The point regular automorphism groups of the Payne derived quadrangle of $W(q)$

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Abstract

In this paper, we completely determine the point regular automorphism groups of the Payne derived quadrangle of the symplectic quadrangle $W(q)$, $q$ odd. As a corollary, we show that the finite groups that act regularly on the points of a finite generalized quadrangle can have unbounded nilpotency class.

1. Introduction

A finite generalized quadrangle $\mathcal{Q}$ of order $(s, t)$ is a point-line incidence structure such that each point is incident with $t + 1$ lines, each line is incident with $s + 1$ points, and for each non-incident point-line pair $(P, \ell)$ there is a unique point $Q$ incident with $\ell$ such that $P, Q$ are concurrent. The quadrangle $\mathcal{Q}$ is called thick if $s > 1$ and $t > 1$. If we interchange the role of points and lines of a generalized quadrangle $\mathcal{Q}$ of order $(s, t)$, we get the dual quadrangle of order $(t, s)$. The classical generalized quadrangles are those that arise as point-line incidence structures of finite classical polar spaces of rank 2. The standard textbook on finite generalized quadrangles is the classical monograph [24].

The study of generalized quadrangles has close connections with group theory and other branches of mathematics. J. Tits [30] introduced the notion of generalized polygons in order to better understand the Chevalley groups of rank 2. A generalized 3-gon is a projective plane, and a generalized 4-gon is a generalized quadrangle. In analogy with the study of finite projective planes, there has been extensive work on the classification of generalized quadrangles that exhibit high symmetry using a blend of geometric arguments and deep results in group theory. Please refer to the monographs [28, 29] for an account of the history and recent developments in this field.

Besides the classical examples and their duals, there are three further types of known finite generalized quadrangles up to duality: translation generalized quadrangles, flock generalized quadrangles, and the generalized quadrangles of order $(q - 1, q + 1)$. Most of the known examples except those with parameter $(q - 1, q + 1)$ can be described by their respective Kantor family or 4-gonal family, a concept first introduced by Kantor in [17]. The first examples of generalized quadrangles of order $(q - 1, q + 1)$, $q$ a prime power, are due to Ahrens and Szekeres [11], and independently to M. Hall, Jr. [16] for $q$ even. Payne has a general construction method, now known as the Payne derivation, that...
yields a generalized quadrangle $Q^P$ of order $(q - 1, q + 1)$ from a generalized quadrangle $Q$ of order $(q, q)$ with a regular point $P$, cf. [21, 22, 23]. The resulting quadrangle $Q^P$ is called a Payne derived quadrangle. All the known generalized quadrangles of order $(q - 1, q + 1)$ arise as Payne derived quadrangles. The automorphism groups of Payne derived quadrangles were studied in [15] and [9].

Ghinelli [13] was the first to study the generalized quadrangles admitting a point regular group, where she used representation theory and difference sets to study the case where the generalized quadrangle has order $(s, s)$ with $s$ even. Further progress was made in [31] and [27]. For instance, it is shown in [31] that finite thick generalized quadrangles of order $(t^2, t)$ does not admit a point regular automorphism group. By combining the results in [14] and [31], it was shown in [4] that any skew-translation generalized quadrangle of order $(q, q)$, $q$ odd, is isomorphic to the classical symplectic quadrangle $W(q)$. Swartz [27] initiated the study of generalized quadrangles admitting an automorphism group that acts regularly on both points and lines. Up till 2011, all the known finite generalized quadrangles admitting a point regular group arise by Payne derivation from a thick elation quadrangle $Q$ of order $(s, s)$ with a regular point, and their point regular groups are induced from the elation groups of $Q$. Motivated by this observation, De Winter, K. Thas and Shult attempted to show that there are no other examples in a series of papers [8, 9, 10, 11]. In the preprint [11], the known generalized quadrangles admitting a point regular automorphism group were classified in a combinatorial fashion.

A breakthrough in this direction is the classification of thick classical generalized quadrangles admitting a point regular group that leads to the discovery of three sporadic examples and the construction of new point regular groups of the derived quadrangles of the symplectic quadrangle $W(q)$ in [3]. This corrected a small error in [11] and disproved some conjectures in [10]. In the same paper, the authors calculated and listed the point regular automorphism groups of the derived quadrangle of $W(q)$ for small $q$ by Magma [5], and the results suggest that “the problem is wild”. In the case $q$ is odd, Chen [6], K. Thas and De Winter [7] independently classified the linear case, i.e., the group is induced by a linear group of the ambient projective space of $W(q)$. There are also some constructions in the even characteristic case in [7]. All the known finite groups that act regularly on the point set of a finite generalized quadrangle so far have nilpotency class at most 3 except for some groups of small order. The point regular groups of the derived quadrangles have applications in the constructions of uniform lattices in $\tilde{\Delta}_2$-buildings, cf. [7, 12].

In this paper, we systematically study the point regular automorphism groups of the Payne derived quadrangle $Q^P$ of the symplectic quadrangle $Q = W(q)$ with respect to a regular point $P$. Every point of $W(q)$ is regular, and different choices of $P$’s yield isomorphic derived quadrangles. By [15, Corollary 2.4], the full automorphism group of $Q^P$ is the stabilizer of $P$ in $\text{PGL}(4,q)$ when $q \geq 5$. We call a point regular group $G$ of $Q^P$ linear if $G$ is a subgroup of $\text{PGL}(4,q)$, and call it nonlinear otherwise. We completely determine all the point regular groups of $Q^P$ for odd $q$ and all the linear point regular groups for even $q$. This leads to four (resp. two) new constructions in the odd (resp. even) characteristic case. We derive relatively tight upper and lower bounds on the nilpotency classes of the resulting groups in the case $q$ is odd. Our result contributes to the fundamental problem as for which finite groups can act regularly on the points of a finite generalized quadrangle. In particular, we see that such a group can have arbitrarily
large nilpotency class.

This paper is organized as follows. In Section 2, we present some preliminary results on the arithmetic of the finite field $\mathbb{F}_q$ and on the regular point groups of the derived quadrangle $Q^p$ of $Q = W(q)$ with respect to a regular point $P$. In Section 3, we completely determine the linear point regular groups of $Q^p$. In Section 4, we analyze the structure of a putative nonlinear point regular group $G$ of $Q^p$, which leads to the complete classification in the odd characteristic case in Section 5. In Section 6, we first summarize all the constructions in the odd characteristic case, and then study some group invariants to distinguish their isomorphism classes. In Section 7, we conclude the paper.

2. Preliminaries

The group theoretical terminology that we use is standard, cf. [2, 25, 26]. Let $G$ be a finite group. The exponent of $G$, denoted $\exp(G)$, is the smallest positive integer $n$ such that $g^n = 1$ for all $g \in G$. For $g, h \in G$, their commutator is $[g, h] = g^{-1}h^{-1}gh$. For $g \in G$, the centralizer $C_G(g)$ of $g$ is the set of elements $h \in G$ such that $[g, h] = 1$. The center of $G$ is $Z(G) = \{g \in G : [g, h] = 1, \forall h \in G\}$. For two subgroups $H_1, H_2$ of $G$, we use $[H_1, H_2]$ for the subgroup $\langle [h_1, h_2] : h_1 \in H_1, h_2 \in H_2 \rangle$. In particular, the derived subgroup $G'$ is the subgroup $[G, G]$. We use the symbol $\gamma_i(G)$ for the $i$-th term of the upper central series of $G$. Inductively, we have $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for $i \geq 1$. The group $G$ is nilpotent if $\gamma_{c+1}(G) = 1$ for some integer $c$ and the smallest such integer is its nilpotency class. Similarly, we use the symbol $Z_i(G)$ for the $i$-th term of the lower central series of $G$, where $Z_0(G) = 1$, and $Z_{i+1}(G)$ is defined by the property $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$ for $i \geq 0$. The upper central series and the lower central series of a finite nilpotent group have the same length. Let $d$ be the maximal size of an abelian subgroup of a finite $p$-group $G$. The Thompson subgroup of $G$ is generated by all abelian subgroups of order $d$, and is denoted by $J(G)$.

2.1. The arithmetic of the field $\mathbb{F}_q$

Throughout this paper, we write $\mathbb{F}_q$ for the finite field with $q$ elements, where $q = p^m$ with $p$ prime. We write $\text{Aut}(\mathbb{F}_q)$ for the Galois group of the field $\mathbb{F}_q$, consisting of the Frobenius maps $x \mapsto x^p$, $0 \leq i \leq m - 1$. For a divisor $d$ of $m$, the trace function from $\mathbb{F}_q$ to the subfield $\mathbb{F}_{q^d}$ is

$$\text{Tr}_{\mathbb{F}_q/\mathbb{F}_{q^d}}(x) := x + x^{p^d} + \cdots + x^{p^{m-d}}, \quad x \in \mathbb{F}_q.$$  

For $g \in \text{Aut}(\mathbb{F}_q)$, we write both $x^g$ and $g(x)$ for the action of $g$ on $x \in \mathbb{F}_q$. The first notation avoids the excessive use of brackets in certain circumstances, and the latter is better for working with group rings.

Theorem 2.1. [18, Theorem 2.25] Let $d$ be a divisor of a positive integer $m$. For $\alpha \in \mathbb{F}_{p^m}$,

$$\text{Tr}_{\mathbb{F}_{p^m}/\mathbb{F}_{p^d}}(\alpha) = 0 \text{ if and only if } \alpha = \beta^{p^d} - \beta \text{ for some } \beta \in \mathbb{F}_{p^m}.$$  

Lemma 2.2. Suppose that $q = p^m$ with $p$ prime, and $g \in \text{Aut}(\mathbb{F}_q)$ has order $p^r$ with $r \geq 0$. If $K = \{x \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu x) = 0\}$ is $g$-invariant, i.e., $g(K) = K$, then $g(\mu) = \mu$.  

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Proof. The cases \( r = 0 \) and \( \mu = 0 \) are both trivial, so assume that \( r \geq 1 \) and \( \mu \neq 0 \). Since \( g(K) = \{ x \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(g(x)) = 0 \} \), we deduce from \( K = g(K) \) that \( g(\mu) = \lambda \mu \) for some \( \lambda \in \mathbb{F}_p^* \). Taking the relative norm to the subfield fixed by \( g \), we see that \( \lambda^{p^r} = 1 \), i.e., \( \lambda = 1 \). This completes the proof.

A linearized polynomial over \( \mathbb{F}_q \) is a polynomial \( f \) of the form \( f(X) = \sum_{i=0}^n a_i X^{p^i} \), \( a_i \in \mathbb{F}_q \). It is reduced if \( n \leq m - 1 \), where \( q = p^m \). There is a bijection between \( \mathbb{F}_p \)-linear transformations of \( \mathbb{F}_q \) and the reduced linearized polynomials over \( \mathbb{F}_q \).

**Lemma 2.3.** Let \( x \mapsto L(x) \) be a linear transformation of \( \mathbb{F}_q \), and set \( K = \{ x \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\eta x) = 0 \} \) for some \( \eta \in \mathbb{F}_q \). If \( \text{Im}(L|_K) = \mathbb{F}_p \cdot \omega \), then there exist \( u, \mu \in \mathbb{F}_q^* \) such that

\[
L(x) = \omega \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu x) + u \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\eta x), \quad \text{for } x \in \mathbb{F}_q.
\]

**Proof.** The case \( \eta = 0 \), i.e., \( K = \mathbb{F}_q \), follows by [13, Theorem 2.24], so we assume that \( \eta \neq 0 \). Take \( \beta \in \mathbb{F}_q \) such that \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\eta \beta) = 1 \), and define \( M(x) := L(x) - L(\beta) \cdot \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\eta x) \). Then \( M \) is \( \mathbb{F}_p \)-linear, and \( M(a + \lambda \beta) = L(a) \) for \( a \in K \) and \( \lambda \in \mathbb{F}_p^* \), so \( \text{Im}(M) = \mathbb{F}_p \cdot \omega \). The claim now follows by applying [13, Theorem 2.24] to \( M \). □

**Lemma 2.4.** Let \( q = p^m \) be a prime power with \( m = p^e l \), and take \( g \in \text{Aut}(\mathbb{F}_q) \) such that \( g(x) = x^{p^d} \). Take \( \eta \in \mathbb{F}_p^* \), and define \( K := \{ x \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\eta x) = 0 \} \). If \( L \) is a linear transformation of \( \mathbb{F}_q \) such that \( g(L(g^{-1}(x))) = L(x) \) for \( x \in K \), then there exists a reduced linearized polynomial \( L_1(X) \) with coefficients in \( \mathbb{F}_{p^l} \) such that \( L(x) = L_1(x) \) for \( x \in K \).

**Proof.** Suppose that \( L(x) = \sum_{i=0}^{m-1} d_i x^{p^i} \) for \( x \in \mathbb{F}_q \). We have

\[
D(x) := g(L(g^{-1}(x))) - L(x) = \sum_{i=0}^{m-1} (g(d_i) - d_i) x^{p^i},
\]

(2.1)

which is zero for \( x \in K \). It follows that \( \dim_{\mathbb{F}_p}(\text{Im}(D)) \leq 1 \). If \( \text{Im}(D) = 0 \), then we have \( g(d_i) = d_i \) for \( 0 \leq i \leq m - 1 \), i.e., \( d_i \in \mathbb{F}_p^* \), and the claim follows. Therefore, we assume that \( \text{Im}(D) \) has dimension 1 over \( \mathbb{F}_p^* \); in particular, \( K \neq \mathbb{F}_q \), i.e., \( \eta \neq 0 \). By Lemma 2.3 \( D(x) = \omega \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\eta x) \) for some \( \omega \in \mathbb{F}_p^* \). Therefore, we have a polynomial equation

\[
\sum_{i=0}^{m-1} (g(d_i) - d_i) X^{p^i} = \sum_{i=0}^{m-1} \omega \eta^{p^i} X^{p^i}.
\]

By comparing the coefficients of both sides, we deduce that \( g(d_i) - d_i = \omega \eta^{p^i} \), \( 0 \leq i \leq m - 1 \). By assumption we have \( g(\eta) = \eta \), so \( \eta \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\omega) = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(g(d_0) - d_0) = 0 \). By Theorem 2.1 \( \omega = g(u) - u \) for some \( u \in \mathbb{F}_q \). We deduce from \( g(d_i) - d_i = \omega \eta^{p^i} \) that \( f_i := d_i - \omega \eta^{p^i} \) is fixed by \( g \), i.e., lies in \( \mathbb{F}_p^l \). Now define \( L_1(X) := \sum_{i=0}^{m-1} f_i X^{p^i} \). For \( x \in K \), we have

\[
L(x) - L_1(x) = \sum_{i=0}^{m-1} \omega \eta^{p^i} x^{p^i} = u \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\eta x) = 0,
\]

so \( L_1 \) is the desired polynomial. This completes the proof. □
Let $L(X)$ be a reduced linearized polynomial over $\mathbb{F}_q$. Its trace dual is the (unique) reduced linearized polynomial $\tilde{L}(X)$ such that $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(L(x)y) = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}((\tilde{L})(y)x)$ for $x, y \in \mathbb{F}_q$. If $L(X) = \sum_{i=0}^{m-1} s_i X^{p^i}$, then we have

$$\tilde{L}(X) = \sum_{i=0}^{m-1} s_{m-i} X^{p^i}. \quad (2.2)$$

**Lemma 2.5.** Suppose that $q = p^m$ with $p$ prime. Let $L(X) = \sum_{i=0}^{m-1} s_i X^{p^i}$ be a reduced linearized polynomial over $\mathbb{F}_q$ and $\tilde{L}$ be its trace dual. Take $\mu \in \mathbb{F}_q$. Suppose that $B(c, z) := \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu L(z))$ is a symmetric bilinear form on the $\mathbb{F}_p$-subspace $K = \{x \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\eta x) = 0\}$ for some $\eta \in \mathbb{F}_q$. Then there exists $u \in \mathbb{F}_q$ such that $\tilde{L}(\mu x) = \mu L(x) - \eta \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(ux) + u \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\eta x)$ for $x \in \mathbb{F}_q$ and

$$\mu s_i - s_{m-i} \mu p^i = \eta u p^i - u \eta p^i, \quad 0 \leq i \leq m - 1. \quad (2.3)$$

Moreover, if $q$ is even and $K = \mathbb{F}_q$, then $B(c, c) = 0$ for all $c \in \mathbb{F}_q$ if and only if $\mu s_0 = 0$.

**Proof.** The proofs for the case $\eta = 0$ and the case $\eta \neq 0$ are similar, and we only prove the more complicated case $\eta \neq 0$ here. If $\mu = 0$, then we can simply take $u = 0$. We assume that $\mu \neq 0$ in the sequel. It is clear that $B$ is bilinear. By the preceding paragraph, we have $B(c, z) = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\tilde{L}(\mu c)z)$, where $\tilde{L}$ is the trace dual of $L$. It follows from $B(c, z) = B(z, c)$ that $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu L(c) - \tilde{L}(\mu c))z = 0$ for $c, z \in K$. It follows that $\mu L(c) - \tilde{L}(\mu c) \in \mathbb{F}_p \cdot \eta$ for $c \in K$. By Lemma 2, there exist $u, v \in \mathbb{F}_q$ such that $\mu L(c) - \tilde{L}(\mu c) = u \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(uc) + v \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\eta x)$. The first part of the claim then follows by comparing the coefficients of the corresponding polynomial identity $\mu L(X) - \tilde{L}(\mu X) = \sum_{i=0}^{m-1} \eta u p^i X^{p^i} + \sum_{i=0}^{m-1} v \eta p^i X^{p^i}$. Here, by comparing the coefficient of $X$, which is 0 on the left hand side, we deduce that $v = -u$.

Now assume that $q$ is even and $K = \mathbb{F}_q$, i.e., $\eta = 0$. We only handle the case $m$ is even, since the case $m$ is odd is similar. In this case, by taking $i = m/2$ in Eqn. (2.3), we get $\mu s_{m/2} \in \mathbb{F}_{2^{m/2}}$. We calculate that

$$B(c, c) = \text{Tr}(\mu s_0 c^2) + \text{Tr}(\mu s_{m/2} c^{2^{m/2} + 1}) + \sum_{i=1}^{m/2-1} \text{Tr}(\mu s_i c^{2^i + 1} + \mu s_{m-i} c^{2^{m-i} + 1}) = \text{Tr}(\mu s_0 c^2),$$

where $\text{Tr} = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}$. The second sum is 0 since $\mu s_{m/2} c^{2^{m/2} + 1} \in \mathbb{F}_{2^{m/2}}$, and each summand in the third sum vanishes by Eqn. (2.3) and the fact $\text{Tr}(\mu s_{m-i} c^{2^{m-i} + 1}) = \text{Tr}((\mu s_{m-i})^2 c^{2^{m-i} + 1})$. The second part of the claim now follows.

For two subsets $A, B$ of $\mathbb{F}_q$, we define $A \cdot B := \{xy : x \in A, y \in B\}$.

**Lemma 2.6.** Suppose that $q = p^m$ with $p$ prime and $m \geq 2$, and let $A, B$ be two $\mathbb{F}_p$-subspaces of $(\mathbb{F}_q, +)$ of codimension 1. Then $\langle A \cdot A \rangle_{\mathbb{F}_p} = \langle A \cdot B \rangle_{\mathbb{F}_p} = \mathbb{F}_q$. 
Proof. We assume that \( 1 \in A \cap B \) without loss of generality. Write \( W = \langle A \cdot A \rangle_{\mathbb{F}_p} \), and assume that \( W \neq \mathbb{F}_q \). Since \( A \leq W \leq \mathbb{F}_q \), we have \( W = A \). It follows that \( A \) is a proper subfield of \( \mathbb{F}_q \) and thus \( q \geq |A|^2 \). On the other hand, \( A \) has codimension 1 by assumption, i.e., \( q = p \cdot |A| \). We deduce that either \( (|A|, q) = (p, p^2) \) or \( (|A|, q) = (1, p) \), both contradicting the assumption that \( m > 2 \). To sum up, we have shown that \( \langle A \cdot A \rangle_{\mathbb{F}_p} = \mathbb{F}_q \).

Now assume that \( A \neq B \). Write \( U = \langle A \cdot B \rangle_{\mathbb{F}_p} \). It contains both \( A \) and \( B \), so contains the subspace \( A + B = \mathbb{F}_q \). This completes the proof. \( \square \)

One of the key ingredients for our classification is the \( \text{Aut}(\mathbb{F}_q) \)-module structure of \( \mathbb{F}_q \).

Assume that \( m = p^e l \) with \( e \geq 1 \). Take \( h \in \text{Aut}(\mathbb{F}_q) \) such that \( g(x) = x^p \) for \( x \in \mathbb{F}_q \). Write \( G := \langle g \rangle \leq \text{Aut}(\mathbb{F}_q) \), so that \( |G| = p^e \). Let \( \mathbb{F} \) be a subfield of \( \mathbb{F}_q \). Here we record some basic facts about the group ring \( \mathbb{F}[G] \), cf. \([19, 20]\).

(i) The ring \( \mathbb{F}[G] \) is a uniserial local ring. The collection of ideals, \( (1 - g)^i \mathbb{F}[G] \), \( 0 \leq i \leq p^e \), is the set of all its ideals, and they form a chain under containment. The dimension of \( (1 - g)^i \mathbb{F}[G] \) over \( \mathbb{F} \) is \( p^e - i \); in particular, \( (1 - g)^i \mathbb{F} = 1 - g^i = 0 \).

(ii) \( (1 - g)^{p^e - 1} = 1 + g + \cdots + g^{p^e - 1} \) by binomial expansion, and inductively

\[
(1 - g)^{p^i - 1} = 1 + g + \cdots + g^{p^i - 1}, \quad 1 \leq i \leq e. \tag{2.4}
\]

Lemma 2.7. Take notation as above. If \( 1 \leq i \leq p^e - 1 \) and \( \gcd(i, p) = 1 \), then \( 1 + g + \cdots + g^{i - 1} \) is invertible in \( \mathbb{F}[G] \).

