RAMIFICATION LOCI OF NON-ARCHIMEDEAN CUBIC RATIONAL FUNCTIONS

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Abstract. For a cubic rational function with coefficients in a non-archimedean field \( K \) whose residue characteristic is 0 or greater than 3, there are 2 possibilities for the shape of its Berkovich ramification locus, considered as an endomorphism of the Berkovich projective line: one is the connected hull of all the critical points, and the other is consisting of 2 disjoint segments. In this paper, we list up all the possible forms of cubic rational functions and calculate their ramification loci.

1. Introduction

1.1. Main results. Let \( K \) be an algebraically closed field with complete non-archimedean and non-trivial valuation. We assume that the residue characteristic of \( K \) is 0 or greater than 3. \( \mathcal{O}_K \) be the valuation ring, and \( \phi \) be a cubic rational function with coefficients in \( K \). Rational functions can be considered as endomorphisms of the Berkovich projective line \( \mathbb{P}^{1,an} \) (for definition, see [1]). The Berkovich ramification locus, or simply the ramification locus of \( \phi \) is defined to be the following set

\[ R_\phi = \{ x \in \mathbb{P}^{1,an} | m_\phi(x) > 1 \}, \]

where the symbol \( m_\phi(x) \) is the multiplicity of \( \phi \) at \( x \), i.e., the degree of the field extension \([\kappa(x) : \kappa(\phi(x))]\), where the field \( \kappa(x) \) is the complete residue field at \( x \) (for details and another description of \( m_\phi \), see [2]). The ramification locus is a closed subset of \( \mathbb{P}^{1,an} \).

The aim of this paper is to give a complete description of the shape of the ramification locus of any cubic rational function. Rational functions \( \phi \) and \( \psi \) are conjugate if there exists Möbius transformations \( \tau \) and \( \sigma \) such that \( \phi = \tau \circ \psi \circ \sigma \). Since automorphisms do not change the shape of ramification loci, we will describe it for the following representative of each conjugate class.

First, if there exists a critical point of \( \phi \) whose multiplicity is 3, then the rational function \( \phi \) is conjugate to a polynomial. This can be done by taking \( \tau \) and \( \sigma \) so that the critical point with multiplicity 3 of \( \tau \circ \phi \circ \sigma \) is \( \infty \). Otherwise, taking a suitable \( \tau \) and \( \sigma \), we may assume the following conditions:

1. 0 and 1 are fixed critical points,
2. \( \infty \) is fixed but not critical, and
3. the other 2 critical points are distinct.

The cubic rational function \( \phi \) with the above conditions can be put

\[ \phi(z) = \frac{a_3z^3 + a_2z^2}{b_2z^2 + b_1z + b_0} = \frac{(1 - \alpha)(1 - \beta)z^2(z - \gamma)}{(1 - \gamma)(z - \alpha)(z - \beta)}, \]

where \( a_2, a_3, b_0, b_1, b_2 \in \mathcal{O}_K \) and \( \alpha, \beta, \gamma \in K \). Set \( f(z) = a_3z^3 + a_2z^2 \) and \( g(z) = b_2z^2 + b_1z + b_0 \). To satisfy the above 3 conditions, we further assume several conditions on them; for details, see the next subsection. Throughout this paper, we consider polynomials or rational functions of this form to calculate the ramification locus. Our result is briefly stated as follows:

Theorem 1.1. The ramification locus of a cubic rational function \( \phi \) is connected if and only if \( \phi \) is conjugate to a polynomial or a rational function of the above forms with the following conditions:

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Therefore, in degree 3 case, since the ramification is always tame when the residue characteristic of non-trivial case; ramification loci of polynomials, rational functions of good reduction, and quadratic general strategy.

