Phase Transitions in Asymptotically Singular Anderson Hamiltonian and Parabolic Model

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Abstract Let $\xi$ be a Gaussian white noise on $\mathbb{R}^d$ ($d = 1, 2, 3$). Let $(\xi_\epsilon)_{\epsilon > 0}$ be continuous Gaussian processes such that $\xi_\epsilon \to \xi$ as $\epsilon \to 0$, defined by convolving $\xi$ against a mollifier. We consider the asymptotics of the parabolic Anderson model (PAM) with noise $\xi_\epsilon(t)$ for large time $t \gg 1$, and the Dirichlet eigenvalues of the Anderson Hamiltonian (AH) with potential $\xi_\epsilon(t)$ on large boxes $(-t,t)^d$, where the parameter $\epsilon(t)$ vanishes as $t \to \infty$. We prove that the asymptotics in question exhibit a phase transition in the rate at which $\epsilon(t)$ vanishes, which distinguishes between the behavior observed in the AH/PAM with continuous Gaussian noise and white noise. By comparing our main theorems with previous results on the AH/PAM with white noise, our results show that some asymptotics of the latter can be accessed with solely elementary methods, and we obtain quantitative estimates on the difference between the AH/PAM with white noise and its continuous-noise approximations as $t \to \infty$.

Keywords Parabolic Anderson model · Anderson Hamiltonian · white noise · asymptotically singular noise · phase transition

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1 Introduction

1.1 Continuous PAM and AH

The continuous parabolic Anderson model (PAM) is defined as the solution $u(t,x)$ of a random heat equation of the form

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2} \Delta u(t,x) + \xi(x)u(t,x) \\ u(0,x) = u_0(x) \end{cases}, \quad t \geq 0 \text{ and } x \in \mathbb{R}^d, \quad (1.1)$$

where $\xi$ is a random potential called the noise. A closely associated object is the Anderson Hamiltonian (AH), defined as the operator

$$Af(x) := \frac{1}{2} \Delta f(x) + \xi(x)f(x) \quad (1.2)$$

1 In terms of physical terminology, one should instead define the Anderson Hamiltonian with random potential $\xi$ as $-\frac{1}{2} \Delta + \xi$. That said, in this paper we use (1.2) for convenience.
acting on some domain of functions $f : \mathbb{R}^d \to \mathbb{R}$ on which $A$ is self-adjoint.

Starting from the pioneering work of Gärtner and Molchanov [21], the PAM literature has mostly been concerned with understanding the occurrence of intermittency in (1.1) for large times (e.g., [31, Section 1.4]). Given the connection between the AH and PAM via semigroup theory, a closely related problem is that of localization in the AH’s spectrum (e.g., [31, Sections 2.2.1–2.2.4]). We refer to [8, 31] and references therein for surveys of the field. As it turns out, a few features of the AH/PAM have been the subject of the majority of investigations to date, arguably due to the fact that they are amenable to computation and encode useful information about the geometry of intermittency: Let us denote the Dirichlet eigenvalues of $A$ on a bounded open set $\Omega \subset \mathbb{R}^d$ as

$$\Lambda_1(A, \Omega) \geq \Lambda_2(A, \Omega) \geq \Lambda_3(A, \Omega) \geq \cdots . \quad (1.3)$$

Let us define the total mass of the PAM as

$$U(t) := \mathbb{E}_0^0 \left[ \exp \left( \int_0^t \xi \left( B(s) \right) \, ds \right) \right], \quad t \geq 0, \quad (1.4)$$

where $B$ is a standard Brownian motion on $\mathbb{R}^d$ and $\mathbb{E}_0^0$ denotes the expectation with respect to $B$ with initial value of zero (i.e., $B(0) = 0$), conditional on $\xi$. (Equivalently, we can write the total mass $U(t) = u(t, 0)$ as the solution of the PAM at $x = 0$ with flat initial condition $u(0, x) = 1$.)

**Problem 1.1 (Annealed Total Mass)** Understand the $t \to \infty$ behavior of the moments of the total mass $\mathbb{E}[U(t)^p]$ ($p \geq 0$).

**Problem 1.2 (Quenched Total Mass)** Understand the almost-sure $t \to \infty$ behavior of the total mass $U(t)$.

**Problem 1.3 (Eigenvalues)** Understand the almost-sure $t \to \infty$ behavior of the eigenvalues $\Lambda_k(A, Q_t)$ for fixed $k \geq 1$ on large boxes $Q_t := (-t, t)^d$.

We refer to [9, 19, 20] for a derivation of the first- and second-order asymptotics of the above in the AH/PAM with certain continuous noises (including continuous Gaussian processes) and an explanation of how these computations shed light on the geometry of intermittency. See [21, 22] for similar results in the discrete setting.

### 1.2 AH/PAM with White Noise

In this paper, we are interested in understanding intermittency in the PAM with white noise (WN). WN is formally defined as a centered Gaussian process on $\mathbb{R}^d$ with delta Dirac covariance

$$\mathbb{E} \left[ \xi(x) \xi(y) \right] = \delta_0(x - y), \quad x, y \in \mathbb{R}^d. \quad (1.5)$$

Although WN is among the most natural examples of noises to consider on $\mathbb{R}^d$ (e.g., [31, Section 1.5.2]), the rigorous treatment of the AH/PAM in this setting is made difficult by the fact that WN is a Schwartz distribution. Most notably, the “pointwise products” $\xi(x)u(t, x)$ and $\xi(x)f(x)$ in (1.1) and (1.2) are ill posed, making the very definition of the AH/PAM nontrivial.

To overcome this technical issue, an intuitive approach is to proceed as follows: Using classical theory, define a family of approximate AHs and PAMs $(A_\varepsilon)_{\varepsilon > 0}$ and $(u_\varepsilon)_{\varepsilon > 0}$ with
smoothed noises \((\xi_\varepsilon)_{\varepsilon>0}\) that approach \(\xi\) as \(\varepsilon \to 0\). Then, the hope is that we can obtain universal (i.e., independent of the particular way in which we define \(\xi_\varepsilon\)) limits

\[
A := \lim_{\varepsilon \to 0} A_\varepsilon \quad \text{and} \quad u(t, x) := \lim_{\varepsilon \to 0} u_\varepsilon(t, x),
\]

which we take as the definitions of the AH/PAM with WN. In one dimension \((d = 1)\), this procedure works and there is a straightforward sense in which the limits (1.6) can be interpreted as the AH/PAM using quadratic forms and stochastic calculus; see [5, 12, 18, 23, 24, 25, 30, 33, 41]. In contrast, in higher dimensions \((d \geq 2)\) the limits (1.6) blow up. While the AH/PAM with WN are not expected to make sense for \(d \geq 4\) (e.g., [27, 33]), for \(d = 2, 3\) nontrivial limits can be obtained if one considers renormalizations of \(u_\varepsilon\) and \(A_\varepsilon\). The limits thus obtained can be interpreted in a rigorous sense as the AH/PAM with WN using sophisticated solution theories for SPDEs with irregular noise, such as regularity structures or paracontrolled calculus; e.g., [1, 26, 27, 29, 33]. (Though, in some cases, simpler constructions can be used, e.g., [28].)

Due to these technical difficulties, the understanding of intermittency in the AH/PAM with WN is much less advanced than that with continuous Gaussian noise (c.f., [9, 19, 20]). More specifically, for \(d = 1\), first-order asymptotics for Problems 1.1–1.3 have been obtained in [12, 30] (see also [6, 7, 17, 37]). For \(d = 2, 3\), it is understood that the total mass moments blow up in finite time [1, 13, 33] (and thus Problem 1.1 is intractable), first-order asymptotics for Problems 1.2 and 1.3 when \(d = 2\) were proved in [15, 32] using paracontrolled calculus, and Problems 1.2 and 1.3 for \(d = 3\) are open.

1.3 Main Results

We now proceed to an exposition of our main results (Theorems 1.7, 1.11, and 1.17 below). For the remainder of this paper, unless otherwise mentioned, we assume that \(d \in \{1, 2, 3\}\).

1.3.1 Asymptotically Singular Noise

Throughout the paper, we consider the following type of smoothed noise:

**Definition 1.4** \(\xi_1\) is a continuous, centered, and stationary Gaussian process on \(\mathbb{R}^d\) with covariance

\[
E[\xi_1(x)\xi_1(y)] = R(x - y), \quad x, y \in \mathbb{R}^d.
\]

(1.7)

We assume that we can write \(R = \bar{R} \ast \bar{R}\), where the function \(\bar{R} : \mathbb{R}^d \to \mathbb{R}\) satisfies the following conditions:

1. \(\bar{R}\) is a probability density function,
2. \(\bar{R}\) is an even function,
3. \(\bar{R}\) is compactly supported, and
4. there exists some \(h > 0\) and \(C > 0\) such that

\[
|\bar{R}(x) - \bar{R}(y)| \leq C|x - y|^h \quad \text{for every } x, y \in \mathbb{R}^d.
\]

(1.8)

Then, for every \(\varepsilon \in (0, 1]\), we define the approximate AH and PAM total mass as

\[
A_\varepsilon := \frac{1}{2} \Delta + \xi_\varepsilon \quad \text{and} \quad U_\varepsilon(t) := \mathbb{E}^0 \left[ \exp \left( \int_0^t \xi_\varepsilon(B(s)) \, ds \right) \right],
\]

(1.9)

where \(\xi_\varepsilon(x) := \varepsilon^{-d/2} \xi_1(x/\varepsilon)\). We denote the Dirichlet eigenvalues of \(A_\varepsilon\) on some bounded open set \(\Omega \subset \mathbb{R}^d\) as \(\Lambda_1(A_\varepsilon, \Omega) \geq \Lambda_2(A_\varepsilon, \Omega) \geq \cdots\).
Remark 1.5 If we denote
\[ R_\varepsilon(x) := e^{-dR(x/\varepsilon)} \quad \text{and} \quad R_\varepsilon(x) := e^{-dR(x/\varepsilon)}, \]
then \( R_\varepsilon = \tilde{R}_\varepsilon * \tilde{R}_\varepsilon \), and \( \tilde{R}_\varepsilon \) has covariance \( R_\varepsilon \). Hence, \( \tilde{\xi}_\varepsilon \overset{\text{distr}}{=} \tilde{\xi} * \tilde{R}_\varepsilon \), where \( \tilde{\xi} \) is a WN. Though it is more common to use \( \tilde{\xi} * \tilde{R}_\varepsilon \) as the definition of the smoothed noise, in this paper we use the coupling \( \tilde{\xi}_\varepsilon(x) = e^{-d/2 \xi_t(x/\varepsilon)} \) for convenience, as doing so does not affect our main results (see Remark 1.16).

Our aim in this paper is to propose to study the large-\( t \) asymptotics of the AH/PAM with WN by considering asymptotically singular noise. That is, we study the behavior of \( \Lambda_k(A_{\varepsilon(t)} Q_t) \) and \( U_{\varepsilon(t)}(t) \) as \( t \to \infty \), where the approximation parameter \( \varepsilon(t) \) goes to zero as \( t \to \infty \). The hope is that

1. if \( \varepsilon(t) \to 0 \) at a fast enough rate, then the asymptotics of \( \Lambda_k(A_{\varepsilon(t)} Q_t) \) and \( U_{\varepsilon(t)}(t) \) carry insight into those of the AH/PAM with WN, and
2. since we are only ever considering objects with continuous noise, the asymptotics of \( \Lambda_k(A_{\varepsilon(t)} Q_t) \) and \( U_{\varepsilon(t)}(t) \) can be accessed with elementary methods (at least comparatively to regularity structures/paracontrolled calculus).

1.3.2 Quenched Phase Transitions

In this paper, we take the first steps in actualizing the above-described program: Using only elementary methods (i.e., standard operator/semigroup theory, suprema of continuous Gaussian processes, etc.), we prove that the first-order asymptotics in Problems 1.2 and 1.3 exhibit a “phase transition” in the rate \( \varepsilon(t) \) at which \( \tilde{\xi}_{\varepsilon(t)} \) becomes singular as \( t \to \infty \). To this effect, our first main result states that if \( \varepsilon(t) \) is not too small, then the first-order quenched total mass and eigenvalue asymptotics behave as though \( \varepsilon(t) \) is constant. We call this regime of \( \varepsilon(t) \) the regular phase.

Definition 1.6 (Regular Phase) The function \( \varepsilon(t) \in (0, 1] \) \( (t \geq 0) \) is in the regular phase if \( \varepsilon(t) \gg (\log t)^{-1/(4-d)} \) as \( t \to \infty \).

Theorem 1.7 (Regular Phase) Let \( \varepsilon(t) \) be in the regular phase.

\[ \lim_{t \to \infty} \frac{\Lambda_k(A_{\varepsilon(t)} Q_t)}{\varepsilon(t)^{-d/2} \sqrt{\log t}} = \frac{\sqrt{2dR(0)}}{t} \quad \text{in probability} \] \hspace{1cm} (1.10)

for every \( k \in \mathbb{N} \), and

\[ \lim_{t \to \infty} \frac{\log U_{\varepsilon(t)}(t)}{t \varepsilon(t)^{-d/2} \sqrt{\log t}} = \frac{\sqrt{2dR(0)}}{t} \quad \text{in probability}. \] \hspace{1cm} (1.11)

Remark 1.8 If we take \( \varepsilon(t) = 1 \) in Theorem 1.7, then we recover the first-order asymptotics for the PAM with continuous Gaussian noise \( \tilde{\xi}_1 \) in [9, Theorem 5.1].

Our second main result states that if \( \varepsilon(t) \to 0 \) at a fast enough rate, then the quenched total mass and eigenvalue asymptotics are universal (i.e., independent of the choice of \( R \)), and are given by a variational constant. Moreover, this result identifies \( (\log t)^{-1/(4-d)} \) as the critical rate of decay at which this transition occurs. We call this second regime the singular phase.

Definition 1.9 (Singular Phase) \( \varepsilon(t) \in (0, 1] \) is in the singular phase if one of the following holds:

1. \( d = 1 \) and \( \varepsilon(t) \ll (\log t)^{-1/(4-d)} \) as \( t \to \infty \); or
2. $d = 2, 3$, the Hölder exponent $h$ in (1.8) satisfies $h > d/4$, and
\[
(\log t)^{-1/(4-d)-\epsilon_d} \ll \epsilon(t) \ll (\log t)^{-1/(4-d)}
\]
as $t \to \infty$, where we define the constant
\[
\epsilon_d := \frac{h}{d(d + h)}.
\]  

**Definition 1.10 (Variational Constant)** Let $\Theta_d \in (0, \infty)$ be the smallest possible constant in the Gagliardo-Nirenberg-Sobolev (GNS) inequality
\[
\|\varphi\|_4^4 \leq \Theta_d \left( \int_{\mathbb{R}^d} |\nabla \varphi(x)|^2 \, dx \right)^{d/2} \|\varphi\|_2^{d-4} \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d)
\]
(since $2(d-2) \leq d$ holds for $d = 1, 2, 3$, we know that $\Theta_d < \infty$; e.g., [10, (C.1)]). Then, we define the associated Lyapunov exponent
\[
\mathcal{L}_d := \frac{4-d}{4} \left( \frac{d}{2} \right)^{(d/(4-d))} (2d\Theta_d)^{2/(4-d)}.
\]  

**Theorem 1.11 (Singular Phase)** Let $\epsilon(t)$ be in the singular phase.
\[
\lim_{t \to \infty} \frac{\Lambda_k(A_{\epsilon(t)}; \mathcal{Q}_t)}{(\log t)^{2/(4-d)}} = \mathcal{L}_d \quad \text{in probability}
\]
for every $k \in \mathbb{N}$, and
\[
\lim_{t \to \infty} \frac{\log U_{\epsilon(t)}(t)}{(\log t)^{2/(4-d)}} = \mathcal{L}_d \quad \text{in probability}.
\]

We now end the statement of Theorems 1.7 and 1.11 with some remarks:

**Remark 1.12** When $d = 1, 2$, the asymptotics in Theorem 1.11 match that of the AH/PAM with WN proved by Chen, Chouk, König, Perkowski, and van Zuijlen in [11, 15, 32]. We refer to Section 1.4.3 for more details on the applications of our results to the AH/PAM with WN.

**Remark 1.13** When $d \geq 4$, we can prove that no phase transition occurs. More specifically, the asymptotics remain in the regular phase no matter how quickly $\epsilon(t) \to 0$ (see Remark 2.4 for a heuristic and Theorem 4.7 and Sections 4.2.2 and 4.2.4 for specifics). In particular, this lack of phase transition provides a different point of view with which to explain that the AH/PAM with WN do not make sense in $d \geq 4$.

**Remark 1.14** The lower bound of $(\log t)^{-1/(4-d)-\epsilon_d} \ll \epsilon(t)$ in the singular phase for $d = 2, 3$ is due to the fact that a technical argument fails when $\epsilon(t)$ is too small (see (4.25) in Proposition 4.11). While we make no claim that (1.12) is optimal for Theorem 1.11 to hold, some kind of lower bound is to be expected in $d = 2, 3$, since in those cases the AH/PAM must be renormalized to obtain nontrivial $\epsilon \to 0$ limits. We point to Section 1.4.3 below for more details on this point.

**Remark 1.15** It would be interesting to see if asymptotics that interpolate between Theorems 1.7 and 1.11 could be obtained in a “critical phase” of the form
\[
\epsilon(t) = C(\log t)^{-1/(4-d)}(1 + o(1)), \quad t \to \infty
\]
for some $C > 0$. As we were unable to obtain matching upper and lower bounds in this regime, we leave it as an open question.
Remark 1.16 When comparing Theorems 1.7 and 1.11 with the corresponding results in [9,11,15,32], it can be noted that the former all prove almost sure convergence, whereas in this paper we only prove convergence in probability. The argument typically used to prove almost sure convergence in this context relies on the monotonicity of \( A_k(H, Q_t) \) in \( t \) for every fixed operator \( H \). Given that, in this paper, the operator \( A_\varepsilon(t) \) changes with \( t \), this argument can no longer be used (except in the special case of the subcritical phase where \( \varepsilon(t) \) is constant).

1.3.3 Annealed Total Mass

As mentioned earlier in this introduction, the moments of the PAM total mass with WN are not finite for all \( t > 0 \) in \( d = 2, 3 \). That said, it is nevertheless natural to ask if the moments \( \mathbb{E}[U_\varepsilon(t)^p] \) carry meaningful information about intermittency in the PAM with WN when \( \varepsilon(t) \) is very small. The following result suggests that this may not the case when \( d = 2, 3 \):

**Theorem 1.17** Let \( \varepsilon(t) \in (0, 1] \) for \( t \geq 0 \). On the one hand, if

1. \( d = 1 \) and \( \varepsilon(t) \gg t^{-1} \), or
2. \( d \geq 2 \),

then for every \( p \in \mathbb{N} \),

\[
\lim_{t \to \infty} \frac{\log \mathbb{E}[U_\varepsilon(t)^p]}{\varepsilon(t)^{-d}t^{2}} = \frac{p^2 R(0)}{2}.
\]

On the other hand, if \( d = 1 \) and \( \varepsilon(t) \ll t^{-1} \) as \( t \to \infty \), then there exists some constants \( 0 < \theta_1 \leq \theta_2 < \infty \) independent of \( R \) such that for every \( p \in \mathbb{N} \),

\[
\theta_1 p^3 \leq \liminf_{t \to \infty} \frac{\log \mathbb{E}[U_\varepsilon(t)^p]}{t^3} \leq \limsup_{t \to \infty} \frac{\log \mathbb{E}[U_\varepsilon(t)^p]}{t^3} \leq \theta_2 p^3.
\]

**Remark 1.18** When \( \varepsilon(t) = 1 \), Theorem 1.17 reduces to the moment asymptotics [9, Theorem 4.1] for the PAM with continuous Gaussian noise \( \xi_1 \). In particular, when \( d \geq 2 \), Theorem 1.17 shows that no matter how small we take \( \varepsilon(t) \), the moment asymptotics never transition to a universal limit independent of \( R \). In contrast, if \( d = 1 \) and \( \varepsilon(t) \ll t^{-1} \), then we recover in (1.18) the annealed asymptotics for the one-dimensional PAM with WN proved in [30, (6.8)].

