NEW APPROACH TO $q$-EULER, GENOCCHI NUMBERS AND THEIR INTERPOLATION FUNCTIONS

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ABSTRACT. In [1], Cangul-Ozden-Simsek constructed a $q$-Genocchi numbers of higher order and gave Witt’s formula of these numbers by using a $p$-adic fermionic integral on $\mathbb{Z}_p$. In this paper, we give another constructions of a $q$-Euler and Genocchi numbers of higher order, which are different than their $q$-Genocchi and Euler numbers of higher order. By using our $q$-Euler and Genocchi numbers of higher order, we can investigate the interesting relationship between $q$-$w$-Euler numbers and $q$-$w$-Genocchi numbers. Finally, we give the interpolation functions of these numbers.

§1. Introduction/ Preliminaries

In [1], Cangul-Ozden-Simsek presented a systematic study of some families of the $q$-Genocchi numbers of higher order. By applying the $q$-Volkenborn integral in the sense of fermionic, they also constructed the $q$-extension of Euler zeta function, which interpolates these numbers at negative integers. Some properties of Genocchi numbers of higher order were treated in [1, 11, 12, 13, 14, 15, 16, 17]. So, we will construct the appropriate the $q$-Genocchi numbers of higher order to be related $q$-Euler numbers of higher order for doing study the $q$-extension of Genocchi numbers of higher order in this paper. Let $p$ be a fixed odd prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$.

The $q$-basic natural numbers are defined by $[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$ for $n \in \mathbb{N}$, and the binomial coefficient is defined as

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$  

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The binomial formulas are well known that
\[(1 - b)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i b^i, \text{ and } \frac{1}{(1 - b)^n} = \sum_{i=0}^{n} \binom{n + i - 1}{i} b^i, \text{ (see [1-20]).}\]

In this paper, we use the notation
\[\lfloor x \rfloor_q = \frac{1 - q^x}{1 - q}, \text{ and } \lfloor x \rfloor_{-q} = \frac{1 - (-q)^x}{1 + q}, \text{ (see [1-22]).}\]

We say that \(f\) is uniformly differentiable function at a point \(a \in \mathbb{Z}_p\), and write \(f \in UD(\mathbb{Z}_p)\), if the difference quotient \(F_f(x, y) = \frac{f(x) - f(y)}{x - y}\) have a limit \(f'(a)\) as \((x, y) \to (a, a)\).

For \(f \in UD(\mathbb{Z}_p)\), \(q \in \mathbb{C}_p\) with \(|1 - q|_p < 1\), an invariant \(p\)-adic \(q\)-integral is defined as
\[I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1 + q}{1 + q^{pN}} \sum_{x=0}^{pN-1} f(x)(-q)^x, \text{ see [9].}\]

Thus, we have the following integral relation:
\[qI_{-q}(f_1) + I_{-q}(f) = (1 + q)f(0), \text{ where } f_1(x) = f(x + 1).\]

The fermionic \(p\)-adic invariant integral on \(\mathbb{Z}_p\) is defined as
\[I_{-1}(f) = \lim_{q \to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x).\]

For \(n \in \mathbb{N}\), let \(f - n(x) = f(x + n)\). From (1), we can derive
\[q^nI_{-q}(f_n) = (-1)^nI_{-q}(f) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l).\]

It is known that the \(w\)-Euler polynomials are defined as
\[2e^{xt} \frac{e^{2e^{xt}}}{we^t + 1} = \sum_{n=0}^{\infty} E_{n,w}(x) \frac{t^n}{n!}, \text{ (see [20, 21, 22]).}\]

Note that \(E_{n,w}(0) = E_{n,w}\) are called the \(w\)-Euler numbers. \(w\)-Genocchi polynomials are defined as
\[2t \frac{e^{2t} e^{xt}}{we^t + 1} = \sum_{n=0}^{\infty} G_{n,w}(x) \frac{t^n}{n!}, \text{ (see [1]).}\]
In the special case $x = 0$, $G_{n,w}(0) = G_{n,w}$ are called $w$-Genocchi numbers. In [1], Cangul-Ozden-Simsek have studied $w$-Genocchi numbers of order $r$ as follows. For $w \in \mathbb{C}_p$ with $|1 - w|_p < 1$, the $w$-Genocchi numbers of order $r$ are given by

$$
\left( \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{t(x_1 + \cdots + x_r)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \right)^r = 2^r \left( \frac{t}{we^t + 1} \right)^r = \sum_{n=0}^{\infty} G_{n,w}^{(r)} \frac{t^n}{n!},
$$

They also considered the $q$-extension of (3) as follows.

$$
\left( \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q \sum_{v=0}^{r} (q^h - v)x_v e^{t(x_1 + \cdots + x_r)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \right)^r = 2^r \frac{t^r}{(q^h e^t + 1)(q^{h-1} e^t + 1) \cdots (q^{h-r+1} e^t + 1)} = \sum_{n=0}^{\infty} G_{n,q}^{(h,r)} \frac{t^n}{n!}, \text{ (see [1])}.
$$

The purpose of this paper is to give another construction of $q$-Euler numbers and $q$-Genocchi numbers of higher order, which are different than a $q$-Genocchi numbers of Cangul-Ozden-Simsek.

