EXISTENCE AND ASYMPTOTIC BEHAVIOR OF BOUND STATES FOR A CLASS OF NONAUTONOMOUS
SCHRÖDINGER-POISSON SYSTEM

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Abstract. This paper is concerned with the following Schrödinger-Poisson system
\( (P_\mu) : \begin{cases} 
-\Delta u + u + K(x)\phi u = |u|^{p-1}u + \mu h(x)u, \\
-\Delta \phi = K(x)u^2,
\end{cases} \quad x \in \mathbb{R}^3, \)
where \( p \in (3, 5), K(x) \) and \( h(x) \) are nonnegative functions, and \( \mu \) is a positive parameter. Let \( \mu_1 > 0 \) be an isolated first eigenvalue of the eigenvalue problem
\( -\Delta u + u = \mu h(x)u, \quad u \in H^1(\mathbb{R}^3). \)
As \( 0 < \mu \leq \mu_1 \), we prove that \( (P_\mu) \) has at least one nonnegative bound state with positive energy. As \( \mu > \mu_1 \), there is \( \delta > 0 \) such that for any \( \mu \in (\mu_1, \mu_1 + \delta) \), \( (P_\mu) \) has a nonnegative ground state \( u_{0, \mu} \) with negative energy, and \( u_{0, \mu(n)} \to 0 \) in \( H^1(\mathbb{R}^3) \) as \( \mu(n) \downarrow \mu_1 \). Besides, \( (P_\mu) \) has another nonnegative bound state \( u_{2, \mu} \) with positive energy, and \( u_{2, \mu(n)} \to u_{\mu_1} \) in \( H^1(\mathbb{R}^3) \) as \( \mu(n) \downarrow \mu_1 \), where \( u_{\mu_1} \) is a bound state of \( (P_{\mu_1}) \).

1. Introduction

In this paper, we study a class of Schrödinger-Poisson system with the following version
\( (1) \begin{cases} 
-\Delta u + u + K(x)\phi u = |u|^{p-1}u + \mu h(x)u, \\
-\Delta \phi = K(x)u^2,
\end{cases} \quad x \in \mathbb{R}^3, \)
where \( p \in (3, 5), \mu > 0, K(x) \) and \( h(x) \) are nonnegative functions. System \( (1) \) can be looked on as a non-autonomous version of the system
\( (2) \begin{cases} 
-\Delta u + u + \phi u = f(u), \\
-\Delta \phi = u^2,
\end{cases} \quad x \in \mathbb{R}^3, \)
which has been derived from finding standing waves of the Schrödinger-Poisson system
\( \begin{cases} 
i\psi_t - \Delta \psi + \phi \psi = f(\psi), \\
-\Delta \phi = |\psi|^2,
\end{cases} \quad x \in \mathbb{R}^3. \)

A starting point of studying system \( (1) \) is the following fact. For any \( u \in H^1(\mathbb{R}^3) \) and \( K \in L^\infty(\mathbb{R}^3) \), there is a unique \( \phi_u \in D^{1,2}(\mathbb{R}^3) \) with
\[ \phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)|u(y)|^2}{|x-y|} \, dy \]
such that $-\Delta \phi_a = K(x)u^2$, see e.g. [11, 20]. Inserting this $\phi_a$ into the first equation of the system (1), we get that

$$\tag{3} -\Delta u + u + K(x)\phi_a u = |u|^{p-1}u + \mu h(x)u, \quad u \in H^1(\mathbb{R}^3).$$

Problem (3) can be also looked on as a usual semilinear elliptic equation with an additional nonlocal perturbation $K(x)\phi_a u$. Our aim here is to prove some new phenomenon of (3) due to the presence of the term $K(x)\phi_a u$. Before giving the main results, we state the following assumptions.

- **(A1):** $h(x) \geq 0$, $h(x) \neq 0$ in $\mathbb{R}^3$ and $h(x) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.
- **(A2):** There exist $b > 0$ and $H_0 > 0$ such that $h(x) \geq H_0 e^{-b|x|}$ for all $x \in \mathbb{R}^3$.
- **(A3):** $K(x) \geq 0$ and $K(x) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.
- **(A4):** There exist $a > 0$ and $K_0 > 0$ such that $K(x) \leq K_0 e^{-a|x|}$ for all $x \in \mathbb{R}^3$.

From Lemma 2.1, we know that under the condition (A1), the following eigenvalue problem

$$-\Delta u + u = \mu h(x)u, \quad u \in H^1(\mathbb{R}^3)$$

has a first eigenvalue $\mu_1 > 0$ and $\mu_1$ is simple. Denote

$$F(u) := \int_{\mathbb{R}^3} K(x)\phi_a(x)|u(x)|^2 dx$$

and introduce the energy functional $I_\mu : H^1(\mathbb{R}^3) \to \mathbb{R}$ associated with (3)

$$I_\mu(u) = \frac{1}{2} \|u\|^2 + \frac{1}{4} F(u) - \int_{\mathbb{R}^3} \left( \frac{1}{p+1} |u|^{p+1} + \frac{\mu}{2} h(x)u^2 \right) dx,$$

where $\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx$. From [11] and the Sobolev inequality, $I_\mu$ is well defined and $I_\mu \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$. Moreover, for any $v \in H^1(\mathbb{R}^3)$,

$$\langle I'_\mu(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv + K(x)\phi_a uv - |u|^{p-1}uv + \mu h(x)uv) dx.$$

It is known that there is a one to one correspondence between solutions of (3) and critical points of $I_\mu$ in $H^1(\mathbb{R}^3)$. Note that if $u \in H^1(\mathbb{R}^3)$ is a solution of (3), then $(u, \phi_a)$ is a solution of the system (1). If $u \geq 0$ and $u$ is a solution of (3), then $(u, \phi_a)$ is a nonnegative solution of (1) since $\phi_a$ is always nonnegative. We call $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ a bound state of (3) if $I'_\mu(u) = 0$. At this time $(u, \phi_a)$ is called a bound state of (1). A bound state $u$ is called a ground state of (3) if $I'_\mu(u) = 0$ and $I_\mu(u) \leq I_\mu(w)$ for any bound state $w$. In this case, we call $(u, \phi_a)$ a ground state of (1). The first result is about $\mu$ less than $\mu_1$.

**Theorem 1.1.** Suppose that the assumptions of (A1) - (A4) hold and $0 < b < a < 2$. If $0 < \mu < \mu_1$, then problem (3) has at least one nonnegative bound state.

The second result is about $\mu$ in a small right neighborhood of $\mu_1$.

**Theorem 1.2.** Under the assumptions of (A1) - (A4), if $0 < b < a < 2$, then there exists $\delta > 0$ such that, for any $\mu \in (\mu_1, \mu_1 + \delta)$,

1. problem (3) has at least one nonnegative ground state $u_{0,\mu}$ with $I_\mu(u_{0,\mu}) < 0$. Moreover, $u_{0,\mu^{(n)}} \to 0$ strongly in $H^1(\mathbb{R}^3)$ for any sequence $\mu^{(n)} > \mu_1$ and $\mu^{(n)} \to \mu_1$;

2. problem (3) has another nonnegative bound state $u_{2,\mu}$ with $I_\mu(u_{2,\mu}) > 0$. Moreover, $u_{2,\mu^{(n)}} \to u_1$ strongly in $H^1(\mathbb{R}^3)$ for any sequence $\mu^{(n)} > \mu_1$ and $\mu^{(n)} \to \mu_1$, where $u_{\mu_1}$ satisfies $I'_{\mu_1}(u_{\mu_1}) = 0$ and $I_{\mu_1}(u_{\mu_1}) > 0$. 


The proofs of Theorem 1.1 and Theorem 1.2 are based on critical point theory. There are several difficulties in the road of getting critical points of $I_\mu$ in $H^1(\mathbb{R}^3)$ since we are dealing with the problem in the whole space $\mathbb{R}^3$, the embedding from $H^1(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ ($2 < q < 6$) is not compact, the appearance of a nonlocal term $K(x)\phi_\mu u$ and the non coercive linear part. To explain our strategy, we review some related known results. For the system (2), under various conditions of $f$, there are a lot of papers dealing with the existence and nonexistence of positive solutions $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, see for example [2, 23] and the references therein. The lack of compactness from $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ ($2 < q < 6$) was overcome by restricting the problem in $H^1_0(\mathbb{R}^3)$ which is a subspace of $H^1(\mathbb{R}^3)$ containing only radial functions. The existence of multiple radial solutions and non-radial solutions have been obtained in [2, 13]. See also [6, 15, 16, 17, 18, 19, 24, 29, 30] for some other results related to the system (2).

