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POSITIVE ENERGY REPRESENTATIONS OF THE LOOP GROUPS OF NON SIMPLY CONNECTED LIE GROUPS

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Abstract. We classify and construct all irreducible positive energy representations of the loop group of a compact, connected and simple Lie group and show that they admit an intertwining action of Diff(S^1).

1. Introduction

Let K be a compact, connected and simple Lie group and \( LK = C^\infty(S^1, K) \) its loop group. We shall be concerned with the study of positive energy representations of \( LK \), i.e. projective unitary representations

\[
\pi : LK \rightarrow PU(\mathcal{H}) = U(\mathcal{H})/T
\]

extending to the semi–direct product \( LK \rtimes \text{Rot}(S^1) \) in such a way that \( \text{Rot}(S^1) \) acts by non–negative characters only and with finite–dimensional eigenspaces. In other words,

\[
\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}(n)
\]

where \( \mathcal{H}(n) = \{ \xi \in \mathcal{H} \mid \pi(R_\theta)\xi = e^{in\theta}\xi \} \), the subspace of energy \( n \), supports a finite–dimensional projective representation of \( K \).

Positive energy representations are completely reducible and, when \( K \) is simply connected, have been classified by several authors \[\text{PS, Wa}\]. The irreducible ones are then uniquely determined by their level \( \ell \in \mathbb{N} \) and lowest energy subspace \( \mathcal{H}(0) \). The former classifies the corresponding central extension of \( LK \) or, equivalently, the cocycle associated to the infinitesimal representation of its Lie algebra. The latter is an irreducible \( K \)–module the highest weight \( \lambda \) of which is bound by the requirement that

\[
\langle \lambda, \theta \rangle \leq \ell
\]

where \( \theta \) is the highest root of \( K \) and \( \langle \cdot, \cdot \rangle \) is the basic inner product of \( K \), i.e. the multiple of the Killing form such that \( \langle \theta, \theta \rangle = 2 \).

The aim of the present paper is to extend the above classification to the case of groups which are not simply connected. Write \( K = G/Z \) where \( G \) is the universal covering group of \( K \) and \( \pi_1(K) \cong Z \subseteq Z(G) \). It will be more convenient to consider positive energy representations of the group of discontinuous loops

\[
L_ZG = \{ \zeta \in C^\infty(\mathbb{R}, G) \mid \zeta(x + 2\pi)\zeta(x)^{-1} \in Z \}
\]

deferring to later the determination of those which factor through \( LK \).

Since \( L_ZG/LG \cong Z \), these may be studied with Mackey’s machine \[\text{Ma1, Ma2}\], paying however due care to the fact that the representations in question are genuinely projective and form a strict subclass of those of \( L_ZG \). With these provisos, the analysis carries over essentially unchanged and is dealt with in the following sections. We summarise them below.

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In section 3, we classify the central extensions of $L_2G$ and $L(G/Z)$ by $T$. We show in particular the existence of an obstruction to the extension of the level $\ell$ central extension of $LG$ to $L_2G$. This appears only at odd $\ell$ and for some $G$, the complete list of which is given in an appendix. It comprises $SU_n$ with $n$ even. Thus, surprisingly perhaps, $L_2SU_2$, and a fortiori $LSO_3$ do not have any odd level positive energy representations. A further obstruction on the level appears when demanding that a given central extension of $L_2G$ descend to $L(G/Z)$. $\ell$ must then be a multiple of a given non–negative integer $\ell_0$, the basic level of $G/Z$, of which we compute the value for all simple groups. It is $n$ for $SU_n/\mathbb{Z}_n$.

In section 4, we show that the category $\mathcal{P}_\ell$ of positive energy representations of $LG$ at a given level $\ell$ is closed under conjugation by elements of $L_2G$ and therefore that $\mathcal{Z} \cong L_2G/LG$ acts on the positive energy dual of $LG$. We also compute the geometric counterpart of this action on the alcove of $G$ which parametrises the irreducibles in $\mathcal{P}_\ell$. Aside from rendering the action of $Z$ more explicit, this shows that it operates by automorphisms of the extended Dynkin diagram of $G$.

In section 3, we compute the Mackey obstruction for a subgroup $Y \subseteq Z$ stabilising a given positive energy representation $H$ of $LG$. This vanishes for most groups since $Y$ is cyclic unless $G/Y = PSO_4n$ but, somewhat surprisingly, doesn’t in the latter case.

Section 3 contains our main results. We construct all irreducible positive energy representations of $L_2G$ and show that they are classified by the central extension of $L_2G$ they induce and their isomorphism class as $LG$–modules. Moreover, we prove that they admit an intertwining action of $\text{Diff}_+(S^1)$ and identify it with the Segal–Sugawara representation obtained by regarding them as positive energy $LG$–modules. Finally, in section 5 we determine those representations which factor through $L(G/Z)$, thereby obtaining all positive energy representations of the latter group. They are exactly those the level of which is a multiple of the basic level of $G/Z$.

Remark. In physical terms, we classify in this paper all inequivalent quantisations of the chiral Wess–Zumino–Witten model with target group $G/Z$. Related results have been obtained in the non–chiral case by Gepner–Witten [GW], Felder–Gawędzki–Kupiainen [FGK1, FGK2] and Gaberdiel [G].

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2. The coroot and coweight lattices of $G$

We begin by gathering some elementary properties of the lattices canonically associated to $G$. The present discussion follows [GO]. Throughout this paper, $G$ denotes a compact, connected and simply connected simple Lie group with Lie algebra $\mathfrak{g}$. Let $T \subseteq G$ be a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. By the roots of $G$ we shall always mean its infinitesimal roots, namely the set $\Delta$ of linear forms $\alpha \in \mathfrak{t}^\ast = \text{Hom}(\mathfrak{t},i\mathbb{R})$ such that the subspace $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}_{\mathfrak{c}} | [h,x] = \alpha(h)x \ \forall h \in \mathfrak{c}\}$ is non–zero. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a basis of $\Delta$ and $\theta$ the corresponding highest root. The basic inner product $\langle \cdot, \cdot \rangle$, i.e. the unique multiple of the Killing form such that $\langle \theta, \theta \rangle = 2$, is positive definite on $\mathfrak{t}$ and gives an identification $\mathfrak{t}^\ast \cong \mathfrak{t}$ of which we shall make implicit use. The coroots of $G$ are the elements of $\mathfrak{t}$ given...
by $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$. They form the dual root system $R^\vee$.

The root and coroot lattices $\Lambda_R \subset i\mathbb{R}^*$, $\Lambda^\vee_R \subset i\mathbb{R}$ are the lattices spanned by $R$ and $R^\vee$ respectively. They have $\mathbb{Z}$-basis given by $\Delta$ and $\Delta^\vee = \{\alpha_i^\vee, \ldots, \alpha_n^\vee\}$. Since $\theta$ is a long root and there are at most two root lengths in $R$ with the ratio of the squared length of a long root by that of a short one equal to 2 or 3, rewriting $\alpha^\vee = \frac{(\theta, \theta)}{\langle \alpha, \alpha \rangle} \alpha$ we see that $\Lambda_R^\vee \subset \Lambda_R$.

Notice that $\langle \alpha^\vee, \alpha^\vee \rangle = \frac{1}{\langle \alpha, \alpha \rangle} = 2(\theta, \theta)$ so that $\Lambda_R^\vee$ is an even, and therefore integral lattice.

The weight and coroweight lattices $\Lambda_\mathbb{Z}^\vee$, $\Lambda_\mathbb{Z}$ respectively. They have $Z$–basis given by the fundamental (co)weights $\lambda_i$, $\lambda_i^\vee$ defined by

$$\langle \lambda_i, \lambda_j^\vee \rangle = \langle \lambda_i^\vee, \alpha_j \rangle = \delta_{ij} \quad (2.1)$$

Clearly, $\Lambda^\vee_\mathbb{W} \subset \Lambda_\mathbb{W}$. Moreover, by the integrality properties of root systems, $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for any root $\alpha$ and coroot $\beta^\vee$ so that $\Lambda_R \subset \Lambda^\vee_\mathbb{W}$ and, dually, $\Lambda^\vee_R \subset \Lambda^\vee_\mathbb{W}$. Graphically,

$$\Lambda_R \subset \Lambda_\mathbb{W} \subset i\mathbb{R}^*$$

$$\Lambda^\vee_R \subset \Lambda^\vee_\mathbb{W} \subset i\mathbb{R}$$

(2.2)

Let $Z(G)$ be the centre of $G$ and $\widehat{Z(G)} = \text{Hom}(Z(G), \mathbb{T})$ its Pontryagin dual. The following is well–known

**Lemma 2.1.**

(i) The map $e(h) = \exp_T(-2\pi i h)$ induces an isomorphism $\Lambda^\vee_\mathbb{W}/\Lambda^\vee_R \cong Z(G)$.

(ii) The pairing $\mu(\exp_T(h)) = e^{\langle \mu, h \rangle}$ induces an isomorphism $\Lambda_\mathbb{W}/\Lambda_R \cong \widehat{Z(G)}$.

**Remark.** When $G$ is simply–laced, i.e. with all roots of equal length, the basic inner product identifies roots and coroots and the vertical inclusions in (2.2) are equalities. Moreover, lemma 2.1 yields a canonical isomorphism $\widehat{Z(G)} \cong Z(G)$.

The Weyl group $W$ of $G$ is the finite group generated in $\text{End}(i\mathbb{R}^*)$ by the orthogonal reflections $\sigma_\alpha$ corresponding to the roots $\alpha \in R$. Since

$$\sigma_\alpha(\mu) = \mu - 2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha = \mu - \langle \mu, \alpha^\vee \rangle \alpha = \mu - \langle \mu, \alpha \rangle \alpha^\vee \quad (2.3)$$

the action of $W$ preserves $\Lambda^\vee_R$–cosets in $\Lambda^\vee_\mathbb{W}$ and $\Lambda_R$–cosets in $\Lambda_\mathbb{W}$. Call $\mu \in \Lambda_\mathbb{W}$ (resp. $\mu \in \Lambda^\vee_\mathbb{W}$) minimal if it is of minimal length in its $\Lambda_R$ (resp. $\Lambda^\vee_R$)–coset. The following gives a characterisation of minimal (co)weights.

**Proposition 2.2.** There is, in each $\Lambda^\vee_\mathbb{W}/\Lambda^\vee_R$–coset (resp. $\Lambda_\mathbb{W}/\Lambda_R$–coset) a unique $W$–orbit of elements of minimal length. These may equivalently be characterised as those $\lambda$ such that

$$\langle \lambda, \alpha \rangle \in \{0, \pm1\} \quad (\text{resp. } \langle \lambda, \alpha^\vee \rangle \in \{0, \pm1\}) \quad (2.4)$$

for any root $\alpha$ (resp. coroot $\alpha^\vee$).

**Proof.** It is sufficient to consider the case of $\Lambda^\vee_\mathbb{W}/\Lambda^\vee_R$ since $\Lambda_R, \Lambda_\mathbb{W}$ are the coroot and coweight lattices of the dual root system $R^\vee$. Let $\mu \in \Lambda^\vee_\mathbb{W}$ be of minimal length in its $\Lambda^\vee_R$–coset. Then, for any root $\beta$ and corresponding coroot $\beta^\vee = \frac{2\beta}{\langle \beta, \beta \rangle}$, we have $\|\mu \pm \beta^\vee\|^2 \geq \|\mu\|^2$ and, expanding $|\langle \mu, \beta \rangle| \leq 1$. Assume that $\lambda \in \Lambda^\vee_\mathbb{W}$ satisfies $\langle \lambda, \alpha \rangle$ and $\nu = \lambda \mod \Lambda^\vee_R$ is of minimal length in its coset. We claim that $w\lambda = \nu$ for an appropriate $w \in W$. To see this, write $\nu = \lambda + \beta^\vee_i + \cdots + \beta^\vee_j$ where the $\beta^\vee_i$ are (possibly repeated) coroots. Clearly, one cannot have $\langle \lambda, \beta_i \rangle \geq 0$ for all $i$ otherwise

$$\langle \nu, \nu \rangle = \langle \lambda, \lambda \rangle + \langle \sum \beta^\vee_i, \sum \beta^\vee_i \rangle + 2\langle \lambda, \sum \beta^\vee_i \rangle > \langle \lambda, \lambda \rangle \quad (2.5)$$
in contradiction with the minimality of \( \nu \). Thus, by (2.4) there exists an \( i \in \{1, \ldots, r\} \) such that \( (\lambda, \beta_i) = -1 \) and therefore \( \lambda_1 := \sigma_{\beta} \lambda = \lambda + \beta_i \). Moreover, \( \lambda_1 \) satisfies (2.4) since \( W \) permutes the roots and preserves \( \langle \cdot, \cdot \rangle \). We may therefore iterate the above step to find a permutation \( \tau \) of \( \{1, \ldots, r\} \) such that

\[
\lambda_i := \lambda + \beta_{\tau(1)} + \cdots + \beta_{\tau(i)} = \sigma_{\beta_{\tau(i)}} \cdots \sigma_{\beta_{\tau(1)}} \lambda
\]  

(2.6)

In particular, \( \lambda_r = \nu \) and therefore \( \nu \in W \lambda \) whence \( \| \nu \| = \| \lambda \| \).

Recall that a weight \( \mu \in \Lambda_W \) is dominant if it lies in the cone

\[
\Lambda_W^+ = \{ \nu \in \Lambda_W \mid \langle \nu, \alpha_i^\vee \rangle \geq 0 \ \forall \alpha_i^\vee \in \Delta^\vee \} = \bigoplus \lambda_i \cdot \mathbb{N}
\]  

(2.7)

Since \( \Lambda_W^+ \) is a fundamental domain for the action of \( W \) on \( \Lambda_W \), lemma 2.1 and proposition 2.2 establish a bijective correspondence between elements in \( \mathbb{Z}(G) \) and minimal dominant weights. Dually, the elements of \( Z(G) \) correspond to special roots, i.e. those \( \alpha \) root expansion

\[
\text{non–zero minimal dominant coweights are those corresponding to a subgroup} \ Z \subseteq Z(G) \ \text{of minimal length in their} \ \Lambda_R^– \text{–coset. The following gives another characterisation of minimal dominant coweights.}
\]

**Lemma 2.3.** The non–zero minimal dominant coweights are exactly the fundamental coweights corresponding to special roots, i.e. those \( \alpha_i \in \Delta \) bearing the coefficient 1 in the expansion

\[
\theta = \sum m_i \alpha_i
\]  

(2.8)

**Proof.** By proposition 2.2, \( \mu \in (\Lambda_W^+)^\vee \) is minimal iff \( \langle \mu, \theta \rangle \leq 1 \). Indeed, for any positive root \( \alpha \), we get \( 0 \leq \langle \mu, \alpha \rangle \leq \langle \mu, \theta \rangle - \langle \mu, \theta - \alpha \rangle \leq \langle \mu, \theta \rangle \). Since \( \langle \mu, \theta \rangle = 0 \) implies \( \mu = 0 \), the non–zero minimal dominant coweights are those \( \mu \in (\Lambda_W^+)^\vee \) such that \( \langle \mu, \theta \rangle = 1 \). Writing \( \mu = \sum k_i \lambda_i^\vee \), \( k_i \geq 0 \) and using (2.8), we find \( \langle \mu, \theta \rangle = \sum k_i m_i \). Since \( \theta - \alpha_i \) is a sum of positive roots, \( m_i \geq 1 \) for any \( i \) and result follows \( \Diamond \).

3. Central extensions of \( L_2G \)

This section is devoted to the study of the central extensions by \( T \) of the group of discontinuous loops

\[
L_2G = \{ \zeta \in C^\infty(\mathbb{R}, G) \mid \zeta(x + 2\pi)\zeta(x)^{-1} \in Z \}
\]  

(3.1)

corresponding to a subgroup \( Z \subseteq Z(G) \). These are uniquely determined by their restrictions to \( LG = C^\infty(S^1, G) \) and to \( \text{Hom}(T, T/Z) \), the integral lattice of \( G/Z \). The former are classified by their level \( \ell \in \mathbb{Z} \) and the latter by their commutator map, a \( T \)–valued, skew–symmetric bilinear form \( \omega \) on \( \text{Hom}(T, T/Z) \). We shall prove below that \( \ell \) and \( \omega \) are bound by the requirement that

\[
\omega(\lambda, \mu) = (-1)^{\ell(\lambda, \mu)}
\]  

(3.2)

whenever \( \lambda \) lies in the coroot lattice \( \text{Hom}(T, T) \) and therefore that central extensions of \( LG \) do not necessarily extend to \( L_2G \) since a suitable \( \omega \) satisfying (3.2) for a given \( \ell \) need not exist. In particular, \( L_{2\mathbb{Z}}SU_2 \), and more generally \( L_{2\mathbb{Z}}SU_{2n} \), do not possess central extensions of odd level. For compatible \( \ell \) and \( \omega \), we construct the corresponding central extension of \( L_2G \) and show that the action of \( \text{Diff}_+(S^1) \) on \( L_2G \) lifts uniquely to it. The classification of central extensions of \( L(G/Z) \) follows easily from this and is described at the end of this section.
3.1. Central extensions of $L G$.

