Anisotropic spacetimes in $f(T, B)$ theory I: Bianchi I universe

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Abstract That is the first part of a series of studies on analyzing the higher-order teleparallel theory of gravity known as the $f(T, B)$-theory. This work attempts to understand how the anisotropic spacetimes are involved in the $f(T, B)$-theory. In this work, we review the previous analysis of $f(T, B)$-gravity, and we investigate the global dynamics in the case of a locally rotational Bianchi I background geometry. We focus in the case of $f(T, B) = T + f(B)$ theory and we determine the criteria where $f(T, B)$-theory solves the homogeneity problem. Finally, the integrability properties for the field equations are investigated by applying the Painlevé analysis. The analytic solution is expressed by a right Painlevé expansion.

I Introduction

The theory of General Relativity is well-tested and predicts astrophysical objects and phenomena which observations have confirmed such as the black holes and the gravitational waves \cite{1–3}. Successful though General Relativity may be in the description of astrophysical gravitational systems, it is challenged by the analysis of the recent cosmological observations \cite{4–10}.

At present, the Universe is going under an acceleration phase driven by an exotic matter source known as dark energy with negative pressure, which provides repulsive forces and anti-gravity phenomena. Cosmological inflation \cite{11}, on the other hand, was introduced to solve various cosmological problems. For example, the observable Universe’s inhomogeneity and flatness. According to the cosmic “no-hair” conjecture \cite{12, 13}, an asymptotic solution of the Universe described by the de Sitter solution provides a rapid expansion of the size of the Universe such that the latter effectively loses its memory of the initial conditions, which means that the de Sitter expansion solves the “flatness”, “horizon” and monopole problem \cite{14, 15}.

The modification of the Einstein-Hilbert action with the introduction of a cosmological constant term is the most straightforward mechanism for the description of the acceleration phases of the Universe. Indeed, a positive cosmological constant in Bianchi cosmologies leads to expanding Bianchi spacetimes, evolving toward the de Sitter universe \cite{16}. However, that conclusion is valid for the case of anisotropic background geometries such as that of the Szekeres universes \cite{17, 18}. However, the cosmological constant suffers from other issues, for extended discussions on the cosmological constant problems we refer the reader to \cite{19, 20}.

Over the last years, cosmologists have proposed various modifications of the Einstein-Hilbert action led to the modified theories of gravity, by introducing geometric invariants and modifying the gravitational field equations \cite{21–23}. In modified theories of gravity, the new terms in the Einstein-Hilbert Action introduce new dynamical degrees of freedom in the field equations which drive the dynamics such that to explain the observational phenomena. There are various families of modified theories of gravity, categorized according to the geometric invariants, which are introduced in the Action Integral, see for instance \cite{25–38} and references therein.

The fundamental invariant in General Relativity is the Ricciscalar $R$ of the Levi-Civita symmetric connection. However, in the formulation of teleparallelism and specifically in the Teleparallel Equivalent of General Relativity (TEGR) is the fundamental geometric invariant which is used for the definition of the gravitational Action Integral. That is, the torsion scalar $T$ defined by the antisymmetric connection of the nonholonomic basis \cite{39–44}, that is, of the curvature-less Weitzenböck connection \cite{45}.

The equivalence of teleparallelism and General Relativity does not hold on the modified theories of gravity based on the Ricciscalar and that based on the torsion scalar. For instance, the $f(T)$-theory \cite{46} inspired by the $f(R)$-theory is a second-order theory while $f(T)$-theory is a fourth-order theory \cite{25}. The latter is because the torsion scalar $T$ admits terms with first-order derivatives, instead of the Ricciscalar $R$, which admits second-order derivatives. Moreover, while $f(R)$-gravity is equivalent to a scalar-tensor theory, that is not true for the $f(T)$ theory. The two theories are different and have different properties with that of General Relativity \cite{47, 48}. Applications of $f(T)$ theory in the dark energy problem and in the description of the cosmological history are presented in \cite{49–54}.

Similarly to General Relativity and the modifications of the Einstein-Hilbert Action Integral, there have been proposed various modified theories of gravity based on the modification of the Action Integral for the TEGR, which extends the $f(T)$ modification...
with the use of other invariants constructed by the antisymmetric connection [55]. The various families of the modified gravitational theories based on teleparallelism are summarized in the recent review [56].

We are interested in a higher-order teleparallel theory known as \( f(T, B) \) where \( B \) is the boundary term relating the torsion \( T \) with the Ricciscalar \( R \), that is \( B = T + R \) [55, 57]. Because \( B \) includes second-order derivatives, the \( f(T, B) \)-theory is a fourth-order theory for a nonlinear function \( f \) on the variable \( B \). \( f(T, B) \) provides the limits of other modified theories for specific functional forms of \( f \). Indeed, when \( f(T, B) = f_1 T + f_2 B \), General Relativity is recovered, while when \( f(T, B) = f(T) + f_2 B \), \( f(T) \) teleparallel theory is recovered. Finally, when \( f(T, B) = f(T - B) \), the theory gives the limit for the \( f(R) \)-theory of gravity.

There are a plethora of studies which deal with \( f(T, B) \)-theory in cosmology. Exact and analytic cosmological solutions with an isotropic background space were investigated in [58–60], while the reconstruction of the cosmological history in \( f(T, B) \) theory was the subject of study in [61–64]. The presence of nonzero spatial curvature was recently considered in [65]. The minisuperspace quantization in \( f(T, B) \) was studied in [66] while an inhomogeneous exact solution was recently found in [67]. The recent work [68] deals with the study of anisotropic solutions in \( f(T, B) \). Specifically, the Bianchi I background space was considered, and it was found that Kasner and Kasner-like solutions are asymptotic solutions for the field equations. However, anisotropic exponential solutions are not preferred by the theory.

Kasner Universe [69] is a well-known anisotropic closed-form exact solution of General Relativity. The Kasner four-dimensional metric has three parameters, namely the Kasner indices, which must satisfy the two so-called Kasner algebraic relations. There are a plethora of cosmological applications of the Kasner solution, which makes the solution important, see, for instance [70–77]. Anisotropic and homogeneous cosmologies described by the Bianchi class spacetimes contain several cosmological models that have been applied to discuss the anisotropies of the primordial Universe and for its evolution towards the isotropy [78–80].

This work is the first part of a series of studies where anisotropic and homogeneous spacetimes are studied in the context of \( f(T, B) \)-theory of gravity. For the background space, we consider the locally rotational (LRS) spacetimes, the so-called Bianchi I, Bianchi III and Kantowski-Sachs spacetimes. These are LRS spacetimes admitting a four-dimensional isometry group. This isometry group has a three-parameter subgroup whose orbits are 2-surfaces of constant curvature [81]. The novelty of these spaces is that in the isotropic limit, the Friedmann–Lemaître–Robertson–Walker (FLRW) Universe is recovered. Indeed, the Bianchi I reduces to the spatially flat FLRW spacetime, while the open and closed FLRW spacetimes follow from the isotropic Bianchi III and Kantowski-Sachs spacetimes, respectively. The Kantowski-Sachs spacetime can follow from the LRS Bianchi type IX metric by a Lie contraction [81]. Additionally, these spacetimes can be seen as the limit of the homogeneity in the case of the silent Universe and specifically of the Szekeres spacetimes [82].

