Monomialization of singular metrics on real surfaces

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Abstract
We present a result of monomialization of regular 2-symmetric tensors over regular real surfaces. For the inner metric of an embedded real surface singularity, the result obtained for the regular extension of the pull-back of the inner metric over a resolved surface is more precise for applications than a Hsiang and Pati type resolution of singularities.

Keywords Singular metrics · Singular surfaces · Resolution of singularities · Singular foliations · Simple singularities of foliations · Hsiang and Pati property

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To celebrate Felipe Cano on his sixtieth birthday.

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1 Introduction and statement of a simpler version of the result

Any smooth 2-symmetric tensor over a smooth Riemannian manifold \( M \) is point-wise diagonalizable. The collection of these diagonalizing bases rarely forms a smooth local orthogonal frame everywhere on \( M \).

Any real analytic family \((A(u))_{u \in \mathcal{U}}\) of \( n \times n\)-symmetric matrices over an open subset \( \mathcal{U} \) of an Euclidean space \( \mathbb{R}^p \), can be re-parameterized by a real analytic proper surjective mapping \( \pi : \mathcal{V} \rightarrow \mathcal{U} \), a finite composition of geometrically admissible blowings-up, which locally simultaneously diagonalizes everywhere on \( \mathcal{V} \) the pulled-back family \((A \circ \pi(u))_{u \in \mathcal{V}}\), see Kurdyka and Paunescu [21] and our generalization [11] (with a different proof).

Our primary motivation is two fold: locally describe, as simply as can be \( (1) \) real analytic 2-symmetric tensors over a real analytic Riemannian manifold which may degenerate somewhere, \( (2) \) the restriction of real analytic 2-symmetric tensors over a real analytic Riemannian (or Hermitian) ambient manifold \( M \) to (the regular part of) a given real (or complex) analytic singular sub-variety \( S \).

The model example to have in mind is the inner metric on \( S \), that is the restriction of the ambient metric to the regular part of the analytic subset \( S \). The problem at hands in this model example is understanding how the inner Riemannian structure on the regular part of the singular sub-variety accumulates at the singular part. An accepted scheme to treat this problem, which hopefully will yield a local description of the inner metric nearby the singular locus, is to re-parameterize the (embedded) singular sub-variety as a regular manifold by a (bi-rational like) regular surjective mapping (the resolution mapping of an embedded desingularization of the singular sub-variety). The pull back of the inner metric onto the resolved manifold then becomes a regular semi-Riemannian metric, losing positive-definiteness along the pre-image of the singular locus of the original sub-variety (the exceptional locus of the resolution). Further carefully chosen blowings-up should yield a better description of the iterated pull-back of the metric. The meaning of carefully chosen is yet to be systematically developed in regard of the control that can be guaranteed after pull-back.

Hsiang and Pati were first to provide a local description of inner metric of normal complex surface singularity germs [19, Sects. II and III] along the proposed scheme. Later Pardon and Stern conceptualized this point of view [24, Sect. 3]. Grieser also found a real analytic version (unpublished) of Hsiang and Pati’s result [15]. Almost fifteen years after Hsiang and Pati, Youssin announced the existence of such a local description on some resolved manifold of any given complex algebraic singularity [28]. The short version can be stated as follows:

**Youssin Conjecture** [2,28]: Given a singular complex algebraic sub-variety \( X_0 \) of pure dimension \( n \) embedded in a complex (algebraic) manifold \( M_0 \), there exists a resolution of singularities \( \sigma : (X, E) \rightarrow X_0 \) of \( X_0 \), which is a locally finite composition of blowings-up with regular (algebraic) centers, such that at each point \( a \) of the exceptional divisor \( E \), the pull-back, onto the resolved manifold \( X \), of the inner metric of \( X_0 \) (inherited from the ambient Hermitian metric in \( M_0 \)) by the resolution mapping \( \sigma \) is (locally) quasi-isometric to a sum of the “Hermitian squares” of (exactly \( n \) independent) differentials of monomials in the exceptional divisor \( E \). Furthermore, the integer vectors made of the power of each monomials must be linearly independent and totally ordered.

Our joint article [2] proposes a problem equivalent to Youssin’s, which is better adapted to resolution singularities techniques, proving Youssin’s Conjecture in the case of real and complex (algebraic or analytic) surfaces and threefolds singularities.

The present paper focuses only on real analytic surface singularities from the above point of view. Each of our joint works, [12] deals with the geodesics of cuspidal surface singularities.
ending at the singular point, while [13] and [10] deal with germs of gradient differential equations over real surface singularities, requires a tailored description of a singular metric over a regular surface nearby an exceptional curve, finer than Hsiang and Pati’s normal form.

Given a real analytic surface singularity \( S \) in an ambient real analytic manifold \( M \) endowed with a real analytic metric, given a 2-symmetric tensor \( \kappa \) on \( M \), can we find, as in the case of the inner metric, a useful and simple local presentation of the pull-back, on a resolved surface, of the restriction of \( \kappa \) to the regular part of \( S \)?

Considered from the point of view of a resolved manifold, the problems presented above are occurrences of the following general situation: Let \( X_1 \) be a regular (i.e. real or complex algebraic or analytic) manifold and let \((B_1, g_1)\) be a regular vector bundle over \( X_1 \) of finite rank and equipped with a regular fiber-metric (Riemannian or Hermitian) \( g_1 \). Let \( \kappa_1 \) be a 2-symmetric tensor field on \( B_1 \), (later shortened as 2-symmetric tensor on \( B_1 \)). Some features of \( \kappa_1 \) can be investigated following two similar but different approaches:

Parameterization Problem. Find a regular and surjective mapping \( \sigma_2 : X_2 \to X_1 \), and describe the features of \( \kappa_1 \circ \sigma_2 \) as objects with regular variations on \( X_2 \). Note that \( \kappa_1 \circ \sigma_2 \) is a 2-symmetric tensor on the regular vector bundle \( \sigma_2^*B_1 \).

Resolution Problem. When there exists a regular mapping of vector bundles \( TX_1 \to B_1 \) which is an isomorphism outside a proper sub-variety of \( X_1 \), find a regular and surjective mapping \( \sigma_2 : X_2 \to X_1 \) such that the features of the pulled-back \( \kappa_2 \) of the 2-symmetric tensor \( \kappa_1 \), now a 2-symmetric tensor on \( X_2 \), are as good as can be.

The expression “as good as can be” is vague, indeed. But, depending on motivations, the end product of a solution of the resolution problem may vary. An instance of “as good as can be” is obtaining Hsiang and Pati’s local form.

Resolution Problem refined. The pulled-back tensor \( \kappa_2 \) can locally be written as a weighted sum of symmetric squares of two 1-forms such that, at each point of a (explicit) simple normal crossing divisor \( D \) containing the exceptional divisor, the push-forward of the kernels of these two 1-forms by the resolution mapping are orthogonal. Moreover these two 1-forms admit only simple singularities (if any), and the corresponding coefficients (weights) in the diagonal presentation of \( \kappa_2 \) are monomial in \( D \) times a unit.

The Parameterization Problem consists mostly of regularizing some functions (roots, components of vector fields, minors,...). Kurdyka and Paunescu’s results, generalizing Rellich’s real and Kato’s complex one dimensional case, provide almost immediately an answer to the Parameterization Problem when \( X_1 \) is an open subset of an Euclidean space. Our generalization [11] provides the full answer.

The Resolution Problem, in the described refinement form of Hsiang and Pati, has never been addressed. It is significantly harder than the associated Parameterization Problem, although in practice we very likely have to start with solving this latter one.

The present paper proposes an answer to the Resolution Problem refined for a “singular” 2-symmetric tensor of a real analytic vector bundle of rank two over a regular real analytic surface, isomorphic to the tangent bundle of the surface outside a sub-variety of positive codimension. This applies after preliminary preparations to the inner metric of a real analytic surface singularity embedded in real analytic Riemannian manifold. Below, our results are stated for the inner metric of a real analytic singular surface, the simplest possible situation.

Let \( X_0 \) be a real analytic surface singularity of a real analytic Riemannian manifold \((M_0, g_0)\), supporting a real analytic space structure \((X_0, \mathcal{O}_{X_0} := \mathcal{O}_{M_0}/I_0)\), for a coherent \( \mathcal{O}_{M_0} \)-ideal sheaf \( I_0 \). Let \( \gamma_0 : T(X_0 \setminus Y_0) \to G(2, TM_0|X_0 \setminus Y_0) \) be the Gauss mapping of \( X_0 \), for \( Y_0 \) the (non-empty) singular locus of \( X_0 \).
Our first result (see Proposition 2 for the general case) is a parameterization result of a global simultaneous diagonalization, namely:

**Proposition 1** There exists \( \sigma_1 : (X_1, E_1) \rightarrow (X_0, Y_0) \), a locally finite composition of geometrically admissible blowings-up, such that \( X_1 \) is a regular surface, the Gauss mapping \( \gamma_0 \circ \sigma_1 \) extends as a real analytic mapping \( \gamma_1 : X_1 \rightarrow \mathbb{G}(2, \sigma_1^* TM|_{X_1}) \) over \( X_1 \) and \( E_1 \) is a simple normal crossing divisor. Let \( B_1 \) be the real analytic vector sub-bundle of rank 2 of \( \sigma_1^* TM|_{X_1} \) corresponding to \( \gamma_1 \), extending \( \sigma_1^* T(X_0 \setminus Y_0) \) over \( X_1 \).

There exists an \( \mathcal{O}_{X_1} \)-invertible sheaf \( \mathcal{Q}_1 \) of sections of \( S^2 B_1^\vee \) with empty co-support such that at each point \( a_1 \) of \( X_1 \) there exists an open neighbourhood \( \mathcal{U}_1 \) of \( a_1 \) with the following properties:

0) there exists, up to permutation, a pair (unique if the inner metric is not constant along the fibers) \( \theta_1, \theta_2 \) of regular sections of \( B_1^\vee \mid_{\mathcal{U}_1} \) both nowhere vanishing in \( \mathcal{U}_1 \) such that \( \mathcal{Q}_1 \mid_{\mathcal{U}_1} \) is generated by \( \theta_1 \cdot \theta_2 \), and

i) for each \( a \in \mathcal{U}_1 \), the vector lines \( \ker \theta_1(a) \) and \( \ker \theta_2(a) \) of \( T_{\sigma_1(a)} M_0 \) intersect orthogonally;

ii) If \( \sigma_1(a) \) is not a singular point of \( X_0 \), then \( \ker \theta_i(a) \) is contained in \( T_{\sigma_1(a)} X_0 \);

iii) The metric parameterized by \( \sigma_1 \) writes on \( \mathcal{U}_1 \) as

\[
(\mathfrak{g}_0 \circ \sigma_1|_{\mathcal{U}_1})|_{\mathcal{U}_1} = (\alpha_1 \cdot \theta_1)(\alpha_1 \cdot \theta_1) + (\alpha_2 \cdot \theta_2)(\alpha_2 \cdot \theta_2) = \alpha_1^2 \theta_1 \otimes \theta_1 + \alpha_2^2 \theta_2 \otimes \theta_2,
\]

where \( \alpha_i \) is a unit for \( i = 1, 2 \).

In other words, a global regular object, the quadratic ideal \( \mathcal{Q}_1 \), locally induces an orthogonal directional frame on the regular vector bundle \( B_1 \) over \( X_1 \), locally simultaneously diagonalizing \( (\mathfrak{g}_0|_{X_0}) \circ \sigma_1 \), so that the size of the (local) generators of the diagonalizing frame are monomials (here local units - compare with [21]). This form is stable under further points blowings-up. Then, we get the desingularization of the pulled-back metric, leading to Theorem 6 presented in the following simpler form:

**Theorem 1** There exists a locally finite composition of geometrically admissible blowings-up

\[
\sigma_2 : (X_2, E_2) \xrightarrow{\beta_2} (X_1, E_1) \xrightarrow{\sigma_1} (X_0, Y_0)
\]

such that each point \( a_2 \) of \( X_2 \) admits an open neighbourhood \( \mathcal{U}_2 \) and, up to permutation, a pair (unique if not constant along the fiber) \( \omega_1, \omega_2 \) of regular sections of \( TX_2^\vee \mid_{\mathcal{U}_2} \) such that

i) Each foliation \( \mathcal{D}_1 \) over \( \mathcal{U}_2 \) generated by \( \omega_i, i = 1, 2 \), admits only simple singularities adapted to \( E_2 \);

ii) If \( \sigma_2(a_2) \) is a regular point of \( X_0 \), the lines \( D\sigma_2(a_2) \cdot \ker \omega_i(a_2) \), for \( i = 1, 2 \) are orthogonal in \( T_{\sigma_2(a_2)} X_0 \);

iii) The pull-back of the metric \( \mathfrak{g}_0|_{X_0} \) by the resolution mapping \( \sigma_2 \) extends on \( X_2 \) as a real analytic semi-Riemannian metric \( \mathfrak{g}_2 \), which on \( \mathcal{U}_2 \) is written as

\[
\mathfrak{g}_2 = (\mathcal{M}_1 \cdot \omega_1)(\mathcal{M}_1 \cdot \omega_1) + (\mathcal{M}_2 \cdot \omega_2)(\mathcal{M}_2 \cdot \omega_2) = \mathcal{M}_1^2 \omega_1 \otimes \omega_1 + \mathcal{M}_2^2 \omega_2 \otimes \omega_2
\]

where \( \mathcal{M}_i \) is a monomial in \( E_2 \), and \( i = 1, 2 \).

Without further blowings-up, we obtain the following (cf. Corollary 4 and Corollary 5):

**Corollary** Each point of \( E_2 \) admits Hsiang and Pati coordinates:
i) For \(a_2\) a regular point of \(E_2\), there are local coordinates \((u, v)\) at \(a_2\) such that \((E_2, a_2) = \{u = 0\}\) and the extension of the pulled-back metric \(g_2\) is quasi-isometric nearby \(a_2\) to

\[du^{k+1} \otimes du^{k+1} + d(u^{l+1}v) \otimes d(u^{l+1}v)\]

for non-negative integer numbers \(l \geq k\).

ii) For \(a_2\) a corner point of \(E_2\), there are local coordinates \((u, v)\) at \(a_2\) such that \((E_2, a_2) = \{uv = 0\}\) and the extension of the pulled-back metric \(g_2\) is quasi-isometric nearby \(a_2\) to

\[d(u^m v^n) \otimes d(u^m v^n) + d(u^k v^l) \otimes d(u^k v^l)\]

for positive integer numbers \(m \leq k, n \leq l\) and such that \(ml - kn \neq 0\).

This article is organized as follows: Sect. 2 presents basic material needed throughout the paper and set some notations. Theorem 2 states Hironaka’s resolution of singularities \([1,17]\) in a form best suited for our purpose. Section 3 introduces further classical definitions. The parameterization problem strictly speaking is solved in Sect. 4 by Proposition 2 (given above in a special case). Section 5 recalls what is a resolution of singularities of a real analytic plane singular foliation. Section 6 investigates pairs of generically transverse plane singular foliations, gives some technical results about the behavior of such a pair with respect to a prescribed normal crossing divisor. We recall the logarithmic point of view of the reduction of singularities of a plane foliation \([5–7]\). What is developed in this section is essential to obtain the announced normal form. Our main result, Theorem 6 presented in Sect. 7, solves the resolution problem in the refined version presented above. The local normal form of the pull-back of the initial 2-symmetric tensor is stable under further blowings-up. Section A.1 and Sect. A.2 deal with the notion of the restriction of 2-symmetric tensor onto a singular sub-variety. To that end, we show Proposition 5, stating the existence of Gauss-regular resolution of singularities (see also \([2]\)). Appendix B describes the unexpected consequence of our detailed Theorem 6 and its completion Corollary 3, for inner metrics on singular surfaces. Proposition 11 gives a normal form of the pull-back of the ambient metric on the resolved surface at any regular point of the exceptional divisor.