We begin by reviewing the construction of central extensions of $L G$, and more generally of a connected and simply-connected Fréchet Lie group $G$, following chapter 4 of [PS]. All central extensions considered in this section are understood to be smooth and have $T$ as their extending group. Let $L$ be the Lie algebra of $G$ and $\beta$ a two-cocycle on $L$, i.e. a continuous, skew-symmetric, bilinear map $\beta : L \times L \to \mathbb{R}$ satisfying

$$\beta([X,Y], Z) + \beta([Y,Z], X) + \beta([Z,X], Y) = 0$$ \hspace{1cm} (3.1.1)

$\beta$ may be regarded as a right-invariant, closed two-form on $G$ and we assume that $(2\pi)^{-1}\beta$ is integral, i.e. such that its integral over any two-cycle in $G$ is an integer. Then, there exists a unique central extension

$$1 \to T \to \tilde{G} \xrightarrow{\pi} G \to 1$$ \hspace{1cm} (3.1.2)

the Lie algebra of which is $\tilde{L} = L \oplus i\mathbb{R}$ with bracket

$$[X \oplus it, Y \oplus is] = [X, Y] \oplus i\beta(X, Y)$$ \hspace{1cm} (3.1.3)

$\tilde{G}$ may be constructed using the following path group description. Assume $\tilde{G}$ exists and regard it as a principal $T$-bundle over $G$ with connection given by the splitting $\tilde{L} = L \oplus i\mathbb{R}$. In other words, the horizontal subspace at $\tilde{g} \in \tilde{G}$ is $L\tilde{g}$. The pull-back of $\tilde{G}$ to the space

$$\mathcal{PG} = \{ p : I \to G | p(0) = 1 \}$$ \hspace{1cm} (3.1.4)

of piece-wise smooth paths via the end-point fibration $\mathcal{PG} \xleftarrow{\pi} G$ is topologically trivial, the identification of the fibre at the constant path $1$ with that at $p$ being simply given by parallel transport along $p$. Explicitly, if $X = \tilde{p}p^{-1} : I \to L$ is the right logarithmic derivative of $p$, the identification maps $z \in T = e^s\pi^{-1}(1)$ to the end point of the path $\tilde{p}$ in $\tilde{G}$ obtained by solving $\tilde{p} = X\tilde{p}$, $\tilde{p}(0) = z$. If $p$ is closed, and therefore contractible in $G$, the corresponding identification is simply multiplication by the holonomy $e^{\int_{\tilde{p}} \beta}$ where $\sigma$ is any two-cycle in $G$ with boundary $p$.

The concatenation of pointed paths defined by

$$p \vee q(t) = \begin{cases} q(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ p(2t - 1)q(1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$ \hspace{1cm} (3.1.5)

induces a monoidal structure on $\mathcal{PG}$ which, combined with the group law on $\tilde{G}$ makes $e^s\tilde{G}$ a monoid. The crucial feature of the corresponding multiplication law is that it becomes the canonical one when transported to $\mathcal{PG} \times T \cong e^s\tilde{G}$, a direct consequence of the $\tilde{G}$-invariance of the connection on $\tilde{G}$. It follows that, as a group, $\tilde{G}$ may be described, or indeed defined as the quotient of $\mathcal{PG} \times T$ with law $(p, z) \star (q, w) = (p \vee q, zw)$ by the equivalence relation

$$(p, z) \sim (q, w) \iff p(1) = q(1) \text{ and } e^{\int_{\tilde{p}} \beta} = \omega e^{\int_{\tilde{q}} \beta}$$ \hspace{1cm} (3.1.6)

where $\sigma$ is a two-cycle with boundary $p \vee q$ and $\tilde{q}(t) = q(1 - t)q(1)^{-1}$.

**Lemma 3.1.1.** An automorphism $A$ of $G$ lifts to $\tilde{G}$ if, and only if it leaves the cohomology class of $\beta$ invariant, i.e. iff there exists a linear map $F : L \to \mathbb{R}$ such that for any $X, Y \in L$

$$\beta(AX, AY) = \beta(X, Y) + F([X, Y])$$ \hspace{1cm} (3.1.7)

The lift is then unique up to multiplication by a character of $G$ and is given infinitesimally by

$$\tilde{A}(X \oplus it) = AX \oplus i(F(X) + t)$$ \hspace{1cm} (3.1.8)

and in the path group description of $\tilde{G}$ by

$$\tilde{A}(p, z) = (Ap, ze^{\int_{\tilde{p}} F})$$ \hspace{1cm} (3.1.9)
where $F$ is regarded as a right–invariant one–form, so that $\int_0^1 F = \int_0^1 \dot{F}(\dddot{p} \dddot{p}^{-1})$.

**Proof.** The necessity of (3.1.7) is straightforward. Indeed, a lift $\tilde{A}$ acts on $\tilde{L}$ by $\tilde{A}(X \oplus i t) = AX \oplus i G(X) + t$ for some linear map $G : L \to \mathbb{R}$. Requiring that $\tilde{A}[X \oplus i, Y \oplus i] = [\tilde{A}(X \oplus i), \tilde{A}(Y \oplus i)]$ and expanding both members yields

$$A[X, Y] + i(G([X, Y]) + \beta(X, Y)) = [AX, AY] + i\beta(AX, AY) \quad (3.1.10)$$

and therefore (3.1.7). Conversely, (3.1.9) is a well–defined lift of $A$. Indeed, when regarded as an identity between right–invariant forms in $G$, (3.1.7) reads $A^*\beta = \beta – dF$. It follows that if $(p, z) \sim (q, w)$ and $\sigma$ is a two–cycle in $G$ with $\partial \sigma = p \vee \dot{q}$, then, $\partial A\sigma = A(p \vee \dot{A}q)$ and

$$e^{i\int_{\sigma} A^*\beta} = e^{i\int_{\sigma} A^*\beta} e^{i\int_{\sigma} e^{-i\int_{\sigma} F} e^{i\int_{\sigma} F}} = we^{i\int_{\sigma} F} \exp(i\int_{\sigma} e^{F}) \quad (3.1.11)$$

so that $(Ap, ze^{i\int_{\sigma} F}) \sim (Aq, we^{i\int_{\sigma} F})$. The uniqueness of $\tilde{A}$ is clear for if $\tilde{A}_i, i = 1, 2$ are two lifts of $A$, then $\tilde{A}_2\tilde{A}_1^{-1}$ is a lift of the identity and fixes $T$ so that it is given by $\tilde{\chi} \circ \pi$ for some $\chi \in \text{Hom}(G, T)$.

**Remark.** The phase factor in (3.1.9) may be derived from (3.1.8) as follows. For any $p \in PG$, denote by $\tilde{p}$ its unique horizontal lift through $1 \in G$ so that $p = \tilde{p} \tilde{p}^{-1}$ and $\tilde{p}(0) = 1$. Then $Q(t) = \tilde{A}(pt)$ solves

$$\dot{Q} = \tilde{A}(\dddot{p} \dddot{p}^{-1})Q = (\dddot{A}p)(Ap)^{-1}Q + iF(\dddot{p} \dddot{p}^{-1})Q \quad (3.1.12)$$

Set $Q(t) = \phi(t)\tilde{A}p(t)$ where $\phi(t) \in T$, then (3.1.12) reduces to $\dot{\phi} = iF(\dddot{p} \dddot{p}^{-1})\phi$ and therefore $\phi(t) = e^{i\int_{0}^{t} F(\dddot{p} \dddot{p}^{-1})d\tau}$. Conversely, (3.1.8) may be obtained from (3.1.9) by taking $p$ as the path $s \to \exp_{G}(stX)$ and differentiating at $t = 0$.

Let now $G = LG = C^\infty(S^1, G)$ with Lie algebra $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$. The basic inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ determines a right–invariant, closed two–form on $LG$ given by

$$B(X, Y) = \int_{0}^{2\pi} \langle X, Y \rangle \frac{d\theta}{2\pi} \quad (3.1.13)$$

and such that $(2\pi)^{-1}B$ is integral. For any $\ell \in \mathbb{Z}$, denote by $\tilde{L}G^\ell$ and $\tilde{L}\mathfrak{g}^\ell$ the central extension of $LG$ corresponding to $\ell B$ and its Lie algebra. Then, any central extension of $LG$ is isomorphic to some $\tilde{L}G^\ell$ for a uniquely determined $\ell \in \mathbb{Z}$ called its level [PS, thm. 4.4.1].

Since $B$ is invariant under the action of the group $\text{Diff}_+(S^1)$ of orientation–preserving diffeomorphisms of $S^1$ given by $\phi \gamma = \gamma \circ \phi^{-1}$, and $\text{Hom}(LG, T) = \{1\}$ [PS, prop. 3.4.1], this action lifts uniquely to any central extension of $LG$. Similarly, the action by conjugation of $LZG$ on $LG$ lifts to any $\tilde{L}G^\ell$. Indeed, for $\zeta \in LZG$ we have

$$B(\zeta X \zeta^{-1}, \zeta Y \zeta^{-1}) = \int_{0}^{2\pi} \langle \zeta X \zeta^{-1}, \zeta Y \zeta^{-1} \rangle \frac{d\theta}{2\pi} + \int_{0}^{2\pi} \langle \zeta X \zeta^{-1}, \zeta [\zeta^{-1} \zeta^* Y] \zeta^{-1} \rangle \frac{d\theta}{2\pi} \quad (3.1.14)$$

where we used the $\text{Ad}$–invariance of $\langle \cdot, \cdot \rangle$ and the fact that $\zeta^{-1} \zeta + \zeta^{-1} \zeta = (\zeta^{-1} \zeta) = 0$. Therefore, by lemma $3.1.4$ on $\tilde{L}\mathfrak{g}$

$$\tilde{\text{Ad}}(\zeta) X \oplus it = \zeta X \zeta^{-1} + i \left( t - \ell \int_{0}^{2\pi} \langle \zeta^{-1} \zeta^* X, \zeta \rangle \frac{d\theta}{2\pi} \right) \quad (3.1.15)$$

In particular, the adjoint action of $\tilde{L}G^\ell$ factors through $LG \subset LZG$ and, by the uniqueness of lifts, is given by (3.1.15).
3.2. The compatibility requirement.

By lemma [3], the subgroup $Z \subseteq Z(G)$ is isomorphic to $\Lambda^\vee_Z/\Lambda^\gamma_Z$ where $\Lambda^\gamma_Z \subset \Lambda^\vee_Z \subset \Lambda^\vee_W$ is the integral lattice of $G/Z$, i.e. $\Lambda^\gamma_Z \cong \text{Hom}(T, T/Z)$. We will regard $\Lambda^\vee_Z$ as a subgroup of $L_ZG$ by associating to $\mu \in \Lambda^\vee_Z$ the discontinuous loop $\zeta_\mu(\theta) = \exp_T(-i\theta\mu)$. Since any character of $\Lambda^\gamma_Z$ extends to $\Lambda^\vee_Z$, the connecting homomorphism in the five term sequence

$$\text{Hom}(\Lambda^\vee_Z, T) \to \text{Hom}(\Lambda^\gamma_Z, T) \to H^2(Z, T) \to H^2(\Lambda^\vee_Z, T)$$

is the zero map. Using the five term sequence for the inclusion $LG \subset L_ZG$ and the fact that $\text{Hom}(LG, T) = \{1\}$ [PS, Prop. 3.4.1], we therefore obtain the following commutative diagram with exact row

\[
\begin{array}{cccc}
0 & & & 0 \\
\downarrow & & & \downarrow \\
0 & \longrightarrow & H^2(Z, T) & \longrightarrow & H^2(L_ZG, T) & \longrightarrow & H^2(LG, T) & \longrightarrow & 0 \\
\downarrow & & & \downarrow & & & \downarrow & & & \downarrow \\
& & & H^2(\Lambda^\vee_Z, T) & & & & & \\
\end{array}
\]

which shows that a central extension of $L_ZG$ by $T$ is entirely determined by its restrictions to $LG$ and to $\Lambda^\vee_Z$. The former is classified by its level $\ell$ and the latter by its commutator map $\omega$ defined by

$$\omega(\lambda, \mu) = \tilde{\omega}_\lambda \tilde{\omega}_\mu \tilde{\omega}_\lambda^{-1} \tilde{\omega}_\mu^{-1}$$

(3.2.3)

where $\tilde{\omega}_\lambda, \tilde{\omega}_\mu \in \tilde{L}_ZG$ are arbitrary lifts of $\omega(\lambda, \mu)$. $\omega$ is a skew–symmetric, $T$–valued, $Z$–bilinear form on $\Lambda^\vee_Z$. Since the identification $\Lambda^\vee_Z \cong \text{Hom}(T, T/Z)$ maps $\Lambda^\gamma_Z$ to $\text{Hom}(T, T) \subset LG$, $\omega$ is bound by the requirement that $\omega(\alpha, \beta) = (-1)^{\ell(\alpha, \beta)}$ whenever $\alpha, \beta \in \Lambda^\gamma_Z$ [PS, prop. 4.8.1].

We shall presently establish that $\omega$ is constrained by a more astringent identity, the proof of which gives an alternative derivation of proposition 4.8.1 of [PS].

**Theorem 3.2.1.** Let $\tilde{L}_ZG$ be a central extension of $L_ZG$ by $T$, the restrictions to $LG$ and $\Lambda^\vee_Z$ of which have level $\ell$ and commutator map $\omega$ respectively. Then, for any $\lambda \in \Lambda^\gamma_Z$ and $\mu \in \Lambda^\vee_Z$

$$\omega(\mu, \lambda) = (-1)^{\ell(\mu, \lambda)}$$

(3.2.4)

Theorem 3.2.1 is an immediate corollary of the following

**Proposition 3.2.2.** For any $\mu \in \Lambda^\vee_W$, denote by $\tilde{A}_\mu$ the unique lift of the conjugation action of $\zeta_\mu$ on $LG$ to $L^\ell_ZG$. Then, for any $\lambda \in \Lambda^\gamma_Z$ and lift $\tilde{\zeta}_\lambda \in \tilde{L}^\ell_ZG$ of $\zeta_\lambda$,

$$\tilde{A}_\mu(\tilde{\zeta}_\lambda) = (-1)^{\ell(\mu, \lambda)} \tilde{\zeta}_\lambda$$

(3.2.5)

**Proof (of Theorem 3.2.1).** Let $\tilde{\zeta}_\lambda, \tilde{\zeta}_\mu \in \tilde{L}_ZG$ be lifts of $\zeta_\lambda$ and $\tilde{\zeta}_\mu \in \tilde{L}_ZG$ respectively. By lemma 3.1.1 $\text{Ad}(\zeta_\mu) = \tilde{A}_\mu$ as automorphisms of $L^\ell_ZG \cong L^\ell_ZG_{LG}$, since both are lifts of $\text{Ad}(\zeta_\mu)$ and $\text{Hom}(LG, T) = \{1\}$. Thus, by (3.2.3) and (3.2.3)

$$\omega(\mu, \lambda) = \tilde{A}_\mu(\tilde{\zeta}_\lambda)\tilde{\zeta}_\lambda^{-1} = (-1)^{\ell(\mu, \lambda)}$$

(3.2.6)

\[
\diamond
\]

**Proof (of Proposition 3.2.2).** Since $\zeta_\mu \zeta_\lambda \zeta_\mu^{-1} = \zeta_\lambda$ in $LG$, the left hand–side of (3.2.3) is equal to $\omega(\mu, \lambda)\tilde{\zeta}_\lambda$, where $\omega(\mu, \lambda) \in T$ is independent of the choice of the lift $\tilde{\zeta}_\lambda$ and is
bilinear in $\mu, \lambda$. Moreover, if $\mu \in \Lambda^*_{R'}$, lemma 3.1.1 implies that $\bar{A}_\mu = \text{Ad}(\bar{\zeta}_\mu)$ and $\omega$ is therefore skew–symmetric when restricted to $\Lambda^*_{R'} \times \Lambda^*_{R'}$. We begin by establishing that

$$\omega(\mu, \lambda) = (-1)^{\ell(\mu, \lambda)}$$  \hspace{1cm} (3.2.7)

when $\lambda = \alpha^\vee$ is the coroot corresponding to a positive root $\alpha$ and $\mu \in \Lambda^*_{W'}$ is such that $\langle \alpha, \mu \rangle \in \{0, 1\}$. The loop $\zeta_{\alpha^\vee}(\theta) = \exp(-i\theta \alpha^\vee)$ may then be written as a product of two exponentials in $LG$ \cite{PS, 4.8.1}, namely

$$\zeta_{\alpha^\vee} = \exp_{LG}\left(-\frac{\pi}{2}(e_\alpha(0) - f_\alpha(0))\right) \exp_{LG}\left(\frac{\pi}{2}(e_\alpha(1) - f_\alpha(-1))\right)$$  \hspace{1cm} (3.2.8)

Here, using standard notation, $e_\alpha, f_\alpha$ and $h_\alpha = \alpha^\vee$ span the $\mathfrak{sl}_2(\mathbb{C})$–subalgebra of $\mathfrak{g}_c$ corresponding to $\alpha$ and, for any $x \in \mathfrak{g}_c$ and $n \in \mathbb{N}$, $x(n) = x \otimes e^{in\theta} \in LG_c$. To see that (3.2.8) holds, consider the homomorphism $\sigma_\alpha : SU_2 \rightarrow G$ mapping the standard basis of $\mathfrak{sl}_2(\mathbb{C})$ given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hspace{1cm} f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \hspace{1cm} h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  \hspace{1cm} (3.2.9)

to $\{e_\alpha, f_\alpha, h_\alpha\}$. This induces a homomorphism $LSU_2 \rightarrow LG$ sending

$$\theta \rightarrow \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$  \hspace{1cm} (3.2.10)

to $\zeta_{\alpha^\vee}$ and (3.2.8) reduces to a simple matrix check.

