DERIVED EQUIVALENCE AND FIBRATIONS OVER CURVES AND SURFACES

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Abstract. We prove that the bounded derived category of coherent sheaves on a smooth projective complex variety reconstructs the isomorphism classes of fibrations onto smooth projective curves of genus \(g \geq 2\). Moreover, in dimension at most four, we prove that the same category reconstructs the isomorphism classes of fibrations onto normal projective surfaces with positive holomorphic Euler characteristic and admitting a finite morphism to an abelian variety. Finally, we study the derived invariance of a class of fibrations with minimal base-dimension under the condition that all the Hodge numbers of type \(h^{p,q}(X)\) are derived invariant.

1. Introduction

The bounded derived category \(D(X) \overset{\text{def}}{=} D^b(Coh(X))\) of coherent sheaves on a smooth projective complex variety \(X\) is a homological object that encodes several geometric properties of the variety itself. For instance, if the (anti)canonical bundle of \(X\) is ample (resp. big), then \(D(X)\) reconstructs \(X\) up to isomorphism (resp. \(K\)-equivalence). We recall that two smooth projective varieties \(X\) and \(Y\) are \(K\)-equivalent if there exists a third smooth projective variety \(U\) and two birational morphisms \(X \overset{p}{\leftarrow} U \overset{q}{\rightarrow} Y\) such that \(p^*\omega_X \simeq q^*\omega_Y\). A leading conjecture concerning the role of \(D(X)\) in birational geometry is the \(DK\)-hypothesis (cf. [BO02, Conjecture 4.4], [Kaw18, Conjecture 1.2] and [Huy06, Conjecture 6.24]). This predicts that \(K\)-equivalent varieties are \(D\)-equivalent, that is there exists an equivalence of triangulated categories \(D(X) \simeq D(Y)\). In this paper, we prove that some classes of fibrations of projective varieties that are invariant under \(K\)-equivalence, are also invariant under \(D\)-equivalence.

To begin with, we study the behavior of fibrations onto smooth projective curves under \(D\)-equivalence. Let \(X\) be a smooth projective complex variety and let \(g \geq 1\) be an integer. An irrational pencil of genus \(g\) of \(X\) is a surjective morphism with connected fibers \(f: X \rightarrow C\) onto a smooth projective curve of genus \(g\). We say that two irrational pencils \(f_i: X \rightarrow C_i\) are isomorphic if there exists an isomorphism \(\gamma: C_1 \overset{\sim}{\rightarrow} C_2\) such that \(f_2 = \gamma \circ f_1\). We set

\[
F^g_X = F^{1,g}_X \overset{\text{def}}{=} \{ \text{isomorphism classes of irrational pencils of genus } g \text{ of } X \}.
\]

Our first result regards the derived invariance of irrational pencils onto smooth projective curves \(C\) of genus \(g \geq 2\), or equivalently satisfying \(\chi(\omega_C) > 0\).

Theorem 1. Let \(X\) and \(Y\) be smooth projective complex varieties. If \(\Phi: D(X) \rightarrow D(Y)\) is an equivalence of triangulated categories, then for any integer \(g \geq 2\) the equivalence \(\Phi\) induces a bijection of sets \(\mu_g: F^g_X \rightarrow F^g_Y\) preserving the bases of the fibrations. More specifically, if \(\mu_g(f: X \rightarrow C) = (h: Y \rightarrow D)\), then the curves \(C\) and \(D\) are isomorphic.

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The proof of Theorem \ref{thm1} employs the results of \cite{LP15} concerning the behavior of the non-vanishing loci
\begin{equation}
V^i(\omega_X)_0 \overset{\text{def}}{=} \{ \alpha \in \text{Pic}^0(X) \mid H^i(X, \omega_X \otimes \alpha) \neq 0 \}, \quad i = 0, \ldots, n \overset{\text{def}}{=} \dim X
\end{equation}
attached to the canonical bundle under D-equivalence (the subscript 0 denotes the union of the irreducible components passing through the origin of an algebraic subset; cf. \cite{GL87} and \cite{GL91}). Less informally, the definition of $\mu_g$ relies on the following two ingredients: the derived invariance of $V^{n-1}(\omega_X)_0$, and the existence of a bijection between $F^q_X$ and the set of $g$-dimensional irreducible components of $V^{n-1}(\omega_X)_0$ (cf. Theorems \ref{thm1} and \ref{thm2}). On the other hand, the isomorphism between the base curves of the fibrations is constructed by manipulating the support of the kernel of the equivalence $\Phi$ in the style of Kawamata \cite{Kaw02}. Earlier attempts at proving Theorem \ref{thm1} appear in the base curves of the fibrations is constructed by manipulating the support of the kernel of the equivalence $\Phi$ in the style of Kawamata \cite{Kaw02}. Earlier attempts at proving Theorem \ref{thm1} appear in \cite{LP15} Question 13 (cf. also \cite{Pop13, Corollary 3.4}), and carries applications towards the behavior of the properties of the fundamental group under derived equivalence (cf. \cite{5}).

In the second part of this paper we study the derived invariance of a class of fibrations over normal projective surfaces, which extends the case of irrational pencils of genus $g \geq 2$. A $\chi$-positive 2-higher irrational pencil is a surjective morphism with connected fibers $f : X \rightarrow S$ such that: (i) $S$ is a normal projective surface, (ii) $\chi(\omega_S) > 0$ for some resolution of singularities $\tilde{S} \rightarrow S$, and (iii) there exists a morphism $S \rightarrow \text{Alb}(\tilde{S})$ finite onto its image such that the composition $\tilde{S} \rightarrow S \rightarrow \text{Alb}(\tilde{S})$ equals the Albanese map of $\tilde{S}$. Two such $\chi$-positive 2-higher irrational pencils are isomorphic if there exists an isomorphism between the bases of the fibrations that commutes with the structure morphisms (cf. \cite{3}). We consider the following sets of fibrations attached to $X$:
\begin{equation}
F^2,q_X \overset{\text{def}}{=} \{ \text{isomorphism classes of } \chi \text{-positive 2-higher irrational pencils } f : X \rightarrow S \text{ such that } h^0(\tilde{s},\Omega^1_{\tilde{S}}) = q \}.
\end{equation}

It turns out that the behavior of $F^2,q_X$ under derived equivalence is related to the conjectured derived invariance of the Hodge number $h^{0,2}(X) = \dim H^2(X, O_X)$. In general, all Hodge numbers are expected to be invariant under D-equivalence. We refer to \cite{Huy06}, \cite{PS11} and \cite{Abu17} for partial results in this direction (cf. also Remark \ref{rem5} for a preview).

**Conjecture** ($H^k_n$). Let $n \geq 1$ and $k \geq 0$ be integers such that $k \leq n$. Then we have $h^{0,k}(Z_1) = h^{0,k}(Z_2)$ for all pairs of smooth projective D-equivalent complex varieties $Z_1$ and $Z_2$ of dimension $n$.

**Theorem 2.** Suppose that Conjecture ($H^2_n$) holds for some $n \geq 2$, and let $X$ and $Y$ be smooth projective complex varieties of dimension $n$. If $\Phi : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ is an equivalence of triangulated categories, then for any $q \geq 2$ the equivalence $\Phi$ induces a bijection of sets $\nu_q : F^2,q_X \rightarrow F^2,q_Y$ preserving the bases of the fibrations.

A key tool in the proof of Theorem \ref{thm2} is the derived invariance of the non-vanishing locus $V^{n-2}(\omega_X)_0$. By the results of \cite{LP15}, this invariance is guaranteed as soon as the Hodge number $h^{0,2}(X)$ is invariant. Apart from this fact, the proof of Theorem \ref{thm2} follows the general strategy of Theorem \ref{thm1}. It is important to note that the result \cite{Abu17, Theorem 1.3} by Abuaf proves Conjecture ($H^k_n$) for all $0 \leq k \leq n \leq 4$; therefore Theorem \ref{thm2} holds unconditionally in dimension at most four.

Finally, we extend the notion of $\chi$-positive 2-higher irrational pencil to fibrations with base varieties of arbitrary dimension (cf. \cite{3}). Let $n = \dim X$ and define the non-negative integer
\begin{equation}
b_{\chi > 0}(X) \in \{0, \ldots, n-1\}
\end{equation}
as the minimal dimension of a base variety of a $\chi$-positive $k$-higher irrational pencil of $X$ with $k \in \{1, \ldots, n-1\}$. We declare $b_{\chi>0}(X) = 0$ if and only if $X$ does not carry any $\chi$-positive $k$-higher irrational pencil for all $k \in \{1, \ldots, n-1\}$. The following result studies the derived invariance of $b_{\chi>0}(X)$.

**Theorem 3.** Assume that Conjecture ($H^k_{\chi}$) holds for some $n \geq 2$ and all integers $k \in \{1, \ldots, n\}$. Let $X$ and $Y$ be smooth projective complex varieties of dimension $n$. If $\Phi: \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ is an equivalence of triangulated categories, then we have $b_{\chi>0}(X) = b_{\chi>0}(Y)$. Moreover, if $b \overset{\text{def}}{=} b_{\chi>0}(X) > 0$, then for any integer $q \geq b$ the equivalence $\Phi$ induces a bijection of sets $\sigma_q: F^{b,q}_X \rightarrow F^{b,q}_Y$ preserving the bases of the fibrations.

As in a higher dimension the correspondence between $\chi$-positive higher irrational pencils and irreducible components of non-vanishing loci is not in general one-to-one, the techniques of this paper are not enough to establish the $D$-invariance of all the sets $F^{b,q}_X$ even if Conjecture ($H^k_{\chi}$) would hold for all $k \geq 0$. Along these lines, Caucci and Pareschi have interesting results concerning the derived invariance of fibrations of varieties of maximal Albanese dimension (cf. [CP19 §4]). Finally we refer to §3 for partial results about the invariance of fibrations over threefolds of $D$-equivalent fourfolds.

