Exact accidental $U(1)$ symmetries for the axion

Luc Darmé$^{1, *}$ and Enrico Nardi$^{1, †}$

$^1$INFN, Laboratori Nazionali di Frascati, C.P. 13, 100044 Frascati, Italy

We study a class of gauge groups that can automatically yield a perturbatively exact Peccei-Quinn symmetry, and we outline a model in which the axion quality problem is solved at all operator dimensions. Gauge groups belonging to this class can also enforce and protect accidental symmetries of the clockwork type, and we present a toy model where an ‘invisible’ axion arises from a single breaking of the gauge and global symmetries.

Introduction. The non-trivial structure of the vacuum of Yang-Mills theories [1] implies that CP violation is a built-in feature in QCD [2, 3]. Strong CP violation is parametrized in terms of an angular variable $\theta \in [0, 2\pi]$ whose value is not determined by the theory, but is experimentally bounded to lie surprisingly close to zero $|\theta| \lesssim 10^{-10}$. It is hard to believe that this could occur simply as whim of nature, especially because any value $\theta \lesssim 10^{-1}$ would leave our Universe basically unaffected [4–6], precluding an anthropic explanation. A convincing rationale for $\theta \approx 0$ is provided by the Peccei-Quinn (PQ) mechanism [7, 8], which postulates the existence of a global Abelian symmetry, endowed with a mixed $U(1)_{PQ} \times SU(3)_C$ anomaly and broken spontaneously. This unavoidably implies a quasi-massless spin zero boson, the axion [9, 10], whose central role is to relax dynamically $\theta$ to 0. Remarkably, the axion also provides a novel solution to the apparently unrelated puzzle of the origin of dark matter [11–13], as well as a plethora of other implications for astrophysics and cosmology (for a recent review see [14]). However, it also raises various new issues. Among the deepest new questions stands the very origin of the axion or, more precisely, which is ‘the origin of the PQ symmetry’? There are in fact good reasons to believe that global symmetries cannot be fundamental, and this is especially true for a symmetry that, being anomalous, does not survive at the quantum level. A satisfactory explanation would arise if, in some suitable extension of the Standard Model (SM), the PQ symmetry occurs accidentally, in the sense that all renormalizable Lagrangian terms respecting first principles (Lorentz and local gauge invariance) preserve automatically also a global $U(1)$ with the required properties. A second problem emerges because to comply with the bound $|\theta| < 10^{-10}$, $U(1)_{PQ}$ must be respected by all effective operators acquiring a vacuum expectation value (VEV) up to dimension $D \gtrsim 11$. This is at odd with the well founded belief that all global symmetries are eventually violated by operators of all types and dimensions induced by quantum gravity [15–25]. This is known as ‘the PQ symmetry quality problem’. A third issue is related with ‘the axion scale’. The axion is a periodic field that, to comply with phenomenological constraints, must take values over a compact space of rather large radius $v_a \sim 10^{10 \pm 2}$ GeV. In benchmark models this is generally engineered by identifying $v_a$ with the PQ spontaneous symmetry breaking (SSB) scale $v_{PQ}$. This, however, brings in the usual problem of stabilising the electroweak scale against $\mathcal{O}(v_{PQ})$ corrections. Various strategies have been put forth to explain the origin of the PQ symmetry and protect it up to a suitable operator dimension $D$: discrete gauge symmetries $\mathbb{Z}_D$ [26–32], multiple scalars with values of $U(1)$ gauge charges of order $D$ [16], non-Abelian gauge symmetries, which generally have degree not less than $D$ [33, 34], often assisted by supersymmetry [35–37] or by higher dimensional constructions [38–41]. However, an unsatisfactory aspect of all these solutions is that if the scale of PQ-breaking (PQ) effects lies below $m_P$, if PQ SSB occurs at a scale $v_{PQ} \gg 10^{10}$ GeV, or if future experimental limits will hint to $\theta \ll 10^{-10}$, the value of $D$ will have to be accordingly increased. As regards the axion scale problem, certain solutions have been attempted exploiting the so-called clockwork mechanism [42–50]. Clockwork PQ symmetries allow to boost selectively some axion couplings [51–54], and to exponentially enhance [55] or suppress [56] the ratio $v_a/v_{PQ}$. Clearly, also these symmetries call for an explanation of their origin and required high quality. However, devising ways to generate and protect clockwork symmetries employing first principles is an even more challenging task.