Proof. The ring \( \mathbb{F}[G] \) is a local ring with maximal ideal \( I = (1 - g)\mathbb{F}[G] \), and an element \( a \) is invertible if and only if its image in the quotient ring \( \mathbb{F}[G]/I \cong \mathbb{F} \) is invertible. The claim is now an easy consequence. \( \square \)

From now on, set \( R' = \mathbb{F}_{p^e}[G] \), \( R = \mathbb{F}_{p^e}[G] \), and define \( R_i := (1 - g)^i (1 - x)^{p^e} \) for \( 0 \leq i \leq p^e - 1 \).

We have a chain of ideals of \( R \):

\[
R = R_0 \supseteq R_1 \supseteq \cdots \supseteq R_{p^e - 1} \supseteq R_{p^e} = 0, \tag{2.5}
\]

with \( \dim_{\mathbb{F}_{p^e}}(R_i/R_{i+1}) = 1 \) for \( 0 \leq i \leq p^e - 1 \). In particular, \( \dim_{\mathbb{F}_{p^e}}(R_i) = p^e - i \).

There exists a (normal) basis of \( \mathbb{F}_q \) over \( \mathbb{F}_{p^e} \) of the form \( \{g^i(\eta) : 0 \leq i \leq e - 1 \} \) with \( \eta \in \mathbb{F}_q \) by \([18]\), so \( \mathbb{F}_q \) is a free \( \mathbb{F}_{p^e}[G] \)-module with generator \( \eta \), i.e., \( \mathbb{F}_q = R' \cdot \eta \). Take \( \xi_1, \xi_2, \ldots, \xi_i \) to be a basis of \( \mathbb{F}_{p^e} \) over \( \mathbb{F}_p \). Then \( R' = \xi_1 R \oplus \xi_2 R \oplus \cdots \oplus \xi_i R \) and

\[
\mathbb{F}_q = R \cdot \xi_1 \eta \oplus R \cdot \xi_2 \eta \oplus \cdots \oplus R \cdot \xi_i \eta. \tag{2.6}
\]

Here, each \( R \cdot \xi_1 \eta \) is a free \( R \)-module with generator \( \xi_i \eta \). The submodules of \( R \cdot \xi_i \eta \) are \( R_k \cdot \xi_i \eta \), \( 0 \leq k \leq p^e - 1 \), which form a chain as in Eqn. \((2.5)\).

Lemma 2.8. Suppose that \( q = p^{e-1} \) with \( e \geq 1 \), and take \( g \in \text{Aut}(\mathbb{F}_q) \) such that \( g(x) = x^{p^i} \) for \( x \in \mathbb{F}_q \). Then there exists a pair \((W, t)\) such that

(1) \( W \) is a \( g \)-invariant \( \mathbb{F}_{p^e} \)-subspace of codimension \( h \) in \( \mathbb{F}_q \) with \( 0 < h \leq e \),

(2) \( t \) is an element of \( \mathbb{F}_q \) such that \( W_i := W + t_i, 0 \leq i \leq p^h - 1 \), are pairwise disjoint, where \( t_0 = 0 \) and \( t_i = (1 + \cdots + g^{i-1})(t) \), \( 1 \leq i \leq p^h - 1 \),
if and only if \( h \geq p^{h-1} \), i.e., \( h = 1 \) if \( p \) is odd and \( h = 1 \) or 2 if \( p = 2 \).

Proof. If \((W, t)\) is such a desired pair, then by the condition (1) \( W \) is a \( R \)-submodule of \( \mathbb{F}_q \), where \( G = \langle g \rangle \), \( R = \mathbb{F}_p[G] \). Let \( R_i \)'s be the ideals of \( R \) as in Eqn. \((2.5)\).

We claim that the condition (2) can be simplified to the single condition \( t_{p^h-1} \not\in W \). For \( 0 \leq i < j \leq p^h - 1 \), we have \( t_j - t_i = g^i(t_{j-i}) \). Since \( W \) is \( g \)-invariant, \( W + t_i \cap W + t_j = \emptyset \) iff \( t_{j-i} \not\in W \). If \( j - i = p^ku \) with \( \gcd(u, p) = 1 \), then

\[
  t_{j-i} = (1 + g_k + \cdots + g_k^{p^h-1})(t_{p^k}). \quad g_k := g^{p^k}.
\]

By Lemma \((2.7)\), \( 1 + g_k + \cdots + g_k^{p^h-1} \) is invertible, so \( t_{j-i} \not\in W \) if and only if \( t_{p^k} \not\in \mathbb{F}_q \). It thus suffices to show that \( t_{p^k} \not\in W \) for \( 0 \leq k \leq h - 1 \). By Eqn. \((2.4)\), \( t_{p^k} = (1 - g^{p^k})^{h-1}(t) \), which is a generator of the submodule \( R_{p^k-1} \cdot t = R \cdot t_{p^k} \). By applying the chain \((2.5)\) to \( t \), we see that it suffices to show that \( t_{p^h-1} \not\in W \). This proves the claim.

By Eqn. \((2.3)\), \( t_{p^h-1} = (1 - g)^{p^h-1-1}(t) \). The existence of \( t \) with \( (1 - g)^{p^h-1-1}(t) \not\in W \) is equivalent to that \( W^* \subsetneq W \), where

\[
  W^* := (1 - g)^{p^h-1-1}(\mathbb{F}_q) = R_{p^h-1-1} \cdot \xi_1 \eta + \cdots + R_{p^h-1-1} \cdot \xi_i \eta,
\]

where \( \eta \) and \( \xi_i \)'s are the same as in Eqn. \((2.6)\). Each component \( W^*_i = R_{p^h-1-1} \cdot \xi_i \eta \) of \( W^* \) has dimension \( p^e - p^{h-1} + 1 \) over \( \mathbb{F}_p \).

With all the preparations, we are ready to prove the claim. If \( h \geq p^{h-1} \), take \( W \) to be the direct sum of \( R \cdot \xi_1 \eta \oplus \cdots \oplus R \cdot \xi_i \eta \) and a \( R \)-submodule of \( R \cdot (\xi_1 \eta) \) of dimension

\[
  p^e l - h - p^e (l - 1) = p^e - h < \dim_{\mathbb{F}_p}(W^*_i).
\]

It follows that \( W^*_i \) is not contained in the chosen \( W \). Take \( t \in \mathbb{F}_q \) such that \( (1 - g)^{p^{h-1}-1}(t) \in W^*_i \setminus W \), and \((W, t)\) is a desired pair. If \( h < p^{h-1} \), then

\[
  \dim_{\mathbb{F}_p}(W \cap (R \cdot \xi_i \eta)) \geq \dim_{\mathbb{F}_p} W + \dim_{\mathbb{F}_p}(R \cdot \xi_i \eta) - \dim_{\mathbb{F}_p}(\mathbb{F}_q) = p^e l - h + p^e - p^e l = p^e - h \geq \dim_{\mathbb{F}_p}(W^*_i),
\]

so \( W \) contains \( W^*_i \) by the fact that \( R \cdot \xi_i \eta \) is a uniserial \( R \)-module. It follows that \( W^* \subsetneq W \), and thus there is no pair \((W, t)\) with the desired properties. This completes the proof. \( \Box \)

2.2. The Payne derived quadrangle \( Q^P \) of \( Q = W(q) \)

We adopt the standard notions on generalized quadrangles as can be found in the monograph \([24]\). Let \( q = p^m \) be a prime power with \( p \) prime. Let \( \perp \) be a fixed symplectic polarity of \( \text{PG}(3, q) \). The classical generalized quadrangle \( Q = W(q) \) has the same point set as \( \text{PG}(3, q) \) and has the totally isotropic lines of \( \text{PG}(3, q) \) as its lines. The Payne derived quadrangle \( Q^P \) of \( Q \) with respect to a point \( P \) has points of \( \text{PG}(3, q) \setminus P^\perp \) as its points and has two types of lines: the totally isotropic lines not containing \( P \), and the lines \((P, Q)\) with \( Q \not\in P^\perp \). The parameter of \( Q^P \) is \((q-1, q+1)\). The automorphism group of \( Q \) acts transitively on its point set, so the different choices of \( P \) lead to isomorphic derived quadrangles. By \([13, \text{Corollary 2.4}]\), the stabilizer of \( P \) in \( \text{P}TS\text{p}(4, q) \) is the full automorphism group of \( Q^P \) when \( q \geq 5 \).
Let $V := \mathbb{F}_q^4$ be a vector space over $\mathbb{F}_q$ equipped with the alternating form $(x, y) := x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4$. For $A \in \text{Sp}(4, q)$ and $\sigma \in \text{Aut}(\mathbb{F}_q)$, we write $(A, \sigma)$ for the element of $\Gamma\text{Sp}(4, q)$ that maps $x \in V$ to $x^\sigma A$, where $x^\sigma = (x_1^\sigma, \ldots, x_4^\sigma)$. We refer to $A$ as the matrix part and refer to $\sigma$ as the Frobenius part of the element $(A, \sigma)$ respectively. Take a fixed projective point $P := \langle (1, 0, 0, 0) \rangle$. The stabilizer $\text{Sp}(4, q)_P$ of $P$ in $\text{Sp}(4, q)$ consists of the matrices

$$
\begin{pmatrix}
\lambda & 0 & 0 & 0 \\
-HJv^T & H & 0 & 0 \\
z & v & \lambda^{-1}
\end{pmatrix}, \quad H \in \text{SL}(2, q), \ v \in \mathbb{F}_q^2, \ z \in \mathbb{F}_q, \ \lambda \in \mathbb{F}_q^*,
$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The group $\text{Aut}(\mathbb{F}_q)$ also stabilizes $P$, and together with $\text{Sp}(4, q)_P$ they generate $\Gamma\text{Sp}(4, q)_P$.

Let $G$ be a subgroup of $\Gamma\text{Sp}(4, q)_P$ that acts regularly on the points of $Q^P$. The point set of $Q^P$ is $\{ (a, b, c, T) : a, b, c \in \mathbb{F}_q \}$. Without loss of generality, we can assume that $G$ is contained in the Sylow $p$-subgroup of $\Gamma\text{Sp}(4, q)_P$ generated by the Sylow $p$-subgroup of $\text{Aut}(\mathbb{F}_q)$ and the lower triangular matrices with diagonal entries 1 in $\text{Sp}(4, q)_P$, i.e.,

$$E(a, b, c, t) := \begin{pmatrix} 1 & 0 & 0 & 0 \\
-c & 1 & 0 & 0 \\
b - ct & t & 1 & 0 \\
a & b & c & 1 \end{pmatrix}, \quad a, b, c, t \in \mathbb{F}_q. \quad (2.7)$$

In order for $G$ to be point regular, for each triple $(a, b, c)$, there should be exactly one element of $G$ that maps $(0, 0, 0, 1)$ to $(a, b, c, 1)$, i.e., whose matrix part has $(a, b, c, 1)$ as its last row. We denote such an element by $g_{a,b,c} = (M_{a,b,c}, \theta_{a,b,c})$, where $\theta_{a,b,c} \in \text{Aut}(\mathbb{F}_q)$ and $M_{a,b,c} := E(a, b, c, T(a, b, c))$. Here, $T$ is a function from $\mathbb{F}_q^3$ to $\mathbb{F}_q$. The group multiplication $\circ$ of $G$ is

$$g_{a,b,c} \circ g_{x,y,z} = (M_{a,b,c}^{\theta_{x,y,z}} \cdot M_{x,y,z}, \theta_{a,b,c}^{\theta_{x,y,z}}), \quad (2.8)$$

where $M^{\theta}$ is the matrix obtained by applying $\theta$ to each entry of the matrix $M$, and $\cdot$ is the usual matrix multiplication. To summarize, up to conjugacy in $\Gamma\text{Sp}(4, q)_P$ a point regular subgroup $G$ of $Q^P$ is of the form $G = \{ g_{a,b,c} : a, b, c \in \mathbb{F}_q \}$ for some functions $T : \mathbb{F}_q^3 \to \mathbb{F}_q$ and $\theta : \mathbb{F}_q^3 \to \text{Aut}(\mathbb{F}_q)$, where $\theta(x, y, z) = \theta_{x,y,z}$.

**Theorem 2.9.** Let $T : \mathbb{F}_q^3 \to \mathbb{F}_q$ and $\theta : \mathbb{F}_q^3 \to \text{Aut}(\mathbb{F}_q)$ be two functions. Set $M_{a,b,c} := E(a, b, c, T(a, b, c))$, $\theta_{a,b,c} := \theta(a, b, c)$, and $g_{a,b,c} := (M_{a,b,c}, \theta_{a,b,c})$. Define $G := \{ g_{a,b,c} : a, b, c \in \mathbb{F}_q \}$. Then $G$ is a point regular group of the derived quadrangle $Q^P$ if and only if for any triples $(a, b, c)$ and $(x, y, z)$ we have $g_{a,b,c} \circ g_{x,y,z} = g_{u,v,w}$, which is equivalent to

$$\theta_{a,b,c} \theta_{x,y,z} = \theta_{u,v,w}, \quad (2.9)$$

$$T(a, b, c)^{\theta_2} + T(x, y, z) = T(u, v, w), \quad (2.10)$$

where $\theta_2 = \theta_{x,y,z}$, $w = c^{\theta_2} + z$ and

$$u = a^{\theta_2} + x - b^{\theta_2}z + c^{\theta_2}y - c^{\theta_2}zT(x, y, z),$$

$$v = b^{\theta_2} + y + c^{\theta_2}T(x, y, z).$$
\textbf{Proof.} The element \( g_{a,b,c} \) maps \((0, 0, 0, 1)\) to \((a, b, c, 1)\), so \( G \) acts regularly on the points of \( Q^3 \) provided that it is indeed a group. Since \( G \) is finite, it suffices to make sure that \( G \) is closed under the multiplication \( \circ \) as defined in Eqn. (2.8). By direct calculations, we deduce that the last row of the matrix part of \( g_{a,b,c} \circ g_{x,y,z} \) is \((u, v, w, 1)\) with \( u, v, w \) as in the statement of the theorem. Therefore, we need to have \( g_{a,b,c} \circ g_{x,y,z} = g_{u,v,w} \).

In general, the condition in Theorem 2.9 is fairly complicated. We consider some special cases, where the conditions can be simplified. We introduce the following symbols which will be used throughout this paper: \( \sigma_c := \theta_{0,0,c} \) and

\[
L(x) := T(x, 0, 0), \quad M(y) := T(0, y, 0), \quad S(z) := T(0, 0, z).
\]

\begin{corollary}
Take notation as in Theorem 2.9, and assume that \( G \) is a point regular group of \( Q^3 \). Then for \( a, b, c, x, y, z \in \mathbb{F}_q \), it holds that

\begin{enumerate}
\item \[
\theta_{a,0,0} \circ \theta_{x,0,0} = \theta_{a \circ x,0,0} \quad \text{and} \quad L(a) \theta_{0,0,0} + L(x) = L(a \theta_{0,0,0} + x);
\]
\item \[
\theta_{0,b,0} \circ \theta_{y,0,0} = \theta_{0,(b \circ y,0,0+y)} \quad \text{and} \quad M(b) \theta_{0,0,0} + M(y) = L(b \theta_{0,0,0} + y);
\]
\item \[
\sigma_c \circ \sigma_x = \sigma_{a \circ u, v, w} \quad \text{and} \quad S(c) \sigma_x + S(z) = T(u, v, w), \quad \text{where} \quad u = -\sigma_z(c) z S(z), \quad v = \sigma_z(c) S(z)
\]
\end{enumerate}

\end{corollary}

\textbf{Proof.} The claims (1)-(5) are special cases of Theorem 2.9. The first equation in claim (6) can be directly verified by checking the action of both sides on the point \((0, 0, 0, 1)\), and the second part follows by Theorem 2.9. It remains to prove (7). Fix a triple \((a, b, c)\). By (6), there exists a triple \((u, v, w)\) such that

\[
g_{0,0,c} \circ g_{a,b,0} = g_{u,v,w} \circ g_{0,0,0}.
\]

The left hand side equals \( g_{x,y,w} \) with \((x, y, w) = \left(a + b \theta_{a,b,0}, b + c \theta_{a,b,0} T(a, b, 0), c \theta_{a,b,0}\right)\) by Theorem 2.9. We derive the expressions for \( u, v \) as shown in the statement of (7) by applying (6). We directly compute both sides of Eqn. (2.11), and the claim follows by comparing their Frobenius parts and the \((3, 2)\)-nd entry of their matrix parts (alternatively, by passage through \( g_{x,y,w} \) by using Theorem 2.9).

\begin{corollary}
Take notation as in Theorem 2.9 and assume that \( G \) is a point regular group of \( Q^3 \). Then \( G_A := \{g_{a,0,0} : a \in \mathbb{F}_q\}\), \( G_B := \{g_{0,b,0} : b \in \mathbb{F}_q\}\) and \( G_{A,B} := \{g_{a,b,0} : a, b \in \mathbb{F}_q\}\) are subgroups of \( G\).

\textbf{Proof.} It is straightforward to verify that these subsets are closed under multiplication, so they are subgroups of the finite group \( G\).
Remark 2.12. Let $E$ be as defined in Eqn. (2.7). We shall make extensive use of the following calculations throughout the paper:

$$E(a, b, c, t) \cdot E(x, y, z, w) = E(a + x - bz + cy - cw, b + y + cw, c + z, t + w),$$

$$E(a, b, c, t)^{-1} = E(-a, -b + ct, -c, -t).$$

In each equation, the last two coordinates, i.e., the $(4,3)$-rd and the $(3,2)$-nd entries of the matrix, on the right hand side has a very simple form. This is particularly helpful in many circumstances where we are only concerned with these two coordinates.

3. The point regular groups of $Q^p$ in $\text{PGL}(4, q)$

Let $G$ be a point regular group of the Payne derived quadrangle $Q^p$, where $Q = W(q)$, $P = (1, 0, 0, 0)$. In this section, we consider the case where the group $G$ is linear, i.e., $G$ is a subgroup of $\text{PGL}(4, q)$. In [3], the authors enumerated all the point regular groups of the quadrangle $Q^p$ for $q \leq 25$ by Magma [5], so we only consider the case $q \geq 5$ below. The main result of this section is the following theorem.

Theorem 3.1. Let $G$ be a subgroup of $\text{PGL}(4, q)$ that acts regularly on the points of the derived quadrangle $Q^p$ of $Q = W(q)$, $q \geq 5$. Then $G$ is conjugate to one of the groups in Construction 3.3 and Construction 3.6 below.

The odd characteristic case of the above theorem is also due to [6] and [7] independently. The rest of this section is devoted to the proof of Theorem 3.1. By the analysis in Section 2.2, we can assume that $G$ is as defined in Theorem 2.9 for some functions $T$ and $\theta$ up to conjugacy. Take the same notation as in Theorem 2.9 and write $q = p^m$ with $p$ prime. In this case, $\theta_{a,b,c} \equiv 1$. Recall that $L(x) = T(x, 0, 0)$, $M(y) = T(0, y, 0)$. By (1), (2), (4) and (5) of Corollary 2.10, the maps $L$ and $M$ are both additive and $T(a, b, 0) = L(a) + M(b)$ for $a, b \in \mathbb{F}_q$. Also, by (7) it holds for all $a, b, c \in \mathbb{F}_q$ that

$$L(2bc + c^2L(a) + c^2M(b)) + M(cL(a) + cM(b)) = 0. \quad (3.1)$$

By setting $b = 0$ and $a = 0$ respectively, we get

$$L(c^2L(a)) + M(cL(a)) = 0, \text{ for } a \in \mathbb{F}_q, \quad (3.2)$$

$$L(2bc + c^2M(b)) + M(cM(b)) = 0, \text{ for } b, c \in \mathbb{F}_q. \quad (3.3)$$

Lemma 3.2. Take notation as above. If $q$ is odd, then $L(x) \equiv 0$ and $M(y) \equiv 0$. If $q$ is even, there exist $\omega, \mu \in \mathbb{F}_q$ such that

$$L(x) = \omega \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mu^2 \omega x), \quad M(y) = \omega \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mu y). \quad (3.4)$$

Proof. Write $q = p^m$ with $p$ prime. Let $L(X) := \sum_{i=0}^{m-1} u_iX^{2^i}$, $M(X) := \sum_{i=0}^{m-1} v_iX^{2^i}$ be the corresponding reduced polynomials for the additive maps $x \mapsto L(x)$ and $y \mapsto M(y)$ respectively. The subscripts of $u_i$’s and $v_i$’s are taken modulo $m$.

First assume that $q$ is odd. Since $2p^{m-1} \leq q - 1$, we deduce from Eqn. (3.2) that $L(X^2L(a)) + M(XL(a)) = 0$ in $\mathbb{F}_q[X]$. By comparing the coefficients of $X^{2p^i}$ we get

10
\( u_i L(a)^{p^i} = 0 \) for \( 0 \leq i \leq m - 1 \) and \( a \in \mathbb{F}_q \). If \( L(a) = 0 \) for all \( a \in \mathbb{F}_q \), then \( L(X) = 0 \); otherwise, take \( a \in \mathbb{F}_q \) such that \( L(a) \neq 0 \), and we deduce that \( u_i = 0 \) for each \( i \), i.e., \( L(X) = 0 \); a contradiction. Hence we always have \( L(X) = 0 \). Then Eqn. (3.3) reduces to \( M(XM(b)) = 0 \), and the same argument yields \( M(X) = 0 \).