If $h \in \mathfrak{t}$, then $[h, e_\alpha] = \langle h, \alpha \rangle e_\alpha$ whence $\text{Ad}(\exp_{\mathfrak{t}}(h))e_\alpha = \exp(\text{ad}(h))e_\alpha = e^{(h, \alpha)}e_\alpha$. Therefore, since $\zeta_\mu(\theta) = \exp_{\mathfrak{t}}(-i\theta \mu)$, we have

$$\zeta_\mu e_\alpha(n)\zeta_\mu^{-1}(\theta) = e_\alpha \otimes e^{i\theta(n - \langle \alpha, \mu \rangle)} = e_\alpha(n - \langle \alpha, \mu \rangle)(\theta)$$  \hspace{1cm} (3.2.11)

so that, in $LG$

$$\zeta_\mu e_\alpha(n)\zeta_\mu^{-1} = e_\alpha(n - \langle \alpha, \mu \rangle)$$  \hspace{1cm} (3.2.12)

$$\zeta_\mu f_\alpha(n)\zeta_\mu^{-1} = f_\alpha(n + \langle \alpha, \mu \rangle)$$  \hspace{1cm} (3.2.13)

Since $\zeta_\mu^{-1} \bar{\zeta}_\mu = -i\mu \in \mathfrak{t}$ and this subspace is orthogonal to $\mathbb{C}e_\alpha \oplus \mathbb{C}f_\alpha$ with respect to the Killing form, no correction term arises from (3.1.13) and the same holds in $\bar{LG}$\textsuperscript{f}. It follows that (3.2.7) holds if $\langle \alpha, \mu \rangle = 0$. If, on the other hand $\langle \alpha, \mu \rangle = 1$, then

$$\bar{A}_\mu(\bar{\zeta}_{\alpha^\vee})\bar{\zeta}_{\alpha^\vee}^{-1} = \exp_{\bar{LG}}\left(-\frac{\pi}{2}(e_\alpha(-1) - f_\alpha(1))\right) \exp_{\bar{LG}}\left(\frac{\pi}{2}(e_\alpha(0) - f_\alpha(0))\right)$$

$$\cdot \exp_{\bar{LG}}\left(-\frac{\pi}{2}(e_\alpha(1) - f_\alpha(-1))\right) \exp_{\bar{LG}}\left(\frac{\pi}{2}(e_\alpha(0) - f_\alpha(0))\right)$$

$$= \exp_{\bar{LG}}\left(-\frac{\pi}{2}(e_\alpha(-1) - f_\alpha(1))\right)$$

$$\cdot \text{Ad}\left(\exp_{\bar{LG}}\left(\frac{\pi}{2}(e_\alpha(0) - f_\alpha(0))\right)\right) \exp_{\bar{LG}}\left(-\frac{\pi}{2}(e_\alpha(1) - f_\alpha(-1))\right)$$

$$\cdot \exp_{\bar{LG}}\left(\frac{\pi}{2}(e_\alpha(0) - f_\alpha(0))\right)$$  \hspace{1cm} (3.2.14)

As is readily checked using $\sigma_\alpha$, we have

$$\text{Ad}\left(\exp_{LG}\left(\frac{\pi}{2}(e_\alpha(0) - f_\alpha(0))\right)\right) (e_\alpha(1) - f_\alpha(-1)) = e_\alpha(-1) - f_\alpha(1)$$  \hspace{1cm} (3.2.15)

Moreover, since we are conjugating by a constant loop, no correction term arises from (3.1.15) and (3.2.14) is therefore equal to

$$\exp_{\bar{LG}}\left(-\pi(e_\alpha(-1) - f_\alpha(1))\right) \exp_{\bar{LG}}\left(\pi(e_\alpha(0) - f_\alpha(0))\right)$$  \hspace{1cm} (3.2.16)
To proceed, we seek to diagonalise the above elements. This is best done in \( L\text{SU}_2 \) using the identity
\[
e_\alpha(-m) - f_\alpha(m) = V(m)ih_\alpha(0)V(m)^* \tag{3.2.17}
\]
where \( V(m) \in L\text{SU}_2 \) is given by \( \theta \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & ie^{im\theta} \\ ie^{-im\theta} & 1 \end{pmatrix} \). Since
\[
V^{-1}(m)V(m) = \frac{m}{2}\left(i_\alpha(0) + e_\alpha(-m) - f_\alpha(m)\right) \tag{3.2.18}
\]
we have
\[
\int_0^{2\pi} \langle V^{-1}(m)V(m),i_\alpha(0) \rangle \frac{d\theta}{2\pi} = -\frac{m}{2} \|h_\alpha\|^2 = -\frac{m}{2} \langle \alpha,\alpha \rangle \tag{3.2.19}
\]
and therefore, using (3.1.15), (3.2.16) is equal to
\[
e^{-\pi\ell_\alpha(\alpha,\alpha)} \widetilde{V}(1) \exp_{\widetilde{LG}^\ell}(-i\pi h_\alpha(0)) \widetilde{V}(1)^{-1} \widetilde{V}(0) \exp_{\widetilde{LG}^\ell}(i\pi h_\alpha(0)) \widetilde{V}(0)^{-1} \tag{3.2.20}
\]
where \( \widetilde{V}(0), \widetilde{V}(1) \) are arbitrary lifts of \( V(0), V(1) \) in \( \widetilde{LG}^\ell \). Since \( \exp_{\text{SU}_2}(-i\pi h_\alpha(0)) = -1 \) lies in the centre of any central extension of \( L\text{SU}_2 \), the above is equal to \((-1)^{\ell_\alpha(\alpha,\alpha)} = (-1)^{\ell(\alpha^\vee,\mu)} \) and (3.2.7) holds if \( (\alpha,\mu) = 1 \).

Let now \( \lambda = \alpha^\vee \) and \( \mu = \beta^\vee \) be coroots. Then, either \( |(\alpha,\beta^\vee)| \leq 1 \) or \( |(\alpha^\vee,\beta)| \leq 1 \) [Hu, Table 1, p.45]. Using the bilinearity and skew–symmetry of both sides of (3.2.7), we may assume, up to a permutation and a sign change, that \( \alpha \) is positive and that \( (\alpha,\beta^\vee) \in \{0,1\} \) so that, by the computation above, (3.2.7) holds whenever \( \lambda \) and \( \mu \) lie in the coroot lattice. To complete the proof, it is sufficient to check (3.2.7) when \( \lambda = \alpha^\vee \) is a positive coroot and \( \mu \) varies in a set of representatives of \( \Lambda_R^\vee \)-cosets in \( \Lambda_W^\vee \). A convenient choice is given by the minimal dominant coweights. If \( \mu \) is one such then, by proposition 2.2, \( (\mu,\alpha) \in \{0,1\} \) and therefore (3.2.7) holds by our previous computation.

3.3. Construction of central extensions of \( L\text{G} \).

Define the level \( \ell \) and commutator map \( \omega \) of a central extension \( \widetilde{L\text{G}} \) of \( L\text{G} \) by restriction to \( LG \) and \( \Lambda_R^\vee \) respectively. By theorem 3.2.1, no such \( \widetilde{L\text{G}} \) exists unless \( \ell \) and \( \omega \) are compatible, i.e. satisfy (3.2.4). In particular, \( \widetilde{L\text{SU}_2} \), and a fortiori \( L\text{SO}_3 \), do not possess any central extensions of odd level since in this case \( \Lambda_R^\vee = \alpha\mathbb{Z} \) and \( \Lambda_W^\vee = \frac{\mathbb{Z}}{2} \mathbb{Z} \) with \( (\alpha,\alpha) = 2 \). On the other hand, (3.2.4) requires \( \omega(\alpha,\frac{\alpha}{2}) = -1 \) in contradiction with the skew–symmetry of \( \omega \).

Let now \( \ell \in \mathbb{Z} \) and \( \omega \) be a skew–symmetric, \( \mathbb{T} \)-valued bilinear form on \( \Lambda_R^\vee \). Then,

**Proposition 3.3.1.** There exists a (necessarily unique) central extension \( \widetilde{L\text{G}} \) of \( L\text{G} \) of level \( \ell \) and commutator map \( \omega \) if, and only if
\[
\omega(\mu,\lambda) = (-1)^{\ell(\mu,\lambda)} \tag{3.3.1}
\]
whenever \( \lambda \in \Lambda_R^\vee \).

**Proof.** The necessity of (3.3.1) is the contents of theorem 3.2.1 and the uniqueness of \( \widetilde{L\text{G}} \) that of (3.2.2). Let \( LG \) and \( \Lambda_R^\vee \) be the central extensions of \( LG \) and \( \Lambda_R^\vee \) with level \( \ell \) and commutator map \( \omega \) respectively. Following [PS, Prop. 4.6.9], we shall construct \( \widetilde{L\text{G}} \) as a quotient of \( LG \times \Lambda_R^\vee \). Lift the conjugation action of \( \Lambda_R^\vee \subset L\text{G} \) on \( LG \) to \( \tilde{LG} \) by using lemma 3.1.1 and denote the corresponding automorphisms of \( LG \) by \( \tilde{\alpha}_\mu, \mu \in \Lambda_R^\vee \). Form the semi–direct product \( \tilde{LG} \rtimes \Lambda_R^\vee \) where the action of \( \Lambda_R^\vee \) factors through \( \Lambda_W^\vee \). By theorem 3.2.1...
and the compatibility of \( \ell \) and \( \omega \), \( \tilde{LG} \) and \( \tilde{\Lambda}_Z \) restrict to isomorphic central extensions \( \tilde{\Lambda}_R \) of \( \Lambda_R \). We may therefore consider the subgroup
\[
N = \{ (\zeta_\alpha, \zeta_\alpha^{-1}) \} \subset \tilde{LG} \rtimes \tilde{\Lambda}_Z
\]  
where \( \zeta_\alpha \) varies in \( \tilde{\Lambda}_R \). We claim that \( N \) is normal. By lemma 3.1.1, for any \( \alpha \in \Lambda_R \), \( \tilde{A}_\alpha = \text{Ad}(\tilde{\zeta}_\alpha) \) since both are lifts of \( \text{Ad}(\zeta_\alpha) \). Therefore, for any \( \gamma \in LG \),
\[
(\gamma, 1)(\zeta_\alpha, \zeta_\alpha^{-1})(\gamma^{-1}, 1) = (\gamma, 1)((\zeta_\alpha, \tilde{A}_\alpha^{-1}(\gamma^{-1}), \zeta_\alpha^{-1}) = (\zeta_\alpha, \zeta_\alpha^{-1})
\]
Moreover, by proposition 3.2.2 and (3.3.1)
\[
(1, \tilde{\zeta}_\mu)(\zeta_\alpha, \zeta_\alpha^{-1})(1, \tilde{\zeta}_\mu^{-1}) = (\tilde{A}_\mu(\zeta_\alpha), [\tilde{\zeta}_\mu, \zeta_\alpha^{-1}]\zeta_\alpha^{-1}) = ((-1)^{\ell(\mu, \alpha)}\tilde{\zeta}_\alpha, (-1)^{-\ell(\mu, \alpha)}\zeta_\alpha^{-1})
\]
Thus, \( N \) is normal, and the quotient \( \tilde{LG} \rtimes \tilde{\Lambda}_Z / N \) is the required central extension of \( L_ZG \).

**Remark.** Theorem 3.2.1 and proposition 3.3.1 prove the exactness of
\[
1 \to H^2(Z, T) \to H^2(L_ZG, T) \to H^2(LG, T) \to \mathbb{Z} / \ell_f \mathbb{Z} \to 0
\]
where \( \ell_f \) is 1 if \( \Lambda^\vee \) possesses a commutator map satisfying (3.2.4) with \( \ell = 1 \) and \( \ell_f = 2 \) otherwise. We call \( \ell_f \) the fundamental level of \( G / Z \). The fundamental levels of all compact simple groups are given in [3.3].

### 3.4. Automorphic action of \( \text{Diff}_+(S^1) \) on \( L_ZG \).

Let \( \text{Diff}_+(S^1) \) be the group of orientation-preserving diffeomorphisms of \( S^1 \) and \( D \) its universal covering group. \( D \) may be realised as the subgroup of diffeomorphisms \( \phi \) of \( \mathbb{R} \) such that \( \phi(x + 2\pi) = \phi(x) + 2\pi \) and \( \text{Diff}_+(S^1) \cong D / (T_{2\pi}) \) where \( T_y \) is translation by \( y \). \( D \) acts automorphically on \( L_ZG \) by
\[
\phi \zeta = \zeta \phi^{-1} = \zeta \circ \phi^{-1}
\]
and this action factors to one of \( D / (T_{2\pi k}) \) where \( k \) is the order of the largest cyclic subgroup of \( Z \). The Lie algebra of \( D \) is the Lie algebra \( \text{Vect}(S^1) \) of all smooth vector fields on \( S^1 \) with bracket \( \{ \cdot, \cdot \} \)
\[
[f \frac{d}{d\theta}, g \frac{d}{d\theta}] = (f g - f \dot{g}) \frac{d}{d\theta}
\]
If \( \xi = f \frac{d}{d\theta} \in \text{Vect}(S^1) \), the action of \( \xi \) on \( L_G \) corresponding to \( 3.4.1 \) is simply
\[
\xi X = -f \dot{X}
\]
so that the Lie algebra of \( D \rtimes L_ZG \) is \( \text{Vect}(S^1) \rtimes L_G \) with bracket
\[
[X \oplus f \frac{d}{d\theta}, Y \oplus g \frac{d}{d\theta}] = ([X, Y] - f \dot{Y} + g \dot{X}) \oplus (f g - f \dot{g}) \frac{d}{d\theta}
\]

**Proposition 3.4.1.** Let \( k \) be the order of the largest cyclic subgroup of \( Z \). Then, the action of the universal \( k \)–covering of \( \text{Diff}_+(S^1) \) on \( L_ZG \) lifts uniquely to any central extension \( \tilde{L}_ZG \).

**Proof.** The uniqueness is easily settled for two lifts necessarily differ by some
\[
\chi \in \text{Hom}(D, \text{Hom}(L_ZG, T)) \cong \text{Hom}(D, \tilde{Z}) = 1
\]
where the first isomorphism follows from \( \text{Hom}(L_G, T) = 1 \) and the second by connectedness of \( D \). Let \( \tilde{L}_G, \tilde{\Lambda}_Z \) be the restrictions of \( L_ZG \) to \( LG \) and \( \Lambda^\vee \) respectively and \( \ell, \omega \) the corresponding level and commutator map. We shall describe, as in the proof of proposition 3.3.1.

\(^2\)The bracket (3.4.2) is the Lie–theoretic bracket on \( \text{Vect}(S^1) \) satisfying
\[
[X, Y] = \frac{d}{dt} \bigg|_{t=0} \exp(tX)Y \exp(-tX)
\]
and is the opposite of the differential geometric one defined by the action of \( \text{Vect}(S^1) \) on \( C^\infty(S^1) \).
\( \widetilde{LG} \) as a quotient of \( \Lambda_Z \ltimes \widetilde{LG} \). Since \( LZG \) isn’t connected, the action of \( D \) on it cannot be lifted to \( \widetilde{LG} \) by using lemma 3.1.1. Following the proof of [PS, prop. 4.7.1], we shall regard it instead as one of \( LZG \) on \( LG \ltimes D \) in the following way.

Consider first the connected component of the identity of \( \widetilde{LG} \), i.e. \( \widetilde{LG} \) and form the semi-direct product \( \widetilde{LG} \ltimes D \) where \( D \) acts on \( \widetilde{LG} \) as in 3.1.1. Since \( D \) is contractible, \( \widetilde{LG} \ltimes D \) may equivalently be described as the central extension of \( LG \ltimes D \) corresponding to the Lie algebra cocycle

\[
\beta(X \oplus f \frac{d}{d\theta}, Y \oplus g \frac{d}{d\theta}) = \ell B(X, Y) = \ell \int_0^{2\pi} \langle X, Y \rangle \frac{d\theta}{2\pi} \quad (3.4.6)
\]

We claim that \( LZG \) acts on \( LG \ltimes D \) and \( \widetilde{LG} \ltimes D \). The first action simply stems from the fact that \( LG \ltimes D \) is a normal subgroup of \( LZG \ltimes D \) and is given explicitly by

\[
\zeta(\gamma, \phi) = (\zeta_1 \gamma^{-1} \zeta_0^{-1}, \phi) \quad (3.4.7)
\]

and infinitesimally by

\[
\zeta(X \oplus f \frac{d}{d\theta}) = (\zeta X \zeta^{-1} + f \zeta \zeta^{-1}) \oplus f \frac{d}{d\theta} \quad (3.4.8)
\]

To see that this action lifts to \( \widetilde{LG} \ltimes D \), we compute

\[
\beta(\zeta(X \oplus f \frac{d}{d\theta}), \zeta(Y \oplus g \frac{d}{d\theta})) = \ell B(\zeta X \zeta^{-1} + f \zeta^{-1}, \zeta Y \zeta^{-1} + g \zeta \zeta^{-1}) \quad (3.4.9)
\]

By (3.1.14),

\[
B(\zeta X \zeta^{-1}, \zeta Y \zeta^{-1}) = B(X, Y) - \int_0^{2\pi} ([X, Y], \zeta^{-1} \zeta) \frac{d\theta}{2\pi} \quad (3.4.10)
\]

On the other hand,

\[
B(f \zeta \zeta^{-1}, \zeta Y \zeta^{-1}) + B(\zeta X \zeta^{-1}, g \zeta \zeta^{-1}) = \int_0^{2\pi} f(\zeta \zeta^{-1}, \zeta Y \zeta^{-1}) + f(\zeta \zeta^{-1}, \zeta \zeta^{-1}, \zeta Y \zeta^{-1}) \frac{d\theta}{2\pi}
\]

\[
-\int_0^{2\pi} g(\zeta \zeta^{-1}, \zeta X \zeta^{-1}) - g(\zeta \zeta^{-1}, \zeta \zeta^{-1}, \zeta X \zeta^{-1}) \frac{d\theta}{2\pi}
\]

\[
= \int_0^{2\pi} \langle \zeta^{-1} \zeta, f Y - g X \rangle \frac{d\theta}{2\pi} \quad (3.4.11)
\]

Finally, anti-symmetrising, we find

\[
B(f \zeta \zeta^{-1}, g \zeta \zeta^{-1}) = \frac{1}{2} \int_0^{2\pi} (f \dot{g} - \dot{f} g)(\zeta \zeta^{-1}, \zeta \zeta^{-1}) \frac{d\theta}{2\pi} \quad (3.4.12)
\]

