A BLOWING-UP FORMULA FOR THE INTERSECTION COHOMOLOGY OF THE MODULI OF RANK 2 HIGGS BUNDLES OVER A CURVE WITH TRIVIAL DETERMINANT

SANG-BUM YOO

ABSTRACT. We prove that a blowing-up formula for the intersection cohomology of the moduli space of rank 2 Higgs bundles over a curve with trivial determinant holds. As an application, we derive the Poincaré polynomial of the intersection cohomology of the moduli space under a technical assumption.

1. INTRODUCTION

Let $X$ be a smooth complex projective curve of genus $g \geq 2$ and let $G$ be $\text{GL}(r, \mathbb{C})$ or $\text{SL}(r, \mathbb{C})$. Let $M^{d}_{\text{Dol}}(G)$ be the moduli space of $G$-Higgs bundles $(E, \phi)$ of rank $r$ and degree $d$ on $X$ (with fixed $\text{det} E$ and traceless $\phi$ in the case $G = \text{SL}(r, \mathbb{C})$) and let $M^{d}_{B}(G)$ be the character variety for $G$ defined by

$$M^{d}_{B}(G) = \{(A_{1}, B_{1}, \cdots, A_{g}, B_{g}) \in G \mid [A_{1}, B_{1}] \cdots [A_{g}, B_{g}] = e^{2\pi \sqrt{-1}d/r}I_{r}\}/G.$$ 

By the theory of harmonic bundles ([8], [39]), we have a homeomorphism $M^{d}_{\text{Dol}}(G) \cong M^{d}_{B}(G)$ as a part of the nonabelian Hodge theory. If $r$ and $d$ are relatively prime, these moduli spaces are smooth and their underlying differentiable manifold is hyperkähler. But the complex structures do not coincide under this homeomorphism.

Under the assumption that $r$ and $d$ are relatively prime, motivated by this fact, there have been several works calculating invariants of these moduli spaces on both sides over the last 30 years. The Poincaré polynomial of the ordinary cohomology is calculated, for $M^{d}_{\text{Dol}}(\text{SL}(2, \mathbb{C}))$ by N. Hitchin in [22], and for $M^{d}_{\text{Dol}}(\text{SL}(3, \mathbb{C}))$ by P. Gothen in [21]. The compactly supported Hodge polynomial and the compactly supported Poincaré polynomial for $M^{d}_{\text{Dol}}(\text{GL}(4, \mathbb{C}))$ can be obtained by the motivic calculation in [16]. By counting the number of points of these moduli spaces over finite fields with large characteristics, the compactly supported Poincaré polynomials for $M^{d}_{\text{Dol}}(\text{GL}(r, \mathbb{C}))$ and $M^{d}_{B}(\text{GL}(r, \mathbb{C}))$ are obtained in [37]. By using arithmetic methods, T. Hausel and F. Rodriguez-Villegas expresses the E-polynomial of $M^{d}_{B}(\text{GL}(r, \mathbb{C}))$ in terms of a simple generating function in [24]. By the same way, M. Mereb calculates the E-polynomial of $M^{d}_{B}(\text{SL}(2, \mathbb{C}))$ and expresses the E-polynomial of $M^{d}_{B}(\text{SL}(r, \mathbb{C}))$ in terms of a generating function in [32].

Without the assumption that $r$ and $d$ are relatively prime, there have been also some works calculating invariants of $M^{d}_{B}(\text{SL}(r, \mathbb{C}))$. For $g = 1, 2$ and any $d$, explicit formulas for the E-polynomials of $M^{d}_{B}(\text{SL}(2, \mathbb{C}))$ are obtained by a geometric technique in [30]. The E-polynomial of $M^{d}_{B}(\text{SL}(2, \mathbb{C}))$ is calculated, for $g = 3$ and any $d$ by a further geometric technique in [33], and for any $g$ and $d$ in [34].

2000 Mathematics Subject Classification. 14D20, 14D22, 14E15, 14F08, 14F43.

Key words and phrases. Higgs bundles, Kirwan’s algorithm, intersection cohomology, blowing-up formula.
When we deal with a singular variety $M^d_{\text{Dol}}(G)$ under the condition that $r$ and $d$ are not relatively prime, the intersection cohomology is a natural invariant. Our interest is focused on the intersection cohomology of $M := M^d_{\text{Dol}}(\text{SL}(2, \mathbb{C}))$.

For a quasi-projective variety $V$, $IH^i(V)$ and $\text{IC}^*(V)$ denote the $i$-th intersection cohomology of $V$ of the middle perversity and the complex of sheaves on $V$ whose hypercohomology is $IH^*(V)$ respectively. $IP_t(V)$ denotes the Poincaré polynomial of $IH^*(V)$ defined by

$$IP_t(V) = \sum_i \dim IH^i(V).$$

Recently, $IP_t(M)$ was calculated in [31] by using ways different from ours. First of all, the $E$-polynomial of the compactly supported intersection cohomology of $M$ was calculated in [13] over a smooth curve of genus 2 and in [31] over a smooth curve of genus $g \geq 2$. Then the author of [31] proved the purity of $IH^*(M)$ from the observation of the semiprojectivity of $M$. He used the purity of $IH^*(M)$ and the Poincaré duality for the intersection cohomology (Theorem 3.2) to calculate $IP_t(M)$.

From now on, $\text{GL}(n, \mathbb{C}), \text{SL}(n, \mathbb{C}), \text{PGL}(n, \mathbb{C}), \text{O}(n, \mathbb{C})$ and $\text{SO}(n, \mathbb{C})$ will be denoted by $\text{GL}(n), \text{SL}(n), \text{PGL}(n), \text{O}(n)$ and $\text{SO}(n)$ respectively for the simplicities of notations.

1.1. Main result. In this paper, we prove that a blowing-up formula for $IH^*(M)$ holds.

It is known that $M$ is a good quotient $R//\text{SL}(2)$ for some quasi-projective variety $R$ (Theorem 2.5, Theorem 2.6). $M$ is decomposed into

$$M = \bigcup (T^*/Z_2 - Z_2^g) \bigcup Z_2^g,$$

where $M^s$ denotes the stable locus of $M$ and $J := \text{Pic}^0(X)$. The singularity along the locus $Z_2^g$ is the quotient $\Upsilon^{-1}(0)/\text{SL}(2)$, where $\Upsilon^{-1}(0)$ is the affine cone over a reduced irreducible complete intersection of three quadrics in $\mathbb{P}(\mathbb{C}^g \otimes \text{sl}(2))$ and $\text{SL}(2)$ acts on $\mathbb{C}^g \otimes \text{sl}(2)$ as the adjoint representation. The singularity along the locus $T^*/Z_2 - Z_2^g$ is $\Psi^{-1}(0)/\mathbb{C}^*$, where $\Psi^{-1}(0)$ is the affine cone over a smooth quadric in $\mathbb{P}((\mathbb{C}^g)^{-1})$ and $\mathbb{C}^*$ acts on $(\mathbb{C}^g)^{-1}$ with weights $-2, 2, 2$ and $-2$. Let us consider the Kirwan’s algorithm consisting of three blowing-ups $K := R_2^s//\text{SL}(2) \rightarrow R_2^s//\text{SL}(2) \rightarrow R_1^s//\text{SL}(2) \rightarrow \mathbb{R} = M$ induced from the composition of blowing-ups $\pi_{R_1} : R_1 \rightarrow R$ along the locus $Z_2^g$ with the exceptional divisor $E_1$, $\pi_{R_2} : R_2 \rightarrow R_1^s$ along the strict transform $\Sigma$ of the locus of $R_1^s$ over $T^*/Z_2 - Z_2^g$ with the exceptional divisor $E_2$ and $R_3 \rightarrow R_2$ along the locus of points with stabilizers larger than the center $Z_2$ in $\text{SL}(2)$ (Section 4).

We also have local pictures of the Kirwan’s algorithm mentioned above. For any $x \in Z_2^g$, we have $\pi_{R_1}^{-1}(x) = \mathbb{P}\Upsilon^{-1}(0)$ which is a subset of $\mathbb{P}\text{Hom}(\text{sl}(2), \mathbb{C}^2)$ and $\pi_{R_1}^{-1}(x) \cap \Sigma$ is the locus of rank 1 matrices (Section 5). Thus the strict transform of $\mathbb{P}\Upsilon^{-1}(0)^{ss}//\text{PGL}(2)$ in the second blowing-up of the Kirwan’s algorithm is just the blowing-up

$$B\mathbb{P}\text{Hom}_1(\mathbb{P}\Upsilon^{-1}(0)^{ss}//\text{PGL}(2) \rightarrow \mathbb{P}\Upsilon^{-1}(0)^{ss}//\text{PGL}(2)$$

along the image of the locus of rank 1 matrices in $\mathbb{P}\Upsilon^{-1}(0)^{ss}//\text{PGL}(2)$.

In this setup, we have the following main result.

**Theorem 1.1 (Theorem 6.11).** Let $I_{2g-3}$ be the incidence variety given by

$$I_{2g-3} = \{(p, H) \in \mathbb{P}^{2g-3} \times \mathbb{P}^{2g-3}[p \in H]\}.$$

Then we have
(1) \( \dim IH^i(R_1^{ss}/\SL(2)) = \dim IH^i(M) \)
+ \(2g^2 \dim IH^i(\PP^{*-3}/\PGL(2)) - 2g^2 \dim IH^i(\PP^{*-3}/\PGL(2)) \)
for all \(i \geq 0\).

(2) \( \dim H^i(R_2^{ss}/\SL(2)) = \dim IH^i(R_1^{ss}/\SL(2)) \)
+ \( \sum_{p+q=i} \dim [H^p(T*J) \otimes H^q(I_{2g-3})] \)
for all \(i \geq 0\), where \(t(q) = q - 2\) for \(q \leq \dim I_{2g-3} = 4g - 7\) and \(t(q) = q\) otherwise, where \(\alpha : T*J \to T*J\) is the blowing-up along \(\ZZ^2g\).

(3) \( \dim IH^i(Bl_{\Hom_1}\PP^{*-3}/\SL(2)) = \dim IH^i(\PP^{*-3}/\SL(2)) \)
+ \( \sum_{p+q=i} \dim [H^p(\PP^{2g-1}) \otimes H^q(I_{2g-3})] \)
for all \(i \geq 0\), where \(t(q) = q - 2\) for \(q \leq \dim I_{2g-3} = 4g - 7\) and \(t(q) = q\) otherwise.

It is an essential process to apply this blowing-up formula to calculate \( IP_t(M) \).

1.2. Method of proof of Theorem 1.1 We follow the same steps as in the proof of [27] Proposition 2.1, but we give a proof in each step because our setup is different from that of [27]. We start with the following formulas coming from the decomposition theorem (Proposition 3.4-(1)) and the same argument as in the proof of [27] Lemma 2.8:

\[
\dim IH^i(R_1^{ss}/\SL(2)) = \dim IH^i(M) + \dim IH^i(U_1) - \dim IH^i(U_1),
\]
\[
\dim IH^i(R_2^{ss}/\SL(2)) = \dim IH^i(R_1^{ss}/\SL(2)) + \dim IH^i(U_2) - \dim IH^i(U_2)
\]
and

\[
\dim IH^i(Bl_{\Hom_1}\PP^{*-3}/\SL(2)) = \dim IH^i(\PP^{*-3}/\SL(2)) + \dim IH^i(\UU) - \dim IH^i(U),
\]
where \(U_1\) is a disjoint union of sufficiently small open neighborhoods of each point of \(\ZZ^2g\) in \(M\), \(\UU_1\) is the inverse image of the first blowing-up, \(U_2\) is an open neighborhood of the strict transform of \(T*J/\ZZ^2\) in \(R_2^{ss}/\SL(2)\), \(\UU_2\) is the inverse image of the second blowing-up, \(U\) is an open neighborhood of the locus of rank 1 matrices in \(\PP^{*-3}/\PGL(2)\) and \(\UU\) is the inverse image of the blowing-up map \(Bl_{\Hom_1}\PP^{*-3}/\PGL(2) \to \PP^{*-3}/\PGL(2)\). By Proposition 6.2 we can see that \(U_1, U_2, \UU_2, U\) and \(\UU\) are analytically isomorphic to relevant normal cones respectively. By Section 4 and Lemma 6.3 these relevant normal cones are described as free \(\ZZ^2\)-quotients of nice fibrations with concrete expressions of bases and fibers. By the calculations of the intersection cohomologies of fibers (Lemma 6.6 and Lemma 6.8) and applying the Leray spectral sequences (Proposition 3.4-(2)) of intersection cohomologies associated to these fibrations, we complete the proof.

1.3. Towards a formula for the Poincaré polynomial of \( IH^*(M) \). For a topological space \(W\) on which a reductive group \(G\) acts, \(H^*_G(W)\) and \(P^*_G(W)\) denote the \(i\)-th equivariant cohomology of \(W\) and the Poincaré series of \(H^*_G(W)\) defined by

\[
P^*_G(W) = \sum_i \dim H^*_G(W).
\]

We start with the formula of \(P^*_{\SL(2)}(\RR)\) that comes from that of [12]. Then we use the following conjectural formulas.
Conjecture 1.2 (Conjecture 7.12, Proposition 7.14). 

1. \( P_t^{SL(2)}(R_{1\,ss}) = P_t^{SL(2)}(R) + 2g(P_t^{SL(2)}(\mathbb{P}Y^{-1}(0)) - P_t^{BSL(2)}) \).

2. \( P_t^{SL(2)}(R_2) = P_t^{SL(2)}(R_{1\,ss}) + P_t^{SL(2)}(E_{2\,ss}) - P_t^{SL(2)}(\Sigma) \) under some technical assumptions, where \( E_{2\,ss} = E_2 \cap R_{2\,ss} \).

We use this conjectural blowing-up formula for the equivariant cohomology to get \( P_t^{SL(2)}(R_2) \) from \( P_t^{SL(2)}(R) \). Since \( R_2/SL(2) \) has at worst orbifold singularities (Section 4), \( P_t^{SL(2)}(R_2) = P_t(R_2/SL(2)) \) (Section 7). Now we use the blowing-up formula for the intersection cohomology (Theorem 1.1) to get \( IP_t(M) \) from \( P_t(R_2/SL(2)) \).

Theorem 1.3 (Theorem 7.26). Assume that Conjecture 7.8, Conjecture 7.12 and Conjecture 7.13 hold. Then

\[
IP_t(M) = \frac{(1 + t^3)^{2g} - (1 + t)^{2g}t^{2g+2}}{(1 - t^2)(1 - t^4)} - t^{4g-4} + \frac{t^{2g+2}(1 + t)^{2g}}{(1 - t^2)(1 - t^4)} + \frac{(1 - t)^{2g}t^{4g-4}}{4(1 + t^2)} + \frac{(1 + t)^{2g}t^{4g-4}}{2(1 - t^2)}
\]

\[+ \frac{2g}{t + 1} + \frac{1}{t^2 - 1} - \frac{1}{2} + (3 - 2g) \]

\[+ \frac{1}{2}(2^{2g} - 1)t^{4g-4}((1 + t)^{2g-2} + (1 - t)^{2g-2} - 2) \]

\[+ 2^{2g} \left[ \frac{1 - t^{12}}{1 - t^2} - \frac{1 - t^6}{1 - t^2} + \frac{1 - t^{6g}}{1 - t^2} \right] \]

\[+ \frac{2g}{t^2(1 - t^4)(1 - t^4)} \cdot \frac{1}{1 - t^2} + \frac{2g}{1 - t^4} \cdot \frac{1 - t^2}{1 - t^4} \]

\[+ \frac{1}{2}(1 + t)^{2g} + (1 - t)^{2g} \]

\[+ 2^{2g} \left( \frac{1 - t^{4g-4}}{1 - t^2} - 1 \right) \]

\[+ \frac{1}{2}(1 + t)^{2g} + (1 - t)^{2g} \]

\[+ 2^{2g} \left( \frac{1 - t^{4g}}{1 - t^2} - 1 \right) \]

\[- \frac{t^2}{(1 - t^4)} \frac{1}{2}((1 + t)^{2g} - (1 - t)^{2g}) \]

\[- \frac{1}{2}(1 + t)^{2g} + (1 - t)^{2g} + 2^{2g} \left( \frac{1 - t^{4g}}{1 - t^2} - 1 \right) \frac{t^2(1 - t^{4g-4})(1 - t^{4g-6})}{(1 - t^2)(1 - t^4)} \]

\[- \frac{1}{2}((1 + t)^{2g} - (1 - t)^{2g}) \frac{t^4(1 - t^{4g-4})(1 - t^{4g-10})}{(1 - t^2)(1 - t^4)} + t^{4g-6} \]

\[- 2^{2g} \left( \frac{1 - t^{8g-8}}{1 - t^{4g}} - \frac{1 - t^{4g}}{1 - t^4} \right) \]

which is a polynomial with degree \( 6g - 6 \).

This conjectural formula for \( IP_t(M) \) coincides with that of [31].

Notations. Throughout this paper, \( X \) denotes a smooth complex projective curve of genus \( g \geq 2 \) and \( K_X \) the canonical bundle of \( X \).
2. Higgs bundles

In this section, we introduce two kinds of constructions of the moduli space of Higgs bundles on $X$. For details, see [22], [40] and [41].

2.1. Simpson’s construction. An $SL(2)$-Higgs bundle on $X$ is a pair of a rank 2 vector bundle $E$ with trivial determinant on $X$ and a section $\phi \in H^0(X, \text{End}_0(E) \otimes K_X)$, where $\text{End}(E)$ denotes the bundle of endomorphisms of $E$ and $\text{End}_0(E)$ the subbundle of traceless endomorphisms of $\text{End}(E)$. We must impose a notion of stability to construct a separated moduli space.

Definition 2.1 ([22], [40]). An $SL(2)$-Higgs bundle $(E, \phi)$ on $X$ is stable (respectively, semistable) if for any $\phi$-invariant line subbundle $F$ of $E$, we have

$$\deg(F) < 0$$

(respectively, $\leq$).

Let $N$ be a sufficiently large integer and $p = 2N + 2(1 - g)$. We list C.T. Simpson’s results to construct a moduli space of $SL(2)$-Higgs bundles.

Theorem 2.2 (Theorem 3.8 of [40]). There is a quasi-projective scheme $Q$ representing the moduli functor which parametrizes the isomorphism classes of triples $(E, \phi, \alpha)$ where $(E, \phi)$ is a semistable $SL(2)$-Higgs bundle and $\alpha$ is an isomorphism $\alpha : \mathbb{C}^p \to H^0(X, E \otimes \mathcal{O}_X(N))$.

Theorem 2.3 (Theorem 4.10 of [40]). Fix $x \in X$. Let $\hat{Q}$ be the frame bundle at $x$ of the universal object restricted to $x$. Then the action of $\text{GL}(p)$ lifts to $\hat{Q}$ and $SL(2)$ acts on the fibers of $\hat{Q} \to Q$ in an obvious fashion. Every point of $\hat{Q}$ is stable with respect to the free action of $\text{GL}(p)$ and

$$R = \hat{Q}/\text{GL}(p)$$

represents the moduli functor which parametrizes triples $(E, \phi, \beta)$ where $(E, \phi)$ is a semistable $SL(2)$-Higgs bundle and $\beta$ is an isomorphism $\beta : E|_x \to \mathbb{C}^2$.

Theorem 2.4 (Theorem 4.10 of [40]). Every point in $R$ is semistable with respect to the action of $SL(2)$. The closed orbits in $R$ correspond to polystable $SL(2)$-Higgs bundles, i.e. $(E, \phi)$ is stable or $(E, \phi) = (L, \psi) \oplus (L^{-1}, -\psi)$ for $L \in \text{Pic}^0(X)$ and $\psi \in H^0(K_X)$. The set $R^*$ of stable points with respect to the action of $SL(2)$ is exactly the locus of stable $SL(2)$-Higgs bundles.

Theorem 2.5 (Theorem 4.10 of [40]). The good quotient $R/\!/SL(2)$ is $M$.

Theorem 2.6 (Theorem 11.1 of [41]). $R$ and $M$ are both irreducible normal quasi-projective varieties.

2.2. Hitchin’s construction. Let $E$ be a complex Hermitian vector bundle of rank 2 and degree 0 on $X$. Let $\mathcal{A}$ be the space of traceless connections on $E$ compatible with the Hermitian metric. $\mathcal{A}$ can be identified with the space of holomorphic structures on $E$ with trivial determinant. Let

$$B = \{(A, \phi) \in \mathcal{A} \times \Omega^0(\text{End}_0(E) \otimes K_X) : d''_A \phi = 0\}.$$

Let $\mathcal{G}$ (respectively, $\mathcal{G}_2$) be the gauge group of $E$ with structure group $SU(2)$ (respectively, $SL(2)$). These groups act on $B$ by

$$g \cdot (A, \phi) = (g^{-1} A'' g + g^* A'(g^*)^{-1} + g^{-1} d''_A g - (d'g^*)(g^*)^{-1}, g^{-1} \phi g),$$

where $A''$ and $A'$ denote the $(0, 1)$ and $(1, 0)$ parts of $A$ respectively.
The cotangent bundle $T^*A \cong A \times \Omega^0(\text{End}_0(E) \otimes K_A)$ admits a hyperkähler structure preserved by the action of $\mathcal{G}$ with the moment maps for this action

$$\mu_1 = F_A + [\phi, \phi^*]$$
$$\mu_2 = -i(d_A^t \phi + d_A^t \phi^*)$$
$$\mu_3 = -d_A^t \phi + d_A^t \phi^*.$$

$\mu_C = \mu_2 + i\mu_3 = -2id_A^t \phi$ is the complex moment map. Then

$$\mathcal{B} = \mu_2^{-1}(0) \cap \mu_3^{-1}(0) = \mu_C^{-1}(0).$$

Consider the hyperkähler quotient

$$\mathcal{M} := T^*A//\mathcal{G} = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/\mathcal{G} = \mu_1^{-1}(0) \cap \mathcal{B}/\mathcal{G}.$$ 

Let $\mathcal{B}_{ss} = \{(A, \phi) \in \mathcal{B} : ((E, d_A^t), \phi) \text{ is semistable}\}.$

**Theorem 2.7** (Theorem 2.1 and Theorem 4.3 of [22], Theorem 1 and Proposition 3.3 of [38]).

$$\mathcal{M} \cong \mathcal{B}_{ss}/\mathcal{G}_C \cong \mathcal{M}.$$

### 3. Intersection cohomology theory

In this section, we introduce some basics on the intersection cohomology ([18], [19]) and the equivariant intersection cohomology ([4], [17]) of a quasi-projective complex variety. Let $V$ be a quasi-projective complex variety of pure dimension $n$ throughout this section.

**3.1. Intersection cohomology.** It is well-known that $V$ has a Whitney stratification

$$V = V_n \supseteq V_{n-1} \supseteq \cdots \supseteq V_0$$

which is embedded into a topological pseudomanifold of dimension $2n$ with filtration

$$W_{2n} \supseteq W_{2n-1} \supseteq \cdots \supseteq W_0,$$

where $V_j$ are closed subvarieties such that $V_j - V_{j-1}$ is either empty or a nonsingular quasi-projective variety of pure dimension $j$ and $W_{2k} = W_{2k+1} = V_k.$

Let $\bar{p} = (p_2, p_3, \ldots, p_{2n})$ be a perversity. For a triangulation $T$ of $V$, $(C^T_i(V), \partial)$ denotes the chain complex of chains with respect to $T$ with coefficients in $\mathbb{Q}$. We define $I^\bar{p}C^T_i(V)$ to be the subspace of $C^T_i(V)$ consisting of those chains $\xi$ such that

$$\dim_{\mathbb{R}}(|\xi| \cap V_{n-c}) \leq i - 2c + p_{2c}$$

and

$$\dim_{\mathbb{R}}(|\partial \xi| \cap V_{n-c}) \leq i - 1 - 2c + p_{2c}.$$ 

Let $IC_i^\beta(V) = \lim_{T} I^\bar{p}C^T_i(V)$. Then $(IC_i^\beta(V), \partial)$ is a chain complex. The $i$-th intersection homology of $V$ of perversity $\bar{p}$, denoted by $IH_i^\beta(V)$, is the $i$-th homology group of the chain complex $(IC_i^\beta(V), \partial)$. The $i$-th intersection cohomology of $V$ of perversity $\bar{p}$, denoted by $IH^i_{\text{cl},\beta}(V)$, is the $i$-th homology group of the chain complex $(IC_i^\beta(V) \vee, \partial^\vee)$.

When we consider a chain complex $(IC_i^\beta(V), \partial)$ of chains with closed support instead of usual chains, we can define the $i$-th intersection homology with closed support (respectively, intersection cohomology with closed support) of $V$ of perversity $\bar{p}$, denoted by $IH^i_{\text{cl},\beta}(V)$ (respectively, $IH^i_{\text{cl},\beta}(V)$).
There is an alternative way to define the intersection homology and cohomology with closed support. Let $IC^i_p(V)$ be the sheaf given by $U \mapsto IC^{i,p}_1(U)$ for each open subset $U$ of $V$. Then $IC^i_p(V)$ is a complex of sheaves as an object in the bounded derived category $D^b(V)$. Then we have $IH^{i-p}(V) = H^{-i}IC^{i,p}(V)$ and $IH^{i,q}_\text{cl}(V) = H^{i-2\dim(V)}IC^{i,p}(V)$, where $H^i(A^*)$ is the $i$-th hypercohomology of a complex of sheaves $A^*$.

When $\bar{p}$ is the middle perversity $\bar{m}$, $IH^{\bar{m}}_i(V)$, $IH^{\bar{m}}_m(V)$, $IH^{d,\bar{m}}_i(V)$, $IH^{d,\bar{m}}_m(V)$ and $IC^\ast_{\bar{m}}(V)$ are denoted by $IH^i(V)$, $IH^q(V)$, $IH^{d}(V)$, $IH^{d}_{G}(V)$ and $IC^\ast(V)$ respectively.

3.2. Equivariant intersection cohomology. Assume that a compact connected algebraic group $G$ acts on $V$ algebraically. For the universal principal bundle $EG \to BG$, we have the quotient $V \times_G EG$ of $V \times EG$ by the diagonal action of $G$. Let us consider the following diagram

$$V \xleftarrow{p} V \times EG \xrightarrow{q} V \times_G EG.$$  

**Definition 3.1** (2.1.3 and 2.7.2 in [4]). The *equivariant derived category* of $V$, denoted by $D^b_G(V)$, is defined as follows:

1. An object is a triple $(F_V, \bar{F}, \beta)$, where $F_V \in D^b(V)$, $\bar{F} \in D^b(V \times_G EG)$ and $\beta : p^*(F_V) \to q^*(\bar{F})$ is an isomorphism in $D^b(V \times EG)$.
2. A morphism $\alpha : (F_V, \bar{F}, \beta) \to (G_V, \bar{G}, \gamma)$ is a pair $\alpha = (\alpha_V, \bar{\alpha})$, where $\alpha_V : F_V \to G_V$ and $\bar{\alpha} : \bar{F} \to \bar{G}$ such that $\bar{\alpha} \circ p^*(\alpha_V) = q^*(\bar{\alpha}) \circ \beta$.

