In this paper we have considered a five-dimensional warped product spacetime with spacelike extra dimension. We have decomposed the geodesic equations to get the motion along the extra dimension and have studied the dynamical system associated with it. The relation between the total energy along the phase path and the extra-dimensional coordinate have been established for different types of warping function and the nature of confinement of particle trajectories in these cases have been analyzed. We have also calculated the energy associated with the phase path under a linear coordinate perturbation independent of the affine parameter and have analyzed the nature of confinement and the geodesic motions in such a case. Finally, the stability of the dynamical system under a perturbation due to an extra-dimension dependent additional force and under the effect of scalar fields in the bulk, have been examined.

I. INTRODUCTION

Recent years have observed a notable upswing of interest in the non-compact higher-dimensional theories of physics, mostly due to the advances in string theory, which postulated that we are living on a 3+1-dimensional hypersurface embedded within a higher-dimensional manifold [1, 2]. These so-called "braneworld" scenarios have been extensively studied and have been used to address issues such as the hierarchy problem of particle physics. Among the several higher-dimensional models developed over the years, the warped braneworld model of Randall and Sundrum (RS) with a single extra dimension has turned out to be the most popular for a number of reasons [3, 4]. In this model, matter fields are localized on a 4-dimensional hypersurface in a constant curvature five-dimensional bulk furnished with mirror symmetry. The exponential warp factor is responsible for "squeezing" the gravitational field closer to the 4-dimensional hypersurface. The metric is non-factorizable and the fifth dimension is large or infinite. The field equations on the corresponding 4-dimensional universe, get modified by the effect of the extra dimension [5].

The point of interest in such a model is the study of the geodesic motions and in particular the relation between the geodesics of the higher-dimensional space and those belonging to the lower-dimensional hypersurface. Much effort has been given in recent times to study the geodesic motions in five-dimensional spaces [6–13]. Generally, during the study of the classical geodesic motions of nonzero rest mass test particles and zero rest mass test particles in 5-dimensional warped product spaces, we decouple the motions along the extra dimensions from the motion in the 4D submanifolds. Under these circumstances, phase space analysis using the method of dynamical systems, can be an important tool to investigate the possibility of confinement and the stability of motion of the particles in the submanifolds of such 5-dimensional spaces. The dynamical systems analysis have been popular for more than a decade [14, 15] due to the reason that very few ordinary differential equations have explicit solutions expressible in finite terms. As a result, the repertory of standard functions in terms of which the solutions may be expressed, is often too limited to accommodate the variety of differential equations that we encounter in practice. The qualitative study of differential equations is concerned with the deduction of important characteristics of the solutions of differential equations without actually solving them [16–19]. Therefore we can introduce a geometrical device in the form of the phase plane, which is used extensively for obtaining directly from the differential equation, the properties of the dynamical system such as equilibrium points, periodicity, stability, and so on.

The paper is organized as follows: Having introduced the mathematical preliminaries in Section II, we have calculated the total energy along the phase path in terms of the extra-dimensional coordinate and have analyzed the nature of confinement of particle trajectories for different types of warping function for the unperturbed state, in Section III. This is followed by a similar analysis under a linear coordinate perturbation which is independent of the affine parameter and a study of the effect of this perturbation on geodesic motions, in Section IV. In Section V, we have examined the stability of the dynamical system under a perturbation due to an extra-dimension dependent additional force and followed it up in Section VI for the case where the system is under the influence of scalar fields present in the bulk. The summary of this whole exercise has been presented in Section VII.
II. MATHEMATICAL PRELIMINARIES

In general a warped product space is defined in the following way. Let \((M^m, h)\) and \((N^n, k)\) be two Riemannian (or pseudo-Riemannian) manifolds of dimension \(m\) and \(n\), with metrics \(h\) and \(k\), respectively. Let \(f : N^n \rightarrow \mathbb{R}\) be a smooth function, known as the warping function. Thus one can construct a new warped product Riemannian (pseudo-Riemannian) space by setting \(M = M^m \times N^n\) and defining a metric \(g = e^{2f} h \otimes k\). Here we shall take \(M = M^4 \times \mathbb{R}\) and identify \(M^4\) with the (3+1)-dimensional spacetime, a four-dimensional Lorentzian manifold with signature (- + + +). Here the class of warped geometries which we shall consider can be characterized by the following line element

