COSMOLOGICAL PAIR CREATION OF CHARGED AND ROTATING BLACK HOLES

I. S. N. Booth†, R. B. Mann†

†Dept. of Physics, University of Waterloo, Waterloo, ON
N2L 3G1, CANADA

Abstract. In this paper, we investigate the creation of charged and rotating black hole pairs in a background with a positive cosmological constant. Instantons to describe this situation are constructed from the Kerr-Newmann deSitter solution to the Einstein-Maxwell equations and the actions of these instantons are calculated in order to estimate pair creation rates and the entropy of the created spacetimes.

1. Introduction

In recent years there has been a considerable amount of interest in black hole pair production. Inspired by the more mundane and well understood particle pair production of quantum field theory (for example $2\gamma \rightarrow e^+ + e^-$), studies have been made to investigate the probability that a spacetime with a source of excess energy will quantum tunnel into a spacetime containing a pair of black holes. The earliest investigations studied pair creation due to background electromagnetic fields [1] but since then people have studied pair creation powered by domain walls [2], cosmic strings [3], and the cosmological constant [4]. To date, the studies have focussed on the creation of static black holes. In this paper we seek to extend the work on cosmological constant pair creation to include the creation of rotating black holes.

As in the extant studies, we shall conduct this investigation within the framework of the path integral formulation of quantum gravity, of which we now present a lightning review. The standard problem of quantum mechanics is to calculate the probability that
a system passes from an initial state $X_1$ to a final state $X_2$. If the system in question is a four dimensional spacetime, then these states are each characterized by a three manifold $\Sigma$, with Riemannian metric $h_{ij}$, and a symmetric tensor field $K_{ij}$ (the extrinsic curvature of the surface) which describes how the three manifold is embedded in the wider space time (or on a more physical level, roughly describes how the three manifold is evolving at that “instant” in time). Matter fields may be present on the spacetime in which case their values on the three manifold must also be specified. Then, in the path integral approach, we consider all four manifolds, $M$, and metrics and fields, $g$ and $A$, on those manifolds that smoothly interpolate between the initial and final conditions (not just those that are solutions to the equations of motion). The probability amplitude that a space time evolves from the initial condition $X_1$ to a final condition $X_2$ may be written as a functional integral over all such manifolds,

$$\Psi_{12} = \int d[M]d[g]d[A]e^{-iI(M,g,A)},$$

(1)

where $I$ is an appropriate action for the situation being studied. In this context, appropriate means an action that may be varied to give the correct set of field equations for classical motion plus an acceptable set of fixed quantities on the boundaries of the regions $M$ (for a more complete discussion, see section 4). In analogy with flat-space calculations, it is then argued [5] that to lowest order, this amplitude may be approximated by,

$$\Psi_{12} \approx e^{-I_c}$$

(2)

where $I_c$ is the real action of a (not necessarily real) Riemannian solution to the Einstein-Maxwell equations that smoothly interpolates between the given initial and final conditions. Such a solution is referred to as an instanton.

With this review of the path integral formalism in mind, we now turn to a preview of how it will be applied to the work at hand. First off, we need a metric that describes a deSitter spacetime containing a pair of charged and rotating black holes. These requirements are fulfilled by the Kerr Newmann deSitter metric, which we present in section 2. Secondly, an instanton solution is needed that interpolates between this solution and some initial condition. In section 3 we construct such a solution from the KNdS solution, and along the way choose our initial condition to be the no boundary condition. With the instanton in hand, we must then calculate its action in order to estimate the probability of pair creation and as a bonus the entropy of the spacetimes. This will be the subject of section 4.

