The Husimi distribution, the Wehrl entropy and the superradiant phase in spin–boson interactions

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Abstract

We study the Husimi distribution of the ground state in the Dicke model of field–matter interactions to visualize the quantum phase transition, from normal to superradiant, in phase space. We follow an exact numerical and variational analysis, without making use of the usual Holstein–Primakoff approximation. We find that the Wehrl entropy of the Husimi distribution provides an indication of the sharp changes in symmetry through the critical point. Additionally, we note that the zeros of the Husimi distribution characterize the Dicke model quantum phase transition.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The study of quantum phase transitions (QPTs) is an important subject in many-body quantum physics [1]. If we consider a quantum system described by the Hamiltonian $H = H_0 + \lambda H_1$, where $H_0$ and $H_1$ have different symmetries and $\lambda$ is a control parameter, then QPT occurs when $\lambda$ reaches a critical value $\lambda_c$ for which the properties of the system change suddenly.

In this work, we will analyze phase-space properties for a QPT and, for this purpose, we will consider the representative Dicke model of spin–boson interactions (see, e.g., [2–5]). There are several distributions to analyze phase-space properties [6]; the most popular one is the Wigner distribution, but there is another important one, the Husimi distribution, which has the interesting property of non-negativity and is defined as the overlap between a minimal uncertainty (coherent) state and the wavefunction. Recently, we proposed the Husimi distribution as a tool for a phase-space visualization of QPTs using two algebraic models to exemplify the study: the Dicke model [7] and the vibron model [8]; the latter is used for studying rotational and vibrational spectra in diatomic and polyatomic molecules, which also exhibit a (shape) QPT. In [7], we made use of the Holstein–Primakoff approximation [9] (large spin $j$) to approximate the atomic sector by a harmonic oscillator for a large number of atoms $N = 2j$. Here we will not use this approximation and we will work with finite $N$ in an exact manner.

The advantage of working in phase space is that we can analyze contributions in position and momentum space jointly. Additionally, we have characterized QPTs using the zeros of the Husimi distribution. Other information theoretic measures for QPTs in the Dicke and vibron models have recently been studied in position and/or momentum spaces separately. In particular, it has been shown that there is an abrupt change in the Rényi entropy [10], Fisher information [11] and complexity measures [12] at the transition point in the Dicke model. Moreover, it has been found that the uncertainty Shannon [13] and Rényi [14, 15] entropic relations account for the QPTs better than other variance-based uncertainty relations. See also [16] for a recent
paper on vibration–rotation entanglement measures of vibron models in the “rigidly bent” phase.

The structure of the paper is as follows. In section 2, we briefly recall the Dicke model, boson and spin–j coherent states and present the Husimi distribution (without the Holstein–Primakoff approximation) and the Wehrl entropy. In section 3, we present numerical and variational results in terms of symmetry-adapted coherent states. Three-dimensional (3D) plots, contour lines and the Wehrl entropy of the Husimi distribution reveal a drastic change in the symmetry of the ground-state wave function and provide a signature of the QPT even for a finite number of particles. Finally, zeros of the Husimi distribution (in the variational approximation) are also computed and graphically represented to characterize the QPT.

2. The Dicke Hamiltonian, the Husimi distribution and the Wehrl entropy

The single-mode Dicke model is a well-studied object in the field of QPTs [2, 3, 5]. In this case the Hamiltonian is given by

\[ H = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{2}} \left( (a^\dagger + a) J_z + J_+ \right), \] (1)

describing an ensemble of \( N \) two-level atoms with level-splitting \( \omega_0 \), with \( J_z, J_+ \) the angular momentum operators for a pseudospin of length \( j = N/2 \), and \( a \) and \( a^\dagger \) are the bosonic operators of the field with frequency \( \omega \). It is well known that there is a QPT at the critical value of the coupling parameter \( \lambda = \lambda_c = \frac{\omega_0}{\sqrt{2} \omega} \) from the so-called normal phase \( (\lambda < \lambda_c) \) to the superadiabatic phase \( (\lambda > \lambda_c) \).

