Curvature estimates for convex solutions of some fully nonlinear Hessian-type equations

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Abstract
In the research of fully nonlinear elliptic partial differential equations, the concavities of the nonlinear operators is always essential. The concavity is the key to establish the curvature estimates of these equations. However, very few concave operators are known. In the present paper, some new examples are provided, which are the sum Hessian operators and the linear combination of \( k \) Hessian operators. We obtain the concavities and the quotient concavities of these new operators. As an application of our concavities, we establish the curvature estimates of convex solutions for the equations with general right-hand side defined by these new operators.

Mathematics Subject Classification 35J60 · 53C42

1 Introduction
For fully nonlinear elliptic partial differential equations(PDEs), one of the central issues is establishing the curvature(\( C^2 \)) estimates. The most popular fully nonlinear elliptic equations may be \( k \)-Hessian equations. This section presents the context of our study before providing a
short review of the curvature estimates for Hessian equations. Suppose \( M \) is an \( n \)-dimensional compact hypersurface in the Euclidean space \( \mathbb{R}^{n+1} \). We let \( \nu(X), \kappa(X) \) be the outer-normal and principal curvatures of the hypersurface \( M \subset \mathbb{R}^{n+1} \) at position vector \( X \), respectively. The prescribed \( k \)-Hessian curvature equations are

\[
\sigma_k(\kappa(X)) = \psi(X, \nu),
\]

for \( 1 \leq k \leq n \). When \( k = 1 \), the curvature estimate comes from the theory of quasilinear PDEs. If \( k = n \), the curvature estimate for general \( \psi(X, \nu) \) is based on the work of Caffarelli et al. [7]. For \( \psi \) independent of the normal vector, the \( C^2 \) estimate was proven by Caffarelli et al. [9,10] for a general class of fully nonlinear operators, including Hessian type and quotient Hessian type. Ivochkina [32,33] considered the Dirichlet problem of Eq. (1.1) on domains in \( \mathbb{R}^n \), and the \( C^2 \) estimate was proved under several extra conditions on the dependence of \( \psi \) on \( \nu \). Chou and Wang [13] conducted an in-depth study on \( C^2 \) estimates for Hessian equations. In [47], the interior curvature estimates have been established for a large class of curvature equations including \( k \)-Hessian equations by Sheng, Urbas and Wang. The \( C^2 \) estimate was also proved for the equation used in prescribing curvature measures problem in [24,25]. If the function \( \psi \) is convex with respect to the normal \( \nu \), the global \( C^2 \) estimate has been obtained by Guan [18]. For general right-hand sides, it is still a conjecture formally proposed by Guan et al. [24]. Recently, this conjecture has made some progress. Guan et al. [26] obtained global \( C^2 \) estimates for all closed convex hypersurfaces and partially solved this long-standing problem. In the same paper [26], they also established the estimate for all star-shaped 2-convex hypersurfaces. In [36], Li, Ren and Wang showed that the condition “convex” in [26] can be substituted by the condition “\( k + 1 \)- convex” for any \( k \)-Hessian equation. In [46], Ren and Wang completely solved the case \( k = n - 1 \), specifically by obtaining the global curvature estimate of \( n - 1 \) convex solutions for \( n - 1 \) Hessian equations. In [11], Chen, Li, and Wang attempted to derive these estimates for the non-Codazzi second fundamental form of a hypersurface in the general warped product space. For 2-Hessian equations, Spruck and Xiao [48] extended the Euclidean space in [26] to space forms and provided a simple proof for the curvature estimates. We also note the recently important work on the curvature estimates and \( C^2 \) estimates developed by Guan [19] and Guan et al. [27].

The Hessian equations have numerous applications in the literature. The famous Minkowski problem, namely, prescribed Gauss–Kronecker curvature on the outer normal, has been widely discussed in [12,39,43,44]. Alexandrov also posed the problem of prescribing general Weingarten curvature on outer normals, seeing [2,20]. The prescribing curvature measures problem in convex geometry also has been extensively studied in [1,24,25,43]. In [3,9,49], the prescribing mean curvature problem and Weingarten curvature problem also have been considered and obtained fruitful results.

The estimates of Hessian equations with the general right-hand side have been used in many new geometric applications recently. Phong et al. [40,41] have considered Fu–Yau equations in high-dimensional spaces, which is a complex 2-Hessian equations with the right-hand side depending on the gradient terms. Phong et al. [41,42] obtained their \( C^2 \) estimates using the idea of [26]. Guan and Lu [22] considered the curvature estimate for the isometric embedding system in general Riemannian manifolds, which can reduce to a 2-Hessian equation with the right-hand side depending on the normal condition. The estimate in [26] is also applied in [5,54].

In the previous mentioned papers, the concavities and the quotient concavities of \( k \)-Hessian operators are very important to establish the curvature estimates. In the papers studying the fully nonlinear partial differential equations, the assumption of the concavity appears almost in every paper [8,10,17,18,37,50–52]. Until to now, the well known nonlinear concave
operators only are $k$-Hessian operators $\sigma_k^1$ and quotient Hessian operators $\left(\frac{\sigma_k}{\sigma_l}\right)^{1/(k-l)}$. If we aim to study more general fully nonlinear elliptic partial differential equations, the first step seems to understand the concavities and admissible sets of these operators.

In the present paper, we consider the following general curvature equations:

$$Q(\kappa(X)) = \psi(X, \nu(X)), \quad (1.2)$$

where $Q$ is a symmetric function with respect to $\kappa_1, \ldots, \kappa_n$. The function $Q$ needs certain restrictions. We mainly discuss the convex solutions in the present paper. $Q$ is elliptic in the convex cone, namely,

$$Q > 0, \quad \text{and} \quad Q_{ii} = \frac{\partial Q}{\partial \kappa_i} > 0 \quad \text{for} \quad \kappa_i \geq 0.$$ 

Hence, as the first step, the simplest choice for $Q$ to satisfy the above requirements may be the linear combination of $k$ Hessian operators

$$Q(\kappa) = \sum_{s=0}^{k} \alpha_s \sigma_s(\kappa), \quad (1.3)$$

where $\alpha_s$ are non-negative constants and $\alpha_k > 0, k \leq n$.

We mainly consider the “quotient concavity” of $Q(\kappa)$, which corresponds to the $k$ Hessian equations. Thus, we introduce the following concept:

**Definition 1 (Quotient Concavity).** Suppose that $k-1$ polynomials $S_1, \ldots, S_{k-1}$ are defined by

$$S_l(\kappa) = \sigma_l + \sum_{s=0}^{l-1} \beta_s^l \sigma_s(\kappa), \quad \text{for} \quad 1 \leq l \leq k - 1,$$

where $\beta_s^l \geq 0$ are constants depending on indices $s, l$. If the function $Q$ defined by (1.3) satisfies that $(Q/S_l)^{1/(k-l)}(\kappa)$ are concave functions with respect to variables $\kappa = (\kappa_1, \ldots, \kappa_n)$, we call that the operator $Q$ is quotient concave.

For the simplest case in which the left hand side consists of two terms in (1.5),

$$Q_k^l(X) := \alpha_\sigma_{k-1}(\kappa(X)) + \sigma_l(\kappa(X)) = f(X, \nu(X)), \quad \forall X \in M, \quad (1.4)$$

where $1 \leq k \leq n$ and the constant $\alpha$ is non-negative. The equations of this type are called sum Hessian equations. In Sect. 3, we will study the concavity of these equations. In fact, we will prove the following:

**Theorem 2** The sum Hessian equations satisfy the quotient concavities in their admissible sets.

The concept of the admissible set, which including the convex cone, will be defined in Sect. 3.

For the general case, the quotient concavity may not hold. For example, the operator $\sigma_k + \sigma_{k-1} + \sigma_{k-2}$ is not quotient concave, even in the convex cone. Further more, the operator $(\sigma_k + \sigma_{k-1} + \sigma_{k-2})^{1/n}$ is not a concave function. Therefore, we introduce the following Condition (C) to recover quotient concavity in Sect. 5.
Condition (C) There is some “non-negative” vector $b \in \mathbb{R}^N$, where $N \geq k$ is a positive integer such that for $m = 0, 1, \ldots, k$, we have

$$
\alpha'_m = \frac{(n-k)!\alpha_{k-m}}{(n-k+m)!} = \sigma_m(b).
$$

Here, the “non-negativity” means that every component of the vector $b$ is non-negative. If $N = k$, the non-negativity of $\alpha_0, \alpha_1, \ldots, \alpha_k$ implies the non-negativity of $b$. In other words, our condition is equivalent to the following polynomial with real constant coefficients $\alpha_0', \alpha_1', \ldots, \alpha_k'$:

$$
\sum_{m=0}^{N} \alpha'_m t^m = 0
$$

only having negative real roots.

Note that a polynomial that satisfies Condition (C) is similar to hyperbolic polynomials [16,31]. Using Condition (C), in Sect. 5, we will prove the following:

**Theorem 3** If the coefficients of a $k$-th degree polynomial $Q$ defined by (1.3) satisfy Condition (C), then $Q$ satisfies the quotient concavities in cone $\Gamma_k$.

Here, $\Gamma_k$ is the Gårding cone which is defined in Sect. 2. Theorems 2 and 3 study the concavity of the sum Hessian operators and the linear combination of the Hessian operators.

Using the quotient concavity, we obtain the curvature estimates of convex solutions for the equation defined by (1.3) in Sect. 3.

**Theorem 4** Suppose $M \subset \mathbb{R}^{n+1}$ is a closed convex hypersurface that satisfies the curvature equation

$$
Q(\kappa(X)) = \sum_{s=0}^{k} \alpha_s \sigma_s(\kappa(X)) = \psi(X, \nu(X))
$$

with coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \geq 0$ and $\alpha_k > 0$ for some positive function $\psi(X, \nu) \in C^2(\Gamma)$, where $\Gamma$ is an open neighborhood of unit normal bundle of $M$ in $\mathbb{R}^{n+1} \times S^n$. We further assume that the polynomial $Q$ satisfies the quotient concavity, then a constant $C$ depends only on $n$, $k$, $\|M\|_{C^1}$, $\inf \psi$ and $\|\psi\|_{C^2}$, such that

$$
\max_{X \in M, i=1, \ldots, n} \kappa_i(X) \leq C.
$$

The equations defined by linear combinations of the $k$ Hessian operators have been studied and applied in geometrical problems. Harvey and Lawson [29] considered the special Lagrangian equations among this category. Krylov [34] and Dong [14] also considered equations close to the above type and obtained the curvature estimates using the concavity of the operators. Guan and Zhang [23] studied the curvature estimates for equations of such type with the right-hand side not depending on gradient term but with coefficients depending on the position of the hypersurfaces. The geometrical problems in the hyperbolic space also reduce to equations of this type [15].

According to Theorem 4, the curvature estimates only require a suitable condition to satisfy the quotient concavity.

Combining Theorems 4 and 3, we obtain the following main result for convex solutions in the present paper:
Theorem 5 Suppose $M \subset \mathbb{R}^{n+1}$ is a closed convex hypersurface satisfying curvature Eq. (1.5) with coefficients $\alpha_0, \ldots, \alpha_{k-1} \geq 0$ and $\alpha_k > 0$ satisfying Condition (C) for some positive function $\psi(X, v) \in C^2(\Gamma)$, where $\Gamma$ is an open neighborhood of unit normal bundle of $M$ in $\mathbb{R}^{n+1} \times S^n$, then a constant $C$ depends only on $n, k, \|M\|_{C^1}, \inf \psi$ and $\|\psi\|_{C^2}$, such that

$$\max_{X \in M, j=1, \ldots, n} \kappa_i(X) \leq C. \quad (1.7)$$

Given an appropriate barrier, we obtain the following existence result for the sum Hessian equations from the preceding theorem:

Theorem 6 For Eq. (1.4), assume that the coefficient $\alpha \geq 0$. Suppose $\psi \in C^2(\mathbb{R}^{n+1} \times S^n)$ is a positive function and suppose a constant $r > 1$ exists such that

$$\psi\left(X, \frac{X}{|X|}\right) \leq \frac{Q(1, \ldots, 1)}{r^k} \quad \text{for} \quad |X| = r, \quad (1.8)$$

and $\psi^{-1/k}(X, v)$ is locally convex in $X \in B_r(0)$ for any fixed $v \in S^n$, then the Eq. (1.4) has a strictly convex $C^{3,\alpha}$ solution inside $\tilde{B}_r$.

