DTC ultrafilters on groups

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Abstract
We say that an ultrafilter on an infinite group $G$ is DTC if it determines the topological centre of the semigroup $\beta G$. If $G$ has a subgroup of finite index in which conjugacy classes are all finite and uniformly bounded in size, then $G$ does not admit a DTC ultrafilter. On the other hand, if $G$ has no subgroup of finite index in which all conjugacy classes are finite, then $G$ does admit a DTC ultrafilter. It follows that an infinite finitely generated group admits a DTC ultrafilter if and only if it has no abelian subgroup of finite index.

Keywords Ultrafilter · Topological centre · Virtually abelian group · FC group

1 Introduction

When $G$ is an infinite discrete group, its binary group operation extends to the Čech–Stone compactification $\beta G$ in two natural ways. They are defined in Sect. 2 and, as in [3, Ch.6], denoted by $\boxplus$ and $\triangledown$. Say that $v \in \beta G$ is a DTC ultrafilter for $\beta G$ if $u \boxplus v \neq u \triangledown v$ for every $u \in \beta G \setminus G$. DTC stands for determining the (left) topological centre. This is a special case of a small set determining the topological centre of $\beta G$ [3, Def.12.4].

Dales et al. [3] investigate the DTC notion in the context of their study of Banach algebras on semigroups and their second duals. They prove that the free group $F_2$ admits a DTC ultrafilter [3, 12.22], and that no abelian group does. Here we address the problem of characterizing those groups that admit DTC ultrafilters, the class of groups we call DTC(1). It is not obvious how good such a characterization can be, even for countable groups, as it is not clear from the definition whether the class of
countable DTC(1) groups, suitably encoded, is even within the projective hierarchy. However, it follows from our results that the class of finitely generated DTC(1) groups is a Borel set. In fact, we prove that an infinite finitely generated group is virtually abelian if and only if it does not belong to DTC(1). This provides a partial answer to question (13) in [3, Ch.13].

The algebraic property used above to define a DTC ultrafilter is equivalent to a topological one: \( v \in \beta G \) is a DTC ultrafilter for \( \beta G \) if and only if for every \( u \in \beta G \setminus G \) the mapping \( w \mapsto u \sqcap w \) from \( G \cup \{ v \} \) to \( \beta G \) is discontinuous at \( v \). In other words, if and only if for every \( u \in \beta G \) the property

- the mapping \( w \mapsto u \sqcap w \) from \( G \cup \{ v \} \) to \( \beta G \) is continuous at \( v \)

implies that \( u \in G \) and therefore \( u \) is in the topological centre of \( \beta G \). The equivalence of the algebraic and the topological property is noted in the comment after Definition 12.4 in [3].

In this paper we do not deal with another notion of sets determining the topological centre, in which for every \( u \in \beta G \setminus G \) the mapping \( w \mapsto u \sqcap w \) from the whole \( \beta G \) to \( \beta G \) is required to be discontinuous at \( v \). Every infinite group admits a DTC ultrafilter in that sense. Budak et al [1] discuss and compare the two notions.

Section 2 contains definitions and other prerequisites. Our new results are then in the next two sections. We show that there is a close connection between the existence of DTC ultrafilters on a group \( G \) and the structure of conjugacy classes in \( G \). In Sect. 3 we derive a sufficient condition for a group not to admit a DTC ultrafilter. In Sect. 4 we construct a DTC ultrafilter for certain countable groups. In particular, we show that an infinite finitely generated group admits a DTC ultrafilter if and only if it is not virtually abelian. In Sect. 5 we apply our results to several familiar countable groups, and in the last section we propose several open problems.

2 Preliminaries

When \( X \) is a set, \( \beta X \) is the Čech–Stone compactification of \( X \), the set of ultrafilters on \( X \) with the usual compact topology [5, §3.2]. We identify each element of \( X \) with the corresponding principal ultrafilter, so that \( X \subseteq \beta X \). When \( Y \subseteq X \), identify each ultrafilter on \( Y \) with its image on \( X \), so that \( \beta Y \subseteq \beta X \).

If \( \{ x_\xi \}_{\xi \in I} \) is a net of elements of \( X \) indexed by a directed partially ordered set \( I \), then \( \{ x_\xi \mid \xi \geq \eta, \eta \in I \} \) is a base of a filter \( F \) of subsets of \( X \). An ultrafilter \( u \in \beta X \) is a cluster point of the net \( \{ x_\xi \}_{\xi \in I} \) in \( \beta X \) if and only if \( F \subseteq u \).

