ON SEMIFREE SYMPLECTIC CIRCLE ACTIONS
WITH ISOLATED FIXED POINTS

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Abstract. Let $M$ be a symplectic manifold, equipped with a semifree symplectic circle action with a finite, nonempty fixed point set. We show that the circle action must be Hamiltonian, and $M$ must have the equivariant cohomology and Chern classes of $(P^1)^n$.

1. Introduction

Let $(M^{2n}, \omega)$ be a compact, connected symplectic manifold of dimension $2n$. A circle action on $M$ is symplectic if it preserves the symplectic form; that is, if its generating vector field $X$ satisfies $L_X \omega = d i_X \omega = 0$. A particular case is that of a Hamiltonian circle action, where $i_X \omega$ is exact; in this case $i_X \omega = d \mu$ where $\mu \in C^\infty(M)$ is the moment map. A great deal is known about Hamiltonian actions. For example, the quantization and the push-forward measure are determined by fixed point data, and other manifold invariants such as cohomology and Chern classes are constrained by this information.

A Hamiltonian circle action on a compact manifold must have fixed points (one way to see this is that the minimum of the moment map must be a fixed point). In the case of a Kahler manifold [1], or of a four-dimensional symplectic manifold [2], the existence of a fixed point guarantees that a symplectic circle action must in fact be Hamiltonian. This is not true in higher dimensions: McDuff [3] has constructed a symplectic six-manifold with a symplectic circle action which has fixed points, but is not Hamiltonian. In this example the fixed point sets are tori; and there are no known examples of symplectic, non-Hamiltonian circle actions with isolated fixed points. A natural question is whether any such examples can exist.

In this paper we focus on the case of semi-free circle actions, and show that any semi-free, symplectic circle action with isolated fixed
points is Hamiltonian if and only if it has a fixed point. Recall that an action of a group $G$ on a manifold $M$ is **semi-free** if the action is free on $M \setminus M^G$. Our main result is the following.

**Theorem 1.** Let $(M^{2n}, \omega)$ be a compact, connected symplectic manifold, equipped with a semifree, symplectic circle action with isolated fixed points. Then if $M^{S^1}$ is nonempty, the circle action must be Hamiltonian.

We prove this in Section 3 by an argument using integration in equivariant cohomology.

Theorem 1 brings us to the realm of Hamiltonian circle actions. Though a great deal is known about these, the general classification problem for Hamiltonian $S^1$ spaces remains open. However, they have been classified in dimensions 2 and 4. In particular, up to symplectomorphism, $P^1$ is the only two dimensional example, and $(P^1)^2$ is the only four dimensional symplectic manifold with a semi-free Hamiltonian circle action with isolated fixed points. In higher dimensions, much less is known: in the semi-free case with isolated fixed points, the only example we know of is $(P^1)^n$, equipped with the diagonal circle action. As it turns out, the classical manifold invariants—that is, cohomology and Chern classes—of any such space must concord with those of $(P^1)^n$.

Specifically, we prove the following theorem, which is an equivariant version of Theorem 1 below, due to Hattori [H].

**Theorem 2.** Let $(M, \omega)$ be a compact, connected symplectic manifold, equipped with a semifree, Hamiltonian circle action with isolated fixed points. Let $i : M^{S^1} \to M$, $j : (P^1)^n^{S^1} \to (P^1)^n$ denote the natural inclusions of the fixed point sets of $M$ and $(P^1)^n$, respectively.

There exists a map from $M^{S^1}$ to $((P^1)^n)^{S^1}$ which identifies the images of $i^* : H^*_S(M, \mathbb{Z}) \to H^*_S(M^{S^1}, \mathbb{Z})$ and $j^* : H^*_S((P^1)^n, \mathbb{Z}) \to H^*_S(((P^1)^n)^{S^1}, \mathbb{Z})$. This map sends the images of the equivariant Chern classes of $M$ to those of $(P^1)^n$.

