RESONANCES AS VISCOSITY LIMITS FOR EXPONENTIALLY DECAYING POTENTIALS

HAOREN XIONG

Abstract. We show that the complex absorbing potential (CAP) method for computing scattering resonances applies to the case of exponentially decaying potentials. That means that the eigenvalues of 

$$-\Delta + V - i\varepsilon x^2, \quad |V(x)| \leq Ce^{-2\gamma|x|}$$

converge, as \(\varepsilon \to 0^+\), to the poles of the meromorphic continuation of \((-\Delta + V - \lambda^2)^{-1}\) uniformly on compact subsets of \(\Re \lambda > 0, \Im \lambda > -\gamma, \arg \lambda > -\pi/8\).

1. Introduction

The complex absorbing potential (CAP) method has been used as a computational tool for finding scattering resonances – see Riss–Meyer [RiMe] and Seideman–Miller [SeMi] for an early treatment and Jagau et al [JZBRK] for some recent developments. For potentials \(V \in L^\infty_{\text{comp}}\) the method was justified by Zworski [Z2]. In [Xi] the author extended it to potentials which are dilation analytic near infinity. In this paper we show that the CAP method is also valid for potentials which are exponentially decaying. While the key component of [Z2] and [Xi] was the method of complex scaling (see Hislop–Sigal [HS], Dyatlov–Zworski [DyZ] for an account and references), here we use complex scaling on the Fourier transform side following Nakamura [Na] and Kameoka–Nakamura [KaNa].

Thus, we consider the Schrödinger operator \(P := -\Delta + V\) acting on \(L^2(\mathbb{R}^n)\) whose potential is exponentially decreasing, this means that there exist \(C, \gamma > 0\) such that

$$|V(x)| \leq Ce^{-2\gamma|x|}. \quad (1.1)$$

Let \(R_V(\lambda) = (P - \lambda^2)^{-1}\) be the resolvent of \(P\), initially defined for \(\Im \lambda > 0\). The exponentially weighted resolvent \(\sqrt{V} R_V(\lambda) \sqrt{V}\) can be meromorphically continued to the strip \(\Im \lambda > -\gamma\), see Froese [F], Gannot [Ga] and a review in §2. Resonances of \(P\) are the poles in this meromorphic continuation.

We now introduce a regularized operator,

$$P_\varepsilon := -\Delta - i\varepsilon x^2 + V, \quad \varepsilon > 0. \quad (1.2)$$

(We write \(x^2 := x_1^2 + \cdots + x_n^2\).) It is easy to see, with details reviewed in §4, that \(P_\varepsilon\) is a non-normal unbounded operator on \(L^2(\mathbb{R}^n)\) with a discrete spectrum. When \(V \equiv 0\), \(P_\varepsilon\) is reduced to the rescaled Davies harmonic oscillator, whose spectrum is
given by \( \sqrt{\varepsilon} e^{-i\pi/4}(2|\ell| + n), \ell \in \mathbb{N} \) – see §3. Thus we will restrict our attentions to \( \arg z > -\pi/4 \). Suppose that
\[
\sigma(P_\varepsilon) \cap \mathbb{C} \setminus e^{-i\pi/4}[0, \infty) = \{ \lambda_j(\varepsilon)^2 \}_{j=1}^\infty, \quad -\pi/8 < \arg \lambda_j(\varepsilon) < 7\pi/8.
\] (1.3)

Zworski [Z2] proved that resonances can be defined as the limit points of \( \{ \lambda_j(\varepsilon) \}_{j=1}^\infty \) as \( \varepsilon \to 0^+ \), in the case of compactly supported potentials. We generalize this result to the case of exponentially decaying potentials. More precisely, we have

**Theorem 1.** For any \( 0 < a' < a < b \) and \( \gamma' < \gamma \) such that the rectangle
\[
\Omega := (a', a) + i(-\gamma', b) \subseteq \{ \lambda \in \mathbb{C} : -\pi/8 < \arg \lambda < 7\pi/8 \},
\] (1.4)
we have, uniformly on \( \Omega \),
\[
\lambda_j(\varepsilon) \to \lambda_j, \quad \varepsilon \to 0^+,
\]
where \( \lambda_j \) are the resonances of \( P \).

**Notation.** We use the following notation: \( f = \mathcal{O}_\ell(g) \) means that \( \|f\|_H \leq C_\ell g \) where the norm (or any seminorm) is in the space \( H \), and the constant \( C_\ell \) depends on \( \ell \). When either \( \ell \) or \( H \) are absent then the constant is universal or the estimate is scalar, respectively. When \( G = \mathcal{O}_\ell(g) : H_1 \to H_2 \) then the operator \( G : H_1 \to H_2 \) has its norm bounded by \( C_\ell g \). Also when no confusion is likely to result, we denote the operator \( f \mapsto gf \) where \( g \) is a function by \( g \).

**Acknowledgments.** The author would like to thank Maciej Zworski for helpful discussions. This project was supported in part by the National Science Foundation grant 1500852.

## 2. Meromorphic Continuation

In this section we will introduce a meromorphic continuation of the weighted resolvent \( \sqrt{\mathcal{V}} R_V(\lambda) \sqrt{\mathcal{V}} \) from \( \text{Im} \lambda > 0 \) to the strip \( \text{Im} \lambda > -\gamma \) under the assumption (1.1). As in [F], we define the resonances of \( P \) as the poles of this meromorphic continuation, with agreement of multiplicities. For a detailed presentation, we refer to [F].

Let \( R_0(\lambda) := (-\Delta - \lambda^2)^{-1} \) be the free resolvent. For \( \text{Im} \lambda > 0 \), the resolvent equation
\[
R_0(\lambda) - R_V(\lambda) = R_V(\lambda) V R_0(\lambda) = 0
\]
implies
\[
(I - \sqrt{\mathcal{V}} R_V(\lambda) \sqrt{\mathcal{V}})(I + \sqrt{\mathcal{V}} R_0(\lambda) \sqrt{\mathcal{V}}) = I.
\]
Since \( R_0(\lambda) = \mathcal{O}(\|\text{Im} \lambda\|^{-1}) : L^2 \to L^2 \), then for \( \text{Im} \lambda \) large, \( I + \sqrt{\mathcal{V}} R_0(\lambda) \sqrt{\mathcal{V}} \) is invertible by a Neumann series argument and
\[
I - \sqrt{\mathcal{V}} R_V(\lambda) \sqrt{\mathcal{V}} = (I + \sqrt{\mathcal{V}} R_0(\lambda) \sqrt{\mathcal{V}})^{-1}.
\] (2.1)
We will show that the right side of (2.1) has a meromorphic continuation. For that, we recall some bounds of the free resolvent with exponential weights, see [Ga] for details, to prove the following lemma:

**Lemma 1.** For any \( a > 0 \) and \( \gamma' < \gamma \),

\[
\lambda \mapsto (I + \sqrt{V}R_0(\lambda)\sqrt{V})^{-1}, \quad \text{Re} \lambda > a, \ \text{Im} \lambda > -\gamma',
\]

is a meromorphic family of operators on \( L^2(\mathbb{R}^n) \) with poles of finite rank.