In this Letter we show that a so far uncharted type of ‘flavor’ gauge symmetries of the form $G_{MN} = SU(M) \times SU(N)$ with $M \neq N$, that we henceforth denote as ‘rectangular’ symmetries, allow to solve at the root the PQ origin and quality problems, by enforcing automatically global $U(1)$ symmetries that are either perturbatively exact at the Lagrangian level, or that become exact on the vacuum.\footnote{In this work ‘flavor’ refers to a replication of exotic quarks.} We outline a simple example where axion protection is enforced by $SU(4) \times SU(2)$. Finally, we speculate how rectangular symmetries might prove useful to solve also the axion scale problem. To illustrate this we construct a toy model wherein a clockwork PQ symmetry arises automatically, \footnote{This term refers to global symmetries that are broken explicitly solely by operators whose VEV vanishes. Vacua having more global symmetries than the Lagrangian yield additional massless scalars besides the usual Nambu-Goldstone-Bosons (NGB) [57].}
and a large axion scale \( v_a \) results from a gauge/global symmetry spontaneously broken by VEVs \( v \ll v_a \).

**Rectangular gauge groups and accidental U(1)\(_{PQ}\).**

Consider a scalar multiplet \( Y \) transforming in the bi-fundamental representation \((M, N)\) of the gauge group \( G_{MN} = SU(M) \times SU(N) \) with \( M > N \geq 2 \). Let us denote a generic component as \( Y_{\alpha i} \), where Greek indices span \( SU(M) \) and Latin indices span \( SU(N) \). Each group factor has a pair of Kronecker and Levi-Civita invariant tensors \((\delta_M, \epsilon_M), (\delta_N, \epsilon_N)\) which can be used to construct invariants by contracting the indices of field components. The renormalizable Lagrangian always contains the two invariants \( T = Tr(Y^\dagger Y) \) and \( T_4 = Tr(Y^4) \) constructed from \( \delta_M \) and \( \delta_N \). Being Hermitian, they are manifestly invariant under a global \( U(1)_{\xi Y} \) phase redefinition \( Y \to e^{i\xi Y} \). Let us denote the trace of the matrix of the minors of order \( k \) of \( Y^\dagger Y \) as \( C_k = Tr[\text{Min} (Y^\dagger Y, k)] \). We have \( T = C_1 \) and we replace \( T_4 \) with \( \frac{1}{2} T^2 - T_4 = C_2 \) [58]. The \( C_k \)’s up to \( C_N = \det [Y^\dagger Y] \) form a fundamental set of \( G_{MN} \times U(1)_{\xi Y} \) invariants: it can be proven [59] that any higher order invariant \( \tilde{T}_k = Tr(Y^\dagger Y^k) \) can be expressed in terms of this set.\(^3\) The accidental \( U(1)_{\xi Y} \) can only be broken by non-Hermitian invariants, which are monomials with an unequal number of \( Y \) and \( Y^\dagger \) components, which must then involve the \( \epsilon \) tensors. However, all invariants involving \( \epsilon_M \) and a single scalar multiplet vanish symmetrically. Consider in fact the SU(M) singlet

\[
e^{\epsilon_1 \cdots \epsilon_M Y_{\alpha_1 i_1} \cdots Y_{\alpha_M i_M}} = (\epsilon_M Y^M)_{i_1 \cdots i_M} \tag{1}
\]

where the right hand side (r.h.s) defines a shorthand notation for the contraction of \( SU(M) \) indices with \( \epsilon_M \). Since \( M > N \) at least two components have the same \( SU(N) \) index, so that the string vanishes symmetrically.\(^4\)

Thus the Lagrangian for a scalar multiplet \( Y \) transforming under a rectangular gauge symmetry automatically enjoys a global \( U(1)_{\xi Y} \), which is perturbatively exact.