This research comes from the works of Faber [3] and [4]. There he studies the shape of the ramification locus by [3, Corollary 6.6]. Also, in general, a component containing a point (not necessarily classical) (be the residue field of \( \mathcal{O}_K \) is denoted by \( \wp \in \mathbb{K} \) denotes its reduction. In the same way, the reduction of any rational function \( \psi \in \mathcal{O}_K[z] \) is denoted by \( \overline{\psi} \). By definition, \( \overline{\psi} = 0 \) is equivalent to the condition \( |a| < 1 \). For a fixed coordinate of \( \mathbb{P}^1, \) denote its Gauss point by \( \zeta_0,1 \).

For a rational function as in \( \triangledown \) in Section [4.1] we may assume that

- \( a_3 \neq 0 \),
- \( b_2 \neq 0 \),
- at least one of \( a_3, a_2, b_2, b_1 \) or \( b_1 \) is invertible, and
- polynomials \( f \) and \( g \) have no common root i.e. \( \alpha \neq 0 \), \( \gamma \) and \( \beta \neq 0 \), \( \gamma \).

Also we have the following equations about the coefficients:

\[
\phi(1) = 1, \quad \text{and} \quad \text{Wr}_\phi(1) = 0,
\]

where the \( \text{Wr}_\phi(z) \) is the Wronskian of \( \phi \):

\[
\text{Wr}_\phi(z) = (3a_2z^2 + 2a_2z)(b_2z^2 + b_1z + b_0) - (2b_2z + b_1)(a_3z^3 + a_2z^2),
\]

The first condition is equivalent to

\[
(\triangledown) \quad a_3 + a_2 = b_2 + b_1 + b_0.
\]
The second condition is equivalent to \((3a_3 + 2a_2)(b_2 + b_1 + b_0) - (a_3 + a_2)(2b_2 + b_1) = 0\), i.e.,

\[
3a_3 + 2a_2 - 2b_2 - b_1 = 0
\]

under the condition (\(\heartsuit\)). We can then list up all the possible cases for the coefficients under these conditions.

(1-1): When \(\overline{a}_3 = \overline{a}_2 = 0\), we have \(\phi(\zeta_{0,1}) \neq \zeta_{0,1}\), which is treated in Section 2.1. This situation is divided into the following 3 cases:

(1-1-1-1): \(|\gamma| \leq 1\) i.e. \(|a_2| \leq |a_3|\), and \(g(\gamma) = 0\),

(1-1-1-2): \(|\gamma| \leq 1\) and \(g(\gamma) \neq 0\),

(1-1-2): \(|\gamma| > 1\) i.e. \(|a_2| > |a_3|\).

(1-2-1-1): When \(\overline{a}_3 = \overline{b}_0 = \overline{b}_1 = 0\) and \(\overline{a}_2 \neq 0\), we have \(\phi(\zeta_{0,1}) \neq \zeta_{0,1}\), which is treated in Section 2.1. This situation is divided into the following 3 cases:

(1-2-1-1-1): \(|b_1| > |a_3|\),

(1-2-1-1-2): \(|b_1| \leq |a_3|\) and \(|a_3| > |b_0|\),

(1-2-1-1-3): \(|a_3| = |b_0| = |b_1|\).

Any other condition on \(a_3, b_1\) and \(b_0\) is impossible by (\(\heartsuit\)) and (\(\diamondsuit\)).

(1-2-1-2): When \(\overline{a}_3 = \overline{b}_0 = 0\), \(\overline{a}_2 \neq 0\) and \(\overline{b}_1 \neq 0\), the degree of \(\phi\) is 1. This case is treated in Section 2.2.

(1-2-2): When \(\overline{a}_3 = 0\), \(\overline{a}_2 \neq 0\) and \(\overline{b}_0 \neq 0\), the degree of \(\overline{\phi}\) is 2, which is treated in Section 2.3.

(2-1-1): When \(\overline{a}_3 \neq 0\) and \(\overline{a}_2 = \overline{b}_0 = 0\), the degree of \(\overline{\phi}\) is 1. It is treated in Section 2.2.