**Proof.** Choose \( \varphi \in C^\infty(\mathbb{R}^n) \) satisfying \( \varphi(x) = |x| \) for large \( |x| \), it is well known that for each \( c > 0 \), the weighted resolvent:

\[
e^{-c\varphi}R_0(\lambda)e^{-c\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)
\]

extends analytically across \( \text{Re} \lambda > 0 \) to the strip \( \text{Im} \lambda > -c \), see [Ga, §1] and references given there. Moreover, Gannot [Ga, §1] proved that for any \( a, c, \varepsilon > 0 \) and \( \alpha \in \mathbb{N}^n, |\alpha| \leq 2 \) there exists \( C_\alpha = C_\alpha(a, c, \varepsilon) \) such that

\[
\|D^\alpha(e^{-c\varphi}R_0(\lambda)e^{-c\varphi})\|_{L^2 \to L^2} \leq C_\alpha |\lambda|^{|\alpha|-1}, \quad \text{for} \ \text{Im} \lambda > -c + \varepsilon, \ \text{Re} \lambda > a.
\] (2.2)

In particular, for \( \text{Re} \lambda > a \) and \( \text{Im} \lambda > -\gamma' \),

\[
\lambda \mapsto e^{-\gamma'\varphi}R_0(\lambda)e^{-\gamma'\varphi}
\]

is an analytic family of operators \( L^2 \to H^2 \). Since \( \lim_{|x| \to \infty} \sqrt{V}(x)e^{\gamma'\varphi(x)} = 0 \) by (1.1), it is easy to see that \( \sqrt{V}e^{\gamma'\varphi} : H^2 \to L^2 \) is compact. Hence,

\[
\lambda \mapsto \sqrt{V}R_0(\lambda)\sqrt{V} = \sqrt{V}e^{\gamma'\varphi}(e^{-\gamma'\varphi}R_0(\lambda)e^{-\gamma'\varphi})\sqrt{V}e^{\gamma'\varphi}
\]

is an analytic family of compact operators \( L^2 \to L^2 \) for \( \text{Re} \lambda > a, \ \text{Im} \lambda > -\gamma' \). Recalling that \( I + \sqrt{V}R_0(\lambda)\sqrt{V} \) is invertible for \( \text{Im} \lambda \gg 1 \), then by the analytic Fredholm theory – see [DyZ, §C.4], \( \lambda \mapsto (I + \sqrt{V}R_0(\lambda)\sqrt{V})^{-1} \) is a meromorphic family of operators in the same range of \( \lambda \). \( \square \)

From now on, we identify the resonances \( \lambda_j \), in \( \Omega \) given in (1.4), with the poles of \( (I + \sqrt{V}R_0(\lambda)\sqrt{V})^{-1} \), with agreement of multiplicities. More precisely, the multiplicity of resonance \( \lambda \) is given by

\[
m(\lambda) := \frac{1}{2\pi i} \text{tr} \oint_{\lambda} (I + \sqrt{V}R_0(\zeta)\sqrt{V})^{-1} \partial_\zeta (\sqrt{V}R_0(\zeta)\sqrt{V}) d\zeta,
\] (2.3)

where the integral is over a positively oriented circle enclosing \( \lambda \) and containing no poles other than \( \lambda \).
3. Resolvent Estimates for the Davies Harmonic Oscillator

The operator $H_c := -\Delta + cx^2$, $-\pi < \arg c \leq 0$, was used by Davies [Da] to illustrate properties of non-normal differential operators. We recall some known facts about $H_c$ and its resolvent. As established in [Da], $H_c$ is an unbounded operator on $L^2(\mathbb{R}^n)$ with the discrete spectrum given by

$$\sigma(H_c) = e^{1/2}(n + 2|N_0|), \quad |\alpha| = \alpha_1 + \cdots + \alpha_n. \quad (3.1)$$

In particular $\sigma(H_{-i\varepsilon}) \subset e^{-\varepsilon^4/4}[0, \infty)$, then one can study the resolvent of $H_{-i\varepsilon}$ outside $e^{-i\varepsilon/4}[0, \infty)$. Unlike the normal operators, there does not exist any constant $C$ such that $\|(-\Delta - i\varepsilon x^2 - z)^{-1}\|_{L^2 \to L^2} \leq C \text{dist}(z, e^{-i\varepsilon/4}[0, \infty))^{-1}$. Instead, according to Hitrik–Sjöstrand–Viola [HSV], [Z2, §3] and references given there, for $\Omega \in \{z : -\pi/2 < \arg z < 0\} \setminus e^{-i\varepsilon/4}[0, \infty)$, there exists $C = C(\Omega)$ such that

$$\frac{1}{C} e^{-\frac{i}{2}/C} \leq \|(-\Delta - i\varepsilon x^2 - z)^{-1}\|_{L^2 \to L^2} \leq C e^{-\frac{i}{2}}, \quad z \in \Omega. \quad (3.2)$$

In this section we will show how exponential weights dramatically improve the bound (3.2) for $(-\Delta - i\varepsilon x^2 - \lambda^2)^{-1}$ in the rectangle $\Omega$ given by (1.4), which will be crucial in the proof of Theorem 1.

First, note that $-\Delta_x - i\varepsilon x^2 = F^{-1}(\xi^2 + i\varepsilon \Delta_x)F$, where $F$ denotes the Fourier transform $F u(\xi) = \hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-i\xi \cdot x} u(x) \, dx$. Inspired by [Na] and [KaNa], we introduce a family of spectral deformations in the Fourier space as follows. For a more detailed introduction of the spectral deformation theory, we refer to [HS, §17, §18].

For any fixed $\Omega$ given in (1.4), we choose $\rho \in C^\infty((0, \infty); \mathbb{R})$ with $\rho \equiv 0$ near 0 and $\rho(t) \equiv 1$ for $t \gg 1$ such that

$$0 \leq \rho'(t) < \gamma^{-1} \tan \frac{\pi}{8}, \quad \forall t \geq 0; \quad \Omega \in \{x + iy : x > 0, \ y > -\gamma \rho(x)\}, \quad (3.3)$$

and define the map

$$\psi : \mathbb{R}^n \to \mathbb{R}^n, \quad \psi(\xi) = |\xi|^{-1} \rho(|\xi|) \xi, \quad (3.4)$$

then $\psi$ is smooth with the Jacobian:

$$D\psi(\xi) = |\xi|^{-1} \rho(|\xi|) I + (|\xi|^{-2} \rho'(|\xi|) - |\xi|^{-3} \rho(|\xi|)) \xi \cdot \xi^T. \quad (3.5)$$

Let $A$ be an orthogonal matrix with $n$-th column $|\xi|^{-1} \xi$, then we have

$$A^T D\psi(\xi) A = \text{diag}[|\xi|^{-1} \rho(|\xi|), \cdots, |\xi|^{-1} \rho(|\xi|), \rho'(|\xi|)]. \quad (3.6)$$

For $\theta \in \mathbb{R}$, we consider a family of deformations:

$$\phi_\theta(\xi) = \xi + \theta \psi(\xi), \quad (3.7)$$

and the corresponding unitary operators $U_\theta$, $\theta \in \mathbb{R}$ defined by

$$U_\theta u(\xi) := (\det D\phi_\theta(\xi))^{\frac{1}{2}} u(\phi_\theta(\xi)). \quad (3.8)$$
Using (3.6), we can compute \( \det D\phi_\theta(\xi) \) explicitly, i.e. 

\[
J_\theta(\xi) \equiv \det D\phi_\theta(\xi) = \det(I + \theta D\psi(\xi)) = (1 + \theta \rho'(|\xi|))(1 + \theta|\xi|^{-1}\rho(|\xi|)^{n-1}),
\]

then by (3.3), \( U_\theta \) is invertible as \( \det D\phi_\theta(\xi) \neq 0 \) for \( \theta \in \mathbb{R}, |\theta| < \gamma \), the inverse is given by 

\[
U^{-1}_\theta v(\xi) = (\det D\phi_\theta(\phi^{-1}(\xi)))^{-\frac{1}{2}}v(\phi^{-1}(\xi)).
\]

Now we consider the deformed operators of \( \xi^2 + i\varepsilon\Delta_\xi \):

\[
Q_{\varepsilon,\theta} := U_\theta(\xi^2 + i\varepsilon\Delta_\xi)U^{-1}_\theta
\]

\[
= \phi_\theta(\xi)^2 - i\varepsilon J_\theta(\xi)^{-\frac{1}{2}}D\xi_j J_\theta(\xi) J^{lj}(\xi) D\xi_l J_\theta(\xi)^{-\frac{1}{2}}
\]

where \( D\xi_k = -i\partial_{\xi_k} \), \( J_\theta(\xi) = \det D\phi_\theta(\xi) \), \( J^{lj}(\xi) = [D\phi_\theta(\xi)^{-1}]_{jl} \), see [HS, §18.1] for a detailed calculation. To extend \( Q_{\varepsilon,\theta} \) to \( \theta \in \mathbb{C} \), we define

\[
D_\gamma := \{ \theta \in \mathbb{C} : |\text{Re}\theta| + |\text{Im}\theta| < \gamma \}.
\]

In view of (3.3) and (3.9), \( D\phi_\theta^{-1} \) and \( \det D\phi_\theta \) extend analytically to \( \theta \in D_\gamma \). As in [HS, §18.1], we obtain that \( Q_{\varepsilon,\theta} \), given by the second equation in (3.11), extends analytically to \( \theta \in D_\gamma \).

