Research Article

Best Approximation of Data Distributed in a Band

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Received 7 June 2011; Accepted 14 July 2011

Academic Editor: J. Jahn

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We study the problem to approximate a data set which are affected in a such way that they present us as a band in the plane. We introduce a deviation measure, and we research the asymptotic behavior of the best approximants when the band shrink in some sense.

1. Introduction

In some situations, we find us with the problem to approximate a given function of physical origin which is contaminated by different causes. For example, it occurs when we receive a signal, and we observe at the screen of an electronic oscilloscope a band produced by noise or other factors. Here, a criterion of selecting is necessary in order to approximate to that band. More precisely, we must choose a measure of deviation from one band to a given approximant class. A way could be to approximate a segment value multivalued function using the Hausdorff metric in the plane (see [1]); another could be to consider the best simultaneous approximation to the set of every functions whose graphics live in the band determined by them (see [2–4]). In this paper, we give an alternative deviation measure, and we establish a relation with the best simultaneous approximation.

Let $1 \leq p < \infty$, and $F : [a, b] \to 2^\mathbb{R}$, be a multivalued function with $F(x)$ a Lebesgue measurable set for all $x \in [a, b]$. Given an approximant class $S$, we consider the following function as measure of deviation of $F$ to $Q \in S$:

$$
\Phi_{[a,b]}(F, Q) = \left( \int_a^b \int_{F(x)} |y - Q(x)|^p dy dx \right)^{1/p}.
$$

(1.1)

Let $D := \{(x, y) : x \in [a, b], y \in F(x)\}$. It is easy to see that (1.1) is a special case to approximate a function $I$ from a given class $C$ with a norm $\| \cdot \|$ over the space of functions
such a linear subspace. In this case, \( f \) of the Taylor polynomials of \( S \) is a finite dimensional linear subspace, and the existence of \( Q \) is well known (see [5]).

We consider that the band shrinks to a curve in two situations:

(i) The functions \( f \) and \( g \) are replaced by a family of functions \( f_\epsilon, g_\epsilon \), where \( f_\epsilon \) and \( g_\epsilon \) converge to a function \( h \), as \( \epsilon \) tends to 0. That is, the band shrinks vertically.

(ii) The interval \([a,b]\) is substituted by \([x_0 - \epsilon, x_0 + \epsilon]\), where \( \epsilon \) tends to 0; that is, the band shrinks horizontally.

If there exists the limit of \( Q \) minimizing (1.1) when the band shrinks to a curve, as such, it provides useful qualitative and approximation analytic information concerning the approximants on small bands, which is difficult to obtain from a strictly numerical treatment. The existence of the limit of \( Q \) is close to the best local approximation problem (see [6–8]).

In Section 2, we prove that if the band shrinks vertically to a given function, then the set of closure points of \( Q \) is contained in the set of best approximants to that function, with a suitable seminorm. Moreover, we see that the limit of \( Q \) exists when \( p > 1 \).

In Section 3, we prove that if the band shrinks horizontally, the limit of \( Q \) is the mean of the Taylor polynomials of \( f \) and \( g \) at \( x_0 \). We also show that this approximation problem is related with the subject of best simultaneous local approximation which was studied in [8].

We assume conditions about the functions \( f \) and \( g \) in order to be (1.1) finite for all \( Q \in S \). Henceforward, \( f, g \in L^q([a,b]), \ S \subseteq L^q([a,b]) \), and \( g - f \in L^{q/(q-p)}([a,b]) \), \( p \leq q \). In this case, using Hölder’s inequality, we have

\[
\Phi_{[a,b]}(F, Q) \leq \int_a^b \max \{|f(x) - Q(x)|, |g(x) - Q(x)|\}^p (g(x) - f(x))dx \\
\leq \|\max\{|f - Q|, |g - Q|\}\|_{q,[a,b]}^p \|g - f\|_{q/(q-p),[a,b]} < \infty,
\]

for all \( Q \in S \). If \( q \geq p + 1 \), the condition \( g - f \in L^{q/(q-p)}([a,b]) \) is automatically satisfied.
2. The Band Shrinks Vertically

Let $g$ be a measurable nonnegative function in $L^q([a, b])$, $p \leq q$, and $\{f_e\}, \{g_e\}, e > 0$, two net of measurable functions such that $|f_e|, |g_e| \leq g$, a.e. on $[a, b]$. We write $F_e(x) = [f_e(x), g_e(x)]$, $x \in [a, b]$. Given $S \subset L^q([a, b])$ a finite dimensional lineal subspace, let $Q_e \in S$ which minimizes $\Phi_{|a,b|}(F_e, Q)$, $Q \in S$.

