ENUMERATION OF GRAPHS WITH GIVEN WEIGHTED
NUMBER OF CONNECTED COMPONENTS

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Abstract. We give a generating function for the number of graphs with given num-
erical properties and prescribed weighted number of connected components. As
an application, we give a generating function for the number of bipartite graphs of
given order, size and number of connected components.

1. Introduction

In this paper, we consider the generating function for the number of graphs with
prescribed numerical properties and the weighted number of connected components:
Given a weight vector $\omega = (\omega_1, \omega_2, \ldots) \in \mathbb{Q}^\infty$, the $\omega$-weighted number of connected
components of a graph $G$ with connected components $G_1, \ldots, G_s$ is defined to be

$$h^\omega_0(G) := \sum_{i=1}^s \omega_i |G_i| |G_i|.$$ 

The notion of weighted number of components frequently arises in everyday life, such
as group or bundle discount. More complex and sophisticated applications may be
found in network analysis.

In this paper, we give a generating function for the number of graphs of given order,
size, and $h^\omega_0$. (In fact, order, size can be replaced by homogeneous properties). When
$\omega$ is the uniform trivial weight $(1, 1, 1, \ldots)$, $h^\omega_0$ is simply the number of connected
components. Our method is a slight modification of the exponential formula [Sta99]. We
first define an auxiliary multi-variabed exponential generating function Equation (1)
which enumerates the number of graphs with prescribed number of connected compo-
nents of given order, and use a ring homomorphism $\tau_\omega$ to induce the desired generating
function. Let $f_i$ be additive functions on graphs (Definition 1) and $\mathcal{P}$ be a collection
of homogeneous properties (Definition 2).

Theorem. The number of graphs with properties $\mathcal{P}$, given $f_i$ values and $h^\omega_0$ is generated
by

$$\tau_\omega \left( \exp \sum_{n, k_i} g_{n, k_1, \ldots, k_s} \frac{1}{n!} y^n \prod_{i=1}^s y_i^{k_i} z_n \right)$$

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where \( g_{n,k_1,\ldots,k_s} \) is the number of connected graphs with properties \( P \) and given numerical values of \( f_i, i = 1,\ldots,s \), and \( \tau_\omega \) is the ring homomorphism determined by mapping \( \prod z_i^{\alpha_i} \) to \( z^{\sum \omega_i \alpha_i} \) (Definition 3).

As an application, we shall enumerate the number \( \beta_{n,k,\nu} \) of bipartite graphs of order \( n \), size \( k \) with \( \nu \) connected components. We specifically chose this trivial weight \( \omega = (1,1,1,\ldots) \) case because of the importance of bipartite graphs and because it will be used in our forthcoming paper [Son15b]. Enumeration of bipartite graphs have been studied by many authors ([Har58] [HP63] [HP73] [Han79] [Sta99] [UT14] to name just several), but our search did not turn up a table of \( \beta_{n,k,\nu} \), and we put it in the appendix.

All graphs are assumed to be labeled.

2. Exponential Formula

In this section, we quickly review the exponential formula [Sta99] and give a few variant forms of it that will be useful for our purpose here and in [Son15b].

Definition 1. We say that a graph property \( P \) is homogeneous if \( G \) satisfies \( P \) then all of its components satisfy \( P \).

For example, order, size and rank (of the incidence matrix) are all additive.

Definition 2. We say that a graph property \( P \) is additive if \( f(G + G') = f(G) + f(G') \) for any two graphs \( G, G' \) with disjoint vertex sets.

Let \( f_1,\ldots,f_s \) be additive functions on graphs. Say one is interested in enumerating graphs with given order and given values of \( f_i, i = 1,\ldots,s \). Then the exponential formula (see for instance, [Sta99]) allows one to relate the number of graphs with the desired numerical properties and the number of connected graphs with the same properties. Let \([n] = \{1,2,\ldots,n\}\).

