Learning stereometry in a secondary school within GeoGebra’s Augmented Reality app

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Abstract. GeoGebra dynamic mathematics system has earned a public acceptance as a software product to support educational and even scientific activities in the field of mathematics. Since 2017, GeoGebra developers have provided users of this program with a new feature for creating and using AR. In this article a series of tasks for modeling of real objects using GeoGebra AR is presented. They assigned to different levels of secondary geometric education: basic, specialized, and advanced. Setting such tasks for students contributes to the formation of visual thinking and skills of using of stereometry knowledge for solving practical problems.

1. Introduction
The idea of augmented reality happens, doesn’t belong to the scientist, but to the science-fiction writer Lyman Frank Blaum (1856–1919), who is the author of the well-known book for children «The Wizard of Oz» [1]. He expressed this idea in another of his books, The Master Key [2]. In the book he described the glasses-marker of the personality properties which allow you to see on the forehead of each person you meet a letter denoting the main feature of his character: G-good, E-evil, W-wise, F-foolish, etc. The term «augmented reality» was first offered by T. Kodel, an employee of the American company Boeing [3]. He named so a computer device attached to a person’s head. It allows a computer-produced diagram to be superimposed and stabilized on a specific position on a real-world object. Today AR defined as «…a system that fulfills three basic features: a combination of real and virtual worlds, real-time interaction, and accurate 3D registration of virtual and real objects [4]. The ability to create an augmented reality effect is provided by various applications for tablets and phones, also some specialized hands-free devices. One of these devices is the Google glass smartphone headset, which testing has began in the spring of 2012. Another example is the holographic car navigation system Navigon, released in 2016.

Augmented reality technology is gaining popularity today, providing quick and convenient access to the necessary information for specialists in various fields.

Meanwhile, the assessment of educational opportunities of this technology is just beginning. Volosyuk [5] believes that the use of augmented reality will solve the problem of clip thinking and clip representations of students about the reality. These negative qualities are generated by the subject nature of learning, the rapid growth of information volumes, and the habit of students to get information by «web surfing». The integrity and completeness of the perception of the studied object is ensured by the overlay of various educational content (text, 3D model, video, audio). In this regard, textbooks and
workbooks with augmented reality are being created, Museum expositions and excursion routes with augmented reality are becoming very popular, and so on. Shelev [6] believes that augmented reality technology opens up the possibility of integrating educational activities with gaming, practical or professional ones by immersing the student in an environment which simulates the conditions for applying knowledge in the past, present or future. Grishkun [7] believes that augmented reality allows students to learn about real objects through experimenting with their virtual models in situations where conducting a full-scale experiment is impossible for one reason or another (for security reasons, limited school time, due to the lack of the necessary material and technical base, etc.).

2. GeoGebra's capabilities in creating and using AR models
In all cases described in the literature, AR models of real objects are created by adults for students, and they also determine possible scenarios for their use in education. In 2017, the developers of the GeoGebra dynamic mathematics system provided users with the ability to independently construct and reproduce geometric AR models of real objects. They created the GeoGebra Augmented Reality mobile app, as well as a tool for projecting models created in GeoGebra 3D Calculator to real space. Initially, these features were implemented only for IOs. Users just had to install one of the above applications on their IPad or IPhone, create a 3D model in them, and then click the AR button in the lower-right corner of the screen to localize the virtual object in real space (figure 1).

![Figure 1. 3D Model created in GeoGebra 3D Calculator and ready to be projected using the AR button.](image)

The detailed mechanism for using this feature is presented in the move [8]. Currently, users of mobile devices running on Android are also able to create and reproduce AR models. These devices must meet two conditions: have an operating system not lower than Android 7.0, have an AR CORE developer certificate. The list of devices [9] that support this software is published on the developer's website.

One of the active popularizers of AR modeling in GeoGebra is Tim Brzezinski, who created several methodological guides on the use of AR models in the education: GeoGebra 3D with AR (Google): Explorations & Lesson Ideas [10], GeoGebra 3D with AR: Quick Setup Instructions [11], Augmented Reality: Ideas for Student Explorations [12].

