\textbf{L}^p\text{-bounds for semigroups generated by non-elliptic quadratic differential operators}

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\textbf{Abstract}

In this note, we establish \( L^p \)-bounds for the semigroup \( e^{-tq^w(x,D)} \), \( t \geq 0 \), generated by a quadratic differential operator \( q^w(x,D) \) on \( \mathbb{R}^n \) that is the Weyl quantization of a complex-valued quadratic form \( q \) defined on the phase space \( \mathbb{R}^{2n} \) with non-negative real part \( \text{Re} \, q \geq 0 \) and trivial singular space. Specifically, we show that \( e^{-tq^w(x,D)} \) is bounded \( L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \) for all \( t > 0 \) whenever \( 1 \leq p \leq q \leq \infty \), and we prove bounds on \( \|e^{-tq^w(x,D)}\|_{L^p \to L^q} \) in both the large \( t \gg 1 \) and small \( t \ll 1 \) time regimes. Regarding \( L^p \to L^q \) bounds for the evolution semigroup at large times, we show that \( \|e^{-tq^w(x,D)}\|_{L^p \to L^q} \) is exponentially decaying as \( t \to \infty \), and we determine the precise rate of exponential decay, which is independent of \((p,q)\). At small times \( 0 < t \ll 1 \), we establish bounds on \( \|e^{-tq^w(x,D)}\|_{L^p \to L^q} \) for \((p,q)\) with \( 1 \leq p \leq q \leq \infty \) that are polynomial in \( t^{-\epsilon} \).

\section{Introduction and Statement of Results}

In this note, we prove \( L^p \)-bounds for the solution operator \( e^{-tq^w(x,D)} \) of the Schrödinger initial value problem

\[
\begin{align*}
\partial_t u(t,x) + q^w(x,D)u(t,x) &= 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n, \\
u(0,x) &= u_0(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

where \( u_0 \in L^2(\mathbb{R}^n) \) is the initial data, \( q = q(x,\xi) \) is a complex-valued quadratic form on the phase space \( \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \) with non-negative real part \( \text{Re} \, q \geq 0 \), and \( q^w(x,D) \) is the Weyl quantization of \( q(x,\xi) \), defined by

\[
q^w(x,D)v(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} q \left( \frac{x+y}{2},\xi \right) v(y) \, dy \, d\xi, \quad v \in S'(\mathbb{R}^n),
\]

in the sense of distributions. Operators of the form \( q^w(x,D) \) are quadratic differential operators with a simple, explicit expression. This is because the Weyl quantization of a quadratic monomial of the form \( x^\alpha\xi^\beta \), where \( \alpha, \beta \in \mathbb{N}^n, |\alpha + \beta| = 2 \), is

\[
\frac{x^\alpha D^\beta + D^\beta x^\alpha}{2}, \quad D := \frac{1}{i} \partial.
\]

The class of evolution equations of the form \( q^w(x,D) \) contains a number of familiar examples, such as the free Schrödinger equation where \( q(x,\xi) = i|\xi|^2 \), \( (x,\xi) \in \mathbb{R}^{2n} \), the quantum harmonic oscillator, where \( q(x,\xi) = i(|x|^2 + |\xi|^2) \), \( (x,\xi) \in \mathbb{R}^{2n} \), the heat equation, where \( q(x,\xi) = |\xi|^2 \), \( (x,\xi) \in \mathbb{R}^{2n} \), and the Kramers-Fokker-Planck equation with a quadratic potential, where \( q(x,v,\xi,\eta) = \eta^2 + \frac{1}{2} v^2 + i(v \cdot \xi - ax \cdot \eta) \), for

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\((x, v, \xi, \eta) \in \mathbb{R}^{4n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n}\) and \(a \in \mathbb{R} \setminus \{0\}\) a constant. From the work \([14]\), it is known that the operator \(q^w(x, D)\), regarded as an unbounded operator on \(L^2(\mathbb{R}^n)\) equipped with the maximal domain

\[
D_{\text{max}} = \{ u \in L^2(\mathbb{R}^n) : q^w(x, D)u \in L^2(\mathbb{R}^n) \},
\]

is maximally accretive and generates a strongly continuous contraction semigroup \(G(t) := e^{-tq^w(x, D)}, t \geq 0\), on \(L^2(\mathbb{R}^n)\). We may regard \(G(t)\) as the solution operator for the problem (1.1). Given that a wide range of physical processes may be modeled by equations of the form (1.1), it is of interest to understand the \(\parallel G(t)\parallel_{L^p \rightarrow L^q}\) mapping properties of the evolution semigroup \(G(t)\) and to obtain bounds for the operator norm \(\parallel G(t)\parallel_{L^p \rightarrow L^q}\) at various time regimes. Let us mention that the study of \(L^p\)-bounds for semigroups generated by self-adjoint Schrödinger operators has a long and rich tradition in the field of mathematical physics. We refer to \([7, 8, 23]\), and \([24]\) for some fundamental results in this area. In particular, \(L^p\)-bounds for the propagator \(G(t)\) were obtained in \([17]\) in the case when \((1.1)\) is the time evolution of the quantum harmonic oscillator.

In this note, we shall be primarily interested in obtaining \(L^p \rightarrow L^q\) bounds for \(G(t)\) in the case when the quadratic form \(q\) is non-elliptic. In order to recount the known results in this direction, we pause to recall the notion of the singular space of a complex-valued quadratic form \(q\) on \(\mathbb{R}^{2n}\) with non-negative real part \(\text{Re } q \geq 0\). Let \(\mathbb{R}^{2n}\) be equipped with the standard symplectic form

\[
\sigma((x, \xi), (y, \eta)) = \xi \cdot y - x \cdot \eta, \quad (x, \xi), \ (y, \eta) \in \mathbb{R}^{2n}.
\]

(1.5)

Suppose \(q : \mathbb{R}^{2n} \rightarrow \mathbb{C}\) is a complex-valued quadratic form with \(\text{Re } q \geq 0\) and let \(q(\cdot, \cdot)\) denote its symmetric \(\mathbb{C}\)-bilinear polarization. Because \(\sigma\) is nondegenerate, there is a unique \(F \in \text{Mat}_{2n \times 2n}(\mathbb{C})\) such that

\[
q((x, \xi), (y, \eta)) = \sigma((x, \xi), F(y, \eta))
\]

(1.6)

for all \((x, \xi), (y, \eta) \in \mathbb{R}^{2n}\). This matrix \(F\) is called the Hamilton map or Hamilton matrix of \(q\) (see Section 21.5 of \([16]\)). Explicitly, the Hamilton map of \(q\) is given by

\[
F = \frac{1}{2}H_q,
\]

(1.7)

where \(H_q = \begin{pmatrix} q^\xi & -q_x \end{pmatrix}\) is the Hamilton vector field of \(q\), viewed as a linear map \(\mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}\). Let

\[
\text{Re } F = \frac{F + \overline{F}}{2}, \quad \text{Im } F = \frac{F - \overline{F}}{2i}
\]

be the real and imaginary parts of \(F\) respectively. The singular space \(S\) of \(q\) is defined as the following finite intersection of kernels:

\[
S = \left( \bigcap_{j=0}^{2n-1} \ker \left[ (\text{Re } F)(\overline{\text{Im } F})^j \right] \right) \cap \mathbb{R}^{2n}.
\]

(1.8)

The singular space was first introduced by M. Hitrik and K. Pravda-Starov in \([9]\) where it arose naturally in the study of spectra and semi-group smoothing properties for non-self adjoint quadratic differential operators. The concept of the singular space has since been shown to play a key role in the understanding of hypoelliptic and spectral properties of non-elliptic quadratic differential operators. See for instance \([10, 11, 19, 20, 27, 28]\). Recent work has also shown that the singular space is vital for the description of the propagation of microlocal singularities for the evolution \((1.1)\). We refer the reader to \([15, 21, 22, 24, 30]\), as well as \([2]\) and \([3]\).