§2. New approach to $q$-Euler, Genocchi numbers and polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$ and let $w \in \mathbb{C}_p$ with $|1 - w|_p < 1$. The $w$-Euler polynomials of order $r$, denoted $E_{n,w}^{(r)}(x)$, are defined as

$$
e^{xt} \left( \frac{2}{we^t + 1} \right)^r = \left( \frac{2}{we^t + 1} \right) \cdots \left( \frac{2}{we^t + 1} \right) = \sum_{n=0}^{\infty} E_{n,w}^{(r)}(x) \frac{t^n}{n!}, \text{ (see [1])}.
$$

The values of $E_{n,w}^{(r)}(x)$ at $x = 0$ are called $w$-Euler number of order $r$: when $r = 1$ and $w = 1$, the polynomials or numbers are called the ordinary Euler polynomials or numbers. When $x = 0$ or $r = 1$, we use the following notation: $E_{n,w}^{(r)}$ denote $E_{n,w}^{(r)}(0)$, $E_{n,w}(x)$ denote $E_{n,w}^{(1)}(x)$, and $E_{n,w}$ denote $E_{n,w}^{(1)}(0)$.

It is known that

$$
\int_{\mathbb{Z}_p} w^x e^{tx} d\mu_{-1}(x) = 2 \frac{t^n}{we^t + 1} = \sum_{n=0}^{\infty} E_{n,w} \frac{t^n}{n!},
$$

and

$$
\int_{\mathbb{Z}_p} w^y e^{(t+y)x} d\mu_{-1}(y) = 2 \frac{t^n}{we^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,w}(x) \frac{t^n}{n!}, \text{ (see [1])}.
$$
The higher order $w$-Euler numbers and polynomials are given by

$$
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_r} e^{(x_1 + \cdots + x_r)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left(\frac{2}{we^t + 1}\right)^r = \sum_{n=0}^{\infty} E_{n,w}^{(r)} \frac{t^n}{n!},
$$

and

$$
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{\sum_{i=1} r x_i} e^{(\sum_{i=1}^r x_i + t)} \prod_{i=1}^r d\mu_{-1}(x_i) = \left(\frac{2}{we^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_{n,w}(x)^{(r)} \frac{t^n}{n!}.
$$

Thus, we have the following theorem.

**Proposition 1.** For $n \in \mathbb{N}$, we have

$$
E_{n,w}^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_r}(x_1 + \cdots + x_r + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r),
$$

and

$$
E_{n,w}^{(r)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_r}(x_1 + \cdots + x_r)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).
$$

From the results of Cangul-Ozden-Simsek (see [1]), we can derive the following $w$-Genocchi polynomials of order $r$ as follows.

$$
t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1 + x_2 + \cdots + x_r} e^{(x_1 + x_2 + \cdots + x_r + x)t} d\mu_{-1}(x_1) d\mu_{-1}(x_2) \cdots d\mu_{-1}(x_r)
$$

$$
= \left(\frac{2}{we^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} G_{n,w}^{(r)}(x) \frac{t^n}{n!}, \text{ (see [1]).}
$$

From (7), we note that

$$
t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1 + x_2 + \cdots + x_r} e^{(x_1 + x_2 + \cdots + x_r + x)t} d\mu_{-1}(x_1) d\mu_{-1}(x_2) \cdots d\mu_{-1}(x_r)
$$

$$
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_r}(x_1 + \cdots + x_r + x)^n r! \binom{n + r}{r} \frac{t^{n+r}}{(n+r)!}.
$$

By (6), (7) and (8), we easily see that $G_{0,w}^{(r)}(x) = G_{1,w}^{(r)}(x) = \cdots = G_{n-1,w}^{(r)}(x) = 0$, and

$$
\frac{G_{n+r,w}^{(r)}(x)}{r! \binom{n+r}{r}} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_r}(x_1 + x_2 + \cdots + x_r + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
$$

$$
= E_{n,w}^{(r)}(x).
$$
In the viewpoint of the $q$-extension of (6), let us define the $w$-$q$-Euler numbers of order $r$ as follows.

\[(9) \quad E_{n,w,q}^{(r)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_r}[x_1+\cdots+x_r]^n_q q^{-(x_1+\cdots+x_r)} d\mu_q(x_1) \cdots d\mu_q(x_r) .\]

From (9), we note that

\[(10) \quad E_{n,w,q}^{(r)} = \frac{[2]^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \left( \frac{[2]^r}{1+wq^l} \right)^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m q^{lm} \]

Therefore, we obtain the following theorem.

**Theorem 2.** For $n \in \mathbb{N}$, we have

\[E_{n,w,q}^{(r)} = [2]^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m [m]_q^n .\]

Let $F(t, x|q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(r)} \frac{t^n}{n!}$. Then we have

\[F(t, w|q) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_r} q^{-(x_1+\cdots+x_r)} e^{[x_1+\cdots+x_r]q} d\mu_q(x_1) \cdots d\mu_q(x_r) \]

\[= [2]^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m e^{[m]_q t} .\]

Thus, we obtain the following corollary.

