TWO NOTES ON MULTIFORK EXTENSIONS OF SLIM RECTANGULAR LATTICES WITH APPLICATIONS TO THE CONGRUENCE LATTICES OF THESE LATTICES

GÁBOR CZÉDLI

Dedicated to my grandchildren, Péter, Adélia, Valentina Blanka, and Liliána

ABSTRACT. Slim rectangular lattices were introduced by G. Grätzer and E. Knapp in 2009. According to a 2014 result of the author, these lattices can be obtained from direct products of two chains by adding multiforks to them at distributive 4-cells. Here we relax the assumption of distributivity and, moreover, we can even thrust multiforks at certain elements. Lamps, introduced in 2021 by the author, are tools that give the best insight into the congruence lattice of a slim rectangular lattice. Applying lamps and thrusting multiforks, we prove three new properties of the class of congruence lattices of slim rectangular lattices. The Dioecious Maximal Elements Property, published by the author in 2021, becomes a corollary.

1. INTRODUCTION

1.1. A heuristic introduction. Following Grätzer and Knapp [17], a slim semimodular lattice is a finite, planar, $M_3$-free, semimodular lattice. Our goal is two-fold.

First, we are going to present a new construction for slim rectangular lattices. (A slim rectangular lattice is a slim semimodular lattice in which there are two doubly irreducible elements that are complements of each other.) Before getting acquainted with the necessary definitions, a first impression about this construction can be obtained by looking at Figures 1 and 2. These figures show that by thrusting a 3-fold multifork into a slim rectangular lattice $L$ at the peak of an internal lamp $J$, we obtain a new slim rectangular lattice, $\text{Thr}_3(L, J)$. As opposed to the multifork extension introduced in Czédli [1] and used at several papers thereafter, now the old lattice, $L$, is not a sublattice of the new lattice, $\text{Thr}_3(L, J)$. For any slim rectangular lattice $K$, $(\text{Lamp}(K); \leq)$ (to be defined in due course) is isomorphic to the poset (that is, partially ordered set) $(J(\text{Con } K); \leq)$ of the join-irreducible congruences of $K$. Thus, thrusting multiforks is useful at studying the congruence lattices of slim rectangular lattices; this will be clear from the paper.

Second, we are going to find some new properties of the class of congruence lattices of slim rectangular (or slim semimodular) lattices.

2020 Mathematics Subject Classification. 06C10

Key words and phrases. Slim rectangular lattice, slim semimodular lattice, planar semimodular lattice, multifork, thrusting multiforks, congruence lattice, join-irreducible element, lamp, order filter, Dioecious Maximal Elements Property.

This research was supported by the National Research, Development and Innovation Fund of Hungary, under funding scheme K 134851.
1.2. Outline. See Theorem 1.1, Corollary 1.2, Observation 1.4, Note 2.6, and Note 2.8 for the main results; in particular, see the two raised statements for new properties of the congruence lattices of slim semimodular lattices. There are also some new auxiliary statements about lamps scattered in the paper.

1.3. More detailed introduction. Let $L$ be a finite lattice. Then
\[ J(L) = \{ x \in L : x \text{ has exactly one lower cover} \} \] (1.1)
denotes the poset (that is, partially ordered set) of (nonzero) join-irreducible elements of $L$. The ordering of $J(L) = (J(L); \preceq)$ is inherited from $L$. Going after Czédli and Schmidt [9], a slim semimodular lattice is a finite semimodular lattice $L$ such that $J(L)$ is the union of two chains. These lattices are necessarily planar.
An equivalent definition of these lattices is the original one by Grätzer and Knapp: a slim planar semimodular lattice is a planar semimodular lattice without $M_3$-sublattices (equivalently: without cover-preserving $M_3$-sublattices); here $M_3$ denotes the five-element modular lattice that is not distributive. Frequently, these lattices are shortly called SPS lattices. So, to conclude this paragraph,

$$\text{slim semimodular lattice} = \text{slim planar semimodular lattice} = \text{SPS lattice}. \quad (1.2)$$

By Grätzer and Knapp [18], a slim semimodular lattice $L$ is a slim rectangular lattice if it has two doubly irreducible elements (that is, two elements each with exactly one cover and exactly one lower cover) and these two elements are complements of each other. These two elements are called the left corner of $L$ and the right corner of $L$; both are on the boundary of (any planar diagram) of $L$. Clearly, a slim rectangular lattice has at least four elements.

The congruence lattice of a lattice $L$ is denoted by $\text{Con} L = (\text{Con} L; \subseteq)$. If $|L|$, the number of elements of $L$ is at most 2, then $\text{Con} L \cong L$. Hence, the congruence lattices of the at most 2-element slim semimodular lattices are well understood and we do not loose anything by excluding them from the following notation. Let

$$\text{Con}(\text{SPS}_{\geq 3}) := I\{\text{Con} L : L \text{ is a slim semimodular lattice and } |L| \geq 3\}, \quad (1.3)$$

$$\text{Con}(\text{SPS}) := I\{\text{Con} L : L \text{ is a slim semimodular lattice}\}, \quad (1.4)$$

$$\text{Con}(\text{SR}) := I\{\text{Con} L : L \text{ is a slim rectangular lattice}\}, \quad (1.5)$$

where $I$ is the operator of taking isomorphic lattices. We know from Grätzer and Knapp [18] (and it is cited in [7, Lemma 3.7]) that

if $K$ is a slim semimodular lattice with more than two elements,

then $\text{Con} K \cong \text{Con} L$ for some slim rectangular lattice $L$. \quad (1.6)

Therefore,

$$\text{Con}(\text{SR}) = \text{Con}(\text{SPS}_{\geq 3}). \quad (1.7)$$

In 2016, Grätzer [14] Problem 24.1 and [15] asked for a description of $\text{Con}(\text{SPS})$. The class $\text{Con}(\text{SR})$ only slightly differs from $\text{Con}(\text{SPS})$. Clearly, both classes only contain finite distributive lattices. We know infinitely many independent properties (of the members) of these two classes that do not follow from distributivity and finiteness, see Czédli [4]. We also know from Czédli and Grätzer [7, Theorem 1.2] that $\text{Con}(\text{SR})$ is closed with respect to finite direct products. Moreover, if $D \in \text{Con}(\text{SR})$ and $B_2$ is the four-element boolean lattice, then the ordinal sum $D \mathbin{+_{\text{ord}}} B_2 \in \text{Con}(\text{SR})$; see [7, Theorem 1.2] again. (This fact is also a consequence of Observation 1.4[A], which we are going to prove here.)

Our goal here is to present another construction under which $\text{Con}(\text{SPS})$ and $\text{Con}(\text{SR})$ are closed. This construction and the corresponding theorem seem to be useful to explore properties of $\text{Con}(\text{SPS})$ and $\text{Con}(\text{SR})$.

Let $P = (P; \leq)$ be a poset. For $u \in P$,

we denote $\{x \in P : x \geq u\}$ and $\{x \in P : x \leq u\}$ by $\uparrow u$ and $\downarrow u$, \quad (1.8)

respectively.

A subset $Y$ of $P$ is called an order filter of $P$ if for every $x \in Y$, $\uparrow x \subseteq Y$. \quad (1.9)

Order filters are also called up-sets. Empty order filters will not occur in the paper.

Recall from, say, Grätzer [14, Corollary 108] that each finite distributive lattice $D$ is determined by the poset $J(D)$, \quad (1.10)
uniquely up to isomorphism.

Based on (1.1), ..., (1.10), now we formulate the main result of the paper.

**Theorem 1.1.** Let $D$ be a finite distributive lattice and let $F = (F; \leq)$ be a nonempty order filter of $J(D) = (J(D); \leq)$.

(A) If $D \in \text{Con}(\text{SPS})$ and $D'$ is a finite distributive lattice such that $J(D') \cong F$, then $D' \in \text{Con}(\text{SPS})$.

(B) If $D \in \text{Con}(\text{SR})$, $|F| \geq 2$, and $D'$ is a finite distributive lattice such that $J(D') \cong F$, then $D' \in \text{Con}(\text{SR})$.

It is immediate to see that this theorem implies the following statement.

**Corollary 1.2.** Let $D'$ be a finite distributive lattice and let $F = (F; \leq) := (J(D'); \leq)$.

(A) If $D' \notin \text{Con}(\text{SPS})$, then every $D \in \text{Con}(\text{SPS})$ has the property that no order filter of $J(D)$ is isomorphic to $F$.

(B) If $D' \notin \text{Con}(\text{SR})$ and $|F| \geq 2$, then every $D \in \text{Con}(\text{SR})$ has the property that no order filter of $J(D)$ is isomorphic to $F$.

