The probability of exceeding a piecewise deterministic barrier by the heavy-tailed renewal compound process

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Abstract

We analyze the asymptotics of crossing a high piecewise linear barriers by a renewal compound process with the subexponential jumps. The study is motivated by ruin probabilities of two insurance companies (or two branches of the same company) that divide between them both claims and premia in some specified proportions when the initial reserves of both companies tend to infinity.

Key words: First time passage problem, ruin probabilities, subexponential distribution.

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1 Motivation

The study of boundary crossing probabilities of a stochastic process with heavy tailed increments has applications in fields such as queuing theory, insurance and finance. In this paper we consider the problem of a renewal process crossing a high piecewise linear boundary. This study is in particular motivated by ruin probabilities of two insurance companies with proportional claims (see [1]) and the steady state distribution of a tandem queue with two servers (see [2]). To be more precise, we set

\[ S_t = \sum_{i=1}^{N_t} \sigma_i \]  

for \( N_t \) a renewal process with i.i.d. inter-arrival times \( \zeta_i \), and the claims \( \sigma_i \) are i.i.d. random variables independent of \( N(t) \), with the distribution function \( F(x) \). We shall denote by \( \lambda \) and \( \mu \) the reciprocals of the means of \( \zeta_i \) and \( \sigma_i \), respectively. Let the boundaries \( b_1, b_2 \) given by

\[ b_1(t) = b_1(t; x_1) = x_1 + p_1 t, \quad b_2(t) = b_2(t; x_2) = x_2 + p_2 t, \]

where we assume that \( p_1 > p_2 \), \( p_2 > \rho := \frac{\lambda}{\mu} = E[\sigma]/E[\zeta] \)

and consider the following boundary crossing probabilities:

\[ \psi_\land(x_1, x_2) = P(\exists t \geq 0 : S_t > b_1(t) \land b_2(t)) \]
\[ \psi_\lor(x_1, x_2) = P(\exists t \geq 0 : S_t > b_1(t) \lor b_2(t)) \]
\[ \psi_\times(x_1, x_2) = P(\exists t \geq 0, u \geq 0 : S_t > b_1(t), S_u > b_2(u)), \]

where \( x \lor y = \max\{x, y\} \) and \( x \land y = \min\{x, y\} \).

The \( \psi_\land(x_1, x_2) \) describes the ruin probability of at least one insurance company. \( \psi_\lor(x_1, x_2) \) corresponds to the simultaneous ruin of the insurance companies. Finally, \( \psi_\times(x_1, x_2) \) describes probability that both companies will have ruin. First assumption in (2) means that the second company receives less premium per amount paid out and second one is the stability condition under which reserves of both insurance companies tend to infinity. The solutions of the "degenerate two-dimensional" ruin problems strongly depend on the relative position of the vector of premium rates \( p = (p_1, p_2) \) with respect to the proportions vector \((1, 1)\). Namely, if the initial reserves satisfy \( x_2 \leq x_1 \), the two lines do not intersect. It follows therefore that the barriers the "\( \land \)" and "\( \lor \)" ruin always happen for the second and first company respectively. In this case the asymptotics follows from one-dimensional ruin theory – see e.g. Rolski et al. (1999). Therefore we focus here on the opposite case when \( x_1 < x_2 \).

In this paper we derive the exact first order asymptotics of these ruin probabilities if \( x_1, x_2 \) tend to infinity according to a ray in the positive quadrant (i.e. \( x_1/x_2 \) is constant) if the claims follow a subexponential distribution. We model
the claims by subexponential distributions since many catastrophic events like earthquakes, storms, terrorist attacks etc are used in their description. Insurance companies use e.g. the lognormal distribution (which is subexponential) to model car claims – see Rolski et al. (1999) or Embrechts et al. (1997) for the further background.

2 Main result

In order to state our results we start with recalling some notions. A distribution function \( G \) on \([0, \infty)\) is subexponential \((G \in S)\) if

\[
\lim_{x \to \infty} \frac{G^{\ast 2}(x)}{G(x)} = 2 ,
\]

where \( G^{\ast 2} \) is the fold of \( G \) with itself. The integrated tail distribution \( G_I \) of \( G \) is defined as

\[
G_I(x) = \nu - 1 \int_0^x G(y) \, dy ,
\]

where \( \nu = \int_0^\infty G(x) \, dx \).

In fact we will impose the stronger condition, that is,

\[
G(x) \sim Ax^{-\alpha} , \quad 1 < \alpha < 2 . \tag{3}
\]

Note that \( G \) is long-tailed:

\[
G(x) > 0 \text{ for all } x, \quad \lim_{x \to \infty} \frac{G(x - h)}{G(x)} = 1 , \quad \text{for all fixed } h > 0 . \tag{4}
\]

Let \( F \) denote the distribution of the claim size \( \sigma \) and define

\[
H(aK, K) = \int_0^\infty F(\max \{aK + m_1t, K + m_2t\}) \, dt \tag{5}
\]

and

\[
J(aK, K) = \int_0^\infty F(\min \{aK + m_1t, K + m_2t\}) \, dt \tag{6}
\]

for \( m_i = p_iE[\zeta] - E[\sigma] \) \((i = 1, 2)\).