But for Hironaka’s resolution of singularities and reduction of singularities of plane vector fields germs, the paper is self contained, at the low cost of reproving already known results (or their variations).

2 Setting—resolution of singularities theorems

In what follows the adjective analytic only means real analytic.

A regular manifold is an analytic manifold. A regular sub-manifold of a given regular manifold is an analytic sub-manifold. A regular mapping \(M \to N\) is an analytic mapping between regular manifolds \(M\) and \(N\). A sub-variety is a closed real analytic subset of a given regular manifold. A regular sub-variety is a sub-variety and a regular sub-manifold. A point \(a\) of a given irreducible sub-variety \(X\) is regular if the germ \((X, a)\) is a regular sub-manifold germ of dimension \(\dim X\). A point of a sub-variety is regular if it is a regular point of the irreducible component containing the given point. A point of a sub-variety is singular if it is not regular. The singular locus of an irreducible sub-variety is a sub-variety of dimension strictly lower, if not empty. Let \(\mathcal{O}_M\) be the sheaf of analytic function germs on the regular manifold \(M\). We will speak of \(\mathcal{O}_M\)-ideals and \(\mathcal{O}_M\)-modules to mean \(\mathcal{O}_M\)-ideal sheaves and \(\mathcal{O}_M\)-module sheaves.
A normal crossing divisor of a regular manifold $M$ of dimension $n$ is the co-support of a principal $\mathcal{O}_M$-ideal of finite type which is locally monomial at each point of $M$. It is called a simple normal crossing divisor if furthermore each of its irreducible component is regular. We will shorten as (s)nc-divisor.

Let $D$ be a normal crossing divisor of $M$. Each point $a \in M$ admits local regular coordinates $(u, v) = (u_1, \ldots, u_s; v)$ centered at $a$, with $0 \leq s \leq n$, such that the germ of $D$ at $a$ writes $(D, a) = \{ u_1 \cdots u_s = 0 \}$. Such local coordinates are said adapted to the (simple) normal crossing divisor $D$ at $a$.

Let $a$ be a point of $M$ and let $\mathcal{O}_a := \mathcal{O}_{M, a}$ be the regular local ring of the germ $(M, a)$. Let $\mathfrak{m}_a$ be its maximal ideal and let $n = \dim(M, a) := \dim_{\mathcal{O}_a}/\mathfrak{m}_a$.

A local monomial $\mathcal{M}$ (at $a$) in the nc-divisor $(D, a)$ is a function germ of $\mathcal{O}_a$ such that there exist local regular coordinates $(u, v) = (u_1, \ldots, u_s; v)$ adapted to $D$ at $a$ in which

$$\mathcal{M} = \pm \prod_{i=1}^{n} u_i^{p_i},$$

for non-negative integer numbers $p_1, \ldots, p_s$.

A principal $\mathcal{O}_M$-ideal of finite type $I$ is monomial in the nc-divisor $D$ if at each point of $M$ there exists local coordinates $(u, v) = (u_1, \ldots, u_s; v)$ adapted to $D$ at $a$ such that the local generator at $a$ of the ideal $I$ is a local monomial in the nc-divisor $D$.

Two local monomials $\mathcal{M}_1 = \prod_{i=1}^{n} u_i^{p_i}$ and $\mathcal{M}_2 = \prod_{i=1}^{n} u_i^{q_i}$ in the nc-divisor $(D, a)$ are said ordered if, either $p_i \geq q_i$ for each $i$ or $p_i \leq q_i$ for each $i$. Any finite set of local monomials in $(D, a)$ is well ordered if any pair of distinct monomials is ordered. Any finite set of monomials in the nc-divisor is well ordered if it is well ordered.

Let $D$ be a nc-divisor of $M$. A regular sub-manifold $C$ of $M$ is normal crossing with $D$ at $a$ if up to a change of coordinates adapted to $D$ at $a$, we have

$$\{ C, a \} = \{ u_1 = \cdots = u_r = v_1 = \cdots = v_t = 0 \}$$

for $0 \leq r \leq s$ and $0 \leq t \leq n - s$.

Let $(M, D)$ be a regular manifold with a nc-divisor $D$ (possibly empty). A blowing-up with center a regular sub-variety $C$ is geometrically admissible if it is normal crossing with $D$ at each point (see [4, p. 213] for a more restrictive definition). Assume the center $C$ is of codimension greater than or equal to 2, and let $\beta_C : (M', E') \to (M, D)$ be such a blowing-up, then $E' := \beta_C^{-1}(D \cup C)$ is a nc-divisor, and is a nc-divisor if $D$ were.

Let $Z$ be any subset of $M$. The strict transform of $Z$ by $\beta_C$ is defined as the analytic Zariski closure of $\beta_C^{-1}(Z \setminus C)$ and is denoted $Z^{\text{str}}$. When a sub-variety $Z$ can be equipped with a real analytic space structure, the strict transform admits an algebraic description (see below). If $\gamma : (M'', E'') \to (M', E')$ is a locally finite sequence of geometrically admissible blowings-up, we write again $Z^{\text{str}}$ for the strict transform of $Z$ by $\beta_C \circ \gamma$. Suppose furthermore that the (s)nc-divisor $D$ is the exceptional divisor of a locally finite sequence of geometrically admissible blowings-up $\pi : (M, D) \to N$, for some regular manifold $N$. We find $E' = D^{\text{str}} \cup \beta_C^{-1}(C)$. Nevertheless strict transforms of an existing exceptional divisor will be denoted by the same letter as the exceptional divisor: $E' = D \cup \beta_C^{-1}(C)$.

We will use, almost systematically, the resolution of singularities of Hironaka, in the following embedded real setting.

**Theorem 2** (Embedded resolution of singularities [1,4,17]) Let $M$ be a (connected) regular manifold.

1) Let $I$ be a (non-zero and coherent) $\mathcal{O}_M$-ideal sheaf. There exists a locally finite composition of geometrically admissible blowings-up $\pi : (\widetilde{M}, E_{\widetilde{M}}) \to M$ such that the total transform $\pi^* I$ is a principal ideal, monomial in the nc-divisor $E_{\widetilde{M}}$.
2) Let $X$ be a sub-variety of $M$, of positive codimension, for which there exists a coherent $\mathcal{O}_M$-ideal sheaf with co-support $X$. Let $Y$ be the singular locus of $X$. There exists $\pi: (\tilde{M}, \tilde{X}, E_{\tilde{M}}) \to (M, X, Y)$, a locally finite composition of geometrically admissible blowings-up, such that $\tilde{X} := \pi^{-1}(X \setminus Y)^{\text{str}}$ is a regular sub-variety of $\tilde{M}$, normal crossing with the snc-divisor $E_{\tilde{M}} := \pi^{-1}(Y)$ and such that $\tilde{X} \cap E_{\tilde{M}}$ is a snc-divisor of $\tilde{X}$.

Although real algebraic sub-varieties can always be equipped with a ringed space structure induced by a coherent ideal sheaf, in order to be desingularized, real analytic singular sub-varieties (see [16, Sect. 4]) must also be equipped with a real analytic space structure, which, unlike their complex counter-part, is not always possible. (See [4, Sect. 10] for a proper account on the category of ringed spaces that can be desingularized). Therefore we ask for the following condition to be satisfied:

**Hypothesis.** Any singular sub-variety to be desingularized admits a coherent ideal sheaf of the structural sheaf of the ambient regular manifold with co-support the given sub-variety, so that the sub-variety is equipped with the corresponding real analytic space structure.

Let $X$ be a sub-variety of a regular analytic manifold $M$, equipped with a real analytic space structure given by a coherent $\mathcal{O}_M$-ideal sheaf $I$. Let $\beta: (M', E') \to (M, C)$ be a geometrically admissible blowing-up with center $C$ which does not contain all irreducible components of $X$. The coherent $\mathcal{O}_{M'}$-ideal $\beta^* I$ gives rise to $I^{\text{str}}$, a coherent $\mathcal{O}_{M'}$-ideal obtained locally as follows: factors out of any local section of $\beta^* I$ any term that vanishes along the germ of $E'$. The co-support of this ideal is the strict transform $X^{\text{str}}$ of $X$ we mentioned earlier (see [4, Sect. 3, p. 237]). In contrast the weak transform of $X$ is obtained as follows: the ideal $\beta^* I$ factors out as $I_E^m \cdot I'$, where $I_E$ is the coherent $\mathcal{O}_{M'}$-ideal of function germs vanishing along $E'$ and $m$ is a non-negative integer number and $I'$ is a coherent $\mathcal{O}_{M'}$-ideal sheaf with co-support the weak transform $X^{\text{weak}}$ of $X$, which does not contain $E'$. The weak transform $X^{\text{weak}}$ always contains the strict transform $X^{\text{str}}$, and they may differ because of their respective intersection with the exceptional divisor $E'$. Note that if $I$ is principal, then both notions coincide.

We recall and will use the following result of ordering any finite family of Monomials.

**Theorem 3** (Ordering Monomials [4]) Let $M$ be a (connected) regular manifold. Let $D$ be a (s)nc-divisor such that each of its component is regular. Let $I_1, \ldots, I_k$ be principal $\mathcal{O}_M$-ideals monomial in the nc-divisor $D$. There exists $\pi: N \to M$, a locally finite sequence of geometrically admissible blowings-up (with centers normal crossing with $D$ and its iterated total transforms), such that the pulled-back ideals $\pi^* I_1, \ldots, \pi^* I_k$ are principal $\mathcal{O}_N$-ideals monomial in the (s)nc-divisor $\pi^{-1}(D)$ such that at each point of $N$ the local generators of $\pi^* I_1, \ldots, \pi^* I_k$ are ordered.

Some notations.

We will use $\text{Unit}$ to mean any regular function germ which is a local unit and for which a more specific notation is not necessary.

We will write $\text{const}$ to mean a non-zero constant we do not want to precise further.

We will write (...) to mean a regular function germ we do not want to denote specifically.

If $z$ is a component of some local coordinates system centered at some given point then $z^{+\infty}$ means the null function germ. Let $\mathbb{N} := \mathbb{N} \cup \{+\infty\}$ and $\mathbb{N}_{\geq t} := \{n \in \mathbb{N} : n \geq t\}$. 
3 Bilinear symmetric forms

Let $V$ be a real vector space of finite dimension.

The real vector space tensor product $V \otimes V$ decomposes as the direct sum of the real vector subspaces $S^2(V) \oplus \wedge^2 V$, where $S^2(V)$ is the 2-nd symmetric power of $V$ and $\wedge^2 V$ is the 2-nd exterior power of $V$. A symmetric bilinear form of $V$ is just an element of $S(2, V) := S^2(V^\vee)$ for $V^\vee$ the dual vector space of $V$.

Let $Q(V)$ be the real vector space of real quadratic forms on $V$. There is a canonical isomorphism $Q(V) \to S(2, V)$ mapping a quadratic form $\kappa$ onto its polar form $\kappa_D$.

Let $M$ be a connected regular manifold of finite dimension. Let $F$ be a regular vector bundle of finite rank over $M$. Let $F^\vee$ be the dual bundle and $\mathcal{P}F$ be the projective bundle associated with $F$. Let $\Gamma_M(F)$ be the $O_M$-module of regular sections of $F$. Let $S(2,F) := S^2(F^\vee)$, respectively $Q(F)$, be the vector bundle over $M$ of 2-symmetric tensors, respectively quadratic forms, over the fibers of $F$. These two vector bundles are canonically isomorphic via a regular mapping of vector bundles over $M$. A regular quadratic form on $F$ is a regular section $M \to Q(F)$ and a 2-symmetric regular tensor field on $F$ is a regular section $M \to S(2,F)$, shortened as 2-symmetric tensor on $F$. When $F = TM$ we just say 2-symmetric tensor on $M$, respectively quadratic form on $M$.

A (regular) fiber-metric on $F$ is a regular section $M \to S(2,F)$ such that the corresponding quadratic form is everywhere positive definite. Such a regular fiber-metric always exists by Grauert and Morrey Theorem [14,22]. When $F$ is equipped with a (regular) fiber metric $g$ we write $| - |_g$ for the norm and $(\cdot, \cdot)_g$ for the corresponding scalar product.

As for foliations (see Sect. 5), it is more convenient to study invertible sub-modules of 2-symmetric tensors than sections.

An invertible $O_M$-sub-module $\mathcal{L}$ of $\Gamma_M(S(2,F))$ vanishes at the point $a$ of $M$, if the bilinear symmetric form $\kappa(a)$ over $T_aM$ is the null form, for $\kappa$ is a local generator of $\mathcal{L}$ nearby $a$. The vanishing locus $V(\mathcal{L})$ of $\mathcal{L}$ is the co-support of $\mathcal{L}$ and is a sub-variety of $M$. Since $M$ is connected, we say that $\mathcal{L}$ is non-zero (or non-null) if $V(\mathcal{L})$ is of positive codimension. We say that $\mathcal{L}$ does not vanish over a subset $S$ if $V(\mathcal{L}) \cap S$ is empty.

4 Good parameterization of 2-symmetric tensors on regular surfaces

Proposition 2, the main result of this section, is the first important step towards our main result. It is essentially a precise local simultaneous diagonalization result for a global orthogonal (directional) frame. It is a consequence of Theorem 4 ([21] for the local case and our paper [11] for the global case).

Since there will be competitive notations further down in the paper, we do take the following

Notations for pull-backs: Let $B$ be a smooth vector bundle over a smooth manifold $N$. Let $\sigma : M \to N$ be a smooth mapping. The pulled-back vector bundle $\sigma^*B$ of $B$ on $M$ along the mapping $\sigma$ is denoted from now on by $B^\sigma$.

If $A$ is a sub-module or sub-module sheaf over $N$ of sections of $B$ or $B^\vee$ or $\wedge^2 B$ or $S(2,B)$ (more generally of sections of tensors over $B$), we will denote $A^\sigma$ for $A \circ \sigma$.

Let $\kappa$ be any section which is tensorial in/over the fibers of $B$. We will denote the pull-back $\kappa \circ \sigma$, tensorial in/over the fibers of $B^\sigma$, by $\kappa^\sigma$.

If $I$ is an ideal or ideal sheaf over $N$ the pull-back of this ideal by $\sigma$ is still denoted in the standard way: $\sigma^*I = I \circ \sigma$. 

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We recall the following

**Theorem 4** ([11,21]) Let \( M \) be a regular connected manifold, and let \( B \) be a regular vector bundle over \( M \) of rank \( r \geq 1 \) equipped with a regular fiber-metric \( \mathbf{g} \). Let \( \kappa : M \to S(2, B) \) be a regular symmetric operator with respect to the fiber-metric \( \mathbf{g} \).

There exists \( \sigma : M' \to M \) a locally finite composition of geometrically admissible blowings-up such that every point \( a' \) of \( M' \) admits a neighborhood \( U' \) in \( M' \) and a regular frame \((\xi_1, \ldots, \xi_r)\) of \((B^\sigma)|_{U'}\) orthonormal for \( g^\sigma|_{U'} \) in which \( \kappa^\sigma(b') \) is diagonal at every point \( b' \) of \( U' \).

Truth be told, the conclusion of Theorem 4 still holds for a coherent invertible \( O_S \)-sheaf of sections of \( S(2, B) \) by our method [11].

