Fares Gherbi and Tarek Rouabhi

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FARES GHERBI
TAREK ROUABHI

Abstract

The main result of this note is that a finitely generated hyper–(Abelian–by–finite) group $G$ is finite–by–nilpotent if and only if every infinite subset contains two distinct elements $x, y$ such that $\gamma_n(\langle x, x^y \rangle) = \gamma_{n+1}(\langle x, x^y \rangle)$ for some positive integer $n = n(x, y)$ (respectively, $\langle x, x^y \rangle$ is an extension of a group satisfying the minimal condition on normal subgroups by an Engel group).

1. Introduction and results

Let $\mathcal{X}$ be a class of groups. Denote by $(\mathcal{X}, \infty)$ (respectively, $(\mathcal{X}, \infty)^*$) the class of groups $G$ such that for every infinite subset $X$ of $G$, there exist distinct elements $x, y \in X$ such that $\langle x, y \rangle \in \mathcal{X}$ (respectively, $\langle x, x^y \rangle \in \mathcal{X}$). Note that if $\mathcal{X}$ is a subgroup closed class, then $(\mathcal{X}, \infty) \subseteq (\mathcal{X}, \infty)^*$.

In answer to a question of Erdős, B.H. Neumann proved in [16] that a group $G$ is centre–by–finite if and only if $G$ is in the class $(\mathcal{A}, \infty)$, where $\mathcal{A}$ denotes the class of Abelian groups. Lennox and Wiegold showed in [13] that...
that a finitely generated soluble group is in the class $(N, \infty)$ (respectively, $(P, \infty)$) if and only if it is finite-by-nilpotent (respectively, polycyclic), where $N$ (respectively, $P$) denotes the class of nilpotent (respectively, polycyclic) groups. Other results of this type have been obtained, for example in [1]—[3], [4]—[6], [7], [8], [13], [14]—[16], [21], [22] and [23].

We say that a group $G$ has finite depth if the lower central series of $G$ stabilises after a finite number of steps. Thus if $\gamma_n(G)$ denotes the $n^{th}$ term of the lower central series of $G$, then $G$ has finite depth if and only if $\gamma_n(G) = \gamma_{n+1}(G)$ for some positive integer $n$. Denote by $\Omega$ the class of groups which has finite depth. Moreover, if $k$ is a fixed positive integer, let $\Omega_k$ denotes the class of groups $G$ such that $\gamma_k(G) = \gamma_{k+1}(G)$.

Clearly, any group in the class $FN$ is of finite depth, where $F$ denotes the class of finite groups. From this and the fact that $FN$ is a subgroup closed class, we deduce that finite-by-nilpotent groups belong to $(\Omega, \infty)^*$. Here we shall be interested by the converse. In [5], Boukaroura has proved that a finitely generated soluble group in the class $(\Omega, \infty)$ is finite-by-nilpotent. We obtain the same result when $(\Omega, \infty)$ is replaced by $(\Omega, \infty)^*$ and soluble by hyper-(Abelian-by-finite). More precisely we shall prove the following result.

**Theorem 1.1.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group. Then, $G$ is in the class $(\Omega, \infty)^*$ if, and only if, $G$ is finite-by-nilpotent.

Note that Theorem 1.1 improves the result of [12] which asserts that a finitely generated soluble-by-finite group whose subgroups generated by two conjugates are of finite depth, is finite-by-nilpotent.

It is clear that an Abelian group $G$ in the class $(\Omega_1, \infty)^*$ is finite. For if $G$ is infinite, then it contains an infinite subset $X = G \setminus \{1\}$. Therefore there exist two distinct elements $x, y \neq 1$ in $X$ such that $\gamma_1(\langle x, x^y \rangle) = \gamma_2(\langle x, x^y \rangle) = 1$; so $x = 1$, which is a contradiction. From this it follows that a hyper-(Abelian-by-finite) group $G$ in the class $(\Omega_1, \infty)^*$ is hyper-(finite) as $(\Omega_1, \infty)^*$ is a subgroup and a quotient closed class. But it is not difficult to see that a hyper-(finite) group is locally finite [17, Part 1, page 36]. So $G$ is locally finite. Now if $G$ is infinite, then it contains an infinite Abelian subgroup $A$ [17, Theorem 3.43]. Since $A$ is in the class $(\Omega_1, \infty)^*$, it is finite; a contradiction and $G$, therefore, is finite. As consequence of Theorem 1.1, we shall prove other results on the class $(\Omega_k, \infty)^*$.