In view of (2) we note that if \( a \geq 1 \), \( b_1 \lor b_2 = b_1 \) and \( b_1 \land b_2 = b_2 \), so that the crossing problems are one-dimensional. If \( F_I \) follows a subexponential distribution, Veraverbeke’s theorem (see Veraverbeke (1977), Embrechts and Veraverbeke (1982) and also Zachary (2004) for a short proof) implies that

\[
\psi_\land(aK, K) \sim \frac{1}{m_2\mu} F_I(K) , \quad \psi_\lor(aK, K) \sim \frac{1}{m_1\mu} F_I(aK) , \quad K \to \infty ,
\]

where we write \( f(x) \sim g(x) \) \((x \to \infty)\) if \( \lim_{x \to \infty} f(x)/g(x) = 1 \). In the opposite case that \( a < 1 \) the asymptotics of \( \psi_\land/\psi_\lor/\psi_\times \) are as follows:

**Theorem 1** Let \( a < 1 \). Assume that \( E[\zeta^3] < \infty \). If \( F \) satisfies (3) with \( 1 < \alpha < 2 \), then it holds that, as \( K \to \infty \),

\[
\psi_\land(aK, K) \sim \frac{1}{m_1\mu} F_I(aK) + \frac{1}{m_2\mu} F_I(K) - H(aK, K) \sim J(aK, K) , \tag{7}
\]

\[
\psi_\lor(aK, K) \sim \psi_\times(aK, K) \sim H(aK, K) . \tag{8}
\]
It is worth observing that, in contrast to the case of light-tails (see [1] Thm. 1), the asymptotic probability of crossing both $b_1$ and $b_2$ at the same time appears in the asymptotics for $\psi_\lambda(aK, K)$ (and is equal to $H(aK, K)$). In the setting of (re)insurance companies with proportional claims (see [1] for details) this result agrees with what we might expect - "large" claims cause often bankruptcy not only of the insurance company but of a whole chain of reinsurers.

The proof of the main result follows from the Lemmas ... The paper is organized as follows. In Section 3 we present useful facts and in Section 4 main (lower and upper) bounds giving the Theorem 1.

3 Key Lemmas

Let $T_0 = 0, T_n = \sum_{i=1}^{n} \zeta_i$ and $\Xi_0 = 0, \Xi_n = \sum_{i=1}^{n} \sigma_i$ denote the two random walks of claims and interarrival times and define the associated random walks $(S_n^{(1)})_{n \geq 0}$ and $(S_n^{(2)})_{n \geq 0}$ by

$$S_n^{(i)} = \Xi_n - p_i T_n \quad i = 1, 2.$$ 

Let $M_n^{(i)} = \max_{m \leq n} S_m^{(i)}$ and $M_{[n, \infty)}^{(i)} = \max_{m > n} S_m^{(i)}$ with $M_{\infty}^{(i)} = M_{[0, \infty)}^{(i)}$. Also let

$$T = \frac{x_2 - x_1}{m_1 - m_2} = \frac{x_2 - x_1}{(p_1 - p_2)E[\zeta]}$$

be the epoch that the lines $\tilde{b}_1(t) = x_1 + m_1 t$ and $\tilde{b}_2(t) = x_2 + m_2 t$ cross and let

$$\tilde{T} = \frac{x_2 - x_1}{p_1 - p_2} = \frac{T}{\lambda}$$

be the epoch that the lines $b_1(t) = x_1 + p_1 t$ and $b_2(t) = x_2 + p_2 t$ cross. We denote

$$c = \frac{1 - a}{m_1 - m_2}$$

and note that $T = cK$ if $x_1 = aK$ and $x_2 = K$. Define

$$N^* = \min\{n \geq 1 : T_n > \tilde{T}\} - 1$$

and note that $N^*$ is a stopping time w.r.t. $\sigma(\{\Xi_n, T_n\}_{n})$. Further, in view of the strong law of large numbers $N^*/T \to 1$. From the Theorem V.5.14 of Petrov (1972) we have the following upper bound:

** Lemma 1** Let $\zeta^3 < \infty$. Denote $\rho = E|\zeta - E\zeta|^3/s^2$ for $s^2 = E(\zeta - E\zeta)^2$. Then

$$\left| P \left( \frac{T_n - nE\zeta}{\sqrt{n}} < x \right) - \Phi(x) \right| \leq A \frac{\rho}{\sqrt{n(1 + |x|)^3}}$$

for all $x$ and fixed $A$. 

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Now, we have
\[
P(N^* > (1 + \epsilon)T) = P(T_{(1+\epsilon)T} < \tilde{T})
= P \left( \frac{T_{(1+\epsilon)T} - (1 + \epsilon)T}{\sqrt{T(1 + \epsilon)}} < -\sqrt{T} \frac{\epsilon}{\lambda \sqrt{(1 + \epsilon)}} \right) \leq \Phi \left( -\sqrt{T} \frac{\epsilon}{\lambda \sqrt{(1 + \epsilon)}} \right) + A \frac{(1 + \epsilon)\lambda^3}{T^{2\epsilon^3}} = O(T^{-2}) = o(T^{-\alpha}).
\]
Hence
\[
P(N^* > (1 + \epsilon)T) = o(F(T)), \quad (9)
\]
Similarly,
\[
P(N^* < (1 - \epsilon)T) = o(F(T)). \quad (10)
\]
and
\[
P(T_T < (1 - \epsilon)E\zeta T) = o(F(T)), \quad (11)
P(T_T > (1 + \epsilon)E\zeta T) = o(F(T)). \quad (12)
\]