## 2. Background material

### 2.1. Notation.
Throughout the paper we work over the field of complex numbers. By the term *fibration* we mean a morphism of varieties that is surjective and with connected fibers. We denote by $q(X) = h^1(X, \mathcal{O}_X) = \dim \text{Pic}^0(X)$ the *irregularity* of a smooth projective variety $X$, and denote by $\text{alb}_X : X \rightarrow \text{Alb}(X)$ the Albanese map. The morphism $\text{alb}_X$ is only defined up to the choice of a point in $X$ and any two Albanese maps differ by a translation on $\text{Alb}(X)$. We say that $X$ is of *maximal Albanese dimension* if $\text{alb}_X$ is generically finite onto its image. The irregularity of a normal variety $V$ is defined as $q(V) \overset{\text{def}}{=} q(\tilde{V})$ where $\tilde{V}$ is any resolution of singularities. We say that a fibration $\tilde{f} : \tilde{X} \rightarrow \tilde{V}$ is a non-singular representative of a fibration $f : X \rightarrow V$ if there exist resolutions of singularities $\pi : \tilde{X} \rightarrow X$ and $\rho : \tilde{V} \rightarrow V$ such that $f \circ \pi = \rho \circ \tilde{f}$ (cf. [Mor87 (1.10)]).

### 2.2. The Rouquier isomorphism.
Let $X$ and $Y$ be smooth projective varieties of dimension $n$. We denote by $\mathbf{D}(X) = D^b(\text{Coh}(X))$ and $\mathbf{D}(Y) = D^b(\text{Coh}(Y))$ the bounded derived categories of coherent sheaves on $X$ and $Y$, respectively. We say that $X$ and $Y$ are *$D$-equivalent* if there exists an equivalence of triangulated categories $\Phi : \mathbf{D}(X) \xrightarrow{\sim} \mathbf{D}(Y)$. By [Orl97 Theorem 2.2], this equivalence is of Fourier–Mukai type $\Phi(-) \simeq \Phi_\mathcal{E}(-) \overset{\text{def}}{=} \mathbf{R}p_{2*}(p_1^*(-) \boxtimes \mathcal{E})$. Here the morphisms $p_1$ and $p_2$ are the projections from $X \times Y$ onto the first and second factor, respectively, and $\mathcal{E}$ is an object in $\mathbf{D}(X \times Y) \overset{\text{def}}{=} D^b(\text{Coh}(X \times Y))$, uniquely determined by $\Phi$ up to isomorphism. Any equivalence $\Phi_\mathcal{E}$ induces an isomorphism of algebraic groups

$$F_\mathcal{E} : \text{Aut}^0(X) \times \text{Pic}^0(X) \xrightarrow{\sim} \text{Aut}^0(Y) \times \text{Pic}^0(Y)$$

called *Rouquier's isomorphism* (cf. [Rou11 Théorème 4.18] and [PS11 Equation (3.1)]). We refer to equation (3) for the action of $F_\mathcal{E}$.

Let

$$V^k(\omega_X) \overset{\text{def}}{=} \{ \alpha \in \text{Pic}^0(X) \mid H^k(X, \omega_X \otimes \alpha) \neq 0 \}$$
be the $k$-th non-vanishing locus attached to the canonical bundle. The following theorem describes the images of the loci $V^k(\omega_X)_0$ under $F_\xi$ (cf. \cite{2} for the definition of $V^k(\omega_X)_0$). We refer to \cite[Theorem 12]{LP15} for a more general result.

**Theorem 4** (Lombardi–Popa). Suppose that Conjecture $(H^k)^i$ holds for some integers $n \geq 1$ and $k \in \{0, \ldots, n\}$. If $X$ and $Y$ are smooth projective varieties of dimension $n$ and $\Phi_\xi : \mathbf{D}(X) \xrightarrow{\sim} \mathbf{D}(Y)$ is an equivalence, then we have

\[
F_\xi(\text{id}_X, V^{n-k}(\omega_X)_0) = (\text{id}_Y, V^{n-k}(\omega_Y)_0).
\]

In particular, $F_\xi$ induces an isomorphism of algebraic sets $V^{n-k}(\omega_X)_0 \cong V^{n-k}(\omega_Y)_0$.

**Remark 5.** Conjecture $(H^k)^i$ holds for all integers $n \geq 1$ and $k \in \{0, 1, n-1, n\}$ (cf. \cite[Remark 5.40]{Huy06}, \cite{PS11} and \cite{LP15}). Moreover, it holds for all $n \leq 4$ and $k \in \{0, \ldots, n\}$ (cf. \cite[Theorem 1.3]{Abu17}).

The above Theorem \ref{thm:4} plays an important role in the construction of the bijections $\mu_g$ of Theorem \ref{thm:1}. This goes as follows. By Green–Lazarsfeld’s Theory \cite[Theorem 0.1]{GL91}, the positive-dimensional irreducible components of $V^{n-1}(\omega_X)_0$ are abelian varieties that give rise to irrational pencils $f : X \to C$ of genus $g \geq 1$ (see Lemma \ref{lemma:10} and Remark \ref{remark:17}). Moreover, in \cite[Corollaire 2.3]{Bea92}, Beauville proves that the irrational pencils $f$ must be of genus $g \geq 2$, and hence that

\[
V^{n-1}(\omega_X)_0 = \{O_X\} \cup \bigcup_{g \geq 2} \bigcup_{(f : X \to C) \in F^g_X} \left( f^* \text{Pic}^0(C) \right)
\]

if $q(X) > 0$ (see \cite{11} for the definition of $F^g_X$). In Theorem \ref{thm:25} we will observe that for any integer $g \geq 2$ there are one-to-one correspondences of sets

\[
u_{X,1,g} : F^g_X \to \pi_0^g \left( V^{n-1}(\omega_X)_0 \right), \quad (f : X \to C) \mapsto f^* \text{Pic}^0(C)
\]

between $F^g_X$ and the set of $g$-dimensional irreducible components of $V^{n-1}(\omega_X)_0$. In view of Theorem \ref{thm:4} and Remark \ref{remark:5} we define the bijections $\mu_g : F^g_X \to F^g_Y$ as $\mu_g = \nu_{Y,1,g}^{-1} \circ F_\xi \circ \nu_{X,1,g}$.

### 3. Non-vanishing loci and fibrations

We denote by $X$ a smooth projective variety of dimension $n$.

**3.1. Non-vanishing loci.** The *non-vanishing loci* attached to a coherent sheaf $\mathcal{F}$ on $X$ are the algebraic closed subsets of $\text{Pic}^0(X)$ defined by

\[
V^i(\mathcal{F}) \overset{\text{def}}{=} \{\alpha \in \text{Pic}^0(X) \mid H^i(X, \mathcal{F} \otimes \alpha) \neq 0\} \quad (i \geq 0).
\]

We denote by $V^i(\mathcal{F})_0$ for the union of all the components of $V^i(\mathcal{F})$ passing through the origin.

Of particular interest is the case of the canonical bundle $\mathcal{F} = \omega_X$. Fundamental theorems of Green–Lazarsfeld and Simpson (cf. \cite[Theorem 0.1]{GL91} and \cite{Sim93}) prove that every irreducible component of $V^i(\omega_X)$ is a translate of an abelian subvariety of $\text{Pic}^0(X)$ by a point of finite order. Furthermore, in \cite[Theorem 1]{GL87}, the authors prove that if $X$ is of maximal Albanese dimension, then the **generic vanishing condition**

\[
\text{codim} V^i(\omega_X) \geq i \quad \text{for all} \quad i > 0
\]

holds.

**Lemma 6.** Suppose that $X$ is of maximal Albanese dimension. Then we have $\chi(\omega_X) \geq 0$. Moreover, we have $\chi(\omega_X) > 0$ if and only if $V^0(\omega_X) = \text{Pic}^0(X)$.
Proof. By the inequalities in (4) we have that $V^i(\omega_X) \neq \text{Pic}^0(X)$ for all $i > 0$. Hence, by the deformation invariance of the holomorphic Euler characteristic, one deduces that
\[
\chi(\omega_X) = \chi(\omega_X \otimes \alpha) = h^0(X, \omega_X \otimes \alpha) \geq 0 \quad \text{for any} \quad \alpha \in \text{Pic}^0(X) \backslash \left( \bigcup_{i > 0} V^i(\omega_X) \right).
\]

\[\Box\]

**Proposition 7.** Let $U$ be a normal projective variety, and let $\vartheta: \tilde{U} \to U$ be a resolution of singularities. If there exists a morphism $\alpha: U \to \text{Alb}(\tilde{U})$ such that the composition $\tilde{U} \xrightarrow{\vartheta} U \xrightarrow{\alpha} \text{Alb}(\tilde{U})$ equals the Albanese map $\text{alb}_{\tilde{U}}$, then the morphisms $\vartheta^*$ and $\vartheta_*$ induce an isomorphism of algebraic groups $\text{Pic}^0(\tilde{U}) \simeq \text{Pic}^0(U)$.

Proof. The morphism $\vartheta^*: \text{Pic}^0(U) \to \text{Pic}^0(\tilde{U})$ is surjective as any line bundle $\alpha \in \text{Pic}^0(\tilde{U})$ is a pullback of a topologically trivial line bundle on $\text{Alb}(\tilde{U})$. Suppose now that $\vartheta^*(\alpha_1 \otimes \alpha_2^{-1}) \simeq \mathcal{O}_{\tilde{U}} \simeq \vartheta^* \mathcal{O}_U$ where $\alpha_1$ and $\alpha_2$ are in $\text{Pic}^0(U)$. As $\vartheta_* \mathcal{O}_{\tilde{U}} \simeq \mathcal{O}_U$, the projection formula yields $\alpha_1 \otimes \alpha_2^{-1} \simeq \mathcal{O}_U$.

\[\Box\]

**Remark 8.** If $\vartheta: \tilde{U} \to U$ is a birational morphism between smooth projective varieties, then $\vartheta^*: \text{Pic}^0(U) \to \text{Pic}^0(\tilde{U})$ induces isomorphisms $V^i(\omega_U) \simeq V^i(\omega_{\tilde{U}})$ for all $i \geq 0$.

The following proposition will be useful in the sequel.

