Nonlinear Fractional Dynamics on a Lattice with Long Range Interactions

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Abstract

A unified approach has been developed to study nonlinear dynamics of a 1D lattice of particles with long-range power-law interaction. A classical case is treated in the framework of the generalization of the well-known Frenkel-Kontorova chain model for the non-nearest interactions. Quantum dynamics is considered following Davydov’s approach for molecular excitons.

In the continuum limit the problem is reduced to dynamical equations with fractional derivatives resulting from the fractional power of the long-range interaction. Fractional generalizations of the sine-Gordon, nonlinear Schrödinger, and Hilbert-Schrödinger equations have been found.

There exists a critical value of the power $s$ of the long-range potential. Below the critical value ($s < 3, s \neq 1, 2$) we obtain equations with fractional derivatives while for $s \geq 3$ we have the well-known nonlinear dynamical equations with space derivatives of integer order.

Long-range interaction impact on the quantum lattice propagator has been studied. We have shown that the quantum exciton propagator exhibits transition from the well-known Gaussian-like behavior to a power-law decay due to the long-range interaction. A link between 1D quantum lattice dynamics in the imaginary time domain and a random walk model has been discussed.

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1 Introduction

Discrete nonlinear lattice models have become one of the most popular models of nonlinear physics, often used to investigate a rather broad set of physical phenomena and systems. The first simplest 1D lattice model was introduced a long time ago by Frenkel and Kontorova [1] to study the structure and dynamics of dislocations in metals (for more details see review [2] and references therein). The Frenkel and Kontorova discrete classical dynamics is described as a 1D chain of atoms with the nearest-neighbor intersite/interatomic interaction and a periodic on-site potential. Despite its simplicity the model has many applications in solid-state and nonlinear physics, including propagation of charge-density waves, the dynamics of absorbed layers of atoms on crystal surfaces, commensurable-incommensurable phase transitions, and domain walls in magnetically ordered structures, see for details [2]. Moreover the model is surprisingly attractive because it provides exactly integrable cases in discrete and continuous approximations including the well-known sine-Gordon nonlinear equation (see, for example [3]).

A specific, but not less important application, is the case of long-range interatomic interaction, which is the main subject of this paper and which has been studied for different physical systems. Examples include ferromagnetic chains [4], exciton transfer in molecular crystals [5], [6], commensurable-incommensurable phase transitions [7], theory of Josephson junctions [8], [9]. To model the long-range interaction potential in 1D lattice, few potentials have been applied: power-law interaction [7], [10], Lennard-Jones long-range coupling [11] and exponential interaction of Kac-Beker form [12].

Since the Fermi, Pasta and Ulam work [13] analytical developments and numerical simulation on discrete nonlinear dynamical equations have provided numerous types of nonlinear excitations with various properties: solitons, continuous and discrete breathers [2], [14], [15], self-trapping states [16] and others.

Another broad area of applications is related to the coupled system of oscillators with long-range interactions and the phenomenon of synchronization in the system [17]-[19]. In many situations, impact of long-range coupling
can be compared to a phase transition [20]-[22]. Long-range coupling can be presented as nonlocality with finite space scale $a$, when the coupling energy is proportional to $\exp(-|x_n - x_m|/a)$ for atoms located at the points $x_n$ and $x_m$, or with scale free interaction when the coupling energy is proportional to $|x_n - x_m|^{-s}$ $(s > 0)$. In addition to the well known interaction with integer values of $s$, some complex media can be described by fractional values of $s$ (see, for example, references in [23]).

The goal of this paper is to focus on analytical developments on classical and quantum 1D lattice dynamics with fractional power-law long-range interaction. The long-range interaction leads in general to a nonlocal integral term in the equation of motion if we go from discrete to continues space. We show that in some particular cases for power-law interaction with non-integer power $s$ the integral term can be expressed through a fractional order derivative. In other words, nonlocality originating from the long-range interaction is revealed in the dynamics in the form of space derivatives of fractional order. The appearance of fractional differential equations in the continuum space limit of lattice dynamics leads to a possibility to apply powerful tools of random processes theory [24], [25] and fractional calculus [26]-[29] to the lattice dynamics and kinetics. Particularly, it helps to obtain some qualitative results based on the known features of fractional derivatives. As an example, let us mention that the fractional oscillator appears to have power law dissipation at $t \to \infty$ and, being perturbed, gives rise to a new type of stochastic attractor [23]. In the case of a fractional space derivative, one can expect new types of synchronizations and coherent structures [30], where our approach to obtain equations with fractional derivative is applied to discrete model of oscillating media with long-range power-type coupling between oscillators. It also worthwhile to mention that fractional dynamical equations prove to be an adequate approach to study chaotic dynamical systems and their anomalous transport properties [31].

We present new developments on a 1D lattice model with fractional power-law interatomic interaction defined by fractional values of the parameter $s$ and non-linear self-interaction potential for classical and quantum mechanical consideration. We show that the dynamics of the 1D lattice can be equivalently presented by the corresponding fractional nonlinear equation in the long-wave limit. We concentrate on the conditions of such equivalence, type of the equations with fractional derivatives and some related properties. As example of our developments fractional sine-Gordon and wave-Hilbert nonlinear equations have been found for classical lattice dynam-
ics, and fractional nonlinear Schrödinger and nonlinear Hilbert-Schrödinger equations have been obtained for quantum lattice dynamics in the long wave limit.

The paper is organized as follows. In Sec. 2 we study classical 1D lattice non-linear dynamics. It has been found that depending on the long-range interaction parameter $s$, the interaction term can be presented in long-wave limit in differential or fractional differential form which results either as a nonlinear wave equation or as a nonlinear fractional wave equation. Examples are fractional sine-Gordon and wave-Hilbert nonlinear equations.

Sec. 3 focuses on quantum 1D lattice dynamics. Depending on the long-range interaction parameter $s$, the interaction term can be presented in the long-wave limit in differential or fractional differential form which results in the nonlinear Schrödinger equation or in a nonlinear Hilbert-Schrödinger equation.

Quantum lattice propagator is introduced in Sec. 4. It has been observed that depending on the parameter $s$ the quantum lattice propagator exhibits a transition from the well-known Gaussian-like behavior to power-law decay due to long-range interaction. We show a link between the quantum lattice dynamics and random walk in the imaginary time domain.

In conclusion we outline the new developments and discuss their applicability to other fields.

In Appendices we briefly review the definition of the polylogarithm and its integral and power series representations.

## 2 Classical nonlinear lattice dynamics

Nonlinear classical discrete dynamics of a 1D infinite lattice can be described by Hamiltonian function $H(u)$, which depends on particle ("atom") displacement $u_n(t)$ at the site $n$

$$H(u) = \frac{1}{2} M \sum_{n=-\infty}^{\infty} \dot{u}_n^2 + \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} V_{n-m}(u_n - u_m)^2 + \sum_{n=-\infty}^{\infty} U(u_n), \quad (1)$$

where the first term is the kinetic energy of the chain, $M$ is particle mass, and $V_{n-m}$ is interaction matrix which describes long-range interaction between sites $n$ and $m$. 