Then we introduce some preliminary results about the spectrum of \( Q_{\varepsilon,\theta} \):

**Proposition 1.** There exists constant \( \varepsilon_0 = \varepsilon_0(\Omega, \gamma) \) such that for all \( 0 < \varepsilon < \varepsilon_0 \) and \( \theta \in D_\gamma \),

\[
\sigma(Q_{\varepsilon,\theta}) \cap \{ z \in \mathbb{C} : |z| > 1, \pi/2 < \arg z < \pi \} = \emptyset.
\]

**Proof.** We note that for \( \theta \in D_\gamma \), by (3.3),

\[
1 - \tan \frac{\pi}{8} < 1 - |\theta||\rho'(t)| \leq 1 + |\theta||\rho'(t)| \leq 1 + |\theta||\rho'(t)| < 1 + \tan \frac{\pi}{8}, \quad \forall t \geq 0.
\]

Thus, (3.9) implies that \( C^{-1} < |J_\theta(\xi)| < C \) for some constant \( C > 0 \). Since

\[
[D\phi_\theta(\xi)]_{jl} = \left(1 + \theta^\rho(|\xi|)/|\xi|\right)\delta_{jl} + \frac{\theta|\xi|\rho(|\xi|)}{|\xi|^3} \xi_j \xi_l
\]

by (3.5), and \( \rho' \in C^\infty_c((0, \infty)) \), together with (3.9), we conclude that

\[
J_\theta, J^{-1}_\theta, J^{lj} \in C^\infty_b(\mathbb{R}^n), \quad 1 \leq j, l \leq n.
\]

Here \( C^\infty_b(\mathbb{R}^n) := \{ u \in C^\infty(\mathbb{R}^n) : |\partial^\alpha u| \leq C_\alpha \text{ for all } \alpha \in \mathbb{N}^n_0 \} \). Hence we have

\[
Q_{\varepsilon,\theta} = \phi_\theta(\xi)^2 - i\varepsilon J^{lj}(\xi) D\xi_j D\xi_l + \varepsilon a_j(\xi) D\xi_j + \varepsilon b(\xi),
\]

where \( a_j, b \in C^\infty_b(\mathbb{R}^n) \). Let \( h = \sqrt{\varepsilon} \), then \( Q_{\varepsilon,\theta} = q_\theta(\xi, hD\xi; h) \) is a semiclassical differential operator – see Zworski [Z1, §4], with the symbol

\[
q_\theta(\xi, \xi^*; h) = \phi_\theta(\xi)^2 - i(D\phi_\theta(\xi)^{-2}\xi^* - \xi^*) \cdot \xi^* + ha_j(\xi)\xi_j^* + h^2 b(\xi),
\]

where \( \phi_\theta(\xi, \xi^*; \gamma) \) is the indicial equation for the elliptic operator

\[
D(1 + \theta \rho'(\xi), J^{lj}(\xi) D\xi_j D\xi_l) - \varepsilon a_j(\xi) D\xi_j + \varepsilon b(\xi).
\]
where \((\xi, \xi^*)\) are coordinates of \(T^*\mathbb{R}^n\). Choose \(m(\xi, \xi^*) = 1 + \xi^2 + \xi^*^2\) as an order function, we recall the symbol class \(S(m)\) from [Z1, §4.4],
\[
S(m) := \{a \in C^\infty : |\partial^\alpha a| \leq C_{\alpha} m \quad \text{for } \forall \alpha \in \mathbb{N}_0^{2n}\}. \tag{3.16}
\]
Then by (3.3), (3.7) and (3.13), we have \(q_\theta \in S(m)\). Hence it suffices to show that there exists constant \(h_0 > 0\) such that for \(h < h_0\),
\[
q_\theta - z \text{ is elliptic in } S(m) \text{ for } |z| > 1, \ \pi/2 < \arg z < \pi.
\]
For a detailed introduction of general elliptic theory, we refer to [Z1, §4].

Using (3.4) we calculate:
\[
\phi_\theta(\xi)^2 = (\xi + \theta \psi(\xi)) \cdot (\xi + \theta \psi(\xi)) = (|\xi| + \theta \rho(|\xi|))^2. \tag{3.17}
\]
Then for \(\theta \in D_\gamma\), by (3.3), we have
\[
-\pi/4 < \arg \phi_\theta(\xi)^2 < \pi/4, \quad |\phi_\theta(\xi)^2| > \left(1 - \tan \frac{\pi}{8}\right)^2 |\xi|^2. \tag{3.18}
\]
To obtain similar bounds for the argument and modulus of \((D\phi_\theta(\xi)^{-2}\xi^*) \cdot \xi^*\), we recall (3.6) to compute
\[
(D\phi_\theta^{-2}\xi^*) \cdot \xi^* = (1 + \theta \rho(|\xi|)|\xi|^{-1})^{-2}(\eta_1^* + \cdots + \eta_{n-2}^* - 2) + (1 + \theta \rho'(|\xi|)^{-2} \eta_n^*, \tag{3.19}
\]
where \(\eta^* = A^T \xi^* \in \mathbb{R}^n\) with the same orthogonal matrix \(A\) as in (3.6). By (3.3), for \(\theta \in D_\gamma\), we have
\[
\pm \text{Im} \theta \geq 0 \implies 0 \leq \pm \arg(1 + \theta \rho(|\xi|)|\xi|^{-1}), \quad \pm \arg(1 + \theta \rho'(|\xi|)) < \pi/8.
\]
Hence, for all \(\theta \in D_\gamma\),
\[
\pm \text{Im} \theta \geq 0 \implies 0 \leq \mp \text{arg} (D\phi_\theta^{-2}\xi^*) \cdot \xi^* < \pi/4, \tag{3.20}
\]
and by applying the following basic inequality with (3.3) to (3.19),
\[
|r_1 e^{i\theta_1} + r_2 e^{i\theta_2}|^2 = r_1^2 + r_2^2 + 2r_1r_2 \cos(\theta_1 - \theta_2) \geq \frac{1 - |\cos(\theta_1 - \theta_2)|}{2}(r_1 + r_2)^2, \tag{3.21}
\]
we also obtain that for all \(\theta \in D_\gamma\),
\[
|(D\phi_\theta^{-2}\xi^*) \cdot \xi^*| \geq C|\eta^*|^2 = C|\xi^*|^2. \tag{3.22}
\]
Since \(\text{arg}(\phi_\theta(\xi)^2 - z) \in (-\pi/2, \pi/4)\) for \(\pi/2 < \arg z < \pi\) and \(\arg -i(D\phi_\theta^{-2}\xi^*) \cdot \xi^* \in (-3\pi/4, -\pi/4)\) by (3.20), using (3.21) together with (3.18) and (3.22), we have
\[
|\phi_\theta(\xi)^2 - z - i(D\phi_\theta^{-2}\xi^*) \cdot \xi^*| \geq C|\phi_\theta(\xi)^2 - z| + C - i(D\phi_\theta^{-2}\xi^*) \cdot \xi^*| \geq C|\phi_\theta(\xi)^2| + C|z| + C|\xi^*|^2 \tag{3.23}
\]
\[
\geq C(1 + |\xi|^2 + |\xi^*|^2) = Cm.
\]
Then by (3.15), we conclude that there exists \(h_0 > 0\) such that for all \(h < h_0\), \(|q_\theta - z| \geq Cm\), which completes the proof. \(\square\)
Figure 1. An illustration of the results of Proposition 2 in the case of dimension $\dim = 1$, $\beta = 0.4$, which shows that the numerical range of the principal symbol of $Q_{\varepsilon,-0.4i}$ avoids the region $\{\lambda^2 : \lambda \in \Omega\}$. We choose $\rho = 0.4 \tanh$ to compute the numerical range of $(\phi_{-0.4i}(\xi)^2 - i(\phi'_{-0.4i}(\xi))^{-2}\xi^*^2)^{1/2}$.