$IC^\ast_{G,\bar{p}}(V)$ (respectively, $QG^\ast_V$) denotes $(IC^\ast_p(V), IC^\ast_G(V \times_G EG), \beta)$ (respectively, $(QV, QV \times_G EG, id)$) as an object of $D^b_G(V)$. The $i$-th equivariant cohomology of $V$ can be obtained by $IH^i_G(V) = H^{-i}(IC^\ast_p(V \times_G EG))$. The *equivariant intersection cohomology* of $V$ of perversity $\bar{p}$, denoted by $IH^i_{G,\bar{p}}(V)$, is defined by $IH^i_{G,\bar{p}}(V) := H^{-i}(IC^\ast_p(V \times_G EG))$.

When $\bar{p}$ is the middle perversity $\bar{m}$, $IH^i_{G,\bar{m}}(V)$ and $IC^\ast_{G,\bar{m}}(V)$ are denoted by $IH^i_G(V)$ and $IC^\ast_G(V)$ respectively.

The equivariant cohomology and the equivariant intersection cohomology can be described as a limit of a projective limit system coming from a sequence of finite dimensional submanifolds of $EG$. Let us consider a sequence of finite dimensional submanifolds $EG_0 \subset EG_1 \subset \cdots \subset EG_n \subset \cdots$ of $EG$, where $G$ acts on all of $EG_n$ freely, $EG_n$ are $n$-acyclic, $EG_n \subset EG_{n+1}$ is an embedding of a submanifold, $\dim EG_n < \dim EG_{n+1}$ and $EG = \bigcup_n EG_n$. Since $G$ is connected, such a sequence exists by [4] Lemma 12.4.2]. Then we have a sequence of finite dimensional subvarieties $V \times_G EG_0 \subset V \times_G EG_1 \subset \cdots \subset V \times_G EG_n \subset \cdots$ of $V \times_G EG$. Hence we have $H^*_G(V) = \lim_n H^*(V \times_G EG_n)$ and $IH^*_{G,\bar{p}}(V) = \lim_n IH^i_{G,\bar{p}}(V \times_G EG_n)$.

3.3. The generalized Poincaré duality and the decomposition theorem. In this subsection, we state two important theorems. One is the generalized Poincaré duality and the other is the decomposition theorem.

**Theorem 3.2** (The generalized Poincaré duality). If $\bar{p} + \bar{q} = \bar{i}$, then there is a non-degenerate bilinear form

$$IH^\bar{p}_i(V) \times IH^\bar{q}_{2\dim(V) - i}(V) \to \mathbb{Q}.$$
Theorem 3.3 (The decomposition theorem). Suppose that \( \varphi : W \to V \) is a projective morphism of quasi-projective varieties. Then there exist closed subvarieties \( V_\alpha \) of \( V \), local systems \( L_\alpha \) on the non-singular parts \( (V_\alpha)_{\text{nonsing}} \) of \( V_\alpha \) and integers \( l_\alpha \) such that there is an isomorphism
\[
R\varphi_* \mathbf{IC}^\bullet(W) \cong \bigoplus_{\alpha} \mathbf{IC}^\bullet(V_\alpha, L_\alpha)[l_\alpha]
\]
in the derived category \( D^b(V) \), where \( \mathbf{IC}^\bullet(V_\alpha, L_\alpha) \) is the complex of sheaves of intersection chains with coefficients in \( L_\alpha \).

There are three special important consequences of the decomposition theorem.

Proposition 3.4. 
1. Suppose that \( \varphi : W \to V \) is a resolution of singularities. Then \( \mathbf{IC}^\bullet(V) \) (respectively, \( IH^*(V) \)) is a direct summand of \( R\varphi_* \mathbf{IC}^\bullet(W) \) (respectively, \( IH^*(W) \)).
2. Suppose that \( \varphi : W \to V \) is a projective morphism which is topologically a fibration whose fiber is a projective variety \( F \). Then there is a Leray spectral sequence \( E_2^{ij} \) converging to \( IH^{i+j}(W) \) with \( E_2 \) term \( E_2^{ij} = IH^i(V, IH^j(F)) \), where \( IH^j(F) \) denotes the local system \( \mathcal{L} \) on \( V \) with stalk \( \mathcal{L}_x = IH^j(\varphi^{-1}(x)) \cong IH^j(F) \). The decomposition theorem for \( \varphi \) is equivalent to the degeneration of \( E_2 \) at the \( E_2 \) term.
3. Suppose that \( \varphi : W \to V \) is a \( G \)-equivariant resolution of singularities. Then \( \mathbf{IC}^\bullet_G(W) \) (respectively, \( IH^*_G(V) \)) is a direct summand of \( R\varphi_* \mathbf{IC}^\bullet_G(W) \) (respectively, \( IH^*_G(W) \)).

Proof. 
1. Applying Theorem 3.3 to \( \varphi \) and to the shifted constant sheaf \( \mathbb{Q}_W[\dim W] \), we get the result. The details of the proof can be found in [10, Corollary 5.4.11].
2. The statement is from [29, Proposition 8.4.5].
3. We know that \( \mathbf{IC}^\bullet_G(V) = (\mathbf{IC}^\bullet(V), \mathbf{IC}^*(V \times_G EG), \alpha) \) and \( \mathbf{IC}^\bullet_G(W) = (\mathbf{IC}^\bullet(W), \mathbf{IC}^*(W \times_G EG), \beta) \). It follows from item (1) that \( \mathbf{IC}^\bullet(W) \) is a direct summand of \( R\varphi_* \mathbf{IC}^\bullet(W) \) and that \( \mathbf{IC}^\bullet(V \times G EG) \) is a direct summand of \( R\varphi_* \mathbf{IC}^\bullet(W \times G EG) \) for all \( n \). Since \( \mathbf{IC}^*(V \times G EG) = \lim_n \mathbf{IC}^*(V \times_G EG_n) \) and \( \mathbf{IC}^*(W \times G EG) = \lim_n \mathbf{IC}^*(W \times_G EG_n) \), \( \mathbf{IC}^*(V \times G EG) \) is a direct summand of \( R\varphi_* \mathbf{IC}^*(W \times G EG) \).

Let \( i : \mathbf{IC}^\bullet(V) \to R\varphi_* \mathbf{IC}^\bullet(W) \) and \( \tilde{i} : \mathbf{IC}^\bullet(V \times G EG) \to R\varphi_* \mathbf{IC}^\bullet(W \times G EG) \) be the inclusions from the decomposition theorem. It is easy to see that the following diagram
\[
\begin{array}{ccc}
p^*_V \mathbf{IC}^\bullet(V) & \longrightarrow & q^*_V \mathbf{IC}^\bullet(V \times G EG) \\
p^*(i) \downarrow & & \downarrow q^*(i) \\
R\varphi_* p^*_V \mathbf{IC}^\bullet(W) & \cong & R\varphi_* q^*_V \mathbf{IC}^\bullet(W \times G EG)
\end{array}
\]
commutes, where \( p_V : V \times EG \to V \) (respectively, \( p_W : W \times EG \to W \)) is the projection onto \( V \) (respectively, \( W \)) and \( q_V : V \times EG \to V \times G EG \) (respectively, \( q_W : W \times EG \to W \times G EG \)) is the quotient.

\[\square\]

4. Kirwan’s desingularization of \( M \)

In this section, we briefly explain how \( M \) can be desingularized by three blowing-ups by the Kirwan’s algorithm introduced in [26]. For details, see [28] and [36].

We first consider the loci of type (ii) of \( (L, 0) \oplus (L, 0) \) with \( L \cong L^{-1} \) in \( M \setminus M^s \) and in \( R \setminus R^s \), where \( R^s \) is the stable locus of \( R \). The loci of type (ii) in \( M \) and in \( R \) are both isomorphic to the set
of $\mathbb{Z}_2$-fixed points $\mathbb{Z}_2^{2g}$ in $J := \text{Pic}^0(X)$ by the involution $L \mapsto L^{-1}$. The singularity of the locus $\mathbb{Z}_2^{2g}$ of type (i) in $M$ is the quotient
\[ \Upsilon^{-1}(0) / \text{SL}(2) \]
where $\Upsilon : [H^0(K_X) \oplus H^1(O_X)] \otimes \text{sl}(2) \to H^1(K_X) \otimes \text{sl}(2)$ is the quadratic map given by the Lie bracket of $\text{sl}(2)$ coupled with the perfect pairing $H^0(K_X) \oplus H^1(O_X) \to H^1(K_X)$ and the $\text{SL}(2)$-action on $\Upsilon^{-1}(0)$ is induced from the adjoint representation $\text{SL}(2) \to \text{Aut}(\text{sl}(2))$.

Next we consider the loci of type (iii) of $R$ of type (i). Let $R_1$ be the blowing-up of $\mathbb{R}$ along the locus $\mathbb{Z}_2^{2g}$ of type (i). Let $R_2$ be the blowing-up of $\mathbb{R}_1^{s\text{ss}}$ along the strict transform $\Sigma$ of the locus of type (iii), where $\mathbb{R}_1^{s\text{ss}}$ is the locus of semistable points in $R_1$. Let $R_2^{s\text{ss}}$ (respectively, $R_2^s$) be the locus of semistable (respectively, stable) points in $R_2$. Then it follows from the same argument as in [36, Claim 1.8.10] that
\[ \begin{align*}
(a) \quad & R_2^{s\text{ss}} = R_3^s, \\
(b) \quad & R_2^s \text{ smooth.}
\end{align*} \]

In particular, $R_2^s / \text{SL}(2)$ has at worst orbifold singularities. When $g = 2$, this is smooth. When $g \geq 3$, we blow up $R_2^s$ along the locus of points with stabilizers larger than the center $\mathbb{Z}_2$ of $\text{SL}(2)$ to obtain a variety $R_3$ such that the orbit space $K := R_3^s / \text{SL}(2)$ is a smooth variety obtained by blowing up $M$ along $\mathbb{Z}_2^{2g}$, along the strict transform of $T^*J / \mathbb{Z}_2 - \mathbb{Z}_2^{2g}$ and along a nonsingular subvariety contained in the strict transform of the exceptional divisor of the first blowing-up. $K$ is called the Kirwan’s desingularization of $M$.

Throughout this paper, $\pi_{R_1} : R_1 \to \mathbb{R}$ (respectively, $\pi_{R_2} : R_2 \to R_3^{s\text{ss}}$) denotes the first blowing-up map (respectively, the second blowing-up map). $\pi_{R_1} : R_1^{s\text{ss}} / \text{SL}(2) \to \mathbb{R} / \text{SL}(2)$ and $\pi_{R_2} : R_2^{s\text{ss}} / \text{SL}(2) \to R_3^{s\text{ss}} / \text{SL}(2)$ denote maps induced from $\pi_{R_1}$ and $\pi_{R_2}$ respectively.

5. Local pictures in Kirwan’s algorithm on $\mathbb{R}$

In this section, we list local pictures that appear in Kirwan’s algorithm on $\mathbb{R}$ for later use. For details, see [36, 1.6 and 1.7].

We first observe that $\pi_{R_1}^{-1}(x) = \mathbb{P} \Upsilon^{-1}(0)$ for any $x \in \mathbb{Z}_2^{2g}$. We identify $\mathbb{H}^g$ with $T_x(T^*J) = H^1(O_X) \oplus H^0(K_X)$ for any $x \in T^*J$, where $\mathbb{H}$ is the division algebra of quaternions. Since the adjoint representation gives an identification $\text{PGL}(2) \cong \text{SO}(\text{sl}(2))$, $\text{PGL}(2)$ acts on both $\Upsilon^{-1}(0)$.
and \( \mathbb{P} \Omega^{-1}(0) \). Since \( \text{PGL}(2) = \text{SL}(2)/\{\pm \text{id}\} \) and the action of \( \{\pm \text{id}\} \) on \( \Omega^{-1}(0) \) and \( \mathbb{P} \Omega^{-1}(0) \) are trivial,

\[
\Omega^{-1}(0)/\text{SL}(2) = \Omega^{-1}(0)/\text{PGL}(2) \text{ and } \mathbb{P} \Omega^{-1}(0)^{ss}/\text{SL}(2) = \mathbb{P} \Omega^{-1}(0)^{ss}/\text{PGL}(2).
\]

We have an explicit description of semistable points of \( \mathbb{P} \Omega^{-1}(0) \) with respect to the \( \text{PGL}(2) \)-action as following.

**Proposition 5.1** (Proposition 1.6.2 of [36]). A point \( [\varphi] \in \mathbb{P} \Omega^{-1}(0) \) is \( \text{PGL}(2) \)-semistable if and only if:

\[
\text{rk} \varphi \begin{cases} 
\geq 2, & \text{or} \\
= 1 & \text{and } [\varphi] \in \text{PGL}(2) \cdot \mathbb{P} \{ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} | \lambda \in \mathbb{H}^g \setminus \{0\} \}.
\end{cases}
\]

Let \( \text{Hom}^\omega(sl(2), \mathbb{H}^g) := \{ \varphi : sl(2) \to \mathbb{H}^g|\varphi^* \omega = 0 \} \), where \( \omega \) is the Serre duality pairing on \( \mathbb{H}^g \). Let \( (m, n) = 4\text{Tr}(mn) \) be the killing form on \( sl(2) \). The killing form gives isomorphisms

\[
\mathbb{H}^g \otimes sl(2) \cong \text{Hom}(sl(2), \mathbb{H}^g) \text{ and } sl(2) \cong \wedge^2 sl(2)^\vee.
\]

By the above identification, \( \Omega : \text{Hom}(sl(2), \mathbb{H}^g) \to \wedge^2 sl(2)^\vee \) is given by \( \varphi \mapsto \varphi^* \omega \). Then we have

\[
\Omega^{-1}(0) = \text{Hom}^\omega(sl(2), \mathbb{H}^g).
\]

Let

\[
\text{Hom}_k(sl(2), \mathbb{H}^g) := \{ \varphi \in \text{Hom}(sl(2), \mathbb{H}^g)|\text{rk} \varphi \leq k \}
\]

and

\[
\text{Hom}_k^\omega(sl(2), \mathbb{H}^g) := \text{Hom}_k(sl(2), \mathbb{H}^g) \cap \text{Hom}^\omega(sl(2), \mathbb{H}^g).
\]

We have a description of points of \( E_1 \cap \Sigma \) as following.

**Proposition 5.2** (Lemma 1.7.5 of [36]). Let \( x \in \mathbb{Z}_2^{2g} \). Then

\[
\pi_{R_1}^{-1}(x) \cap \Sigma = \mathbb{P} \text{Hom}_1(sl(2), \mathbb{H}^g)^{ss},
\]

where \( \mathbb{P} \text{Hom}_1(sl(2), \mathbb{H}^g)^{ss} \) denotes the set of semistable points of \( \mathbb{P} \text{Hom}_1(sl(2), \mathbb{H}^g) \) with respect to the \( \text{PGL}(2) \)-action.

Assume that \( \varphi \in \text{Hom}_1(sl(2), \mathbb{H}^g) \). Since the Serre duality pairing is skew-symmetric, we can choose bases \( \{e_1, \cdots, e_{2g}\} \) of \( \mathbb{H}^g \) and \( \{v_1, v_2, v_3\} \) of \( sl(2) \) such that \( \varphi = e_1 \otimes v_1 \) and so that

\[
< e_i, e_j > = \begin{cases} 
1 & \text{if } i = 2q - 1, j = 2q, q = 1, \cdots, g, \\
-1 & \text{if } i = 2q, j = 2q - 1, q = 1, \cdots, g, \\
0 & \text{otherwise}.
\end{cases}
\]

Every element in \( \text{Hom}(sl(2), \mathbb{H}^g) \) can be written as \( \sum_{i,j} Z_{ij} e_i \otimes v_j \). Then we have a description of the normal cone \( C_{\mathbb{P} \text{Hom}_1(sl(2), \mathbb{H}^g)} \mathbb{P} \Omega^{-1}(0) \) to \( \mathbb{P} \text{Hom}_1(sl(2), \mathbb{H}^g) \) in \( \mathbb{P} \Omega^{-1}(0) \).

**Proposition 5.3.** Let \( [\varphi] \in \mathbb{P} \text{Hom}_1(sl(2), \mathbb{H}^g) \) and let \( \omega \varphi \) be the bilinear form induced by \( \omega \) on \( \text{im} \varphi^\perp/\text{im} \varphi \). There is a \( \text{Stab}([\varphi]) \)-equivariant isomorphism

\[
(C_{\mathbb{P} \text{Hom}_1(sl(2), \mathbb{H}^g)} \mathbb{P} \Omega^{-1}(0))_{|[\varphi]} \cong \text{Hom}^{\omega \varphi}(\ker \varphi, \text{im} \varphi^\perp/\text{im} \varphi)
\]

where

\[
\text{Hom}^{\omega \varphi}(\ker \varphi, \text{im} \varphi^\perp/\text{im} \varphi) = \{ \chi \in \text{Hom}(\ker \varphi, \text{im} \varphi^\perp/\text{im} \varphi)|\chi^* \omega \varphi = 0 \}\]
Proof. Following the idea of proof of [36, Lemma 1.7.13], both sides are defined by the equation
\[
\sum_{2 \leq q \leq g} (Z_{2q-1,2}Z_{2q,3} - Z_{2q,2}Z_{2q-1,3}) = 0.
\]
under the choice of basis as above.

We now explain how \textup{Stab}([\varphi]) acts on \((C_{\textup{PHom}_1(sl(2),\mathbb{H}^g)}\mathbb{P}Y^{-1}(0))[\varphi]\). If we add the condition that
\[
(v_1, v_i) = -\delta_{1i}, \\
(v_j, v_j) = 0, \quad j = 2, 3, \\
(v_2, v_3) = 1,
\]
and \(v_1 \wedge v_2 \wedge v_3\) is the volume form, where \(\wedge\) corresponds to the Lie bracket in \(sl(2)\), then \textup{Stab}([\varphi]) = O(ker \varphi) = O(2) is generated by
\[
\{\theta_\lambda := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} | \lambda \in \mathbb{C}^*\} \text{ and } \tau := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]
as a subgroup of \(SO(sl(2))\). \(O(2)\) can be also realized as the subgroup of \(PGL(2)\) generated by
\[
SO(2) = \{\theta_\lambda := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} | \lambda \in \mathbb{C}^*\}/\{\pm id\}, \quad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
The action of \textup{Stab}([\varphi]) on \((C_{\textup{PHom}_1(sl(2),\mathbb{H}^g)}\mathbb{P}Y^{-1}(0))[\varphi]\) is given by
\[
\theta_\lambda \left( \sum_{i=3}^{2g} (Z_{i,2}e_i \otimes v_2 + Z_{i,3}e_i \otimes v_3) \right) = \sum_{i=3}^{2g} (\lambda Z_{i,2}e_i \otimes v_2 + \lambda^{-1}Z_{i,3}e_i \otimes v_3),
\]
\[
\tau \left( \sum_{i=3}^{2g} (Z_{i,2}e_i \otimes v_2 + Z_{i,3}e_i \otimes v_3) \right) = \sum_{i=3}^{2g} (-Z_{i,3}e_i \otimes v_2 - Z_{i,2}e_i \otimes v_3).
\]

Let us consider the blowing-up
\[
\pi : Bl_{\textup{PHom}_1}\mathbb{P}Y^{-1}(0)^{ss} \to \mathbb{P}Y^{-1}(0)^{ss}
\]
of \(\mathbb{P}Y^{-1}(0)^{ss}\) along \(\textup{PHom}_1(sl(2),\mathbb{H}^g)^{ss}\) with the exceptional divisor \(E_\pi\), where \(\mathbb{P}Y^{-1}(0)^{ss}\) is the locus of semistable points of \(\mathbb{P}Y^{-1}(0)\) with respect to the \(PGL(2)\)-action. It is obvious that \((\pi_{R_1} \circ \pi_{R_2})^{-1}(x) = Bl_{\textup{PHom}_1}\mathbb{P}Y^{-1}(0)^{ss}\) for any \(x \in \mathbb{Z}_2^{2g}\).

**Proposition 5.4** (Lemma 1.8.5 of [36]). \(Bl_{\textup{PHom}_1}\mathbb{P}Y^{-1}(0)^{ss}\) is smooth.

**Proposition 5.5** (Lemma 1.8.6 of [36]). All semistable points of \(Bl_{\textup{PHom}_1}\mathbb{P}Y^{-1}(0)^{ss}\) is stable. Explicitly:

1. Semistable points in \(E_\pi\) are given by
   \[
   \{(\varphi, [\alpha]) | [\varphi] \in \textup{PHom}_1(sl(2),\mathbb{H}^g)^{ss}, [\alpha] \in \textup{Hom}^{\omega}(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi), \alpha(v_2) \neq 0 \neq \alpha(v_3)\}.
   \]
2. Semistable points not in \(E_\pi\) are given by
   \[
   \{[\varphi] \in \textup{PHom}^{\omega}(sl(2),\mathbb{H}^g) | \text{rk } \varphi = 3 \text{ or rk } \varphi = 2 \text{ and ker } \varphi \text{ non-isotropic}\}.
   \]
6. BLOWING-UP FORMULA FOR INTERSECTION COHOMOLOGY

In this section, we prove that a blowing-up formula for the intersection cohomology holds in Kirwan’s algorithm introduced in Section 4.

Let $E_1$ (respectively, $E_2$) be the exceptional divisor of $\pi_{R_1}$ (respectively, $\pi_{R_2}$). Let $C_1$ be the normal cone to $Z_2^g$ in $R$, $\tilde{C}_1$ the normal cone to $E_1^{ss} := E_1 \cap R_1^{ss}$ in $R_1^{ss}$, $C_2$ the normal cone to $\Sigma$ in $R_1$, $\tilde{C}_2$ the normal cone to $E_2^{ss} := E_2 \cap R_2^{ss}$ in $R_2^{ss}$, $\tilde{C}$ the normal cone to $E^{ss} := E_\pi \cap (Bl_{\text{Hom}_1} \mathbb{P}^{-1}(0)^{ss})^{ss}$ in $(Bl_{\text{Hom}_1} \mathbb{P}^{-1}(0)^{ss})^{ss}$, where $(Bl_{\text{Hom}_1} \mathbb{P}^{-1}(0)^{ss})^{ss}$ is the locus of semistable points of $Bl_{\text{Hom}_1} \mathbb{P}^{-1}(0)^{ss}$ with respect to the lifted $\text{PGL}(2)$-action. Note that

$$(Bl_{\text{Hom}_1} \mathbb{P}^{-1}(0)^{ss})^{ss} = (Bl_{\text{Hom}_1} \mathbb{P}^{-1}(0)^{ss})^s,$$

where $(Bl_{\text{Hom}_1} \mathbb{P}^{-1}(0)^{ss})^s$ is the locus of stable points of $Bl_{\text{Hom}_1} \mathbb{P}^{-1}(0)^{ss}$ with respect to the $\text{PGL}(2)$-action (Lemma 1.8.6). Then we have the following formulas.

**Lemma 6.1.**

(1) $\dim IH^i(R_1^{ss}/\text{SL}(2)) = \dim IH^i(R/\text{SL}(2))$

$$+ \dim IH^i(\tilde{C}_1/\text{SL}(2)) - \dim IH^i(C_1/\text{SL}(2))$$

$$= \dim IH^i(R/\text{SL}(2)) + 2^{2g} \dim IH^i(Bl_0 \mathbb{P}^{-1}(0)/\text{PGL}(2)) - 2^{2g} \dim IH^i(\mathbb{P}^{-1}(0)/\text{PGL}(2))$$

for all $i \geq 0$, where $Bl_0 \mathbb{P}^{-1}(0)$ is the blowing-up of $\mathbb{P}^{-1}(0)$ at the vertex.

(2) $\dim IH^i(R_2^{ss}/\text{SL}(2)) = \dim IH^i(R_1^{ss}/\text{SL}(2))$

$$+ \dim IH^i(\tilde{C}_2/\text{SL}(2)) - \dim IH^i(C_2/\text{SL}(2))$$

for all $i \geq 0$.

(3) $\dim IH^i(Bl_{\text{Hom}_1} \mathbb{P}^{-1}(0)^{ss}/\text{SL}(2)) = \dim IH^i(\mathbb{P}^{-1}(0)^{ss}/\text{SL}(2))$

$$+ \dim IH^i(\tilde{C}/\text{SL}(2)) - \dim IH^i(C/\text{SL}(2))$$

for all $i \geq 0$.

For the proof, we need to review a useful result by C.T. Simpson. Let $A^i$ (respectively, $A^{i,j}$) be the sheaf of smooth $i$-forms (respectively, $(i,j)$-forms) on $X$. For a polystable Higgs bundle $(E, \phi)$, consider the complex

$$(6.1) \quad 0 \to \text{End}_0(E) \otimes A^0 \to \text{End}_0(E) \otimes A^1 \to \text{End}_0(E) \otimes A^2 \to 0$$
whose differential is given by $D'' = \partial + \phi$. Because $A^1 = A^{1,0} \oplus A^{0,1}$ and $\phi$ is of type $(1,0)$, we have an exact sequence of complexes with (6.1) in the middle

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \text{End}_0(E) \otimes A^{1,0} & \overset{-\eta}{\longrightarrow} & \text{End}_0(E) \otimes A^{1,1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{End}_0(E) \otimes A^0 & \overset{D''}{\longrightarrow} & \text{End}_0(E) \otimes A^1 & \overset{D''}{\longrightarrow} & \text{End}_0(E) \otimes A^2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{End}_0(E) \otimes A^{0,0} & \overset{-\eta}{\longrightarrow} & \text{End}_0(E) \otimes A^{0,1} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

This gives us a long exact sequence

$$
0 \longrightarrow T^0 \longrightarrow H^0(\text{End}_0(E)) \stackrel{[\phi,-]}{\longrightarrow} H^0(\text{End}_0(E) \otimes K_X) \longrightarrow \bigg\vert_T^1 \bigg\longrightarrow H^1(\text{End}_0(E)) \stackrel{[\phi,-]}{\longrightarrow} H^1(\text{End}_0(E) \otimes K_X) \longrightarrow T^2 \longrightarrow 0
$$

where $T^i$ is the $i$-th cohomology of (6.1). The Zariski tangent space of $M$ at polystable $(E, \phi)$ is isomorphic to $T^1$.

