On the parton picture of Froissart asymptotic behavior

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Abstract

The Froissart $\mathcal{F}$ asymptotic behavior of high energy cross-sections, if considered in a parton picture, is usually represented as a kind of behavior that occurs in the process of a collision of two almost black disks filled with partons, when radiiuses of these $\mathcal{F}$-disks grow proportional to log’s of there energies. In this article we briefly summarize the main asymptotic properties of $\mathcal{F}$-disks that one can expect in QCD. Then we consider if it is possible to guarantee the boost-invariance of transparency $T(s, b) = 1 - \sigma_{\text{in}}(s, b)$, where $\sigma_{\text{in}}(s, b)$ is the total inelastic cross-section at a definite impact parameter $b$, in process of collision of two such $\mathcal{F}$-disks. Such a question arise because the mean transverse area of the overlapping of colliding $\mathcal{F}$-disks, at the same impact parameter $b$ and total energy $\sqrt{s}$, is varying with the Lorentz frame. We find that in the simple picture of $\mathcal{F}$-disks, that on can expect in QCD, with confinement and parton saturation, the value of $T$ is not boost-invariant, and as result the $\mathcal{F}$ type behavior seems contradictory.

1. Introduction

The Froissart ($\mathcal{F}$) limitation on the asymptotic behavior of total cross sections takes place in all local renormalisable field theories and was proved on rather general grounds [1].

Not in all such theories the expected behavior of cross-sections coincides with the Froissart behavior $\sigma_{\text{tot}}(E) \sim (1/m^2) \ln^2(E/m)$ at $E \to \infty$, where $E = m \exp(y)$ is the fast particles energy.

For example, in theories containing only spinor and scalar particles at week coupling the total cross-sections decrease asymptotically with energy.

But in the theories containing vector particles, such as QCD we can have perturbatively growing cross-sections. This is especially evident in the parton picture. Here we have the primary bremsstrahlung-like contribution to the vector parton spectra of the type

$$dn \sim \alpha_s \frac{d\omega}{\omega} \frac{d^2 k_\perp}{k_\perp^2 + \mu^2},$$

where $\omega, k_\perp$ are partons energy and transverse momenta, and $\mu$ is some infrared (or confinement) scale, $\alpha_s = g_{\text{QCD}}^2/4\pi$. In the first order in $\alpha_s$ we already have asymptotically constant cross sections. Then the parton cascading of such primary vector partons transforms these spectra to the power growing form

$$dn \sim \alpha_s (E/\omega)^\Delta \frac{d\omega}{\omega} \frac{d^2 k_\perp}{k_\perp^2 + \mu^2},$$

where

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with the particles initial energy $E$ and where the parton splitting factor $\Delta \sim \alpha_s$. The number of partons with low energies $\sim \mu$ grows like $\sim (E/\mu)^{\Delta}$. They are distributed in a small transverse disk area (region, where the parton density $> 1/\mu^2$), whose radius linearly grows with the number of cascading steps $\sim (\alpha_c/\mu) \ln E/\mu$. Interaction of such a $F$ disk ($\equiv \mathcal{F}$ disk) with target or another fast particle naturally leads to the Froissart behavior of $\sigma_{tot}(E) \sim (\alpha_s/\mu)^2 \ln^2(E/\mu)$. In such a simple picture the very fast particle can be represented by almost black $F$ disk, because the parton density inside the $\mathcal{F}$ disk infinitely grows with $E$.

The same picture takes place in the regge approach. Here in QCD we have the BFKL pomeron [2] with the intercept $\alpha_P(0) = 1 + \Delta$, $\Delta \sim \alpha_s$, which gives the growing cross-section $\sigma_{tot}(E) \sim (E/m)^{\Delta}$. Then by the eikonal unitarisation we come directly to $F$ behavior of $\sigma_{tot}(E)$.

When the parton density inside $\mathcal{F}$ becomes big the process of parton gluing becomes important even at small $\alpha_s$, and the soft parton density saturates at values $\sim 1/\alpha_s$. In the regge approach the transition to the saturation is connected with a contribution of diagrams with pomeron interactions. And these energies mark the beginning of the energy region where the regge approach becomes not safe, because here the average energies on interacting reggeons become not asymptotically large.

The real process of the parton saturation in $\mathcal{F}$ is more complicated in QCD, because here the saturation of partons of different virtualities(transverse momenta) takes place on a different energy and density scales, so that the full parton density continues to grow, so that it is decreasing from $\mathcal{F}$ center to its border. We consider the main details of this picture in section 3.

The parton(gluon) density (even saturated) inside some part of $\mathcal{F}$ does not increases with energy. This means that low energy particle (or parton from other $\mathcal{F}$ ) moving toward this disk can pass through such part of disc without interaction with a finite probability. This, in particular, is reflected on that $\sigma_{\text{ell}} < \sigma_{\text{tot}}/2$, and the difference $\sigma_{\text{in}} - \sigma_{\text{ell}}$ is connected with the $F$ disk transparency.

The value of the $\mathcal{F}$ transparency can be characterized by the quantity $T(s, b) = 1 - |S(s, b)|$ which gives the probability that the colliding particles (or two $\mathcal{F}$) penetrate one trough another without any interaction at the given impact parameter $b$. This quantity must be boost-invariant, that is should not depend on the coordinate system in which it is calculated.

In the regge approach the $S(s, b)$-matrix is boost-invariant by construction, but this approach is rather inaccurate for description of details of $F$ asymptotic, because here the average rapidity on pomerons, entering essential reggeon diagrams, becomes small.

In parton approach we don’t meet such a problem with a high density. But here we have only the Fock wave function (WF) of a fast particle, in which partons in the dominant components are arranged in the $\mathcal{F}$ - like configuration. And by itself this WF is evidently not boost invariant. Using these WF we can calculate the cross-sections for various high-energy collision processes, in particular the $\sigma_{\text{in}}(s, b)$ and which may be not boost invariant.

In the same time on can use the requirement of such a invariance of $T(s, b)$ and of other quantities as a condition on the property of system, in our case the structure of $\mathcal{F}$. Such a condition is connected with the fact that the boost-invariance for all interaction cross-sections in the parton description can imitate the t-channel unitary conditions.

In this article we consider a number of questions connected with the structure of the Froissart behavior in QCD, and there consistency with the requirements of boost-invariance.

In section 2 we consider a simple example of $(2 + 1)$D dimensional QCD witch is soft and can contain 1-dim saturated $\mathcal{F}$. In this case the corresponding $\mathcal{F}$ is asymptotically gray and the
transparency is not boost-invariant.

In the section 3 we briefly summarize what can be expected about the parton structure of $\mathcal{R}$ in 4D.

In the section 4 we consider the collision of two such $\mathcal{R}$’s and describe the properties of some amplitudes and cross-sections.

In the section 5 we calculate the transparency of $\mathcal{R}$ in process of their collision. We find that it is not boost invariant in configurations when two $\mathcal{R}$ collide with their gray borders.

Section 6 contains some concluding comments.

In Appendix we consider in some details the fluctuation of $\mathcal{R}$ border.

2. Froissart behavior in 2+1 dimensional gauge theory

It is instructive to consider the asymptotic behavior of cross-sections in $(2 + 1_\perp)D$ because here we have only one transverse direction and the analysis is much simpler. Also Yang-Mills (YM) theory in 3 dimensions is soft, and the corresponding BFKL like pomeron is the supercritical regge pole [4]. Therefore one can expect that by the same mechanism as in 4D YM we come to the Froissart-type behavior. In $(2 + 1_\perp)D$ the Froissart behavior has the form $\sigma_{tot}(E) \sim (1/m) \ln E/m$.

