GROUP COCYCLES ON THE VOLUME-PRESERVING
DIFFEOMORPHISM GROUP

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ABSTRACT. We construct two kinds of group cocycles on the volume-preserving
diffeomorphism group. We show that, for the volume-preserving diffeomorphism
group of the sphere, one of the cocycles gives the Euler class of flat sphere bundles.

1. Introduction

Let $M$ be a connected manifold and $n$ denote the minimum positive number that
the homology $H_n(M; \mathbb{Z})$ is non-zero. Let $\Omega$ denote a closed $n$-form on $M$ such that
the cohomology class $[\Omega] \in H^n(M; \mathbb{R})$ is a non-zero image of the map $H^n(M; \mathbb{Z}) \to
H^n(M; \mathbb{R})$. Let $G$ denote the group of $\Omega$-preserving diffeomorphisms of $M$. The
normalized volume form and the group of volume-preserving diffeomorphisms is an
example. In this paper, we construct two kinds of group-cocycles on $G$; group
$(n + 1)$-cocycles $c_k$ with coefficients in the trivial $G$-module $\mathbb{Z}$ (Definition 2.1), and
group $n$-cocycles $b_k$ with coefficients in the trivial $G$-module $S^1 = \mathbb{R}/\mathbb{Z}$ (Definition
2.8). We show that, if the manifold is the $n$-sphere $S^n$ and the $n$-form $\Omega$ is the
normalized standard volume form on $S^n$, the cohomology class $[c_k]$ is equal to the
Euler class of flat sphere bundle up to sign (Theorem 3.1). By using this, we also
show that the group cohomology classes $[c_k]$ and $[b_k]$ are non-zero for the $n$-sphere
$S^n$ (Theorem 3.2).

Some group cocycles have been constructed on groups of diffeomorphisms that
preserve a fixed differential form, such as the symplectomorphism group and the
volume-preserving diffeomorphism group. On the symplectomorphism group of an
exact symplectic manifold, Ismagilov, Losik, and Michor constructed in [3] a group
two-cocycle with coefficients in $\mathbb{R}$. On the symplectomorphism group of an integral
symplectic manifold, a group three-cocycle with coefficients in $\mathbb{Z}$ is constructed in [6],
which is a variant of Ismagilov, Losik, and Michor’s one. Our group cocycles $c_k$ are
considered as generalizations of the cocycle in [6]. On the group of diffeomorphisms
that preserve a fixed exact form, Losik and Michor constructed in [4] a group cocycle
with coefficients in $\mathbb{R}$. Our group cocycles $b_k$ are analogous to Losik and Michor’s
one.

2. Group cocycles

2.1. Group cohomology. Let $G$ be a group and $A$ be a $G$-module. The set of all
maps $C_{\mathrm{grp}}^p(G; A) = \{ c : G^p \to A : \text{map} \}$ is called the group $p$-cochains of $G$
with coefficients in $A$. The coboundary operator $\delta : C_{\mathrm{grp}}^p(G; A) \to C_{\mathrm{grp}}^{p+1}(G; A)$ is defined
by
\[ \delta c(g_1, \ldots, g_{p+1}) = g_1 c(g_2, \ldots, g_{p+1}) + \sum_{i=1}^{p} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1}) + (-1)^{p+1} c(g_1, \ldots, g_p) \]
if $A$ is a left $G$-module and
\[ \delta c(g_1, \ldots, g_{p+1}) = c(g_2, \ldots, g_{p+1}) + \sum_{i=1}^{p} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1}) + (-1)^{p+1} c(g_1, \ldots, g_p)g_{p+1} \]
if $A$ is a right $G$-module. The group cohomology $H^*_\text{grp}(G; A)$ of $G$ with coefficients in $A$ is the cohomology of the cochain complex $(C^*_\text{grp}, \delta)$.

Let $G^\delta$ denote the group $G$ with discrete topology. Then the group cohomology of $G$ is isomorphic to the singular cohomology of the classifying space $BG^\delta$ of $G^\delta$ (see [2]). Under this identification, a group cohomology class gives a universal characteristic class of flat $G$-bundles.

