Estimating tidal Love number of a class of compact stars

Shyam Das\textsuperscript{1,\textit{a}}, Bikram Keshari Parida\textsuperscript{2,\textit{b}}, Ranjan Sharma\textsuperscript{3,\textit{c}}

\textsuperscript{1} Department of Physics, Malda College, Malda, West Bengal 732101, India
\textsuperscript{2} Department of Physics, Pondicherry University, Kalapet, Puducherry 605014, India
\textsuperscript{3} Department of Physics, Cooch Behar Panchanan Barma University, Cooch Behar, West Bengal 736101, India

Received: 25 October 2021 / Accepted: 24 January 2022 / Published online: 12 February 2022
© The Author(s) 2022

Abstract

Tidal deformability of a star in the presence of an external tidal field provides an important avenue to our understanding about the structure and properties of neutron stars. The deformation of the star is characterized by the tidal Love number (TLN). In this paper, we propose a technique to measure the TLN of a particular class of compact stars. In particular, we analyze the impact of anisotropy and compactness on the TLN.

1 Introduction

Compact objects provide extreme conditions in terms of gravity and density and thus are unique astrophysical laboratories for studying general relativity and interactions at the super nuclear density. In general, compact objects exist in binaries comprising either two neutron stars (NS–NS binaries) or a black hole (BH) and a neutron star (NS) (BH–NS binaries). The merger of these objects generates huge gravitational waves which have been experimentally verified in the recent times.

A neutron star provides a perfect place for investigating the nature of particle interactions at very high densities in a natural way [1]. Neutron stars (NS) are compact objects of very high energy density having approximate masses $1.5 \ M_\odot$ and radii $10^5$ times smaller than the Sun’s radius. Therefore, they are perfectly natural systems to study nuclear matter properties at high densities. In fact, density inside the core of an NS can be as high as several times the density that is reached inside a heavy atomic nucleus [2]. Despite attempts of several decades, we still lack a proper understanding of the thermodynamical behaviour inside a compact star. The extreme conditions at the interior of a compact star comprising matter of uncertain composition have prompted many investigators to study its gross macroscopic properties within the framework of General Relativity. In order to understand the microscopic properties, physical quantities such as NS masses and radii have been used as important tools to constrain its EOS.

This article explores the possibility of introducing tidal deformation as one of the astrophysically observable macroscopic properties that can be used to study the interior of a NS [3]. Like any other extended object, a NS is tidally deformed under the influence of an external tidal field. The tidal deformability measures the star’s quadrupole deformation in response to a companion perturbing star [4]. The induced quadrupole moment of the neutron star affects the binding energy of the system and increases the rate of emission of gravitational waves [5–7]. Tidal deformability plays an important role in the observation of coalescing NS with gravitational waves and has been used to probe the internal structure of NS. The TLN characterizes how easy or difficult it would be to deform a NS away from sphericity [8,9]. The TLN can be computed by following the standard methods available in the literature [10–14].

In gravitational wave astronomy, the tidal deformability characterized by TLN [10], can be used to analyze the physical features of the merging objects [30]. The TLN, in par-
ticular, is used to constrain the EOS of the NS [28,30]. To
understand the methods of estimating the TLN, we refer to
the citations [14,31–33]. The algorithm can also be extended
to slowly rotating extended compact objects [33–38]. Note
that even though the TLN of a Schwarzschild black hole is
zero [31,32,39–41], it does not vanish for a Kerr BH [42].
For relatively less compact objects, the dominant contribu-
tion to the tidal deformability comes from the even parity
quadrupole term \( l \), which starts to impact the phase of the GW
signal emitted in a binary at the fifth post-Newtonian (5PN)
order [43]. The leading order (6PN) term of even-parity tidal
deformability has also been calculated [44]. A method to cal-
culate the odd-parity (or gravitomagnetic or mass-current)
tidal deformability was proposed independently by Damour
and Nagar [31] and Binnington and Poisson [32]. The choice
of fluid properties also affects the odd-parity tidal deforma-
bility [33] as shown by Pani et al. [45]. The pioneer in this
field was Yagi [46] who, for the first time, estimated the
impact of odd-parity tidal deformability on the gravitational
waves phase evolution and then extended the work by ana-
lyzing the signal from GW170817 [47].

In this paper, we develop a method to estimate the TLN
for a spherically symmetric and anisotropic relativistic star
in static equilibrium. In a compact object, pressures may be dif-
ferent in radial and transverse directions and the difference of
radial pressure \( (p_r) \) and tangential pressure \( (p_\theta) \) is defined as
pressure anisotropy. Incorporating anisotropy into the mat-
ter distribution of compact objects, numerous anisotropic
stellar models have been developed and investigated which
include the investigations carried out in references [48–72]
and Raposo et al. [73]; amongst others. Ruderman [74] and
Canuto [75] have shown that anisotropy may develop inside
highly dense, compact stellar objects due to a variety of fac-
tors. Kippenhahn and Weigert [76] showed that in relativis-
tics stars, anisotropy might occur due to the existence of a
solid core or type 3A superfluid. Strong magnetic fields can
also generate an anisotropic pressure inside a self-gravitating
body [77]. Anisotropy may also develop due to the slow rota-
tion of fluids [78]. A mixture of perfect and a null fluid may
also be represented by an effective anisotropic fluid model
[79]. Local anisotropy may occur in astrophysical objects for
various reasons such as viscosity, phase transition [80], pion
condensation [81] and the presence of strong electromagnetic
field [82]. The factors contributing to the pressure anisotropy
have also been discussed by Dev and Gleiser [83,84] and
Gleiser and Dev [85]. Ivanov [86] pointed out that influences
of shear, electromagnetic field etc. on self-bound systems
can be absorbed if the system is considered to be anisotropic.
Self-bound systems composed of scalar fields, the so-called
‘boson stars’ are naturally anisotropic [87]. Wormholes [88]
and gravastars [89,90] are also naturally anisotropic. The
shearing motion of the fluid can be considered as one of the
reasons for the presence of anisotropy in a self-gravitating
body [91]. Bowers and Liang [92] have extensively discussed
the underlying causes of pressure anisotropy in the stellar
interior and analyzed the effects of anisotropic stress on the
equilibrium configuration of relativistic stars. Therefore, we
find it worthwhile to investigate the impacts of anisotropic
stress on sources of gravitational waves. Alternatively, for
an estimated TLN of the source, the technique can also be
used to constrain the anisotropy of the source of a given mass
and radius. Earlier, Biswas and Bose [93] used the gravita-
tional wave (GW) and electromagnetic (EM) observation of
GW170817 to constrain the extent of pressure anisotropy.
Many applications of TLN in neutron stars have also been
explored by Yagi and Yunes [8].

