Factoring Perfect Reconstruction Filter Banks into Causal Lifting Matrices: A Diophantine Approach

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Abstract

The elementary theory of bivariate linear Diophantine equations over polynomial rings is used to construct causal lifting factorizations (elementary matrix decompositions) for causal two-channel FIR perfect reconstruction transfer matrices and wavelet transforms. The Diophantine approach generates causal factorizations satisfying certain polynomial degree-reducing inequalities, enabling a new factorization strategy called the Causal Complementation Algorithm. This provides a causal (i.e., polynomial, hence realizable) alternative to the noncausal lifting scheme developed by Daubechies and Sweldens using the Extended Euclidean Algorithm for Laurent polynomials. The new approach replaces the Euclidean Algorithm with Gaussian elimination employing a slight generalization of polynomial division that ensures existence and uniqueness of quotients whose remainders satisfy user-specified divisibility constraints. The Causal Complementation Algorithm is shown to be more general than the causal version of the Euclidean Algorithm approach by generating additional causal lifting factorizations beyond those obtainable using the polynomial Euclidean Algorithm.

Keywords: Causality, causal complementation, digital signal processing, Diophantine equation, elementary matrix decomposition, Euclidean Algorithm, filter bank, lifting factorization, polyphase matrix, wavelet transform

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1. Introduction

Multirate filter banks are the digital signal processing constituents of wavelet transforms. A discrete wavelet transform (DWT) is a cascade of filter banks that decomposes discrete-time signals into time-frequency components, such as the exponentially scaled Mallat decomposition. Under suitable conditions DWTs correspond to analog signal representations called multiresolution analyses [1, 2, 3, 4]; such decompositions offer a wealth of data representations featuring joint time-frequency localization and fast digital implementations. For one measure of their success, the number of U.S. patents containing the term “wavelet[s]” is now in the thousands. For more examples of success, [5, §II] surveys multirate filter banks in digital communication coding standards.

Figure 1 depicts the Z-transform (complexified frequency domain) representation of a two-channel multirate filter bank with input \(X(z) = \sum_i x(i) z^{-i}\) [2, 6, 7, 8, 4]. (The author recommends

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references [6, 8] to readers who are less familiar with, and seek a guide to, the engineering language and concepts used in this subject area.) The downarrows represent 2:1 subsampling, which halves the sampling rate of the analysis-filtered output from $H_0$ and $H_1$. Uparrows restore the original sampling rate to subbands $Y_0$ and $Y_1$ by inserting zeros prior to synthesis filtering with $G_0$ and $G_1$, then summing to get the reconstructed signal, $\hat{X}(z)$. Figure 1 is a perfect reconstruction (PR) filter bank if the transfer function $X(z)/X(z)$ is a monomial, i.e., a constant multiple of a delay.

For suitably defined polyphase transfer matrices $H(z)$ and $G(z)$ (denoted in boldface) the system in Figure 1 is mathematically equivalent to the polyphase-with-delay (PWD) representation in Figure 2 [6, 8]. The polyphase analysis matrix, $H(z)$, is the frequency-domain representation of a linear translation-invariant operator acting boundedly on vector-valued discrete-time signals in $\ell^2(\mathbb{Z}, \mathbb{C}^2)$. By Cramer’s Rule, a Laurent polynomial matrix $H(z)$ is the polyphase matrix of a finite impulse response (FIR) PR filter bank with FIR inverse if and only if, for some gain $\hat{a} \neq 0$ and delay $\hat{d} \in \mathbb{Z}$, it satisfies

$$|H(z)| \overset{\text{def}}{=} \det H(z) = \hat{a}z^{-\hat{d}}. \quad (1)$$

In this paper all filter banks are assumed to be FIR systems satisfying (1).

1.1. Background and Relation to Other Work

Many structures for fast, customizable implementations of PR filter banks decompose the transfer matrices $H(z)$ and $G(z)$ into cascades (matrix products) of simpler building blocks. Examples include decompositions for particular classes such as paraunitary or linear phase filter banks [9, 10, 11, 12, 13, 14, 15, 16, 17] and low-complexity structures like cosine-modulated filter banks [6, 8, 18, 19, 20]. The cascade structures studied in this paper are lifting factorizations [21, 22, 23] of $H(z)$ and $G(z)$ into elementary (“lifting”) matrices $S(z)$ of the form

$$S(z) = \begin{bmatrix} 1 & 0 \\ S(z) & 1 \end{bmatrix} \quad \text{or} \quad S(z) = \begin{bmatrix} 1 & S(z) \\ 0 & 1 \end{bmatrix}. \quad (2)$$

Lifting figures prominently in image communication standards like the ISO/IEC JPEG 2000 image coding standards [24, 25, 26, 27, 28, 29] and CCSDS Recommendation 122.0 for Space Data System Standards [30, 31].
The lifting scheme of Daubechies and Sweldens [23] factors unimodular polyphase matrices (matrices of determinant 1) using computational byproducts of the Extended Euclidean Algorithm (EEA) [32, 33, 34, 35] for computing greatest common divisors (gcds) over the Laurent polynomials in $z$ and $z^{-1}$, denoted $\mathbb{C}[z, z^{-1}]$. The term “lifting” seems to originate with Sweldens [21], although the result now known as the Lifting Theorem had appeared earlier in the dissertation of Herley [36, 37]. The EEA was not used in [21], which used lifting to synthesize high-order wavelets rather than for decomposing a given wavelet transform. The Euclidean Algorithm does appear, however, in Daubechies’ proof of Bezout’s Theorem for polynomials [2, Theorem 6.1.1]. Daubechies credits Y. Meyer [2, Chapter 6, endnote 2] for suggesting the use of Bezout’s Theorem to factor trigonometric polynomials when constructing smooth wavelets, so Meyer deserves some credit for the closely related use of the Euclidean Algorithm in lifting factorization.

Lifting was anticipated by others, including Tolhuizen et al. [38], who studied existence of elementary matrix decompositions for multivariate matrix polynomials. Cohn [39] constructed an example of a $2 \times 2$ bivariate matrix polynomial with no elementary matrix decompositions, but Suslin [40] subsequently proved that elementary matrix decompositions exist for all multivariate matrix polynomials of size $3 \times 3$ or greater. A constructive proof using Gröbner bases was found by Park and Woodburn [41]. The univariate challenge is not the existence of elementary matrix decompositions; as noted in [38], univariate polynomial (resp., Laurent polynomial) matrices can always be factored into elementary matrices using the polynomial (resp., Laurent polynomial) EEA. The univariate challenge is, instead, to define and systematically generate “good” lifting decompositions well-suited for computational applications.

Since the appearance of [23], some work on lifting has studied two-channel filter banks [42, 43, 44, 45, 46] but much has focused on $M > 2$ channels [47, 48, 49, 13, 50, 51, 52, 53, 54] and related methods like linear predictive transform coding [55, 56]. The limitation of mathematical technique in [23] (and in most of the literature since [23]) to linear algebra and the Euclidean Algorithm strikes the author as unduly restrictive. Herley’s dissertation [36, 37] noted a connection to Diophantine equations, but that idea was not followed up in subsequent papers, and the use of abstract algebra in filter bank theory has remained largely unexplored with only occasional exceptions [57, 58, 59, 60, 61, 62, 63].

We shall focus on gaining a deeper understanding of the two-channel case, which has yielded significant applications to date such as digital communications coding and which informs our understanding of the more difficult $M$-channel case. The two-channel case has also proven amenable to nonlinear algebraic methods. After serving on the JPEG standards committee (ISO/IEC JTC1/SC29/WG1) the author developed a group-theoretic approach to lifting for two-channel unimodular linear phase FIR PR filter banks [64, 65, 66]. Surprisingly, it was shown that factoring linear phase transfer matrices using linear phase lifting filters produces elementary matrix decompositions that are unique within “universes” of factorizations that the author called group lifting structures. These uniqueness results were used [5] to characterize the group of unimodular whole-sample symmetric (WS, or odd-length linear phase) filter banks up to isomorphism. It was also shown that the class of unimodular half-sample symmetric (HS, or even-length linear phase) filter banks, which is not a group, can nonetheless be partitioned into cosets of similar groups. An overview is given in [67].

The present paper is the author’s response to Daubechies and Sweldens [23], who sacrificed causality to exploit nonuniqueness of Laurent polynomial division. A time-dependent operator is called causal if its output at time $t$ depends only on input data received at times prior to $t$; i.e., the operator does not need to “see into the future” to compute its output. A discrete-time FIR filter or
filter bank is causal if its transfer function is \textit{polynomial} in $z^{-1}$ since terms with positive powers of $z$ would indicate a dependence on “future” inputs. The transfer functions of noncausal systems are given by \textit{Laurent} polynomials in both $z$ and $z^{-1}$. Causality is physically necessary for “realizing” engineering systems that operate continuously on streaming input, but it also has advantages in computing contexts that enjoy random access to preloaded memory.

A fundamental limitation of modern High Performance Computing platforms is the latency incurred while writing and reading data to and from the huge amounts of memory provided on such machines. In many cases memory I/O dominates computing performance over and above the runtime cost of the actual calculations being performed on the CPUs. Causal realizations of mathematical operators ensure that the input data need only be read into “near” memory (buffers or cache) once, as opposed to a noncausal realization that accesses the same data in “far” memory multiple times as both “future” and “past” samples. Causality also ensures that the memory holding the input data is effectively “freed up” once it has been read, allowing the algorithm to be performed \textit{in situ} by writing the latest output samples into the freed memory that previously held the latest inputs. This is a significant advantage for applications with extremely large input arrays since it eliminates the need to allocate memory for an equally large target array to hold the computation’s output.

Converting noncausal unimodular factorizations into “equivalent” minimal causal realizations suitable for implementation is not straightforward, so sacrificing realizability as Daubechies and Sweldens have done is a big price to pay for factorization options. It begs the question of just how many factorizations Laurent division creates; there appears to be no systematic way to enumerate “all possible” clever uses of Laurent division. It also highlights the lack of a definition in \[23\] of what, exactly, makes an elementary matrix decomposition a lifting factorization. E.g., what is the difference between the “nice” factorizations in \[23\] and pathological factorizations like \[65, Proposition 1 and Example 1\] or \[5, Example 1\]?

Given that the goal is to obtain elementary matrix decompositions of FIR transfer matrices, it seems natural to approach the problem via elementary row and column operations. By taking an algebraic perspective we develop lifting from basic properties of linear Diophantine equations (LDEs) over polynomial rings. We show that much of lifting factorization, including the Lifting Theorem \[36, 37, 21\], follows from basic factorization theory in commutative rings and does not even involve polynomials per se. One is led naturally from abstract algebraic considerations to issues that really require polynomials and causality, such as degree inequalities and uniqueness results. This leads to a new lifting strategy based on elementary row and column operations, demonstrated by example in this paper and developed formally in work in progress \[68\] that we call the \textit{Causal Complementation Algorithm} (CCA).

### 1.2. Degree-Reducing Causal Complements and Degree-Lifting Factorizations

Lifting factorization of a transfer matrix $H(z)$ involves factoring off elementary matrices (2). E.g., left-factorization of an upper-triangular elementary matrix reduces row 0 (the top row) of $H$; suppressing the $z$ for clarity,

$$H(z) \overset{\text{def}}{=} \begin{bmatrix} E_0 & E_1 \\ F_0 & F_1 \end{bmatrix} = \begin{bmatrix} 1 & S \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_0 & R_1 \\ F_0 & F_1 \end{bmatrix}. \quad (3)$$

To get (3), pick a column index $j \in \{0, 1\}$ and divide the pivot $F_j$ into $E_j$ using polynomial division to get $E_j = F_j S + R_j$, where remainder $R_j$ satisfies the degree-reducing condition $\text{deg}(R_j) < \text{deg}(F_j)$. 


In the language of Definition 1.1, the CCA constructs factorizations of the form (3) by polynomial division to compute degree-reducing causal complements. Remarks. The polyphase notion of complementary filters in the causal case (i.e., for polynomials in \( z \)) are complementary if and only if their polyphase matrix satisfies (1), motivating the following.

Given a filter \( H_i(z) \), \( i \in \{0,1\} \), Herley and Vetterli [36, 37] call a second filter \( H'_{i'}(z) \), where \( i' \overset{\text{def}}{=} 1 - i \), a complementary filter if \( \{H_0(z), H_1(z)\} \) is a PR filter bank. FIR filters \( H_0 \) and \( H_1 \) are complementary if and only if their polyphase matrix satisfies (1), motivating the following polyphase notion of complementary filters in the causal case (i.e., for polynomials in \( z^{-1} \)).