Such a behavior in $(2 + 1_\perp)D$ case is rather evident, because the primary gluon emission spectrum (over their energy $\omega$ and transverse momenta $k_\perp$ is

$$dn \sim \alpha_s \frac{d\omega}{\omega} \frac{dk_\perp}{k_\perp^2 + \mu^2}, \quad (2.1)$$

where $\mu$ is some infrared confining scale or the effective gluon mass. From here, by the parton cascading, we come to the parton spectrum

$$dn \sim \alpha_s (E/\omega)^\Delta \frac{d\omega}{\omega} \frac{dk_\perp}{k_\perp^2 + \mu^2} \quad (2.2)$$

and to the power growth $\sim (E/\mu)^\Delta$ of the number of low energy partons with the full particles energy, and to the soft BFKL like pomeron [4] with intercept $\alpha_P(0) \simeq 1 + \Delta$, $\Delta \sim (\alpha_s/\mu)$ . Then, if we use the eikonal unitarisation of elastic reggesed amplitude, corresponding to (2.2), we become the Froissart type behavior $\sigma_{tot}(E) \sim (\alpha_s/\mu^2) \ln E/\mu$. In the parton language this corresponds to a power growth of the number of partons, which fill the $\mathcal{R}^3$ as $\sim (E/\mu)^\Delta \exp(-(x_{\perp} \mu)^2)$, and to their screening during the interaction.

Including also the pomeron interactions, one usually expects that we come to the saturation of the parton density, with almost stops to grow with energy inside the $F$ disk 4. In the parton language such a saturation corresponds to the situation when the local rate of parton gluing becomes the same as the local rate of their splinting. These partons are soft, with average transverse momenta $\sim \mu/\alpha_s^2$. Their density is $\sim 1/\alpha_s$, and the mean transverse size of $R(E)$ of the one-dimensional F-disk ($\mathcal{R}^1$) is $\sim (\alpha_s/\mu^2) \ln E/\mu$. The transparency of this disk when colliding with one low energy parton is $T_0(E) \sim \exp(-\alpha_s/\mu)$. This is the qualitative picture of a mean state of the fast particle, and at $y \gg 1$ such $\mathcal{R}$ looks as an almost classical gray object.

The shape of elastic amplitude corresponding to such $\mathcal{R}$ at $y \gg 1$ as a function of transverse distance $b = x_{\perp}$ is $i\theta(R(y) - b)$ where $\theta$ is close to the $\theta$-function with smoothed border. At $b \gg R(y)$ one can expect the behavior of the tail of $\theta(R(y) - b) \sim \exp(R(y) - b\mu)$.

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3 Here we use for the region filled with saturated partons the name F-disk $\equiv \mathcal{F}^d$, although it is in this case one dimensional.

4 It grows only linearly with $Y$ due to contribution of saturated soft partons with higher energies, so that the transverse density inside $\mathcal{F}^d$ is $f(y, b) \simeq (y - b/r_0)f_0 : f_0 \sim 1/\alpha_s$.
**Parton density fluctuations in \( R \)**

One can expect that in a small region inside the saturated part of \( R \) the fluctuation of parton density are Poisson like as in a dense gas.

But the behavior of the \( R \) border fluctuations are different. Here we meet with the scale, connected with the distribution of the position in \( b \) of the \( R \) border. This position fluctuates from “event to event”, and defines the variation of the size of the \( R \). In the case of one dimensional \( R \) the growth of \( R(y) \) with rapidity \( y \) looks like a random ”brownian process”, and this leads to the dispersion of the \( R \) borders position on \( \lambda(y) \sim (\bar{R}(y)/\mu)^{1/2} \sim r_0 \sqrt{y} \).

The big components of the Fock wave function of fast particle at \( y \gg 1 \) is given by the superposition of saturated \( R \) states with different size. In the inelastic interaction the interference from components with different \( \lambda(y) \) is small. On the contrary, in elastic diffractive scattering all components fully interfere.

**Collision of two \( R \)**

Firstly let us consider processes with large cross-sections. When two \( R \) with rapidity \( y_1 \) and \( y_2 \) collide, secondary particles are created from the region of two \( R \) intersection. And this defines the behavior of total inclusive cross-section, which is proportional to the value of area of this intersection

\[
\frac{d\sigma(Y, y_1, b)}{dy_1} \sim \int dB \, \tilde{\theta} \left( \bar{R}(y_1) - b \right) \tilde{\theta} \left( \bar{R}(Y - y_1) - b - B \right) , \quad Y = y_1 + y_2
\]

at given rapidity \( y_1 \) and impact parameter \( b \) and, and full rapidity interval = \( Y \).

The total inelastic cross-section \( \sigma_{in}(y_1, y_2) \approx \bar{R}(y_1) + \bar{R}(y_2) = r_0(y_1 + y_2) = r_0Y \) is boost invariant. The elastic diffraction cross-section at \( Y \gg 1 \) for almost black \( R \) is equal to inelastic cross section \( \approx 2\bar{R}(Y) \). The elastic amplitude is purely imaginary \( A(y, b) \approx i\tilde{\theta}(r_0y - b) \) and universal - it should not depend on quantum numbers of colliding hadrons.

As it was discussed above on can expect that the long range fluctuations of border are gaussian with the distribution \( w(R, Y) = (1/\sqrt{\pi}) \exp(-(R - \bar{R}(Y))^2/\lambda(Y)^2) \). So the elastic amplitude is the sum of the black disk elastic amplitudes with weight \( w(R) \). This leads to

\[
A(Y, b) \approx i \int \theta(R - b)w(R, Y) dR = \frac{i}{\sqrt{\pi}} \int_{(b-R)/\lambda}^{\infty} \exp(-z^2)dz ,
\]

\[
A(Y, k_\perp) \approx \frac{2i}{k_\perp} \left( e^{i\bar{R}(Y) - k_\perp^2/4} - 1 \right) , \quad (2.3)
\]

\[
A(Y, k_\perp = 0) = 2i \bar{R} = 2ir_0Y , \quad A(Y, k_\perp \gg 1/(r_0\sqrt{Y}) ) \approx -2i/k_\perp .
\]

Note that the soft spreading of the \( R \) border leads to a more smooth (less oscillatory) behavior of \( d\sigma_{el}/dk_\perp^2 \).

The processes of diffraction generation come from configurations when two colliding \( R \) intersect with each other with borders. This is so because only on the border different components of particles wave function have different probabilities to interact. So one can expect that the corresponding cross section \( \sigma_{tot}(Y) \sim \lambda \sim \sqrt{Y} \), where \( \lambda(Y) \) is the width of \( R \) border in lab. frame of one particle.

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5In fact, one must distinguish two transverse scales \( r_0 \sim 1/\mu, \) connected with distances on which parton moves when they split and distance \( r_1 \sim r_0/\alpha_s \) on which parton density grows from an unsaturated value to a saturated. Here , for simplification, we do not distinguish these scales.
Transparency of $\mathcal{F}d$ in $(2 + 1\perp)D$ case

Consider the collision of two such F-disks with energies $E_1 = \mu \exp y_1$ and $E_2 = \mu \exp y_2$ with large full energy, so that $Y = y_1 + y_2 \gg 1$ and at small impact parameter $b \sim 1/\mu$. Then it is simple to see that their mutual transparency is not boost-invariant if considered only in the states close to the mean one.

In the laboratory frame of one of colliding particles the transparency $T_{\text{lab}} \sim T_0 \sim \exp(-\alpha_s/\mu)$, because it is determined by the probability of tunnelling without interaction of only a few $\sim 1$ slow partons through the finite density saturated F-disc $^6$. This probability remains finite at $Y \to \infty$.

And in an arbitrary longitudinal frame many partons $\sim \nu(y_1) \sim (\alpha_s/\mu)y_1$ from one disk must tunnel without interaction through another disk. As a result we have $T(y_1, y_2) \sim (\tau)^\nu(y_1) \to 0$, when $y_2 > y_1 \to \infty$.

