Quantum Spin Chains with Nonlocally-Correlated Random Exchange Coupling and Random-Mass Dirac Fermions

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Abstract

$S = \frac{1}{2}$ quantum spin chains and ladders with random exchange coupling are studied by using an effective low-energy field theory and transfer matrix methods. Effects of the nonlocal correlations of exchange couplings are investigated numerically. In particular we calculate localization length of magnons, density of states, correlation functions and multifractal exponents as a function of the correlation length of the exchange couplings. As the correlation length increases, there occurs a “phase transition” and the above quantities exhibit different behaviors in two phases. This suggests that the strong-randomness fixed point of the random spin chains and random-singlet state get unstable by the long-range correlations of the random exchange couplings.

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1 Introduction

The properties of low-energy excitations in disordered systems in low dimensions have been intensively studied in recent several years. It is believed that in one-dimensional (1D) systems almost all states are exponentially localized regardless of amount of disorder if the random variables have $\delta$-function like short-range correlations. However recently, existence of phases of extended states was reported for some 1D models, which include the Anderson model, the aperiodic Kronig-Penny model, etc., if the disorder has long-range correlations\cite{1}. In the previous papers\cite{2, 3} motivated by the above studies, we studied a field-theory model, the random-mass Dirac fermions in 1D, which describe low-energy excitations of the random-hopping tight-binding model. In particular we considered the effects of the power-law spatial correlations of the random Dirac mass and found that properties of the states near the band center change as the correlation is increased, though extended states exist only at the band center.

In this paper we apply the previous field-theory model to the systems of spin chains and ladders with random exchange coupling. In particular, we shall numerically calculate the localization length (LL) of the magnons, their density of states (DOS) and correlations of low-energy wave functions as a function of the correlation length of the random exchange couplings. We shall employ the Fourier filtering method (FFM) and its modified techniques for generating random variables with long-range correlations. It is known that in the random quantum spin chain with $S = \frac{1}{2}$ there exists a nontrivial low-energy fixed point called strong-randomness fixed point and at that fixed point random-singlet (RS) phase is realized\cite{4}. In particular there the spin correlation function behaves as $\langle S_n^z S_0^z \rangle \propto \frac{1}{n^2}$ where $n$ is the lattice site index. Stability of the fixed point has been discussed from various aspects\cite{5}. In this paper we study effects of the long-range correlation of the exchange couplings of the spin chains on the stability problem. In order to investigate this problem, we focus on low-energy modes and study a more tractable field-theory model, i.e., random-mass Dirac fermions.

In Sec.2, we shall briefly review the derivation of the effective field-theory model, i.e., the random-mass Dirac fermions, for low-energy excitations in $S = \frac{1}{2}$ quantum spin chains and ladders. We specify the random exchange couplings and random mass with nonlocal correlations. In Sec.3, we explain FFM and modified FFM (MFFM) and show the random variables generated by the MFFM actually have power-law correlations. This result is very important for the present study because the original FFM is not suitable for generating random numbers with long-range correlations. In Sec.4, we calculate the LL and DOS for various random variables with nonlocal correlations by the MFFM. These quantities change their behavior as the correlation of the random variables is varied.
This result agrees with our previous calculations which employed the FFM and the above conclusion about the “existence of phase transition” is confirmed though its validity was questioned sometimes. In Sec.5, we shall study properties of the low-energy wave functions and the multifractal exponents. These quantities also change their behavior for the various correlations of the random mass. This result strongly indicates that the long-range correlation of the exchange couplings changes behavior of the spin-spin correlation functions from that of the RS phase. Section 6 is devoted for conclusion.