These results can be extended to the Dirichlet problem of the equations defined by the linear combination of the Hessian operators, which we will discuss in the last section.

The rest of the paper is organized as follows: In Sect. 2, we explain more notations and list several needed lemmas. In Sect. 3, we establish the curvature estimates of convex solutions of some fully nonlinear parabolic equations from the preceding theorem: In Sect. 4, we study the admissible set and concavity of the sum Hessian equations. Section 5 mainly studies the “quotient concavity” of a polynomial defined by (1.3) if its coefficients satisfy Condition (C). The last section proves Theorem 6 and discusses the parallel Dirichlet problems.

2 Preliminary

The $k$-th elementary symmetric function is defined by, for $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{R}^n$ and $1 \leq k \leq n$,

$$\sigma_k(\kappa) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}. \quad (2.1)$$

We also set $\sigma_0(\kappa) = 1$ and $\sigma_k(\kappa) = 0$ for $k > n$. Based on [8], a suitable definition domain of $\sigma_k$ is Gårding’s cone $\Gamma_k$, which is an open, convex, symmetric (invariant under the interchange of any two $\kappa_i$) cone with its vertex at the origin, containing the positive cone $\{\kappa \in \mathbb{R}^n | \text{each component } \kappa_i > 0, 1 \leq i \leq n\}$.

Definition 7 For a domain $\Omega \subset \mathbb{R}^n$, a function $v \in C^2(\Omega)$ is called $k$-convex if the eigenvalues $\lambda(x) = (\lambda_1(x), \ldots, \lambda_n(x))$ of the Hessian $\nabla^2 v(x)$ is in $\Gamma_k$ for all $x \in \Omega$, where $\Gamma_k$ is Gårding’s cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n | \sigma_m(\lambda) > 0, \quad m = 1, \ldots, k\}. \quad (2.2)$$

A $C^2$ regular hypersurface $M \subset \mathbb{R}^{n+1}$ is $k$–convex if its principal curvature $\kappa(X) \in \Gamma_k$ for all $X \in M$.

Korevaar [35] showed that the cone $\Gamma_k$ also can be characterized as

$$\left\{ \kappa \in \mathbb{R}^n; \sigma_k(\kappa) > 0, \frac{\partial \sigma_k(\kappa)}{\partial \kappa_{i_1}} > 0, \cdots, \frac{\partial^k \sigma_k(\kappa)}{\partial \kappa_{i_1} \cdots \partial \kappa_{i_k}} > 0, \text{for all } 1 \leq i_1 < \cdots < i_k \leq n \right\}. \quad (2.3)$$
The definition of $\Gamma_k$ shows that

$$
\Gamma_n \subset \cdots \subset \Gamma_k \cdots \subset \Gamma_1.
$$

We explain additional notations. We let $\kappa(A)$ represent the eigenvalues of a matrix $A = (a_{ij})$. Suppose $F$ is a function defined on a set of symmetric matrices. We let

$$
f(\kappa(A)) = F(A).
$$

Thus, we denote

$$
F_{pq} = \frac{\partial F}{\partial a_{pq}}, \text{ and } F_{pq,rs} = \frac{\partial^2 F}{\partial a_{pq}\partial a_{rs}}.
$$

For a local orthonormal frame, if $A$ is diagonal at a point, then at this point, we have

$$
F_{pp} = \frac{\partial f}{\partial \kappa_p} = f_p, \text{ and } F_{pq-qq} = \frac{\partial^2 f}{\partial \kappa_p\partial \kappa_q} = f_{pq}.
$$

Thus, the definition of the $k$-th elementary symmetric function can be extended to symmetric matrices. Suppose $W$ is an $n \times n$ symmetric matrix and $\kappa(W)$ represents its eigenvalues. We define

$$
\sigma_k(W) = \sigma_k(\kappa(W)),
$$

which is the summation of the $k$-th principal minors of the matrix $W$.

We present some algebraic identities and properties of $\sigma_k$. For an index $1 \leq l \leq n$, the notation $\sigma_l(\kappa|ab\ldots)$ means the $l$-th elementary symmetric function of $\kappa_1, \kappa_2, \ldots, \kappa_n$ with $\kappa_a = 0, \kappa_b = 0, \ldots$. Thus, we have

(i) $\sigma_k^{pp}(\kappa) := \frac{\partial \sigma_k(\kappa)}{\partial \kappa_p} = \sigma_{k-1}(\kappa|p)$ for any $p = 1, \ldots, n$;

(ii) $\sigma_k^{pp,qq}(\kappa) := \frac{\partial^2 \sigma_k(\kappa)}{\partial \kappa_p\partial \kappa_q} = \sigma_{k-2}(\kappa|pq)$ for any $p, q = 1, \ldots, n$ and $\sigma_k^{pp,pp}(\kappa) = 0$;

(iii) $\sigma_k(\kappa) = \kappa_1\sigma_{k-1}(\kappa|1) + \sigma_k(\kappa|1)$ for any fixed index $i$;

(iv) $\sum_{i=1}^n \kappa_i\sigma_{k-1}(\kappa|i) = k\sigma_k(\kappa)$.

Thus, for a Codazzi tensor $W = (w_{ij})$, if $W$ is diagonal, we have

(v) $-\sum_{p,q,r,s} \sigma_k^{pq,rs} w_{pql}w_{rsl} = \sum_{p,q} \sigma_k^{pp,qq} w_{pq}^2 - \sum_{p,q} \sigma_k^{pp,qq} w_{ppl}w_{qql}$,

where $w_{pq}$ means the covariant derivative of $w_{pq}$ and $\sigma_k^{pq,rs} = \frac{\partial^2 \sigma_k(W)}{\partial w_{pq}\partial w_{rs}}$. The meaning of the Codazzi tensor can be found in [26]. For $\kappa \in \Gamma_k$, if we let $\kappa_1 \geq \cdots \geq \kappa_n$, then we have

(vi) $\sigma_{k-1}(\kappa|n) \geq \cdots \geq \sigma_{k-1}(\kappa|1) > 0$;

(vii) $\kappa_1\sigma_{k-1}(\kappa|1) \geq C_{n,k}\sigma_k(\kappa)$,

where $C_{n,k}$ is a positive constant depending only on $n, k$. More details of the proof of these formulas can be found in [28,53].

Two important concavities exist for $\sigma_k$, which is essential to our proof. In [8,51], the function $\sigma_k^{1/k}(\kappa)$ is a concave function in $\Gamma_k$ and the function $\left(\frac{\sigma_k(\kappa)}{\sigma_{l}(\kappa)}\right)^{1/(k-l)}$ for $l < k$ is also a concave function in $\Gamma_k$.

Now, we give the following two Lemmas, which will be needed in our proof. Suppose $Q$ is a polynomial defined by (1.3) with coefficients $\alpha_0, \ldots, \alpha_{k-1} \geq 0$ and $\alpha_k > 0$. If $Q$ satisfies the quotient concavity, we have the following lemma similar to the $k$-Hessian operators.
Lemma 8 Assume that $k > l$, $W = (w_{ij})$ is a Codazzi tensor in $\Gamma_k$. Denote $\beta = \frac{1}{k - l}$. Suppose the operator $Q$ defined by (1.3) satisfies the quotient concavity, then, for $h = 1, \ldots, n$, we have the following inequality:

\[-\sum_{p, q} \frac{Q^{pp,qq}}{Q}(W)w_{pph}w_{qqh} + \sum_{p, q} \frac{S_l^{pp,qq}}{S_l}(W)w_{pph}w_{qqh} \geq \left( \frac{(Q(W))_h}{Q(W)} - \frac{(S_l(W))_h}{S_l(W)} \right) \left( \beta - 1 \right) \frac{(Q(W))_h}{Q(W)} - (\beta + 1) \frac{(S_l(W))_h}{S_l(W)} \right). \tag{2.3}\]

Here, $S_l$ is a $l$-th degree polynomial defined in the definition of the quotient concavity. Furthermore, for any $\delta > 0$, we have

\[-\sum_{p, q} \frac{Q^{pp,qq}}{Q}(W)w_{pph}w_{qqh} + \left( 1 - \beta + \frac{\beta}{\delta} \right) \frac{(Q(W))^2}{Q(W)} \geq Q(W)(\beta + 1 - \delta\beta) \left[ \frac{(S_l(W))_h}{S_l(W)} \right]^2 - \frac{Q(W)}{S_l(W)} \sum_{p, q} \frac{S_l^{pp,qq}}{S_l}(W)w_{pph}w_{qqh}. \tag{2.4}\]

A symmetric tensor being in $\Gamma_k$ means that its corresponding eigenvalues are in $\Gamma_k$. The other necessary lemma is

Lemma 9 Denote $\text{Sym}(n)$ as the set of all $n \times n$ symmetric matrices. Let $F$ be a $C^2$ symmetric function defined in some open subset $\Psi \subset \text{Sym}(n)$. At any diagonal matrix $A \in \Psi$ with distinct eigenvalues, let $\tilde{F}(B, B)$ be the second derivative of $C^2$ symmetric function $F$ in the direction $B \in \text{Sym}(n)$, and then we have

\[\tilde{F}(B, B) = \sum_{j, k=1}^{n} \tilde{f}^{jk} B_{jj} B_{kk} + 2 \sum_{j<k} \frac{\tilde{f}^{j} - \tilde{f}^{k}}{\kappa_j - \kappa_k} B_{jk}^2. \tag{2.5}\]

Here $\kappa_1, \ldots, \kappa_n$ are the distinct eigenvalues of the matrix $A$. $\tilde{f}^{j}$, $\tilde{f}^{jk}$ are the first and second derivatives of the function $f(\kappa_1, \ldots, \kappa_n) = F(A)$ with respect to $\kappa_j$ and $\kappa_j, \kappa_k$.

The proof of the first lemma is similar to the proof in [24,26]. The proof of the second one can be found in [4] and originally stated in a preliminary version of [8].

3 Sum Hessian equation

In this section, we discuss the sum Hessian equations defined by (1.4), where we always assume that the integer $n \geq k \geq 1$ and the constant $\alpha \geq 0$.

Corresponding to our problem, we need a new convex cone $\tilde{\Gamma}_k$ compatible with our equations. We define

\[\tilde{\Gamma}_k = \Gamma_{k-1} \cap \{ \lambda \in \mathbb{R}^n; \alpha \sigma_{k-1}(\lambda) + \sigma_k(\lambda) > 0 \}. \tag{3.1}\]

Here, $\Gamma_k$ is the $k$-convex Gårding’s cone. Through the definition of $\Gamma_k$ and $\tilde{\Gamma}_k$, it is clear that

$\Gamma_{k-1} \supseteq \tilde{\Gamma}_k \supseteq \Gamma_k$.

This section consists of two parts. In the first part, we prove that the cone $\tilde{\Gamma}_k$ is a suitable set for admissible solutions. In the second part, we prove the quotient concavity that implies the curvature estimate for the sum Hessian equation.
Theorem 10 The set $\tilde{\Gamma}_k$ is a convex set. In cone $\tilde{\Gamma}_k$, Eq. (1.4) with $1 \leq k \leq n$ and $\alpha \geq 0$ is elliptic.