**Definition 1.1.** Let \( F_0, F_1 \) be causal FIR polyphase filters satisfying \( \gcd(F_0, F_1) = z^{-d} \). Given constant \( \hat{a} \neq 0 \) and integer \( \hat{d} \geq d \), an ordered pair of causal filters \( \left(R_0, R_1\right) \) is a causal complement to \( \left(F_0, F_1\right) \) for inhomogeneity \( \hat{a}z^{-\hat{d}} \) if it satisfies the linear Diophantine polynomial equation

\[
R_0(z)F_1(z) - R_1(z)F_0(z) = \hat{a}z^{-\hat{d}}. \tag{4}
\]

We say a causal complement \( \left(R_0, R_1\right) \) is degree-reducing in \( F_\ell \), \( \ell \in \{0,1\} \), if

\[
\deg(R_\ell) < \deg(F_\ell) - \deg \gcd(F_0, F_1). \tag{5}
\]

**Remarks.** In the language of Definition 1.1, the CCA constructs factorizations of the form (3) by using polynomial division to compute degree-reducing causal complements \((R_0, R_1)\) for inhomogeneity \( \hat{a}z^{-\hat{d}} = |H(z)| \) without using the EEA. The reason for the correction term \( \deg \gcd(F_0, F_1) \) in (5) is explained in Section 4.1 following Definition 4.1; existence and uniqueness of degree-reducing causal complements are addressed by Theorem 4.5.

In the course of factorization, the degree-reducing property (5) eventually drives one of the remainders \( R_\ell \) to zero, causing factorization to terminate. The result can then be put into standard causal lifting form (cf. Figure 3),

\[
H(z) = \text{diag}(\kappa_0 z^{-\kappa_0}, \kappa_1 z^{-\kappa_1}) U_{N-1}(z) A_{N-1}(z) \cdots U_1(z) A_1(z) U_0(z) P_0 \text{diag}(z^{-c_0}, z^{-c_1}). \tag{6}
\]

The causal matrices \( U_n(z) \) in (6) are alternating upper- and lower-triangular lifting matrices (2) with causal lifting filters \( U_n(z) \). The matrices \( A_n(z) \) are diagonal delay matrices. They have a single delay factor \( z^{-m_n} \), \( m_n \geq 0 \), in the upper channel, \( A_n(z) = \text{diag}(z^{-m_n}, 1) \), or (resp.) the.
lower channel, $\text{diag}(1, z^{-m_n})$, if and only if $U_n(z)$ is upper-triangular (resp., lower-triangular). $P_0$ is either the identity, $I$, or the swap matrix,

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad J^{-1} = J. \tag{7}$$

Using the CCA, every causal FIR PR filter bank has a factorization (many, in fact) in standard causal lifting form. We shall see that the ability to factor off diagonal delay matrices $A_n(z)$ is a major advantage the CCA holds over the causal version of the EEA lifting factorization method.

The ancient Euclidean Algorithm was a clever idea for recursively reducing the gcd of “large” arguments to the gcd of “smaller” arguments. The notion of degree-reducing solutions to LDEs over polynomial rings (Definition 1.1) captures the size-reducing aspect of the EEA without its misleading focus on gcds. Moreover, the degree-reducing notion applies to factorizations of arbitrary FIR PR filter banks whereas the polyphase order-increasing property employed in [65, 66] was found to be useful only for linear phase filter banks. A degree-reducing decomposition corresponds to a degree-increasing synthesis, so we use the neutral term degree-lifting to encompass both decomposition and synthesis. The author holds that this degree-lifting character of both the EEA and the CCA is what distinguishes lifting factorizations within the much bigger universe of elementary matrix decompositions, a distinction not made in [23].

1.3. Overview of the Paper

Section 2 introduces LGT(5,3), the LeGall-Tabatabai 5-tap/3-tap piecewise-linear spline wavelet filter bank [69, 24], and uses it to illustrate a connection between causality and uniqueness of lifting factorizations. It is shown that the Causal Complementation Algorithm (CCA) generates all of the causal lifting factorizations formed by the causal version of the EEA plus other factorizations not generated by it, generalizing the degree-lifting aspects of the EEA.

To understand the connection between causality and uniqueness of factorizations, Section 3 defines linear Diophantine equations (LDEs) and reviews the ring theory needed to characterize the solution sets of homogeneous LDEs and inhomogeneous LDEs (the Abstract Lifting Theorem). Necessary and sufficient conditions are given for existence (but not uniqueness) of solutions to inhomogeneous LDEs in principal ideal domains (the Abstract Bezout Theorem). Section 3.2 reviews the differences between polynomials and Laurent polynomials. Readers not interested in these algebraic underpinnings should skip ahead to Theorem 4.5.

Degree-reducing causal complements are generalized in Definition 4.1 to include LDEs over the Laurent polynomials. By Lemma 4.4 degree-reducing solutions to polynomial LDEs are unique thanks to a max-additive inequality that fails for the Laurent order, explaining why filters can have multiple Laurent-order-reducing complements. Existence and uniqueness of degree-reducing solutions (in either unknown) to polynomial LDEs are given by the Linear Diophantine Degree-Reduction Theorem (LDDRT, Theorem 4.5), along with necessary and sufficient conditions for both solutions to agree. Although quite elementary, the LDDRT seems to have escaped attention in the literature, perhaps for lack of a motive for finding degree-reducing solutions.

Section 5 uses the LDDRT to prove existence and uniqueness of quotients whose remainders satisfy degree-reducing inequalities and user-specified divisibility constraints (the Generalized Polynomial Division Theorem, which also appears to have escaped previous notice). This is specialized to the case of remainders divisible by monomials and given a constructive proof in the Slightly Generalized Division Theorem, yielding a Slightly Generalized Division Algorithm (SGDA).
Section 6 returns to the filter bank setting and presents a case study using CDF(7,5), a 7-tap/5-tap cubic B-spline wavelet filter bank [23]. It is shown that the causal EEA and CCA factorizations in column 1 using classical polynomial division are the same. To produce a causal analogue of a unimodular linear phase lifting factorization obtained by Daubechies and Sweldens [23, §7.8], we use the SGDA to factor off a diagonal delay matrix with the first CCA lifting step. This factorization is not produced by running the causal EEA in any row or column of CDF(7,5).

Section 7 summarizes the paper’s contributions.

2. Case Study: The LeGall-Tabatabai Filter Bank

We begin by contrasting the EEA and CCA approaches to factoring LGT(5,3), the 5-tap/3-tap LeGall-Tabatabai biorthogonal linear phase filter bank [69], whose analog synthesis scaling function and mother wavelet generate piecewise-linear B-splines. JPEG 2000 Part 1 [24] specifies LGT(5,3) via a lifting factorization of the unimodular polyphase-with-advance (PWA) matrix representation $A(z)$ [23, 64] for its noncausal whole-sample symmetric analysis filters. This lifting factorization is its unique WS lifting factorization in the WS group of unimodular whole-sample symmetric transfer matrices [65, Definition 8]. The factorization halves the number of multiplications per unit input when implementing the filter bank, a big motivation for using lifting decompositions.

$$A_0(z) = -z^2/8 + z/4 + 3/4 + z^{-1}/4 - z^{-2}/8$$
$$A_1(z) = -z^2/2 + z - 1/2$$

$$A(z) = \begin{bmatrix} (-z+6-z^{-1})/8 & (1+z^{-1})/4 \\ -(z+1)/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (1+z^{-1})/4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(z+1)/2 & 1 \end{bmatrix}$$

We will work instead with the causally normalized filters

$$H_0(z) = (-1 + 2z^{-1} + 6z^{-2} + 2z^{-3} - z^{-4})/8,$$
$$H_1(z) = (-1 + 2z^{-1} - z^{-2})/2$$

and their causal polyphase-with-delay (PWD) analysis matrix [6, 8],

$$H(z) = \begin{bmatrix} (-1 + 6z^{-1} - z^{-2})/8 & (1 + z^{-1})/4 \\ -(1 + z^{-1})/2 & 1 \end{bmatrix}, \quad |H(z)| = z^{-1}.$$  

Laurent polynomials can have multiple reduced-Laurent-order unimodular complements (shorter filters with which they form a unimodular filter bank). For instance, $A_0(z)$ also has the reduced-order nonlinear phase unimodular complement $A'_0(z) = -7/2 - z^{-1} + z^{-2}/2$, which happens to be causal. $A_0(z)$ and $A'_0(z)$ have noncausal unimodular PWA matrix $A'(z)$,

$$A'(z) = \begin{bmatrix} (-z + 6 - z^{-1})/8 & (1 + z^{-1})/4 \\ (-7 + z^{-1})/2 & -z^{-1} \end{bmatrix},$$

while $H_0(z)$ and $H'_0(z) \equiv A'_0(z)$ have causal PWD counterpart $H'(z)$,

$$H'(z) = \begin{bmatrix} (-1 + 6z^{-1} - z^{-2})/8 & (1 + z^{-1})/4 \\ (-7 + z^{-1})/2 & -1 \end{bmatrix}.$$
The determinants of $A(z)$ and $A'(z)$ are both 1, but $|H(z)| = z^{-1}$ and $|H'(z)| = 1$ distinguish between the two reduced-degree causal complements of $H_0(z)$. Theorem 4.5 will imply that $H_1(z)$ and $H'_1(z)$ are the unique reduced-degree causal complements to $H_0(z)$ for these determinants. This shows that the unimodular normalization employed by Daubechies and Sweldens [23] is discarding useful information about filter banks. Using Theorem 4.5 and the connection between lifting factorization and degree-lifting causal complementation, the Diophantine perspective uncovers a connection between causality and uniqueness of lifting factorizations that allows the CCA to enumerate and generate all causal degree-lifting factorizations of a causal filter bank.

2.1. Causal Factorization via the Causal Extended Euclidean Algorithm

There are four possible factorizations based on running the causal EEA in either row or column of $H(z)$. Our notation for the EEA is a compromise between several sources, including [23], [33], [34], and [35].

2.1.1. EEA in Column 0

Initialize remainders $r_0 \defeq H_{00}(z) = (-1 + 6z^{-1} - z^{-2})/8$ and $r_1 \defeq H_{10}(z) = -(1 + z^{-1})/2$. Iterate using the polynomial division algorithm, $r_0 = q_0 r_1 + r_2,$

$$q_0 = (-7 + z^{-1})/4, \quad r_2 = -1, \quad \text{and} \deg(r_2) = 0 < \deg(r_1) = 1. \quad (12)$$

Define the matrix

$$M_0 \defeq \begin{bmatrix} q_0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{so that} \quad \begin{pmatrix} r_0 \\ r_1 \end{pmatrix} = M_0 \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \quad (13)$$

Remainder $r_2$ is invertible so the next division step yields

$$r_3 = r_1 - q_1 r_2 = 0 \quad \text{with} \quad q_1 = r_1/r_2 = (1 + z^{-1})/2, \quad (14)$$

where $\deg(r_3) = \deg(0) \defeq -\infty < \deg(r_2)$. Define

$$M_1 \defeq \begin{bmatrix} q_1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{so that} \quad \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = M_1 \begin{pmatrix} r_2 \\ r_3 \end{pmatrix}. \quad (15)$$

Iteration terminates since $r_3 = 0$, and (13) and (15) imply

$$\begin{pmatrix} r_0 \\ r_1 \end{pmatrix} = M_0 M_1 \begin{pmatrix} r_2 \\ 0 \end{pmatrix}. \quad (16)$$

Put two swap matrices $J$ (7) between $M_0$ and $M_1$ to get lifting matrices $S_0$ and $S_1$,

$$M_0 M_1 = (M_0 J)(J M_1) = \begin{bmatrix} 1 & q_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_1 & 1 \end{bmatrix} = S_0 S_1.$$ As in the unimodular approach [23], form an augmentation matrix $H'(z)$ by augmenting (16) with causal filters $a_0$ and $a_1$ determined by the condition

$$H'(z) \defeq \begin{bmatrix} r_0 & a_0 \\ r_1 & a_1 \end{bmatrix} = S_0 S_1 \begin{bmatrix} r_2 & 0 \\ 0 & |H(z)|/r_2 \end{bmatrix} = \begin{bmatrix} 1 & (-7 + z^{-1})/4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (1 + z^{-1})/2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -z^{-1} \end{bmatrix} = \begin{bmatrix} (1 + 6z^{-1} - z^{-2})/8 & (7z^{-1} - z^{-2})/4 \\ -(1 + z^{-1})/2 & -z^{-1} \end{bmatrix}. \quad (17)$$
$|H'(z)| = |H(z)|$ by (17) and the matrices agree in column 0, so the Lifting Theorem [36, 37, 21] (Corollary 3.6 below) says that $H(z)$ can be lifted from $H'(z)$ by a causal lifting update to column 1, $H(z) = H'(z)S(z)$,

\[
H(z) = \begin{bmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{bmatrix} = \begin{bmatrix} r_0 & a_0 \\ r_1 & a_1 \end{bmatrix} \begin{bmatrix} 1 & S \\ 0 & 1 \end{bmatrix} \text{ iff } \begin{cases} H_{01} = r_0 S + a_0 \\ H_{11} = r_1 S + a_1 \end{cases}.
\]

