EFFECTIVE ACTION AND EXACT GEOMETRY IN
CHIRAL GAUGED WZW MODELS †

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ABSTRACT

Following recent work on the effective quantum action of gauged WZW models, we suggest such an action for chiral gauged WZW models which in many respects differ from the usual gauged WZW models. Using the effective action we compute the conformally exact expressions for the metric, the antisymmetric tensor, and the dilaton fields in the \( \sigma \)-model arising from a general chiral gauged WZW model. We also obtain the general solution of the geodesic equations in the exact geometry. Finally we consider in some detail a three dimensional model which has certain similarities with the three dimensional black string model.

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1. Introduction

There are numerous solutions to string theory, all corresponding to some conformal field theory. One wants to use such solutions for phenomenological considerations. The traditional and older approach was to consider string solutions in manifolds of the type $M^4 \times K$, where $M^4$ denotes the four dimensional Minkowski space-time represented by a free field theory with central charge $c = 4$ (or $\hat{c} = 4$) and $K$ denotes some internal space represented by a conformal field theory of appropriate central charge. Therefore in such constructions only the internal part requires a non-trivial conformal field theory and strings propagate in flat space-times.

However, to describe the very early Universe with a string inspired cosmological model or to use string theory to shed light into the singularities of general relativity a non-trivial conformal field theory is required to describe the curved space-time the strings propagate on. In a bosonic string theory it is very interesting to study the massless excitations of the string, namely the graviton, the axion and the massless scalar because they govern the geometry of space-time. In a two dimensional $\sigma$-model they are represented by a symmetric tensor (the metric), its antisymmetric partner (the axion), and the dilaton (the scalar). These fields obey a set of generalized Einstein’s equations, the beta functions equations (for instance see [1]), which follow by requiring that the $\sigma$-model is conformally invariant.

The older avenue [2] that was followed to solve these equations was to specialize to the solutions that have special symmetries, hoping that the equations will become solvable. The main problem of this approach is that the general form of the beta function equations is unknown and it is determined order by order in perturbation theory. Therefore with the old approach it is very hard to find the exact solution to all orders in perturbation theory, except in some very special circumstances, i.e. for instance plane fronted solutions [3].

In order to resolve these problems exact conformal field theories in the form of coset models based on non-compact groups, i.e. $G_{-k}/H_{-k}$, $k$ being the central extension of the current algebra, were introduced in [4] as exact string theories. Based on the equivalence of the cosets models with the gauged WZW models [5] [6] the authors of [4] argued that after integrating out the gauge fields a $\sigma$-model arises in $\text{dim}(G/H)$ space-time dimensions. The signature of the resulting space-time is intimately connected to the properties of the group $G$ and the subgroup $H$. In [7] the coset model $SU(2)_k/U(1)$ was considered in a $\sigma$-model approach to classical parafermions, and in [8] the coset model $SL(2, \mathbb{R})_{-k}/\mathbb{R}$ was found to have, in the semi-classical limit ($k \to \infty$), a two dimensional black hole.
interpretation. The latter discovery led to a flurry of activity and many other solutions were found, corresponding to black holes, black strings, and other intricate gravitational singularities, or corresponding to cosmological solutions [9] [10] [11] [12] [13] [14] [15]. All of these solutions satisfy the perturbative equations for conformal invariance up to one loop in perturbation theory (to the leading order in the 1/k expansion). However, the interest in these models stems from the fact that conformal invariance is an exact symmetry and therefore there must be a way to compute the fields in the σ-model to all orders in the expansion parameter 1/k. For the metric and dilaton fields this was achieved for the coset $SL(2, \mathbb{R})_k/\mathbb{R}$ in [16], and for the general gauged WZW model in [17], by following an algebraic Hamiltonian approach to gauged WZW models. With this method the background fields in the σ-model corresponding to many abelian and non-abelian cosets were explicitly computed [16] [17] [18] [19], thus providing the first examples of string theory solutions with non-trivial dependence in the expansion parameter. Using the Hamiltonian approach type-II supersymmetric and heterotic string models in curved space-time have also been discussed [17] [19] (For the type-II superstring based on $SL(2, \mathbb{R})/\mathbb{R}$ see also [20]). It has been shown that the background corresponding to the simplest case $SL(2, \mathbb{R})_k/\mathbb{R}$ verifies the beta function equations in string perturbation theory up to three [21] and four [20] loops (five loops in the type-II supersymmetric case [21]), up to field redefinitions.

To compute the axion with the Hamiltonian method seems difficult. To overcome this problem and to understand the exact results obtained with the Hamiltonian method from a field theoretical point of view, an effective action for gauged WZW models which incorporates all the quantum effects in the σ-model was recently suggested [22] [23]. Using this effective action the exact metric, antisymmetric tensor and dilaton fields in the general σ-model were obtained in [23]. The exact metric and dilaton fields one derives are identical [22] [23] [24] to those obtained with the Hamiltonian method. However, for the antisymmetric tensor the results are totally new [23]. In particular, explicit expressions for the cases of the three dimensional $SL(2, \mathbb{R}) \otimes \mathbb{R}/\mathbb{R}$ (black string) and $SO(2, 2)/SO(2, 1)$ coset models were given in [23].

In this paper we consider another class of exactly conformal models, the so called chiral gauged WZW (CGWZW) models which were introduced in [25]. These models have similarities but also major differences with the usual gauged WZW models which will be pointed out in the appropriate places. This paper is organized as follows: In section 2 we discuss general properties of the CGWZW models and we obtain the semiclassical expressions of the various fields in the σ-model. We also explain in detail how to
solve the particle geodesic equations for the general semi-classical metric. In section 3 we present an effective quantum action for CGWZW models by making correspondence with the analogous situation for gauged WZW models. Using this action the exact expressions for the various fields in the $\sigma$-model and the particle geodesics in the exact metric are obtained. In section 4 we apply the mechanism developed in the previous sections to the case of the three dimensional model with $G = SL(2, \mathbb{R})$ and $H = SO(1,1)$ (It will be explained shortly why this is a three dimensional and not a two dimensional model). We find that it has certain similarities with the three dimensional black string. Finally, we end this paper in section 5 with concluding remarks and discussion.