Let \(q\) be a complex-valued quadratic form on \(\mathbb{R}^{2n}\) with non-negative real part \(\text{Re } q \geq 0\). Let \(S\) be the singular space of \(q\). The quadratic form \(q\) is said to be elliptic if

\[
q(X) = 0, \quad X \in \mathbb{R}^{2n} \implies X = 0,
\]

(1.9)
If (1.9) fails to hold, then we say that \( q \) is non-elliptic. To the best of our knowledge, there are currently only two general results regarding \( L^p \to L^q \) bounds for the semigroup \( G(t) \) in the case when \( q \) is non-elliptic. First, in Theorem 1.2.3 of [9], it was established that \( \|G(t)\|_{L^2 \to L^2} \) decays exponentially as \( t \to \infty \) whenever \( S \) is symplectic and distinct from the entire phase space \( \mathbb{R}^{2n} \). In other words, if \( S \) is symplectic and \( S \neq \mathbb{R}^{2n} \), then there are \( C, c > 0 \) such that

\[
\|G(t)\|_{L^2 \to L^2} \leq Ce^{-ct}, \quad t \geq 0. \tag{1.10}
\]

Thanks to the subsequent work [18], it is also known that if \( S \) is trivial, i.e. \( S = \{0\} \), then the optimal rate of exponential decay of \( \|G(t)\|_{L^2 \to L^2} \) is the quantity \( \gamma \) defined below in Theorem 1.1. The second general result concerning \( L^p - L^q \) bounds for \( G(t) \) is Theorem 1.2 of [12], which yields the following \( L^2 - L^\infty \) estimate: if \( S = \{0\} \), then, for every \( s > n/2 \), there is \( C > 0 \) such that

\[
\|G(t)\|_{L^2 \to L^\infty} \leq Ct^{-\frac{1}{2}(2k_0+1)(2n+s)}, \quad 0 < t \ll 1, \tag{1.11}
\]

where \( k_0 \in \{0, 1, \ldots, 2n-1\} \) is the smallest non-negative integer such that

\[
\bigcap_{j=0}^{k_0} \ker [(\text{Re } F)(\text{Im } F)^2] \cap \mathbb{R}^{2n} = \{0\}. \tag{1.12}
\]

Our goal in the present work is to prove bounds for the operator norm \( \|G(t)\|_{L^p \to L^q} \) with \( (p, q) \) more general than \((2, 2)\) and \((2, \infty)\). The main result of this note refines and extends the bounds (1.10) and (1.11) under the assumption that \( S = \{0\} \). We recall from Theorem 1.2.2 of [9] that when \( S = \{0\} \) the spectrum of the quadratic differential operator \( q^w(x, D) \) is only composed of eigenvalues of finite algebraic multiplicity with

\[
\text{Spec}(q^w(x, D)) = \left\{ \sum_{\lambda \in \text{Spec}(F) \atop \text{Im}(\lambda) > 0} (r_\lambda + 2k_\lambda)(-i\lambda) : k_\lambda \in \mathbb{N} \right\}, \tag{1.13}
\]

where \( r_\lambda \) is the dimension of the space of generalized eigenvectors of the Hamilton matrix \( F \) of \( q \) in \( \mathbb{C}^{2n} \) corresponding to the eigenvalue \( \lambda \in \mathbb{C} \). In particular, the eigenvalue of \( q^w(x, D) \) obtained by setting \( k_\lambda = 0 \) for all \( \lambda \in \text{Spec}(F) \) in (1.13) is

\[
\rho = \sum_{\lambda \in \text{Spec}(F) \atop \text{Im}(\lambda) > 0} -ir_\lambda \lambda. \tag{1.14}
\]

We may think of \( \rho \) as the ‘lowest eigenvalue’ or ‘ground state energy’ of the operator \( q^w(x, D) \).

**Theorem 1.1.** Let \( q, q^w(x, D), G(t), S, \) and \( F \) be as above. Assume that \( S = \{0\} \).

1. Let \( \gamma = \text{Re}(\rho) > 0 \). For every \( 1 \leq p \leq q \leq \infty \) and \( \epsilon > 0 \), there are constants \( C = C_{\epsilon, p, q} > 0 \) and \( c = c_{p, q} > 0 \), such that

\[
ce^{-\gamma t} \leq \|G(t)\|_{L^p \to L^q} \leq Ce^{-\gamma t}, \quad t \geq \epsilon. \tag{1.15}
\]

2. Let \( k_0 \in \{0, 1, \ldots, 2n-1\} \) be the smallest non-negative integer such that (1.12) holds. There is a time \( 0 < t_0 \ll 1 \) such that for any \( 1 \leq p \leq q \leq \infty \) we have

\[
\|G(t)\|_{L^p \to L^q} \leq C t^{-(2k_0+1)n}, \quad 0 < t \leq t_0, \tag{1.16}
\]

for some constants \( C = C_{p, q} > 0 \) and \( c = c_{p, q} > 0 \).
Let us make some general comments regarding Theorem 1.1. First, the bounds (1.15) show that for any $1 \leq p \leq q \leq \infty$ the operator norm $\|G(t)\|_{L_p \to L_q}$ decays exponentially as $t \to \infty$, with $\gamma$ being the precise rate of decay, independent of $(p,q)$. To prove that $\gamma$ is the exact rate of exponential decay, one may examine the action of the propagator $G(t)$ on the ‘ground state’ eigenfunction of $q^{\omega}(x,D)$ corresponding to the eigenvalue $\rho$ (see Section 4 below). Regarding the short time $0 < t \ll 1$ bounds in Theorem 1.1 it is clear that (1.16) is not sharp for all $1 \leq p \leq q \leq \infty$. For instance, (1.16) fails to reproduce (1.10) when $p = q = 2$. However, one may interpolate (1.16) with the bound (1.10) gives $G(t) = \mathcal{O}_{L^2 \to L^\infty}(t^{-\frac{(2k_0+1)n}{n}})$ as $t \to 0^+$, which is an improvement over (1.11).