Next assume that \( q \) is even. If \( L(X) = 0 \), then we deduce from Eqn. (3.3) that \( M(X) = 0 \) in the same way, and we can take \( \omega = \mu = 0 \). So assume that \( L(X) \neq 0 \). Fix \( a \in \mathbb{F}_q \) such that \( L(a) \neq 0 \). By converting Eqn. (3.2) into the reduced polynomial form and comparing the coefficients, we get \( u_i L(a)^{2^i} + v_{i+1} L(a)^{2^{i+1}} = 0 \), i.e., \( u_i = v_{i+1} L(a)^{2^i} \) for \( 0 \leq i \leq m - 1 \). It follows that \( \text{Im}(L) \) has only one nonzero element, say \( \omega \). By Lemma 2.3, there exists \( \mu_A \in \mathbb{F}_q^* \) such that \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mu_A x) \), i.e., \( u_i = \omega \mu_A^{2^i} \). It follows that \( v_{i+1} = \omega^{1-2^i} \mu_A^{2^i} \), and \( M(X) = \omega \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\omega^{-1/2} \mu_A^{1/2} X) \). By setting \( \mu = \omega^{-1/2} \mu_A^{1/2} \), we get the desired form for \( L \) and \( M \).

In the sequel, we consider the two cases in Lemma 3.2 separately. First consider the case \( L(a) = M(b) = 0 \) for all \( a, b \in \mathbb{F}_q \). In this case, \( T(a, b, c) = S(c) \) by (6) of Corollary 2.10, where we recall that \( S(c) = T(0, 0, c) \). By (3) of Corollary 2.10 \( S \) is additive. The condition in Theorem 2.9 trivially holds in this case. Therefore, we have the following construction, which is also implicit in [1] and [7].

**Construction 3.3.** For a prime power \( q \), set \( \theta_{x,y,z} := 1 \) and \( T(x, y, z) := S(z) \) for an additive map \( S \) on \( \mathbb{F}_q \). Then the corresponding set \( G \) as defined in Theorem 2.7 for the prescribed functions \( T \) and \( \theta \) is a point regular subgroup of \( \mathbb{Q}^p \).

**Example 3.4.** Write \( q = p^m \) with \( p \) prime. For \( \alpha \in \mathbb{F}_q \), define \( \theta_{\alpha} := E(0, 0, \alpha, 0) \) and \( t_{0,0,\alpha} := E(0, 0, \alpha, 0) \), where \( E \) is defined in Eqn. (2.1). Let \( \{ \alpha_1, \cdots, \alpha_m \} \) be a basis of \( \mathbb{F}_q \) over \( \mathbb{F}_p \). Take \( \{ \beta_1, \cdots, \beta_m \} \) to be the dual basis, i.e., \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\alpha_i \beta_j) = 1 \) if \( i = j \) and \( = 0 \) otherwise. For \( 1 \leq k \leq m - 1 \), set \( T_k(x, y, z) := \sum_{j=0}^{k} \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\beta_i z) \alpha_i \), and let \( G_k \) be the group arising from Construction 3.3 with \( T = T_k \). Then in \( G_k \) we have \( g_{0,0,\alpha_i} = \theta_{\alpha_i} \), for \( 1 \leq i \leq k \) and \( g_{0,0,\alpha_i} = t_{0,0,\alpha_i} \), for \( k + 1 \leq i \leq m \). In the notation of [3], the group \( G_k \) is \( S_{U,W} \) with \( U = (\alpha_1, \cdots, \alpha_k) \) and \( W = (\alpha_{k+1}, \cdots, \alpha_m) \).

Next we consider the remaining case of Lemma 3.2, i.e., \( q \) is even and Eqn. (3.4) holds for some nonzero elements \( \omega, \mu \). Set

\[
B(c, z) := S(c + z) + S(c) + S(z),
\]

which is symmetric in \( c, z \). By (6) and (3) of Corollary 2.10 we have \( S(c) + S(z) = T(czS(z), cS(z), c + z) \) and \( T(u, v, w) = T(u + vw, v, 0) + S(w) \), so

\[
B(c, z) = S(c + z) + T(czS(z), cS(z), c + z)
= T(czS(z) + cS(z)(c + z), cS(z), 0)
= L(c^2 S(z)) + M(cS(z)) = \omega \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mu^2 c^2 (\omega S(z) + S(z)^2)).
\]

It is additive in \( c \), so is also additive in \( z \). Set \( F(z) := (\omega S(z) + S(z)^2)^{1/2} \), so that \( B(c, z) = \omega \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mu^2 F(z)) \). From \( B(c, z_1 + z_2) = B(c, z_1) + B(c, z_2) \) for all \( c \in \mathbb{F}_q \) we deduce that \( z \mapsto F(z) \) is additive. Moreover, \( B(c, c) = S(2c) + 2S(c) = 0 \). Let
\( F(X) = \sum_{i=0}^{m-1} f_i X^{2^i} \) be the corresponding reduced polynomial of the map \( z \mapsto F(z) \). By Lemma 2.5, we have \( f_0 = 0 \), and
\[
\mu f_i + (\mu f_{m-i})^{2^{i+1}} = 0 \quad \text{for } 1 \leq i \leq m - 1
\] (3.6)
We define an auxiliary function \( H(x) := \sum_{0 \leq i < j \leq m-1} \mu^{2^i} f_{j-i}^{2^{i+1}} \). Its value lies in \( \mathbb{F}_2 \), since
\[
H(x) + H(x)^2 = \sum_{0 \leq i < j \leq m-1} \mu^{2^i} f_{j-i}^{2^{i+1}+2^i} + \sum_{1 \leq i < j \leq m} \mu^{2^i} f_{j-i}^{2^{i+1}+2^j} = \sum_{1 \leq j \leq m} \mu f_j x^{1+2^j} + \sum_{1 \leq i \leq m-1} (\mu f_{m-i})^{2^i} x^{2^i+1} = 0,
\]
where we used Eqn. (3.6) to get the last equality. Now set \( S_1(z) := S(z) + \omega \cdot H(z) \). Then \( S_1(c + z) + S_1(c) + S_1(z) \) equals
\[
B(c, z) + \omega \cdot \sum_{i < j} \mu^{2^i} f_{j-i}^{2^{i+1}} \left( (c + z)^{2^{i+1}+2i} + c^{2^{i+1}} + z^{2^{i+1}} \right)
= B(c, z) + \omega \cdot \sum_{i < j} \mu^{2^i} f_{j-i}^{2^{i+1}} c^{2^i} z^{2^i} + \omega \cdot \sum_{i < j} \mu^{2^i} f_{j-i}^{2^{i+1}} z^{2^i} c^{2^i}
= B(c, z) + \omega \cdot \sum_{i < j} (\mu f_{j-i})^{2^i} c^{2^i} z^{2^i} + \omega \cdot \sum_{i < j} (\mu f_{i-j})^{2^i} z^{2^i} c^{2^i}
= B(c, z) + \omega \cdot \sum_{i, j} (\mu f_{j-i})^{2^i} c^{2^i} z^{2^i} = B(c, z) + \omega \cdot \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mu F(z)) = 0.
\]
Here, in the second equality we used the fact \( \mu f_{j-i} = (\mu f_{i-j})^{2^{j-i}} \) in Eqn. (3.6), and in the third equality we interchanged the label of \( i, j \) in the last summation and used the fact \( f_0 = 0 \). Therefore, \( S_1 \) is additive. Let \( S_1(X) := \sum_{i=0}^{m-1} s_i X^{2^i} \) be the corresponding reduced polynomial. To summarize, we have \( S(x) = S_1(x) + \omega \cdot H(x) \), i.e.,
\[
S(x) = \sum_{i=0}^{m-1} s_i x^{2^i} + \omega \cdot \sum_{0 \leq \ell < j \leq m-1} \mu^{2^i} f_{j-i}^{2^{i+1}+2^j}.
\] (3.7)
This expression involves the coefficients of \( S_1(X) \) and \( F(X) \) only. We now consider the relation \( F(x)^2 = \omega S(x) + S(x)^2 \). The left hand side is \( \sum_{i=0}^{m-1} f_i X^{2^{i+1}} \). The right hand side is \( \sum_{i=0}^{m-1} (\omega s_i + s_i^2) x^{2^i} \). Both expressions have degree not exceeding \( q - 1 \), so they are equal as polynomials. By comparing coefficients, we get \( \omega s_{i+1} + s_i^2 = f_i^2 \) for \( 0 \leq i \leq m-1 \). Inductively, we deduce that
\[
s_{i+1} = \sum_{j=1}^{i} \omega^{-2^{i+1}} f_{i+1-j}^{2^j} + \sum_{j=1}^{i} w^{-2^{i+1}+1} s_{i+1}^{2^{i+1}+1}, \quad 0 \leq i \leq m - 1.
\] (3.8)
In this way we express \( s_i, 1 \leq i \leq m - 1 \), in terms of the \( f_i \)'s and \( s_0 \), and the fact \( s_m = s_0 \) yields an restriction
\[
\sum_{1 \leq j \leq m-1} \omega^{-2^{i+1}} f_{m-j}^{2^j} = 0.
\] (3.9)
Lemma 3.5. For fixed nonzero elements $\omega$, $\mu$, the number of $m$-tuples $(f_0, \ldots, f_{m-1}, s_0)$ such that $f_0 = 0$ and all the conditions in Eqns. (3.6) and (3.9) hold is $2q^{\frac{m(m-1)}{2}}$.

Proof. First, assume that $m$ is odd. By Eqn. (3.6), we can express $f_{(m+1)/2}, \ldots, f_{m-1}$ in terms of $f_1, \ldots, f_{(m-1)/2}$ as follows: $f_{m-i} = \mu^{2m-i-1}f_i^{2m-i}$ for $(m+1)/2 \leq i \leq m-1$. Plugging them into Eqn. (3.9) and dividing both sizes by $\omega \mu$, we get

$$\beta + \beta^2 = \sum_{i=2}^{(m-1)/2} (\mu^{-2i} \omega^{-1-2i} f_i + \mu^{-1} \omega^{-2m-i} f_i^{2m-i})$$

with $\beta = \mu^{-1} \omega^{-1-2m-1} f_1^{2m-1}$. It is routine to check that the right hand side has absolute trace 0. Therefore, by Lemma 2.1, there exist two possible solutions $\beta$'s for any chosen tuple $(f_2, \ldots, f_{(m-1)/2})$. The claim now follows in this case.

Next, consider the case $m$ is even. The argument is basically the same, and the only distinction is that in showing that the right hand side has absolute trace 0 we need the observation that $f_{m/2} \mu$ is in the subfield $\mathbb{F}_{2m/2}$. This completes the proof. \hspace{1cm} $\square$

It turns out that the conditions that we have derived so far are also sufficient. This leads to the following construction.

Construction 3.6. Suppose that $q = 2^m$ with $m > 1$, and let $\omega$, $\mu$ be two nonzero elements of $\mathbb{F}_q$. Take any tuple $(f_0, \ldots, f_{m-1}, s_0)$ satisfying the conditions in Lemma 3.5 and define $s_1, \ldots, s_{m-1}$ by Eqn. (3.8). Set $\theta_{a,b,c} \equiv 1$, and

$$T(a,b,c) = \omega \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mu^2 \omega (a + bc) + \mu b) + \sum_{i=0}^{m-1} s_i c^{2i} + \omega \sum_{0 \leq i < j \leq m-1} \mu^2 f_{i-j} f_{j}^{2i+2j}.$$ 

Then the set $G$ as defined in Theorem 2.9 with the prescribed functions $T$ and $\theta$ is a point regular group of $Q^P$.

Proof. Let $L$ and $M$ (resp. $S$) be the corresponding functions as defined in Eqn. (3.4) (resp. Eqn. (3.7)). Then we have $T(a,b,c) = L(a + bc) + M(b) + S(c)$. Set $F(x) := \sum_{i=0}^{m-1} f_i x^{2i}$, $B(c,z) := S(c + z) + S(c) + S(z)$. By reversing the preceding arguments we obtain $F(x)^2 = \omega S(x) + S(x)^2$, $B(c,z) = \omega \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mu c F(z))$.

By Theorem 2.9 we need to verify that $T(a,b,c) + T(x,y,z) = T(u,v,w)$, where $w = c + z$, $v = b + y + c T(x,y,z)$ and $u = a + x + b z + c y + c z T(x,y,z)$. We have $u + v w + a + b c + x + y z = c^2 T(x,y,z)$, $v + b + y = c T(x,y,z)$, so $T(u,v,w) + T(a,b,c) + T(x,y,z)$ equals $L(c^2 T(x,y,z)) + M(c T(x,y,z)) + B(c,z)$. By plugging in the expressions of $L$, $M$, $T$, and $B$, we deduce that it equals 0 after simplification. This completes the proof. \hspace{1cm} $\square$

To summarize, we have now completed the proof of Theorem 3.1.

Remark 3.7. Suppose that $G$ is a point regular group of $Q^P$ obtained from either of Construction 3.5 or Construction 3.6 and assume that $q$ is even. If $T(a,b,c) \equiv 0$, then $G$ is elementary abelian, so we assume that $T(a,b,c) \neq 0$. It is routine to deduce that $G$ has exponent 4 and nilpotency class 2 and its center is $Z(G) = \{0_{a,b,0} : a, b \in \mathbb{F}_q, T(a,b,0) = 0\}$ in both cases. In the case of Construction 3.5, $T(a,b,c) \equiv 0$ and $Z(G)$ has size $q^2$; in the case of Construction 3.6, $T(a,b,0) = \omega \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mu^2 \omega a + \mu b)$ and $Z(G)$ has size $q^2/2$. Therefore, the two constructions yield non-isomorphic groups in the even characteristic case.
4. The structure of a nonlinear point regular group $G$ of $Q^P$

Throughout this section, we assume that $G$ is a nonlinear point regular group of the quadrangle $Q^P$ as defined in Theorem 2.9 with associated functions $T$ and $\theta$. We aim to establish some structural theorems about the group $G$. We first introduce some notation that will be used throughout this and the next section.

Notation. Write $q = p^m$ with $p$ prime. Since $G$ is nonlinear, we have $p|m$. For the subgroup $G_A$, we define

$$r_A := \log_p (\max \{ o(\theta_{a,0,0}) : a \in \mathbb{F}_q \}),$$
$$K_A := \{ a \in \mathbb{F}_q : \theta_{a,0,0} = 1 \}.$$

We fix an element $g_A \in \text{Aut}(\mathbb{F}_q)$ of order $p^{r_A}$, and let $t_A$ be an element of $\mathbb{F}_q$ such that $\theta_{t_A,0,0} = g_A$. Set $t_{A,0} := 0$, and for $i \geq 1$ define

$$t_{A,i} := (1 + g_A + \cdots + g_A^{i-1})(t_A).$$

Let $r_B$, $g_B$, $t_B$ and $t_{B,i}$’s be the corresponding notions for $G_B$, and set

$$r_{A,B} := \max \{ r_A, r_B \}.$$

Similarly, define $r_C := \log_p (\max \{ o(\sigma_c) : c \in \mathbb{F}_q \})$, and define $g_C$, $t_C$ and $t_{C,i}$’s in a similar way. Set

$$s := \max \{ 0, r_C - r_{A,B} \}.$$

Also, for each $i \geq 0$, define

$$\mathcal{K}_i := \{ z \in \mathbb{F}_q : \sigma_z = g_C^i \},$$
$$\mathcal{K}_i^* := \{ z \in \mathbb{F}_q : \sigma_z^{p^{r_{A,B}}} = g_C^{p^{r_{A,B}}} \}.$$

Finally, recall that we have defined $\sigma_c := \theta_{0,0,c}$ and

$$L(x) := T(x,0,0), M(y) := T(0,y,0), S(z) := T(0,0,z).$$

4.1. The Frobenius part of the group $G$

**Theorem 4.1.** Take notation as above, and take $g_2 \in \text{Aut}(\mathbb{F}_q)$ of order $p$.

1. The subset $K_A$ is a $g_A$-invariant $\mathbb{F}_p$-subspace of codimension $r_A$ in $\mathbb{F}_q$, and the map $x \mapsto L(x)$ is additive on $K_A$. Moreover, we have $t_{A,p^{r_A}} \in K_A$, and for $x \in K_A$ it holds that $L(x)g_A = L(g_A(x))$ and

$$L(g_A^i(x) + t_{A,i}) = g_A^i(L(x)) + \sum_{j=0}^{i-1} g_A^j(L(t_A)), \quad i \geq 1. \quad (4.1)$$

2. The subset $G_{A,K} := \{ \theta_{a,0,0} : a \in K_A \}$ is a normal subgroup of $G_A$ of index $p^{r_A}$, and

$$G_A = \langle G_{A,K}, g_{t_A,0,0} \rangle = \cup_{i=0}^{p^{r_A}-1} G_{A,K} \circ g_{t_A,0,0}^i.$$
(3) If $q$ is odd (resp. even), then $r_{A,B} \leq 1$ (resp. $r_{A,B} \leq 2$).
(4) If $r_A \leq 1$, there exists $\mu_A$ such that $g_2(\mu_A) = \mu_A$ and $\theta_{a,0,0} = g_2^{\text{TF}_{q/p}(\mu_A)}$.

The above statements also hold when the $A$'s in the subscripts are all replaced by $B$ and the map $L$ is replaced by $M$.

**Proof.** Since the arguments for $G_A$ and $G_B$ are the same, we only give the proof for $G_A$ below. In the case $r_A = 0$, we have $K_A = \mathbb{F}_q$ and the map $L$ is additive by (1) of Corollary 2.10. All the claims are trivial in this case: take $\mu_A = 0$ in the part (4). We assume that $r_A \geq 1$ in the sequel. For each $i \geq 0$ define

$$K_{A,i} := \{a \in \mathbb{F}_q : \theta_{a,0,0} = g_A^i\}.$$  

In particular, we have $t_A \in K_{A,1}$. For the ease of notation, in this proof we ignore the $A$'s in the subscripts and write

$$r = r_A, \ g_1 = g_A, \ t_1 = t_A, \ t_i = t_{A,i}, \ K_i = K_{A,i}.$$  

**Step 1.** We define the map $\theta^* : \mathbb{F}_q \to \text{Aut}(\mathbb{F}_q), a \mapsto \theta_{a,0,0}$. By (1) of Corollary 2.10 we have $L(a)_{\theta^*(x)} + L(x) = L(a_{\theta^*(x)} + x)$, and

$$\theta^*(a)\theta^*(x) = \theta^*(a_{\theta^*(x)} + x) \text{ for } a, x \in \mathbb{F}_q.$$  

(4.2)

In particular, by restricting to $K_0 = K_A$, it follows readily that $K_0$ is an $\mathbb{F}_p$-subspace of $\mathbb{F}_q$ and $L$ is additive on $K_0$.

The set $G_{A,K}$ is exactly the set of elements of $G_A$ that has a trivial Frobenius part, so it is indeed a normal subgroup of $G_A$. For $a \in K_0$ we compute that $g_{i_1,0,0}^{-1}g_{a,0,0}g_{i_1,0,0} = (\mathcal{M}_{-t_1,0,0} \cdot \mathcal{M}_{a,0,0}^{g_1} \cdot \mathcal{M}_{t_1,0,0}, 1)$. The last row of its matrix part is $(g_1(a), 0, 0, 1)$, so it equals $g_1(a_{t_1,0,0})$. Comparing the $(3,2)$-nd entry of their matrix parts, we get $L(a)^{g_1} = L(g_1(a))$. It has a trivial Frobenius part, so $g_1(a) \in K_0$. To sum up, we have

$$g_1(K_0) = K_0, \ L(a)^{g_1} = L(g_1(a)) \text{ for } a \in K_0.$$  

For $a \in K_0$ and $1 \leq i \leq p^r$, set $g_i := g_{a,0,0} \circ g_{t_1,0,0}^i$. The last row of the matrix part of $g_i$ is $(g_1^i(a) + t_i, 0, 0, 1)$ by induction, so it equals $g_{g_1^i(a)+t_i,0,0}$.

(1) The Frobenius part of $g_i$ is $g_1^i$, so $K_0 + t_i \subseteq K_i$ for $i \geq 1$; in particular, with $i = p^r$ we have $K_0 + t_{p^r} = K_0$ by the fact $g_1^{p^r} = 1$. It follows that $t_{p^r} \in K_0$.

(2) From Eqn. (12) with $x \in K_1$ and $a \in K_i$ we deduce that $t_1 + g_1(K_i) \subseteq K_{i+1}$, so $|K_i| = |g_1(K_i)| \leq |K_{i+1}|$ for each $i$. It follows that all the $K_i$'s have the same size. As a corollary, $K_0 + t_i = K_i$ by item (1). It follows that $G_A = \bigcup_{i=0}^{p^r-1} G_{A,K} \circ g_{i,A,0,0}$, i.e., $G_A$ is generated by $G_{A,K}$ and $g_{a,0,0}$.

(3) By comparing the $(3,2)$-nd entry of the matrix parts of $g_{a,0,0} \circ g_{t_1,0,0} = g_{g_1(a)+t_1,0,0}$, we get $g_1^i(L(a)) + \sum_{j=0}^{i-1} g_1^j(L(t_1)) = L(g_1^i(a) + t_i)$. In the case $i = p^r$, by the fact $t_{p^r} \in K_0$ and $L$ is linear on $K_0$, we deduce that $L(t_{p^r}) = \sum_{j=0}^{p^r-1} g_1^j(L(t_1))$. 

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This completes the proof of the first two parts of the theorem. All the conditions in Lemma 2.8 are satisfied for the pair \((K_0, t_1)\) with \(e = h = r\), so \(r \geq p^{r-1}\), which is the third part.

