Evaluations of certain Catalan-Hankel Pfaffians via classical skew orthogonal polynomials

Bo-Jian Shen¹, Shi-Hao Li²,∗ and Guo-Fu Yu¹

¹ School of Mathematical Sciences, Shanghai Jiaotong University, People’s Republic of China
² Department of Mathematics, Sichuan University, Chengdu, 610064, People’s Republic of China

E-mail: JOHN-EINSTEIN@sjtu.edu.cn, lishihao@lsec.cc.ac.cn and gfyu@sjtu.edu.cn

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Abstract

This paper is to evaluate certain Catalan-Hankel Pfaffians by the theory of skew orthogonal polynomials. Due to different kinds of hypergeometric orthogonal polynomials underlying the Askey scheme, we explicitly construct the classical skew orthogonal polynomials and then give different examples of Catalan-Hankel Pfaffians with continuous and q-moment sequences.

Keywords: Catalan-Hankel Pfaffian, classical orthogonal polynomials, skew orthogonal polynomials

1. Introduction

Hankel determinant as a specific determinant has attracted much attention from researchers in many different subjects such as orthogonal polynomials, random matrix theory and combinatorics (see, for example [5, 10, 18, 30] and references therein). It is well known that if \( \{ \mu_n \}_{n \geq 0} \) is a moment sequence taking the form

\[
\mu_n = \int_{\mathbb{R}} x^n \omega(x) \, dx,
\]

then Hankel determinant \( \det (\mu_{i+j})_{i,j=0}^{n-1} \) has a nice integral formula due to the Andriejéf formula [13].

∗Author to whom any correspondence should be addressed.
\[
\det(\mu_{ij})_{i,j=0}^{n-1} = \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{1 \leq i < j \leq n} |x_i - x_j|^2 \prod_{i=1}^n \omega(x_i) dx_i.
\]

Moreover, if \( \omega(x) \) is a classical weight, e.g. the Gaussian weight, Laguerre weight, Jacobi weight and circular Jacobi weight, then Hankel determinant with classical moments can be explicitly evaluated due to the Selberg integrals [12, chapter 4]. Moreover, \( q \)-versions of Selberg integrals with discrete measures also play important roles in modern mathematical physics as well [17].

In recent years, the Pfaffian version of Hankel determinant, called the Catalan-Hankel Pfaffian, was proposed due to its potential applications in the theory of combinatorics [19]. In the evaluations of Catalan-Hankel Pfaffian, the Pfaffian version of Andrei de Bruijn formula, which is called the de Bruijn formula, has been applied with the help of Selberg integrals and their \( q \)-versions [19, 20]. Catalan-Hankel Pfaffian has the form

\[
\text{Pf}\left((j-i)\mu_{i+j-1}(-1)^{2N-1}i,j=0\right),
\]

and its \( q \)-version is expressed by

\[
\text{Pf}\left([j]_q - [i]_q \mu_{i+j-1}(-1)^{2N-1}i,j=0\right), \quad [j]_q = \frac{1 - q^j}{1 - q},
\]

where \( \{\mu_n\}_{n \geq 0} \) is a moment sequence related to discrete \( q \)-measures. The specific little \( q \)-Jacobi case were considered in [19] and some recent developments like the Al-Salam & Carlitz I case was given in [20].

Regarding evaluations of \((q-)Catalan-Hankel Pfaffian\), one application is to give formula to the weighted enumeration of some specific plane partitions. For example, in [19], the authors considered a little \( q \)-Jacobi case and enumerated a special family of shifted reverse plane partitions with weights that resemble the weight in the inner product of little \( q \)-Jacobi polynomials. The importance of these evaluations lies in the random matrix theory as well. In an earlier work of Mehta and Wang [25], they gave an evaluation of the Catalan-Hankel Pfaffian

\[
\text{Pf}\left((j-i)\Gamma(i + j + a)\left(-1\right)^{2N-1}i,j=0\right),
\]

which was related to the skew orthogonal Laguerre polynomials given in [1]. Moreover, the shifted Catalan-Hankel Pfaffian

\[
\text{Pf}\left((j-i)\mu_{i+j}(-1)^{2N-1}i,j=0\right) \quad \text{or} \quad \text{Pf}\left([j]_q - [i]_q \mu_{i+j+r}\left(-1\right)^{2N-1}i,j=0\right), \quad r \in \mathbb{Z}^+, \]

is closely related to the adjacent families of skew-orthogonal polynomials considered in [23]. These evaluations may give more hints in exploring novel examples in integrable systems and random matrices.

As mentioned, one way to evaluate these Pfaffians is based on the de Bruijn formula and Selberg integral [19, 20]. In this paper, we will investigate another way, namely by using the theory of classical \((q-)skew orthogonal polynomials\), to make these evaluations and demonstrate the effectiveness—by simply considering different classical weights underlying Askey scheme, we can get different examples of \((q-)Catalan-Hankel Pfaffians\). Since Hankel determinant is closely related to the theory of classical orthogonal polynomials, then if we know normalization constants of these classical orthogonal polynomials, the evaluations of Hankel determinant can be made. It is natural to ask whether we can apply the theory of classical \((q-)skew orthogonal polynomials\) into evaluations of the certain \((q-)Catalan-Hankel Pfaffians\). The answer is affirmative. If we know the normalization factors of skew orthogonal
polynomials, then these Catalan-Hankel Pfaffians can be explicitly given by those normalization factors. Due to different expressions in discrete and continuous cases (cf equations (1.1) and (1.2)), we discuss them separately. Continuous cases including Hermite-type, Laguerre-type and Jacobi-type skew orthogonal polynomials were firstly constructed by Adler et al [1] and the Cauchy case was considered later by Forrester and Nagao [16]. We give a brief review in section 2. Some evaluations of Catalan-Hankel Pfaffians related to these weights including formula (1.3) will be demonstrated. In section 3, evaluations of $q$-Catalan-Hankel Pfaffians will be based on the theory of classical $q$-skew orthogonal polynomials. With the help of discrete Pearson relation, we construct different examples of classical ($q$-)skew orthogonal polynomials, thus obtaining different kinds of skew orthogonal polynomials and normalization factors with respect to different classical $q$-weights, which extend previous results given in [15, 20].