For any compact symplectic manifold $M$ with a Hamiltonian circle action, a theorem of Kirwan [K] states that the natural inclusion map $i : M^{S^1} \to M$ of the fixed set $M^{S^1}$ into $M$ induces an injection $i^* : H^*_S(M, \mathbb{Q}) \to H^*_S(M^{S^1}, \mathbb{Q}) = H^*_S(M^{S^1}, \mathbb{Q}) \otimes H^*(BS^1, \mathbb{Q})$. In the case of a circle action with isolated fixed points, this injectivity theorem

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1 An action of a group $G$ on a manifold $M$ is **quasi-free** if the stabilizers of points are connected; in the case $G = S^1$ this is the same as the action being semi-free.

2 The condition that the fixed points be isolated implies the manifold is simply connected.
holds for integral cohomology as well (see [TW]). Thus Theorem 2 has the following corollaries:

**Theorem 1.1.** There is an isomorphism between $H^*_S(M, \mathbb{Z})$ and $H^*_S((P^1)^n, \mathbb{Z})$ which takes the equivariant Chern classes of $M$ to the equivariant Chern classes of $(P^1)^n$.

Thus the equivariant cohomology ring is given by $H^*_S(M, \mathbb{Z}) = \mathbb{Z}[a_1, \ldots, a_n, y]/(a_i y - a_i^2)$, and the equivariant Chern series $c_i(M) = \sum_i t^i c_i(M)$ is given by

$$c_i(M) = \prod_i (1 + t(2a_i - y)).$$

**Theorem 1.2.** (Hattori [H]) There is an isomorphism between $H^*(M, \mathbb{Z})$ and $H^*((P^1)^n, \mathbb{Z})$ which takes the Chern classes of $M$ to the Chern classes of $(P^1)^n$.

Thus, the cohomology ring is given by $H^*(M, \mathbb{Z}) = \mathbb{Z}[a_1, \ldots, a_n]/(a_i^2)$, and the Chern series is given by

$$c_i(M) = \prod_i (1 + t2a_i).$$

**Remark 1.3.** In dimension 6, $M$ must be diffeomorphic to $(P^1)^3$. This follows by a theorem of Wall [W] since $M$ is a simply connected symplectic manifold with the cohomology ring and Chern classes of $(P^1)^3$.

The basic idea underlying the proof of Theorem 2 is the use of Morse theory and integration in equivariant cohomology to obtain a good picture of the cohomology ring of the manifold in terms of the fixed point set. This argument is given in Section 4.

Our final result shows that the reduced spaces of any semifree Hamiltonian $S^1$-space with isolated fixed points have the cohomology rings and Chern classes of the reduced spaces of $(P^1)^n$. This corollary of Theorem 2 and of the results of [TW] is stated in Section 5 (Corollary 1).

2. Equivariant Cohomology

In this section we review equivariant cohomology. The integration formula (Proposition 2.1) is the main technical tool in this paper, and will occur repeatedly.

Let $ES^1$ denote a contractible space on which $S^1$ acts freely. Let $BS^1 = ES^1/S^1$, and note that $H^*(BS^1, \mathbb{Z})$ is the polynomial ring in a single generator $x \in H^2(BS^1, \mathbb{Z})$. 

If $S^1$ acts on a manifold $M$, define $H^*_S(M,\mathbb{Z}) = H^*(M \times S^1, ES^1, \mathbb{Z})$. In particular, if $P$ is a point, $H^*_S(P,\mathbb{Z})$ is naturally isomorphic to $H^*(BS^1,\mathbb{Z})$; we will slightly abuse notation by identifying these.

The projection $p : M \times S^1 \to BS^1$ induces a pull-back map $H^*(BS^1,\mathbb{Z}) \to H^*_S(M,\mathbb{Z})$; this makes $H^*_S(M,\mathbb{Z})$ into a $H^*(BS^1,\mathbb{Z})$ module. The projection $p$ also induces a push forward map $p_* : H^*_S(M,\mathbb{Z}) \to H^*(BS^1,\mathbb{Z})$ given by “integration over the fibre”; we will usually denote $p_*$ by the symbol $\int_M$. When $M$ is compact and the fixed points are isolated, this has the following expression:

**Proposition 2.1.** Let $M$ be a compact manifold equipped with an action of $S^1$ with isolated fixed points. Let $\alpha \in H^*_S(M,\mathbb{Z})$. Then as elements of $\mathbb{Q}(x)$,

$$\int_M \alpha = \sum_F \frac{\alpha|_F}{e(\nu_F)},$$

where the sum is taken over all fixed points $F$, $\nu(F)$ is the normal bundle to $F$ and $e(\nu_F)$ is the equivariant Euler class of this bundle.