1.4 Other Results and Applications

We now discuss how our paper relates to the wider literature, taking this opportunity to showcase an application of our results to the study of the AH/PAM with WN in Corollary 1.20.

1.4.1 Other Noise Scalings

We note that the present paper is not the first to study spectral asymptotics of AH- and PAM-type objects whose noise depends on the parameter being sent to infinity (or zero), including the occurrence of a phase transition.

One the one hand, in [38,39] Merkl and Wüthrich consider the largest eigenvalue of the operator \( \frac{1}{\sigma^2} \Delta - V_t \) on the box \( Q_t \), where \( V_t \) is of the form

\[
V_t(x) := \frac{\beta}{\varphi(t)^2} \sum_i W(x - x_i), \quad x \in \mathbb{R}^d
\]
for some scale function $\varphi$, shape function $W$, $\beta > 0$, and Poisson point process $(\xi_i)_i$. In particular, they identify the presence of a phase transition in the asymptotics when $d \geq 4$ and $\varphi(t) = (\log t)^{1/d}$ in terms of the parameter $\beta$ (i.e., there exists a dimension-dependent critical $\beta_c > 0$ such that the asymptotics differ if $\beta < \beta_c$ or $\beta > \beta_c$). Then, they study the behavior of the so-called Brownian motion in the scaled Poissonian potential $V_t$ (the analog of the PAM in their setting). Although the broad outline of the strategies used in those papers (especially [38]) is similar to the present paper, the details are very different due to the nature of the random potentials and their scaling in $t$ (i.e., a Poissonian potential with a multiplicative factor versus a Gaussian potential with both a multiplicative factor and a space scaling).

On the other hand, in [3,4], Biskup, Fukushima, and König study the $\epsilon \to 0$ asymptotics of the top eigenvalues of discrete operators of the form $\epsilon^{-2} \Delta - \xi^{(\epsilon)}$ on lattices that approximate some bounded domain $D \subset \mathbb{R}^d$ (i.e., the space between lattice points is of order $\epsilon$). Here, $\Delta$ is the lattice Laplacian, and $\xi^{(\epsilon)}$ is an independent random field with a properly scaled expectation and variance. More specifically, they establish convergence in probability of the eigenvalues of $\epsilon^{-2} \Delta - \xi^{(\epsilon)}$ to that of a deterministic “homogenized” continuum Schrödinger operator on $D$ whose potential is determined by $E[\xi^{(\epsilon)}]$. They also prove Gaussian fluctuations of the eigenvalues about their expectations, where the limiting covariance depends on $\text{Var}[\xi^{(\epsilon)}]$. While the setting in [3,4] differs from the present paper in various significant ways (in particular, both the domain of the operators and the variance $\text{Var}[\xi^{(\epsilon)}]$ are bounded as $\epsilon \to 0$), it nevertheless raises the interesting question of whether analogs of Theorems 1.7 and 1.11 can be proved if the approximations $A_{\epsilon(t)}$ and $U_{\epsilon(t)}$ are constructed using properly scaled lattice WN instead of a continuous smoothing of the WN. If that is possible, then some ideas from [3,4] would be expected to be fruitful; we leave this direction open.

1.4.2 Finer Asymptotics

As shown in (1.10) and (1.15), the eigenvalues $A_k(A_{\epsilon(t)}, Q_t)$ are all asymptotically equivalent in the first order (as $t \to \infty$). We expect that, with a finer scaling, one could identify the fluctuations of the eigenvalues and thus uncover a nontrivial point process limit. Such a result would complement previous investigations in this direction, such as the Poisson point process limits uncovered by Dumaz and Labbé in [17] for the one-dimensional WN (see also [2] and references therein for a survey of such results in the discrete setting).

In a similar vein, it is natural to wonder if a phase transition also occurs in the second order asymptotics of the PAM. If that is the case, then we expect that, in similar fashion to [19,20], understanding the transition of the smaller order asymptotics would provide new information on intermittency in the PAM with WN; more specifically, in clarifying the connection (if any) between the variational constant $\Sigma_d$ in (1.14) and the local geometry of intermittent peaks.

We leave both of these questions open for future investigations.

1.4.3 Renormalizations and Rate of Convergence

As mentioned in Remark 1.14, the approximate AH and PAM need to be renormalized in order to give a nontrivial $\epsilon \to 0$ limit in $d = 2, 3$. More specifically, as shown in [28,29,33], for every fixed $t > 0$, one has

$$A_k(A, Q_t) := \lim_{\epsilon \to 0} \left( A_k(A_{\epsilon}, Q_{t\epsilon}) - c_\epsilon \right) \quad \text{and} \quad U(t) := \lim_{\epsilon \to 0} U_{\epsilon(t)} e^{-3c_\epsilon},$$

(1.19)

where the renormalization constant $c_\epsilon$ blows up as $\epsilon \to 0$ on the order of $|\log \epsilon|$ when $d = 2$ and $\epsilon^{-1}$ when $d = 3$ (up to constants and lower order terms).
Given that $\epsilon_d < 1/d$ by (1.12), when $d = 2, 3$ in the singular phase we always assume that $\epsilon(t) \gg (\log t)^{-1/(4-d)-1/d}$, which implies that $\epsilon(t) \ll (\log t)^{2/(4-d)}$. Thus, the fact that we do not include renormalization constants in Theorem 1.11 does not contradict (1.19). That said, if $\epsilon(t)$ vanishes so quickly that $\epsilon(t) \gg (\log t)^{2/(4-d)}$, then it is not clear that we can expect the asymptotics of

$$\frac{\Lambda_k(A_{\epsilon(t)}, Q_t)}{(\log t)^{2/(4-d)}}, \quad \frac{\log U_{\epsilon(t)}(t)}{t(\log t)^{2/(4-d)}}$$

to be meaningful without the renormalizations $\Lambda_k(A_{\epsilon(t)}, Q_t) - c_{\epsilon(t)}$ and $U_{\epsilon(t)}(t)e^{-tc_{\epsilon(t)}}$.

In light of this, one of the main insights of this paper is that the asymptotics of the AH/PAM with WN can be accessed even without requiring the use of renormalizations, so long as $\epsilon(t)$ is not too big or small. In fact, a comparison of Theorem 1.19 with known results for the AH/PAM with WN provides quantitative upper bounds on the difference between the latter and their smooth approximations for large $t$: Among the main results of [12] (for $d = 1$) and [15, 32] (for $d = 2$) are the following:

**Theorem 1.19** Let $A$ and $U(t)$ be the AH and PAM total mass with WN in $d = 1, 2$.

$$\lim_{t \to \infty} \frac{\Lambda_k(A, Q_t)}{(\log t)^{2/(4-d)}} = \Omega_d \quad \text{in probability}$$

for every $k \in \mathbb{N}$, and

$$\lim_{t \to \infty} \frac{\log U(t)}{t(\log t)^{2/(4-d)}} = \Omega_d \quad \text{in probability}.$$ 

By combining the above with Theorem 1.19, we obtain the following:

**Corollary 1.20** Let $d = 1, 2$ and $\epsilon(t)$ be in the singular phase.

$$\lim_{t \to 0} \frac{|\Lambda_k(A, Q_t) - \Lambda_k(A_{\epsilon(t)}, Q_t)|}{(\log t)^{2/(4-d)}} = 0 \quad \text{in probability} \quad (1.20)$$

for every $k \in \mathbb{N}$, and

$$\lim_{t \to 0} \frac{|\log U(t) - \log U_{\epsilon(t)}(t)|}{t(\log t)^{2/(4-d)}} = 0 \quad \text{in probability}. \quad (1.21)$$

In particular, if the estimates (1.20) and (1.21) can be independently established, then this would provide a new elementary proof of Theorem 1.19 in $d = 2$. It would also be interesting to see if similar results can be proved in $d = 3$; although to the best of our knowledge, an analog of Theorem 1.19 is not yet proved in this case. We leave such questions open for future investigations.

1.5 Organization

The remainder of this paper is organized as follows. In Section 2, we discuss the strategy of proof for Theorems 1.7 and 1.11, including an intuitive explanation of why the phase transition therein occurs at the critical rate $(\log t)^{-1/(4-d)}$. In Section 3, we introduce the notation used in our paper and state various classical results that lie at the heart of our proof. In Section 4 we prove the eigenvalue asymptotics (1.10) and (1.15), in Section 5 we prove the total mass asymptotics (1.11) and (1.16), and in Section 6 we prove Theorem 1.17.
2 Proof Strategy for Theorems 1.7 and 1.11

2.1 PAM Total Mass Reduces to Leading Eigenvalue Asymptotics

The main ingredient of the proofs of (1.11) and (1.16) consists of the heuristic
\[ U_{E(t)}(t) \approx e^{\Lambda_1(A_{\varepsilon(t)}, Q_t)} \quad \text{as } t \to \infty, \]  
(2.1)
which completely reduces the quenched total mass asymptotics to the exponential of the leading eigenvalue. A rigorous version of this heuristic can be achieved by using semigroup theory/the Feynman-Kac formula, as shown in Sections 3.4 and 5.

**Remark 2.1** As per (2.1), the PAM asymptotics are only determined by the behavior of the leading eigenvalue \( \Lambda_1(A_{\varepsilon(t)}, Q_t) \). That said, we nevertheless include a statement for \( \Lambda_k(A_{\varepsilon(t)}, Q_t) \), \( k \geq 2 \), in Theorems 1.7 and 1.11 for the following reasons:

1. The asymptotics of \( \Lambda_k(A_{\varepsilon(t)}, Q_t) \) are of independent interest from the point of view of the spectral theory of random Schrödinger operators; and
2. as we will show in Section 4.1, once the asymptotics of \( \Lambda_1(A_{\varepsilon(t)}, Q_t) \) are established, those of \( \Lambda_k(A_{\varepsilon(t)}, Q_t) \) more or less immediately follow using a simple argument; hence the additional statement does not require a more involved proof.

This type of argument for computing the asymptotics of the total mass dates back to at least the work of Gärtner and Molchanov [22, Sections 2.4 and 2.5], and was used in several more papers since then (e.g., [11, 12, 19, 20]). The particular implementation of the argument used in this paper most closely resembles that of [12] (more specifically, see [12, Sections 3 and 4]). From the technical point of view, the argument deployed in this paper is simultaneously simpler and more involved than that of [12]: On the one hand, the fact that we do not deal with noises that are Schwartz distributions allows to sidestep a number of technical hurdles encountered in [12], such as the approximation arguments [12, (2.22) and Sections 3, 4, and A.1]. On the other hand, the need to consider a different noise (namely, \( \xi_{\varepsilon(t)} \)) for every value of \( t \) and to distinguish between two regimes of \( \varepsilon(t) \) increases the complexity of some arguments.

2.2 Eigenvalue Asymptotics

We now discuss how the eigenvalue asymptotics (1.10) and (1.15) are obtained. In Section 4.1, we show that the eigenvalues \( \Lambda_k(A_{\varepsilon(t)}, Q_t) \) for \( k \geq 2 \) have the same asymptotics as \( \Lambda_1(A_{\varepsilon(t)}, Q_t) \); hence we only need to prove asymptotics for the leading eigenvalue. By the min-max principle, we can write
\[ \Lambda_1(A_{\varepsilon(t)}, Q_t) = \sup_{\phi \in C_0^\infty(Q_t), \|\phi\|_2 = 1} \left( \langle \xi_{\varepsilon(t)}, \phi^2 \rangle - \frac{1}{2} \hat{E}^e(\phi) \right), \]  
(2.2)
where \( C_0^\infty(Q_t) \) denotes the set of smooth and compactly supported functions on the box \( Q_t \), and we use \( \hat{E}^e \) as shorthand for the Dirichlet form induced by \( \Delta \) (i.e., (3.1)). In this expression, we note that there is a competition between two terms:

On the one hand, maximizing \( \langle \xi_{\varepsilon(t)}, \phi^2 \rangle \) provides an incentive for \( \phi \) to allocate all of its mass at the maximum of \( \xi_{\varepsilon(t)} \) on \( Q_t \). More specifically, by the \( L^1/L^\infty \) Hölder inequality, we have that
\[ \sup_{\phi \in C_0^\infty(Q_t), \|\phi\|_2 = 1} \langle \xi_{\varepsilon(t)}, \phi^2 \rangle \leq \sup_{x \in Q_t} |\xi_{\varepsilon(t)}(x)|, \]
where the supremum over \( \phi \) is achieved (at least formally) at any delta Dirac distribution \( \delta_{x_{\varepsilon(t)}} \) such that the point \( x_{\varepsilon(t)} \) achieves \( \xi_{\varepsilon(t)} \)'s supremum on \( Q_t \)'s closure. On the other hand,
the term $-\frac{1}{2} \mathcal{E}(\varphi)$ penalizes functions with very substantial variations, such as functions that are very close to a Dirac distribution. Thus, while we expect that the eigenfunction that achieves the supremum in (2.2) is localized near $\xi^\varepsilon$’s maximum on $Q_t$, its gradient cannot be too large.

Then, understanding the asymptotics of $A_1(A_{\varepsilon(t)}, Q_t)$ is a matter of identifying the contributions of both of these effects in the large-$t$ limit. In this paper, this is carried out using so-called localization bounds (see Section 3.3 for the details as well as references identifying previous works where this idea has already appeared). That is, we partition $Q_t$ into smaller sub-boxes $B_1(t), B_2(t), \ldots, B_n(t)$, hoping that one of the $B_i(t)$’s will contain the bulk of the mass of the leading eigenfunction (which itself is localized near $\xi^\varepsilon(t)$’s maximizer on $Q_t$); hence

$$A_1(A_{\varepsilon(t)}, Q_t) \approx \max_{1 \leq i \leq n} A_1(A_{\varepsilon(t)}, B_i(t)). \quad (2.3)$$

The task of understanding the eigenvalue asymptotics is now reduced to

1. identifying the size that the $B_i(t)$’s must have as $t \to \infty$ to capture the bulk of the mass of the leading eigenfunction (which relies on understanding the tradeoff between $\langle \xi^\varepsilon, \varphi^2 \rangle$ and $-\frac{1}{2} \mathcal{E}(\varphi)$ in (2.2)); and
2. analyzing the asymptotics of the maximum on the right-hand side of (2.3). As $\varepsilon(t) \to 0$, $\xi^\varepsilon(t)$ becomes uncorrelated over very small distances).

We now provide a heuristic based on a variety of previous results that serves as the main guide in carrying out the above, that explains the asymptotics obtained in Theorems 1.7 and 1.11, and that explains why a transition occurs at $\varepsilon(t) \sim (\log t)^{-1/(4-d)}$.

### 2.3 Eigenvalue Scaling Heuristics

#### 2.3.1 Case $\varepsilon(t) = 1$

The starting point of our heuristic is the second order asymptotics for the total mass found by Gärtner, König, and Molchanov in [20]. Combining Theorem 1.1 in that paper with the asymptotic equivalence in (2.1), we have the following result, which settles the special case where $\varepsilon(t) = 1$ in Theorem 1.7:

**Theorem 2.2** ([20]) Let us denote

$$L_t := \sqrt{2dR(0) \log t} \quad (2.4)$$

and

$$l_t := \frac{\text{Tr} \left[ \left( -R''(0) \right)^{1/2} \right]}{2} \left( \frac{2d}{R(0) \log t} \right)^{1/4}, \quad (2.5)$$

where $R''$ denotes the Hessian matrix of the covariance. As $t \to \infty$, one has

$$A_1(A_1, Q_t) = L_t - l_t + o(l_t). \quad (2.6)$$

On the one hand, the leading order term $L_t$ is solely determined by the tendency of the eigenfunction to localize near $\xi^{1}$’s maximum on $Q_t$. Indeed, the specific form of (2.4) is due to the classical extreme value theory of Gaussian processes:
Lemma 2.3 Suppose that $X$ is a centered stationary Gaussian process on $\mathbb{R}^d$ or $\mathbb{Z}^d$, assuming that $X$ has continuous sample paths if it is on $\mathbb{R}^d$. If the covariance $\text{E}[X(0)X(x)]$ of $X$ vanishes as $|x|_2 \to \infty$, then

$$
\lim_{t \to 0} \sup_{x \in Q_t} \frac{X(x)}{\sqrt{\log t}} = \sqrt{2d\text{E}[X(0)^2]} \quad \text{almost surely.} 
$$

(2.7)

Proof. We refer to [9, Section 2.1] and references therein for (2.7) in the continuous case. For the discrete case, we point to [40, Theorem 3.4] ([40] is only stated in $d = 1$, but its argument can adapted to $d \geq 2$ with only trivial modifications).

On the other hand, as explained in [20, Section 1.6], the second order term $-l_t$ comes from the contribution of $-\frac{1}{2} \xi'(\phi)$. More specifically, the leading eigenfunction will localize in a box of approximate size

$$
s_t := (2dR(0)\log t)^{-1/2},
$$

(2.8)

and the trace in (2.5) contains information about the local geometry of the leading eigenfunction near its maximum.

In summary, if $\epsilon(t) = 1$, then $\langle \xi_1, \phi^2 \rangle$ dominates $-\frac{1}{2} \xi'(\phi)$; hence the only contribution in the first order asymptotics comes from $\xi_1$’s maximum over $Q_t$.