2. Continuous measure

Let us consider a skew inner product related to (1.1)

$$\langle \phi(x), \psi(x) \rangle_{4,\omega} = \frac{1}{2} \int_{\mathbb{R}} \left[ \phi(x)\psi(x) - \phi'(x)\psi(x) \right] \omega(x) dx,$$

(2.1)

and skew moments $\{m_{ij}\}_{i,j=0}^{\infty}$ are given by

$$m_{ij} := \langle x^i x^j \rangle_{4,\omega} = \frac{1}{2} (j-i) \int_{\mathbb{R}} x^{i+j-1} \omega(x) dx.$$

The skew Borel decomposition [2] of moment matrix $\{m_{ij}\}_{i,j=\mathbb{N}}$ could then give rise to a family of monic skew orthogonal polynomials $\{Q_n(x)\}_{n=\mathbb{N}}$ satisfying skew orthogonal relation

$$\langle Q_{2n}(x), Q_{2m+1}(x) \rangle_{4,\omega} = u_n \delta_{n,m},$$

$$\langle Q_{2n}(x), Q_{2m}(x) \rangle_{4,\omega} = \langle Q_{2n+1}(x), Q_{2m+1}(x) \rangle_{4,\omega} = 0,$$

(2.2)

for certain $u_n > 0$. Moreover, these skew orthogonal polynomials have following Pfaffian expressions [8]

$$Q_{2n}(x) = \frac{1}{\tau_{2n}} \text{Pf}(0, \ldots, 2n, x), \quad Q_{2n+1}(x) = \frac{1}{\tau_{2n}} \text{Pf}(0, \ldots, 2n-1, 2n+1, x),$$

(2.3)

with $\tau_{2n} = \text{Pf}(0, \ldots, 2n-1)$ and the Pfaffian elements are given by $\text{Pf}(i,j) = m_{ij}$ and $\text{Pf}(i,x) = x^i$. Please refer to the appendix for a detailed explanation of this formula. By putting $m_{ij}$ into the expression of $\tau_{2n}$, we have

$$\tau_{2n} = \frac{1}{2^n} \text{Pf}[(j-i)\mu_{i+j-1}]_{i,j=0}^{2n-1}, \quad \mu_n = \int_{\mathbb{R}} x^n \omega(x) dx.$$ 

(2.4)

Interestingly, $u_n$ in the skew orthogonal condition (2.2) is the ratio $\tau_{2n+2}/\tau_{2n}$, and therefore, if we can explicitly compute $\{u_n\}_{n=\mathbb{N}}$ in the skew orthogonal condition, then we directly have $\tau_{2n} = \prod_{i=0}^{n} u_i$. In fact, normalization factors $\{u_n\}_{n=\mathbb{N}}$ can be explicitly computed when $\{Q_n(x)\}_{n=\mathbb{N}}$ are classical skew orthogonal polynomials.

Starting from the symmetric inner product

$$\langle \phi(x), \psi(x) \rangle_{2,\rho} = \int_{\mathbb{R}} \phi(x)\psi(x)\rho(x) dx,$$
one can construct a family of monic orthogonal polynomials $\{p_j(x)\}_{j \in \mathbb{N}}$ satisfying the orthogonal relation
\[
\langle p_j(x), p_k(x) \rangle_{2,\rho} = h_j \delta_{j,k}.
\]
We call these orthogonal polynomials classical if the weight function $\rho(x)$ satisfies Pearson equation
\[
\frac{\rho'(x)}{\rho(x)} = -\frac{g(x)}{f(x)} \quad \text{with } \deg f(x) \leq 2 \text{ and } \deg g(x) \leq 1.
\]
From this relation, one can construct an operator
\[
A = f \frac{\partial}{\partial x} + \frac{f' - g}{2},
\]
such that
\[
\langle \phi(x), A \psi(x) \rangle_{2,\rho} = \langle \phi(x), \psi(x) \rangle_{4,\omega}, \quad \omega(x) = \rho(x)f(x),
\]
where the skew inner product $\langle \cdot, \cdot \rangle_{4,\omega}$ is given in (2.1). This is a key formula to establish the relation between classical orthogonal polynomials and classical skew orthogonal polynomials. Moreover, the normalization factor $u_n$ could be computed via
\[
A p_k(x) = -\frac{c_k}{h_{k+1}} p_{k+1}(x) + \frac{c_{k-1}}{h_{k-1}} p_{k-1}(x), \quad u_n = c_{2n},
\]
with $h_k$ the normalization factor of orthogonal polynomials. Therefore, to obtain normalization factors of skew orthogonal polynomials, our attention should be paid to the computation of $c_k$ in the above equation. The fastest method to compute $c_k$ is to compare the coefficients of $x^{k+1}$ on both sides and the following are examples of the continuous classical weights including Hermite, Laguerre, Jacobi and Cauchy weights.