Proof We first prove the convexity of $\tilde{\Gamma}_k$. Suppose $\lambda, \tilde{\lambda} \in \tilde{\Gamma}_k$ and $0 \leq t \leq 1$. We denote
\[ \lambda_t = t\lambda + (1-t)\tilde{\lambda}. \]
By the convexity of $\Gamma_{k-1}$, we have
\[ \sigma_{k-1}(\lambda_t) > 0. \]
In order to prove that
\[ Q^k_S(\lambda_t) = \sigma_k(\lambda_t) + \alpha \sigma_{k-1}(\lambda_t) > 0, \]
one only needs to show
\[ \alpha + \frac{\sigma_k(\lambda_t)}{\sigma_{k-1}(\lambda_t)} > 0. \]
By the concavity of the function $\frac{\sigma_k}{\sigma_{k-1}}$ in $\Gamma_{k-1}$, we have
\[ \alpha + \frac{\sigma_k(\lambda_t)}{\sigma_{k-1}(\lambda_t)} \geq t\alpha + (1-t)\alpha + \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)} + (1-t) \frac{\sigma_k(\tilde{\lambda})}{\sigma_{k-1}(\tilde{\lambda})} > 0. \]

Then, we prove the ellipticity. Guan and Lin [21] observed that the function $\frac{\sigma_k}{\sigma_{k-1}}$ is degenerated elliptic in $\Gamma_{k-1}$ and it is only degenerate on the set $\{\sigma_k = 0\}$, i.e., it is strongly elliptic even in the set $\{\sigma_k < 0\} \cap \Gamma_{k-1}$. In fact, the property of the degenerate ellipticity can be easily obtained by the following calculation. Since we have
\[
\left( \frac{\sigma_k}{\sigma_{k-1}} \right)^i = \frac{\sigma_k^i \sigma_{k-1} - \sigma_k \sigma_{k-1}^{i+1}}{\sigma_{k-1}^2} = \frac{\sigma_{k-1}(\lambda|i)\sigma_{k-1}(\lambda|i) - \sigma_k(\lambda|i)\sigma_k-2(\lambda|i)}{\sigma_{k-1}^2},
\]
then, by Newton’s inequality, it is non-negative. Furthermore, if $\sigma_{k-1}(\lambda|i) \neq 0$, it is positive. Therefore, for Eq. (1.4), a straightforward calculation shows
\[
(\hat{Q}^k_S)^i = \sigma_k^i + \alpha \sigma_k^{i+1} = \sigma_{k-1} \left( \frac{\sigma_k^i}{\sigma_{k-1}} + \alpha \frac{\sigma_k^{i+1}}{\sigma_{k-1}} \right) = \sigma_{k-1} \left( \frac{\sigma_k}{\sigma_{k-1}} \right)^i + \left( \frac{\sigma_k^{i+1}}{\sigma_{k-1}} \right) \left( \alpha + \frac{\sigma_k}{\sigma_{k-1}} \right) > 0,
\]
which provides the ellipticity in $\tilde{\Gamma}_k$. 

Before we discuss the quotient concavity, we need the following lemma:

Lemma 11 $\frac{\sigma_k(\lambda)}{Q^k_S(\lambda)}$ is concave for all $\lambda \in \tilde{\Gamma}_k$ and integer $1 \leq k \leq n$. Here, $Q^k_S$ is defined in (1.4) for $\alpha \geq 0$. Springer
Proof The function \( \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)} \) is concave for \( \lambda \in \Gamma_{k-1} \supseteq \tilde{\Gamma}_k \), which implies that
\[
\alpha + \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)} = \frac{\alpha \sigma_{k-1}(\lambda) + \sigma_k(\lambda)}{\sigma_{k-1}(\lambda)}
\]
is also a concave function. Thus, we obtain that the function
\[
\frac{\sigma_k(\lambda)}{Q^k_S(\lambda)} = 1 - \frac{\alpha \sigma_{k-1}(\lambda)}{\alpha \sigma_{k-1}(\lambda) + \sigma_k(\lambda)}
\]
is concave for \( \lambda \in \tilde{\Gamma}_k \) and non-negative \( \alpha \). \( \square \)

Therefore, we have the following quotient concavity similar to the Hessian equations. Our idea comes from \([28,38]\).

Lemma 12 The quotient function
\[
q_k(\lambda) = \frac{Q_{S}^{k+1}(\lambda)}{Q_{S}^{k}(\lambda)}
\]
is a concave function for all \( \lambda \in \tilde{\Gamma}_k \) and each integer \( 1 \leq k \leq n-1 \). Here, \( Q_S^k \) is defined in (1.4) for \( \alpha \geq 0 \).

Proof We prove our assertion by induction. We initially consider \( q_1 \). Since \( \lambda \in \tilde{\Gamma}_1 \), it is equivalent to
\[
\tilde{\lambda} = \lambda + \frac{\alpha}{n} e \in \Gamma_1,
\]
where \( e = (1, 1, \ldots, 1) \in \mathbb{R}^n \) is an \( n \)-dimensional vector and every entry of \( e \) is 1. One may read off from this expression that
\[
q_1(\lambda) = \frac{\sigma_2(\tilde{\lambda})}{\sigma_1(\tilde{\lambda})} - \frac{n+1}{2n} \alpha^2 + \frac{\alpha}{n}.
\]
Since \( \tilde{\lambda} \in \Gamma_1 \), using the concavity of the quotient functions, \( q_1 \) is a concave function by the above formula.

Now, we assume that \( q_{k-1} \) is concave and we will prove that \( q_k \) is also concave. Using
\[
\sum_i \lambda_i \sigma_k(\lambda_i) = (k + 1) \sigma_{k+1} \quad \text{and} \quad \sigma_k(\lambda_i) = \sigma_k - \lambda_i \sigma_{k-1}(\lambda_i),
\]
we obtain
\[
(k + 1)q_k(\lambda) - \frac{\alpha \sigma_k(\lambda)}{Q^k_S(\lambda)} = \sum_i \lambda_i \frac{Q^k_k(\lambda_i)}{Q^k_S(\lambda)}
\]
\[
= \sum_i \lambda_i \frac{Q^k_S(\lambda) - \lambda_i Q^k_S(\lambda_i)}{Q^k_S(\lambda)}
\]
\[
= \sum_i \left( \lambda_i - \frac{\lambda_i^2}{\lambda_j + q_{k-1; i}(\lambda_j)} \right).
\]
(3.2)
For simplicity, the notation \( q_{k-1}(\lambda|i) \) is replaced by the notation \( q_{k-1;i}(\lambda) \). Take a vector \( \xi \in R^n \) with sufficiently small norm satisfying \( \lambda \pm \xi \in \Gamma_k \). By (3.2), we get
\[
(k + 1)[2q_k(\lambda) - q_k(\lambda + \xi) - q_k(\lambda - \xi)] = \left[ \frac{2\sigma_k}{Q_S^k}(\lambda) - \frac{\sigma_k}{Q_S^k}(\lambda + \xi) - \frac{\sigma_k}{Q_S^k}(\lambda - \xi) \right] - \sum_i \left( \frac{(\lambda_i + \xi_i)^2}{\lambda_i + \xi_i + q_{k-1;i}(\lambda + \xi)} + \frac{(\lambda_i - \xi_i)^2}{\lambda_i - \xi_i + q_{k-1;i}(\lambda - \xi)} \right) - \frac{(2\lambda_i)^2}{2\lambda_i + q_{k-1;i}(\lambda + \xi) + q_{k-1;i}(\lambda - \xi)} + \sum_i \left( \frac{(2\lambda_i)^2}{2\lambda_i + q_{k-1;i}(\lambda + \xi) + q_{k-1;i}(\lambda - \xi)} - \frac{2\lambda_i^2}{\lambda_i + q_{k-1;i}(\lambda)} \right) = \sum_i \left( \frac{[(\lambda_i + \xi_i)q_{k-1;i}(\lambda - \xi) - (\lambda_i - \xi_i)q_{k-1;i}(\lambda + \xi)]^2}{[\lambda_i + \xi_i + q_{k-1;i}(\lambda + \xi)][\lambda_i - \xi_i + q_{k-1;i}(\lambda - \xi)][2\lambda_i + q_{k-1;i}(\lambda + \xi) + q_{k-1;i}(\lambda - \xi)]} \right) + 2 \sum_i \lambda_i^2 \left[ \frac{2q_{k-1;i}(\lambda) - q_{k-1;i}(\lambda + \xi) - q_{k-1;i}(\lambda - \xi)}{[\lambda_i + q_{k-1;i}(\lambda)][2\lambda_i + q_{k-1;i}(\lambda + \xi) + q_{k-1;i}(\lambda - \xi)]} \right].
\]

Using Lemma 11, we have
\[
\frac{2\sigma_k}{Q_S^k}(\lambda) - \frac{\sigma_k}{Q_S^k}(\lambda + \xi) - \frac{\sigma_k}{Q_S^k}(\lambda - \xi) \geq 0.
\]

Thus, combining the two previous formulas, we obtain
\[
(k + 1)[2q_k(\lambda) - q_k(\lambda + \xi) - q_k(\lambda - \xi)] \geq 2 \sum_i \lambda_i^2 \left[ \frac{2q_{k-1;i}(\lambda) - q_{k-1;i}(\lambda + \xi) - q_{k-1;i}(\lambda - \xi)}{[\lambda_i + q_{k-1;i}(\lambda)][2\lambda_i + q_{k-1;i}(\lambda + \xi) + q_{k-1;i}(\lambda - \xi)]} \right],
\]
which implies that, for all \( \xi \), we have
\[
-\frac{\partial^2 q_k}{\partial \xi^2}(\lambda) = \lim_{\varepsilon \to 0} \frac{2q_k(\lambda) - q_k(\lambda + \varepsilon \xi) - q_k(\lambda - \varepsilon \xi)}{\varepsilon^2} \geq \lim_{\varepsilon \to 0} \sum_i \frac{2\lambda_i^2}{k + 1} \frac{2q_{k-1;i}(\lambda) - q_{k-1;i}(\lambda + \varepsilon \xi) - q_{k-1;i}(\lambda - \varepsilon \xi)}{\varepsilon^2[\lambda_i + q_{k-1;i}(\lambda)][2\lambda_i + q_{k-1;i}(\lambda + \varepsilon \xi) + q_{k-1;i}(\lambda - \varepsilon \xi)]} \geq \sum_i \frac{\lambda_i^2}{k + 1} \frac{\partial^2 q_{k-1;i}}{\partial \xi^2}(\lambda) \geq \sum_i \frac{\lambda_i^2}{(k + 1)[\lambda_i + q_{k-1;i}(\lambda)]^2} \geq 0.
\]

Here, \([\xi]_i\), \([\lambda]_i\) denote two new \( n \)-dimensional vectors of which each component is the same as \( \xi \), \( \lambda \) except the \( i \)-th component, which is zero. We obtain the concavity of \( q_k \).

A corollary of the above theorem is that
Corollary 13  For any two integers \( 1 \leq l < k \leq n \), the two functions

\[
\left( \frac{Q^k_S}{Q^l_S}(\lambda) \right)^{\frac{1}{k-l}} \quad \text{and} \quad (Q^k_S(\lambda))^{\frac{1}{k}}
\]

are concave functions in the admissible cone \( \tilde{\Gamma}_k \). Here, \( Q^k_S \) is defined in (1.4) for \( \alpha \geq 0 \).

Proof  By using the definition of \( q_k \) in the previous lemma, we observe that

\[
\left( \frac{Q^k_S}{Q^l_S} \right)^{\frac{1}{k-l}} = (q_{k-1}q_{k-2} \cdots q_l)^{\frac{1}{k-l}},
\]

and

\[
(Q^k_S)^{\frac{1}{k}} = (q_{k-1}q_{k-2} \cdots q_1 Q^1_S)^{\frac{1}{k}}.
\]

Again by the previous lemma, \( q_{k-1}, \ldots, q_1 \) and \( Q^1_S \) are all concave functions; thus, corollary is based on a well-known fact that the geometric mean value of finite numbers of positive concave functions is also a concave function. \( \square \)

Thus, the Corollary 13 implies Theorem 2.

4 Quotient concavity

In view of Sect. 3, the quotient concavity is the key assumption for the curvature estimates. The previous section provides the quotient concavity of the sum Hessian equation. In this section, we try to study the quotient concavity of the general form, namely, linear combination of the Hessian operators.