The paper is organized as follows: In Sect. 2, the methodol-
ogy of determining the TLN is discussed. Section 3 provides
a particular stellar model which is used to get an estimate of
the TLN. In Sect. 4, the TLN \( k_2 \) for a wide range of masses and radii is provided. The range of values of \( k_2 \) for a fixed
compactness \( \epsilon \) possessing anisotropic stress is investigated.
Section 5 summarizes the main results and provides some
prospects of future investigation in this direction.

2 Tidal Love number

We consider a static spherically symmetric neutron star (NS)
immersed in an external tidal field. In response to the tidal
field, the star will be deformed by the tidal force by devel-
oping a multipolar structure. This kind of situation occurs in
coalescing binary systems where the gravitational field of its
companion tidally deforms each component. The TLN char-
acterizes the deformability of the NS away from sphericity
[94]. For mathematical simplicity, in our calculation, we shall
restrict ourselves to quadrupole moments \( \mathcal{Q}_{ij} \) only. This is
reasonable if the two binary neutron stars remain sufficiently
far away from each other. In such a situation, the quadrupole
moment \( (l = 2) \) dominates over the multiple moments. \( \mathcal{Q}_{ij} \)
can be related to the external tidal field \( \delta_{ij} \) as [95]

\[
\mathcal{Q}_{ij} = -\Lambda \delta_{ij},
\]

where \( \Lambda \) is the tidal deformability of the neutron star and it
is related to the TLN \( k_2 \) as [95]

\[
k_2 = \frac{3}{2} \Lambda R^{-5}.
\]

The TLN is dimensionless. The quadrupole fields \( \mathcal{Q}_{ij} \) and
\( \delta_{ij} \) can be expanded in tensor spherical harmonics \( \mathcal{Y}^{lm}_{ij} \) as:

\[
\delta_{ij} = \sum_{m=-2}^{2} \delta_{m} \mathcal{Y}^{2m}_{ij} = \delta_{0} \mathcal{Y}^{20}_{ij} = \mathcal{C} \mathcal{Y}^{20}_{ij},
\]

\[
\mathcal{Q}_{ij} = \sum_{m=-2}^{2} \mathcal{Q}_{m} \mathcal{Y}^{2m}_{ij} = \mathcal{Q}_{0} \mathcal{Y}^{20}_{ij} = \mathcal{C} \mathcal{Y}^{20}_{ij},
\]
\( \mathcal{D}_{ij} = \sum_{m=-2}^{2} \mathcal{D}_m \Psi_{ij}^{2m} = \mathcal{D}_0 \Psi_{ij}^{20} = \mathcal{D} \Psi_{ij}^{20}. \) (4)

In the second equality, the coordinate system was so oriented that the term became symmetric in \( \phi \). Subsequently, the only component that is non-vanishing is the \( m = 0 \) component. We can rewrite Eq. (1) as

\[ \mathcal{D} = -\Lambda \delta \epsilon. \] (5)

Now the background metric \( (0)g_{\mu\nu}(x^\nu) \) corresponding to the neutron star, with a small perturbation \( h_{\mu\nu}(x^\nu) \) due to external tidal field, gets modified as

\[ g_{\mu\nu}(x^\nu) = (0)g_{\mu\nu}(x^\nu) + h_{\mu\nu}(x^\nu). \] (6)

We write the background geometry of the spherical static star in the standard form

\[ (0)d s^2 = g_{\mu\nu}(x^\nu) dx^\mu dx^\nu = -e^{2r(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(\theta^2 + \sin^2 \theta d\phi^2). \] (7)

For the linearized metric perturbation \( h_{\mu\nu} \), using the method as in Refs. [93, 96], we restrict ourselves to static \( l = 2, m = 0 \) even parity perturbation. With these assumptions, the perturbed metric becomes

\[ h_{\mu\nu} = \text{diag} \left[ H_0(r)e^{2r}, H_2(r)e^{2\lambda(r)}, r^2K(r), r^2\sin^2 \theta K(r) \right] Y_{20}(\theta, \phi) \] (8)

where \( H_0(r), H_2(r) \) and \( K(r) \) are radial functions to be determined by the perturbed Einstein field equations.

For the spherically static metric (7), the stress-energy tensor is given by [97–100]

\[ (0)T^\xi_\chi = (\rho + p_r)u^\xi u_\chi + p_r^\xi + (p_t - p_r)\eta^\xi \eta_\chi, \] (9)

where \( \eta^r \) is the space-like vector and the vector \( u^r \) represents fluid 4-velocity. The quantities satisfy the relations \( u^\xi u_\xi = -1, \eta^\xi \eta_\xi = 1 \) and \( \eta^\xi u_\xi = 0 \). The quantities \( \rho, p_r \) and \( p_t \) represent density, radial pressure and tangential pressure, respectively.