From weight functions
\[
\rho(x) = \begin{cases} 
  e^{-x^2} \chi(-\infty, +\infty), & \text{Hermite}, \\
  x^a e^{-x^2} \chi(0, +\infty), & \text{Laguerre}, \\
  x^a (1-x)^b \chi(0,1), & \text{Jacobi}, \\
  (1 + x^2)^{-a} \chi(-\infty, +\infty), & \text{Cauchy},
\end{cases}
\]
one can obtain the Pearson pairs
\[
(f, g) = \begin{cases} 
  (1, 2x), & \text{Hermite}, \\
  (x, x-a), & \text{Laguerre}, \\
  (x(1-x), (a+b)x-a), & \text{Jacobi}, \\
  (1 + x^2, 2ax), & \text{Cauchy}.
\end{cases}
\]
This table was given by [12, equation (5.58)] and it should be remarked that we use the weight function of Jacobi polynomials as $x^a(1-x)^b$ supported in $[0, 1]$ to make the moments easily
written down. From $\omega(x) = f(x)\rho(x)$, one knows that weights of classical skew orthogonal polynomials are

$$\omega^{(H)}(x) = e^{-x^2}, \quad \omega^{(L)}(x) = x^{a+1} e^{-x},$$

$$\omega^{(J)}(x) = x^{a+1}(1 - x^{b+1}), \quad \omega^{(C)}(x) = (1 + x^2)^{-a+1}.$$ 

Therefore, we have the following expressions for moments (here we assume that parameter $a$ in the Cauchy case is large enough so that those moments are finite)

$$\mu^{(H)}_n = \frac{1 + (-1)^n}{2} \Gamma\left(\frac{n + 1}{2}\right), \quad \mu^{(L)}_n = \Gamma(n + a + 2),$$

$$\mu^{(J)}_n = \frac{\Gamma(n + a + 2)\Gamma(b + 2)}{\Gamma(n + a + b + 4)}, \quad \mu^{(C)}_n = \frac{1 + (-1)^n \Gamma((n + 1)/2)\Gamma(a - 1 - (n + 1)/2)}{2 \Gamma(a - 1)}.$$ 

Moreover, we could obtain

$$c_k = \begin{cases} h^{(H)}_{k+1}, & \text{Hermite,} \\ h^{(L)}_{k+1}/2, & \text{Laguerre,} \\ (k + 1 + (a + b)/h^{(J)}_{k+1}, & \text{Jacobi,} \\ (a - 1 - k)h^{(C)}_{k+1}, & \text{Cauchy} \end{cases}$$

where $\{h_k\}_{k \in \mathbb{N}}$ are normalization factors with respect to different weights [12, chapter 5]

$$h^{(H)}_k = \pi^{1/2}2^{-k}k!, \quad h^{(L)}_k = \Gamma(k + 1)\Gamma(a + k + 1),$$

$$h^{(J)}_k = \frac{\Gamma(k + 1)\Gamma(a + b + 1 + k)\Gamma(a + 1 + k)\Gamma(b + 1 + k)}{\Gamma(a + b + 2k + 1)\Gamma(a + b + 2k + 2)},$$

$$h^{(C)}_k = \pi 2^{2k-2a+2} \frac{(k + 1)\Gamma(2a - 2k)\Gamma(2a - 2k - 1)}{\Gamma(2a - k)\Gamma(a - k)^2}.$$ 

By using above results, we could state the following proposition.

**Proposition 2.1.** We have following evaluations of certain Catalan-Hankel Pfaffians

$$Pf\left[(j - i)\mu^{(H)}_{i+j-1}\right]_{i,j=0}^{2N-1} = 2^{-N(N-1)\sqrt{\pi}} \prod_{i=0}^{N-1} \Gamma(2i + 2),$$

$$Pf\left[(j - i)\mu^{(L)}_{i+j-1}\right]_{i,j=0}^{2N-1} = \prod_{i=0}^{N-1} \Gamma(2i + 2)\Gamma(2i + a + 2),$$

$$Pf\left[(j - i)\mu^{(J)}_{i+j-1}\right]_{i,j=0}^{2N-1} = \prod_{i=0}^{N-1} \Gamma(a + 2i + 2)\Gamma(b + 2i + 2)\Gamma(2i + 2)}{\Gamma(a + b + 2i + 2N + 2)},$$

$$Pf\left[(j - i)\mu^{(C)}_{i+j-1}\right]_{i,j=0}^{2N-1} = (\sqrt{\pi})^N \prod_{i=0}^{N-1} \frac{\Gamma(2i + 2)\Gamma(2a - 2i - 1)\Gamma(a - 2i - 3/2)}{\Gamma(2a - 2i - 1)\Gamma(a - 2i - 1)}.$$
Remark 2.2. The first formula is a special case of [19, corollary 3.5]. The second one is exactly equation (1.3) with $a \to a - 1$ and also a special case of [19, corollary 3.4].

Note that by (A.1), the first and the last equation can also be viewed as determinant evaluations

$$
\begin{align*}
\det \left[ (2j - 2i + 1) \frac{\Gamma(i + j + 1/2)}{\Gamma(a - 1)} \right]_{i,j=0}^{N-1} &= 2^{-N(N-1)}(\sqrt{\pi})^N \prod_{i=0}^{N-1} (2i + 2), \\
\det \left[ (2j - 2i + 1) \frac{\Gamma(i + j + 1/2)\Gamma(a - (i + j) - 3/2)}{\Gamma(a - 1)} \right]_{i,j=0}^{N-1} &= (\sqrt{\pi})^N \prod_{i=0}^{N-1} \frac{\Gamma(2i + 2)\Gamma(2a - 4i - 1)\Gamma(a - 2i - 3/2)}{\Gamma(2a - 2i - 1)\Gamma(a - 2i - 1)}.
\end{align*}
$$

3. Discrete measure: $q$-case

This part is devoted to the evaluations of $q$-Catalan Hankel Pfaffians given by formula (1.2). The $q$-case corresponds to a special discrete measure distributed on exponential lattices $\{t(i) = q^i\}$ with $0 < q < 1$ and $i \in \mathbb{Z}$. By using the definition of Jackson’s $q$-integral\(^3\)

$$
\int_{0}^{\infty} f(x) \, dq_x = (1 - q) \sum_{s=\infty}^{\infty} f(q^s)q^s,
$$

one can define following inner product

$$
\langle \phi(x), \psi(x) \rangle_{z,\rho} := \int_{0}^{\infty} \phi(x)\psi(x) \rho(x) \, dq_x = (1 - q) \sum_{x \in \mathbb{Z}} \phi(q^x)\psi(q^x) \rho(q^x)q^x. \quad (3.1)
$$