The right hand side of this equation is particularly simple in the case where $M$ has an invariant almost complex structure. The computation of the Chern classes of $\nu_F$ is then given by the following lemma.

**Lemma 2.2.** Let $E$ be a a representation of $S^1$, considered as complex $S^1$-equivariant vector bundle over a point $P$. The representation $E$ decomposes as a direct sum $E = \bigoplus E_i(w_i)$, where $E_i(w_i)$ is a complex line on which the circle acts with weight $w_i$. The equivariant Chern series $c_t(E) = \sum_i t^i c_i(E)$ is given by

$$c_t(E) = \prod_i (1 + tw_ix),$$

where $x$ is the generator of $H^2_S(P,\mathbb{Z})$.

In particular, the Euler class of a representation is $x^n$ times the product of the weights of the circle action, and the first Chern class is $x$ times the sum of the weights.

### 3. Existence of the Moment Map: Proof of Theorem II

In this section, we prove Theorem II. We do this by applying the integration formula of Proposition 2.1 to the characteristic classes of our manifold to prove that if the manifold has any fixed points, then it has a fixed point with no negative weights. The following argument of McDuff [M] then shows that this implies that the action is Hamiltonian.

First, the symplectic form can be deformed to a rational invariant symplectic form. Furthermore, a symplectic manifold equipped
with a symplectic circle action can be given a compatible invariant almost-complex structure. Although this almost-complex structure is not unique, both the Chern classes and the weights at every fixed point are well-defined. By Lemma 3.1 below, if \( M \) has any fixed points it has a fixed point \( F \) with no negative weights.

In the case of a manifold with an invariant rational symplectic form, the circle action will possess a circle valued moment-map. Since the fixed point \( F \) is a local minimum for this circle-valued moment map, it must lift to a real-valued moment map. This real-valued moment map is a Morse function with only even index critical points, so our manifold is simply connected. Therefore any symplectic circle action on \( M \) is Hamiltonian.

It remains to prove Lemma 3.1.

Note that since the fixed points are all isolated, none of the weights on the normal bundles to the fixed points can be zero. Additionally, since the action is semifree, all these weights are \( \pm 1 \).

**Lemma 3.1.** Let \( M^{2n} \) be a compact manifold equipped with a semifree circle action with isolated fixed points. Suppose \( M \) has an invariant almost-complex structure. Let \( N_k \) denote the number of fixed points of the circle action with \( k \) negative weights. Then for all \( k \) with \( 0 \leq k \leq n \)

\[
N_k = N_0 \frac{n!}{k!(n-k)!}.
\]

**Proof.** Let \( P \) be a point, and let \( x \) be the generator of \( H^2_\mathbb{S}^1(P, \mathbb{Z}) \). Let \( y \in H^2_\mathbb{S}^1(M, \mathbb{Z}) \) be the pull-back of \( x \) to \( M \) by the map \( p : M \to P \). Consider the equivariant cohomology class \( \gamma = \frac{1}{2}(ny - c_1(TM)) \).

For dimensional reasons, for any \( l \) with \( 0 \leq l < n \),

\[
\int_M \gamma^l = 0.
\]

Applying the equivariant integration formula of Proposition 2.1 yields

\[
\int_M \gamma^l = \sum_F \frac{\gamma^l|_F}{e(\nu_F)},
\]

where \( e(\nu_F) \) is the equivariant Euler class of the tangent bundle at \( F \) and the sum is over all fixed points \( F \).