In accordance with the standard set theoretic notation, each ordinal is the set of all smaller ordinals, and the least infinite ordinal is \( \omega = \{0, 1, 2, \ldots \} \). The cardinality of a set \( X \) is \( |X| \). The set of all subsets of \( X \) of cardinality \( \kappa \) is \( [X]^\kappa \), and the set of all subsets of cardinality less than \( \kappa \) is \( [X]^{<\kappa} \).
When $G$ is a group, denote by $e_G$ its identity element. When $G$ is an infinite group and $u, v \in \beta G$, define \[ u \Box v := \{ A \subseteq G \mid \{ x \in G \mid x^{-1} A \in v \} \in u \} \]
\[ u \Diamond v := \{ A \subseteq G \mid \{ x \in G \mid Ax^{-1} \in u \} \in v \} \]
The operations $\Box$ and $\Diamond$ are associative, and $u \Box v, u \Diamond v \in \beta G$ for $u, v \in \beta G$. Thus $(\beta G, \Box)$ and $(\beta G, \Diamond)$ are semigroups. When $u \in \beta G$, define $u \Box^1 := u$ and $u \Box^{(n+1)} := u \Box u \Box^n$ for $n \in \omega, n > 0$.

Say that $D \subseteq \beta G$ is a (left) DTC set for $\beta G$ (in the sense of Dales–Lau–Strauss [3, 12.4]) if \[ \forall u \in \beta G \setminus G \exists v \in Du \Box v \neq u \Diamond v. \]
Thus $v \in \beta G$ is a DTC ultrafilter for $\beta G$ (defined in the introduction) if and only if the singleton $\{ v \}$ is a DTC set.

The next lemma gathers the elementary properties of $\Box$ and $\Diamond$ needed in the sequel.

**Lemma 2.1** The following hold for any infinite group $G$ and $x, y \in G, u, v \in \beta G$:

(i) $x \Box y = x \Diamond y = xy$.
(ii) $x \Box u = x \Diamond u$.
(iii) $u \Box x = u \Diamond x$.
(iv) If $U \in u$ and $V \in v$ then $UV \in u \Box v$ and $UV \in u \Diamond v$.
(v) If $H$ is a subgroup of $G$ then $(\beta H, \Box)$ is a subsemigroup of $(\beta G, \Box)$ and $(\beta H, \Diamond)$ is a subsemigroup of $(\beta G, \Diamond)$.
(vi) If $G$ is abelian then $u \Box v = v \Diamond u$, and in particular $u \Box u = u \Diamond u$.
(vii) If $v$ is a DTC ultrafilter then $v \in \beta G \setminus G$.
(viii) If $u, v \in \beta G \setminus G$ then $u \Box v, u \Diamond v \in \beta G \setminus G$.

**Proof** Parts (i)–(vi) follow directly from the definition of $\Box$ and $\Diamond$, and (vii) follows from (iii). Part (viii) is a special case of Corollary 4.29 in [5].

Say that a group is DTC(0) if it is finite. When $G$ is an infinite group, say $G$ is DTC($\kappa$) if $\kappa$ is the least cardinality of a DTC set for $\beta G$. By the next theorem every infinite group is either DTC(1) or DTC(2).

**Theorem 2.2** Let $G$ be an infinite group. Then there is a two-point DTC subset of $\beta G$.

This is a special case of Theorem 12.15 in [3] and of Theorem 1.2 in [1]. For the reader’s convenience we include a direct proof using the following lemma.

**Lemma 2.3** Let $G$ be an infinite group, and let $A$ be an index set with $|A| = |G|$. For each $\alpha \in A$ let $F_\alpha$ be a finite subset of $G$. Then there are $z_\alpha \in G$ for $\alpha \in A$ such that
\[ F_\alpha z_\alpha z_\gamma^{-1} \cap F_\beta z_\delta z_\beta^{-1} = \emptyset \quad \text{for} \quad \{\alpha, \beta, \gamma, \delta\} \in [A]^4. \tag{1} \]
Proof Without loss of generality, assume $A$ is the cardinal $\kappa = |G|$. Define $z_0 = z_1 = z_2 = e_G$. Then proceed by transfinite recursion: For $\beta \in \kappa \setminus \{0, 1, 2\}$, when $z_\alpha$ have been defined for all $\alpha \in \beta$, take any

$$z_\beta \in G \setminus \bigcup_{[\alpha, \gamma, \delta] \in \beta} \left( F_\beta^{-1} F_\alpha z_\alpha^{-1} F_\delta z_\delta \cup z_\gamma z_\delta^{-1} F_\alpha^{-1} F_\delta^{-1} z_\delta \right).$$

\[ \square \]