To promote \( U(1)_{\xi Y} \) to a PQ symmetry, it must be endowed with a QCD anomaly. This requires assigning \( U(1)_{\xi Y} \) charges to fermions that carry color and couple to \( Y \). Let us introduce two sets of chiral exotic quarks in the fundamental of \( SU(3)_C \), singlets under the electroweak gauge group, and transforming under \( G_{MN} \) as \( Q_L \sim (M, 1) \) and \( Q_R \sim (1, N) \) so that the Yukawa operator \( \overline{Q}_L Y Q_R \) is gauge invariant. To prevent a color gauge anomaly we add \( P = M - N \) quarks \( q_R \), and a new scalar multiplet \( Z \) acquiring a VEV so that all the quarks can be massive. This step can be arranged in different ways, the two extreme possibilities are:

(I) Add a set of \( G_{MN} \)-singlets \( q_{Ra} \) \((a = 1, \ldots, P)\) which couple to a scalar multiplet \( Z \sim (M, 1) \) via \( P \) Yukawa operators \( \sum_{a=1}^{P} \overline{q}_L Z q_{Ra} \).

(II) Assign the \( q_R \)'s to the fundamental representation of a new gauge factor \( SU(P) \), and \( Z \) to the bi-fundamental \((M, P)\) of \( G_{MP} \), so that there is a single Yukawa operator \( \overline{q}_L Z q_R \).

Note that for \( M = N + 1 \) the two cases coincide, hence we restrict case (II) to \( P \geq 2 \). \( G_{MN(P)} \) gauge anomalies can be canceled by adding three copies of \( M, N, (P) \)-plets of colorless ‘leptons’ of chirality opposite to that of the quarks, which can acquire mass from the VEVs of the same multiplets \( Y \) and \( Z \), e.g. \( \sum_{r=1}^{3} \overline{P}_R Y \ell_L^c \) etc.

Scalar terms involving only \( Z \) also enjoy an exact accidental symmetry \( U(1)_{\xi Z} \), i.e. \( V(Z) = V(Z^\dagger Z) \). However, by contracting the \( SU(M) \) indices of \( Y \) and \( Z \) it is possible to construct certain mixed non-Hermitian operators that break \( U(1)_{\xi Y} \times U(1)_{\xi Z} \) to a single \( U(1) \) that is defined by some specific condition between the \( U(1) \) charges \( \chi_Y \) and \( \chi_Z \). As it will become clear below, depending if \( SU(M) \) index contraction is performed with \( \delta_3^\alpha \) or with \( \epsilon^{\alpha_1 \cdots \alpha_M} \) the two possibilities are

\[
\delta_M : U(1)_{\xi Y} \times U(1)_{\xi Z} \to U(1)_{\xi}, \quad \chi_Y - \chi_Z = 0 \tag{2}
\]

\[
\epsilon_M : U(1)_{\xi Y} \times U(1)_{\xi Z} \to U(1)_{\xi}, \quad N \chi_Y + P \chi_Z = 0 \tag{3}
\]

The charge relation in Eq. (3) implies that \( U(1)_{\xi} \) has no QCD anomaly. Hence the symmetry preserved by the operators constructed with \( \epsilon_M \) cannot be promoted to a PQ symmetry. To see this let us consider a chiral transformation with generic quark charges \( \chi_{Q_L}, \chi_{Q_R}, \chi_{q_R} \). The \( U(1) \)-QCD anomaly coefficient is precisely

\[
|2N| = MX_{Q_L} - NX_{Q_R} - PX_{q_R} = N \chi_Y + P \chi_Z, \tag{4}
\]

where the relation with the charges of the scalars follows from requiring \( U(1) \) invariance of the Yukawa terms.

- **U(1)-breaking operators.** Eqs. (2) and (3) show that operators involving \( \delta_M \) break \( U(1)_{\xi} \), while \( \epsilon_M \)-type of operators break \( U(1)_{\xi} \), so that in the presence of both no \( U(1) \) would survive. To see which operators can arise, let us start with case (I) where the multiplets have components \( Y_{\alpha i}, Z_{\alpha i} \). Let us define a set of \( SU(N) \) vectors \( (X_n)_i = (Z^i (Y^\dagger Y^i)^n Y)_i, n = 1, \ldots, N \). The operator

\[
O_I(X_n) = \epsilon_N \prod_{n=1}^{N} X_n \tag{5}
\]