We perform a detailed analysis of the dynamics of the field equations by using normal coordinates to investigate their asymptotic behaviour. Indeed, we shall study the existence of exact anisotropic solutions. We also construct the cosmological history by assuming an anisotropic background space, with or without spatial curvature. Such an analysis will provide us with necessary information about \( f(T, B) \)-theory. We shall investigate if isotropy and flatness problems can be solved by \( f(T, B) \)-theory, as well as if the limit of General Relativity can be recovered. This approach has been widely studied in various gravitational theories with many exciting results, for instance [81, 83–99].

In this study, we present a brief review of \( f(T, B) \)-theory and we focus on \( f(T, B) = T + F(B) \) theory for the anisotropic and homogeneous Bianchi I geometry. \( f(T, B) = T + F(B) \) theory is of special interest because it is a fourth-order theory of gravity with the same degrees of freedom as \( f(R) \)-theory. Moreover, there exists a scalar field description related to scalar-torsion theory. Anisotropic and homogeneous background geometries are investigated in the studies [100], and [101]. Indeed, the Kantowski-Sachs spacetime and the LRS Bianchi III spacetimes are considered, respectively. As we have mentioned before there is a one-to-one relation of the anisotropic geometries of our consideration with the FLRW spacetimes in the isotropization limit. Because the three different geometries provide different topological spaces in terms of the dynamical systems we decided to split this piece of study in a series of works. In addition, substantial differences are expected in the cosmological evolution of the physical variables according to whether the spacelike surface has open, closed or flat topology, which make necessary the need to separate the present analysis in a series of studies according to the admitted topology, see also the discussion in [100].

Finally, in [102], we focus on the existence of the minisuperspace description of the field equations. We apply the Noether symmetry analysis [103] to construct conservation laws and determine exact and analytic solutions for the field equations. The plan of the paper is as follows.

In Sect. 2 we present a brief review of the teleparallel \( f(T, B) \)-theory and we focus on \( f(T, B) = T + F(B) \) theory. For the Bianchi I geometry, in Sect. 3 we derive the field equations and explain how the function form of \( f(T, B) \) theory affects the geometrodynamic degrees of freedom which are involved in the cosmological dynamics. The dynamical system analysis of \( f(T, B) = T + F(B) \) theory is provided in Sect. 4. We apply the \( H \)-normalization approach and write the field equations in an equivalent form using dimensionless variables. For the new system, we determine the stationary points. Each stationary point corresponds to a specific asymptotic solution. The physical properties of the asymptotic solutions are investigated, along with the stability properties. They are an essential analysis in order to reconstruct the cosmological history and answer the question if \( f(T, B) = T + F(B) \) can solve the isotropy of the Universe for anisotropic initial conditions. In Sect. 5, we investigate the integrability properties of the field equations and the existence of an analytic solution by using the Painlevé analysis. Finally, Sect. 6 summarises the results and conclusions.
2 Teleparallel theory of gravity

In teleparallelism the vierbein fields $e_\mu(x^\nu)$ introduce the dynamical degrees of freedom [41, 42]. They form an orthonormal basis for the tangent space at each point $P$ such that $g(e_\mu, e_\nu) = e_\mu \cdot e_\nu = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the line element of the Minkowski spacetime, $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$. The commutator relations for the vierbein fields are $[e_\mu, e_\nu] = c^\rho_\mu e_\rho$ where $c^\rho_\mu = 0$.

In the nonholonomic coordinates the covariant derivative $\nabla_\mu$ is defined with the connection

$$\hat{\Gamma}_{\nu\beta}^\mu = \{\nu\beta\} + \frac{1}{2} \delta^\mu_\beta (c_{\nu\sigma,\beta} + c_{\sigma\beta,\nu} - c_{\mu\beta,\sigma}),$$

where $\{\nu\beta\}$ is the symmetric Levi-Civita connection of Riemannian geometry which is used in General Relativity. For the case where $e_\mu \cdot e_\nu = \eta_{\mu\nu}$, the Weitzenböck connection [45]

$$\hat{\Gamma}_{\nu\beta}^\mu = \frac{1}{2} \eta_{\mu\sigma} (c_{\nu\sigma,\beta} + c_{\sigma\beta,\nu} - c_{\mu\beta,\sigma}),$$

where $\hat{\Gamma}_{\nu\beta}^\mu$ are antisymmetric in the two first indices, that is

$$\hat{\Gamma}_{\mu\nu\beta} = -\hat{\Gamma}_{\nu\mu\beta}, \quad \hat{\Gamma}_{\mu\nu\beta} = \eta_{\mu\sigma} \hat{\Gamma}_{\nu\beta}^\sigma,$$

and they describe the Ricci rotation coefficients.

The nonnull torsion tensor is defined by the relation

$$T_{\mu\nu\beta} = \hat{\Gamma}_{\nu\beta}^\mu - \hat{\Gamma}_{\mu\nu}^\beta,$$

while the torsion scalar $T$ is given by the expression

$$T = \frac{1}{2} \left( K_{\mu\nu}^\beta + \delta^\beta_\beta T_{\nu\beta} - \delta^\beta_\beta T_{\mu\nu\beta} \right) T^\beta_{\mu\nu}.$$

where

$$K_{\beta}^{\mu\nu} = -\frac{1}{2} \left( T_{\mu\nu}^\beta - T_{\nu\mu}^\beta - T_{\beta}^{\mu\nu} \right).$$

In TEGR the gravitational Action Integral is as follows [39, 40]

$$S_T = \frac{1}{16\pi G} \int d^4x T + S_m, \quad e = \det(e_\mu),$$

in which $S_m$ is the Action Integral component for the matter source.

2.1 $f(T)$-theory

The simplest extension of TEGR is the $f(T)$. The gravitational Action Integral (7) is modified as follow [46]

$$S_{f(T)} = \frac{1}{16\pi G} \int d^4x e(f(T)) + S_m,$$

where $f$ is a function which should be considered. In the limit where $f(T)$ is a linear function TEGR is recovered. Because $T$ admits only first order derivatives of the vierbeins, then the gravitational field equations are of second-order.

Variation with respect to the vierbein provides the modified field equations [56]

$$4\pi G e T_a^{(m)\lambda} = ef_{,T} G_a^{\lambda} + \left[ \frac{1}{4} \left( T f_{,T} - f \right) e h^\lambda_a + e f_{,T} S_a^{\mu\lambda} \right],$$

with $T_a^{(m)\lambda}$ the energy–momentum tensor of the matter source $S_m$ while $f_T$ and $f_{TT}$ denote the first and second derivatives of the function $f(T)$ with respect to $T$, $S_a^{\mu\lambda}$ is the superpotential tensor

$$S_a^{\mu\nu} = \frac{1}{2} \left( K_{\mu\nu}^\beta + \delta^\beta_\beta T_{\mu\nu\beta} - \delta^\beta_\beta T_{\beta}^{\mu\nu} \right),$$

and $e_\alpha = h_\alpha^\lambda(x) \partial_\lambda$ is the coordinate basis for the vierbein fields.