In Sect. 2, we introduced the notions of strict transform and weak transform. Let \( S \) be a regular surface and let \( I \) be a coherent \( O_S \)-ideal with co-support \( X \) of dimension 1. If \( \sigma : (S', E') \to S \) is any sequence of points blowings-up, the difference between the strict transform \( X^{str} \) and its weak transform \( X^{weak} \) may consist of an isolated subset of the exceptional divisor \( E' \). We blow-up to principalize and monomialize some ideals, making the support of their total transform a snc-divisor. At this stage weak transform and total transform of the co-support of the initial ideal are equal. It is a snc-divisor which is normal crossing with the exceptional divisor of the principalization and monomialization process.

Let \( S \) be a connected regular surface and let \( B \) be a regular vector bundle of rank two over \( S \) equipped with a regular fiber-metric \( \mathbf{g} \).

Suppose given \( L \) a non-zero invertible \( O_S \)-sub-module of sections of \( S(2, B) \). The ideal \( \mathcal{C}_L \) of coefficients of \( L \) is the coherent \( O_S \)-ideal obtained by the evaluation of \( L \) from the \( (\text{local}) \) regular sections \( S \to B \times S B \). The vanishing locus \( V(L) \) is the co-support of \( \mathcal{C}_L \). The ideal \( \mathcal{C}_L \) is of finite type by definition, thus \( O_S \)-coherent.

The degeneracy locus \( D(L) \) of \( L \) is the sub-variety of \( S \) consisting of the points at which any local generator \( \kappa \) gives rise to a degenerate bilinear symmetric form on the corresponding fiber, which contains \( V(L) \). If \( D(L) \) is empty, we say that \( L \) is non-degenerate. If \( D(L) \) is whole of \( S \), we say that \( L \) is everywhere degenerate. If \( D(L) \) is everywhere of positive local codimension, we say that \( L \) is generically non-degenerate.

Given local (and trivializing) coordinates \((u, v; X, Y)\) of \( B \) at \( a \) in \( S \), we can write \( a \) a local generator of \( L \) at \( a \) as

\[
(\kappa(a))((X_1, Y_1), (X_2, Y_2)) = a X_1 X_2 + b(X_1 Y_2 + Y_1 X_2) + c Y_1 Y_2,
\]

The \( O_S \)-ideal locally generated at \((a)\) by \( a c + b^2 \) is also of finite type, whose co-support is exactly the degeneracy locus \( D(L) \), and where \( a, b, c \) lie in \( O_{S, a} \).

**Notation.** We denote this latter \( O_S \)-ideal by \( I^D_L \).

Let \( \sigma : R \to S \) be a locally finite composition of point blowings-up and let \( E \) be the exceptional divisor. By definition, the pull-back \( L^\sigma \) of \( L \) by \( \sigma \) is the invertible \( O_R \)-module of \( \Gamma_R(S(2, B^\sigma)) \) locally generated by \( \kappa^\sigma \) (observe that \( S(2, B^\sigma) = S(2, B^\sigma) \)).

Since \( L \) is non-zero, the total transform of \( \mathcal{C}_L \) factors out as \( \sigma^* \mathcal{C}_L = J \cdot K \), for a principal \( O_S \)-ideal \( J \) monomial in \( E \) and an \( O_S \)-ideal \( K \) whose co-support does not contain any component of \( E \). The invertible \( O_R \)-submodule of \( \Gamma_R(S(2, B^\sigma)) \) defined as

\[
L^\sigma, div := (L^\sigma)^{div} := J^{-1} \cdot (L^\sigma)
\]

is called the divided pull-back of \( L \) by \( \sigma \). If \( \kappa' \) is a local generator of \( L^\sigma, div \) then the form \( \kappa' \) is the null bilinear symmetric form at \( a \) of \( R \) if and only if \( a \) lies in co-supp(\( K \)).
As we did for \( \mathcal{L} \), we also define \( D(\mathcal{L}^{\sigma,\text{div}}) \), the degeneracy locus of \( \mathcal{L}^{\sigma,\text{div}} \), that is the set of points \( \underline{a} \) of \( R \) where the bilinear symmetric form \( \kappa'(\underline{a}) \) is degenerate, for \( \kappa' \) a local generator of \( \mathcal{L}^{\sigma,\text{div}} \) at \( \underline{a} \). If \( \mathcal{L} \) is non-degenerate (respectively everywhere-degenerate, respectively generically degenerate), so is \( \mathcal{L}^{\sigma,\text{div}} \).

When \( \mathcal{L} \) is generically non-degenerate, we find out that the ideal \( \sigma^* I^D_\mathcal{L} \) decomposes as \( \sigma^* I^D_\mathcal{L} = J^D \cdot K^D \), where \( J^D \) is a principal \( \mathcal{O}_R \)-ideal monomial in \( E \), while \( K^D \) is a \( \mathcal{O}_R \)-ideal whose co-support does not contain any component of \( E \). Since \( I^D_\mathcal{L} \subset \mathcal{O}_\mathcal{L}^2 \), the ideal \( J^D \) is contained in the ideal \( J^2 \), and thus deduces that
\[
D(\mathcal{L}^{\sigma,\text{div}}) = \text{co-sup}(J^{-2} J^D K^D).
\]

We need to prepare a little bit our situation with the following

**Lemma 1** There exists \( \sigma_1 : (S_1, E_1) \to S \), a locally finite sequence of points blowings-up, so that the total transform \( \sigma_1^* \mathcal{L}_\mathcal{L} \) is a principal and monomial \( \mathcal{O}_{S_1} \)-ideal in the snc-divisor \( V_\mathcal{L} := \sigma_1^{-1}(V(\mathcal{L})) \), which contains the exceptional divisor \( E_1 \). Furthermore, if \( \mathcal{L} \) is generically non-degenerate, the total transform \( \sigma_1^* I^D_\mathcal{L} \) is principal and monomial in the snc-divisor \( D_\mathcal{L} := (\sigma_1)^*^{-1}(D(\mathcal{L})) \) which is normal crossing with the snc-divisor \( E_1 \cup V_\mathcal{L}^{\text{str}} \).

**Proof** It is straightforward from principalization and monomialization of ideals [3,17], as quoted in point 1) of Theorem 2.

We need some further preparatory material. We will work mostly locally, with germs. These local data will be gathered in an appropriate module or ideal sheaf.

Let \( B_1 := B^{\sigma_1} \) and let \( \mathcal{L}_1 \) be the invertible \( \mathcal{O}_{S_1} \)-module
\[
\mathcal{L}_1 := \mathcal{L}^{\sigma_1,\text{div}}
\]
(by Lemma 1) and let \( \kappa_1 \) be a local generator of \( \mathcal{L}_1 \) over some open neighbourhood \( \mathcal{U}_1 \) (of some given point). The 2-tensor \( \kappa_1 \) vanishes nowhere in \( \mathcal{U}_1 \). Let \( g_1 \) be the fiber-metric \( g_1^\sigma \). Let \( (u, v; X, Y) \) be regular trivializing coordinates of \( B_1 \) over \( \mathcal{U}_1 \). Let \( G_1(\underline{a}) \) be the matrix of \( g_1(\underline{a}) \) and \( K_1(\underline{a}) \) that of \( \kappa_1 \) for any \( \underline{a} \in \mathcal{U}_1 \). The mappings \( \underline{a} \to G_1(\underline{a}) \), \( K_1(\underline{a}) \) are regular over \( \mathcal{U}_1 \). Up to shrinking \( \mathcal{U}_1 \) and applying Gram-Schmidt orthonormalization to the frame over \( \mathcal{U}_1 \), we can assume that \( G_1 \) is the identity matrix at every point of \( \mathcal{U}_1 \). Thus for any \( U, V \) sections of \( B_1|_{\mathcal{U}_1} \) we get
\[
\kappa_1(U, V) = g_1(K_1U, V) = \langle K_1U, V \rangle.
\]

At any point \( \underline{a} = (u, v) \) of \( \mathcal{U}_1 \) we can write
\[
\kappa_1(\underline{a})(\langle X_1, Y_1 \rangle, \langle X_2, Y_2 \rangle) = a_1 X_1 X_2 + b_1(X_1 Y_2 + X_2 Y_1). + c_1 Y_1 Y_2.
\]

By Lemma 1 we find that \( a_1 c_1 - b_1^2 = M_1 \cdot \varphi_1 \), where \( M_1 \) is a monomial and \( \varphi_1 \) is a unit over \( \mathcal{U}_1 \), or \( a_1 c_1 - b_1^2 \) is identically null, while the ideal generated by \( (a_1, b_1, c_1) \) is \( \mathcal{O}_{S_1}|_{\mathcal{U}_1} \).

Following the idea of [11], the quadratic form in \( B_1|_{\mathcal{U}_1} \)
\[
K_1U \wedge U = b_1(Y^2 - X^2) + (a_1 - c_1)XY =: P_1(\underline{a})(X, Y)
\]
vanishes either along two distinct lines or along the full plane \( B_1|_{\underline{a}} \). Note that
\[
\{U \in B_1|_{\underline{a}} : K_1U \wedge U = 0\} = \{U \in B_1|_{\underline{a}} : d_U K_1(\underline{a}) \wedge d_U g_1(\underline{a}) = 0\},
\]
therefore is independent on the choice of local coordinates on \( B_1 \). Let \( Q_1(\underline{a}) \) be the line of \( S(2, B_1|_{\underline{a}}) \) generated by the quadratic polynomial \( P_1(\underline{a}) \). This vector line depends only on
\( \mathcal{L}_1 \) at \( a \) and that is why we can glue all the local \( Q_1(a) \) together to construct a \( \mathcal{O}_{S_1} \)-coherent invertible sheaf \( \mathcal{Q}_{\mathcal{L}_1} \) of \( \Gamma_{S_1}(S(2, B_1)) \).

Let \( \mathcal{V} D_{\mathcal{L}_1} \) be the vertical discriminant locus of \( \mathcal{L}_1 \), that is the set of points \( a \) in \( S_1 \) at which the principal quadratic ideal \( Q_1(a) \) is null. Let \( \mathcal{I}^{VD}_{\mathcal{L}_1} \) be the \( \mathcal{O}_{S_1} \)-ideal sheaf locally generated by the coefficients of the homogeneous quadratic polynomial \( P_1 \). Therefore the co-support of \( \mathcal{I}^{VD}_{\mathcal{L}_1} \) is exactly \( \mathcal{V} D_{\mathcal{L}_1} \).

The invertible module \( \mathcal{L} \) is said constant along the fibers if its vertical discriminant locus is \( B \).

**Lemma 2**  
If \( \mathcal{L} \) is not constant along the fibers, there exists a locally finite sequence of points blowings-up \( \beta_2 : (S_2, E_2) \to (S_1, E_1) \) so that the total transform \( \beta_2^* \mathcal{I}^{VD}_{\mathcal{L}_1} \) is a principal \( \mathcal{O}_{S_2} \)-ideal sheaf which is monomial in the vertical discriminant, the snc-divisor,

\[
\mathcal{V} D_{\mathcal{L}} := \beta_2^{-1}(\text{co-sup}(\mathcal{I}^{VD}_{\mathcal{L}_1}))
\]

which is normal crossing with the snc-divisor \( E_2 \cup \mathcal{V} \mathcal{E}^{\text{str}}_{\mathcal{L}_1} \cup \mathcal{D}^{\text{str}}_{\mathcal{L}_1} \).

If \( \mathcal{L} \) is constant along the fibers then we define \( S_2 := S_1, E_2 := E_1 \) and \( \beta_2 \) is the identity map of \( S_1 \).

**Proof** It is again straightforward from principalization and monomialization of ideals. \( \square \)

Let \( \mathcal{X} \) be a \( \mathcal{O}_S \)-invertible sheaf of regular sections of \( B \). Assume that the co-support \( \text{sing}(\mathcal{X}) \) of \( \mathcal{X} \) has codimension 2 if not empty. Since \( B \) admits a fibered structure, the data of such a \( \mathcal{X} \) is equivalent to have \( \Theta := \mathcal{X}^\vee \) the \( \mathcal{O}_S \)-invertible sheaf of sections of \( B^\vee \) dual to \( \mathcal{X} \). Although we may not be able to define an invertible sheaf \( \mathcal{X}^\perp \) of \( B \), the orthogonal of \( \mathcal{X} \), we can find an invertible sheaf of sections of \( \mathcal{P}B|_{\text{sing}(\mathcal{X})} \) orthogonal to that generated by \( \mathcal{X} \), say \( \mathcal{X}^\perp : \mathcal{P} \), the orthogonal direction to \( \mathcal{X} \). After a locally finite composition of points blowings-up \( \gamma : (S', E') \to S \) we know that \( \mathcal{X}^\vee = \mathcal{J}' \cdot \mathcal{X}' \) where \( \mathcal{J}' \) is an \( \mathcal{O}_{S'} \)-ideal principal an monomial in \( E' \) and \( \mathcal{X}' \) is an invertible \( \mathcal{O}_{S'} \)-sub-modules of regular sections of \( B^\vee \) with empty co-support. The orthogonal direction invertible sheaf of \( \mathcal{X}' \) is thus well defined over \( S' \).

We now can state the main result of this section

**Proposition 2** Let \( \mathcal{L} \) be a non-zero invertible \( \mathcal{O}_S \)-module of \( \Gamma_S(S(2, B)) \). There exists a locally finite sequence of points blowings-up \( \sigma_R : (R, E_R) \to S \) such that,

1) Assuming that \( \mathcal{L} \) is not constant along the fibers.

i) The conclusions of Lemmas 1 and 2 hold true for \( \sigma_R := \sigma_1 \circ \beta_2 \).

ii) Let \( B_R := B_R^{\text{str}} \). At each point \( a_R \) of \( R \) there exist an open neighbourhood \( \mathcal{U}_R \) of \( a_R \) and a unique pair (up to permutation) \( \theta_1, \theta_2 \) of regular sections of \( B_R^{\text{str}}|\mathcal{U}_R \) such that

(a) For each point \( b \) of \( \mathcal{U}_R \) the kernels \( \ker(\theta_2(\cdot \cdot b)) \) \( \ker(\theta_2(\cdot \cdot b)) \) are both a line of \( B_R|_{\mathcal{U}_R} \) and are orthogonal for the metric \( g^{\text{str}} \).

(b) For each point \( b \) of \( \mathcal{U}_R \) the lines \( \ker(\theta_1(\cdot \cdot b)) \) \( \ker(\theta_1(\cdot \cdot b)) \) are the orthogonal “eigen-lines” of \( K_R \), a local generator of \( \mathcal{L}_R := \mathcal{L}^{\sigma_R, \text{div}} \) over \( \mathcal{U}_R \) (up to shrinking \( \mathcal{U}_R \)).