2 Random spin chains and random-mass Dirac fermions

Let us start with the random XX spin chain whose Hamiltonian is given as

\[
H_{XX} = -\sum_n J_n \left( S_n^x S_{n+1}^x + S_n^y S_{n+1}^y \right)
\]

where \( n \) is site index and the magnitude of the quantum spin \( S_n^i (i = x, y, z) \) is \( 1/2 \). The exchange couplings \( J_n \) are random variables which have a spatial nontrivial correlation as we specify shortly. It is well-known that the Hamiltonian (1) is mapped to a fermion system through the Jordan-Wigner transformation,

\[
S_n^z = C_n^\dagger C_n - \frac{1}{2}, \quad S_n^+ = C_n^\dagger \exp(i\pi \sum_{i=-\infty}^{n-1} N_i),
\]

where \( C_1 \) is a spinless fermion operator and \( N_i = C_i^\dagger C_i \),

\[
H_{XX} = -\sum_n \frac{J_n}{2} \left( C_n^\dagger C_{n+1} + C_{n+1}^\dagger C_n \right).
\]

In the case of the non-random XX model with a constant exchange coupling \( J_n = J \), the ground state of the Hamiltonian (3) is the half-filled state of the fermion \( C_n \) and low-energy excitations are described by the right and left-moving modes near the Fermi points \( k_F = \pm \frac{\pi}{2} \).

\[
C_n = (i)^n \psi_R(n) + (-i)^n \psi_L(n).
\]

In this paper we consider the random exchange couplings \( J_n \) which have a uniform constant part \( J \) and a random fluctuating part \( \delta J_n = (-)^n m(n) \), where \( m(n) \) is a “smooth function” of site index \( n \);

\[
J_n = J + \delta J_n = J + (-)^n m(n).
\]
By substituting Eqs. (4) and (5) into (3), we obtain the low-energy effective field theory of the random XX model,

\[ \mathcal{H} = \int dx \psi^\dagger h \psi, \]

\[ h = -i \sigma^z \partial_x + m(x) \sigma^y, \]

where \( \sigma \) are the Pauli spin matrices, \( m(x) \) is the “continuum limit” of \( m(n) \) \( (x = na \) with the lattice spacing \( a ) \), \( \psi(x) = (\psi_R(x), \psi_L(x))^t \) and we have put \( J = 1 \) without loss of generality. The above Hamiltonian is nothing but the random-mass Dirac field in 1D which we studied in the previous papers[2, 3]. The LL and DOS were calculated by the transfer-matrix methods and imaginary vector potential methods. From (6), it is obvious that an energy gap appears for the constant nonvanishing mass \( m(x) = m_0 \). This corresponds to the energy gap in the dimer or spin-Peierls state of the XX spin chain which appears as a result of the alternating exchange couplings \( J_n = J + (-)^n m_0 \). There the spin-spin correlation function (SSCF) \( \langle S_z^n S_z^0 \rangle \) \( \text{ens} \) decays exponentially as \( e^{-m_0 |n|} \) \( \text{ens} \) means the ensemble average for the random variables). We are interested in how this spin-Peierls state changes when \( J_n \)’s become random variables.

It is not so difficult to show that the following random Heisenberg spin chain has the same low-energy effective theory with the random XX spin chain[6],

\[ H = \sum_n J_n \vec{S}_n \cdot \vec{S}_{n+1}. \]

This result comes from the fact that in the spin-Peierls state for \( J_n = J + (-)^n m_0 \) the term \( S_z^n S_z^{n+1} \) becomes irrelevant because of the existence of the spin gap. In the uniform case \( J_n = J \) (i.e., \( m_0 = 0 \)), the above term is marginal and renormalize the “speed of light” and the exponent of the spin correlation[7]. In the random spin chains on the other hands, it is believed that the RS phase dominates at low energies and the SSCF behaves as

\[ \langle S_z^n S_z^0 \rangle \text{ens} \propto \frac{1}{n^2} \]

regardless of the (relatively small) value of the \( z \) component of spin coupling. It can be also shown that the spin ladder system like

\[ H_{SL} = \sum_{n,j=1,2} J_{\perp n,j} \vec{S}_{n,j} \cdot \vec{S}_{n+1,j} + \sum_n J_{\perp n} \vec{S}_{n,1} \cdot \vec{S}_{n,2} \]

has a low-energy excitations which is described by the random-mass Dirac fermions[6, 8]. There the Dirac mass is proportional to the inter-ladder coupling \( J_{\perp n} \).

