The generalized chiral Schwinger model on the two-sphere

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Abstract

A family of theories which interpolate between vector and chiral Schwinger models is studied on the two-sphere $S^2$. The conflict between the loss of gauge invariance and global geometrical properties is solved by introducing a fixed background connection. In this way the generalized Dirac–Weyl operator can be globally defined on $S^2$. The generating functional of the Green functions is obtained by taking carefully into account the contribution of gauge fields with non–trivial topological charge and of the related zero–modes of the Dirac determinant. In the decompactification limit, the Green functions of the flat case are recovered; in particular the fermionic condensate in the vacuum vanishes, at variance with its behaviour in the vector Schwinger model.
1 Introduction

Quantum field theories in 1–space, 1–time dimensions are intensively studied in recent years owing to their peculiarity of being exactly solvable both by functional and by operatorial techniques. From a practical point of view they find interesting applications in string models, while behaving as useful theoretical laboratories in which many features, present also in higher dimensional theories, can be directly tested. In addition 2–dimensional models possess a quite peculiar infrared structure on their own.

Historically the first 2–dimensional model was proposed by Thirring [1], describing a pure fermionic current–current interaction. The interest suddenly increased 4 years later, when Schwinger [2] was able to obtain an exact solution for 2–dimensional electrodynamics with massless spinors.

Chiral generalizations of this model were studied by Hagen [3] and, more recently, by Jackiw and Rajaraman [4]. The last authors draw very important conclusions concerning theories with “anomalies”, i.e. the occurrence of symmetry breakings by quantum effects [5], [6], [7], [8]. They were able to show that, taking advantage of the arbitrariness in the (non perturbative) regularization of the fermionic determinant, it was possible to recover a unitary theory even in the presence of a gauge anomaly (gauge non–invariant formulation of an anomalous gauge theory).

In ref.[9] a family of theories which interpolate between vector and chiral Schwinger models according to a parameter $r$, which tunes the ratio of the axial to vector coupling, has been studied. We call it generalized chiral Schwinger model. The treatment depends on two parameters: $r$ and $a$, $a$ being the constant involved in the regularization of the fermionic determinant.

In the Minkowski space, using a non–perturbative approach, we first obtained, by means of a functional formalism, the correlation functions for bosons, fermions and fermionic composite operators. We found two allowed ranges for the parameters $r$ and $a$. The first range was also partially studied in a similar context in [10], [11]. In this range the bosonic sector consists of two “physical” quanta, a free massive and a free massless excitation. The fermionic sector is much more interesting: both left and right spinors exhibit a propagator decreasing at very large distances, indicating the presence of asymptotic states which however feel the long range interaction mediated by the massless boson. The asymptotic fermions are described by a massless Thirring model.

The solution interpolates between two conformal invariant theories at small and large distances, respectively, with different critical exponents. The c–theorem [12] is explicitly verified, confirming that $\Delta c = 1$, as one could expect on the basis of the structure of the bosonized theory.

The second range is characterized in the bosonic sector by a “physical” massive excitation and by a massless negative norm state (“ghost”). Both quanta are free: one can define a stable Hilbert space of states in which the “ghost” does not appear. However no asymptotic states for fermions are available in this case; their correlation function increases with distance, giving rise to a confinement phenomenon.

All those features were confirmed and further elucidated by a treatment based on operators which are canonically quantized according to a Dirac bracket formalism [13].

A perturbative approach to this generalized chiral Schwinger model has also been proposed in ref.[14].

First the perturbative expansion for the boson propagator is resummed, starting from the Feynman diagrams: in order to develop the Feynman rules we had to introduce a gauge fixing.

In the non–perturbative context, where gauge invariance is naturally broken by the anomaly, this amounts to studying different theories for different gauge fixings. Therefore
the limit of vanishing gauge fixing was performed after resummation. A lot of interesting features was hidden in this limit: studying the bosonic spectrum, the decoupling of ghost particles from the theory was followed, to recover the previous non-perturbative results.

The fermionic correlation functions were also examined, leading to the correct Thirring behaviour in the non-perturbative limit; nevertheless we found very different ultraviolet scaling laws before and after the gauge-fixing removal, related to the appearance of an ultraviolet renormalization constant. Decoupling of heavy states is indeed not trivial when anomalies are present [15].

In this paper we study the euclidean version of this generalized chiral Schwinger model on the two-sphere $S^2$: the problem is of interest by itself because there is a conflict between the loss of gauge invariance and the globality properties of the model.

It is well known that gauge anomalies in presence of non-trivial fiber-bundle depend on some “fixed” background connection [16]: the global meaning of the cohomological solution requires the presence of these connections. From the functional point of view we will show that the determinant of the generalized Dirac–Weyl operator is globally defined on $S^2$ only after the introduction of a classical external field.

We discuss its physical meaning and we obtain the generating functional of the Green functions: the contribution of gauge fields with non-trivial topological charge and of the related zero-modes of the Dirac determinant is carefully taken into account. As an application we derive the fermionic propagator and the condensate in the vacuum: we find that, in the decompactification limit, the latter vanishes, at variance with the Schwinger model case, confirming the conjecture [17] about the triviality of the vacuum of the chiral Schwinger model.

In sect.2 we recall the basic results of [9], following the path-integral approach, and establish our notations.

Sect.3 deals with the geometrical background necessary to properly define and treat the Dirac-Weyl determinant, which is studied in sect.4. The chiral gauge theory on the sphere is defined in sect.5 and its generating functional is constructed in sect.6. The fermionic two-point function and the vacuum expectation value for the scalar density are computed and discussed in sect.7, and the decompactification limit is performed. Conclusions are drawn in sect.8.
2 The non–perturbative solution of the generalized chiral Schwinger model

The model, characterized by the classical lagrangian

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \gamma^\mu \left[ i \partial_\mu + e \left( \frac{1 + r\gamma^5}{2} \right) A_\mu \right] \psi,
\]

is quantized following the path–integral method. In eq.(1) \( F_{\mu\nu} \) is the usual field tensor, \( A_\mu \) the vector potential and \( \psi \) a massless spinor. The quantity \( r \) is a real parameter interpolating between the vector (\( r = 0 \)) and the chiral (\( r = \pm 1 \)) Schwinger models. Our notations are

\[
\begin{align*}
g_{00} = -g_{11} &= 1, & \epsilon_{01} = -\epsilon_{01} &= 1, \\
\gamma^0 &= \sigma_1, & \gamma^1 &= -i \sigma_2, \\
\gamma^5 &= \sigma_3, & \tilde{\partial}_\mu &= \epsilon_{\mu\nu} \partial^\nu,
\end{align*}
\]

\( \sigma_i \) being the usual Pauli matrices.

The classical lagrangian eq.(1) is invariant under the local transformations

\[
\psi'(x) = \exp \left[ ie \left( \frac{1 + r\gamma^5}{2} \right) \Lambda(x) \right] \psi(x)
\]

\[
A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda.
\]

However, as is well known, it is impossible to make the fermionic functional measure simultaneously invariant under vector and axial vector gauge transformations; as a consequence, for \( r \neq 0 \) the quantum theory will exhibit anomalies.

The Green function generating functional is

\[
\mathcal{Z}[J_\mu, \bar{\eta}, \eta] = \mathcal{Z}_0^{-1} \int \mathcal{D}(A_\mu, \bar{\psi}, \psi) \exp \left[ i \int d^2 x (\mathcal{L} + \mathcal{L}_s) \right],
\]

and

\[
\mathcal{L}_s = J_\mu A^\mu + \bar{\eta} \psi + \bar{\psi} \eta
\]

\( J_\mu, \eta \) and \( \bar{\eta} \) being vector and spinor sources respectively.

The integration over the fermionic degrees of freedom can be performed, leading to the expression

\[
\mathcal{Z}[J_\mu, \eta, \bar{\eta}] = \mathcal{Z}_0^{-1} \int \mathcal{D}(A_\mu, \phi) \exp \left[ i \int d^2 x \mathcal{L}_{eff}(A_\mu, \phi) + d^2 x J_\mu A^\mu \right]
\]

\[
\exp \left[ - i \int d^2 x \int d^2 y \bar{\eta}(x) S(x, y; A_\mu) \eta(y) \right]
\]

where

\[
\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a e^2}{8 \pi} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{e}{2 \sqrt{\pi}} A^\mu (\tilde{\partial}_\mu - r \partial_\mu) \phi.
\]

\( \phi \) being a scalar field we have introduced in order to have a local \( \mathcal{L}_{eff} \) and \( a \) the subtraction parameter reflecting the well–known regularization ambiguity of the fermionic determinant \([4]\).
The quantity $S(x, y; A_\mu)$ in eq.(3) is the fermionic propagator in the presence of the potential $A_\mu$, which will be computed later on by using standard decoupling techniques.

For the moment we let the sources $\eta$ and $\bar{\eta}$ vanish and consider the bosonic sector of the model for different values of the parameters $r$ and $a$. In this sector the effective lagrangian is quadratic in the fields; this means an essentially free (although non local) theory. First functionally integrating over $\phi$ and then over $A_\mu$, we easily obtain

$$Z[J_\mu, 0, 0] = \exp\left[-\frac{1}{2} \int d^2 x J^\mu(K^{-1})_{\mu\nu} J^\nu\right],$$

where

$$K_{\mu\nu} = g_{\mu\nu}(\Box + \frac{e^2}{4\pi}(1 + a) - (1 + \frac{e^2}{4\pi} \frac{1 + r^2}{\Box}) \partial_\mu \partial_\nu + \frac{e^2}{4\pi} \frac{r}{\Box} (\bar{\partial}_\mu \partial_\nu + \bar{\partial}_\nu \partial_\mu)$$

and, consequently,

$$(K^{-1})_{\mu\nu} \equiv D_{\mu\nu} = \frac{1}{\Box + m^2} [g_{\mu\nu} + \frac{\Box + \frac{e^2}{4\pi}(1 + r^2)}{a - r^2} \partial_\mu \partial_\nu + \frac{r}{a - r^2} \Box (\bar{\partial}_\mu \partial_\nu + \bar{\partial}_\nu \partial_\mu)].$$

We have introduced the quantity

$$m^2 = \frac{e^2}{4\pi} \frac{a(1 + a - r^2)}{a - r^2},$$

which is to be interpreted as a dynamically generated mass in the theory; $D_{\mu\nu}$ has a pole there $\sim (k^2 - m^2 + i\epsilon)^{-1}$, with causal prescription, as usual. We note that $D_{\mu\nu}$ exhibits also a pole at $k^2 = 0$.

Eqs. (11) and (12) generalize the well-known results of the vector and chiral Schwinger models. As a matter of fact, setting first $r = 0$ and then $a = 0$ we recover for $m^2$ the value $\frac{e^2}{4\pi}$ of the (gauge invariant version of the) vector Schwinger model. The kinetic term $K_{\mu\nu}$ becomes a projection operator

$$K_{\mu\nu}(a = 0, r = 0) = (\Box + m^2)(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}),$$

which can only be inverted after imposing a gauge fixing. In other words the limit $r = 0$, $a = 0$ in eq.(11) is singular, as it should, as gauge invariance is indeed recovered.

When $r = \pm 1$, we obtain the two equivalent formulations of the chiral Schwinger model; eq.(10) becomes

$$m^2 = \frac{e^2}{4\pi} \frac{a^2}{a - 1},$$

To avoid tachyons, we must require $a > 1$. Gauge invariance is definitely lost, and eq.(10) becomes

$$D_{\mu\nu} = \frac{1}{\Box + m^2} [g_{\mu\nu} + \frac{1}{a - 1} (\frac{\pi}{e^2} + \frac{2}{\Box}) \partial_\mu \partial_\nu \mp \frac{1}{a - 1} \bar{\partial}_\mu \partial_\nu \pm \bar{\partial}_\nu \partial_\mu].$$
Going back to the general expression eq.(11) we remark that the condition \( m^2 > 0 \), which is necessary to avoid the presence of tachyons in the theory, allows two ranges:

\[
\begin{align*}
1) & \quad a > r^2, \\
2) & \quad 0 < a < r^2 - 1 \quad \text{or} \quad r^2 - 1 < a < 0,
\end{align*}
\]

for the parameters \((a, r)\). Only the first range has been considered so far in the literature, to our knowledge.

By taking in eq.(10) the residue at the pole \( k^2 = m^2 \), one gets

\[
\text{Res} \, D_{\mu\nu} |_{k^2=m^2} = \frac{1}{m^2} T_{\mu\nu}(k),
\]

\( T_{\mu\nu} \) being a positive semidefinite degenerate quadratic form in \( k_\mu \), involving the parameters \((a, r)\). One eigenvalue vanishes, corresponding to a decoupling of the would-be related excitation, the other is positive and can be interpreted in both ranges as the presence of a vector particle with a rest mass given by the positive square root of eq.(11) and positive residue at the pole in agreement with the unitary condition. This state decouples in the limit \( a = r^2 \). There is also a massless degree of freedom with

\[
\text{Res} \, D_{\mu\nu} |_{k^2=0} = \frac{\pi}{e^2 a (1 + a - r^2)} [(1 + r^2) k_\mu k_\nu - r (\bar{k}_\mu k_\nu + \bar{k}_\nu k_\mu)] |_{k^2=0}.
\]

One can easily realize that again the quadratic form at the numerator is positive semidefinite for any value of \( r \). The poles at \( k^2 = m^2 \) and \( k^2 = 0 \) exhaust the singularities of \( D_{\mu\nu} \).