While for nonautonomous version of Schrödinger-Poisson system, only a few results are known in the literature. Jiang et.al.[21] have studied the following Schrödinger-Poisson system with non constant coefficient

$$\begin{cases}
-\Delta u + \left(1 + \lambda g(x)\right) u + \theta \phi(x) u = |u|^{p-2} u & \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \quad \lim_{|x| \to \infty} \phi(x) = 0,
\end{cases}$$

in which the authors prove the existence of ground state solution and its asymptotic behavior depending on $\theta$ and $\lambda$. The lack of compactness was overcome by suitable assumptions on $g(x)$ and $\lambda$ large enough. The Schrödinger-Poisson system with critical nonlinearity of the form

$$\begin{cases}
-\Delta u + u + \phi u = V(x) |u|^4 u + \mu P(x) |u|^{q-2} u & \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \quad 2 < q < 6, \mu > 0
\end{cases}$$

has been studied by Zhao et al. [31]. Besides some other conditions, Zhao et al. [31] assume that $V(x) \in C(\mathbb{R}^3, \mathbb{R})$, $\lim_{|x| \to \infty} V(x) = V_\infty \in (0, \infty)$ and $V(x) \geq V_\infty$ for $x \in \mathbb{R}^3$ and prove the existence of one positive solution for $4 < q < 6$ and each $\mu > 0$. It is also proven the existence of one positive solution for $q = 4$ and $\mu$ large enough. Cerami et al. [11] study the following type of Schrödinger-Poisson system

$$\begin{cases}
-\Delta u + u + L(x) \phi u = g(x, u) & \text{in } \mathbb{R}^3, \\
-\Delta \phi = L(x) u^2 & \text{in } \mathbb{R}^3.
\end{cases}$$

Besides some other conditions and the assumption $L(x) \in L^2(\mathbb{R}^3)$, they prove the existence and nonexistence of ground state solutions. We emphasize that $L(x) \in L^2(\mathbb{R}^3)$ will imply suitable compactness property of the coupled term $L(x)\phi$. Huang et al. [20] have used this property to prove the existence of multiple solutions of (4) when $g(x, u) = a(x)|u|^{p-2} u + \mu h(x) u$ and $\mu$ stays in a right neighborhood of $\mu_1$. The lack of compactness was overcome by suitable assumptions on the sign changing function $a(x)$. While for (3), none of the aboved mentioned properties can be used. We have to analyze the energy level of the functional $I_\mu$ such that the Palais-Smale ($PS$ for short) condition may hold at suitable interval. Also for (3), another difficulty is to find mountain pass geometry for the functional $I_\mu$ in the case of $\mu \geq \mu_1$. We point out that for the semilinear elliptic equation

$$-\Delta u = a(x)|u|^{p-2} u + \mu k(x) u, \quad \text{in } \mathbb{R}^N,$$

Costa et.al.[14] have proven the mountain pass geometry for the related functional of (5) when $\mu \geq \mu_1$, where $\mu_1$ is the first eigenvalue of $-\Delta u = \mu k(x) u$ in $D^{1,2}(\mathbb{R}^N)$. Costa et. al. have managed to do these with the help of the condition
$\int_{\mathbb{R}^N} a(x) \tilde{\varphi}_1^2 \, dx < 0$, where $\tilde{\varphi}_1$ is a positive eigenfunction corresponding to $\tilde{\mu}_1$. In the present paper, it is not possible to use such kind of condition. We will develop further the techniques in [20] to prove the mountain pass geometry. A third difficulty is present paper, it is not possible to use such kind of condition. We will develop further the techniques in [20] to prove the mountain pass geometry. A third difficulty is

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Using (6) and (7), we obtain that

$$\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3).$$

Clearly $\phi_u(x) \geq 0$ for any $x \in \mathbb{R}^3$. We also have that

$$\phi(u) \in \mathcal{D}^{1,2}(\mathbb{R}^3),$$

one may deduce from the H"older and the Sobolev inequalities that

$$|L_u(v)| \leq C \|u\|_{L^4}^2 \|v\|_{L^6} \leq C \|u\|_{L^4}^2 \|v\|_{D^{1,2}}.$$

Hence, for any $u \in L^1(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta \phi = K(x)u^2$ in $D^{1,2}(\mathbb{R}^3)$. Moreover it holds that

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|} \, dy.$$
Hence on $H^1(\mathbb{R}^3)$, both the functional

$$F(u) = \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2(x)dx$$

and

$$I_\mu(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4} F(u) - \int_{\mathbb{R}^3} \left( \frac{1}{p+1}|u|^{p+1} + \frac{\mu}{2} h(x)u^2 \right) dx$$

are well defined and $C^1$. Moreover, for any $v \in H^1(\mathbb{R}^3)$,

$$\langle I'_\mu(u), v \rangle = \int_{\mathbb{R}^3} \left( \nabla u \nabla v + K(x)\phi_u(x)uv - |u|^{p-1}uv - \mu h(x)uv \right) dx.$$ 

The following Lemma 2.1 is a direct consequence of [28, Lemma 2.13].

**Lemma 2.1.** Assume that the hypothesis (A1) holds. Then the functional $u \in H^1(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} h(x)u^2dx$ is weakly continuous and for each $v \in H^1(\mathbb{R}^3)$, the functional $u \in H^1(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} h(x)uvdx$ is weakly continuous.

Using the spectral theory of compact symmetric operators on Hilbert space, the above lemma implies the existence of a sequence of eigenvalues $(\mu_n)_{n \in \mathbb{N}}$ of

$$-\Delta u + u = \mu h(x)u,$$

in $H^1(\mathbb{R}^3)$ with $\mu_1 < \mu_2 \leq \cdots$ and each eigenvalue being of finite multiplicity. The associated normalized eigenfunctions are denoted by $e_1, e_2, \cdots$ with $\|e_i\| = 1$, $i = 1, 2, \cdots$. Moreover, one has $\mu_1 > 0$ with an eigenfunction $e_1 \in L^2(\mathbb{R}^3)$. In addition, we have the following variational characterization of $\mu_n$:

$$\mu_1 = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \frac{\|u\|^2}{\int_{\mathbb{R}^3} h(x)u^2dx}, \quad \mu_n = \inf_{u \in S_n^{n-1} \setminus \{0\}} \frac{\|u\|^2}{\int_{\mathbb{R}^3} h(x)u^2dx},$$

where $S_n^{n-1} = \{\text{span}\{e_1, e_2, \cdots, e_{n-1}\}\}^\perp$.

Next we analyze the $(PS)$ condition of the functional $I_\mu$ in $H^1(\mathbb{R}^3)$. The following definition is standard.

**Definition 2.2.** For $d \in \mathbb{R}$, the functional $I_\mu$ is said to satisfy $(PS)_d$ condition if for any $(u_n)_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^3)$ with $I_\mu(u_n) \to d$ and $I'_\mu(u_n) \to 0$, the $(u_n)_{n \in \mathbb{N}}$ contains a convergent subsequence in $H^1(\mathbb{R}^3)$. The functional $I_\mu$ is said to satisfy $(PS)$ conditions if $I_\mu$ satisfies $(PS)_d$ condition for any $d \in \mathbb{R}$.

**Lemma 2.3.** Let $(u_n)_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^3)$ be such that $I_\mu(u_n) \to d \in \mathbb{R}$ and $I'_\mu(u_n) \to 0$, then $(u_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$.

**Proof.** For $n$ large enough, we have that

$$d + 1 + o(1)\|u_n\| = I_\mu(u_n) - \frac{1}{4}\langle I'_\mu(u_n), u_n \rangle = \frac{1}{4}\|u_n\|^2 - \frac{\mu}{4}\int_{\mathbb{R}^3} h(x)u^2_n dx + \frac{p - 3}{4(p + 1)} \int_{\mathbb{R}^3} |u_n|^{p+1} dx.$$ 

Note that $\frac{p+1}{p-1} > \frac{3}{2}$ for $p \in (3, 5)$. Then for any $\vartheta > 0$, we obtain from $h \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ that

$$\int_{\mathbb{R}^3} h(x)u^2_n dx \leq \left( \int_{\mathbb{R}^3} |u_n|^{p+1} dx \right)^{\frac{p+1}{p+1}} \left( \int_{\mathbb{R}^3} |h(x)|^{\frac{p+1}{p-1}} dx \right)^{\frac{p-1}{p+1}} \leq \frac{2^\vartheta}{p + 1} \int_{\mathbb{R}^3} |u_n|^{p+1} dx + \frac{p - 1}{p + 1} \vartheta^{-\frac{p+1}{p-1}} \int_{\mathbb{R}^3} |h(x)|^{\frac{p+1}{p-1}} dx.$$
Choosing \( \theta = \frac{p-3}{2p+1} \), we get

\[
d + 1 + o(1)\|u_n\| \geq \frac{1}{4}\|u_n\|^2 - D(p,h)\mu^{\frac{p+1}{p+1}},
\]

where \( D(p,h) = \frac{p-1}{4(p+1)} \left( \frac{p-3}{2} \right)^{\frac{p}{p+1}} \int_{\mathbb{R}^3} |h(x)|^{\frac{p+1}{p+1}} \, dx \).

Hence \((u_n)_{n \in \mathbb{N}}\) is bounded in \( H^1(\mathbb{R}^3) \). \( \square \)

The following lemma is a variant of Brezis-Lieb lemma. One may find the proof in [20].

Lemma 2.4. [20] If a sequence \((u_n)_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^3)\) and \(u_n \rightharpoonup u_0\) weakly in \( H^1(\mathbb{R}^3)\), then

\[
\lim_{n \to \infty} F(u_n) = F(u_0) + \lim_{n \to \infty} F(u_n - u_0).
\]

Lemma 2.5. There is a \( \delta_1 > 0 \) such that for any \( \mu \in [\mu_1, \mu_1 + \delta_1] \), any solution \( u \) of (3) satisfies

\[
I_{\mu}(u) > -\frac{p-1}{2(p+1)} S_{\frac{p+1}{p+1}}^\frac{p+1}{p+1}.
\]

Proof. Since \( u \) is a solution of (3), we get that

\[
I_{\mu}(u) = \frac{1}{2} \left( \|u\|^2 - \mu \int_{\mathbb{R}^3} h(x)u^2 \, dx \right) + \frac{1}{4} F(u) - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \, dx
\]

\[
= \frac{p-1}{2(p+1)} \left( \|u\|^2 - \mu \int_{\mathbb{R}^3} h(x)u^2 \, dx \right) + \frac{p-3}{2(p+1)} F(u).
\]

Noticing that \( \|u\|^2 \geq \mu_1 \int_{\mathbb{R}^3} h(x)u^2 \, dx \) for any \( u \in H^1(\mathbb{R}^3) \), we deduce that for any \( u \neq 0 \),

\[
I_{\mu_1}(u) \geq -\frac{p-3}{4(p+1)} F(u) > 0.
\]

Next, we claim: there is a \( \delta_1 > 0 \) such that for any \( \mu \in [\mu_1, \mu_1 + \delta_1] \), any solution \( u \) of (3) satisfies

\[
I_{\mu}(u) > -\frac{p-1}{2(p+1)} S_{\frac{p+1}{p+1}}^\frac{p+1}{p+1}.
\]

