A CLOSED ANALYTICAL FORMULA
FOR TWO-LOOP MASSIVE TADPOLES WITH ARBITRARY TENSOR NUMERATORS

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ABSTRACT

Using the integration by parts method we derive a closed analytical expression for the result of the integration of an arbitrary dimensionally regulated tadpole diagram composed of a massless propagator and two massive ones, each raised into an arbitrary power, and including an arbitrary tensor numerator. We also briefly discuss the implementation of the formula in the algebraic manipulation language of FORM.

1. Introduction

Feynman integrals (FI) with complicated tensor numerators are usually difficult to work with. Even in the cases when they are known to be analytically calculable in principle, in practical terms their evaluation often implies a tedious, time-consuming and error-prone labour of reducing the problem to the calculation of a host of properly constructed scalar integrals. Moreover, the number of the latter integrals grows very fast as the numerator's structure gets more complicated.

There exist only a few examples when the task is completely solved. A good example is provided with the so-called $p$-integrals, that is completely massless Feynman integrals depending on only one external momentum. Here the explicit result is known for one-loop integrals\(^1\), while, say, a given three-loop tensor $p$-integral can be done by employing a rather cumbersome and time-costly method of harmonic projections\(^2\).

In this talk we discuss another useful class of Feynman integrals — integrals without external momenta at all but comprising massive lines\(^2\) as well as massless ones. They will be referred to as $m$-integrals. Such integrals naturally appear in many problems where the mass $m$ may be treated as a ’heavy’ one, much larger than all other mass scales involved.

In the one-loop case $m$-integrals are rather trivial and we shall concentrate on two-loop $m$-integrals pictured in Fig. 1. 2-loop $m$-integrals with only one massive line (Fig. 1a) may be reduced to 1-loop case after firstly integrating the 1-loop $p$-subintegral. 2-loop $m$-integrals with more than 1 massive lines (Fig. 1b,c) are not

\(^1\)Published in in: New Computing Techniques in Physics Research III, eds. K.-H. Becks and D. Perret-Gallix (World Scientific, Singapore, 1994), p. 559.

\(^2\)It is understood that all the massive propagators depend on one and the same mass $m$. 
so easy to do. We show how the use of the integration by parts method leads to a simple and general result for arbitrary (not necessarily scalar) two-loop $m$-integral with two massive and one massless line (see Fig. 1b). In principle, our method allows also to reduce a tensor integral with 3 massive lines of Fig. 1c to a similar scalar integral. The latter can probably be done (at least in some particular cases) again through integration by parts\(^3\). But to the best of our knowledge no explicit integration formula for this integral exists if the powers of all three propagators are arbitrary.

2. Setting the problem

We begin with a bit more complicated FI of the same topology as shown in Fig. 1b but with a non-zero external momentum $q$. The corresponding analytical expression reads (in Euclidean space)

$$\frac{1}{(\pi^2)^{2-\epsilon}} \int \frac{d^D\ell_1 d^D\ell_2 P(p)}{(p_1^2 + m^2)^\alpha (p_2^2 + m^2)^\beta (p_3^2 + m^2)^\gamma}$$

where $\ell_1$ and $\ell_2$ are the loop momenta, $p = \{p_1, p_2, p_3\}$ are the propagator momenta and $P()$ is a tensor nominator. We shall deal with three possibilities of expressing the momenta $p$ in terms of the loop momenta and the external momentum, viz.,

$$p_1 = \ell_1 + q, \quad p_2 = \ell_2, \quad p_3 = -(\ell_1 + \ell_2),$$

$$p_1 = \ell_1, \quad p_2 = \ell_2 + q, \quad p_3 = -(\ell_1 + \ell_2),$$

$$p_1 = \ell_1, \quad p_2 = \ell_2, \quad p_3 = q - (\ell_1 + \ell_2).$$

In a particular case of $P() \equiv 1$, $q = 0$ the result for Eq. (1) is known\(^4\)