**Corollary 1.2.** Let $k$ be a positive integer and let $G$ be a finitely generated hyper-(Abelian-by-finite) group. We have:
A condition on infinite subsets

(i) If \( G \) is in the class \((\Omega_k, \infty)^*\), then there exists a positive integer \( c = c(k) \), depending only on \( k \), such that \( G/Z_c(G) \) is finite.

(ii) If \( G \) is in the class \((\Omega_2, \infty)^*\), then \( G/Z_2(G) \) is finite.

(iii) If \( G \) is in the class \((\Omega_3, \infty)^*\), then \( G \) is in the class \( \mathcal{FN}_3^{(2)} \), where \( \mathcal{N}_3^{(2)} \) denotes the class of groups whose 2-generator subgroups are nilpotent of class at most 3.

Let \( k \) be a fixed positive integer, denote by \( \mathcal{M}, \varepsilon_k \) and \( E \) respectively the class of groups satisfying the minimal condition on normal subgroups, the class of \( k \)-Engel groups and the class of Engel groups. Using Theorem 1.1, we will prove the following results concerning the classes \((\mathcal{M}\varepsilon, \infty)^*\) and \((\mathcal{M}\varepsilon_2, \infty)^*\)

**Theorem 1.3.** Let \( G \) be a finitely generated hyper-(Abelian-by-finite) group. Then, \( G \) is in the class \((\mathcal{M}\varepsilon, \infty)^*\) if, and only if, \( G \) is finite-by-nilpotent.

Note that this theorem improves Theorem 3 of [23] (respectively, Corollary 3 of [5]) where it is proved that a finitely generated soluble group in the class \((\mathcal{CN}, \infty)^*\) (respectively, \((\mathcal{XN}, \infty))\) is finite-by-nilpotent, where \( C \) (respectively, \( X \)) denotes the class of Chernikov groups (respectively, the class of groups satisfying the minimal condition on subgroups).

**Corollary 1.4.** Let \( k \) be a positive integer and let \( G \) be a finitely generated hyper-(Abelian-by-finite) group. We have:

(i) If \( G \) is in the class \((\mathcal{M}\varepsilon_k, \infty)^*\), then there exists a positive integer \( c = c(k) \), depending only on \( k \), such that \( G/Z_c(G) \) is finite.

(ii) If \( G \) is in the class \((\mathcal{MA}, \infty)^*\), then \( G/Z_2(G) \) is finite.

(iii) If \( G \) is in the class \((\mathcal{M}\varepsilon_2, \infty)^*\), then \( G \) is in the class \( \mathcal{FN}_3^{(2)} \).

Note that these results are not true for arbitrary groups. Indeed, Golod [9] showed that for each integer \( d > 1 \) and each prime \( p \), there are infinite \( d \)-generator groups all of whose \((d-1)\)-generator subgroups are finite \( p \)-groups. Clearly, for \( d = 3 \), we obtain a group \( G \) which belongs to the class \((\mathcal{F}, \infty)^*\). Therefore, \( G \) belongs to the classes \((\Omega, \infty)^*\), \((\Omega_k, \infty)^*\), \((\mathcal{M}\varepsilon, \infty)^*\) and \((\mathcal{M}\varepsilon_k, \infty)^*\), but it is not finite-by-nilpotent.

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2. Proofs of Theorem 1.1 and Corollary 1.2

Let $E(\infty)$ the class of groups in which every infinite subset contains two distinct elements $x, y$ such that $[x, n y] = 1$ for a positive integer $n = n(x, y)$. In [15], it is proved that a finitely generated soluble group in the class $E(\infty)$ is finite-by-nilpotent. We will extend this result to finitely generated hyper-(Abelian-by-finite) groups (Proposition 2.5).

Our first lemma is a weaker version of Lemma 11 of [23], but we include a proof to keep our paper reasonably self contained.

**Lemma 2.1.** Let $G$ be a finitely generated Abelian-by-finite group. If $G$ is in the class $(\mathcal{FN}, \infty)$, then it is finite-by-nilpotent.

