Position-dependent mass harmonic oscillator: classical-quantum mechanical correspondence and ordering-ambiguity

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We recycle Cruz et al.’s (Phys. Lett. A 369 (2007) 400) work on the classical and quantum position-dependent mass (PDM) oscillators. To elaborate on the ordering ambiguity, we properly amend some of the results reported in their work and discuss the classical and quantum mechanical correspondence for the PDM harmonic oscillators. We use a point canonical transformation and show that one unique quantum PDM oscillator Hamiltonian (consequently, one unique ordering-ambiguity parametric set \( j = l = -1/4 \) and \( k = -1/2 \)) is obtained. To show that such a parametric set is not just a manifestation of the quantum PDM oscillator Hamiltonian, we consider the classical and quantum mechanical correspondence for quasi-free PDM particles moving under the influence of their own PDM force fields.

**Keywords:** Position-dependent-mass, ordering-ambiguity parameters, Classical and quantum correspondence.

I. INTRODUCTION

The assumption that the information on the material properties is encoded in the mass of the non-relativistic quantum particle suggests the well known position-dependent mass (PDM) von Roos Hamiltonian (cf. e.g., [1])

\[
\hat{H} = \frac{1}{4} \left[ m(x)^{j} p_{x} m(x)^{k} p_{x} m(x)^{l} + m(x)^{l} p_{x} m(x)^{k} p_{x} m(x)^{j} \right] + V(x),
\]

where the ordering-ambiguity parameters \( j, k, \) and \( l \) are subjected to the von Roos constraint \( j + k + l = -1 \). Changing the values of \( j, k, \) and \( l \) would change the profile of the kinetic energy term (hence changing the profile of the effective potential) and therefore an ordering ambiguity conflict emerges in the process. This has inspired many research contributions over the last few decades, some of which were developed from theoretical points of view [2-3] and others were developed to generate exactly solvable problems [6-7]. On the theoretically acceptable sides, nevertheless, it is found that the continuity conditions at the abrupt heterojunction between two crystals enforce the parametric condition \( j = l \) (cf., e.g., Koc et al. [4]). A condition that makes the parametric proposals of Ben Daniel and Duke \((j = l = 0, k = -1)\), Zhu and Kroemer \((j = l = -1/2, k = 0)\), and Mustafa and Mazharimousavi \((j = l = -1/4, k = -1/2)\) (cf., e.g., [3, 4]) survive among others available in the literature.

Very recently, on the other hand, Mazharimousevi and Mustafa [5] have argued that the fixation of the ordering-ambiguity parameters may be sought through the classical observations of a given free PDM-particle moving under the influence of its own internally byproducted force field. They have observed that Zhu and Kroemer’s [1] \((j = l = -1/2, k = 0)\), and Mustafa and Mazharimousavi’s [4] \((j = l = -1/4, k = -1/2)\) orderings have provided consistent quantum mechanical correspondence to the classical observations. They have also suggested that the Gora and William’s [4] \((k = l = 0, j = -1)\) and Li and Kuhn’s [4] \((k = l = -1/2, j = 0)\) orderings should be disqualified, not only on the grounds of the continuity conditions at the abrupt heterojunction but also on the grounds of failing to provide a consistent quantum correspondence to classical observations. Such classical and quantum correspondence related observations form the motivation of the current proposal.

In this communication we recycle Cruz et al.’s [2] work on the classical and quantum position-dependent mass oscillators. To elaborate on the ordering ambiguity, we properly amend some of the results reported in their work and discuss the classical-quantum mechanical correspondence for the PDM harmonic oscillators. In section II, we recollect the quantum position-dependent mass (PDM) oscillator [2]. We emphasis that the quantum harmonic oscillator’s creation \( \hat{A}^+ \) and annihilation \( \hat{A}^- \) operators not only satisfy the commutation relation \( [\hat{A}^-, \hat{A}^+] = 1 \) (as in [2]), but also they satisfy the well known complementary textbook condition \( \hat{H} = \hat{A}^+ \hat{A}^- + 1/2 = \hat{A}^- \hat{A}^+ + 1/2 \). Such a condition results a unique quantum PDM oscillator Hamiltonian, unlike the two partner-like oscillator Hamiltonians reported by Cruz et al. [2]. In section III, we use a point canonical transformation (PCT) and show that such a