Finally, let us briefly touch on the main ideas involved in the proof of Theorem 1.1. In the recent work [30], we showed that if $T_c$ is a global metaplectic FBI transform on $\mathbb{R}^n$, in the sense of either Chapter 13 of [31] or the minicourse [13], then the conjugated propagator $\tilde{G}(t) := T_c \circ G(t) \circ T_c^*$ is, for each $t \geq 0$, a metaplectic Fourier integral operator acting on the Bargmann space $H_{\Phi_0}(\mathbb{C}^n)$, which is the unitary image of $L^2(\mathbb{R}^n)$ under $T_c$. In particular, we showed that the ‘Bergman form’ (9, 28) of $\tilde{G}(t)$ is given by

$$\tilde{G}(t)u(z) = \tilde{a}(t) \int_{\mathbb{C}^n} e^{2\Phi_t(z,w)} u(w) e^{-2\Phi_0(w)} L(dw), \quad z \in \mathbb{C}^n, \quad u \in H_{\Phi_0}(\mathbb{C}^n), \quad t \geq 0,$$

where $L(dw)$ is the Lebesgue measure on $\mathbb{C}^n$, $\Phi_0(w) := \sup_{y \in \mathbb{R}^n} (-\text{Im } \varphi(w,y)), \quad w \in \mathbb{C}^n$, is the strictly plurisubharmonic form on $\mathbb{C}^n$ associated to $\varphi$, $\Psi_t$ is a holomorphic quadratic form on $\mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$ depending analytically on $t \geq 0$, and $\tilde{a} \in C^\infty([0,\infty); \mathbb{C})$ is a non-vanishing amplitude. Moreover, we showed that $\Psi_t$ and $\tilde{a}$ are the solutions of an eikonal equation and a transport equation, respectively. In particular, we did not attempt to solve these equations explicitly for $\Psi_t$ and $\tilde{a}$. Now, thanks to the work [1], it is known that when the singular space is trivial $S = \{0\}$ it is possible to choose a metaplectic FBI transform $T_c$ so that conjugated semigroup has the simple form

$$\tilde{G}(t)u(z) = e^{\frac{4\pi}{(M)t}} u(e^{itM}z), \quad u \in H_{\Phi_0}(\mathbb{C}^n), \quad t \geq 0,$$

where $M \in \text{Mat}_{n \times n}(\mathbb{C})$ is a suitable matrix. In the present work, we show that this choice of $T_c$ leads to equations for $\Psi_t$ and $\tilde{a}$ that may be easily solved. One may then show that (1.17) coincides with (1.18), giving an alternative derivation of (1.18). Once the Bergman form of $\tilde{G}(t)$ is known and a basic estimate for the real part of its phase function is established, the bounds (1.15) and (1.16) follow easily by writing down a formal expression for the Schwartz kernel of the composition $T_c^* \circ \tilde{G}(t) \circ T_c$ using (1.17) and applying Young’s integral inequality.

The plan for this note is as follows. In Section 2, we recall how to choose the FBI transform $T_c$ so that (1.18) holds. In Section 3, we determine the Bergman form (1.17) of $\tilde{G}(t)$ for $t \geq 0$ and prove some basic estimates. In Section 4, we conclude the proof of Theorem 1.1 as outlined in this introduction.

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## 2 Reduction to a Normal Form on the FBI Transform Side

In this section, we follow the approach of [14] and [28] for reducing $q^{\omega}(x,D)$ to a normal form via a metaplectic FBI transform. We provide additional references where convenient. Let $q$ be a complex-valued quadratic form on $\mathbb{R}^{2n}$ with non-negative real part $\text{Re } q \geq 0$ and trivial singular space $S = \{0\}$. Let $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_{\bar{z}}$ be equipped with the standard complex symplectic form $\sigma = dz \wedge d\bar{z}$. Let $F$ be the Hamilton matrix of $q$ introduced in (1.6). From the work [9], it is known that the matrix $F$ has no real eigenvalues. Consequently,

$$\# \{ \lambda \in \text{Spec}(F) : \text{Im } \lambda > 0 \} = \# \{ \lambda \in \text{Spec}(F) : \text{Im } \lambda < 0 \},$$

(2.1)
counting algebraic multiplicities. For $\lambda \in \text{Spec}(F)$, let
\[ V_\lambda = \ker \left((F - \lambda)^{2n}\right) \subset \mathbb{C}^{2n} \]  
(2.2)
be the generalized eigenspace of $F$ corresponding to $\lambda$. Let us also introduce the stable outgoing and stable incoming manifolds for the quadratic form $-iq$ given by
\[ \Lambda^+ = \bigoplus_{\lambda \in \text{Spec}(F), \text{Im} \lambda > 0} V_\lambda, \quad \Lambda^- = \bigoplus_{\lambda \in \text{Spec}(F), \text{Im} \lambda < 0} V_\lambda, \]  
(2.3)
respectively. By Proposition 2.1 of [28], $\Lambda^+$ is a strictly positive $\mathbb{C}$-Lagrangian subspace of $\mathbb{C}^{2n}$ in the sense that $\Lambda^+$ is Lagrangian with respect to the complex symplectic form $\sigma$ and
\[ \frac{1}{i} \sigma(Z, \bar{Z}) > 0, \quad Z \in \Lambda^+ \setminus \{0\}, \]  
(2.4)
and $\Lambda^-$ is a strictly negative $\mathbb{C}$-Lagrangian subspace of $\mathbb{C}^{2n}$ in the sense that $\Lambda^-$ is Lagrangian for the form $\sigma$ and (2.4) holds for all $Z \in \Lambda^- \setminus \{0\}$ with ‘$>$’ replaced by ‘$<$’. For background information regarding positive and negative $\mathbb{C}$-Lagrangian subspaces of $\mathbb{C}^{2n}$, we refer to either [13] or [6]. In particular, from the discussion on pages 488-489 of [13], we know that there exists a holomorphic quadratic form $\varphi = \varphi(z, y)$ on $\mathbb{C}^{2n} = \mathbb{C}_z \times \mathbb{C}_y$ with
\[ \det \varphi''_{zy} \neq 0, \quad \text{Im} \varphi''_{yy} > 0, \]  
(2.5)
such that the complex linear canonical transformation
\[ \kappa_{\varphi} : \mathbb{C}^{2n} \ni (y, -\varphi'_y(z, y)) \mapsto (z, \varphi'_z(z, y)) \in \mathbb{C}^{2n}, \quad (z, y) \in \mathbb{C}^{2n}, \]  
(2.6)
generated by $\varphi$ satisfies
\[ \kappa_{\varphi}(\Lambda^+) = \{(z, 0) : z \in \mathbb{C}_n\}, \quad \kappa_{\varphi}(\Lambda^-) = \{(0, \zeta) : \zeta \in \mathbb{C}_n\}. \]  
(2.7)
Let
\[ \Phi_0(z) = \sup_{y \in \mathbb{R}^n} (-\text{Im} \varphi(z, y)), \quad z \in \mathbb{C}_n, \]  
(2.8)
be the strictly plurisubharmonic quadratic form on $\mathbb{C}_n$ associated to the phase $\varphi$ (see Chapter 13 of [31] or Section 1.3 of [13]), and let
\[ \Lambda_{\Phi_0} = \left\{ z, \frac{2}{i} \Phi'_{0,z}(z) : z \in \mathbb{C}_n \right\}. \]  
(2.9)
From either Theorem 13.5 of [31] or Proposition 1.3.2 of [13], we have
\[ \kappa_{\varphi}(\mathbb{R}^{2n}) = \Lambda_{\Phi_0}, \]  
(2.10)
and thus $\Lambda_{\Phi_0}$ is $I$-Lagrangian and $R$-symplectic for the complex symplectic form $\sigma$. Also, the strict positivity of $\Lambda^+$ in conjunction with (2.7) gives that the base $\{(z, 0) : z \in \mathbb{C}_n\}$ is strictly positive relative to $\Lambda_{\Phi_0}$ (see e.g. [6]). It then follows, as explained in Chapter 11 of [25], that the quadratic form $\Phi_0$ is strictly convex. Let
\[ \tilde{q} = q \circ \kappa_{\varphi}^{-1}, \]  
(2.11)
regarded as a holomorphic quadratic form on $\mathbb{C}^{2n}$. Since $\Lambda^+$ and $\Lambda^-$ are invariant under $F$ and Lagrangian with respect to $\sigma$, we have
\[ q(X) = \sigma(X, FX) = 0, \quad X \in \Lambda^+ \cup \Lambda^- . \]  
(2.12)
From (2.7) and (2.11), it follows that \( \tilde{q} \) must be of the form
\[
\tilde{q}(z, \zeta) = Mz \cdot \zeta, \quad (z, \zeta) \in \mathbb{C}^{2n},
\] (2.13)
for some \( M \in \text{Mat}_{n \times n} (\mathbb{C}^n) \). In particular, the complex Hamilton vector field of \( \tilde{q} \) with respect to \( \sigma \) is
\[
H_\tilde{q} = (Mz, -MT \zeta), \quad (z, \zeta) \in \mathbb{C}^{2n}. \tag{2.14}
\]
The Hamilton map of \( \tilde{q} \) is thus given by \( \tilde{F} = \frac{1}{2} H_\tilde{q} \), and we have
\[
\tilde{F} = \frac{1}{2} \begin{pmatrix} M & 0 \\ 0 & -MT \end{pmatrix}. \tag{2.15}
\]
As a consequence of (2.11), (1.6), and the invariance of \( \tilde{q} \), it is true that \( \tilde{F} = \kappa_\varphi \circ F \circ \kappa_\varphi^{-1} \). Since also \( \tilde{F} \) maps \( (z, 0) \in \kappa_\varphi (\Lambda^+) \) to \( \frac{1}{2} (Mz, 0) \in \kappa_\varphi (\Lambda^+) \), we have
\[
\text{Spec}(M) = \text{Spec}(2F) \cap \{ \lambda \in \mathbb{R} \mid \lambda > 1 \}, \tag{2.16}
\]
with agreement of algebraic multiplicities.

