On Seeley-type universal extension operators for the upper half space

Haowen Lu | Liding Yao

1School of Mathematics, Sun Yat-sen University, Guangzhou, Guangdong, China
2Department of Mathematics, The Ohio State University, Columbus, Ohio, USA

Correspondence
Liding Yao, Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA.
Email: yao.1015@osu.edu

Abstract
Modified from the standard half-space extension via the reflection principle, we construct a linear extension operator for the upper half space $\mathbb{R}_+^n = \{(x', x_n) : x_n > 0\}$ that has the form $E f(x) = \sum_{j=\infty}^{\infty} a_j f(x', -b_j x_n)$ for $x_n < 0$. We prove that $E$ is bounded in all $C^k$-spaces, Sobolev and Hölder spaces, Besov and Triebel–Lizorkin spaces, along with their Morrey generalizations. We also give an analogous construction on bounded smooth domains.

KEYWORDS
Besov and Triebel–Lizorkin spaces, Seeley’s extension operator, universal extension operator

1 | INTRODUCTION

Given a function $f$ defined on the upper half space $\mathbb{R}_+^n = \{(x', x_n) : x_n > 0\}$, the reflection principle gives a well-known construction to extend $f$ to the total space $\mathbb{R}^n$ while preserving its regularity property: we can define the extension $E f$ that depends linearly on $f$ by

$$E f(x', x_n) = E^{a,b} f(x', x_n) := \sum_j a_j f(x', -b_j x_n), \quad \text{for } x_n < 0, \quad \text{where } b_j > 0. \quad (1)$$

Here, $(a_j, b_j)_j$ are (finitely or infinitely many) real numbers that satisfy certain algebraic conditions (see below). This method traces back to Lichtenstein [10] for $C^1$-extensions and [7] for $C^k$-extensions. See also [2, 11] for the corresponding Sobolev extensions.

In the spaces of distributions, a linear extension operator $E$ is viewed as a right inverse of the natural restriction map $\mathcal{D}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}_+^n)$. Triebel [21] and Franke [5] showed that for every $\varepsilon > 0$ there is a $m = m(\varepsilon) > 0$, such that, if the finite collection $(a_j, b_j)_j$ satisfy $\sum_j a_j (-b_j)^k = 1$ for all integers $-m \leq k \leq m$, then for $E$ in Equation (1) have boundedness in Besov spaces $E : \mathcal{B}_{pq}^s(\mathbb{R}_+^n) \rightarrow \mathcal{B}_{pq}^s(\mathbb{R}^n)$ and in Triebel–Lizorkin spaces $E : \mathcal{F}_{pq}^s(\mathbb{R}_+^n) \rightarrow \mathcal{F}_{pq}^s(\mathbb{R}^n)$, for all $\varepsilon < p, q \leq \infty$ and $-\varepsilon^{-1} < s < \varepsilon^{-1}$ ($p < \infty$ in $\mathcal{F}$-cases). In [23, Section 1.11.5] such $E$ is called a common extension operator which depends on $\varepsilon(>0)$.

In contrast, we call $E$ a universal extension operator, if we have the boundedness $E : \mathcal{A}_{pq}^s(\mathbb{R}_+^n) \rightarrow \mathcal{A}_{pq}^s(\mathbb{R}^n)$ for all $\mathcal{A} \in \{\mathcal{B}, \mathcal{F}\}$, $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ ($p < \infty$ in $\mathcal{F}$-cases), simultaneously (i.e., we can take $\varepsilon = 0$).

Note that the collections $(a_j, b_j)_j$ discussed above are all finite. In those cases, the range of spaces on which $E$ is bounded would inevitably be finite and depends on $\varepsilon > 0$. Hence, such $E$ is never universal.

By taking some infinitely nonzero sequences $(a_j, b_j)_j$, Seeley [17] constructed one such operator (1) such that $E : C^k(\mathbb{R}_+^n) \rightarrow C^k(\mathbb{R}^n)$ for all $k = 0, 1, 2, \ldots$. In particular, Seeley’s extension perverse $C^\infty$-smoothness. However, to
the best of authors’ knowledge, there was no proof of the boundedness of Seeley’s operator on general Besov and Triebel–Lizorkin spaces.

In this paper, we generalize Seeley’s construction by extending the boundedness to the spaces of the negative index, and we show that such operator is a universal extension operator.

**Theorem 1.**

(i) There exists an extension operator $E : C^0(\mathbb{R}_n^+) \to C^0(\mathbb{R}_n^+)$ that has the form

$$Ef(x', x_n) = E_{a, b}f(x', x_n) := \begin{cases} \sum_{j=-\infty}^{\infty} a_j f(x', -b_j x_n) & x_n < 0 \\ f(x) & x_n > 0 \end{cases},$$

where $b_j > 0$ such that

$$\sum_{j \in \mathbb{Z}} 2^{\delta_k j} |a_j| |b_j^k| < \infty$$

for some $\delta_k > 0$. \tag{3}

Moreover, when Equation (3) is satisfied, then Equation (2) always defines an extension operator $E$ for $\mathbb{R}_n^+$, that has the following boundedness (simultaneously):

(ii) (Sobolev, Hölder, and $C^k$) $E : W^{k, p}(\mathbb{R}_n^+) \to W^{k, p}(\mathbb{R}_n^+)$ and $E : C^{k, s}(\mathbb{R}_n^+) \to C^{k, s}(\mathbb{R}_n^+)$ are defined and bounded for all $k \in \mathbb{Z}$, $0 < p \leq \infty$ and $0 < s < 1$.

(iii) (Besov and Triebel–Lizorkin) $E : S'(\mathbb{R}_n^+) \to S'(\mathbb{R}_n^+)$ is continuous. Moreover we have boundedness in Besov spaces $E : \mathcal{B}^{s, p, q}(\mathbb{R}_n^+) \to \mathcal{B}^{s, p, q}(\mathbb{R}_n^+)$ and in Triebel–Lizorkin spaces $E : \mathcal{F}^{s, p, q}(\mathbb{R}_n^+) \to \mathcal{F}^{s, p, q}(\mathbb{R}_n^+)$ for all $s \in \mathbb{R}$ and $0 < p, q \leq \infty$.

(iv) (Morrey-type) More generally $E$ has boundedness on all Besov-type spaces $\mathcal{B}^{s, \tau, p, q}$, Triebel–Lizorkin-type spaces $\mathcal{F}^{s, \tau, p, q}$ and Besov–Morrey spaces $\mathcal{N}^{s, \tau, p, q}$. That is, $E : \mathcal{A}^{s, \tau, p, q}(\mathbb{R}_n^+) \to \mathcal{A}^{s, \tau, p, q}(\mathbb{R}_n^+)$ for $\mathcal{A} \in \{\mathcal{B}, \mathcal{F}, \mathcal{N}\}$ and all $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $0 \leq \tau < 1/p$ ($p < \infty$ for the $\mathcal{F}$-cases).

See Definitions 3, 6, and 8 for the spaces in the theorem. Here, the Sobolev spaces $W^{k, p}$ for $p < 1$ can be defined and are discussed in [12], see also Remark 4.

The analog of Theorem 1 on smooth domains is also true; see Theorem 26. The operator is given by Equation (39).

**Remark 2.** Here, we allow $\delta_k$ in Equation (3) to tend to 0 as $|k| \to \infty$. In practice, we can choose $(a_j, b_j)_j$ such that the sum $\sum_j 2^{\delta_j |j|} |a_j| |b_j^k| < \infty$ holds for all $k$ and all arbitrarily large $\delta$, see Proposition 14.

It is not known to the authors whether the results still hold if we remove $2^{\delta_k j|j|}$ in Equation (3), that is, if we only assume $\sum_j |a_j| |b_j^k| < \infty$ for all $k \in \mathbb{Z}$. In the proof of Theorem 1, the terms $2^{\delta_k j|j|}$ are used only when we consider $p < 1$ in (ii) and $\min(p, q) < 1$ in (iii) and (iv).

We should remark that the universal extension operator exists not only on smooth domains but also on Lipschitz domains. This is done by Rychkov [16] using Littlewood–Paley decompositions. Rychkov’s extension operator $E^R$ has the form $E^R f = \sum_{j=0}^{\infty} \psi_j * (1_{\mathbb{R}_n^+} \cdot (\phi_j * f))$, where $1_{\mathbb{R}_n^+}$ is the characteristic function of $\mathbb{R}_n^+$ and $(\phi_j, \psi_j)_{j=0}^{\infty}$ is a carefully chosen family of Schwartz functions. The shape of $E^R$ is totally different from Equation (1) and the construction follows a different methodology.

Our result shows that the construction (1) can also be made to be universal extension, cf. the comment below [9, Corollary 5.7].

In fact, our extension operator is “more universal” than the Rychkov extension $E^R$ [16]. Although $E^R$ is bounded in all Besov and Triebel–Lizorkin spaces (see [16, Theorem 4.1]), it is not known whether $E^R$ is bounded on the endpoint Sobolev spaces $W^{k, 1}$, $W^{k, \infty}$, and the $C^k$-spaces. On the other hand, our extension operator is also defined on the measurable function space $L^p(\mathbb{R}_n^+)$ for $0 < p < 1$, whose element may not be realized as a distribution on $\mathbb{R}_n^+$.

The range of the extension operator is important if we want to construct some operator that has the form

$$Tf(x) = \int_{\Omega} K(x, y) E f(y) dy,$$

where $E$ is an extension operator for the domain $\Omega \subset \subset \Omega'$. \tag{4}

Here, $K(x, y)$ can be some kernel of the integral operator or singular integral operator.
This method has been used extensively in solving Cauchy–Riemann problem (the \( \bar{\partial} \)-equation) on certain domains in \( \mathbb{C}^n \). For example, Wu [27] used the Seeley’s extension to construct a solution operator that is \( W^{k,p} \) bounded for all \( k \geq 0 \). If \( E \) is some common extension operator that is only bounded in a finite range of \( k \), then \( T \) is only bounded in a finite range as well.

By replacing the Seeley’s extension with our extension operator and doing more analysis on \( K(x, y) \), it might be possible that one can show that \( T \) in Equation (4) is bounded in \( W^{k,p} \) for \( k < 0 \) as well.

The estimates on negative Sobolev spaces are discussed in [19, 29], where we took \( E \) to be Rychkov’s extension operator. In the case of smooth domains, if we replace Rychkov’s extension by our Seeley-type extension, then it is possible not only to get some estimates related to \( L^1 \)-Sobolev spaces, but also to simplify some of the proofs in these papers (see, e.g., [29, version 1, Proposition 5.1 and Remark 5.2 (iii)]).

2 FUNCTION SPACES AND NOTATIONS

In the paper, we use the following definitions for Sobolev spaces and Hölder spaces, including both positive and negative indices. We also include the case \( p < 1 \) for Sobolev spaces.

**Definition 3.** Let \( U \subseteq \mathbb{R}^n \) be an arbitrary open subset. Let \( k \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} \).

We use \( C^k(U) = C^k(\overline{U}) \) for the space of all continuous functions \( f : U \to \mathbb{R} \) such that \( \partial^\alpha f \) are bounded and uniformly continuous for all \( |\alpha| \leq k \). We use \( \|f\|_{C^k(U)} := \max_{|\alpha| \leq k} \|\partial^\alpha f\|_{C^0(U)} \).

We let \( C^\infty(U) = C^\infty(\overline{U}) \) be the space of smooth functions.

For \( 0 < s < 1 \), we define the Hölder space \( C^{k,s}(U) \) to be the space of all functions \( f \in C^k(U) \) such that \( \|f\|_{C^{k,s}(U)} := \|f\|_{C^k(U)} + \max_{|\alpha| \leq k} \sup_{x,y\in U} |x-y|^{-s} |\partial^\alpha f(x) - \partial^\alpha f(y)| < \infty \).

For \( 0 < p \leq \infty \), we define the Sobolev space \( W^{k,p}(U) \) by the following:

(i) For \( 1 \leq p \leq \infty \), \( W^{k,p}(U) \) consists of all \( f \in L^p(U) \) whose distributional derivatives \( \partial^\alpha f \) belong to \( L^p(U) \) for all \( |\alpha| \leq k \).

We use the norm \( \|f\|_{W^{k,p}(U)} = \left( \sum_{|\alpha| \leq k} \int_U |\partial^\alpha f|^p \right)^{\frac{1}{p}} \) for \( 1 \leq p < \infty \) and \( \|f\|_{W^{k,\infty}(U)} := \sup_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty(U)} \).

(ii) For \( 0 < p < 1 \), the quasi-Banach space \( W^{k,p}(U) \) is the abstract completion of \( C^\infty(\overline{U}) \) under the quasi-norm \( \|f\|_{W^{k,p}(U)} = \left( \sum_{|\alpha| \leq k} \int_U |\partial^\alpha f|^p \right)^{\frac{1}{p}} \).

For \( k > 0 \), \( 0 < s < 1 \) and \( 1 \leq p \leq \infty \), we define

\[
\|f\|_{C^{-k,s}(U)} := \inf \left\{ \max_{|\alpha| \leq k} \|g_\alpha\|_{C^{0,s}(U)} : \{g_\alpha\}_{|\alpha| \leq k} \subseteq C^{0,s}(U), \quad f = \sum_{|\alpha| \leq k} \partial^\alpha g_\alpha \text{ as distributions} \right\}. \tag{5}
\]

\[
\|f\|_{W^{-k,p}(U)} := \inf \left\{ \left( \sum_{|\alpha| \leq k} \|g_\alpha\|_{L^p(U)}^p \right)^{1/p} : \{g_\alpha\}_{|\alpha| \leq k} \subseteq L^p(U), \quad f = \sum_{|\alpha| \leq k} \partial^\alpha g_\alpha \text{ as distributions} \right\}. \tag{6}
\]

(iii) We define \( C^{-k,s}(U) \) and \( W^{-k,p}(U) \) \((1 \leq p \leq \infty)\) to be the subset of distributions such that the above norms are finite, respectively.

(iv) For \( 0 < p < 1 \), we define \( W^{k,p}(U) \) to be the abstract completion of \( C^\infty(\overline{U}) \) under the quasi-norm \( (6) \) with \( \{g_\alpha\}_{|\alpha|} \subseteq C^\infty(\overline{U}) \) (where we only consider \( f \in C^\infty(\overline{U}) \) on both sides).

**Remark 4.** When \( 0 < p < 1 \) and \( k \geq 1 \), the natural (continuous) mapping \( \iota : W^{k,p}(U) \to L^p(U) \) is not injective. The proof can be modified from [12, Proposition 3.1]. If one use \( W^{k,p}(U)/ \ker \iota \) instead, then the elements can always be realized as measurable functions (in \( L^p(U) \)).

One can check that Theorem 1 (ii) implies the boundedness \( E : W^{k,p}(\mathbb{R}^n) / \ker \iota_{\mathbb{R}^n} \to W^{k,p}(\mathbb{R}^n) / \ker \iota_{\mathbb{R}^n} \).
Remark 5. When $U$ satisfies the nice boundary condition, particularly when $U \in \{\mathbb{R}^n_+, \mathbb{R}^n\}$ or when $U$ is a bounded smooth domain, we have the following:

(a) Definition 3 (i) and (ii) coincide, in the sense that $C_c^\infty(\overline{U})$ is dense in $W^{k,p}(U)$ for $p \in [1, \infty)$. For the proof see [4, Section 5.3] or [1, Theorem 3.17]. Taking the distribution derivatives, we get the density of $C_c^\infty(\overline{U})$ in $W^{k,p}(U)$ for $p \in [1, \infty)$ as well.

(b) When $1 \leq p < \infty$, $W^{-k,p'}(U)$ is the dual of $W_0^{k,p}(U) := C_c^\infty(\overline{U}) \subset W^{k,p}(U)$. See [1, Theorem 3.9] for example.