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\[ V_{n-m} = \frac{V_0}{|n-m|^s}. \] (2)

Here \( V_0 \) is interaction constant, parameter \( s \) covers different physical models, and \( U(u) \) characterizes nonlinear interaction of the lattice atoms with external on-site potential. Integer values of \( s \) can be used to describe the well-known physical models: the nearest-neighbor approximation \( (s = \infty) \), the dipole-dipole interaction \( (s = 3) \), the Coulomb potential \( (s = 1) \).

Our main interest will be in fractional values of \( s \) that can appear for more sophisticated interaction potentials attributed to complex media. The equation of motion follows from Eq. (1)

\[
M\ddot{u}_n + \sum_{m=\infty}^{\infty} V_{n-m}(u_n - u_m) + \frac{\partial U(u)}{\partial u_n} = 0, \tag{3}
\]

To go from the discrete version Eq. (3) to the continuum one, let us define

\[
v(k,t) = \sum_{n=-\infty}^{\infty} e^{-ikn}u_n(t), \tag{4}
\]

where \( u_n(t) \) is related to \( v(k,t) \) as follows

\[
u(k,t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk e^{ikn}v(k,t), \tag{5}
\]

and \( k \) can be considered as a wave number.

Then the second term in Eq. (1) reads

\[
\frac{1}{2} \sum_{n,m} V_{n-m}(u_n - u_m)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk V(k)|v(k,t)|^2, \tag{6}
\]

where function \( V(k) \) has been defined by

\[
V(k) = V(0) - V(k), \tag{7}
\]

and \( V(k) \) is

\[
V(k) = \sum_{n=-\infty}^{\infty} e^{-ikn}V_n. \tag{8}
\]
In the long wave limit when the wavelength exceeds the intersite scale we may consider \( v(k,t) \) as a \( k \)th Fourier component of function \( u(x,t) \), \( u_n(t) \xrightarrow{k \to 0} u(x,t) \). That is the functions \( v(k,t) \) and \( u(x,t) \) are related each other by the Fourier transform

\[
 u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} v(k,t), \quad \quad v(k,t) = \int_{-\infty}^{\infty} dx e^{-ikx} u(x,t). \tag{9}
\]

Therefore, from Eq.(6) we have

\[
\frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} V_{n-m}(u_n - u_m)^2 = \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy u(x,t) W(x-y) u(y,t), \tag{10}
\]

with the kernel

\[
 W(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dk e^{ikx} V(k). \tag{11}
\]

that comes from the long-range interaction matrix \( V_{n-m} \) and it possesses the properties:

(i) \( W(x) \) is an even function;

(ii) \( \int_{-\infty}^{\infty} dx W(x) = 0 \).

One can express Eq.(10) in a non-local kinematic form

\[
\frac{1}{2} \sum_{n,m \neq m} J_{n-m}(u_n - u_m)^2 = \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy (\partial_x u(x,t)) K(x-y) \partial_y u(y,t), \tag{12}
\]

where notation \( \partial_x = \partial/\partial x \) has been introduced and the relationship between kernel \( K(x) \) and function \( V(k) \) (see Eq.(7)) is given by

\[
 K(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dk e^{ikx} \frac{V(k)}{k^2}. \tag{13}
\]

In other words, the kernels \( W(x) \) and \( K(x) \) are related to each other as
\( W(x) = -\partial_x^2 K(x). \) \( (14) \)

Thus, in continuum limit the equation of motion becomes

\[ M\ddot{u}(x,t) - \int_{-\infty}^{\infty} dy \partial_x K(x-y) \partial_y u(y,t) + \frac{\partial U(u)}{\partial u(x,t)} = 0, \] \( (15) \)

or in a symbolic form

\[ M\ddot{u}(x,t) - \partial_x(\hat{K}\partial_x u) + \frac{\partial U(u)}{\partial u(x,t)} = 0, \] \( (16) \)

and the operator \( \hat{K} \), as it follows from Eq.(13), can be considered as operator of multiplication in wave number (momentum) space

\[ \hat{K}u(x,t)(k) = \frac{\mathcal{V}(k)}{k^2}v(k,t). \] \( (17) \)

Equation (15) is integro-differential equation of motion. The integral part comes from the long-range interaction term Eq.(12). To get the differential equation of motion we use the properties of function \( \mathcal{V}(k) \) at \( k \to 0 \), which can be obtained from the asymptotics of polylogarithm (we provide the properties of the polylogarithm function and the expression for \( \mathcal{V}(k) \) in the Appendices),

\[ \mathcal{V}(k) \sim \frac{\pi V_0}{\Gamma(s) \sin\left(\frac{\pi (s-1)}{2}\right)} k^{s-1} = D_s|k|^{s-1}, \quad 2 \leq s < 3, \] \( (18) \)

\[ \quad \quad \quad \mathcal{V}(k) \sim -\frac{V_0 k^2}{2} \ln k^2, \quad s = 3, \] \( (19) \)

\[ \quad \quad \quad \mathcal{V}(k) \sim \frac{V_0 \zeta(s-2)}{2} k^2, \quad s > 3, \] \( (20) \)

where \( \Gamma(s) \) is \( \Gamma \)-function, \( \zeta(s) \) is Riemann zeta function and coefficient \( D_s \) is defined by

\[ D_s = \frac{\pi V_0}{\Gamma(s) \sin\left(\frac{\pi (s-1)}{2}\right)}. \] \( (21) \)

It is seen from Eq.(18) that fractional powers of \( k \) occurs for the interactions with \( 2 \leq s < 3 \) only.

Going back to the coordinate space yields for fractional powers of \( |k| \) fractional Riesz derivative of order \( \alpha \) \[25], \[26\]
\[ \partial^x_\alpha u(x,t) = -\frac{1}{2\cos(\pi\alpha/2)} \left( \frac{d^{\alpha}}{dx^{\alpha}} + \frac{d^{\alpha}}{d(-x)\alpha} \right) u(x,t), \quad 1 < \alpha \leq 2, \quad (22) \]

where \( d^{\alpha}/d(\pm x)^\alpha \) are the Riemann-Liouville derivatives

\[
\frac{d^{\alpha}}{dx^{\alpha}} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{x} \frac{dy f(y)}{(x-y)^{\alpha-n+1}},
\]

\[
\frac{d^{\alpha}}{d(-x)^{\alpha}} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{x}^{\infty} \frac{dy f(y)}{(y-x)^{\alpha-n+1}},
\]

with \( n-1 < \alpha < n \) and integer \( n \). Substitution of Eq. (18) into Eq. (15) yields a fractional nonlinear wave equation

\[ M \ddot{u}(x,t) - D_s \partial^{s-1}_x u(x,t) + \frac{\partial U(u)}{\partial u(x,t)} = 0, \quad (23) \]

where the Riesz fractional derivative has been transformed into [25], [32]

\[ \partial^{s-1}_x u(x,t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} |k|^{s-1} v(k,t), \quad (24) \]

and coefficient \( D_s \) is defined by Eq. (21).