2.3.2 Subcritical Phase

The general statement of (1.10) and (1.15) can be seen as an extension of the argument for $\epsilon(t) = 1$ to its most general incarnation in the setting of asymptotically singular noise: By a straightforward rescaling (i.e., (3.4)), it can be seen that for every $t > 0$, $\Lambda_1(A_\epsilon(t), Q_t)$ is equal to the leading Dirichlet eigenvalue of the operator $\epsilon(t)^{-2}(\frac{1}{2}\Delta + \epsilon(t)^{4-d}/2\xi_1)$ on the box $Q_t/\epsilon(t)$. If we apply this rescaling to the quantities in (2.6) (i.e., replace $R$ by $\epsilon(t)^{(4-d)/2}R$ and scale space by $\epsilon(t)$), then this suggests the following: If we denote

$$
\bar{L}_t := \epsilon(t)^{(4-d)/2} \sqrt{2dR(0)\log t},
$$

(2.9)

$$
\bar{l}_t := \epsilon(t)^{(4-d)/4} \text{Tr}(\frac{(-R''(0))^{1/2}}{2}) \left( \frac{2d}{R(0)} \log t \right)^{1/4},
$$

(2.10)

$$
\bar{s}_t = \epsilon(t)^{-4-d/2}(2dR(0)\log t)^{-1/2} \cdot \epsilon(t),
$$

(2.11)

then we expect from (2.6) that

$$
\Lambda_1(A_\epsilon(t), Q_t) = \epsilon(t)^{-2}(\bar{L}_t - \bar{l}_t + o(\bar{l}_t)),
$$

where $\bar{L}_t$ is determined by the maximum of $\xi_{\epsilon(t)}$ on $Q_t$. $-\bar{l}_t$ comes from the contribution of $-\frac{1}{2} \xi'(\phi)$, and the leading eigenfunction is localized near this maximum in a box of size $\bar{s}_t$. Then, given that for $d = 1, 2, 3$,

$$
\epsilon(t)^{(4-d)/2} \sqrt{\log t} \gg \epsilon(t)^{(4-d)/4}(\log t)^{1/4}
$$

(2.12)

if and only if $\epsilon(t) \gg (\log t)^{-1/(4-d)}$, we infer that the subcritical phase coincides precisely with the regime where $\bar{L}_t \gg \bar{l}_t$, that is, the regime where the leading order asymptotics of $\Lambda_1(A_\epsilon(t), Q_t)$ are solely determined by the maximum of $\xi_{\epsilon(t)}$ on $Q_t$. At this point, by noting that

$$
\epsilon(t)^{-2}\bar{L}_t = \epsilon(t)^{-d/2} \sqrt{2dR(0)}(1 + o(1)) \quad \text{as } t \to \infty,
$$

we recover the subcritical eigenvalue asymptotics claimed in (1.10).

Remark 2.4 The fact that this argument relies crucially on (2.12) also explains why there is no phase transition when $d \geq 4$ in Theorems 1.7 and 1.11: Indeed, if $d \geq 4$, then (2.12) always holds, and thus we expect that the leading order asymptotics of $\Lambda_1(A_\epsilon(t), Q_t)$ should be determined by the maximum of $\xi_{\epsilon(t)}$ on $Q_t$ no matter how small $\epsilon(t)$ is. See Theorem 4.7 and Sections 4.2.2 and 4.2.4 for the details.
2.3.3 Supercritical Phase

If \( \varepsilon(t) \) vanishes faster than the critical threshold \( (\log t)^{-1/(4-d)} \), then (2.12) is no longer true. In particular, the terms \( \tilde{L}_s \) and \( \tilde{I}_t \) in (2.9) and (2.10) coalesce, and thus the leading order asymptotics of \( \Lambda_1(A_{\varepsilon(t)}, Q_t) \) can no longer be expected to be solely determined by the maximum of \( \xi_{\varepsilon(t)} \) on \( Q_t \).

We expect that the asymptotics of \( \Lambda_1(A_{\varepsilon(t)}, Q_t) \) should in some sense stabilize to that of \( \Lambda(A, Q_t) \) when \( \varepsilon(t) \) is very small. Thus, it is natural to hypothesize that the magnitude of the terms corresponding to \( \varepsilon(t)^{-2}(\tilde{L}_s - \tilde{I}_t) \) and \( \tilde{f}_t \) in this regime can be obtained by replacing \( \varepsilon(t) \) in (2.9)–(2.11) by the value of the critical threshold \( a(t) := (\log t)^{-1/(4-d)} \).

Following this hypothesis, we expect that, up to a constant,

\[
\Lambda_1(A_{\varepsilon(t)}, Q_t) \asymp a(t)^{-2} \cdot a(t)^{(4-d)/2} \sqrt{\log t} = (\log t)^{2/(4-d)},
\]

(2.13)

and that the corresponding eigenfunction localizes in a box of size (up to a constant)

\[
a(t)^{-(4-d)/2}(\log t)^{-1/2} \cdot a(t) = (\log t)^{-1/(4-d)}.
\]

(2.14)

This heuristic is further corroborated by the facts that

1. The known asymptotics for the AH/PAM with WN found in [12] for \( d = 1 \) and [15, 32] for \( d = 2 \) correspond to (2.13) (c.f., Theorem 1.19); and
2. in the one-dimensional case [12], Chen showed that the asymptotics in question can be obtained with a localization argument of the form (2.3) with sub-boxes \( B_i(t) \) of size \( (\log t)^{-1/3} \) (up to a constant) as \( t \to \infty \).

From the technical standpoint (see the outline in steps (1)–(3) in [12, Page 583]), a crucial innovation in [12] lies in relating the maxima on the right-hand side of (2.3) for WN in \( d = 1 \) to the extreme value theory of the function-valued Gaussian process

\[
\varphi \mapsto \langle \varphi, \xi \rangle, \quad \varphi : \mathbb{R}^d \to \mathbb{R}
\]

on carefully chosen function spaces that are amenable to computation. Thus, one of the insights of this paper is that this type of argument can be extended in \( d = 2, 3 \), so long as we consider asymptotically singular noise \( \xi_{\varepsilon(t)} \) where \( \varepsilon(t) \) is not too large (or too small when \( d = 2, 3 \)).

3 Setup and Notation

3.1 Basic Notations

**Definition 3.1** Given \( 1 \leq p \leq \infty \), we use \( \|f\|_p \) to denote the \( L^p \) norm of a function \( f : \mathbb{R}^d \to \mathbb{R} \), and \( |x|_p \) to denote the \( \ell^p \)-norm of a vector \( x \in \mathbb{R}^d \). We use \( \langle f, g \rangle \) to denote the Euclidean inner product on \( L^2(\mathbb{R}^d) \), and \( f * g \) to denote the convolution.

**Definition 3.2** Given an open set \( \Omega \subset \mathbb{R}^d \), we use \( C^0_c(\Omega) \) to denote the set of smooth functions \( \varphi : \mathbb{R}^d \to \mathbb{R} \) with compact support in \( \Omega \). We denote the Dirichlet form of such functions as

\[
\mathcal{E}(\varphi) := \int_{\mathbb{R}^d} |\nabla \varphi(x)|_2^2 \, dx,
\]

(3.1)

where \( \nabla \) denotes the gradient, and we denote the function spaces

\[
S(\Omega) := \{ \varphi \in C^0_c(\Omega) : \|\varphi\|_2 = 1 \},
\]

\[
W(\Omega) := \{ \varphi \in C^0_c(\Omega) : \|\varphi\|_2^2 + \frac{1}{2} \mathcal{E}(\varphi) = 1 \}.
\]

(3.2)
Definition 3.3 For every $z \in \mathbb{R}^d$, we define the translation operator
\[ \tau_z \varphi(x) := \varphi(x - z), \quad \varphi \in C^0_0(\mathbb{R}^d). \]  
(3.3)

For every $\eta > 0$ and $\varphi \in C^0_0(\mathbb{R}^d)$, we define the rescaled function
\[ \varphi^{(\eta)}(x) := \eta^{d/2} \varphi(\eta x), \quad x \in \mathbb{R}^d. \]  
(3.4)

Remark 3.4 It is easy to see that for every domain $\Omega \subset \mathbb{R}^d$ and $\eta > 0$,

1. $\varphi \in S(\eta \Omega)$ if and only if $\varphi^{(\eta)} \in S(\Omega)$, and
2. $\mathcal{E}(\varphi^{(\eta)}) = \eta^d \mathcal{E}(\varphi)$.

3.2 Covariance Semi Inner Product

Definition 3.5 For every $\varepsilon > 0$, we denote the covariance semi inner product by
\[ \langle f, g \rangle_{\varepsilon} := \langle f \ast \varepsilon, g \ast \varepsilon \rangle = \int_{(\mathbb{R}^d)^2} f(x) \varepsilon(x - y) g(y) \, dx \, dy \]
for $f, g : \mathbb{R}^d \to \mathbb{R}$, and we denote the associated seminorm by
\[ \|f\|_{\varepsilon} := \sqrt{\langle f, f \rangle_{\varepsilon}}. \]
In particular, since $\varepsilon \to \delta_0$, one has
\[ \lim_{\varepsilon \to 0} \langle \varphi, \psi \rangle_{\varepsilon} = \langle \varphi, \psi \rangle, \quad \varphi, \psi \in C^0_0(\mathbb{R}^d). \]  
(3.5)

3.3 Operator Theory and Localization Bounds

For every $\varepsilon, \sigma > 0$, let us denote the operator
\[ A^{(\varepsilon)} := \frac{1}{2} \Delta + \sigma \xi_{\varepsilon}, \]  
(3.6)
so that, in particular, $A_{\varepsilon} = A^{(1)}_{\varepsilon}$. For any bounded and connected open set $\Omega \subset \mathbb{R}^d$, the operator $-A^{(\varepsilon)}_{\sigma}$ with Dirichlet boundary conditions on $\Omega$ is self-adjoint on $L^2(\Omega)$ and has compact resolvent, and $C^0_0(\Omega)$ is a core for its quadratic form:
\[ -\langle \varphi, A^{(\varepsilon)} \varphi \rangle := -\sigma(\xi_{\varepsilon}, \varphi^2) + \frac{1}{2} \mathcal{E}(\varphi), \quad \varphi \in C^0_0(\Omega) \]
(e.g., [43, Example 3.16.4]; we refer more generally to [43, Section 7.5] for the operator-theoretic terminology used here). In particular, it follows from the min-max principle (e.g., [43, Theorem 7.8.10]) that
\[ A_k(A^{(\varepsilon)}_{\sigma}, \Omega) = \sup_{\varphi_1, \ldots, \varphi_k \in C^0_0(\Omega)} \inf_{(\varphi, \varphi_j) = 0 \forall \jmath \\ j \neq k} \left( \sigma(\xi_{\varepsilon}, \varphi^2) - \frac{1}{2} \mathcal{E}(\varphi) \right), \quad k \in \mathbb{N}, \]  
(3.7)
with matching eigenfunctions forming an orthonormal basis of $L^2(\Omega)$. The following localization bounds are the main technical tools in our analysis of the asymptotics of the $A^{(\varepsilon)}_{\sigma}$’s eigenvalues:

Lemma 3.6 For every $\varepsilon > 0$ and bounded open sets $\Omega \subset \mathbb{R}^d$, $\Omega_1, \ldots, \Omega_n \subset \Omega$,
\[ \Lambda_1(A^{(\varepsilon)}_{\sigma}, \Omega) \geq \max_{i=1,\ldots,n} \Lambda_1(A^{(\varepsilon)}_{\sigma}, \Omega_i). \]
Lemma 3.7 There exists a constant \( C > 0 \) such that for every \( \varepsilon, \kappa > 0 \) and \( r > \kappa \), if we let \( Z := 2\kappa Z_d \cap Q_r \), then

\[
A_1(A^{(\sigma)}_\varepsilon, Q_r) \leq \frac{C}{\kappa} + \max_{z \in Z} \Lambda_1(A^{(\sigma)}_\varepsilon, z + Q_{\kappa+1}).
\]

The use of localization bounds such as Lemmas 3.6 and 3.7 date back to at least to work of Gärtner, König, and Molchanov [19, 20]. While Lemma 3.6 is a trivial consequence of the min-max principle (3.7), Lemma 3.7 is more delicate; we refer to the proof of [12, (2.27)] for the latter.

3.4 Semigroup Theory and Feynman-Kac Formula

Definition 3.8 For every open set \( \Omega \subset \mathbb{R}^d \), we let \( T_\Omega \) denote the first exit time of \( \Omega \) by the Brownian motion \( B \), that is, \( T_\Omega := \inf\{t \geq 1 : B(t) \notin \Omega \} \).

For every \( x, y \in \mathbb{R}^d \) and \( t > 0 \), we use \( \mathbb{E}^x \) to denote the expectation with respect to the law of the Brownian motion \( \{B_s/B(0) = x\} \), and \( \mathbb{E}_t^{x,y} \) to denote the expectation with respect to the law of the Brownian bridge \( \{B_s/B(0) = x \text{ and } B(t) = y\} \), both conditional on \( \xi_\varepsilon \).

We use \( \mathbb{G}_t \) to denote the Gaussian kernel, that is,

\[
\mathbb{G}_t(x) := \frac{e^{-|x|^2/2t}}{(2\pi t)^{d/2}}, \quad t > 0, \ x \in \mathbb{R}^d.
\]

Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set. The semigroup of the operator \( A^{(\sigma)}_\varepsilon \) with Dirichlet boundary conditions on \( \Omega \) is defined as the family of operators

\[
\mathcal{S}_t^{A^{(\sigma)}_\varepsilon}(\cdot, \Omega) := \sum_{k=1}^\infty e^{tA_k(A^{(\sigma)}_\varepsilon, \Omega)} \langle \Psi_k(A^{(\sigma)}_\varepsilon, \Omega), f \rangle \Psi_k(A^{(\sigma)}_\varepsilon, \Omega), \quad t > 0
\]

acting on \( f \in L^2(\Omega) \), where \( \Psi_k(A^{(\sigma)}_\varepsilon, \Omega) \) (\( k \in \mathbb{N} \)) denote the orthonormal eigenfunctions associated with \( \Lambda_k(A^{(\sigma)}_\varepsilon, \Omega) \). According to the Feynman-Kac formula (e.g., [16, (34) and Theorem 3.27]), \( \mathcal{S}_t^{A^{(\sigma)}_\varepsilon}(\cdot, \Omega) \) is an integral operator on \( L^2(\Omega) \) with kernel

\[
\mathcal{S}_t^{A^{(\sigma)}_\varepsilon}(x, y) := \mathbb{G}_t(x-y) \mathbb{E}_t^{x,y} \left[ \exp \left( \int_0^t \xi_\varepsilon(B(s)) \ ds \right) : T_\Omega \geq t \right]
\]

for all \( t > 0 \) and \( x, y \in \mathbb{R}^d \), where, for any random variable \( Y \) and event \( E \), we denote \( \mathbb{E}[Y; E] := \mathbb{E}[1_E Y] \). Since \( \Omega \) is bounded, the semigroup \( \{\mathcal{S}_t^{A^{(\sigma)}_\varepsilon}(\cdot, \Omega)\}_{t \geq 0} \) is Hilbert-Schmidt/trace class, and for every \( t > 0 \) (e.g., [16, Theorem 3.17]),

\[
\sum_{k=1}^\infty e^{tA_k(A^{(\sigma)}_\varepsilon, \Omega)} = \text{Tr} \left[ \mathcal{S}_t^{A^{(\sigma)}_\varepsilon}(\cdot, \Omega) \right] = \int_\Omega \mathcal{S}_t^{A^{(\sigma)}_\varepsilon}(x, x) \ dx = \left\| \mathcal{S}_t^{A^{(\sigma)}_\varepsilon}(\cdot, \Omega) \right\|^2 < \infty.
\]

One of the main ingredients in the proof of Theorems 1.7 and 1.11 consists of the observation that if \( \Omega \) is very large and contains the origin, then we expect that

\[
U_\varepsilon(t) = \mathbb{E}^0 \left[ \exp \left( \int_0^t \xi_\varepsilon(B(s)) \ ds \right) \right] \approx \mathbb{E}^0 \left[ \exp \left( \int_0^t \xi_\varepsilon(B(s)) \ ds \right) : T_\Omega \geq t \right].
\]

Thanks to (3.9) and (3.10), this then creates a connection between the asymptotics of \( U_\varepsilon(t) \) and that of \( A_1(A_\varepsilon, Q_r) \) as \( t \to \infty \), allowing to formalize the heuristic (2.1). In order to make this precise, we use the following two technical results, which are the statements of [11, (4.2) and (4.5)] and [11, (4.3) and (4.6)], respectively (and which are proved using a variety of time-cutoff arguments on sub-intervals \([0, \eta] \subset [0, t]\) and Hölder’s inequality with \( p, q \geq 1 \)).
**Proposition 3.9** Let us denote, for every $\varepsilon, \sigma > 0$ and $t \geq 0$, the quantity

$$U^{(\sigma)}_\varepsilon(t) := E^0 \left[ \exp \left( \sigma \int_0^t \xi_\varepsilon(B(s)) \, ds \right) \right].$$

Let $r > 0$ and $p, q > 1$ be such that $1/p + 1/q = 1$. For every $t \geq 1$ and $0 < \eta < t$, it holds that

$$E^0 \left[ \exp \left( \int_0^t \xi_\varepsilon(B(s)) \, ds ; T_{Q_t} \geq t \right) \right] \leq U^{(q)}_\varepsilon(\eta)^{1/q} \cdot \left( \frac{1}{(2\pi \eta)^{d/2}} \right) \int_{Q_t} E^x \left[ \exp \left( p \int_{t-\eta}^t \xi_\varepsilon(B(s)) \, ds ; T_{Q_t} \geq t - \eta \right) \right] \, dx \right)^{1/p}. \tag{3.12}$$

and for every $i \geq 1$ and $\theta > 0$,

$$\int_{Q_t} E^x \left[ \exp \left( \theta \int_0^i \xi_\varepsilon(B(s)) \, ds ; T_{Q_t} \geq i \right) \right] \, dx \leq (2r)^d e^{\Lambda_1(A^{(q)}_\varepsilon, Q_t)}. \tag{3.13}$$

**Proposition 3.10** Let $r > 0$ and $p, q > 1$ be such that $1/p + 1/q = 1$. For every $t \geq 1$ and $0 < \eta < t$, it holds that

$$U^{(-q/p)}_\varepsilon(\theta) = \theta \int_{Q_t} E^x \left[ \exp \left( \frac{1}{p} \int_0^t \xi_\varepsilon(B(s)) \, ds ; T_{Q_t} \geq t - \eta \right) \right] \, dx \right]^p. \tag{3.14}$$

For every $\tilde{t} > 0, 0 < \eta < \tilde{t}$, and $\theta > 0$, it holds that

$$\int_{Q_t} E^x \left[ \exp \left( \theta \int_0^{\tilde{t}} \xi_\varepsilon(B(s)) \, ds ; T_{Q_t} \geq \tilde{t} \right) \right] \, dx \geq (2\pi)^{pd/2} \eta^{d/2} \theta^{d/2q} (2r)^{-2p/q} e^{-\eta(p/q)A_t(\lambda^{(q/p)}_\varepsilon, Q_t)} \epsilon^{\theta(t + \eta)} \Lambda_1(A^{(q/p)}_\varepsilon, Q_t). \tag{3.15}$$

### 4 Eigenvalue Asymptotics

Our purpose in this section is to prove the eigenvalue asymptotics in Theorems 1.7 and 1.11, namely, (1.10) and (1.15). For this purpose, in this section the main result we prove is the following:

**Theorem 4.1** Let $w : [0, \infty) \to \mathbb{R}^d$ be an arbitrary function. Let $\theta, \alpha > 0$ and $\beta \geq 0$ be fixed constants, and define

$$r(t) := \begin{cases} \theta t^\alpha (e(t)^{-d/2} \sqrt{\log t})^\beta & \text{if } e(t) \text{ is in the regular phase}, \\ \theta t^\alpha \left((\log t)^{2/(4-d)}\right)^\beta & \text{if } e(t) \text{ is in the singular phase}. \end{cases} \tag{4.1}$$

If $e(t)$ is in the regular phase, then for every $\sigma > 0$,

$$\lim_{t \to \infty} \frac{\Lambda_1 \left(A^{(\sigma)}_e, w(t) + Q_r(t)\right)}{e(t)^{-d/2} \sqrt{\log r(t)}} = \sigma \sqrt{2dR(0)} \quad \text{in probability.} \tag{4.2}$$

If $e(t)$ is in the singular phase, then for every $\sigma > 0$,

$$\lim_{t \to \infty} \frac{\Lambda_1 \left(A^{(\sigma)}_e, w(t) + Q_r(t)\right)}{(\log r(t))^{2/(4-d)}} = \sigma^{4/(4-d)} \Omega_d \quad \text{in probability.} \tag{4.3}$$
Remark 4.2  Since $\xi(t)$ is translation invariant, if Theorem 4.1 holds for $w(t) = 0$, then it immediately follows that the same is also true for any other choice of $w(t)$. We nevertheless state Theorem 4.1 with general $w(t)$, since this statement is used in the proof of the eigenvalue asymptotics of $\Lambda_k(A_{\xi(t)}, Q_t)$ for $k \geq 2$ (see Section 4.1).

