INFINITE TRANSITIVITY FOR CALOGERO-MOSER SPACES

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Abstract. We prove the conjecture of Berest-Eshmatov-Eshmatov by showing that the group of automorphisms of a product of Calogero-Moser spaces $C_{n_i}$, where the $n_i$ are pairwise distinct, acts $m$-transitively for each $m$.

1. Introduction

For affine algebraic varieties, their automorphism groups are usually small. However, if they are rich, such varieties and their automorphism groups become objects of intensive study. If an automorphism group is infinite dimensional, it may satisfy the property called infinite transitivity: for any $m \in \mathbb{N}$ the group can send any $m$-tuple of points of the variety to any other $m$-tuple of points. We study Calogero-Moser spaces and their products and show that their automorphism groups are infinitely transitive.

Definition 1. The Calogero-Moser space $C_n$ is

$C_n := \{(X, Y) \in \text{Mat}_n(\mathbb{C}) \times \text{Mat}_n(\mathbb{C}) : \text{rk}([X, Y] + I_n) = 1\}/PGL_n(\mathbb{C})$,

where $PGL_n(\mathbb{C})$ acts via $g.(X, Y) = (gXg^{-1}, gYg^{-1})$.

Calogero-Moser spaces play an important role in Representation Theory. It is known that $C_n$ is a smooth irreducible affine algebraic variety of dimension $2n$, see Wilson [14]. It is proved by Popov [11] that it is unirational. It carries a symplectic structure, see [7]. It is a particular case of a Nakajima quiver variety. It appears as a partial compactification of the Calogero-Moser integrable system.

Definition 2. We denote by $G$ the group generated by two kinds of transformations.

(1) $(X, Y) \mapsto (X + p(Y), Y)$, $p$ is a polynomial in one variable,

(2) $(X, Y) \mapsto (X, Y + q(X))$, $q$ is a polynomial in one variable.

It is isomorphic to the group of automorphisms of the first Weyl algebra [6] [10].

Formulae (1) and (2) can be used to define the $G$-action on $\text{Mat}_n(\mathbb{C}) \times \text{Mat}_n(\mathbb{C})$. This action descends to $C_n$. To verify this, check two things. First, formulae (1) and (2) agree with the $PGL_n(\mathbb{C})$-action. Second, the obtained points remain inside $C_n$. Indeed, $[X + p(Y), Y] = [X, Y] = [X, Y + q(X)]$, hence,

$\text{rk}([X + p(Y), Y] + I_n) = \text{rk}([X, Y + q(X)] + I_n) = \text{rk}([X, Y] + I_n) = 1$.

Theorem 1. ([5, Theorem 1]) For each $n \geq 1$, the action of $G$ on $C_n$ is doubly transitive.

The conjecture in [5] says, in particular, that $C_n$ has an infinitely transitive action of its automorphism group. It is proved below in Theorem 3.

There is a more general class of varieties: for any pairwise distinct integers $n_1, n_2, \ldots, n_k$ one can consider the product of the corresponding Calogero-Moser spaces

(3) $C_{n_1} \times C_{n_2} \times \ldots \times C_{n_k}$.

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The group $G$ acts diagonally on this product. It also acts on

$$(4) \quad C_{n_1} \sqcup C_{n_2} \sqcup \ldots \sqcup C_{n_k}.$$ 

Moving a finite number of points on the product $(3)$ can be seen as moving a finite number of points on $C_{n_1} \sqcup C_{n_2} \sqcup \ldots \sqcup C_{n_k}$. For these actions, we consider the property of collective infinite transitivity.

**Definition 3.** We say that the $G$-action on $(3)$ or on $(4)$ is **collectively infinite transitive** if for any integers $m_1, m_2, \ldots, m_k$ and for any two tuples of $m_1$ points on the first variety $C_{n_1}$, $m_2$ points on the second variety $C_{n_2}$, etc., $m_k$ points on the $k$th variety $C_{n_k}$ there exists an element of $G$ which simultaneously sends the first tuple to the second tuple.

**Theorem 2.** ([5] Theorem 2) For any pairwise distinct natural numbers $(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$, the diagonal action of $G$ on $C_{n_1} \times C_{n_2} \times \ldots \times C_{n_k}$ is transitive.

If $n_i = n_j$, then $C_{n_i} \times C_{n_j}$ has a diagonal subvariety which remains invariant under the diagonal $G$-action.