does not vanish symmetrically, is non-renormalizable \((D = N(N + 1) \geq 6 \text{ for } N \geq 2)\) and preserves \( U(1)_{\xi} \). Since for \( M - N \geq 2 \) all \( \epsilon_M \) contractions must involve at least two \( Z_{\alpha i} \), they vanish symmetrically, and thus \( U(1)_{\xi} \) survives as a perturbatively exact accidental symmetry, broken only by the anomaly with coefficient \( |2N| = (N + P) \chi_Y \). For \( M - N = 1 \) instead we can write

\[
O_I(Y, Z) = (N!)^{-1} \epsilon^{\alpha_1 \cdots \alpha_N M} (\epsilon_N Y^M)_{\alpha_1 \cdots \alpha_N} Z_{\alpha M}, \tag{6}
\]
that has dimension $D = M$ (and hence is renormalizable for $G_{32}$ and $G_{33}$). Then, in this particular case $U(1)_{\xi'}$ gets broken at $D = M \cdot N$ and no protected $U(1)$ survives.

In case (II) the multiplets components are $Y_{a1}$, $Z_{aa}$ where $a, b, \ldots$ span $SU(P)$. Let us take $N \geq P \geq 2$ ($N \leq P$ amounts to interchange $Y \leftrightarrow Z$) and let us consider the $SU(P)$ and $SU(N)$ singlets $(\epsilon_P Z^P)_{\alpha \ldots \alpha P}$ and $(\epsilon_N Y^N)_{\beta_1 \ldots \beta_N}$. Since $M = P + N$ the SU($M$) indices of their product can be exactly saturated with $\epsilon_M$, yielding the $\mathcal{G}_{MN_P}$ invariant operator of dimension $D = M$

$$\mathcal{O}'_H(Y, Z) = (P! N!)^{-1} \epsilon_M (\epsilon_P Z^P) (\epsilon_N Y^N), \quad (7)$$

which preserves $U(1)_{\xi'}$ (and is renormalizable for $G_{322}$). $\delta_M$-type of operators can be constructed starting from $(\epsilon_P Z^P)_{\alpha \ldots \alpha P}$ and by contracting the $SU(M)$ indices with $P$ components of $Y$. Defining $X_i = (Z^i Y^i)^a$ this yields $(\epsilon_P X_P)_i = (Z^i Y^i)^a$ this yields $X_i$ only if $N = P = M/2$, that is when $X_1$ is a $N \times N$ square matrix. The $D = M$ operator

$$\mathcal{O}_H(X_1) = (P!)^{-1} \epsilon_N \epsilon_P X_N^i = \det X_1, \quad (8)$$

is also renormalizable only for $G_{322}$, and is invariant under $U(1)_{\xi'}$. For $P < N \leq 2P$, adding $N - P$ new objects $X_i = (Z^i Y^i)^a$ allows for the contraction $\epsilon_N (\epsilon_P X_P)_i X_N^{-P} \cdots X_N^{-P}$. However, unless $N = 2P$ this cannot be constructed into a $P$-singlet. We thus need to consider the least common multiplicity $L \equiv \text{len}(P, N)$, in terms of which the structure of these operators is

$$\mathcal{O}_H(X_n) \sim (\epsilon_N)^{\frac{1}{2}} (\epsilon_P)^{\frac{1}{2}} (X^P_1 \ldots X^P_n X^{-P}_{n+1})^{\frac{1}{2}}, \quad (9)$$

where $F \equiv \text{floor}(N/P)$ denotes the greatest integer less or equal to $N/P$. Operators of this type preserve the symmetry defined by $X(X_n) = 0$, that is $U(1)_{\xi'}$ of Eq. (2), while they break $U(1)_{\xi'}$. However, the dimension $D(L) = (L/N)(F+1)(2N-FP)$ grows rapidly with $L$ (for $N = 4$ and $P = 3, D = 30$) so that in most cases $U(1)_{\xi'}$ breaking remains an academic issue. The dimension of the effective operators of lowest order that break respectively $U(1)_{\xi}$ and $U(1)_{\xi'}$ are given in Table I.

