NONDIVERGENCE ON HOMOGENEOUS SPACES AND RIGID TOTALLY GEODESICS

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Abstract. Let $G/\Gamma$ be the quotient of a semisimple Lie group by an arithmetic lattice. We show that for reductive subgroups $H$ of $G$ that is large enough, the orbits of $H$ on $G/\Gamma$ intersect nontrivially with a fixed compact set. As a consequence, we deduce finiteness result for totally geodesic submanifolds of arithmetic quotients of symmetric spaces that do not admit nontrivial deformation and with bounded volume. Our work generalizes previous work of Tomanov–Weiss and Oh on this topic.

Contents

1. Introduction
   1.1. Main results
   1.2. Generalizations
   1.3. Geometric consequences
   1.4. Organizations

2. Non-divergence in the space of unimodular lattices

3. Non-divergence in the general case

4. Proof of Theorem 1.3 and 1.4

5. Geometric consequences
   5.1. Arithmetic quotients of symmetric spaces of noncompact type
   5.2. Totally geodesic submanifolds
   5.3. Rigid totally geodesic submanifolds
   5.4. Proof of Proposition 5.2
   5.5. Proof of Theorem 1.5

References

1. Introduction

1.1. Main results. Let $G$ be a semisimple linear algebraic group defined over $\mathbb{Q}$ and $\Gamma \leq G(\mathbb{Q})$ an arithmetic lattice. Let $H$ be a subalgebraic group of $G_{\mathbb{R}}$ (over $\mathbb{R}$). Let the Roman letter $G$ (resp. $H$) denote the identity connected component of $G(\mathbb{R})$ (resp. $H(\mathbb{R})$) in the analytic topology. Without loss of generality assume $\Gamma$ is contained in $G$.

Definition 1.1. Let $A$ be a subgroup of $G$, the action of $A$ on $G/\Gamma$ is said to be uniformly non-divergent if there exists a compact set $C \subset G/\Gamma$ such that for all $x \in G/\Gamma$, there exists $g \in A$ such that $g \cdot x \in C$, or equivalently, $G/\Gamma = A \cdot C$. 

1
We are interested in conditions on $H$ that would guarantee the action of $H$ on $G/\Gamma$ is uniformly non-divergent.

On the one hand, by [DM91], if $H$ is semisimple and has no compact factor, then the action of $H$ on $G/\Gamma$ is uniformly non-divergent if the centralizer of $H$ in $G$ is finite (see [EMV09, Lemma 3.2]). On the other hand, if $H$ contains a maximal $\mathbb{R}$-split torus, then this is also true and is due to [TW03, Theorem 1.3] extending the idea of Margulis (see [TW03, Appendix]).

In the present article we find a common generalization of both theorems.

**Theorem 1.2.** Assume that $H$ is connected, reductive, $\mathbb{R}$-split and the centralizer of $H$ in $G$ is $\mathbb{R}$-anisotropic modulo the center of $H$, then the action of $H$ on $G/\Gamma$ is uniformly non-divergent.

Recall that for an $\mathbb{R}$-linear algebraic group $F$, it is $\mathbb{R}$-anisotropic iff its real point $F(\mathbb{R})$ is compact. We shall write $Z_G H$ for the centralizer of $H$ in $G$ and $Z(H)$ for the center of $H$.

1.2. Generalizations. In this subsection we discuss some further generalization of our main theorems.

Combining with [SW00, Corollary 1.3], we have

**Theorem 1.3.** Let $F$ be a connected $\mathbb{R}$-subgroup of $G$. Assume that the epimorphic closure of $F$ in $G$ contains a subgroup $H$ satisfying the condition in Theorem 1.2. Then the action of $F$ on $G/\Gamma$ is uniformly non-divergent.

By using [RS18, Theorem 1.1], we have a uniform version of our main theorem.

**Theorem 1.4.** Consider $\mathcal{H}$, the set of all connected reductive $\mathbb{R}$-subgroups of $G$ that are $\mathbb{R}$-split and whose centralizer in $G$ is $\mathbb{R}$-anisotropic modulo its center, then we can find a compact set $C \subset G/\Gamma$ such that for all $x \in G/\Gamma$, for all $H \in \mathcal{H}$, there exists $h \in H$ such that $hx \in C$.

1.2.1. Example. Let $G = \text{SL}_4(\mathbb{R})$, $\Gamma = \text{SL}_4(\mathbb{Z})$ and

$$S := \begin{bmatrix} t^3 & t^{-1} & & \\ & t & t^{-1} & \\ & & t & \\ & & & t^{-1} \end{bmatrix}, \quad M := \begin{bmatrix} 1 & & & \\ & M_0 & & \\ & & & \\ & & & \\ & & & \end{bmatrix},$$

where $M_0$ is $SO(2,1) \subset SL_3$. Let $H := S \cdot H$. Then it is not hard to check that our theorem applies and hence the action of $H$ on $\text{SL}_4(\mathbb{R})/\text{SL}_4(\mathbb{Z})$ is uniformly non-divergent. To get a better result, consider

$$B := \begin{bmatrix} 1 & & & \\ & B_0 & & \\ & & & \\ & & & \end{bmatrix},$$

where $B_0$ is a Borel subgroup of $SO(2,1)$. Let $F := S \cdot B$, then the epimorphic closure of $F$ in $\text{SL}_4$ is $H$ (because $B_0$ is a parabolic subgroup of $SO(2,1)$, see [Gro97] for details). Hence the action of $F$ on $\text{SL}_4(\mathbb{R})/\text{SL}_4(\mathbb{Z})$ is also uniformly non-divergent. Using Lie algebras, it is not hard to show that no proper connected Lie subgroup of $F$ has this property.
1.3. Geometric consequences. Let $X$ be an arithmetic quotient of a symmetric space of noncompact type. Let $\mathcal{T}G^N$ be the space of embedded totally geodesic finite-volume submanifolds in $X$ of dimension $N$. Equip $\mathcal{T}G^N$ with the Chabauty topology. Let $\mathcal{T}G^N_{rigid}$ be those that do not admit nontrivial deformation in $X$. More precisely, $Y \in \mathcal{T}G^N_{rigid}$ iff $\{Y\}$ is open in $\mathcal{T}G^N$. We have the following finiteness result generalizing [Oh04, Theorem 1.1].

**Theorem 1.5.** For every natural number $N$ and positive number $T > 0$, the set

$$\mathcal{T}G^N_{\leq T, rigid} := \{Y \in \mathcal{T}G^N_{rigid} \mid \text{Vol}(Y) \leq T\}$$

is finite.

1.4. Organizations. In section 2 we prove Theorem 1.2 in the special case of unimodular lattices. The general case is treated in section 3. Theorem 1.4 and 1.3 are proved in section 4. In the last section 5 we prove Theorem 1.5 and in Proposition 5.2 we give a characterization of rigid totally geodesic submanifolds in terms of Lie algebras.

2. Non-divergence in the space of unimodular lattices

In this section we provide a proof of Theorem 1.2 in the special case when $G = \text{SL}_N$ and $\Gamma$ is commensurable with $\text{SL}_N(\mathbb{Z})$. Compared to the general case to be treated in section 3, the proof here is more elementary (though still relies on the non-divergence criterion of Dani–Margulis) and does not rely on [DGUL19]. Yet it still illustrates some key ideas also appearing in the general case.

Without loss of generality assume $\Gamma = \text{SL}_N(\mathbb{Z})$ and identify $\text{SL}_N(\mathbb{R})/\text{SL}_N(\mathbb{Z})$ as the space of unimodular lattices of $\mathbb{R}^N$. Fix a reductive subgroup $H$ of $\text{SL}_N$ over $\mathbb{R}$ such that $\mathbb{Z}_{\text{SL}_N}H/\mathbb{Z}(H)$ is $\mathbb{R}$-anisotropic. Write $H = S \cdot M$ as an almost direct product of an $\mathbb{R}$-split torus and an $\mathbb{R}$-split semisimple group $M$.

Let $\mathbb{R}^N = \bigoplus_{i \in A_0} V_i$ be a decomposition of $\mathbb{R}^N$ into $\mathbb{R}$-irreducible representations of $H$.

**Lemma 2.1.** For distinct $i_1, i_2 \in A_0$, $V_{i_1}$ and $V_{i_2}$ are not isomorphic as representations of $H$.

**Proof.** Assume otherwise that $\psi : V_{i_2} \rightarrow V_{i_1}$ gives an $H$-equivariant isomorphism. Consider for $s \in \mathbb{R}$

$$\begin{bmatrix} 1 & s\psi \\ 0 & 1 \end{bmatrix}$$

as operators on $V_{i_1} \oplus V_{i_2}$. And write $v_s$ for the image of its embedding in $\text{SL}(V)$ by asking that it acts as identity on $V_i$'s for $i \neq i_1, i_2$. Then $\{v_s\}_{s \in \mathbb{R}}$ is a noncompact subgroup of $G$ centralizing $H$, and hence has to be contained in $H$. But it does not preserve $V_{i_2}$, which is a contradiction. □

In light of this lemma, $S$ can be described more concretely. On the one hand, every $s \in S$ acts as a positive scalar $s_i$ when restricted to each $V_i$. On the other hand, if $s \in G$ acts as a positive scalar when restricted to each $V_i$, then $s \in S$ because it centralizes $H$, is $\mathbb{R}$-diagonalizable and also is connected to the identity via some one-parameter flow.

Let $\|\cdot\|$ be the standard Euclidean metric on $\mathbb{R}^N$ and by abuse of notation also the induced metrics $\wedge \|\cdot\|$ for all $i$'s. For a lattice $\Lambda \leq \mathbb{R}^N$, an $\mathbb{R}$-linear subspace (will be abbreviated as an $\mathbb{R}$-subspace) $W$ of $\mathbb{R}^N$ is said to be $\Lambda$-rational iff $\Lambda \cap W \leq W$ is
a lattice, in which case we let $\Lambda_W := \Lambda \cap W \leq W$ and let $\|\Lambda_W\|$ denote the volume of $W/\Lambda_W$. If $v_1, ..., v_k$ is a set of $\mathbb{Z}$-basis of $\Lambda_W$ then $\|\Lambda_W\| = \|v_1 \wedge ... \wedge v_k\|$. A subspace $W$ is said to be $(M, \Lambda)$-eligible if $W$ is both $M$-stable and $\Lambda$-rational.