When the nonlinear external potential \( U(u) = U_0(1 - \cos u) \), we get the fractional sine-Gordon equation [33]

\[ M \ddot{u}(x,t) - D_s \partial^{s-1}_x u(x,t) + U_0 \sin u(x,t) = 0. \quad (25) \]

To get differences for the phonon spectrum of fractional and standard sine-Gordon equations let’s consider a linear equation of the form

\[ \ddot{u}(x,t) - c_0^2 \partial^2_{xx} u(x,t) + m^2 u(x,t) = 0, \quad (26) \]

where \( c_0 \) is velocity of wave and \( m \) is the particle mass. Plane wave solution of this equation has the form

\[ u(x,t) = Ae^{i(\omega t + kx)}, \quad (27) \]
with constant $A$, frequency $\omega$, and wave number $k$. Then the dispersion law is
\[ \omega(k) = \pm \sqrt{m^2 + c_0^2 k^2}. \] (28)

For the fractional linear wave equation
\[ \ddot{u}(x,t) - b^{s-3} c_0^2 \partial_x^{s-1} u(x,t) + m^2 u(x,t) = 0, \] (29)
where $\partial_x^{s-1}$ is defined by Eq.(24), $b$ is the scale constant attributed to the fractional wave equation, $c_0$ is wave velocity, the dispersion law is
\[ \omega(k) = \pm \sqrt{m^2 + b^{s-3} c_0^2 |k|^{s-1}}, \quad 2 < s < 3. \] (30)

For fairly small values of $m$ and not too small $k$ we obtain
\[ \omega(k) \approx b^{(s-3)/2} c_0 |k|^{(s-1)/2}, \] (31)
which results in phase $v_{ph}$ and group $v_g$ velocities equal
\[ v_{ph} = \frac{\omega(k)}{k} = b^{(s-3)/2} c_0 / |k|^{(3-s)/2}, \quad 2 < s < 3, \] (32)
\[ v_g = \frac{\partial \omega(|k|)}{\partial |k|} = \frac{s-1}{2} b^{(s-3)/2} c_0 / |k|^{(3-s)/2} = \frac{s-1}{2} v_{ph}, \quad 2 < s < 3. \] (33)

These expressions tend to infinity for $k \to 0$, and we arrive at new physical properties of the lattice of particles just because of fractality of the long-range interaction. For small enough $k$ the mass $m$ can not be neglected if $m \neq 0$. Several other examples of equations can be easily obtained using the general scheme of Eq.(23).

It follows from Eq.(18) that in the case $s = 2$ the function $V(k)$ takes the form
\[ V(k) = \pi V_0 |k|. \] (34)

Substitution of this expression into Eqs.(13) and (16) yields a nonlinear wave-Hilbert equation
\[ M \ddot{u}(x,t) - \pi V_0 \mathcal{H} \{ \partial_x u(x,t) \} + \frac{\partial U(u)}{\partial u(x,t)} = 0, \] (35)
where $\mathcal{H}$ is the Hilbert integral transform defined by
\[ \mathcal{H}\{\phi(x)\} = P \int_{-\infty}^{\infty} dy \frac{\phi(y)}{y-x}. \] 

(36)

and \( P \) stands for the Cauchy principal value of the integral.

When \( U(u) = U_0(1 - \cos u) \) we find the sine-Hilbert equation \[34], \[35\]

\[ M\ddot{u}(x,t) - \pi V_0 \mathcal{H}\{\partial_x u(x,t)\} + U_0 \sin u(x,t) = 0. \]

(37)

Examples of physical problems with nonlocal interaction like \( V(k) \) given by Eq.(34) can be found in nonlocal Josephson electrodynamics \[36\] where Eq.(37) is one of the basic model equations. Let’s note that some exact solutions of this equation have been found and their stability under weak perturbations has been analyzed \[35\]–\[37\].

Finally, when \( s > 3 \) and \( U(u) = U_0(1 - \cos u) \) we get the sine-Gordon equation \[3\],

\[ M\ddot{u}(x,t) - \frac{V_0\zeta(s-2)}{2} \partial_{xx}^2 u(x,t) + U_0 \sin u(x,t) = 0, \]

(38)

where \( \sqrt{\frac{V_0\zeta(s-2)}{2M}} \) can be interpreted as the wave velocity.

3 Quantum lattice dynamics

3.1 Lattice Hamiltonian with long-range interaction

To model a 1D quantum lattice dynamics we follow \[5], \[6\] and consider a linear, rigid arrangement of sites with one molecule at each lattice site. To describe creation (annihilation) of molecular excitation or, for simplicity exciton, at the site \( n \) we introduce exciton creation \( b_n^+ \) and annihilation \( b_n \) operators. Operators \( b_n^+ \) and \( b_n \) satisfy the commutation relations \([b_n^+, b_m] = \delta_{n,m}, \ [b_n, b_m] = 0, \ [b_n^+, b_m^+] = 0\) that is \( b_n^+ \) and \( b_n \) are the Bose operators.

With the help of the operators \( b_n^+ \) and \( b_n \) the well-known Hamiltonian operator of excitons is expressed as \[5], \[6\], \[38\],

\[ \hat{H}_D = \varepsilon \sum_{n=-\infty}^{\infty} b_n^+ b_n - J \sum_{n=-\infty}^{\infty} (b_n^+ b_{n+1} + b_n b_{n-1}) = \varepsilon \sum_{n=-\infty}^{\infty} b_n^+ b_n - \sum_{n,m=-\infty}^{\infty} J_{n,m} b_n^+ b_m, \]

(39)
where $\varepsilon$ is constant exciton energy, $J$ is interaction constant and the excitation transfer matrix element $J_{n,m}$ between sites $n$ and $m$ can be written as

$$J_{n,m} = J(\delta_{(n+1),m} + \delta_{(n-1),m}), \quad (40)$$

the Kroneker symbols $\delta_{m,n\pm 1}$ mean that only the nearest-neighbor interaction had been considered. The interaction term $J_{n,m}b_n^+b_m$ in Eq.(39) is responsible for exciton transfer from site $n$ to the nearest neighbor sites $n \pm 1$.

To go beyond the nearest-neighbor interaction we introduce the long-range excitation transfer matrix $J_{n,m}^{LR}$ which describes power-law interaction between sites $n$ and $m$ (see Eq.(2))

$$J_{n,m}^{LR} \equiv V_{n-m} = \frac{V_0}{|n-m|^s}, \quad n \neq m. \quad (41)$$

Then we can rewrite the Hamiltonian operator $\hat{H}_{LR}$ with long-range excitation transfer,

$$\hat{H}_{LR} = \varepsilon \sum_n b_n^+b_n - \sum_{n,m \neq m} J_{n,m}^{LR}b_n^+b_m. \quad (42)$$