Proof of Theorem 2.2. Put $A := \{0, 1\} \times [G]^{<\aleph_0}$ and $F_{i, K} := K$ for every $(i, K) \in A$. By Lemma 2.3 there are $z_{i, K}$ for $(i, K) \in A$ such that (1). For $i = 0, 1$ the elements $z_{i, K}$ form a net indexed by the directed poset $([G]^{<\aleph_0}, \subseteq)$; let $v_i \in \beta G$ be a cluster point of the net $\{z_{i, K}\}_K$. We shall prove that $\{v_0, v_1\}$ is a DTC set.

For $i = 0, 1$ define

$$W_i := \bigcup \{Kz_{i, K} \mid K \in [G]^{<\aleph_0}\} \quad S_i := \bigcup \{Kz_{i, K}z_{i, L}^{-1} \mid K, L \in [G]^{<\aleph_0}, K \neq L\}$$

Since $x^{-1}W_i \subseteq v_i$ for every $x \in G$, it follows that $W_i \subseteq u \bar{v}_i$ for every $u \in \beta G$.

Now take any $u \in \beta G \setminus G$. From (1) we have $S_0 \cap S_1 = \emptyset$, hence there is $i \in \{0, 1\}$ such that $S_i \neq u$. For every $K \in [G]^{<\aleph_0}$ we have $W_i z_{i, K}^{-1} \subseteq K \cup S_i$, hence $W_i z_{i, K}^{-1} \notin u$. Moreover, $\{z_{i, K} \mid K \in [G]^{<\aleph_0}\} \subseteq v_i$, and from the definition of $\diamond$ we get $W_i \neq u \diamond v_i$. We have proved that $u \bar{v}_i \neq u \diamond v_i$. \[ \square \]

For any group $G$ and $y, z \in G$ define

$$G_{y, z} := \{x \in G \mid x^{-1}yx = z\}$$

$$[y]_G := \{x^{-1}yx \mid x \in G\}$$

$$\text{FC}(G) := \{y \in G \mid [y]_G \text{ is finite}\}$$

Conjugacy classes $[y]_G$ have an important role in our results in the next two sections. Each $G_{y, z}$ is a subgroup of $G$. Clearly $G_{y, z} \neq \emptyset$ if and only if $[y]_G = [z]_G$, and in that case $G_{y, z}$ is a right coset of $G_{y, y}$. If $x^{-1}yx = z$ then $G_{y, z} = G_{y, y}x$. For a fixed $y \in G$, $\varphi(x) := x^{-1}yx$ defines a mapping $\varphi$ from $G$ onto $[y]_G$ such that $\varphi^{-1}(z) = G_{y, z}$ for each $z \in [y]_G$. Hence the cardinality of $[y]_G$ equals the index of $G_{y, y}$ in $G$.

$\text{FC}(G)$ is a normal subgroup of $G$, the FC-centre of $G$ [6, §4.3]. Say that $G$ is an FC group if $[y]_G$ is finite for every $y \in G$; in other words, $\text{FC}(G) = G$. Say that $G$ is a BFC group if $\sup_{y \in G} |[y]_G| < \infty$. The acronyms FC and BFC stand for finite and bounded finite, respectively, (number of) conjugates. Properties of FC and BFC groups are surveyed in Section 4.3 of [6].

Say that $G$ is an ICC group if it is nontrivial and the conjugacy class of every element $y \neq e_G$ in $G$ is infinite [2, Ex.8.3]; in other words, $G \neq \text{FC}(G) = \{e_G\}$. The acronym ICC stands for infinite conjugacy classes.

When $P$ is a property of groups, a group is said to be virtually $P$ if it has a subgroup of finite index that has property $P$.

The next lemma lists several known results that we need in Sect. 4. They are respectively Lemma 4.17, Theorem 1.41 and a corollary of Theorem 4.32 in [6].
Lemma 2.4

(i) (B.H. Neumann’s theorem) Let $G$ be a group such that $G = \bigcup_{i=0}^{n} G_i x_i$ where $x_i \in G$ and $G_i$ is a subgroup of $G$ for $i = 0, 1, \ldots, n$. Then at least one of the groups $G_i$ has finite index in $G$.