Suppose \( \alpha_0, \ldots, \alpha_k \) are the coefficients in (1.3). We denote \( \tilde{\alpha}_{k-m} = \alpha_m \), where \( m = 0, 1, \ldots, k \). Without loss of generality, we assume that \( \alpha_k = \tilde{\alpha}_0 = 1 \). Let \( \theta = (1, 1, \ldots, 1) \in \mathbb{R}^n \). For any \( t \in \mathbb{R} \) and any \( x \in \mathbb{R}^n \), we have

\[
\sigma_m(t\theta + x) = \frac{(n-k)!}{(n-m)!} \frac{d^{k-m}}{dt^{k-m}} \sigma_k(t\theta + x).
\]

Therefore, by (1.3), we have

\[
Q(t\theta + x) = \sum_{m=0}^{k} \frac{(n-k)!\tilde{\alpha}_m}{(n-k+m)!} \frac{d^m}{dt^m} \sigma_k(t\theta + x).
\]

Using Condition (C), we find a “non-negative” vector \( b \in \mathbb{R}^N \) and \( N \geq k \) such that for \( m = 0, 1, \ldots, k \), we have

\[
\alpha'_m = \frac{(n-k)!\tilde{\alpha}_m}{(n-k+m)!} = \sigma_m(b). \tag{4.1}
\]

Using the preceding formula, we obtain

\[
Q(t\theta + x) = \prod_{m=1}^{N} \left( 1 + b_m \frac{d}{dt} \right) \sigma_k(t\theta + x),
\]

where \( b_1, \ldots, b_N \) are the components of the vector \( b \).
Lemma 14 For any \( x \in \mathbb{R}^n \), if Condition (C) holds, then the \( k \)-th degree polynomial \( Q(t\theta + x) \) defined by (1.3) always has \( k \) real roots. Furthermore, if \( x \in \Gamma_k \), then all of the \( k \) roots are non-positive.

**Proof** By induction, we only need to prove that the polynomial

\[
\left( 1 + b_m \frac{d}{dt} \right) \sigma_k(t\theta + x)
\]

has \( k \) real roots. For \( 1 \leq m \leq N \), if \( b_m \neq 0 \), since

\[
\left( 1 + b_m \frac{d}{dt} \right) \sigma_k(t\theta + x) = b_m e^{-\frac{t}{b_m}} \frac{d}{dt} \left( e^{\frac{t}{b_m}} \sigma_k(t\theta + x) \right),
\]

then by the generalized Roll’s theorem, we obtain the desired result. In fact, if \( b_m > (\leq) 0 \), then \( e^{\frac{t}{b_m}} \sigma_k(t\theta + x) \) has \( k + 1 \) roots, because \( \sigma_k(t\theta + x) \) has \( k \) roots and \(-\infty(+\infty)\) is another root. If \( b_m = 0 \), it is obvious.

If \( x \in \Gamma_k \) and in fact the vector \( b \) is “non-negative,” then the roots of the equation \( \sigma_k(t\theta + x) = 0 \) are non-positive. In view of the above proof, we have our result.

We define another related polynomial

\[
Q_t(at + x) = \sum_{s=0}^{k} t^{k-s} \alpha_s \sigma_s(at + x).
\]

Thus, we have the following Proposition:

**Proposition 15** Suppose Condition (C) holds. For any \( a \in \Gamma_k \subset \mathbb{R}^n \) and any \( x \in \mathbb{R}^n \), we can write that

\[
Q_t(at + x) = Q(a) \prod_{m=1}^{k} (t + \lambda_m(x, a)),
\]

where \(-\lambda_1(x, a), \ldots, -\lambda_k(x, a)\) are the \( k \) real roots of \( Q_t(at + x) \).

**Proof** The proof is based on the work of [30]. We consider the polynomial

\[
Q_t(s\theta + at + x) = \sum_{m=0}^{k} t^{k-m} \alpha_m \sigma_m(s\theta + at + x)
\]

for \( s \in \mathbb{R} \). The following is clearly demonstrated:

\[
\frac{(n - k)! m^{m} \tilde{\alpha}_m}{(n - k + m)!} = \sigma_m(tb).
\]

Thus, by Lemma 14, we obtain

\[
Q_t(s\theta + at + x) = \sigma_k(\theta) \prod_{m=1}^{k} (s + \lambda_m(at + x)).
\]

Here, every \( \lambda_m(at + x) \) further depends on \( \tilde{\alpha}_0, t\tilde{\alpha}_1, \ldots, t^k\tilde{\alpha}_k \), but we will clarify the meaning of this phrase, namely, using the notion \( \lambda_m(at + x) \) to express \( \lambda_m(at + x, \tilde{\alpha}_0, t\tilde{\alpha}_1, \ldots, t^k\tilde{\alpha}_k) \) without ambiguity.
We know that each $\lambda_m(at + x)$ is a continuous function with respect to $t$. By a same argument as the proof of Lemma 2.10 in [30], we can also prove that $\lambda_m(at + x)$ is real analytic with respect to the variable $t$ by an arrangement of the indices $\{1, 2, \ldots, k\}$. We include their proof in the next lemma. Here, we further explain the meaning of the “arrangement”: If two different functions $\lambda_m(at + x)$ and $\lambda_n(at + x)$ take the same values at some $t_0$, we can define a new function

$$\lambda(at + x) = \begin{cases} 
\lambda_m(at + x) & \text{for } t \leq t_0 \\
\lambda_n(at + x) & \text{for } t > t_0 
\end{cases}, \quad (4.2)$$

which is still a continuous function with respect to $t$, but may not be real analytic. Thus, we need an arrangement of the indices here to let every function become analytic. From now on, the indices for $\lambda_m(at + x)$ is the arrangement for maintaining analyticity.

Now, we follow the argument of Theorem 2.9 in [30]. For $t \neq 0$, we have

$$Q_t(s \theta + at + x) = \sum_{m=0}^{k} t^{k-m} \alpha_{k-m} \sigma_m(s \theta + at + x)$$

$$= t^k \sum_{m=0}^{k} \tilde{\alpha}_{k-m} \sigma_m \left( \frac{s}{t} \theta + a + \frac{x}{t} \right). \quad (4.3)$$

By Lemma 14 and Eq. (4.3), we have

$$Q_t(s \theta + at + x) = t^k \sigma_k(\theta) \prod_{m=1}^{k} \left( s + t \mu_m \left( a + \frac{x}{t}, \tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k \right) \right)$$

$$= \sigma_k(\theta) \prod_{m=1}^{k} s + t \mu_m \left( a + \frac{x}{t}, \tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k \right). \quad (4.4)$$

On the other hand, we have

$$\sum_{m=0}^{k} \tilde{\alpha}_{k-m} \sigma_m(s \theta + a) = \sigma_k(\theta) \prod_{m=1}^{k} \left( s + \mu_m(a) \right).$$

Taking derivatives with respect to $s$ for both sides of the above equality and then letting $s = 0$, we obtain the following:

$$\sigma_l(\mu_1(a), \mu_2(a), \ldots, \mu_k(a)) > 0$$

for $l = 1, 2, \ldots, k$, which implies $\mu_l(a) > 0$. Thus, obviously, if $|t|$ is sufficiently large, for $l = 1, \ldots, k$, by the continuity of every function $\mu_l$, we have

$$\mu_l \left( a + \frac{x}{t}, \tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k \right) > 0.$$ 

Thus, if $t \to +\infty(-\infty)$, then have

$$\lambda_m(at + x) \to +\infty(-\infty).$$ 

The continuity of $\lambda_m(at + x)$ implies that it always has one real solution $t_m$, namely, $\lambda_m(at_m + x) = 0$. Define a set $S = \{t_1, t_2, \ldots, t_k\}$. We claim that if $t_{m_1} = t_{m_2} = \cdots = t_{m_l} = \tau$ in $S$, then

$\tau$ is at least $l$ multiple roots of the polynomial $Q_t(at + x)$. 

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The following result is obtained.

\[ Q_t(at + x) = \sigma_k(\theta) \prod_{m=1}^{k} \lambda_m(at + x). \]

We know that \( Q_\tau(at + x) = 0 \). As \( \lambda_m(at + x) \) is real analytic with respect to variable \( t \), then

\[ \frac{d^\alpha Q_t}{dt^\alpha}(at + x) = 0, \]

for \( 0 \leq \alpha \leq l - 1 \). The claim has been proved. By our claim, we know that \( t_1, \ldots, t_k \) are the exact \( k \) real roots of the polynomial \( Q_t(at + x) \).

\[ \Box \]

The following lemma and proof are adopted from Lemma 2.10 in [30].

**Lemma 16** For any \( a \in \Gamma_k \subset \mathbb{R}^n \) and any \( x \in \mathbb{R}^n \), the functions \( \lambda_m(at + x) \) for \( m = 1, 2, \ldots, k \) are defined by

\[ Q_t(s \theta + at + x) = \sigma_k(\theta) \prod_{m=1}^{k} (s + \lambda_m(at + x)), \]

is real analytic with respect to \( t \).

**Proof** We consider a complex algebraic curve \( C \) in 2D complex space \( \mathbb{C}^2 \),

\[ C = \{(s, t) : Q_t(s \theta + ta + x) = 0\}. \]

At some point \((s_1, t_0)\) of \( C \), we select a local irreducible component \( C_1 \) of \( C \) along with a local uniformizing parameter \( z \). This condition means that we have holomorphic functions

\[ s(z) = s_1 + z^p g(z) \quad \text{and} \quad t(z) = t_0 + z^q h(z) \quad \text{for} \quad |z| < \delta, \]

with \( g(0) \neq 0, h(0) \neq 0 \), such that \( z \to (s(z), t(z)) \) is a homeomorphism of a neighborhood of \( 0 \in \mathbb{C} \) onto a neighborhood of \((s_1, t_0)\) in \( C_1 \). By extracting a \( q \)-th root of \( h(z) \), we may assume that \( h(z) = 1 \), i.e.,

\[ t(z) = t_0 + z^q. \]

It will suffice to show that \( q = 1 \), for then \( s(z) \) is a root of \( Q_{t_0 + \epsilon}(s \theta + (t_0 + \epsilon)a + x) \), and, as such, must be real if \( z \) is real. Setting

\[ \lambda_1(at + x) = -s(t - t_0), \]

we obtain a local real analytic function. Repeating this construction for each of the \( k \) real points \((s_1, t_0), \ldots, (s_k, t_0) \in C \) (counting multiplication) completes the proof.

We now show that \( q = 1 \). As \( s(z) \) is a real root of \( s \to Q_{t(z)}(s \theta + t(z)a + x) \), we find that

\[ z^q \quad \text{real implies} \quad s(z) \quad \text{real}. \quad (4.5) \]

In particular, \( z \) real implies \( s(z) \) real, so that the power series

\[ s(z) = \sum_{k=0}^{\infty} a_k z^k \]
for \( s(z) \) has real coefficients \( a_k \). We set \( \omega = e^{\frac{2\pi i}{q}} \) and \( z = r \omega \). Then, \( \omega^k \) is not real unless \( k \) is a multiple of \( q \). Now, \( z^q \) is real and thus, \( \sum_{k=0}^{\infty} a_k \omega^k r^k \) is real by Eq. (4.5). It follows that \( a_k = 0 \) unless \( k \) is a multiple of \( q \). Thus,

\[
s(z) = s_1 + \sum_{k=1}^{\infty} b_k z^{kq} = s_1 + f(z^q).
\]

\( z \mapsto (f(z^q), z^q) = (s(z) - s_1, t(z) - t_0) \) is one to one, this condition proves that \( q = 1 \).

Now, we can prove the concavity of \( Q \).

**Lemma 17** If the condition (C) holds, then for \( Q(x) \) defined by (1.3), \( (Q(x))^{1/k} \) is a concave function in \( \Gamma_k \).