2. The Kerr-Newmann deSitter Metric

The Kerr-Newmann deSitter solution to the Einstein-Maxwell equations has the following form [6]:

$$ds^2 = -\frac{Q}{G\chi^4} (dt - \alpha \sin^2 \theta d\phi)^2 + \frac{G}{Q} dr^2 + \frac{G}{\mathcal{H}} d\theta^2 + \frac{\mathcal{H} \sin^2 \theta}{G\chi^4} \left(\alpha dt - [\gamma^2 + \alpha^2] d\phi\right)^2,$$

(3)

where, $G = r^2 + \alpha^2 \cos^2 \theta$, $\mathcal{H} = 1 + \frac{\Lambda}{3} \alpha^2 \cos^2 \theta$, $\chi^2 = 1 + \frac{\Lambda}{3} \alpha^2$,

$$Q = -\frac{\Lambda}{3} r^4 + \left(1 - \frac{\Lambda}{3} \alpha^2\right) r^2 - 2Mr + \left(\alpha^2 + E_0^2 + G_0^2\right),$$

(4)
and the electromagnetic field is,

\[ F = d \left\{ \frac{E_0 r}{G \chi^2} \left[ dt - \alpha \sin^2 \theta d\phi \right] + \frac{G_0 \cos \theta}{G \chi^2} \left[ dt - (r^2 + \alpha^2) d\phi \right] \right\} . \tag{5} \]

In this metric, \( \Lambda \) is the cosmological constant (which for deSitter solutions will be assumed to be positive), \( \alpha \) is the rotation parameter, \( M \) parameterizes the mass, and \( E_0 \) and \( G_0 \) are respectively the effective electric and magnetic charge of the solution. The roots of \( Q \) determine the locations of the black hole and cosmological horizons. The absence of the cubic term in \( Q \) means that \( Q \) will have at most three real and positive roots. We shall assume that three such roots exist. In ascending order they are the inner, outer, and cosmological horizons.

In the context of pair creation studies, it is important to note that this metric, which apparently describes a single black hole in deSitter space, may be analytically continued through the cosmological horizon to describe a pair of black holes sitting at “opposite ends” of the universe, and separated by the cosmological horizon. Essentially two identical such solutions are joined together by identifying their cosmological horizons (figure 1). For computational reasons, it will later be necessary to formally identify their outer horizons (figure 2) as well. These analytic extensions are discussed further in [7].

3. Constructing the Instantons

An instanton with complex metric may be constructed from the single hole version of the KNdS solution by restricting the radial coordinate to lie between the outer black hole and cosmological horizons, sending \( t \to i\tau \) and periodically identifying the new imaginary time with some period \( T \). One half of this instanton may then be wedded to the two hole KNdS solution, in order to describe pair creation within the context of the no-boundary condition (see figure 2).

Things are not quite that simple however. In the previous paragraph, we ignored the possibility that conical singularities can, and usually will, show up in the metric if the period \( T \) of \( \tau \) is chosen in an arbitrary way. In order to avoid such singularities, the period must be chosen with some care, and even then fairly strong additional restrictions must be placed on the metric.

Briefly, conical singularities are defined in the context of two-manifolds with \( SO(2) \) symmetry groups that have a fixed point under the group action. In that context, the two-manifold may be foliated into a family of concentric “circles” generated by the
Figure 2. The KNdS instanton cut in half, and attached to the two hole KNdS solution. The view is of the $r - \tau/t$ sector.

action of the group. This family may be definitely ordered, and so a coordinate system may be set up, with a coordinate radius $\zeta$ labelling which “circle” a point is on, and an angular coordinate giving the position on the “circle”. Then, bringing the metric into play, a physical radius $R(\zeta)$ and circumference $C(\zeta)$ for each circle may be calculated. If $\lim_{\zeta \to 0} R = 0$ (ie the point is not a point at infinite proper distance from all other points), and $\lim_{\zeta \to 0} C_R \neq 2\pi$, then a conical singularity is said to exist at the fixed point.