From (9) it follows that the limit of TEGR is recovered for a linear Lagrangian function $f_{TT} = 0$. However, that is not the only possible case. Vacuum solutions of General Relativity follow in the case of a nonlinear function $f(T)$ when [54]

$$T = 0, \quad f(T)_{T \to 0} = 0,$$

and

$$S_{\rho}^{\mu\nu} \partial_\mu(T) f_{TT} = 0.$$
Similarly, solutions with cosmological constant term follow when [104]

\[ f(T)|_{T=\Lambda} = 0, \quad T = -\Lambda. \]  

### 2.2 \( f(T, B) \)-theory

\( f(T, B) \)-theory is a natural extension of \( f(T) \) teleparallel theory where the boundary term

\[ B = 2e^{-1}\partial_\nu(eT_\rho^{\nu}), \]  

which connects the torsion scalar \( T \) with the Riccscalar \( R \), i.e.

\[ B = T + R, \]  

is used in the gravitational Action Integral.

The modified gravitational Action Integral is [55]

\[ S_{f(T, B)} = \frac{1}{16\pi G} \int d^4x ef(T, B). \]  

Variation for the vierbein fields provides the field equations

\[ 4\pi GeT^{(m)s}_a = ef(T)eT^{s}_a + \left[ \frac{1}{4}(Tf,T - f)eh_a^{\lambda} + e(f,T)_a \right] \]

\[ + \left[ e(f,B)_a T^{\mu\lambda} - \frac{1}{2} e(h_\mu^{\alpha}(f,B)_{,\sigma} - h_\mu^{\alpha}(f,B))^{\mu\nu}g_{\nu\sigma} + \frac{1}{4} eBh_a^{\nu}eB, \right]. \]

Where the terms

\[ T^{(B)}_a = e(f,B)_a T^{\mu\lambda} - \frac{1}{2} e(h_\mu^{\alpha}(f,B)_{,\sigma} - h_\mu^{\alpha}(f,B))^{\mu\nu}g_{\nu\sigma} + \frac{1}{4} eBh_a^{\nu}eB, \]

are related to a nonlinear function \( f \) on the variable \( B \), which includes the higher-order derivatives.

Similarly, to before the limit of TEGR is recovered when \( f(T, B) \) is a linear function, that is, \( f(T, B) = f_1T + f_2B \). The special case \( f(T, B) = T + F(B) \) has drawn the attention before because for small values of \( F(B) \) there exists a small derivation from TEGR.

In addition the additional degrees of freedom of the field equations can be attributed to a scalar field. Thus is for \( \phi = f_B \), terms \( T^{(B)}_a \) become

\[ T^{(B)}_a = e(f,B)_a T^{\mu\lambda} - \frac{1}{2} e(h_\mu^{\alpha}(f,B)_{,\sigma} - h_\mu^{\alpha}(f,B))^{\mu\nu}g_{\nu\sigma} + \frac{1}{4} eBh_a^{\nu}eB, \]

with \( V(\phi) = -Bf_B \). At this point it is important to mention that \( f(T, B) \) gravity under a conformal transformation is equivalent to the scalar-torsion theory [105, 106]. That is analogue to the relation of \( f(R) \)-gravity with the scalar field theories. \( f(T, B) \) theory and \( f(R) \)-gravity are of the same order, while the \( f(T) = T + F(B) \) theory has the same degrees of freedom as the \( f(R) \)-theory and canonical quantization in the minisuperspace approach can be applied [66].

### 3 Bianchi I universe

In the Misner variables the LRS Bianchi I spacetime is described by the line element

\[ ds^2 = -N^2(t)dt^2 + e^{2a(t)}(e^{2\beta(t)}dx^2 + e^{-\beta(t)}(dy^2 + dz^2)), \]

where \( \alpha(t) \) is the scale factor for the three-dimensional hypersurface and \( \beta \) is the anisotropic parameter while \( N(t) \) is the lapse function. For \( \beta(t) \to 0 \), the line element (20) reduces to the spatially flat FLRW geometry.

In order to calculate the torsion scalar \( T \), we should consider the vierbein fields [56], thus we assume the basis

\[ e^1 = NdN, \quad e^2 = e^{a^{\rho}}dx, \quad e^3 = e^{a^{t}}P, \quad e^4 = e^{a^{z}}dz, \]

which provides

\[ T = \frac{1}{N^2} \left( 6\dot{\alpha}^2 - \frac{3}{2}\dot{\beta}^2 \right), \]

and the boundary term

\[ B = \frac{6}{N^2} \left( \ddot{\alpha} - \dot{N} \frac{\dot{N}}{N} + 3\ddot{\alpha} \right). \]
while the Lagrangian of General Relativity, i.e. the Ricci scalar, is calculated

\[ R = \frac{1}{N^2} \left( 6\dot{\alpha}^2 - 6\dot{\alpha} \frac{\dot{N}}{N} + 12\dot{\alpha}^2 + \frac{3}{2} \dot{\beta}^2 \right). \]  

(23)

Bianchi I spacetime admits a minisuperspace description, which means that a point-like Lagrangian function reproduces the existing field equations. In order to write the point-like Lagrangian, we shall use the method of Lagrange multiplier. The latter approach is a powerful method widely applied in various modified theories of gravity; see, for instance, [107–109] and references therein.

In the case of vacuum the Action Integral (16) with the use of two Lagrange multipliers \( \lambda_1 \) and \( \lambda_2 \) for the Bianchi I background space can be written in the equivalent form

\[
S_f(T, B) = \frac{1}{16\pi G} \int d^4x N e^{3\alpha} \left( f(T, B) \right) + \frac{1}{16\pi G} \int d^4x N e^{3\alpha} \left( T - \frac{1}{N^2} \left( 6\dot{\alpha}^2 - \frac{3}{2} \dot{\beta}^2 \right) \right) + \frac{1}{16\pi G} \int d^4x N e^{3\alpha} \left( B - \frac{6}{N^2} \left( \ddot{\alpha} - \dot{\alpha} \frac{\dot{N}}{N} + 3\dot{\alpha}^2 \right) \right). 
\]  

(24)

Variation with respect to the variables \( T \) and \( B \), it follows \( \frac{\delta}{\delta T} S_f(T, B) = 0, \frac{\delta}{\delta B} S_f(T, B) = 0 \), that is,

\[ \lambda_1 = -f_T \text{ and } \lambda_2 = -f_B. \]  

(25)

Thus, by replacing in (24) and integrating by parts we end with the point-like Lagrangian

\[
\mathcal{L}(\alpha, \dot{\alpha}, \beta, \dot{\beta}, T, B, \dot{B}) = \frac{1}{N} \left( f(T) e^{3\alpha} \left( 6\dot{\alpha}^2 - \frac{3}{2} \dot{\beta}^2 \right) - 6e^{3\alpha} f_{BB} \dot{\alpha} \dot{B} \right) + Ne^{3\alpha} \left( f - T f_T - B f_B \right). 
\]  

(26)

In the following analysis we focus on the case where \( f(T, B) \) is separable, that is,

\[ f(T, B) = T + K(T) + F(B), \]  