**Vacuum structure of the operators.** The PQ solution is endangered when the minimum of the axion potential is shifted away from the one selected by the non-perturbative QCD effects. Therefore, operators that break explicitly $U(1)_{PQ}$ in the Lagrangian but have vanishing VEVs are harmless, since they do not contribute to determine the minimum. Thus we need to study the behaviour of $\langle \mathcal{O} \rangle$, $\langle \mathcal{O}' \rangle$ at the potential minimum. Let us consider the renormalizable potential for $Y$. It reads

$$V(Y) = \kappa (T - \mu_Y^2)^2 + \lambda A, \quad (10)$$

where $T$ and $A$ are the two invariants introduced above, we require $\kappa > 0$ and $\lambda > -2N/\kappa$ to ensure a potential bounded from below, and $\mu_Y^2 > 0$ to trigger SSB. Let us write $Y(x)$ in its singular value decomposition (SVD):

$$\sqrt{v_Y} Y = U \tilde{Y} V^\dagger = U \tilde{Y} e^{i\varphi_Y} V^\dagger \rightarrow \tilde{Y} e^{i\varphi_Y}, \quad (11)$$

where $v_Y = \sqrt{2(T)}$, $U$ and $V$ are $U(M)$ and $U(N)$ unitary matrices, $U$ and $V$ are the corresponding special unitary (det $(U, V) = +1$), $\varphi_Y \equiv \arg(V(x)/v_Y) = \frac{1}{M} \arg \det U - \frac{1}{N} \arg \det V$ is the NGB of the global $U(1)_{\xi'}$, and $\tilde{Y}$ is the matrix of real non-negative singular values, which can be taken to lie in the diagonal upper $N \times N$ block, while all other entries vanish. We will henceforth denote as $Y|_{N \uparrow}$ the $N \times N$ upper left block of a matrix $Y$. The last form in Eq. (11) is obtained by gauging away $U(x)$ and $V(x)$. In this case the two invariants read:

$$T(\tilde{Y}) = \sum_{i=1}^{N} y_i^2, \quad A(\tilde{Y}) = \sum_{i<j} y_i^2 y_j^2. \quad (12)$$

It is now easy to identify the vacuum configurations $Y_{\xi} \equiv (Y)$ that minimize $V(Y)$ [58]: $T$ is blind to specific orientations of $\tilde{Y}$ in field space. This is because it carries a $SO(2 \times M \times N)$ symmetry much larger than $\mathcal{G}_{MN}$ that allows to rotate different configurations into each other. Adopting the classification of ref. [61] it is a ‘flavour irrelevant’ operator. The structure of $Y_{\xi}$ is then determined by the extrema of $A(Y)$. Since $A$ is non-negative, its minimum occurs at $\langle A \rangle = 0$, that is when all $y_i$’s but one vanish. The maximum instead occurs at the point of enhanced symmetry $y_i^2 = 1/\sqrt{N}$, $\forall i$. The sign of $\lambda$ thus determines which minimum is selected. We take $\lambda < 0$ so that $Y_{\xi}|_{N \uparrow} = \text{diag}(1, \ldots , 1)/\sqrt{N}$. The little group is $H = SU(N)_{\nu} \times SU(M-N)$ with $SU(N)_{\nu}$ the ‘diagonal’ combination of $SU(N)$ and of $(SU(N) \subset SU(M)$, while the value of $\varphi_Y$ is left undetermined. As regards the renormalizable potential for $Z$, in case (I) it has the form Eq. (10) (with $\mu_Y \leftrightarrow \mu_Z$) but with $A(Z_0) = 0$. In the SVD Eq. (11) $V \rightarrow V_Z = I$ while $\tilde{Z}$ has a single non-zero entry in some row $\alpha$ with VEV $v_{Z_0}^\alpha = 1$. In case (II) $V_Z$ can be gauged away via a $SU(P)$ transformation, so that $\sqrt{v_Z} Z \rightarrow U_Z \tilde{Z} e^{i\varphi_Z}$ where $\tilde{Z}$ has $P$ singular values located.
in different rows/columns. For \( \lambda_Z < 0 \) the potential is lowered when \( \langle A(Z) \rangle \) is maximum, which corresponds to \( z_a^c = 1/\sqrt{p'}, \forall a \). The relative orientation of \( \langle Y \rangle \) and \( \langle Z \rangle \) is determined by the \( D = 4 \) Hermitian operator

\[
O_{ZY} = \text{Tr} (Z Y^\dagger Y Z)^\dagger.
\] (13)