Whatever trivial it is to derive Corollary 1.2 from Theorem 1.1, this corollary has applications. Indeed, Czédi [1, Corollary 3.5] gives an algorithm to decide whether a finite distributive lattice belongs to $\text{Con}(\text{SPS})$ or $\text{Con}(\text{SR})$. So, for a small $D'$, it is possible to check whether $D'$ is in $\text{Con}(\text{SR})$ or $\text{Con}(\text{SPS})$; if not then we obtain a property of $\text{Con}(\text{SR})$ or $\text{Con}(\text{SPS})$ by Corollary 1.2.

For example, let $F$ be the two-element chain. Then $F = J(D')$ where $D'$ is the three-element chain. It was proved by Grätzer [16] (and it is easy to verify by lamps introduced in Czédi [3] later) that $D' \notin \text{Con}(\text{SR})$. Thus, by Corollary 1.2, $\text{Con}(\text{SPS}_{\geq 3})$ has the following property:

\[
\text{for } \forall D \in \text{Con}(\text{SR}) = \text{Con}(\text{SPS}_{\geq 3}), \text{ no order filter of } J(D) \text{ is a two-element chain.}
\]

(1.11)

Since (1.11) is clearly a reformulation of Czédi [3 Corollary 3.5], we have obtained a new proof of the fact that $\text{Con}(\text{SR}) = \text{Con}(\text{SPS}_{\geq 3})$ satisfies (that is, all elements of $\text{Con}(\text{SR}) = \text{Con}(\text{SPS}_{\geq 3})$ satisfy) the Dioecious Maximal Elements Property.

![Figure 3. Illustration for Definition 1.3](image)

To state two properties of different nature, we need the following definition.

\[1\] grew out from a part of this arXiv paper. The remaining part with a new title will be soon submitted to a journal.
Definition 1.3. Assume that $P_1$ and $P_2$ are finite posets and $j \in P_1$ is not a maximal element of $P_1$. Here, exceptionally, we allow that $P_2 = \emptyset$. We define a new poset, $P^\sharp(P_1, j, P_2)$ as follows; see Figure 3 for illustration. First, we assume that $P_1 \cap P_2 = \emptyset$ and $i \notin P_1 \cup P_2$. Let $P^\sharp = P^\sharp(P_1, j, P_2) := P_1 \cup \{i\} \cup \{P_2\}$. The edges of $P^\sharp(P_1, j, P_2)$ are the following:

- if $x < y$ in $P_1$ or $x < y$ in $P_2$, then $x < y$ in $P^\sharp(P_1, j, P_2)$;
- if $x$ is a maximal element of $P_2$, then $x < i$ in $P^\sharp(P_1, j, P_2)$;
- if $j < x$ in $P_1$, then $i < x$ in $P^\sharp(P_1, j, P_2)$;
- and nothing else.

If we identify $i$ and $j$ in $P^\sharp(P_1, j, P_2)$, the we obtain $P^+(P_1, j, P_2)$. Equivalently, $P^+(P_1, j, P_2)$ is defined by $\downarrow j \setminus \{j\} = P_2$ and $(P^+(P_1, j, P_2) \setminus \downarrow j) \cup \{j\} = P_1$. Note that $P^\sharp(P_1, j, P_2) = P^+(P^\sharp(P_1, j, \emptyset), i, P_2)$ where $\{i\} = P^\sharp(P_1, j, \emptyset) \setminus P_1$.

These two constructions are quite artificial and the observation below is almost trivial in view of other results and proofs of the paper; our excuse for presenting these constructions is that they fit into the present paper and we do not spend much space or effort on them.

Observation 1.4. Assume that $D_1, D_2 \in \text{Con}(\text{SPS}_{\geq 3})$, $j \in J(D_1)$ is not a maximal element of $J(D_1)$, and $D$ is a finite distributive lattice. Then the following two assertions hold.

(A) If $J(D) = P^\sharp(J(D_1), j, J(D_2))$, then $D \in \text{Con}(\text{SPS}_{\geq 3}) = \text{Con}(\text{SR})$.

(B) If $J(D) = P^+(J(D_1), j, J(D_2))$, then $D \in \text{Con}(\text{SPS}_{\geq 3}) = \text{Con}(\text{SR})$.

2. Notes and proofs

This section contains the proof of the main theorem. It also contains two notes on multiforks that explain the title of the paper. We need lots of concepts, notations, and auxiliary statements from Czédi [3] and, although less frequently, from Czédi and Grätzer [7]. At the time of writing, each of these two papers have an open access view and they can be downloaded. Whenever something is needed from [3] or [7], we will exactly tell where (which page, definition, or statement) to find it. Due to these “high precision coordinates”, it suffices to only read some small parts of these two papers, which should be kept near. (The arXiv versions of [3] and [7] are also available at http://arxiv.org/abs/2101.02929 and http://arxiv.org/abs/2103.04458 but the above-mentioned coordinates for them are could be less precise in some cases.) From now on,

$L$ stands for a slim rectangular lattice with a fixed $C_1$-diagram; (2.1)

see [3] 3rd paragraph in page 384 or Grätzer and Knapp [18] for rectangularity and $C_1$-diagrams.

The left corner and the right corner of $L$ are denoted by $\text{lc}(L)$ and $\text{rc}(L)$; (2.2)

they are the only doubly irreducible elements of $L$ and they are on the boundary of $L$. The crucial role of rectangularity in the paper is explained by the main result of Grätzer and Knapp [18] implying that

$$\text{Con}(\text{SPS}_{\geq 3}) = \text{Con}(\text{SR}).$$

(2.3)
2.1. Anatomy of lamps. Recall from \[3, \text{Definition 2.3}\] that neon tubes are the edges (prime intervals) with meet-irreducible bottoms. In notation, a prime interval \( n = [a, b] = [0_n, 1_n] \) (here \( a < b \), of course) is a neon tube if and only if \( 0_n \) belongs to \( M(L) \), the set of non-unit meet irreducible elements. If \( 0_n \) is on the (upper) boundary of \( L \) then \( n \) is a boundary neon tube; otherwise it is an internal neon tube. The boundary neon tubes are of normal slopes while the internal neon tubes are precipitous. The boundary neon tubes are boundary lamps; their feet and peaks are the bottoms and the tops of the corresponding neon tubes. Note that mostly (but not always) we prefer the notation \( \text{Peak}(n) \) and \( \text{Foot}(n) \) to \( 1_n \) and \( 0_n \). If \( x \) is the peak (that is, the top) of an internal neon tube, then all internal neon tubes with the same peak form a lamp \( I \) with peak = \( x \) and foot = the meet of the bottoms of these neon tubes. We say that a lamp has its neon tubes but a lamp \( I \) is the interval \( [\text{Foot}(I), \text{Peak}(I)] \). Following \[3, \text{Convention 2.4}\], the feet of lamps in our figures of slim rectangular lattices are black-filled and the thick edges are the neon tubes. As in \[3, \text{Definition 2.6}\], \( \text{Body}(I) \) denotes the body of a lamp \( I \); it is the geometric region determined by the interval \( I \). It is either a quadrangle whose upper edges are precipitous and the lower edges are of normal slopes or it is just a precipitous line segment. If \( I \) is an internal lamp, then it has a circumscribed rectangle \( \text{CircR}(I) \); see \[3, \text{Definition 2.6}\].

There are also line segments associated with lamps. Let \( \text{Lamp}(L) \) stand for the set of lamps of \( L \). For \( I \in \text{Lamp}(L) \), the (left and right) roof and the (left and right) floor of \( I \) are defined in \[3, \text{Figure 3 and the paragraph below it}\], and they are denoted by \( \text{LRoot}(I), \text{RRoot}(I), \text{Roof}(I), \text{LFloor}(I), \text{RFloor}(I), \) and \( \text{Floor}(I) \), respectively.

The rectangle formed by the boundaries of \( L \) is called the full geometric rectangle of \( L \); it is denoted by \( \text{FullRect}(L) \). For a polygon \( W \) (that is, a line segment or a sequence of connected line segments) from the left boundary to the line boundary of \( L \), we define the following two geometric areas.

\[
\downarrow_g W := \{(x, y) \in \text{FullRect}(L) : (\exists y') (y' > y \text{ and } (x, y') \in W)\}, \\
\uparrow_g W := \{(x, y) \in \text{FullRect}(L) : (\exists y') (y' < y \text{ and } (x, y') \in W)\}.
\]

The (left and right) illuminated sets \( \text{LeftLit}(I) \), \( \text{RightLit}(I) \), and \( \text{Lit}(I) \) of \( I \) are defined in \[3, \text{Definition 2.8}\]. Note that

\[
\text{Lit}(I) = \downarrow_g \text{Roof}(I) \cap \uparrow_g \text{Floor}(I). \tag{2.4}
\]

As in \[3, \text{Definition 2.9}\], we define the following five relations for \( I, J \in \text{Lamp}(L) \):

\[
(I, J) \in \rho_{\text{Body}} \overset{\text{def}}{\iff} \left( I \neq J, \text{Body}(I) \subseteq \text{Lit}(J), \text{ and } I \text{ is an internal lamp} \right). \tag{2.5}
\]

\[
(I, J) \in \rho_{\text{LRBody}} \overset{\text{def}}{\iff} \left( I \neq J, I \text{ is an internal lamp, and } \text{Body}(I) \subseteq \text{LeftLit}(J) \text{ or Body}(I) \subseteq \text{RightLit}(J) \right). \tag{2.6}
\]

\[
(I, J) \in \rho_{\text{foot}} \overset{\text{def}}{\iff} \left( I \neq J, I \text{ is an internal lamp, and } \text{Foot}(I) \in \text{Lit}(J) \right). \tag{2.7}
\]
\[(I, J) \in \rho_{\text{infoot}} \ \overset{\text{def}}{\iff} \ (I \neq J, \ I \text{ is an internal lamp, and Foot}(I) \ \text{is in the topological interior of Lit}(J)). \tag{2.8}\]

\[(I, J) \in \rho_{\text{LRcircR}} \ \overset{\text{def}}{\iff} \ (I \neq J, \ I \text{ is an internal lamp, and CircR}(I) \subseteq \text{LeftLit}(J) \text{ or CircR}(I) \subseteq \text{RightLit}(J)). \tag{2.9}\]

Below, we recall a part of [3, Lemma 2.11]. For \(x, y \in L\), \(\text{con}(x, y)\) stands for the least congruence collapsing \(x\) and \(y\).