(27)

such that Lagrangian (25) to become

\[
\mathcal{L}(\alpha, \dot{\alpha}, \beta, \dot{\beta}, T, B, \dot{B}) = \frac{1}{N} \left( (1 + K, T) e^{3\alpha} \left( 6\dot{\alpha}^2 - \frac{3}{2} \dot{\beta}^2 \right) - 6e^{3\alpha} F_{BB} \dot{\alpha} \dot{B} \right) + Ne^{3\alpha} \left( K - T K, T + F(B) - B F_B \right). 
\]  

(28)

or equivalently

\[
\mathcal{L}(\alpha, \dot{\alpha}, \beta, \dot{\beta}, T, \phi, \dot{\phi}) = \frac{1}{N} \left( (1 + K, T) e^{3\alpha} \left( 6\dot{\alpha}^2 - \frac{3}{2} \dot{\beta}^2 \right) - 6e^{3\alpha} \dot{\phi} \right) + Ne^{3\alpha} \left( K - T K, T + V(\phi) \right), 
\]  

(29)

with \( \phi = F_B \) and \( V(\phi) = F(B) - BF_B \).

Thus, the gravitational field equations are

\[
0 = \left( 1 + K, T \right) e^{3\alpha} \left( 6\dot{\alpha}^2 - \frac{3}{2} \dot{\beta}^2 \right) - 6e^{3\alpha} \dot{\phi} - e^{3\alpha} \left( K - T K, T + V(\phi) \right), 
\]  

(30)

\[ 0 = (1 + K, T) \left( \ddot{\alpha} + \frac{3}{2} \dot{\alpha}^2 + \frac{3}{8} \dot{\beta}^2 \right) - \frac{1}{4} \left( K - T K, T \right) + K, T T \dot{\alpha} \dot{T} - \frac{1}{2} \left( \ddot{\phi} + \frac{1}{2} V(\phi) \right), \]  

(31)

and the two constraint equations

\[ 0 = V_{,\alpha} + 6(3\dot{\alpha}^2 + \ddot{\alpha}), \]  

(33)

\[ 0 = \left( 6\dot{\alpha}^2 - \frac{3}{2} \dot{\beta}^2 \right) - T, \]  

(34)

where without loss of generality we have selected the lapse function \( N(t) \) to be constant, that is, \( N(t) = 1 \).

We define the Hubble variable \( H = \dot{\alpha} \) and we write Eqs. (30), (31) in the equivalent form

\[
\left( 3H^2 - \frac{3}{4} \dot{\beta}^2 \right) = G_{\text{eff}} \rho_{f(T, B)}, 
\]  

(35)

\[
\left( 2H + 3H^2 - \frac{3}{4} \dot{\beta}^2 \right) = -G_{\text{eff}} P_{f(T, B)}, 
\]  

(36)
in which the geometrodynamical fluid source has the following components

\[ \rho_{f(T,B)} = \left( 6H\dot{\phi} + \frac{1}{2}(K - TK,T + V(\phi)) \right), \]

\[ p_{f(T,B)} = -\frac{1}{2}(K - TK,T) + 2K,TTH\dot{T} + \left( \dot{\phi} + \frac{1}{2}V(\phi) \right). \]

(37)  

(38)

and \( G_{\text{eff}} = (1 + K,T)^{-1} \) is the effective gravitational constant, which varies on time for a nonlinear function \( K(T) \).

However, that is not the unique way to define the geometrodynamical fluid. Indeed, without loss of generality we can write the field equations

\[ \left( 3H^2 - \frac{3}{4}\beta^2 \right) = \tilde{\rho}_{f(T,B)}, \]

\[ \left( 2\dot{H} + 3H^2 + \frac{3}{4}\beta^2 \right) = -\tilde{p}_{f(T,B)}, \]

(39)

where now

\[ \tilde{\rho}_{f(T,B)} = \left( 3H\dot{\phi} - \frac{1}{2}(K - 2TK,T + V(\phi)) \right), \]

\[ \tilde{p}_{f(T,B)} = -\frac{1}{2}(K - TK,T) + 2K,TTH\dot{T} \]

\[ + \left( \dot{\phi} + \frac{1}{2}V(\phi) \right) + K,T(2\dot{H} + 3H^2 + \frac{3}{4}\beta^2). \]

(40)  

(41)

In the special case of \( f(T,B) = T + F(B) \), where \( G_{\text{eff}} = \text{cont} \), the modified field equations read

\[ \left( 3H^2 - \frac{3}{4}\beta^2 \right) = \left( 3H\dot{\phi} + \frac{1}{2}V(\phi) \right), \]

\[ \left( 2\dot{H} + 3H^2 + \frac{3}{4}\beta^2 \right) = -\left( \dot{\phi} + \frac{1}{2}V(\phi) \right), \]

\[ 0 = \ddot{\beta} + 3\dot{\beta}\dot{\alpha}, \]

(42)  

(43)  

(44)

where now the geometrodynamical terms follow only from the boundary term \( F(B) \).

Furthermore, for \( \phi = \text{const} \), \( f(T) \)-theory is recovered and the field equations are

\[ \left( 3H^2 - \frac{3}{4}\beta^2 \right) = \frac{1}{4}(K - TK,T), \]

\[ \left( 2\dot{H} + 3H^2 + \frac{3}{4}\beta^2 \right) = -\frac{1}{4}(K - TK,T) + 2K,TTH\dot{T}, \]

(45)  

(46)

or equivalently

\[ \left( 3H^2 - \frac{3}{4}\beta^2 \right) = \frac{1}{2}(K - 2TK,T), \]

\[ -\left( 2\dot{H} + 3H^2 + \frac{3}{4}\beta^2 \right) = -\frac{1}{2}(K - TK,T) \]

\[ + 2K,TTH\dot{T} + K,T(2\dot{H} + 3H^2 + \frac{3}{4}\beta^2). \]

(47)  

(48)

while for the anisotropic parameter it holds

\[ 0 = \ddot{\beta} + \beta(3\dot{\alpha} + \dot{T}K,TT). \]

(49)

4 Dynamical analysis for \( f(T,B) = T + F(B) \) gravity

We want to extend the analysis presented in [58, 61] for the case of anisotropic spacetimes. That is, we focus on the case where \( f(T,B) = T + F(B) \). This specific theory has many interesting properties. It is a modified teleparallel theory with many degrees of freedom as \( f(R) \)-theory, and it admits a scalar field description. From the previous analysis, it is clear that this specific theory can describe interesting epochs of cosmological history in the case of isotropy.

In order to perform a detailed analysis of the dynamics we define new dimensionless variables

\[ \Sigma = \frac{\dot{\beta}}{2H}, \quad x = \frac{\dot{\phi}}{H}, \quad y = \frac{V(\phi)}{6H^2}, \quad \lambda = -\ln(V(\phi)). \]

(50)  

(51)
Fig. 1 Phase-space portrait for the two-dimensionless dynamical system (55, 56) in the finite regime for different values of the free parameter $\lambda$. The Figs. of the first and second row are for $\lambda < 6$ where $P_2$ is the unique attractor. However, for $\lambda \geq 6$ the dynamical system goes at infinity and $P_2$ is a source point. The solid line corresponds to the family of points $P_1$ where it is clear that the asymptotic solutions at the points $P_1$ are always unstable.