Let $\delta_M : M \setminus \text{SL}_N(\mathbb{R})/\text{SL}_N(\mathbb{Z}) \to (0, \infty)$ be defined by

$$\delta_M([\Lambda]) := \inf \left\{ \|\Lambda_W\|^{\frac{1}{\text{dim} W}} \mid W \text{ is (M,}\Lambda)-\text{eligible} \right\}.$$  

When $M = \mathbb{H}$, $\delta_M([\Lambda])$ is always equal to 1. When $M = \{e\}$ is the trivial group, write $\delta := \delta_M$. By Mahler’s criterion, to prove Theorem 1.2, it suffices to show that there exists $\eta > 0$ such that for every $\Lambda \in M \setminus \text{SL}_N(\mathbb{R})/\text{SL}_N(\mathbb{Z})$ there exists some $h \in H$ such that $\delta(h\Lambda) > \eta$. This would in turn follow from the following proposition by [DGU20, Theorem 4.6] which is based on the work of [DM91] (see also [Kle10, Corollary 3.3, Theorem 3.4]).

**Proposition 2.2.** There exist $0 < \eta_0 < 1$ and $C > 1$ such that for all $[\Lambda] \in M \setminus \text{SL}_N(\mathbb{R})/\text{SL}_N(\mathbb{Z})$ with $\delta_M([\Lambda]) < \eta_0$, there exists $s \in S$ such that $\delta_M([s\Lambda]) \geq C\delta_M([\Lambda])$. As a result, there exists $s \in S$ such that $\delta_M([s\Lambda]) \geq \eta_0$.

The proof of this proposition will be based on the two lemmas below.

**Lemma 2.3.** There exist, and we fix, $C_1, C_2 > 1$ and a finite subset $\mathcal{F} \subset S$ such that for every proper $\mathbb{R}$-subspace $W$ of $\mathbb{R}^N$ that is $M$-stable, there exists $s \in \mathcal{F}$ such that

1. $\|sv\| > \frac{1}{C_1} \|v\|$ for all pure wedges $v$ in $\mathbb{R}^N$;
2. $\|sv\| > C_2 \|v\|$ for all pure wedges $v$ with $L_v$ contained in $W$.

By a pure wedge $v$, we mean some non-zero vector of the form $v = v_1 \wedge ... \wedge v_i$ in $\wedge^i \mathbb{R}^N$ for some $i$. For such a pure wedge, write $L_v$ for the $\mathbb{R}$-subspace of $\mathbb{R}^N$ spanned by $v_1, ..., v_i$.

**Proof.** The first part comes for free as long as $\mathcal{F}$ is a finite set. We shall focus on the second part. It suffices to show that for each $M$-stable subspace $W$, there exists $s_W \in S$ such that $s_W \cdot w > \|w\| \quad \forall w \in W$

Then the same thing would be true replacing $w$ by any pure wedge $w$ with $L_w$ contained in $W$. A continuity argument applied to the unit vectors would then finish the proof.

Recall the decomposition of $\mathbb{R}^N = \bigoplus_{i \in A_0} V_i$ into irreducible representations with respect to $H$. Also, for each $i$, every $s \in S$ acts as $s_i \text{id}_{V_i}$, for some $s_i > 0$, when restricted to $V_i$.

For each $I \subset A_0$, define $\pi_I : \mathbb{R}^N \to V_I := \bigoplus_{i \in I} V_i$ to be the corresponding $H$-equivariant projection. We claim that for any $M$-stable $W$, there exists $I$ such that $\pi_I|_W$ is bijective.

Fix $W$ and we define $I = \{i_1, ..., i_k\}$ inductively. Firstly we pick $i_1 \in A_0$ such that $\pi_{i_1}|_W \neq 0$ and let $W_1 := \ker(\pi_{i_1}|_W)$. Secondly we pick $i_2 \in A_0$ such that $\pi_{i_2}|_{W_1} \neq 0$ and let $W_2 := \ker(\pi_{i_2}|_{W_1})$. Continue this until $\pi_i|_{W_i} = 0$ for all $i \in A_0$, in which case $W_k = \{0\}$. So for $j \leq k$ we have the exact sequence

$$0 \longrightarrow W_{j_0} \longrightarrow W_{j-1} \longrightarrow \pi_{j_0}(W_{j-1}) \longrightarrow 0.$$  

Now $\pi_{j}|_W$ is injective. Indeed if $w \in W$ is such that $\pi_j(w) = 0$, then $w \in \cap_i W_i = \{0\}$.
The map \( \pi_l|_W \) is also surjective. Write \( W_0 := W \). For each \( i = 1, \ldots, k \),

\[
\pi_i|_{W_{l-1}} \neq 0 \implies \pi_i|_{W_{l-1}} \text{ is surjective on } V_i
\]
as \( V_i \) is actually an irreducible representation with respect to \( M \) (though \( V_i \) may be isomorphic to \( V_{j_0} \) as \( M \)-representations). As \( W_{l-1} \) is defined to vanish under \( \pi_{i-1} \) (for \( i > 1 \)), an induction argument shows that \( \pi_{\{i_1, \ldots, i_l\}}|_W \) is surjective for \( l = 1, \ldots, k \).

Once \( \pi_l|_W : W \to V_l \) is bijective, we see that \( W \) is the graph of some linear map \( \phi_W : V_l \to V_r \). That is to say, for any \( w \in W \), there exists a unique \( v \in V_l \) such that \( w = v + \phi_W(v) \). Find \( C_W > 0 \) such that

\[
\|v + \phi_W(v)\| \leq C_W \|v\|, \quad \forall v \in V_l.
\]

Then we take \( s_W \in S \) such that \( s_w|_{V_l} = \lambda \text{id}_{V_l} \) for some \( \lambda > C_W^2 \). Thus

\[
\|s_w \cdot w\| \geq \frac{1}{C_W} \|\lambda v\| > C_W \|v\| \geq \|v + \phi_W(v)\| = \|w\|.
\]

Now the proof is complete. \( \square \)

**Lemma 2.4.** For every \( [\Lambda] \in M \setminus \text{SL}_N(\mathbb{R})/\text{SL}_N(\mathbb{Z}) \) with \( \delta_M([\Lambda]) < \eta_0 := 1/(C_1 C_2)^N \), there exists a proper \((M, \Lambda)\)-eligible subspace \( W_\infty \leq \mathbb{R}^N \) such that for any \((M, \Lambda)\)-eligible subspace \( W \) not contained in \( W_\infty \) we have

\[
\|\Lambda W + \Lambda W_\infty\| \geq (C_1 C_2) \|\Lambda W_\infty\|.
\]

**Proof:** Take such a \( \Lambda \) as in the statement. Find an \((M, \Lambda)\)-eligible \( \mathbb{R} \)-subspace \( W_1 \) such that \( \|\Lambda W_1\| < \eta_0 \). If \( W_1 \) satisfies the conclusion above, then we take \( W_\infty = W_1 \). Otherwise there exists \( W'_1 \leq \mathbb{R}^N \), \((M, \Lambda)\)-eligible, such that

\[
\|\Lambda W'_1 + \Lambda W_1\| < (C_1 C_2) \|\Lambda W_1\|.
\]

Let \( W_2 := W'_1 + W_1 \), then \( W_2 \) is still \((M, \Lambda)\)-eligible. If \( W_2 \) satisfies the conclusion, we stop. Otherwise we define \( W_2 \) and \( W_3 \) similarly as above. As the dimension is finite, this process has to stop at some index \( l \) not exceeding \( N \). We let \( W_\infty := W_l \) and it only remains to show that \( W_\infty \) is a proper subspace. Indeed \( \Lambda W_i + \Lambda W'_l \leq \Lambda W_{l+1} \), so

\[
\|\Lambda W_i\| \leq \|\Lambda W'_l + \Lambda W_{l-1}\| \leq (C_1 C_2) \|\Lambda W_{l-1}\| \leq \ldots \leq (C_1 C_2)^l \|W_1\| < 1.
\]

As \( \|\mathbb{R}^N\|_\Lambda = \|\Lambda\| = 1 \), \( W_\infty \neq \mathbb{R}^N \). \( \square \)

Now finally we come to the

**Proof of Proposition 2.2 with \( C := C_2 \).** Recall \( \eta_0 = 1/(C_1 C_2)^N \). We take \( [\Lambda] \in M \setminus \text{SL}_N(\mathbb{R})/\text{SL}_N(\mathbb{Z}) \) such that \( \delta_M([\Lambda]) < \eta_0 \). By Lemma 2.4, pick a proper \((M, \Lambda)\)-eligible subspace \( W_\infty \leq \mathbb{R}^N \) such that for any \((M, \Lambda)\)-eligible subspace \( W \) not contained in \( W_\infty \) we have

\[
\|\Lambda W + \Lambda W_\infty\| \geq (C_1 C_2) \|\Lambda W_\infty\|.
\]

Then by applying Lemma 2.3 to \( W_\infty \), we get some \( s \in S \) such that

1. \( \|sv\| > \frac{1}{C_1^2} \|v\| \) for all pure wedges \( v \) in \( \mathbb{R}^N \);
2. \( \|sv\| > C_2 \|v\| \) for all pure wedges \( v \) with \( \mathcal{L}_e \) contained in \( W_\infty \).
Now we prove our assertion with this $s \in S$. It suffices to show that for every $(M, s\Lambda)$-eligible $W'$, we have $\|\Lambda_{W'}\| \geq C_2 \|\Lambda_{W''}\|$ for some $(M, \Lambda)$-eligible $W''$.

First let $W := s^{-1}W'$, then $W$ is $(M, \Lambda)$-eligible and $s \cdot \Lambda_W = (s\Lambda)_{W'}$. There are two cases to consider.

Case I, $W \subset W_\infty$. Then
$$\|(s\Lambda)_{W'}\| = \|s\Lambda_W\| \geq C_2 \|\Lambda_W\|.$$ 
So setting $W'' := W$ concludes the proof.

Case II, $W \notin W_\infty$. Let $W'' := W \cap W_\infty$. Then $W''$ is $(M, \Lambda)$-eligible and $\Lambda_{W''} = \Lambda_W \cap \Lambda_{W_\infty}$. We have
$$\|\Lambda_{W''}\| (C_1 C_2) \|\Lambda_{W_\infty}\| \leq \|\Lambda_W \cap \Lambda_{W_\infty}\| \cdot \|\Lambda_W + \Lambda_{W_\infty}\| \leq \|\Lambda_W\| \cdot \|\Lambda_{W_\infty}\| \implies \|\Lambda_W\| \geq (C_1 C_2) \|\Lambda_{W''}\|.$$ 
So
$$\|(s\Lambda)_{W'}\| \geq \frac{1}{C_1} \|\Lambda_W\| \geq C_2 \|\Lambda_{W''}\|$$
and we are done. \hfill \Box

3. Non-divergence in the general case

In this section we prove Theorem 1.2 in general.

So let $G$ be a $\mathbb{Q}$-semisimple group of dimension $N$ and $\Gamma \leq G$ be an arithmetic lattice. Let $H \leq G$ be a connected reductive subgroup defined over $\mathbb{R}$ without compact factors. Write $H = S \cdot M$ as an almost direct product of some $\mathbb{R}$-split torus $S$ and some connected $\mathbb{R}$-split semisimple group $M$. We assume that $S \leq Z_G M$ is a maximal $\mathbb{R}$-split torus.