**Proposition 6.2** (Theorem 10.4 and 10.5 of [41]). Let $C$ be the quadratic cone in $T^1$ defined by the map $T^1 \to T^2$ which sends an $\text{End}_0(E)$-valued 1-form $\eta$ to $[\eta, \eta]$. Let $y = (E, \phi, \beta : E|_x \to \mathbb{C}^2) \in R$ be a point with closed orbit and $\bar{y} \in M$ the image of $y$. Then the formal completion $(R, y)^{\hat{\gamma}}$ is isomorphic to the formal completion $(C \times h^+, 0)^{\hat{\gamma}}$ where $h^+$ is the perpendicular space to the image of $T^0 \to H^0(\text{End}_0(E)) \to \text{sl}(2)$. Furthermore, if we let $Y$ be the étale slice at $y$ of the SL(2)-orbit in $R$, then $(Y, y)^{\hat{\gamma}} \cong (C, 0)^{\hat{\gamma}}$ and $(M, \bar{y})^{\hat{\gamma}} = (Y/\text{Stab}(y), \bar{y})^{\hat{\gamma}} \cong (C/\text{Stab}(y), v)^{\hat{\gamma}}$ where Stab$(y)$ is the stabilizer of $y$ and $v$ is the cone point of $C$.

**Proof of Lemma 6.7**

(1) Let $U_x$ be a sufficiently small open neighborhood of $x \in \mathbb{Z}_2^{2g}$ in $R://\text{SL}(2)$, let $U_1 = \bigcup_{x \in \mathbb{Z}_2^{2g}} U_x$ and $\bar{U}_1 = \pi^{-1}_{R_1}(U_1)$. By the same argument as in the proof of [27] Lemma 2.8, we have

$$\dim IH^i(\mathbb{R}^*_1://\text{SL}(2)) = \dim IH^i(R://\text{SL}(2)) + \dim IH^i(\bar{U}_1) - \dim IH^i(U_1)$$

for all $i \geq 0$. By [20] Theorem 3.1 and Proposition 6.2 there is an analytic isomorphism $U_1 \cong \tilde{C}_1://\text{SL}(2)$. Since $\tilde{C}_1://\text{SL}(2)$ is naturally isomorphic to the blowing-up of $\tilde{C}_1://\text{SL}(2)$ along $\mathbb{Z}_2^{2g}$, we also have an analytic isomorphism $\bar{U}_1 \cong \tilde{C}_1://\text{SL}(2)$. Since $\tilde{C}_1://\text{SL}(2) (\text{respectively, } \tilde{C}_1://\text{SL}(2))$ is the $2^{2g}$ copy of $\mathbb{Y}^{-1}(0)://\text{PGL}(2)$ (respectively, of $Bl_0 \mathbb{Y}^{-1}(0)://\text{PGL}(2)$), we get the formula.

(2) Let $U_2$ be a sufficiently small open neighborhood of the strict transform of $T^*J//\mathbb{Z}_2$ in $\mathbb{R}^*_1://\text{SL}(2)$ and let $\bar{U}_2 = \pi^{-1}_{R_2}(U_2)$. By the same argument as in the proof of [27] Lemma 2.8, we have

$$\dim IH^i(\mathbb{R}^*_2://\text{SL}(2)) = \dim IH^i(\mathbb{R}^*_1://\text{SL}(2)) + \dim IH^i(\bar{U}_2) - \dim IH^i(U_2)$$
for all $i \geq 0$. By [20] Theorem 3.1 and Proposition 6.2 and Luna’s étale slice theorem, there is an analytic isomorphism $U_2 \cong C_2/\text{SL}(2)$. Since $\tilde{C}_2/\text{SL}(2)$ is naturally isomorphic to the blowing-up of $C_2/\text{SL}(2)$ along the strict transform of $T^*J/\mathbb{Z}_2$ in $\mathbb{R}^2_+^{\text{ss}}/\text{SL}(2)$, we also have an analytic isomorphism $\tilde{U}_2 \cong \tilde{C}_2/\text{SL}(2)$. Hence we get the formula.

(3) Let $\pi : B\ell_{\text{Hom}_1} \mathbb{P}^{\mathcal{Y}(1)(0)^{ss}}/\text{PGL}(2) \to \mathbb{P}^{\mathcal{Y}(1)(0)^{ss}}/\text{PGL}(2)$ be the map induced from $\pi$. Since $C = C_2|_{\Sigma \mathbb{P}^{\mathcal{Y}(1)(0)^{ss}}}$ and $\tilde{C} = \tilde{C}_2|_{E_2 \cap B\ell_{\text{Hom}_1} \mathbb{P}^{\mathcal{Y}(1)(0)^{ss}}}$, it follow from the argument of the proof of item (2) that $C/\text{SL}(2)$ (respectively, $\tilde{C}/\text{SL}(2)$) can be identified with an open neighborhood $U$ of $\mathbb{P}^{\text{Hom}_{1}(sl(2), \mathbb{H}^9)^{ss}}/\text{PGL}(2)$ (respectively, with $\pi^{-1}(U)$). Again by the same argument as in the proof of [27] Lemma 2.8, we get the formula.

We give a computable formula from Lemma 6.1 by more analysis on $B\ell_0 \mathbb{P}^{\mathcal{Y}(1)(0)^{ss}}/\text{PGL}(2), C_2/\text{SL}(2), \tilde{C}_2/\text{SL}(2)$ and $\tilde{C}/\text{SL}(2)$.

We first give explicit geometric descriptions for $C_2/\text{SL}(2), \tilde{C}_2/\text{SL}(2)$, $\tilde{C}/\text{SL}(2)$ and $\tilde{C}/\text{SL}(2)$.

Let $\alpha : \tilde{T}^*J \to T^*J$ be the blowing-up along $\mathbb{Z}_2^{2g}$. Let $(\mathcal{L}, \psi_{\mathcal{L}})$ be the pull-back to $\tilde{T}^*J \times X$ of the universal pair on $T^*J \times X$ by $\alpha \times 1$ and let $p : \tilde{T}^*J \times X \to T^*J$ the projection onto the first factor.

**Lemma 6.3.**

1. $C_2|_{\Sigma \setminus E_1}/\text{SL}(2)$ is a free $\mathbb{Z}_2$-quotient of $\Psi^{-1}(0)/C^*$-bundle over $\tilde{T}^*J \setminus \alpha^{-1}(\mathbb{Z}_2^{2g})$.

2. $\tilde{C}_2|_{\Sigma \setminus E_1}/\text{SL}(2)$ is a free $\mathbb{Z}_2$-quotient of $B\ell_0 \Psi^{-1}(0)/C^*$-bundle over $\tilde{T}^*J \setminus \alpha^{-1}(\mathbb{Z}_2^{2g})$, where $B\ell_0 \Psi^{-1}(0)$ is the blowing-up of $\Psi^{-1}(0)$ at the vertex.

3. $C_2|_{\Sigma \cap E_1}/\text{SL}(2)$ is a free $\mathbb{Z}_2$-quotient of $\text{Hom}^{\omega^*}(\ker \varphi, \text{im} \varphi^* / \text{im} \varphi)/C^*$-bundle over $\alpha^{-1}(\mathbb{Z}_2^{2g})$, where $[\varphi] \in \Sigma \cap E_1$.

4. $\tilde{C}_2|_{\Sigma \cap E_1}/\text{SL}(2)$ is a free $\mathbb{Z}_2$-quotient of $B\ell_0 \text{Hom}^{\omega^*}(\ker \varphi, \text{im} \varphi^* / \text{im} \varphi)/C^*$-bundle over $\alpha^{-1}(\mathbb{Z}_2^{2g})$, where $[\varphi] \in \Sigma \cap E_1$ and $B\ell_0 \text{Hom}^{\omega^*}(\ker \varphi, \text{im} \varphi^* / \text{im} \varphi)$ is the blowing-up of $\text{Hom}^{\omega^*}(\ker \varphi, \text{im} \varphi^* / \text{im} \varphi)$ at the vertex.

5. $C/\text{SL}(2)$ is a free $\mathbb{Z}_2$-quotient of $\text{Hom}^{\omega^*}(\ker \varphi, \text{im} \varphi^* / \text{im} \varphi)/C^*$-bundle over $\mathbb{P}^{2g-1}$, where $[\varphi] \in \mathbb{P}^{\text{Hom}_{1}(sl(2), \mathbb{H}^9)^{ss}}$.

6. $\tilde{C}/\text{SL}(2)$ is a free $\mathbb{Z}_2$-quotient of $B\ell_0 \text{Hom}^{\omega^*}(\ker \varphi, \text{im} \varphi^* / \text{im} \varphi)/C^*$-bundle over $\mathbb{P}^{2g-1}$, where $[\varphi] \in \mathbb{P}^{\text{Hom}_{1}(sl(2), \mathbb{H}^9)^{ss}}$.

**Proof.** Let $x$ be a point of $X$.

1. Consider the principal PGL(2)-bundle

$$q : \mathbb{P} \text{Isom}(O^2_{T^*J}, \mathcal{L}|_x \oplus \mathcal{L}^{-1}|_x) \to \tilde{T}^*J.$$ 

PGL(2) acts on $O^2_{T^*J}$ and $O(2)$ acts on $\mathcal{L}|_x \oplus \mathcal{L}^{-1}|_x$. By the same argument as in the proof of [36] Proposition 1.7.10],

$$\Sigma \cong \mathbb{P} \text{Isom}(O^2_{T^*J}, \mathcal{L}|_x \oplus \mathcal{L}^{-1}|_x)/O(2).$$

$C_2|_{\Sigma \setminus E_1}/\text{SL}(2)$ is the quotient of $q^*\Psi_{L_{III}}^{-1}(0)/O(2)$ by the PGL(2)-action, where $\mathcal{L}_{III} = \mathcal{L}|_{T^*J \setminus \alpha^{-1}(\mathbb{Z}_2^{2g})}$ and

$$\Psi_{L_{III}} : [p_*(\mathcal{L}_{III}^2 K_X) \oplus R^1 p_*(\mathcal{L}_{III}^2)] \oplus [p_*(\mathcal{L}_{III}^2 K_X) \oplus R^1 p_*(\mathcal{L}_{III}^{-2})] \to R^1 p_*(K_X)$$

is the sum of perfect pairings $p_*(\mathcal{L}_{III}^2 K_X) \oplus R^1 p_*(\mathcal{L}_{III}^2) \to R^1 p_*(K_X)$ and $p_*(\mathcal{L}_{III}^2 K_X) \oplus R^1 p_*(\mathcal{L}_{III}^{-2}) \to R^1 p_*(K_X)$. Since the actions of PGL(2) and O(2) commute and $q$ is the
principal PGL(2)-bundle,
\[ C_2|_{\Sigma \setminus E_1} // SL(2) = \Psi^{-1}_{L_{III}}(0) // O(2) = \frac{\Psi^{-1}_{L_{III}}(0) // SO(2)}{O(2) // SO(2)} = \frac{\Psi^{-1}_{L_{III}}(0) // C^*}{\mathbb{Z}_2}. \]

Hence we get the description.

(2) Since \( \tilde{C}_2|_{\Sigma \setminus E_1} // SL(2) \) is isomorphic to the blowing-up of \( C_2|_{\Sigma \setminus E_1} // SL(2) \) along \( T^*J \setminus \mathbb{Z}_2 \), it is isomorphic to \( \frac{\Psi^{-1}_{L_{III}}(0) // C^*}{\mathbb{Z}_2} \), where \( \Psi^{-1}_{L_{III}}(0) \) is the blowing-up of \( \Psi^{-1}_{L_{III}}(0) \) along \( T^*J \setminus \alpha^{-1}(\mathbb{Z}_2) \) isomorphic to \( T^*J \setminus \mathbb{Z}_2 \).

(3) Note that \( E_1 \) is a disjoint union of \( \mathbb{P}^g \). It follows from Proposition 5.2 that \( \Sigma \cap E_1 \) is a disjoint union of \( \text{PHom}_1(sl(2), H^g)^{ss} \). By Proposition 5.1, we have
\[ \text{PHom}_1(sl(2), H^g)^{ss} = \text{PGL}(2)Z^{ss} \cong \text{PGL}(2) \times O(2) Z^{ss} \]
and
\[ \text{PHom}_1(sl(2), H^g)^{ss} // \text{PGL}(2) \cong Z // O(2) = Z_1 = \mathbb{P}^{2g-1}, \]
where \( Z = Z_1 \cup Z_2 \cup Z_3 \), \( Z^{ss} \) is the set of semistable points of \( Z \) for the action of \( O(2) \), \( Z_1 = \mathbb{P}\{v_1 \otimes H^g\} = Z^{ss} \), \( Z_2 = \mathbb{P}\{v_2 \otimes H^g\} \) and \( Z_3 = \mathbb{P}\{v_3 \otimes H^g\} \) for the basis \( \{v_1, v_2, v_3\} \) of \( sl(2) \) chosen in Section 5. Then we have
\[ C_2|_{\Sigma \cap \mathbb{P}^{-1}(0)} = \text{PGL}(2) \times O(2) C_2|_{Z^{ss}} \]
and
\[ C_2|_{\Sigma \cap \mathbb{P}^{-1}(0)} // SL(2) = C_2|_{\Sigma \cap \mathbb{P}^{-1}(0)} // \text{PGL}(2) = C_2|_{Z^{ss}} // O(2) = \frac{C_2|_{Z^{ss}} // SO(2)}{\mathbb{Z}_2}. \]
Since \( C_2|_{Z^{ss}} \) is a Hom\( (\ker \varphi, im\varphi^\perp // im\varphi) \)-bundle over \( Z^{ss} \) by Proposition 5.3
\[ C_2|_{Z^{ss}} // SO(2) \]
is a Hom\( (\ker \varphi, im\varphi^\perp // im\varphi) // \mathbb{C}^* \)-bundle over \( Z // SO(2) = Z_1 = \mathbb{P}^{2g-1} \). Since \( \alpha^{-1}(\mathbb{Z}_2) \) is a disjoint union of \( \mathbb{P}^{2g-1} \), we get the description.

(4) Since \( \tilde{C}_2|_{\Sigma \setminus E_1} // SL(2) \) is isomorphic to the blowing-up of \( C_2|_{\Sigma \setminus E_1} // SL(2) \) along \( 2^{2g} \) disjoint union of \( \text{PHom}_1(sl(2), H^g)^{ss} // \text{PGL}(2) \cong \mathbb{P}^{2g-1} \), it is isomorphic to \( 2^{2g} \) disjoint union of \( \tilde{C}_2|_{Z^{ss}} // SO(2) \), where \( \tilde{C}_2|_{Z^{ss}} \) is the blowing-up of \( C_2|_{Z^{ss}} \) along \( Z^{ss} \).

(5) Since \( C = C_2|_{\Sigma \cap \mathbb{P}^{-1}(0)^{ss}} \), we get the description from (3).

(6) Since \( \tilde{C} = \tilde{C}_2|_{E_2 \cap \text{PGL}_1(\mathbb{P}^{-1}(0))^{ss}} \), we get the description from (4).

\[ \square \]

We next explain how to compute the terms
\[ \dim IH^i(Bl_0 \mathbb{T}^{-1}(0) // \text{PGL}(2)) - \dim IH^i(\mathbb{T}^{-1}(0) // \text{PGL}(2)), \]
and
\[ \dim IH^i(\tilde{C}_2 // SL(2)) - \dim IH^i(C_2 // SL(2)) \]
that appear in Lemma 6.1. We start with the following technical lemma.

**Lemma 6.4** (Lemma 2.12 in [27]). Let \( V \) be a complex variety on which a finite group \( F \) acts. Then
\[ IH^*(V // F) \cong IH^*(V)^F \]
where \( IH^*(V)^F \) denotes the invariant part of \( IH^*(V) \) under the action of \( F \).
Now recall that $\mathcal{Y}^{-1}(0)/\text{SL}(2) = Y^{-1}(0)/\text{PGL}(2)$ and $\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2) = \mathbb{P}Y^{-1}(0)^{ss}/\text{PGL}(2)$ from section 5.

We can compute $IH^*(\mathcal{Y}^{-1}(0)/\text{SL}(2))$ (respectively, $IH^*(\Psi^{-1}(0)/\mathbb{C}^*)$) and

$$IH^*(\text{Hom}^{\omega_{\psi}}(\ker \varphi, \text{im} \varphi^\perp/\text{im} \varphi)/\mathbb{C}^*)$$

in terms of $IH^*(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2))$ (respectively, $IH^*(\mathbb{P}\Psi^{-1}(0)^{ss}/\mathbb{C}^*)$) and

$$IH^*(\text{PHom}^{\omega_{\psi}}(\ker \varphi, \text{im} \varphi^\perp/\text{im} \varphi)^{ss}/\mathbb{C}^*)$$

In order to explain this, we need the following lemmas. The first lemma shows the surjectivities of the Kirwan maps on the fibers of normal cones and exceptional divisors.

**Lemma 6.5.** (1) The Kirwan map

$$IH^*_{\text{SL}(2)}(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}) \to IH^*(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2))$$

is surjective.

(2) The Kirwan map

$$IH^*_{\text{SL}(2)}(\mathcal{Y}^{-1}(0)) \to IH^*(\mathcal{Y}^{-1}(0)/\text{SL}(2))$$

is surjective.

(3) The Kirwan map

$$H^*_{\mathbb{C}^*}(\mathbb{P}\Psi^{-1}(0)^{ss}) \to IH^*(\mathbb{P}\Psi^{-1}(0)^{ss}/\mathbb{C}^*)$$

is surjective.

(4) The Kirwan map

$$IH^*_{\mathbb{C}^*}(\Psi^{-1}(0)) \to IH^*(\Psi^{-1}(0)/\mathbb{C}^*)$$

is surjective.

(5) The Kirwan map

$$H^*_{\mathbb{C}^*}(\text{PHom}^{\omega_{\psi}}(\ker \varphi, \text{im} \varphi^\perp/\text{im} \varphi)^{ss}/\mathbb{C}^*)$$

is surjective.

(6) The Kirwan map

$$IH^*_{\mathbb{C}^*}(\text{Hom}^{\omega_{\psi}}(\ker \varphi, \text{im} \varphi^\perp/\text{im} \varphi)) \to IH^*(\text{Hom}^{\omega_{\psi}}(\ker \varphi, \text{im} \varphi^\perp/\text{im} \varphi)/\mathbb{C}^*)$$

is surjective.

**Proof.** (1) Consider the quotient map

$$f : \mathbb{P}\mathcal{Y}^{-1}(0)^{ss} \to \mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2).$$

In [4, section 6], Bernstein and Lunts define a functor

$$Qf_* : D_{\text{SL}(2)}(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}) \to D(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2))$$

that extends the pushforward of sheaves $f_*$. By the same arguments as those of [44, §2 and §3], we can obtain morphisms

$$\lambda_{\mathcal{Y}^{-1}(0)} : \text{IC}^*(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2))[3] \to Qf_* \text{IC}^*_{\text{SL}(2)}(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss})$$

and

$$\kappa_{\mathcal{Y}^{-1}(0)} : Qf_* \text{IC}^*_{\text{SL}(2)}(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}) \to \text{IC}^*(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2))[3]$$

such that $\kappa_{\mathcal{Y}^{-1}(0)} \circ \lambda_{\mathcal{Y}^{-1}(0)} = \text{id}$. $\lambda_{\mathcal{Y}^{-1}(0)}$ induces a map

$$IH^*(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2)) \to IH^*_{\text{SL}(2)}(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss})$$
which is an inclusion.

Hence \( \kappa_{\Phi^{-1}(0)} \) induces a map
\[
\tilde{\kappa}_{\Phi^{-1}(0)} : IH^*_{\text{SL}(2)}(\mathbb{P} \Phi^{-1}(0)^{ss}) \to IH^*(\mathbb{P} \Phi^{-1}(0)^{ss} // \text{SL}(2))
\]
which is split by the inclusion
\[
IH^*(\mathbb{P} \Phi^{-1}(0)^{ss} // \text{SL}(2)) \to IH^*_{\text{SL}(2)}(\mathbb{P} \Phi^{-1}(0)^{ss}).
\]

(2) Let \( R := \mathbb{C}[T_0, T_1, \cdots, T_{6g-1}] \). For an \( \text{SL}(2) \)-invariant ideal \( I \subset R \) generated by three quadratic homogeneous polynomials in \( R \) defining \( \Phi^{-1}(0) \), we can write
\[
\Phi^{-1}(0) = \text{Spec}(R/I).
\]
Let \( \Phi^{-1}(0) \) be the Zariski closure of \( \Phi^{-1}(0) \) in \( \mathbb{P}^{6g} \). Since the homogenization of \( I \) equals to \( I \), we can write
\[
\Phi^{-1}(0) = \text{Proj}(R[T]/I \cdot R[T])
\]
where \( \text{SL}(2) \) acts trivially on the variable \( T \). Thus
\[
\Phi^{-1}(0) // \text{SL}(2) = \text{Spec}(R^{\text{SL}(2)} / I \cap R^{\text{SL}(2)})
\]
and
\[
\Phi^{-1}(0)^{ss} // \text{SL}(2) = \text{Proj}(R[T]/I \cdot R[T])^{\text{SL}(2)}.
\]
Since \( \text{SL}(2) \) acts trivially on the variable \( T \),
\[
\Phi^{-1}(0)^{ss} // \text{SL}(2) = \text{Proj}(R^{\text{SL}(2)}[T]/(I \cap R^{\text{SL}(2)}) \cdot R^{\text{SL}(2)}[T])
\]
Hence we have an open immersion \( \Phi^{-1}(0) // \text{SL}(2) \to \Phi^{-1}(0)^{ss} // \text{SL}(2) \) given by \( p \mapsto p^{\text{hom}} \) where \( p^{\text{hom}} \) is the homogenization of \( p \).

Note that
\[
\Phi^{-1}(0) \setminus \Phi^{-1}(0) = \text{Proj}(R[T]/I \cdot R[T] + (T)) \cong \text{Proj}(R/I) = \mathbb{P} \Phi^{-1}(0)
\]
and
\[
[\Phi^{-1}(0)^{ss} // \text{SL}(2)] \setminus [\Phi^{-1}(0) // \text{SL}(2)]
= \text{Proj}(R^{\text{SL}(2)}[T]/(I \cap R^{\text{SL}(2)}) \cdot R^{\text{SL}(2)}[T] + (T) \cap R^{\text{SL}(2)}[T])
\cong \text{Proj}(R^{\text{SL}(2)} / I \cap R^{\text{SL}(2)}) = \mathbb{P} \Phi^{-1}(0)^{ss} // \text{SL}(2)
\]
where \( (T) \) is the ideal of \( R[T] \) generated by \( T \).

Consider the quotient map
\[
f : \Phi^{-1}(0)^{ss} \to \Phi^{-1}(0)^{ss} // \text{SL}(2).
\]
In [4] section 6], Bernstein and Lunts define a functor
\[
Qf_* : \text{D}_{\text{SL}(2)}(\Phi^{-1}(0)^{ss}) \to \text{D}(\Phi^{-1}(0)^{ss} // \text{SL}(2))
\]
that extends the pushforward of sheaves \( f_* \).

By the same arguments as those of [4], §2 and §3, we can obtain morphisms
\[
\lambda_{\Phi^{-1}(0)} : \text{IC}^* (\Phi^{-1}(0)^{ss} // \text{SL}(2))[3] \to Qf_* \text{IC}^*_{\text{SL}(2)}(\Phi^{-1}(0)^{ss})
\]
and
\[
\kappa_{\Phi^{-1}(0)} : Qf_* \text{IC}^*_{\text{SL}(2)}(\Phi^{-1}(0)^{ss}) \to \lambda_{\Phi^{-1}(0)} : \text{IC}^* (\Phi^{-1}(0)^{ss} // \text{SL}(2))[3]
\]
such that \( \kappa_{\Phi^{-1}(0)} \circ \lambda_{\Phi^{-1}(0)} = \text{id.} \lambda_{\Phi^{-1}(0)} \) induces a map
\[
IH^*(\Phi^{-1}(0)^{ss} // \text{SL}(2)) \to IH^*_{\text{SL}(2)}(\Phi^{-1}(0)^{ss})
\]
which is an inclusion.
Hence \( \kappa_{\bar{\tau}-1(0)} \) induces a map

\[
\tilde{\kappa}_{\bar{\tau}-1(0)} : IH^i_{SL(2)}(\bar{\Upsilon}^{-1}(0)^{ss}) \to IH^i(\bar{\Upsilon}^{-1}(0)^{ss}/SL(2))
\]

which is split by the inclusion \( IH^* (\bar{\Upsilon}^{-1}(0)^{ss}/SL(2)) \to IH^*_{SL(2)}(\bar{\Upsilon}^{-1}(0)^{ss}) \).

Consider the following commutative diagram:

\[
\begin{array}{c}
\vdots \\
IH^{i-2}_{SL(2)}(P\Upsilon^{-1}(0)^{ss}) \quad \xrightarrow{\tilde{\kappa}_{\bar{\tau}-1(0)}} \quad IH^{i-2}(P\Upsilon^{-1}(0)^{ss}/SL(2)) \\
\downarrow \\
IH^i_{SL(2)}(\Upsilon^{-1}(0)^{ss}) \quad \xrightarrow{\tilde{\kappa}_{\bar{\tau}-1(0)}} \quad IH^i(\Upsilon^{-1}(0)^{ss}/SL(2)) \\
\downarrow \\
IH^i_{SL(2)}(\Upsilon^{-1}(0)) \quad \xrightarrow{\tilde{\kappa}_{\bar{\tau}-1(0)}} \quad IH^i(\Upsilon^{-1}(0)/SL(2)) \\
\vdots \\
\end{array}
\]

Vertical sequences are Gysin sequences and \( \tilde{\kappa}_{\bar{\tau}-1(0)} \) is induced from \( \tilde{\kappa}_{P\Upsilon^{-1}(0)} \) and \( \tilde{\kappa}_{\Upsilon^{-1}(0)} \).

Since \( \tilde{\kappa}_{P\Upsilon^{-1}(0)} \) and \( \tilde{\kappa}_{\Upsilon^{-1}(0)} \) are surjective, \( \tilde{\kappa}_{\bar{\tau}-1(0)} \) is surjective.

(3) Following the idea of the proof of item (1), we get the result.
(4) Following the idea of the proof of item (2), we get the result.
(5) Following the idea of the proof of item (1), we get the result.
(6) Following the idea of the proof of item (2), we get the result.

\[ \square \]

The second lemma shows how to compute the intersection cohomologies of the fibers of the normal cones of the singularities of \( M \) via those of the projectivizations of the fibers.

It is well known that there is a very ample line bundle \( L \) (respectively, \( M_1 \) and \( M_2 \)) on

\[ P\Upsilon^{-1}(0)^{ss}/SL(2) \] (respectively, \( P\Psi^{-1}(0)^{ss}/\mathbb{C}^* \) and \( PHom^{\omega\varphi}(ker \varphi, im\varphi^\perp/im\varphi)^{ss}/\mathbb{C}^* \)),

whose pullback to \( P\Upsilon^{-1}(0)^{ss} \) (respectively, \( P\Psi^{-1}(0)^{ss} \) and \( PHom^{\omega\varphi}(ker \varphi, im\varphi^\perp/im\varphi)^{ss} \)) is the \( M \)th (respectively, \( N_1 \)th and \( N_2 \)th) tensor power of the hyperplane line bundle on \( P\Upsilon^{-1}(0) \) (respectively, \( P\Psi^{-1}(0) \) and \( PHom^{\omega\varphi}(ker \varphi, im\varphi^\perp/im\varphi) \)) for some \( M \) (respectively, \( N_1 \) and \( N_2 \)).