**Proposition 2.** For any $\beta \in (\gamma, \gamma')$ satisfying
\begin{equation}
\Omega \in \{x + iy : x > 0, y > -\beta \rho(x)\},
\end{equation}
there exists $\varepsilon_0 = \varepsilon_0(\Omega, \gamma, \beta)$ such that for all $0 < \varepsilon < \varepsilon_0$,
\[\sigma(Q_{\varepsilon,-i\beta}) \cap \{\lambda^2 : \lambda \in \Omega\} = \emptyset.\]

**Proof.** As in the proof of Proposition 1, it suffices to show that there exists $h_0 = h_0(\Omega, \gamma, \beta)$ such that for $0 < h < h_0$,
\[q_{-i\beta}(\xi, \xi^*; h) - \lambda^2 \text{ is elliptic in } S(m) \text{ for } \lambda \in \Omega.\]

Recalling $\arg -i(D\phi_{-i\beta}^2 \xi^*) \cdot \xi^* \in [-\pi/4, -\pi/2]$ by (3.20), in order to apply (3.21), we claim that
\[\exists \delta > 0 \text{ s.t. } \arg(\phi_{-i\beta}(\xi)^2 - \lambda^2) \leq \pi/2 - \delta \text{ or } \geq 3\pi/4 + \delta, \text{ for all } \lambda \in \Omega, \xi \in \mathbb{R}^n. \quad (3.25)\]
We notice that for $|\xi| \gg 1$, $\phi_{-i\beta}(\xi)^2 = (|\xi| - i\beta)^2$ by (3.17), thus $\arg(\phi_{-i\beta}(\xi)^2 - \lambda^2) \in (-\pi/4, 0)$, in other words, there exists some large $R$ such that (3.25) holds for $|\xi| > R$ with $\delta = \pi/2$. It remains to show that (3.25) holds for all $|\xi| \leq R$ and $\lambda \in \Omega$. We
argue by contradiction: if it does not hold, there must exist $\lambda \in \Omega$, $\xi \in \mathbb{R}^n$ such that
\[
\arg(\phi_{-i\beta}(\xi)^2 - \lambda^2) \in [\pi/2, 3\pi/4],
\]
i.e.
\[
0 \leq -\text{Re} ((|\xi| - i\beta \rho(|\xi|))^2 - \lambda^2) \leq \text{Im} ((|\xi| - i\beta \rho(|\xi|))^2 - \lambda^2),
\]
which immediately implies $\text{Im} \lambda \leq 0$. Let $t = |\xi|$ and write $\lambda = x - iy$, then we have
\[
x^2 - y^2 - t^2 + \beta^2 \rho(t)^2 \leq 2xy - 2\beta t \rho(t)
\]
(3.26)
\[
\beta t \rho(t) \leq xy
\]
(3.27)
Since $x > 0$ and $0 \leq y < \beta x$ by (3.24), then (3.26) implies that
\[
x^2 - 2\beta x \rho(x) - \beta^2 \rho(x)^2 < t^2 - 2\beta t \rho(t) - \beta^2 \rho(t)^2.
\]
Let $S(x) = x^2 - 2\beta x \rho(x) - \beta^2 \rho(x)^2$, by (3.3),
\[
S'(x) = 2x \left( 1 - \beta \frac{\rho(x)}{x} - \beta' \rho'(x) - \beta \frac{\rho(x)}{x} \cdot \beta' \rho'(x) \right)
\]
\[
> 2x \left( 1 - 2 \tan \frac{\pi}{8} - \tan^2 \frac{\pi}{8} \right) = 0,
\]
thus $S(x) < S(t) \implies x < t$. Recalling that $\rho$ is non-decreasing, we have $\beta t \rho(t) \geq \beta x \rho(x) > xy$, which contradicts (3.27). Hence (3.25) holds, using (3.21) and (3.22), we obtain that
\[
|\phi_{-i\beta}(\xi)^2 - \lambda^2 - i(D\phi_{-i\beta}^2 \xi^*) \cdot \xi^*| \geq C(\delta)(|(|\xi| - i\beta \rho(|\xi|))^2 - \lambda^2| + |\xi^*|^2).
\]
Since for $|\xi| \gg 1$,
\[
|(|\xi| - i\beta \rho(|\xi|))^2 - \lambda^2| = |(|\xi| - i\beta)^2 - \lambda^2| \geq |\xi|^2 - \beta^2 - |\lambda|^2,
\]
there exists $R = R(\Omega, \beta) > 0$ such that $|(|\xi| - i\beta \rho(|\xi|))^2 - \lambda^2| \geq (1 + |\xi|^2)/2$ whenever $|\xi| > R$. We also note that, by (3.24),
\[
\text{dist} \left( \{t - i\beta \rho(t) : t \geq 0 \} \cup \Omega \right) \geq C = C(\Omega, \gamma, \beta) > 0,
\]
thus $|(|\xi| - i\beta \rho(|\xi|))^2 - \lambda^2| \geq C^2 \geq C^2(1 + R^2)^{-1}(1 + |\xi|^2)$ for $|\xi| \leq R$. Hence $|\phi_{-i\beta}(\xi)^2 - \lambda^2 - i(D\phi_{-i\beta}^2 \xi^*) \cdot \xi^*| \geq C(1 + |\xi|^2 + |\xi^*|^2)$, where $C$ determined by $\Omega, \gamma, \beta$. Then by (3.15), we conclude that there exist $h_0 = h_0(\Omega, \gamma, \beta)$ and $C = C(\Omega, \gamma, \beta) > 0$ such that
\[
|q_{-i\beta}(\xi, \xi^*; h) - \lambda^2| \geq Cm(\xi, \xi^*),
\]
(3.28)
which completes the proof.

Now we state the main result of this section:

**Lemma 2.** For any $0 < a' < a < b$ and $\gamma' < \gamma$ such that the rectangle
\[
\Omega := (a', a) + i(-\gamma', b) \subseteq \{ \lambda \in \mathbb{C} : -\pi/8 < \arg \lambda < 7\pi/8 \},
\]
there exist constant $C = C(\Omega, \gamma) > 0$ and $\varepsilon_0 = \varepsilon_0(\Omega, \gamma) > 0$ such that uniformly for $0 < \varepsilon < \varepsilon_0$,

$$\|e^{-\gamma|x|}(-\Delta - i\varepsilon x^2 - \lambda^2)^{-1}e^{-\gamma|x|}\|_{L^2 \to L^2} \leq C, \quad \forall \lambda \in \Omega.$$ 

**Proof.** We consider the matrix element

$$B_{f,g}^\varepsilon(\lambda) := \langle e^{-\gamma|x|}(-\Delta - i\varepsilon x^2 - \lambda^2)^{-1}e^{-\gamma|x|}f, g \rangle_{L^2_x},$$

for $f, g \in L^2(\mathbb{R}^n)$, where $\langle u, v \rangle_{L^2} = \int_{\mathbb{R}^n} u\overline{v} \, dx$ is the standard $L^2$ inner product. It suffices to show that there exist $C, \varepsilon_0$ such that uniformly for $0 < \varepsilon < \varepsilon_0$,

$$|B_{f,g}^\varepsilon(\lambda)| \leq C\|f\|_{L^2}\|g\|_{L^2}, \quad \text{for all } f, g \in L^2, \lambda \in \Omega. \quad (3.29)$$