Fix a maximal compact subgroup $K$ of $G$ and an $\text{Ad}(K)$-invariant metric on $G$, defined as the Lie algebra of $G$. Only in this subsection we follow the convention of [TW03] to use script letters for Lie algebras. We fix an integral structure $\mathcal{G}_\mathbb{Z}$ on $G$ that is contained in the Lie algebra of $G$ and is preserved by $\text{Ad}(\Gamma)$. For each $g \in G$, write $\mathcal{G}_g := \text{Ad}(g) \cdot \mathcal{G}_\mathbb{Z}$. For $\eta > 0$, let $\mathcal{N}_\eta := \{v \in \mathcal{G} \mid \|v\| < \eta\}$. For a discrete subgroup $\Lambda$ of $G$, we let $\|\Lambda\|$ be the covolume of $\Lambda$ in $\mathcal{A}_\mathbb{R}$, the $\mathbb{R}$-span of $\Lambda$.

For each $\eta > 0$, let $X_\eta$ be a compact subset of $G/\Gamma$ defined by
$$X_\eta = \{[g] \in G/\Gamma \mid \mathcal{G}_g \cap \mathcal{N}_\eta = \{0\}\}.$$ 
As the map $[g] \mapsto \mathcal{G}_g$ is a proper map from $G/\Gamma$ to the space of lattices in $G$ with some fixed volume, the union of interiors of $\{X_\eta\}_{\eta>0}$ covers $G/\Gamma$ by Mahler’s criterion.

To take into consideration of $M$, we define
$$X_\eta^M = \{[g] \in G/\Gamma \mid M[g] \cap X_\eta \neq \emptyset\}.$$ 
We need to introduce some terminologies from reduction theory. For more detailed expositions one may consult [Bor19] (see also [BS73, BJ06, DGU20, Zha20]).

There exists a finite collection of $\mathbb{Q}$-parabolic subgroups $\{P_i\}_{i \in \mathcal{A}_1}$ of $G$ such that any $\mathbb{Q}$-parabolic subgroup $P$ of $G$ is conjugate to one of $P_i$ by $\Gamma$. Let $U_i$ be the unipotent radical of $P_i$, then $P_i/U_i$ is a $\mathbb{Q}$-reductive group. Let $S'_i$ denote the $\mathbb{Q}$-split part of its center. The lift of $S'_i$ to $P_i$ is not unique, but we fix one $S_i$ that is defined over $\mathbb{Q}$. On the other hand we take another lift $A_i$ of $S'_i$ that is invariant under the Cartan involution on $G$ associated with $K$. We let $\Delta_i$ be the
simple roots for \((A_i, P_i)\). As \(A_i\) is conjugate to \(S_i\) in a unique way, we are safe to think of \(\Delta_i\) also as simple roots for \((S_i, P_i)\), in which case consists of \(\mathbb{Q}\)-characters. For a \(\mathbb{Q}\)-parabolic subgroup \(P\), let \(\mathcal{P}\) be the subgroup of \(P\) defined by the common kernel of all \(\mathbb{Q}\)-characters of \(P\). And let \(\mathcal{P}\) be the identity connected component, in the analytic topology, of \(\mathcal{P}(\mathbb{R})\). Associated with \((K, P)\), we write \(g = k_i a_i p_i\) for the horospherical coordinates of \(g \in G\) (see for instance [Zha20, Section 2.3], one should take inverse of everything happening in the reference and combine the \(M, U\) term together to get \(p_i \in \mathcal{I}\)). Note \(k_i^0 \in K\), \(a_i^0 \in A_i\) and \(p_i^0 \in \mathcal{P}\).

Now we define generalized Siegel sets taking into concerns of \(M\). When \(M = \{e\}\), this specializes to the usual Siegel set. For each index \(i \in A_1\), a bounded set \(B \subset \mathcal{P}\) and \(\theta, \varepsilon > 0\), define

\[
\Sigma_{i, B, \theta}^M := \left\{ g \in G \mid g^{-1}M g \subset P_i; \alpha(a_i^0) < \theta, \forall \alpha \in \Delta_i; \exists m \in M, \exists \gamma \in \Delta_i \right\}.
\]

and

\[
\Sigma_{i, B, \theta, \varepsilon}^M := \left\{ g \in \Sigma_{i, B, \theta}^M \mid \alpha(a_i^0) < \varepsilon, \exists \alpha \in \Delta_i \right\}.
\]

We need the following proposition, which is a corollary to the main result of [DGUL19].

**Proposition 3.1.** For each \(0 < \theta < 1\), there exist \(B \subset G\) bounded and \(\eta' > 0\) such that

1. for every \(g \in G\),

\[
g \notin X_{\eta'}^M \implies g \in \bigcup_{i} \Sigma_{i, B \cap \mathcal{P} \cap \theta, \varepsilon}^M G;
\]

2. fix such a set of \(\theta = \theta_0\), \(B = B_0\) and \(\eta' = \eta_0\), then there exists a function \(\varepsilon_0 : (0, \eta_0) \to (0, \infty)\) with \(\lim_{\eta \to 0} \varepsilon_0(\eta) = 0\) such that for every \(g \in G\),

\[
g \notin X_{\eta}^M \implies g \in \bigcup_{i} \Sigma_{i, B \cap \mathcal{P} \cap \theta, \varepsilon_0(\eta)}^M G.
\]

**Proof of (1).** For each positive integer \(n\), we let \(B_n := X_{\frac{1}{n}}^M\) and \(\eta_n := \frac{1}{n}\). If (1) were not true, then there exists \(\theta_0 \in (0, 1)\) and a sequence of \(g_n \notin X_{\eta_n}^M\), yet \(g_n \notin \bigcup_{i} \Sigma_{i, B_n \cap \mathcal{P} \cap \theta_0}^M G\). By definition of \(X_{\eta_n}^M\), \((M, g_n)\) diverges topologically in \(G / \Gamma\).

By [DGUL19, Section 5.3] (where they proved the hypothesis of [DGUL19, Theorem 4.2] is met) and after passing to a subsequence, there exists \(i_0 \in A_1\) and \(\gamma_n \in \Gamma\) such that \(M_n := \gamma_n^{-1}g^{-1}Mg\gamma_n\) is contained in \(\mathcal{P}_{i_0}\) and if we write

\[
g_n \gamma_n = k_n a_n p_n,
\]

the horospherical coordinate of \(g_n \gamma_n\) with respect to \(P_{i_0}\) and \(K\), then

1. \(d_n := \max_{\alpha \in \Delta_{i_0}} \alpha(a_n) \to 0\);
2. \((p_n M_n)\) is non-divergent in \(\mathcal{P}_{i_0} / \mathcal{P}_{i_0} \cap \Gamma\). That is to say, there exist a sequence \((m_n)\) in \(M\), a bounded sequence \((b_n)\) in \(\mathcal{P}_{i_0}\) and \((\lambda_n)\) in \(\mathcal{P}_{i_0} \cap \Gamma\) such that

\[
p_n \gamma_n^{-1}g_n^{-1}m_n g_n \gamma_n = b_n \lambda_n.
\]

Now we let \(B := \{b_n\}\), which is a bounded set. Then we have

\[
g_n \gamma_n \in \Sigma_{i_0, B, d_n}^M, \quad \text{so} \quad g_n \in \Sigma_{i_0, B, d_n}^M G.
\]

When \(n\) is large enough such that \(B\) is contained in \(B_n \cap \mathcal{P}_{i_0}\) and \(d_n < \theta_0\), we have a contradiction. \(\square\)
Proof of (2). By (1) for each \( g \in G \) with \( [g] \notin X^M_0 \), choose \( i_g \in \mathcal{A}_i \) such that \( g \in \Sigma^M_{i_g,B_0,\theta_0} \Gamma \). The choice of \( i_g \) may not be unique, but we just fix one.

Define for \( g \in G \) with \( [g] \notin X^M_0 \),

\[
\varepsilon_0(g) := \inf \{ \varepsilon > 0 \mid g \in \Sigma_{i_g,B_0,\theta_0,\varepsilon} \Gamma \}.
\]

From the definition we see that \( 0 < \varepsilon_0(g) \leq \theta_0 \) and \( g \in \Sigma_{i_g,B_0,\theta_0,2\varepsilon_0(g)} \Gamma \). So (2) amounts to saying that (there exists some choice of \( i_g \) such that)

\[
\lim_{\eta \to \theta_0} \sup_{[g] \notin X^M_0} \varepsilon_0(g) = 0.
\]

Hence if (2) is not true, then there exists \( \eta_n \geq \varepsilon_0 > 0 \) and a sequence \( g_n \notin X^M_0 \) such that \( \varepsilon_0(g_n) \geq \varepsilon_0 \) for all positive integers \( n \). By passing to a subsequence we may assume that \( i_{g_n} \) are identically equal to some \( i_\theta \in \mathcal{A}_i \) for all \( n \). So there exists \( \gamma_n \in \Gamma \) such that if \( g_n \gamma_n = k_n a_n p_n \) is the horospherical coordinate of \( g_n \gamma_n \) with respect to \( P_{i_\theta} \) then \( (k_n a_n) \) is bounded by the assumption that \( \varepsilon_0(g_n) \geq \varepsilon_0 \). Also \( p_n \gamma_n \in B_0 \Gamma \neq \emptyset \). So there exist a bounded set \( B \subset G/\Gamma \) such that \( g_n \gamma_n \in B \) for all \( n \). This contradicts against \( g_n \notin X^M_0 \) for some \( \eta_n \to 0 \).