Let us consider the situation in the two ranges of parameters. The first range does not deserve particular comments at this stage. No ghost is present at \( k^2 = 0 \), as one eigenvalue of the residue matrix vanishes and the other is positive, corresponding to a “physical” excitation. The second range does entail no news concerning the state with mass \( m \). The situation is different however when considering the pole at \( k^2 = 0 \). We have indeed a negative residue in this case corresponding to a “ghost” excitation (particle with a negative probability). The theory can be accepted only if this excitation can be consistently excluded from a positive norm Hilbert space of states, which is stable under time evolution. This can be done, as shown in ref.[9] by means of a canonical approach.

The bosonic world is rather dull, consisting only of free excitations; therefore it is worth considering at this stage the fermionic sector.

We go back to the general expression eq.(6) in which fermionic sources are on. We have now to consider the fermionic propagator in the field \( A_\mu \), which obeys the equation

\[
[i \partial / + e (1 - r \gamma^5) / 2 A] S(x; y; A_\mu) = \delta^2(x - y),
\]

with causal boundary conditions. Let us also introduce the free propagator \( S_0 \)

\[
i \partial S_0(x) = \delta^2(x)
\]

with the solution

\[
S_0(x) = \int \frac{d^2k}{(2\pi)^2} \frac{k}{ik} e^{-ikx} = \frac{1}{2\pi} \frac{\gamma_\mu x^\mu}{x^2 - i\epsilon}.
\]
If we remember that any vector fields in two dimensional Minkowski space can be written as a sum of a gradient and a curl part

$$A_\mu = \partial_\mu \alpha + \tilde{\partial}_\mu \beta,$$

the following change of variables in eq.(1)

$$\psi = \exp[\frac{ie}{2} (\alpha + \gamma^5 \beta + r \beta + r \alpha \gamma^5)] \chi$$

realizes the decoupling of the fermions, leading to the expression for the “left” propagator:

$$S^L(x - y) \equiv \int \mathcal{D}(A_\mu, \phi) S^L(x, y; A_\mu) \exp[i \int d^2z \mathcal{L}_{eff}(A_\mu, \phi)] =$$

$$= S^L_0(x - y) Z_L \exp\{-\frac{1}{4} \frac{(1 - r^2)^2}{a(a + 1 - r^2)} \ln[\tilde{m}^2(-(x - y)^2 + i\epsilon)] -$$

$$-i\pi \frac{a + 1 - r^2}{a(a - r^2)}(r - \frac{a}{a + 1 - r^2})^2 D(x - y, m)\},$$

(23)

where $\tilde{m} = m\gamma^5$, $D$ is the scalar Feynman propagator: $D \equiv D_0$, with

$$D_{1-\omega}(x, m) = -(\lambda^2)^{1-\omega} \int \frac{d^2k}{(2\pi)^2} \frac{e^{-ikx}}{k^2 - m^2 + i\epsilon} =$$

$$= \frac{2i}{(4\pi)^\omega} \left(\frac{\lambda^2 \sqrt{-x^2}}{2m}\right)^{1-\omega} K_{1-\omega}(m \sqrt{-x^2 + i\epsilon}),$$

(24)

$\gamma$ being the Euler–Mascheroni constant. For further developments it is useful to consider $2\omega$ dimensions and to introduce a mass parameter $\lambda$ to balance dimensions. $Z_L$ is a (dimensionally regularized) ultraviolet renormalization constant for the fermion wave function

$$Z_L = \exp[i \pi (r - 1)^2 \frac{a}{a - r^2} D_{1-\omega}(0, m)].$$

(25)

The “right” propagator can be obtained from eq.(23) simply by replacing $S^L_0$ with $S^R_0$ and changing the sign of the parameter $r$.

The Fourier transform in the momentum space of eq.(24) cannot be obtained in closed form; however it exhibits the singularities related to the thresholds at $p^2 = 0$ and $p^2 = (nm)^2$, $n = 1, 2, 3, \ldots$.

Now we show how to derive the left propagator eq.(23) in the path–integral formalism; all the other Green functions can be obtained in the same way. The first step is to integrate the fermions in eq.(4) to give eq.(6) (we put $J_\mu = 0$). The change of variables eq.(22) decouples the spinors from $A_\mu$ but has a non trivial Jacobian $\mathcal{J}[A_\mu]$

$$\mathcal{J}[A_\mu] = \exp \int d^2x \frac{e^2}{8\pi} A_\mu \left[(1 + a) g^{\mu\nu} - (1 + r^2) \frac{\partial^\mu \partial^\nu}{\Box} - 2r e^{a\mu} \partial_\alpha \partial^\nu \right] A_\nu.$$

(26)

This result can be obtained, in euclidean space, using $\zeta$–function technique for functional determinants.
\[
\det[D] = \exp[-\frac{d}{ds}\zeta_D(s)]_{s=0},
\]
\[
\zeta_D(s) = \int d^2x \text{Tr}[K_s(D;x,x)],
\]
where \(K_s(D;x,x)\) is the kernel of \(D^{-s}\), \(D\) being a pseudoelliptic operator [19].

The fermionic action is now
\[
\int d^2x \left[ i\bar{\chi}\d\chi + \bar{\eta} \exp\left(\frac{ie}{2}[\alpha + \gamma^5\beta + r\beta + r\alpha\gamma^5]\right)\chi + \right.
\]
\[
+ \left. \bar{\chi} \exp\left(\frac{ie}{2}[-\alpha + \gamma^5\beta - r\beta + r\alpha\gamma^5]\right)\eta \right],
\]
where \(\chi\) is a free fermion and \(\alpha, \beta\) are linked by eq.(21) to \(A_{\mu}\). The diagonalization of eq.(29) gives the propagator \(S(x,y;A_{\mu})\):
\[
S(x,y;A_{\mu}) = S^L_0(x-y) \exp\left[i\int d^2z \xi^L_\mu(z;x,y)A_{\mu}(z)\right] +
\]
\[
+ S^R_0(x-y) \exp\left[i\int d^2z \xi^R_\mu(z;x,y)A_{\mu}(z)\right],
\]
\[
\xi^{L,R}_\mu(z;x,y) = \frac{e}{2}(r \pm 1)(\partial^2_{\mu} \pm \tilde{\partial}^2_{\mu})[D(z-x) - D(z-y)],
\]
where \(D(x)\) is the free massless scalar propagator in \(d = 1 + 1\) and \(S^L_0, S^R_0\) the free left and right fermion propagators.

To obtain the left propagator eq.(23) we derive with respect to \(\bar{\eta}_L\) and \(\eta_L\) (the left component of the sources eq.(1)) and get
\[
S^L(x,y) = S^L_0(x-y) \int \mathcal{D}A_{\mu} \mathcal{J}[A_{\mu}] \exp i \int d^2z \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}(z) + \xi^L_\mu(z;x,y)A_{\mu}(z)\right].
\]

Using the explicit form of \(\mathcal{J}[A_{\mu}]\) (eq.(28)), we can write the path–integral over \(A_{\mu}\) as
\[
\int \mathcal{D}A_{\mu} \exp(i \int d^2z \left[\xi^L_\mu A_{\mu} + \frac{1}{2}A_{\mu}K^{\mu\nu}A_{\nu}\right]),
\]
\(K^{\mu\nu}\) being defined in eq.(3). The Gaussian integration is trivial and gives
\[
S_L(x,y) = S^L_0(x-y) \exp(-\frac{1}{2} \int d^2zd^2w \xi^L_\mu(z;x,y)\{K^{-1}\}^{\mu\nu}(z,w)\xi^L_\nu(w;x,y)).
\]

The explicit computation of the exponential factor gives the renormalization constant \(Z_L\) and the interaction contribution in eq.(23).

We can now begin the study of the fermionic propagator. First of all, we notice that for \(r = 1\) the “left” fermion is free. The same happens to the “right” fermion when \(r = -1\). Moreover we notice from eq.(23) that the long range interaction completely decouples for \(r^2 = 1\). As a consequence the interacting fermion (for instance the “right” one for \(r = 1\)) asymptotically behaves like a free particle.
In general, at small values of $x^2$, the propagator $S^L$ has the following behaviour

$$S^L \sim_{x^2 \to 0} C_0 x^+ (-x^2 + i\epsilon)^{-1-A}$$

(34)

with

$$A = \frac{1}{4} \frac{(1 - r)^2}{a - r^2}$$

(35)

and $C_0$ a suitable constant.

We remark that the ultraviolet behaviour of the left fermion propagator can be directly obtained from the ultraviolet renormalization constant

$$\gamma_{\psi_L} = \lim_{\omega \to 1} \frac{1}{2} \left( \lambda \frac{\partial}{\partial \lambda} \ln Z_L \right) = -\frac{(1 - r)^2}{4(a - r^2)}$$

(36)

and, of course, it coincides with the one of the explicit solution eq.(23). It can be obtained from the renormalization group equation in the ultraviolet limit in which the mass dependent term is disregarded.

For large values of $x^2$ we get instead

$$S^L \sim_{x^2 \to -\infty} C_\infty x^+ (-x^2 + i\epsilon)^{-1-B}$$

(37)

where

$$B = \frac{1}{4} \frac{(1 - r)^2}{a(a + 1 - r^2)}$$

(38)

and $C_\infty$ another constant.

We have shown in [9] that eq.(37) exactly coincides with the behavior of the fermionic propagator of the massless Thirring model in the spin-$\frac{1}{2}$ representation, with a coupling constant:

$$g = \frac{1 - r^2}{a}.$$  

(39)

We remark that in the first region of the parameter space, where unitarity holds without subsidiary condition, it happens that

$$g > -1,$$  

(40)

as required to have a consistent solution for the model [20].

In the first range ($a > r^2$), both $A$ and $B$ are positive. The propagator decreases at infinity indicating the possible existence of asymptotic states for fermions, which however feel the long range interaction mediated by the massless excitation which is present in the bosonic spectrum. The situation in the second range is much more intriguing. Here both $A$ and $B$ are negative. Moreover

$$1 + B = \frac{(2a + 1 - r^2)^2}{4a(a + 1 - r^2)} < 0$$

(41)

leading to a propagator which increases when $x^2 \to -\infty$. We interpret this phenomenon as a sign of confinement. We recall indeed that gauge invariance is broken and therefore the fermion propagator is endowed of a direct physical meaning. The unphysical massless bosonic excitation, which occurs in this region of parameters, produces an anti-screening effect of a long range type. Nevertheless no asymptotic freedom is expected ($A \neq 0$).
There is another interesting quantity which can be easily discussed in a path–integral approach. Let us introduce the scalar fermion composite operator

$$\hat{S}(x) = N[\bar{\psi}(x)\psi(x)]$$ (42)

where \( N \) means the finite part, after divergencies have been (dimensionally) regularized and renormalized. By repeating standard techniques, it is not difficult to get the expression

$$<0|T(\hat{S}(x)\hat{S}(0))|0>= -\frac{Z^{-1}}{2\pi^2(x^2-\imath\epsilon)}\mathcal{K}(x)$$ (43)

where

$$\mathcal{K}(x) = \exp\{-4i\pi\frac{a}{(a-r^2)(a-r^2+1)}(D(x,m) - D_{1-\omega}(0,m)) +$$

$$+ \frac{1-r^2}{a-r^2+1}(D_{1-\omega}(0,0) - D_{1-\omega}(x,0))\} \}$$ (44)

and

$$Z = \exp\{4i\pi\frac{r^2}{a-r^2}D_{1-\omega}(0,m)\}. \quad (45)$$

Dimensional regularization is again understood.

The expressions for the renormalization constants \( Z_L \) and \( Z \) we have just found, will be recovered in the decompactification limit from the corresponding quantities on \( S^2 \).

We end this section by remarking that conformal invariance is recovered both in the ultraviolet and in the infrared limit, with different scale coefficients.

## 3 Compactification of \( R^2 \) to \( S^2 \): non–trivial principal bundle and problems of globality

In the previous section we have considered the generalized chiral Schwinger model on the Minkowski space: now we want to study its euclidean version on a compact riemannian manifold, namely on the two–dimensional sphere \( S^2 \).

There are many reasons for this investigation: it is well known that in the (vector) Schwinger model gauge field configurations with non–trivial topology (winding number different from zero) and zero modes of the Dirac operator play an important role in order to identify the vacuum structure of the theory. More precisely, we can consider Q.E.D. on \( S^2 \) which, in the limit of the radius \( R \) going to infinity, becomes Q.E.D. in the euclidean two–dimensions. One may say that the two–dimensional plane is compactified to \( S^2 \). This kind of compactification is particularly suited for studying the mentioned problems. Because of the non–trivial topology of \( S^2 \), the gauge fields fall into classes characterized by the winding number \( n \), defined as:

$$n = \frac{e}{2\pi} \int_{S^2} d^2x F_{01}(x), \quad (46)$$

which is an integer. \( A_\mu \) belongs to a non–trivial principal bundle over \( S^2 \). The number of zero modes of the Dirac operator, linked to \( A_\mu \), turns out to be equal to \( |n| \). Thus to neglect the zero modes is equivalent to neglecting all non–trivial topological sectors and leads to
an incorrect result even in the limit $R \to \infty$. In particular it entails the vanishing of the fermionic condensate $\langle \bar{\psi} \psi \rangle$, in disagreement with an operatorial analysis.

We show that in order to define the determinant of the Dirac–Weyl operator an external fixed background connection must be introduced: the determinant shall depend on it. This is a new feature that appears in an anomalous gauge theory: as the coboundary terms become relevant, even the background connection plays an important role. The quantum theory has indeed an intrinsic arbitrariness which depends this time not just on a parameter, but on a field configuration (the choice of the coboundary).

