Faster Recovery of Approximate Periods over Edit Distance

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Abstract

The approximate period recovery problem asks to compute all approximate word-periods of a given word $S$ of length $n$: all primitive words $P$ ($|P| = p$) which have a periodic extension at edit distance smaller than $\tau_p$ from $S$, where $\tau_p = \lceil \frac{n}{3.75 + \epsilon} \rceil$ for some $\epsilon > 0$. Here, the set of periodic extensions of $P$ consists of all finite prefixes of $P^\infty$.

We improve the time complexity of the fastest known algorithm for this problem of Amir et al. [Theor. Comput. Sci., 2018] from $O(n^{4/3})$ to $O(n \log n)$. Our tool is a fast algorithm for Approximate Pattern Matching in Periodic Text. We consider only verification for the period recovery problem when the candidate approximate word-period $P$ is explicitly given up to cyclic rotation; the algorithm of Amir et al. reduces the general problem in $O(n)$ time to a logarithmic number of such more specific instances.

1 Introduction

The aim of this work is computing periods of words in the approximate pattern matching model (see e.g. [8, 10]). This task can be stated as the approximate period recovery (APR) problem that was defined by Amir et al. [2]. In this problem, we are given a word; we suspect that it was initially periodic, but then errors might have been introduced in it. Our goal is to attempt to recover the periodicity of the original word. If too many errors have been introduced, it might be impossible to recover the period. Hence, a requirement is imposed that the distance between the original periodic word and the word with errors is upper bounded, with the bound being related to the period length. Here, edit distance is used as a metric. The fastest known solution to the APR problem is due to Amir et al. [1].

A different version of the APR problem was considered by Sim et al. [16], who bound the number of errors per occurrence of the period. The general problem of computing approximate periods over weighted edit distance is known to be NP-complete; see [15, 16]. Other variants of approximate periods have also been introduced. One direction is the study of approximate repetitions, that is, subwords of the given word that are approximately periodic in some sense (and, possibly, maximal); see [3, 12, 17, 18]. Another is the study of quasiperiods, occurrences of which may overlap in the text; see, e.g., [4, 5, 6, 11, 14].

Let $\text{ed-dist}(S, W)$ be the edit distance (or Levenshtein distance) between the words $S$ and $W$, that is, the minimum number of edit operations (insertions, deletions, or substitutions) necessary to transform $S$ to $W$. A word $P$ is called primitive if it cannot be expressed as $P = Q^k$ for a word $Q$ and an integer $k \geq 2$. The APR problem can now formally be defined as follows.

| Approximate Period Recovery (APR) Problem |
|---|
| **Input:** A word $S$ of length $n$ |
| **Output:** All primitive words $P$ (called approximate word-periods) for which the infinite word $P^\infty$ has a prefix $W$ such that $\text{ed-dist}(S, W) < \tau_p$, where $p = |P|$ and $\tau_p = \lceil \frac{n}{3.75 + \epsilon} \rceil$ with $\epsilon > 0$ |

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Approximate Pattern Matching in Periodic Text (APM Problem)

Input: A word $S$ of length $n$, a word $P$ of length $p$, and a threshold $k$

Output: For every cyclic shift $U$ of $P$, compute $ED(S, U^\infty)$ or report that this value is greater than $k$

Amir et al. [1] use two solutions to the APM problem that work in $O(np)$ time and $O(n + k(k + p))$ time, respectively. The main tool of the first algorithm is wrap-around dynamic programming [9] that solves the APM problem without the threshold constraint $k$ in $O(np)$ time. The other solution is based on the Landau–Vishkin algorithm [13]. For each $p$ and $k < \tau_p$, either algorithm works in $O(n^{4/3})$ time.

Our results. We show that:

- The APM problem can be solved in $O(n + kp)$ time.
- The APR problem can be solved in $O(n \log n)$ time.

Our solution to the APM problem involves a more efficient combination of wrap-around dynamic programming with the Landau–Vishkin algorithm.