Theorem 1. ([Rea58] Theorem 1), [Sta99]) Let \( P \) be a homogeneous graph property. Let \( g_{n,1,\ldots,k_s} \) (resp. \( \bar{g}_{n,1,\ldots,k_s} \)) be the number of (resp. connected) graphs \( G \) on \([n]\) with property \( P \) and \( f_i(G) = k_i, \forall i \). Then the generating function for \( g_{n,k_1,\ldots,k_s} \) is given by

\[
g(x,y_1,\ldots,y_s) = \exp \left( \sum \bar{g}_{n,k_1,\ldots,k_s} \frac{1}{n!} x^n y_1^{k_1} \cdots y_s^{k_s} \right)
\]

or equivalently, the generating function for \( \bar{g}_{n,k_1,\ldots,k_s} \) is given by the formal logarithm

\[
\bar{g}(x,y) = \log \left( \sum \bar{g}_{n,k_1,\ldots,k_s} \frac{1}{n!} x^n y_1^{k_1} \cdots y_s^{k_s} \right).
\]
By using this theorem, one can also sort out unwanted components when enumerating graphs. Suppose that the generating function \( g(x, y_1, \ldots, y_s) \) as in Theorem 1 is known and we want to enumerate the graphs with the same numerical features but without certain components.

**Corollary 1.** Let \( \mathcal{I} \subset \mathbb{Z}^{s+1} \). The number of graphs of given order and given values of \( f_i \) without a connected component of order \( n^* \) and \( f_i \) value of \( k_i^* \) is generated by

\[
\exp \left( \log \left( g(x, y_1, \ldots, y_s) - \sum_{(n^*, k_1^*, \ldots, k_s^*) \in \mathcal{I}} \frac{1}{n^*!} x^{n^*} y_1^{k_1^*} \cdots y_s^{k_s^*} \right) \right)
\]

**Proof.** By construction, the \( x^n \prod_{i=1}^s y_i^{k_i} \)-coefficient comes from the partitions of \( n \) and \( k_i \) that do not involve \( n^* \) and \( k_i^* \). The assertion follows immediately. \( \square \)

In particular, the number of graphs with desired numerical properties but without an isolated vertex is generated by \( \exp (\log (g(x, y_1, \ldots, y_s) - x)) \).

### 3. Graphs with a given number of weighted connected components

Suppose we have the generating function \( g(x, y) = \sum_{n \geq 1, k_1, k_2, \ldots, k_s} \frac{1}{n!} x^n y_1^{k_1} \cdots y_s^{k_s} \) for the number of connected graphs with certain homogeneous properties \( P \) and given numerical values of \( f_i, i = 1, \ldots, s \).

To control the weighted number of connected components, we define an auxiliary generating function

\[
(1) \quad \exp \tilde{g}(x, y, z) = \exp \sum_{n, k_1} g_{n, k_1, \ldots, k_s} \frac{1}{n!} x^n y_1^{k_1} \cdots y_s^{k_s} z_n \in \mathbb{Q}[x, y_1, \ldots, y_s, z_1, z_2, z_3, \ldots,]
\]

which involves infinitely many variables \( z_i, i \in \mathbb{N} \). The upshot is that the auxiliary \( z_i \)'s allow one to keep track of the orders of connected components.

**Definition 3.** Let \( \omega = (\omega_1, \omega_2, \omega_3, \ldots) \in \mathbb{Q}^\infty \) and define \( \tau_\omega : \mathbb{Q}[z_1, z_2, z_3, \ldots] \to \overline{\mathbb{Q}(z)} \) to be the ring homomorphism determined by

\[
z^\omega := \prod z_i^{\omega_i} \mapsto z^{\sum \omega_i \alpha_i}.
\]

Here, \( \overline{\mathbb{Q}(z)} \) denotes the algebraic closure of the function field \( \mathbb{Q}(z) \), employed just to make sure that \( \tau_\omega \) is a well-defined ring homomorphism. In practice, computations involving the fractional powers of \( z \) are done purely symbolically.

**Theorem 2.** The number of graphs with properties \( P \), given \( f_i \) values and \( h_0^\omega \) is generated by \( \tau_\omega(\exp \tilde{g}) \). That is, if

\[
\tau_\omega(\exp \tilde{g}) = \sum T_{n, k, \nu} \frac{1}{n!} x^n y_1^{k_1} \cdots y_s^{k_s}, \quad y_k = \prod_{i=1}^s y_i^{k_i}
\]
then $T_{n,k,\nu}$ is precisely the number of graphs with properties $\mathcal{P}$, $f_i$-value $k_i$ and $h_0^i = \nu$.