With this inner product, a family of monic $q$-orthogonal polynomials $\{p_n(x; q)\}_{n \in \mathbb{N}}$ are defined by the orthogonal relation

$$
\langle p_n(x; q), p_m(x; q) \rangle_{z,\rho} = h_n(q)\delta_{n,m}, \quad (3.2)
$$

with respect to a $q$-weight $\rho(x; q)$. Moreover, if $\rho(x; q)$ is a classical weight, then $\{P_n(x; q)\}_{n=0}^{\infty}$ in (3.2) are called as classical $q$-orthogonal polynomials. Classical $q$-orthogonal polynomials include many interesting examples underlying the $q$-Askey scheme. For details, please refer to [3, 21, 22]. One important property of classical $q$-orthogonal polynomials is that their weight functions satisfy an analogy of Pearson relation given by Nikiforov and Suslov [26]

$$
\frac{\rho(qx)}{\rho(x)} = \frac{f(x) - q^{-\frac{1}{2}}(1 - q)xg(x)}{f(qx)} \quad \text{with deg } f(x) \leq 2 \text{ and deg } g(x) \leq 1. \quad (3.3)
$$

In the following, we demonstrate how to connect $q$-inner product with $q$-skew inner product.

Let us define a $q$-analog of the skew inner product (2.1)

$$
\langle \phi(x), \psi(x) \rangle_{\omega} = \int_{0}^{\infty} [\phi(x)D_q\psi(x) - D_q\phi(x)\psi(x)]\omega(x) \, dq_x, \quad (3.4)
$$

\(^3\)The definition of $q$-integral in the interval $[0, \infty)$ is different from the one defined in $[0, 1]$, see [21]. However, these two cases can be treated similarly and we just consider the former one here.
with $q$-difference operator

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}. $$

From the skew inner product (3.4), one has following ($q$-)skew moments

$$m_{i,j} := ([l]_q - [l]_q) \int_0^\infty x^{i+j-1} \omega(x)dx, \quad [l]_q = \frac{1 - q^l}{1 - q}. \quad (3.5)$$

Similarly, it is known that a family of **monic** skew orthogonal polynomials \( \{Q(x; q)\}_{i \in \mathbb{N}} \) could be constructed by those moments if \( \text{Pf}(m_{i,j})_{i,j=0}^{2n} \neq 0 \) for all \( n \in \mathbb{N}_+ \) [15]. Furthermore, polynomials \( \{Q_i(x; q)\}_{i \in \mathbb{N}} \) admit following Pfaffian expressions

$$Q_{2n}(x; q) = \frac{1}{\tau_{2n}(q)} \text{Pf}(0, \ldots, 2n, x),$$

$$Q_{2n+1}(x; q) = \frac{1}{\tau_{2n}(q)} \text{Pf}(0, \ldots, 2n - 1, 2n + 1, x)$$

$$\tau_{2n}(q) = \text{Pf}(0, \ldots, 2n - 1), \quad \text{Pf}(i, j) = m_{i,j}, \quad \text{Pf}(i, x) = x^i,$$

and they satisfy the following skew orthogonal relations

$$\langle Q_{2n}(x; q), Q_{2n+1}(x; q) \rangle_{4,\omega} = \frac{\tau_{2n}(q)}{\tau_{2n+1}(q)} \delta_{n,m},$$

$$\langle Q_{2n}(x; q), Q_{2n}(x; q) \rangle_{4,\omega} = \langle Q_{2n+1}(x; q), Q_{2n+1}(x; q) \rangle_{4,\omega} = 0.$$ 

Therefore, one knows that

$$\text{Pf} \left[ ([l]_q - [l]_q) \int_0^\infty x^{i+j-1} \omega(x)dx \right] = \text{Pf}(0, \ldots, 2N - 1) = N! u_i(q).$$

Interestingly, there is a method to evaluate the value of \( u_i(q) \) quickly by taking advantage of \( q \)-Pearson relation. By defining an operator \( \mathcal{A}_q \) [15]

$$\mathcal{A}_q = q^\frac{1}{2} g(x)T_q + q^{-1} f(x)D_q^{-1} + f(x)D_q, \quad T_q f(x) = f(qx). \quad (3.6)$$

one can find a connection formula

$$\langle \phi(x; q), \mathcal{A}_q \psi(x; q) \rangle_{2,\omega} = \langle \phi(x; q), \psi(x; q) \rangle_{4,\omega}, \quad \omega(x) = \rho(qx) f(qx). \quad (3.7)$$

As an analogy of equation (2.6), there holds the formula

$$\mathcal{A}_q p_k(x; q) = -\frac{c_k(q)}{h_{k+1}(q)} p_{k+1}(x; q) + \frac{c_{k-1}(q)}{h_{k-1}(q)} p_{k-1}(x; q), \quad (3.8)$$

in the \( q \)-case due to the property of \( \mathcal{A}_q \) [15, equation (4.28)]. Moreover, the quantity \( h_k(q) \) is the normalization factor of the orthogonal relation (3.2) and \( c_i(q) \) is closely related to \( u_i(q) \) via the relation \( u_i(q) = c_{2i}(q) \). Therefore, it is the point to compute \( c_i(q) \) from above equation and
The first example considered here is the Al-Salam & Carlitz I case with weight function 
\[ \rho(x; q) = (qx, a^{-1}qx; q)_\infty, \quad a < 0. \]