(c) For every $k \in \mathbb{Z}$, we have equivalent norms $\| \cdot \|_{W^{k+1,p}(U)} \approx \| \cdot \|_{W^{k,p}(U)} + \sum_{j=1}^n \| \partial^j f \|_{W^{k,p}(U)}$ for $1 < p < \infty$, and $\| \cdot \|_{C^{k+1,s}(U)} \approx \| \cdot \|_{C^{k,s}(U)} + \sum_{j=1}^n \| \partial^j f \|_{C^{k,s}(U)}$ for $0 < s < 1$.

However, they fail for $p = 1, \infty$: in both cases we have $\| f \|_{L^p(U)} \gg \| f \|_{W^{-1,p}(U)} + \sum_{j=1}^n \| \partial^j f \|_{W^{-1,p}(U)}$ but the converse inequalities are false. This can be done by using (b) and [13, Theorem 5].

We use the standard convention for spaces of tempered distributions.

Definition 6. We use $\mathcal{S}(\mathbb{R}^n)$ for the Schwartz space and $\mathcal{S}'(\mathbb{R}^n)$ for the space of tempered distributions.

For an arbitrary open subset $U \subseteq \mathbb{R}^n$, we define $\mathcal{S}'(U) := \{ ̃f \vert_U : ̃f \in \mathcal{S}'(\mathbb{R}^n) \}$ to be the space of distributions in $U$ that has tempered distributional extension.

We let $\mathcal{S}(U) := \{ f \in \mathcal{S}(\mathbb{R}^n) : f \vert_U \equiv 0 \}$ to be the space of Schwartz functions supported in $U$.

Remark 7.

(i) By definition $\mathcal{S}'(U) = \mathcal{S}'(\mathbb{R}^n) / \{ ̃f \in \mathcal{S}'(\mathbb{R}^n) : ̃f \vert_U = 0 \}$ is the quotient space. Since $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ are dual to each other with respect to their standard topologies, we see that $\mathcal{S}'(U) = \mathcal{S}(U)'$. See also [21, Theorem 2.10.5/1].

(ii) We see that $\mathcal{S}'(U)$ is always the subspace of extendable distributions, that is, $\mathcal{S}'(U) \subseteq \{ ̃f \vert_U : ̃f \in \mathcal{D}'(U) \}$. They are equal when $U$ is a bounded domain: indeed, take a $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi \vert_U \equiv 1$, if $f \in \mathcal{D}'(U)$ extends a $f \in \mathcal{S}'(U)$, then $\chi f \in \mathcal{S}'(\mathbb{R}^n)$ extends such $f$ as well.

The Besov and Triebel–Lizorkin spaces, along with their Morrey analogies, can be defined using Littlewood–Paley decomposition. Note that we do not use these characterizations directly in the proof.

Definition 8. Let $\lambda = (\lambda_j)_{j=0}^\infty$ be a sequence of Schwartz functions satisfying:

- The Fourier transform $\hat{\lambda}_0(\xi) = \int_{\mathbb{R}^n} \lambda_0(x) e^{-2\pi i x \cdot \xi} \, dx$ satisfies $\text{supp} \hat{\lambda}_0 \subset \{ |\xi| < 2 \}$, $\hat{\lambda}_0 |_{|\xi|<1} \equiv 1$.
- $\lambda_j(x) = 2^{jn} \lambda_0(2^j x) - 2^{(j-1)n} \lambda_0(2^{j-1} x)$ for $j \geq 1$.

Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $0 \leq r \leq \frac{1}{p}$, we define the Besov-type norm $\| \cdot \|_{\mathcal{B}^{s,r}_{pq}(\lambda)}$, the Triebel–Lizorkin-type norm $\| \cdot \|_{\mathcal{F}^{s,r}_{pq}(\lambda)}$ and the Besov–Morrey norm $\| \cdot \|_{\mathcal{N}^{s,r}_{pq}(\lambda)}$ as follows:

$$\| f \|_{\mathcal{B}^{s,r}_{pq}(\lambda)} := \sup_{x \in \mathbb{R}^n, j \in \mathbb{Z}} 2^{nJr} \left( \sum_{j=\max(0,J)}^\infty 2^{jsq} \left( \int_{B(x,2^{-j})} |\lambda_j * f(x)|^p \, dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}};$$

$$\| f \|_{\mathcal{F}^{s,r}_{pq}(\lambda)} := \sup_{x \in \mathbb{R}^n, j \in \mathbb{Z}} 2^{nJr} \left( \int_{B(x,2^{-j})} \left( \sum_{j=\max(0,J)}^\infty 2^{js} |\lambda_j * f(x)|^q \right)^{\frac{p}{q}} \, dx \right)^{\frac{1}{p}}, \quad \text{provided } p < \infty;$$

$$\| f \|_{\mathcal{N}^{s,r}_{pq}(\lambda)} := \sum_{j=0}^\infty \sup_{x \in \mathbb{R}^n, j \in \mathbb{Z}} 2^{nJr+jsq} \left( \int_{B(x,2^{-j})} |\lambda_j * f(x)|^p \, dx \right)^{\frac{q}{p}};$$
Here for \( q = \infty \) we take natural modifications by replacing the \( \ell^q \)-sums with the suprema over \( j \).

We define the corresponding spaces \( \mathcal{A}_{pq}^s(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f \| \mathcal{A}_{pq}^s(\mathbb{R}^n) := \| f \| \mathcal{A}_{pq}(\mathbb{R}^n) < \infty \} \) where \( \mathcal{A} \in \{ \mathcal{B}, \mathcal{F}, \mathcal{N} \} \), for a fixed choice of \( \lambda (p < \infty \) for the \( \mathcal{F} \)-cases).

For arbitrary open \( U \subseteq \mathbb{R}^n \), we define \( \mathcal{A}_{pq}^s(U) := \{ \tilde{f} |_U : \tilde{f} \in \mathcal{A}_{pq}^s(\mathbb{R}^n) \} \) with \( \| f \| \mathcal{A}_{pq}^s(U) := \inf \{ f \|_{\mathcal{A}_{pq}^s(U)} : f \in \mathcal{A}_{pq}^s(U) \} \).

For the classical Besov and Triebel–Lizorkin spaces, we use

\[
\mathcal{B}^s_{pq}(U) := \mathcal{B}^s_{pq}(\mathbb{R}^n) \quad \text{and} \quad \mathcal{F}^s_{pq}(U) := \mathcal{F}^s_{pq}(\mathbb{R}^n),
\]

for \( p, q \in (0, \infty] \) and \( s \in \mathbb{R} \) (\( p < \infty \) for the \( \mathcal{F} \)-cases).

We shall see that \( \mathcal{A}_{pq}^s(U) \) is always a (quasi-)Banach space, and different choices of \( (\lambda_j)_{j=0}^\infty \) result in equivalent norms. See [24, Proposition 2.3.2] and [26, Propositions 1.3 and 1.8].

Remark 9. In the definition, we only consider \( \tau \leq \frac{1}{p} \). If we extend them to \( \tau > \frac{1}{p} \) then by [32, Theorem 2] and [18, Lemma 3.4] we have

\[
\mathcal{B}^s_{pq}(U) = \mathcal{F}^s_{pq}(U) = \mathcal{B}^{s+\frac{1}{p}}_{\infty, \infty}(\mathbb{R}^n), \quad \mathcal{N}^s_{p, q}(U) = \mathcal{N}^{s+\frac{1}{p}}_{\infty, q}(\mathbb{R}^n), \quad \forall 0 < p, q \leq \infty, s \in \mathbb{R}.
\]

See also [26, Proposition 1.18].

Our notation \( \mathcal{A}^s_{pq} \) corresponds to the \( \mathcal{B}^s_{pq} \) in [18, Definition 5]. For the usual conventions of Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces: we have \( \mathcal{N}^s_{u,q,p}(\mathbb{R}^n) = \mathcal{N}^{s,p-1}_{u,q,p}(\mathbb{R}^n) \) and \( \mathcal{L}^s_{u,q,p}(\mathbb{R}^n) = \mathcal{L}^{s,p-1}_{u,q,p}(\mathbb{R}^n) \) see [18, Remark 13(iii) and Proposition 3.3(iii)].

For the notations in [8, 25], we have the correspondence \( L^{\frac{s+1}{p}}_{pq}(\mathbb{R}^n) = \Lambda^{p r}_{pq}(\mathbb{R}^n) \) for \( \mathcal{A} \in \{ \mathcal{B}, \mathcal{F} \} \) and \( \Lambda^{p r}_{pq}(\mathbb{R}^n) = \mathcal{A}^{p r}_{pq}(\mathbb{R}^n) \) for \( \mathcal{A} = \mathcal{N} \).

For more discussions, we refer [8, 18, 25, 30] to readers.

In the following, we will use the notation \( x \lesssim y \) to mean \( x \leq C y \), where \( C \) is a constant independent of \( x, y, \) and \( x \approx y \) for \( x \lesssim y \) and \( y \lesssim x \). We use \( x \lesssim y \) to emphasize the dependence of \( C \) on the parameter \( \varepsilon \).

For \( r \neq 0 \), we use \( \vartheta^r \) as the dilation operator given by

\[
\vartheta^r f(x', x_n) := f(x', r x_n)
\]

We use \( M_{pq}^{ns} \) and \( \tilde{M}_{pq}^{ns} \) for some fixed positive constants in Proposition 22 and Theorem 23, both of which depend on \( n \geq 1, 0 < p, q \leq \infty \) and \( s \in \mathbb{R} \).

We let \( S = S_n \) be the zero extension operator of \( \mathbb{R}_+^n \), that is,

\[
S f(x) = S_n f(x) := 1_{\mathbb{R}_+^n}(x) \cdot f(x) = \begin{cases} f(x', x_n) & x_n > 0 \\ 0 & x_n < 0 . \end{cases}
\]

Remark 10. We have the definedness and boundedness \( S_n : \mathcal{A}_{pq}^s(\mathbb{R}^n) \to \mathcal{A}_{pq}^s(\mathbb{R}^n) \) for \( \max(n(\frac{1}{p} - 1), \frac{p}{p} - 1) < s < \frac{1}{p} \), see [26, Theorem 2.48]. We will use the case \( n = 1 \) and the case \( p = \infty \) in the proof of Theorem 1 (iii).

To clarify, the statement of [26, Theorem 2.48] requires \( q \geq 1 \) for the \( \mathcal{F}_{pq}^{s_{\text{eq}}} \)-case. We do not need this restriction as we do not use the unboundedness of \( S_n \) for the case \( s \not\in (\max(n(\frac{1}{p} - 1), \frac{p}{p} - 1), \frac{1}{p}) \). See also [26, Theorem 2.48, Proof Step 2].
THE CONSTRUCTION OF UNIVERSAL SEELEY’S EXTENSION: PROOF OF THEOREM 1 (I) AND (II)

First, we need to fulfill condition (3). In [17], Seeley took \( b_j := 2^j \) for \( j \geq 0 \) and found \((a_j)_{j=0}^\infty \) such that \( \sum_{j=0}^\infty a_j(-b_j)^k = 1 \) for all \( k \geq 0 \). This can be done by constructing an entire function \( F(z) = \sum_{j=0}^\infty a_jz^j \) such that \( F(2^k) = (-1)^k \), we recall the construction below.

Lemma 11.

(i) For \( \beta > 1 \), \( W_\beta(z) := \prod_{j=1}^\infty (1 - z/\beta^j) \) (\( z \in \mathbb{C} \)) defines a nonzero entire function \( W_\beta \), all of whose zeros are simple.

(ii) Let \( u = (u_k)_{k=0}^\infty \in l^\infty \). The function \( F^\beta_u \) below is entire and satisfies \( F^\beta_u(\beta^k) = u_k \), for all \( k \geq 0 \):

\[
F^\beta_u(z) := \sum_{k=0}^\infty \frac{u_k}{W^\prime_\beta(\beta^k)} \cdot \frac{W_\beta(z)}{z - \beta^k}.
\]

(iii) When \( \beta > 2 \), we have the following quantitative estimate for \( l = 1, 2, 3,... \):

\[
\sum_{k=1}^\infty \left| \frac{W_\beta(\beta^{-l})}{W^\prime_\beta(\beta^k)(\beta^{-l} - \beta^k)} \right| \leq \frac{\beta^3 \exp \frac{2}{\beta-1}}{(\beta^2 - 1)(\beta^2 - \beta - 1)}.
\]

Proof. (i) is the result of the Weierstrass factorization theorem, see [14, Theorem 15.9] for example.

For (ii), one can see that for \( k \geq 0 \),

\[
W^\prime_\beta(\beta^k) = \frac{W_\beta(z)}{z - \beta^k} \bigg|_{z=\beta^k} = -\beta^{-k} \prod_{j=0, j \neq k}^\infty (1 - \beta^k/\beta^j) = -\beta^{-k}W_\beta(\beta^{-1}) \cdot \prod_{l=1}^k (1 - \beta^l).
\]

Since \( \beta^l - 1 \geq \beta^{l-1}(\beta - 1) \) for \( l \geq 1 \), we see that for \( k \geq 0 \):

\[
|W^\prime_\beta(\beta^k)| \geq |W_\beta(\beta^{-1})|\beta^{-k} \prod_{l=1}^k \beta^{l-1}(\beta - 1) = |W_\beta(\beta^{-1})|(\beta - 1)^k \beta^{-k} \cdot \beta^{\frac{k^2-k}{2}}.
\]

Since \( \beta > 1 \), we see that the sum \( \sum_{k=0}^\infty u_k/W^\prime_\beta(\beta^k) \) absolutely converges. Therefore, the sum in Equation (10) converges absolutely and locally uniformly in \( z \). We conclude that \( F^\beta_u \) is indeed an entire function.

For each \( j \geq 0 \), we have \( W_\beta(z) \bigg|_{z=\beta^k} = W^\prime_\beta(\beta^k) \) when \( j = k \) and \( W_\beta(z) \bigg|_{z=\beta^k} = 0 \) when \( j \neq k \). Therefore, \( F^\beta_u(\beta^k) = u_k \), as for each \( z = \beta^k \) there is only one nonzero term in the sum (ii).

For (iii), note that for each \( l \geq 1 \), \( W_\beta(\beta^{-l}) = |W_\beta(\beta^{-l})| < 1 \) because \( |1 - \beta^{-l}/\beta^j| < 1 \) for each \( j \geq 0 \). Therefore using (12) and (13) we see that for \( k, l \geq 1 \),

\[
\left| \frac{W_\beta(\beta^{-l})}{W^\prime_\beta(\beta^k)(\beta^{-l} - \beta^k)} \right| \leq \frac{W_\beta(\beta^{-1})^{-1} \cdot W_\beta(\beta^{-l}) \cdot \beta^k}{(\beta - 1)^k \beta^{\frac{k^2-k}{2}} \cdot \beta^{-l} - \beta^k} \leq \frac{W_\beta(\beta^{-1})^{-1}}{(\beta^2 - \beta)^k} \cdot \frac{1}{1 - \beta^{-k-1}} \leq \frac{W_\beta(\beta^{-1})^{-1}}{(\beta^2 - \beta)^k} \cdot \frac{\beta}{1 - \beta^{-2}}.
\]

On the other hand

\[
(W_\beta(\beta^{-1}))^{-1} = \prod_{j=1}^\infty (1 - \beta^{-j})^{-1} = \exp \sum_{j=1}^\infty \log \frac{1}{1 - \beta^{-j}} < \exp \sum_{j=1}^\infty \log(1 + 2\beta^{-j}) < \exp \sum_{j=1}^\infty 2\beta^{-j} = \exp \frac{2}{\beta - 1} \quad \text{when} \quad 0 < \beta < 1.
\]
Therefore taking sum over \( k \geq 1 \), we obtain Equation (11):

\[
\sum_{k=1}^{\infty} \left| \frac{W_\beta(\beta^{-1})}{W'_\beta(\beta^k)(\beta^{-1} - \beta^k)} \right| \leq \frac{\beta}{1 - \beta^{-2}} \sum_{k=1}^{\infty} \left( \frac{\exp \frac{2}{\beta-1}}{\beta^2 - \beta} \right)^k = \frac{\beta^3}{(\beta^2 - 1)(\beta^2 - \beta - 1)}.
\]

\( \square \)

**Remark 12.** If one takes \( u_k := (-1)^k \) in Lemma 11 (ii), then the corresponding sequence \( a_j := \frac{1}{j!} (F_4^u(u))(0) \) (\( j = 0, 1, 2, \ldots \)) satisfies \( \sum_j a_j \beta^j = (-1)^k \) for all \( k \geq 0 \). This is enough to prove the boundedness results in [17] \( C^k \) for \( k \geq 0 \), but in our case we also need the equality to be true for \( k < 0 \). However, one cannot construct an entire function \( F \) such that \( F(2^{-k}) = (-1)^{-k} \) for all \( k \geq 0 \).

