Information Content in the Angular Power Spectrum of Weak Lensing: Wavelet Method

Hao-Ran Yu\textsuperscript{1,2}, Joachim Harnois-Déraps\textsuperscript{2,3}, Tong-Jie Zhang\textsuperscript{*1,4} and Ue-Li Pen\textsuperscript{2}

\textsuperscript{1}Department of Astronomy, Beijing Normal University, Beijing, 100875, P. R. China; tjzhang@bnu.edu.cn
\textsuperscript{2}Canadian Institute for Theoretical Astrophysics, University of Toronto, M5S 3H8, Ontario, Canada
\textsuperscript{3}Department of Physics, University of Toronto, M5S 1A7, Ontario, Canada
\textsuperscript{4}Center for High Energy Physics, Peking University, Beijing, 100871, P. R. China

ABSTRACT

We quantify the performance of a non-linear Wiener filter, constructed in wavelet space, at recovering some of the Fisher information that was lost in the weak lensing convergence field. The proposed method consists in a separation of the original field into the sum of a Gaussian and a non-Gaussian contribution. After filtering an ensemble of such fields, which are obtained from \(N\)-body simulations, we find that we can recapture about four times more Fisher information, an effect that can potentially improve by a significant amount the constraining power of weak lensing surveys on cosmological parameters, including the dark energy equation of state \(\omega\). We compare this performance with that of a logarithmic mapping and find that the wavelet method can recover up to three times more information.

Key words: cosmology: theory—dark matter—weak lensing—methods: statistical

1 INTRODUCTION

Mapping the mass distribution of matter in the universe has been a major challenge and focus of modern observational cosmology [Kaiser & Squires 1993; Bartelmann & Schneider 1999; Mellier 1999; Bartelmann & Schneider 2001; Refregier 2003; Massey et al. 2011; Huterer 2010]. Because the light to mass bias is rather uncertain, the only direct procedure to weigh the universe is by measuring the deflection of light caused by the presence of matter between the source and the observer. In particular, the statistics of gravitational lensing can serve as a powerful probe of the mass distribution of the universe (Van Waerbeke et al. 2001; Vafaei et al. 2010; Benjamin et al. 2007).

It was recently realized that weak lensing could also provide insight on dark energy via the measurement of the growth function [Huterer 2002; Albrecht et al. 2006; Huterer 2010], and an international effort was put into motion in order to measure weak lensing signal with unprecedented accuracy and resolution (LSST Science Collaborations et al. 2007). Alternatives techniques such as the redshift distance measurements of supernovae [Perlmutter et al. 1999], the detection of baryonic acoustic oscillations (BAO) in galaxy surveys [Percival et al. 2007; Eisenstein et al. 2005], the measurement of the growth factor from clusters [Voit 2003] and the detection of weak lensing signal [Hoekstra et al. 2006] have already set tight constraints on the dark energy equation of state \(\omega\) (Oguri et al. 2008), and the weak lensing contribution provides a complimentary approach as it is sensitive to different systematic uncertainties. The goal of the combined dark energy experiments is to maximize a collective “figure-of-merit”, which influenced design of most of the future experiments like LSST\textsuperscript{1} (LSST Science Collaborations et al. 2007), EUCLID\textsuperscript{2} (Beaulieu et al. 2010), JDEM\textsuperscript{3} (Gehrels 2010), CHIME\textsuperscript{4} (Peterson et al. 2006), SKA\textsuperscript{5} (Schilizzi 2007), DES\textsuperscript{6} (Dewdney et al. 2009), BOSS\textsuperscript{6} (Schlegel et al. 2009) and STARS\textsuperscript{7}.

It was soon realized in the BAO community that the constraining strength of these surveys depends directly on the amount of Fisher information [Fisher 1935; Teegmark et al. 1997], i.e. statistically independent Fourier modes contained in the measurements of the power spectrum bands, and one needs to maximize that information in order to minimize the uncertainty on cosmological parame-

---

\* E-mail: tjzhang@bnu.edu.cn

1 http://www.lsst.org/lsst/
2 http://www.congrex.nl/09c08/
3 http://science.nasa.gov/missions/jdem/
4 http://www.physics.ubc.ca/chime/
5 http://www.skatelescope.org/
6 http://cosmology.lbl.gov/BOSS/
7 http://pan-starrs.ifa.hawaii.edu/public/
Counting these Fourier modes is straightforward when the underlying density field is Gaussian, however departures from Gaussianity are to be expected, as arising from the non-linear gravitational collapse of the density field (Huterer 2002). In the theory of structure formation, large scales structures grow from an initially linear Gaussian random field, which progressively becomes non-linear through gravitational instabilities, starting from the smallest scales. Only the largest scales of the field remains intrinsically Gaussian, while the non-linear Fourier modes start to couple together (Zhang et al. 2003).

In the theory of structure formation, large scales structures grow from an initially linear Gaussian random field, which progressively becomes non-linear through gravitational instabilities, starting from the smallest scales. Only the largest scales of the field remains intrinsically Gaussian, while the non-linear Fourier modes start to couple together (Zhang et al. 2003).