We assume a nontrivial spatial correlation for the random Dirac mass \( m(x) \) in (6),

\[ \langle m(x) m(y) \rangle \text{ens} = \chi(|x - y|). \]
For the short-range white-noise correlation, which is often employed in studies of the random systems, \( \chi(|x-y|) \propto \delta(x-y) \). In this paper we are interested in the effects of nonlocal correlations, i.e., the exponential-decay correlation like

\[
\langle m(x) m(y) \rangle_{\text{ens}} = \frac{g}{2\lambda} \exp\left(-\frac{|x-y|}{\lambda}\right),
\]  

(10)

and also the power-decay correlation like,

\[
\langle m(x) m(y) \rangle_{\text{ens}} = \frac{C}{|x-y|^\gamma},
\]  

(11)

where \( g, \lambda, C \) and \( \gamma \) are parameters.

We shall consider various nonlocally-correlated random exchange couplings or random Dirac masses and calculate the LL of the magnon, DOS and SSCF. Behaviors of these quantities change as the correlation of the random mass is varied. As in the previous papers[3], we shall consider multi-kink configurations of the mass \( m(x) \). In the practical calculation, we fix the distances between kinks and vary the magnitudes of \( m(x) \) as random variables (see Fig.1). In the previous papers, we used the FFM for generating random numbers with nontrivial correlation. The FFM has some disadvantage for large systems. To avoid this, we first generated a large sequence of random numbers and used only small fraction of them for practical calculations of the LL and DOS. In this paper we shall employ a modified method of the FFM which was proposed by Makse et.al.[9]. Let us first explain the MFFM briefly.

3 Long-range Correlated Disorders

We explain the methods for generating long-range correlated random variables numerically in this section. The FFM is a well-known numerical method for generating random numbers with correlations. However, this method is not suitable for generating power-law correlated random numbers. Random numbers generated by this method are power-law correlated within only 0.1% of the whole
system. The MFFM was proposed by Makse et al.[9] in order to modify this flaw. They showed the random numbers generated by the MFFM have the power-law correlation almost over the whole system.

Here we briefly review the FFM and the MFFM. First we consider the random numbers \( u_i \) which have white-noise correlation, \( [u_i u_{i+l}]_{\text{ens}} = \delta_{l,0} \). Our goal is to generate random numbers \( \{\eta_i\} \) which has the power-law correlation like,

\[
C(l) = [\eta_i \eta_{i+l}]_{\text{ens}} \sim l^{-\gamma} \quad (12)
\]

where \( \gamma \) is the exponent of the power decay. It is straightforward to show that variables \( \eta_q \), the Fourier transformation of \( \eta_i \) in Eq.(12), also has the power-law correlation with respect to \( q \) as follows,

\[
S(q) = [\eta_q \eta_{-q}]_{\text{ens}} \sim q^{\gamma - 1}. \quad (13)
\]

Then one can easily show that Eq.(13) is satisfied if we put,

\[
\eta_q = \{S(q)\}^{1/2} u_q \quad (14)
\]

where \( u_q \) is the Fourier transformation of \( u_i \).

¿From the above observation, the FFM is given as follows; First we generate white-noise random numbers \( u_i \) and calculate its Fourier transformation \( u_q \). Secondly we calculate \( \eta_q \) using Eq.(14). Thirdly we calculate the inverse Fourier transformation of \( \eta_q \) and obtain the power-law correlated random numbers \( \eta_i \).

The modified FFM changes the following part of the original FFM. The power-law correlation \( C(l) \) has the singularity at the origin \( l = 0 \). This singularity causes the difficulty of large-distance correlation which we mentioned above. Therefore we replace it as follows,

\[
C(l) = (1 + l^2)^{-\gamma/2} \quad (15)
\]

which has the same power-law correlation with Eq.(12) for large \( l \). We also impose periodic boundary condition on \( C(l) \),

\[
C(l) = C(l + L) \quad (16)
\]

where \( L \) is system size, and we define \( C(l) \) by Eq.(15) only in the range \(-L/2 < l < L/2\). In practical calculations using the MFFM, we obtain \( S(q) \) from \( C(l) \) in Eq.(15) by using numerical Fourier transformation.

