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Optimal control of hybrid variable-order fractional coronavirus (2019-nCov) mathematical model; numerical treatments

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ABSTRACT
A novel coronavirus is a serious global issue and has a negative impact on the economy of Egypt. According to the publicly reported data, the first case of the novel corona virus in Egypt was reported on 14 February 2020. Total of 96753 cases were recorded in Egypt from the beginning of the pandemic until the eighteenth of August, where 96,581 individuals were Egyptians and 172 were foreigners. Recently, many mathematical models have been considered to better understand coronavirus infection. Most of these models are based on classical integer-order derivatives which can not capture the fading memory and crossover behavior found in many biological phenomena. Therefore, we study the coronavirus disease in this paper by exploring the dynamics of COVID-19 infection using new variable-order fractional derivatives. This paper presents an optimal control problem of the hybrid variable-order fractional model of Coronavirus. The variable-order fractional operator is modified by an auxiliary parameter in order to satisfy the dimensional matching between the both sides of the resultant variable-order fractional equations. Existence, uniqueness, boundedness, positivity, local and global stability of the solutions are proved. Two control variables are considered to reduce the transmission of infection into healthy people. To approximate the new hybrid variable-order operator, Grünwald-Letnikov approximation is used. Finite difference method with a hybrid variable-order operator and generalized fourth order Runge-Kutta method are used to solve the optimality system. Numerical examples and comparative studies for testing the applicability of the utilized methods and to show the simplicity of these approximation approaches are presented. Moreover, by using the proposed methods we can concluded that, the model given in this paper describes well the confirmed real data given by WHO about Egypt.

1. Introduction
The new emerged infectious disease that is known the coronavirus disease (COVID-19), which is a high contagious viral infection that started in December 2019 in China city Wuhan and spread very fast to the rest of the world. According to the publicly reported data, the first case of the novel corona virus in Egypt was reported on 14 February 2020. It has a negative impact on the economy of Egypt.

Fractional calculus is the generalization of classical calculus. Mathematical models with non-integer order operators provide a better understanding of a phenomena. Further, models with fractional order derivatives are capable to capture the fading memory and crossover behavior and provides a greater degree of accuracy. Mathematical models with fractional derivative gives more insights about a disease under consideration (Arenas et al. (2016), Ali et al. (2020), Singh (2020), Singh et al. (2021)). One of the most effective and reliable operators is the hybrid fractional operator, it can be expressed as a linear combination of the Riemann-Liouville fractional integral and the Caputo fractional derivative, this operator is general than Caputo fractional operator Baleanu et al. (2020). Also, the hybrid fractional operator is more suitable to describe the biological phenomena than the Caputo operator Sweilam et al. (2020b).

Recently, many mathematical models have been considered to better understand the coronavirus infection. Most of these models are based on classical integer-order derivative which can not capture the fading memory and crossover behavior found in many biological phenomena. The variable-order fractional derivatives (VOFDs) can describe the memory and hereditary properties in several processes and materials because the fractional derivatives are defined in terms of an integral form over all history of the interval (Samko et al. (1993), Sun et al.

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Therefore the derivatives models that described using variable order fractional derivative are useful and proper for the epidemics models. In the epidemic applications models which have been constructed depending on the variable order fractional derivatives have been shown to produce a better appropriate real data conclusions than the models which structured depending on the integer order derivative (Sweilam et al. (2020b), Madubueze et al. (2020)) and the references cited therein. Recently, Sweilam and Al-Mekhlafi introduced some numerical studies for variable-order optimal control (VOC) models, for more details see (Sweilam et al. (2020a)-Sweilam et al. (2020c)).

The design of the present study is to extend the model of COVID-19 which is given as integer order model in Eikenberry et al. (2020) to hybrid variable-order fractional model. The positivity, boundedness, existence, uniqueness and the basic reproduction number of the present model will be discussed. We consider the available data of daily ongoing COVID-19 pandemic see for example (Machado and Lopes (2020), Ali et al. (2020), Bonyah et al. (2020), Danane et al. (2021), Yadav et al. (2021)).

The analytical exact solution of like differential models of fractional variable-order is very difficult to be obtained Area et al. (2016). To approximate the solutions of these models, it is important to develop some numerical method. The finite difference methods (FDM) have been mostly updated to approximate the solution of the variable-order fractional differential equations (Area et al. (2016)-Chen et al. (2012)). The fractional optimal control (FOC) of diseases treatment has become popular in biology refers to the minimization (maximization) of functional objective with dynamic constraints, on state and control variables, such that this conditions have derivative of fractional order. Some numerical methods to find approximation solutions of some types of FOCs were recorded (Bonyah et al. (2020), Ameen et al. (2022), Sweilam et al. (2020b), Madubueze et al. (2020)) and the references cited therein. Recently, Sweilam and Al-Mekhlafi introduced some numerical studies for variable-order optimal control (VOC) models, for more details see (Sweilam et al. (2020a)-Sweilam et al. (2020c)).

The aim of the present study is to extend the model of COVID-19 which is given as integer order model in Eikenberry et al. (2020) to hybrid variable-order fractional model. The positivity, boundedness, existence, uniqueness and the basic reproduction number of the present model will be discussed. We consider the available data of daily confirmed cases in Egypt from 9 March to 13 June, 2020 WHO (2021). Furthermore, we transformed the classical model by newly developed hybrid variable-order time fractional model. Moreover, we extended the necessary optimality conditions in Agrawal (2008) using hybrid variable-order time fractional derivatives. Then we constructed numerical method using the discretization of new hybrid variable-order operator of Caputo proportional constant Baleanu et al. (2020) and FDM (CPC-FDM) to solve the obtained optimality system numerically. The FDM using the discretization of Caputo variable-order operator (C-FDM) is derived from CPC-FDM as spatial case. Also, we extended the generalized fourth order Runge-Kutta method (GRK4M) to solve the variable-order model in Caputo case. Comparative study with the three proposed methods will be given.