**Proposition 9.** Let $U$ be a proper variety and let $\alpha: U \to A$ be a morphism to an abelian variety, finite onto its image. Then any rational map $Y \dashrightarrow U$ from a smooth projective variety $Y$ extends to a morphism.

Proof. If $Y \dashrightarrow U$ is not a morphism, then by [KM98, Corollary 1.5] the variety $U$ contains a rational curve $E$. Therefore $a(E)$ is a rational curve in $A$, which is a contradiction.

\[\Box\]

3.2. **Fibrations.** In this subsection we establish the existence of the bijection $u_{X,1,g}$ mentioned in §2 together with its extension to fibrations over higher-dimensional bases. We begin by defining the following class of varieties (resp. fibrations) which represents a higher-dimensional analog of that of smooth projective curves (resp. irrational pencils) of genus $g \geq 2$.

**Definition 10.** A normal projective variety $V$ is $\chi$-positive if there exists a resolution of singularities $\rho: \tilde{V} \to V$ and a morphism $b: V \to \text{Alb}(\tilde{V})$ finite onto its image such that the following two conditions hold: (i) the composition $\tilde{V} \xrightarrow{\rho} V \xrightarrow{b} \text{Alb}(\tilde{V})$ equals the Albanese map $\text{alb}_{\tilde{V}}$, and (ii) $\chi(\omega_{\tilde{V}}) > 0$.

**Definition 11.** Let $0 < k < n$ be an integer. A $\chi$-positive $k$-higher irrational pencil of $X$ is a fibration $f: X \to V$ onto a $\chi$-positive variety of dimension $k$. Two such fibrations $f_1: X \to V_1$ and $f_2: X \to V_2$ are isomorphic if there exists an isomorphism $\varphi: V_1 \xrightarrow{\sim} V_2$ such that $f_2 = \varphi \circ f_1$.

**Remark 12.** Let $\tilde{V}_i \xrightarrow{\nu_i} V_i \xrightarrow{b_i} \text{Alb}(\tilde{V}_i)$ ($i = 1, 2$) be varieties and morphisms as in Definition 10. Any birational map $\psi: V_1 \dashrightarrow V_2$ induces a birational map $\tilde{V}_1 \dashrightarrow \tilde{V}_2$, and thus an isomorphism $\psi_*: \text{Alb}(\tilde{V}_1) \to \text{Alb}(\tilde{V}_2)$ such that $b_2 \circ \psi = \psi_* \circ b_1$ ([Lev73, Lemma 2.6]). As $b_1$ and $b_2$ are finite morphisms, the map $\psi$ extends to an isomorphism. We conclude that in the definition of an isomorphism of $\chi$-positive higher irrational pencils we may just require $\varphi$ to be a birational map.
Remark 13. Let \( \widehat{V} \xrightarrow{b} V \xrightarrow{\rho} \text{Alb}(\widehat{V}) \) be varieties and morphisms as in Definition 10. If \( V' \rightarrow V \) is another resolution of singularities, then \( b \) induces a morphism \( b': V \rightarrow \text{Alb}(V') \) finite onto its image such that the composition \( V' \rightarrow V \xrightarrow{b'} \text{Alb}(V') \) equals the Albanese map of \( V' \). This fact is again an application of [Lombardi 73, Lemma 2.6]. Moreover, once a resolution \( \widehat{V} \) of \( V \) has been fixed, a finite map \( b: V \rightarrow \text{Alb}(\widehat{V}) \) as in Definition 10 is uniquely determined up to a translation on \( \text{Alb}(\widehat{V}) \).

Remark 14. A \( \chi \)-positive variety \( V \) is necessarily of general type (cf. [CH01, Theorem 1]). Moreover, if \( V \) is non-singular, then \( \widehat{V} \) does not admit any non-trivial Fourier–Mukai partner.

Remark 15. In [Cat91], the author considers fibrations over normal projective varieties of maximal Albanese dimension and with non-surjective Albanese map. We notice that our classes of \( \chi \)-positive higher irrational pencils are not sub-classes of Cataneo’s fibrations. In fact, there are smooth projective varieties with positive holomorphic Euler characteristic such that their Albanese maps are finite and surjective (cf. [BPS19, Example 7.2]).

For any pairs of integers \( q \geq k \geq 1 \) we define the following sets of fibrations:

\[ F_{X}^{k,q} \overset{\text{def}}{=} \{ \text{isomorphism classes of } \chi \text{-positive } k \text{-higher irrational pencils } f: X \rightarrow V \text{ such that } q(V) = q \}. \]

Note that, for any integer \( g \geq 2 \), the set \( F_{X}^{1,g} \) coincides with the set \( F_{X}^{g} \) of the Introduction. The main result of this section is Theorem 25 which builds upon the results of [GL91] and [Par17]. To begin with, we recall the result [Par17, Lemma 5.1].

Lemma 16 (Green–Lazarsfeld, Pareschi). Let \( 0 < i < n \) be an integer and let \( Z \subset V^{i}(\omega_{X}) \) be an irreducible component of positive dimension. Then \( Z \) induces a fibration \( p: X \rightarrow V \) onto a normal projective variety \( V \) of dimension \( 0 < \dim V \leq n - i \) (see Remark 17 for the definition of \( p \)). Moreover, if \( (\pi: \tilde{X} \rightarrow X, \rho: \tilde{V} \rightarrow V, \tilde{\rho}: \tilde{X} \rightarrow \tilde{V}) \) is a non-singular representative of \( p \), then \( V \) admits a morphism \( b: V \rightarrow \text{Alb}(\tilde{V}) \) finite onto its image such that \( b \circ \rho = \text{alb}(\tilde{V}) \). Moreover, we have

\[ \chi(R^{i}_{\tilde{\rho},*}\omega_{\tilde{X}}) > 0, \quad V^{0}(R^{i}_{\tilde{\rho},*}\omega_{\tilde{X}}) = \text{Pic}^{0}(\tilde{V}), \quad \text{and} \quad Z = \pi_{*}\tilde{\rho}^{*}V^{0}(R^{i}_{\tilde{\rho},*}\omega_{\tilde{X}}) = \pi_{*}\tilde{\rho}^{*} \text{Pic}^{0}(\tilde{V}). \]

Remark 17. The fibration \( p: X \rightarrow V \) of Lemma 16 is defined as the Stein factorization of the composition \( (q \circ \text{alb}_{X}): X \rightarrow \tilde{Z} \) where \( q: \text{Alb}(X) \rightarrow \tilde{Z} \) is the dual map of the inclusion \( Z \subset \text{Pic}^{0}(X) \). In particular, \( V \) admits a morphism \( b: V \rightarrow \tilde{Z} \) which is finite onto its image. In addition, if \( \tilde{V} \rightarrow V \) is any resolution of singularities, then \( \tilde{Z} \simeq \text{Alb}(\tilde{V}) \) and the composition \( \tilde{V} \rightarrow V \xrightarrow{b} \text{Alb}(\tilde{V}) \) equals \( \text{alb}(\tilde{V}) \) (cf. [Par17, Lemma 5.1]). It follows that \( q(V) = \dim Z \).

Remark 18. As the Albanese map is defined up to the choice of a point, the fibration \( p: X \rightarrow V \) of Lemma 10 is only determined up to isomorphism of fibrations.

Corollary 19. Assume that the hypotheses of Lemma 16 hold. If \( \dim V = n - i \), then we have \( \chi(\omega_{\tilde{V}}) > 0 \) and the fibration \( f: X \rightarrow V \) determines a class in \( F_{X}^{n-i,q(V)} \).

Proof. By [Kol86, Proposition 7.6] there is an isomorphism \( R^{i}_{\tilde{f},*}\omega_{\tilde{X}} \simeq \omega_{\tilde{V}} \). It follows that \( \chi(\omega_{\tilde{V}}) = \chi(R^{i}_{\tilde{f},*}\omega_{\tilde{X}}) > 0 \).

Lemma 20. Let \( f_{i}: X \rightarrow V_{i} \) (\( i = 1, 2 \)) be two fibrations onto normal projective varieties such that \( \dim V_{2} \leq \dim V_{1} \). Let \( (\pi_{i}: \tilde{X}_{i} \rightarrow X, \rho_{i}: \tilde{V}_{i} \rightarrow V, \tilde{f}_{i}: \tilde{X}_{i} \rightarrow \tilde{V}_{i}) \) be non-singular representatives of \( f_{i} \) for \( i = 1, 2 \), and let \( b_{i}: \tilde{V}_{i} \rightarrow \text{Alb}(\tilde{V}_{i}) \) be morphisms finite onto their images such that \( b_{i} \circ \rho_{i} = \text{alb}_{\tilde{V}_{i}} \)

If \( \pi_{1*}f_{1}^{*}\text{Pic}^{0}(V_{1}) \subset \pi_{2*}f_{2}^{*}\text{Pic}^{0}(V_{2}) \) in \( \text{Pic}^{0}(X) \), then there exists an isomorphism \( \gamma: V_{2} \simeq V_{1} \) such that \( f_{1} = \gamma \circ f_{2} \).
Proof. We consider the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & V_1 \\
\downarrow alb_X & & \downarrow \gamma \\
\Alb(X) & \xrightarrow{\tilde{f}_i} & \Alb(V_i) \\
\Alb(V_2) & \xrightarrow{i_*} & \Alb(V_1), \\
\end{array}
\]

where the morphisms \(\tilde{f}_{i*} (i = 1, 2)\) are induced by the inclusions \(\pi_{i*}\tilde{f}_i^*\Pic^0(V_i) \subset \Pic^0(X)\), or, equivalently, by the morphisms \(\tilde{f}_i\). Moreover, the morphism \(i_*\) is induced by the inclusion \(\pi_{1*}\tilde{f}_1^*\Pic^0(V_1) \subset \pi_{2*}\tilde{f}_2^*\Pic^0(V_2)\). Also note both the inner and outer squares of the above diagram commute. As the general fiber of \(f_2\) is contracted by \(f_1\) (recall that \(b_1\) is finite onto its image), by [Deb01] Lemma 1.15 there exists a dominant rational map \(\gamma: V_2 \dashrightarrow V_1\) such that \(f_1 = \gamma \circ f_2\). Hence \(q(V_1) = q(V_2)\) and \(i_*\) is an isomorphism. Moreover \(\gamma\) is birational as \(\dim V_1 = \dim V_2\) and \(f_1, f_2\) have connected fibers. Finally, \(b_1 \circ \gamma = i_* \circ b_2\), so that \(\gamma\) extends to an isomorphism as \(b_1, b_2\) have finite fibers.

