Backward Orbit Conjecture for Lattès Maps

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Abstract For a Lattès map \( \phi: \mathbb{P}^1 \to \mathbb{P}^1 \) defined over a number field \( K \), we prove a conjecture on the integrality of points in the backward orbit of \( P \in \mathbb{P}(\overline{K}) \) under \( \phi \).

Keywords: backward orbit conjecture, Lattès maps

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1. Introduction

Let \( \phi: \mathbb{P}^1 \to \mathbb{P}^1 \) be a rational map of degree \( \geq 2 \) defined over a number field \( K \), and write \( n_{\phi} \) for the \( n \)th iterate of \( \phi \). For a point \( P \in \mathbb{P}^1 \), let \( \phi^+(P) = \{ P, \phi(P), \phi^2(P), \ldots \} \) be the forward orbit of \( P \) under \( \phi \), and let

\[
\phi^-(P) = \bigcup_{n \geq 0} \phi^{-n}(P)
\]

be the backward orbit of \( P \) under \( \phi \). We say \( P \) is \( \phi \)-preperiodic if and only if \( \phi^+(P) \) is finite.

Viewing the projective line \( \mathbb{P}^1 \) as \( \mathbb{A}^1 \cup \{ \infty \} \) and taking \( P \in \mathbb{A}^1(K) \), a theorem of Silverman [4] states that if \( \infty \) is not a fixed point for \( \phi^2 \), then \( \phi^+(P) \) contains at most finitely many points in \( \mathcal{O}_K \), the ring of algebraic integers in \( K \). If \( S \) is the set of all archimedean places for \( K \), then \( \mathcal{O}_K \) is the set of pointsDefinition of Integrality

In [6], Conjecture 1.1 was shown true for the powering map \( \phi(z) = z^d \) with degree \( d \geq 2 \), and consequently for Chebyshev polynomials. A generalized version of this conjecture, which is stated over a dynamical family of maps \( [\phi] \), is given in [1], Sec. 4. Along those lines, our goal is to prove a general form of Conjecture 1.1 where \( [\phi] \) is the family of Lattès maps associate to a fixed elliptic curve \( E \) defined over \( K \) (see Section 3).

2. The Chordal Metric and Integrality

2.1. The Chordal Metric on \( \mathbb{P}^N \)

Let \( M_K \) be the set of places on \( K \) normalized so that the product formula holds: for all \( \alpha \in K^\times \),

\[
\prod_{v \in M_K} |\alpha|_v = 1.
\]

For points \( P = [x_0 : x_1 : \cdots : x_N] \) and \( Q = [y_0 : y_1 : \cdots : y_N] \) in \( \mathbb{P}^N(\overline{\mathbb{F}}) \), define the \( v \)-adic chordal metric as

\[
\Delta_v(P, Q) = \max_{i,j} \left( \frac{|x_i y_j - x_j y_i|_v}{\max_i \left( |x_i|_v \right) \cdot \max_j \left( |y_j|_v \right)} \right).
\]

Note that \( \Delta_v \) is independent of choice of projective coordinates for \( P \) and \( Q \), and \( 0 \leq \Delta_v(P, \cdot) \leq 1 \) (see [2]).

2.2. Integrality on Projective Curves

Let \( C \) be an irreducible curve in \( \mathbb{P}^N \) defined over \( K \) and \( S \) a finite subset of \( M_K \) which includes all the archimedean places. A divisor on \( C \) defined over \( \overline{K} \) is a formal sum \( \sum n_i Q_i \) with \( n_i \in \mathbb{Z} \) and \( Q_i \in C(\overline{K}) \). The divisor is effective if \( n_i > 0 \) for each \( i \), and its support is the set \( \text{Supp}(D) = \{ Q_1, \cdots, Q_r \} \).

A conjecture for finiteness of integral points in backward orbits was stated in [6], Conj. 1.2.