2. Chiral gauged WZW models

The two dimensional action for the general CGWZW model is \[ S_{cgwzw} = -k I_0(g) - k I_1(g, A_+, A_-), \] \[ (2.1) \]
where $I_0(g)$ is the WZW action \[ I_0(g) = \frac{1}{8\pi} \int_M d^2\sigma Tr(\partial_+ g^{-1} \partial_- g) + \frac{1}{24\pi} \int_B Tr(g^{-1} dg)^3, \] \[ (2.2) \]
and

\[ I_1(g, A_+, A_-) = \frac{1}{4\pi} \int_M d^2\sigma Tr(A_- \partial_+ gg^{-1} - A_+ g^{-1} \partial_- g + A_- g A_+ g^{-1}). \] \[ (2.3) \]
In the above $g(\sigma^+, \sigma^-)$ is a group element of $G$, and $A_\pm(\sigma^+, \sigma^-)$ are the gauge fields associated with the subgroup $H$ of $G$. The gauge fields $A_+$ and $A_-$ may belong to two different subgroups of $G$, but for simplicity we will only consider the case where the subgroup is the same. It is important to notice that a term of the type $Tr(A_+ A_-)$ is absent in (2.3) in contrast with the cases of vectorially (and axially) gauged WZW models \[ [5] \] \[ [10] \] or deformed gauged WZW models \[ [11] \], which we will collectively call gauged WZW models.

The action (2.1) is invariant under the following gauge-type transformations

\[ g \rightarrow \Lambda_-^{-1} g \Lambda_+, \quad A_\pm \rightarrow \Lambda_\pm^{-1} (A_\pm - \partial_\pm) \Lambda_\pm, \] \[ (2.4) \]
where \( \Lambda^\pm = \Lambda_\pm(\sigma^\pm) \in H \). We see that the parameters of the gauge transformation are functions of only \( \sigma^+ \) or \( \sigma^- \).\footnote{This is a reason for the terminology chiral gauged WZW models. Also notice that \( I_0(g) \) by itself is not invariant under the transformation \( g \rightarrow \Lambda_-^{-1} g \Lambda_+ \). For this to be true \( \Lambda_+ \) (\( \Lambda_- \)) must be a function of \( \sigma^- \) (\( \sigma^+ \)), i.e. \( \Lambda_\pm = \Lambda_\pm(\sigma^\mp) \).} Consequently this gauge symmetry cannot be used to remove degrees of freedom, by the usual procedure of gauge fixing. This is again in contrast with the case of gauged WZW models where one gauge fixes \( \text{dim}(H) \) variables due a gauge symmetry of the type \((2.4)\), but with gauge transformation parameters which are functions of both variables \( \sigma^+ \) and \( \sigma^- \).

The classical equations of motion for the CGWZW action \((2.1)\) follow by varying the \( A_-^\pm, A_+^\pm \) and \( g \). One then obtains the following equations

\[
(D_+ gg^{-1})_H = 0, \quad (D^-_L (D_+ gg^{-1}))_{G/H} = 0 \tag{2.5}
\]

\[
(g^{-1} D_- g)_H = 0, \quad (D^-_R (g^{-1} D_+ g))_{G/H} = 0
\]

\[
\partial_- A_+ = \partial_+ A_- = 0,
\]

where the subscripts \( H, G/H \) imply a projection to the \( H \)-subspace or the \( G/H \)-subspace.

The covariant derivatives are defined as

\[
D_+ g = \partial_+ g + g A_+, \quad D_- g = \partial_- g - A_- g
\]

\[
D_+^R = \partial_+ - [A_+, \quad ], \quad D_-^L = \partial_- - [A_-, \quad ]. \tag{2.6}
\]

In order to obtain the \( \sigma \)-model action from the action \((2.1)\) one needs to integrate out the gauge fields \( A_\pm \). This integration is easy to perform since the gauge fields appear mostly quadratically in the action \((2.1)\), and the corresponding measure in the functional path integral is just the flat measure \( dA_+ dA_- \). To eliminate the gauge fields through the equations of motion \((2.5)\), one has to solve them for \( A_\pm \). To do that and for further convenience it is useful to introduce a set of matrices \( \{ t_A \} \) in the Lie algebra of \( G \) which obey \( Tr(t_A t_B) = \eta_{AB} \), \( [t_A, t_B] = if_{AB}^C t_C \), where \( \eta_{AB} \) is the Killing metric and \( f_{AB}^C \) are the structure constants of the Lie algebra of \( G \). The subset of matrices belonging to the Lie algebra of the subgroup \( H \) will be denoted by \( \{ t_a \} \) with lower case subscripts or superscripts. Then we define the following quantities

\[
L^H_\pm = (g^{-1} \partial_\pm g)_H , \quad L^A_\mu \partial_\pm X^\mu = Tr(g^{-1} \partial_\pm g t^A) \]

\[
R^H_\pm = (- \partial_\pm gg^{-1})_H , \quad R^A_\mu \partial_\pm X^\mu = -Tr(\partial_\pm gg^{-1} t^A) \tag{2.7}
\]

\[
C_{ab} = Tr(t_a g t_b g^{-1}) ,
\]
where $X^\mu$, $\mu = 0, 1, \ldots, d - 1$ are $d = \text{dim}(G)$ independent parameters in $g(\sigma^+, \sigma^-) \in G$ which will become the string coordinates in the $\sigma$-model. In contrast the $\sigma$-model arising from the corresponding gauged WZW model would have $\text{dim}(G/H)$ string coordinates because in that case the gauge symmetry enables us to gauge fix $\text{dim}(H)$ parameters in $g(\sigma^+, \sigma^-)$. The solution of the classical equations of motion (2.5) for $A^a_\pm \equiv \text{Tr}(t^a A_\pm)$ is

$$A^a_+ = (C^{-1})^a b R^b_\mu \partial_+ X^\mu, \quad A^a_- = L^b_\mu (C^{-1})_b^a \partial_- X^\mu. \quad (2.8)$$