\[
dS^2 = e^{2f(y)} g_{AB}(x) dx^A dx^B - dy^2
\]

where 'y' is the extra-dimensional coordinate and \(f(y)\) is the extra dimension dependent warping function. We shall assume that the five-dimensional manifold \(M\) can be foliated by a family of hypersurfaces \(\Sigma\) defined by the equation \(y = \text{constant}\). Then the geometry of each hypersurface, say \(y = y_0\), will be determined by the induced metric

\[
dS^2 = e^{2f(y_0)} g_{ab}(x) dx^a dx^b.
\]

III. TOTAL ENERGY ALONG THE PHASE PATH

Let us consider the equations of the 5-dimensional geodesics

\[
g_{AB} \frac{dy^A}{d\lambda} \frac{dy^B}{d\lambda} = \varepsilon_5.
\]

With the help of this equation and equation (3), the geodesic motion along the extra dimension can be written as

\[
\ddot{y} = -f'(y)(\varepsilon_5 + \dot{y}^2)
\]

where \(\varepsilon_5 = 0\) for null geodesics and \(\varepsilon_5 = 1\) for timelike geodesics and \(\dot{y}\) denotes differentiation with respect to the affine parameter \(\lambda\). If we consider \(\dot{y} = q\) then equation (5) can be written as

\[
\dot{q} = -f'(y)(\varepsilon_5 + q^2).
\]

Let \(y\) and \(q\) be the dynamical variables of the system. For timelike geodesics the system may have critical points at \(q = 0\) and \(y = l\) (i.e at \(y = l, f'(l) = 0\)). If so, then this critical point will be a saddle point for \(f''(l) < 0\), and it will be a center for \(f''(l) > 0\). On the other hand for null geodesics, the equilibrium point will lie on the line \(q = 0\). Whatever be the situation, we can determine the phase path for the system, which is given by the relation

\[
f(y) = -\ln \frac{\sqrt{\varepsilon_5 + q^2}}{K_0}
\]

where \(K_0\) is the integration constant. Finally, after doing some basic mathematics this equation takes the form

\[
Ke^{-2f(y)} = (\varepsilon_5 + q^2)
\]

where \(K\) is a positive constant. Thus along the phase path we can write equation (5) as

\[
\ddot{y} = \frac{d}{dy} \left( \frac{Ke^{-2f(y)}}{2} \right).
\]

Here \(\left(\frac{Ke^{-2f(y)}}{2}\right)\) can be treated as a potential along the phase path. Thus the total energy along the phase path can be written as

\[
E = \frac{\dot{y}^2}{2} + \frac{Ke^{-2f(y)}}{2}
\]
which gives us the relation

$$e^{2f(y)} = \frac{2K}{2E + \varepsilon_5}.$$  \hspace{1cm} (8)

Since we are focussing only on the timelike geodesics now, we consider $\varepsilon_5 = 1$. To analyze this relation we will consider three different types of warping functions:

1. $f(y) = A \log \cosh(By)$,
2. $f(y) = -A \log \cosh(By)$ (where $A$ and $B$ are positive constants) and
3. $f(y) = \frac{1}{2} \ln \frac{\Lambda y^2}{3}$.

A. Case 1

For this warping function, the phase path and the plot between the extra dimension and energy are represented in fig 1 and fig 2 respectively. It shows that the energy sharply increases as the path approaches $y = 0$ and becomes maximum at $y = 0$.

B. Case 2

Here the decaying warp function is $f(y) = -A \log \cosh(By)$. The phase path and the plot between extra dimension and energy are indicated in fig 3 and fig 4 respectively. It shows that the energy decreases as the path approaches $y = 0$. It is minimum at $y = 0$. We note that the minimum energy of the phase path is at $y = 0$. But there the
energy is not zero, which resembles the nature of motion of a harmonic oscillator in quantum mechanics where the zeroth level energy does not mean a zero value for energy. As the particle moves away from the hypersurface it will gather more energy i.e it will lose its stability.

C. Case 3

For \( f(y) = \frac{1}{2} \ln \frac{\Lambda y^2}{3} \) the dynamical system will not have any equilibrium point. But still we can determine the phase path, which is shown in fig 5. If we feed this warp function into equation (8) we will get fig 6, which shows that the energy sharply increases as the path approaches towards the hypersurface \( y = 0 \), indicating an instability.