(2-1-2): When \(\overline{a}_3 \neq 0\), \(\overline{a}_2 = 0\) and \(\overline{b}_0 \neq 0\), the function \(\phi\) has good reduction i.e., the ramification locus is connected.

(2-2): When \(\overline{a}_3 \neq 0\) and \(\overline{a}_2 \neq 0\), the degree of \(\overline{\phi}\) depends on whether \(g(\gamma)\) is zero or not, and whether \(\overline{b}_0\) is zero or not.

(2-2-1-1): When \(\overline{b}_0 = \overline{b}_2 = 0\), the degree of \(\phi\) is 2. This case is treated in Section 2.3.

(2-2-1-2): When \(\overline{b}_0 = g(\gamma) = 0\) and \(\overline{b}_2 \neq 0\), the degree of \(\overline{\phi}\) is 1. It is treated in Section 2.2.

(2-2-2-1): When \(\overline{b}_2 = g(\gamma) = 0\) and \(\overline{b}_0 \neq 0\), the degree of \(\overline{\phi}\) is 2. It is treated in Section 2.3.

The numbering is due to Figure 11 and Figure 2.

Since the Wronskian \(Wr_\phi(z)\) vanishes at 0 and 1, we have

\[
Wr_\phi(z) = z(z - 1)\psi(z),
\]

where

\[
\psi(z) = a_3b_2z^2 + (2a_3b_1 + a_3b_2)z - 2a_3b_0.
\]

In each of the above cases, we compare the zeros of \(\overline{\psi}(z)\) and \(Wr_{\overline{\phi}}(z)\) to calculate the ramification locus.
2. Calculation

2.1. The case $\phi(\zeta_{0,1}) \neq \zeta_{0,1}$. Cases in (1-1) requires $\alpha_3 = \alpha_2 = 0$. It follows from $\clubsuit$ and $\heartsuit$ that

$\bar{b}_2 + \bar{b}_1 + \bar{b}_0 = 0$, and

$2\bar{b}_2 + \bar{b}_1 = 0$,  

$\bar{a}_3 = 0$

\[\begin{align*}
\alpha_3 &= 0 \\
\pi_2 &= 0 \\
|\gamma| &\leq 1 \\
g(\gamma) &= 0 \\
case(1-1-1) \\
|\gamma| &> 1 \\
g(\gamma) &\neq 0 \\
case(1-1-1-2) \\
\bar{b}_1 &= 0 \\
\alpha_3 &\leq |a_3| \\
|a_3| &> |b_0| \\
case(1-2-1-1-1) \\
|a_3| = |b_0| = |b_1| \\
case(1-2-1-1-3) \\
\bar{b}_1 &\neq 0 \\
case(1-2-1-2) \\

\begin{align*}
\alpha_2 &= 0 \\
\pi_3 &\neq 0 \\
\bar{b}_0 &= 0 \\
\bar{b}_2 &= 0 \\
case(2-1-1) \\
g(\gamma) &= 0 \\
case(2-2-1-1-1) \\
\alpha &= \gamma = 1, \\
\beta &= 1/2 \\
case(2-2-2-1-1) \\
\alpha &= \gamma \neq 1, \\
\beta &\neq 1/2 \\
case(2-2-2-2-1) \\

\end{align*}

\[\begin{align*}
\alpha_2 &= 0 \\
\pi_3 &\neq 0 \\
\bar{b}_0 &\neq 0 \\
\bar{b}_2 &\neq 0 \\
case(2-1-2) \\
g(\gamma) &\neq 0 \\
case(2-2-1-2) \\
\alpha &= \gamma = 1, \\
\beta &= 1/2 \\
case(2-2-2-1-2) \\
\alpha &= \gamma \neq 1, \\
\beta &\neq 1/2 \\
case(2-2-2-2-2) \\
\end{align*}

