The solid angle and the Burgers formula in the theory of gradient elasticity: line integral representation

Markus Lazar \textsuperscript{a,*} and Giacomo Po \textsuperscript{b}

\textsuperscript{a} Heisenberg Research Group, Department of Physics, Darmstadt University of Technology, Hochschulstr. 6, D-64289 Darmstadt, Germany

\textsuperscript{b} Mechanical and Aerospace Engineering, University of California, Los Angeles, Los Angeles, CA, 90095, USA

December 11, 2013

Abstract

A representation of the solid angle and the Burgers formula as line integral is derived in the framework of the theory of gradient elasticity of Helmholtz type. The gradient version of the Eshelby-deWit representation of the Burgers formula of a closed dislocation loop is given. Such a form is suitable for the numerical implementation in 3D dislocation dynamics (DD).

Keywords: dislocation loops; gradient elasticity; Burgers formula; solid angle; Dirac monopole.

1 Introduction

The Burgers formula and the solid angle play an important role in the dislocation theory (e.g. \cite{1,2,4,5}) and in the simulation of dislocation dynamics (e.g. \cite{6,7,8}). The original formulas are given in the form of surface integrals. The transformation of the

\textsuperscript{*}E-mail address: lazar@fkp.tu-darmstadt.de (M. Lazar).
surface integrals into line integrals was proposed by deWit [9] and Eshelby [10] adopting Dirac’s theory of magnetic monopoles [11, 12, 13]. In particular, it turned out that the representation as line integral is more appropriate for numerical implementation of these equations into the dislocation dynamics. The classical expressions for the Burgers formula and for the solid angle are singular at the line of the dislocation loop. Moreover, the Burgers formula is discontinuous on the slip surface.

Non-singular expressions for the Burgers formula and the solid angle have been recently found by Lazar [14, 15] using the theory of gradient elasticity of Helmholtz type. The theory of gradient elasticity of Helmholtz type is a special version of Mindlin’s gradient elasticity theory [16] (see also [17, 15]) with only one characteristic length parameter. Lazar and Maugin [18] have shown that, for straight dislocations, the gradient parameter leads to a smoothing of the displacement profile, in contrast to the jump occurring in the classical solution. Lazar [14, 15] has given the generalized solid angle and the corresponding part of the Burgers formula in the form of surface integrals. In this letter, we recast the Burgers formula and the solid angle of gradient elasticity in compact form as line integrals over the closed dislocation loop. The results have a direct application to the numerical implementation and the computer simulation of non-singular dislocations within the so-called (discrete) dislocation dynamics. In Section 2, we discuss and point out the basics of the line integral form of the solid angle and of the associated vector potential in the framework of classical elasticity and their relation to Dirac’s solution of a magnetic monopole. In Section 3, we derive the corresponding expressions in the framework of gradient elasticity.

2 Classical elasticity

In the theory of classical elasticity, the solid angle is given as a surface integral (see, e.g., [1])

\[ \Omega^0(r) = \int_S v_0^0(R) dS'_i = \int_V v_0^0(R) \delta_i(S') dV' = v_0^0(r) \ast \delta_i(S), \]  

where the vector field \( v_0^0 \) is

\[ v_0^0 = \frac{1}{2} \partial_i R = -\partial_i \left( \frac{1}{R} \right) = \frac{R_i}{R^3}, \]  

while the Dirac \( \delta \)-function on a surface \( S \) [19, 20] is defined as

\[ \delta_i(S) \equiv \int_S \delta(R) dS'_i. \]  