Suppose this claim is not true, then there is a sequence \( \mu^{(n)} \rightarrow \mu_1 \) with \( \mu^{(n)} \rightarrow \mu_1 \) and solutions \( u_{\mu^{(n)}} \) of (3) such that

\[
I_{\mu^{(n)}}(u_{\mu^{(n)}}) \leq -\frac{p-1}{2(p+1)} S_{\frac{p+1}{p+1}}^\frac{p+1}{p+1}.
\]

Note that \( I_{\mu^{(n)}}'(u_{\mu^{(n)}}) = 0 \). Then we deduce that for \( n \) large enough,

\[
I_{\mu^{(n)}}(u_{\mu^{(n)}}) + o(1)\|u_{\mu^{(n)}}\| \geq I_{\mu^{(n)}}(u_{\mu^{(n)})}) - \frac{1}{4} (I_{\mu^{(n)}}'(u_{\mu^{(n)}}), u_{\mu^{(n)}})
\]

\[
\geq \frac{1}{4} \|u_{\mu^{(n)}}\|^2 - D(p,h) \left( \mu^{(n)} \right)^{\frac{p+1}{p+1}}.
\]

This implies that \((u_{\mu^{(n)}})_{n \in \mathbb{N}}\) is bounded in \( H^1(\mathbb{R}^3) \). Since for any \( n \in \mathbb{N}, \|u_{\mu^{(n)}}\|^2 \geq \mu_1 \int_{\mathbb{R}^3} h(x)(u_{\mu^{(n)}})^2 \, dx \), we obtain that as \( \mu^{(n)} \rightarrow \mu_1 \)

\[
\|u_{\mu^{(n)}}\|^2 - \mu^{(n)} \int_{\mathbb{R}^3} h(x)(u_{\mu^{(n)}})^2 \, dx \geq \left( 1 - \frac{\mu^{(n)}}{\mu_1} \right) \|u_{\mu^{(n)}}\|^2 \rightarrow 0
\]
because \((u_{\mu(n)})_{n \in \mathbb{N}}\) is bounded in \(H^1(\mathbb{R}^3)\). Noting that
\[
I_{\mu(n)}(u_{\mu(n)}) = \frac{p-1}{2(p+1)} \left( \|u_{\mu(n)}\|^2 - \mu^{(n)} \int_{\mathbb{R}^3} h(x)(u_{\mu(n)})^2 dx \right)
+ \frac{p-3}{4(p+1)} F(u_{\mu(n)}),
\]
we deduce that
\[
\liminf_{n \to \infty} I_{\mu(n)}(u_{\mu(n)}) \geq \frac{p-3}{4(p+1)} \liminf_{n \to \infty} F(u_{\mu(n)}) \geq 0,
\]
which contradicts to the
\[
I_{\mu(n)}(u_{\mu(n)}) \leq -\frac{p-1}{2(p+1)} \left[\frac{p+1}{p+1}\right].
\]
This proves the claim and the proof of Lemma 2.5 is complete. \(\square\)

**Lemma 2.6.** If \(\mu \in (\mu_1, \mu_1 + \delta_1)\), then \(I_{\mu}\) satisfies \((PS)_d\) condition for any \(d < 0\).

**Proof.** Let \((u_n)_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^3)\) be a \((PS)_d\) sequence of \(I_{\mu}\) with \(d < 0\). Then for \(n\) large enough,
\[
d + o(1) = \frac{1}{2} \|u_n\|^2 - \frac{\mu}{2} \int_{\mathbb{R}^3} h(x)u_n^2 dx + \frac{1}{4} F(u_n) - \frac{1}{p+1} \int_{\mathbb{R}^3} |u_n|^{p+1} dx
\]
and
\[
\langle I'_{\mu}(u_n), w_n \rangle = \|u_n\|^2 - \mu \int_{\mathbb{R}^3} h(x)u_n^2 dx + F(u_n) - \int_{\mathbb{R}^3} |u_n|^{p+1} dx.
\]
Then we can prove that \((u_n)_{n \in \mathbb{N}}\) is bounded in \(H^1(\mathbb{R}^3)\). Without loss of generality, we may assume that \(u_n \rightharpoonup u_0\) weakly in \(H^1(\mathbb{R}^3)\) and \(u_n \to u_0\) a. e. in \(\mathbb{R}^3\). Denoting \(w_n := u_n - u_0\), we obtain from Brezis-Lieb lemma and Lemma 2.4 that for \(n\) large enough,
\[
\|u_n\|^2 = \|u_0\|^2 + \|w_n\|^2 + o(1),
\]
and
\[
F(u_n) = F(u_0) + F(w_n) + o(1)
\]
then using Lemma 2.1, we also have that \(\int_{\mathbb{R}^3} h(x)u_n^2 dx \to \int_{\mathbb{R}^3} h(x)u_0^2 dx\) as \(n \to \infty\). Therefore
\[
d + o(1) = I_{\mu}(u_n) = I_{\mu}(u_0) + \frac{1}{2} \|w_n\|^2
+ \frac{1}{4} F(u_0) - \frac{1}{p+1} \int_{\mathbb{R}^3} |w_n|^{p+1} dx.
\]
Noticing \(\langle I'_{\mu}(u_n), \psi \rangle \to 0\) for any \(\psi \in H^1(\mathbb{R}^3)\), we obtain that \(I'_{\mu}(u_0) = 0\). From which we deduce that
\[
\|u_0\|^2 - \mu \int_{\mathbb{R}^3} h(x)u_0^2 dx + F(u_0) = \int_{\mathbb{R}^3} |u_0|^{p+1} dx.
\]
Since \((u_n)_{n \in \mathbb{N}}\) is bounded in \(H^1(\mathbb{R}^3)\), we obtain from \(I'_{\mu}(u_n) \to 0\) that
\[
o(1) = \|u_n\|^2 - \mu \int_{\mathbb{R}^3} h(x)u_n^2 dx + F(u_n) - \int_{\mathbb{R}^3} |u_n|^{p+1} dx.
\]
Combining this with (15) as well as Lemma 2.1, we obtain that
\[
o(1) = \|w_n\|^2 + F(w_n) - \int_{\mathbb{R}^3} |w_n|^{p+1} dx.
\]
Recalling the definition of $S_{p+1}$, we have that $\|u\|^{2} \geq S_{p+1}\|u\|_{L^{p+1}}^{2}$, for any $u \in H^{1}(\mathbb{R}^{3})$. Now we distinguish two cases:

(i): $\int_{\mathbb{R}^{3}}|w_{n}|^{p+1}dx \not\to 0$ as $n \to \infty$;

(ii): $\int_{\mathbb{R}^{3}}|w_{n}|^{p+1}dx \to 0$ as $n \to \infty$.

Suppose that the case (i) occurs. We may obtain from (16) that
\[
\|w_{n}\|^{2} \geq S_{p+1}(\|w_{n}\|^{2} + F(w_{n}) - o(1))^{\frac{2}{p+1}}.
\]
Hence we get that for $n$ large enough,
\[
\|w_{n}\|^{2} \geq S_{p+1}^{\frac{p+1}{p}} + o(1).
\]
Therefore using (14), (16) and (17), we deduce that for $n$ large enough,
\[
d + o(1) = I_{\mu}(u_{n})
= I_{\mu}(u_{0}) + \frac{1}{2}\|w_{n}\|^{2} + \frac{1}{p+1}\int_{\mathbb{R}^{3}}|w_{n}|^{p+1}dx.
\]
\[
\geq \frac{p-1}{2(p+1)}S_{p+1}^{\frac{p+1}{p}} + \frac{p-1}{2(p+1)}\|w_{n}\|^{2} + \frac{p-3}{4(p+1)}F(w_{n}) > 0,
\]
which contradicts to the condition $d < 0$. This means that the case (i) does not occur. Therefore the case (ii) occurs. Using (16), we deduce that $\|w_{n}\|^{2} \to 0$ as $n \to \infty$. Hence we have proven that $u_{n} \to u_{0}$ strongly in $H^{1}(\mathbb{R}^{3})$.

Next we give a mountain pass geometry for the functional $I_{\mu}$.

**Lemma 2.7.** There exist $\delta_{2} > 0$ with $\delta_{2} \leq \delta_{1}$, $\rho > 0$ and $\alpha > 0$, such that for any $\mu \in [\mu_{1}, \mu_{1} + \delta_{2})$, $I_{\mu}|_{\partial B_{\rho}} \geq \alpha > 0$.

**Proof.** For any $u \in H^{1}(\mathbb{R}^{3})$, there exist $t \in \mathbb{R}$ and $v \in S_{1}^{\perp}$ such that
\[
u = te_{1} + v, \quad \int_{\mathbb{R}^{3}}(\nabla v\nabla e_{1} + ve_{1})dx = 0.
\]
Hence we deduce that
\[
\|u\|^{2} = \|\nabla(te_{1} + v)\|_{L^{2}}^{2} + \|te_{1} + v\|_{L^{2}}^{2} = t^{2} + \|v\|^{2},
\]
\[
\mu_{2}\int_{\mathbb{R}^{3}}h(x)v^{2}dx \leq \|v\|^{2}, \quad \mu_{1}\int_{\mathbb{R}^{3}}h(x)e_{1}^{2}dx = \|e_{1}\|^{2} = 1
\]
and
\[
\mu_{1}\int_{\mathbb{R}^{3}}h(x)e_{1}vdx = \int_{\mathbb{R}^{3}}(\nabla v\nabla e_{1} + ve_{1})dx = 0.
\]
We first consider the case of $q = 4$. We know that
\[ I_{\mu_1}(u) = \frac{1}{2}||u||^2 + \frac{1}{4} F(u) - \frac{\mu_1}{2} \int_{\mathbb{R}^3} h(x)u^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} dx \]
and
\[ F(te_k + v) = \frac{1}{4} \left( 1 - \frac{\mu_1}{\mu_2} \right) \|v\|^2 + \frac{1}{4} F(te_k + v) - \frac{1}{p+1} \int_{\mathbb{R}^3} |t e_k + v|^{p+1} dx \]
\[ \geq \frac{1}{2} \left( 1 - \frac{\mu_1}{\mu_2} \right) \|v\|^2 + \frac{1}{4} F(te_k + v) - \frac{1}{p+1} \int_{\mathbb{R}^3} |t e_k + v|^{p+1} dx \]
\[ \geq \theta_1 \|v\|^2 + \frac{1}{4} F(te_k + v) - C_1 |t|^{p+1} - C_2 \|v\|^{p+1}. \]