To the best of our knowledge, the numerical studies of optimal control for hybrid variable-order fractional Coronavirus (2019-nCov) mathematical model in Egypt have never been explored.

The rest of this paper is structured as follows: Some mathematical tools of VOFD are given in section 2. The hybrid variable-order fractional Coronavirus model with control is introduced and some properties of the proposed model such as the boundedness, positivity existence, uniqueness, local and global stability are proved in section 3. Formulation of hybrid variable-order optimal control of the proposed model is given in section 4. Numerical schemes for solving the optimality system are given and the stability and convergence of the proposed method are proved in section 5. We simulated numerically the proposed model and we stated the results to display the applicability and the efficiency the proposed methods in section 6. Finally, the conclusion of the paper is offered in the last section 7.

2. Preliminaries and notations

In this section, we recall some important definitions of the fractional calculus used throughout the remaining sections of this paper.

**Definition 2.1.** Let \( \Omega = [a, b] \), \(-\infty < a < b < +\infty, a \in \mathbb{C}, b \in \mathbb{C}, \Re(a) > 0, \) \( f(t) \) be a continuous function. The left-side and the right-side Riemann-Liouville’s derivatives of order \( \alpha \) for a continuous function \( f(t) \) are defined respectively by Podlubny (1999)

\[
D^\alpha_L f(t) = \left( \frac{d}{dt} \right)^\alpha \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > a, \tag{1}
\]

\[
D^\alpha_R f(t) = \left( \frac{d}{dt} \right)^\alpha \int_t^b \frac{f(s)}{(s-t)^{1-\alpha}} ds, \quad t < b. \tag{2}
\]

Where \( n = \lfloor \Re(a) \rfloor + 1 \).

**Definition 2.2.** Let \( \Omega = [a, b] \), \(-\infty < a < b < +\infty, a \in \mathbb{C}, b \in \mathbb{C}, \) the left-side and the right-side Caputo’s derivatives of order \( \alpha \) for a function \( f(t), f \in AC^n[a, b] \) are defined, respectively by Podlubny (1999)

\[
\left( ^c D^\alpha_L f(t) \right)(t) = \left( ^c D^\alpha_L g(t) \right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{1-\alpha}} ds, \quad t > a, \tag{3}
\]

\[
\left( ^c D^\alpha_R f(t) \right)(t) = \left( ^c D^\alpha_R g(t) \right)(t) = \frac{(1-n)}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(s)}{(s-t)^{1-\alpha}} ds, \quad t < b. \tag{4}
\]

Where \( n = \lfloor \Re(a) \rfloor + 1, \Re(a) \in \mathbb{R}, \).

- If \( 0 < \alpha < 1 \), we obtain Almeida and Torres (2010):

\[
\int_a^b g(t)^{\frac{d}{dt}} D^\alpha f(t) dt = \int_a^b f(t)^{\frac{d}{dt}} g(t)^{\frac{d}{dt}} (t) dt + l^\alpha^{-1} g(t) f(t)^{\frac{d}{dt}}. \tag{5}
\]

\[
\int_a^b g(t)^{\frac{d}{dt}} D^\alpha f(t) dt = \int_a^b f(t)^{\frac{d}{dt}} g(t)^{\frac{d}{dt}} (t) dt - l^\alpha^{-1} g(t) f(t)^{\frac{d}{dt}}. \tag{6}
\]

**Proposition 2.1.** Podlubny (1999) Let \( D^\alpha_L f \), \( D^\alpha_R f \) are the left-side and right-side Riemann-Liouville’s fractional derivatives of \( f(t) \) and \( ^c D^\alpha_L f \), \( ^c D^\alpha_R f \) are the left-side and right-side Caputo’s fractional derivatives of \( f(t) \), \( \alpha \notin \mathbb{N} \), then for \( n = \lfloor \Re(a) \rfloor + 1, \)

\[
D^\alpha_L f(t) = ^c D^\alpha_L f(t) + \sum_{k=0}^{n-1} \frac{(t-a)^{(k-\alpha)}}{\Gamma(k-\alpha+1)} f^{(k)}(a), \tag{5}
\]

\[
D^\alpha_R f(t) = ^c D^\alpha_R f(t) + \sum_{k=0}^{n-1} \frac{(b-t)^{(k-\alpha)}}{\Gamma(k-\alpha+1)} f^{(k)}(b). \tag{6}
\]

**Proposition 2.2.** If

\[
f(a) = f'(a) = f''(a) = \ldots = f^{n-1}(a) = 0, \text{ then } \]

\[
D^\alpha_L f(t) = ^c D^\alpha_L f(t), \]

and

\[
f(b) = f'(b) = f''(b) = \ldots = f^{n-1}(b) = 0, \text{ then } \]

\[
D^\alpha_R f(t) = ^c D^\alpha_R f(t). \]
Definition 2.3. Baleanu et al. (2020) The Caputo proportional
variable-order fractional hybrid operator (CP) is deﬁned either as general way:

\[
C^\alpha_0 D^\alpha_C y(t) = \left( \int_0^t (y(s)K_i(s, \alpha(t)) + y'(s)K_0(s, \alpha(t))) (t-s)^{-\alpha(t)} ds \right) \frac{1}{\Gamma(1-\alpha(t))} \\
= (K_i(t, \alpha(t))y(t) + K_0(t, \alpha(t))y'(t)) \left( \frac{t^{-\alpha(t)}}{\Gamma(1-\alpha(t))} \right).
\]