**Theorem 2.1.** Assume that there are two functions $W(e) : (0, \infty) \to (0, \infty)$ and $h \in L^{1/(q-p)}([a, b])$ such that almost everywhere on $[a, b]$

$$\frac{g_e - f_e}{W(e)} \leq h,$$

with $|(w > 0)| > 0$. If $f_e$ and $g_e$ converge to $f$ almost everywhere on $[a, b]$, then the set of closure points of $Q_e$ is a nonempty set, and it is contained in the set of best approximants to $f$ from $S$ with the seminorm $\|\cdot\|_{p,w,a,b}$. In particular, if $p > 1$, the net $Q_e$ converges to the unique best approximant to $f$, as $e \to 0$.

**Proof.** For $e > 0$, we denote $w_e^* = w_e/W(e)$, where $w_e = g_e - f_e$. Let

$$J_{Q,e}(x) := \int_{f_e(x)}^{g_e(x)} |y - Q(x)|^p dy, \quad x \in [a, b], \; Q \in S. \quad (2.3)$$

Let $Q \in S$. By integral mean value theorem, for each $x \in [a, b]$, there exists

$$\alpha_e(x), \xi_e(x) \in [f_e(x), g_e(x)], \quad (2.4)$$

such that $J_{Q,e} = |\alpha_e - Q_e|^p w_e$ and $J_{Q,e} = |\xi_e - Q|^p w_e$. Now, the Fubbini Theorem implies that $J_{Q,e}$ and $J_{Q,e}$ are measurable functions on $[a, b]$, and hence, $\alpha_e$ and $\xi_e$ are measurable functions on $[a, b]$. Consequently, $\alpha_e, \xi_e \in L^q([a, b])$, because $|\alpha_e|, |\xi_e| \leq g$, a.e. on $[a, b]$.

On the other hand,

$$\|f - Q_e\|_{p,w,a,b} - \|\alpha_e - f\|_{p,w,a,b} \leq \|\alpha_e - Q_e\|_{p,w,a,b} = \Phi_{|a,b|}(F_e, Q_e) \leq \Phi_{|a,b|}(F_e, Q) = \|\xi_e - Q\|_{p,w,a,b}. \quad (2.5)$$

In consequence, we get

$$\|f - Q_e\|_{p,w,a,b} \leq \|\alpha_e - f\|_{p,w,a,b} + \|\xi_e - Q\|_{p,w,a,b}. \quad (2.6)$$

As $|f| \leq g$ a.e. on $[a, b]$, from (2.1) and the Hölder inequality, we have

$$|\alpha_e - f|^p w_e^* \leq (2g)^p h \in L^1([a, b]), \quad |\xi_e - Q|^p w_e^* \leq (g + |Q|)^p h \in L^1([a, b]). \quad (2.7)$$
implies that

\[ \lim_{\epsilon \to 0} (\| \alpha_{\epsilon} - f \|_{p,w_{\epsilon}[a,b]} + \| \xi_{\epsilon} - Q \|_{p,w_{\epsilon}[a,b]}) = \| f - Q \|_{p,w[a,b]}. \]  (2.8)

By hypothesis, there is \( k > 0 \) satisfying \( \{ \| w > k \| \} > 0 \). By the Egoroff Theorem (see [9]), there exists a set \( A \subset \{ w > k \}, |A| > 0 \), where we have that \( w_{\epsilon}^* \) uniformly converges to \( w \) on \( A \). So, we can choose a positive constant \( \epsilon_0 \) such that for all \( x \in A \) and all \( 0 < \epsilon < \epsilon_0 \), \( w_{\epsilon}^*(x) \geq k/2 \). Hence,

\[ \| f - Q_{\epsilon} \|_{p,A} \leq \left( \frac{k}{2} \right)^{-1/p} \| f - Q_{\epsilon} \|_{p,w_{\epsilon}^*[a,b]} \quad 0 < \epsilon < \epsilon_0. \]  (2.9)

According to (2.6)–(2.9), \( \{ Q_{\epsilon} \} \) is a uniformly bounded net. Then, \( \{ Q_{\epsilon} \} \) have a subsequence that we again denote by \( \{ Q_{\epsilon} \} \) converging to \( T \in S \). Again, the Lebesgue dominated convergence theorem implies \( \lim_{\epsilon \to 0} \| f - Q_{\epsilon} \|_{p,w_{\epsilon}^*[a,b]} = \| f - T \|_{p,w_{\epsilon}[a,b]} \). Finally, from (2.6) and (2.8), we conclude that

\[ \| f - T \|_{p,w[a,b]} \leq \| f - Q \|_{p,w[a,b]}. \]  (2.10)

As \( Q \in S \) is arbitrary, the theorem immediately follows.