It is well known that Al-Salam & Carlitz polynomials \( \{U_n^{(a)}(x; q)\}_{n \geq 0} \) have the orthogonality
\[ \int_a^1 U_m^{(a)}(x; q)U_n^{(a)}(x; q)\rho(x; q)dx = (-a)^n(1-q)(q; a, a^{-1}q; q)_\infty q^{\binom{n}{2}}\delta_{n,m}, \]
and canonical moments are given by [27]
\[ \mu_n = \int_a^1 x^n\rho(x; q)dx = (1-q)(q, a, a^{-1}q; q)_\infty \sum_{i=0}^{n} \binom{n}{i} a^i. \]

Specifically, Al-Salam & Carlitz I polynomials have \( q \)-hypergeometric function expressions
\[ U_n^{(a)}(x; q) = (-a)^nq^{\binom{n}{2}}\phi_1 \left( \binom{n-1}{0}; q, \frac{qx}{a} \right). \]

and the Pearson pair in this case is
\[ (f, g) = \left( x^2 - (1+a)x + a, \frac{q^{1/2}}{1-q}(x-(1+a)) \right). \]

The coefficient \( c_n \) in (3.8) could be explicitly computed as \( c_n = -q^{-n}h_{n+1}/(1-q) \). Thus we have following evaluation
\[ \text{Pf}(m_{i,j})_{i,j=0}^{2N-1} = \prod_{k=0}^{N-1} a^{2k+1}(q; q)_{2k+1}q^{\binom{2k}{2}}(q, a, a^{-1}q; q)_\infty = a^{N^2}q^{\delta_{N(2N-1)(4N^2-5)}}(q, a, a^{-1}q; q)_\infty \sum_{k=0}^{N-1} (q, q)_{2k+1}, \]
where \( m_{i,j} = ([j]_q - [i]_q)\int_0^1 x^{i+j-1}\omega(x; q)dx \). Since \( \omega(x; q) = f(qx; q)\rho(qx; q) = a\rho(x; q) \), we have
\[ m_{i,j} = a([j]_q - [i]_q)\mu_{i+j-1} = (q' - q')(q, a, a^{-1}q; q)_\infty \sum_{k=0}^{i+j-1} \binom{i+j-1}{k} a^k q^k. \]

Dividing both sides by \( (a(q, a, a^{-1}q; q)_\infty)^N \) leads to
\[ \text{Pf}\left( (q' - q')^{i+j-1} \sum_{k=0}^{i+j-1} \binom{i+j-1}{k} a^k \right)_{i,j=0}^{2N-1} 8^{2N-1} = a^{N(2N-1)}q^{\delta_{N(2N-1)(4N^2-5)}} \prod_{k=0}^{N-1} (q; q)_{2k+1}. \]
Remark 3.1. With $a = -1$, Al-Salam & Carlitz I polynomials reduce to the $q$-Hermite I polynomials [22, section 14.28], and therefore the above mentioned method could be applied to the $q$-Hermite I case as well. The moment of $q$-Hermite I polynomials takes the form [27]

$$
\mu_n = (1 - q)(q, -1, -q; q)_\infty \frac{1 + (-1)^n}{2} (q; q^2)_{n/2},
$$

and we have following evaluation of $q$-Catalan-Hankel Pfaffian

$$
\text{Pf} \left( (q^j - q^{j'}) \frac{1 + (-1)^{j+j'-1}}{2} (q; q^2)^{j+j'-1} \right)_{l,j=0}^{2N-1} = q^{k_{N(N-1)(4N-5)}} \prod_{k=0}^{N-1} (q; q^2)^{2k+1}.
$$

In view of (A.1), this indicates an evaluation for the determinant

$$
\det \left( (q^n - q^{n+1}) (q; q^3)_{n+1} \right)_{l,j=0}^{N-1} = q^{k_{N(N-1)(4N-5)}} \prod_{k=0}^{N-1} (q; q^2)^{2k+1}.
$$

3.2. Stieltjes–Wigert case

The Stieltjes–Wigert polynomials are well studied in random matrix theory of the so-called Stieltjes–Wigert ensemble, which firstly appeared in the study of non-intersecting Brownian walks, and subsequently in quantum many body systems, etc. For a detailed review, please refer to [14] and references therein. The weight of Stieltjes–Wigert polynomials was first given by Stieltjes as an example of indeterminate moment problems [29] and further studied by Wigert [31]. There are several different expressions for the Stieltjes–Wigert’s weight function [11]. The original one corresponding to a continuous measure is

$$
w(x) = \frac{1}{\sqrt{\pi}} x^{-k^2} \log x, \quad x > 0,
$$

with the moment $\mu_n = \int_0^\infty x^n w(x) dx = e^{(\sigma + 1)^2/4k^2}$. If we set $q = e^{-1/2k^2}$, then $\mu_n$ can be written as $q^{-(\sigma + 1)^2/2}$. With this measure, Wigert found the following expression for Stieltjes–Wigert polynomials [31]

$$
P_n(x) = (-1)^n q^{n/2 + 1/4} \sqrt{q; q^2} \sum_{k=0}^{n} \binom{n}{k_q} (-1)^k q^{2k/2} x^k,
$$

with orthogonality

$$
\int_0^\infty P_n(x) P_m(x) w(x) dx = \delta_{nm}.
$$

In [9], Chihara proposed a discrete weight on the exponential lattices admitting the form

$$
\xi(x) = \begin{cases} 
\frac{1}{\sqrt{2\pi}} q^{n+\pi^2/2} & , x = q^n, \\
0 & , x \neq q^n
\end{cases}, \quad M = (-q \sqrt{q}, -q^{-1/2}, q; q)_{\infty}, \quad n \in \mathbb{Z}.
$$

Then the corresponding inner product becomes

$$
\langle p_n(x), p_m(x) \rangle_{2L} = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} p_n(q^k) p_m(q^k) q^k + \pi^2/2.
$$