A key step is to note that the ruin probabilities can be defined in terms of these quantities as follows:

**Proposition 1** It holds that
\[
\psi_{\vee}(aK,K) = P(M^{(2)}_\infty > K) + P(M^{(2)}_N \leq K, M^{(1)}_{N^*+1,\infty} > aK),
\]
\[
\psi_{\wedge}(aK,K) = P(M^{(1)}_N > aK) + P(M^{(1)}_N \leq aK, M^{(2)}_{N^*+1,\infty} > K),
\]
\[
\psi_{\times}(aK,K) = P(M^{(2)}_\infty > K) - P(M^{(2)}_\infty > K, M^{(1)}_\infty \leq aK)
= P(M^{(2)}_\infty > K) + P(M^{(1)}_\infty \leq aK, M^{(1)}_{N^*+1,\infty} > aK)
- P(M^{(1)}_N \leq aK, M^{(2)}_{N^*+1,\infty} > K).
\]

Indeed note that
\[
P(M^{(2)}_\infty > K, M^{(1)}_\infty \leq aK) = P(\max\{M^{(2)}_N, M^{(2)}_{N^*+1,\infty}\} \geq K, M^{(1)}_N \leq aK, M^{(1)}_{N^*+1,\infty} \leq aK)
= P(M^{(2)}_{N^*+1,\infty} > K, M^{(1)}_N \leq aK, M^{(1)}_{N^*+1,\infty} \leq aK)
= P(M^{(1)}_{N^*} \leq aK, M^{(2)}_{N^*+1,\infty} > K) - P(M^{(1)}_N \leq aK, M^{(1)}_{N^*+1,\infty} > aK).
\]

Note that in view of Veraverbeke’s theorem
\[
P(M^{(2)}_\infty > K) \sim \frac{1}{m_{2\mu}} \overline{F}_1(K).
\]

To prove the Theorem we will estimate the other terms.
4 Main estimates

Before we proceed we first introduce some extra notation and collect some auxiliary results. Let

\[ H(x, y) = I^{(2)}_{[0,T]}(x) + I^{(1)}_{[T,\infty]}(y), \]  
\[ J(x, y) = I^{(1)}_{[0,T]}(x) + I^{(2)}_{[T,\infty]}(y), \]

where \( I^{(i)}_{[x,y]}(w) = \int_x^y F(w + mt) \, dt \) for \( i = 1, 2 \). Note that definition of \( H(ak, k) \) and \( J(aK, K) \) agree with (5) and (6) respectively.

The following result relates \( H(ak, k) \) to \( I^{(2)}_{[0,\infty]}(ak) \) and \( J(aK, K) \) to \( I^{(1)}_{[0,\infty]}(K) \):

**Lemma 1** The following hold true:

(i) \( I^{(1)}_{[T,\infty]}(ak) = (m_2/m_1)I^{(2)}_{[0,\infty]}(K) \),

(ii) \( H(ak, K) \leq I^{(2)}_{[0,\infty]}(ak) \).

(iii) \( J(aK, K) \leq (m_1/m_2)I^{(1)}_{[0,\infty]}(aK) \).

(iv) For \( i = 1, 2 \) it holds that \( I^{(i)}_{[\alpha,\beta]}(\zeta) \leq \frac{\beta-\alpha}{\alpha}I^{(i)}_{[0,\alpha]}(\zeta) \) where \( \alpha, \beta, \zeta > 0 \) are constants with \( \alpha < \beta \). In particular, \( I^{(1)}_{[T,T(1+\beta)]}(ak) \leq \beta I^{(1)}_{[0,T]}(ak) \) and \( I^{(2)}_{[T,T(1+\beta)]}(K) \leq \beta I^{(2)}_{[0,T]}(K) \).

**Proof:** (i) follows by a straightforward calculation. The bounds in (ii) and (iii) follow then in view of (i), equations (13) and (14) and since \( m_2 \leq m_1 \).

(iv) Since \( F \) is decreasing it follows that \( I^{(i)}_{[\alpha,\beta]}(\zeta) \) is bounded above by \( (\beta - \alpha)F(\zeta + m_\alpha) \) and \( I^{(i)}_{[0,\alpha]}(\zeta) \) is bounded below by \( \alpha F(\zeta + m_\alpha) \). Combining these two estimates proves the statement. \( \square \)

Since \( \psi_\times \geq \psi_\vee \), we note that it will be sufficient to prove the following estimates:

\[ \liminf_{K \to \infty} \frac{\psi_\times(ak, K)}{J(ak, K)} \geq 1, \]  
\[ \liminf_{K \to \infty} \frac{\psi_\vee(ak, K)}{H(ak, K)} \geq 1, \]

and

\[ \limsup_{K \to \infty} \frac{\psi_\times(ak, K)}{J(ak, K)} \leq 1, \]  
\[ \limsup_{K \to \infty} \frac{\psi_\vee(ak, K)}{H(ak, K)} \leq 1. \]
Lemma 2 Suppose that $F$ satisfies $\Box$. Given $\eta > 0$, it holds for $K$ large enough that
\[ P(M_{N,K}^{(i)} \leq a_i K, M_{N,K+1}^{(i)} > a_i K) \sim I_{[T,\infty]}^{(i)}(a_i K). \]

Proof It follows from Lemmas $\Box \Box \Box$ and $\Box$.