Let us consider a basis set \( \{|n; j, m\} \equiv \{|n\} \otimes |j, m\rangle \) of the Hilbert space, with \( \{|n\}\) the number states of the field \( n \) and \( \{|j, m\}\) the so-called Dicke states of the atomic sector. The matrix elements of the Hamiltonian in this basis are

\[ \langle n'; j', m' | H | n; j, m \rangle = (\text{ constant terms}) \delta_{n', n} \delta_{m', m} \]

\[ + \frac{\lambda}{\sqrt{2} j} \left( \sqrt{n+1} \delta_{n', n+1} + \sqrt{n} \delta_{n', n-1} \right) \]

\[ \times \left( \sqrt{j(j+1)} - m(m+1) \delta_{m', m+1} + \sqrt{j(j+1)} - m(m-1) \delta_{m', m-1} \right). \]

(2)

At this point it is important to note that time evolution preserves the parity \( e^{i\pi(n+m+1)/2} \) of a given state \( |n; j, m\rangle \). That is, the parity operator \( \hat{P} = e^{i\pi(a^\dagger a + J_+ J_-)/2} \) commutes with \( H \) and both operators can then be jointly diagonalized. In particular, the ground state must be even (see equation (13)).

Let us denote by

\[ |\alpha\rangle = e^{-|\alpha|^2/2} e^{i\alpha a^\dagger} |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \]

\[ |\alpha\rangle = (1 + |z|^2)^{-j/2} e^{iJ_z} |j, -j\rangle \]

\[ = (1 + |z|^2)^{-j/2} \sum_{m=-j}^{j} \binom{2j}{j+m}^{1/2} i^{j+m} z^{j+m} |j, m\rangle, \]

(3)

(with \( \alpha, z \in \mathbb{C} \)) the standard (canonical or Glauber) and spin–j coherent states for the photon and the particle sectors, respectively. It is well known (see, e.g., [17]) that coherent states form an overcomplete set of the corresponding Hilbert space and fulfill the closure relations or resolutions of the identity:

\[ 1 = \frac{1}{\pi} \int_{\mathbb{R}^2} |\alpha\rangle \langle \alpha| \, d^2\alpha, \]

\[ 1 = \frac{2j+1}{\pi} \int_{\mathbb{R}^2} |z| |\alpha\rangle \langle \alpha| \, d^2z, \]

(4)

with \( d^2w \equiv d\text{Re}(w) \, d\text{Im}(w) \) (or \( d^2w = r \, dr \, d\theta \) in polar coordinates \( w = r e^{i\theta} \)) the Lebesgue measure on \( \mathbb{C} \). The complex parameters \( \alpha \) and \( z \) are related to the mean number of photons, as \( \langle \alpha^\dagger \alpha \rangle = |\alpha|^2 \), and the mean fraction of excited atoms, as \( \langle z J_z + j \rangle = N|z|^2/(1 + |z|^2) \), respectively. It is also clear that the probability amplitude of detecting \( n \) photons and \( j + m \) excited atoms in \( |\alpha, z\rangle \equiv |\alpha\rangle \otimes |z\rangle \) is given by

\[ \psi_{n,m}^{(j)}(\alpha, z) = \langle n|\alpha\rangle \langle j, m|z\rangle = \frac{e^{-|\alpha|^2/2}|\alpha|^m}{\sqrt{n!} \sqrt{m!}} \sqrt{\frac{2j+1}{(1 + |z|^2)^j}}. \]

(5)

The ground-state vector \( \psi \) will be given as an expansion

\[ |\psi\rangle = \sum_{m=0}^{n} \sum_{j=0}^{j} c_{nm}^{(j)} |n; j, m\rangle, \]

(6)

where the coefficients \( c_{nm}^{(j)} \) are calculated by numerical diagonalization of (2), with a given cutoff \( n_c \), and depend on the control parameter \( \lambda \). The Husimi distribution of \( \psi \) is then given by

\[ \Psi(\alpha, z) = |\langle \alpha, z|\psi\rangle|^2 = \sum_{n,n'=0}^{n_c} \sum_{m=0}^{m} c_{nm}^{(j)} c_{n'm}^{(j)} \psi_{n,m}(\alpha, z) \psi_{n',m}(\tilde{\alpha}, \tilde{z}), \]

and normalized according to

\[ \int_{\mathbb{R}^2} \Psi(\alpha, z) \, d\mu(\alpha, z) = 1, \]

(8)

with integration measure

\[ d\mu(\alpha, z) = \frac{2j+1}{\pi} \frac{d^2\alpha \, d^2z}{(1 + |z|^2)^j}. \]

(9)

An important quantity to visualize the QPT in the Dicke model across the critical point \( \lambda_c \) will be the Wehrl entropy

\[ W_j(\lambda) = -\int_{\mathbb{R}^2} \Psi(\alpha, z) \ln(\Psi(\alpha, z)) \, d\mu(\alpha, z), \]

(10)

where the dependence of \( W_j \) on \( \lambda \) comes from the dependence of \( c_{nm}^{(j)} \) on \( \lambda \).

