NOTE ON THE $X_1$-LAGUERRE ORTHOGONAL POLYNOMIALS.

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Abstract. This note supplements the results in the paper on $X_1$-Laguerre orthogonal polynomials written by David Gómez-Ullate, Niky Kamran and Robert Milson.

1. Introduction

This note reports on, the $X_1$-Laguerre polynomials, one of the two new sets of orthogonal polynomials considered in the papers [3] and [1], written by David Gómez-Ullate, Niky Kamran and Robert Milson. The other set is named the $X_1$-Jacobi polynomials and is discussed, in similar terms, in the note [2].

These two papers are remarkable and invite comments on the results therein which have yielded new examples of Sturm-Liouville differential equations and their associated differential operators.

The two sets of these orthogonal polynomials are distinguished by:

(i) Each set of polynomials is of the form \( \{ P_n(x) : x \in \mathbb{R} \text{ and } n \in \mathbb{N} \equiv \{1, 2, 3, \ldots \} \} \) with \( \deg(P_n) = n \); that is there is no polynomial of degree 0.

(ii) Each set is orthogonal and complete in a weighted Hilbert function space.

(iii) Each set is generated as a set of eigenvectors from a self-adjoint Sturm-Liouville differential operator.

2. $X_1$-Laguerre polynomials

These polynomials and the associated differential equation are detailed in [3, Section 2]. In [3, Section 2, (21)] the second-order linear differential equation concerned is given as

\[
(2.1) \quad - xy''(x) + \left( \frac{x - k}{x + k} \right) ((x + k + 1)y'(x) - y(x)) = \lambda y(x) \text{ for all } x \in (0, \infty)
\]

where the parameter \( \lambda \in \mathbb{C} \) plays the role of a spectral parameter for the differential operators defined below, and the parameter \( k \in (0, \infty) \).

This equation (2.1) is not a Sturm-Liouville differential equation; such equations take the form, in this case taking the interval to be \((0, \infty)\),

\[
(2.2) \quad - (p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in (0, \infty),
\]

but can be transformed into this form on using the information in [3, Section 2]. In particular let the coefficients \( p_k, q_k, w_k \) be defined as follows;

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\[2000 \text{ Mathematics Subject Classification. Primary: 34B24; 34L05, 33C45: Secondary: 05E35, 34B30.}\]
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(i) $p_k, q_k, w_k : (0, \infty) \to \mathbb{R}$

(ii) $p_k(x) := \frac{x^k}{(x+k)^2} \exp(-x)$ for all $x \in (0, \infty)$

(iii) $q_k(x) := -\frac{(x-k)x^k}{(x+k)^3} \exp(-x)$ for all $x \in (0, \infty)$

(iv) $w_k(x) := \frac{x^k}{(x+k)^2} \exp(-x)$ for all $x \in (0, \infty)$.

Let the Sturm-Liouville differential expression $M_k$ have the domain

$$D(M_k) := \{ f : (0, \infty) \to \mathbb{C} : f^{(r)} \in AC_{\text{loc}}(0, \infty) \text{ for } r = 0, 1 \}$$

and be defined by, for all $f \in D(M_k)$,

$$M_k[f](x) := -(p_k(x)f'(x))' + q_k(x)f(x) \text{ for almost all } x \in (0, \infty).$$

Now define the Sturm-Liouville differential equation by, for all $k \in (0, \infty)$,

$$M_k[y](x) = \lambda w_k(x)y(x) \text{ for all } x \in (0, \infty)$$

where $\lambda \in \mathbb{C}$ is a complex valued spectral parameter.

For an account of Sturm-Liouville theory of differential operators and equations, see [1] Sections 2 to 6.

It is important to notice that the differential equation (2.8) is equivalent to, and is derived from the differential equation (2.1), see again [3] Section 2.2, (22a)).

The differential equation (2.8) is to be studied in the Hilbert function space $L^2((0, \infty); w_k)$.

The symplectic form for $M_k$ is defined by, for all $k \in (0, \infty)$ and for all $f, g \in D(M_k)$,

$$[f, g]_k(x) := f(x)(p_kg')(x) - (p_kf')(x)g(x) \text{ for all } x \in (0, \infty).$$

The maximal operator $T_{k,1}$ is defined by, for all $k \in (0, \infty)$,

$$T_{k,1} : D(T_{1,k}) \subset L^2((0, \infty); w_k) \to L^2((0, \infty); w_k)$$

$$D(T_{k,1}) := \{ f \in D(M_k) : f, w^{-1}M_k[f] \in L^2((0, \infty); w_k) \}$$

$$T_{k,1}f := w^{-1}M_k[f] \text{ for all } f \in D(T_{1,k}).$$

All self-adjoint differential operators in $L^2((0, \infty); w_k)$ generated by $M_k$ are given by restrictions of the maximal operator $T_{k,1}$; these restrictions are determined by placing boundary conditions at the endpoints 0 and $\infty$, on the elements of $D(T_{k,1}).$ The number and type of boundary conditions depends upon the endpoint classification of $M_k$ in $L^2((0, \infty); w_k)$; see [1] Section 5.