Let \( C_L(P\Upsilon^{-1}(0)^{ss}/SL(2)) \) (respectively, \( C_{M_1}(P\Psi^{-1}(0)^{ss}/\mathbb{C}^*) \) and

\[ C_{M_2}(PHom^{\omega\varphi}(ker \varphi, im\varphi^\perp/im\varphi)^{ss}/\mathbb{C}^*) \)

be the affine cone on \( P\Upsilon^{-1}(0)^{ss}/SL(2) \) (respectively, \( P\Psi^{-1}(0)^{ss}/\mathbb{C}^* \) and

\[ PHom^{\omega\varphi}(ker \varphi, im\varphi^\perp/im\varphi)^{ss}/\mathbb{C}^* \)

with respect to the projective embedding induced by the sections of \( L \) (respectively, \( M_1 \) and \( M_2 \)).
Lemma 6.6. (1) $IH^*(\mathbb{Y}^{-1}(0)//\text{SL}(2)) = IH^*(C_L(\mathbb{P}\mathbb{Y}^{-1}(0)^s//\text{SL}(2)))$ and  
$IH^*(Bl_0\mathbb{Y}^{-1}(0)//\text{SL}(2)) = IH^*(\mathbb{P}\mathbb{Y}^{-1}(0)^s//\text{SL}(2))$, 

(2) $IH^*(\Psi^{-1}(0)//\mathbb{C}^*) = IH^*(C_{\mathcal{M}_1}(\mathbb{P}\Psi^{-1}(0)^s//\mathbb{C}^*))$ and  
$IH^*(Bl_0\Psi^{-1}(0)//\mathbb{C}^*) = IH^*(\mathbb{P}\Psi^{-1}(0)^s//\mathbb{C}^*)$. 

(3) $IH^*(\text{Hom}^{\omega_{\varphi}}(\ker \varphi, \text{im} \varphi/\text{im} \varphi)//\mathbb{C}^*) = IH^*(C_{\mathcal{M}_2}(\mathbb{P}\text{Hom}^{\omega_{\varphi}}(\ker \varphi, \text{im} \varphi/\text{im} \varphi)^s//\mathbb{C}^*))$ and  
$IH^*(Bl_0\text{Hom}^{\omega_{\varphi}}(\ker \varphi, \text{im} \varphi/\text{im} \varphi)//\mathbb{C}^*) = IH^*(\mathbb{P}\text{Hom}^{\omega_{\varphi}}(\ker \varphi, \text{im} \varphi/\text{im} \varphi)^s//\mathbb{C}^*)$.

Proof. (1) We first follow the idea of the proof of [27] Lemma 2.15 to see that  
$C_L(\mathbb{P}\mathbb{Y}^{-1}(0)^s//\text{SL}(2)) \cong \mathbb{Y}^{-1}(0)//(\text{SL}(2) \times F) \cong (\mathbb{Y}^{-1}(0)//\text{SL}(2))/F$, 
where $F$ is the finite subgroup of $\text{GL}(6g)$ consisting of all diagonal matrices $\text{diag}(\eta, \cdots, \eta)$ such that $\eta$ is an $M$th root of unity.

The coordinate ring of $C_L(\mathbb{P}\mathbb{Y}^{-1}(0)^s//\text{SL}(2))$ is the subring $(\mathbb{C}[Y_0, \cdots, Y_{6g-1}]/I)^{\text{SL}(2)}_M$ of the coordinate ring $\mathbb{C}[Y_0, \cdots, Y_{6g-1}]/I$ of $\mathbb{Y}^{-1}(0)$ which is generated by homogeneous polynomials fixed by the natural action of $\text{SL}(2)$ and of degree $M$. Since  
$(\mathbb{C}[Y_0, \cdots, Y_{6g-1}]/I)_M = \mathbb{C}[Y_0, \cdots, Y_{6g-1}]/I \cap \mathbb{C}[Y_0, \cdots, Y_{6g-1}]_M$, 
we have  
$(\mathbb{C}[Y_0, \cdots, Y_{6g-1}]/I)^{\text{SL}(2)}_M = \mathbb{C}[Y_0, \cdots, Y_{6g-1}]^{\text{SL}(2)}_M / I \cap \mathbb{C}[Y_0, \cdots, Y_{6g-1}]^{\text{SL}(2)}_M = \mathbb{C}[Y_0, \cdots, Y_{6g-1}]^{\text{SL}(2)}_M / I \cap \mathbb{C}[Y_0, \cdots, Y_{6g-1}]^{\text{SL}(2)}_M = (\mathbb{C}[Y_0, \cdots, Y_{6g-1}]/I)^{\text{SL}(2)}_M$. 
Thus we get  
$C_L(\mathbb{P}\mathbb{Y}^{-1}(0)^s//\text{SL}(2)) \cong \mathbb{Y}^{-1}(0)//(\text{SL}(2) \times F) \cong (\mathbb{Y}^{-1}(0)//\text{SL}(2))/F$ 
and then  
$IH^*(\mathbb{Y}^{-1}(0)//\text{SL}(2))/F = IH^*(C_L(\mathbb{P}\mathbb{Y}^{-1}(0)^s//\text{SL}(2)))$. 

It remains to show that the action of $F$ on $IH^*(\mathbb{Y}^{-1}(0)//\text{SL}(2))$ is trivial. Since the Kirwan map  
$IH^*_\text{SL}(2)(\mathbb{Y}^{-1}(0)) \to IH^*(\mathbb{Y}^{-1}(0)//\text{SL}(2))$ 
is surjective by Lemma 6.5-(2), it suffices to show that the action of $F$ on $IH^*_\text{SL}(2)(\mathbb{Y}^{-1}(0))$ is trivial. Let  
$\pi_1 : Bl_0\mathbb{Y}^{-1}(0) \to \mathbb{Y}^{-1}(0)$ 
be the blowing-up of $\mathbb{Y}^{-1}(0)$ at the vertex and let  
$\pi_2 : Bl_{\text{Hom}_1} Bl_0\mathbb{Y}^{-1}(0) \to Bl_0\mathbb{Y}^{-1}(0)$ 
be the blowing-up of $Bl_0\mathbb{Y}^{-1}(0)$ along $\text{Hom}_1(\mathbb{H}(2), \mathbb{H}^g)$, where $\text{Hom}_1(\mathbb{H}(2), \mathbb{H}^g)$ is the strict transform of $\text{Hom}_1(\mathbb{sl}(2), \mathbb{H}^g)$. Since the centers of the blowing-ups are $F$-invariant, the action of $F$ on $\mathbb{Y}^{-1}(0)$ lifts to an action of $F$ on $Bl_{\text{Hom}_1} Bl_0\mathbb{Y}^{-1}(0)$. Since $\pi_1 \circ \pi_2$ is proper and $Bl_{\text{Hom}_1} Bl_0\mathbb{Y}^{-1}(0)$ is smooth (See the proof of Lemma 1.8.5 in [35]), by Proposition 3.4-(3), $IH^*_\text{SL}(2)(\mathbb{Y}^{-1}(0))$ is a direct summand of  
$IH^*_\text{SL}(2)(Bl_{\text{Hom}_1} Bl_0\mathbb{Y}^{-1}(0)) = H^*_\text{SL}(2)(Bl_{\text{Hom}_1} Bl_0\mathbb{Y}^{-1}(0))$. 
Since $Bl_{\text{Hom}_1} Bl_0\mathbb{Y}^{-1}(0)$ is homotopically equivalent to $Bl_{\text{Hom}_1} \mathbb{P}\mathbb{Y}^{-1}(0)$, it suffices to show that the action of $F$ on $H^*_\text{SL}(2)(Bl_{\text{Hom}_1} \mathbb{P}\mathbb{Y}^{-1}(0))$ is trivial. But this is true because the action
of $F$ on $\mathbb{P}\mathcal{Y}^{-1}(0)$ is trivial and it lifts to the trivial action of $F$ on $Bl_{\mathbb{P}\text{Hom}}(\mathbb{P}\mathcal{Y}^{-1}(0))$. Hence $F$ acts trivially on $IH^*(\mathcal{Y}^{-1}(0)/\text{SL}(2))$.

Similarly, we next see that $Bl_v(C_L(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2)))$ is naturally isomorphic to

$$(Bl_0\mathcal{Y}^{-1}(0)/\text{SL}(2))/F,$$

where $v$ is the vertex of $C_L(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2))$.

Let $J$ be the ideal of $\mathbb{C}[Y_0, \cdots, Y_{6g-1}]/I$ corresponding to the vertex $O$ of $\mathcal{Y}^{-1}(0)$. Then we have $Bl_0\mathcal{Y}^{-1}(0) = \text{Proj}\left(\bigoplus_{m \geq 0} J^m\right)$. Then

$$(Bl_0\mathcal{Y}^{-1}(0)/\text{SL}(2))/F = Bl_0\mathcal{Y}^{-1}(0)/(\text{SL}(2) \times F) = \text{Proj}\left(\bigoplus_{m \geq 0} (J^m)^{\text{SL}(2)\times F}\right).$$

Since $(J^m)^{\text{SL}(2)\times F} = J^m \cap (\mathbb{C}[Y_0, \cdots, Y_{6g-1}]/I)^{\text{SL}(2)\times F} = (J \cap (\mathbb{C}[Y_0, \cdots, Y_{6g-1}]/I)^{\text{SL}(2)\times F})^m = (J^{\text{SL}(2)\times F})^m$

and $J^{\text{SL}(2)\times F}$ is the ideal corresponding to $v = O/(\text{SL}(2) \times F)$, we have

$$\text{Proj}\left(\bigoplus_{m \geq 0} (J^m)^{\text{SL}(2)\times F}\right) = \text{Proj}\left(\bigoplus_{m \geq 0} (J^{\text{SL}(2)\times F})^m\right) = Bl_v(\mathcal{Y}^{-1}(0)/(\text{SL}(2) \times F))$$

Thus

$$IH^*(Bl_0\mathcal{Y}^{-1}(0)/\text{SL}(2))^F = IH^*(Bl_v(C_L(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2)))).$$

By the same idea of the proof of the first statement, $F$ acts trivially on $IH^*(Bl_0\mathcal{Y}^{-1}(0)/\text{SL}(2))$ and then

$$IH^*(Bl_0\mathcal{Y}^{-1}(0)/\text{SL}(2))^F = IH^*(Bl_0(\mathcal{Y}^{-1}(0)/\text{SL}(2))).$$

Since $Bl_v(C_L(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2)))$ is homeomorphic to the line bundle $L^\vee$ over $\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2)$, there is a Leray spectral sequence $E_p^q$ converging to

$$IH^*(Bl_v(C_L(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2))))$$

with

$$E_2^{pq} = IH^p(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2), IH^q(\mathbb{C})) = \begin{cases} IH^p(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2)) & \text{if } q = 0 \\ 0 & \text{otherwise}. \end{cases}$$

Hence we get

$$IH^*(Bl_0\mathcal{Y}^{-1}(0)/\text{SL}(2)) = IH^*(\mathbb{P}\mathcal{Y}^{-1}(0)^{ss}/\text{SL}(2)).$$

(2) Following the idea of the proof of item (1) and using Lemma 6.5-(4), we get the result.

(3) Following the idea of the proof of item (1) and using Lemma 6.5-(6), we get the result.

By the standard argument of [29, Proposition 4.7.2], we get the third lemma as follows. It gives a way to compute the intersection cohomology of affine cones of projective GIT quotients.
Lemma 6.7. (1) Let $n = \dim_{\mathbb{C}} C_{\mathcal{L}}(\mathbb{P}^{-1}(0)/\mathbb{SL}(2))$. Then

$$IH^i(C_{\mathcal{L}}(\mathbb{P}^{-1}(0)/\mathbb{SL}(2))) \cong \begin{cases} 0 & \text{for } i \geq n \\ IH^i(C_{\mathcal{L}}(\mathbb{P}^{-1}(0)/\mathbb{SL}(2)) - \{0\}) & \text{for } i < n. \end{cases}$$

(2) Let $n = \dim_{\mathbb{C}} C_{\mathcal{M}_1}(\mathbb{P}\Psi^{-1}(0)/\mathbb{C}^*)$. Then

$$IH^i(C_{\mathcal{M}_1}(\mathbb{P}\Psi^{-1}(0)/\mathbb{C}^*)) \cong \begin{cases} 0 & \text{for } i \geq n \\ IH^i(C_{\mathcal{M}_1}(\mathbb{P}\Psi^{-1}(0)/\mathbb{C}^*)) - \{0\} & \text{for } i < n. \end{cases}$$

(3) Let $n = \dim_{\mathbb{C}} C_{\mathcal{M}_2}(\mathbb{P}\text{Hom}_{\omega}^*(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^*)$. Then

$$IH^i(C_{\mathcal{M}_2}(\mathbb{P}\text{Hom}_{\omega}^*(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^*))$$

$$\cong \begin{cases} 0 & \text{for } i \geq n \\ IH^i(C_{\mathcal{M}_2}(\mathbb{P}\text{Hom}_{\omega}^*(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^*)) - \{0\} & \text{for } i < n. \end{cases}$$

The following lemma explains how $IH^*(\mathcal{Y}^{-1}(0)/\mathbb{SL}(2))$ (respectively, $IH^*(\mathcal{Y}^{-1}(0)/\mathbb{C}^*)$ and $IH^*(\mathcal{M}_1(\mathbb{P}\Psi^{-1}(0)/\mathbb{C}^*))$ can be computed in terms of $IH^*(\mathcal{Y}^{-1}(0)/\mathbb{SL}(2))$ (respectively, $IH^*(\mathcal{M}_1(\mathbb{P}\Psi^{-1}(0)/\mathbb{C}^*))$ and $IH^*(\mathcal{M}_2(\mathbb{P}\text{Hom}_{\omega}^*(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^*))$ as desired.

Lemma 6.8. (1) \begin{align*}
IH^i(\mathcal{Y}^{-1}(0)/\mathbb{SL}(2)) &= 0 \quad \text{for } i \geq \dim \mathcal{Y}^{-1}(0)/\mathbb{SL}(2) \\
IH^i(\mathcal{Y}^{-1}(0)/\mathbb{C}^*) &= 0 \quad \text{for } i \geq \dim \mathcal{Y}^{-1}(0)/\mathbb{C}^* \\
IH^i(\mathcal{M}_1(\mathbb{P}\Psi^{-1}(0)/\mathbb{C}^*)) &= \ker \dim \mathcal{M}_1(\mathbb{P}\Psi^{-1}(0)/\mathbb{C}^*) \quad \text{for } i < \dim \mathcal{Y}^{-1}(0)/\mathbb{SL}(2),
\end{align*}

where $\lambda : IH^{i-2}(\mathbb{P}\Psi^{-1}(0)/\mathbb{C}^*) \rightarrow IH^i(\mathcal{Y}^{-1}(0)/\mathbb{SL}(2))$ is an injection.

(2) \begin{align*}
IH^i(\mathcal{Y}^{-1}(0)/\mathbb{C}^*) &= 0 \quad \text{for } i \geq \dim \mathcal{Y}^{-1}(0)/\mathbb{C}^* \\
IH^i(\mathcal{M}_1(\mathbb{P}\Psi^{-1}(0)/\mathbb{C}^*)) &= \ker \dim \mathcal{M}_1(\mathbb{P}\Psi^{-1}(0)/\mathbb{C}^*) \quad \text{for } i < \dim \mathcal{Y}^{-1}(0)/\mathbb{C}^*,
\end{align*}

where $\lambda : IH^{i-2}(\mathbb{P}\Psi^{-1}(0)/\mathbb{C}^*) \rightarrow IH^i(\mathcal{M}_1(\mathbb{P}\Psi^{-1}(0)/\mathbb{C}^*))$ is an injection.

(3) \begin{align*}
IH^i(\text{Hom}_{\omega}^*(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^*) &= 0 \quad \text{for } i \geq \dim \text{Hom}_{\omega}^*(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^* \\
IH^i(\text{Hom}_{\omega}^*(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^*)) &= \ker \dim \text{Hom}_{\omega}^*(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^* \quad \text{for } i < \dim \text{Hom}_{\omega}^*(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^*,
\end{align*}

where $\lambda : IH^{i-2}(\mathbb{P}\text{Hom}_{\omega}^*(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^*) \rightarrow IH^i(\mathbb{P}\text{Hom}_{\omega}^*(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^*)$ is an injection.

Proof. We follow the idea of the proof of [27, Corollary 2.17]. We only prove item (1) because the proofs of item (2) and item (3) are similar to that of item (1).

By Lemma 6.6(1),

$$IH^*(\mathcal{Y}^{-1}(0)/\mathbb{SL}(2)) = IH^*(C_{\mathcal{L}}(\mathbb{P}^{-1}(0)/\mathbb{SL}(2))).$$

Let $n = \dim_{\mathbb{C}} \mathcal{Y}^{-1}(0)/\mathbb{SL}(2)$. By Lemma 6.7(1),

$$IH^i(C_{\mathcal{L}}(\mathbb{P}^{-1}(0)/\mathbb{SL}(2))) \cong \begin{cases} 0 & \text{for } i \geq n \\ IH^i(C_{\mathcal{L}}(\mathbb{P}^{-1}(0)/\mathbb{SL}(2)) - \{0\}) & \text{for } i < n. \end{cases}$$

Since $C_{\mathcal{L}}(\mathbb{P}^{-1}(0)/\mathbb{SL}(2)) - \{0\}$ fibers over $\mathbb{P}^{-1}(0)/\mathbb{SL}(2)$ with fiber $\mathbb{C}^*$, there is a Leray spectral sequence $E_\infty$ converging to

$$IH^*(C_{\mathcal{L}}(\mathbb{P}^{-1}(0)/\mathbb{SL}(2)) - \{0\}).$$
with
\[ E_2^{pq} = IH^p(\mathbb{P}^q - 1(0)^{ss}/\text{SL}(2), IH^q(\mathbb{C}^*)) = \begin{cases} IH^p(\mathbb{P}^q - 1(0)^{ss}/\text{SL}(2)) & \text{if } q = 0, 1 \\ 0 & \text{otherwise.} \end{cases} \]

It follows from \([5, 5.1]\) and \([15, \text{p.462–p.468}]\) that the differential
\[ \lambda : IH^{i-2}(\mathbb{P}^q - 1(0)^{ss}/\text{SL}(2)) \to IH^i(\mathbb{P}^q - 1(0)^{ss}/\text{SL}(2)) \]
is given by the multiplication by \(c_1(\mathcal{L})\). By the Hard Lefschetz theorem for intersection cohomology, \(\lambda\) is injective for \(i < n\). Hence we get the result.

The quotients \(\mathbb{P}^q - 1(0)^{ss}/\mathbb{C}^*\) and \(\text{PHom}^\omega(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^*\) can be identified with some incidence variety.

**Lemma 6.9.** Let \(I_{2g-3}\) be the incidence variety given by
\[ I_{2g-3} = \{(p, H) \in \mathbb{P}^{2g-3} \times \mathbb{P}^{2g-3} | p \in H \}. \]

1. \(\mathbb{P}^q - 1(0)^{ss}/\mathbb{C}^* \cong I_{2g-3},\)
2. \(\text{PHom}^\omega(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^* \cong I_{2g-3}.\)

**Proof.**
(1) Consider the map \(f : \mathbb{P}^q - 1(0) \to I_{2g-3}\) given by
\[ (a, b, c, d) \mapsto ((b, c), (-a, d)). \]
Since \(f\) is \(\mathbb{C}^*\)-invariant, we have the induced map
\[ \bar{f} : \mathbb{P}^q - 1(0)^{ss}/\mathbb{C}^* \to I_{2g-3}. \]
We claim that \(\bar{f}\) is injective. Assume that \(\bar{f}([a_1, b_1, c_1, d_1]) = \bar{f}([a_2, b_2, c_2, d_2])\) where \([a, b, c, d]\) denotes the closed orbit of \((a, b, c, d)\). Then there are nonzero complex numbers \(\lambda\) and \(\mu\) such that \((b_1, c_1) = \lambda(b_2, c_2)\) and \((-a_1, d_1) = \mu(-a_2, d_2)\). Then
\[ \begin{align*}
[a_1, b_1, c_1, d_1] &= [\mu a_2, \lambda b_2, \lambda c_2, \mu d_2] = [\lambda \mu b_2, (\lambda \mu)^{1/2} \lambda c_2, (\lambda \mu)^{1/2} \mu d_2] \\
&= [a_2, b_2, c_2, d_2].
\end{align*} \]
Thus \(\bar{f}\) is injective.
Since the domain and the range of \(\bar{f}\) are normal varieties with the same dimension and the range \(I_{2g-3}\) is irreducible, \(\bar{f}\) is an isomorphism.
(2) Consider the map \(g : \text{PHom}^\omega(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi) \to I_{2g-3}\) given by
\[ (Z_{12}, \ldots, Z_{2g,2}, Z_{13}, \ldots, Z_{2g,3}) \mapsto ((Z_{12}, Z_{22}, \ldots, Z_{2g-1,2}, Z_{2g,2}), (Z_{23}, -Z_{13}, \ldots, Z_{2g,3}, -Z_{2g-1,3})). \]
Since \(g\) is \(\mathbb{C}^*\)-invariant, we have the induced map
\[ \bar{g} : \text{PHom}^\omega(\ker \varphi, \text{im}\varphi^\perp/\text{im}\varphi)^{ss}/\mathbb{C}^* \to I_{2g-3}. \]
We can see that \(\bar{g}\) is injective by the similar way as in the proof of (1). Since the domain and the range of \(\bar{g}\) are normal varieties with the same dimension and the range \(I_{2g-3}\) is irreducible, \(\bar{g}\) is an isomorphism.

By the proof of Lemma 6.3,
\[ C_2/\text{SL}(2) = (Y/\mathbb{C}^*)/\mathbb{Z}_2 \text{ and } \hat{C}_2/\text{SL}(2) = (Bl_{\hat{T}^1} Y/\mathbb{C}^*)/\mathbb{Z}_2, \]
where $Y$ is either a $\Psi^{-1}(0)$-bundle or a $\text{Hom}^{\omega_N}(\ker \varphi, \text{im} \varphi^*)$-bundle over $\widetilde{T^*J}$.

To give computable formulas from Lemma 6.1, we need the following technical statements for $Y//C^*$ and $Bl_{\widetilde{T^*J}} Y//C^*$.

**Lemma 6.10.** Let $g : Y//C^* \to \widetilde{T^*J}$ be the map induced by the projection $Y \to \widetilde{T^*J}$ and let $h : Bl_{\widetilde{T^*J}} Y//C^* \to \widetilde{T^*J}$ be the map induced by the composition of maps $Bl_{\widetilde{T^*J}} Y \to Y \to \widetilde{T^*J}$. Then $R^ig_*IC^*(Y//C^*)$ and $R^ih_*IC^*(Bl_{\widetilde{T^*J}} Y//C^*)$ are constant sheaves for each $i \geq 0$.

**Proof.** Following the idea of proof of [27, Proposition 2.13] in the case that $G = SU(2)$, $R = S = C^*$, $N = N_0 = G$ and $Z^*_R//N_0 = \widetilde{T^*J}$ and using Lemma 6.6-(2), (3) and Lemma 6.8-(2), (3), we can see that $R^ig_*IC^*(Y//C^*)$ and $R^ih_*IC^*(Bl_{\widetilde{T^*J}} Y//C^*)$ are locally constant sheaves for each $i \geq 0$. Since $\widetilde{T^*J}$ is irreducible, we get the conclusion. □

Then we have the following computable blowing-up formula.

**Theorem 6.11.** (1) $\dim IH^i(\mathbb{R}^s_{\mathcal{A}}//SL(2)) = \dim IH^i(\mathbb{R}^s//SL(2))$

$$+ 2^{2g} \dim IH^i(\mathbb{P}Y^{-1}(0)^{ss}//\text{PGL}(2)) - 2^{2g} \dim IH^i(Y^{-1}(0)//\text{PGL}(2))$$

for all $i \geq 0$.

(2) $\dim IH^i(\mathbb{R}_{\mathcal{A}}^s//SL(2)) = \dim IH^i(\mathbb{R}_{\mathcal{A}}^s//SL(2))$

$$+ \sum_{p+q = i} \dim[H^p(\mathcal{T}^*J) \otimes H^q(I_{2g-3})]^\mathbb{Z}_2$$

for all $i \geq 0$, where $t(q) = q - 2$ for $q \leq \dim I_{2g-3} = 4g - 7$ and $t(q) = q$ otherwise.

(3) $\dim IH^i(Bl_{\text{Hom}_{\mathcal{A}}}(\mathbb{P}Y^{-1}(0)^{ss}//\text{SL}(2)) = \dim IH^i(\mathbb{P}Y^{-1}(0)^{ss}//\text{SL}(2))$

$$+ \sum_{p+q = i} \dim[H^p(\mathbb{P}^{2g-1}) \otimes H^q(I_{2g-3})]^\mathbb{Z}_2$$

for all $i \geq 0$, where $t(q) = q - 2$ for $q \leq \dim I_{2g-3} = 4g - 7$ and $t(q) = q$ otherwise.

**Proof.** (1) Since it follows from Lemma 6.6-(1) that $IH^i(Bl_0 Y^{-1}(0)//\text{PGL}(2)) = IH^i(\mathbb{P}Y^{-1}(0)^{ss}//\text{PGL}(2))$, we get the formula.