3. Numerical versus variational results

In figure 1, we present a 3D plot of the exact Husimi distribution of the ground state \( \Psi(\alpha, z) \) in ‘position’
Figure 1. 3D plot of the exact Husimi distribution in (left) ‘position space’ (α and z real) and (right) ‘momentum space’ (α and z imaginary) for different values of λ. (from top to bottom: λ = 0, λ = 0.6 and λ = 1) for j = 3 and ω = ω0 = 1 ⇒ λc = 0.5. Atomic units are used.

(α and z real) and ‘momentum’ (α and z imaginary) spaces. We observe that the Husimi distribution in position space is concentrated around α = 0 = z at the normal phase λ < λc (no photons and no excited atoms) but splits into two differentiated packets at the superradiant phase λ > λc. In momentum space, the Husimi distribution becomes more and more delocalized with the emergence of multiple modulations above the critical point λc (see also figure 3 for a contour line of the variational case).

This delocalization of the exact Husimi distribution is captured by the Wehrl entropy Wj(λ) as a function of λ for different values of j. The computed results are given in figure 2, where we present Wj(λ) for j = 5 and j = 10 (solid lines) and for ω = ω0 = 1 (for which λc = 0.5), together with the variational results (see later). The Wehrl entropy tends to 2 (for high j) in the normal phase and to 2 + ln 2 in the superradiant phase, with an abrupt change (more abrupt as j increases) around the critical point.

The exact values of Wj(λ) for λ ≪ λc and λ ≫ λc are nicely reproduced by the following trial states expressed in terms of the ‘parity-symmetry-adapted’ coherent states introduced by Castaños et al [18, 19], which turn out to be an excellent approximation of the exact quantum solution of the ground (+) and first excited (−) states of the Dicke model.

Using the direct product |α, z⟩ ≡ |α⟩ ⊗ |z⟩ as a ground-state ansatz, one can easily compute the mean energy

\[ \mathcal{H}(\alpha, z) = \langle \alpha, z | H | \alpha, z \rangle \]

\[ = \omega |\alpha|^2 + j \omega \sqrt{\frac{2}{\lambda}} \sqrt{1 - \left( \frac{\lambda}{\lambda_c} \right)^2} \]

which defines a 4D ‘energy surface’. Minimizing with respect to these four coordinates gives the equilibrium points (see [18, 19]).

\[ \alpha_e = \begin{cases} 0, & \text{if } \lambda < \lambda_c, \\ -\sqrt{2j} \sqrt{\frac{\omega}{\lambda}} & \sqrt{1 - \left( \frac{\lambda}{\lambda_c} \right)^2}, & \text{if } \lambda \geq \lambda_c. \end{cases} \]

\[ z_e = \begin{cases} 0, & \text{if } \lambda < \lambda_c, \\ \frac{\pi \pm (\lambda - \lambda_c)^2}{\lambda}, & \text{if } \lambda \geq \lambda_c. \end{cases} \]  

Note that αe and zc are real and non-zero above the critical point λc (i.e. in the superradiant phase).

Although the direct product |α, z⟩ gives a good variational approximation to the ground state mean energy in the thermodynamic limit j → ∞, it does not capture the correct behavior for other ground-state properties sensitive to the parity symmetry P of the Hamiltonian (1) such as, for instance, uncertainty and entropy measures [14, 19]. That is why parity-symmetry-adapted coherent states are introduced. Indeed, a far better variational description of the ground (resp. first-excited) state is given in terms of the even (resp. odd) parity coherent states [18, 19].