Suppose that \( p > 1 \). If we have two functions \( W_i : (0, \infty) \to (0, \infty) \) and \( w_i, i = 1,2 \), fulfilling the hypothesis of Theorem 2.1, we conclude that \( Q_{\epsilon} \) converges to the best approximant to \( f \) from \( S \) when we consider \( \| \cdot \|_{p,w_{\epsilon},[a,b]}\) and \( \| \cdot \|_{p,w_{\epsilon}^*,[a,b]} \), respectively. The next lemma shows that it is not surprising, because these norms differing by a constant.

**Lemma 2.2.** Let \((\Omega, \Sigma, \mu)\) be a finite measure space. Let \( w_{\epsilon} : \Omega \to \mathbb{R}, \epsilon > 0 \) be a net of nonnegative measurable functions. Assume that there are two functions \( W_i : (0, \infty) \to (0, \infty), i = 1, 2 \) such that almost everywhere on \( \Omega \)

\[ \lim_{\epsilon \to 0} \frac{w_{\epsilon}(x)}{W_i(\epsilon)} = w_i(x) \]  (2.11)

is finite. If \( A_i := \{ x \in \Omega : w_i(x) > 0 \} \) and \( \mu(A_i) > 0, i = 1,2 \), then there exists \( k > 0 \) satisfying \( \mu(A_1) = k \mu A_2, a.e. \) on \( \Omega \).

**Proof.** We denote by \( I_i, i = 1,2 \), the subsets of \( \Omega \), where \( w_i \) is finite, and we write \( K = I_1 \cap I_2 \). Clearly, \( \mu(K) = \mu(\Omega) \). Let \( J_i = A_i \cap K \).

If \( \mu(J_1 \cap J_2) = 0 \), then \( \mu(J_2 \cap J_1) = k \mu(A_1) > 0 \) and \( \mu(J_2 - J_1) = k \mu(A_2) > 0 \). We take \( x_1 \in J_1 - J_2 \) and \( x_2 \in J_2 - J_1 \). The equality

\[ \frac{w_{\epsilon}(x)}{W_i(\epsilon)} = \frac{w_{\epsilon}(x)}{W_1(\epsilon)} \frac{W_1(\epsilon)}{W_2(\epsilon)} \]  (2.12)

implies that \( W_i(\epsilon)/W_j(\epsilon) \) tends to zero if \( x = x_1 \), and it tends to infinite if \( x = x_2 \), a contradiction. So, we get \( \mu(J_1 \cap J_2) = 0 \).
For all \( x \in J_1 \cap J_2 \), we have
\[
0 < k := \lim_{\varepsilon \to 0} \frac{W_1(\varepsilon)}{W_2(\varepsilon)} = \frac{w_2(x)}{w_1(x)}.
\] (2.13)

Next, we prove that \( \mu(J_1 - J_2) = \mu(J_2 - J_1) = 0 \). If \( \mu(J_1 - J_2) > 0 \) the above argument implies that \( k = 0 \), and if \( \mu(J_2 - J_1) > 0 \), we obtain \( k = \infty \). In either case, it contradicts (2.13). Therefore, \( \mu(J_1 \cap J_2) = \mu(J_1 \cup J_2) = \mu(A_1) = \mu(A_2) \) and
\[
w_2(x) = kw_1(x), \quad \text{a.e. on } A_1 \cup A_2.
\] (2.14)

It immediately follows that (2.14) holds a.e. on \( \Omega \).

We also observe in the next example that (2.2) in Theorem 2.1 is essential.