We construct the Green functions generating functional for finite $R$, defining the theory on the two–sphere. As an application, we shall eventually compute the fermionic propagator and the value of the condensate $\langle \bar{\psi} \psi \rangle$ in the limit $R \to \infty$.

Before examining the generalized model we are interested in, some geometrical considerations are in order and the simpler cases of the vector and chiral models have to be first thoroughly discussed.

Our first step is to give a geometrical description of $U(1)$–valued one–forms on the two–sphere, using angular and stereographical coordinates for $S^2$ (of radius $R$).

We can parametrize the two–sphere by angular coordinates:

$$\theta, \phi \quad 0 \leq \theta < \pi \quad 0 \leq \phi < 2\pi$$

or by the use of the stereographical projection (the south pole is identified with the $\infty$):

$$\hat{x}_1, \hat{x}_2 \quad \hat{x}_1 = 2R \tan \frac{\theta}{2} \cos \phi,$$

$$\hat{x}_2 = 2R \tan \frac{\theta}{2} \sin \phi.$$  \hfill (48)

The natural metric is:

$$g_{\theta \theta} = R^2, \quad g_{\phi \phi} = R^2 \sin^2 \theta, \quad g_{\theta \phi} = 0.$$  \hfill (49)

An orthonormal basis for the tangent space can be chosen:

$$e_1 = \frac{1}{R} \frac{\partial}{\partial \theta}, \quad e_2 = \frac{1}{R \sin \theta} \frac{\partial}{\partial \phi}.$$  \hfill (50)

A basis for the one–forms is obviously:

$$\hat{e}_1 = R \, d\theta, \quad \hat{e}_2 = R \sin \theta \, d\phi.$$  \hfill (51)

We define the $U(1)$–valued one–form:

$$A = A_\theta (R \, d\theta) + A_\phi (R \sin \theta \, d\phi).$$  \hfill (52)
The corresponding object in stereographical coordinates is:

\[
A = A_\theta R \left[ \frac{d\theta}{d\hat{x}_1} d\hat{x}_1 + \frac{d\theta}{d\hat{x}_2} d\hat{x}_2 \right] + \\
A_\varphi R \sin \theta \left[ \frac{d\varphi}{d\hat{x}_1} d\hat{x}_1 + \frac{d\varphi}{d\hat{x}_2} d\hat{x}_2 \right].
\]  

(53)

Now

\[
\theta = 2 \arctan \left[ \frac{1}{2R} \sqrt{\hat{x}_1^2 + \hat{x}_2^2} \right],
\]

(54)

leading to

\[
A = \hat{A}_1 d\hat{x}_1 + \hat{A}_2 d\hat{x}_2,
\]

\[
\hat{A}_1 = \frac{1}{1 + \frac{\hat{x}_2^2}{4R^2}} \left[ A_\theta \frac{\hat{x}_1}{\sqrt{\hat{x}^2}} - A_\varphi \frac{\hat{x}_2}{\sqrt{\hat{x}^2}} \right],
\]

\[
\hat{A}_2 = \frac{1}{1 + \frac{\hat{x}_2^2}{4R^2}} \left[ A_\theta \frac{\hat{x}_2}{\sqrt{\hat{x}^2}} + A_\varphi \frac{\hat{x}_1}{\sqrt{\hat{x}^2}} \right].
\]  

(55)

The stereographical projection establishes a one to one correspondence between the points of the plane and the points of the sphere except the \( \infty \) that is identified with the south pole. Let us consider (in stereo–coordinates) the connection:

\[
\hat{A}^{(n)}_\mu = -\frac{n}{e} \epsilon_{\mu\nu} \frac{\hat{x}_\nu}{4R^2 + \hat{x}^2};
\]  

(56)

in angular coordinates:

\[
A_\theta^{(n)} = 0,
\]

\[
A_\varphi^{(n)} = \frac{n}{2eR} \tan \frac{\theta}{2},
\]  

(57)

namely

\[
A^{(n)} = \frac{n}{2e} \tan \frac{\theta}{2} (\sin \theta d\varphi).
\]  

(58)

We observe that the one–form \((\sin \theta d\varphi)\) has a global meaning while a singularity arises in the \(\varphi\)–component: in order to understand if this singularity is meaningful or it is only an artifact of our coordinate system (we stress that at least two patches are needed to describe a sphere and therefore a singularity might be a spurious effect) , we study the situation in another patch.

The previous result can be rewritten as:

\[
A^{(n)} = \frac{n}{2e} (1 - \cos \theta) d\varphi,
\]  

(59)
being regular in a region containing $\theta = 0$ and excluding $\theta = \pi$:

$$0 \leq \theta < \pi. \quad (60)$$

Now we consider stereographical coordinates derived from a north pole projection: it is a simple exercise to show that the relation between the “northern” and the “southern” coordinates is

$$\hat{x}_S = \frac{4R^2}{\hat{x}_N^2}(-\hat{x}_N^1, \hat{x}_N^2) \quad (61)$$

and that the connection eq.$(56)$ has the same form. We can repeat all the calculations finding the expression of $A'$:

$$A'^{(n)} = \frac{n}{2e}(1 + \cos \theta)d\varphi, \quad (62)$$

that, this time, is well defined on $0 < \theta \leq \pi$ (the south pole, $\theta = \pi$ is safe being mapped in a finite plane coordinate).

We immediately notice that in the intersection of the patches, $0 < \theta < \pi$, $A^{(n)}$ and $A'^{(n)}$ do not coincide: we have two different expressions for $A$, that cannot be globally defined on the sphere. Nevertheless in the patch intersection:

$$A^{(n)} - A'^{(n)} = \frac{n}{e}d\varphi = \frac{1}{e}i g^{-1}dg, \quad \quad g = \exp[-in\varphi], \quad (63)$$

$g$ being a map from the intersection region to $U(1)$ (notice that this is possible only if $n$ is an integer); $A'^{(n)}$ differs from $A^{(n)}$ by a gauge transformation. Gauge invariant objects possess a global definition on $S^2$: $A^{(n)}$ belongs to a non–trivial principal bundle on $S^2$. Then:

$$dA^{(n)} = \frac{n}{2e}\sin \theta d\theta d\varphi, \quad \quad \frac{e}{2\pi} \int dA^{(n)} = n, \quad (64)$$

and $A^{(n)}$ carries the non–trivial winding number $n$.

All the connections on the plane that can be considered as derived by a process of stereographic projection, carry integer winding number and belong, on the sphere, to a $U(1)$–bundle characterized by the same integer. In general we can represent any connection as

$$A_\mu = A_\mu^{(n)} + a_\mu, \quad (65)$$

where obviously:

$$\frac{e}{2\pi} \int_{S^2} d^2x (\partial_0 a_1 - \partial_1 a_0) = 0.$$ 

All the topological charge is carried by $A_\mu^{(n)}$ and $a_\mu$ admits a global representation on $S^2$. 

13
4 Dirac and Dirac–Weyl operators on $S^2$

Let us take for the moment angular coordinates: to build the Dirac operator we need the zwei-bein related to the metric eq.(49):

\[
\begin{align*}
    e_\theta^1 &= R \cos \varphi, \\
    e_\varphi^1 &= -R \sin \varphi \sin \theta, \\
    e_\theta^2 &= R \sin \varphi, \\
    e_\varphi^2 &= R \cos \varphi \sin \theta.
\end{align*}
\] (66)

The Dirac operator is (we choose $A_\mu = A_\mu^{(n)}$):

\[
D = i \gamma_a e_\mu^a \left[ \partial_\mu + \frac{1}{4} \gamma_c \gamma_d \omega_{\mu cd} + i e A_\mu^{(n)} \right],
\] (67)

$\omega_{\mu cd}$ being the usual spin-connection. We give the only non zero component (we are in the “southern” patch $0 \leq \theta < \pi$):

\[
\omega_{\varphi 12} = 1 - \cos \theta.
\] (68)

The Dirac operator is:

\[
D = \begin{pmatrix} 0 & D_{12} \\ D_{21} & 0 \end{pmatrix}
\] (69)

where

\[
\begin{align*}
    D_{12} &= i \exp(-i\varphi) \left[ \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi - \frac{1 - n}{2} \frac{1 - \cos \theta}{\sin \theta} \right], \\
    D_{21} &= i \exp(i\varphi) \left[ \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi - \frac{1 + n}{2} \frac{1 - \cos \theta}{\sin \theta} \right].
\end{align*}
\] (70)

It is very simple to derive the explicit expression of this operator in the second patch:

\[
D' = \begin{pmatrix} 0 & D'_{12} \\ D'_{21} & 0 \end{pmatrix},
\] (71)

\[
\begin{align*}
    D'_{12} &= i \exp(i\varphi) \left[ \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{1 + n}{2} \frac{1 + \cos \theta}{\sin \theta} \right], \\
    D'_{21} &= i \exp(-i\varphi) \left[ \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{1 + n}{2} \frac{1 + \cos \theta}{\sin \theta} \right].
\end{align*}
\] (72)

In the intersection of the patches the two operators are related by a unitary transformation:

\[
D' = U^{-1} D U, \\
U = \begin{pmatrix} \exp[-i(n + 1)\varphi] & 0 \\ 0 & \exp[i(n + 1)\varphi] \end{pmatrix}.
\] (73)
$D$ maps a globally defined Dirac field into a new one; the eigenvalue equation, that is essential to obtain the Dirac determinant,

$$D\psi = E\psi,$$

has therefore a well defined meaning with all the eigenvalues $E$’s being invariant under gauge transformations and local frame rotations.

The situation for a Dirac–Weyl operator

$$D = i\gamma_a e^\mu_a [\partial_\mu + \frac{1}{4} \gamma_c \gamma_d \omega_{\mu cd} + ie(\frac{1 + \gamma_5}{2})A_\mu^{(n)}],$$

(74)

is completely different owing to the relation:

$$D' = U_1 D U_2,$$

$$U_1 = \begin{pmatrix} \exp[-i\varphi] & 0 \\ 0 & \exp[i(n + 1)\varphi] \end{pmatrix},$$

(75)

$$U_2 = \begin{pmatrix} \exp[-i(n + 1)\varphi] & 0 \\ 0 & \exp[i\varphi] \end{pmatrix}.$$ (76)

The eigenvalue equation in this case has no global meaning: for a generic $A_\mu$ of winding number $n$ (see eq.(65)) the situation does not change.

It is well known that, in presence of a non–trivial fiber bundle, globality considerations force the dependence of the anomaly on a fixed background gauge connection [16], [21], [22]: the vertex functional is assumed to involve both a dynamical gauge field $A$ and an “external” one $A_0$. From the geometrical point of view this property is very simple to be understood: the transgression formula [23] relies on the fact that a symmetric polynomial on the Lie algebra, invariant under the adjoint action of the group, usually denoted as $P(F_n)$, is an exact form defined on the whole principal bundle while its projection, considered as a form on the base manifold, is only closed:

$$P(F^n) - P(F^n_0) = d\omega^0_{2n-1}(A, A_0).$$

(77)

The anomaly, being derived from $\omega^0_{2n-1}(A, A_0)$, depends on $A_0$: we recall that in this approach [16] $A_0$ does not transform under the B.R.S.T. action, that defines the cohomological problem, and its introduction makes the solution globally defined. A change of the background connection reflects itself in a change of the coboundaries of the cohomological solution [16]: in this sense the choice of $A_0$ does not change the anomaly because the cohomology class remains the same.

We are therefore induced to solve the problem of globality of the Dirac–Weyl operator in a similar way: we introduce a fixed background connection, belonging to the same bundle, in order to recover the transformation property eq.(73) in passing from a patch to another:

$$D = i\gamma_a e^\mu_a [\partial_\mu + \frac{1}{4} \gamma_c \gamma_d \omega_{\mu cd} + ie(\frac{1 + \gamma_5}{2})A_\mu + ie(\frac{1 - \gamma_5}{2})A^0_\mu].$$

(78)

It is rather clear that

$$D' = U^{-1} D U.$$
and hence the global meaning of the Dirac–Weyl determinant is safe \[22\]. It is an exercise to compute the gauge anomaly from the operator eq.(78): one can use the $\zeta$-function technology to recover the infinitesimal variation of the $D$–determinant, that coincides with the result of $\[14\]$. Different choices of $A_0$ reflect themselves into different representatives of the cohomology class: a change of $A_0$ changes the local terms of the determinant. This is a new feature we find in studying an anomalous model on a compact surface: we stress again that an anomalous theory strictly depends on the choice of the coboundary so that the quantum theory looks sensitive to this background connection. In the next section we will try to understand this dependence, and to arrive to a reasonable definition of a chiral gauge theory on the sphere.