(ii) (Schreier’s subgroup lemma) Every subgroup of finite index in a finitely generated group is finitely generated.

(iii) Every finitely generated FC group is virtually abelian.

3 Sufficient condition

In this section we prove a sufficient condition for a group to be DTC(2). We start by showing that the DTC(1) and DTC(2) properties are inherited by subgroups of finite index.

Theorem 3.1 Let $G$ be a group and $H$ a subgroup of finite index in $G$. Then $G$ is DTC(1) if and only if $H$ is.

Proof As $G$ is the finite union of the left cosets of $H$, for every ultrafilter $u \in \beta G$ there is $y \in G$ such that $yH \in u$, and then $y^{-1} \square u = y^{-1} \Diamond u \in \beta H$.

When $G$ is DTC(1), let $v \in \beta G$ be a DTC ultrafilter for $\beta G$, and let $y \in G$ be such that $y^{-1} \square v = y^{-1} \Diamond v \in \beta H$. Take any $u \in \beta H \setminus H$. Then $u \square y^{-1} = u \Diamond y^{-1} \in \beta G \setminus H$ and

$$u \square (y^{-1} \square v) = (u \square y^{-1}) \square v \neq (u \Diamond y^{-1}) \Diamond v = u \Diamond (y^{-1} \Diamond v).$$

Thus $y^{-1} \square v$ is a DTC ultrafilter for $\beta H$, and $H$ is DTC(1).

When $H$ is DTC(1), let $v \in \beta H \subseteq \beta G$ be a DTC ultrafilter for $\beta H$. Take any $u \in \beta G \setminus H$, and let $y \in G$ be such that $y^{-1} \square u = y^{-1} \Diamond u \in \beta H \setminus H$. Then

$$y^{-1} \square (u \square v) = (y^{-1} \square u) \square v \neq (y^{-1} \Diamond u) \Diamond v = y^{-1} \Diamond (u \Diamond v) = y^{-1} \square (u \Diamond v),$$

hence $u \square v \neq u \Diamond v$. Thus $v$ is a DTC ultrafilter for $\beta G$, and $G$ is DTC(1). \hfill \square

By 2.1(vi) and 3.1, every infinite virtually abelian group is DTC(2). Next we shall prove that even every infinite virtually BFC group is DTC(2).

Lemma 3.2 Let $G$ be an infinite group, $H$ a finite group, and $\alpha : G \to H$ a surjective homomorphism. Then $\alpha$ extends to a homomorphism $\overline{\alpha} : (\beta G, \square) \to H$ such that for every $u \in \beta G$ and every $h \in H$ we have $\alpha^{-1}(h) \in u$ if and only if $\overline{\alpha}(u) = h$.

In fact $\overline{\alpha}$ is the unique continuous extension of $\alpha$ to $\beta G$ [5, 4.22]. This observation is not needed in the sequel.

Proof Write $\overline{\alpha}(u) := h$ when $u \in \beta G$, $h \in H$ and $\alpha^{-1}(h) \in u$. That defines a mapping from $\beta G$ onto $H$, because the sets $\alpha^{-1}(h)$, $h \in H$, form a finite partition of $G$, and therefore for every $u \in \beta G$ there is a unique $h \in H$ such that $\alpha^{-1}(h) \in u$.

Obviously $\overline{\alpha}(x) = \alpha(x)$ for $x \in G$. To prove $\overline{\alpha}$ is a homomorphism, take any $u, v \in \beta G$ and let $f := \overline{\alpha}(u)$, $h := \overline{\alpha}(v)$. Then $\alpha^{-1}(fh) = \alpha^{-1}(f)\alpha^{-1}(h) \in u \square v$ by 2.1(iv), which means that $\overline{\alpha}(u \square v) = fh$. \hfill \square
Theorem 3.3 Let $G$ be an infinite FC group for which there exists $n \in \omega$, $n \geq 1$, such that $x^n y = y x^n$ for all $x, y \in G$. Then $G$ is DTC(2).