(2) Let $g : Y//C^* \to \widetilde{T^*J}$ and $h : Bl_{\widetilde{T^*J}} Y//C^* \to \widetilde{T^*J}$ be the maps induced by the projections $Y \to \widetilde{T^*J}$ and $Bl_{\widetilde{T^*J}} Y \to \widetilde{T^*J}$. By Proposition 3.4-(2), the Leray spectral sequences of intersection cohomology associated to $g$ and $h$ have $E_2$ terms given by

$$E_2^{pq} = H^p(\mathcal{T}^*J) \otimes IH^q(I_{2g-3})$$

and

$$E_2^{pq} = H^p(\mathcal{T}^*J) \otimes H^q(I_{2g-3})$$

by Lemma 6.6, Lemma 6.9 and Lemma 6.10 where $I_{2g-3}$ is the affine cone of $I_{2g-3}$. Since the spectral sequence associated to $h$ is identical to that associated to the projection $E_h//C^* \to T^*J$, where $E_h$ is the exceptional divisor of the blowing-up map $Bl_{\widetilde{T^*J}} Y \to Y$, it follows from Proposition 3.4-(2) that the decomposition theorem for the projection $E_h//C^* \to T^*J$ implies that the Leray spectral sequence of intersection cohomology associated to $h$ degenerates at the $E_2$ term. Since $IH^q(I_{2g-3})$ embeds in $IH^q(I_{2g-3})$ by Lemma 6.8-(2), (3) and Lemma 6.9, the Leray spectral sequence of intersection cohomology associated...
to \( g \) also degenerates at the \( E_2 \) term. Since \( \mathcal{C}_2/ SL(2) = (Y//\mathbb{C}^*)/\mathbb{Z}_2 \) and \( \tilde{\mathcal{C}}_2/ SL(2) = (Bl_{T^m} Y//\mathbb{C}^*)/\mathbb{Z}_2 \), we have

\[
IH^i(\mathcal{C}_2/ SL(2)) = \bigoplus_{p+q=i} [H^p(T^*J) \otimes IH^q(\tilde{I}_{2g-3})]^{\mathbb{Z}_2}
\]

and

\[
IH^i(\tilde{\mathcal{C}}_2/ SL(2)) = \bigoplus_{p+q=i} [H^p(T^*J) \otimes H^q(\tilde{I}_{2g-3})]^{\mathbb{Z}_2}
\]

by Lemma 6.4. Applying Lemma 6.8(2), (3) again, we get the formula.

3. Note that \( \tilde{\mathcal{C}}/ SL(2) = (Y|p_{2g-1}//\mathbb{C}^*/)/\mathbb{Z}_2 \) and \( \tilde{\mathcal{C}}_2/ SL(2) = (Bl_{p_{2g-1}} Y|p_{2g-1}///\mathbb{C}^*)/\mathbb{Z}_2 \). Let

\[
g' : Y|p_{2g-1}///\mathbb{C}^* \rightarrow \mathbb{P}^{2g-1} \quad \text{and} \quad h' : Bl_{p_{2g-1}} Y|p_{2g-1}///\mathbb{C}^* \rightarrow \mathbb{P}^{2g-1}
\]

be the maps induced by the projections \( Y|p_{2g-1} \rightarrow \mathbb{P}^{2g-1} \) and \( Bl_{p_{2g-1}} Y|p_{2g-1} \rightarrow \mathbb{P}^{2g-1} \). Since \( \mathbb{P}^{2g-1} \) is simply connected, \( R^ig'_* IC^*(Y|p_{2g-1}///\mathbb{C}^*) \) and \( R^ih'_* IC^*(Bl_{p_{2g-1}} Y|p_{2g-1}///\mathbb{C}^*) \) are constant sheaves for each \( i \geq 0 \) by the same argument as in the proof of Lemma 6.10 and then the Leray spectral sequences of intersection cohomology associated to \( g' \) and \( h' \) have \( E_2 \) terms given by

\[
E_2^{pq} = H^p(\mathbb{P}^{2g-1}) \otimes IH^q(\tilde{I}_{2g-3})
\]

and

\[
E_2^{pq} = H^p(\mathbb{P}^{2g-1}) \otimes H^q(\tilde{I}_{2g-3})
\]

by Lemma 6.6 and Lemma 6.9. By the same argument as in the remaining part of the proof of item (2), we get the formula.

\[\square\]

7. A STRATEGY TO GET A FORMULA FOR THE POINCARÉ POLYNOMIAL OF \( IH^*(M) \)

Since \( R_2^2/ SL(2) \) has an orbifold singularity, we have \( H^i(R_2^2/ SL(2)) \cong H^i_{SL(2)}(R_2^2) \) for each \( i \geq 0 \). If we have a blowing-up formula for the equivariant cohomology that can be applied to get \( \dim H^i_{SL(2)}(R_2^2) \) from \( \dim H^i_{SL(2)}(R) \) for each \( i \geq 0 \), Theorem 6.11 can be used to calculate \( \dim IH^i(M) \) from \( \dim IH^i(R_2^2/ SL(2)) \) for each \( i \).

7.1. Towards a blowing-up formula for the equivariant cohomology. In this subsection, we give a strategy to get a blowing-up formula for the equivariant cohomology in Kirwan’s algorithm starting with \( R \) and prove that a blowing-up formula for the equivariant cohomology in the blowing-up \( \pi : Bl_{\varphi} Hom_{1} \mathbb{P}T^{-1}(0)^{ss} \rightarrow \mathbb{P}T^{-1}(0)^{ss} \) holds.

There are the \( \mathbb{C}^* \)-actions on \( M \) by \( \lambda \cdot (E, \phi) = (E, \lambda \phi) \) and on \( R \) by \( \lambda \cdot (E, \phi, \beta) = (E, \lambda \phi, \beta) \). Then the \( SL(2) \)-action on \( R \) commutes with the \( \mathbb{C}^* \)-action on \( R \). Since \( \mathbb{Z}_2^g \) is invariant under the \( \mathbb{C}^* \)-action, the \( \mathbb{C}^* \)-action on \( R \) lifts to \( R_1 \) and the \( \mathbb{C}^* \)-action on \( M \) lifts to \( R_1^{ss} // SL(2) \). Since \( \Sigma \) is invariant under the \( \mathbb{C}^* \)-action on \( R_1^{ss} \), the \( \mathbb{C}^* \)-action on \( R_1^{ss} \) lifts to \( R_1 \). The fixed loci of the \( \mathbb{C}^* \)-action on \( M \) and \( R_1^{ss} // SL(2) \) can be described as in the following two Propositions.

**Proposition 7.1 (4.1) of [23].** The fixed locus of the \( \mathbb{C}^* \)-action on \( M \) is

\[
M^{\mathbb{C}^*} = N \sqcup \bigsqcup_{d=1}^{g-1} F_d
\]

where

1. \( N \) is the moduli space of semistable rank 2 vector bundles with trivial determinant on \( X \), which parametrizes \( SL(2) \)-Higgs bundles of the form \( (E, 0) \);
2. \( F_d = S^{2g-2-2d}X \times_{\text{Pic}^{2d}(X)} \text{Pic}^{d}(X) \), which parametrizes \( \text{SL}(2) \)-Higgs bundles \((E, \phi)\) of the form \( E = L \oplus L^{-1}, \phi = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} \), where \( S^m X \) is the \( m \)-fold symmetric product of \( X \), \( F_d = S^{2g-2-2d}X \times_{\text{Pic}^{2d}(X)} \text{Pic}^{d}(X) \), the map \( S^{2g-2-2d}X \rightarrow \text{Pic}^{2d}(X) \) is given by \( D \mapsto K_X(-D) \), the map \( \text{Pic}^{d}(X) \rightarrow \text{Pic}^{2d}(X) \) is given by \( L \mapsto L^2 \) and \( K_X L^{-2} = O_X(\text{div}(s)) \).

**Proposition 7.2.** The fixed locus of the \( \mathbb{C}^* \)-action on \( \mathbb{R}^s_{1s}/\text{SL}(2) \) is

\[
(\mathbb{R}^s_{1s}/\text{SL}(2))^{\mathbb{C}^*} = \mathbb{N}_1 \sqcup \bigcup_{d=1}^{g-1} \pi_{\mathbb{R}^s_1}(F_d) \sqcup \bigcup_{j=1}^{2g} t_j,
\]

where \( \mathbb{N}_1 \) is the strict transform of \( \mathbb{N} \) under \( \pi_{\mathbb{R}^s_1} \) and \( t_j = \mathbb{P}(H^0(K_X) \otimes \text{sl}(2))^{s s}/\text{SL}(2) \).

**Proof.** The component of the fixed locus \((\mathbb{R}^s_{1s}/\text{SL}(2))^{\mathbb{C}^*}\) not contained in \( \pi_{\mathbb{R}^s_1}(\mathbb{N} - \mathbb{Z}_{2}^{2g}) \cup \bigcup_{d=1}^{g-1} \pi_{\mathbb{R}^s_1}(F_d) \) lies over \( \mathbb{Z}_{2}^{2g} \).

The fiber of \( \pi_{\mathbb{R}^s_2} \) over the points of \( \mathbb{Z}_{2}^{2g} \) is \( \mathbb{P}Y^{-1}(0)^{s s}/\text{SL}(2) \). The \( \mathbb{C}^* \)-action on \( \mathbb{P}Y^{-1}(0)^{s s}/\text{SL}(2) \) is given by

\[
\lambda \cdot \begin{pmatrix} d & e \\ f & -d \end{pmatrix}, \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} \lambda d & \lambda e \\ \lambda f & -\lambda d \end{pmatrix}, \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.
\]

Thus

\[
(\mathbb{P}Y^{-1}(0)^{s s}/\text{SL}(2))^{\mathbb{C}^*} = \left\{ \begin{pmatrix} d & e \\ f & -d \end{pmatrix}, \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathbb{P}Y^{-1}(0)^{s s}/\text{SL}(2) \mid d = e = f = 0 \right\} \\
\sqcup \left\{ \begin{pmatrix} d & e \\ f & -d \end{pmatrix}, \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathbb{P}Y^{-1}(0)^{s s}/\text{SL}(2) \mid a = b = c = 0 \right\} \\
\cong \mathbb{P}(H^1(O_X) \otimes \text{sl}(2))^{s s}/\text{SL}(2) \sqcup \mathbb{P}(H^0(K_X) \otimes \text{sl}(2))^{s s}/\text{SL}(2).
\]

Since the disjoint union of \( \pi_{\mathbb{R}^s_1}^{-1}(\mathbb{N} - \mathbb{Z}_{2}^{2g}) \) and \( 2^{2g} \) copies of \( \mathbb{P}(H^1(O_X) \otimes \text{sl}(2))^{s s}/\text{SL}(2) \) is isomorphic to the strict transform \( \mathbb{N}_1 \) of \( \mathbb{N} \), we get the conclusion. \( \Box \)

We consider special kinds of normal complex quasi-projective varieties. Let \( R \) is a normal complex quasi-projective variety with a \( \mathbb{C}^* \)-action. If \( \lim_{\lambda \to 0} \lambda \cdot x \) uniquely exists in \( R^{\mathbb{C}^*} \) for every \( x \in R \), we can give a decomposition of Białynicki–Birula type on \( R \).

**Proposition 7.3** (Białynicki–Birula decomposition, [1], [2], 4.4 of [43], Theorem 5.5 of [14]). Let \( R \) be a normal complex quasi-projective variety with a \( \mathbb{C}^* \)-action. \( R^{\text{sm}} \) denotes the smooth locus of \( R \). Assume that \( \lim_{\lambda \to 0} \lambda \cdot x \) uniquely exists in \( R^{\mathbb{C}^*} \) for every \( x \in R \). Then

1. \( R \) decomposes into \( \mathbb{C}^* \)-invariant locally closed subsets

\[
R = \bigsqcup_{C \in \pi_0(R^{\mathbb{C}^*})} R^+_C,
\]

where \( R^+_C \) is the attracting set \( \{ x \in R \mid \lim_{\lambda \to 0} \lambda \cdot x \in C \} \) for each connected component \( C \subset R^{\mathbb{C}^*} \).

2. The limit map

\[
R^+_C \to C, \quad x \mapsto \lim_{\lambda \to 0} \lambda \cdot x
\]

is a morphism of algebraic varieties, and it is a bundle of affine spaces if \( C \subset R^{\text{sm}} \).

3. Each connected component of \( (R^{\text{sm}})^{\mathbb{C}^*} \) is smooth.
4. There exists an ordering $C_0, C_1, \cdots$ of the components of $R^{C^*}$ such that
- $i < j \iff \dim R^+_{C_i} \geq \dim R^+_{C_j}$;
- the sets

$$R_i = \bigcup_{i \leq j} R^+_{C_i}$$

yield a filtration of $R$ by closed subvarieties.

Assume that the $i$-th connected component $C_i$ of $R^{C^*}$ is smooth. Applying the local-global spectral sequence ([42] Proof of Proposition 3.4.4), [43] §4.4) associated to the filtration $R_i \times_{SL(2)} ESL(2)$ of $R \times_{SL(2)} ESL(2)$

$$E_1^{ij} = H^{i+j}(R_i \times_{SL(2)} ESL(2), u_i^1Q_{R \times_{SL(2)} ESL(2)}) \Rightarrow H^{i+j}(R \times_{SL(2)} ESL(2), Q)$$

where $u_i : R^+_{C_i} \times_{SL(2)} ESL(2) \hookrightarrow R \times_{SL(2)} ESL(2)$ is the inclusion.

If $R$ is smooth and $R^+_{C_i}$ is smooth of codimension $c_i$ in $R$, we have

$$u_i^1Q_{R \times_{SL(2)} ESL(2)} = Q_{R^+_{C_i} \times_{SL(2)} ESL(2)}[2c_i]$$

and we can rewrite the terms of our spectral sequence as

$$E_1^{ij} = H^{i+j}(R^+_{C_i} \times_{SL(2)} ESL(2), u_i^1Q_{R \times_{SL(2)} ESL(2)})$$

$$= H^{i+j-2c_i}(R^+_{C_i} \times_{SL(2)} ESL(2), Q) = H^{i+j-2c_i}(C_i \times_{SL(2)} ESL(2), Q).$$

**Proposition 7.4** (Proof of Proposition 3.4.4 of [42], §4.4 of [43]). Let $R$ be a normal complex quasi-projective variety with a $C^*$-action. $R^{sm}$ denotes the smooth locus of $R$. Assume that $\lim_{\lambda \to 0} \lambda \cdot x$ uniquely exists in $R^{C^*}$ for every $x \in R$. Then

1. The decomposition of $R$ in Proposition 7.3 yields the spectral sequence

$$E_1^{ij} = H^{i+j}(R^+_{C_i} \times_{SL(2)} ESL(2), u_i^1Q_{R \times_{SL(2)} ESL(2)}) \Rightarrow H^{i+j}_{SL(2)}(R, Q)$$

where $u_i : R^+_{C_i} \times_{SL(2)} ESL(2) \hookrightarrow R \times_{SL(2)} ESL(2)$ is the inclusion.

2. If $R$ is smooth and $R^+_{C_i}$ is smooth of codimension $c_i$ in $R$, then we can rewrite the spectral sequence (7.1) as

$$E_1^{ij} = H^{i+j-2c_i}(C_i, Q) \Rightarrow H^{i+j}_{SL(2)}(R, Q)$$

3. The spectral sequence (7.2) degenerates at the first page $E_1$.

Let $r_R : R \to R//SL(2) = M$, $r_{R_1} : R^{C^*}_1 \to R^{C^*}_1//SL(2)$ and $r_{R_2} : R^{C^*}_2 \to R^{C^*}_2//SL(2)$ be the good quotient maps which are $C^*$-equivariant. We have the following lemma useful later.

**Lemma 7.5.** (1) $R^{C^*} = R^{-1}_R(N) \sqcup \bigsqcup_{d=1}^{g-1} r^{-1}_R(F_d)$;

(2) $R^1_{C^*} = \overset{\sim}{R^{-1}_R(N)} \sqcup \bigsqcup_{d=1}^{g-1} (\pi_R \circ r_{R_1})^{-1}(F_d) \sqcup \bigsqcup_{j=1}^{2g} \mathbb{P}(H^0(K_X) \otimes sl(2))$, where $\overset{\sim}{R^{-1}_R(N)}$ is the strict transform of $r^{-1}_R(N)$ under $\pi_{R_1}$;
(3) $\mathbb{R}_2^{\mathbb{C}^*} = \mathbb{R}_1^\mathbb{C}^*(N_1) \sqcup \bigcup_{d=1}^{g-1} (\pi_{r_1} \circ \pi_{r_2})^{-1}(F_d) \sqcup \bigcup_{j=1}^{2g} \pi_{r_2}^{-1}(\mathbb{P}(H^0(K_X) \otimes \text{sl}(2))^{ss}) \sqcup T^+$, where $\mathbb{R}_1^\mathbb{C}^*(N_1)$ is the strict transform of $r_1^{-1}(N_1)$ under $\pi_{r_2}$ and $T^+$ is a $\mathbb{P}^{2g-3}$-bundle over $\mathbb{R}_2^{\mathbb{C}^*} \cap \Sigma$.

**Proof.** (1) It is an immediate consequence from Proposition 7.1 the fact that $(E, \phi)$ is fixed under the $\mathbb{C}^*$-action on $\mathbb{M}$ if and only if $(E, \phi, \beta)$ is fixed under the $\mathbb{C}^*$-action on $\mathbb{R}$ for all $\beta$.

(2) The component of the fixed locus $\mathbb{R}_1^{\mathbb{C}^*}$ not contained in

$$\pi_{r_1}^{-1}(r_1^{-1}(N) - \mathbb{Z}_2^{2g}) \sqcup \bigcup_{d=1}^{g-1} \pi_{r_1}^{-1}(r_1^{-1}(F_d))$$

lies over $\mathbb{Z}_2^{2g}$.

The fiber of $\pi_{r_1}$ over the points of $\mathbb{Z}_2^{2g}$ is $\mathbb{P}Y^{-1}(0)$. The $\mathbb{C}^*$-action on $\mathbb{P}Y^{-1}(0)$ is given by

$$\lambda \cdot \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} \lambda d & \lambda e \\ \lambda f & -\lambda d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

Thus

$$(\mathbb{P}Y^{-1}(0))^{\mathbb{C}^*} = \{ \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathbb{P}Y^{-1}(0) \mid d = e = f = 0 \}$$

$$\sqcup \{ \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathbb{P}Y^{-1}(0) \mid a = b = c = 0 \}$$

$$\cong \mathbb{P}(H^1(O_X) \otimes \text{sl}(2)) \sqcup \mathbb{P}(H^0(K_X) \otimes \text{sl}(2)).$$

Since the disjoint union of $\pi_{r_1}^{-1}(r_1^{-1}(N) - \mathbb{Z}_2^{2g})$ and $2^{2g}$ copies of $\mathbb{P}(H^1(O_X) \otimes \text{sl}(2))$ is isomorphic to the strict transform $\mathbb{R}_1^\mathbb{C}^*(N)$, we get the conclusion.

(3) Let $T := \mathbb{R}_1^{\mathbb{C}^*} \cap \Sigma$. The component of the fixed locus $\mathbb{R}_2^{\mathbb{C}^*}$ not contained in

$$\pi_{r_2}^{-1}(r_1^{-1}(N) - T) \sqcup \bigcup_{d=1}^{g-1} \pi_{r_2}^{-1}((\pi_{r_1} \circ \pi_{r_1})^{-1}(F_d)) \sqcup \bigcup_{j=1}^{2g} \pi_{r_2}^{-1}(\mathbb{P}(H^0(K_X) \otimes \text{sl}(2))^{ss})$$

lies over $T$.

The fiber of $\pi_{r_2}$ over the points of $T \setminus \pi_{r_1}^{-1}(\mathbb{Z}_2^{2g})$ is $\mathbb{P}Y^{-1}(0)$. Since $\mathbb{C}^*$ acts on $\mathbb{P}Y^{-1}(0)$ by $\mu \cdot [a, b, c, d] = [\mu a, b, \mu c, d]$, we have

$$(\mathbb{P}Y^{-1}(0))^{\mathbb{C}^*} = \{ [a, b, c, d] \in \mathbb{P}Y^{-1}(0) \mid b = d = 0 \} \sqcup \{ [a, b, c, d] \in \mathbb{P}Y^{-1}(0) \mid a = c = 0 \}$$

$$\cong P_1 \sqcup P_2,$$

where $P_1 = \mathbb{P}(H^0(L^{-2}K_X) \oplus H^0(L^2K_X)) \cong \mathbb{P}^{2g-3}$ and $P_2 = \mathbb{P}(H^1(L^2) \oplus H^1(L^{-2})) \cong \mathbb{P}^{2g-3}$.

The fiber of $\pi_{r_2}$ over the points $[\varphi]$ of $T \setminus \pi_{r_1}^{-1}(\mathbb{Z}_2^{2g})$ is $\mathbb{P}\text{Hom}^{\omega_\varphi} (\ker \varphi, \text{im} \varphi / \text{im} \varphi)$. The $\mathbb{C}^*$-action on $\mathbb{P}\text{Hom}^{\omega_\varphi} (\ker \varphi, \text{im} \varphi / \text{im} \varphi)$ is given by

$$\mu \cdot [Z_{32}, \ldots, Z_{2g, 2}, Z_{33}, \ldots, Z_{2g, 3}]$$

$$= [Z_{32}, \mu Z_{32}, \ldots, Z_{2g-1, 2}, \mu Z_{2g, 2}, Z_{33}, \mu Z_{33}, \ldots, Z_{2g-1, 3}, \mu Z_{2g, 3}].$$

Thus

$$(\mathbb{P}\text{Hom}^{\omega_\varphi} (\ker \varphi, \text{im} \varphi / \text{im} \varphi))^{\mathbb{C}^*}$$

$$= \{ [Z_{32}, \ldots, Z_{2g, 2}, Z_{33}, \ldots, Z_{2g, 3}] \in \mathbb{P}\text{Hom}^{\omega_\varphi} (\ker \varphi, \text{im} \varphi / \text{im} \varphi) \mid Z_{32} = Z_{52} = \cdots = Z_{2g-1, 2} = Z_{33} = Z_{53} = \cdots = Z_{2g-1, 3} = 0 \}.$$
\[ \square \{ [Z_{32}, \ldots, Z_{2g}, Z_{33}, \ldots, Z_{2g}, 3] \in \mathbb{P} \text{Hom}^{\omega \nu}(\ker \varphi, \im \varphi^\perp / \im \varphi) \mid Z_{42} = Z_{62} = \cdots = Z_{2g, 2} = Z_{43} = Z_{63} = \cdots = Z_{2g, 3} = 0 \} = P_1 \sqcup P_2, \]

where

\[ P_1 = \{ [Z_{32}, \ldots, Z_{2g}, Z_{33}, \ldots, Z_{2g}, 3] \in \mathbb{P} \text{Hom}^{\omega \nu}(\ker \varphi, \im \varphi^\perp / \im \varphi) \mid Z_{32} = Z_{52} = \cdots = Z_{2g-1, 2} = Z_{33} = Z_{53} = \cdots = Z_{2g-1, 3} = 0 \} \cong \mathbb{P}^{2g-3} \]

and

\[ P_2 = \{ [Z_{32}, \ldots, Z_{2g}, Z_{33}, \ldots, Z_{2g}, 3] \in \mathbb{P} \text{Hom}^{\omega \nu}(\ker \varphi, \im \varphi^\perp / \im \varphi)^{ss} \mid Z_{42} = Z_{62} = \cdots = Z_{2g, 2} = Z_{43} = Z_{63} = \cdots = Z_{2g, 3} = 0 \} \cong \mathbb{P}^{2g-3}. \]

Let \( T^+ \) is the \( P_1 \)-bundle over \( T \). Since the \( P_2 \)-bundle over \( T \) is contained in the strict transform \( r_{R_1}^{-1}(N_1) \), we get the conclusion.

\[ \square \]

**Lemma 7.6.**

1. \( \mathbb{R}^+ \frac{r_{R_1}(N)}{r_{R_1}(N)} \) is open in \( \mathbb{R} \).
2. \( (\mathbb{R}_1)^+ \frac{r_{R_1}(N)}{r_{R_1}(N)} \) is open in \( \mathbb{R}_1 \).

**Proof.**

1. We claim that \( \mathbb{R}^+ \frac{r_{R_1}(N)}{r_{R_1}(N)} \) has codimension 0 in \( \mathbb{R} \). Let \( N^s \) denote the stable locus of \( N \).

Since \( r_{R}^{-1}(T^*N^s) \) has codimension 0 in \( \mathbb{R} \), it suffices to show that \( r_{R}^{-1}(T^*N^s) \subset \mathbb{R}^+ \frac{r_{R_1}(N)}{r_{R_1}(N)} \).

If \( (E, \phi, \beta) \in r_{R}^{-1}(T^*N^s) \) and \( (E, \phi, \beta) \) is fixed under the \( C^* \)-action on \( R \), then there exists \( g \in \text{Aut}(E) \) such that \( (g \otimes \text{id}_{K_X})^{-1} \phi g = \lambda \phi \) for all \( \lambda \in \mathbb{C}^* \). Since \( E \) is stable, \( g \) should be of the form \( k \cdot \text{id} \) for some \( k \in \mathbb{C}^* \). Then we have \( \phi = \lambda \phi \) for all \( \lambda \in \mathbb{C}^* \). Thus \( \phi = 0 \) and \( (E, \phi, \beta) \in r_{R}^{-1}(N) \). Hence \( r_{R}^{-1}(T^*N^s) \subset \mathbb{R}^+ \frac{r_{R_1}(N)}{r_{R_1}(N)} \).

2. We claim that \( (\mathbb{R}_1)^+ \frac{r_{R_1}(N)}{r_{R_1}(N)} \) has codimension 0 in \( \mathbb{R}_1 \). Since \( r_{R}^{-1}(T^*N^s) \) doesn’t change under \( \pi_{R_1} \), we have \( (r_{R} \circ \pi_{R_1})^{-1}(T^*N^s) \subset (\mathbb{R}_1)^+ \frac{r_{R_1}(N)}{r_{R_1}(N)} \). Since \( r_{R}^{-1}(T^*N^s) \) has codimension 0 in \( \mathbb{R} \), we get the conclusion.

\[ \square \]

**Lemma 7.7.** \( \mathbb{R}^+ \frac{r_{R_1}(F_d)}{r_{R_1}(F_d)} \) has codimension \( g - 1 + 2d \) in \( \mathbb{R} \).

**Proof.** We claim that \( \mathbb{R}^+ \frac{r_{R_1}(F_d)}{r_{R_1}(F_d)} \) is equal to the locus of stable triples \( (E, \phi, \beta) \in \mathbb{R} \) with unstable \( E \) that fits into an exact sequence

\[ 0 \to L \to E \to L^{-1} \to 0 \]

with \( \deg L = d \) and \( 1 \leq d \leq g - 1 \). Note that the locus of such triples has dimension \( 5g - 5 - 2d \) from [28 3.3].

- Assume that \( (E, \phi, \beta) \in \mathbb{R} \) is semistable with respect to \( \text{SL}(2) \)-action with semistable \( E \).

Then \( \lim_{\lambda \to 0} \lambda \cdot (E, \phi, \beta) = (E, 0, \beta) \). Hence \( (E, \phi, \beta) \in \mathbb{R}^+ \frac{r_{R_1}(N)}{r_{R_1}(N)} \), which means that \( (E, \phi, \beta) \not\in \mathbb{R}^+ \frac{r_{R_1}(F_d)}{r_{R_1}(F_d)} \).
A blowing-up formula for intersection cohomology of moduli of Higgs

Conjecture 7.8. Assume that \((E, \phi, \beta) \in \mathbb{R}\) is stable with respect to \(\text{SL}(2)\)-action with unstable \(E\) that fits into an exact sequence

\[ 0 \to L \to E \to L^{-1} \to 0 \]

with \(\deg L = d\) and \(1 \leq d \leq g - 1\) and that \((E, \phi, \beta)\) is fixed under the \(\mathbb{C}^*\)-action on \(\mathbb{R}\). Note that \((E, \phi)\) is also stable.