5 The chiral gauge theory on $S^2$

The gauge fields on $S^2$ fall into classes characterized by the topological charge $n$:

$$n = \frac{e}{2\pi} \int_{S^2} d^2x \sqrt{g} \epsilon^{\mu\nu} F_{\mu\nu}$$

$$\epsilon^{01} = \frac{1}{\sqrt{g}} = -\epsilon^{10}. \quad (79)$$

Let us consider the field $A_\mu^{(n)}$, defined in stereographical coordinates by eq.(56): $F^{(n)}_{\mu\nu}$ turns out to be

$$F^{(n)}_{\mu\nu} = \frac{n}{2eR^2} \epsilon_{\mu\nu} \quad (80)$$

and satisfies the equation of motion ($D_\mu$ is the covariant derivative with respect to the usual Levi–Civita connection)

$$D_\mu F^{(n)}_{\mu\nu} = 0. \quad (81)$$

Obviously eq.(81) has a global meaning due its gauge invariance, while $A_\mu^{(n)}$ does not possess a global expression. In the same way any field of the type

$$\tilde{A}_\mu^{(n)} = A_\mu^{(n)} + \frac{1}{ie} u \partial_\mu u^{-1}, \quad (82)$$

where $u$ is a $U(1)$–valued map, is a solution of eq.(81). In particular it happens that:

$$F_{01}^{(n)} = \frac{n}{2eR^2} \sqrt{g}. \quad (83)$$

Now let us suppose that $A_\mu$ is any gauge potential with topological charge $n$: a field $\phi$ can be defined through

$$-\Delta \phi = \frac{F_{01}^{(n)}}{\sqrt{g}} - \frac{n}{2eR^2},$$

$$\int_{S^2} d^2x \sqrt{g} \phi = 0. \quad (84)$$
\( \frac{F^{(n)}}{\sqrt{g}} \) being a scalar field on \( S^2 \); the Laplace–Beltrami operator is:

\[
\Delta = \frac{1}{\sqrt{g}} \partial_{\mu} \sqrt{g} g^{\mu \nu} \partial_{\nu}.
\]  

(85)

If we expand the function \( \frac{F^{(n)}}{\sqrt{g}} \) in a complete orthonormal set of eigenfunctions of \( \Delta \), the term \( \frac{n}{2eR^2} \) corresponds to its zero mode. Hence the function \( \frac{F^{(n)}}{\sqrt{g}} - \frac{n}{2eR^2} \) has no projection on the zero mode and the laplacian can be inverted to obtain \( \phi \), that is gauge invariant and scalar under diffeomorphism. The most general form of a generic \( A_{\mu} \) is thereby:

\[
A_{\mu} = A^{(n)}_{\mu} + \epsilon_{\mu \nu} g^{\nu \rho} \partial_{\rho} \phi + \frac{i}{e} h \partial_{\mu} h^{-1},
\]

(86)

\( h \in U(1) \). The winding number is carried by \( A^{(n)} \), \( \phi \) and \( h \) describing the topologically trivial part.

Now we define the chiral Schwinger model on the sphere by:

\[
S^{(n)}_{\text{Class.}} = \int d^2 x \sqrt{g} \left[ \frac{1}{4} F^{(n)}_{\mu \nu} F^{(n)}_{\nu \lambda} g^{\mu \rho} g^{\nu \lambda} + \bar{\psi} \gamma_a e^\mu_a \left[ iD_{\mu} + e A^{(n)}_\mu + e \left( \frac{1 + \gamma_5}{2} \right) a_\mu \right] \psi \right] + \bar{\psi} \gamma_5 e^\mu_a \left[ iD_{\mu} + e A^{(n)}_\mu + e \left( \frac{1 + \gamma_5}{2} \right) a_\mu \right] \psi + \bar{\psi} \gamma_5 e^\mu_a \left[ iD_{\mu} + e A^{(n)}_\mu + e \left( \frac{1 + \gamma_5}{2} \right) a_\mu \right] \psi
\]

(87)

\( S^{(n)}_{\text{Class.}} \) is the action on the \( n \)-topological sector: now we define any expectation value of quantum operators \( O(\bar{\psi}, \psi, A) \) as:

\[
< O(\bar{\psi}, \psi, A) > = \mathcal{Z}_0^{-1} \sum_{n=-\infty}^{+\infty} \int D a_\mu D \bar{\psi} D \psi O(\bar{\psi}, \psi, A) \exp \left[ -S^{(n)}_{\text{Class.}} \right],
\]

(88)

In so doing we represent the \( A_{\mu} \) connection as the sum of a classical instantonic solution \( A^{(n)}_\mu \) and a quantum fluctuation \( a_\mu \) and we choose the fixed background connection \( A^{(0)}_\mu \) to be equal to \( A^{(n)}_\mu \). We notice that \( S^{(n)}_{\text{Class.}} \) does not change under the transformation:

\[
A^{(n)}_\mu \to A^{(n)}_\mu, \\
\bar{\psi} \to \bar{\psi} u^{-1}, \\
\psi \to u \psi.
\]
Quantum theory is unchanged too, since Jacobian is unity (as one could check using \( \zeta \)-function regularization) for this transformation.

The quantum fluctuation \( a_\mu \) couples chirally to the spinor field: no problem of globality arises, having \( a_\mu \) a global expression. Our definition is the most natural generalization of the flat case (or \( n = 0 \)) where:

\[
S_{\text{class.}}^{(0)} = \int d^2 x \left[ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} i \gamma^\mu [\partial_\mu + ie \left( \frac{1 + \gamma_5}{2} \right) A_\mu] \psi \right].
\] (89)

The field \( A_\mu \) is the candidate to represent quantum fluctuations whereas one chooses, in absence of spinors, as solution of the equation of motion \( A_\mu^{(0)} = 0 \). The fluctuation couples chirally to \( \psi \).

We argue that in a topological sector the vacuum is described by the classical \( A_\mu^{(n)} \) solution, and both components of the spinor have to interact with it. But one could also consider eq. (88) as the very definition of our model.

It is rather simple to show that:

\[
\int d^2 x \sqrt{g} \frac{1}{4} F_{\mu\nu}^{(n)} F_{\rho\lambda}^{(n)} g^{\mu\rho} g^{\nu\lambda} = \frac{\pi n^2}{2eR^2} + \int d^2 x \sqrt{g} \frac{1}{4} f_{\mu\nu} f_{\rho\lambda} g^{\mu\rho} g^{\nu\lambda},
\]

\[
f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu.
\] (90)

No coupling between quantum fluctuation \( a_\mu \) and classical background \( A_\mu^{(n)} \) arises, due to eq. (81). To get the action for the generalized chiral Schwinger model we have only to couple \( a_\mu \) with:

\[
e(\frac{1 - r\gamma_5}{2}).
\]

6 The Green’s function generating functional of the generalized chiral Schwinger model on \( S^2 \)

In this section we obtain the Green’s function generating functional of the model. The action eq. (87) is again quadratic in fermion fields, therefore the fermionic integration can be performed for many important operators. Before performing this integration, let us note that it is more convenient to use a dimensionless operator in the action. This can be achieved by setting:

\[
\psi_A = \frac{\psi}{\sqrt{R}},
\]

\[
\bar{\psi}_A = \frac{\bar{\psi}}{\sqrt{R}},
\]

\[
\hat{D} = R D.
\] (91)

The operator \( \hat{D} \) is not hermitian and possesses a non-trivial kernel; the result of the fermionic integration depends crucially on the number of zero modes. In order to work with an hermitian operator (that in \( S^2 \) admits a complete set of eigenstates and eigenvalues)
and to bypass problems with the Berezin integration, we define the Green’s function eq. (88) through a process of analytic continuation on a parameter that we call $\lambda$, a trick essentially due to Andrianov and Bonora [24]:

$$<O(\bar{\psi}, \psi, A) > = \lim_{\lambda \to i\gamma} \int_{\mathcal{Z}_0^{-1}(\lambda)} D\psi_A D\bar{\psi}_A O(\sqrt{R}\bar{\psi}_A, \sqrt{R}\psi_A, A) \exp\left[ -\frac{\pi n^2}{2e^2 R^2} - \int d^2 x \sqrt{\bar{g}} \frac{1}{4} f_{\mu\nu} f^{\mu\nu} \right] \exp\left[ -\int d^2 x \sqrt{\bar{g}} R \bar{\psi}_A \gamma_5 \gamma^\mu e^\mu_a \left[ iD_\mu + eA^{(n)}_\mu + e\left(\frac{1 + i\lambda_5}{2}\right)a_\mu \right] \psi_A \right].$$

We use the properties of the two–dimensional Dirac algebra to get:

$$\gamma_a e^\mu_a \left( \frac{1 + i\lambda_5}{2}\right) a_\mu = \gamma_a e^\mu_a \left( \frac{1}{2}(g_{\mu\nu} + \lambda e_{\mu\nu})a^\nu. \right. \quad \text{(93)}$$

Redefining the (topologically trivial) fluctuation $\hat{a}_\mu$:

$$\frac{1}{2}(g_{\mu\nu} + \lambda e_{\mu\nu})a^\nu = \hat{a}_\mu,$$

$$\frac{2}{1 + \lambda^2}(g_{\mu\nu} - \lambda e_{\mu\nu})\hat{a}_\mu = a_\mu,$$

we can change variables of integration from $a_\mu$ to $\hat{a}_\mu$ (up to an overall constant Jacobian that disappears in eq. (92)) and write:

$$<O(\bar{\psi}, \psi, A) > = \lim_{\lambda \to i\gamma} \int_{\mathcal{Z}_0^{-1}(\lambda)} D\hat{\psi}_A D\bar{\psi}_A O(\sqrt{R}\bar{\psi}_A, \sqrt{R}\psi_A, A[\hat{a}]) \exp\left[ \int d^2 x \sqrt{\bar{g}} R \bar{\psi}_A \gamma_5 \gamma^\mu e^\mu_a \left[ iD_\mu + eA^{(n)}_\mu + e\hat{a}_\mu \right] \psi_A \right].$$

$R_{\mu\nu}$ being the Ricci tensor on $S^2$. In this way we have a fermionic integration linked to the Dirac operator (no chiral couplings); the price we have paid is to work with a more complicated action for the gauge fluctuation. We remark that the fixed background connection was not touched by our definition; the geometrical structure of the theory is unchanged.

Let us study some properties of the operator

$$\hat{D}^{(n)} = R \gamma_a e^\mu_a \left[ iD_\mu + eA^{(n)}_\mu + e\hat{a}_\mu \right];$$

it can be proved that to every eigenfunction $\Phi_i$ of the operator $\hat{D}^{(n)}$ with a non-zero eigenvalue $E_i$ another eigenfunction $\Phi_{-i} = \gamma_5 \Phi_i$ corresponds, with eigenvalue $E_i = -E_{-i}$. Furthermore, the zero modes $\chi^{(n)}_m$ have definite chirality i.e.

$$\gamma_5 \chi^{(n)}_m = \pm \chi^{(n)}_m,$$
and the number of zero modes is \( n = n_+ + n_- \), \( n_+ \) corresponding to positive chirality and \( n_- \) to the negative one \([23]\):

\[
\begin{align*}
n_+ &= 0 & n_- &= |n| & n \geq 0, \\
n_+ &= |n| & n_- &= 0 & n \leq 0.
\end{align*}
\]

We can now compute the partition function by performing the quadratic fermionic integration:

\[
\mathcal{Z}_0(\lambda) = \sum_{n=-\infty}^{+\infty} \int \mathcal{D}\hat{a}_\mu \exp\left[ -\frac{\pi n^2}{2\epsilon^2 R^2} - S_{\text{Bos.}}(\lambda) \right] \det[\hat{D}(n)],
\]

(97)

\( S_{\text{Bos.}}(\lambda) \) being the bosonic part of the action:

\[
S_{\text{Bos.}}(\lambda) = -\frac{2}{1 + \lambda^2} \int d^2 x \sqrt{g} \hat{a}_\mu \left[ (g^{\mu\nu} \Delta - D^\mu D^\nu - R^{\mu\nu}) + \lambda^2 D^\mu D^\nu + 2\lambda \epsilon^{\mu\nu} D_\mu D_\nu \right] \hat{a}_\nu.
\]

(98)

The presence of the zero modes for \( n \neq 0 \) leads to a vanishing contribution of the topological sectors to the partition function: the determinant, defined as the product of the eigenvalues, is zero in this case,

\[
\mathcal{Z}_0(\lambda) = \int \mathcal{D}\hat{a}_\mu \exp\left[ -S_{\text{Bos.}}(\lambda) \right] \det[\hat{D}(0)].
\]

(99)

It follows that any operator of the type \( O(A) \) takes contribution only from the \( n = 0 \) sector; on the other hand the correlation functions involving fermions feel the presence of topological charged configurations: they manifest themselves through parity violating amplitudes \([20]\). We will not consider in the following the general problem of mixed correlation functions, being satisfied with understanding the pure fermionic and bosonic sectors. To this aim we introduce sources to build the generating functional:

\[
\mathcal{Z}[J_\mu, \eta, \bar{\eta}] = \mathcal{Z}_0^{-1} \sum_{n=-\infty}^{+\infty} \int \mathcal{D}\hat{a}_\mu \exp\left[ -\frac{\pi n^2}{2\epsilon^2 R^2} - S_{\text{Bos.}}(\lambda) - J^\mu \frac{2}{1 + \lambda^2} (g_{\mu\nu} - \lambda \epsilon_{\mu\nu}) \hat{a}_\nu \right] \\
\int \mathcal{D}\bar{\psi}_A \mathcal{D}\psi_A \exp\left[ -\int d^2 x \sqrt{g} \bar{\psi}_A \hat{D}^{(n)}(x) \psi_A + \sqrt{R} \bar{\eta} \psi_A + \sqrt{R} \bar{\psi}_A \eta \right].
\]

(100)

Let \( \Phi^{(n)} \) be any eigenstate of \( \hat{D}^{(n)} \) corresponding to a non vanishing eigenvalue: we can construct the kernel

\[
\sum_{i \neq 0} \frac{\Phi^{(n)}_i(x) \Phi^{(n)\dagger}_i(y)}{E_i} = S^{(n)}(x, y),
\]