IV. MOTION UNDER COORDINATE PERTURBATION WHERE THE PERTURBED PART IS INDEPENDENT OF AFFINE PARAMETER

The equation

\[
\ddot{y}_0 = -f'(y_0)(\varepsilon_5 + y_0^2)
\]

is the equation for the unperturbed system. We are now introducing a coordinate perturbation and we like to see the significant changes that take place in the system. Let us consider

\[
y = y_0 + \epsilon y_1,
\]

so that the new perturbed equation becomes

\[
\ddot{y} = -f'(y_0 + \epsilon y_1)(\varepsilon_5 + y^2)
\] (9)
where \( \epsilon \) is a very small number and \( y_1 \) is independent of the affine parameter. Using Taylors formula we can write

\[
\ddot{y} = -(f'(y_0) + (\epsilon y_1)f''(y_0) + (\epsilon y_1)^2 f'''(y_0)/2 + \ldots)(\epsilon_5 + \dot{y}^2). 
\]

(10)

Thus we are getting an additional term in righthand side of the equation if we consider only the first order perturbation. This term can be described as additional force term. It is to be noted that if we put \( y_0 = 0 \) i.e if we consider the brane, then also the additional force will be present provided \( f''(y_0) \neq 0 \). The condition \( f''(y_0) = 0 \) implies that the critical point is degenerate. This may happen if we choose the warping function to be a constant one.

For example if we consider a growing or decaying warp function i.e \( f(y) = \pm a \ln \cosh(y) \), then for time like geodesics, considering the first order perturbation, equation (10) can be written as

\[
\ddot{y} = \mp a \tanh(y_0)(1 + \dot{y}^2) \mp 4a \epsilon y_1 \cosh^{-2}(y_0)(1 + \dot{y}^2) 
\]

(11)

Let us now consider the perturbed energy along the phase path as

\[
E = E_0 + \epsilon E_1
\]

and proceed to find the nature of the equation (8) under this perturbed condition for different warp factors.

1. Growing warp function

Let us first consider the warping function \( f(y) = A \log \cosh(By) \). For the simplicity we will consider \( A = 1/2 \) and \( B = 1 \). Thus the equation (8) will look like

\[
E = \frac{2K\left(\frac{1}{\cosh y}\right) - 1}{2}.
\]
Since we are considering only 1st order perturbation, the perturbed equation can be written as

$$E_0 + \epsilon E_1 = 2K\left(\frac{1}{\cosh y_0} - \frac{\epsilon y_1 \sinh y_0}{(\cosh y_0)^2}\right) - 1. \quad (12)$$

Thus

$$E_0 = \frac{2K}{\cosh y_0} - \frac{1}{2}$$

and

$$E_1 = -2Ky_1 \left(\frac{\sinh y_0}{\cosh y_0}\right),$$

where $E_0, E_1$ are the zeroth and the 1st order terms for the energy under the perturbed condition. Fig 7 demonstrates how the perturbed energy varies along the extra dimension.

### 2. Decaying warp function

Here the warp function is $f(y) = -A \log \cosh(By)$, where $A$ and $B$ are positive constants. As in the previous case we choose $A = 1/2$ and $B = 1$. Equation (12) now looks like

$$E = \frac{2K \cosh y - 1}{2}$$

Considering only the 1st order perturbation, the perturbed equation is

$$E_0 + \epsilon E_1 = 2\left(\cosh y_0 - \frac{\epsilon y_1 \sinh y_0}{\cosh y_0}\right) - 1. \quad (13)$$

Therefore

$$E_0 = \frac{2K \cosh y_0 - 1}{2}$$

and

$$E_1 = -Ky_1 \sinh y_0,$$

where $E_0, E_1$ are the zeroth and the 1st order terms for the energy under the perturbation. Fig 8 demonstrates the variation of the perturbed energy along the extra dimension.
FIG. 8: Variation of Energy with extra dimension along the phase path for decaying warp function under perturbed condition

FIG. 9: Variation of Energy with extra dimension along the phase path for warp function $f(y) = \frac{1}{2} \ln \frac{\Lambda y^2}{\Lambda}$ under perturbed condition

3. Warp function $f(y) = \frac{1}{2} \ln \frac{\Lambda y^2}{\Lambda}$

For this particular choice of the warp function, we obtain a system with no critical point and the nature of the energy along the phase path is described in fig 5. However if we consider the same nature of perturbation as mentioned earlier we will get fig 9 which shows a very interesting result that the nature of the energy changes at the junction $y = 0$.

