Abstract
The question of the measurements on the Bell states by making use of mode change (from mixed to pure) of one qubit is considered. Such a mode change cannot be taken advantage of for superluminal communication in teleportation, and it may define constraints on the size of the gates.

Introduction
Underlying protocols for dense coding and teleportation[1, 3] is the idea of distinguishing between the four orthogonal Bell states

\begin{align*}
|B_1\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\
|B_2\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\
|B_3\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\
|B_4\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{align*}

by operations on the entangled qubits. One may wish to consider these protocols also to account for effects of mode change for each individual particle from their original mixed state represented by the density matrix

\[
\begin{bmatrix}
0.5 & 0 \\
0 & 0.5
\end{bmatrix} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)
\]

to some pure state upon observation of one particle[2]. This raises the question if this transition can be detected gainfully. This note examines this question.
Dense coding

In the dense coding protocol[1], the use of an XOR gate drives the second qubit of $|B_1\rangle$ and $|B_2\rangle$ into 0 and the second qubit of $|B_3\rangle$ and $|B_4\rangle$ into 1. For example, $|B_1\rangle$ is transformed into $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle$.

The measurement of the second qubit reduces the state into either $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ or $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Since the state may now be factored, each of the two qubits are transformed into the pure form.

The transformation

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

is now applied to this state to distinguish between the terms with plus and minus.

The capacity of $H$ to distinguish between the two states (requiring that the state presented to the operator be pure) rests on $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ becoming $|0\rangle$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ becoming $|1\rangle$. This provides the second classical bit of information.

As mentioned before, the application of the XOR changes the individual qubit state functions from mixed to pure, and this change of mode may be taken advantage of to communicate information about the gate operation without collapsing the state function.

However this communication will only occur across the size of the physical gate. Since no communication can be faster than the speed of light, and the transition of the state function from mixed to pure is considered to take place instantaneously, this may imply limits on the size of the quantum gate.

Teleportation

This protocol is the converse of the dense coding protocol. Here Alice and Bob start with the entangled pair $|B_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Alice wishes to send to Bob the unknown qubit $|\phi\rangle$. Without loss of generality, $|\phi\rangle = a|0\rangle + b|1\rangle$, where $a$ and $b$ are unknown coefficients. The initial state of the three qubits is:

$$a|000\rangle + b|100\rangle + a|011\rangle + b|111\rangle$$
(For convenience, we leave out the constant factor in this and other expressions that follow.)

Alice now applies the XOR operator on the first two qubits, obtaining the state:

\[ a|000\rangle + b|110\rangle + a|011\rangle + b|101\rangle \]

This is now followed by the \( H \) operator on the first qubit, giving us:

\[
\begin{align*}
& a(|000\rangle + |100\rangle) + b(|010\rangle - |110\rangle) \\
& + a(|011\rangle + |111\rangle) + b(|001\rangle - |101\rangle)
\end{align*}
\]

Simplifying, we obtain:

\[
\begin{align*}
& |00\rangle(a|0\rangle + b|1\rangle) + |01\rangle(a|1\rangle + b|0\rangle) \\
& + |10\rangle(a|0\rangle - b|1\rangle) + |11\rangle(a|1\rangle - b|0\rangle)
\end{align*}
\]

Alice now measures the first two qubits (the unknown one and the first of the two entangled ones). The state of the remaining qubit collapses to one of the four states:

\[ a|0\rangle \pm b|1\rangle, \ a|1\rangle \pm b|0\rangle. \]

The information of the two qubits is used in the protocol to determine which of the operators \( I = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \), \( A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \), \( B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \), \( C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \) should be applied to his qubit to place it in the state \( |\phi\rangle \).

Although the operators XOR and H did not interact with this third qubit, its state changed because of the entanglement and the measurements of the other qubits.

**Teleportation of mixed state**

The state \( |\phi\rangle \) need not be pure. If the original state is mixed then the teleported state will also be mixed.
Consider Alice can pick photons in states $|0\rangle$ and $|1\rangle$ randomly and send them to Bob using the teleportation protocol.

If Alice picks $|0\rangle$, the final state is:

$$|00\rangle|0\rangle + |01\rangle|1\rangle + |10\rangle|0\rangle + |11\rangle|1\rangle$$

If she picks $|1\rangle$, the final state is:

$$|00\rangle|1\rangle + |01\rangle|0\rangle - |10\rangle|1\rangle - |11\rangle|0\rangle$$

Although any specific photon will be in a pure state after the measurement, Bob will represent it as a mixture, indicating that the teleported mixed state may not necessarily represent the state of the specific copy of the input. Should Alice inform Bob later and tell which kind of photon had been transmitted, the state function of that photon will change so far as Bob is concerned.

**Mode change in teleportation**

Suppose one begins with the state function $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. The mode change occurs in Bob’s qubit as soon as the measurements of the first two qubits are made by Alice. But Bob’s qubit, when it switches to the pure mode, will be in one of the two different orthogonal states $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, making it impossible to distinguish these from the earlier mixed state before measurements by Alice have been communicated to Bob on the classical channel to help him distinguish between the two orthogonal states.

**Conclusion**

This note takes a look at the dense coding and teleportation protocols from the point of view of exploiting mode change, when the state function changes from mixed to pure, or from one mixed state to another (if the original state is mixed in the teleportation protocol). Expectedly, such a transition cannot be taken advantage of to transmit information faster than the speed of light. It may define constraints on the physical size of the quantum gates.
References

1. C.H. Bennett, “Quantum information and computation,” Phys. Today 48 (10), 24-30 (1995).

2. S. Kak, “Paradox of quantum information,” quant-ph/0304060.

3. A. Peres, Quantum Theory: Concepts and Methods. Kluwer, 1995.