**Remark 13.** Alternatively, one can construct the \( (a_j) \) using Vandermonde determinant, which is the method of [17]. Let \( M_1, M_2 \in \mathbb{N}_0 \) and let \( b_{-M_1}, \ldots, b_{M_2} > 0 \) be distinct numbers. By Cramer’s rule, we see that the solutions of the equation system \( \sum_{j=-M_1}^{M_2} a_j b_j^k = (-1)^k \) for \( -M_1 \leq k \leq M_2 \) are (uniquely) determined by the following (see [1, p. 149], for example):

\[
a_j = (-b_j)^{M_1} \frac{V(b_{-M_1}, \ldots, b_j - 1, \ldots, b_{M_2})}{V(b_{-M_1}, \ldots, b_{M_2})} \quad (j = -M_1, \ldots, M_2).
\]

\( (14) \)

Here, \( V(x_{-M_1}, \ldots, x_{M_2}) = \prod_{-M_1 \leq u < v \leq M_2} (x_v - x_u) \).

Now, we set \( b_j := \beta^j \) for some \( \beta > 1 \). If \( \nu = 0 \) and let \( M_2 \rightarrow +\infty \), we see that Equation (14) converges and produces the same sequence \( (a_j)_{j=0}^{\infty} \) to Remark 12.

We find \( (a_j)_{j=\infty}^{\infty} \) through the following:

**Proposition 14.** There are real numbers \( (\tilde{a}_j)_{j=0}^{\infty} \) such that \( \sum_{j=0}^{\infty} \tilde{a}_j(4^j + 4^{-j}) = (-1)^k \) for all \( k = 0, 1, 2, \ldots \), with all the summations converging absolutely.

Therefore, the sequences \( a_j := \tilde{a}_j \) (\( j \neq 0 \)) and \( \tilde{a}_0 := 2\tilde{a}_0 \) fulfill condition (3).

The idea is to construct an iterated sequence \( \tilde{a}^\nu = (\tilde{a}^\nu_0, \tilde{a}^\nu_1, \tilde{a}^\nu_2, \ldots) \) such that

\[
\sum_{j=0}^{\infty} \tilde{a}^\nu_j 4^j = (-1)^k - \sum_{j=0}^{\infty} \tilde{a}^{\nu-1}_j 4^{-j}, \quad \forall k \geq 0.
\]

The limit \( \lim_{\nu \rightarrow \infty} \tilde{a}^\nu \) will be our \( \tilde{a}_j \), provided that it is convergent. In practice, it is more convenient to prove the convergence of \( (u^\nu)_{\nu=0}^{\infty} \subset \ell^\infty \) via the correspondence \( u^\nu := \sum_{j=0}^{\infty} \tilde{a}^\nu_j 4^j \) and \( \tilde{a}^\nu_j = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{d}{dz} F^4_{u^\nu-1}(\nu, z)|_{z=0} \).

\( u^\nu \) is the entire function in Lemma 11 (ii) with \( \beta = 4 \) and \( u = u^\nu \).

**Proof.** Let \( u^0 = (u^0_k)_{k=0}^{\infty} \in \ell^\infty \) be \( u^0_k := \frac{1}{2} \) and \( u^0_k := (-1)^k \) for \( k \geq 1 \). For each \( \nu \geq 1 \), we let \( u^\nu_0 := \frac{1}{2} \) and

\[
\tilde{a}^\nu_j := (-1)^k - \sum_{j=0}^{\infty} \left( \frac{1}{j!} \frac{d}{dz} F^4_{u^{\nu-1}}(\nu, z)|_{z=0} \right) 4^{-j} = (-1)^k - F^4_{u^{\nu-1}}(4^{-k}), \quad \text{for } k \geq 1.
\]

By Lemma 11 (ii), if \( u^{\nu-1} \in \ell^\infty \) then \( F_{u^{\nu-1}}(z) \) is entire in \( |z| < 1 \). Since \( u^0 \in \ell^\infty \), by recursion we get \( \|u^\nu\|_{\ell^\infty} \leq 1 + \|F_{u^{\nu-1}}\|_{L^\infty([-1, 1])} < \infty \) for every \( \nu \geq 1 \). Thus, we have a sequence of bounded sequences \( (u^\nu)_{\nu=0}^{\infty} \subset \ell^\infty \).

Moreover since \( u^0 \equiv \frac{1}{2} \), by Equation (11),

\[
\|u^{\nu+1} - u^{\nu}\|_{\ell^\infty} = \sup_{l \geq 1} |u^{\nu+1}_l - u^{\nu}_l| \leq \sup_{l \geq 1} |F^4_{u^{\nu}}(4^{-l}) - F^4_{u^{\nu-1}}(4^{-l})| \leq \sup_{l \geq 1} \sum_{k=1}^{\infty} \left| \frac{(u_k - u^{\nu-1}_k) \cdot W_4(4^{-l})}{W_4(4^k) \cdot (4^{-l} - 4^k)} \right| \leq \|u^{\nu} - u^{\nu-1}\|_{\ell^\infty} \frac{4^3 \exp \frac{2}{4-1}}{(4^2 - 1)(4^2 - 4 - 1)} = \frac{64}{165} \cdot \|u^{\nu} - u^{\nu-1}\|_{\ell^\infty}.
\]
Here, \( \frac{64e^{1/3}}{165} \approx 0.755 < 1 \). We conclude that \((u^\nu)^\infty_{\nu=0} \subset \ell^\infty\) is a contraction sequence and must have a limit \( u^\infty \in \ell^\infty \).

By Lemma 11 (ii), the function \( F^4_{u^\infty}(z) \) is entire. We define \( \bar{a}_j \) in the way that \( F^4_{u^\infty}(z) = \sum_{j=0}^{\infty} \bar{a}_j z^j \), thus

\[
\sum_{j=0}^{\infty} \bar{a}_j u^\infty_0 = 1 - u^\infty_0 = (-1)^0 - \sum_{j=0}^{\infty} \bar{a}_j,
\]

\[
\sum_{j=0}^{\infty} \bar{a}_j 4^j k = u^\infty_k = (-1)^k - F^4_{u^\infty}(4^{-k}) = (-1)^k - \sum_{j=0}^{\infty} \bar{a}_j 4^{-jk},
\]

\( k \geq 1 \).

Since \( F^4_{u^\infty} \) is entire, the sum \( \sum_{j=0}^{\infty} \bar{a}_j 4^j k (= F^4_{u^\infty}(4^k)) \) always converge absolutely. Therefore, \((\bar{a}_j)^\infty_{j=0}\) is as desired.

Let \( a_j := \bar{a}_j |_{j} \) for \( j \neq 0 \) and \( a_0 := 2 \bar{a}_0 \), we see that \( \sum_{j=0}^{\infty} |a_j| 4^j |k| < \infty \) for all \( k \geq 1 \), which means that \( \sum_{j=0}^{\infty} |a_j| 2^j |4^j k| < \infty \) for all \( \delta > 0 \) and \( k \in \mathbb{Z} \). Therefore, \((a_j, 4^j)^\infty_{j=-\infty}\) satisfies the condition (3) and we prove Theorem 1(i). \( \square \)

One can see that Seeley-type extensions have the following structures:

**Lemma 15.**

(i) Let \((a, b) = (a_j, b_j)^\infty_{j=-\infty}\) be the collection satisfying the condition (3). Then for every \( k \in \mathbb{Z} \) the collections \((a(-b)^k, b)^\infty_{j=-\infty}\) and \((a(-b)^k, 1/b_j)^\infty_{j=-\infty}\) also satisfy (3).

(ii) For multi-index \( \gamma = (\gamma', \gamma_n) \in \mathbb{N}^n_0 \) we have \( \partial^\gamma \circ E a, b = E a(-b)^{\gamma_n}, b \circ \partial^\gamma \) and \( E a, b \circ \partial^\gamma = \partial^\gamma \circ E a(-b)^{-\gamma_n}, b \).

(iii) Let \( E = E a, b \) be the extension operator defined in Equation (2). The formal adjoint \( E^* \) has the following expression:

\[
E^* g(x', x_n) = g(x', x_n) + \sum_{j=-\infty}^{\infty} \frac{a_j}{b_j} g(x', -\frac{1}{b_j} \cdot x_n), \quad x' \in \mathbb{R}^{n-1}, \quad x_n > 0.
\]

(15)

**Proof.** The results follow from direct computations.

(i): By assumption, \( \sum_j a_j(-b)^k(-b)^l = \sum_j a_j(-b)^{k+l} = 1 \) and \( \sum_j a_j(-b)^k(-1/b_j)^l = \sum_j a_j(-b)^{k-l} = 1 \), both hold for every \( l \in \mathbb{Z} \). Thus, \((a(-b)^k, b)\) and \((a(-b)^k, 1/b_j)^\infty_{j=-\infty}\) both satisfy Equation (3).

(ii): Clearly, \( \partial_{x_j} \circ E = E \circ \partial_{x_j} \) for every \( j \neq n \). By the chain rule \( \partial_{x_n}^k (f(x', -b j x_n)) = (-b j)^k \cdot (\partial_{x_n}^k f)(x', -b j x_n) \) holds for every \( k \geq 0 \). Therefore, \( \partial^\gamma (f(x', -b j x_n)) = (-b j)^{\gamma_n} \cdot (\partial^\gamma f)(x', -b j x_n) \). Taking sum over \( j \) we get \( \partial^\gamma \circ E a, b = E a(-b)^{-\gamma_n}, b \circ \partial^\gamma \). Replacing \( a, b \) by \((a(-b)^{-\gamma_n}, b)\) we see that \( \partial^\gamma \circ E a(-b)^{-\gamma_n}, b \circ \partial^\gamma = E a, b \circ \partial^\gamma \).

(iii): For every \( f \in C^\infty_c(\mathbb{R}^n_+) \) and \( g \in C^\infty_c(\mathbb{R}^n) \),

\[
\int_{\mathbb{R}^n_+} E f(x) g(x) dx = \int_{\mathbb{R}^n_+} f(x) g(x) dx + \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} a_j f(x', -b j x_n) g(x', x_n) dx' dx_n
\]

\[
= \int_{\mathbb{R}^n_+} f(x) g(x) dx + \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} a_j f(x', y_n) g\left(x', -\frac{1}{b_j} y_n\right) dx' \frac{dy_n}{-b_j}
\]

\[
= \int_{\mathbb{R}^n_+} f(x) g(x) dx + \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \frac{a_j}{b_j} f(x', y_n) g\left(x', -\frac{1}{b_j} y_n\right) dx' dy_n.
\]

Replacing \( y_n \) by \( x_n \) we get expression (15). \( \square \)
Remark 16. In the notations of Equations (8) and (9), we can write \( E^{a,b} = S + \sum_j a_j \delta^{-b} o S \). For \( r < 0 \), we have
\[
\partial x_n (\delta' S f)(x) = r (S \partial x_n f)(x', rx_n) - \delta_0 (x_n) f(x', rx_n),
\]
where \( \delta_0 \) is the Dirac measure at 0 \( \in \mathbb{R} \).

Therefore, \( \delta_0 o E^{a,b} = E^{a(-b),b} o \delta \) (in the domain where either side is defined) if and only if \( \sum_j a_j = 1 \). Taking higher order derivatives we see that \( \delta'^r o E^{a,b} = E^{a(-b)^r,n,b} o \delta'^r \) still holds if we only assume the condition \( \sum_j a_j (-b_j)^k = 1 \) for \( 0 \leq k \leq y_n \).

Lemma 17. Let \( (X, \| \cdot \|) \) be a quasi-Banach space such that \( (x, y) \mapsto \|x - y\|^q \) is a metric for some \( 0 < q \leq 1 \), that is, \( \|x + y\|^q \leq \|x\|^q + \|y\|^q \) for all \( x, y \in X \). Then for any \( \delta > 0 \) there is a constant \( K_{\delta,q} > 0 \) such that
\[
\| \sum_{j=-\infty}^\infty x_j \| \leq K_{\delta,q} \left( \sum_{j=-\infty}^\infty 2^{j|\|x_j\|} \right) \quad \text{for all sequence } (x_j)_{j \in \mathbb{Z}} \subset X \text{ such that right-hand summation converges.}
\]
In this case, the sum \( \sum_{j=-\infty}^\infty x_j \) converges with respect to the quasi-norm topology of \( X \).

Remark 18. The couple \( (X, \| \cdot \|) \) is also called a \( q \)-convex quasi Banach space (\( q \)-Banach space for short).

Proof of Lemma 17. When \( (x_j)_{j \in \mathbb{Z}} \) are finitely nonzero, by Hölder’s inequality we have
\[
\left\| \sum_j x_j \right\|_X \leq \frac{1}{1/q} \left( \left\| \left( 2^{j|\|x_j\|} \right)_{j \in \mathbb{Z}} \right\|_1 \right)^{1/q}.
\]

Here, \( K_{\delta,q} := \left( \frac{2^{j|\|x_j\|} + 1}{2^{j|\|x_j\|} + 1} \right)^{1/q} \) if \( q < 1 \) and \( K_{\delta,q} := 1 \) if \( q = 1 \).

When \( \sum_{j=-\infty}^\infty 2^{j|\|x_j\|} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, the sum must converge to a unique limit element and the same inequality \( \left\| \sum_j x_j \right\|_X \leq K_{\delta,q} \left( \sum_j 2^{j|\|x_j\|} \right) \) holds.

The proof of boundedness on Sobolev spaces is standard:

Proof of Theorem 1 (ii). We only need to prove the Sobolev boundedness \( W^{k,p} \) for \( k \geq 0, 0 < p \leq \infty \).

Indeed suppose we get \( E : W^{k,\infty}(\mathbb{R}^n_+) \to W^{k,\infty}(\mathbb{R}^n) \), then \( C^{\infty}(U) = \bigcap_{k=0}^\infty W^{k,\infty}(U) \) for \( U \in \{ \mathbb{R}^n, \mathbb{R}^n_+ \} \) gives \( E : C^{\infty}(\mathbb{R}^n_+) \to C^{\infty}(\mathbb{R}^n) \). Clearly, \( C^k(U) = C^{\infty}(U) \) for \( U \in \{ \mathbb{R}^n, \mathbb{R}^n_+ \} \), thus \( E : C^{k}(\mathbb{R}^n) \to C^k(\mathbb{R}^n) \) holds for all \( k \geq 0 \).

The \( C^{k,s} \) boundedness \( (k \in \mathbb{Z}, 0 < s < 1) \) can follow either from the same argument for \( W^{k,p} \) below, or the Besov correspondence \( C^{k,s}(U) = B^{k+1}_{p,\infty}(U) \), for \( U \in \{ \mathbb{R}^n, \mathbb{R}^n_+ \} \) (see Proposition 21 (ii)). When \( k \geq 0 \) one can also obtain the boundedness by the interpolation \( C^{k,s}(U) = (C^k(U), C^{k+1}(U))_{s,\infty} \) (see, e.g., [21, Theorem 2.7.2/1] and the proof of [21, Theorem 4.5.2/1]).