We show the two-point correlation function of random numbers \( \eta_i \) generated by the MFFM in Fig.2. The averaged correlation functions show power-law behavior almost exactly over half size of the system. This result is much better than those obtained by the old-fashioned FFM. In this
paper we apply the MFFM to generate the random exchange coupling of the quantum spin model or random mass of Dirac fermion with power-law correlations. The correlator $[m(x)m(y)]_{\text{ens}}$ has the same behavior with $[\eta_i\eta_{i+l}]_{\text{ens}}$ in Fig.2. From this result it is obvious that $[m(x)m(y)]_{\text{ens}}$ has the desired behavior in almost the whole system.

4 Localization Length and DOS

In this section we show the result of numerical calculations of the LL and DOS in the system with the nonlocally correlated random exchange couplings or random-mass Dirac fermions. By using the modified FFM, we first calculate the LL and DOS as a function of energy and reexamine the previous results obtained by the old-fashioned FFM.

We calculate energy dependence of the averaged (typical) localization length of eigenstates and DOS first. In one-dimensional disorder system, the DOS and localization length are related by the Thouless formula \[10\]. In the case of random mass Dirac fermion (or random spin chain), this formula is given as,

$$\frac{1}{\xi(E)} \propto \int_0^{\lvert E \rvert} \log(\frac{E}{E'})\rho(E')dE',$$

(17)

where $\xi(E)$ and $\rho(E)$ are the LL and DOS at energy $E$, respectively. It is not so difficult to show that the DOS is parametrized as \[11\]

$$\rho(E) \sim E^{-\eta}\lvert \log E \rvert^{-\delta},$$

(18)

where $\eta$ and $\delta$ are exponents. Then the Thouless formula Eq.(17) gives the energy dependence of the LL as follows,

$$\xi(E) \sim E^{-1+\eta}\lvert \log E \rvert^{-1+\delta}, \text{ for } \eta \neq 1,$$

$$\xi(E) \sim \lvert \log E \rvert^{-2+\delta}, \text{ for } \eta = 1.$$ 

(19)
In the case of white-noise disorder,

\[ [m(x)m(y)]_{\text{ens}} = g \delta(x-y), \]  

(20)

(where \( g \) is the constant which controls the strength of randomness) the values of \( \eta \) and \( \delta \) are 1 and 3, respectively.

Let us briefly explain the numerical methods to calculate the DOS and LL\([3]\). We set the random mass \( m(x) \) in Eq.(6) as multi-kink configuration and vary the height of kinks randomly (see Fig.1). Our choice of the random-mass configurations simplifies the problem to solve the Dirac equation for Eq.(6) and we can calculate the energy eigenfunctions systematically using transfer-matrix methods. We also introduce an imaginary vector potential into this system. By varying the magnitude of vector potential, we can obtain the localization length of the states without imaginary vector potential as a function of energy and the correlation length of the random mass. This idea is based on the work by Hatano and Nelson\([12]\). Localized state has the wave function like,

\[ \Psi_0(x) \simeq \exp\left(-\frac{|x-x_c|}{\xi_0}\right), \]  

(21)

where \( \xi_0 \) is the localization length. Wave function in the presence of an imaginary vector potential \( iA \) is obtained as follows by the “imaginary” gauge transformation,

\[ \Psi_A(x) \simeq \exp\left(-\frac{|x-x_c|}{\xi_0} - A(x-x_c)\right). \]  

(22)

Then if \( A \) exceeds inverse of the localization length of the original wave function \( \Psi_0 \), \( \Psi_A \) becomes unnormalizable. By increasing the value of \( A \) and searching normalizable wave function with the same energy, we can obtain the value of \( \xi_0 \).