(18)

Compute $H_{01} - a_0 = (1 - 6z^{-1} + z^{-2})/4 = -2r_0$, so $S = -2$. The resulting factorization in standard causal lifting form (6) based on (17)–(18) is

\[
H(z) = -\begin{bmatrix} 1 & (-7 + z^{-1})/4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (1 + z^{-1})/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-2}.
\]

(19)

2.1.2. EEA in Column 1

Initialize $r_0 \overset{\text{def}}{=} H_{01}(z) = (1 + z^{-1})/4$ and $r_1 \overset{\text{def}}{=} H_{11}(z) = 1$. Polynomial division yields $q_0 = r_0/r_1 = (1 + z^{-1})/4$, and $r_2 = 0$ so

\[
\begin{pmatrix} r_0 \\ r_1 \end{pmatrix} = M_0 \begin{pmatrix} r_1 \\ 0 \end{pmatrix} \text{ with } M_0 \overset{\text{def}}{=} \begin{bmatrix} q_0 & 1 \\ 1 & 0 \end{bmatrix}, |M_0| = -1.
\]

(20)

Augment (20) in column 0 with causal filters $a_0$ and $a_1$ defined by

\[
M_0 J = H'/r_1 = M_0 \begin{bmatrix} 0 & r_1 \\ 1 & 0 \end{bmatrix} = M_0 \begin{bmatrix} 0 & r_1 \\ 1 & 0 \end{bmatrix}
\]

(21)

$|H'(z)| = |H(z)|$ and the matrices agree in column 1 so $H(z)$ can be lifted from $H'(z)$ by a causal lifting update to column 0,

\[
H(z) = \begin{bmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{bmatrix} = \begin{bmatrix} a_0 & r_0 \\ a_1 & r_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ S & 1 \end{bmatrix} \text{ iff } \begin{cases} H_{00} = r_0 S + a_0 \\ H_{10} = r_1 S + a_1 \end{cases}.
\]

(22)

$H_{00} - a_0 = -(1 + 2z^{-1} + z^{-2})/8 = r_0 S$ for $S(z) = -(1 + z^{-1})/2$ so by (22)

\[
H(z) = \begin{bmatrix} 1 & (1 + z^{-1})/4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -(1 + z^{-1})/2 & 1 \end{bmatrix}.
\]

(23)

Remarks. This is a causal analogue of the noncausal linear phase lifting (9) for the unimodular LGT(5,3) analysis bank; its causal linear phase lifting filters differ from the corresponding lifting filters in (9) by at most delays.

2.1.3. EEA in Row 0

Initialize $r_0 \overset{\text{def}}{=} H_{00}(z)$ and $r_1 \overset{\text{def}}{=} H_{01}(z)$. Division yields $q_0 = (7 - z^{-1})/2$ and $r_2 = r_0 - r_1 q_0 = -1,$

\[
(r_0, r_1) = (r_1, r_2) M_0, \text{ with } M_0 \overset{\text{def}}{=} \begin{bmatrix} q_0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

(24)
Since $r_2 = -1$ is invertible, we get $q_1 = r_1 / r_2 = -(1 + z^{-1}) / 4$, $r_3 = 0$, and

$$(r_1, r_2) = (r_2, 0) M_1,$$ with $M_1 \overset{\text{def}}{=} \begin{bmatrix} q_1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{25}$$

Combine (24) and (25) and define $S_0 \overset{\text{def}}{=} JM_0$, $S_1 \overset{\text{def}}{=} M_1 J$ to express the top row as $(r_0, r_1) = (r_2, 0) S_1 S_0$. Augment with filters $a_0$ and $a_1$,

$$H'(z) \overset{\text{def}}{=} \begin{bmatrix} r_0 & r_1 \\ a_0 & a_1 \end{bmatrix} = \begin{bmatrix} r_2 & 0 \\ 0 & |H| / r_2 \end{bmatrix} \begin{bmatrix} -1 + 6z^{-1} - z^{-2} / 8 & (1 + z^{-1}) / 4 \\ -7z^{-1} + z^{-2} / 2 & -z^{-1} \end{bmatrix}.$$

$H$ can be lifted from augmentation matrix $H'$ by updating row 1, $H = SH'$,

$$\begin{bmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ S & 1 \end{bmatrix} \begin{bmatrix} r_0 & r_1 \\ a_0 & a_1 \end{bmatrix} \iff \begin{cases} H_{10} = r_0 S + a_0 \\ H_{11} = r_1 S + a_1. \end{cases}$$

This implies $S(z) = 4$, and the causal lifting factorization is

$$H(z) = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & -(1 + z^{-1}) / 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (7 - z^{-1}) / 2 & 1 \end{bmatrix}. \tag{26}$$

2.1.4. EEA in Row 1

This option also yields the causal linear phase lifting factorization (23).

2.2. Causal Factorization via the Causal Complementation Algorithm

We begin by using the CCA to reproduce all of the EEA factorizations.

2.2.1. CCA With Division in Column 0

Initialize quotient matrix $Q_0(z) \overset{\text{def}}{=} H(z)$ and find an initial lifting of the form

$$Q_0(z) = \begin{bmatrix} -1 + 6z^{-1} - z^{-2} / 8 & (1 + z^{-1}) / 4 \\ -7z^{-1} + z^{-2} / 2 & -z^{-1} \end{bmatrix} = \begin{bmatrix} E_0 & E_1 \\ F_0 & F_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ S & 1 \end{bmatrix} \begin{bmatrix} R_0 & R_1 \end{bmatrix}. \tag{27}$$

Set $E_0 \leftarrow H_{00}$, $F_0 \leftarrow H_{10}$, and divide $F_0$ into $E_0$ to get $E_0 = F_0 S + R_0$ with

$$S(z) = -(7 + z^{-1}) / 4 \text{ and } R_0(z) = -1, \text{ as in the EEA (12).} \tag{28}$$

Set $R_1 \leftarrow E_1 - F_1 S = 2$ to get a row reduction, $(R_0, R_1) = (E_0, E_1) - S(F_0, F_1)$. $R_0$ and $R_1$ are coprime, so the first lifting step (27) is

$$Q_0(z) = \begin{bmatrix} 1 & -7 + z^{-1} / 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -(1 + z^{-1}) / 2 & 1 \end{bmatrix} = V_0(z) Q_1(z). \tag{29}$$

Reset $E_j \leftarrow F_j$ and $F_j \leftarrow R_j$ in $Q_1(z)$ and divide again in column 0 to get

$$Q_1(z) = \begin{bmatrix} -1 & 2 \\ -(1 + z^{-1}) / 2 & 1 \end{bmatrix} = \begin{bmatrix} F_0 & F_1 \\ E_0 & E_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ S & 1 \end{bmatrix} \begin{bmatrix} R_0 & R_1 \end{bmatrix}. \tag{30}$$

Dividing $F_0 = -1$ into $E_0 = -(1 + z^{-1}) / 2$ as in (14) gives $E_0 = F_0 S + R_0$,

$$S(z) = (1 + z^{-1}) / 2 \text{ and } R_0 = 0, \text{ deg}(R_0) \overset{\text{def}}{=} -\infty < \text{deg}(F_0). \tag{31}$$
Set $R_1 \leftarrow E_1 - F_1 S = -z^{-1}$; \(\gcd(R_0, R_1) = z^{-1}\) so factor it out of \(Q_2(z)\),

\[
Q_1(z) = \begin{bmatrix}
\frac{1}{(1+z^{-1})/2} & 0 & -1 & 1 \\
0 & 1 & -z^{-1}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{(1+z^{-1})/2} & 0 & 1 & -2 \\
0 & z^{-1} & 0 & 1
\end{bmatrix}
= V_1(z) \Delta_1(z) Q_2(z), \text{ where } \Delta_1(z) = \text{diag}(1, z^{-1}).
\]  

Combining (29) and (32) and dividing \(Q_2(z)\) by \(-1\) to get a proper lifting step yields the same factorization (19) obtained using the EEA in column 0,

\[
H(z) = -\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 1 & -2
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & z^{-1} & 0 & 1
\end{bmatrix}.
\]  

2.2.2. CCA With Division in Column 1

Initialize \(E_1 \leftarrow H_{01}\) and \(F_1 \leftarrow H_{11}\). Divide \(F_1\) into \(E_1\) in (27) to get \(E_1 = F_1 S + R_1\) where \(S = (1 + z^{-1})/4\) and \(R_1 = 0\). Set \(R_0 \leftarrow E_0 - F_0 S = z^{-1}\); factoring out \(\gcd(R_0, R_1) = z^{-1}\) yields the lifting (23) obtained by the EEA,

\[
H(z) = \begin{bmatrix}
1 & (1 + z^{-1})/4 & 0 & 0 \\
0 & 1 & z^{-1} & 0 \\
0 & 0 & 1 & -(1 + z^{-1})/2
\end{bmatrix}.
\]  

2.2.3. CCA With Division in Row 0

Initialize \(E_0 \leftarrow H_{00}\) and \(F_0 \leftarrow H_{01}\) and divide \(F_0\) into \(E_0\) to get

\[
Q_0(z) = \begin{bmatrix}
E_0 & F_0 \\
E_1 & F_1
\end{bmatrix} = \begin{bmatrix}
R_0 & F_0 \\
R_1 & F_1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
S & 1
\end{bmatrix}, \quad S = (7 - z^{-1})/2, \quad R_0 = -1.
\]  

Set \(R_1 \leftarrow E_1 - F_1 S = -4\). The first factorization step is

\[
Q_0(z) = Q_1(z) V_0(z) = \begin{bmatrix}
-1 & (1 + z^{-1})/4 \\
-4 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
(7 - z^{-1})/2 & 1
\end{bmatrix}.
\]  

Reset the labels \(E_j \leftarrow F_j\) and \(F_j \leftarrow R_j\) in \(Q_1(z)\), so that

\[
Q_1(z) = \begin{bmatrix}
-1 & (1 + z^{-1})/4 \\
-4 & 1
\end{bmatrix} = \begin{bmatrix}
F_0 & E_0 \\
F_1 & E_1
\end{bmatrix}.
\]  

Divide \(F_0 = -1\) into \(E_0 = (1 + z^{-1})/4\),

\[
E_0 = F_0 S + R_0, \quad S = -(1 + z^{-1})/4, \quad \text{and } R_0 = 0.
\]  

The first two division steps (35) and (38) are identical to the first two steps in the row 0 EEA calculation, (24) and (25). Set \(R_1 \leftarrow E_1 - F_1 S = -z^{-1}\),

\[
Q_1(z) = \begin{bmatrix}
F_0 & R_0 \\
F_1 & R_1
\end{bmatrix} \begin{bmatrix}
1 & S \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
-4 & -1
\end{bmatrix} \begin{bmatrix}
1 & -(1 + z^{-1})/4 \\
0 & 1
\end{bmatrix}.
\]  

Factor out \(\gcd(R_0, R_1) = z^{-1}\); the resulting factorization agrees with (26),

\[
H(z) = -\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 1 & -2
\end{bmatrix}\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & z^{-1} & 0 & 1
\end{bmatrix}\begin{bmatrix}
1 & -(1 + z^{-1})/4 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
(7 - z^{-1})/2 & 1
\end{bmatrix}.
\]
2.2.4. CCA With Division in Row 1

The factorization is identical to the linear phase factorization (23), (34).

2.3. Other Degree-Lifting Factorizations via the CCA

It is convenient to have two algebraic tools for manipulating lifting cascades.