Let \(\{U_i\}\) be an open cover of \(X\) so that both \(E\) and \(K_X\) are trivialized on each \(U_i\). Let

\[
\begin{pmatrix}
 a_{ij} & b_{ij} \\
 0 & a_{ij}^{-1}
\end{pmatrix}
\]

be the transition matrices of \(E\) and

\[
\begin{pmatrix}
p_i & q_i \\
r_i & -p_i
\end{pmatrix}
\]

the local Higgs fields, where \(a_{ij}\) are the transition functions of \(L\). Since \(L\) is not \(\phi\)-invariant, \(r_i \neq 0\) for some \(i\). Then for each \(\lambda \in \mathbb{C}^*\) there exists \(A_\lambda \in \text{Aut}(E)\) such that

\[
A_\lambda|_{U_i} = \begin{pmatrix}
e(\lambda) & f_i(\lambda) \\
g_i(\lambda) & h(\lambda)
\end{pmatrix} \in \text{Aut}(E|_{U_i}),
\]

(7.3)

and

\[
\begin{pmatrix}
e(\lambda) & f_j(\lambda) \\
g_j(\lambda) & h(\lambda)
\end{pmatrix} \begin{pmatrix}
a_{ij} & b_{ij} \\
0 & a_{ij}^{-1}
\end{pmatrix} \begin{pmatrix}
e(\lambda) & f_i(\lambda) \\
g_i(\lambda) & h(\lambda)
\end{pmatrix} .
\]

It follows from (7.4) that \(b_{ij}g_i(\lambda) = 0\) for \(i, j\) and

(7.5)

\[ b_{ij}(e(\lambda) - h(\lambda)) = a_{ij}f_i(\lambda) - f_j(\lambda)a_{ij}^{-1}. \]

If \(e(\lambda) = h(\lambda)\) for all \(\lambda\), then \(\{f_i(\lambda)\}\) gives a section \(f(\lambda)\) of \(H^0(L^2)\) by (7.5). We may assume that \(g_i(\lambda) = 0\) for all \(i\). Then from (7.3), we have \(p_i = q_i = r_i = 0\), which is a contradiction to the assumption that \(E\) is unstable. Thus we have \(e(\lambda) \neq h(\lambda)\) for some \(\lambda\). Since \(r_i \neq 0, \lambda \neq 1\) from (7.3). Now we may assume that \(f(\lambda) = 0\). Then \(p_i = 0\) from (7.3). Then \(b_{ij} = 0\) from (7.4). Thus \(E = L \oplus L^{-1}\) and \(\phi = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}\) with \(r \neq 0\). Further since

\[
\begin{pmatrix}
\lambda & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
0 & q \\
r & 0
\end{pmatrix} \begin{pmatrix}
\lambda^{-1} & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & \lambda^2 q \\
r & 0
\end{pmatrix},
\]

\(\phi\) is of the form \(\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}\) with \(r \neq 0\) after taking limit as \(\lambda \to 0\). Hence we can conclude that \((E, \phi, \beta) \in \mathbb{R}^+_r(F_d)\).

\[\square\]

Now we give a conjecture and its corollary that guarantee the assumption of Proposition 7.3 and Proposition 7.4.

**Conjecture 7.8.** \(\lim_{\lambda \to 0} \lambda \cdot x\) uniquely exists in \(\mathbb{R}^{C^*_x}\) for every \(x \in \mathbb{R}\).

**Corollary 7.9.** Assume that Conjecture 7.8 holds.

1. \(\lim_{\lambda \to 0} \lambda \cdot x\) uniquely exists in \(\mathbb{R}^{C^*_x}\) for every \(x \in \mathbb{R}_1\).
2. \(\lim_{\lambda \to 0} \lambda \cdot x\) uniquely exists in \((\mathbb{R}_1^{ss})^{C^*_x}\) for every \(x \in \mathbb{R}_1^{ss}\).
3. \(\lim_{\lambda \to 0} \lambda \cdot x\) uniquely exists in \(\mathbb{R}^{C^*_x}\) for every \(x \in \mathbb{R}_2\).
Proof. (1) Since \( \pi_{R_1} : R_1 \to R \) is proper, it follows from Conjecture \ref{conj:7.8} that \( \lim_{\lambda \to 0} \lambda \cdot x \) uniquely exists in \((R_1)^{C^*} \) for every \( x \in R_1 \).

(2) Since \( R^{ss}_{1} \) is invariant under the \( C^* \)-action on \( R_1 \), the statement follows from the item (1).

(3) Since \( \pi_{R_2} : R_2 \to R^{ss}_{1} \) is proper, it follows from Conjecture \ref{conj:7.8} that \( \lim_{\lambda \to 0} \lambda \cdot x \) uniquely exists in \((R_2)^{C^*} \) for every \( x \in R_2 \).

\[ \square \]

Conjecture \ref{conj:7.8} and Corollary \ref{cor:7.9} guarantee the existences of the spectral sequences \((\ref{eq:7.2})\) in the cases that \( R = R_1, R_1 = R^{ss}_{1} \) and \( R = R_2 \). So we have the following injective pullbacks \( \pi_{R_1}^* \) and \( \pi_{R_2}^* \) on the equivariant cohomologies.

**Lemma 7.10.** Assume that Conjecture \ref{conj:7.8} holds.

1. \( \pi_{R_1}^* : H^i_{\text{SL}(2)}(R) \to H^i_{\text{SL}(2)}(R_1) \) is injective for each \( i \).
2. \( \pi_{R_2}^* : H^i_{\text{SL}(2)}(R^{ss}_{1}) \to H^i_{\text{SL}(2)}(R_2) \) is injective for each \( i \).

**Proof.** (1) By Theorem \ref{thm:7.1} and \cite{11} Corollary 6.12, we have

\[
M^{C^*} = N \sqcup \bigsqcup_{d=1}^{g-1} F_d,
\]

where \( F_0 = N \). Let \( r_R : R \to R/\text{SL}(2) = M \) be the good quotient map. By Lemma \ref{lem:7.5}(1), we have

\[
R^{C^*} = r^{-1}_R(N) \sqcup \bigsqcup_{d=1}^{g-1} r^{-1}_R(F_d) = C_0 \sqcup \bigsqcup_{d=1}^{g-1} \bigsqcup_{k} C_{d,k},
\]

where \( C_0 = r^{-1}_R(N) \) and \( C_{d,k} \) are connected components of \( r^{-1}_R(F_d) \).

Then it follows from Proposition \ref{prop:7.4}, Lemma \ref{lem:7.6}, Lemma \ref{lem:7.7} and Conjecture \ref{conj:7.8} that the spectral sequence

\[
E_1^{ij} = H^i_{\text{SL}(2)}(R^+_{C_0}, \mathbb{Q}) \text{ or } E_1^{ij} = H^i_{\text{SL}(2)}(C_0, \mathbb{Q}) \Rightarrow H^j_{\text{SL}(2)}(R, \mathbb{Q})
\]

which degenerates at \( E_1 \), where \( c_i = g - 1 + 2i \). Further, since the limit map \( R^{C_0} \to C_0 \) is affine by \cite{11} Theorem 1.4.2] and its fiber is connected, the Leray spectral sequence for the limit map gives

\[
H^j_{\text{SL}(2)}(R^+_{C_0}, \mathbb{Q}) = H^j_{\text{SL}(2)}(C_0, \mathbb{Q}) = H^j_{\text{SL}(2)}(r^{-1}_R(N), \mathbb{Q}).
\]

Let \( r_{R_1} : R^{ss}_{1} \to R^{ss}_{1}/\text{SL}(2) = M \) be the good quotient map. By Lemma \ref{lem:7.5}(2), we have

\[
R^{ss}_{1} = r^{-1}_R(N) \sqcup \bigsqcup_{d=1}^{g-1} (\widetilde{r_{R_1}} \circ r_{R_1})^{-1}(F_d) \sqcup \bigsqcup_{j=1}^{2^g} \mathbb{P}(H^0(K_X \otimes sl(2))
\]

which gives

\[
\widetilde{C}_0 \sqcup \bigsqcup_{l=1}^{2^g} \widetilde{C}_{1,l} \sqcup \bigsqcup_{d=1}^{g-1} \bigsqcup_{k} \widetilde{C}_{1+d,k}
\]

where \( \widetilde{C}_0 = r^{-1}_R(N) \), \( \widetilde{C}_{1,l} = \mathbb{P}(H^0(K_X \otimes sl(2))) \) for \( 1 \leq l \leq 2^g \) and \( \widetilde{C}_{1+d,k} \) are connected components of \((\widetilde{r_{R_1}} \circ r_{R_1})^{-1}(F_d) \) for \( 1 \leq d \leq g - 1 \). Note that \((R_1)^{C^*}_{\otimes_1} \cong \widetilde{C}_{1,l} \). Then it
follows from Proposition 7.4, Lemma 7.6, Lemma 7.7, Conjecture 7.8 and Corollary 7.9 that the spectral sequence

\[ E_1^{0j} = H^j_{\text{SL}(2)}((R_1)^+_{C_0}, \mathbb{Q}), E_1^{1j} = H^{1+j-2\tau}_{\text{SL}(2)}(\bigcup_{l=1}^{2^{2g}} \tilde{C}_{1,l}, \mathbb{Q}) \]

or \( E_1^{ij} = H^{i+j-2d_i}(\bigcup_k \tilde{C}_{i,k}, \mathbb{Q}) \) for \( i \geq 2 \)

\[ \Rightarrow H^{i+j}_{\text{SL}(2)}(R_1, \mathbb{Q}) \]

which degenerates at \( E_1 \), where \((R_1)^+_{C_0}\) is a subvariety of \( R_1 \) of codimension 0, \((R_1)^+_{\tilde{C}_{1,l}}\) are smooth subvarieties of \( R_1 \) of codimension \( c = 3g - 2 \) and \((R_1)^+_{\tilde{C}_{1,k}}\) are smooth subvarieties of \( R_1 \) of codimension \( d_i = g - 1 + 2(i - 1) \). Further, since the limit map \((R_1)^+_{\tilde{C}_{0}} \to \tilde{C}_{0}\) is affine by [11, Theorem 1.4.2] and its fiber is connected, the Leray spectral sequence for the limit map gives

\[ H^j_{\text{SL}(2)}((R_1)^+_{C_0}, \mathbb{Q}) = H^j_{\text{SL}(2)}(\tilde{C}_{0}, \mathbb{Q}) = H^j_{\text{SL}(2)}(r_{R_1}^{-1}(N), \mathbb{Q}). \]

Fix \( x \in X \). By [3, Proposition 3.3], \( r_{R_1}^{-1}(N) \) is isomorphic to the moduli space \( N \) of \( \tau \)-stable pairs \((E, \beta)\) where \( \tau \) is a real number, \( E \) is a rank 2 vector bundle and \( \beta \) is an isomorphism \( \beta : E|_x \to \mathbb{C}^2 \). Further \( N \) is a nonsingular irreducible quasi-projective variety of dimension \( 3g \) for some \( \tau \) by [3, Lemma 1.2] and it consists of semistable points with respect to the \( \text{SL}(2) \)-action \( g \cdot (E, \beta) = (E, g \circ \beta) \) on \( N \) by [3, Lemma 3.2]. Moreover we see that \( r_{R_1}^{-1}(N) \) is isomorphic to the blowing-up \( N'_1 \) of \( N \) along \( \mathbb{Z}^{2g} \). Thus

\[ \pi_{R_1}^*: H^*_{\text{SL}(2)}(r_{R_1}^{-1}(N)) \to H^*_{\text{SL}(2)}(r_{R_1}^{-1}(N)) \]

is injective.

Moreover since \( \pi_{R_1}^*: H^*_{\text{SL}(2)}(r_{R_1}^{-1}(F_d)) \to H^*_{\text{SL}(2)}((\pi_{R_1} \circ r_{R_1})^{-1}(F_d)) \) is an isomorphism for each \( 1 \leq d \leq g - 1 \), \( \pi_{R_1}^*: H^*_{\text{SL}(2)}(R_1) \to H^*_{\text{SL}(2)}(R_1) \) is also injective for each \( i \).

(2) By Lemma 7.7, we have

\[ (R_1^{ss})^{C^*} = \bigcup_{d=1}^{g-1} (\pi_{R_1} \circ r_{R_1})^{-1}(F_d) \cup \bigcup_{j=1}^{2^{2g}} \mathbb{P}(H^0(K_X) \otimes sl(2))^{ss} \]

\[ = \bigcup_{l=0}^{2^{2g}} \tilde{C}_{0,l} \cup \bigcup_{d=1}^{g-1} \bigcup_{k=1}^{2^{2g}} \tilde{C}_{1+l,k}. \]

Then it follows from Proposition 7.4, Lemma 7.6, Lemma 7.7, Conjecture 7.8 and Corollary 7.9 that the spectral sequence

\[ E_1^{0j} = H^j_{\text{SL}(2)}((R_1^{ss})^{C^*}_{C_0}, \mathbb{Q}), E_1^{1j} = H^{1+j-2e}_{\text{SL}(2)}(\bigcup_{l=1}^{2^{2g}} \tilde{C}_{1,l}, \mathbb{Q}) \]

or \( E_1^{ij} = H^{i+j-2d_i}(\bigcup_k \tilde{C}_{i,k}, \mathbb{Q}) \) for \( i \geq 2 \)

\[ \Rightarrow H^{i+j}_{\text{SL}(2)}(R_1^{ss}, \mathbb{Q}) \]
which degenerates at $E_1$, where $(R_1^{ss})_C^*$ is a subvariety of $R_1^{ss}$ of codimension 0, $(R_1^{ss})_C^*$ are smooth subvarieties of $R_1^{ss}$ of codimension $c = 3g - 2$ and $(R_1^{ss})^+_{C_i}$ are smooth subvarieties of $R_1^{ss}$ of codimension $d_i = g - 1 + 2(i - 1)$. Further, since the limit map $(R_1^{ss})^+_{C_0} \rightarrow \tilde{C}_0$ is affine by [11] Theorem 1.4.2 and its fiber is connected, the Leray spectral sequence for the limit map gives

$$H^j_{SL(2)}((R_1^{ss})^+_{C_0}, Q) = H^j_{SL(2)}(\tilde{C}_0, Q) = H^j_{SL(2)}(r_{\tilde{R}_1}^{-1}((N_1), Q)).$$

Let $r_{R_2} : R_2^+ \rightarrow R_2^+/SL(2)$ be the good quotient map. By Lemma 7.5 and Corollary 7.7, we have

$$R_2^C = r_{\bar{R}_1}^{-1}(N_1) \cup \bigcup_{d=1}^{g-1} (\pi_{R_1} \circ \pi_{R_2} \circ r_{R_2})^{-1}(F_d) \cup \bigcup_{j=1}^{2^{2g}} \pi_{R_2}^{-1}((\mathbb{P}(H^0(K_X) \otimes sl(2))^{ss}) \cup T^+$$

$$= \tilde{C}_0 \cup \tilde{C}_1 \cup \bigcup_{l=0}^{2^{2g}+1} \tilde{C}_{2,l} \cup \bigcup_{d=1}^{g-1} \tilde{C}_{2+d,k},$$

where $\tilde{C}_0 = r_{\bar{R}_1}^{-1}(N_1)$, $\tilde{C}_1 = T^+$, $\tilde{C}_{2,l}$ and $\tilde{C}_{2+d,k}$ are connected components of $(\pi_{R_1} \circ \pi_{R_2} \circ r_{R_2})^{-1}(F_d)$. Note that $(R_2)_C \cong \tilde{C}_1$. Then it follows from Proposition 7.4, Lemma 7.5, Lemma 7.6, Lemma 7.7 Conjecture 7.8 and Corollary 7.9 that the spectral sequence

$$E_{1j} = H^j_{SL(2)}((R_2)_C^*, Q), E_{1j}^1 = H^{1-j-2d}_{SL(2)}(\tilde{C}_1, Q),$$

$$E_{2j} = H^{2+j-2d}_{SL(2)}(\bigcup_{l=1}^{2^{2g}} \tilde{C}_{2,l}, Q)$$

for $i \geq 3$

$$\Rightarrow H^{1+j}_{SL(2)}(R_2, Q)$$

which degenerates at $E_1$, where $(R_2)^+_C$ is a subvariety of $R_2$ of codimension 0, $(R_2)^+_C$ is a smooth subvariety of $R_2$ of codimension $c = 3g - 3$, $(R_2)^+_C$ are smooth subvarieties of $R_2$ of codimension $d = 3g - 2$ and $(R_2)^+_C$ are smooth subvarieties of $R_2$ of codimension $d_i = g - 1 + 2(i - 2)$. Further, since the limit map $(R_2)^+_C \rightarrow \tilde{C}_0$ is affine by [11] Theorem 1.4.2 and its fiber is connected, the Leray spectral sequence for the limit map gives

$$H^j_{SL(2)}((R_2)^+_C, Q) = H^j_{SL(2)}(\tilde{C}_0, Q) = H^j_{SL(2)}(r_{\bar{R}_1}^{-1}(N_1), Q).$$

We see that $r_{\bar{R}_1}^{-1}(N_1)$ is isomorphic to the blowing-up of $N_1^{ss}$ along $N_1^{ss} \cap \Sigma$. Thus $\pi_{R_2}^*: H^*_C((r_{\bar{R}_1}^{-1}(N_1))) \rightarrow H^*_C((r_{\bar{R}_1}^{-1}(N_1)))$ is injective.

Moreover since both

$$\pi_{R_2}^*: H^*_C((\pi_{R_1} \circ r_{R_1})^{-1}(F_d)) \rightarrow H^*_C((\pi_{R_1} \circ r_{R_1} \circ r_{R_2})^{-1}(F_d))$$

and

$$\pi_{R_2}^*: H^*_C((\mathbb{P}(H^0(K_X) \otimes sl(2))^{ss}) \rightarrow H^*_C((\mathbb{P}(H^0(K_X) \otimes sl(2))^{ss}))$$

are isomorphisms for each $1 \leq d \leq g - 1$ and $1 \leq l \leq 2^{2g}$, $\pi_{R_2}^*: H^*_C((R_1^{ss})) \rightarrow H^*_C((R_2))$ is also injective for each $i$. 

With Lemma 7.10, we can use a standard argument to get the following formula.

**Lemma 7.11.** Assume that Conjecture 7.8 holds.

1. $P_t^{SL(2)}(R_1) = P_t^{SL(2)}(R) + 2g(P_t^{SL(2)}(\mathbb{P}\mathcal{Y}^{-1}(0)) - P_t(\mathbb{B}SL(2)))$.
2. $P_t^{SL(2)}(R_2) = P_t^{SL(2)}(R_1^{ss}) + P_t^{SL(2)}(E_2) - P_t^{SL(2)}(\Sigma)$.

**Proof.** (1) Let $U_x$ be a sufficiently small open neighborhood of $x \in \mathbb{Z}_2^{2g}$ in $R$, let $U_1 = \bigcup_{x \in \mathbb{Z}_2^{2g}} U_x$ and let $\tilde{U}_1 = \pi_{R_1}^{-1}(U_1)$. Let $V_1 = R \setminus \mathbb{Z}_2^{2g}$. We can identify $V_1$ with $R_1 \setminus E_1$ under $\pi_{R_1}$. Then we have the following commutative diagram

$$
\cdots \to H_{SL(2)}^{i-1}(U_1 \cap V_1) \xrightarrow{\alpha} H_{SL(2)}^i(R) \to H_{SL(2)}^i(U_1) \oplus H_{SL(2)}^i(V_1) \xrightarrow{\beta} H_{SL(2)}^i(U_1 \cup V_1) \to \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
\cdots \to H_{SL(2)}^{i-1}(U_1 \cap V_1) \xrightarrow{\tilde{\alpha}} H_{SL(2)}^i(R_1) \to H_{SL(2)}^i(\tilde{U_1}) \oplus H_{SL(2)}^i(V_1) \xrightarrow{\tilde{\beta}} H_{SL(2)}^i(U_1 \cup V_1) \to \cdots ,
$$

where the horizontal sequences are Mayer-Vietoris sequences and the vertical maps are $\pi_{R_1}$. It follows from Conjecture 7.8 and Lemma 7.10 (1) that the vertical maps are injective. So $\ker \alpha = \ker \tilde{\alpha}$ and then $\text{im} \beta = \text{im} \tilde{\beta}$. Thus we have

$$P_t^{SL(2)}(R_1) = P_t^{SL(2)}(R) + P_t^{SL(2)}(\tilde{U}_1) - P_t^{SL(2)}(U_1).$$

By [20] Theorem 3.1 and Proposition 6.2, $U_1$ is analytically isomorphic to $2^{2g}$ copies of $\mathcal{Y}^{-1}(0)$ and then $\tilde{U}_1$ is analytically isomorphic to $2^{2g}$ copies of $B\mathbb{B}L_{0}\mathcal{Y}^{-1}(0)$. Since $B\mathbb{B}L_{0}\mathcal{Y}^{-1}(0)$ is the tautological line bundle $\mathcal{O}_{\mathcal{Y}^{-1}(0)}(-1)$ over $\mathbb{P}\mathcal{Y}^{-1}(0)$ and the cohomology of the fiber is trivial, the Leray spectral sequence of equivariant cohomology associated to the projection $B\mathbb{B}L_{0}\mathcal{Y}^{-1}(0) \to \mathbb{P}\mathcal{Y}^{-1}(0)$ has $E_2$-term given by

$$E_2^{g0} = H_{SL(2)}^p(\mathbb{P}\mathcal{Y}^{-1}(0)) \otimes H^0(\mathbb{C}) = H_{SL(2)}^p(\mathbb{P}\mathcal{Y}^{-1}(0)).$$

Thus the spectral sequence degenerates at the $E_2$-term and then we have

$$H_{SL(2)}^i(B\mathbb{B}L_{0}\mathcal{Y}^{-1}(0)) \cong H_{SL(2)}^i(\mathbb{P}\mathcal{Y}^{-1}(0))$$

for each $i \geq 0$.

Hence we get

$$P_t^{SL(2)}(R_1) = P_t^{SL(2)}(R) + 2^{2g}(P_t^{SL(2)}(\mathbb{P}\mathcal{Y}^{-1}(0)) - P_t(\mathbb{B}SL(2))).$$

(2) Let $U_2$ be a sufficiently small open neighborhood of $\Sigma$ and let $\tilde{U}_2 = \pi_{R_2}^{-1}(U_2)$. Let $V_2 = R_1^{ss} \setminus \Sigma$. We can identify $V_2$ with $R_2 \setminus E_2$ under $\pi_{R_2}$. By Conjecture 7.8, Lemma 7.10 (2) and the same way as in the proof of item (1), we have

$$P_t^{SL(2)}(R_2) = P_t^{SL(2)}(R_1^{ss}) + P_t^{SL(2)}(\tilde{U}_2) - P_t^{SL(2)}(U_2).$$

By [20] Theorem 3.1 and Proposition 6.2, $U_2$ is analytically isomorphic to $C_{\Sigma}R_1$ and then $\tilde{U}_2$ is analytically isomorphic to $B\mathcal{C}_{\Sigma}(C_{\Sigma}R_1)$. Since $C_{\Sigma}R_1$ is the fibration on $\Sigma$ whose fiber is an affine cone and the cohomology of the fiber is trivial, the Leray spectral sequence of equivariant cohomology associated to the projection $C_{\Sigma}R_1 \to \Sigma$ has $E_2$-term given by

$$E_2^{g0} = H_{SL(2)}^p(\Sigma) \otimes H^0(\mathbb{C}) = H_{SL(2)}^p(\Sigma).$$
Thus the spectral sequence degenerates at the $E_2$-term and then we have

$$H^i_{\text{SL}(2)}(C_2 \mathbf{R}_1) \cong H^i_{\text{SL}(2)}(\Sigma)$$

for each $i \geq 0$.

Since $Bl_{C_2}(C_2 \mathbf{R}_1)$ is the tautological line bundle $\mathcal{O}_{E_2}(-1)$ over $E_2$ and the cohomology of the fiber is trivial, the Leray spectral sequence of equivariant cohomology associated to the projection $Bl_{C_2}(C_2 \mathbf{R}_1) \to E_2$ has $E_2$-term given by

$$E_2^{p0} = H^p_{\text{SL}(2)}(E_2) \otimes H^0(\mathbb{C}) = H^p_{\text{SL}(2)}(E_2).$$

Thus the spectral sequence degenerates at the $E_2$-term and then we have

$$H^i_{\text{SL}(2)}(Bl_{C_2}(C_2 \mathbf{R}_1)) \cong H^i_{\text{SL}(2)}(E_2)$$

for each $i \geq 0$.

Hence we get

$$P_t^{\text{SL}(2)}(\mathbf{R}_2) = P_t^{\text{SL}(2)}(\mathbf{R}_1^{ss}) + P_t^{\text{SL}(2)}(E_2) - P_t^{\text{SL}(2)}(\Sigma).$$

\[\square\]

The following blowing-up formula for the first blowing-up $\pi_{\mathbf{R}_1} : \mathbf{R}_1 \to \mathbf{R}$ is what we desire.

**Conjecture 7.12.** $P_t^{\text{SL}(2)}(\mathbf{R}_1^{ss}) = P_t^{\text{SL}(2)}(\mathbf{R}_1^{ss}) + 2^{2g}(P_t^{\text{SL}(2)}(\mathbb{P}^1 - (0)^{ss}) - P_t(\text{BSL}(2))).$

Since $\mathbf{R}_1$ is neither smooth nor projective, we cannot directly apply the Morse theory of [25] developed by F. Kirwan for a proof of Conjecture 7.12.

On the other hand, since $\Sigma$ is smooth (See the proof of Lemma 6.3(1) and [36, Proposition 1.7.10]), $\mathbf{R}_2$ is smooth. So a blowing-up formula for the equivariant cohomology on the second blowing-up

$$\pi_{\mathbf{R}_2} : \mathbf{R}_2 \to \mathbf{R}_1^{ss}$$

follows from the same argument as in the proof of [26, Proposition 7.4] under an assumption suggested in [25, 9.5].