Lemma 3 Suppose $F_I$ is long-tailed. Fix $0 < \eta < 1$. For $K$ large enough it holds that
\[ P(M_{N,K}^{(i)} > a_i K) \sim I_{[0,T]}^{(i)}(a_i K), \quad i = 1, 2. \]

Proof By Theorem 2 of Foss et al. (2006) for sufficiently large $K$ and small $\epsilon$ we have,
\[ P(M_{N,K}^{(i)} > a_i K) \geq P(M_{T(1-\epsilon)}^{(i)} > a_i K, N^* > T(1-\epsilon)) \]
\[ \geq P(M_{T(1-\epsilon)}^{(i)} > a_i K) - P(N^* < T(1-\epsilon)) = P(M_{T(1-\epsilon)}^{(i)} > a_i K) - o(I_{[0,T]}^{(i)}(a_i K)) \]
\[ \geq (1 - \eta/2) I_{[0,T]}^{(i)}(a_i K) - (1 - \eta/2) I_{[T,\infty]}^{(i)}(a_i K) \]
\[ \geq (1 - \eta/2) I_{[0,T]}^{(i)}(a_i K) - \eta I_{[0,T]}^{(i)}(a_i K) \geq (1 - \eta) I_{[0,T]}^{(i)}(a_i K). \]

Similarly,
\[ P(M_{N,K}^{(i)} > a_i K) \leq P(M_{T(1+\epsilon)}^{(i)} > a_i K, N^* < T(1+\epsilon)) + P(N^* > T(1+\epsilon)) \]
\[ \leq P(M_{T(1+\epsilon)}^{(i)} > a_i K) + P(N^* > T(1+\epsilon)) = P(M_{T(1+\epsilon)}^{(i)} > a_i K) + o(I_{[0,T]}^{(i)}(a_i K)) \]
\[ \leq (1 + \eta/2) I_{[0,T]}^{(i)}(a_i K) + (1 + \eta/2) I_{[T,\infty]}^{(i)}(a_i K) \]
\[ \leq (1 + \eta/2) I_{[0,T]}^{(i)}(a_i K) + (1 + \eta/2) \epsilon I_{[0,T]}^{(i)}(a_i K) \leq (1 + \eta) I_{[0,T]}^{(i)}(a_i K). \]

4.1 Lower bounds

Let
\[ a_i = \begin{cases} 
  a & \text{for } i = 1, \\
  1 & \text{for } i = 2. 
\end{cases} \]

Lemma 4 Suppose $F_I \in S$. Given $\eta > 0$, it holds for $K$ large enough that
\[ P(M_{N,K}^{(i)} > a_i K) \geq (1 - \eta) I_{[T,\infty]}^{(i)}(a_i K) - \eta I_{[0,T]}^{(i)}(a_i K). \]

Proof Fix $1 > \epsilon > 0$ and $0 < \delta < \epsilon$. By the law of large numbers it holds for $K$ large enough that
\[ P(S_{N,K}^{(i)} > -m_i(1+\epsilon)N^*, N^* < T(1+\epsilon)) > 1 - \delta. \] (19)
Denote by $	ilde{M}^{(i)}$ an independent copy of $M^{(i)}$. It follows by an application of Veraverbeke’s theorem and Lemma 1 that for $a_i K$ large enough it holds that

\[
P(M^{(i)}_{[N^*+1, \infty)} > a_i K) 
\geq P(M^{(i)}_{[N^*+1, \infty)} > a_i K, S^{(i)}_{N^*} > -(m_i + \epsilon)N^*, N^* < T(1 + \epsilon)) 
\geq P(\tilde{M}^{(i)}_{[0, \infty)} > a_i K + m_i(1 + \epsilon)T)P(S^{(i)}_{N^*} > -(m_i + \epsilon)N^*, N^* < T(1 + \epsilon)) 
\geq (1 - \delta) \mathbb{I}^{(i)}_{[0, \infty]}(a_i K + m_i(1 + 3\epsilon)T) 
\geq (1 - 2\delta) \int_0^\infty \mathcal{F}(a_i K + m_i(1 + 3\epsilon)T + m_i t)dt 
\geq (1 - 2\delta) \int_T^\infty \mathcal{F}(a_i K + m_i(1 + 3\epsilon)t)dt 
= (1 - 2\delta) \frac{m_i}{m_i(1 + 3\epsilon)} \mathbb{I}^{(i)}_{[T, \infty]}(a_i K) 
\geq (1 - 2\delta) \frac{1}{1 + 3\epsilon} \left( \mathbb{I}^{(i)}_{[T, \infty]}(a_i K) - 3\epsilon \mathbb{I}^{(i)}_{[0, T]}(a_i K) \right).
\]

\[\square\]

**Lemma 5** Suppose $F_I \in S$. For $i, j = 1, 2$, given $\eta > 0$, it holds for $K$ large enough that

\[
P(M^{(j)}_{N^*+1} \leq a_j K, M^{(i)}_{[N^*+1, \infty)} > a_i K) \geq \mathbb{I}^{(i)}_{[T, \infty]}(a_i K) - \eta \mathbb{I}^{(i)}_{[0, T]}(a_i K).
\]