Moreover, we define the new independent variable $d\tau = Hdt$ and the exponential potential $V(\phi) = V_0 e^{-\lambda \phi}$ such that the field equations to be

$$
\frac{d\Sigma}{d\tau} = -\lambda y \Sigma, \quad (51)
$$

$$
\frac{dx}{d\tau} = (2\lambda - 3)y + x(3 - \lambda y) + 3(\Sigma^2 - 1), \quad (52)
$$

$$
\frac{dy}{d\tau} = -y(\lambda(x + 2y) - 6), \quad (53)
$$
with constraint equation

\[ 1 - x - y - \Sigma^2 = 0. \]  
\[ (54) \]

For the exponential potential we find that \( F(B) \) function is \( F(B) = -\frac{1}{2}B\ln B \) \[ [58] \].

With the use of the constraint equation \((54)\) we replace variable \( y \) \( = 1 - x - \Sigma^2 \) in the dynamical system and we end with the set of differential equations

\[ \frac{d\Sigma}{d\tau} = -\lambda (1 - x - \Sigma^2) \Sigma, \]
\[ (55) \]
\[ \frac{dx}{d\tau} = (2\lambda - 3)(1 - x - \Sigma^2) + x (3 - \lambda (1 - x - \Sigma^2)) + 3 (\Sigma^2 - 1). \]
\[ (56) \]

Variables \((\Sigma, x)\) are not constrained, thus, they can take values at all the region of the real values, i.e. \((x, \Sigma) \in \mathbb{R}\). Consequently, we shall study the existence of stationary points for the two-dimensional dynamical system \((55)\) and \((56)\) in the finite and infinity regimes.

Each stationary point of the dynamical system corresponds to a specific era of cosmological evolution. The dependent variables’ values determine the asymptotic solution’s physical properties. For each stationary point, we investigate the stability properties. The latter is a fundamental analysis to construct the cosmological evolution and determine the future attractors.

4.1 Analysis at the finite regime

The stationary points \(P(\Sigma(P), x(P))\) for the dynamical system \((55, 56)\) are

\[ P_1 = (\Sigma, 1 - \Sigma^2), \quad P_2 = \left(0, 2 - \frac{6}{\lambda}\right). \]
\[ (57) \]

Points \(P_1\) describe a family of anisotropic spacetimes where there is not any contribution of the potential term to the cosmological solution, that is, \(y(P_1) = 0\). The eigenvalues of the linearized system around \(P_1\) are \(e_1(P_1) = 0\) and \(e_2(P_1) = \lambda (\Sigma^2 - 1) + 6\). For \(e_2(P_1) > 0\) the stationary point is always a source; however, for \(e_2(P_1) < 0\) the center manifold theorem should be applied because may there exist a stable submanifold. However, the application of the center manifold theorem does not contribute in the physical discussion of the theory, such that we omit it.

Point \(P_2\) corresponds to an asymptotic solution which describes an isotropic spatially flat FLRW Universe where the cosmological fluid has the equation of state parameter \(w_{\text{eff}}(P_2) = -3 + \frac{2}{3}\lambda\). Thus for \(\lambda = 3\) the de Sitter universe is recovered. The eigenvalues of the linearized system are \(e_1(P_2) = \lambda - 6\) and \(e_2(P_2) = \lambda - 6\), thus for \(\lambda < 6\) the stationary point is an attractor.

In Fig. 1, we present the two-dimensional phase-space portrait for the dynamical system \((55, 56)\) in the finite regime. We observe that points \(P_1\) always describe unstable anisotropic spacetimes. Furthermore, we conclude that for \(\lambda < 6\), the cosmological model has the homogeneous and isotropic spatially flat Universe as an attractor.

The results are summarized in Table 1.

4.2 Analysis at the infinity

In order to understand the global dynamics of the dynamic system \((55, 56)\), we investigate the existence of stationary points at the infinity.

We define the new Poincaré variables

\[ \Sigma = \frac{Z}{\sqrt{1 - X^2 - Z^2}}, \quad x = \frac{X}{\sqrt{1 - X^2 - Z^2}}, \quad d\eta = \sqrt{1 - X^2 - Z^2} d\tau. \]
\[ (58) \]

Thus, the dynamical system \((55, 56)\) reads

\[ \frac{1}{Z} \frac{dZ}{d\eta} = X^3 (\lambda - 6) - X (\lambda - 6 - 3(\lambda - 4)Z^2) + \sqrt{1 - X^2 - Z^2 (3(\lambda - 2)X^2 + \lambda (2Z^2 - 1))}, \]
\[ (59) \]
\[ \frac{dX}{d\eta} = X^3 (\lambda - 6) + 3X^2 (\lambda - 4)(Z^2 - 1) - 2(\lambda - 3)(2Z^2 - 1) \]
Fig. 2 Phase-space portrait for the two-dimensionless dynamical system (58, 59) for different values of the free parameter $\lambda$. Figures of the first and second row are for $\lambda < 6$ where $P_2$ is the unique attractor. However, for $\lambda \geq 6$ the dynamical system goes at infinity and $P_2$ is a source point

$$\begin{aligned}
+ \sqrt{1 - X^2 - Z^2 (3(\lambda - 2)(X^2 - 1) + 2\lambda Z^2)} X.
\end{aligned}
$$

At the infinity, it holds $1 - X^2 - Z^2 = 0$, thus the stationary points $Q(Z, X)$ at the infinity are

$$Q_1 = (0, 1) \text{ and } Q_2 = (0, -1).$$

Hence, there are not any anisotropic solutions at the infinity value of $\lambda$.

However, $\lambda = 3$, there exists the stationary point

$$D^{\pm} = \left( \pm \sqrt{1 - Z^2}, Z \right)$$

where the asymptotic solution is anisotropic. That is a very interesting point because $\lambda = 3$ is the case where the future attractor in the finite regime describes the de Sitter Universe.
On the surface $1 - X^2 - Z^2 = 0$, the effective equation of state parameter is expressed as $w_{\text{eff}}(X) = -1 + \frac{2}{3} \lambda X$. Thus, the asymptotic solutions at the infinity are always scaling solutions, which describe acceleration when $\lambda < 1$ for point $Q_1$ or $\lambda > -1$ for point $Q_2$.

The eigenvalues of the linearized system at the stationary points are calculated $e_1(Q_1) = 0$, $e_2(Q_1) = -2\lambda$ and $e_1(Q_2) = 0$, $e_2(Q_2) = 2\lambda$.

From the phase-space portraits presented in Fig. 2, we observe that for $\lambda < 0$ the stationary point $Q_2$ is a saddle point, and $Q_1$ is a source, while for $\lambda > 0$, $Q_1$ is a saddle point and $Q_2$ is a source. For $\lambda < 6$, the unique attractor is point $P_1$, while the trajectories have origin the infinity and the surface of stationary points $P_1$. For $\lambda > 6$, we observe that the trajectories start from the infinity, $Q_2$, and the source $P_2$, and they reach the surface of points $P_1$ and the saddle point $Q_1$. Then through these lines, the trajectories end at the infinity.