Thus

\[
\beta(\zeta(X \oplus f \frac{d}{d\theta}), \zeta(Y \oplus g \frac{d}{d\theta})) = \beta(X \oplus f \frac{d}{d\theta}, Y \oplus g \frac{d}{d\theta}) - \ell F([X \oplus f \frac{d}{d\theta}, Y \oplus g \frac{d}{d\theta}]) \quad (3.4.13)
\]

where \( F : LG \ltimes \text{Vect}(S^1) \rightarrow \mathbb{R} \) is given by

\[
F(X \oplus f \frac{d}{d\theta}) = \int_0^{2\pi} \langle X, \zeta^{-1} \zeta \rangle \frac{d\theta}{2\pi} + \frac{1}{2} \int_0^{2\pi} f(\zeta^{-1} \zeta, \zeta^{-1} \zeta) \frac{d\theta}{2\pi} \quad (3.4.14)
\]

Since \( D \) is perfect [Eq], it follows from lemma 3.1.1 that the action of \( LZG \) on \( LG \ltimes D \) lifts uniquely to \( \widetilde{LG} \ltimes D \) and is given by

\[
F(X \oplus f \frac{d}{d\theta}) = \int_0^{2\pi} \langle X, \zeta^{-1} \zeta \rangle \frac{d\theta}{2\pi} + \frac{1}{2} \int_0^{2\pi} f(\zeta^{-1} \zeta, \zeta^{-1} \zeta) \frac{d\theta}{2\pi} \quad (3.4.14)
\]
\begin{equation}
\zeta(X \oplus f \frac{d}{d\theta} \oplus it) = (\zeta X \zeta^{-1} + f \zeta \zeta^{-1}) \oplus f \frac{d}{d\theta} + i \left(t - \ell \int_0^{2\pi} (X, \zeta^{-1} \zeta) \frac{d\theta}{2\pi} - \ell \int_0^{2\pi} f(\zeta^{-1} \zeta, \zeta^{-1} \zeta) \frac{d\theta}{2\pi}\right) \tag{3.4.15}
\end{equation}

Consider now the semi-direct product \( \tilde{\Lambda}_Z \rtimes (\tilde{L}G \times \mathbb{D}) \) where the action of \( \tilde{\Lambda}_Z \) factors through \( \Lambda_Z \) and the subgroup

\[ N = \{ (\zeta_0, \zeta^{-1}_0, 1) \} \subset \Lambda_Z \rtimes (\tilde{L}G \times \mathbb{D}) \tag{3.4.16} \]

where \( \zeta_0 \) varies in \( \Lambda_R = \tilde{L}G|_{\Lambda_R^H} \). \( N \) lies in the centraliser of \( \tilde{L}G \times \mathbb{D} \) since, by uniqueness, \( \text{Ad}((\zeta_0, 1, 1)) = \text{Ad}((1, \zeta_0, 1)) \) on \( \tilde{L}G \times \mathbb{D} \) as both automorphisms are lifts of \( \text{Ad}(\zeta_0) \). It follows that the quotient \( \tilde{\Lambda}_Z \rtimes \tilde{L}G/N \cong \tilde{L}G \) is acted upon by \( \mathbb{D} \).

To conclude, we need only show that translations by multiples of \( 2\pi k \) act trivially on \( \tilde{L}G \), where \( k \) is the order of the largest cyclic subgroup of \( Z \). It is sufficient to check this on a representative of each connected component of \( \tilde{L}G \) since, by uniqueness, \( T_{2\pi \gamma} = \gamma \) for any \( \gamma \in \tilde{L}G \). Let \( \tilde{\zeta}_\lambda \) be a lift of the discontinuous loop \( \zeta_\lambda = \exp_T(-i\lambda \theta), \lambda \in \Lambda_Z \). By (3.4.15),

\[ \tilde{\zeta}_\lambda T_{2\pi k} \tilde{\zeta}_\lambda^{-1} = -i\lambda \frac{d}{d\theta} + i \frac{\ell}{2}(\lambda, \lambda) \tag{3.4.17} \]

and therefore, since \( T_y = \exp_D(y \frac{d}{d\theta}) \)

\[ \tilde{\zeta}_\lambda T_{2\pi k} \tilde{\zeta}_\lambda^{-1} = e^{i\ell(\lambda, \lambda)} \exp_T(-2\pi ik\lambda) T_{2\pi k} = (-1)^{k\ell(\lambda, \lambda)} T_{2\pi k} \tag{3.4.18} \]

Notice that \( k\lambda \in \Lambda_R^H \) since its image in \( Z \) is 1. Thus, by the skew-symmetry of \( \omega \) and its compatibility with \( \ell \), we have

\[ 1 = \omega(k\lambda, \lambda) = (-1)^{k\ell(\lambda, \lambda)} \tag{3.4.19} \]

whence

\[ \tilde{\zeta}_\lambda T_{2\pi k} \tilde{\zeta}_\lambda^{-1} = T_{2\pi k} \tag{3.4.20} \]

as claimed \( \diamond \)

Let us record the following by-product of the proof of proposition 3.4.3 since it extends formula 4.9.4 of [PS]

Corollary 3.4.2. The action of \( \tilde{L}G \) on the Lie algebra of \( \tilde{L}G \rtimes \text{Diff}_+(S^1) \), where \( \tilde{L}G \) is the central extension of \( L_G \) of level \( \ell \), is given by

\[ \zeta(X \oplus f \frac{d}{d\theta} \oplus it) = (\zeta X \zeta^{-1} + f \zeta \zeta^{-1}) \oplus f \frac{d}{d\theta} + i \left(t - \ell \int_0^{2\pi} (X, \zeta^{-1} \zeta) \frac{d\theta}{2\pi} - \ell \int_0^{2\pi} f(\zeta^{-1} \zeta, \zeta^{-1} \zeta) \frac{d\theta}{2\pi}\right) \tag{3.4.21} \]

3.5. Central extensions of \( L(G/Z) \).

We now classify the central extensions of \( L(G/Z) \). The five term sequence corresponding to

\[ 1 \to Z \to \tilde{L}G \xrightarrow{\pi} L(G/Z) \to 1 \tag{3.5.1} \]

and the fact that \( \text{Hom}(L(G, \mathbb{T}) = 1 \) yield the exactness of

\[ 1 \to \text{Hom}(Z, \mathbb{T}) \to H^2(L(G/Z), \mathbb{T}) \xrightarrow{\pi^*} H^2(\tilde{L}G, \mathbb{T}) \tag{3.5.2} \]
The image of $\pi^*$ is easily described. Let the basic level $\ell_b$ of $G/Z$ be the smallest integer $\ell$ such that the restriction of $\ell(\cdot, \cdot)$ to $\Lambda^*_G$ is integral, i.e. such that
\[
\ell(\lambda^\vee, \lambda^\vee_j) \in \mathbb{Z}
\] (3.5.3)
for all fundamental coweights $\lambda^\vee_j$ lying in $\Lambda^*_G$. Then,

**Proposition 3.5.1.** A central extension of $L\tilde{Z}G$ is the pull–back of one of $L(G/Z)$ only if its level $\ell$ is a multiple of the basic level of $G/Z$. Conversely, if $\ell_b|\ell$, the subgroup $Z \subset \hat{L}_ZG$ corresponding to the canonical embedding $G \hookrightarrow \hat{L}_ZG$ is central and
\[
\hat{L}_ZG \cong \pi^*(\hat{L}_ZG/Z)
\] (3.5.4)

**Proof.** As readily verified, a central extension $\hat{L}_ZG$ of $LZG$ is the pull–back of one of $L(G/Z)$ only if its restriction to $Z$ lies in its centre. Conversely, since $G$ is simple and simply-connected, the restriction of $\hat{L}_ZG$ to $G$, and therefore to $Z$, is canonically split. If $s : Z \rightarrow Z = \hat{L}_ZG|_{Z}$ is the corresponding section and $\hat{Z}$ is central, then $\hat{L}_ZG/s(Z)$ is a central extension of $L(G/Z)$ which pulls back to $\hat{L}_ZG$. We therefore need to determine those $\hat{L}_ZG$ for which $Z$ is a central subgroup. Notice first that $\hat{Z}$ lies in the centre of $\hat{L}G = \hat{L}_ZG|_{\hat{L}G}$. Indeed, for any $z \in Z$, $\gamma \in \hat{L}G$ and lifts $\tilde{z}, \tilde{z} \in \hat{L}G$,
\[
\tilde{z}\tilde{z}^{-1}\tilde{z}^{-1} = \chi(\gamma, z)
\] (3.5.5)
where $\chi(\gamma, z) \in \mathbb{T}$ is independent of the lifts and multiplicative in each variable. Since $\text{Hom}(\hat{L}G, T) = \{1\}$ however, $\chi = 1$. Thus, we need only check that $\hat{Z}$ commutes with the lifts of the discontinuous loops $\zeta_{\lambda}(\theta) = \exp(-i\lambda\theta)$, $\lambda \in \Lambda^*_Z$. For any $h \in t$, we have by (3.1.15),
\[
\tilde{\zeta}_{\lambda}h\tilde{\zeta}_{\lambda}^{-1} = h + \ell(h, \lambda)
\] (3.5.6)
whence
\[
\tilde{\zeta}_{\lambda}\tilde{z}\tilde{z}^{-1} = \tilde{z}e^{-2\pi i\ell(\mu, \lambda)}
\] (3.5.7)
where $\mu \in \Lambda^*_Z$ is such that $\exp(-2\pi i\mu) = z$, and it follows that $\hat{Z}$ is central if and only if $\ell$ is a multiple of $\ell_b$.

Proposition 3.5.1 shows that
\[
H^2(LG/Z, \mathbb{T}) \cong \text{Ker } \ell \oplus \text{Hom}(Z, T) \subseteq H^2(L\tilde{Z}G, T) \oplus \text{Hom}(Z, T)
\] (3.5.8)
where $\ell$ is the map giving the residue mod $\ell_b$ of the level of a central extension of $L\tilde{Z}G$. The isomorphism is simply given by associating to a central extension $\hat{L}_ZG$ of level $\ell \in \ell_b\mathbb{Z}$ and $\chi \in \hat{Z}$ the central extension
\[
\hat{L}_ZG/(\cdot \cdot \cdot \chi(z))_{z \in Z}
\] (3.5.9)
The list of basic levels for all compact, connected and simple Lie groups is given in §3.6.

**Remark.** If the central extension $\hat{L}_ZG$ has level $\ell \in \ell_b\mathbb{Z}$, the action of $D$ on $\hat{L}_ZG$ clearly descends to $\hat{L}(G/Z) = \hat{L}_ZG/Z$. Surprisingly perhaps, it does not then necessarily factor to one of $\text{Diff}^+(S^1)$. Indeed, (3.4.17) yields
\[
T_{2\pi}\tilde{\zeta}_{\lambda}T_{-2\pi} = \tilde{\zeta}_{\lambda}\exp(2\pi i\lambda)e^{2\pi i\ell(\lambda, \lambda)}
\] (3.5.10)
which equals 1 in $\hat{L}_ZG/Z$ if, and only if $\ell(\lambda, \lambda) \in 2\mathbb{Z}$ for any $\lambda$ i.e. if $\Lambda^*_Z$, endowed with $\langle \cdot , \cdot \rangle$, is an even lattice.

---

3. The ’only if’ implication of proposition 3.5.1 is essentially the contents of lemma 4.6.3 of [PS].
Remark. The basic level of \( G/Z \) is a multiple of the fundamental one for \( \ell_b | \ell \), the form
\[
\omega(\lambda, \mu) = (-1)^{\ell(\lambda, \mu) + \ell^2(\lambda, \lambda) - (\mu, \mu)}
\]  
(3.5.11)
is a commutator map on \( \Lambda^\vee_Z \) satisfying the hypothesis of proposition 3.3.1. In particular, \( L_ZG \) possesses a canonical central extension at level \( \ell_b \).

3.6. Appendix: fundamental and basic levels of simple Lie groups.

Let \( Z \subseteq Z(G) \) and \( \Lambda_R^Z \subseteq \Lambda^\vee_Z \subseteq \Lambda^\vee_W \) be the fundamental group and integral lattice of \( G/Z \).

Lemma 3.6.1. If \( Z \cong \Lambda^\vee_R/\Lambda^\vee_R \) is cyclic of order \( k \), then \( G/Z \) has fundamental level 1 if and only if \( k(\lambda, \lambda) \in 2\mathbb{Z} \) where \( \lambda \in \Lambda^\vee_Z \) is a generator.

Proof. If \( \omega \) is a commutator map on \( \Lambda^\vee_Z \) satisfying
\[
\omega(\alpha, \mu) = (-1)^{(\alpha, \mu)}
\]  
(3.6.1)
whenever \( \alpha \in \Lambda^\vee_R \), then, by skew–symmetry, \( 1 = \omega(k\lambda, \lambda) = (-1)^{k(\lambda, \lambda)} \) since \( k\lambda \in \Lambda^\vee_R \).

Conversely, if \( k(\lambda, \lambda) \in 2\mathbb{Z} \), the form \( \tilde{\omega}(\alpha + a\lambda, \beta + b\lambda) = (-1)^{(\alpha, \beta) + (a\lambda, b\lambda)} \) on \( \Lambda^\vee_R + \mathbb{Z}\lambda \) descends to one on \( \Lambda^\vee_R + \mathbb{Z}\lambda/ - k\lambda + k\lambda \cong \Lambda^\vee_Z \) satisfying (3.6.1) \( \lozenge \)

Proposition 3.6.2. The following is the list of fundamental and basic levels \( \ell_f, \ell_b \) for all compact, connected and simple Lie groups with universal cover \( G \) and fundamental group \( Z \neq \{1\} \).

| \( G \) | \( Z(G) \) | \( Z \) | \( G/Z \) | \( \ell_f \) | \( \ell_b \) |
|---|---|---|---|---|---|
| \( SU_n \) | \( n \geq 2 \) | \( \mathbb{Z}_n \) | \( \mathbb{Z}_k \) | 1 for \( n \) odd or \( \frac{n}{k} \) even \( \frac{n(n-1)}{2k} \) \( \ell \) in \( \mathbb{Z} \) |
| \( Spin_{2n+1} \) | \( n \geq 2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( SO_{2n+1} \) | 1 |
| \( Sp_n \) | \( n \geq 1 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | 1 for \( n \) even | 1 for \( n \) even |
| \( Spin_{4m} \) | \( m \geq 2 \) | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) | \( SO_{4m} \) | 1 |
| \( Spin_{4m+2} \) | \( m \geq 1 \) | \( \mathbb{Z}_4 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) | \( PSO_{4m+2} \) | 1 |
| \( E_6 \) | \( \mathbb{Z}_3 \) | \( \mathbb{Z}_3 \) | \( SO_{4m+2} \) | 1 |
| \( E_7 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | | 2 |

Proof. We proceed by enumeration according to the Lie–theoretic type of \( G \), using the tables [Bou planches I–IX] and lemmas 3.6.1 and 2.3. For \( G \) simply–laced, we identify the coroot and coweight lattices with the root and weight lattices respectively. In what follows, \( \theta_i, i = 1 \ldots n \) and \( \langle \cdot, \cdot \rangle \) are the standard basis and inner product in \( \mathbb{R}^n \). Unless otherwise indicated, the basic inner product is the standard one.

\( SU_n, n \geq 2 \)

\( SU_n \) is simply–laced and the quotient \( \Lambda_W/\Lambda_R \cong \mathbb{Z}_n \) is generated by \( \lambda_i^\vee = \theta_1 - \frac{1}{n}(\sum_{i=1}^n \theta_i) \) corresponding to the special root \( \alpha_1 = \theta_1 - \theta_2 \). For any \( k|n \), the subgroup of \( Z(SU_n) \) isomorphic to \( \mathbb{Z}_k \) is generated by \( \frac{1}{k} \lambda_i^\vee \) and \( \langle \frac{1}{k} \lambda_i^\vee, \frac{1}{k} \lambda_j^\vee \rangle = \frac{n}{k} $$(
Spin\(_{2n+1}\), \(n \geq 2\)

\(Z(\text{Spin}_{2n+1}) \cong \mathbb{Z}_2\) is generated by the coweight \(\lambda^\vee_i = \theta_i\) corresponding to the unique special root \(\alpha_i = \theta_1 - \theta_2\). Since \(\langle \lambda^\vee_i, \lambda^\vee_j \rangle = 1\), \(\ell_b = \ell_f = 1\).

\(\text{Sp}_n, n \geq 1\)

\(\text{Sp}_1\) is the group of unit quaternions and is therefore isomorphic to \(SU_2\). For \(n \geq 2\), \(Z(\text{Sp}_n) \cong \mathbb{Z}_2\) is generated by \(\lambda^\vee_1 = \theta_1 + \cdots + \theta_n\) corresponding to the unique special root \(\alpha_n = 2\theta_n\).

Since the basic inner product is half the standard one on \(\mathbb{R}^n\), we have \(\langle \lambda^\vee_1, \lambda^\vee_n \rangle = \frac{2}{n}\) whence \(\ell_b = \ell_f = 1\) for \(n\) even and 2 for \(n\) odd. This is consistent with the isomorphism \(\text{Sp}_2 \cong \text{Spin}_5\).

\(\text{Spin}_{2n}, n \geq 3\)

\(\text{Spin}_{2n}\) is simply–laced with minimal dominant coweights \(\lambda^\vee_1 = \theta_1, \lambda^\vee_{n-1} = \frac{1}{2}(\theta_1 + \cdots + \theta_{n-1} - \theta_n)\) and \(\lambda^\vee_n = \frac{1}{2}(\theta_1 + \cdots + \theta_n)\) corresponding to the special roots \(\alpha_1 = \theta_1 - \theta_2, \alpha_{n-1} = \theta_{n-1} - \theta_n\) and \(\alpha_n = \theta_n - \theta_{n-1}\).