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unique quantum PDM oscillator Hamiltonian represents the quantum mechanical correspondence of the classical PDM oscillator Hamiltonian. In connection with the ordering-ambiguity associated with the PDM von Roos Hamiltonian (1) \(^1\), we observe that one "unique" ordering-ambiguity parametric set \(j = l = -1/4\) and \(k = -1/2\) is obtained (namely, that of Mustafa and Mazharimousavi's \(^4\)). To show that such a parametric set is not just a manifestation of the quantum PDM oscillator Hamiltonian, we consider (in section VI) the classical and quantum mechanical correspondence for quasi-free PDM particles moving under the influence of their own PDM force fields. We conclude in section V.

II. QUANTUM POSITION-DEPENDENT MASS HARMONIC OSCILLATOR

In their general considerations of the quantum position-dependent mass oscillator, Cruz et al. \(^2\) have used a PDM Hamiltonian of the form

\[
\hat{H} = -\frac{1}{2} m (x)^a \partial_x m (x)^{2b} \partial_x m (x)^a + V (x) ; \quad a + b = -\frac{1}{2},
\]

(2)

Which is in fact obtained from the PDM von Roos Hamiltonian (1) using the continuity conditions at the abrupt heterojunction \(j = l\), where \(j = l = a\) and \(k = 2b\) are the parametric mappings between Hamiltonian (1) and (2). Their construction of the harmonic oscillator creation (i.e., equation (26) of \(^2\)) and annihilation (i.e., equation (25) of \(^2\)) and consequently their corresponding Hamiltonians

\[
\hat{A}^+ = -\frac{1}{\sqrt{2}} m^a \partial_x m^b + W_a (x),
\]

(3)

and

\[
\hat{A}^- = \frac{1}{\sqrt{2}} m^b \partial_x m^a + W_a (x),
\]

(4)

operators have led them to two Hamiltonians

\[
\hat{H}^+ = \hat{A}^+ \hat{A}^- = -\frac{1}{2} m^a \partial_x m^{2b} \partial_x m^a + V_a^+ (x) = \hat{T}_a^+ + V_a^+ (x),
\]

(5)

(i.e., equation (25) of \(^2\)) and

\[
\hat{H}^- = \hat{A}^- \hat{A}^+ = -\frac{1}{2} m^b \partial_x m^{2a} \partial_x m^b + V_a^- (x) = \hat{T}_a^- + V_a^- (x),
\]

(6)

(i.e., equation (26) of \(^2\)). Where their \(V_a^+ (x)\) and \(V_a^- (x)\) are defined in equations (27) and (28) of their paper, respectively. At this point, the last term in their expression for \(V_a^- (x)\) in (28) should be removed (no such term should be there) and consequently their corresponding \(V_a^\pm (x)\) in (30) should read

\[
V_a^\pm (x) = \frac{1}{2} \left( \int \sqrt{m (u) du} \right)^2 - \frac{(4a + 1)^2}{8} \left[ \left( \frac{1}{\sqrt{m}} \right)'' + \frac{(4a + 1)}{4} \frac{1}{\sqrt{m}} \left( \frac{1}{\sqrt{m}} \right)'' + \frac{1}{2} \right].
\]

(7)

Obviously, our third term in (7) does not agree with their second term in (30).

Indeed one would use the oscillator commutation relation \([\hat{A}^- , \hat{A}^+] = 1\) to obtain \(W_a (x)\) given in their equation (29). However, the harmonic oscillator is also well known to have one unique Hamiltonian given by the textbook complementary relation

\[
\hat{H} = \hat{A}^+ \hat{A}^- + \frac{1}{2} = \hat{A}^- \hat{A}^+ - \frac{1}{2}.
\]

(8)