For every fixed point \( F \), let \( k_F \) be the number of negative weights of the circle action on the normal bundle \( \nu_F \). Applying Lemma 2.2, we see that

\[
\gamma|_F = k_F x \quad \text{and} \quad e(\nu_F) = (-1)^{k_F} x^n.
\]
Thus for all \( l \) with \( 0 \leq l < n \),
\[
\sum_{F} (k_F)^l (-1)^{k_F} = 0.
\]
Equivalently, for all \( l \) with \( 0 \leq l < n \),
\[
(3.2) \quad \sum_{k=0}^{n} N_k k^l (-1)^k = 0.
\]
Define a \( n \times (n + 1) \) dimensional matrix \( V \) whose entries are given by
\[
V_{i,j} = j^i
\]
for \( 0 \leq i < n \) and \( 0 \leq j \leq n \). Since this matrix consists of the first \( n \) rows of a nonsingular \( (n + 1) \times (n + 1) \) Vandermonde matrix, it has rank \( n \).

Let \( A \) be the column vector whose entries are \( A_k = (-1)^k N_k \). Equation 3.2 can be written as a matrix equation
\[
VA = 0.
\]
Up to multiplication by a constant, there is a unique solution to equation 3.2. Moreover, the fixed point data of \( (P^1)^n \) is a solution.

Remark 3.3. Our methods generalize to the case of non-semifree circle actions with finite non-empty fixed point sets and give constraints on the possible fixed point data of such actions. In the semifree case above these constraints eliminate all non-Hamiltonian examples. In the general case, some possibilities cannot be ruled out by our methods. For example, we cannot rule out the existence of a symplectic six-manifold, equipped with a symplectic circle action which has two fixed points, with the action having weights \((1,1,-2)\) on the normal bundle to one fixed point and \((-1,-1,2)\) on the normal bundle to the other.

4. The cohomology ring: Proof of Theorem 2

In this section we prove Theorem 2. The method of proof again involves the equivariant fixed point formula (Proposition 2.1), but this time we apply the formula not only to the Chern classes of the manifold \( M \) but also to additional classes. To construct these classes we use the following theorems, which are proved using Morse theory.

Theorem 4.1. (Frankel, Kirwan) Let a circle act on a a symplectic manifold \( M \) with moment map \( \mu : M \to \mathbb{R} \). Then the moment map \( \mu \) is a perfect Morse function on \( M \) (for both ordinary and equivariant cohomology). The critical points of \( \mu \) are the fixed points of \( M \), and
the index of a critical point $F$ is precisely twice the number of negative weights of the circle action on $TM_F$.

**Theorem 4.2.** (Kirwan) Let a circle act on a a symplectic manifold $M$ in a Hamiltonian fashion. Let $i : M^{S^1} \rightarrow M$ denote the natural inclusion. The induced map $i^* : H^{S^1}_*(M^{S^1}, Q) \rightarrow H^{S^1}_*(M, Q)$ is injective.

More precisely, we need to take the following result from Kirwan’s proof of Theorem 4.1 and Theorem 4.2 ([K], see also [TW]). For simplicity, we restrict our attention to the case of circle actions with isolated fixed points.

**Theorem 4.3.** Let a circle act on a a symplectic manifold $M$ in a Hamiltonian fashion with isolated fixed points. Let $F$ be any fixed point of index $2k$. Let $w_1, \ldots, w_k$ be the negative weights of the circle action on $TM_F$. Then there exists a class $a_F \in H^{2k}_{S^1}(M, \mathbb{Z})$ such that $a_F|_F = (-1)^k x^k \prod_{i=1}^k w_i$ and $a_F|_{F'} = 0$ for all fixed points $F'$ of index less than $2k$. Moreover, taken together over all fixed points, these classes are a basis for the cohomology $H^{S^1}_*(M, \mathbb{Z})$ as a $H^*(BS^1, \mathbb{Z})$ module.

When the action is semi-free, there is a unique way to choose the class $a_F$ so that $a_F|_{F'} = 0$ for all other fixed points $F'$ of index less than or equal to $2k$.