Definition 2.1 (cf. [65], eq. (27)). Let \( D_{\kappa_0,\kappa_1} \) be \( \text{diag}(\kappa_0, \kappa_1) \); \( \kappa_0, \kappa_1 \neq 0 \). The diagonal inner automorphism \( \gamma_{\kappa_0,\kappa_1} \) is the matrix automorphism \( \gamma_{\kappa_0,\kappa_1} A \) such that

\[
\gamma_{\kappa_0,\kappa_1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \kappa_0 & 0 \\ 0 & \kappa_1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \kappa_0^{-1} & 0 \\ 0 & \kappa_1^{-1} \end{bmatrix} = \begin{bmatrix} a \kappa_0 \kappa_1^{-1} b \\ \kappa_0^{-1} \kappa_1 c d \end{bmatrix},
\]

and \( D_{\kappa_0,\kappa_1} = (\gamma_{\kappa_0,\kappa_1} A) D_{\kappa_0,\kappa_1} A D_{\kappa_0,\kappa_1}^{-1} \), which satisfies

\[
\begin{align}
D_{\kappa_0,\kappa_1} A &= (\gamma_{\kappa_0,\kappa_1} A) D_{\kappa_0,\kappa_1} A D_{\kappa_0,\kappa_1}^{-1} A.
\end{align}
\]

Definition 2.2. The double transpose automorphism is \( A^{\dagger\dagger} = J A J \), which satisfies

\[
\begin{align}
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\dagger\dagger} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix},
\end{align}
\]

\[
JA = A^{\dagger\dagger} J \quad \text{and} \quad AJ = JA^{\dagger\dagger}.
\]

2.3.1. Switching Between Rows

We now construct CCA factorizations that are different from those obtained using the EEA by exploiting options that have no obvious EEA analogues. Consider (36), \( Q_0(z) \) is \( \text{H}(z) = Q_1(z) V_0(z) \), obtained by dividing in row 0. In Section 2.2.3 we factored \( Q_1(z) \) by dividing \( F_0 = -1 \) into \( E_0 = (1 + z^{-1}) / 4 \) in (37). We are not obliged to continue dividing in row 0, however. Since

\[
\deg Q_0(z) = 1 > \deg F_0 + \deg F_1 = 0,
\]

Theorem 4.5 (below) implies that division in row 1 will produce a different lifting step than (38). Therefore, divide \( F_1 \) into \( E_1 \) in \( Q_1(z) \) to get \( E_1 = F_1 S + R_1 \), where \( S = -1/4 \) and \( R_1 = 0 \). Set \( R_0 \leftarrow E_0 - F_0 S = z^{-1} / 4 \) and factor \( \gcd(R_0, R_1) = z^{-1} \) out of column 1 of the quotient matrix,

\[
Q_1(z) = \begin{bmatrix} F_0 & R_0 \\ F_1 & R_1 \end{bmatrix} \begin{bmatrix} 1 & S \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 z^{-1/4} & 1 -1/4 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 1/4 & 1 0 \\ -4 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Q_2(z) \Delta_1(z) V_1(z), \quad \text{where} \quad \Delta_1(z) = \text{diag}(1, z^{-1}).
\]

\( Q_2(z) \) factors into a diagonal gain matrix, a lifting step, and a swap matrix,

\[
Q_2(z) = \begin{bmatrix} -1 1/4 \\ -4 0 \end{bmatrix} = \begin{bmatrix} 1/4 0 \\ 0 -4 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = \text{diag}(1/4, -4) V_2(z) J.
\]

Include the other matrices to get a causal lifting factorization (44) that is different from those obtained using the EEA in Section 2.1.

\[
\begin{align}
\text{H}(z) &= \text{diag}(1/4, -4) V_2(z) J \Delta_1(z) V_1(z) V_0(z) \\
&= \text{diag}(1/4, -4) V_2(z) \Delta_1(z) V_1(z) V_0(z) J \quad \text{by (43)} \\
&= \begin{bmatrix} 1/4 0 \\ 0 -4 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & (7-z^{-1})/2 \\ 0 & 1 \end{bmatrix}, \quad \text{(44)}
\end{align}
\]
2.3.2. Switching Between Columns

Consider (29), \( Q_0(z) = V_0(z)Q_1(z) \), the initial lifting step obtained by dividing in column 0. In (30) we see that the first quotient \( Q_1(z) \) satisfies \( \deg(Q_1(z)) = 1 > \deg(F_0) + \deg(F_1) = 0 \) so Theorem 4.5 implies that division in column 1 of \( Q_1(z) \) yields a different lifting step than (31). Division in column 1 yields \( S = E_1/F_1 = 1/2, R_1 = 0, \) and \( R_0 \leftarrow E_0 - F_0 S = -z^{-1}/2 \) so

\[
Q_1(z) = \begin{bmatrix} -1 & 2 \\ -(1 + z^{-1})/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ S & 1 \end{bmatrix} \begin{bmatrix} F_0 & F_1 \\ R_0 & R_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1/2 & 0 \end{bmatrix}
\]

\( = V_1(z)\Delta_1(z)Q_2(z) \).

\( Q_2(z) \) can be factored into a gain matrix, a lifting step, and a swap matrix,

\[
Q_2(z) = \begin{bmatrix} -1 & 2 \\ -1/2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \text{diag}(2, -1/2)V_2(z)J.
\]

The complete lifting factorization for \( H(z) \), which differs from (44) and from the EEA factorizations in Section 2.1, is

\[
H(z) = V_0(z)V_1(z)\Delta_1(z)\text{diag}(2, -1/2)V_2(z)J
\]

\[
= \text{diag}(2, -1/2)(\gamma_{-1/2}^{-1}V_0(z))((\gamma_{-1/2}^{-1}V_1(z))\Delta_1(z)V_2(z)J \text{ by (41)},
\]

\[
= \begin{bmatrix} 2 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 7z^{-1}/16 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

2.3.3. Extracting Diagonal Delay Matrices

CCA schemes that mix row and column updates are also possible, but a more significant ability will be a “generalized polynomial division” technique, which we now demonstrate, for factoring off diagonal delay matrices at arbitrary points in the process, rather than waiting for \( R_0 \) and \( R_1 \) to have a nontrivial gcd. Divide in column 0 of LGT(5,3) to get a factorization of the form (27), but this time generate a remainder divisible by \( z^{-1} \) when dividing \( F_0 \) into \( E_0 \). Killing the highest-order term in \( E_0 \) with \( S(z) \leftarrow z^{-1}/4 \) leaves

\[
R_0(z) = E_0(z) - F_0(z)(z^{-1}/4) = (-1 + 7z^{-1})/8.
\]

Instead of subtracting 7/4 from \( S(z) \) to get \( R_0 = -1 \) as in (28), add 1/4 to kill the constant term, \( S(z) \leftarrow (z^{-1} + 1)/4 \). This leaves \( R_0(z) = z^{-1} \), which satisfies \( \deg(R_0) < \deg(F_0) + 1 \). The motivation, generating a remainder divisible by \( z^{-1} \), is new but the arithmetic is comparable to how Daubechies and Sweldens [23, §7.8] generated a linear phase lifting factorization for a unimodular linear phase filter bank using the Laurent polynomial EEA.

Now set \( R_1 \leftarrow E_1 - F_1S = 0 \). Interestingly, \( R_1 \) also happens to be divisible by \( z^{-1} \); this allows the resulting factorization to be written

\[
H(z) = \begin{bmatrix} 1 & S \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_0 & R_1 \\ F_0 & F_1 \end{bmatrix} = \begin{bmatrix} 1 & (1 + z^{-1})/4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(1 + z^{-1})/2 & 1 \end{bmatrix}.
\]

This is the causal linear phase lifting factorization (23) so we have obtained a different factorization than (33), though it is not a new factorization.
3. Linear Diophantine Equations

We now focus on the theory behind the CCA to understand the differences between unimodular and causal lifting. Given \( a, b, \) and \( c \), a linear Diophantine equation (LDE) in unknowns \( x \) and \( y \) is an equation of the form

\[
ax + by = c. \tag{46}
\]

Indeterminate equations of this type have been studied over the integers (albeit not using modern notation) at least as far back as Diophantus of Alexandria. The Indian astronomer Aryabhata (fifth–sixth centuries C.E.) and his colleagues and successors found the general solution to (46) over the integers using the Euclidean Algorithm. Indeed, judging from van der Waerden [70, Chapter 5], it appears that the general solution found by these ancient Indian scholars consisted of the integer version of the Lifting Theorem (Corollary 3.6).

3.1. Factorization in Commutative Rings

While we are mainly interested in LDEs over rings of (causal) polynomials and (noncausal) Laurent polynomials, we first determine which aspects of lifting follow from general factorization theory and thus do not distinguish between the causal and noncausal cases. We follow standard terminology for commutative rings \( \mathcal{R} \) [32, 71, 72, 73]. Divisibility of \( b \) by \( a \neq 0 \) is denoted \( a \mid b \); \( a \) and \( b \) are associates if \( a \mid b \) and \( b \mid a \). A multiplicative identity for \( \mathcal{R} \) is denoted 1. A unit is any element \( u \) with a multiplicative inverse, \( u^{-1} \), such that \( uu^{-1} = 1 \). A nonzero nonunit, \( c \), is irreducible if its only divisors are units and associates. A subset \( A \subset \mathcal{R} \) is coprime if the only common divisors of all \( a_i \in A \) are units. An integral domain is a commutative ring with identity that contains no zero divisors: nonzero elements \( a, b \) for which \( ab = 0 \).

3.1.1. Unique Factorization Domains

This is the least restrictive relevant class of commutative rings. A unique factorization domain is any integral domain in which each nonzero nonunit can be factored into irreducibles that are “unique modulo associates.” This means that if \( a = \Pi a_i = \Pi b_j \) are two (finite) factorizations of \( a \) into irreducibles then each \( a_i \) is an associate of a distinct \( b_j(i) \), a requirement satisfied by polynomial rings. A common divisor \( h \) for a subset \( A \subset \mathcal{R} \) is a greatest common divisor (gcd) of \( A \) if all common divisors of \( A \) necessarily divide \( h \). In unique factorization domains every finite subset with a nonzero element has a gcd [72, Theorem III.3.11(iii)]. Gcds are not unique, and we write \( h = \text{gcd}(A) \) if \( h \) is any gcd of \( A \). The next two lemmas are straightforward exercises.

**Lemma 3.1.** Let \( \mathcal{R} \) be a unique factorization domain and let \( A \) be a finite subset with common divisor \( d: a_i = \tilde{a}_i d \) for all \( a_i \in A \). Then \( d = \text{gcd}(A) \) if and only if the set of all \( \tilde{a}_i \) is coprime.

**Lemma 3.2.** Let \( \mathcal{R} \) be a unique factorization domain and let \( a, b \in \mathcal{R}, a \neq 0 \). Then \( a \) and \( b \) are coprime if and only if, for all \( c \in \mathcal{R} \), \( a \mid bc \) implies \( a \mid c \).

**Definition 3.3.** Let \( \mathcal{R} \) be a unique factorization domain with \( a, b \in \mathcal{R} \) not both zero and consider an LDE (46) involving \( a, b \). If \( c \) in (46) is nonzero then we call (46) an inhomogeneous \( LDE \) for \( a, b \) with inhomogeneity \( c \). If \( c \) in (46) is zero,

\[
ax + by = 0, \tag{47}
\]

then (47) is called the homogeneous \( LDE \) for \( a, b \).
Let \( h \overset{\text{def}}{=} \gcd(a, b) \) so that, by Lemma 3.1, \( a = \tilde{a}h \) and \( b = \tilde{b}h \) with \( \tilde{a} \) and \( \tilde{b} \) coprime. The reduced homogeneous LDE for \( a, b \) is
\[
\tilde{a}x + \tilde{b}y = 0. \tag{48}
\]

**Lemma 3.4.** If \( \mathcal{R} \) is a unique factorization domain with \( a, b \) not both zero then (47) and (48) have identical solutions.

We can now characterize the solution sets to homogeneous and inhomogeneous LDEs.

**Theorem 3.5 (Homogeneous LDE Theorem).** Let \( \mathcal{R} \) be a unique factorization domain with \( a, b \) not both zero, \( h \overset{\text{def}}{=} \gcd(a, b) \), \( a = \tilde{a}h \) and \( b = \tilde{b}h \). Conditions (i) and (ii) below are equivalent.

(i) The pair \( (x, y) \) satisfies (47).

(ii) There exists a unique \( s \in \mathcal{R} \) with \( x = s\tilde{b} \) and \( y = -s\tilde{a} \).

If (i) and (ii) hold then the following are equivalent.

(iii) \( x \) and \( y \) are coprime.

(iv) \( s \) is a unit.

**Proof.** (i\( \Rightarrow \)ii) Let \( (x, y) \) satisfy (47); by Lemma 3.4 \((x, y)\) also satisfies (48). Suppose \( \tilde{a} = 0 \); \( \tilde{a} \) and \( \tilde{b} \) are coprime by Lemma 3.1 so coprimality implies that \( \tilde{b} \) is a unit. By (48), \( y = 0 \) and the unique solution to (ii) is \( s = xb^{-1} \). Clause (ii) is similarly satisfied if \( b = 0 \) so assume \( \tilde{a}, \tilde{b} \neq 0 \). We have \( \tilde{a} \mid by \) by (48) so \( \tilde{a} \mid y \) by Lemma 3.2. Thus, \( y = r\tilde{a} \) for some uniquely determined \( r \in \mathcal{R} \). Similarly, \( x = s\tilde{b} \) for a unique \( s \in \mathcal{R} \). One can therefore write (48) as \( 0 = \tilde{a}s\tilde{b} + \tilde{b}r\tilde{a} = \tilde{a}\tilde{b}(s + r) \). This implies \( r = -s \) since \( \tilde{a}\tilde{b} \neq 0 \), proving (i\( \Rightarrow \)ii).

