Some Results in Grünwald-Letnikov Fractional Derivative and its Best Approximation

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Abstract. In this paper, we have approximated Grünwald-Letnikov Derivative of a function having m continuous derivatives by Bernstein Chlodowsky polynomials with proving its best approximation. As well as we have solved Bagley-Torvik equation and Fokker–Planck equation where the derivative is in Grünwald-Letnikov sense.

Keywords: Best Approximation, Bernstein-Chlodowsky polynomials, Grünwald-Letnikov derivatives, Fractional differential equations.

1. Introduction

The topic of fractional calculus has a tall history whose earliest stages go back to the start of classical calculus and it is a field having intriguing applications with regards to genuine issues, this kind of calculus has its roots in the popularizations of the differential and integral calculus [1].

By the second half of the twentieth century, the field of fractional calculus had grown to such extent that in 1974 the first conference “The First Conference on Fractional Calculus and its Applications” concerned solely with the theory and applications of fractional calculus was held in New Haven. In the same year, the first book on fractional calculus by Oldham and Spanier [2] was published after a joint collaboration started in 1968.

Podlubny [3], proposed a solution of more than 300 years old problem of geometric and physical explanation of fractional integration and differentiation in 2002. J. A. Tenreiro Machado in 2003[4], given a probabilistic explanation of fractional order derivative, by using Grünwald – Letnikov definition of fractional order differentiation.

Here we suggest the utilization of Grünwald-Letnikov Derivative of the classical Bernstein Chlodowsky polynomials to find the best approximation of the Grünwald-Letnikov Derivative of f for every $f \in C^m$.

We solve the Bagley-Torvik equation and Fokker–Planck equation where the derivative is in Grünwald-Letnikov sense. When an equation Bagley and Torvik (1984) [5] was discovered we were able to define fractional calculus in solving the problem of viscous fluid motion and it emerged that the
resulting shear stress at any point is made through a fractional derivative of the fluid velocity profile. When they applied their work to describe the motion of some physical systems and arrived at the conclusion, they were surprised to find the fractional derivative in the differential equation. Whereas we use the Fokker-Planck fractional space equation (SFFPE) of immediate origin to describe Brownian motion of particles is the equation that describes the change in the probability of a random function in space and time, and is then used naturally to describe the dissolved transport [6].

**Remark:** Along the paper, we use \( \| \cdot \| \) to denote the uniform norm over the interval \([0,1]\), that is defined by:

\[
\| f \| = \max_{0 < x < 1} |f(x)| \text{ for } f \in C[0,1].
\]

2. Grünwald-Letnikov fractional derivative

**Definition 1** [1]: Let \( \alpha \) is a positive real number, the Grünwald-Letnikov \( n \)th order fractional derivative of the function \( f(x) \) with respect to \( x \) and defined by:

\[
\frac{d^n}{dx^n} f(x) = \lim_{h \to 0} \frac{h^{-\alpha}}{a h} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(x - k h)
\]

(2.1)

Where

\[
\binom{\alpha}{k} = \frac{\Gamma(\alpha + 1)}{k! \Gamma(\alpha - k + 1)}
\]

\[
\frac{d^n}{dx^n} f(x) = \lim_{h \to 0} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \left( \frac{\Gamma(\alpha + 1)}{k! \Gamma(\alpha - k + 1)} \right) f(x - k h)
\]

(2.2)

Where \( \Gamma \) is the gamma function defined by the Euler limit expression:

\[
\Gamma(x) = \lim_{a \to x} a^x \frac{\Gamma(x + 1)}{x(x + 1)(x + 2) ... (x + \alpha)}, x > 0
\]

Where the left-sided derivative:

\[
\frac{d^n}{dx^n} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \left( \frac{\Gamma(\alpha + 1)}{k! \Gamma(\alpha - k + 1)} \right) f(t - k h), ah = x - a
\]

And the right-sided derivative:

\[
\frac{d^n}{dx^n} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \left( \frac{\Gamma(\alpha + 1)}{k! \Gamma(\alpha - k + 1)} \right) f(t + k h), ah = b - x
\]
3. Best Approximation and the classical Bernstein-Chlodowsky polynomials

By elements from $\mathcal{P}_n$ the subspace of algebraic polynomials the best (uniform) approximation of a given function $f \in C[a,b]$ of degree at most $n$ in $C[a,b]$ which we’ve chosen to write as $p_n$ defined by:

$$E_n(f) = \min_{p \in \mathcal{P}_n} \|f - p\| = \|f - p_n\|$$

(3.1)

Because $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n$, it’s easy to show that:

$$E_n(f) \geq E_{n+1}(f), \text{ for each } n \geq 6.$$

Theorem 1 (The Weierstrass approximation theorem) [6]:

Let $f \in C[a,b]$. $\forall \varepsilon > 0, \exists p_n \in \mathcal{P}_n$ such that $|f(x) - p_n(x)| \leq \varepsilon$, $\forall x \in [0,1]$. 