Substitution of these expressions back into (2.1) gives (to the leading order in the $1/k$ expansion) the following $\sigma$-model type of action

$$S_\sigma = \frac{k}{8\pi} \int d^2 \sigma \left( G_{\mu\nu} + B_{\mu\nu} \right) \partial_- X^\mu \partial_+ X^\nu, \quad (2.9)$$

where the explicit forms for the semi-classical metric $G_{\mu\nu}$ and the semi-classical antisymmetric tensor (axion) $B_{\mu\nu}$ are

$$G_{\mu\nu} = g_{\mu\nu} + (C^{-1})_{ab} L^a_\mu R^b_\nu, \quad (2.10)$$

$$B_{\mu\nu} = b_{\mu\nu} + (C^{-1})_{ab} L^a_\mu R^b_\nu.$$

The brackets denote symmetrization (when curly) or antisymmetrization of the appropriate indices. The $g_{\mu\nu}$ and $b_{\mu\nu}$ are the parts of the metric and axion due to the kinetic and Wess-Zumino terms in $I_0(g)$ respectively. It can easily be shown that

$$g_{\mu\nu} = L^A_\mu L^B_\nu \eta_{AB} = R^A_\mu R^B_\nu \eta_{AB}, \quad (2.11)$$

and that $b_{\mu\nu}$ are the components of a 2-form defined through the relation $h = -\frac{3}{2} db$, where the components of the 3-form $h$ are

$$h_{\mu\nu\rho} = \frac{1}{2} f_{ABC} L^A_\mu L^B_\nu L^C_\rho = -\frac{1}{2} f_{ABC} R^A_\mu R^B_\nu R^C_\rho. \quad (2.12)$$

However, the most efficient way to compute $g_{\mu\nu}$ and $b_{\mu\nu}$ is to use the Polyakov-Wiegman formula [27] repeatedly until the Wess-Zumino term in $I_0(g)$ vanishes identically. In order to preserve conformal invariance in the $\sigma$-model approach we need to take into account the dilaton field [28] and to satisfy the perturbative beta function equations [1]. Up to one loop in perturbation theory the dilaton can be identified as the finite part of the determinant of the matrix we obtain by integrating out the gauge fields [10][29] (see also [30] for further
clarifications). Therefore in our case the semi-classical \((k \to \infty)\) expression for the dilaton is

\[
\Phi(X) = \ln(\det C) + \text{const.} .
\]  

\text{(2.13)}

Consequently the total \(\sigma\)-model action which is conformally invariant up to one loop in perturbation theory takes the form

\[
S = S_{\sigma} - \frac{1}{8\pi} \int_M d^2\sigma \sqrt{\gamma} R^{(2)}(\gamma) \Phi(X) ,
\]  

\text{(2.14)}

where \(\gamma, R^{(2)}(\gamma)\) are the determinant of the world-sheet metric and the world-sheet curvature respectively. The infinitesimal part of the determinant \(\det(C)\), combines with the Haar measure for the group \(G\) (which together with the flat measure \(dA_+dA_-\) for the gauge fields provides the correct measure in the path integral for the action \((2.1)\)) to give the correct measure for the \(\sigma\)-model which is none other but \(\sqrt{-G}\), where \(G = \det(G_{\mu\nu})\). Namely the following relation must be true

\[
(\text{Haar})/\det(C) = \sqrt{-G} .
\]  

\text{(2.15)}

Furthermore because of the identification \((2.13)\) one can rewrite \((2.15)\) as

\[
e^\Phi \sqrt{-G} = (\text{Haar}) .
\]  

\text{(2.16)}

One now notices that the right hand side of the previous relation is purely group theoretical and therefore one expects that \((2.16)\) will be true even when we include all the \(1/k\) corrections. Namely, although \(G_{\mu\nu}\) and \(\Phi\) would be nontrivial functions of \(k\) the following combination would remain \(k\)-independent

\[
e^\Phi \sqrt{-G} \ (\text{any } k) = e^\Phi \sqrt{-G} \ (\text{at } k = \infty) .
\]  

\text{(2.17)}

We will prove \((2.16)\) and \((2.17)\) for any CGWZW model at the end of section 3. Similar relations to \((2.16), (2.17)\) hold for the \(\sigma\)-model arising from a general gauged WZW model as well. These were conjectured in \([29]\) for the abelian \(SL(2, \mathbb{R})_{-k}/\mathbb{R}\) coset case, and in \([10]\) where they were generally formulated for any gauged WZW model, by using the path integral measure argument that led to \((2.16), (2.17)\). Subsequently their validity was explicitly checked for many abelian and non-abelian cases in \([11],[17],[18],[19]\) and proved generally for any gauged WZW model in \([24]\).
As we have already mentioned the string coordinates $X^\mu$ in the $d$-dimensional $\sigma$-model arising from a general CGWZW model are functions of the $d = \text{dim}(G)$ parameters in the group element $g(\sigma^+, \sigma^-) \in G$. Therefore the ranges in which they take values are completely determined by the group theory for $G$ and in that sense the $X^\mu$'s are global coordinates. In the case of gauged WZW models finding global coordinates was a rather delicate procedure. In that case the corresponding $\sigma$-model depends on $\text{dim}(G/H)$ string coordinates which are the $\text{dim}(G/H)$ independent $H$-invariant combinations one can form out of the $\text{dim}(G)$ parameters in $g(\sigma^+, \sigma^-)$. The necessary techniques for finding the global space in this case were developed and applied in various examples in [31][17][19].