Next we estimate the term $F(te_k + v)$. Using the expression of $F(u)$, we have that
\[ F(te_k + v) = \frac{1}{4\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} K(x)K(y)(te_k(y) + v(y))2(te_k(x) + v(x))^2 dy dx. \]

Since
\[ (te_k(y) + v(y))^2 = t^4 (e_k(y))^2 (e_k(x))^2 + (v(y))^2 (v(x))^2 \]
\[ + 2t^2 (e_k(y)e_k(x))^2 v(x) + e_k(x)(e_k(y))^2 v(x) \]
\[ + 2t (e_k(x)v(x)) (v(y))^2 + e_k(y)v(y) (v(x))^2 \]
\[ + t^2 ((e_k(x))^2 (v(y))^2 + 4e_k(y)e_k(x)v(y)v(x) + (e_k(y))^2 (v(x))^2), \]
we know that
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} K(x)K(y) (e_k(y) e_k(x))^2 v(y) + e_k(x)(e_k(y))^2 v(x) \]
\[ \|v\| \leq C \|v\|^2; \]
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} K(x)K(y) (2(e_k(x))^2 (v(y))^2 + 4e_k(y)e_k(x)v(y)v(x)) \]
\[ \|v\| \leq C \|v\|^3; \]
and
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} K(x)K(y) (e_k(x)v(x) (v(y))^2 + e_k(y)v(y) (v(x))^2) \]
\[ \|v\| \leq C \|v\|^3. \]

Hence
\[ I_{\mu_1}(u) \geq \theta_1 \|v\|^2 + \theta_2 |t|^4 - C_1 |t|^{p+1} - C_2 \|v\|^{p+1} \]
\[ - C_3 |t|^3 \|v\| - C_4 |t|^2 \|v\|^2 - C_5 |t| \|v\|^3 + \frac{1}{4} F(v), \]
where $\theta_2 = \frac{1}{4} \int_{\mathbb{R}^d} K(x)\phi_k dx$. Note that
\[ t^2 \|v\|^2 \leq \frac{2}{p+1} |t|^{p+1} + \frac{p-1}{p+1} \|v\|^{\frac{2(p+1)}{p+1}}, \]
\[ |t| \|v\| \leq \frac{1}{p+1} |t|^{p+1} + \frac{p}{p+1} \|v\|^{\frac{3(p+1)}{p+1}}, \]
and for some $q_0$ with $2 < q_0 < 4$, we also have that
\[ |t|^3 \|v\| \leq \frac{1}{q_0} \|v\|^{q_0} + \frac{q_0-1}{q_0} |t|^{\frac{3q_0}{3q_0-1}}. \]
Lemma 2.7.

Let the assumptions of Proposition 3.1. \( \mu \) be a term that we have seen in Lemma 2.7, with the help of the competing between the Poisson term \( \theta \) and the nonlinear term, the 0 is a local minimizer of the functional \( I_\mu \) and \( \tilde{\theta}_5 \) is a critical value of \( I_\mu \).

(26)
\[
I_\mu (u) \geq \theta_3 \|v\|^2 + \theta_4 |t|^4 - \frac{2C_4}{p+1} |t|^{p+1} - \frac{3(4p+1)}{p+1} \|v\|_p^{2p+1} - \frac{C_5}{p+1} |t|^{p+1} - \frac{C}{p+1} |t|^{p+1}.
\]

From \( q_0 > 2 \) and \( \frac{3q_0}{q_0 - 1} > 4 \) (since \( q_0 < 4 \)), we know that there are positive constants \( \theta_3, \theta_4 \) and \( \tilde{\theta}_3, \tilde{\theta}_4 \) such that

(27)
\[
I_\mu (u) \geq \theta_3 \|v\|^2 + \theta_4 |t|^4
\]
provided that \( \|v\| \leq \tilde{\theta}_3 \) and \( |t| \leq \tilde{\theta}_4 \). Hence there are positive constants \( \theta_5 \) and \( \tilde{\theta}_5 \) such that

Set \( \delta := \min\{ \frac{\theta_3}{2} \tilde{\theta}_3, \mu_2 - \mu_1 \} > 0 \) and \( \delta_2 := \min\{ \delta, \delta_1 \} \). Then for any \( \mu \in [\mu_1, \mu_1 + \delta_2] \), we deduce from (27) that

\[
I_\mu (u) = I_\mu (u) + \frac{1}{2} (\mu_1 - \mu) \int h(x) u^2 \, dx \\
\geq \theta_5 \|u\|^4 - \frac{2 \mu_1}{\mu_1} \|u\|^2 \\
= \|u\|^2 \left( \theta_5 \|u\|^2 - \frac{\mu - \mu_1}{2 \mu_1} \right) \\
\geq \|u\|^2 \left( \frac{1}{2} \theta_5 \theta_5^2 - \frac{1}{4} \theta_5 \theta_5^2 \right) = \frac{1}{4} \theta_5 \theta_5^2 \|u\|^2
\]
for \( \frac{1}{4} \theta_5 \theta_5^2 \leq \|u\|^2 \leq \tilde{\theta}_5^2 \). Choosing \( \rho^2 = \frac{1}{2} \theta_5 \theta_5^2 \) and \( \alpha = \frac{1}{4} \theta_5 \theta_5^2 \rho^2 \), we finish the proof of Lemma 2.7. \( \square \)

3. Proof of Theorem 1.1

In this section, our aim is to prove Theorem 1.1. For \( 0 < \mu < \mu_1 \), it is standard to prove that the functional \( I_\mu \) contains mountain pass geometry. For \( \mu = \mu_1 \), as we have seen in Lemma 2.7, with the help of the competing between the Poisson term \( K(x) \phi_n u \) and the nonlinear term, the 0 is a local minimizer of the functional \( I_\mu \) and \( I_\mu \) contains mountain pass geometry. To get a mountain pass type critical point of the functional \( I_\mu \), it suffices to prove the \((PS)_{\delta}\) condition by the mountain pass theorem of [3]. In the following we will focus our attention to the case of \( \mu = \mu_1 \), since the case of \( 0 < \mu < \mu_1 \) is similar.

Proposition 3.1. Let the assumptions (A1)–(A4) hold and \( 0 < b < a < 2 \). Define

\[
d_{\mu_1} = \inf_{\gamma \in \Gamma_1} \sup_{t \in [0,1]} I_{\mu_1} (\gamma(t))
\]
with

\[
\Gamma_1 = \{ \gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, I_{\mu_1} (\gamma(1)) < 0 \}.
\]
Then \( d_{\mu_1} \) is a critical value of \( I_{\mu_1} \).

Before proving Proposition 3.1, we analyze the \((PS)_{d_{\mu_1}}\) condition of \( I_{\mu_1} \). Let \( U(x) \) be the unique positive solution of \(-\Delta u + u = |u|^{p-2} u \) in \( H^1(\mathbb{R}^3) \). We know that for any \( \varepsilon \in (0, 1) \), there is a \( C \equiv C(\varepsilon) > 0 \) such that \( U(x) \leq Ce^{-(1-\varepsilon)|x|} \).
Lemma 3.2. If the assumptions (A1) – (A4) hold and $0 < b < a < 2$, then the $d_{\mu_1}$ defined in Proposition 3.1 satisfies
\[ d(29) \leq \frac{p - 1}{2(p + 1)} S_{p+1}^{\frac{p+1}{p}}. \]

Proof. It suffices to find a path $\gamma(t)$ starting from 0 such that
\[ \sup_{t \in [0,1]} I_{\mu_1}(\gamma(t)) < \frac{p - 1}{2(p + 1)} S_{p+1}^{\frac{p+1}{p}}. \]
Define $U_R(x) = U(x - R\theta)$ with $\theta = (0,0,1)$. Note that for the $U_R$ defined as above, the $I_{\mu_1}(tU_R) \to -\infty$ as $t \to +\infty$ and $I_{\mu_1}(tU_R) \to 0$ as $t \to 0$. We know that there is a unique $T_R > 0$ such that $\frac{d}{dt} I_{\mu_1}(tU_R)|_{t=T_R} = 0$, which is
\[ \|U_R\|^2 - \mu_1 \int h(x)U_R^2 dx + T_R^2 F(U_R) - T_R^{p-1} \int U_R^{p+1} dx = 0. \]
If $T_R \to 0$ as $R \to \infty$, then $\|U_R\|^2 - \mu_1 \int h(x)U_R^2 dx \to 0$ as $R \to \infty$, which is impossible. If $T_R \to \infty$ as $R \to \infty$, then as $R \to \infty$,
\[ \frac{1}{T_R^{p-1}} \left( \|U_R\|^2 - \mu_1 \int h(x)U_R^2 dx \right) + F(U_R) = T_R^{p-3} \int U_R^{p+1} dx \to \infty, \]
which is impossible either. Hence we only need to estimate $I_{\mu_1}(tU_R)$ for $t$ in a finite interval and we may write
\[ I_{\mu_1}(tU_R) \leq g(t) + CF(U_R), \]
where
\[ g(t) = \frac{t^2}{2} \left( \|U_R\|^2 - \mu_1 \int h(x)U_R^2 dx \right) - \frac{|t|^{p+1}}{p + 1} \int U_R^{p+1} dx. \]
Noting that under the assumptions (A1) – (A4), we obtain that for $R$ large enough,
\[ F(U_R) \leq C \left( \int \phi_{U_R}^6 dx \right) \leq C \left( \int \phi_1^6 dx \right) \leq Ce^{-aR} \]
since $0 < a < 2$. We can also prove that
\[ \int h(x)U_R^2 dx = \int h(x + R\theta)U^2(x)dx \geq C \int e^{-b|x + R\theta|} U^2(x)dx \geq Ce^{-bR} \int U^2(x)dx \geq Ce^{-bR}. \]
It is now deduced from (28) and (29) that
\[ \sup_{t \geq 0} I_{\mu_1}(tU_R) \leq \sup_{t \geq 0} g(t) + Ce^{-aR} \]
and
\[ \sup_{t \geq 0} I_{\mu_1}(tU_R) \leq \frac{p - 1}{2(p + 1)} (\|U_R\|^2 - \mu_1 \int h(x)U_R^2 dx) + \frac{p+1}{p} (\|U_R\|_{L^{p+1}}) \]
for $R$ large enough since $0 < b < a$. The proof is complete. \qed
Lemma 3.3. Under the assumptions (A1)–(A4), $I_{\mu_1}$ satisfies $(PS)_d$ condition for any $d < \frac{p-1}{2(p+1)}S^{\frac{p+1}{p}}_{p+1}$.

