One-loop approximation of Møller scattering in Krein-space quantization

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Abstract

It has been shown that the negative-norm states necessarily appear in a covariant quantization of the free minimally coupled scalar field in de Sitter spacetime.1,2 In this process ultraviolet and infrared divergences have been automatically eliminated.3 A natural renormalization of the one-loop interacting quantum field in Minkowski spacetime (λφ4) has been achieved through the consideration of the negative-norm states defined in Krein space. It has been shown that the combination of quantum field theory in Krein space together with consideration of quantum metric fluctuation, results in quantum field theory without any divergences.4 Pursuing this approach, we express Wick’s theorem and calculate Møller scattering in the one-loop approximation in Krein space. The mathematical consequence of this method is the disappearance of the ultraviolet divergence in the one-loop approximation.

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I. INTRODUCTION

In view of the appearance of infrared divergence in the two-point function for the minimally coupled scalar field in de Sitter spacetime, a new method of field quantization called “Krein QFT” has been proposed, which uses negative-norm states [1, 2, 3, 4].

Consideration of the negative-norm states was proposed by Dirac in 1942 [5]. In 1950 Gupta applied the idea in QED [6]. The presence of higher derivatives in the Lagrangian also leads to ghosts; states with negative norm [7]. The auxiliary negative-norm states were primarily introduced in de Sitter spacetime to achieve covariant quantization. However, their presence has other consequences, too. For example, in QED the negative-energy photon disappears [6], and in de Sitter spacetime the infrared divergence of minimally coupled scalar field is eliminated [8]. In similarity with linear gravity, this divergence does not manifest itself in the quadratic part of the effective action in the one-loop approximation. This means that the pathological behavior of the graviton propagator is gauge dependent and so should not appear in an effective way as a physical quantity [9, 10]. Ignoring the positivity condition (for norm and energy), similar to Gupta-Bleuler quantization of the electrodynamics in Minkowski spacetime, the quantization of free boson and spinor fields has been performed in Krein space [11, 12]. Following this scheme, the normal ordering procedure is rendered useless since the vacuum energy remains convergent [11, 12].

It is worthwhile to note that the unphysical (negative-energy) states for boson fields have both negative and positive norms and are, therefore, defined in Krein space. In Gupta-Bleuler quantization, the unphysical states with positive and negative norms are introduced in order to preserve the Lorentz invariance. The negative-norm states appear due to the sign of the Minkowskian metric. However, in Krein space quantization the appearance of the additional negative-norm states owes itself to the negative-frequency solutions. The two sets of solutions for boson fields in Krein space are complex conjugates. However, for spinor fields the unphysical states are positive-norm states (moving forward in time). The unphysical positive-norm states of spinor fields are different from physical antiparticles. The latter, although having negative energy, move backward in time whereas the former move forward in time and are not observable. In other words, in the spinor case the unphysical state may be regarded as the unphysical particle and antiparticle in the inverse time direction. It is noteworthy that the unphysical states play no role in the physical world and they are just
used as a mathematical tool.

The most interesting result of the new construction is the convergence of the Green’s function at large distances, which means that the infrared divergence is gauge dependent \[2, 3\]. The ultraviolet divergence in the stress tensor disappears as well, so that the quantum free fields are automatically renormalized by this method. The role of unphysical states appears in the method as a natural renormalization tool in the one-loop approximation. It is worthwhile to note that via this method, a natural renormalization of the following problems are attained:

- The massive free field in de Sitter spacetime \[2\].
- The graviton two-point function in de Sitter spacetime \[13\].
- The one-loop effective action for scalar field in a general curved spacetime \[14\].
- Tree level scattering amplitude for scalar field with one graviton exchange in de Sitter spacetime \[15\].
- Casimir effect in Krein space quantization \[16\].
- Free fields Quantization in Krein space \[11, 12\].

Following the above works, in this paper we express Wick’s theorem and calculate the Møller (electron-electron) scattering matrix \(S\) in the one-loop approximation in Krein space. Again, it is seen that the presence of negative-norm states plays the role of an automatic renormalization tool for the theory of quantum fields. A number of works already published about Krein space and quantum field theories with indefinite metric could be found in \[17, 18, 19\].