Consider $\mathbf{R}_2$ as a subvariety of $\mathbb{P}^N$ acted on linearly by $\text{SL}(2)$. Assume that $\mathbb{C}^*$ acts on $\mathbb{P}^N$ by $t \cdot x = \text{diag}(x_0(t), \ldots, x_N(t))x$ for $t \in \mathbb{C}^*$, where $x_0, \ldots, x_N$ are characters of $\mathbb{C}^*$ identified with points of $\text{Lie}(\mathbb{C}^*)^*$. For the Morse stratification $\{S_\beta \mid \beta \in \mathcal{B}\}$ of $\mathbb{P}^N$ obtained from the norm square of the moment map $\mu : \mathbb{P}^N \to \text{sl}(2)^*$, $\{\mathbf{R}_2 \cap S_\beta \mid \beta \in \mathcal{B}\}$ is the induced stratification of $\mathbf{R}_2$ and we have

$$\mathbf{R}_2 \cap S_\beta \cong \text{SL}(2) \times p_\beta(\mathbf{R}_2 \cap Y_\beta^{ss})$$

with a retraction $p_\beta : Y_\beta^{ss} \to Z_\beta^{ss}$ of $Y_\beta^{ss}$, where $\mathcal{B}$ is a subset of $\text{Lie}(\mathbb{C}^*)^* = \mathbb{C}$ as the indexing set for the Morse stratification, $Y_\beta = \{(x_0 : \cdots : x_N) \in \mathbb{P}^N \mid x_j = 0 \text{ unless } \alpha_j \cdot \beta = ||\beta||^2 \text{ and } x_j \neq 0 \text{ for some } j \text{ with } \alpha_j \cdot \beta = ||\beta||^2\}, Z_\beta = \{(x_0 : \cdots : x_N) \in \mathbb{P}^N \mid x_j = 0 \text{ unless } \alpha_j \cdot \beta = ||\beta||^2\}$,

$p_\beta : Y_\beta \to Z_\beta$ is a retraction defined as the limit map by the path of steepest descent under the function $\mu_\beta : \mathbb{P}^N \to \mathbb{R}, x \mapsto \mu(x) \cdot \beta$, $Z_\beta^{ss}$ is the set of the semistable points of $Z_\beta$ under the action of a subgroup of $\text{SL}(2)$ and $Y_\beta^{ss} = p_\beta^{-1}(Z_\beta^{ss})$.

**Conjecture 7.13.** $p_\beta(x) \in \mathbf{R}_2$ whenever $x \in \mathbf{R}_2 \cap Y_\beta^{ss}$ for each $\beta \in \mathcal{B}$.

Conjecture 7.13 implies that $\{\mathbf{R}_2 \cap S_\beta \mid \beta \in \mathcal{B}\}$ is equivariantly perfect (See [25, Section 5 and 8]). If Conjecture 7.13 holds, we can use the argument of the proof of [26, Proposition 7.4] to get the following blowing-up formula.
**Proposition 7.14.** Assume that Conjecture 7.8 and Conjecture 7.13 holds. Then
\[
P_t^{SL(2)}(R_2^s) = P_t^{SL(2)}(R_1^s) + P_t^{SL(2)}(E_2^{ss}) - P_t^{SL(2)}(\Sigma).
\]

**Proof.** Assume that Conjecture 7.8 and Conjecture 7.13 holds. Then by [25 Section 5 and 8], \( R_2 \cap S_\beta | \beta \in B \) and \( R_2 \cap S_\beta \cap E_2 | \beta \in B \) are equivariantly perfect. Thus we have
\[
P_t^{SL(2)}(R_2) = P_t^{SL(2)}(R_2^s) + \sum_{\beta \neq 0} t^{2d(\beta)} P_t^{SL(2)}(R_2 \cap S_\beta)
\]
and
\[
P_t^{SL(2)}(E_2) = P_t^{SL(2)}(E_2^{ss}) + \sum_{\beta \neq 0} t^{2d(\beta)} P_t^{SL(2)}(R_2 \cap S_\beta \cap E_2),
\]
where \( d'(\beta) \) (respectively, \( d(\beta) \)) is the codimension of \( R_2 \cap S_\beta \) (respectively, \( R_2 \cap S_\beta \cap E_2 \)) in \( R_2 \) (respectively, \( E_2 \)). Since \( p_\beta : Y^{ss}_\beta \to Z^{ss}_\beta \) is a retraction, we also have
\[
P_t^{SL(2)}(R_2 \cap S_\beta) = P_t^{SL(2)}(SL(2)(R_2 \cap Z^{ss}_\beta))
\]
and
\[
P_t^{SL(2)}(R_2 \cap S_\beta \cap E_2) = P_t^{SL(2)}(SL(2)(R_2 \cap Z^{ss}_\beta) \cap E),
\]
where \( Z^{ss}_\beta \) denotes the set of points of \( S_\beta \) fixed by the one-parameter subgroup generated by \( \beta \). Since \( R_2 \cap Z^{ss}_\beta \subseteq E_2 \) by [26 Lemma 7.6] and \( R_2 \cap S_\beta \not\subseteq E_2 \) for any \( \beta \in B \) by [26 Lemma 7.11], it follows from Lemma 7.11(2) that
\[
P_t^{SL(2)}(R_2) = P_t^{SL(2)}(R_1^s) + P_t^{SL(2)}(E_2^{ss}) - P_t^{SL(2)}(\Sigma)
\]
\[+ \sum_{\beta \neq 0} (t^{2d'(\beta)} - t^{2d(\beta)}) P_t^{SL(2)}(SL(2)(R_2 \cap Z^{ss}_\beta))
\]
\[= P_t^{SL(2)}(R_2^s) + P_t^{SL(2)}(E_2^{ss}) - P_t^{SL(2)}(\Sigma).
\]

\[\square\]

For the blowing-up \( \pi : Bl_{\mathbb{P}Hom_1} \mathbb{P}Y^{-1}(0)^{ss} \to \mathbb{P}Y^{-1}(0)^{ss} \), the assumption of Proposition 7.4 can be verified. So we have the following injective pullback \( \pi^* \) on the equivariant cohomology.

**Proposition 7.15.**
\[\pi^* : H^*_{SL(2)}(\mathbb{P}Y^{-1}(0)^{ss}) \to H^*_{SL(2)}(Bl_{\mathbb{P}Hom_1} \mathbb{P}Y^{-1}(0)^{ss})\]
is injective.

**Proof.** Recall that
\[\pi : Bl_{\mathbb{P}Hom_1} \mathbb{P}Y^{-1}(0)^{ss} \to \mathbb{P}Y^{-1}(0)^{ss}\]
is the blowing-up of \( \mathbb{P}Y^{-1}(0)^{ss} \) along \( \mathbb{P}Hom_1(sl(2), \mathbb{H}^g)^{ss} \). Let \( \tilde{\pi} : (Bl_{\mathbb{P}Hom_1} \mathbb{P}Y^{-1}(0)^{ss})/\!/SL(2) \to \mathbb{P}Y^{-1}(0)^{ss}/\!/SL(2) \) be the blowing-up of \( \mathbb{P}Y^{-1}(0)^{ss}/\!/SL(2) \) along \( \mathbb{P}Hom_1(sl(2), \mathbb{H}^g)^{ss}/\!/SL(2) \) induced from \( \pi \).

We showed that
\[\mathbb{P}Y^{-1}(0)^{ss})^{\mathbb{C}^*} \cong \mathbb{P}(H^1(\mathcal{O}_X) \otimes sl(2))^{ss} \sqcup \mathbb{P}(H^0(K_X) \otimes sl(2))^{ss}\]
in the proof of Lemma 7.3(2).

Let \( s := (\mathbb{P}Y^{-1}(0)^{ss})^{\mathbb{C}^*} \cap \mathbb{P}Hom_1(sl(2), \mathbb{H}^g)^{ss} \). Then we can see that
\[Bl_{\mathbb{P}Hom_1} \mathbb{P}(H^1(\mathcal{O}_X) \otimes sl(2))^{ss} \sqcup \pi^{-1}(\mathbb{P}(H^0(K_X) \otimes sl(2))^{ss}) \sqcup s^+\]
as in the proof of Lemma 7.5-(3), where \( s^+ \) is \( \mathbb{P}^{2g-3} \)-bundle over \( s \).

Now we need to check the assumption of Proposition 7.4. It is easy to check that \( \lim_{\lambda \to 0} \lambda \cdot x \) uniquely exists in \( (\mathbb{P}^1)^{\mathbb{C}^*} \) for every \( x \in \mathbb{P}^1 \). Moreover since \( \pi : Bl_{\text{Phom}_1} \mathbb{P}^1 \to \mathbb{P}^1 \) is proper, we see that \( \lim_{\lambda \to 0} \lambda \cdot \tilde{x} \) uniquely exists in \( (Bl_{\text{Phom}_1} \mathbb{P}^1)^{\mathbb{C}^*} \) for every \( \tilde{x} \in Bl_{\text{Phom}_1} \mathbb{P}^1 \). Thus the assumption of Proposition 7.4 is verified.

Since the pullback \( \pi^* : H^*_\text{SL}(2)(\mathbb{P}(H^1(\mathcal{O}_X) \otimes \mathfrak{s}(2))^s) \to H^*_\text{SL}(2)(Bl_{\text{Phom}_1} \mathbb{P}(H^1(\mathcal{O}_X) \otimes \mathfrak{s}(2))^s) \) is injective, it follows from Proposition 7.4 that \( \pi^* : H^*_\text{SL}(2)(\mathbb{P}(H^1(\mathcal{O}_X))) \to H^*_\text{SL}(2)(Bl_{\text{Phom}_1} \mathbb{P}(H^1(\mathcal{O}_X))) \) is injective.

**Proposition 7.16.** \( P_t^{\text{SL}(2)}(Bl_{\text{Phom}_1} \mathbb{P}^1 \mathbb{P}^1)^{s} = P_t^{\text{SL}(2)}(\mathbb{P}^1)^{s} + P_t^{\text{SL}(2)}(E_{\pi}) - P_t^{\text{SL}(2)}(\text{Phom}_1(s(2), \mathbb{H}^g)^s) \).

**Proof.** By the same way as in the proof of Lemma 7.11-(i), we have
\[
P_t^{\text{SL}(2)}(Bl_{\text{Phom}_1} \mathbb{P}^1)^{s} = P_t^{\text{SL}(2)}(\mathbb{P}^1)^{s} + P_t^{\text{SL}(2)}(\mathcal{U}) - P_t^{\text{SL}(2)}(\mathcal{U})
\]
for some sufficiently open neighborhood \( U \) of \( \text{Phom}_1(s(2), \mathbb{H}^g)^s \) and \( \mathcal{U} = \pi^{-1}(U) \). By [20, Theorem 3.1] and Proposition 6.2 \( \mathcal{U} \) is analytically isomorphic to
\[
C_{\text{Phom}_1(s(2), \mathbb{H}^g)^s} \mathbb{P}^1 \mathbb{P}^1
\]
and then \( \mathcal{U} \) is analytically isomorphic to
\[
Bl_{\text{Phom}_1}(s(2), \mathbb{H}^g)^s(C_{\text{Phom}_1(s(2), \mathbb{H}^g)^s} \mathbb{P}^1 \mathbb{P}^1).
\]
By using the Leray spectral sequence as in the proof of Lemma 7.11, we see that
\[
H^*_\text{SL}(2)(\mathbb{P}^1)^{s} \cong H^*_\text{SL}(2)(\text{Phom}_1(s(2), \mathbb{H}^g)^s)
\]
and
\[
H^*_\text{SL}(2)(Bl_{\text{Phom}_1}(s(2), \mathbb{H}^g)^s(C_{\text{Phom}_1(s(2), \mathbb{H}^g)^s} \mathbb{P}^1 \mathbb{P}^1)) \cong H^*_\text{SL}(2)(E_{\pi}).
\]
Hence we get
\[
P_t^{\text{SL}(2)}(Bl_{\text{Phom}_1} \mathbb{P}^1)^{s} = P_t^{\text{SL}(2)}(\mathbb{P}^1)^{s} + P_t^{\text{SL}(2)}(E_{\pi}) - P_t^{\text{SL}(2)}(\text{Phom}_1(s(2), \mathbb{H}^g)^s).
\]

The blowing-up formula for the equivariant cohomology on the blowing-up
\[
\pi : Bl_{\text{Phom}_1} \mathbb{P}^1 \mathbb{P}^1 \to \mathbb{P}^1 \mathbb{P}^1
\]
is obtained from the same argument as in the proof of [26, Proposition 7.4].

**Proposition 7.17.** \( P_t^{\text{SL}(2)}(Bl_{\text{Phom}_1} \mathbb{P}^1 \mathbb{P}^1)^{s} = P_t^{\text{SL}(2)}(E_{\pi}) - P_t^{\text{SL}(2)}(\text{Phom}_1(s(2), \mathbb{H}^g)^s).\)

**Proof.** By Proposition 5.4, \( Bl_{\text{Phom}_1} \mathbb{P}^1 \mathbb{P}^1 \) is a smooth projective variety. Thus the same argument as in the proof of [26, Proposition 7.4] can be applied.

There is a Morse stratification \( \{ S_\beta \mid \beta \in B \} \) of \( Bl_{\text{Phom}_1} \mathbb{P}^1 \mathbb{P}^1 \) associated to the lifted action of \( \text{SL}(2) \). Then \( \{ S_\beta \mid \beta \in B \} \) is the Morse stratification of \( E_{\pi} \). By [25, Section 5 and 8], \( \{ S_\beta \mid \beta \in B \} \) and \( \{ S_\beta \mid \beta \in B \} \) are equivariantly perfect. Thus we have
\[
P_t^{\text{SL}(2)}(Bl_{\text{Phom}_1} \mathbb{P}^1 \mathbb{P}^1)^{s} = P_t^{\text{SL}(2)}(Bl_{\text{Phom}_1} \mathbb{P}^1 \mathbb{P}^1)^{s} + \sum_{\beta \neq 0} t^{2d(\beta)} P_t^{\text{SL}(2)}(S_\beta)\]
and

\[ P_t^{SL(2)}(E_\pi) = P_t^{SL(2)}(E_{\pi}^{ss}) + \sum_{\beta \neq 0} t^{2d(\beta)} P_t^{SL(2)}(S_\beta \cap E_\pi), \]

where \( d'(\beta) \) (respectively, \( d(\beta) \)) is the codimension of \( S_\beta \) (respectively, \( S_\beta \cap E_\pi \)) in \( Bl_{\mathbb{P}Hom_1} \mathbb{P}Y^{-1}(0)^{ss} \) (respectively, \( E_\pi \)). We also have

\[ P_t^{SL(2)}(S_\beta) = P_t^{SL(2)}(\text{SL}(2)Z_\beta^{ss}) \]

and

\[ P_t^{SL(2)}(S_\beta \cap E_\pi) = P_t^{SL(2)}(\text{SL}(2)Z_\beta^{ss} \cap E_\pi), \]

where \( Z_\beta^{ss} \) denotes the set of points of \( S_\beta \) fixed by the one-parameter subgroup generated by \( \beta \). Since \( Z_\beta^{ss} \subseteq E_\pi \) by \([26, \text{Lemma 7.6}]\) and \( S_\beta \not\subseteq E_\pi \) for any \( \beta \in B \) by \([26, \text{Lemma 7.11}]\), we have

\[ P_t^{SL(2)}(\mathbb{P}Hom_1 \mathbb{P}Y^{-1}(0)^{ss}) = P_t^{SL(2)}(\mathbb{P}Y^{-1}(0)^{ss}) + P_t^{SL(2)}(E_\pi) \]

\[ -P_t^{SL(2)}(\mathbb{P}Hom_1(\text{sl}(2), \mathbb{H}^g)^{ss}) + \sum_{\beta \neq 0} (t^{2d'(\beta)} - t^{2d(\beta)}) P_t^{SL(2)}(\text{SL}(2)Z_\beta^{ss}) \]

\[ = P_t^{SL(2)}(\mathbb{P}Y^{-1}(0)^{ss}) + P_t^{SL(2)}(E_\pi) - P_t^{SL(2)}(\mathbb{P}Hom_1(\text{sl}(2), \mathbb{H}^g)^{ss}). \]

\[ \square \]

7.2. Intersection Poincaré polynomial of the deepest singularity of \( M \). In order to use Theorem \([6.11] (1)\), we must calculate \( IP_t(\mathbb{P}Y^{-1}(0)^{ss} // \text{PGL}(2)) \) and \( IP_t(Y^{-1}(0) // \text{PGL}(2)) \).

Recall that it follows from Proposition \([5.1] \) that

\[ \mathbb{P}Hom_1(\text{sl}(2), \mathbb{H}^g)^{ss} = \text{PGL}(2)Z^{ss} \cong \text{PGL}(2) \times \text{O}(2) Z^{ss} \]

and

\[ \mathbb{P}Hom_1(\text{sl}(2), \mathbb{H}^g)^{ss} // \text{PGL}(2) \cong Z // \text{O}(2) = Z // \text{SO}(2) = Z_1 = \mathbb{P}^{2g-1}, \]

where \( Z = Z_1 \cup Z_2 \cup Z_3 \), \( Z^{ss} \) is the set of semistable points of \( Z \) for the action of \( \text{O}(2) \), \( Z_1 = \mathbb{P}\{v_1 \otimes \mathbb{H}^g\} = Z^{ss} \), \( Z_2 = \mathbb{P}\{v_2 \otimes \mathbb{H}^g\} \) and \( Z_3 = \mathbb{P}\{v_3 \otimes \mathbb{H}^g\} \). Here \( \{v_1, v_2, v_3\} \) is the basis of \( \text{sl}(2) \) chosen in Section \([5] \).

We see that

\[ P_t^+(Z // \text{SO}(2)) = P_1(\mathbb{P}^{2g-1}) = 1 + t^2 + \cdots + t^{2(2g-1)} = \frac{1 - t^{4g}}{1 - t^2} \]

and

\[ P_t^-(Z // \text{SO}(2)) = 0, \]

where \( P_t^+(Z // \text{SO}(2)) \) and \( P_t^-(Z // \text{SO}(2)) \) are Poincaré polynomials of the invariant part and variant part of \( H^*(Z // \text{SO}(2)) \) with respect to the action of \( \mathbb{Z}_2 = \text{O}(2) / \text{SO}(2) \) on \( Z // \text{SO}(2) \).

By \([31, \text{Proposition 3.10}] \) and Theorem \([6.11] (3)\),

\[ IP_t(Bl_{\mathbb{P}Hom_1} \mathbb{P}Y^{-1}(0)^{ss} // \text{PGL}(2)) = IP_t(\mathbb{P}Y^{-1}(0)^{ss} // \text{PGL}(2)) \]

\[ + \sum_{p+q=i} \text{dim}[H^p(\mathbb{P}^{2g-1}) \otimes H^q(I_{2g-3}) \mathbb{Z}_2] t^i \]

\[ = IP_t(\mathbb{P}Y^{-1}(0)^{ss} // \text{PGL}(2)) + \sum_{p+q=i} \text{dim}[H^p(\mathbb{P}^{2g-1}) \mathbb{Z}_2 \otimes H^q(I_{2g-3}) \mathbb{Z}_2] t^i \]

(7.6)

\[ \quad = IP_t(\mathbb{P}Y^{-1}(0)^{ss} // \text{PGL}(2)) + \frac{1 - t^{4g}}{1 - t^2} \cdot \frac{t^2(1 - t^{4g-6})(1 - t^{4g-4})}{(1 - t^2)(1 - t^4)}. \]
Let $\widetilde{\text{Hom}}_2^\omega(sl(2), \mathbb{H}^g)$ be the blowing-up of $\text{Hom}_2^\omega(sl(2), \mathbb{H}^g)$ along $\text{Hom}_1(sl(2), \mathbb{H}^g)^{ss}$ and let

$$\widetilde{Bl}_{\text{Hom}_2^\omega} \text{Hom}_1^\omega \mathbb{P}^{Y-1(0)^{ss}}$$

be the blowing-up of $Bl_{\text{Hom}_1^\omega} \mathbb{P}^{Y-1(0)^{ss}}$ along $\text{Hom}_2^\omega(sl(2), \mathbb{H}^g)^{ss}$.

Assume that $g \geq 3$. Denote $D_1 = Bl_{\text{Hom}_2^\omega} \text{Hom}_1^\omega \mathbb{P}^{Y-1(0)^{ss}}//\text{PGL}(2)$. By [28 Proposition 4.2], $D_1$ is a $\mathbb{P}^5$-bundle over $\text{Gr}^\omega(2, 2g)$ where $\mathbb{P}^5$ is the blowing-up of $\mathbb{P}^5$ (projectivization of the space of $3 \times 3$ symmetric matrices) along $\mathbb{P}^2$ (the locus of rank 1 matrices). Since $D_1$ is a nonsingular projective variety over $\mathbb{C}$,

$$\dim \mathbb{C} H^k(D_1; \mathbb{C}) = \sum_{p+q=k} h^{p,q}(H^k(D_1; \mathbb{C})).$$

Thus it follows from [28 Proposition 5.1] that

$$IP_t(D_1) = P_t(D_1) = E(D_1; -t, -t) = \left( \frac{1 - t^{12}}{1 - t^2} - \frac{1 - t^6}{1 - t^2} + \frac{1 - t^6}{1 - t^2} \right) \cdot \prod_{1 \leq i \leq 3} \frac{1 - t^{4g-12+4i}}{1 - t^{2i}}.$$

Moreover by the proof of [35 Proposition 3.5.1]

$$\text{Hom}_1^\omega (\mathbb{P}^{2g-5})(\mathbb{P}(S^2 \mathcal{A}))$$

where $\mathcal{A}$ is the tautological rank 2 bundle over $\text{Gr}^\omega(2, 2g)$. Following the proof of [35 Lemma 3.5.4], we can see that the exceptional divisor of $D_1$ is a $\mathbb{P}^{2g-5}$-bundle over $\mathbb{P}(S^2 \mathcal{A})$.

By the usual blowing-up formula mentioned in [15 p.605], we have

$$(7.7) \quad IP_t(D_1) = P_t(D_1) = E(D_1; -t, -t) = \left( \frac{1 - t^{12}}{1 - t^2} - \frac{1 - t^6}{1 - t^2} + \frac{1 - t^6}{1 - t^2} \right) \cdot \prod_{1 \leq i \leq 2} \frac{1 - t^{4g-8+4i}}{1 - t^{2i}}.$$

Since $\mathbb{P}(S^2 \mathcal{A})$ is the $\mathbb{P}^2$-bundle over $\text{Gr}^\omega(2, 2g)$,

$$P_t(\mathbb{P}(S^2 \mathcal{A})) = P_t(\mathbb{P}^2)P_t(\text{Gr}^\omega(2, 2g)) = \left( 1 - \frac{t^6}{1 - t^2} \right) \cdot \prod_{1 \leq i \leq 2} \frac{1 - t^{4g-8+4i}}{1 - t^{2i}}$$

by Deligne’s criterion (see [9]).

Therefore it follows from (7.7) that

$$IP_t(Bl_{\text{Hom}_1^\omega} \mathbb{P}^{Y-1(0)^{ss}}//\text{PGL}(2)) = \left( 1 - \frac{t^{12}}{1 - t^2} - \frac{1 - t^6}{1 - t^2} + \frac{1 - t^6}{1 - t^2} \right) \cdot \prod_{1 \leq i \leq 3} \frac{1 - t^{4g-12+4i}}{1 - t^{2i}} \cdot \prod_{1 \leq i \leq 2} \frac{1 - t^{4g-8+4i}}{1 - t^{2i}} \times \frac{t^2(1 - t^{2(2g-5)})}{1 - t^2}.$$

Assume that $g = 2$. In this case, we know from [35 Proposition 2.0.1] that

$$Bl_{\text{Hom}_1^\omega} \mathbb{P}^{Y-1(0)^{ss}}//\text{PGL}(2)$$

is already nonsingular and that it is a $\mathbb{P}^2$-bundle over $\text{Gr}^\omega(2, 4)$. Then by Deligne’s criterion (See [9]),

$$IP_t(Bl_{\text{Hom}_1^\omega} \mathbb{P}^{Y-1(0)^{ss}}//\text{PGL}(2)) = P_t(Bl_{\text{Hom}_1^\omega} \mathbb{P}^{Y-1(0)^{ss}}//\text{PGL}(2))$$

is $\mathbb{P}^2$-bundle over $\text{Gr}^\omega(2, 4)$. Then by Deligne’s criterion (See [9]),
For an arbitrary point $y$ ending at $\gamma$

Theorem 7.20

Combining these with (7.6), we obtain

Proposition 7.18.

$$IP_t(\mathbb{P}^n(0)^{ss} // \text{PGL}(2)) = \frac{(1 - t^{8g-8})(1 - t^{4g})}{(1 - t^2)(1 - t^4)}.$$  

By Lemma 6.8-(1), we also obtain

Proposition 7.19.

$$IP_t(\mathbb{P}^n(0) // \text{PGL}(2)) = \frac{1 - t^{4g}}{1 - t^4}$$

7.3. Intersection Poincaré polynomial of $M$. In this subsection, we compute a conjectural formula for $IP_t(M)$.

7.3.1. Computation for $P^{SL(2)}_t(\mathbb{R})$. We start with the following result.

Theorem 7.20 (Corollary 1.2 in [12]).

$$P^G_C(B^{ss}) = \frac{(1 + t^3)^2 - (1 + t)^2 t^2 g}{(1 - t^2)(1 - t^4)}$$

$$- t^{4g-4} + \frac{t^{2g+2}(1 + t)^2 g}{(1 - t^2)(1 - t^4)} + \frac{(1 - t)^2 t^{4g-4}}{4(1 + t^2)}$$

$$\frac{(1 + t)^2 g t^{4g-4}}{2(1 - t^2)} \left( \frac{2g}{t + 1} + \frac{1}{t^2 - 1} - \frac{1}{2} + (3 - 2g) \right)$$

$$\frac{1}{2} (2g - 1) t^{4g-4} ((1 + t)^2 g^2 - 2 + (1 - t)^2 g^2 - 2).$$

In this subsection, we show that $P^{SL(2)}_t(\mathbb{R}) = P^G_C(B^{ss})$. To prove this, we need some technical lemmas.

Choose a base point $x \in X$ as in Theorem 2.3. Let $E$ be a complex Hermitian vector bundle of rank 2 and degree 0 on $X$. Let $p : E \to X$ be the canonical projection. Let $(G_C)_0$ be the normal subgroup of $G_C$ which fixes the fiber $E|_x$.

We first claim that $(G_C)_0$ acts freely on $B^{ss}$. In fact, assume that $g \cdot (A, \phi) = (A, \phi)$ for $g \in (G_C)_0$. For an arbitrary point $y \in X$ and for any smooth path $\gamma : [0, 1] \to X$ starting at $\gamma(0) = x$ and ending at $\gamma(1) = y \in X$, there is a parallel transport mapping $p_{\gamma} : E|_x \to E|_y$ defined as follows. If $v \in E|_x$, there exists a unique path $\gamma_v : [0, 1] \to V$ such that $p \circ \gamma_v = \gamma$, $\gamma_v(0) = v$ given by $A$. Define $p_{\gamma}(v) = \gamma_v(1)$. By the assumption, $p_{\gamma} \circ g|_x = g|_y \circ p_{\gamma}$. Since $g|_x$ is the identity on $E|_x$, $g|_y$ is also the identity on $E|_y$. Therefore $g$ is the identity on $E$.