**Lemma 2.1.** Let \(L\) be as in (2.1), and let \(\leq\) be the reflexive transitive closure of \(\rho_{\text{Body}}\). Then \(\rho_{\text{Body}} = \rho_{\text{LRBody}} = \rho_{\text{foot}} = \rho_{\text{infoot}} = \rho_{\text{LRcircR}}\), \(\text{Lamp}(L) = (\text{Lamp}(L); \leq)\) is a poset, and

\[\varphi: (\text{Lamp}(L); \leq) \rightarrow (J(\text{Con} L); \leq) \text{ defined by } I \mapsto \text{con}(\text{Foot}(I), \text{Peak}(I))\]  

(2.10) is an order isomorphism. Furthermore, if \(I \prec J\) in \(\text{Lamp}(L)\), then \((I, J) \in \rho_{\text{Body}}\).

For a lamp \(I \in \text{Lamp}(L)\), let \(\text{Num}(I)\) denote the number of neon tubes of \(I\).  

(2.11)

As usual (outside lattice theory), \(\mathbb{R}\) and \(\mathbb{R}^+\) will stand for the set of real numbers and \(\{x \in \mathbb{R} : x \geq 0\}\), respectively. Assume that the usual coordinate system of the plane is chosen so that \(0_L = (0, 0)\) and each edge of normal slope of the diagram is of slope \((1, 1)\) or \((1, -1)\). For a planar point \(u = (u_x, u_y)\), let \(\sqrt{u}\) and \(\sqrt{u}\) denote the half-lines \(\{(u_x - t, u_y - t) : t \in \mathbb{R}^+\}\) and \(\{(u_x + t, u_y - t) : t \in \mathbb{R}^+\}\), respectively; they are half-lines of normal slopes. Let \(\text{lx}(u) \in \mathbb{R}\) and \(\text{rx}(u) \in \mathbb{R}\) be defined by the rules that \((\text{lx}(u), 0)\) and \((\text{rx}(u), 0)\) are the intersection point of the \(x\)-axis with \(\sqrt{u}\) and that with \(\sqrt{u}\), respectively. If \(H\) is an interval, in particular, if it is a lamp or a neon tube, then the **coordinate quadruple** of \(H\) is

\[(p_H, q_H, r_H, s_H) := (\text{lx}([\text{Peak}(H)]), \text{lx}([\text{Foot}(H)]), \text{rx}([\text{Foot}(H)]), \text{rx}([\text{Peak}(H)]))\].  

(2.12)

For example, see [21, Figure 8] or [6, Figure 6].

Next, we extend Czédli and Grätzer [7, Definition 4.1.(iii)] and Czédli [3, (4.1)].

**Definition 2.2.** For \(L\) as in (2.1) and \(I, J \in \text{Lamp}(L)\), we define five relations.

- \(I \sim J\), that is, \(I\) is to the left of \(J\) if \(q_I \leq p_J\) and \(s_I \leq r_J\);
- \(I \delta J\), that is, \(I\) is geometrically under \(J\) if \(q_J \leq p_I\) and \(s_J \leq r_I\);
- \(I \beta_{\text{mid}} J\) if \(p_J < p_I < q_I < q_J\) and \(r_J < r_I < s_I < s_J\);
- \(I \beta_{\text{left}} J\) if \(p_J \leq p_I < q_I < q_J\), \(s_I \leq r_J\), and \(I\) is an internal lamp;
- \(I \beta\text{right} J\) if \(q_J \leq p_I, r_J < r_I < s_I \leq s_J, \text{ and } I\) is an internal lamp.

**Lemma 2.3** (Czédli [4, Lemma 3.2]). Let \(L\) be as in (2.1). The relations defined in Definition 2.2 are **irreflexive**. Furthermore, for any two distinct lamps \(G\) and \(H\) of \(L\), there are a unique \(I \in \{G, H\}\) such that with \(\{J\} = \{G, H\} \setminus \{I\}\), exactly one of \(I \sim J, I \delta J, I \beta_{\text{mid}} J, I \beta_{\text{left}} J, \text{ and } I \beta\text{right} J\) holds.

**2.2. Lamps and normal multifork series.** In this subsection, \(L\) is still as in (2.1). Let \(C = [0_C, 1_C]\) be a **4-cell** of \(L\); that is, \(C\) is a 4-element interval of length 2. We know from Kelly and Rival [19] that each interval \(I\) of a planar lattice diagram determines a geometric area,

the region determined by \(I\); it is denoted by \(\text{Reg}(I)\) and it is bordered by edges. For example, \(\text{FullRect}(L) = \text{Reg}([0_L, 1_L])\).  

(2.13)
Note that Reg(I) can be a line segment with area 0 (with respect to the Lebesgue measure on the plane); (2.16) provides examples for this. Sometimes, for a 4-cell I, we speak of I instead of the more precise Reg(I). We say that C is a distributive 4-cell if the principal ideal ↓1_C is distributive. Note that in this case ↓1_C contains no precipitous edge and it is a grid. By [3, Lemma 2.12 and Figure 6], quoted from Czédli [1, Theorem 3.7], each slim rectangular lattice is obtained from a grid by a sequence of multifork extensions at distributive 4-cells, and every lattice obtained in this way is a slim rectangular lattice. (Note that this fact offers an easy way to verify the our lattice diagrams give slim rectangular lattices.) The sequence in this result will be called a multifork sequence of L. It is not unique in general, but we are going to fix one. Then each internal lamp comes to existence by a multifork extension; see [3, (2.10)]. So combining [3, (2.9) and (2.10)], we can fix a sequence L_0, L_1, ..., L_k = L, a repetition-free list I_1, I_2, ..., I_k of the internal lamps of L, a distributive 4-cell H_i of L_{i-1} for i ∈ {1, ..., k}, and a positive integer m_i for i ∈ {1, ..., k} such that L_0 is a grid and, for i ∈ {1, ..., k}, L_i is obtained from L_{i-1} by performing an m_i-fold multifork extension at H_i, the lamp I_i comes to existence at the i-th multifork extension, CircR(I_i) is the geometric region Reg(H_i) determined by H_i, and Foot(I_i) ∈ L_i \ L_{i-1}.

It follows easily from (2.14) that for an L as in (2.1) and I, J ∈ Lamp(L),

\[
\text{if } n ⊆ \text{Lit}(J) \text{ for some neon tube } n \text{ of } J, \text{ then } (I, J) ∈ \rho_{\text{Body}}. \tag{2.15}
\]

Based on (2.14), the definition of multifork extensions, and the definition of C_i-diagrams, a trivial induction yields the following statement. It also follows from, say, the main result of Czédli [25].

If L is as in (2.1) and x ∈ L, then x ∧ lc(L) and x ∧ rc(L) are on the boundary of L, the interval \([x ∧ lc(L), x]\) is a chain with all edges of normal slope (1, 1), and the interval \([x ∧ rc(L), x]\) is a chain with all edges of normal slope (1, −1).

Intervals, lamps, and 4-cells are different entities, which can only coincide in exceptional cases. However, it seems reasonable to extend some concepts from lamps to intervals and, in particular, to 4-cells. Let I be an interval of L from (2.1). LRoof(I) is the line segment between 1_I ∧ lc(L) and 1_I; it is of slope (1, 1) by (2.16). We define RRooft(I) analogously and let Roof(I) be the polygon LRoof(I) ∪ RRooft(I). Using 0_I instead of 1_I, we get LFloor(I), RFloor(I), and Floor(I) = LFloor(I) ∪ RFloor(I). Finally, the illuminated set of I is defined to be

\[
\text{Lit}(I) := \downarrow g \text{Roof}(I) \cap \uparrow g \text{Floor}(I). \tag{2.17}
\]

It is clear by (2.4) that for a lamp I, (2.17) gives the same as [3, Definition 2.8]. We also need the following concept. For a non-singleton interval I = \([a ∧ b, a ∨ b]\) such that a || b, we let

\[
\text{LitD}(I) = (\text{Lit}(I) \cap \downarrow a) \cup (\text{Lit}(I) \cap \downarrow b); \tag{2.18}
\]

“D” in the notation comes from “down”. Although LitD(I) is not defined for all intervals I, it is defined when, in particular, I is a 4-cell.

