RANDOM WALKS ON GRAPHS: IDEAS, TECHNIQUES AND RESULTS

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Abstract

Random walks on graphs are widely used in all sciences to describe a great variety of phenomena where dynamical random processes are affected by topology. In recent years, relevant mathematical results have been obtained in this field, and new ideas have been introduced, which can be fruitfully extended to different areas and disciplines. Here we aim at giving a brief but comprehensive perspective of these progresses, with a particular emphasis on physical aspects.

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1 Introduction

A graph is the most general mathematical description of a set of elements connected pairwise by some kind of relation. Therefore, it is not surprising that graph theory has been successfully applied to a wide range of very different disciplines, from biology to social science, computing, psychology, economy, chemistry and physics.

In recent times, physicists have been mainly interested in graphs as models of complex systems, in condensed matter and in network theory. Indeed, these structures have proven to be very useful to describe inhomogeneous structures such as disordered materials, glasses, polymers, biomolecules as well as electric circuits, communication networks, statistical models of algorithms, and applications of statistical mechanics to different (non physical) systems.

The function of graphs in physics, however, is not purely descriptive. Geometry and topology have a deep influence on the physical properties of complex systems, where the presence of a large number of interacting degrees of freedom typically matters more than the interaction details. In fact, the specific interest of a physicist concerns the properties of a graph which most affect the dynamical and thermodynamical behaviour of the system it describes. On the other hand, the study of complex systems requires the introduction of statistical methods, to give an effective description of a number of quantities which, otherwise, would be too difficult to control.

Random walks are probably the simplest stochastic process affected by topology and, at the same time, the basic model of diffusion phenomena and non-deterministic motion. They have been extensively studied for decades on regular structures such as lattices, and most of the common wisdom concerning them relies on the results obtained in this particular geometry. The richer topology of a generic graph can have a dramatic effect on the properties of random walks, especially when considering infinite graphs, which are introduced to describe macroscopic systems in the thermodynamic limit. There, the asymptotic behaviour at long time typically exhibits universal features, only depending on large scale topology. On lattices, such features are known to be related to the Euclidean dimension only. On general graphs, universality allows to generalize
the concept of dimension to inhomogeneous structures, providing a very power-
ful tool to investigate a large class of different physical models, apparently not
connected to diffusion processes. On the other hand, a new and unexpected
phenomenon arises in presence of strong inhomogeneity, namely the splitting
between local and average properties. This provides a fundamental conceptual
framework to investigate complex systems even from an experimental point of
view.

Most results concerning random walks on graphs in physics have been obtained
in the last two decades and are scattered in a large number of technical papers.
This review is intended to provide the reader with a rigorous, self-contained
and up to date account of the present knowledge about this subject. Particu-
lar attention has been paid to give a simple and general framework effectively
resuming rather different results. As for specific calculations, we refer to bib-
liography, unless required by clarity reasons. The emphasis is always put on
the physical meaning. The reader more interested in formal aspects can find a
presentation focused on mathematics in another recent review [1].

The paper is organized as follows: In the first sections we give a brief mathem-
atical description of graphs and random walks, introducing the language and
the formalism we will use through the whole article. Then we present a simple
treatment of the finite graphs case, before dealing with infinite graphs. The
latter require the introduction of specific concepts, which are fully discussed in
an introductory section. Then, the asymptotic behaviour of random walks on
infinite graphs is studied and used to define the type problem and the spectral
dimension. The difference between local and average properties is evidenced in
the following sections. The concluding chapters are devoted to the analysis of a
large class of specific graphs and to the relations of random walks with different
physical problems.

2 Mathematical description of graphs

Let us begin by introducing the basic mathematical definitions and results con-
cerning graphs [2].

A graph $G$ is a countable set $V$ of vertices (or sites) $(i)$ connected pairwise by
a set $E$ of unoriented links (or bonds) $(i, j) = (j, i)$. If the set $V$ is finite, $G$
is called a finite graph and we will denote by $N$ the number of vertices of $G$.
Otherwise, when $V$ is infinite, $G$ is called an infinite graph. A subgraph $S$ of $G$
is a graph whose set of vertices $S \subseteq V$ and whose set of links $E' \subseteq E$.

A path $C_{i \rightarrow j}$ in $G$ connecting points $i$ and $j$ is a sequence of consecutive links
$\{(i, k)(k, h) \ldots (n, m)(m, j)\}$ and a graph is said to be connected, if for any two
points $i, j \in V$ there is always a path joining them. In the following we will
consider only connected graphs.

Every connected graph $G$ is endowed with an intrinsic metric generated by the
chemical distance $r_{ij}$ which is defined as the number of links in the shortest
path connecting vertices $i$ and $j$.

A particular class of graphs, often occurring in physical applications, is char-
acterized by the absence of closed paths containing an odd number of links. These graphs are called bipartite, since we can divide their sites into 2 sets $V_1$ and $V_2$ such that the points of $V_1$ are connected by a link only to points $V_2$ and vice versa. Square and hypercubic lattices are the most typical examples of bipartite graphs, as well as all trees (graphs without closed self-avoiding paths).

The graph topology can be algebraically represented introducing its adjacency matrix $A_{ij}$ given by:

$$A_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{if } (i, j) \notin E \end{cases}$$

(1)

The Laplacian matrix $\Delta_{ij}$ is defined by:

$$\Delta_{ij} = z_i \delta_{ij} - A_{ij}$$

(2)

where $z_i = \sum_j A_{ij}$, the number of nearest neighbours of $i$, is called the coordination number of site $i$.

In order to describe disordered structures we introduce a generalization of the adjacency matrix given by the ferromagnetic coupling matrix $J_{ij}$, with $J_{ij} \neq 0 \iff A_{ij} = 1$ and $\sup_{(i,j)} J_{ij} < \infty$, $\inf_{(i,j)} J_{ij} > 0$. One can then define the generalized Laplacian:

$$L_{ij} = I_i \delta_{ij} - J_{ij}$$

(3)

where $I_i = \sum_j J_{ij}$.

3 The random walk problem

Let us now introduce the so called simple random walk on a graph $G$. Assuming the time ($t$) to be discrete, we define at each time step $t$ the jumping probability $p_{ij}$ between nearest neighbour sites $i$ and $j$:

$$p_{ij} = \frac{A_{ij}}{z_i} = (Z^{-1} A)_{ij}$$

(4)

where $Z_{ij} = z_i \delta_{ij}$.

This is the simplest case we can consider: the jumping probabilities are isotropic at each point and they do not depend on time; in addition the walker is forced to jump at every time step. As we will see later, the last condition, i.e. the impossibility of staying on site, although crucial for the short time behaviour, has no significant influence on the long time regime.

Usually, the random walk problem is considered to be completely solvable if, for any $i, j \in G$ and $t \in N$, we are able to calculate the functions $P_{ij}(t)$, each representing the probability of being in site $j$ at time $t$ for a walker starting from site $i$ at time 0. These probabilities are the elements of a matrix $P = \ |P_{ij}(t)|$ which is equal to the $t$-th power of the jumping probabilities matrix $p = \ |p_{ij}|$:

$$P_{ij}(t) = (p^t)_{ij}.$$
The relation (5) can be easily proven by induction on \( t \). It also has an interesting physical interpretation as a sum over paths; developing the matrix products term by term we can write the whole expression as

\[
P_{ij}(t) = (p^t)_{ij} = \sum_{C_{i\rightarrow j}(t)} w(C_{i\rightarrow j}(t))
\]

where the sum is over all \( t \)-steps paths between \( i \) and \( j \). The weight \( w(C_{i\rightarrow j}) \) is the probability for the walker of going from \( i \) to \( j \) following exactly the path \( w(C_{i\rightarrow j}) \):

\[
w(C_{i\rightarrow j}(t)) = \prod_{(k,l) \in C_{i\rightarrow j}(t)} p_{kl}
\]

the product being over all the \( t \) links belonging to the path.

The calculation of all \( P_{ij}(t) \), which is straightforward as far as relatively small graphs are concerned, for large or infinite graphs becomes practically impossible and, above all, little significant. In fact, for large systems we are mainly interested in global and collective properties as it typically happens in statistical physics. Therefore, a small subset of all these quantities is usually chosen, together with some other related to them, which give an effective physical description of the random walker behaviour. The most relevant of them is from many points of view the probability \( P_{ii}(t) \) of returning to the starting point after \( t \) steps, also called the random walk autocorrelation function. As we will see, its asymptotic behaviour gives the most direct characterization of the large scale topology for infinite graphs. A related quantity is the average number \( P_{ii} \) of returns to the starting point \( i \), which can be generalized to the average number \( P_{ij} \) of passages through \( j \) starting from \( i \):

\[
P_{ij} \equiv \lim_{t \to \infty} \sum_{k=0}^{t} P_{ij}(k), \tag{8}
\]

where the limit can be infinite.

The mean displacement \( r_i(t) \) from the starting site \( i \) after \( t \) steps is deeply related to the diffusion properties and is defined as

\[
r_i(t) \equiv \sum_j r_{ij} P_{ij}(t) \tag{9}
\]

Notice that, unlike the case of random walks in continuous Euclidean space, here we consider \( r \) instead of \( r^2 \), the latter having no particular significance in absence of Euclidean metric.

The quantities introduced up to now are not "sensible to the history". Indeed, we can in principle determine all of them simply by considering the situation of the walker at time \( t \) regardless of his previous behaviour. In order to keep track of what happened before the instant \( t \), a different class of functions is introduced, starting with the first passage probability \( F_{ij}(t) \). The latter denotes the conditional probability for a walker starting from \( i \) of reaching for the first
time the site \( j \neq i \) in \( t \) steps. For \( i = j \) the previous definition would not be interesting, being the walker in \( i \) at \( t = 0 \) by definition. Therefore, one defines \( F_{ii}(t) \) to be the probability of returning to the starting point \( i \) for the first time after \( t \) steps and one sets \( F_{ii}(0) = 0 \). In spite of the deeply different nature of \( P \) and \( F \), a fundamental relation can be established between them if all time steps form 0 to \( t \) are taken into account (in other words, we have to give up the time locality):

\[
P_{ij}(t) = \sum_{k=0}^{t} F_{ij}(k)P_{jj}(t-k) + \delta_{ij} \delta_{t0}.
\]

(10)

This can be easily obtained by considering that each walker which is in \( j \) at time \( t \) only has two possibilities: either it gets there for the first time, or it has reached \( j \) for the first time at a previous time \( k \) and then it has returned there after \( t - k \) steps. The first passage probability is in turn connected to other meaningful history dependent quantities. The probability \( F_{ij} \) of ever reaching the site \( j \) starting from \( i \) (or of ever returning to \( i \), if \( i = j \)) is given by

\[
F_{ij} = \sum_{t=0}^{\infty} F_{ij}(t)
\]

(11)

By \( S_i(t) \) we denote the average number of different sites visited after \( t \) steps by a walker starting from \( i \). Its relation to \( F_{ij}(t) \) is

\[
S_i(t) = 1 + \sum_{k=1}^{t} \sum_{j} F_{ij}(k)
\]

(12)

Finally, the first passage time \( t_{ij} \), i.e. the average time at which a walker starting from \( i \), and passing at least once through \( j \), reaches \( j \) for the first time (or returns for the first time to \( i \), if \( i = j \)) is

\[
t_{ij} = \lim_{t \to \infty} \frac{\sum_{k=0}^{t} kF_{ij}(k)}{F_{ij}}
\]

(13)

The simple random walk can be modified to give a richer behaviour and to describe more general physical problems. Indeed, one can introduce anisotropic jumping probabilities by substituting in (4) the adjacency matrix with a ferromagnetic coupling matrix:

\[
p_{ij} = \frac{J_{ij}}{I_i} = (I^{-1}J)_{ij}
\]

(14)

where \( I_i = I_i \delta_{ij} \) and \( I_i = \sum_k J_{ik} \). Depending on the specific properties of \( J_{ij} \), this can produce only local effects or introduce a global bias which destroys the leading diffusive behaviour giving rise to transport phenomena. Moreover, one can relax the constraint of jumping at each time step by introducing waiting and traps on the sites. The jumping probabilities are then modified to:

\[
p_{i,j} = \frac{J_{i,j} + w_i \delta_{i,j}}{I_i + w_i + d_i}
\]

(15)
where both \( w_i \) and \( d_i \) are real positive numbers. From (15), \( w_i/(I_i + w_i + d_i) \) is the probability for the walker to stay on site \( i \) instead of jumping away and \( d_i/(I_i + w_i + d_i) \) the probability of disappearing (or dying, or being trapped forever) at site \( i \). As we will see later, waiting only affects the short time behaviour, while traps can dramatically modify also the long time asymptotic properties.