\[\begin{align*}
\alpha_2 &= 0 \\
\pi_3 &\neq 0 \\
\bar{b}_0 &\neq 0 \\
\bar{b}_2 &\neq 0 \\
case(2-2-1) \\
g(\gamma) &= 0 \\
case(2-2-1-1) \\
\alpha &= \gamma = 1, \\
\beta &= 1/2 \\
case(2-2-2-1-1) \\
\alpha &= \gamma \neq 1, \\
\beta &\neq 1/2 \\
case(2-2-2-2-1) \\
\end{align*}

\[\begin{align*}
\alpha_2 &= 0 \\
\pi_3 &\neq 0 \\
\bar{b}_0 &\neq 0 \\
\bar{b}_2 &\neq 0 \\
case(2-2-2) \\
g(\gamma) &\neq 0 \\
case(2-2-2-1) \\
\alpha &= \gamma = 1, \\
\beta &= 1/2 \\
case(2-2-2-2-1) \\
\alpha &= \gamma \neq 1, \\
\beta &\neq 1/2 \\
case(2-2-2-2-2) \\
\end{align*}\]
from which we have $\overline{b}_0 = \overline{b}_2$ and $\overline{b}_1 = -2\overline{b}_2$.

In case (1-1-1-1), we have $g(\gamma) = \overline{b}_2\gamma^2 - 2\overline{b}_2\gamma + \overline{b}_2 = \overline{b}_2(\gamma - 1)^2 = 0$, i.e., $(a_2/a_3)\gamma = -1$. The polynomial $\psi(z)$ in (♣) is

$$\psi(z) = a_3(b_2z^2 + (2b_1 + b_2)z - 2a_2b_0/a_3),$$

The reduction of $\psi/a_3$ is

$$\overline{\psi/a_3}(z) = \overline{b}_2z^2 - 3\overline{b}_2z - 2\overline{b}_0(a_2/a_3) = \overline{b}_2(z^2 - 3z + 2).$$

The solutions of $\overline{\psi/a_3}(x) = 0$ are $\overline{\gamma}_1 = -2$ and $\overline{\gamma}_2 = -1$.

On the other hand, since $\phi(\zeta_{0,1}) = \zeta_{0,|a_3|}$ in this case, we have

$$\overline{\phi/a_3}(z) = \frac{z^2(z - \overline{\gamma})}{\overline{b}_2(z - 1)^2} = \frac{z^2}{\overline{b}_2(z - 1)}.$$

The Wronskian is

$$W_{\overline{\gamma}}(z) = \overline{b}_2z(z - 2).$$

Therefore, the ramification locus has 2 connected components; one is the segment connecting 0 and $c_1$ and the other is the one connecting 1 and $c_2$, as shown in Figure 3.

The case (1-1-1-2) is when $\phi$ has potentially good reduction; the ramification locus is always connected in this case.

In case (1-1-2), we can calculate $c_1$, $c_2$ and zeros of $W_{\overline{\gamma}}(z)$ in the similar way as above by replacing $\phi/a_3$ and $\psi/a_2$ by $\phi/a_2$ and $\psi/a_2$ respectively; the zeros of $\overline{\psi/a_2}$ are $\overline{\gamma}_1 = \overline{\gamma}_2 = \infty$ i.e. they have absolute value greater than 1, and the zeros of $W_{\overline{\gamma}}(z)$ are 0 and 1. The ramification locus has hence two connected components; one is the segment connecting 0 and 1, and the other is the one connecting $c_1$ and $c_2$.

In cases (1-2-1-1), we have $\overline{\gamma}_2 = \overline{b}_2$.

In case (1-2-1-1), consider

$$\phi'(z) = \frac{\phi(z) - a_2/b_2}{b_1} = \frac{b_2a_3z^3/b_1 - a_2z - a_2b_0/b_1}{b_2(b_2z^2 + b_1z + b_0)}.$$

Since $2b_0/b_1 = -1$ by (♠), we have

$$W_{\phi'}(z) = -\overline{b}_2z^2 + 2\overline{b}_2z(z - 1/2) = \overline{b}_2z(z - 1).$$

By Newton polygon argument, the two zeros of $\psi$ have absolute value greater than 1. Therefore, the ramification locus has two connected components; one is the segment connecting 0 and 1, and the other is the one connecting the remaining two critical points.