**Proof** As the previous Lemma, replacing (19) by

\[
P(M^{(j)}_{N^*} \leq a_j K, S^{(i)}_{N^*} > -(m_i + \epsilon)N^*, N^* < T(1 + \epsilon)) > 1 - \delta.
\]

\[\square\]

**Lemma 6** Suppose that $F$ satisfies (3). Given $\eta > 0$, it holds for $K$ large enough that

\[
P(M^{(i)}_{[N^*+1, \infty)} > a_i K) \geq (1 - \eta) T \mathcal{F}(a_i K + m_i T) + (1 - \eta) \mathbb{I}^{(i)}_{[T, \infty]}(a_i K)
\]

**Proof** Note that

\[
P(M^{(i)}_{[N^*+1, \infty)} > a_i K) = P(S^{(i)}_{N^*+1} + \tilde{M}^{(i)}_{[0, \infty)} > a_i K),
\]

\[
\]
where $\hat{M}_{[0, \infty)}^{(i)}$ is all-time maximum from the independent random walk distributed as $S_{n_i}^{(i)}$. Moreover, for any $\epsilon > 0$,

\[
P(S_{N*+1} + \hat{M}_{[0, \infty)}^{(i)} > a_i K) \geq P(\Xi_{N*+1} + \hat{M}_{[0, \infty)}^{(i)} > a_i K + p_i T_{N*+1}, T(1+\epsilon) > N* > T(1-\epsilon))
\]

\[
\geq P(\Xi_{[T(1-\epsilon)+1]} + \hat{M}_{[0, \infty)}^{(i)} > a_i K + p_i T_{[T(1+\epsilon)+1]})
\]

\[
\geq P(\Xi_{[T(1-\epsilon)+1]} + \hat{M}_{[0, \infty)}^{(i)} > a_i K + p_i T(1+\epsilon)^2 E\zeta) - P(N* < T(1-\epsilon)) - P(T_{[T(1+\epsilon)+1]} > (1+\epsilon)^2 E\zeta T)
\]

\[
\geq P(\Xi_{[T(1-\epsilon)+1]} + \hat{M}_{[0, \infty)}^{(i)} > a_i K + p_i T(1+\epsilon)^2 E\zeta) - o(I_{[T, \infty]}(a_i K)).
\]

Further, for $1 < \alpha < 2$ and

\[
C_\alpha = \frac{1 - \alpha}{\Gamma(2-\alpha) \cos(\pi\alpha/2)}
\]

we derive

\[
P(\Xi_{[T(1-\epsilon)+1]} + \hat{M}_{[0, \infty)}^{(i)} > a_i K + p_i T(1+\epsilon)^2 E\zeta)
\]

\[
= P \left( \frac{\Xi_{[T(1-\epsilon)+1]} - E\sigma T(1-\epsilon)}{(A/C_\alpha)^{1/\alpha} T^{1/\alpha} (1-\epsilon)^{1/\alpha}} + \hat{M}_{[0, \infty)}^{(i)} > (a_i + \kappa) K + m_i T \right),
\]

where $\kappa = \epsilon(E\sigma - p_i E\zeta(2+\epsilon))(1-\alpha)/(m_1 - m_2)$. If $F$ satisfies $[3]$ with $1 < \alpha < 2$, then by Theorem 4.5.2 of Whitt (2000) it is in a normal domain of attraction of the stable law $S_{n_i}(1, 1, 0)$. Denoting $F_S(dx)$ the distribution of the stable law $S_{n_i}(1, 1, 0)$ and writing $a_i(K) = (A/C_\alpha)^{1/\alpha} T^{1/\alpha} (1-\epsilon)^{1/\alpha}$, we have

\[
P(\Xi_{[T(1-\epsilon)+1]} - E\sigma T(1-\epsilon)) a_i(K) + \hat{M}_{[0, \infty)}^{(i)} > (a_i + \kappa) K + m_i T
\]

\[
= \int_{-\infty}^{\infty} P \left( x a_i(K) + \hat{M}_{[0, \infty)}^{(i)} > (a_i + \kappa) K + m_i T \right) dF_S(x)
\]

\[
\geq \int_{(a_i + \kappa) K + m_i T/a_i(K)}^{\infty} P \left( x a_i(K) + \hat{M}_{[0, \infty)}^{(i)} > (a_i + \kappa) K + m_i T \right) dF_S(x)
\]

\[
+ \int_{(a_i + \kappa) K + m_i T/a_i(K)}^{\infty} P \left( x a_i(K) + \hat{M}_{[0, \infty)}^{(i)} > (a_i + \kappa) K + m_i T \right) dF_S(x).
\]