Since the surjective map $G_C \to SL(2)$ given by $g \mapsto g|_x$ has the kernel $(G_C)_0$, we have

$$G_C/(G_C)_0 \cong SL(2).$$

Let $G_C/(G_C)_0 \times_{G_C} B^{ss}$ be the quotient space of $G_C/(G_C)_0 \times B^{ss}$ by the action of $G_C$ given by

$$h \cdot (\bar{g}, (A, \phi)) = (\bar{g}h^{-1}, h \cdot (A, \phi))$$
where $\overline{f}$ is the image of $f \in G_C$ under the quotient map $G_C \to G_C/(G_C)_0$. Since $(G_C)_0$ acts freely on $B^{ss}$, $G_C$ acts freely on $G_C/(G_C)_0 \times B^{ss}$.

**Lemma 7.21.** There exists a homeomorphism between $\text{SL}(2) \times_{G_C} B^{ss}$ and $B^{ss}/(G_C)_0$.

**Proof.** By (7.8), it suffices to show that there exists a homeomorphism between $G_C/(G_C)_0 \times G_C B^{ss}$ and $B^{ss}/(G_C)_0$.

Consider the continuous surjective map

\[ q : G_C \times B^{ss} \to G_C/(G_C)_0 \times B^{ss} \]

given by $(g, (A, \phi)) \mapsto ([g], (A, \phi))$. If $G_C$ acts on $G_C \times B^{ss}$ by $h \cdot (g, (A, \phi)) = (gh^{-1}, h \cdot (A, \phi))$, $q$ is $G_C$-equivariant.

Taking quotients of both spaces by $G_C$, $q$ induces the continuous surjective map

\[ q : G_C \times G_C B^{ss} \to G_C/(G_C)_0 \times G_C B^{ss} \]

given by $[g, (A, \phi)] \mapsto [\overline{g}, (A, \phi)]$.

If $(G_C)_0$ acts on $G_C \times G_C B^{ss}$ by $h \cdot [g, (A, \phi)] = [g, h \cdot (A, \phi)]$, $\overline{q}$ is $(G_C)_0$-invariant. Precisely for $g_0 \in (G_C)_0$, $\overline{q}(g, g_0 \cdot (A, \phi)) = [\overline{g}, g_0 \cdot (A, \phi)] = [\overline{g}, (A, \phi)] = [\overline{q}(g, (A, \phi))]$.

Thus $\overline{q}$ induces the continuous surjective map

\[ \overline{q} : \frac{G_C \times G_C B^{ss}}{(G_C)_0} \to G_C/(G_C)_0 \times G_C B^{ss} \]

given by $[g, (A, \phi)] \mapsto [\overline{g}, (A, \phi)]$.

Furthermore $\overline{q}$ is injective. In fact, assume that

\[ \overline{q}([g_1, (A_1, \phi_1)]) = \overline{q}([g_2, (A_2, \phi_2)]) \]

that is,

\[ [\overline{g}_1, (A_1, \phi_1)] = [\overline{g}_2, (A_2, \phi_2)]. \]

Then there is $k \in G_C$ such that $([\overline{g}_1, (A_1, \phi_1)] = [\overline{g}_2 \overline{k}^{-1}, k \cdot (A_2, \phi_2)]$. Then $g_1 = g_2 k^{-1} l$ for some $l \in (G_C)_0$. Thus $[g_1, (A_1, \phi_1)] = [g_2, k^{-1} l \cdot k \cdot (A_2, \phi_2)] = [g_2, k^{-1} l \cdot (A_2, \phi_2)] = [g_2, (A_2, \phi_2)]$ because $(G_C)_0$ is the normal subgroup of $G_C$.

On the other hand, since both $q$ and the quotient map $G_C/(G_C)_0 \times B^{ss} \to G_C/(G_C)_0 \times G_C B^{ss}$ are open, $\overline{q}$ is open. Moreover since the quotient map $G_C \times_{G_C} B^{ss} \to \frac{G_C \times_{G_C} B^{ss}}{(G_C)_0}$ is open, $\overline{q}$ is also open.

Hence $\overline{q}$ is a homeomorphism. Since there is a homeomorphism $\frac{G_C \times_{G_C} B^{ss}}{(G_C)_0} \cong B^{ss}$ given by $[g, (A, \phi)] \mapsto g \cdot (A, \phi)$, we get the conclusion. \qed

**Lemma 7.22.** There is an isomorphism of complex analytic spaces

\[ \text{SL}(2) \times_{G_C} B^{ss} \cong \mathbb{R}. \]

**Proof.** There is a bijection between $\text{SL}(2) \times_{G_C} B^{ss}$ and $\mathbb{R}$. In fact, consider a map

\[ f : \text{SL}(2) \times B^{ss} \to \mathbb{R} \]

given by $(\beta, (A, \phi)) \mapsto ((E, A''), (A, \phi))$, where $(\beta, (A, \phi))$ is the image of $(\beta, (A, \phi))$ of the quotient map $\text{SL}(2) \times B^{ss} \to \text{SL}(2) \times_{G_C} B^{ss}$. $f$ induces the map $\overline{f} : \text{SL}(2) \times_{G_C} B^{ss} \to \mathbb{R}$ given by $[\beta, (A, \phi)] \mapsto ((E, A''), (A, \phi))$. Now we claim that $\overline{f}$ is bijective. Since $M \cong B^{ss}/G_C$ by Theorem 2.7, we can see that $\overline{f}$ is surjective. If $((E_1, A''_1), \phi_1, \beta_1) \cong ((E_2, A''_2), \phi_2, \beta_2)$, then there exists $g \in G_C$ such that
\((A_2, \phi_2) = g \cdot (A_1, \phi_1)\) and \(\beta_2 = \beta_1 g = \beta_1\). Then \(g \in (G_C)_0\) and \([\beta_2, (A_2, \phi_2)] = [\beta_1, g \cdot (A_1, \phi_1)] = [\beta_1 g, (A_1, \phi_1)] = [\beta_1, (A_1, \phi_1)]\). Thus \(\mathcal{F}\) is injective.

Further, the family \(E \times (\text{SL}(2) \times G_C B^{ss})\) over \(X \times (\text{SL}(2) \times G_C B^{ss})\) gives a complex analytic map \(g : \text{SL}(2) \times G_C B^{ss} \to \mathbb{R}\) by [40, Lemma 5.7], and \(\mathcal{F}((\beta, (A, \phi))) = g((\beta, (A, \phi)))\) for all \([\beta, (A, \phi)] \in \text{SL}(2) \times G_C B^{ss}\).

Hence \(f\) is an isomorphism of complex analytic spaces \(\text{SL}(2) \times G_C B^{ss} \cong \mathbb{R}\). \(\square\)

There is a technical lemma for equivariant cohomologies.

**Lemma 7.23.** Let \(H\) be a closed normal subgroup of \(G\) and \(M\) be a \(G\)-space on which \(H\) acts freely. Then \(G/H\) acts on \(M/H\) and

\[H^*_G(M) = H^*_{G/H}(M/H).\]

**Proof.** Use the fibration \(EG \times_G M \cong (EG \times E(G/H)) \times_G M \to E(G/H) \times_G M \cong E(G/H) \times_{G/H} (M/H)\) whose fibers \(EG\) is contractible. \(\square\)

The following equality is an immediate consequence from Lemma 7.21, Lemma 7.22 and Lemma 7.23.

**Proposition 7.24.**

\[P^\text{SL}(2)_{\text{t}}(\mathbb{R}) = P^{G_C}_{\text{t}}(B^{ss})\]

Thus we get the same formula for \(P^\text{SL}(2)_{\text{t}}(\mathbb{R})\) as Theorem 7.20.

7.3.2. **Computation for \(P^\text{SL}(2)_{\text{t}}(\Sigma)\).** In the proof of Lemma 6.3-(1), we observed that

\[\Sigma \cong \mathbb{P}\text{Isom}(O^2_{T^* J}, L_x \oplus L^{-1} x)/O(2).\]

Since \(\{\pm \text{id}\} \subset \text{SL}(2)\) acts trivially on \(\mathbb{P}\text{Isom}(O^2_{T^* J}, L_x \oplus L^{-1} x)\), \(\{\pm \text{id}\} \subset \text{SL}(2)\) also acts trivially on \(\Sigma\). Then

\[ESL(2) \times_{\text{SL}(2)} \Sigma \cong \text{EPGL}(2) \times_{\text{PGL}(2)} \Sigma.\]

Since \(O(2)\) acts on \(\mathbb{P}\text{Isom}(O^2_{T^* J}, L_x \oplus L^{-1} x)\) freely and both actions of \(\text{PGL}(2)\) and \(O(2)\) commute,

\[\text{EPGL}(2) \times_{\text{PGL}(2)} \Sigma \sim \text{EPGL}(2) \times_{\text{PGL}(2)} (\mathbb{P}\text{Isom}(O^2_{T^* J}, L_x \oplus L^{-1} x))\]

\[\cong \text{EO}(2) \times_{O(2)} (\text{EPGL}(2) \times_{\text{PGL}(2)} \mathbb{P}\text{Isom}(O^2_{T^* J}, L_x \oplus L^{-1} x)) \sim \text{EO}(2) \times_{O(2)} T^* J\]

\[\cong (\text{ESO}(2) \times_{\text{SO}(2)} T^* J)/(O(2)/\text{SO}(2)) \cong (\text{BSO}(2) \times T^* J)/\mathbb{Z}_2,\]

where \(\sim\) denotes the homotopic equivalence. Thus

\[P^\text{SL}(2)_{\text{t}}(\Sigma) = P^+_{\text{t}}(\text{BSO}(2))P^+_{\text{t}}(T^* J) + P^-_{\text{t}}(\text{BSO}(2))P^-_{\text{t}}(T^* J),\]

where \(P^+_{\text{t}}(W)\) (respectively, \(P^-_{\text{t}}(W)\)) denotes the Poincaré polynomial of the invariant (respectively, variant) part of \(H^*(W)\) with respect to the action of \(\mathbb{Z}_2\) on \(W\) for a \(\mathbb{Z}_2\)-space \(W\).

**Lemma 7.25.**

\[P^\text{SL}(2)_{\text{t}}(\Sigma) = \frac{1}{(1-t^4)} \left( \frac{1}{2} ((1+t)^{2g} + (1-t)^{2g}) + 2^g \left( \frac{1-t^4}{1-t^2} - 1 \right) + \frac{t^2}{(1-t^4)} \frac{1}{2} ((1+t)^{2g} - (1-t)^{2g}) \right).\]
Proof. Note that $\mathrm{BSO}(2) \cong \mathbb{P}^\infty$. Since the action of $\mathbb{Z}_2 \setminus \{\text{id}\}$ on $H^*(\mathrm{BSO}(2))$ represents reversing of orientation and $\mathbb{P}^n$ possess an orientation-reversing self-homeomorphism only when $n$ is odd, we have $P^+_t(\mathrm{BSO}(2)) = \frac{1}{1-t^4}$ and $P^-_t(\mathrm{BSO}(2)) = \frac{t^2}{1-t^4}$.

Further, by the computation mentioned in [6] Lemma 4.3 and [7] Section 5], we have

$$P^+_t(T^*J) = \frac{1}{2}((1+t)^2g + (1-t)^2g) + 2^g\left(\frac{1-t^4g}{1-t^2} - 1\right)$$

and

$$P^-_t(T^*J) = \frac{1}{2}((1+t)^2g - (1-t)^2g).$$

\[\square\]

7.3.3. Computation for $P^\mathrm{SL(2)}_t(\mathbb{P}Y^{-1}(0)^{ss})$. Since $E//\mathrm{SL}(2)$ has an orbifold singularity and $E//\mathrm{SL}(2) \cong \mathbb{P}C//\mathrm{SL}(2)$ is a free $\mathbb{Z}_2$-quotient of $I_{2g-3}$-bundle over $\mathbb{P}^{2g-1}$ by Lemma 5.3-(5) and Lemma 6.9, we use [31] Proposition 3.10 to have

$$P^\mathrm{SL(2)}_t(E^{ss}) = P_t(E//\mathrm{SL}(2)) = P^+_t(I_{2g-3})P_t(\mathbb{P}^{2g-1})$$

$$= \frac{(1-t^{4g-4})^2}{(1-t^2)(1-t^4)} \cdot \frac{1-t^4g}{1-t^2}.$$ 

By Proposition 7.17,

$$P^\mathrm{SL(2)}_t((B\mathrm{l}_{\text{Hom}_{1}}\mathbb{P}Y^{-1}(0)^{ss})^s)$$

$$= P^\mathrm{SL(2)}_t(\mathbb{P}Y^{-1}(0)^{ss}) + P^\mathrm{SL(2)}_t(E^{ss}) - P^\mathrm{SL(2)}_t(\text{Hom}_{1}(\mathbb{sl}(2), \mathbb{H}^g)^{ss})$$

$$= P^\mathrm{SL(2)}_t(\mathbb{P}Y^{-1}(0)^{ss}) + \frac{(1-t^{4g-4})^2}{(1-t^2)(1-t^4)} \cdot \frac{1-t^4g}{1-t^2} - \frac{1-t^4}{1-t^2} - \frac{1-t^2}{1-t^4} \cdot \frac{1-t^2}{1-t^2}$$

On the other hand

$$P^\mathrm{SL(2)}_t((B\mathrm{l}_{\text{Hom}_{1}}\mathbb{P}Y^{-1}(0)^{ss})^s) = P_t(B\mathrm{l}_{\text{Hom}_{1}}\mathbb{P}Y^{-1}(0)^{ss} //\mathrm{SL}(2))$$

$$= \left(\frac{1-t^{12}}{1-t^2} - \frac{1-t^6}{1-t^2} + \frac{(1-t^6)^2}{1-t^2}\right) \frac{(1-t^{4g-8})(1-t^{4g-4})(1-t^{4g})}{(1-t^2)(1-t^4)(1-t^6)}$$

$$- \frac{1-t^6}{1-t^2} \frac{(1-t^{4g-4})(1-t^{4g}) t^2(1-t^{2(2g-5)})}{(1-t^2)(1-t^4)}$$

$$- \frac{1-t^6}{1-t^2} \frac{(1-t^{4g-4})(1-t^{4g}) t^2(1-t^{2(2g-5)})}{(1-t^2)(1-t^4)}$$

$$- \frac{(1-t^{4g-4})^2}{(1-t^2)(1-t^4)} \cdot \frac{1-t^4g}{1-t^4} + \frac{1-t^4}{1-t^4} \cdot \frac{1-t^4}{1-t^4}.$$
7.3.4. Computation for $P_t^{\text{SL(2)}}(E_2^{ss})$. Since $E_2^{ss}/\text{SL}(2)$ has an orbifold singularity and $E_2^{ss}/\text{SL}(2) \cong \mathbb{P}C_2^{ss}/\text{SL}(2)$ is a free $\mathbb{Z}_2$-quotient of a $I_{2g-3}$-bundle over $\tilde{T}^*J$ by Lemma 6.3(1) and Lemma 6.3(3), we use [31 Proposition 3.10] to have

$$P_t(E_2^{ss}/\text{SL}(2)) = P_t^+(\tilde{T}^*J)P_t^+(I_{2g-3}) + P_t^-(\tilde{T}^*J)P_t^-(I_{2g-3})$$

$$= \left(\frac{1}{2}((1 + t)^{2g} + (1 - t)^{2g}) + 2^{2g}\left(\frac{1 - t^{4g}}{1 - t^2} - 1\right)\right)\left(\frac{1 - t^{4g-4}}{(1 - t^2)(1 - t^4)}\right)$$

$$+ \frac{1}{2}\left(\frac{1}{(1 + t)^{2g}} - \frac{1}{(1 - t)^{2g}}\right) + 2^{2g}\left(\frac{1 - t^{12}}{1 - t^2} + \frac{1 - t^6}{1 - t^2} + \frac{1 - t^6}{1 - t^2}\right)$$

7.3.5. A conjectural formula for $IP_t(M)$. Combining Theorem 6.11 Lemma 7.11 Proposition 7.19 section 7.3.1 section 7.3.2 section 7.3.3 and section 7.3.4 we get a conjectural formula for $IP_t(M)$ as following. The residue calculations show that the coefficients of the terms of $t^i$ are zero for $i > 6g - 6$ and the coefficient of the term of $t^{6g-6}$ is nonzero.

**Theorem 7.26.** Assume that Conjecture 7.8 Conjecture 7.12 and Conjecture 7.13 hold. Then

$$IP_t(M) = \frac{(1 + t^3)^{2g} - (1 + t)^{2g}\ell_{2g}^{g+2}}{(1 - t^2)(1 - t^4)} - t^{4g-4} + \frac{t^{2g+2}(1 + t)^{2g}}{(1 - t^2)(1 - t^4)} + \frac{(1 - t)^{2g}t^{4g-4}}{4(1 + t^2)}$$

$$+ \frac{(1 + t)^{2g}t^{4g-4}}{2(1 - t^2)} - \left(\frac{2g}{t + 1} + \frac{1}{t^2 - 1} - \frac{1}{2} + (3 - 2g)\right)$$

$$+ \frac{1}{2}(2^{2g} - 1)t^{4g-4}((1 + t)^{2g-2} + (1 - t)^{2g-2} - 2)$$

$$+ 2^{2g}\left(\frac{1 - t^{12}}{1 - t^2} - \frac{1 - t^6}{1 - t^2} + (1 - t^{6})\left(\frac{1 - t^{4g-8}}{1 - t^2}\right)\right)$$

$$\frac{1 - t^6}{(1 - t^2)(1 - t^4)} + \frac{t^2(1 - t^{2g-5})}{1 - t^2}$$

$$- \frac{(1 - t^{4g-4})^2}{(1 - t^2)(1 - t^4)} + \frac{1 - t^6}{(1 - t^2)(1 - t^4)} - \frac{1 - t^6}{(1 - t^2)(1 - t^4)}$$

$$- \frac{1 - t^6}{(1 - t^2)(1 - t^4)} - \frac{1 - t^6}{(1 - t^2)(1 - t^4)} - \frac{1 - t^6}{(1 - t^2)(1 - t^4)}$$

which is a polynomial with degree $6g - 6$. 
In low genus, we have $IP_t(M)$ as follows:

- $g = 2 : IP_t(M) = 1 + t^2 + 17t^4 + 17t^6$
- $g = 3 : IP_t(M) = 1 + t^2 + 6t^3 + 2t^4 + 6t^5 + 17t^6 + 6t^7 + 81t^8 + 12t^9 + 396t^{10} + 6t^{11} + 66t^{12}$
- $g = 4 : IP_t(M) = 1 + t^2 + 8t^3 + 2t^4 + 8t^5 + 30t^6 + 16t^7 + 31t^8 + 72t^9 + 59t^{10} + 72t^{11} + 385t^{12} + 80t^{13} + 3955t^{14} + 80t^{15} + 3885t^{16} + 16t^{17} + 259t^{18}$
- $g = 5 : IP_t(M) = 1 + t^2 + 10t^3 + 2t^4 + 10t^5 + 47t^6 + 20t^7 + 48t^8 + 140t^9 + 93t^{10} + 150t^{11} + 304t^{12} + 270t^{13} + 349t^{14} + 522t^{15} + 1583t^{16} + 532t^{17} + 2941t^{18} + 532t^{19} + 72170t^{20} + 280t^{21} + 28784t^{22} + 30t^{23} + 1028t^{24}$

ACKNOWLEDGMENTS

I would like to thank Young-Hoon Kiem for suggesting problem, and for his help and encouragement. This work is based and developed on the second topic in my doctoral dissertation [45].

REFERENCES

[1] A. Białynicki–Birula: Some theorems on actions of algebraic groups, Ann. of Math. (2) 98 (1973), 480–497.
[2] A. Białynicki–Birula: Some properties of the decompositions of algebraic varieties determined by actions of a torus, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24(9) (1976), 667–674.
[3] I. Biswas, T. Gómez and V. Muñoz: Torelli theorem for the moduli space of framed bundles, Math. Proc. Cambridge Philos. Soc. 148 (2010), Issue 3, 409–423.
[4] J. Bernstein and V. Lunts: Equivariant sheaves and functors, Lecture Notes in Math. 1578, Springer-Verlag, Berlin, 1994.
[5] J. Cheeger, M. Goresky and R. MacPherson: $L^2$-cohomology and intersection homology for singular algebraic varieties, Ann. of Math. Stud. 102 (1982), 303–340.
[6] J. Choy and Y-H. Kiem: On the existence of a crepant resolution of some moduli spaces of sheaves on an abelian surface, Math. Z. 252 (2006), no. 3, 557–575.
[7] J. Choy and Y-H. Kiem: Nonexistence of a crepant resolution of some moduli spaces of sheaves on a K3 surface, J. Korean Math. Soc. 44 (2007), no. 1, 35–54.
[8] K. Corlette: Flat $G$-bundles with canonical metrics, J. Differential Geom. 28 (1988), 361–382.
[9] P. Deligne: Théorème de Lefschetz et critères de dégénérescence de suites spectrales, Publ. Math. Inst. Hautes Études Sci. 35 (1968), 107–126.
[10] A. Dimca: Sheaves in Topology, Universitext, Springer-Verlag, 2004.
[11] V. Drinfeld: On algebraic spaces with an action of $G_m$, math.AG/1308.2604.
[12] G.D. Daskalopoulos, J. Weitsman and G. Wilkin: Morse theory and hyperkähler Kirwan surjectivity for Higgs bundles, J. Differential Geom. 87 (2011), no. 1, 81–115.
[13] C. Felisetti: Intersection cohomology of the moduli space of Higgs bundles on a genus 2 curve, J. Inst. Math. Jussieu 22(3) (2023), 1037–1086.
[14] C. Felisetti and M. Mauri: $P=W$ conjectures for character varieties with symplectic resolution, Journal de l’École polytechnique – Mathématiques, 9 (2022), 853–905.
[15] P. Griffiths and J. Harris: Principles of algebraic geometry, Wiley, New York, 1978.
[16] O. García-Prada, J. Heinloth and A. Schmitt: On the motives of moduli of chains and Higgs bundles, J. Eur. Math. Soc. (JEMS) 16 (2014), 2617–2668.
[17] M. Goresky, R. Kottwitz and R. MacPherson: Equivariant cohomology, Koszul duality, and the localization theorem, Invent. Math. 131 (1998), 25–83.
[18] M. Goresky and R. MacPherson: Intersection homology theory, Topology 19 (1980), 135–162.
[19] M. Goresky and R. MacPherson: Intersection Homology II, Invent. Math. 71 (1983), 77–129.
[20] W.M. Goldman and J.J. Milson: The deformation theory of representations of fundamental groups of compact Kähler manifolds, Publ. Math. Inst. Hautes Études Sci. 67 (1988), 43–96.
[21] P. Gothen: The Betti numbers of the moduli space of stable rank 3 Higgs bundles on a Riemann surface, Internat. J. Math. 5 (1994), no. 6, 861–875.
[22] N.J. Hitchin: The self-duality equations on a Riemann surface, Proc. Lond. Math. Soc. (3) 55 (1987), 59–126.
[23] T. Hausel and M. Thaddeus: Relations in the cohomology ring of the moduli space of rank 2 Higgs bundles, J. Amer. Math. Soc. 16 (2003), 303–329.
[24] T. Hausel and F. Rodriguez-Villegas: *Mixed Hodge polynomials of character varieties*, Invent. Math. **174** (2008), 555–624.

[25] F. Kirwan: Cohomology of Quotients in Symplectic and Algebraic Geometry, Math. Notes, Princeton University Press, 1985.

[26] F. Kirwan: *Partial Desingularisations of Quotients of Nonsingular Varieties and their Betti Numbers*, Ann. of Math. (2) **122** (1985), 41–85.

[27] F. Kirwan: *Rational intersection cohomology of quotient varieties*, Invent. Math. **86** (1986), no. 3, 471–505.

[28] Y.-H. Kiem and S.-B. Yoo: *The stringy E-function of the moduli space of Higgs bundles with trivial determinant*, Math. Nachr. **281** (2008), no. 6, 817–838.

[29] F. Kirwan and J. Woolf: An introduction to intersection homology theory (second edition), Chapman & Hall/CRC, Boca Raton, FL, 2006.

[30] M. Logares, V. Muñoz and P.E. Newstead: *Hodge polynomials of $SL(2, \mathbb{C})$-character varieties for curves of small genus*, Rev. Mat. Complut. **26** (2013), 635–703.

[31] M. Mauri: *Intersection cohomology of rank two character varieties of surface groups*, J. Inst. Math. Jussieu (2021), 1–40, doi:10.1017/S1474748021000487.

[32] M. Mereb: *On the E-polynomials of a family of $SL_n$-character varieties*, Math. Ann. **363** (2015), 857–892.

[33] J. Martínez and V. Muñoz: *E-polynomials of $SL(2, \mathbb{C})$-character varieties of complex curves of genus 3*, Osaka J. Math. **53** (2016), 645–681.

[34] J. Martínez and V. Muñoz: *E-Polynomials of the $SL(2, \mathbb{C})$-character varieties of surface groups*, nt. Math. Res. Not. IMRN **2016** (2016), no. 3, 926–961.

[35] K.G. O’Grady: *Desingularized moduli spaces of sheaves on a K3*, math.AG/9708009, 1997.

[36] K.G. O’Grady: *Desingularized moduli spaces of sheaves on a K3*, J. Reine Angew. Math. **512** (1999), 49–117.

[37] O. Schiffmann: *Indecomposable vector bundles and stable Higgs bundles over smooth projective curves*, Ann. of Math. **183** (2016), 297–362.

[38] C.T. Simpson: *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), no. 4, 867–918.

[39] C.T. Simpson: *Higgs bundles and local systems*, Publ. Math. Inst. Hautes Études Sci. **75** (1992), 5–95.

[40] C.T. Simpson: *Moduli of representations of the fundamental group of a smooth projective variety. I*, Publ. Math. Inst. Hautes Études Sci. **79** (1994), 47–129.

[41] C.T. Simpson: *Moduli of representations of the fundamental group of a smooth projective variety. II*, Publ. Math. Inst. Hautes Études Sci. **80** (1994), 5–79.

[42] W. Soergel: *Langlands’ philosophy and Koszul duality*, Algebra – Representation theory (Constanta, 2000), NATO Sci. Ser. II Math. Phys. Chem. **28** (2001), 379–414.

[43] G. Williamson: *Modular representations and reflection subgroups*, arXiv:2001.04569, 2020.

[44] J. Woolf: *The decomposition theorem and the intersection cohomology of quotients in algebraic geometry*, J. Pure Appl. Algebra. **182** (2003), no. 2-3, 317–328.

SANG-BUM YOO, DEPARTMENT OF MATHEMATICS EDUCATION, GONGJU NATIONAL UNIVERSITY OF EDUCATION, GONGJU-SI, CHUNGCHEONGNAM-DO, 32553, REPUBLIC OF KOREA

Email address: sbyoo@gjue.ac.kr