(ii\( \Rightarrow \)i) \((s\tilde{b}, -s\tilde{a})\) satisfies (48) and therefore satisfies (47) by Lemma 3.4.

Next, assume that (i) and (ii) hold.

(iii\( \Rightarrow \)iv) \( s \) is a common divisor of \( x \) and \( y \) by (ii) so if \( x \) and \( y \) are coprime then \( s \) must be a unit, proving (iii\( \Rightarrow \)iv).

(iv\( \Rightarrow \)iii) Let \( c \) be a common divisor of \( x = s\tilde{b} \) and \( y = -s\tilde{a} \); if \( s \) is a unit then \( c \) is also a common divisor of \( \tilde{a} \) and \( \tilde{b} \). By Lemma 3.1 \( \tilde{a} \) and \( \tilde{b} \) are coprime so \( c \) must be a unit, implying \( x \) and \( y \) are coprime. \( \square \)

**Corollary 3.6 (Abstract Lifting Theorem; cf. [36, 37, 21]).** Let \( \mathcal{R} \) be a unique factorization domain with \( a, b, c \in \mathcal{R} \), \( a \) and \( b \) not both zero. Let \( h \overset{\text{def}}{=} \gcd(a, b), a = \tilde{a}h, b = \tilde{b}h, \) and let \((x, y)\) satisfy (46). Then \((x', y')\) is another solution to (46) if and only if \( x' = x + sb \) and \( y' = y - s\tilde{a} \) for some \( s \in \mathcal{R} \).

**Proof.** (Sufficiency) If both \((x', y')\) and \((x, y)\) satisfy (46) then their difference, \((x' - x, y' - y)\), satisfies the homogeneous LDE (47) so Theorem 3.5(i\( \Rightarrow \)ii) provides \( s \in \mathcal{R} \) such that \( x' - x = s\tilde{b} \) and \( y' - y = -s\tilde{a} \).

(Necessity) Let \((x, y)\) satisfy (46) and suppose that \( x' = x + sb, y' = y - s\tilde{a} \) for some \( s \in \mathcal{R} \). Theorem 3.5(ii\( \Rightarrow \)i) implies that the pair \((x' - x, y' - y)\) satisfies (47) so, by bilinearity, the pair \((x', y')\) also satisfies (46). \( \square \)
Remarks. The Abstract Lifting Theorem (Corollary 3.6) provides all solutions to (46) in terms of the homogeneous solution set and any one particular solution. The homogeneous solution set \{(sb, -s\tilde{a}) : s \in \mathcal{R}\} is never just \{(0,0)\}, so LDEs over unique factorization domains never have unique solutions. Finding a particular solution requires more work, but unlike [37, Fact 4.1], which appeals to the polynomial Euclidean Algorithm [74], we now prove existence of inhomogeneous solutions via more abstract methods by specializing the rings \mathcal{R} under consideration a bit more.

3.1.2. Principal Ideal Domains

The subset \mathcal{I} \subset \mathcal{R} is a principal ideal of \mathcal{R} if it is generated by a single element,

\[ \mathcal{I} = (a) = a\mathcal{R} \overset{\text{def}}{=} \{ ar : r \in \mathcal{R} \}. \tag{49} \]

The ideal generated by a finite set of elements, \( A \overset{\text{def}}{=} \{ a_i \}_{i=0}^n \), is the subset

\[ \mathcal{I} = (A) = (a_0) + \cdots + (a_n) = \{ a_0 r_0 + \cdots + a_n r_n : r_i \in \mathcal{R} \}. \tag{50} \]

Principal ideal domains are integral domains in which every ideal is principal. A classical (but non-trivial) fact is that principal ideal domains are unique factorization domains [72, Theorem III.3.7]. We need the following basic result.

Lemma 3.7. [72, Theorem III.3.11] In a principal ideal domain any finite subset \( A \) with a nonzero element has a gcd, and \( h \) is a gcd for \( A \) if and only if \( (A) = (h) \).

Theorem 3.8 (Abstract Bezout Theorem). Let \( \mathcal{R} \) be a principal ideal domain with \( a, b, c \in \mathcal{R} \), \( a \) and \( b \) not both zero, \( h \overset{\text{def}}{=} \gcd(a, b) \). There exists a solution \((x, y)\) to the LDE (46) for inhomogeneity \( c \) if and only if \( h \mid c \).

Proof. If (46) has a solution then any common divisor of \( a \) and \( b \) divides \( c \). Conversely, if \( h \mid c \) then, by (49) and Lemma 3.7, \( c \in (h) = \{ a, b \} \). By (50) this means \( c \in (a) + (b) \) so there exist \( x, y \in \mathcal{R} \) such that \( c = ax + by \). \( \square \)

3.2. Factorization in Euclidean Domains

Per [65], the goal of lifting is usually to find factorizations that are “size-reducing” in some sense. While the matrix polynomial order was shown to be a useful measure of “size” in [65, 66] for linear phase liftings, that approach does not work with more general lifting factorizations. Instead, the present paper exploits the strong factorization theory for scalar polynomials. Their natural “size” function leads us to Euclidean domains [72, Definition III.3.8], which are automatically principal ideal domains [72, Theorem III.3.9].

Definition 3.9. A Euclidean domain, \( \mathcal{R} \), is an integral domain equipped with a Euclidean size function, \( \sigma : \mathcal{R}\setminus\{0\} \to \mathbb{N} \overset{\text{def}}{=} \{0, 1, 2, \ldots\} \), that satisfies the following two axioms for \( a, b \in \mathcal{R} \).

i) (Monotonicity) If \( a, b \neq 0 \) then \( \sigma(a) \leq \sigma(ab) \).

ii) (Division algorithm) If \( b \neq 0 \) then there exist \( q, r \in \mathcal{R} \) such that \( a = qb + r \), where either \( r = 0 \) or \( \sigma(r) < \sigma(b) \).
3.2.1. Examples and Special Properties

**Example 3.10.** \( \mathbb{F}[\zeta] \) is the ring of polynomials in \( \zeta \), \( f(\zeta) = \sum_{i=0}^{\deg(f)} f_i \zeta^i \), with coefficients \( f_i \in \mathbb{F} \) for some field \( \mathbb{F} \) (e.g., the field of real or complex numbers). Let

\[
\sigma(f) \overset{\text{def}}{=} \deg(f), \text{ the polynomial degree of } f. \tag{51}
\]

\( \mathbb{F}[\zeta] \) with polynomial division and the size function (51) is a Euclidean domain. Moreover, division of polynomials yields a unique quotient and remainder satisfying axiom (ii) \[32, \text{Theorem IV.21}], \[72, \text{Theorem III.6.2}]\]. The units in \( \mathbb{F}[\zeta] \) are the nonzero constant polynomials, which have size (degree) zero. As is commonly done in algebra, we set \( \deg(0) \overset{\text{def}}{=} -\infty \). With this convention, the degree function enjoys two additional important properties.

Homomorphism property of \( \deg(f) \): \( \deg(fg) = \deg(f) + \deg(g) \). \tag{52}

Max-additive bound on \( \deg(f) \): \( \deg(f + g) \leq \max\{\deg(f), \deg(g)\} \). \tag{53}

**Example 3.11.** Write Laurent polynomials, \( f \in \mathbb{F}[\zeta, \zeta^{-1}] \), over a field \( \mathbb{F} \) as

\[
f(\zeta) = \sum_{i=m}^{n} f_i \zeta^i, \text{ where } f_m, f_n \neq 0; -\infty < m \leq n < \infty, \text{ and}
\]

\[
\text{ord}_L(f) \overset{\text{def}}{=} n - m \geq 0, \text{ the Laurent order (or "length") of } f. \tag{54}
\]

\( \mathbb{F}[\zeta, \zeta^{-1}] \) equipped with \( \sigma(f) \overset{\text{def}}{=} \text{ord}_L(f) \) and Laurent polynomial division satisfies the axioms of a Euclidean domain, but the quotient and remainder satisfying axiom (ii) are not unique in the Laurent case. For instance, if \( a(\zeta) \overset{\text{def}}{=} 1 + \zeta + \zeta^2 \) and \( b(\zeta) \overset{\text{def}}{=} 1 + \zeta \) then the unique solution to (ii) over the polynomials \( \mathbb{F}[\zeta] \) is

\[
q(\zeta) = \zeta, \ r(\zeta) = 1; \ \deg(r) < \deg(b).
\]

This is also a solution over the Laurent polynomials, but so is

\[
q(\zeta) = 1, \ r(\zeta) = \zeta^2; \ \text{ord}_L(r) < \text{ord}_L(b).
\]

The units in \( \mathbb{F}[\zeta, \zeta^{-1}] \) are the nonzero Laurent monomials, which have Laurent order zero. As with the polynomial degree, we define \( \text{ord}_L(0) \overset{\text{def}}{=} -\infty \). The Laurent order satisfies the homomorphism property (52),

\[
\text{ord}_L(fg) = \text{ord}_L(f) + \text{ord}_L(g). \tag{55}
\]

It does not satisfy (53), though: \( \text{ord}_L(\zeta + \zeta^2) = 1 \) but \( \text{ord}_L(\zeta^2) = \text{ord}_L(\zeta) = 0 \). This shows that the Laurent polynomials have “too many units” since each term in a Laurent polynomial is a Laurent unit, of size 0.

4. Factorization in Polynomial Rings

The max-additive bound (53) implies uniqueness of quotients and remainders in polynomial division \[32, 72]\]. In fact, division in a Euclidean domain produces unique quotients and remainders if and only if the size function satisfies an inequality like (53) \[75, \text{Proposition II.21}]\]. We will see that (53) also yields unique “degree-reducing” solutions to polynomial LDEs, implying unique causal lifting results that do not hold over the Laurent (noncausal) polynomials.
4.1. Size-Reducing Solutions to LDEs

We introduce terminology for “size-reducing” solutions to LDEs in Euclidean domains, generalizing Definition 1.1. The size-reducing concept is important enough in the context of lifting factorization to warrant precise definitions, which seem to be lacking in the algebra literature.

**Definition 4.1 (Size-Reducing Solutions).** Let \( R \) be a Euclidean domain whose size function satisfies \( \sigma(0) = -\infty \) and the homomorphism property,

\[
\sigma(ab) = \sigma(a) + \sigma(b).
\]

(56)

Let \( a, b, c \in R \) with \( a \) and \( b \) not both zero. Let \( h \overset{\text{def}}{=} \gcd(a, b) \) and assume \( h \mid c \). A solution \((x, y)\) to

\[
ax + by = c
\]

will be called size-reducing in \( a \neq 0 \) (resp., size-reducing in \( b \neq 0 \)) if

\[
\sigma(y) < \sigma(a) - \sigma(h) \quad \text{(resp., } \sigma(x) < \sigma(b) - \sigma(h)).
\]

(58)

**Remarks.** By (56), the “correction” terms \( \sigma(h) \) ensure that \( \sigma(a) - \sigma(h) \) and \( \sigma(b) - \sigma(h) \) are invariant under cancellation of common divisors for \( a, b, \) and \( c \) in (57), a manipulation that leaves the solution set unchanged. We now show that size-reducing solutions in \( a \) and \( b \) may or may not agree.

**Example 4.2.** Let \( R = \mathbb{F}[\zeta] \) for some field \( \mathbb{F} \), and define

\[
a(\zeta) \overset{\text{def}}{=} 1 + \zeta, \quad b(\zeta) \overset{\text{def}}{=} 1, \quad c(\zeta) \overset{\text{def}}{=} \zeta, \quad h \overset{\text{def}}{=} \gcd(a, b) = 1, \quad \text{with } \deg(h) = 0.
\]

The pair \((x, y) = (1, -1)\) is the unique solution to (57) that is degree-reducing in \( a, \deg(y) < \deg(a) \), while \((0, \zeta)\) is the unique degree-reducing solution in \( b \).