Furthermore, the energy–momentum tensor is perturbed by a perturbation tensor \( \delta T^\xi_\chi \) which is defined as

\[ T^\xi_\chi = (0)T^\xi_\chi + \delta T^\xi_\chi. \] (10)

The non-zero components of \( T^\xi_\chi \) are:

\[ T_r^\rho = \delta p_r(r)Y(\theta, \phi) + p_r(r), \] (12)

\[ T_\theta^\phi = \frac{dp_r}{dp_r} \delta p_r(r)Y(\theta, \phi) + p_r(r), \] (13)

\[ T^\phi_\phi = \frac{dp_r}{dp_r} \delta p_r(r)Y(\theta, \phi) + p_r(r). \] (14)

With these perturbed quantities, we write the perturbed Einstein Field Equations as

\[ G^\xi_\chi = 8\pi T^\xi_\chi, \] (15)

where we assume \( G = c = 1 \) and the Einstein tensor \( G^\xi_\chi \) is calculated using the metric \( g_{\chi\xi} \).

2.1 Derivation of the master equation and expression for TLN

Using the background field equations \( (0)G^\xi_\chi = 8\pi (0)T^\xi_\chi \), we obtain the following results:

\[ (0)G^\rho_\chi = 8\pi (0)T^\rho_\chi, \]

\[ \Rightarrow \lambda'(r) = \frac{8\pi r^2 e^{2\lambda(r)} \rho(r) - e^{2\lambda(r)} + 1}{2r}, \] (16)

\[ (0)G^r_\rho = 8\pi (0)T^r_\rho, \]

\[ \Rightarrow v'(r) = \frac{8\pi r^2 p_r(r)e^{2\lambda(r)} + e^{2\lambda(r)} - 1}{2r}. \] (17)

Note that \( \nabla^\xi_\chi T^\xi_\chi = 0. \) Choosing \( \xi = r \), we obtain

\[ p'_r(r) = \frac{-rp_r(r)v'(r) - 2p_r(r) + 2p_t(r) - r \rho(r)v'(r)}{r}. \] (18)

For the perturbed metric, using Einstein equations (15), we get the following results:

\[ G^\rho_\rho + G^\phi_\phi = 0 \Rightarrow H_0(r) = H_2(r) = H(r), \] (19)

\[ G^r_r = 0 \Rightarrow K' = H' + 2Hv', \] (20)

\[ G^\rho_\rho + G^\phi_\phi = 8\pi (T^\rho_\rho + T^\phi_\phi), \]

\[ \Rightarrow \delta p_r = \frac{H(r)e^{-2\lambda(r)}(\lambda'(r) + v'(r))}{8\pi \frac{dp_r}{dp_r} r}. \] (21)

Now, using the identity

\[ \frac{\partial^2 Y(\theta, \phi)}{\partial \theta^2} + cot(\theta) \frac{\partial Y(\theta, \phi)}{\partial \theta} + \csc^2(\theta) \frac{\partial^2 Y(\theta, \phi)}{\partial \phi^2} = -6Y(\theta, \phi), \]

\[ \mathcal{D} \text{ Springer} \]
and Eqs. (16), (17), (18), (19), (20) and (21), we obtain the master equation for $H(r)$ as

$$
- \frac{1}{e^{-2\lambda(r)}Y(\theta, \phi)} \left[ G^t_t - G^r_r \right] \\
= \frac{8\pi}{e^{-2\lambda(r)}Y(\theta, \phi)} \left[ T^t_t - T^r_r \right] \\
\Rightarrow H''(r) + \mathcal{B} H'(r) + \mathcal{J} H(r) = 0,
$$

where,

$$
\mathcal{B} = \left( \frac{-e^{2\lambda(r)} - 1}{r} - 4\pi r e^{2\lambda(r)} (p_r(r) - \rho(r)) \right),
$$

$$
\mathcal{J} = \left( 4e^{2\lambda(r) + \frac{e^{4\lambda(r)}}{r^2}} + 1 + 64\pi^2 r^2 p_r(r)^2 e^{4\lambda(r)} + 16\pi e^{2\lambda(r)} \left( p_r(r) \left( e^{2\lambda(r)} - 2 \right) - p_t(r) - \rho(r) \right) + 4\pi \frac{dp_r}{d\rho} (p_r(r) + \rho(r)) \right). 
$$

The exterior region of the static spherically symmetric star will be described by the Schwarzschild metric and hence by setting, $\rho = 0$, $p_r = 0$, $p_t = 0$ and $e^{2\lambda} = 1/(1 - 2M/r)$, the master equation (22) takes the form

$$
H''(r) - \frac{2(M - r)H'(r)}{r(2M - r)} + \frac{2H(r)(2M^2 - 6Mr + 3r^2)}{r^2(r - 2M)^2} = 0.
$$

The solution to this second-order differential equation (25) is obtained as [9]

$$
H(r) = \frac{1}{2M^2r(2M - r)} \left[ c_1 \left( -2M \left( 2M^3 + 4Mr \\ -9M^2 + 3r^3 \right) \right) - 3r^2(r - 2M)^2 \log \left( \frac{r}{M} \right) \right] + \frac{3c_1(r(2M^2 - r))}{M^2},
$$

where, $c_1$ and $c_2$ are integration constants. In order to get the expression for these constants, we make a series expansion of Eq. (26) as

$$
H(r) = -\frac{3c_1 r^2}{M^2} + \frac{6c_1 r}{M} - \frac{c_2 (8M^3)}{5r^3} + \mathcal{O} \left( \left( \frac{1}{r} \right)^4 \right).
$$