The Conjecture in [5] states that the $G$-action on $C_{n_1} \times C_{n_2} \times \ldots \times C_{n_k}$ is collectively infinitely transitive. We prove this conjecture in Theorem (5). The key ingredient of the proof is that, whenever $X$-components of the given points have pairwise coprime minimal polynomials, the given points can be moved independently via automorphisms of form $(2)$.

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2. **Geometry of Calogero-Moser spaces and their automorphisms**

We recall here some facts on the geometry of $C_n$ from [14, Sec. 1] and then strengthen them to apply to the products of Calogero-Moser spaces of form $(3)$ and $(4)$. We also use results of Berest and Wilson [4].

**Lemma 1.** ([14] Prop. 1.10) If $(X, Y) \in C_n$ and if $X$ is diagonal, then the eigenvalues of $X$ are distinct. The non-diagonal entries of $Y$ have the form

$$y_{ij} = 1/(x_i - x_j).$$

**Proof.** The proof is straightforward. \hfill \Box

If $X$ is diagonalizable but not diagonal, the point $(X, Y)$ of $C_n$ has another representative $(AXA^{-1}, AYA^{-1})$ where the new $X$ is diagonal and we can express all the non-diagonal entries of $Y$ in entries of $X$.

**Lemma 2.** ([14] Lemma 10.2) If $(X, Y) \in C_n$ and $(X, Y') \in C_n$ with $X$ diagonal, then there exist a polynomial $p$ in one variable such that $(X, Y) \mapsto (X, Y + p(X)) = (X, Y')$.

**Proof.** By Lemma \[1\] matrices $Y$ and $Y'$ may differ only in diagonal entries, denote them by $y_{11}, \ldots, y_{nn}$ and $y'_{11}, \ldots, y'_{nn}$. Let $X = \text{Diag}(x_1, \ldots, x_n)$. Since all the $x_i$ are different, there exists an interpolation polynomial $p(x)$ such that $p(x_i) = y'_{ii} - y_{ii}$. But $f(X) = \text{Diag}(f(x_1), \ldots, f(x_n))$ and hence $Y + p(X) = Y'$. \hfill \Box

**Remark 1.** Let $(X_0, Y_0) \in C_n$. If a polynomial in one variable $q(x)$ is divisible by the minimal polynomial of $Y_0$, then the automorphisms $(X, Y) \mapsto (X + p(Y), Y)$ and $(X, Y) \mapsto (X + p(Y) + q(Y), Y)$ send $(X_0, Y_0)$ to the same point.
Lemma 3. Suppose that square matrices $X_1$, $X_2$, $X_m$ (possibly of different sizes) have pairwise coprime minimal polynomials. Take $(X_1, Y_1), (X_2, Y_2), \ldots, (X_m, Y_m)$, where each $Y_i$ is a square matrix of the same size as $X_i$, and polynomials $p_1, p_2, \ldots, p_m \in k[x]$. Then there exists a polynomial $p \in k[x]$ such that for each $i$ we have

$$Y_i + p_i(X_i) = Y_i + p(X_i).$$

Proof. By Remark 1, each $p_i$ is defined modulo the minimal polynomial $\chi_i$ of $X_i$. Since $\chi_1, \chi_2, \ldots, \chi_m$ are pairwise coprime, by the Chinese remainder theorem there exists a polynomial $p$ such that for each $i = 1, 2, \ldots, m$ the polynomial $p - p_i$ is divisible by $\chi_i$. \[\square\]

Lemma 4 (Refinement of Lemma [2]). Take two $m$-tuples of points of $C_{n_1} \sqcup C_{n_2} \sqcup \ldots \sqcup C_{n_k}$

$$(X_1, Y_1), (X_2, Y_2), \ldots, (X_m, Y_m)$$

and

$$(X_1, Y_1'), (X_2, Y_2'), \ldots, (X_m, Y_m')$$

(so, each $C_{n_i}$ contains an even number of chosen points). Suppose that $X_1, \ldots, X_m$ are diagonalizable and have pairwise coprime minimal polynomials (equivalently, diagonalizable and with disjoint spectra). Then there exists a polynomial $p(x) \in k[x]$ such that for each $i$ we have

$$Y_i + p_i(X_i) = Y_i'.$$

Proof. First, by Lemma 2 we choose a polynomial $p_i(x) \in k[x]$ such that $Y_i' = Y_i + p(X_i)$ for each $i = 1, 2, \ldots, m$. Then by Lemma 3 we find a polynomial $p(x)$ which works for all $i$. \[\square\]

The following lemma is a refined Lemma 10.3 from [4]. Its proof is explained in [14, Lemma 5.6] and also in [13, 12 Prop. 8.6].