Corollary 21. Let \(0 < i < n\) be an integer and let \(Z \subset V^i(\omega_X)\) be an irreducible component of positive dimension inducing a fibration \(p: X \to V\) as in Lemma 16. Let \(f: X \to Q\) be a fibration onto a normal projective variety such that \(Z = \pi_*\tilde{f}_i^*\Pic^0(Q)\) for some non-singular representative \((\pi: \tilde{X} \to X, \rho: \tilde{Q} \to Q, \tilde{f}: \tilde{X} \to \tilde{Q})\) of \(f\). Assume there exists a morphism \(b: Q \to \Alb(Q)\) finite onto its image such that \(b \circ \rho = alb_{\tilde{Q}}\). Then there exists an isomorphism \(\gamma: Q \xrightarrow{\sim} V\) such that \(p = \gamma \circ f\).

Proof. By Remark 17 the Stein factorization of the composition \(X \to \Alb(X) \to \tilde{Z}\) factors through a finite morphism \(b': V \to \tilde{Z}\). We apply Lemma 20.

Corollary 22. Let \(f: X \to V\) be a fibration onto a normal projective variety of dimension \(k\), and let \((\pi: \tilde{X} \to X, p: \tilde{V} \to V, \tilde{f}: \tilde{X} \to \tilde{V})\) be a non-singular representative of \(f\). Assume there exists a morphism \(b: V \to \Alb(V)\) finite onto its image such that \(b \circ \rho = alb_{\tilde{V}}\). If \(\chi(\omega_{\tilde{V}}) > 0\), then \(\pi_*\tilde{f}_i^*\Pic^0(\tilde{V})\) is an irreducible component of \(V^{n-k}(\omega_X)_0\) of dimension \(q(V)\). Moreover \(\pi_*\tilde{f}_i^*\Pic^0(\tilde{V})\) does not depend on the particular choice of the non-singular representative \((\pi, \rho, \tilde{f})\). Finally, the fibration induced by the component \(\pi_*\tilde{f}_i^*\Pic^0(\tilde{V})\) by Lemma 16 is precisely the fibration \(f: X \to V\) up to isomorphism.

Proof. By Lemma 16 we have \(V^0(\omega_{\tilde{V}}) = \Pic^0(\tilde{V})\). Moreover, by [Lom14] Lemma 6.3 and Remark 8 the variety \(Z = \pi_*\tilde{f}_i^*\Pic^0(\tilde{V})\) is contained in \(V^{n-k}(\omega_X)_0\). We now show \(Z\) is an irreducible component of \(V^{n-k}(\omega_X)_0\). We argue by contradiction and assume \(Z\) is strictly contained in an irreducible component \(Z'\) of \(V^{n-k}(\omega_X)_0\). By Lemma 16 \(Z'\) gives rise to both a fibration \(X \to U\) such that \(0 < \dim U \leq k\) as well as a morphism \(U \to \Alb(U)\) finite onto its image satisfying the properties listed in Lemma 16 (here \(\tilde{U}\) is a resolution of singularities of \(U\)). Because of Lemma 20 this is impossible. The other assertions of the corollary follow by Lemma 20.

In the following results we denote by \((\pi: \tilde{X} \to X, \tilde{f}: \tilde{X} \to \tilde{V})\) a non-singular representative of a \(\chi\)-positive higher irrational pencil \(f: X \to V\), and by \(\pi_0^q(V^{n-k}(\omega_X)_0)\) the set of \(q\)-dimensional irreducible components of \(V^{n-k}(\omega_X)_0\).
Proposition 23. Let \( q \geq k \geq 1 \) be integers. Then the assignment

\[
u_{X,k,q}: F_{X}^{k,q} \rightarrow \pi_{0}^{q}(V^{n-k}(\omega_{X})_{0}), \quad (f: X \rightarrow V) \mapsto \pi_{*}f^{*}\text{Pic}^{0}(\tilde{V})
\]
is well-defined and injective. In particular the sets \( F_{X}^{k,q} \) are finite.

Proof. Let \((f: X \rightarrow V) \in F_{X}^{k,q}\) be an isomorphism class determined by a \( \chi \)-positive \( k \)-higher irrational pencil \( f: X \rightarrow V \) with \( q(V) = q \). By Corollary [22], the abelian variety \( \pi_{*}f^{*}\text{Pic}^{0}(\tilde{V}) \) is an irreducible component of \( V^{n-k}(\omega_{X})_{0} \) of dimension \( q \). This variety is independent both of the non-singular representative \((\pi, f)\), and the choice of a representative of the class of \( f \). Hence the map \( u_{X,k,q} \) is injective by Corollary [21]. □

Proposition 24. Let \( q \geq k \geq 1 \) be integers and suppose \( \dim V^{i}(\omega_{X})_{0} \leq 0 \) for all \( i > n - k \). Then the assignment

\[
u_{X,k,q}: F_{X}^{k,q} \rightarrow \pi_{0}^{q}(V^{n-k}(\omega_{X})_{0}), \quad (f: X \rightarrow V) \mapsto \pi_{*}f^{*}\text{Pic}^{0}(\tilde{V})
\]
of Proposition [23] is surjective.

Proof. Let \( Z \subset V^{n-k}(\omega_{X})_{0} \) be an irreducible component of dimension \( q \). We will show that \( Z \) arises from an isomorphism class of a \( \chi \)-positive \( k \)-higher irrational pencil of irregularity \( q \). By Lemma [16] the component \( Z \) determines a \( \chi \)-positive \( k \)-higher irrational pencil \( f: X \rightarrow V \) up to isomorphism onto a normal projective variety \( V \) such that \( 0 < \dim V \leq k \) and \( q(V) = q \). Moreover, we have \( Z = \pi_{*}f^{*}\text{Pic}^{0}(\tilde{V}) \); also, \( V \) admits a finite morphism, onto its image, to \( \text{Alb}(\tilde{V}) \). Hence, by Corollary [19] it is enough to show \( \dim V = k \). We proceed by contradiction and suppose that \( \dim V = k - s \) for some integer \( s > 0 \). We distinguish two cases: \( \dim V^{0}(\omega_{\tilde{V}})_{0} > 0 \) and \( \dim V^{0}(\omega_{\tilde{V}})_{0} = 0 \) (notice that in either case \( \mathcal{O}_{\tilde{V}} \in V^{0}(\omega_{\tilde{V}})_{0} \)). If \( \dim V^{0}(\omega_{\tilde{V}})_{0} > 0 \), then by pulling topologically trivial line bundles back to \( \tilde{X} \), we have \( V^{0}(\omega_{\tilde{V}})_{0} \subset V^{n-k+s}(\omega_{\tilde{X}})_{0} \simeq V^{n-k+s}(\omega_{X})_{0} \) (cf. [Lom14, Lemma 6.3]) which is a contradiction. Suppose now the second case \( V^{0}(\omega_{\tilde{V}})_{0} = \{ \mathcal{O}_{\tilde{V}} \} \).

Then, by [EL97, Proposition 2.2], the Albanese map \( \text{alb}_{\tilde{V}}: \tilde{V} \rightarrow \text{Alb}(\tilde{V}) \) is surjective and we have \( q = q(V) = q(\tilde{V}) \leq \dim \tilde{V} = k - s < k \). This is again a contradiction.

□

By putting together the previous two propositions we obtain the following result.

Theorem 25. Let \( q \geq k \geq 1 \) be integers and suppose \( \dim V^{i}(\omega_{X})_{0} \leq 0 \) for all \( i > n - k \). Then the assignment

\[
u_{X,k,q}: F_{X}^{k,q} \rightarrow \pi_{0}^{q}(V^{n-k}(\omega_{X})_{0}), \quad (f: X \rightarrow V) \mapsto \pi_{*}f^{*}\text{Pic}^{0}(\tilde{V})
\]
is well-defined and yields a bijection of sets.

4. Proofs

4.1. Proof of Theorem [1].

Proof of Theorem [1] Set \( n = \dim X = \dim Y \) and let \( p_{1} \) and \( p_{2} \) be the projections from \( X \times Y \) onto \( X \) and \( Y \), respectively. The equivalence \( \Phi \) is of Fourier–Mukai type, that is \( \Phi(-) \simeq \Phi_{\mathcal{E}}(-) = R_{p_{2}}(p_{1}^{*}(\mathcal{L} \otimes \mathcal{E})) \) for some object \( \mathcal{E} \) in \( D(X \times Y) \). By [PS11, Lemma 3.1] the induced Rouquier isomorphism

\[F_{\mathcal{E}} : \text{Aut}^{0}(X) \times \text{Pic}^{0}(X) \rightarrow \text{Aut}^{0}(Y) \times \text{Pic}^{0}(Y)\]
acts as
\[(5) \quad F_{\mathcal{E}}(\varphi, \alpha) = (\psi, \beta) \iff p_1^* \alpha \otimes (\varphi \times \text{id}_Y)^* \mathcal{E} \simeq p_2^* \beta \otimes (\text{id}_X \times \psi)_* \mathcal{E}.\]
Moreover, by Theorem 3.1 and Remark 5, \(F_{\mathcal{E}}\) induces an isomorphism of algebraic sets \(V^{n-1}(\omega_X)_0 \cong V^{n-1}(\omega_Y)_0\). By (2) and Theorem 3.1 it follows that the composition
\[\mu_g \overset{\text{def}}{=} (u_{Y,1,g} \circ F_{\mathcal{E}} \circ u_{X,1,g}) : F_{\mathcal{E}}^g \to F_{\mathcal{E}}^g\]
defines a bijection of sets for any integer \(g \geq 2\) such that, if \(\mu_g(f : X \to C) = (h : Y \to D)\), then
\[(6) \quad F_{\mathcal{E}}(\text{id}_X, f^* \text{Pic}^0(C)) = (\text{id}_Y, h^* \text{Pic}^0(D)).\]
Thus we only need to show that the curves \(C\) and \(D\) are isomorphic. This is achieved by constructing a complex in \(\mathbf{D}(C \times D) \overset{\text{def}}{=} \mathbf{D}(\text{Coh}(C \times D))\) whose support dominates both \(C\) and \(D\).