Certainly one must check that there may be not mean, but some rare particles states, containing small number of partons, which can give the needed ($\sim \text{const}$) contribution to transparency also in the center of mass system. Such are the states of the fast particle which contain only small number of partons, or do not contain the F-disc. From (2.1) one can estimate the probability of such a component of the wave function of a fast particle as

$$w_0(E) \sim \exp(-\nu(E)), \quad \nu(E) \sim \int_E dn \sim (\alpha_s/\mu) \ln(E/\mu)$$  \hspace{1cm} (2.4)

This corresponds to the condition that partons in the primary spectrum of one of the colliding particles are at all not emitted. So anyway, the contribution to transparency in c.m.s, coming from this a mechanism is such that

$$T_{\text{scm}} \sim w_0(E_1) \to 0 \quad \text{for} \quad E_1 \to \infty,$$

and it does not remain finite as is $T_{\text{lab}}$ when $\mathcal{F}d$ is saturated. This estimate take place when one or both colliding particles are in the bare states.

Therefore, in $(2 + 1\perp)D$ it is probably impossible to have a consistent boost-invariant behavior of $T(y_1, y_2)$ in the case when the saturation of parton density takes place at finite value inside of the $\mathcal{F}d$.

How can this be cured? There exists a number of possibilities, but they all look rather artificial.

* One possibility is that the full saturation does not takes place, and the mean parton density does not fully stabilize on a fixed value, but continue to grow with energy as $\sim (E/\mu)^\Delta_1$ with some $0 < \Delta_1 < \Delta$. In this case the transparency in an arbitrary Lorentz system in a mean sate is approximately given by

$$T(E_1, E_2) \sim w_0(E_1)w_0(E_2) \sim \exp(-(E_1/\mu)^\Delta_1)(E_2/\mu)^\Delta_1) \hspace{1cm} (2.5)$$

which is boost invariant but small in all systems. And then the main contribution to transparency comes from states without $\mathcal{F}d$.

* At high parton densities the parton system goes into the strong coupling regime near the critical point with the large density fluctuation in the mean states of $\mathcal{F}d$. In this case the transparency even in c.m.s (and $b = 0$) can be large ($\sim 1$). Also such fluctuations are reflected in the behavior of diffractive-generation cross section, with can become of the same order in $Y$ as the total cross-section. Note that for the saturated $\mathcal{F}d$ such processes are generated only on the border of $\mathcal{F}d$. But here the entire disc can look like “border”. This type of soft $\mathcal{F}$ behavior was

\footnote{Note that only the interactions with the low energy partons from another $\mathcal{F}d$ are essential, because the local interaction cross-section decreases $\sim 1/\epsilon$ with partons relative energy $\epsilon$.}
briefly discussed in [3]. However, it should be noted that this case seems to require a fine tuning of the system parameters in the regge approach.

* It is not excluded that if at very high energies, when in the \((2+1)\) case one takes into account all high order corrections, we become the effective \(\alpha_s(0) \lesssim 1\), and then the total cross-sections decrease \(\sim E^{-(1-\alpha_s(0))}\) and the transparency becomes boost invariant.

### 3. Parton structure of \(\mathcal{F}\) disk in 4D QCD

The main difference of 4D QCD parton spectrum, in comparison with \((2+1)\) case, as follows from (2.1), is that partons with high \(k_\perp\) are also generated and the hard component of \(\mathcal{R}\) grows with particles energy. In this section we briefly summarize the main properties of such a \(\mathcal{R}\) that can be expected in QCD.

**Mean picture of \(\mathcal{R}\)**

The evolution with \(y\) of the mean parton density \(f(y,u,b)\) in the \(\mathcal{R}\) can be represented schematically by the non linear generalization

\[
\frac{1}{\alpha_s(u)} \frac{\partial f(y,u,b)}{\partial y} \simeq \Phi[f(y,u,b)] + c_0 e^{-u} \mu^{-2} \left[ \frac{\partial^2 f(y,u,b)}{\partial^2 b} \right] + c_1 \frac{\partial^2 f(y,u,b)}{\partial^2 u} + \ldots, \quad (3.1)
\]

\[
\Phi[f] = c_2 f(y,u,b) - c_3 e^{-u} \mu^{-2} f(y,u,b)^2 + \ldots + c_4 \alpha_s(u) e^{-2u} \mu^{-4} f(y,u,b)^3 + \ldots
\]

of the BFKL-like equation with running \(\alpha_s(u) \sim 1/u\), where \(u = \ln k_\perp^2/\mu^2\) is the partons virtuality, and where \(\mu \sim r_0^{-1}\) and all \(c_i \sim 1\). All main qualitative properties of \(\mathcal{R}\) can be simply found from this equation \(^7\).

The structure of mean \(\mathcal{R}\) can be schematically represented as a system of inserted into each other saturated disks(sub-disks) with a different virtuality \(u\). The soft disk is the largest and its average size grows with rapidity as \(R(Y) = r_0 Y\) with small corrections \(^8\). Note that evolution equation (3.1) does not take explicitly into account the nonperturbative QCD effects, which are essential for grow of the soft \(\mathcal{F}\) sub-disk, and which in fact operates near the border of \(\mathcal{R}\) where the parton density is low. The distribution of the parton density \(f(y,u,b)\) in \(\mathcal{R}\), corresponding to Eq. (3.1), is illustrated by Fig.1

The positions of more hard sub-disks borders can be marked by the saturation boundary - the line \(u_s(y,b)\) which divides regions of \(\mathcal{R}\), so that partons with \(u < u_s(y,b)\) are in the saturated state. Parametrically this boundary can be found from Eq.(3.1) by the condition \(\Phi[f(y,u,b)] = 0\), which gives \(f_s(y,u,b) \sim e^u\) for the value of saturated density. To estimate the growth of \(f(y,u,b)\) with \(y\) one can take into account that the parton diffusion in the transverse coordinate \(b\) is small for large \(u\). At the given \((u,b)\) the parton density, before it reaches the saturation level, grows as \(f \sim \exp(c y_b/u)\). This \(f\) reaches \(f_s \sim e^u\) at \(y_b \sim u^2\). From here one finds the simple expression for the partons saturation border

\[
u_s(y,b) \sim \sqrt{y_b} \simeq a \sqrt{y - b/r_0} \quad (3.2)
\]

\(^7\)We use here such a simplified equation (3.1), in which some gluon properties and effects connected with gluon coherence are not taken into account explicitly, because they are not essential for our qualitative picture. In (3.1) we also included term \(\sim \partial^2 f(y,u,b)/\partial^2 b\), corresponding to parton propagation in the transverse plane \(b \equiv |\vec{x}|\). This is consistent if parton density is not small.

\(^8\)In fact there can be small corrections [7, 9] to the dependence of the average \(\mathcal{F}\) radius from particles rapidity, so that \(R_F(Y) = r_0(Y - c \ln Y)\). See also Appendix.
The value of \( y_b = y - b/r_0 \) can be seen on Fig.1 - as the "length" of path in rapidity on which the partons with high virtuality \( u \) are created.

\[
y - b/r_0
\]

\[
N(y) = \int d^2b \rho(y, b) \sim e^{\sqrt{y}}
\]

\[
f_s(y, u, b) \sim f_0 e^{u} (y - u^2 - b/r_0) \tilde{\theta}(y - u^2 - b/r_0), \quad f_0 \sim r_0^{-2}
\]

\[
\text{Statistical properties of } \mathcal{F}_d
\]

Usually we represent a fast particle with its parton cloud as a pure state, but when the mean parton number in cloud is very big then in some cases one can consider this particle (or big part of it) as a macroscopic object with the finite entropy and temperature. At \( y \gg 1 \) the average parton
number $N(y)$ in $\mathcal{F}$ is big and in the description of various inelastic processes the $\mathcal{F}$ can be approximately considered as a macroscopic body, whose entropy $S(y)$ is growing with $y$.