To prove the \( W^{k,p} \)-boundedness, we apply Lemma 17. For every \( f \in L^p(\mathbb{R}^n_+) \) if \( 1 \leq p \leq \infty \), and every \( f \in C^{\infty}_{c}(\mathbb{R}^n_+) \) if \( 0 < p < 1 \), we have (recall the notation \( \delta' \) in Equation (8))
\[
\| E^{a,b} f \|_{L^p(\mathbb{R}^n_+)} \leq 2^{\max(0, \frac{1}{p}-1)} \left( \| f \|_{L^p(\mathbb{R}^n_+)} + \| \sum_{j \in \mathbb{Z}} a_j \cdot \delta^{-b} f \|_{L^p(\mathbb{R}^n_+)} \right).
\]

(by Lemma 17)
\[
= \| f \|_{L^p(\mathbb{R}^n_+)} + \sum_{j \in \mathbb{Z}} 2^{j|\|a_j\|} \left( b_j^{-1} \int_{\mathbb{R}^n_+} |f(x', y_n)|^p \, dx' \, dy_n \right)^{\frac{1}{p}}
\]
\[
= \left( 1 + \sum_{j \in \mathbb{Z}} 2^{j|\|a_j\|} b_j \right) \| f \|_{L^p(\mathbb{R}^n_+)},
\]

(16)
When \( k \geq 0 \), also for every \( f \in L^p(\mathbb{R}^n_+) \) if \( 1 \leq p \leq \infty \), and every \( f \in C_c^\infty(\overline{\mathbb{R}^n_+}) \) if \( 0 < p < 1 \), applying Lemma 15 (ii) to (16) we have

\[
\|E^{a,b} f\|_{W^{k,p}(\mathbb{R}^n)} \lesssim_{k,p} \sum_{|\gamma| \leq k} \|\partial^{\gamma} E^{a,b} f\|_{L^p(\mathbb{R}^n)} = \sum_{|\gamma| \leq k} \|E^{(-b)^{\gamma_n},b} \partial^{\gamma} f\|_{L^p(\mathbb{R}^n)} \\
\lesssim_{p,\delta} \sum_{|\gamma| \leq k} \sum_{j \in \mathbb{Z}} \gamma |a_j| b_j^{\gamma_n - \frac{1}{p} - \frac{1}{p}} \|\partial^{\gamma} f\|_{L^p(\mathbb{R}^n)} \lesssim_{k,p} \sum_{j \in \mathbb{Z}} \gamma |a_j| (b_j^{\frac{k}{p} - \frac{1}{p}} + b_j^{\frac{k-1}{p}}) \|f\|_{W^{k,p}(\mathbb{R}^n)}.}
\]

By the assumption (3), we can take \( \delta > 0 \) small such that \( \sum_{j \in \mathbb{Z}} \gamma |a_j| b_j^{\frac{k-1}{p}} < \infty \) and \( \sum_{j \in \mathbb{Z}} \gamma |a_j| b_j^k < \infty \). We now get the boundedness of \( W^{k,p} \) for \( k \geq 0 \).

When \( k < 0 \), applying Lemma 15 (ii) to Equation (16) for

\(-\) every \( f \in W^{k,p}(\mathbb{R}^n_+) \) and \( \{g_\gamma\}_{|\gamma| \leq -k} \subset L^p(\mathbb{R}^n_+) \) such that \( f = \sum_{|\gamma| \leq -k} \partial^{\gamma} g_\gamma \), if \( 1 \leq p \leq \infty \);

\(-\) every \( f \in C_c^\infty(\overline{\mathbb{R}^n_+}) \) and \( \{g_\gamma\}_{|\gamma| \leq -k} \subset C_c^\infty(\overline{\mathbb{R}^n_+}) \) such that \( f = \sum_{|\gamma| \leq -k} \partial^{\gamma} g_\gamma \), if \( 0 < p < 1 \),

we have the following:

\[
\|E^{a,b} f\|_{W^{k,p}(\mathbb{R}^n)} \lesssim_{k,p} \sum_{|\gamma| \leq -k} \|E^{a,b} \partial^{\gamma} g_\gamma\|_{W^{k,p}(\mathbb{R}^n)} = \sum_{|\gamma| \leq -k} \|\partial^{\gamma} E^{a^{(-b)^{\gamma_n},b}} g_\gamma\|_{W^{k,p}(\mathbb{R}^n)} \\
\leq \sum_{|\gamma| \leq -k} \|E^{(-b)^{\gamma_n},b} g_\gamma\|_{L^p(\mathbb{R}^n)} \lesssim_{p,\delta} \sum_{|\gamma| \leq -k} \sum_{j \in \mathbb{Z}} \gamma |a_j| b_j^{-\gamma_n - \frac{1}{p}} \|g_\gamma\|_{L^p(\mathbb{R}^n)} \\
\leq \sum_{j \in \mathbb{Z}} \gamma |a_j| (b_j^{-\frac{1}{p}} + b_j^{\frac{k-1}{p}}) \sum_{|\gamma| \leq -k} \|g_\gamma\|_{L^p(\mathbb{R}^n)}.}
\]

Taking the infimum over all \( \{g_\gamma\}_{|\gamma| \leq -k} \) for \( f \), and taking \( \delta \) from Equation (3) such that \( \sum_{j \in \mathbb{Z}} \gamma |a_j| (b_j^{-\frac{1}{p}} + b_j^{\frac{k-1}{p}}) < \infty \) as well, we get the \( W^{k,p} \) -boundedness for \( k < 0 \), finishing the proof. \( \square \)

**Remark 19.** The result can be extended to vector-valued functions. Indeed let \( 0 < q \leq 1, 0 < p < \infty \), and let \( (X, |.|_X) \) be a quasi Banach space such that \( |x + y|_X^{1/q} \leq |x|_X^{1/q} + |y|_X^{1/q} \). Then for every \( L^p \) (strongly measurable) function \( f : \mathbb{R}^n_+ \to X \), by the same calculation to Equation (16),

\[
\left( \int_{\mathbb{R}^n_+} \sum_{j \in \mathbb{Z}} a_j f(x', -b_j x_n) |x|_X^{1/q} dx'dx_n \right)^{\frac{1}{p}} \leq K_{q,\delta,p} \left( \int_{\mathbb{R}^n} \left( \sum_{j \in \mathbb{Z}} \gamma |a_j| |f(x', -b_j x_n)|_X \right)^p dx'dx_n \right)^{\frac{1}{p}} \\
\leq K_{q,\delta,p} \sum_{j \in \mathbb{Z}} \gamma |a_j| b_j^{-\frac{1}{p}} \left( \int_{\mathbb{R}^n_+} b_j^{-1} |f(x', y_n)|_X^p dx'dy_n \right)^{\frac{1}{p}} = \left( K_{q,\delta,p} \sum_{j \in \mathbb{Z}} \gamma |a_j| b_j^{-\frac{1}{p}} \right) \|f\|_{L^p(\mathbb{R}^n_+;X)}.
\]

Here \( K_{q,\delta/2} \) and \( K_{p,\delta/2} \) are the constants in Lemma 17. Also by Lemma 17 the sum of functions \( \sum_{j \in \mathbb{Z}} a_j f(x', -b_j x_n) \) converges in \( L^p(\mathbb{R}^n_+;X) \).

Therefore, \( E^{a,b} \) defines a bounded linear map \( E^{a,b} : L^p(\mathbb{R}^n_+;X) \to L^p(\mathbb{R}^n_+;X) \) with the control of operator norms

\[
\|E^{a,b}\|_{L^p(\mathbb{R}^n_+;X) \to L^p(\mathbb{R}^n_+;X)} \lesssim_{p,\delta} \sum_{j \in \mathbb{Z}} \gamma |a_j| b_j^{-1/p}.
\]

In practice, we will take \( X = \mathcal{B}_{pq}^p(\mathbb{R}^{n-1}) \) in the proof of Theorem 23, Step A3.
Before we discuss the boundedness on Besov and Triebel–Lizorkin spaces, for completeness we prove that $E : \mathcal{S}' \to \mathcal{S}'$ is defined. Recall Definition 6 that $\mathcal{S}'(\mathbb{R}_+^n) = \mathcal{S}'(\mathbb{R}^n)/\{\tilde{f} : \tilde{f}|_{\mathbb{R}_+^n} = 0\}$.

**Lemma 20.** $E : \mathcal{S}'(\mathbb{R}_+^n) \to \mathcal{S}'(\mathbb{R}^n)$ is defined and bounded.

**Proof.** Since $\mathcal{S}'(\mathbb{R}_+^n) = \mathcal{S}(\mathbb{R}_+^n)'$ (see Remark 7 (i)), by duality it is equivalent to show that $E^* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}_+^n)$ is continuous.

Let $g \in \mathcal{S}(\mathbb{R}^n)$, using Equation (15) we see that for every $\alpha, \beta \in \mathbb{N}_0^n$ and $x \in \mathbb{R}_+^n$,

$$|x\alpha \partial^\beta E^* g(x)| \leq |x\alpha \partial^\beta g(x)| + \sum_{j \in \mathbb{Z}} |a_j(-b_j)a_{\alpha,n-\beta,n-1}(x', -\frac{1}{b_j} \cdot x_n) (\partial^\beta g)(x', -\frac{x_n}{b_j})|.$$

By Equation (3), $\sum_{j \in \mathbb{Z}} |a_j| \cdot b_j^{\alpha_n-\beta_n-1} < \infty$. We conclude that $|x\alpha \partial^\beta E^* g(x)| \lesssim a, b, \alpha, \beta \sup_{y \in \mathbb{R}^n} |y\alpha \partial^\beta g(y)|$ for all $x \in \mathbb{R}_+^n$.

On the other hand, by the condition $\sum_j a_j(-b_j)^{-k} = 1$ for $k = 1, 2, 3, \ldots$ in Equation (3), we have

$$\lim_{x_n \to 0^+} \partial^\beta E^* g(x) = \partial^\beta g(x', 0) - \sum_{j \in \mathbb{Z}} a_j(-b_j)^{-\beta_n-1}(\partial^\beta g)(x', 0) = 0, \quad \forall \beta \in \mathbb{N}_0^n.$$

Therefore along with the zero extension to $\mathbb{R}_+^n$, $E^* g$ defines a Schwartz function in $\mathbb{R}^n$ which has support $\mathbb{R}_+^n$. Moreover, $\sup_{x \in \mathbb{R}^n} |x\alpha \partial^\beta E^* g(x)| \lesssim a, b, \alpha, \beta \sup_{y \in \mathbb{R}^n} |y\alpha \partial^\beta g(y)|$ with the implied constant being independent of $g$.

Since the seminorms $\{\sup_{x \in \mathbb{R}^n} |x\alpha \partial^\beta E^* g(x)|\}_{\alpha, \beta \in \mathbb{N}_0^n}$ define the topology of $\mathcal{S}(\mathbb{R}^n)$, we conclude that $E^* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}_+^n)$ is continuous. Therefore, $E : \mathcal{S}'(\mathbb{R}_+^n) \to \mathcal{S}'(\mathbb{R}^n)$ is continuous as well. \qed

We now turn to prove Theorem 1(iii) and (iv).

To prove Theorem 1(iii), it might be possible to use the mean oscillation approach in [20, Chapter 4.5.5]. However, [20, (4.5.2/9)] fails when $(a_j)_j$ has infinite nonzero terms. This is critical if one wants to prove the Besov and Triebel–Lizorkin boundedness of Seeley’s operator (see Remark 12) using such method. The authors do not know what the correct modification of [20, (4.5.2/9)] is.

Instead, we adapt the arguments from [24, Section 2.9]. The proof involves characteristic function multiplier [24, Section 2.8.7], the Fubini decomposition [24, Section 2.5.13], and interpolations. We recall these facts from the literature.

**Proposition 21.** Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $0 \leq \tau \leq \frac{1}{p}$. Let $U \subseteq \mathbb{R}^n$ be an arbitrary open subset.

(i) For $\mathcal{A} \in \{\mathcal{B}, \mathcal{F}\}$ and for every integer $m \geq 1$, $\mathcal{A}^{s+\tau}_{pq}(U) = \left\{ \sum_{|\alpha| \leq m} \partial^\alpha g_\alpha : g_\alpha \in \mathcal{A}^{s+\tau}_{pq}(U) \right\}$ and has equivalent (quasi-)norm $\|f\|_{\mathcal{A}^{s+\tau}_{pq}(U)} \approx_{pq,x,m} \inf \left\{ \sum_{|\alpha| \leq m} \|g_\alpha\|_{\mathcal{A}^{s+\tau}_{pq}(U)} : f = \sum_{|\alpha| \leq m} \partial^\alpha g_\alpha \text{ as distributions} \right\}$.

(ii) If $U \in \{\mathbb{R}^n, \mathbb{R}_+^n\}$ or $U$ is a bounded smooth domain, then $C^{k,s}(U) = \mathcal{B}^{s}_{\infty\infty}(U)$ for all $k \in \mathbb{Z}$ and $0 < s < 1$, with equivalent norms.

(iii) (Fubini) When $p < \infty$ or $p = \infty$, and when $s > n \max(0, \frac{1}{p} - 1, \frac{1}{q} - 1)$, we have decomposition $\mathcal{F}^{s}_{pq}(\mathbb{R}^n) = L^p(\mathbb{R}_x; \mathcal{F}^{s}_{pq}(\mathbb{R}^{n-1}_x)) \cap L^p(\mathbb{R}^{n-1}_x; \mathcal{F}^{s}_{pq}(\mathbb{R}_x))$, with equivalent (quasi-)norms:

$$\|f\|_{\mathcal{F}^{s}_{pq}(\mathbb{R}^n)} \approx_{n,p,q,s} \int_{\mathbb{R}} \|f(\cdot, x_n)\|_{\mathcal{F}^{s}_{pq}(\mathbb{R}^{n-1}_x)}^{p} dx_n + \int_{\mathbb{R}^{n-1}} \|f(x', \cdot)\|_{\mathcal{F}^{s}_{pq}(\mathbb{R})}^{p} dx'. \quad (17)$$
(iv) (Interpolations) For \( s_0, s_1 \in \mathbb{R} \) and \( \theta \in (0, 1) \) such that \( s_0 \neq s_1 \) and \( s = (1 - \theta)s_0 + \theta s_1 \), we have
\[
\langle T^{s_0}_{pq}(\mathbb{R}^n), T^{s_1}_{pq}(\mathbb{R}^n), \theta \rangle = T^{s}_{pq}(\mathbb{R}^n);
\]
\[
(\langle F^{s_0}_{pq}(\mathbb{R}^n), F^{s_1}_{pq}(\mathbb{R}^n), \theta \rangle)_{\infty} = \mathcal{B}_{pq}(\mathbb{R}^n);
\]
\[
(\mathcal{F}^{s_0}_{pp}(\mathbb{R}^n), \mathcal{F}^{s_1}_{pp}(\mathbb{R}^n))_{\infty} = \mathcal{N}^{s}_{pq}(\mathbb{R}^n), \quad \text{for} \, \tau \in [0, \frac{1}{p}) \quad \text{(provided} \, p < \infty).\]

(v) (Local representations) For \( s < 0 \), we have equivalent norms
\[
\|f\|_{\mathcal{B}^s_{pq}(\mathbb{R}^n)} \approx_{n,p,q,s,\tau} \sup_{x \in \mathbb{R}^n} R^{-n\tau} \|f\|_{\mathcal{B}^s_{pq}(B(x, R))};
\]
\[
\|f\|_{\mathcal{B}^s_{pq}(\mathbb{R}^n)} \approx_{n,p,q,s,\tau} \sup_{x \in \mathbb{R}^n} \|f\|_{\mathcal{B}^s_{pq}(B(x, R))}, \quad \text{provided} \, p < \infty.\]

Proof. For (i), by [26, Theorem 1.24] \( \|\Delta^{a}g\|_{\mathcal{B}^0_{pq}(\mathbb{R}^n)} \leq \|g\|_{\mathcal{B}^{m}_{pq}(\mathbb{R}^n)} \) holds for \( |\alpha| \leq m \). Note that \( \|g\|_{U} = \|g\| \) implies \( \|\Delta^{a}g\|_{U} = \|\Delta^{a}g\| \), thus taking restrictions we get \( \|\sum_{|\alpha| \leq m} \Delta^{a}g_{\alpha}\|_{\mathcal{B}^{m}_{pq}(U)} \leq \|\sum_{|\alpha| \leq m} \Delta^{a}g_{\alpha}\|_{\mathcal{B}^{m}_{pq}(\mathbb{R}^n)} \). Therefore, \( \|f\|_{\mathcal{B}^s_{pq}(U)} \leq \sum_{|\alpha| \leq m} \|\Delta^{a}g_{\alpha}\|_{\mathcal{B}^{m}_{pq}(U)} \) whenever \( f = \sum_{|\alpha| \leq m} \Delta^{a}g_{\alpha} \).