4. Effects on confinement

To study the nature of confinement under this perturbed situation, we consider the three warping functions mentioned earlier. We know that the extrinsic curvature of a hypersurfaces $\Sigma$ is given by $\Omega_{\alpha\beta} = -f' e^{\frac{2}{3}h_{\alpha\beta}}(x)$. Naturally for the first two types of warping function, the submanifold i.e the hypersurface called brane (at $y = 0$) will be totally geodesic, and as a result there will be confinement owing to the condition $f'(0) = 0$. But the third type of warping function does not produce any confinement.

For the first two types of warping function (growing and decaying warp functions), we have $f'(y) = \pm \tanh(y)$ in the unperturbed state. But under perturbation we have

$$f'(y_0 + y_1) = \pm \frac{\sinh(y_0) + \epsilon y_1 \cosh(y_0)}{\cosh(y_0) + \epsilon y_1 \sinh(y_0)}$$

which is not zero under any situation other than $y_0 = 0$ and $y_1 = 0$. In fact, such a situation corresponds to the unperturbed state. Thus we can say that owing to the perturbation, the hypersurfaces are no longer totally geodesic and as a result we will not get confinement, although for the third type of warping function, the nature of confinement remains unchanged in the perturbed state, as was in the unperturbed condition.
5. Effects on geodesic motion

To study the effects of this perturbation on the 4D part of the geodesic motion, we will assume a FRW-type metric with extra dimension dependent warp factor. Let that metric be

$$dS^2 = e^{2f(y)} (dt^2 - R^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2)) - T^2(t,y)dy^2.$$ 

The induced metric will be

$$dS^2 = e^{2f(y_0)} (dt^2 - R^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2)).$$

The 4D part of the geodesic equations are

$$\frac{d^2t}{d\lambda^2} + (\dot{R}R)\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2 + r^2\sin^2\theta \left(\frac{d\phi}{d\lambda}\right)^2 = 0,$$

$$\frac{d^2r}{d\lambda^2} + (\dot{R}/R)\left(\frac{dt}{d\lambda}\right)\left(\frac{dr}{d\lambda}\right) - r\left(\frac{d\theta}{d\lambda}\right)^2 - r\sin^2\theta \left(\frac{d\phi}{d\lambda}\right)^2 = 0,$$

$$\frac{d^2\theta}{d\lambda^2} + (\dot{R}/R)\left(\frac{dt}{d\lambda}\right)\left(\frac{d\theta}{d\lambda}\right) + (1/r)\left(\frac{d\theta}{d\lambda}\right)\left(\frac{dr}{d\lambda}\right) - \cos\theta \sin\theta \left(\frac{d\phi}{d\lambda}\right)^2 = 0,$$

$$\frac{d^2\phi}{d\lambda^2} + (\dot{R}/R)\left(\frac{dt}{d\lambda}\right)\left(\frac{d\phi}{d\lambda}\right) + (1/r)\left(\frac{d\phi}{d\lambda}\right)\left(\frac{dr}{d\lambda}\right) + \cot\theta \left(\frac{d\theta}{d\lambda}\right)\left(\frac{d\phi}{d\lambda}\right) = 0.$$ 

With the help of the equation

$$g_{\alpha\beta} \frac{dZ^\alpha}{d\lambda} \frac{dZ^\beta}{d\lambda} = \varepsilon_4,$$

where $\varepsilon_4 = 1$ for timelike geodesics and $\varepsilon_4 = 0$ for null geodesics, the temporal part of the geodesic motion can be written as

$$\frac{d^2t}{d\lambda^2} = -a \left(\frac{dt}{d\lambda}\right)^2 + a\left(\frac{\varepsilon_4}{e^{2f(y_0)}}\right)$$

where $a = \frac{\dot{R}}{R}$. It is clear that this kind of perturbation will not affect the spatial and temporal part of the 4D null geodesic motion. For timelike geodesics, the spatial part still remains unchanged, but the temporal part will change. To measure the change we consider a specific warping function, say a decaying warping function, $-\ln \cosh(y)$. So the temporal part of the geodesic equations will be

$$\frac{d^2t}{d\lambda^2} = -a \left(\frac{dt}{d\lambda}\right)^2 + a(\cosh(y_0) - \epsilon_1 \sinh(y_0))$$

instead of

$$\frac{d^2t}{d\lambda^2} = -a \left(\frac{dt}{d\lambda}\right)^2 + a \cosh(y_0).$$