Proof First we shall prove that
\[ v \Box^n u = u \Box v \Box^n \quad \text{for all } u, v \in \beta G. \tag{2} \]

Take any $y \in G$. Denote by $\text{Sym}([y]_G)$ the group of all permutations of the finite set $[y]_G$. Define the homomorphism $\alpha: G \to \text{Sym}([y]_G)$ by
\[ \alpha(x)(z) := x^{-1} z x, \quad x \in G, z \in [y]_G, \]
and let $H := \alpha(G) \subseteq \text{Sym}([y]_G)$. Write $E := a^{-1}(e_H)$. Then $xy = yx$ for $x \in E$, hence
\[ E \cap y^{-1} A = E \cap Ay^{-1} \]
for every $A \subseteq G$.

Take any $v \in \beta G$ and $A \subseteq G$. Let $\bar{\alpha}: (\beta G, \Box) \to H$ be the extension of $\alpha$ as in Lemma 3.2. As $x^{-n} z x^n = z$ for all $x \in G$ and $z \in [y]_G$, we get $h^n = e_H$ for every $h \in H$. It follows that $\bar{\alpha}(v \Box^n) = \bar{\alpha}(v)^n = e_H$, and thus $E \in v \Box^n$. From that we obtain $y^{-1} A \in v \Box^n$ if and only if $E \cap y^{-1} A = E \cap Ay^{-1}$ if and only if $Ay^{-1} \in v \Box^n$.

We have proved
\[ y^{-1} A \in v \Box^n \iff Ay^{-1} \in v \Box^n \]
for all $y \in G$, $v \in \beta G$ and $A \subseteq G$. Now (2) follows from the definition of $\Box$ and $\Diamond$.

Next take any $v \in \beta G \setminus G$. Then $v \Box^n \in \beta G \setminus G$ by 2.1(viii), and from (2) we get
\[ v \Box^n \Box v = v \Box^{n+1} = v \Box v \Box^n = v \Box^n \Diamond v. \]

Corollary 3.4 Every infinite virtually BFC group is DTC(2).

Proof In view of Theorem 3.1 it is enough to prove that every infinite BFC group is DTC(2). For any such $G$, apply Theorem 3.3 with $n$ equal to the factorial of $\max_{y \in G} |[y]_G|$. \qed

In Example 5.4 we produce a countable group that is DTC(2) but not virtually BFC.

4 Countable groups

Let $G$ be a countable infinite group, $G = \bigcup_{n \in \omega} F_n$ where $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ are finite sets. Let $v \in \beta G$ be a cluster point of a sequence $\{x_n\}_{n \in \omega}$ in $G$, and set $W := \bigcup_n F_n x_n$. From the definition of $\Box$ we get $W \in u \Box v$ for every $u \in \beta G$. The next theorem describes a condition that allows a choice of $x_n$ for which $W \notin u \Diamond v$ for every $u \in \beta G \setminus G$, so that $v$ is a DTC ultrafilter.

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Theorem 4.1 Each of the following conditions for a countable infinite group $G$ implies the next:

(i) There is $V \subseteq G$ such that for every finite $F \subseteq G$ there is $x \in G$ for which

$$x \notin FV \cup Fx(F \setminus V).$$

(ii) There are finite $F_n \subseteq G$, $n \in \omega$, such that $e_G \in F_n = F_n^{-1}$ and $F_n F_m \subseteq F_{n+1}$ for all $n$, and $G = \bigcup_{n \in \omega} F_n$. There is a sequence $(x_n)_{n \in \omega}$ in $G$ such that

$$F_n x_n x_i^{-1} \cap F_k x_k x_j^{-1} = \emptyset \text{ for } i, j < k < n \quad (3)$$

$$F_n x_n x_i^{-1} \cap F_n x_n x_j^{-1} = \emptyset \text{ for } i < j < n \quad (4)$$

(iii) There are $v \in \beta G \setminus G$ and $W \subseteq G$ such that $W \subseteq u \uparrow v$ for all $u \in \beta G$ and $W \notin u \uparrow v$ for all $u \in \beta G \setminus G$.

(iv) $G$ is DTC(1).