Recalling (3.1), both $-\Delta_x - i\varepsilon x^2 - \lambda^2$ and $\xi^2 + i\varepsilon \Delta_\xi - \lambda^2$ are invertible for $\lambda \in \Omega$. Then we have

$$B_{f,g}^\varepsilon(\lambda) = \langle (-\Delta_x - i\varepsilon x^2 - \lambda^2)^{-1}e^{-\gamma|x|}f, e^{-\gamma|x|}g \rangle_{L^2_x}$$

$$= \langle \mathcal{F}^{-1}(\xi^2 + i\varepsilon \Delta_\xi - \lambda^2)^{-1}\mathcal{F}e^{-\gamma|x|}f, e^{-\gamma|x|}g \rangle_{L^2_x}$$

$$= \langle (\xi^2 + i\varepsilon \Delta_\xi - \lambda^2)^{-1}\mathcal{F}(e^{-\gamma|x|}f) (\xi), \mathcal{F}(e^{-\gamma|x|}g)(\xi) \rangle_{L^2_x}. \quad (3.30)$$

Let $F_\gamma(\xi) := \mathcal{F}(e^{-\gamma|x|}f)(\xi)$ and $G_\gamma(\xi) := \mathcal{F}(e^{-\gamma|x|}g)(\xi)$, recalling the formula

$$\mathcal{F}(e^{-|x|})(\xi) = c_n(1 + \xi^2)^{-\frac{n+1}{2}}, \quad c_n = (2\pi)^\frac{n}{2}\Gamma\left((n + 1)/2\right)\pi^{-\frac{n+1}{2}},$$

then $F_\gamma = K_\gamma \ast \hat{f}$ and $G_\gamma = K_\gamma \ast \hat{g}$, where $K_\gamma(\xi) = c_n \gamma (\gamma^2 + \xi^2)^{-\frac{n+1}{2}}$.

First we consider, for $\theta \in \mathbb{R}$, $|\theta| < \gamma$ and $U_\theta$ defined by (3.8), the integral kernel of the map $U_\theta \circ (K_\gamma \ast )$:

$$K(\xi, \eta; \theta) := (\det D\phi_\theta(\xi))^{\frac{1}{2}}K_\gamma(\phi_\theta(\xi) - \eta), \quad \xi, \eta \in \mathbb{R}^n.$$ 

We claim that $K(\xi, \eta; \theta)$ has an analytic extension to $\theta \in D_\gamma$. Since $K_\gamma$ extends analytically to the strip $\{\xi \in \mathbb{C}^n : |\text{Im} \xi| < \gamma\}$, it suffices to show that $|\text{Im} \psi(\xi)| < \gamma$, which is a direct consequence of $\theta \in D_\gamma$ and $|\psi(\xi)| \leq 1$ by (3.4). Then for $\theta \in D_\gamma$, using (3.3) and (3.9), we can estimate $K(\xi, \eta; \theta)$ as follows:

$$|K(\xi, \eta; \theta)| \leq C\gamma |\gamma^2 + (\xi + \theta \psi(\xi) - \eta)^2|^{-\frac{n+1}{2}}$$

$$\leq C\gamma |\gamma^2 - |\text{Im} \theta|^2|\psi(\xi)|^2 + (\xi - \eta + \text{Re} \theta \psi(\xi))^2|^{-\frac{n+1}{2}}$$

$$\leq C\gamma (\gamma^2 - |\text{Im} \theta|^2 + (|\xi - \eta| - |\text{Re} \theta|^2)^2)^{-\frac{n+1}{2}}$$

thus

$$\max \left\{ \sup_{\xi \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(\xi, \eta; \theta)| \, d\eta, \sup_{\eta \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(\xi, \eta; \theta)| \, d\xi \right\}$$

$$\leq C\gamma \int_{\mathbb{R}^n} (\gamma^2 - |\text{Im} \theta|^2 + (|\xi| - |\text{Re} \theta|^2)^2)^{-\frac{n+1}{2}} \, dx \leq C(\gamma, \theta). \quad (3.31)$$
Hence, by Schur’s criterion, $U_\theta \circ (K_\gamma \ast \cdot)$, first defined for $\theta \in D_\gamma \cap \mathbb{R}$, with the integral kernel $K(\xi, \eta; \theta)$, extends to $\theta \in D_\gamma$ as an analytic family of operators $L^2 \to L^2$. In particular,

$$D_\gamma \ni \theta \mapsto U_\theta F_\gamma = U_\theta (K_\gamma \ast \hat{f}) \text{ and } U_\theta G_\gamma = U_\theta (K_\gamma \ast \hat{g}),$$

are two analytic families of functions in $L^2(\mathbb{R}^n)$.

Now we define

$$B^\varepsilon_{f,g}(\lambda; \theta) = \langle (Q_{\varepsilon, \theta} - \lambda^2)^{-1} U_\theta F_\gamma, U_\theta G_\gamma \rangle$$

for $\theta \in D_\gamma$, with $Q_{\varepsilon, \theta}$ given by (3.11), where we write $U_\theta G_\gamma$, instead of $U_\theta G_\gamma$. Then by Proposition 1, there exists $\varepsilon_0 = \varepsilon_0(\Omega, \gamma)$ such that for all $0 < \varepsilon < \varepsilon_0$, and $|\lambda| > 1$ with $\pi/4 < \arg \lambda < \pi/2$,

$$D_\gamma \ni \theta \mapsto B^\varepsilon_{f,g}(\lambda; \theta) \text{ is analytic.}$$

However, for $\theta \in D_\gamma \cap \mathbb{R}$, since $U_\theta$ is unitary, by (3.30) we have

$$B^\varepsilon_{f,g}(\lambda; \theta) = \langle U_\theta (\xi^2 + i\varepsilon \Delta_\xi - \lambda^2)^{-1} U_\theta^{-1} U_\theta F_\gamma, U_\theta G_\gamma \rangle = \langle (\xi^2 + i\varepsilon \Delta_\xi - \lambda^2)^{-1} F_\gamma, U_\theta G_\gamma \rangle = \langle (\xi^2 + i\varepsilon \Delta_\xi - \lambda^2)^{-1} F_\gamma, G_\gamma \rangle = B^\varepsilon_{f,g}(\lambda).$$

Thus by analyticity, $B^\varepsilon_{f,g}(\lambda; \theta) \equiv B^\varepsilon_{f,g}(\lambda)$, $\forall \theta \in D_\gamma$ whenever $|\lambda| > 1$, $\pi/4 < \arg \lambda < \pi/2$. In particular, for fixed $\beta \in (\gamma', \gamma)$ satisfying (3.24),

$$B^\varepsilon_{f,g}(\lambda) = B^\varepsilon_{f,g}(\lambda; -i\beta) \text{ whenever } |\lambda| > 1, \pi/4 < \arg \lambda < \pi/2.$$ 

In view of Proposition 2 and (3.1), both $B^\varepsilon_{f,g}(\lambda)$ and $B^\varepsilon_{f,g}(\lambda; -i\beta)$ are analytic in $\Omega$. Without loss of generality, we may assume that $a > 1$ in (1.4), then

$$\Omega \cap \{ \lambda : |\lambda| > 1, \pi/4 < \arg \lambda < \pi/2 \} \neq \emptyset,$$

where $B^\varepsilon_{f,g}(\lambda)$ and $B^\varepsilon_{f,g}(\lambda; -i\beta)$ coincide. Hence by analyticity, we conclude that for each $0 < \varepsilon < \varepsilon_0$,

$$B^\varepsilon_{f,g}(\lambda) = B^\varepsilon_{f,g}(\lambda; -i\beta) = \langle (Q_{\varepsilon, -i\beta} - \lambda^2)^{-1} U_{-i\beta} F_\gamma, U_{i\beta} G_\gamma \rangle, \quad \forall \lambda \in \Omega. \quad (3.32)$$