Having global coordinates is sometimes not sufficient to get a feeling for the geometry; one also needs to know the behavior of the particle geodesics by solving the usual geodesic equations $\ddot{X}^\mu + \Gamma^\mu_{\nu\rho} \dot{X}^\nu \dot{X}^\rho = 0$. However, these equations may seem completely unmanageable if the metric that emerges from (2.10) is complicated. To get around this problem we first, as in [31], dimensionally reduce the CGWZW action (2.1) by taking all fields to be functions of only $\tau$, instead of $\tau$ and $\sigma$. This corresponds to a string shrunk to a point particle. Therefore the action we consider is

$$S = \frac{-k}{4\pi} \int d\tau \text{Tr} \left( \frac{1}{2} \partial_\tau g^{-1} \partial_\tau g + a_- \partial_\tau gg^{-1} - a_+ g^{-1} \partial_\tau g + a_- ga_+ g^{-1} \right),$$

(2.18)

where $g(\tau) \in G$ is a group element and $a_\pm(\tau)$ are two gauge potentials in the Lie algebra of $H$. Two gauge potentials are needed for our purposes. The model is invariant for rigid ($\tau$ and $\sigma$ independent) gauge transformations of the form (2.4). This was expected since already for the full two dimensional action (2.1) the parameters of the gauge transformation were functions of only $\sigma^+$ or only $\sigma^-$. Consider the equations of motion

$$\left(g^{-1} D_- g\right)_H = 0 = \left(D_+ gg^{-1}\right)_H, \quad D_+^R(g^{-1} D_- g) = 0, \quad \dot{a}_+ = \dot{a}_- = 0,$$

(2.19)

where we have defined the “covariant” derivatives on the worldline $D_+ g = \dot{g} + ga_+, D_- g = \dot{g} - a_- g, D_+^R = \partial_\tau - [a_+, \cdot]$. The solution of these equations for $g(\tau)$ will provide the required geodesics by virtue of the fact that $g(\tau)$ contains all particle coordinates $X^\mu(\tau)$. From (2.19) one can see that $a_\pm(\tau) = \alpha_\pm$, where $\alpha_\pm$ are two constant matrices in the Lie algebra of $H$. The first and third equation yield the equation $\dot{p} = [\alpha_+, p]$, where $p(\tau) \equiv (g^{-1} D_- g)_{G/H}$, whose solution is $p(\tau) \equiv \exp(\alpha_+ \tau)p_0 \exp(-\alpha_+ \tau)$, with $p_0$ a constant
matrix in the Lie algebra of $G/H$. From the definition of the matrix $p(\tau)$ and the first equation one determines $g(\tau)$ to be

$$g(\tau) = \exp (\alpha_- \tau) \, g_0 \, \exp ( (\alpha_+ + p_0) \tau) \, \exp (-\alpha_+ \tau),$$

(2.20)

where $g_0$ is a constant group element that characterizes the initial conditions. Finally, replacing this form into the remaining second equation in (2.19) yields a constraint among the constants of integration which completely determines the constant matrix $\alpha_-$ in terms of the constant matrices $g_0$, $\alpha_+$ and $p_0$

$$\alpha_- = - (g_0 (\alpha_+ + p_0) g_0^{-1})_H.$$

(2.21)

The number of independent parameters in (2.20) is: $\dim(G)$ parameters from $g_0$, plus $\dim(H)$ parameters from $\alpha_+$, plus $\dim(G/H)$ parameters from $p_0$, giving a total number of $2 \dim(G)$ parameters. This is precisely the number of initial positions $X^\mu(0)$ and velocities $\dot{X}^\mu(0)$ needed for the general geodesic in $\dim(G)$ dimensions. Therefore (2.20), with the condition (2.21), is the general geodesic solution. The Lagrangian defined in (2.18) is rewritten in terms of the line element $(ds/d\tau)^2 = \frac{k}{8\pi} \dot{X}^\mu \dot{X}^\nu G_{\mu\nu}(X)$, because all the other string modes drop out in the point particle limit. Therefore, if we substitute the solution (2.20) in the Lagrangian defined in (2.18) we find the value of $(ds/d\tau)^2$ for the geodesic solution. This gives

$$(\frac{ds}{d\tau})^2 = \frac{k}{8\pi} \text{Tr}(p_0^2 - \alpha_-^2).$$

(2.22)

Now by choosing the various constant matrices we have control on whether the geodesic is light-like, time-like or space-like.

The model with $G = SL(2, \mathbb{R})$ and $H = SO(1,1)$ is perhaps the simplest model one may consider. We will discuss it in section 4 where we will derive the corresponding conformally exact metric, antisymmetric tensor and dilaton fields using the methods of section 3.
3. The effective action for chiral gauged WZW models

The effective quantum action for any field theory is derived by introducing sources and then applying a Legendre transform \[32\]. The effective action, which is then used as a classical field theory, incorporates all the higher loop effects. In this section we suggest such an effective action for the CGWZW models. The main idea we follow was developed for gauged WZW models in \[22\][23].