Proof. Let $(u_n)_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^3)$ be a $(PS)_d$ sequence of $I_{\mu_1}$ with $d < \frac{p-1}{2(p+1)}S^{\frac{p+1}{p}}_{p+1}$. Then we have that for $n$ large enough,

\[ d + o(1) = \frac{1}{2} \Vert u_n \Vert^2 - \frac{\mu_1}{2} \int_{\mathbb{R}^3} h(x)u_n^2 \, dx + \frac{1}{4} F(u_n) - \frac{1}{p+1} \int_{\mathbb{R}^3} |u_n|^{p+1} \, dx \]

and

\[ (I'_{\mu_1}(u_n), u_n) = \Vert u_n \Vert^2 - \mu_1 \int_{\mathbb{R}^3} h(x)u_n^2 \, dx + F(u_n) - \int_{\mathbb{R}^3} |u_n|^{p+1} \, dx. \]

Hence we can deduce that $(u_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence, we may assume that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^3$. Denote $w_n := u_n - u_0$. We then obtain from Brezis-Lieb lemma and Lemma 2.4 that for $n$ large enough,

\[ \|u_n\|^2 = \|u_0\|^2 + \|w_n\|^2 + o(1), \quad F(u_n) = F(u_0) + F(w_n) + o(1) \]

and

\[ \|u_n\|_{L^{p+1}}^{p+1} = \|u_0\|_{L^{p+1}}^{p+1} + \|w_n\|_{L^{p+1}}^{p+1} + o(1). \]

Since $\int_{\mathbb{R}^3} h(x)u_n^2 \, dx \to \int_{\mathbb{R}^3} h(x)u_0^2 \, dx$ as $n \to \infty$, we deduce that

\[ (30) \quad d + o(1) = I_{\mu_1}(u_n) = I_{\mu_1}(u_0) + \frac{1}{2} \Vert w_n \Vert^2 + \frac{1}{p+1} \int_{\mathbb{R}^3} |w_n|^{p+1} \, dx. \]

From $\langle I'_{\mu_1}(u_n), \psi \rangle \to 0$ for any $\psi \in H^1(\mathbb{R}^3)$, one may deduce that $I'_{\mu_1}(u_0) = 0$. Therefore

\[ \|u_0\|^2 - \mu_1 \int_{\mathbb{R}^3} h(x)u_0^2 \, dx + F(u_0) = \int_{\mathbb{R}^3} |u_0|^{p+1} \, dx \]

and then

\[ I_{\mu_1}(u_0) \geq \frac{p-1}{2(p+1)} \left( \|u_0\|^2 - \mu_1 \int_{\mathbb{R}^3} h(x)u_0^2 \, dx \right) + \frac{p-3}{4(p+1)} F(u_0) \geq 0. \]

Now using an argument similar to the proof of (16), we obtain that

\[ (31) \quad o(1) = \|w_n\|^2 + F(w_n) - \int_{\mathbb{R}^3} |w_n|^{p+1} \, dx. \]

By the relation $\|u\|^2 \geq S_{p+1} \|u\|_{L^{p+1}}^2$ for any $u \in H^1(\mathbb{R}^3)$, we proceed our discussion according to the following two cases:

(I) \[ \int_{\mathbb{R}^3} |w_n|^{p+1} \, dx \not\to 0 \text{ as } n \to \infty; \]

(II) \[ \int_{\mathbb{R}^3} |w_n|^{p+1} \, dx \to 0 \text{ as } n \to \infty. \]

Suppose that the case (I) occurs. Then up to a subsequence, we may obtain from (31) that

\[ \|w_n\|^2 \geq S_{p+1} \left( \|w_n\|^2 + F(w_n) - o(1) \right)^{\frac{2}{p+1}}, \]

which implies that for $n$ large enough,

\[ \|w_n\|^2 \geq S_{p+1}^{\frac{p+1}{p}} + o(1). \]
It is deduced from this and (30) that \( d \geq \frac{p-1}{2(p+1)} S_{p+1} \), which is a contradiction. Therefore the case (II) must occur. This and (31) imply that \( \|u_n\| \to 0 \). Hence we have proven that \( I_{\mu_1} \) satisfies \((PS)_d\) condition for any \( d < \frac{p-1}{2(p+1)} S_{p+1} \).

\[\square\]

Proof of Proposition 3.1. Since 0 is a local minimizer of \( I_{\mu_1} \) and for \( v \neq 0 \), \( I_{\mu_1}(sv) \to -\infty \) as \( s \to +\infty \), Lemma 3.2, Lemma 3.3 and the mountain pass theorem [3] imply that \( d_{\mu_1} \) is a critical value of \( I_{\mu_1} \).

Proof of Theorem 1.1. By Proposition 3.1, the \( d_{\mu_1} \) is a critical value of \( I_{\mu_1} \) and \( d_{\mu_1} > 0 \). The proof of nonnegativity for at least one of the corresponding critical point is inspired by the idea of [1]. In fact, since \( I_{\mu_1}(u) = I_{\mu_1}(|u|) \) for any \( u \in H^1(\mathbb{R}^3) \), for every \( n \in \mathbb{N} \), there exists \( \gamma_n \in \Gamma_1 \) with \( \gamma_n(t) \geq 0 \) (a.e. in \( \mathbb{R}^3 \)) for all \( t \in [0,1] \) such that

\[d_{\mu_1} \leq \max_{t \in [0,1]} I_{\mu_1}(\gamma_n(t)) < d_{\mu_1} + \frac{1}{n}.
\]

By Ekeland’s variational principle [5], there exists \( \gamma^*_n \in \Gamma_1 \) satisfying

\[
\begin{cases}
    d_{\mu_1} \leq \max_{t \in [0,1]} I_{\mu_1}(\gamma^*_n(t)) \leq \max_{t \in [0,1]} I_{\mu_1}(\gamma_n(t)) < d_{\mu_1} + \frac{1}{n}; \\
    \max_{t \in [0,1]} \|\gamma_n(t) - \gamma^*_n(t)\| < \frac{1}{n}; \\
    \text{there exists } t_n \in [0,1] \text{ such that } z_n = \gamma^*_n(t_n) \text{ satisfies: } I_{\mu_1}(z_n) = \max_{t \in [0,1]} I_{\mu_1}(\gamma^*_n(t)) \text{ and } \|I'_{\mu_1}(z_n)\| \leq \frac{1}{n}.
\end{cases}
\]

By Lemma 3.2 and Lemma 3.3 we get a convergent subsequence (still denoted by \( (z_n)_{n \in \mathbb{N}} \)). We may assume that \( z_n \to z \) in \( H^1(\mathbb{R}^3) \) as \( n \to \infty \). On the other hand, by (33), we also arrive at \( \gamma_n(t_n) \to z \) in \( H^1(\mathbb{R}^3) \) as \( n \to \infty \). Since \( \gamma_n(t) \geq 0 \), we conclude that \( z \geq 0, z \not\equiv 0 \) in \( \mathbb{R}^3 \) with \( I_{\mu_1}(z) > 0 \) and it is a nonnegative bound state of (3) in the case of \( \mu = \mu_1 \).

4. Ground state and bound states for \( \mu > \mu_1 \)

In this section, we always assume the conditions \((A1)-(A4)\). We will prove the existence of ground state and bound states of (3) as well as their asymptotical behavior with respect to \( \mu \). We emphasize that if \( 0 < \mu < \mu_1 \), then one may consider a minimization problem like

\[\inf \{ I_\mu(u) : u \in \mathcal{M} \}, \quad \mathcal{M} = \{u \in H^1(\mathbb{R}^3) : \langle I'_\mu(u), u \rangle = 0 \}\]

to get a ground state solution. But for \( \mu \geq \mu_1 \), we can not do like this because for \( \mu > \mu_1 \), we do not know if \( 0 \not\in \partial \mathcal{M} \). To overcome this difficulty, we define the set of all nontrivial critical points of \( I_\mu \) in \( H^1(\mathbb{R}^3) \):

\[\mathcal{N} = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : I'_\mu(u) = 0 \}.
\]

And then we consider the following minimization problem

\[c_{0,\mu} = \inf \{ I_\mu(u) : u \in \mathcal{N} \}.
\]

Lemma 4.1. Let \( \delta_2 \) and \( \rho \) be as in Lemma 2.7 and \( \mu \in (\mu_1, \mu_1 + \delta_2) \). Define the following minimization problem

\[d_{0,\mu} = \inf_{\|u\| < \rho} I_\mu(u).
\]