**Conjecture 1.1.** If \( Q \in \mathbb{P}^1(\overline{K}) \) is not \( S \)-preperiodic, then \( \phi^-(P) \) contains at most finitely many points in \( \mathbb{P}^1(\overline{K}) \) which are \( S \)-integral relative to \( Q \).
Let $\lambda_{D,v}(P) = -\log \Delta_v(P,Q)$ and $\lambda_{D,v}(P) = \sum n_i \lambda_{Q_i,v}(P)$ when $D = \sum n_i Q_i$. This makes $\lambda_{D,v}$ an arithmetic distance function on $C$ (see [3]) and as with any arithmetic distance function, we may use it to classify the integral points on $C$.

For an effective divisor $D = \sum n_i Q_i$ on $C$ defined over $\overline{K}$, we say $P \in C(\overline{K})$ is $S$-integral relative to $D$, or $P$ is a $S$-integral point, if and only if $\lambda_{D,v}(P) = 0$ for all embeddings $\sigma, \tau : K \to \overline{K}$ and for all places $v \notin S$. Furthermore, we say the set $\mathcal{R} \subset C(\overline{K})$ is $S$-integral relative to $D$ if and only if each point in $\mathcal{R}$ is $S$-integral relative to $D$.

As an example, let $C$ be the projective line $\mathbb{P}^1 \cup \{\infty\}$, $S$ be the Archimedean place of $K = \mathbb{Q}$, and $D = \infty$. For $P = x/y$, with $x$ and $y$ relatively prime in $\mathbb{Z}$, we have $\lambda_{D,v}(P) = -\log |y|_v$ for each prime $v$. Therefore, $P$ is $S$-integral relative to $D$ if and only if $y = \pm 1$; that is, $P$ is $S$-integral relative to $D$ and only if $P \in \mathbb{Z}$. From the definition we find that if $S_1 \subset S_2$ are finite subsets of $M_K$ which contains all the archimedean places, then $P$ is a $(D,S_2)$-integral point implies that $P$ is a $(D,S_1)$-integral point. Similarly, if $\text{Supp}(D_1) \subset \text{Supp}(D_2)$, then $P$ is a $(D_2,S)$-integral point implies that $P$ is also a $(D_2,S)$-integral point. Therefore enlarging $S$ or $\text{Supp}(D)$ only enlarges the set of $(D,S)$-integrals points on $C(\overline{K})$.

For $\phi : C_1 \to C_2$ a finite morphism between projective curves and $P \in C_2$, write

$$\phi^* P = \sum_{Q \in \phi^{-1}(P)} e_{\phi}(Q) \cdot Q$$

where $e_{\phi}(Q) \geq 1$ is the ramification index of $\phi$ at $Q$. Furthermore, if $D = \sum n_i Q_i$ is a divisor on $C$, then we define $\phi^* D = \sum n_i \phi^* Q_i$.

**Theorem 2.1** (Distribution Relation). Let $\phi : C_1 \to C_2$ be a finite mor-phism between irreducibly smooth curves in $\mathbb{P}^N(\overline{K})$. Then for $Q \in C_1$, there is a finite set of places $S$, depending only on $\phi$ and containing all the archimedean places, such that $\lambda_{\phi^* P,v} = \lambda_{\phi,v}(\phi^* P)$ for all $v \notin S$.

**Proof.** See [3], Prop. 6.2b and note that for projective varieties the $\lambda_{\overline{\phi^* P},v}$ term is not required, and that the big-O constant is an $M_K$-bounded constant not depending on $P$ and $Q$.

**Corollary 2.2.** Let $\phi : C_1 \to C_2$ be a finite morphism between irreducibly smooth curves in $\mathbb{P}^N(\overline{K})$, let $P \in C_1(\overline{K})$, and let $D$ be an effective divisor on $C_2$ defined over $K$. Then there is a finite set of places $S$, depending only on $\phi$ and containing all the archimedean places, such that $\phi(P)$ is $S$-integral relative to $D$ if and only $P$ is $S$-integral relative to $\phi^* D$.