Now, in the star’s local asymptotic rest frame, at large $r$ the metric coefficient $g_{tt}$ is given by [95,101,102]

$$
\frac{(1 - g_{tt})}{2} = -\frac{M}{r} - \frac{36\pi}{2r^3} \left( n^i n^j \frac{1}{3} \delta^{ij} \right) + \mathcal{O} \left( \frac{1}{r^3} \right) \\
+ \frac{1}{2} \mathcal{E}_{ij} x^i x^j + \mathcal{O}(r^3),
$$

where $n^i = x^i/r$. Matching the asymptotic solution using Eq. (27) together with the expansion of Eq. (28) and using Eq. (1), we obtain

$$
c_1 = -\frac{M^2 \mathcal{E}}{3}, \quad c_2 = \frac{15\mathcal{E}}{8M^3}.
$$

Subsequently, the expression for TLN $k_2$ can be obtained by using Eqs. (2), (26), (29) and also using the expression for $H(r)$ and its derivatives at the star’s surface $r = R$ as

$$
k_2 = \left[8(1 - 2\mathcal{E})^{2/3} \mathcal{E}^5 (2\mathcal{E}(y - 1) - y + 2)\right]/X,
$$

where,

$$
X = 5(2\mathcal{E}(2\mathcal{E}(2\mathcal{E}(2\mathcal{E}(y + 1) + 3y - 2) \\
- 11y + 13) + 3(5y - 8)) \\
- 3y + 6 + 3(1 - 2\mathcal{E})^2 (2\mathcal{E}(y - 1) \\
- y + 2) \log \left( \frac{1}{\mathcal{E}} \right) \\
- 3(1 - 2\mathcal{E})^2 (2\mathcal{E}(y - 1) - y + 2) \log \left( \frac{1}{y} \right) \right).
$$

Note that $\mathcal{E} = \left( -\frac{M}{R} \right)$ and $y$ depend on $r$, $H(r)$ and it’s derivatives evaluated at $R$ in the form

$$
y = \left. \frac{r H'(r)}{H(r)} \right|_{r=R}.
$$

To calculate the TLN $k_2$ for a particular compact star, we need to specify a model which we can be utilized to calculate $y$ and subsequently $k_2$ for a particular NS of given mass $M$ and radius $R$.

### 3 Choice of a physically acceptable model

#### 3.1 Einstein field equations

To describe the interior of a static and spherically symmetric relativistic star, we take the line element in coordinates $(x^a) = (t, r, \theta, \phi)$ as given in Eq. (7).

We also assume an anisotropic matter distribution for which the energy–momentum tensor is assumed in the form as given in the Eq. (9).
The energy density $\rho$, the radial pressure $p_r$ and the tangential pressure $p_t$ are measured relative to the comoving fluid velocity $u^i = e^{-\psi} \delta^i_0$. For the line element (7), the independent set of Einstein field equations are then obtained as

$$8\pi \rho = \frac{1}{r^2} \left[ r(1 - e^{-2\lambda}) \right]',$$  
(33)

$$8\pi p_r = -\frac{1}{r^2} \left( 1 - e^{-2\lambda} \right) + \frac{2\nu}{r} e^{-2\lambda},$$  
(34)

$$8\pi p_t = e^{-2\lambda} \left( \nu'' + \nu^2 + \frac{\nu'}{r} - \nu' \lambda' - \frac{\lambda'}{r} \right).$$  
(35)

where primes (') denote differentiation with respect to $r$. The system of equations determines the behaviour of the gravitational field of an anisotropic imperfect fluid sphere. The mass contained within a radius $r$ of the sphere is defined as

$$m(r) = 4\pi \int_0^r \omega^2 \rho(\omega) d\omega.$$  
(36)

We define, $\Delta = p_t - p_r$, as the measure of anisotropy. The anisotropic stress will be directed outward (repulsive) when $p_t > p_r$ (i.e., $\Delta > 0$) and inward when $p_t < p_r$ (i.e., $\Delta < 0$).

3.2 A particular anisotropic model

Any well-behaved, physically viable stellar model can be used to find the TLN $k_2$ in our construction. For example, Jiang and Yagi [103] have used the Tolman VII model to analyze the relationship between the TLN with the moment of inertia and compactness of the star. The same model was also used by them for the description of neutron star interiors, where the authors introduced central density as an input to fine-tune the observables [104]. The authors have also used the tidal measurement of binary stars for probing the GW propagation [105]. To calculate the TLN, we choose a particular model, which is an anisotropic generalization of the Korkina and Orlyanskii solution III obtained earlier by [106]. To examine the physical acceptability of the solution, we first write the variables which are obtained as

$$e^{2\nu} = A^2 (1 + aCr^2)^2,$$  
$$e^{2\lambda} = \left[ 1 - BCr^2 (1 + 3aCr^2)^{-2/3} - aCr^2 (1 + aCr^2)^{-1} \right] (1 + 3aCr^2)^{-2/3},$$  
(37)

$$8\pi \Delta = \frac{aC^2 r^2 \eta_1^2}{\eta_2^2}.$$  
(38)

The line element (7) then takes the form

$$ds^2 = -A^2 (1 + aCr^2)^2 dt^2 + \left[ 1 - BCr^2 (1 + 3aCr^2)^{-2/3} - aCr^2 (1 + aCr^2)^{-1} \right]^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$  
(39)