It is useful to make a change of variables for the gauge fields \(A_\pm = \partial_\pm h_\pm^{-1}\), where \(h_\pm(\sigma^+, \sigma^-) \in H\). After picking up a determinant and an anomaly from the measure, the path integral is rewritten with a new form for the action \[27\][6][25]

\[
S_{\text{cgwzw}} = -kI_0(h_+^{-1}gh_+) + (k - 2g_H)I_0(h_-^{-1}) + (k - 2g_H)I_0(h_+) ,
\]

(3.1)

which is gauge invariant under \(g \rightarrow \Lambda_+^{-1}g\Lambda_+\) and \(h_\pm \rightarrow \Lambda_\pm^{-1}h_\pm\), where \(\Lambda_\pm = \Lambda_\pm(\sigma^\pm)\) as in \(2.4\). The new path integral measure is the Haar group measure \(Dg \ Dh_+ \ Dh_-\). The action (3.1) is similar to the classical WZW action (2.2): the first term is appropriate for the group \(G\) with central extension \((-k)\), and the second and third terms are appropriate for the subgroup \(H\) with central extension \((k - 2g_H)\). Defining the new fields \(g' = h_-^{-1}gh_+, h' = h_-^{-1}, h'' = h_+\) and taking advantage of the properties of the Haar measure, we can rewrite the measure and action in decoupled form \(Dg' \ Dh' \ Dh''\) and \(S = -kI_0(g') + (k - 2g_H)(I_0(h') + I_0(h''))\). This decoupled form emphasizes the close connection to the WZW path integral, and gives us a clue for how to guess the effective quantum action.

However, \(g', h', h''\) are not really decoupled, since we must consider sources coupled to the original fields. Indeed, to derive the quantum effective action one must introduce source terms and perform a Legendre transformation. Since these coupled \(g', h', h''\) integrations are not easy to perform, we will introduce, as in the case of gauged WZW models \[33\], sources only for the gauge invariant combinations \(g', h', h''\). Since for each one of these fields the action is that of a WZW model the effect is a shift in \(k\) (as it was argued in \[22\] based on the perturbative analysis of \[34\][35]), which however is different in the various terms in \(3.1\). For the first term, \((-k) \rightarrow (-k + g_G)\), and for the second and third term \((k - 2g_H) \rightarrow (k - 2g_H) + g_H = k - g_H\), where \(g_G, g_H\) are the dual Coxeter numbers for the group \(G\) and the subgroup \(H\). Therefore the effective action for the CGWZW models we suggest, is

\[
S_{\text{cgwzw}}^{\text{eff}} = (-k + g_G)I_0(h_-^{-1}gh_+) - (-k + g_H)I_0(h_-^{-1}) - (-k + g_H)I_0(h_+) .
\]

(3.2)
The similar effective action for the vectorially gauged WZW models can be obtained by omitting the third term and replacing $h^{-1}$ by $h^{-1}h_+$ in the second term [33][22][23] (For axial gauged WZW models see [23]). We should point out that in (3.2) we have neglected possible field renormalizations [36] for the group elements $g, h$ ± since they give rise to non-local terms in the $\sigma$-model action [24]. The action (3.2) may now be rewritten back in terms of classical fields $g, A_+, A_-$ by using the definitions given before and the Polyakov-Wiegman formula [27]. We obtain

$$S_{e\text{ff}_{\text{cgwzw}}} = (-k + gG)[I_0(g) + I_1(g, A_+, A_-) + \frac{gG - gH}{-k + gG}I_2(A_+, A_-)],$$

$$I_2(A_+, A_-) = I_0(h^{-1}) + I_0(h_+) .$$

Note that $I_2(A_+, A_-)$ is gauge invariant. Our proposed effective action differs from the purely classical action (2.1) by the overall renormalization $(-k + gG)$ and by the additional term proportional to $(gG - gH)$. In the large $k$ limit (which is equivalent to small $\hbar$) the effective quantum action reduces to the classical action, as it should. This is not yet the end of the story, because what we are really interested in is the effective action for the $\sigma$-model after the gauge fields are integrated out. At the outset, with the classical action, the path integral over $A_\pm$ was purely Gaussian, and therefore it could be performed by simply substituting the classical solutions for $A_\pm = A_\pm(g)$ back into the action. This integration also introduces an anomaly which can be computed exactly as a one loop effect. The anomaly gives the dilaton piece to be added to the effective action. In order to obtain the exact dilaton we need to perform the $A_\pm$ integrals with the effective action, not the classical one. However, in (3.3) the terms in $I_2(A_+, A_-)$ are non-local in the $A_\pm$ (although they are local in $h_\pm$). For instance, $I_0(h_+) \sim \int Tr(A_+ \partial_- h_+ h_+^{-1}) + \ldots$, and we cannot write $\partial_- h_+ h_+^{-1}$ as a local function of $A_+$. Furthermore, if $H$ is non-abelian $I_2(A_+, A_-)$ has additional non-linear terms. So, if we believe that the quantum effective action is indeed (3.3), then the effective $\sigma$-model action we are seeking seems to be generally non-local even in the abelian case. This was also true for the effective action for gauged WZW models, as it was discussed in [22][23]. As in [23], we can isolate the local contribution to the $\sigma$-model by concentrating on the zero mode sector of (3.3). To restrict ourselves to the zero mode sector we employ the same dimensional reduction technique as before by taking all the fields as functions of only $\tau$ (i.e. worldline rather than world-sheet). This extracts the low energy point particle content of the string and it captures the entire local contribution to the $\sigma$-model. The derivatives $\partial_\pm$ get replaced by $\partial_\tau$ and $A_\pm$ get replaced
by \( a_\pm = \partial_\tau h_\pm h_\pm^{-1} \). Then all non-local and non-linear terms drop out and we obtain the effective action in the zero mode sector

\[
S_{\text{eff}} = \frac{-k + g G}{4\pi} \int d\tau \ Tr\left(\frac{1}{2} \partial_\tau g^{-1} \partial_\tau g + a_- \partial_\tau g g^{-1} - a_+ g^{-1} \partial_\tau g + a_- g a_+ g^{-1}\right) \\
- \frac{g G - g H}{8\pi} \int d\tau \ Tr\left(a_+^2 + a_-^2\right).
\]