**Step 2.** We claim that \(g_1(t_1) \equiv t_1 \pmod{K_0}\), and \(t_i \equiv it_1 \pmod{K_0}\) for \(0 \leq i \leq p - 1\) in the case \(r = 1\). Assume that \(r = 1\). In this case, \(g_1\) has order \(p\) and so \(g_1 = g_2^d\), \(1 \leq d \leq p - 1\). The \(\mathbb{F}_p\)-subspace \(K_0\) has codimension \(1\) in \(\mathbb{F}_q\) and \(t_1 \notin K_0\), so we have a decomposition \(\mathbb{F}_q = \bigcup_{\lambda \in \mathbb{F}_p} K_0 + \lambda t_1\). Since \(K_0 + g_1(t_1)\) is a coset of \(K_0\), there is \(\lambda \in \mathbb{F}_p\) such that \(K_0 + g_1(t_1) = K_0 + \lambda t_1\). Since \(K_0\) is \(g_1\)-invariant, we get \(K_0 + g_1^i(t_1) = K_0 + \lambda^i t_1\) inductively. In the quotient space \(\mathbb{F}_q/K_0\), we thus have \(g_1^i(t_1) = \lambda^i t_1\) for \(i \geq 0\). From \(t_p = \sum_{i=0}^{p-1} g_1^i(t_1) \in K_0\), we deduce that \(\sum_{i=0}^{p-1} \lambda^i \cdot t_1 = 0\), which yields that \(\lambda = 1\). This proves the claim.

Take \(\mu_A \in \mathbb{F}_q\) such that \(\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_A x) = 0\) for \(x \in K_0\) and \(\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_A t_1) = 1\). Then \(K_i = K_0 + it_1 = \{x \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_A x) = i\}\) for \(i \geq 0\). By the definition of the \(K_i\)'s, we thus have \(\theta_{0,0,0} = g_{2\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_A)} = g_{2\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu A)}\) for \(a \in \mathbb{F}_q\). Also, we have \(g_1(\mu_A) = \mu_A\) by Lemma 2.2. This proves the fourth part. \(\square\)

**Corollary 4.2.** Take notation as above, and assume that \(q\) is odd. Let \(g_2\) be an element of \(\text{Aut}(\mathbb{F}_q)\) of order \(p\). Then there exists \(\mu_A\) and \(\mu_B\) that are \(g_2\)-invariant such that \(\theta_{0,0,0} = g_{2\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_{A+B})}\).

**Proof.** Since \(q\) is odd, we have \(r_A \leq 1\), \(r_B \leq 1\), and there exist \(\mu_A\) and \(\mu_B\) that are fixed by \(g_2\) such that \(\theta_{0,0,0} = g_{2\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu A)}\), \(\theta_{0,0,0} = g_{2\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu B)}\) by Theorem 4.1. Here, we take \(\mu_A = 0\) if \(r_A = 0\) and take \(\mu_B = 0\) if \(r_B = 0\). By (4) of Corollary 2.10, we have \(\theta_{0,0,0} = \theta_{0,b,0}(a,0) \cdot \theta_{0,0,0} = g_{2\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_{A+B})}\). In the second equality, we used the fact that \(\theta_{0,0,0}\) is in \(\langle g_2 \rangle\) and thus fixes \(\mu_A\). \(\square\)

**Theorem 4.3.** Take notation as above. If \(q\) is even, then \(r_C \leq r_{A,B} + 2\). If \(q\) is odd, then \(r_C \leq r_{A,B} + 1\). \(K_i^* = \mathbb{F}_q\) for each \(i \geq 0\), and the claims are trivial. Therefore, we assume that \(r_C \geq r_{A,B} + 1\), i.e., \(s \geq 1\). Let \(g_C\), \(t_C\), \(c, \sigma_i\)'s be as defined in the beginning of this section. For brevity, we write

\[
r := r_{A,B},\ g := g_C,\ g_0 := g^{p^r},\ g_1 := g^{p^r},\ t_i := t_{C,i}.
\]

Also, define \(H_0^* := \{x \in \mathbb{F}_q : x + K_0^* \subseteq K_0^*\}\). It is clear that \(H_0^*\) is an \(\mathbb{F}_p\)-subspace. Since \(0 \in K_0^*\), we have \(H_0^* \subseteq K_0^*\).

Combining (3) and (6) of Corollary 2.10, we have \(\sigma_{c}z = \theta_{0,0}(\sigma_{c}z,S(z))\), \(\sigma_{c}z = \theta_{0,0}(\sigma_{c}z,S(z))\), \(\sigma_{c}z = \theta_{0,0}(\sigma_{c}z,S(z))\), where \(c' = c^{p^r} + z\), \(a = \sigma_{c'}^{-1}(c^{p^r}S(z))\) and \(b = \sigma_{c'}^{-1}(c^{p^r}S(z))\). Raising both sides to the \(p^r\)-th power, we deduce that

\[
g^{i+p^rk}(c) + z \in K_i^* + K_j^*,\ \text{for} \ c \in K_i^*,\ z \in K_i^* + K_j^*.
\]

This holds for all nonnegative integers \(i, j, k\)'s. By a similar argument to that in the proof of Theorem 4.1, we obtain: \(|K_i^*| = q/p^s\), and \(K_i^* = t_i + g^i(K_0^*)\) for \(i \geq 0\). In the case \(i = p^sk\), we deduce from \(0 \in K_0^*\) and \(K_i^* = K_0^*\) that \(t_{p^sk} \in K_0^*\) for each \(k\).
We rewrite Eqn. (4.3) as \( K_{i+p^s} = g^i(K^*_0) + t_i \), where equality holds by comparing sizes. Set \( i = 0 \) and use \( K^*_0 = g^i(K^*_0) + t_j \) to obtain

\[ K_{i+p^s} + g^i(g^{p^s}(K^*_0)) + g^{p^s}(t_j) = g^i(K^*_0) + t_j. \]

Further use \( g^{p^s}(K^*_0) = K^*_0 - t_{p^s} \) and \(-g^i(t_{p^s}) + g^{p^s}(t_j) - t_j = -t_{p^s}, \) i.e.,

\[
-p^{s+k+j-1} \sum_{l=j}^{p^s+k+j-1} g^l(t_1) - \sum_{l=0}^{p^s+k-1} g^l(t_1) = -\sum_{l=0}^{p^s+k-1} g^l(t_1),
\]

to deduce that \( K_{i+p^s} - t_{p^s} + g^j(K^*_0) = g^j(K^*_0). \) It follows that \( K_{i+p^s} - t_{p^s} \subseteq g^j(H^*_0) \) for all nonnegative integers \( j \) and \( k. \)

We now introduce the set \( W := \cap_{i=0}^{p^{s+k}-1} g^i(H^*_0). \) Since \( H^*_0 \) is a subspace over \( \mathbb{F}_p, W \) is a \( g \)-invariant \( \mathbb{F}_p \)-subspace of \( \mathbb{F}_q. \) In the last paragraph we showed that \( K_{p^s+k} \subseteq W + t_{p^s} \) for each \( k \geq 0. \) Combining the two special cases \( k = 0 \) and \( k = p^s \) and using the fact that \( 0 \in K_0 = K_{p^s+s}, \) we deduce that \( t_{p^s+s} \subseteq W. \) Since \( W \subseteq H^*_0 \) and \( t_{p^s+k} \subseteq K^*_0, \) we have

\[
K^*_0 = \bigcup_{i=0}^{p^s-1} K_{p^s+i} \subseteq \bigcup_{i=0}^{p^s-1} (W + t_{p^s+i}) \subseteq K^*_0.
\]

Therefore, each equality holds in the above equation.

Let \( d \) be the smallest positive integer such that \( t_{p^s+d} \in W. \) We just showed that \( t_{p^s+i} \in W, \) so \( d \leq p^s. \) From \( t_{p^s+i} = g^i(t_{p^s+i}) \) for \( i \leq j \) and \( g(W) = W \) we deduce that: \( t_{p^s+i} \in W \) for \( i \geq 0, \) and the \( d \) cosets \( W + t_{p^s+i}, 0 \leq i \leq d-1, \) are pairwise disjoint. Moreover, we have

\[
W + t_{p^s+i} = t_{p^s+i} + g^i(t_{p^s+i} + W) = W + t_{p^s+i}, \quad 0 \leq i \leq d-1.
\]

By Eqn. (4.4) we have a partition of \( K^*_0 \) as follows:

\[
K^*_0 = \bigcup_{i=0}^{d-1} W + t_{p^s+i}.
\]

Further, we plug the decomposition (4.5) into \( K^*_i = g^i(K^*_0) + t_i \) and simplify by the fact \( t_{i+p^s} = t_i + g^i(t_{p^s}) \) to obtain \( K^*_i = \bigcup_{j=0}^{d-1} (W + t_{i+p^s}), 0 \leq i \leq p^s - 1. \) Therefore, from \( \mathbb{F}_q = \bigcup_{i=0}^{p^s-1} K^*_i \) we get a decomposition \( \mathbb{F}_q = \bigcup_{i=0}^{p^s-1} \bigcup_{j=0}^{d-1} W + t_{i+p^s+j} \) as a disjoint union of \( p^s d \) distinct cosets of \( W. \) It follows that \( d = p^{d_0} \) for some nonnegative integer \( d_0. \) We have \( 0 \leq d_0 \leq r \) by the fact \( d \leq p^s. \) To summarize, we have shown that the pair \( (W, t_C) \) satisfies all the conditions in Lemma (2.8) with \( h = d_0 + s \) and \( e = r_C = r + s, \) so \( p^{d_0+s-1} \leq d_0 + s. \) It follows that \( d_0 + s \leq 1 \) if \( q \) is odd, \( d_0 + s \leq 2 \) if \( q \) is even. In the case \( q \) is odd, we must have \( d_0 = 0, s = 1 \) by the assumption \( s \geq 1, \) in which case \( K^*_0 = W \) by Eqn. (4.5). This completes the proof.

**Corollary 4.4.** Take notation as above, and assume that \( q \) is odd. The set

\[
G_{K^*_0} := \{ g_{a,b,c} : a, b \in \mathbb{F}_q, c \in K^*_0 \}
\]

is a normal subgroup of \( G \) of index \( p^s \) and \( G = \bigcup_{i=0}^{p^s-1} G_{K^*_0} \circ g_{0,0,i} \), where \( s = \max \{0, r_C - r_{A,B} \}. \)

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We thus have $G_{K_0}$ is a normal subgroup. By (4) and (5) of Corollary 2.10 we deduce that $\theta_{a,b,0} : a, b \in \mathbb{F}_q$ is a group of order $p^s$. By (6) of Corollary 2.10 $\theta_{a,b,c} = \theta_{c^{-1}(a+bc),c^{-1}(b),c}$, so the Frobenius part of each element in $G_{K_0}$ lies in $\langle g_1 \rangle$, where $g_1 = g^{p^s}$ has order $p^s$. By Theorem 2.9 $g_{a,b,c} \circ g_{x,y,z} = g_{a',b',0}$, where $a', b'$ are certain elements in $\mathbb{F}_q$. Since $K_0$ is a $g$-invariant $\mathbb{F}_p$-subspace by Theorem 1.3 we see that $\theta_{x,y,z}(c) + z \in K_0^*$ for $c, z \in K_0^*$. We conclude that $G_{K_0}$ is closed under multiplication and is thus a subgroup of $G$. We directly compute that 

$$g_{x,y,z}^{-1} \circ g_{a,b,c} \circ g_{x,y,z} = g_{a',b',-\theta_{a,b,c}(z)+\theta_{x,y,z}(c)+z},$$

where $a', b'$ are certain elements of $\mathbb{F}_q$. Again, if $c, z \in K_0^*$, then $-\theta_{a,b,c}(z) + \theta_{x,y,z}(c) + z$ lies in $K_0^*$ for the same reason. Therefore, $G_{K_0}$ is normal in $G$.

The subgroup $G_{K_0}$ has size $q^2 \cdot |K_0^*| = q^3/p^s$. For $0 \leq i \leq p^s - 1$, the coset $G_{K_0} \circ g_{i,0,t_C}$ consists of elements of $G$ whose Frobenius parts lie in $g^i(g^{p^s})$, so they are pairwise disjoint. We thus have $G = \bigcup_{i=0}^{p^s-1} G_{K_0} \circ g_{i,0,t_C}$ by comparing sizes. This completes the proof. 

4.2. The matrix part of $G$ in the odd characteristic case

In this subsection, we take the notation as introduced in the beginning of this section and assume that $q$ is odd. By (4) and (6) of Corollary 2.10 the group $G$ is linear, i.e., contained in $\text{Sp}(4,q)$, if and only if $r_{A,B} = r_C = 0$. Since the linear case has been settled in Section 3, we assume that $G$ is nonlinear. In this case, $\text{Aut}(\mathbb{F}_q)$ must have a nontrivial Sylow $p$-subgroup, i.e., $p|m$, where $q = p^m$. By Theorem 1.1 and Theorem 4.3 we have $r_{A,B} = \max\{r_A, r_B\} \leq 1$ and $s = \max\{0, r_C - r_{A,B}\} \leq 1$.

In this subsection, we write $q = p^m$, $g = gc$, $r = r_{A,B}$ and let $g_2$ be an element of $\text{Aut}(\mathbb{F}_q)$ of order $p$. Recall that $K_A = \{a \in \mathbb{F}_q : \theta_{a,0,0} = 1\}$, $K_B = \{b \in \mathbb{F}_q : \theta_{0,b,0} = 1\}$, and $K_i^* := \{z \in \mathbb{F}_q : \sigma_{g^i} = g^{i}\}$ for $i \geq 0$. We now collect some known facts:

(F1) By Corollary 1.2 $\theta_{a,b,0} = g_{\text{Tr}_{\mathbb{F}_q/F_p}(\mu_A + \mu_B b)}^{\text{Tr}_{\mathbb{F}_q/F_p}(\mu_A a + \mu_B b)}$ for $a, b \in \mathbb{F}_q$, where both $\mu_A$ and $\mu_B$ are $g_2$-invariant. In particular, $K_A = \{x \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/F_p}(\mu_A x) = 0\}$, and $r_A = 0$ if and only if $\mu_A = 0$, which also holds if the $A$’s in the subscripts are replaced by $B$’s.

(F2) By (1), (2), (4) and (5) of Corollary 2.10 we have $T(a, b, 0) = L(a) + M(b)$ if either $a \in K_A$ or $b \in K_B$, and $L$ (resp. $M$) is additive on the subspace $K_A$ (resp. $K_B$).

(F3) By (7) of Corollary 2.10 we have

$$\sigma_c \theta_{a,b,0} = \theta_{a',b',0} \sigma_c;$$

$$S(c)\theta_{a,b,0} + T(a, b, 0) = T(a', b', 0)^{\sigma_{c'}} + S(c');$$

where $c' = \theta_{a,b,0}(c)$, $b' = \sigma_{c'}^{-1}(b + c \theta_{a,b,0}T(a, b, 0))$, and

$$a' = \sigma_{c'}^{-1}(a + 2bc \theta_{a,b,0} + c^2 \theta_{a,b,0}T(a, b, 0)).$$

The main result of this subsection is the following theorem.

**Theorem 4.5.** Take notation as above, and assume that $q$ is odd. Then $\mu_A = 0$, $L \equiv 0$, $g(\mu_B) = \mu_B$ and $M(b) = 0$ for $b \in K_B$. Moreover, if $r_{A,B} = 1$, then $M(y) = (1 + g_B + \cdots + g_B^{i-1})(M(t_B))$, if $\theta_{0,0,0} = g_B^i$, $1 \leq i \leq p - 1$. 

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Proof. We divide the proof into five steps, which involves the repeated use of the two equations in the fact (F3) in the special cases where \( a \in K_A, \ b \in K_B \). We make the general observation that \( \sigma_c \in \langle g_2 \rangle \) for \( c \in K_0^* \), since \( r = r_{A,B} \leq 1 \). By Theorem 4.3 \( K_0^* \) is a \( g \)-invariant \( \mathbb{F}_p \)-subspace of codimension \( s \leq 1 \) in \( \mathbb{F}_q \). By Lemma 2.2, there exists a \( g \)-invariant element \( \mu c \) such that \( K_0^* = \{ x \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu c x) = 0 \} \). Since \( \mu_A \) and \( \mu_B \) are both fixed by \( g_2 \) by the same theorem, they are also fixed by \( \sigma_c, c \in K_0^* \).

**Step 1.** We show that \( \mu_A = 0 \), so that \( K_A = \mathbb{F}_q \). In (F3), with \( a = 0, b \in K_B \) and \( c \in K_0^* \), we have \( c' = c^{\theta_{0,b,0}}_c = c \), and by comparing exponents Eqn. (4.7) reduces to

\[
\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_A(2bc + c^3M(b)) + cM(b)) = 0.
\] (4.9)

Taking the difference of Eqn. (4.9) for \( c = c_1, c_2 \in K_0^* \), we obtain

\[
\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}((2\mu_Ab + \mu_BM(b)) \cdot v) = -2\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_A M(b)uv).
\] (4.10)

where \( u = \frac{1}{2}(c_1 + c_2), \ v = c_1 - c_2 \). We observe that \( (u, v) \) ranges over \( K_0^* \times K_0^* \), as \( c_1, c_2 \) varies in \( K_0^* \). Take the difference of both sides of Eqn. (4.10) for \( u_1, u_2 \in K_0^* \), and we deduce that \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_A M(b) v u'') = 0 \) for \( v \) and \( v' = u_1 - u_2 \in K_0^* \). Since \( \{ vv'' : v, v' \in K_0^* \} \) spans \( \mathbb{F}_q \) over \( \mathbb{F}_p \), we deduce that \( \mu_A M(b) = 0 \) for \( b \in K_B \). It follows that the left hand side of Eqn. (4.10) equals 0, which yields \( \dim_{\mathbb{F}_p}(2\mu_A b + \mu_B M(b)) : b \in K_B \mathbb{F}_q \leq s \leq 1 \). Since \( K_B \) has size at least \( q/p \) and \( q \geq p^s \), this is possible only if \( \mu_A = 0 \). This completes the proof of the claim.

**Step 2.** We show that \( g(\mu_B) = \mu_B, M(b) = 0 \) for \( b \in K_B \) in the case \( \mu_B \neq 0 \). Assume that \( \mu_B \neq 0 \). In (F3), with \( a = 0, b \in K_B \) and \( c \in \mathbb{F}_q \), by comparing exponents Eqn. (4.7) takes the simpler form

\[
\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B^c (b + cM(b))) = 0, \quad b \in K_B, c \in \mathbb{F}_q.
\] (4.11)

If \( K_0^* = \mathbb{F}_q \), then \( \mu_B^c = \mu_B \) and Eqn. (4.11) reduces to \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B c M(b)) = 0 \) for \( c \in \mathbb{F}_q \), which yields that \( M(b) = 0 \) for \( b \in K_B \) as desired. So we assume that \( K_0^* \neq \mathbb{F}_q \). In this case, \( r = 1, s = 1 \), and we have \( \langle g_2 \rangle = \langle g^p \rangle \). By Theorem 4.3 \( K_i^* = t_{C,i} + K_0^* \) for \( i \geq 0 \). For \( c \in K_i^* \), we have \( \mu_B^c = g^i(\mu_B) \) by the fact that \( g_B(\mu_B) = \mu_B \). By taking the difference of Eqn. (4.11) over \( c = c_1, c_2 \in K_i^* \), we deduce that \( g^i(\mu_B) M(b) \in \mathbb{F}_p, c \mu_C \) for \( i \geq 0 \) and \( b \in K_B \). With these preparations, we are now ready to prove the claim.

1. We first show that \( M(b) = 0 \) for \( b \in K_B \). Suppose that there is \( b_0 \in K_B \) such that \( M(b_0) \neq 0 \). Then we deduce from \( \mathbb{F}_p \cdot g(\mu_B) M(b_0) = \mathbb{F}_p \cdot \mu_B M(b_0) = \mathbb{F}_p \cdot c \mu_C \) that \( g(\mu_B) = \lambda \mu_B \) for some \( \lambda \in \mathbb{F}_p^* \). Taking the relative norm to the fixed subfield of \( g \), we deduce that \( \lambda = 1 \), i.e., \( g(\mu_B) = \mu_B \). Eqn. (4.11) then simplifies to \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B M(b)c) = 0 \) for \( c \in \mathbb{F}_q \), from which we deduce that \( M(b) = 0 \) for \( b \in K_B \); a contradiction.

2. We then show that \( g(\mu_B) = \mu_B \). Eqn. (4.11) now reduces to \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B b) = 0 \) for \( b \in K_B \). It follows that \( \sigma_c(\mu_B) \in \mathbb{F}_p \cdot \mu_B \) for \( c \in \mathbb{F}_q \), so \( g(\mu_B) \in \mathbb{F}_p \cdot \mu_B \). We deduce that \( g(\mu_B) = \mu_B \) in the same way.
Step 3. We show that \( L(a) = 0 \) for \( a \in \mathbb{F}_q \). We consider two cases.

1. If \( \mu_B \neq 0 \), take \( a \in K_A = \mathbb{F}_q, \ b \in K_B \) and \( c \in \mathbb{F}_q \) in Eqn. (4.17), and by comparing exponents we get \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B L(a)) = 0 \). The claim then follows.

2. If \( \mu_B = 0 \), then \( \theta_{a,b,0} \equiv 1 \), and \( r = r_{A,B} = 0 \). In this case, \( s = 1 \), \( K_0^* \) has codimension 1, and both \( L \) and \( M \) are additive over \( \mathbb{F}_q \). Take \( b = 0, \ c \in K_0^* \) in Eqn. (4.8), and it yields \( L(c^2 L(a)) = -M(cL(a)) \). Taking the difference over \( c = c_1, c_2 \in K_0^* \), we derive that \( L(uvL(a)) = -M(vL(a)) \) for \( u, v \in K_0^* \). The right hand side is independent of \( u \), so both sides equals \( L(0 \cdot vL(a)) = 0 \). It follows that \( L \equiv 0 \) as desired, since \( \{ uv : u, v \in K_0^* \} \) spans \( \mathbb{F}_q \) by Lemma 2.6.