**Example 4.3.** Now redefine \( a(\zeta) \) slightly:

\[
a(\zeta) \overset{\text{def}}{=} 1 + \zeta^2, \quad b(\zeta) \overset{\text{def}}{=} 1, \quad c(\zeta) \overset{\text{def}}{=} \zeta, \quad h \overset{\text{def}}{=} \gcd(a, b) = 1.
\]

The pair \((x, y) = (0, \zeta)\) from Example 4.2 is still the unique solution to (57) that is degree-reducing in \( b \), but it is now the unique solution that is degree-reducing in \( a \), too. If we reinterpret (57) as a problem over the *Laurent* polynomials, however, then a second solution that is size-reducing in \( a \) is

\[
(x, y) = (\zeta^{-1}, -\zeta^{-1}), \quad \text{ord}_L(y) = 0 < \text{ord}_L(a) = 2.
\]

4.2. LDEs in Polynomial Domains

We now show, using inequality (53), that an LDE over a *polynomial* domain has exactly one solution that is degree-reducing in \( a \neq 0 \) or \( b \neq 0 \). It is possible, as in Example 4.2, to have different degree-reducing solutions in \( a \) and in \( b \) or, as in Example 4.3, to have one solution that is degree-reducing in both \( a \) and \( b \). The question of which alternative holds is answered by Theorem 4.5.

**Lemma 4.4.** Let \( a, b, c \in \mathbb{F}[\zeta] \) with \( a \) and \( b \) not both zero. Let \( h \overset{\text{def}}{=} \gcd(a, b) \) and assume \( h \mid c \). Let \((x, y)\) be a solution to (57), and suppose \((x, y)\) is degree-reducing in a (resp., \( b \)). If \((x', y')\) is another solution to (57) that is degree-reducing in a (resp., \( b \)) then \( x' = x \) and \( y' = y \).
Proof. Assume \((x, y)\) and \((x', y')\) are both degree-reducing in \(a \neq 0\) (the proof is similar if both are degree-reducing in \(b\)),
\[
\deg(y), \quad \deg(y') < \deg(a) - \deg(h).
\]
Factor \(h\) out of \(a, b,\) and \(c:\)
\[
a = h\tilde{a}, \quad b = h\tilde{b},\quad \text{and} \quad c = h\tilde{c},\quad \text{where (52) implies}
\]
\[
\deg(a) = \deg(h) + \deg(\tilde{a}),\quad \text{etc.}
\]
Let \(x'' \overset{\text{def}}{=} x - x'\) and \(y'' \overset{\text{def}}{=} y - y'\); then \((x'', y'')\) satisfies the corresponding reduced homogeneous LDE (48), \(\tilde{a}x'' + \tilde{b}y'' = 0\). Theorem 3.5 supplies \(s\) such that \(x'' = s\tilde{b}\) and \(y'' = -s\tilde{a}\). By inequality (53), assumption (59), and (60)
\[
\deg(s\tilde{a}) = \deg(y'') \leq \max\{\deg(y), \deg(y')\} < \deg(a) - \deg(h) = \deg(\tilde{a}) \leq \deg(s\tilde{a}) \quad \text{if } s \neq 0,
\]
which is a contradiction unless \(s = 0\). Thus, \(x - x' = x'' = s\tilde{b} = 0\) and \(y - y' = y'' = -s\tilde{a} = 0\), proving uniqueness of degree-reducing solutions. \(\square\)

The main result in this section, the Linear Diophantine Degree-Reduction Theorem (or LD-DRT, Theorem 4.5), is a strengthened version of [35, Theorem 4.10(iii)] and Bezout’s Theorem for polynomials, e.g. [2, Theorem 6.1.1]. It uses the Abstract Lifting Theorem (Corollary 3.6), the Abstract Bezout Theorem (Theorem 3.8), Lemma 4.4, and the polynomial division algorithm to characterize existence and uniqueness of degree-reducing solutions to inhomogeneous LDEs over polynomial domains. Theorem 4.5(i) is also used to prove Corollary 5.1, which the author has been unable to find in the literature.

**Theorem 4.5 (Linear Diophantine Degree-Reduction Theorem).** Let \(a, b, c \in \mathbb{F}[\zeta], \ a\) and \(b\) not both zero. Let \(h \overset{\text{def}}{=} \gcd(a, b)\); assume \(h | c\).

i) If \(a \neq 0\) then there exists a unique solution, \((x, y)\), to the LDE
\[
a x + by = c
\]
that is degree-reducing in \(a\),
\[
\deg(y) < \deg(a) - \deg(h).
\]

ii) If \(b \neq 0\) then there exists a unique solution, \((x', y')\), to (61) that is degree-reducing in \(b\),
\[
\deg(x') < \deg(b) - \deg(h).
\]

iii) Let \(a, b \neq 0\) and let \((x, y)\) and \((x', y')\) be the solutions in clauses (i) and (ii). These two solutions are the same, \(x = x'\) and \(y = y'\), if and only if
\[
\deg(c) < \deg(a) + \deg(b) - \deg(h).
\]

**Proof.** As in the proof of Lemma 4.4, if \(a\) and \(b\) are not coprime then we factor \(h\) out of \(a, b,\) and \(c\) in (61), leaving an equivalent LDE with \(\tilde{a}\) and \(\tilde{b}\) coprime,
\[
\tilde{a} x + \tilde{b} y = \tilde{c}.
\]
By (60) the degree-reducing conditions (62) (resp., (63)) are equivalent to
\[ \deg(y) < \deg(\tilde{a}) \quad (\text{resp., } \deg(x') < \deg(\tilde{b})). \] (66)

Formula (60) also implies that (64) is equivalent to
\[ \deg(\tilde{c}) < \deg(\tilde{a}) + \deg(\tilde{b}). \] (67)

Since \( h | c \), by Theorem 3.8 there exists a solution \((x^*, y^*)\) to (61) and (65).

(i) Divide \( \tilde{a} \neq 0 \) into \( y^* \) to get \( q \) and \( y \) satisfying
\[ y^* = q\tilde{a} + y, \quad \deg(y) < \deg(\tilde{a}). \] (68)

Let \( x \overset{df}{=} x^* + q\tilde{b} \), then \((x, y) = (x^*, y^*) + (q\tilde{b}, -q\tilde{a})\). By Corollary 3.6 \((x, y)\) is also a solution to (61). Inequality (68) is precisely (66) so \((x, y)\) is degree-reducing in \( a \) and uniqueness follows from Lemma 4.4.

(ii) The proof of clause (ii) is similar to the proof of (i).

(iii) Let \((x, y)\) and \((x', y')\) be the unique degree-reducing solutions given in clauses (i) and (ii), respectively,
\[ \deg(y) < \deg(\tilde{a}) \text{ and } \deg(x') < \deg(\tilde{b}). \] (69)

Suppose \((x, y) = (x', y')\); applying (53) and (52) to (65) proves (67),
\[ \deg(\tilde{c}) \leq \max\{\deg(\tilde{a}x), \deg(\tilde{b}y)\} = \max\{\deg(\tilde{a}) + \deg(x), \deg(\tilde{b}) + \deg(y)\} \]
\[ < \deg(\tilde{a}) + \deg(\tilde{b}) \text{ by (69) since } x' = x. \]

Conversely, assume (67). By (52) and (69) we have
\[ \deg(\tilde{b}y) = \deg(\tilde{b}) + \deg(y) < \deg(\tilde{b}) + \deg(\tilde{a}). \] (70)

Therefore, by (52), (65), and (53),
\[ \deg(\tilde{a}) + \deg(x) = \deg(\tilde{a}x) = \deg(\tilde{c} - \tilde{b}y) \]
\[ \leq \max\{\deg(\tilde{c}), \deg(\tilde{b}y)\} < \deg(\tilde{a}) + \deg(\tilde{b}) \text{ by (67), (70).} \]

This implies \( \deg(x) < \deg(\tilde{b}) = \deg(b) - \deg(h) \), which says that \((x, y)\) is also degree-reducing in \( b \).

But \((x', y')\) is the unique solution to (61) that is degree-reducing in \( b \), so \((x, y) = (x', y')\).

\(\square\)

5. Generalized Polynomial Division

We now generalize the polynomial division algorithm using the LDDRT to handle divisibility constraints on remainders. A specialization of this will be used in the CCA to factor diagonal delay matrices off of causal PR filter banks.
5.1. Ideal-Theoretic Interpretation

Given polynomials $e$ and $f \neq 0$ in $\mathbb{F}[\zeta]$, the polynomial division algorithm yields a unique quotient $q$ whose remainder, $r \overset{\text{def}}{=} e - fq$, satisfies $\deg(r) < \deg(f)$. This forms the unique solution $(q, r)$ that is degree-reducing in $f$ for the LDE

$$f q + 1 r = e. \quad (71)$$

Consequently, the coset $e + (f)$ of the ideal generated by $f$ contains a unique element, $r = e - fq$, that satisfies $\deg(r) < \deg(f)$, making it the unique element of minimum degree in $e + (f)$.

Theorem 4.5 implies a far-reaching generalization of classical polynomial division. Let polynomials $e$ and $f, g \neq 0$ be given such that $h \overset{\text{def}}{=} \gcd(f, g) \mid e$. Theorem 4.5(i) yields a unique solution $(q, p)$ that is degree-reducing in $f$ to

$$f q + g p = e, \text{ with } \deg(p) < \deg(f) - \deg(h). \quad (72)$$

This makes $r = e - fq = gp$ the unique element of minimum degree in the coset-ideal intersection $[e + (f)] \cap (g)$ since it is the unique element that satisfies

$$\deg(r) = \deg(p) + \deg(g) < \deg(f) - \deg(h) + \deg(g) \text{ by (52) and (72).}$$

This proves the following result.

**Corollary 5.1 (Generalized Polynomial Division Theorem).** Let $e, f, g \in \mathbb{F}[\zeta], f, g \neq 0$. If $h \overset{\text{def}}{=} \gcd(f, \zeta^M) \mid e$ then there exists a unique quotient $q$ whose remainder, $r \overset{\text{def}}{=} e - fq$, is divisible by $g$ and satisfies

$$\deg(r) < \deg(f) - \deg(h) + \deg(g). \quad (73)$$

5.2. A Constructive Generalized Division Algorithm

In the lifting context, $e$ and $f$ in Corollary 5.1 are the polynomials in one row or column of a transfer matrix, $\zeta = z^{-1}$, and $g(\zeta) = z^{-M}$. The proof via the (nonconstructive) LDDRT (Theorem 4.5) will be replaced by a constructive proof (Theorem 5.3) and a computational algorithm. But first, one more definition.

**Definition 5.2.** Let $e, f \in \mathbb{F}[\zeta], f \neq 0$, and $M \geq 0$. Given quotient $q$, the remainder $r \overset{\text{def}}{=} e - fq$ has (a root at $\zeta = 0$ of) multiplicity $M$ if $\zeta^M \mid r(\zeta)$. We say $r$ is *degree-reducing modulo* $M$ if $r$ has multiplicity $M$ and satisfies

$$\deg(r) < \deg(f) - \deg(g) + \deg(M).$$

**Remarks.** In our usage, “$r$ has multiplicity $M$” includes the possibility that $\zeta^{M+1} \mid r$. The classical division algorithm [35, Algorithm 2.5] provides existence and uniqueness of quotients whose remainders are degree-reducing modulo 0.

**Theorem 5.3 (Slightly Generalized Division Theorem).** Let $M \geq 0, e, f \in \mathbb{F}[\zeta], f \neq 0$. If $\gcd(f, \zeta^M) \mid e$ then there exists a unique quotient $q^{(M)}$ whose remainder, $r^{(M)} \overset{\text{def}}{=} e - fq^{(M)}$, is degree-reducing modulo $M$,

$$\zeta^M \mid r^{(M)} \text{ and } \deg(r^{(M)}) < \deg(f) - \deg(g) + \deg(M). \quad (74)$$

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Proof. Induction on $M$.

Case: $M = 0$. This is the classical polynomial division algorithm.

Case: $M > 0$. Assume the theorem holds whenever $\gcd(f, \zeta^{M-1}) \mid e$. Suppose $e, f$ are such that $\gcd(f, \zeta^{M}) \mid e$; then $\gcd(f, \zeta^{M-1}) \mid e$ so by hypothesis there exists a unique $q(M-1)$ whose remainder, $r(M-1) \overset{\text{def}}{=} e - f q(M-1)$, satisfies

$$\zeta^{M-1} \mid r(M-1) \quad \text{and} \quad \deg(r(M-1)) < \deg(f) - \deg(\gcd(f, \zeta^{M-1}) + M - 1. \tag{76}$$

(Existence of $q(M)$ and $r(M)$.) Note that

$$\deg(\gcd(f, \zeta^{M})) \leq \deg(\gcd(f, \zeta^{M-1}) + 1. \tag{77}$$

Apply (77) to (76) to get a different bound on $\deg(r(M-1))$,

$$\deg(r(M-1)) < \deg(f) - (\deg(\gcd(f, \zeta^{M-1}) + 1) + M \leq \deg(f) - \deg(\gcd(f, \zeta^{M}) + M. \tag{78}$$

If $r_{M-1}(M-1) = 0$ set $q(M) \overset{\text{def}}{=} q(M-1)$ and get $r(M) = r(M-1)$. $r_{M-1}(M-1) = 0$ implies $\zeta^{M} \mid r(M-1) = r(M)$, yielding (73). Since $r(M) = r(M-1)$, (78) implies (74), proving existence of $q(M)$ and $r(M)$ satisfying (73)–(74) when $r_{M-1}(M-1) = 0$.