(3.4)

This action is gauge invariant for rigid (\( \sigma^\pm \)-independent) gauge transformations \( \Lambda_\pm \). Most notably the path integral over \( a_\pm \) is now Gaussian, and this permits the elimination of \( a_\pm \) through the classical equations of motion

\[
(D_+ g g^{-1})_H = -\frac{g G - g H}{k - g G} \ a_-, \quad (g^{-1} D_- g)_H = \frac{g G - g H}{k - g G} \ a_+ ,
\]

(3.5)

with the same “covariant” derivatives \( D_\pm \) on the worldline as before. The system of equations (3.3) is linear and algebraic in \( a_\pm \) and therefore it can easily be solved. Its solution for \( a_\pm \) is

\[
a_+ = (C^t C - \lambda^2 I)^{-1} \left(C^t R^H - \lambda L^H \right) ,
\]

\[
a_- = (C C^t - \lambda^2 I)^{-1} \left(C L^H - \lambda R^H \right) ,
\]

(3.6)

where \( C^t \) denotes the transpose matrix of \( C \) and \( \lambda = \frac{g G - g H}{k - g G} \). The rest of the quantities appearing in (3.6) were defined in (2.7) (\( L^H, R^H \) are the point particle analogs of \( L^H_\pm, R^H_\pm \)). Substitution of these expressions back into (3.4) gives

\[
S_{\text{eff}}^\text{point} = \frac{k - g G}{8\pi} \int d\tau \ G_{\mu\nu} \partial_\tau X^\mu \partial_\tau X^\nu ,
\]

(3.7)

where the metric \( G_{\mu\nu} \) is defined as

\[
G_{\mu\nu} = g_{\mu\nu} + \left(\tilde{V}^{-1} C^t\right)_{ab} L^a_{\mu} R^b_{\nu} - \lambda \tilde{V}^{-1}_{ab} L^a_{\mu} L^b_{\nu} - \lambda V^{-1}_{ab} R^a_{\mu} R^b_{\nu} ,
\]

(3.8)

where \( g_{\mu\nu} \) was defined in (2.11) and for convenience we have defined the symmetric matrices

\[
V_{ab} = \left( C C^t - \lambda^2 I \right)_{ab} , \quad \tilde{V}_{ab} = \left( C^t C - \lambda^2 I \right)_{ab} .
\]

(3.9)

In the \( k \) large limit the metric (3.8) tends to the corresponding semi-classical expression in (2.10) as it should.
To obtain the axion $B_{\mu\nu}$ we need to retain the $\partial_\pm$ on the worldsheet and then read
off the coefficient of $\frac{1}{2}(\partial_- X^\mu \partial_+ X^\nu - \partial_- X^\nu \partial_+ X^\mu)B_{\mu\nu}(X)$. As already explained above we
cannot do this fully because of the non-local terms and non-abelian non-linearities, but
we can still obtain the local contribution to the axion as follows. We formally replace
the $R^H, L^H$ in the expressions for $a_\pm$ and elsewhere by $R^{\pm}_H, L^{\pm}_H$, where $R^H_\pm$ and $L^H_\pm$ were
defined in (2.7). We justify this step by the conformal transformation properties for left
and right movers. We then substitute these forms of $A_\pm$ back into the action (3.3) and
extract the desired metric and axion from the quadratic part. The expression we find for
the metric $G_{\mu\nu}$ is of course the same as in (3.8), whereas for the axion $B_{\mu\nu}(X)$ we find the
following result

$$B_{\mu\nu} = b_{\mu\nu} + (\tilde{V}^{-1}C^t)_{ab}L^a_\mu R^b_\nu,$$

where $b_{\mu\nu}$ was defined in (2.12). As it was the case with the metric (3.8) the axion (3.10)
tends to the corresponding semi-classical expression in (2.10) for large $k$.

To obtain the exact dilaton we must compute the anomaly in the integration over
$A_\pm$. However, as it was the case with the metric and the axion, the local part of the dilaton
can be obtained by going to the point particle limit. The effective action (3.4) contains a
quadratic part in the gauge fields which can be rewritten as

$$-\frac{k + g_G}{4\pi} \int d\tau Tr(a_- C a_+ + \frac{\lambda}{2}(a_-^2 + a_+^2)).$$

(3.11)

Integrating out the gauge fields $a_\pm$ gives a determinant that produces the exact dilaton by
identifying, as in section 2, $e^\Phi = (\text{determinant})$. The result we obtain is

$$\Phi(X) = \frac{1}{2} \ln(\det(V)) + \text{const.}.$$  

(3.12)

Again it is easy to see that the above expression for the dilaton tends, for large $k$, to the
semi-classical result (2.13). The expressions for the metric, the antisymmetric tensor and
the dilaton in the $\sigma$-model arising from a general gauged WZW model were found in [23].

It is worth pointing out that they can be obtained from the corresponding expressions in
(3.8), (3.10), (3.12) if we formally make the substitution $C \rightarrow C - (\lambda + 1)I$ and, among the
$\dim(G)$ string coordinates, restrict to the $\dim(G/H)$ combinations that are $H$-invariant.