(c) The local generator \( K_R \) of \( \mathcal{L}_R \) writes over \( \mathcal{U}_R \) as

\[
K_R = \varepsilon_1 M_1 \theta_1 \otimes \theta_1 + \varepsilon_2 M_2 \theta_2 \otimes \theta_2,
\]

with \( \varepsilon_1, \varepsilon_2 \in \{-1, 1\} \) and \( M_1, M_2 \) are monomials in \( V^{\text{str}}_{\mathcal{L}_1} \cup E_R \) locally generating \( \sigma_R^{\text{str}} \mathcal{L}_1 \).
2) Assume $\mathcal{L}$ is constant along the fibers. Let $\Theta$ be any invertible $\mathcal{O}_S$-sub-module of $\Gamma_S(B^{\vee})$, not everywhere co-linear to $\mathcal{L}$ and with co-support of codimension two if not empty. Let $\mathcal{L}_\Theta$ be its ideal of coefficients.

i) The mapping $\sigma_R$ (depending on $\Theta$) factors as $\sigma_R = \sigma_1 \circ \beta_R$, for a locally finite sequence of point blowings-up $\beta_R : (R, E_R) \to (S_1, E_1)$, so that Lemma 1 holds true.

ii) The ideal $\sigma_R^*\mathcal{L}_\Theta$ is principal and monomial in the snc-divisor $V_{\Theta} := \text{co-supp}(\sigma_R^*\mathcal{L}_\Theta)$ which is normal crossing with $E_R \cup V_{\mathcal{L}}^\text{str} \cup D_{\mathcal{L}}^\text{str}$.

iii) Let $\mathcal{L}_R := \mathcal{L}_R^{\text{div}}$ and $B_R := B_R^{\sigma_R}$. Let $\Theta_1$ be the $\mathcal{O}_R$-module of $\Gamma_R(B_R^\sigma)$ defined as $(\sigma_R^*\mathcal{L}_\Theta)^{-1} \Theta_R$ (with empty co-support). Let $\Theta_1^\perp \mathcal{P}$ be the sub-module of $\Gamma_R(\mathcal{P}B_R^{\vee})$ orthogonal to $\Theta_1^\perp$ of $\Gamma_R(\mathcal{P}B_R^{\vee})$ generated by $\Theta_1$, for the fiber-metric $g^{\vee} \circ \sigma_R$ where $g^{\vee}$ is the fiber metric on $B^{\vee}$ induced by the metric $g$.

Points (a), (b) and (c) of 1-ii) are satisfied by $\Theta_1$, $\Theta_1^\perp \mathcal{P}$ substituting $D_{\mathcal{L}}^\text{str}$ by $V_{\Theta}$.

**Proof** Suppose that $\mathcal{L}$ is not constant along the fibers. We check that $\sigma_R := \sigma_1 \circ \beta_R$ satisfies all the properties. We start after Lemma 2.

Let $\mathcal{L}_R := \mathcal{L}_R^{\beta_R} = (\mathcal{L}_R^{\sigma_R})^\text{div}$.

If $\mathcal{L}$ is not constant along the fibers, then the ideal $\beta_R^*\mathcal{L}_1^{VD}$ is a principal $\mathcal{O}_R$-ideal monomial in $V D_{\mathcal{L}}$ while $Q_R^{\beta_R} = J_R \cdot Q_R$, where $J_R$ is a principal $\mathcal{O}_R$-ideal and monomial in $V D_{\mathcal{L}}$ and $Q_R$ is a $\mathcal{O}_R$-invertible sheaf of regular sections of $S(2, B_R)$ whose zero locus is a regular two sheeted ramified covering of $R$. Indeed: Let $a_R$ be a point of $R$ and let $U_R \times \mathbb{R}^2$ be a locally trivializing chart of $B_R$ with coordinates $(w, z, X, Y)$. We find that $\beta_R^{-1}(V D_{\mathcal{L}}) := \{Mb[X^2 - Y^2] + aXY = 0\}$ and $V D_{\mathcal{L}_R} := \{b[(Y - X) + aXY = 0]\}$ for a regular function $c$ over $U_R$. In other words $X^2 + 2cXY - Y^2$ is a local generator of $Q_R$ over $U_R$ and $X^2 + 2cXY - Y^2 = \theta_1(X, Y) \cdot \theta_2(X, Y)$ for regular sections $\theta_1, \theta_2$ of $B_R^\sigma|_{U_R}$, whose kernels are orthogonal with respect to the metric $g_R = g \circ \sigma_R$ and are respectively directed by $[1 : \pm c \pm \sqrt{1 + c^2}] \in \mathcal{P}B_R$. These kernels correspond to the “eigen-directions” of any local generator of $\mathcal{L}_R$.

Since $\mathcal{L}$ is not constant along the fibers, so is $\mathcal{L}_R|_{U_R}$. Thus the regular directions fields $[1 : -c(w, z) + \sqrt{1 + c^2}(w, z)]$ and $[1 : -c(w, z) - \sqrt{1 + c^2}(w, z)]$ diagonalize simultaneously over $U_R$ any generator of $\mathcal{L}_R|_{U_R}$, so that over $U_R$ we get points (a), (b) and (c) immediately.

Point 2). Since $\mathcal{L}$ is constant along the fibers, so is $\mathcal{L}_1$ implying that the quadratic ideal sheaf $Q_{\mathcal{L}_1}$ is the null section of $S(2, B_2)$.

The properties announced for $\Theta_1$ and then $\Theta_1^\perp \mathcal{P}$ are straightforward since the additional blowings-up $\beta_R : (R, E_R) \to (S_1, E_1)$ are just to “make nice” the module $\Theta^{\beta_R, \text{div}}$. $\square$

**Remark 1** In the body of the proof of 1-ii), we show the existence of the invertible $\mathcal{O}_R$-sheaf $Q_R$ of regular sections of $S(2, R)$, such that the restriction $Q_R|_{U_R}$ has a global section over
$U_R$, namely $\theta_1 \cdot \theta_2$. The sub-module $Q_R$ is a global object, unlike its local generators $\theta_1$ and $\theta_2$, any of which may not be glued to yield a global object since there may be topological obstructions, such as the Euler class of $B$.

A careful reading of the above proof, that is tracking down which ideals have been principalized and monomialized, also provides the additional following properties.

**Corollary 1** If $L$ is not constant along the fibers, we see that in point c), we further obtain

If $L_R$ is not degenerate then $M_1 = M_2$ is a monomial in $V^\text{str}_L \cup E_R$ locally generating the ideal $\sigma^*_R C_L$.

If $L_R$ is generically non-degenerate, one of the function germ $M_1$, $M_2$ is a local monomial in $V^\text{str}_L \cup E_R$ locally generating the ideal $\sigma^*_R C_L$, while the other one is a monomial in $V^\text{str}_L \cup D^\text{str}_L \cup E_R$ and is locally generating the $O_R$-ideal $\sigma^*_R I^D_R$.

If $L_R$ is degenerate, one of germs $M_1$, $M_2$ is a monomial in $V^\text{str}_L \cup E_R$ locally generating the ideal $\sigma^*_R C_L$, and the other germ $M_1$, $M_2$ is identically zero.

As a final consequence of Proposition 2 (see [11,21]) we find

**Corollary 2** (see [11]) Let $L$ be a non-zero invertible $O_S$-module of $\Gamma_S(S(2, B))$. There exists a locally finite sequence of points blowings-up $\gamma : (S', E') \to S$ such that for each point $a'$ in $S'$, there exist a neighbourhood $U'$ of $a'$ and two orthonormal and non-vanishing local sections $\xi_1, \xi_2 : U' \to B'$ such that at each point $b'$ of $U'$, the 2-symmetric tensor $\kappa'$ locally generating $E'$ is a sum of squares in the basis $\xi_1(B'), \xi_2(B')$.

Consequently when $L$ admits a global section $\kappa$ over $S$, each “eigen-value” of the 2-symmetric tensor $\kappa^\gamma$, that is the size of each generator $(M_i(\theta_i))^2$ for $i = 1, 2$, is a monomial times a local unit nearby $a'$, thus analytic.

## 5 Resolution of singularities of plane singular foliations

We present here some elements of the classical reduction of singularities of plane foliations, aiming at our paper be as self-contained as can reasonably be. The material presented below is well known and can be found for instance in [5,7–9,26]

Let $O_2 := O_{\mathbb{R}^2,0}$ be the local $\mathbb{R}$-algebra of regular function germs at $0$ the origin of $\mathbb{R}^2$, with maximal ideal $m_2$. Let $\Omega_2^1$ be the $O_2$-module of regular differential 1-form germs at $0$.

Let $\xi$ be a germ of regular vector field at the origin $0$ of $\mathbb{R}^2$. Given any regular local coordinates system $(x, y)$ centered at $0$, the vector field writes as $\xi = a(x, y) \partial_x + b(x, y) \partial_y$, where $a, b \in O_2$. Since we are only interested in foliations (phase portraits), up to dividing $\xi$ by $\gcd(a, b)$, we can assume that $a$ and $b$ have no common factor so that any vector field of the form $\text{Unit} \cdot \xi$ gives the same foliation as $\xi$. The vector field $\xi$ comes with (up to the multiplication by a regular unit) a unique dual regular differential form defining the same foliation, namely $\omega = bdx - ady$. Since $\gcd(a, b) = 1$, we have

$$\iota(a, b, 0) := \dim_{\mathbb{R}} O_2/(a, b) < +\infty. \quad (2)$$

**Definition 1** 1) A germ of a plane foliation $\mathcal{F}$ at the origin of $\mathbb{R}^2$ is the data of an invertible $O_2$-sub-module $D_\mathcal{F}$ of $\Omega_2^1$, which is finite co-dimenisonal at the origin, that is satisfying Eq. (2). If $D_\mathcal{F}$ is generated by $\omega = bdx - ady$, the number $\iota(a, b, 0)$ depends only on the invertible sheaf generated by $\omega$, therefore we write $\iota(\mathcal{F}, 0)$ instead.

2) Let $S$ be a regular surface. A foliation $\mathcal{F}$ on $S$ is the data of a non-zero $O_S$-invertible sub-module $D_\mathcal{F}$ of $\Omega_S^1$ such that at each point $p$ of $S$ there exists a regular diffeomorphism

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germ \( \phi : (S, p) \rightarrow (\mathbb{R}^2, 0) \) such that the \( \mathcal{O}_2 \)-submodule \( \phi_* \mathcal{D}_F \) of \( \Omega^1_2 \) is generating a germ of plane foliation at \( 0 \).

The invertible sub-module \( \mathcal{D}_F \) of \( \Omega^1_2 \) corresponding to the germ of foliation \( F \) generated by the vector field germ \( \xi \) is \( \mathcal{O}_{2\omega} \), for \( \omega \) its dual form. Point 1) of the Definition implies that the germ of vector field \( \xi \) (equivalently \( \omega \)) may only vanish at \( 0 \), so that for \( p \) close enough to \( 0 \) but not \( 0 \), we find \( \iota(F, p) = 0 \).

Let \( F \) be a foliation on a regular surface \( S \). The \textit{ideal of coefficients of} \( F \) is the \( \mathcal{O}_S \)-ideal \( \mathcal{C}_F \) obtained by the evaluation of any local generator of \( \mathcal{D}_F \) along the germs of regular vector field on \( S \). The \textit{singular locus of} \( F \) is the \textit{sub-variety} defined as

\[
\text{sing}(F) := \text{co-supp} \mathcal{C}_F,
\]

and is of codimension 2, if not empty. A plane foliation will be said \textit{singular} if the sub-variety \( \text{sing}(F) \) is not empty.

**Definition 2** ([7–9,20,26]) Let \( F \) be a germ of plane foliation with an isolated singularity at the origin. The origin \( 0 \) is called a \textit{simple singularity} of \( F \), if there exist local coordinates \((x, y)\) centered at the origin such that the local generator \( \omega \) writes

\[
\omega = \lambda ydx - \mu xdy + \theta,
\]

with \( \lambda \in \mathbb{R}, \mu \in \mathbb{R}^* \), and \( \mu^{-1} \lambda \notin \mathbb{Q}_{>0} \), and \( \theta \in \mathfrak{m}_2^2 \Omega^1_2 \),

**Definition 3** Let \( S \) be a regular surface and \( F \) be a foliation on \( S \). Let \( p \) be a point of \( S \).

- Let \((H, p)\) be a hypersurface germ, regular at \( p \). It is called \textit{invariant (or non-di-critical) at} \( p \) \textit{with respect to} \( F \), if it is a union of leaves of \((F, p)\). If it is not invariant at \( p \), it is called \textit{di-critical at} \( p \).
- A normal crossing divisor \( D \) of \( S \) containing \( p \) is \textit{invariant at} \( p \) \textit{w.r.t.} \( F \) if each irreducible component of \((D, p)\) is invariant at \( p \) \textit{w.r.t.} \( F \).
- If \( p \) is a regular point of \( F \), the foliation \( F \) is \textit{normal crossing with} \( D \) at \( p \) if the germ \((D, p)\) is not invariant, and the union \( L_p \cup D \), of the leaf \( L_p \) through \( p \) with \( D \), is the germ of a normal crossing divisor at \( p \).

We introduce the following well known convenient

**Definition 4** A foliation \( F \) of \( S \) is \textit{adapted to the nc-divisor} \( D \) of \( S \) if:

- Each singular point of \( F \) is simple;
- At a regular point \( p \) of \( D \), regular point of \( F \), the foliation \( F \) is normal crossing with the germ \((D, p)\);
- At a corner point \( p \) of \( D \), regular point of \( F \), one local component of \((D, p)\) is invariant at \( p \) \textit{w.r.t.} \( F \) and the other one is normal crossing with \( F \) at \( p \);
- At a point \( p \) of \( D \), singular point of \( F \), the germ \((D, p)\) is invariant at \( p \) \textit{w.r.t.} \( F \).

**Reminder on notations.** In this section and in the following one, we recall that when \( \theta \) is a differential 1-form on a manifold \( N \) (or a sub-module of \( \Omega^1_N \)), the notation \( \sigma^*\theta \) means the pull back by a given regular mapping \( \sigma : M \rightarrow N \) in the sense of differential topology, that is \( \sigma^*\theta := \theta \circ D\sigma \) where \( D\sigma \) is the differential mapping of \( \sigma \).

Let \( F \) be a singular foliation at the origin \( 0 \) of \( \mathbb{R}^2 \).

Let \( \pi : S_0 := [\mathbb{R}^2, 0] \rightarrow \mathbb{R}^2 \) be the blowing-up of the origin \( 0 \) and let \( E_0 \) be the exceptional curve \( \pi^{-1}(0) \). Let \( I_{E_0} \) be the \( \mathcal{O}_{S_0} \)-ideal of the regular function germs vanishing on \( E_0 \). There exists a largest positive integer \( m \) such that \( I_{E_0}^m \cdot \pi^* \mathcal{D}_F \) is a \( \mathcal{O}_{S_0} \)-invertible sub-module of \( \Omega^1_{S_0} \), which is finite co-dimensional everywhere.
**Remark 2** Let \( \beta : (S, E) \to (\mathbb{R}^2, 0) \) be a finite composition of point blowings-up, where \( S \) is a regular surface and \( E := \beta^{-1}(0) \) is a snc divisor. The ideal \( \mathcal{C}_{\beta*D_{\mathcal{F}}} \) of coefficients of the \( O_S \)-sub-module \( \beta^*D_{\mathcal{F}} \) decomposes into a product of \( O_S \)-ideals \( J \cdot K \), where the ideal \( J \) is principal and monomial in the exceptional divisor \( E \), while the ideal \( K \) (with co-support in \( E \)) has finite co-dimension, namely \( \dim_{\mathbb{R}} O_{S,p}/K < +\infty \), for any point \( p \) of \( S \).

Remark 2 leads to the following

**Definition 5** Let \( \beta : (S, E) \to (\mathbb{R}^2, 0) \) be a finite composition of point blowings-up.

Using the notations of Remark 2, the pulled-back foliation \( \beta^*\mathcal{F} \) of \( \mathcal{F} \), is given by the invertible \( O_S \)-sub-module \( D_{\beta^*\mathcal{F}} := J^{-1}\beta^*D_{\mathcal{F}} \) of \( \Omega^1_S \). For a local generator \( \omega \) of \( \mathcal{F} \) at \( 0 \), every point \( p \) of \( S \) admits an open neighbourhood \( U \) over which there exist a monomial \( M \) in the exceptional divisor \( E \) (generating \( J \) locally over \( U \)) and a local generator \( \theta \) of \( D_{\beta^*\mathcal{F}} \) over \( U \) such that \( M\theta = \beta^*\omega \). The local generator \( M^{-1}\beta^*\omega \) is called the strict transform of \( \omega \) under the blowing-up \( \beta \).