If we look at $r - \tau$ slices of the instanton metric, then there are potential conical singularities at the cosmological and outer black hole horizons (they are fixed points under $\tau$ translations). If the inner black hole horizon coincides with the outer, then these lie at a double root of $Q$ and it is not hard to see that the horizon is at an infinite distance from all other points of the manifold, and so is not a candidate for a conical singularity. If we consider a single root however, calculation shows that a conical singularity exists unless,

$$T = 4\pi \chi^2 (r_i^2 + \alpha^2)/Q'(r_i),$$  \hspace{1cm} (6)$$

where $r_i$ is the radial coordinate of the root.

Thus, we can see that there are two ways to eliminate the conical singularities. Either we need a double root at the black hole horizon, in which case we can adjust the period of $\tau$ to match that required by the cosmological horizon, or we need to pick the physical parameters in $Q$ such that both the black hole horizon and cosmological horizons are single roots, and have a common required period. The first case will pair create extreme black holes. We label such instantons as cold instantons in analogy with the equivalent Reissner-Nördstom deSitter instanton \[4\]. There are two ways to satisfy the second condition. The first provides distinct horizons, while the second has the cosmological and black hole horizons coincident. These are labelled as luke-warm and Nariai instantons, again in analogy with the RNdS instantons \[4\].
Figure 3. Allowed range of the KNdS instanton parameters. The closest sheet represents the cold instantons, the middle the lukewarm, and the farthest the Nariai.

Figures 3 and 4 show the allowed range of the physical parameters for each type of instanton. Due to space constraints, the calculations leading up to these diagrams are omitted here. They will be included in [8].

The proper physical interpretation of the Nariai instantons is not so clear as for the other two types of instantons. In fact, at first inspection they appear to be degenerate. They may be rescued from degeneracy and to physical relevancy by considering a combination of a mathematical limiting process and a quantum correction as was originally done in [9] and discussed in more detail in [10]. Due to spatial constraints, we will omit further consideration of the Nariai metric in this summary.

Finally, before moving on to a calculation of the actions, we should note that by eliminating conical singularities from the instantons we have equivalently ensured that the cosmological and black hole horizons have the same surface gravity and temperature in the Lorentzian counterpart solutions. That is, we have ensured that the created space time will be in thermal equilibrium. Equivalently, there is no net particle flux between the two horizons. In the absence of charge and rotation, the system would then be in true thermodynamic equilibrium and would manifest no time dependence. With charge and/or rotation however, the black hole will evolve in time by discharging and/or spinning down.

We have neglected both discharge and spin-down effects for the classes of black holes we consider. Discharge effects are small if the temperature of a black hole is small relative to the mass of the lightest charged particle. Spin-down effects resulting from the quantum evaporation of spinning particles from the hole will also tend to be small for black holes in thermal equilibrium with the cosmological horizon. However for massless bosons of angular momentum $m$ with frequencies $\omega$ in the range $m\Omega_C < \omega < m\Omega_H$ where $\Omega_{C,H} = \frac{\alpha}{r_{C,H}^2 + \alpha^2}$ is the angular velocity of the cosmological/outer black hole horizon, these effects will be amplified by superradiance. The relative rate of angular momentum loss due to scalar fields for near extremally rotating, uncharged, Kerr de-Sitter holes (not in thermal equilibrium with the cosmological horizon) as amplified by superradiance, is $< \approx 2 \times 10^{-4}$ [11], and is expected to be larger for bosons of higher spin, increasing by as much as 2 orders of magnitude for graviton emission. The effect is
strongest for rapidly rotating black holes, but decreases rapidly as the angular momentum parameter of the black hole decreases (at least in the asymptotically flat case [12]). Now, uncharged lukewarm and cold holes are necessarily rapidly rotating (see figures 3 and 4), but with the inclusion of charge they need not be (figures 3 and 4 again), and in fact they may rotate at arbitrarily slow rates. Thus, it appears that while neither the cold nor lukewarm holes are in complete thermodynamic equilibrium, at least some subset of them will be close enough to equilibrium that we may treat them as being in that state within the approximations inherent in the path integral techniques. For the Nariai spacetime, $r_C = r_H$ and so the spin down effect is absent. A more complete discussion of these issues will appear in a forthcoming paper [8].