**Proof.** Extend $S$ so that the conclusion of Theorem 2.1 holds. Then for $D = \sum n_i Q_i$ with each $n_i > 0$ and $Q_i \in C_2(\overline{K})$, we have that

$$\lambda_{\phi^* D,v}(P) = \lambda_{D,v}(\phi(P)) = \sum n_i \lambda_{Q_i,v}(\phi(P)).$$

So $\lambda_{\phi^* D,v}(P) = 0$ if and only if $\lambda_{Q_i,v}(\phi(P)) = 0$.

**3. Main Result**

Let $E$ be an elliptic curve, $\psi : E \to E$ a morphism, and $\pi : E \to \mathbb{P}^1$ be a finite covering. A Lattès map is a rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ making the following diagram commute:

$$\begin{array}{ccc}
E & \xrightarrow{\psi} & E \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \xrightarrow{\pi} & \mathbb{P}^1
\end{array}$$

For instance, if $E$ is defined by the Weierstrass equation $y^2 = x^3 + ax^2 + bx + c$, $\psi = [2]$ is the multiplication-by-2 endomorphism on $E$, and $\pi(x,y) = x$, then

$$\phi(x) = \frac{x^4 - 2bx^2 + 2cx + b^2 - 4ac}{4x^3 + 4ax^2 + 4bx + 4c}.$$ 

Fix an elliptic curve $E$ defined over a number field $K$, and for $P \in \mathbb{P}^1(\overline{K})$, define:

$$[\phi] = \left\{ \phi : \mathbb{P}^1 \to \mathbb{P}^1 \text{ and finite covering } \pi : E \to \mathbb{P}^1 \text{ such that } \pi \circ \psi = \phi \circ \pi \right\}$$

$$\Gamma_0 = \bigcup_{\phi \in [\phi]} \phi^*(P)$$

$$\Gamma = \bigcup_{\phi \in [\phi]} \phi^*(\Gamma_0)$$

A point $Q$ is $[\phi]$-preperiodic if and only if $Q$ is $\phi$-preperiodic for some $\phi \in [\phi]$. We write $\mathbb{P}^1(\overline{K})_{[\phi]}^{-\text{preper}}$ for the set of $[\phi]$-preperiodic points in $\mathbb{P}^1(\overline{K})$.

**Theorem 3.1.** If $Q \in \mathbb{P}^1(\overline{K})$ is not $[\phi]$-periodic, then $\Gamma$ contains at most finitely many points in $\mathbb{P}^1(\overline{K})$ which are $S$-integral relative to $Q$.

**Proof.** Let $\Gamma_0'$ be the $\text{End}(E)$-submodule of $E(\overline{K})$ that is finitely generated by the points in $\pi^{-1}(P)$, and let $\Gamma' = \{ \xi \in E(\overline{K}) \mid \lambda(\xi) \in \Gamma_0' \text{ for some non-zero } \lambda \in \text{End}(E) \}$.
Then $\pi^{-1}(\Gamma) \subset \Gamma'$. Indeed, if $\pi(\xi) \in \Gamma$ is not $[\phi]$-preperiodic, then $\xi$ is non torsion and $(\phi \circ \pi)(\xi) \in \Gamma_0$ for some Lattès map $\phi$. So $(\phi \circ \pi)(\xi) \in \Gamma_0$ for some morphism $\psi : E \to E$, and this gives $(\pi \circ \psi_1)(\xi) \in \phi_1(\mathcal{P})$ for some Lattès map $\phi_1$. Therefore $\psi_1(\xi) \in \left(\pi^{-1} \circ \phi_1\right)(\mathcal{P}) = (\psi_2 \circ \pi^{-1})(\mathcal{P})$ for some morphism $\psi_2 : E \to E$. Since any morphism $\psi : E \to E$ is of the form $\psi(X) = \alpha(X) + T$ where $\alpha \in \text{End}(E)$ and $T \in E_{\text{tors}}$ (see [5, 6.19]), we find that there is a $\lambda \in \text{End}(E)$ such that $\lambda(\xi)$ is in $\Gamma_0$, the $\text{End}(E)$-submodule generated by $1$. So $\psi_2(\xi) \in \Gamma_0$ for some morphism $\psi_2 : E \to E$. Since $\lambda(\xi)$ is in $\Gamma_0$, the $\text{End}(E)$-submodule generated by $1$, we find that there is a $\lambda \in \text{End}(E)$ such that $\lambda(\xi)$ is in $\Gamma_0$, the $\text{End}(E)$-submodule generated by $1$. Hence $\pi^{-1}(\Gamma) \subset \Gamma'$.