The first step is to analyze these cohomology classes on $(P^1)^n$. Our strategy will then be to show their counterparts on $M$ mimic their behavior. Each fixed point in $(P^1)^n$ is on the north pole of some set of spheres, and on the south pole for all the other spheres. Thus, if we identify the spheres with the integers $1$ through $n$, the fixed points $q_k^j$ of index $2k$ are in one-to-one correspondence with subsets $J \subset \{1, \ldots, n\}$ with $k$ elements. Consider two fixed points $F$ and $F'$, which correspond to subsets $J$ and $J'$ of $\{1, \ldots, n\}$, with $k$ and $k'$ elements, respectively. Let $\alpha_F$ be the cohomology class associated to $F$ as in Theorem 1.3. Then it is easy to see that $\alpha_F|_{F'} = x^k$ if $J \subset J'$, otherwise $\alpha_F|_{F'} = 0$.

We now study the cohomology group $H^{S^1}_*(M, \mathbb{Z})$. Denote by $p^j_i$ the critical points of index $2k$ in $M$. By Theorem 1.3, for each $1 \leq j \leq n$, we can find a class $a_j$ such that $a_j|_{p^j_i} = x$ and

\[(4.4) \quad a_j|_F = 0 \text{ for all other critical points } F \text{ of index 0 or 1.}\]

In the remainder of this section, we will prove that these forms satisfy the following Proposition:
Proposition 4.5. Let $J$ be a subset of \{1, \ldots , n\} with $k$ elements. There exists a unique fixed point $F$ of index $2k$ such that $a_j|_F = x$ if and only if $j \in J$, and $a_j|_F = 0$ otherwise.

By identifying the fixed points of $M$ with subsets of \{1, \ldots , n\}, this Proposition gives an isomorphism between the fixed point set of $M$ and the fixed point set of $(P^1)^n$. We claim that this map identifies the images of $i^*: H^*_{S^1}(M, \mathbb{Z}) \to H^*_{S^1}(M^{S^1}, \mathbb{Z})$ and $j^*: H^*_{S^1}((P^1)^n, \mathbb{Z}) \to H^*_{S^1}(((P^1)^n)^{S^1}, \mathbb{Z})$, and also identifies the images of the equivariant Chern classes.

First, this map sends the classes $a_i$ to the classes $\alpha_i := \alpha_{q_i}$. More generally, let $F$ be any fixed point of index $2k_F$, and let $J_F$ be the corresponding subset of \{1, \ldots , n\}. The cohomology class $a_F = \prod_{j \in J_F} a_j$ is the unique class which restricts to $x^k$ on $F$, and to 0 on all other fixed points of index less than or equal to $2k_F$. Now for any other fixed point $F'$ with an associated subset $J_{F'}$, $a_F|_{F'} = x^{k_F}$ exactly if $J_F \subset J_{F'}$, and otherwise $a_F|_{F'} = 0$. Since the same is true for the $\alpha_F$, this map also takes the classes $a_F$ to the classes $\alpha_F$. Since these classes form a basis for the cohomology as a $H^*(BS^1, \mathbb{Z})$ module, this proves the first claim.

Moreover, the restriction of an equivariant Chern class of any manifold to the fixed point set is the equivariant Chern class of the normal bundle to that fixed set. For isolated fixed points, this class is determined by the weights of the circle action on the normal bundle. Since this map takes every fixed point in $M$ to fixed point in $(P^1)^n$ with the same weights, it takes equivariant Chern classes to equivariant Chern classes.

Thus, it remains to prove Proposition 4.5. To do this we will repeatedly use the following generalization of the technique used in the last section.

Lemma 4.6. Let $M$ be a compact symplectic manifold with a semi-free Hamiltonian circle action with isolated fixed points. Let $d \in H^2_{S^1}(M, \mathbb{Z})$ and $\delta \in H^2_{S^1}((P^1)^n, \mathbb{Z})$ be equivariant cohomology classes of degree $2l$.

If there exist $l + 1$ integers $k$ with $0 \leq k \leq n$ such that

$$\sum_i d|_{p_k} = \sum_i \delta|_{q_k},$$

then for all integers $k$ with $0 \leq k \leq n$,

$$\sum_i d|_{p_k} = \sum_i \delta|_{q_k}.$$

Here, the sum is always taken over all fixed points of index $2k$. 