2.2. The group cocycles with coefficients in $\mathbb{R}$. Let $M$ be a connected manifold and $n$ denote the minimum positive number that the homology $H_n(M; \mathbb{Z})$ is non-zero. Let $\Omega$ denote a closed $n$-form on $M$ such that the cohomology class $[\Omega] \in H^n(M; \mathbb{R})$ is a non-zero image of the map $H^n(M; \mathbb{Z}) \to H^n(M; \mathbb{R})$. The typical example of $(M, \Omega)$ is a homology $n$-sphere and its (normalized) volume form. Let $G = \text{Diff}_\Omega(M)$ denote the $\Omega$-preserving diffeomorphism group. We regard the integers $\mathbb{Z}$ as the trivial $G$-module. Then we define group $(n+1)$-cocycles $c_k$ in $C^{n+1}_\text{grp}(G; \mathbb{Z})$ as follows.

Let $(C_*(M; \mathbb{Z}), \partial)$ and $(C^*(M; \mathbb{Z}), d)$ denote the singular chain complex and the singular cochain complex respectively. We regard $C_q(M; \mathbb{Z})$ and $C^*(M; \mathbb{Z})$ as the left $G$-module and the right $G$-module respectively. Let us consider the double complexes $C^*_\text{grp}(G; C_q(M; \mathbb{Z}))$ and $C^*_\text{grp}(G; C^*(M; \mathbb{Z}))$. Take a point $\Delta_0 = x \in M = C_0(M; \mathbb{Z})$ and a singular $n$-cocycle $w_n \in C^n(M; \mathbb{Z})$ that the cohomology class $[w_n] \in H^n(M; \mathbb{Z})$ corresponds to the class $[\Omega]$ in $H^n(M; \mathbb{R})$. By the assumption of $M$, we take elements $\Delta_k \in C^k_\text{grp}(G; C_k(M; \mathbb{Z}))$ for $0 \leq k < n$ satisfying
\[ \delta \Delta_k = \partial \Delta_{k+1} \in C^{k+1}_\text{grp}(G; C_k(M; \mathbb{Z})). \]
Since the map $H^n(M; \mathbb{Z}) \to H^n(M; \mathbb{R})$ is injective, any element in $G$ preserves the cohomology class $[w_n]$, that is, $\delta w_n(g) = w_n - g^* w_n$ is a coboundary for any $g \in G$. Thus we take an element $w_{n-1} \in C^1_\text{grp}(G; C^{n-1}(M; \mathbb{Z}))$ such that
\[ \delta w_n = -(1)^{n} w_{n-1} \in C^1_\text{grp}(G; C^n(M; \mathbb{Z})). \]
Since $H^k(M; \mathbb{Z}) = 0$ for $0 < k < n$ by the universal coefficients theorem, we take elements $w_k \in C^{n-k}_\text{grp}(G; C^k(M; \mathbb{Z}))$ such that
\[ \delta w_k = -(1)^{n-k} w_{k-1} \in C^{n-k+1}_\text{grp}(G; C^k(M; \mathbb{Z})). \]
Let \( \langle \cdot, \cdot \rangle : C^r(M; \mathbb{Z}) \times C_r(M; \mathbb{Z}) \rightarrow \mathbb{Z} \) denote the pairing. This map induces the map
\[
\langle \cdot, \cdot \rangle : C^p_{\operatorname{grp}}(G; C^r(M; \mathbb{Z})) \times C^q_{\operatorname{grp}}(G; C_r(M; \mathbb{Z})) \rightarrow C^{p+q}_{\operatorname{grp}}(G; \mathbb{Z}).
\]

**Definition 2.1.** For \( 0 \leq k \leq n \), define \( c_k \in C^{n+1}_{\operatorname{grp}}(G; \mathbb{Z}) \) by \( c_k = \langle \delta w_k, \Delta_k \rangle \).