**Proof.** Let $G$ be a finitely generated infinite Abelian-by-finite group in the class $(\mathcal{FN}, \infty)$. Hence there is a normal torsion-free Abelian subgroup $A$ of finite index. Let $x$ be a non trivial element in $A$ and let $g$ in $G$. Then the subset $\{x^i g : i \text{ a positive integer}\}$ is infinite, so there are two positive integers $m, n$ such that $\langle x^m g, x^n g \rangle$ is finite-by-nilpotent, hence $\langle x^r, x^n g \rangle$ is finite-by-nilpotent where $r = m - n$. Thus there are two positive integers $c$ and $d$ such that $[x^r, x^c]^d = 1$. The element $x$ being in $A$ which is Abelian and normal in $G$, we have $[x^r, x^c] = [x^r, x^c]^r$; so $[x^r, x^c]^r d = 1$. Now $[x, c g]$ belongs to the torsion-free group $A$, so $[x, c g] = 1$. It follows that $x$ is a right Engel element of $G$. Since $G$ is Abelian-by-finite and finitely generated, it satisfies the maximal condition on subgroups; so the set of right Engel elements of $G$ coincides with its hypercentre which is equal to $Z_i(G)$, the $(i + 1)$-th term of the upper central series of $G$, for some integer $i > 0$ [17, Theorem 7.21]. Hence, $A \leq Z_i(G)$; and since $A$ is of finite index in $G$, $G/Z_i(G)$ is finite. Thus, by a result of Baer [10, Theorem 1], $G$ is finite-by-nilpotent. \hfill $\square$

**Lemma 2.2.** Let $G$ be a finitely generated Abelian-by-finite group. If $G$ is in the class $E(\infty)$, then it is finite-by-nilpotent.

**Proof.** Let $G$ be an infinite finitely generated Abelian-by-finite group in $E(\infty)$, and let $A$ be an Abelian normal subgroup of finite index in $G$. It is clear that all infinite subsets of $G$ contains two different elements $x, y$ such that $x A = y A$; so $y = xa$ for some $a$ in $A$ and $\langle x, y \rangle = \langle x, a \rangle$. Thus $\langle x, y \rangle$ is a finitely generated metabelian group in the class $E(\infty)$. It follows by the result of Longobardi and Maj [15, Theorem 1], that $\langle x, y \rangle$...
is finite-by-nilpotent. Hence $G$ is in the class $(\mathcal{FN}, \infty)$. Now, by Lemma 2.1, $G$ is finite-by-nilpotent; as required. □

**Lemma 2.3.** A finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$ is nilpotent-by-finite.

**Proof.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$. Since $\mathcal{E}(\infty)$ is a quotient closed class of groups and since finitely generated nilpotent-by-finite groups are finitely presented, we may assume that $G$ is not nilpotent-by-finite but every proper homomorphic image of $G$ is in the class $\mathcal{NF}$. Since $G$ is hyper-(Abelian-by-finite), $G$ contains a non-trivial normal subgroup $H$ such that $H$ is finite or Abelian; so we have $G/H$ is in $\mathcal{NF}$. If $H$ is finite then $G$ is nilpotent-by-finite, a contradiction. Consequently $H$ is Abelian and so $G$ is Abelian-by-(nilpotent-by-finite) and therefore it is (Abelian-by-nilpotent)-by-finite. Hence, $G$ is a finite extension of a soluble group; there is therefore a normal soluble subgroup $K$ of $G$ of finite index. Now, $K$ is a finitely generated soluble group in the class $\mathcal{E}(\infty)$; it follows, by the result of Longobardi and Maj [15, Theorem 1], that $K$ is finite-by-nilpotent. By a result of P. Hall [10, Theorem 2], $K$ is nilpotent-by-finite and so $G$ is nilpotent-by-finite, a contradiction. Now, the Lemma is shown. □

Since finitely generated nilpotent-by-finite groups satisfy the maximal condition on subgroups, Lemma 2.3 has the following consequence:

**Corollary 2.4.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$. Then $G$ satisfies the maximal condition on subgroups.