4 The generating functions

Even if we consider only the few fundamental quantities devised at the end of the last section, their direct calculation can be in practice a hard or impossible task on general graphs. However, a powerful indirect mathematical technique exists allowing overcoming a series of typical difficulties: this is the discrete Laplace transform, which maps a time function into its generating function. The generating function \( \tilde{f}(\lambda) \) of \( f(t) \) is defined by:

\[
\tilde{f}(\lambda) = \sum_{t=0}^{\infty} \lambda^t f(t) \quad (16)
\]

where \( \lambda \) is a complex number. The inverse equation giving \( f(t) \) from \( \tilde{f}(\lambda) \) is

\[
f(t) = \frac{\partial^t \tilde{f}(\lambda)}{\partial \lambda^t} \bigg|_{\lambda=0} . \quad (17)
\]

This equation is useful as fare as we are interested in small \( t \) behaviour, but it becomes absolutely ineffectual in the study of asymptotic regimes for \( t \to \infty \). In this case a very powerful tool is provided by the Tauberian theorems, relating the singularities of \( \tilde{f}(\lambda) \) to the leading large \( t \) behaviour of \( f(t) \). We give here a rather general Tauberian theorem, which is particularly useful when dealing with random walks. The main assumption we make concerns the analytical form of the leading singularity: we only consider power laws and logarithmic behaviours, since all cases discussed in this paper as well as all physically meaningful cases belong to this class. Suppose that \( \tilde{f}(\lambda) \) has its singularity nearest to \( \lambda = 0 \) in \( \lambda = 1. \) and that \( \tilde{f}(1-\epsilon) \), for \( \epsilon \to 0^+ \) goes as

\[
\tilde{f}(1-\epsilon) \sim h(\epsilon) + \text{const} \prod_{i=0}^{\infty} (i \ln(1/\epsilon))^{\alpha(i)} \quad (18)
\]

where \( ^i \ln x \equiv \ln^{i-1} \ln x \), with \( ^0 \ln x \equiv x \) and \( h(\epsilon) \) is finite for \( \epsilon \to 0^+ \).

Then, for \( t \to \infty \)

\[
f(t) \sim \text{const}' r^{-t} \prod_{i=0}^{\infty} i \ln^{\beta(i)}(t) \quad (19)
\]

where \( \beta(i) \) are related to \( \alpha(i) \) by

\[
\beta(i) = \begin{cases} 
\alpha(0) - 1 & \text{for } i = 0 \\
\theta(i-m)(\alpha(i) + 1) - 1 - \delta_{i,m} I(d/2) & \text{otherwise}
\end{cases}
\quad (20)
\]

\[
\theta(i-m)(\alpha(i) + 1) - 1 - \delta_{i,m} I(d/2)
\]

\[
\delta_{i,m} I(d/2)
\]
where
\[ m = \min \{ i \geq 0 | \beta(i) \neq -1 \} \]  \hspace{1cm} (21)
and
\[ I \left( \frac{\bar{d}}{2} \right) = \begin{cases} 1 & \text{if } \frac{\bar{d}}{2} \text{ is integer} \\ 0 & \text{otherwise} \end{cases} \]  \hspace{1cm} (22)

The constant \( const' \) is in general a function of \( const \) and of all the exponents appearing in the previous formulas. We don’t give here its rather involved explicit expression, since it is not relevant to the purposes of this paper.

The generating functions are usually easier to calculate, since they allow exploiting some peculiar properties of random walks functions. Moreover, a series of relevant random walk parameters which are non-local in time, can be obtained directly from generating function, without calculating the corresponding time dependent quantities. A good example is given by \( P_{ij}, F_{ij}, \) and \( t_{ij} \) which are related to \( \tilde{P}_{ij}(\lambda) \) and \( \tilde{F}_{ij}(\lambda) \) by

\[ P_{ij} = \lim_{\lambda \to 1} \tilde{P}_{ij}(\lambda) \]  \hspace{1cm} (23)
\[ F_{ij} = \tilde{F}_{ij}(1) \]  \hspace{1cm} (24)
\[ t_{ij} = \lim_{\lambda \to 1} \frac{\partial \log \tilde{F}_{ij}(\lambda)}{\partial \lambda} \]  \hspace{1cm} (25)

The basic property of random walks generating functions arise from the deconvolution of eq.(10), which after some straightforward steps becomes

\[ \tilde{P}_{ij}(\lambda) \tilde{F}_{ij}(\lambda) \tilde{P}_{jj}(\lambda) + \delta_{ij} \]  \hspace{1cm} (26)

In other words, the relation which was non-local in time becomes local in \( \lambda \). As we will see in practical application, many iteration techniques for the analytical calculation of generating functions are based on this property.

5 Random walks on finite graphs

Finite graphs consist of a finite number of sites and links. In principle, every physical structure is composed of a finite number of elements, but it is well known that a series of behaviours occurring in macroscopic systems are better described in the thermodynamic limit. Indeed, the typical singularities and power laws characterizing phase transitions and asymptotic regimes, such as large scale, long times, low temperature and low frequency behaviours, can only be found on infinite graphs.

However, finite graphs are appropriate when dealing with mesoscopic structures and finite size effects. The random walk problem on finite graphs is simplified by the finiteness of the adjacency matrix. In fact, the analytical study is reducible to a spectral problem on a real finite-dimensional vector space, and numerical simulations are easily implemented by Monte Carlo techniques.
Let us first consider the case of random walk without traps, whose jumping probabilities are given by (15) with $d_i = 0 \quad \forall \quad i$. The matrix elements $p_{ij}$ satisfy the relations

\[ p_{ij} \geq 0 \quad \forall \quad i, j \quad (27) \]

\[ \sum_{j=1}^{N} p_{ij} = 1 \quad \forall \quad i \quad (28) \]

defining a stochastic matrix.

The stochastic matrices we are considering exhibits different properties according to the some general features of the graph and of the jumping probabilities [3]. We distinguish two cases:

1. \textbf{c1} If $G$ is not bipartite, or if it has a staying probability on at least one site, then it has only one eigenvalue $p_{\text{max}}$ with maximal modulus and $p_{\text{max}} = 1$. Moreover, the eigenvector corresponding to $p_{\text{max}}$ has the same entry on each site (usually one chooses $v_{\text{max}} = (1, 1, 1, \ldots, 1)$ for simplicity)

2. \textbf{c2} If $G$ is bipartite without staying probabilities, then the spectrum of $p$ is symmetric with respect to the origin of the complex plane. Therefore, in addition to $p_{\text{max}}$ it has a second maximal modulus eigenvalue $p_{\text{min}} = -1$. The eigenvector $v_{\text{max}}$ has the same properties of the previous case, while $v_{\text{min}}$ has all entries on $V_1$ equal to the same number $v$ and all entries on $V_2$ equal to $-v$ (usually one chooses $v = 1$)

In case c1, one can easily show that the random walk is \textit{ergodic}, i.e. that it admits limit probabilities for $t \to \infty$:

\[ P_{ij}^\infty = \lim_{t \to \infty} P_{ij}(t) \quad \forall \quad i, j \quad (29) \]

and that

\[ P_{ij}^\infty = \frac{1}{N} \quad \forall \quad i, j. \quad (30) \]

This means that, independently of the initial conditions, the asymptotic probabilities are the same over all the graph sites. Moreover, this uniform limit value is reached exponentially and the exponential decay of each matrix element is no slower than $p_{\text{max}}^2$, $p_2$ being the second greatest eigenvalue of $p_{ij}$.

In case c2, the random walk is not ergodic. In particular, we have

\[ P_{ij}(t) = 0 \quad (31) \]

for all $t$ such that $t - r_{ij}$ is odd. On the other hand, considering for each couple of sites $i$ and $j$ only the values $t'_{ij}$ of $t$ having the same parity as $r_{ij}$, one can show that

\[ P_{ij}^\infty = \lim_{t'_{ij} \to \infty} P_{ij}(t'_{ij}) \quad \forall \quad i, j \quad (32) \]

with

\[ P_{ij}^\infty = \frac{2}{N} \quad \forall \quad i, j \quad (33) \]
and the limit is reached exponentially as in case 1.
Similarly, one can easily prove that
\[ \lim_{t \to \infty} F_{ij}(t) = 0 \quad \forall \ i, j \quad (34) \]

and
\[ \lim_{t \to \infty} S_i(t) = N \quad \forall \ i \quad (35) \]
the limit values being reached exponentially.
Moreover,
\[ P_{ij} = \infty \quad \forall \ i, j \quad (36) \]
\[ F_{ij} = 1 \quad \forall \ i, j \quad (37) \]
and
\[ t_{ij} < \infty \quad \forall \ i, j \quad (38) \]

The introduction of at least one trap, setting \( d_i > 0 \) for at least one site \( i \) in (15), dramatically changes the random walk behaviour. The jumping probabilities matrix \( p \) is no longer stochastic, since condition (28) is not satisfied. However, condition (27) (i.e. non-negativity) still holds, implying relevant properties. We can still distinguish between case 1 and 2, but the corresponding properties are modified as follows:

1. If \( G \) is not bipartite, or if it has a staying probability on at least one site, then it has only one eigenvalue \( p_{\text{max}} \) with maximal modulus and \( p_{\text{max}} < 1 \). Moreover, the entries \( v_{\text{max}i} \) of the eigenvector \( v_{\text{max}} \), corresponding to \( p_{\text{max}} \), have the same sign, and \( v_{\text{max}} \) is the only eigenvector having such a property.

2. If \( G \) is bipartite without staying probabilities, then the spectrum of \( p \) is symmetric with respect to the origin of the complex plane. Therefore, in addition to \( p_{\text{max}} < 1 \) it has a second maximal modulus eigenvalue \( p_{\text{min}} = -p_{\text{max}} \). The eigenvector \( v_{\text{max}} \) has the same properties of the previous case, while \( v_{\text{min}} \) can be chosen in such a way that all its entries on \( V_1 \) are equal to \( v_{\text{max}i} \) and all its entries on \( V_2 \) equal to \(-v_{\text{max}j}\).

In both cases the random walk is ergodic and the limit probabilities vanish:
\[ P_{ij}^\infty = 0 \quad \forall \ i, j. \quad (39) \]

However, in case 2 the time parity still has to be taken into account and holds. Moreover, the asymptotic decay is exponential and no slower than \( p_{\text{max}}^t \).

Finally, as for the other random walk functions we get
\[ \lim_{t \to \infty} F_{ij}(t) = 0 \quad \forall \ i, j \quad (40) \]
\[ \lim_{t \to \infty} S_i(t) < N \quad \forall \ i \quad (41) \]
the limit values being reached exponentially,

\[ P_{ij} < \infty \quad \forall \quad i, j \] (42)

\[ F_{ij} < 1 \quad \forall \quad i, j \] (43)

\[ t_{ij} < \infty \quad \forall \quad i, j \] (44)

6 Infinite graphs

When dealing with macroscopic systems, composed of a very large number \( N \) of sites, one usually takes the thermodynamic limit \( N \to \infty \). This means that we have to consider infinite graphs, i.e. graphs composed by an infinite number of sites. This is particularly convenient for two main reasons:

- first of all, a single infinite structure effectively describes a very large (infinite, indeed) number of large structures having different sizes, but similar geometrical features

- the singularities in thermodynamic potentials typical of critical phenomena as well as a series of universal asymptotic behaviours only occur on infinite structures.