4. Action and Entropy of the Instantons

We now turn to the problem of calculating the action of these instantons. These calculations will mostly be done within the context of the quasi-local formulation for gravitational thermodynamics that has been developed by Brown and York [13]. For simplicity we will only consider the lukewarm and cold Kerr deSitter instantons. A complete discussion for all three types of instantons including electromagnetic fields, corner terms, and reference spacetimes will be presented in [8]. Here however, we shall just present the general ideas.

Let us begin by considering a region of spacetime $M$ bounded by two time-like boundaries $B_i$ and $B_o$, and two spacelike boundaries $\Sigma_i$ and $\Sigma_f$. Further we foliate the spacetime regions with spacelike hypersurfaces such that the initial and final hypersurfaces coincide with $\Sigma_i$ and $\Sigma_f$ respectively. Specializing to our situation, the timelike boundaries will be the ($r = \text{constant}$) hypersurfaces just outside/inside the black hole/cosmological horizons, while the spacelike foliations/boundaries will be ($t = \text{constant}$) hypersurfaces. With this region in mind we turn to considering the action.

As mentioned in the introduction, before a calculation of the actions can be made, it is necessary to decide what specific action is to be calculated. A variety of actions
may be chosen that, upon extremization, will yield the Einstein field equations in the interior of a spacetime region, \( M \). They will differ however, in what quantities will be required to remain fixed on the boundaries \( \partial M \).

As a concrete and pertinent example consider the usual action.

\[
I_{\text{Stan}} = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left( R - 2\Lambda \right) + \frac{1}{8\pi} \int_{\Sigma_f - \Sigma_i} d^3x \sqrt{h} K - \frac{1}{8\pi} \int_{B_o - B_i} d^3x \sqrt{-\gamma} \Theta ,
\]

where \( g \) is the determinant of the spacetime metric \( g_{ab} \) and \( h \) and \( \gamma \) are the determinants of the induced boundary metrics. \( R \) is the Ricci scalar, \( \Lambda \) is the cosmological constant, and \( K \) and \( \Theta \) are the traces of the extrinsic curvatures of the boundaries. The variation of this action is given by,

\[
\delta I_{\text{Stan}} = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left( G_{ab} + \Lambda g_{ab} \right) \delta g^{ab} + \int_{\Sigma_f - \Sigma_i} d^3x P^{ab} \delta h_{ab} + \int_{B_o - B_i} d^3x \Pi^{ab} \delta \gamma_{ab},
\]

where \( P^{ab} = \frac{1}{8\pi} \sqrt{h} \left( Kh^{ab} - K^{ab} \right) \), and \( \Pi^{ab} = \frac{1}{8\pi} \sqrt{-\gamma} \left( \Theta h^{ab} - \Theta^{ab} \right) \). In this case, it can be seen that the choice of action requires that the components of the induced boundary metric be fixed on the boundaries.

To investigate the relevance of this action to our situation, let us now perform the standard ADM [14] analysis, as extended by Brown and York [13], to get the following form for the variation of the action,

\[
\delta I_{\text{Stan}} = \left( \text{terms that are zero for solutions of the field equations} \right) + \int_{\Sigma_f - \Sigma_i} d^3x P^{ab} \delta h_{ab} + \int_{B_o - B_i} d^3x \sqrt{\sigma} \left( -\varepsilon_E \delta N + j_a \delta V^a + \frac{N}{2} s^{ab} \delta \sigma_{ab} \right).
\]

In the above, \( \sigma_{ab} \) is the induced metric on the intersection of timelike boundary with the spacelike foliations, \( \varepsilon_E, j_a, \) and \( s_{ab} \) are respectively a quasilocal energy density, a quasilocal momentum density, and a quasilocal stress density defined on those boundaries, while \( N \) and \( V^a \) are respectively the lapse and shift functions, also defined on those boundaries.