For the endpoint classification of the differential expression $M_k$ in $L^2((0, \infty); w_k)$ we have the results, see again [1] Section 5;
(i) At $0^+$ the classification is:

\[
\begin{array}{|c|c|}
\hline
k \in (0, 3] & \text{limit-circle non-oscillatory} \\
\hline
k \in (3, \infty) & \text{limit-point.} \\
\hline
\end{array}
\]

(ii) At $+\infty$ the classification is:

\[
\begin{array}{c}
\text{For all } k \in (0, \infty) \text{ limit point.}
\end{array}
\]

To establish these properties we have the following results:

1. For $\lambda = 0$ the function

\[
\varphi_1(x) := x + k + 1 \text{ for all } x \in [0, \infty),
\]

is a solution of the differential equation (2.8), for all $k \in (0, \infty)$; see [3, Section 2, (14)].

2. We have $\varphi_1 \in L^2((0, \infty); w_k)$ for all $k \in (0, \infty)$.

3. For $\lambda = 0$ the function

\[
\varphi_2(x) := \varphi_1(x) \int_1^x \frac{1}{\varphi_1^2(t)p_k(t)} \, dt \text{ for all } x \in (0, \infty),
\]

is a solution of the differential equation (2.8), for all $k \in (0, \infty)$; $\varphi_2$ is independent of $\varphi_1$.

4. Asymptotic analysis shows that

\[
\varphi_1 \in L^2((0, \infty); w_k) \text{ for all } k \in (0, \infty)
\]

and

\[
\varphi_2 \notin L^2((1, \infty); w_k) \text{ for all } k \in (0, \infty)
\]

\[
\varphi_2 \in L^2((0, 1]; w_k) \text{ for all } k \in (0, 3]
\]

\[
\varphi_2 \notin L^2((0, 1]; w_k) \text{ for all } k \in (3, \infty).
\]

The endpoint classifications (2.11) and (2.12) follow from the results items 1 to 4 above; see [1, Section 5].

We can now define the restriction $A_k$ of the maximal operator $T_{k,1}$, see (2.10), which is self-adjoint in the Hilbert function space $L^2((0, \infty); w_k)$, and which has the $X_1$-Laguerre polynomials as eigenvectors. To obtain this result it is essential:

(i) To apply the general theory of such restrictions as given in the Naimark text [5, Chapter V, Sections 17 and 18].

(ii) To apply the detailed results on the properties of the $X_1$-Laguerre polynomials given in [3, Section 2].

At any limit-point endpoint no boundary condition is required; at the limit-circle endpoint $0^+$ the boundary condition for any $f \in D(T_{k,1})$ takes the form

\[
\lim_{x \to 0^+} [f, \alpha_1 \varphi_1 + \alpha_2 \varphi_2](x) = 0,
\]

\[
\frac{1}{\varphi_1^2(t)p_k(t)} = \frac{1}{\varphi_1^2(t)} - \frac{p_k(t)}{\varphi_1^2(t)p_k(t)}
\]

\[
\cdot \frac{1}{p_k(t)} = \frac{1}{\varphi_1^2(t)} - \frac{1}{p_k(t)}
\]

\[
\alpha_1 \varphi_1 + \alpha_2 \varphi_2
\]

\[
\text{and}
\]

\[
\frac{1}{\varphi_1^2(t)p_k(t)} = \frac{1}{\varphi_1^2(t)} - \frac{p_k(t)}{\varphi_1^2(t)p_k(t)}
\]

\[
\cdot \frac{1}{p_k(t)} = \frac{1}{\varphi_1^2(t)} - \frac{1}{p_k(t)}
\]

\[
\alpha_1 \varphi_1 + \alpha_2 \varphi_2
\]
where \( \alpha_1, \alpha_2 \in \mathbb{R} \). Since the \( X_1 \)-Laguerre polynomials are to be in the domain of the operator \( A_k \) we take \( \alpha_1 = 1 \) and \( \alpha_2 = 0 \).