Follow Davydov’s ansatz [5], [6] we define the eigenstate $|\phi(t)\rangle$ of a quantum exciton as a superposition:

$$|\phi(t)\rangle = \sum_n \psi_n(t)b_n^+|0\rangle, \quad (43)$$

where $|0\rangle$ is the vacuum state of exciton system and $\psi_n(t)$ is the exciton wave function. In the $|\phi(t)\rangle$ basis the Hamiltonian $\hat{H}_{LR}$ becomes the Hamiltonian function $H_{LR}(\psi, \psi^*)$

$$H_{LR}(\psi, \psi^*) = \langle \phi(t)|\hat{H}_{LR}|\phi(t)\rangle = \varepsilon \sum_n \psi_n^*(t)\psi_n(t) - \sum_{n,m \neq m} V_{n-m}\psi_n^*(t)\psi_m(t), \quad (44)$$

where $V_{n-m}$ is given by Eq.(41). Considering nonlinearity in addition to the long-range interaction, we add a nonlinear term $U(|\psi|)$ to the right hand of Eq.(44), and, thus, we obtain for the Hamiltonian $H(\psi, \psi^*)$ of 1D lattice

$$H(\psi, \psi^*) = \varepsilon \sum_n \psi_n^*(t)\psi_n(t) - \sum_{n,m \neq m} V_{n-m}\psi_n^*(t)\psi_m(t) + \sum_n U(|\psi_n|). \quad (45)$$
The nonlinear Schrödinger equation in discrete space $i\hbar \partial \psi_n(t)/\partial t = \delta H(\psi)/\delta \psi^*_n$ generated by the above Hamiltonian has the form

$$i\hbar \frac{\partial \psi_n(t)}{\partial t} = \varepsilon \psi_n(t) - \sum_{m \neq n} V_{n-m} \psi_m(t) + \frac{\delta U(|\psi|)}{\delta \psi^*_n},$$

(46)

where $\hbar$ is the Planck constant.

To get continuum space equation let us follow the same approach as in Sec.2 and introduce wave function $\varphi(k, t)$,

$$\varphi(k, t) = \sum_{n=-\infty}^{\infty} e^{-ikn} \psi_n(t),$$

(47)

where $\psi_n(t)$ is related to $\varphi(k, t)$ as

$$\psi_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk e^{ikn} \varphi(k, t),$$

(48)

and $k$ can be considered as a wave number. Substitution of Eq.(48) into Eq.(44) yields

$$H_{LR} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk (\omega + V(k)) |\varphi(k, t)|^2,$$

(49)

with the function $V(k)$ given by Eq.(7) and the energy parameter $\omega$ which is

$$\omega = \varepsilon - V(0),$$

(50)

here $V(0) = \sum_{n=-\infty}^{\infty} V_n$ (see the definition given by Eq.(8)).

At this point we may consider $\varphi(k, t)$ as the wave number representation of continuous space wave function $\psi(x, t)$. That is the functions $\varphi(k, t)$ and $\psi(x, t)$ are related to each other by the Fourier transform

$$\psi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \varphi(k, t), \quad \varphi(k, t) = \int_{-\infty}^{\infty} dx e^{-ikx} \psi(x, t).$$

(51)

Therefore, we have for $H_{LR}$ in coordinate space
\[ H_{LR}(\psi) = \omega \int_{-\infty}^{\infty} dx |\psi(x,t)|^2 + \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \psi^*(x,t)W(x-y)\psi(y,t), \quad (52) \]

where the kernel \( W(x) \) is defined by Eq.(11). To get kinematic form of the second term in the right side of Eq.(52) we follow Eqs.(13) and (14) and find,

\[ H_{LR}(\psi) = \omega \int_{-\infty}^{\infty} dx |\psi(x,t)|^2 + \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \partial_x(\partial_x\psi^*(x,t))K(x-y)\partial_y\psi(y,t), \quad (53) \]

where the kernel \( K(x) \) is defined by Eq.(13).

Further, by adding a nonlinear term to right hand of (53) we obtain the integro-differential (non local) nonlinear Schrödinger type equation which can written in the following compact way

\[ i\hbar \frac{\partial \psi(x,t)}{\partial t} = \omega \psi(x,t) + \partial_x(\hat{K}\partial_x\psi) + \frac{\delta U(|\psi|)}{\delta \psi^*(x,t)}. \quad (54) \]

where the operator \( \hat{K} \) acts on wave function \( \psi(x,t) \) as

\[ \hat{K}\psi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \frac{\mathcal{V}(k)}{k^2} \varphi(k,t), \quad (55) \]

or in wave number (momentum) representation

\[ \hat{K}\psi(x,t)(k) = \frac{\mathcal{V}(k)}{k^2} \varphi(k,t). \quad (56) \]

Using the properties of function \( \mathcal{V}(k) \) in the limit \( k \to 0 \) (see Eqs.(18)-(20)) we can get different special forms of general Eq.(54).

### 3.2 Fractional nonlinear Schrödinger equation

Substitution Eq.(18) into Eqs.(54) and (55) yields a new nonlinear fractional differential equation

\[ i\hbar \frac{\partial \psi(x,t)}{\partial t} = \omega \psi(x,t) - D_s\partial_x^{-1}\psi(x,t) + \frac{\delta U(|\psi|)}{\delta \psi^*(x,t)}, \quad 2 \leq s < 3, \quad (57) \]
where we use the definition of the Riesz fractional derivative \[25, 26\]

\[
\partial_x^{s-1} \psi(x, t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} |k|^{s-1} \varphi(k, t).
\]  

(58)

In Eq. (57) coefficient \(D_s\) is given by Eq. (21) and \(\omega\) is the constant defined by Eq. (50). For the case when

\[U(|\psi|) = U(x)|\psi|^2,\]

(59)

Eq. (57) is reduced to the linear fractional Schrödinger equation \[32, 39, 40\]

\[
i\hbar \frac{\partial \phi(x, t)}{\partial t} = -D_s \partial_x^{s-1} \phi(x, t) + U(x)\phi(x, t),
\]

(60)

where the wave function \(\phi(x, t)\) is related to the wave function \(\psi(x, t)\) by

\[\phi(x, t) = \exp\{i\frac{\omega t}{\hbar}\} \psi(x, t).
\]

(61)

Generalization of Eq. (60) for 3D space and the applications of the 1D and 3D linear fractional Schrödinger equation to quantum mechanical problems have been developed in \[40\]. In these papers, a particle in infinite potential well, fractional oscillator, and fractional Bohr atom have been studied and the energy spectra for these three quantum mechanical problems have been obtained using Eq. (60) and its 3D generalization. New physical issues following from quantum mechanical applications of Eq. (60) have been discussed in \[40\].

For nonlinearity of the form

\[U(|\psi|) = \frac{U_0}{2} |\psi|^4,\]

(62)

with a constant \(U_0\), Eq. (57) is reduced to the fractional nonlinear Schrödinger equation \[41, 42, 43\]

\[
i\hbar \frac{\partial \phi(x, t)}{\partial t} = -D_s \partial_x^{s-1} \phi(x, t) + U_0 |\phi(x, t)|^2 \phi(x, t),
\]

(63)

where \(\phi(x, t)\) is related to the wave function \(\psi(x, t)\) by means of Eq. (61). The equation (63) and its 3D generalization were proposed in \[41\] to study wave propagation or kinetics in a nonlinear media with fractal properties (see also \[42, 43\]). Following from Eq. (63) a fractional generalization of the nonlinear Ginzburg-Landau equation has been developed and square integrability of its solution has been established as well in \[41\].

