I = 2 ππ scattering using G-parity boundary condition

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To make the ππ state with non-zero relative momentum as the leading exponential, we impose anti-periodic boundary condition on the pion, which is implemented by imposing G-parity or H-parity on the quark fields at the boundary. With this, we calculate the I = 2 ππ phase shift from lattice simulation by using Lüscher’s formula.

1. Introduction

Lattice gauge theory provides a way to investigate low energy physics of QCD, which cannot be done using any perturbative method. One of the interesting physical quantities is related to ππ system. The K → ππ WME which violates CP symmetry is of particular interest. Since lattice calculations easily extract ground state, the need to generate the final ππ state with non-trivial relative momentum is a serious difficulty. We proposed to use G-parity boundary conditions to overcome this difficulty[2]. In the present paper, we have implemented this G-parity idea and a new H-parity boundary condition in numerical simulations and calculated the I = 2, ππ phase shift.

2. G parity boundary condition

Since the G-parity operation on a pion gives
\[ G|\pi^\pm > = -|\pi^\pm >, \quad G|\pi^0 > = -|\pi^0 >, \]
by applying this operation on the boundary, we can impose anti-periodic boundary condition on pion. To implement this condition on lattice, we have to use the G-parity operation on the quark fields:
\[ G \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} -d^C \\ u^C \end{pmatrix}. \]
In actual calculation, we impose this boundary condition only in the z-direction so that we have a pion with non-zero z-momentum. Because at the boundary there are terms such as ψψ and ψψ, it requires some special care to implement. First, we have to impose a charge-conjugate boundary condition on the gauge field to keep gauge invariance and we have to virtually double the box size for the Dirac operator inversion. Since isospin has an important role in the two pion system, it is worth noting that G-parity commutes with isospin, which means that under this unusual boundary condition isospin is still a good quantum number.

3. H parity boundary condition

An easier way to impose anti-periodic boundary conditions on a pion is to apply the following operation on the quark fields,
\[ H \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} -u \\ d \end{pmatrix}. \]
We will call this operation H-parity. Then, the operation on the pions will be,
\[ H|\pi^\pm > = -|\pi^\pm >, \quad H|\pi^0 > = |\pi^0 >. \]
Under this boundary condition, isospin is not a good quantum number anymore, but \( I_z \) is still good. Since we know that the \( I_z = 2, \pi\pi \) state is composed of two \( \pi^\pm \), this state must have non-zero relative momentum. This is not true for the \( I = 0 \) state. So the utility of this boundary condition is more limited than the G-parity boundary condition. However it has the advantage that it doesn’t require any modification of the gauge field boundary condition. This allows us to use existing lattices, including dynamical ones.
4. Results

4.1. Single pion

We first investigate the properties of the one-pion system. We expect to find a one-pion state with momentum $\frac{\pi}{T}$, unlike the conventional $2\pi L$. Figure 1 shows a graph of energy versus pion mass. As expected, the single pion state with smallest energy has $E(m_\pi) = \sqrt{m_\pi^2 + \sin^2(\pi/8)}$.

4.2. Two pions

Figure 2 shows the effective mass plot for the two-pion state. We have a very nice plateau in the time range from 3 to 11, but after that we have suspicious fall-off. This fall-off can be explained by considering Fig. 3. A pion created at $t = 0$ can propagate in either direction. Because we have two particles, there is a state in which the two particles propagate in opposite directions. The correlation function for this state is

$$G(t) \approx e^{-E_\pi(T-t)} e^{-E_\pi t} = e^{-2E_\pi T}$$

(5)

where $E_\pi$ means the energy of one pion with momentum. This is just a constant, and will be dominant near $t = \frac{T}{2}$ because the energy of the $I = 2$, two-pion state is bigger than $2E_\pi$ and can cause the abrupt fall-off in this region. We confirmed this idea by fitting the effective mass plot with the function “cosh + const.”, the solid line in Fig. 2.

Figure 4 shows the same two-pion state effective mass plot for the G-parity boundary condition. This G-parity effective mass plot is quite different from the one for H-parity. Instead of an abrupt fall off, it has gradual decrease. It even looks like it has two plateaus. A simulation with more time slices ($N_t = 48$) also shown in Fig. 4 demonstrates this two-plateau structure. Since the spatial lattice volume was small ($\approx (1.7\text{fm})^3$) for this calculation, we guessed that it might be a finite volume effect and performed the same simulation with a bigger volume. Figure 5 shows the result for a spatial volume $\approx 1.7\text{fm} \times 1.7\text{fm} \times 3.4\text{fm}$.

We notice that the gradual decrease of G-parity plot has disappeared and it is almost identical to that seen with H-parity. Therefore, we can conclude that it is a finite volume effect that caused the two plateau behaviour. This might mean that we have discovered a new finite volume $\bar{q}q\bar{q}q\bar{q}$ state.

After becoming convinced from these tests that we have a two-pion state with non-zero relative momentum, we extended this simulation and cal-
calculated the $I = 2 \pi\pi$ phase shift. Since we are using Lüscher’s formalism\cite{1}, this is nothing but spectroscopy. Figure 5 shows our phase shift calculation including CP-PACS\cite{4} and experimental results. The following are our $\delta_{\pi\pi}$ simulation parameters ($1/a$ is in GeV and $N_t=32$):

Domain Wall Fermions, $L_s=10$, $M_5=1.65$

- $p \approx 250\text{MeV}$
  - G $8^2 \times 16$ $1/a = 0.978(14)$ Wilson 91conf
  - H $8^2 \times 16$ $1/a = 0.978(14)$ Wilson 172conf
- $p \approx 450\text{MeV}$
  - H $8^3$ $1/a = 0.978(14)$ Wilson 269conf
  - H $16^3$ $1/a = 1.98(3)$ DBW2 156conf

5. Conclusion

We have tested the idea of imposing anti-periodic boundary conditions on the pion by applying a G-parity or H-parity operation on the quark field at the boundary. For the one pion case, we found that the energy of the particle is given by $\sqrt{m_{\pi}^2 + \sin^2(\frac{\pi}{L})}$ as expected. We can achieve a two-pion state with non-zero momentum, and from this relative momentum, we can calculate the $I = 2, \pi\pi$ phase shift. Since the H-parity boundary condition can be applied to existing lattices, it will be more convenient than G-parity. In particular, we plan to use the H-parity boundary condition for a $\Delta I = 3/2 K \to \pi\pi$ calculation on existing lattices. However, for the $I = 0, \pi\pi$ state only the G-parity boundary condition with dynamical lattices will work. Since the G-parity boundary condition is more vulnerable to finite volume effects, e.g. the $\bar{q}q\bar{q}q$ state which we need to study further, it may require more resources.

REFERENCES

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