First we show the energy dependence of the DOS and LL for the white-noise disorder for comparison with the power-law decaying cases (Fig.3). We fit the energy dependence of the DOS in the log-log plot by the following function\(^3\),

\[ f(x) = A + Bx + C\log(x) \quad (x = |\log E|), \]  

(23)

where \( B \) and \( C \) are parameters which correspond to the exponents \( \eta \) and \( \delta \) in Eq.(18), respectively. The result is shown in the figure of the DOS data, and \( B \) and \( C \) are estimated as \( B = 1.25 \) and \( C = 3.1 \) (see Fig.3). From the analytical result, we have \( \eta = 1 \) and \( \delta = 3 \), and therefore the numerical calculations are in good agreement with the analytical ones.

Let us turn to the LL. From the linearity of data in the log-linear plot (top left figure), we expect that \( \xi(E) \) for the white-noise disorder is parametrized as follows,

\[ \xi(E) = F|\log E/2g| = F|\log E| + G, \]  

(24)

\(^3\)Here we set the constant \( g \) unity in the term \( |\log E/2g| \) for the convenience of the fitting.
Figure 3: Energy dependence of the LL and DOS in the system with white-noise disorder: We set \( L(\text{system size}) = 102.4 \) and 1024 kinks in these systems. The constant \( g \) in Eq.(20) is unity in this calculation. The lines in the top two figures are the results of linear fitting. The result of non-linear fitting for DOS is shown as the curve in the bottom figure.

where \( g \) is defined in Eq.(20), and \( F \) and \( G \) are constants. The constant \( g \) is estimated as 0.90 from the linear fitting, and this is consistent with our setting of the numerical calculation in which \( g = 1 \). We also show the LL data with the log-log plot in Fig.3 (top right). From this, it is obvious that the parametrization in terms of Eq.(24) gives better fit for the LL than any parametrization with \( \eta \neq 1 \) as expected. This means that our calculation gives reliable results.

In the previous papers[3], we found that the value of the parameter \( \eta \) in Eq.(18) is about 0.5 for the power-law correlated disorders. We also showed that the the value of \( \eta \) is rather insensitive to the power-law exponent of correlation, \( \gamma \) in Eq.(12). Since \( \eta = 1 \) for the white-noise disorder, we concluded that there is a “phase transition” as the correlation of the random variables is varied from the short-range to long-range correlations. However in Ref.[3], we used the old-fashioned FFM for generating random correlated numbers and therefore we suspect that long-range behavior of correlated random numbers was not produced correctly. In this paper we calculate energy dependence of the LL and DOS for the power-law correlated disorder by using modified FFM in order to examine our previous results.

Numerical results of the LL and DOS obtained by using MFFM are given in Fig.4 for \( \gamma = 0.4 \).
Figure 4: Energy dependence of the LL and DOS in the system with power-law correlated disorders: We set $L$ (system size) = 102.4 and 1024 kinks in these systems. $\gamma$'s in the figures are defined in Eq.(12). We show the energy dependences of LL in log-linear and log-log plots. We also show the energy dependence of DOS in log-log plot. The fitting results are shown as the lines and the curves in each figure (see also the text).

and $\gamma = 0.2$. We show the LL data in log-linear and also log-log plots as in the previous white-noise case. In the both cases of $\gamma = 0.2$ and $\gamma = 0.4$, the results in Fig.4 show that $\eta \sim 0.5$ gives better fit than that of the white-noise parameters as we found in the previous paper[3]. Therefore we have confirmed the validity of the previous result, i.e., the LL and DOS exhibit different behaviors as the correlation of the random variables is changed from the short-range to long-range ones. We hope that this “phase transition” is observed in future by experiments of the random spin chains.

