The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map

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January 5, 2009

Abstract
Motivated by the need for parametric families of rich and yet tractable distributions in financial mathematics, both in pricing and risk management settings, but also considering wider statistical applications, we investigate a novel technique for introducing skewness or kurtosis into a symmetric or other distribution. We use a “transmutation” map, which is the functional composition of the cumulative distribution function of one distribution with the inverse cumulative distribution (quantile) function of another. In contrast to the Gram-Charlier approach, this is done without resorting to an asymptotic expansion, and so avoids the pathologies that are often associated with it. Examples of parametric distributions that we can generate in this way include the skew-uniform, skew-exponential, skew-normal, and skew-kurtotic-normal.

Presented at the First IMA Computational Finance Conference. Date: Friday 23rd March 2007.

1 Introduction
In this paper we consider composite maps of the following two forms: sample transmutation maps (STMs) \( y = G^{-1}[F(x)] \), and rank transmutation maps (RTMs) \( v = G[F^{-1}(u)] \), where \( F \) and \( G \) are cumulative distribution functions (CDFs). Such maps have attracted the attention of statisticians in the past, in particular of Gilchrist [11], who refers to STMs and RTMs as \( Q \)-transformations and \( P \)-transformations, respectively.

In this paper our primary focus is the RTM, which we use as a tool for the discovery of new families of non-Gaussian distributions. We use it to modulate a given base distribution for the purposes of modifying the moments, in particular the skew and kurtosis. An important example will be to take the base distribution to be normal, but there is wide latitude in the choice of the base distribution. An attraction of the approach is that if the CDF and inverse CDF (or quantile function (QF)) are tractable for the base distribution, they will remain so for the transmuted distribution.

Applications of the STM include sampling from exotic distributions, e.g. Student’s T [22, 25, 21, 24]; and approximating the quantile function for the normal distribution [23, 26].

1.1 Asymptotic approaches
The Cornish Fisher (CF) series provides a means to approximate the QF of a distribution in terms of its cumulants. In finance it finds applications in the calculation of value at risk. Conversely the Edgeworth or Gram-Charlier (EGC) expansion is used to approximate the CDF, also based on the cumulants of the distribution. This idea was first introduced to financial economics by Jarrow and Rudd [14], who used the approach to find corrections to the Black-Scholes price of vanilla options. The CF and EGC

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series can be thought of as the asymptotic analogues of the STMs and RTMs, respectively. For further details see [2] (also available on-line at [1] and [15]).

Deficiencies and problems with such expansions include: the requirement that moments be finite, in particular higher moments when greater accuracy is sought through additional terms in the expansion, the tendency of the GC method to give negative values for the PDF, leading to strict bounds on permissible moment values, and the difficulty in dealing with the special functions that arise.

In common with the asymptotic methods, our goal is to discover families of parametric distributions with the ability to fit to levels of third and fourth moments that depart from those of a given base distribution (e.g. normal, uniform, exponential). However, in contrast, we do so exactly, and so without encountering the pitfalls that plague the EGC series.

1.2 Plan of paper

Section 2 contains a survey of the skew normal literature and demonstrations of how to introduce (additional) skew into the uniform, exponential and normal distributions. In Section 3 we introduce skew and kurtosis to the normal distribution. In Section 4 we conclude.

1.3 Acknowledgments

The authors would like to thank the organisers of the First IMA Computational Finance Conference and Professors Samuel Kotz and Marc Genton for helpful comments and explaining the history and current issues associated with the skew-normal and related distributions. We also thank Prof Warren Gilchrist for numerous comments on quantile methods.

2 Rank transmutation, skewness and kurtosis

Given two distributions with a common sample space with CDFs, $F_1$, $F_2$, we can define a pair of general RTMs as follows:

$$G_{R_{12}}(u) = F_2(F_1^{-1}(u)), G_{R_{21}}(u) = F_1(F_2^{-1}(u)).$$

The functions $G_{R_{12}}(u)$ and $G_{R_{21}}(u)$ both map the unit interval $I = [0, 1]$ into itself, and under suitable assumptions are mutual inverses. Naturally they satisfy $G_{R_{12}}(0) = 0$, and $G_{R_{21}}(1) = 1$. To ensure that transmuted densities are continuous we optionally make the additional assumption that the RTMs be continuously differentiable, and in general we require that the maps be monotone. In the later examples our convention will be to take $F_1$ as the base distribution and $F_2$ as the modulated distribution.