Proof. We shall give a proof for the case $s = 1$ (single numerical feature). One will see immediately that the general case can be proved in the exact same fashion, but it is just more cumbersome to write. Fix $\nu$ and choose a set of partitions

\begin{equation}
\sum_{i=1}^{\ell} (\sum_j m_{ij}) n_i, \quad k = \sum_{i,j} m_{ij} k_{ijj}, \quad \text{such that } \nu = \sum_i (\omega_i \sum_j m_{ij}), \quad n_1 < n_2 < \cdots < n_\ell \text{ and } k_{11} < k_{22} < \cdots, \forall i.
\end{equation}

We first count the number of graphs with the properties $\mathcal{P}$ which has precisely $m_{ij}$ connected components of order $n_i$ and numerical value $k_{ij}$. We first choose $m_i = \sum_j m_{ij}$ many sets $V_{ij}$ of vertices of cardinality $n_i$ out of $[n]$. Note that we have triply indexed the vertex sets since for each fixed $i$ and $j$, we need to choose $m_{ij}$ many sets of vertices i.e. $V_{ij1}, V_{ij2}, \ldots, V_{ijm_{ij}}$. There are $(n_1 \cdot n_1 \cdot \cdots n_\ell \cdot \ell)$ ways to do this. Once we have chosen $V_{ijr}$, on each $V_{ijr}$, there are $g_{n_i,k_{ijj}}$ many ways to draw the graphs (connected components) on $V_{ijr}$ with the desired properties. If the last index of $V_{ij1}, \ldots, V_{ijm_{ij}}$ mattered, then we would simply have $g_{n_i,k_{ijj}}$ many graphs on $V_{ijr}$. It does not, so we divide out by $m_{ij}!$ to remove the repetition. All in all, there are

\begin{equation}
\sum_{m_1}^{n_1} \cdots \sum_{m_\ell}^{n_\ell} \frac{1}{m_{ij}!} \prod_{i,j} m_{ijj} g_{n_i,k_{ijj}}
\end{equation}

many graphs with $\ell$ components, where the sum runs over all partitions of $n$ and $k$ as in Equation (2) (note that $\ell$ is fixed here).

On the other hand, consider $\exp \tilde{g} = \prod \exp \left( g_{n,k,\nu} \frac{1}{n} z_n y^{k} x^n \right)$. We shall compute its $\prod(z_n, y^{k_{ij}}, x^{n_i})^{m_{ij}}$ coefficient where $\{n_i, k_{ij}\}$ gives a partition as in Equation (2). By definition of $\prod \exp \left( g_{n,k,\nu} \frac{1}{n} z_n y^{k} x^n \right)$, the product is over the partitions Equation (2).

Consider a general term

\[ \prod_{u,v} \left( z_{n_u} y^{k_{uv}} x^n u v \right)^{m_{uv}} \]

coming from a partition $n = \sum_{u,v} m_{uv} n_u^i$ and $k = \sum_{u,v} m_{uv} k_{uv}$. Suppose that it is equal to $\prod(z_n, y^{k_{ij}}, x^{n_i})^{m_{ij}}$. Then by considering the $z_n$ factor, we conclude that for each $u$ and $v$, $n_u^i = n_i$ and $m_{uv} \leq m_{ij}$. By symmetry, we conclude that the two sets of partitions ($n = \sum m_{ij} n_i, k = \sum m_{ij} k_{ij}$) and ($n = \sum m_{uv} n_u, k = \sum m_{uv} k_{uv}$) are the same. Hence the $\prod(z_n, y^{k_{ij}}, x^{n_i})^{m_{ij}}$ coefficient is simply the product of the coefficients of $z_n, y^{k_{ij}}, x^{n_i}$:

\begin{equation}
\prod_{m_{ij}} \frac{1}{m_{ij}!} \left( \frac{1}{(n_i)!} g_{n_i,k_{ij}j} \right)^{m_{ij}} = \frac{1}{(n_1)! m_1 \cdots (n_\ell)! m_\ell} \prod_{m_{ij}} \frac{1}{m_{ij}!} \prod_{m_{ij}} g_{n_i,k_{ij}j}.
\end{equation}
Let $\tau_\omega$ be as in Definition 3 but extended to the rings with $\mathbb{Q}$ replaced by $\mathbb{Q}[x, y_1, \ldots, y_s]$. We have

$$\tau_\omega \left( \prod (z_{i,j}^{y_{i,j} x^{n_i}})^{m_{ij}} \right) = z^{\sum_i (\omega_i \sum_j m_{ij})} \prod y^{\sum m_{ij} k_{ij} x^{\sum m_{ij} n_i}} = z^{\sum_i (\omega_i \sum_j m_{ij})} y^k x^n.$$ 

Hence the $x^n y^k z^\nu$-coefficient of $\tau_\omega(\exp \tilde{g})$ is

$$\sum (\prod (z_{i,j}^{y_{i,j} x^{n_i}})^{m_{ij}} \text{ coefficient of } \exp \tilde{g})$$

where the sum runs over all partitions as in Equation (2). Substituting Equation (4), we obtain

$$T_{n,k,\nu} = \sum_{n!} \left( \frac{n!}{(n_1)! m_1 \cdots (n_l)! m_l} \prod \frac{1}{m_{ij}} \prod g_{n_{i,j} k_{ij}}^{m_{ij}} \right)$$

which precisely equals Equation (3). □

4. APPLICATION: BIPARTITE GRAPHS OF GIVEN ORDER, SIZE AND RANK

Our motivation for studying the enumerative combinatorics of bipartite graphs comes from certain hyperplane arrangements [Son15c, Son15a], but they are certainly interesting on their own [Har58, HP63, HP73, Han79, UT14]. Following these earlier works, we shall first enumerate bi-colored graphs and from it, we shall obtain a method for enumerating bipartite graphs of given order and size.

**Definition 4.** A graph is bi-colored if its vertices are colored black and white so that no two adjacent vertices have the same color.

The number of bi-colored graphs of order $n$ and size $k$ is easy to enumerate: any such graph has precisely two blocks of orders, say $i$ and $n-i$. Assume one is colored black and the other, white. There are $i(n-i)$ possible edges between the two blocks. Hence the total number of bi-colored graphs of order $n$ and size $k$ is

$$\sum_{i=0}^{n} \binom{n}{i} \binom{i(n-i)}{k}.$$ 

The number of connected bi-colored graphs of given order and size is then generated by the formal logarithm

$$\log \left( 1 + \sum_{n \geq 1, k \geq 0} \left( \sum_i \binom{n}{i} \binom{i(n-i)}{k} \frac{1}{n!} x^n y^k \right) \right).$$

Since a connected bipartite graph admits precisely two bi-colorings, the number of connected bipartite graphs of given order and size is exactly half of the number of
connected bi-colored graphs of the same order and size. That is, if we let \( b_{n,k} \) denote the number of connected bipartite graphs of order \( n \) and size \( k \), then we have

\[
B(x, y) := \sum_{n \geq 0, k \geq 0} b_{n,k} \frac{1}{n!} x^n y^k = \frac{1}{2} \log \left( 1 + \sum_{n \geq 1, k \geq 0} \left( \sum_{i} \binom{n}{i} \binom{n-i}{k} \frac{1}{n!} x^n y^k \right) \right).
\]

By Theorem 2 we have

**Corollary 2.** The number of bipartite graphs of given order, size and number of connected components is given by \( \tau_n(\exp B(x, y, z)) \) with \( \omega = (1, 1, \ldots) \). That is, if

\[
\tau_n(\exp B) = \sum b_{n,k,\nu} \frac{1}{n!} x^n y^k z^\nu,
\]

then \( b_{n,k,\nu} \) is precisely the number of bipartite graphs of order \( n \), size \( k \) with \( \nu \) connected components.