II. KREIN QFT CALCULATION

The origin of divergence in QFT lies in the singular character of the Green’s function at short relative distances. In Krein QFT, although the Green’s function is changed due to the presence of negative-frequency states, we see that these unphysical states disappear as far as observable average values are concerned \[11, 20\].

One of the interesting results of constructing field theory in Krein space is that all ultraviolet divergences of QFT with the exception of the light cone singularity are removed. It was conjectured long ago \[21, 22\] that quantum metric fluctuations does smear out the light cone singularities, but it does not remove other ultraviolet divergences. Indeed, it has
been shown that quantum metric fluctuations remove the singularities of Green’s functions on the light cone \[23\].

In a previous work \[4\], it has been established that the combination of QFT in Krein space together with consideration of quantum metric fluctuations results in a QFT without any divergence (\(T\) means time-ordered):

\[
< G_T(x - x') > = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2 \sigma_0^2}} \exp\left(-\frac{\sigma_0^2}{2 \sigma_1^2}\right) + \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}}
\]

(1)

where \(2\sigma_0 = \eta_{\mu\nu}(x^\mu - x'^\mu)(x'^\nu - x^\nu)\), and \(< \sigma_1^2 >\) is related to the density of gravitons. When \(\sigma_0 = 0\), due to the metric quantum fluctuation \(< \sigma_1^2 >\) ≠ 0, we have

\[
< G_T(0) > = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2 \sigma_0^2}} + \frac{m^2}{8\pi}.
\]

The field operators and their associated divergence-free Green’s functions are \[11\] (\(K\) shall, hereafter, always stand for quantities in Krein space),

(i) Klein-Gordon field:

\[
\phi_K(x) = \frac{1}{\sqrt{2}} \int d^3k \left[ (a_k^\dagger + b_k^\dagger) u_p(k, x) + (a_k + b_k) u_n(k, x) \right]
\]

where

\[
u_p(k, x) = \frac{e^{-ik.x}}{\sqrt{(2\pi)^3 2\omega_k}}, \quad u_n(k, x) = \frac{e^{ik.x}}{\sqrt{(2\pi)^3 2\omega_k}}, \quad \omega_k = \sqrt{k.k + m^2}
\]

and the Green’s function is given by (1).

(ii) Maxwell field:

\[
A^K_{\mu}(x) = \frac{1}{\sqrt{2}} \int d^3k \sum_{\lambda=0}^3 \left[ e^{\lambda}_{\mu}(k)[(a_k^\dagger + b_k^\dagger) u_p(k, x) + (a_k + b_k) u_n(k, x)] \right]
\]

and

\[
D^T_{\mu\nu}(x, x') = -\eta_{\mu\nu} G_T(x, x')
\]

is the time-ordered propagator.

(iii) Dirac field (Krein space):

\[
\psi_K(x) = \frac{1}{\sqrt{2}} \int d^3k \sum_{s=1,2} \left[ (b_{ks} + c_{ks}^\dagger) \mathcal{U}(k, x) + (d_{ks}^\dagger + a_{ks}) \mathcal{V}(k, x) \right]
\]

where

\[
\mathcal{U}(k, x) = \sqrt{\frac{m}{(2\pi)^3 \omega_k}} u_s^*(k)e^{-ik.x} \quad \text{(positive energy)}
\]
\[ V^s(k, x) = \sqrt{\frac{m}{(2\pi)^3 \omega_k}} v^s(\vec{k}) e^{ik \cdot x} \quad \text{(negative energy)} \]

and

\[ S_T(x, x') = (i \not \partial + m) G_T(x, x') \]

is the time-ordered propagator, that is

\[ S_T(x, x') = \frac{1}{8\pi} i \gamma^\mu (x_\mu - x'_\mu) \left\{ \frac{\pi}{2 < \sigma_1^2 >} e^{-\frac{\sigma_0^2}{2 < \sigma_1^2 >}} + \frac{m^2 J_1(\sqrt{2m^2 \sigma_0})}{\sqrt{2m^2 \sigma_0}} \right\} + \frac{m}{2\sqrt{2}} \theta(\sigma_0) \left[ \sqrt{2m^2 \sigma_0} J_0(\sqrt{2m^2 \sigma_0}) - 2J_1(\sqrt{2m^2 \sigma_0}) \right] + \frac{m}{8\pi} \left[ -\frac{\pi}{2 < \sigma_1^2 >} e^{-\frac{\sigma_0^2}{2 < \sigma_1^2 >}} + m^2 \theta(\sigma_0) \frac{J_1(\sqrt{2m^2 \sigma_0})}{\sqrt{2m^2 \sigma_0}} \right] \]