The relative radius vector \( R = r - r' \) connects a source point \( r' \) on the loop to a field point \( r \) and \( R = |R| \) denotes the norm of \( R \). Here \( S \) denotes an arbitrary smooth surface enclosed by the loop \( L \), \( dS'_i \) is an oriented surface element, \( \Omega^0(r) \) is the solid angle under which the loop \( L \) is seen from the point \( r \), and \( \ast \) denotes the spatial convolution. The vector field (2) is analogous to the magnetic field of a magnetic monopole fixed at the
The divergence of the vector field \( \mathbf{v}_0 \) yields
\[
\partial_i v_0^i = -\frac{1}{2} \Delta \Delta R = 4\pi \delta(R),
\] (4)
since
\[
\Delta \Delta R = -8\pi \delta(R).
\] (5)
The solid angle \( \Omega^0 \) is a multi-valued quantity with the residue \( 4\pi \). Thus, the solid angle \( \Omega \) changes by \( 4\pi \) when the field point crosses the surface \( S \). In particular, this happens for a Burgers circuit that encircles \( L \). In other words, \( S \) represents the surface of discontinuity. Notice that, in classical elasticity, the plastic distortion caused by a dislocation loop is concentrated at the surface \( S \). From a physical viewpoint, \( S \) represents the area swept by the loop \( L \) during its motion and may be called the slip surface. Thus, the surface \( S \) is what determines the history of the plastic distortion of a dislocation loop (see, e.g., \[1, 19\]).

We may use the Stokes theorem to arrive at a line integral over \( L \) for the solid angle. To do so, it is necessary to express \( v_0^i \) as the curl of a “vector potential” \( A_0^0 \). However, Eq. (4) shows that the divergence of the vector field \( v_0^i \) is not identically zero, and therefore it becomes impossible to write \( v_0^i \) everywhere as the curl of a vector potential. Nevertheless, introducing a so-called fictitious vector field \( v_i^{(f)0} \), which is sometimes called “string of singularity”, (see, e.g., \[12, 13\]) having the property
\[
\partial_i v_i^{(f)0} = -\partial_i v_i^0,
\] (6)
a vector potential \( A_k^0 \) may be introduced for the divergenceless sum \( v_i^0 + v_i^{(f)0} \):
\[
v_i^0 + v_i^{(f)0} = \epsilon_{ijk}\partial_j A_k^0.
\] (7)
Subtraction of the fictitious vector field in Eq. (7) leads to the physical vector field \( v_i^0 \) given by
\[
v_i^0 = \epsilon_{ijk}\partial_j A_k^0 - v_i^{(f)0}.
\] (8)
Taking the curl of Eq. (7) and imposing the “Coulomb gauge” \( \partial_k A_k^0 = 0 \), we find an inhomogeneous Laplace equation for the vector potential
\[
\Delta A_k^0 = -\epsilon_{klm}\partial_l v_m^{(f)0},
\] (9)
where the fictitious vector field \( v_m^{(f)0} \) is the source term of the vector potential. Using the 3D Green function of the Laplace equation, \(-1/(4\pi R)\), the solution of Eq. (9) reads
\[
A_k^0(r) = \frac{1}{4\pi} \epsilon_{klm} \int_V \partial_l \frac{1}{R} v_m^{(f)0}(r') \, dV' = -\frac{1}{4\pi} \epsilon_{klm} \int_V v_l^0(R) v_m^{(f)0}(r') \, dV'.
\] (10)
The fictitious singular vector field \( v_i^{(f)0} \) can be taken as \[13, 20\]
\[
v_i^{(f)0}(r) = \int_C v_{k,i}(r-s) \, ds_i = 4\pi \int_C \delta(r-s) \, ds_i \equiv 4\pi \delta_i(C),
\] (11)
where \(C\) is a curve, called the “Dirac string”, starting at \(-\infty\) and ending at the origin and \(\delta_i(C)\) is the \(\delta\)-function along the Dirac string. The divergence of this field is concentrated at the endpoint of the string:
\[
\partial_i v_i^{(f)0}(r) = -4\pi \delta(r) = -\partial_i v_i^0(r). \tag{12}
\]

Then the vector potential of the monopole \((10)\) is given as a line integral along the path \(C\) (see, e.g., [12]):
\[
A^0_k(r) = \epsilon_{klm} \int_C v^0_m(r - s) \, ds = -\epsilon_{klm} \int_C \frac{1}{|r - s|} \, ds. \tag{13}
\]

The fictitious vector field \(v_i^{(f)0}\) is a singular field which vanishes everywhere except along the Dirac string \(C\).