Lemma 1 [7]: If the Weierstrass theorem holds for $C[0,1]$, then it also holds for $C[a,b]$, and conversely.

Definition 2 [8]: The classical Bernstein-Chlodowsky polynomials were presented by Chlodowsky as a speculation of Bernstein polynomials, have the following form:

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}$$

(3.2)

Where $0 \leq x \leq b_n$ and $b_n$ is a sequence of positive numbers such that

$$\lim_{n \to \infty} b_n = \infty, \lim_{n \to \infty} \frac{b_n}{n} = 0.$$

4. Fractional derivative of the classical Bernstein-Chlodowsky polynomials and its best approximation:

Now, we compute the Grünwald-Letnikov fractional derivatives of the classical Bernstein-Chlodowsky polynomials with respect to a function $f$:

$$^{G_L}D_0^\alpha [B_n(f,x)] = ^{G_L}D_0^\alpha \left[ \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} \right]$$

$$= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k \left(\frac{1}{b_n}\right)^k \left[D_0^\alpha x^k (b_n - x)^{\alpha-k}\right]$$

$$= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k \left(\frac{1}{b_n}\right)^k \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{\alpha-k} \frac{\Gamma(\alpha-k+1)}{\Gamma(\alpha-k+1-\alpha)} (b_n - x)^{\alpha-k-\alpha}$$
\[
\sum_{k=0}^{n} {\binom{\alpha}{k} c_k \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} f\left(\frac{k}{\alpha} b_n\right) x^{k-\alpha}}
\]

**Theorem 2:** Let \(\alpha\) be a nonnegative real number and \(m \in \mathbb{N}\) be such that \(m - 1 \leq \alpha \leq m\). If \(f \in C^m[0,1]\) and \(\varepsilon > 0\), then:

\[
\left\| D_{x}^{\alpha} f - D_{x}^{\alpha} B_n(f) \right\| < \varepsilon
\]

**Proof:**

We wish to show that, given \(\varepsilon > 0\) and \(f \in C^m[0,1]\) such that

\[
\left\| D_{x}^{\alpha} f - D_{x}^{\alpha} B_n(f) \right\| < \varepsilon
\]

Thus,

\[
\left| g_{\alpha} D_{x}^{\alpha} f (x) - g_{\alpha} D_{x}^{\alpha} B_n(f; x) \right|
\]

\[
= \left| \lim_{n \to \infty} N \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f \left(1 - \frac{k}{\alpha}\right) - \lim_{n \to \infty} N \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} B_n \left(1 - \frac{k}{\alpha}\right) \right|
\]

\[
\leq \lim_{n \to \infty} N \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} \left| f \left(1 - \frac{k}{\alpha}\right) - B_n \left(1 - \frac{k}{\alpha}\right) \right|
\]

\[
\leq \lim_{n \to \infty} N \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} \varepsilon
\]

This theorem shows that \(g_{\alpha} D_{x}^{\alpha} B_n(f; x) \to g_{\alpha} D_{x}^{\alpha} f(x)\) as \(n \to \infty\) uniformly on \([0,1]\).

5. **Application of Bagley-Torvik and Fokker–Planck equations:**

- **Bagley-Torvik Equation**

Starting with the diffusion equation:

\[
\frac{\alpha}{\alpha} \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial z^2}
\]

Where \(\alpha\) is the fluid density, \(\mu\) is the viscosity of time \(t\), \(z\) is the distance from the (wetted plate), after that, they found that the differential equation is:
\[ m \dddot{x} = F_x = -Gx - 2A \delta(t, 0), \]  

(They think about rigid plate of mass \( m \) immersed in a Newtonian fluid of infinite extent and connected by a massless spring of stiffness \( G \) to a fixed point).

Finally,

\[ m \ddot{x} + 2A \sqrt{\mu \alpha} D^{3/2} x + Gx = 0, \]

Where,

\[ D^{3/2} x = \frac{d}{dt} D^{1/2} x, \]

(More details can be found in [5]).

In the following, we solve Bagley-Torvik equation where the derivative is in Grünwald sense.