Rimes & Hamilton (2005, 2006) first measured the amount of Fisher information about the power spectrum amplitude contained in the matter field, as a function of scale, from an ensemble of 400 N-body simulations. They found that in the largest scales, the Fisher information grows in a manner consistent with a Gaussian random field. However, they observed departures from Gaussianity in the trans-linear regime, in the form of an information plateau, followed by a second rise on much smaller scales. This was later interpreted in terms of the halo model as a transition between the two-halo and the one-halo terms (Neyrinck et al. 2006; Neyrinck & Szapudi 2007).

This loss of Fisher information is an undesired effect, in the sense that it is equivalent to a reduction of the survey effective volume, and many strategies have been proposed to recover some of the leaked cosmic information. Weinberg (1992) used a method called Gaussianization, which is a monotonic transformation of the smoothed galaxy distribution that reconstructs primordial density fluctuations. Running N-body simulations backwards in time (Goldberg & Spergel 2000), or density field reconstruction from linear theory (Eisenstein et al. 2007; Noh et al. 2004; Padmanabhan et al. 2004; Ngan et al. 2011) has also been successful at recovering parts of the lost information. More recently, it was found that a logarithmic transformation of density fields appears to be effective on trans-linear scales (Neyrinck et al. 2009; Neyrinck et al. 2011) have successfully recovered some of the Fisher information with a Gaussianization method that takes the Poisson noise into account. Non-linear Wiener filters, which can be designed to decompose a density field into Gaussian and non-Gaussian parts, are also among the best techniques found so far (Zhang et al. 2011). All of these techniques somehow attempt to diagonalize the covariance matrix of the matter power spectra by bringing back to the 2-point function some of the information that had leaked to higher order terms.

The next step was to measure the impact of these non-Gaussianities on cosmological parameters. It was first shown that their inclusion has only a minor impact on the constraining power about the BAO dilation scale, when both the power spectrum and the covariance matrix are obtained from N-body simulations (Takahashi et al. 2011). In current data analyses, however, the power spectrum is often estimated with techniques that assume Gaussianity in the matter field (Feldman et al. 1994; Vogeley & Szalay 1996), which have the unfortunate effect of producing sub-optimal estimates of the mean (Teegmark et al. 2006). In the case where the survey selection function is complex, the estimate is likely to be biased (Harnois-Déraps & Pen 2011). In regards with the sub-optimal measurement, it was recently shown that the error bars on the BAO dilation scale that are consistent with a sub-optimal measurement of the power spectrum might be significantly larger, compared to those obtained under standard Gaussian prescription (Ngan et al. 2011). In the era of precision cosmology, these few percent level effects need to be considered when constraining dark energy parameters.

In the pursuit of robustness and accuracy in weak lensing analyses, equal considerations must be granted to non-Gaussianities. Shear and convergence maps are indeed sensitive to the non-linear regime, typically at low and intermediate distances. For instance, it was recently shown that Fisher information – about the amplitude of the lensing power spectrum – contained in convergence fields was also departing from the information of a Gaussian field (Doré et al. 2009; Lu et al. 2010). It was furthermore, shown that these deviations could potentially impact the constraining power on the dark energy equation of state. The covariance matrix in the weak lensing angular power spectrum indeed shows strong correlations across different scales, as confirmed by Seo et al. 2011).

Since then, much efforts have been made to Gaussianize the lensing fields as well, in an attempt to recover some information and thus improve the precision on current and future measurements of cosmological parameters. Seo et al. 2011a have also shown that the method of logarithmic mapping is able to pump back some of the information lost. Such transformation was shown to suppress the non-Gaussian contribution in the statistics of higher order cumulants, and can also suppress the bispectrum (Yu et al. 2011). Joachimi et al. 2011 also showed that a Cox-Box transformation can also Gaussianize the fields and restore about the same amount of information lost in lensing fields, compared to logarithmic mapping. Those techniques basically reconstruct a probability distribution function (PDF) of the κ field which is much closer to that of a Gaussian (Seo et al. 2011a; Joachimi et al. 2011).

In the effort to Gaussianize the fields, non-linear Wiener filters are also a promising technique and work especially well with wavelet transforms which are well suited for extracting multiscalar information (Fang & Thews 1998). Moreover, they were found to offer better performances over the standard Fourier basis if the data are intermittent in nature (Percy 1999). They are proved to be successful in recovering Fisher information about the amplitude of the power spectrum of the dark matter field (Zhang et al. 2011), and could potentially outperform other Gaussianization techniques of weak lensing maps. In this paper, we thus construct a non-linear filtering method similar to that of Zhang et al. 2011 to Gaussianize the κ fields, and we compare the increase the Fisher information contained in simulated maps with other methods.

The outline of the paper is as follows: we first describe our strategy to construct weak lensing maps from N-body simulations in 3. We discuss in 3 the application of the discrete wavelet transform (DWT) and Bayesian theory to filter out the non-Gaussian component of these fields. In 4 we calculate the information content about the power spectrum in the Gaussianized κ fields. Discussion and conclusion are presented in 5.
2 SIMULATIONS AND CONSTRUCTION OF CONVERGENCE MAPS

We ran a series of 185 N-body simulations to generate convergence maps, using the fast parallel cubep3m (Merz et al. 2004), a particle-mesh Poisson solver that has sub-grid resolution, thanks to the calculation of the exact Newton force between particles. It was run with 1024\(^3\) particles on eight IBM nodes of the Tightly Coupled System on SciNet, each node being equipped with 32 cores, 128Gb of RAM, with an infiniband connection across nodes (Loken et al. 2010). We output a series of periodic surface density on 2048\(^2\) pixel maps, along the three orthogonal directions of the cube at each specified redshifts.