Proof Write $G = \bigcup_{n \in \omega} F_n$ with finite sets $F_n \subseteq G$ such that $e_G \in F_n = F_n^{-1}$ and $F_n F_n \subseteq F_{n+1}$ for all $n$. Assuming (i), by recursion we shall construct a sequence of $x_n \in G$, $n \in \omega$, such that

$$x_n \notin x_i V \quad \text{for } i < n \quad (5)$$

$$x_n \notin F_{n+1} x_k x_i^{-1} x_j \quad \text{for } i, j < k < n \quad (6)$$

$$x_n \notin F_{n+1} x_n x_i^{-1} x_j \quad \text{for } i < j < n \quad (7)$$

Start with any $x_0 \in G$. By (i) there is $x_1 \in G \setminus x_0 V$. For the recursive step, assume that $m \geq 2$ and that there are $x_0, \ldots, x_{m-1}$ such that (5), (6) and (7) hold for $n < m$. For the finite set

$$F := \{x_i \mid i < m\} \cup F_{m+1} \{x_k x_i^{-1} x_j \mid i, j < k < m\}$$

$$\cup F_{m+1} \cup \{x_i^{-1} x_j \mid i < j < m\}$$

there exists $x_m \in G$ such that $x_m \notin FV \cup Fx_m(F \setminus V)$.

Note that $e_G \in V$ because otherwise we would have $x \in [e_G] x ((e_G) \setminus V)$ for all $x \in G$. Hence $F \subseteq FV$, and from $x_m \notin FV$ we get (5) and (6) for $n = m$. Moreover, from (5) we have $x_i^{-1} x_j \notin V$, and therefore $x_i^{-1} x_j \in F \setminus V$, for $i < j < m$. Hence (7) for $n = m$ follows from $x_m \notin Fx_m(F \setminus V)$.

Since (3) follows from (6) and (4) follows from (7), this proves (i) $\Rightarrow$ (ii).

Now assume (ii). From (4) we get $x_i \neq x_j$ for $i \neq j$. Let $v \in \beta G \setminus G$ be a cluster point of the sequence $(x_n)_{n \in \omega}$, and put $W := \bigcup_{n \in \omega} F_n x_n$. Then $W \subseteq u \uparrow v$ for every $u \in \beta G$.

Take any $u \in \beta G \setminus G$. From (3) and (4), for $i < j$ we get

$$W x_i^{-1} \cap W x_j^{-1} = \left(\bigcup_{n \in \omega} F_n x_n x_i^{-1}\right) \cap \left(\bigcup_{k \in \omega} F_k x_k x_j^{-1}\right) \subseteq \bigcup_{k=0}^{j} F_k x_k x_j^{-1}$$
Thus the intersection $Wx_i^{-1} \cap Wx_j^{-1}$ is finite for $i \neq j$. Hence there is at most one $i \in \omega$ for which $Wx_i^{-1} \in u$. Since $v \in \beta G \setminus G$ and $\{x_i \mid i \in \omega\} \in v$, it follows that $\{x \mid Wx_i^{-1} \in u\} \notin v$, and $W \notin u \diamond v$ from the definition of $\diamond$. That proves (ii) $\Rightarrow$ (iii).

Obviously (iii) $\Rightarrow$ (iv).

We do not know if condition (iv) in Theorem 4.1 is inherited from quotients, but condition (i) is:

**Proposition 4.2** Let condition 4.1(i) hold for a group $G$. Let $H$ be a group with a surjective homomorphism $\pi : H \to G$. Then condition 4.1(i) holds also for $H$ in place of $G$ and $\pi^{-1}(V)$ in place of $V$.

**Proof** Write $U := \pi^{-1}(V)$. Then $\pi(F \setminus U) = \pi(F) \setminus V$ for every $F \subseteq H$.

Take any finite $F \subseteq H$. By the assumption there is $x \in G$ such that

$$x \notin \pi(F) \cup \pi(F)x \pi(F) \setminus V).$$

Let $y \in H$ be such that $\pi(y) = x$. Then

$$\pi(y) \notin \pi(F) \pi(U) \cup \pi(F) \pi(y) \pi(F) \setminus U),$$

hence $y \notin FU \cup Fy(F \setminus U)$.

**Theorem 4.3** Let $G$ be a countable group such that $|G/FC(G)| = \aleph_0$. Then $G$ satisfies condition 4.1(i), and hence is DTC(1).

**Proof** We shall prove that $G$ satisfies condition 4.1(i) with $V = FC(G)$. Take any finite set $F \subseteq G$. If $y \in V$ and $z \in F \setminus V$ then $G_{y,z} = \emptyset$, and for $y \in F^{-1} \setminus V$ the index of $G_{y,z}$ is infinite. Therefore by 2.4(i) there is

$$x \notin FY \cup \bigcup_{y \in F^{-1}} \bigcup_{z \in F \setminus V} G_{y,z}$$

which means that $x \notin FY \cup FX(F \setminus V)$.