Proof. For dimensional reasons, for all $i$ with $0 \leq i < n - l$,

$$\int_M d \cdot \gamma^i = 0$$

On the other hand, applying the formula in Proposition 2.1,

$$\int_M d \cdot \gamma^i = \sum_{k=0}^{n} \sum_{i} (-1)^k k^i x^i.$$

Thus, for all $i$ with $0 \leq i < n - l$,

$$\sum_{k=0}^{n} \sum_{i} (-1)^k d|_{p_k^i} k^i = 0. \quad (4.7)$$

Define a matrix $V^{n-l}$ which has $n - l$ columns and $n + 1$ rows, and with entries given by $V^{n-l}_{ij} = j^i$, where $0 \leq i \leq n$ and $0 \leq j < n - l$. The matrix $V^{n-l}$ consists of the first $n - l$ rows of an $(n + 1) \times (n + 1)$ Vandermonde matrix.

Define a column vector $D$ by

$$D_k = (-1)^k \sum_{i} d|_{p_k^i}$$

for all $0 \leq k \leq n$, where the sum is taken over all fixed points of index $2k$. Then equation (4.7) is equivalent to the matrix equation

$$V^{n-l} D = 0.$$

Similarly, define a column vector $\Delta$ by $\Delta_k = (-1)^k \sum_{i} \delta|_{q_k^i}$ for all $k$ with $0 \leq k \leq n$, where the sum is taken over all fixed points of index $2k$. An argument identical to that given above shows that the vector $\Delta$ satisfies the matrix equation $V^{n-l} \Delta = 0$.

Finally, by assumption, there are $l + 1$ integers $0 \leq k \leq n$ such that $D_k = \Delta_k$. The remaining entries of the difference $D - \Delta$ form an $(n - l)$-dimensional column vector which lies in the kernel of a nonsingular $(n - l) \times (n - l)$-dimensional Vandermonde matrix. Thus $D = \Delta$. $\square$

Lemma 4.8. For any $1 \leq j \leq n$ and $0 \leq k \leq n$,

$$\sum_{i} a_j|_{p_k^i} = \frac{(n-1)!}{(k-1)!(n-k)!} x,$$

where the sum is taken over all fixed points of index $2k$. In particular, since there is only one fixed point of index $2n$,

$$a_j|_{p_k^i} = x. \quad (4.9)$$
Proof. We apply Lemma 4.6 to the cohomology classes $a_j$ and $\alpha_j$. By construction, $a_j|_{p^i_k} = 0$ and $\sum_i a_j|_{p^i_k} = a_j|_{q^i_k} = x = \sum_i \alpha_j|_{q^i_k}$. This gives us the necessary agreement on two coordinates. \hfill $\square$

Lemma 4.10. For any $0 \leq j \leq n$ and $0 \leq k \leq n$,
$$\sum_i (a_j|_{p^i_k})^2 = \frac{(n-1)!}{(k-1)!(n-k)!} x^2,$$
where the sum is taken over all fixed points of index $2k$.

Proof. We apply Lemma 4.6 to the cohomology classes $(a_j)^2$ and $(\alpha_j)^2$. By construction, $(a_j|_{p^i_k})^2 = 0 = (\alpha_j|_{q^i_k})^2$ and $\sum_i (a_j|_{p^i_k})^2 = (a_j|_{p^i_k})^2 = x^2 = \sum_i (\alpha_j|_{q^i_k})^2$. Finally, by equation 4.9, $(a_j|_{p^i_n})^2 = x^2$. This gives us the necessary agreement on three coordinates. \hfill $\square$

Combining Lemmas 4.8 and 4.10, we see:

Lemma 4.11. The restriction $a_i|_{F}$ is equal to 0 or $x$ for all $i$ and all fixed points $F$.