Step 4. We show that \( M(b) = 0 \) for \( b \in \mathbb{F}_q \) if \( \mu_B = 0 \). Assume that \( \mu_B = 0 \) and \( M \) is not constantly zero. In this case, \( s = 1, \ o(g) = p, \ \theta_{a,b,0} \equiv 1 \), and \( M \) is additive over \( K_B = \mathbb{F}_q \) by the facts (F1) and (F2). Take \( a = 0, \ b \in \mathbb{F}_q \) and \( c \in \mathbb{F}_q \) in Eqn. (4.8), and we get

\[
M(\sigma_c^{-1}(b))^{\sigma_c} - M(b) + M(\sigma_c^{-1}(cM(b)))^{\sigma_c} = 0. \tag{4.12}
\]

By taking difference over \( c = c_1, c_2 \in K_0^* \), we deduce that \( M(g^{-i}(vM(b))) = 0 \) for \( v \in K_0^* \) and \( i \geq 0 \). Therefore, \( K_0^* \cdot g^i(M(b)) \subseteq \ker(M) \) for \( i \geq 0 \). By comparing sizes, we see that \( \ker(M) = K_0^* \cdot g^i(M(b)) \) so long as \( M(b) \neq 0 \). It follows that \( \text{Im}(M) \) has dimension 1, say, \( \text{Im}(M) = F_p \cdot \omega \) with \( \omega \neq 0 \). Then \( \ker(M) = K_0^* \cdot \omega \) and is \( g \)-invariant. By Lemma 2.2 we deduce that \( g(\omega) = \omega \).

By Lemma 2.3 there exists \( \eta \in F_q^* \) such that \( M(x) = \omega \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\eta x) \). Plugging it into Eqn. (4.12), we get \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\Delta b) = 0 \) with \( \Delta = \sigma_c(\eta) - \eta + \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\sigma_c(\eta)c\omega)\eta \). Since \( b \) is arbitrary, we have \( \Delta = 0 \) for \( c \in \mathbb{F}_q \). It follows that \( \sigma_c(\eta)\eta^{-1} = 1 - \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\sigma_c(\eta)c\omega) \) lies in \( F_p \). We deduce that \( \sigma_c(\eta) = \eta \) by taking relative norm. Now \( \Delta = 0 \) reduces to \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\eta c\omega) = 0 \) for \( c \in \mathbb{F}_q \), and so \( \omega \eta = 0 \): a contradiction. This proves the claim.

Step 5. Finally, we show that if \( r_B = 1 \) and \( \theta_{a,y,0} = g_B^i, 1 \leq i \leq p - 1 \), then \( M(y) = (1 + g_B + \cdots + g_B^{i-1})(M(t_B)) \). In this case, \( \mu_B \neq 0 \). Let \( y \) be such an element as in the claim. In terms of the notation in the proof of Theorem 1, this means that \( y \in K_{B,i} \). In the same proof, we showed that \( K_{B,i} = K_B + t_{B,i}, g_B(K_B) = K_B \), so there exists \( x \in K_B \) such that \( y = g_B^i(x) + t_{B,i} \). The claim then follows from the \((B,M)\)-version of Eqn. (4.1).

5. The nonlinear point regular groups of \( Q^P \) with \( q \) odd

In this section, we determine all the nonlinear point regular group \( G \)'s of \( Q^P \) in the odd characteristic case, i.e., \( G \) is contained in \( PTL(4,q) \) but not contained in \( PGL(4,q) \). The main result of this section is as follows.

**Theorem 5.1.** Let \( G \) be a group that acts regularly on the points of the derived quadrangle \( Q^P \) of \( Q = W(q), q \) odd and \( q \geq 5 \). If \( G \) is not contained in \( PGL(4,q) \), then \( G \) is conjugate to one of the groups in Constructions 5.3, 5.7 and 5.11 below.
By the analysis in Section 2.2, we can assume that $G$ is as defined in Theorem 2.9 with associated functions $T$ and $\theta$. The function $\theta : \mathbb{F}_q \to \text{Aut}(\mathbb{F}_q)$ is not the trivial map, so we have $p|m$, where $q = p^m$ with $p$ prime.

Notation. Take the same notation as introduced in the beginning of Section 4. Set $r := r_{A,B} = \max\{r_A, r_B\}$, $s := \max\{0, r_C - r_{A,B}\}$, and write $q = p^{r+s+1}$ for some positive integer $l$. Take $g \in \text{Aut}(\mathbb{F}_q)$ such that $g(x) = x^p$, and set $g_1 := g^{p^r}$. By Theorem 1.5 we have $\mu_A = 0$, so $r_A = 0$, $r = r_B$. If $r = 1$, then take $g_B = g_1$ and take $t_B$ such that $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B t_B) = 1$. By Theorem 4.3 and Lemma 2.2 there exists $\mu_C$ such that

$$K^*_0 = \{x \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_C x) = 0\}$$

and $g(\mu_C) = \mu_C$, where we recall that $K^*_0 = \{c \in \mathbb{F}_q : \sigma^p c = 1\}$.

We divide the proof of Theorem 5.1 into two cases according as $r_{A,B} = 0$ or $r_{A,B} = 1$. In the former case, we must have $r_C = 1$ since $G$ is nonlinear. We shall frequently make use of the following facts:

$$\sum_{i=0}^{p^r-1} g_1^i(x) = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x), \quad \sum_{i=0}^{p^{r+s}-1} g_1^i(x) = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x).$$

We make some preparations before we proceed to the proof. We first reformulate (F1)-(F3) in Section 4.2 by utilizing the results in Theorem 1.5

(F4) $\theta_{a,b,0} = g_2^{\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B b)}$, where $g_2 \in \text{Aut}(\mathbb{F}_q)$ has order $p$ and $g(\mu_B) = \mu_B$. In the case $r = 1$, we take $g_2 = g_1$. As a corollary, $K_A = \mathbb{F}_q, K_B = \{b \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B b) = 0\}$, and $\theta_{a,g(b),0} = \theta_{a,b,0}.

(F5) $T(a, b, 0) = M(b)$ for $a, b \in \mathbb{F}_q$, and $M$ vanishes on $K_B$. If $r = 0$, then $M \equiv 0$. If $r = 1$, then $M(b) = N_C(\theta_{0,b,0})$, where $\nu_B := M(t_B)$ and

$$N_C(g_1^i) := \begin{cases} 0, & \text{if } i = 0, \\ (1 + g_1 + \cdots + g_1^{i-1})(\nu_B), & \text{if } 1 \leq i \leq p - 1. \end{cases} \quad (5.1)$$

It is routine to verify that $M(g(b)) = M(b)$ and

$$g_1^i(N_C(g_1^i)) + N_C(g_1^i) = N_C(g_1^{i+j}), \quad 0 \leq i, j \leq p - 1. \quad (5.2)$$

(F6) For $b, c$ in $\mathbb{F}_q$, $c' = \theta_{0,b,0}(c)$, we claim that

$$\sigma_c = \sigma_{c'b\theta_{0,b,0}} = \sigma_{c'b^{\nu_B}}.$$

We sketch the proof. Write $b'' = b + c'M(b)$. The first is just Eqn. (4.4), which can also be written as $\sigma_c \theta_{0,b,0} = \sigma_{c'b^{\nu_B}}$. Combining (F5) and Eqn. (1.8), Eqn. (5.4) reduces to $N_C(\theta_{0,b,0}) + N_C(\sigma_c) \theta_{0,b,0} = N_C(\theta_{0,b,0}) \sigma_c + N_C(\sigma_c)$, which follows from Eqn. (5.2) and the fact $\sigma_c \theta_{0,b,0} = \sigma_{c'b^{\nu_B}}$. This completes the proof.

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Lemma 5.2. Take notation as above. For \( a, b, c, x, y, z \) in \( \mathbb{F}_q \) and \( v = \sigma_z(c)S(z) \), \( w = \sigma_z(c) + z \), it holds that
\[
\theta_{a,b,c} = \theta_{0,0,0} \sigma_c, \quad (5.5)
\]
\[
T(a, b, c) = M(b)^{\sigma_c} + S(c), \quad (5.6)
\]
\[
\sigma_c \sigma_z = \theta_{0,0,0} \sigma_w, \quad (5.7)
\]
\[
S(c)^{\sigma_z} + S(z) = M(v)^{\sigma_w} + S(w). \quad (5.8)
\]

Proof. The first four equations are reformulations of (6) and (3) of Corollary 2.10 using the facts (F4) and (F5).

5.1. The classification of the case \( r_{A,B} = 0, r_{C} = 1 \)

In this subsection, we consider the case \( r_{A,B} = 0, r_{C} = 1 \). In this case, \( r = 0, s = 1 \), \( K_A = K_B = \mathbb{F}_q \), \( o(g) = p \) and \( \mu_C \in \mathbb{F}_p^\ast \). We deduce that \( T(a, b, 0) \equiv 0 \) by (F5), \( S \) is additive on \( K_0^\ast \) by Eqn. (5.8), and \( T(a, b, c) = S(c) \) by Eqn. (5.6). Take \( t_C \in \mathbb{F}_q \) such that \( \sigma_t = g \), and write \( \nu_C := S(t_C) \).

We now explore the condition that \( G_{K_0^\ast} \) is a normal subgroup of index \( p \) in \( G \), cf. Corollary 4.11. First, we have \( g_{0,0,0,t_C}^{-1} \circ g_{a,b,c} \circ g_{0,0,0,t_C} \in G_{K_0^\ast} \) for \( a, b \in \mathbb{F}_q, c \in K_0^\ast \). By the multiplication rule (2.8), it equals \( (M', 1) \) with \( M' = M_{0,0,t_C} \cdot M_{a,b,c} \cdot M_{0,0,t_C}. \) The (4,3)-rd entry of \( M' \) is \( e^g \), and the (3,2)-nd entry of \( M' \) is \( S(c)^g \) by Remark 2.12. It follows that \( S(c)^g = S(c)g \) for \( c \in K_0^\ast \). By Lemma 2.4 there is a reduced linearized polynomial \( S_1(X) \in \mathbb{F}_p^\ast[X] \) such that \( S(c) = S_1(c) \) for \( c \in K_0^\ast \).

Second, we have \( g_{0,0,0,t_C} \in G_{K_0^\ast} \), where \( g_{0,0,0,t_C} = (M_{0,0,t_C}, g) \) with \( M_{0,0,t_C} = E(0,0,t_C, \nu_C) \). By the multiplication rule (2.8) we deduce that the (3,2)-nd entry and the (4,3)-rd entry of its matrix part is \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\nu_C) \) and \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(t_C) \) respectively. We have \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(t_C) \in K_0^\ast \) by the fact \( \mu_C \in \mathbb{F}_p^\ast \). It follows that \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\nu_C) = S(\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(t_C)) \). Since \( S(c) = S_1(c) \) for \( c \in K_0^\ast \) and \( S_1 \) has coefficients in \( \mathbb{F}_p^\ast \), we deduce that \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\nu_C - S_1(t_C)) = 0 \).

It turns out that the conditions that we have derived so far are also sufficient. This leads to the following construction.

Construction 5.3. Suppose that \( q = p^d \) with \( p \) an odd prime and \( l \) a positive integer, and let \( g \in \text{Aut}(\mathbb{F}_q) \) be such that \( g(x) = x^p \). Take \( \mu_C \in \mathbb{F}_p^\ast \), and define \( K := \{ x \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_C x) = 0 \} \). Take an element \( t_C \in \mathbb{F}_q \setminus K \) and a linearized polynomial \( S_1(X) \in \mathbb{F}_p^\ast[X] \). Let \( \nu_C \) be an element of \( \mathbb{F}_q^\ast \) such that \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\nu_C - S_1(t_C)) = 0 \).

Set \( M_{a,b,c} = E(a, b, c, S_1(c)) \) for \( a, b \in \mathbb{F}_q, c \in K \), and set \( M_{0,0,t_C} := E(0,0,t_C, \nu_C) \), where \( E \) is as defined in Eqn. (2.7). Then \( G_K := \{ g_{a,b,c} : a, b \in \mathbb{F}_q, c \in K \} \) is a group of order \( q^3/p \), where \( g_{a,b,c} = (M_{a,b,c}, 1) \). Let \( G \) be the group generated by \( G_K \) and \( g_{0,0,t_C} := (M_{0,0,t_C}, g) \). Then \( G \) is a point regular group of \( Q_p \).

Proof. The verification is routine, and we only give a sketch. First, we verify that \( G_K \) forms a group of order \( q^3/p \) by showing that it is closed under multiplication. Then we show that \( g_{0,0,t_C}^{-1} \circ g_{a,b,c} \circ g_{0,0,t_C} \in G_K \) for \( a, b \in \mathbb{F}_q, c \in K \), so \( G \) is a group of order \( q^3 \) with \( G_K \) as a normal subgroup of index \( p \). Finally, by applying the elements of \( G = \bigcup_{t_C} G_K \circ g_{0,0,t_C} \) to the point \( ((0,0,0,1)) \), we deduce that \( G \) acts regularly on the points of \( Q_p^3 \).
To summarize, we have shown that if $G$ is a point regular subgroup of $Q^p$ with $r_{A,B} = 0$ and $r_C = 1$, then it is conjugate to the ones in Construction 5.3.

### 5.2. The classification of the case $r_{A,B} = 1$

In this subsection, we consider the case $r_{A,B} = 1$. The proof splits into two parts. In the first part, we derive some general results on $\sigma_c$ and $S(c)$ by exploring the group structure of $G_{K_0^*}$, and as a byproduct we complete the proof for the case $r_C \leq 1$. In the second part, we explore the fact that $G_{K_0^*}$ is a normal subgroup of index $p^s$ in $G$, and complete the proof for the case $r_C = 2$.

**Part 1: the group structure of $G_{K_0^*}$**

In this case, $g_1 = g^p$ has order $p$ and we take $g_2 = g_1$ in the fact (F4). Recall that $t_B$ is chosen such that $\text{Tr}_{F_q/F_p}(\mu_B t_B) = 1$, and we set $\nu_B := M(t_B)$.

**Lemma 5.4.** Let the function $N_C : \langle g_1 \rangle \to F_q$ be as defined in Eqn. (5.1). We define $B : K_0^* \times K_0^* \to F_p$ as follows:

$$B(z, c) := \text{Tr}_{F_q/F_p}(\mu_B c (N_C(\sigma_z) - S(z)))$$

and set $Q(x) := B(x, x)$. Then $B$ is a symmetric bilinear form, and $Q(g_1(c)) = Q(c)$ for $c \in K_0^*$. Moreover, there is $\alpha \in F_q$ such that $\alpha - g_1(\alpha) + \mu_B \nu_B \in \mu_F$ and

$$\sigma_z = g_1^{\frac{1}{2} Q(\alpha z) + \text{Tr}_{F_q/F_p}(\alpha z)} \quad \text{for} \quad z \in K_0^*.$$  

**Proof.** We first establish the following equation:

$$\sigma_{c+z}(\sigma_c \sigma_z)^{-1} = g_1^{\text{Tr}_{F_q/F_p}(\mu_B c (N_C(\sigma_z) - S(z)))}, \quad c \in F_q, \ z \in K_0^*.$$  

(5.10)

Fix an element $z \in K_0^*$, and take $b \in F_q$ such that $\theta_{0,b,0} = \sigma_z$. By replacing $c$ with $\sigma_{z}^{-1}(c)$ in Eqn. (5.10), we deduce that

$$\sigma_{\theta_{0,b,0}^{-1}(c)} = \sigma_c g_1^{\text{Tr}_{F_q/F_p}(\mu_B c M(b))}.$$  

(5.11)

Similarly, by replacing $c$ with $\sigma_{z}^{-1}(c)$ in Eqn. (5.7) we get

$$\sigma_{\sigma_z^{-1}(c)} = \sigma_{c+z} \sigma_z^{-1} g_1^{\text{Tr}_{F_q/F_p}(\mu_B c S(z))}.$$  

Since $\theta_{0,b,0} = \sigma_z$, the right hand sides are equal. The claim then follows from the fact that $M(b) = N_C(\theta_{0,b,0}) = N_C(\sigma_z)$, cf. (F5).

By Eqn. (5.10), $B$ is both symmetric and linear in $c$, so is a symmetric bilinear form on $K_0^*$. We can rewrite Eqn. (5.10) as follows:

$$\sigma_{c+z} g_1^{-\frac{1}{2} Q(c+z)} = \left(\sigma_c g_1^{-\frac{1}{2} Q(c)}\right) \cdot \left(\sigma_z g_1^{-\frac{1}{2} Q(z)}\right), \quad c, \ z \in K_0^*.$$  

That is, $z \mapsto \sigma_z g_1^{-\frac{1}{2} Q(z)}$ is a group homomorphism from $K_0^*$ to $\langle g_1 \rangle$. Therefore, there exists an element $\alpha \in F_q$ such that $\sigma_z = g_1^{-\frac{1}{2} Q(z) + \text{Tr}_{F_q/F_p}(\alpha z)}$ for $z \in K_0^*$.  

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Recall that $\mathcal{K}_0^*$ is $g$-invariant and $g(\mu_C) = \mu_C$, so $\theta_{0,b,0}^{-1}(c) \in \mathcal{K}_0^*$ if $c \in \mathcal{K}_0^*$ and $b \in \mathbb{F}_q$, we plug the expressions of $\sigma_c$ and $\theta_{0,b,0}$ into Eqn. (5.11) and compare the exponents to get

$$\frac{1}{2} Q(c^{\theta_{0,b,0}}) + \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\alpha^{\theta_{0,b,0}}c) = \frac{1}{2} Q(c) + \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\alpha c) + \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B c M(b)).$$

By replacing $c$ by $\lambda c$ with $\lambda \in \mathbb{F}_p$ and comparing the coefficients of $\lambda$ and $\lambda^2$, we derive that $Q(c^{\theta_{0,b,0}}) = Q(c)$ and $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}((\alpha^{\theta_{0,b,0}} - \alpha - \mu_B M(b)) \cdot c) = 0$. Take $b = t_B$, and we get the remaining claims in the lemma.

**Lemma 5.5.** There is a reduced linearized polynomial $S_1(X)$ over $\mathbb{F}_q$ and a map $H : \mathcal{K}_0^* \to \mathbb{F}_p$ such that

$$S(z) = S_1(z) + N_C(\sigma_z) + \mu_B^{-1} \mu_C H(z), \quad z \in \mathcal{K}_0^*. \quad (5.12)$$

**Proof.** Each bilinear form over $\mathbb{F}_q$ can be written in the form $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B x L(y))$ for some reduced linearized polynomial $L$ over $\mathbb{F}_q$. If $\mu_C \neq 0$, take $c \in \mathbb{F}_q \setminus \mathcal{K}_0^*$, and define $B(x, y) := B(x, y)$ for $x, y \in \mathcal{K}_0^*$ and $\lambda_1, \lambda_2 \in \mathbb{F}_p$. Then $B$ is a bilinear form over $\mathbb{F}_q$. Therefore, whether $\mu_C = 0$ or not, there is a reduced linearized polynomial $S_1(X)$ such that $B(x, y) = -\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B x S_1(y))$ for $x, y \in \mathcal{K}_0^*$. Together with Eqn. (5.9), we deduce that $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B x (N_C(\sigma_y) - S(y) + S_1(y))) = 0$ for $x, y \in \mathcal{K}_0^*$. It follows that $N_C(\sigma_y) - S(y) + S_1(y) \in \mathbb{F}_p : \mu_B^{-1} \mu_C$ for $y \in \mathcal{K}_0^*$. This completes the proof.

Let $H$ be as introduced in Lemma 5.5 and set

$$S_2(c) := S_1(c) + \mu_B^{-1} \mu_C H(c), \quad c \in \mathcal{K}_0^*. \quad (5.13)$$

Then $S(c) = S_2(c) + N_C(\sigma_c)$ for $c \in \mathcal{K}_0^*$ by Eqn. (5.12). By Eqn. (5.9), we have $B(c, z) = -\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B c S_2(z))$ for $c, z \in \mathcal{K}_0^*$. In particular, $Q(c) = -\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B c S_2(c))$ for $c \in \mathcal{K}_0^*$ and $a, b \in \mathbb{F}_q$ and $c \in \mathcal{K}_0^*$, we plug the expressions of $\theta_{0,b,0}$ and $\sigma_c$ in (F4) and Lemma 5.4 into Eqns. (5.5) and (5.6) to obtain:

$$\begin{align*}
\theta_{a,b,c} &= g_1^i Q(c) + \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\alpha c + \mu_B b) \\
T(a, b, c) &= S_2(c) + N_C(\theta_{a,b,c}), \quad (5.14)
\end{align*}$$

where $N_C$ is as defined in Eqn. (5.1).

**Lemma 5.6.** The map $x \mapsto S_2(x)$ as defined in Eqn. (5.13) is additive on $\mathcal{K}_0^*$.

**Proof.** We first show that $S_2(c)^{\sigma_z} + S_2(z) = S_2(c^{\sigma_z} + z)$ for $c, z \in \mathcal{K}_0^*$ Fix $c, z \in \mathcal{K}_0^*$, and set $v = \sigma_z(c) S(z), w = \sigma_z(c) + z$. By Eqn. (5.7), we have $\sigma_v \sigma_w = \theta_{0,v,0} \sigma_w$. Since $M(v) = N_C(0, 0)$, it readily follows by Eqn. (5.4) that $N_C(\sigma_v) \sigma_z + N_C(\sigma_w) = M(v) \sigma_w + N_C(\sigma_w)$. We plug $S(z) = S_2(z) + N_C(\sigma_z)$ into Eqn. (5.8) and obtain the desired equation.