Then the two Hamiltonians in (5) and (6) are not two different Hamiltonians but they represent the constituents of one Hamiltonian \(\hat{H} = \hat{H}^\pm \pm 1/2\). This would, in turn, suggest that the corresponding kinetic energy operators satisfy the relation

\[
\hat{T}_a^+ = \hat{T}_a^- \implies -\frac{1}{2} m^a \partial_x m^{2b} \partial_x m^a = -\frac{1}{2} m^b \partial_x m^{2a} \partial_x m^b.
\]

(9)
Clearly, this relation can only be satisfied if and only if \( a = b \). This result, when substituted in the corresponding von Roos constraint \( a + b = -1/2 \), would immediately imply that \( a = b = -1/4 \). Consequently, the quantum harmonic oscillator PDM-Hamiltonian in (8) is unique and represented by

\[
\hat{H} = -\frac{1}{2} \frac{1}{\sqrt{m}} \frac{\partial}{\partial x} \frac{1}{\sqrt{m}} \frac{\partial}{\partial x} + \frac{1}{2} \left( \int \frac{x}{\sqrt{m(u)du}} \right)^2.
\]

This result not only shows that one unique quantum PDM oscillator’s Hamiltonian is obtained but also exactly represents the quantum mechanical correspondence of the classical PDM harmonic oscillator Hamiltonian. We discuss this in the following section.

III. HARMONIC OSCILLATOR CLASSICAL AND QUANTUM MECHANICAL CORRESPONDENCE: POINT CANONICAL TRANSFORMATION

Let us consider a classical PDM-particle moving under the influence of a potential field \( V(x) \). Then the corresponding Hamiltonian would read

\[
\mathcal{H}_x = \frac{P_x^2}{2m(x)} + V(x) = \frac{1}{2} m(x) \dot{x}^2 + V(x) \quad ; \quad \dot{x} = \frac{dx}{dt}.
\]

Which, under PCT

\[
q'(x) = \sqrt{m(x)} \implies q(x) = \int \sqrt{m(u)du},
\]

would be transformed into \( \mathcal{H}_q \) such that

\[
\mathcal{H}_q = \frac{1}{2} \dot{q}^2 + V(q) \quad ; \quad \dot{q} = \frac{dq}{dt}.
\]

Where \( \mathcal{H}_q \) represents a classical particle with a ”constant unit mass” moving under the influence of a potential field \( V(q) = V(q(x)) \) with a momentum \( P_q = \dot{q} \). We may now safely recollect that \( \mathcal{H}_q \) can be factorized into

\[
\mathcal{H}_q = a^- a^+ = a^+ a^- = \frac{1}{2} P_q^2 + V(q),
\]

such that

\[
a^\pm = \mp i \frac{P_q}{\sqrt{2}} + G(q),
\]

satisfy the Poisson bracket

\[
\{ a^-, a^+ \} = \frac{\partial a^-}{\partial P_q} \frac{\partial a^+}{\partial q} - \frac{\partial a^-}{\partial q} \frac{\partial a^+}{\partial P_q} = i
\]

for the harmonic oscillator with a ”constant unit mass”. This condition on \( a^\pm \) would yield that

\[
G(q) = \frac{q}{\sqrt{2}} \implies V(q) = \frac{1}{2} q^2 = \frac{1}{2} \left( \int \frac{x}{\sqrt{m(u)du}} \right)^2.
\]

Of course, for a given PDM \( m(x) \) one may then find the corresponding \( V(x) \). However, this readily lies far beyond our current proposal.

The quantum mechanical correspondence of such a classical model would, moreover, transform Hamiltonian (2), using substitution \( \psi(x) = m(x)^{1/4} \varphi(q) \) in \( \hat{H}_\psi(x) = E \psi(x) \), into

\[
\hat{H}_q = -\frac{1}{2} \frac{\partial^2}{\partial q^2} + V_{eff}(q(x)) = \frac{1}{2} \frac{P_q^2}{2} + V_{eff}(q),
\]

(18)
where \( \hat{P}_q = -i\partial_q \) analogous to the linear momentum operator (with \( \hbar = 1 \)) for a quantum particle with a constant unit mass,

\[
V_{\text{eff}} (q) = \frac{1}{8} (1 + 4b) F_1 (q) - \frac{1}{2} \left[ \frac{9}{16} + a (a + 2b + 1) + 2b \right] F_2 (q) + V(q),
\]