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3.3 Nonlinear Hilbert-Schrödinger equation

For the case when $s = 2$ substitution Eq. (18) into Eqs. (54) and (55) yields a new nonlinear integro-differential equation

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = \omega \psi(x, t) - \pi V_0 \mathcal{H} \{ \partial_x \psi(x, t) \} + \frac{\delta U(|\psi|)}{\delta \psi^*(x, t)}, \quad (64)$$

where $\mathcal{H}$ is defined in Eq. (36).

When $U(|\psi|)$ is given by Eq. (59) we find from Eq. (67) the linear Hilbert-Schrödinger equation

$$i\hbar \frac{\partial \phi(x, t)}{\partial t} = -\pi V_0 \mathcal{H} \{ \partial_x \phi(x, t) \} + U(x) \phi(x, t). \quad (65)$$

When $U(|\psi|)$ is given by Eq. (62) we obtain from Eq. (67) the nonlinear Hilbert-Schrödinger equation

$$i\hbar \frac{\partial \phi(x, t)}{\partial t} = -\pi V_0 \mathcal{H} \{ \partial_x \phi(x, t) \} + U_0 |\phi(x, t)|^2 \phi(x, t). \quad (66)$$

Finally, we note that for $s > 3$ it follows from Eq. (20) that Eq. (57) takes the form of the nonlinear Schrödinger-like equation with extra term $\omega \psi(x, t)$ originated because of the energy parameter $\omega$

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = \omega \psi(x, t) - \left( \frac{V_0 \zeta(s - 2)}{2} \right) \partial_x^2 \psi(x, t) + \frac{\delta U(|\psi|)}{\delta \psi^*(x, t)}, \quad s > 3, \quad (67)$$

here $\zeta(s)$ is the Riemann zeta function.

4 Quantum exciton propagator

To get insight on impact of long-range interaction on 1D quantum dynamics let us focus on the discrete linear problem associated with Eq. (46)

$$i\hbar \frac{\partial \psi_n(t)}{\partial t} = \varepsilon \psi_n(t) - \sum_{m \neq n} V_{n-m} \psi_m(t), \quad (68)$$

with long-range potential $V_{n-m}$ given by Eq. (2).
Suppose that we know the solution $\psi_{n'}(t')$ of Eq. (68) at some time instant $t'$ at the site $n'$. Then solution $\psi_n(t)$ at later time $t$, $(t > t')$, and site $n$ will be

$$
\psi_n(t) = \sum_{n'} G_{n,n'}(t|t') \psi_{n'}(t'),
$$

(69)

where $G_{n,n'}(t|t')$ is quantum exciton propagator, that is probability of exciton transition from site $n'$ at the moment $t'$ to site $n$ at the moment $t$.

It follows from Eq. (68) and Eq. (69) that $G_{n,n'}(t|t')$ is governed by the equation

$$
i\hbar \frac{\partial G_{n,n'}(t|t')}{\partial t} = \varepsilon G_{n,n'}(t|t') - \sum_{(n \neq m)} V_{n-m} G_{m,n'}(t|t'), \quad t > t',
$$

(70)

with the initial condition $G_{n,n'}(t|t) = \delta_{n,n'}$, here $\delta_{n,n'}$ is the Kronecker symbol. Let us put for simplicity $n' = 0$ and $t' = 0$ and introduce a notation

$$
G_n(t) \equiv G_{n,0}(t|0).
$$

(71)

It yields

$$
i\hbar \frac{\partial G_n(t)}{\partial t} = \varepsilon G_n(t) - \sum_{(n \neq m)} V_{n-m} G_n(t), \quad t \geq 0,
$$

(72)

with the initial condition

$$
G_n(t)|_{t=0} = \delta_{n,0}.
$$

(73)

To get the continuum version of Eq. (72) we apply transformations similar to Eq. (4) and Eq. (5). That is we define $G(k, t)$ as

$$
G(k, t) = \sum_{n=-\infty}^{\infty} e^{-ikn} G_n(t),
$$

(74)

$$
G_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk e^{ikn} G(k, t),
$$

(75)

and $G(k, t)$ satisfies the equation
\[ i\hbar \frac{\partial G(k,t)}{\partial t} = (\omega + \mathcal{V}(k)) G(k,t), \quad t \geq 0, \quad (76) \]

where \( \mathcal{V}(k) \) is the same as in Eq.(4) and the energy parameter \( \omega \) is defined by Eq.(50). The initial condition Eq.(73) now becomes

\[ G(k,t=0) = 1. \quad (77) \]

For further consideration it is convenient to introduce the quantum propagator \( g(k,t) \) which is

\[ g(k,t) = \exp\{i\frac{\omega t}{\hbar}\} G(k,t). \quad (78) \]

Thus, \( G(k,t) \) can be expressed in terms of \( g(k,t) \) as follow

\[ G(k,t) = \exp\{-i\frac{\omega t}{\hbar}\} g(k,t). \quad (79) \]

It follows from Eq.(76) that the propagator \( g(k,t) \) is governed by

\[ i\hbar \frac{\partial g(k,t)}{\partial t} = \mathcal{V}(k) g(k,t), \quad t \geq 0, \quad (80) \]

with the initial condition

\[ g(k,t = 0) = 1. \quad (81) \]

The solution of the problem Eqs.(80) and (81) is

\[ g(k,t) = \exp\left\{i\frac{\mathcal{V}(k)t}{\hbar}\right\}. \quad (82) \]

By substituting this solution into Eqs.(79) and (75) we find

\[ G_n(t) = \frac{1}{(2\pi)} \int_{-\pi}^{\pi} dk \exp(i kn - i \frac{\omega + \mathcal{V}(k)t}{\hbar}), \quad (83) \]

or, after extraction of the energy-time factor \( \exp\{-i\frac{\omega t}{\hbar}\} \),

\[ G_n(t) = \exp\{-i\frac{\omega t}{\hbar}\} g_n(t), \quad (84) \]

where the lattice propagator \( g_n(t) \) has been introduced by
\begin{equation}
    g_n(t) = \frac{1}{(2\pi)} \int_{-\pi}^{\pi} dk \exp(ikn - i\frac{\mathcal{V}(k)t}{\hbar}).
\end{equation}

Generalization to 1D lattice quantum exciton propagator \( g_{n,n'}(t|t') \) which describes transition from site \( n' \) at the time instant \( t' \) to site \( n \) at the time instant \( t \), is obvious

\begin{equation}
    g_{n,n'}(t|t') = \frac{1}{(2\pi)} \int_{-\pi}^{\pi} dk \exp\{ik(n - n') - i\frac{\mathcal{V}(k)(t - t')}{\hbar}\}.
\end{equation}

This is the quantum exciton propagator which describes 1D transport discrete in space and continuous in time. Let us note that \( g_{n,n'}(t|t') \) satisfies the following conditions:

1. Normalization condition

\begin{equation}
    \sum_{n=-\infty}^{\infty} g_{n,n'}(t|t') = 1,
\end{equation}

2. Consistency condition

\begin{equation}
    g_{n,n'}(t_1|t_2) = \sum_{m=-\infty}^{\infty} g_{n,m}(t_1|t') \cdot g_{m,n'}(t'|t_2).
\end{equation}

The last condition means that exciton propagator \( g_{n,n'}(t|t') \) can be considered as a transition quantum amplitude, and for propagations occurring in succession in time transition amplitudes are multiplied.