Now, only some of these boundary conditions are correct for the work that we are doing. Ultimately we are going calculate the action of a particular instanton, and the boundary terms will define what ensemble we are studying (recall from the introduction, that the instanton gives a first approximation to the full path integral calculation involving all possible interpolating spacetimes). We will be interested in the microcanonical ensemble, and so will want boundary conditions that fix the quasilocal energy and momentum densities. On the other hand however, we will want the induced spatial metrics on the spatial boundaries to be fixed so that the instantons may be cleanly matched to the Lorentzian solutions. Finally, removal of conical singularities requires a fixing of the surface stress tensor density, \( s_{ab} \). With these considerations in mind, it is a trivial matter to modify the action so that the conditions will be fulfilled. The new action is,

\[
I_{\text{Micro}} = I_{\text{Stan}} + \int_{B_o - B_i} d^3x \sqrt{\sigma} \left( N \varepsilon_E - V^a j_a - \frac{N}{2} s^{ab} \sigma_{ab} \right).
\]

We may now calculate instanton actions - or rather the actions of $\frac{1}{2}$ of the instantons since it is only half instantons that we connect to the Lorentzian spacetime (figure 2). Performing the calculations, we obtain, from (13)

$$I_{\text{Micro-Lukewarm}} = \frac{1}{8} (A_{BH} + A_C)$$

(14)

$$I_{\text{Micro-Cold}} = \frac{1}{8} A_C,$$

(15)

where $A_{BH}$ is the area of the black hole horizon, and $A_C$ is the area of the black hole horizon.

Now, having chosen microcanonical boundary conditions the partition function $Z = \Psi^2$ can be interpreted as the density of states, and therefore the entropy of the spacetimes is $\ln Z$ [4]. Thus, this calculation gives the standard results for entropies of spacetimes. Entropy is one quarter of the sum of the areas of the horizons for the luke warm case, and one quarter of the area of the cosmological horizon for the cold case.

The pair creation rates are given by,

$$\Gamma = \Psi^2 \approx e^{-2I}.$$  

(16)

Thus we see that the rates are always suppressed by an amount equal to the entropy of the created spacetime.

Acknowledgements

The authors would like to thank the Natural Sciences and Engineering Council of Canada for financial support. We gratefully acknowledge correspondence by R. Bousso and S. Ross.

REFERENCES

[1] Dowker, H.F., et. al., 1994, Phys. Rev. D 49, 2909.
[2] Caldwell, R.R., Chamblin, A., Gibbons, G.W., 1996, Phys. Rev. D 53, 7103–7114.
[3] Hawking, S.W., and Ross, S.F., 1995, Phys. Rev. Lett. 75, 3382–3385.
[4] Mann, R.B. and Ross, S.F., 1995, Phys. Rev. D 52, 2254–2265.
[5] Gibbons, G.W. and Hawking, S.W., 1977, Phys. Rev. D 15, 2752–2756.
[6] Mellor, F., Moss, I., 1989, Class. Quantum Grav. 6, 1379–1384.
[7] Brill, D.R. and Hayward, S.A., 1994, Class. Quantum Grav. 11, 359–370.
[8] Booth, I.S.N., and Mann, R.B., (manuscript in preparation)
[9] Ginsparg, P. and Perry, M.J., 1983, Nucl. Phys. B222, 245.
[10] Bousso, R., 1997, Phys. Rev. D 55, 3614–3621.
[11] Tachizawa, T., and Maeda, K., 1993, Phys. Lett. A 172, 325–330.
[12] Page, D.N., 1976, Phys. Rev. D 12, 3260–3273.
[13] Brown, J.D., and York, J.W., 1993, Phys. Rev. D 47, 1407–1420; 1420–1431; gr-qc/9405024
[14] Arnowitt, R., Deser, S., Misner, C.W., 1962, Gravitation: an introduction to current research, (John Wiley and Sons).