Lemma 5. Let $(X, Y) \in C_n$. Then there exists a polynomial $p$ such that the matrix $X + p(Y)$ is diagonalizable.

By almost all we mean a cofinite subset of the set of complex numbers, i.e., all complex numbers but finitely many. The following fact can be deduced from Lemma 5.

Lemma 6. a) Let $(X, Y) \in C_n$. Then there exists a polynomial $p$ such that the matrix $X + t \cdot p(Y)$ is diagonalizable for almost all $t$.

b) Let us fix $m, m \in \mathbb{N}$, and take an $m$-tuple of points of $C_n$. Then one can make all the $2m$ matrices diagonalizable via $2m$ automorphisms of form (1) and (2).

c) Let us take $m_1$ points on the first variety $C_{n_1}$, $m_2$ points on the second variety $C_{n_2}$, etc., $m_k$ points on the $k$th variety $C_{n_k}$. Then all the matrices (i.e., $X$- and $Y$-components of our points) can be made diagonalizable via $2(m_1 + m_2 + \ldots + m_k)$ automorphisms of form (1) and (2).

Proof. a) Take a polynomial $p$ as in Lemma 5. For this polynomial consider $X + t \cdot p(Y)$. For $t = 1$ it is diagonalizable, hence, it is diagonalizable for almost all $t$.

b) Using Lemma 5, make $X_1$ diagonalizable. Then, acting as in a), find a polynomial $p_2$ such that $X_2 + p_2(Y_2)$ is diagonalizable. Consider an automorphism $(X, Y) \mapsto (X + t \cdot p_2(Y))$. For $t = 0$ the matrix $X_1$ maps to a diagonalizable matrix, hence, it also maps to a diagonalizable matrix for almost all $t$. For $t = 1$ the matrix $X_2$ maps to a diagonalizable matrix, hence, it also maps to a diagonalizable matrix for almost all $t$. Now $X_1$ and $X_2$ are diagonalizable. In this way we can make all the $X_i$ diagonalizable. Then in the same way we make all the $Y_i$ diagonalizable, while the $X_i$ remain unchanged and hence diagonalizable.

The proof is exactly the same as in b). \[\square\]
There is a map $\Upsilon: \mathcal{C}_n \to (\mathbb{C}^n/S_n) \times (\mathbb{C}^n/S_n)$ which sends $X$ and $Y$ to their spectra, where $S_n$ stands for the symmetric group on an $n$-element set. By $\Upsilon_1$ and $\Upsilon_2$ we mean projections to the first and to the second components, respectively. One of the key statements is

**Lemma 7** (Prop. 4.15 and Theorem 11.16 in [8]). The map $\Upsilon$ is surjective.

**Lemma 8.** Take an $n \times n$ matrix $Y$ with a simple spectrum $(\mu_1, \mu_2, \ldots, \mu_n)$. Fix pairwise distinct $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. Then there exists $(X, Y) \in \mathcal{C}_n$ such that $X$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

**Proof.** Since $\Upsilon$ is surjective, there is a point $(X', Y')$ such that

$$\Upsilon((X', Y')) = ((\lambda_1, \lambda_2, \ldots, \lambda_n), (\mu_1, \mu_2, \ldots, \mu_n)).$$

Since $m_i$ are pairwise distinct, $Y'$ is conjugate to $Y$, that is, there exists a matrix $A$ such that $Y = AY'A^{-1}$. Take $X = AX'A^{-1}$. Clearly, $(X, Y)$ is the same point of $\mathcal{C}_n$ as $(X', Y')$ and $X$ has the prescribed spectrum. \hfill \Box

**Remark 2.** In [5], the fibers of $\Upsilon_1$ over nilpotent Jordan blocks are used. The advantage is that $X^n = 0$. We use the fibers over diagonalizable $X$ (hence having simple spectra) since they can be easily described.

## 3. Main results

We are ready to prove our main result.

**Theorem 3.**

1. The group of automorphisms of a Calogero-Moser space $\mathcal{C}_n$ acts infinitely transitively.
2. The group of automorphisms of a product of Calogero-Moser spaces $\mathcal{C}_{n_1} \times \mathcal{C}_{n_2} \times \ldots \times \mathcal{C}_{n_k}$, where $n_1, n_2, \ldots, n_k$ are pairwise distinct, acts collectively infinitely transitively.

We prove these statements together since their proofs are almost identical.