Conversely for \( f \in \mathcal{B}^0_{pq}(U) \), take an extension \( \tilde{f} \in \mathcal{B}^s_{pq}(\mathbb{R}^n) \), we have \( \tilde{f} = (I - \Delta)(I - \Delta)^{-1}f = \tilde{g}_{0} + \sum_{j=1}^{n} \delta j \tilde{g}_{j} \) where \( \tilde{g}_{0} : = (I - \Delta)^{-1}f \) and \( \tilde{g}_{j} := -\delta j (I - \Delta)^{-1}f \) \((1 \leq j \leq n) \). Thus by [26, Theorems 1.22 and 1.24] we have \( \sum_{j=0}^{n} \|\tilde{g}_{j}\|_{\mathcal{B}^s_{pq}(\mathbb{R}^n)} \leq \|\tilde{f}\|_{\mathcal{B}^s_{pq}(\mathbb{R}^n)}. \) Taking infimum of all extensions \( \tilde{f} \) over \( f \) we get (i) for \( m = 1 \). Doing this recursively we get all \( m \geq 1 \).

For (ii), we have \( \mathcal{C}^{k,b}(\mathbb{R}^n) = \mathcal{B}_{\infty \infty}^{k+\infty}(\mathbb{R}^n) \) from [24, Section 2.5.7] for \( k \geq 0 \) and \( 0 < s < 1 \). Thus in this case \( \mathcal{B}_{\infty \infty}^{k+\infty}(U) = \{f \in \mathcal{C}^{k,b}(\mathbb{R}^n) \} \subseteq \mathcal{C}^{k,b}(U) \) for arbitrary open subset \( U \). When \( U = \mathbb{R}^n \) or \( U \) is bounded smooth, by the existence of extension operator \( \mathcal{C}^{k,b}(U) \to \mathcal{C}^{k,b}(\mathbb{R}^n) \) (see [6, Lemma 6.37]) we get \( \mathcal{C}^{k,b}(U) \subseteq \mathcal{B}_{\infty \infty}^{k+\infty}(U) \). This proves the case \( k \geq 0 \).

Comparing the result (i) and (5), we see that \( \mathcal{C}^{k,b}(\mathbb{R}^n) = \mathcal{B}_{\infty \infty}^{k+\infty}(\mathbb{R}^n) \) for \( k < 0 \) as well.

For (iii) see [26, Theorem 3.25] and [22, Theorem 4.4] in the reference, a further decomposition \( \mathcal{F}^{s}_{pq}(\mathbb{R}^n) = \bigcap_{k=1}^{n} L^{p}(\mathbb{R}^{n-1}; \mathcal{F}^{s}_{pq}(\mathbb{R}) \big) \) is given, where \( \mathcal{F}^{s}_{pq}(\mathbb{R}^{n-1}) \) is the slide of \( f \) with variable \( x_k \in \mathbb{R} \). Similar notations for \( g \). Therefore, we get Equation (17) by the following:
\[
\|f\|_{\mathcal{F}^{s}_{pq}(\mathbb{R}^n)} \approx \sum_{k=1}^{n-1} f_{R^{n-1}} \|f_{x_k}\|_{\mathcal{F}^{s}_{pq}(\mathbb{R})} d\mathcal{S}^n_n + \int_{R^{n-1}} f(x', \cdot) \|f_{x_k'}\|_{\mathcal{F}^{s}_{pq}(\mathbb{R})} d\mathcal{S}^n_k
\]
\[
\|g\|_{\mathcal{F}^{s}_{pq}(\mathbb{R}^n)} \approx \sum_{k=1}^{n-1} f_{R^{n-1}} \|f_{x_k}\|_{\mathcal{F}^{s}_{pq}(\mathbb{R})} d\mathcal{S}^n_n + \int_{R^{n-1}} f(x', \cdot) \|f_{x_k'}\|_{\mathcal{F}^{s}_{pq}(\mathbb{R})} d\mathcal{S}^n_k.
\]
For (iv), we refer the reader to [31].

Here, Equation (18) follows from [31, Theorem 2.13 and Remark 2.13 (i), (iii)]. Here \((X_0, X_1, \theta)\) is call the \( \pm \)-method, which is different from the standard complex interpolation \([X_0, X_1]\). See [31, Definitions 2.9 and 2.50], for example.

See [31, Theorem 2.80] for Equation (20) and [31, Remark 2.81 (i)] for Equation (20) when \( p < \infty \). Recall from Remark 9 that we use \( \mathcal{N}^{s}_{pq} = N_{p,q,s} \).
The endpoint case $p = \infty$ of Equation (19) follows from the standard real interpolation of Besov spaces. Indeed we have $\mathcal{F}_s^{\infty} = \mathcal{B}_s^{\infty}$ and $(\mathcal{B}_s^{\infty}(\mathbb{R}^n), \mathcal{B}_s^{1}(\mathbb{R}^n))_{\infty, q} = \mathcal{B}_s^{q}(\mathbb{R}^n)$. See [24, Theorem 2.4.2 (i)], for example.

See [25, Theorem 3.64] for (v). In the reference, the collection $\{B(x, 2^{-J}) : x \in \mathbb{R}^n, J \in \mathbb{Z}\}$ is replaced by \{2Q_{J,v} := 2^{-J}(v + (-\frac{1}{2}, \frac{3}{2})^n) : J \in \mathbb{Z}, v \in \mathbb{Z}^n\}, but there is no difference between the proofs.

Recall the dilation operator $(\mathcal{D}^s f)(x) := f(x', r x_n)$ for $r \in \mathbb{R} \setminus \{0\}$ from Equation (8).

**Proposition 22.** For any $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ there is a $M = M_{p,q,s}^n > 0$ and $C = C_{n,p,q,s,M} > 0$ such that

$$\|\mathcal{D}^s f\|_{\mathcal{F}_p^q(\mathbb{R}^n)} \leq C(|r|^{-M} + |r|^M)\|f\|_{\mathcal{F}_p^q(\mathbb{R}^n)},$$

for $r \neq 0$, $\mathcal{D} \in \mathcal{B}, \mathcal{F}$, $f \in \mathcal{F}_p^q(\mathbb{R}^n)$.

**Proof.** We only need to prove the Triebel–Lizorkin case, that is, $\mathcal{D} = \mathcal{F}$. Indeed suppose we get the case $\mathcal{D} = \mathcal{F}$, then by real interpolation (19), for each $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ there is a $C_{p,q,s} > 0$ such that (see [21, Definition 1.2.2/2 and Theorem 1.3.3], in fact $C_{p,q,s} = C_q$)

$$\|T\|_{\mathcal{F}_p^q(\mathbb{R}^n) \rightarrow \mathcal{F}_p^q(\mathbb{R}^n)} \leq C_{n,p,q,s,M} \|T\|_{\mathcal{F}_p^q(\mathbb{R}^n) \rightarrow \mathcal{F}_p^q(\mathbb{R}^n)} + \|T\|_{\mathcal{F}_p^q(\mathbb{R}^n) \rightarrow \mathcal{F}_p^q(\mathbb{R}^n)},$$

whenever $T : \mathcal{F}_p^q(\mathbb{R}^n) \rightarrow \mathcal{F}_p^q(\mathbb{R}^n)$ is a bounded linear map such that $T : \mathcal{F}_p^q(\mathbb{R}^n) \rightarrow \mathcal{F}_p^q(\mathbb{R}^n)$ is also bounded. Taking $T = \mathcal{D}$ and replacing $M_{p,q}^n$ (obtained from the $\mathcal{F}$-cases) with $\max(M_{p,q}^n, M_{p,q}^{n-1}, M_{p,q}^{n+1})$, we complete the proof.

Step 1: The case $n = 1, 0 < p < \infty$, $0 < q \leq \infty$ and $s > \frac{1}{p} - 1$. We use [3, Section 2.3.1].

Recall $\lambda = (\lambda_j)_{j=0}^\infty$ from Definition 8. When $s > \max(0, 1 - \frac{1}{p})$, by [20, Theorem 2.3.3], we have $\|f\|_{\mathcal{F}_p^q(\mathbb{R}^n)} \approx_{p,q,s} \|f\|_{\mathcal{F}_p^q(\mathbb{R}^n)}$ shows that

$$\|\mathcal{D}^s f\|_{\mathcal{F}_p^q(\mathbb{R}^n)} \approx \|r|^{-\frac{1}{p}} f\|_{\mathcal{F}_p^q(\mathbb{R}^n)} + |r|^\frac{1}{p} \|x \mapsto \left(\int_0^\infty t(\lambda_1(t) * f)(x) \frac{dt}{t^{\frac{1}{p}+1}}\right)^{\frac{1}{p}}\|_{\mathcal{F}_p^q(\mathbb{R}^n)}.$$
In particular, we can take \( M_{n,s}^{p,q} = M_{n,s+p,q} \) in this case.

Step 4: The case \( p = \infty, 0 < q \leq \infty \) and \( s \in \mathbb{R} \). We use Proposition 21 (v).

For \( x \in \mathbb{R}^n \) and \( R > 0 \), we see that \( f \in \mathcal{F}_{n,q}^d (B \left( (x', \frac{x_n}{r}), \max \left( \frac{R}{r}, \frac{R}{|x|} \right) \right)) \) implies \( \hat{\sigma} f \in \mathcal{F}_{n,q}^d (B(x, R)) \). Therefore

\[
\|\hat{\sigma} f\|_{\mathcal{F}_{n,q}^d (\mathbb{R}^n)} \leq \sup_{x \in \mathbb{R}^n, R > 0} R^{-\frac{n}{q}} \sup_{\mathbb{R}^{n,q}} \left\| \hat{\sigma} f\right\|_{B \left( (x', \frac{x_n}{r}), \max \left( \frac{R}{r}, \frac{R}{|x|} \right) \right)}
\]

\[
\leq \sup_{x \in \mathbb{R}^n, R > 0} R^{-\frac{n}{q}} \sup_{\mathbb{R}^{n,q}} \left\| \hat{\sigma} f\right\|_{B \left( (x', \frac{x_n}{r}), \max \left( \frac{R}{r}, \frac{R}{|x|} \right) \right)}
\]

\[
\leq (|r|^{-m_{n,s}^{p,q}} + |r|^{m_{n,s}^{p,q}}) \sup_{x \in \mathbb{R}^n, R > 0} R^{-\frac{n}{q}} \left\| f\right\|_{\mathcal{F}_{n,q}^d (B(x, R))} \lesssim \left( |r|^{-m_{n,s}^{p,q}} + |r|^{m_{n,s}^{p,q}} \right) \| f\|_{\mathcal{F}_{n,q}^d (\mathbb{R}^n)}
\]

Taking \( M_{n,s}^{p,q} := M_{n,s+p,q} + n/q \), we complete the proof. \( \square \)

We now start to prove Theorem 1(iii). In order to handle Theorem 1(iv) later, we consider a more quantitative version of Theorem 1(iii). It will contain an estimate of the operator norm of \( E \).

**Theorem 23** (Quantitative common extensions). For every \( 0 < p, q \leq \infty \), \( s \in \mathbb{R} \), there is a constant \( \mathcal{M} = \mathcal{M}_{p,q}^{n,s} \), such that the following holds.

Let \( m_1, m_2 \in \mathbb{Z}_{\geq 0} \) be two non-negative integers such that \( n \max(0, \frac{2}{p} - 1, \frac{1}{q} - 1) - m_1 < s < \frac{1}{p} + m_2 \). Let \( a = (a_j)_{j=-\infty}^{\infty} \subset \mathbb{R} \) and \( b = (b_j)_{j=-\infty}^{\infty} \subset \mathbb{R}_+ \) be two sequences satisfying

\[
\sum_{j=-\infty}^{\infty} a_j (-b_j)^k = 1 \quad \text{for } -m_1 \leq k \leq m_2; \quad \sum_{j \in \mathbb{Z}} 2^{\delta j} |a_j| (b_j^{-\max(m_1, \hat{M})} + b_j^{\max(m_2, \hat{M})}) < \infty \quad \text{for some } \delta > 0.
\]

Then, \( E = E^{a,b} \) given by (2) has the boundedness \( E : \mathcal{S}_{p,q}^d (\mathbb{R}^n) \rightarrow \mathcal{S}_{p,q}^d (\mathbb{R}^n) \) for \( d \in \{\mathcal{B}, \mathcal{F}\} \). Moreover for the operator norms of \( E \), we have

\[
\|Ef\|_{\mathcal{S}_{p,q}^d (\mathbb{R}^n)} \leq C_{n,p,q,s,\hat{M}} \sum_{j \in \mathbb{Z}} 2^{\delta j} |a_j| (b_j^{-\hat{M}^{p,q} + b_j^{\hat{M}^{p,q}}} ) \| f\|_{\mathcal{S}_{p,q}^d (\mathbb{R}^n)}.
\]

Here, \( C_{n,p,q,s,\hat{M}} \) does not depend on the choices of \( (a_j)_j \) and \( (b_j)_j \).

**Remark 24.** By Proposition 14, the sequences \( (a_j)_j \) and \( (b_j)_j \) can be chosen independently of \( p, q, n, s, m_1, m_2 \). Note that \( \delta \) in Equation (25) is allowed to depend on \( m_1, m_2, \hat{M}_{n,s}^{p,q} \) (cf. the \( \delta_k \) in Equation (3) and Remark 2).

Therefore, letting \( m_1, m_2 \rightarrow +\infty \) we obtain all boundedness properties of \( E \) in Theorem 1(iii).

Additionally, Theorem 23 also gives the following:

- If one take \( m_1 = 0 \) and let \( m_2 \rightarrow +\infty \), we see that Seeley’s extension operator (see Remark 12) has boundedness \( E : \mathcal{S}_{p,q}^d (\mathbb{R}^n) \rightarrow \mathcal{S}_{p,q}^d (\mathbb{R}^n) \) for \( d \in \{\mathcal{B}, \mathcal{F}\}, p, q \in (0, \infty) \) and \( s > n \max(0, \frac{2}{p} - 1, \frac{1}{q} - 1) \). It is possible that the range for \( s \) is not optimal.
- If one take \( (a_j)_j \) to be finitely nonzero (depending on the upper bound of \( m_1, m_2 \)), then the qualitative result \( E : \mathcal{S}_{p,q}^d (\mathbb{R}^n) \rightarrow \mathcal{S}_{p,q}^d (\mathbb{R}^n) \) is well-known for common extension operators. See [26, Remark 2.72].

