Apparent superluminal velocities and random walk in the velocity space

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We conjecture that the random walk and the corresponding diffusion in the relativistic velocity space is an adequate method for describing the acceleration process in relativistic jets. Considering a simple toy model, the main features of diffusion in the velocity space are demonstrated in both non-relativistic and relativistic regimes.

Keywords: Special relativity; Velocity space; Random walk; Apparent superluminal velocities

I. INTRODUCTION

Apparent superluminal motion has been found to occur in many astrophysical phenomena with the relativistic ejecta such as active galactic nuclei (quasars, radio galaxies, blazars), gamma-ray bursts and micro-quasars [1–4]. Although these objects are significantly different in their sizes and properties, the common feature that makes possible the apparent superluminal motion is that they all support relativistic jets.

In fact, the possibility of apparent superluminal motion was predicted by Rees [5] well before the phenomenon was actually observed thanks to a new very long baseline interferometry technique, first in the quasar 3C273, and then in many other sources [2].

The basic idea behind the apparent superluminal motion is rather elementary: this effect arises from the Doppler contraction of the arrival times of photons due to the finite speed of light [6, 7]. This Doppler contraction is most easily explained by the space-time diagram in Fig. I.

Suppose a radio-emitting blob moves along world-line $AB$. Two consecutive radio-emissions at $A$ and $B$ are separated by a time interval $\Delta t_c$. Arrival times of the corresponding radio-waves at $D$ and $C$ are separated by

![Diagram of Doppler contraction](image)

FIG. 1: Illustration of the Doppler contraction of the arrival times of photons.
another time interval $\Delta t_r$. Then $BF = c \Delta t_e$ and $CD = BE = c \Delta t_r$. If the velocity of the blob is $V$, and its radial component is $V \cos \theta$, then $AF = EF = V \cos \theta \Delta t_e$. It is clear from Fig. [1] that $c \Delta t_e = BF - EF = c \Delta t_e - V \cos \theta \Delta t_e$. Therefore $\Delta t_e = \Delta t_r(1 - \beta \cos \theta)$. Since the lateral displacement of the blob is $V \sin \theta \Delta t_e$, the apparent transverse velocity of the blob (divided by the light velocity $c$) equals to

$$\beta_{\text{app}} = \frac{V \sin \theta \Delta t_e}{c \Delta t_r} = \frac{\beta \sin \theta}{1 - \beta \cos \theta}. \quad (1)$$

The apparent lateral motion is superluminal if

$$\beta > \frac{1}{\sin \theta + \cos \theta} = \frac{1}{\sqrt{2} \sin \left(\frac{\pi}{4} + \theta\right)} \geq \frac{1}{\sqrt{2}} \approx 0.71. \quad (2)$$

Consequently, apparent superluminal motion is possible only if the source moves with relativistic speed.

Relativistic jets are produced by accreting compact objects such as neutron stars or black holes, although the exact mechanism how relativistic outflows are launched is not yet clear [9]. The most promising candidate is the so-called Blandford–Znajek mechanism [10, 11], which suggests that the jets are driven by the rotational energy of the black hole, extracted electromagnetically in the form of a Poynting flux through magnetic fields penetrating the event horizon of the central black hole.

However, there is still no consensus on the mechanisms that cause acceleration and collimation of jets. Observations show that jets are subluminal near the core, then they are gradually accelerated and become relativistic at a distance relatively remote from the central engine (about a thousand Schwarzschild radii) [12].

Possible acceleration mechanisms are mostly related to the propagation of relativistic shock waves and turbulent structures in the jets and include diffusive shock acceleration (first-order Fermi acceleration), fast magnetic reconnection, second-order Fermi acceleration (stochastic acceleration), Compton rocket [9, 13–15]. There is, probably, a competition between different acceleration mechanisms, with stochastic acceleration playing a significant role [15]. Due to the stochastic nature of the acceleration process, it was proposed to describe it as diffusion in momentum space [10], assuming that the isotropic and homogeneous phase-space density $f(p, t)$ evolves in accordance with the equation

$$\frac{\partial f(p, t)}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 D(p, t) \frac{\partial f(p, t)}{\partial p} \right], \quad (3)$$

where $D(p, t)$ is the momentum-diffusion coefficient.