By the elliptic theory of semiclassical differential operators – see [Z1, §4.7], (3.28) implies that there exists $\varepsilon_0 = \varepsilon_0(\Omega, \gamma, \beta)$ such that for all $0 < \varepsilon < \varepsilon_0$,

$$\|(Q_{\varepsilon, -i\beta} - \lambda^2)^{-1}\|_{L^2 \to L^2} \leq C(\Omega, \gamma, \beta), \quad \forall \lambda \in \Omega. \quad (3.33)$$

Recalling (3.31), by Schur’s criterion, we obtain that

$$\|U_{-i\beta} F_\gamma\|_{L^2} = \|U_{-i\beta} \circ (K_\gamma \ast \hat{f})\|_{L^2} \leq C(\gamma, \beta) \|\hat{f}\|_{L^2} = C(\gamma, \beta) \|f\|_{L^2},$$

$$\|U_{i\beta} G_\gamma\|_{L^2} = \|U_{i\beta} \circ (K_\gamma \ast \hat{g})\|_{L^2} \leq C(\gamma, \beta) \|\hat{g}\|_{L^2} = C(\gamma, \beta) \|g\|_{L^2}. \quad (3.34)$$

Combining (3.32), (3.33) and (3.34), also noticing that $\beta$ can be determined by $\Omega, \gamma$, we obtain (3.29) with $C = C(\Omega, \gamma)$, which completes the proof. \qed
4. Eigenvalues of the Regularized Operator

In this section we will review the meromorphy of the resolvent

\[ R_{\varepsilon}(\lambda) := (P_\varepsilon - \lambda^2)^{-1}, \quad \varepsilon > 0, \]

with \( P_\varepsilon \) in (1.2), in a similar form to the meromorphic continuation of the weighted resolvent \( \sqrt{V}R_\varepsilon(\lambda)\sqrt{V} \) given by (2.1).

First we write \( R_\varepsilon(\lambda) := (-\Delta - i\varepsilon x^2 - \lambda^2)^{-1} \) and recall

\[ R_\varepsilon(\lambda) = O_\delta(1/|\lambda|) : L^2 \to L^2, \quad \delta < \arg \lambda < 3\pi/4 - \delta, \quad |\lambda| > \delta, \quad (4.1) \]

which follows from (semiclassical) ellipticity. Then

\[ (P_\varepsilon - \lambda^2)R_\varepsilon(\lambda) = I + VR_\varepsilon(\lambda), \quad -\pi/8 < \arg \lambda < 7\pi/8, \quad (4.2) \]

In view of (4.1), \( I + VR_\varepsilon(\lambda) \) is invertible for \( \pi/4 < \arg \lambda < \pi/2, |\lambda| \gg 1 \). Since \( R_\varepsilon(\lambda) : L^2 \to H^2 \) is analytic in \( \{ \lambda : -\pi/8 < \arg \lambda < 7\pi/8 \} \), see (3.1), \( V : H^2 \to L^2 \) is compact by (1.1), we have \( \lambda \mapsto VR_\varepsilon(\lambda) \) is an analytic family of compact operators for \( -\pi/8 < \arg \lambda < 7\pi/8 \). Hence \( \lambda \mapsto (I + VR_\varepsilon(\lambda))^{-1} \) is a meromorphic family of operators in the same range of \( \lambda \). Using (4.2), we conclude that \( R_{\varepsilon}(\lambda) = R_\varepsilon(\lambda)(I + VR_\varepsilon(\lambda))^{-1} \)

is meromorphic for \( -\pi/8 < \arg \lambda < 7\pi/8 \) (in fact \( R_{\varepsilon}(\lambda) \) is meromorphic for \( \lambda \in \mathbb{C} \) by the Gohberg–Sigal factorization theorem - see \[ \text{DyZ, §C.4} \]), with poles \( \{\lambda_j(\varepsilon)\}_{j=1}^\infty \), i.e. \( \{\lambda_j(\varepsilon)^2\}_{j=1}^\infty \) are the eigenvalues of \( P_\varepsilon \) in \( \{z \in \mathbb{C} : \arg z \neq -\pi/4 \} \).

Then we have

**Lemma 3.** For each \( \varepsilon > 0 \),

\[ \lambda \mapsto (I + \sqrt{V}R_\varepsilon(\lambda)\sqrt{V})^{-1}, \quad -\pi/8 < \arg \lambda < 7\pi/8, \]

is a meromorphic family of operators on \( L^2(\mathbb{R}^n) \) with poles of finite rank. Moreover,

\[ m_\varepsilon(\lambda) := \frac{1}{2\pi i} \text{tr} \int_\lambda (I + \sqrt{V}R_\varepsilon(\zeta)\sqrt{V})^{-1}\partial_\zeta (\sqrt{V}R_\varepsilon(\zeta)\sqrt{V}) \, d\zeta, \quad (4.3) \]

where the integral is over a positively oriented circle enclosing \( \lambda \) and containing no poles other than possibly \( \lambda \), satisfies

\[ m_\varepsilon(\lambda) = \frac{1}{2\pi i} \text{tr} \int_\lambda (\zeta^2 - P_\varepsilon)^{-1}2\zeta \, d\zeta. \quad (4.4) \]

**Remark.** The multiplicity of an eigenvalue \( \lambda^2 \) of \( P_\varepsilon \) can be defined by the right side of (4.4), thus Lemma 3 implies that the poles of \( (I + \sqrt{V}R_\varepsilon(\lambda)\sqrt{V})^{-1} \) coincide with \( \{\lambda_j(\varepsilon)\}_{j=1}^\infty \) given in (1.3), with agreement of multiplicities.

**Proof.** Following the above argument, it easy to see that \( \lambda \mapsto \sqrt{V}R_\varepsilon(\lambda)\sqrt{V} \) is an analytic family of compact operators for \( -\pi/8 < \arg \lambda < 7\pi/8 \). Then

\[ \lambda \mapsto (I + \sqrt{V}R_\varepsilon(\lambda)\sqrt{V})^{-1}, \quad -\pi/8 < \arg \lambda < 7\pi/8, \]
is a meromorphic family of operators, since $I + \sqrt{V} R_\varepsilon(\lambda) \sqrt{V}$ is invertible for $\pi/4 < \arg \lambda < \pi/2$, $|\lambda| \gg 1$ by (4.1). In this range of $\lambda$, $I + VR_\varepsilon(\lambda)$ is also invertible by the Neumann series argument, thus we have

\[(P_\varepsilon - \lambda^2)^{-1} = R_\varepsilon(\lambda)(I + VR_\varepsilon(\lambda))^{-1}\]

\[= R_\varepsilon(\lambda) \sum_{j=0}^{\infty} (-1)^j (VR_\varepsilon(\lambda))^j\]

\[= R_\varepsilon(\lambda)(I - \sqrt{V} \sum_{j=0}^{\infty} (-1)^j(\sqrt{VR_\varepsilon(\lambda)}\sqrt{V})^j \sqrt{VR_\varepsilon(\lambda)})\]

\[= R_\varepsilon(\lambda)[I - \sqrt{V}(I + \sqrt{VR_\varepsilon(\lambda)}\sqrt{V})^{-1}\sqrt{VR_\varepsilon(\lambda)}].\]

Since both sides of (4.5) are meromorphic for $-\pi/8 < \arg \lambda < 7\pi/8$, by meromorphy, we conclude that (4.5) holds for all $-\pi/8 < \arg \lambda < 7\pi/8$, as an identity between meromorphic families of operators.