We can now present the theorem of reduction of singularities of singular plane foliation ((5,7,8,26)) in the form which is most convenient for our later use. The global reduction of singularities of plane foliation being deduced from the local one, we only present the latter one in the following classical form:

**Theorem 5** ([5,7–9,20,26]) Let \( \mathcal{F} \) be a germ of singular plane foliation at the origin \( 0 \) of \( \mathbb{R}^2 \). There exists a finite composition of points blowings-up \( \pi : (S', E') \to (\mathbb{R}^2, 0) \) such that each point of \( \operatorname{sing}(\mathcal{F}', E') \) of the pulled-back foliation \( \mathcal{F}' := \pi^*\mathcal{F} \) is a simple singularity of \( \mathcal{F}' \) adapted to \( E' \).

Moreover if \( \beta \) is the point blowing-up \( (S'', E'') \to (S', E' \cup \{p'\}) \) with center \( p' \), then \( \mathcal{F}'' := \beta^*\mathcal{F}' \) only admits simple singularities adapted to \( E'' \).

We end the section with the normal form of a local generator of a “reduced” germ of singular plane foliation \( \mathcal{F} \) with simple singularities adapted to an exceptional divisor \( E \) as in Theorem 5. By Remark 2, the exceptional divisor \( E \) contains the singular locus \( \operatorname{sing}(\mathcal{F}) \).

- Let \( p \notin E \), then \( \omega(p) \neq 0 \).
- Suppose \( p \in E \setminus \operatorname{sing}(\mathcal{F}) \). There exist local coordinates \((u, v)\) centered at \( p \) such that \( \{u = 0\} \subset (E, p) \subset \{uv = 0\} \) and \( \omega(p) \neq 0 \).

If \((E, p) = \{u = 0\}\) is invariant for \( \mathcal{F} \), a local generator is of the form \( \omega = du + u(\cdots)dv \).

If \((E, p) = \{uv = 0\}\) is normal crossing with \( \mathcal{F} \), a local generator is of the form \( \omega = dv \) (up to a change of coordinate of the form \( \tilde{v} = v + F \), with \( F \in m_p \)).

If \((E, p) = \{uv = 0\}\) and if a local generator is of the form \( du + \text{Unit} \cdot dv \), we check that blowing-up \( p \) will give a local generator of the pulled-backed foliation such that at each of the new two corners one of the new exceptional divisor and the strict transform of the corresponding old component through \( p \), is invariant and the other one is di-critical. Thus (up to blowing-up the point \( p \)), we deduce that, up to permuting \( u \) and \( v \), a local generator is given by \( du + u(\cdots)dv \) (see Lemma 3 and 4 for details).

- Suppose \( p \in E \cap \operatorname{sing}(\mathcal{F}) \). Thus each component of \( E \) must be invariant. There exist local coordinates \((u, v)\) centered at \( p \) such that \( \{u = 0\} \subset (E, p) \subset \{uv = 0\} \) and there exists another set of local coordinates \((x, y)\) centered at \( p \) such that \( \omega = \lambda xdy - ydx + \theta \) where \( \theta \in m_p^2\Omega^1_S \) and \( \lambda \notin \mathbb{Q}_{>0} \).

If \((E, p) = \{uv = 0\}\), since it is non-di-critical, we deduce also that, up to permuting \( u \) and \( v \), we get \( \omega = vdu + u(\cdots)dv \).
If \((E, p) = \{u = 0\}\), then a local generator writes as \(\omega = uAdv + (uB + v^k\phi(v))du\), with \(A, B \in \mathcal{O}_p\), where \(\phi\) is an analytic unit in a single variable and \(k = 1\) if \(A(p) = 0\). When \(A(p) \neq 0\), a local generator is of the form

\[
u dv + (v^k\phi(v) + uB)du
\]  
and when \(A(p) = 0\), it is of the form

\[
(v + b_0u)du + u\theta
\]

where \(\theta \in \mathfrak{m}_p\Omega^1_p\) and \(B \in \mathcal{O}_p\) and \(b_0 \in \mathbb{R}\).

### 6 Pairs of singular foliations and singular foliation adapted to nc-divisors

The material presented in this section is known folklore of desingularization of singular plane foliations (see [5,7,20]). In order to have our paper reasonably self-contained, we present the results, that we will use in the demonstration of Theorem 6, with proofs.

Any singular foliation can be resolved into a singular foliation with simple singularities adapted to the exceptional divisor of the resolution (Sect. 5). Simple adapted singularities transform under blowings-up either in simple adapted singularities or in regular points.

An additional property of the resolution of a singular foliation to wish for is a good behavior of the foliation with respect to (the pull-back of) a given curve. A consequence of Definition 4 is the next

**Lemma 3** (see also [20, Sect. 8]) Let \(D\) be a nc-divisor of a regular surface \(S\) and let \(\mathcal{F}\) be a foliation on \(S\) adapted to \(D\). Let \(\beta : (S', E') \to S\) be the blowing-up with center a given point \(p\) of \(S\). The pulled-back foliation \(\beta^*\mathcal{F}\) is adapted to the nc-divisor \(\beta^{-1}(D)\).

**Proof** If \(p\) does not belong to \(D\), it is true.

Let \((u, v)\) be local coordinates centered at the corner point \(p\) of \(D\) and adapted to \(D\), thus \((D, p) = \{uv = 0\}\). We use the normal forms presented at the end of Sect. 5.

If \(p\) is a singular point of \(\mathcal{F}\) adapted to \(D\), then, up to permuting \(u\) and \(v\), a local generator of \(\mathcal{F}\) at \(p\) is \(\theta = vdu + uAdv\) with \(A \in \mathcal{O}_p\) such that \(-A(p)\) is not in \(\mathbb{Q}_{>0}\).

In the chart \((x, y) \to (x, xy)\) of the blowing-up \(\beta\) of the point \(p\), we find

\[
\beta^{-1}(D) = \{xy = 0\}, \quad \text{and} \quad \beta^*\theta = x \cdot \text{Unit} \cdot (ydx + xBdy),
\]

so that in this chart \(\beta^*\mathcal{F}\) is adapted to \(\beta^{-1}(D)\). In the chart \((x, y) \to (xy, y)\), we check similarly that \(\beta^*\mathcal{F}\) is also adapted to \(\beta^{-1}(D)\).

If \(p\) is a regular point of \(\mathcal{F}\) then, up to permuting \(u\) and \(v\), a local generator of \(\mathcal{F}\) at \(p\) is \(\theta = du + uAdv\) with \(A \in \mathcal{O}_p\) such that \(-A(p) \notin \mathbb{Q}_{>0}\).

In the chart \((x, y) \to (x, xy)\) of the blowing-up \(\beta\) of the point \(p\), we find

\[
\beta^{-1}(D) = \{xy = 0\}, \quad \text{and} \quad \beta^*\theta = \text{Unit} \cdot (dx + xBdy)
\]

so that in this chart \(\beta^*\mathcal{F}\) is adapted to \(\beta^{-1}(D)\). In the chart \((x, y) \to (xy, y)\), we get

\[
\beta^{-1}(D) = \{xy = 0\}, \quad \text{and} \quad \beta^*\theta = \text{Unit} \cdot (ydx + xBdy) \quad \text{with} \quad B(0, 0) = 1
\]

so that in this chart \(\beta^*\mathcal{F}\) is also adapted to \(\beta^{-1}(D)\).

When \(p\) is regular point of \(D\), similar computations will lead to the stated conclusion, using the normal forms at the end of Sect. 5. \(\Box\)
Let $D$ be a (s)nc-divisor of $S$ and let $\mathcal{F}$ be a foliation on $S$. Let $NA(\mathcal{F}, D)$ be the subset of points of $S$ where the foliation $\mathcal{F}$ is not adapted to the germ of $D$ at this point. It is a real analytic set which is isolated, when not empty.

**Lemma 4** (see also [20, Sect. 8]) There exists $\sigma_1 : (S_1, E_1) \to S$, a locally finite sequence of points blowings-up, such that the pulled-back foliation $\sigma_1^* \mathcal{F}$ is adapted to the (s)nc-divisor $E_1 \cup D^{\text{str}}$.

**Proof** We resolve the singularities of $\mathcal{F}$ with $\beta : (S', E') \to S$ a locally finite sequence of blowings-up, so that $\mathcal{F}'$ the pulled-back foliation $\beta^* \mathcal{F}$ is adapted to $E'$. The strict transform $D'$ of $D$ is a (s)nc-divisor which is normal crossing with $E'$ (up to possibly finitely many further points blowings-up). The intersection $D' \cap E'$ consists only of isolated points.

Since the locus $NA(\mathcal{F}', D')$ of non-adapted point is isolated, we can suppose that it is reduced to the single point $\{p\}$. Let $(u', v')$ be local coordinates centered at $p'$ and adapted to $D'$ so that $\{v' = 0\} \subset (D', p') \subset \{u'v' = 0\}$.

1) Let $p'$ be a point of $D' \cap E'$, so that it is a regular point of $D'$. Suppose that $u'$ is such that $(E', p') := \{u' = 0\}$.

a) Suppose that $p'$ is a regular point of $\mathcal{F}'$. Thus the leaf through $p'$ is tangent to $D'$ (thus normal crossing with $E'$) while all the nearby ones are normal crossing with $D'$. A local generator of $\mathcal{F}'$ is of the form

$$\text{Unit} \cdot ([u']^l + v' \cdots) du' + dv'$$

for $l$ a positive integer.

The exceptional curve $C''$ obtained by blowing-up the point $p$ is a maximal invariant curve of the strict transform $\mathcal{F}''$ of $\mathcal{F}'$ and is normal crossing with $D''$, the strict transform of $D'$. In the chart $(u'', v'') \to (u', u''v'')$, we find $D'' = \{v'' = 0\}$ with $C'' = \{u'' = 0\}$, so that a local generator at $p'' = (0, 0)$ of $\mathcal{F}''$ is of the form

$$\text{Unit} \cdot ([u'']^{l-1} + v'' \cdots) du'' + dv''.$$
First we resolve the singularities of the curve $C$ by means of a locally finite sequence of points blowings-up, namely $\sigma_1 : (S_1, E_1) \to S$.

Second we principalize and monomialize the ideal $\sigma_1^* C_{1,2}$ by means of a locally finite sequence of point blowings-up $\beta_2 : (S_2, E_2) \to (S_1, E_1)$ so that $\text{co-support}(\sigma_2^* C_{1,2})$ is a snc-divisor which is normal crossing with $E_1 \cup C^{\text{str}}$ and where $\sigma_2 := \sigma_1 \circ \beta_2$.

Third, by means of $\beta_3 : (S_3, E_3 = E_2 \cup E'_3) \to (S_2, E_2)$, a locally finite sequence of point blowings-up where $E'_3$ is the new exceptional divisor (keeping denoting $E_2$ for the strict transform $E_2^{\text{str}}$), the pulled-back foliation $F_i^3 := \beta_3^* (\sigma_2^* F_i)$ only have singularities adapted to $E_3'$ where $i = 1, 2$ (and is regular at each point of $E_2 \setminus E'_3$).

We do further point blowings-up $\beta_4 : (S_4, E_4) \to (S_3, E_3)$ so that the foliations $\beta_4^* F_i$ are adapted to the snc-divisor $E_4 \cup C^{\text{str}} \cup \Sigma(F_1, F_2)^{\text{str}}$, which is possible thanks to Lemma 3 and Lemma 4. Then we define $\tau_i^* := \beta_4^* \mathcal{D}_i^3$ and $\tau := \sigma_2 \circ \beta_3 \circ \beta_4$.

Last to get point iv) we have

$$\tau^* \mathcal{D}_1 = \mathcal{I}_1 \mathcal{D}_1^1, \quad \tau^* \mathcal{D}_2 = \mathcal{I}_2 \mathcal{D}_2, \quad \tau^* (\mathcal{D}_1 \cap \mathcal{D}_2) = \tau^* (\mathcal{C}_{1,2} \Omega^2_S) = \tau^* \mathcal{C}_{1,2} \Omega^2_{S_4}$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are principal and monomial ideals in some snc divisors contained in $E_4 \cup C^{\text{str}} \cup \Sigma^{\text{str}}$. From point ii) we deduce point iv) since $\tau^* C_{1,2} = \mathcal{I}_1 \cdot \mathcal{I}_2 \cdot \mathcal{I}_{1,2}$. \hfill $\Box$

The next proposition is of interest for our main result, but requires a few preparations.
We denote $\Omega^2_S(\log E)$ the $\mathcal{O}_S$-module of $q$-logarithmic forms along $E$.

If $p$ does not lie in $E$, there exists a neighbourhood $U$ of $p$ such that

$$\Omega^1_S(\log E)|_U = \Omega^1_S|_U = \Omega^1_S(\log E)|_U.$$ 

If $p$ is a regular point of $E$, we can find local coordinates $(u, v)$ at $p$ adapted to $E$ for which $(E, p) = (u = 0)$, there exists a neighbourhood $U$ of $p$ such that

$$\Omega^1_S(\log E)|_U = \mathcal{O}_U du + \mathcal{O}_U d\log u,$$

where $d\log u := \frac{du}{u}$.

If $p$ is a corner point of $E$, there are local coordinates $(u, v)$ at $p$ adapted to $E$ such that $(E, p) = (uv = 0)$, there exists a neighbourhood $U$ of $p$ such that

$$\Omega^1_S(\log E)|_U = \mathcal{O}_U d\log u + \mathcal{O}_U d\log v.$$ 

Since $\Omega^1_S$ is a sub-module of $\Omega^1_S(\log E)$, any sub-module $\Theta$ of $\Omega^1_S$ is a sub-module of $\Omega^1_S(\log E)$. A local logarithmic generator of $\Theta$ is a local generator of $\Theta$ as a sub-module of $\Omega^1_S(\log E)$. The ideal of logarithmic coefficients of $\Theta$ is the ideal $C^\log_\Theta$ locally generated by the logarithmic generator of $\Theta$ evaluated along local regular vector fields on $S$. Note that if $C^\log_\Theta$ is the ideal of coefficients of $\Theta$, then $C^\log_\Theta \subseteq C^\log_\Theta$.

Assume Proposition 3 is satisfied.

Let $p'$ be a corner point of $E'$. Let $(u, v)$ be local coordinates centered at $p'$ and adapted to $E'$. For each $i$, the foliation $\mathcal{F}'_i$ has a local generator at $p'$ either of the form $du + u(\cdots)dv$ (up to permuting $u$ and $v$) or $p'$ is an adapted singularity of $\mathcal{F}'_i$. Let $\theta_i$ be a local generator of $\mathcal{D}'_i$. The logarithmic generator $\theta'^{\log}_i$ of $\mathcal{D}'_i$ associated to $\theta_i$ is defined as follows:

- if $\theta_i = du + u(\cdots)dv$ then $\theta'^{\log}_i := u^{-1}\theta_i = d\log u + v(\cdots)d\log v$,
- if $\theta_i = vdu + u(\cdots)dv$ then $\theta'^{\log}_i := u^{-1}v^{-1}\theta_i = d\log u + (\cdots)d\log v$.

Note that, in each case, the logarithmic 1-form $\theta'^{\log}_i$ is nowhere vanishing.