Furthermore, the low energy partons, corresponding to last stages of parton cascade are entangled with vacuum partons (fluctuations), and this also can be considered as a source of entropy in the parton description of a fast particle. If interpreted in such a way, the entropy of the large soft $\mathcal{F}$ sub-disk is $\sim (R(y)/r_0)^2 \sim y^2$. Partons in more hard $\mathcal{R}$ sub-disks, although they are strongly correlated with the soft $\mathcal{F}$, can give an additional contribution to fast particles $S(y)$.

To “discover” that isolated fast particle is in a pure state one needs a large time. This time grows with the energy for particles states containing $\mathcal{R}$, and it is of the order of the Poincare’ recurrence time $t_P \sim \exp (S(y)) \sim \exp y^2$, which for $y \gg 1$ is much bigger than the characteristic “Compton” time, that is proportional to the particles energy $m^{-1}\exp y$, and during which the final state is prepared after the interaction.

The process of the particles elastic diffraction also operates on the time scale $\sim m^{-1}\exp(y)$, that is short relative to $t_P$. But this process “measures” only the part of the wave function connected with the particles full momentum, while the $\mathcal{F}$ entropy is connected with the “internal part” of the wave function, and all these components diffract in the equal way. The difference in absorption of various components of the wave function comes mainly from the borders interaction of $\mathcal{F}$, and this gives the contribution to the diffraction generation.

### Fluctuations of $\mathcal{F}$

The physical state of a fast particle is the complicated superposition of multiparton states which in mean give the almost black $\mathcal{F}$. One can expect that there are Poisson-like local fluctuations of the parton density around the mean values in the interior parts of $\mathcal{F}$, as in every dense gas or liquid. But in processes with high mean multiplicities their contribution to various effects is not so essential.

The main global big fluctuations in the parton population of $\mathcal{F}$ come from the fluctuations on early stages of the parton cascade. For example, if, according to Eq. (2.1), the primary partons are not emitted in the rapidity interval $Y > y > y_1$, then the mean size of $\mathcal{F}$ in this component of the parton wave function is $r_0 y_1$. The probability of such a fluctuation (it is the weight of this component of the parton wave function) is

$$\sim \exp(-(Y-y_1)\alpha_s). \quad (3.5)$$

That is the probability that in the wave function of incoming particle there is no $\mathcal{F}$ can be estimated as $\sim \exp(-\alpha_s Y)$.

The “opposite” type fluctuations, which highly enlarge the number of partons in $\mathcal{F}$ compared with the mean value, can be generated by the following type parton mechanism. Firstly the valent partons should emit a secondary parton which is far in transverse plane - on the distance $b > r_0 Y$, and with the energy $\sim m(r_0 Y)$ primary particles energy $E$. The probability of such an emission is $\sim \exp(-b\mu) \sim \exp(-(r_0\mu) Y)$. Then this isolated parton will give with the probability $\simeq 1$ an independent cascade with $\simeq$ the same number of low energy partons as in the parent cascade. So, the mean number of partons in this state will approximately double. In the same way emitting two such far partons on the first stages of cascading (the probability $\sim \exp(-2(r_0\mu) Y)$) we come to the tripling of the number of low energy partons, etc.

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9In principle, one can not exclude that such a parton system is in a critical point and in this case all fluctuations in $\mathcal{F}$ are big. Such possibility was discussed in [3]. But, to this behavior one needs some special symmetry which is not seen now.

10Or more accurately $\sim \exp(-c Y \ln Y)$, if one takes into account the contribution of primary high $k_{\perp}$ partons and the running of $\alpha_s$. Evidently for the not too high energies the additional factor $\ln Y$ seems not so essential because $\ln Y$ “almost does not varies with energy”. But it can be essential if we consider the consistency of the full picture.
From this one can simply conclude that the tail of the multiplicity distribution in Froissart asymptotic collisions will have the KNO form of the type

$$\sigma_N \sim \exp \left( -c \frac{N}{N(Y)} \right) , \quad c \sim 1 . \quad (3.6)$$

where \(< N(Y) >\) is the mean multiplicity in an inelastic interaction. It is interesting that this behavior of the multiplicity distribution is of the same type as it comes from the simplest summing of pomeron contributions and also of that type as it is seen in experiments at high accelerator energies.

**Fluctuations of the \( F \) disk borders shape**

The \( y \)-development of the parton cascade contains also such fluctuations which contribute to a distortion of the shape of \( F \) border - so that it is not a pure round, but a randomly oscillating curve. In the physical state of a fast particle this can be represented as a superposition of \( F \) with various shape.

In \((2 + 1)D \) QCD the spectrum of shape distortion of \( F \)'s as discussed in Section 2 has the universal gaussian form with the average amplitude of the border fluctuation of order \( r_0 \sqrt{Y} \) and this almost does not depend from the details of parton dynamics. In \( 4D \) QCD the structure of the \( F \) shape fluctuation is more complicated but of the similar type (See details in Appendix).

At large \( Y \gg 1 \) the average amplitude of the radial border oscillation is \( \lambda(Y) \sim r_0 \sqrt{Y} \) as in the \( 3D \) QCD.

So are the shape fluctuations of the soft sub-disk. The shape fluctuations of the more hard saturated sub-disks have a close structure. This is the consequence of that hard partons almost do not move in the transverse plane in the process of their creation in a cascade from a partons of lower virtuality. As a result, the fluctuations of the shape of the sub-disk with the virtuality \( u \) do not move in the transverse plane in the process of their creation in a cascade from a partons.

Because the distributions of the border radius are gaussian and the diffractive scattering from the different parts of the target are additive, one can expect that the smoothed \( F \) elastic amplitude can be approximately represented in the form

$$A(y, b) \simeq i \tilde{\theta}(r_0 y - b) , \quad \tilde{\theta}(\cdot) \simeq \theta(\cdot) , \quad T(y, b) \simeq 1 - \tilde{\theta}(r_0 y - b) . \quad (4.1)$$

Because the distributions of the border radius are gaussian and the diffractive scattering from the different parts of the target are additive, one can expect that the smoothed \( F \) elastic amplitude can be approximately represented in the form

$$A(y, k_\perp) \simeq \frac{i \tilde{R}(y)^2}{\lambda(y) \sqrt{\pi}} \int e^{-\frac{\rho^2}{\lambda(y)}} \frac{J_1 \left( k_\perp (\tilde{R}(y) + \rho) \right)}{k_\perp (\tilde{R}(y) + \rho)} d\rho , \quad (4.2)$$

which corresponds to the superposition of contributions to \( A \) from scattering on \( \theta \)-like profiles distributed around the mean \( \tilde{R}(y) \). For an almost black \( F \), when the transparency \( T \ll 1 \) we have \( \sigma_{el} \simeq \sigma_{in} \).

For the hard scattering when \( k_\perp \gg 1/R \) the main mechanism of the elastic scattering is different. It comes from components of the wave function of the colliding particles that contain
minimum of partons and no $\mathcal{R}l$. This leads to a power decrease of $d\sigma_{el}/dk_{\perp}^{2}$ with $k_{\perp}$ and the cross sections contain an additional factor $\sim \exp(-\Delta y)$ representing the probability that the colliding particles are in bare states that contain no $\mathcal{R}l$.