Now assume that $r_{M-1}(M-1) \neq 0$. Define $m_{f} = \max\{i \geq 0 : \zeta^{i} \mid f\}$; then

$$f(\zeta) = \sum_{i=m_{f}}^{m} f_{i} \zeta^{i} \quad \text{where} \quad f_{m_{f}} \neq 0 \quad \text{and} \quad m = \deg(f). \tag{79}$$

Suppose $m_{f} \geq M$; then $\zeta^{M} = \gcd(f, \zeta^{M})$ divides both $f$ and $e$ (by hypothesis) so $\zeta^{M} \mid r(M-1)$ since $r_{M-1}(M-1) = e - f q(M-1)$, contradicting the assumption that $r_{M-1}(M-1) \neq 0$. This forces $m_{f} < M$, implying

$$m_{f} = \deg(\gcd(f, \zeta^{M}). \tag{80}$$

It also ensures that the following definition is a polynomial,

$$q(M)(\zeta) \overset{\text{def}}{=} q(M-1)(\zeta) + f_{m_{f}}^{-1} r_{M-1}(M-1) \zeta^{M-1-m_{f}}. \tag{81}$$

The remainder, $r(M) \overset{\text{def}}{=} e - f q(M)$, can now be written

$$r(M)(\zeta) = r(M-1)(\zeta) - f(\zeta) f_{m_{f}}^{-1} r_{M-1}(M-1) \zeta^{M-1-m_{f}}. \tag{82}$$

Using (79), its $(M - 1)^{th}$ (i.e., lowest-order) coefficient is

$$r_{M-1}(M-1) = r_{M-1}(M-1) - f_{m_{f}}^{-1} f_{m_{f}}^{-1}(M-1) = 0. \tag{83}$$

This implies $\zeta^{M} \mid r(M)$ so (73) is satisfied. To prove (74) apply (53) to (82),

$$\deg(r(M)) \leq \max\{\deg(r(M-1)), \deg(f) + M - 1 - m_{f}\}. \tag{83}$$

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\[\text{deg}(r^{(M-1)}) \text{ satisfies (78) and by (80) the second argument in (83) is also strictly less than } \text{deg}(f) - \text{deg gcd}(f, \zeta^M) + M \text{ so (74) is satisfied,}
\]
\[\text{deg}(r^{(M)}) < \text{deg}(f) - \text{deg gcd}(f, \zeta^M) + M.\]

(Uniqueness.) Suppose \((q', \, r')\) is another solution that is degree-reducing modulo \(M\). Subtract \(r' = e - f q'\) from \(r^{(M)} = e - f q^{(M)}\),
\[(q' - q^{(M)}) f = r^{(M)} - r'.\] (84)
Take the degree of both sides,
\[\text{deg}(q' - q^{(M)}) + \text{deg}(f) = \text{deg}(r^{(M)} - r') \leq \max\{\text{deg}(r^{(M)}), \text{deg}(r')\} < \text{deg}(f) - \text{deg gcd}(f, \zeta^M) + M,\]
where the last inequality is hypothesis (74). Simplify this to get
\[\text{deg}(q' - q^{(M)}) < M - \text{deg gcd}(f, \zeta^M).\] (85)
Define \(m_f \overset{\text{def}}{=} \max\{i \geq 0 : \zeta^i | f\}\) and write \(f(\zeta) = \zeta^{m_f} f'(\zeta),\) where \(\zeta \not| f'\). If \(\text{deg gcd}(f, \zeta^M) = M\) then \(q' = q^{(M)}\) by (85) and we’re done.

Otherwise, assume \(\text{deg gcd}(f, \zeta^M) < M\), which implies \(m_f = \text{deg gcd}(f, \zeta^M)\). The hypothesis (73) for \(r^{(M)}\) and \(r'\) means that \(\zeta^M | (r^{(M)} - r')\), so (84) implies \(\zeta^M | f(q' - q^{(M)})\). Factor \(\zeta^{m_f}\) out of \(f(\zeta)\) and cancel to infer that \(\zeta^{M-m_f} | f'(q' - q^{(M)})\). Since \(\zeta \not| f'\) this implies
\[\zeta^{M-m_f} | (q' - q^{(M)}),\] where \(M - m_f = M - \text{deg gcd}(f, \zeta^M) > 0\).
This contradicts (85) unless \(q' - q^{(M)} = 0\), proving uniqueness of \(q^{(M)}\).
\[
\square
\]

**Remarks.** If \(r^{(M)}\) is degree-reducing modulo \(M\) but has multiplicity \(M + k, \, k > 0\), then it follows from (74) that it is also degree-reducing modulo \(M + k\), implying \(q^{(M+k)} = q^{(M)}\) and \(r^{(M+k)} = r^{(M)}\).

### 5.3. A Computational Generalized Division Algorithm

Theorem 5.3 yields an algorithm that generalizes the classical polynomial division algorithm. For technical reasons, the formal statement of the CCA assumes that \(f\) is coprime to \(\zeta^M\), i.e., that \(f_0 \neq 0\) if \(M > 0\), so we simplify our Slightly Generalized Division Algorithm (SGDA) accordingly. We also reduce the complexity of the initial loop (the classical division algorithm) a bit when \(M > 0\). Since we are assuming that \(\text{deg gcd}(f, \zeta^M) = 0\), (74) simplifies to \(\text{deg}(r') < \text{deg}(f) + M\) so the SGDA only needs to reduce \text{deg}(r) by this much.

Polynomials \(f(\zeta) = \sum_{i=0}^{m} f_i \zeta^i, \, f_m \neq 0\), are represented by coefficient vectors in bold italics, \(\mathbf{f} = (f_0, \ldots, f_m)\), where \(\text{deg}(\mathbf{f}) \overset{\text{def}}{=} \text{deg}(f) = m\) and where different-length vectors are implicitly extended with zeros in sums. Multiplication of \(f(\zeta)\) by \(\zeta^k\) right-shifts its coefficient vector by \(k\),
\[\tau_k \mathbf{f} = \begin{cases} 0, & 0 \leq n < k, \\ f_{n-k}, & n \geq k. \end{cases}\] (86)
Thus, \(\text{deg}(\tau_k \mathbf{f}) = \text{deg}(\mathbf{f}) + k\). The output condition \(r^{(M)}_k = 0\) for \(0 \leq k \leq M - 1\) in Algorithm 1 is equivalent to \(\zeta^M | r^{(M)}(\zeta)\). Left-arrows represent assignments to registers (e.g., \(k \leftarrow k + 1\) or \(q \leftarrow 0\)) that may be overwritten later.
Algorithm 1 (Slightly Generalized Division Algorithm).

Input: Integer $M \in \mathbb{N}$. Dividend vector $e$ of degree $n$. Divisor $f \neq 0$ of degree $m \leq n$, with $f_0 \neq 0$ whenever $M > 0$.

Output: Quotient vector $q(M)$ of degree $\leq n - m$. Remainder vector $r(M)$ with $r_k(M) = 0$ for $0 \leq k \leq M - 1$ satisfying the bound $\deg(r(M)) < \deg(f) + M$.

1. initialize $q \leftarrow 0$ \hspace{1cm} \text{(vector of $n - m + 1$ zeros)}
2. initialize $r \leftarrow e$
3. for $(k \leftarrow n - m, k \geq M, k \leftarrow k - 1)$ do
   4. \hspace{1cm} if $r_{m+k} \neq 0$ then
   5. \hspace{2cm} $q_k \leftarrow f_m^{-1}r_{m+k}$
   6. \hspace{2cm} $r \leftarrow r - q_k(r_kf)$ \hspace{1cm} \text{(renders $r_{m+k} = 0$)}
7. for $(k \leftarrow 0, k < M, k \leftarrow k + 1)$ do
8. \hspace{1cm} if $r_k \neq 0$ then
9. \hspace{2cm} $q_k \leftarrow f_0^{-1}r_k$
10. \hspace{2cm} $r \leftarrow r - q_k(r_kf)$ \hspace{1cm} \text{(renders $r_k = 0$)}
11. return $q(M) = q$, $r(M) = r$

Remarks. Line 3: When $M = 0$ this loop implements the classical division algorithm. When $M > 0$ it only zeros out enough high-order terms to ensure

$$\deg(r(M)) < \deg(f) + M = m + M.$$  \hspace{1cm} (87)

If $n - m < M$ then (87) is satisfied by the initial condition $\deg(r) = \deg(e) = n$ so this loop is not traversed.

Line 7: This loop is only traversed if $M > 0$, which assumes $f_0 \neq 0$. It ensures that $\zeta^M | r(M)(\zeta)$ while preserving the bound $\deg(r(M)) < \deg(f) + M$.

6. Case Study: The Cubic B-Spline Binomial Filter Bank

CDF(7,5) is a biorthogonal wavelet filter bank constructed by Cohen, Daubechies, and Feauveau [76, §6.A], [2, §8.3.4]; the analog synthesis scaling function and mother wavelet generate cubic B-splines. Reverting to Z-transform notation ($\zeta \leftarrow z^{-1}$), the noncausal synthesis filters are

$$P_0(z) = (z^2 + 4z + 6 + 4z^{-1} + z^{-2})/8,$$

$$P_1(z) = (-3z^2 - 12z - 5 + 40z^{-1} - 5z^{-2} - 12z^{-3} - 3z^{-4})/32.$$

Daubechies and Sweldens [23, §7.8] factored the unimodular PWA synthesis matrix $P(z)$ into linear phase lifting steps using the Laurent polynomial EEA. The corresponding PWA analysis matrix $A(z)$ and its factorization are

$$A(z) = \begin{bmatrix} (-3z^2 + 10z^{-1} - 3z^{-2})/8 & (3z^2 + 5z^{-1} + 3z^{-2})/32 \\ -(z+1)/2 & (z^2 + 1z^{-1})/8 \end{bmatrix}, \quad |A(z)| = 1,$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 3(1+z^{-1})/16 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(z+1) & 1 \end{bmatrix} \begin{bmatrix} 1 & -(1+z^{-1})/4 \\ 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (88)
As with (9), this is the unique WS group lifting factorization of \( A(z) \) \cite{65, 66}; its lifting filters require just two multipliers. We will now factor the 7-tap/5-tap causal PWD analysis matrix, \[ H(z) = \begin{bmatrix} (3+5z^{-1}+5z^{-2}+3z^{-3})/32 & (-3+10z^{-1}-3z^{-2})/8 \\ (1+6z^{-1}+z^{-2})/8 & -(1+z^{-1})/2 \end{bmatrix}, \quad |H(z)| = -z^{-2}. \quad (89) \]

### 6.1. Causal EEA Factorization in Column 1

Initialize the remainders for the EEA, \( r_0 \equiv H_{01}(z) \) and \( r_1 \equiv H_{11}(z) \). Using the division algorithm, \( r_0 = q_0 r_1 + r_2 \) where \( q_0 = (-13+3z^{-1})/4, r_2 = -2, \) and

\[
\begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = M_0 \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \quad \text{for} \quad M_0 \equiv \begin{bmatrix} q_0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Next, \( r_1 = q_1 r_2 + r_3 \) where \( q_1 = (1+z^{-1})/4, r_3 = 0, \) and

\[
\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = M_1 \begin{bmatrix} r_2 \\ 0 \end{bmatrix} \quad \text{for} \quad M_1 \equiv \begin{bmatrix} q_1 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Now define an augmentation matrix with causal filters \( a_0 \) and \( a_1, \)

\[
H'(z) \equiv \begin{bmatrix} a_0 & r_0 \\ a_1 & r_1 \end{bmatrix} = M_0 M_1 \begin{bmatrix} 0 & r_2 \\ -|H|/r_2 & 0 \end{bmatrix} = \begin{bmatrix} z^{-2}(13-3z^{-1})/8 & (-3+10z^{-1}-3z^{-2})/8 \\ -z^{-2}/2 & -(1+z^{-1})/2 \end{bmatrix}.
\]

Apply the Lifting Theorem and transform to standard causal lifting form.