### III. WICK'S THEOREM IN KREIN SPACE

Let \( A \) and \( B \) be two linear operators in Krein space:

\[ A_K(x) = A_p(x) + A_n(x) = (A^+_p + A^-_p) + (A^+_n + A^-_n) \]
\[ B_K(x) = B_p(x) + B_n(x) = (B^+_p + B^-_p) + (B^+_n + B^-_n) \]

where \( A_p(x) \), \( B_p(x) \) are the physical (positive-frequency) and \( A_n(x) \), \( B_n(x) \) the unphysical (negative-frequency) parts of the operators. We have

\[ A^+_p |0> = 0, \quad A^+_n |0> = 0, \quad A^-_p |0> = |1_{\vec{k}}>, \quad A^-_n |0> = |\bar{1}_{\vec{k}}> \]
\[ B^+_p |0> = 0, \quad B^+_n |0> = 0, \quad B^-_p |0> = |1_{\vec{k}}>, \quad B^-_n |0> = |\bar{1}_{\vec{k}}> \]
\[ A_K B_K =: A_K B_K : + [A^+_p, B^-_p] + [A^+_n, B^-_n] \quad (2) \]
\[ < 0|A_K B_K|0> = [A^+_p, B^-_p] + [A^+_n, B^-_n] \quad (3) \]

Comparing (2) and (3) yields

\[ A_K B_K =: A_K B_K : + < 0|A_K B_K|0> \]

and

\[ < 0|T[A_K(x_1)B_K(x_2)]|0> = [A_K(x_1)B_K(x_2)] \quad (4) \]
Since the contractions of physical and unphysical parts are zero, (4) reads

\[ < 0|T[A_K(x_1)B_K(x_2)]|0 >= [A_p(x_1)\widehat{B}_p(x_2)] + [A_n(x_1)\widehat{B}_n(x_2)] \]

Thus, the contraction of two operators in Krein space splits into the contraction of physical parts, and the contraction of unphysical parts. We have

\[ < 0|T[\phi_K(x_1)\phi_K(x_2)]|0 >= iG_T(x_1, x_2) \]
\[ < 0|T[A^K_\mu(x_1)A^K_\nu(x_2)]|0 >= iD^T_{\mu\nu}(x_1, x_2) = -i\eta_{\mu\nu}G_T(x_1, x_2) \]
\[ < 0|T[\psi_K(x_1)\psi_K(x_2)]|0 >= iS_T(x_1, x_2) = i(i\beta + m)G_T(x_1, x_2) \]

and Wick’s theorem for the time-ordered product of some linear operators \( A_K, B_K, \ldots, Z_K \)
in Krein space is expressed by

\[ T[A_K(x_1)B_K(x_2)\ldots] = [ : A_pB_p\ldotsZ_p : + : A_nB_n\ldotsZ_n :] + [(A_p\widehat{B}_p) : C_p\ldotsZ_p :
\quad + (A_n\widehat{B}_n) : C_n\ldotsZ_n :] + \ldots + [(A_p\widehat{B}_p)(C_p\widehat{D}_p) : E_p\ldotsZ_p :
\quad + (A_n\widehat{B}_n)(C_n\widehat{D}_n) : E_n\ldotsZ_n :] + \ldots \]

IV. FEYNMAN RULES IN KREIN SPACE

In Krein space, the graphical “translation rules” introduced by Feynman can be written as [24):

1. Each point of interaction \( x \) is associated with a vertex. This corresponds to the algebraic factor \( -i\gamma^\mu \).
2. Each electron field operator is associated with an external fermion line and an external unphysical fermion line glued to a vertex. The physical lines are full while the unphysical lines are dashed. These lines carry an arrow; however, if the physical lines are forward in time, the unphysical lines run backward in time. The field operators in Krein space are

\[ \psi_K(x) = \psi_p(x) + \psi_n(x) = [\psi^+_p(x) + \psi^-_p(x)] + [\psi^+_n(x) + \psi^-_n(x)] \]
\[ \bar{\psi}_K(x) = \bar{\psi}_p(x) + \bar{\psi}_n(x) = [\bar{\psi}^+_p(x) + \bar{\psi}^-_p(x)] + [\bar{\psi}^+_n(x) + \bar{\psi}^-_n(x)] \]
\[ A^K_\mu(x) = A^K_\mu^p(x) + A^K_\mu^n(x) = [A^+_\mu(x) + A^-_\mu(x)] + [A^+_\mu^p(x) + A^-_\mu^p(x)] \]