From now on we fix a choice of \( \theta_0, \eta_0 \in (0, 1) \), \( B_0 \) bounded in \( G \) and \( \varepsilon_0 : (0, \eta_0) \to (0, \infty) \) satisfying the above proposition. We choose \( \theta_0 > 0 \) small enough such that

- \( \alpha(a_\theta) < 1 \) for all \( g \in \bigcup_{i} \Sigma^M_{i,B_0,\gamma_0} \) and all \( \alpha \in \Phi^{-}_i \)

where \( \Phi^{-}_i \) denotes all nontrivial characters of \( A_i \) that appears in the Lie algebra of \( P_i \). Elements of \( \Phi^{-}_i \) are positive linear combinations of those from \( \Delta_i \), thus such a choice of \( \theta_0 \) exists. By choosing a smaller \( \eta_0 \), we assume that \( 0 < \varepsilon_0(\eta) < 1 \) for all \( 0 < \eta < \eta_0 \). Define a function \( \delta^M : M \setminus \Gamma \to (0, \infty) \) by

\[
\delta^M([g]) := \inf \left\{ \| \mathcal{L} \cap \mathcal{G}_\alpha \|_{\text{proj}_{\pi_\alpha}} \mid \mathcal{L} \leq \mathcal{G} \text{ is } \mathcal{G} \text{-rational and } M \text{-stable} \right\}.
\]

For each \( i \in \mathcal{A}_i \), fix (the unique) \( \omega_i \in U_i \) such that \( \omega_i A_i \omega_i^{-1} = S_i \) and decompose \( \mathcal{G} \) according to the Adjoint action of \( A_i, S_i \):

\[
\mathcal{G} = \bigoplus_{\alpha \in \Phi^{-}_i(S_i)} \mathcal{G}^{S_i}_\alpha, \quad \mathcal{G} = \bigoplus_{\alpha \in \Phi^{-}_i(A_i)} \mathcal{G}^{A_i}_\alpha.
\]

We identify \( \Phi^{-}_i(S_i) \) with \( \Phi^{-}_i(A_i) \) via \( \text{Ad}(w_i) \) and will simply refer to them as \( \Phi_i \). By definition non-zero weights appearing in the Lie algebra of \( P_i \), or equivalently in \( \mathcal{H}_i \), the Lie algebra of \( U_i \), have been called negative, and write \( \Phi^*_i \) for the negative, zero or positive weights for \( * = -, 0, + \) respectively. Also \( \Phi^{-}_i := \Phi^*_i \cup \Phi^0_i \) and \( \Phi^{0+}_i := \Phi^+_i \cup \Phi^0_i \). For each \( \alpha \in \Phi_i \), let \( \pi^{S_i}_\alpha \) and \( \pi^{A_i}_\alpha \) denote the corresponding projections to the weight space. Note \( \text{Ad}(w_i)(\mathcal{G}^{A_i}_\alpha) = \mathcal{G}^{S_i}_\alpha \), so we have

\[
\text{Ad}(w_i) \circ \pi^{A_i}_\alpha = \pi^{S_i}_\alpha \circ \text{Ad}(w_i).
\]

Also for each \( i \in \mathcal{A}_i \), define \( \pi^{A_i}_0^+ : \mathcal{G} \to \bigoplus_{\alpha \in \Phi^{-}_i(S_i)} \mathcal{G}^{S_i}_\alpha \) to be the natural projection and similarly define \( \pi^{A_i}_0^- \) and \( \pi^{S_i}_0^- \). They are related in the same manner as above.

Note that \( \pi^{A_i}_0^- \) is also the orthogonal projection onto \( \mathcal{H}^{A_i}_i \), which is denoted by \( \pi^{A_i}_0 \). Actually, the usefulness of \( A_i \) comes from the fact it is invariant under the Cartan involution and hence \( \pi^{A_i}_0^- \)'s are all orthogonal projections whereas the rationality of \( S_i \)'s makes \( \pi^{S_i}_0 \)'s defined over \( \mathbb{Q} \).
To be prepared for the upcoming corollary, we define some constants. For each $i \in A_1$ and $b \in B_0 \cap P_i$, let $h_i^b \in Z_G(A_i)$ and $u_i^b \in U_i$ such that $b = h_i^b u_i^b$. Then the set of all possible $\{h_i^b\}$ as $i$ and $b$ vary is also bounded. Define

$$C_3 := \max \{1, \|\mathcal{U}_i \cap \mathcal{E}_2\| : i \in A_1\}$$

$$C_4 := \sup \{1, \|\text{Ad}(\omega_i)^{-1}\|, \|\text{Ad}(\omega_i)\|, \|\text{Ad}(\omega_i h_i^b \omega_i^{-1})^{-1}\| : i \in A_1, b \in B_0\}.$$  

where $\|\text{Ad}(g)\|$ denotes the operator norm of $\text{Ad}(g)$.

We also choose $C_5 > 1$ such that for each $i \in A_1$,

1. for every $v \in \mathcal{E}_2$ and $\alpha \in \Phi_i$, either $\|\pi_0^S\| = 0$ or $\|\pi_0^S\| \geq 1/C_5$;
2. for every $v \in \mathcal{E}$ and $\alpha \in \Phi_i$, $\|v\| \geq 1/C_5 \|\pi_0^S\|;$
3. for every $v \in \mathcal{E}$ and $\alpha \in \Phi_i^0$, $\|\pi_0^S\| \geq 1/C_5 \|\pi_0^S\|$.

As a result of Proposition 3.1 we obtain the following:

**Corollary 3.2.** There exist $\eta_1 > 0$ and a function $\varepsilon_1 : (0, \eta_1) \to (0, \infty)$ with $\lim_{\eta \to 0} \varepsilon_1(\eta) = 0$ such that for all $0 < \eta < \eta_1$ and $g \in G$ such that $|g| \notin X_{\eta}^M$, there exists $m \in M$ and an $\mathbb{R}$-parabolic subgroup $P$ of $G$ containing $M$ such that $q^{-1}Pq$ is defined over $\mathbb{Q}$ and if we let $\mathcal{U}$ be the Lie algebra of $U$, which is the real points of the unipotent radical of $P$, then

1. $\|\mathcal{U} \cap \mathcal{E}_2\|^\frac{1}{\delta_{\mathcal{U}}} < \varepsilon_1(\eta);$
2. for all $v \in \mathcal{E}_m \setminus \mathcal{U}$, the orthogonal projection of $v$ to the orthogonal complement of $\mathcal{U}$ satisfies $\|\pi_{\mathcal{U}^\perp}(v)\| \geq \left(\frac{1}{\delta_{\mathcal{U}}}\right)^2$;
3. for any $\mathcal{E}_m$-rational, $M$-stable subspace $\mathcal{L}$ that is not contained in $\mathcal{U}$, we have $\|\mathcal{L} \cap \mathcal{E}_m\|^\frac{1}{\delta_{\mathcal{U}}} \geq 2C_8 \delta_M(|g|)$ with $C_8 > 1$ as in Lemma 3.3 below.

As the reader will see, $C_8$ could be replaced by any positive constant except that one needs to modify $\eta_1$ accordingly.

**Proof.** Take $\eta \in (0, \eta_0)$ and $g \notin X_{\eta}^M$. By Proposition 3.1, find $i = i_g$ such that $g \in \Sigma_i^{M, B_0 \cap P_i, \theta_0, \varepsilon_0(\eta)} \Gamma$.

By unwrapping the definition, there exists $\gamma_g \in \Gamma$ such that $\gamma_g^{-1} g^{-1} M g \gamma_g$ is contained in $P_i$ (as $M$ is semisimple and $M$ is connected, any conjugate of $M$ being contained in $P_i$ automatically implies being contained in $P_i$) and if $g \gamma_g = k_{a_g} p_g$ is the horospherical coordinate of $g \gamma_g$ with respect to $P_i$, then

1. $\alpha(a_g) < \theta_0, \forall \alpha \in \Delta_i;$
2. $\alpha_0(a_g) < \varepsilon_0(\eta), \exists a_g \in \Delta_i;$
3. $p_g \gamma_g^{-1} g^{-1} m_g g \gamma_g = b_g \lambda_g$ for some $m_g \in M, b_g \in B_0 \cap P_i$ and $\lambda_g \in P_i \cap \Gamma$.

Recall that we have chosen $\theta_0$ such that $\alpha(a_g) < 1$ for all $\alpha \in \Phi_i^{-1}$. Hence $\alpha(a_g) \geq 1$ for all $\alpha \in \Phi_i^{+0}$.

Now take $m := m_g, 0 < \eta_1 < \eta_0$, which will be determined later at Equation 1 in the proof of (3). Let $P := \gamma_g P \gamma_g^{-1} g^{-1}$. Then $P$ contains $M$ and $g^{-1} P g = \gamma_g P \gamma_g^{-1}$ is defined over $\mathbb{Q}$. Also let $U$ be the unipotent radical $P$. Define $\varepsilon_1(\eta) := \varepsilon_0(\eta)^{1/N} C_3^{1/N}$. It remains to prove the three claims.
Proof of (1). Recall that $N$ denotes the dimension of $G$.

$$
\|\mathcal{U} \cap \mathcal{G}\| = \|(g_{\gamma g} \cdot \mathcal{U}) \cap (g_{\gamma g} \cdot \mathcal{G})\|
= \|(g_{\gamma g}) \cdot (\mathcal{U} \cap \mathcal{G})\|
= \|(k_{g}\alpha_{g}p_{g}) \cdot (\mathcal{U} \cap \mathcal{G})\|.
$$

First note that $p_{g}$ preserves $\mathcal{U}$ and preserves the (co)volume. On the other hand $a_{g}$ also preserves $\mathcal{U}$ but $\alpha(a_{g}) < 1$ for all $\alpha$ appearing in $\mathcal{U}$ and for $\alpha = \alpha_{g}$, which appears in $\mathcal{U}$, $\alpha(a_{g}) < \varepsilon_{0}(\eta)$. So $|\det(\text{Ad}(a_{g})|\mathcal{U})| < \varepsilon_{0}(\eta)$. Hence

$$
\|\mathcal{U} \cap \mathcal{G}\| \leq \varepsilon_{0}(\eta) \|\mathcal{U} \cap \mathcal{G}\| \leq \varepsilon_{0}(\eta)C_{3}.
$$

This proves (1). Note this also shows that $\delta_{M}(\|g\|) \leq \varepsilon_{1}(\eta)$.