While (2.14) has been used successfully in many cases, it does not seem to be appropriate for the present task in itself. Hence, we need to introduce two other constructions based on multiforks. Since we are going to construct slim semimodular lattices, it is important that we recognize them. The following lemma
can often help. A planar lattice is called a \textit{4-cell lattice} if it has a planar diagram in which the minimal regions (of positive area) are “squares”, that is, 4-element boolean sublattices. Note that a $C_1$-diagram is a 4-cell diagram. Remember that according to Grätzer and Knapp \cite{17}, a planar semimodular lattice is \textit{slim} if it contains no cover-preserving $M_3$; this is the key how the following lemma is included in \cite{17}. Planar lattices are finite by definition.

\textbf{Lemma 2.4} (Grätzer and Knapp \cite{17} Lemmas 4 and 5) and \cite{18}). \textit{Let $L$ be a planar lattice.}

(A) If $L$ is semimodular, then it is a 4-cell lattice.

(B) If $L$ is a 4-cell lattice such that no two distinct 4-cells have the same bottom, then $L$ is a slim semimodular lattice.

(C) If $L$ has two complementary doubly irreducible elements and the assumptions of (B) hold, then $L$ is a slim rectangular lattice.

We know from Czédli and Schmidt \cite{10} Theorem 11] that we can perform a fork extension of a slim semimodular lattice at any of its 4-cells and the lattice we obtain in this way is again a slim semimodular lattice. On the other hand, a multifork extension can always be obtained by a finite sequence of fork extensions. Putting these two facts together and using that corners remain corners, we conclude the following.

Let $L$ be as in \eqref{2.1}, and let $H$ be a 4-cell of $L$. Then, for any $k \in \mathbb{N}^+$, we can perform a \textit{k}-fold multifork extension at $H$ and the lattice we obtain in this way is a slim rectangular lattice. \hfill \eqref{2.19}

We should note that, in general, \eqref{2.14} does not extend to the case when $H$ in \eqref{2.19} is not distributive. Indeed, if, say, the two upper edges of $H$ are neon tubes, then the multifork extension mentioned in \eqref{2.19} does not increase the number of lamps. However, there is a luckier case, which is discussed in Lemma 2.6. But first we need some preparations.

By a \textit{rectangular interval} of $L$ from \eqref{2.1}, we mean a non-chain interval that it bordered by four line segments of normal slopes in the fixed $C_1$-diagram of $L$, and each of these four line segments consists of edges of $L$. We warn the reader: an interval that is a rectangular lattice on its own right need not be a rectangular interval in our terminology. For example, if $n$ and $p$ are two neighboring neon tubes of an internal lamp, then the 4-cell $[\text{Foot}(n) \land \text{Foot}(p), \text{Peak}(n)]$ is a rectangular lattice but not a rectangular interval. However, the converse is true (but will not be used): a rectangular interval is a rectangular lattice.

\textbf{Observation 2.5.} \textit{Let $L$ be as in \eqref{2.1}, and let $H$ be a rectangular interval of $L$. Let $U$ stand for $\text{Lit}(I)$ or $\text{LitD}(I)$. Then the following three conditions are equivalent.}

(i) $U$ contains no precipitous edge, i.e., no precipitous edge lies entirely in $U$.

(Then necessarily\footnote{since $U$ is bordered by line segments of normal slopes and these line segments consist of edges by \eqref{2.16}, a precipitous edge cannot cross the line segments bordering $\text{Lit}(U)$ or $\text{LitD}(U)$, by the same reason} no part of positive length of a precipitous edge lies in $U$.)

(ii) $\text{Body}(J) \subseteq U$ holds for no internal $J \in \text{Lamp}(L)$. (Then necessarily\footnote{by the same reason} there is no internal lamp $J$ such that $\text{Body}(J) \cap U$ is of a positive area.)
(iii) There is no 3-element antichain \( \{x_0, x_1, x_2\} \) of lattice elements lying in \( U \) such \( x_0, x_1, \) and \( x_2 \) have a common cover \( y \in L \) that also lies in \( U \).

Proof. To show that \( \neg (i) \Rightarrow \neg (ii) \), let \( n \) be a precipitous edge in \( U \). Then \( n \) is a neon tube of an internal lamp \( J \). We can assume that \( n \) is not the only neon tube of \( J \) as the opposite case is trivial. Since any other neon tube \( p \) with \( 1_p = 1_n \) is precipitous, \( U \) is bordered by line segments of normal slopes, and these line segments consist of edges by (2.16) and (2.17), we obtain that \( p \) is also in \( U \). (Indeed, otherwise \( p \) would cross an edge, contradicting planarity.) The neon tubes \( p \) mentioned above and \( n \) form an internal lamp \( J \). Now the upper boundary of Body(\( J \)) is formed by the leftmost of its neon tubes and the rightmost one. We know, say, from Czédi [5] or from (2.14) that the lower boundary of Body(\( J \)) is formed by two line segments of normal slopes and these line segments consist of edges; one of these line segments connects the foot of the leftmost neon tube of \( J \) and Foot(\( J \)) while the other one connects the foot of the rightmost neon tube of \( J \) and Foot(\( J \)). Again by planarity, we have that the boundary of Body(\( J \)) cannot cross the boundary of \( U \). Hence, Body(\( J \)) \( \subseteq \) \( U \), that is, \( \neg (i) \).

To show that \( \neg (ii) \Rightarrow \neg (iii) \), assume that \( \text{Body}(J) \subseteq U \) for an internal lamp \( U \). Using a neon tube of \( J \) and the top edges of CircR(\( J \)), which are also in \( U \), we obtain that \( \neg (iii) \).

Finally, assume that \( \neg (iii) \). Since there are only two normal slopes, one of the edges \( [x_0, y], [x_1, y], \) and \( [x_1, y] \) is precipitous. Hence, \( \neg (i) \). That is, \( \neg (iii) \Rightarrow \neg (i) \), and the proof of Observation 2.5 is complete.

Next, let \( q \) be a prime interval (that is, an edge) of \( L \); \( L \) is still taken from (2.1). Then, with reference to (2.10),

\[
\text{the color of } q \text{ is } \text{col}(q) := \varphi^{-1}(\text{con}(q)) \in \text{Lamp}(L).
\]

This makes sense since it is well known that a prime interval generates a join-irreducible congruence. It follows from Czédi [1] Theorem 4.4 that, for \( q \) and \( L \) as above and \( I \in \text{Lamp}(L) \),

\[
\text{col}(q) = I \text{ if and only if the top edge } n \text{ of the (unique) trajectory containing } q \text{ is a neon tube of } I;
\]

note that this fact was heavily used (implicitly) when we introduced the concept of lamps in [3].

We know from, say, Grätzer and Knapp [17, Lemma 5] and their definition of sliminess or, alternatively, even from (2.14) that

\[
L \text{ from (2.1) is distributive if and only if each of its elements has at most two lover covers.}
\]

This motivates the following definition. Let \( H \) be a rectangular interval of \( L \) from (2.1), and let \( U \) be one of \( \text{Lit}(H) \) and \( \text{LitD}(H) \).

We say that \( H \) is \( U \)-\text{distributive} if one of the three equivalent conditions occurring in Observation 2.5 holds. (Equivalently, if all the three conditions hold.)

Note at this point that

for any internal lamp \( I \), CircR(\( I \)) is a rectangular interval; (2.24)

this follows prompt from (2.14). Hence the following statement makes sense.
Note 2.6. For $L$ as in (2.1), let $I$ be an internal lamp of $L$ and let $H := \text{CircR}(I)$. Assume that $H$ is $\text{LitD}(H)$-distributive and $\{J \in \text{Lamp}(L) : \text{Body}(J) \subseteq \text{Lit}(H)\}$ and $J$ is an internal lamp $H$. Then there exists a slim rectangular lattice $L'$ with a $\mathcal{C}_1$-diagram and a $\mathcal{C}$-cell $H'$ of $L'$ such that the following hold.