5 Zero-energy wave functions and multifractality

In this section we study the multifractality of zero-energy wave functions in the system with nonlocally correlated disorders. Behavior of the wave functions must give us more detailed information on the effects of the nonlocal correlations of the random disorders. In the case of the white-noise disorder, the zero-energy wave functions are exactly obtained and it is not so difficult to calculate
the ensemble-averaged correlation of them[13, 14]

\[ W_q(l, L) = [\langle \psi^\dagger(x)\psi(x + l)\rangle_q]_{\text{ens}}, \quad (25) \]

where \( L \) is the system size, \( q \) is a number and \( \psi(x) \) is the zero-energy wave function. We shall calculate \( W_q(l, L) \) numerically for various nonlocally correlated disorders. It is easy to see that the above \( W_2(l, L) \) is related with the spin-spin correlation function,

\[ [\langle S_z^z(x)S_z^z(y)\rangle]_{\text{ens}} = [\langle :\psi^\dagger\psi(x) : \psi^\dagger\psi(y) : \rangle]_{\text{ens}}, \quad (26) \]

where the normal ordering \( :O:\) is taken with respect to the half-filled state. The above spin-spin correlation function is dominated by the nodeless lowest-energy mode for each configuration of the random mass. For the white-noise correlation, relationship between the averaged Green function and the lowest-energy wave functions is discussed rather in detail in Ref.[13]. By using the transfer-matrix methods, it is not so difficult to obtain the wave function of the lowest energy and to calculate the correlation (25) numerically. In the non-random Heisenberg spin chain with a uniform exchange coupling \( J_n = J \) for all \( n \), the above correlator decays as

\[ [\langle S_z^z(x)S_z^z(y)\rangle]_{\text{ens}} \propto \frac{1}{|x - y|^3}, \quad (27) \]

whereas in the spin-Peierls state with \( J_n = J + (-)^n m_0 \),

\[ [\langle S_z^z(x)S_z^z(y)\rangle]_{\text{ens}} \propto \exp(-m_0|x - y|). \quad (28) \]

Furthermore for the random exchange of the white-noise correlation, which corresponds to the limit \( \lambda \to 0 \) of Eq.(10), the correlator \( W_q(l, L) \) was calculated both analytically[13] and numerically[2],

\[ [\langle \psi^\dagger\psi(x)\psi^\dagger\psi(y)\rangle]_{\text{ens}} \propto \frac{1}{|x - y|^2} \quad (29) \]

\( W_q(l, L) \) essentially measures the spatial correlation of the first and second peaks of the wave functions. On the other hand, the spin-spin correlator behaves as

\[ [\langle S_z^z(x)S_z^z(y)\rangle]_{\text{ens}} \propto \frac{1}{|x - y|^2} \quad (30) \]

Relation between the spin-spin correlation function \([\langle S_z^z(x)S_z^z(y)\rangle]_{\text{ens}} \) and \( W_q(l, L) \) is discussed in Ref.[6]. Therefore if the long-range behavior of \( W_q(l, L) \) changes as a result of the long-range correlation of the random variables, then we can expect that the spin correlation functions also behave differently from those of the white-noise case.

We must notice one technical difference between analytical and numerical studies. Normalization of the zero-energy wavefunctions plays an important role in the numerical study because they are
genuinely normalizable only for specific \( m(x) \). More explicitly, in solving the Dirac equation we imposed the periodic boundary condition on the system. In contrast, boundary condition is not imposed on the wave function in the analytical calculation\[6, 13, 14\].

It is expected that the correlation functions of zero-energy wavefunction exhibit multi-fractal scaling,

\[
W_q(l, L) \sim L^{-1-\tau(q)}|l|^{-y(q)}. \tag{31}
\]

For the system of the white-noise random mass \([m(x)m(y)]_{\text{ens}} \propto \delta(x-y)\), \( W_q(l, L) \) is obtained as follows for \( L \to \infty \) by the analytical calculation\[13\],

\[
W_q(l, L) \sim \frac{\tilde{W}(q^2 l)}{L} \tag{32}
\]

where

\[
\tilde{W}(l) = \int_0^\infty dk \frac{k^2}{(1+k^2)^{4}} e^{-lk^2}. \tag{33}
\]

For large \( l \)

\[
W_q(l, L) \sim 1/(l^{3/2}L) \tag{34}
\]

and therefore it is expected that \( \tau(q) \) and \( y(q) \) are independent of \( q \) and \( \tau(q) = 0 \) and \( y(q) = 3/2 \) for the white-noise disorder. In this paper, we shall numerically obtain the zero-energy wave functions and calculate \( y(q) \) under various types of disorder using the MFFM. Typical example of the zero-energy wave function obtained numerically is given in Fig.5.