\(2\lambda^\vee_1 = 0 \mod \Lambda_R\) and \(\langle \lambda^\vee_1, \lambda^\vee_n \rangle = 1\) so that the corresponding quotient \(\text{Spin}_{2n}/Z_2 \cong \text{SO}_{2n}\) has \(\ell_b = \ell_f = 1\). We must now distinguish two cases:

\(n\) odd. Then \(2\lambda^\vee_{n-1} = 2\lambda^\vee_n = \lambda^\vee_1 \mod \Lambda_R\) and \(Z(\text{Spin}_{2n}) \cong \mathbb{Z}_4\) with \(\lambda^\vee_{n-1}, \lambda^\vee_n\) of order 4.

Since \(\langle \lambda^\vee_1, \lambda^\vee_{n-1} \rangle = 2\), we get \(\ell_b = 4\) and \(\ell_f = 2\) for \(\text{Spin}_{2n}/Z_4\). This is in agreement with the isomorphism \(\text{Spin}_6 \cong SU_4\).

\(n\) even. Then \(2\lambda^\vee_{n-1} = 2\lambda^\vee_n = 0 \mod \Lambda_R\) and \(Z(\text{Spin}_{2n}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2\). Since \(\langle \lambda^\vee_{n-1}, \lambda^\vee_1 \rangle = \langle \lambda^\vee_n, \lambda^\vee_1 \rangle = \frac{1}{2}\) we get \(\ell_b = \ell_f = 1\) and \(\ell_f = 2\) for the quotients \(\text{Spin}_{2n}/Z_2^2\) corresponding to \(\lambda^\vee_{n-1}\) and \(\lambda^\vee_n\) according to whether \(n\) is divisible by 4 or not. For \(Z = \mathbb{Z}_2 \times \mathbb{Z}_2\), we get \(\langle \lambda^\vee_1, \lambda^\vee_{n-1} \rangle = \frac{1}{2}\) so that \(\ell_b = 2\). To determine \(\ell_f\), notice that if there exists a commutator map \(\omega\) on \(\Lambda^\vee = \Lambda_W\) satisfying (3.2.4) with \(\ell = 1\), the fundamental level of \(\text{Spin}_{2n}/Z_2^2\) is one and therefore \(n\) is divisible by 4. Conversely, if \(4|\ell\), the form \(\tilde{\omega}(\alpha + q^\vee \beta) = i\alpha^\vee \omega(\alpha) + (\alpha^\vee q^\vee \beta + \beta^\vee q\alpha^\vee + q^\vee \beta^\vee + \alpha^\vee q\beta^\vee + q^\vee \beta^\vee)\) defined on \(\Lambda_R^\vee \oplus \lambda^\vee_1 \mathbb{Z} \oplus \lambda^\vee_2 \mathbb{Z}\) descends to a suitable form on \(\Lambda_W = \Lambda_R^\vee \oplus \lambda^\vee_1 \mathbb{Z} \oplus \lambda^\vee_2 \mathbb{Z}/(-2\lambda^\vee_1 \oplus 2\lambda^\vee_2 \oplus 0)\mathbb{Z} + (-2\lambda^\vee_1 \oplus 0 \oplus 2\lambda^\vee_2)\mathbb{Z}\).

\(\text{E}_6\)

\(\text{E}_6\) is simply–laced and \(\Lambda_W/\Lambda_R \cong \mathbb{Z}_3\) is generated by any of its non–zero elements and therefore by \(\lambda^\vee_6 = \theta_5 - \frac{1}{2}(\theta_6 + \theta_7 - \theta_8)\) corresponding to the special root \(\alpha_6 = -\theta_4 + \theta_5\).

Since \(\langle \lambda^\vee_6, \lambda^\vee_7 \rangle = \frac{4}{3}\), \(\ell_b = 3\) and \(\ell_f = 1\).

\(\text{E}_7\)

\(\text{E}_7\) is simply–laced and \(\Lambda_W/\Lambda_R \cong \mathbb{Z}_2\) is generated by \(\lambda^\vee_7 = \theta_6 - \frac{1}{2}(\theta_7 - \theta_8)\) corresponding to the unique special root \(\alpha_7 = -\theta_4 + \theta_5\). Since \(\langle \lambda^\vee_7, \lambda^\vee_8 \rangle = \frac{4}{3}\), \(\ell_b = \ell_f = 2\).

4. The action of \(L_\ell G\) on the positive energy dual of \(LG\)

We show in this section that the category \(\mathcal{P}_\ell\) of positive energy representations of \(LG\) at a given level \(\ell\) is closed under conjugation by elements of \(L_\ell G\). We also identify the corresponding abstract action of \(Z \cong L_\ell G/LG\) on the alcove of \(G\) parametrising the irreducibles in \(\mathcal{P}_\ell\) with the geometric one obtained by realising \(Z\) as a distinguished subgroup of the automorphisms of the extended Dynkin diagram of \(G\). We begin by studying the latter.

4.1. Geometric action of \(Z(G)\) on the level \(\ell\) alcove.

This subsection is essentially an expanded version of [Bon, ch. VI, §2.3]. The notation follows that of section [3]. Denote \(-\theta\) by \(\alpha_0\), then

**Lemma 4.1.1.** For any special root \(\alpha_i\), the set \(\Delta_i = \Delta \setminus \{\alpha_i\} \cup \{\alpha_0\}\) is a basis of \(R\) with highest root \(-\alpha_i\) and dual basis

\[
\lambda^\vee_0' = -\lambda^\vee_i \\
\lambda^\vee_i' = \lambda^\vee_i - \langle \theta, \lambda^\vee_i \rangle \lambda^\vee_0
\]

**Proof.** Let \(x \in t_C\), then, by (2.8)

\[
x = \sum \langle x, \lambda^\vee_i \rangle \alpha_j = \langle x, \lambda^\vee_i \rangle \theta + \sum_{j \neq i} \langle \langle x, \lambda^\vee_j \rangle - \langle x, \lambda^\vee_i \rangle \langle \theta, \lambda^\vee_j \rangle \rangle \alpha_j
\]
so that $\Delta_i$ is a vector space basis of $t_C$ with dual basis given by (1.1.1)-(1.1.2). If $\beta$ is a positive root, then either $\langle \beta, \lambda_i^\vee \rangle = 0$, in which case $\langle \beta, \lambda_i^\vee \rangle$ and $\langle \beta, \lambda_i^\vee \rangle$ are all non-negative, or $\langle \beta, \lambda_i^\vee \rangle = 1$ since $\lambda_i^\vee$ is a minimal dominant coweight. In the latter case $\langle \beta, \lambda_i^\vee \rangle = -1$ and $\langle \beta, \lambda_i^\vee \rangle = \langle \beta - \theta, \lambda_i^\vee \rangle \leq 0$. Thus, $\Delta_i$ is a basis of $R$. Next, for any $\beta \in R$, $\langle -\alpha_i - \beta, \lambda_i^\vee \rangle = 1 + \langle \beta, \lambda_i^\vee \rangle \geq 0$ since $\lambda_i^\vee$ is minimal. Moreover, for $j \neq i$

$$\langle -\alpha_i - \beta, \lambda_j^\vee \rangle = \langle \theta, \lambda_j^\vee \rangle - \langle \beta, \lambda_j^\vee \rangle + \langle \theta, \lambda_i^\vee \rangle \langle \beta, \lambda_i^\vee \rangle = \langle \theta - \beta, \lambda_j^\vee \rangle + \langle \theta, \lambda_j^\vee \rangle \langle \beta, \lambda_i^\vee \rangle$$

(4.1.4)

The above is clearly non-negative if $\langle \beta, \lambda_i^\vee \rangle \geq 0$. If, on the other hand, $\langle \beta, \lambda_i^\vee \rangle = -1$ then $\beta$ is negative and (1.1.4) is equal to $-\langle \beta, \lambda_i^\vee \rangle \geq 0$. Thus, $-\alpha_i$ is the highest root relative to $\Delta_i$.

**Proposition 4.1.2.** Let $\Sigma = \Delta \cup \{\alpha_0\}$. Then, for any special root $\alpha_i$, there exists a unique $w_i \in W_0 = \{w \in W | w\Sigma = \Sigma\}$ (4.1.5)

such that $w_i\alpha_0 = \alpha_i$. The resulting map $i : Z(G) \to W_0$ obtained by identifying $Z(G) \setminus \{1\}$ with the set of special roots is a group isomorphism.

**Proof.** The existence of $w_i$ follows from the previous lemma since $W$ acts transitively on the set of basis of $R$ and maps highest roots to highest roots. $w_i$ is unique because an element $w \in W_0$ is determined by $w\alpha_0$. Indeed, if $w_i\alpha_0 = \alpha_j = w_2\alpha_0$, then $w_2^{-1}w_1$ is a permutation of $\Delta$ and is therefore the identity since $W$ acts merely on basis. $i$ is injective because $w_0\alpha_0 = \alpha_i$. Let now $w \in W_0$. We claim that $\alpha_i = w\alpha_0$ is a special root. It then follows by uniqueness that $w = w_i$ and therefore that $i$ is surjective. To see this, we apply $w$ to (2.8) and get $-\alpha_i = \sum_j m_j w_0w_j$ while at the same time $-\alpha_i = m_i^{-1}(\alpha_0 + \sum_j m_j \alpha_j)$. Equating the coefficients of $\alpha_0$, we get $m_i = 1$. To prove that $i$ is a homomorphism, let $\alpha_i$ and $\alpha_j$ be special roots. Then either $w_iw_j = 1$ or $w_iw_j = w_k$ where $w_k$ is another special root. In the former case, $w_i\alpha_j = \alpha_0$ and therefore, by (4.1.1)

$$w_i\lambda_j^\vee = \lambda_0^\vee = -\lambda_i^\vee$$

(4.1.6)

so that $\lambda_j^\vee = -\lambda_i^\vee$ mod $\Lambda_R^\vee$ since $W$ leaves $\Lambda_R^\vee$–cosets invariant. In the latter, $w_i\alpha_j = \alpha_k$ and therefore, using (1.1.2)

$$w_i\lambda_j^\vee = \lambda_k^\vee = \lambda_0^\vee - \langle \theta, \lambda_j^\vee \rangle \lambda_i^\vee = \lambda_k^\vee - \lambda_i^\vee$$

(4.1.7)

whence $\lambda_j^\vee + \lambda_i^\vee = \lambda_k^\vee$ mod $\Lambda_R^\vee$ .

The following is well-known and often rediscovered [OT, Ga].

**Corollary 4.1.3.** $Z(G)$ is canonically isomorphic to the group of automorphisms of the extended Dynkin diagram of $G$ induced by Weyl group elements.

For any $\ell \in \mathbb{N}$, recall that the level $\ell$ alcove is the set defined by

$$A_\ell = \{\lambda \in \Lambda_W | \langle \lambda, \alpha_i \rangle \geq 0, \langle \lambda, \theta \rangle \leq \ell\}$$

(4.1.8)

**Proposition 4.1.4.** For any $\ell \in \mathbb{N}$, there is a canonical action of $Z(G)$ on the level $\ell$ alcove $A_\ell$ given by

$$z \mapsto A_i = \tau(\ell\lambda_i^\vee)w_i$$

(4.1.9)

where $i$ is the label of the special root corresponding to $z$ via lemma 2.3, $\tau$ denotes translation and $w_i = i(z)$ corresponds to $z$ via proposition 3.1.3.

**Proof.** If $\lambda \in A_\ell$ then for $j \neq i$, $\langle A_i\lambda, \alpha_j \rangle = \langle \lambda, w^{-1}_i\alpha_j \rangle \geq 0$ since $w^{-1}_i\alpha_j \neq \alpha_0$. On the other hand, $\langle A_i\lambda, \alpha_i \rangle = \ell + \langle \lambda, \alpha_0 \rangle \geq 0$. Finally, $\langle A_i\lambda, \theta \rangle = \ell - \langle \lambda, w^{-1}_i\theta \rangle \leq \ell$ so that the $A_i$ leave $A_\ell$ invariant. Next, $A_iA_j = \tau(\ell(\lambda_i^\vee + w_i\lambda_j^\vee))w_iw_j$. If $w_iw_j = 1$, we get by (4.1.4) and the previous proposition $A_iA_j = 1$. If, on the other hand $w_iw_j = w_k$, (4.1.7) yields $A_iA_j = A_k$.

**Remark.** The explicit action of $Z(G)$ on $A_\ell$ for all classical groups is given in §4.4.
4.2. Positive energy representations of LG.

We outline the classification of positive energy representations of LG following Wassermann to which we refer for more details. Let $\pi$ be a projective unitary representation of $LG \rtimes \text{Rot}(S^1)$ on a complex Hilbert space $\mathcal{H}$, i.e. a strongly continuous homomorphism

$$\pi : LG \rtimes \text{Rot}(S^1) \longrightarrow PU(\mathcal{H}) = U(\mathcal{H})/T$$  \hfill (4.2.1)

Over $\text{Rot}(S^1)$, $\pi$ lifts to a unitary representation which we denote by the same symbol. By definition, $\mathcal{H}$ is of positive energy if

$$\mathcal{H} = \bigoplus_{n \geq n_0} \mathcal{H}(n)$$  \hfill (4.2.2)

where each $\mathcal{H}(n) = \{ \xi \in \mathcal{H} | \pi(R_\theta)\xi = e^{in\theta}\xi \}$ is finite-dimensional. The lift is unique up to multiplication by a character of $\text{Rot}(S^1)$ and we normalise it by choosing $n_0 = 0$ and $\mathcal{H}(0) \neq \{0\}$.

The classification of positive energy representation is obtained via the associated infinitesimal action of the Lie algebra of $\mathfrak{g}$–valued trigonometric polynomials $L^{pol}\mathfrak{g} \subset L\mathfrak{g}$ in the following way. Consider the subspace $\mathcal{H}^{\text{fin}} \subset \mathcal{H}$ of finite energy vectors for $\text{Rot}(S^1)$, that is the algebraic direct sum of the $\mathcal{H}(n)$. The latter is a core for the normalised self–adjoint generator of rotations which we denote by $d$. Thus

$$d|_{\mathcal{H}(n)} = n \quad \text{and} \quad \pi(R_\theta) = e^{i\theta d} \quad (4.2.3)$$

For any $X \in L^{pol}\mathfrak{g}$, the one–parameter projective group $\pi(\exp_{LG}(tX))$ possesses a continuous lift to $U(\mathcal{H})$, unique up to multiplication by a character of $\mathbb{R}$. It is therefore given, via Stone’s theorem by $e^{i\pi(X)}$ where $\pi(X)$ is a skew–adjoint operator determined up to an additive constant.

**Theorem 4.2.1** (Wassermann). The subspace $\mathcal{H}^{\text{fin}}$ of finite energy vectors is an invariant core for the operators $\pi(X), X \in L^{pol}\mathfrak{g}$. The operators $\pi(X)$ may be chosen uniquely so as to satisfy $[d, \pi(X)] = i\pi(X)$ on $\mathcal{H}^{\text{fin}}$ and then $X \to \pi(X)$ gives a projective representation of $L^{pol}\mathfrak{g}$ on $\mathcal{H}^{\text{fin}}$ such that

$$[\pi(X), \pi(Y)] = \pi([X, Y]) + i\ell B(X, Y) \quad (4.2.4)$$

where $B(X, Y) = \int_0^{2\pi} \langle X, Y \rangle \frac{d\theta}{2\pi}$ and $\ell$ is a non–negative integer called the level of $\mathcal{H}$.

We denote the restriction of the operators $\pi(X), X \in L^{pol}\mathfrak{g}$ to $\mathcal{H}^{\text{fin}}$ by the same symbol and extend $\pi$ to a projective representation of $L^{pol}\mathfrak{g}$ on $\mathcal{H}^{\text{fin}}$ satisfying (4.2.4) as well as the formal adjunction property $\pi(X)^* = -\pi(X^*)$. The operators $\pi(X)$ and $d$ then give rise to a unitarisable representation of the Kac–Moody algebra $\tilde{\mathfrak{g}}_c$ at level $\ell$ such that $d$ is diagonal with finite–dimensional eigenspaces and spectrum in $\mathbb{N}$. Such representations split into a direct sum of irreducibles, each of which is an integrable highest weight representation, that is, a module generated over the enveloping algebra $\mathcal{U}(\mathfrak{g}_c)$ by a vector $v$ uniquely determined by the requirement that it be annihilated by $\mathfrak{g}_c$ and diagonalises the action of $T \rtimes \text{Rot}(S^1)$. Here $\mathfrak{g}_\leq$ (resp. $\mathfrak{g}_\geq$) is the nilpotent Lie algebra spanned by the $x(n)$ with $n < 0$ (resp. $n > 0$) and $x \in \mathfrak{g}_c$ or $n = 0$ and $x$ lying in a negative (resp. positive) root space of $\mathfrak{g}_c$. Thus, for any $h \in \mathfrak{t}_c$

$$dv = nv \quad (4.2.5)$$

$$\pi(h)v = \lambda(h)v \quad (4.2.6)$$

for some $n \in \mathbb{N}$ and dominant integral weight $\lambda$ of $G$ which satisfies $\langle \lambda, \theta \rangle \leq \ell$. The pair $(\ell, \lambda)$ classifies the integrable representation uniquely [Ka]. The finite collection $A_\ell$ of dominant integral weights of $G$ satisfying $\langle \lambda, \theta \rangle \leq \ell$ is called the level $\ell$alcove of $G$. 