As for random walks on large real structures, the time dependence of physical quantities exhibits different features according to the time scale which is considered. For very long times, the walker can explore every site and its behaviour is described by the finite graphs laws introduced in the previous section. However, if the time is long enough to explore large portions of the system, but still too short to experience the finite size effects, many significant quantities are quite insensitive to local details and exhibit power law time dependence with universal exponents. Often, this is the most interesting regime in physical applications. On infinite graphs, this is the true asymptotic regime even for very large times; therefore, we can reproduce the universal behaviours of a huge variety of finite large structures simply by considering infinite graphs with similar topological features.

To deal with infinite graphs, some further mathematics has to be introduced. In particular, we need tools to "explore" large scale topology. To this purpose, we define the Generalized Van Hove Spheres (GVHS): a GVHS \( S_{o,r} \) of centre \( o \) and radius \( r \) is the subgraph of \( G \), given by the set of vertices \( V_{o,r} = \{ i \in V | r_{i,o} \leq r \} \) and by the set of links \( E_{o,r} = \{ (i,j) \in E | i \in V_{o,r}, j \in V_{o,r} \} \).

Let us use \( |S| \) to denote the number of elements of a set \( S \). Then \( |V_{o,r}| \), as a function of the distance \( r \), describes the growth rate of the graph at the
large scales \[4\]. In particular, a graph is said to have a polynomial growth if 
\[\forall o \in V_X \exists c, k, \text{ such that } |V_{o,r}| < c r^k.\] (45)
For a graph satisfying (45), we define the upper growth exponent \(d_g^+\) and the lower growth exponent \(d_g^-\) as
\[d_g^+ = \inf\{k | |V_{o,r}| < c_1 r^k, \forall o \in V\}\] (46)
and
\[d_g^- = \sup\{k | |V_{o,r}| > c_2 r^k, \forall o \in V\}.\] (47)
If \(d_g^+ = d_g^-\), which usually happens on physically interesting structures, we call them the growth exponent \(d_g\), or the connectivity dimension.
The connectivity dimension \(d_g\) is known for a large class of graphs: on lattices, it coincides with the usual Euclidean dimension \(d\), and for many fractals it has been exactly evaluated \[5\]. In general, we can think of it as the analogous of the fractal dimension, when the chemical distance metric is considered instead of the usual Euclidean metric.

Infinite graphs are too general to describe systems of physical interest. Indeed, the discrete structures usually studied in physics are characterized by some important properties, often implicitly assumed in literature, which can be translated in mathematical requirements:

A We consider only connected graphs, since any physical model on disconnected structures can be reduced to the separate study of the models defined on each connected component and hence to the case of connected graphs.

B Since physical interactions are always bounded, the coordination numbers \(z_i\), representing the number of neighbours interacting with the site \(i\), have to be bounded; i.e. \(\exists z_{max} \ | \ z_i \leq z_{max} \forall i \in V\).

C Real systems are always embedded in finite dimensional spaces. This constraint requires for the graph \(\mathcal{G}\) the conditions:

(a) \(\mathcal{G}\) has a polynomial growth (Definition \[4\])
(b) \[\lim_{r \to \infty} \frac{|\partial V_{o,r}|}{|V_{o,r}|} = 0\] (48)
where \(\partial V_{o,r}\) denotes the border of \(V_{o,r}\), i.e. the set of points of \(V_{o,r}\) not belonging to \(V_{o,r-1}\) (the existence itself of the limit is a physical requirement on \(\mathcal{G}\)). This condition is equivalent to require that boundary conditions are negligible in the thermodynamic limit.

Notice that some graphs studied in physical literature, such as the Bethe lattice, do not satisfy (a) and (b), while many random graphs do not fulfil B.
For a large class of physically interesting graphs we have considered so far, conditions (a) and (b) appear to be equivalent. However for the equivalence of the two conditions a rigorous result is still lacking. A graph satisfying A, B and C will be called physical graph. Conditions A and B represent strong constraints on G and, as we will see later, they have very important consequences.

7 Random walks on infinite graphs

Considering random walks on infinite structures, some further mathematical constraints are to be introduced to describe physical situations. First of all, the problem of uniform boundedness comes into play. Indeed, in (14) the conditions

\[ \exists J_{\text{min}}, J_{\text{max}} > 0 \mid J_{\text{min}} \leq J_{i,j} \leq J_{\text{max}} \quad \forall i, j \] (49)

together with B are usually required to exclude the presence of a global bias, which would generate a non-diffusive behaviour.

Moreover, in (15), in presence of waiting and traps, analogous considerations lead to the following conditions

\[ \exists w_{\text{min}}, w_{\text{max}} > 0 \mid \text{either } w_i = 0, \text{ or } w_{\text{min}} \leq w_i \leq w_{\text{max}} \quad \forall i \] (50)

\[ \exists d_{\text{min}}, d_{\text{max}} > 0 \mid \text{either } d_i = 0, \text{ or } d_{\text{min}} \leq d_i \leq d_{\text{max}} \quad \forall i \] (51)

In the case of finite graphs, the possibility of associating to any matrix an operator acting on a finite dimensional vector space allowed to obtain very general and rigorous results. In the infinite case, it is in general impossible to associate a linear operator acting on a Hilbert space to any matrix. However, when B holds, the jumping probabilities matrix is quite particular: indeed, it only has a limited number of non-vanishing entries in each row and column. Due to this property, the elements of a matrix product are given by finite sums, as in the finite graphs case, instead of being sums of series. Therefore, the typical convergence problems of infinite dimension space do not arise, allowing for a simple and effective study of random walks properties.

Despite the increased mathematical complexity, many general results about infinite graphs have been rigorously proven. Some have correspondents in the finite graphs case, but most of them concern quantities and properties which cannot be even defined on finite structures. The rest of this section is devoted to the former: following the same format used in section 5 we resume the main differences with respect to the finite case. The new properties arising in the thermodynamic limit will be discussed in the following sections.

First of all, let us consider random walks without traps on infinite graphs satisfying A, B, and (49) and (50). It can be shown that

\[ P_{ij}^\infty = \lim_{t \to \infty} P_{ij}(t) = 0 \quad \forall \quad i, j \] (52)
even for bipartite graphs. However, for bipartite graphs without staying probabilities, we still have
\[ P_{ij}(t) = 0 \quad (53) \]
for all \( t \) such that \( t - r_{ij} \) is odd. Therefore, to study the large times asymptotic behaviours, as in the finite case we usually consider, for any given couple of sites \( i \) and \( j \), only the values \( t'_{ij} \) of \( t \) having the same parity as \( r_{ij} \). Unlike the finite case, the limit in (52) in general is not reached exponentially. Indeed, if \( C \) also holds, i.e. for physical graphs, the asymptotic behaviour is typically a power law, whose exponent only depends on topology, as we will discuss in details in the next sections. Notice that the widely studied case of Bethe lattices, not satisfying \( C \), is still characterized by an exponential decay.

Similarly, one can easily prove that
\[ \lim_{t \to \infty} F_{ij}(t) = 0 \quad \forall \ i, j \quad (54) \]
and
\[ \lim_{t \to \infty} S_i(t) = \infty \quad \forall \ i \quad (55) \]
the asymptotic behaviour being always bounded from above by \( t \).

As for the quantities concerning the number of visits and the first visit probabilities, the situation is far more complex. Indeed, dramatically different behaviours can occur, according to the graph topology. In particular,
\[ P_{ij} = \infty \quad \forall \ i, j \quad \text{or} \quad < \infty \quad \forall \ i, j \quad (56) \]
\[ F_{ij} = 1 \quad \forall \ i, j \quad \text{or} \quad < 1 \quad \forall \ i, j \quad (57) \]
and
\[ t_{ij} < \infty \quad \forall \ i, j \quad \text{or} \quad = \infty \quad \forall \ i, j \quad (58) \]

The classification of infinite graphs according to these possible behaviours is the subject of the next section.

8 Recurrence and transience: the type problem

On finite graphs, in absence of traps, the probability of ever reaching (or returning to) a site, \( F_{ij} \), is always 1. This means that the walker surely visits each site. This probability can be lowered only by adding traps, but in this case the total probability is not conserved, i.e. the walker asymptotically disappears. On infinite graphs, a third possibility arises, which is expressed in (56) and (57): the walker can escape forever from its starting point, or never reach a given site, even in absence of traps.

This phenomenon was first noticed by Polya in 1921 on lattices: he showed that, while in 1 and 2 dimensions \( F_{ij} = 1 \), for \( d \geq 3 \) \( F_{ij} < 1 \). Since him, the former case has been called recurrent and the latter transient. Transience is an exclusive property of infinite graphs and it is fundamentally due to large scale
topology. In other words, in the transient case, it happens that the number of paths leading the walker away from its starting point is large enough, with respect to the number of returning paths, to act as an asymptotic trap (still conserving the total probability).

As we will see in a while, transience and recurrence of random walks, when (49) and (50) are satisfied, only depend on the graph topology. Therefore they are intrinsic properties of a discrete structure and the classification of infinite graphs according to them is also known as the type problem.

Let us define the problem mathematically. First of all, a very general theorem on Markov chains states the following:

$$\exists i, j \in G \ | \ F_{ij} = 1 \ \Rightarrow \ F_{hk} = 1 \ \forall h, k \in G \quad (59)$$

this means that recurrence is point independent, or, in other words, that if a walker surely reaches a point $j$ starting from a given point $i$, then it surely reaches any point $k$ starting from any point $h$. It is straightforward to see that an analogous result follows for the case $F_{ij} < 1$. Therefore, recurrence and transience are global properties of a random walk.

Another important result relates $F_{ij}$ and $P_{ij}$. Indeed, from (23), (24), and (26), it follows that

$$F_{ij} = 1 \ \Leftrightarrow \ P_{ij} = \infty \quad (60)$$

and

$$F_{ij} < 1 \ \Leftrightarrow \ P_{ij} < \infty \quad (61)$$

i.e., a walk is recurrent (transient) if and only if any site is visited an infinite (finite) number of times. The latter can be taken as an alternative definition of recurrence and transience. However, as we will see, the situation is more complex when considering averages over all the sites. A consequent property concerns the way the walker explores the sites of $G$. Indeed, it can be shown that

$$F_{ij} = 1 \ \Leftrightarrow \ \lim_{t \to \infty} \frac{S_i(t)}{t} = 0 \quad (62)$$

while

$$F_{ij} < 1 \ \Leftrightarrow \ 0 < \lim_{t \to \infty} \frac{S_i(t)}{t} < 1 \quad (63)$$

In the first situation, where the number of distinct visited sites increases slower than the number of steps, is sometimes called compact exploration, since the subgraphs of the visited sites presents a negligible number of "holes".

Recurrent graphs exhibit a further relevant property: one can show that

$$\lim_{\lambda \to 1^-} \frac{\tilde{P}_{ij} (\lambda)}{\tilde{P}_{kk} (\lambda)} = \lim_{t \to \infty} \frac{P_{ij}(t)}{P_{kk}(t)} = \frac{z_j}{z_k} \ \forall i, j, h, k \quad (64)$$

The most important properties in the type problem concern its invariance with respect to a wide class of dynamical and topological transformations, establishing its independence of the graph details.
First of all, consider two different random walks (without traps) on the same graph $G$, one (W) defined by the ferromagnetic coupling matrix $J_{ij}$ and by the waiting probabilities $w_i$, and one (W') by $J'_{ij}$ and $w'_i$. It can be shown \cite{7} that, if both satisfy (49) and (50), then W is recurrent if and only if W' is. In other words, any local bounded rescaling of ferromagnetic couplings and waiting probabilities leaves the random walk type unchanged. Therefore, provided the previously mentioned boundedness conditions are satisfied, the walk type only depends on the graph topology.