(101)

that satisfies

\[
\hat{D}^{(n)}(x) S^{(n)}(x, y) = \frac{\delta^2(x, y)}{\sqrt{g}} - \sum_{m=1}^{n} \chi^{(n)}_m(x) \chi^{(n)\dagger}_m(y).
\]

(102)

After the translation

\[
\psi_A = \psi'_A + \int d^2 y \sqrt{g} \sqrt{R} S^{(n)}(x, y) \eta(y),
\]

\[
\bar{\psi}_A = \bar{\psi}'_A + \int d^2 x \sqrt{g} \sqrt{R} \bar{\eta}(y) S^{(n)}(x, y),
\]

(103)

\[
\psi'_A + \int d^2 y \sqrt{g} \sqrt{R} S^{(n)}(x, y) \eta(y),
\]

\[
\bar{\psi}'_A + \int d^2 x \sqrt{g} \sqrt{R} \bar{\eta}(y) S^{(n)}(x, y),
\]

(104)
the Berezin integration gives:

$$Z[J_\mu, \eta, \bar{\eta}] = Z_0^{-1} \sum_{n=-\infty}^{+\infty} \int D\hat{a}_\mu \exp\left[-\frac{\pi n^2}{2e^2 R^2} - S_{\text{Bos}}(\lambda) - J_\mu \frac{2}{1+\lambda^2}(g_{\mu\nu} - \lambda e_{\mu\nu})\hat{a}_\nu\right]$$

$$\exp\left[\int d^2x \sqrt{g} d^2y \sqrt{g} R \bar{\eta}(x) S^{(n)}(x, y) \eta(y)\right]$$

$$\det'[\hat{D}^{(n)}] \Pi_{m=1}^{n} \left[\int d^2x \sqrt{g} \sqrt{R} \chi_m^{(n)\dagger}\eta\right] \left[\int d^2x \sqrt{g} \sqrt{R} \eta \chi_m^{(n)}\right]; \quad (105)$$

$$\det'[\hat{D}^{(n)}]$$ is the (regularized) product of the non–vanishing eigenvalues. One immediately realizes that the correlation functions different from zero are of the type:

$$\langle \bar{\psi}(x_1) \bar{\psi}(x_2) \ldots \bar{\psi}(x_N) \psi(x_{N+1}) \psi(x_{N+2}) \ldots \psi(x_{2N}) \rangle \quad (106)$$

and, at fixed $N$, all the sectors $|n| \leq N$ contribute. At this point we use the representation for $\hat{a}_\mu$:

$$\hat{a}_\mu = e_{\mu\rho} g^{\rho\phi} \partial_\phi + \frac{1}{ie} h \partial_\mu h^{-1}, \quad (107)$$

to obtain

$$\hat{D}^{(n)} = h \exp[e \mu \gamma_5] \hat{D}_0^{(n)} \exp[e \mu \gamma_5] h^{-1}, \quad (108)$$

$$\hat{D}_0^{(n)} = R \gamma_a e_a^\mu \left[i D_\mu + e A_\mu^{(n)}\right].$$

The zero modes of $\hat{D}^{(n)}$ are related to the ones of $\hat{D}_0^{(n)}$:

$$\chi_m^{(n)} = h \exp[-e \mu \gamma_5] \sum_{j=1}^{n} B_{mj} \chi_{0j}^{(n)}, \quad (109)$$

where we have used a $n \times n$ matrix $B$ to have an orthonormal basis for the null space of $\hat{D}^{(n)}$ ($\chi_{0j}^{(n)}$ are chosen to be orthonormal). The role of the matrix $B$ is discussed in Appendix A. In eq.(103) one can prove that:

$$\exp\left[\int d^2x \sqrt{g} d^2y \sqrt{g} R \bar{\eta}(x) S^{(n)}(x, y) \eta(y)\right]$$

$$\Pi_{m=1}^{n} \left[\int d^2x \sqrt{g} \sqrt{R} \chi_m^{(n)\dagger}\eta\right] \left[\int d^2x \sqrt{g} \sqrt{R} \eta \chi_m^{(n)}\right] =$$

$$\exp\left[\int d^2x \sqrt{g} d^2y \sqrt{g} R \bar{\eta}'(x) S_0^{(n)}(x, y) \eta'(y)\right]$$

$$\Pi_{m=1}^{n} \left[\int d^2x \sqrt{g} \sqrt{R} \chi_{0m}^{(n)\dagger}\eta'\right] \left[\int d^2x \sqrt{g} \sqrt{R} \eta' \chi_{0m}^{(n)}\right] \det B|^2, \quad (110)$$

where:

$$S_0^{(n)}(x, y) = \sum_{i=-\infty}^{+\infty} \frac{\phi_i^{(n)}(x) \phi_i^{(n)\dagger}(y)}{E_i^{n}},$$

$$\bar{\eta}' = \bar{\eta} \exp[e \mu \gamma_5] h^{-1},$$

$$\eta' = h \exp[e \mu \gamma_5] \eta, \quad (111)$$
$E_i^{(0)}$ being the eigenvalues of $\hat{D}_0^{(n)}$ and $\phi_0^{(n)}(x)$ the related eigenfunctions: an explicit expression for $S_0^{(n)}(x, y)$ is given in [27].

To calculate the generating functional for the fermionic fields we are left with computing $\det'[\hat{D}^{(n)}]$. The standard $\zeta$–function calculation will give us a result that does not take into account the Jackiw–Rajaraman ambiguity: in order to implement correctly the freedom in the choice of the local term, carefully considering the global meaning of the determinant, we use the usual definition [14], generalized to the case of a non–trivial connection:

$$\det'[\hat{D}^{(n)}] = \frac{\det'[\hat{\mathcal{D}}^{(n)}]}{\det[\hat{D}_0^{(n)}]}, \quad (112)$$

where

$$\hat{D}_0^{(n)} = R \gamma_a e^\mu_a [i D_\mu + e A_\mu^{(n)} + e a \hat{a}_\mu], \quad (113)$$

$\alpha$ being a real number. The parameter ambiguity only affects the topologically trivial part of the gauge connection: only in this way the operator eq.(113) is well defined on $S^2$, because the associated winding number is still an integer. Moreover it is easy to prove that:

$$Ker[\hat{D}_0^{(n)}] = Ker[\hat{D}_0^{(n)}] = n,$$

$$Ker[\hat{D}_\alpha^{(n)}] = Ker[\hat{D}_\alpha^{(n)}] = n. \quad (114)$$

In our approach $A_\mu^{(n)}$ is a classical field; the ambiguity of regularization can only affect the quantum fluctuations, that depend on quantum loops (determinant calculation): we expect that, with our definition, the terms depending on $\alpha$ do not involve $A_\mu^{(n)}$. More precisely they must be local polynomials in the quantum fluctuations and their derivatives.

The computation of eq.(112) is rather involved from the technical point of view: essentially we have applied to the present situation the theorems derived in [28] and [29], to obtain (details are given in Appendix B)

$$\det'[\hat{D}^{(n)}\hat{D}_0^{(n)}] = \det[(\hat{D}_0^{(n)})^2] \exp \int_0^1 dt \omega'(t) \quad (115)$$

and

$$\omega'(t) = \int d^2 x \sqrt{g} \text{Tr} \left[ K_0 \left( \hat{D}_\alpha^{(n)}(t); x, x \right) [e(1 + \alpha) \phi \gamma_5 + (1 - \alpha)h(t) \partial_t h^{-1}(t)] + \right.$$  

$$+ K_0 \left( \hat{D}_\alpha^{(n)}(t); x, x \right) [e(1 + \alpha) \phi \gamma_5 - (1 - \alpha)h^{-1}(t) \partial_t h(t)] \right] -$$

$$- \sum_{m=1}^{[n]} \int d^2 x \sqrt{g} \varphi_0^{(n)}(x, t)[e(1 + \alpha) \phi \gamma_5 + (1 - \alpha)h(t) \partial_t h^{-1}(t)] \varphi_0^{(n)}(x, t) -$$

$$- \sum_{m=1}^{[n]} \int d^2 x \sqrt{g} \chi_0^{(n)}(x, t)[e(1 + \alpha) \phi \gamma_5 - (1 - \alpha)h^{-1}(t) \partial_t h(t)] \chi_0^{(n)}(x, t), \quad (116)$$

where $K_0(A; x, y)$ is the analytic continuation in $s = 0$ of the kernel of the $s$–complex power of the operator $A$ [15], (see eq.(28)); $h(t)$ interpolates along the $U(1)$–valued functions between $h$ and the identity, (remember that $\pi_2(S^1) = 0$), and

$$\hat{D}^{(n)}(t) = \hat{D}^{(n)}(\hat{a}_\mu(t)), \quad \hat{D}_\alpha^{(n)}(t) = \hat{D}_\alpha^{(n)}(\hat{a}_\mu(t)), \quad (117)$$
\[
\hat{a}_\mu(t) = t \epsilon_{\mu\nu}g^{\mu\nu} \partial_\nu \phi + \frac{1}{ie} h(t) \partial_\mu h^{-1}(t),
\]
(118)

\[
\dot{D}^{(n)}(t) \chi^{(n)}_{0m}(x, t) = 0,
\]
\[
\dot{D}^{(n)}_\alpha(t) \phi^{(n)}_{0m}(x, t) = 0,
\]
(119)

\[\chi^{(n)}_{0m}(x, t), \phi^{(n)}_{0m}(x, t)\] being the orthonormal bases of the kernels of the operators in eq. (117), smoothly interpolating between \(\dot{D}^{(n)}\) and \(\dot{D}^{(n)}_0\) and between \(\dot{D}^{(n)}_\alpha\) and \(\dot{D}^{(n)}_0\) respectively.

In the same way:

\[
\det' \left[ \hat{D}^{(n)}_\alpha \right] = \det' \left[ \hat{D}^{(n)}_0 \right] \exp \int_0^1 dt \omega''(t)
\]
(120)

and

\[
\omega''(t) = \int d^2 x \sqrt{g} Tr \left[ K_0 \left( \dot{D}^{(n)}_\alpha(t); x, x \right) [e \alpha \phi \gamma_5] \right] - \sum_{m=1}^{\left| n \right|} \int d^2 x \sqrt{g} \left[ \phi^{(n)}_{0m}(x, t) [e \alpha \phi \gamma_5] \right] \phi^{(n)}_{0m}(x, t).
\]
(121)

The trace of the heat–kernel coefficients can be easily performed, as in the previous section; on the other hand the computation of the integrals over the “interpolating” zero modes is more involved and subtler; for this reason we give the final result, deferring the technical procedure to the Appendix B:

\[
\det' \left[ \hat{D}^{(n)} \right] = \det' \left[ \hat{D}^{(n)}_0 \right] \exp \left[ \int_0^1 dt \omega''(t) \right]
\]
(122)

The determinant of the zero mode matrix disappears from the generating functional. The parameter \(\gamma\) is linked to \(\alpha\) by:

\[
\gamma = \frac{1}{2} (1 - \alpha)^2.
\]
(123)

We remark that eq. (122) exhibits an ambiguity only in the quantum fluctuations and, with respect to them, is local:

\[
\int d^2 x \sqrt{g} \left[ -\phi \Delta \phi + \frac{1}{e^2} \partial_\mu h \partial^\mu h^{-1} \right] = \int d^2 x \sqrt{g} \hat{a}_\mu \hat{a}_\mu.
\]
(124)

Moreover it is quite natural to have in any sector the same ambiguity: it is related to the quantum fluctuation and not to the classical background. The last point is the calculation of \(\det' \left[ \hat{D}^{(n)}_0 \right]\), that we present in Appendix C. The result is:

\[
\det' \left[ \hat{D}^{(n)}_0 \right] = \exp \left[ -4 \zeta_R'(-1) + \frac{n^2}{2} + \left| n \right| \log \left| n \right|! - \sum_{m=1}^{\left| n \right|} 2m \log m \right].
\]
(125)
7 The fermionic propagator

The generating functional (101) for $J_\mu = 0$ becomes:

$$Z[\eta, \bar{\eta}] = Z_0^{-1}(\lambda) \sum_{n=-\infty}^{+\infty} \exp\left[-\frac{\pi n^2}{2e^2 R^2}\right] \int D\phi Dh \exp \left[ \int d^2x \sqrt{g} \int d^2y \sqrt{g} \bar{R} \eta'(x) S_0^{(n)}(x,y) \eta'(y) \right]$$

$$\det'[\hat{D}_0^{(n)}] \exp \left[ \frac{e^2}{2\pi} \int d^2x \sqrt{g} \left[ (1-\gamma) \phi \Delta \phi + \frac{1}{e^2} \partial_\mu h \partial^\mu h^{-1} \right] \right]$$

$$\exp \left[ -S_{Bos.}[\lambda; \phi, h] \right] \Pi_{m=1}^{\eta^{(n)}} \int d^2x \sqrt{g} \sqrt{R} \chi^{(n)}_{0m} \eta^{(n)} \chi^{(n)}_{0m} \right].$$

(126)

All the fermionic correlation functions of the theory can be derived: the functional integration over $\phi$ and $h$ is gaussian, as in the flat case, and the model is still constructed by means of free fields (on curved background). Chirality-violating correlation functions can be different from zero only in the non-trivial winding number sectors, as in the case of the vector Schwinger model: actually, in that model, this feature survives the limit $R \to \infty$, changing in this way the vacuum structure of the theory [30]. In particular the vacuum fermionic condensate is seen to be different from zero [27]:

$$\langle \bar{\psi} \psi \rangle = \frac{e \exp[C]}{2\pi \sqrt{\pi}}$$

(127)

recovering, by a path-integral procedure, the well known operatorial result [31] ($C$ is the Euler–Mascheroni constant). As a matter of fact the fermionic propagator on the sphere has a different algebraic structure, correlations being now possible between $\bar{\psi}_L$ and $\psi_L$ and between $\bar{\psi}_R$ and $\psi_R$. The relevant contributions come from the $n = \pm 1$ sectors, while the sector $n = 0$ leads to the usual chirality conserving correlation.