Currently on the site GeoGebra.org there are more than 400 models created by authors from different countries: Brazil, Germany, Italy, USA, Russia, Austria, etc.

Our analysis of models allowed us to identify the following relatively independent areas of their use:
• demonstration of the possibilities of geometry in modeling real objects of the surrounding reality (3 Rung Bike Rack: Quick GeoGebra 3D with Augmented Reality Demo [13]; Pringles Modeling in GeoGebra 3D: Quick Demo [14]);
• experimenting with models of real-world objects (Campbell's Test: Maximizing Volume - Exploration in GeoGebra 3D with AR [15]);
• visualization of the transformation results of the analytical dependencies (Baby Step Modeling in 3D GC with AR [16]);
• illustration of mathematical ideas related to the transformation of images: changing the position in space, building projections, scans, completion, etc. (Exploring surface area of a rectangular prism [17], Quick AR Illustration: Projection of a Surface onto the xy-Plane [18]).

3. A series of tasks for teaching stereometry to students with different levels of geometric training
The studying of stereometry in Russian schools belongs to the senior classes, when students are self-determined in the professional sphere, decide whether to continue their mathematical education and at what level.

The approximate basic educational program [19] provides three levels of mathematics: basic (for those students who do not intend to continue their mathematical education), specialized (for those students who plan to develop their knowledge in the field of mathematics applications) and advanced (for those students who plan to specialize in mathematics).

Here we present examples of a series of augmented reality tasks assigned to these levels.

3.1. A series of tasks for students with a basic level of training
The main goals of teaching stereometry to students who have chosen this level is to form a geometric vision of the real world (the ability to see the place of application of the knowledge obtained at school) and practical skills.

AR modeling provides unique opportunities to achieve the first of these goals. Architectural structures that have a well-recognized geometric shape are well-suited for modeling objects (figure 2).

Figure 2. Architectural structures that are easy to model.

For this purpose, you can also use items that students encounter in everyday life: florariums, boxes, vases, Christmas decorations, hats, etc.

A special feature of AR models of real objects is that they preserve not only the basic shape, but also the proportions of the real object.

Classes related to setting and performing tasks on AR modeling of architectural structures are better conducted in the form of excursions. This will give students the opportunity to get reference data by directly measuring objects, as well as to check the correctness of problem solving by visually combining the AR model with a real object.

Here is a series of tasks related to the topic «Pyramids».

The preliminary task. It is known that the perimeter of the base of the pyramid of Cheops (figure 3) is equal to the length of a circle with a radius equal to the height of the pyramid. Perform calculations
to convert a ready-made dynamic model of a regular quadrilateral pyramid into a scale model of the Cheops pyramid (figure 4).

![Figure 3. Top-down, aerial view of The Pyramid of Cheops.](image)

![Figure 4. Dynamic model of a regular quadrilateral pyramid.](image)

This task is calculated in nature. The result is the formula: \( R = \frac{\pi H \sqrt{2}}{4} \), where \( R \) – radius of the circumscribed circle of the base; \( H \) – the height of the pyramid. To convert a dynamic model, enter this formula in the «Enter» line. A new variable appears. Then you need to replace the \( R \) parameter with the name of this variable.

**The Excursion task** (for students of Vladikavkaz, Russia). In Vladikavkaz, at 2B Markus street, there is a cafe «Pyramid» (figure 5).

- Describe the geometric shape of this building using the names of the stereometric shapes you know.
- Use the ready-made AR-model to check whether its pyramidal part can be considered a model of The Pyramid of Cheops.
- Set how many times the height, surface area, and volume of the pyramid part of this building is less than the corresponding values of The Pyramid of Cheops, if it is known that the height of The Pyramid of Cheops is 136.5 m.
- Create an AR-model of this building by first setting the height ratio of its parts using a virtual ruler.
- Prepare the resulting model for 3D printing a souvenir.