2.1 Skew-normal literature

There is an extensive literature on the notion of modulating a given distribution by the introduction of skewness, dating back to the work of the early pioneer, Fernando de Helguero (1908) [13]. A survey of the history of continuous skewed distributions in general has been made recently by Kotz and Vicari [16]. Another good entry point to the literature is the article [7], and an extensive bibliography has been made available at [4]. Azzalini’s own recent survey is available at [6], and forms part of a trio of articles with Genton [10, 5] that well demonstrates that this is a vigorous area of research. See also the 2006 paper by Arellano-Valle et al [3]. Briefly, Azzalini et al consider distributions whose densities are the product of the density $\phi(x)$ and the distribution functions $\Phi(x)$ for some base distribution (e.g. normal)

$$f(x, \alpha) = 2\phi(x)\Phi(\alpha x),$$

where $\alpha$ is a parameter that measures the intensity of the modulation.
2.2 Quadratic transmutation

A natural RTM to consider, which we term the *quadratic RTM* (QR TM), has the following simple quadratic form, for \(|\lambda| \leq 1\):

\[ G_{R_{12}}(u) = u + \lambda u (1 - u), \]  

(3)

from which it follows that the CDFs obey the relationship

\[ F_2(x) = (1 + \lambda) F_1(x) - \lambda F_1(x)^2. \]  

Because the inverse RTM is available in closed-form, the sampling algorithm remains tractable:

\[ F_2^{-1}(u) = F_1^{-1}(G_{R_{21}}(u)), \quad G_{R_{21}}(u) = \frac{1 + \lambda - \sqrt{(\lambda + 1)^2 - 4\lambda u}}{2\lambda}. \]  

(4)

The effect of the QR TM is to introduce skew to a symmetric base distribution. There is no specific requirement that the base distribution \(F_1\) be symmetric. However, if the \(F_1\) distribution is symmetric about the origin, in the sense that

\[ F_1(x) = 1 - F_1(-x), \]

we have the result that the distribution of the square of the transmuted random variable is identical to that of the distribution of the square of the original random variable. A consequence of this is that if the original distribution is symmetric, then the quadratic RTM preserves all even moments. Frameworks within which the distribution of the square is preserved under the skew transformation are well documented. See for example, Roberts & Gesser [19], Gupta & Cheng [12], Kotz & Vicari [16], and Wang et al [27]. We proceed by applying the QR TM to the cases in which the base distribution is uniform, exponential and normal.

2.3 Skew-uniform

We apply the QR TM to the uniform as the base distribution, initially under the assumption that \(|\lambda| \leq 1\)

\[ F_1(x) = \begin{cases} 0 & x < 0 \\ (1 + \lambda) x - \lambda x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}. \]  

(5)

This may be compared with the skew-uniform distribution obtained within the Azzalini framework that is described in [17]. We can relax the bounds on \(\lambda\) if we truncate the modulated distribution in the following manner:

\[ G_{R_{12}}(u) = \min[\max[u + \lambda u (1 - u), 0], 1]. \]  

(6)

The PDF for the skew-uniform distribution is shown in Figure 1, top left.

2.4 Skew-exponential

To make the point that when adopting the rank transmutation approach it is not necessary that the base distribution be centred, symmetric or even defined for negative values, we consider an exponential base distribution \(f_1\) for \(\beta > 0\)

\[ f_1(x) = \begin{cases} 0 & x < 0 \\ \beta e^{-\beta x} & 0 \leq x \end{cases}. \]  

(7)

which has the transmuted PDF

\[ f_2(x) = \begin{cases} 0 & x < 0 \\ \beta e^{-\beta x} (1 - \lambda) + 2\beta \lambda e^{-2\beta x} & 0 \leq x \end{cases}. \]  

(8)