Moreover, even the local topological details are irrelevant to determine the type of a graph. Indeed, it is possible to show that recurrence and transience are left invariant by adding and cutting of links satisfying the quasi-isometry conditions. More precisely, two graphs $G$ and $G'$ are called quasi-isometric if there are a mapping $\varphi: G \to G'$ and constants $A > 0$, $B \geq 0$ such that
\[
A^{-1}r_{ij} - B \leq r'_{\varphi(i), \varphi(j)} \leq Ar_{ij} + B
\]
for all $i, j \in G$, and
\[
r'_{i', \varphi(i)} \leq B
\]
for every $i' \in G'$.

If $B = 0$ then we say that $G$ and $G'$ are metrically equivalent. Quasi-isometries can be defined between arbitrary metric spaces and represent the most general local topology deformations. Typical examples of them are given by the decimation transformations used on fractals and in real-space renormalization. In some sense, we can consider quasi-isometries as their extension to general networks.

All the results presented so far refer to random walks without traps, i.e. to jumping probabilities given by (15) with $d_i = 0$. The introduction of at least one trap, setting $d_i > 0$ for at least one site $i$, has a very general and simple influence on the random walk behaviour: indeed transience is left unchanged, while recurrent random walks always become transient.

9 The local spectral dimension

As well as it happens for recurrence and transience properties, large scale topology affects the long time dependence of random walks quantities on infinite graphs. Indeed, it has been known for many years that, on regular (translation invariant) lattices, the exponents of the asymptotic power laws of random walks only depend on the lattice (Euclidean) dimension $d$. For example,
\[
P_{ii}(t) \sim t^{-d/2} \quad \text{for } t \to \infty, \forall i
\]
(65)

and
\[
S_i(t) \sim t^{\min(1,d/2)} \quad \text{for } t \to \infty, \forall i, \text{ for } d \neq 2
\]
(66)

(while, for $d = 2$, $S_i(t) \sim t/\ln t$). As we mentioned before, these laws typically present power behaviour even on general physical graphs, and the exponents of such powers can be used to define a generalized dimension.
Let us consider a random walk withoutWaiting and traps satisfying (49), and suppose that, for a given $i \in G$

$$P_{ii}(t) \sim t^{-d/2} \quad \text{for} \quad t \to \infty$$

(67)

then it can be shown that

$$P_{hk}(t) \sim t^{-\tilde{d}/2} \quad \text{for} \quad t \to \infty, \quad \forall h, k$$

(68)

(for bipartite graphs, the usual assumptions on the parity $t$ are understood).

This means that the exponent of the power law is site independent and, therefore, it is a parameter characterizing the whole random walk. Since $\tilde{d} = d$ on regular lattices, we can consider it as a dimension associated to the random walk on $G$. More precisely, we shall call local spectral dimension the limit

$$\tilde{d} = -2 \lim_{t \to \infty} \frac{\ln P_{ii}(t)}{\ln t}$$

(69)

when it exists.

Notice that the existence of this limit for a given $i$ implies it exists and has the same value for any $j \in G$. Moreover, the definition given in (69) is more general than (67), since it includes the case of possible multiplicative correction to the asymptotic behaviour, provided they are slower than any power law (e.g. logarithmic corrections).

\[\text{From an historical point of view, the term ”spectral dimension” was first introduced by Alexander and Orbach in 1982 [8], studying the anomalous vibrational dynamics on fractals. In the same work, they suggested that even the random walks should be ruled by the same parameter and wrote eq. (67). Then the definition was generalized to general networks by Hattori, Hattori and Watanabe [9]. Later, it has been shown that the anomalous dimension involved in vibrational dynamics is the average spectral dimension (10), we shall discuss in further sections, which coincides with }\tilde{d}\text{ only for particular graphs, such as exactly decimable fractals. As for the existence of the limit (69), a general theorem is still lacking, but it can be easily proven that the asymptotic decay of }P_{ii}(t)\text{ is always bounded from above and from below by power laws. In any case, on all known cases of random walks on physical graphs, the local spectral dimension has been shown to exist. Notice that for the Bethe lattice, which does not fulfill the polynomial growth condition, the limit (69) is infinite. }\text{From now on, we shall consider random walks on graphs where }\tilde{d}\text{ is defined. Then, one can easily derive the following results:}

- Random walks are recurrent if $\tilde{d} < 2$ and transient if $\tilde{d} > 2$. For $\tilde{d} = 2$, if (67) holds, random walks are recurrent. However, subleading corrections to the power law can change the type to transient.

- When (67) holds,

$$S_i(t) \sim t^{\min(1, \tilde{d}/2)} \quad \text{for} \quad t \to \infty, \quad \forall i, \text{ for } \tilde{d} \neq 2$$

(70)
otherwise, in general,
\[
\lim_{t \to \infty} \frac{\ln S_i(t)}{\ln t} = \min(1, \tilde{d}/2) \quad \forall i, \text{ for } \tilde{d} \neq 2 \quad (71)
\]

- When (67) holds,
\[
F_{ij}(t) \sim t^{\min(\tilde{d}/2 - 2, -\tilde{d}/2)} \quad \text{for } t \to \infty, \quad \forall i, j \text{ for } \tilde{d} \neq 2 \quad (72)
\]

otherwise, in general,
\[
\lim_{t \to \infty} \frac{\ln F_{ij}(t)}{\ln t} = \min(\tilde{d}/2 - 2, -\tilde{d}/2) \quad \forall i, j \text{ for } \tilde{d} \neq 2 \quad (73)
\]

The case \( \tilde{d} = 2 \) is rather particular. Indeed \( \tilde{d} = 2 \) is a critical dimension for random walks, discriminating recurrence from transience. The asymptotic behaviours of \( S_i(t) \) and \( F_{ij}(t) \) have a different dependence on \( \tilde{d} \) for \( \tilde{d} < 2 \) and \( \tilde{d} > 2 \). In particular, the probability of first visit has the same time decay of \( P_{ij} \) for \( \tilde{d} > 2 \) while it decays faster for \( \tilde{d} < 2 \). When \( \tilde{d} = 2 \), the behaviours of \( S_i(t) \) and \( F_{ij}(t) \) are strongly affected by subleading corrections.

As for the type problem, local spectral dimension presents interesting invariance properties. First of all, it can be shown that waitings satisfying (50) do not affect its value [10]. Moreover, for \( \tilde{d} < 2 \), on a given \( G \) it is the same for all ferromagnetic couplings satisfying (49) [9]. Unfortunately, an analogous result has not been proven for \( \tilde{d} > 2 \). However, we will see in later sections that the average spectral dimension has also this universality property.

The introduction of a finite number of traps do not affect \( \tilde{d} \) if \( \tilde{d} > 2 \). If \( \tilde{d} < 2 \) a finite number of traps (even only one) changes \( \tilde{d} \) to \( \tilde{d} + 1 \). If the traps are infinite the behaviour is more complex and depends on their distribution.

### 10 Averages on infinite graphs

Usually, infinite graphs describing real systems are inhomogeneous, i.e., in mathematical terms, they are not invariant with respect to a transitive symmetry group. In simpler words, this means that the topology is seen in a different way from every site. The main effect of inhomogeneity is that the numerical values of physical quantities are site dependent. Therefore, one is typically interested in taking averages over all sites. This requires the introduction of suitable mathematical tools.

First of all, the average in the thermodynamic limit \( \bar{\phi} \) of a function \( \phi_i \) defined on each site \( i \) of the infinite graph \( G \) is defined by:

\[
\bar{\phi} \equiv \lim_{r \to \infty} \sum_{i \in S_0,r} \frac{\phi_i}{N_{o,r}}.
\]
The measure $|S|$ of a subset $S$ of $V$ is the average value $\overline{\chi(S)}$ of its characteristic function $\chi_i(S)$ defined by $\chi_i(S) = 1$ if $i \in S$ and $\chi_i(S) = 0$ if $i \notin S$. The measure of a subset of links $E' \subseteq E$ is given by:

$$|E'| \equiv \lim_{r \to \infty} \frac{E'_r}{N_{o,r}} \quad (75)$$

where $E'_r$ is the number of links of $E'$ contained in the sphere $S_{o,r}$. The normalized trace $\overline{Tr} B$ of a matrix $B_{ij}$ is:

$$\overline{Tr} B \equiv \overline{b} \quad (76)$$

where $b_i \equiv B_{ii}$. If condition $C$ holds, then we can prove [13] that the averages of a bounded from below function $\phi_i$ are independent from the centre $o$ of the spheres sequence, using the fact that $\chi_i(S)$ is bounded and that measures of subsets are always well defined.

Now, due to this site independence, we have a good definition of averages which we will use in dealing with properties of random walks on infinite graphs. As we shall see in the next section, on inhomogeneous networks the averages of site dependent functions can have a very different behaviour from their local counterparts, giving rise to rather unexpected phenomena.

### 11 The type problem on the average

In the last few years it has become clear that bulk properties are affected by the average values of random walks return probabilities over all starting sites: this is the case for spontaneous breaking of continuous symmetries [11], critical exponents of the spherical model [12], harmonic vibrational spectra [10]. Therefore the classification of discrete structure in terms of recurrence on the average and transience on the average appears to be the most suitable. Unfortunately, while for regular lattices the two classifications are equivalent, on more general networks they can be different and one has to study a Type-Problem on the Average [13].

This is defined using the return probabilities on the average $\overline{P}$ and $\overline{F}$, which are given by:

$$\overline{P} = \lim_{\lambda \to 1^-} \overline{P(\lambda)} \equiv \lim_{\lambda \to 1^-} \overline{Tr}\overline{P(\lambda)} \quad (77)$$

$$\overline{F} = \lim_{\lambda \to 1^-} \overline{F(\lambda)} \equiv \lim_{\lambda \to 1^-} \overline{Tr}\overline{F(\lambda)} \quad (78)$$

A graph $G$ is called recurrent on the average (ROA) if $\overline{F} = 1$, while it is transient on the average (TOA) when $\overline{F} < 1$.

Recurrence and transience on the average are in general independent of the corresponding local properties. The first example of this phenomenon occurring on inhomogeneous structures was found in a class of infinite trees called NTD (see sect. [13]) which are locally transient but recurrent on the average [14].
Moreover, while for local probabilities (26) gives:

\[  \tilde{P}_{ii}(\lambda) \tilde{F}_{ii}(\lambda) \tilde{P}_{ii}(\lambda) + 1 \]

(79)

an analogous relation for (78) and (77) does not hold since averaging (79) over all sites \( i \) would involve the average of a product, which due to correlations is in general different from the product of the average. Therefore the double implication \( \tilde{F}_i(1) = 1 \Leftrightarrow \lim_{\lambda \to 1} \tilde{P}_i(\lambda) = \infty \) is not true. Indeed there are graphs for which \( \bar{F} < 1 \) but \( \bar{P} = \infty \) (an example is shown in Fig.1) and the study of the relation between \( \bar{P} \) and \( \bar{F} \) is a non trivial problem.

A detailed study of this relation \[13\] shows that a complete picture of the behaviour of random walks on graphs can be given by dividing transient on the average graphs into two further classes, which are called pure and mixed transient on the average (TOA).