An example of the result of these tasks is shown in figure 6–7.

![Figure 5. The cafe «Pyramid», Vladikavkaz.](image)

![Figure 6. An AR-model.](image)

![Figure 7. The model in STL format for 3D printer.](image)
By itself, taking geometry in nature with a tablet contributes to the development of students’ interest in its study. Setting the last task to create a souvenir increases the responsibility of students for the results of their educational activities.

3.2. The series of tasks for students with advanced level training

A special feature of the stereometry course for students with advanced training is to increase attention to the disclosure of connections between algebra and geometry by studying elements of analytical geometry in space and generalizing knowledge about functional dependencies. This allows students to set tasks for creating AR models of objects that have a complex geometric nature.

For example, the study of the topic “Surfaces of rotation” in the geometry course 11 in academic classes can be organized using the following series of tasks.

**Task 1.** Establish a correspondence between real objects (A–E) and equations (1–5). Add to the equations a description of the meaning of the parameters and the range of their acceptable values. To solve this problem, use AR modeling in the GeoGebra program.

![Images of objects](image)

A Hat box  A Satellite plate  A Donut  An Asian hat  A Rugby ball

\[
\begin{align*}
z &= \sqrt{k^2(x^2 + y^2) + c^2} \\
x^2 + y^2 &= R^2 \\
z &= h - k\sqrt{x^2 + y^2} \\
z &= \pm\sqrt{c^2 - k^2(x^2 + y^2)} \\
z &= \pm\sqrt{r^2 - (\sqrt{x^2 + y^2} - R)^2}
\end{align*}
\]

After completing the task, it is useful to draw students’ attention to the fact that all these equations have the form: \( z = f(x^2 + y^2) \). The next step in mastering the analytical method for determining rotation surfaces is to perform the task of converting known functions into equations of rotation surfaces.

**Task 2.** Find out which rotation surface is generated by substitution: \( x \to \sqrt{x^2 + y^2} \) into the formula \( z = f(x) \), where: 1) \( f(x) = kx \) – a directly proportional; 2) \( f(x) = \frac{k}{x} \) – an inversely proportional; 3) \( f(x) = x^2 \) – a quadratic function.

Name the surfaces you know.

To recognize dependencies, students have two ways: find the suitable general equation, construct it in GeoGebra (figure 8).

For example, \( f(x) = kx \to z = k\sqrt{x^2 + y^2} \) corresponds to the conic surface equation (3), when \( h = 0 \).
Figure 8. A conical surface constructed in GeoGebra.

All these tasks prepare students to apply knowledge in practice for AR modeling.  

**Task 3.** Select the values of the equation parameters (1) and (3) to get an AR model of the donut on a plate. Check the correct construction by combining it with the original.

Figure 9. The object and its AR model.

Students are offered a creative task to construct an analytical dependency.  

**Task 4.** Before you Sombrero it is a type of wide-brimmed hat from Mexico, used to shield from the sun (figure 10). In the language of geometry, it can be described as a surface of rotation given by an equation $z = f(x^2 + y^2)$ on the set $x^2 + y^2 \leq R^2$, where R is a parameter.

- List the properties that the function $f$ should have.  
- Come up with an equation for a function that has these properties.  
- Construct an AR-model of the Sombrero.

Figure 10. Sombrero and its AR-model.
During the problem solving, students at first select a function \( f(x) \) which has the following properties: it is even; it has a maximum at \( x = 0 \), and one minimum at \( x > 0 \). Then they use substitution: 
\[
x \mapsto x^2 + y^2.
\]

One of the results of solving the problem is the construction of a fractional-rational function:
\[
if \ (x^2 + y^2) \leq 20, \ z = \frac{11}{2 + 0.8(x^2 + y^2) + 0.1(x^2 + y^2)}
\]

3.3. A series of tasks for students of specialized engineering classes

A special feature of the stereometry course for students of engineering classes is the increased attention to the issues of geometric design - the creation and use of projection images. Stereometry here serves as a theoretical basis for the study of descriptive geometry at the University. It will be interesting and useful for students of engineering classes to solve problems for creating AR models of future objects.