Let $\mathcal{M}_i$ be a local monomial (in $E'$) generating the ideal $\tau^*\mathcal{C}_{\mathcal{D}_i}$ and let $\mathcal{M}_i^{\log}$ be a local generator of $\mathcal{C}^{\log}_{\tau^*\mathcal{D}_i}$, the logarithmic coefficient ideal of the total transform $\tau^*\mathcal{D}_i$. According to the two cases to distinguish, we either find that $\mathcal{M}_i^{\log} = uv \cdot \mathcal{M}_i$ or, respectively, that $\mathcal{M}_i^{\log} = uv \cdot \mathcal{M}_i$.

Proposition 4 Continuing Proposition 3, let $p'$ be a corner point of the exceptional divisor $E'$.

v) If $p'$ is an adapted singular point of $\mathcal{F}'_1$ and $\mathcal{F}'_2$ such that the ideals $\mathcal{C}_{\tau^*\mathcal{D}_1}$ and $\mathcal{C}_{\tau^*\mathcal{D}_2}$ are not ordered at $p'$, there exists a locally finite sequence of point blowings-up $\beta'' : (S'', E'' = E' \cup E''_{\beta''}) \rightarrow (S', E')$ such that at each corner point of $E''_{\beta''}$ the ideals $\mathcal{C}^{\log}_{(\tau_0\beta'')^*\mathcal{D}_1}$ and $\mathcal{C}^{\log}_{(\tau_0\beta'')^*\mathcal{D}_2}$ are ordered. Consequently so are the ideals of logarithmic coefficients $\mathcal{C}^{\log}_{(\tau_0\beta'')^*\mathcal{D}_1}$ and $\mathcal{C}^{\log}_{(\tau_0\beta'')^*\mathcal{D}_2}$.

vi) If $p'$ is a regular point of $\mathcal{F}'_1$ such that the ideals $\mathcal{C}_{\tau^*\mathcal{D}_1}$ and $\mathcal{C}_{\tau^*\mathcal{D}_2}$ are not ordered at $p'$, there exists $\beta'' : (S'', E'' = E' \cup E''_{\beta''}) \rightarrow (S', E')$, a locally finite sequence of point blowings-up, such that at each corner point of $E''_{\beta''}$ the ideals $\mathcal{C}^{\log}_{(\tau_0\beta'')^*\mathcal{D}_1}$ and $\mathcal{C}^{\log}_{(\tau_0\beta'')^*\mathcal{D}_2}$ are ordered. Thus the ideals $\mathcal{C}^{\log}_{(\tau_0\beta'')^*\mathcal{D}_1}$ and $\mathcal{C}^{\log}_{(\tau_0\beta'')^*\mathcal{D}_2}$ are also ordered.
Proof We recall that $\mathcal{J}_i := \mathcal{C}_{+D_i}$, so that $\mathcal{C}_i^{\log} = (uv) \cdot \mathcal{J}_i$.

Suppose $p'$ is an adapted singularity of both foliations. Let $(u, v)$ be local coordinates adapted to $E'$ at $p'$, so that $(E', p') = \{uv = 0\}$. Thus $\mathcal{J}_i$ is locally generated by $u^{p_i}v^{q_i}$ for non-negative integers $p_i, q_i$ and $i = 1, 2$. Suppose that $p_1 < p_2$ and $q_2 < q_1$.

Let $\theta \in \Omega^1_{p'}$ such that $p'$ is a singularity adapted to $(E', p')$. Thus we can assume (up to permuting $u$ and $v$) that $\theta = vdu - \lambda udv + uv\eta$ where $\eta \in \Omega^1_{p'}$ and $\lambda \notin \mathbb{Q}_{>0}$. Let $\gamma$ be the blowing-up of $p'$. At any (of the two) corner point $p''$ of $E'' := E' \cup \gamma^{-1}(p')$, let $z$ be a reduced equation of $\gamma^{-1}(p')$. We find that $\sigma^*\theta = z \cdot \theta'$ for $\theta' \in \Omega^1_{p''}$ and (following Lemma 3) such that $p''$ is an adapted singularity of $\theta'$.

Let $\omega_i$ be a local generator of $\mathcal{J}_i$. Let $p''$ be a corner point of $E''$. Let $(t, z)$ be local coordinates at $p''$ adapted to $E''$ such that $(\gamma^{-1}(p'), p'') = \{z = 0\}$. Thus we deduce that $(\tau \circ \gamma)^*\omega_i = t^{ri}z^{(s_i+1)}w_i'$ where $(r_i, s_i) = (p_i, q_i)$ or $(q_i, p_i)$ and for a local generator $\omega_i'$ of $\gamma^*\mathcal{J}_i$ which has an adapted singularity at $p''$. Thus either $(r_1 + s_1 + 1) - (r_2 + s_2 + 1)$ and $r_1 - r_2$ have the same sign or $|(r_1 + s_1 + 1) - (r_2 + s_2 + 1)| < |r_1 - r_2|$.

With further finitely many point blowings-up, at any corner point $q$ of the exceptional divisor lying over $p'$, the coefficients ideals of the total transform of $D_1$ and $D_2$ are ordered.

Let $p'$ be a corner point of $E'$ such that it is an adapted singularity of only one of the two foliations, say $\mathcal{F}_2$. Let $(u, v)$ be local coordinates adapted to $E'$, thus $(E', p') = \{uv = 0\}$ and such that a local generator $\theta_1$ of $\mathcal{F}_1'$ is of the form $du + uCdv$, so that $G := \{u = 0\}$ is invariant for $\mathcal{F}_1'$ and $H := \{v = 0\}$ is di-critical. Let just write a local generator $\theta_2$ of $\mathcal{F}_2'$ as $\phi(u)du + uBdv$ and (at least) one of the function germ $A, B$ is a local analytic unit.

Suppose that for $i = 1, 2$ the ideal $\mathcal{J}_i$ is locally generated by $u^{p_i}v^{q_i}$ for non-negative integers $p_i, q_i$ and $i = 1, 2$ with $(p_1 - p_2)(q_1 - q_2) < 0$.

Let $\gamma$ be the blowing-up with center $p'$. Note that for $i = 1, 2$ the logarithmic generator $\theta_i^{\log}$ is of the form $a_id\log u + b_id\log v$, and at least one of the function germs $a_i, b_i$ is a local analytic unit.

We observe that for $i = 1, 2$ the pull-back $\gamma^*\theta_i^{\log}$ of $\theta_i^{\log}$ is of the form $a_i'\log u + b_i'\log v$ and one of the function germs $a_i', b_i'$ is a local analytic unit. Thus the ideal of logarithmic coefficients of $(\tau \circ \gamma)^*\mathcal{D}_i$ is the pull-back $\gamma^*(\theta_i^{\log})$ of the ideal of logarithmic coefficients of $\tau^*\mathcal{D}_i$. We have thus replaced a problem of foliations into a problem on (principal and monomial) ideals to order, which can be achieved by finitely many point blowings-up.

If $p'$ is a corner point of $E'$ such that it is a regular point of both foliations $\mathcal{F}_1'$ and $\mathcal{F}_2'$, then we reach the conclusion as in the case of a single regular foliation. 

We end-up the section with some pairs of normal forms at points of $\Sigma(\mathcal{F}_1', \mathcal{F}_2')$.

* Let $p$ be a regular point of $\Sigma(\mathcal{F}_1', \mathcal{F}_2')$ and let $H$ be the component containing $p$. Let’s pick local coordinates $(u, v)$ at $p$ such that $(H, p) = \{u = 0\}$. Let $\theta_1$ be a local generator of $\mathcal{F}_1'$, for $i = 1, 2$. We find that $\theta_1 \wedge \theta_2 = Unit \cdot u^mdu \wedge dv$, for a positive integer $m$. Note that $H$ is either invariant for $\mathcal{F}_1'$ and $\mathcal{F}_2'$ or is di-critical for $\mathcal{F}_1'$ and $\mathcal{F}_2'$.

**Case 1.** Suppose $\theta_1(p) \neq 0$ and $\theta_2(p) \neq 0$. Then we check that $\theta_2 = Unit \cdot \theta_1 + u^m\omega$ with $\omega$ such that $\omega = dv$ if $H$ is invariant for $\theta_1$ and $\omega = du$ if $H$ is di-critical for $\theta_1$.

**Case 2.** Suppose $\theta_1(p) \neq 0$ and $\theta_2(p) = 0$. Thus $H$ is invariant for $\mathcal{F}_2'$. We get that $\theta_1 = du + uB_1dv$ while $\theta_2 = (v^k\phi(v) + uA_2)du + uB_2dv$, for function germs $A_2, B_1, B_2$ such that $uB_2 - uB_1(v^k\phi(v) + uA_2) = u^m$, so that $\theta_2 = (v^k\phi(v) + uA_2)\theta_1 + u^m\omega$. Note that $m = 1$ if and only if $B_2$ is a unit.
Case 3. Suppose $\theta_1 = udv + (v^k\phi_1(v) + uB_1)\,du$ and $\theta_2(p) = 0$. Thus $H$ is invariant for both foliations and necessarily $m \geq 2$. We write $\theta_2 = u(v^k\phi_2(v) + uB_2)\,du + uA_2\,dv$ so that $(v^k\phi_2 + uB_2) - A_2(v^k\phi_1 + uB_1) = u^{m-1}$, thus $\theta_2 = u^{m-1}\,dv + A_2\theta_1$.

There would be another case to consider, but the situation in which we will use these normal forms and their behavior is, as we will see in Appendix B, not generic. We will deal with this last situation in due time.

• Let $p$ be a corner point of $\Sigma(F'_1, F'_2)$. Let $(u, v)$ be local coordinates centered at $p$ such that $(\Sigma(F'_1, F'_2), p) = \{uv = 0\}$. Let $\theta_i$ be local generator of $F'_i$, for $i = 1, 2$. Thus we find that $\theta_1 \land \theta_2 = \text{Unit} \cdot u^m v^m \,du \land dv$, for positive integers $m$ and $n$.

Case 4. Suppose $\theta_1(p) \not= 0$ and $\theta_2(p) \not= 0$. Up to permuting $u$ and $v$, we can find $\theta_1$ and $\theta_2$ such that $\theta_1 = du + u(\cdots)\,dv$ and $\theta_2 = \text{Unit} \cdot \theta_1 + u^m v^m \,dv$.

Case 5. Suppose $\theta_1(p) \not= 0$ and $\theta_2(p) = 0$. We find $\theta_1$ such that $\theta_1 = du + u(\cdots)\,dv$, up to permuting $u$ and $v$. The point $p$ is a singularity of $F'_2$ adapted to $\Sigma(F'_1, F'_2)$ and $\theta_2 = wdz + (\cdots)\,dv$ for $(w, z) = (u, v)$ or $(w, z) = (v, u)$. We deduce that $\theta_2 = \text{Unit} \cdot v\theta_1 + u^m v^n \,dv$. The other case is not possible since $n$ must be positive.

Case 6. Suppose $\theta_1(p) = 0$ and $\theta_2(p) = 0$. The point $p$ is a singularity of $F'_1$ and of $F'_2$ adapted to $\Sigma(F'_1, F'_2)$. Up to permuting $u$ and $v$ we find $\theta_1$ such that $\theta_1 = vdu + u(\cdots)\,dv$. We know that $\theta_2 = wdz + (\cdots)\,dv$ for $(w, z) = (u, v)$ or $(w, z) = (v, u)$. We deduce, up to a multiplication by a local unit, we can find $\theta_2$ such that $\theta_2 = \text{Unit} \cdot \theta_1 + u^m v^{n-1} \,dv$.

7 Main result: monomialization of $2$-symmetric tensors on regular surfaces

We present here the main result of the paper. At the very end of the section, we will recall the two situations we want to apply the main result to, starting points of the paper.

We recall some well known facts about morphisms between vector-bundles.

Let $\sigma : M \rightarrow N$ be a regular mapping between regular manifolds. Any fiber-bundle considered below will be a regular fiber-bundle, unless explicitly mentioned otherwise.

Let $F$ be a vector bundle of finite rank over $N$. The base change $\sigma : M \rightarrow N$ induces a regular mapping of vector bundle $\sigma B : F^\sigma \rightarrow F$, induces $\sigma$ on the 0-section and identity in the fibers.

If $A : F \rightarrow F'$ is a regular mapping of vector bundles (both of finite rank) over $N$, the base-change $\sigma$ induces a regular mapping $A^\sigma : F^\sigma \rightarrow (F')^\sigma$ of vector bundles over $M$.

Let $F$ be a vector bundle over $N$ and $E$ be a vector bundle over $M$, both of finite rank. Let $\Phi : E \rightarrow F$ be a regular vector bundles mapping along $\sigma$, that is such that $\pi E^F \circ \Phi = \sigma \circ \pi M^E$, where $\pi E^F$ denotes the projection of the vector bundle $E$ onto its basis $B$.

The regular mapping $A \circ \Phi : E \rightarrow F'$ of vector bundles along $\sigma$ is thus well defined.

There exists also a unique regular mapping $\Phi^\sigma : E \rightarrow F^\sigma$ of regular vector bundles over $M$ factoring $\Phi$ through $\sigma B^F$, namely $\sigma B^E \circ \Phi^\sigma = \Phi$.

The differential mapping $D\sigma : TM \rightarrow TN$ is a regular mapping of vector bundle along $\sigma$. Thus it factors as $D\sigma = \sigma T^N \circ \Delta\sigma$, where

$$\Delta\sigma := (D\sigma)^\sigma : TM \rightarrow TN^\sigma.$$

This allows to pull-back any $O_M$-section $\theta : M \rightarrow (TN)^\sigma$ as the regular $O_M$-section $\theta \circ \Delta\sigma : M \rightarrow TM^\nu$, in other words, a regular 1-form on $M$. 
Important reminder about notations. Let $\sigma : M \to N$ be a regular mapping.

Let $\theta$ be a regular differential $1$-form over $N$.

- The notation $\theta^\sigma$ will just mean $\theta \circ \sigma$ section of $(TN^\vee)^\sigma$.
- The notation $\sigma^* \theta$ will mean the pull-back of $\theta$ in the usual sense of differential topology, that is

$$\sigma^* \theta = \theta \circ D\sigma \in \Omega^1_M.$$

The relation between these notations being:

$$\theta \circ D\sigma = \sigma^* \theta = (\theta^\sigma) \circ \Delta \sigma = (\theta \circ \sigma) \circ \Delta \sigma.$$

Let $S$ be a regular surface. Let $F$ be a regular vector bundle of rank $2$ over $S$ equipped with a fiber metric $g$. Let $\mathcal{L}$ be a non-zero invertible $\mathcal{O}_S$-sub-module of $T_S(S(2, F))$. Let $\tau : (T, E_T) \to S$ be a finite composition of point blowings-up so that $\mathcal{L} := \mathcal{L}^{T, \text{div}}$ satisfies the conclusions of Proposition 2. We denote $\mathcal{L}$ the snc-divisor $V_L \cup D_L \cup \Delta_L$ (see Sect. 4). Either $\Delta_L := V_D \cup D_L$ if $E_T$ is not constant along the fibers, or $\Delta_L := V_{\mathcal{L}}$ if $E_T$ is constant along the fibers (knowing then that $D_L = \emptyset$ once $L$ is not generically non-degenerate).

Let $\lambda : (T', E_T') \to (T, \Lambda)$ be a locally finite sequence of point blowings-up, where $\Lambda = E_T \cup C$ for $C$ an isolated set of points of $T$. Therefore the exceptional divisor $E_T'$ decomposes as the union $E_T' \cup E_{\lambda}$, where $E_{\lambda}$ is the exceptional divisor of $\lambda$, that is $\lambda^{-1}(C)$.