\[
H(z) = H'(z) V(z) = H'(z) \begin{bmatrix} 1 & 0 \\ V(z) & 1 \end{bmatrix}, \quad \text{where} \quad V(z) = -(1+5z^{-1})/4,
\]

\[
= (M_0 J)(M_1 J) \text{diag}(-2, -1/2) \text{diag}(1, z^{-2}) J V(z); \quad \text{use} \ (41), \ (43),
\]

\[
= \begin{bmatrix} -2 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & (-13+3z^{-1})/16 \\ 1+z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -z^{-2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.
\quad (90)
\]

Unlike what happened when factoring LGT(5,3) in Section 2.1.2, factoring CDF(7,5) using the causal EEA in column 1 does not produce a causal analogue of the unimodular linear phase lifting factorization (88). Note how the entire determinantal delay of \( H(z) \), which was introduced via the augmentation matrix \( H'(z) \), winds up in a single diagonal delay matrix, \( \text{diag}(1, z^{-2}). \)

### 6.2. CCA Factorization in Column 1

Initialize \( Q_0(z) \equiv H(z); \) set \( E_1 \leftarrow H_{01} \) and \( F_1 \leftarrow H_{11}, \deg(F_1) < \deg(E_1), \) and seek a lifting factorization,

\[
Q_0(z) = \begin{bmatrix} (3+5z^{-1}+5z^{-2}+3z^{-3})/32 & (-3+10z^{-1}-3z^{-2})/8 \\ (1+6z^{-1}+z^{-2})/8 & -(1+z^{-1})/2 \end{bmatrix} = \begin{bmatrix} E_0 & E_1 \\ F_0 & F_1 \end{bmatrix} = \begin{bmatrix} 1 & S \\ R_0 & R_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ R_0 & F_1 \end{bmatrix}.
\quad (91)
\]
Divide $F_1$ into $E_1$ using classical polynomial division, $E_1 = F_1 S + R_1$, where

$$S(z) = (-13 + 3z^{-1})/4$$

and $R_1(z) = -2, \deg(R_1) < \deg(F_1)$.

Define $R_0 \overset{\text{def}}{=} E_0 - F_0 S = (1 + 5z^{-1})/2$; the first factorization step is

$$Q_0(z) = \begin{bmatrix} 1 & (1+5z^{-1})/2 \\ 0 & (1+6z^{-1}+z^{-2})/8 \end{bmatrix} = V_0(z)Q_1(z). \quad (92)$$

Reset the labels $E_j \leftarrow F_j$ and $F_j \leftarrow R_j$ in $Q_1(z)$,

$$Q_1(z) = \begin{bmatrix} (1+5z^{-1})/2 \\ (1+6z^{-1}+z^{-2})/8 \end{bmatrix} = \begin{bmatrix} F_0 & F_1 \\ E_0 & E_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ S & 1 \end{bmatrix} \begin{bmatrix} F_0 & F_1 \\ 0 & 1 \end{bmatrix}. \quad (93)$$

Dividing in column 1, $E_1 = F_1 S + R_1$ for $S(z) = (1 + z^{-1})/4$ with $R_1 = 0$. Set $R_0 \overset{\text{def}}{=} E_0 - F_0 S = -z^{-2}/2$; the second step is

$$Q_1(z) = \begin{bmatrix} 1 & 0 \\ (1+z^{-1})/4 & 1 \end{bmatrix} = \begin{bmatrix} (1+5z^{-1})/2 \\ -1/2 \end{bmatrix} = V_1(z)\Delta_1(z)Q_2(z). \quad (94)$$

Factor a diagonal gain matrix and a swap off of $Q_2(z)$,

$$Q_2(z) = \begin{bmatrix} -2 & 0 \\ 0 & -1/2 \end{bmatrix} = D_{-2,-1/2}V_2(z)J. \quad (95)$$

Combine (92), (94), and (95) to get the same factorization as (90),

$$H(z) = V_0(z) V_1(z) \Delta_1(z) D_{-2,-1/2} V_2(z) J$$

$$= -D_{2,1/2} \left( \gamma_{2,1/2}V_0(z) \right) \left( \gamma_{2,1/2}V_1(z) \right) \Delta_1(z) V_2(z) J,$$

using (41),

$$= \begin{bmatrix} -2 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & (1+5z^{-1})/4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$  

6.2.1. Enumerating Degree-Lifting Factorizations

Theorem 4.5 allows us, in principle, to enumerate all possible degree-lifting decompositions of a given filter bank. E.g., what are the possible degree-reducing causal complements to $(F_0, F_1)$ for inhomogeneity $\hat{a}z^{-d} = -z^{-2}$ in (91)? Since

$$\deg |Q_1| = \deg |R_0 R_1| = \deg |F_0| = 2 < \deg(F_0) + \deg(F_1) = 3,$$

Theorem 4.5(iii) predicts that division in column 0 of $Q_0$ will yield the same causal complement $(R_0, R_1)$ (the top row of $Q_1$ in (92)) that was obtained by division in column 1 of $Q_0$. This is confirmed by noting that the remainders in the top row of $Q_1$ are degree-reducing in both $F_0$ and $F_1$ and are therefore, by Theorem 4.5(i–ii), the unique remainders given by division in either column.

Factoring $Q_1(z)$ as in (93) is another matter, however, because

$$\deg \begin{bmatrix} F_0 & F_1 \\ R_0 & R_1 \end{bmatrix} = \deg |Q_1| = 2 > \deg(F_0) + \deg(F_1) = 1.$$
By Theorem 4.5(iii) the causal complements that are degree-reducing in $F_0$ and $F_1$ are different. Division in column 1 of $Q_1(z)$ produces (94), which leads to (90), so dividing in column 0 produces a different result than (90). There is no EEA analogue of “switching columns” like this.

This notion of enumerating degree-lifting decompositions of a given filter bank is pursued in [68, §2] where we introduce a bookkeeping schema for keeping track of the options for forming different lifting factorizations. This schema is incorporated into the formulation of the Causal Complementation Algorithm [68, Algorithm 1]. A provably complete example using the Daubechies 4-tap/4-tap paraunitary filter bank is presented [68, Table 2] that enumerates all “left degree-lifting” factorizations (i.e., factorizations constrained by partial pivoting to employ only elementary row reductions).

### 6.3. CCA Factorization Using the SGDA in Column 1

Another serious limitation of the causal EEA method is that it puts the entire determinant in a single diagonal delay matrix, e.g., $\text{diag}(1, z^{-2})$ in (90). The CCA allows one to factor out delays at arbitrary points in the factorization using the SGDA, which lets us construct CCA factorizations not obtainable using the causal EEA. We now derive a causal version of the linear phase factorization (88); as in Section 6.2 we seek a factorization of the form (91),

$$\begin{bmatrix}
    (3+5z^{-1}+5z^{-2}+3z^{-3})/32 \\
    (1+6z^{-1}+z^{-2})/8 \\
    -(1+z^{-1})/2
\end{bmatrix} = \begin{bmatrix} 1 & S \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_0 & R_1 \\ F_0 & F_1 \end{bmatrix}. \tag{96}
$$

This time, however, dividing $F_1$ into $E_1$ using the SGDA with $M = 1$ yields

$$S(z) = 3(1 + z^{-1})/4 \quad \text{with} \quad R_1 = 2z^{-1}, \ deg(R_1) < \deg(F_1) + 1. \tag{97}$$

Set $R_0 \leftarrow E_0 - F_0S = -z^{-1}(1 + z^{-1})/2$ and note that $z^{-1}$ also divides $R_0(z)$ (this follows from [68, Theorem 2.3]). The first lifting step can be written

$$Q_0(z) = \begin{bmatrix} 3(1+z^{-1})/4 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -(1+z^{-1})/2 \\ 2 \\ -(1+6z^{-1}+z^{-2})/8 \end{bmatrix} \tag{98}$$

Next, seek a factorization of $Q_1(z)$ of the form

$$Q_1(z) = \begin{bmatrix} -(1+z^{-1})/2 \\ (1+6z^{-1}+z^{-2})/8 \\ -(1+z^{-1})/2 \end{bmatrix} = \begin{bmatrix} F_0 & F_1 \\ E_0 & E_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ S & 1 \end{bmatrix} \begin{bmatrix} F_0 & F_1 \\ R_0 & R_1 \end{bmatrix}. \tag{99}$$

Dividing in column 1 using the classical polynomial division algorithm produces $S(z) = -(1+z^{-1})/4$ and $R_1 = 0$. Set $R_0 \leftarrow E_0 - F_0S = z^{-1}/2$; both $R_0$ and $R_1$ are divisible by $z^{-1}$ so the second lifting step can be written

$$Q_1(z) = \begin{bmatrix} 1 \\ -(1+z^{-1})/4 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -(1+z^{-1})/2 \\ 2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} V_1(z) & \Delta_1(z) & Q_2(z) \end{bmatrix}. \tag{100}$$

Factor a diagonal gain matrix and a swap matrix off of $Q_2(z)$ to get

$$Q_2(z) = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \\ 1/z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -(1+z^{-1})/4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = D_{2,1/2}V_2(z)J. \tag{101}$$
Combine (98), (100), and (101) and put in standard causal lifting form (6),

\[ H(z) = V_0(z) \Delta_0(z) V_1(z) \Delta_1(z) D_{2,1/2} V_2(z) J \]

\[ = D_{2,1/2} U_2(z) \Delta_2(z) U_1(z) \Delta_1(z) U_0(z) J \] using (41),

\[ = 2 0 \begin{bmatrix} 1 & 3(1+z^{-1})/16 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \] (102)

This is a causal analogue of the unimodular WS group lifting factorization (88); its causal linear phase lifting filters differ from those in (88) by at most delays. The reader can confirm that (102) is also obtained using the SGDA with \( M = 1 \) in row 1 of (89) when computing the lifting by column reduction. In comparison, (102) is not produced by the causal EEA method in any row or column of (89). One might wonder what happens if one uses the SGDA with \( M = 1 \) (as in (97)) to perform the first division operation in the EEA calculation in Section 6.1. This cannot possibly produce (102), however, because the augmentation matrix will still put the entire determinantal delay \( (z^{-2}) \) into a single diagonal delay matrix. Indeed, instead of (90) or (102) the EEA method using the SGDA in this manner yields the factorization

\[ H(z) = 2 0 \begin{bmatrix} 1 & -3(1+z^{-1})/16 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \] (103)

Since both use the SGDA division in (97), formulas (102) and (103) have the same leftmost lifting filter, modulo a sign difference attributable to using (41) with the diagonal matrix in (103).

7. Conclusions

We introduce a new method, the Causal Complementation Algorithm (CCA), for factoring causal FIR perfect reconstruction transfer matrices into causal lifting steps (elementary matrices). Daubechies and Sweldens [23] factored unimodular filter banks using the Extended Euclidean Algorithm (EEA) for Laurent polynomials, but the CCA is more general, using Gaussian elimination to generate more causal lifting factorizations than those produced using the causal version of the EEA method. This includes a causal linear phase analogue of a unimodular linear phase lifting factorization for the CDF(7,5) cubic B-spline wavelet filter bank that is not generated by the causal EEA approach.

The causal EEA works within a single row or column of \( H(z) \), effectively limiting the EEA to constructing at most four lifting factorizations of a given filter bank (using classical polynomial division), no matter how big the matrix polynomial order of \( H(z) \) is. The EEA method constructs an augmentation matrix that is in turn lifted to \( H(z) \), further limiting possible factorizations by putting the entire determinant of \( H(z) \) into a single diagonal delay matrix, which can lead to ill-conditioned factorizations [68]. The CCA, in comparison, factors the whole matrix \( H(z) \) from the outset and enables strategies like switching the row or column in which division takes place that have no EEA analogue. The CCA allows the user to factor off diagonal delay matrices at will using the Slightly Generalized Division Algorithm, expanding the choice of lifting factorizations that can be obtained.

A max-additive inequality connects causality to uniqueness of degree-reducing solutions to linear Diophantine equations over causal polynomial rings via a Linear Diophantine Degree-Reduction Theorem; this inequality fails for Laurent polynomials. The LDDRT uses the determinantal degree
information that is missing from unimodular normalizations) to decide whether a causal lifting step can be factored off in exactly one or two distinct ways. This allows users in principle to systematically generate all possible degree-lifting factorizations of a given causal filter bank, a capability not provided by the EEA approach. The author maintains that the degree-reducing properties of such factorizations should distinguish lifting factorizations amongst all elementary matrix decompositions of polyphase matrices, a definition not made in [23].

Work in progress includes specification of the CCA in algorithmic form, complexity analysis showing the advantages of the CCA over the EEA [68], specializations to linear phase filter banks, and realization theory for causal lifting.

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