The classification of a positive energy representation \((\pi, \mathcal{H})\) as an \(LG\)-module is equivalent as that of \(\mathcal{H}^{\text{fin}}\) as an \(L^{\text{pol}}g\)-module. In particular, \(\mathcal{H}\) is topologically irreducible under \(LG\) iff \(\mathcal{H}^{\text{fin}}\) is irreducible under \(L^{\text{pol}}g\) and is then uniquely determined by its level \(\ell\) and highest weight \(\lambda\). Moreover, for any \(\ell \in \mathbb{N}\) and \(\lambda \in A_{\ell}\), there exists a unique unitarisable highest weight \(g\)-module with level \(\ell\) and highest weight \(\lambda\) \cite{Ka}. The corresponding action of \(L^{\text{pol}}g\) may then be exponentiated to yield a positive energy representation of \(LG\) on its Hilbert space completion \cite{JoWa, TL2}.

### 4.3. Action of \(L_{Z}G\) on positive energy representations of \(LG\).

The group of discontinuous loops \(L_{Z}G\) acts on \(LG\) by conjugation and therefore on the irreducible projective unitary representations of \(LG\) by \(\zeta, \pi(\gamma) = \pi(\zeta^{-1}\gamma\zeta), \zeta \in L_{Z}G\). The following result shows that positive energy ones are stable under this action

**Proposition 4.3.1.** If \((\pi, \mathcal{H})\) is an irreducible positive energy representation of \(LG\) and \(\zeta \in L_{Z}G\), the conjugated representation \(\zeta_{*}\pi\) is of positive energy.

**Proof.** Any intertwining action of \(\text{Rot}(S^{1})\) for \(\zeta_{*}\pi\), whether of positive energy or not is necessarily given by

\[
\theta \mapsto V_{\theta} = \pi(\zeta^{-1}\zeta_{0})U_{\theta} \tag{4.3.1}
\]

where \(\zeta_{0}(x) = \zeta(x - \theta)\) and \(U_{\theta}\) is the positive energy intertwining action of \(\text{Rot}(S^{1})\) for \(\pi\). Indeed, \(\zeta^{-1}\zeta_{0} \in LG\) and \(V_{\theta}\) yields a projective action of \(\text{Rot}(S^{1})\) satisfying \(V_{\theta}\zeta_{*}\pi(\gamma)V_{\theta}^{*} = \zeta_{*}\pi(\zeta_{0})\). Moreover, if \(V_{\theta}^{i}, i = 1, 2\) are two actions of \(\text{Rot}(S^{1})\) intertwining \(\zeta_{*}\pi\), then \(W_{\theta} = (V_{\theta}^{1})^{*}V_{\theta}^{2}\) commutes projectively with \(LG\) and the following holds in \(U(\mathcal{H})\)

\[
W_{\theta}\pi(\gamma)W_{\theta}^{*}\pi(\gamma)^{*} = \chi(\gamma, \theta) \tag{4.3.2}
\]

where \(\chi(\gamma, \theta) \in \mathbb{T}\) depends multiplicatively on either variable. Since \(LG\) is perfect \cite[Prop. 3.4.1.]{PS}, \(\chi \equiv 1\) and by Shur’s lemma \(W_{\theta} = 1\) in \(PU(\mathcal{H})\).

Thus, \(\zeta_{*}\pi\) is of positive energy iff \(V_{\theta}\) is a positive energy representation of \(\text{Rot}(S^{1})\). It is sufficient to check this for a given set of representations of \(LG\)-cosets in \(L_{Z}G\) and therefore for the discontinuous loops \(\zeta_{\mu}(\theta) = \exp_{\nu}(-i\theta\mu), \mu \in \Lambda_{Z}^{\nu}\). If \(\mu \in \Lambda_{\ell}^{\nu}\), then \(\zeta_{\mu} \in LG\) and the action of \(\text{Rot}(S^{1})\) given by \(\zeta_{\mu}^{-1}\zeta_{0}\) may be rewritten as \(\pi(\zeta_{\mu}^{-1}\zeta_{0})U_{\theta} = \pi(\zeta_{\mu})^{*}U_{\theta}\pi(\zeta_{\mu})\) which is of positive energy. The general case \(\mu \in \Lambda_{Z}^{\nu}\) is settled by the following simple observation. Notice that \(\zeta_{\mu}^{-1}\zeta_{0} = \zeta_{\mu}(-\theta) = \exp_{\nu}(i\theta\mu)\) since \(\zeta_{\mu}\) is a homomorphism and write \(\mu\) as a convex combination of elements in the coroot lattice, \(\mu = \sum_{i=1}^{m} t_{i}\mu_{i}, t_{i} \in [0, 1], \sum_{i} t_{i} = 1, \mu_{i} \in \Lambda_{\ell}^{\nu}\). Since \(\pi\) lifts to a unitary representation \(\tilde{\pi}\) over \(G \times \text{Rot}(S^{1}) \supset \text{T} \times \text{Rot}(S^{1})\),

\[
\tilde{\pi}(\exp_{\nu}(i\theta\mu))\tilde{\pi}(R_{\theta}) = \prod_{j}^{\nu}(\exp_{T}(i\theta t_{j}\mu_{j}))\tilde{\pi}(R_{t_{j}\theta}) \tag{4.3.3}
\]

is a lift of \(\pi(\zeta_{\mu}^{-1}\zeta_{0})U_{\theta}\) and the product of \(m\) commuting representations of \(\mathbb{R}\) which by our previous argument are of positive energy. It follows that \(\pi(\zeta_{\mu}^{-1}\zeta_{0})U_{\theta}\) is of positive energy \(\diamondsuit\).

**Proposition 4.3.2.** Let \((\pi, \mathcal{H})\) be a positive energy representation of \(LG\) of level \(\ell\) and \(\zeta \in L_{Z}G\). Then,

(i) \(\zeta_{*}\pi\) is of level \(\ell\).

(ii) If \(\zeta(\phi) = \exp_{\nu}(-i\phi\mu)\) is the discontinuous loop corresponding to \(\mu \in \Lambda_{Z}^{\nu}\), the subspaces of finite energy vectors of \(\pi\) and \(\zeta_{*}\pi\) coincide.

(iii) If \(\text{Ad}(\zeta)L^{\text{pol}}g = L^{\text{pol}}g\) and the finite energy subspaces of \(\pi\) and \(\zeta_{*}\pi\) coincide, the conjugated action of \(L^{\text{pol}}g\) on \(\mathcal{H}^{\text{fin}}\) is given by

\[
\zeta_{*}\pi(X) = \pi(\zeta^{-1}X\zeta) + i\ell \int_{0}^{2\pi} \langle \dot{\zeta}^{-1}X, \dot{\zeta} \rangle \frac{d\theta}{2\pi} \tag{4.3.4}
\]
Proof. It is sufficient to check (i) for the discontinuous loops $\zeta_\mu$, $\mu \in \Lambda^\vee_{Z}$. This will be done in the course of the proof of (iii).

(ii) It was remarked in the proof of the previous proposition that the conjugated action of rotations \((4.3.1)\) corresponding to \(\zeta_\mu\) is given by

\[
\pi(\zeta_\mu^{-1}\zeta_{\mu_0})U_\theta = \pi(\exp_T(i\theta \mu))U_\theta
\]

which commutes with the original action of \(\text{Rot}(S^1)\) given by \(U_\theta\). Since both are of positive energy, their finite energy subspaces coincide.

(iii) Let \(h \in \mathbb{N}\) be the level of \(\zeta_\ast\pi\) and denote by \(\pi\) and \(\zeta_\ast\pi\) the projective representations of \(L^{\text{ad}}g\) on \(H^\text{fin}\) given by theorem 4.2.1, so that

\[
\pi(X, Y) = \pi([X, Y]) + i\ell B(X, Y) = \zeta_\ast\pi([X, Y]) + i\ell B(X, Y)
\]

Evidently, \(\zeta_\ast\pi(X) = \pi(\zeta^{-1}X\zeta) + iF(X)\) for some \(F(X) \in \mathbb{R}\) since \(\zeta_\ast\pi(\exp_{L^G}(X)) = \pi(\exp_{L^G}(\zeta^{-1}X\zeta)) = e^{i\zeta^{-1}X\zeta}\) in \(PU(H)\). It follows, by (3.1.14) that

\[
[\zeta_\ast\pi(X), \zeta_\ast\pi(Y)] = [\pi(\zeta^{-1}X\zeta), \pi(\zeta^{-1}Y\zeta)]
\]

\[
= \pi(\zeta^{-1}[X, Y]\zeta) + i\ell B(\zeta^{-1}X\zeta, \zeta^{-1}Y\zeta)
\]

\[
= \zeta_\ast\pi([X, Y]) - iF([X, Y]) + i\ell B(X, Y) + i\ell \int_0^{2\pi} \langle \zeta^{-1}, [X, Y] \rangle \frac{d\theta}{2\pi}
\]

Since \(hB\) and \(\ell B\) lie in the same cohomology class iff \(h = \ell\), we find by equating the above with \((4.3.7)\) that the level of \(\zeta_\ast\pi\) is \(\ell\). Moreover, \((4.3.4)\) holds since \([L^{\text{ad}}g, L^{\text{ad}}g] = L^{\text{ad}}g\) \\(\circ\)

Theorem 4.3.3. Let \((\pi, H)\) be an irreducible positive energy representation of \(L^G\) of level \(\ell\) and highest weight \(\lambda\) and \(\zeta \in L^G G\). Then, the conjugated representation \(\zeta_\ast\pi(\gamma) = \pi(\zeta^{-1}\gamma\zeta)\) on \(H\) is of positive energy, level \(\ell\) and highest weight \(\zeta \lambda\) where the notation refers to the geometric action of \(Z(G) \cong L^G G / L^G\) on the level \(\ell\) alcove defined by proposition 4.1.4.

Proof. It suffices to prove the result for a given choice of representatives of \(L^G\)-cosets in \(L^G G\). Let \(z \in Z^\ast = \{1\}\) correspond to the special root \(\alpha_j\) by lemma 2.3 and consider the discontinuous loop \(\zeta = \zeta_{\alpha_j} w_j\) where \(\alpha_j^\vee\) is the associated fundamental coweight and \(w_j \in G\) a representative of the Weyl group element corresponding to \(z\) by proposition 1.1.2. Since \(G\) commutes \(\text{Rot}(S^1)\), the subspace of finite energy vectors of \(\pi\) coincides with that of \(w_j\pi\) and, by the previous proposition, with that of \(\zeta_\ast\pi\). We may therefore compare the infinitesimal actions of \(L^{\text{ad}}g\) corresponding to \(\pi\) and \(\zeta_\ast\pi\) on \(H^\text{fin}\).

If \(\alpha\) is a root and \(x_\alpha \in g_\alpha\), then \([\alpha^\vee, x_\alpha\] = \(\langle \alpha^\vee, \alpha\rangle x_\alpha\). Since \(\zeta_{\alpha^\vee}(\theta) = \exp_T(-i\lambda^\vee_{\alpha^\vee})\), this gives

\[
\zeta_{\alpha^\vee}^{-1} x_\alpha(n)\zeta_{\alpha^\vee}(\theta) = x_\alpha \otimes e^{i\theta(n + \langle \lambda^\vee_{\alpha^\vee}, \alpha\rangle)} = x_\alpha(n + \langle \lambda^\vee_{\alpha^\vee}, \alpha\rangle)(\theta)
\]

Therefore, up to a non–zero multiplicative constant

\[
\zeta_\ast\pi(x_\alpha(n)) = \pi(e_{w_j^{-1}\alpha}(n + \langle \lambda^\vee_{\alpha^\vee}, \alpha\rangle))
\]

since no additional term arises from \((4.3.4)\) because \(\dot{\zeta_{\alpha^\vee}^{-1}} = -i\lambda^\vee_{\alpha^\vee}\) lies in \(t_\alpha\) which is orthogonal to \(g_\alpha\). If, on the other hand \(h \in t_\alpha\), then \(\zeta^{-1} h(n)\zeta = w_j^{-1} h(n)\) and \((4.3.4)\) reads

\[
\zeta_\ast\pi(h(n)) = \pi(w_j^{-1} h(n)) + \ell \delta_{n,0}(h, \lambda^\vee_{\alpha^\vee})
\]

Let \(\Omega \in H^\text{fin}\) be the highest weight vector for \(\zeta_\ast\pi\). We claim that, up to a scalar factor, \(\Omega = 1\), the highest weight vector for \(\pi\). To see this, recall that \(\Omega\) is the unique element of \(H^\text{fin}\) annihilated by the subalgebra \(g_{\geq}\) spanned by the \(x(n), x \in g_\alpha\) and \(n > 0\) and the \(x_{\alpha}(0)\) with \(\alpha > 0\). \(g_{\geq}\) is generated by the elements corresponding the simple affine roots,
namely $e_{\alpha_i}(0)$ and $e_{\alpha_0}(1)$ where $\alpha_0 = -\theta$. Recalling from proposition 1.1.2 that $w_j^{-1}$ acts as a permutation of $\Sigma = \{\alpha_0, \ldots, \alpha_n\}$ and maps $\alpha_j$ to $\alpha_0$, we get, using (4.3.10)

$$
\zeta_\ast \pi(e_{\alpha_k}(0)) = \begin{cases} 
\pi(e_{w_j^{-1}\alpha_k}(0)) & \text{if } k \neq j \\
\pi(e_{\alpha_k}(1)) & \text{if } k = j 
\end{cases}
$$

(4.3.12)

whence $\Omega = \Upsilon$. To find the weight of $\Omega$ and therefore the highest weight of $\xi$, we use (4.3.11) and the fact that $\pi(h(0))\Upsilon = \langle \lambda, h \rangle \Upsilon$ whenever $h \in \mathfrak{t}_C$ so that

$$
\zeta_\ast \pi(h(0))\Omega = \langle h, w_j \lambda + e_\lambda \rangle \Omega = \langle h, \xi \rangle \Omega
$$

(4.3.14)

\△

The proof of the above theorem has the following useful

**Corollary 4.3.4.** Let $z \in Z(G) \setminus \{1\}$ and $\lambda^\vee_j \in \Lambda^\vee_W$ and $w_j \in G$ the fundamental coweight and representative of the Weyl group element corresponding to $z$ by lemma 2.3 and proposition 4.1.2 respectively. Then, conjugation by $z = \xi \lambda^\vee_j w_j$ induces an automorphism of $L^\vee \mathfrak{g}_C$ preserving its triangular decomposition.

4.4. Appendix : explicit action of $Z(G)$ on the level $\ell$ alcove.

We describe explicitly the action of $Z(G)$ given by proposition 4.1.4 for all classical groups, using the tables [Bou, Tables I–IV]. For each special root $\alpha_i$, we denote the corresponding element of $W_0 \subset W$ by $w_i$ and note that the fundamental coweight $\lambda_i^\vee$ and weight $\lambda_i$ coincide since $\alpha_i$ is long. We let moreover $\theta_i$, $i = 1 \ldots n$ be the standard basis in $\mathbb{R}^n$ and $I$ the lattice $\bigoplus \theta_i \cdot \mathbb{Z}$. Unless otherwise stated, the basic inner product is the standard one on $\mathbb{R}^n$.

**SU$_n$, $n \geq 2$**

simple roots : $\alpha_i = \theta_i - \theta_{i+1}$, $i = 1 \ldots n - 1$.

highest root : $\theta = \theta_1 - \theta_n = \alpha_1 + \cdots + \alpha_{n-1}$.

minimal dominant coweights : $\lambda_i^\vee = \theta_1 + \cdots + \theta_i - \frac{1}{2} \sum_j \theta_j$, $i = 1 \ldots n - 1$.

Weyl group : $\mathfrak{S}_n$ acting by permutation of the $\theta_i$.

$W_0 : w_k$ is the cyclic permutation $(\theta_1 \cdots \theta_n)^k = (\alpha_0 \cdots \alpha_{n-1})^k$.

level $\ell$ alcove : $A_\ell = \{ \mu \in I \mid \mu_1 \geq \cdots \geq \mu_n, \mu_1 - \mu_n \leq \ell \} / (\sum_j \theta_j)$.

action of the centre : $A_k(\mu_1, \ldots, \mu_n) = (\ell + \mu_{n+1-k}, \ldots, \ell + \mu_n, \mu_1, \ldots, \mu_{n-k})$.

**Spin$_{2n+1}$, $n \geq 2$**

simple roots : $\alpha_i = \theta_i - \theta_{i+1}$, $i = 1 \ldots n - 1$ and $\alpha_n = \theta_n$.

highest root : $\theta = \theta_1 + \theta_2 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_n)$.

minimal dominant coweight : $\lambda_i^\vee = \theta_1$.

Weyl group : $\mathfrak{S}_n \ltimes \mathbb{Z}_2^n$ acts by permutations and sign changes of the $\theta_i$.

$W_0 : w_1$ is the sign change $\theta_1 \rightarrow -\theta_1$ permuting $\alpha_0$ and $\alpha_1$.

level $\ell$ alcove : $A_\ell = \{ \mu \in I \mid \mu_1 \geq \cdots \geq \mu_n, \mu_1 - \mu_n \leq \ell \}$. 

action of the centre : $A_1(\mu_1, \mu_2, \ldots, \mu_n) = (\ell - \mu_1, \mu_2, \ldots, \mu_n)$.

**Sp$_n$, $n \geq 2$**

simple roots : $\alpha_i = \theta_i - \theta_{i+1}$, $i = 1 \ldots n - 1$ and $\alpha_n = 2\theta_n$.

highest root : $\theta = 2\theta_1 = 2(\alpha_1 + \cdots + \alpha_{n-1}) + \alpha_n$.

basic inner product : half the standard one on $\mathbb{R}^n$.

minimal dominant coweight : $\lambda_i^\vee = \theta_1 + \cdots + \theta_n$.

Weyl group : $\mathfrak{S}_n \ltimes \mathbb{Z}_2^n$ acts by permutations and sign changes of the $\theta_i$.