In case (1-1-2) and (1-2-1-1), the shape of the ramification locus looks like Figure 4.

We can do the similar calculation for the cases (1-2-1-2) and (1-2-1-1-3) by replacing $b_1$ by $a_3$.

In case (1-2-1-2), $\overline{\gamma}_1 = 0$ and $\overline{\gamma}_2 = -1$. The Wronskian of the reduction of $(\phi - a_2/b_2)/a_3$ is $\overline{b}_2(z + 1)(z - 1)$. Therefore, the ramification locus has two components; one is the segment connecting 0 and $c_1$, and the other is the one connecting 1 and $c_2$ i.e. as shown in Figure 3.

In case (1-2-1-1-3), The reduction of $(\phi - a_2/b_2)/a_3$ has degree 3 i.e. of good reduction. The ramification locus is always connected.
2.2. **The case** $m_\phi(\zeta_{0,1}) = 1$. In this case, the ramification locus must have two connected components and neither of them contains the Gauss point $\zeta_{0,1}$. The remaining critical points $c_1$ and $c_2$ must satisfy that $\overline{c}_1 = 0$ and $\overline{c}_2 = 1$. Figure 5 shows its shape.

2.3. **The case** $m_\phi(\zeta_{0,1}) = 2$. In this case, the following three cases are possible:

1. the ramification locus has two connected components, one of which is the segment connecting 0 and $c_1$ and the other is the one connecting 1 and $c_2$;
2. the ramification locus has two connected components, one of which is the segment connecting 0 and 1 and the other is the one connecting the two remaining critical points i.e. as shown in Figure 4;
3. the ramification locus is connected.

If the two remaining critical points are of absolute value greater than 1 i.e. the case (2) above, we must have $|a_3b_2| < 1$ and $|a_3b_1 + a_3b_0| < 1$ in (♠). In the list in Section 1, it is possible only in case (1-2-2).

By a straightforward calculation similar to that in Section 2.1, the case (1) happens in any other cases except for the cases (2-2-2-2-1-1) and (2-2-2-2-1-2).

In each of these cases where (1) occurs, the reduction of the zeros of $\psi$ is as follows:

- **(1-2-2):** $|c_1| = |c_2| > 1$ i.e. $\overline{\tau}_1 = \overline{\tau}_2 = \infty$, and the shape looks like Figure 9
- **(2-2-1-1):** $\overline{\tau}_1 = 0$ and $\overline{\tau}_2 = \infty$, and the shape looks like Figure 8
- **(2-2-1-2-2):** $\overline{\tau}_1 = \infty$ and $\overline{\tau}_2 = 1$, and the shape looks like Figure 7

Therefore, we calculate the ramification locus when $\overline{a}_3, \overline{a}_2, \overline{b}_0, \overline{b}_2 \neq 0$ and $g(\gamma) = 0$. In this case, we have from $\overline{\text{Wr}}_{\phi}(1) = 0$ that $\overline{\beta} = 1/2$ or $\overline{\alpha} = \overline{\gamma} = 1$. When either of these two equations fails to hold, we have the case (1). The only non-trivial case is when $\phi$ satisfies the both equations i.e. the case (2-2-2-2-1-1), which is treated in the next subsection.

For the case (2-2-2-2-1-2), the reduction of the remaining critical points are

- when $\overline{\beta} = 1/2$: $\overline{\tau}_1 = \overline{\tau}_2 = \overline{\alpha}$ i.e. as shown in Figure 9
- when $\overline{\alpha} = \overline{\gamma} = 1$: $\overline{\tau}_1 = 2\overline{\beta}$ and $\overline{\tau}_2 = 1$ i.e. as shown in Figure 10

2.4. **The case** (2-2-2-2-1-1). The reduction of the Wronskian is

$$\overline{\text{Wr}}_{\phi}(z) = z(z - 1)^3,$$

i.e. $\overline{\tau}_1 = \overline{\tau}_2 = 1$.