The latter means that for $\lambda > 6$, the Universe moves from isotropy to anisotropy and vice versa. While for $\lambda < 6$, the final stage of the Universe is the isotropy. Hence, in order $f(T, B) = T + F(B)$ to solve the isotropic problem $\lambda < 6$.

Finally, for $\lambda = 3$, from Fig. 2 we remark that the family of points $D^{\pm}$ are saddle points, and in this case, the future attractor is the de Sitter Universe.

5 Integrability and analytic behaviour

We proceed with our analysis with the investigation of the integrability properties for the two dynamical systems (55), (56). This kind of analysis is essential because we can relate the numerical behaviour of the trajectories presented in Figs. 1 and 2 to actual solutions of the dynamical system.

An equivalent way to write the dynamical system is to the second-order ordinary differential equation

$$
\Sigma \left( \frac{d^2 \Sigma}{d\tau^2} + (\lambda - 6) \frac{d\Sigma}{d\tau} \right) - 2 \left( \frac{d\Sigma}{d\tau} \right)^2 - 8 \Sigma^3 \left( \frac{d\Sigma}{d\tau} \right) = 0.
$$

We perform the change of variable $\Sigma = Y^{-1}$, thus the latter equation becomes

$$
\frac{d^2 Y}{d\tau^2} + \left( \lambda - 6 \right) \frac{Y}{Y^2} \frac{dY}{d\tau} = 0,
$$

that is,

$$
\frac{dY}{d\tau} + \left( \lambda - 6 \right) Y + \frac{\lambda}{Y} = 0,
$$

where the latter can be solved by quadrature.

For large values of $Y$, the term $(\lambda - 6)Y$ dominates in Eq. (65) such that the analytic solution to be approximated as

$$
Y(\tau) \simeq Y_1 e^{-(\lambda - 6)\tau},
$$

which means that for $\lambda < 6$ when $\tau > 0$, $Y(\tau)$ increases and $\Sigma(\tau)$ becomes zero. That describes the behaviour near the isotropic universe. On the other hand for $\lambda > 6$, the solution is going far from the isotropic solution.

Furthermore, for small values of $Y(\tau)$, it follows

$$
Y(\tau) \simeq Y_1 + Y_2 \tau
$$

which holds only for small values of $\tau$. The latter solution describes the evolution of the trajectories near to the infinity.

5.1 Painlevé analysis

The Singularity analysis, which Sophie Kowalewski introduced [110] and established by the French school of Painlevé [111–113], is a powerful method in order to make inferences about the integrability properties of a given dynamical system and write the analytic solution by using analytic functions. Indeed, a differential equation which possesses the Painlevé property, its solution is written in terms of Laurent expansions.

At this point, it is essential to clarify that in this section, the term singularity, we mean movable singularities of the differential equations and the term singularity should not be confused with the cosmological singularity of the physical space.

The modern treatment of the singularity analysis is summarized in a simple algorithm proposed by Ablowitz, Ramani and Segur (ARS algorithm) [114–116]. A pedagogical discussion on the ARS algorithm can be found in the review [117].

Applications of the singularity analysis in cosmology cover various subjects, from the study of anisotropic models [118–120] to the investigation of the integrability properties of dark energy theories [121, 122].

Hence, the application of the ARS algorithm in Eq. (63) gives that the leading-order term is

$$
\Sigma(\tau) = \Sigma_0 \tau^{-\frac{1}{2}}, \quad 2\lambda(\Sigma_0)^2 = 1
$$
with resonances

\[ r_1 = -1 \text{ and } r_2 = \frac{1}{2}, \]

(69)

The analytic solution of Eq. (63) is presented in terms of Right Puiseux Series, that is,

\[ \Sigma(\tau) = \Sigma_0 \tau^{-\frac{1}{2}} + \Sigma_1 + \Sigma_2 \tau^{\frac{1}{2}} + \Sigma_3 \tau + \ldots \]

(70)

in which \( \Sigma_1 \) is arbitrary, the second integration constant, and

\[ \Sigma_2 = -\frac{1 + 12\Sigma_0^2 + 3\Sigma_1^2}{4\Sigma_0}, \quad \Sigma_3 = -\frac{\Sigma_1(1 + 12\Sigma_0^2 - 2\Sigma_1^3)}{5\Sigma_0^2}, \text{ etc.} \]

Solution (70) for small values of \( \tau \) indicates that \( \Sigma(\tau) \to \infty \), which means that the leading-order behaviour describes an anisotropic universe at the infinite regime of the original dynamical system.

Furthermore, we observe that the analytic solution is expressed by a Right Painlevé expansion, from the latter, we can infer that the leading-order behaviour describes an unstable solution [120]. That is in agreement with the analysis presented before.

6 Conclusions

The dynamical system analysis for the field equations for the anisotropic Bianchi I spacetime in modified teleparallel \( f(T,B) \)-theory was performed. We determined the stationary points for the field equations in dimensionless variables, in the so-called \( H \)-normalization, in the finite and infinity regimes. Moreover, we used a Lagrange multiplier and introduced a scalar field to attribute the higher-order degrees of freedom provided by the theory. Thus a scalar field potential \( V(\phi) \) has been introduced. For our analysis we considered the exponential potential function \( V(\phi) = V_0 e^{-\lambda \phi} \).

In the finite regime, we derived two families of stationary points, \( P_1 \) and \( P_2 \). Points \( P_1 \) describe a family of anisotropic Bianchi I spacetimes. The spacetimes do not describe acceleration, while the asymptotic solutions are always unstable. The points are sources or saddle points. On the other hand, point \( P_2 \) corresponds to a spatially flat FLRW geometry which describes acceleration for \( \lambda < 4 \). \( P_2 \) is an attractor for values of \( \lambda < 6 \). We remark that for \( \lambda = 3 \), the de Sitter Universe is recovered.

We introduced a new set of variables by defining a Poincare map and investigating the evolution of the trajectories at the infinity. It was found that for an arbitrary value of \( \lambda \), two stationary points at the infinity describe isotropic spatially flat FLRW geometries exist. However, for the particular case in which \( \lambda = 3 \), anisotropic asymptotic solutions exist at the infinity. However, it was found that the stationary points at the infinity always describe unstable solutions, and the points are sources or saddle points.

According to the above analysis and with the numerical results presented in Fig. 2, for the evolution of the trajectories for the field equations, we remark that for \( \lambda < 6 \), the trajectories have the origin at the infinity or on the surface of anisotropic solutions described by \( P_1 \), and the final attractor is point \( P_2 \). Hence, in order to solve the isotropic problem in \( f(T,B) \)-theory with initial conditions that of Bianchi I geometry, the parameter \( \lambda \) should be constrained as \( \lambda < 6 \).

Furthermore, we investigated the integrability properties of the field equations by using the singularity analysis. Such analysis is essential because we know that the numerical trajectories correspond to actual solutions to the problem for an integrable dynamical system. In the context of the \( H \)-normalization, the field equations can be written as one second-order ordinary differential equation.