Recall that

\[
\mathcal{F}_S(x) \sim C_\alpha x^{-\alpha}, \quad F_S(-x) = o(x^{-\alpha}) \quad \text{as} \quad x \to \infty.
\]
Hence using integration-by-parts or monotone density theorem, result of Embrechts and Veraverbeke (1982), for large $K$ we have

\[
P \left( \Xi_{T(1-\epsilon)} - E\sigma T(1-\epsilon) \right. \left. a_m(K) + M^{(i)}_{0,\infty} > (a_i + \kappa)K + m_i T \right)
\]

\[
\geq (1 - \eta/2) \left[ \frac{A}{m_i(\alpha-1)} \int_{-(\kappa(a_i + \kappa)K + m_i T)/a_m(K)}^{\kappa(a_i + \kappa)K + m_i T/a_m(K)} ((a_i + \kappa)K + m_i T - xa_m(K))^{-\alpha+1} x^{-\alpha-1} \,dx \right]
\]

\[
\geq (1 - \eta/2) \left[ \frac{A}{m_i(\alpha-1)} \left( (1+\kappa)((a_i + \kappa)K + m_i T)^{-\alpha+1} (1 - F_S((a_i + \kappa)K + m_i T)/a_m(K)) \right. \right.
\]

\[
\left. \left. - F_S(-\kappa((a_i + \kappa)K + m_i T)/a_m(K))) \right) \right)
\]

\[
\geq (1 - \eta)AT(a_i K + m_i T)^{-\alpha} + (1 - \eta) \frac{A}{m_i(\alpha-1)} \left( a_i K + m_i T \right)^{-\alpha+1},
\]

which completes the proof.

\[\square\]

**Lemma 7** Suppose that $F$ satisfies (3). Given $\eta > 0$, it holds for $K$ large enough that

\[
P(M^{(j)}_{N^* > 1,\infty} > a_i K) \geq (1 - \eta)T\bar{F}(a_i K + m_i T) = (1 - \eta)T\bar{F}(a_i K + m_i T).
\]

**Proof** We have

\[
P(M^{(j)}_{N^* > 1,\infty} > a_i K) \geq P(M^{(j)}_{T(1-\epsilon)} > a_i K, M^{(i)}_{[T(1-\epsilon),\infty]} > a_i K, N^* > T(1-\epsilon))
\]

\[
\geq P(M^{(j)}_{T(1-\epsilon)} > a_i K, M^{(i)}_{[T(1-\epsilon),\infty]} > a_i K) - P(N^* > T(1-\epsilon))
\]

\[
\geq P(M^{(j)}_{T(1-\epsilon)} > a_i K, S^{(i)}_{[T(1-\epsilon)} > a_i K) - o(T\bar{F}(a_i K + m_i T))
\]

\[
\geq P(M^{(j)}_{T(1-\epsilon)} > a_i K, \Xi_{T(1-\epsilon)} > a_i K + p_i E\zeta T(1-\epsilon^2), T_{[T(1-\epsilon)} < E\zeta T(1-\epsilon^2))
\]

\[
-o(T\bar{F}(a_i K + m_i T))
\]

\[
\geq P(M^{(j)}_{T(1-\epsilon)} > a_i K, \Xi_{T(1-\epsilon)} > E\sigma T(1-\epsilon) > (1 + \epsilon)(a_i K + m_i T) + \epsilon(1-p_i)E\zeta T)
\]

\[
-P(T_{[T(1-\epsilon)} > E\zeta T(1-\epsilon^2)) - o(T\bar{F}(a_i K + m_i T))
\]

\[
\geq P(M^{(j)}_{T(1-\epsilon)} > a_i K, \Xi_{T(1-\epsilon)} > E\sigma T(1-\epsilon) > (1 + \epsilon)(a_i K + m_i T) + \epsilon p_i E\zeta T)
\]

\[
-o(T\bar{F}(a_i K + m_i T))
\]

\[
\geq P(M^{(j)}_{T(1-\epsilon)} > a_i K, S^{(i)}_{[T(1-\epsilon)} > (1 + \epsilon)a_i K + \epsilon(p_i + p_j)E\zeta T - 2\epsilon E\sigma T, T_{[T(1-\epsilon)} > E\zeta T(1-\epsilon^2))
\]

\[
-o(T\bar{F}(a_i K + m_i T))
\]

\[
\geq P(M^{(j)}_{T(1-\epsilon)} > a_i K, S^{(i)}_{[T(1-\epsilon)} > (1 + \epsilon + \xi)a_i K) - o(T\bar{F}(a_i K + m_i T))
\]

\[\square\]
Lemma 8

4.2 Upper bounds

Suppose that $F$ satisfies \([\mathbf{5}]\). Given $\eta > 0$, it holds for $K$ large enough that

\[
P(M_{[N^*+1,\infty]}^{(i)} > a_i K) \leq (1 + \eta) T \bar{F}(a_i K + m_i T) + (1 + \eta) I_{[T,\infty]}^{(i)}(a_i K)
\]

where $\xi = \epsilon((p_i + p_j)E\zeta - 2E\sigma(1-a))/a_j(m_1 - m_2)$. Denote by $X_k^{(i)}$ the $k$th increment of the random walk $S_n^{(i)}$. Then

\[
P(M^{(j)}_{[T(1-\epsilon)]-1} > a_j K, S^{(j)}_{[T(1-\epsilon)]} > (1 + \epsilon + \xi)a_j K)
\]

\[
\geq P(M^{(j)}_{[T(1-\epsilon)]-1} > a_j K, S^{(j)}_{[T(1-\epsilon)]} > (1 + \epsilon + \xi)a_j K, X_{[T(1-\epsilon)]} < \epsilon a_j K)
\]