Remark 4.3  Following-up on the previous remark, we note that apart from the presence of $w(t)$, the statement of Theorem 4.1 has several differences with (1.10) and (1.15), making it simultaneously more and less general than the latter.

Firstly, Theorem 4.1 is less general, since it only concerns the leading eigenvalue $\Lambda_1$. This is due to the fact that the localization bounds in Lemmas 3.6 and 3.7, which are the main technical tools with which we prove Theorem 4.1, only apply to the first eigenvalue. The fact that (1.10) and (1.15) follow from (4.2) and (4.3) comes from one aspect of Theorem 4.1 that is more general, namely, that we consider the asymptotics of the leading eigenvalue on off-centered boxes $w(t) + Q_{\eta(t)}$ with side length $2r(t)$ instead of $2r$.

Secondly, Theorem 4.1 is more general in the sense that we consider boxes of side length $r(t)$, as well as the scaling factor $\sigma$ (which is equal to 1 in (1.10) and (1.15)). These more general aspects are used in the proof of the total mass asymptotics in Theorems 1.7 and 1.11; we refer to Section 5 for the details.

The remainder of this section is organized as follows. In Section 4.1, we use Theorem 4.1 to prove (1.10) and (1.15). In Section 4.2, we prove Theorem 4.1.

4.1 Proof of (1.10) and (1.15)

We only prove the eigenvalue asymptotics in the regular phase, as the proof in the singular case follows from the same argument. We begin with the following statement (see, e.g., [15, Theorem 8.5]), which connects the non-leading eigenvalues to $\Lambda_1$:

Lemma 4.4  Let $k \in \mathbb{N}$ and $t \geq 1$ be fixed. If $z_1, \ldots, z_k \in \mathbb{R}^d$ and $\kappa > 0$ are such that $z_i + Q_\kappa \subset Q_t$ for every $1 \leq i \leq k$, and

$$\langle 1_{z_i+Q_\kappa}, 1_{z_j+Q_\kappa} \rangle = 0$$

for every $1 \leq i < j \leq k$, then for every $\varepsilon > 0$ one has

$$\Lambda_k(A_{\xi^t}, Q_t) \geq \min_{1 \leq i \leq k} \Lambda_1(A_{\xi^t}; z_i + Q_\kappa).$$

We also make the following simple remark:

Remark 4.5  $r(t)$ in (4.1) satisfies the following as $t \to \infty$:

$$\varepsilon(t)^{-d/2} \sqrt{\log r(t)} = \sqrt{\alpha} \varepsilon(t)^{-d/2} \sqrt{\log t (1 + o(1))} \quad \text{regular phase},$$

$$\left( \log r(t) \right)^{2/(4-d)} = \alpha^2 (4-d) (\log t)^{2/(4-d)} (1 + o(1)) \quad \text{singular phase}. \quad (4.4) \quad (4.5)$$

Thanks to Theorem 4.1 (with $\alpha = 1$, $\beta = 0$, and $\sigma = 1$) and Remark 4.5, every sequence of $t > 0$ such that $t \to \infty$ has a subsequence $(t_n)_{n \in \mathbb{N}}$ along which

$$\lim_{n \to \infty} \frac{\Lambda_1(A_{\xi(t_n)}; w(t_n) + Q_{\theta t_n})}{\varepsilon(t_n)^{-d/2} \sqrt{\log t_n}} = \sqrt{2dR(0)}$$

almost surely for every choice of $\theta \in (0, 1] \cap \mathbb{Q}$ and $w(t_n) \in \mathbb{Q}$.
On the one hand, by taking \( w(t) = 0 \) and \( \theta = 1 \) and using the trivial inequality \( \Lambda_k(A_\epsilon, \Omega) \leq \Lambda_1(A_\epsilon, \Omega) \) for all \( k \in \mathbb{N} \) and \( \Omega \subset \mathbb{R}^d \), we get

\[
\limsup_{n \to \infty} \frac{\Lambda_k(A_\epsilon(t_n), Q_{t_n})}{\epsilon(t_n)^{-d/2} \sqrt{\log t_n}} \leq \sqrt{2dR(0)} \quad \text{almost surely.} \quad (4.6)
\]

On the other hand, we can find a small enough constant \( \theta \in (0, 1] \cap \mathbb{Q} \) (that only depends on \( k \)) such that for every \( t \geq 0 \), there exists \( w^1(t), \ldots, w^k(t) \in \mathbb{Q} \) such that the sets \( w^i(t) + Q_{\theta t} \) are mutually disjoint and inside \( Q_t \). Consequently, up to taking a further subsequence of \( (t_n)_{n \in \mathbb{N}} \) in (4.6), we obtain from Theorem 4.1, Remark 4.5, and Lemma 4.4 that

\[
\limsup_{n \to \infty} \frac{\Lambda_k(A_\epsilon(t_n), Q_{t_n})}{\epsilon(t_n)^{-d/2} \sqrt{\log t_n}} \geq \min_{1 \leq i \leq k} \liminf_{n \to \infty} \frac{\Lambda_1(A_\epsilon(t_n); w^i(t_n) + Q_{\theta t_n})}{\epsilon(t_n)^{-d/2} \sqrt{\log t_n}} = \sqrt{2dR(0)}
\]

almost surely. Combined with (4.6), this completes the proof of the asymptotic for \( \Lambda_k(A_\epsilon(t), Q_t) \) in Theorem 1.7.

4.2 Proof of Theorem 4.1

We now prove Theorem 4.1. Given that \( \tilde{\xi}_\epsilon \) is stationary, there is no loss of generality in assuming that \( w(t) = 0 \); hence we need only prove asymptotics for \( \Lambda_1(A_\epsilon(t), Q_{r(t)}) \). We separate the proof of Theorem 4.1 into four steps, namely, matching lower and upper bounds for (4.2) and (4.3):

4.2.1 Step 1. Lower Bound for (4.2)

Let \( \epsilon(t) \) be in the regular phase. Let \( \kappa > 0 \) be large enough so that \( R \) is supported on \( Q_{\kappa/2} \).

For large \( t > 0 \), let us define

\[
Z_t := 3\kappa \epsilon(t) \mathbb{Z}^d \cap Q_{r(t) - \kappa \epsilon(t)}.
\]

By Lemma 3.6 (with the sets \( \Omega_t \) given by \( z + Q_{\kappa \epsilon(t)} \) for all \( z \in Z_t \)), we have

\[
\Lambda_1(A_\epsilon(t), Q_{r(t)}) \geq \max_{z \in Z_t} \sup_{\phi \in S(z + Q_{\kappa \epsilon(t)})} \left( \sigma(\tilde{\xi}_\epsilon(t), \phi^2) - \frac{1}{\tau} \delta(\phi) \right)
\]

\[
= \max_{\phi \in S(Q_{\kappa \epsilon(t)}) \setminus \mathbb{Z}_t} \left( \max_{z \in Z_t} \sigma(\tilde{\xi}_\epsilon(t), (\tau \phi)^2) - \frac{1}{\tau} \delta(\phi) \right),
\]

where we recall that \( \tau \) is the translation operator defined in (3.3), and we used in (4.8) the fact that \( \delta \) is translation invariant. The set of functions \( \phi \) over which the supremum (4.8) is taken depends on \( t \). When considering the large-\( t \) limit of this expression, however, it is more convenient to consider \( \phi \) from a fixed function space. For this reason, we consider rescaled functions: By Remark 3.4 (with \( \eta = \epsilon(t)^{-1} \)), we find that for every \( \phi \in S(Q_{\kappa}) \), one has

\[
\Lambda_1(A_\epsilon(t), Q_{r(t)}) \geq \left( \max_{z \in Z_t} \sigma(\tilde{\xi}_\epsilon(t), (\tau \phi^{(1/\epsilon(t))})^2) \right) - \epsilon(t)^{-2} \frac{1}{\tau} \delta(\phi).
\]

Until further notice, we assume that we are considering a single fixed function \( \phi \in S(Q_{\kappa}) \). By a straightforward change of variables,

\[
\int_{\mathbb{R}^d} \tilde{\xi}_\epsilon(t)(x) \epsilon(t)^{-d} \phi \left( \frac{x - z}{\epsilon(t)} \right)^2 \, dx = \epsilon(t)^{-d/2} \int_{\mathbb{R}^d} \xi_1(x) \phi \left( \frac{x - z}{\epsilon(t)} \right)^2 \, dx,
\]

which we can use to obtain the following identity for \( \tilde{\xi}_\epsilon(t)(x) \epsilon(t)^{-d} \phi \left( \frac{x - z}{\epsilon(t)} \right)^2 \, dx \).
and thus (4.9) yields
\[
A_1(A_{\varepsilon(t)}^{(\sigma)}, Q_{r(t)}) \geq \left( \sigma \varepsilon(t)^{-d/2} \max_{z \in Z_k} \langle \xi_1, (\tau_{z/\varepsilon(t)} \varphi)^2 \rangle \right) - \varepsilon(t)^{-2} \frac{1}{2} \delta^2(\varphi). \tag{4.11}
\]

On the one hand, since \( \varepsilon(t)^{-4d/2} \ll \sqrt{\log r} \) in the regular phase, it follows from (4.4) that
\[
\frac{\varepsilon(t)^{-2}}{\varepsilon(t)^{-d/2} \sqrt{\log r}} \delta^2(\varphi) = O \left( \varepsilon(t)^{-4d/2} / \sqrt{\log r} \right) = o(1) \quad \text{as } t \to \infty. \tag{4.12}
\]

On the other hand, we note that
\[
\langle (\tau_{z/\varepsilon(t)} \varphi)^2, (\tau_{z'/\varepsilon(t)} \varphi)^2 \rangle, \quad z \in Z_k
\]
is a Gaussian process with mean zero and covariance
\[
\langle (\tau_{z/\varepsilon(t)} \varphi)^2, (\tau_{z'/\varepsilon(t)} \varphi)^2 \rangle_R, \quad z, z' \in Z_k.
\]

By definition of \( Z_k \) and our assumption that \( \varepsilon(t) \leq 1 \), if \( z, z' \in Z_k \) are distinct, then the supports of \( (\tau_{z/\varepsilon(t)} \varphi)^2 \) and \( (\tau_{z'/\varepsilon(t)} \varphi)^2 \) are separated by at least \( \kappa \) in \( \infty \)-norm. Therefore, since we have assumed \( \kappa \) to be large enough so that \( R \) (and therefore \( R_{r(t)} \) for all \( t \geq 0 \)) is supported in \( Q_{r(t)} \), we conclude that (4.13) are i.i.d. Gaussians with variance \( \sigma^2 \| \varphi^2 \|^2_R \).

In particular, given that by definition of \( Z_k \) in (4.7) and the fact that \( \varepsilon(t) \) is in the regular phase, we have
\[
\log |Z_k| = (d \log r(t)) (1 + o(1)),
\]
then it follows from Lemma 2.3 (by coupling the \( \langle X_t(z) \rangle_{z \in Z_k} \) with a collection of i.i.d. Gaussians with mean zero and variance \( \sigma^2 \| \varphi^2 \|^2_R \) on \( Z_k \)) that
\[
\lim_{t \to \infty} \frac{1}{\sqrt{\log r(t)}} \max_{z \in Z_k} \langle \xi_1, (\tau_{z/\varepsilon(t)} \varphi)^2 \rangle = \sigma \| \varphi^2 \|^2_R \sqrt{2d} \quad \text{in probability.} \tag{4.14}
\]

By combining the limits (4.12) and (4.14) with the lower bound (4.11), our argument so far can be summarized as follows: For every \( \kappa > 0 \) and \( \varphi \in S(Q_k) \), every sequence of \( t > 0 \) such that \( t \to \infty \) has a subsequence \( (t_n)_{n \in \mathbb{N}} \) along which
\[
\lim_{n \to \infty} \frac{A_1(A_{\varepsilon(t_n)}^{(\sigma)}, Q_{r(t_n)})}{\varepsilon(t_n)^{-d/2} \sqrt{\log r(t_n)}} \geq \sigma \| \varphi^2 \|^2_R \sqrt{2d} \quad \text{almost surely.}
\]
Since \( \varphi^2 \in L_1(Q_k) \), we can take a sequence \( (\varphi_n)_{n \in \mathbb{N}} \subset S(Q_k) \) such that \( \varphi_n \to \delta_0 \) as \( n \to \infty \); hence \( \| \varphi_n^2 \|^2_R \to R(0) \). Up to taking further subsequence of \( t_n \), this concludes the proof of the lower bound for (4.2).

4.2.2 A Remark on \( d \geq 4 \), Part 1

In the proof of the lower bound for (4.2) that we have just provided, the only manifestation of the assumption that \( d \leq 3 \) comes from the requirement that \( \varepsilon(t)^{d-4}/2 \ll \sqrt{\log r(t)} \), which is equivalent to the assumption that \( \varepsilon(t) \) is in the regular phase. If \( d \geq 4 \), then
1. \( \log |Z_k| = d \log (r(t) / \varepsilon(t)) (1 + o(1)) \) with \( Z_k \) as in (4.7), and
2. \( \varepsilon(t)^{(d-4)/2} \ll \sqrt{\log (r(t) / \varepsilon(t))} \) no matter how quickly \( \varepsilon(t) \) vanishes.

Therefore, we get the following by using the same arguments as in Section 4.2.1:

**Proposition 4.6** Let \( d \geq 4 \) and \( \varepsilon(t) \in (0, 1] \) arbitrary. Every sequence of \( t > 0 \) such that \( t \to \infty \) has a subsequence \( (t_n)_{n \in \mathbb{N}} \) along which
\[
\liminf_{n \to \infty} \frac{A_1(A_{\varepsilon(t_n)}^{(\sigma)}, Q_{r(t_n)})}{\varepsilon(t_n)^{-d/2} \sqrt{\log (r(t_n) / \varepsilon(t_n))}} \geq \sigma \sqrt{2dR(0)} \quad \text{almost surely.}
\]
4.2.3 Step 2. Upper Bound for (4.2)

Let \( \varepsilon(t) \) be in the regular phase. Since \( \varepsilon \) is nonnegative and \( \varphi \in S(Q_{r(t)}) \) are such that \( \int_{\mathbb{R}^d} \varphi(x)^2 \, dx = 1 \), we have that

\[
\Lambda_1(A^{(\varepsilon)}_{r(t)}, Q_{r(t)}) \leq \sigma \sup_{|x| \leq r(t)/\varepsilon(t)} \varepsilon(t)^{-d/2} \xi_1(x).
\]

We then get the desired bound by a direct application of Lemma 2.3.

4.2.4 A Remark on \( d \geq 4 \), Part 2

Carrying on from Section 4.2.2, the simple argument in Section 4.2.3 does not depend on the assumption that \( d \geq 3 \). The only difference is that if \( d \geq 4 \) and we do not assume a lower bound on the vanishing rate of \( \varepsilon(t) \), then \( \log \left( r(t)/\varepsilon(t) \right) \) need not be asymptotically equivalent to \( \log r(t) \). Consequently, by combining Proposition 4.6 with the argument presented in Section 4.1, we obtain the following result, which states that no phase transition occurs in the eigenvalue asymptotics when \( d \geq 4 \):

**Theorem 4.7** Let \( d \geq 4 \), and let \( \varepsilon(t) \in (0, 1) \) be arbitrary. For every \( k \in \mathbb{N} \),

\[
\lim_{t \to \infty} \frac{\Lambda_k(A^{(\varepsilon)}_{r(t)}, Q_{r(t)})}{\varepsilon(t)^{-d/2} \sqrt{\log \left( t/\varepsilon(t) \right)}} = \sqrt{2dR(0)} \quad \text{in probability.}
\]

4.2.5 Lower Bound for (4.3)

Let \( \varepsilon(t) \) be in the singular phase. Let \( \kappa > 0 \) be large enough so that \( R \) is supported in \( Q_{\kappa/2} \). Following the heuristic in (2.14), we expect that the leading eigenfunction of \( A^{(\varepsilon)}_{r(t)} \) on \( Q_{r(t)} \) should localize in a sub-box of size (up to a constant) \( \left( \log r(t) \right)^{-1/(4-d)} \). Thus, in order to set up the localization lower bound in this regime, we introduce the following lattice: Fix some \( \kappa_1, \kappa_2 > 0 \), and for every \( t > 0 \), let

\[
a(t) := \left( \kappa_1 \log r(t) \right)^{1/(4-d)} \quad \text{and} \quad Z_t := (2\kappa_2 + \kappa)a(t)\mathbb{Z}^d \cap Q_{r(t)} - \kappa_2a(t). \tag{4.15}
\]

**Remark 4.8** It is easy to see that there exists some \( c > 0 \) such that \( |Z_t| \geq cr(t)^d \) for all large enough \( t \).