**Proof of Theorem 23** (hence Theorem 1(iii)). Similar to the proof of Proposition 22, using Equation (23) we only need to prove the case \( d = \mathcal{F} \).

To make the notation clear, we use \( E_{n,a,b} \) for the extension operator of \( \mathbb{R}^n_+ \) given in Equation (2) associated with sequences \( a = (a_j)_{j=-\infty}^{\infty} \) and \( b = (b_j)_{j=-\infty}^{\infty} \). Recall the zero extension operator \( S_n f(x) := 1_{\mathbb{R}_n^+} (x) f(x) \) from Equation (9).

Case \( \hat{A} \): We consider \( p < \infty \) and \( s \) suitably large. We sub-divide the discussion into three parts.
Step A1: The case \( n = 1, 0 < p < \infty, 0 < q \leq \infty \) and \( \frac{1}{p} - 1 < s < \frac{1}{p} \), with arbitrary \( m_1, m_2 \geq 0 \).

Indeed by [26, Theorem 2.48] we have boundedness \( S_1 : F^s_{pq}(\mathbb{R}^n) \to F^s_{pq}(\mathbb{R}) \) at this range. Clearly, \( E^{a,b}_1 = S_1 + \sum_{j=-\infty}^{\infty} a_j \cdot \partial^{-b} o S_1 \). Therefore by Lemma 17 and Proposition 22,

\[
\|E^{a,b}_1 f\|_{F^s_{pq}(\mathbb{R})} \leq \min(p,q,1,\delta) \|S_1 f\|_{F^s_{pq}(\mathbb{R})} + \sum_{j \in \mathbb{Z}} 2^{s/p} |a_j| \|\partial^{-b} S_1 f\|_{F^s_{pq}(\mathbb{R})}
\]

\[
\leq \sum_{j \in \mathbb{Z}} \left( b_j \cdot (\tilde{M}_1^{s-1})^p + b_j \cdot (\tilde{M}_1^{s})^p \right) \sum_{j \in \mathbb{Z}} 2^{s/p} |a_j| \left( b_j \cdot (\tilde{M}_1^{s-1})^q + b_j \cdot (\tilde{M}_1^{s})^q \right) \|f\|_{F^s_{pq}(\mathbb{R}^n)}.
\]

We conclude the case by taking \( \tilde{M}_1^{1s} := M_1^{1s} \), the constant in Proposition 22.

Step A2: The case \( n = 1, 0 < p < \infty, 0 < q \leq \infty \) and \( s > \frac{1}{p} - 1 \), with \( m_1 \geq 0 \) and \( m_2 > s - \frac{1}{p} \).

By [24, Theorem 2.38 and Remark 3.3.5/2] \( \|f\|_{F^s_{pq}(\mathbb{R})} \approx_{p,q,s,m_0} \sum_{k=0}^{m_0} \|\partial^k f\|_{F^s_{pq}(\mathbb{R})} \) holds for every \( m_0 \geq 1 \) and \( U \in \{\mathbb{R}^n, \mathbb{R}^n_+\} \). Recall from Remark 16 that \( \partial^k o E^{a,b}_1 = E^{a(-b),b}_1 \partial^k \) holds for \( 0 \leq k \leq m_2 \).

When \( s - \frac{1}{p} \not\in \mathbb{Z} \), we let \( m_0 := [s - \frac{1}{p}] \). Therefore by Step A1 (since \( m_2 \geq m_0 \)),

\[
\|E^{a,b}_1 f\|_{F^s_{pq}(\mathbb{R})} \leq \sum_{j \in \mathbb{Z}} |a_j| \left( b_j \cdot (\tilde{M}_1^{s-1})^p + b_j \cdot (\tilde{M}_1^{s})^p \right) \|f\|_{F^s_{pq}(\mathbb{R}^n)}.
\]

In this case, we can take \( \tilde{M}_1^{1s} := \tilde{M}_1^{1s,m_0} + m_0 = \tilde{M}_1^{1s} + [s - \frac{1}{p}] \).

The case \( s - \frac{1}{p} \in \mathbb{Z}_+ \) follows from the \( \pm \) interpolation (18), where we can take \( \tilde{M}_1^{1s} := \max(\tilde{M}_1^{1s+1}, \tilde{M}_1^{1s-\frac{1}{p}}) \).

Step A3: The case \( n \geq 1, 0 < p < \infty, 0 < q \leq \infty \) and \( s > n \max(0, \frac{1}{p} - 1, \frac{1}{q} - 1) \), with \( m_1 \geq 0 \) and \( m_2 > s - \frac{1}{p} \).

Indeed we have \( (E^{a,b}_n f)(x) = E^{a,b}_1 (f(x', \cdot))(x_n) \). Therefore, for any extension \( \tilde{f} \in F_{pq}^s(\mathbb{R}^n) \) of \( f \in F_{pq}^s(\mathbb{R}^n_+) \),

\[
\|E^{a,b}_n f\|_{F_{pq}^s(\mathbb{R}^n)} \approx_{p,q,s} \left( \int_{\mathbb{R}^{n-1}} \|f(x', \cdot)(x_n)\|_{F_{pq}^s(\mathbb{R}^n)} \right)^{1/p} \left( \int_{\mathbb{R}^n} \|f(x', \cdot)(x_n)\|_{F_{pq}^s(\mathbb{R}^n)} \right)^{1/p} \quad \text{(by Proposition 21 (iii))}
\]

\[
\leq \|E^{a,b}_1 f\|^p_{L^p(\mathbb{R}^{n-1};F_{pq}^s(\mathbb{R}^n))} + \|E^{a,b}_1 f\|_{L^p(\mathbb{R}^{n-1};F_{pq}^s(\mathbb{R}^{n-1+1}))} \|
\]

\[
\leq \sum_{j \in \mathbb{Z}} |a_j| \left( b_j \cdot (\tilde{M}_1^{s-1})^p + b_j \cdot (\tilde{M}_1^{s})^p \right) \|f\|_{L^p(\mathbb{R}^{n-1};F_{pq}^s(\mathbb{R}^n))} + \sum_{j \in \mathbb{Z}} 2^{s/p} |a_j| \left( b_j \cdot (\tilde{M}_1^{s-1})^q + b_j \cdot (\tilde{M}_1^{s})^q \right) \|f\|_{F_{pq}^s(\mathbb{R}^n_+)} \quad \text{(by Step A2 and Remark 19)}
\]

Taking the infimum over all extensions \( \tilde{f} \) of \( f \), we get the claim with \( \tilde{M}_1^{ns} := \tilde{M}_1^{1s} + 1/p \).

Case B: We consider \( p = \infty \), that is, the \( F_{\infty q}^s \)-cases, for \( s > -1 \).

Step B1: The case \( n \geq 1, p = \infty, 0 < q \leq \infty \) and \( -1 < s < 0 \), with arbitrary \( m_1, m_2 \geq 0 \).

The proof is identical to Step A1, where we use \( E^{a,b}_n = S_n + \sum_{j=-\infty}^{\infty} a_j \cdot \partial^{-b} o S_n \) and \( S_n : F_{pq}^s(\mathbb{R}^n_+) \to F_{pq}^s(\mathbb{R}^n) \) (see [26, Theorem 2.48]). We get the result with \( \tilde{M}_1^{ns} := \tilde{M}_1^{1s} \), the constant in Proposition 22.
Step B2: The case $n \geq 1$, $p = \infty$, $0 < q \leq \infty$ and $s > -1$, with $m_1 \geq 0$ and $m_2 > s$.

It remains to prove the case $s \geq 0$. The proof is similar to Step A2, except we use (see [26, Theorem 1.24])

$\|f\|_{\mathcal{F}_{\infty,0}^{p,q}(\mathbb{R}^n)} \approx_{p,q,s,m_1} \sum_{|\gamma| \leq m_0} \|\partial^\gamma f\|_{\mathcal{F}_{\infty,q}^{p,q}(\mathbb{R}^n)}$. When $s \not\in \mathbb{Z}_+$, since $m_2 \geq [s]$ we get from Step B1 that

$$
\|E_n^{a,b} f\|_{\mathcal{F}_{pq}^p(\mathbb{R}^n)} \lesssim_{p,q,s} \sum_{j \in \mathbb{Z}} 2^{|j|/\delta} |a_j| \left( |b_j|^{\delta^{n,s-1}} + |b_j|^{s} + |\hat{M}_{n,s-1}^{n,s-1} f| \right) \|f\|_{\mathcal{F}_{\infty,q}^{p,q}(\mathbb{R}^n)}, \quad \text{whenever } f|_{\mathbb{R}^n} = f.
$$

Taking infimum of the extensions $\hat{f}$ over $f$, we get the claim with $\hat{M}_{pq}^{ns} = \hat{M}_{n,s-1}^{n,s-1} + [s]$.

The case $s \in \mathbb{Z}_+$ follows from the $\pm$ interpolation (18), where we can take $\hat{M}_{pq}^{ns} = \max(\hat{M}_{n,s-1}^{n,s-1}, \hat{M}_{n,s+1}^{n,s+1})$.

Final step: the general case: $n \geq 1$, $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, with $m_1 > n \max(0, \frac{1}{p} - 1, \frac{1}{q} - 1) - s$ and $m_2 > s - \frac{1}{p}$. Here, we can take $m_1 := [n \max(0, \frac{1}{p} - 1, \frac{1}{q} - 1) - s] + 1$.

By Remark 16 again, we have $E_n^{a,b} \partial^\gamma f = \partial^\gamma o E_n^{a,-b} \partial^\gamma f$ for all $|\gamma| \leq m_1$, where $a(-b)^{-\gamma_n} = (a_j(-b_j)^{-\gamma_n})_{j=-\infty}$. Therefore, when $f \in \mathcal{F}_{pq}^s(\mathbb{R}^n)$ and $g \in \mathcal{F}_{pq}^{s+m_1}(\mathbb{R}^n)$ satisfies $f = \sum_{|\gamma| \leq m_1} \partial^\gamma g$, by Steps A3 and B2 we have

$$
\|E_n^{a,b} f\|_{\mathcal{F}_{pq}^p(\mathbb{R}^n)} = \|\sum_{|\gamma| \leq m_1} \partial^\gamma o E_n^{a,-b} \partial^\gamma g\|_{\mathcal{F}_{pq}^p(\mathbb{R}^n)} 
$$

$$
\lesssim_{p,q,s,m_1} \sum_{|\gamma| \leq m_1} \|\partial^\gamma o E_n^{a,-b} \partial^\gamma b\|_{\mathcal{F}_{pq}^p(\mathbb{R}^n)} 
$$

$$
\lesssim_{p,q,s,m_1} \sum_{|\gamma| \leq m_1} \|E_n^{a,-b} \partial^\gamma b\|_{\mathcal{F}_{pq}^{s+m_1}(\mathbb{R}^n)} \quad \text{(see (24, (2.3.8/3)))}
$$

Taking the infimum over the decompositions $f = \sum_{|\gamma| \leq m_1} \partial^\gamma g$, by Proposition 21 (i) we obtain the claim with $\hat{M}_{pq}^{ns} := \hat{M}_{pq}^{n,s+m_1} + m_1 = \hat{M}_{pq}^{n,s+m_1} + [n \max(0, \frac{1}{p} - 1, \frac{1}{q} - 1) - s] + 1$. This completes the proof of $\mathcal{A} = \mathcal{F}$. The case $\mathcal{A} = \mathcal{B}$ is done by real interpolations, as discussed in the beginning of the proof.

We now prove Theorem 1 (iv), the boundedness on $\mathcal{B}^{\infty}_{pq}$, $\mathcal{F}^{\infty}_{pq}$, and $\mathcal{A}^{\infty}_{pq}$, using Theorem 23.

In the following, for $m \in \mathbb{Z}_+$ and $r > 0$, we define a sequence $A_{mr} = (A_{j,mr})_{j=0}^{2m}$ to be the unique sequence such that

$$
\sum_{j=0}^{2m} A_{j,mr} (-2^{-j} r)^k = 1 \text{ for integers } -m \leq k \leq m. \quad \text{Recall from Remark 13 we have}
$$

$$
A_{j,mr} = (-r 2^{-j})^m \prod_{0 \leq k \leq 2m; k \neq j} \frac{r 2^{-k} + 1}{r 2^{-k} - r 2^{-j}} = (-1)^m 2^{-j m} \prod_{0 \leq k \leq 2m; k \neq j} \frac{r 2^{-k} + 1}{r 2^{-k} - r 2^{-j}}, \quad 0 \leq j \leq 2m. \quad \text{(28)}
$$

We see that $|A_{j,mr}| \lesssim_m r^m + r^{-m}$ for $0 \leq j \leq 2m$, where the implicit constant does not depend on $r > 0$.

We define

$$
E^{mr} f(x', x_n) := E_n^{mr} f(x', x_n) := \begin{cases} 
\sum_{j=0}^{2m} A_{j,mr} f(x', -r 2^{-j} x_n) & x_n < 0 \\
f(x) & x_n > 0
\end{cases}. \quad \text{(29)}
$$

By Theorem 23, for every $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $m > \max(s - \frac{1}{p} - 1, s, n(\frac{1}{p} - 1) - s, n(\frac{1}{q} - 1) - s)$, there is a $C = C_{n,p,q,s,m} > 0$, such that $E^{mr} : \mathcal{B}^{s}_{pq}(\mathbb{R}^n) \to \mathcal{B}^{s}_{pq}(\mathbb{R}^n)$ is bounded for $\mathcal{A} \in \{\mathcal{B}, \mathcal{F}\}$ and for all $r > 0$, with

$$
\|E^{mr}\|_{\mathcal{B}^{s}_{pq}(\mathbb{R}^n) \to \mathcal{B}^{s}_{pq}(\mathbb{R}^n)} \leq C_{n,p,q,s,m} (r^{-m} \hat{M}_{pq}^{ns} + r^m \hat{M}_{pq}^{ns}), \quad \forall r > 0. \quad \text{(30)}
$$

Here, $\hat{M}_{pq}^{ns} > 0$ is the corresponding constant in Theorem 23.
Proof of Theorem 1 (iv). We only need to prove the case $\mathcal{A} \in \{\mathcal{B}, \mathcal{F}\}$. Indeed, when $\tau < \frac{1}{p}$ (hence $p < \infty$), the boundedness of $\mathcal{N}_{\tau, pq}^{\mathcal{A}}$ can be deduced from real interpolations (20). When $\tau = \frac{1}{p}$ we have coincidence $\mathcal{N}_{\tau, pq}^{\mathcal{A}} = \mathcal{R}_{\mathcal{A}}^{\mathcal{A}}$ (see Equation (7)), which is treated in Theorem 1 (iii).

We begin with the case $s < 0$.

Recall from Proposition 21 (v), \( \|f\|_{d_{pq}^s(B(0,\mathbb{R}^n))} \approx_{p,q,s} \sup_{x \in \mathbb{R}^n} R^{-nt} \|f\|_{d_{pq}^s(B(x,\mathbb{R}^n))} \) for $\mathcal{A} \in \{\mathcal{B}, \mathcal{F}\} (p < \infty$ if $\mathcal{A} = \mathcal{F})$ under the assumption $s < 0$.

Clearly if $x_n \geq R$, then $\|Ef\|_{d_{pq}^s(B(x,\mathbb{R}^n))} = \|f\|_{d_{pq}^s(B(x,\mathbb{R}^n))}$.