With \(\beta = 1\) and \(\lambda\) varying from \(-1\) to \(+1\) in steps of \(1/3\) we obtain the pleasing set of curves shown in Figure 1, top right.
2.5 Skew-normal

Of special interest is the case of a normal base distribution,

\[ F_1(x) = \Phi(x) := \frac{1}{2} \left( 1 + \text{erf} \left( \frac{z}{\sqrt{2}} \right) \right), \quad (10) \]

where \( \phi \) and \( \Phi \) are the PDF and CDF for a standard normal variable, respectively. This gives rise to a transmuted distribution with density

\[ f_2(x) = \phi(x) (1 + \lambda - 2\lambda \Phi(x)). \quad (11) \]

Closed-form expressions for the moments can be found easily and used for calibration. The standardized PDF (with mean 0 and variance 1) takes the form

\[ f_3(x) = \sqrt{1 - \lambda^2/\pi} f_2(x \sqrt{1 - \lambda^2/\pi} - \lambda/\pi, \lambda). \quad (12) \]

By expanding in \( x \) and \( \lambda \) it would be possible to recover a form of the ECG expansion. However, as our goal is to find similarly tractable and yet more robust ways to generate exotic distributions we prefer not to.

The PDFs for the skew-uniform distribution as \( \lambda \) is varied from \(-1\) to \(+1\) are shown in Figure 1, lower two plots. The raw transmuted and normalised versions of the distributions are shown to the left and the right of the figure, respectively. By applying the QRTM to other base distributions we can similarly discover novel skew-Student, skew-Cauchy, and so forth, distributions. A desirable feature of the approach is that Monte Carlo sampling via quantile mechanism is as tractable as for the base distribution in each case.

![Figure 1: PDFs for skew-uniform (top left), skew-exponential (top right) and skew-normal (bottom) distributions. For the skew-normal case raw (bottom left) and standardized (bottom right) forms are shown.](image)

2.6 General rank transmutation maps

Whilst our emphasis so far has been on perturbations of the symmetry in order to introduce skewness, we can also consider other perturbations. If we stay within the polynomial structure we can consider maps of the form

\[ G_{R_{12}}(u) = u + u(1 - u)P(u), \quad (13) \]

where \( P \) is a polynomial with various parameters. It is of particular interest to place natural constraints on the structure of \( P(u) \).
Table 1: Special cases for parameter values \(\{\alpha_1, \alpha_2\}\). Descriptions refer to the relationship between the transmuted distribution and its base distribution.

| \(\{\alpha_1, \alpha_2\}\) | Description         |
|--------------------------|---------------------|
| \(\{1,0\}\)            | maximum of two      |
| \(\{0,1\}\)            | minimum of two      |
| \(\{\frac{3}{2}, 1\}\) | maximum of three    |
| \(\{-\frac{3}{2}, 1\}\)| minimum of three    |
| \(\{0, 4\}\)           | extreme bimodal     |
| \(\{0, -2\}\)          | middle of three     |

2.6.1 Symmetric cubic rank transmutation

The simplest possible type of symmetric mapping is obtained by choosing \(P(u) = \gamma(u - \frac{1}{2})\) for some constant \(\gamma\). This leads us to define a function that we shall term the \textit{symmetric cubic rank transmutation mapping} (SCRTM), symmetric in the sense that \(G_{R_{12}}(1 - u) = 1 - G_{R_{12}}(u)\). We could restrict the range of \(\gamma\) appropriately but will project to the unit interval to obtain a map valid for all \(\gamma\) as follows:

\[
G_{R_{12}}(u) = \min[\max[u + \gamma u (1 - u) \left( u - \frac{1}{2} \right)], 0, 1]. \tag{14}
\]

The densities of the distributions that result for various values of the parameter \(\gamma\) are given in Figure 2 for base distributions that are uniform (left) and normal (right).

![Figure 2](image)

Figure 2: Probability density functions obtained when symmetric cubic rank transformation is applied to uniform (left) and normal (right) base distributions.