$W_0 : w_n$ is the transformation $\theta_i \rightarrow -\theta_{n+1-i}$.

level $\ell$ alcove : $A_\ell = \{ \mu \in I \mid \ell \geq \mu_1 \geq \cdots \geq \mu_n \geq 0 \}$.

action of the centre : $A_n(\mu_1, \ldots, \mu_n) = (\ell - \mu_1, \ldots, \ell - \mu_1)$.
We determine below the central extensions of $\mathfrak{L}$ invariant under $\mathfrak{L}$. In particular, it is uniquely determined by its commutator map which satisfies $\alpha Z$.

We begin by computing a number of commutators related to (5.2). Recall that the roots of $\text{Spin}_{\theta}$ such that $C$ are the vectors $\alpha$.

5. The Mackey Obstruction Corresponding to $\text{LG} \subset \mathbb{LZ}G$

We determine below the central extensions of $\mathbb{LZ}G$ corresponding to positive energy representations which remain irreducible when restricted to $\mathbb{LZ}G$. More precisely, let $\mathcal{H}$ be an irreducible, level $\ell$ positive energy representation of $\mathbb{LZ}G$ the isomorphism class of which is invariant under $\mathbb{LZ}G$. $\mathcal{H}$ gives rise to a projective action $\pi$ of $\mathbb{LZ}G$ extending that of $\mathbb{LZ}G$ and therefore by pull–back of
to a central extension $\mathbb{LZ}G$ of $\mathbb{LZ}G$. Since the restriction of $\mathbb{LZ}G$ to $\mathbb{LZ}G$ is smooth and of level $\ell$ in chap. II, §2.4], $\mathbb{LZ}G$ is smooth and therefore falls within the classification of section 3. In particular, it is uniquely determined by its commutator map

which satisfies

whenever $\alpha \in \Lambda_R^+ \subset \Lambda_R^+$. This binds $\omega$ uniquely if $Z$ is cyclic, for if $\lambda \in \Lambda_R^+$ is a generator of $Z \cong \Lambda_R^+/\Lambda_R^-$, then $\omega(\alpha + a\lambda, \beta + b\lambda) = (-1)^{\ell(\alpha,\beta)+(\lambda,\alpha+\beta)}$ for any $\alpha, \beta \in \Lambda_R^+$. We therefore only need to investigate the case of $G/Z = \text{Spin}_{4n}/\mathbb{Z}_2 \times \mathbb{Z}_2$, $n \geq 2$. The main result of this section is that in this case $\ell$ is even and $\omega \equiv 1$.

We begin by computing a number of commutators related to (5.2). Recall that the roots of $\text{Spin}_{4n}$ are the vectors $\pm \theta_i \pm \theta_j$, $1 \leq i \neq j \leq 2n$, where $\{\theta_i\}$ is the canonical basis of $\mathbb{R}^{2n}$. The simple roots are $\alpha_i = \theta_i - \theta_{i+1}$, $i = 1, \ldots, n-1$ and $\alpha_n = \theta_{n-1} + \theta_n$.

The simple roots are $\alpha_i = \theta_i - \theta_{i+1}$, $i = 1, \ldots, n-1$ and $\alpha_n = \theta_{n-1} + \theta_n$ and the highest root is $\theta = \theta_1 + \theta_2 = \alpha + 2(\alpha_2 + \cdots + \alpha_n) + \alpha_{n-1} + \alpha_n$ so that the minimal dominant coweights are $\lambda_1 = \theta_1$, $\lambda_{2n-1} = \frac{1}{2}(\theta_1 + \cdots + \theta_{n-1} - \theta_n)$ and $\lambda_{2n} = \frac{1}{2}(\theta_1 + \cdots + \theta_{2n})$. Fix, for any positive root $\alpha$, a basis $e_\alpha, f_\alpha, h_\alpha = \alpha^*$ of the corresponding $\mathfrak{sl}_2(C)$-subalgebra of $\mathfrak{so}_{4n,C}$ such that $e_\alpha^* = f_\alpha$ where $^*$ is the canonical anti–linear anti–involution acting as -1 on $\mathfrak{so}_{4n}$. Then,
Lemma 5.1. Let \( w_j \) be the Weyl group elements corresponding to the minimal dominant coweights \( \lambda_j^\vee, j \in \{1, 2n - 1, 2n\} \) by proposition 4.1.3. Then, the following elements may be taken as representatives of \( w_j \) in Spin\(_{4n}\):

\[
\begin{align*}
\mathfrak{w}_1 &= \exp_{\text{Spin}_{4n}} \left( \frac{\pi}{2} (e_{\theta_1 + \theta_2} - f_{\theta_1 + \theta_2}) \right) \\
\mathfrak{w}_{2n-1} &= \prod_{i=2}^{n} \exp_{\text{Spin}_{4n}} \left( \frac{\pi}{2} (e_{\theta_1 + \theta_{2n-i+1}} - f_{\theta_1 + \theta_{2n-i+1}}) \right) \\
\mathfrak{w}_{2n} &= \prod_{i=1}^{n} \exp_{\text{Spin}_{4n}} \left( \frac{\pi}{2} (e_{\theta_1 + \theta_{2n-i+1}} - f_{\theta_1 + \theta_{2n-i+1}}) \right)
\end{align*}
\]

Moreover, the group commutators \( [\mathfrak{w}_j, \mathfrak{w}_k] = \mathfrak{w}_j \mathfrak{w}_k \mathfrak{w}_j^{-1} \mathfrak{w}_k^{-1} \) are all equal to one.

Proof. It is readily seen that the \( w_j \) may be expressed as

\[
\mathfrak{w}_1 = \sigma_{\theta_1 + \theta_2, \theta_1 - \theta_2} \quad \mathfrak{w}_{2n-1} = \prod_{i=2}^{n} \sigma_{\theta_1 + \theta_{2n-i+1}, \theta_1 - \theta_{2n-i+1}} \quad \mathfrak{w}_{2n} = \prod_{i=1}^{n} \sigma_{\theta_1 + \theta_{2n-i+1}}
\]

where \( \sigma_\alpha \) is the orthogonal reflection corresponding to the root \( \alpha \) and acts on \( t_C \) as

\[
\sigma_\alpha (h) = h - \langle h, \alpha \rangle \alpha \vee
\]

Each \( \sigma_\alpha \) may be lifted in Spin\(_{4n}\) to \( \mathfrak{g}_\alpha = \exp_{\text{Spin}_{4n}} \left( 2 (e_\alpha - f_\alpha) \right) \). Indeed, a power series expansion shows that \( \text{Ad}(\mathfrak{g}_\alpha) \) leaves \( t_C \) invariant and coincides with the right hand–side of (5.5). Thus, (5.4) and (5.6) hold. The last claim is a consequence of the fact that the lifts of \( w_j \) and \( w_k \) only involve roots \( \alpha_{jp}, \beta_{kq} \) such that \( \alpha_{jp} \pm \beta_{kq} \) is either zero or not a root. Thus, \( [e_{\alpha_{jp}}, f_{\beta_{kq}}, e_{\beta_{kq}}, f_{\alpha_{jp}}] = 0 \) and \( \mathfrak{g}_{\alpha_{jp}} \) and \( \mathfrak{g}_{\beta_{kq}} \) commute.

Lemma 5.2. Let \( (\pi, \mathcal{H}) \) be an irreducible positive energy representation of \( L \text{Spin}_{4n} \) the isomorphism class of which is invariant under \( LZ_{2} \times Z_{2} \text{Spin}_{4n} \) and denote by the same symbol its unique extension to the latter group. Then, if \( \mathfrak{w}_j \) are as in lemma 5.1 and \( \zeta_j (\theta) = \exp_T (-i \theta \lambda_j^\vee) \) are the discontinuous loops corresponding to the minimal dominant coweights \( \lambda_j^\vee \)

\[
\pi (\mathfrak{w}_j) \pi (\mathfrak{w}_k) \pi (\mathfrak{w}_j)^* \pi (\mathfrak{w}_k)^* = 1
\]

Proof. Let \( \lambda_i^\vee \) be the minimal dominant coweight with corresponding Weyl group element \( w_i \) such that \( w_i w_k = w_i \). By (4.1.7), \( w_i \lambda_k^\vee = \lambda_j^\vee - \lambda_j^\vee \) so that

\[
\mathfrak{w}_j \zeta_j (\theta) \mathfrak{w}_j^{-1} = \exp_T (-i \theta w_j \lambda_k^\vee) = \zeta_i \zeta_j^{-1} (\theta)
\]

and similarly

\[
\mathfrak{w}_k \zeta_j \mathfrak{w}_k^{-1} = \zeta_i \zeta_j^{-1}
\]

Thus,

\[
[\zeta_j \mathfrak{w}_j, \zeta_k \mathfrak{w}_k] = \zeta_j \mathfrak{w}_j \zeta_k \mathfrak{w}_k^{-1} \mathfrak{w}_j \mathfrak{w}_k \zeta_j^{-1} \mathfrak{w}_k^{-1} = \zeta_i [\mathfrak{w}_j, \mathfrak{w}_k] \zeta_i^{-1}
\]

which, by lemma 5.1, equals 1. It follows that the group commutator \( [\pi (\mathfrak{w}_j), \pi (\mathfrak{w}_k)] \) acts as a scalar \( \chi \) on \( \mathcal{H} \). To evaluate \( \chi \), recall that by corollary 4.3.4, conjugation by \( \zeta_j \mathfrak{w}_j \) and \( \zeta_k \mathfrak{w}_k \) induces an automorphism of \( L^\infty s\text{so}_{4n} \), preserving its triangular decomposition so that the unitaries \( \pi (\mathfrak{w}_j), \pi (\mathfrak{w}_k) \) leave the highest weight vector in \( \mathcal{H} \) invariant and \( \chi = 1 \).

Theorem 5.3. Let \( \pi \) be an irreducible positive energy representation of \( L \text{Spin}_{4n} \) the isomorphism class of which is invariant under \( LZ_{2} \times Z_{2} \text{Spin}_{4n} \) and extend it uniquely to a projective representation of the latter group. Then, the corresponding central extension of \( LZ_{2} \times Z_{2} \text{Spin}_{4n} \) has even level and trivial commutator map.
By lemma 5.1 and the fact that \( \pi \) lifts to a unitary representation of \( \text{Spin}_{4n} \)

\[
[\pi(\zeta_j \omega_k), \pi(\zeta_k \omega_j)] = \pi(\zeta_j) \pi(\zeta_k) [\pi(\zeta_j)^* \pi(\zeta_k)^* \pi(\zeta_j)^* \pi(\zeta_k)] = \pi(\zeta_j) \pi(\zeta_k) [\pi(\zeta_j)^* \pi(\zeta_k)^* \pi(\zeta_j)^* \pi(\zeta_k)]
\]

(5.13)

By (5.11), \( [\zeta_k^{-1}, \omega_j] = \zeta_k^{-1} \zeta_j \zeta_k^{-1} \) so that \( [\pi(\zeta_k)^*, \pi(\omega_j)] \) is proportional to \( \pi(\zeta_k)^* \pi(\zeta_j)^* \). Similarly, by (5.11), \( [\pi(\omega_k), \pi(\zeta_j)] \) is proportional to \( \pi(\zeta_k)^* \pi(\zeta_j) \) so that, by 5.2, (5.13) is equal to

\[
\omega(\lambda_j^\vee, \lambda_k^\vee) [\pi(\zeta_k)^*, \pi(\omega_j)] [\pi(\omega_k), \pi(\zeta_j)^*]
\]

(5.14)

and, by lemma 5.2,

\[
\omega(\lambda_j^\vee, \lambda_k^\vee) = [\pi(\zeta_j)^*, \pi(\omega_k)] [\pi(\omega_j), \pi(\zeta_k)^*]
\]

(5.15)

We shall now show that the right hand–side of (5.14) is equal to one.

Set \( j = 1 \) and \( k = 2n \) so that the corresponding coweights are \( \lambda_1^\vee = \theta_1 \) and \( \lambda_{2n}^\vee = \frac{1}{2}(\theta_1 + \cdots + \theta_n) \). As previously noted, if \( x, \beta \in \text{so}_{4n, \mathbb{C}} \) is a root vector corresponding to \( \beta \) and \( \zeta_{\lambda}(\theta) = \exp_{\mathbb{T}}(-i\theta \lambda) \), then

\[
\zeta_{\lambda}^{-1} x, \beta(n) \zeta_{\lambda} = x, \beta(n + \langle \lambda, \beta \rangle)
\]

(5.16)

and therefore

\[
\zeta_{\lambda}^{-1} \omega_{2n} \zeta_{\lambda} = \exp_{\text{Spin}_{4n}} \left( \frac{\pi}{2}(e_{\theta_1 + \theta_{2n}}(1) - f_{\theta_1 + \theta_{2n}}(-1)) \right) \prod_{i=2}^{n} \exp_{\text{Spin}_{4n}} \left( \frac{\pi}{2}(e_{\theta_i + \theta_{2n-i+1}}(1) - f_{\theta_i + \theta_{2n-i+1}}(-1)) \right)
\]

(5.17)

\[
\zeta_{\lambda}^{-1} \omega_{2n}^{-1} \zeta_{\lambda} = \exp_{\text{Spin}_{4n}} \left( -\frac{\pi}{2}(e_{\theta_1 - \theta_{2n}} - f_{\theta_1 - \theta_{2n}}) \right) \exp_{\text{Spin}_{4n}} \left( -\frac{\pi}{2}(e_{\theta_1 + \theta_{2n}}(1) - f_{\theta_1 + \theta_{2n}}(-1)) \right)
\]

(5.18)

Denoting by \( \rho \) the infinitesimal action of \( L(\text{so}_{4n, \mathbb{C}}) \) on \( \mathcal{H}^{4n} \) corresponding to \( \pi \) via theorem 4.2.1, we therefore get by proposition 4.3.3

\[
[\pi(\zeta_1)^*, \pi(\omega_{2n})] = \exp \left( \frac{\pi}{2} \rho(e_{\theta_1 + \theta_{2n}}(1) - f_{\theta_1 + \theta_{2n}}(-1)) \right) \exp \left( -\frac{\pi}{2} \rho(e_{\theta_1 + \theta_{2n}} - f_{\theta_1 + \theta_{2n}}) \right)
\]

(5.19)

\[
[\pi(\omega_1), \pi(\zeta_{2n})] = \exp \left( \frac{\pi}{2} \rho(e_{\theta_1 + \theta_{2n}} - f_{\theta_1 + \theta_{2n}}) \right) \exp \left( -\frac{\pi}{2} \rho(e_{\theta_1 + \theta_{2n}}(1) - f_{\theta_1 + \theta_{2n}}(-1)) \right)
\]

(5.20)

where the exponentials are given by the spectral theorem. Thus,

\[
[\pi(\zeta_1)^*, \pi(\omega_{2n})][\pi(\omega_1), \pi(\zeta_{2n})] = 1
\]

(5.21)

as claimed ◇
6. Positive energy representations of $L_2G$

Let $k$ be the order of the largest cyclic subgroup of $Z$ and consider the action of the universal $k$–coverings of $\text{Diff}_+(S^1)$ and $\text{Rot}(S^1)$ on $L_2G$ by reparametrisation, as in §3.3. We denote these coverings by $\text{Diff}_k^+(S^1)$ and $\text{Rot}^k(S^1)$ respectively. Define a positive energy representation of $L_2G$ to be a strongly continuous homomorphism

$$\pi : L_2G \to PU(\mathcal{H})$$

(6.1)

extending to $L_2G \times \text{Rot}^k(S^1)$ in such a way that $\text{Rot}^k(S^1)$ acts by non–negative characters only and with finite–dimensional eigenspaces.

**Theorem 6.1.** An irreducible positive energy representation $(\pi, \mathcal{H})$ of $L_2G$ yields by restriction

(i) A positive energy representation of $LG$ of level

$$\ell \in \ell_f : \mathbb{N}$$

(6.2)

where $\ell_f \in \{1, 2\}$ is the fundamental level of $G/Z$.

(ii) A projective representation of $\Lambda^\ast_{\mathbb{Z}} \cong \text{Hom}(\mathbb{T}, T/Z)$, the commutator map of which, defined by $\omega(\lambda, \mu) = \pi(\zeta_\lambda)\pi(\zeta_\mu)^\ast \pi(\zeta_\mu)^\ast$, satisfies

$$\omega(\alpha, \mu) = (-1)^{\ell(\alpha, \mu)}$$

(6.3)

whenever $\alpha$ lies in the coroot lattice $\Lambda^\ast_{\mathbb{R}} \cong \text{Hom}(\mathbb{T}, T)$.

As an $LG$–module,

$$\mathcal{H} = \bigoplus_{\mu \in \mathbb{Z}_\lambda} \mathcal{H}_\mu \otimes \mathbb{C}^{m_\lambda}$$

(6.4)

where $\lambda$ lies in the level $\ell$ alcove of $G$, $Z\lambda$ is its orbit under the action of $Z \subseteq Z(G)$ defined by proposition 4.1.4 and $\mathcal{H}_\mu$ is the irreducible level $\ell$ positive energy representation of $LG$ with highest weight $\mu$. Moreover, $m_\lambda = 1$ unless $G/Z = \text{PSO}_{4n}$, $Z\lambda = \{\lambda\}$, $\ell$ is even and $\omega$ is the pull–back of the non–trivial, skew–symmetric form on $Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, in which case $m_\lambda = 2$. The triple $(\ell, \omega, Z\lambda)$ classifies $\mathcal{H}$ uniquely and for any $(\ell, \omega, Z\lambda)$ satisfying 6.2 and 6.3, there exists an irreducible positive energy representation of $L_2G$ realising it. Lastly, the action of $\text{Rot}^k(S^1)$ on $\mathcal{H}$ extends uniquely to a projective unitary representation $\rho$ of $\text{Diff}_k^+(S^1)$ satisfying

$$\rho(\phi)\pi(\zeta)\rho(\phi)^\ast = \pi(\zeta \circ \phi^{-1})$$

(6.5)

which coincides with the Segal–Sugawara representation obtained by regarding $\mathcal{H}$ as a positive energy representation of $LG$.