**Inclusive spectra and multiplicity**

Inclusive spectra of secondary particles $\rho_1(y,Y)$ as a function of rapidity $y$ can be simply estimated as being proportional to the mean transverse area of the two $\mathcal{R}l$ intersection in the system in which rapidities of the colliding particles are $y$ and $Y-y$. This gives the contribution from the collision of soft sub-disks to the created particles density

$$\rho_1(y,Y,u \sim 1) \sim \rho_0 \cdot y^2 (Y - y)^2 / Y^2,$$

where $\rho_0$ is the parton density in the soft sub-disk. The corresponding soft particle multiplicity is

$$N = \int \rho dy \sim Y^3.$$  

The collision of hard sub-disks produces secondary particles with larger transverse momenta $k_{\perp} = \exp(u/2)$. The corresponding hard inclusive cross-section can again be estimated to be proportional to the average intersection area of the sub-disc with the virtuality $u$ with sub-disc with a lower virtuality. It must be also proportional to the parton density in $u$-sub-disk $\sim \exp(u)$. On this way we become

$$\rho_1(y,Y,u) \sim \frac{e^u}{Y^2} \left[ (y - u^2)^2 (Y - y)^2 \theta(y - u^2) + y^2 (Y - y - u^2)^2 \theta(Y - y - u^2) \right].$$  

Here the main contribution comes from the collision of saturated sub-disk with the big virtuality $u$ from the one particle with the soft disk from the other particle.

The full multiplicity of created hard particles with transverse momenta $k_{\perp} = \exp(u/2)$ is $N(Y,u) \sim e^u (Y - u^2)^3$. So the spectra of these particles grow with $u$ up to $u_{\text{max}} \sim \sqrt{Y}$ and at larger $u$ it rapidly decreases $\sim \exp(-(u - u_{\text{max}})^2 / Y)$. The mean virtuality of these particles is

$$\bar{u} \sim \int N(Y,u) u du / \bar{N} \sim Y^{1/2} \sim u_{\text{max}},$$

where $\bar{N} = \int N(Y,u) du \sim Y^{1.5} \exp \sqrt{Y}$ is the full mean particles multiplicity.

These are the primary particles, created when the two $\mathcal{R}l$’s move one through another. The interaction between these particles in the final state can largely thermalize the created system with the mean temperatures $\sim \exp \left(Y^{1/2}\right)$.

**Multiplicity distribution**

The shape of the multiplicity distribution of the created particles in the $\mathcal{F}$ limit is defined mostly by the geometrical reasons (the main dependence comes from the impact parameter $B$), as in the case of collision of the heavy nuclei. As it can be seen from (3.4) the mean multiplicity $\bar{N}(B) \sim \exp \left(\sqrt{Y}\right)$ only changes slowly for $0 < B < \bar{R}(Y)/2$ and decreases as $\bar{N}(B) \sim \exp \left(\sqrt{Y - (2B - \bar{R}(Y))/r_0}\right)$ for $\bar{R}/2 < B < \bar{R}(Y)$.

The tail of the multiplicity distribution, as discussed above, comes from the rare long range fluctuations (the creation of energetic partons with $x_{\perp} > \bar{R}(Y)$) when several ($n$) mean $\mathcal{R}l$’s are created. The probability of these fluctuations is $\sim \exp(-n\bar{R}(Y)/r_0)$. Such fluctuations correspond to the components of the wave function of a fast particle containing $\approx n* < N$ > partons, and this leads to the KNO type form (3.6) of the multiplicity distribution.
**Diffraction generation**

Processes of the diffraction generation (DG) take place on the borders of the $\mathcal{R}$, when two colliding $\mathcal{R}$'s only touch each other. In this case, the different components of the wave functions of the colliding particles have different transparencies and this is the source of the diffraction generation of various final states. Because the shape fluctuations of $\mathcal{R}$ are big $\lambda \sim r_0 \sqrt{y}$, this will reflect in the full diffraction generation cross-section $\sigma_{dg}(Y) \sim \bar{R}(Y) \lambda(Y) \sim Y^{3/2}$.

Depending on the rapidity $y_1, y_2$ of colliding particles this process will look differently. In lab. frame of one particle $y_1 \simeq 0$, $y_2 \simeq Y$ the probability for the particle 1 to convert diffractively to an other state is $\sim r_0 \bar{R}(Y)$. This is because the saturated soft border length of $\mathcal{R}$ does not almost fluctuate, and only on the border of width $\sim r_0$ the different components of the particle 1 have the different probabilities to interact with the particle 2.

This corresponds to the diffraction of the particle 1 in small mass beams with the cross-section $\sim Y$.

At the same time, the probability for the particle 2 to diffract is $\sim \bar{R}(y_2) \lambda(y_2)$. This comes from different probabilities of the interaction for the components of the particle 2 with different shapes at the given impact parameter vector. Here we have a diffraction to the states with large masses. The configuration of particles created in such a collision is probably with the inclusive spectra of a multiperipheral type.

The hard diffraction generation originates from the collision of borders of hard $\mathcal{R}$ sub-disks with $\sigma \sim R(y, u) \simeq r_0(y - u^2)$. This process arises together with the soft production of particles coming from the collision of the more soft sub-disks.

**Duality between high energy QCD and gravity**

If some form of duality between the high energy QCD and the gravity [11] takes place - then the described above picture of $\mathcal{R}$ can have some reflection in the gravitational parton structure of particles with superplanck energies, and in details of the black holes creation in the superplanck gravitational collision as suggested in [12].

In this picture the $\mathcal{R}$ of radius $\bar{R}(y) = r_0 y$ can be considered as a dual AdS projection of the black hole of radius $\sim r_0 e^y$. The $\mathcal{R}$ entropy $\sim (R(y)/r_0)^2$ can be mapped on the BH entropy $\sim (R_{BH}(e^y)m_{planck})^2$.

The border fluctuation in $\mathcal{R}$ can have some reflection in the BH horizon fluctuations. The $\mathcal{R}$ border width $\lambda \sim r_0 \sqrt{y}$ can be mapped on the BH horizon width $\delta R_{BH}$ in such a way that $(\delta r(Y))^2/R(Y) \sim \delta (R_{BH})^2/R_{BH}$. This corresponds to the BH horizon width $\sim \sqrt{R_{BH}/m_{planck}}$. Such a smearing of the BH horizon was discussed earlier [13].

5. Transparency of Froissart disk and boost-invariance of cross-sections

The condition of the boost-invariance (the frame independence) of various amplitudes and cross-sections calculated in the parton approach is rather strong. It imitates the t-unitarity and it can essentially restrict the calculated quantities. Consider for example a high energy collision of fully black disks (particles) whose radii $R(y_i)$ somehow depend on their rapidity. Then the total inelastic cross-section can be determined from purely geometrical conditions as

$$\sigma_{in}(Y) = \pi \left( R(y) + R(Y - y) \right)^2.$$  

(5.1)
From the condition of the independence of the right-hand side of Eq.(5.1) on \( y \) it clearly follows the unique solution for \( R(y) = r_1 \cdot y + r_2 \). So, in the case of black disks we immediately come to the \( \mathcal{F} \) behavior of cross-sections.

But if disks are gray the picture changes. For example, for the constant disk transparency \( T = \text{const} > 0 \) we have in lab. frame \( \sigma_{in}(Y) = \pi R(Y)^2(1 - T) \) and in c.m.s frame \( \sigma_{in}(Y, B) = \pi R(Y)^2(1 - O(\exp(-TR(Y)^2/r_0^2))) \). This phenomenon is illustrated in Fig.2, and it shows that the gray disk picture of a fast particle is contradictory if the parton density is approximately equal in all parts of disk.

Because at arbitrary high but finite energies these particles-disks are always gray (at least, near there borders), one must consider what the distribution of the parton density inside such disks can lead to a consistent picture.

The transparency in a high energy interaction of particles \( a \) and \( b \) with rapidity \( y_1, y_2 \) can be expressed as

\[
T(y_1, y_2, B) = \sum_{i,j} w_i^{(a)}(y_1) \cdot w_j^{(b)}(y_2) \cdot T_{ij}(y_1, y_2, B),
\]

where we sum over all parton configurations of particles \( a \) and \( b \). In (5.2) \( w_i^{(a)} \) and \( w_j^{(b)} \) are the probabilities of these configurations, and \( T_{ij} \) - the corresponding transparency in a \( |i > * |j > \) colliding state. One can expect that for a majority of many parton configurations \( |i > * |j > \) the transparencies are Poisson-like \( T_{ij} \sim \exp(-cN_{ij}), \) where \( N_{ij} \) is the mean number of parton collisions in a \( |i > * |j > \) scattering.