3 Skew-kurtotic transmutations

3.1 Polynomial family and valid parameter set

We standardize a family of polynomial rank transmutation maps and with calibration issues in mind discuss the moment structure. For parameters \(\alpha_1\) and \(\alpha_2\) we consider the polynomial family

\[
P(z, \alpha_1, \alpha_2) = z - z (1 - z) \left( \alpha_1 + \left( z - \frac{1}{2} \right) \alpha_2 \right), \tag{15}
\]

with the restriction on \(P\) that it be a monotone increasing, 1:1 mapping of the unit interval into itself. To achieve this it is sufficient to impose non-negativity of \(P'\) at the end-points and at \(z = 1/2\), and that \(\min_{0 \leq z \leq 1} P'(z) \geq 0\). At particular points in the parameter space \(\{\alpha_1, \alpha_2\}\) the modulated distribution reduces to particular simple forms. These are listed in Table 1.
Table 2: First five moments for skew-kurtosis transmutation applied to a normal base distribution.

| \( k \) | \( \mathbb{E}[X^k] \) |
|---|---|
| 1 | \( \frac{1}{\sqrt{\pi}} \alpha_1 \) |
| 2 | \( 1 + \sqrt{\frac{3}{\pi}} \alpha_2 \) |
| 3 | \( \frac{5}{2\sqrt{\pi}} \alpha_1 \) |
| 4 | \( 3 + \frac{13}{2\pi\sqrt{3}} \alpha_2 \) |

Whilst awkward, these conditions guarantee a globally valid density function, and the payback is the simplicity of the moment structure. This region of admissible parameters (i.e. those giving rise to a well-defined PDF), shown in Figure 3 (left) in \((\alpha_1, \alpha_2)\) space, can be extended by applying a floor and a cap, but within this region we have a simple polynomial mapping. Importantly the region contains a large open set around the origin, which is all that is needed for many practical purposes where the introduction of a modest amount of skewness and kurtosis is all that is required. Some typical skew-kurtotic RTMs are shown in Figure 3 (right).

Figure 3: Schematic diagram to show valid parameters in \((\alpha_1, \alpha_2)\)-space (left) and permissable skew-kurtotic rank transmutation maps (right).

3.2 Monte Carlo sampling algorithm

To simulate using the transmuted distribution it is necessary to solve the cubic equation \( P(z, \alpha_1, \alpha_2) = u \) for \( z \) given \( u \). Whilst this can be done in closed form using the approach of Tartaglia, as described in [20], the logic is non-trivial. A suitable general cubic solver is described in [18]. The final step is of course to apply the quantile function for the base case to the samples of \( z \): \( X = F_1^{-1}(G_{R_1}^{-1}(U)) \).

3.3 Normal case

When taking \( z \) to be the CDF of the normal distribution, the transmuted density function takes the form

\[
F_2(x) = \phi(x)P' (\Phi(x), \alpha_1, \alpha_2).
\]

The first few moments when the base distribution is the normal distribution are provided in Table 2.

4 Conclusions

Transmutation maps provide a powerful technique for turning the ranks of one distribution into the ranks of another, e.g. to introduce skewness in a universal way. These techniques are well adapted for
quasi-Monte-Carlo and copula simulation methods, and may be extended to include a degree of kurtosis, in contrast to the traditional approach to distributional modulation. We have given explicit formulae to allow a skew-kurtotic-normal distribution to be simulated. Proposals for parameter estimation will be provided in a forthcoming publication. Clearly further work is needed to

- make more detailed comparisons with the Azzalini framework;
- look carefully at the details of the relationship with series of Gram-Charlier type;
- identify optimal parameter estimation methods;
- consider multivariate extensions.

However, initial results from our “alchemy” studies are very encouraging. The proposals for skewness adjustments are very simple and may be applied to any base distribution irrespective of whether it is symmetric or even defined for $x < 0$. Furthermore, as we have seen, the skewness adjustments may be extended to manage kurtosis adjustments as well. Our proposals also contain the basic order statistics (mix, min, middle) as special cases, and give elegant expressions for the CDFs of the relevant distributions within a univariate framework. We are also able to work out moments for the skew-kurtotic-normal developed within this framework, and these moments are all simple linear functions of the transmutation parameters. Our techniques are also very well adapted to Monte Carlo simulation as they make use of the quantile function of the base distribution composed with an elementary mapping.

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