The model contains five constants namely, $a$, $A$, $B$, $C$ and $\alpha$ three of which do get fixed by the boundary conditions. The parameter $\alpha$ that appears as a free parameter in the solutions provided by [106], without any loss of generality, can be set to $\alpha = 1$. The other free parameter $\alpha$ provides the measure of anisotropy. For an isotropic sphere ($\alpha = 0$), if we set $B = 0$ and $C = 1$, the metric (40) reduces to

$$ds^2 = A^2 (1 + ar^2)^2 dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$  
(41)

which is the Korkina and Orlyanskii solution III [107]. In other words, the solution (40) obtained by [106] is an anisotropic generalization of the solution of [107]. Consequently, this particular solution provides a tool to investigate the anisotropic effects on the TLN. Physical quantities in this model are obtained as

$$8\pi \rho = \frac{BC \eta_1^2 (3 + 5aCr^2) + C (3 + aCr^2 (6 - aCr^2)) \alpha}{\eta_2^2},$$  
(42)

$$8\pi p_r = \frac{C (B \eta_1 \psi_1 + \alpha + a(-4\eta_2^2 + Cr^2(-4a\eta_2^2 + 5a)))}{\eta_1^2 \eta_2^2},$$  
(43)

$$8\pi p_t = \frac{1}{\eta_1^2 \eta_2^2} \left[ C (-B \eta_1^2 \psi_1 + a(4\eta_2^2 + Cr^2(2a(4\eta_2^2 + Cr^2(2a(4\eta_2^2 + Cr^2 - \alpha)) - 5a)) - \alpha)) \right].$$  
(44)

$$8\pi \Delta = \frac{aC^2 r^2 \eta_1^2 \alpha}{\eta_2^2},$$  
(45)

$$m(r) = \frac{Cr^3 (B + aBCr^2 + \alpha)}{2\eta_1 \eta_2^2},$$  
(46)

where, $\psi_1 = (1 + 5aCr^2)$, $\eta_1 = (1 + aCr^2)$ and $\eta_2 = (1 + 3aCr^2)$.

3.3 Physical acceptability of the solution

Before using the solutions, let us first examine the physical acceptability of the solution:

(i) In this model, we have $(e^{2\nu(r)})_{r=0} = (e^{2\lambda(r)})_{r=0} = 0$ and $e^{2\psi(0)} = A^2$, $e^{2\lambda(0)} = 1$; these imply that the metric is regular at the centre $r = 0$.

(ii) Since $8\pi \rho(0) = 3 \frac{C (B + \alpha)}{8\pi}$ and $8\pi p_r(0) = 8\pi p_t(0) = C (4a - B - \alpha)$, the energy density, radial pressure and tangential pressure will be non-negative at the centre if we choose the parameters satisfying the condition $a > \frac{B + \alpha}{4}$.
(iii) The interior solution (7) should be matched to the exterior Schwarzschild metric
\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (47) \]
across the boundary of the star \( r = R \), where \( M \) is the total mass of the sphere which can be obtained directly from Eq. (36) as
\[ M = m(R) = \frac{CR^3(B + aBCR^2 + \alpha)}{2(1 + 2aCR^2)(1 + 3aCR^2)^\frac{3}{2}}. \]

Matching of the line elements (40) and (47) at the boundary \( r = R \) yields,
\[ \left(1 - \frac{2M}{R}\right) = 1 - \frac{aCR^2}{(1 + aCR^2)(1 + 3aCR^2)^\frac{3}{2}} \]
\[ B = \frac{(5M - 2R)[2^5 a(M - 2R)^1\frac{3}{2}(2M - R) + Ra]}{2R(2M - R)}. \]

Using the junction conditions, we determine the constants \( A, B, C \) as
\[ A = \frac{(5M - 2R)}{2\sqrt{R(R - 2M)}}. \quad (50) \]
\[ C = \frac{M}{aR^2(2R - 5M)}. \quad (51) \]
\[ B = \frac{(5M - 2R)[2^8 a(M - 2R)^1\frac{3}{2}(2M - R) + Ra]}{2R(2M - R)}. \quad (52) \]

(iv) The gradient of density, radial pressure and tangential pressure are respectively obtained as
\[ 8\pi \frac{d\rho}{dr} = \frac{1}{A_1} \times \left(2aC^2 r(-10B\eta_1^4 + (-15 + aCr^2(-53 + aCr^2(-49 + 5aCr^2)))\alpha)\right). \quad (53) \]
\[ 8\pi \frac{dp_r}{dr} = -\frac{1}{A_2} \times (2aC^2 r(-2B\eta_1\eta_3 + a(4\eta_2^2 + Cr^2(a(16\eta_2^2/3) + Cr^2(12\alpha) - 25\alpha) - 8\alpha)) + \alpha), \quad (54) \]
\[ 8\pi \frac{dp_t}{dr} = \frac{1}{\eta_1^3\eta_2^3} \times (4aC^2 r(B\eta_1^2\eta_3 + a(-2\eta_2^2 + 4aC^2 (5\eta_3^2 + a(16\eta_2^2/3) + Cr^2(12\alpha) - 25\alpha)) - 8\alpha) + \alpha), \quad (55) \]

where, \( \eta_3 = (-1 + 5a^2C^2r^4) \), \( A_1 = \eta_1^3\eta_2^3 \) and \( A_2 = \eta_1^3\eta_2^3 \).
The decreasing nature of these quantities is shown graphically.

(v) Within a stellar interior, it is expected that the speed of sound should be less than the speed of light i.e., \( 0 \leq \frac{dp_r}{d\rho} \leq 1 \) and \( 0 \leq \frac{dp_t}{d\rho} \leq 1 \).
In this model, we have
\[ \frac{dp_r}{d\rho} = \frac{1}{(-10B\eta_1^4 + (-15 + aCr^2(-53 + aCr^2\eta_3)))\alpha} \times \left[\eta_2(-2B\eta_1\eta_3 + a(4\eta_2^2 + Cr^2(a(16\eta_2^2/3) + Cr^2(12\alpha) - 25\alpha)) - 8\alpha)) + \alpha)\right], \quad (56) \]
\[ \frac{dp_t}{d\rho} = \frac{-1}{(10B\eta_1^4 + 15\alpha + aCr^2(68 + aCr^2(102 + aCr^2\eta_3)))\alpha}\]
Fig. 3 Radial and transverse component of sound speed at the stellar interior

Fig. 4 Fulfillment of energy condition at the stellar interior

Fig. 5 Radial variation of anisotropy at the stellar interior

Fig. 6 Radial variation of adiabatic index at the stellar interior

Fig. 7 Radial variation of mass at the stellar interior

Fig. 8 Nature of Equation of state (EOS) at the stellar interior

\[
\begin{align*}
\times & \left[2\eta_2(B\eta_1^2\eta_3 + a(-2\eta_2^2 + C r^2(6\alpha + a(-10\eta_2^2 + Cr^2(-6\alpha\eta_2^2 + 5\alpha)))\right)\right].
\end{align*}
\]

\[57\]

where, \(\eta_4 = (-49 + 5aCr^2)\) and \(\eta_5 = (44 - 5aCr^2)\).

