EXTRA STATES IN $c < 1$ STRING THEORY

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ABSTRACT: A construction of elements of the BRS cohomology of ghost number $\pm 1$ in $c < 1$ string theory is described, and their two-point function computed on the sphere. The construction makes precise the relation between these extra states and null vectors. The physical states of ghost number $+1$ are found to be exact forms with respect to a “conjugate” BRS operator.

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1. Introduction

The no-ghost theorem[1] for critical string theory states that the BRS cohomology classes can be represented by

\[ c(z)\bar{c}(\bar{z}) \, V(z, \bar{z}) \]  

(1)

where \( V(z, \bar{z}) \) is a dimension (1, 1) primary field of the \( c = 26 \) matter conformal field theory, \( c(z) \) is the holomorphic spin \(-1\) ghost and \( \bar{c}(\bar{z}) \) is its antiholomorphic counterpart. There is an interesting exception to this theorem: the identity field of the combined matter-ghost theory is certainly in the BRS cohomology, but is not of the form given in Eq.(1). Indeed, if the ghost number of the states in Eq.(1) is chosen by convention to be \((0, 0)\), then that of the identity becomes \((-1, -1)\). In addition, one finds a state of ghost number \((1, 1)\) and a few more of mixed ghost number \((1, 0)\), \((-1, 0)\) and so on.

The chiral BRS operator for bosonic string theory in a given background is

\[ Q_B = \oint dz : c(z) \left( T^M(z) + \frac{1}{2} T^G(z) \right) : \]

(2)

where \( T^M(z), T^G(z) \) are the holomorphic stress-energy tensors of the \( c = 26 \) matter theory and the \( c = -26 \) ghost system respectively, and \( L_n^M \) are the modes of \( T^M(z) \). In the closed string theory, we are interested in the cohomology of \( Q_B + \bar{Q}_B \), the sum of the chiral and antichiral BRS charges. More precisely, we must restrict to the cohomology on the subspace annihilated by \( b_0^- \equiv b_0 - \bar{b}_0 \) where \( b(z) \) is the chiral antighost field. This is an example of a “relative” cohomology.

It can be shown that the cohomology for the closed string can be reconstructed from a knowledge of that for the open string, in other words, the cohomology of the chiral BRS operator alone. Even more, one can restrict to the “relative” chiral cohomology, where we consider only the subspace annihilated by \( b_0 \). The no-ghost theorem for the relative chiral cohomology of the critical string says that the only physical states are of the form \( c_1 |V\rangle \) where \( |V\rangle \) is a chiral primary state of dimension 1. The vacuum state \( |0\rangle \), along with a finite number of other states, provides an exception to the theorem. It is important to note that all these states have zero 26-momentum. Such exceptional states are very few within the enormous classical phase space of the critical string, and do not seem to be associated with significant physical effects.

The situation is very different for non-critical string theories in conformal backgrounds with \( c \leq 1 \). In the present article I will concentrate only on the case \( c < 1 \), where the background matter theory is a minimal model[2]. In this case, the full Hilbert space is
obtained by taking the direct product of the matter CFT, the Liouville theory (which is believed to describe the effect of quantized two-dimensional gravity when the matter is non-critical), and the ghost system. The BRS operator described above is generalized by adding the Liouville stress-energy tensor to the matter one, wherever the latter appears.

We parametrize the matter and Liouville central charges as

\[ c_M = 13 - 6/t - 6t \]
\[ c_L = 26 - c_M = 13 + 6/t + 6t \]  (3)

with \( t = q/p > 0 \). Here, \( p \) and \( q \) are two positive, coprime integers. The Liouville stress-energy tensor is

\[ T^{(L)}(z) = -\frac{1}{2}(\partial \Phi^{(L)} \partial \Phi^{(L)} + Q_L \partial^2 \Phi^{(L)}) \]  (4)

where

\[ Q_L = \sqrt{\frac{25 - c^{(M)}}{3}} \]
\[ = \sqrt{2}(\sqrt{t} + 1/\sqrt{t}) \]  (5)

The Virasoro generators following from this are

\[ L_n^{(L)} = \frac{1}{2} \sum_{m=-\infty}^{\infty} : \alpha_{n-m}^{(L)} \alpha_m^{(L)} : + \frac{iQ}{2} (1+n) \alpha_0^{(L)} \]  (6)

We follow the convention that Liouville vertex operators are defined as

\[ V_{k_L}^{(L)} = : e^{i k_L \Phi^{(L)}} : \]

with conformal dimension \( \Delta_{k_L}^{(L)} = -\frac{1}{2} k_L (k_L - Q_L) \) and \( \alpha_0^{(L)} \) eigenvalue \(-i k_L\). Thus a Liouville primary of given conformal dimension can have two possible values of momentum, denoted \( k_L^\pm \), where \( k_L^+ > Q_L/2 \), \( k_L^- < Q_L/2 \) and \( k_L^+ + k_L^- = Q_L \). We will denote the Fock space above a momentum \( k_L > Q_L/2 \) as the “(+)-Fock space”, and the other one as the “(-)-Fock space”.