If we choose for the path \(C\) a straight line in the direction of a constant unit vector \(n_i\), the fictitious vector field reads
\[
v_i^{(f)0}(r) = 4\pi n_i \int_{-\infty}^0 \delta(r - ns) \, ds, \tag{14}
\]
and the vector potential of the “magnetic monopole” reduces to
\[
A^0_k(r) = \epsilon_{klm} \frac{n_i r_m}{r(r + r_i n_i)}. \tag{15}
\]

Eq. \((15)\) has the original form of the vector potential of Dirac’s magnetic monopole (see, e.g., [11, 12]) which was adopted by deWit [9] and Eshelby [10] for the representation of the solid angle as a line integral.

Substituting Eqs. \((8)\) and \((11)\) into \((11)\) and using the Stokes theorem, we find
\[
\Omega^0(r) = v_i^0(r) \star \delta_i(S) = \epsilon_{ijk} \partial_j A^0_k(r) \star \delta_i(S) - v_i^{(f)0}(r) \star \delta_i(S)
= A^0_k(r) \star \epsilon_{kji} \partial_j \delta_i(S) - 4\pi \delta_i(C) \star \delta_i(S)
= A^0_k(r) \star \delta_k(L) - 4\pi \delta_i(C) \star \delta_i(S), \tag{16}
\]
where
\[
A^0_k(r) \star \delta_k(L) = \int_V A^0_k(R) \delta_k(L') \, dV' = \int_L A^0_k(R) \, dL'_k, \tag{17}
\]
the \(\delta\)-function on a closed line \(L\) [19, 20]
\[
\delta_i(L) \equiv \int_L \delta(R) \, dL'_i, \tag{18}
\]
and \(\epsilon_{kji} \partial_j \delta_i(S) = \delta_k(L)\). Here \(dL'_i\) denotes the line element at \(r'\). For the contribution of the fictitious vector field we used the formula [19, 21]
\[
\delta_i(L) \star \delta_i(S) = \int_S \int_L \delta(r - r') \, dL'_i \, dS_i = \begin{cases} 1, & \text{if } L \text{ crosses } S \text{ positively} \\ 0, & \text{if } L \text{ does not cross } S \\ -1, & \text{if } L \text{ crosses } S \text{ negatively} \end{cases}. \tag{19}
\]
Finally, the solid angle reduces to a line integral of the monopole vector potential \((13)\) or \((15)\) and a constant
\[
\Omega^0(\mathbf{r}) = \oint_L A_k^0(\mathbf{R}) \, dL_k' - 4\pi \begin{cases} 
1, & \text{if } C \text{ crosses } S \text{ positively} \\
0, & \text{if } C \text{ does not cross } S \\
-1, & \text{if } C \text{ crosses } S \text{ negatively}
\end{cases}.
\tag{20}
\]
We mention that deWit \([9]\) and Eshelby \([10]\) have neglected the subtraction of the fictitious vector field of the semi-infinite solenoid of the monopole field in their calculations. Of course, if it is chosen that the path \(C\) of the solenoid does not cross \(S\), the fictitious vector field does not give a contribution to the solid angle in Eq. \((20)\). An equation like \((20)\) was also derived by Asvestas \([22]\) for physical optics.

### 3 Gradient elasticity

In the theory of gradient elasticity of Helmholtz type, the solid angle is defined as the following surface integral \([14, 15]\)
\[
\Omega(\mathbf{r}) = \int_S v_i(\mathbf{R}) \, dS'_i = \int_V v_i(\mathbf{R}) \delta_i(S') \, dV' = v_i(\mathbf{r}) \ast \delta_i(S),
\tag{21}
\]
where the non-singular vector field is given by\(^1\)
\[
v_i = -\frac{1}{2} \Delta \partial_i A(\mathbf{R}) = -\partial_i \left[ \frac{1}{R} \left( 1 - e^{-R/\ell} \right) \right] = \frac{R_i}{R^3} \left( 1 - \left( 1 + \frac{R}{\ell} \right) e^{-R/\ell} \right)
\tag{22}
\]
with the “regularization function”
\[
A(\mathbf{R}) = R + \frac{2\ell^2}{R} (1 - e^{-R/\ell}),
\tag{23}
\]
and \(\ell\) denotes the characteristic internal length of gradient elasticity. In the framework of gradient elasticity, the displacement vector of a closed dislocation loop in an isotropic elastic material is given by the generalized Burgers formula \([14, 15]\)
\[
u_i(\mathbf{r}) = -\frac{b_i}{4\pi} \Omega(\mathbf{r}) + \frac{b_i \epsilon_{ijkl}}{8\pi} \oint_L \left\{ \delta_{ij} \Delta - \frac{1}{1 - \nu} \partial_i \partial_j \right\} A(\mathbf{R}) \, dL'_k,
\tag{24}
\]
where \(\mu\) and \(\nu\) are the shear modulus and Poisson’s ratio, respectively, and \(b_i\) is the Burgers vector.