**Proof** For any \( x, y \in \Gamma_k \) and \( 0 \leq t \leq 1 \), we need to prove that \( (Q(yt + (1 - t)x))^{1/k} \) is a concave function. Let \( a = y - x \), and by Proposition 15, we have

\[
(Q(yt + (1 - t)x))^{1/k} = (Q(at + x))^{1/k} = \left( r_1 Q_1^2 \left( a + \frac{x}{t} \right) \right)^{1/k} = \left( \left( r_k Q(x) \prod_{m=1}^{k} \left( \frac{1}{t} + \lambda_m(x, a) \right) \right)^{1/k} \right) = \left( Q(x) \prod_{m=1}^{k} (1 + t\lambda_m(x, a)) \right)^{1/k} = (Q(x))^{1/k} (\sigma_k(\theta_k + t\lambda(x, a)))^{1/k}.
\]

Here, \( \theta_k = (1, 1, \ldots, 1) \in \mathbb{R}^k \) and \( \lambda(x, a) = (\lambda_1(x, a), \ldots, \lambda_k(x, a)) \). Then, by Newton–Maclaurin inequality, we obtain

\[
\frac{d^2}{dt^2} (Q(at + x))^{1/k} = \frac{(Q(x))^{1/k}}{k} \left( \sigma_k(\theta_k + t\lambda(x, a)) \right)^{\frac{1}{2}} \left( 2\sigma_2 \left( \frac{\lambda(x, a)}{1 + t\lambda(x, a)} \right) \right) - \frac{k - 1}{k} \sigma_1^2 \left( \frac{\lambda(x, a)}{1 + t\lambda(x, a)} \right) \leq 0.
\]

Here, \( \frac{\lambda(x, a)}{1 + t\lambda(x, a)} \) means the vector \( \left( \frac{\lambda_1(x, a)}{1 + t\lambda_1(x, a)}, \ldots, \frac{\lambda_k(x, a)}{1 + t\lambda_k(x, a)} \right) \). We obtain our result.

Now, we study the quotient concavity of these polynomials. If Condition (C) holds, then for \( x \in \mathbb{R}^n \), we know that

\[
Q(t\theta + x) = \prod_{m=1}^{N} \left( 1 + b_m \frac{d}{dt} \right) \sigma_k(t\theta + x).
\]

Thus, we denote, for any integer \( 1 \leq N' \leq N \),

\[
Q_k^{N'}(t\theta + x) = \prod_{m=1}^{N'} \left( 1 + b_m \frac{d}{dt} \right) \sigma_k(t\theta + x), \tag{4.7}
\]
which is a $l$-th degree polynomial with respect to $t$. We define, for $x \in \mathbb{R}^n$,
\[
Q^{N'}_l(x) = Q^{N'}_l(t\theta + x)\bigg|_{t=0}.
\] (4.8)

**Lemma 18** For all $x \in \Gamma_k$, if Condition (C) holds, for $Q(x)$ defined by (1.3) and $Q^{N}_{k-1}(x)$, $Q^{N-1}_{k-1}(x)$ defined by (4.8), functions
\[
\frac{Q(x)}{Q^{N}_{k-1}(x)}, \text{ and } \frac{Q(x)}{Q^{N-1}_{k-1}(x)}
\]
are concave with respect to $x$.

**Proof** For $x, y \in \Gamma_k$, let $ty + (1-t)x = at + x$, where $a = y - x$. By Proposition 15, we have
\[
Q((s\theta + a)t + x) = t^k Q^1_\frac{x}{t} (\frac{x}{t} + s\theta + a)
= Q(x) \prod_{m=1}^{k} (1 + t\lambda_m(s\theta + a, x)).
\] (4.9)

Based on Eq. (4.7), the following is clear:
\[
\frac{Q(st\theta + at + x)}{Q^{N}_{k-1}(st\theta + at + x)} = (n - k + 1) \frac{Q(st\theta + at + x)}{d(st) Q(st\theta + at + x)},
\]
thus, using (4.9) and taking $s = 0$, we get
\[
\frac{Q(at + x)}{Q^{N}_{k-1}(at + x)} = \frac{(n - k + 1) Q(x) t \prod_{m=1}^{k} (1 + t\lambda_m(a, x))}{Q(x) \sum_{l=1}^{k} \frac{d\lambda_i}{ds}(a, x) \prod_{m \neq l} (1 + t\lambda_m(a, x))}
= \frac{n - k + 1}{\sum_{l=1}^{k} \frac{d\lambda_i}{ds}(a, x) \frac{1}{1 + t\lambda_i(a, x)}}.
\]

By an argument the same as the proof of Lemma 16, we also can prove that $\lambda_m(s\theta + a, x)$ is real analytic with respect to the variable $s$ by a proper arrangement of the indices. Thus,
\[
\frac{d\lambda_i}{ds}(a, x) = \frac{d}{ds} (\lambda_i(s\theta + a, x))\bigg|_{s=0}
\]
is well defined.

For $0 < t \leq 1$, we let $\tau = \frac{1}{t} \geq 1$. For any fixed $|s| < 1$, we consider
\[
Q_{\tau+s}((\tau + s)x + a) = Q(x) \prod_{m=1}^{k} (\tau + s + \lambda_m(a, x)) = Q(x) \sigma_k ((\tau + s)\theta_k + \lambda(a, x)).
\]

Here, in the second equality, we have used Proposition 15 and the notion $\lambda(a, x)$ means $(\lambda_1(a, x), \ldots, \lambda_k(a, x))$. Thus, letting $s = 0$, we have
\[
\frac{1}{t^k} Q(at + x) = Q_\tau(\tau x + a) = Q(x) \sigma_k (\tau\theta_k + \lambda(a, x)) > 0,
\]
because $at + x \in \Gamma_k$. We also have

$$\frac{d^l}{ds^l} (Q_{\tau+s}((\tau + s)x + a)) = \sum_{i+j=l, i,j \geq 0} \sum_{m=0}^k d^i_i \sum_{m=0}^k \alpha_m (x) d^j_m (\tau + s)^{k-m} \sigma_m ((\tau + s)x + a).$$

As $\tau x + a \in \Gamma_k$, we have

$$\frac{d^j}{ds^j} \sigma_m (sx + \tau x + a) \bigg|_{s=0} \geq 0,$$

for $0 \leq m \leq k$. Thus, we obtain

$$\frac{d^l}{ds^l} (Q_{\tau+s}((\tau + s)x + a)) \bigg|_{s=0} > 0,$$

for $1 \leq l \leq k - 1$, which implies

$$\sigma_1 (\tau \theta_k + \lambda(a, x)) > 0, \ldots, \sigma_k (\tau \theta_k + \lambda(a, x)) > 0.$$

Thus, we obtain $\tau \theta_k + \lambda(a, x) \in \Gamma_k$, which implies that every $1 + t\lambda_m(a, x) > 0$ for $t > 0$.

On the other hand, by Proposition 15, we have

$$Q_{\tau}(\tau x + s\theta + a) = Q(x) \prod_{m=1}^k (\tau + \lambda_m(s\theta + a, x)).$$

For $s \neq 0$, we have

$$Q_{\tau}(\tau x + s\theta + a) = s^k Q_{\tau} \left( \frac{\tau}{s} x + \frac{a}{s} + \theta \right)$$

$$= s^k Q(x) \prod_{m=1}^k \left( \frac{\tau}{s} + \mu_m \left( \theta + \frac{a}{s}, x \right) \right)$$

$$= Q(x) \prod_{m=1}^k \left( \tau + s \mu_m \left( \theta + \frac{a}{s}, x \right) \right).$$

Similar to the argument of Proposition 15, if $|s|$ is sufficiently large, then for every $m = 1, \ldots, k$, we obtain

$$\mu_m \left( \theta + \frac{a}{s}, x \right) > 0.$$}

Thus, if $s \to +\infty (-\infty)$, we have

$$\lambda_m(s\theta + a, x) \to +\infty (-\infty).$$

By the continuity of the function $\lambda_m(s\theta + a, x)$ and a similar argument of the Proposition 15, for any given number $\mu \in \mathbb{R}$, the equation

$$\lambda_m(s\theta + a, x) = \mu$$

has one and only one solution, which implies that every $\lambda_m(s\theta + a, x)$ is a strictly monotonic increasing function with respect to $s$. Thus, we have

$$\frac{d\lambda_m}{ds} (a, x) > 0.$$
We prove the concavity by defining a function
\[ g_l(t) = \frac{1 + t\lambda_l(x, a)}{d\lambda_l(x, a)} > 0, \]
for \( l = 1, \ldots, k \) and \( g_l''(t) = 0 \). Then, we have
\[ \frac{1}{n-k+1} \frac{Q(at+x)}{Q_{k-1}^{N}(at+x)} = \frac{1}{\sum_{m} g_m(t)}. \] (4.10)

The second derivative of the above function is
\[ -\frac{2}{\left(\sum_{m} \frac{1}{g_m(t)}\right)^3} \left[ \sum_{m} \frac{1}{g_m(t)} \sum_{m} \left(\frac{g_m'(t)}{g_m(t)}\right)^2 - \left(\sum_{m} \frac{g_m'(t)}{g_m(t)}\right)^2 \right] \leq 0. \]

Thus, the first function defined in our lemma is a concave function. The second function can be rewritten as
\[ \frac{Q_l(x)}{Q_{l-1}^{N}(x)} = \frac{Q_k^{N-k+1}(x)}{Q_{k-1}^{N-k+1}(x)} + (n-k+1)b_N. \]

Thus, it is also a concave function. □

A corollary of the above lemma is the following result:

**Corollary 19** Given any two integers \( 1 \leq \ell < \kappa \leq n \), in \( \Gamma_k \), if Condition (C) holds, for \( Q(x) \) defined by (1.3) and \( Q_{\ell}^{N}(x), Q_{k-\ell-1}^{N}(x) \) defined by (4.8), functions
\[ \left( \frac{Q(x)}{Q_{\ell}^{N}(x)} \right)^{1/(k-\ell)} \quad \text{and} \quad \left( \frac{Q(x)}{Q_{k-\ell-1}^{N}(x)} \right)^{1/\ell} \]
are concave. Specifically, if \( N = k \), functions
\[ \left( \frac{Q(x)}{\sigma_1(x) + n \sum_{m} b_m} \right)^{1/(k-1)} \quad \text{and} \quad \left( \frac{Q(x)}{\sigma_1(x) + nb_1} \right)^{1/(k-1)} \]
are concave, where \( b_0, b_1, \ldots, b_N \) are components of vector \( b \) appearing in (4.1).

**Proof** Obviously, we have
\[ \frac{Q(x)}{Q_{\ell}^{N}(x)} = \frac{Q(x)}{Q_{k-1}^{N}(x)} \quad \text{and} \quad \frac{Q_{\ell}^{N}(x)}{Q_{k-1}^{N}(x)} \]
and
\[ \frac{Q(x)}{Q_{k-\ell-1}^{N}(x)} = \frac{Q(x)}{Q_{k-1}^{N-k-1}(x)} \quad \text{and} \quad \frac{Q_{k-\ell-1}^{N}(x)}{Q_{k-1}^{N-k-1}(x)}. \]

Thus, by Lemma 18 and the proof of Corollary 13, we obtain our first result. If \( N = k \), because we have
\[ Q_1^{k}(x) = \sigma_1(x) + n \sum_{m} b_m, \quad Q_1^{1} = \sigma_1(x) + nb_1, \]
then we obtain the second result using the concavity of the first two functions. □
We can use the previous result to revisit the quotient concavity of the sum Hessian equation.

**Corollary 20** The function

\[ q_k(x) = \frac{\sigma_{k+1}(x) + \alpha \sigma_k(x)}{\sigma_k(x) + \alpha \sigma_{k-1}(x)} \]

is a concave function in \( \Gamma_{k+1} \) for \( 1 \leq k \leq n - 1 \) and \( \alpha \geq 0 \).

**Proof** As we have

\[ \sigma_k(t\theta + x) = \frac{1}{n-k} \frac{d}{dt} \sigma_{k+1}(t\theta + x), \quad \text{and} \quad \sigma_{k-1}(t\theta + x) = \frac{1}{n-k+1} \frac{d}{dt} \sigma_k(t\theta + x), \]

we obtain

\[ q_k(x) = \frac{\sigma_{k+1}(t\theta + x) + \alpha \frac{d}{dt} \sigma_{k+1}(t\theta + x)}{\sigma_k(t\theta + x) + \alpha \frac{d}{dt} \sigma_k(t\theta + x)} \bigg|_{t=0} + \alpha \frac{\sigma_k(x)}{n-k+1} \sigma_k(x) + \alpha \sigma_{k-1}(x). \]

Both of the above two functions are concave functions by Corollary 19 and Lemma 11. Thus, we have the concavity of \( q_k(x) \) in \( \Gamma_{k+1} \).