(7)

where, \(K_0(a(t), t) = a(t)t^{\alpha(t)}, K_1(a(t), t) = (1 - a(t))t^{\alpha(t)}\), 0 < \(a(t) < 1\). Or as the Caputo proportional constant variable-order fractional
hybrid operator (CPC) Baleanu et al. (2020):

\[
C_{0}^{\alpha} D_{C}^{\alpha} y(t) = \left( \int_0^t (t-s)^{-\alpha(t)} (y(s)K_i(\alpha(t)) + y'(s)K_0(\alpha(t))) ds \right) \frac{1}{\Gamma(1-\alpha(t))} \\
= K_i(\alpha(t))_{0}^{\alpha} D_{C}^{\alpha} y(t) + K_0(\alpha(t))_{0}^{\alpha} D_{C}^{\alpha} y(t),
\]

(8)

where, \(K_0(\alpha(t)) = a(t)Q^{1-\alpha(t)}, K_1(\alpha(t)) = (1 - a(t))Q^{\alpha(t)}\), \(Q\) is a constant.

Definition 2.4. The inverse operators to the variable-order fractional
CPC derivatives is given by Baleanu et al. (2020):

\[
C_{0}^{\alpha} D_{C}^{\alpha} y(t) = \left( \int_0^t \exp \left( \frac{K_i(\alpha(t))}{K_0(\alpha(t))} (t-s) \right) \frac{1}{\Gamma(1-\alpha(t))} \right) \frac{1}{\Gamma(1-\alpha(t))} \\
= \left( \int_0^t \exp \left( \frac{K_i(\alpha(t))}{K_0(\alpha(t))} (t-s) \right) \frac{1}{\Gamma(1-\alpha(t))} \right) \frac{1}{\Gamma(1-\alpha(t))}.
\]

(9)

3. A hybrid variable-order mathematical model

The COVID-19 model which presented in Elkenberry et al. (2020)
will be developed here to hybrid variable-order fractional model. Two controls variables \(u_1\), and \(u_2\) are added; \(u_1\) represents the precautionary measures such as social distancing, abundant use of face mask etc to reduce the transmission of infection into healthy people. \(u_2\) represents the isolation rate for infectious not hospitalized individuals at time. The variable-order fractional operator is modiﬁed by an auxiliary parameter \(\mu\) in order to satisfy the dimensional matching between the both sides of the resultant variable-order fractional equations. In this way, the left side possesses the dimension of \(day^{-1}\) Ullah and Baleanu (2021).

\[
\begin{align*}
1-\mu^{-\alpha(t)} & C_{0}^{\alpha} D_{C}^{\alpha} S = -\beta(1-u_1)I + \eta A, \\
1-\mu^{-\alpha(t)} & \frac{C_{0}^{\alpha} D_{C}^{\alpha} E}{N} = \beta((1-u_1)I + \eta A), \\
1-\mu^{-\alpha(t)} & \frac{C_{0}^{\alpha} D_{C}^{\alpha} I}{N} = \alpha_1 E - \phi I - \gamma_1 R - r_1 u_2 I, \\
1-\mu^{-\alpha(t)} & \frac{C_{0}^{\alpha} D_{C}^{\alpha} A}{N} = (1-\alpha_1) E - \gamma_2 A, \\
1-\mu^{-\alpha(t)} & \frac{C_{0}^{\alpha} D_{C}^{\alpha} H}{N} = \phi I - \delta H - \gamma_3 H, \\
1-\mu^{-\alpha(t)} & \frac{C_{0}^{\alpha} D_{C}^{\alpha} R}{N} = \gamma_1 I + \gamma_2 A + \gamma_3 H + r_1 u_2 I, \\
1-\mu^{-\alpha(t)} & \frac{C_{0}^{\alpha} D_{C}^{\alpha} D}{N} = \delta H, \\
I + R + A + S + E = N.
\end{align*}
\]

with the initial conditions

\[
\begin{align*}
S(0) = s_0 \geq 0, & \quad E(0) = e_0 \geq 0, & \quad I(0) = i_0 \geq 0, & \quad A = a_0 \geq 0, & \quad H = h_0 \geq 0, \\
R(0) = r_0 \geq 0, & \quad D = d_0 \geq 0.
\end{align*}
\]

(11)

where, \((R)\) denotes the recovered class, \((A)\) denotes the hospitalized
asymptomatic infectious class, \((I)\) denotes the symptomatic infectious
class, (D) denotes the cumulative deaths, (S) denotes the class of susceptible, (E) denotes the exposed class, and (H) denotes the hospitalized class, we assume that some fraction of symptomatic infectious individuals progress to the hospitalized class H are unable to pass the disease to the general public Eikenberry et al. (2020).

Boundness of the proposed model solution can be verified by adding all equations of system (10) as follows:

\[ cpcD^{\mu(t)}_0N_0(t) = 0, \quad N_0(0) = A, \geq 0, \]  

(12)

where, \( N_0 \) is total summation of population in (10) and A is a constant. The solution of (12) is given as follows Baleanu et al. (2020):

\[ N(t) = \begin{cases} 0 & \text{if } t \geq 0 \end{cases} \]

The inequality (13) implies that \( \lim_{t \to \infty} N(t) = A \). Therefore, the conclusion \( N(t) \geq 0 \), at \( t \to \infty \). Then the solutions of the system (10) are bounded.

**Lemma 3.1.** Under the initial conditions (11), all the solutions of the system (10) remain nonnegative for \( t \geq 0 \).