**Example 2.3.** Let \( a = 0, b = 1, \) and \( p = 2 \). We consider \( S \) the space of constant functions,
\[
f_m(x) = x, \quad g_m = \begin{cases} 
    x + \frac{1}{m} & \text{if } m \text{ is even}, \\
    x - \frac{1}{m} & \text{if } m \text{ is odd}.
\end{cases}
\] (2.15)

Here, \( m(g_m - f_m) \), is equal to 1 if \( m \) is even, and it is equal \( x \) in otherwise. Applying Theorem 2.1, we get \( Q_{2m} \to 1/2 \) and \( Q_{2m-1} \to 2/3 \), as \( m \to \infty \).

### 3. The Band Shrinks Horizontally

Let \( x_0 \in [a, b] \) and \( I_\varepsilon := [x_0 - \varepsilon, x_0 + \varepsilon] \subset [a, b], \varepsilon > 0 \). In this section, we assume that \( f \) and \( g \) have continuous derivatives up to order \( n \) at \( x_0 \). From now on, \( S = \Pi^n \) is the space of algebraic polynomials of degree at most \( n \), and \( Q_\varepsilon \) denotes an element of \( \Pi^n \) that minimizes \( \Phi_{I_\varepsilon}(F, Q), Q \in \Pi^n \).

We recall (see [10, 11]) the Newton divided difference formula for the interpolation polynomial of a function \( h \) of degree \( n \) at \( x_1, x_2, \ldots, x_{n+1} \),
\[
P(x) = h(x_1) + (x - x_1)h[x_1, x_2] + \cdots + (x - x_1)\cdots(x - x_{n})h[x_1, \ldots, x_{n+1}].
\] (3.1)

Here, \( h[x_1, \ldots, x_{m+1}] \) denotes the \( m \)-th order Newton divided difference. If \( h \) has continuous derivatives up to order \( m \), on an interval \([a, b]\) containing to \( x_1, \ldots, x_{m+1} \), then the \( m \)-th divided difference can be expressed as
\[
h[x_1, \ldots, x_{m+1}] = \frac{h^{(m)}(\xi)}{m!},
\] (3.2)

for some \( \xi \) in the interval \([x_1, x_{m+1}]\). It is well known that the \( m \)-th divided difference is a continuous function as function of their arguments \( x_1, \ldots, x_{m+1} \).
For simplicity of notation, we write $T(f)$ and $T(g)$ the Taylor polynomials of $f$ and $g$ at $x_0$ of order $n$, respectively.

**Lemma 3.1.** Let $q \geq 0$, and let $f_i : [a, b] \to \mathbb{R}, 1 \leq i \leq 3$ be continuous functions with $f_1 \leq f_2$. If

$$
    h(x) = \int_{f_i(x)}^{f_2(x)} |y - f_3(x)|^q \text{sgn}(y - f_3(x)) \, dy, \quad a \leq x \leq b,
$$

(3.3)

then $h$ is a continuous function.

**Proof.** If $q > 0$, the continuity of $h$ follows from the continuity of $f_i, 1 \leq i \leq 3$ and the uniform continuity of $t^q \text{sgn}(t)$ on compact sets. If $q = 0$, we have

$$
    h(x) = \begin{cases} 
        f_1(x) - f_2(x), & \text{if } f_3(x) \geq f_2(x), \\
        f_2(x) - f_1(x), & \text{if } f_3(x) \leq f_1(x), \\
        f_1(x) + f_2(x) - 2f_3(x), & \text{if } f_1(x) \leq f_3(x) \leq f_2(x).
    \end{cases}
$$

(3.4)

Thus, the continuity of $h$ is immediate. \hfill \Box

**Theorem 3.2.** If $p \geq 1$, then any net $Q_e$ converges to $(T(f) + T(g))/2$, as $e \to 0$.

**Proof.** It is easy to see that $Q_e$ is characterized by

$$
    \int_{L_e} h(x)Q(x) \, dx = 0, \quad Q \in \Pi^n,
$$

(3.5)

where

$$
    h(x) = \int_{f(x)}^{g(x)} |y - Q_e(x)|^{p-1} \text{sgn}(y - Q_e(x)) \, dy.
$$

(3.6)

By Lemma 3.1, $h$ is continuous. Therefore, $h$ interpolates to zero in at least $n + 1$ different points of the interval $L_e$, say $x_1 < x_2 < \cdots < x_{n+1}$. In fact, if $h$ has $m$ different zeros, $m \leq n$, we can find an element $Q \in \Pi^n$ such that $h(x)Q(x) > 0$ for all $x \neq x_i, 1 \leq i \leq m$. It contradicts (3.5). So,

$$
    \int_{f(x_i)}^{g(x_i)} |y - Q_e(x_i)|^{p-1} \text{sgn}(y - Q_e(x_i)) \, dy = 0, \quad 1 \leq i \leq n + 1.
$$