We applied the ARS algorithm, and we found that the second-order ordinary differential equation admits the Painlevé property with leading-order term \( \tau^{-\frac{1}{2}} \) and resonances \( r_1 = -1 \) and \( r_2 = \frac{1}{2} \). Hence, the analytic solution is expressed by a Right Puiseux Series. Indeed, the leading-order term describes an anisotropic solution on the surface of points \( P_1 \). Because the solution is written in the form of a Right Puiseux Series, we can conclude that the singular solution \( \tau^{-\frac{1}{2}} \) is unstable, as was found by the analysis of the stationary points.

Data Availability Statements Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

References

1. B.P. Abbot et al., Phys. Rev. Lett. 123, 011102 (2019)
2. E. Berti et al., Class. Quantum Gravity 32, 243001 (2015)
3. D. Ayzenberg, C. Bambi, Tests of general relativity using black hole X-ray data, in Handbook of X-Ray and Gamma-Ray Astrophysics. ed. by C. Bambi, A. Santangelo (Springer, Singapore, expected in 2022) [arXiv:2111.13918]
4. M. Tegmark et al., Astrophys. J. 606, 702 (2004)
5. M. Kowalski et al., Astrophys. J. 686, 749 (2008)
6. E. Komatsu et al., Astrophys. J. Suppl. Ser. 180, 330 (2009)
7. N. Suzuki et al., Astrophys. J. 746, 85 (2012)
8. Planck Collaboration: P.A.R. Ade et al., A&A 594, A13 (2016)
9. Planck Collaboration: Y. Akrami et al., A&A 641, A10 (2020)
10. E. Di Valentino, O. Mena, S. Pan, L. Visinelli, W. Yang, A. Melchiorri, D.F. Mota, A.G. Riess, J. Silk, Class. Quantum Gravity 38, 153001 (2021)
11. A. Guth, Phys. Rev. D 23, 347 (1981)
12. G.W. Gibbons, S.W. Hawking, Phys. Rev. D 15, 2738 (1977)
13. S.W. Hawking, J.G. Moss, Phys. Lett. B 110, 35 (1982)
14. K. Sato, MNRAS 195, 467 (1981)
15. J.D. Barrow, A. Ottewill, J. Phys. A 16, 2757 (1983)
16. R. Wald, Phys. Rev. D 28, 2118 (1983)
17. J.D. Barrow, J. Stein-Schabes, Phys. Lett. A 103 (6–7), 315 (1984)
18. K. Bolejko, Phys. Rev. D 97, 083515 (2018)
19. V. Sahni, Class. Quantum Gravity 19, 3435 (2002)
20. L. Perivolaropoulos, Six Puzzles for LCDM Cosmology, [arXiv:0811.4684]
21. T. Clifton, P.G. Ferreira, A. Padilla, C. Skordis, Phys. Rep. 513, 1 (2012)
22. S. Nojiri, S.D. Odintsov, IJGMP 4, 115 (2007)
23. S. Nojiri, S.D. Odintsov, V.K. Oikonomou, Phys. Rep. 692, 1 (2017)
24. A. Starobinsky, Phys. Lett. B 91, 99 (1980)
25. H.A. Buchdahl, Mon. Not. R. Astron. Soc. 150, 1 (1970)
26. TP. Sotiriou, V. Faraoni, Rev. Mod. Phys. 82, 451 (2010)
27. S. Nojiri, S.D. Odintsov, Phys. Rep. 505, 59 (2011)
28. A.A. Starobinsky, Phys. Lett. B 91, 99 (1980)
29. G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov, L. Sebastiani, S. Zerbini, Phys. Rev. D. 77, 046009 (2007)
30. D. Glavan, C. Lin, Phys. Rev. Lett. 124, 081301 (2020)
31. M.V. de S. Silva, M.E. Rodrigues, Eur. Phys. J. C 78, 638 (2018)
32. Y. Zhang, D.S.C. Gomez, Symmetry 10, 170 (2018)
33. T. Harko, F.S.N. Lobo, S. Nojiri, S.D. Odintsov, Phys. Rev. D 84, 024020 (2011)
34. J. Wu, G. Li, T. Harko, S.-D. Liang, EPJC 78, 430 (2018)
35. S. Nojiri, S.D. Odintsov, V.K. Oikonomou, Phys. Rep. 692, 1 (2017)
36. S.D. Odintsov, V.K. Oikonomou, F.P. Frinomis, Nucl. Phys. B 958, 115135 (2020)
37. S.D. Odintsov, V.K. Oikonomou, Phys. Lett. B 805, 135437 (2020)
38. F.K. Anagnostopoulos, S. Basilakos, E.N. Saridakis, Phys. Rev. D 100, 083517 (2019)
39. A. Heitner, S. Potraz, Philipp. Wiss. 217, 224 (1928)
40. A. Unzicker, T. Case, Translation of Einstein’s attempt of a unified field theory with teleparallelism. arXiv:physics/0503046
41. K. Hayashi, T. Shirafuji, Phys. Rev. D 19, 3524 (1979)
42. K. Hayashi, T. Shirafuji, Addendum-ibid. Phys. Rev. D 24, 3312 (1982)
43. J.W. Maluf, J. Math. Phys. 35, 335 (1994)
44. H.I. Arcos, J.G. Pereira, Int. J. Mod. Phys. D 13, 2193 (2004)
45. R. Weitzenböck, Invarianten Theorie (Nordhoff, Groningen, 1923)
46. S. Nojiri, S.D. Odintsov, V.K. Oikonomou, Phys. Lett. B 80, 217 (1921)
47. B. Li, T.P. Sotiriou, J.D. Barrow, Phys. Rev. D 83, 064035 (2011)
48. N. Tamanini, C.G. Bohmer, Phys. Rev. D 86, 044009 (2012)
49. S.H. Chen, J.B. Dent, S. Dutta, E.N. Saridakis, Phys. Rev. D 83, 023508 (2011)
50. J.B. Dent, S. Dutta, E.N. Saridakis, JCAP 01, 009 (2011)
51. K. Bamba, C.-Q. Geng, C.-C. Lee, L.-W. Luo, JCAP 11, 021 (2011)
52. W. El Hanafy, E.N. Saridakis, JCAP 09, 019 (2011)
53. C. Xu, E.N. Saridakis, G. Leon, JCAP 07, 005 (2012)
54. A. Paliathannas, J.D. Barrow, P.G.L. Leach, Phys. Rev. D 94, 023525 (2016)
55. S. Bahamonte, C.G. Boehmer, M. Wight, Phys. Rev. D 92, 104042 (2015)
56. S. Bahamonte, K.F. Dialektopoulos, C. Escamilla-Rivera, V. Gakis, M. Hendry, J.L. Said, J. Mifsud, E. Di Valentino, Teleparallel Gravity: From Theory to Cosmology, arXiv:2106.13793 (2021)
57. R. Myrzakulov, EPJC 72, 1 (2012)
58. A. Paliathanass, JCAP 1708, 027 (2017)