We calculate the explicit form of the propagator; in the limit for $R \to \infty$ we recover the expression found in the theory on the plane. As a particular case we get the vacuum expectation value of the fermionic scalar density: at variance with the eq.(127) we find a vanishing result, confirming the conjecture of [17] about the triviality of the vacuum in the chiral Schwinger model and showing how the breaking of the gauge invariance completely changes the structure of the theory.

In order to perform the calculation it is useful to write

$$\frac{1}{ie} \partial_\mu h^{-1} = \partial_\mu \beta,$$

(128)

what is possible, thanks to the triviality of $\pi_2(S_1)$. We change variable from $h$ to $\beta$ in eq.(126) and rescale

$$\phi \to \left( \frac{1 + \lambda^2}{2} \right) \phi,$$

$$\beta \to \left( \frac{1 + \lambda^2}{2} \right) \beta,$$

$$e^2 \to \frac{(1 + \lambda^2) e^2}{4\pi} = \tilde{e}^2.$$

(129)
We use the notation:
\[
\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix},
\]
(130)
\[
\bar{\psi} = (\bar{\psi}_R, \bar{\psi}_L).
\]
(131)
We can easily derive from the generating functional the fermionic propagator:
\[
<\psi(x)\bar{\psi}(y)> = \begin{pmatrix} S_{RR}(x,y) & S_{RL}(x,y) \\ S_{LR}(x,y) & S_{LL}(x,y) \end{pmatrix};
\]
(132)
where
\[
S_{RL,LR}(x,y) = S^0_{RL,LR}(x,y) \exp\left[-\frac{1}{2}G_{\pm}(x,y)\right]
\]
(133)
\[
S_{RR,LL}(x,y) = \frac{1}{4\pi R} \exp\left[\frac{1}{2} - \frac{\pi}{2e^2 R^2}\right] \exp\left[\frac{1}{2}G_{R,L}(x,y)\right]
\]
(134)
\[
\exp\left[-\frac{1}{2}G_{\pm}(x,y)\right] = \frac{\int D\phi D\beta \exp\left[-\frac{1}{2}\Gamma(\phi,\beta)\right]}{\int D\phi D\beta \exp\left[-\frac{1}{2}\Gamma(\phi,\beta)\right]}.
\]
(135)
\[
\exp\left[-\frac{1}{2}G_{R,L}(x,y)\right] = \frac{\int D\phi D\beta \exp\left[-\frac{1}{2}\Gamma(\phi,\beta)\right]}{\int D\phi D\beta \exp\left[-\frac{1}{2}\Gamma(\phi,\beta)\right]}.
\]
(136)
\[
\Gamma(\phi,\beta) = \int d^2x \sqrt{g} \left[\phi \Delta^2 \phi + \lambda^2 \beta \Delta^2 \beta - 2\lambda \phi \Delta^2 \beta - \hat{e}^2(1 - \gamma)\phi \Delta \phi + \hat{e}^2 \gamma \beta \Delta \beta\right].
\]
(137)
The explicit form of the zero mode of \(n = \pm 1\) [32] was taken into account; \(S^0_{RL,LR}(x,y)\) are the nonvanishing components of the Green function defined in eq. (111) that takes a very simple form in stereographical coordinates [27]:
\[
S^0(\hat{x},\hat{y}) = \frac{1}{4\pi R}
\]
(138)
\[
\int d^2x \sqrt{g} \left(1 + \frac{\hat{x}^2}{4R^2}\right)^{\frac{1}{4}} \left(1 + \frac{\hat{y}^2}{4R^2}\right)^{\frac{1}{4}} \gamma_a(\hat{x}_a - \hat{y}_a) \]
\[
\left(\hat{x} - \hat{y}\right)^2.
\]
Now the integration over \(\phi\) and \(\beta\) is quadratic and can be easily performed expanding, for example, the scalar fields in spherical harmonics, that are (up to a scale factor) a complete set of orthogonal eigenfunctions for the laplacian. No problem arises with the zero mode thanks to the properties:
\[
-\Delta \phi = f_{01}
\]
(139)
\[
\int d^2x \sqrt{g} \epsilon^{\mu\nu} f_{\mu\nu} = 0
\]
\[
\Delta \beta = D_\mu (h \partial^\mu h^{-1})
\]
(140)
\[
\int d^2x \sqrt{g} D_\mu (h \partial^\mu h^{-1}) = 0.
\]
The final result is:

\[
G_{\pm}(\omega) = G^{U.V.}_{\pm} + \hat{G}_{\pm}(\omega)
\]
\[
G_{R,L} = G^{U.V.}_{R,L} + \hat{G}_{R,L}(\omega)
\] (141)

\[
G^{U.V.}_{\pm} = \lim_{\alpha \to 0} \frac{1}{2} \frac{(1 \pm i\lambda)^2}{[\gamma(1 + \lambda^2) - \lambda^2]} \sum_{l=1}^{\infty} \frac{(2l + 1)}{[l(l + 1) + m^2(\lambda)R^2]} P_l(\cos \alpha),
\] (142)

\[
\hat{G}_{\pm}(\omega) = \frac{m^2(\lambda)R^2}{2(1 - \gamma)\gamma} \sum_{l=1}^{\infty} \frac{(2l + 1)}{[l(l + 1) + m^2(\lambda)R^2]} [1 - P_l(\cos \theta)]
\]
\[ - \frac{1}{2} \frac{(1 \mp i\lambda)^2}{[\gamma(1 + \lambda^2) - \lambda^2]} \sum_{l=1}^{\infty} \frac{(2l + 1)}{[l(l + 1) + m^2(\lambda)R^2]} P_l(\cos \theta),
\] (143)

\[
G^{U.V.}_{R,L} = \lim_{\alpha \to 0} \frac{1}{2} \frac{(1 - \lambda^2)}{[\gamma(1 + \lambda^2) - \lambda^2]} \sum_{l=1}^{\infty} \frac{(2l + 1)}{[l(l + 1) + m^2(\lambda)R^2]} P_l(\cos \alpha),
\] (144)

\[
\hat{G}_{R,L}(\omega) = \frac{m^2(\lambda)R^2}{2\gamma} \sum_{l=1}^{\infty} \frac{(2l + 1)}{[l(l + 1) + m^2(\lambda)R^2]} [1 - P_l(\cos \theta)]
\]
\[ + \frac{m^2(\lambda)R^2}{2(1 - \gamma)} \sum_{l=1}^{\infty} \frac{(2l + 1)}{[l(l + 1) + m^2(\lambda)R^2]} [1 + P_l(\cos \theta)]
\]
\[ + \frac{1}{2} \frac{(1 + \lambda^2)}{[\gamma(1 + \lambda^2) - \lambda^2]} \sum_{l=1}^{\infty} \frac{(2l + 1)}{[l(l + 1) + m^2(\lambda)R^2]} P_l(\cos \theta),
\] (145)

\(P_l\) being the Legendre polynomials and \(\omega\) the angle between \(\hat{r}(x)\) and \(\hat{r}(y)\), the three–vectors representing the points \(x\) and \(y\) on \(S^2\), embedded in \(R^3\); the mass \(m^2(\lambda)\) is

\[
m^2(\lambda) = \frac{\hat{e}^2\gamma(\gamma - 1)}{\gamma(1 + \lambda^2) - \lambda^2}.
\] (146)

In order to perform the limit on the plane we require that:

\[
\lim_{R \to \infty} Z(\lambda = ir) = Z,
\]

the effective action obtained in eq.(13); this entails a relation between \(\gamma\) and \(a\):

\[
\gamma = \frac{a}{(1 + \lambda^2)}.
\] (147)

We recognize that

\[
m^2(\lambda) = m^2,
\]
m^2 being the bosonic mass we have found in the calculation on the plane.
Actually the series defining $G_\pm$ and $G_{RL}$ can be summed in terms of special functions. We can choose the point $y$ as the north pole, $\cos \omega = \cos \theta$ without losing any information. Then $\hat{G}(\omega)$ depends only on the polar angle $\theta$: following [27] we introduce the function

$$\Delta_1(\theta) = \sum_{l=1}^{\infty} \frac{(2l + 1)P_l(\cos \omega)}{l(l+1) + m^2 R^2} + \frac{1}{m^2 R^2},$$

(148)

that satisfies the differential equation

$$[-\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + m^2] \Delta_1(\theta) = 0$$

(149)

for $\theta \neq 0$. The solution are the associated Legendre functions [33], usually denoted by $Q_\nu(x)$, $x = \cos \theta$. It is not difficult to prove that with the usual normalization for $Q_\nu(x)$:

$$\Delta_1(\theta) = Q_{\nu_1}(\cos \theta) + Q_{\nu_2}(\cos \theta),$$

$$\nu_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - m^2 R^2},$$

(150)

obtaining

$$\sum_{l=1}^{\infty} \frac{(2l + 1)P_l(\cos \omega)}{l(l+1) + m^2 R^2} = \left[ (Q_{\nu_1}(\cos \theta) + Q_{\nu_2}(\cos \theta)) - \frac{1}{m^2 R^2} \right].$$

(151)

Then we use the following result [33]:

$$\sum_{l=1}^{\infty} \frac{(2l + 1)P_l(\cos \omega)}{l(l+1)} = - \ln\left(\frac{1 - \cos \theta}{2}\right) - 1,$$

(152)

to get

$$\sum_{l=1}^{\infty} \frac{m^2 R^2 (2l + 1)}{l(l+1)[l(l+1) + m^2(\lambda) R^2]} P_l(\cos \theta) = - \left[ Q_{\nu_1}(\cos \theta) + Q_{\nu_2}(\cos \theta) \right]$$

$$+ \ln\left(\frac{1 - \cos \theta}{2}\right) - \frac{1}{m^2 R^2} + 1]$$

(153)

and

$$\sum_{l=1}^{\infty} \frac{m^2 R^2 (2l + 1)}{l(l+1)[l(l+1) + m^2(\lambda) R^2]} = \left[ \psi(1 + \nu_1) + \psi(1 + \nu_2) + 2\psi(1) + \frac{1}{m^2 R^2} - 1 \right].$$

(154)

Eqs.(142-145) become

$$G^{U.V.}_\pm = -\frac{1}{2} \frac{(1 \pm r)^2}{[a - r]} \lim_{\alpha \to 0} \left[ (Q_{\nu_1}(\cos \alpha) + Q_{\nu_2}(\cos \alpha)) - \frac{1}{m^2 R^2} \right]$$

(155)
\[
\hat{G}(\omega) = \frac{1}{2} \left( \frac{1}{a(a+1-r^2)(a-r^2)} \right) [Q_{\nu_1}(\cos \theta) + Q_{\nu_2}(\cos \theta)] \\
+ \frac{1}{a(a+1-r^2)} \ln \left( \frac{1-\cos \theta}{2} \right) \\
+ \frac{1}{2a(a+1-r^2)} \left[ \psi(1+\nu_1) + \psi(1+\nu_2) + 2C \right], \\
(156)
\]

\[
G^{U.V.}_{R,L} = \frac{1}{2} \frac{1+r^2}{a-r^2} \left[ Q_{\nu_1}(\cos \alpha) + Q_{\nu_2}(\cos \alpha) \right] - \frac{1}{m^2 R^2}, \\
(157)
\]

\[
\hat{G}_{R,L}(\omega) = -\frac{1}{2} \frac{1}{a-r^2} \left[ Q_{\nu_1}(\cos \theta) + Q_{\nu_2}(\cos \theta) \right] \\
- \frac{1}{2} \frac{(1-r^2)^2}{a(a+1-r^2)} \left[ \psi(1+\nu_1) + \psi(1+\nu_2) + 2C \right] \\
+ \frac{1}{2} \frac{1}{a(a+1-r^2)} \left[ Q_{\nu_1}(\cos \theta) + Q_{\nu_2}(\cos \theta) + \ln \left( \frac{1-\cos \theta}{2} \right) \right]. \\
(158)
\]

The term \( G^{U.V.}_{\pm} \) and \( G^{U.V.}_{R,L} \) are obviously related to the ultraviolet divergencies we have found in the flat calculation: they characterize the short-range behaviour of the theory and therefore are present no matter the global topology. One can directly check the independence from \( R \) of the divergent part. Moreover the relation

\[
G^{U.V.}_{R,L} = \frac{1}{2} \left[ G^{U.V.}_{+} + G^{U.V.}_{-} \right], \\
(159)
\]

is consistent with the redefinition

\[
\psi^{\text{Ren.}}_R = Z^{-\frac{1}{2}}_R \psi_R \\
\psi^{\text{Ren.}}_L = Z^{-\frac{1}{2}}_L \psi_L, \\
(160)
\]

that leads to a meaningful expression for the correlation functions:

\[
S^{\text{Ren.}}_{RL,LR}(\theta) = S^0_{RL,LR}(\theta) \exp \left[ -\frac{1}{2} \hat{G}_\pm \right], \\
S^{\text{Ren.}}_{RR,LL}(\theta) = \frac{1}{4\pi R} \exp \left[ \frac{1}{2} - \frac{\pi}{2\epsilon^2 R^2} \right] \exp \left[ -\frac{1}{2} \hat{G}_{R,L} \right]. \\
(162)
\]