By choosing the model parameters appropriately, it can
be shown that this requirement is also satisfied in this
model.

(vi) Fulfillment of the energy conditions for an anisotropic
fluid i.e., \(\rho + p_r + 2p_t \geq 0\) \(\rho + p_r \geq 0\) and \(\rho + p_t \geq 0\) can
also be shown graphically to be satisfied in this model.

3.4 Physical behaviour of the model

The simple elementary functional forms of the physical
quantities help us to make a detailed study of the physical
behaviour of the star. Most importantly, the solution contains
an ‘anisotropic switch’ \(\alpha\), which allows us to investigate
the impact of anisotropy. We analyze the physical behaviour of
the model by using the values of masses and radii of observed
pulsars as input parameters. We consider the data available
from the pulsar PSR J0030+0451 whose estimated mass
and radius are \(M = 1.34\, M_\odot\), and \(R = 12.71\, \text{km}\), resep-
tively [108]. Even though systematic errors in the measure-
ments of neutron star masses and radii can not be ignored
[109] for the assumed set of values, we determine the con-
stants for two different values of the anisotropic factor \(\alpha\).
For an isotropic case \((\alpha = 0)\), we obtain the constants
\(B = 3.03291\), \(C = 0.000787453\), \(A = 0.736378\); assum-
ing the star to be composed of an anisotropic fluid distribution
(we assume \(\alpha = 1.5\)), the constants are calculated as
\(A = 0.736378\), \(B = 1.70219\), \(C = 0.000787453\). Making
use of these values, we show graphically the nature of all the
physically meaningful quantities in Figs. 1, 2, 3, 4, 5, 6, 7 and
8. The plots clearly show that all the quantities comply with
the requirements of a real star. In particular, the figures high-
light the effect of anisotropy on the gross physical behaviour
of the compact star. Figure 9 shows that the configuration is
stable under the combined effects of three different types of
forces.

![Fig. 9 Forces in equilibrium at the stellar interior](image)

![Fig. 10 \(k_2\) is plotted against \(C\) for different \(\alpha\)](image)

![Fig. 11 \(V_r^2\) and \(V_t^2\) plotted against \(C\) for different \(\alpha\) values](image)

| Anisotropy \(\alpha\) | Compactness \(C\) | TLN \(k_2\) |
|---------------------|-----------------|-----------|
| 2.5                 | 0.31405         | 0.144954  |
| 3                   | 0.2288          | 0.173846  |
| 3.5                 | 0.1800          | 0.221919  |
| 4                   | 0.1384          | 0.269166  |

4 Numerical calculation of TLN

Using the method employed in reference [110], we now cal-
culate the numerical value of \(k_2\) for a particular neutron star.
We first rewrite the master equation (22) using the Eq. (32)
as
\[
ry' + y^2 + (r\mathcal{R} - 1)y + r^2\mathcal{F} = 0.
\]  

(58)
Fig. 12 \( k^2 \) is plotted against \( \mathcal{C} \) for different \( \alpha \) values. Only physically allowed range of \( \alpha \) and \( \mathcal{C} \) are considered here.

Fig. 13 (Top) \( k^2 \) plotted against different masses \( M \) for different \( \alpha \) at \( R = 10 \text{ km} \), (Bottom) \( k^2 \) plotted against different masses \( M \) with different radii \( R \) for a fixed value of \( \alpha = 2 \).

From Eq. (30), at \( \mathcal{C} = 0 \) we expect \( k^2 = 0 \). This implies that \( y(0) = 2 \). Moreover, in the horizon formation limit \( \mathcal{C} = 0.5 \), the TLN \( k^2 \) vanishes for all values of \( y \).

In order to solve the differential equation (58), we use the initial condition \( y(0) = 2 \) in addition to the expression for \( \mathcal{R} \) using Eqs. (23) and (24), respectively. Using the initial condition and Eqs. (38), (42), (43) and (44) for a particular NS, Eq. (58) can be solved and subsequently using Eq. (30), the TLN \( k^2 \) can be calculated. One can also find the analytical expression for \( y(r) \) in terms of compactness factor \( \mathcal{C} \) and anisotropy \( \alpha \). Employing this technique, we plot the relation between \( k^2 \) and compactness factor \( \mathcal{C} \) for different values of \( \alpha \).

Fig. 14 (Top) \( k^2 \) plotted against different radii \( R \) for different values of \( \alpha \) of a star of fixed mass \( M = 1.5 \, M_{\odot} \), (Bottom) \( k^2 \) plotted against different radii \( R \) and masses \( M \) for a fixed value of \( \alpha = 2 \).

Fig. 15 \( y \) is plotted against \( \alpha \) assuming different compact star masses and radii provided in the Table 2. Only physically allowed range of \( \alpha \) is considered.

Fig. 16 \( k^2 \) is plotted against \( \alpha \) assuming different compact star masses and radii provided in the Table 2. Only physically allowed range of \( \alpha \) has been considered.
\( \alpha \) as shown in the Fig. 10. We note that \( k_2 \) increases gradually with increasing \( \mathcal{C} \) up to a certain value and then decreases with further increase of \( \mathcal{C} \). The range numerical value of \( k_2 \) resembles with the Ref. [111]. For a NS having compactness \( \mathcal{C} < 0.34 \) and \( \alpha \leq 2 \), the above scheme can be used to calculate the TLN in this model.