Then, a class of physical states in the (relative, chiral) cohomology is again given by states like \( c_1 |V\rangle \), where this time \( |V\rangle \) is the direct product of a matter primary and a Liouville momentum state:

\[ |V\rangle = |\Psi\rangle_M \otimes |k_L\rangle_L \]  (7)

Here, the Liouville momentum is adjusted such that the Liouville dimension \( \Delta_L \) and the matter primary dimension \( \Delta_M \) satisfy \( \Delta_L + \Delta_M = 1 \). Such states (two for every matter primary, because of the two possible values of the Liouville momenta) are known as “DDK states” [3].

Just like the critical string, the non-critical string also has exceptional states which are not of the above kind, among which one example is the vacuum state. However, it has been
shown by Lian and Zuckerman[4] that in this case there are infinitely many exceptional physical states (we will call them “LZ states”) in the relative chiral cohomology, at all positive and negative values of the ghost number. At the same time, the DDK states, analogous to the dimension 1 primaries of the critical string, are finite in number for minimal model backgrounds, since each minimal model has only finitely many primary fields. Thus in these theories, the number of exceptional states is a lot larger than that of the “normal” states, a situation quite different from the critical string. One may expect these states to play an important physical role in the analysis of non-critical string theories. (Various arguments have been put forward of late to support the idea that there are actually infinitely many DDK states in minimal backgrounds, because the decoupling of null vectors fails in the present of gravity due to “contact terms” at boundaries of moduli space. It remains true that the proportion of exceptional states of non-trivial ghost number to the states of ghost number 0 is significantly greater than for the critical string.)

The DDK states are often thought of as matter primaries “dressed” by a Liouville momentum state to have total dimension 1. The principal observation in Ref. 4 is that given a primary field of the matter CFT, exceptional physical states appear in the module whenever the Liouville momentum is one which “dresses” a null vector over the matter primary. In minimal models, each primary has an infinite chain of null vectors over it. The ghost number of the physical state turns out to be equal in magnitude to the distance of the associated null vector in this chain from the original matter primary. The sign of the ghost number is positive or negative if the Liouville momentum lies in the (+) or (−) Fock space respectively. The dimension of the relative chiral BRS cohomology is precisely 1 in every such case. Thus there is an infinite set of physical states for each matter primary, one for each null vector over it and for each sign of the ghost number.

In the rest of this article I will summarise an explicit method of construction for a large class of LZ states, in arbitrary minimal models coupled to gravity. The details, including proofs of a number of theorems, can be found in Ref.[5].

2. Null Vectors

Degenerate fields in a $c < 1$ CFT are those which have null vectors in the Verma module above them. Their dimensions are given by the Kac formula:

$$\Delta_{r,s}^{(M)} = \frac{(r^2 - 1)}{4} \frac{1}{t} + \frac{(s^2 - 1)}{4} t - \frac{(rs - 1)}{2} t (r,s)$$

with $t = q/p$, $1 \leq r \leq q - 1$, $1 \leq s \leq p - 1$.

Each such field has infinitely many null vectors above it. We will refer to the lowest of these null vectors in a module as “primitive”. The primitive $(r, s)$ null vector over $\Psi_{r,s}^{(M)}$
has dimension
\[ \Delta_{r,s}^{(M)(null)} = \Delta_{r,s}^{(M)} + rs \\
= \frac{(r^2 - 1)}{4} \frac{1}{t} + \frac{(s^2 - 1)}{4} \frac{1}{t} + \frac{(rs + 1)}{2} \] (9)

Now, an LZ state will occur whenever we consider a Liouville momentum which “dresses” the above dimension:
\[ \Delta_{r,s}^{(L)} = 1 - \Delta_{r,s}^{(M)(null)} \\
= \frac{1 - rs}{2} - \frac{(r^2 - 1)}{4} \frac{1}{t} - \frac{(s^2 - 1)}{4} \frac{1}{t} \] (10)

From the dimension formula above for Liouville vertex operators, it follows that the two Liouville momenta are
\[ k_L^{\pm} = \frac{1}{\sqrt{2}} \left( \frac{(1 \pm r)}{\sqrt{t}} + (1 \pm s) \sqrt{t} \right) \] (11)

According to the Lian-Zuckerman theorem, a physical state of ghost number +1 occurs in the Fock module above \( k_L^+ \), while a state of ghost number \(-1\) occurs above \( k_L^- \).