To show that the cochains \( c_k \) are cocycles, we use the following proposition.

**Proposition 2.2.** For any \( (a, b) \in C^p_{\operatorname{grp}}(G; C^r(M; \mathbb{Z})) \times C^q_{\operatorname{grp}}(G; C_r(M; \mathbb{Z})) \), we have
\[
\delta \langle a, b \rangle = \langle \delta a, b \rangle + (-1)^p \langle a, \delta b \rangle.
\]

Since the proof is the straightforward calculation, we omit it.

**Proposition 2.3.** The group cochains \( c_k \) are cocycles and cohomologous to each other.

**Proof.** By Proposition 2.2, we have
\[
d c_k = \langle \delta \delta w_k, \Delta_k \rangle + (-1)^{n-k+1} \langle \delta w_k, \delta \Delta_k \rangle = 0
\]
for any \( 0 \leq k \leq n \). Thus the group cochain \( c_k \in C^{n+1}_{\operatorname{grp}}(G; \mathbb{Z}) \) is a cocycle for any \( 0 \leq k \leq n \). For \( 0 < k \leq n \), we have
\[
c_k = \langle \delta w_k, \Delta_k \rangle = -(-1)^{n-k+1} \langle \delta w_k, \Delta_k \rangle = -(-1)^{n-k+1} \langle w_{k-1}, \delta \Delta_k \rangle = -(-1)^{n-k+1} \langle w_{k-1}, \delta \Delta_{k-1} \rangle = -(-1)^{n-k+1} \langle \delta w_{k-1}, \Delta_{k-1} \rangle = c_{k-1} - \delta \langle w_{k-1}, \Delta_{k-1} \rangle.
\]
Thus the cocycles \( c_k \) are cohomologous to each other. \( \square \)

**Remark 2.4.** If \( n = 2 \) and the 2-form \( \Omega \) is an integral symplectic form on \( M \), then the cocycle \( c_2 \) is, up to sign, equal to the cocycle introduced in [6].

Let \( E^{p,q} \) denote the spectral sequence of the double complex \( C^p_{\operatorname{grp}}(G; C^q(M; \mathbb{Z})) \). Then \( E^{p,q} \) is isomorphic to \( H^p_{\operatorname{grp}}(G; H^q(M; \mathbb{Z})) \), where we consider the coefficients \( H^q(M; \mathbb{Z}) \) as the right \( G \)-module by pullback. Since \( H^q(M; \mathbb{Z}) = 0 \) for \( 0 < q < n \), we have
\[
E^{0,n}_{n+1} = E^{0,n}_n = \cdots = E^{0,n}_2 = H^n(M; \mathbb{Z})^G,
\]
where \( H^n(M; \mathbb{Z})^G \) denotes the \( G \)-invariant part, and
\[
E^{n+1,0}_{n+1} = E^{n+1,0}_n = \cdots = E^{n+1,0}_2 = H^{n+1}_{\operatorname{grp}}(G; \mathbb{Z}).
\]
Thus the transgression map \( d_{n+1}^{0,n} : E^{0,n}_{n+1} \rightarrow E^{n+1,0}_{n+1} \) defines the map
\[
d_{n+1}^{0,n} : H^n(M; \mathbb{Z})^G \rightarrow H^{n+1}_{\operatorname{grp}}(G; \mathbb{Z}).
\]
Since the cohomology class \( [w_n] \) is in \( H^n(M; \mathbb{Z})^G \), we obtain the class \( d_{n+1}^{0,n}[w_n] \in H^{n+1}_{\operatorname{grp}}(G; \mathbb{Z}) \).