Further we will study behavior of \( g_n(t) \) at large |\( n \)|, when the main contribution to the integral Eq. (85) comes from small \( k \). Therefore, we expand integral over \( k \) from \(-\infty\) up to \( \infty\)

\begin{equation}
    g_n(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp(ikn - i\frac{\gamma_s k^{\nu(s)} t}{\hbar}),
\end{equation}

with

\begin{equation}
    \nu(s) = \begin{cases} 
    2, & \text{for } s > 3, \\
    s - 1, & \text{for } 2 < s < 3,
    \end{cases}
\end{equation}

and
\[ \gamma_s = \begin{cases} \frac{V_0 \zeta(s-2)}{2} , & \text{for } s > 3 , \\ D_s , & \text{for } 2 < s < 3 , \end{cases} \]  

(91)

where \( D_s \) is given by Eq.(21).

Asymptotic behavior of the quantum 1D propagator \( g_n(t) \) at large \( |n| \) depends on the parameter \( s \). Indeed, when \( s > 3 \) Eq.(89) goes to

\[ g_n(t) = \frac{\hbar}{2\pi i V_0 \zeta(s-2)t} \exp \left\{ -\frac{\hbar|n|^2}{2iV_0 \zeta(s-2)t} \right\} , \quad s > 3 , \]  

(92)

if we take into account Eqs.(90) and (91).

When \( 2 < s < 3 \) the integral in right-hand side of Eq.(89) can be expressed in terms of the Fox’s \( H \)-function [44] and we have

\[ g_n(\tau) = \frac{1}{|n|(s-1)} H_{1.1}^{1.2} \left[ \left( \frac{\hbar}{iD_s \tau} \right)^{1/(s-1)} |n| \left( \begin{array}{c} 1, 1/(s-1), (1, 1/2) \\ (1, 1), (1, 1/2) \end{array} \right) \right] , \]  

(93)

where \( H_{1.1}^{1.2} \) is the Fox’s function (for definition see, for example [44], [45]). From other hand, in the long-wave limit for \( 2 < s < 3 \) the integral in Eq.(89) can be estimated as

\[ g_n(t) \approx \frac{1}{\pi} \Gamma(s) \sin \left( \frac{\pi(s-1)}{2} \right) \left( \frac{iD_st}{\hbar} \right)^{s/(s-1)} \frac{1}{|n|^2} , \quad 2 < s < 3 . \]  

(94)

Thus, the long-wave asymptotics at large \( |n| \) of the quantum exciton propagator \( g_n(t) \) exhibits the power-law behavior for \( 2 < s < 3 \). Transition from Gaussian-like behavior Eq.(92) to power-law decay Eq.(94) is due to long-range interaction (second term in the right hand of Eqs.(68), (72)). This transition can be interpreted as phase transition. In fact, for \( s > 3 \) propagator \( g_n(t) \) given by Eq.(92) has correlation length (or characteristic scale) which can be describe by

\[ \Delta n \approx \left( \frac{2V_0 \zeta(s-2)t}{\hbar} \right)^{1/2} , \]

as far as in all above considerations we put the lattice constant \( a \) equal 1 for simplicity. As it follows from Eq.(94), the correlation length is infinite,
is the correlation length doesn’t exist for long-wave excitons in 1D lattice with $2 < s < 3$ power-law interaction.

Finally, let us note that the above discussed 1D quantum lattice dynamics for imaginary time is similar to the random walk model, that is similar to CTRW [24]. Indeed, if we put $it \rightarrow \tau$, Eq. (86) is transformed to

$$P_{n,n'}(\tau|\tau') = \frac{1}{(2\pi)} \int dk \exp\{ik(n - n') - \frac{\mathcal{V}(k)(\tau - \tau')}{\hbar}\}, \quad (95)$$

where $P_{n,n'}(\tau|\tau')$ is transition probability, i.e. the probability to walk from site $n$ to site $n'$ at the time interval $\tau - \tau'$. It is easy to see that $P_{n,n'}(\tau|\tau')$ is normalized

$$\sum_{n=-\infty}^{\infty} P_{n,n'}(\tau|\tau') = 1, \quad (96)$$

and satisfies the discrete version of the Kolmogorov-Chapmen equation

$$P_{n,n'}(\tau_1|\tau_2) = \sum_{m=-\infty}^{\infty} P_{n,m}(\tau_1|\tau') \cdot P_{m,n'}(\tau'|\tau_2). \quad (97)$$

That is we came to a continuous in time and discrete in space (1D lattice) random walk model. The obtained random walk model exhibits a phase transition from the Brownian random walk ($s > 3$) with finite correlation length to the symmetric $\alpha$-stable (with $\alpha = s - 1, 2 < s < 3$) or the Lévy random process with an infinite correlation length.

In the case of continuum space the integral over $k$ Eq. (95) can be expanded from $-\infty$ up to $\infty$ and we obtain transition probability distribution function $P(x, \tau|x', \tau')$

$$P(x, \tau|x', \tau') = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} dk \exp\left(ik(x - x') - \frac{\gamma_s k^\nu(s)(\tau - \tau')}{\hbar}\right), \quad (98)$$

where $\nu(s)$ and $\gamma_s$ are defined by Eq. (90) and Eq. (91) reciprocally. This probability distribution function exhibits transition from Gaussian-like behavior at $s > 3$ to power-law decay at $2 < s < 3$ because of the power-law long-range interaction.
5 Conclusions

We have presented analytical developments on classical and quantum 1D lattice dynamics with fractional power-law long-range interaction. The classical case has been treated in the framework of the well-known Frenkel-Kontorova chain model. Quantum lattice dynamics is considered following Davydov’s approach to dynamics of molecular excitons. It has been shown that the long-range power-law interaction leads, in general, to a nonlocal integral term in the equation of motion if we go from discrete to continuous space. In some particular cases with non-integer power $s$ for power-law interaction the integral term can be expressed through the fractional order derivative. In other words nonlocality from the long-range interaction reveals the dynamics in the form of space derivatives of fractional order.

As useful examples of classical lattice dynamics in the continuum media limit we were able to derive in a unified form fractional sine-Gordon and wave-Hilbert nonlinear equations.

In the quantum case we obtained linear a fractional Schrödinger equation, fractional nonlinear Schrödinger equation, linear Hilbert-Schrödinger equation, and nonlinear Hilbert-Schrödinger equation.

This applied approach permits one to see, in an explicit form, the interplay between second order and fractional order space derivatives both in classical and quantum dynamics.

We have observed that there exists critical value for the power of the long-range potential such that separates the cases of fractional equations from the well-known nonlinear dynamical equations with integer order derivatives.