(i) $L$ and its $\mathcal{C}_1$-diagram are obtained from $L'$ by a $\text{Num}(I)$-fold multifork extension at $H'$.
(ii) $I$ is a minimal element of $\text{Lamp}(L)$.
(iii) Denoting the upper edges of $H$ by $a$ and $b$, if $\text{col}(a)$ and $\text{col}(b)$ are incomparable, then $I$ in $\text{Lamp}(L)$ has exactly two covers, $\text{col}(a)$ and $\text{col}(b)$. If $\text{col}(a)$ and $\text{col}(b)$ are comparable, then $I$ in $\text{Lamp}(L)$ has exactly one cover and this cover is in $\{\text{col}(a), \text{col}(b)\}$.
(iv) $\{\text{Body}(J) : J \in \text{Lamp}(L')\}$ and $\{\text{Body}(J) : J \in \text{Lamp}(L \setminus \{I\})\}$ are exactly the same sets of geometric shapes.
(v) $(\text{Lamp}(L') ; \leq ) \cong (\text{Lamp}(L \setminus \{I\}) ; \leq )$.

Proof. Since there is no $J \in \text{Lamp}(L \setminus \{I\})$ such that $\text{Body}(J) \subseteq \text{Lit}(I)$, that is, $(J, I) \in \mathcal{P}_{\text{body}}$, the last sentence of Lemma 2.1 yields Part (i) of Note 2.6.

The next task is to construct $L'$. To do so, take a neon tube $n$ of $I$, and delete all elements of the intervals $[\text{Foot}(n) \land \text{lc}(L), \text{Foot}(n)]$ and $[\text{Foot}(n) \land \text{rc}(L), \text{Foot}(n)]$. Note that (2.16) applies to these two intervals. After deleting the elements of $F_n := [\text{Foot}(n) \land \text{lc}(L), \text{Foot}(n)] \cup [\text{Foot}(n) \land \text{rc}(L), \text{Foot}(n)]$, we obtain a subposet and a subdiagram. Note that, in the terminology of Czédli and Schmidt [10], $F_n$ is a so-called fork of $L$. Since every planar poset diagram with a least element and a largest element is a lattice diagram by Kelly and Rival [19, Corollary 2.4], we can apply Lemma 2.4 to conclude that the poset $L \setminus F_n$ we have just obtained is a slim rectangular lattice. It is easy to see that this lattice is a sublattice of $L$. In the next step, take another neon tube $n'$ of $I$ and omit the elements of the fork $F_{n'}$ from $L \setminus F_n$. And so on; after omitting the elements of the forks of the neon tubes of $I$, we obtain $L'$. Since the forks we have obtained can be put back, one by one, and these fork extensions result in one multifork extension, we conclude Part (i) of Note 2.6.

Before dealing with Part (iii), we formulate the Swing Lemma from Grätzer [13]. For $L$ as in (2.1) and prime intervals (that is, edges) $p$ and $q$ of $L$, we define the following four relations.

\[
p \triangleright q \quad \text{def} \quad p \text{ is a lower edge of a 4-cell and } q \text{ is the opposite edge of this 4-cell};
\]
\[
p \triangleright q \quad \text{def} \quad q \triangleright p;
\]
\[
p \triangleright_{\text{ext}} q \quad \text{def} \quad p \text{ there is an internal lamp } I \in \text{Lamp}(L) \text{ such that is an upper edge of } \text{CircR}(I) \text{ and } q \text{ is a neon tube of } I;
\]
\[
p \triangleright_{\text{int}} q \quad \text{def} \quad p \text{ there is an internal lamp } I \text{ such that both } p \text{ and } q \text{ are neon tubes of } I.
\]
We know from Grätzer [13] that, for $L$ as in (2.1) and prime intervals $p$ and $q$ of $L$,

\[
\text{con}(p) \geq \text{con}(q) \quad \text{if and only if there is an } n \in \mathbb{N}^+ \text{ and there are edges } t_0 = p, t_1, \ldots, t_n = q \text{ such that}
\]

(i) for each $i \in \{1, \ldots, n\}$, $t_{i-1} \wedge t_i, t_{i-1} \vee t_i, t_{i-1} \cup \text{ext } t_i,$

\[
\text{or } t_{i-1} \cup \text{int } t_i, \text{ and}
\]

(ii) for each $i < j \in \{1, \ldots, n\}$, if $t_{j-1} \wedge t_j$, then $t_{i-1} \wedge t_i.$

Furthermore, $\text{con}(p) > \text{con}(q)$ if and only if no $\cup \text{ext}$ occurs in (i) above.

Note that, based on Czédli and Makay [8], we slightly modified the Swing Lemma of Grätzer [13] and its terminology.

Next, we deal with Part (iii). Let $\alpha := \text{col}(I) \in J(\text{Con } L)$. Based on definitions, that is, on (2.10), (2.20), and (2.21), and Lemma (2.1), it suffices to show that $\text{con}(a) > \alpha$, $\text{con}(b) > \alpha$, and whenever $\beta \in J(\text{Con } L)$ such that $\beta > \alpha$, then $\alpha \geq \text{con}(a)$ or $\beta \geq \text{con}(a)$.

Since any two neon tubes of $I$ generate the same congruence by (2.29), we can fix a neon tube $q$ of $I$; then $\alpha = \text{con}(q)$. Since $a \cup \text{ext } q$ and $b \cup \text{ext } q$, (2.29) gives that $\text{con}(a) > \text{con}(q) = \alpha$ and $\text{con}(b) > \alpha$. It is well known from, say, Grätzer [14] Page xxviii) that the join-irreducible congruences of $L$ (or any finite lattice) are the same as the congruences generated by prime intervals, by one prime interval each. Thus, there is a prime interval $c$ such that $\beta = \text{con}(c)$. With the terminology of Czédli and Schmidt [9] or Czédli [1], let $p$ be the uppermost edge of the trajectory containing $c$.

In other words, starting from $p_0 := c$, take the longest sequence (possibly of length 0) of the form $p_0 \nearrow p_1 \nearrow \cdots \nearrow p_i$; then $p := p_i$.

So we have that $\beta = \text{con}(c) = \text{con}(p)$ by (2.29). Clearly, $p$ is a neon tube since otherwise $\text{Foot}(p_i)$ would be meet reducible and we could continue the just-mentioned sequence by $p_{i+1}$. Now since $\text{con}(p) = \beta > \alpha = \text{con}(q)$, we can take a sequence $t_0 = p, t_1, \ldots, t_{n-1}, t_n = q$ of edges of $L$ according to (2.29) (i).

Since $p$ is a lamp and so $\text{Foot}(p)$ is meet-irreducible, this sequence cannot begin with $\nearrow$. Hence, by (2.29)(ii), $\nearrow$ cannot occur in the sequence at all. Similarly, $\text{Foot}(q) \in M(L)$ gives that the sequence cannot terminate with $\searrow$. In fact, for any neon tube $n$ of $I$, we cannot arrive at $n$ by a “$\searrow$ step”. Let $i$ be the smallest subscript such that $t_i$ is a neon tube of $I$. Since any other possibility has been excluded, we have that $t_{i-1} \cup \text{ext } t_i$. This is only possible if $t_{i-1} \in \{a, b\}$. By (2.29), the first part of the sequence shows that $\beta = \text{con}(p) \geq \text{con}(t_{i-1}) \in \{a, b\}$, as required. This completes the argument for Part (iii).

Part (iv) is trivial by the construction of $L'$.

For $J \in \text{Lamp}(L)$, $\text{Body}(J)$ determines $\text{Foot}(J)$, $\text{Peak}(J)$, and so $\text{Lit}(J)$. Thus, in view of Lemma 2.1 (vi) follows from Part (iv). We have proved Note 2.6.

Next, we describe an entirely new kind of multifork “extensions”. The new construction is not a real extension since the original lattice will not be a sublattice of the new lattice. The following definition is illustrated by Figures 1 and 2.

**Definition 2.7.** For $L$ as in (2.1), let $J$ be an internal lamp of $L$, and let $k \in \mathbb{N}^+$. We can thrust a $k$-fold multifork into $L$ at $\text{Peak}(J)$ as follows.

First, let $G(J) = \{J_0 = J, J_1, \ldots, J_t\}$ be the set of those internal lamps $F \in \text{Lamp}(L)$ for which $\text{Peak}(F)$ lies on the polygon $\text{Roof}(J)$, and choose a geometric point $z$ in the (geometric) interior of $\text{CircR}(J)$ such that

- $\text{lx}(\text{Peak}(J)) < \text{lx}(z)$ and $\forall u \in L$, $\text{lx}(\text{Peak}(J)) < \text{lx}(u) \Rightarrow \text{lx}(z) < \text{lx}(u)$.
• $\text{rx}(\text{Peak}(J)) > \text{rx}(z)$ and $\forall u \in L$, $\text{rx}(\text{Peak}(J)) > \text{rx}(u) \Rightarrow \text{rx}(z) > \text{rx}(u)$.

Note that any $z$ of the interior of $\text{CircR}(J)$ that is near enough to $\text{Peak}(J)$ satisfies these two conditions on $z$. Note also that in Figure 1, $G(J) = \{J, J_1, J_2\}$.