The simulations were designed to optimize the usage of the simulated comoving volume in the construction of the past light cone, which consist of a juxtaposition of cubes of 231.1Mpc/h per side for \(z > 1\), and of 147.0Mpc/h per side for lower redshifts. The light cone opening angle was set to 3.58 degrees, which exactly touches the edges of the small box at \(z = 1\), then propagates into the larger boxes until \(z = 2\), beyond which we used the periodicity of the simulations to populate the volumes. This creates repeated structures, which increase the correlation across different Fourier modes in the mass density at the percent level. But this has negligible impact on our results since the lensing kernel strongly suppresses the contributions from such high redshifts. At most, such increased correlation indeed accentuates the non-Gaussian features, but this effect gets strongly suppressed by the projection along the line of sight.

Simulations started at an initial redshift \(z_i = 40\) for the lower redshift boxes and 200 otherwise. The cosmological parameters used are \(\Omega_M = 0.279\), \(\Omega_k = 0.721\), with Hubble constant \(h = 0.701\), and we obtained a transfer function from CAMB with \(n_s = 0.96\). The power spectrum normalization was then specified to be \(\sigma_8 = 0.815\).

The convergence \(\kappa\) field is obtained from the projection of the matter overdensity \(\delta\) along the line of sight \(\theta\), weighted by the lensing geometry and, potentially, a source-galaxy distribution. It can be expressed as

\[
\kappa(\theta, \chi) = \int_0^\chi W' \delta(\chi', r(\chi') \theta) d\chi',
\]

where \(\chi\) is the comoving distance in unit of \(c/H_0\), and \(H_0 = 100 h \, \text{km s}^{-1} \, \text{Mpc}^{-1}\). The weight function \(W(\chi)\) is

\[
W(\chi) = \frac{3}{2} \Omega_M g(\chi)(1 + z)
\]

where \(g(\chi)\) is determined by the source galaxy distribution function \(n(z)\) and the lensing geometry:

\[
g(\chi) = r(\chi) \int_\chi^\infty n(\chi') \frac{r(\chi') - r(\chi)}{r(\chi')} d\chi'.
\]

Here, \(r(\chi)\) is the radial coordinate and is equal to \(\chi\) for the flat geometry we consider.

The density fields are converted into convergence maps by stacking the images with the appropriate geometrical weights \(W(z)\), through the comoving volume contained in the past light cone. This tiling method was developed in Seljak (1998) and assumes both the thin lens and Born approximations. At each lens plane, we choose randomly one of the \(x\)-, \(y\)- and \(z\)- directions, and we shift the centre of the plane. This effectively suppresses much of the correlation that exists across the lenses. We then interpolate the lens onto a pixel map of constant angular size. Since we chose a uniform galaxy distribution, and a source placed at \(z_s = 3.0\), most of the lensing contribution comes from \(z \sim 1.5\). One of these \(\kappa\) map is shown in the left panel of Fig. 1.

3 WAVELET NON-LINEAR WIENER FILTERING OF \(\kappa\) FIELDS

In this section, we first briefly review the properties of the wavelet transform that are relevant to our discussion, then we describe our non-linear Wiener filter, and finally we quickly present the log transform, which has been previously used to Gaussianize the \(\kappa\) fields and against which we compare our results in section 4.
3.1 Discrete Wavelet Transform (DWT)

Similar to the Fast Fourier Transform (FFT), the Discrete Wavelet Transform (DWT) is an invertible, linear operation, and can be considered as a rotation in function space. We consider in this paper the Daubechies-4 (hereafter DB-4) wavelet basis [Daubechies, 1992], which contains families of scaling functions $\phi$ and difference functions (or wavelet functions) $\psi$ that are orthogonal, continuous and have compact support. In a DWT decomposition, the data which, in our case, have 2D grid elements per dimension, are thus expanded into a combinations of these orthogonal basis, and weighted by wavelet coefficients $\epsilon$.

In two dimensions, each wavelet coefficient thus depend on two scale indices $(j_1, j_2)$ – controlling the dilation of the wavelet DB-4 functions – and two location indices $(l_1, l_2)$ – controlling its translation. On a given dimension, the grid scale corresponding to a specified dilation is $L/2^j$, where $L = 2048$ in our case. Similarly, the index $l$ corresponds to the spatial location comprised in the range $lL/2^j < x < (l+1)/2^j$. The 2-dimensional convergence field $\kappa(\hat{x}, x_2)$ can be expanded as