In order to prove Proposition 4.5, we will need to introduce some new cohomology classes. Given a fixed point $F$ of index $2k_F$, there exists a unique cohomology class $b_F$ such that $b_F|_{F} = x^{n-k_F}$, and $b_F|_{F'} = 0$ for all other fixed points $F'$ whose index is greater than or equal to $2k_F$. (The proof of this function is essentially identical to the proof of Theorem 4.3, except in this case the Morse function is $-\mu$.)

Lemma 4.12. Let $F$ be any fixed point of index $2k_F$, and let $b_F \in H^{n-k_F}_{S_1}(M)$ be defined as above. Then $a_j|_{F} x^{n-k_F} = b_F|_{p^i_k} x$.

Proof. Pick any fixed point $\tilde{F}$ of index $2k_F$ in $(P^1)^n$ such that $\alpha_j|_{\tilde{F}} = a_j|_{F}$, and let $\beta_{\tilde{F}} \in H^{n-k_F}_{S_1}((P^1)^n), \mathbb{Z})$ be the corresponding cohomology class. We apply Lemma 4.6 to $a_j \cdot b_F \in H^{2n-2k_F+2}(M, \mathbb{Z})$ and $\alpha_j \cdot \beta_{\tilde{F}}$. By construction, $a_j$ vanishes on the unique fixed point of index 0, whereas $b_F$ vanishes on all fixed points of index greater than $k_F$. Thus
$$\sum_i a_j|_{p^i_k} \cdot b_F|_{p^i_k} = \sum_i \alpha_j|_{q^i_k} \cdot \beta_{\tilde{F}}|_{q^i_k} = 0$$
for $k = 0$, or $k_F < k \leq n$. Also,
$$\sum_i a_j|_{p^i_k} \cdot b_F|_{p^i_k} = a_j|_{F} \cdot b_F|_{F} = a_j|_{F} = \alpha_j|_{\tilde{F}}$$
This gives the required agreement on $n - k_F + 2$ coordinates.

Therefore
$$\sum_i a_j|_{p^i_k} \cdot b_F|_{p^i_k} = a_j|_{p^i_k} \cdot b_F|_{p^i_k} = a_j|_{p^i_k} \cdot \beta_{\tilde{F}}|_{p^i_k},$$
and the result follows.

Proposition 4.5 will follow from the two lemmas below.

**Lemma 4.13.** For each fixed point $F$ of index $2k_F$, there exist precisely $k_F$ numbers $j \in \{0, \ldots, n\}$ such that $(a_j)_F = x$.

**Proof.** Pick any fixed point $\tilde{F}$ in $(P^n_1)^n$ of index $2k_F$. Let $b_F \in H^2_{S^1}(M)$ and $\beta_F \in H^{2(n-k_F)}_{S^1}(\mathbb{C}P^n)$ be the classes associated to $F$ and $\tilde{F}$, respectively. We apply Lemma 4.6 to $b_F$ and $\beta_F$. By construction, both forms vanish on all points of index greater than $2k_F$, and the sum of each over points of index $2k_F$ is $x^{n-k_F}$. This gives agreement on $n-k_F+1$ coordinates. Therefore $\sum_i b_F|_{p^i} = k_Fx^{n-k_F}$. Combining this with the previous lemmas, the result follows.

**Lemma 4.14.** Let $J \subset \{1, \ldots, n\}$ be a subset with $|J|$ elements. There exists a unique fixed point $F$ of index $2|J|$ so that $a_j|_F = x$ for all $j \in J$.

**Proof.** We apply Lemma 4.6 to $a_J = \prod_{j \in J} a_j$ and $\alpha_J = \prod_{j \in J} \alpha_j$. The restriction $\alpha_J|_F = 0$ for all critical points $F$ of index less than $2|J|$. By the previous lemma, the same is true for $a_J$. Also, $\alpha_J|_{q^i_h} = a_J|_{p^i_h} = x^{|J|}$. This gives the necessary agreement on $|J|+1$ coordinates.

Therefore, $\sum a_J|_{p^i_{|J|}} = \sum \alpha_J|_{q^i_{|J|}} = x^{|J|}$.