First we numerically calculate \( W_q(l, L) \) in the case of the white-noise disorder. Values of \( y(1) \) and \( y(2) \) are obtained from the calculations of \( W_q(l, L) \) as a function of \( l/L \). (See Fig.6.) The value of \( y(2) \) is related with the power of algebraically decaying \([S^z(x)S^z(y)]_{\text{ens}}\) in the crossover region\[6\]. From Fig.6, the values of \( y(1) \) and \( y(2) \) for white-noise disorder are both about 1.5 and this result is in agreement with Eq.(34). Similarly we also calculate \( y(1) \) and \( y(2) \) for exponentially-correlated
disorder. Numerical calculations are shown in Fig.6 ($\lambda = 0.5$ and 25) and the results are summarized in Table 1. For small values of $\lambda/L$, $y(2)$ is also about 1.5. On the other hand for $\lambda = 25$, $y(2) \sim 2.2$. This reflects the effect of the nonlocal correlations. The above result suggests that nontrivial change occurs in the behavior of $W_q(l, L)$ for the power-decay correlated mass $[m(x)m(y)]_{\text{ens}} \propto \frac{1}{|x-y|^{\gamma}}$.

| $\lambda$ | 0(white) | 0.05 | 0.5 | 5 | 25 |
|-----------|----------|------|-----|---|----|
| $y(1)$    | 1.19     | 1.05 | 1.01| 1.27| 1.25|
| $y(2)$    | 1.55     | 1.46 | 1.36| 1.62| 2.25|

Table 1: Fractal exponent $y(1)$ and $y(2)$: The parameters are set as follows; $[m(x)m(y)]_{\text{ens}} = \exp(|x-y|/\lambda)$, $L = 102.4$ and 2048 kinks. The range of the data is $-2.3 < \log(l/L) < -0.7$.

Next we calculate $y(1)$ and $y(2)$ for the case of the long-range correlated disorders with power-law decay. We show the results in Table 2. (Examples of correlation are shown in Fig.7.) The values of $y(1)$ are about $\frac{1}{2}$ in almost all cases except $[m(x)m(y)]_{\text{ens}} = 0.01/|x-y|^{\gamma}$. This result may indicate universal properties of the single-particle Green’s function. However in all cases, $y(1) \neq y(2)$. This is in sharp contrast to the case of the white-noise disorder. In each cases, the value of $y(2)$ is larger than $\frac{1}{2}$. More precisely, the estimated value $y(2)$ is larger for smaller value of $\gamma$ and some of the $y(2)$
Figure 7: Examples of correlation function $W_q(l, L)$ under the power-law correlated disorder $[m(x)m(y)]_{ens} = C/|x - y|^{\gamma}$: We set L(system size) = 102.4, 2048 kinks and C = 0.1 here. 

exceeds 2. On the other hand for large $\gamma$, $y(2)$ is getting close to $\frac{4}{3}$. This reflects the fact that the effect of correlation becomes larger for smaller value of $\gamma$. The long-range correlation of the random mass suppresses the appearance of a large variety of wave functions. However our numerical study indicates that the system is still in a critical region with the power-decaying correlators.

Table 2: Fractal exponent $y(1)$ and $y(2)$: The parameters are shown in the table. $C$ and $\gamma$ are defined as $[m(x)m(y)]_{ens} = C/|x - y|^{\gamma}$.

| L(size) | # of kinks | $C$ | range of data log($l/L$) | $\gamma = 0.2$ | $\gamma = 0.4$ | $\gamma = 0.6$ | $\gamma = 0.8$ |
|---------|------------|-----|------------------------|----------------|----------------|----------------|----------------|
| I       | 51.2       | 1024| $-2.3 \sim -0.7$      | 1.37 | 2.39           | 1.47 | 2.41           | 1.45 | 2.20           | 1.27 | 2.03           |
| II      | 102.4      | 2048| $-2.5 \sim -2$       | 1.70 | 2.29           | 1.60 | 2.13           | 1.46 | 1.97           | 1.33 | 1.85           |
| III     | 204.8      | 4096| $-2.3 \sim -1.5$     | 1.41 | 2.27           | 1.26 | 2.03           | 1.12 | 1.85           | 0.97 | 1.70           |