First, considering a ROA graph, it can be proven that if \( \bar{F} = 1 \) then \( \bar{P} = \infty \). The proof can be easily generalized to graphs in which there is a positive measure subset \( S \) such that: \( \lim_{\lambda \to 1} \chi(S) \tilde{F}(\lambda) = |S| \). Indeed in an analogous way it can be proven that:

\[ \bar{P} \geq \lim_{\lambda \to 1} \chi(S') \tilde{P}(\lambda) = \infty \quad \forall S' \subseteq S, |S'| > 0 \]

(80)

We call mixed transient on the average a TOA graph having a positive measure subset \( S \) such that:

\[ \lim_{\lambda \to 1} \chi(S) \tilde{F}(\lambda) = |S| \]

(81)

while a graph is called pure TOA, if:

\[ \lim_{\lambda \to 1} \chi(S) \tilde{F}(\lambda) < |S| \quad \forall S \subseteq V, |S| > 0 \]

(82)

Examples of pure TOA graphs are all the \( d \)-dimensional cubic lattices with \( d > 2 \), while the "haired cube" of Fig.1 is a typical mixed TOA graph. Notice that a relevant theorem \[13\] establishes that for mixed TOA graphs we have \( \bar{P} = \infty \), while for pure TOA graphs \( \bar{P} \) is finite. A further important property, characterizing mixed TOA graphs, allows simplifying the study of statistical models on these very inhomogeneous structures. It can be shown \[13\] that in this case, the graph \( \mathcal{G} \) can be always decomposed in a pure TOA subgraph \( \bar{S} \) and a ROA subgraph \( S \) with independent jumping probabilities by cutting a zero measure set of links \( \partial S \equiv \{(i,j) \in E|i \in S, j \in \bar{S}\} \). The separability property implies that the two subgraphs are statistically independent and that their thermodynamic properties can be studied separately. Indeed, in the thermodynamic limit, the partition functions referring to the two subgraphs factorize \[15\].

To conclude this section, we note that the same invariance properties of the local type problem under addition of waiting probabilities, coupling rescaling and quasi-isometries still hold for the type problem on the average. This means, as for the local case, that recurrence and transience on the average are intrinsic
properties of a graph and not only of a specific random walk defined on it. On the other hand, the introduction of a finite number of traps does not change the type on the average. Notice also that a slightly different definition of the type problem on the average can be found in mathematical literature [16]; it is more convenient for the formal development of the theory, but it is not directly related to statistical models on graphs.

12 The average spectral dimension

The asymptotic time dependence of the return probability on the average can be used to define a new intrinsic dimension which turns out to be very strictly related to the physical behaviour of statistical models on graphs [11,10,17], as we will briefly discuss in the last section.

Indeed, even if the asymptotic time decay of $P_i(t)$ is always the same for all sites $i$, when the graph topology is strongly inhomogeneous it happens that its average over all the sites decays according to a different law. The average spectral dimension $\bar{d}$ is defined for physical graphs, like the local one in (67) and (69), by

$$\bar{P}(t) \sim t^{-\bar{d}/2} \quad \text{for} \quad t \to \infty$$

when the asymptotic behaviour is a power law without subleading corrections, or, more generally, by

$$\bar{d} = -2 \lim_{t \to \infty} \frac{\ln \bar{P}(t)}{\ln t}$$

Notice, however, that, differently from the local case, no physical graphs are known, up to now, where the long time decay is not given by (83). Considerations analogous to those presented for the local case hold here, concerning the existence of the limit (84). Obviously, in all cases where the local type is different from the average type, also the local spectral dimension differs from the average spectral dimension. A typical example, and, historically, the first one, is given again by NTD (see sect. 13.3 for a detailed account). However, the relations between $\bar{d}$ and the type problem on the average are not the same as in the local case. Indeed, while if $\bar{d} > 2$ the walk is always pure TOA, random walks with $\bar{d} < 2$ can be either pure ROA or mixed TOA.

The most relevant property of $\bar{d}$ is without any doubt its strong invariance with respect to a very large class of dynamical and topological transformations, making it a unique universal parameter associated to a graph $\mathcal{G}$ [11,12].

These transformations can be divided into three main classes:

1. **Dynamical transformations leaving the graph topology unchanged.** These consist in addition of waitings and of a finite number of traps, as well as in bounded local rescaling of ferromagnetic couplings.

2. **Topological transformations modifying the number of links but leaving the sites unchanged.** These include "addition transformations" and "cutting transformations". The additions transforms consist in adding links joining
sites up to an arbitrary but finite chemical distance from any site, while the cutting transform are defined to be their inverse. The most general transformations consist in a combination of addition and cutting. Notice that even an infinite number of links can be modified with respect to the original graph.

3. **Topological rescaling, i.e. topological transformations modifying both links and sites.** The most general topological rescaling can be realized through two independent steps. The first one is the *partition* and consists in dividing the graph $G$ in an infinite family of connected subgraphs $G_\alpha$, with uniformly bounded number of points. The second one is the *substitution* and consists in generating a new graph $G'$ by replacing some or all $G_\alpha$ by a different (connected) graph $S_\alpha$, whose number of points ranges from 1 to a fixed $N_{max}$, and by adding links connecting different $S_\alpha$ in such a way that two generic $S_\alpha$ and $S_\beta$ are connected by some links if and only if $G_\alpha$ and $G_\beta$ were. The simplest topological rescaling occurs when every $S_\alpha$ is composed by just one point. In this case the resulting graph $G'$ is called the *minimal structure* of the partition $\{G_\alpha\}$.

These three very general classes of geometrical transformations (together with even more general ones violating conditions B and C and therefore not discussed here [18]) can be applied in all possible sequences to a graph, leading to an overall transformation on coupling strength, number of links and degrees of freedom which does not change its spectral dimension $\bar{d}$. We will call such a transformation an *isospectrality*. Notice that isospectrality includes quasi-isometries as a particular case. Indeed, isospectrality includes most part of currently used transformations.

As an example, the usual decimation procedure on fractals is a topological rescaling. In particular, for all exactly decimable fractals (such as e.g. Sierpinski gaskets and T-fractals, as discussed in the next sections), the minimal structure of the graph coincides with the graph itself. Again, an isospectrality relates the usual two dimensional square lattice, the hexagonal lattice and the triangular lattice, which therefore all have dimension 2. In other words, isospectrality is the theoretical formalization of the intuitive idea of invariance with respect to bounded scale perturbations and disorder and the *isospectrality classes*, defined as the classes of graphs related by such transformations, are the practical realization of the apparently abstract concept of non integer dimension.

Now, since most dynamical and thermodynamical properties of generic discrete structures depend only on $\bar{d}$, isospectrality provides a very powerful tool to reduce a very complicated geometrical structure to the simplest one having the same $\bar{d}$. The latter turns out to be much simpler to study and still presents the same universal properties.

Moreover, not only an isospectrality can be used to reduce and simplify structures and problems. It can also be applied, with the opposite aim, to build complicated structures with controlled dynamical and thermodynamical properties, starting from simple deterministic geometrical models. This is the point of view
of **spectral dimension engineering**, providing a very interesting field of possibilities to polymer physicists and material scientists dealing with non-crystalline materials. In Fig. 2 we give explicit examples of isospectral structures obtained applying isospectral transformations (without long range couplings) to the T-fractal and to the square lattice.

On macroscopically inhomogeneous graphs, it can happen that the average value of $P_{ii}(t)$ on infinite subgraphs of $G$ with positive measure decays with a power law different from (83) [19]. In such cases, it is interesting to look for the maximal (positive-measure) subgraphs having no (positive measure) parts with different power law decay. These are called **spectral classes** and each is characterized by its own spectral dimension. A theorem rather relevant in physical applications establishes that spectral classes can be separated from each other by cutting a zero-measure set of links, implying the same statistical independence property we discussed for mixed TOA graphs [19].

### 13 A survey of analytical results on specific networks

Apart from the well known case of regular lattices, where it is completely solved [20], the random walk problem has been studied analytically only on some specific classes of infinite graphs. In these cases, one usually focuses on the asymptotic properties of random walks autocorrelation functions and on the calculation of the local and average spectral dimension. As we discussed in the previous sections, these are the most important quantities in statistical physics and thermodynamics. On lattices, the random walk problem is solved by using the translation invariance of the structure, and this allows applying powerful mathematical tools, such as the Fourier transform. On general graphs these methods do not apply. Therefore due to the lack of translation invariance, one has to introduce new and alternative techniques, which can be grouped in three main classes: renormalization techniques, combinatorial techniques and mixed techniques. In the next subsections we will review recent and significant results obtained with these techniques.

#### 13.1 Renormalization techniques

Renormalization techniques have been successfully applied on deterministic fractals networks, where one can take advantage of the decimation transformations which connect two consecutive generations. In particular, a well studied class of fractals is that of **exactly decimable** fractals. On these structures, exact renormalization group calculations based on a real space decimation procedure allow obtaining all the relevant random walks quantities.

Let us consider a random walk without traps and sources defined by the jumping probabilities [4] and let us write the master equation for the probability $P_{0i}$ of being at site $i$ after $t$ steps for a random walker starting from an origin site 0.
at time 0:

\[ P_{0i}(t + 1) - P_{0i}(t) = \sum_j A_{0j} \left( \frac{P_{0j}(t)}{z_j} - \frac{P_{0j}(t)}{z_i} \right) + \delta_{i0} \delta_{0i} \]  

(85)

Equation (85) can be written in terms of the generating function \( \tilde{P}_{ij}(\lambda) \) when \( \lambda \to 1^- \) by setting \( \lambda = 1 - \epsilon \), writing:

\[ \tilde{P}_{ij}(\epsilon) = \sum_{t=0}^{\infty} (1 + \epsilon)^{-t} P_{ij}(t) \]  

(86)

and taking \( \epsilon \to 0 \):

\[ \epsilon \tilde{P}_{0i}(\epsilon) = \sum_j A_{0j} \left( \frac{\tilde{P}_{0j}(\epsilon)}{z_j} - \frac{\tilde{P}_{0j}(\epsilon)}{z_i} \right) + \delta_{0i} \]  

(87)

Notice that the system (87) is inhomogeneous and corresponds to a Cauchy problem, which has only one solution. The behaviour of such solution for \( \epsilon \to 0 \) is what we need to obtain the local spectral dimension \( \tilde{d} \), as defined in (67), through the Tauberian Theorems. On the other hand, to calculate the average spectral dimension \( \bar{d} \) we will need to average over all starting points the solution of equation (87), strongly modifying its asymptotic behaviour on inhomogeneous graphs, as we will see in the following.

Exactly decimable fractals are a restricted class of self similar structures (i.e. not all self similar structures are exactly decimable) which are geometrically invariant under site decimation. This invariance is explicitly applied in analytical calculations for random walks. A geometrical structure is decimation invariant if it is possible to eliminate a subset of points (and all the bonds connecting these points) obtaining a network with the same geometry of the starting one. From a mathematical point of view this corresponds to the possibility of eliminating by substitution a set of equations from system (85) or (87) obtaining a system which is similar to the initial one after a suitable redefinition of the coupling parameters. Examples of exactly decimable fractals are the Sierpinski Gasket (Fig.3), 25, 26, 27, 28, the \( T^- \)-fractal, shown in Fig.4 29, 30, the branched Koch curves, in Fig.5 31. In general, all deterministic finitely-ramified fractals are exactly decimable. Notice that exact decimation is a particular case of isospectrality, as we discussed in previous sections.

Let us consider now the general procedure to decimate the set of equation (85). After eliminating a set of points and substituting the corresponding equation, one finds:

\[ \epsilon \to \epsilon'(\epsilon) \sim a^2 \epsilon \]  

(88)

The presence of the term \( \delta_{i0} \) in (87) requires a redefinition of the quantities \( \tilde{P}_{ij}(\epsilon) \) to assure that, even after the decimation, the initial condition will correspond to the probability of being in a fixed site equal to 1. One introduces a new parameter \( c \) and writes the transformation law for \( \tilde{P}_{ij}(\epsilon) \) as:

\[ \tilde{P}_{ij}(\epsilon) \to \tilde{P}_{ij}'(\epsilon') \sim \frac{1}{c} \tilde{P}_{ij}(\epsilon) \]  

(89)
From the rescaling of $\epsilon$ and $\tilde{P}_{ij}(\epsilon)$, the local spectral dimension $\tilde{d}$ is obtained by using a suitable expression for $\tilde{P}_{00}(\epsilon)$:

$$\tilde{P}_{00}(\epsilon) \sim e^ {\tilde{d}/2 - 1}$$

(90)

which holds only for $\tilde{d} < 2$. This is always the case for exactly decimable fractals. Using expression (90) one easily finds:

$$\tilde{d} = 2 \frac{\log a^2/c}{\log a^2}$$

(91)

As for the average spectral dimension $\bar{d}$, by using the relation between equation (87) and the equation for harmonic oscillations to be discussed later [21], one has that

$$\bar{d} = \frac{\log r}{\log a}$$

(92)

where $r$ is the decimation ratio used in the renormalization procedure. Therefore $\tilde{d} = \bar{d}$ if

$$r = a^2/c$$

(93)

This can be shown to be the case for all exactly decimable fractals, using results obtained [9] for the Gaussian model.