Then the \( d_{0,\mu} \) is achieved by a nonnegative function \( w_{0,\mu} \in H^1(\mathbb{R}^3) \). Moreover this \( w_{0,\mu} \) is a nonnegative solution of (3).
Proof. Firstly, we prove that $-\infty < d_{0,\mu} < 0$ for $\mu \in (\mu_1, \mu_1 + \delta_2)$. Keeping the expression of $I_{\mu}(u)$ in mind, we obtain from the Sobolev inequality that

$$I_{\mu}(u) = \frac{1}{2} \|u\|^2 - \frac{\mu}{2} \int_{\mathbb{R}^3} h(x)u^2 \, dx + \frac{1}{4} F(u) - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \, dx$$

$$\geq \frac{1}{2} \|u\|^2 - \frac{\mu}{2\mu_1} \|u\|^2 - C\|u\|^{p+1} > -\infty$$

as $\|u\| < \rho$. Next, for any $t > 0$, we have that

$$I_{\mu}(te_1) = \frac{t^2}{2} \|e_1\|^2 - \frac{\mu t^2}{2} \int_{\mathbb{R}^3} h(x) e_1^2 \, dx + \frac{t^4}{4} F(e_1) - \frac{t^{p+1}}{p+1} \int_{\mathbb{R}^3} |e_1|^{p+1} \, dx.$$

It is now deduced from $I_{\mu} \int_{\mathbb{R}^3} h(x) e_1^2 \, dx = \|e_1\|^2$ that

$$I_{\mu}(te_1) = \frac{t^2}{2} \left( 1 - \frac{\mu}{\mu_1} \right) \|e_1\|^2 + \frac{t^4}{4} F(e_1) - \frac{t^{p+1}}{p+1} \int_{\mathbb{R}^3} |e_1|^{p+1} \, dx.$$

Since $\mu > \mu_1$, we obtain that for $t$ small enough, the $I_{\mu}(te_1) < 0$. Thus we have proven that $-\infty < d_{0,\mu} < 0$ for $\mu \in (\mu_1, \mu_1 + \delta_2)$.

Secondly, let $(v_n)_{n \in \mathbb{N}}$ be a minimizing sequence, that is, $\|v_n\| < \rho$ and $I_{\mu}(v_n) \to d_{0,\mu}$ as $n \to \infty$. By the Ekeland’s variational principle, we can obtain that there is a sequence $(u_n)_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^3)$ with $\|u_n\| < \rho$ such that as $n \to \infty$,

$$I_{\mu}(u_n) \to d_{0,\mu} \quad \text{and} \quad I'_{\mu}(u_n) \to 0.$$

Then we can prove that $(u_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Using Lemma 2.6, we obtain that $(u_n)_{n \in \mathbb{N}}$ contains a convergent subsequence, still denoted by $(u_n)_{n \in \mathbb{N}}$, such that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$. Noticing the fact that if $(v_n)_{n \in \mathbb{N}}$ is a minimizing sequence, then $(|v_n|)_{n \in \mathbb{N}}$ is also a minimizing sequence, we may assume that for each $n \in \mathbb{N}$, the $u_n \geq 0$ in $\mathbb{R}^3$. Therefore we may assume that $u_0 \geq 0$ in $\mathbb{R}^3$. The $I'_{\mu}(u_n) \to 0$ and $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$ imply that $I'_{\mu}(u_0) = 0$. Hence choosing $w_{0,\mu} \equiv u_0$, we know that $w_{0,\mu}$ is a nonnegative solution of the (3). \qed

We emphasize that the above lemma does NOT mean that $w_{0,\mu}$ is a ground state of (3). But it does imply that $\mathcal{N} \neq \emptyset$ for any $\mu \in (\mu_1, \mu_1 + \delta_2)$. Now we are in a position to prove that the $c_{0,\mu}$ defined in (34) can be achieved.

Lemma 4.2. For $\mu \in (\mu_1, \mu_1 + \delta_2)$, the $c_{0,\mu}$ is achieved by a nontrivial $v_{0,\mu} \in H^1(\mathbb{R}^3)$, which is a nontrivial critical point of $I_{\mu}$ and hence a solution of the (3).

Proof. By Lemma 4.1, we know that $\mathcal{N} \neq \emptyset$ for $\mu \in (\mu_1, \mu_1 + \delta_2)$. Hence we have that $c_{0,\mu} < 0$. Next we prove that $c_{0,\mu} > -\infty$.

For any $u \in \mathcal{N}$, since $I'_{\mu}(u) = 0$, then $(I'_{\mu}(u), u) = 0$. Then we can deduce that

$$I_{\mu}(u) = I_{\mu}(u) - \frac{1}{4}(I'_{\mu}(u), u) \geq \frac{1}{4} \|u\|^2 - D(p, h) \frac{|u|^{p+1}}{p+1}.$$

Therefore the $c_{0,\mu} > -\infty$.

Now let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{N}$ be a sequence such that

$$I_{\mu}(u_n) \to c_{0,\mu} \quad \text{and} \quad I'_{\mu}(u_n) = 0.$$

Since $-\infty < c_{0,\mu} < 0$, we know from Lemma 2.6 that $(u_n)_{n \in \mathbb{N}}$ contains a convergent subsequence in $H^1(\mathbb{R}^3)$ and then we may assume without loss of generality that $u_n \to v_0$ strongly in $H^1(\mathbb{R}^3)$. Therefore we have that $I_{\mu}(v_0) = c_{0,\mu}$ and $I'_{\mu}(v_0) = 0$. Choosing $v_{0,\mu} \equiv v_0$ and we finish the proof of the Lemma 4.2. \qed
Next, to analyze further the \((PS)_d\) condition of the functional \(I_\mu\), we have to prove a relation between the minimizer \(w_{0,\mu}\) obtained in Lemma 4.1 and the minimizer \(v_{0,\mu}\) obtained in Lemma 4.2.

**Lemma 4.3.** There exists \(\delta_3 \in (0, \delta_2]\) such that for any \(\mu \in (\mu_1, \mu_1 + \delta_3)\), the \(v_{0,\mu}\) obtained in Lemma 4.2 can be chosen to coincide the \(w_{0,\mu}\) obtained in Lemma 4.1.

**Proof.** The proof is divided into two steps. In the first place, for \(u \neq 0\) and \(I'_{\mu_1}(u) = 0\), we have that

\[
\|u\|^2 - \mu_1 \int_{\mathbb{R}^3} h(x)u^2\,dx + F(u) = \int_{\mathbb{R}^3} |u|^{p+1}\,dx
\]

and hence

\[
I_{\mu_1}(u) = \frac{p-1}{2(p+1)} \left(\|u\|^2 - \mu_1 \int_{\mathbb{R}^3} h(x)u^2\,dx\right) + \frac{p-3}{4(p+1)} F(u).
\]

Since \(\|u\|^2 \geq \mu_1 \int_{\mathbb{R}^3} h(x)u^2\,dx\) for any \(u \in H^1(\mathbb{R}^3)\), we obtain that

\[
I_{\mu_1}(u) \geq \frac{p-3}{4(p+1)} F(u) > 0.
\]

In the second place, denoted by \(u_{0,\mu}\) a ground state obtained in Lemma 4.2. For any sequence \(\mu^{(n)} > \mu_1\) and \(\mu^{(n)} \rightarrow \mu_1\) as \(n \rightarrow \infty\), we have that \(u_{0,\mu^{(n)}}\) satisfies

\[
I'_{\mu^{(n)}}(u_{0,\mu^{(n)}}) = 0
\]

and we also have that

\[
c_{0,\mu^{(n)}} = I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) < 0.
\]

Hence we deduce that \((u_{0,\mu^{(n)}})_{n \in \mathbb{N}}\) is bounded in \(H^1(\mathbb{R}^3)\). Since \(I'_{\mu^{(n)}}(u_{0,\mu^{(n)}}) = 0\), one also has that

\[
I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) = \frac{p-1}{2(p+1)} \left(\|u_{0,\mu^{(n)}}\|^2 - \mu^{(n)} \int_{\mathbb{R}^3} h(x)(u_{0,\mu^{(n)}})^2\,dx\right)
\]

\[
+ \frac{p-3}{4(p+1)} F(u_{0,\mu^{(n)}}).
\]

Using the definition of \(\mu_1\), we obtain that, as \(n \rightarrow \infty\),

\[
\|u_{0,\mu^{(n)}}\|^2 - \mu^{(n)} \int_{\mathbb{R}^3} h(x)(u_{0,\mu^{(n)}})^2\,dx \geq \left(1 - \frac{\mu^{(n)}}{\mu_1}\right) \|u_{0,\mu^{(n)}}\|^2 \rightarrow 0
\]

because \((u_{0,\mu^{(n)}})_{n \in \mathbb{N}}\) is bounded in \(H^1(\mathbb{R}^3)\). Next since \((u_{0,\mu^{(n)}})_{n \in \mathbb{N}}\) is bounded in \(H^1(\mathbb{R}^3)\), we may assume without loss of generality that \(u_{0,\mu^{(n)}} \rightharpoonup \tilde{u}_0\) weakly in \(H^1(\mathbb{R}^3)\).

**Claim.** As \(n \rightarrow \infty\), the \(u_{0,\mu^{(n)}} \rightarrow \tilde{u}_0\) strongly in \(H^1(\mathbb{R}^3)\) and \(\tilde{u}_0 = 0\).

**Proof of the Claim.** From \(u_{0,\mu^{(n)}} \rightharpoonup \tilde{u}_0\) weakly in \(H^1(\mathbb{R}^3)\), we may assume that \(u_{0,\mu^{(n)}} \rightarrow \tilde{u}_0\) a. e. in \(\mathbb{R}^3\). Using these and the fact of \(I'_{\mu^{(n)}}(u_{0,\mu^{(n)}}) = 0\), we deduce that \(I'_{\mu_1}(\tilde{u}_0) = 0\). Then similar to the proof in Lemma 2.6, we obtain that

\[
o(1) + I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) = I_{\mu^{(n)}}(\tilde{u}_0) + \frac{1}{2} ||\tilde{w}_n||^2
\]

\[
+ \frac{1}{4} F(\tilde{w}_n) - \frac{1}{p+1} \int_{\mathbb{R}^3} |\tilde{w}_n|^{p+1}\,dx,
\]

where \(\tilde{w}_n := u_{0,\mu^{(n)}} - \tilde{u}_0\).