2 Approximate Pattern Matching in Periodic Texts

We assume that the length of a word $U$ is denoted by $|U|$ and the letters of $U$ are numbered 0 through $|U| - 1$, with $U[i]$ representing the $i$th letter. By $U[i \ldots j]$ we denote the subword $U[i] \ldots U[j]$; if $i > j$, it denotes the empty word. A prefix of $U$ is a subword $U[0 \ldots i]$ and a suffix of $U$ is a subword $U[i \ldots |U| - 1]$, denoted also as $U[i \ldots]$. The length of the longest common prefix of words $U$ and $V$ is denoted by $lcp(U, V)$. The following fact specifies a well-known efficient data structure answering such queries over suffixes of a given text; see, e.g., [7].

Fact 3. Let $S$ be a word of length $n$ over an integer alphabet of size $\sigma = n^{O(1)}$. After $O(n)$-time preprocessing, given indices $i$ and $j$ ($0 \leq i, j < n$) one can compute $lcp(S[i \ldots], S[j \ldots])$ in $O(1)$ time.

2.1 Wrap-Around Dynamic Programming

Following [9], we introduce a table $T[0 \ldots n, 0 \ldots p - 1]$ whose cell $T[i, j]$ denotes the minimum edit distance between $S[0 \ldots i - 1]$ and some subword of the periodic word $P^\infty$ ending on the $(j - 1)$th character of the period. More formally, for $i \in \{0, \ldots, n\}$ and $j \in \mathbb{Z}_p$, we define

$$T[i, j] = \min\{\text{ed-dist}(S[0 \ldots i - 1], P^\infty[i' \ldots j']) : i' \in \mathbb{N}, j' \equiv j - 1 \pmod{p}\};$$

see Fig. 1. The following fact characterizes $T$ in terms of $ED$.

\footnote{Also the APR problem under the Hamming distance was considered [2] for which an $O(n \log n)$-time algorithm was presented [1] for the threshold $\left\lceil \frac{p}{2\epsilon + \epsilon^2 p} \right\rceil$ with $\epsilon > 0$.}
Fact 4. We have \( \min \{ \text{ED}(S, U^\infty) : U \approx P \} = \min \{ T[n, j] : j \in \mathbb{Z}_p \} \).  

Proof. First, let us observe that the definition of \( T \) immediately yields

\[
\min \{ T[n, j] : j \in \mathbb{Z}_p \} = \min \{ \text{ed-dist}(S, P^\infty[i \ldots j']) : i', j' \in \mathbb{N} \}.
\]

In other words, \( \min \{ T[n, j] : j \in \mathbb{Z}_p \} \) is the minimum edit distance between \( S \) and any subword of \( P^\infty \). On the other hand, \( \min \{ \text{ED}(S, U^\infty) : U \approx P \} \) by definition of \( \text{ED} \) is the minimum edit distance between \( S \) and a prefix of \( U^\infty \) for a cyclic shift \( U \) of \( P \). Finally, it suffices to note that the sets of subwords of \( P^\infty \) and of prefixes of \( U^\infty \) taken over all \( U \approx P \) are the same. \( \square \)

Below, we use \( \oplus \) and \( \ominus \) to denote operations in \( \mathbb{Z}_p \).

Lemma 5 ([9]). The table \( T \) is the unique table satisfying the following formula:

\[
T[0, j] = 0,
\]

\[
T[i + 1, j \oplus 1] = \min \left\{ \frac{T[i, j \ominus 1]}{T[i + 1, j]} + \frac{1}{T[i, j]} \begin{cases} 1 & [S[i] \neq P[j]] \\ 0 & [S[i] = P[j]] \end{cases} \right\}.
\]

Let us mention that the above formula contains cyclic dependencies that emerge due to wrapping (the third value in the minimum). Nevertheless, the table can be computed using a graph-theoretic interpretation. With each \( T[i, j] \) we associate a vertex \((i, j)\). The arcs are implied by the formula in Theorem 5: the arcs pointing to \((i + 1, j \oplus 1)\) are from \((i, j \ominus 1)\) with weight 1 (deletion), from \((i, j)\) with weight 0 or 1 (match or substitution), and from \((i + 1, j)\) with weight 1 (insertion). Then \( T[i, j] \) is the length of the shortest path from any vertex \((0, j')\) to the vertex \((i, j)\). With this interpretation, the table \( T \) is computed using Breadth-First Search, with the 0-arcs processed before the 1-arcs.