If the coupling is negative, the potential is lowered when \( \langle O_{ZY} \rangle \) is maximum. Since \( \langle Y Y^\dagger \rangle_{|N^+} \propto I_{N \times N} \) with all other entries vanishing (in particular in the lower \( P \times P \) block) this occurs when the \( P \) entries \( z_a \) fall in the upper \( N \) positions of \( Z^c \), while \( U_{Z_2} \), restricted to the block corresponding to these entries, is unitary. Thus \( \langle Y \rangle \) and \( \langle Z \rangle \) get maximally aligned, and in this case all \( \epsilon_M \)-type of operators \( \mathcal{O} \) vanish on the vacuum. If the coupling is positive, then \( \langle O_{ZY} \rangle \to 0 \) which is obtained when the entries \( z_a \) fill the lower \( P \) positions of \( Z^c \), and only \( U_{Z_2}^{a\dagger} \in U_{Z_2} \) is non-trivial (i.e. with off-diagonal entries). The two VEVs are maximally misaligned, which implies that all \( \delta_M \)-type of operators have \( \langle \mathcal{O} \rangle = 0 \). \( U_{Z_2}^{a\dagger} \) is unitary but otherwise undetermined. However, in case (II) the \( D = M \) operator \( O_{II} \), Eq. (7) is always allowed, and its VEV would lower the potential proportionally to \( \propto |\langle O_{II} \rangle| \) that is maximum for \( U_{Z_2}^{a\dagger} \to I_{P \times P} \).

**A SU(4) \times SU(2) model.** As a concrete application of our study let us outline a model in which the PQ symmetry arises automatically and remains perturbatively exact (details of the phenomenology will be discussed elsewhere). Case (I) with \( M - N > 1 \) is particularly favorable, since it does not allow for \( \epsilon_M \)-operators that could endanger the anomalous \( U(1)_\xi \) (see Table I). The minimal symmetry of this class is \( G_{42} = SU(4) \times SU(2) \).

We take \( Y \sim (4, \overline{2}) \), \( Z \sim \overline{(4, 1)} \), \( Q_L \sim (4, 1) \), \( Q_R \sim (1, 2) \) and \( q_R^u = (1, 1) \) with \( a = 1, 2 \). The flavor relevant scalar terms and the quark Yukawa operators are

\[
V_f = -\lambda A(Y) + \eta O_{ZY} + \left[ \eta \mathcal{O}_y^{i(6)} + \text{h.c.} \right],
\]

\[
V_q = \kappa_Q \mathcal{Q}_L Y Q_R + \sum_{a=1,2} \kappa_a \mathcal{Q}_L Z q_R^a + \text{h.c.}
\]

with \( \lambda, \eta > 0 \). \( A(Y) \) drives \( \hat{Y} \to \hat{Y}^{\dagger} |_{Z^c} \sim \text{diag}(1, 1) \) at the minimum, while \( O_{ZY} \) misaligns \( \langle Z \rangle \sim (0, 0, z_1, z_2)^T \) and \( \langle Y \rangle \). \( G_{42} \to SU(2)_V \) and all the quarks are massive. As regards the global symmetries \( U(1)_{\xi_{17}} \times U(1)_{\xi_2} = U(1)_{\xi_1} \times U(1)_{\xi_2} \), the \( D = 6 \) operator \( \mathcal{O}_y^{i(6)} \) preserves \( U(1)_{\xi_1} \) (see Eq. (5)) and breaks \( U(1)_{\xi_2} \). However, VEVs misalignment implies \( \langle \mathcal{O}_y^{i(6)} \rangle = 0 \) which yields two NGB:

\[
a = \frac{1}{v_a} (v_Y a_Y + v_Z a_Z), \quad a' = \frac{1}{v_a} (v_Y a_Y - v_Z a_Z),
\]