Let \( T_\varphi : \mathcal{S}'(\mathbb{R}^n) \to \text{Hol}(\mathbb{C}^n) \) be the metaplectic FBI transform on \( \mathbb{R}^n \) associated to \( \varphi \), given in the sense of distributions by
\[
T_\varphi u(z) = c_\varphi \int_{\mathbb{R}^n} e^{i \varphi(z, y)} u(y) \, L(dy), \quad u \in \mathcal{S}'(\mathbb{R}^n),
\] (2.17)
where
\[
c_\varphi = 2^{-n/2} \pi^{-3n/4} (\det \varphi''_{yy})^{-1/4} |\det \varphi''_{zy}|. \tag{2.18}
\]
By Theorem 13.7 of [31], \( T_\varphi \) is unitary \( L^2(\mathbb{R}^n) \to H_{\Phi_0}(\mathbb{C}^n) \), where
\[
H_{\Phi_0}(\mathbb{C}^n) := L^2(\mathbb{C}^n, e^{-2\Phi_0(z)} L(dz)) \cap \text{Hol}(\mathbb{C}^n)
\] (2.19)
is the Bargmann space associated to the weight \( \Phi_0 \), equipped with the natural Hilbert space structure inherited from \( L^2(\mathbb{C}^n, e^{-2\Phi_0(z)} L(dz)) \). Here \( L(dz) \) denotes the Lebesgue measure on \( \mathbb{C}^n \). Let \( \tilde{\varphi}^w(z, D) \) denote the complex Weyl quantization of the symbol \( \tilde{q} \) with respect to the weight \( \Phi_0 \). We recall that \( \tilde{\varphi}^w(z, D) \) is defined as an unbounded operator on \( H_{\Phi_0}(\mathbb{C}^n) \) that acts on suitable \( u \in H_{\Phi_0}(\mathbb{C}^n) \) by
\[
\tilde{\varphi}^w(z, D) u(z) = \frac{1}{(2\pi)^n} \int_{\Gamma_{\Phi_0}(z)} e^{i(z - w) \cdot \zeta} \tilde{q}^w \left( \frac{z + w}{2}, \zeta \right) u(w) \, dw \wedge d\zeta, \quad z \in \mathbb{C}^n,
\] (2.20)
for the contour of integration
\[
\Gamma_{\Phi_0}(z) : w \mapsto \zeta = \frac{2}{i} \Phi_0'(z) \left( \frac{z + w}{2} \right), \quad w \in \mathbb{C}^n, \quad z \in \mathbb{C}^n.
\] (2.21)
For more information on Weyl quantization in the complex domain, see Chapter 13 of [31] or Section 1.4 of [13]. By Egorov’s theorem (see Theorem 13.9 in [31] or Theorem 1.4.2 of [13]), we have
\[
\tilde{\varphi}^w(x, D) = T_\varphi^* \circ \tilde{\varphi}^w(z, D) \circ T_\varphi \tag{2.22}
\]
when both sides are viewed as operators acting on the maximal domain of \( \tilde{\varphi}^w(x, D) \),
\[
D_{\text{max}} = \left\{ u \in L^2(\mathbb{R}^n) : \tilde{\varphi}^w(x, D) u \in L^2(\mathbb{R}^n) \right\}. \tag{2.23}
\]
Let
\[ D_{\text{max}} = \{ u \in H_{\Phi_0}(\mathbb{C}^n) : \tilde{q}^w(z, D)u \in H_{\Phi_0}(\mathbb{C}^n) \} \]  
(2.24)
be the maximal domain of \( \tilde{q}^w(z, D) \), and let us view \( \tilde{q}^w(z, D) \) as an unbounded operator on \( H_{\Phi_0}(\mathbb{C}^n) \) with the domain \( D_{\text{max}} \). Thanks to (2.22), we have
\[ D_{\text{max}} = T_\varphi (D_{\text{max}}). \]  
(2.25)

Let \( G(t) = e^{-t\tilde{q}^w(x, D)} \), \( t \geq 0 \), be the strongly continuous semigroup on \( L^2(\mathbb{R}^n) \) generated by \( q^w(x, D) \) (see (1.6)). From (2.22), (2.25), and the unitarity of \( T_\varphi \), it follows that \( \tilde{q}^w(z, D) \) generates a strongly continuous semigroup \( \tilde{G}(t) = e^{-t\tilde{q}^w(z, D)} \), \( t \geq 0 \), on \( H_{\Phi_0}(\mathbb{C}^n) \). The semigroups \( G(t) \) and \( \tilde{G}(t) \) are related by
\[ G(t) = T_\varphi^* \circ \tilde{G}(t) \circ T_\varphi \]  
(2.26)
for all \( t \geq 0 \).

We have established the following proposition, which summarizes the discussion in this section.