The Wronskian of $\overline{\psi}$ is

$$\overline{\text{Wr}}_{\overline{\psi}}(z) = z(z - 1).$$
The reduction of the 2 remaining critical points are both 1, from which we need more detailed analysis in order to determine the ramification locus. Since $\tilde{\alpha} = \tilde{\gamma} = 1$ and $\tilde{\beta} = 1/2$, we have some $p$, $q \in \{z \in K : |z| < 1\}$ such that

$$\beta = \frac{1}{2} + p, \quad \text{and} \quad \gamma = 1 + q.$$ 

Since $W_{r,\phi}(1) = 0$, we have that

$$\alpha = \frac{1 - 2p + q + 2pq}{1 - 2p + 4pq}.$$ 

The solution $c_{\pm}$ other than $\psi$ is

$$c_{\pm} = \alpha + \beta - \frac{1}{2} \pm \sqrt{\left(\alpha + \beta - \frac{1}{2}\right)^2 - 2\alpha\beta\gamma}$$

$$= \frac{1 - p + q + 2pq - 2p^2 + 4p^2q}{1 - 2p + 4pq} \pm \frac{\sqrt{R}}{1 - 2p + 4pq},$$

where we put $R$ to be the terms inside of the root i.e.

$$R = (1 - 2p + 4pq)^2 \left(\left(\alpha + \beta - \frac{1}{2}\right)^2 - 2\alpha\beta\gamma\right)$$

$$= p^2 - 2pq - 4p^3 + 8p^2q - 6pq^2 + 4p^4 - 4pq^3 - 16p^4q + 24p^3q^2 - 16p^2q^3 + 16p^4q^2 - 16p^3q^3.$$

\[\text{Figure 6.}\]
\[\text{Figure 7.}\]
\[\text{Figure 8.}\]
\[\text{Figure 9.}\]
\[\text{Figure 10.}\]
Therefore, $1 - c_{\pm}$ is

$$1 - c_{\pm} = \frac{p + q - 2pq - 2p^2 + 4p^2q}{1 - 2p + 4pq} \pm \sqrt{R}$$

We compare the absolute value of the terms appeared in $1 - c_{\pm}$ for each of the following 5 cases:

- **Case 1:** $|p| < |q|$;
- **Case 2:** $|p| = |q|$ and $|p + q| < |p|$;
- **Case 3:** $|p| > |q|$;
- **Case 4:** $|p| = |q|$ and $|p - 2q| < |p|$;
- **Case 5:** $|p| = |q|$ and $|4p + q| < |p|$;
- **Case 6:** $|p| = |q| = |p + q| = |p - 2q| = |4p + q|$;

Before analyzing them, let us state a lemma which is used several times in the following arguments.

**Lemma 2.1.** *In the above notation, the ramification locus is connected if $|1 - c_-| = |1 - c_+|$.*

**Proof.** If not, the ramification locus consists of 2 segments. If one segment connects 0 and 1, then it must intersect with the other one at $\zeta_{1,|1-c_+|}$. By the same argument, in any other possibilities of the 2 segments, they must intersect at $\zeta_{1,|1-c_+|}$, too. This is contradiction. $\Box$

**Case 1.** In this case, the result is the following:

**Proposition 2.2.** *In Case 1, the ramification locus is connected. We have $c_{\pm} = 1$ and $|1 - c_{\pm}| = |q|$. The shape is as shown in Figure 11.*