We expect that the small distance behaviour of the theory is the same of the one in the flat case: let us study the limit for \( \theta \to 0 \) in eq. (162). The fermionic Green function in this limit has the expression

\[
S^0_{RL,LR} = \frac{1}{4\pi R} \frac{1}{\theta}. 
\]
that exhibits the canonical scaling in $R\theta$; the small-$\theta$ expansions of the exponents lead to

$$<\psi_{R,L}^{\text{Ren.}}(0)\bar{\psi}_{R,L}^{\text{Ren.}}(\theta)> = (R\theta)^{-\frac{1}{2} \left(1 + \frac{(1\pm r)^2}{2 R^2}\right)},$$

$$<\psi_{R,L}^{\text{Ren.}}(0)\bar{\psi}_{R,L}^{\text{Ren.}}(\theta)> = (R\theta)^{\frac{1}{2} \frac{(1\pm r)^2}{2 R^2}}.$$

(163)

We notice that the scaling exponent of the chirality-conserving part of the fermionic correlation function is exactly the same as in the flat case: we have indeed the correct singularity as $\theta \to 0$. The contribution of the $n = \pm 1$ sectors agree with our intuitive arguments for $r^2 < 1$: the chirality-violating part goes to zero as $\theta$, in this range. For $r^2 > 1$ a singularity arises, changing drastically the analytical structure of ultraviolet limit. In the very special case of $r^2 = 1$ we have:

$$<\psi_{R,L}^{\text{Ren.}}(0)\bar{\psi}_{R,L}^{\text{Ren.}}(\theta)> = \frac{1}{4\pi R}\exp\left[\frac{1}{2} \frac{\pi}{2e^2 R^2}\right].$$

By the way there is another potential singularity in eqs.(156,158): the Legendre function $Q_{\nu}$ has branch points at $x = \pm 1$ and $x = \infty$, so for $\theta = \pi$ one could expect some critical behaviour. A careful use of the asymptotic expansion near $x = -1$ [33]:

$$Q_{\nu}(x) = \frac{1}{2} \cos(\nu \pi) \ln\left(\frac{1 + x}{2}\right) + O(1)$$

shows that the singularities cancel and the correlation function goes to a constant value in this limit.

At this point we try to decompactify the theory: we have essentially to discuss the large $R$ behaviour of two functions:

$$Q_{\nu_1}(\cos \theta) + Q_{\nu_2}(\cos \theta)$$

and

$$\psi(1 + \nu_1) + \psi(1 + \nu_2).$$

In order to check the flat-space limit let us define the geodesic distance:

$$\rho = R\theta$$

(164)

and let $R \to \infty$, keeping $R\theta$ fixed. We define $\tau = \sqrt{m^2 R^2 - \frac{1}{4}}$ which is real for large $R$. Now:

$$Q_{\nu_1}(\cos \theta) + Q_{\nu_2}(\cos \theta) = \left[\frac{\pi}{\cosh(\pi \tau)}P_{\frac{1}{2} + i\tau}(\cos \theta) - \frac{1}{m^2 R^2}\right].$$

(165)

An asymptotic expansion for large $\tau$ and small $\theta$ can be found [33]:

$$P_{\frac{1}{2} + i\tau}(\cos \theta) = \frac{1}{\pi} \exp[\tau \pi]K_0(\tau \theta) + \frac{\rho}{R} + O\left(\frac{\rho}{R^2}\right).$$

(166)

Furthermore:

$$\tau \theta = \sqrt{m^2 \rho} - \frac{\rho}{8\sqrt{m^2}}$$

$$K_0(\tau \theta) = K_0(\sqrt{m^2 \rho}) + K_1(\sqrt{m^2 \rho}) \frac{\rho}{8\sqrt{m^2}} + O\left(\frac{1}{R}\right).$$

(167)
leading, in the limit $R \to \infty$, to

$$Q_{\nu_1}(\cos \theta) + Q_{\nu_2}(\cos \theta) = 2K_0(\sqrt{m^2 \rho}) + O\left(\frac{1}{R^2}\right). \quad (168)$$

In the same way we use the asymptotic expansion for $\psi(z) \ (|z| \to \infty)$:

$$\psi(z) = \ln z - \frac{1}{2z} + O\left(\frac{1}{z^2}\right), \quad (169)$$

leading to

$$\psi\left(\frac{1}{2} + i\tau\right) + \psi\left(\frac{1}{2} - i\tau\right) = 2 \ln(mR) + O\left(\frac{1}{R^2}\right). \quad (170)$$

The first point to check is that the renormalization constants coincide in the limit with the ones in the flat case: in order to compare the two expressions we perform the limit $\rho \to 0$ in eqs.(156,158), after the decompactification. We get:

$$Z_{R,L} = \lim_{\rho \to 0} \exp\left[\frac{-(1 \pm r)^2}{2(a - r^2)}K_0(\sqrt{m^2 \rho})\right], \quad (171)$$

that corresponds to eq.(25). The renormalized two–point function becomes:

$$<\psi_{R,L}^{\text{Ren.}}(0)\bar{\psi}_{L,R}^{\text{Ren.}}(\rho)> = S^0(\rho) \exp\left[\frac{1}{4a(a + 1 - r^2)} \ln(m^2 \rho^2)\right] + \frac{(1 + r^2)(1 + 2a - r^2)}{2a(a + 1 - r^2)(a - r^2)}K_0(\sqrt{m^2 \rho}) + O\left(\frac{1}{R^2}\right), \quad (172)$$

that coincides with eq.(23), and

$$<\psi_{R,L}^{\text{Ren.}}(0)\bar{\psi}_{L,R}^{\text{Ren.}}(\rho)> = \frac{1}{4\pi} R^{-(1+g)} \exp\left[\frac{1}{2}F(\rho)\right], \quad (173)$$

$$F(\rho) = \exp\left[-\frac{1}{4} (1 - r^2)^2 K_0(\sqrt{m^2 \rho}) - \frac{1 - r^2}{2a} \ln(m^2 \rho^2)\right] + \frac{(1 - r^2)(1 + 2a - r^2)}{a(a + 1 - r^2)}[K_0(\sqrt{m^2 \rho}) + \frac{1}{2} \ln(m^2 \rho^2)]. \quad (174)$$

The $n = 0$ sector reproduces in the large $R$ limit the result of the flat case. The behaviour of the chirality–violating part is very different: it is tuned by the effective Thirring coupling $g$, we have found to describe the infrared regime of the fermionic operators. In the first unitarity region, where no ghost is present, it turns out to be $g > -1$, as it should for the consistency of the Thirring theory [20]. The contribution of $n = \pm 1$ is therefore suppressed as $R \to \infty$; the situation is very different from the Schwinger model, where topological sectors contribute.

We expect a similar result for the vacuum expectation value of the fermionic scalar density: we go back to eqs.(143,145) and take the limit $\theta \to 0$

$$<\bar{\psi}\psi> = \frac{\exp\left[\frac{1}{2} - \frac{\pi}{2\rho^2 R^2}\right]}{2\pi R} \exp\left[\frac{1}{2}[H_{U.V.} + H]\right], \quad (175)$$
where

\[ H^{U.V.} = \lim_{\theta \to 0} \frac{r^2}{a - r^2} [Q_{\nu_1}(\cos \theta) + Q_{\nu_2}(\cos \theta) - \frac{1}{m^2 R^2}], \]

\[ H = \frac{1 - r^2}{a + 1 - r^2} [\psi(1 + \nu_1) + \psi(1 + \nu_2) + 2C - 1 + \frac{1}{m^2 R^2}], \]  

(176)

\( H^{U.V.} \) is not a finite quantity, it is the ultraviolet divergence related to the renormalization constant for the composite operator \( \bar{\psi} \psi \), found in the flat space. It is very easy to prove, using the large \( R \) expansion, that

\[ Z = \exp \left[ \frac{1}{2} H^{U.V.} \right] \]

coincides with eq.(15). After the renormalization:

\[ < (\bar{\psi} \psi)_{\text{Ren.}} > = \exp \left[ \frac{1 - \frac{\pi^2}{2}}{2 \pi R} \right] \exp \left[ \frac{1}{2} H \right]. \]  

(177)

If we perform the limit \( R \to \infty \), we end up with:

\[ < (\bar{\psi} \psi)_{\text{Ren.}} > = \lim_{R \to \infty} \frac{P^{\delta_1}}{2 \pi R} m^{\delta_1} \exp [\delta_1 C + \delta_2], \]

\[ \delta_1 = \frac{1 - r^2}{(a + 1 - r^2)}; \]

\[ \delta_2 = \frac{a}{(a + 1 - r^2)} \]  

(178)

The power of \( R \) is:

\[ R^{-\delta_2}; \]  

(179)

for \( a > r^2 \) (in the first unitarity region, where no ghost is present) the limit \( R \to \infty \) is zero. The vacuum expectation value of the scalar density vanishes for all the generalized chiral Schwinger models, in the first unitarity region: in particular for the chiral Schwinger model, confirming the conjecture proposed in [17].

We end this section with some remarks concerning the bosonic spectrum: as we have previously pointed out, the bosonic Green functions take contribution only from the \( n = 0 \) sector. The calculation of the propagator for \( A_{\mu} \) is straightforward and it can be expressed in terms of the functions \( Q_{\nu}(\cos \theta) \), taking also eq.(152) into account; in the limit of large \( R \) we recover exactly the propagator we have found in the flat case.

8 Conclusions

In conclusion we have thoroughly studied on the two-sphere \( S^2 \) a vector–axial vector theory characterized by a parameter which interpolates between pure vector and chiral Schwinger models: the generalized chiral Schwinger model. The theory has been completely solved by means of non perturbative techniques, obtaining explicit expressions for its correlators.
We have defined the theory, respecting its global character, a non-trivial task because it was an anomalous one on a non-trivial principal bundle. At least at our knowledge, it is the first time that an anomalous gauge theory is quantized on a non-trivial topology, taking into account the contributions of winding numbers different from zero and zero-modes of the relevant fermionic operator.

We have discussed the definition of the Dirac–Weyl determinant on $S^2$, in presence of topological charged gauge connections, showing the appearance, in an analytical approach, of the well-known fixed background connection of the cohomological solution for the anomaly. We have explained its physical meaning and we have carefully computed the Green’s functions generating functional.

The bosonic spectrum is the same while parity-violating fermionic correlators are seen to be different from zero for a finite radius of the sphere; ultraviolet divergencies are present but their character is the same as in the flat limit.

As an application we have calculated the fermionic propagator and the fermionic vacuum condensate: in the flat-limit, the latter vanishes, at variance with the behaviour in the vector Schwinger model: no vacuum degeneracy is present in our general case, confirming the very different structure between an anomalous (but still unitary) theory and a gauge invariant one.

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9 Appendix A

In this appendix we discuss the dependence of our results on the zero mode matrix eq.(109): in the standard gauge invariant situation, the regularized product of the non vanishing eigenvalues develops a term depending on the zero modes themselves [26]. This term induces a non local and non linear self–interaction on the gauge fields, seemingly jeopardizing the solubility of the theory. In ref.[29] the decoupling of this complicated interaction was proved on the correlation function: it turns out that it can be expressed as the determinant of the inverse matrix of the zero modes, appearing as an effect of the source term in the generating functional eq.(110). Our problem is slightly different: in order to introduce the Jackiw–Rajaraman parameter, we use a gauge–non invariant definition of the fermionic determinant eq.(112), so that an explicit calculation is needed to prove the decoupling of such a kind of interaction from the correlation functions.