2.2 Wrap-Around DP with Kangaroo Jumps

Our next goal is to compute all the values \( T[n, j] \) not exceeding \( k \). In the algorithm, we exploit two properties that our dynamic programming array has.

First of all, let us consider a diagonal modulo length of the period, that is, cells of the form \( T[i, j \oplus i] \) for a fixed \( j \in \mathbb{Z}_p \). We can notice that the sequence of values on every diagonal is non-decreasing. This stems from the fact that on each diagonal the alignment of the pattern is the same and extending a prefix of \( S \) and a subword of \( P^\infty \) by one letter does not decrease their edit distance. This results in a conclusion that if we would like to iteratively compute the set of reachable cells within a fixed distance,
then we can convey this information with just the indices of the furthermost reachable cells in each of the diagonals. Our task is to check whether we can reach some cell in the last row within the distance \( k \). To achieve this, we can iteratively find the set of cells reachable within subsequent distances \( 0, 1, \ldots \).

More formally, for \( d \in \{0, \ldots, k\} \) and \( j \in \mathbb{Z}_p \), we define

\[
D[d, j] = \max\{i : T[i, j \oplus i] \leq d\};
\]

see Fig. 1.

Secondly, we observe that it is cheap to check how many consecutive cost-0 transitions can be made from a given cell. Let us remind ourselves that our only free transition checks whether the next letters of the pattern and the periodic word are equal. To know how far we can stack this transition is, in other words, finding the longest common prefix of appropriate suffixes of \( S \) and \( P^\infty \). We obtain the following recursive formulae for \( D[d, j] \); see Fig. 2.

**Fact 6.** The table \( D \) can be computed using the following formula:

\[
D[0, j] = \text{lcp}(S, P^\infty[j..]),
D[d + 1, j] = i + \text{lcp}(S[i..], P^\infty[i \oplus j..]),
\]

where \( i = \min(u, \max\{D[d, j] + 1, D[d, j \ominus 1], D[d, j \oplus 1] + 1\}) \).

**Proof.** We will prove the fact by considering the interpretation of \( T[i, j] \) as distances in a weighted graph (see Section 2.1). By Theorem 5, from every vertex \((i, j)\) we have the following outgoing arcs:

- \((i, j) \xrightarrow{1} (i + 1, j)\),
- \((i, j) \xrightarrow{S[i] \neq P[j]} (i + 1, j \oplus 1)\),
- \((i, j) \xrightarrow{0} (i, j \ominus 1)\).

Moreover, the value \( T[i, j] \) is equal to the minimum distance to \((i, j)\) from some vertex \((0, j')\). The only arc of cost 0 is \((i, j) \xrightarrow{0} (i + 1, j \ominus 1)\) when \( S[i] = P[j] \). Therefore, when we have reached a vertex \((i, j)\), the only vertices we can reach from it by using only 0-arcs are \((i, j), (i + 1, j \oplus 1), \ldots, (i + k, j \oplus k)\), where \( k \)
is the maximum number such that \( S[i] = P[j], \ S[i + 1] = P[j \oplus 1], \ldots, S[i + (k - 1)] = P[j \oplus (k - 1)]. \) Therefore, \( k = \text{lcp}(S[i \ldots], P^{\infty}[j \ldots]). \)

Hence, \( D[0, j] = \text{lcp}(S, P^{\infty}[j \ldots]) \) holds for distance 0. Taking advantage of monotonicity of distances on each diagonal, we know that full information about reachable vertices at distance \( d \) can be stored as a list of the furthest points on each diagonal. Moreover, to reach a vertex of distance \( d + 1 \) we need to pick a vertex of distance \( d \), follow a single 1-arc and then zero or more 0-arcs. Combining this with the fact that arcs changing diagonal can be arbitrarily used at any vertex, it suffices to consider only the bottom-most point of each diagonal with the distance \( d \) as the starting point of the 1-arc, as we can greedily postpone following an arc that switches diagonals. \( \square \)