Equation (92) allows calculating the spectral dimension on all exactly decimable fractals, once the decimation procedure is identified, recovering known results.

One of the most studied fractal is without any doubt the Sierpinski gasket [25, 26, 27, 28] and its generalizations. For the simplest case one has $r = 3$ and $a = \sqrt{3}$, leading to:

$$\tilde{d} = 2 \frac{\log 3}{\log 5}$$

(94)

For $d$-dimensional generalized Sierpinski gaskets, which are built from a $d$-dimensional hypertetrahedron of side length $b$ filled with $b$ layers of smaller hypertetrahedra of unit site length, Hilfer and Blumen [26] have shown that for $b = 2$

$$\tilde{d} = 2 \frac{\log d + 1}{\log d + 3}$$

(95)

and for $b = 3$

$$\tilde{d} = 2 \frac{\log((d + 1)(d + 2)/2)}{\log((d + 2)(2d^2 + 9d + 19)/(4d + 6))}$$

(96)

Due to the self-similarity of the structure, the return probabilities on the Sierpinski gasket show a remarkable effect, which has been pointed out in [32]. Indeed, the coefficients have an oscillatory behaviour, which is given by:

$$P_{00}(t) = t^{-\tilde{d}/2} F\left(\frac{\log t}{\log 5}\right)$$

(97)
where \( F \) is a periodic \( C^\infty \)-function of period 1 whose Fourier series is given by

\[
F(x) = \sum_{k=-\infty}^{\infty} \Gamma(1 - \frac{\log 3}{\log 5} + \frac{2\pi ki}{\log 5})^{-1} \exp(2\pi kix)
\]  

Interestingly, it can be shown that the oscillation of the coefficients disappears in the probability of return on the average.

The renormalization techniques can be applied to all exactly decimable fractals. For example, for the \( T \)-fractal [29, 30], which is a particular case of hierarchical combs [33], one has \( r = 3 \) and \( a = \sqrt{6} \).

### 13.2 Combinatorial techniques

Renormalization procedures cannot be applied on non self-similar graphs. Therefore one has to develop alternative techniques to study the random walk problem. This is the case of bundled structures [22, 23], a large class of very interesting graphs used in condensed matter as realistic models for the geometry and dynamics of polymers and other inhomogeneous systems. Given two graphs \( \mathcal{B} \) and \( \mathcal{F} \), not necessarily different, and a site \( F \) of \( \mathcal{F} \), we call bundled graph with base \( \mathcal{B} \) and fibre \( \mathcal{F} \) the graph built by joining to each site of \( \mathcal{B} \) a copy of \( \mathcal{F} \) in such a way that \( F \) is the only site \( \mathcal{B} \) and \( \mathcal{F} \) have in common (Fig.6). Examples of bundled structures are comb polymers [34] (Fig.7), brush polymers, shown in Fig.8, and many kinds of branched aggregates (Fig.9). For these graphs a purely combinatorial technique allows calculating of the asymptotic properties of the random walk autocorrelation functions.

Let us consider a walker starting from a point belonging to the base and let us restrict ourselves to base graphs with constant coordination number \( z_B \). By decomposing the motion of the walker on the fibre and on the base, one can obtain:

\[
P_0(t) = \sum_{t_B=0}^{\infty} \sum_{t_1=0}^{\infty} \ldots \sum_{t_{B+1}=0}^{\infty} P_B(t_B) \left( \frac{z_B}{z_B + z_F} \right)^{t_B} P_F'(t_1) \cdots P_F'(t_{B+1}) \delta_{t,t_B + \sum_{i=1}^{t_{B+1}} t_i}
\]

(99)

where \( P_F' \) refers to a random walk on \( \mathcal{F} \) with a trap in the starting point of \( \mathcal{F} \).

In terms of the generating functions equation, (99) becomes:

\[
\tilde{P}_0(\lambda) = \sum_{t_B=0}^{\infty} P_B(t_B) \left( \frac{\lambda z_B}{z_B + z_F} \right)^{t_B} (\tilde{P}_F'(\lambda))^{t_B+1} = \tilde{P}_F'(\lambda) \tilde{P}_B(\lambda')
\]

(100)

with

\[
\lambda' = \frac{\lambda z_B}{z_B + z_F} \tilde{P}_F'(\lambda)
\]

(101)

and

\[
\tilde{P}_F'(\lambda) = \left( 1 - \frac{z_F}{z_B + z_F} \left( 1 - \left( \tilde{P}_F(\lambda)^{-1} \right) \right)^{-1}\right)^{-1}
\]

(102)
with $\tilde{P}_F(\lambda)$ being the generating function of the probability of returning to the starting point $F$ on $F$ without the trap. From this relation one obtains the values for the local spectral dimension on general bundled graphs:

$$
\tilde{d} = \begin{cases} 
\tilde{d}_x & \text{if } \tilde{d}_x \geq 2 \\
4 - \tilde{d}_x & \text{if } \tilde{d}_x \leq 2 \text{ and } \tilde{d}_B \geq 4 \\
\tilde{d}_x + \tilde{d}_B - \frac{\tilde{d}_x \tilde{d}_B}{2} & \text{if } \tilde{d}_x \leq 2 \text{ and } \tilde{d}_B \leq 4
\end{cases}
$$

(103)

where $\tilde{d}_B$ and $\tilde{d}_x$ are the local spectral dimension of the base and of the fibre. If the coordination number of the base is not constant, it can be shown that this amounts to introduce waiting probabilities on the points connecting the fibre and the base, which, as shown in the previous section, does not change the value of the spectral dimension.

As for the average spectral dimension, it is easy to show that if the fibre is an infinite graph, the average spectral dimension of the whole graph is the spectral dimension of the fibre. On the other hand, if the fibre is a finite graph, the average spectral dimension coincides with that of the base.

From equations (100), (102) one also obtains the asymptotic laws for the probability of returning to the starting point, which on these structures can contain logarithmic corrections. Indeed, writing

$$
P_0(t) \sim \prod_{i=0}^{\infty} i^{\beta(i)}(t)
$$

(104)

and setting

$$
m = \min\{i \geq 0 | \beta(i) \neq -1\}
$$

(105)

and

$$
I\left(\tilde{d}/2\right) = \begin{cases} 
1 & \text{if } \tilde{d}/2 \text{ is an integer} \\
0 & \text{otherwise}
\end{cases}
$$

(106)

one has:

a) if $\tilde{d}_B < 4$ and $\tilde{d}_x < 2$

$$
\beta(i) = \begin{cases} 
-1 & \text{for } 0 < i < m \\
\left(1 - \frac{\tilde{d}_x}{2}\right) \left[\beta_x(m_x) + I\left(\tilde{d}_x^{m_x}\right) - I\left(\tilde{d}_x^{m_x}\right)\right] & \text{for } i = mm_x \\
\left(1 - \frac{\tilde{d}_B}{2}\right) \beta_x(i) + \theta(i-m_x-m_B) \beta_B(i-m_x) + \\
\delta_{i-m_x-m_B} I\left(\tilde{d}_B^{i-m_x-m_B}\right) - \delta_{i-m} I\left(\tilde{d}_x^{i-m}\right) & \text{otherwise}
\end{cases}
$$

(107)

where $m_B$ and $m_B$ refers to the base and to the fibre respectively while $m$ refers to the whole graph and is determined by:

$$
m = m_x + \delta_{\tilde{d}_B^{-1}m_B} m_B
$$

(108)
b) if $\tilde{d}_B > 4$ and $\tilde{d}_F < 2$

$$\beta(i) = \begin{cases} -\beta_F(i) - 2\delta_{i,m_F}(\tilde{d}_F/2) & \text{for } i \geq m_F \\ \beta_F(i) & \text{for } 0 < i < m_F \end{cases}$$ (109)

c) if $\tilde{d}_B = 4$ and $\tilde{d}_F < 2$ and $d_B > -1$

$\beta(i)$ has to be determined as in a).

d) if $\tilde{d}_B = 4$ and $\tilde{d}_F < 2$ e $m_B < -1$

$\beta(i)$ has to be determined as in b).

e) if $\tilde{d}_F > 2$

$$\beta(i) = \beta_F(i) \quad \forall i$$ (110)

The case $\tilde{d}_F = 2$ has to be treated separately, as the case $\tilde{d}_F < 2$ if the fibre is a recurrent graph or as the case $\tilde{d}_F > 2$ if it is transient.

Another interesting way of combining together two graphs to obtain a more complex structure is the Cartesian product. The Cartesian product of two graphs $X, Y$ has vertex set $X \times Y$, and two pairs $xy, x'y'$ are adjacent if $x \sim x'$ and $y = y'$, or $x = x'$ and $y \sim y'$. An example of an interesting Cartesian product is that of the Toblerone graph [35], shown in Fig.10, which is obtained from the product of a line with a Sierpinski gasket. Using combinatorial techniques analogous to those presented for bundled graphs, it can be show that the local and the average spectral dimension on the whole graph are the sum of the corresponding dimensions of the two initial graphs [7].

13.3 Mixed techniques

The random walk problem on some very interesting cases of graphs cannot be studied simply by one of the above cited techniques and it requires instead a ”mixed” use of the two, which gives rise to very interesting phenomena. Indeed, the first example of a difference between the local and the average spectral dimension, the ”dynamical dimension splitting”, was observed on the quasi self-similar graphs $NT_D$, where the asymptotic properties of the random walk were found by a mixed technique [6].

The fractal trees known as $NT_D$ [36] can be recursively defined as follows: an origin point $O$ (Fig.11) is connected to a point 1 by a link, of unitary length; from 1, the tree splits in $k$ branches of length 2 (i.e. consisting of two consecutive links); the ends of these branches split in $k$ branches of length 4 and so on; each endpoint of a branch of length $2^n$ splits in $k$ branches of length $2^{n+1}$.

As one can easily verify, $NT_D$ are not exactly decimable and therefore the simple decimation techniques cite above cannot be applied. Indeed, after a simple decimation starting from the origin $O$, one obtains $k$ copies of the original structure joined together in a point instead of the same $NT_D$. However, $NT_D$ are invariant under a more complex transformation $T = D \cdot C$, consisting of the
product of a cutting transform $C$ and a decimation $D$, that can be described as follows. Let us cut the log of the tree in point 1 and separate the $k$ branches (cutting transform). Now, each branch can be obtained from the initial $NT_D$ by a dilatation with a factor 2. Eliminating all branches but one and decimating it (decimation transform), one obtains the original $NT_D$.