**Proof.** By the initial conditions (11), it was discovered that:

\[ \frac{1}{\mu^{(\alpha(t))}(t)} cpcD^{\mu(t)}_0S |_{t=0} \leq 0, \]

\[ \frac{1}{\mu^{(\alpha(t))}(t)} cpcD^{\mu(t)}_0E |_{t=0} \leq (I + \eta A) S \leq 0, \]

\[ \frac{1}{\mu^{(\alpha(t))}(t)} cpcD^{\mu(t)}_0I |_{t=0} \leq \alpha t \eta E + 0, \]

\[ \frac{1}{\mu^{(\alpha(t))}(t)} cpcD^{\mu(t)}_0A |_{t=0} \leq (1 - \alpha t) \eta E - \eta A, \]

\[ \frac{1}{\mu^{(\alpha(t))}(t)} cpcD^{\mu(t)}_0H |_{t=0} \leq \phi I \leq 0, \]

\[ \frac{1}{\mu^{(\alpha(t))}(t)} cpcD^{\mu(t)}_0R |_{t=0} \leq \phi I + \phi \eta A + \phi \eta H \geq 0, \]

\[ \frac{1}{\mu^{(\alpha(t))}(t)} cpcD^{\mu(t)}_0D |_{t=0} \leq \delta H \geq 0. \]

**3.1. Uniqueness and existence**

Here, we will use the fixed point theory. Let the system (10) is written as follows:

\[ cpcD^{\mu(t)}_0y(t) = q(y(t), t), \quad y(0) = y_0 \geq 0. \]

(15)

\[ y(t) = (I.H.S.A.D.E.R. u_1(t), u_2(t))^T \] denotes the state variables and q is a vector of continuous function such that:

\[ \| B[y_1(t)] - B[y_2(t)] \|_{K} \leq \frac{1}{K_0(a(t))} \int_0^t \exp \left( - \frac{K_1(a(t))}{K_0(a(t))} (t - s) \right) \frac{RL}{\omega} D^{\alpha(t)}_t q(y(s), s) ds \|_K \]

\[ \leq \frac{Y^{(\alpha(t))}_{max}}{\Gamma(a(t) - 1)K_0(a(t))} \int_0^t (t - s)^{\alpha(t) - 2} q(y_1(s), s) - q(y_2(s), s) ds \|_K, \]

\[ \leq \frac{Y^{(\alpha(t))}_{max}}{\Gamma(a(t) - 1)K_0(a(t))} \| y_1(t) - q(y_1(t), t) \|_K, \]

\[ \leq \frac{M^3Y^{(\alpha(t))}_{max}}{\Gamma(a(t) - 1)K_0(a(t))} \| y_1(t) - q(y_1(t), t) \|_K. \]

(20)

\[ B[y_1(t)] - B[y_2(t)] \|_{K} \leq \frac{1}{K_0(a(t))} \int_0^t \exp \left( - \frac{K_1(a(t))}{K_0(a(t))} (t - s) \right) \frac{RL}{\omega} D^{\alpha(t)}_t q(y(s), s) ds \|_K \]

\[ \leq \frac{Y^{(\alpha(t))}_{max}}{\Gamma(a(t) - 1)K_0(a(t))} \int_0^t (t - s)^{\alpha(t) - 2} q(y_1(s), s) - q(y_2(s), s) ds \|_K, \]

\[ \leq \frac{Y^{(\alpha(t))}_{max}}{\Gamma(a(t) - 1)K_0(a(t))} \| y_1(t) - q(y_1(t), t) \|_K, \]

\[ \leq \frac{M^3Y^{(\alpha(t))}_{max}}{\Gamma(a(t) - 1)K_0(a(t))} \| y_1(t) - q(y_1(t), t) \|_K. \]
Then, we have:
\[ \| B[ y_1(t)] - B[ y_2(t)] \|_k \leq L \| y_1(t) - y_2(t) \|_k, \]  
(21)

where
\[ L = \frac{M^T \theta(\alpha(t)) \omega(\alpha(t))}{\Gamma(\alpha - 1)K_i(\alpha(t))} \]

The operator B is called a contraction, if L < 1. Then (10) a unique solution.

3.2. Local stability and existence of positive equilibrium point

In the following, the basic reproduction number is calculated. The local stability of the disease-free equilibrium (DFE) is explored using the next generation operator method. It follows that the matrices F of new infection terms and V of the remaining transfer terms associated with the baseline model are given, respectively, by

\[ F = \begin{pmatrix} 0 & \beta y_2(t) \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ V = \begin{pmatrix} \sigma & 0 & 0 \\ -\alpha \sigma (\phi + y_1(t)) & 0 & 0 \\ -1 - \alpha \sigma & 0 & 0 \end{pmatrix} \]

The basic reproduction number of the model, denoted by \( R_0 \), is given by

\[ \rho(FV^{-1}) = R_0 = \mu^{-\omega(t)} \left( \begin{pmatrix} \beta y_1(t) + \beta y_2(t)(1 - \alpha_1) \end{pmatrix} \right) \]  
(22)

Theorem 3.3. The DFE (\( E_0 \)) of the model (10) is locally asymptotically stable (LAS) if \( R_0 < 1 \) and unstable if \( R_0 > 1 \).

Proof. The local stability of model (10) is analyzed by the Jacobian matrix of the system (10) evaluated at the Cov-19 DFE given by

\[ J(E_0) = \begin{pmatrix} 0 & 0 & -\beta \\ -\sigma & \beta & \beta y_2(t) \\ 0 & -\alpha \sigma (\phi + y_1(t)) & 0 \end{pmatrix} \]

After compute the characteristic equation of the above matrix we have the eigenvalues, \( \lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0 \) and \( \lambda_4 < 0 \), if \( R_0 < 1 \). So the system (10) is locally stable if \( R_0 < 1 \) and unstable if \( R_0 > 1 \).