We next show that $S_2$ is additive on $\mathcal{K}_0^*$. By Eqn. (5.4), we have $S_2(c^{\sigma_i}) = S_2(c)^{g_1}$ for $c \in \mathcal{K}_0^*$ and $i \geq 0$. Since $\sigma_z \in \langle g_1 \rangle$ for $z \in \mathcal{K}_0^*$ and $\mathcal{K}_0^*$ is $g$-invariant, the claim then follows. This completes the proof.
We are now ready to complete the classification in the case $r_C \leq r_{A,B} = 1$. In this case, $s = 0$, $g \equiv g_1$, and $K_0^s = \mathbb{F}_q$. Lemma 5.5 yields that $S(z) = S_1(z) + N_C(\sigma_z)$ for $z \in \mathbb{F}_q$, where $S_1(X) = \sum_{i=0}^{pl-1} s_i X^i$ is a reduced linearized polynomial over $\mathbb{F}_q$. It follows that $Q(x) = -\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B x S_1(x))$. By Eqn. (5.4) we have $S_1(c^g) = S_1(c)^{g^i}$ for $c \in \mathbb{F}_q$. It follows that $g_1(s_i) = s_i$, i.e., $s_i \in \mathbb{F}_p$, for each $i$. The function $B$ is a symmetric bilinear form on $\mathbb{F}_q$ if and only if $\mu_B s_i - s_{p^{i-1}l^m} = 0$ for $0 \leq i \leq pl - 1$ by Lemma 2.5. Set $S_2(x) := S_1(x)$. The expressions of $T$ and $\theta$ are as defined in (5.11).

It turns out that the conditions that we have derived so far are also sufficient. This leads to the following construction.

**Construction 5.7.** Suppose that $q = p^m$ with $p$ an odd prime and $l$ a positive integer, and let $g_1 \in \text{Aut}(\mathbb{F}_q)$ be such that $g_1(x) = x^p$.

(i) Take $\mu_B \in \mathbb{F}_p^*$, and take an element $t_B$ such that $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B t_B) = 1$.

(ii) Take a tuple $(s_0, s_1, \ldots, s_{pl-1})$ with entries in $\mathbb{F}_p$, such that $\mu_B s_i - s_{pl-i}^l = 0$ for $1 \leq i \leq pl - 1$.

(iii) Take $\nu_B$ and $\alpha$ such that $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}((\nu_B) = 0, g_1(\alpha) - \alpha = \mu_B \nu_B$.

Set $S_2(x) := \sum_{i=0}^{pl-1} s_i x^{i}$, and $Q(x) := -\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B x S_2(x))$ for $x \in \mathbb{F}_q$. Let $N_C$ be as defined in Eqn. (5.1) with the prescribed $\nu_B$. Let $\theta$ and $T$ be as defined in Eqn. (5.14). Then the set $G$ as defined in Theorem 2.9 with the prescribed functions $T$ and $\theta$ is a point regular group of $Q^p$.

**Proof.** The condition $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\nu_B) = 0$ in (iii) is to guarantee the existence of the desired $\alpha$, cf. Theorem 2.1. The number of tuples satisfying the conditions in (ii) equals $p^{(pl+1)/2}$ by a similar argument to that in the proof of Lemma 3.3.

We only give a sketch as for how to verify the two conditions in Theorem 2.9. First, we check that $S_2(g_1(x)) = S_2(x)^{g_1}, Q(g_1(x)) = Q(x)$. Eqn. (2.9) reduces to $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\alpha - \alpha^{\theta_2} + \mu_B N_C(\theta_2)(c^{\theta_2}) = 0$ upon expansion and simplification, where $\theta_2 = \theta_{x,y,z}$. It reduces to showing that $g_1(\alpha) - \alpha = \mu_B N_C(g_1)$ for $1 \leq i \leq p-1$, which follows from $g_1(\alpha) - \alpha = \mu_B \nu_B$ by induction. Second, from $S_2(g_1(x)) = S_2(x)^{g_1}$ and Eqn. (5.2) we deduce that Eqn. (2.10) holds. This completes the proof.

To summarize, we have shown that in the case $r_C \leq r_{A,B} = 1$, the group $G$ must be conjugate to one of those arising from Construction 5.7.

**Part 2: the normality of $G_{K_0^s}$**

From now on, assume that $r_{A,B} = 1$ and $r_C = 2$, i.e., $r = s = 1$. In this case, $g$ has order $p^2$, $g_1 = g^p$ has order $p$ and we take $g_2 = g_1$. Recall that $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B t_B) = 1$, $\nu_B = M(t_B)$. Take $t_C$ such that $\sigma_{t_C} = g$, and set

$$
\lambda_C := \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_C t_C), \quad \nu_C := S(t_C).
$$

In particular, we have $\mathcal{M}_{0,0,t_C} = E(0,0,t_C,\nu_C)$, where $E$ is as defined in Eqn. (2.7). Since $t_C \not\in K_0^s$, we have $\mu_C \neq 0, \lambda_C \neq 0$. 

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We now explore the condition that $G^*_{K_0^*}$ is normal in $G$. To be specific, we consider the special case where $a, b \in \mathbb{F}_q$, $c \in \mathcal{K}_0^*$ satisfy that $\theta_{a,b,c} = 1$, i.e.,

$$- \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B b) = \frac{1}{2} Q(c) + \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\alpha c).$$  \hspace{1cm} (5.15)

For any given $c \in \mathcal{K}_0^*$, we can find $b \in \mathbb{F}_q$ such that Eqn. (5.15) holds by the assumption $\mu_B \neq 0$, i.e., $r \neq 0$. Recall that $\mathcal{M}_{0,0,tC} = E(0,0,t_C,\nu_C)$. By the multiplication rule (2.7) we compute that

$$g_{0,0,tC}^{-1} \circ g_{0,0,tC} = g_{a',b',c'}. \hspace{1cm} (5.16)$$

where $a'$ is irrelevant, $b' = b^q - tCS_2(c)\mu + c^\alpha \nu_C$ and $c' = c^g$. We have $c' \in \mathcal{K}_0^*$, since $\mathcal{K}_0^*$ is $g$-invariant. By comparing the Frobenius part and the (3,2)-nd entry of the matrix part of both sides of Eqn. (5.16), we get $\theta_{a',b',c'} = 1$ and $T(a, b, c) = T(a', b', c')$. By plugging in the expressions in (5.14), we deduce that $S_2(c)^g = S_2(c^q)$ and

$$- \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B b) = \frac{1}{2} Q(c^q) + \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\alpha c^q - \mu_B tCS_2(c)^q + \mu_B \nu_C c^q)). \hspace{1cm} (5.17)$$

**Lemma 5.8.** $S_2(c)^q = S_2(c^q)$, $Q(c^q) = Q(c)$ for $c \in \mathcal{K}_0^*$.

*Proof.* Recall that $g(\mu_B) = \mu_B$. The first equality has just been proved; observe that it is stronger than Eqn. (5.15), since $g_1 = g^p$ in this case. The second equality follows from the fact $Q(c) = -\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B tCS_2(c))$ and $\mathcal{K}_0^*$ is $g$-invariant. \hfill \square

From Eqn. (5.15) and Eqn. (5.17), we deduce that

$$\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B tCS_2(c^q)) = \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}((-\alpha^q + \alpha + \mu_B \nu_C) c^q)) \text{ for } c \in \mathcal{K}_0^*. \hspace{1cm} (5.18)$$

**Lemma 5.9.** The map $x \mapsto S_2(x)$ as defined in Eqn. (5.13) can be extended to an additive map on $\mathbb{F}_q$ whose corresponding reduced linearized polynomial has coefficients in $\mathbb{F}_p$. Denote this polynomial by $S_2(X)$ by abuse of notation, and write $S_2(X) = \sum_{i=0}^{p^2l-1} s_i X^p^i$ with each $s_i \in \mathbb{F}_p$. Write $\lambda_C = \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu CT)$, $\nu_C = S(t_C)$.

(i) There is $u \in \mathbb{F}_q$ such that $u - g(u) \in \mathbb{F}_p \cdot \mu_C$, $(1-g)^2(u) = 0$ and

$$- \mu_B s_i + s_{m-i}^i \mu_B^i = \mu_C u^i - \mu_C^i, \hspace{1cm} 0 \leq i \leq p^2l - 1. \hspace{1cm} (5.19)$$

(ii) For $x, y \in \mathbb{F}_q$ and $\operatorname{Tr} = \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$, it holds that

$$\operatorname{Tr}(\mu_B xS_2(y)) = \operatorname{Tr}(\mu_B yS_2(x)) + \operatorname{Tr}(ux) \cdot \operatorname{Tr}(\mu_C y) - \operatorname{Tr}(\mu_C x) \cdot \operatorname{Tr}(uy). \hspace{1cm} (5.20)$$

(iii) There exists $\lambda \in \mathbb{F}_p$ such that

$$\alpha^q - \alpha = \mu_B \nu_C - \mu_B S_2(t_C) + \lambda_C u + \lambda \mu_C. \hspace{1cm} (5.21)$$

As a corollary, we have $\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\nu_C - S_2(t_C)) = 0$ and

$$\sum_{i=0}^{p^2-1} \alpha^{i(q)} := - \sum_{i=1}^{p^2-1} i \left( \mu_B \nu_C^{q^{-1}} - \mu_B S_2(t_C)^{q^{-1}} + \lambda_C u^{q^{-1}} \right). \hspace{1cm} (5.22)$$
**Proof.** By Lemma 5.8, we have \( S_2(c)^g = S_2(c^g) \) for \( c \in K_0^* \). By Lemma 5.6 the map \( x \mapsto S_2(x) \) is additive on \( K_0^* \). The existence of the desired polynomial \( S_2(X) \) follows from Lemma 2.4.

(i). Eqn. (5.19) follows by applying Lemma 2.5 in the case \( \mu = -\mu_B, \eta = \mu_C \) and \( L = S_2 \). Since \( \mu_B \) and each \( s_i \) are in \( F_p \), the left hand side of Eqn. (5.19) lies in \( F_p \). It follows that \( \mu C u^p - u \mu C^p = \mu C g(u)^p - g(u) \mu C^p \), i.e., \( \mu C (u - g(u))^p = (u - g(u)) \mu C^p \). In the case \( i = 1 \), it follows that \( u - g(u) \in F_p, \mu C = \mu C \), we deduce that \( (1 - g)^2(u) = 0 \) as desired.

(ii). Let \( \tilde{S}_2 \) be the trace dual of \( S_2(X) \). Also by Lemma 2.5 we have

\[
\tilde{S}_2(\mu_B x) = \mu_B S_2(x) + \mu C \text{Tr}_{F_q/F_p}(ux) - u \text{Tr}_{F_q/F_p}(\mu Cx)
\]

for \( x \in F_q \). Multiplying both sides by \( y \) and taking the absolute trace, we obtain Eqn. (5.20) by observing that \( \text{Tr}_{F_q/F_p}(\mu_B x S_2(y)) = \text{Tr}_{F_q/F_p}(\tilde{S}_2(\mu_B x)y) \).

(iii). First observe that \( \text{Tr}_{F_q/F_p}(u) = (1 - g)^{p-1}(u) = 0 \) by Eqn. (5.21) and the fact \( (1 - g)^2(u) = 0 \). Take \( c \in K_0^* \), i.e., \( \text{Tr}_{F_q/F_p}(\mu C c) = 0 \). Applying Eqn. (5.20) to the pair \( x, y = (t_C, c^g) \), we deduce that the left hand side of Eqn. (5.18) equals \( \text{Tr}_{F_q/F_p}(\mu_B c^g S_2(t_C) - \lambda_C c^g) \). By collecting terms, it reduces to

\[
\text{Tr}_{F_q/F_p}((\alpha^g - \alpha - \mu_B \nu C + \mu_B S_2(t_C) - \lambda_C u)c^g) = 0.
\]

This holds for all \( c \in K_0^* \), so there exists \( \lambda' \in F_p \) such that Eqn. (5.21) holds. Taking the relative trace to \( F_p \) on both sides of Eqn. (5.21), we deduce that \( \text{Tr}_{F_q/F_p}(\nu C - S_2(t_C)) = 0 \).

By binomial expansion, we have \( (1 - g)^{p-2} = \sum_{i=0}^{p-1} i g^{i-1} \), \( (1 - g)^{p-1} = 1 + g + \cdots + g^{p-1} \). Eqn. (5.22) follows by applying \( -(1 - g)^{p-2} \) to both sides of Eqn. (5.21). \( \Box \)

Finally, we explore the condition \( \mathfrak{g}_{0,0,t_C}^p \in G_{K_0} \). Recall that \( \mathcal{M}_{0,0,t_C} = E(0,0,t_C, \nu C) \), and \( \mathfrak{g}_{0,0,t_C} = (\mathcal{M}_{0,0,t_C}, \mathfrak{g}) \). For \( i \geq 0 \), set

\[
\mathfrak{g}_{a_i,b_i,c_i} := \mathfrak{g}_{0,0,t_C}^i = (\mathcal{M}_{0,0,t_C}^{g^{i-1}}, \cdots, \mathcal{M}_{0,0,t_C}, \mathfrak{g}^i).
\]

The value of \( a_i \) is irrelevant, and \( b_{i+1} = b_i + c_i \nu C, c_{i+1} = c_i + t_C \). From \( b_1 = 0, c_1 = t_C \), we deduce that \( b_{i+1} = \sum_{j=1}^{i} \sum_{k=1}^{i} g^{i-j} (t_C^j \nu C), c_{i+1} = \sum_{j=0}^{i} t_C^j \). In particular, we have \( \mathfrak{g}_{0,0,t_C}^p = \mathfrak{g}_{a''',b''',c'''} \), where \( a'' \) is irrelevant and

\[
b'' = b_p = \sum_{i=1}^{p-1} \sum_{k=1}^{i} g^{p-1-i} (t_C^k \nu C), c'' = c_p = \sum_{i=0}^{p-1} t_C^i.
\]

Observe that \( c'' \in K_0^* \), since \( \text{Tr}_{F_q/F_p}(\mu C c'') = 0 \) by the fact \( \mu C \in F_p \).

**Lemma 5.10.** Take notation as in Lemma 5.4 and write \( \nu_B = M(t_B) \). Then \( p = 3, \lambda_C = \text{Tr}_{F_q/F_p}((u - u^g)t_C), \nu_B = \sum_{i=0}^{p-1} (\nu C - S_2(t_C)) g^i \), and \( \mu C = u - u^g \).
Proof. We continue with the analysis preceding the lemma. Since $\sigma t'_c = g^{p_1}$, the Frobenius part of $g_{0,0,t_c}'$ is $g_1$. The Frobenius part of $g_{\alpha''',\nu',\nu''}$ is $g_1^{P_1}$ with $D = \frac{1}{2} Q(c'') + \text{Tr}_{F_q/F_p} (\alpha'' + \mu g b''')$ by Eqn. (5.14). By comparing the Frobenius parts and the $(3,2)$-nd entry of the matrix parts of both sides of $g^p_{0,0,t_c} = g_{\alpha''',\nu',\nu''}$, we obtain $D = 1$ and

\[ \nu_B = \nu_C + \nu_C^{p-1} + \cdots + \nu_C^{p-1} - S_2(c''). \]  

(5.24)

For the ease of notation, we write $T_c = T_{F_q/F_p}$ in this proof. We now compute $D$ explicitly. Since $S_2$ has coefficients in $F_{p'}$, we have $S_2(x^{g'}) = S_2(x)^g'$ for $i \geq 0$. Since $c'' = \sum_{i=0}^{p-1} t_c^g$ and $Q(c'') = -\text{Tr}(\mu_B t''_c S_2(c''))$, we compute that

\[
Q(c'') = -\sum_{i,j=0}^{p-1} \text{Tr} \left( \mu_B t'^{g}_c S_2(t_c)^{g'} \right) = -\sum_{i<j} \text{Tr} \left( \mu_B t'^{g}_c S_2(t_c)^{g'} \right) - \sum_{j<i} \text{Tr} \left( \mu_B t'^{g}_c S_2(t_c)^{g'} \right) = -\sum_{k=1}^{p-1} (p-k) \left( \text{Tr}(\mu_B t_c S_2(t_c^k)) + \text{Tr}(\mu_B t'_c S_2(t_c^k)) \right) = \sum_{k=1}^{p-1} k \left( 2\text{Tr}(\mu_B t'_c S_2(t_c^k)) + \lambda_C \text{Tr}(t_c^k) - \lambda_C \text{Tr}(t_c^k) \right) = \sum_{k=1}^{p-1} k \cdot \text{Tr}(\mu_B t'_c S_2(t_c) - \lambda_C t_c^k).
\]

Here, in the second equality the $p$ terms with $i = j$ are equal and sum to 0, in the fourth equality we used Eqn. (5.20) with $(x, y) = (t_c, t'_c)$, and in the fifth we used the fact $\sum_{i=0}^{p-1} i \equiv 0 \pmod{p}$. Similarly, we have

\[
\text{Tr}(\alpha c'') = \text{Tr}(t_c^{g^p-1} (\alpha + \cdots + \alpha^{g^p-1})) = -\sum_{i=1}^{p-1} \text{Tr} \left( i t_c^{g^p-1} (\mu_B t'_c - \mu_B S_2(t_c)^{g^p-1} + \lambda_C t_c^{g^p-1}) \right) = \sum_{i=1}^{p-1} i \cdot \text{Tr} \left( -t_c^{g^p-1} \mu_B t'_c + \mu_B t'_c S_2(t_c) - \lambda_C t_c^{g^p-1} \right) = \sum_{k=1}^{p-1} k \cdot \text{Tr}(\mu_B t'_c t_c^k - \mu_B t'_c S_2(t_c) + \lambda_C t_c^k),
\]

where in the second equality we used Eqn. (5.22), and in the last one we did a change of variable $p - i \mapsto k$. Finally, it is straightforward to check that

\[
\text{Tr}(\mu_B b'') = -\sum_{i=1}^{p-1} i \cdot \text{Tr}(\mu_B t_c^{g^p} \nu_C).
\]

Putting these pieces together, we get $D = \frac{p}{2} \cdot \sum_{i=1}^{p-1} i \cdot \text{Tr}(u t_c^g)$.

Set $v := u - u^g$, which is in $F_{p'} \cdot \mu_C$ by Lemma 5.9. In particular, $v \in F_{p'}$. From $u^g = u - v$, we have $u^{g^i} = u - iv$ for $i \geq 0$ by induction. Recall that $\lambda_C = \text{Tr}(\mu_C t_c) \neq 0$. We compute that

\[
2\lambda_C^{-1} D = \sum_{i=1}^{p-1} i \cdot \text{Tr}(u t_c^{g^p-1} t_c^{g^p-1}) = \sum_{i=1}^{p-1} i \cdot \text{Tr} \left( (u - (p - i - 1)v) t_c^{g^p-1} \right) = \sum_{i=1}^{p-1} (i^2 + i) \cdot \text{Tr}(v t_c^{g^p-1}) = \frac{p(p-1)(p+1)}{3} \cdot \text{Tr}(vt_c).
\]

This value is 0 if $p > 3$, so we must have $p = 3$. Since $D = 1$ and $\lambda_C \in \{ \pm 1 \} = F_3^*$, $\lambda_C = \text{Tr}(vt_c)$ follows from the above equation.

Combining the facts $\lambda_C = \text{Tr}(\mu_C t_c)$, $\lambda_C = \text{Tr}(vt_c)$, and $v \in F_{p'} \cdot \mu_C$, we deduce that $\mu_C = v$. Since $S_2(x^{g'}) = S_2(x)^g$ for $x \in F_q$, the expression of $\nu_B$ follows from Eqn. (5.24). This completes the proof. □
It turns out that the conditions that we have derived so far are also sufficient. This leads to the following construction.

Construction 5.11. Suppose that \( q = 3^{2l} \) with \( l \) a positive integer, and take \( g \in \text{Aut}(\mathbb{F}_q) \) such that \( g(x) = x^q \). Set \( g_1 := g^3 \).

(i) Take \( u \in \mathbb{F}_q \) such that \( \mu_C := u - u^q \in \mathbb{F}_{q^2} \).
(ii) Take \( t_C \in \mathbb{F}_q^* \) such that \( \lambda_C := \text{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(\mu_C t_C) \neq 0 \).
(iii) Take \( \mu_B \in \mathbb{F}_q^* \) and \( t_B \in \mathbb{F}_q^* \) such that \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(\mu_B t_B) = 1 \).
(iv) Take a tuple \( (s_0, s_1, \ldots, s_{9l-1}) \) with entries in \( \mathbb{F}_{q^l} \) that satisfies
\[
-\mu_B s_i + s_{9l-i}^i \mu_B^i = \mu_C u^{3^i} - u \mu_C^{3^i}, \quad 1 \leq i \leq 9l - 1;
\]
Set \( S_2(x) := \sum_{i=0}^{3^{2l-1}} s_i x^{3^i} \), \( Q(x) := -\text{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(\mu_B x S_2(x)) \) for \( x \in \mathbb{F}_q \).
(v) Take \( \nu_C \in \mathbb{F}_q \) such that \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(\nu_C - S_2(t_C)) = 0 \), and take \( \alpha \in \mathbb{F}_q \), \( \lambda \in \mathbb{F}_p \) such that \( g(\alpha) - \alpha = \mu_B \nu_C - \mu_B S_2(t_C) + \lambda_C u + \lambda \mu_C \).

Set \( \nu_B := \sum_{i=0}^{9l-1} g^i(\nu_C - S_2(t_C)) \), and let \( N_C \) be as defined in Eqn. (5.1) with the prescribed \( \nu_B \). Set \( K := \{ z \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(\mu_C z) = 0 \} \).