(Eq. (1) with $P(p) \equiv 1$, and $q = 0) = (m^2)^{D-\alpha-\beta-\gamma} \frac{\Gamma(D/2 - \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D/2)} M(\alpha, \beta, \gamma)$

with

$$M(\alpha, \beta, \gamma) = \frac{\Gamma(\alpha + \gamma - D/2)\Gamma(\beta + \gamma - D/2)\Gamma(\alpha + \beta + \gamma - D)}{\Gamma(\alpha + \beta + 2\gamma - D)}.$$  

Our aim is to generalize this result on the case of arbitrary tensor polynomial
\(\mathcal{P}(p)\), still keeping \(q = 0\). To simplify the formulas we shall set \(m = 1\) below.

2. Solution through the integration by parts method

As is well known within dimensional regularization the value of the FI Eq. (1) does not depend on specifying the propagator momenta and all three choices (2-4) are completely equivalent. The fact can be conveniently expressed through some differential identities. The essence of the integration by part method for dimensionally regulated Feynman integrals consists of the use of such identities in order to simplify integrals to be computed.\(^1\)

Let us try to apply the method in our case. It is convenient to write eq. Eq. (3) in a condensed form as follows

\[
eq \mathcal{P}(p)I(\alpha, \beta, \gamma)
\]

and to introduce four differential operators acting on Eq. (6)

\[
\hat{p}_i^\mu = \frac{1}{2} \frac{\partial}{\partial p_i^\mu}, \quad i = 1, 2, 3 \quad \text{and} \quad \hat{q}^\mu = \frac{1}{2} \frac{\partial}{\partial q^\mu}.
\]

Explicitly, one has

\[
\int I(\alpha, \beta, \gamma) = \frac{\Gamma(D/2 - \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D/2)} M(\alpha, \beta, \gamma),
\]

\[
\hat{p}_i^\mu (\mathcal{P}(p)I(\alpha, \beta, \gamma)) = \left( \frac{1}{2} \frac{\partial}{\partial p_i^\mu} \mathcal{P}(p) \right) I(\alpha, \beta, \gamma) - \alpha_i \mathcal{P}(p) I(\alpha + \delta_1, \beta + \delta_2, \gamma + \delta_3) p_i^\mu,
\]

\[
\hat{q}^\mu (\mathcal{P}(p)I(\alpha, \beta, \gamma)) = \left( \frac{1}{2} \frac{\partial}{\partial q^\mu} \mathcal{P}(p) \right) I(\alpha, \beta, \gamma) - \gamma \mathcal{P}(p) I(\alpha, \beta, \gamma + 1) q^\mu,
\]

with \(\alpha_1 = \alpha, \alpha_2 = \beta, \alpha_3 = \gamma\). The equivalence of the three momentum patterns (2-4) may now be expressed as a chain of identities

\[
\int \hat{p}_i^\mu \mathcal{P}(p)I(\alpha, \beta, \gamma) = \int \hat{p}_2^\mu \mathcal{P}(p)I(\alpha, \beta, \gamma) = \int \hat{p}_3^\mu \mathcal{P}(p)I(\alpha, \beta, \gamma) = \int \hat{q}^\mu \mathcal{P}(p)I(\alpha, \beta, \gamma).
\]

These equalities allow us to evaluate immediately an integral of the form

\[
\int \hat{p}_{2n}(\hat{p})I(\alpha, \beta, \gamma)|_{q=0} = \frac{\Gamma(D/2 - \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D/2)} \left( \frac{\nabla^2}{4} \right)^n \hat{p}(\hat{q})_{2n} \left( \frac{(-)^n(\gamma)n}{n!(2 - \epsilon)n} \right) M(\alpha, \beta, \gamma + n),
\]

with the Pochhammer symbol \((a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}\), \(D = 4 - 2\epsilon\) and \(\hat{p}_{2n}(\hat{p})\) being an arbitrary tensor in \(\hat{p} = \{\hat{p}_1, \hat{p}_2, \hat{p}_3\}\) of rank \(2n\). Indeed, in view of Eq. (8) we may freely replace \(\hat{p}_{2n}(\hat{p})\) by \(\hat{p}_{2n}(\hat{q})\) for any \(\hat{p}_{2n}(\hat{q}) = \hat{p}_{2n}(\hat{q}_1 = \hat{q}, \hat{p}_2 = \hat{q}, \hat{p}_3 = \hat{q})\). Now the result Eq. (9) comes from three simple observations:

(i) The integral \(\int I(\alpha, \beta, \gamma)\) is a scalar function of \(q\) and thus only the scalar component of the polynomial \(\hat{p}_{2n}(\hat{q})\) (that is proportional to \(\hat{q}^{2n}\)) will survive after setting \(\hat{q} = 0\) in the very end of the calculation.

(ii) \(\left( \frac{\nabla^2}{4} \right)^n (\hat{q})^n = n!(2 - \epsilon)_n\).
(iii) \((\frac{\partial^4}{\partial^4})^n \frac{1}{q^4} = (\gamma)n(\gamma - 1 + \epsilon)n \frac{1}{(q^2)^{2+n}} = (-)^n(\gamma)n(2-\epsilon-\gamma-n)n \frac{1}{(q^2)^{2+n}}.\)

Thus we are left with the task of finding a representation of the initial integral Eq. (1) as a linear combination of integrals of the form displayed in Eq. (3). The problem is solved by the use of the following identity
\[
\mathcal{P}_n(q)f(q^2) = \sum_{\sigma=0}^{[n/2]} \frac{(-)^\sigma}{4^\sigma \sigma!} \{ \mathcal{P}_n(q) \} f(-n+\sigma)(q^2)
\]
where \(f(x)\) is an arbitrary smooth function of \(x\) and \(f^{(n)}(x)\) is defined in such a way that
\[
\frac{d}{dx} f^{(n)}(x) \equiv f^{(n+1)}(x).
\]

It should be clear now that once the initial integral Eq. (1) has been expressed as a linear combination of integrals of the form Eq. (9) it may be done without any problem. Indeed, without essential loss of generality we may assume that the polynomial \(\mathcal{P}(p) = \mathcal{P}_{n_1,n_2}(p)\) does not depend on \(p_3\) and meets the following homogeneity equation:
\[
\mathcal{P}_{n_1,n_2}(\lambda_1 p_1, \lambda_2 p_2) \equiv \lambda_1^{n_1} \lambda_2^{n_2} \mathcal{P}_{n_1,n_2}(p_1, p_2).
\]

Now a direct application of Eq. (8), Eq. (9) and Eq. (10) gives
\[
\int \mathcal{P}_{n_1,n_2}(p_1, p_2) I(\alpha, \beta, \gamma) = \frac{\Gamma(D/2-\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D/2)} \sum_{\sigma_1=0}^{\lfloor (n_1+n_2)/2 \rfloor} \sum_{\sigma_2=0}^{\lfloor n_2/2 \rfloor} \sum_{\sigma_3=0}^{\lfloor n_1/2 \rfloor} \frac{(\gamma)_{\sigma_3}}{\sigma_3!(2-\epsilon)_{\sigma_3} M(\alpha+\sigma_1-n_1, \beta+\sigma_2-n_2, \gamma+\sigma_3)}
\]
\[
\frac{(-)^{(n_1+n_2+\sigma_3)}}{4^{\sigma_1+\sigma_2+\sigma_3} \sigma_1! \sigma_2!} \{ \mathcal{P}_{p_3} \{ \mathcal{P}_{p_2} \mathcal{P}_{p_1} \mathcal{P}_{n_1,n_2}(p_1, p_2) \} \}|_{p_1=p_2=p_3} \}
\]
which is the formula we wanted.

The algebraic structure of Eq. (12) is very similar to that of the corresponding formula for 1-loop \(p\)-integrals in \(^1\). This observation has allowed us to perform a simple algebraic programming of Eq. (12) in FORM\(^5\) by closely following the routine ONE.PRC from the package MINCER\(^2\).

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6. References

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