Let $D$ be an effective divisor whose support lies entirely in $\pi^{-1}(\mathcal{Q})$, let $\mathcal{R}_D$ be the set of points in $\Gamma$ which are $S$-integral relative to $\mathcal{Q}$, and let $\mathcal{R}_D'$ be the set of points in $\Gamma'$ which are $S$-integral relative to $D$. Extending $S$ so that Theorem 2.1 holds for the map $\pi : E \to \mathbb{P}^1$, and since $\text{Supp}(D) \subset \text{Supp}(\pi \ast D)$, we have: if $\gamma \in \Gamma$ is $S$-integral relative to $\mathcal{Q}$, then $\pi^{-1}(\gamma)$ is $S$-integral relative to $D$. Therefore $\pi^{-1}(\mathcal{R}_D) \subset \mathcal{R}_D$. Now $\pi$ is a finite map and $\pi(E(\mathcal{K})) = \mathbb{P}^1(\mathcal{K})$, so to complete the proof, it suffices to show that $D$ can be chosen so that $\mathcal{R}_D'$ is finite.

From [5, Prop. 6.37], we find that if $\Lambda$ is a nontrivial subgroup of $\text{Aut}(E)$, then $E/\Lambda \cong \mathbb{P}^1$ and the map $\pi : E \to \mathbb{P}$ can be determined explicitly. The four possibilities for $\pi$, which are $\pi(x, y) = x, x^2, x^3$, or $y$ correspond respectively to the four possibilities for $\Lambda$, which are $\Lambda = \mu_2, \mu_4, \mu_6$, or $\mu_3$, which in turn depends only on the $j$-invariant of $E$ (Here, $\mu_N$ denotes the $N$th roots of unity in $\mathbb{C}$.)

First assume that $\pi(x, y) \neq y$. Since $\mathcal{Q}$ is not $[\phi]$-preperiodic, take $\xi \in \pi^{-1}(\mathcal{Q})$ to be non torsion. Then $-\xi \in \pi^{-1}(\mathcal{Q})$ since $\Lambda = \mu_2, \mu_4, or \mu_6$, and $\xi - (-\xi) = 2\xi$ is non-torsion. Taking $D = (\xi) + (-\xi)$, [11], Thm. 3.9(i) again gives that $\mathcal{R}_D'$ is finite.

Suppose that $\pi(x, y) = y$. Then $\pi(x, y) = [\xi, \xi', \xi'']$ where $\xi + \xi' + \xi'' = 0$ and $\xi$ is non-torsion since $\mathcal{Q}$ is not $[\phi]$-preperiodic. Assuming that both $\xi - \xi'$ and $\xi - \xi''$ are torsion give that $3\xi$ is torsion, and this contradicts the fact that $\xi$ is torsion. Therefore, we may assume that $\xi - \xi'$ is non-torsion. Now taking $D = (\xi) + (\xi')$, [11], Thm. 3.9(i) again gives that $\mathcal{R}_D'$ is finite. Hence $\mathcal{R}_D'$, the set of points in $\Gamma$ which are $S$-integral relative to $D$, is finite.

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