We consider the discrete measure. Denote
\[ \rho(x; q) = \frac{1}{\sqrt{q(1 - q)M}} \frac{h_n}{q^{n^2 + n + 1/4}}. \]
to be the weight function and define
\[ p_n(x) = \frac{\sqrt{(q; q)_n}}{q^{n^2 + n + 1/4}} P_n(x), \]
to be the monic Stieltjes–Wigert polynomials. We can rewrite the orthogonality in terms of \( \rho(x; q) \) and \( \{p_n(x)\}_{n \geq 0} \) as
\[ \int_0^\infty p_n(x)p_m(x)\rho(x; q)dx = \frac{(q; q)_n}{q^{2n^2 + 2n + 1/4}} \delta_{mn} := h_n \delta_{n,m}. \]
By using Jacobi’s triple product identity, one could verify that these two definitions are the same by checking that moments are the same. Assuming \( f(x) = x \) and solving Pearson equation (3.3), we get \( g(x) = (q^2x - q^{1/2})/(q - 1) \) and
\[ c_n(q) = \frac{q^{n^2 + n + 1/4}}{1 - q^2}. \]
As a consequence, we have the following evaluation of the moment Pfaffian
\[ \text{Pf}(m_{ij})_{i,j=0}^{2N-1} = q^{-\frac{1}{2}N(8N^2 + 3N^2 - 2)}(1 - q)^{-N}\prod_{k=0}^{N-1} (q; q)_{2k + 1}. \]
On the other hand, we could compute the skew moments (3.5) as a scalar product of canonical moments \( m_n \)
\[ \langle [j]_q - [i]_q \rangle = \int_0^\infty x^{i+j-1} \rho(x; q)dx = (q; q)_j q^{1/2} \mu_{n,j} \]
\[ = (q; q)_j q^{1/2} \mu_{n,j+1} = \langle [j]_q - [i]_q \rangle q^{-3i(j+2j^2 - j)/2}. \]
By combining above results, we have the evaluation
\[ \text{Pf}([j]_q - [i]_q)q^{-(i+j+2j^2)/2} \prod_{i,j=0}^{2N-1} = q^{-\frac{1}{4}N(16N^2 + 6N + 1)} \prod_{k=0}^{N-1} (q^2; q)_{2k}. \]

### 3.3. Little q-Jacobi case

Little q-Jacobi polynomials are important in many mathematical fields such as polynomials theory [22] and quantum group [24]. These polynomials have following series form (cf [24, equation (2.21)])
\[ p_n^{(\alpha, \beta)}(z; q) = \sum_{r \geq 0} \frac{(q^{-n}; q)_{r}\{q^{\alpha + \beta + n + 1}; q\}_{r} \{qz\}_{r}}{(q; q)r(q^{\alpha + 1}; q)_{r}}. \]
and they obey orthogonal relation [24, proposition 3.9]
\[
\int_0^1 p_n^{(\alpha,\beta)}(z; q)p_m^{(\alpha,\beta)}(z; q)z^\alpha(qz; q)_\beta dq
= \delta_{n,m} \frac{(1 - q)q(q; q)_\infty(q; q)_{n+\alpha}(q; q)_{\alpha + \beta + n}}{(1 - q^{\alpha + \beta + 2n+1})(q; q)_{\alpha + \beta + n}(q; q)_{\alpha + \beta + n+\alpha}}.
\]
In this case, canonical moments are given by [27]
\[
\mu_n^{(\alpha,\beta)} = \int_0^1 z^{n-\alpha}(q; q)_\beta dq = \frac{(1 - q)(q; q)_\infty(q; q)_{n+\alpha}(q; q)_{\alpha + \beta + n}}{(q; q)_{2n+2}\alpha + \beta} \delta_{n,m} := h_n \delta_{n,m}.
\]
Regarding the weight of little \( q \)-Jacobi polynomials
\[
\rho^{(\alpha,\beta)}(z; q) = z^\alpha(qz; q)_\beta,
\]
by solving (3.3), it admits the Pearson pair
\[
(f, g) = \left(-x^2 + x, -q^2 \left( [\alpha + \beta + 2]_q x - [\alpha + 1]_q \right) \right).
\]
Therefore, the coefficient \( c_n \) has the expression
\[
c_n = q^{-n} [2n + 2 + \alpha + \beta]_q h_{n+1},
\]
where \( h_n \) is the normalization constant in the orthogonal relation of monic polynomials (3.9).
Then we have the following expression for the moment Pfaffian
\[
Pf(m_{i,j}^{(\alpha,\beta)})_{i,j=0}^{N-1} = q^{\frac{N(N^2 - 3N + 2) + \alpha N^2}{4}} \prod_{k=0}^{N-1} \frac{(1 - q^{\alpha + \beta + 4k+2})(q; q)_{a+1}(q; q)_{\alpha + \beta + 1}(q; q)_{2k+1}(q; q)_{\alpha + \beta + 1}(q)_{4k+2}(q^{\alpha + \beta + 1}; q)_{4k+2}}{(q^{\alpha + 1}; q)_{\alpha + \beta + 1}(q^{\alpha + \beta + 1}; q)_{4k+2}(q^{\alpha + \beta + 1}; q)_{4k+2}}.
\]
Since the weight of the corresponding skew orthogonal little \( q \)-Jacobi polynomials is
\[
\omega(x; q) = f(x; q)\rho^{(\alpha,\beta)}(x; q) = q^{\alpha + 1} \rho^{(\alpha + 1,\beta + 1)}(x; q) = q^{\alpha + 1} x^{\alpha + 1}(qx; q)_{\beta + 1},
\]
skew moments \( m_{ij}^{(\alpha,\beta)} \) for \( i, j \in \mathbb{N} \) are related to the canonical moments by

\[
m^{(\alpha,\beta)}_{ij} = (\lfloor l \rfloor_q - [l]_q) \int_0^1 x^{i+j-1} \omega(x; q) d_q x = (\lfloor l \rfloor_q - [l]_q) q^{i+j} \mu_{i+j}^{(\alpha,\beta)}
\]

\[
= (\lfloor l \rfloor_q - [l]_q) q^{i+j} \frac{(1 - q)(q^{\alpha+\beta+1}; q)_\infty (q; q)_\infty (q^{\alpha+1}; q)_{i+j-1}}{(1 - q^{\alpha+\beta+1})(q^{\alpha+1}; q)_\infty (q^{\beta+1}; q)_\infty (q^{\alpha+\beta+2}; q)_{i+j-1}}.
\]