**Proof.** By the straightforward calculation of the absolute values, we have

$$|\sqrt{R}| = |pq| < |q|,$$

and

$$\left|\frac{p + q - 2pq - 2p^2 + 4p^2q}{1 - 2p + 4pq}\right| = |q|.$$ 

Therefore, both of $c_+$ and $c_-$ satisfies

$$|c_{\pm} - 1| = |q| < 1.$$ 

In this case, the ramification locus must be connected by Lemma 2.1. $\Box$

**Case 2.** In this case, the result is the following:

**Proposition 2.3.** *In Case 2, then the ramification locus consists of two connected components; one is the segment connecting 0 and $c_-$ and the other is the one connecting 1 and $c_+$. The points $c_{\pm}$ satisfies $c_{\pm} = 1$, $|1 - c_-| = |p|$ and $|1 - c_+| < |p|$.*

![Figure 11](image1.png) ![Figure 12](image2.png) ![Figure 13](image3.png)
Proof. Since
\[ R = p^2 - 2pq - 4p^3 + 8p^2q - 6pq^2 + 4p^4 - 4pq^3 - 16p^4q + 24p^3q^2 - 16p^2q^3 + 16p^4q^2 - 16p^3q^3 \]
= \( p^2(1-x) \)
where \( |x| < 1 \), we have
\[
\sqrt{R} = p\sqrt{1-x} = p\left(1 - \frac{x}{2} + (\text{h.o.t. of } x)\right).
\]

Therefore, we have
\[
1 - c_+ = \frac{p + q - 2pq - 2p^2 + 4p^2q}{1 - 2p + 4pq} = \frac{\sqrt{R}}{1 - 2p + 4pq}
\]
so \( |1 - c_+| < |p| \). A similar calculation shows that \( |1 - c_-| = |p| \).

Next, to have the shape of the ramification locus, we calculate the multiplicity of \( \rho \) at \( \zeta_{1,|p|} \). To calculate it, we consider the following rational function \( \rho \):

\[
\rho(z) := \phi(1 + z) - \phi(1) = -\frac{(1 - 2p)^2z^3 + (4p + q + 4pq - 8p^2 + 4p^2q)z^2}{(1 - 2p + 4pq)z - (q - 2pq))(2z + 1 - 2p)}
\]
By setting \( \sigma(z) = z - 1 \), we have \( \rho(z) = \sigma \circ \phi \circ \sigma^{-1} \). Hence \( \rho \) is conjugation of \( \phi \) by \( \sigma \). To calculate the multiplicity of \( \phi \) at \( \zeta_{1,|p|} \), we need to calculate the multiplicity of \( \rho \) at \( \zeta_{0,|p|} \). For \( |z| \leq 1 \),

\[
|\rho(pz)| = \frac{|\rho(pz)|}{|pz|^2} = \frac{|(1 - 2p)^2z^3 + (4 + \frac{q}{p} + 4q - 8p + 4pq)z^2|}{(1 - 2p + 4pq)z - (q - 2pq))(2pz + 1 - 2p)}
\]

from which we have \( \rho(\zeta_{0,|p|}) = \zeta_{0,|p|^2} \). Therefore, \( m_{\rho}(\zeta_{0,|p|}) = \deg \rho(pz)/p^2 \).

Next, to have the shape of the ramification locus, we calculate the multiplicity of \( \phi \) at \( \zeta_{1,|p|} \). To calculate it, we consider the following rational function \( \rho \):

\[
\rho(z) := \phi(1 + z) - \phi(1) = -\frac{(1 - 2p)^2z^3 + (4p + q + 4pq - 8p^2 + 4p^2q)z^2}{(1 - 2p + 4pq)z - (q - 2pq))(2z + 1 - 2p)}
\]
By setting \( \sigma(z) = z - 1 \), we have \( \rho(z) = \sigma \circ \phi \circ \sigma^{-1} \). Hence \( \rho \) is conjugation of \( \phi \) by \( \sigma \). To calculate the multiplicity of \( \phi \) at \( \zeta_{1,|p|} \), we need to calculate the multiplicity of \( \rho \) at \( \zeta_{0,|p|} \). For \( |z| \leq 1 \),

\[
|\rho(pz)| = \frac{|\rho(pz)|}{|pz|^2} = \frac{|(1 - 2p)^2z^3 + (4 + \frac{q}{p} + 4q - 8p + 4pq)z^2|}{(1 - 2p + 4pq)z - (q - 2pq))(2pz + 1 - 2p)}
\]

which is of degree 2.