(19)

and

\[
F_1 (q) = \frac{m'' (x)}{m (x)^2}; \quad F_2 (q) = \frac{m' (x)^2}{m (x)^3}.
\]

(20)

Where \( F_1 (q) \) and \( F_2 (q) \) are two smooth functions manifestly introduced by the ordering-ambiguity in (1). Obviously, \( V_{\text{eff}} (q) \) in (19) represents an effective potential field produced by the PDM particle itself (represented by the first two terms) and the interaction potential. Let us now use the harmonic oscillator creation and annihilation operators

\[
\hat{b}^+ = G (q) - \frac{i \hat{P}_q}{\sqrt{2}} = G (q) - \frac{1}{\sqrt{2}} \frac{d}{dq}, \quad \hat{b}^- = G (q) + \frac{i \hat{P}_q}{\sqrt{2}} = G (q) + \frac{1}{\sqrt{2}} \frac{d}{dq},
\]

(21)

(22)

respectively, for a quantum particle with a constant unit mass. These operators satisfy the commutation relation \( [\hat{b}^-, \hat{b}^+] = 1 \) and imply that

\[
G (q) = \frac{q}{\sqrt{2}} \implies V(q) = \frac{1}{2} q^2.
\]

(23)

Moreover, the condition \( \hat{H}_q = \hat{b}^+ \hat{b}^- + \frac{1}{2} = \hat{b}^- \hat{b}^+ - \frac{1}{2} \) implies that

\[
\hat{H}_q = -\frac{1}{2} \hat{q}^2 + G (q)^2 = -\frac{1}{2} \hat{q}^2 + \frac{1}{2} q^2.
\]

If the classical Hamiltonian in (14) (along with \( V(q) \) of (17)) is to find its quantum mechanical counterpart in (18) (along with \( V_{\text{eff}} (q) \) of (19) and (23)) then the first two terms of the effective potential in (19) should vanish identically to yield \( b = -1/4 \) and \( a = -1/4 \) (i.e., Mustafa and Mazharimousavi’s ordering parametric set). Now, using the substitution \( \varphi(q) = m(x)^{-1/4} \psi(x) \) in \( \hat{H}_q \varphi(q) = E \varphi(q) \), one may easily show that

\[
\hat{H}_q = -\frac{1}{2} \partial_q^2 + \frac{1}{2} q^2 \implies \hat{H} = -\frac{1}{2} \frac{1}{\sqrt{m}} \partial_x \frac{1}{\sqrt{m}} \partial_x \frac{1}{\sqrt{m}} + \frac{1}{2} \left( \int \sqrt{m(u)} du \right)^2.
\]

Which is indeed in exact accord with Hamiltonian (10). This result not only shows that one unique quantum PDM oscillator’s Hamiltonian is obtained but also exactly represents the quantum mechanical correspondence of the classical PDM harmonic oscillator Hamiltonian given in (10).

Hereby, a question of delicate nature arises in the process as to whether such a parametric set is only associated to the PDM harmonic oscillator problem. One would therefore invest similar procedure in a different model to test it. This is done in the following section.

**IV. QUASI-FREE PDM-PARTICLE; CLASSICAL AND QUANTUM CORRESPONDENCE**

Consider a free PDM quantum particle (i.e., \( V(x) = 0 \)) moving under the influence of its own PDM-field (hence quasi-free PDM-particle) [3, 7]. Using the point canonical transformation (12) along with the substitutions \( \psi(x) = m(x)^{1/4} \varphi(q) \) and \( V(x) = 0 \) in \( \hat{H} \psi(x) = E \psi(x) \), then \( \hat{H} \) of (2) would transform into \( \hat{H}_q \) so that

\[
\hat{H}_q = -\frac{1}{2} \partial_q^2 + V_{\text{eff}}(q(x)) = \frac{1}{2} \hat{P}_q^2 + V_{\text{eff}}(q),
\]