So, to calculate the full transparency on must sum over all possible parton collisions in the region where two \( \mathcal{F}d \) ’s intersect. With the exponential precision the transparency can be expressed through the saturated parton densities \( f_s(y, u, b) \) in \( \mathcal{F}d \) as:

\[
T(y_1, y_2, B) \sim \exp\left(-\tau(y_1, y_2, B)\right),
\]

where the expression

\[
\tau(y_1, y_2, B) \sim \int d^2x_{\perp} \int du_1 du_2 \sigma(u_1, u_2) \cdot f_s(y_1, u_1, |x_{\perp}|) \cdot f_s(y_2, u_2, |\vec{B} - x_{\perp}|),
\]

is proportional to the mean number of the parton scatterings when two \( \mathcal{F}d \) penetrate one trough another during their collision at the impact parameter \( B \), and

\[
\sigma(u_1, u_2) \sim 1/(k_{1\perp} \cdot k_{2\perp}) \sim \exp(-(u_1 + u_2)/2)
\]
is the cross-section for the parton interaction with virtualities \( u_1 \) and \( u_2 \).

If in the expression (5.4) for \( \tau \) we leave only the terms with small values of the parton virtuality \( u_i = u_0 \sim 1 \), we come to the case of the of gray \( F \) collision when

\[
\tau(y_1, y_2, B) \sim \int d^2x_\perp f_s(y_1, u_0, |x_\perp|) \cdot f_s(y_2, u_0, |\vec{B} - x_\perp|) .
\]  

(5.5)

Because inside \( F \) the parton density \( f_s(y_1, u_0, x) \) is \( \simeq \) constant the expression (5.5) is evidently not a boost invariant. From (5.5) follows that the value of \( \tau(Y, 1, B) \sim 1 \) in the lab.frame and \( \tau(Y/2, Y/2, B) \sim Y^2 \) in the c.m.s system. This is the same type picture as in the case of QCD in \((2 + 1)D\), considered in Section 2.

The expression (5.5) can be boost invariant only for very special forms of \( f_s \), for example, for the Gaussian form of \( f_s \) like

\[
f_s(y, u_0, |x_\perp|) \sim \frac{1}{y} e^{\lambda y - x_\perp^2 / yr_0^2} ,
\]  

(5.6)

which corresponds to the parton distribution arising in the parton cascade (without saturation!), and also in the amplitudes corresponding to a regge pole exchange with intercept \( \Delta > 0 \). But, in fact, this expression corresponds again to black disk of radius \( n_0y\Delta \) with a tin border, because the parton density changes fast from the small to the big values on the distances \( \delta x_\perp \sim r_0/\sqrt{\Delta} \).

**Central \( F \) - disk collision**

Consider the central-like collision of two \( F \) with small \( B \), when

\[
|\vec{R}(y_1) + \vec{R}(y_2) - B| \gg \lambda(y_1) + \lambda(y_2) .
\]  

(5.7)

The geometry of the collision of two \( F \) in various systems and with the same impact parameter and total energy looks like shown in Fig.2.

Firstly we estimate the contribution to the transparency from the mean parton configurations. In the expression (5.4) the hard partons give the main contribution to the value of \( \tau \), and using for \( f_s \) the expressions (3.3), we become:

\[
\tau(y_1, y_2, B) \sim \exp \left( \frac{c}{\sqrt{y_1 + y_2 - B/r_0}} \right) , \quad c \sim 1 ,
\]  

(5.8)

where the main contribution to \( \tau \) comes from the collision of the most hard sub-disks\(^{11}\) at given \( y_i \) and the impact parameter \( B \). It follows from (5.8-5.3) that the mean states of colliding \( F \) give no essential contribution to \( T \) in all Lorentz systems, including the laboratory frame.

At \( Y = y_1 + y_2 \gg 1 \) this is much less than the contribution in \( T \) that comes from some rare components of the parton wave function in (5.2), so that in these states particles have smaller transverse sizes and do not interact at the same \( B \). These are basically the states containing no \( F \), or \( F \) of smaller radius. For a collision at the impact parameter \( B \) one needs the states in (5.2) with \( (R(E_1) + R(E_2)) < B \). It is simple to estimate their probability \( w \), because these states are basically created by a fluctuation in which the primary soft partons are not emitted at the rapidity intervals \( \delta y_1, \delta y_2 \), close to the colliding particles valent zones, so that \( (Y - \delta y_1 - \delta y_2) r_0 < B \). This condition is boost-invariant, and the corresponding transparency is

\[
T \sim w \sim \exp \left( -c_1 \left( Y - B/r_0 \right) \right) , \quad c_1 \sim \alpha_s ,
\]  

(5.9)

\(^{11}\)The expression (5.8) is estimated with an exponential precession and it is boost-invariant. The correction to (5.8) are not boost invariant, but this does not change the main conclusion, that in this case the mean density states of \( F \) in 4D case are not essential for the transparency
and has the same property.

Therefore, for the collision with such B, we probably do not have a contradiction of the behavior of T with boost-invariance.

**Disk collision close to there borders**

Note that expressions (5.3, 5.8) become unapplicable for large impact parameters where

\[ r_0 Y - B \sim \lambda \sim r_0 Y^{1/2} \]  

(5.10)

that is in the stripe where the \( F \) border fluctuates. Here the different mechanisms for the transparency T can operate.

In the **lab.frame** of one particle and for such a B we have \( T_{lab} \sim 1 \), and it approximately do not change with grow of \( Y \). This is because there are such curved configurations of \( F \) border of other particle (and which appear with the probability \( \sim 1 \)) that particles do not touch each other (Fig.3a).

\[ F \]

\[ Y \]

\[ B \]

\[ \lambda \]

\[ r_0 \]

\[ R \]

\[ Y \]

\[ l \]

\[ \bar{R}(Y) \]

\[ \lambda(Y) \]

\[ T_{cms} \sim \exp \left(-c l \lambda/Y_0^2\right) \sim \exp \left(-c \lambda^{3/2}(Y) R^{1/2}(Y)/r_0^2\right) \sim \exp \left(-c Y^{5/4}\right), \quad c \sim 1 \]

(5.11)

in the mean \( F \) configurations.

Next, one must estimate the probability of a big fluctuation of the part of the border of \( F \) of length \( l \), so that there the border line shifts to smaller radii on the value \( \sim \lambda(Y) \).

In such configurations \( T \sim 1 \) also in c.m.s at the same impact parameter (Fig.3c). It can be estimated (from above !) in such a way. In mean \( F \) configurations the radial position of the \( F \) border randomly fluctuates around the average \( \bar{R}(Y) \) when we move along the border. It moves in both sides, \( R > \bar{R}(Y) \) and \( R < \bar{R}(Y) \), shifting on the mean radial distance \( \sim \lambda(Y) \).
During this “motion” it crosses the line $R = \bar{R}(Y)$, and the average number of such “long wave” intersection on distance $l$ is $\sim \sqrt{l/\lambda(Y)}$. Because such crosses are independent their number is Poisson distributed. So the probability that on the length $\sim l$ there are no intersections is $w \sim \exp\left(-\sqrt{l/\lambda}\right)$.

One can take this $w$ as an estimation of the probability of Fock component of one $\mathcal{F}$, in which the part of the parton disk is shifted inside on the length l so that two $\mathcal{F}$ do not collide also in c.m.s. This gives for

$$T_{c.m.s.} < w \sim \exp\left(-\sqrt{l/\lambda}\right) \sim \exp\left(-Y^{1/8}\right).$$

(5.12)

This contribution to the transparency $T_{c.m.s.}$ is bigger than that is coming from the mean parton configurations of $\mathcal{F}$, but it is also decreasing with $Y$, and so it cannot coincide with the transparency in the lab frame at $B \simeq \bar{R}(Y) \pm \lambda(Y)$, which is const(Y).