If $x_n \leq -R$, then $(E^{a,b}f)(B(x,\mathbb{R}^n)) = \sum_{j=-\infty}^{\infty} a_j \cdot \tilde{\vartheta}^{-b} f$. Therefore, for every extension $\tilde{f} \in d_{pq}^s(\mathbb{R}^n)$ of $f$,

\[
R^{-nt} \|E^{a,b}f\|_{d_{pq}^s(B(x,\mathbb{R}^n))} \lesssim_{p,q,\delta} R^{-nt} \sum_{j=-\infty}^{\infty} 2^{\frac{j}{b}} \|a_j\| \|\tilde{\vartheta}^{-b} f\|_{d_{pq}^s(B(x,\mathbb{R}^n))}
\]

(by Lemma 17)

\[
\leq R^{-nt} \sum_{j=-\infty}^{\infty} 2^{\frac{j}{b}} \|a_j\| (b_j^{-M_{pq}^{a,b}} + b_j^{M_{pq}^{a,b}}) \|f\|_{d_{pq}^s(B(x',\mathbb{R}^n))}
\]

(by Proposition 22)

\[
\leq \sum_{j=-\infty}^{\infty} 2^{\frac{j}{b}} \|a_j\| (b_j^{-M_{pq}^{a,b}} + b_j^{M_{pq}^{a,b}}) \max(R, \frac{b_j}{R}) \sup_{y \in \mathbb{R}^n} \|f\|_{d_{pq}^s(B(y,\max(R,b_j)))}
\]

\[
\lesssim_{p,q,s,a,b} \sup_{y \in \mathbb{R}^n, R > 0} \varrho^{-nt} \|\tilde{f}\|_{d_{pq}^s(B(y,\mathbb{R}^n))} \approx_{n,p,q,s,\tau} \|\tilde{f}\|_{d_{pq}^s(\mathbb{R}^n)}
\]

(by Proposition 21 (v))

Here, $M_{pq}^{a,b}$ is the constant in Proposition 22. Taking the infimum of the extension $\tilde{f}$ over $f$ we see that

\[
R^{-nt} \|Ef\|_{d_{pq}^s(B(0,\mathbb{R}^n))} \lesssim_{p,q,s,a,b} \|f\|_{d_{pq}^s(\mathbb{R}^n)} \quad \text{uniformly for } R > 0 \text{ and } x_n \leq -R, \quad \text{and hence for all } |x_n| \geq R.
\]

The difficult part is to prove the estimate for $|x_n| \leq R$. In this case, $B(x, R) \subset B(x', 0, 2R)$, thus $R^{-nt} \|Ef\|_{d_{pq}^s(B(x,\mathbb{R}^n))} \leq 2^nt R^{-nt} \|Ef\|_{d_{pq}^s(B(x',3R))}$. Since $E$ is translation invariant in $x'$, we can assume $x' = 0$ without loss of generality.

To summarize the discussion above, to prove the boundedness $E : d_{pq}^s(\mathbb{R}^n) \to d_{pq}^s(\mathbb{R}^n)$ for $0 < p, q \leq \infty$, $s < 0$ and $0 \leq \tau \leq \frac{1}{p}$ ($p < \infty$ if $\mathcal{A} = \mathcal{F}$), it remains to show the existence of $C = C_{n,p,q,s,\tau,a,b} > 0$ such that

\[
R^{-nt} \|Ef\|_{d_{pq}^s(B(0,\mathbb{R}^n))} \leq C \sup_{\varrho > 0} \varrho^{-nt} \|\tilde{f}\|_{d_{pq}^s(B(0,\mathbb{R}^n))} \quad \forall R > 0, \quad \forall \tilde{f} \in d_{pq}^s(\mathbb{R}^n) \text{ such that } \tilde{f}|_{\mathbb{R}^n_+} = f.
\]

In the following, we let

\[
B_+(0, R) := B(0, R) \cap \mathbb{R}^n_+; \quad J_+ := \{j \in \mathbb{Z} : b_j > 1\}; \quad J_- := \{j \in \mathbb{Z} : b_j < 1\}, \quad \text{thus } \mathbb{Z} = J_+ \cup J_-.
\]

We fix $R > 0$ temporarily.

For each $j \in J_+$, take an extension $\tilde{f}_j = \tilde{f}_j^R \in d_{pq}^s(\mathbb{R}^n)$ of $f|_{B_+(0,b_jR)}$ such that $\|\tilde{f}_j\|_{d_{pq}^s(\mathbb{R}^n)} \leq 2\|f\|_{d_{pq}^s(B_+(0,b_jR))}$. We let $\tilde{f}_- = \tilde{f}_-^R \in d_{pq}^s(\mathbb{R}^n)$ be an extension of $f|_{B_-(0,0)}$ such that $\|\tilde{f}_-\|_{d_{pq}^s(\mathbb{R}^n)} \leq 2\|f\|_{d_{pq}^s(B_-(0,0))}$.

Clearly, we have the following extension of $Ef|_{B_-(0,0)}$:

\[
\tilde{f}_< = \tilde{f}_<^R := \sum_{j \in J_+} a_j \cdot \tilde{\vartheta}^{-b} \tilde{f}_j + \sum_{j \in J_-} a_j \cdot \tilde{\vartheta}^{-b} \tilde{f}_-.
\]

However, $\tilde{f}_<^R$ is not the extension of $f|_{B_+(0,0)} = Ef|_{B_+(0,0)}$. To modify it on the domain $B_+(0, R)$, we consider

\[
\tilde{f}_> = \tilde{f}_>^R := \tilde{f}_- + \sum_{j \in J_+} \vartheta^{-1} \mathcal{E}^{\frac{1}{p}, b_j} \left[ \vartheta^{\frac{1}{p}} (\tilde{f}_j - \tilde{f}_-) \right] = \tilde{f}_- + \sum_{j \in J_+} \sum_{k=0}^{2m} A_k^{\frac{1}{p}, b_j} \left[ \vartheta^{-k} \tilde{f}_j - \vartheta^{-k} \tilde{f}_- \right]
\]

in $\mathbb{R}^n_+$,

where we have chosen $m > \max(s - \frac{1}{p}, -s, n(\frac{1}{q} - 1) - s, n(\frac{1}{q} - 1) - s)$, and $\mathcal{E}^{\frac{1}{p}, b_j}$ is given by Equation (29).
When \( j \in J_+ \) (i.e., \( b_j > 1 \)) we have \( (\tilde{f}_j - \tilde{f}_-)_B(0,R) = 0 \). Thus in this case \( E^{m, \frac{1}{b_j}}_{\partial\tilde{f}_j -} \| \mathcal{S}_{\tilde{f}_j -} \|_{B(0,R)} = 0 \) (since \( 2^{-k}/b_j < 1 \) for \( j \in J_+ \) and \( k \geq 0 \)). We conclude that \( \tilde{f}_{>0}|_{B_+(0,R)} = \tilde{f}_-|_{B_+(0,R)} = f|_{B_+(0,R)} \).

On the other hand, we can rewrite \( \tilde{f}_{<0} \) as

\[
\tilde{f}_{<0} = \sum_{j=-\infty}^{\infty} a_j \cdot \mathcal{S}_j \tilde{f}_- + \sum_{j \in J_+} a_j \cdot \mathcal{S}_j (\tilde{f}_j - \tilde{f}_-) = E^{a,b}_{\partial\tilde{f}_-} \sum_{j \in J_+} a_j \cdot \mathcal{S}_j (\tilde{f}_j - \tilde{f}_-) \quad \text{on } \mathbb{R}_+^n. (33)
\]

Here for global defined function \( \tilde{g} \), we use the notation \( E\mathcal{F} \) for \( E((\tilde{g}|_{\mathbb{R}_+^n}) \).

Therefore, the function \( F = F^R := (\tilde{f}_{>0}|_{\mathbb{R}_+^n}) \cup (\tilde{f}_{<0}|_{\mathbb{R}_-^n}) \) is an extension of \( E\mathcal{F}_{\partial B(0,R)} \) and has the expression

\[
\tilde{F} = E^{a,b}_{\partial\tilde{F}_-} + \sum_{j \in J_+} a_j \cdot \mathcal{S}_j (\tilde{f}_j - \tilde{f}_-) \in \mathcal{S}'(\mathbb{R}^n).
\]

We see that

\[
\| \tilde{F} \|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)} \leq \sum_{j \in J_+} 2^{\delta j} |a_j| \| E^{m, \frac{1}{b_j}}_{\partial\tilde{f}_j -} \| \mathcal{S}_{\tilde{f}_j -} \|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)} (\text{by Lemma 17})
\]

\[
\leq n, p, q, m C'_{\delta, a, b, \tilde{M}} \| \tilde{f}_- \|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)} + \sum_{j \in J_+} 2^{\delta j} |a_j| b_j^{m, \frac{M_{n s}}{p q}} \| \mathcal{S}_{\tilde{f}_j -} \|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)} (\text{by Theorem 23 and (30)})
\]

\[
\leq n, p, q, s, a, b, \tilde{M} \| \tilde{f}_- \|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)} + \sum_{j \in J_+} 2^{\delta j} |a_j| b_j^{m, \frac{M_{n s}}{p q}} \| \tilde{f}_j - \tilde{f}_- \|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)} (\text{by Proposition 22})
\]

\[
\leq \min(1, p, q) \| \tilde{f}_- \|_{d^s_{\partial\tilde{F}_-}(B_+(0,R))} + \sum_{j \in J_+} 2^{\delta j} |a_j| b_j^{m, \frac{M_{n s}}{p q}} \left( \| f \|_{d^s_{\partial\tilde{F}_-}(B_+(0,R))} + \| f \|_{d^s_{\partial\tilde{F}_-}(B_+(0,b_j R))} \right).
\]

Here, \( C' = C'_{\delta, a, b, \tilde{M}} = \sum_{j \in \mathbb{Z}} 2^{\delta j} |a_j| \left( b_j^{-\frac{M_{n s}}{p q}} + b_j^{-\frac{M_{n s}}{p q}} \right) \). Note that all implied constants in the above computation do not depend on the \( R > 0 \).

Therefore for general \( R > 0 \), applying Equation (3) again with a small \( \delta \), we see that for every extension \( \tilde{f} \) of \( f \),

\[
R^{-\frac{n \tau}{s t}} \| E f \|_{d^s_{\partial\tilde{F}_-}(B_+(0,R))} \leq R^{-\frac{n \tau}{s t}} \| F \|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)}
\]

\[
\leq p, q, s, m C''_{\delta, a, b, \tilde{M}} \| f \|_{d^s_{\partial\tilde{F}_-}(B_+(0,R))} + \sum_{j \in J_+} 2^{\delta j} |a_j| b_j^{m, \frac{M_{n s}}{p q}} \left( R^{-\frac{n \tau}{s t}} \| f \|_{d^s_{\partial\tilde{F}_-}(B_+(0,R))} + (b_j R)^{-\frac{n \tau}{s t}} \| f \|_{d^s_{\partial\tilde{F}_-}(B_+(0,b_j R))} \right)
\]

\[
\leq a, b, n, s, t, \tilde{M}, M, \tilde{M} \sup_{\rho \in [1,b_j : b_j > 1]} \rho^{-\frac{n \tau}{s t}} \| f \|_{d^s_{\partial\tilde{F}_-}(B(0,\rho))} \approx n, p, q, s, t, M \| f \|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)}.
\]

Here, \( C'' = C''_{n, p, q, s, t, m} := m \cdot M_{n s}^{\frac{n}{p q}} + M_{n s}^{\frac{n}{p q}} + n \tau \) and we let \( \delta > 0 \) be such that \( \sum_{j \in \mathbb{Z}} 2^{\delta j} |a_j| b_j^{C''} < \infty \). Taking the infimum over the extension \( \tilde{f} \), we obtain Equation (32) and conclude the proof of the case \( s < 0 \).

For \( s \geq 0 \), we use the following equivalent norm (see [28, Theorem 1.6]):

\[
|\tilde{f}|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)} \approx n, p, q, s, t, m \sum_{|\partial^\gamma \tilde{f}|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)}} \sum_{|\partial^\gamma \tilde{f}|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)}} \forall m \geq 1. (34)
\]

Using Lemma 15 (ii), and taking \( m \geq s + 1 \), we have

\[
\| E^{a,b} f \|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)} \leq \sum_{|\partial^\gamma \tilde{f}|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)}} \sum_{|\partial^\gamma \tilde{f}|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)}} \| E^{a,b} \tilde{f} \|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)} \leq \sum_{|\partial^\gamma \tilde{f}|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)}} \| \partial^\gamma \tilde{f} \|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)} \leq \| f \|_{d^s_{\partial\tilde{F}_-}(\mathbb{R}^n)}, (35)
\]

This completes the proof of \( E : d^s_{\partial\tilde{F}_-}(\mathbb{R}_+^n) \rightarrow d^s_{\partial\tilde{F}_-}(\mathbb{R}^n) \) for \( \partial \in \{\mathcal{B}, \mathcal{F}\} \) and all \( 0 < p, q \leq \infty, s \in \mathbb{R} \) and \( 0 \leq \tau \leq \frac{1}{p} \) (\( p < \infty \)).

The case \( \partial = \mathcal{N} \) follows from real interpolations, as mentioned in the beginning of the proof.

\[\square\]
Remark 25. By keeping track of the constants, and possibly enlarging the constant $\tilde{M} = \tilde{M}_{p,q}^{s,t}$ in Theorem 23, it is likely to prove that Equation (26) is also true if we replace $\mathcal{A}_{s}^{p,q}$ by $\mathcal{A}_{s,t}^{p,q}$ for every $0 \leq \tau \leq \frac{1}{p}$ and $\mathcal{A} \in \{\mathcal{B}, \mathcal{F}, \mathcal{N}\}$ ($p < \infty$ for $\mathcal{F}$-cases).

5 THE ANALOGY ON BOUNDED SMOOTH DOMAINS

On bounded smooth domains the analogy of Theorem 1 is the following:

**Theorem 26** (Universal Seeley’s extension on smooth domains). Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain with a smooth defining function $\sigma$, and let $\Omega' \subset \mathbb{R}^n$ be a bounded neighborhood of $\overline{\Omega}$. Then, there exists an extension operator $E : S'((0, \infty)) \rightarrow \mathcal{D}'(\mathbb{R})$ that maps extendable distributions in $\mathcal{D}'(\mathbb{R})$ to compactly supported distributions in $\mathcal{D}'(\mathbb{R})$, such that

(i) (One-parameter dependence) For every $x \in \Omega' \setminus \Omega$ and $f \in C^0(\overline{\Omega})$, the value $Ef(x)$ depends only on the value of $f$ on the (unique) integral curve for $\nabla \sigma$ passing through $x$ that contains in $\Omega$, which is, the set

$$\{ y^x(t) : t \in \mathbb{R}, y^x(t) \in \Omega \subset U', \text{ where } y^x(t) = \nabla \sigma(y^x(t)) \forall t \in \mathbb{R} \}.$$

(ii) (Universal boundedness) $E : W^{k,p}(\Omega) \rightarrow W^{k,p}(U')$ and $E : C^{k,s}(\Omega) \rightarrow C^{k,s}(U')$ are defined and bounded for all $k \in \mathbb{Z}$, $0 < p \leq \infty$ and $0 < s < 1$. In particular, $E : C^k(\Omega) \rightarrow C^k(U')$ for $k \geq 0$.

In particular, $E : \mathcal{A}_{s,t}^{p,q}(\Omega) \rightarrow \mathcal{A}_{s,t}^{p,q}(U')$ is bounded for $\mathcal{A} \in \{\mathcal{B}, \mathcal{F}, \mathcal{N}\}$, $p, q \in (0, \infty)$, $s \in \mathbb{R}$ and $\tau \in [0, \frac{1}{p}]$ ($p < \infty$ for $\mathcal{F}$-cases). In particular, $E : \mathcal{A}_{s}^{p,q}(\Omega) \rightarrow \mathcal{A}_{s}^{p,q}(\mathbb{R})$ is bounded for $\mathcal{A} \in \{\mathcal{B}, \mathcal{F}\}$, $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.

Note that (i) is false if one uses the usual patching construction, see, for example, [24, (3.3.4/6)].

In the proof, we need the standard partition of unity argument.