There exists a rather neat formula, due to Benoit and Saint-Aubin[6] for the null vector in the Verma module of any CFT above a primary satisfying the Kac formula(8) for the special case \( r = 1, \) \( s \) arbitrary:
\[ |\Psi_{1,s}^{(null)(Vir)}\rangle = \sum_j \sum_{p_1+p_2+\cdots+p_j=s, \ p_i \geq 1} \sum_{j=1}^{s-j} \frac{(s-1)!^2}{\prod_{i=1}^{j-1} (p_1 + \cdots + p_i) (r - p_1 - \cdots - p_i)} \frac{1}{L-p_1 L-p_2 \cdots L-p_j} |\Psi_{1,s}\rangle \] (12)

where \( t \) parametrizes the central charge via the second equation in Eq.(3). We will ultimately apply this to Liouville theory with \( c > 25 \), hence positive \( t \).

One can ask how this null vector descends to the Fock space, for theories with a Fock space description. It may in principle vanish identically when re-expressed in oscillators, in which case it is in the kernel of the projection map from the Verma module to Fock space. In this case, from well-known arguments, there must be a state in the Fock space which is not in the image of the projection. Alternatively, the projection to Fock space may have no kernel in this module, in which case every state in the Fock space lies in the image of the projection. This situation was analyzed some years ago by Kato and Matsuda[7]. For our purposes we need a stronger result than that of Ref.[7], namely:

**Theorem 1:** For \( t > 0 \),
\[ |\Psi_{1,s}^{(null)(Vir)}\rangle \rightarrow 0, \ k_L < \frac{Q_L}{2} \]
\[ |\Psi_{1,s}^{(null)(Fock)}\rangle \rightarrow \prod_{k=1}^{s} (kt + 1) |\Psi_{1,s}^{(null)(Fock)}\rangle, \ k_L > \frac{Q_L}{2} \]
where the state $|\Psi_{1,s}^{null}(Fock)\rangle$ is a null state in the Fock module which is non-vanishing for all values of $t$, and the arrow indicates the projection map.

The proof of this theorem is given in Ref.[5]. It shows that in general, for $t > 0$, the projection to the $(-)$ Fock space has a kernel, while the projection to the $(+)$ Fock space does not. But in fact we learn something even for $t < 0$. In this case, the result of Theorem 1 holds with a possible interchange of $(+)$ and $(-)$ Fock spaces, so that for the values $t = -1/k, k = 1, \cdots, s$, the projection to each Fock space has a kernel. In other words, for these special values of $t$, null vectors vanish when expressed in oscillators, in both Fock spaces. This result will be crucial in the subsequent analysis. Note that this in particular holds for $t = -1$ at every value of $s$, confirming the well-known result that Virasoro null vectors in $c = 1$ CFT vanish identically in terms of oscillators.

3. Construction of LZ States

Although the LZ states occur in correspondence with matter null vectors, they are not themselves null in any sense. They are to be found in the module above a matter primary and a Liouville momentum which has the right dimension to dress a matter null vector. This means that the Liouville momentum $k_L$ corresponds to a conformal dimension $\Delta^{(L)}$ which satisfies

$$\Delta^{(L)} + \Delta^{(M)(null)} = 1$$

(13)

where $\Delta^{(M)(null)}$ is the dimension of some null vector above the chosen matter primary.

Because of the well-known relation[8] between null vectors for two theories of central charge $c$ and $26 - c$, we can re-state the above result in a complementary way: whenever there is a Liouville null vector (in the Fock module above a given momentum state) of dimension $\Delta^{(L)(null)}$ satisfying

$$\Delta^{(L)(null)} + \Delta^{(M)} = 1$$

(14)

for some matter primary of dimension $\Delta^{(M)}$, the module contains an LZ state.