5. Cohomology rings of reduced spaces

One corollary of Theorem 2 is that the cohomology of the reduced spaces can also be computed. Again the classical manifold invariants are given by the same formulas as in the case of $(P^n_1)^n$.

**Corollary 1.** Let $M$ be a compact, connected symplectic manifold equipped with a semifree Hamiltonian circle action with isolated fixed points. Denote the moment map for the circle action by $\mu$, and suppose $0$ is a regular value of the moment map. Then the cohomology ring of $M_{\text{red}}$, the reduced space at zero, is given by

$$H^*(M_{\text{red}}; \mathbb{Z}) = \mathbb{Z}[a_1, \ldots, a_n, y]/R,$$

where $R$ is generated by the elements

1. $a_iy - a_i^2$ for all $i \in \{1, \ldots, n\}$,
2. $\prod_{j \in J} a_j$ for all $J \subset \{1, \ldots, n\}$ such that $\mu(F) > 0$ for the corresponding fixed point $F$, and
3. $\prod_{j \notin J} (y - a_j)$ for all $J \subset \{1, \ldots, n\}$ such that $\mu(F) < 0$ for the corresponding fixed point $F$. 
The Chern series of the reduced space is given by
\[ c_t = \prod_i 1 + t(2a_i - y), \]

For the case that \( M = (P^1)^n \), the reduced space is a toric variety, and the result above was proved by Haussmann and Knutson [HK]. For the general case, we use the following proposition from [TW]:

**Proposition 5.1.** Let \( S^1 \) act on a compact symplectic manifold \( M \) with moment map \( \mu : M \to \mathbb{R} \). Assume that 0 is a regular value of the moment map. Let \( F \) denote the set of fixed points. Assume that for every prime \( p \), one of the following two conditions is satisfied:

1. The integral cohomology of \( F \) has no \( p \)-torsion, or:
2. For every point \( m \in M \) which is not fixed by the \( S^1 \)-action, there exists a subgroup of \( S^1 \) congruent to \( \mathbb{Z}/p \) which acts freely on \( m \).

Define
\[ K_+ := \{ \alpha \in H^*_S(M; \mathbb{Z}) \mid \alpha|_{F_+} = 0 \}, \quad \text{where } F_+ := F \cap \phi^{-1}(0, \infty); \]
\[ K_- := \{ \alpha \in H^*_S(M; \mathbb{Z}) \mid \alpha|_{F_-} = 0 \}, \quad \text{where } F_- := F \cap \phi^{-1}(-\infty, 0); \quad \text{and} \]
\[ K := K_+ \oplus K_. \]

Then there is a short exact sequence:
\[ 0 \to K \to H^*_S(M; \mathbb{Z}) \xrightarrow{\kappa} H^*(M_{\text{red}}; \mathbb{Z}) \to 0, \]
where \( \kappa : H^*_S(M; \mathbb{Z}) \to H^*(M_{\text{red}}; \mathbb{Z}) \) is the Kirwan map.

Note the case of a manifold equipped with a semi-free circle action which has isolated fixed points is one where there is no torsion at all in the cohomology of the fixed point set, and where any \( \mathbb{Z}/k \)-subgroup of the circle acts freely outside fixed points. Hence both of the conditions in Proposition 5.1 are satisfied.

By another slight variation of Theorem 4.3, for each fixed point \( F \) of index 2\( k \) such that \( \mu(F) > 0 \), there exists a cohomology class \( A_F \in K_+ \) such that \( A_F|_F = x^k \) and so that \( A_F|_{F'} = 0 \) for all other fixed points \( F' \) such that the index of \( F' \) is less than or equal to \( k \). Moreover, taken together over all fixed points, there classes are a basis for \( K_+ \) as a module. Because these classes are unique, it is clear that \( A_F = \prod_{j \in J} a_j \), where \( J \) is the corresponding subset of \( \{1, \ldots, n\} \).

Similarly, \( K_- \) is generated by cohomology classes of the form \( \prod_{j \notin J} (y - a_j) \) for all \( J \subset \{1, \ldots, n\} \) such that \( \mu(F) < 0 \) for the corresponding fixed point \( F \).
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