**Proposition 2.1.** Let \( q \) be a complex-valued quadratic form on \( \mathbb{R}^{2n} \) with non-negative real part \( \text{Re} q \geq 0 \) and trivial singular space \( S = \{ 0 \} \). Let \( F \) be the Hamilton matrix of \( q \), and let \( q^w(x, D) \) be the Weyl quantization of \( q \), viewed as an unbounded operator on \( L^2(\mathbb{R}^n) \) equipped with its maximal domain \( D_{\text{max}} \) defined in (2.23). Let \( G(t) = e^{-tq^w(x, D)} \), \( t \geq 0 \), be the strongly continuous semigroup on \( L^2(\mathbb{R}^n) \) generated by \( q^w(x, D) \).

1. There exists a holomorphic quadratic form \( \varphi \) on \( \mathbb{C}^{2n} \) satisfying (2.2) such that the quadratic form \( \Phi_0 \) defined by (3.8) is strictly convex and the complex linear canonical transformation \( \kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n} \) defined implicitly by (2.6) has the property that
\[ \tilde{q}(z, \zeta) := (q \circ \kappa^{-1}_\varphi)(z, \zeta) = Mz \cdot \zeta, \quad (z, \zeta) \in \mathbb{C}^{2n}, \]  
(2.27)
where \( M \in \text{Mat}_{n \times n}(\mathbb{C}) \) is such that \( \text{Spec}(M) = \text{Spec}(2F) \cap \{ \lambda > 0 \} \) with agreement of algebraic multiplicities.

2. Let \( \tilde{q}^w(z, D) \) be the complex Weyl quantization (2.1) of \( \tilde{q} \) with respect to the weight \( \Phi_0 \), realized as an unbounded operator on the Bargmann space \( H_{\Phi_0}(\mathbb{C}^n) \) introduced in (2.19) equipped with the maximal domain \( \tilde{D}_{\text{max}} \) defined in (2.23). The operator \( \tilde{q}^w(z, D) \) generates a strongly continuous semigroup \( \tilde{G}(t) = e^{-t\tilde{q}^w(z, D)} \), \( t \geq 0 \), on \( H_{\Phi_0}(\mathbb{C}^n) \) that is unitarily equivalent to \( G(t) \) for each \( t \geq 0 \). This unitary equivalence is given by the FBI transform \( T_\varphi \) introduced in (2.17), i.e.
\[ G(t) = T_\varphi^* \circ \tilde{G}(t) \circ T_\varphi, \quad t \geq 0. \]  
(2.28)

3 **The Evolution Semigroup on the FBI Transform Side**

We now study the semigroup \( \tilde{G}(t) \), \( t \geq 0 \). Let \( \Psi_0 \) be the polarization of \( \Phi_0 \), i.e. \( \Psi_0 \) is the unique holomorphic quadratic form on \( \mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n \) such that \( \Psi_0(z, \overline{z}) = \Phi_0(z) \) for all \( z \in \mathbb{C}^n \). Since
\[ \Phi_0(z) = \frac{1}{2} \Phi^\prime_{0,zz}z \cdot z + \Phi^\prime_{0,\overline{z}z}z \cdot \overline{z} + \frac{1}{2} \Phi^\prime_{0,\overline{z}z}\overline{z} \cdot \overline{z}, \quad z \in \mathbb{C}^n, \]  
(3.1)
we see that \( \Psi_0 \) is given explicitly by
\[ \Psi_0(z, \theta) = \frac{1}{2} \Phi^\prime_{0,zz}z \cdot z + \Phi^\prime_{0,\overline{z}z}z \cdot \theta + \frac{1}{2} \Phi^\prime_{0,\overline{z}z}\theta \cdot \theta, \quad (z, \theta) \in \mathbb{C}^{2n}. \]  
(3.2)
In the work [30], we showed that for every \( t \geq 0 \) the semigroup \( \tilde{G}(t) \) is a metaplectic Fourier integral operator in the complex domain whose underlying complex canonical transformation is the Hamilton flow \( \tilde{\kappa}_t \) of the symbol \( \tilde{q} \) at time \( t/i \), i.e.

\[
\tilde{\kappa}_t = \exp \left( \frac{t}{i} H_{\tilde{q}} \right), \quad t \geq 0.
\]  

(3.3)

In view of (2.14), we have

\[
\tilde{\kappa}_t(z, \zeta) = \begin{pmatrix} e^{-itM} z, e^{itM^*} \zeta \end{pmatrix}, \quad (z, \zeta) \in \mathbb{C}^{2n}, \quad t \geq 0.
\]  

(3.4)

For background information regarding metaplectic Fourier integral operators in the complex domain, see Appendix B of [4]. In particular, in the work [6], it was shown that every such metaplectic Fourier integral operator in \( \mathbb{C}^n \) possesses a unique `Bergman form.' In Section 6 of [30], we proved that the Bergman form of \( \tilde{G}(t) \) is given by

\[
\tilde{G}(t)u(z) = \hat{a}(t) \int_{\mathbb{C}^n} e^{2\Psi_t(z, \zeta)} u(w) e^{-2\Phi_0(w)} L(dw), \quad z \in \mathbb{C}^n, \quad u \in H_{\Phi_0}(\mathbb{C}^n),
\]  

(3.5)

where \( \Psi_t \) is a holomorphic quadratic form on \( \mathbb{C}^{2n} \), depending analytically on \( t \geq 0 \), and \( \hat{a} \in C^\infty([0, \infty); \mathbb{C}) \) is a non-vanishing amplitude. In addition, we showed that \( \Psi_t, t \geq 0 \), is the unique solution of the eikonal equation

\[
\begin{cases}
2\partial_t \Psi_t(z, \theta) + \tilde{q} \left( z, 2\Psi_t'(z, \theta) \right) = 0, & (z, \theta) \in \mathbb{C}^{2n}, \quad t \geq 0, \\
\Psi_t(z, \theta) \big|_{t=0} = \Psi_0(z, \theta), & (z, \theta) \in \mathbb{C}^{2n},
\end{cases}
\]  

(3.6)

and \( \hat{a} \) is the unique solution of the transport equation

\[
\begin{cases}
\hat{a}'(t) + \frac{1}{i} \beta(t) \hat{a}(t) = 0, & t \geq 0, \\
\hat{a}(0) = C_{\Phi_0},
\end{cases}
\]  

(3.7)

where

\[
\beta(t) = \text{tr} \left( \tilde{q}_{zz} + \tilde{q}_{\zeta\zeta} \cdot \frac{2}{i} \Psi_t''(z, \theta) \right), \quad t \geq 0,
\]  

(3.8)

and

\[
C_{\Phi_0} = 2^n \pi^{-n} \det \Phi_0''(z, \zeta).
\]  

(3.9)

We note that the initial conditions in (3.6) and (3.7) are chosen so that when \( t = 0 \) the righthand side of (3.5) coincides with the orthogonal projector \( \Pi_{\Phi_0} : L^2(\mathbb{C}^n, e^{-2\Phi_0(z)} L(dz)) \rightarrow H_{\Phi_0}(\mathbb{C}^n) \), which has the explicit integral representation

\[
\Pi_{\Phi_0} u(z) = C_{\Phi_0} \int_{\mathbb{C}^n} e^{2\Phi_0(z, \zeta)} u(w) e^{-2\Phi_0(w)} L(dw), \quad u \in L^2(\mathbb{C}^n, e^{-2\Phi(z)} L(dz)).
\]  

(3.10)

In the literature, the operator \( \Pi_{\Phi_0} \) is known as the ‘Bergman projector’ associated to the weight \( \Phi_0 \). For a proof of (3.10), see Theorem 13.6 of [31] or Proposition 1.3.4 of [13].