\[ |\psi_{\pm}\rangle = |\alpha, z, \pm\rangle = \frac{|\alpha\rangle \otimes |z\rangle ± |\alpha\rangle \otimes |z\rangle}{\mathcal{N}_\pm(\alpha, z)}, \]

obtained by applying projectors of even and odd parities \( \hat{P}_\pm = (1 \pm \hat{P}) \) to the direct product |α⟩ ⊗ |z⟩. Here

\[ \mathcal{N}_\pm(\alpha, z) = \sqrt{2} \left( 1 \pm e^{-2|\alpha|^2} \right) \left( 1 \pm e^{-2|z|^2} \right)^j \]
is a normalization factor. These even and odd coherent states are ‘Schrödinger cat states’ in the sense that they are a quantum superposition of quasi-classical, macroscopically distinguishable states. The new energy surface $\mathcal{H}_\pm(\alpha, z) = (\alpha, z, \pm |H(\alpha, z, \pm) \text{ (see [18, 19] for an explicit expression of it) is more involved than } H(\alpha, z) \text{ in (11) and makes it much more difficult to obtain the new critical points } \alpha^{(\pm)}_e, z^{(\pm)}_e \text{ minimizing the corresponding energy surface. See [20]} \text{ in this volume for a numerical computation of the new critical points. It should be emphasized that the equilibrium points given in expression (12) are correct only in the thermodynamic limit } j \to \infty \text{ or far from } \lambda = \lambda_c \text{ for finite } j. \text{ Otherwise the minimization of } \mathcal{H}_\pm(\alpha, z) \text{ should be done (see [18–20] for more details). In this paper, instead of carrying out a numerical computation of } \alpha^{(\pm)}_e, z^{(\pm)}_e \text{ for different values of } j \text{ and } \lambda, \text{ we shall use the approximation } \alpha^{(\pm)}_e \approx \alpha_e, z^{(\pm)}_e \approx z_e, \text{ which turns out to be quite good except in a close neighborhood around } \lambda_c, \text{ which diminishes as the number of particles } N = 2j \text{ increases (see [19, 20]). With this approximation, we expect a rather good agreement between our numerical and variational results except perhaps in the close vicinity of } \lambda_c. \text{ Indeed, see figure 2.}

Taking into account the coherent state overlaps

$$\langle \alpha \pm \alpha_e | z \rangle = e^{-\frac{1}{2}|z|^2 + i z \alpha},$$

$$\langle z \pm z_e | \rangle = \frac{1}{(1 + |z|^2) + i z z_e},$$

the Husimi distribution for the variational states $|\alpha_e, z_e, \pm \rangle$ can be simply written as

$$\Psi(\alpha, z) = \frac{(|\alpha | |\alpha_e, z_e, \pm \rangle (\alpha, z) - |\alpha_e, z_e, \pm \rangle \langle \alpha, z |)}{\mathcal{N}_\pm^2(\alpha_e, z_e)}.$$  

(15)

From now on, we shall restrict ourselves to the even case and simply denote by $\Psi = \Psi_1$, the Husimi distribution of the variational ground state. Figure 3 shows a contour line of the variational Husimi distribution. Note that, in position space, it reproduces the packet splitting across the critical point depicted in figure 1, with two differentiated packets located around the equilibrium points $|\alpha_e, z_e \rangle$ and its antipode $(-\alpha_e, -z_e)$ in the superradiant phase. In momentum space, it exhibits a delocalization and ‘modulation’ for increasing values of $\lambda$.

We can easily compute the Wehrl entropy of (16), which gives

$$W_j(\lambda) = \begin{cases} \frac{1 + 2j}{2j+1}, & \text{if } \lambda < \lambda_c, \\ \frac{2j+1}{2j} + \ln 2, & \text{if } \lambda \gg \lambda_c. \end{cases}$$

(17)

denoting an entropy excess of $\ln(2)$ in the superradiant phase. In the normal phase we have $W_j(\lambda) = 1 + 2j/(2j+1)$, as corresponds to a coherent state according to the (still unproved) Lieb’s conjecture. Indeed, as conjectured by Wehrl [21] and proved by Lieb [22], any Glauber coherent state $|\alpha \rangle$ has a minimum Wehrl entropy of 1. In the same paper by Lieb [22], it was also conjectured that the extension of Wehrl’s definition of entropy for coherent spin-1 states $|z \rangle$ will yield a minimum entropy of $2j/(2j+1)$. For the joint system of radiation field plus atoms, we would have $W_j(\lambda) = 1 + 2j/(2j+1)$ in the normal phase $(\lambda < \lambda_c)$, and therefore

$$\Psi_\pm(\alpha, z) = \frac{(|\alpha | |\alpha_e, z_e, \pm \rangle (\alpha, z) - |\alpha_e, z_e, \pm \rangle \langle \alpha, z |)}{\mathcal{N}_\pm^2(\alpha_e, z_e)}.$$  