Second, for each neon tube $p$ of $J$, change $\text{Peak}(p)$ to $z$. Then $z$ will be the peak of the modified $J$. The assumption on $z$ guarantees that the modified neon tubes are precipitous. (Think of $n_4$ in Figures 1 and 2.) Although we will not always make a notational distinction between the old and the modified version of a lamp, sometimes, in case of ambiguity, we indicate by superscripts “old” or “shr” if the original (old) lamp or the modified (shrunk) lamp is taken. For example $\text{Peak}(J^{\text{shr}}) \neq \text{Peak}(J^{\text{old}})$.

Third, for each $J_i \in G(J) \setminus \{J\}$, we add a new lattice element to the geometric intersection point $\text{Roof}(J^{\text{shr}}) \cap \text{Roof}(J^{\text{old}})$. Then we change the peak of each neon tube of $J_i$ to this intersection point, which becomes $\text{Peak}(J_i^{\text{shr}})$. Again, it follows from the choice of $z$ that the neon tubes of $J_i^{\text{shr}}$ are still precipitous; think of $n_4$ in Figures 1 and 2.

Fourth, let $I$ denote the (unique) internal lamp of the lattice (diagram) $S_7^{(k)}$ defined in Czédli [1]; $S_7^{(3)}$ is also given here in Figure 1. This $I$ is an interval, so it is a lattice (diagram) on its own right. We can re-scale and resize $k$ the diagram of $S_7^{(k)}$ (to change the diagram of $I$) if necessary, but the diagram of $S_7^{(k)}$ should be a $C_1$-diagram. (Note that the subdiagram $I$ of $S_7^{(k)}$ never a $C_1$-diagram. However, every edge of its lower boundary is of normal slope.) Glue a copy of the diagram of $I$ into the diagram of $L$ at its present stage so that $\text{Peak}(I) = \text{Peak}(J^{\text{shr}})$ and $\text{Foot}(I) = z$; that is, $\text{Foot}(I) = \text{Peak}(J^{\text{shr}})$; this gluing is usually only possible after rescaling $^4$.

Fifth, from each element $x \neq \text{Foot}(I)$ on the lower left boundary chain of $I$, that is, for each $x \in [\text{Foot}(I), \text{lc}(I)] \setminus \{\text{Foot}(I)\}$, we draw a line segment to the “south-west” at normal slope $(1, 1)$ that ends at the lower left boundary of $L$.

Sixth, from each element $x \neq \text{Foot}(I)$ on the lower right boundary chain of $I$, that is, for each $x \in [\text{Foot}(I), \text{rc}(I)] \setminus \{\text{Foot}(I)\}$, we draw a line segment to the “south-east” at normal slope $(1, -1)$ that ends at the lower right boundary of $L$.

Seventh, wherever a new line segment (drawn to the southwest or southeast) has a common planar point with an edge of the diagram, then we add a new lattice element there. This completes the description of a new poset diagram, which we denote by $\text{Thr}_k(L, J)$.

Instead of using the terminology “thrusting a multifork”, we can also say that we thrust a lamp (with $k$ neon tubes) at the peak of $J$.

**Note 2.8.** (i) For $L$ as in (2.1), $k \in \mathbb{N}^+$, and an internal lamp $J$ of $L$, the poset $\text{Thr}_k(L, J)$ described above is a slim rectangular lattice.

(ii) For $L$, $k$, and $J$ as above, let $I$ denote the new lamp. Then, up to isomorphism, $(\text{Lamp}(L); \leq)$ is a subposet of $(\text{Lamp}(\text{Thr}_k(L, J)); \leq)$; namely, $\text{Lamp}(I) = \text{Lamp}(\text{Thr}_k(L, J)) \setminus \{I\}$. The lamp $I$ is a minimal element of $\text{Lamp}(\text{Thr}_k(L, J))$. Furthermore, $J$ has the same covers in $\text{Lamp}(\text{Thr}_k(L, J))$ as it has in $\text{Lamp}(L)$, and the covers of $I$ and $J$ in $\text{Lamp}(\text{Thr}_k(L, J))$ are the same.

$^4$ This is possible by the two-three sentences following (5.8) in Czédli [2].

$^5$ Instead of $S_7^{(k)}$, we can re-scale the diagram of $L$. In Figures 1 and 2, no rescaling was necessary.
(iii) For \( L \) as in (2.1), \( k \in \mathbb{N}^+ \), and an internal lamp \( J \) of \( L \), assume that \( I \in \text{Lamp}(L) \) is another internal lamp such that \( I \) is a minimal element of \( (\text{Lamp}(L), \leq) \) and \( \text{Foot}(I) = \text{Peak}(J) \). Then there is a slim rectangular lattice \( L' \) such that \( L \) and its diagram can be obtained as \( \text{Thr}_{\text{Num}}(I)(L', J) \).

**Proof.** To verify Part (i), we apply Kelly and Rival [19, Corollary 2.4], stating that a planar poset with 0 and 1 is a lattice. So \( \text{Thr}_k(L, J) \) is a lattice. We conclude by Lemma 2.4 that this lattice is slim and rectangular.

To verify Part (ii), observe that \( \text{Body}(X) \subseteq \text{Lit}(I) \) holds for no lamp \( X \) of \( \text{Thr}_k(L, J) \). Hence, for every \( X \in \text{Lamp}(\text{Thr}_k(L, J)) \), we know by Lemma 2.1 that neither \((X, I) \in \rho_{\text{Body}} \) nor \((X, I) \in \rho_{\text{LRBody}} \) holds. Thus, \( I \) is a minimal element of \((\text{Lamp}(\text{Thr}_k(L, J)); \leq) \) by Lemma 2.1. Clearly, \(|\text{Lamp}(\text{Thr}_k(L, J))| = 1 + |\text{Lamp}(L)| \) and \( \text{Lamp}(\text{Thr}_k(L, J)) = \text{Lamp}(L) \cup \{I\} \). It follows from the minimality of \( I \) in \( \text{Lamp}(\text{Thr}_k(L, J)) \) and Lemma 2.1 that, in order to see that \((\text{Lamp}(L); \leq) \) is a subposet of \((\text{Lamp}(\text{Thr}_k(L, J)), \leq) \), it suffices to show that

\[
\text{for distinct } J_1, J_2 \in \text{Lamp}(L), (J_1, J_2) \in \rho_{\text{fooot}} \text{ holds in } \text{Lamp}(L)
\text{ if and only if it holds in } (\text{Lamp}(\text{Thr}_k(L, J)), \leq).
\]

To avoid even the least chance of confusion, we are going to write \( J_i^{\text{old}} \) and \( J_i^{\text{shrink}} \) when the context is \( L \) and \( \text{Thr}_k(L, J) \), respectively. By construction, \( \text{Foot}(J_i^{\text{shrink}}) \subseteq \text{Lit}(J_i^{\text{old}}) \). This implies the “if” part of (2.30). To show the “only if” part by way of contradiction, suppose that \((J_1^{\text{old}}, J_2^{\text{old}}) \in \rho_{\text{foot}} \) but \((J_1^{\text{shrink}}, J_2^{\text{shrink}}) \notin \rho_{\text{foot}} \). Denote \( \text{Foot}(J_i^{\text{shrink}}) = \text{Foot}(J_i^{\text{old}}) \) by \( f_i \). It is an element of \( L \) but we often think of it as a geometric point. We know that \( f_1 \in \text{Lit}(J_2^{\text{old}}) \) but \( f_1 \notin \text{Lit}(J_2^{\text{shrink}}) \). Hence, \( J_2 \in G(J) \) (since otherwise \( J_2 \) would not be shrunk) and \( f_1 \notin \text{Lit}(J_2^{\text{shrink}}) \). It follows by the construction of \( \text{Thr}_k(L, J) \) that \( \text{Lit}(J_2^{\text{old}}) \setminus \text{Lit}(J_2^{\text{shrink}}) \subseteq \text{Lit}(I) \); here \( \text{Lit}(I) \) is understood in \( \text{Thr}_k(L, J) \), of course. By the choice of \( z \), \( \text{Lit}(I) \) contains no element of \( L \) except those lying on \( \text{Roof}(J_i^{\text{old}}) \). So \( f_1 \) lies on \( \text{Roof}(J_i^{\text{old}}) \). By \( J_2 \in G(J) \) and the definition of \( G(J) \) (and since \( \text{Roof}(J_i^{\text{old}}) \) consists of line segments of normal slopes and \( \text{Lit}(J_2^{\text{old}}) \) is formed by line segments of normal slopes), it follows that \( f_1 \) is not in the geometric interior of \( \text{Lit}(J_2^{\text{old}}) \). Hence, \((J_1^{\text{old}}, J_2^{\text{old}}) \notin \rho_{\text{infot}} \), which contradicts our assumption that \((J_1^{\text{old}}, J_2^{\text{old}}) \in \rho_{\text{foot}} \) by Lemma 2.1. We have shown that \((\text{Lamp}(L); \leq) \) is a subposet of \((\text{Lamp}(\text{Thr}_k(L, J)), \leq) \), as required.