Proof of (2). Take $v \in \mathcal{G}_{m_{g}} \setminus \mathcal{U}$. As $\mathcal{G}_{m_{g}} = m_{g} \cdot \mathcal{G} = m_{g}\gamma_{g}\lambda_{g}^{-1} \cdot \mathcal{G}$ and $\mathcal{U} = m_{g}\gamma_{g}\lambda_{g}^{-1} \cdot \mathcal{U}$, we can find $v_{g} \in \mathcal{G} \setminus \mathcal{U}$ such that

$$
v = m_{g}\gamma_{g}\lambda_{g}^{-1} \cdot v_{g}.
$$

Hence

$$
v = (g_{\gamma g}) \cdot (\gamma_{g}^{-1}g_{1}m_{g}\gamma_{g}) \cdot \lambda_{g}^{-1} \cdot v_{g}
= (k_{g}\alpha_{g}) \cdot (p_{g}\gamma_{g}^{-1}g_{1}m_{g}\gamma_{g}) \cdot \lambda_{g}^{-1} \cdot v_{g}
= k_{g}\alpha_{g}b_{g} \cdot v_{g}.
$$

Note that $\mathcal{U} = g_{\gamma g} \cdot \mathcal{U}_{i} = k_{g}a_{g}p_{g} \cdot \mathcal{U}_{i} = k_{g} \cdot \mathcal{U}_{i}$ and $\text{Ad}(k_{g})$ acts by isometry, we have

$$
\text{Ad}(k_{g}) \circ \pi_{\mathcal{U}_{i}} = \pi_{\mathcal{U}_{i}} \circ \text{Ad}(k_{g}).
$$

Also recall that $\pi_{\mathcal{U}_{i}} = \pi_{0+}^{A_{1}}$ and

$$
\text{Ad}(w_{i}) \circ \pi_{0+}^{A_{1}} = \pi_{0+}^{S_{1}} \circ \text{Ad}(w_{i}).
$$

Now if we write $s_{g} = w_{i}a_{g}w_{i}^{-1} \in S_{i}$ and $b_{g} = h_{g}^{i}u_{g}^{i}$ for some $h_{g}^{i} \in Z_{G}(A_{i})$ and $u_{g}^{i} \in U_{i}$, then

$$
\|\pi_{\mathcal{U}_{i}}(v)\| = \|\pi_{\mathcal{U}_{i}}(k_{g}a_{g}b_{g} \cdot v_{g})\|
= \|k_{g} \cdot \pi_{\mathcal{U}_{i}}(a_{g}b_{g} \cdot v_{g})\|
= \|\pi_{0+}^{A_{1}}(a_{g}b_{g} \cdot v_{g})\|
= \|w_{i}^{-1} \cdot \pi_{0+}^{A_{1}}((w_{i}a_{g}w_{i}^{-1})(w_{i}b_{g}) \cdot v_{g})\|
\geq \frac{1}{C_{4}} \|\pi_{0+}^{S_{1}}((s_{g})(w_{i}b_{g}w_{i}^{-1})(w_{i}u_{g}^{i}) \cdot v_{g})\|
$$

Let $\alpha_{g} \in \Phi_{i}^{0+}$ be a maximal element (our convention about the partial order is that $\alpha - \beta$ is contained in positive combinations of $\Delta_{i}$ iff $\alpha \leq \beta$) such that $\pi_{\alpha_{g}}(v_{g}) \neq 0$. Then for any $u \in U_{i}$, $\pi_{\alpha_{g}}^{S_{1}}(u \cdot v_{g}) = \pi_{\alpha_{g}}(v_{g})$. Also the reader is reminded that $\|\pi_{\alpha_{g}}^{S_{1}}(v_{g})\| \geq 1/C_{5}$ and $C_{4}$ bounds the operator norm of some elements. Recall that $\alpha_{g}(s_{g}) > 1$ as we have chosen $\theta_{0}$ small enough.
We may continue the above inequalities as
\[
\|\pi_{\mathcal{W}}(v)\| \geq \frac{1}{C_4 C_5^4} \|\pi_{\alpha_0}^{S_i}(w_{i\mathcal{H}}^i w_{i\mathcal{G}}^{-1}) (w_{i\mathcal{H}}^i / v_g)\|
\geq \frac{1}{C_4 C_5^4} (w_{i\mathcal{H}}^i / v_g) \cdot \pi_{\alpha_0}^{S_i}(w_{i\mathcal{H}}^i / v_g)
\geq \frac{1}{C_4 C_5^4} \pi_{\alpha_0}^{S_i}(v_g) \geq \left(\frac{1}{C_4 C_5^4}\right)^2.
\]

Proof of (3). We are going to use both (1) and (2) here. As \(\mathcal{L}\) is M-stable, \(\mathcal{L}\) is also \(\mathcal{G}_{\text{mg}}\)-rational and \(\|\mathcal{L} \cap \mathcal{G}_{\text{mg}}\| = \|\mathcal{L} \cap \mathcal{G}_{\text{mg}}\|\). This is the only place we need \(\mathcal{L}\) to be M-stable.

As \(\mathcal{W}\) is also \(\mathcal{G}_{\text{mg}}\)-rational, we have that \(\pi_{\mathcal{W}}(\mathcal{L} \cap \mathcal{G}_{\text{mg}})\) is a lattice in \(\mathcal{W}^\perp\) (we are not claiming that \(\pi_{\mathcal{W}}\) is \(\mathcal{G}_{\text{mg}}\)-rational, but \(\pi_{\mathcal{W}}\) is a lift of the map of quotient by \(\mathcal{W}\), which is \(\mathcal{G}_{\text{mg}}\)-rational) and
\[
\|\mathcal{L} \cap \mathcal{G}_{\text{mg}}\| = \|\mathcal{L} \cap \mathcal{W} \cap \mathcal{G}_{\text{mg}}\| \cdot \|\pi_{\mathcal{W}}(\mathcal{L} \cap \mathcal{G}_{\text{mg}})\|.
\]

By (2), \(\pi_{\mathcal{W}}(\mathcal{L} \cap \mathcal{G}_{\text{mg}})\) has its shortest non-zero vector with length at least \(1/(C_4 C_5^4)^2\). Therefore it has a fundamental domain containing a cube of size \(1/2N(C_4 C_5^4)^2\).

Hence
\[
\|\pi_{\mathcal{W}}(\mathcal{L} \cap \mathcal{G}_{\text{mg}})\| \geq \left(\frac{1}{2NC_4^2 C_5^4}\right)^N.
\]

For simplicity we let
\[
C_6 := (2NC_4^2 C_5^4)^N,
\]
\[
C_7^{-1} := \min \left\{ \left| \frac{1}{x} - \frac{1}{y} \right| \left| x, y \in \{1, \ldots, N\} \right\}ight. \}
\]

And we choose \(\eta_1\) small enough such that for any \(0 < \eta < \eta_1\),
\[
2^N C_8^N C_6^N \varepsilon_1(\eta) < 1,
\]
\[
C_6^{-1} (2^N C_8^N C_6^N \varepsilon_1(\eta))^{-C_7^{-1}} \geq 2C_8.
\]

where \(C_8\) is as in Lemma 3.3.

Now there are two cases depending on how large \(\|\mathcal{L} \cap \mathcal{W} \cap \mathcal{G}_{\text{mg}}\|\) is.

Case I, \(\|\mathcal{L} \cap \mathcal{W} \cap \mathcal{G}_{\text{mg}}\|^{1/\dim(\mathcal{L} \cap \mathcal{W})} \leq \|\mathcal{L} \cap \mathcal{G}_{\text{mg}}\|^{N \delta_M([g])}\).

Note by our assumption \(\dim \mathcal{L} > \dim (\mathcal{L} \cap \mathcal{W})\) and hence \((\dim \mathcal{L})^{-1} - (\dim \mathcal{L} \cap \mathcal{W})^{-1} < 0\). Also, \(\delta_M([g]) \leq \varepsilon_1(\eta)\) by part (1).
\[
\|\mathcal{L} \cap \mathcal{G}_{\text{mg}}\|^{\frac{1}{\dim \mathcal{L}}} \geq C_6^{-1/\dim \mathcal{L}} \cdot \|\mathcal{L} \cap \mathcal{W} \cap \mathcal{G}_{\text{mg}}\|^{\frac{1}{\dim \mathcal{L}}} \cdot \|\mathcal{L} \cap \mathcal{W} \cap \mathcal{G}_{\text{mg}}\|^{-\frac{1}{\dim \mathcal{L}}}
\geq C_6^{-1} \cdot (2^N C_8^N C_6^N \varepsilon_1(\eta))^n \cdot \delta_M([g])
\]

By assumption \(0 < 2^N C_8^N C_6^N \varepsilon_1(\eta) < 1\) and \(\dim(\mathcal{L} \cap \mathcal{W}) \cdot (\frac{1}{\dim \mathcal{L}} - \frac{1}{\dim \mathcal{L} \cap \mathcal{W}}) \leq -C_7^{-1} < 0\), hence we may continue:
\[
\|\mathcal{L} \cap \mathcal{G}_{\text{mg}}\|^{\frac{1}{\dim \mathcal{L}}} \geq C_6^{-1} \cdot (2^N C_8^N C_6^N \varepsilon_1(\eta))^{-C_7^{-1}} \cdot \delta_M([g]) \geq 2C_8 \delta_M([g]),
\]

which completes the case I.

Case II, \(\|\mathcal{L} \cap \mathcal{W} \cap \mathcal{G}_{\text{mg}}\|^{1/\dim(\mathcal{L} \cap \mathcal{W})} \geq \|\mathcal{L} \cap \mathcal{G}_{\text{mg}}\|^N \delta_M([g])\).
This case is more direct.
\[ \|L \cap \mathcal{F}_mg\|_{\dim Z} \geq C_6^{-1/\dim L} (2N C_8 N C_6 \delta_M([g]))^{\dim (L \cap \mathcal{F})} \geq C_6^{-1} 2C_8 C_6 \delta_M([g]) = 2C_8 \delta_M([g]). \]

Now the proof of the corollary is complete.

Lemma 3.3. There exist two constants $C_8, C_9 > 1$ and a finite set $\mathcal{F} \subset S$ such that for any $\mathbb{R}$-parabolic subgroup $P$ of $G$ containing $M$, there exists $s \in \mathcal{F}$ such that

1. $\|sv\| > \frac{1}{C_8} \|v\|$ for all pure wedges $v$ in $\mathcal{F}$;
2. $\|sv\| > C_9 \|v\|$ for all pure wedges $v$ with $L \subset \mathcal{F}$.

Proof. The proof is similar to that of Lemma 2.3 where the key is to produce a bijective projection onto some ‘coordinate plane’ $V_i$. Here we will produce a bijective projection to some ‘coordinate horospherical Lie subalgebra’.

Let $\mathcal{P}_M$ be the collection of $\mathbb{R}$-parabolic subgroups of $G$ containing $M$. For each $a_t \in X_+(S)$, we let
\[ P_n := \left\{ g \in G \mid \lim_{t \to 0} a_t g a_t^{-1} \text{ exists} \right\}. \]

This way we get a finite collection of elements in $\mathcal{P}_M$. Label this set as $\{P_i\}_{i \in B}$ for some finite set $B$. As $S \leq Z_G M$ is a maximal $\mathbb{R}$-split torus, for any other $P \in \mathcal{P}_M$, there exists $h \in Z_G(M)$ such that $P = hP_i h^{-1}$ for some $i \in B$. Hence $\mathcal{P}_M$ is a finite union of compact homogeneous spaces of $Z_G(M)$. Indeed, the stabilizer of each $P_i$ in $Z_G(M)$ is a parabolic subgroup of $Z_G(M)$.

Now fix $P \in \mathcal{P}_M$ and $h \in Z_G(M)$ with $P = hP_i h^{-1}$. By Bruhat decomposition (see [Bor91, 21.15]), there exists $w \in N_{Z_G(M)} S$, $u \in U$ and $p \in P_i$ such that $h = uw p$ where $U$ is a maximal $\mathbb{R}$-unipotent subgroup contained in $P_i$ that is normalized by $S$. Then $P = h \cdot P_i = uw p \cdot P_i = u \cdot P_{j_0}$ for some $j_0 \in B$, where $\cdot$ denotes the conjugation by a group element.