$$K(x_1, x_2) = \sum_{l_1=0}^{J-1} \sum_{l_2=0}^{J-1} \int_{l_1L/2^j}^{(l_1+1)L/2^j} \int_{l_2L/2^j}^{(l_2+1)L/2^j} \epsilon_{j_1, j_2, l_1, l_2} \phi_{j_1, l_1} (x_1) \phi_{j_2, l_2} (x_2) \phi_{0, l_2} (x_2) + \sum_{l_1=0}^{J-1} \sum_{l_2=0}^{J-1} \sum_{j_1=0}^{j_1-1} \sum_{j_2=0}^{j_2-1} \epsilon_{j_1, j_2, l_1, l_2}^{(1)} \psi_{j_1, l_1} (x_1) \phi_{j_2, l_2} (x_2) \phi_{0, l_2} (x_2) + \sum_{l_1=0}^{J-1} \sum_{l_2=0}^{J-1} \sum_{j_1=0}^{j_1-1} \sum_{j_2=0}^{j_1-1} \epsilon_{j_1, j_2, l_1, l_2}^{(2)} \psi_{j_1, l_1} (x_1) \psi_{j_2, l_2} (x_2) \phi_{0, l_2} (x_2) + \sum_{j_1=0}^{J-1} \sum_{l_1=0}^{j_1-1} \sum_{j_2=0}^{j_1-1} \sum_{l_2=0}^{j_2-1} \epsilon_{j_1, j_2, l_1, l_2}^{(3)} \psi_{j_1, l_1} (x_1) \psi_{j_2, l_2} (x_2) \phi_{0, l_2} (x_2). \quad (4)$$

where

$$\phi_{j,l}(x) = \sqrt{2^j} \phi(2^j x / L - l) \quad (5)$$

$$\psi_{j,l}(x) = \sqrt{2^j} \psi(2^j x / L - l). \quad (6)$$

With the combination of these two basis functions, scaling function coefficients (hereafter SFC’s) $\epsilon$’s and three kinds of wavelet function coefficients (hereafter WFC’s) $\epsilon$’s can be calculated by

$$\epsilon_{j_1, j_2, l_1, l_2} = \int \int K(x_1, x_2) \phi_{j_1, l_1} (x_1) \phi_{j_2, l_2} (x_2) dx_1 dx_2. \quad (7)$$

$$\epsilon_{j_1, j_2, l_1, l_2}^{(1)} = \int \int K(x_1, x_2) \psi_{j_1, l_1} (x_1) \phi_{j_2, l_2} (x_2) dx_1 dx_2. \quad (8)$$

$$\epsilon_{j_1, j_2, l_1, l_2}^{(2)} = \int \int K(x_1, x_2) \phi_{j_1, l_1} (x_1) \psi_{j_2, l_2} (x_2) dx_1 dx_2. \quad (9)$$

and

$$\epsilon_{j_1, j_2, l_1, l_2}^{(3)} = \int \int K(x_1, x_2) \psi_{j_1, l_1} (x_1) \psi_{j_2, l_2} (x_2) dx_1 dx_2. \quad (10)$$

For each simulation, the $\kappa$ field is thus wavelet transformed, and each of the four kinds of coefficients found in Eqs. (7)-(10) are stored in a 2-dimensional field, preserving the grid resolution (see [Fang & Thews, 1998; Press et al., 1992] for more details).

3.2 Non-linear Wiener Filtering

Our strategy to construct a non-linear Wiener filter relies on the fact that in wavelet basis, the non-Gaussianities are clearly characterized in the PDF of the WFCs $\epsilon_{j_1, j_2, l_1, l_2}$, which we obtained from Eq. (10). We thus construct our filter by splitting the wavelet transform of the original map, which we label $K$, into two components: a Gaussian $(G)$ and a non-Gaussian $(N)$ map. Namely, in wavelet space, we have

$$K = G + N. \quad (11)$$

Since wavelet transforms are linear operations, we can inverse wavelet transform the above equation and write, in real space,

$$K = \kappa_G + \kappa_N \quad (12)$$

where the original map ($\kappa$) is expressed as the sum over a Gaussian contribution (hereafter $\kappa_G$) and a non-Gaussianized contribution (hereafter $\kappa_N$). Our goal is thus to design a filter that concentrates most of the collapsed structure in $\kappa_N$, and thus produces $\kappa_G$ that are closer to linear theory. We perform this operation on our simulated maps, compute their power spectrum, construct a covariance matrix and measure the Fisher information of both components separately. Then we can finally compare our results with the unfiltered maps.

The algorithm we use to devise the filter is a 2-dimensional version of that presented in [Zhang et al., 2011], which we briefly describe here again for completeness. The filter acts on each wavelet mode, which is defined as a collection of all WFCs having the same two scale indices $(j_1, j_2)$. Let us focus on a given wavelet mode, or a pair of scale components, or dilations $(j_1, j_2)$, whose value at location $(l_1, l_2)$ is labeled $x$ for simplicity. We construct the PDF of the WFCs in this wavelet mode, $f(k)$, by looping over the other two indices $(l_1, l_2)$, and express this measurement as a convolution of the PDF of two components. The Gaussian contribution $G(x) = K(x) - N(x)$ has a Gaussian PDF by assumption. We label the PDF of the non-Gaussian contribution as $\Theta(n)$, and write

$$f(k) = \frac{1}{\sqrt{2\pi}} \int \Theta(n) \exp\left[-\frac{1}{2}(k - n)^2\right] dn \quad (13)$$

This description also assumes that all the measurements of $x$ are statistically independent. Thanks to Bayes’s theorem, we can then calculate the conditional probability $P(N|K) = P(K|N)P(N)/P(K)$. The posterior conditional expectation value is written as

$$\langle N|K = k \rangle = \frac{1}{\sqrt{2\pi} f(k)} \int \exp\left[-\frac{1}{2}(n - k)^2\right] \Theta(n) dn = K + \frac{1}{\sqrt{2\pi} f(k)} \int \exp\left[-\frac{1}{2}(n - k)^2\right] \Theta(n) dn = K + \langle \ln f(k) \rangle \quad (14)$$

The expectation value of the Gaussian distribution is simply recovered from Eq. (11). In other words, knowing $f(k)$ allows one to solve for both $G$ and $N$ given $K$.