The following associations are made:

(i) \( \psi^+_p(x) \): a line pointing upward ending at \( x \) (electron absorption)
(ii) $\psi^+_n(x)$: a dashed line pointing downward starting at $x$ (unphysical positron absorption)
(iii) $\psi^+_p(x)$: a line pointing downward ending at $x$ (positron emission)
(iv) $\psi^-_n(x)$: a dashed line pointing upward starting at $x$ (unphysical electron emission)
(v) $\bar{\psi}^+_p(x)$: a line pointing downward starting at $x$ (positron absorption)
(vi) $\bar{\psi}^+_n(x)$: a dashed line pointing upward ending at $x$ (unphysical electron absorption)
(vii) $\bar{\psi}^-_p(x)$: a line pointing upward starting at $x$ (electron emission)
(viii) $\bar{\psi}^-_n(x)$: a dashed line pointing downward ending at $x$ (unphysical positron emission)

3. Each photon field operator is associated with a wiggly external photon line and each unphysical photon line is associated with a dashed wiggly external photon line:
(i) $A^+_{\mu p}(x)$: a line starting at $x$ pointing downward (photon absorption)
(ii) $A^+_{\mu n}(x)$: a dashed line ending at $x$ pointing upward (unphysical photon absorption)
(iii) $A^-_{\mu p}(x)$: a line starting at $x$ pointing upward (photon emission)
(iv) $A^-_{\mu n}(x)$: a dashed line ending at $x$ pointing downward (unphysical photon emission)

4. The contraction of two fermion operators in Krein space,

$$[\psi_K(x_1)\bar{\psi}_K(x_2)] = iS_T(x, x'),$$

is associated with a directed internal fermion line from $x_1$ to $x_2$, and a directed unphysical fermion line from $x_2$ to $x_1$.

5. The contraction of two photon operators in Krein space,

$$[A^K_\mu(x_1)A^K_\nu(x_2)] = iD^T_{\mu\nu}(x_1, x_2),$$

is associated with a wiggly internal photon line connecting $x_1, x_2$ and a dashed wiggly internal unphysical photon line connecting $x_1, x_2$. It is remarkable that the photon lines have no sense of direction (they bear no arrow) since the photon “is its own antiparticle”. This is reflected in the symmetry of the photon propagator.

Since the physical and unphysical states are orthogonal to each other, the expectation values of unphysical operators on the physical initial and final states will be zero. Therefore, to evaluate an $S$ matrix element of $n$th order, all topologically distinct graphs with $n$ vertices and within the desired configuration of external physical lines are drawn. Each vertex is assigned a coordinate variable $x_i$, and the Feynman rules of QED in coordinate space in Krein-space quantization read:

1. Vertex: $-ie(\gamma_\mu)_{\alpha\beta}$
2. Internal photon and unphysical photon lines: \( iD_{\mu\nu}^T(x_k, x_l) \)

3. Internal fermion and unphysical fermion lines: \( iS_T(x_k, x_l) \)

4. External fermion line:
   \( u^s(\vec{k})u_p(k, x) \) (incoming electron)
   \( \bar{v}^s(\vec{k})u_p(k, x) \) (incoming positron)
   \( \bar{u}^s(\vec{k})u_n(k, x) \) (outgoing electron)
   \( v^s(\vec{k})u_n(k, x) \) (outgoing positron)

5. External photon line:
   \( \epsilon^\lambda_{\mu}(\vec{k})u_p(k, x) \) (incoming photon)
   \( \epsilon^\lambda_{\mu}(\vec{k})u_n(k, x) \) (outgoing photon)

6. All coordinates \( x_i \) are integrated over. The integration can be carried out owing to the simple plane-wave factors.