Figure 3. Contour lines of the variational Husimi distribution $\Psi_1(\alpha, z)$ in ‘position space’ ($\alpha$ and $z$ real; left panel) and ‘momentum space’ ($\alpha$ imaginary; right panel) for different values of $\lambda$ (from top to bottom: $\lambda = 0, \lambda = 0.6$ and $\lambda = 1$) for $j = 3$ and $\omega = \omega_0 = 1 \Rightarrow \lambda_c = 0.5$. Atomic units are used.

$$W_j \to 2 \text{ in the thermodynamic limit } j \to \infty, \text{ in agreement with our result.}$$

To finish, we would like to comment on the zeros of the Husimi distribution as a fingerprint for the QPT (see [7] for more information). From (16) we obtain

$$\Psi(\alpha, z) = 0 \Rightarrow 2d\alpha e + 2j \ln \frac{1 + z\alpha}{1 - \alpha z} = i\pi(2l + 1), \quad l \in \mathbb{Z},$$

(18)

which defines a 2D surface (for each value of $l$) in a 4D manifold with parametric equations

$$\alpha = f_j(l)(\alpha, z) = \frac{i}{\alpha e} \ln \frac{1 - z\alpha}{1 + \alpha z} + i\pi(2l + 1).$$

(19)

This expression gives, in particular, the ‘less probable mean photon number $|\alpha |^2$ for each mean atom fraction $|z|^2/(1 + |z|^2)$’ in phase space (note the comment before equation (5)).

In figure 4, we represent this surface as a conformal mapping of a regular grid in the $z$-plane. That is, for $z = z_1 + i z_2$, we present the image of the vertical lines $z_1 = \text{constant} \text{ (red solid curves)}$ and the horizontal lines $z_2 = \text{constant} \text{ (blue dotted curves).}$ We see from (18) that, in the normal phase $(\alpha_e = 0 = z_e)$, the Husimi distribution $\Psi(\alpha, z)$ has no zeros. In the
In this respect).

Lambert N, Emary C and Brandes T 2004

Lambert N, Emary C and Brandes T 2005

Emary C and Brandes T 2003

Brandes T 2005

Gerry C C and Knight P L 2005

Romera E, del Real R and Calixto M 2012

Calixto M, del Real R and Romera E 2012

Romera E and Nagy ´A 2011

375

Holstein T and Primakoff H 1940

Romera E, Calixto M and Nagy ´A 2012

Romera E, Sen K D and Nagy ´A 2012

85

Romera E, del Real R, Calixto M, Nagy S and Nagy ´A 2013

Calixto M, Nagy ´A, Paraleda I and Romera E 2012

Conclusions

We found that the Wehrl entropy of the Husimi distribution provides a sharp indication of a QPT in the Dicke model even for finite \( j \). This uncertainty measure detects a delocalization of the Husimi distribution across the critical point \( \lambda_c \) and we have employed it, together with 3D plots and contour lines of the Husimi distribution, to quantify and visualize the phase-space spreading of the ground state.

Calculations have been performed numerically and through a variational approximation. The variational approach, in terms of symmetry-adapted coherent states, complements and enriches the analysis, providing explicit analytical expressions for the Husimi distribution and Wehrl entropies, which remarkably coincide with the numerical results, especially in the thermodynamic limit and far from \( \lambda = \lambda_c \), where the approximate equilibrium points (12) fail. A more accurate calculation could perhaps be done by using the ‘true’ equilibrium points of [20], although we think that our variational approach still captures the qualitative behavior near \( \lambda_c \) and the quantitatively exact values far from \( \lambda_c \) (see again figure 2 in this respect).

In the superradiant phase, the Wehrl entropy undergoes an entropy excess of \( \ln(2) \). This fact implies that the Husimi distribution splits up into two identical subpackets with negligible overlap in passing from normal to superradiant phase. In general, for \( s \) identical subpackets with negligible overlap, one would expect an entropy excess of \( \ln(s) \).