**Proposition 2.5.** A finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$ is finite-by-nilpotent.

**Proof.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in $\mathcal{E}(\infty)$. According to Corollary 2.4, $G$ satisfies the maximal condition on subgroups. Now, since $\mathcal{E}(\infty)$ is a quotient closed class, we may assume that every proper homomorphic image of $G$ is in $\mathcal{FN}$, but $G$ itself is not in $\mathcal{FN}$. Our group $G$ being hyper-(Abelian-by-finite), contains a non-trivial normal subgroup $H$ such that $H$ is finite or Abelian; so by hypothesis $G/H$ is in the class $\mathcal{FN}$. If $H$ is finite, then $G$ is finite-by-nilpotent, a contradiction. Consequently $H$ is Abelian and so $G$ is in the class $\mathcal{A}(\mathcal{FN})$, hence $G$ is in $(\mathcal{AF})\mathcal{N}$. Now, since $G$ satisfies the maximal condition on
subgroups, it follows from Lemma 2.2, that $G$ is in $(\mathcal{FN})\mathcal{N}$, so it is in $\mathcal{F}(\mathcal{NN})$. Consequently, there is a finite normal subgroup $K$ of $G$ such that $G/K$ is soluble. The group $G/K$, being a finitely generated soluble group in the class $\mathcal{E}(\infty)$, is in $\mathcal{FN}$, by the result of Longobardi and Maj [15, Theorem 1]. So $G$ is in the class $\mathcal{FN}$, which is a contradiction and the Proposition is shown.

The remainder of the proof of Theorem 1.1 is adapted from that of Lennox’s Theorem [11, Theorem 3]

**Lemma 2.6.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega, \infty)^*$. If $G$ is residually nilpotent, then $G$ is in the class $\mathcal{FN}$.

**Proof.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega, \infty)^*$ and assume that $G$ is residually nilpotent. Let $X$ be an infinite subset of $G$, there are two distinct elements $x$ and $y$ of $X$ such that $\langle x, x^y \rangle \in \Omega$. It follows that there exists a positive integer $k$ such that $\gamma_k(\langle x, x^y \rangle) = \gamma_{k+1}(\langle x, x^y \rangle)$. The group $\langle x, x^y \rangle$, being a subgroup of $G$, is residually nilpotent, so $\bigcap_{i \in \mathbb{N}} \gamma_i(\langle x, x^y \rangle) = 1$. Hence $\gamma_k(\langle x, x^y \rangle) = \bigcap_{i \in \mathbb{N}} \gamma_i(\langle x, x^y \rangle) = 1$. Since $\langle x, x^y \rangle = \langle [y, x], x \rangle$, $\gamma_k([y, x], x) = 1$, thus $[y, k] x = 1$. We deduce that $G$ is a finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$. It follows, by Proposition 2.5, that $G$ is in the class $\mathcal{FN}$, as required.

**Lemma 2.7.** If $G$ is a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega, \infty)^*$, then it is nilpotent-by-finite.

**Proof.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in $(\Omega, \infty)^*$. Since finitely generated nilpotent-by-finite groups are finitely presented and $(\Omega, \infty)^*$ is a quotient closed class of groups, by [17, Lemma 6.17], we may assume that every proper quotient of $G$ is nilpotent-by-finite, but $G$ itself is not nilpotent-by-finite. Since $G$ is hyper-(Abelian-by-finite), it contains a non-trivial normal subgroup $K$ such that $K$ is finite or Abelian; so $G/K$ is in $\mathcal{NF}$. In this case, $K$ is Abelian and so $G$ is in the class $A(\mathcal{NF})$ and therefore it is in the class $(AN)\mathcal{F}$. Consequently, $G$ has a normal subgroup $N$ of finite index such that $N$ is Abelian-by-nilpotent. Moreover, $N$ being a subgroup of finite index in a finitely generated group, is itself finitely generated, and so $N$ is a finitely generated Abelian-by-nilpotent group. It follows, by a result of Segal [19,
Corollary 1], that \( N \) has a residually nilpotent normal subgroup of finite index. Thus, \( G \) has a residually nilpotent normal subgroup \( H \), of finite index. Therefore, \( H \) is residually nilpotent and it is a finitely generated hyper-(Abelian-by-finite) group in the class \((\Omega, \infty)^*\). So, by Lemma 2.6, \( H \) is in the class \( \mathcal{FN} \), hence \( H \) is in the class \( \mathcal{NF} \). Thus \( G \) is in the class \( \mathcal{NF} \), a contradiction which completes the proof.

\[\square\]

Lemma 2.8. Let \( G \) be a finitely generated group in the class \((\Omega, \infty)^*\) which has a normal nilpotent subgroup \( N \) such that \( G/N \) is a finite cyclic group. Then \( G \) is in the class \( \mathcal{FN} \).