3.3. Global asymptotic stability analysis of the disease-free equilibrium

Theorem 3.4. If \( R_0 < 1 \), then the system (10) is globally stable.

Proof. Let us consider the following Lyapunov function for the purpose of investigating the stability of the disease free equilibrium for the sub model

\[ L_1 = \mu^{-\omega(t)} \left( \begin{pmatrix} \beta y_1(t) + \beta y_2(t)(1 - \alpha_1)(\phi + y_1(t)) \end{pmatrix} \right) \]  
(23)

where

\[ \frac{d}{dt} D^\theta \mu(t) L_2 = \mu^{-\omega(t)} \begin{pmatrix} \beta y_1(t) \left( 1 - \frac{A^T}{A} \right) c_{D^\theta} D^\theta \mu(t) + \frac{\beta}{(\phi + y_1(t))} \left( 1 - \frac{P}{T} \right) c_{D^\theta} D^\theta \mu(t) \end{pmatrix}. \]  
(24)

Substituting the right hand sides of (10) into the above, we have

\[ L_1 = \mu^{-\omega(t)} \left( \begin{pmatrix} \beta y_1(t) + \beta y_2(t)(1 - \alpha_1)(\phi + y_1(t)) \end{pmatrix} \right) \]  
(25)

which can be simplified into:

\[ L_1 = \mu^{-\omega(t)} \left( \begin{pmatrix} \beta y_1(t)(1 - \alpha_1)\sigma E + \frac{\beta y_2(t)}{(\phi + y_1(t))} \end{pmatrix} \right) \]  
(26)

Simplifying further, we have that

\[ L_1 \leq R_0 - \mu^{-\omega(t)} \rho(y + I) \]  
(27)

Clearly \( R_0 < 1 \) is satisfied since the above inequality with positive parameters shows that \( c_{D^\theta} D^\theta \mu(t) L_2 \) is negative. Thus we can now say that the disease free equilibrium is globally asymptotically stable if \( R_0 < 1 \) and unstable if \( R_0 > 1 \).

4. Prerequisite optimality conditions for the studied problem

Consider the system (10) in \( \mathbb{R}^2 \), let

\[ \Omega = \{ (u_1(t), u_2(t)) | u_1, u_2 | \in [0, T], \} \]

be the admissible control set. We will define the objective functional as follows:

\[ J(u_1, u_2) = \int_0^T \left( f(t) + B_1 u_1^2(t) + B_2 u_2^2(t) \right) dt, \]  
(28)

where, \( B_1 \) and \( B_2 \) are weight constants. The aim now is to find \( u_1(t), u_2(t) \) such that the following cost functional is minimum:

\[ J(u_1, u_2) = \int_0^T \eta(t, S, E, I, A, R, D, u_1, u_2) dt, \]  
(29)

subject to the constraints

\[ c_{D^\theta} D^\theta \mu(t) = 0. \]  
(30)

We will use a kind of Pontryagin’s maximum principle in variable order fractional case, this idea in fractional order is given by Agrwal in Agrawal (2008) and by variable order using caputo definition are given in some references (Sweilam et al. (2020a)-Sweilam et al. (2020e)), here we will extended the necessary optimal control numerically to hybrid variable-order fractional derivatives as follows:

The Hamiltonian is define as follows:

\[ H(t, S, E, I, A, R, D, u_1, u_2, \lambda) = \eta(t, S, E, I, A, R, D, u_1, u_2, \lambda) + \sum_{i=1}^{7} \lambda_i E(t, S, E, I, A, R, D, u_1, u_2). \]  
(31)

Then the necessary optimal control conditions are given as follows:

\[ c_{D^\theta} D^\theta \mu(t) \frac{\partial H}{\partial \theta_i} = \frac{\partial H}{\partial \theta_i}, \quad i = 1, \ldots, 7, \]  
(32)

where

\[ \theta_i = \{ t, S, E, I, A, R, D, u_1, u_2, \lambda \}, \quad i = 1, \ldots, 7 \].

(33)

\[ \frac{\partial H}{\partial \theta_i} = \begin{pmatrix} 1, 2, \end{pmatrix} \]  
(34)
and it is also required that the Lagrange multipliers satisfies:
\[ \lambda_i(T_j) = 0, \quad i = 1, 2, \ldots, 7. \] (35)

**Theorem 4.1.** There exists optimal control variables \( u_1, u_2 \) with the corresponding solutions \( S', E', I', A', R', D', \) that minimizes \( J(u_1, u_2) \) over \( \Omega \). Furthermore, there exists adjoint variables \( \lambda_i, i = 1, 2, 3, \ldots, 7 \), satisfy the following:

(i) **Adjoint equations:**
\[
\frac{1}{\mu^{1-\alpha_0}} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_1 = \left( -\beta \left( u_i(T) \right) \right) \frac{\mu^{1-\alpha_0}}{N} + \beta \left( u_i(T) \right) \frac{\mu^{1-\alpha_0}}{N},
\]
\[
\frac{1}{\mu^{1-\alpha_0}} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_2 = ( -\sigma \lambda_2 + a_1 \sigma \lambda_3 + (1 - a_1) \sigma \lambda_4 ),
\]
\[
\frac{1}{\mu^{1-\alpha_0}} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_3 = \left( 1 - \beta S \lambda_1 \left( 1 - u_1 \right) / N + \beta S \lambda_2 \left( 1 - u_1 \right) / N - \phi \lambda_3 - \gamma_4 \lambda_3 ight),
\]
\[
\frac{1}{\mu^{1-\alpha_0}} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_4 = \left( -\beta \lambda_1 \mu S + \beta \lambda_2 \mu S - \gamma_4 \lambda_4 + \gamma_4 \lambda_3 \right),
\]
\[
\frac{1}{\mu^{1-\alpha_0}} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_5 = ( -\delta \lambda_5 - \gamma_4 \lambda_5 + \gamma_4 \lambda_4 + \delta \lambda_7 ),
\]
\[
\frac{1}{\mu^{1-\alpha_0}} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_6 = \left( 1 - \mu^{1-\alpha_0} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_1 = 0 \right).
\]

(ii) **The transversality conditions**
\[ \lambda_i(T_j) = 0, \quad i = 1, 2, \ldots, 7. \] (37)

(iii) **Optimality conditions:**
\[ H(t, S, E, I, A, R, D, u_1, u_2, \lambda_1, \lambda_2) = \min_{0 \leq t, u_1, u_2} \left( H(t, S, E, I, A, R, D, u_1, u_2, \lambda_1, \lambda_2) \right). \] (38)

Moreover:
\[
u_1 = \min \left\{ \lambda_1, \lambda_2 \right\}
\]
\[
u_2 = \min \left\{ \lambda_1, \lambda_2 \right\}
\]

Proof. Equations (36) can be obtained from (32), where:
\[
H = \frac{1}{2} \mu^{1-\alpha_0} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_1 + \frac{1}{2} \mu^{1-\alpha_0} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_2 + \frac{1}{2} \mu^{1-\alpha_0} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_3 + \frac{1}{2} \mu^{1-\alpha_0} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_4 + \frac{1}{2} \mu^{1-\alpha_0} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_5 + \frac{1}{2} \mu^{1-\alpha_0} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_6 + \frac{1}{2} \mu^{1-\alpha_0} \mu^{1-\alpha_0} D^{\alpha_0}_{0} \lambda_7 = 0.
\]

The definition of all parameters of system (10).

| Name | Definition | Value (per day$^{-1}$) | Reference |
|------|------------|------------------------|-----------|
| $\sigma$ | Transition exposed to infectious | (1/1.5) | Eikenberry et al. (2020) |
| $\eta$ | Coefficient transmission due to super-spreaders | 1.9 | Fitting |
| $\beta$ | Infectious contact rate | Dimensionless | 2.5 |
| $\gamma_A$ | Recovery rate, Asymptomatic | (1/7) | Eikenberry et al. (2020) |
| $\gamma_H$ | Recovery rate, hospitalized | (1/14) | Eikenberry et al. (2020) |
| $\delta$ | Rate of disease induced death of infected class | 0.015 | Eikenberry et al. (2020) |
| $\gamma_R$ | Recovery rate, symmetric | (1/7) | Eikenberry et al. (2020) |
| $\alpha_1$ | A part of infections that will be symptomatic | 0.5 | Eikenberry et al. (2020) |
| $\gamma$ | The rate of recovery | Dimensionless | (1/7) |
| $\phi$ | Hospitalization rate | 0.0025 | Eikenberry et al. (2020) |

5. Numerical method for solving VOC

5.1. Generalized fourth order runge-kutta method

Consider the following variable-order fractional derivatives equation:
\[
\frac{D^\alpha_0 u(t)}{D^\alpha_0 t^\alpha} = f(t, u(t)), \quad 1 \geq \alpha > 0, \quad T_j \geq t > 0,
\]
\[
y(0) = y_0.
\]

Using GR4M Milici et al. (2019), the approximate solution of (43) is given as follows:
\[
y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4),
\]
\[
K_1 = \kappa(T_n, y_n),
\]
\[
K_2 = \kappa \left( T_n + \frac{1}{2} K_1, y_n + \frac{1}{2} K_1 \right),
\]
\[
K_3 = \kappa \left( T_n + \frac{1}{2} K_2, y_n + \frac{1}{2} K_2 \right),
\]
\[
K_4 = \kappa(T_n + K_3, y_n + K_3).
\]
where \( \kappa = \frac{r^{(\alpha/2)}}{\Gamma(\alpha/2+1)} \).

5.2. Stability of generalized fourth order runge-kutta method

For simplicity we will use the following test problem to study the stability of GRK4M:

\[
\begin{equation}
\frac{d^0 y(t)}{d^0 t} = \alpha y(t), \quad \nu < 0, \quad 1 \geq \alpha(t) > 0, \quad T_f = T > 0.
\end{equation}
\]

Using GRK4M Milici et al. (2019), equation (45) written as follows:

\[
y(t_{i+1}) = y(t_i) + \frac{1}{6} \frac{\omega^0 T_i}{\Gamma(1 + \alpha(t_i))} y(t_i), \quad i = 0, 1, ..., n - 1.
\]

(46)

The stability analysis of GRK4M is given similar to the GEM method Sweilam et al. (2020a), when the terms are regrouped, then we have the following equation:

\[
K_1(\alpha(t_i)) \frac{y_{i+1} - \sum_{j=0}^{i} \omega_0 y_{i+1-j}}{t^{(\alpha/2)} - \sum_{j=0}^{i} \omega_1 y_{i+1-j}} + K_0(\alpha(t_i)) \frac{y_{i+1} - \sum_{j=0}^{i+1} \omega_1 y_{i+1-j}}{t^{(\alpha/2)} - \sum_{j=0}^{i+1} \omega_0 y_{i+1-j}} = \dot{\xi}(t_{i+1}, t_i),
\]