The equations (5) and (6) define an isomorphism of abelian varieties \(\zeta : \text{Pic}^0(C) \to \text{Pic}^0(D)\) such that
\[(7) \quad p_1^* f^* \alpha \otimes \mathcal{E} \simeq p_2^* h^* \zeta(\alpha) \otimes \mathcal{E} \quad \text{for all} \quad \alpha \in \text{Pic}^0(C).\]
We fix a sufficiently ample line bundle \(L\) on \(X \times Y\) and define
\[\mathcal{E}' \overset{\text{def}}{=} R(f \times h)_*(\mathcal{E} \otimes L).\]
Moreover, we denote by \(t_1\) and \(t_2\) the projections from \(C \times D\) onto the first and second factor, respectively. An application of the projection formula yields isomorphisms
\[R(f \times h)_*(p_1^* f^* \alpha \otimes \mathcal{E} \otimes L) \simeq R(f \times h)_*((f \times h)^* t_1^* \alpha \otimes \mathcal{E} \otimes L) \simeq t_1^* \alpha \otimes \mathcal{E}',\]
\[R(f \times h)_*(p_2^* h^* \zeta(\alpha) \otimes \mathcal{E} \otimes L) \simeq R(f \times h)_*((f \times h)^* t_2^* \zeta(\alpha) \otimes \mathcal{E} \otimes L) \simeq t_2^* \zeta(\alpha) \otimes \mathcal{E}',\]
so that
\[(8) \quad t_1^* \alpha \otimes \mathcal{E}' \simeq t_2^* \zeta(\alpha) \otimes \mathcal{E}' \quad \text{for all} \quad \alpha \in \text{Pic}^0(C).\]
Let \(i_b : t_2^{-1}(b) \hookrightarrow C \times D\) be the closed immersion of the fiber \(t_2^{-1}(b) \simeq C\), and set \(\mathcal{E}'_b \overset{\text{def}}{=} \mathcal{E}_b \otimes \mathcal{E}'\). By restricting the isomorphisms (8) to every fiber of \(t_2\), we obtain new isomorphisms:
\[(9) \quad \mathcal{E}'_b \otimes \alpha \simeq \mathcal{E}'_b \quad \text{for all} \quad \alpha \in \text{Pic}^0(C) \quad \text{and} \quad b \in D.\]
We denote by \(H^q(\mathcal{F})\) the \(q\)-th cohomology sheaf of a complex \(\mathcal{F}\), and define the support of \(\mathcal{F}\) by \(\text{Supp}(\mathcal{F}) = \bigcup_q \text{Supp}(H^q(\mathcal{F}))\). We endow \(\text{Supp}(\mathcal{F})\) with its reduced scheme structure. By taking cohomology in (4), we find \(H^q(\mathcal{E}'_b) \otimes \alpha \simeq H^q(\mathcal{E}'_b)\) for all \(q \in \mathbb{Z}\), \(\alpha \in \text{Pic}^0(C)\) and \(b \in D\). Thus by [Muk81 Lemma 3.3] we obtain
\[\dim \text{Supp} \left( H^q(\mathcal{E}'_b) \right) = \dim \text{Supp} \left( (jC)_* H^q(\mathcal{E}'_b) \right) \leq 0\]
for all \(q \in \mathbb{Z}\) and \(b \in D\) (here \(j_C : C \hookrightarrow \text{Pic}^0(C)\) denotes the Abel–Jacobi embedding). We conclude \(\dim \text{Supp} \left( \mathcal{E}'_b \right) \leq 0\) for all \(b \in D\) since \(\mathcal{E}'_b\) is a bounded complex. Moreover, from the isomorphisms
\[\text{Supp}(\mathcal{E}'_b) = \text{Supp}(\mathcal{E}') \cap t_2^{-1}(b) \quad \text{for all} \quad b \in D\]
(cf. [Huy06 Lemma 3.29]), we conclude that \(t_2 : \text{Supp}(\mathcal{E}') \to D\) is a finite morphism and \(\dim \text{Supp}(\mathcal{E}') \leq 1\). Also, in a complete analogy, also \(t_1 : \text{Supp}(\mathcal{E}') \to C\) is a finite morphism.

Now consider the following commutative diagram:
\[
\begin{array}{ccc}
\text{Supp} (\mathcal{E} \otimes L) & \overset{p_1}{\longrightarrow} & X \\
\downarrow f \times h & & \downarrow f \\
\text{Supp} (\mathcal{E}') & \overset{t_1}{\longrightarrow} & C.
\end{array}
\]
Thus, follows that \( \text{Supp}_{\text{alb}} \) equals \( \text{Supp}(\mathcal{E}) \).

**Claim 26.** The restriction of \( f \times h \) to \( \text{Supp}(\mathcal{E} \otimes L) = \text{Supp}(\mathcal{E}) \) defines a morphism \( (f \times h): \text{Supp}(\mathcal{E}) \to \text{Supp}(\mathcal{E}') \) of algebraic sets.

**Proof.** As \( L \) is sufficiently ample, on the one hand we have the following vanishing:

\[
R^p(f \times h)_*(H^q(\mathcal{E}) \otimes L) = 0 \quad \text{for all} \quad p > 0 \quad \text{and} \quad q \in \mathbb{Z}.
\]

On the other hand there are surjections

\[
(f \times h)^*(f \times h)_*(H^q(\mathcal{E}) \otimes L) \to H^q(\mathcal{E} \otimes L) \quad \text{for all} \quad q.
\]

Therefore, by the degeneration of the spectral sequence

\[
E^{p,q}_2 := R^p(f \times h)_*(H^q(\mathcal{E}) \otimes L) \Rightarrow R^{p+q}(f \times h)_*(\mathcal{E} \otimes L),
\]

we obtain the following isomorphisms for all \( q \):

\[
H^q(\mathcal{E}') = R^q(f \times h)_*(\mathcal{E} \otimes L) \simeq (f \times h)_*(H^q(\mathcal{E}) \otimes L)
\]

and

\[
\text{Supp}((f \times h)_*(H^q(\mathcal{E}) \otimes L)) = (f \times h)(\text{Supp}(H^q(\mathcal{E}) \otimes L)).
\]

We conclude \( \text{Supp}(H^q(\mathcal{E}')) \simeq (f \times h)(\text{Supp}(H^q(\mathcal{E}) \otimes L)) \), from which we deduce

\[
\text{Supp}(\mathcal{E}') = \bigcup_q \text{Supp}(H^q(\mathcal{E}')) \simeq (f \times h)\left(\bigcup_q \text{Supp}(H^q(\mathcal{E}) \otimes L)\right) = (f \times h)(\text{Supp}(\mathcal{E} \otimes L)).
\]

To conclude the proof, we observe the projection \( p_1 \) is surjective with connected fibers (cf. \cite{Huy06} Lemmas 6.4 and 6.11). Thus, \( t_1: \text{Supp}(\mathcal{E}') \to C \) is surjective with connected fibers. It follows that \( \text{Supp}(\mathcal{E}') \) is irreducible, and that \( t_1 \) is an isomorphism by \cite[Liu02, Corollary 4.4.3]{Liu02}. In complete analogy, we have \( \text{Supp}(\mathcal{E}') \simeq D \).

**Remark 27.** As noted in \cite{CL}, \cite{CLP23}, the argument of the proof of Theorem \ref{thm:main} can be employed to study the derived invariance of the finite part of the Stein factorization of the composition between the Itaka fibration and the Albanese map of its base variety.

### 4.2. Proof of Theorem \ref{thm:main2}

**Proof of Theorem \ref{thm:main2}.** As in the proof of Theorem \ref{thm:main} we have \( \Phi(-) \simeq \Phi_{\mathcal{E}}(-) = Rp_{2*}(p_1^*(-) \otimes \mathcal{E}) \) where \( p_1, p_2 \) are the projections from \( X \times Y \) onto \( X \) and \( Y \), respectively, and \( \mathcal{E} \) is an object in \( D(X \times Y) \). Let \( (v: X \to S) \in F^\chi_{X,Y} \) be the isomorphism class of a \( \chi \)-positive 2-higher irrational pencil \( v: X \to S \) with \( q(S) = q \). Moreover, let \( (\pi_X: \tilde{X} \to X, \tilde{v}: \tilde{X} \to \tilde{S}) \) be a non-singular representative of \( v \). By Corollary \ref{cor:irr} the abelian variety \( Z \overset{\text{def}}{=} \pi_X^*\tilde{v}^*\text{Pic}^0(\tilde{S}) \) is an irreducible component of \( V^{n-2}(\omega_X)_0 \) which does not depend on the choice of the non-singular representative \( (\pi_X, \tilde{v}) \). By Theorem \ref{thm:rouquier} the Rouquier isomorphism \( F_{\mathcal{E}} \) induced by \( \Phi_{\mathcal{E}} \) yields an irreducible component \( Z' \overset{\text{def}}{=} F_{\mathcal{E}}(\text{id}_X, Z) \subset V^{n-2}(\omega_Y)_0 \). This in turn induces, by means of Lemma \ref{lem:proj} a fibration \( w: Y \to T \) onto a normal projective variety \( T \) such that \( 0 < \dim T \leq 2 \) and \( q(T) = q \geq 2 \). If \( (\pi_Y: \tilde{Y} \to Y, \tilde{w}: \tilde{Y} \to \tilde{T}) \) is a non-singular representative of \( w \), then the following properties hold: (i) there exists a morphism \( b: T \to \text{Alb}(\tilde{T}) \) finite onto its image such that the composition \( \tilde{T} \to T \xrightarrow{b} \text{Alb}(\tilde{T}) \) equals \( \text{alb}_{\tilde{T}} \), (ii) \( \chi(R^{n-2}\tilde{w}_*\omega_{\tilde{T}}^0) > 0 \), and (iii) \( Z' = \pi_{Y*}\tilde{w}^*\text{Pic}^0(\tilde{T}) = \pi_{Y*}\tilde{w}^*V^0(R^{n-2}\tilde{w}_*\omega_{\tilde{T}}^0) \).
We claim $\dim T = 2$. If, by contradiction, we had $\dim T = 1$, then the following facts would be true: (i) $\widetilde{T} \simeq T$, (ii) $V^0(\omega_T) = \text{Pic}^0(T)$ and (iii) $w^* \text{Pic}^0(T) = Z' \subset V^{n-1}(\omega_T)$ (cf. [Lom14, Lemma 6.3]). Hence, by Remark 5 and Theorem 4, the component $Z$ is contained in $V^{n-1}(\omega_X)$, and by Lemma 10 it induces an irrational pencil $u: X \to B$ of genus $q$ such that $Z = u^* \text{Pic}^0(B)$. By Lemma 20 we get an isomorphism between $B$ and $S$ which is impossible. We conclude $T$ is a surface, and the fibration $w: Y \to T$ defines a class in $F^2_\omega$ (cf. Corollary 19). Moreover, the argument also shows the assignments $u_{X,2,q}: F^2_X \to \pi^q_0(V^{n-2}(\omega_X)_0)$ and $u_{Y,2,q}: F^2_Y \to \pi^q_0(V^{n-2}(\omega_Y)_0)$ are surjective, and hence bijective by Proposition 23. Finally, by Theorem 3 the composition