Since \( \tilde{q} \) has the simple form (3.11), we may determine \( \Psi_t \) and \( \hat{a} \) by solving (3.6) and (3.7) explicitly. We begin by studying the transport equation (3.11). Thanks to (2.13), we see that

\[
\beta(t) = \text{tr} (M), \quad t \geq 0.
\]  

(3.11)
The unique solution of (3.7) is
\[ \hat{a}(t) = C_\Phi_0 e^{\frac{i}{2} \text{tr}(M^t)} , \quad t \geq 0. \] (3.12)

Next, we solve (3.6) for \( \Psi_t \). We search for a solution to (3.6) of the form
\[ \Psi_t(z, \theta) = \frac{1}{2} A_t z \cdot z + B_t z \cdot \theta + \frac{1}{2} D_t \theta \cdot \theta, \quad (z, \theta) \in \mathbb{C}^{2n}, \quad t \geq 0, \] (3.13)
where \( A_t, B_t, D_t \in \text{Mat}_{n \times n}(\mathbb{C}) \) depend smoothly on \( t \) and \( A_t = A_0^t \) and \( D_t = D_0^t \) for all \( t \geq 0 \). Inserting (3.13) into (3.6) and using (2.13) and (3.2), we see that \( \Psi_t \) will be a solution of (3.6) provided \( A_t, B_t, \) and \( D_t \) satisfy
\[ \begin{cases} \partial_t A_t z \cdot z + \frac{1}{4} A_t M z \cdot z = 0, & z \in \mathbb{C}^n, \quad t \geq 0, \\ A_0 = \Phi''_{0,zz}, \end{cases} \] (3.14)
and
\[ \begin{cases} \partial_t B_t z \cdot \theta + \frac{1}{4} B_t M z \cdot \theta = 0, & z, \theta \in \mathbb{C}^n, \quad t \geq 0, \\ B_0 = \Phi''_{0,\overline{zz}}, \end{cases} \] (3.15)
respectively. The symmetry of \( A_t \) implies that
\[ 2A_t M z \cdot z = (A_t M + M^T A_t) z \cdot z, \quad z \in \mathbb{C}^n, \quad t \geq 0. \] (3.17)
Thus (3.14) holds if and only if
\[ \begin{cases} \partial_t A_t + \frac{1}{4} A_t M + \frac{1}{4} M^T A_t = 0, & t \geq 0, \\ A_0 = \Phi''_{0,zz}. \end{cases} \] (3.18)
The unique solution of (3.18) is
\[ A_t = e^{i(\text{tr}(M^t) t)} \Phi''_{0,zz} e^{iM t} , \quad t \geq 0. \] (3.19)

By inspection, the solutions of (3.15) and (3.16) are
\[ B_t = \Phi''_{0,\overline{zz}} e^{itM} , \quad C_t = \Phi''_{0,zz}, \quad t \geq 0, \] (3.20)
respectively. Using (3.2), we get
\[ \Psi_t(z, \theta) = \Psi_0(e^{itM} z, \theta) , \quad (z, \theta) \in \mathbb{C}^{2n}, \quad t \geq 0. \] (3.21)
From (3.5), (3.10), (3.12), and (3.21), we deduce that
\[ \tilde{G}(t) u(z) = e^{\frac{i}{2} \text{tr}(M^t) t} u(e^{itM} z) , \quad u \in H_{\Phi_0}(\mathbb{C}^n), \quad t \geq 0. \] (3.22)

The formula (3.22) for the semigroup \( \tilde{G}(t) \) was obtained by a different method in [1].

For \( t \geq 0 \), let us define
\[ \Phi_t(z) = \Phi_0(e^{itM} z) , \quad z \in \mathbb{C}^n, \quad t \geq 0. \] (3.23)
Since $\Phi_0$ is strictly convex, $\Phi_t$ is a strictly convex quadratic form on $\mathbb{C}^n$ for all $t \geq 0$. In addition, we have $\Phi_t|_{t=0} = \Phi_0$. For $t \geq 0$, let

$$H_{\Phi_0}(\mathbb{C}^n) = L^2\left(\mathbb{C}^n, e^{-2\Phi_t(z)} L(\text{d}z)\right) \cap \text{Hol}(\mathbb{C}^n)$$

be the Bargmann space associated to $\Phi_t$, equipped with the natural Hilbert space structure induced from $L^2\left(\mathbb{C}^n, e^{-2\Phi_t(z)} L(\text{d}z)\right)$. From (3.22), it is clear that $\tilde{G}(t)$ is bounded $H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$ for every $t \geq 0$, and a direct computation using (3.22), (3.23), and (2.16) gives

$$\gamma > 0.$$  

Let $\Phi_t = \Phi_0 e^{-i t M z}$, where $C, c > 0$. Moreover, there are constants $C, c > 0$, independent of $t$, such that

$$-C|w - e^{i t M z}|^2 \leq 2 \text{Re} \Psi_t(z, w) - \Phi_t(z) - \Phi_0(w) \leq -c|w - e^{i t M z}|^2, \quad z, w \in \mathbb{C}^n, \ t \geq 0. \quad (3.34)$$

The following proposition summarizes the discussion so far in this section and establishes some basic estimates that will be necessary for the proof of Theorem 1.1 in Section 4.

**Proposition 3.1.** Let $q, \tilde{q}, M, \Phi_0, H_{\Phi_0}(\mathbb{C}^n)$, and $\tilde{G}(t)$ be as in Proposition 2.1.

1. For every $t \geq 0$, we have

$$\tilde{G}(t)u(z) = e^{\frac{i}{2} tr(M) t} u(e^{i t M} z), \quad u \in H_{\Phi_0}(\mathbb{C}^n). \quad (3.26)$$

In addition,

$$\|\tilde{G}(t)u\|_{H_{\Phi_0}(\mathbb{C}^n)} = e^{\gamma t} \|u\|_{H_{\Phi_0}(\mathbb{C}^n)}, \quad t \geq 0, \quad (3.27)$$

where

$$\Phi_t(z) = \Phi_0(e^{i t M} z), \quad z \in \mathbb{C}^n, \ t \geq 0, \quad (3.28)$$

the norm $\|\|_{H_{\Phi_0}(\mathbb{C}^n)}$ is the norm on the Bargmann space $H_{\Phi_0}(\mathbb{C}^n)$ introduced in (3.24), and $\gamma > 0$ is as in the statement of Theorem 1.1.

2. Let $R_t = \Phi_0 - \Phi_t, t \geq 0$, and let $\alpha : [0, \infty) \to \mathbb{R}$ be the continuous function defined by

$$\alpha(t) = \min_{|z|=1} R_t(z), \quad (3.29)$$

so that

$$R_t(z) \geq \alpha(t) |z|^2, \quad z \in \mathbb{C}^n, \ t \geq 0. \quad (3.30)$$

The function $\alpha$ has the following properties:

(a) $\alpha(0) = 0$ and $\alpha(t) > 0$ for all $t > 0$,

(b) $\alpha$ is non-decreasing,

(c) there is $0 < t_0 \ll 1$ and $c > 0$ such that

$$\alpha(t) \geq ct^{2k_0 + 1}, \quad 0 \leq t \leq t_0, \quad (3.31)$$

where $k_0 \in \{0, 1, \ldots, 2n - 1\}$ is the smallest non-negative integer such that (1.14) holds, and

(d) $\alpha(t) \to \min_{|z|=1} \Phi_0(z) > 0$ as $t \to \infty$.