6 Conclusion and Discussion

In this paper, we studied $S = \frac{1}{2}$ spin chain with nonlocally-correlated random exchange coupling via its low-energy effective field theory, random-mass Dirac fermions. In particular we investigated the effects of the long-range correlation of the random variables. In order to generate random numbers with power-decaying correlation, we employed the MFFM. We calculated the DOS, LL and correlators of the zero-energy functions $W_q(l, L)$, which give important information about low-energy states of the system. The results indicate that there appear nontrivial effects of the long-range correlation in various physical quantities. Energy dependence of the DOS and LL and the long-
distance behavior of $W_q(l, L)$ are changed by the correlations. As $W_q(l, L)$ is closely related with the spin correlation function $\langle S^z(x)S^z(y) \rangle_{\text{ens}}$, the above result indicates that the spin correlators also exhibit different long-distance behavior from that of the short-range correlated random variables.

Let us briefly discuss qualitative nature of the “new phase” which appears as a result of the long-range correlation of the random variables. In the short-range white-noise case, the DOS behaves as $\rho(E) \sim E^{-1}|\log E|^{-3}$ and the LL $\xi(E) \sim |\log E|$. On the other hand for the long-range correlated case, $\rho(E) \sim \xi(E) \sim E^{-1/2}$. This means that the low-energy states become more extended by the effect of the long-range correlations whereas the low-energy DOS becomes smaller because of it through the Thouless formula. As the localization length $\xi_0$ parametrizes the typical wave functions as in (21), it measures the width of the largest peak of the wave functions. Furthermore we found that $W_q(l, L)$, in particular $q = 2$, tends to decay more rapidly by the long-range correlations. As $W_q(l, L)$ measures the correlations between the first and second (or third) largest peaks of the wave functions, this result indicates that the second largest peak becomes smaller compared to the first one and/or the distances between the first and second largest peaks tend smaller by the long-range correlations. In the picture of the RS phase of the spin chains, the above results imply that amplitude of the spin-singlet pair becomes a smooth decreasing function of the distance between spins in the pair as a result of the long-range correlations of the exchange couplings. In other words, the “randomness” or “variety of configurations” of the RS phase is suppressed by the long-range correlations. Actually in the Dirac fermions with the mass of the single-kink configuration like $m(x) \propto \theta(x)$, where $\theta(x)$ is the step function, an exactly massless mode appears in the vicinity of the kink of $m(x)$ where $m(x)$ changes its sign[17]. This localized massless mode corresponds to an unpaired spin in the dimer state of the spin chains which appears as a result of the sign change of $m(n)$ in (5). In the random-mass Dirac fermions, the random mass $m(x)$ vanishes at various points $x = x_i$ ($i =$ an integer) and roughly speaking low-energy modes are given by linear combinations of the localized modes in the vicinity of the kinks of $m(x)$[2] like $\sum_i C_i |i\rangle$ where $|i\rangle$ is the localized state at $x_i$ and $C_i$’s are coefficients. The long-range correlation of the random mass keeps distance between adjacent kinks $|x_i - x_{i+1}|$ longer and as a result it makes the LL of the low-energy states larger. On the other hand, mixing amplitude of the modes localized in the vicinity of different kinks of $m(x)$ tends smaller by the smooth behavior of $m(x)$, in other words some coefficient $C_{i_0}$ dominates the others $C_{i \neq i_0}$ and $C_i$ decreases smoothly as $|x_{i_0} - x_i|$ increases. This consideration indicates that the low-energy DOS tends smaller by the long-range correlation as we observed by the numerical calculation. In the new phase, most of spins make a singlet pair with their nearest-neighbor spin. Spins in kinks of $m(n)$ embedded in the dimer state have weak antiferromagnetic correlations with each other and generate low-energy spin excitations.
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