Let's consider as an example a series of tasks that can be offered to students to bring them to the project activity when studying the topic «Regular Polyhedra».

All these tasks are united by a historical theme: «According to a story recounted in 1693 by English mathematician John Wallis, Prince Rupert (1619–1682) wagered that a hole could be cut through a cube, large enough to let another cube of the same size pass through it. Wallis showed that in fact such a hole was possible and Prince Rupert won his wager. For a long time, it was considered that this problem can only be solved theoretically» [20].

**Task 1.** In Wikipedia, the solution to the Prince Rupert problem is shown on figure 11. Create an AR-model of the hole with the same proportions.

During solving this problem, students should carefully read the article to find information about the proportions in it. Then they must look at the image through the eyes of a mathematician to highlight the familiar polyhedra, and choose the right GeoGebra tools (figure12).

**Figure 11.** The hole model for solving the Prince Rupert problem.  **Figure 12.** Result of construction the 3D model by GeoGebra tools.

**Task 2.** Prove that a cube of the same size can pass through the hole.

Solving this problem requires a higher level of mathematical training. Students need to find points on the model that define the secant plane, perpendicular to which the cube should move (points \( N, L \) and \( F_1 \) on figure 11). Then they must prove that the cross-section of the hole is a square \( (NLL_1P_1) \) with a side larger than the side of the cube (the main calculations are shown in figure 13).
\[ AB = a \Rightarrow SN = \frac{3}{4} a \sqrt{2} \quad \text{and} \quad SF = \frac{1}{4} a \sqrt{2} \]
\[ SF_1 = \sqrt{a^2 + SF^2} = \frac{3}{4} a \sqrt{2} = SN \]
\[ \angle ALF = \angle CLN = 45^\circ \Rightarrow \angle F_1LN = 180^\circ - (\angle ALF + \angle CLN) = 90^\circ \]

\[ NL \perp LF \quad \text{and} \quad NL \perp LF_1 \Rightarrow NL \perp (LF_1L) \Rightarrow NL \perp LF_1 \]

**Figure 13.** Solving the task 2.

**Task 3.** Calculate the angle of inclination of the trajectory of the cube to the plane found in the solution of the task 2. Using this data, build a dynamic AR model that proves the practical feasibility of the Rupert's hypothesis (figure 14).

**Figure 14.** The AR-model for solving the Rupert’s problem [21].

Solving this problem with GeoGebra tools requires knowledge about parallel transfer and how to set it (coordinates or vector).

**Task 4.** Create an AR model and propose a computer experiment that proves that the solution presented in Wikipedia is not the only possible one. This is a research task. To solve it, students need to «release» the points \( L, F_1 \) and \( N \) that fix the position of the hole, and conduct a computer experiment, the purpose of which is to establish the dependence of the size of the square inscribed in the section on the position of the points on the edges.

Students’ project work can be directed to AR-modeling of holes in spatial bodies, which, like a cube, have the «Rupert property»: tetrahedron, octahedron [22]; dodecahedron, icosahedron [23]; eight Archimedean bodies [24].

**4. Conclusions**

Despite the different level of training of students, each series of tasks presented by us includes tasks for independent creation of AR models by students. Using ready-made models in training, to our opinion, should be very limited, since the application of mathematical knowledge occurs to a greater extent at the time of their creation. Performing these tasks is not difficult for those students who regularly use GeoGebra during their studies, starting from primary school. These examples show that augmented reality allows you to make stereometry training practice-oriented, regardless of the profile and level of study chosen by students. The temporary problem of AR GeoGebra is the impossibility of long-term localization of an augmented reality GeoGebra in the real space. This feature would allow you to create textbooks, workbooks, control and measurement materials with dynamic augmented reality, and create resources for conducting geometry lessons on the ground (in parks, estates, and museums).
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