Therefore, here, as in $(2 + 1_{\perp})$, case we have a contradiction between the expected “usual” picture of $\mathcal{F}$ in 4D QCD and the boost-invariance (t-unitarity) of some amplitudes.

**Partons and reggeon diagrams**

At the first sight, it is natural to expect that the calculations of various cross-sections in the parton approach should, in principal, give an answer that is the same as that one can become using the reggeon approach. For example, so is the transparency $T(Y, B) = 1 - \sigma_{tot}(Y, B) + \sigma_{el}(Y, B)$, which we estimated in the parton approach for the case of the collision of $\mathcal{F}$, and have found that it is not boost-invariant for some values of $B, Y$. But if we calculate such quantities as $\sigma_{tot}(Y, B)$ and $\sigma_{el}(Y, B)$ using reggeon diagrams only for elastic amplitude we find that they are by construction boost-invariant, and so here we can expect the same property for $T(Y, B)$.

It should be noted that there is no contradiction here. The Regge approach can be consistently applied only when the reggeon(pomeron) density in the transverse plane is small. Otherwise, the mean energies on pomeron lines entering dominant reggeon diagrams are small. This is what takes place near the saturation point. And here, in fact, we are out of region of applicability of regge approach, but for parton approach the big QCD-parton density can be even in advantage.

Despite the fact that the large quantities like $\sigma_{tot}(Y, B)$ have in both approaches the same approximate values, the small corrections to them (that contribute to $T$) can differ.

At the same time, when the pomeron density is small (in this case $T(Y, B)$ is large) the parton calculation gives the same boost invariant answer as in the regge case. This is the consequence of the Gauss distribution $(5.6)$ of transverse parton density.

6. Conclusion

In this article we qualitatively described the structure of the Froissart disk ($\mathcal{F}$) and the properties of some processes in the Froissart ($\mathcal{F}$) limit which one can expect in QCD.

The $\mathcal{F}$ type behavior is not so different from that we see now in the scattering data at high accelerator energies. At LHC energies $\sqrt{s} \sim 7 \div 13$ TeV the transverse profile function estimated

\footnote{In fact, the mean number of all such intersection is much larger $\sim \sqrt{l/r_0}$. Note that for $l \sim R(Y) \sim r_0 Y$ this gives $w \sim \exp\left(-Y^{1/2}\right)$. This coincides with the estimate (3.5) of the probability that the size of $\mathcal{F}$ is smaller than the average $R(Y)$ on $\Delta R \sim \lambda(Y) \sim \sqrt{r_0 Y}$.}

\footnote{Another estimation of the probability of such a large variation of $\mathcal{F}$ border is given by the expression (A.3). Substituting there the dependence of $\lambda$ and $l$ from $Y$ we become $w \sim \exp\left(-Y^{1/4}\right)$.}
from the behavior of $d\sigma/dt$ gives the transparency $\leq 0.1$ at $B \leq (1 \pm 2) GeV^{-1}$. In such a case we already have in the middle of the fast hadron a clearly seen embryo of a $F$ disk. The gaussian shape of the border of such a $F$ disk can also be seen from the shape of the profile functions corresponding to the high energy data for $d\sigma/dt$.

The soft component of $F$ and its size are mainly defined by a nonperturbative QCD dynamics. And one can estimate the growth of the $F$ radius from the multi-$P$ contribution to the elastic amplitude, which will dominate the near $F$ border behavior, where the contribution of enhanced diagrams is small. This gives $r_0 = R(y)/y = 2\sqrt{\alpha_p \Delta}$, where the $P$ intercept $\Delta$ and slope $\alpha'_P$ can be found from the fit of the data at not too high energies and give $r_0 \sim 0.5 GeV^{-1}$.

The elastic scattering in $F$ limit at small transverse momenta $k_\perp \sim 1/R(Y)$ is diffractive. And at large $k_\perp \gg 1/R(Y)$ the main contribution to amplitudes comes from the components of the wave function without $F$ and minimal number partons in the state (in fact only the valent quarks), and it decreases only power-like in $k_\perp$. Such a behavior is similar to what we see at TeV energies.

The main aim of this article was to consider if it was possible to have a boost-invariant picture of the collision of two $F$, and to avoid problems with the boost-invariance of the high energy interaction, which appears in the case of grey $F$. For this we calculated the transparency $T(p_1,p_2,B) = 1 - \sigma_{in}(s,B) = |S(s,B)|$ in the process of the collision of two $F$ with momenta $p_1$ and $p_2$ at various systems at the same impact parameter $B$ and total invariant energy $s = (p_1+p_2)^2$.

The requirement is that $T$ should be boost invariant - i.e. depends only on $s$. This condition reflects in particular the t-unitarity of scattering amplitudes.

* The $F$ behavior in the $(2+1)$D case which we considered in Section 2, as a simple example, is probably contradictory. It corresponds to gray $F$ and so leads to a not boost invariant S-matrix. It signals that such a behavior breaks the t-unitarity.

* In 4D QCD the $F$ becomes almost black in the central part, but the borders of $F$ are gray and exhibit large fluctuations. As a result, for such impact parameters when two $F$ impact only by their borders during their collision the value of $\sigma_{in}(s,B)$ calculated in the parton model is not boost invariant and essentially differs in lab. and c.m.s. frames.

To avoid this contradiction $F$ border must be thin and specially arranged $^{14}$ or oppositely include all the $F$ disk. The standard QCD picture of $F$ is different.

* All this shows that the high energy parton structure of fast hadron in QCD must have some very special properties, otherwise the Froissart type behavior should not take place asymptotically. In principal there is a number of possibilities, such as mentioned at the end of Section 2 for the case of $(2+1)$ dimensions, but they all look rather artificial.

If for a while forget about the hard component of $F$, then the case of the critical Froissaron with gray $F$ seems more promising. Here we have a big fluctuation of the density on a scale of all $F$. But, moreover, the soft components of $F$ evolve independently from hard one, and the distribution of hard hard component fluctuation can “repeat” the distribution of soft one. If this really takes place one can have also the c.m.s. transparency $T \sim 1$. But the critical $F$ meets with the fine tuning of various pomeron parameters in the regge approach. Maybe, some effects in nonperturbative QCD can make such a model of $F$ natural. It would be interesting if there were traces of such fluctuations in the data at accelerator energies.

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$^{14}$For example, if the black $F$ profile is such as results from the summation of the contributions eikonal diagrams with supercritical pomerons.
Appendix

Structure of the $F$-disk border

The filling of the $F$ disc with partons can be represented as a parton cascade resulting from the emission of the additional less energetic partons by more energetic ones when rapidity $y$ increases.

In the process of the parton cascading new partons randomly move in the transverse plane. For a big number of steps they fill the transverse area which is approximately round, but with small fluctuations of $R$ radius $R(y, \varphi)$ in the different transverse directions $\varphi$ of the $\mathcal{R}$. Here we discuss only the fluctuations in the soft part of $F_d$ at large $y$, and also suppose that the amplitude of these fluctuations $\lambda(y, x)$ is small compared to the mean $\bar{R}(y) = r_0 y$. The random function $\lambda(y, x)$ depends on $y$ and the transverse coordinate $x = \bar{R}(y) \ast \varphi$, $0 < \varphi < 2\pi$ which varies in the transverse plane along the border line of the mean $\mathcal{R}$. This variable $x$ is more appropriate, and so we represent the $R(y, x) = \bar{R}(y) + \lambda(y, x)$. In fact the $F_d$ border if considered as a continuous function randomly growing with $y$ can have the complicated fractal structure [14], but we use here the smooth approximation.