To find these extra physical states, we start with a primitive $(1, s)$ Liouville null vector (in the Liouville Verma module) and the $(1, s)$ matter primary, combined into the state

$$|X^{(0)}\rangle = |\Psi^{(null)}_{L}(k_{L}^{\pm})\rangle_{L} \otimes |\Psi_{1,s}^{(M)}\rangle_{M} \otimes |c_{1}\rangle_{G}$$

(15)

The total dimension of this state is zero, by virtue of Eq.(14) and the fact that the ghost mode $c_{1}$ has dimension $-1$. It has the same ghost number as DDK states, which we have chosen to call 0 by convention.
We will construct physical states of ghost number $\pm 1$ starting from the state defined above (this state is itself null, since its Liouville sector is null.) Let us first define a conjugation operator

$$\mathcal{C} : (\pm) \text{ Fock space } \rightarrow (\mp) \text{ Fock space}$$

(16)

which, acting on a state, simply replaces $k_L$ by $Q_L - k_L$. Clearly, $\mathcal{C}$ acts in either direction, and $\mathcal{C}^2 = 1$. Using this operator we define a “conjugate” BRS charge:

$$Q_B^* \equiv \mathcal{C} Q_B \mathcal{C}$$

(17)

which is nilpotent:

$$(Q_B^*)^2 = 0$$

(18)

by virtue of the nilpotence of $Q_B$ and the fact that $\mathcal{C}^2 = 1$.

Now, the state in Eq.(15) has the following property: because of Theorem 1, $|X^{(0)}\rangle$ is in the kernel of the projection to the $(\mp)$ Fock space, so it vanishes identically when expressed in oscillators. On the other hand, as long as $t \neq -1/k$ (in fact $t$ is positive for Liouville theory), this state is not in the kernel of the projection to the $(\pm)$ Fock space. Hence it descends to a non-vanishing Fock space state, which is, however, both primary and secondary and hence null in the usual sense. In particular this means that it is $Q_B$-exact, and hence of course closed. Nevertheless, the operator $Q_B^*$ that we have just defined has a non-trivial action on it. Indeed, define

$$|X^{(1)\mp}\rangle \equiv Q_B^* |X^{(0)\pm}\rangle$$

(19)

Here, the $\mp$-superscript indicates that we are dealing with states in the $(\mp)$ Fock space. The action of $Q_B^*$ is to first conjugate the state to the $(\pm)$ Fock space, (where it becomes a non-primary, non-secondary state with respect to the Virasoro algebra!), then act with $Q_B$, which produces a nonzero result on such a state, and finally conjugate back to the $(\pm)$ Fock space. The result is a state of ghost number $+1$ in the $(\pm)$ Fock space, and we have:

**Theorem 2:** The state $|X^{(1)\mp}\rangle$ defined in Eq.(19) above is a LZ state of ghost number $+1$.

This is a remarkably simple result, and clarifies a conceptual point: although LZ states are not themselves null, they are $Q_B^*$-variations of null states. The fact that $Q_B^*$ has very different properties from $Q_B$ reflects a deep property of minimal matter coupled to gravity: the $(\pm)$ and $(\mp)$ Liouville Fock spaces are very different, a fact which has played an important role in several contexts[7][9][10][11][5].
Before discussing the proof of this theorem, we state the corresponding result for the conjugate LZ state, which according to Ref.[4] should lie in the $(-)$ Fock space and have ghost number $-1$. Let us define the operator

$$K \equiv \sum_{n=1}^{\infty} nc_n c_n$$

This operator is produced by anticommuting $Q_B$ and $Q_B^*$, as one can easily check by explicit computation:

$$\{Q_B, Q_B^*\} = (k_L^+ - k_L^-)^2 K$$

A property of $K$ that can be checked is that it has no kernel on states of ghost number $-1$, hence its inverse exists on ghost number $+1$, and is given by

$$K^{-1} = \sum_{n=1}^{\infty} \frac{1}{n} b_{-n} b_n$$

Consider now the Fock space state

$$|X^{(-1)+}\rangle = K^{-1} |X^{(1)+}\rangle$$

of ghost number $-1$. Since it is in the $(+)$ Fock space, where the projection from the Verma module has no kernel, we can rewrite the left hand side of this equation in terms of (Liouville and matter) Virasoro secondaries acting on suitable primary states. Then, project this to the $(-)$ Fock space. This procedure defines a state which we call $|X^{(-1)-}\rangle$, of ghost number $-1$. This brings us to