(51)

The stability condition Sweilam et al. (2020a) given as follows:

\[-1 < \left(1 + \frac{1}{6} \frac{\omega^0 T_i}{\Gamma(1 + \alpha(t_i))}\right) < 1.\]

(47)

5.3. CPC-FDM

Consider the following hybrid variable-order fractional derivatives equation:

\[
\begin{equation}
\frac{d^0 \text{CPC} y(t)}{d^0 t} = \frac{d^0 \xi(t, y(t))}{d^0 t}, \quad 1 \geq \alpha(t) > 0, \quad y(0) = y_0.
\end{equation}
\]

(48)

The relation (8) can be written as follows:

\[
\frac{C_{\text{CPC}} D_{\text{CPC}}^\alpha(y(t))}{d^0 t} = \frac{1}{\Gamma(\alpha(t))} \int_0^t (t-s)^{-\alpha(t)}(K_1(\alpha(t))y(s) + K_0(\alpha(t))y'(s))ds,
\]

\[
= K_1(\alpha(t)) D_{\text{GL}}^\alpha y(t) + K_0(\alpha(t)) y_{\text{GL}}^\alpha y(t),
\]

\[
= K_1(\alpha(t)) D_{\text{CPC}}^\alpha y(t) + K_0(\alpha(t)) y_{\text{CPC}}^\alpha y(t),
\]

(49)

where, \( K_1(\alpha(t)), K_0(\alpha(t)) \) are depending only on \( \alpha(t) \). Using GL-approximation and FDM, we can discretize (49) as follows:

\[
\begin{equation}
\frac{C_{\text{CPC}} D_{\text{CPC}}^\alpha(y(t))}{d^0 t} = \frac{K_1(\alpha(t))}{t^{\alpha(t)}} \left( y_{n+1} + \sum_{i=1}^{n+1} \omega_0 y_{n+1-i} \right)
\]

\[+ \frac{K_0(\alpha(t))}{t^{\alpha(t)}} \left( y_{n+1} - \sum_{i=1}^{n+1} \mu_i y_{n+1-i} - q_i y_n \right),
\]

(50)

\[
\begin{equation}
K_1(\alpha(t_i)) \frac{y_{i+1} - \sum_{j=0}^{i} \omega_0 y_{i+1-j}}{t^{(\alpha/2)} - \sum_{j=0}^{i} \omega_1 y_{i+1-j}} + K_0(\alpha(t_i)) \frac{y_{i+1} - \sum_{j=0}^{i+1} \omega_1 y_{i+1-j}}{t^{(\alpha/2)} - \sum_{j=0}^{i+1} \omega_0 y_{i+1-j}} = \dot{\xi}(t_{i+1}, t_i),
\end{equation}
\]

(51)

where, \( \omega_0 = 1, \omega_i = \left(1 - \frac{\omega^0 T_i}{\Gamma(1 + \alpha(t_i))}\right)\omega_0, \rho = \frac{T_f}{T}, \quad N_0 \) is a natural number, \( \mu_i = (-1)^{i-1} \left(\frac{\omega^0 T_i}{\Gamma(1 + \alpha(t_i))}\right) \mu_0 = \alpha(t_n), \quad q_i = \frac{t^{(\alpha/2)}}{\Gamma(1 - \alpha(t_i))} \) and \( i = 1, 2, ..., n + 1 \). Also, let us assume that Scherer et al. (2011):

\[0 < \mu_{i+1} < \mu_i < ... < \mu_1 \leq \alpha(t_n) < 1,
\]

\[0 < q_{i+1} < q_i < ... < q_i = \frac{1}{\Gamma(1 - \alpha(t_n))}.
\]

Remark 1. We have the discretization of finite difference method with Caputo operator (C-FDM), if \( K_1(\alpha(t)) = 0 \) and \( K_0(\alpha(t)) = 1 \) in (51).

5.4. Stability of CPC-FDM

In the following to investigate the stability of the proposed method, consider the test problem of linear fractional differential equation:

\[
\begin{equation}
\left(\frac{d^0 \text{CPC} y(t)}{d^0 t}\right) = A y(t), \quad t > 0, \quad 0 < \alpha(t) \leq 1, \quad A < 0.
\end{equation}
\]

(52)

Using GL-approximation and FDM (49) we can discretize (52) as follows:

Fig. 1. Fitting model (10) verses real data with different \( a(t) \).
Fig. 2. Approximations of the solutions model (10) using GRK4M and final time 58.

Fig. 3. Comparison between the solutions of the proposed model with control and without control using GRK4M.
\[ K_1(\alpha(t_n)) \left( y_{n+1} + \sum_{i=1}^{n+1} \theta_i y_{n+1-i} \right) + K_0(\alpha(t_n)) \left( y_{n+1} - \sum_{i=1}^{n+1} \mu_i y_{n+1-i} \right) - q_{n+1} y_0 \right) = A y_n, \]

put \( C = \frac{K_1(\alpha(t_n))}{\theta_1}, \) \( B = \frac{K_0(\alpha(t_n))}{\mu_1}. \) Then from boundness theorem Arenas et al. (2016) we have:

\[ y_{n+1} = \frac{1}{C + B} \left( A y_n - C \sum_{i=1}^{n+1} \theta_i y_{n+1-i} + B \left( \sum_{i=1}^{n+1} \mu_i y_{n+1-i} + q_{n+1} y_0 \right) \right) \leq y_n, \]

then we have: \( y_1 \leq y_0 \) and \( y_0 \geq y_1 \geq \ldots \geq y_{n-1} \geq y_n \geq y_{n+1}. \) So the proposed scheme is stable.