(24)
where,
\[ V_{\text{eff}} (q) = V_{\text{eff}} (q(x)) = \frac{1}{8} (1 + 4b) F_1 (q) - \frac{1}{2} \left[ \frac{9}{16} + a (a + 2b + 1) + 2b \right] F_2 (q) , \]
(25)

Now consider the classical Hamiltonian for a free PDM-particle moving under the influence of its own PDM-field
\[ \mathcal{H}_x = \frac{p_x^2}{2m(x)} = \frac{1}{2} m(x) \dot{x}^2 ; \quad \dot{x} = \frac{dx}{dt} . \]
(26)

Which, under the PCT would be transformed into \( \mathcal{H}_q \) such that
\[ \mathcal{H}_q = \frac{1}{2} \dot{q}^2 . \]
(27)

The transformed Hamiltonian \( \mathcal{H}_q \) represents a free particle with a constant unit mass moving in \( q \)-space with a conserved momentum

\[ P_q = \dot{q} \rightarrow \dot{q} (x) = \dot{q} (x_0) . \]
(28)

(cf., e.g., Mazhariousavi and Mustafa [3] for more details on this issue). Under such settings, we recast \( \mathcal{H}_q \) to read
\[ \mathcal{H}_q = \frac{1}{2} P_q^2 . \]
(29)

If the quantum mechanical Hamiltonian in (24) is to correspond to the classical Hamiltonian in (29), then the two terms of the effective potential in (25) should vanish identically. That is,
\[ (1 + 4b) = 0 , \quad \text{and} \quad \frac{9}{16} + a (a + 2b + 1) + 2b = 0 , \]
(30)

which would, again, immediately suggest that \( b = a = -1/4 \) (i.e., Mustafa and Mazhariousavi’s [4] ordering-ambiguity parametric set).

V. CONCLUSION

We have recycled the work of Cruz et al.’s [2] on the classical and quantum position-dependent mass oscillators. In addition to the amendments reported above, we have discussed the classical and quantum mechanical correspondence for the PDM harmonic oscillators to conjecture on the feasibility of the ordering ambiguity parametrization. We have used the creation and annihilation operators of the quantum PDM harmonic oscillator along with the commutation relation \( [\hat{A}^-, \hat{A}^+] = 1 \) and the complementary relation \( \hat{H} = \hat{A}^+ \hat{A}^- + \frac{1}{2} = \hat{A}^- \hat{A}^+ - \frac{1}{2} \) to show that the harmonic oscillator PDM-Hamiltonian in (10) is unique and represented by
\[ \hat{H} = -\frac{1}{2} \frac{1}{\sqrt{m(x)}} \frac{\partial}{\partial x} \frac{1}{\sqrt{m(x)}} \frac{\partial}{\partial x} - \frac{1}{2} \left( \int \sqrt{m(u) du} \right)^2 . \]
(31)

We have shown that this quantum PDM oscillator’s Hamiltonian is not only unique but also exactly represents the quantum mechanical correspondence of the classical PDM harmonic oscillator Hamiltonian (documented in section III). Moreover, we have reported that such unique representation of the PDM kinetic energy term in (31) is not only a by-product of the quantum PDM oscillator, but also a by-product of the quantum mechanical correspondence of the classical quasi-free PDM Hamiltonians (documented in sections III and IV). In fact, one may go further and conjecture (from both PCT-transformed harmonic oscillator and quasi-free PDM-Hamiltonians) that such unique representation holds true also for any potential \( V(x) \) (which would PCT-transform into \( V(q(x)) = V(q) \) leaving the first two terms of the effective potential (19) to vanish identically).

In short, the examples discussed above show that the classical and quantum mechanical correspondence leaves no doubt that the kinetic energy operator in the von Roos Hamiltonian (1) now finds its unique representation through Mustafa and Mazharariousavi’s [4] ordering-ambiguity parametric set to read
\[ \hat{T}_x = -\frac{1}{2} \frac{1}{\sqrt{m(x)}} \frac{\partial}{\partial x} \frac{1}{\sqrt{m(x)}} \frac{\partial}{\partial x} \frac{1}{\sqrt{m(x)}} . \]
(32)
This should very likely settle down the ordering-ambiguity conflict.

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