) \). As \( 3 = 0 \) in \( RG \), \( U(RG) = 1 + \text{idem}(RG) \) by Lemma 3.13. It follows that \( g^2 = 1 \) for all \( g \in G \).

(\( \Leftarrow \)). Since \( G \) is a group of exponent 2, \( G \) is locally finite. So it suffices to show that for any finite subgroup \( H \) of \( G \), \( RH \) is a strong UII-ring. But \( 2 \in U(R) \) by Lemma 3.13, so \( RH \) is a direct sum of finitely many copies of \( R \). Hence \( RH \) is a strong UII-ring. \( \square \)

By Hirano and Tominaga [11], the ring \( R \) must be commutative if every element of \( R \) is a sum of two commuting idempotents. However, a strong UII-ring need not be commutative. In fact, by Henriksen [10, pp.86], the Weyl algebra over \( \mathbb{Z}_2 \) in the noncommuting indeterminants \( x, y \) subject to the relation \( yx = xy + 1 \) is a strong UII-ring that is not commutative.

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**References**

[1] A. Alhevaz, A. Moussavi, and M. Habibi, On rings having McCoy-like conditions, *Comm. Algebra* 40 (2012), 1195-1221.
[2] S. Annin, Associated primes over skew polynomials rings, *Comm. Algebra* 30 (2002), 2511-2528.
[3] G. Călugăreanu, UU rings, *Carpathian J. Math.* 31 (2) (2015), 157-163.
[4] H. Chen and M. Sheibani, Strongly 2-nil-clean rings, *J. Algebra Appl.* 16 (8) (2017), 1750178 (12 pages).
[5] W. Chen, On constant products of elements in skew polynomial rings, *Bull. Iranian Math. Soc.* 41 (2) (2015), 453-462.
[6] I.G. Connell, On the group ring, *Canad. J. Math.* 15 (1963), 650-685.
[7] P. Danchev and T.Y. Lam, Rings with unipotent units, *Publ. Math. Debrecen* 88 (3-4) (2016), 449-466.
[8] A.J. Diesl, Nil clean rings, *J. Algebra* 383 (2013), 197-211.
[9] M. Ferrero, E.R. Puczyłowski, S. Sidki, On the representation of an idempotent as a sum of nilpotent elements, *Canad. Math. Bull.* 39 (2) (1996), 178-185.
[10] M. Henrïksen, Rings with a unique regular element, *Rings, modules and radicals* (Hobart, 1987), 7887, Pitman Res. Notes Math. Ser., 204, Longman Sci. Tech., Harlow, 1989.
[11] Y. Hirano and H. Tominaga, Rings in which every element is the sum of two idempotents, *Bull. Austral. Math. Soc.* 37 (2) (1988), 161-164.
[12] T. Kosan, T-K. Lee and Y. Zhou, When is every matrix over a division ring a sum of an idempotent and a nilpotent? *Linear Alg. Appl.* 450 (2014), 7-12.
[13] T. Kosan, S. Sahinkaya, Y. Zhou, On weakly clean rings, *Comm. Alg.* 45(8) (2017), 3494-3502.
[14] T. Kosan, Z. Wang and Y. Zhou, Nil-clean and strongly nil-clean rings, *J. Pure Appl. Algebra* 220 (2) (2016), 633-646.
[15] T.Y. Lam, A First Course in Noncommutative Rings, *Graduate Texts in Mathematics*, 131, Springer-Verlag, New York, 1991.
[16] J. Levitzki, On the structure of algebraic algebras and related rings, *Trans. AMS.* 74 (1953), 384-409.
[17] W.Wm. McGovern, S. Raja and A. Sharp, Commutative nil clean group rings, *J. Alg. Appl.* 14 (6) (2015), 1550094.
[18] W.K. Nicholson, Lifting idempotents and exchange rings, *Trans. AMS.* 229 (1977), 269-278.
[19] S. Sahinkaya, G. Tang and Y. Zhou, Nil-clean group rings, *J. Algebra Appl.* 16 (5) (2017), 1750135.
[20] J. Ster, Rings in which nilpotents form a subring, *Carpathian J. Math.* 32 (2) (2016), 251-258.
[21] A. Steger, Diagonability of idempotent matrices, *Pacific J. Math.* 19 (3) (1966), 535-541.
[22] Z. Ying, T. Kosan and Y. Zhou, Rings in which every element is a sum of two tripotents, *Canad. Math. Bull.* 59 (3) (2016), 661-672.

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