7. Each closed fermion loop leads to a factor \(-1\).

It is seen that in calculations the unphysical states are eliminated in the external legs and are introduced only in the propagators. They eliminate the divergences of the theory automatically and without any renormalization procedure. The physical meaning of the negative-frequency states is not clear. However, by analogy with the standard QFT, one can interpret them as negative-energy unphysical particles (or antiparticles) by changing the time direction.

V. MÔLLER SCATTERING IN KREIN SPACE

In Møller scattering, two electrons with momenta and spins \( k_1, s_1 \) and \( k_2, s_2 \) in the initial state are scattered into the final state with \( k'_1, s'_1 \) and \( k'_2, s'_2 \). According to the Wick expansion of the \( S \) operator, for the one-loop approximation of Møller scattering in Krein-space quantization, we have up to fourth order

\[
S = \frac{(-ie)^4}{4!} \int d^4x_1 \, d^4x_2 \, d^4x_3 \, d^4x_4 : \bar{\psi}^-_{K}(x_1) \gamma^\mu \tilde{\psi}^+_K(x_1) \, iD_{\mu\alpha}^T(x_1 - x_3) \, (-Tr)[iS_T(x_4 - x_3)\gamma^\alpha iS_T(x_3 - x_4)\gamma^\beta] \, iD_{\beta\nu}^T(x_2 - x_4) \, \bar{\psi}^-_{K}(x_2) \gamma^\nu \tilde{\psi}^+_K(x_2) :
\]

The \( S \)-matrix element of Møller scattering is given by

\[
S_{fi} = < 0 | b_{k'_2 s'_2}^\dagger b_{k'_1 s'_1}^\dagger S \, b_{k_1 s_1}^\dagger \, b_{k_2 s_2}^\dagger | 0 >
\]
Insert the field operators of Krein space and discard the products of the physical and unphysical states because of the separation of their spaces; \( S_{fi} \) splits into the sum of two physical and unphysical terms. Because of the orthogonality of the physical and unphysical states, the expectation values of unphysical operators on physical \( |i> \) and \( <f| \) states are zero. Therefore the unphysical term of \( S_{fi} \) vanishes, and the final result for the \( S \)-matrix element of Møller scattering in the one-loop approximation in Krein space reads:

\[
S_{fi} = \frac{(-ie)^4}{2!} \frac{1}{(2\pi)^6} \int d^4x_1 \, d^4x_2 \, d^4x_3 \, d^4x_4 \sqrt{m \omega_{k_1}} \sqrt{m \omega_{k_2}} \sqrt{m \omega_{k_1}'} \sqrt{m \omega_{k_2}' \, \frac{iD^T_{\mu\alpha}(x_1-x_3)(-Tr)\left[ iS_T(x_4-x_3)\gamma^\alpha iS_T(x_3-x_4)\gamma^\beta iD^T_{\mu\nu}(x_2-x_4)[e^{i(k_2'-k_2)\cdot x_2}e^{i(k_1'-k_1)\cdot x_1} \bar{u}^{s_2}(k_2')\gamma^\mu u^{s_2}(k_2)\bar{u}^{s_1}(k_1')\gamma^\nu u^{s_1}(k_1) - e^{i(k_2'-k_2)\cdot x_2}e^{i(k_1'-k_1)\cdot x_1} \bar{u}^{s_1}(k_2)\gamma^\mu u^{s_2}(k_2)\bar{u}^{s_2}(k_1)\gamma^\nu u^{s_1}(k_1)\right]\right]}.
\]

which is free of any divergence. Of course, it is not enough to say that some amplitudes are finite. It is also necessary to show that it has something to do with the real stuff and processes which occur in nature. So, We are going to calculate it in the forthcoming papers.

VI. CONCLUSION

The negative-frequency solutions of the field equation are indispensable for the covariant quantization of minimally coupled scalar field in de Sitter spacetime. Contrary to the Minkowski spacetime, discarding the negative-frequency states in this case breaks the de Sitter invariance. In other words, for restoration of de Sitter invariance, one must take the negative-frequency states into account in the Krein-space quantization. Free boson and spinor fields are investigated in Krein space quantization. In the present paper the method has been applied to QED, and the one-loop approximation of Møller scattering is investigated in Krein space. Once again it is found that the theory is automatically renormalized.

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