To conclude, assuming we know the indices of furthest reachable cells in each of the diagonals for an edit distance \( d \), we can easily compute indices for the next distance. In the beginning, we update the indices by applying available 1-arcs and afterwards, we increase indices by the results of appropriate lcp-queries. In the end, we have computed the furthest reachable cells in each of the diagonals within distance \( d + 1 \) and achieved that in linear time with respect to the number of diagonals, i.e., in \( O(p) \) time. This approach is shown as Algorithm 1.

| Algorithm 1: Compute all values \( T[n, j] \) not exceeding \( k \) |
|-------------------------------------------------------------------|
| \( T[n, 0 \ldots p - 1] := (\perp, \ldots, \perp) \) |
| for \( d := 0 \) to \( k \) do |
| foreach \( j \in \mathbb{Z}_p \) do |
| \( i := 0 \) |
| else |
| \( i := \min(n, \max(D[d - 1, j] + 1, D[d - 1, j \oplus 1], D[d - 1, (j \oplus 1) + 1])) \) |
| \( D[d, j] := i + \text{lcp}(S[i \ldots], P^{\infty}[i \oplus j \ldots]) \) |
| if \( D[d, j] = n \) and \( T[n, j \oplus n] = \perp \) then |
| \( T[n, j \oplus n] := d \) |

**Lemma 7.** Algorithm 1 for each \( j \in \mathbb{Z}_p \) computes \( T[n, j] \) or reports that \( T[n, j] > k \). It can be implemented in \( O(n + pk) \) time.

**Proof.** We use Theorem 3 to answer each lcp-query in constant time, by creating a data structure for lcp queries for the word \( S \# P^r \) where \( \# \) is a sentinel character and \( r \) is an exponent large enough so that \( |P^r| \geq n + p \). \( \square \)

### 2.3 Main Results

The table \( T \) specifies the last position of an approximate match within the period of the periodic word. However, in our problem we need to know the starting position, which determines the sought cyclic shift of the period. Thus, let \( T^R \) be the counterpart of \( T \) defined for the reverse words \( S^R \) and \( P^R \). Its last row satisfies the following property:

**Fact 8.** For every \( j \in \mathbb{Z}_p \), we have

\[
T^R[n, p \oplus j] = \text{ED}(S, U^\infty) \quad \text{where} \quad U = P[j \ldots p - 1] \cdot P[0 \ldots j - 1].
\]

Here, \( U \) a cyclic shift of \( P \) with the leading \( j \) characters moved to the back.

**Proof.** By definition of \( T^R \) and \( T \), for \( 0 \leq i \leq n \) and \( j \in \mathbb{Z}_p \), we have

\[
T^R[n, j] = \min \{ \text{ed-dist}(S, (P^R)^{\infty}[i' \ldots]) : i' \in \mathbb{N}, j' \equiv j - 1 \pmod{p} \}
\]

\[
= \min \{ \text{ed-dist}(S, P^{\infty}[j' \ldots i']) : i' \in \mathbb{N}, j' \equiv j - 1 \pmod{p} \}
\]

\[
= \min \{ \text{ed-dist}(S, P^{\infty}[p \oplus j \ldots i']) : i' \in \mathbb{N} \}
\]

\[
= \text{ED}(S, P^{\infty}[p \oplus j \ldots]).
\]
Consequently,
\[ T^{R}[n, p \odot j] = \text{ED}(S, P^{\infty}[j..]) = \text{ED}(S, (P[j..p-1] \cdot P[0..j-1])^{\infty}) \]
holds as claimed. \( \square \)

Example 9. If \( P = \text{ABCA} \) and \( S = \text{CBAACAABCA} \) (see Fig. 1), then \( T^{R}[10, 2] = \text{ED}(\text{CBAACAABCA}, (\text{CAAB})^{\infty}) = \text{ed-dist}(\text{CBAACAABCA}, \text{CAABCAABCA}) = 2 \).

Running Algorithm 1 for the reverse input, we obtain the solution to the APM problem.

**Theorem 10.** The Approximate Pattern Matching in Periodic Text problem can be solved in \( \mathcal{O}(n + kp) \) time.

By combining Theorem 2 and Theorem 10 with \( k < \tau_p \), we arrive at an improved solution to the APR problem.

**Theorem 11.** The Approximate Period Recovery problem can be solved in \( \mathcal{O}(n \log n) \) time.

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