This gives rise to an additional non-gravitational force.
V. PERTURBATION DUE TO EXTRA-DIMENSION DEPENDENT ADDITIONAL FORCE

Let us consider that the system is under an additional external force $\epsilon y$, so that the dynamical system can be written as

$$\dot{y} = q = P(q, y)$$

and

$$\dot{q} = -f'(y)(\varepsilon q + q^2) + \epsilon y = Q(q, y)$$

Thus the critical point of this system for the null geodesics is $y = 0$ and $q = 0$, which is a saddle point and is unstable, although the critical point is degenerate and lies on the line $q = 0$ in the absence of this additional force. On the other hand, critical points for timelike geodesics can be achieved if $f'(l) = -\epsilon l$ and $q = 0$.

To analyze the stability of the system, we find that the following conditions for the necessary parameters lead us to meaningful results:

$$\alpha = \left. \frac{\partial P}{\partial y} \right|_{y=l, q=0} = 0 \quad (14)$$

$$\beta = \left. \frac{\partial P}{\partial q} \right|_{y=l, q=0} = 1 \quad (15)$$

$$\zeta = \left. \frac{\partial Q}{\partial y} \right|_{y=l, q=0} = -f''(y) + \epsilon \quad (16)$$

$$\varsigma = \left. \frac{\partial Q}{\partial q} \right|_{y=l, q=0} = 0 \quad (17)$$

Thus when $-f''(l) + \epsilon > 0$ i.e $f''(l) < \epsilon$ the critical point will be a saddle point and when $-f''(l) + \epsilon < 0$ i.e $f''(l) > \epsilon$ the critical point will be a center. A very important feature of this system is revealed if we choose the warping function $f(y) = \frac{1}{2} \ln \frac{\Lambda y^2}{3}$. The unperturbed system for this warping function does not possess any critical point and as a result confinement is not achieved. But under perturbation, the system possesses critical points with coordinates $(\pm \frac{1}{\sqrt{\epsilon}}, 0)$. As $\epsilon$ is very small, the location of the critical points will be far away from the brane and it will be a saddle point.

VI. CONFINEMENT IN PRESENCE OF SCALAR FIELDS

In this section we shall check for the confinement of particles and the stability of the trajectories in the presence of scalar fields in the bulk. For simplicity we will consider the scalar field to be a function of the extra-dimension only. The equation of motion is given by

$$\frac{1}{\sqrt{-g}} \partial_A (\sqrt{-g} g^{AB} \partial_B \Phi(y)) = 0$$

which reduces to the following form:

$$\Phi''(y) + 4f'(y)\Phi'(y) = 0.$$  

In subsection (4) of section (IV) we have indicated the role of the term $f'(y)$ in confinement. Here we can express $f'(y)$ in terms of the scalar field as

$$f'(y) = -\frac{\Phi''(y)}{4\Phi'(y)}.$$  

Let us introduce a small linear coordinate perturbation represented by $y = y_0 + \epsilon y_1$. As a result the expression for the scalar field will change to the form $\Phi(y_0 + \epsilon y_1) = \Phi(y_0) + \epsilon y_1 \Phi'(y_0)$, where $y_0$ is the unperturbed part. The expression for $f'(y_0 + \epsilon y_1)$ can be written as

$$f'(y_0 + \epsilon y_1) = -\frac{\Phi''(y_0)}{4\Phi'(y_0)} + \frac{\epsilon y_1}{4} \left[ \left( \frac{\Phi''(y_0)}{\Phi'(y_0)} \right)^2 - \left( \frac{\Phi'''(y_0)}{\Phi'(y_0)} \right) \right].$$
To keep the nature of the confinement unaltered, we must assume the 1st order perturbation to be zero, which gives us the following differential equation for the scalar field:

\[(\Phi''(y_0))^2 = \Phi'''(y_0)\Phi'(y_0).\]

The solution of this equation is \(\Phi(y_0) = C_1 e^{y_0} - C_2\). If we adopt the confinement mechanism proposed by Rubakov and Shaposhnikov \([20, 21]\) we will get the plot shown in fig 10, which indicates the probability, \(P\), of finding the fermion near the hypersurface.