When measuring the PDF of a given wavelet mode
$(j_1, j_2) = (7, 7), \quad \alpha = 1399.8508, \quad s = 0.071879$

$(j_1, j_2) = (5, 5), \quad \alpha = 58.2517, \quad s = 0.40664$

$(j_1, j_2) = (3, 3), \quad \alpha = 6.5073, \quad s = 1.6906$

$(j_1, j_2) = (2, 2), \quad \alpha = 2.2338, \quad s = 4.6318$

Figure 2. WFC PDFs (red dots) and their analytical fits by Eq. (15) (black solid lines). Their corresponding Gaussian PDFs, which have the same standard deviations with the central regions of the fitted PDFs, are also shown in blue dashed lines. The upper left, upper right, bottom left and bottom right panels show four different wavelet modes, from smaller (more non-Gaussian) to larger scales (more Gaussian), as indicated by the value of $(j_1, j_2)$'s in the titles. They are roughly corresponding to multipole moment $\ell \approx 9.1 \times 10^3$, $2.3 \times 10^3$, $5.7 \times 10^2$ and $2.8 \times 10^2$, respectively. The two parameters $\alpha$ and $s$ for Eq. (15) are also shown in the title of each panel.

$(j_1, j_2)$ in practice, we encounter the difficulty of poorly determined PDFs, and thus even worse are the filter functions. This condition is rooted in the scarceness of the WFCs (especially for larger scale wavelet modes, containing less WFCs) and in the instability of numerical differentiation. In order to resolve this issue, we construct $f(k)$ for each wavelet mode by looping over the spacial indices $(l_1, l_2)$, and also over all the realizations. We further improve the resolution of the PDF of each mode with a two parameter analytical function:

$$f_{PDF}(x) = \frac{1}{\sqrt{\pi s^4}} \frac{\Gamma\left(\frac{3}{2}s^2\right)}{\Gamma\left(\frac{3}{2}s^2 - \frac{1}{2}\right)} \left(s^2 - x^2\right)^{-\frac{3}{2}}$$

which follow the PDF curves (see Fig. 2), where the $\Gamma$’s are the usual Gamma functions. The two parameters $\alpha$ and $s$ are actually extracted via their relationship to the second moment $m_2$ and the fourth moment $m_4$ of the PDF:

$$\alpha = \frac{5m_4 - 9m_2^2}{2m_2m_4} \quad (16)$$

and

$$s = \sqrt{|m_4 - 3m_2^2|}. \quad (17)$$

Having measured the expectation value of both components, we can now loop back over all spatial indices $(l_1, l_2)$ and separate each coefficient into the two components, by applying the following (non-linear) Wiener filters:

$$w_g(x) = \frac{G/K}{(\ln f)'(k)} \quad (18)$$
which are functions of $K$ only, and, we recall, apply solely to the current wavelet mode $(j_1, j_2)$.

With the above parametrization in terms of $(\alpha, s)$, the Gaussian filter function can be expressed by combining Eq. (15) with Eq. (15):

$$w_{g}(x) = \left(1 + \frac{x^2}{s^2}\right)^{-1}. \quad (20)$$

Note that the final filter function depends only on $s$, which characterizes the extent of the departure from a Gaussian PDF, i.e., the greater the $s$, the smaller departure from Gaussian.

In the limit of a pure Gaussian PDF

$$f_{PDF}^G = N(0, \sigma^2) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{x^2}{2\sigma^2}}, \quad (21)$$

the second moment and the fourth moment are given by $m_2^G = \sigma^2$ and $m_4^G = 3\sigma^4$. From Eq. (16) we have $\alpha = \sigma^{-2}$ and $s \to +\infty$, which is understandable. In this case, from Eq. (15) the filter function $w_{g}(x)$ reduces to a constant unity function $w_{g}(x) = 1$. This is to be expected since for those Gaussian or nearly Gaussian distributed PDFs, we do not need to filter at all.

In Fig. 2 we select four wavelet modes, characterizing four different scales of the $\kappa$ fields, and plot the PDF and fitted PDF for each of their WFCs. In order to visualize better the non-Gaussianities of the PDF, we also plot for each panel a Gaussian PDF which fits the central region of the fitted PDF Eq. (15). As indicated by the dilations $(j_1, j_2)$ in each of the panel’s title, the corresponding scale increases when one looks from the upper left panel to the bottom right panel. The corresponding multipole moments for those four chosen $(j_1, j_2)$’s are roughly $\ell \simeq 9.1 \times 10^3$, $2.3 \times 10^4$, $5.7 \times 10^2$ and $2.8 \times 10^2$, respectively. One can also see that, as the scale goes up (or $\ell$ goes down), the parameter $s$ grows larger, and the PDF (black solid line) of that wavelet mode becomes more Gaussian, and closer to the Gaussian distribution (blue dashed line).