Our hypothesis is that probably the better description of the stochastic acceleration process is given by a random walk on the relativistic velocity space. Even a very simple type of such random walk, an inebriated-astronaut model, is surprisingly effective in producing relativistic velocities [17] in the sense that rapidity grows linearly with the number of steps, not as the square root of the number of steps, as expected from non-relativistic experience.

II. RANDOM WALK ON THE NON-RELATIVISTIC VELOCITY SPACE

We suppose that in the comoving rest frame of the particle under stochastic acceleration it receives isotropic kicks and after each kick its velocity changes by $\Delta \vec{V}$, such that $\delta V = |\Delta \vec{V}| = \text{const}$. If at the $(n - 1)$-th step the velocity was $\vec{X}$ and after the $n$-th kick it became $\vec{V}$, then $\vec{X} + \Delta \vec{V} = \vec{V}$ and

$$\vec{X} = \sqrt{V^2 + (\delta V)^2} = V - \delta V \cos \alpha + \frac{(\delta V)^2}{2V} \sin^2 \alpha, \quad (4)$$

where $\alpha$ is the angle between $\Delta \vec{V}$ and $\vec{V}$. The probability that $\alpha$ is in the range $(\alpha, \alpha + \Delta \alpha)$ is $\frac{\Delta \alpha}{\pi} = \frac{1}{2} \sin \alpha \, d\alpha$. Therefore, if $P(V, n)$ is the probability density at the $n$-th step $^2$, then we can write

$$P(V, n) = \int_0^\pi P \left( V - \delta V \cos \alpha + \frac{\delta V^2}{2V} \sin^2 \alpha, n - 1 \right) \frac{1}{2} \sin \alpha \, d\alpha. \quad (5)$$

1 This formula is valid only for nearby sources. The apparent transverse velocity of a distant radio source depends on its redshift $z$ and is reduced by a factor of $1 + z$ [2, 8].
2 That is, $P(V, n) \, d\vec{V}$ is the probability that the velocity at the $n$-th step will be in the range $(\vec{V}, \vec{V} + d\vec{V})$. 
Now we assume that \( \delta V \ll V \) and expand up to the quadratic terms in \( \delta V \):
\[
P \left( V - \delta V \cos \alpha + \frac{(\delta V)^2}{2V} \sin^2 \alpha, n - 1 \right) \approx P(V, n - 1) + \frac{\partial P}{\partial V} \left( -\delta V \cos \alpha + \frac{(\delta V)^2}{2V} \sin^2 \alpha \right) + \frac{1}{2} \frac{\partial^2 P}{\partial V^2} (\delta V)^2 \cos^2 \alpha. \tag{6}
\]
We can substitute \(6\) into \(5\), evaluate elementary integrals and obtain
\[
P(V, n) = P(V, n - 1) + \frac{1}{3} (\delta V)^2 \frac{\partial P}{\partial V} + \frac{1}{6} (\delta V)^2 \frac{\partial^2 P}{\partial V^2}. \tag{7}
\]
If \( n \gg 1 \), then \( P(V, n) - P(V, n - 1) \approx \frac{\partial P}{\partial n} \) and \(7\) can be rewritten as follows
\[
\frac{\partial P}{\partial n} = \frac{(\delta V)^2}{6V^2} \frac{\partial}{\partial V} \left( V^2 \frac{\partial P}{\partial V} \right) = \frac{(\delta V)^2}{6} \nabla^2 P, \tag{8}
\]
where \( \nabla^2 \) is the Laplacian in the velocity space, and the last equality follows from the fact that \( P \) depends only on the magnitude \( V \) of the velocity, not on the angular variables characterizing its direction.

The equation \(8\) is the same diffusion equation \(4\), provided that the momentum is given by its non-relativistic expression \( p = mV \) and the diffusion coefficient \( D(p, t) = (\delta V)^2/6 \) is a constant. We need a solution of this diffusion equation with the initial condition \( P(V, 0) = f_0 \delta(V) \), which assumes that the initial velocity was zero. Here \( f_0 \) is the normalization constant to be determined from the normalization condition \( \int P(V, n) dV = 1, n \neq 0 \).

The solution is given by the well-known Euclidean heat kernel \(18\) and has the form of a normalized Gaussian
\[
P(V, n) = \left(4\pi\right)^{-3/2} \left( \frac{n (\delta V)^2}{6} \right)^{-3/2} \exp \left( - \frac{3V^2}{2n (\delta V)^2} \right). \tag{9}
\]
As we see, the likelihood that the speed increase will far exceed \( \sqrt{n} \delta V \) is vanishingly small. As expected, the random walk efficiency for producing high speeds increases as \( \sqrt{n} \).