Proof. We prove by induction on the order of \( G/N \) that \( G \) is in the class \( \mathcal{FN} \). Let \( n = |G/N| \); if \( n = 1 \), then \( G = N \) and \( G \) is nilpotent. Now suppose that \( n > 1 \) and let \( q \) be a prime dividing \( n \). Since \( G/N \) is cyclic, it has a normal subgroup of index \( q \). Thus \( G \) has a normal subgroup \( H \) of index \( q \) containing \( N \). Since \( |H/N| < |G/N| \), then by the inductive hypothesis, \( H \) is in the class \( \mathcal{FN} \). Let \( T \) be the torsion subgroup of \( H \). Since \( H \) is finitely generated, \( T \) is finite. So \( H/T \) is a finitely generated torsion-free nilpotent group. Therefore, by Gruenberg [18, 5.2.21], \( H/T \) is residually a finite \( p \)-group for all primes \( p \) and hence, in particular, \( H/T \) is residually a finite \( q \)-group. But \( H \) has index \( q \) in \( G \) from which we get that \( G/T \) is residually a finite \( q \)-group [20, Exercise 10, page 17]. This means that \( G/T \) is residually nilpotent. It follows, by Lemma 2.6, that \( G/T \) is in the class \( \mathcal{FN} \). So \( G \) itself is in \( \mathcal{FN} \). \[\square\]

Proof of Theorem 1.1. Let \( G \) be a finitely generated hyper-(Abelian-by-finite) group in the class \((\Omega, \infty)^*\). Hence, by Lemma 2.7, \( G \) is in the class \( \mathcal{NF} \). Let \( K \) be a normal nilpotent subgroup of \( G \) such that \( G/K \) is finite. Since \( K \) is a finitely generated nilpotent group, it has a normal torsion-free subgroup of finite index [18, 5.4.15 (i)]. Thus, \( G \) has a normal torsion-free nilpotent subgroup \( N \) of finite index. Let \( x \) be a non-trivial element of \( G \). Since \( N \) is finitely generated, \( \langle N, x \rangle \) is a finitely generated hyper-(Abelian-by-finite) group in the class \((\Omega, \infty)^*\). Furthermore, \( \langle N, x \rangle \) is in the class \( \mathcal{FN} \). Consequently, there is a finite normal subgroup \( H \) of \( \langle N, x \rangle \) such that \( \langle N, x \rangle/H \) is nilpotent. Therefore \( \gamma_{k+1}(\langle N, x \rangle) \leq H \) for some positive integer \( k \); so \( \gamma_{k+1}(\langle N, x \rangle) \) is finite. Hence, there is a positive integer \( m \) such that \( [g, k \ x]^m = 1 \), for all \( g \in N \). Since \( [g, k \ x] \) is an element of the torsion-free group \( N \), we get that \( [g, k \ x] = 1 \). Thus, \( g \) is a right Engel element of \( G \); so \( N \subseteq R(G) \),
where $R(G)$ denotes the set of right Engel elements of $G$. Moreover, since $G$ is a finitely generated nilpotent-by-finite group, it satisfies the maximal condition on subgroups. Therefore, from Baer [17, Theorem 7.21], $R(G)$ coincides with the hypercentre of $G$ which equal to $Z_n(G)$ for some positive integer $n$. Thus $N \leq Z_n(G)$, so $Z_n(G)$ is of finite index in $G$. It follows, by a result of Baer [10, Theorem 1], that $G$ is in the class $\mathcal{FN}$.