Therefore, the ramification locus in this case has 2 components; one connects 0 and \( c_- \) and the other connects 1 and \( c_+ \). □

Case 3-Case 6.

Proposition 2.4. In Case 3, Case 4, Case 5 and Case 6, the ramification locus is connected. In Case 3, Case 4 and Case 5, we have \( |1 - c_{\pm}| = |p| \) i.e. as shown in Figure 13, and in Case 6, we have exactly one of \( |1 - c_{\pm}| \) is smaller than \( |p| \) and the other is equal to \( |p| \), i.e., as shown in Figure 14.

Proof. By the straightforward calculation of the absolute values, we have \( |1 - c_{\pm}| = |p| \) in Case 3 and Case 4, where we have the connected ramification locus by Lemma 2.1. Hence we consider Case 5 and Case 6. In these cases,

\[
R = p^2 \left(1 - \frac{2q}{p} - 4p + 8q - \frac{6q^2}{p} + 4p^3 - \frac{4q^3}{p} - 16p^2q + 24pq^2 - 16pq^3 + 16p^2q^2 - 16pq^3 \right)
\]
= \( p^2 \left(1 - \frac{2q}{p} + x\right) \),
where \( |x| < 1 \). Hence we have

\[
\sqrt{R} = p\sqrt{\frac{1 - 2q}{p} + x} = p\left(\sqrt{\frac{1 - 2q}{p} + \frac{x}{2\sqrt{1 + 2q/p}}} + \text{h.o.t. of } x\right),
\]

By straightforward calculation, we have

\[
1 - c_{\pm} = \frac{p + q - 2pq - 2p^2 + 4q^2 p}{1 - 2p + 4pq} \pm \frac{\sqrt{R}}{1 - 2p + 4pq}
\]

\[
= \frac{p}{1 - 2p + 4pq} \cdot \left(1 + \frac{q}{p} - 2q - 2p + 4pq \pm \sqrt{1 - \frac{2q}{p} + \frac{x}{2\sqrt{1 + 2q/p}}} + \text{h.o.t. of } x\right)
\]

\[
= \frac{p}{1 - 2p + 4pq} \cdot \left(1 + \frac{q}{p} \pm \sqrt{1 - \frac{2q}{p} + y}\right),
\]

where \( |y| < 1 \). Therefore, we have \( |1 - c_+| < |p| \) or \( |1 - c_-| < |p| \) happens when

\[
\left|1 + \frac{q}{p} \pm \sqrt{1 - \frac{2q}{p}}\right| < 1.
\]

This is equivalent to the condition that \( 1 + \frac{2q}{p} + \left(\frac{q}{p}\right)^2 = 1 - \frac{2q}{p} \), whence

\[
\frac{q}{p}(4 + \frac{q}{p}) = 0.
\]

Since \( \frac{q}{p} \neq 0 \) by \( |p| = |q| \), This occurs when \( |4p + q| < |p| \) i.e. Case 5. In Case 6, we have \( |1 - c_{\pm}| = |p| \) i.e. the ramification locus is connected by Lemma 2.1.

In Case 5,

\[
\frac{\rho(qz)}{q^2} = \frac{-q^3(1 - 2p)^2z^3 + q^3(4p/q + 1 + 4p - 8p^2/q + 4p^2)z^2}{q^3((1 - 2p + 4pq)z - (1 - 2p))2qz + 1 - 2p)}.
\]

Therefore,

\[
\bar{\rho}(z) = -\frac{z^3 + (4p/q)z^2}{z - 1} = \frac{z^3}{z - 1}.
\]

Since \( m_{\phi}(\zeta_{1,|p|}) = \deg \bar{\rho} = 3 \), the ramification component is always connected in this case, too. \( \square \)

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