**Theorem 3:** The state $|X^{(-1)-}\rangle$ defined above is a LZ state of ghost number $-1$.

Indeed, it is clear that $|X^{(-1)-}\rangle$ and $|X^{(1)+}\rangle$ are built on the same Liouville primary, but the Liouville momenta are respectively $k_L^-$ and $k_L^+$, whose sum is precisely $Q_L$. Thus this pair of states can have a non-vanishing inner product between them. The computation of this inner product proves Theorems 2 and 3, according to which these states are genuinely in the cohomology, for the following reason. The state $|X^{(1)+}\rangle$ is closed, because of its definition Eq.(19), the anticommutation relation Eq.(21), and the fact that $|X^{(0)}\rangle$ is annihilated by both $Q_B$ and $K$. This in turn implies that $|X^{(-1)-}\rangle$ is closed, since $Q_B$ commutes with $K$ (an immediate consequence of Eq.(21)). Now if both states are closed, and if their inner product is nonvanishing, it follows that neither of them is exact, which proves Theorem 2 and 3 above. We find that in fact the inner product is:

$$\langle l |X^{(-1)-}c_0|X^{(1)+}\rangle = s(s-1)!^2 \prod_{n=1}^{s-1} \left( (nt)^2 - 1 \right)$$
which is in fact nonvanishing for all \( t \) other than \( \pm 1, \pm \frac{1}{2}, \ldots, \pm \frac{1}{s-1} \). Apart from the first pair of values, which correspond to \( c = 1, 25 \), the remaining are “unphysical” minimal models. Hence for all genuine \( c < 1 \) minimal models, the inner product above is nonvanishing, and our construction gives the LZ states associated to the matter primaries of type \((1, s)\), at ghost number \( \pm 1 \). For \( c = 1 \) it does not work, a fact which may merit further investigation.

The detailed proof of Eq.(24), which is the principal result of this work, is given in Ref.[5], along with various other expressions for the physical states, and the inter-relations among them. Here I will just sketch the proof. Starting from the Benoit-Saint-Aubin formula, Eq.(12), one can convince oneself that all the states that we have constructed above, and their inner products, are polynomials in \( t \). (This is not true for null vectors of type \((r, s)\) for \( r \neq 1 \), where one finds polynomials in \( t \) and \( \frac{1}{t} \).) From the asymptotic behaviour of the formula, one can evaluate the asymptotic behaviour, for large \( t \), of the inner product. This suffices to fix the degree of the polynomial and the leading coefficient. It remains only to determine the zeroes. Precisely half of them are obtained from Theorem 1 above, since the special values of \( t \) at which the null vector vanishes on projecting to both Fock spaces are clearly values for which the states we constructed above, and hence their inner product, vanish. The remaining zeroes follow from the fact that the transformation \( t \to -t \) interchanges matter and Liouville sectors, and one can argue that the inner products have a definite parity under this transformation. This completes the derivation.

Although this derivation only works starting from the special null vectors of type \((1, s)\), this appears to be only a technical limitation. One can check in explicit examples that the same construction works also in situations where \( r \neq 1 \), but a general proof for this is not available at present. Another, more serious, limitation is the restriction to ghost numbers \( \pm 1 \).

4. Conclusions

More than a year after the discovery of infinitely many extra physical states of every ghost number in the \( c < 1 \) string, their physical interpretation remains unclear. (Recently there has been very interesting progress in the corresponding problem for the \( c = 1 \) string[12][13].) An understanding of the role played by these states is likely to help clarify the situation regarding correlation functions in \( c < 1 \) strings, on which a lot of work has been done but a clear understanding reconciling all the different approaches remains to be achieved (see the lecture of V. Dotsenko in this volume, and references therein.)

The present understanding of these correlation functions from the continuum approach is based on analytic continuation and the eventual insertion of a fractional and/or negative
number of screening charges in the Liouville sector. The fact that even vertex operators outside the minimal table acquire non-vanishing correlators in this framework, described by the heuristic expression that “null vectors do not decouple in the presence of gravity”, tends to agree with the qualitative feature that there are infinitely many scaling fields in the matrix model and topological approaches. On the other hand, we now have infinitely many LZ physical states, which one might also be tempted to identify with this infinity of scaling fields. It needs to be understood whether these states are in some sense an alternate representation of the null vectors which “do not decouple”, or should be interpreted in some different way. (Some aspects of the relation between matrix model and continuum CFT fields are discussed in Ref.[14].)

The present work is an attempt not at addressing this question directly, but rather at formulating and analysing the explicit form of LZ states, which are much more non-trivial to write down than DDK states. An explicit algorithm (quite distinct from the “brute-force” method that one can always use) was obtained to construct the extra LZ states. Explicit examples are worked out in Ref.[5]. The algebraic structure related to the conjugate BRS operator $Q^*_B$ remains somewhat mysterious, and perhaps once this is clarified then the physical interpretation of LZ states for $c < 1$ will become more evident.

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