For the convenience of readers, we present in the appendix the derivation of the Euclidean heat kernel. Although originally associated with the heat equation \(19\), physical motivation soon receded far into the background, and heat kernels in their general form became ubiquitous in mathematics \(20,23\).

### III. RANDOM WALK ON THE RELATIVISTIC VELOCITY SPACE

If \( \delta V \ll c \), in the reference frame where initially the particle velocity is non-relativistic, it will remain non-relativistic for many consecutive kicks. Therefore, equation \(8\) remains valid in this frame locally. It will be globally valid if written in the covariant form \(17\)
\[
\frac{\partial P}{\partial n} = \frac{(\delta V)^2}{6} \nabla^2 P, \quad \nabla^2 P = \frac{1}{\sqrt{g}} \frac{\partial}{\partial V^\alpha} \left( \sqrt{g} g^{\alpha\beta} \frac{\partial P}{\partial V^\beta} \right), \tag{10}
\]
where Einstein’s summation convention is accepted, \( g_{\alpha\beta} \) is the metric tensor in the relativistic velocity space with determinant \( g \), \( g^{\alpha\beta} \) is the inverse of the metric tensor, and the expression used for the Laplacian in curved spaces can be found, for example, in \(21\).

Given two relativistic velocities, \( \vec{V} \) and \( \vec{V} + d\vec{V} \), the square of the relative velocity can be considered as a line element in three-dimensional velocity space \(25,27\). If \( \vec{V}_1 \) and \( \vec{V}_2 \) are two relativistic velocities, and \( u_1 \) and \( u_2 \) are the corresponding four-velocities, then it is easy to see that the square of the relative velocity is \(17,28\) (to simplify expressions, \( c = 1 \) is assumed throughout the rest of the paper)
\[
[\vec{V}_1 \odot \vec{V}_2]^2 = 1 - \frac{1}{(u_1 \cdot u_2)^2} = \frac{[\vec{V}_1 - \vec{V}_2]^2 - [\vec{V}_1 \times \vec{V}_2]^2}{(1 - \vec{V}_1 \cdot \vec{V}_2)^2} \tag{11}
\]
Substituting \( \vec{V} \) and \( \vec{V} + d\vec{V} \), instead of \( \vec{V}_1 \) and \( \vec{V}_2 \) in this formula, we get for the line element of the relativistic velocity frame the following expression

\[
ds^2 = \frac{(d\vec{V})^2 - [\vec{V} \times d\vec{V}]^2}{(1 - \vec{V} \cdot \vec{V})^2} = \frac{dV^2}{1 - V^2} (d\theta^2 + \sin^2 \theta \, d\phi^2) + \frac{V^2}{1 - V^2} (d\eta^2 + \sin^2 \varphi \, d\phi^2),
\]

where the last equality follows from \([\vec{V} \times d\vec{V}]^2 = V^2 d\vec{V} \cdot d\vec{V} - (\vec{V} \cdot d\vec{V})^2 \) and \( \vec{V} = V(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \).

It turns out that (12) is the Riemannian line element corresponding to the Beltrami–Klein model \( \mathbb{H}^3 \) of three-dimensional hyperbolic geometry [26, 29].

The natural parameter for Lorentz boosts is rapidity, not velocity [30]. It is not surprising therefore that (11) simplifies if, instead of the velocity \( V \), we introduce the rapidity \( \psi \) through \( V = \tanh \psi \):

\[
ds^2 = d\psi^2 + \sinh^2 \psi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right).
\]

Thus, non-zero components of the metric tensor and its determinant are

\[
g_{\psi\psi} = 1, \quad g_{\theta\theta} = \sinh^2 \psi, \quad g_{\varphi\varphi} = \sinh^2 \psi \sin^2 \theta, \quad g = \sinh^4 \psi \sin^2 \theta.
\]

Because of spherical symmetry, \( P(\psi, n) \) does not depend on \( \theta \) and \( \varphi \). Therefore, from (10), we get

\[
\frac{\partial P}{\partial n} \frac{(\delta V)^2}{6} \frac{1}{\sinh^2 \psi} \frac{\partial}{\partial \psi} \left( \sinh^2 \psi \frac{\partial P}{\partial \psi} \right) = \frac{(\delta V)^2}{6} \left[ \frac{\partial^2 P}{\sinh^2 \psi} \frac{\partial \psi}{\partial \psi} + 2 \cosh \psi \frac{\partial P}{\partial \psi} \right].
\]