The QPT fingerprints in the Dicke model have also been tracked by exploring the distribution of zeros of the Husimi density within the analytical variational approximation. Now, we have corroborated that the zeros of the Husimi distribution evidence the QPT without the Holstein–Primakoff approximation, confirming again that there are no zeros in the normal phase and a larger number of zeros as \( j \) and \( \lambda \) increase in the superradiant phase. This interesting result supports the assertion that the emergence of zeros of the Husimi distribution can be an indication of QPTs [7, 8].

References

[1] Sachdev S 2000 Quantum Phase Transitions (Cambridge: Cambridge University Press)
[2] Lambert N, Emary C and Brandes T 2004 Phys. Rev. Lett. 92 073602
[3] Lambert N, Emary C and Brandes T 2005 Phys. Rev. A 71 053804
[4] Emary C and Brandes T 2003 Phys. Rev. E 67 066203
[5] Brandes T 2005 Phys. Rep. 408 315
[6] Gerry C C and Knight P L 2005 Introductory Quantum Optics (Cambridge: Cambridge University Press)
[7] Romera E, del Real R and Calixto M 2012 Phys. Rev. A 85 053831
[8] Calixto M, del Real R and Romera E 2012 Phys. Rev. A 86 032508
[9] Holstein T and Primakoff H 1940 Phys. Rev. 58 1098
[10] Romera E and Nagy Á 2011 Phys. Lett. A 375 3066
[11] Nagy Á and Romera E 2012 Physica A 391 3650
[12] Romera E, Sen K D and Nagy Á 2012 J. Stat. Mech. P09016
[13] Romera E, Calixto M and Nagy Á 2012 Europhys. Lett. 97 20011
[14] Calixto M, Nagy Á, Paraleda I and Romera E 2012 Phys. Rev. A 85 053813
[15] Romera E, del Real R, Calixto M, Nagy S and Nagy Á 2013 J. Math. Chem. 51 620

Figure 4. Surface of zeros \( \alpha = f_j^0(z, \lambda) \) of the variational Husimi distribution \( \Psi_j(\alpha, z) \) for \( \lambda = 1 \), \( j = 10 \) and \( l = 0 \) (\( \lambda_c = 0.5 \)) seen as a conformal mapping of a regular grid in the \( z \)-plane.

superradiant phase (\( \lambda > \lambda_c \)) there are more and more zeros as \( j \) and \( \lambda \) increase. To study the high-\( j \) limit, we can redefine \( \beta = \sqrt{j} \sqrt{z} \), which simplifies the expression of

\[
2j \ln \frac{1 + z \beta}{1 - z \beta} = 2 \beta \alpha, \quad \text{for} \quad j \gg 1, \quad (20)
\]

where we have made use of the definition of the Euler number at some stage. Therefore, equation (19) becomes

\[
\alpha_1 = -\frac{\beta}{\alpha} \beta_1, \quad \alpha_2 = -\frac{\beta}{\alpha} \beta_2 + \frac{\pi}{2 \alpha} (2l + 1), \quad (21)
\]

for \( \alpha = \alpha_1 + i \alpha_2 \) and \( \beta = \beta_1 + i \beta_2 \). Therefore, in the high-\( j \) limit, and in the superradiant phase (\( \lambda > \lambda_c \)), the zeros are localized along straight lines (‘dark fringes’) in the \( \alpha_1 \beta_1 \) (position) and \( \alpha_2 \beta_2 \) (momentum) planes. In the momentum plane, the number of dark fringes grows with \( \lambda \) and \( j \). In the thermodynamic limit \( j \to \infty \), zeros densely fill the momentum plane \( \alpha_2 \beta_2 \) (see [7] for a graphical representation of zeros in the high-\( j \) limit).

4. Conclusions

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[16] Calixto M, Romera E and del Real R 2012 J. Phys. A: Math. Theor. 45 365301
[17] Perelomov A 1986 Generalized Coherent States and Their Applications (Berlin: Springer)
[18] Castaños O, Nahmad-Achar E, López-Peña R and Hirsch J G 2011 Phys. Rev. A 83 051601
[19] Castaños O, Nahmad-Achar E, López-Peña R and Hirsch J G 2011 Phys. Rev. A 84 013819
[20] Hirsch J G, Castaños O, Nahmad-Achar E and López-Peña R 2013 Phys. Scr. T153 014033
[21] Wehrl A 1979 Rep. Math. Phys. 16 353
[22] Lieb E H 1978 Commun. Math. Phys. 62 35