We claim that the $S$-equivariant projection $\pi_{j_0} : \mathcal{G} \to \mathcal{G}_{j_0}$, when restricted to $\mathcal{V}$, is bijective (but usually $\mathcal{V}$ is not $S$-stable).

Decompose $\mathcal{G} = \bigoplus_{a \in \Phi} \mathcal{G}_a^S$ with respect the Ad($S$)-action. And let $\pi_{j_0}^S$ be the associated projection onto $\mathcal{G}_{j_0}^S$. Fix a minimal $\mathbb{R}$-parabolic $P_{\min} \subset S \cdot U \leq P_i$. A partial order \( \leq \) is defined on $\Phi$ by demanding that $\alpha \leq \beta$ iff $\beta - \alpha$ is a weight that appears in $P_{\min}$, the Lie algebra of $P_{\min}$.

Now take $v \in \mathcal{V}$, so $v = u \cdot w$ for some $w \in \mathcal{V}_{j_0}$. Take $\alpha_0 \in \Phi$ to be a minimum element such that $\pi_{\alpha_0}^S(w) \neq 0$. Then
\[ \pi_{\alpha_0}^S(u \cdot w) = \pi_{\alpha_0}^S(u \cdot \pi_{\alpha_0}^S(w)) = \pi_{\alpha_0}^S(w) \neq 0. \]

As $\mathcal{V}_{j_0}$ is defined by a cocharacter of $S$ and projects to $\mathcal{G}_{j_0}^S$ nontrivially, $\mathcal{V}_{j_0}$ necessarily contains $\mathcal{G}_{\alpha_0}^S$. Thus $\pi_{j_0}(v) = \pi_{j_0}(u \cdot w) \neq 0$. This proves that $\pi_{j_0}|_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}_{j_0}$ is injective. And hence it is bijective because $\mathcal{V}$ and $\mathcal{V}_{j_0}$ have the same dimension.

The rest of the proof follows the same lines as in Lemma 2.3 and is omitted.

Proposition 3.4. There exists $\eta_2 > 0$ and $C_{10} > 1$ such that for all $[g] \not\in X_{\eta_2}^M$, there exists $s \in S$ such that $\delta_M([sg]) > C_{10} \delta_M([g])$. Consequently, there exists $s \in S$ such that $[sg] \in X_{\eta_2}^M$. 
Theorem 1.2 would follow from this proposition by [DGU20, Theorem 4.6] just as in last section.

Proof. We prove the proposition with \( \eta_2 := \eta_1 \) and \( C_{10} := \min\{C_g^{3/N}, 2\} \).

Take any \( g \notin X^M_n \).

Find \( P \) an \( \mathbb{R} \)-parabolic subgroup of \( G \) according to Corollary 3.2. Then choose \( s \in S \) with Lemma 3.3 being applied to \( P \). It suffices to prove that \( \delta_M([sg]) \geq C_{10}\delta_M([g]) \).

Take an \( \mathbb{R} \)-subspace \( L' \) of \( G \) that is \( G_{sg} \)-rational and \( M \)-stable. Then \( L := s^{-1}L' \) is \( G_g \)-rational and \( M \)-stable. Also, \( L' \cap G_{sg} = s \cdot (L \cap G_g) \). There are two cases.

If \( L \subseteq H \), then

\[
\|L' \cap G_{sg}\|_{\operatorname{dim} L'} \geq C_g^{\frac{1}{3-2\varepsilon}} \|L \cap G_g\|_{\operatorname{dim} L} \geq C_g^{\frac{1}{3-2\varepsilon}} \delta_M([g]) \geq C_g^{\frac{1}{3-2\varepsilon}} \delta_M([g]).
\]

If \( L \not\subseteq H \), then

\[
\|L' \cap G_{sg}\|_{\operatorname{dim} L'} \geq C_g^{\frac{1}{3-2\varepsilon}} \|L \cap G_g\|_{\operatorname{dim} L} \geq C_g^{\frac{1}{3-2\varepsilon}} 2C_g \delta_M([g]) \geq 2\delta_M([g]).
\]

Hence our proof is complete. \( \Box \)

4. Proof of Theorem 1.3 and 1.4

Proof of Theorem 1.3. Indeed, by [SW00, Corollary 1.3], for every \( x \in G/\Gamma \), \( \overline{Fx} \) is \( H \)-invariant, thus has to intersect some bounded set independent of \( x \) nontrivially. \( \Box \)

Proof of Theorem 1.4. First we claim that up to \( G \)-conjugacy, there are only finitely many elements from \( H \). For every dimension, up to isomorphism, there are only finitely many real semisimple algebras (see [Kna02]). And for each of them, the Lie subalgebras of \( G \) that are isomorphic to this one form a finite orbit under \( G \) by [EMV09, Lemma A.1] (see also [Ric67]). Now we enumerate the corresponding real algebraic subgroups as \( \{M_1, \ldots, M_n\} \). Also fix a maximal \( \mathbb{R} \)-split torus \( S_i \) of \( Z_G M_i \).

By shortening the list, we assume that each \( H_i := M_i \cdot S_i \) belongs to \( H \). We now argue that each \( H \in H \) is isomorphic to one of \( H_i \). Indeed, write \( H \) as an almost direct product \( M \cdot S \). Then there exists \( g \in G \) such that \( qMg^{-1} = M_i \) for some \( i \). Then \( gSg^{-1} \) is a maximal \( \mathbb{R} \)-split torus of \( Z_G M_i \); thus there exists some \( h \in Z_G M_i \) such that \( hgh^{-1}g^{-1}h^{-1} = S_i \) (see [Spr98, 15.2.6]). Thus \( hgh^{-1}g^{-1}h^{-1} = H_i \).

As this is a finite list, we find a compact set \( C \subseteq G/\Gamma \) such that for every \( i \in C \) and every \( x \in G/\Gamma \), there exists \( h \in H_i \), with \( hx \in C \). For each \( i \), find a compact subgroup \( K_i \) such that \( Z_G H_i = K_i \cdot S_i \). Let \( C' \) be a larger compact subset containing \( K_i \cdot C \) for all \( i \)'s. We fix some embedding of \( G \to \operatorname{SL}_N(\mathbb{R})/\operatorname{SL}_N(\mathbb{Z}) \). For each \( i \), fix a nonempty bounded open set \( \Omega_i \) in \( H_i \).

Thus by [RS18, Theorem 1.1], for each \( i \), there exists a closed subset \( Y_i \) of \( G \) such that

1. \( G = Y_i \cdot K_i \cdot S_i \);
2. there exists some \( c > 0 \) such that

\[
\sup_{\omega \in \Omega_i} \|y_\omega \cdot v\| \geq c \|v\| \quad \forall i, y \in Y, v \in \mathbb{R}^N.
\]
Through the work of [EMS97] (or [KM98]), this implies that there exists a compact set $C'' \subset G/\Gamma$ such that for every $x \in C'$ and every $y \in Y_i$,

$$y\Omega_i \cdot x \cap C'' \neq \emptyset.$$ 

By further using $G = Y_i \cdot K_i \cdot S_i$ and $K_i \cdot C \subset C'$, we have that for every $x \in C$ and every $y \in Y_i$,

$$y\Omega_i \cdot x \cap C'' \neq \emptyset.$$ 

Now take $x_0 \in G/\Gamma$ and $H \in \mathcal{H}$, we wish to show $H \cdot x_0 \cap C'' \neq \emptyset$. First find $g_H \in G$ such that $H = g_H H_i g_H^{-1}$ for some $i$. Then by Theorem 1.2,

$$H \cdot x_0 = g_H H_i g_H^{-1} \cdot x_0 = g_H H_i \cdot x' \exists x'_0 \in C,$$

which intersect with $C''$ nontrivially. Hence we are done. □

5. Geometric consequences

Main results in this section is a proof of Theorem 1.5 and a characterization of rigid totally geodesic submanifolds in Proposition 5.2. For backgrounds on symmetric spaces, our main references here are [Hel01] and [KM18].

5.1. Arithmetic quotients of symmetric spaces of noncompact type. Let $\tilde{X}$ be a connected global Riemannian symmetric space of noncompact type. Thus the identity connected component of isometry group of $\tilde{X}$ is a connected semisimple Lie group $G$ and $\tilde{X}$ is identified with the space of maximal compact subgroups of $G$. By fixing a maximal compact subgroup $K_0$ of $G$, $\tilde{X}$ is identified with $K_0 \backslash G$. We are going to assume $G = G := G(\mathbb{R})^\circ$ for a connected $\mathbb{Q}$-algebraic group $G$ and take $\Gamma \leq G \cap G(\mathbb{Q})$ to be an arithmetic lattice. Call a locally symmetric space $X$ of the form $X = \tilde{X}/\Gamma = K_0 \backslash G/\Gamma$ arithmetic. We assume $\Gamma$ to be neat so that $X$ is a Riemannian manifold.

For a maximal compact subgroup $K$ of $G$, there exists a unique algebraic Cartan involution $\iota_K$ over $\mathbb{R}$ associated with $K$. If $K$ is the fixed point of $\iota_K$ in $G_{\mathbb{R}}$, then $K = K := K(\mathbb{R})^\circ$. By abuse of notation we also use $\iota_K$ to denote its induced action on the Lie algebra. Thus the Lie algebra $\mathfrak{t}$ of $K$ is identified with those fixed by $\iota_K$. Let $\mathfrak{p}_K$ be the $(-1)$-eigenspace of $\iota_K$ in $\mathfrak{g}$, the Lie algebra of $G$. Then

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}_K.$$

For simplicity we write $\iota_0 := \iota_{K_0}$ and $\mathfrak{p}_0 := \mathfrak{p}_{K_0}$.

Let

$$B(v, w) := - \text{Tr} (\text{ad}(v) \text{ad}((\iota_0 w)))$$

be the positive definite bilinear form on $\mathfrak{g}$, identified with the tangent space of $G$ at $id$, associated with $K_0$. By right translation, we get a right $G$-invariant Riemannian metric on $G$. This metric is also left $K_0$-invariant. This metric thus induces metrics on $G/\Gamma$, $K_0 \backslash G/\Gamma$ and their closed submanifolds. All the “Vol” appearing below will be referred to measures induced from this metric. Up to scalars (to be more precise, for each irreducible factor there is a positive scalar), the original Riemannian metric on $X$ coincides this one.
5.2. Totally geodesic submanifolds. A totally geodesic submanifold $\tilde{Y}$ of $\tilde{X}$ is again a symmetric space. By [Hel01, Theorem 7.2], there exists a triple system $s_0 \subset p_0$, i.e. $[x, [y, z]] \in s_0$ if $x, y, z \in s_0$ and $g \in G$, such that $\tilde{Y} = K_0 \backslash K_0 \exp(s_0)g$. Then $\mathfrak{h}_Y := s_0 + [s_0, s_0]$ is a $\iota_0$-stable Lie subalgebra. By writing $H_0$ for the corresponding Lie subgroup, we have $\tilde{Y} = K_0 \backslash K_0 H_0 g$. The choice of $s_0$ is not unique but it is understood when we write $\tilde{Y} = K_0 \backslash K_0 H_0 g$.