3. Let $\Psi_0$ be the polarization of $\Phi_0$ given by (3.3). For any $t \geq 0$ and $u \in H_{\Phi_0}(\mathbb{C}^n)$, we have

$$\tilde{G}(t)u(z) = C_{\Phi_0} e^{\frac{i tr(M) t}{2}} \int_{\mathbb{C}^n} e^{2 \Psi_t(z, \overline{w})} u(w) e^{-2 \Phi_0(w)} L(\text{d}w), \quad z \in \mathbb{C}^n, \quad (3.32)$$

where

$$\Psi_t(z, \theta) = \Psi_0(e^{i t M} z, \theta), \quad (z, \theta) \in \mathbb{C}^{2n}, \ t \geq 0. \quad (3.33)$$

Moreover, there are constants $C, c > 0$, independent of $t$, such that

$$-C|w - e^{i t M} z|^2 \leq 2 \text{Re} \Psi_t(z, \overline{w}) - \Phi_t(z) - \Phi_0(w) \leq -c|w - e^{i t M} z|^2, \quad z, w \in \mathbb{C}^n, \ t \geq 0. \quad (3.34)$$

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Proof. It remains to establish Point 2 and the estimate (3.34). To this end, let
\[ R_t(z) = \Phi_0(z) - \Phi_t(z), \quad z \in \mathbb{C}^n, \quad t \geq 0, \quad (3.35) \]
and let \( \alpha : [0, \infty) \to \infty \) be as in (3.29). We will begin by showing that
\[ R_t(z) \geq 0, \quad z \in \mathbb{C}^n, \quad t \geq 0. \quad (3.36) \]
Let \( \tilde{\kappa}_t, t \geq 0, \) be as in (3.3). A straightforward computation using (2.9), (3.4), and (3.28) gives that
\[ \tilde{\kappa}_t (\Lambda \Phi_0) = \Lambda \Phi_t := \left\{ \left( z, \frac{2}{i} \Phi_{t,z}(z) \right) : z \in \mathbb{C}^n \right\}, \quad t \geq 0. \quad (3.37) \]
From either the discussion in Section 6 of [30] or a direct computation, we know that the family \((\Phi_t)_{t \geq 0}\) satisfies the eikonal equation
\[ \begin{cases} \partial_t \Phi_t(z) + \text{Re} \tilde{q} \left( z, \frac{2}{i} \Phi_{t,z}(z) \right) = 0, \quad z \in \mathbb{C}^n, \quad t \geq 0, \\ \Phi_t|_{t=0} = \Phi_0 \quad \text{on} \quad \mathbb{C}^n. \end{cases} \quad (3.38) \]
As a consequence of (3.37), for every \( z \in \mathbb{C}^n \) and \( t \geq 0, \) there is a point \( Z \in \Lambda \Phi_0 \) such that
\[ \left( z, \frac{2}{i} \Phi_{0,z}(z) \right) = \tilde{\kappa}_t(Z). \quad (3.39) \]
Since \( \tilde{q} \) is invariant under the flow \( \tilde{\kappa}_t, \) for every \( t \geq 0 \) and \( z \in \mathbb{C}^n, \) there is \( Z \in \Lambda \Phi_0 \) such that
\[ \partial_t \Phi_t(z) = -\text{Re} \tilde{q}(Z). \quad (3.40) \]
Because \( \text{Re} q \geq 0, \) (2.10) and (2.11) imply that \( \text{Re} \tilde{q} \geq 0 \) on \( \Lambda \Phi_0, \) and we have
\[ \partial_t \Phi_t(z) \leq 0, \quad z \in \mathbb{C}^n, \quad t \geq 0. \quad (3.41) \]
Thus, for any fixed \( z \in \mathbb{C}^n, \) the function
\[ t \mapsto \Phi_0(z) - \Phi_t(z) \quad (3.42) \]
is non-decreasing. It follows that \( R_t \geq 0 \) for all \( t \geq 0 \) and that the function \( \alpha \) is non-decreasing.
We next recall from Proposition 6.1 of [30] that
\[ \Lambda \Phi_0 \cap \Lambda \Phi_t = \pi_1 (\kappa_\varphi(S)), \quad t > 0, \quad (3.43) \]
where \( S \) is the singular space of \( q, \kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n} \) is the complex linear canonical transformation defined by (2.6), and \( \pi_1 : \mathbb{C}^{2n} \to \mathbb{C}^n \) is the projection \( \pi_1 : (z, \zeta) \mapsto z. \) Since we assume that \( S = \{0\}, \) we deduce from (3.43) that
\[ \Lambda \Phi_0 \cap \Lambda \Phi_t = \{0\}, \quad t > 0. \quad (3.44) \]
Thus, for every \( t > 0 \) and \( z \in \mathbb{C}^n, \)
\[ \frac{2}{i} \Phi_{0,z}(z) - \frac{2}{i} \Phi_{t,z}(z) = 0 \quad \iff \quad z = 0. \quad (3.45) \]
Because \( R_t \) is a non-negative quadratic form for each \( t \geq 0, \) we have
\[ R_t(z) = 0, \quad z \in \mathbb{C}^n, \quad t > 0 \quad \iff \quad \nabla_{\text{Re} z, \text{Im} z} R_t(z) = 0 \quad \iff \quad \frac{2}{i} \Phi_{0,z}(z) - \frac{2}{i} \Phi_{t,z}(z) = 0. \quad (3.46) \]
Hence, for any $z \in \mathbb{C}^n$ and $t > 0$,

$$R_t(z) = 0 \iff z = 0. \quad (3.47)$$

Thus $\alpha(t) > 0$ for all $t > 0$.

To establish (3.31), we recall the main result of Section 2 of [12], which states that if the singular space of $q$ is trivial, $S = \{0\}$, then there is a small time $0 < t_0 \ll 1$ and a constant $c > 0$ such that

$$R_t(z) \geq ct^{2k_0+1} |z|^2, \quad z \in \mathbb{C}^n, \quad 0 \leq t \leq t_0, \quad (3.48)$$

where $k_0 \in \{0, 1, \ldots, 2n - 1\}$ is the smallest non-negative integer such that (1.12) holds. It is therefore true that

$$\alpha(t) \geq ct^{2k_0+1}, \quad 0 \leq t \leq t_0. \quad (3.49)$$

To prove the claim regarding the behavior of $\alpha(t)$ as $t \to \infty$, we note that (2.16) implies that $\text{spec}(iM) \subset \{\Re \lambda < 0\}$. Thus there is $c > 0$ such that

$$R_t(z) = \Phi_0(z) + \mathcal{O}(e^{-ct |z|^2}) \text{ as } t \to \infty. \quad (3.50)$$

It follows that

$$\alpha(t) \to \min_{|z|=1} \Phi_0(z) \text{ as } t \to \infty. \quad (3.51)$$

The proof of Point 2 is complete.