where \( v_a^2 = v_Y^2 + v_Z^2 \) and, given that all the fields have the same periodicity, we have set \( \lambda_Y = \lambda_Z = 1 \). \( a(x) \) gets a mass \( m_a \sim m_\pi f_\pi / f_a \) from the QCD anomaly, with \( f_a = v_a/|2N| \) and \( |2N| = 2(\lambda_Y + \lambda_Z) = 4 \). There are, however, only two domain walls because under the \( Z_4 \) center of \( SU(2)_V \) \( \langle a \rangle \to \langle a \rangle + \pi \). At this stage \( a'(x) \) remains massless. However, considering that breaking \( U(1)_\xi \) does not imply breaking the gauge symmetry, it might acquire a mass à la Coleman-Weinberg \([62]\) once all the effects, including those of the fermions, are included in the effective potential.

**A gauge symmetry for a clockwork axion.** We now discuss a construction based on rectangular gauge symmetries that enforces a mechanism for a highly protected ‘clockwork’ \( U(1)_{pq} \). Although we use suggestive names for some group factors, this should be regarded as a toy model not intended to describe real phenomenology.

Consider the gauge group \( U(1)_Y \times [SU(2)_2 \times SU(3)]^{n+1} \). We call \( U(1)_Y \) hypercharge, and the first \( SU(2)_2 \times SU(3) \) isospin and flavor. We introduce three sets of quarks in the fundamental of color transforming under these factors as \( Q_L \sim (2, 3)_q \), \( u_R \sim (1, 1)_q \), \( d_R \sim (1, 1)_q \) \( (a = 1, 2, 3) \) (we leave understood that gauge anomalies are compensated by suitable sets of ‘leptons’) and two scalar multiplets \( Y_{d,u} \sim (3, \overline{2})_c \) which acquire VEVs \( \langle T(Y_{d,u}) \rangle = v_{d,u}^2 / 2 \). The Yukawa Lagrangian reads:

\[
\mathcal{L}_q = -\sum_{a=1}^3 \left[ \kappa_{u,a} \mathcal{Q}_L Y U_R^a + \kappa_{d,a} \mathcal{Q}_L Y d_R^a \right] + \text{h.c.}
\]

where \( \kappa_{u,d} \) are coupling constants. Note that a coupling \( (\epsilon_2 Y_u Y_d)_{a,b} \) is forbidden because of unsaturated flavor indices, so that the potential involving the two scalars has the form \( V(Y_u^a Y_u^b Y_d^a Y_d^b) \) and carry an accidental global symmetry \( U(1)_{\xi_a} \times U(1)_{\xi_b} = U(1)_{\xi_2} \times U(1)_{\xi_2} \). Orthogonality with hypercharge \( Y_u X_u v_{d,u}^2 + Y_d X_d v_{d,u}^2 = 0 \) fixes the ratio of the \( U(1)_{\xi_2} \) charges of the scalars as \( X_u / X_d = v_{d,u}^2 / v_{d,u}^2 \), and we normalize their sum to \( X_u + X_d = 2 \). We now add two sets of hyperchargeless fields \( \Sigma_p, Y_p \) \( (p = 1, \ldots, n) \) which transform under the additional gauge factors. For \( SU(3) \times SU(2)_1 \times SU(3)_1 \) we add \( \Sigma_{1,1} \sim (3, 2, 1) \) and \( Y_{1,1,1} \sim (1, 2, 1) \), and for the successive factors \( \Sigma_{p,1} \sim (3, p-1, 2, p-1) \) and \( Y_{p,1,1} \sim (1, 2, p, 3) \) with \( p > 1 \). This allows to write a chain of \( n \) renormalizable operators

\[
(\epsilon_3 \epsilon_2 Y_u Y_d \Sigma_{1,1,1} + \sum_{p=2}^n (\epsilon_3 \epsilon_2 Y_{p-1,1,1}^{a,p} + \text{h.c.}) Y_{p-1,1,1}^{a,p} \Sigma_{p,1,1}^{a,p})
\]