Clearly, if \( X \in \text{Lamp}(L) \) and \( \text{Peak}(X) \subseteq \downarrow \text{Roof}(J_i^{\text{old}}) \), then none of \((I, X^{\text{shrink}}) \in \rho_{\text{LRCircR}}, (J_i^{\text{old}}, X^{\text{old}}) \in \rho_{\text{LRCircR}}, \text{ and } (J_i^{\text{shrink}}, X^{\text{shrink}}) \in \rho_{\text{LRCircR}} \) is possible. This fact and the last sentence of Lemma 2.1 yield that when we look for covers of \( I \), \( J_i^{\text{old}} \), and \( J_i^{\text{shrink}} \), then it suffices to deal with \( X \) such that \( \text{Peak}(X) \notin \downarrow \text{Roof}(J_i^{\text{old}}) \) and so \( X^{\text{shrink}} \) is not shrunk and \( \text{Lit}(X^{\text{old}}) \subseteq \text{Lit}(X^{\text{shrink}}) \). Since \( \text{CircR}(I) = \text{CircR}(J_i^{\text{old}}) \), we have by Lemma 2.1 that \( \uparrow I \setminus \{I\} \) in \( \text{Lamp}(\text{Thr}_k(L, J)) \) is the same as \( \uparrow J_i^{\text{old}} \setminus \{J_i^{\text{old}}\} \). In particular, the covers of \( I \) in \( \text{Lamp}(\text{Thr}_k(L, J)) \) are exactly the same as the covers of \( J_i^{\text{old}} \) in \( \text{Lamp}(L) \). To see that the covers of \( J_i^{\text{old}} \) and \( J_i^{\text{shrink}} \) are the same, it is sufficient (again by Lemma 2.1) to see that \((J_i^{\text{old}}, X^{\text{old}}) \in \rho_{\text{foot}} \iff (J_i^{\text{shrink}}, X^{\text{shrink}}) \in \rho_{\text{foot}} \), that is, \( \text{Foot}(J_i^{\text{old}}) \subseteq \text{Lit}(X^{\text{old}}) \iff \text{Foot}(J_i^{\text{shrink}}) \subseteq \text{Lit}(X^{\text{shrink}}) \). But this is true since \( \text{Foot}(J_i^{\text{old}}) = \text{Foot}(J_i^{\text{shrink}}) \) and, as mentioned above, \( \text{Lit}(X^{\text{old}}) = \text{Lit}(X^{\text{shrink}}) \). We have proved Part (ii).

To prove Part (iii), let \( z := \text{Foot}(I) \). It follows easily from Lemma 2.3 that \( \rho_{\text{Body}} \) is the union of \( \beta_{\text{left}}, \beta_{\text{left}}, \) and \( \beta_{\text{mid}} \); see Definition 2.2. Thus, using that \( I \) is minimal in \((\text{Lamp}(L); \leq) \), it follows from Lemma 2.1 that for any lamp \( K \neq I \),...
none of $K \beta_{\text{left}} I$, $K \beta_{\text{right}} I$, and $K \beta_{\text{mid}} I$ holds. Hence, it follows from (2.15) and Lemma 2.3 that,

with the exception of the neon tubes of Lamp($I$), there is no precipitous edge in Lit($I$).

In this aspect, Figure 2 reflects generality. Now let $M$ be the collection of those lamps $J$, for which Peak($J$) $\in$ Floor($I$). Motivated by the (hypothetical) passage from Figure 2 to Figure 1, we do the following. For each lamp $n$ of $I$ that is neither the leftmost lamp nor the rightmost lamp of $I$, delete the elements of the intervals $[\text{lc}(L) \wedge \text{Foot}(n), \text{Foot}(n)]$ and $[\text{rc}(L) \wedge \text{Foot}(n), \text{Foot}(n)]$. Note that these intervals are chains of normal slopes; see (2.16). For the leftmost lamp $p_0$ of $I$, if $p_0$ is not the only lamp of $I$, then delete the elements $[\text{lc}(L) \wedge \text{Foot}(p_0), \text{Foot}(p_0)]$. Similarly, for the rightmost lamp $p_1$ of $I$, if $p_1$ is not the only lamp of $I$, then delete the elements $[\text{rc}(L) \wedge \text{Foot}(p_1), \text{Foot}(p_1)]$. Then for each edge $u \prec v$, if at least one of $u$ and $v$ has been deleted, then remove this edge. Next, delete the edges and the vertices of the intervals $[\text{lc}(L) \wedge \text{Foot}(I), \text{Foot}(I)]$ and $[\text{rc}(L) \wedge \text{Foot}(I), \text{Foot}(I)]$. (These intervals, again, are chains of normal slopes.)

For $J_i \in M \setminus \{J\}$, if Peak($J_i$) is on the left line segment, that is, on the line segment of slope $(1, 1)$ of Floor($I$), then we shift Peak($J_i$) to the northwest, that is, with slope $(1, -1)$ to the line segment $[\text{lc}(L) \wedge \text{Peak}(J_i), \text{Peak}(I)]$; this line segment is of normal slope by (2.16). The neon tubes of $J_i$ remain precipitous since their feet are below the carrier line of this line segment. Similarly, if Peak($J_i$) is on the right line segment of Floor($I$), then we shift Peak($J_i$) to the northeast to the line segment $[\text{rc}(L) \wedge \text{Peak}(J_i), \text{Peak}(I)]$. Finally, we shift Peak($J$) to the geometric point where Peak($I$) was. Using Lemma 2.4, we can conclude that the lattice $L'$ we have just obtained is a slim rectangular lattice. It is clear by construction that (the diagram of) $L$ is (the diagram of) Thr$_{\text{Num}(I)}(L', J)$, as required. This completes the proof of Note 2.8. \hfill \Box

Proof of Theorem 1.1. We begin with, and mostly we deal with, Part (B) of the theorem. (At the end, we will derive Part (A) from Part (B) quite easily.) Observe that it suffices to show Part (B) for the particular case where $|\text{Lamp}(L) \setminus F| = 1$, that is,

\begin{equation}
\text{it suffices to prove that whenever } L \text{ is as in (2.1), } |\text{Lamp}(L)| \geq 3, \text{ and } I \text{ is a minimal element of } (\text{Lamp}(L); \leq), \text{ then there exists a slim rectangular lattice } L' \text{ such that } (\text{Lamp}(L')); \leq \cong (\text{Lamp}(L) \setminus \{I\}; \leq). \end{equation}

Indeed, if we manage to prove (2.32), then we can conclude Part (B) by induction on $|\text{Lamp}(L)|$ as follows:

If $F$ is distinct from $J(D) = J(\text{Con } L)$ (which can be assumed since otherwise there is nothing to prove), then take a minimal element $I$ in $\text{Lamp}(L) \setminus F$; then (2.32) yields a slim rectangular lattice $L'_1$ with $(\text{Lamp}(L'_1); \leq) \cong (\text{Lamp}(L) \setminus \{I\}; \leq)$. After identifying $F$ with its image under the isomorphism denoted by $\cong$ in the previous sentence, $F$ is an order filter in $\text{Lamp}(L'_1)$. Since $|\text{Lamp}(L'_1)| = |\text{Lamp}(L)| - 1$, the induction hypothesis yields a slim rectangular lattice $L'$ with $(\text{Lamp}(L'); \leq) \cong (F; \leq)$. Hence, by Lemma 2.4 (J(Con $L'$); $\leq) \cong (F, \leq)$, as required by Part (B) of the theorem.

---

\footnote{That is, if Num($I$) > 1 before deleting any neon tube.}

\footnote{That is, if Num($I$) > 1 before deleting any neon tube.}
To prove the existence of \( L' \) satisfying the requirements of (2.32), there are three cases to consider. But first of all, we make some preparations; see (2.23) for the concept it involves. It follows by a trivial induction from (2.14) that for every lamp \( I' \), \( \text{CircR}(I') \) is a rectangular interval. This allows us to observe by Lemma 2.1 that if \( I' \) is a minimal element of \( \text{Lamp}(L) \), then \( I' \) is LitD(\( I' \))-distributive. \( (2.33) \)

Now we are ready to deal with the above-mentioned three cases.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4.png}
\caption{Removing a minimal boundary lamp}
\end{figure}

First, assume that \( I \), in addition to being minimal, is also a maximal element of \( \text{Lamp}(L) \). Then \( I \) is a boundary lamp; say, a left boundary lamp. By (2.33) or Lemma 2.1, the geometric interior of \( \text{Lit}(I) \) contains no element of any lamp. In Figure 4, \( \text{Lit}(I) \) is grey-filled; the task is to get rid of this grey-filled area. As the figure shows, \( \uparrow \text{Roof}(I) \) can be shifted to the southwest, to the direction \((-1, -1)\), so that (the former) \( \text{Roof}(I) \) overlaps (the former) \( \text{Floor}(I) \). Then \( I \) disappears but all other lamps remain. These other lamps not only remain lamps but their bodies remain the same (apart from a geometric translation in some cases). Let \( L' \) denote the new lattice we obtain. (Note that it is not a problem if \( I \) is geometrically the uppermost lamp or the lowest lamp on the left upper boundary of \( L \).) Since we have shifted the upper half of the diagram in the direction \((-1, -1)\) and this direction is parallel or perpendicular to the paths of the photons, the relation \( \rho_{\text{Body}} \) does not change\(^8\) for the members of \( \text{Lamp}(L) \setminus \{I\} \). Since \( \rho_{\text{Body}} \) determines the ordering by Lemma 2.1 \((\text{Lamp}(L') \leq) \cong (\text{Lamp}(L) \setminus \{I\}) \leq)\), as required.