Now let $Y \subset X$ be an embedded totally geodesic submanifold, by choosing a lift of some point of $y \in Y$, we have a unique totally geodesic submanifold $\tilde{Y}$ of $\tilde{X}$ which projects to $Y$ as a local isometry. Thus $Y = K_0 \backslash K_0 H_0 g \Gamma / \Gamma$. It is not hard to verify that $H_0 g \Gamma / \Gamma$ is also closed in $G / \Gamma$.

We are going to be interested in finite-volume embedded totally geodesic submanifold $Y$ of $X$. Using the notation as in the last paragraph, one can verify from the definition that for such a $Y$,

$$\text{Vol}(H_0 g \Gamma / \Gamma) = \text{Vol}(Y) \cdot \text{Vol}(K_0 \cap H_0).$$

(2)

5.3. Rigid totally geodesic submanifolds.

**Definition 5.1.** Fix a natural number $N$, let

$$\mathcal{T}G^N := \left\{ Y \subset X \mid Y \text{ is a embedded totally geodesic submanifold of } X, \ \text{Vol}(Y) < \infty, \ \dim Y = N \right\}$$

be equipped with the Chabauty topology (see [BP92, E.1]). We say that $Y \in \mathcal{T}G^N$ is rigid if $\{Y\}$ is open in $\mathcal{T}G^N$. The collection of such $Y$'s are denoted as $\mathcal{T}G^N, \text{rigid}$.

Take $Y := K_0 \backslash K_0 H_0 g \Gamma / \Gamma \in \mathcal{T}G^N$. We can write $\mathfrak{z}_0(\mathfrak{h}_Y) = \mathfrak{t}_Y \oplus \mathfrak{z}(\mathfrak{h}_Y)$ for some $\iota_0$-stable subalgebra $\mathfrak{t}_Y$ centralizing $\mathfrak{z}(\mathfrak{h}_Y)$ where $\mathfrak{z}_0(\mathfrak{h}_Y)$ is the centralizer of $\mathfrak{h}_Y$ in $\mathfrak{g}$ and $\mathfrak{z}(\mathfrak{h}_Y)$ is the center of $\mathfrak{h}_Y$. If $Y$ is rigid, then $\mathfrak{t}_Y$ is contained in $\mathfrak{t}_0$ for otherwise there exists $v_{\neq 0} \in \mathfrak{t}_Y \cap \mathfrak{p}_0$ and

$$Y_t := K_0 \backslash K_0 \exp(\nu t) H_0 g \Gamma / \Gamma \to Y \quad \text{as } t \to 0$$

in $\mathcal{T}G^N$ and $Y_t \neq Y$ for $t \neq 0$. The converse is also true.

**Proposition 5.2.** Notations as above. Then $Y \in \mathcal{T}G^N$ is rigid iff $\mathfrak{t}_Y$ is contained in $\mathfrak{t}_0$. In this case $\mathfrak{h}_Y$ is algebraic.

The second claim follows from the following lemma. The proof of the rest of the claim is delayed to the next section. Recall that a Lie subalgebra of the Lie algebra of a linear algebraic group over $\mathbb{R}$ is said to be algebraic iff its the Lie algebra of some algebraic subgroup over $\mathbb{R}$ (see [Bor91, Chapter II.7]).

**Lemma 5.3.** Let $\mathfrak{h}$ be a $\iota_0$-stable Lie subalgebra of $\mathfrak{g}$ with no compact factors. Assume that all noncompact factors of $\mathfrak{z}_0 \mathfrak{h}$ are contained in $\mathfrak{h}$, then

1. $\mathfrak{h}$ is algebraic;
2. identity coset is the unique $x \in K_0 \backslash G$ such that $x \mathfrak{h}$ is totally geodesic;
3. there exists a finite list (only depends on $G$, $K_0$) $\{\mathfrak{h}_1, \ldots, \mathfrak{h}_k\}$ satisfying the same condition as $\mathfrak{h}$ does such that $\mathfrak{h}$ is conjugate to one of them via $K_0$.

**Proof.** Write $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{m}$ for some abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}_0$ and semisimple Lie subalgebra $\mathfrak{m}$. Moreover $\mathfrak{a}$ commutes with $\mathfrak{m}$. By [Bor91, II.7.9], $\mathfrak{m}$ is algebraic. As $\mathfrak{z}_0 \mathfrak{h}$ is algebraic and $\mathfrak{a}$ is characterized as the $(-1)$-eigenspace of $\iota_0$, which is algebraic, in $\mathfrak{z}_0 \mathfrak{h}$, we have $\mathfrak{a}$ is algebraic. Hence $\mathfrak{h}$ is algebraic.
Item 2. and 3. have been proved in [KM18, Section 2] under the additional assumption that \( \mathfrak{h} \) is semisimple. But the same proof presented there also works without this assumption.

Item 2. and 3., together with Equa. 2 imply the following

**Lemma 5.4.** There exists a constant \( C_{11} > 1 \) such that

\[
C_{11}^{-1} \text{Vol}(Y) \leq \text{Vol}(H_0gG/\Gamma) \leq C_{11} \text{Vol}(Y)
\]

for all \( Y \) in \( T^{-N}_{\text{rigid}} \).

### 5.4. Proof of Proposition 5.2
It remains to show that, assuming \( \mathfrak{t}_Y \) is contained in \( \mathfrak{t}_0 \), for a sequence of \( Y_i = K_0 \backslash K_0H_i\gamma_i\Gamma/\Gamma \) converging to \( Y = K_0 \backslash K_0H_0\delta_0\Gamma/\Gamma \), then \( Y_i = Y \) for \( i \) sufficiently large. Replacing \( \Gamma \) by \( g_0\Gamma g_0^{-1} \), assume \( g_0 = id \). Also write \( \mathfrak{h}_0 = (\mathfrak{t}_0 \cap \mathfrak{h}_0) \oplus \mathfrak{z}_0 \) where \( \mathfrak{z}_0 \) is a triple system associated with \( Y \). Also write \( H_0 = A_0 \cdot M_0 \) (at the level of Lie algebra, \( \mathfrak{h}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0 \)) as an almost product between its center and the semisimple part.

For an element \( g \in G \), let \( K_0[g]\Gamma \) (resp. \( [g]\Gamma, K_0[g] \)) denote its image in \( K_0 \backslash G/\Gamma \) (resp. \( G/\Gamma, K_0 \backslash G \)).

From definition, 

\[
k_i h_{Y_i} g_i \gamma_i = \varepsilon_i
\]

for some sequences of \( k_i \in K_0, h_{Y_i} \in H_{Y_i}, \gamma_i \in \Gamma \) and \( \varepsilon_i \in G \) with \( \varepsilon_i \to id \). Thus \( Y_i = K_0 \backslash K_0H_i\varepsilon_i\Gamma/\Gamma \), \( \text{with } H_i := k_i h_{Y_i} k_i^{-1} \).

Let \( \mathfrak{s}_i := \text{Ad}(k_i) \cdot \mathfrak{s}_{Y_i} \) (recall \( \mathfrak{s}_{Y_i} \subset \mathfrak{p}_0 \) is the triple system associated with \( Y_i \)). For \( \delta > 0 \), let \( B_{\mathfrak{p}_0, \delta} \) be the open ball of radius \( \delta \) in \( \mathfrak{p}_0 \). We choose \( \delta > 0 \) small enough such that 

\[
B_{\mathfrak{p}_0, \delta} \to K_0 \backslash G/\Gamma : v \mapsto k_0[\exp(v)]\Gamma
\]

is a homeomorphism onto its image. By passing to a subsequence assume \( \mathfrak{s}_i \) converges to \( \mathfrak{s}_\infty \). Then \( \mathfrak{s}_\infty \) is still a triple system and hence \( \mathfrak{h}_\infty := \mathfrak{h}_\infty \oplus [\mathfrak{s}_\infty, \mathfrak{s}_\infty] \) is a \( \mathfrak{t}_0 \)-stable Lie subalgebra. Also \( K_0[\exp(\mathfrak{s}_0 \cap B_{\mathfrak{p}_0, \delta})] \Gamma \) converges to \( K_0[\exp(\mathfrak{s}_\infty \cap B_{\mathfrak{p}_0, \delta})] \Gamma \), which must be contained in \( \mathfrak{s}_0 \) by assumption. Since they share the same dimension we conclude that \( \mathfrak{h}_\infty = \mathfrak{h}_0 \). Thus \( \mathfrak{s}_i \)'s are all algebraic by Lemma 5.3.

We would like to understand the limiting behavior of 

\[
\bar{Y}_i := H_i\varepsilon_i\Gamma/\Gamma
\]

in \( G/\Gamma \). So far we know that

1. \( \lim \bar{Y}_i \) is a closed \( H_0 \)-invariant set;
2. \( \lim \bar{Y}_i \supset \bar{Y} := H_0\Gamma/\Gamma \);
3. \( \lim \bar{Y}_i \subset K_0H_0\Gamma/\Gamma \).

Assume \( [k_1 g_1] \Gamma \in \bar{Y}_i \) for some \( k_1 \in K_0 \) and \( g_1 \in H_0 \). Then by item 1. above, 

\[
\lim \bar{Y}_i \supset H_0 k_1 g_1 \Gamma/\Gamma
\]

and in particular

\[
K_0 \backslash K_0H_0\Gamma/\Gamma = \lim K_0 \backslash K_0H_i\varepsilon_i\Gamma/\Gamma \supset K_0 \backslash K_0H_0 k_1 g_1 \Gamma/\Gamma.
\]

Thus \( \mathfrak{s}_0 \) contains \( \text{Ad}(k_1^{-1}) \cdot \mathfrak{s}_0 \). And since they have the same dimensions, \( \mathfrak{s}_0 = \text{Ad}(k_1^{-1}) \cdot \mathfrak{s}_0 \), which implies that \( k_1 \in N_G(H_0) \), the normalizer of \( H_0 \) in \( G \). Therefore item 3. above is upgraded to

4. \( \lim \bar{Y}_i \subset (K_0 \cap N_G(H_0))H_0\Gamma/\Gamma \subset N_G(H_0)\Gamma/\Gamma \).
Let $H'_i := \varepsilon_i^{-1}H_i\varepsilon_i$ and $\mathfrak{h}'_i := \text{Ad}(\varepsilon_i^{-1})\mathfrak{h}_i$. Decompose $\mathfrak{h}'_i = a'_i \oplus m'_i$ into an abelian ideal $a'_i$ and a semisimple ideal $m'_i$. Both $a'_i$ and $m'_i$ are $\varepsilon_0$-stable. Write $A'_i$ and $M'_i$ for the associated Lie subgroups. By Borel density lemma (use the version in \cite[Corollary 4.2]{dan80}), $H'_i$ is defined over $\mathbb{Q}$. Thus $M'_i$ and $A'_i$ are also defined over $\mathbb{Q}$. As $Y_i$ has finite volume, it follows that $M'_i\Gamma/\Gamma$ and $A'_i\Gamma/\Gamma$ have finite volume.