Note that the wavelet non-linear Wiener filter is a parameter-free method. In each wavelet mode, the filter is determined only by the 1-point PDF of all the WFCs. The method is stable in that, if we make little changes in data in real space, then all the WFC’s PDFs will have little change, but will not have much effect on the final fit, thus will not change the non-Gaussian decomposition. In contrast, if one adds in a lot of non-Gaussian features on certain scales, then the PDF of the WFCs, especially on that scale, will be more non-Gaussian. This results in a stronger filtering and those structures will be filtered out.

We finally repeat this process for all wavelet modes, constructing the filters $w_{g}$ and $w_{ng}$ and separate each $\kappa$ map into a Gaussian and non-Gaussian part. As an example, one decomposition among those 185 realizations is shown in the middle and right panel of Fig. 1.

### 3.3 Logarithmic Mapping of $\kappa$ Fields

In order to quantify the performance of our method, we wish to compare the recovery of Fisher information in our Gaussianized fields with that of a competing method. We follow the prescription of (Seo et al. 2011a), and “log-map” each of our $\kappa$ fields

$$ \kappa_{lin} \equiv \kappa_{0} \ln \left(1 + \frac{\kappa}{\kappa_{0}}\right), \quad (22)$$

where $\kappa_{0}$ is defined so as to keep the argument of logarithm positive, i.e. $\kappa_{0} < \min(\kappa)$, with $\min(\kappa)$ being the minimum value of the $\kappa$ field. This transformation is designed such that the non-linear peaks, who show strong convergence values, are attenuated. As a result, higher-order statistics become less important, which translates in a decrease of the non-Gaussian contribution. We can characterize the degree of alteration of the field by defining $r \equiv \kappa_{lin}/\min(\kappa)$, where $1 < r < \infty$. The smaller the $r$, the more the log-mapping alters the field. In order to draw general conclusions, we sample the $r$ space and try different values in the following sections. These log-transforms are also applied onto wavelet filtered $\kappa$ fields:

$$ \kappa_{lin+g} \equiv \kappa_{0} \ln \left(1 + \frac{\kappa_{g}}{\kappa_{0}}\right), \quad (23)$$

The results of both methods and of their combination are discussed in the next section.

### 4 ANALYSIS

In Fig. 1 we can see by eye that the peaks in the unfiltered $\kappa$ maps are mostly transferred into $\kappa_{ng}$, leaving $\kappa_{g}$ with much less structure. We also plot the real-space 1-point probability distribution function in Fig. 3. The result shows that the original convergence map has a very large skewness. We observe that the logarithmic mapping method have the best effect in recovering a Gaussian PDF, while the non-linear Wiener filtering mildly removes some of the skewness. This shows that the wavelet techniques is optimized for restoring...
4.1 Angular Power Spectra

We calculate for each map their dimensionless angular power spectra $C(\ell)$, which is defined as

$$C(\ell) \equiv \frac{\ell(\ell + 1)P(\ell)}{2\pi},$$

where $\ell$ is the multipole characterizes the scale in the two-dimensional $\kappa$-fields. We construct 20 $\ell$-bins and for each wavenumber $|\vec{\ell}|$, the power spectrum $P(\ell)$ is estimated by averaging all the Fourier modes within the bin, while $\ell \equiv \langle |\vec{\ell}| \rangle_{\text{bin}}$ is determined by averaging all $|\vec{\ell}|$ that fall into the bin. In Fig. 4 we plot the mean angular power spectra and error bars of $\kappa$, $\kappa_g$, and $\kappa_{ng}$ fields respectively. We can see that on linear scales, where $\ell \lesssim 10^3$, the Gaussianized power spectrum is nearly unchanged. On non-linear scales, however, it drops by a factor of two, while the error bars are reduced. Hence, in the presence of moderate noise, such a lowering of the power spectrum shall not affect significantly the detectability.

4.2 Cumulative Information

We measure the covariance matrix of the $C(\ell)$, which captures the correlation between the variance, or the error bars, of the power spectra at different scales $\ell$. If the measurements were completely uncorrelated, the diagonal of the covariance matrix would be the variance at each value of $\ell$, and all the off-diagonal entries would be zero. Mathematically, the covariance matrix $C$ is defined as

$$C(\ell, \ell') \equiv \frac{1}{N-1} \sum_{i=1}^{N} [C_i(\ell) - \langle C(\ell) \rangle][C_i(\ell') - \langle C(\ell') \rangle],$$

where $N$ is the number of realizations and $\langle C(\ell) \rangle$ is the mean angular power spectrum over all realizations. Note that the covariance matrix $C(\ell, \ell')$ is not to be confused with angular power spectrum $C(\ell)$.