\[
\geq P(M^{(j)}_{[T(1-\epsilon)]-1} > a_j K, S^{(j)}_{[T(1-\epsilon)]-1} > (1 + 2\epsilon + \xi)a_j K, X_{[T(1-\epsilon)]} > -\epsilon a_j K)
\]

\[
P(S^{(j)}_{[T(1-\epsilon)]-1} > (1 + 2\epsilon + \xi)a_j K)\]

\[
P(S^{(j)}_{[T(1-\epsilon)]-1} > (1 + 2\epsilon + \xi)a_j K) - P(X_{[T(1-\epsilon)]} < -\epsilon a_j K)
\]

\[
P(S^{(j)}_{[T(1-\epsilon)]-1} > (1 + 2\epsilon + \xi)a_j K) - o(T \bar{F}(a_j K + m_j T)).
\]

The assertion of the lemma follows now from the similar considerations as previously and given in the proof of Lemma 6.

\[
P(S^{(j)}_{[T(1-\epsilon)]-1} > (1 + 2\epsilon + \xi)a_j K)
\]

\[
\geq P((\Xi_{[T(1-\epsilon)]-1} - E\sigma(1-\epsilon))/a_\alpha(K) > (1 + \epsilon)(1 + 2\epsilon + \xi)(a_j K + m_j T)/a_\alpha(K))
\]

\[
-P(T_{[T(1-\epsilon)]-1} > TE\zeta(1+\epsilon)^2)
\]

\[
\geq \bar{F}_S((1 + \epsilon)(1 + 2\epsilon + \xi)(a_j K + m_j T)/a_\alpha(K)) - o(T \bar{F}(a_j K + m_j T))
\]

\[
\geq (1 - \eta) \bar{F}_S((a_j K + m_j T)/a_\alpha(K))
\]

\[
= (1 - \eta) T \bar{F}(a_j K + m_j T).
\]

\[\square\]
Proof Similarly like in the proof of the lower bound given in Lemma 6 for large $K$ and some $\kappa > 0$ we have,

\[ P(\tilde{S}_{N^{*}+1}^{(i)} + \tilde{M}_{[0,\infty)}^{(i)} > a_iK) \leq P(\tilde{\Xi}_{N^{*}+1}^{(i)} + \tilde{M}_{[0,\infty)}^{(i)} > a_iK + p_iT_{N^{*}+1}, N^{*} < T(1 + \epsilon)) \]

\[ + P(N^{*} > T(1 + \epsilon)) \]

\[ \leq P(\tilde{\Xi}_{[T(1+\epsilon)+1]}^{(i)} + \tilde{M}_{[0,\infty)}^{(i)} > a_iK + p_i\tilde{T}) + o(I_{[T,\infty]}^{(i)}(a_iK)) \]

\[ \leq P \left( \frac{\tilde{\Xi}_{[T(1+\epsilon)+1]}^{(i)} - E\sigma T(1 + \epsilon)}{b_{\alpha}(K)} + \tilde{M}_{[0,\infty)}^{(i)} > (a_i - \kappa)K + m_iT \right) \]

\[ + o(I_{[T, \infty]}^{(i)}(a_iK)) \]

\[ \leq (1 + \eta/2) \tilde{F}_S((a_i - \kappa)K + m_iT)/b_{\alpha}(K)) \]

\[ + \frac{C_{\alpha}A}{m_i(\alpha - 1)} \int_{K((a_i - \kappa)K + m_iT)/b_{\alpha}(K))}^{(a_i - \kappa)K + m_iT - x/\alpha} (\alpha - 1) x^{-\alpha - 1} dx \]

\[ + F_S(-\kappa((a_i - \kappa)K + m_iT)/b_{\alpha}(K)) \]

where $b_{\alpha}(K) = (A/C_{\alpha})^{1/\alpha}T^{1/\alpha}(1 + \epsilon)^{1/\alpha}$. The first two increments give required asymptotics since the second one can be estimated above by

\[ \frac{A}{m_i(\alpha - 1)} \left( (1 - \kappa) ((a_i - \kappa)K + m_iT) \right)^{-\alpha} \]

\[ (1 - \tilde{F}_S \kappa((a_i - \kappa)K + m_iT)/b_{\alpha}(K)) - F_S(-\kappa((a_i - \kappa)K + m_iT)/b_{\alpha}(K))) \]

The last increment is $o(I_{[T, \infty]}^{(i)}(a_iK))$ by (21). Finally, the third increment could be estimated in the following way,

\[ \frac{C_{\alpha}A}{m_i(\alpha - 1)} \int_{K((a_i - \kappa)K + m_iT)/b_{\alpha}(K))}^{(a_i - \kappa)K + m_iT - x/\alpha} (\alpha - 1) x^{-\alpha - 1} dx \]

\[ = b_{\alpha}(K)^{\alpha} \frac{C_{\alpha}A}{m_i(\alpha - 1)} \int_{K((a_i - \kappa)K + m_iT)/b_{\alpha}(K))}^{(a_i - \kappa)K + m_iT - w/\alpha} (\alpha - 1) w^{-\alpha - 1} dw \]

\[ = ((a_i - \kappa)K + m_iT)^{-2\alpha+1}T \frac{A^2(1 + \epsilon)}{m_i(\alpha - 1)} \int_{\kappa}^{1} (1 - t)^{-\alpha + 1} t^{-\alpha - 1} dt \]

\[ = O(K^{-2\alpha+2}) = o(K^{-\alpha+1}) = o(I_{[T, \infty]}^{(i)}(a_iK)) \]

\[ \square \]