Now we distinguish two cases:
(i): \[ \int_{\mathbb{R}^3} |\tilde{w}_n|^{\nu+1} \, dx \not\to 0 \text{ as } n \to \infty; \]
(ii): \[ \int_{\mathbb{R}^3} |\tilde{u}_n|^{\mu+1} \, dx \to 0 \text{ as } n \to \infty. \]

Suppose that the case (i) occurs. We may deduce from a proof similar to Lemma 2.6 that
\[ I_{\mu(n)}(u_{0,\mu(n)}) + o(1) \geq I_{\mu}(\tilde{u}_0) + \frac{p-1}{2(p+1)} S^{\frac{p+1}{p}}_{p+1}, \]
which is a contradiction because \( I_{\mu}(\tilde{u}_0) > -\frac{p-1}{2(p+1)} S^{\frac{p+1}{p}}_{p+1} \) by Lemma 2.5 and the fact of \( I_{\mu(n)}(u_{0,\mu(n)}) < 0 \). Therefore the case (ii) occurs, which implies that \( u_{0,\mu(n)} \to \tilde{u}_0 \) strongly in \( H^1(\mathbb{R}^3) \) (the proof is similar to those in Lemma 2.6). From this we also deduce that \( F(\tilde{w}_n) \to F(\tilde{u}_0) \).

Next we prove that \( \tilde{u}_0 = 0 \). Arguing by a contradiction, if \( \tilde{u}_0 \neq 0 \), then we know from \( I'_{\mu(n)}(u_{0,\mu}) = 0 \) that
\[ \liminf_{n \to \infty} I_{\mu(n)}(u_{0,\mu(n)}) \geq \frac{p-3}{4(p+1)} F(\tilde{u}_0) > 0, \]
which is also a contradiction since \( I_{\mu(n)}(u_{0,\mu(n)}) < 0 \). Therefore \( \tilde{u}_0 = 0 \).

Hence there is \( \delta_3 \in (0, \delta_2] \) such that for any \( \mu \in (\mu_1, \mu_1 + \delta_3) \), \( \|u_{0,\mu}\| < \rho \), which implies that \( c_{0,\mu} = d_{0,\mu} \). Using Lemma 4.1, we can get a nonnegative ground state of (3), called \( w_{0,\mu} \) and \( c_{0,\mu} = d_{0,\mu} = I_{\mu}(w_{0,\mu}) \). The proof is complete. \( \square \)

**Remark 4.4.** The proof of Lemma 4.3 implies that (1) of Theorem 1.2 holds.

In the following, we are going to prove the existence of another nonnegative bound state solution of (3). To obtain this goal, we have to analyze further the \((PS)_d\) condition of the functional \( I_{\mu} \).

**Lemma 4.5.** Under the assumptions of (A1) – (A4), if \( \mu \in (\mu_1, \mu_1 + \delta_3) \), then \( I_{\mu} \) satisfies \((PS)_d\) condition for any \( d < c_{0,\mu} + \frac{p-1}{2(p+1)} S^{\frac{p+1}{p}}_{p+1} \).

**Proof.** Let \( (u_n)_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^3) \) be a \((PS)_d\) sequence of \( I_{\mu} \) with \( d < c_{0,\mu} + \frac{p-1}{2(p+1)} S^{\frac{p+1}{p}}_{p+1} \). Then we have that for \( n \) large enough,
\[ d + o(1) = \frac{1}{2} \|u_n\|^2 - \frac{\mu}{2} \int_{\mathbb{R}^3} h(x) u_n^2 \, dx + \frac{1}{4} F(u_n) - \frac{1}{p+1} \int_{\mathbb{R}^3} |u_n|^{p+1} \, dx \]
and
\[ \langle I'_{\mu}(u_n), u_n \rangle = \|u_n\|^2 - \mu \int_{\mathbb{R}^3} h(x) u_n^2 \, dx + F(u_n) - \int_{\mathbb{R}^3} |u_n|^{p+1} \, dx. \]

Similar to the proof in Lemma 2.3, we can deduce that \( (u_n)_{n \in \mathbb{N}} \) is bounded in \( H^1(\mathbb{R}^3) \). Going if necessary to a subsequence, we may assume that \( u_n \to u_0 \) weakly in \( H^1(\mathbb{R}^3) \) and \( u_n \to u_0 \) a. e. in \( \mathbb{R}^3 \). Denote \( w_n := u_n - u_0 \). We then obtain from Brezis-Lieb lemma and Lemma 2.4 that for \( n \) large enough,
\[ \|u_n\|^2 = \|u_0\|^2 + \|w_n\|^2 + o(1), \]
\[ F(u_n) = F(u_0) + F(w_n) + o(1) \]
and
\[ \|u_n\|^{p+1} = \|u_0\|^{p+1} + \|w_n\|^{p+1} + o(1). \]
Using Lemma 2.1, we also have that \( \int_{\mathbb{R}^3} h(x)u_n^2 dx \to \int_{\mathbb{R}^3} h(x)u_0^2 dx \) as \( n \to \infty \). Therefore we deduce that

\[
(36) \quad d + o(1) = I_\mu(u_n) = I_\mu(u_0) + \frac{1}{2} \|w_n\|^2 + \frac{1}{4} F(w_n) - \frac{1}{p+1} \int_{\mathbb{R}^3} |w_n|^{p+1} dx.
\]

Since \( (I_\mu(u_n), \psi) \to 0 \) for any \( \psi \in H^1(\mathbb{R}^3) \), we know that \( I_\mu'(u_0) = 0 \). Moreover we have that

\[ I_\mu(u_0) \geq c_{0,\mu} \]

and

\[ \|u_0\|^2 - \mu \int_{\mathbb{R}^3} h(x)u_0^2 dx + \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 = \int_{\mathbb{R}^3} |u_0|^{p+1} dx. \]

Note that \( (u_n)_{n \in \mathbb{N}} \) is bounded in \( H^1(\mathbb{R}^3) \). The Brezis-Lieb lemma, Lemma 2.4 and

\[ o(1) = \|u_n\|^2 - \mu \int_{\mathbb{R}^3} h(x)u_n^2 dx + F(u_n) - \int_{\mathbb{R}^3} |u_n|^{p+1} dx \]

imply that

\[
(37) \quad o(1) = \|w_n\|^2 + F(w_n) - \int_{\mathbb{R}^3} |w_n|^{p+1} dx.
\]

Using \( \|u\|^2 \geq S_{p+1} \|u\|_{L_{p+1}}^2 \) for any \( u \in H^1(\mathbb{R}^3) \), we distinguish two cases:

(I): \( \int_{\mathbb{R}^2} |w_n|^{p+1} dx \neq 0 \) as \( n \to \infty \);

(II): \( \int_{\mathbb{R}^3} |w_n|^{p+1} dx \to 0 \) as \( n \to \infty \).

Suppose (I) occurs. Up to a subsequence, we may obtain from (37) that

\[ \|w_n\|^2 \geq S_{p+1} \left( \|w_n\|^2 + F(w_n) - o(1) \right)^{\frac{2}{p+1}}. \]

Hence we get that for \( n \) large enough,

\[
(38) \quad \|w_n\|^2 \geq S_{p+1}^{\frac{p+1}{p+1}} + o(1).
\]

Therefore using (36) and (38), we deduce that for \( n \) large enough,

\[
(39) \quad d + o(1) = I_\mu(u_n) = I_\mu(u_0) + \frac{1}{2} \|w_n\|^2 + \frac{1}{4} F(w_n) - \frac{1}{p+1} \int_{\mathbb{R}^3} |w_n|^{p+1} dx
\]

\[ = I_\mu(u_0) + \frac{p-1}{2(p+1)} \|w_n\|^2 + \frac{p-3}{4(p+1)} F(w_n) \]

\[ \geq c_{0,\mu} + \frac{p-1}{2(p+1)} \|w_n\|^2 + \frac{p-3}{4(p+1)} F(w_n) \]

\[ > c_{0,\mu} + \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p+1}}, \]

which contradicts to the assumption \( d < c_{0,\mu} + \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p+1}} \). Therefore the case (II) must occur, i.e., \( \int_{\mathbb{R}^3} |u_n|^{p+1} dx \to 0 \) as \( n \to \infty \). This and (37) imply that \( \|w_n\| \to 0 \). Hence we have proven that \( I_\mu \) satisfies \((PS)_d\) condition for any \( d < c_{0,\mu} + \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p+1}} \). \( \square \)
Lemma 4.6. Suppose that the conditions \( \mu \) hold and \( 0 < b < a < 1 \). If 
\[
d_{2,\mu} = \inf_{\gamma \in \Gamma_2} \sup_{t \in [0,1]} I_{\mu}(\gamma(t))
\]
with 
\[
\Gamma_2 = \{ \gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = w_{0,\mu}, I_{\mu}(\gamma(1)) < c_{0,\mu} \}.
\]

Lemma 4.6. Suppose that the conditions (A1) – (A4) hold and \( 0 < b < a < 1 \). If 
\[
d_{2,\mu} < c_{0,\mu} + \frac{p-1}{2(p+1)} S_{p+1}^{1/2}. \]