After eliminating the constant \( (q^{\alpha+1}(q^{\alpha+\beta+1}; q)_\infty (q; q)_\infty \) \), we have

\[
Pr \left( (\lfloor l \rfloor_q - [l]_q) (1 - q)(q^{\alpha+2}; q)_{i+j-1} \right)_{i,j=0}^{2N-1} = q^{\frac{1}{2}(N(N-1)(4N+1) + aN(N-1))} \prod_{k=0}^{N-1} \frac{(q; q)_{2k+1}(q^{\alpha+2}; q)_{2k}}{(q^{\alpha+\beta+2}; q)_k(q^{\alpha+\beta+1}; q)_{2k}},
\]

(3.10)

which coincides with the result in [19, corollary 3.2] [20, theorem 5.1].

### 3.4. Big \( q \)-Jacobi case

Big \( q \)-Jacobi polynomials were introduced by Andrews and Askey as an infinite-dimensional version of \( q \)-Hahn polynomials [4]. In addition, big \( q \)-Jacobi polynomials were also contained in the Bannai–Ito scheme of dual systems of orthogonal polynomials as an infinite dimension analogue of the \( q \)-Racah polynomials [6]. These polynomials take the hypergeometric function form

\[
P_n(x; a, b, c; q) = _3\phi_2 \left( q^{-n}, \frac{abq^{n+1}}{aq, cq}, x^1 \right| q; q)_\infty
\]

which are orthogonal with respect to the weight

\[
\rho^{(a,b,c)}(x; q) = \frac{(a^{-1}x, c^{-1}x; q)_\infty}{(x, bc^{-1}x; q)_\infty},
\]

and have the orthogonality

\[
\int_c^a P_n(x; a, b, c; q) P_m(x; a, b, c; q) \rho^{(a,b,c)}(x; q) d_q x
\]

\[
= aq(1 - q) \left( q, a^{-1}c, ac^{-1}q, abq; q \right)_\infty \left( 1 - abq \right) \left( aq, bq, cq, abc^{-1}q, q; q \right)_\infty \left( 1 - abq^{2n+1} \right)
\]

\[
\times \frac{(q, bq, abc^{-1}q; q)_n}{(aqbq, aq, cq; q)_n} (-acq^2)^n \delta_{nm}.
\]

One can easily check that the normalization constant for the monic polynomials is

\[
h_n = \frac{aq(1 - q)(-acq^2)^n q^{(3)}(q, a^{-1}c, ac^{-1}q, abq; q)_\infty (q, aq, bq, cq, abc^{-1}q; q)_n}{(1 - abq^{2n+1})(aqbq, aq, cq, abc^{-1}q; q)_\infty (aqbq, aq, cq, abc^{-1}q; q)_n^2}.
\]
and the canonical moments are given by [27]

\[
\mu_n^{(a,b,c)} = \int_{cq}^{aq} x^n \rho^{(a,b,c)}(x; q) d_q x
\]

\[
= aq(1 - q)^{\frac{(q, abq^2, a^{-1}c, ac^{-1}q; q)_{\infty}}{(aq, bq, cq, abc^{-1}q; q)_{\infty}}} \times \sum_{m=0}^{n} (-1)^m \frac{n!}{m! q^{-m+n+\left(\frac{m+1}{2}\right)}} \frac{(aq, cq; q)_m}{(aq^2, q)_m}
\]

Solving equation (3.3) gives the Pearson pair

\[
(f, g) = \left(1 - \frac{x}{aq}, 1 - \frac{x}{cq} \cdot \frac{q^2}{1 - q} \left(\frac{1}{acq^2} - \frac{b}{c} \right) x + \frac{b}{c} + 1 - \frac{1}{aq} - \frac{1}{cq}\right).
\]

Then, by comparing the leading coefficients on both sides of (3.8) we have

\[
c_n = \frac{abq^{2n+2} - 1}{ac(1 - q)q^{2n+2} h_{n+1}},
\]

which leads to the following evaluation of the moment Pfaffian

\[
\text{Pf}(m_{ij}^{(a,b,c)})_{i,j=0}^{2N-1} = c^{N(N-1)} a^{N(N+1)2} q^{-\frac{1}{2}N(N+1)(4N-1)} \times \prod_{k=0}^{N-1} (q, a^{-1}c, ac^{-1}q, abq; q)_{\infty}(aq, bq, cq, abc^{-1}q; q)_{2k+1}.
\]

where \(m_{ij}^{(a,b,c)}\) is given by

\[
m_{ij}^{(a,b,c)} = ([j]_q - [1]_q) \int_{cq}^{aq} x^{i-j-1} \omega^{(a,b,c)}(x; q) d_q x,
\]

and \(\omega^{(a,b,c)}(x; q)\) is expressed by

\[
\omega^{(a,b,c)}(x; q) = f(qx; q) \rho^{(a,b,c)}(qx; q)
\]

\[
= (1 - x)(1 - bc^{-1}x) \rho(x; q)^{a,b,c}
\]

\[
= \rho^{aq, abq, qe}(qx; q).
\]