**Remark 21** The difference between Corollary 20 and Lemma 12 is the definition domain of function \( q_k \). As \( \Gamma_{k+1} \subset \tilde{\Gamma}_k \), Lemma 12 is better.

By using Corollary 19, we can conclude the main result of this section in Theorem 3.

## 5 Curvature estimates

In this section, we derive the global curvature estimates for the curvature Eq. (1.5) defined by the linear combination of the Hessian operators, namely, proving Theorem 4. We will use the Einstein conversion from this point on.

Denote \( X, v \) as the position vector and the outer normal vector of \( M \). Set \( u(X) = \langle X, v(X) \rangle \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product of the ambient space. By the assumption that \( M \) is a convex hypersurface with \( C^1 \) bound, \( u \) is bounded from below and above by two positive constants. At every point in the hypersurface \( M \), select a local coordinate \( \{x_1, x_2, \ldots, x_{n+1}\} \) of \( \mathbb{R}^{n+1} \) such that the first \( n \) vectors \( \partial/\partial x_1, \ldots, \partial/\partial x_n \) are a local frame of the hypersurface and the last one \( \partial/\partial x_{n+1} \) is the unit outer normal vector. We let \( h_{ij} \) be the second fundamental form of the hypersurface \( M \). The following geometric formulas are well known (e.g., [24, 26]).

\[ h_{ij} = \langle \partial_i X, \partial_j v \rangle, \]  

(5.1)

and

\[ X_{ij} = -h_{ij} v \]  

(Gauss formula)

\[ (v)_i = h_{ij} \partial_j \]  

(Weigarten equation)

\[ h_{ijk} = h_{ikj} \]  

(Codazzi formula)

\[ R_{ijkl} = h_{iklj} - h_{ijlk} \]  

(Gauss equation),

(5.2)
where $R_{ijkl}$ is the $(4, 0)$-Riemannian curvature tensor. We also have

$$h_{ijkl} = h_{ijlk} + h_{mj} R_{imlk} + h_{im} R_{jmlk}$$

$$= h_{klij} + (h_{mj} h_{il} - h_{ml} h_{ij}) h_{mk} + (h_{mj} h_{kl} - h_{ml} h_{kj}) h_{mi}.$$  \hspace{1cm} (5.3)

Suppose $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of $M$. As in [36], we consider the following test function:

$$\varphi = \log P_m - mZ \log u, \quad \text{where} \quad P_m = \sum_j \kappa_j^m.$$  \hspace{1cm} (5.4)

Here, $m, Z$ are two undetermined constants. Suppose that the function $\varphi$ achieves its maximum value on $M$ at some point $X_0$. Through proper rotation of the coordinates, we assume that $(h_{ij})$ is diagonal at $X_0$, and $\kappa_1 \geq \kappa_2 \cdots \geq \kappa_n$.

Covariant differentiating our test function twice and using Lemma 9, at $X_0$, we have

$$\sum_j \frac{\kappa_j^{m-1} h_{jji}}{P_m} - Z \frac{(X, \partial_i)}{u} h_{ii} = 0,$$  \hspace{1cm} (5.5)

and

$$0 \geq \frac{1}{P_m} \left( \sum_j \kappa_j^{m-1} h_{jji} + (m - 1) \sum_j \kappa_j^{m-2} h_{jji}^2 + \sum_{p \neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} h_{pqi}^2 \right)$$

$$- \frac{m}{P_m^2} \left( \sum_j \kappa_j^{m-1} h_{jji} \right)^2 - \frac{Z}{u} \sum_l h_{iil}(\partial_l, X) - Z h_{ii}^2 + Z h_{ii}^2 (X, \partial_l)^2 \left( \frac{u^2}{u^2} \right) \hspace{1cm} (5.6)$$

At $X_0$, covariant differentiating Eq. (1.5) twice, we have

$$Q^{ij} h_{iik} = dX \psi(\partial_k) + h_{kk} d_v \psi(\partial_k),$$  \hspace{1cm} (5.7)

and

$$Q^{ij} h_{iikk} + Q^{pq,rs} h_{pqk} h_{rsk} \geq -C - C h_{11}^2 + \sum_l h_{lkk} d_v \psi(\partial_l),$$ \hspace{1cm} (5.8)

where $C$ is a uniform constant depending on $C^0$ and $C^1$ bound of the hypersurface $M$. Using (5.5) and (5.7), we have

$$\frac{1}{P_m} \sum_l \sum_j \kappa_j^{m-1} d_v \psi(\partial_l) h_{iij} - \frac{Z}{u} \sum_l Q^{ij} h_{iil}(\partial_l, X)$$

$$= -\frac{Z}{u} \sum_l dX \psi(\partial_l)(X, \partial_l).$$

On the other hand, using Lemma 9, we have

$$Q^{pq,rs} h_{pql} h_{rsl} = Q^{pp,qq} h_{ppl} h_{qql} - \sum_{p \neq q} Q^{pp,qq} h_{pql}^2.$$
Then, contacting $Q^{ii}$ on both sides of (5.6), and using (5.7) and (5.8), we have

$$0 \geq \frac{1}{P_m} \left[ \sum_l \kappa_l^{m-1} \left( -C - C(K)h_{11}^2 + K Q_l^2 - Q^{pp.qq} h_{ppi}h_{qqi} + \sum_{p \neq q} Q^{pp.qq} h_{pqi}^2 \right) \right]$$

$$+ (m - 1) Q^{ii} \sum_j \kappa_j^{m-2} h_{jjj}^2 + Q^{ii} \sum_{p \neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} h_{pqi}^2$$

$$- \frac{m Q^{ii}^2}{P_m^2} \left( \sum_j \kappa_j^{m-1} h_{jjj} \right)^2 + (Z - 1) Q^{ii} h_{ii}^2 + Z Q^{ii} h_{ii}^2 X, \partial_i^2 - C(Z, K) \kappa_1.$$ 

We denote

$$A_i = \frac{\kappa_i^{m-1}}{P_m} \left( K Q_i^2 - \sum_{p,q} Q^{pp.qq} h_{ppi}h_{qqi} \right), \quad B_i = \frac{2}{P_m} \sum_j \kappa_j^{m-1} Q^{jj.ii} h_{jjj}^2,$$

$$C_i = \frac{m - 1}{P_m} Q^{ii} \sum_j \kappa_j^{m-2} h_{jjj}^2, \quad D_i = \frac{2 Q^{ij}}{P_m} \sum_{j \neq i} \frac{\kappa_j^{m-1} - \kappa_i^{m-1}}{\kappa_j - \kappa_i} h_{jjj}^2,$$

$$E_i = \frac{m Q^{ii}}{P_m^2} \left( \sum_j \kappa_j^{m-1} h_{jjj} \right)^2.$$

We divide these third derivatives into two cases to deal with, $i \neq 1$ and $i = 1$.

**Lemma 22** For any $i \neq 1$, we have

$$A_i + B_i + C_i + D_i - E_i \geq 0,$$

if $m$ is sufficiently large.

**Proof** Initially, by Lemma 8, if the constant $K$ is sufficiently large, we have

$$K Q_i^2 - Q^{pp.qq} h_{ppi}h_{qqi} \geq Q \left( 1 + \frac{\beta}{2} \right) \left[ \frac{(S_1)h}{S_1} \right]^2 \geq 0. \quad (5.9)$$

Thus, $A_i \geq 0$.

Then, we have

$$P_m^2 (B_i + C_i + D_i - E_i)$$

$$= \sum_{j \neq i} P_m \left( 2 \kappa_j^{m-1} Q^{jj.ii} + (m - 1) \kappa_j^{m-2} Q^{ii} + 2 Q^{ij} \sum_{l=0}^{m-2} \kappa_i^{m-2-l} \kappa_j^l \right) h_{jjj}^2$$

$$+ P_m (m - 1) Q^{ii} \kappa_i^{m-2} h_{iii}^2$$

$$- m Q^{ii} \left( \sum_j \kappa_j^{2m-2} h_{jjj}^2 + \kappa_i^{2m-2} h_{iii}^2 + \sum_{p \neq q} \kappa_p^{m-1} \kappa_q^{m-1} h_{ppi}h_{qqi} \right). \quad (5.10)$$

Note that

$$\kappa_j Q^{jj.ii} + Q^{jj} = \kappa_i Q^{ii.jj} + Q^{ii} \geq Q^{ii}.$$
For any index $j \neq i$, using the above inequality, we have
\[
P_m \left(2\kappa_j^{m-1} Q_{jj,ii} + (m - 1)\kappa_j^{m-2} Q_{ii}^j + 2 Q_{jj}^j \sum_{l=0}^{m-2} \kappa_i^{m-2-l} k_j^l \right) h_{jji}^2
\]
\[
- m Q_{ii}^j k_j^{2m-2} h_{jji}^2
\]
\[
\geq (m + 1)(P_m - \kappa_j^m) Q_{ii}^j k_j^{m-2} h_{jji}^2 + 2 P_m Q_{jj}^j \left(\sum_{l=0}^{m-3} \kappa_i^{m-2-l} k_j^l \right) h_{jji}^2. \quad (5.11)
\]

Using Cauchy–Schwarz inequality, we have
\[
2 \sum_{j \neq i} \sum_{p \neq i, j} \kappa_j^{m-2} \kappa_p h_{jji}^2 \geq 2 \sum_{p \neq q; p, q \neq i} \kappa_p^{m-1} \kappa_q^{m-1} h_{ppi} h_{qqi}. \quad (5.12)
\]
Thus, using (5.11) and (5.12) in (5.10), we obtain
\[
P_m^2 (B_i + C_i + D_i - E_i)
\]
\[
\geq \sum_{j \neq i} \left((m + 1)\kappa_i^m \kappa_j^{m-2} Q_{ii}^j + 2 P_m Q_{jj}^j \sum_{l=0}^{m-3} \kappa_i^{m-2-l} k_j^l \right) h_{jji}^2
\]
\[
+ [(m - 1)(P_m - \kappa_i^m) - \kappa_i^m] \kappa_i^{m-2} Q_{ii}^j h_{iii}^2 - 2 m Q_{ii}^j \kappa_i^{m-1} h_{iii} \sum_{j \neq i} \kappa_i^{m-1} h_{jji}.
\]
\[
(5.13)
\]

Obviously, $Q_{jj}^i > Q_{ii}^j$ if $\kappa_j < \kappa_i$ and $\kappa_j Q_{jj}^i > \kappa_i Q_{ii}^j$ if $\kappa_j > \kappa_i$. Thus, for $m \geq 6$, we obtain
\[
2 P_m Q_{jj}^j \sum_{l=0}^{m-3} \kappa_i^{m-2-l} k_j^l = 2 P_m Q_{jj}^j (\kappa_i \kappa_j^{m-3} + \kappa_i^2 \kappa_j^{m-4}) + 2 P_m Q_{jj}^j \sum_{l=0}^{m-5} \kappa_i^{m-2-l} k_j^l
\]
\[
\geq 4 \kappa_i^m \kappa_j^{m-2} Q_{ii}^j + 2 P_m Q_{jj}^j \sum_{l=0}^{m-5} \kappa_i^{m-2-l} k_j^l,
\]
Then, (5.13) becomes
\[
P_m^2 (B_i + C_i + D_i - E_i)
\]
\[
\geq \sum_{j \neq i} \left((m + 5)\kappa_i^m \kappa_j^{m-2} Q_{ii}^j h_{jji}^2 + [(m - 1)(P_m - \kappa_i^m) - \kappa_i^m] \kappa_i^{m-2} Q_{ii}^j h_{iii}^2
\]
\[
- 2 m Q_{ii}^j \kappa_i^{m-1} h_{iii} \sum_{j \neq i} \kappa_j^{m-1} h_{jji} + 2 P_m \sum_{j \neq i} Q_{jj}^j \sum_{l=0}^{m-5} \kappa_i^{m-2-l} k_j^l h_{jji}^2
\]
\[
\geq (m + 5)\kappa_i^m \kappa_j^{m-2} Q_{ii}^j h_{jji}^2 + [(m - 1)\kappa_i^m - \kappa_i^m] \kappa_i^{m-2} Q_{ii}^j h_{jji}^2
\]
\[
- 2 m Q_{ii}^j \kappa_i^{m-1} h_{iii} \kappa_j^{m-1} h_{jji} + [(m - 1)\kappa_i^m - \kappa_i^m] \kappa_i^{m-2} Q_{ii}^j h_{jji}^2
\]
\[
\geq 0. \quad (5.14)
\]

For $m \geq 6$, we use the following inequality:
\[
(m + 5)(m - 1) \geq m^2 \quad \text{and} \quad (m + 5)(m - 2) \geq m^2.
\]
\[\Box\]
We continue to deal with the other case $i = 1$ in the following lemma, which is modified from [26].