**Corollary 4.4** Every countable ICC group satisfies condition 4.1(i), and hence is DTC(1).

**Corollary 4.5** Every countable group that has an ICC quotient is DTC(1).

**Proof** Apply Proposition 4.2 and Corollary 4.4.

**Corollary 4.6** An infinite finitely generated group is DTC(2) if and only if it is virtually abelian.

**Proof** Let $G$ be an infinite finitely generated group. If $G$ is virtually abelian then it is DTC(2) by Theorem 3.1. If $G$ is DTC(2) then $|G/FC(G)| < \aleph_0$ by Theorem 4.3. In that case $FC(G)$ is finitely generated by 2.4(ii), hence $G$ is virtually abelian by 2.4(iii).

Example 5.4 shows that the assumption that the group is finitely generated cannot be omitted in Corollary 4.6.
5 Examples

Dales et al. [3, 12.22] prove that the free group $\mathbb{F}_2$ is DTC(1). This follows from Corollary 4.4, since non-commutative free groups are ICC [2, Ex.8.3].

The comment after [3, 12.22] asks whether there is an amenable semigroup $S$ and an ultrafilter in $\beta S$ that determines the topological centre of $M(\beta S)$. We now exhibit several examples of a slightly weaker property: An amenable group $G$ and an ultrafilter in $\beta G$ that determines the topological centre of $\beta G$; that is, a DTC ultrafilter. The first such example is the group $\text{Sym} < \aleph_0 (\omega)$ of finite permutations of $\omega$. This group is ICC [2, Ex.8.3], hence again DTC(1) by 4.4. More generally we obtain other subgroups of $\text{Sym} < \aleph_0 (\omega)$ that are DTC(1):

Example 5.1 Subgroups of $\text{Sym} < \aleph_0 (\omega)$ that act transitively on $\omega$.

Let $G$ be a subgroup of $\text{Sym} < \aleph_0 (\omega)$ that acts transitively on $\omega$. We shall prove that $G$ is ICC and therefore DTC(1).

For $x \in \text{Sym} < \aleph_0 (\omega)$ write $\text{supp}(x) := \{ a \in \omega \mid x(a) \neq a \}$. Take any $y \in G \backslash \{ e_G \}$ and finite $F \subseteq G$ for which $y \in F$. There are $a, b \in \omega$ such that $y(a) = b \neq a$. By transitivity there is $x \in G$ such that $x(a) \neq \bigcup z \in F \text{supp}(z)$. Then $xy^{-1}(x(a)) = (x(b) \neq (x(a)$, hence $x(a) \in \text{supp}(xy^{-1})$, hence $xy^{-1} \notin F$. Thus $[y]_G$ is infinite.

Example 5.2 A finitely generated metabelian group of exponential growth and generalizations.

Let $R$ be a countable infinite integral domain, and $P$ an infinite multiplicative subgroup of $R$. Let $G$ be the set $P \times R$ with multiplication defined by

$$(x, r)(y, s) := (xy, r + sx) \quad \text{for} \quad x, y \in P, r, s \in R.$$ 

The mapping

$$(x, r) \mapsto \begin{pmatrix} x & r \\ 0 & 1 \end{pmatrix}$$

is an isomorphism between $G$ and a group of $2 \times 2$ matrices with the usual matrix multiplication. A particular instance, in which $R$ is the ring of dyadic rationals and $P$ is the multiplicative group of integer powers of 2, is a metabelian group with two generators and exponential growth [2, 6.7.1].

Write $0 := 0_R$ and $1 := 1_R$ and note that $e_G = (1, 0)$ and $(x, r)^{-1} = (x^{-1}, -rx^{-1})$.

We shall prove that $G$ is ICC and therefore DTC(1). Take any $(y, s) \in G \backslash \{ e_G \}$ and finite $F \subseteq G$. Write $S := \{ t \in R \mid (y, t) \in F \}$. By cancellability in $R$ we get:

- If $s \neq 0$ then there exists $x \in P$ such that $sx \notin S$. In that case let $r := 0$.
- If $s = 0$ then $y \neq 1$, and there exists $r \in R$ such that $r(1 - y) \notin S$. In that case let $x := 1$.