$$\nu_q \overset{\text{def}}{=} \left( u_{Y,2,q} \circ F \circ u_{X,2,q} \right): F^2_X \to F^2_Y$$

defines a bijection of sets for all $q \geq 2$ such that, if $(v: X \to S) \in F^2_X$ and $\nu_q(v) = (w: Y \to T)$, then

$$F_E \left( \text{id}_X, \pi_X \cdot \pi_Y \cdot v^* \cdot \text{Pic}^0(\widetilde{T}) \right) = \left( \text{id}_Y, \pi_Y \cdot w^* \cdot \text{Pic}^0(\widetilde{T}) \right).$$

In the rest of the proof we will show the surfaces $S$ and $T$ are isomorphic.

We consider non-singular representatives of $v$ and $w$ as follows:

\[
\begin{array}{ccc}
\tilde{X} & \overset{\pi_X}{\longrightarrow} & X \\
\downarrow \tilde{\nu} & & \downarrow v \\
\tilde{S} & \overset{\rho_S}{\longrightarrow} & S \\
\end{array}
\quad \begin{array}{ccc}
\tilde{Y} & \overset{\pi_Y}{\longrightarrow} & Y \\
\downarrow \tilde{w} & & \downarrow w \\
\tilde{T} & \overset{\rho_T}{\longrightarrow} & T \\
\end{array}
\]

By Remarks 17 and 13 the surfaces $S$ and $T$ are equipped with morphisms $a: S \to \text{Alb}(\tilde{S})$ and $b: T \to \text{Alb}(\tilde{T})$, respectively, finite onto their images, such that $a \circ \rho_S = a b_{\tilde{S}}$ and $b \circ \rho_T = a b_{\tilde{T}}$. We set $\pi = \pi_X \times \pi_Y$ and denote by $\xi: \text{Pic}^0(\tilde{S}) \to \text{Pic}^0(\tilde{T})$ the isomorphism induced by (10). By Proposition 7, $\xi$ induces an isomorphism $\text{Pic}^0(S) \simeq \text{Pic}^0(T)$, which we continue to denote, with a slight abuse of notation, by $\xi$. In this way we have $\xi \circ \rho_S^* = \rho_T^* \circ \xi$. Finally, let $\tilde{p}_1$ and $\tilde{p}_2$ be the projections from $\tilde{X} \times \tilde{Y}$ onto the first and second factor, respectively. By the equality (10) and the description of the action of $F_E$ in (5), we obtain a collection of isomorphisms:

$$\pi_* \cdot \tilde{p}_1^* \cdot \tilde{\nu}^* \cdot \alpha \otimes E \simeq \pi_* \cdot \tilde{p}_2^* \cdot \tilde{w}^* \cdot \xi(\alpha) \otimes E \quad \text{for all} \quad \alpha \in \text{Pic}^0(\tilde{S}).$$

(11)

By Proposition 7 the isomorphisms (11) are equivalent to the following isomorphisms:

$$\pi_* \cdot \tilde{p}_1^* \cdot \tilde{\nu}^* \cdot \rho_S^* \gamma \otimes E \simeq \pi_* \cdot \tilde{p}_2^* \cdot \tilde{w}^* \cdot \rho_T^* \xi(\gamma) \otimes E \quad \text{for all} \quad \gamma \in \text{Pic}^0(S).$$

Denote now by $t_1$ and $t_2$ the projections from $S \times T$ onto $S$ and $T$, respectively. By tensoring the previous isomorphisms by a sufficiently ample line bundle $L$ on $X \times Y$, and by pushing them forward to $S \times T$, we obtain new isomorphisms:

$$t_1^* \gamma \otimes E' \simeq t_2^* \xi(\gamma) \otimes E' \quad \text{for all} \quad \gamma \in \text{Pic}^0(S),$$

(12)

where

$$E' \overset{\text{def}}{=} R(v \times w)_*(E \otimes L).$$

We set

$$E'' \overset{\text{def}}{=} (a \times b)_* E'.$$

1Any smooth projective curve of genus $q \geq 2$ satisfies $V^q(\omega_C) = \text{Pic}^q(C)$. On the contrary, in higher dimension there are varieties $Z$ of maximal Albanese dimension such that $q(Z) > \dim Z$ and $V^q(\omega_Z) \not\subseteq \text{Pic}^q(Z)$. For instance, isotrivial elliptic surfaces $S$ fibered over curves of genus at least two provide counterexamples for any $q(S) \geq 3$. This is the reason why the arguments of this proof do not extend to fibrations in $F^{k,q}_X$ with $k \geq 3$. 

and denote by $q_1$ and $q_2$ the two projections from $\text{Alb}(\bar{S}) \times \text{Alb}(\bar{T})$ onto the first and second factor, respectively. By pushing the isomorphisms (12) forward to $\text{Alb}(\bar{S}) \times \text{Alb}(\bar{T})$, we obtain new isomorphisms:

$$(a \times b)_*(t_1^* \gamma \otimes \mathcal{E}') \simeq (a \times b)_*(t_2^* \xi(\gamma) \otimes \mathcal{E}') \quad \text{for all} \quad \gamma \in \text{Pic}^0(S).$$

By noting that every $\gamma \in \text{Pic}^0(S)$ can be written as $\gamma = a^* \beta$ for a unique $\beta \in \text{Pic}^0(\text{Alb}(\bar{S}))$, the projection formula yields (again with a slight abuse of notation)\(^2\)

$$q_1^* \beta \otimes \mathcal{E}'' \simeq q_2^* \beta \otimes \mathcal{E}'' \quad \text{for all} \quad \beta \in \text{Pic}^0(\text{Alb}(\bar{S})).$$

Denote by $\mathcal{E}'' \overset{\text{def}}{=} \mathcal{E}''_{q_1^{-1}(s)}$ the derived restriction of $\mathcal{E}''$ to the fiber $q_1^{-1}(s)$. By restricting the isomorphisms (13) to every fiber of $q_1$, we have isomorphisms

$$\mathcal{E}''_{q_1^{-1}(s)} \simeq \mathcal{E}'' \otimes \xi(\beta) \quad \text{for all} \quad \beta \in \text{Pic}^0(\text{Alb}(\bar{S})) \quad \text{and} \quad s \in \text{Alb}(\bar{S}).$$

As the complex $\mathcal{E}''$ has only a finite number of non-zero cohomology sheaves, Mukai’s lemma [Muk81, Lemma 3.3] implies dim $\text{Supp}(\mathcal{E}''_{q_1^{-1}(s)}) \leq 0$ for all $s$. Similarly, we also have dim $\text{Supp}(\mathcal{E}''_{q_2^{-1}(t)}) \leq 0$ for all $t \in \text{Alb}(\bar{T})$. By [Huy06, Lemma 3.29] the projections

$$q_1: \text{Supp}(\mathcal{E}'') \to \text{Alb}(\bar{S}) \quad \text{and} \quad q_2: \text{Supp}(\mathcal{E}'') \to \text{Alb}(\bar{T})$$

are finite morphisms and dim $\text{Supp}(\mathcal{E}'') \leq 2$. Moreover, by the argument of Claim 26 there are commutative diagrams

$$\begin{array}{ccc}
\text{Supp}(\mathcal{E}') & \xrightarrow{\ t_1\ } & S \\
\downarrow a \times b & & \downarrow a \\
\text{Supp}(\mathcal{E}'') & \xrightarrow{\ q_1\ } & \text{Alb}(\bar{S})
\end{array} \quad \begin{array}{ccc}
\text{Supp}(\mathcal{E}') & \xrightarrow{\ t_2\ } & T \\
\downarrow a \times b & & \downarrow b \\
\text{Supp}(\mathcal{E}'') & \xrightarrow{\ q_2\ } & \text{Alb}(\bar{T})
\end{array}$$

from which we also infer that also the morphisms $t_1$ and $t_2$ are finite. In order to conclude the proof, we consider the following commutative diagrams:

$$\begin{array}{ccc}
\text{Supp}(\mathcal{E}) & \xrightarrow{\ p_1\ } & X \\
\downarrow v \times w & & \downarrow v \\
\text{Supp}(\mathcal{E}') & \xrightarrow{\ t_1\ } & S
\end{array} \quad \begin{array}{ccc}
\text{Supp}(\mathcal{E}) & \xrightarrow{\ p_2\ } & Y \\
\downarrow v \times w & & \downarrow w \\
\text{Supp}(\mathcal{E}') & \xrightarrow{\ t_2\ } & T.
\end{array}$$

As $p_1: \text{Supp}(\mathcal{E}) \to X$ and $p_2: \text{Supp}(\mathcal{E}) \to Y$ are surjective with connected fibers (cf. [Huy06, Lemmas 6.4 and 6.11]), the morphisms $t_1: \text{Supp}(\mathcal{E}') \to S$ and $t_2: \text{Supp}(\mathcal{E}') \to T$ also are surjective with connected fibers. Therefore $\text{Supp}(\mathcal{E}')$ is irreducible and $S \simeq \text{Supp}(\mathcal{E}') \simeq T$. \hfill \square

In the second part of the previous proof the dimensions of the varieties $S$ and $T$ did not play a role. In fact the argument extends to fibrations of higher-dimensional bases as follows.