Let us determine the particle geodesic equations for the exact metric (3.8). So, we
seek a solution to the classical equations of motion given by (3.3) and
\[ D^R_+ (g^{-1} D_- g) = -\partial_\tau a_+ , \]  

which follows from varying \( g \), and where \( D^R_+ \) was defined just below equation (2.19). The method for solving these equations is identical to the one that led to equations (2.20), (2.22) and it will not be repeated. We will only give the solution which as a function of proper time \( \tau \) is

\[
g(\tau) = \exp\left(\frac{k-g}{k-g_H} \alpha_- \tau \right) g_0 \exp\left( (\alpha_+ + p_0) \tau \right) \exp\left( -\frac{k-g}{k-g_H} \alpha_+ \tau \right),
\]

with \( \alpha_- = -(g_0(\alpha_+ + p_0)g_0^{-1})_H \),

where \( \alpha_\pm, p_0 \) are constant matrices in the Lie algebra of \( H \) and \( G/H \) respectively, and \( g_0 \) is a constant group element in \( G \). These matrices, define the initial conditions for any geodesic at \( \tau = 0 \). The line element evaluated at this general solution becomes

\[
\left( \frac{ds}{d\tau} \right)^2 = \frac{k-g}{8\pi} Tr \left( p_0^2 - \alpha_+^2 + \frac{\lambda}{(\lambda+1)^2} \alpha_-^2 + \frac{\lambda}{\lambda+1} \alpha_+^2 \right).
\]

For the particular example considered in section 4 we have verified that (3.14), (3.15) indeed solve the geodesic equations which are obtained from the exact metric in (4.5) below.

In the rest of this section we prove the theorems (2.16), (2.17). Let us denote the inverses of \( L^A_\mu, R^A_\mu \), which were defined in (2.7), by \( L^A_\mu \) and \( R^A_\mu \) respectively. The following properties are helpful in the algebraic manipulations needed for the proof

\[
L^A_\mu L^B_\mu = \eta^A_\mu B, \quad L^A_\mu L^B_\nu = \delta^A_\mu \nu, \quad R^A_\mu R^B_\mu = \eta^A_\mu B, \quad R^A_\mu R^B_\nu = \delta^A_\nu B.
\]

V\(^{-1}\)C = CV\(^{-1}\), \quad \tilde{V}\(^{-1}\)C\(^t\) = C\(^t\)V\(^{-1}\).

Let us rewrite the exact metric (3.8) in the following way

\[
G_{\mu\nu} = g_{\mu\rho} \tilde{G}^\rho_\nu ,
\]

where \( g_{\mu\rho} \) was defined in (2.11) and

\[
\tilde{G}^\rho_\nu = \delta^\rho_\nu + (\tilde{V}\(^{-1}\)C\(^t\))_{ab} \left( L^a_\rho R^b_\nu + L^a_\nu R^b_\rho \right) - \lambda \tilde{V}\(^{-1}\) L^a_\rho L^b_\nu - \lambda \tilde{V}\(^{-1}\) V\(^{-1}\) R^a_\rho R^b_\nu
= \delta^\rho_\nu + \tilde{C}_{a'b'} S^a_\rho S^b_\nu ,
\]
where $S^a_{\mu} = (L^a_{\mu}, R^a_{\mu})$ and the $2\dim(H) \times 2\dim(H)$ dimensional symmetric matrix $(\tilde{C}_{a'b'})$ is defined as
\[
(\tilde{C}_{a'b'}) = \begin{pmatrix} -\lambda \tilde{V}^{-1} & \tilde{V}^{-1} C^t \\ \tilde{V}^{-1} C & -\lambda V^{-1} \end{pmatrix}.
\]
(3.19)

In order to compute $\det(G_{\mu\nu})$ we need $\det(\tilde{G}_{\rho\nu})$. We have
\[
\det(\tilde{G}_{\rho\nu}) = \det(\eta_{a'b'} + \tilde{C}_{a'c'} S^c_{\mu} S^b_{\mu})
\]
\[
= \det\left( I - \tilde{V}^{-1}(C^t C + \lambda I) \right) = (\lambda + 1)^{2\dim(H)} \det(\tilde{V}^{-1} C \quad \tilde{V}^{-1} C^t -\lambda V^{-1})
\]
\[
= (\lambda + 1)^{2\dim(H)} \det\left( \begin{array}{cc} -\lambda \tilde{V}^{-1} & \tilde{V}^{-1} C^t \\ \tilde{V}^{-1} C & -\lambda V^{-1} \end{array} \right)
\]
\[
= (\lambda + 1)^{2\dim(H)} \det(\tilde{V}^{-1} \quad 0 \quad 0 \quad V^{-1}) \det(\begin{array}{cc} -\lambda I & C^t \\ C & -\lambda I \end{array})
\]
\[
= -(\lambda + 1)^{2\dim(H)} \det(V^{-1})
\]
(3.20)

The Haar measure for the group $G$ is given by $\det^{1/2}(g_{\mu\rho})$. Therefore by using (3.17), (3.20), (3.12) one easily establishes the validity of the theorems (2.16), (2.17) for any CGWZW model.

4. Chiral gauging with $G = SL(2, \mathbb{R})$ and $H = SO(1,1)$

Let us work out explicitly the details in the simple case where $G = SL(2, \mathbb{R})$ and $H = SO(1,1)$. If one considers the corresponding gauged WZW model one obtains a two dimensional black hole [8]. However in our case we will find a three dimensional model which is related to the three dimensional black string model [9].