For \( a, b \in \mathbb{F}_q \) and \( c \in K \), let \( \theta_{a,b,c} \) and \( T(a,b,c) \) be as defined in Eqn. (5.14), and set \( M_{a,b,c} = E(a,b,c,T(a,b,c)) \), where \( E \) is as defined in Eqn. (2.7). Then \( G_K := \{ g_{a,b,c} : a, b \in \mathbb{F}_q, c \in K \} \) is a subgroup of \( g^q/3 \), where \( g_{a,b,c} = (M_{a,b,c}, \theta_{a,b,c}) \). Take \( g_{0,0,t_C} = (M_{0,0,t_C}, g) \) with \( M_{0,0,t_C} := E(0,0,t_C, \nu_C) \). Then \( G := \langle G_K, g_{0,0,t_C} \rangle \) is a point regular group of \( Q^p \).

Proof. We first check that the desired elements in the construction exist. The condition on \( \nu_C \) in (v) guarantees the existence of the desired \( \alpha \) by Theorem 2.1. The number of tuples satisfying the conditions in (iv) equals \( 3^{(9l+1)/2} \) by a similar argument to that in the proof of Lemma 3.5.

Here is the sketch of the proof. By the same argument as in the proof of Construction 5.7, we can show that \( G_K \) is a group of order \( q^3/3 \). Let \( G_1 \) be the subgroup of \( G_K \) of index 3 with a trivial Frobenius part. Then we can check that \( g_{0,0,t_C} \) normalizes \( G_1 \) and \( g_{0,0,t_C}^3 \in G_K \) by reversing the preceding arguments. Since \( g_{0,0,t_C}^3 \) has Frobenius part \( g_1 = g^3 \), we deduce that \( G_K = \langle G_1, g_{0,0,t_C}^3 \rangle \). It follows that \( G_K \) is normalized by \( g_{0,0,t_C} \).

We thus conclude that \( G \) is a group of order \( q^3 \). It is point regular by Theorem 2.9.

To summarize, we have thus shown that in the case \( r_{A,B} = 1, r_C = 2 \), the point regular subgroup \( G \) must be conjugate to one of those in Construction (5.11).

To conclude, this completes the proof of Theorem 5.1.

6. The isomorphism issue in the odd characteristic case

In this section, we first give a summary of all the constructions for the odd characteristic case. The constructions are simplified upon conjugacy within \( \text{PSp}(4, q) \) for the ease of calculating their invariants. Then we consider the isomorphism issue amongst the
groups arising from different constructions. This is achieved by considering such group invariants as exponents and Thompson subgroups. Finally, we give a relatively tight bound on the nilpotency class for each construction.

Recall that for \( a, b, c, t \) in \( \mathbb{F}_q \), we define

\[
E(a, b, c, t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-c & 1 & 0 & 0 \\
b - ct & t & 1 & 0 \\
a & b & c & 1
\end{pmatrix},
\]

which lies in \( \text{PSp}(4, q) \) and thus fixes the derived quadrangle \( Q^P \).

### 6.1. Summary of the known constructions

In this subsection, we summarize the known constructions in the odd characteristic case. For a group \( G \) arising from either of these constructions, we write \( g_{a,b,c} \) for the element of \( G \) that maps \( \langle (0, 0, 0, 1) \rangle \) to \( \langle (a, b, c, 1) \rangle \). Let \( \theta_{a,b,c} \) be the Frobenius part of \( g_{a,b,c} \) and \( T(a, b, c) \) be the \( (3, 2) \)-nd entry of its matrix part. This is consistent with the notation we introduced in Section 2. We will freely make use of the various properties of \( T \) and \( \theta \) that we have proved so far.

**Construction 6.1** (Construction 3.3). For an odd prime power \( q = p^m \), set \( \theta_{a,b,c} \equiv 1 \) and \( T(a, b, c) := S_1(c) \) for a reduced linearized polynomial \( S_1 \) over \( \mathbb{F}_q \). The set \( G \) as defined in Theorem 2.9 for the prescribed functions \( T \) and \( \theta \) is a point regular subgroup of \( Q^P \).

In Construction 5.3 set \( \nu := \nu_{c} - S_1(t_c) \). Since \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_{p^m}}(\nu) = 0 \), there exists \( u \in \mathbb{F}_q \) such that \( u^q - u = \nu \) by Theorem 2.1. Take \( g_1 := (E(0, 0, 0, u), 1) \). It is clear that \( \tilde{G} := g_1^{-1} \circ G \circ g_1 \) is also a point regular group of \( Q^P \). The group \( \tilde{G} \) is of the form as defined in Theorem 2.9 for some functions \( T' \) and \( \theta' \) by the analysis in Section 2. Let \( g_{a,b,c}' \) be the element of \( \tilde{G} \) that maps \( \langle (0, 0, 0, 1) \rangle \) to \( \langle (a, b, c, 1) \rangle \), and set \( \sigma_c' := \theta_{0,0,c}' \), \( M'(y) := T'(0, y, 0) \) and \( S'(z) := T'(0, 0, z) \). For \( a, b \in \mathbb{F}_q \) and \( c \in K \), we check that

\[
g_{a,b,c}' = g_{a,b-cw,c}' \circ g_1 = (E(a, b, c, S_1(c)), 1).
\]

It follows that \( T'(a, b, c) = S_1(c), \theta'(a, b, c) = 1 \) if \( c \in K \). In particular, \( M' \equiv 0, \theta_{0,y,0}' \equiv 1 \). Similarly, we have

\[
g_{0,ut_c,t_c}' = g_1^{-1} \circ g_{0,0,t_c}' \circ g_1 = (E(0, ut_c, t_c, S_1(t_c)), g).
\]

That is, \( T'(0, ut_c, t_c) = S_1(t_c), \theta_{0,ut_c,t_c}' = g \). By Eqns. (5.5) and (5.6), we have \( \sigma_{t_c}' = g \) and \( S'(t_c) = S_1(t_c) \). It follows that \( g_{0,0,t_c}' = (E(0, 0, t_c, S_1(t_c)), g) \).

Set \( \tilde{G} := g_1^{-1} \circ G_K \circ g_1 \), the kernel of the group homomorphism \( g_{a,b,c}' \mapsto \theta_{a,b,c}' \). Then \( \tilde{G} = \langle \tilde{G}', g_{0,0,t_c}' \rangle \). It follows that for each triple \( (x, y, z) \) there exist \( a, b \in \mathbb{F}_q, c \in K \) and \( i \geq 0 \) such that \( g_{x,y,z}' = g_{a,b,c}' \circ g_{0,0,t_c}' \). By the multiplication rule (2.8) we can directly check that \( z = g^t(c) + \sum_{k=0}^{i-1} k^k(t_c), T'(x, y, z) = S_1(z) \) and \( \theta'_{x,y,z} = g_{\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mu_c t_c)}^{-1} \). Alternatively, we can derive the expressions of \( T' \) and \( \theta' \) by repeated use of the equations in Lemma 5.2. We thus have the following simplified version of Construction 3.3 up to conjugacy in \( \text{PGL}(4, q) \).
Construction 6.2 (Construction 5.3'). Suppose that $q = p^d$ with $p$ an odd prime and $l$ a positive integer, and let $g \in \text{Aut}(\mathbb{F}_q)$ be such that $g(x) = x^q$. Let $S_1(X)$ be a reduced linearized polynomial whose coefficients lie in $\mathbb{F}_{q^l}$. Take $\mu_C \in \mathbb{F}_{q^l}^*$. Set $\theta_{a,b,c} := g^{\text{Tr}_{q^l/q}(\mu_C)}$, and $T(a,b,c) := S_1(c)$ for $a, b, c \in \mathbb{F}_q$. Then the set $G$ as defined in Theorem 2.4 with the prescribed functions $T$ and $\theta$ is a point regular group of $\mathbb{Q}^P$.

In Construction 5.7, we have $\text{Tr}_{q^l/q}(\nu_B) = 0$, so there exists $u \in \mathbb{F}_q$ such that $\nu_B = u\theta - u$ by Theorem 2.1. We deduce that $N_C(\theta_{a,b,c}) = u\theta_{a,b,c} - u$ from $\nu_B = u\theta - u$, where $N_C$ is as defined in Eqn. (5.1). Taking conjugation by $g_1 = (E(0, 0, 0, u), 1)$, we calculate that $g_1'_{a,b,c} := g_1^{-1} \circ g_{a,b,c} \circ g_1$ equals $(E(a, b, c, S_2(c)), g_1^{Q(c) + \text{Tr}_{q^l/q}(\alpha c + \mu_Bb)})$, where $\alpha' = \alpha - u\mu_B$. Since $\nu_B - \alpha' = \mu_B\nu_B, \mu_B = \mu_B$ and $\nu_B = u\theta - u$, we deduce that $\alpha' = 0, \alpha' \in \mathbb{F}_p^*$. By changing the notation $S_2$ to $S_1$ (to be consistent with previous constructions), we have the following simplified version of Construction 5.7 upon conjugation within $\text{PGSp}(4, q)$.

Construction 6.3 (Construction 5.7). Suppose that $q = p^d$ with $p$ an odd prime and $l$ a positive integer, and let $g \in \text{Aut}(\mathbb{F}_q)$ be such that $g(x) = x^q$. Take $\mu_B \in \mathbb{F}_{q^l}^*$, $\alpha \in \mathbb{F}_{q^l}$, and a tuple $(s_0, s_1, \ldots, s_{pl-1})$ with entries in $\mathbb{F}_{q^l}$ such that $\mu_B s_i - s_i^{p^l} \mu_B = 0$ for $1 \leq i \leq pl - 1$. Set $S_1(x) := \sum_{i=0}^{pl-1} s_i x^{p^i}$, and $Q(x) := -\text{Tr}_{q^l/q}(\mu_B x S_1(x))$ for $x \in \mathbb{F}_q$. Set $\theta_{a,b,c} := g_1^{Q(c) + \text{Tr}_{q^l/q}(\mu_Bc + \alpha)}$, and $T(a,b,c) := S_1(c)$ for $a, b, c \in \mathbb{F}_q$. Then the set $G$ as defined in Theorem 2.4 with the prescribed functions $T$ and $\theta$ is a point regular subgroup of $\mathbb{Q}^P$.

In Construction 5.11, let $\nu := \nu_C - S_2(t_C)$. In the construction we need $\text{Tr}_{q^l/q}(\nu B) = 0$, so there exists $u_0 \in \mathbb{F}_q$ such that $\nu = u_0^q - u_0$ by Theorem 2.1. Set $\lambda := \lambda_C \text{Tr}_{q^l/q}(\mu_B u_0 t_C)$ and $u_1 := u_0 - \lambda \mu_B^1 \mu_C$, so that $\text{Tr}_{q^l/q}(\mu_B t_C u_1) = 0$. Since $\mu_B$ and $\mu_C$ are in $\mathbb{F}_{q^l}^*$, we deduce that $u_0^q - u_1 = \nu$. It follows that $\nu_B = \sum_{i=0}^{2^i \frac{q}{2}} g_1^{i}(\nu) = g_1(u_1) - u_1$, where $g_1 = g^3$. The following are some facts that we need below:

1. Set $\alpha' := \alpha - u_1 \mu_B$. We deduce that $g(\alpha') - \alpha' = \lambda_C u + \mu_C$ from the expression of $\alpha' = \alpha - u_1$. Since $u - u_0^q \in \mathbb{F}_{q^l}$ and $(g - 1)^3 = g_1 - 1$, we further deduce that $g_1(\alpha') - \alpha' = 0, \alpha' \in \mathbb{F}_{q^3}$.]

2. Since $\text{Tr}_{q^l/q}(\mu_B t_C u_1) = 0$, by Eqn. (5.5) we have

$$\theta_{0,-t_C u_1, t_C} = \theta_{0,-t_C u_1, 0} \cdot \sigma_{t_C} = g_1^{\text{Tr}_{q^l/q}(-\mu_B t_C u_1)}.$$ 

3. By Eqn. (5.6), $T(0, -t_C u_1, t_C) = M(b')^\nu + \nu_C$ with $b' = -t_C u_1$. We have $\theta_{0,b',0} = 1$, so $M(b') = N_C(1) = 0$ by (F5). It follows that $T(0, -t_C u_1, t_C) = \nu_C$.

Let $G := g_1^{-1} \circ G \circ g_1$ with $g_1 = (E(0, 0, 0, u_1), 1)$, and let $g'_{a,b,c}$ be the element of $G$ that maps $(0, 0, 0, 1)$ to $(a, b, c, 1))$. By the analysis in Section 2, the group $G$ arises from Theorem 2.9 for some functions $T'$ and $\theta'$. By exactly the same argument as in the preceding case, we deduce that the subgroup $G_{K'} := g_1^{-1} \circ G \circ g_1$ consists of $g'_{a,b,c} = (\mathcal{M}'_{a,b,c}, \theta'_{a,b,c})$ with $a, b \in \mathbb{F}_q, c \in K$, where $\theta'_{a,b,c} = g_1^{\frac{Q(c)}{2} + \text{Tr}_{q^l/q}(\alpha' c + \mu_B b)}$ and $\mathcal{M}'(a, b, c) = E(a, b, c, S_2(c))$. Similarly, $g'_{0,0,t_C} = g_1^{-1} \circ g_{0,-t_C u_1, t_C} \circ g_1$, and its Frobenius part is $g$ by the
fact (2) above. We compare their matrix parts and get \( T'(0,0,t_C) = \nu_C - g(u_1) + u_1 = S_2(t_C) \), where we used the fact (3) above. By changing the notation \( S_2 \) to \( S_1 \), we have the following simplified version of Construction 5.11 up to conjugacy in \( \text{PGSp}(4,q) \).

**Construction 6.4** (Construction 5.11). Suppose that \( q = 3^d \) with \( l \) a positive integer, and take \( g \in \text{Aut}(\mathbb{F}_q) \) such that \( g(x) = x^3 \). Set \( g_1 := g^3 \).

(i) Take \( u \in \mathbb{F}_{3^d} \) such that \( \mu_C := u - u^9 \in \mathbb{F}_3^* \).

(ii) Take \( t_C \in \mathbb{F}_q^* \) such that \( \lambda_C := \text{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(\mu_C t_C) \neq 0 \).

(iii) Take \( \mu_B \in \mathbb{F}_3^* \) and \( t_B \in \mathbb{F}_q^* \) such that \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(\mu_B t_B) = 1 \).

(iv) Take a tuple \( (s_0, s_1, \ldots, s_{9l-1}) \) with entries in \( \mathbb{F}_{3^d} \) that satisfies

\[-\mu_B s_i + s^3_{9l-i} \mu_B = \mu_C u^i - u^3 \lambda_C, \quad 1 \leq i \leq 9l - 1; \]

(v) Take \( \alpha \in \mathbb{F}_{3^d}, \lambda \in \mathbb{F}_p \) such that \( g(\alpha) - \alpha = \lambda_C u + \lambda \mu_C \).

Set \( S_1(x) := \sum_{i=0}^{9l-1} s_i x^{p^i}, Q(x) := -\text{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(\mu_B x S_1(x)) \) for \( x \in \mathbb{F}_q \). Set \( K := \{ z \in \mathbb{F}_q : \text{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(\mu_C z) = 0 \} \). For \( a, b \in \mathbb{F}_q \) and \( c \in K \), define

\[ \mathcal{M}_{a,b,c} = E(a, b, c, S_1(c)), \quad \theta_{a,b,c} := \frac{1}{3^\frac{1}{2}} \left( Q(c) + \text{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(ac + \mu_B b) \right), \]

and \( \mathfrak{g}_{a,b,c} = (\mathcal{M}_{a,b,c}, \theta_{a,b,c}) \). Then \( G_K := \{ \mathfrak{g}_{a,b,c} : a, b \in \mathbb{F}_q, c \in K \} \) is a group of \( q^3/3 \).

Set \( \mathfrak{g}_{0,0,t_C} := (\mathcal{M}_{0,0,t_C}, g) \) with \( \mathcal{M}_{0,0,t_C} := E(0,0,t_C,S_1(t_C)) \). Then \( G := \langle G_K, \mathfrak{g}_{0,0,t_C} \rangle \) is a point regular group of \( Q^3 \).

**Remark 6.5.** In Construction 6.4, we show that \( T(x,y,z) = S_1(z) \) for \( x, y, z \in \mathbb{F}_q \) as in the other constructions. For each triple \( (x, y, z) \), there exist \( a, b, c \in \mathbb{F}_q \), \( c \in K \) and \( i \geq 0 \) such that \( \mathfrak{g}_{x,y,z} = \mathfrak{g}_{a,b,c} \circ \mathfrak{g}_{0,0,t_C}^i \), since \( G = \langle G_K, \mathfrak{g}_{0,0,t_C} \rangle \). The equation yields \( z = g^i(c) + \sum_{k=0}^{i-1} g^k(t_C) \) and \( T(x,y,z) = S_1(z) \) by direct check using Remark 2.12.

Let \( G \) be the group in either of the above constructions. For later use, we define \( U := \{ c \in \mathbb{F}_q : \theta_{a,b,c} = 1 \} \) and

\[ G_F := \{ \mathfrak{g}_{a,b,c} : \theta_{a,b,c} = 1 \}. \tag{6.1} \]

We now collect some basic properties for quick reference.

(P1) The set \( U = \mathbb{F}_q \) for Constructions 6.1 and 6.3 and \( |U| = q/p \) for Constructions 6.2 and 6.4. This follows by a case by case check.

(P2) For \( \mathfrak{g}_{a,b,c} \in G_F \), we have \( \mathfrak{g}_{a,b,c}^p = \mathfrak{g}_{x,0,0} \) with \( x = -\frac{(p^2-1)p}{6} c^2 S_1(c) \). We sketch the proof:

Set \( T_1 := T(a, b, c) \) and \( \mathfrak{g}_{a_0,b_0,c_0} := \mathfrak{g}_{a,b,c}^1 \). We check that \( c_{i+1} = c_i + c, b_{i+1} = b_i + cT_1 \) and \( a_{i+1} = a_i - b_i c + c_i b - c_i c T_1 \), from which we deduce that \( c_i = ic, b_i = ib + \frac{(i-1)i}{2} c T_1 \) and \( a_i = ia - \frac{i(i^2-1)}{6} c^2 T_1 \). The claim then follows.
6.2. The group invariants of the point regular groups

Our first goal in this subsection is to show that the four constructions in general yield non-isomorphic point regular groups. This is achieved by considering the exponents and the Thompson subgroups.

**Theorem 6.6.** Let $G$ be the group in either of Constructions 6.1–6.4

1. In Construction 6.1, $\exp(G) = p$ or $p^2$, and the latter occurs if and only if $p = 3$ and $S_1$ is not the zero map.
2. In Construction 6.2 or 6.3, $\exp(G) = p^2$ or $p^3$, and the latter occurs if and only if $p = 3$ and the restriction of $S_1$ to $\mathbb{F}_q$ is not the zero map.
3. In Construction 6.4, $\exp(G) = 3^3$ or $3^4$, and the latter occurs if and only if the restriction of $S_1$ to $\mathbb{F}_q$ is not the zero map.

**Proof.** Let $G_F$ be as defined in Eqn. (6.1). For $g_{a,b,c} \in G_F$, we have $g_{a,b,c}^p = g_{u,0,0}$ with $u = -(p^2-1)p/6$, by the property (P2). Another application of the same property yields that $g_{a,b,c}^2 = 1$. Therefore, an nonidentity element $g_{a,b,c} \in G_F$ has order $p$ or $p^2$, and the latter occurs if and only if $p = 3$ and $S_1(c) \neq 0$.

The analysis of each construction is similar, so we only explain the case of Construction 6.4 in detail. Take any triple $(x, y, z)$, and write $g_{a,b,c} = g_{x,y,z}^3$, which clearly lies in $G_F$. If $\theta_{x,y,z} = g^i$, then we calculate that $w = \sum_{k=0}^8 g^{ik}(z)$ by the multiplication rule (2.8). The element $g_{a,b,c}$ has order at most $3^3$ by the preceding paragraph, so $\exp(G) \leq 3^3$. By taking $z = t_C + g(c)$ with $c \in K$, we have $\theta_{x,y,z} = g^{1+3j}$ for some $0 \leq j \leq 2$, and correspondingly $w = \text{Tr}_{q/\mathbb{F}_q}(t_C) + \text{Tr}_{q/\mathbb{F}_q}(c)$. Here we used the fact $\sum_{k=0}^8 g^k(z) = \text{Tr}_{q/\mathbb{F}_q}(z)$. As $t_C \notin K$, we have $\mathbb{F}_q = \langle K, t_C \rangle_{\mathbb{F}_q}$, so there exists $c \in K$ such that $w \neq 0$. It follows that $\exp(G) \geq 3^3$. If the restriction of $S_1$ to $\mathbb{F}_q$ is constantly zero, then $g_{x,y,z}^3 = 1$ by the preceding paragraph, and so $\exp(G) = 3^3$; otherwise, there exists $c \in K$ such that the corresponding $S_1(w) \neq 0$, and $g_{0,0,tc+g(c)}$ has order $3^4$, so $\exp(G) = 3^4$. This completes the proof. \hfill \Box

Theorem 6.6 does not help to distinguish Construction 6.2 and Construction 6.3. We compute their Thompson subgroups which will do the work in general. Take $g_{a,b,c} \in G$, and write $\theta_1 = \theta_{a,b,c}$. For $g_{x,y,z} \in C_G(g_{a,b,c})$, by Theorem 2.9 we deduce from $g_{a,b,c} \circ g_{x,y,z} = g_{x,y,z} \circ g_{a,b,c}$ that

$$a^{\theta_2} + x - b^{\theta_2}z + c^{\theta_2}y - d^{\theta_2}z S_1(z) = x^{\theta_1} + a - cy^{\theta_1} + bz^{\theta_1} - cz^{\theta_1} S_1(c);$$

$$b^{\theta_2} + y + c^{\theta_2} S_1(z) = b + y^{\theta_1} + z^{\theta_1} S_1(c),$$

$$c^{\theta_2} + z = c + z^{\theta_1},$$

$$S_1(c)^{\theta_2} + S_1(z) = S_1(z)^{\theta_1} + S_1(c).$$

(6.2)

where $\theta_2 = \theta_{x,y,z}$. Here we used the fact $T(x, y, z) = S_1(z)$, cf. Remark 6.5.