5.5. Convergence of the results

We can discretize (49) as follows:

\[ \text{Fig. 4. Comparison between the solutions of } I \text{ and } D \text{ with control and without control using CPC-FDM.} \]

\[ \text{Fig. 5. Approximations of the solutions of the proposed model using CPC-FDM in controlled case at different } \alpha(t). \]
\[ CPC \] \[ 0 \]
\[ D \]
\[ \alpha (t) \]
\[ t \]
\[ y(t) \]
\[ t = t_n \]
\[ \tau \]
\[ \alpha (t_n) \]
\[ 1 \]
\[ y_n + 1 \]
\[ \omega_i y_n + 1 \]
\[ \tau \]
\[ \alpha (t_n) \]
\[ K_0 (\alpha (t_n)) \]
\[ \tau \]
\[ \alpha (t_n) \]
\[ 1 \]
\[ y(t_n) \]
\[ 0 \]
\[ 1 \]
\[ \xi (y(t_n), t_n) = T_{KR} \]

\[ \xi (y(t_n), t_n) = T_{KR} \]
where,
\[ \| T_{\mathbf{x}} \|_{\infty} < M, \quad M = C \max_{0 \leq t \leq \tau \leq \infty} |y_{i}(\tau)|, \]
where, \( C = \left( e^{\theta(1)} - 1 + \theta(0) \right) \).

The proposed method is stable and consistent, so it is convergent. Thanks to (Yuste and Quintana-Murillo (2012), Section 4), then the scheme (56) will be convergent.

6. Numerical experiments

In order to perform the numerical simulations in this section, we adopted CPC-FDM (51) and GRK4M (44) to solve the hybrid variable-order fractional optimization systems (42) and (36) numerically with the transversality conditions (37). For the real data fitting of model (10) we have taken some parameter values from the literature and the remaining values are fitted for the data collected for Egypt. We have fitted data of (10) solutions with WHO data for Egypt from 9th March to 13 June, 2020 WHO (2021). According to the publically reported data, the total population of Egypt for the year 2020 is 100500159. Let \( H(0) = 3, E(0) = 0, A(0) = 5, I(0) = 2, D(0) = 0, S(0) = \left( \frac{100500159}{27193} \right) - 10 \) and \( R(0) = 0 \).

We get different parameter values as shown in Table 1. The role of the positive parameters \( B_1, B_2 \) is only to balance the terms size in the equations. Fig. 1 shows the comparison between real data from model (10) and versus the proposed model at different values of \( \mu(t) \). Clear from this figure that the data from WHO and the results of (10) are agreement. Fig. 2 shows the approximations of the solutions model (10) (obtained by GRK4M) and final time 58 at \( \alpha = 0.1 \). Fig. 3 compares the solutions of the proposed model with and without control case using GRK4M at \( \alpha = 0.1 + 0.1 \mathrm{e}^{-\mu(t)}. \) We noted that the number of infected people is reducing in control case. Fig. 4 shows the behavior of the variables \( I \) and \( D \) with and without controls using CPC-FDM at \( \alpha = 1 - \sin(0.008)t \). Fig. 5 depicts the approximations of the solutions of the proposed model using CPC-FDM in controlled case at different \( \alpha(t) \). Fig. 6 shows the approximations of control variables \( u_1, u_2 \) (obtained by CPC-FDM and C-FDM) at \( \alpha = 1 - \sin(0.008)t \). Fig. 7 shows the approximations of the variables \( I, E, A, H \) (obtained by CPC-FDM and C-FDM) at \( \alpha = 0.9 - 0.01t \). We noted that the results which obtained by CPC-FDM is better than the results obtained by C-FDM. Table 2 contains the value of the cost functional for different values of \( \alpha(t) \) using three different methods. We noted that the results obtained by CPC-FDM is better than C-FDM and GRK4M. Table 3 shows the the CPU time for CPC-FDM and GRK4M. We noted that GRK4M is the fastest than CPC-FDM.

7. Conclusions

This article contributes on applying hybrid variable-order fractional optimal control techniques on epidemic models. The combination of variable order fractional derivative and optimal control in the model improves the dynamics and increases complexity of the model. In this work a novel model of hybrid variable-order fractional Coronavirus is presented. The new hybrid variable order fractional CPC operator is more general than the variable order fractional Caputo operator. Moreover, we can derive the variable order fractional Caputo operator as spatial case from CPC operator. Clear the proposed COVID-19 model describes well the WHO data from Egypt WHO (2021). uniqueness, existence and boundedness of (10) are proved. A new parameter \( \mu \) is presented in order to be consistent with the physical model problem. We have successfully applied a kind of Pontryagin’s maximum principle to reduce the transmission of infection into healthy people. CPC-FDM and GRK4M are used to study numerically the control problem. Mathematical analysis for CPC-FDM and GRK4M are given. The results obtained by CPC-FDM in Table 2 is the better than the results obtained by C-FDM and GRK4M. Numerical outcomes are introduced to show the validity and applicability of the proposed scheme. In future, the present study can be extended to distributed order fractional and examines the impact of multiple vaccination strategies on the dynamics of COVID-19 in a population.

CRediT authorship contribution statement

N.H. Sweilam: Data curation, Methodology, Resources, Supervision, Writing – review & editing. S.M. AL-Mekhlafi: Conceptualization, Data curation, Formal analysis, Investigation, Visualization, Writing – original draft. T.M. Al-Ajami: Methodology, Writing – original draft.

Declaration of Competing Interest

The authors have declared no conflict of interest.

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