The $T$ transform can now be used to solve the random walks problem. Let us sketch the main points of the calculation. The cutting transform gives a relation between random walks on the whole tree and random walks on one of its branches; more precisely one relates $\tilde{P}^{\text{tree}}_O(\lambda)$, the generating function of the probability of returning to point $O$ after a random walk on the $NT_D$ tree, and $\tilde{P}^{\text{branch}}_1(\lambda)$, the generating function of the probability of returning to the starting point 1 after a random walk on one of the branches. This relation is given by (111):

$$\tilde{P}^{\text{tree}}_O(\lambda) = \frac{\tilde{P}^{\text{branch}}_1(\lambda) + k}{2\lambda \tilde{P}^{\text{branch}}_1(\lambda) + k}$$

Now, the decimation transformation is performed using a time-rescaling technique. Indeed, the motion of the random walker on the branch considered only after an even number of steps can be exactly mapped in the motion of a random walker on the tree after the introduction of a staying probability $p_{ii} = 1/2$ in every site $i$. This equivalence can be translated in terms of generating functions through the substitutions:

$$\tilde{P}_O(\lambda) \rightarrow \frac{\lambda}{2 - \lambda} \tilde{P}_O \left( \frac{2}{2 - \lambda} \right)$$

$$\lambda \rightarrow \lambda^2$$

Equations (112) and (113) can be used to rewrite (111) as:

$$\tilde{P}^{\text{tree}}_O(\lambda) = \frac{2 - \lambda^2}{2 - \lambda} \tilde{P}^{\text{tree}}_O \left( \frac{\lambda^2}{2 - \lambda^2} \right) + k$$

Choosing a suitable power law expression for the singularity of $P^{\text{tree}}_{OO}(\lambda)$ for $\lambda \rightarrow 1^-$ [14] we obtain:

$$\bar{d} = 1 + \frac{\log k}{\log 2}$$

To obtain the average spectral dimension, one has to calculate the normalized trace of the return probability $\tilde{P}^{\text{tree}}_O(\lambda)$. It can be shown that $\bar{d} = 1$ and this can be intuitively understood noting that the topology of $NT_D$ is dominated by linear chains which become longer and longer in the outer branches [21]. Therefore, while $NT_D$ are locally transient if the ramification $k$ is greater than 2, they are always recurrent on the average. This result has been generalized. Indeed, recently it has been shown that all physical trees, satisfying conditions A, B and C are recurrent on the average [37].
The cutting decimation transform can be applied to a large class of non-exactly
decimable fractals which correspond to more general cases of the $NT_D$. This
are built with the same recurrence procedure as the $NT_D$ and we shall call them
$2^m NT_D$, $nNT_D$ and $p-polygonNT_D$, depending on the growth rules for the
branches [24].

The first generalization is that of $2^m NT_D$. The $2^m NT_D$ are infinite fractal
trees that can be recursively built using the same recipe as for $NT_D$ but, from
point 1, the log splits in $k$ branches of length $2^m$ (i.e. made of $2^m$ consecutive
links) which, in turn, split in $k$ branches of length $2^{2m}$ and so on in such a way
that each branch of length $2^{nm}$ splits in $k$ branches of length $2^{(n+1)m}$. The
case $m = 1$ corresponds to the usual $NT_D$ previously studied. If $m > 1$ the
time rescaling procedure which led to (112) and (113) must be iterated $m$ times
obtaining:

$$\tilde{P}_O^{tree}(\lambda) = \frac{\left(\prod_{i=1}^{m} \frac{2}{2-\lambda^2_i}\right) \tilde{P}_O^{tree}(\lambda_{i+1}) + k}{(1-\lambda^2) \left(\prod_{i=1}^{m} \frac{2}{2-\lambda^2_i}\right) \tilde{P}_O^{tree}(\lambda_{i+1}) + k}$$

with

$$\lambda_i = \begin{cases} \lambda & i = 1 \\ \frac{\lambda_i^2}{2-\lambda_i^2} & i > 1 \end{cases}$$

$i$ being the iteration step. This gives, with the same steps as for $m = 1$:

$$\tilde{d}_{2^m} = 1 + \frac{\ln k}{\ln 2^m}$$

which represent the generalization of the result obtained for $m = 1$.

The previous results can be extended to $nNT_D$, where now $n$ is an integer and
not necessarily a power of 2, and to $p-polygonNT_D$, where the branches of
$NT_D$ are replaced by $p$-vertices regular polygons (Fig.12).

Let us consider $nNT_D$ first. While relation (111) for the cutting transform still
holds, the exact time-rescaling procedure can not be applied to the branch of
generic length $n$. However even in this case it is possible to obtain an asympto-
tic recursion relation applying the Renormalization Group techniques usually
implemented on exactly decimable fractals. Although this procedure cannot
give an exact equation for $\tilde{P}_O^{tree}(\lambda)$ as in the previous case, nevertheless it can
be used to obtain the exact value of $\tilde{d}$ via an asymptotic estimation.

Indeed, in this case the branch of the $nNT_D$ can be considered as a tree with
a dilatation factor equal to $n$. The log of this tree can be reduced to a unitary
length log after the suppression of the $n-2$ sites between the edges and intro-
ducing a new link connecting the edges. The same operation can be repeated for
branches of every length suppressing the inner $n-2$ consecutive sites in every
sequence of $n$ sites and introducing a new link between the surviving points.
The final structure is equal to the original tree and the generating function
\( \tilde{P}_{1}^{\text{branch}}(\lambda) \) becomes \( \tilde{P}_{1}^{\text{'branch}}(\lambda') \) where:

\[
\lambda' = n^2 \lambda
\]

(118)

\[
\tilde{P}_{1}^{\text{'branch}}(\lambda') = \frac{1}{n} \tilde{P}_{1}^{\text{branch}}(\lambda)
\]

(119)

Now \( \tilde{P}_{1}^{\text{'branch}}(\lambda') \) coincides with \( \tilde{P}_{O}^{\text{tree}}(\lambda') \) since our branch has been transformed into a tree and (111) can be rewritten as:

\[
\tilde{P}_{O}^{\text{tree}}(\lambda) = \frac{n \tilde{F}_{1}^{\text{branch}}(n^2 \lambda) + k}{2n \tilde{F}_{O}^{\text{tree}}(n^2 \lambda) + k}
\]

(120)

Using the procedure described in the previous section for \( 2^m NT_D \), from (120) it follows that for an \( n - NT_D \) the spectral dimension is given by:

\[
\tilde{d}_n = 1 + \frac{\ln k}{\ln n}
\]

(121)

An analogous technique can be used for \( p - polygon NT_D \) (Fig.12). The log polygon has now \( p \) faces of unitary length; from each of \( p - 1 \) of its vertices \( k \) polygons depart, whose faces have length \( n \) and so on. These structures, though similar to \( NT_D \) are no longer loopless structures nor necessarily bipartite graphs (e.g. the \( 3 - polygon \) tree). The Cutting-Decision transform can be applied to \( p - polygon NT_D \) as in the case of \( NT_D \) with the same substitutions (115) and (118). Indeed, even if (111) does not hold in this case, a new relation between the generating functions of the tree and that of one of its branches can be obtained using bundled structures theory discussed above [38]. Let us consider a \( p - polygon NT_D \) and suppose to attach \( k \) branches also in the free vertex of the log (the root of the tree): we obtain a bundled structure having the log polygon as base and the graph made of \( k \) branches as fibre. Since for a \( p \)-polygon:

\[
\tilde{P}_{O}(\lambda) \sim \frac{1}{p(1 - \lambda)}
\]

(122)

as \( \lambda \to 1 \), we obtain for our bundled structure:

\[
\tilde{P}_{O}^{b.s.}(\lambda) = \frac{1}{1 - \frac{k}{k+1} \tilde{F}_{1}^{\text{branch}}(\lambda)} \left( 1 - \frac{\lambda}{k+1} \right)^p \left( 1 - \frac{1}{k+1} \tilde{F}_{1}^{\text{branch}}(\lambda) \right)^{-1}
\]

(123)

where \( \tilde{P}_{O}^{b.s.}(\lambda) \) is the generating function of the probability of returning to point \( O \) (one of the vertices of the log polygon) after a random walk on the bundled structure and \( \tilde{F}_{1}^{\text{branch}}(\lambda) \) is the generating function of the probability of returning for the first time to the point of connection with the base after a random walk on the fibre. Now,

\[
\tilde{F}_{O}^{b.s.}(\lambda) = \frac{k}{k+1} \tilde{F}_{1}^{\text{branch}}(\lambda) + \frac{1}{k+1} \tilde{F}_{O}^{\text{tree}}(\lambda)
\]

(124)
where $F_{O_{tree}}(\lambda)$ refers to the $p-polygonNT_D$. From (123), and (124) and using the usual relation between $\tilde{F}_{branch}^{1}(\lambda)$ and $\tilde{P}_{branch}^{1}(\lambda)$, a relation between $P_{O_{tree}}(\lambda)$ and $\tilde{P}_{branch}^{1}(\lambda)$ follows, which represents the cutting transformation. It is now possible to perform the Cutting-Decimation transform to $p-polygonNT_D$ and get:

$$\tilde{d}_p = 1 + \frac{\ln k(p - 1)}{\ln n}$$

In the same way we can calculate the spectral dimension of an $NT_D$ built with $d-$dimensional simplexes instead of $p-polygons$. A $d$-dimensional simplex is a complete graph of $d+1$ points i.e. a graph where each point is nearest neighbour of all other points. The 2-dimensional case is the triangle, the 3-dimensional one is the tetrahedron and so on. Since for $d-$simplex $P_{O}(\lambda) \sim 1/(d+1)(1-\lambda)$ the spectral dimension is:

$$\tilde{d}_d = 1 + \frac{\ln kd}{\ln n}$$

### 14 Relation with other physical problems

As we have shown in previous chapters, the random walk problem is strictly related to graph topology. Indeed, the main physical quantities are simple functions of the adjacency matrix $A$, which algebraically describes the graph structure. Now, the Hamiltonians of a series of fundamental statistical models are linear in $A$, therefore even their behaviour is deeply influenced by topology and it can be expressed in terms of random walks functions. Due to this reason, the main concepts and parameters characterizing random walks, such as recurrence and transience, as well as the spectral dimension, determine also the properties of these models, which have very different physical origins. This provides a very powerful tool to investigate and classify geometrically disordered and inhomogeneous systems, where the usual techniques and ideas developed for lattices do not apply.

#### 14.1 The oscillating network

Probably, the physical model whose connection with random walks has been most extensively explored is the so-called oscillating network.

The harmonic oscillations of a generic network of masses $m$ linked by springs of elastic constant $K$ can be studied by writing the equations of motion of the displacements $x_i$ of each mass from its equilibrium position:

$$m \frac{d^2}{dt^2} x_i = -K \sum_j A_{ij} (x_i - x_j) = -K \sum_j \Delta_{ij} x_j$$

which after Fourier transforming with respect to the time reads:

$$\frac{\omega^2}{\omega_0^2} \tilde{x}_i = \sum_j \Delta_{ij} \tilde{x}_j$$
where $\omega_0^2 \equiv K/m$. In other words, the determination of the normal modes and of the normal frequencies of the oscillating network reduces to the diagonalization of the Laplacian operator $\delta$.

Noticing that $\Delta = Z(1 - P)$, where $1$ the identity matrix and $P$ is given by (4), it is not difficult to establish mathematical correspondences with random walks. In particular, using the universality properties discussed in the previous sections, one can show a fundamental result concerning the density $\rho(\omega)$ of normal modes at low frequencies:

$$\rho(\omega) \sim \omega^{d-1} \quad \text{for} \quad \omega \to 0 \quad (129)$$

This basic connection between random walks and harmonic oscillations was first introduced by Alexander and Orbach in 1982 for the case of fractals. Notice that at that time the splitting between local and average spectral dimension on inhomogeneous structures was not yet known and the exponent describing the scaling of the density of states at low frequencies was simply called spectral dimension, since it was related to the vibrational spectrum. Due to the already mentioned universality properties, the above result hold for the very general case where oscillating masses and elastic constants may have different values on different sites and links, provided they are bounded by positive numbers. More precisely, considering the equations of motions

$$m_i \frac{d^2}{dt^2} x_i = -K \sum_j J_{ij}(x_i - x_j) = -K \sum_j L_{ij} x_j \quad (130)$$

for the same graph of (127), if (49) holds together with

$$\exists m_{\text{min}}, m_{\text{max}} > 0 \mid m_{\text{min}} \leq m_i \leq m_{\text{max}} \quad \forall i \quad (131)$$

then the asymptotic behaviour of the density of vibrational states is still given by (129).

From all the above properties, it follows that also the spectrum of the Laplacian operator $L$ depends on $\bar{d}$: indeed it can be shown [10] that the spectral density $\rho(l)$ of $L$ at low eigenvalues behaves as $\rho(l) \sim l^{\bar{d}/2-1}$.