Thus in both cases there exists $(x, r) \in G$ such that $r + sx - ry \notin S$, hence

$$(x, r)(y, s)(x, r)^{-1} = (y, r + sx - ry) \notin F.$$ 

That proves $[(y, s)]_G$ is infinite.
Example 5.3 Discrete Heisenberg group and generalizations.

Let $R$ be a countable infinite integral domain. Let $G$ be $R \times R \times R$ with the multiplication

$$(a, b, c)(p, q, r) := (a + p, b + q, c + r + aq) \text{ for } a, b, c, p, q, r \in R.$$ 

The mapping

$$(a, b, c) \mapsto \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

is an isomorphism between $G$ and a group of $3 \times 3$ matrices with the usual matrix multiplication. For the special case $R = \mathbb{Z}$, the ring of integers, this is the discrete Heisenberg group. In that case $G$ is finitely generated and nilpotent, hence has no ICC quotients by the Duguid–McLain theorem [4]. Nevertheless Theorem 4.3 applies to $G$, as will now be shown, so that $G$ is DTC(1).

Write $0 := 0_R$ and $1 := 1_R$ and note that $e_G = (0, 0, 0)$ and $(a, b, c)^{-1} = (-a, -b, ab - c)$.

Put $V := \{(0, 0, c) \mid c \in R\}$. We shall prove that $\mathrm{FC}(G) = V$.

Clearly $[(0, 0, c)]_G = \{(0, 0, c)\}$ for every $c \in R$, hence $V \subseteq \mathrm{FC}(G)$. Take any $(p, q, r) \notin V$ and finite $F \subseteq G$. Write $S := \{t \in R \mid (p, q, t) \in F\}$. By cancellability in $R$ we get:

- If $p \neq 0$ then there exists $b \in R$ such that $r - bp \notin S$. In that case let $a := 0$.
- If $p = 0$ then $q \neq 0$, and there exists $a \in R$ such that $r + aq \notin S$. In that case let $b := 0$.

Thus in both cases there exists $(a, b, 0) \in G$ such that $r + aq - bp \notin S$, hence

$$(a, b, 0)(p, q, r)(a, b, 0)^{-1} = (p, q, r + aq - bp) \notin F.$$ 

That proves $[(p, q, r)]_G$ is infinite. Thus $\mathrm{FC}(G) = V$, and so $|G/\mathrm{FC}(G)| = \aleph_0$.

Example 5.4 Reduced power of a finite group.

Let $K$ be a nontrivial finite group, and $I$ an infinite index set. Let $K^I$ be the product group, and for $g = (g_i)_{i \in I} \in K^I$ write $\mathrm{supp}(g) := \{i \in I \mid g_i \neq e_K\}$. Let $G \subseteq K^I$ be the reduced product, i.e. the subgroup of those $g \in K^I$ for which $\mathrm{supp}(g)$ is finite. If $g, h \in G$ are conjugates then $\mathrm{supp}(g) = \mathrm{supp}(h)$. Hence $G$ is an FC group. Since $G$ satisfies the assumption of Theorem 3.3 with $n = |K|$, it is DTC(2).

Now assume that $K$ is not abelian. Then $G$ is not BFC because for any conjugacy class $C$ in $K$ the subsets of $G$ of the form $C^F \times \{e_K\}^{I \setminus F}$, for $F \subseteq I$ finite, are conjugacy classes in $G$. Moreover, $G$ is not even virtually BFC. Indeed, if $H$ is a subgroup of finite index in $G$ then there is a finite set $F \subseteq I$ such that $G_F = \{g \in G \mid \mathrm{supp}(g) \subseteq I \setminus F\}$ is a subgroup of $H$, and every such $G_F$ is isomorphic to $G$. 

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6 Open problems

In view of Corollary 3.4 it is natural to ask

**Question 6.1** Is it true that every infinite (or at least every countable infinite) FC group is DTC(2)?

A positive answer would yield an improvement of Corollary 4.6: It would then follow that a countable infinite group is DTC(2) if and only if it is virtually FC. However, as mentioned in the introduction, we do not even know if countable DTC(1) groups form a projective set. If the answer to the following question is positive, then countable DTC(1) groups form an analytic set.

**Question 6.2** Does Condition (iv) of Theorem 4.1 imply Condition (i)?

The results in Sect. 4 are specific to countable groups. That raises

**Question 6.3** Which results in Sect. 4 generalize to uncountable groups?

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