Proof of Corollary 1.2. (i) Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega_k, \infty)^*$; from Theorem 1.1, $G$ is in the class $\mathcal{FN}$. Let $H$ be a normal finite subgroup of $G$ such that $G/H$ is nilpotent. It is clear that $G/H$ is in the class $(\Omega_k, \infty)^*$. Let $\bar{X}$ be an infinite subset of $G/H$; there are therefore two distinct elements $\bar{x} = xH$, $\bar{y} = yH$ $(x, y \in G)$ of $\bar{X}$ such that $\langle \bar{x}, \bar{y} \rangle \in \Omega_k$, so $\gamma_k(\langle \bar{x}, \bar{y} \rangle) = \gamma_{k+1}(\langle \bar{x}, \bar{y} \rangle)$. Now, since $\langle \bar{x}, \bar{y} \rangle$ is nilpotent, there is an integer $i$ such that $\gamma_i(\langle \bar{x}, \bar{y} \rangle) = 1$; so $\gamma_k(\langle \bar{x}, \bar{y} \rangle) = 1$. Since $\langle \bar{x}, \bar{y} \rangle = \langle \bar{y}, \bar{x} \rangle$, we have $\gamma_k(\langle \bar{y}, \bar{x} \rangle) = 1$ and thus $[\bar{y}, k \bar{x}] = 1$. Consequently, $G/H$ is in the class $\mathcal{E}_k(\infty)$ of groups in which every infinite subset contains two distinct elements $g, h$ such that $[g, k h] = 1$. The group $G/H$, being a finitely generated soluble group in the class $\mathcal{E}_k(\infty)$; it follows by a result of Abdollahi [2, Theorem 3], that there is an integer $c = c(k)$, depending only on $k$, such that $(G/H)/Z_c(G/H)$ is finite. By a result of Baer [10, Theorem 1], $\gamma_{c+1}(G/H) = \gamma_{c+1}(G)/H/H$ is finite; and since $H$ is finite, $\gamma_{c+1}(G)$ is finite. According to a result of P. Hall [10, 1.5], $G/Z_c(G)$ is finite.

(ii) If $G$ is in the class $(\Omega_2, \infty)^*$, then by Theorem 1.1 $G$ is finite-by-nilpotent. Therefore, $G$ has a finite normal subgroup $H$ such that $G/H$ is nilpotent. Since $G/H$ is in the class $(\Omega_2, \infty)^*$, it is in the class $\mathcal{E}_2(\infty)$. Hence, by Abdollahi [1, Theorem], $(G/H)/Z_2(G/H)$ is finite, so $\gamma_3(G/H)$ is finite. Since $H$ is finite, $\gamma_3(G)$ is finite. It follows, by P. Hall [10, 1.5], that $G/Z_2(G)$ is finite.

(iii) Now if $G$ is in the class $(\Omega_3, \infty)^*$, then by Theorem 1.1 $G$ has a finite normal subgroup $H$ such that $G/H$ is nilpotent. Since $G/H$ is in the class $(\Omega_3, \infty)^*$, it is in the class $\mathcal{E}_3(\infty)$. Hence, by Abdollahi [2, Theorem 1] $G/H$ is in the class $\mathcal{FN}_3^{(2)}$; consequently $G$ is in the class $\mathcal{FN}_3^{(2)}$. □

3. Proofs of Theorem 1.3 and Corollary 1.4

We start by showing a weaker version of Theorem 1.3:
Lemma 3.1. A finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{MN}, \infty)^*$ is finite-by-nilpotent.

Proof. Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{MN}, \infty)^*$, and let $X$ be an infinite subset of $G$. There are therefore two distinct elements $x, y$ of $X$ such that $\langle x, x^y \rangle$ is in the class $\mathcal{MN}$, so there exists a normal subgroup $N$ of $\langle x, x^y \rangle$ such that $N$ is in $\mathcal{M}$ and $\langle x, x^y \rangle / N$ is nilpotent. Now, $\gamma_{i+1}(\langle x, x^y \rangle) \leq N$ for some positive integer $i$, therefore $\gamma_{i+1}(\langle x, x^y \rangle) \leq \gamma_{i+2}(\langle x, x^y \rangle) \geq \ldots$ is an infinite descending sequence of normal subgroups of $N$; however $N$ is in $\mathcal{M}$, therefore there exists a positive integer $n \geq i + 1$ such that $\gamma_n(\langle x, x^y \rangle) = \gamma_{n+1}(\langle x, x^y \rangle)$. Hence, $G$ is in the class $(\Omega, \infty)^*$; it follows, by Theorem 1.1, that $G$ is finite-by-nilpotent. □

Lemma 3.2. A finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{ME}, \infty)^*$ is nilpotent-by-finite.