The function \(A(\mathbf{R})\) satisfies the following relations \([15]\)
\[
L \Delta \Delta A(\mathbf{R}) = -8\pi \, \delta(\mathbf{R}),
\tag{25}
\Delta \Delta A(\mathbf{R}) = -8\pi \, G(\mathbf{R}),
\tag{26}
L A(\mathbf{R}) = R,
\tag{27}
\]

\(^1\) It interesting to note, from the historical point of view, that also Panofsky and Phillips \([23]\) considered such a non-singular vector field \([22]\) for a solid angle.
with the Helmholtz operator
\[ L = 1 - \ell^2 \Delta. \]

The Green function of the three-dimensional Helmholtz equation is defined by
\[ L G(R) = \delta(R), \quad G(R) = \frac{e^{-R/\ell}}{4\pi \ell^2 R}. \]

If we multiply Eq. (22) by the Helmholtz operator (28) and use the relation (27), we obtain
\[ L v_i = -\frac{1}{2} \Delta \partial_i L A(R) = -\frac{1}{2} \Delta \partial_i R. \]

Comparing Eqs. (2) and (30), we find an inhomogeneous Helmholtz equation for \( v_i \):
\[ L v_i = v_i^0, \]
where \( v_i^0 \) gives the inhomogeneous part. Using Eq. (26), the divergence of the vector field (22) is calculated as
\[ \partial_i v_i = -\frac{1}{2} \Delta \Delta A(R) = 4\pi G(R). \]

Multiplying Eq. (32) by the Helmholtz operator \( L \) and using Eqs. (25) and (29) or performing the divergence of Eq. (31), we get
\[ L \partial_i v_i = -\frac{1}{2} L \Delta \Delta A(R) = 4\pi L G(R) = 4\pi \delta(R). \]

In order to introduce a “vector potential” \( A_k \) in the framework of gradient elasticity, we introduce a fictitious vector field \( v_i^{(f)} \) with the property that
\[ \partial_i v_i^{(f)} = -\partial_i v_i. \]

A vector potential can now be introduced for the divergenceless sum \( v_i + v_i^{(f)} \):
\[ v_i + v_i^{(f)} = \epsilon_{ijk} \partial_j A_k. \]

The field \( A_k \) in (35) is a monopole vector field in gradient theory. If we subtract the fictitious vector field, we obtain the vector field
\[ v_i = \epsilon_{ijk} \partial_j A_k - v_i^{(f)}, \]
which is independent of the Dirac string. Only, the vector potential \( A_k \) and the fictitious vector field \( v_i^{(f)} \) depend on the Dirac string \( C \). If we now take the curl of Eq. (35) and impose the “Coulomb gauge” \( \partial_k A_k = 0 \), we find that the vector potential satisfies the following inhomogeneous Laplace equation
\[ \Delta A_k = -\epsilon_{klm} \partial_l v_m^{(f)}, \]
and therefore has solution
\[ A_k(r) = \frac{1}{4\pi} \epsilon_{klm} \int_{V} \partial_l \frac{1}{R} v_{m}^{(f)}(r') \, dV'. \]  
(38)