To obtain the multiplicity formula, we fix any $\lambda$ with $-\pi/8 < \arg \lambda < 7\pi/8$, then there exists a neighborhood $\lambda \in U$ in this half plane and finite rank operators $A_j$, $1 \leq j \leq J$ such that $(I + \sqrt{VR_\varepsilon(\zeta)}\sqrt{V})^{-1} - \sum_{j=1}^{J} \frac{A_j}{(\zeta - \lambda)^j}$ is analytic in $\zeta \in U$. Let $C_\lambda \subset U$ be a positively oriented circle enclosing $\lambda$ and containing no poles of $(I + \sqrt{VR_\varepsilon(\zeta)}\sqrt{V})^{-1}$ other than possibly $\lambda$, thus it also contains no poles of $(\zeta^2 - P_\varepsilon)^{-1}$ other than possibly $\lambda$ as a consequence of (4.5). On the one hand, we can compute

\[m_\varepsilon(\lambda) = \frac{1}{2\pi i} \text{tr} \int_{C_\lambda} (I + \sqrt{VR_\varepsilon(\zeta)}\sqrt{V})^{-1} VR_\varepsilon(\zeta)^2 \sqrt{V} 2\zeta d\zeta\]

\[= \frac{1}{2\pi i} \text{tr} \int_{C_\lambda} \sum_{j=1}^{J} \frac{A_j \sqrt{VR_\varepsilon(\zeta)^2} 2\zeta \sqrt{V}}{(\zeta - \lambda)^j} d\zeta\]

\[= \sum_{j=1}^{J} \sum_{k=0}^{j-1} \frac{1}{k!(j-1-k)!} \text{tr} A_j \sqrt{V} \partial_\zeta^k R_\varepsilon(\zeta) \partial_\zeta^{j-1-k} (R_\varepsilon(\zeta)2\zeta) \sqrt{V}.\]

On the other hand, by (4.5), we have

\[\frac{1}{2\pi i} \text{tr} \int_{\lambda} (\zeta^2 - P_\varepsilon)^{-1} 2\zeta d\zeta\]

\[= \frac{1}{2\pi i} \text{tr} \int_{C_\lambda} \sum_{j=1}^{J} \frac{R_\varepsilon(\zeta)2\zeta \sqrt{V} A_j \sqrt{VR_\varepsilon(\zeta)}}{(\zeta - \lambda)^j} d\zeta\]

\[= \sum_{j=1}^{J} \sum_{k=0}^{j-1} \frac{1}{k!(j-1-k)!} \text{tr} \partial_\zeta^{j-1-k} (R_\varepsilon(\zeta)2\zeta) \sqrt{V} A_j \sqrt{VR_\varepsilon(\zeta)} \partial_\zeta^k R_\varepsilon(\zeta).\]

Now we compare (4.6) and (4.7), since each $A_j$ has finite rank, we can apply cyclicity of the trace to obtain the multiplicity formula (4.4).
5. Proof of convergence

The proof of convergence is based on Lemma 1, Lemma 3, with an application of the Gohberg–Sigal–Rouché theorem, see Gohberg–Sigal [GS] and [DyZ, Appendix C].

We now state a more precise version of Theorem 1 involving the multiplicities given in (2.3) and (4.3) as follows:

**Theorem 2.** For any \( \Omega \) given in (1.4), there exists \( \delta_0 = \delta_0(\Omega) \) satisfying the following: for any \( 0 < \delta < \delta_0 \), there exists \( \varepsilon_\delta > 0 \) such that for any \( \lambda \in \Omega \) with \( m(\lambda) > 0 \),

\[
\# \{ \lambda_j(\varepsilon) \}_{j=1}^\infty \cap B(\lambda, \delta) = m(\lambda), \quad \text{for all } 0 < \varepsilon < \varepsilon_\delta,
\]

where \( \{ \lambda_j(\varepsilon) \}_{j=1}^\infty \) given in (1.3) is counted with multiplicity, \( B(\lambda, \delta) \) is the ball in \( \mathbb{C} \).

**Proof.** In view of Lemma 1, the poles of \( (I + \sqrt{V} R_0(\lambda) \sqrt{V})^{-1} \) are isolated in the region \( \{ \lambda \in \mathbb{C} : \Re \lambda > 0, \Im \lambda > -\gamma \} \), thus there are finitely many \( \lambda \in \Omega \) with \( m(\lambda) > 0 \), denoted by \( \lambda_1, \ldots, \lambda_J \). We choose \( \delta_0 > 0 \) such that \( B(\lambda_j, \delta_0), j = 1, \ldots, J \) are disjoint discs in \( \Omega \), then for any fixed \( 0 < \delta < \delta_0 \) and each \( \lambda \in \Omega \) with \( m(\lambda) > 0 \), we have

\[
\|(I + \sqrt{V} R_0(\zeta) \sqrt{V})^{-1}\|_{L^2 \to L^2} < C(\delta), \quad \forall \zeta \in \partial B(\lambda, \delta),
\]

for some constant \( C(\delta) > 0 \).

In order to apply the Gohberg–Sigal–Rouché theorem, we need to estimate:

\[
\|I + \sqrt{V} R_\varepsilon(\zeta) \sqrt{V} - (I + \sqrt{V} R_0(\zeta) \sqrt{V})\|_{L^2 \to L^2}, \quad \text{for any } \zeta \in \Omega.
\]

1. Choose \( \chi \in C^\infty_c(\mathbb{R}^n) \) satisfying \( \chi \equiv 1 \) in \( B_{\mathbb{R}^n}(0, 1) \) and \( \text{supp} \chi \subset B_{\mathbb{R}^n}(0, 2) \), we define \( \chi_R(x) = \chi(R^{-1}x) \) and calculate:

\[
I + \sqrt{V} R_\varepsilon(\zeta) \sqrt{V} - (I + \sqrt{V} R_0(\zeta) \sqrt{V})
= \sqrt{V} R_\varepsilon(\zeta) \sqrt{V} - \chi_R \sqrt{V} R_\varepsilon(\zeta) \chi_R \sqrt{V} + \sqrt{V} \chi_R (R_\varepsilon(\zeta) - R_0(\zeta)) \chi_R \sqrt{V}
- (\sqrt{V} R_0(\zeta) \sqrt{V} - \chi_R \sqrt{V} R_0(\zeta) \chi_R \sqrt{V}).
\]

2. The first term can be written as \( (1 - \chi_R) \sqrt{V} R_\varepsilon(\zeta) \sqrt{V} + \chi_R \sqrt{V} R_\varepsilon(\zeta)(1 - \chi_R) \sqrt{V} \).

Let \( \tilde{\gamma} = (\gamma + \gamma')/2 \), then

\[
(1 - \chi_R) \sqrt{V} R_\varepsilon(\zeta) \sqrt{V} = (1 - \chi_R) \sqrt{V} e^{\tilde{\gamma} |x|} (e^{-\tilde{\gamma} |x|} R_\varepsilon(\zeta) e^{-\tilde{\gamma} |x|}) \sqrt{V} e^{\tilde{\gamma} |x|},
\]

where \( |\sqrt{V} e^{\tilde{\gamma} |x|}| \leq C e^{-(\gamma - \gamma') |x|} = C e^{-(\gamma - \gamma') |x|}/2 \). By Lemma 2, there exists \( \varepsilon_0 = \varepsilon_0(\Omega, \tilde{\gamma}) \) such that for any \( 0 < \varepsilon < \varepsilon_0, \|e^{-\tilde{\gamma} |x|} R_\varepsilon(\zeta)\|_{L^2 \to L^2} \leq C(\Omega, \tilde{\gamma}) \). Thus,

\[
\|(1 - \chi_R) \sqrt{V} R_\varepsilon(\zeta) \sqrt{V}\|_{L^2 \to L^2} \leq C(\Omega, \gamma) e^{-(\gamma - \gamma') |x|}, \quad \forall \zeta \in \Omega.
\]