By \cite[Theorem 1.1]{ms95}, there exists a connected $\mathbb{R}$-split subgroup $F$ of $G$ such that $[F]\Gamma$ has finite volume, $[M'_i]\Gamma$ converges to $[F]\Gamma$ and moreover, there exists $\delta_i \in G$ converging to $id$ such that $[\delta_i M'_i]\Gamma$ is contained in $[F]\Gamma$ for $i$ large enough. From the latter, it can be shown that $M'_i$ is contained in $F$ for $i$ large enough. On the other hand, the limit of $[M'_i]\Gamma$ is contained in $[N_G(H_0)]\Gamma$, thus $F$ is contained in $N_G(H_0)$. As $F$ is connected and the Lie algebra of $N_G(H_0)$ is the same as that of $Z_G(H_0)^0 H_0$. We conclude that $F$ is contained in $Z_G(H_0)^0 H_0$. But $F$ is semisimple and contains $M'_i$, thus we must have $M_0$, the semisimple part of $H_0$, is exactly equal to $F$. Hence $M'_i = F = M_0$.

Thus we have seen that (write $\tilde{Y}'_i := \varepsilon_i^{-1}\tilde{Y}_i$)

5. $\tilde{Y}'_i = A'_i M_0 \Gamma / \Gamma \subset Z_G(M_0)^0 M_0 \Gamma / \Gamma$ for $i$ large enough.

So to find the limit of $\tilde{Y}'_i$, it suffices to consider

$$\lim [A'_0 M_0]_{\Gamma\Gamma} \text{ inside } Z_G(M_0)^0 M_0 / \Gamma \cap (Z_G(M_0)^0 M_0).$$

Now we have all the ingredients to conclude the proof. Some definitions and notations are introduced to ease the argument.

Let $L := Z_G(M_0)^0 M_0$ and $\Gamma_L := \Gamma \cap L$. Let $\pi : L \to L / M_0$ be the natural quotient map and it induces $\pi' : L / \Gamma_L \to \pi(L) / \pi(\Gamma_L)$. Note that $\pi(\Gamma_L)$ is still a lattice in $\pi(L)$. Let $K_L := K_0 \cap Z_G(H_0)^0 = (K_0 \cap Z_G(H_0))^0$. For an element (or a subset) $x$ of $\pi(L)$, as before, we let $\pi(x)[x]_{\pi(\Gamma_L)}$ be its image in $\pi(K_L) \cap \pi(L) / \pi(\Gamma_L)$. Other similar notations are also defined. Note that since $A_0$ commutes with $K_0$, $\pi(A_0)$ acts from the left on $\pi(K_L) / \pi(L) / \pi(\Gamma_L)$.

Let $\{x_1, ..., x_c\} \subset N_G(H_0) \cap K_0$ be a set of representatives for the quotient $N_G(H_0) \cap K_0 / (N_G(H_0) \cap K_0)^0$. In light of item 4. and 5. above, only those $x_s$'s contained in $\Gamma_L$ are interesting to us. By rearranging the order, we find $1 \leq c_0 \leq c$ such that for each $1 \leq s \leq c_0$, there exists $l_{x_s} \in L$, $\gamma_{x_s} \in \Gamma$ such that

$$x_s = l_{x_s} \gamma_{x_s}.$$

Now we start the argument. Item 4. above implies that

$$\lim \tilde{Y}'_i = \lim A'_i M_0 \Gamma / \Gamma \subset \bigcup_{s=1, \ldots, c_0} H_0 (N_G(H_0) \cap K_0)^0 l_{x_s} \Gamma / \Gamma$$

$$= \bigcup_{s=1, \ldots, c_0} H_0 K_Z l_{x_s} \Gamma / \Gamma$$

because $H_0 K_Z = H_0 (Z_G(H_0) \cap K_0)^0 = H_0 (N_G(H_0) \cap K_0)^0$.

As $(a'_i)$ converges to $a_0$ and they correspond to maximal $\mathbb{R}$-split subtori of $Z_G(M_Y)$, there exists $\varepsilon'_i \to id$ in $L$ such that

$$\varepsilon'_i A'_i \varepsilon'^{-1}_i = A_0$$

where we are using the fact that an orbit of an algebraic group is open in its closure, see \cite[2.3.3]{spr98}. So we have

$$\lim A_0 \varepsilon'_i M_0 \Gamma / \Gamma \subset \bigcup_{s=1, \ldots, c_0} H_0 K_Z l_{x_s} \Gamma / \Gamma.$$
Now this is a convergence inside $L \Gamma / \Gamma$, which we identify with $L / \Gamma_L$. Thus
\[
\lim [A_0 \varepsilon_i^2 M_0]_{\Gamma_L} \subset \bigcup_{s=1,\ldots,c_0} [H_0 K Z l_{x_s}]_{\Gamma_L} \quad \text{inside } L / \Gamma_L.
\]
By applying $\pi'$ we have
\[
\lim [\pi(A_0 \varepsilon_i^2)]_{\pi(\Gamma_L)} \subset \bigcup_{s=1,\ldots,c_0} [\pi(A_0 K Z l_{x_s})]_{\pi(\Gamma_L)} \quad \text{inside } \pi(L) / \pi(\Gamma_L).
\]
By further projecting to $\pi(K_Z) / \pi(L) / \pi(\Gamma_L)$, we have
\[
\lim \pi(A_0) \pi(K_Z) [\pi(\varepsilon_i^2)]_{\pi(\Gamma_L)} \subset \bigcup_{s=1,\ldots,c_0} \pi(A_0) \pi(K_Z) [\pi(l_{x_s})]_{\pi(\Gamma_L)}.
\]
As the right hand side is a union of closed orbits of $\pi(A_0)$, we may write it as a disjoint union by identifying certain indices,
\[
\lim \pi(A_0) \pi(K_Z) [\pi(\varepsilon_i^2)]_{\pi(\Gamma_L)} \subset \bigsqcup \pi(A_0) \pi(K_Z) [\pi(l_{x_s})]_{\pi(\Gamma_L)}.
\]
As each of $\pi(A_0) \pi(K_Z) [\pi(l_{x_s})]_{\pi(\Gamma_L)}$ is a compact set, there exists bounded neighborhood $N_s$ such that the closures of $N_s$ as $s$ varies are disjoint. But for all $i$, $\pi(A_0) \pi(K_Z) [\pi(\varepsilon_i^2)]_{\pi(\Gamma_L)}$ are connected, so they are contained in exactly one $N_i$ for $i$ large enough and therefore
\[
\lim \pi(A_0) \pi(K_Z) [\pi(\varepsilon_i^2)]_{\pi(\Gamma_L)} \subset \pi(A_0) \pi(K_Z) [id]_{\pi(\Gamma_L)}.
\]
Now take a small neighborhood $N$ of
\[
\pi(A_0) \pi(K_Z) [id]_{\pi(\Gamma_L \cap A_0)} \subset N \subset \pi(K_Z) / \pi(L) / \pi(\Gamma_L \cap A_0).
\]
Let $p : \pi(K_Z) / \pi(L) / \pi(\Gamma_L) \to \pi(K_Z) / \pi(L) / \pi(\Gamma_L \cap A_0)$ denote the natural projection. Choose $N$ small enough such that $p$ restricted to $N$ is a homeomorphism onto its image. Then for $i$ large enough,
\[
\pi(A_0) \pi(K_Z) [\pi(\varepsilon_i^2)]_{\pi(\Gamma_L)} \subset p(N).
\]
Hence for $i$ large enough,
\[
\pi(A_0) \pi(K_Z) [\pi(\varepsilon_i^2)]_{\pi(\Gamma_L \cap A_Y)} \subset N.
\]
This shows, in particular, that $\pi(A_0 \varepsilon_i^2)$ is bounded in $\pi(L) / \pi(A_0)$. Consequently,
\[
\pi(\varepsilon_i^{2-1} A_0 \varepsilon_i^2) = \pi(A_0), \quad H_1 = H_0, \quad Y_1 = K_0 \setminus K_0 H_0 \Gamma / \Gamma.
\]
But by item 2 of Lemma 5.3, $H_0$ only has one totally geodesic orbit on $K_0 \setminus G$. So $K_0 \setminus K_0 H_0 = K_0 \setminus K_0 H_0$ and the proof completes.

5.5. **Proof of Theorem 1.5.** The proof follows the same line as in [Oh04].

By Lemma 5.3 and use the notation there, $Y = K_0 \setminus K_0 H_0 g_Y \Gamma / \Gamma$ for some $i \in \{1,\ldots,s\}$ and $g_Y \in G$. By Theorem 1.2, we may assume $g_Y$ belongs to a fixed compact set $C \subset G$. Depending on $C$, we find a number $C_{12} > 0$ such that
\[
\Vol(h^{-1} H_1 g \Gamma / \Gamma) < C_{12} \cdot \Vol(H_1 g \Gamma / \Gamma)
\]
for all $i \in \{1,\ldots,s\}$ and $g \in G$ such that $H_1 g \Gamma / \Gamma$ has finite volume. Let $C_{11}$ be as in Lemma 5.4. Thus
\[
\Vol(g_Y^{-1} H_1 g_Y \Gamma / \Gamma) \leq C_{12} \Vol(H_1 g_Y \Gamma / \Gamma) \leq C_{11} C_{12} \Vol(Y).
\]
Thus for a fixed $T > 0$, $\Vol(g_Y^{-1}H_i g_Y \Gamma / \Gamma)$ as $Y$ varies in $\mathcal{G}_{\leq T}^{N, \text{rigid}}$ is bounded. By [DM93, Theorem 5.1], the set

$$\left\{ g_Y^{-1}H_i g_Y \cap \Gamma \mid Y \in \mathcal{G}_{\leq T}^{N, \text{rigid}} \right\}$$

is finite. To recover $Y$ from $g_Y^{-1}H_i g_Y \cap \Gamma$, it suffices to note that $g_Y^{-1}H_i g_Y \cap \Gamma$ is Zariski dense in $g_Y^{-1}H_i g_Y$ and that $K_0 \setminus K_0 H_i g_Y$ is the unique orbit of $g_Y^{-1}H_i g_Y$ on $K_0 \setminus G$ that is totally geodesic. So we are done.

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