**Lemma 23** Suppose that the operator $Q$ defined by (1.3) satisfies the quotient concavity. For $μ = 1, \ldots, k - 1$, suppose that positive constant $δ \leq 1$ exists, satisfying $κ_μ/κ_1 \geq δ$ and $α_1 = \cdots = α_μ = 0$ in (1.5). Then, another sufficiently small positive constant $δ'$ exists on $δ$, such that if $κ_{μ+1}/κ_1 \leq δ'$ and $κ_1$ is sufficiently large, then we have

$$A_1 + B_1 + C_1 + D_1 - E_1 \geq 0.$$  

**Proof** Initially, note that $Q_{jj}^j \geq Q_{11}^j$ for all $j > 1$, which is the same as (5.14) in Lemma 22, and for $m \geq 6$, we have

$$P_m^2(B_1 + C_1 + D_1 - E_1) \geq -κ_1^{2m-2}Q_{11}^j h_{111}^2 + 2P_m \sum_{j \neq 1} Q_{jj}^j h_{jj1}^2 \sum_{l=0}^{m-5} κ_1^{m-2-l} k_l^j h_{jj1}^2 + 2P_m^{m-2} \sum_{j \neq 1} Q_{jj}^j h_{jj1}^2.$$  

Using the quotient concavity and Lemma 8, we have

$$A_1 \geq \frac{κ_1^{m-1} Q}{P_m S_2^μ} \left[ \left(1 + \frac{β}{2}\right) \sum_a (S_{aa}^μ h_{aa1})^2 + \frac{β}{2} \sum_{a \neq b} S_{aa}^μ S_{bb}^μ h_{aa1} h_{bb1} \right] + \sum_{a \neq b} (S_{aa}^μ S_{bb}^μ - S_{aa}^μ S_{bb}^μ) h_{aa1} h_{bb1}.  \tag{5.16}$$

For $μ = 1$, notice that $S_{aa}^1 = 1$ and $S_{bb}^{aa,bb} = 0$. Then, we have

$$\left(1 + \frac{β}{2}\right) \sum_{a,b} h_{aa1} h_{bb1} \geq (1 + \frac{β}{4}) h_{111}^2 - C_β \sum_{a \neq 1} h_{aa1}^2. \tag{5.17}$$

Then, using Eq. (1.5), we obtain

$$P_m^2 A_1 \geq \frac{P_m^{m-2} Q_{11}^j}{(1 + \sum_{j \neq 1} κ_j/κ_1 + P_0^1/κ_1)^2} \left(1 + \frac{β}{4}\right) h_{111}^2 - \frac{C_β P_m^{m-1}}{S_2^μ} \sum_{a \neq 1} h_{aa1}^2. \tag{5.18}$$

The last two inequalities come from

$$Q \geq κ_1 Q_{11}^j, \quad and \quad \frac{κ_j}{κ_1} \leq δ'; \quad 1 + \frac{β}{4} \geq (1 + nδ')^2.$$  

For $μ \geq 2$ and $a \neq b$, we have

$$S_{aa}^μ S_{bb}^μ - S_{aa}^μ S_{bb}^μ = S_{μ-1}^2 (κ|ab) - S_μ (κ|ab) S_{μ-2} (κ|ab). \tag{5.19}$$

Thus, for any $a, b \leq μ$, we have

$$S_{aa}^μ \geq \frac{κ_1 \cdots κ_μ}{κ_a}; \quad S_{bb}^μ \geq \frac{κ_1 \cdots κ_μ}{κ_b}. \tag{5.20}$$
For $a, b \leq \mu$, if $\kappa_1$ is sufficiently large, then we have

$$S_{\mu-1}(\kappa|ab) \leq C\left(\frac{\kappa_1 \ldots \kappa_{\mu}}{\kappa_0 \kappa_b} + \frac{\kappa_1 \ldots \kappa_{\mu+1}}{\kappa_0 \kappa_b} \right) \leq \frac{C}{\kappa_b} \frac{1 + \kappa_{\mu+1}}{\kappa_{\mu+1}} S_{\mu}^{aa}.$$  

$$S_{\mu}(\lambda|ab) \leq C\left(\frac{\kappa_1 \ldots \kappa_{\mu}}{\kappa_0 \kappa_b} + \frac{\kappa_1 \ldots \kappa_{\mu+1}}{\kappa_0 \kappa_b} + \frac{\kappa_1 \ldots \kappa_{\mu+2}}{\kappa_0 \kappa_b} \right) \leq \frac{C}{\kappa_b} \frac{1 + \kappa_{\mu+1} + \kappa_{\mu+1} \kappa_{\mu+2}}{\kappa_{\mu+2}} S_{\mu}^{aa},$$

$$S_{\mu-2}(\kappa|ab) \leq C\frac{\kappa_1 \ldots \kappa_{\mu}}{\kappa_0 \kappa_b} \leq C\frac{C}{\kappa_b} S_{\mu}^{aa}. \quad (5.21)$$

Then, by using (5.21) and for any undetermined positive constant $\epsilon$, we obtain

$$\sum_{a \neq b; a, b \leq \mu} (S_{\mu}^{aa} S_{\mu}^{bb} - S_{\mu}^{aa, bb}) h_{aa1} h_{bb1} \geq -\sum_{a \neq b; a, b \leq \mu} (S_{\mu-1}^{2}(\kappa|ab) + S_{\mu}(\kappa|ab) S_{\mu-2}(\kappa|ab)) h_{aa1}^2 \geq -\frac{C_2}{\delta^2} (1 + \delta^2 \kappa_1)^2 + 1 + \delta^2 \kappa_1 + (\delta')^2 \kappa_1 \sum_{a \leq \mu} (S_{\mu}^{aa} h_{aa1})^2 \geq -\epsilon \sum_{a \leq \mu} (S_{\mu}^{aa} h_{aa1})^2. \quad (5.22)$$

Here, the constant $\delta'$ is selected to be sufficiently small to satisfy

$$\delta' \leq \frac{\epsilon \delta^2}{2C_2}, \quad \frac{1}{\kappa_1} \leq \delta \sqrt{\frac{\epsilon}{5C_2}}. \quad (5.23)$$

For $a \leq \mu$ and $b > \mu$, we have

$$S_{\mu}^{aa} \geq \frac{\kappa_1 \ldots \kappa_{\mu}}{\kappa_a}; \quad S_{\mu}^{bb} \geq \kappa_1 \ldots \kappa_{\mu-1}. \quad (5.24)$$

Then, for $a \leq \mu, b > \mu$, if $\kappa_1$ is sufficiently large, we have

$$S_{\mu-1}(\kappa|ab) \leq C\frac{\kappa_1 \ldots \kappa_{\mu}}{\kappa_a} \leq C S_{\mu}^{aa} \text{ or } C S_{\mu}^{bb},$$

$$S_{\mu}(\lambda|ab) \leq C \left(\frac{\kappa_1 \ldots \kappa_{\mu}}{\kappa_a} + \frac{\kappa_1 \ldots \kappa_{\mu+1}}{\kappa_a} \right) \leq C (1 + \kappa_{\mu+1}) S_{\mu}^{aa},$$

$$S_{\mu-2}(\kappa|ab) \leq C \kappa_1 \ldots \kappa_{\mu-2} \leq C S_{\mu}^{bb}. \quad (5.24)$$

By (5.24), we also have

$$2 \sum_{a \leq \mu; b > \mu} (S_{\mu}^{aa} S_{\mu}^{bb} - S_{\mu}^{aa, bb}) h_{aa1} h_{bb1} \geq -2 \sum_{a \leq \mu; b > \mu} [S_{\mu-1}^{2}(\kappa|ab) + S_{\mu}(\kappa|ab) S_{\mu-2}(\kappa|ab)]|h_{aa1} h_{bb1}| \geq -\epsilon \sum_{a \leq \mu; b > \mu} (S_{\mu}^{aa} h_{aa1})^2 - \frac{C}{\epsilon} \sum_{a \leq \mu; b > \mu} (S_{\mu}^{bb} h_{bb1})^2. \quad (5.25)$$

For $a, b > \mu$, we have

$$S_{\mu}^{aa} \geq \kappa_1 \ldots \kappa_{\mu-1}; \quad S_{\mu}^{bb} \geq \kappa_1 \ldots \kappa_{\mu-1}.$$
Then, for \( a, b > \mu \), if \( \kappa_1 \) is sufficiently large, we have
\[
S_{\mu-1}(\kappa|ab) \leq C \kappa_1 \ldots \kappa_{\mu-1}, \quad S_{\mu}(\lambda|ab) \leq C \kappa_1 \ldots \kappa_{\mu},
\]
\[
S_{\mu-2}(\kappa|ab) \leq C \kappa_1 \ldots \kappa_{\mu-2}.
\] (5.26)

By (5.26), we have
\[
\sum_{a \neq b; a, b > \mu} \left( S_{\mu}^{aa} S_{\mu}^{bb} - S_{\mu}^{aa,bb} \right) h_{aa} h_{bb} \geq - C \sum_{a \neq b; a, b > \mu} (S_{\mu}^{aa} h_{aa})^2.
\] (5.27)

Therefore, combining (5.16), (5.22), (5.25) and (5.27), we obtain
\[
A_1 \geq \frac{\kappa_1^{m-1}}{P_m S_{\mu}^2} Q^{11} \left( 1 + \frac{\beta}{2} - 2\epsilon \right) \sum_{a \leq \mu} \left( 1 - \frac{C_3(1 + \kappa_{\mu+1})}{\kappa_a} \right)^2 h_{aa}^2 - C \epsilon \sum_{a > \mu} (S_{\mu}^{aa} h_{aa})^2.
\] (5.28)

Here, the last inequality holds if we select \( \delta' \) and \( \epsilon \) satisfies
\[
\delta' C_3 \leq \epsilon \delta, \quad \frac{C_3}{\kappa_1} \leq \epsilon \delta \quad \text{and} \quad \left( 1 + \frac{\beta}{2} - 2\epsilon \right)^2 (1 + \delta^m) \geq 1.
\]

Using (5.15) and (5.18) or (5.28), we have
\[
P_m^2 (A_1 + B_1 + C_1 + D_1 - E_1) \geq 2 P_m \kappa_1^{m-2} \sum_{j \neq 1} Q_{jj}^1 h_{jj}^2 - \frac{C_3 P_m \kappa_1^{m-3}}{\delta^2} \sum_{j > \mu} h_{jj}^2.
\] (5.29)
Now, for \( k \geq j > \mu \), we have
\[
\kappa_1 Q^{jj} = \kappa_1 \sum_{s > j} \alpha_s \sigma_s^{jj} + \kappa_1 \sum_{\mu < s \leq j} \alpha_s \sigma_s^{jj} \\
\geq \sum_{s > j} \frac{\alpha_s \kappa_1 \ldots \kappa_s \cdot \kappa_1}{\kappa_s} + \sum_{\mu < s \leq j} \frac{\alpha_s \kappa_1 \ldots \kappa_s \cdot \kappa_1}{\kappa_s} \\
\geq \frac{\kappa_1}{C_4 \kappa_j} \sum_{s > j} \alpha_s Q_s + \frac{\kappa_1}{C_4 \kappa_s} \sum_{\mu < s \leq j} \alpha_s Q_s \\
\geq \frac{Q}{C_4 \delta'}.
\] (5.30)

For \( j \geq k + 1 \), similar to the above argument, we have
\[
\kappa_1 Q^{jj} \geq \frac{\kappa_1}{C_4} \sum_{s > \mu} \frac{\alpha_s Q_s}{\kappa_s} \geq \frac{Q}{C_4 \delta'}.
\]

For both of the above cases, we set \( \delta' \) small enough to satisfy
\[
\delta' < \frac{Q \delta^2}{C_4 C_\epsilon},
\]
then (5.29) is non-negative. Thus, we have our desired results.