The cross-correlation coefficient matrix, or for short the correlation matrix, is a normalized version of the covariance...
matrix, where each value is divided by the square root of the diagonal values as follows:

\[ \rho(\ell, \ell') = \frac{C(\ell, \ell')}{\sqrt{C(\ell, \ell)C(\ell', \ell')}} \]  

(26)

The three correlation matrices for our unfiltered, Gaussianized and non-Gaussianized \( \kappa \) fields are shown in the three panels in Fig.5. As expected, the original correlation matrix shows cross-correlation greater than 60% and up to 80% where \( \ell \gtrsim 10^2 \), while on larger scales, which is still in the linear regime, the matrix is diagonal. In the Gaussianized correlation matrix, the correlated region is much smaller, so the correlations between different Fourier modes are suppressed to some extent. In contrast, in the non-Gaussianized matrix, the cross-correlation is even higher and even spread into linear region.

The cumulative, or Fisher, information, is a measurement of the number of independent Fourier modes presented in a field up to a given scale, here \( \ell_n \). It is obtained as follows: for a given wavenumber \( \ell_n \), we select the subsection of the covariance matrix up to that scale, then we invert this sub-matrix and sum over all its elements. The results are meaningful when using a normalized covariance matrix, which is defined as

\[ C_{\text{norm}}(\ell, \ell') = \frac{C(\ell, \ell')}{\sqrt{C(\ell, \ell)C(\ell', \ell')}} \]  

(27)

and the function of cumulative information is

\[ I(< \ell_n) = \sum_{i,j=1}^{n} C_{\text{norm}}^{-1}(\ell_i, \ell_j). \]  

(28)

We plot the cumulative information contained in the unfiltered, Gaussianized and non-Gaussianized angular power spectra respectively in Fig[6]. We also over-plot the information curve obtained from \( \kappa_{\text{lin}} \) with \( r = 1.5 \) which seems to have the best result among log transforms, and the theoretical Gaussian information. In analogy with the Fisher information measured in the density field, \( \rho(\ell, \ell') \) also deviates from the Gaussian prediction towards \( \ell = 800 \). However, in the weak lensing case, the departure from Gaussian predictions is somehow attenuated, an effect of the angular projection across multiple scales.

Note that because we are using a finite number of simulations to measure several bins of data, even for a Gaussian random field, the inverse of the covariance matrix is biased \cite{Hartlap2007}. The biases are bin-dependent and are up to 10%, and would be applied to all the information curves, but we are interested only in the difference between these curves. For this reason we do not include this bias in all calculations.

We also note that our unfiltered Fisher information is somewhat lower than that presented by \cite{Dor2004}, but this apparent discrepancy is caused by a different choice of galaxy window function. \cite{Dor2009} opted for a series of top hat windows, extending from \( z \sim 1.0 \) to \( z \sim 3.0 \), while we include galaxy counts all the way down to \( z = 0 \). This effectively enhances the amount of non-linearities in our fields, thus reducing the Fisher information.

After performing the wavelet non-linear Wiener filtering and logarithmic mapping method, both the \( \kappa_g \) and \( \kappa_{\text{lin}} \) fields recovered information, however, the wavelet technique restores nearly 4 times more information than the logarithmic mapping for \( \ell > 10000 \). This performance gap gets even larger for \( r = 3.0 \), \( r = 2.0 \) or \( r = 1.05 \). We should also mention that wavelet method has the advantage to be a parameter-free method, in the sense that degrading the resolution by factors of two does not affect the WFC's PDF estimation of other scales. For the logarithmic method, however, one needs to re-optimize the value of \( r \) for different resolution, which affects the Fisher information at all scales. For example, there are likely coarser resolutions where the log transform would perform better (and worse) than the resolution shown here.

We also investigated whether these two methods could be combined together such that the total information recover exceeds that of the two methods separately and found that these two methods should not be used in conjuction. When we perform a logarithmic mapping after wavelet filtering, the recovered Fisher information is lower than for the wavelets alone. This is likely caused by the fact that a logarithmic mapping applied on a nearly Gaussian field transforms the field into something else, even less Gaussian. Because the wavelet transform is resource consuming, we have not tried the other way around, i.e. performing the logarithmic mapping first, which would involve a filtering for each sampling of \( \kappa_0 \). However, we have strong reasons to believe that the gain would also be minimal, if not worse than the wavelet on its own. As discussed before, we observe that the PDFs of the log-transformed fields, in both real and wavelet spaces, are nearly Gaussian. This means that the wavelet filtering would have a very small impact, as most of the contribution would directly fall in the Gaussian filter. As a result, the information that is recovered by the wavelet non-linear Wiener filter, but not by the log-transform, will not reappear with an additional Wiener filtering. However, other Gaussianization methods, especially those that attempt to go back in time – and thus only leave some non-Gaussian features – are likely to combine much better with the wavelet non-linear Wiener filters. For example, we are in the process of testing whether wavelet filtering can be used in conjuction with a density reconstruction algorithm to optimize the Fisher information.

In addition, it has been shown that the logarithmic mapping has very little gain in Fisher information when one includes realistic levels of Gaussian noise \cite{Seo2011}, which are inherent to weak lensing observations. Although we have not yet tested the performance of wavelet filtering in such a noisy environment, it has for sure the advantage of being able to extract a non-Gaussian component, which still contains most of the collapsed structures, and give us a novel handle in the data analysis.