(3.7)
It follows that \( Q_{\epsilon}(x_i) \) is a best constant approximant to the identity function with norm \( \| \cdot \|_{p, |f(x_i), g(x_i)|, 1 \leq i \leq n + 1} \). A straightforward computation shows that \( Q_{\epsilon}(x_i) = (f(x_i) + g(x_i))/2, 1 \leq i \leq n + 1 \); that is, \( Q_{\epsilon} \) interpolates to the function \((f + g)/2\) at \( x_i, 1 \leq i \leq n + 1 \). From (3.1) and (3.2), it follows that

\[
Q_{\epsilon}(x) = h(x_1) + (x - x_1)h^{(1)}(\xi_1) + \cdots + (x - x_1)\cdots(x - x_n) \frac{h^{(n)}(\xi_n)}{n!},
\]

(3.8)

where \( h = (f + g)/2, \xi_i \in I_{\epsilon}, 1 \leq i \leq n \). Taking limit for \( \epsilon \to 0 \) in (3.8) and using the continuity of the derivatives of the functions \( f \) and \( g \), we get the theorem.

**Corollary 3.3.** Let \( p \geq 1 \). Suppose that \( f \) and \( g \) have continuous derivatives up to order \( n + 1 \) at \( x_0 \) and \( f(x_0) < g(x_0) \). Then, for sufficiently small \( \epsilon \), \( Q_{\epsilon} \) is the best \( L^{p+1} \)-simultaneous approximant in \( L^{p+1}(I_{\epsilon}) \); that is, \( Q_{\epsilon} \) minimizes

\[
\left( \| f - Q \|_{p+1, I_{\epsilon}}^{p+1} + \| g - Q \|_{p+1, I_{\epsilon}}^{p+1} \right)^{1/(p+1)}, \quad Q \in \Pi^n.
\]

(3.9)

**Proof.** By hypothesis, there is \( \epsilon_0 > 0 \) such that \( f < g \) on \( I_{\epsilon_0} \). Let \( P_{\epsilon} \) be the best \( L^{p+1} \)-simultaneous approximant to \( f \) and \( g \) in \( L^{p+1}(I_{\epsilon}) \). From Theorem 3.2, and [8, Theorem 3.4], there exists \( 0 < \epsilon_1 < \epsilon_0 \) such that

\[
Q_{\epsilon}(x), P_{\epsilon}(x) \in [f(x), g(x)], \quad x \in I_{\epsilon}, \quad \epsilon < \epsilon_1.
\]

(3.10)

On the other hand, we have

\[
(p + 1)\Phi^p_{I_{\epsilon}}(F, Q) = \int_{I_{\epsilon}} |f(x) - Q(x)|^{p+1} \text{sgn}(Q(x) - f(x)) dx
\]

\[
+ \int_{I_{\epsilon}} |g(x) - Q(x)|^{p+1} \text{sgn}(g(x) - Q(x)) dx, \quad Q \in \Pi^n,
\]

(3.11)

From (3.10) and (3.11), we get

\[
\| f - Q_{\epsilon} \|_{p+1, I_{\epsilon}}^{p+1} + \| g - Q_{\epsilon} \|_{p+1, I_{\epsilon}}^{p+1} = \| f - P_{\epsilon} \|_{p+1, I_{\epsilon}}^{p+1} + \| g - P_{\epsilon} \|_{p+1, I_{\epsilon}}^{p+1}
\]

(3.12)

for all \( \epsilon < \epsilon_1 \). So, \( Q_{\epsilon} = P_{\epsilon} \) for all \( \epsilon < \epsilon_1 \). 

**Remark 3.4.** If \( \Phi_{[a,b]}(F, Q) = \sup_{x \in [a,b]} \sup_{y \in F(x)} |y - Q(x)| \), then

\[
\Phi_{[a,b]}(F, Q) = \sup_{x \in [a,b]} \max \{ |f(x) - Q(x)|, |g(x) - Q(x)| \}
\]

(3.13)

implies that \( Q_{\epsilon} \) is the best \( L^\infty \)-simultaneous approximant in \( L^\infty([a,b]) \), for all \( \epsilon > 0 \).
Acknowledgments

This work was supported by Universidad Nacional de Rio Cuarto, CONICET, and ANPCyT.

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