For each field \( \Sigma_p \) there is an operator \( \epsilon_3^2 \epsilon_2 \Sigma_{p,1}^{a,p} \) of dimension \( D = 6 \) which, together with the operators in Eq. (18), breaks the global symmetry \( U(1)_{\xi_2} \times U(1)_{\xi_2} \) to \( U(1)_{pq} \), under which \( \hat{X}_Y = (-p)^2 \) and \( \hat{X}_Z = 0 \). Let us now assume that all dimensional parameters in the scalar potential have values of order \( v_{u,d} \) so that there is no large scale in the model. The operators in Eq. (18) have multifold effects. First, non vanishing VEVs would lower the potential by an amount \( \sim |\langle Y Y' \Sigma Y' \rangle| \) so that the VEVs of the fields tend to align in specific directions. The combination \( \epsilon_2 Y_u Y_d \) in the first operator Eq. (18) misaligns \( \langle Y_u \rangle \) and \( \langle Y_d \rangle \) in isospin space, in such a way that after \( U(1)_Y \times SU(2) \) breaking
a $U(1)$ gauge factor is preserved in the usual way. At the same time $\epsilon_3 Y_a Y_b \Sigma_1$ rotates $(\Sigma_1)$ in the direction in flavor space orthogonal to the plane $(Y_a)(Y_b)$ while, of course, of $\delta_2$ and $\delta_3$, index-contraction, $(\Sigma_1)$ and $(Y_1)$ tend to get aligned in $SU(3)_c \times SU(2)_L$ space. Isospin breaking provides a negative mixed-term $-v_u v_d \Sigma_1 Y_1$ and thus there are regions in parameter space where these two fields acquire a VEV proportional to $v_{u,d}$ even if their squared masses are non-negative. Hence, regions exist in which all the VEVs of the chain vanish if isospin is unbroken and $v_{u,d} \to 0$. Let us verify if $\bar{U}(1)_{\text{PQ}}$ remains preserved by higher order operators. For each pair $(\Sigma_{p+1}, Y_{p+1})$ let us define $(X_n)_{\gamma_p} = [\Sigma(\Sigma|\Sigma)^{n-1}Y]_{\gamma_p}$ with $n = 1, 2, \ldots$. It is indeed possible to write $\bar{U}(1)_{\text{PQ}}$ breaking operators like $\epsilon_3, X_1 X_2 X_3$ etc. However, since $(\Sigma_{p+1})$ is orthogonal in $SU(3)_c$ space to the plane $(Y_p)_{\alpha}(Y_p)_\beta$, it has only one non-zero $\gamma_p$ component, and thus all these operators vanish on the vacuum. Thus the accidental $\bar{U}(1)_{\text{PQ}}$ is perturbatively exact, and is broken by the QCD anomaly with $|2\lambda| = 3(A_u + A_d) = 6$. The corresponding NGB is

$$\tilde{a}(x) = \frac{1}{v_3} (v_u a_u + v_d a_d + \sum_{p=1}^n v_p a_p),$$

(19)

where $v_{u,d,p}$ and $v_{u,d,p}$ are the VEVs and orbital modes and of $Y_{u,d,p}$ and $v^2 = \lambda_u^2 v_u^2 + \lambda_d^2 v_d^2 + \sum_p \lambda_p^2 v_p^2 \approx \frac{\sqrt{5}}{3} 4 n^{1+1}$, where the approximation holds if all $v_{u,d,p} \approx v$. If we now take the VEVs that break isospin and PQ symmetries at $v \sim 100$ GeV, then for $n \sim 20$ the radius of the axion compact space is boosted to $v \approx 10^8$ GeV without the need of introducing any large fundamental parameter.

**Conclusions.** The ‘origin’ and ‘quality’ problems of the PQ symmetry can be solved by assigning the scalar multiplets hosting the axion to representations of semi-simple gauge groups with a ‘rectangular’ structure. No group factors of large degree are required, which renders the solution particularly elegant. It should have not gone unnoticed that such constructions require that (exotic) quarks must replicate, with some ‘generations’ obtaining a mass from different VEVs than others. Admittedly, the embedding into the SM of rectangular symmetries to play the role of flavor symmetries appears to be a challenging undertaking, but hopefully not insurmountable. Succeeding in this venture might uncover unexpected implications for the SM flavor problem.

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* luc.darme@lnf.infn.it
  enrico.nardi@lnf.infn.it

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