**Theorem 6.7.** Let $G$ be the group in either of Construction 6.2 or Construction 6.3, and assume that $1 < \deg(S_1) < q/p$. Then the Thompson subgroup of $G$ is $J(G) = \{g_{a,b,0} : a, b \in \mathbb{F}_q, \theta_{a,b,0} = 1\}$, which has size $q^2$ in the former case and $q^2/p$ in the latter case.
Proof. We only deal with the case of Construction 6.3 here, and the other case is similar. Let $d$ be the largest order of an abelian subgroup of $G$. It is routine to check that $G_{ab} := \{ (a,b,0) : a, b \in \mathbb{F}_q, \theta_{a,b,0} = 1 \}$ is an abelian subgroup of order $q^2/p$, so $d \geq q^2/p$.

We first show that $J(G) \leq G_F$, where $G_F$ is the set of elements of $G$ with a trivial Frobenius part. This is achieved by showing that any abelian subgroup of order $d$ is contained in $G_F$. Suppose to the contrary that $H$ is an abelian subgroup of order $d$ which contains an element $g \in G$ with $\theta_{a,b,c} = g^i \neq 1$. The subgroup $g$ contains $H$, so should have size at least $d \geq q^2/p$. We now estimate the size of $G_{ab}$ in an alternative way. Fix an integer $j$, $0 \leq j \leq p-1$. Suppose that $g \in G_{ab}$. Then its centralizer in $G_F$ has at most $p^j$ solutions, since $o(g^i) = p$ and $q = p^d$. For a given $g$, the second equation in (6.2) has at most $p^j$ solutions in the variable $y$ for the same reason. Similarly, for a given pair $(y, z)$, the first equation in (6.2) has at most $p^j$ solutions in $x$. In total, we see that $|G_{ab}| \leq p^{2j+1}$. This number is less than $q^2/p = p^{2j+1}$ by the fact $(2p-3)l \geq 3$. This proves the claim.

We next show that $J(G) \leq G_{ab}$. As in the previous paragraph, it suffices to show that for any $g \in G_F$ with $c \neq 0$ its centralizer in $G_F$ (not $G$) has size smaller that $q^2/p$. For $g \in G_{ab}$, it has a trivial Frobenius part, and the equations in (6.2) reduce to $cS_1(z) - zS_1(c) = 0$ and $2y = 2bc^{-1}z + zS_1(z) - zS_1(c)$. Recall that $S_1$ is $F_p$-linear. By the restriction on $\deg(S_1)$, we see that there are at most $q/p^2$ such $(y, z)$ pairs. Therefore, $G_{ab}$ has size at most $q^2/p^2$ as desired. This proves the claim.

Since $G_{ab}$ is abelian, we conclude that $J(G) = G_{ab}$ and it is the unique maximal abelian subgroup of order $q^2/p$. This completes the proof.

Our next goal is to bound the nilpotency class of $G$. We start with the center of $G$. Recall that $G_A = \{ (a,0,0) : a \in \mathbb{F}_p^* \}$, cf. Corollary 2.11.

Lemma 6.8. Let $G$ be the group in either of Constructions 6.1-6.4. Then its center is $Z(G) = \{ (a,0,0) : a \in \mathbb{F}_p^* \}$.

Proof. Take $(a,b,c) \in Z(G)$. First, we claim that $\theta_1 = \theta_{a,b,c} = 1$. Suppose that $\theta_1 \neq 1$. For a triple $(a,b,c)$ such that $\theta_2 = \theta_{a,b,c} = 1$, the third equation in Eqns. (6.2) reduces to $z^q = z^{\theta_1}$, i.e., $z$ lies in a proper subfield of $\mathbb{F}_q$. In particular, there are at most $q^2$ z’s such that there is a triple $(x, y, z)$ with $\theta_{x,y,z} = 1$. On the other hand, this number is at least $q/p$ by the property (P1): a contradiction. This proves the claim.

Next, we show that $c = 0$. Take $(x,y,z) = (0, y_0, 0)$ for a nonzero element $y_0$ such that $\theta_{0,y_0,0} = 1$. The first equation in (6.2) reduces to $2cy_0 = 0$, which gives $c = 0$.

By the facts $\theta_1 = 1$ and $c = 0$, the equations in (6.2) reduce to $a^{\theta_2} - b^{\theta_2}z = a+bz$ and $b^{\theta_2} = b$, where $\theta_2 = \theta_{x,y,z}$. For any triple $(x,y,z)$ with $\theta_{x,y,z} = 1$, we deduce from the first equation that $2bz = 0$, so $b = 0$. The equations in (6.2) further reduce to $a^{\theta_{x,y,z}} = a$ for all $x, y, z \in \mathbb{F}_q$. That is, $a^2 = a$. This completes the proof.

Set $R_0 := \mathbb{F}_q$, and for $1 \leq i \leq p^e$ set $R_i := (1-g)^i(\mathbb{F}_q)$, i.e., $R_i = \{ (1-g)^i(x) : x \in \mathbb{F}_q \}$. By Theorem 2.1 and the fact that $T_{\mathbb{F}_q/\mathbb{F}_{p^e}}(x) = (1-g)^{p^e-1}(x)$, we have $R_{p^e-1} = \mathbb{F}_{p^e}$. There is a short exact sequence

$$0 \longrightarrow \mathbb{F}_{p^e} \longrightarrow R_i \overset{1-g}{\longrightarrow} R_{i+1} \longrightarrow 0,$$
for each $0 \leq i \leq p^e - 1$. It follows that $\dim_{\mathbb{F}_q} R_i = (p^e - i)l$; in particular, $R_{p^e} = 0$. In the sequel, we write $x \equiv y \pmod{R_i}$ if $x - y \in R_i$.

**Lemma 6.9.** Let $G$ be the group in either of Constructions $[6.1][6.4]$, and let $p^e$ be the order of $g$. For the last three constructions, further assume that $l > 1$. For $1 \leq i \leq p^e$, we have $Z_i(G) := \{g_{a,0,0} : a \in (1 - g)^{p^e - i}(\mathbb{F}_q)\}$. In particular, $Z_{p^e}(G) = G_A$.

**Proof.** We prove by induction. In the case $i = 1$, it follows from Lemma 6.8 the fact that $R_{p^e - 1} = \mathbb{F}_q$. Suppose that the claim is true for $1 \leq i \leq p^e - 1$, and we aim to establish the claim for $i + 1$. By a similar argument to that preceding Theorem 6.7 we see that $g_{a,b,c} \in Z_{i+1}(G)$ if and only if

$$a^{\theta_2} + x - b^{\theta_2}z + c^{\theta_2}y - c^{\theta_2}zS_1(z) \equiv x^{\theta_1} + a - cy^{\theta_1} + bz^{\theta_1} - cz^{\theta_1}S_1(c) \pmod{R_{p^e - i}}.$$  

and the last three equations of (6.2) hold for all $x, y, z$, where $\theta_1 = \theta_{a,b,c}, \theta_2 = \theta_{x,y,z}$. By the same argument as in the proof of Lemma 6.8 we deduce that $\theta_1 = 1$.

In the first two constructions, $\theta_{0,y_0,0} = 1$ for all $y_0 \in \mathbb{F}_q$; in the last two constructions, there are $q/p$ $y_0$'s in $\mathbb{F}_q$ such that $\theta_{0,y_0,0} = 1$. Take $(x, y, z) = (0, y_0, 0)$ for such an element $y_0$. Eqn. (6.3) reduces to $2cy_0 \in R_{p^e - i}$. If $c \neq 0$, this leads to a contradiction by comparing sizes. Here we used the assumption $l > 1$ for the last two constructions.

By the facts $\theta_1 = 1$ and $c = 0$, the conditions now reduce to $b^{\theta_2} = b$ and $2bz \equiv a^{\theta_2} - a \pmod{R_{p^e - i}}$. Recall that $U = \{c \in \mathbb{F}_q : g_{a,b,c} \in G_F\}$. We consider two separate cases according as $i = p^e - 1$ or not.

1. First consider the case $i < p^e - 1$. For each $z \in U$, take a triple $(x, y, z)$ with $\theta_{x,y,z} = 1$, and Eqn. (6.3) reduces to $2bz \in R_{p^e - i}$. By the property (P1), we deduce a contradiction by comparing sizes if $b \neq 0$. Hence $b = 0$. The conditions further reduce to $a^{\theta_2} - a \in R_{p^e - i}$ for all $x, y, z$, or equivalently, $a^{\theta_2} - a \in R_{p^e - i}$. This holds if and only if $a \in R_{p^e - i - 1}$ as desired.

2. Next consider the case $i = p^e - 1$. In this case, $a^{\theta_2} - a = (g^k - 1)(a)$ is always contained in $R_1 = R_{p^e - 1} \leq R_{p^e - 1}$ for $k \geq 0$, so the conditions reduce to $b^{\theta_2} = b$ and $2bz \in R_1$ for all $x, y, z \in \mathbb{F}_q$. We thus conclude that $b = 0$ by comparing sizes. There is no restriction on $a$.

This completes the proof.

Take the same notation and assumption as in Lemma 6.9. By [2, (9.7)] and Lemma 6.9, the nilpotency class of $G$ equals $p^e$ plus the nilpotency class of $G/G_A$, where $p^e = o(g)$. We now consider the nilpotency class of $\overline{G} := G/G_A = \{\overline{g}_{b,c} : b, c \in \mathbb{F}_q\}$, where $\overline{g}_{b,c}$ stands for the quotient image $\overline{g}_{b,c}$. We make the observation that $\overline{g}_{b,c}$ lies in $Z(\overline{G})$ if and only if the last three equations in (6.2) hold for all $y, z \in \mathbb{F}_q$.

**Theorem 6.10.** If $G$ is the group in Construction $[6.1]$, then its nilpotency class is 2 or 3, and the latter occurs if and only if $\deg(S_1) > 1$.

**Proof.** In this case, $p^e = 1$. Suppose that $\overline{g}_{b,c} \in Z(\overline{G})$. Since $\theta_{x,y,z} = 1$, the last three equations in (6.2) reduce to $cS_1(z) = zS_1(c)$ for all $z \in \mathbb{F}_q$. If $c \neq 0$, then $S_1(z) = \overline{g}_{b,c}$ lies in $Z(\overline{G})$ if and only if $\overline{g}_{b,c}$ lies in $Z(\overline{G})$.
$c^{-1}S_1(c)z$, so $\deg(S_1) \leq 1$. Therefore, $Z(\bar{G}) = \{\bar{g}_{b,0} : b \in \mathbb{F}_q\}$ if $\deg(S_1) > 1$; moreover, $\bar{G}/Z(\bar{G})$ is abelian in this case, i.e., $Z_2(\bar{G}) = \bar{G}$. If $\deg(S_1) \leq 1$, then $Z(\bar{G}) = \bar{G}$, since $cS_1(z) = zS_1(c)$ holds for all $c, z \in \mathbb{F}_q$ in this case. This completes the proof.

For the last three constructions, the nilpotency class of the group $G$ varies in a large range as we will see soon. It is infeasible to give an explicit description of the nilpotency class in general, so instead we give a reasonably tight bound and give some examples under the assumption $l > 1$. In the sequel, we change our strategy and consider the lower central series of $\bar{G} = G/G_A$. We introduce a chain of subgroups of $\bar{G}$ as follows:

$\bar{G}_i := \{\bar{g}_{b,c} : b \in \mathbb{F}_q, c \in R_i\}, \quad 1 \leq i \leq p^e,$

(6.4)

where $R_i = (1 - g)^i(\mathbb{F}_q)$. In particular, we have $\bar{G}_{p^e} = \bar{G}_B$, where $\bar{G}_B := \{\bar{g}_{b,0} : b \in \mathbb{F}_q\}$. They will help to distinguish the $\gamma_i(\bar{G})$'s.

Take $\bar{g}_{b,c} \in \bar{G}$. For $y, z \in \mathbb{F}_q$, we set $\bar{g}_{v,w} := \bar{g}_{b,c}^{-1} \circ \bar{g}_{y,z}^{-1} \circ \bar{g}_{b,c} \circ \bar{g}_{y,z}$. Here we are only concerned with the coordinate $w$ which we compute as

$w = (\theta_{0,y,z} - 1)(c) - (\theta_{0,b,c} - 1)(z).$

(6.5)

**Theorem 6.11.** Let $G$ be the group in either of Constructions [6.2][6.4], let $p^e$ be the order of $g$, and assume $l > 1$. Then the nilpotency class of $G$ lies in the range $[2p^e, 3p^e]$.

**Proof.** By Lemma [6.9], it suffices to show that the nilpotency class of $\bar{G} = G/G_A$ lies in the range $[p^e, 2p^e]$. Recall that $U = \{c \in \mathbb{F}_q : \theta_{a,b,c} = 1\}$, $\bar{g}_{b,c} = \bar{g}_{a,b,c}$, and let $\bar{G}_i$ be as defined in Eqn. (6.4). We have $e \geq 1$ in each of the three constructions.

We claim that $\gamma_i(\bar{G})$ is contained in $\bar{G}_{i-1}$ but not in $\bar{G}_i$ for $2 \leq i \leq p^e$, and $\gamma_{p^e+1}(\bar{G})$ is contained in $\bar{G}_{p^e} = \bar{G}_B$. We prove this by induction.

1. First consider the case $i = 2$. By the property (P1) and the assumption $l > 1$, there exists $c \in U \setminus R_1$ by comparing sizes. Since $\theta_{a,b,c}$ is independent of $a$ by Eqn. (5.5), there exists $b \in \mathbb{F}_q$ such that $\theta_{0,b,c} = 1$ for such a $c$. For such a pair $(b, c)$ and $(y, z) = (0, t_c)$, we have $w = c^a - c$, where $w$ is as defined in Eqn. (6.5). The element $w$ lies in $R_1 \setminus R_2$, i.e., $[\bar{g}_{b,c}, \bar{g}_{0,t_c}] \in \bar{G}_1 \setminus \bar{G}_2$. On the other hand, for any $(b, c)$ and $(y, z)$, the corresponding $w$ always lies in $R_1$. This proves the case $i = 2$.

2. Suppose that the claim has been established for $2 \leq i \leq p^e - 1$. Take $\bar{g}_{b,c} \in \gamma_i(\bar{G})$. We have $\theta_{0,b,c} = 1$, since $\gamma_i(\bar{G})$ is contained in $\bar{G}_i$. Eqn. (6.5) reduces to $w = (\theta_{0,y,z} - 1)(c)$, which always lies in $R_i$ by induction. There exists $\bar{g}_{b,c} \in \gamma_i(\bar{G})$ with $c \in R_{i-1} \setminus R_i$ by induction. Take $(y, z) = (0, t_c)$, and correspondingly $w = (g-1)(c)$. This element $w$ lies in $R_i \setminus R_{i+1}$. This established the case $i + 1$.

The claim on $\gamma_{p^e+1}(\bar{G})$ is proved in the same way as in (2). This completes the proof of the claim. In particular, $\gamma_{p^e+1}(\bar{G})$ is contained in the group $\bar{G}_B$, while $\gamma_{p^e}(\bar{G})$ is not. This gives the lower bound on the nilpotency class of $\bar{G}$.

Since $\gamma_{p^e+1}(\bar{G})$ has a trivial Frobenius part, it is contained in the subgroup $H := \{\bar{g}_{b,0} : \theta_{0,b,0} = 1\}$ of $\bar{G}_B$. Let $K_B$ be the set of $b \in \mathbb{F}_q$ such that $\bar{g}_{b,0} \in H$. In Construction 6.2, we have $K_B = \mathbb{F}_q$. In Construction 6.3 and 6.4, we have $K_B = \{b : Tr_{\mathbb{F}_q/\mathbb{F}_p}(\mu_B b) = 0\}$. We define $\varepsilon_0(H) := H$, and inductively $\varepsilon_{i+1}(H) := [\varepsilon_i(H), G]$ for $i \geq 1$. Then by induction we have $\gamma_{p^e+1}(\bar{G}) \leq \varepsilon_{i-1}(H)$ for $i \geq 1$. It is routine to check that $\bar{g}_{b,0}^{-1} \circ \bar{g}_{y,z}^{-1} \circ \bar{g}_{b,0} \circ \bar{g}_{y,z} = \bar{g}_{b,0} \circ \bar{g}_{y,z}^{-1} = \bar{g}_{b,0} \circ \bar{g}_{y,z}$.
It readily follows by induction that \( e_i(H) = \{ g_{b,0} : b \in (1-g)^i(K_B) \} \) for \( 1 \leq i \leq p^k \). In particular, \( e_{p^k}(H) = \{1\} \), and so \( \gamma_{2p^k+1}(G) = 1 \). This gives the upper bound on the nilpotency class of \( \bar{G} \).

In Table 1, we give some explicit values of nilpotency classes for some special cases of Construction 6.2 in some special cases that are listed in Table 1. The terms \( \alpha \equiv 0 \), \( \mu_C = 1 \), and \( \mu_B = 1 \). In both cases, we assume that \( l > 1 \). From the table we see that in general the nilpotency class of \( \bar{G} \) is larger if \( \ker(S_1) \) has a smaller size. The details for the calculations of the data in Table 1 are omitted.

Table 1: The nilpotency class of the point regular group \( G \). In Construction 6.2 (resp. 6.3), take \( \mu_C = 1 \) (resp. \( \alpha = 0 \), \( \mu_B = 1 \)) and assume that \( l > 1 \).

| Construction | \( S_1(z) \) | nilpotency class | Condition |
|--------------|-------------|-----------------|-----------|
| 6.2          | 0           | 2 \( p \)       |           |
|              | \( z^{p^k} \) | 3 \( p \)       | \( l \nmid k \) |
|              | \( (1-g)^k(z) \) | 3 \( p - k \)   | 1 \( \leq k \leq p - 1 \) |
| 6.3          | 0           | 2 \( p \)       |           |
|              | \( z \)      | 3 \( p - 1 \)   |           |
|              | \( (1-g)^{2k}(z^{b^{p^k}}) \) | 3 \( p - 2k \) | 2 \( \leq 2k \leq p - 1 \) |

Example 6.12. In this example, we give the upper central series of the group \( G \) arising from Construction 6.3 in some special cases that are listed in Table 1. The terms \( Z_i(G) \), \( 1 \leq i \leq p \), have been determined in Lemma 6.4, so we do not list them below. Recall that \( q = p^{d_0} \), \( g(x) = x^{p^l} \), and we set \( \mu_C = 1 \), \( K = \{ x \in \mathbb{F}_q : \Tr_{\mathbb{F}_q/\mathbb{F}_p}(x) = 0 \} \). Further assume that \( l > 1 \). As usual, set \( R_0 := \mathbb{F}_q \) and \( R_i := (1-g)^i(\mathbb{F}_q) \) for \( 1 \leq i \leq p \).

1. If \( S_1(z) \equiv 0 \), then
   \[
   Z_{p+i}(G) = \{ g_{a,b,c} : a \in \mathbb{F}_q, b, c \in (1-g)^{p^i}(\mathbb{F}_q) \}, \quad 0 \leq i \leq p.
   \]

2. If \( S_1(z) = z^{p^k} \) with \( l \nmid k \), then
   \[
   Z_{p+i}(G) = \{ g_{a,b,0} : a \in \mathbb{F}_q, b \in R_{p-i} \}, \quad 0 \leq i \leq p,
   \]
   \[
   Z_{2p+i}(G) = \{ g_{a,b,c} : a, b \in \mathbb{F}_q, c \in R_{p-i} \}, \quad 0 \leq i \leq p.
   \]

3. If \( S_1(z) = (1-g)^k(z) \) for \( 1 \leq k \leq p-1 \), then the expression of \( Z_{p+i}(G) \), \( 1 \leq i \leq p-k \), is the same as in case (2), and
   \[
   Z_{2p-k+j}(G) = \{ g_{a,b,c} : a \in \mathbb{F}_q, b \in R_{k-j}, c \in R_{p-j} \}, \quad 1 \leq j \leq k,
   \]
   \[
   Z_{2p+j}(G) = \{ g_{a,b,c} : a, b \in \mathbb{F}_q, c \in R_{p-k-j} \}, \quad 1 \leq j \leq p - k.
   \]

The upper central series of the group \( G \) in the other cases in Table 1 can be derived in a similar fashion, and we omit the details here.
7. Concluding remarks

In this paper, we have determined all the point regular groups of the Payne derived quadrangle $Q^P$ of the classical symplectic quadrangle $Q = W(q)$ in the case $q$ is odd. We have considered the isomorphism issues between the different constructions by calculating their group invariants such as exponents and Thompson subgroups. We also have obtained tight upper and lower bounds on the nilpotency classes of the resulting groups. As a corollary, we see that the finite groups that act regularly on the points of a finite generalised quadrangle can have unbounded nilpotency class. Prior to our work, the only known such groups have nilpotency class at most 3 except for computer data for some groups of small order.

In Section 3, we have also determined all the point regular subgroups of $Q^P$ in $\text{PGL}(4, q)$ in the case $q$ is even. In an ongoing project, we are able to completely classify the cases where $r_{A,B} \leq 1$ and $r_C \leq 2$ by the same method as we have used in the odd characteristic case, where $r_{A,B}$ and $r_C$ are defined as in the beginning of Section 4. There are parallel results to Construction 5.3, Construction 5.7 and Construction 5.11 and more constructions show up with Construction 3.6 as a prelude. It seems a challenging problem to fully classify the even characteristic case due to the wild nature of the problem.

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