The long-range interaction impact on quantum lattice propagators has been studied. It was shown that the exciton propagator exhibits a transition for the same critical value of the power-law exponent from the well-known Gaussian-like behavior to a power-law space decay. This transition can be treated as a phase transition accompanying by an infinite growth of the exciton correlation length.

It has been shown that in the imaginary time domain 1D quantum lattice dynamics with long-range interaction can be considered as random walk. The corresponding equations (Eq. (95) and Eq. (98)) for the transition probability of this random walk have been obtained. The random walk model also exhibits a phase transition from the Brownian random walk ($s > 3$) with finite correlation length to the Lévy random walk ($2 < s < 3$) with an infinite correlation length.
Finally, let us note that this approach can be generalized to 2D and 3D classical and quantum lattice dynamics.

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7 Appendix

7.1 Polylogarithm as a power series

The polylogarithm $\text{Li}_s(z)$ is defined by \[46]-\[48]

$$\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = \frac{z}{\Gamma(s)} \int_0^{\infty} dt \frac{t^{s-1}}{e^t - z},$$

(99)

here $s$ is real parameter and argument $z$ is the complex argument. It is easy to see that for $z = 1$ the polylogarithm $\text{Li}_s(1)$ reduces to the well-known Riemann zeta function $\zeta(s)$,

$$\text{Li}_s(1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (100)$$

It seems that the quantum statistical mechanics is the best known field where the polylogarithm arises in natural way. Indeed, the integral of the Bose-Einstein distribution is expressed in terms of a polylogarithm as,

$$\text{Li}_s(z) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt \frac{t^{s-1}}{e^{t/z} - 1}.$$ 

(101)

In statistical mechanics handbooks the above integral is referred as a Bose integral or a Bose-Einstein integral (see, for example \[19\]).

Next, the integral of the Fermi-Dirac distribution is also expressed in term of a polylogarithm,
This integral is referred as a Fermi-Dirac integral \[49\].

It is convenient to write the argument as \(e^\mu\), that is consider \(Li_s(e^\mu)\). Then one can use the Hankel contour integral for the polylogarithm \[50\],

\[
Li_s(e^\mu) = -\frac{\Gamma(1-s)}{2\pi} \oint_H dt \frac{(-t)^{s-1}}{e^{t-\mu} - 1}.
\]

where \(H\) represent the Hankel contour and \(s \neq 1, 2, 3, ...,\) and pole of the integrand does not lie on non-negative real axis at the \(t = \mu\). The integrand has a cut along the real axis from zero to infinity, with the real axis being on the lower half of the sheet (\(\text{Im} t \leq 0\)). For the case where \(\mu\) is real and non-negative, contribution of the pole has to be count,

\[
Li_s(e^\mu) = -\frac{\Gamma(1-s)}{2\pi} \oint_H dt \frac{(-t)^{s-1}}{e^{t-\mu} - 1} + 2\pi i R,
\]

where \(R\) is the residue of the pole \(R = \frac{\Gamma(1-s)(-\mu)^{s-1}}{2\pi}\).

To get power series (about \(\mu = 0\)) representation of the polylogarithm \(Li_s(e^{-\mu})\) we apply the Mellin transform following Ref. \[51\]

\[
M_s(r) = \int_0^{\infty} du u^{r-1} Li_s(e^{-u}) = \frac{1}{\Gamma(s)} \int_0^{\infty} du \int_0^{\infty} dt \frac{u^{r-1} e^{t} e^{-u} - 1}{t^{s-1}}.
\]

Then we observe that changing of variables \(t = xy, u = x(1-y)\) separates the integrals and we obtain

\[
M_s(r) = \frac{1}{\Gamma(s)} \int_0^{1} dy y^{r-1}(1-y)^{s-1} \int_0^{\infty} dx \frac{x^{s+r-1}}{e^x - 1} = \Gamma(r) Li_{s+r}(1) = \Gamma(r) \zeta(s+r).
\]

Now through the inverse Mellin transform we have
\[
\text{Li}_s(e^{-\mu}) = \frac{1}{2\pi i} \int_{\gamma} dr \mu^{-r} \Gamma(r) \zeta(s + r),
\]

(108)

Here \(c\) is a constant to the right of the poles of the integrand. The path of integration may be converted into a closed contour, and the poles of the integrand are those of the gamma function \(\Gamma(r)\) at \(r = -l\) with residues \((-1)^l/l!\) \((l = 0, -1, -2,\ldots)\), and of the Riemann zeta function \(\zeta(s + r)\) at \(r = 1 - s\) with residue +1. Summing the residues yields, for \(|\mu| < 2\pi\) and \(s \neq 1, 2, 3,\ldots\)

\[
\text{Li}_s(e^{-\mu}) = \Gamma(1-s)(\mu)^{s-1} + \sum_{l=0}^{\infty} \frac{\zeta(s-l)}{l!} (-\mu)^l.
\]

(109)

This equation gives us get power series (about \(\mu = 0\)) representation of the polylogarithm \(\text{Li}_s(e^{-\mu})\).

Let’s note that if the parameter \(s\) is a positive integer \(n\), both the \(l = n-1\) term and the gamma function \(\Gamma(1-n)\) become infinite, although their sum does not. For integer \(l > 0\) we have

\[
\lim_{s \to n+1} \left[ \frac{\zeta(s-l)}{l!} \mu^l + \Gamma(1-s)(-\mu)^{s-1} \right] = \frac{\mu^l}{l!} \left( \sum_{m=1}^{l} \frac{1}{m} - \ln(-\mu) \right)
\]

(110)

and for \(l = 0\),

\[
\lim_{s \to 1} [\zeta(s) + \Gamma(1-s)(-\mu)^{s-1}] = -\ln(-\mu).
\]

(111)

The power series representation Eq. (109) will be used to get Eqs. (114)-(116).

### 7.2 Properties of function \(V(k)\)

It is easy to see that \(V(k)\) given by Eq. (8) with \(V_n\) defined by Eq. (2) can be expressed in terms of the polylogarithm \(\text{Li}_s(z)\)

\[
V(k) = 2V_0 \sum_{n=1}^{\infty} \frac{\cos kn}{n^s} = 2V_0 \cdot \text{Re}\{\text{Li}_s(e^{-ik})\}.
\]

Further, the function \(V(k)\) defined by Eq. (7) becomes
\[ V(k) = 2V_0 \zeta(s) \operatorname{Re}(1 - \frac{\text{Li}_s(e^{-ik})}{\zeta(s)}). \]  

(113)

Taking into account power series representation given by Eq.(109) we find from Eq.(113) at the limit \( k \to 0 \) that

\[ V(k) \sim \frac{V_0 \pi}{\Gamma(s) \sin\left(\frac{\pi(s-1)}{2}\right)} |k|^{s-1}, \quad 2 \leq s < 3, \]  

(114)

\[ V(k) \sim -\frac{V_0 k^2}{2} \ln k^2, \quad s = 3, \]  

(115)

\[ V(k) \sim \frac{V_0 \zeta(s-2)}{2} k^2, \quad s > 3. \]  

(116)