Proof. It suffices to find a path starting from \( w_{0,\mu} \) and the maximum of the energy 
functional over this path is strictly less than \( c_{0,\mu} + \frac{p-1}{2(p+1)} S_{p+1}^{1/2} \). To simplify the 
notation, we denote \( w_0 := w_{0,\mu} \), which corresponds to the critical value \( c_{0,\mu} \). We 
will prove that there is a \( T_0 \) such that the path \( \gamma(t) = w_0 + t T_0 U_R \) is what we 
need, here \( U_R(x) = U(x - R\theta) \) is defined as before. Similar to the discussion in the 
proof of Lemma 3.2, we only need to estimate \( I_{\mu}(w_0 + t U_R) \) for positive \( t \) in a finite 
time interval. By direct calculation, we have that 
\[
I_{\mu}(w_0 + t U_R) = \frac{1}{2} \left( \|w_0 + t U_R\|^2 - \mu \int_{\mathbb{R}^3} h(x)|w_0 + t U_R|^2 dx \right) 
+ \frac{1}{4} F(w_0 + t U_R) - \frac{1}{p+1} \int_{\mathbb{R}^3} |w_0 + t U_R|^{p+1} dx 
= I_{\mu}(w_0) + A_1 + A_2 + A_3 + \frac{t^2}{2} \int_{\mathbb{R}^3} h(x) U_R^2 dx,
\]
where 
\[
A_1 = \langle w_0, t U_R \rangle - \mu t \int_{\mathbb{R}^3} h(x) w_0 U_R dx,
\]
\[
A_2 = \frac{1}{4} \left( F(w_0 + t U_R) - F(w_0) \right)
\]
and 
\[
A_3 = \frac{1}{p+1} \int_{\mathbb{R}^3} (|w_0|^{p+1} - |w_0 + t U_R|^{p+1}) dx.
\]
Since \( w_0 \) is a solution of (3), we have that 
\[
A_1 = \int_{\mathbb{R}^3} (w_0)^p t U_R dx - \int_{\mathbb{R}^3} K(x) \phi_{w_0} w_0 t U_R dx.
\]
From an elementary inequality: 
\[
(a + b)^q - a^q \geq b^q + qa^{q-1}b, \quad q > 1, \quad a > 0, b > 0,
\]
we deduce that 
\[
|A_3| \leq \frac{1}{p+1} \int_{\mathbb{R}^3} |t U_R|^{p+1} dx - \int_{\mathbb{R}^3} |w_0|^p t U_R dx.
\]
For the estimate of \( A_2 \), using the expression of \( F(u) = \int_{\mathbb{R}^3} K(x) \phi_u u^2 dx \) and the 
symmetry property of the integral with respect to \( x \) and \( y \), we can obtain that 
\[
|A_2| \leq t \int_{\mathbb{R}^3} K(x) \phi_{w_0} w_0 U_R dx + \frac{t^2}{2} \int_{\mathbb{R}^3} K(x) \phi_{w_0} (U_R)^2 dx 
+ \frac{t^4}{4} \int_{\mathbb{R}^3} K(x) \phi_{U_R} (U_R)^2 dx + t^3 \int_{\mathbb{R}^3} K(x) \phi_{U_R} w_0 U_R dx 
+ t^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{K(x) K(y) w_0(x) w_0(y) U_R(x) U_R(y)}{|x - y|} dx dy.
\]
Proposition 4.7. Under the conditions (A1)-(A4), if for $R$ large enough since $0 < a < 1$.

Similarly we can deduce that for $R$ large enough,

$$\int_{\mathbb{R}^3} K(x)\phi_{w_0} U_R dx \leq Ce^{-aR}, \quad \int_{\mathbb{R}^3} K(x)\phi_{w_0}(U_R)^2 dx \leq Ce^{-aR},$$

$$\int_{\mathbb{R}^3} K(x)\phi_{U_R}(U_R)^2 dx \leq Ce^{-aR} \quad \text{and} \quad \int_{\mathbb{R}^3} K(x)\phi_{U_R} w_0 U_R dx \leq Ce^{-aR}.$$  

Since $\int_{\mathbb{R}^3} h(x)(U_R)^2 dx \geq Ce^{-bR}$ for $R$ large enough, we obtain that

$$I_{\mu}(w_0 + tU_R) \leq I_{\mu}(w_0) + \frac{t^2}{2}\|U_R\|^2 dx - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)U_R^2 dx$$

$$- \frac{1}{p + 1}\int_{\mathbb{R}^3} |U_R|^p dx + Ce^{-aR}$$

$$\leq I_{\mu}(w_0) + \frac{p - 1}{2(p + 1)}S_{p+1}^{\frac{p+1}{2}} + Ce^{-aR} - Ce^{-bR} + o(e^{-bR})$$

$$< c_{0,\mu} + \frac{p - 1}{2(p + 1)}S_{p+1}^{\frac{p+1}{2}}$$

for $R$ large enough since $0 < b < a < 1$. The proof is complete. \hfill \Box

**Proposition 4.7.** Under the conditions (A1)-(A4), if $\mu \in (\mu_1, \mu_2 + \delta_4)$ and $w_{0,\mu}$ be the minimizer obtained in Lemma 4.3, then the $d_{2,\mu}$ is a critical value of $I_{\mu}$.

**Proof.** Since for $\mu \in (\mu_1, \mu_2 + \delta_3)$, we know from Lemma 4.1 and Lemma 4.3 that the $w_{0,\mu}$ is a local minimizer of $I_{\mu}$. Moreover, one has that $I_{\mu}(w_{0,\mu} + sU_R) \to -\infty$ as $s \to +\infty$. Therefore Lemma 4.5, Lemma 4.7 and the mountain pass theorem of [3] imply that $d_{2,\mu}$ is a critical value of $I_{\mu}$. \hfill \Box

**Proof of Theorem 1.2.** The conclusion (1) of Theorem 1.2 follows from Lemma 4.3 and Remark 4.4. It remains to prove (2) of Theorem 1.2. By Proposition 4.7, the $d_{2,\mu}$ is a critical value of $I_{\mu}$ and $d_{2,\mu} > 0$. The proof of nonnegativity for at least one of the corresponding critical point is inspired by the idea of [1]. In fact, since $I_{\mu}(u) = I_{\mu}(|u|)$ for any $u \in H^1(\mathbb{R}^3)$, for every $n \in \mathbb{N}$, there exists $\gamma_n \in \Gamma_2$ with $\gamma_n(t) \geq 0$ (a.e. in $\mathbb{R}^3$) for all $t \in [0, 1]$ such that

$$d_{2,\mu} \leq \max_{t \in [0, 1]} I_{\mu}(\gamma_n(t)) < d_{2,\mu} + \frac{1}{n}. \quad (40)$$
By Ekeland’s variational principle, there exists $\gamma_n^* \in \Gamma_2$ satisfying
\begin{equation}
\begin{aligned}
d_{2, \mu} &\leq \max_{t \in [0, 1]} I_\mu(\gamma_n^*(t)) \leq \max_{t \in [0, 1]} I_\mu(\gamma_n(t)) < d_{2, \mu} + \frac{1}{n}; \\
\max_{t \in [0, 1]} \|\gamma_n(t) - \gamma_n^*(t)\| &< \frac{1}{\sqrt{n}};
\end{aligned}
\end{equation}
there exists $t_n \in [0, 1]$ such that $z_n := \gamma_n^*(t_n)$ satisfies:
\end{array}
\right.
\end{equation}

By Lemma 4.6 we get a convergent subsequence (still denoted by $(z_n)_{n \in \mathbb{N}}$). We may assume that $z_n \to z$ strongly in $H^1(\mathbb{R}^3)$ as $n \to \infty$. On the other hand, by (41), we also arrive at $\gamma_n(t_n) \to z$ strongly in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Since $\gamma_n(t) \geq 0$, we conclude that $z \geq 0, z \not\equiv 0$ in $\mathbb{R}^3$ with $I_\mu(z) > 0$ and it is a nonnegative solution of problem (3).

Next, let $u_{2, \mu}$ be the nonnegative solution given by the above proof, that is, $I_\mu'(u_{2, \mu}) = 0$ and $I_\mu(u_{2, \mu}) = d_{2, \mu}$. We claim that for any sequence $\mu^{(n)} > \mu_1$ and $\mu^{(n)} \to \mu_1$, there exist a sequence of solution $u_{2, \mu^{(n)}}$ of (3) with $\mu = \mu^{(n)}$ and a $u_{\mu_1}$ with $I_\mu'(u_{\mu_1}) = 0$ such that $u_{2, \mu^{(n)}} \to u_{\mu_1}$ strongly in $H^1(\mathbb{R}^3)$. In fact, denoted by $u_{0, \mu^{(n)}}$ the minimizer corresponding to $d_{0, \mu^{(n)}}$, according to the definition of $d_{2, \mu}$ and the proof of Lemma 4.6, we deduce that for $n$ large enough,
\[0 < \alpha \leq d_{2, \mu^{(n)}} \leq \max_{s > 0} I_\mu(\mu^{(n)}(w_{0, \mu^{(n)}}) + sU_R)
\]
and
\[I_\mu(u_{0, \mu^{(n)}}) + sU_R) \leq \frac{p-1}{2(p+1)} S^{\frac{p+1}{p}} + C e^{-\alpha R} - C e^{-bR} + o(e^{-bR}),
\]
\[\lim_{n \to \infty} d_{2, \mu^{(n)}} \leq \frac{p-1}{2(p+1)} S^{\frac{p+1}{p}}.
\]

Next, similar to the proof in Lemma 2.3, we can deduce that $(u_{2, \mu^{(n)}})_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence, we may assume that $u_{2, \mu^{(n)}} \to \tilde{u}_2$ weakly in $H^1(\mathbb{R}^3)$ and $u_{2, \mu^{(n)}} \to \tilde{u}_2$ a. e. in $\mathbb{R}^3$. Then we have that $I_\mu'(\tilde{u}_2) = 0$. Moreover $I_\mu(\tilde{u}_2) \geq 0$. If $(u_{2, \mu^{(n)}})_{n \in \mathbb{N}}$ does not converge strongly to $\tilde{u}_2$ in $H^1(\mathbb{R}^3)$, then using an argument similar to the proof of Lemma 4.5, we may deduce that
\[I_\mu(u_{2, \mu^{(n)}}) \geq I_\mu(\tilde{u}_2) + \frac{p-1}{2(p+1)} S^{\frac{p+1}{p}},
\]
which contradicts to (42). Hence $u_{2, \mu^{(n)}} \to \tilde{u}_2$ strongly in $H^1(\mathbb{R}^3)$ and hence $I_\mu(\tilde{u}_2) > 0$. The proof is complete by choosing $u_{\mu_1} = \tilde{u}_2$.

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