To solve (15) with \( P(\psi, 0) \sim \delta(\psi) \) initial condition, let us note that

\[
\frac{\psi}{\sinh \psi} \Delta_0 \sinh \psi = \frac{\delta^2}{\partial \psi^2} + 2 \frac{\cosh \psi}{\sinh \psi} \frac{\partial}{\partial \psi} + 1 = \Delta + 1,
\]

and

\[
e^{-an} \frac{\partial}{\partial n} e^{an} = \frac{\partial}{\partial n} + a,
\]

where \( a = \frac{(\delta V)^2}{6} \), and \( \Delta_0, \Delta \) are radial parts of the Euclidean and hyperbolic Laplacians:

\[
\Delta_0 = \frac{1}{\psi^2} \frac{\partial}{\partial \psi} \left( \psi^2 \frac{\partial}{\partial \psi} \right), \quad \Delta = \frac{1}{\sinh^2 \psi} \frac{\partial}{\partial \psi} \left( \sinh^2 \psi \frac{\partial}{\partial \psi} \right).
\]

Relations (16) and (17) imply that

\[
e^{-an} \frac{\psi}{\sinh \psi} \left( \frac{\partial}{\partial n} - a\Delta_0 \right) \frac{\sinh \psi}{\psi} e^{an} = \frac{\partial}{\partial n} - a\Delta.
\]

Therefore, if \( P_0(\psi, n) \) is a solution of the Euclidean heat equation

\[
\left( \frac{\partial}{\partial n} - a\Delta_0 \right) P_0(\psi, n) = 0
\]

with the initial condition \( P_0(\psi, 0) \sim \delta(\psi) \), then

\[
P(\psi, n) = \frac{\psi}{\sinh \psi} e^{-an} P_0(\psi, n)
\]

is the solution of the heat equation

\[
\left( \frac{\partial}{\partial n} - a\Delta \right) P(\psi, n) = 0
\]

in the \( \mathbb{H}^3 \) hyperbolic space with the initial condition \( P(\psi, 0) \sim \delta(\psi) \).
Consequently, the required solution of (15) is

\[ P(\psi, n) = e^{-\frac{(\psi V)^2}{2}} \frac{\sinh \psi}{\psi} \left( 4\pi \right)^{-3/2} \left( \frac{n (\delta V)^2}{6} \right)^{-3/2} \exp \left( -\frac{3\psi^2}{2n (\delta V)^2} \right). \]  

(20)

This result agrees, as it should be, with the well-known expression of the heat kernel in the \( H^3 \) hyperbolic space \[22, 31–33\]. Besides, in the non-relativistic limit \( \psi \approx V \ll 1, n (\delta V)^2 \ll 1 \), (20) turns into (9).

The volume element in \( H^3 \) is \( d\tilde{V} = \sqrt{g} d\psi d\theta d\phi \). Therefore, the probability that \( \psi \) will be between \( \psi \) and \( \psi + d\psi \) is

\[ p(\psi, n) = \psi \sinh \psi \sqrt{4\pi} \left( \frac{n (\delta V)^2}{6} \right)^{-3/2} \exp \left( -\frac{3}{2n (\delta V)^2} \left[ \psi^2 + \frac{n^2 (\delta V)^4}{9} \right] \right). \]  

(21)

In the ultra-relativistic limit \( \psi \gg 1, \sinh \psi \approx \frac{1}{2} e^\psi \) and (21) takes the form

\[ p(\psi, n) = \frac{\psi}{\sqrt{16\pi}} \left( \frac{n (\delta V)^2}{6} \right)^{-3/2} \exp \left( -\frac{3}{2n (\delta V)^2} \left[ \psi - \frac{n (\delta V)^2}{3} \right]^2 \right). \]  

(22)

When \( n \) increases, the rapidity grows linearly with \( n \), not as \( \sqrt{n} \), and becomes more and more concentrated around \( \psi = \frac{n (\delta V)^2}{3} \).