By applying Lemma 3.6 (with \( \Omega_t \) given by \( z + Q_{\kappa_2a(t)} \) for \( z \in Z_t \)) and a rescaling similar to (4.9) (with \( a(t) \) instead of \( \varepsilon(t) \)), we have the lower bound

\[
\Lambda_1(A^{(\varepsilon)}_{r(t)}, Q_{r(t)}) \geq a(t)^{-2} \max_{z \in Z_t} \sigma a(t)^2 \left( \xi_{r(t)}(z), (\tau_z \varphi^{(1/a(t))})^2 \right) - \frac{1}{2} \sigma \langle \varphi \rangle
\]

for every \( \varphi \in S(Q_{\kappa}) \). Until further notice, we fix a \( \varphi \in S(Q_{\kappa}) \). For every \( t > 0 \), denote the \( Z_t \)-indexed stochastic process

\[
X_t(z) := \sigma a(t)^2 \left( \xi_{r(t)}(z), (\tau_z \varphi^{(1/a(t))})^2 \right), \quad z \in Z_t. \tag{4.17}
\]

By a straightforward change of variables similar to (4.10), we see that \( X_t \) is a centered stationary Gaussian process with covariance

\[
\sigma^2 a(t)^{4-2d} \int_{\mathbb{R}^d} \varphi \left( \frac{x - z}{a(t)} \right)^2 R_{r(t)}(x-y) \varphi \left( \frac{y - z'}{a(t)} \right)^2 \, dx \, dy = \sigma^2 a(t)^{4-d} \int_{\mathbb{R}^d} \varphi \left( \frac{x - z}{a(t)} \right)^2 a(t)^d R_{r(t)}(a(t)(x-y)) \varphi \left( \frac{y - z'}{a(t)} \right)^2 \, dx \, dy = \sigma^2 a(t)^{4-d} \left( \tau_{z/a(t)} \varphi, (\tau_{z'/a(t)} \varphi)^2 \right)_{R_{r(t)/a(t)}}.
\]
We note that for any distinct \( z, z' \in Z \) and \( \varphi \in S(Q_{\kappa_2}) \), the supports of \( \tau_{z'(a(t))} \varphi^2 \) and \( \tau_{z(a(t))} \varphi^2 \) are separated by at least \( \kappa \) in \( \ell^{\infty} \) norm. Given the asymptotic (4.5), we have that \( \varepsilon(t)/a(t) \to 0 \) as \( t \to \infty \) when \( \varepsilon(t) \) is in the singular phase. Therefore, at least for large \( t \), \( (X_t(z))_{z \in Z} \) are i.i.d. random variables with variance \( \sigma^2(a(t))^{1-d} \| \varphi^2 \|_{R_{t(a(t))}}^2 \) thanks to the fact that \( R \) is supported in \( Q_{\kappa/2} \). With this said, we claim that a lower bound for (4.3) is a consequence of the following:

**Proposition 4.9** Let \( X_t \) be as in (4.17). For every \( \kappa_1, \kappa_2 > 0 \) and \( \varphi \in S(Q_{\kappa_2}) \),

\[
E[X_t(0)^2] = \sigma^2 a(t)^{4-d} \| \varphi \|^4 \big( 1 + o(1) \big) \quad \text{as } t \to \infty,
\]

noting that \( a(t)^{4-d} = \left( \kappa_1 \log r(t) \right)^{-1} \).

**Proof.** \( \varepsilon(t) \ll a(t) \) in the singular phase; hence \( \| \varphi^2 \|_{R_{t(a(t))}}^2 \to \| \varphi \|^4 \) by (3.5). \( \square \)

To see this, we apply a standard lower tail bound for suprema of i.i.d. Gaussians: For large enough \( t \) the \( X_t(z) \) are i.i.d. copies of \( X_t(0) \), and thus

\[
P \left[ \sup_{z \in Z} X_t(z) \leq \sigma \| \varphi \|^2 \right] = \left( 1 - P[X_t(0) > \sigma \| \varphi \|^2] \right)^{|Z|}.
\]

We recall the classical Gaussian tail lower bound: If \( N \sim N(0, \nu^2) \) for some \( \nu > 0 \), then for every \( \vartheta > 1 \) and large enough \( \lambda > 0 \), one has \( P[N \geq \lambda] \geq e^{-\vartheta \lambda^2/2\nu^2} \). Therefore, if we assume that \( 0 < \kappa_1 < 2\vartheta \) and then take \( t > 0 \) large enough so that

\[
-\vartheta \sigma^2 \| \varphi \|^4 \frac{1}{2E[X_t(0)^2]} = -\vartheta (1 + o(1)) \kappa_1 \log r(t) \geq -\eta d \log r(t)
\]

for some \( 0 < \eta < 1 \) (by Proposition 4.9), then

\[
\left( 1 - P[X_t(0) > \sigma \| \varphi \|^2] \right)^{|Z|} \leq \left( 1 - \frac{1}{r(t)^\eta d} \right)^{|Z|} \leq \left( 1 - \frac{r(t)^{d(1-\eta)}}{r(t)^d} \right) \leq e^{-c r(t)^{d(1-\eta)}},
\]

where the third inequality follows from Remark 4.8. By (4.1), \( r(t) \geq \theta t^\alpha \) for some \( \theta, \alpha > 0 \) for large \( t \), and thus every sequence of \( t > 0 \) such that \( t \to \infty \) has a subsequence \( (r_n)_{n \in \mathbb{N}} \) such that \( \sum_n e^{-c r_n^d (1-\eta)} < \infty \). Thus, by applying the Borel-Cantelli lemma to the maximum of (4.17), and then combining this with (4.16), we conclude the following: For every \( 0 < \kappa_1 < 2\vartheta, \kappa_2 > 0 \), and \( \varphi \in S(Q_{\kappa_2}) \), every sequence of \( t > 0 \) such that \( t \to \infty \) has a subsequence \( (r_n)_{n \in \mathbb{N}} \) along which

\[
\liminf_{n \to \infty} \frac{A_1(A_{\varepsilon(r_n)}(t), Q_r(r_n))}{\alpha(r_n)^2} \geq \sigma \| \varphi \|^2 - \frac{1}{2} \delta' (\varphi) \quad \text{almost surely}.
\]

Given that \( a(t)^{1-d} = \left( \kappa_1 \right)^{2/(4-d)} (\log r(t))^{2/(4-d)} \), up to selecting further subsequences of \( t_n \), if we take \( \kappa_1 \to 2\vartheta, \kappa_2 \to \infty \), and a sequence of \( \varphi \)'s that achieves the supremum of \( \sigma \| \varphi \|^2 - \frac{1}{2} \delta' (\varphi) \) over \( S(\mathbb{R}^d) \), then we obtain that

\[
\liminf_{n \to \infty} \frac{A_1(A_{\varepsilon(r_n)}(t), Q_r(r_n))}{(\log r(t_n))^{2/(4-d)}} \geq (2\vartheta)^{2/(4-d)} \sup_{\varphi \in S(\mathbb{R}^d)} \left( \sigma \| \varphi \|^2 - \frac{1}{2} \delta' (\varphi) \right) \quad \text{almost surely}.
\]

This provides a lower bound for (4.3) by Proposition A.2 and Lemma A.3.
4.2.6 Upper Bound for (4.3)

Let \( \varepsilon(t) \) be in the singular phase. Let \( \kappa_1, \kappa_2 > 0 \) be fixed, and \( a(t) \) be as in (4.15). By a straightforward rescaling (i.e., \( \varphi \mapsto \varphi^{(1/a(t))} \)) by (3.4) and Remark 3.4, see also (4.18), we have that

\[
\Lambda_1(\sigma(t, \varepsilon(t)), Q_{r(t)}) = a(t)^{-2} \Lambda_1(\sigma(t, a(t)), Q_{r(t)/a(t)}).
\]

(4.19)

Let us define \( Z_t := 2\kappa_2 Z_d \cap Q_{r(t)/a(t)} \).

**Remark 4.10** By definition of \( a(t) \), it is clear that there exists constants \( c_1, c_2 > 0 \) such that \( |Z_t| \leq c_1 r(t)^d (\log r(t))^{c_2} \) for large \( t > 0 \).

By applying Lemma 3.7 to the right-hand side of (4.19), we obtain

\[
\Lambda_1(\sigma(t, \varepsilon(t)), Q_{r(t)}) \leq a(t)^{-2} \left( \frac{C}{\kappa_2} + \max_{\varepsilon \in Z_t, \varphi \in S(Q_{\kappa_2+1})} \left( \sigma a(t)^{(4-d)/2} \langle \xi_{\varepsilon(t)/a(t)}, (\tau, \varphi)^2 \rangle - \frac{1}{2} \varepsilon(\varphi) \right) \right)
\]

(4.20)

where the constant \( C > 0 \) is independent of \( t, \kappa_1, \) and \( \kappa_2 \). In order to control this quantity, we use an idea due to Chen [11, Page 593]: For every \( \varphi \in S(Q_{\kappa_2+1}) \), the function \( \psi := \varphi (1 + \frac{1}{4} \varepsilon(\varphi))^{-1/2} \) is an element of \( W(Q_{\kappa_2+1}) \) because

\[
\|\psi\|_2^2 + \frac{1}{2} \varepsilon(\psi) = \int_{\mathbb{R}^d} \psi(x)^2 + \frac{1}{2} |\nabla \psi(x)|^2 \, dx = \int_{\mathbb{R}^d} \frac{\varphi(x)^2 + \frac{1}{4} |\nabla \varphi(x)|^2}{1 + \frac{1}{4} \varepsilon(\varphi)} \, dx = 1
\]

(4.21)

(recall the definition of the latter function space in (3.2)). Therefore,

\[
\sup_{\varphi \in S(Q_{\kappa_2+1})} \left( \sigma a(t)^{(4-d)/2} \langle \xi_{\varepsilon(t)/a(t)}, (\tau, \varphi)^2 \rangle - \frac{1}{2} \varepsilon(\varphi) \right)
\]

\[
\leq \sup_{\varphi \in S(Q_{\kappa_2+1})} \left( \sup_{\psi \in W(Q_{\kappa_2+1})} \sigma a(t)^{(4-d)/2} \langle \xi_{\varepsilon(t)/a(t)}, (\tau, \psi)^2 \rangle (1 + \frac{1}{4} \varepsilon(\varphi)) - \frac{1}{2} \varepsilon(\varphi) \right)
\]

\[
\leq \sup_{\nu > 0} \left( \sup_{\psi \in W(Q_{\kappa_2+1})} \sigma a(t)^{(4-d)/2} \langle \xi_{\varepsilon(t)/a(t)}, (\tau, \psi)^2 \rangle (1 + \nu) - \nu \right).
\]

Given that

\[
\sup_{\nu > 0} (\mathfrak{A}(1 + \nu) - \nu) = \sup_{\nu > 0} (\mathfrak{A} + (\mathfrak{A} - 1)\nu) = \begin{cases} \mathfrak{A} & \text{if } \mathfrak{A} \leq 1, \\ \infty & \text{otherwise}, \end{cases}
\]

(4.22)

we then have the inclusion of events

\[
\left\{ \sup_{\varphi \in S(Q_{\kappa_2+1})} \left( \sigma a(t)^{(4-d)/2} \langle \xi_{\varepsilon(t)/a(t)}, (\tau, \varphi)^2 \rangle - \frac{1}{2} \varepsilon(\varphi) \right) > 1 \right\}
\]

\[
\subset \left\{ \sup_{\psi \in W(Q_{\kappa_2+1})} \sigma a(t)^{(4-d)/2} \langle \xi_{\varepsilon(t)/a(t)}, (\tau, \psi)^2 \rangle > 1 \right\}
\]

for any \( z \). Since \( \xi_{\varepsilon(t)} \) is stationary, the \( W(Q_{\kappa_2+1}) \)-valued Gaussian processes

\[
\psi \mapsto \sigma a(t)^{(4-d)/2} \langle \xi_{\varepsilon(t)/a(t)}, (\tau, \psi)^2 \rangle, \quad z \in Z_t
\]
are identically distributed for all \( z \in Z \), and thus by a union bound this implies that

\[
P \left[ \max_{z \in Z} \sup_{\varphi \in \mathcal{S}(Q_{z+1})} (\sigma a(t)^{(4-d)/2} \langle \xi_{z/(a(t))}, (\tau, \varphi)^2 \rangle - \frac{1}{4} \varphi' (\varphi) ) > 1 \right] \\
\leq |Z| \cdot P \left[ \sup_{\psi \in W(Q_{z+1})} \sigma a(t)^{(4-d)/2} \langle \xi_{z/(a(t))}, (\tau, \psi)^2 \rangle > 1 \right]. \tag{4.23}
\]

With this in hand, the upper bound for (4.3) can be reduced to a standard Gaussian supremum concentration bound (e.g., Lemma A.1) and the following estimates:

**Proposition 4.11** For every \( \kappa_1, \kappa_2 > 0 \), it holds that

\[
\sup_{\psi \in W(Q_{z+1})} E \left[ \sigma^2 a(t)^{(4-d)} \langle \xi_{z/(a(t))}, \psi^2 \rangle^2 \right] \leq \sigma^2 a(t)^{4-d} \sup_{\psi \in W(\mathbb{R}^d)} \| \psi \|_4^4 < \infty \tag{4.24}
\]

and

\[
\lim_{t \to \infty} E \left[ \sup_{\psi \in W(Q_{z+1})} \sigma a(t)^{(4-d)/2} \langle \xi_{z/(a(t))}, \psi^2 \rangle \right] = 0. \tag{4.25}
\]

To see this, let us henceforth denote for simplicity,

\[
\varsigma := \sup_{\psi \in W(\mathbb{R}^d)} \| \psi \|_4
\]

as well as the median of the supremum

\[
m_t := \text{Med} \left[ \sup_{\psi \in W(Q_{z+1})} \sigma a(t)^{(4-d)/2} \langle \xi_{z/(a(t))}, (\tau, \psi)^2 \rangle \right]
\]

and maximal standard deviation

\[
v_t = \sup_{\psi \in W(Q_{z+1})} E \left[ \sigma^2 a(t)^{(4-d)} \langle \xi_{z/(a(t))}, \psi^2 \rangle^2 \right]^{1/2}.
\]

Then, it follows from (A.2) in Lemma A.1 that

\[
m_t = E \left[ \sup_{\psi \in W(Q_{z+1})} \sigma a(t)^{(4-d)/2} \langle \xi_{z/(a(t))}, \psi^2 \rangle \right] + O(v_t) = o(1)
\]

for any \( \kappa_1, \kappa_2 > 0 \), where the last equality follows from Proposition 4.11 and the fact that \( a(t)^{4-d} = o(1) \). In particular, we can find \( t > 0 \) large enough so that

\[
- \frac{(1-m_t)^2}{2v_t^2} = - \frac{1 + o(1)}{2v_t^2} \leq - \frac{1 + o(1)}{2\sigma^2 a(t)^{4-d}} = - \frac{1 + o(1)}{2\sigma^2 a} \kappa_1 \log r(t),
\]

where the third inequality follows from (4.24) and the last equality from the definition of \( a(t) \) in (4.15). Thus, an application of Lemma A.1 to the present setting with \( \lambda = 1 \) implies that

\[
P \left[ \sup_{\psi \in W(Q_{z+1})} \sigma a(t)^{(4-d)/2} \langle \xi_{z/(a(t))}, (\tau, \psi)^2 \rangle > 1 \right] \leq \exp \left( - \frac{1 + o(1)}{2\sigma^2 a} \kappa_1 \log r(t) \right).
\]
Consequently, for every choice of $\kappa_1 > 2d\sigma^2$s and $\kappa_2 > 0$, a combination of the above bound with Remark 4.10 and (4.23) yields

$$
P \left[ \max_{z \in \mathbb{Z}, \varphi \in S(Q_{k_2+1})} \left( \sigma a(t)^{1-d/2} \langle \xi_{\varepsilon(t)/a(t)}, (\tau, \varphi)^2 - \frac{1}{2} \varepsilon(\varphi) \rangle > 1 \right) \right] 
\leq c_1 r(t)^d (\log r(t))^{c_2} e^{-\delta d \log r(t)} = c_1 r(t)^d (1-\delta) (\log r(t))^{c_2}
$$

for some $\delta > 1$ close enough to 1. For any sequence of $t > 0$ going to infinity, by (4.1) we can always extract a sparse enough subsequence $(t_n)_{n \in \mathbb{N}}$ such that

$$
\sum_{n \in \mathbb{N}} c_1 r(t_n)^d (1-\delta) (\log r(t_n))^{c_2} < \infty.
$$

Hence we obtain the following by an application of the Borel-Cantelli lemma: For every choice of $\kappa_1 > 2d\sigma^2$s and $\kappa_2 > 0$, every sequence of $t > 0$ such that $t \to \infty$ has a subsequence $(t_n)_{n \in \mathbb{N}}$ along which

$$
\limsup_{n \to \infty} \max_{z \in \mathbb{Z}, \varphi \in S(Q_{k_2+1})} \left( \sigma a(t)^{1-d/2} \langle \xi_{\varepsilon(t)/a(t)}, (\tau, \varphi)^2 - \frac{1}{2} \varepsilon(\varphi) \rangle \right) \leq 1
$$

almost surely. By (4.20), this then implies that

$$
\limsup_{n \to \infty} \frac{\Lambda_1(A_{\varepsilon(t_n)^2} Q_{r(t_n)})}{a(t_n)^{-2}} \leq \frac{C}{\kappa_2} + 1 \quad \text{almost surely},
$$

which, by taking $\kappa_1 \to 2d$s and $\kappa_2 \to \infty$ and (and further subsequences of $t_n$ if needed), yields

$$
\limsup_{n \to \infty} \frac{\Lambda_1(A_{\varepsilon(t_n)^2} Q_{r(t_n)})}{(\log r(t_n))^{2/(4-d)}} \leq (2d\sigma^2s)^{2/(4-d)} \quad \text{almost surely}.
$$

This then provides an upper bound for (4.3) by Lemma A.4. We now conclude the proof of Theorem 4.1 by proving Proposition 4.11:

**Proof of Proposition 4.11.** We begin by proving (4.24). By definition,

$$
E \left[ \langle \xi_{\varepsilon(t)/a(t)}, \psi^2 \rangle^2 \right] = \| \psi^2 \|_{R_{\varepsilon(t)/a(t)}}^4
$$

for every $\psi \in W(Q_{k_2+1})$. Since $R_{\varepsilon}$ integrates to one for every $\varepsilon > 0$, it follows from Young’s convolution inequality that $\| \psi^2 \|_{R_{\varepsilon(t)/a(t)}}^2 \leq \| \psi \|_{4}^4$. We then get (4.24) by the trivial bound

$$
\sup_{\psi \in W(Q_{k_2+1})} \| \psi \|_{4}^4 \leq \sup_{\psi \in W(\mathbb{R}^d)} \| \psi \|_{4}^4.
$$

(The fact that the above is finite is proved in Lemma A.4.)

We now prove (4.25). Let us define the pseudometrics

$$
P_t(\psi, \tilde{\psi}) := a(t)^{1-d/2} E \left[ \langle \xi_{\varepsilon(t)/a(t)}, \psi^2 - \tilde{\psi}^2 \rangle \right]^{1/2}, \quad \psi, \tilde{\psi} \in W(Q_{k_2+1})
$$

for $t \geq 0$. Since $a(t)^{1-d/2} = (\kappa_1 \log r(t))^{-1/2}$, we have that

$$
P_t(\psi, \tilde{\psi}) = (\kappa_1 \log r(t))^{-1/2} \| \psi^2 - \tilde{\psi}^2 \|_{R_{\varepsilon(t)/a(t)}}.
$$
For every $\zeta > 0$, let us denote by $N_{\zeta}(\zeta)$ the covering number of $W(Q_{\zeta+1})$ with open balls of radius $\zeta$ in $P$. By Dudley’s theorem (e.g., [34, Theorem 11.17]), to prove (4.25) it is enough to show that

$$\lim_{t \to \infty} \int_0^\infty \sqrt{\log N_{\zeta}(\zeta)} \, d\zeta = 0. \quad (4.27)$$

We first prove (4.27) in the case $d = 1$, where we recall that we impose no lower bound on $e(t)$ in the singular phase. Let us define

$$P_\varepsilon(\psi, \tilde{\psi}) = \|\psi^2 - \tilde{\psi}^2\|_2, \quad \psi, \tilde{\psi} \in W(Q_{\zeta+1}).$$

We note that this is the pseudometric associated to the one-dimensional Gaussian white noise. By Young’s convolution inequality, we have that

$$P_t(\psi, \tilde{\psi}) \leq (\kappa_1 \log t)^{-1/2} P_\varepsilon(\psi, \tilde{\psi}), \quad t \geq 1, \ \psi, \tilde{\psi} \in W(Q_{\zeta+1}).$$

Thus, if we let $N_{\zeta}(\zeta)$ denote the covering number of $W(Q_{\zeta+1})$ by $\zeta$-balls in $P_\varepsilon$, we have the inequality

$$\int_0^\infty \sqrt{\log N_{\zeta}(\zeta)} \, d\zeta \leq (\kappa_1 \log t)^{-1/2} \int_0^\infty \sqrt{\log N_{\zeta}(\zeta)} \, d\zeta.$$ 

Thanks to [11, (2.7)], we know that $\int_0^\infty \sqrt{\log N_{\zeta}(\zeta)} \, d\zeta < \infty$, hence the result.