For \( \alpha > 2 \), a discontinuity arises in the plot of \( k_2 \) vs \( \mathcal{C} \). To address this problem, we set a maximum limit on \( \mathcal{C} \) for a particular \( \alpha > 2 \), using the ‘physical acceptability’ conditions discussed earlier. One can check that for all range of values of \( \mathcal{C} < 0.34 \), \( \rho(r = 0), \rho(r = R), p_r(r = 0), p_r(r = R), p_t(r = 0) \geq 0 \). The energy condition \( \rho + p_r + 2p_t \geq 0 \). Therefore, the maximum limit on \( \mathcal{C} \) can be calculated from the condition \( 0 \leq \frac{dp_r}{d\rho} \leq 1 \) and \( 0 \leq \frac{dp_t}{d\rho} \leq 1 \). For different values of \( \alpha \), \( V_r^2(R) = \frac{dp_r}{d\rho} \) and \( V_t^2(R) = \frac{dp_t}{d\rho} \) are plotted against \( \mathcal{C} \) in Fig. 11. It then becomes easy to evaluate the maximum limit on \( \mathcal{C} \) for different \( \alpha \) from the plot. For example, in Fig. 11, we note that for \( 0 \leq \alpha \leq 2 \), the range of compactness is \( 0 \leq \mathcal{C} \leq 0.4 \). For \( 0 \leq \alpha \leq 2 \), Fig. 10 shows that \( 0 \leq \mathcal{C} \leq 0.34 \). In Table 1, the maximum value of \( \mathcal{C} \) for different \( \alpha \) is given.

For \( \alpha > 2 \), variation of the TLN \( k_2 \) against \( \mathcal{C} \) is shown in Fig. 12. In Table 1, the numerical values of the TLN \( k_2 \) is shown for different \( \alpha \) values.

In the Fig. 13, (Top) \( k_2 \) is plotted against mass \( M \) for different values of \( \alpha \) at \( R = 10 \text{ km} \). (Bottom) \( k_2 \) is plotted against mass \( M \) for different radii \( R \) at a fixed value of \( \alpha = 2 \). In the Fig. 14, (Top) \( k_2 \) is plotted against radius \( R \) for different \( \alpha \) values having a fixed mass \( M = 1.5 M_\odot \) and (Bottom) \( k_2 \) is plotted against \( R \) for different masses \( M \) for a fixed value of \( \alpha = 2 \).

In the Fig. 15, variation of \( y \) with respect to \( \alpha \) is plotted for different compact stars. In this case, the range of \( \alpha \) is assumed to be \( 0 \leq \alpha < 4 \). In Fig. 16, \( k_2 \) is plotted against \( \alpha \). Physically acceptable range of \( \alpha \) has been calculated numerically. Obviously, the maximum value of \( \alpha \) is not the same for different class of compact stars. In Table 2, the maximum value of \( \alpha \) is calculated for different neutron stars and the corresponding \( k_2 \) is also shown. In Fig. 16, we note that the TLN increases monotonically with increasing \( \alpha \) for stars having different compactness. In Table 2, \( k_2 \) is calculated for \( \alpha = 0 \) for different compact objects.

### 5 Discussion

In this paper, we have presented a technique to measure the TLN of a compact object when subjected to an external tidal field. Conversely, if the TLN is known, our method can be used to constrain the anisotropic stress of a compact star of a given mass and radius. The possible role of anisotropy vis-à-vis matter distribution of the star on the TLN has been analyzed. Our investigation clearly shows that TLN is influenced by anisotropic stress. It remains to be seen whether such impacts can be observationally realized. In our model, we focused on the quadrupolar ‘even parity TLN \( k_2 \). However, one can also calculate higher-order TLN as well as magnetic TLN. Effects of other factors such as EOS, electromagnetic field etc. on the TLN needs further probe and will carried out elsewhere.

**Acknowledgements** The authors would like to express their gratitude to the anonymous referee for providing valuable suggestions. RS and SD gratefully acknowledge support from the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India, under its Visiting Research Associateship Programme.

**Data Availability Statement** The data underlying this article is available in the public domain as cited in the references.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permit-
References