The average spectral dimension is crucial in determining the behaviour of the oscillating network in equilibrium with a thermal bath at temperature $T$. Considering the Hamiltonian of the system given by (129)

$$H = \sum_i \frac{p_i^2}{2m_i} + \frac{1}{2} m \omega_0^2 \sum_{ij} J_{ij} x_i x_j \quad (132)$$

and calculating the thermodynamic averages with respect to the Gibbs weight $\exp(-H/kT)$, where $k$ denotes the Boltzmann constant, one can show that, for positive $T$,

$$\langle x^2 \rangle = \infty \quad \text{for} \quad \bar{d} \leq 2 \quad (133)$$

while

$$\langle x^2 \rangle < \infty \quad \text{for} \quad \bar{d} > 2. \quad (134)$$
This is the generalization to graphs of the fundamental Peierls result about the thermodynamic instability of oscillating crystals in low dimensions. In other words, for an infinite oscillating network with \( d \leq 2 \) in equilibrium with a thermal bath, the mean square displacement of masses from their equilibrium positions would diverge.

14.2 The Gaussian model

The Gaussian model is the simplest statistical model used to study magnetic systems on lattices. Even if it is not realistic, its properties are fundamental to understand more complex and phenomenologically significant models. In field theory it is also known as ”free scalar field”. The Gaussian model on \( G \) is defined by the Hamiltonian:

\[
H = \frac{1}{2} \sum_{ij} \phi_i (JL_{ij} + m_i^2 \delta_{ij}) \phi_j - h \sum_i \phi_i
\]  

where \( \phi_i \) is a real field, \( J > 0 \) a ferromagnetic coupling, \( h \) an external magnetic field and \( m_i^2 = \alpha_i m_2 \), with \( 1/K < \alpha_i < K \) for some positive \( K \) [10]. Its specific free energy \( f_G \) is given by

\[
f_G(J, m_i^2, h) = \lim_{N \to \infty} \frac{1}{N} F = -\lim_{N \to \infty} \frac{1}{N} \log Z
\]  

where \( Z \) is the partition function calculated according to the Boltzmann weight \( \exp(-H) \). The spectral dimension is related to the singular part of \( f_G \) for \( h = 0 \) and \( m^2 \to 0 \) by:

\[
\text{Sing}(f) \sim m^d.
\]  

The covariance of this Gaussian process reads

\[
\langle \phi_i \phi_j \rangle \equiv C_{ij}(m^2) = (\Delta + m^2 \eta)^{-1}_{ij}
\]  

and hence it satisfies by definition the Schwinger–Dyson (SD) equation

\[
(J_i + m_i^2 \eta_i) C_{ij}(m^2) - \sum_{k \in G} J_{ik} C_{kj}(m^2) = \delta_{ij}
\]  

Setting

\[
C_{ij} = \frac{(1 - W)_{ij}^{-1}}{J_i + m_i^2 \eta_i}, \quad W_{ij} = \frac{J_{ij}}{J_j + m_j^2 \eta_j}
\]  

one obtains the standard connection with the random walk (RW) over \( G \) [9]:

\[
(1 - W)_{ij}^{-1} = \sum_{t=0}^{\infty} (W^t)_{ij} = \sum_{\gamma: i \to j} W[\gamma]
\]  

where the last sum runs over all paths from \( j \) to \( i \), each weighted by the product along the path of the one–step probabilities in \( W \):

\[
\gamma = (i, k_{t-1}, \ldots, k_2, k_1, j) \implies W[\gamma] = W_{ik_{t-1}} W_{k_{t-1} k_{t-2}} \cdots W_{k_2 k_1} W_{k_1 j}
\]
Notice that, as long as \( m > 0 \), we have \( \sum_i (W^t)_{ij} < 1 \) for any \( t \), namely the walker has a non-zero death probability. This implies that \( C_{ij} \) is a smooth function of \( m^2 \) for \( m \geq \epsilon > 0 \). In the limit \( m \to 0 \) the walker never dies and the sum over paths in eq. (141) is dominated by the infinitely long paths which sample the large scale structure of the entire graph (“large scale” refers here to the metric induced by the chemical distance alone). This typically reflects itself into a singularity of \( C_{ij} \) at \( m = 0 \) whose nature does not depend on the detailed form of \( J_{ij} \) or \( \eta_i \), as long these stay uniformly positive and bounded.

Of particular importance is the leading singular infrared behaviour, as \( m^2 \to 0 \), of the average \[ \langle C(m^2) \rangle \] of \( C_{ii}(m^2) \), which is a positive definite quantity, over all points \( i \) of the graph \( G \), which we may write in general as

\[
\text{Sing} \left[ \langle C(m^2) \rangle \right]_G \sim c(m^2)^{d/2-1} \tag{143}
\]

### 14.3 Spherical model and \( O(n) \) models

The spherical model is again a magnetic model with no direct connection to phenomenology. Nevertheless, is a little more complex than the Gaussian one and, most important, it exhibits phase transitions at finite temperature for \( d > 2 \). Moreover, its critical exponents can be exactly determined and they turn out to be simple functions of \( d \), pointing out the crucial role of the average spectral dimension in phase transitions and critical phenomena. The spherical model can be defined on a generic graph through the Hamiltonian (135) with the generalized spherical constraint \( \sum_i z_i \phi_i^2 = N \). We assume the coordination numbers to be bound: \( 1 \leq z_i \leq z_{\text{max}} \). Its free energy and correlation functions can be expressed in terms of the Gaussian ones. Then the critical behaviour is obtained from the infrared singularities of the latter, i.e. in terms of the long time behaviour of random walks. The results concerning the critical exponents are summarized in the following table, where \( T_c = 0 \) for \( d \leq 2 \):

The so called \( O(n) \) models are defined, for positive integer \( n \), by the Boltzmann weight \( \exp(-\beta H_n) \), where

\[
H_n[S] = \frac{1}{2} \sum_{\langle i,j \rangle} J_{ij}(S_i - S_j)^2 \tag{144}
\]

the sum extends to all links of a certain graph \( G \), \( J_{ij} > 0 \) are ferromagnetic interactions, which may vary from link to link, and \( S_i \) is an \( n \)-dimensional vector of fixed length normalized by \( S_i \cdot S_i = n \). They represent more realistic magnetic models, but their exact solution is in general impossible. However, a series of complex but powerful inequalities, relating their correlation functions to the random walks generating functions, allow proving some very general results shedding light on the complicated phenomena concerning phase transitions on graphs. In particular it has been proven that:

- they cannot have phase transitions at \( T > 0 \) if \( G \) is recurrent on the average;
\[
\begin{array}{|c|c|c|c|}
\hline
 & 1 \leq d < 2 & 2 < d < 4 & d > 4 \\
\hline
 T = T_c & \delta \to \infty & \delta = \frac{d+2}{d-2} & \delta = 3 \\
\hline
 T < T_c & - & \gamma' \text{ does not exist} & \gamma' = 1 \\
\hline
 T > T_c & \gamma = -\frac{2}{d-2} & \gamma = \frac{2}{d-2} & \gamma = 1 \\
\hline
 T < T_c & - & c = \frac{1}{2} K_B & c = \frac{1}{2} K_B \\
\hline
 T > T_c & \alpha = \frac{d}{d-2} & \alpha = \frac{d-4}{d-2} & \alpha = 0 \\
\hline
 T < T_c & - & \beta = \frac{1}{2} & \beta = \frac{1}{2} \\
\hline
\end{array}
\]

Table 1: Critical exponents of the spherical model on a graph of spectral dimension \( \bar{d} \)

- they exhibit phase transitions at \( T > 0 \) if \( \mathcal{G} \) is transient on the average;
- for \( n \to \infty \) their critical exponents tend to the spherical ones.

References

[1] R.Burioni, D.Cassi and A. Vezzani, Random walks and physical models on infinite graphs: an introduction, in Random Walks and geometry, V Kaimanovich, K. Schmidt and W. Woess Eds, de Gruyter, Berlin (2004)

[2] F. Harary, Graph theory, Addison-Wesley Reading (1994).

[3] F.Gantmacher, The Theory of Matrices, Chelsea. Publ. New York (1950).

[4] B. Mohar and W. Woess, Bull. London Math. Soc. 21 (1989) 209.

[5] Y. Gefen, A. Aharony and B.B. Mandelbrot, Phys. Rev. Lett. 45 (1980) 855.

[6] G. Polya, Math. Ann. 84 (1921), 149.
[7] W. Woess, *Random walks on infinite graphs and groups*, Cambridge University Press (2000).

[8] S. Alexander and R. Orbach, J. Physique Lett. 43 (1982) L62.

[9] K. Hattori, T. Hattori and H. Watanabe, Prog. Theor. Phys. Suppl. 92 (1987) 108.

[10] R. Burioni and D. Cassi, Phys. Rev. Lett. 76 (1996) 1091.

[11] D. Cassi, Phys. Rev. Lett. 76 (1996) 2941.

[12] D. Cassi and L. Fabbian, Jour. Phys. A 32 (1999) L93.

[13] R. Burioni, D. Cassi and A. Vezzani, Eur. Phys. Jour. B 15 (2000) 665.

[14] R. Burioni, D. Cassi, Phys. Rev. E 51 (1995) 2865.

[15] R. Burioni, D. Cassi and A. Vezzani, Jour. Phys. A 32 (1999) 5539.

[16] D. Bertacchi and F. Zucca, Journ. Stat. Phys. (2004) 947.

[17] R. Burioni, D. Cassi, and C. Destri, Phys. Rev. Lett. 85 (2000) 1496.

[18] R. Burioni and D. Cassi, Mod. Phys. Lett. B 11 (1997) 1095.

[19] R. Burioni, D. Cassi, and C. Destri, Jour. Phys. A 33 (2000) 3627.

[20] E. Montroll and G.H. Weiss, Jour. Math. Phys. 6 (1965) 167.

[21] R. Burioni, D. Cassi and S. Regina, Mod. Phys. Lett. B 10 (1996) 1059.

[22] D. Cassi e S. Regina, Phys. Rev. Lett. 70 (1993) 1647.

[23] D. Cassi e S. Regina, Phys. Rev. Lett. 76 (1996) 2914.

[24] R. Burioni, D. Cassi, A. Pirati and S. Regina, J. Phys. A 31 (1998) 5013.

[25] R. Rammal and G. Toulouse, Jour. Physique – Lettres 44 (1983) L13.

[26] R. Hilfer and A. Blumen, Jour. Phys. A 17 (1984) L537.

[27] S. Milosevic, D. Stassinopoulos and H.E. Stanley, Jour. Phys. A 21 (1988) 1477.

[28] D. Dhar, Jour. Phys. A 21 (1988) 2261.

[29] B. Kahng and S. Redner, Jour. Phys. A 22 (1989) 887.

[30] A. Maritan, G. Sartoni and A. L. Stella, Phys. Rev. Lett. 71 (1993) 1027.

[31] Y. Gefen, A. Aharony and B. B. Mandelbrot, Phys. Rev. Lett. 45 (1980) 855.

[32] P. J. Grabner and W. Woess, Stochastic Proc. Appl. 69 (1997) 127.
[33] A. Giacometti, A. Maritan and A. L. Stella, Inter. Jour. Mod. Phys. B, 5 (1991) 709.

[34] P. G. de Gennes, *Scaling Concepts in Polymer Physics*, Cornell University Press, Ithaca and London, (1979).

[35] R. Hilfer and A. Blumen, in *Fractals in Physics*, Proc. VI Trieste Int. Symp. (July, 1985), Eds. Pietronero L. and Tosatti E., North-Holland, Amsterdam, (1986).

[36] P. G. Doyle and J. L. Snell, *Random Walks and Electric Networks*, (The Mathematical Association of America, 1984)

[37] L. Donetti, condmat 0206149

[38] D. Cassi and S. Regina, Phys. Rev. Lett., 70, (1993) 1647.
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