**Proof.** We shall repeatedly, and without further mention, use the following fact. Let $Y \subseteq Z$ be a subgroup and $(\rho, \mathcal{K})$ a positive energy representation of $L_Y G$. $\rho$ lifts to a unitary representation of the continuous central extension $\rho^*U(\mathcal{K})$ of $L_Y G$ obtained by pulling back

$$1 \to \mathbb{T} \to U(\mathcal{K}) \to P U(\mathcal{K}) \to 1$$

(6.6)

to $L_2G$. Explicitly,

$$\rho^*U(\mathcal{K}) = \{(\zeta, V) \in L_Y G \times U(\mathcal{K})|\rho(\zeta) = p(V)\}$$

(6.7)

acts on $\mathcal{K}$ by $(\zeta, V)\xi = V\xi$. Let $h$ be the level of $\mathcal{K}$ as a positive energy representation of $LG$. Then, the restriction of $\rho^*U(\mathcal{K})$ to $LG$ is smooth and of level $h$ 3.2.4 so that $\rho^*U(\mathcal{K})$ is a smooth central extension of $L_Y G$ and therefore falls within the classification of section 3. Set now $Y = Z$, $\rho = \pi$ and $\mathcal{K} = \mathcal{H}$. Then, the level $\ell$ and commutator map $\omega$ of $\pi^*U(\mathcal{H})$, which is readily seen to be

$$\omega(\lambda, \mu) = \pi(\zeta_\lambda)\pi(\zeta_\mu)^\ast \pi(\zeta_\mu)^\ast$$

(6.8)
are bound by theorem \[3.2.1\] and therefore satisfy (6.2)–(6.3).

As an \(LG \rtimes \text{Rot}^k(S^1)\)-module, \(\mathcal{H}\) decomposes as

\[
\mathcal{H} = \bigoplus_{\mu} \mathcal{H}(\mu) \tag{6.9}
\]

where \(\mu\) spans the level \(\ell\) alcove \(A_\ell\) of \(G\) and \(\mathcal{H}(\mu)\) is the isotypical summand of \(\mathcal{H}\) corresponding to the irreducible, level \(\ell\) positive energy representation \(\mathcal{H}_\mu\) of \(LG\) with highest weight \(\mu\). Evidently, for any \(\zeta \in L_\Sigma G\), \(\pi(\zeta)\mathcal{H}(\mu) = \mathcal{H}(\zeta \mu)\) where the reference is to the abstract action of \(Z \cong L_\Sigma G/\text{LG}\) on \(A_\ell\) given by propositions \[4.3.1\] and \[4.3.2\] which, by theorem \[4.3.3\], coincides with the geometric one given by proposition \[4.1.4\]. Since \(\mathcal{H}\) is irreducible, (6.9) reduces to

\[
\mathcal{H} = \bigoplus_{\mu \in Z\lambda} \mathcal{H}(\mu) \tag{6.10}
\]

for some \(\lambda \in A_\ell\) and the triple \((\ell, \omega, Z\lambda)\) is an invariant of \(\mathcal{H}\).

To proceed, it will be more convenient to consider unitary representations rather than projective ones. For any subgroup \(Y \subseteq Z\) and central extension \(\hat{L}_Y G\) of \(L_Y G\) with level \(h\) and commutator map \(\kappa\), there is a bijective correspondence between positive energy representations of \(L_Y G\) corresponding to the pair \((h, \kappa)\) and unitary representations of \(\hat{L}_Y G \rtimes \text{Rot}^k(S^1)\) such that \(\text{Rot}^k(S^1)\) acts by non–negative characters only and with finite–dimensional eigenspaces and the central subgroup \(T \subseteq \hat{L}_Y G\) acts as multiplication by the character \(z \rightarrow z\), provided that representations differing by a character of \(\hat{L}_Y G\) are identified \[\text{4}\]. We call these positive energy representations of \(\hat{L}_Y G\) and will work with them from now on.

Fix now \(\ell, \omega\) satisfying (6.2)–(6.3) and denote by \(\hat{L}_\Sigma G\) the central extension of \(L_\Sigma G\) with level \(h\) and commutator map \(\omega\), the existence of which is guaranteed by proposition \[3.3.1\].

Recall moreover that by proposition \[3.4.1\] the action of \(\text{Diff}^k_+ (S^1)\) on \(L_\Sigma G\) lifts to \(\hat{L}_\Sigma G\).

For a given orbit \(Z\lambda \subseteq A_\ell\) with isotropy subgroup \(Y \subseteq Z\), denote by \(\hat{L}_Y G\) and \(\hat{L}_G\) the restrictions of \(\hat{L}_Z G\) to \(L_Y G\) and \(LG\) respectively. Then, by Mackey’s theory \[\text{Ma2} thm 3.11\], and in view of the fact that positive energy representations of \(LG\) are invariant under conjugation by \(L_\Sigma G\), the map

\[
i : \mathcal{K} \longrightarrow \text{ind}_{\hat{L}_\Sigma G \rtimes \text{Rot}^k(S^1)}^{L_\Sigma G \rtimes \text{Rot}^k(S^1)} \mathcal{K} \tag{6.11}
\]

gives a bijection between the irreducible positive energy representations \((\rho, \mathcal{K})\) of \(L_\Sigma G\) the restriction to \(\hat{L}_G\) of which is isotypical of type \(\mathcal{H}_\lambda\), and the irreducible positive energy representations of \(\hat{L}_\Sigma G\) with highest weight orbit \(Z\lambda\) \[\text{4}\]. Moreover, since any character \(\chi\) of \(Y\) extends to \(Z\) and \(\text{ind}(\mathcal{K} \otimes \chi) = \text{ind}(\mathcal{K}) \otimes \chi\), \(\mathcal{K}\) and \(\mathcal{K}'\) differ by a character iff \(i(\mathcal{K})\) and \(i(\mathcal{K}')\) do.

Notice also that \(i(\mathcal{K})\) admits an intertwining action of \(\text{Diff}^k_+(S^1)\) if \(\mathcal{K}\) does. Indeed, let \(R\) be a projective representation of \(\text{Diff}^k_+(S^1)\) on \(\mathcal{K}\) satisfying

\[
R(\phi)\rho(\zeta)R(\phi)^* = \rho(\zeta \circ \phi^{-1}) \tag{6.12}
\]

\[\text{4}\]Since \(LG\) is perfect \[\text{PS prop. 3.4.1}\], such characters factor through the group of components \(Y\) of \(L_\Sigma G\).

\[\text{5}\]The induction functor is well–defined in the present context since \(L_\Sigma G \rtimes \text{Rot}^k(S^1) \subseteq \hat{L}_\Sigma G \rtimes \text{Rot}^k(S^1)\) is of finite index, and satisfies the usual properties of its finite–dimensional counterpart which are necessary to prove Mackey’s theorem. Moreover, an elementary application of the induction–restriction theorem shows that \(T \subseteq \hat{L}_Z G\) acts by the required character on \(i(\mathcal{K})\).
for any $\tilde{\zeta} \in \tilde{L}_Y G$, and lift it to a unitary representation of the corresponding central extension $\text{Diff}^+_k(S^1) = R^*U(K)$ of $\text{Diff}^+_k(S^1)$. Then, by induction–restriction,

$$i(\mathcal{K}) = \text{ind}_{\tilde{L}_Y G \ltimes \text{Rot}^k(S^1)}^{\tilde{L}_Y G \rtimes \text{Diff}^+_k(S^1)} \mathcal{K} \cong \text{ind}_{\tilde{L}_Y G \rtimes \text{Diff}^+_k(S^1)}^{\tilde{L}_Y G \times \text{Rot}^k(S^1)} \mathcal{K}$$

(6.13)

as $\tilde{L}_Y G \rtimes \text{Rot}^k(S^1)$–modules, and $i(\mathcal{K})$ admits an intertwining action of $\text{Diff}^+_k(S^1)$.

The relevant representations of $\tilde{L}_Y G$ are obtained in the following way [Ma, §3.10]. Extend the unitary action $\pi_\lambda$ of $\tilde{L}_G \rtimes \text{Rot}^k(S^1)$ on $H_\lambda$ to a projective one of $\tilde{L}_Y G \rtimes \text{Rot}^k(S^1)$ satisfying, and uniquely determined by

$$\pi_\lambda(\tilde{\zeta})\pi_\lambda(\tilde{\gamma})\pi_\lambda(\tilde{\zeta})^* = \pi_\lambda(\tilde{\gamma}\tilde{\zeta}\tilde{\gamma}^{-1})$$

(6.14)

for any $\tilde{\zeta} \in \tilde{L}_Y G$ and $\tilde{\gamma} \in \tilde{L}_G$. The corresponding central extension $\pi_\lambda^U(H_\lambda)$ of $\tilde{L}_Y G$ determines one of $Y$ by

$$\tilde{Y} = \pi_\lambda^U(H_\lambda)/(\tilde{\gamma}, \pi_\lambda(\tilde{\gamma}))_{\tilde{\gamma} \in \tilde{L}_G}$$

(6.15)

Any irreducible unitary representation $\rho$ of $\tilde{Y}$ such that its central subgroup $T$ acts by

$$z \rightarrow z^{-1} \text{ yields one of } \pi_\lambda^U(H_\lambda), \text{ namely } \pi_\lambda \otimes \rho, \text{ where } T \text{ acts trivially, and therefore one of } \tilde{L}_Y G \cong \pi_\lambda^U(U(H_\lambda))/T \text{ and the representations of } \tilde{L}_Y G \text{ in question are exactly of this form.}$$

Moreover, they admit an intertwining action of $\text{Diff}^+_k(S^1)$. Indeed, if $R$ is the Segal–Sugawara representation of $\text{Diff}^+_k(S^1)$ on $H_\lambda$ intertwining $\tilde{L}_G$ [PS prop. 13.4.2], then, for any $\tilde{\zeta} \in \tilde{L}_Y G$ and $\phi \in \text{Diff}^+_k(S^1)$

$$R(\phi)\pi_\lambda(\tilde{\zeta})R(\phi)^* = \pi_\lambda(\tilde{\zeta} \circ \phi^{-1})$$

(6.16)

projectively, since both sides have the same commutation relations with $\tilde{L}_G$. Thus,

$$(R(\phi) \otimes 1)\pi_\lambda \otimes \rho(\tilde{\zeta})(R(\phi) \otimes 1)^* = \kappa(\phi, \tilde{\zeta})\pi_\lambda \otimes \rho(\tilde{\zeta} \circ \phi^{-1})$$

(6.17)

for some $\kappa(\phi, \tilde{\zeta}) \in \mathbb{T}$ which is multiplicative in each variable. Since $\text{Diff}^+_k(S^1)$ is perfect [K, Eq.], $\kappa \equiv 1$ and $\pi_\lambda \otimes \rho$ admits an intertwining action of $\text{Diff}^+_k(S^1)$.

To determine $\tilde{Y}$, notice that the projective unitaries $\pi_\lambda(\tilde{\zeta}), \tilde{\zeta} \in \tilde{L}_Y G$ defined by (6.14) only depend on the image of $\tilde{\zeta}$ in $L_Y G$ and therefore give rise to a central extension of $L_Y G$. Since this extension has level $\ell$, it differs from $\tilde{L}_Y G$ by the pull–back of an extension of $Y$ which is readily seen to be $\tilde{Y}$. We must now distinguish two cases. If $\tilde{Y}$ splits, which is so if $Y$ is cyclic or, by theorem [Ma], if $Y = Z = Z(\text{Spin}_{4n})$ and $\omega \equiv 1$, the relevant representations of $\tilde{Y}$ correspond to the characters $\chi$ of $Y$ and the irreducible, positive energy representations of $\tilde{L}_Y G$ with highest weight orbit $Z\lambda$ are of the form

$$\text{ind}_{\tilde{L}_Y G \rtimes \text{Rot}^k(S^1)}^{\tilde{L}_Y G \times \text{Rot}^k(S^1)}(\pi_\lambda \otimes \chi)$$

(6.18)

so that their restriction to $L_Y G$ only involves isotypical summands of multiplicity one. If, on the other hand, $\tilde{Y}$ doesn’t split, then $Y = Z = Z(\text{Spin}_{4n}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\ell$ is even and $\omega$ is the pull–back of the non–trivial, skew–symmetric form on $Y$. In this case, the relevant representation is the Heisenberg representation of $\tilde{Y}$ on $K \cong \mathbb{C}^2$ and there exists a unique irreducible, positive energy representation of $L_Y G$ with highest weight orbit $Z\lambda = \{\lambda\}$ the restriction to $L_Y G$ of which is isomorphic to $H_\lambda \otimes \mathbb{C}^2$.

This completes the classification of irreducible positive energy representations of $L_Y G$ and shows that any such $(\pi, H)$ admits an intertwining action $\rho$ of $\text{Diff}^+_k(S^1)$. Let now $\rho_0$ be the Segal–Sugawara representation of $\text{Diff}^+_k(S^1)$ on $H$, where $m$ is some positive integer which
we may take to be a multiple of $k$. To see that $\rho$ coincides with $\rho_0$, notice that for any $\phi \in \text{Diff}_+^m(S^1)$, the operator $\rho(\phi)\rho_0(\phi)^*$ commutes with $LG$. Thus, if
\[
\mathcal{H} = \bigoplus_{\mu \in Z} \mathcal{H}_\mu \otimes \mathbb{C}^m
\]
is the decomposition of $\mathcal{H}$ as an $LG$–module, then $\rho_0(\phi) = \bigoplus_\mu \rho_0^\mu(\phi) \otimes 1$ where $\rho_0^\mu(\phi)$ gives the Segal–Sugawara representation of $\text{Diff}_+^m(S^1)$ on $\mathcal{H}_\mu$ and therefore
\[
\rho(\phi) = \bigoplus_\mu \rho_0^\mu(\phi) \otimes T_\mu(\phi)
\]
for some unitary operators $T_\mu(\phi)$. Since both $\rho$ and the $\rho_0^\mu$ are projective representations of $\text{Diff}_+^m(S^1)$, the same is true of the $T_\mu$ which are therefore trivial since $\text{Diff}_+^m(S^1)$ admits no finite–dimensional projective representations [FS prop. 3.3.2] \[\Box\]

7. Positive energy representations of $L(G/Z)$

We now determine those positive energy representations of $L\hat{Z}$ which factor through $L(G/Z) \cong L\hat{Z}/Z$. 

Lemma 7.1. Let $(\pi, \mathcal{H})$ be an irreducible positive energy representation of $L\hat{Z}$ with highest weight $\lambda$ and consider its unique lift to a unitary representation of $G$. Then, $Z(G)$ acts on $\mathcal{H}$ as multiplication by the character $\chi(\exp(h)) = e^{(\lambda, h)}$.

Proof. For any $z \in Z(G)$ and $\gamma \in LG$ we have $\pi(\gamma)\pi(z)\pi(\gamma)^*\pi(z)^* = \kappa(\gamma, z)$ where $\kappa(\cdot, \cdot) \in T$ is independent of the particular choice of lifts. $\kappa$ is continuous and multiplicative in both variables and therefore defines a continuous map $LG \to \mathbb{Z}(G)$ which, by connectedness of $LG$, is trivial. Thus, by Shur’s lemma, $Z(G)$ as multiplication by a character which is easily computed since if $\Omega \in \mathcal{H}$ is the highest weight vector in $\mathcal{H}$, then $\pi(\exp(h))\Omega = e^{(\lambda, h)}\Omega$ \[\Box\]

Let now $Z \cong \Lambda^\vee_Z/\Lambda^\vee_R$ be a subgroup of $Z(G)$. By using the basic inner product $\langle \cdot, \cdot \rangle$ on it, the dual group $\hat{Z}$ may be identified with $\Lambda_W/(\Lambda^\vee_Z)^*$ where $\Lambda_R \subseteq (\Lambda^\vee_Z)^* \subseteq \Lambda_W$ is the dual lattice of $\Lambda^\vee_Z$.

Lemma 7.2. Let $\mathcal{H}$ be an irreducible positive energy representation of $L\hat{Z}$ of level $\ell$ and highest weight orbit $Z\lambda$. Then, the characters of $Z \subset L\hat{Z}$ corresponding to $\mathcal{H}$ are the classes of $\lambda + \ell\Lambda^\vee_i \mod (\Lambda^\vee_Z)^*$ where $\Lambda^\vee_i$ are the minimal dominant coweights corresponding to $Z$.

Proof. When restricted to $LG$, $\mathcal{H}$ decomposes as a direct sum of positive energy representations of $LG$ the highest weights of which lie on the orbit $Z\lambda$. By lemma 7.1 and proposition [4.1.4] these give rise to the characters $\ell\Lambda^\vee_i + w_i\lambda \mod (\Lambda^\vee_Z)^*$ of $Z$ where $\Lambda^\vee_i$ are the minimal dominant coweights corresponding to $Z$. Since $W$ preserves $\Lambda_R$, and $a$ fortiori $(\Lambda^\vee_Z)^*$–cosets in $\Lambda_W$ however, we get $\ell\Lambda^\vee_i + w_i\lambda = \ell\Lambda^\vee_i + \lambda \mod (\Lambda^\vee_Z)^*$ \[\Box\]

Corollary 7.3. An irreducible positive energy representation $\pi$ of $L\hat{Z}$ factors through $L(G/Z)$ if, and only if its level is a a multiple of the basic level $b_0$ of $G/Z$.

Proof. $\pi$ factors through $L(G/Z)$ iff $Z$ acts by the same character on each of its irreducible $LG$–submodules. By definition, $b_0$ is the smallest integer $\ell$ such that $\ell\langle \cdot, \cdot \rangle$ is integral on $\Lambda^\vee_Z$ and therefore such that $\ell\Lambda^\vee_i \in (\Lambda^\vee_Z)^*$ for any fundamental coweight $\Lambda^\vee_i$ corresponding to $Z$ \[\Box\]

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