1. N.K. Glendenning, *Compact stars: Nuclear Physics, Particle Physics and General Relativity* (Springer Science & Business Media, Berlin, 2012)
2. P. Haensel, A.Y. Potekhin, D.G. Yakovlev, *Neutron Stars I: Equation of State and Structure*, vol. 326 (Springer Science & Business Media, Berlin, 2007)
3. K. Chatziioannou, Gen. Relativ. Gravit. 52(11), 109 (2020). https://doi.org/10.1007/s10714-020-02754-3
4. T. Hinderer, B.D. Lackey, R.N. Lang, J.S. Read, Phys. Rev. D 81(12), 123016 (2010). https://doi.org/10.1103/physrevd.81.123016
5. P.C. Peters, J. Mathews, Phys. Rev. 131, 435 (1963). https://doi.org/10.1103/PhysRev.131.435. https://link.aps.org/doi/10.1103/PhysRev.131.435
6. P.C. Peters, Phys. Rev. 136, B1224 (1964). https://doi.org/10.1103/PhysRev.136.B1224. https://link.aps.org/doi/10.1103/PhysRev.136.B1224
7. L. Blanchet, Living Rev. Relativ. 17(1), 1 (2014)
8. K. Yagi, N. Yunes, Phys. Rev. D 88(2), 023009 (2013). https://doi.org/10.1103/physrevd.88.023009
9. D.H. Neumann, *BACHeR—THESIS Tidal Love Numbers and the I-Love-Relations of Second Family Compact Stars* (2019). https://theorie.ikp.physik.tu-darmstadt.de/ nhq/downloads/thesis/bachelor_neumann.pdf
10. E. Poisson, C.M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic*, 1st edn. (Cambridge University Press, Cambridge; New York, 2014)
11. V. Cardoso, E. Franzin, A. Maselli, P. Pani, G. Raposo, Phys. Rev. D 95(8), 084014 (2017). https://doi.org/10.1103/physrevd.95.084014
12. N. Sennett, T. Hinderer, J. Steinhoff, A. Buonanno, S. Ossokine, Phys. Rev. D 96(2), 024002 (2017). https://doi.org/10.1103/physrevd.96.024002
13. A. Maselli, P. Vani, V. Cardoso, T. Abdelsalhin, L. Gualtieri, V. Ferrari, Phys. Rev. Lett. 120(8), 081101 (2018). https://doi.org/10.1103/PhysRevLett.120.081101
14. T. Hinderer, Astrophys. J. 677(2), 1216–1220 (2008). https://doi.org/10.1086/533487
15. J. Aasi, B.P. Abbott, R. Abbott, T. Abbott, M.R. Abernathy, K. Ackley, C. Adams, T. Adams, P. Addesso et al., Class. Quantum Gravity 32(7), 074001 (2015). https://doi.org/10.1088/0264-9381/32/7/074001
16. F. Acernese, M. Agathos, K. Agatsuma, D. Aisa, N. Allemandou, A. Allocca, J. Amarni, P. Astone, G. Balestri, G. Ballardin et al., Class. Quantum Gravity 32(2), 024001 (2014). https://doi.org/10.1088/0264-9381/32/2/024001
17. B. Abbott, R. Abbott, T. Abbott, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, V. Adya et al., Phys. Rev. Lett. 119(16), 161101 (2017). https://doi.org/10.1103/PhysRevLett.119.161101
18. B.P. Abbott, R. Abbott, T.D. Abbott, S. Abraham, F. Acernese, K. Ackley, C. Adams, R.X. Adhikari, V.B. Adya, C. Affeldt et al., Astrophys. J. 892(1), L3 (2020). https://doi.org/10.3847/2041-8213/ab75f5
19. B. Margalit, B.D. Metzger, Astrophys. J. 850(2), L19 (2017). https://doi.org/10.3847/2041-8213/aa991c
20. A. Bauswein, O. Just, H.T. Janka, N. Stergioulas, Astrophys. J. 850(2), L34 (2017). https://doi.org/10.3847/2041-8213/aa9944
21. L. Rezzolla, E.R. Most, L.R. Weih, Astrophys. J. 852(2), L25 (2018). https://doi.org/10.3847/2041-8213/aa4041
22. M. Ruiz, S.L. Shapiro, A. Tsokaros, Phys. Rev. D 97(2), 021501 (2018). https://doi.org/10.1103/physrevd.97.021501
23. E. Annala, T. Gorda, A. Kurkela, A. Vuorinen, Phys. Rev. Lett. 120(17), 172703 (2018). https://doi.org/10.1103/PhysRevLett.120.172703
24. D. Radice, A. Berego, F. Zappa, S. Bernuzzi, Astrophys. J. 852(2), L29 (2018). https://doi.org/10.3847/2041-8213/aaa402
25. E.R. Most, L.R. Weih, L. Rezzolla, J. Schaffner-Bielich, Phys. Rev. Lett. 120(26), 261103 (2018). https://doi.org/10.1103/PhysRevLett.120.261103
26. I. Tews, J. Carlson, S. Gandolfi, S. Reddy, Astrophys. J. 860(2), 149 (2018). https://doi.org/10.3847/1538-4357/aac267
27. S. De, D. Finstad, J.M. Lattimer, D.A. Brown, E. Berger, C.M. Biwer, Phys. Rev. Lett. 121(9), 091102 (2018). https://doi.org/10.1103/PhysRevLett.121.091102
28. B. Abbott, R. Abbott, T. Abbott, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, V. Adya et al., Phys. Rev. Lett. 121(16), 161101 (2018). https://doi.org/10.1103/PhysRevLett.121.161101
29. S. Köppel, L. Bovard, L. Rezzolla, Astrophys. J. 872(1), L16 (2019). https://doi.org/10.3847/2041-8213/ab0210
30. E.E. Flanagan, T. Hinderer, Phys. Rev. D 77(2), 021502 (2008). https://link.aps.org/doi/10.1103/PhysRevD.77.021502
31. T. Damour, A. Nagar, Phys. Rev. D 80(8), 084035 (2009). https://doi.org/10.1103/physrevd.80.084035
32. T. Binnington, E. Poisson, Phys. Rev. D 80(8), 084018 (2009). https://doi.org/10.1103/physrevd.80.084018
33. P. Landry, E. Poisson, Phys. Rev. D 91(10), 104026 (2015). https://doi.org/10.1103/physrevd.91.104026
34. P. Landry, G. Lovelace, Phys. Rev. D 97(4), 043011 (2018). https://doi.org/10.1103/PhysRevD.97.043011
35. P. Landry, G. Lovelace, Phys. Rev. D 98(12), 124003 (2018). https://doi.org/10.1103/physrevd.98.124003
36. J. Vines, E.E. Flanagan, T. Hinderer, Phys. Rev. D 83(8), 084051 (2011). https://doi.org/10.1103/physrevd.83.084051
37. P. Pani, L. Gualtieri, T. Abdelsalhin, X. Jiménez-Forcada, Phys. Rev. D 98(12), 124023 (2018). https://doi.org/10.1103/physrevd.98.124023
38. K. Yagi, Phys. Rev. D 89(4), 043011 (2014). https://doi.org/10.1103/PhysRevD.89.043011