Note that

\[
\mu_n^{(a,b,c)} = \int_{cq}^{aq} x^n \rho^{(a,b,c)}(x; q) d_q x = q^{n+1} \int_{c}^{a} x^n \rho^{(a,b,c)}(qx; q) d_q x,
\]

we have the expression
Hankel Pfaffians. For example, the formula (3.10) can be evaluated by the following:

\[ m_{i,j}^{(a,b,c)} = (\ell_q^{-a}) q^{-\ell_q^{-a}} q^{\ell_q^{-a} (abq^2, a^{-1}c; c, q)_{\infty}} \]

\[ = (\ell_q^{-a}) q^{-\ell_q^{-a}} q^{\ell_q^{-a} (abq^2, a^{-1}c; c, q)_{\infty}} \times \left( \sum_{m=0}^{i+j-1} (-1)^m \binom{i+j-1}{m} q^{m} \right) \]

Dividing both sides by \[ [aq(q, a, c^{-1}; q)] \] and changing variables \((a, b, c) \rightarrow (q^{-1} a, q^{-1} b, q^{-1} c)\), we get

\[ \text{Pr} \left( \begin{array}{c} i+j-1 \sum_{m=0}^{i+j-1} (-1)^m \binom{i+j-1}{m} q^{m} \end{array} \right) = (ac)^{N(N+1)} q^{\frac{1}{2} N(4N^2+3N-7)} \prod_{k=0}^{N-1} \frac{(q; q)_{2k+1} (q, q, c, c^{-1}; q)_{2k}}{(q^2; q)_{2k(N-2)}}. \]

**Remark 3.2.** With \( a = b = 1 \), big \( q \)-Jacobi polynomials reduce to the big \( q \)-Legendre polynomials [23]. In particular, we have the following evaluation of \( q \)-Catalan-Hankel Pfaffian

\[ \text{Pr} \left( \begin{array}{c} i+j-1 \sum_{m=0}^{i+j-1} (-1)^m \binom{i+j-1}{m} q^{m} \end{array} \right) = (c)^{N(N+1)} q^{\frac{1}{2} N(4N^2+3N-7)} \prod_{k=0}^{N-1} \frac{(q, q)_{2k+1} (q, q, c, c^{-1}; q)_{2k}}{(q^2; q)_{2k(N-2)}}. \]

### 4. Further remarks

In this paper, we have developed a method based on the relation between classical \((q\)-orthogonal and \(q\)-skew orthogonal polynomials to evaluate certain \(q\)-Catalan-Hankel Pfaffians whose entries are composed of the moments of classical orthogonal polynomials. Some examples are given to illustrate the approach including the continuous ones (e.g. Hermite, Laguerre, Jacobi and Cauchy) and discrete \(q\)-cases (e.g. Al-Salam & Carlitz I, Little \(q\)-Jacobi, Stieltjes–Wigert and big \(q\)-Jacobi polynomials). Among those examples, the Al-Salam & Carlitz I and Little \(q\)-Jacobi case are compared with the results obtained by Ishikawa and Zeng in [20] and we present alternative proofs of [20, equation (6.7) & theorem 5.2]. Besides, the examples in [20, conjecture 7.1] seem to be related to some discrete measure on the linear lattice. However, as mentioned in [15], the skew moments defined in linear lattices are of the form \( m_{i,j} = (j-i)\mu_{i+j-1} + \left( \frac{i}{2} - \frac{j}{2} \right) \mu_{i+j-2} + \cdots \), it is unclear to us whether it could be written in the Catalan-Hankel Pfaffian form.

As mentioned before, \(q\)-analogues of Selberg integral can be used to evaluate \(q\)-Catalan-Hankel Pfaffians. For example, the formula (3.10) can be evaluated by the following Askey–Habsieger–Kadell formula [20]
In this appendix, we present some properties of Pfaffian. Given an antisymmetric matrix 

\[ \text{Pf}(a_{i,j})_{i,j=0}^{2n-1} = \text{Pf}(1, 2, \ldots, 2n) \]

one can define a Pfaffian of the matrix

\[ \text{Pf}(a_{i,j})_{i,j=0}^{2n-1} := \text{Pf}(1, 2, \ldots, 2n) \]

where \( \sigma \) is any permutation of \( \{1, 2, \ldots, 2n\} \) and element \( a_{i,j} \) is denoted by \( \text{Pf}(i, j) \). Therefore, with regard to equation (2.3), \( \text{Pf}(0, 1, \ldots, 2n, x) \) is actually referred to the following expression [2]

\[ \begin{pmatrix}
0 & m_{0,1} & m_{0,2} & \ldots & m_{0,2n} & 1 \\
-m_{0,1} & 0 & m_{1,2} & \ldots & m_{1,2n} & x \\
-m_{0,2} & -m_{1,2} & 0 & \ldots & m_{2,2n} & x^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-m_{0,2n} & -m_{1,2n} & -m_{2,2n} & \ldots & 0 & x^{2n} \\
-1 & -x & -x^2 & \ldots & -x^{2n} & 0 
\end{pmatrix} \]
Moreover, Pfaffian has many interesting combinatoric and mathematical properties (see [28, 30]). Given a square matrix $A$, there holds (see [32, lemma 2.3])

\[
Pf\left( \begin{array}{cc} 0 & A \\ -A^\top & 0 \end{array} \right) = (-1)^{\frac{n(n-1)}{2}} \det(A).
\]

From this property, we could show that

\[
Pf(a_{i,j})_{2n-1 \leq i,j \leq 2n-1} = \det(a_{2i,2j+1})_{i,j \leq 0}, \quad \text{if } a_{i,j} = 0 \text{ when } i + j \text{ even}, \quad (A.1)
\]

by using row/column transformations. This formula is used to illustrate classic cases including $(q)$-Hermite and Cauchy case.

ORCID iDs

Shi-Hao Li https://orcid.org/0000-0003-2510-4079
Guo-Fu Yu https://orcid.org/0000-0003-4163-8353

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