Confinement in presence of a scalar field can also be explained by introducing the concept of effective potential as in \([22]\), where it was shown that confinement is possible if we admit a direct interaction between test particles and the scalar field \(\Phi(y)\). The action can be taken as

\[S = \int \sqrt{m^2 + h^2 \Phi^2} \sqrt{-g_{AB} x^A x^B} d\tau\]

where \(m\) is the mass of the test particle, \(h\) is the coupling constant and \(\tau\) is the particle’s proper time. The expression for the effective potential, derived from the equation of motion for this action, is found to be

\[V_{\text{eff}} = e^{2f(y)} \left(1 + \frac{h^2 \Phi^2}{m^2}\right) - 1.\]

Let us introduce the linear coordinate perturbation \(y = y_0 + \epsilon y_1\). Thus we get the expression of the perturbed effective potential for the growing warp function as

\[V_{\text{eff,per}} = \cosh(y) \left(1 + \frac{h^2 \Phi^2}{m^2}\right) - 1 + \epsilon y_1 \left[\sinh(y) \left(1 + \frac{h^2 \Phi^2}{m^2}\right) + 2 \cosh(y) \Phi'(y) \frac{h^2}{m^2}\right].\]

After some basic mathematical calculations if we try to find out \(V_{\text{eff,per}}''(0)\), we obtain

\[V_{\text{eff,per}}'' = \text{unperturbed part} + \epsilon y_1 \left[2 \left(1 + \frac{h^2}{m^2}\right) \Phi' + 2\Phi''\right].\]

So according to \([22]\), if this perturbed part becomes zero, then the point \(y = 0\) will represent a stable equilibrium point and therefore, these particles will be stably confined to the brane. Hence, if we consider the perturbed part to be zero, then we get

\[\Phi(y) = \frac{C_1 e^{ky}}{k} - \frac{C_2 e^{-ky}}{k},\]

where \(C_1\) and \(C_2\) are constants and \(k = \sqrt{1 + \frac{h^2}{m^2}}\).
VII. SUMMARY

In this paper we have considered a five-dimensional warped product spacetime with spacelike extra dimension. We decomposed the geodesic equations to get the motion along the extra dimension and studied the dynamical system associated with it. Considering different types of warping function, we established the relation between the total energy along the phase path and the extra-dimensional coordinate and analyzed the nature of confinement of the particles. We found that the confinement of massive particles is unstable (stable) on the hypersurface for a growing (decaying) warp factor. We also calculated the energy associated with the phase path under a linear coordinate perturbation independent of the affine parameter, which gives rise to the effect of an additional force on the particles. The hypersurface is totally geodesic for growing and decaying warp factors and the four-dimensional timelike geodesics experience additional non-gravitational force. In presence of a perturbation due to an extra-dimension dependent additional force, the critical point for the null geodesics is a saddle point, although it was a degenerate one in the absence of such perturbations. However, for timelike geodesics, the critical point will be a saddle point or a center, depending on the nature of stability. For a particular choice of warping function, we find that although the unperturbed system did not possess any critical point and confinement did not occur, but the perturbed system possesses critical points far away from the hypersurface and these points are saddle points. Finally, we have examined the nature of confinement of particles and the stability of the trajectories in the presence of scalar fields in the bulk. Assuming the scalar field to be a function of only the extra-dimensional coordinate, we have found that under a linear coordinate perturbation, the fermions are mostly expected to be confined to the hypersurface, if we assume the confinement mechanism of Rubakov and Shaposhnikov, although some of them will escape into the extra dimension. However, if we consider a direct interaction between the particles and the scalar field, then the hypersurface \( y = 0 \) will represent a stable equilibrium point and therefore, these particles will be stably confined to the brane.

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