**Remark 4.12** In the paper [11], the one-dimensional Gaussian white noise is referred throughout as the “context of Theorem 1.4”. The space $W(\Omega)$ for $\Omega \subset \mathbb{R}^d$ is denoted by $\mathcal{D}(\Omega)$ (see [11, (2.2)]). We note that the argument used to prove [11, (2.7)] cannot be extended to $d = 2, 3$, since a crucial assumption in that result (i.e., [11, (1.9)]) does not hold for Gaussian white noise in $d > 1$.

We now prove (4.27) in the case $d = 2, 3$.

**Definition 4.13** To improve readability, for every $\psi \in W(Q_{\zeta+1})$ and $\varepsilon > 0$, we denote $\psi_\varepsilon^2 := \psi^2 \ast \tilde{R}_\varepsilon$ for the remainder of this proof.

We begin by bounding the upper limit of integration in (4.27). Recalling that, for every $\varepsilon > 0$, $R_\varepsilon = \tilde{R}_\varepsilon$ in $d$ is even, we can write

$$\|\psi^2 - \tilde{\psi}^2\|_{c(\varepsilon)/a(t)} = \|\psi^2_{c(\varepsilon)/a(t)} - \tilde{\psi}^2_{c(\varepsilon)/a(t)}\|_2 \leq \|\psi^2 - \tilde{\psi}^2\|_2 \leq \|\psi\|_2^2 + \|\tilde{\psi}\|_2^2,$$

where the first inequality follows from Young’s convolution inequality. Then, by the GNS inequality (1.13) and the fact that $\|\psi\|_2^2 \approx \sigma(\psi) \leq 1$ for all $\psi \in W(Q_{\zeta+1})$, we have that

$$\|\psi^2 - \tilde{\psi}^2\|_{c(\varepsilon)/a(t)} \leq 2^{1+d/4} \sigma_{d/2}^2.$$ 

Thus,

$$\int_0^\infty \sqrt{\log N_{\zeta}(\zeta)} \, d\zeta = \int_0^{O((\log r(t))^{-1/2})} \sqrt{\log N_{\zeta}(\zeta)} \, d\zeta. \quad (4.28)$$

Since $\varepsilon(t) \ll a(t)$ and $\tilde{R}$ is compactly supported, we can fix a $\kappa > \kappa_2$ such that $\psi^2_{c(\varepsilon)/a(t)} \in C^\infty_0(Q_\kappa)$ for every $\psi \in W(Q_{\zeta+1})$ and $t \geq 0$. In order to estimate the covering number $N_{\zeta}$, we make use of an $\varepsilon$-net argument using the following projections:

**Definition 4.14** For every $\mu, \nu, M > 0$ and nonnegative $\varphi \in C^\infty_0(\mathbb{R}^d)$, define

$$\Pi_{\mu, \nu, M}(\varphi) := \sum_{z \in 2\nu \mathbb{Z}^d} \min \{ [\varphi(z)]_\mu, M \} 1_{z \in \epsilon(-\nu, \nu)^d},$$

where $[x]_\mu := \max \{ y \in \mathbb{M} : y \leq x \}$ for every $x \in \mathbb{R}$. 
Remark 4.15 The image of all nonnegative \( \varphi \in C^0_0(Q_{<}) \) through \( \Pi_{\mu, v, M} \) has cardinality of order \( (M/\mu)^{O(v-d)} = e^{O(v-d \log(M/\mu))} \) as \( \mu, v \to 0 \) and \( M \to \infty \).

We claim that we have the inequality
\[
\sup_{\psi \in W(Q_{<+1})} \| \psi^2_{E(t)/a(t)} - \Pi_{\mu, v, M}(\psi^2_{E(t)/a(t)}) \|_2^2 \\
\leq C \left( \mu + (e(t)/a(t))^{-(d+h)}v^h + M^{-1/2} \right), \tag{4.29}
\]
where \( h > 0 \) is the Hölder exponent in (1.8), and the constant \( C \) only depends on \( d, \kappa \), and the Hölder constant \( C > 0 \) in (1.8). In order to prove (4.29), we use the following decomposition:
\[
\| \psi^2_{E(t)/a(t)} - \Pi_{\mu, v, M}(\psi^2_{E(t)/a(t)}) \|_2^2 = \int_{\{\psi^2_{E(t)/a(t)} > M\}} (\psi^2_{E(t)/a(t)}(x) - M)^2 \, dx \\
+ \int_{\{0 < \psi^2_{E(t)/a(t)} \leq M\}} (\psi^2_{E(t)/a(t)}(x) - \Pi_{\mu, v, M}(\psi^2_{E(t)/a(t)}))(x)^2 \, dx. \tag{4.30}
\]

Definition 4.16 In what follows, we use \( C > 0 \) to denote positive constants that (possibly) only depend on \( d, \kappa \), and the \( C \) in (1.8), and whose exact values may change from line to line.

We begin by controlling the first term on the right-hand side of (4.30). Given that \( (a+b)^2 \leq 2(a^2 + b^2) \), we have
\[
\int_{\{\psi^2_{E(t)/a(t)} > M\}} (\psi^2_{E(t)/a(t)}(x) - M)^2 \, dx \\
\leq 2 \int_{\{\psi^2_{E(t)/a(t)} > M\}} \psi^2_{E(t)/a(t)}(x)^2 \, dx + 2M^2 \int_{\{\psi^2_{E(t)/a(t)} > M\}} \, dx.
\]
An application of Young’s convolution inequality followed by the general \( L^p \)-GNS inequality (e.g., [10, (C.1)]) implies that
\[
\int_{\mathbb{R}^d} \psi^2_{E(t)/a(t)}(x)^3 \, dx \leq \| \psi \|_6^6 \leq C \| \psi \|_2^{\frac{6}{\beta}} C^{\frac{1}{\beta}} (\psi^\beta) \leq C,
\]
where \( p, \beta \geq 0 \) only depend on \( d \). Therefore, by Hölder’s and Markov’s inequalities,
\[
\int_{\{\psi^2_{E(t)/a(t)} > M\}} \psi^2_{E(t)/a(t)}(x)^2 \, dx \leq \left( \int_{\mathbb{R}^d} \psi^2_{E(t)/a(t)}(x)^3 \, dx \right)^{2/3} \left( \int_{\{\psi^2_{E(t)/a(t)} > M\}} \psi^2_{E(t)/a(t)}(x)^2 \, dx \right)^{1/3} \\
\leq C (M^{-3} \int_{\mathbb{R}^d} \psi^2_{E(t)/a(t)}(x)^3 \, dx \right)^{1/3} \leq CM^{-1}. \tag{4.31}
\]
Applying Markov’s inequality once again, we have
\[
M^2 \int_{\{\psi^2_{E(t)/a(t)} > M\}} \, dx \leq M^{-1} \int_{\mathbb{R}^d} \psi^2_{E(t)/a(t)}(x)^3 \, dx \leq CM^{-1}. \tag{4.32}
\]
We now control the second term on the right-hand side of (4.30). For any \( x, y \in \mathbb{R}^d \), since \( \tilde{R} \) is Hölder continuous of order \( h \) and \( \| \psi \|_2^2 \leq 1 \), one has

\[
|\psi_{\varepsilon(t)/a(t)}(x) - \psi_{\varepsilon(t)/a(t)}(y)| = \left| \int_{\mathbb{R}^d} (\tilde{R}_{\varepsilon(t)/a(t)}(x-z) - \tilde{R}_{\varepsilon(t)/a(t)}(y-z)) \psi(z)^2 \, dz \right| \\
\leq C(\varepsilon(t)/a(t))^{-(d+h)} |x-y|^h_2.
\]

Thus, if \( x \) is such that \( |\psi_{\varepsilon(t)/a(t)}(x)| \leq M \) and \( x \in z + [-\nu, \nu]^d \) for some \( z \in 2\nu \mathbb{Z}^d \),

\[
\Pi_{\mu, \nu, M}(\psi_{\varepsilon(t)/a(t)})(x) = |\psi_{\varepsilon(t)/a(t)}(z)| \mu;
\]

hence

\[
|\psi_{\varepsilon(t)/a(t)}(x) - \Pi_{\mu, \nu, M}(\psi_{\varepsilon(t)/a(t)})(x)| \\
\leq |\psi_{\varepsilon(t)/a(t)}(x) - \psi_{\varepsilon(t)/a(t)}(z)| + |\psi_{\varepsilon(t)/a(t)}(z) - |\psi_{\varepsilon(t)/a(t)}(z)| \mu| \\
\leq C \left( (\varepsilon(t)/a(t))^{-(d+h)} \nu^h + \mu \right).
\]

Since \( \{ 0 < \psi_{\varepsilon(t)/a(t)}(x) \leq M \} \subset \text{supp}(\psi_{\varepsilon(t)/a(t)}) \subset Q_K \), we have that

\[
\int_{\{ 0 < \psi_{\varepsilon(t)/a(t)} \leq M \}} (|\psi_{\varepsilon(t)/a(t)}(x) - \Pi_{\mu, \nu, M}(\psi_{\varepsilon(t)/a(t)})(x)|)^2 \, dx \\
\leq C \left( (\varepsilon(t)/a(t))^{-(d+h)} \nu^h + \mu \right)^2.
\]

If we combine the above with (4.31) and (4.32), we conclude that (4.29) holds.

With (4.29) established, we are now ready to conclude the proof of (4.27): Let \( C \) be the constant on the right-hand side of (4.29). Suppose that we take

\[
\mu \leq \frac{\zeta}{6C}, \quad (\varepsilon(t)/a(t))^{-(d+h)} \nu^h \leq \frac{\zeta}{6C}, \quad M^{-1/2} \leq \frac{\zeta}{6C},
\]

which is equivalent to

\[
\mu \leq \frac{\zeta}{6C}, \quad \nu \leq \left( \frac{\zeta (\varepsilon(t)/a(t))^{d+h}}{6C} \right)^{1/h}, \quad M \geq \left( \frac{\zeta}{6C} \right)^{-2}.
\]

Then, (4.29) implies that

\[
\sup_{\psi \in W(Q_{\kappa_2+1})} \| \psi_{\varepsilon(t)/a(t)} - \Pi_{\mu, \nu, M}(\psi_{\varepsilon(t)/a(t)}) \|_2 \leq \frac{\zeta}{2},
\]

and thus any two \( \psi, \tilde{\psi} \in W(Q_{\kappa_2+1}) \) such that \( \Pi_{\mu, \nu, M}(\psi_{\varepsilon(t)/a(t)}) = \Pi_{\mu, \nu, M}(\tilde{\psi}_{\varepsilon(t)/a(t)}) \) will, by the triangle inequality, satisfy \( R(\psi, \tilde{\psi}) \leq (\kappa_1 \log r(t))^{-1/2} \zeta \). Therefore, it follows from Remark 4.15 that, as \( \zeta \to 0 \),

\[
\sqrt{\log N_l(\zeta/(\kappa_1 \log r(t)))^{1/2}} = \sqrt{\log(e^{O(\nu^{-d} \log(\nu/M \mu))})} \\
= O \left( \sqrt{\frac{\zeta^{-d/h} (\varepsilon(t)/a(t))^{-(d+h)}}{\log(6C/\zeta)}} \right).
\]

Consequently, by a change of variables, we are led to the asymptotic

\[
\int_0^{O((\log r(t))^{-1/2})} \sqrt{\log N_l(\zeta)} \, d\zeta \\
= O \left( (\log r(t))^{-1/2} (\varepsilon(t)/a(t))^{-d^2/2h-d/2} \int_0^{1} \sqrt{\frac{\zeta^{-d/h} (\varepsilon(t)/a(t))^{-(d+h)}}{\log(6C/\zeta)}} \, d\zeta \right)
\]

(4.35)
as \( t \to \infty \). Assuming \( 6C \geq 1 \) (which we can always ensure up to increasing the value of \( C \) in the upper bound (4.29)) and \( h > d/4 \), the integral on the right-hand side of (4.35) is real and finite. (Indeed, \( \sqrt{\xi^{-d/h}} = \xi^{-1} \) when \( h = d/4 \).) Therefore,
\[
\int_0^\infty \sqrt{\log N_i(\xi)} \, d\xi = O\left( \left( \log r(t) \right)^{-1/2} / a(t) \right)^{-d/2h-d/2}.
\]

Recalling that \( a(t) = (\kappa_1 \log r(t))^{1/(4-d)} \), and setting \( \varepsilon(t) \ll (\log r(t))^{-\vartheta} \) for some \( \vartheta > 0 \), we obtain that
\[
\int_0^\infty \sqrt{\log N_i(\xi)} \, d\xi = o\left( \left( \log r(t) \right)^{-\vartheta} \right).
\]

This vanishes so long as
\[
\vartheta \left( \frac{d^2 + d}{2h} \right) - \frac{d^2 + 4h}{8h - 2dh} < 0 \iff \vartheta < \frac{d^2 + 4h}{4 - d(d + h)} = \frac{1}{4 - d} + \frac{h}{d(d + h)}.
\]

By definition of the the singular phase when \( d = 2, 3 \) (in particular (1.12) and the requirement \( h > d/4 \)), this concludes the proof of Proposition 4.11.

\[\square\]

5 Quenched Total Mass Asymptotics

In this section, we prove (1.11) and (1.16). We begin with some preliminary technical results in Section 5.1, and then prove the result in two steps in Sections 5.2 and 5.3.

5.1 Preliminary Estimates

**Proposition 5.1** Let the function \( \eta : [0, \infty) \to (0, \infty) \) be such that
\[
\eta(t) = \begin{cases} 
o(1) & d = 1 
o(\varepsilon(t)^{d/2}) & d = 2, 3, 
\end{cases} \quad t \to \infty.
\]

For every \( \theta > 0 \), it holds in both regular and singular phases that
\[
\lim_{t \to \infty} U^{(\theta)}(\eta(t)) = 1 \quad \text{in probability},
\]

where we recall the definition of \( U^{(\theta)} \) in Proposition 3.9.

**Proof.** Since the limit is constant it suffices to show convergence in distribution. Moreover, since constants are determined by their moments, it suffices to prove convergence of the first two moments. For \( n = 1, 2 \), it follows from Fubini’s theorem and (1.4) that
\[
E \left[ t^{(\theta)}_{\varepsilon(t)}(\eta(t))^n \right] = E \left[ E_{\varepsilon(t)} \left[ \exp \left( \theta \sum_{i=1}^n \int_0^{\eta(t)} \xi(\eta(t))(B'(s)) \, ds \right) \right] \right],
\]

where \( (B')_{1 \leq i \leq n} \) are i.i.d. standard Brownian motions started at zero, and \( E_{\varepsilon(t)} \) denotes expectation with respect to \( \varepsilon(t) \) conditional on the \( B' \). Conditional on a fixed realization of the paths of \( B' \), the sum of integrals
\[
\theta \sum_{i=1}^n \int_0^{\eta(t)} \varepsilon(t)(B'(s)) \, ds
\]
is Gaussian with mean zero and variance

\[ \theta^2 \sum_{i,j=1}^{n} \int_{0, \eta(t)|^2} R_{\varepsilon(t)}(B^i(u) - B^j(v)) \, dudv, \quad n = 1, 2. \]  

(5.3)

We begin with the proof in the case \( d \in \{2, 3\} \). Since \( R \) is a positive semidefinite function, \( R_{\varepsilon(t)} \leq R_{\varepsilon(t)}(0) = \varepsilon(t)^{-d} R(0) \). In particular, (5.3) is bounded above by \( R(0) \theta^2 n^2 \eta(t)^2 \varepsilon(t)^{-d} \); hence

\[ 1 \leq E \left[ e^{\left( \int_{0, \eta(t)}^t \eta(r)^2 \right)} \right] = e^{O(\eta(t)/\varepsilon(t)^{d/2})^2} = e^{o(1)} = 1 + o(1), \]

as desired.

We now settle the case \( d = 1 \). For every \( 1 \leq i \leq n \), let \((L^i_j(x))_{t \geq 1, x \in \mathbb{R}}\) denote the continuous version of the local time process of \( B^i \) (e.g., [42, Chapter VI]), so that

\[ \int_{0, \eta(t)|^2} R_{\varepsilon(t)}(B^i(u) - B^j(v)) \, dudv = \int_{\mathbb{R}^2} L^i_{\eta(t)}(x) R_{\varepsilon(t)}(x-y) L^j_{\eta(t)}(y) \, dx dy. \]

Since \( R_{\eta} \) integrates to one for all \( \eta > 0 \), it then follows from Young’s convolution inequality that the variance in (5.3) is bounded above by

\[ \theta^2 \sum_{i,j=1}^{n} \|L^i_{\eta(t)}\|^2 \|L^j_{\eta(t)}\|^2 \leq 2 \theta^2 \sum_{i,j=1}^{n} (\|L^i_{\eta(t)}\|^2_2 + \|L^j_{\eta(t)}\|^2_2). \]  

(5.4)

By Brownian scaling, \( \|L^i_{\eta(t)}\|^2_2 \overset{\text{dist}}{=} \eta(t)^{3/2} \|L^i_t\|^2_2 \) for all \( t \geq 1 \) (e.g., [10, (2.3.8) and Proposition 2.3.5 with \( d = 1 \) and \( p = 2 \)). Thus, (5.4) converges to zero in probability. Given that \( \|L^i_t\|^2_2 \) have finite exponential moments of all orders (e.g., [10, Theorem 4.2.1 with \( p = 2 \)), it follows from the Vitali convergence theorem that

\[ \lim_{t \to \infty} \mathbb{E} \left[ \exp \left( \eta(t)^{3/2} \theta^2 \sum_{i,j=1}^{n} (\|L^i_{\eta(t)}\|^2_2 + \|L^j_{\eta(t)}\|^2_2) \right) \right] = 1, \]

concluding the proof. \( \square \)

**Proposition 5.2** There exists a constant \( C > 0 \) such that for every \( \theta, t, r > 0 \),

\[ \mathbb{E} \left[ e^{\Lambda_1 (A^{(\theta)}_{\varepsilon(t)}, Q_r)} \right] \leq \begin{cases} \frac{2 \pi e^{C \theta^{1/3}}}{\sqrt{2 \pi |x|}} & d = 1 \\ \frac{(2\pi)^d e^{C \theta^2 \varepsilon(t)^{-d/2}}}{(2\pi)^{d/2} e^{C \theta^2 \varepsilon(t)^{-d/2}}} & d = 2, 3. \end{cases} \]

in both regular and singular phases.