- **The general form of Bagley-Torvik equation is:**

\[ A \ddot{u} + B \dddot{u} + Cu = f(x), x \in [0, \infty) \]

(5.3)

Let us solve the special case:

\[ \ddot{u} + u^3 + u = f(x) \]

\[ u(0) = \ddot{u}(0) = 0 \]

where

\[ f(x) = x^2 + x^3 + 2 + 40 \frac{2}{9} x^2 + 2 \frac{1}{\Gamma(2)} x^1 + \frac{\Gamma(11)}{\Gamma(5)} x^{7/3} \]

The exact solution is:

\[ u(x) = x^2 + x^3 \]  

(5.5)

Suppose

\[ u(x) = \sum_{i=0}^{\infty} c_i x^i + \sum_{j=0}^{\infty} d_j x^{3j/2} \]

\[ u(0) = 0 \rightarrow c_0 + d_0 = 0 \]

\[ \ddot{u}(0) = 0 \rightarrow c_1 = 0 \]

\[ \left[ \sum_{i=0}^{\infty} c_i x^i + \sum_{j=0}^{\infty} d_j x^{3j} \right]^" = \left[ \sum_{i=0}^{\infty} c_i x^i + \sum_{j=0}^{\infty} d_j x^{3j} \right] + \left[ \sum_{i=0}^{\infty} c_i x^i + \sum_{j=0}^{\infty} d_j x^{3j} \right] + \left[ \sum_{i=0}^{\infty} c_i x^i + \sum_{j=0}^{\infty} d_j x^{3j} \right] \]

\[ = x^2 + x^3 + 2 + 40 \frac{2}{9} x^2 + 2 \frac{1}{\Gamma(2)} x^1 + \frac{\Gamma(11)}{\Gamma(5)} x^{7/3} \]

After simplifications and equations coefficient we get \( c_1 = d_1 = 1 \) and all other coefficient equal to zero. So the solution is:

\[ u(x) = x^2 + x^3 \]
-Fokker–Planck equation

The general formula of an equation (FPE) for the motion of a concentration; field \( C(x,t) \) of one space variable \( x \) at time \( t \) has the form[6]:

\[
\frac{\partial C}{\partial t} = \left[ -\frac{\partial}{\partial x}D^{(1)}(x) + \frac{\partial^2}{\partial x^2}D^{(2)}(x) \right] C(x,t), \quad 0 < \alpha < 1 \quad (5.6)
\]

Where \( D^{(2)}(x) > 0 \) is the diffusion coefficient and \( D^{(1)}(x) > 0 \) is the drift coefficient. Eq.(5.6) is a linear second-order partial differential equation of parabolic type.

-Statement of the problem

You great number of independent solute particles. A special case of the SFFPE written as:

\[
\frac{\partial C}{\partial t} = -D^{(1)} \frac{\partial^\alpha c}{\partial x^\alpha} + D^{(2)} \frac{\partial^{1+\alpha} c}{\partial x^{1+\alpha}}
\]

As \( \alpha \to 1 \) then \( \to \) eq. (5.6) in the paper.

Suppose that \( c(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} x^{ai} t^j \)

\[
\frac{\partial C}{\partial t} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} j a_{ij} x^{ai} \\
= a_{00} + 2a_{02}t + 3a_{03}t^2 + \ldots + a_{11}x^a + 2a_{12}x^at + 3a_{13}x^at^2 + \ldots + a_{21}x^{2a} + 2a_{22}x^{2at} \nonumber \]

\[
+ a_{23}x^{2at^2} + \ldots + a_{31}x^{3a} + 2a_{32}x^{3at} + 3a_{33}x^{3at^2} + \ldots + a_{41}x^{4a} + 2a_{42}x^{4at} \nonumber \\
+ a_{43}x^{4at^2} + \ldots + a_{51}x^{5a} + 2a_{52}x^{5at} + 3a_{53}x^{5at^2} + \ldots
\]

\[
\frac{\partial^\alpha C}{\partial x^\alpha} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} \frac{\Gamma\left(\alpha i + 1\right)}{\Gamma\left(\alpha i + 1 - \alpha\right)} x^{ai} t^j \\
= a_{00} \frac{1}{\Gamma\left(1 - \alpha\right)} x^{-a} + \ldots + a_{10} \frac{\Gamma\left(\alpha + 1\right)}{\Gamma\left(1 - \alpha\right)} t + a_{11} \frac{\Gamma\left(\alpha + 1\right)}{\Gamma\left(1 - \alpha\right)} t + a_{12} \frac{\Gamma\left(\alpha + 1\right)}{\Gamma\left(1 - \alpha\right)} t^2 + \ldots 
\]

\[
+ a_{20} \frac{\Gamma\left(\alpha + 1\right)}{\Gamma\left(2 - \alpha\right)} x^a + a_{21} \frac{\Gamma\left(\alpha + 1\right)}{\Gamma\left(2 - \alpha\right)} x^at + a_{22} \frac{\Gamma\left(\alpha + 1\right)}{\Gamma\left(2 - \alpha\right)} x^at^2 + \ldots + a_{30} \frac{\Gamma\left(3 - \alpha\right)}{\Gamma\left(2 - \alpha\right)} x^{2a} 
\]