Lemma 9 Suppose that $F$ satisfies [3]. Given $\eta > 0$, it holds for $K$ large enough that

\[ P(M_{N^{*}+1}^{(i)} > a_iK, M_{[N^{*}+1, \infty)}^{(i)} > a_iK) \leq (1 + \eta)T\tilde{F}(a_iK + m_iT) = (1 + \eta)T\tilde{F}(a_iK + m_iT) \]
Proof Similarly like in the proof of the Lemma for large $K$ we have,

\[ P(M_{N^*}^{(j)} > a_j K, M_{[N^*+1,\infty)}^{(j)} > a_j K) \]
\[ \leq P(M_{[T(1+\epsilon)]}^{(j)} - 1 > a_j K, M_{[T(1+\epsilon), \infty)}^{(j)} > a_j K) + P(N^* > T(1+\epsilon)) \]
\[ \leq P(M_{[T(1+\epsilon)]}^{(j)} - 1 > a_j K, \Xi_{[T(1+\epsilon)]} > a_j K + \epsilon \sigma T(1 - \epsilon^2), T_{[T(1+\epsilon)]} > \epsilon \zeta T(1 - \epsilon^2)) \]
\[ + o(T\overline{F}(a_j K + m_j T)) \]
\[ \leq P(M_{[T(1+\epsilon)]}^{(j)} - 1 > a_j K, \Xi_{[T(1+\epsilon)]} - \epsilon \sigma T(1 + \epsilon) > (1 - \epsilon)(a_j K + m_j T) - 2\epsilon \sigma T) \]
\[ + P(T_{[T(1+\epsilon)]} < \epsilon \zeta T(1 - \epsilon^2)) + o(T\overline{F}(a_j K + m_j T)) \]
\[ \leq P(M_{[T(1+\epsilon)]}^{(j)} - 1 > a_j K, S_{[T(1+\epsilon)]}^{(j)} > (1 - \epsilon)a_j K - \epsilon(1 + 3\epsilon)p_j \epsilon \zeta T - 2\epsilon \sigma T, \]
\[ T_{[T(1+\epsilon)]} < \epsilon \zeta T(1 + \epsilon^2) + P(T_{[T(1+\epsilon)]} > \epsilon \zeta T(1 + \epsilon^2) + o(T\overline{F}(a_j K + m_j T)) \]
\[ \leq P(M_{[T(1+\epsilon)]}^{(j)} - 1 > a_j K, S_{[T(1+\epsilon)]}^{(j)} > (1 - \epsilon - \chi)a_j K) + o(T\overline{F}(a_j K + m_j T)), \]

where $\chi = \epsilon[(1 + 3\epsilon)\epsilon \zeta + 2\epsilon \sigma](1 - a)/a_j(m_1 - m_2)$. Then

\[ P(M_{[T(1+\epsilon)]}^{(j)} - 1 > a_j K, S_{[T(1+\epsilon)]}^{(j)} > (1 - \epsilon - \chi)a_j K) \]
\[ \leq P(S_{[T(1+\epsilon)]}^{(j)} - 1 > (1 - 2\epsilon - \chi)a_j K) + P(M_{[T(1+\epsilon)]}^{(j)} - 1 > a_j K, X_{[T(1+\epsilon)]} > \epsilon a_j K) \]
\[ \leq P(S_{[T(1+\epsilon)]}^{(j)} - 1 > (1 - 2\epsilon - \chi)a_j K) + P(M_{[T(1+\epsilon)]}^{(j)} - 1 > a_j K)P(X_{[T(1+\epsilon)]} > \epsilon a_j K) \]
\[ \leq P(S_{[T(1+\epsilon)]}^{(j)} - 1 > (1 - 2\epsilon - \chi)a_j K) + O(K^{2\alpha + 1}) \]
\[ = P(S_{[T(1+\epsilon)]}^{(j)} - 1 > (1 - 2\epsilon - \chi)a_j K) + o(T\overline{F}(a_j K + m_j T)) \]

and

\[ P(S_{[T(1+\epsilon)]}^{(j)} - 1 > (1 - 2\epsilon - \chi)a_j K) \]
\[ \leq P((\Xi_{[T(1+\epsilon)]} - 1 - \epsilon \sigma T(1 + \epsilon))/b_\alpha(K) > (1 - \epsilon)(1 - 2\epsilon \chi)(a_j K + m_j T)/b_\alpha(K)) \]
\[ + P(T_{[T(1-\epsilon)]} < \epsilon \zeta(1 - \epsilon^2)) \]
\[ \leq \overline{F}(1 - \epsilon)(1 - 2\epsilon - \chi)(a_j K + m_j T)/b_\alpha(K)) + o(T\overline{F}(a_j K + m_j T)) \]
\[ \geq (1 - \eta)\overline{F}(a_j K + m_j T)/b_\alpha(K)) \]
\[ = (1 - \eta)T\overline{F}(a_j K + m_j T), \]

which completes the proof. \qed

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