Similarly, we can bound \( \|\chi_R \sqrt{V} R_\varepsilon(\zeta)(1 - \chi_R) \sqrt{V}\|_{L^2 \to L^2} \) by the right side above. Hence for any \( 0 < \varepsilon < \varepsilon_0, \)

\[
\|\sqrt{V} R_\varepsilon(\zeta) \sqrt{V} - \chi_R \sqrt{V} R_\varepsilon(\zeta) \chi_R \sqrt{V}\|_{L^2 \to L^2} \leq C e^{-(\gamma - \gamma') |x|}, \quad \forall \zeta \in \Omega.
\]
3. We can estimate the third term in (5.1) by a similar argument. (2.2) implies that
\[ \|e^{-\gamma|x|} R_0(\zeta)e^{-\gamma|x|}\|_{L^2 \to L^2} \leq C(\Omega, \gamma), \quad \forall \zeta \in \Omega. \]
Hence, arguing as above, we obtain that
\[ \|\sqrt{\gamma} R_0(\zeta)\sqrt{\gamma} - \chi R \sqrt{\gamma} R_0(\zeta) \chi R \sqrt{\gamma}\|_{L^2 \to L^2} \leq C e^{-(\gamma-\gamma')R/2}, \quad \forall \zeta \in \Omega. \quad (5.3) \]
4. We note that
\[ \chi R(R_\varepsilon(\zeta) - R_0(\zeta)) \chi R = i\varepsilon \chi R(-\Delta - i\varepsilon x^2 - \zeta^2)^{-1}x^2(\Delta - \zeta^2)^{-1} \chi R, \]
and recall [Z2] that there exists \( C = C(\Omega, \chi R) \) (independent of \( \varepsilon \)) such that
\[ \|\chi R(-\Delta - i\varepsilon x^2 - \zeta^2)^{-1}x^2(\Delta - \zeta^2)^{-1}\chi R\|_{L^2 \to L^2} \leq C, \quad \forall \zeta \in \Omega, \varepsilon > 0, \]
which is proved using the method of complex scaling, see [Z2, §5] for details. Hence
\[ \|\sqrt{\gamma} \chi R(R_\varepsilon(\zeta) - R_0(\zeta)) \chi R \sqrt{\gamma}\|_{L^2 \to L^2} \leq C(\Omega, \chi R) \varepsilon, \quad \forall \zeta \in \Omega, \varepsilon > 0. \quad (5.4) \]
By (5.2) and (5.3), we can first fix \( R \) sufficiently large such that
\[ \|\sqrt{\gamma} R_\varepsilon(\zeta)\sqrt{\gamma} - \chi R \sqrt{\gamma} R_\varepsilon(\zeta) \chi R \sqrt{\gamma}\|_{L^2 \to L^2} \leq 1/(3C(\delta)), \quad \forall \zeta \in \Omega, \ 0 \leq \varepsilon < \varepsilon_0. \]
Then by (5.4), there exists \( \varepsilon_\delta > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_\delta, \)
\[ \|\sqrt{\gamma} \chi R(R_\varepsilon(\zeta) - R_0(\zeta)) \chi R \sqrt{\gamma}\|_{L^2 \to L^2} \leq 1/(3C(\delta)), \quad \forall \zeta \in \Omega. \]
We may assume that \( \varepsilon_\delta < \varepsilon_0, \) thus by (5.1), we conclude that for each \( 0 < \varepsilon < \varepsilon_\delta, \)
\[ \|(I + \sqrt{\gamma} R_\varepsilon(\zeta)\sqrt{\gamma})^{-1}(I + \sqrt{\gamma} R_\varepsilon(\zeta)\sqrt{\gamma} - (I + \sqrt{\gamma} R_0(\zeta)\sqrt{\gamma}))\|_{L^2 \to L^2} < 1, \]
on \( \partial B(\lambda, \delta) \).
Now we apply the Gohberg–Sigal–Rouché theorem to obtain that
\[
m(\lambda) = \frac{1}{2\pi i} \text{tr} \int_{\partial B(\lambda, \delta)} (I + \sqrt{\gamma} R_\varepsilon(\zeta)\sqrt{\gamma})^{-1}(\sqrt{\gamma} R_0(\zeta)\sqrt{\gamma}) d\zeta \\
= \frac{1}{2\pi i} \text{tr} \int_{\partial B(\lambda, \delta)} (I + \sqrt{\gamma} R_\varepsilon(\zeta)\sqrt{\gamma})^{-1}(\sqrt{\gamma} R_\varepsilon(\zeta)\sqrt{\gamma}) d\zeta,
\]
for each \( 0 < \varepsilon < \varepsilon_\delta. \) Let \( \lambda_1(\varepsilon), \ldots, \lambda_K(\varepsilon) \) be the distinct poles of \( (I + \sqrt{\gamma} R_\varepsilon(\zeta)\sqrt{\gamma})^{-1} \) in \( B(\lambda, \delta) \), then
\[
m(\lambda) = \sum_{k=1}^{K} \frac{1}{2\pi i} \text{tr} \int_{\lambda_k(\varepsilon)} (I + \sqrt{\gamma} R_\varepsilon(\zeta)\sqrt{\gamma})^{-1}(\sqrt{\gamma} R_\varepsilon(\zeta)\sqrt{\gamma}) d\zeta = \sum_{k=1}^{K} m_\varepsilon(\lambda_k(\varepsilon)),
\]
Therefore, with Lemma 3 and (4.4), we obtain
\[ \# \{\lambda_j(\varepsilon)\}_{j=1}^\infty \cap B(\lambda, \delta) = m(\lambda), \quad \forall 0 < \varepsilon < \varepsilon_\delta. \]
RESONANCES AS VISCOSITY LIMITS FOR EXPONENTIALLY DECAYING POTENTIALS

References

[Da] E.B. Davies, *Pseudo–spectra, the harmonic oscillator and complex resonances*, Proc. R. Soc. Lond. A 455(1999), 585–599.

[DyZ] Semyon Dyatlov and Maciej Zworski. *Mathematical theory of scattering resonances*. American Mathematical Society, 2019.

[F] Richard Froese. Upper bounds for the resonance counting function of Schrödinger operators in odd dimensions. *Canadian Journal of Mathematics*, 50(3):538–546, 1998.

[Ga] Oran Gannot. From quasimodes to resonances: exponentially decaying perturbations. *Pacific Journal of Mathematics*, 277(1):77–97, 2015.

[GS] Gohberg, IC U., and E. I. Sigal. An operator generalization of the logarithmic residue theorem and the theorem of Rouché. *Mathematics of the USSR-Sbornik*, 13(4):603, 1971.

[HS] Peter D Hislop and Israel Michael Sigal. *Introduction to spectral theory: With applications to Schrödinger operators*, volume 113. Springer Science & Business Media, 2012.

[HSV] M. Hitrik, J. Sjöstrand, and J. Viola, *Resolvent Estimates for Elliptic Quadratic Differential Operators*, Analysis & PDE 6(2013), 181–196.

[JZBRK] T-C. Jagau, D. Zuev, K. B. Bravaya, E. Epifanovsky, and A.I. Krylov, *A Fresh Look at Resonances and Complex Absorbing Potentials: Density Matrix-Based Approach*, J. Phys. Chem. Lett. 5(2014), 310–315.

[KaNa] Kentaro Kameoka and Shu Nakamura. Resonances and viscosity limit for the wigner-von neumann type hamiltonian. *arXiv preprint arXiv:2003.07001*, 2020.

[Na] Shu Nakamura. Distortion analyticity for two-body schrödinger operators. In *Annales de l’IHP Physique théorique*, volume 53, pages 149–157, 1990.

[RiMe] U.V. Riss and H.D. Meyer, *Reflection-Free Complex Absorbing Potentials*, J. Phys. B 28 (1995), 1475–1493.

[SeMi] T. Seideman and W.H. Miller, *Calculation of the cumulative reaction probability via a discrete variable representation with absorbing boundary conditions*, J. Chem. Phys. 96(1992), 4412–4422.

[Xi] Haoren Xiong. Resonances as viscosity limits for exterior dilation analytic potentials. *arXiv preprint arXiv:2002.12490*, 2020.

[Z1] Maciej Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics 138, AMS, 2012.

[Z2] Maciej Zworski. Scattering resonances as viscosity limits. In *Algebraic and Analytic Microlocal Analysis*, pages 635–654. Springer, 2013.

E-mail address: xiong@math.berkeley.edu

Department of Mathematics, University of California, Berkeley, CA 94720, USA