**Proposition 28.** Suppose $\Phi_{\mathcal{E}}: D(X) \xrightarrow{\sim} D(Y)$ is an equivalence of triangulated categories and let $F_{\mathcal{E}}$ be the induced Rouquier isomorphism. Let $(f: X \to V) \in F^k_X$ and $(h: Y \to W) \in F^k_Y$ be $\chi$-positive trivial irrational pencils with $q \geq k \geq 1$. If

$$F_{\mathcal{E}} \left( \text{id}_X, \pi_X, \tilde{f}^* \text{Pic}^0(\bar{V}) \right) = (\text{id}_Y, \pi_Y, \tilde{h}^* \text{Pic}^0(\bar{W}))$$

\(^2\)We continue to denote by $\xi: \text{Pic}^0(\text{Alb}(\bar{S})) \to \text{Pic}^0(\text{Alb}(\bar{T}))$ the morphism induced by $\xi: \text{Pic}^0(S) \to \text{Pic}^0(T)$. With this notation we find $\xi \circ a^* = b^* \circ \xi$. \hfill \square
for some non-singular representatives \((\pi_X : \tilde{X} \to X, \tilde{f} : \tilde{X} \to \tilde{V})\) and \((\pi_Y : \tilde{Y} \to Y, \tilde{h} : \tilde{Y} \to \tilde{W})\) of \(f\) and \(h\), respectively, then \(V\) and \(W\) are isomorphic.

### 4.3. Proof of Theorem 3

Let \(X\) be a smooth projective variety of dimension \(n\). Before proving Theorem 3 we recall the integer \(b_{X>0}(X) \in \{0, \ldots, n-1\}\) is defined as the minimal dimension of a base variety of a \(\chi\)-positive \(k\)-higher irrational pencil with \(k \in \{1, \ldots, n-1\}\). Moreover, we declare \(b_{X>0}(X) = 0\) if and only if \(X\) does not admit any \(\chi\)-positive \(k\)-higher irrational pencil for all \(k \in \{1, \ldots, n-1\}\). See also \([\text{CP19}], \S 4\).

**Proposition 29.** Let \(0 < k < n\) be an integer and let \(Z\) be an irreducible component of \(V^{n-k}(\omega_X)_0\) of dimension \(\dim Z \geq k\). Denote by \(f : X \to V\) the fibration induced by \(Z\) as in Lemma 16. If \(\dim V < k\), then \(X\) admits a fibration \(p' : X \to U\) in \(F_X^{k',q'}\) for some integers \(0 < k' < k\) and \(2 \leq q' \leq \dim Z\). Moreover, there exists a dominant rational map \(\gamma : V \dashrightarrow U\) such that \(p' = \gamma \circ f\).

**Proof.** Let \(\tilde{f} : \tilde{X} \to \tilde{V}\) be a non-singular representative of \(f\), and note that \(q(\tilde{V}) = q(V) = \dim Z \geq k > \dim \tilde{V}\). Hence \(\text{all} \tilde{V}\) is not surjective and, by \([\text{Li97}, \text{Proposition 2.2}]\), \(\mathcal{O}_{\tilde{V}}\) is not an isolated point in \(V^0(\tilde{V})\). We conclude that \(\dim V^0(\tilde{V})_0 > 0\). If \(V^0(\tilde{V}) = \text{Pic}^0(\tilde{V})\), then by Lemma 6 we have \(\chi(\tilde{V}) > 0\) so that \(f\) defines a class of fibrations in \(F_X^{\dim V, q}\) with \(q = q(V) = \dim Z\). Suppose now that \(V^0(\tilde{V}) \neq \text{Pic}^0(\tilde{V})\) and \(Z'' \subset V^0(\tilde{V})_0\) is a positive-dimensional irreducible component of codimension \(0 < j \equiv \text{codim} Z'' < q(V)\). Then \(Z''\) is also an irreducible component of \(V^j(\tilde{V})_0\) (cf. \([\text{Par17}, \text{Corollary 3.3}]\)). Therefore, by Lemma 16 there exists a further fibration \(f' : \tilde{V} \to V'\) such that

\[
0 < \dim V' \leq \dim V - j < k - j.
\]

Moreover, if \(\tilde{f}' : \tilde{V} \to \tilde{V}'\) is a non-singular representative of \(f'\), then \(\chi(R^j f'^* \omega_{\tilde{V}}) > 0\) and \(Z'' = \tilde{f}'^* \text{Pic}^0(\tilde{V'})\). Moreover, by the previous argument, we still have \(\dim V^0(\tilde{V'})_0 > 0\) since

\[
q(\tilde{V'}) = \dim Z'' = q(\tilde{V}) - j = q(V) - j \geq \dim V - j = k - j > \dim \tilde{V}'.
\]

Hence we obtain a fibration \(l : \tilde{X} \to \tilde{V}'\) with the same properties of \(f\), but satisfying \(\dim V' < \dim V\). Proceeding in this way, we will eventually construct a fibration \(p : X^* \to U\) in \(F_{X^*}^{k',q'}\) where \(X^*\) is a smooth projective variety birational to \(X\) with \(0 < k' = \dim U < k\) and \(\dim U \leq q' = q(U) \leq q(V)\). However, if \(U\) were a curve, then we must have \(q' \geq 2\), as, by \([\text{Dea92}, \text{Corollaire 2.3}]\), the \((n-1)\)-th non-vanishing locus attached to the canonical bundle does not contain any component of dimension one passing through the origin. By Proposition 9 the rational dominant map \(X \dashrightarrow U\) extends to a morphism \(p'\) such that the composition \(X^* \to X \xrightarrow{p'} U\) equals \(p\). The second statement is a consequence of the construction. \(\square\)

The following proposition is inspired by \([\text{CP19}], \text{Claim 4.9}]\).

**Proposition 30.** We have \(b_{X>0}(X) > 0\) if and only if there exists an index \(i \in \{1, \ldots, n-1\}\) such that \(\dim V^i(\omega_X)_0 \geq n - i\). Moreover, if \(b_{X>0}(X) > 0\), then \(n - b_{X>0}(X)\) is the largest integer \(0 < i < n\) such that \(\dim V^i(\omega_X)_0 \geq n - i\).

**Proof.** Suppose there exists an irreducible component \(Z \subset V^i(\omega_X)_0\) of dimension \(\dim Z \geq n - i\) for some \(i > 0\). Let \(f : X \to V\) be the fibration induced by \(Z\), as in Lemma 16 with \(\dim V \leq n - i\). We distinguish two cases: \(\dim V = n - i\) and \(\dim V < n - i\). If \(\dim V = n - i\), then, by Corollary 19 we have \(f \in F_X^{n-i, \dim Z}\). On the other hand, if \(\dim V < n - i\), then, by Proposition 29 we can
construct a fibration in $F^{b,q}_X$ with $0 < k' < n - i$ and $2 \leq q' \leq \dim Z$. In both cases we have $b_{\chi>0}(X) > 0$.

Conversely, if $f: X \to V$ is a $\chi$-positive $b$-higher irrational pencil with $b = b_{\chi>0}(X) > 0$, then, by Corollary\[22\] $\pi \tilde{f}^* \text{Pic}^0(\tilde{V})$ is an irreducible component of $V''(\omega_X)_0$ of dimension $q(V)$ (as usual $(\pi, \tilde{f})$ denotes a non-singular representative of $f$). Since $\tilde{V}$ is of maximal Albanese dimension, we have $q(V) \geq \dim V = b$. The second claim is proved similarly. \[\Box\]

**Proposition 31.** Suppose $b = b_{\chi>0}(X) > 0$ and let $q \geq b$ be an integer. Then the assignment

$$u_{X,b,q}: F^{b,q}_X \to \pi_0^q(V''(\omega_X)_0), \quad (f: X \to V) \mapsto \pi \tilde{f}^* \text{Pic}^0(\tilde{V})$$

as defined in Proposition\[23\] is well-defined and it defines a bijection of sets.

**Proof.** Proposition\[23\] shows the map $u_{X,b,q}$ is well-defined and injective. To show $u_{X,b,q}$ is also surjective, we denote by $Z \subset V''(\omega_X)_0$ an arbitrary irreducible component of dimension $q$. By Lemma\[19\] the component $Z$ induces a fibration $f: X \to V$ onto a normal projective variety $V$ with $\dim V \leq b$. If $\dim V < b$, then by Proposition\[29\] we can construct a fibration in $F^{b,q}_X$ with $0 < k' < b$ and $2 \leq q' \leq q$. This contradicts the definition of $b$. Therefore $\dim V = b$, and $f \in F^{b,q}_X$ by Corollary\[19\]. \[\Box\]

**Proof of Theorem \[3\]** By Proposition\[31\] the integers $b_{\chi>0}(X)$ and $b_{\chi>0}(Y)$ only depend on the dimensions of the non-vanishing loci $V^i(\omega_X)_0$ and $V^i(\omega_Y)_0$ for all $i \geq 0$. Therefore, by Theorem\[4\] we have $b_{\chi>0}(X) = b_{\chi>0}(Y)$. We set $b = b_{\chi>0}(X) = b_{\chi>0}(Y)$. Denote now by $F_\varepsilon$ the Rouquier isomorphism induced by the equivalence $\Phi = \Phi_\varepsilon: D(X) \to D(Y)$. By Proposition\[31\] and Theorem\[4\] the following assignments

$$\sigma_q \overset{\text{def}}{=} (u_{Y,b,q}^{-1} \circ F_\varepsilon \circ u_{X,b,q}): F^{b,q}_X \to F^{b,q}_Y$$

are bijections for all $q \geq b$. Moreover, if $\sigma_q(f: X \to V) = (h: Y \to W)$, then we have

$$F_\varepsilon((\text{id}_X, \pi X \tilde{f}^* \text{Pic}^0(\tilde{V})) = (\text{id}_Y, \pi Y \tilde{h}^* \text{Pic}^0(\tilde{W}))$$

where $(\pi X: \tilde{X} \to X, \tilde{f}: \tilde{X} \to \tilde{V})$ and $(\pi Y: \tilde{Y} \to Y, \tilde{h}: \tilde{Y} \to \tilde{W})$ are non-singular representatives of $f$ and $h$, respectively. The proof that $V$ and $W$ are isomorphic follows by Proposition\[28\]. \[\Box\]

5. Applications to the fundamental group

The fundamental group $\pi_1(X)$ of a smooth projective variety is not in general a derived invariant, as shown for instance by Schnell in \[Sch12\]. Nevertheless, as an application of Theorem\[1\] we deduce the derived invariance of the following property. In this section $X$ and $Y$ denote two smooth projective varieties.