**Lemma 27.** Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a bounded diffeomorphism, that is, $\Phi$ is bijective and

$$\sup_{x \in \mathbb{R}^n} |\partial^\alpha \Phi(x)| + |\partial^\alpha(-1)(x)| < \infty, \quad \forall \alpha \in \mathbb{N}_0^n \setminus \{0\}.$$ (36)

Let $\chi \in C^\infty_c(\mathbb{R})$ be a bounded smooth function. Define a linear map $Tf := (\chi \cdot f) \circ \Phi$.

(i) $T : W^{k,p}(\mathbb{R}^n) \rightarrow W^{k,p}(\mathbb{R}^n)$ and $T : C^{k,s}(\mathbb{R}^n) \rightarrow C^{k,s}(\mathbb{R}^n)$ are both defined and bounded for all $k \in \mathbb{Z}$, $0 < p \leq \infty$ and $0 < s < 1$. In particular, $T : C^k(\mathbb{R}^n) \rightarrow C^k(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$.

(ii) $T : \mathcal{A}_{s,t}^{p,q}(\mathbb{R}^n) \rightarrow \mathcal{A}_{s,t}^{p,q}(\mathbb{R}^n)$ for $\mathcal{A} \in \{\mathcal{B}, \mathcal{F}, \mathcal{N}\}$, $p, q \in (0, \infty)$, $s \in \mathbb{R}$ and $\tau \in [0, \frac{1}{p}]$ ($p < \infty$ if $\mathcal{A} = \mathcal{F}$).

In particular, $T : \mathcal{A}_{s}^{p,q}(\mathbb{R}^n) \rightarrow \mathcal{A}_{s}^{p,q}(\mathbb{R}^n)$ is bounded for $\mathcal{A} \in \{\mathcal{B}, \mathcal{F}\}$, for $p, q \in (0, \infty)$ and $s \in \mathbb{R}$.

**Proof.** The result (i) follows from using chain rules, product rules, and the formula of change of variables:

$$\int_{\mathbb{R}^n} |(\chi f) \circ \Phi(x)|^p \, dx = \int_{\mathbb{R}^n} |\chi(y) f(y)|^p \, |\det(\Phi^{-1})(y)| \, dy, \quad 0 < p < \infty, \quad f \in L^p_{loc}(\mathbb{R}^n).$$

Note that $|\det(\Phi^{-1})|$ is a bounded smooth function, and all nonzero derivatives of $\Phi$ and $\chi$ are bounded. We leave the details to readers.

For (ii), see, for example, [30, Sections 6.1.1 and 6.2] and [8, Sections 4.2 and 4.3].

**Proof of Theorem 26.** In the following, we let $\mathcal{X}$ be the function class that belongs to the following:

$$\{W^{k,p}, C^{k,s} : k \in \mathbb{Z}, p \in (0, \infty), s \in (0, 1)\} \cup \{\mathcal{R}_{p,q}, \mathcal{F}_{p,q}, \mathcal{B}_{p,q}, \mathcal{A}_{p,q}^{s,t} : p, q \in (0, \infty), s \in \mathbb{R}, \tau \in [0, \frac{1}{p}]\} \cup \{C^k : k \in \mathbb{N}_0\} \cup \{\mathcal{F}_{p,q}^{s,t} : p \in (0, \infty), q \in (0, \infty), s \in \mathbb{R}, \tau \in [0, \frac{1}{p}]\}.$$
Recall from Theorem 1 that $E_{a,b} : \mathcal{X}(\mathbb{R}^n_+) \to \mathcal{X}(\mathbb{R}^n)$ is bounded for all such $\mathcal{X}$.

For a function $g(\theta, t)$ defined on $\partial \Omega \times \mathbb{R}_+$, we define its extension $E_0g$ on $\partial \Omega \times \mathbb{R}$ by

$$E_0g(\theta, t) := \begin{cases} \sum_{j=-\infty}^{\infty} a_j g(\theta, -b_j t) & t < 0; \\ g(\theta, t) & t > 0, \end{cases}$$
where $(a_j, b_j)^{\infty}_{j=-\infty}$ satisfies Equation (3). \hspace{1cm} (37)

Recall from Proposition 14 that such $(a_j, b_j)$ exist.

Since $\nabla \sigma$ is nonzero in a neighborhood of $\partial \Omega$, we see that there is a small $\varepsilon_0 > 0$, such that the ordinary differential equation (ODE) flow map $\exp_{\nabla \sigma} : \partial \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ given by

$$\exp_{\nabla \sigma}(\theta, 0) = \theta, \quad \frac{\partial}{\partial t} \exp_{\nabla \sigma}(\theta, t) = \nabla \sigma(\Psi(\theta, t)), \quad \text{for} \ \theta \in \partial \Omega, \ -\varepsilon_0 < t < \varepsilon_0,$$

is defined and is diffeomorphic onto its image. Shrinking $\varepsilon_0$ if necessary we can assume that $\exp_{\nabla \sigma}(\partial \Omega \times (-\varepsilon_0, \varepsilon_0)) \subseteq U'$.

Since $\frac{\partial}{\partial t} \exp_{\nabla \sigma}(\theta, t) = \nabla \sigma(\Psi(\theta, t))$ and $\sigma(\exp_{\nabla \sigma}(\theta, t)) = 0$ we see that $\sigma(\exp_{\nabla \sigma}(\theta, t)) = t$ hence $\{x : -\varepsilon_0 < \sigma(x) < \varepsilon_0\} \subseteq U'$.

We define $\Psi : \{-\varepsilon_0 < \sigma < \varepsilon_0\} \to \partial \Omega \times (-\varepsilon_0, \varepsilon_0)$ by $\Psi := (\exp_{\nabla \sigma})^{-1}$. Since $\sigma(\exp_{\nabla \sigma}(\theta, t)) = t$, we see that

$$\Psi(x) = (\psi'(x), \sigma(x)) \quad \text{for some smooth map} \ \psi' : \{-\varepsilon_0 < \sigma < \varepsilon_0\} \to \partial \Omega. \hspace{1cm} (38)$$

We take the following cutoff functions:

$$\chi_0, \chi_1 \in C^\infty_c(-\frac{2}{3}\varepsilon_0, \frac{2}{3}\varepsilon_0) \text{ such that } \chi_0|_{[-\frac{\varepsilon_0}{2}, \frac{\varepsilon_0}{2}]} \equiv 1, \ \chi_1|_{\text{supp } \chi_0} \equiv 1; \ \text{and } \tilde{\chi}_0 := \chi_0 \circ \sigma, \ \tilde{\chi}_1 := \chi_1 \circ \sigma.$$  

We have $\tilde{\chi}_0, \tilde{\chi}_1 \in C^\infty_c(U')$. We define $\mathcal{E}$ as

$$\mathcal{E}f := (1 - \tilde{\chi}_0)|_{\Omega} \cdot f + \tilde{\chi}_1 \cdot (\mathcal{E}[\tilde{\chi}_0 \cdot f \circ \Psi^{-1}] \circ \Psi). \hspace{1cm} (39)$$

Clearly, $\mathcal{E}f|_{\Omega} = (1 - \tilde{\chi}_0) \cdot f + \tilde{\chi}_1 \cdot \tilde{\chi}_0 \cdot f = f$. Therefore, $\mathcal{E}$ is an extension operator.

From Equation (38), we see that the pullback vector field $\Psi' \frac{\partial}{\partial t}$ is exactly $\nabla \sigma$ in the domain. Since for each $(\theta, t)$, the value $E_0g(\theta, t)$ depends only on the line $\{g(\theta, s) : s \in \mathbb{R}\}$, taking pullback we get (i).

We postpone the proof of $\mathcal{E} : \mathcal{S}'(\Omega) \to \mathcal{E}'(U')$ to the end. It is more convenient to prove (ii) by passing to local coordinate charts.

We consider the following objects:

- Take a coordinate cover $\{\phi^\prime_\lambda : V_\lambda \subseteq \partial \Omega \to \mathbb{R}^{n-1}\}_{\lambda=1}^N$ of $\partial \Omega$, where $V_\lambda$ are open in the submanifold $\partial \Omega$.
- For $1 \leq \lambda \leq N$, define $\Phi_\lambda : V_\lambda \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^{n-1} \times \mathbb{R}^1$ by $\Phi_\lambda(\theta, t) := (\phi^\prime_\lambda(\theta), t)$.
- Take a partition of unity $\{\mu_\lambda \in C^\infty_c(V_\lambda)\}_{\lambda=1}^N$ of $\partial \Omega$, that is, $\sum_{\lambda=1}^N \mu_\lambda \equiv 1$ on $\partial \Omega$.
- Let $\{\rho_\lambda \in C^\infty_c(V_\lambda)\}_{\lambda=1}^N$ be such that $\rho_\lambda|_{\text{supp } \mu_\lambda} \equiv 1$.
- For $1 \leq \lambda \leq N$, we let $\tilde{\rho}_\lambda := (\rho_\lambda \otimes \chi_0) \circ \Psi = (\rho_\lambda \circ \phi^\prime_\lambda) \cdot (\chi_0 \circ \sigma)$ and $\tilde{\rho}_\lambda(x) := (\rho_\lambda \otimes \chi_1)(x) \circ \Phi_\lambda^{-1}(x) = (\rho_\lambda \circ \phi^\prime_\lambda^{-1})(x') \cdot \chi_1(x_n)$.

See the following picture:
Now, \( \Phi_\lambda \circ \Psi : \Psi^{-1}(V_\lambda \times (-\varepsilon_0, \varepsilon_0)) \rightarrow \mathbb{R}^n \) is a coordinate cover of \( \{-\varepsilon_0 < \sigma < \varepsilon_0\} \). By shrinking each \( V_\lambda \) if necessary, we can ensure the following:

- For every \( 1 \leq \lambda \leq N \), \( \Phi_\lambda \circ \Psi : \Psi^{-1}(V_\lambda \times (-\frac{2}{3} \varepsilon_0, \frac{2}{3} \varepsilon_0)) \subseteq U^* \rightarrow \mathbb{R}^n \) can extend to a bounded diffeomorphism. In other words, there are \( \Xi_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n \) that satisfies Equation (36) and \( \Xi_\lambda|_{\Psi^{-1}(V_\lambda \times (-\frac{2}{3} \varepsilon_0, \frac{2}{3} \varepsilon_0))} = \Phi_\lambda \circ \Psi \), for each \( \lambda \).

From the construction of \( \Psi, \Phi_\lambda, \mu_\lambda \), and \( \rho_\lambda \) we see that

- \( \tilde{\mu}_\lambda \in C^\infty_{\text{loc}}(\Psi^{-1}(V_\lambda \times (-\varepsilon_0, \varepsilon_0))) \) and \( \tilde{\rho}_\lambda \in C^\infty_{\text{loc}}(\Phi_\lambda(V_\lambda \times (-\varepsilon_0, \varepsilon_0))) \).
- When \( \text{supp}\ g \subseteq \text{supp}\ \mu_\lambda \times [0, \infty) \), we have \( \text{supp}\ \tilde{\mu}_\lambda \subseteq (\text{supp}\ \rho_\lambda) \times \mathbb{R} \).
- When \( \text{supp}\ g \subseteq V_\lambda \times [0, 1) \), we have \( \text{supp}\ \tilde{\rho}_\lambda \subseteq (\text{supp}\ \mu_\lambda) \times \mathbb{R} \).

Therefore,

\[
\tilde{x}_1 \cdot (E_0((\tilde{x}_0 \cdot f) \circ \Psi^{-1}) \circ \Psi) = \sum_{\lambda=1}^{N} ((\mu_\lambda \cdot \tilde{x}_1) \cdot E_0((\tilde{\mu}_\lambda \cdot f) \circ \Psi^{-1})) \circ \Psi
\]

\[
= \sum_{\lambda=1}^{N} ((\rho_\lambda \cdot \tilde{x}_1) \cdot E((\tilde{\mu}_\lambda \cdot f) \circ \Psi^{-1} \circ \Phi^{-1}) \circ \Phi) \circ \Psi = \sum_{\lambda=1}^{N} (\tilde{x}_1 \cdot E((\rho_\lambda \cdot f) \circ \Phi^{-1})) \circ (\Phi \circ \Psi).
\] (40)

Applying Lemma 27 and Theorem 1(ii)–(iv), we have the boundedness of compositions

\[
\mathcal{X}(\Omega) \xrightarrow{f \mapsto (\tilde{\mu}_\lambda \cdot f) \circ \Phi^{-1}} \mathcal{X}(\mathbb{R}^n) \xrightarrow{E} \mathcal{X}(\mathbb{R}^n) \xrightarrow{g \mapsto \sum_{\lambda=1}^{N} (\tilde{\rho}_\lambda \cdot f) \circ \Phi^{-1}} \mathcal{X}_c(U_0) \subset \mathcal{X}_c(U).
\]

On the other hand, \( \tilde{x}_0 = x_0 \circ \rho \in C^\infty_{\text{loc}}(U_0) \) is identical to 1 near \( \partial \Omega \), therefore \( (1 - \tilde{x}_0)|_\Omega \in C^\infty_{\text{loc}}(\Omega) \). By Lemma 27, \( [f \mapsto (1 - \tilde{x}_0)|_\Omega \cdot f] : \mathcal{X}(\Omega) \rightarrow \mathcal{X}_c(U) \) is bounded. Altogether we conclude that \( \mathcal{E} : \mathcal{X}(\Omega) \rightarrow \mathcal{X}(U) \) is bounded. This completes the proof of (ii).

Finally, we illustrate the definedness of \( \mathcal{E} : \mathcal{S}'(\Omega) \rightarrow \mathcal{E}'(U) \). Note that alternatively one can obtain this statement by using duality and prove the boundedness \( \mathcal{E}^*: C^\infty_{\text{loc}}(U) \rightarrow \mathcal{S}'(\Omega) \) as \( [g \mapsto \sum_{\lambda=1}^{N} (\tilde{\rho}_\lambda \cdot f) \circ \Phi^{-1}] \).

For an \( f \in \mathcal{S}'(\Omega) \), let \( \tilde{f} \) be an extension of \( f \). Since \( \Omega \) is a bounded domain, we can assume \( \tilde{f} \) to have compact support, otherwise we replace \( \tilde{f} \) with \( \kappa \tilde{f} \), where \( \kappa \in C^\infty_{\text{loc}}(\mathbb{R}^n) \) is such that \( \kappa|_\Omega \equiv 1 \).

By the structure theorem of distributions (see [15, Theorem 6.27], for example), there are \( M \geq 0 \) and \( \{g_\alpha\}_{|\alpha| \leq M} \subset C^0_c(\mathbb{R}^n) \) such that \( \tilde{f} = \sum_{|\alpha| \leq M} \partial^\alpha g_\alpha \). By Proposition 21 (i) this is saying that \( \tilde{f} \in B^{-M}_\infty(\mathbb{R}^n) \) for some \( M > 0 \), which means \( f \in B^{-M}_{\infty}(\Omega) \) for the same \( M \).

Now by (ii) \( \mathcal{E}f \in B^{-M}_{\infty}(U) \subset \mathcal{B}'(U) \) has compact support, we conclude that \( \mathcal{E}f \in \mathcal{E}'(U) \). \( \square \)

ORCID

Liding Yao ▼ https://orcid.org/0000-0002-8747-7359

ENDNOTES

1. \( C^\infty_c(\overline{U}) \subset C^\infty(\overline{U}) \) is the set of all bounded smooth functions \( f : U \rightarrow \mathbb{R} \) such that \( f|_{\{x : |x| > R\}} \equiv 0 \) for some \( R \) (depending on \( f \)).

2. A defining function of \( \Omega \) is a real function (at least \( C^1 \)) \( \sigma : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( \Omega = \{x : \sigma(x) < 0\} \) and \( \nabla \sigma|_{\partial \Omega} \neq 0 \).

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