IV. CONCLUDING REMARKS

The observation of apparent superluminal velocities in astrophysical jets indicates that particles move relativistically in these jets. To explain observed features of such jets, in addition to the initial jet acceleration by means of the Blandford-Znajek and/or some other processes, additional acceleration mechanisms operating within the jets are required, with the stochastic acceleration playing probably a significant role \[13\]. We conjecture that in this case random walk and the corresponding diffusion in the relativistic velocity space is an adequate method for describing the acceleration process in relativistic jets.

The formulation of a consistent theory of relativistic diffusion is a long-standing fascinating problem in physics \[34, 35\]. The diffusion equation is a parabolic differential equation and it assumes an infinite propagation speed for the initial data. Therefore, a relativistic diffusion cannot exist on the Minkowski space-time. However, as shown by R. M. Dudley \[36\] it still makes sense at the level of its tangent bundle (the relativistic velocity space). We believe that the idea that particles could follow a Brownian motion on the relativistic velocity space could prove interesting in the context of astrophysics \[37\].

The way we got the heat kernel on \( H^3 \) is similar (but not identical) to the use of shift operators (intertwining operators) \[38–40\] relating radial parts of Laplacians on \( H^{2n+1} \) and \( \mathbb{R}^1 \).

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Appendix A: Derivation of the Euclidean heat kernel

We want to solve the heat equation

\[ \frac{\partial P}{\partial t} = a \nabla^2 P, \]

(A1)

with the initial condition

\[ P(\bar{r}, 0) = \delta(\bar{r}). \]

(A2)
The following method of solution is adapted from [18]. We take
\[ P(\vec{r}, t) = f(x, t)f(y, t)f(z, t) \]
where \( f(x, t) \) satisfies a one-dimensional heat equation
\[ \frac{\partial f}{\partial t} = a \frac{\partial^2 f}{\partial x^2}, \quad (A3) \]
with the initial condition
\[ f(x, 0) = \delta(x). \quad (A4) \]

It is convenient to introduce a dimensionless auxiliary function \( \tilde{f}(x, t) \), such that
\[ f(x, t) = \frac{\partial}{\partial x} \tilde{f}(x, t). \]
If we further assume that this function satisfies the same one-dimensional heat equation \( (A3) \), but with the different initial condition
\[ \tilde{f}(x, 0) = \Theta(x), \quad (A5) \]
where \( \Theta(x) \) is the Heaviside step function, then \( f(x, t) \) will just satisfy \( (A3) \) with the correct initial condition \( (A4) \).

According to Buckingham’s Pi-theorem [41], any physical law can be expressed as a relationship between dimensionless quantities. From \( x, t \) and \( a \), we can construct only one independent dimensionless quantity \( \tau = \frac{x}{\sqrt{at}} \). Therefore, Pi-theorem implies that
\[ \tilde{f}(x, t) = g(\tau). \quad (A6) \]
Substituting \( (A6) \) into the one-dimensional heat equation, we get an ordinary differential equation for the unknown function \( g \):
\[ \frac{d^2 g}{d\tau^2} + \frac{\tau}{2} \frac{dg}{d\tau} = 0. \quad (A7) \]
The general solution of this equation has the form
\[ g(\tau) = g_0 + g_1 \tau \int_0^{\tau} e^{-s^2/4} ds, \quad (A8) \]
where \( g_0 \) and \( g_1 \) are some constants.

The initial condition \( (A5) \) imply that \( g(\infty) = 1 \) and \( g(-\infty) = 0 \). Therefore, \( g_0 \) and \( g_1 \) are determined by a linear system
\[ g_0 + g_1 \int_0^{\infty} e^{-s^2/4} ds = 1, \quad g_0 + g_1 \int_0^{-\infty} e^{-s^2/4} ds = 0. \quad (A9) \]
In particular,
\[ g_1 = \left[ \int_0^{\infty} e^{-s^2/4} ds - \int_0^{-\infty} e^{-s^2/4} ds \right]^{-1} = \frac{1}{2\sqrt{\pi}}. \quad (A10) \]
Then
\[ f(x, t) = \frac{\partial}{\partial x} \left[ g_0 + g_1 \int_0^\tau e^{-s^2/4} ds \right] = \frac{1}{2\sqrt{\pi}} \frac{\partial}{\partial x} e^{-\tau^2/4} = \frac{1}{\sqrt{4\pi at}} \exp \left( -\frac{x^2}{4at} \right), \quad (A11) \]
and
\[ P(\vec{r}, t) = (4\pi at)^{-3/2} \exp \left( -\frac{r^2}{4at} \right). \quad (A12) \]

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