Proof. Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{ME}, \infty)^*$. Since $(\mathcal{ME}, \infty)^*$ is a closed quotient class of groups and since finitely generated nilpotent-by-finite groups are finitely presented, we may assume that $G$ is not nilpotent-by-finite, but every proper homomorphic image of $G$ is nilpotent-by-finite. Since $G$ is hyper-(Abelian-by-finite), there exists a non-trivial normal subgroup $H$ of $G$ such that $H$ is finite or Abelian; so we have $G/H$ is nilpotent-by-finite. If $H$ is finite then $G$ is nilpotent-by-finite, a contradiction. Consequently $H$ is Abelian and so $G$ is Abelian-by-(nilpotent-by-finite) and therefore it is (Abelian-by-nilpotent)-by-finite. Hence, $G$ is a finite extension of a soluble group. Let $K$ be a normal soluble subgroup of $G$ of finite index. Clearly, $K$ is in $(\mathcal{ME}, \infty)^*$, and since all soluble Engel group coincides with its Hirsch-Plotkin radical which is locally nilpotent [17, Theorem 7.34], we deduce that $K$ is in the class $(\mathcal{MN}, \infty)^*$; it follows by Lemma 3.1 that $K$ is finite-by-nilpotent. According to a result of P. Hall [10, Theorem 2], $K$ is nilpotent-by-finite. Thus, $G$ is nilpotent-by-finite, a contradiction. The proof is now complete. □

Since finitely generated nilpotent-by-finite groups satisfy the maximal condition on subgroups, Lemma 3.2 has the following consequence:

Corollary 3.3. Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{ME}, \infty)^*$. Then $G$ satisfies the maximal condition on subgroups.
Proof of Theorem 1.3. It is clear that all finite-by-nilpotent groups are in the class \((\mathcal{MN}, \infty)^*\). Conversely, let \(G\) be a finitely generated hyper-(Abelian-by-finite) group in \((\mathcal{ME}, \infty)^*\). According to Corollary 3.3, \(G\) satisfies the maximal condition on subgroups. Since Engel groups satisfying the maximal condition on subgroups are nilpotent [18, 12.3.7], we deduce that \(G\) is in the class \((\mathcal{MN}, \infty)^*\). It follows, by Lemma 3.1, that \(G\) is in the class \(\mathcal{FN}\); as required.

Proof of Corollary 1.4. (i) Let \(G\) be a finitely generated hyper-(Abelian-by-finite) group in the class \((\mathcal{ME}_k, \infty)^*\); from Theorem 1.3, \(G\) is in the class \(\mathcal{FN}\). Let \(N\) be a normal finite subgroup of \(G\) such that \(G/N\) is nilpotent. Since \(G/N\) is nilpotent and finitely generated, its torsion subgroup \(T/N\) is finite, so \(T\) is finite and \(G/T\) is a torsion-free nilpotent group. Clearly, the property \((\mathcal{ME}_k, \infty)^*\) is inherited by \(G/T\), and since \(G/T\) is torsion-free and soluble, it belongs to \((\mathcal{E}_k, \infty)^*\) [17, Theorem 5.25]. Let \(\bar{X}\) be an infinite subset of \(G/T\); there are therefore two distinct elements \(\bar{x} = xT\), \(\bar{y} = yT\) \((x, y \in G)\) of \(\bar{X}\) such that \(\langle \bar{x}, \bar{x}^{\bar{y}} \rangle\) is a \(k\)-Engel group. Since \(\langle \bar{x}, \bar{x}^{\bar{y}} \rangle = \langle [\bar{y}, \bar{x}], \bar{x} \rangle\), we have \([\bar{y}, k_{k+1} \bar{x}] = [[\bar{y}, \bar{x}], k \bar{x}] = 1\). Hence, \(G/T\) is in the class \(\mathcal{E}_{k+1}(\infty)\). The group \(G/T\), being a finitely generated soluble group in the class \(\mathcal{E}_{k+1}(\infty)\); it follows by a result of Abdollahi [2, Theorem 3], that there is an integer \(c = c(k)\), depending only on \(k\), such that \((G/T)/Z_c(G/T)\) is finite. By a result of Baer [10, Theorem 1], \(\gamma_{c+1}(G/T) = \gamma_{c+1}(G)T/T\) is finite; and since \(T\) is finite, \(\gamma_{c+1}(G)\) is finite. According to a result of P. Hall [10, 1.5], \(G/Z_c(G)\) is finite.