Multiplying Eq. (35) by the Helmholtz operator \( L \) and using Eqs. (31) and (41), we obtain an inhomogeneous Helmholtz equation for the vector potential \( A_k \)
\[ LA_k = A_0^k, \]  
(39)

where the right hand side is given by the singular monopole field \( A_0^k \), and an inhomogeneous Helmholtz equation for the fictitious vector field \( v_i^{(f)} \)
\[ L v_i^{(f)} = v_i^{(f)0} \]  
(40)

with \( v_i^{(f)0} \) as inhomogeneous part. If we multiply Eq. (37) by the Helmholtz operator \( L \) and use Eq. (40), we obtain an inhomogeneous Helmholtz-Laplace equation for the vector potential
\[ L\Delta A_k = -\epsilon_{klm}\partial_l v_{m}^{(f)0}, \]  
(41)

where \( v_i^{(f)0} \) is now the source of the vector potential of gradient elasticity. The solution of (41) is given by
\[ A_k(r) = \frac{1}{4\pi} \epsilon_{klm} \int_{V} \partial_l \frac{1}{R} \left(1 - e^{-R/\ell}\right) v_{m}^{(f)0}(r') \, dV' = -\frac{1}{4\pi} \epsilon_{klm} \int_{V} v_l(R) v_{m}^{(f)0}(r') \, dV'. \]  
(42)

Moreover, the solution of Eq. (39) might be written as a convolution integral
\[ A_k = G * A_0^k, \]  
(43)

which is a regularization of the singular monopole vector potential \( A_0^k \). Thus, it gives a non-singular monopole vector potential. The solution of Eq. (40) is
\[ v_i^{(f)} = G * v_i^{(f)0}. \]  
(44)

Using the Green function (29) of the 3D Helmholtz equation, the formal solutions (43) and (44) read explicitly
\[ A_k(r) = \frac{1}{4\pi\ell^2} \int_{V} \frac{e^{-R/\ell}}{R} A_0^k(r') \, dV', \]  
(45)

and
\[ v_i^{(f)}(r) = \frac{1}{4\pi\ell^2} \int_{V} \frac{e^{-R/\ell}}{R} v_i^{(f)0}(r') \, dV'. \]  
(46)

It is interesting to note that the solution (45) is similar in the formal form to the solution of a “massive” magnetic monopole in massive electrodynamics given in [24].
Substituting Eqs. (11) and (14) into the integral (46) and performing the integration in \( V' \), the following forms of the fictitious vector field are obtained

\[
v^{(f)}_i(r) = \frac{1}{\ell^2} \int_C e^{-|r-s|/\ell} ds_i = 4\pi G * \delta_i(C) \tag{47}
\]

and

\[
v^{(f)}_i(r) = \frac{n_i}{\ell^2} \int_{-\infty}^{0} e^{-|r-ns|/\ell} ds, \tag{48}
\]

respectively. Solutions (47) and (48) consist of a kernel in the form of a thin tube along the Dirac string decreasing exponentially outside this kernel instead of the classical \( \delta \)-string.

The gradient solution of Eq. (39) with the classical monopole fields (13) and (15) is given by

\[
A_k(r) = \epsilon_{klm} \int_C v_m(r-s) ds_l = -\epsilon_{klm} \int_C \partial_m \frac{1}{|r-s|} \left(1 - e^{-|r-s|/\ell}\right) ds_l \tag{49}
\]

and

\[
A_k(r) = \epsilon_{klm} n_l \int_{-\infty}^{0} v_m(r-ns) ds = \epsilon_{klm} n_l \left(\frac{r_m}{r(r+ni)} + \partial_m \int_{-\infty}^{0} e^{-|r-ns|/\ell} ds\right) \tag{50}
\]

Therefore, Eqs. (49) and (50) are the regularized versions of the singular vector potentials (13) and (15). In general, Eqs. (49) and (50) are the non-singular version of a Dirac monopole valid in gradient theory. It is noted that Eqs. (11) and (14) may be substituted directly into Eq. (42) in order to obtain Eqs. (49) and (50), respectively.