We need the weight $W[\lambda(y, x)]$ of the realization of some definite configuration $\lambda(y, x)$, so that various quantities depending on $\lambda$ can be represented by the averaging

$$\int D\lambda(y, x) W[\lambda(y, x)] \left( \lambda(y, x_1)\lambda(y, x_2) \ldots \right)$$

of corresponding functions of $\lambda(y, x)$.

For a smooth border $R(y, x)$, and due to absence of long range interactions in the dense parton media, the only existing parameter on the scale $x \gg r_0$ is the length of the border. In addition, the far regions of the $\mathcal{R}$ border fluctuate independently. Therefore, one can conclude that the amplitude $W[R(y, x)]$ depends on $\Gamma[R] - 2\pi\bar{R}(y)$ in the exponential form

$$W \sim \exp\left( -\frac{\beta}{r_0} (\Gamma[R] - 2\pi\bar{R}) \right), \quad \beta \sim 1, \quad (A.1)$$

where $\Gamma[R]$ is the length of the $\mathcal{R}$ border and $r_1$ is the average radial distance where the parton density in $\mathcal{R}$ passes from the saturated phase to an unsaturated one ( $r_1 \sim r_0/\alpha_s$ in the perturbative QCD). At a small and smooth $\delta(y, x) \ll \bar{R}$ the length of the border is

$$\Gamma[R] = \int_0^{L(y)} dx \sqrt{1 + (\lambda'(y, x))^2} \simeq L(y) + 1/2 \int_0^{L(y)} dx (\lambda'(y, x))^2, \quad (A.2)$$

where $L = 2\pi \bar{R}(y)$.

One can use this expression to estimate the probability $w$ of the large deepening of the $\mathcal{R}$ border with length $l$ and depth $\lambda$. We come to estimate

$$w \sim \exp\left( -\frac{\beta}{r_0} \delta \Gamma \right),$$

where $\delta \Gamma$ is the variation of length of the deformed border. For a long and smooth deformation when $\lambda \ll l$ we have from (A.2) $\delta \Gamma \sim \lambda^2/l$, and we become

$$w \sim \exp\left( -c\lambda^2/l r_0 \right), \quad c \sim 1. \quad (A.3)$$

Decomposing the border line in harmonics

$$\lambda(y, x) = \sum_n a_n e^{i 2\pi n x / L},$$

$$17$$
we have for
\[ \Gamma[R] = L + \frac{\pi^2}{L} \sum_n n^2 a_n^2 , \]
and therefore for the averaging over the border fluctuation we can use the measure
\[
\int D\lambda(y,x) W[\lambda](\ldots) \equiv \int Da(\ldots) = N_1 ! \left( \frac{\beta \pi^2}{r_0 L} \right)^{N/2} \prod_{n=1}^{N_1} da_n \exp \left( - \frac{\beta \pi^2}{r_0 L} \sum_n n^2 a_n^2 \right)(\ldots) ,
\]
where we cut the series of harmonics on large \( N_1 \gg 1 \). Using (A.4) we can calculate various quantities that characterize the \( \mathcal{F} \) border, for example, the mean width of the border
\[
< (R(y) - \bar{R})^2 > \equiv \frac{1}{2} \int Da \sum_{n=1}^{N_1} a_n^2 = \frac{\beta}{4 \pi^3} L r_0 \sum_{n=1}^{N_1} \frac{1}{n^2} \simeq \frac{(2 \beta)}{4 \pi^3/4} (L r_0) \sim y r_0^2 \quad (A.5)
\]
This shows the same behavior of the width of \( \mathcal{F} \) border as in the \( (2 + 1_{\perp})D \).

The instructive quantity that represents the shrinking of the \( \mathcal{F} \) border, is the correlator
\[ G_y(x) =< |R(y,x) - R(y,0)| > , \]
and especially its dependence on \( x \). From the equations above we have:
\[ G_y(x) = \int Da \sum_n \sin \left( \frac{2 \pi n x}{L} \right) |a_n| \simeq \pi^{-3/2} \sqrt{\frac{r_0 L}{\beta}} \sum_n \frac{1}{n} \sin \left( \frac{2 \pi n x}{L} \right) \simeq \pi^{-3/2} \left( \frac{r_0 L}{\beta} \right) \sin \left( \frac{2 \pi x}{L} \right) \simeq \frac{2 x}{\sqrt{\pi \beta}} \sqrt{\frac{r_0 L}{L}} \]
At \( x \sim L \) we have \( G_y(\sim L) \sim \sqrt{L r_0} \), which agrees with (A.5), and show that the shrinking of the \( \mathcal{F} \) border comes from the long range harmonics.

The other approach to this problem is to start from the stochastic parton cascade evolution of \( W[\delta] \) in rapidity, or, even simply, from the evolution of the local (in \( x \)) radius of the \( \mathcal{F} \) border line in rapidity
\[ \frac{\partial R(y,x)}{\partial y} = \dot{r} - c r_0^2 \rho(y,x) , \quad c \sim 1 \quad (A.6) , \]
where the stochastic function \( \dot{r}(y,x) > 0 \), with \( < \dot{r}^2 > \simeq r_0^2 \), represents the random motion of the border due to a parton splitting near the border of \( \mathcal{F} \). The second term in (A.6) is proportional to \( \rho(y,x) \sim \partial^2 R(y,x)/\partial^2 x \) - the local curvature of \( \mathcal{F} \) border line. It takes into account that the rate of the parton creation close to the border is proportional to the number of neighboring partons. For the convex part of the \( \mathcal{F} \) border \( \rho > 0 \) and for the concave part \( \rho < 0 \). Averaging Eq. (A.6) we become the equation for the evolution with \( y \) of the mean \( \mathcal{F} \) radius
\[ \frac{\partial R(y)}{\partial y} = r_0 - c \frac{r_0^2}{R(y)} \quad (A.7) , \]
Its solution
\[ \bar{R}(y) \simeq r_0 (y - c \ln y) , \quad y > 1 \]
contains \( \sim \ln y \) corrections \([7, 9]\), coming from the positive curvature of the average \( \mathcal{F} \) border. One can also use Eq.(A.6) to estimate the growth with \( y \) of
\[ G_y(y) = \langle (R(y,x) - \bar{R}(y))^2 \rangle_{\dot{r}} \]
which gives the dependence of the mean width of \( \mathcal{F} \) border on \( y \). Using (A.6 , A.7) we become
\[ \frac{\partial}{\partial y} G(y) \simeq 2\langle \dot{r} R(y,x) \rangle_{\dot{r}} - r_0 \bar{R}(y) = 2 \langle \dot{r} \dot{R} \rangle_{\dot{r}} = c_3 r_0^2 \quad , \quad c_3 \sim 1 , \]
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where for \( \langle \lambda \hat{r} \rangle \hat{r} \) we used that on the scale \( \sim r_0 \) the growth of \( \delta \) is correlated with the fluctuation of \( \hat{r} \). From here we have

\[
G(y) \simeq c_3 r_0^2 y
\]

which agrees with (A.5).

Note that the form (A.1 - A.2) for \( W[\delta(y,x)] \) can suggest the analogy with the Luscher’s type oscillation of a string of length \( L = 2\pi R \). Here the quantum string broadening is \( \sim r_0 \ln L/r_0 \). This corresponds to the \( \mathcal{R} \) border width \( \sim \ln R(y) \) and it comes from the oscillations with the mean wave length \( \sim \sqrt{r_0 L} \) at the zero temperature. At the finite temperature \( T \) there is an additional contribution to string broadening, which is \( \sim \sqrt{TL}/\kappa \) where \( \kappa \) is the string tension, and this broadening comes from the large wave length \( \sim L \). In our case the effective \( T \sim r_0^{-1} \) and \( \kappa \sim r_0^{-2} \), so we come again to same result as above.

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