Thus the domain \( D(A_k) \) of our self-adjoint operator \( A_k \) restriction of the maximal operator \( T_k \) is defined as follows:

(i) For \( k \in (0, 3) \)

\[
D(A_k) := \{ f \in D(T_{k,1}) : \lim_{x \to 0^+} [f, \varphi_1](x) = 0 \}
\]

and

\[
A_k f := w_k^{-1} M_k[f] \text{ for all } f \in D(A_k).
\]

(ii) For \( k \in (3, \infty) \)

\[
D(A_k) := D(T_{k,1})
\]

and

\[
A_k f := w_k^{-1} M_k[f] \text{ for all } f \in D(A_k).
\]

The spectrum and eigenvectors of \( A_{\alpha,\beta} \) can be obtained from the results given in [3, Section 2]. The spectrum of \( A_{\alpha,\beta} \) contains the sequence \( \{ \lambda_n = n : n \in \mathbb{N}_0 \} \); the eigenvectors are given by \( \{ \hat{L}^{(k)}_{n+1} : n \in \mathbb{N}_0 \} \), the \( X_1 \)-Laguerre orthogonal polynomials.

**Remark 2.1.**

(i) The notation \( \lambda_n = n \) for all \( n \in \mathbb{N}_0 \) makes good comparison with the eigenvalue notation for the classical Laguerre polynomials; this sequence is independent of the parameter \( k \in (0, \infty) \).

(ii) We note that \( \hat{L}^{(k)}_{n+1} \) is a polynomial of degree \( n + 1 \) for all \( n \in \mathbb{N}_0 \) and all \( k \in (0, \infty) \).

(iii) Note that for \( k \in (0, 3) \), when the limit-circle condition holds at \( 0^+ \), it is essential to check that the polynomials \( \{ \hat{L}^{(k)}_{n+1} \} \) all satisfy the boundary condition at \( 0^+ \) as required in (2.18), i.e.

\[
\lim_{x \to 0^+} \left[ \hat{L}^{(k)}_{n+1}, \varphi_1 \right](x) = 0 \text{ for all } n \in \mathbb{N}_0.
\]

This result follows since, using (2.13),

\[
\left[ \hat{L}^{(k)}_{n+1}, \varphi_1 \right](x) = p_k(x) \left[ \hat{L}^{(k)}_{n+1}(x) \varphi_1'(x) - \hat{L}^{(k)'}_{n+1}(x) \varphi_1(x) \right] = \frac{x^k}{(x + k)^2} \exp(-x) \left[ \hat{L}^{(k)}_{n+1} - \hat{L}^{(k)'}_{n+1}(x + k + 1) \right] = O(x^k) \text{ as } x \to 0^+.
\]

It is shown in [3, Section 3, Proposition 3.3] that the sequence of polynomials

\[
\left\{ \hat{L}^{(k)}_{n+1} : n \in \mathbb{N}_0 \right\}
\]

is orthogonal and dense in the space \( L^2((0, \infty); w_k) \), for all \( k \in (0, \infty) \). This result implies that for all \( k \in (0, \infty) \) the spectrum of the operator \( A_k \) consists entirely of the sequence of eigenvalues \( \{ \lambda_n : n \in \mathbb{N}_0 \} \); from the spectral theorem for self-adjoint operators in Hilbert space it follows that no other point on the real line \( \mathbb{R} \) can belong to the spectrum of \( A_k \).
Remark 2.2. It is to be noted that whilst the Hilbert space theory as given in [1] and [5] provides a precise definition of the self-adjoint operator $A_k$, the information about the particular spectral properties of $A_k$ are to be deduced from the classical analysis results in [3]. Without these results it would be very difficult to deduce the spectral properties of the self-adjoint operator $A_k$, as defined above, in the Hilbert function space $L^2((0,\infty);w_k)$.

References

[1] W.N. Everitt. A catalogue of Sturm-Liouville differential equations. *Sturm-Liouville Theory, Past and Present*: Pages 271-331. (Birkhäuser Verlag, Basel: 2005; edited by W.O. Amrein, A.M. Hinz and D.B. Pearson.)

[2] W.N. Everitt. Note on the $W_1$-Jacobi orthogonal polynomials. (Submitted to arXiv [math-ph] 21 November 2008).

[3] D. Gómez-Ullate, N. Kamran and R. Milson. An extended class of orthogonal polynomials defined by a Sturm-Liouville problem. (arXiv:0807.3939v1 [math-ph] 24 July 2008).

[4] D. Gómez-Ullate, N. Kamran and R. Milson. An extension of Bochner’s problem: exceptional invariant subspaces. (arXiv:0805.3376v2 [math-ph] 24 July 2008).

[5] M.A. Naimark. *Linear differential operators*: Part II. (Ungar New York: 1968.)

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