The next statement refers explicitly to Proposition 2 and to notations introduced in Sect. 4. Any strict transform of a given nc-divisor $\Delta$ will be denoted by $\Delta^{\text{str}}$, in the exception of exceptional divisors where strict transforms will still be denoted with the same symbol.

Theorem 6 Let $S$ be a regular surface. Let $B$ be a regular vector bundle of rank $2$ over $S$ equipped with a fiber metric $g$. Assume that there exists a regular mapping $S : TS \to B$ of vector bundles over $S$ and a sub-variety $Z$ of $S$ of positive codimension, such that the restriction $S : TS_{|S \setminus Z} \to B_{|S \setminus Z}$ is a regular isomorphism of vector bundles over $S_{|S \setminus Z}$.

Let $\mathcal{L}_B$ be a non-zero invertible $\mathcal{O}_S$-sub-module of sections of $(S(2, B))$. Let be a finite composition of point blowings-up $\sigma_R : (R, E_R) \to S$ such that $\mathcal{L} := (\mathcal{L}_B \circ \sigma)_R$ satisfies the conclusions of Proposition 2. Then we find

1) There exists $\beta' : (S', E') \to (R, E_R)$, a locally finite composition of point blowings-up, where $E' := E_{\beta'} \cup E_{R}$, such that the sub-variety $Z' = E' \cup \sigma'^{-1}(Z)$ is a snc-divisor normal crossing with $Y^{\text{str}}_{\mathcal{L}}$, where $\sigma' := \sigma_R \circ \beta'$.

2) There exists $\tilde{E} : (\tilde{S}, \tilde{E}) \to (S', E')$, a locally finite sequence of point blowings-up, with $\tilde{E} := E_{\tilde{\beta}} \cup E'$, and let $\tilde{\sigma}$ be the composed mapping $\sigma' \circ \tilde{E}$, such that

i) At each point $\tilde{p}$ of $\tilde{S}$ there exist a neighbourhood $\tilde{U}$ of $\tilde{p}$ in $\tilde{S}$ and $1$-forms $\theta_1, \theta_2$ of $\Omega^1(\tilde{U})$ such that each foliation generated by $\theta_i$ has only simple singularities adapted to the snc divisor $\tilde{U} \cap \tilde{D}$, where $\tilde{D}$ denotes the snc-divisor $Y^{\text{str}}_{\mathcal{L}} \cup \tilde{Z}$ for $\tilde{Z}$ being $\tilde{E} \cup \tilde{\sigma}^{-1}(Z)$.

ii) Each point $\tilde{p}$ in $\tilde{S}$ admits a neighbourhood $\tilde{U}$ of $\tilde{p}$ in $\tilde{S}$ such that, following-up on the notations of i), denoting $\tilde{S} = S \circ D\tilde{\sigma}$ and $\kappa$ a generator of $\mathcal{L}_B$ nearby $\tilde{\sigma}(\tilde{p})$, we find

$$\kappa \circ \tilde{S} = M_{\mathcal{L}}[\epsilon_1 N_1(M_1 \theta_1) \otimes (M_1 \theta_1) + \epsilon_2 N_2(M_2 \theta_2) \otimes (M_2 \theta_2)],$$

where: $\epsilon_1, \epsilon_2 \in \{-1, 1\}$; The germ $\mathcal{M}_{\mathcal{L}}$ is a monomial generating the ideal $\tilde{\sigma}^* C_{\mathcal{L}|\tilde{U}}$; For $i = 1, 2$, the germ $\mathcal{M}_i$ is a monomial in $(\tilde{Z} \cup \Delta^{\text{str}}_{\mathcal{L}}) \cap \tilde{U}$ such that the invertible
sub-module \((\beta' \circ \tilde{\beta})^*(Q_R \circ \Delta_{\sigma_R})\) of regular sections of \(S(2, TS)\) is generated over \(\tilde{U}\) by \(M_1 \cdot \theta_1 \cdot \theta_2\), for \(Q_R\) the invertible \(O_R\)-sheaf of regular sections of \(S(2, (TS)^{\sigma_R})\) of Remark 1.

Before getting into its demonstration, we complement Theorem 6 with the additional properties below, about the monomials appearing in Eq. (5). They are more technical in nature and similar to those stated in Proposition 2:

**Corollary 3** We can further see that

(iii) \(O_{\tilde{U}}(\theta_1 \wedge \theta_2) = J_{1,2} \cdot \Omega^2_{\tilde{U}}\), where \(J_{1,2}\) is a monomial and principal ideal in the snc-divisor \(\tilde{\Sigma}_{1,2} := \text{co-supp}(J_{1,2}O_{\tilde{U}})\) (containing \((\tilde{Z} \cup \Delta^{\text{str}}_E) \cap \tilde{U}\)) which is normal crossing with \(\tilde{D} \cap \tilde{U}\).

(iv) Let \(M_{1,2}^{\log}\) be a local generator of \(J_{1,2}^{\log}\) at \(\tilde{p}\), the ideal of logarithmic coefficients of \(O_{\tilde{U}}(\theta_1 \wedge \theta_2)\) as a sub-module of \(\Omega^2_{\tilde{U}}(\log(\tilde{D} \cup \tilde{\Sigma}_{1,2}))\). For \(i = 1, 2\), let \(M_i^{\log}\) be a local generator of the logarithmic coefficient ideal \(\iota_i^{\log}\) of \(\theta_i\) as a sub-module of \(\Omega^1_{\tilde{U}}(\log(\tilde{D} \cup \tilde{\Sigma}_{1,2}))\).

The monomials \((M_L \cdot N_i \cdot M_i^{\log})\), \((M_L \cdot N_i^2 \cdot M_i^{\log})\), \((M_i^{\log})\) are ordered.

3) We can track the monomials and their vanishing locus:

- if \(L\) is generically non-degenerate, both function germs \(N_1, N_2\) are monomials in \((\tilde{Z} \cup D^{\text{str}}_{E}) \cap \tilde{U}\) which cannot both vanish simultaneously. And one of the monomials \(N_i \cdot M_L\) is a local generator of the ideal \(\tilde{\sigma}^*I^P_{\tilde{L}}|_{\tilde{U}}\).
- if \(L\) is everywhere degenerate one of the function germs \(N_1, N_2\) is a local monomial in \(\tilde{Z} \cup D^{\text{str}}_{E}\) while the other one is identically zero. If \(N_i,\) for \(i = 1 \text{ or } 2,\) is not the zero monomial then \(N_i \cdot M_L\) is a local generator of the ideal \(\tilde{\sigma}^*I^P_{\tilde{L}}|_{\tilde{U}}\).

We now proof both the Theorem and its Corollary.

**Proof** Point 1) is straightforward.

Let \(Q_R\) be the invertible \(O_R\)-sub-module of regular sections of \(S(2, TR)\) of Remark 1, and let \(Q := Q_R \circ \sigma'\) encoding the diagonalizing directions of \(L^{\sigma'}\). We define \(Q^* := Q_R \circ \Delta_{\sigma_R} \circ D\beta' = Q \circ \Delta_{\sigma'\sigma},\) an \(O_S\)-invertible sheaf of sections of \(S(2, TS')\).

Let \(E'_{\beta'}\) be the exceptional divisor produced by \(\beta'\). Let \(\cup_{i \in I'}(p_i)\) be the image \(\beta'(E'_{\beta'})\.

Since these points are isolated, we can assume for the work to come that there is just a single one, this will not change our arguments since there are local in nature.

**Special case:** \(B\) is regularly isomorphic to \(TS\), in other words \(Z = \emptyset\).

Let \(p_R\) be the point \(\beta'(E'_{\beta'})\). To ease slightly the notations, let \(B_R := B^{\sigma_R}\). By the results of Sect. 4, there exist an open neighbourhood \(\mathcal{U}_R\) of \(p_R\) and two regular sections of \(B_R^1|_{\mathcal{U}_R}\), say \(\eta_1, \eta_2,\) vanishing nowhere in \(\mathcal{U}_R\) with orthogonal kernels and such that \(\eta_1 \cdot \eta_2\) generates \(Q_R\) at every point of \(\mathcal{U}_R\). Moreover, up to shrinking \(\mathcal{U}_R\), we find

\[
\kappa_R|_{\mathcal{U}_R} := \kappa_R^{\sigma_R}|_{\mathcal{U}_R} = a_1 \eta_1 \otimes \eta_1 + a_2 \eta_2 \otimes \eta_2
\]

for regular functions \(a_1, a_2\) over \(\mathcal{U}_R\) monomial in some snc divisors. Outside this neighbourhoold all the results of 2) are trivially true, since \(\beta'\) induces a regular isomorphism. In particular at every point of \(\mathcal{U}_R\) we find that \(Q^*\) is generated by \(\beta'^* \eta_1 \cdot \beta'^* \eta_2\).

Let \(\mathcal{U}' := \beta'^{-1}(\mathcal{U}_R)\), a neighbourhood of \(E_{\beta'}\), over which are defined two regular forms

\[
\omega_i := \eta_i \circ (\sigma'^* \circ \Delta \beta') \quad \text{for } i = 1, 2.
\]

In particular we have two singular foliations over \(\mathcal{U}'\), which are orthogonal outside \(E_{\beta'}\), and have singularities along \(E_{\beta'}\).
Let $D_i$ be the sub-module of $\Omega^1_{U'}$ generated by $\omega_i$ for each $i = 1, 2$. Since $\omega_i$ is singular along $E_{\beta'}$, we can factor out a principal $\mathcal{O}_{U'}$-ideal $\mathcal{C}_i$ monomial in $E_{\beta'}$ such that

$$\mathcal{D}_i = \mathcal{C}_i \cdot D_i'$$

where $D_i'$ is an invertible $\mathcal{O}_{U'}$-sheaf of regular sections of $\Omega^1_{U'}$ with isolated support, if not empty. More precisely if $\mathcal{C}_i$ is locally generated by $h_i$, then $D_i'$ is generated locally by $\omega_i' := (h_i)^{-1} \omega_i$. We have actually more than that: The whole invertible sheaf $Q^\ast$ decomposes as $I_i \cdot Q^\ast$, where $I_i$ is principal and monomial in $E_{\beta'}$ and $Q^\ast$ has co-support of dimension 2 or is empty.

**Observation.** Let $\gamma : (S'', E'' = E' \cup E_\gamma) \to (S', E')$, be the blowing-up of the point $p' \in U'$ and $E_\gamma$ be $\gamma^{-1}(p')$, the newly created exceptional hypersurface. Let $I_{E_\gamma}$ be the reduced ideal of $E_{\gamma}$. Let $U''$ the pre-image $\gamma^{-1}(U')$. We observe that $\gamma \ast D_i' = I_{E_\gamma}^{-k_i} \mid U'' D_i''$, where $k_i$ is a positive integer and $D_i''$ is a sub-module of $\Omega^1_{U''}$, which is finite co-dimensional at each point.

The simple observation above guarantees that there exists a locally finite sequence of points blowings-up $\beta'' : (S'', E'' = E' \cup E_{\beta''}) \to (S', E')$ such that for each $i$ the sub-module $\beta'' \mid_{U'} D_i$ factors as $J_i \mid U' \cdot D_i'$, where $J_i$ is a principal ideal monomial in $E_{\beta''}$. and $D_i'$ is an invertible $\mathcal{O}_{U'}$-sub-module of $\Omega^1_{U'}$, of finite co-dimension at each point, where $U''$ is the pre-image $\beta''^{-1}(U')$.

In order to avoid further notations, we can assume that $\sigma'$ is already such that each ideal $\mathcal{C}_i$ is principal and monomial in $E'$, so that each $D_i'$ is also already defining a foliation $\mathcal{F}_i'$ on $U'$ for $i = 1, 2$.

Now, we have to resolve the singularities of $\mathcal{F}_1'$ and $\mathcal{F}_2'$ and do further point blowings-up so that each final pulled-back foliation is in a form as good as it can be with some of the given snc-divisors we want to take care of. But, up to a locally finite sequence of points blowings-up we can already assumed, thanks to the results of Sect. 6, that the mapping $\sigma'$ achieve this. So we get point i). To get the whole of point ii) there is just to carefully track everything we have at the level of Eq. (1) of Proposition 2 for $\kappa_R$ and since

$$\kappa \circ S \circ D\tilde{\sigma} = (\kappa_R \circ S \circ (\sigma_R)^{h_i} \circ \Delta\sigma_R \circ D\zeta),$$

where $\zeta$ is defined as $\tilde{\sigma} = \zeta \circ \sigma_R$, we check we get what is stated (since we have assumed $S$ is an isomorphism).

Now we deal with point iii). Let $\Delta'$ be the snc-divisor $V^\text{str} \cup D^\text{str} \cup \Delta^\text{str}$. Let $\mathcal{J}_{1, 2}'$ be the coefficients ideal of $\sigma'^\ast(\Omega^2_S) = \mathcal{J}_{1, 2}' \cdot \Omega^2_S$. We also find

$$\mathcal{D}_1 \land \mathcal{D}_2 = \mathcal{C}_1 \mathcal{C}_2 J_{1, 2} \cdot \Omega^2_S$$

with $\mathcal{C}_1 \mathcal{C}_2$ principal and monomial in the snc divisor $E_{\beta'}$ while $J_{1, 2}$ has co-support $\Sigma'$ which does not contain any component of $E_{\beta'}$. In particular the sub-variety $\Sigma'$ is the tangency locus, contained in $U'$ of the foliations $\mathcal{F}_1'$ and $\mathcal{F}_2'$. We can assume, up to further point blowings-up, that $\Lambda'$, the co-support of $(\mathcal{J}_{1, 2})'$, and $\Sigma'$ are snc-divisors which are normal crossings with $\Delta'$ and that $\mathcal{J}_{1, 2}'$ is also principal and monomial in $\Lambda'$.

Note that we also have $\Lambda' \subset \Sigma' \cup E_{\beta'}$. We can assume moreover, up to further point blowings-up, that each local component of $E' \setminus \Sigma' \cup \Delta' \cup E'$, contained in $U'$, is either invariant or di-critical for both foliation $\mathcal{F}_i'$, $i = 1, 2$.

At a regular point of $E'$, each monomial in $E'$ under scrutiny is of the form $u^l$ for $u$ a local coordinate and $l$ a non-negative integer. So they are already ordered. Let $X'$ be the subset of corner points of $E'$. Thus at each point $q'$ of $X'$ and for each $i = 1, 2$, each local component
of $E'$ is either invariant or di-critical for $F'$. This fact is important since the proof of point v) and point vi) of Proposition 4 shows that we can always order the “logarithmic” monomials $M_1^{\log}$ and $M_2^{\log}$. Thus, working with the logarithmic 1-forms along the strict divisor $E'$, these logarithmic monomials can be assumed already ordered at any corner point of $E'$.

Let us repeat the argument here: Let $\mu_i$ be a local logarithmic generator of $D_i$ so that the pull-back $\sigma^*\mu_i = M_i^{\log} \theta_{i, log}^*$, where $\theta_{i, log}$ is a local logarithmic generator of $F_i$ and where $\sigma^*D_i$ is seen as a sub-module of $\Omega_{\mathcal{U}}^1(\log E')$. If $\gamma$ is the blowing-up in $S'$ of the point $q'$ of $E'$, we see that at each corner point of $\gamma^{-1}(E') \cap \gamma^{-1}(q')$, we find out that $\gamma^*\theta_{i, log}$ is indeed a logarithmic generator of the pulled-back foliation $\gamma^*F'$, so that a local generator of the ideal of logarithmic coefficients of $(\sigma' \circ \gamma)^*D_i$ is just the pull back by $\gamma$ of a local generator of the ideal of logarithmic coefficients of $\sigma'^*D_i$. Our problem of comparison of monomials is indeed just a problem of comparing monomials, forgetting about the foliations.