Based on Lemmas 22 and 23, an argument similar to the proof in [26,36] provides the following:

**Corollary 24** Suppose that the operator \( Q \) defined by (1.3) satisfies the quotient concavity. A finite sequence of positive numbers \( \{\delta_i\}^k_{i=1} \) exists, such that, if the following two inequalities hold for an index \( 1 \leq r \leq k - 1 \),
\[
\frac{\kappa_r}{\kappa_1} \geq \delta_r, \text{ and } \frac{\kappa_r + 1}{\kappa_1} \leq \delta_{r+1},
\]
and \( \alpha_1 = \alpha_2 = \cdots = \alpha_r = 0 \) in (1.5), then, if constant \( K \) and \( \kappa_1 \) are both sufficiently large, then we have
\[
A_1 + B_1 + C_1 + D_1 - E_1 \geq 0.
\] (5.31)

Now, we prove Theorem 4.

Based on Corollary 24, a sequence \( \{\delta_i\}^k_{i=1} \) exists. We use a proof minor modification from [26,36]. In fact, the only difference here is the case (B) in the proof of [26,36]. If an index \( 1 \leq r \leq k - 1 \) satisfies
\[
\kappa_r \geq \delta_r \kappa_1, \text{ and } \kappa_{r+1} \leq \delta_{r+1} \kappa_1,
\]
and another index \( s \leq r \), such that \( \alpha_s \neq 0 \), then, by using Eq. (1.5), we have
\[
\alpha_s \sigma_s (\kappa_1, \ldots, \kappa_n) \leq Q(\kappa_1, \ldots, \kappa_n) = \psi(X, \nu),
\]
which implies the bound of \( \kappa_1 \).
6 Conclusions

Combining the discussions of Sects. 3 and 5, we obtain the main result of this study, Theorem 5. By using an appropriate barrier, we can prove the existence result Theorem 6 on the basis of Theorem 5.

**Proof of Theorem 6** The proof can be deduced using the degree theory as in [20,26]. We only provide a brief review following [26]. We consider a modified auxiliary equation

\[ Q^k_S(\kappa(X)) = \psi'(X, \nu) = tf^{-\frac{1}{k}}(X, \nu) + (1 - t) \left[ C_n^k \left( \frac{1}{|X|^k} + \varepsilon \left( \frac{1}{|X|^k} - 1 \right) \right) \right]^{\frac{1}{k}}. \]

The equation can be rewritten as

\[ -\left( Q^k_S(\kappa(X)) \right)^{-1/k} = -\left( \psi'(X, \nu) \right)^{-1/k}. \]

The assumptions of both sides of the above equation satisfy the structural conditions of the constant rank theorem (Theorem 1.2 in [6]), which implies our convexity of solutions. We illustrate how our assumptions provide the structural conditions.

At first, for the operator

\[ h(\kappa(X)) = -\left( Q^k_S(\kappa(X)) \right)^{-1/k} \]

and any symmetric matrix \( A \), we demonstrate that the function

\[ H(A) = h(\kappa(A^{-1})) = -\left( \frac{\sigma_{n-k}(k)}{\sigma_n(k)} + \frac{\alpha \sigma_{n-k+1}(k)}{\sigma_n(k)} \right)^{-1/k} \]

\[ = -\left( \frac{\sigma_n(k)}{\alpha \sigma_{n-k+1}(k) + \sigma_{n-k}(k)} \right)^{1/k} \]

is a convex function with respect to \( A \), which is equivalent to showing that the function

\[ \left( \frac{\alpha \sigma_{n-k+1}(k) + \sigma_{n-k}(k)}{\sigma_n(k)} \right)^{1/k} \]

is concave.

In fact, if \( \alpha \neq 0 \), for \( x \in \Gamma_n \), then we have

\[ \frac{\sigma_n(x)}{\alpha \sigma_{n-k+1}(x) + \sigma_{n-k}(x)} = \frac{1}{\alpha} \frac{\sigma_n(x)}{\sigma_n(x) + \frac{1}{\alpha} \sigma_{n-1}(x)} \frac{\sigma_n(x)}{\sigma_n(x) + \frac{1}{\alpha} \sigma_{n-2}(x)} \cdots \frac{\sigma_n(x)}{\sigma_n(x) + \frac{1}{\alpha} \sigma_{n-k}(x)} \]

\[ = \frac{\sigma_n(x)}{\alpha \sigma_{n-k+1}(x) + \sigma_{n-k}(x)} \quad \text{for } x \in \Gamma_n, \quad (6.1) \]

The concavity of the first quotient function comes from Lemma 11 and the concavity of other quotient functions comes from Lemma 12. Thus, the concavity of the following function

\[ \left( \frac{\sigma_n(x)}{\alpha \sigma_{n-k+1}(x) + \sigma_{n-k}(x)} \right)^{\frac{1}{\alpha}} \quad \text{for } x \in \Gamma_n, \quad (6.2) \]

is based on the well-known fact that the geometric mean value of finite numbers of positive concave functions is also a concave function. If \( \alpha = 0 \), concavity of the above function is evident.
We also let 
\[ \phi_t(X, \nu) = - (\psi_t(X, \nu))^{-1/k} \]
\[ = -tf^{-1/k}(X, \nu) - \frac{1}{k}(X, \nu) - (1 - t)\left[ C_n^k \left( \frac{1}{|X|^k} + \varepsilon \left( \frac{1}{|X|^k} - 1 \right) \right) \right]^{1/k}. \]

The concavity of \( \phi_t(X, \nu) \) with respect to \( X \) comes from the convexity of functions \( f^{-1/k}(X, \nu) \) and
\[ \left( \frac{1}{|X|^k} + \varepsilon \left( \frac{1}{|X|^k} - 1 \right) \right)^{-1/k} \]
for any fixed vector \( \nu \). Finally, we prove the convexity of the above function. If it is considered as a function of single variable \( |X| \), then the first and the second derivatives are non-negative. Furthermore, \( |X| \) is a convex function with respect to vector \( X \). Therefore, the convexity of the above function follows.

Theorem 5 provides curvature estimates. If we have \( C^0 \) bound, then the proof of the remaining part of this theorem is the same as the argument in [26]. The \( C^0 \) upper bound comes from our barrier condition using the maximum principal. Similar to the proof in [26], the lower bound only needs the uniform lower bound of the convex body’s volume, which we further need to discuss here. By our equation and the uniform upper bound, \( 1 \leq m \leq k \) such that \( \alpha_m \neq 0 \) and
\[ \sigma_m(\kappa) \leq C, \]
where \( C \) is constant only depending on \( \psi \) and the uniform upper bound. Thus, using Aleksandrov–Fenchel inequality and an argument the same as the proof in [26], we obtain the lower bound of the volume. \( \square \)

**Remark 25**
A straightforward calculation can show that the function
\[ \left( \frac{\sigma_n(x)}{\alpha \sigma_{n-k+1}(x) + \sigma_{n-k}(x)} \right)^{1/k} \quad \text{for} \quad x \in \Gamma_n, \] (6.3)
is a convex function although it is a corollary of the concavity of the function (6.2).

**Proof**
For \( x \in \Gamma_n \), we set two functions
\[ F(x) = \frac{\sigma_{n-k+1}(x)}{\sigma_n(x)}, \quad G(x) = \frac{\sigma_{n-k}(x)}{\sigma_n(x)}. \]

Since the following two functions
\[ \left( \frac{\sigma_n(x)}{\sigma_{n-k+1}(x)} \right)^{1/k}, \left( \frac{\sigma_n(x)}{\sigma_{n-k}(x)} \right)^{1/k} \]
are concave, we have \( F^{1/k}(x), G^{1/k}(x) \), which are convex functions. We take two points \( x, y \in \Gamma_n \) and \( 0 \leq t \leq 1 \), and then we let
\[ f(t) = F(tx + (1 - t)y), g(t) = G(tx + (1 - t)y). \]

Thus, we have
\[ \left( f^{1/k} \right)^{''} = \frac{f^{1/k - 2}}{k} \left( ff^{''} + \left( \frac{1}{k} - 1 \right) (f')^2 \right) \geq 0. \]
and
\[
(g^{1/k})'' = \frac{f^{1/k-2}}{k} \left( gg'' + \left( \frac{1}{k} - 1 \right) (g')^2 \right) \geq 0.
\]

Here, \(^'\) means that the derivative is taken with respect to variable \(t\). Now, using the previous two inequalities, we have
\[
\left( (\alpha f + g)^{1/k} \right)'' = \frac{1}{k} (\alpha f + g)^{1/k-2} \left[ (\alpha f'' + g'')(\alpha f + g) + \left( \frac{1}{k} - 1 \right) (\alpha f' + g')^2 \right] \\
= \frac{1}{k} (\alpha f + g)^{1/k-2} \left[ \alpha^2 \left( ff'' + \left( \frac{1}{k} - 1 \right) (f')^2 \right) + \left( gg'' + \left( \frac{1}{k} - 1 \right) (g')^2 \right) \\
+ \alpha \left( fg'' + gf'' + 2 \left( \frac{1}{k} - 1 \right) f'g' \right) \right] \\
\geq \frac{\alpha}{k} (\alpha f + g)^{1/k-2} \left( fg'' + gf'' + 2 \left( \frac{1}{k} - 1 \right) f'g' \right).
\]
(6.4)

Note that we have
\[
|2 f'g'| = 2 \left[ \left( \frac{f}{g} \right)^{1/2} g' \right] \left[ \left( \frac{g}{f} \right)^{1/2} f' \right] \leq \frac{f (g')^2}{g} + g (f')^2.
\]

Since \(\frac{1}{k} - 1 < 0\) and \(f > 0, g > 0\), we have
\[
fg'' + gf'' + 2 \left( \frac{1}{k} - 1 \right) f'g' \\
\geq fg'' + gf'' + \left( \frac{1}{k} - 1 \right) \left[ f \left( \frac{(g')^2}{g} \right) + g \left( \frac{(f')^2}{f} \right) \right] \\
\geq f \left[ g'' + \left( \frac{1}{k} - 1 \right) \left( \frac{(g')^2}{g} \right) \right] + g \left[ f'' + \left( \frac{1}{k} - 1 \right) \left( \frac{(f')^2}{f} \right) \right] \\
\geq 0.
\]
(6.5)

Combining Eqs. (6.4) and (6.5), we obtain the result.

A corresponding \(C^2\) estimate of convex solutions for the Dirichlet problem of the linear combination of the Hessian operators defined on a certain domain also holds:

**Corollary 26** Suppose a function \(u\) defined on a domain \(\Omega \subset \mathbb{R}^n\) is a convex solution of the linear combination of the Hessian operators
\[
Q(D^2u) = \sum_{m=0}^{k} \alpha_m \sigma_m (D^2u) = \psi(x, u, Du)
\]
(6.6)

with coefficients \(\alpha_0, \ldots, \alpha_{k-1} \geq 0\) and \(\alpha_k > 0\) satisfying Condition (C), for a positive function \(\psi(x, u, Du) \in C^2(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)\), then a constant \(C\) exists depending only on \(n, k, \|u\|_{C^1}, \inf \psi\) and \(\|\psi\|_{C^2}\), such that
\[
\max_{\Omega} D^2u \leq C + \max_{\partial \Omega} |D^2u|.
\]
(6.7)
Here, $Du$, $D^2u$ are the gradient and Hessian of $u$, respectively.

In Sect. 4, the admissible solution sets of the sum Hessian equations have been obtained. Thus, we can obtain some existence results for the sum Hessian Eqs. (1.4) in their admissible sets. At the same time, inspired by [46], the global $C^2$ estimates for $Q_S^m$ Hessian equation in $\tilde{\Gamma}_n$ also can be obtained. A detailed discussion of these results will be presented out in a forthcoming paper by Ren [45].

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