We have essentially to compute the ratio:

$$\frac{\exp[B_1]}{\exp[B_2]},$$

where

$$B_1 = - \sum_{n=1}^{[n]} \int_0^1 dt \int d^2 x \sqrt{g} \varphi_{0m}^{(n)}(x,t) \left[ e(1 + \alpha) \phi \gamma_5 + (1 - \alpha) h(t) \partial_t h^{-1}(t) \right] \varphi_{0m}^{(n)}(x,t) -$$

$$- \sum_{m=1}^{[n]} \int_0^1 dt \int d^2 x \sqrt{g} \chi_{0m}^{(n)}(x,t) \left[ e(1 + \alpha) \phi \gamma_5 - (1 - \alpha) h^{-1}(t) \partial_t h(t) \right] \chi_{0m}^{(n)}(x,t),$$

(180)
and

\[
B_2 = - \sum_{m=1}^{n} \int_{0}^{1} dt \int d^2 x \sqrt{g} \chi_{0m}^{(n)\dagger}(x, t) \left[ \epsilon \alpha \phi \gamma_5 \chi_{0m}^{(n)}(x, t) \right],
\]

(see eq. (110) and eq. (121) respectively). It turns out to be that:

\[
B_1 - B_2 = D_1 + D_2
\]

\[
D_1 = - \sum_{m=1}^{n} \int_{0}^{1} dt \int d^2 x \sqrt{g} \varphi_{0m}^{(n)\dagger}(x, t) \left[ 2 \epsilon \phi \gamma_5 \varphi_{0m}(x, t) \right]
\]

\[
D_2 = - \sum_{m=1}^{n} \int_{0}^{1} dt \int d^2 x \sqrt{g} \chi_{0m}^{(n)\dagger}(x, t) \left[ \epsilon (1 - \alpha) (\phi \gamma_5 + h(t) \partial_t h^{-1}(t)) \right] \varphi_{0m}(x, t) + \sum_{m=1}^{n} \int_{0}^{1} dt \int d^2 x \sqrt{g} \chi_{0m}^{(n)\dagger}(x, t) \left[ \epsilon (1 - \alpha) (\phi \gamma_5 + h^{-1}(t) \partial_t h(t)) \right] \chi_{0m}(x, t).
\]

(183)

Let us consider firstly \( D_1 \): from the orthonormality of \( \chi_{0m}^{(n)}(x, t) \) we get

\[
\sum_{m=1}^{n} \sum_{k=1}^{n} B_{im}(t) B_{jk}(t) \int d^2 x \sqrt{g} \chi_{0k}^{(n)\dagger}(x) \exp \left[ -2 \epsilon t \phi \gamma_5 \right] \chi_{0m}^{(n)}(x) = \delta_{ij},
\]

(184)

where we have defined the interpolating matrix:

\[
\chi_{m}^{(n)}(x, t) = u \exp \left[ -2 \epsilon t \phi \gamma_5 \right] \sum_{j=1}^{n} B_{mj}(t) \chi_{0j}^{(n)}(x),
\]

(185)

\[
B(0) = \mathds{1},
\]

\[
B(1) = B.
\]

(186)

Eq. (185) implies the relations:

\[
B(t) C^T(t) B(t) = \mathds{1},
\]

\[
| \det B(t) |^2 = (\det C(t))^{-1},
\]

(187)

the matrix \( C(t) \) being:

\[
C_{ij}(t) = \int d^2 x \sqrt{g} \chi_{0i}^{(n)\dagger}(x) \exp \left[ -2 \epsilon t \phi \gamma_5 \right] \chi_{0j}^{(n)}(x).
\]

(188)

We are able to express \( D_1 \) in a very compact way:

\[
D_1 = \sum_{m,k,j=1}^{n} \int_{0}^{1} dt B_{mj}(t) B_{mk}(t) \int d^2 x \sqrt{g} \chi_{0j}^{(n)\dagger}(x) \exp \left[ -2 \epsilon t \phi \gamma_5 \right] \chi_{0k}^{(n)}(x),
\]

\[
= Tr \left[ \int_{0}^{1} dt B(t) \frac{d}{dt} \left( C^T(t) \right) B(t) \right].
\]

(189)
Eq. (187) leads to:

\[ D_1 = Tr \left[ \int_0^1 dt \frac{d}{dt} \left( C^T(t) \right) C^{-1}(t) \right] = Tr \log C^T(1), \tag{190} \]

giving

\[ \exp D_1 = \det C(1) = |\det B|^{-2}. \tag{191} \]

This is the term appearing in eq. (124), that cancels the contribution of the sources. We have now to prove the vanishing of \( D_2 \); we define the matrix \( E \):

\[ \varphi_{0m}^{(n)}(x,t) = \sum_{m=1}^{n} E_{mk}(t) \exp[(1 - \alpha) t (\epsilon \phi \gamma_5 - i \beta)] \chi_{0k}^{(n)}(x,t), \tag{192} \]

\( \beta \) being related to the gauge dependent part of \( \hat{a}_\mu \) in eq. (139). We can easily verify the consistency of eq. (59) being

\[ \exp[-(1 - \alpha) t (\epsilon \phi - i \beta)] \varphi_{0m}^{(n)}(x,t) \in \text{Ker}[\hat{D}^{(n)}(t)]. \tag{193} \]

From the relations:

\[ \int d^2 x \sqrt{g} \chi_{0j}^{(n) \dagger}(x,t) \exp[-(1 - \alpha) t (\epsilon \phi \gamma_5 - i \beta)] \varphi_{0m}^{(n)}(x,t) = E_{mj} \tag{194} \]

and

\[ \int d^2 x \sqrt{g} \varphi_{0j}^{(n) \dagger}(x,t) \exp[(1 - \alpha) t (\epsilon \phi \gamma_5 + i \beta)] \chi_{0m}^{(n)}(x,t) = E_{mj}^*, \tag{195} \]

we obtain:

\[ \int d^2 x \sqrt{g} \varphi_{0j}^{(n) \dagger}(x,t) \exp[(1 - \alpha) t (\epsilon \phi \gamma_5 + i \beta)] \varphi_{0m}^{(n)}(x,t) = Tr[\hat{E} \frac{d}{dt} \hat{E}^{-1}]^* \tag{196} \]

and

\[ - \int d^2 x \sqrt{g} \chi_{0j}^{(n) \dagger}(x,t) \exp[(1 - \alpha) t (\epsilon \phi \gamma_5 - i \beta)] \chi_{0m}^{(n)}(x,t) = D_{mj} = Tr[\hat{E}^{-1} \frac{d}{dt} \hat{E}]^*. \tag{197} \]

Taking into account eq. (196) and eq. (197) we find the desired result:

\[ D_2 = \int_0^1 dt Tr \left[ \frac{d}{dt} (\hat{E} \hat{E}^{-1}) \right]^* = 0. \tag{198} \]

10 Appendix B

In this appendix we sketch the procedure to get

\[ \det \left[ \hat{D}^{(n)} \right] \]
as defined in eq. (112): we do not give the details of the computation, relying on a careful use of the techniques derived in [28] and [29], and only show the relevant steps to arrive to eqs. (115, 116) and eqs. (120, 121).

Let us define:

\[ \hat{D}^{(n)}(t) = h(t) \exp[e t \phi \gamma_5] \hat{D}_0^{(n)} \exp[e t \phi \gamma_5] h^{-1}(t) \]  

and

\[ \hat{D}_\alpha^{(n)}(t) = h(\alpha t) \exp[e \phi \alpha t \gamma_5] \hat{D}_0^{(n)} \exp[e \alpha t \phi \gamma_5] h^{-1}(\alpha t). \]  

\( h(t) \) interpolates along the \( U(1) \)-valued functions between \( h \) and the identity (remember that \( \pi_2(S^1) = 0 \)). Then

\[ \omega'(t) = \frac{d}{dt} \ln \left[ \det' \left[ \hat{D}^{(n)} \hat{D}_\alpha^{(n)} \right] \right], \]

so that eq. (113) follows. We notice that the problem is now reduced to compute a derivative, that is the infinitesimal variation of a determinant in the present case, and to an integration over a real parameter. The subtle point relies in the fact that one has to consider only the non-vanishing eigenvalues in the \( \zeta \)-function definition of the determinant.

The correct procedure was found in [28] where the authors define:

\[ \det [O] = \lim_{\epsilon \to 0} \frac{1}{\chi N} \det [O + \epsilon \mathbb{1}], \]

\( N \) being the dimension of the kernel. The applications of the theorems developed there to our operator leads to:

\[
\begin{align*}
\omega'(t) &= Tr \left[ \left[ \hat{D}^{(n)}(t) \hat{D}_\alpha^{(n)}(t) \right]^{-s-1} A_1 \right]_{s=0} \\
&\quad - \int d^2x \sqrt{g} Tr \left[ e(1 + \alpha) \phi \gamma_5 + (1 - \alpha) h(t) \partial_t h^{-1}(t) \right] P_1(x,x) \\
&\quad - \int d^2x \sqrt{g} Tr \left[ e(1 + \alpha) \phi \gamma_5 - (1 - \alpha) h^{-1}(t) \partial_t h(t) \right] P_2(x,x),
\end{align*}
\]

where

\[
\begin{align*}
A_1 &= \left[ e(1 + \alpha) \phi \gamma_5 + h(t(1 - \alpha)) \frac{d}{dt} h^{-1}(t(1 - \alpha)) \right] \hat{D}^{(n)}(t) \hat{D}_\alpha^{(n)}(t) + \\
&\quad + \hat{D}^{(n)}(t)[e(1 + \alpha) \phi \gamma_5 - h(t(1 - \alpha)) \frac{d}{dt} h^{-1}(t(1 - \alpha))] \hat{D}_\alpha^{(n)}(t)
\end{align*}
\]

and \( P_1(x,x), P_2(x,x) \) are respectively the projectors on the kernel of \( \hat{D}^{(n)} \hat{D}_\alpha^{(n)} \) and \( \hat{D}_\alpha^{(n)} \hat{D}^{(n)} \): the computation of the functional and algebraic traces leads to eq. (116). Along the same line the calculation for \( \det' [\hat{D}^{(n)}] \) follows.

11 Appendix C

We report the explicit \( \zeta \)-function calculation of the determinant in eq. (123): we feel that our procedure is clearer than the original one, presented in [23].

The eigenvalue equation for the operator \( \hat{D}_0^{(n)} \) gives the result [92] (we take \( n \) positive for sake of simplicity):
Eigenvalues: \( \pm \sqrt{l(l+1)}, \quad l > 0 \)

Multiplicity: \( 2l + 1 \).

The relevant \( \zeta \)-function is (we compute essentially \( \det' \left[ \hat{D}_0^{(n)} \right]^2 \)):

\[
\zeta(s) = \sum_{l=1}^{\infty} (2l + n)[l(l+n)]^{-s}.
\] (205)

By performing a binomial expansion we get:

\[
\zeta(s) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(s+k)}{k! \Gamma(s)} n^k l^{-s-k} l^{-s}(2l+n)
\]

\[
= 2 \zeta_R(1+2s) + n(1-2s)\zeta_R(2s) + n^2 s^2 \zeta_R(1+2s)
\]

\[
+ s \sum_{k=3}^{\infty} \frac{(-1)^k \Gamma(s+k)}{k! \Gamma(s+1)} 2s + k - 2 s \zeta_R(2s+k-1).
\] (206)

The series converges for \( s = 0 \): the first terms are defined by analytic continuation \( (\zeta_R(s)) \):

\[
F(s,n) = \sum_{k=3}^{\infty} \frac{(-1)^k \Gamma(s+k)}{k! \Gamma(s+1)} 2s + k - 2 s \zeta_R(2s+k-1),
\]

\[
\lim_{s \to 0} F(s,n) = F(0,n),
\] (207)

and

\[
\zeta'(0) = 4 \zeta_R'(1) + [ -2n \zeta_R(0) + 2n \zeta_R'(0) ] +
\]

\[
+ n^2 \frac{d}{ds} \left[ s^2 \left( \frac{1}{2s} - \psi(1) + O(s) \right) \right]_{s=0} + F(0,n),
\]

\[
= 4 \zeta_R'(1) + n - n \log 2\pi + \frac{n^2}{2} + F(0,n).
\] (208)

Let us compute \( F(0,n) \):

\[
F(0,n) = \sum_{k=3}^{\infty} \frac{(-1)^k \Gamma(k)}{k(k-1)} n^k \zeta_R(k-1) =
\]

\[
= \sum_{k=3}^{\infty} \int_0^\infty dt \frac{e^{-t}}{1 - e^{-t}} \left[ t^{k-2}(k-1) \left( \frac{-1}{k!} n^k - t^{k-2} \frac{(-1)^k}{k!} n^k \right) \right] =
\]

\[
= \sum_{k=3}^{\infty} \int_0^\infty dt \frac{e^{-t}}{1 - e^{-t}} \frac{d}{dt} \left[ \frac{1}{k!} \frac{d}{dt} \left( t^{k-1} \frac{(-1)^k}{k!} n^k \right) - t^{k-2} \frac{(-1)^k}{k!} n^k \right] =
\]

\[
= \int_0^\infty dt \frac{e^{-t}}{1 - e^{-t}} \left[ \frac{2}{t^2} - 2 \frac{e^{-nt}}{t} - \frac{n}{t} - \frac{e^{-nt}}{t} \right] =
\] (209)
It is very easy to verify the convergence of the integral for \( t = 0 \). Let us define the function:

\[
G(x) = \int_0^\infty \frac{dt}{t^2} \frac{e^{-t}}{1 - e^{-t}} [2 - 2e^{-xt} - tx - xte^{-xt}]
\]

\[
\frac{dG(x)}{dx} = \int_0^\infty \frac{dt}{t} \frac{e^{-t}}{1 - e^{-t}} [-1 + e^{-xt} + xte^{-xt}]
\]

\[
G(0) = 0.
\] (210)

All this implies that:

\[
F(0, n) = \int_0^n dx \frac{dG(x)}{dx}.
\] (211)

We compute the derivative:

\[
\frac{dG(x)}{dx} = \int_0^\infty \frac{dt}{t} \left[ \frac{e^{-(1+x)t}}{1 - e^{-t}} + xe^{-t} \right] + \int_0^\infty \frac{dt}{t} \left[ xte^{-(1+x)t} - xe^{-t} \right]
\]

\[
= \log \Gamma(1 + x) + x \int_0^\infty \frac{dt}{t} \left[ \frac{te^{-(1+x)t}}{1 - e^{-t}} - e^{-t} \right] = 
\]

\[
= \log \Gamma(1 + x) + x \int_0^1 \frac{dy}{\log y} \left[ \frac{\log y e^x}{1 - y} + 1 \right] = 
\]

\[
= \log \Gamma(1 + x) - x \psi(1 + x).
\] (212)

The remaining integration is not difficult [33]:

\[
F(0, n) = \int_0^n dx \log \Gamma(1 + x) - \int_0^n dx x \psi(1 + x) = 
\]

\[
= \int_0^n dx \log \Gamma(1 + x) - n \log \Gamma(1 + n)
\]

\[
F(0, n) = \sum_{k=1}^n 2k \log k + n \log 2\pi - n - n^2 - n \log n!.
\] (213)

The final result is:

\[
\zeta'(0) = 4\zeta_R(-1) - \frac{n^2}{2} - n \log n + \sum_{k=1}^n 2k \log k.
\] (214)
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