**Proposition 2.5.** The cohomology class \( d_{n+1}^{0,n}[w_n] \) is equal to the class \( [c_k] \).
Proof. The transgression map $d_{n+1}^{b,n}$ is given by the coboundary of the tail of the zig-zag (see, for example, [1, Section 14]). Moreover, the coboundary of the tail of zig-zag is equal to $\delta w_0 = (\delta w_0, \Delta_0) = c_0$. Thus we have $d_{n+1}^{b,n}[w_n] = [c_0]$. By Proposition 2.3, we have $d_{n+1}^{b,n}[w_n] = [c_k] \in H_{\text{grp}}^{n+1}(G; \mathbb{Z})$.

\begin{corollary}
The cohomology class $[c_k]$ is independent of the choice of $w_k$ and $\Delta_k$.
\end{corollary}

Note that the above corollary can be shown by a straightforward calculation.

\begin{remark}
If we replace the coefficients $\mathbb{Z}$ with $\mathbb{R}$, then the class $d_{n+1}^{b,n}[\Omega]$ in $H_{\text{grp}}^{n+1}(G; \mathbb{R})$ is trivial since the zig-zag is trivial. Thus the cohomology class $d_{n+1}^{b,n}[w_n]$ is equal to 0 in $H_{\text{grp}}^{n+1}(G; \mathbb{R})$. By the exact sequence

$$
\cdots \to H_n^{b}(G; S^1) \to H_{n+1}^{b}(G; \mathbb{Z}) \to H_{n+1}^{b}(G; \mathbb{R}) \to \cdots,
$$

we have the class in $H_n^{b}(G; S^1)$ that hits to the class $d_{n+1}^{b,n}[w_n] = [c_k]$. Since the connecting homomorphism $H_n^{b}(G; S^1) \to H_{n+1}^{b}(G; \mathbb{Z})$ factors through the bounded cohomology $H_{n+1}^{b}(G; \mathbb{Z})$, the cohomology class $[c_k]$ is bounded.

2.3. The group cocycles with coefficients in $S^1$. By Remark 2.7 we know the existence of the cohomology class in $H_{\text{grp}}^{n}(G; S^1)$ corresponding to the class $[c_k] \in H_{\text{grp}}^{n+1}(G; \mathbb{R})$ under the connecting homomorphism

$$
\delta : H_{\text{grp}}^{n}(G; S^1) \to H_{\text{grp}}^{n+1}(G; \mathbb{Z}).
$$

In this section, we give cocycles $b_k \in C_{\text{grp}}^{n}(G; S^1)$ such that $\delta[b_k] = [c_k] \in H_{\text{grp}}^{n+1}(G; \mathbb{Z})$.

By the assumption of the $n$-form $\Omega$ and the exact sequence

$$
\cdots \to H^n(M; \mathbb{Z}) \to H^n(M; \mathbb{R}) \xrightarrow{j} H^n(M; S^1) \to \cdots
$$

we have $j[\Omega] = 0$. Here we consider $\Omega$ as the corresponding singular $n$-coycle (if we temporally use the symbol $\Omega_{\text{sing}}$ to denote the corresponding singular cocycle, this cocycle is defined by $\Omega_{\text{sing}}(\sigma) = \int_\sigma \Omega$ for any singular $n$-simplex $\sigma$). We take a singular $(n - 1)$-cochain $\eta_{n-1} \in C^{n-1}(M; S^1)$ such that $d\eta_{n-1} = j\Omega \in C^n(M; S^1)$. By the universal coefficients theorem, the cohomology $H^n(M; S^1)$ is trivial for $0 < k < n$. Thus, as with the definition of $w_k \in C_{\text{grp}}^{n-k}(G; C^k(M; \mathbb{Z}))$, we define group cocycles $\eta_k \in C_{\text{grp}}^{n-1-k}(G; C^k(M; S^1))$ by

$$
\delta \eta_k = -(-1)^{n-k}d\eta_{k-1}
$$

for $0 < k < n$. Let $\Delta_k \in C_{\text{grp}}^k(G; C^k(M; \mathbb{Z}))$ be the cochains defined in Section 2.2.

\begin{definition}
For $0 \leq k \leq n - 1$, define $b_k \in C_{\text{grp}}^n(G; S^1)$ by $b_k = \langle \delta \eta_k, \Delta_k \rangle$.
\end{definition}

As with Proposition 2.3 and Corollary 2.6 we have the following.

\begin{proposition}
The group cochains $b_k$ are cocycles and cohomologous to each other. Moreover, the cohomology class $[b_k]$ is independent of the choice of cochains $\eta_k$ and $\Delta_k$.
\end{proposition}
Theorem 2.10. Let $\delta : H^n_{grp}(G; S^1) \to H^{n+1}_{grp}(G; \mathbb{Z})$ denote the connecting homomorphism, then we have $\delta[b_k] = [c_k]$.