**Proof.** Thanks to (3.10) and (3.11), we note that

\[ e^{\Lambda_1 (A^{(\theta)}_{\varepsilon(t)}, Q_r)} \leq \sum_{k=1}^{\infty} e^{\Lambda_k (A^{(\theta)}_{\varepsilon(t)}, Q_r)} \]

\[ = \frac{1}{(2\pi)^{d/2}} \int_{Q_r} \mathbb{E}_E^{t,s} \left[ \exp \left( \theta \int_0^t \xi_{\varepsilon(t)}(B(s)) \, ds \right) ; T_{Q_r} \geq t \right] \, dx. \]

Once again employing Fubini’s theorem as in (5.2) and (5.3), this yields

\[ \mathbb{E} \left[ e^{\Lambda_1 (A^{(\theta)}_{\varepsilon(t)}, Q_r)} \right] \]

\[ \leq \frac{1}{(2\pi)^{d/2}} \int_{Q_r} \mathbb{E}_E^{t,s} \left[ \exp \left( \frac{\theta^2}{2} \int_{[0,\varepsilon(t)]^2} R_{\varepsilon(t)}(B(u) - B(v)) \, dudv \right) \right] \, dx. \]
Given that the functional \( \int_{[0,t]} R_{\varepsilon(t)}(B(u) - B(v)) \, du \, dv \) is invariant with respect to the starting point of \( B \), we finally get the upper bound
\[
E_t E_0^{\theta_0} \left[ \exp \left( \frac{\theta^2}{2} \int_{[0,t]} R_{\varepsilon(t)}(B(u) - B(v)) \, du \, dv \right) \right] \leq \frac{(2\pi t)^d}{(2\pi t)^{d/2}} E_0^{\theta_0} \left[ \exp \left( \frac{\theta^2}{2} \int_{[0,t]} R_{\varepsilon(t)}(B(u) - B(v)) \, du \, dv \right) \right].
\] (5.5)

In the case where \( d = 2, 3 \), the result then follows from the trivial bound
\[
\int_{[0,t]} R_{\varepsilon(t)}(B(u) - B(v)) \, du \, dv \leq R(0)t^2 e(t)^{-d}.
\]

We now consider the case \( d = 1 \). Using the same local time estimates leading up to (5.4), we have the upper bound
\[
E_t E_0^{\theta_0} \left[ \exp \left( \frac{\theta^2}{2} \int_{[0,t]} R_{\varepsilon(t)}(B(u) - B(v)) \, du \, dv \right) \right] \leq E_t E_0^{\theta_0} \left[ e^{\theta^2 \|L\|_2^2/2} \right].
\]

According to [14, Lemma 2.2 in the case \( d = 1 \) and \( R = \delta_0 \)], for every \( \vartheta > 0 \),
\[
\log E_0^{\theta} \left[ e^{\vartheta \|L\|_2^2} \right] = O(\vartheta^2 t^3) \quad \text{as } t \to \infty.
\]

Then, by arguing as in the last paragraph of the proof of [24, Lemma 5.11] (see also [24, (5.15) and (5.17)–(5.19)]), we have the bound
\[
E_0^{\theta} \left[ e^{\vartheta \|L\|_2^2/2} \right] = O \left( E_0^{\theta} \left[ e^{2\vartheta \|L\|_2^2/2} \right] \right),
\]
thus concluding the proof for \( d = 1 \). \( \Box \)

5.2 Upper Bounds for (1.11) and (1.16)

For every \( k \in \mathbb{N} \) and \( t \geq 0 \), define
\[
r_k(t) := \begin{cases} \left( t e(t) \right)^{-d/2} \sqrt{\log t} & \text{if } \varepsilon(t) \text{ is in the regular phase}, \\ \left( t (\log t)^{2(4-d)} \right)^{k} & \text{if } \varepsilon(t) \text{ is in the singular phase.} \end{cases}
\] (5.6)

It is clear that, for large enough \( t \), \( r_k(t) < r_{k+1}(t) \) for all \( k \in \mathbb{N} \). Consequently, following [20, (4.24)] (see also [12, Pages 596–597]), we have the decomposition
\[
U_{\varepsilon(t)}(t) = E_t \left[ \exp \left( \int_0^t \xi_{\varepsilon(t)}(B(s)) \, ds \right) : T_{Q_{\varepsilon(t)}} \geq t \right] + \sum_{k=1}^{\infty} E_t \left[ \exp \left( \int_0^t \xi_{\varepsilon(t)}(B(s)) \, ds \right) : T_{Q_{\varepsilon(t)}} \leq t \leq T_{Q_{\varepsilon(t+1)}} \right].
\] (5.7)

We begin by controlling the first term on the right-hand side of (5.7). For the remainder of Section 5.2, let us fix a some small constant \( \theta > 0 \) (precisely how small will be determined later in this proof). By applying (3.12) with \( r = r_k(t) \) and \( \eta = t^{-\theta} \), and then following this up by (3.13) with \( r = r_k(t), \; \eta = t^{-\theta}, \; \text{and } \theta = \rho \), we obtain the upper bound
\[
E_t \left[ \exp \left( \int_0^t \xi_{\varepsilon(t)}(B(s)) \, ds \right) : T_{Q_{\varepsilon(t)}} \geq t \right] 
\leq U_{\varepsilon(t)}^{(q)}(t^{-\theta})^{d/2} \left( 2\pi t^{-\theta} \right)^{-d/2} \left( 2r_k(t) \right)^{d/p} e^{(t-t^{-\theta})A \left( \frac{\rho}{\theta} \right) / p}. \] (5.8)
for every $p, q > 1$ such that $1/p + 1/q = 1$. Since $\varepsilon(t) \gg (\log t)^{-1/(4-d)-\delta} \gg t^{-\delta}$ for all $\delta > 0$ when $d = 2, 3$, it follows from Proposition 5.1 that

$$\lim_{t \to \infty} \log U_{\varepsilon(t)}(t^{-\delta})^{1/q} = 0 \quad \text{in probability}.$$ 

By definition of $r_1(t)$, we have that

$$\lim_{t \to \infty} \frac{\log (2\pi t^{-\theta})^{-d/2p}}{t} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{\log r_1(t)}{t} = 0.$$ 

Finally, noting that $t - t^{-\theta} = t\left(1 + \alpha(1)\right)$, and that $r_1(t)$ is of the form (4.1) with $\alpha = 1$, it follows from Theorem 4.1 and Remark 4.5 that

$$\lim_{t \to \infty} \frac{\log e^{(t-t^{-\theta})A_1(\varepsilon(t), Q_1(t))/p}}{t \varepsilon(t)^{-d/2} \sqrt{\log t}} = \sqrt{2dR(0)} \quad \text{in probability}$$

in the regular phase and

$$\lim_{t \to \infty} \frac{\log e^{(t-t^{-\theta})A_1(\varepsilon(t), Q_1(t))/p}}{t (\log t)^{2/(4-d)}} = p^{4/(4-d)-1} \Sigma_d \quad \text{in probability}$$

in the singular phase. Combining these limits with (5.8) and then taking $p \to 1$, we obtain the following statement:

**Proposition 5.3** Every sequence of $t > 0$ such that $t \to \infty$ has a subsequence $(t_n)_{n \in \mathbb{N}}$ along which the following almost-sure limits hold:

$$\limsup_{n \to \infty} \frac{\log E^0 \left\{ \exp \left( \int_0^{t_n} \xi_{\varepsilon(t_n)}(B(s)) \, ds \right) : T_{Q_1(t_n)} \geq t_n \right\}}{t_n \varepsilon(t_n)^{-d/2} \sqrt{\log t_n}} \leq \sqrt{2dR(0)}$$

in the regular phase, and

$$\limsup_{n \to \infty} \frac{\log E^0 \left\{ \exp \left( \int_0^{t_n} \xi_{\varepsilon(t_n)}(B(s)) \, ds \right) : T_{Q_1(t_n)} \geq t_n \right\}}{t_n (\log t_n)^{2/(4-d)}} \leq \Sigma_d$$

in the singular phase.

With Proposition 5.3 in hand, in order to complete the proof of the upper bounds for (1.11) and (1.16), it is enough to show that the sum on the second line of (5.7) converges to zero in probability. By a straightforward application of Hölder’s inequality, this sum is bounded above by

$$\sum_{k=1}^{\infty} \mathbb{P} \left[ \sup_{s \leq t} |B(s)| > r_k(t) \left| B(0) = 0 \right. \right]^{1/2} \cdot E^0 \left[ \exp \left( 2 \int_0^t \xi_{\varepsilon(t)}(B(s)) \, ds \right) : T_{Q_{k+1}(t)} \geq t \right]^{1/2}.$$  \hspace{1cm} (5.9)

Since Brownian motion suprema have sub-Gaussian tails, there exists a $c > 0$ independent of $t$ and $k$ such that, for large enough $t \geq 1$,

$$\mathbb{P} \left[ \sup_{s \leq t} |B(s)| > r_k(t) \left| B(0) = 0 \right. \right]^{1/2} \leq e^{-cr_k(t)^2/t}.$$
Combining this with the upper bound used in (5.8), but replacing \( r(t) \) by \( r_{k+1}(t) \) and \( \xi(t) \) by \( 2\xi(t) \), we then obtain that (5.9) is bounded above by

\[
U^{(2p)}_t (t^{-\vartheta})^{1/2} q \sum_{k=1}^{\infty} (2\pi t^{-\vartheta})^{-d/4p} (2r_{k+1}(t)) \left( \frac{d}{2} \int \right)^{1/2} \left( r_{k+1}(t) \right)^{2p-2\vartheta} / t \quad (5.10)
\]

for any \( p, q > 1 \) such that \( 1/p + 1/q = 1 \). By Proposition 5.1, it suffices to prove that the sum in (5.10) converges to zero in probability. We analyze the terms \( k = 1 \) and \( k \geq 2 \) in this sum separately.

For the term \( k = 1 \), we note that \( r_2(t) \) is of the form (4.1) with exponent \( \alpha = 2 \). Thus, by Theorem 4.1, there exists a random \( \epsilon > 0 \) independent of \( t \) such that for any sparse diverging sequence \( \{t_n\}_{n \in \mathbb{N}} \), we have that

\[
\frac{(t_n - t_n^{-\vartheta}) A_1(A_{\epsilon(t)}^{(2p)} Q_{r_2(t)})}{c r_1(t_n)^2} \leq \begin{cases} 
\frac{t_n c \epsilon(t_n)^{-d/2} \sqrt{\log t_n} - c \epsilon(t_n)^{-d} \log t_n}{t_n} & \text{if } \epsilon(t) \text{ is in the regular phase,} \\
\frac{t_n (\epsilon(t_n))^{2/(4-d)} - c (\log t_n)^{1/(4-d)}}{t_n} & \text{if } \epsilon(t) \text{ is in the singular phase.}
\end{cases}
\]

In particular, for every \( \kappa > 0 \),

\[
(2\pi t_n^{-\vartheta})^{-d/4p} (2r_2(t_n))^{d/2} c (t_n - t_n^{-\vartheta}) A_1(A_{\epsilon(t)}^{(2p)} Q_{r_2(t)}) / 2p - c r_1(t_n)^2 / t_n
\]

almost surely as \( n \to \infty \). By definition of \( r_2(t) \), this vanishes as \( n \to \infty \), and thus the \( k = 1 \) term in the sum in (5.10) converges to zero in probability as \( t \to \infty \).

We now deal with the terms \( k \geq 2 \). By Proposition 5.2 (and \( t - t^{-\vartheta} \leq t \)), there exists a constant \( C > 0 \) such that

\[
E \left[ \sum_{k=2}^{\infty} (2\pi t^{-\vartheta})^{-d/4p} (2r_{k+1}(t))^{d/2} c (t^{-t^{-\vartheta}} A_1(A_{\epsilon(t)}^{(2p)} Q_{r_{k+1}(t)}) / 2p - c r_1(t)^2 / t) \right]
\]

\[
= \begin{cases} 
O \left( \frac{t^{d/4p}}{t^{d/2}} \sum_{k=2}^{\infty} r_{k+1}(t)^{1+d/2} p C t^{-c r_2(t)^2 / t} \right) & \text{if } d = 1, \\
O \left( \frac{t^{d/4p}}{t^{d/2}} \sum_{k=2}^{\infty} r_{k+1}(t)^{d+d/2} p C \epsilon(t)^{-d c r_2(t)^2 / t} \right) & \text{if } d = 2, 3.
\end{cases}
\]

(5.11)

We begin by controlling the right-hand side of (5.11) in the case \( d = 1 \). We note that for every \( \kappa_0 > 0 \), if \( t > 0 \) is large enough, then \( -c r_1(t)^2 / t \leq -\kappa_0 t^{2k-1} \) for every \( k \in \mathbb{N} \). Given that \( C \) is bounded for all \( d \geq 1 \) and \( k \geq 2 \), for every \( \kappa > 0 \), we have that

\[
\sum_{k=2}^{\infty} r_{k+1}(t)^{1+1/2} p C r_1(t) \leq \sum_{k=2}^{\infty} r_{k+1}(t)^{1+1/2} p e^{-\kappa t^{2k-1}}
\]

for large enough \( t \). As \( r_k(t) = O(t^{\vartheta \kappa}) \) for some \( \vartheta > 0 \) independent of \( t \), this sum is uniformly bounded in \( t \gg 1 \). Thus, so long as we choose \( \vartheta > 0 \) small enough relative to \( p > 1 \) so that \( t^{\vartheta / 4p} = o(t^{1/2}) \), we get that (5.11) vanishes as \( t \to 0 \) for \( d = 1 \). For \( d = 2, 3 \), we use the same argument, noting that, since \( \epsilon(t) = (\log t)^{1/(4-d) - c} \gg t^{-\delta} \) for all \( \delta > 0 \),
\( \varepsilon(t)^3 dt^2 = O(t^3) \). In summary, the contribution of the terms \( k \geq 2 \) to the sum in (5.10) converges to zero in probability, which finally concludes the proof of the following statement: Every sequence of \( t > 0 \) such that \( t \to \infty \) has a subsequence \((t_n)_{n \in \mathbb{N}} \) along which

\[
U_{\varepsilon(t_n)}(t_n) \leq \begin{cases} 
\sqrt{2dR(0)t_n \varepsilon(t_n)^{-d/2} \sqrt{\log t_n}} (1 + o(1)) & \text{regular phase}, \\
\Sigma_d t_n (\log t_n)^{2/(4-d)} (1 + o(1)) & \text{singular phase}.
\end{cases} \tag{5.12}
\]

almost surely as \( n \to \infty \), providing upper bounds for (1.11) and (1.16).

### 5.3 Lower Bounds for (1.11) and (1.16)

We now conclude the proofs of (1.11) and (1.16) by providing matching lower bounds to (5.12). Let us fix some \( p, q > 1 \) such that \( 1/p + 1/q = 1 \) and some \( 0 < \vartheta < 1 \). By (3.14) with \( \eta = r = t^{\vartheta} \), we get

\[
U_{\varepsilon(t)}(t) \geq U_{\varepsilon(t)}^{(-q/p)}(t^\vartheta)^{-p/q} \cdot \left( \int_{Q_{t,\vartheta}} \mathcal{G}_{t,\vartheta}(x) \mathbb{P} \left[ \exp \left( \frac{1}{p} \int_0^{t-t^\vartheta} \xi_{\varepsilon(t)}(B(s)) \, ds \right) : T_{Q_{t,\vartheta}} \geq t-t^\vartheta \right] \, dx \right)^p. \tag{5.13}
\]

Firstly, we note that \( -\xi_{\varepsilon(t)} \) is equal in distribution to \( \xi_{\varepsilon(t)} \), and thus the asymptotics of \( U_{\varepsilon(t)}^{(-q/p)}(t^\vartheta)^{-p/q} \) are the same as that of \( U_{\varepsilon(t)}((t^\vartheta)^{-p/q} \). Secondly, it is easy to see that \( \varepsilon(t) \) is in the regular (resp. singular) phase if and only of \( \varepsilon(t) \) is in the regular (resp. singular) phase for every \( \vartheta > 0 \). Consequently, it follows from (5.12) that if the diverging sequence \((t_n)_{n \in \mathbb{N}} \) is sufficiently sparse, then

\[
\log \frac{U_{\varepsilon(t_n)}^{(-q/p)}(t_n^\vartheta)^{-p/q}}{t_n} = \begin{cases}
O \left( t_n^{\vartheta-1} \varepsilon(t_n)^{-d/2} \sqrt{\log t_n} \right) & \text{in the regular phase} \\
O \left( t_n^{\vartheta-1}(\log t_n)^{2/(4-d)} \right) & \text{in the singular phase}
\end{cases}
\]

almost surely as \( n \to \infty \). Since \( \vartheta < 1 \) this vanishes for large \( n \).

We now analyze the term on the second line of (5.13). It is easy to see that there exists a constant \( c > 0 \) such that \( \mathcal{G}_{t,\vartheta}(x) \geq ce^{-ct^\vartheta} \) for every \( x \in (-t^\vartheta, t^\vartheta)^d \). Thus, an application of (3.15) with \( f = t-t^\vartheta \), \( \eta = r = t^\vartheta \), and \( \vartheta = 1/p \) yields that the term on the second line of (5.13) is bounded below by the quantity

\[
\mathcal{F}_{p,q,\vartheta}(t) := c_\varepsilon e^{-c_\varepsilon t^\vartheta} (2\pi)^{d/2} t^d e^{t^\vartheta/2} (1-t^\vartheta)^d (2t^\vartheta)^{-2p^2/q} \cdot e^{-t^\vartheta(p^2/q)\Lambda_1(\Lambda_{\varepsilon(\vartheta)}^2) Q_{t,\vartheta}} e^{p^2 t^\vartheta \Lambda_1(\Lambda_{\varepsilon(\vartheta)}^2) Q_{t,\vartheta}}
\]

Since \( t^\vartheta \) is of the form (4.1) with exponent \( \alpha = \vartheta \), we conclude from Theorem 4.1 and Remark 4.5 that

\[
\lim_{t \to \infty} \frac{\log \mathcal{F}_{p,q,\vartheta}(t)}{t^{\vartheta-1} \varepsilon(t)^{-d/2} \sqrt{\log t}} = \sqrt{2dR(0)\vartheta} \quad \text{in probability}
\]

in the regular phase and

\[
\lim_{t \to \infty} \frac{\log \mathcal{F}_{p,q,\vartheta}(t)}{t(\log t)^{2/(4-d)}} = p^2 - 8/(4-d) \vartheta 2^{2/(4-d)} \Sigma_d \quad \text{in probability}
\]

in the singular phase. By taking \( p, \vartheta \to 1 \), this yields a lower bound for (1.11) and (1.16), thus concluding the proof of Theorems 1.7 and 1.11.
6 Annealed Total Mass

We now prove Theorem 1.17. Our main tool in establishing this is the following moment formula, which is proved using the same Fubini computation as in (5.2): For every $p \in \mathbb{N}$,

$$
E[U_{e(t)}(t)^p] = E \left[ \exp \left( \frac{1}{2} \sum_{i,j=1}^{p} \int_{[0,t]^2} R_{e(t)}(B^i(u) - B^j(v)) \, du \, dv \right) \right],
$$

where $(B_i)_{1 \leq i \leq p}$ are i.i.d. standard Brownian motions on $\mathbb{R}^d$ started at zero.