**Corollary 32.** If $D(X) \simeq D(Y)$ and $g \geq 2$ is an integer, then $\pi_1(X)$ admits a surjective homomorphism onto the fundamental group of a compact Riemann surface of genus $g$ if and only if $\pi_1(Y)$ does.

**Proof.** In the Appendix of \[Cat91\], Beauville proves that, for any $g \geq 2$, there exists a surjective homomorphism $\pi_1(X) \to \Gamma_g$ onto the fundamental group of a compact Riemann surface of genus
g if and only if X admits an irrational pencil of genus greater or equal to g. The corollary follows by Theorem 1.

Remark 33. In [Cat91, Theorem 1.10] (cf. also [KK10, Theorem 1.1]), the author relates the existence of irrational pencils of genus \( g \geq 2 \) to the existence of maximal isotropic subspaces in \( H^1(X, \mathbb{C}) \) (we recall subspace \( U \subset H^1(X, \mathbb{C}) \) is isotropic if the map \( U \cap U \rightarrow H^2(X, \mathbb{C}) \) is the trivial map). As an application of Theorem 1, we deduce \( X \) admits a maximal isotropic subspace \( U \subset H^1(X, \mathbb{C}) \) of dimension \( g \geq 2 \) if and only if \( Y \) does. More refined statements along these lines can be formulated by means of [Cat91, Theorem 2.25].

A further application of Theorem 1 involves the cup product map

\[
\varphi_X : H^1(X, \mathbb{C}) \wedge H^1(X, \mathbb{C}) \rightarrow H^2(X, \mathbb{C}).
\]

In view of Castelnuovo–De Franchis’ theorem, Theorem 1 implies that, if \( \Phi \) is injective on pure forms of type \( p \wedge q \) and only if \( \varphi_Y : H^1(Y, \mathbb{C}) \wedge H^1(Y, \mathbb{C}) \rightarrow H^2(Y, \mathbb{C}) \) has the same property. A more general statement holds if the geometric genus \( p_g(X) \) is equal to one.

Corollary 34. Suppose \( D(X) \cong D(Y) \) and \( p_g(X) = 1 \). Then \( \varphi_X \) is injective if and only if \( \varphi_Y \) is injective.

Proof. By [CJT17, Theorem 1.3], the map \( \varphi_X \) is injective if and only if \( X \) does not carry a irrational pencil of genus two. As \( p_g(X) = p_g(Y) \), the corollary follows from Theorem 1.

6. Further results

In this section we describe the behavior of \( \chi \)-positive 3-higher irrational pencils of \( D \)-equivalent fourfolds. We begin with a general result.

We denote by \( X \) and \( Y \) two smooth projective varieties of dimension \( n \geq 2 \). Suppose \( \Phi : D(X) \xrightarrow{\sim} D(Y) \) is an equivalence of triangulated categories and let \( F_\Phi \) be the induced Rouquier isomorphism. Moreover let \( f : X \rightarrow V \) be a \( \chi \)-positive \( k \)-higher irrational pencil with \( 0 < k < n \) and irregularity \( q = q(V) \geq k \). If \( (\pi : \tilde{X} \rightarrow X, \tilde{f} : \tilde{X} \rightarrow \tilde{V}) \) is a non-singular representative of \( f \), then \( Z \defeq \pi_*\tilde{f}^*\text{Pic}^0(\tilde{V}) \) is an irreducible component of \( V^{n-k}(\omega_X)_0 \) of dimension \( q \). Denote by \( h : Y \rightarrow W \) the fibration induced by the abelian variety \( Z' \defeq F_\Phi(\text{id}_X, Z) \subset V^{n-k}(\omega_Y)_0 \) as in Lemma 16.

Proposition 35. If \( \dim W < \dim V \), then \( Y \) admits a fibration \( p' : Y \rightarrow U \) in \( F_Y^{k',q'} \) for some integers \( 0 < k' < \dim V \) and \( 2 \leq q' \leq q(V) \). Moreover, there exists a dominant rational map \( \gamma : W \dashrightarrow U \) such that \( p' = \gamma \circ h \).

Proof. The proposition follows by Proposition 29 as \( q(W) = \dim Z' = q(V) \geq k \) and \( \dim W < \dim V = k \).

We can say something more about \( W \) when \( n = 4 \) and \( k = 3 \). We denote by \( (\pi' : \tilde{Y} \rightarrow Y, \tilde{h} : \tilde{Y} \rightarrow \tilde{W}) \) a non-singular representative of \( h \).

Proposition 36. If \( \dim W < 3 \), then \( \tilde{W} \) is either birational to an isotrivial elliptic surface fibered over a curve of genus \( q(\tilde{W}) - 1 \), or a smooth projective curve.
Proof. Suppose \( \dim W = 2 \). We will prove that \( \chi(\omega_{\widetilde{W}}) = 0 \), so that \( \widetilde{W} \) is birational to an isotrivial elliptic surface fibered over a curve of genus \( q(W) - 1 \). This fact follows by the classification theory of complex algebraic surfaces as \( \widetilde{W} \) is of maximal Albanese dimension and \( q(W) = q(W) \geq 3 \) (cf. [BHPvdV04, Chapter V Section 12] and [Bea96, Theorem X.4]). If, by contradiction, \( \chi(\omega_{\widetilde{W}}) > 0 \), then by Corollary 22 we have \( Z' = \pi'_* h^* \text{Pic}^0(\widetilde{W}) \) is an irreducible component of \( V^2(\omega_Y)_0 \). In turn, by Theorem 4 the component \( Z = F^{-1}_E(\text{Id}_Y, Z') \) would be an irreducible component of \( V^2(\omega_Y)_0 \) too, which is impossible as \( Z \) induces a fibration onto the threefold \( V \). The case \( \dim W = 1 \) is obvious.

Theorem 37. Let \( X \) and \( Y \) be smooth projective fourfolds, and let \( \Phi_X : D(X) \to D(Y) \) be an equivalence of triangulated categories. If \( F^g_X = \emptyset \) for all \( g \geq 2 \), then for any integer \( q \geq 3 \) the equivalence \( \Phi_X \) induces a bijection of sets \( \eta_q : F^{3,q}_X \to F^{3,q}_Y \) preserving the bases of the fibrations.

Proof. We define \( \eta_q = u^{-1}_{Y,3,q} \circ F_X \circ u_{X,3,q} \). Let \( (f : X \to V) \in F^{3,q}_X \) and suppose \( \eta_q(f) = (h : Y \to W) \). To begin with, we will prove \( \dim W = 3 \). If \( X \) does not carry any irrational pencil of genus \( g \geq 2 \), then so does \( Y \) by Theorem 1. This forces \( \dim W > 1 \). Suppose now \( \dim W = 2 \). By Proposition 36, the variety \( \widetilde{W} \) admits an elliptic fibration onto a smooth curve \( B \) of genus \( g \geq 2 \). By Proposition 9 there exists a rational map \( Y \dashrightarrow B \) which extends to a morphism. Hence \( Y \) admits an irrational pencil of genus \( g \geq 2 \) which is impossible. We conclude \( \dim W = 3 \) and \( \eta_q \) defines a bijection. The proof that \( \eta_q \) preserves the bases of the fibrations up to isomorphism follows by Proposition 28.

7. Higher irrational pencils and \( K \)-equivalence

In this section we study the behavior of the sets of fibrations \( F^{k,q}_X \) under \( K \)- and birational equivalence.

Let \( X \) and \( Y \) be smooth projective varieties. To begin with, we define a \( k \)-higher irrational pencil \( f : X \to V \) as a \( \chi \)-positive \( k \)-higher irrational pencil without the condition on the positivity of \( \chi(\omega_Y) \). For every pair of integers \( q \geq k \geq 1 \), we denote by \( P^{k,q}_X \) the set of isomorphism classes of \( k \)-higher irrational pencils \( f : X \to V \) such that \( \dim V = k \) and \( q(V) = q \). Note \( P^{k,q}_X \supset F^{k,q}_X \). The following proposition holds if \( X \) and \( Y \) are \( K \)-equivalent.

Proposition 38. If \( X \) and \( Y \) are birational, then for every pair of integers \( q \geq k \geq 1 \) there exists a bijection of sets \( \tau_{k,q} : P^{k,q}_X \to P^{k,q}_Y \) preserving the bases of the fibrations up to isomorphism.

Proof. Let \( \varphi : Y \dashrightarrow X \) be a birational map and let \( f : X \to V \) be a \( k \)-higher irrational pencil in \( P^{k,q}_X \). Moreover consider the composition \( h = (f \circ \varphi) : Y \dashrightarrow V \). By Proposition 9 \( h \) extends to a morphism. The assignment \( f \mapsto h \) defines the wanted bijection \( \tau_{k,q} \).

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