Finally, we prove (3.34). Using (3.1), (3.2), (3.21), and (3.23), we obtain the following identity by elementary algebraic manipulations:

$$2 \Re \Psi_t(z, \overline{w}) - \Phi_t(z) - \Phi_0(w) = -\Phi''_{0,zz} (w - e^{itM}z) \cdot (w - e^{itM}z), \quad z, w \in \mathbb{C}^n, \quad t \geq 0. \quad (3.52)$$

Because $\Phi_0$ is a strictly plurisubharmonic quadratic form, the Levi matrix $\Phi''_{0,zz}$ is Hermitian positive-definite. Consequently, there are constants $C, c > 0$, independent of $t$, such that

$$-C |w - e^{itM}z|^2 \leq 2 \Re \Psi_t(z, \overline{w}) - \Phi_t(z) - \Phi_0(w) \leq -c |w - e^{itM}z|^2, \quad z, w \in \mathbb{C}^n, \quad t \geq 0. \quad (3.53)$$

This proves (3.34). \qed

4 \quad \textbf{The Conclusion of the Proof of Theorem 1.1}

In view of (2.26), (2.17), (3.5), and (3.12), the Schwartz kernel $K_t(x, y)$ of $G(t)$ is given, formally, by

$$K_t(x, y) = c_x^2 C_{\Phi_0} e^{4 \text{tr}(M)t} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} e^{P_t(x, y, z, w)} L(dw) L(dz), \quad (x, y) \in \mathbb{R}^{2n}, \quad t \geq 0, \quad (4.1)$$

where

$$P_t(x, y, z, w) := -i \varphi(z, x) - 2 \Phi_0(z) + 2 \Psi_t(z, \overline{w}) - 2 \Phi_0(w) + i \varphi(w, y), \quad (4.2)$$

for $x, y \in \mathbb{R}^n$, $z, w \in \mathbb{C}^n$, and $t \geq 0$. For $z \in \mathbb{C}^n$, let $r(z) \in \mathbb{R}^n$ be the unique point such that

$$\Phi_0(z) = -\text{Im} \varphi(z, r(z)). \quad (4.3)$$

Since $\text{Im} \varphi''_{yy} > 0$, there is $c > 0$ such that

$$-\text{Im} \varphi(z, y) - \Phi_0(z) \leq -c |y - r(z)|^2, \quad z \in \mathbb{C}^n, \quad y \in \mathbb{R}^n. \quad (4.4)$$
Using (4.14) together with the estimate (3.34), we find that
\[
\Re P_t(x, y, z, w) \leq -c|x - r(z)|^2 - R_t(z) - c|w - e^{itM}z|^2 - c|y - r(w)|^2,
\]
for all \( x, y \in \mathbb{R}^n, z, w \in \mathbb{C}^n, \) and \( t \geq 0, \) where \( R_t(z) \) is as in Proposition 3.1. Let \( \alpha : [0, \infty) \rightarrow \mathbb{R} \) be as in (3.30). Since (3.30) holds, there is \( c > 0 \) such that
\[
\Re P_t(x, y, z, w) \leq -c|x - r(z)|^2 - \alpha(t)|z|^2 - c|w - e^{itM}z|^2 - c|y - r(w)|^2
\]
for all \( x, y \in \mathbb{R}^n, z, w \in \mathbb{C}^n, \) and \( t \geq 0. \)

Let \( \gamma \) be as in the statement of Theorem 1.1. Taking the absolute value of (4.11) and using (4.6) and (2.10), we find that there are constants \( C, c > 0 \) such that
\[
|K_t(x, y)| \leq C e^{-\gamma t} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} e^{-c|x - r(z)|^2 - \alpha(t)|z|^2 - c|w - e^{itM}z|^2 - c|y - r(w)|^2} L(dw) L(dz)
\]
for every \( x, y \in \mathbb{R}^n \) and \( t \geq 0. \) Let \( 1 \leq p \leq q \leq \infty \) be given, and let \( 1 \leq r \leq \infty \) be such that
\[
1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}.
\]
Using Minkowski’s integral inequality and the fact that \( \alpha(t) > 0 \) for every \( t > 0, \) we get that
\[
\|K_t(\cdot, y)\|_{L^r} \leq C e^{-\gamma t} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} e^{-c|x - r(z)|^2 - \alpha(t)|z|^2 - c|w - e^{itM}z|^2} \|e^{-c|y - r(w)|^2}\|_{L^r} L(dw) L(dz)
\leq C \alpha(t)^{-n} e^{-\gamma t}, \quad x \in \mathbb{R}^n, \ t > 0,
\]
where \( C = C_{p,q} > 0 \) depends only on \( p \) and \( q. \) By similar reasoning, there is \( C = C_{p,q} > 0 \) such that
\[
\|K_t(\cdot, y)\|_{L^r} \leq C \alpha(t)^{-n} e^{-\gamma t}, \quad y \in \mathbb{R}^n, \ t > 0.
\]
Applying Young’s integral inequality with (4.9) and (4.10) gives
\[
\|G(t)\|_{L^p} \leq C \alpha(t)^{-n} e^{-\gamma t}, \quad t > 0,
\]
for some \( C = C_{p,q} > 0. \)

Let \( \epsilon > 0 \) be arbitrary. From Proposition 3.1, we know that \( \alpha \) is non-decreasing and \( \alpha(t) > 0 \) for all \( t > 0. \) Thus,
\[
\alpha(t) \geq \alpha(\epsilon), \quad t \geq \epsilon.
\]
In view of (4.11), we may deduce that there is \( C = C_{p,q} > 0 \) such that
\[
\|G(t)\|_{L^p} \leq C e^{-\gamma t}, \quad t \geq \epsilon.
\]
To see that the bound (4.13) is sharp as \( t \to \infty, \) we recall from Theorem 2.1 of [18] that the lowest eigenvalue \( \rho \) of \( q^w(x, D), \) introduced in [1.14], is simple and that the eigenspace of \( q^w(x, D) \) corresponding to \( \rho \) is spanned by a ‘ground state’ of the form
\[
u_0(x) = e^{-a(x)}, \quad x \in \mathbb{R}^n,
\]
where \( a \) is a complex-valued quadratic form on \( \mathbb{R}^n \) with positive-definite real part \( \text{Re} a > 0. \) Let \( v = \|u_0\|_{L^1(\mathbb{R}^n)}^{-1} u_0. \) Since \( q^w(x, D)v = \rho v, \) it is clear that
\[
\|e^{-tq^w(x, D)}v\|_{L^1} = e^{-\gamma t}\|v\|_{L^1}, \quad t \geq 0.
\]
Hence there is a constant $c = c_{p,q} > 0$ such that
\[
\|e^{-tq^w(x,D)}\|_{L^p \to L^q} \geq ce^{-\gamma t}, \quad t \geq 0.
\] (4.16)

We conclude that there are constants $C = C_{c,p,q} > 0$ and $c = c_{p,q} > 0$ such that (1.15) holds for all $t \geq \epsilon$.

Finally, we prove the bound (1.16). From (3.31), (4.11), and (4.16), we get that there are constants $C = C_{p,q} > 0$ and $c_{p,q} > 0$ such that
\[
c \leq \|G(t)\|_{L^p \to L^q} \leq Ct^{-(2k_0+1)n}, \quad 0 < t \leq t_0.
\] (4.17)

The proof of Theorem 1.1 is complete.

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