References

[1] Ya. Frenkel, T. Kontorova, Phys. Z. Sowietunion 13, 1 (1938).
[2] O.M. Brawn, Y.S. Kivshar, Phys. Rep. 306, 1-108 (1998).
[3] G. L. Jr. Lamb, Elements of Soliton Theory, (New York: Wiley, 1980).
[4] H. Nakano and H. Takahashi, Phys. Rev. B52, 6606-6610 (1995).
[5] A.S. Davydov, Solitons in Molecular Systems, 2nd Ed. (Reidel, Dordrecht, 1991).
[6] A. Scott, Phys. Rep. 217, 167 (1992).
[7] V.L. Pokrovsky, A.J Virosztek, J. Phys. C16, 4513 (1983).
[8] A. Barone and G. Patemo, Physics and applications of the Josephson effect (Wiley, New York, 1982).
[9] G.L. Alfimov, Nonlinear Klein-Gordon and Schrödinger Systems: Theory and Applications In: Vazquez, L., Streit, L., Perez-Garcia, V.M. (Eds.)World Scientific, Singapore, p. 257 (1996).
[10] A.M. Kosevich, A.S. Kovalev, Radiation and Other Defects in Solids. (Institute of Physics Publ., Tbilisi, 1974), in Russian.
[11] Y. Ishimori, Prog. Theor. Phys. 68, 402 (1982).
[12] M. Kac and E. Helfand, J. Math. Phys. 4, 1078 (1973).
[13] E. Fermi, J. Pasta, and S. Ulam, Studies in Nonlinear Problems, I. Los Alamos report LA 1940, 1955. Reproduced in Nonlinear Wave Motion (Ed. A. C. Newell, Providence, RI: Amer. Math. Soc., 1974).
[14] S. Flach, C. R. Willis, Phys. Rep. 295, 181 (1998)
[15] S. Flach, Phys. Rev. E58, R4116 (1998).
[16] J.A. Tuszyński and M.L.A. Nip, Physica Scripta 51, 423-430 (1995).
[17] Y. Kuramoto, Chemical oscillations, Waves, and Turbulence, (Springer, 1984).
[18] S. Shima and Y Kuromoto, Phys. Rev. E69, 036213 (2004).
[19] A. Pikovsky, Cambridge Nonlinear Science Series #12: Synchronization: A Universal Concept in Nonlinear Sciences, (Cambridge University Press, 2001).
[20] J. Frohlich, R. Israel, E.H. Lieb, and B Simon, Commun. Math. Phys. 62, 1 (1978).
[21] S.K. Sarker and J.A. Krumhansl, Phys. Rev. B23, 2374 (1981).
[22] O.M. Brawn, Y.S. Kivshar, J.Phys.: Conden. Matter 2, 596 (1990); O.M. Brawn, Y.S. Kivshar, and I.I. Zelenskaya, Phys.Rev. B41, 7118 (1990).
[23] G.M. Zaslavsky, A.A. Stanislavsky, and M. Edelman, Chaotic and Pseudochaotic Attractors of Perturbed Fractional Oscillator, ArXiv nlin.CD/0508018 (2005) (http://arxiv.org/PS_cache/nlin/pdf/0508/0508018.pdf).
[24] E. W. Montroll, M. F. Shlesinger, in Nonequilibrium Phenomena II: On the Wonderful World of Random Walks (Studies in Statistical Mechanics, Vol. 11, Eds J L Lebowitz, E W Montroll) (Amsterdam, North-Holland, 1984) p. 1.
[25] A.I. Saichev, G.M. Zaslavsky, Chaos 7, 753-764 (1997).

[26] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives, Theory and Applications (Gordon and Breach, Amsterdam, 1993).

[27] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations (Wiley, New York, 1993).

[28] R. Hilfer, Ed., Applications of Fractional Calculus in Physics (World Scientific, Singapore, 2000).

[29] I. M. Gelfand and G. E. Shilov, Generalized Functions Vol.1, (Academic Press, New York, 1964).

[30] V. Tarasov and G.M. Zaslavsky, To be submitted.

[31] G.M. Zaslavsky, Phys. Rep. 371, 461-580 (2002).

[32] N. Laskin, Phys. Lett. A268, 298 (2000).

[33] G. Alfimov, T. Pierantozzi and L. Vazquez, Numerical study of a fractional Sine-Gordon equation, in "Fractional Differentiation and its Applications", Eds. A. Le Mahaute, J.A. Tenteiro Machado, J.C. Trigeassou, and J. Sabatier ( Bordeaux, France, 2004) page 644-649.

[34] A. Gurevich. Phys.Rev. B46, 3187, 1992.

[35] Yu.M. Aliiev and V.P. Silin, Phys.Lett. A 177, 259 (1993); G.L. Alfimov and V.P. Silin, J.Exp.Theor.Phys., 79, 369 (1994); G.L. Alfimov and V.P. Silin, Phys.Lett. A198, 105 (1995); G.L. Alfimov and V.P. Silin, J.Exp.Theor.Phys., 81, 915 (1995).

[36] Yu.M. Aliiev, V.P. Silin, and S.A. Uryupin, Superconductivity, 5, 230 (1992); R.G. Mintz and I.B. Snapiro, Phys.Rev. B49, 6188 (1994); G.L. Alfimov and A.F. Popkov, Phys.Rev. B52, 4503 (1995).

[37] Yu.M. Aliiev, K.N. Ovchinnikov, V.P. Silin, and S.A. Uryupin, J.Exp.Theor.Phys. 80, 551 (1995).

[38] V.M. Agranovich, M.D. Galanin, Electronic Excitation Energy Transfer in Condensed Matter. (North-Holland, Amsterdam, 1982).
[39] N. Laskin, Phys. Rev. E62, 3135 (2000);
[40] N. Laskin, Phys. Rev. E66, 056108 (2002); N. Laskin, Chaos, 10, 780 (2000).
[41] H. Weitzner, G.M. Zaslavsky, Commun. Nonlinear Sci. and Numer. Simul. 8, 273-281 (2003).
[42] V. Tarasov and G.M. Zaslavsky, Physica A354, 249 (2005).
[43] A.V. Milovanov, J.J. Rasmussen, Phys. Lett. A337, 75-80 (2005).
[44] N. Laskin, Levy flights over quantum paths, ArXiv quant-ph/0504106 (2005) [http://arxiv.org/PS_ache/quant-ph/pdf/0504/0504106.pdf].
[45] A.M. Mathai and R.K. Saxena, The H-function with Applications in Statistics and Other Disciplines (Wiley Eastern, New Delhi, 1978).
[46] L. Lewin, Dilogarithms and Associated Functions, (Macdonald, 1958).
[47] W. Magnus, F. Oberhettinger, and P.R. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics (Springer-Verlag, Berlin, 1966).
[48] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, Higher Transcendental Functions, Vol. 1. New York: Krieger, pp. 30-31, (1981).
[49] L. D. Landau and E. M. Lifshitz, Statistical Physics, 3rd edition, Part 1, (Pergamon Press, Oxford 1980).
[50] E.T. Whittaker and G.N. Watson, A Course of Modern Analysis, Fourth edition, (Cambridge University Press, 1962).
[51] J.E. Robinson, Phys. Rev., 83, 678-679, (1951).