**Proof.** We use the two-transitivity of the $G$-action on $\mathcal{C}_n$ and on $\mathcal{C}_{n_1} \times \mathcal{C}_{n_2} \times \ldots \times \mathcal{C}_{n_k}$ which is established in [5]. In the second case, the two-transitivity can mean two different things. First, when two points are in the same $\mathcal{C}_{n_i}$, then the two-transitivity on the product follows from the two-transitivity on $\mathcal{C}_{n_i}$ proved in [5] Theorem 1. Second, when two points belong to different $\mathcal{C}_{n_i}$ and $\mathcal{C}_{n_j}$, then it follows from [5] Theorem 2.

By Lemma 8, there exists an automorphism making all the matrices diagonalizable. In Step 1 we show that the spectra of all the matrices can be assumed disjoint via several extra automorphisms.

Step 1 for Theorem 3(a). Let us draw a graph on $n$ vertices. An edge $ij$ is drawn if and only if $(X_i$ and $X_j$ have no common eigenvalue) and $(Y_i$ and $Y_j$ have no common eigenvalue). Using the two-transitivity, we obtain at least one edge: we fix two pairs $(X_1^0, Y_1^0)$ and $(X_2^0, Y_2^0) \in \mathcal{C}_n$ with disjoint spectra and send $(X_1, Y_1)$ and $(X_2, Y_2)$ there. Now let us create new edges. If $i$ and $j$ are not joined because $X_i$ and $X_j$ have a common eigenvalue, then find a simple spectrum for $X_i$ disjoint from the spectra of all the other $X_k$. By Lemma 8 there is a pair $(X'_i, Y'_i) \in \mathcal{C}_n$ with the prescribed spectrum for $X'_i$. Using 2-transitivity, find an automorphism mapping $(X_i, Y_i)$ to $(X'_i, Y'_i)$ and $(X_j, Y_j)$ to $(X'_j, Y'_j)$. Include it into a one-parameter family. For $t$ close to 1 the $X$-image of $j$th point will have spectrum disjoint from all the images of the other $X_i$. We do not want to break edges, so if for $t = 1$ an inequation breaks, we choose another $t$ close to 1. Then we perform the same to disconnect spectra of $Y_i$ and $Y_j$. We further assume that all the spectra of $X_i$ are disjoint and all the spectra of $Y_j$ are disjoint.
Step 1 for Theorem 3b) is proved similarly. When we need 2-transitivity for points in one component, we rely on [5, Theorem 1], and when we need it for two points from different components, we use [5, Theorem 2].

Step 2. Now we need the interpolation polynomial. Let us choose representatives with all the \( X_i \) diagonal. We know how a triangular automorphism \((X, Y) \mapsto (X, Y + p(X))\) looks like: the non-diagonal elements of all the \( Y_i \) do not change, and the \( k \)th diagonal element of the corresponding \( Y_i \) increases by \( p(\lambda_{ki}) \), where \( \lambda_{ki} \) is the \( k \)th diagonal element of the matrix \( X_i \).

To obtain the \( m \)-transitivity, let us take two \( m \)-tuples of points and perform with this \( 2m \)-tuple both Steps 1 and 2. Using Lemma [8], find \( m \) intermediate points of \( C_n \)

\[
(X_1, Y_1''), \quad \text{where } Y_1'' \text{ has the same spectrum as } Y_1';
\]

\[
\ldots;
\]

\[
(X_m, Y_m''), \quad \text{where } Y_m'' \text{ has the same spectrum as } Y_m'.
\]

By Lemma [2] there is an automorphism \( Y \mapsto Y + p(X) \) which sends each \( Y_i \) to the chosen matrix \( Y_i'' \).

Now for each point find a representative with \( Y \) diagonal and make the same interpolation with \( X \) and \( Y \) reversed. \( \square \)

Final remarks. For a variety \( X \), one can generate a group by all the one-parameter unipotent subgroups of \( \text{Aut}(X) \). This subgroup denoted by \( \text{SAut}(X) \) is treated in [9, 3, 1, 2]. It is shown in [1] that infinite transitivity of \( \text{SAut}(X) \) on the smooth locus \( \text{reg}(X) \) for \( \dim X \geq 2 \) is equivalent to simple transitivity and is equivalent to flexibility property which means that the tangent space \( T_xX \) in every smooth point \( x \in X \) is generated by tangent vectors to the orbits of one-parameter unipotent subgroups. We fix attention that this fact is not easily applicable to \( C_n \) since natural automorphisms \((X, Y) \mapsto (X + p(Y), Y) \) and \((X, Y) \mapsto (X, Y + q(X)) \) do not come with all their (one-parameter unipotent) rescalings.

On the other hand, it is not known whether the group \( G \) coincides with \( \text{SAut}(C_n) \).

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