59. L. Karpathopoulos, S. Basilakos, G. Leon, A. Paliathanass, M. Tsamparlis, Gen. Relativ. Gravit. 50, 79 (2018)
60. M. Caruana, G. Farrugia, J.L. Said, EPJC 80, 640 (2020)
61. A. Paliathanass, Phys. Rev. D 95, 064062 (2017)
62. A. Paliathanass, G. Leon, Eur. Phys. J. Plus 136, 1092 (2021)
63. G.A. Rave-Franco, C. Escamilla-Rivera, J.L. Said, EPJC 80, 677 (2020)
64. G.A. Rave-Franco, C. Escamilla-Rivera, J.L. Said, Phys. Rev. D 103, 084017 (2021)
65. A. Paliathanass, G. Leon, f(T,B) Gravity in a Friedmann-Lemaitre-Robertson-Walker Universe with Nonzero Spatial Curvature [arXiv:2201.12189] (2022)
66. A. Paliathanass, Universe 7, 150 (2021)
67. S. Nàjera, A. Aguilar, G.A. Rave-Franco, C. Escamilla-Rivera, R.A. Sussman, Inhomogeneous Solutions in f(T,B) Gravity [arXiv:2201.06177] (2022)
68. A. Paliathanass, Kasner Universes in f(T, B) Gravity (2022)
69. E. Kasner, Am. J. Math. 43, 217 (1921)
70. K. Adhav, A. Nimbkar, R. Hole, Int. J. Theor. Phys. 46, 2396 (2007)
71. S.M.M. Rasouli, M. Farhoudi, H.R. Sepangi, Class. Quantum Gravity 28, 155004 (2011)
72. X.O. Camanho, N. Dadhich, A. Molina, Class. Quantum Gravity 32, 175016 (2015)
73. P. Halpern, Phys. Rev. D 63, 024009 (2001)
74. M.V. Battisti, G. Montani, Phys. Lett. B 681, 179 (2009)
75. Y.B. Zeldovich, A.A. Starobinsky, Sov. Phys. JETP 34, 1159 (1972)
76. J.D. Barrow, M.S. Turner, Nature 291, 469 (1981)
77. S.W. Hawking, R.J. Taylor, Nature 209, 1278 (1966)
78. C.W. Misner, Astrophys. J. 151, 431 (1968)
79. C.B. Collins, S.W. Hawking, Astrophys. J. 180, 317 (1973)
80. J.D. Barrow, Mon. Not. R. Astron. Soc. 175, 359 (1976)
81. J. Wainwright, G.F.R. Ellis (eds.), Dynamical Systems in Cosmology (Cambridge University Press, Cambridge, 1997)
82. P. Szekeres, Commun. Math. Phys. 41, 55 (1975)
83. E.J. Copeland, A.R. Liddle, D. Wands, Phys. Rev. D 57, 4686 (1998)
84. A.A. Coley, Dynamical Systems and Cosmology (Springer, Dordrecht, 2003)
85. L. Amendola, S. Tsujikawa, Dark Energy (Cambridge University Press, Cambridge, 2010)
86. G. Leon, A. Paliathanasis, EPJC 78, 753 (2018)
87. A. Cid, F. Izaurieta, G. Leon, P. Medina, D. Narbona, JCAP 1804, 041 (2018)
88. C.R. Fadragas, G. Leon, Class. Quantum Gravity 31, 195011 (2014)
89. M. Abdelwahab, R. Goswani, P.K.S. Dunsby, Phys. Rev. D 85, 083511 (2012)
90. P. Christodoulidou, D. Roest, E.I. Stakianakis, JCAP 1111, 002 (2019)
91. A. Paliathanasis, G. Leon, Class. Quantum Gravity 38, 075013 (2021)
92. A.A. Coley, Phys. Rev. D 62, 023517 (2000)
93. A.A. Coley, R.J. van den Hoogen, Phys. Rev. D 62, 023517 (2000)
94. A. Coley, G. Leon, Gen. Relativ. Gravit. 51, 115 (2019)
95. G. Leon, A. Coley, A. Paliathanasis, Ann. Phys. 412, 168002 (2020)
96. A.A. Coley, G. Leon, P. Sandin, J. Latta, JCAP 12, 010 (2015)
97. B. Alhulaimi, R.J. van den Hoogen, A.A. Coley, JCAP 17, 045 (2017)
98. R.J. van den Hoogen, A.A. Coley, B. Alhulaimi, S. Mohandas, E. Knighton, S. O’Neil, JCAP 18, 017 (2018)
99. G. Leon, Class. Quantum Gravity 26, 035008 (2009)
100. G. Leon, A. Paliathanasis, Anisotropic Spacetimes in \( f(T,B) \) Theory II: Kantowski-Sachs Universe
101. G. Leon, A. Paliathanasis, Anisotropic Spacetimes in \( f(T,B) \) Theory III: LRS Bianchi III Universe
102. A. Paliathanasis, Anisotropic Spacetimes in \( f(T,B) \) Theory I: Noether Symmetry Analysis
103. M. Tsamparlis, A. Paliathanasis, Symmetry 10, 233 (2018)
104. R. Ferraro, F. Fiorini, Phys. Rev. D 84, 083518 (2011)
105. C.-Q. Geng, C.-C. Lee, E.N. Saridakis, Y.-P. Wu, Phys. Lett. B 704, 384 (2011)
106. M. Wright, Phys. Rev. D 93, 103002 (2016)
107. S. Capozziello, J. Miritzis, S. Nojiri, S.D. Odintsov, Phys. Lett. B 693, 198 (2010)
108. A.N. Makarenko, JGMMP 13, 1630006 (2016)
109. D. Sáez-Gómez, Phys. Rev. D 85, 023009 (2012)
110. S. Kowalevski, Acta Math. 12, 177 (1889)
111. P. Painlevé, Leçons sur la théorie analytique des équations différentielles (Leçons de Stockholm, 1895) (Hermann, Paris, Reprinted, Oeuvres de Paul Painlevé, vol. I (Éditions du CNRS, Paris, 1897), p. 1973
112. P. Painlevé, Bull. Math. Soc. Fr. 28, 201 (1900)
113. P. Painlevé, Acta Math. 25, 1 (1902)
114. M.J. Ablowitz, A. Ramani, H. Segur, Lett. Nuovo Cim. 23, 333 (1978)
115. M.J. Ablowitz, A. Ramani, H. Segur, J. Math. Phys. 21, 715 (1980)
116. M.J. Ablowitz, A. Ramani, H. Segur, J. Math. Phys. 21, 1006 (1980)
117. A. Ramani, B. Grammaticos, T. Bountis, Phys. Rep. 180, 159 (1989)
118. S. Cotatsakis, P.G.L. Leach, J. Phys. A Math. Gen. 27, 1625 (1994)
119. J. Demaret, C. Scheen, J. Math. Phys. A Math. Gen. 29, 59 (1996)
120. A. Paliathanasis, P.G.L. Leach, Phys. Lett. A 381, 1277 (2017)
121. J. Miritzis, P.G.L. Leach, S. Cotatsakis, Grav. Cosmol. 6, 282 (2000)
122. A. Paliathanasis, J.D. Barrow, P.G.L. Leach, Phys. Rev. D 94, 023525 (2016)

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