Proof. By Proposition 2.3 and 2.9 it is enough to show the equality $\delta[b_{n-1}] = [c_n]$. Recall that $b_{n-1} = \langle \delta \eta_{n-1}, \Delta_{n-1} \rangle$ and $c_n = \langle \delta w_n, \Delta_n \rangle$. Let $\overline{\eta}_{n-1} \in C^{n-1}(M; \mathbb{R})$ be a lift of $\eta_{n-1} \in C^{n-1}(M; S^1)$, that is, $\overline{\eta}_{n-1}$ satisfies $j\overline{\eta}_{n-1} = \eta_{n-1}$, and put $\overline{b}_{n-1} = \langle \delta \overline{\eta}_{n-1}, \Delta_{n-1} \rangle \in C^n_{grp}(G; \mathbb{R})$.

Then we have

$$\delta \overline{b}_{n-1} = \delta \langle \delta \overline{\eta}_{n-1}, \Delta_{n-1} \rangle = \langle \delta \delta \overline{\eta}_{n-1}, \Delta_{n-1} \rangle - \langle \delta \overline{\eta}_{n-1}, \delta \Delta_{n-1} \rangle$$

$$= -\langle \delta \overline{\eta}_{n-1}, \partial \Delta_{n-1} \rangle = -\langle \delta \overline{\eta}_{n-1}, \Delta_{n-1} \rangle = -\langle \delta \overline{\eta}_{n-1}, \Delta_{n-1} - \delta \eta_{n-1} \rangle.$$

Since the action by $G$ preserves $\Omega$ as the singular $n$-cocycle, we have $\delta \Omega = 0 \in C^1_{grp}(G; C^n(M; \mathbb{R}))$. Thus we have

$$\delta \overline{b}_{n-1} = -\delta (\Omega - d\overline{\eta}_{n-1}) = \langle \delta (\Omega - d\overline{\eta}_{n-1}), \Delta_{n-1} \rangle.$$

Since $j(\Omega - d\overline{\eta}_{n-1}) = j\Omega - dw_{n-1} = 0$, the cocycle $\Omega - d\overline{\eta}_{n-1}$ is in $C^n(M; \mathbb{Z})$. This integer coefficients cocycle satisfies the assumption of $w_n$. Thus, if we put $w_n = \Omega - d\overline{\eta}_{n-1} \in C^n(M; \mathbb{Z})$, we have

$$\delta \overline{b}_{n-1} = \langle \delta w_n, \Delta_n \rangle = c_n$$

and this implies $\delta[b_{n-1}] = [c_n]$. $\square$

3. The Euler class of flat sphere bundles

In this section, for the $n$-sphere $S^n$ and the normalized standard volume form, we show that the class $[c_k]$ is equal to the Euler class of flat sphere bundles up to sign (Theorem 3.1) and show that the group cohomology classes $[c_k]$ and $[b_k]$ are non-trivial (Theorem 3.2).

Let us recall that the construction of the Euler class in terms of the Leray-Serre spectral sequence. Let $E \to B$ be an oriented sphere bundle over a connected base space $B$ and $E^p,q$ denote the Leray-Serre spectral sequence. Since $H^k(S^n; \mathbb{Z}) = 0$ for $0 < k < n$ and the bundle is oriented, we have $E^{0,n}_{n+1} = E^{0,n}_2 = H^n(S^n; \mathbb{Z})$ and $E_{n+1}^{1,0} = E^{1,0}_{n+1} = H^{n+1}(B; \mathbb{Z})$. Let $d_{n+1}^{0,n} : E^{0,n}_{n+1} \to E^{0,n+1,0}_{n+1}$ denote the derivation map and $\theta$ denote the generator of the cohomology $H^n(S^n; \mathbb{Z}) = E^{0,n+1}_n$, then the cohomology class $-d_{n+1}^{0,n} \theta \in H^{n+1}(B; \mathbb{Z})$ is the Euler class of the oriented sphere bundle $E \to B$.