Second, assume that the minimal lamp \( I \) is not maximal and there is no internal lamp \( J \neq I \) such that \( \text{Body}(J) \subseteq \text{CircR}(I) \). By [3, Lemma 3.2], stating that \{boundary lamps\} = \{maximal lamps\}, \( I \) is an internal lamp. We know from (2.33) that \( I \) is LitD(\( I \))-distributive. Therefore, Note 2.6 with particular emphasis on its Part (v), yields an \( L' \) such that \((\text{Lamp}(L') \leq) \cong (\text{Lamp}(L) \setminus \{I\}) \leq)\), as required.

Third, assume that the minimal lamp \( I \) is not maximal and there is a lamp \( J' \neq I \) such that \( \text{Body}(J') \subseteq \text{CircR}(I) \). By [3, Lemma 3.2], \( I \) is an internal lamp again. Since \( \text{Lit}(I) \) is bordered by line segments of normal slopes and these line segments consist of edges by (2.16), the boundary of \( \text{Lit}(I) \) cannot cut the region \( \text{Body}(J') \), see (2.13), into two parts of positive geometric area\(^9\). Therefore, \( \text{Body}(J') \) either a subset of \( \text{Lit}(I) \), or it is a subset of the geometric (topological)

---

\(^8\)Using the terminology introduced in Czédli and Grätzer [7], it is clear that \( \text{Lamp}(L) \setminus \{I\} \) and \( \text{Lamp}(L') \) have the same distance-free geometry.

\(^9\)If \( \text{Body}(J') \) is of a positive geometric area, that is, if \( \text{Num}(J') > 1 \). If \( \text{Num}(J') = 1 \), then \( \text{Body}(J') \) is a line segment and we should say “positive lengths” instead of “positive areas”.

---
thrusting a multifork at Peak($J_{\text{maximal}}$) lamps of Let $J$ respectively, listing them upwards. Later in Figure 5; pick a slim rectangular lattice

Proof of Observation 1.4. For Part (A), we only give a brief outline. For $\kappa \in \{1, 2\}$, pick a slim rectangular lattice $L_{\kappa}$ with $D_{\kappa} \cong \text{Con} L_{\kappa}$, that is, $J(D_{\kappa}) \cong \text{Lamp}(L_{\kappa})$. Let $J \in \text{Lamp}(L_{1})$ correspond to $j$. Denote by $A_{1}, \ldots, A_{s}$ and $B_{1}, \ldots, B_{t}$ the maximal lamps of $L_{2}$ on its upper left boundary and on its upper right boundary, respectively, listing them upwards. Later in Figure 5 $t = s = 3$. Instead of thrusting a multifork at Peak($J$), see Definition 2.7 we thrust the filter\footnote{Instead of thrusting this filter in one step, it is possible to thrust $I$ first and then add the (1-fold multiforks) $A_{s}, A_{s-1}, \ldots, A_{1}, B_{t}, B_{t-1}, \ldots, A_{1}$, one by one.} $\uparrow z$ of

MULTIFORKS IN SLIM RECTANGULAR LATTICES 17
Figure 5 analogously. (Again, this filter is a slim rectangular lattice but its diagram is not a $C_1$-diagram, but this makes no problem.) So $\text{Peak}(J)$ will be shifted down as well as, possibly, some other lamps of $\downarrow J$, and $z$ in Figure 5 will be $\text{Peak}(J^{\text{shr}})$. The lamps $A_1, \ldots, A_s, B_1, \ldots, B_t$ in Figure 5 illuminate the grey-filled area in the same way as they illuminate the full geometric rectangle of $L_2$. After that $\uparrow z$ is thrust to $L_1$ at $\text{Peak}(J)$, we can perform the same sequence of multifork extensions in the grey-filled area as we perform when deriving $L_2$ from its initial grid according to (2.14). The slim rectangular lattice $L$ we obtain at the end witnesses by $\text{Lamp}(L) \equiv P^7(\text{Lamp}(L_1), j, \text{Lamp}(L_2))$ that $D \in \text{Con}(\text{SR})$, as required.

For Part (B), we give even less details. When deriving $L_1$ from a grid by a (2.14)-sequence, increase the multiplicity when adding the multifork that gives birth to the lamp $J$. If we increase this multiplicity by a sufficiently large number, then $J$ will have neighboring unused lamps that allow us to insert $A_1, A_2, \ldots, B_1, B_2, \ldots$ and to establish a grey-filled area as in case of Part (A), and then the continuation will be the same. The main difference is that now $A_1, A_2, \ldots, B_1, B_2, \ldots$ are illuminated and covered by $J$. We admit that lots of technicalities would be needed to present a rigorous proof. 

\begin{proof}

\end{proof}

\section*{Statements and declarations}

\textbf{Data availability statement.} Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

\textbf{Competing interests.} Not applicable as there are no interests to report.

\section*{References}

[1] Czédli, G.: Patch extensions and trajectory colorings of slim rectangular lattices. Algebra Universalis 72, 125–154 (2014)

[2] Czédli, G.: Diagrams and rectangular extensions of planar semimodular lattices. Algebra Universalis 77, 443–498 (2017)

[3] Czédli, G.: Lamps in slim rectangular planar semimodular lattices. Acta Sci. Math. (Szeged) 87, 381–413 (2021) (Open access: https://doi.org/10.14232/actasm-021-865-y or browse http://www.acta.hu/)

\[ \text{Figure 5. Illustration for the proof of 1.4} \]
[4] Czédli, G.: Infinitely many new properties of the congruence lattices of slim semimodular lattices, submitted to Acta. Sci. Math. (Szeged).

[5] Czédli, G.: A property of meets in slim semimodular lattices and its application to retracts. Acta Sci. Math. (Szeged), to appear. For an earlier version, see http://arxiv.org/abs/2112.07594

[6] Czédli, G.: Notes on congruence lattices and lamps of slim semimodular lattices. https://arxiv.org/abs/2206.14769v2

[7] Czédli, G., Grätzer, G.: A new property of congruence lattices of slim, planar, semimodular lattices. Categories and General Algebraic Structures with Applications 16, 1-28 (2022) (Open access: https://cgasa.sbu.ac.ir/article_101508.html)

[8] Czédli, G., Makay, G.: Swing lattice game and a direct proof of the swing lemma for planar semimodular lattices. Acta Sci. Math. (Szeged) 83, 13–29 (2017)

[9] Czédli, G., Schmidt, E.T.: The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices. Algebra Universalis 66, 69–79 (2011)

[10] Czédli, G., Schmidt, E.T.: Slim semimodular lattices. I. A visual approach. Order 29, 481–497 (2012)

[11] Czédli, G., Schmidt, E.T.: Slim semimodular lattices. II. A description by patchwork systems. ORDER 30, 689–721 (2013)

[12] Grätzer, G.: Lattice Theory: Foundation. Birkhäuser, Basel (2011)

[13] Grätzer, G.: Congruences in slim, planar, semimodular lattices: The Swing Lemma. Acta Sci. Math. (Szeged) 81, 381–397 (2015)

[14] Grätzer, G.: The Congruences of a Finite Lattice, A Proof-by-Picture Approach, second edition. Birkhäuser, 2016. xxxii+347. Part I is accessible at https://www.researchgate.net/publication/299594715

[15] Grätzer, G.: Notes on planar semimodular lattices. VIII. Congruence lattices of SPS lattices. Algebra Universalis 81 (2020), Paper No. 15, 3 pp.

[16] Grätzer, G., Knapp, E.: Notes on planar semimodular lattices. III. Rectangular lattices. Acta Sci. Math. (Szeged) 75 (2009), 29–48.

[17] Kelly, D., Rival, I.: Planar lattices. Canad. J. Math. 27, 636–665 (1975)

Email address: czedli@math.u-szeged.hu
URL: http://www.math.u-szeged.hu/~czedli/

UNIVERSITY OF SZEGED, BOLYAI INSTITUTE. SZEGED, ARADIVÉRTANÚK TERE 1, HUNGARY 6720

---

at the time of writing, see the author’s website