The extension of this work, which we postpone to a future paper, is to propagate the uncertainty of the measured angular power spectrum onto that of cosmological parameters, following the Fisher formalism summarized in \cite{Albrecht2006}, and to compare the constraining performances of the covariance matrices from the unfiltered and from the Gaussianized fields to the simple Gaussian analysis. Quantifying the difference in uncertainty on the dark energy figure-of-merit is the final objective, and a significant improvement would lead to a enhanced yet robust precision on \( \omega \).
5 CONCLUSION

We ran a series of N-body simulations to generate weak lensing convergence maps. After analyzing their angular power spectra, as expected, we find that their cross correlation makes the Fisher information contained in the angular power spectra less than Gaussian information. The lost information can be obtained by higher statistics like bi-spectrum (three point correlation function), tri-spectrum etc., but to avoid dealing with tedious higher order calculations, we use two mathematical tools to Gaussianize $\kappa$ fields, namely a logarithmic mapping and non-linear wavelet Wiener filters.

The method of wavelet non-linear Wiener filtering and logarithmic mapping can both increase the Fisher information in the angular power spectra, but wavelet method has better results with a gain of a factor 4 at $\ell = 10000$. However, one needs to keep in mind that with the forecasted level of shot noise of future survey like EUCLID (Beaulieu et al. 2011) and JDEM (Gehrels 2011), the relative importance of non-Gaussianities will be reduced significantly (Doré et al. 2007). We also find that these two methods are not complementary, and that their combination reduces the information recovery compared to a sole Gaussianization of the fields.

ACKNOWLEDGEMENTS

We are very grateful to the anonymous referee for many valuable comments that greatly improved the paper. This work was supported by the National Science Foundation of China (Grants No. 11173006), the Ministry of Science and Technology National Basic Science program (project 973) under grant No. 2012CB821804, and the Fundamental Research Funds for the Central Universities. Computations were performed on the TCS supercomputer at the SciNet HPC Consortium. SciNet is funded by: the Canada Foundation for Innovation under the auspices of Compute Canada; the Government of Ontario; Ontario Research Fund - Research Excellence; and the University of Toronto. UP and JHD would like to acknowledge NSERC for their financial support.

References

Albrecht, A. et al. 2006, ArXiv Astrophysics e-prints, arXiv:astro-ph/0609591
Bartelmann, M., & Schneider, P. 1999, A&A, 345, 17, arXiv:astro-ph/9902152
——. 2001, Phys. Rep., 340, 291, arXiv:astro-ph/9912508
Beaulieu, J. P. et al. 2010, in Astronomical Society of the Pacific Conference Series, Vol. 430, Astronomical Society of the Pacific Conference Series, ed. V. Coudé Du Foresto, D. M. Gelino, & I. Ribas, 266+++, 1001.3349
Benjamin, J. et al. 2007, MNRAS, 381, 702, arXiv:astro-ph/0703570
Daubechies, I. 1992, Ten Lectures on Wavelets (C B M S - N S F Regional Conference Series in Applied Mathematics) (Soc for Industrial & Applied Math)
Dewdney, P. E., Hall, P. J., Schilizzi, R. T., & Lazio, T. J. L. W. 2009, IEEE Proceedings, 97, 1482
Doré, O., Lu, T., & Pen, U. 2009, ArXiv e-prints, 0905.0501
Eisenstein, D. J., Seo, H., Sirko, E., & Spergel, D. N. 2007, ApJ, 664, 675, arXiv:astro-ph/0604362
Eisenstein, D. J., et al. 2005, Astrophys. J., 633, 560, astro-ph/0501171
Fang, L.-Z. e., & Thews, R. L. e. 1998, Wavelets in Physics (Singapur: World Scientific)
Feldman, H. A., Kaiser, N., & Peacock, J. A. 1994, ApJ, 426, 23
Fisher, R. A. 1935, Journal of the Royal Statistical Society, 98, 39
Gehrels, N. 2010, ArXiv e-prints, 1008.4936
Goldberg, D. M., & Spergel, D. N. 2000, in Astronomical Society of the Pacific Conference Series, Vol. 201, Cosmic Flows Workshop, ed. S. Courteau & J. Willick, 282++, arXiv:astro-ph/9909057
Harois-Déraps, J., & Pen, U.-L. 2011, ArXiv e-prints, 1109.5746
Hartlap, J., Simon, P., & Schneider, P. 2007, A&A, 464, 399, arXiv:astro-ph/0608064
Hoekstra, H. et al. 2006, ApJ, 647, 116, arXiv:astro-ph/0511089
Huterer, D. 2002, Phys. Rev. D, 65, 063001, arXiv:astro-ph/0106399
——. 2010, General Relativity and Gravitation, 42, 2177, 1001.1758
Joachimi, B., Taylor, A. N., & Kiessling, A. 2011, ArXiv e-prints, 1104.1399
Kaiser, N., & Squires, G. 1993, ApJ, 404, 441
Loken, C. et al. 2010, Journal of Physics: Conference Series, 256
LSST Science Collaborations et al. 2009, ArXiv e-prints, 0912.0201

Figure 6. Cumulative information contained in the angular power spectra of the $\kappa$, $\kappa_g$, $\kappa_{og}$ and $\kappa_{in}$ fields respectively. The Gaussian (linear) information is shown with the dashed line.