The generating function \( \tau_n(\exp B) \) can be easily computed by computer algebra systems such as Mathematica. In the subsequent section, we shall list the first 100 or so terms of the generating function (order of the graph up to 10).

**Appendices**

**A. Numerical results.**

**A.1. The generating function for the number of bipartite graphs of given order, size and number of connected components [Corollary 2] as computed by Mathematica.**

\[
\tau_n(\exp B) = 1 + \frac{1}{2} x^2 y z + \frac{1}{2} x^3 y^2 z + x^4 \left( \frac{y^2 z}{8} + \frac{2 y^3 z}{3} + \frac{3 y^2 z}{2} \right) + x^5 \left( \frac{y^2 z}{4} + \frac{25 y^4 z}{24} + \frac{7 y^2 z}{2} + \frac{y^2 z}{12} \right) + \ldots
\]

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\]

By Theorem 2 we have

**Corollary 2.** The number of bipartite graphs of given order, size and number of connected components is given by \( \tau_n(\exp B(x, y, z)) \) with \( \omega = (1, 1, \ldots) \). That is, if

\[
\tau_n(\exp B) = \sum b_{n,k,\nu} \frac{1}{n!} x^n y^k z^\nu,
\]

then \( b_{n,k,\nu} \) is precisely the number of bipartite graphs of order \( n \), size \( k \) with \( \nu \) connected components.

The generating function \( \tau_n(\exp B) \) can be easily computed by computer algebra systems such as Mathematica. In the subsequent section, we shall list the first 100 or so terms of the generating function (order of the graph up to 10).
Table 1. Numer of bipartite graphs of given (order, size, number of components)

| n, k, ν | 2,1,1 | 3,2,1 | 4,3,1 | 4,4,1 | 4,2,2 | 5,4,1 | 5,5,1 |
|---------|-------|-------|-------|-------|-------|-------|-------|
| 1       | 1     | 3     | 16    | 3     | 3     | 125   | 60    |
| 2       | 5,3,2 | 6,5,1 | 6,6,1 | 6,7,1 | 6,8,1 | 6,9,1 |
| 3       | 10    | 30    | 1296  | 1140  | 480   | 105   | 10    |
| 4       | 6,10,1| 6,4,2 | 6,5,2 | 6,3,3 | 7,6,1 | 7,7,1 | 7,8,1 |
| 5       | 0     | 330   | 45    | 15    | 16,807| 23,100| 16,800|
| 6       | 7,9,1 | 7,10,1| 7,5,2 | 7,6,2 | 7,7,2 | 7,4,3 | 8,7,1 |
| 7       | 7,770 | 2,331 | 4,305 | 1,575 | 210   | 315   | 262,144|
| 8       | 8,8,1 | 8,9,1 | 8,10,1| 8,6,2 | 8,7,2 | 8,8,2 | 8,9,2 |
| 9       | 513,240| 555,520| 412,440| 66,248| 45,360| 15,435| 2,940 |
| 10      | 8,10,2| 8,5,3 | 8,6,3 | 8,4,4 | 9,7,2 | 9,9,1 | 9,10,1|
| 11      | 280   | 5,880 | 630   | 105   | 4,782,969| 12,551,112| 18,601,380|
| 12      | 9,7,2 | 9,8,2 | 9,9,2 | 9,10,2| 9,6,3 | 9,7,3 | 9,8,3 |
| 13      | 1,183,644| 1,287,090| 768,600| 309,960| 115,290| 34,020| 3,780 |
| 14      | 9,5,4 | 10,9,1| 10,10,1| 10,8,2| 10,9,2| 10,10,2| 10,7,3|
| 15      | 3,780 | 100,000,000| 336,853,440| 24,170,310| 37,948,680| 34,146,000| 2,467,080|
| 16      | 10,8,3| 10,9,3| 10,10,3| 10,6,4| 10,7,4| 10,5,5|
| 17      | 1,379,700| 392,175| 66,150 | 107,100| 9,450 | 945  |

A missing triple \((n, k, \nu)\) (such as \((3, 1, 1)\)) means there are no bipartite graphs of order \(n\), size \(k\) with \(\nu\) connected components.

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