It is convenient to parametrize the group element $g \in SL(2, \mathbb{R})$ as
\[
g = \begin{pmatrix} u & a \\ -b & v \end{pmatrix}, \quad ab + uv = 1.
\]
(4.1)

The subgroup generator is $t_0 = \frac{1}{\sqrt{2}} \sigma_3$, where $\sigma_3$ denotes the usual third Pauli matrix, and the dual Coxeter numbers are $g_G = 2$, $g_H = 0$. Using (2.7) we compute the following quantities necessary for the evaluation of the various fields in the $\sigma$-model
\[
L_\mu^0 = -\sqrt{2} \begin{pmatrix} b \\ 0 \\ u \end{pmatrix}, \quad R_\mu^0 = -\sqrt{2} \begin{pmatrix} b \\ v \\ 0 \end{pmatrix},
\]
(4.2)

\[
C_{00} = uv - ab, \quad \text{where} \quad X^\mu = (a, u, v).
\]

Then using (3.8), (3.10), (3.12) we find the following expression for the line element, the antisymmetric tensor and the dilaton

\[
ds^2 = \frac{k/2}{uv - ab + \lambda} \left( \begin{array}{c} b \\ a \end{array} \right) \left( \begin{array}{c} da \\ d(uv) \end{array} \right) + \frac{k/2}{uv - ab - \lambda} \left( \begin{array}{c} dudv \\ -(d(uv))^2 \end{array} \right),
\]

\[
B = \ln a du \wedge dv + 2 \left( \frac{uv - ab}{(uv - ab)^2 - \lambda^2} (bu da \wedge du - bu da \wedge dv - uv du \wedge dv) \right),
\]

\[
\Phi = \frac{1}{2} \ln((uv - ab)^2 - \lambda^2) + \text{const}.
\]

Next we make the following change of variables

\[
uv = \frac{1}{2}(1 + \lambda - r), \quad \frac{u}{v} = e^{2t}, \quad a = e^x \left( \frac{1}{2}(1 - \lambda + r) \right)^{1/2}.
\]

(4.4)

In this set of variables the various fields in (4.3) take the form (after rescaling \(t, x\) and dropping out total derivative terms in the expression for \(B\))

\[
ds^2 = -(1 - \frac{1 + \lambda}{r}) dt^2 - (1 + \frac{1 + \lambda}{r - 2\lambda}) dx^2 + \frac{dr^2}{4\lambda (r - \lambda)^2 - 1},
\]

\[
B = (1 - \lambda) \frac{r - \lambda}{(r - \lambda)^2 - \lambda^2} dt \wedge dx
\]

(4.5)

\[
\Phi = \frac{1}{2} \ln(r(r - 2\lambda)) + \text{const}.
\]

The semi-classical limit \((k \to \infty, \lambda \to 0)\) of the above expressions was previously obtained in [37]. It is possible to relate the above results to the corresponding exact results for the three dimensional black string model [1] one obtains by axially gauging the subgroup \(H = SO(1, 1)\) of the direct product group \(G = SL(2, \mathbb{R}) \otimes SO(1, 1)\). The exact metric and dilaton for this model have been found in [18] and the exact antisymmetric tensor in [23]
where $r_+ = (\rho^2 + 1) C', r_- = (\rho^2 + 2/k) C', r_q = 2/k C'$. The embedding of the subgroup $H$ into the group $G$ is parametrized by the positive parameter $\rho^2$ and $C'$ is a constant. If $C' = 2(\lambda + 1)$, $\rho^2 = -1/2$ we get $r_+ = \lambda + 1$, $r_- = \lambda - 1$ and $r_q = 2\lambda$. The fact that $\rho^2 < 0$ means that $x$ is space-like and therefore we should also analytically continue $x \rightarrow ix$. Then one can verify that the expressions for the exact metric and dilaton of the black string (4.6) become the corresponding expressions of our model in (4.5). However, the expression one obtains for the antisymmetric tensor is $B = \frac{\lambda + 1 - r}{r - 2\lambda} dt \wedge dx$ which differs from the one in (4.5). There is agreement only in the semi-classical limit ($k \rightarrow \infty$, $\lambda \rightarrow 0$) (up to a constant piece). In the semi-classical limit the above correspondence between the two models was first observed in [37].

5. Conclusion

We have suggested an effective action for CGWZW models, by making contact with the analogous problem for gauged WZW models. The effective action for the latter reproduced correctly the exact geometry derived before in the Hamiltonian formalism and this is essentially the justification of our approach.

Using the effective action we derived the exact expressions for the metric, the antisymmetric tensor and the dilaton fields in the $\sigma$-model arising from the general CG-WZW model. We explicitly considered the three dimensional case with $G = SL(2, \mathbb{R})$ and $H = SO(1, 1)$ and saw that it is related to the three dimensional black string which however arises in a different context, i.e. by axially gauging the four dimensional direct product group $SL(2, \mathbb{R}) \otimes \mathbb{R}$ by a total translation. The correspondence is exact for the metric and dilaton but only semi-classical for the antisymmetric tensor.

We think that it would be interesting to extend this construction to include the supersymmetric case, along the lines of [38][11][24] and to investigate higher dimensional models in the context of CGWZW models.

\[ \text{In the case of CGWZW models the metric~(3.8)~and dilaton~(3.12)~can also be derived via the Hamiltonian formalism by using the Hamiltonian} \]

\[ \mathcal{H} = \frac{J_G^2}{k - g_G} - 2 \frac{J_H^2}{k - g_H} + \frac{J_G^2}{k - g_G} - 2 \frac{J_H^2}{k - g_H}, \]

where $J_G, J_H$ belong in the Lie algebras of $G$ and $H$ respectively and act as first order differential operators in the group parameter space (similarly for $\bar{J}_G, \bar{J}_H$).
The apparent disagreement of the expressions for the exact antisymmetric tensors in the cases of $SL(2, \mathbb{R}) \otimes \mathbb{R}/\mathbb{R}$ gauged WZW model and $SL(2, \mathbb{R})/\mathbb{R}$ chiral gauged WZW model is resolved in [39]. The two backgrounds are related by local field redefinitions.
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