(ii) If \(G\) is in the class \((\mathcal{MA}, \infty)^* = (\mathcal{ME}_1, \infty)^*\), then by Theorem 1.3, \(G\) is finite-by-nilpotent. We proceed as in (i) until we obtain that \(G/T\) is in the class \(\mathcal{E}_2(\infty)\). Hence, by Abdollahi [1, Theorem], \((G/T)/Z_2(G/T)\) is finite, so \(\gamma_3(G/T)\) is finite. Since \(T\) is finite, \(\gamma_3(G)\) is finite. It follows, by P. Hall [10, 1.5], that \(G/Z_2(G)\) is finite.

(iii) Now if \(G\) is in the class \((\mathcal{ME}_2, \infty)^*\), we proceed as in (i) until we obtain that \(G/T\) is in the class \(\mathcal{E}_3(\infty)\). Hence, by Abdollahi [2, Theorem 1] \(G/T\) is in the class \(\mathcal{FN}_3^2\); consequently \(G\) is in the class \(\mathcal{FN}_3^2\).

References

[1] A. Abdollahi – Finitely generated soluble groups with an Engel condition on infinite subsets, Rend. Sem. Mat. Univ. Padova 103 (2000), p. 47–49.

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[2] ______, Some Engel conditions on infinite subsets of certain groups, *Bull. Austral. Math. Soc.* 62 (2000), p. 141–148.

[3] A. Abdollahi & B. Taeri – A condition on finitely generated soluble groups, *Comm. Algebra* 27 (1999), p. 5633–5638.

[4] A. Abdollahi & N. Trabelsi – Quelques extensions d’un problème de Paul Erdos sur les groupes, *Bull. Belg. Math. Soc.* 9 (2002), p. 205–215.

[5] A. Boukaroura – Characterisation of finitely generated finite-by-nilpotent groups, *Rend. Sem. Mat. Univ. Padova* 111 (2004), p. 119–126.

[6] C. Delizia, A. H. Rhemtulla & H. Smith – Locally graded groups with a nilpotence condition on infinite subsets, *J. Austral. Math. Soc. (series A)* 69 (2000), p. 415–420.

[7] G. Endimioni – Groups covered by finitely many nilpotent subgroups, *Bull. Austral. Math. Soc.* 50 (1994), p. 459–464.

[8] ______, Groups in which certain equations have many solutions, *Rend. Sem. Mat. Univ. Padova* 106 (2001), p. 77–82.

[9] E. S. Golod – Some problems of Burnside type, *Amer. Math. Soc. Transl. Ser. 2* 84 (1969), p. 83–88.

[10] P. Hall – Finite-by-nilpotent groups, *Proc. Cambridge Philos. Soc.* 52 (1956), p. 611–616.

[11] J. C. Lennox – Finitely generated soluble groups in which all subgroups have finite lower central depth, *Bull. London Math. Soc.* 7 (1975), p. 273–278.

[12] ______, Lower central depth in finitely generated soluble-by-finite groups, *Glasgow Math. J.* 19 (1978), p. 153–154.

[13] J. C. Lennox & J. Wiegold – Extensions of a problem of Paul Erdos on groups, *J. Austral. Math. Soc. Ser. A* 31 (1981), p. 459–463.

[14] P. Longobardi – On locally graded groups with an Engel condition on infinite subsets, *Arch. Math.* 76 (2001), p. 88–90.

[15] P. Longobardi & M. Maj – Finitely generated soluble groups with an Engel condition on infinite subsets, *Rend. Sem. Mat. Univ. Padova* 89 (1993), p. 97–102.
[16] B. H. Neumann – A problem of Paul Erdos on groups, *J. Austral. Math. Soc. ser. A* 21 (1976), p. 467–472.

[17] D. J. S. Robinson – *Finiteness conditions and generalized soluble groups*, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[18] —– *A course in the theory of groups*, Springer-Verlag, Berlin, Heidelberg, New York, 1982.

[19] D. Segal – A residual property of finitely generated abelian by nilpotent groups, *J. Algebra* 32 (1974), p. 389–399.

[20] —– *Polycyclic groups*, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1984.

[21] B. Taeri – A question of P. Erdos and nilpotent-by-finite groups, *Bull. Austral. Math. Soc.* 64 (2001), p. 245–254.

[22] N. Trabelsi – Finitely generated soluble groups with a condition on infinite subsets, *Algebra Colloq.* 9 (2002), p. 427–432.

[23] —– Soluble groups with many 2-generator torsion-by-nilpotent subgroups, *Publ. Math. Debrecen* 67/1-2 (2005), p. 93–102.