Substituting Eqs. (36), (44) and (11) into (21) and using the Stokes theorem, we obtain

\[
\Omega(r) = v_i(r) * \delta_i(S) = \epsilon_{ijk} \partial_j A_k(r) * \delta_i(S) - v^{(f)}_i(r) * \delta_i(S) = A_k(r) * \epsilon_{kjl} \partial_j \delta_i(S) - G(r) * v^{(f)0}_i(r) * \delta_i(S) = A_k(r) * \delta_k(L) - 4\pi G(r) * \delta_i(C) * \delta_i(S), \tag{51}
\]

where

\[
A_k(r) * \delta_k(L) = \int_V A_k(R) \delta_k(L') dV' = \oint_L A_k(R) dL'_k \tag{52}
\]

and

\[
G(r) * \delta_i(C) * \delta_i(S) = \int_S \int_C G(r-r') dL'_i dS_i. \tag{53}
\]

Multiplying Eq. (51) by the Helmholtz operator (28), it yields the relation between the solid angle in classical elasticity and gradient elasticity

\[
L \Omega(r) = L A_k(r) * \delta_k(L) - 4\pi L G * \delta_i(C) * \delta_i(S) = A^0_k(r) * \delta_k(L) - 4\pi \delta_i(C) * \delta_i(S) = \Omega^0(r). \tag{54}
\]
Finally, the solid angle reduces to a line integral of the monopole vector potential \((49)\) or \((50)\) and a contribution due to the fictitious vector field \(v_i^{(f)}\)

\[
\Omega(r) = \oint_L A_k(r) dL'_k - 4\pi \int_S \int_C G(r-r') dL'_i dS_i ,
\]

(55)

where \(G\) is the Green function of the three-dimensional Helmholtz equation \((29)\). \(G(r-r')\) is non-zero for \(r = r'\) and for \(r\) different than \(r'\) near the Dirac string. Therefore, the contribution of the fictitious vector is not localized at the intersection of the slip surface and the Dirac string, in contrast with the classical theory. It is a continuous contribution which does not give rise to a jump of the slip surface. This result is consistent with the surface representation of the generalized solid angle \((21)\). In gradient elasticity, the solid angle is non-singular and gives rise to a smoothing of the displacement profile in the Burgers formula.

It is noteworthy that if the line direction of the Dirac string can be made orthogonal to the slip surface, then the contribution of the fictitious vector field vanishes identically. This condition is verified for common cases of practical interest. For example, for dislocation loops gliding on planes the slip surface is flat and the Dirac string can be chosen to lie on a plane parallel to the slip plane. For prismatic loops spanning a glide cylinder, the Dirac string can be chosen to be the cylinder axis. Under these and other conditions, the solid angle \((21)\) may be transformed into a line integral of the “vector potential” \((45)\)

\[
\Omega(r) = \oint_L A_k(r) dL'_k
\]

(56)

and the corresponding Burgers formula \((24)\) can be expressed as a line integral performed over the closed dislocation loop \(L\) as

\[
u_i(r) = -\frac{b_i}{4\pi} \oint_L A_k(r) dL'_k + \frac{b_i \epsilon_{klj}}{8\pi} \oint_L \left\{ \delta_{ij} \Delta - \frac{1}{1-\nu} \partial_i \partial_j \right\} A(R) dL'_k .
\]

(57)

Eq. \((57)\) is the line integral representation of the Burgers formula in the framework of gradient elasticity of Helmholtz type. It enjoys three fundamental properties that make it particularly appealing for numerical implementation in dislocation dynamics codes (e.g. \([6, 7, 8]\)). First, it is non-singular on \(L\), second it is continuous on \(S\), and third it involves only line integrals along the dislocation loop. Applications of the theory presented in this article will be presented elsewhere \([25]\). Finally, Eq. \((57)\) may be called the gradient version of the Eshelby-deWit representation of the Burgers formula of a closed dislocation loop.

**Acknowledgement**

M.L. gratefully acknowledges the grants from the Deutsche Forschungsgemeinschaft (Grant Nos. La1974/2-1, La1974/2-2, La1974/3-1).
References

[1] R. deWit, The continuum theory of stationary dislocations, Solid State Physics 10 (1960) 249–292.

[2] R.W. Lardner, Mathematical Theory of Dislocations and Fracture, University of Toronto Press, Toronto (1974).