\[
+ a_{31} \frac{\Gamma\left(3 - \alpha\right)}{\Gamma\left(2 - \alpha\right)} x^{2at} + a_{32} \frac{\Gamma\left(3 - \alpha\right)}{\Gamma\left(2 - \alpha\right)} x^{2at^2} + \ldots + a_{40} \frac{\Gamma\left(4 - \alpha\right)}{\Gamma\left(3 - \alpha\right)} x^{3a} 
\]

\[
+ a_{41} \frac{\Gamma\left(4 - \alpha\right)}{\Gamma\left(3 - \alpha\right)} x^{3at} + a_{42} \frac{\Gamma\left(4 - \alpha\right)}{\Gamma\left(3 - \alpha\right)} x^{3at^2} + \ldots + a_{50} \frac{\Gamma\left(5 - \alpha\right)}{\Gamma\left(4 - \alpha\right)} x^{4a} 
\]

\[
+ a_{51} \frac{\Gamma\left(5 - \alpha\right)}{\Gamma\left(4 - \alpha\right)} x^{4at} + a_{52} \frac{\Gamma\left(5 - \alpha\right)}{\Gamma\left(4 - \alpha\right)} x^{4at^2} + \ldots
\]

\[
\frac{\partial^{1+\alpha} C}{\partial x^{1+\alpha}} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} \frac{\Gamma\left(\alpha i + 1\right)}{\Gamma\left(\alpha i - \alpha\right)} x^{ai} t^j
\]
\[
\begin{align*}
&= a_{00} \frac{1}{\Gamma(-\alpha)} x^{-1-\alpha} + \ldots + a_{10} \frac{\Gamma(\alpha + 1)}{\Gamma(0)} x^{-1} + \ldots + a_{20} \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha)} x^{\alpha-1} + \ldots \\
&\quad + a_{30} \frac{\Gamma(3\alpha + 1)}{\Gamma(2\alpha)} x^{2\alpha-1} + a_{31} \frac{\Gamma(3\alpha + 1)}{\Gamma(2\alpha)} x^{2\alpha-1} t + a_{32} \frac{\Gamma(3\alpha + 1)}{\Gamma(2\alpha)} x^{2\alpha-1} t^2 + \ldots \\
&\quad + a_{40} \frac{\Gamma(4\alpha + 1)}{\Gamma(3\alpha)} x^{3\alpha-1} + a_{41} \frac{\Gamma(4\alpha + 1)}{\Gamma(3\alpha)} x^{3\alpha-1} t + a_{42} \frac{\Gamma(4\alpha + 1)}{\Gamma(3\alpha)} x^{3\alpha-1} t^2 + \ldots 
\end{align*}
\]

By substituting these derivatives in the fractional differential equation * and equating corresponding coefficient we get:

\[
\begin{align*}
a_{00} & \text{ arbitrary} \\
a_{01} &= -D^1 \Gamma(\alpha + 1) a_{10}, \quad 2a_{02} = -D^1 \Gamma(\alpha + 1) a_{11}, \ldots, 3a_{03} = -D^1 \Gamma(\alpha + 1) a_{12}, \ldots \\
a_{11} &= -D^1 \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} a_{20}, \quad 2a_{12} = -D^1 \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} a_{21}, \quad 3a_{13} = -D^1 \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} a_{22}, \ldots \\
a_{30} &= 0, a_{31} = 0, a_{40} = 0, a_{50} = 0, \ldots
\end{align*}
\]

And the remaining coefficient are zero, by substituting all the outputs getting:

\[
\begin{align*}
a_{01} &= -D^1 \Gamma(\alpha + 1) a_{10} \\
a_{02} &= -D^1 \frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} a_{11} \\
a_{20} &= -D^1 \frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} a_{11}
\end{align*}
\]

The approximate solution:

\[
C(x, t) = a_{00} + a_{01} t + a_{02} t^2 + a_{10} x^\alpha + a_{11} x^\alpha t + a_{20} x^{2\alpha} = a_{00} + (-D^1 \Gamma(\alpha + 1) a_{10} t + \frac{-D^1 \Gamma(\alpha + 1)}{2} a_{11} t^2 + a_{10} x^\alpha + a_{11} x^\alpha t + \frac{\Gamma(\alpha + 1)}{-D^1 \Gamma(2\alpha + 1)} a_{11} x^{2\alpha})
\]

6. Conclusions

In this paper, some basic definitions of the partial derivatives that I benefit from in the research are presented. The fractional derivatives are based on the Grünwald–Letnikov equation. The best approximation of fractional derivatives with used the classical Bernstein–Chlodowsky polynomials by Grünwald–Letnikov fractional derivatives. In the finally section we solve the Bagley–Torvik and Fokker–Planck equations, which are a fractional differential equations, where the procedure is extract the exact solution.

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