**Corollary 3.** Let $F(t, x|q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(r)} \frac{t^n}{n!}$. Then we have

\[F(t, w|q) = [2]^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m e^{[m]_q t} .\]

Note that

\[\frac{d^k}{dt^k} F(t, x|q)|_{t=0} = E_{k,w,q}^{(r)} = [2]^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m [m]_q^k .\]
Let us define the $q$-extension of $w$-Genocchi numbers of order $r$ as follows.

$$t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1 + \cdots + x_r)} w^{x_1 + \cdots + x_r} e^{[x_1 + \cdots + x_r]_q} d\mu_q(x_1) \cdots d\mu_q(x_r)$$

$$= t^r [2]_q \sum_{m=0}^{\infty} \binom{m + r - 1}{m} (-1)^m w^m e^{[m]_q} = \sum_{n=0}^{\infty} G^{(r)}_{n,w,q} \frac{t^n}{n!}. \tag{11}$$

From (11), we can derive

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1 + \cdots + x_r)} w^{x_1 + \cdots + x_r} [x_1 + \cdots + x_r]^n \prod_{i=1}^{r} d\mu_q(x_i) \frac{r! \left(\frac{n+r}{r}\right) t^{n+r}}{(n+r)!}$$

$$= \sum_{n=0}^{\infty} G^{(r)}_{n+r,w,q} \frac{t^{n+r}}{(n+r)!} \sum_{n=0}^{\infty} G^{(r)}_{n+r,w,q} \frac{t^{n+r}}{(n+r)!}. \tag{12}$$

By (9), (11) and (12), we obtain the following theorem.

**Theorem 4.** For $n \in \mathbb{Z}_+$, and $r \in \mathbb{N}$, we have

$$\frac{G^{(r)}_{n+r,w,q}}{r! \left(\frac{n+r}{r}\right)} = E^{(r)}_{n,w,q},$$

and $G^{(r)}_{0,w,q} = G^{(r)}_{1,w,q} = \cdots = G^{(r)}_{r-1,w,q} = 0$.

In the sense of the q-extension of (6), we can consider the $w$-$q$-Euler polynomials of order $r$ as follows.

$$E^{(r)}_{n,w,q}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_r} q^{-(x_1 + \cdots + x_r)} [x_1 + \cdots + x_r]^n d\mu_q(x_1) \cdots d\mu_q(x_r)$$

$$= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \left( \frac{[2]_q}{1+wq^l} \right)^r$$

$$= [2]_q \sum_{m=0}^{\infty} \binom{m + r - 1}{m} (-1)^m [m + x]^n w^m.$$

Let $F(t, w, x|q) = \sum_{n=0}^{\infty} E^{(r)}_{n,w,q}(x) \frac{t^n}{n!}$. Then we have

$$F(t, w, x|q) = [2]_q \sum_{m=0}^{\infty} \binom{m + r - 1}{m} (-1)^m w^m e^{[m+x]_q t}$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_r} q^{-(x_1 + \cdots + x_r)} e^{[x_1 + \cdots + x_r]_q} d\mu_q(x_1) \cdots d\mu_q(x_r). \tag{14}$$
In the viewpoint of (7), we can define the \(w\)-Genocchi polynomials of order \(r\) as follows.

\[
\sum_{n=0}^{\infty} G^{(r)}_{n,w,q}(x) \frac{t^n}{n!} = [2]^r t^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m e^{[m+x]_q t}.
\]

By (14) and (15), we obtain the following theorem.

**Theorem 5.** For \(n \in \mathbb{Z}_+, r \in \mathbb{N}\), we have

\[
G^{(r)}_{0,w,q}(x) = \cdots = G^{(r)}_{r-1,w,q}(x) = 0.
\]

§3. Further Remarks and Observation for the multiple \(w\)-\(q\)-zeta function

In this section, we assume that \(q \in \mathbb{C}\) with \(|q| < 1\). For \(s \in \mathbb{C}\), \(w = e^{2\pi i \xi} (\xi \in \mathbb{R})\), let us define the Lerch type \(q\)-zeta function of order \(r\) as follows.

\[
\zeta^{(r)}_{q,w}(s) = [2]^r_q \sum_{m=1}^{\infty} \binom{m+r-1}{m} (-1)^m w^m [m]_q^s.
\]

For \(k \in \mathbb{N}\), by (10) and (16), we have \(\zeta^{(r)}_{q,w}(-k) = E^{(r)}_{k,w,q}\).

We now consider the \(q\)-extension of Hurwitz-Lerch type zeta function of order \(r\) as follows.

\[
\zeta^{(r)}_{q,w}(s, x) = [2]^r_q \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m [m+x]_q^s,
\]

where \(x \in \mathbb{C}\) with \(x \neq 0, -1, -2, \ldots\), and \(s \in \mathbb{C}\).

From (13) and (17), we can also derive

\[
\zeta^{(r)}_{q,w}(-k, x) = E^{(r)}_{n,w,q}(x), \text{ for } k \in \mathbb{N}.
\]
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