Let $\text{Diff}_+(S^n)$ denote the orientation-preserving diffeomorphism group and $e \in H^{n+1}(\text{Diff}_+(S^n); \mathbb{Z}) \cong H^{n+1}_{grp}(\text{Diff}_+(S^n); \mathbb{Z})$ denote the universal Euler class of flat sphere bundles. Let us consider the normalized standard volume form $\Omega$ on $S^n$. Then the volume-preserving diffeomorphism group $G = \text{Diff}_+(S^n)$ is included in $\text{Diff}_+(S^n)$. Let $e_G \in H^{n+1}_{grp}(G; \mathbb{Z})$ denote the pullback of the Euler class. By the naturality of the Euler class, the class $e_G$ is the universal Euler class of flat sphere bundles whose structure group is reduced to $G$. 
Theorem 3.1. The cohomology class \([c_k] \in H^{n+1}_{\text{grp}}(G; \mathbb{Z})\) is equal to the negative of the Euler class \(e_G\).

Proof. Let \(EG^\delta \to BG^\delta\) denote the universal \(G^\delta\)-bundle. Then the Borel construction \(S^n_{G^\delta} = EG^\delta \times_{G^\delta} S^n \to BG^\delta\) is the universal flat sphere bundle. Note that the Leray-Serre spectral sequence of the Borel construction is isomorphic to the spectral sequence used in Section 2.2 (see [5]). Thus, by the construction of the Euler class in the Leray-Serre spectral sequence, the class \(d^{n+1}_0[w_n] \in H^{n+1}_{\text{grp}}(G; \mathbb{Z})\) is equal to the negative of the Euler class of the flat sphere bundle \(S^n_{G^\delta} \to BG^\delta\) under the identification \(H^{n+1}_{\text{grp}}(G; \mathbb{Z}) \cong H^{n+1}_{\text{grp}}(BG^\delta; \mathbb{Z})\). By Proposition 2.5, the group cocycles \(c_k\) give the negative of the Euler class of the universal flat sphere bundle. \(\Box\)

Since the natural action by \(SO(n+1)\) on \(S^n\) preserves the normalized standard volume form \(\Omega\), there is the inclusion \(SO(n+1) \to G\). Let \(e_{SO(n+1)} \in H^{n+1}_{\text{grp}}(SO(n+1); \mathbb{Z})\) denote the pullback of \(e_G\) by the inclusion. By the naturality of the Euler class, the class \(e_{SO(n+1)}\) is the universal Euler class of flat sphere bundles whose structure group is reduced to \(SO(n+1)\). The universal Euler class in \(H^{n+1}(BSO(n+1); \mathbb{Z})\) of vector bundles hits the class \(e_{SO(n+1)}\) under the canonical map

\[ H^{n+1}(BSO(n+1); \mathbb{Z}) \to H^{n+1}(BSO(n+1); \mathbb{Z}) \cong H^{n+1}_{\text{grp}}(SO(n+1); \mathbb{Z}) \]

Since the canonical map is injective (see [7]), the class \(e_{SO(n+1)}\) is non-trivial and so is the class \(e_G\). Thus, we obtain the following theorem.

Theorem 3.2. Let \(M\) be the \(n\)-sphere and \(\Omega\) the normalized standard volume form. Then the classes \([c_k] \in H^{n+1}_{\text{grp}}(G; \mathbb{Z})\) and \([b_k] \in H^n_{\text{grp}}(G; S^1)\) are non-trivial.

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