[3] J.P. Hirth and J. Lothe, Theory of Dislocations, 2nd edition, John Wiley, New York (1982).

[4] C. Teodosiu, Elastic Models of Crystal Defects, Springer-Verlag, Berlin (1982).

[5] L.D. Landau and E.M. Lifschitz, Theory of Elasticity, 3rd ed., Pergamon, Oxford (1986).

[6] N.M. Ghoniem and L.Z. Sun, Fast-sum method for the elastic field of three-dimensional dislocation ensembles, Phys. Rev. B 60 (1999) 128–140.

[7] N.M. Ghoniem, J. Huang and Z. Wang, Affine covariant-contravariant vector forms for the elastic field of parametric dislocations in isotropic crystals, Phil. Mag. Lett. 82 (2002) 55–63.

[8] S. Li and G. Wang, Introduction to Micromechanics and Nanomechanics, World Scientific, Singapore (2008).

[9] R. deWit, The displacement field of a dislocation distribution, in Dislocation Modelling of Physical Systems, Eds. M.F. Ashby, R. Bullough, C.S. Hartley and J.P. Hirth Pergamon press, Oxford (1981), pp. 304–309.

[10] J.D. Eshelby, Aspects of the theory of dislocations, in Mechanics of Solids, The Rodney Hill 60th Anniversary Volume, eds. H.G. Hopkins and M.J. Sewell, pp. 185–225. Pergamon Press, Oxford (1982). Reprinted in Collected Works of J.D. Eshelby, eds. X. Markenscoff and A. Gupta, pp. 861–902. Springer, Dordrecht (2006).

[11] P.A.M. Dirac, Quantised Singularities in the Electromagnetic Field, Proc. R. Soc. Lond. A 133 (1931) 60–72.

[12] G. Wentzel, Comments on Dirac’s Theory of Magnetic Monopoles, Prog. Theor. Phys. Suppl. 37 (1966) 163–174.

[13] M. Blagojević and P. Senjanović, The Quantum Field Theory of Electric and Magnetic Charge, Physics Reports 157 (1988) 233–346.

[14] M. Lazar, Non-singular dislocation loops in gradient elasticity, Physics Letters A 376 (2012) 1757–1758.

[15] M. Lazar, The fundamentals of non-singular dislocations in the theory of gradient elasticity: Dislocation loops and straight dislocations, International Journal of Solids and Structures 50 (2013) 352–362.
[16] R.D. Mindlin, *Micro-structure in linear elasticity*, Arch. Rational. Mech. Anal. **16** (1964) 51–78.

[17] M. Lazar and G.A. Maugin, *Nonsingular stress and strain fields of dislocations and disclinations in first strain gradient elasticity*, Int. J. Engng. Sci. **43** (2005) 1157–1184.

[18] M. Lazar and G.A. Maugin, *Dislocations in gradient elasticity revisited*, Proc. R. Soc. Lond. A **462** (2006) 3465–3480.

[19] R. deWit, *Theory of disclinations II*, J. Res. Nat. Bur. Stand. (U.S.) **77A** (1973) 49–100.

[20] H. Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism, and Gravitation*, World Scientific, Singapore (2008).

[21] C. Teodosiu, *A dynamic theory of dislocations and its applications to the theory of the elastic-plastic continuum*, in *Fundamental Aspects of Dislocation Theory*, Vol. 2, Eds. J.A. Simmons, R. deWit and R. Bullough, Nat. Bur. Stand. (U.S.), Spec. Publ. **317** (1970) pp. 837–876.

[22] J.S. Asvestas, *Line integrals and physical optics. Part I. The transformation of the solid-angle surface integral to a line integral*, J. Opt. Soc. Am. A **2** (1985) 891–895.

[23] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed., Addison Wesley, Reading, MA (1962); Dover, New York (2005).

[24] A.Yu. Ignatiev and G.C. Joshi, *Massive electrodynamics and the magnetic monopoles*, Phys. Rev. D **53** (1996) 984–992.

[25] G. Po, M. Lazar, D. Seif and N. Ghoniem, *Singularity-free dislocation dynamics with strain gradient elasticity*, (2013), submitted for publication.