6.1 Proof of (1.17)

Suppose that $d \geq 2$, or $d = 1$ and $e(t)^{-1} = o(t)$. Informally speaking, the statement of (1.17) is that, under these assumptions, the only meaningful contribution of (6.1) in the large $t$ limit comes from paths of the Brownian motions that are confined to a neighborhood of the origin, whereby

$$
\int_{[0,t]^2} R_{e(t)}(B^i(u) - B^j(v)) \, du \, dv \approx t^2 R_{e(t)}(0) = e(t)^{-d} t^2 R(0).
$$

Given that $R \leq R(0)$ by virtue of being a covariance function, an upper bound to that effect is trivial:

$$
\int_{[0,t]^2} R_{e(t)}(B^i(u) - B^j(v)) \, du \, dv \leq e(t)^{-d} t^2 R(0)
$$

for every $1 \leq i, j \leq p$, which immediately yields an upper bound for (1.17). To prove a matching lower bound, we now argue that confining the paths of the Brownian motions near zero when $d \geq 2$, or $d = 1$ and $e(t)^{-1} = o(t)$ yields a vanishing error.

Let $\kappa > 0$ be fixed. If $|B^i(s)| \leq \kappa e(t)$ for every $1 \leq i \leq p$ and $0 \leq s \leq t$, then

$$
\sum_{i,j=1}^{p} \int_{[0,t]^2} R_{e(t)}(B^i(u) - B^j(v)) \, du \, dv \geq e(t)^{-d} t^2 p^2 \inf_{|x|,|y| \leq \kappa} R(x - y).
$$

Given that $e(t)$ is bounded above by 1, $e(t)/\sqrt{t} \to 0$ as $t \to \infty$, and thus there exists a constant $C > 0$ independent of $t$ such that for each $1 \leq i \leq p$,

$$
P \left[ \sup_{0 \leq s \leq t} |B^i(s)| \leq \kappa e(t) \right] \geq e^{-C t/\kappa^d}.
$$

(6.2)

for large $t > 0$ (e.g., [35, (1.3), Page 535]). Thanks to (6.1), we have the inequality

$$
E[U_{e(t)}(t)^p] \geq E \left[ \exp \left( \frac{1}{2} \sum_{i,j=1}^{p} \int_{[0,t]^2} R_{e(t)}(B^i(u) - B^j(v)) \, du \, dv \right) \right] \cdot \prod_{i=1}^{p} \mathbb{1}_{\{ \sup_{0 \leq s \leq t} |B^i(s)| \leq \kappa e(t) \}}.
$$

(6.3)

consequently,

$$
\liminf_{t \to \infty} \log E[U_{e(t)}(t)^p] \geq \frac{p^2}{2} \inf_{|x|,|y| \leq \kappa} R(x - y) + \liminf_{t \to \infty} \frac{-C p e(t)^d - 2}{\kappa^2 t}.
$$

(6.4)

The liminf on the right-hand side of (6.4) vanishes whenever $d \geq 2$ (since $e(t) \leq 1$), or $d = 1$ and $e(t)^{-1} = o(t)$. Given that (6.4) holds for arbitrarily small $\kappa > 0$, we thus obtain a lower bound for the limit (1.17) by taking $\kappa \to 0$. 
6.2 Proof of (1.18)

Let us henceforth assume that \( d = 1 \) with \( \varepsilon(t) \ll t^{-1} \). As per (6.4), once \( \varepsilon(t) \) passes the threshold of \( t^{-1} \) in one dimension, the space scaling of \( R_{\varepsilon(t)} \) vanishes too quickly for only Brownian paths confined to zero to have a contribution. In the case of the upper bound, we can simply take the \( \varepsilon \to 0 \) limit in (6.1), whereby the asymptotics of the exponential moment reduce to the large deviations of Brownian motion local time. More specifically: Arguing as in (5.4), we see that

\[
\sum_{1 \leq i, j \leq p} \int_{[0,1]^2} R_{\varepsilon(t)}(B^i(u) - B^j(v)) \, du \, dv \leq 2 \sum_{i, j = 1}^p (\|L^i_t\|_2^2 + \|L^j_t\|_2^2) = 4p \sum_{i = 1}^p \|L^i_t\|_2^2.
\]

By independence of the Brownian motions \( B^i \) and (6.1), this means that

\[
E[U_{\varepsilon(t)}(t)^p] \leq E^0 \left[ e^{4p\|L_t\|_2^2} \right]^p,
\]

where \( L_t \) denotes the local time of some Brownian motion \( B \), under the expectation \( E^0 \). According to [14, Lemma 2.2 in the case \( d = 1 \) and \( R = \delta_u \)], there exists a constant \( C > 0 \) such that for every \( \vartheta > 0 \),

\[
\lim_{t \to \infty} \log E^0 \left[ e^{\vartheta \|L_t\|_2^2} \right]^p = C \vartheta^2,
\]

from which we immediately obtain the upper bound in (1.18).

It now remains to show that there is a lower bound of the same order (i.e., \( p^3 t^3 \)). The argument that we use for this is inspired by the proof of [30, (6.8)]. That is, we introduce an additional smoothing of the noise, which allows to simultaneously provide a lower bound for the moment (6.1) and capture the optimal range of the Brownian paths that contribute to the Annealed asymptotics; once this is done the argument follows as in (6.4). For this purpose, we begin with the following Lemma, in which we introduce the additional smoothing:

**Lemma 6.1** For every \( \eta > 0 \), it holds that

\[
\sum_{i, j = 1}^p \int_{[0,1]^2} R_{\varepsilon(t)}(B^i(u) - B^j(v)) \, du \, dv \geq \sum_{i, j = 1}^p \int_{[0,1]^2} (R_{\varepsilon(t)} * \delta_\eta)(B^i(u) - B^j(v)) \, du \, dv, \tag{6.5}
\]

where we recall that \( \delta_\eta \) denotes the Gaussian kernel defined in (3.8).

**Proof.** Recall that we can write

\[
\int_{[0,1]^2} f(B^i(u) - B^j(v)) \, du \, dv = \int_{\mathbb{R}^2} L^i_t(x)f(x-y)L^j_t(y) \, dx \, dy
\]

for any measurable \( f : \mathbb{R} \to \mathbb{R} \), where \( L^i_t \) denotes the local time process of \( B^i \). Thus, if we denote \( L_t(x) := \sum_{i = 1}^p L^i_t(x) \), then we have that

\[
\sum_{1 \leq i, j \leq p} \int_{[0,1]^2} f(B^i(u) - B^j(v)) \, du \, dv = \int_{\mathbb{R}^2} L_t(x)f(x-y)L_t(y) \, dx \, dy. \tag{6.6}
\]
Letting \( \hat{\eta} \) denote the Fourier transform, it follows from the Parseval formula that
\[
\int_{\mathbb{R}^2} \mathbf{L}_n(x) f(x-y) \mathbf{L}_n(y) \, dx \, dy = \int_{\mathbb{R}} |\mathbf{L}_n(x)|^2 f(x) \, dx.
\] (6.7)

For every \( \eta > 0 \), \( \hat{R}_\eta \) and \( \mathcal{G}_\eta \) are both even functions, and \( R_\eta = \hat{R}_\eta \ast \hat{R}_\eta \) and \( \mathcal{G}_\eta = \mathcal{G}_{\eta/2} \ast \mathcal{G}_{\eta/2} \). In particular, \( \hat{R}_\eta \) and \( \hat{\mathcal{G}}_\eta \) are both nonnegative. Given that
\[
\hat{\mathcal{G}}_\eta \leq \|\mathcal{G}_\eta\|_1 = 1
\]
the result then follows from the applying the inequality
\[
R_{\epsilon(t)}(\mathcal{G}_\eta) = R_{\epsilon(t)} \hat{\mathcal{G}}_\eta \leq \mathcal{G}_\eta.
\]
to (6.7) with \( f = R_{\epsilon(t)} \ast \mathcal{G}_\eta \).

Let \( \kappa > 0 \) be large enough so that \( \text{supp}(R_{\epsilon(t)}) \subset [-\kappa \epsilon(t), \kappa \epsilon(t)] \) for every \( t \geq 0 \); in particular, for every \( \kappa, \eta > 0 \), we have that
\[
\inf_{|t| \leq \kappa} (R_{\epsilon(t)} \ast \mathcal{G}_\eta)(x) \geq \inf_{|t| \leq \kappa + \kappa \epsilon(t)} \mathcal{G}_\eta(x) = \frac{e^{-(\kappa + \kappa \epsilon(t))^2/2\eta}}{\sqrt{2\pi\eta}}.
\]

Let us define a function \( \eta(t) \) that vanishes as \( t \to 0 \) in such a way that \( \epsilon(t) = o(\eta(t)^{1/2}) \) (we define \( \eta(t) \) more specifically in a moment; this function is meant to capture the optimal range of Brownian paths that contribute to \( E[U(t)\eta] \)). If \( |B^i(s)| \leq \eta(t)^{1/2} \) for every \( 0 \leq s \leq t \) and \( 0 \leq i \leq p \), then we have the inequality
\[
\sum_{i,j=1}^p \int_{[0,t]^2} (R_{\epsilon(t)} \ast \mathcal{G}_\eta)(B^i(u) - B^j(v)) \, du \, dv \geq p^2 r^2 \mathbb{E} \left[ \frac{e^{-(2\eta(t)^{1/2} + \kappa \epsilon(t))^2/2\eta(t)}}{\sqrt{2\pi\eta(t)}} \right].
\] (6.8)

Since \( \epsilon(t)^2 = o(\eta(t)) \), there exists some \( c > 0 \) such that (6.8) is bounded below by \( cp^2 t / \eta(t)^{1/2} \) for large enough \( t \). Using essentially the same estimates as (6.2) and (6.3), we therefore conclude that
\[
\liminf_{t \to \infty} \frac{\log \mathbb{E}[U_{\epsilon(t)}(t)^p]}{t^3} \geq \liminf_{t \to \infty} \frac{1}{t^3} \left( \vartheta_1 \frac{p^2 t^2}{\eta(t)^{1/2}} - \vartheta_2 \frac{pt}{\eta(t)} \right)
\]
for some \( \vartheta_1, \vartheta_2 > 0 \) independent of \( p \) and \( t \geq 1 \). If we take
\[
\eta(t) = \frac{\widehat{k} \vartheta_2^2}{\vartheta_1^2 p^2 t}
\]
for some fixed \( \widehat{k} > 1 \) (which satisfies \( \epsilon(t) = o(\eta(t)^{1/2}) \) thanks to our assumption that \( \epsilon(t) = o(t^{-1}) \)), we get that
\[
\frac{1}{t^3} \left( \vartheta_1 \frac{p^2 t^2}{\eta(t)^{1/2}} - \vartheta_2 \frac{pt}{\eta(t)} \right) = \frac{\vartheta_2^2 \sqrt{k} - 1}{\vartheta_1 \widehat{k}} p^3,
\]
which yields the lower bound in (1.18).
A Appendix

A.1 Gaussian Maxima Upper Tails

Lemma A.1 (e.g., [36, Theorem 5.4.3 and Corollary 5.4.5]) Let \( \{X(s)\}_{s \in S} \) be a centered Gaussian process such that \( S \) is a countable metric space. Denote the maximal variance and median of \( X \) as

\[
v := \sup_{s \in S} \mathbb{E}[X(s)^2]^{1/2} \quad \text{and} \quad m := \text{Med}\left( \sup_{s \in S} X(s) \right).
\]

For every \( \lambda \geq 0 \)

\[
\mathbb{P}\left( \sup_{s \in S} X(s) > \lambda \right) \leq e^{-(\lambda - m)^2/2v^2}.
\]

Moreover,

\[
\left| m - \mathbb{E}\left( \sup_{s \in S} X(s) \right) \right| \leq \frac{v}{\sqrt{2\pi}}.
\]

A.2 Variations and Best Constants

The proofs of the results in this section are standard in the large deviation literature (e.g., [10, 11]). We nevertheless provide the arguments in full for the reader’s convenience.

A.2.1 Scaling Property

Proposition A.2 Let \( c > 0 \) be fixed. For every \( \eta > 0 \), it holds that

\[
\sup_{\varphi \in \mathcal{S}(\mathbb{R}^d)} \left( c\|\varphi\|^2_2 - \frac{1}{2} \delta(\varphi) \right) = \eta^2 \sup_{\varphi \in \mathcal{S}(\mathbb{R}^d)} \left( \eta^{(d-4)/2} c\|\varphi\|^2_2 - \frac{1}{2} \delta(\varphi) \right).
\]

Proof. This follows from a direct application of Remark 3.4, as \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) if and only if \( \varphi^{(0)} \in \mathcal{S}(\mathbb{R}^d) \). \( \square \)

A.2.2 Equivalence

Lemma A.3 Recall the definition of \( \Theta_d \) as the smallest constant in the inequality

\[
\|\varphi\|^2_2 \leq \Theta_d \delta(\varphi)^{d/2}\|\varphi\|^{2-d}_2 \quad \text{for all} \quad \varphi \in \mathcal{C}_0^0(\mathbb{R}^d).
\]

It holds that

\[
\sup_{\varphi \in \mathcal{S}(\mathbb{R}^d)} \left( \|\varphi\|^2_2 - \frac{1}{2} \delta(\varphi) \right) = \frac{d}{4} \left( \frac{d}{2} \right)^{d/(4-d)} \Theta_d^{2/(4-d)}.
\]

Proof. We begin with an upper bound. By definition of \( \Theta_d \), for every \( \varphi \in \mathcal{S}(\mathbb{R}^d) \),

\[
\|\varphi\|^2_2 - \frac{1}{2} \delta(\varphi) \leq \Theta_d^{1/2} \delta(\varphi)^{d/4} - \frac{1}{2} \delta(\varphi) \leq \sup_{x \geq 0} \left( \Theta_d^{1/2} x^{d/4} - \frac{1}{2} x \right) = \frac{d}{4} \left( \frac{d}{2} \right)^{d/(4-d)} \Theta_d^{2/(4-d)},
\]

where the last equality follows from elementary calculus. For a matching lower bound, let \( 0 < C < \Theta_d \). Then, there exists \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) such that \( \|\varphi\|^2_2 \geq C\delta(\varphi)^{d/2} \). By Remark 3.4, we see that

\[
\|\varphi^{(0)}\|^2_2 - \frac{1}{2} \delta(\varphi^{(0)}) > C^{1/2} (\eta^2 \delta(\varphi))^{d/4} - \frac{1}{2} \eta^2 \delta(\varphi).
\]

Since \( \eta > 0 \) was arbitrary, we conclude that

\[
\sup_{\varphi \in \mathcal{S}(\mathbb{R}^d)} \left( \|\varphi\|^2_2 - \frac{1}{2} \delta(\varphi) \right) \geq \sup_{x \geq 0} \left( C^{1/2} x^{d/4} - \frac{1}{2} x \right),
\]

which yields the desired lower bound by taking \( C \to \Theta_d \). \( \square \)
Lemma A.4 With $\mathfrak{G}_d$ defined as in Lemma A.3, we have that

$$
\sup_{\psi \in \mathcal{W}(\mathbb{R}^d)} \|\psi\|_1^4 \leq \left(\frac{4 - d}{4}\right)^{(4-d)/2} \left(\frac{d}{2}\right)^{d/2} \mathfrak{G}_d.
$$

(A.3)

In particular, recalling the definition of $\mathfrak{L}_d$ in (1.14), we have that

$$
\left(2d \sup_{\psi \in \mathcal{W}(\mathbb{R}^d)} \|\psi\|_1^4 \right)^{2/(4-d)} = \mathfrak{L}_d.
$$

Proof. For simplicity of notation, let us denote

$$
s := \sup_{\psi \in \mathcal{W}(\mathbb{R}^d)} \|\psi\|_1^4.
$$

We first prove that $s$ is finite. $\varphi \in \mathcal{S}(\mathbb{R}^d)$ if and only if $\varphi (1 + \frac{1}{4} \varphi'(\varphi))^{-1/2} \in \mathcal{W}(\mathbb{R}^d)$. Therefore, an application of (1.13) yields

$$
s = \sup_{\varphi \in \mathcal{S}(\mathbb{R}^d)} \|\varphi\|_1^4 \left(1 + \frac{1}{4} \varphi'(\varphi)\right)^{2} \leq \mathfrak{G}_d \sup_{x \geq 0} \frac{1}{(1 + \frac{1}{4} x)^2},
$$

which is finite for $d = 1, 2, 3$. We now prove (A.3): Note that

$$
\|\varphi\|_1^4 \geq \frac{1}{2} s^{1/2} \varphi'(\varphi) \leq s^{1/2} \left(1 + \frac{1}{2} \varphi'(\varphi)\right) - \frac{1}{2} s^{1/2} \varphi'(\varphi) = s^{1/2}.
$$

Thus, by applying Proposition A.2 with $c = s^{-1/2}$ and $\eta = s^{1/2-d/4}$, and then Lemma A.3, we obtain that

$$
1 \geq s^{-1/2} \sup_{\varphi \in \mathcal{S}(\mathbb{R}^d)} \left(\|\varphi\|_2^4 - \frac{1}{2} s^{1/2} \varphi'(\varphi)\right) = \sup_{\varphi \in \mathcal{S}(\mathbb{R}^d)} \left[ s^{-1/2} \|\varphi\|_2^4 - \frac{1}{2} \varphi'(\varphi) \right]
$$

$$
= s^{-2/(4-d)} \left(\frac{4 - d}{4}\right)^{(4-d)/2} \left(\frac{d}{2}\right)^{d/(4-d)} \mathfrak{G}_d^{2/(4-d)}.
$$

Solving for $s$ in the above inequality yields

$$
s \geq \left(\frac{4 - d}{4}\right)^{(4-d)/2} \left(\frac{d}{2}\right)^{d/2} \mathfrak{G}_d.
$$

We now provide a matching upper bound. For every $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$
\|\varphi\|_2^4 \leq \mathfrak{G}_d^{1/2} \varphi'(\varphi)^{d/4} \|\varphi\|_2^{(4-d)/2} = \mathfrak{G}_d^{1/2} \left(\frac{4 - d}{2d}\right)^{-d/2} \left(\frac{4 - d}{2d}\right)^{-d/2} \left(\frac{4 - d}{4}\right)^{(4-d)/4} \|\varphi\|_2^{(4-d)/4}.
$$

Next, we use Young’s classical inequality $|xy| \leq |x|^p/p + |y|^q/q$ for $1/p + 1/q = 1$ in the special case $p = 4/(4 - d)$ and $q = 4/d$, which yields

$$
\|\varphi\|_2^4 \leq \mathfrak{G}_d^{1/2} \left(\frac{4 - d}{2d}\right)^{-d/2} \left(\frac{4 - d}{4}\right)^{(4-d)/4} \left(\|\varphi\|_2^2 + \frac{1}{2} \varphi'(\varphi)\right).
$$

If we divide both sides by $\|\varphi\|_2^2 + \frac{1}{2} \varphi'(\varphi)$ and take a supremum over smooth and compactly supported $\varphi$, then we get that

$$
s \leq \mathfrak{G}_d \left(\frac{4 - d}{2d}\right)^{-d/2} \left(\frac{4 - d}{4}\right)^{(4-d)/2} \left(\frac{d}{2}\right)^{d/2} \mathfrak{G}_d,
$$

concluding the proof. $\square$

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