Thus at such a point $q'$ of $E'$, there exists $\pi : (S'', E'') \rightarrow (S', E')$, a finite sequence of point blowings-up, such that the pull-back of the monomials, we were looking for to order at $p'$, are ordered at each corner point of $\pi^{-1}(q') \cap \pi^{-1}(E')$.

**General case:** Assume that $Z$ is not empty.

In this context $TS$ play the role of $B$ of the special case above and with $\mathcal{L} = \mathcal{L}_B \circ S$ in the stead of $\mathcal{L}_B$ of the special case. What needs to be done in order to get the announced results, is to track down the position of the strict transform $Z^\text{str}$ of $Z$ with the data we already have, and by means of further additional points blowings-up, put the new corresponding strict transform in general position with the pull-back of the data we already have. This can be achieved by the results of Sect. 6.

Point 3) requires just to keep track of the whole process of the monomialization of the corresponding ideal, initiated in Sect. 4.

The à-priori artificial context of Theorem 6 proceeds from finding a formulation for the two, similar but not identical, following situations below. Instead of working only with the tangent bundle, we work on any regular vector bundle of rank 2. Nevertheless, due to a cell decomposition of $S$, there exists always a closed subset $F$ of $S$, complement of the cells of dimension 2 such that $TS|_{S\setminus F}$ is isomorphic to $B|_{S\setminus F}$, since the connected components $S\setminus F$ are open balls, thus contractible. Whether $F$ can be a sub-variety deserves to be discussed.

The first situation is when $S$ is a regular surface and $B = TS$ its tangent bundle. Thus $\kappa$ can be any 2-symmetric (regular) tensor (field) on $S$, and may be degenerate somewhere (see [10,13] for semi-positive definite examples).

The second situation motivated the hypotheses on $Z$, $B$ and $TS$ in the theorem. Suppose the regular surface $S$ resolves the singularities of a surface singularity $S_0$ embedded in a regular manifold $M_0$, such that it factors through an embedded resolution of the singularities of $S_0$ such that the resolution mapping $\sigma : S \rightarrow S_0$ is Gauss-regular, with exceptional divisor $E$, which is possible by Proposition 5 (see Appendix A). Taking $B := T^\sigma S_0$ (See Appendix A), outside the exceptional divisor $E$ the mapping $\sigma$ induces an isomorphism between $T S$ and $B$.

We take $\mathcal{L}$ as generated by the pull-back of $\sigma^*(\kappa|_{S_0})$ of any given invertible $\mathcal{O}_{M_0}$-sub-module $\mathcal{K}$ of $\Gamma_{M_0}(S(2, TM_0))$. As explained in the introduction, we came across such situations when $\mathcal{K}$ is generated by a given regular metric on $M_0$ [12,13].

**Remark 3** The result proved above does not depend on the Riemannian metric $g_0$ but only on its conformal class, in other word depends only on the invertible $\mathcal{O}_{M_0}$-sub-module of $\Gamma_{M_0}(S(2, TM_0))$ generated by $g_0$. Indeed, the choice of the geometrically admissible centers we blow-up (to reach our main result) is not affected at any step, if instead of working with
g_0 we were working with a conformal metric, since the only feature of g_0 we really need to keep track at any time is simply the notion of orthogonality.

**A Gauss regular resolution and 2-tensors on singular sub-varieties**

We present here further ingredients related to resolution of singularities of sub-varieties. The estranged formulation of the main result, Theorem 6, finds some justifications in the complements of this section.

**A.1 Resolution of singularities with Gauss regular mapping**

The material presented here, although part of the known folklore, introduces useful notions and notations. We are very grateful to Pierre Milman for telling us about Gauss regular desingularization.

Let \( G_k(V) \) be the Grassmann-bundle of \( k \)-dimensional real vector subspaces of the finite dimensional real vector space \( V \). Let \( [P] \) be the point of \( G_k(V) \) corresponding to the \( k \)-dimensional vector subspace \( P \) of \( V \).

Let \( F \) be a regular vector bundle of positive finite rank \( r \) over a regular (connected) manifold \( N \) of finite dimension. Let \( G_k(F) \) be the Grassmann bundle of the \( k \)-vector subspaces in the fibers of \( F \). Let \( G(F) = \bigcup_{k=1}^{r} G_k(F) \) the total Grassmann bundle of \( F \).

Let \( M_0 \) be a connected regular manifold of dimension \( n \). Let \( X_0 \) be a singular sub-variety of the regular manifold \( M_0 \). Let \( Y_0 \) be the singular locus of \( X_0 \). The Gauss mapping of \( X_0 \) is defined as

\[
\nu_{X_0} : X_0 \setminus Y_0 \to G(TM_0)
\]

\[
b_0 \in X_0 \setminus Y_0 \to \nu_{X_0}(b_0) = [Tb_0 X_0] \in G_{\dim(X_0,b_0)}(Tb_0 M_0).
\]

**Definition 7** A geometrically admissible resolution of singularities \( \pi : (X,E) \to (X_0,Y_0) \) of \( X_0 \) is said Gauss regular, if the mapping \( \nu_{X_0} \circ \pi \) extends over \( X \) as a regular mapping \( X \to G(TM_0) \).

Composing a geometrically admissible Gauss regular resolution of singularities of \( X_0 \) with any geometrically admissible blowing-up with center in the exceptional divisor will yield another Gauss regular resolution of singularities of \( X_0 \).

**Proposition 5** (see [2]) There exists a Gauss regular resolution of singularities of \( X_0 \).

**Proof** For simplicity we suppose that \( X_0 \) is of pure dimension \( d \).

Let \( \tau_1 : (M_1, X_1, E_{M_1}) \to (M_0, X_0, Y_0) \) be a geometrically admissible embedded resolution of singularities of \( X_1 \). Let \( \sigma_1 \) be the restriction mapping \( \tau_1|_{X_1} \) and let \( E_1 \) be the intersection \( X_1 \cap E_{M_1} \) which is a snc-divisor of the resolved manifold \( X_1 \).

Let \( F_0(\sigma_1) \) be the \( \mathcal{O}_{X_1} \)-ideal sheaf locally generated by the maximal minors of the differential mapping \( D\sigma_1 \) whose co-support is the critical locus of \( \sigma_1 \), contained in \( E_1 \). Given any geometrically admissible blowing-up \( \beta_C \) with center \( C \) contained in \( X_1 \), there exists a non-negative integer \( \alpha \) (depending on \( C \)) such that the ideal \( F_0(\sigma_1 \circ \beta_C) \) factors as \( (I_{EC})^\alpha \cdot F_0(\sigma_1)^{\beta_C} \), where \( E_C \) is the newly created exceptional hypersurface \( \beta_C^{-1}(C) \) and \( I_{EC} \) is its reduced (and principal) ideal. Up to further geometrically admissible blowings-up (with centers in \( E_1 \)), we can assume that \( F_0(\sigma_1) \) is already principal and monomial in \( E_1 \).
For any point $a_1$ of $X_1 \setminus E_1$, we know that $D\sigma_1(a_1) \cdot T_{a_1}X_1 = T_{\sigma_1(a_1)}X_0$. Let $a_1$ be any point of $E_1$ and let $(u, v)$ be local coordinates adapted to $E_1$. Let $(u', v')$ be another system of local coordinates adapted to $E_1$. Thus, in a neighbourhood $U_1$ of $a_1$ in $X_1$,

$$
D\sigma_1 \cdot \partial_{u_1} \wedge \cdots \wedge D\sigma_1 \cdot \partial_{u_s} \wedge D\sigma_1 \cdot \partial_{v_1} \wedge \cdots \wedge D\sigma_1 \cdot \partial_{v_t}
= \text{Unit} \cdot D\sigma_1 \cdot \partial_{u'_1} \wedge \cdots \wedge D\sigma_1 \cdot \partial_{u'_s} \wedge D\sigma_1 \cdot \partial_{v'_1} \wedge \cdots \wedge D\sigma_1 \cdot \partial_{v'_t}
$$

where $s + t = d$ is the dimension of $X_0$.

Since the ideal $F_0(\sigma_1)$ is principal and monomial in $E_1$, there exists a nowhere vanishing regular mapping $\gamma_1 : U_1 \to \wedge^d TM_0$ such that

$$
D\sigma_1 \cdot \partial_{u_1} \wedge \cdots \wedge D\sigma_1 \cdot \partial_{u_s} \wedge D\sigma_1 \cdot \partial_{v_1} \wedge \cdots \wedge D\sigma_1 \cdot \partial_{v_t} = M_1 \cdot \gamma_1
$$

where $M_1$ is a local generator of $F_0(\sigma_1)$. So we deduce that the mapping

$$
U_1 \ni a \mapsto [\gamma_1(a)] \in G_d(TM_0)
$$

(using here the Plücker embedding of $G_d(TM_0)|_{U_0}$, where $U_0$ is a neighbourhood of $\sigma_1(a_1)$ over which $TM_0$ is trivial) where $[\gamma_1(a)]$ is the vector space direction corresponding to the $d$-vector $\gamma_1(a)$. This regular mapping coincides with $\nu_{X_0} \circ \sigma_1$ on $U_1 \setminus E_1$.

When $X_0$ is not of pure dimension, the resolved manifold $X_1$ is a disjoint union of regular manifolds and proceed exactly as above, independently for each dimension. \qed

### A.2 2-symmetric tensors and quadratic forms on singular sub-varieties

Using Sect. A.1 we make sense here of the notion of restriction of 2-symmetric tensor, respectively quadratic forms, on singular sub-varieties.

Let $V$ be a real vector space of finite dimension $r$. The universal Grassmann bundle $\tilde{G}_k(V)$ is the algebraic sub-variety of $G_k(V) \times V$ consisting of the pairs $(\{P\}, v)$ for any vector $v \in P$ with $P$ any $k$-dimensional vector subspace of $V$. It is also an algebraic real vector bundle of rank $k$ over $G_k(V)$.

Let $M_0$ be a connected analytic manifold of finite dimension.

Let $Z$ be any non-empty sub-variety of $M_0$. The Nash “bundle” $N_Z$ of $Z$ in $M_0$ is the closed semi-analytic subset of $G(TM_0)$ obtained as the (topological) closure of the graph of the Gauss mapping of $Z$. Let $C_4(Z)$ be the closure, taken into $TM_0$, of the tangent bundle $TZ_{\text{reg}}$ of the regular part $Z_{\text{reg}}$ of $Z$. We call it the pseudo-tangent “bundle” of $Z$. We denote it by $C_4(Z)$, since point-wise, the fiber $C_4(Z, a)$ over a point $a$ of $X_0$ is the fourth Whitney tangent cone [27] and consists of the union $\bigcup_{[P] \in N_aZ} P$ of all the limits at $a$ of the tangent spaces to $Z$ at regular points of $Z$. When $Z$ is a sub-manifold $C_4(Z)$ is just the usual tangent bundle $TZ$.

Let $X_0$ be a singular sub-variety of $M_0$ with non-empty singular locus $Y_0$. Since $C_4(X_0)$ is a subset of $TM_0$ we can introduce the following

**Definition** Let $\kappa$ be a regular quadratic form on $M_0$. The restriction of $\kappa$ to $X_0$, denoted $\kappa|_{X_0}$, is defined as the restriction $\kappa|_{C_4(X_0)}$ of $\kappa$ to the pseudo-tangent “bundle” of $X_0$.

The restriction of the 2-symmetric tensor $\kappa$ on $M_0$ to $X_0$ is just defined via the polar form of the restriction of $\kappa|_{\Delta}$. Let $T_0 \to G(TM_0)$ be any Gauss regular admissible resolution of singularities of $X_0$. Let $v_0$ be the regularized Gauss mapping of $X_0$, that is the regular mapping $T_0 \to G(TM_0)$.
extending to the whole of $T_0$ the parameterized Gauss mapping $\nu_{X_0} \circ \tau : T_0 \setminus D_0 \to X_0 \setminus Y_0$. We see that

$$\bigcup_{a \in X_0} \bigcup_{b \in \tau^{-1}(a)} (a, \nu_0(b)) = N_{X_0}.$$  

For any point $b$ in $T_0$, let $T^\tau_b X_0$ be the vector subspace of $T^\tau_{\tau(b)} M_0$ whose direction is the value at $b$ of the regular extension $\nu_0$, namely $\nu_0(b) = [T^\tau_b X_0] \in G(T_{\tau(b)} M_0)$. We call the vector sub-space $T^\tau_b X_0$ the tangent space of $X_0$ at $b$ along $\tau$. We deduce that

$$C_4(X_0) = \bigcup_{a \in X_0} \bigcup_{b \in \tau^{-1}(a)} a \times T^\tau_b X_0$$
and that for each point $b$ of $T_0$, the differential mapping $(D\tau)(b) : T_b T_0 \to T_{\tau(b)} M_0$ takes its values in $T^\tau_b X_0$.

Let $\tilde{G}_k(T M_0)$ be the universal bundle associated with $G_k(T M_0)$ and let

$$\tilde{G}(T M_0) := \bigcup_{k=1}^\tau \tilde{G}_k(T M_0),$$

be the corresponding universal bundle. Let $\tilde{\tau} : \tilde{G}(F) \to F$, defined as $(a, [P], v) \to (a, v)$.

Taking the graph of $\nu_0$, embedding it in the fibered product $T_0 \times_{M_0} G(T M_0)$, then lifting it in the fibered product $T_0 \times_{M_0} \tilde{G}(T M_0)$ and eventually projecting this lift in the fibered product $T_0 \times_{M_0} T M_0$ via the mapping $\tilde{\tau}$ shows that the union

$$T^\tau X_0 := \bigcup_{b \in T_0} T^\tau_b X_0,$$

called the tangent bundle of $X_0$ along $\tau$, is a regular vector bundle over the resolved manifold $T_0$. Outside the critical locus of $\tau$ the restricted vector bundle $(T^\tau X_0)|_{T_0 \setminus D_0}$ is just the pull-back $T(X_0 \setminus Y_0)|_{T_0 \setminus D_0}$.

Thanks to Definition 8, the restriction of any submodule of $\Gamma_{M_0}(S(2, T M_0))$ to $X_0$ is well defined.

Suppose given $\pi_1 : (X_1, E_1) \to (X_0, Y_0)$, a Gauss regular resolution of singularities of $X_0$ and let $L_0$ be an invertible sub-module of $\Gamma_{M_0}(S(2, T M_0))$. Thus the regular “section” $L_0^{\pi_1}|_{T^\pi_1 X_0}$ of $S(2, T^\pi_1 X_0)$ coincides with $(L_0|_{X_0})^{\pi_1}$. Namely for $k$ a local generator of $L$ nearby $a_0$ in $X_0$ and for any $a_1$ in $\pi^{-1}_1(a_0)$, we find

$$(k^{\pi_1}|_{T^\pi_1 X_0})(a_1) = k(\pi_1(a_1))|_{T^\pi_1 X_0}.$$  

Let $L_1 := L_0^{\pi_1}|_{T^\pi_1 X_0}$ and let $C_{L_1}$ be the $O_{X_1}$-ideal of coefficients of $L_1$ obtained by evaluating the “2-symmetric tensor” $L_1$ along the regular section germs of $S(2, T^\pi_1 X_0)$. With further locally finite geometrically admissible blowings-up we can assume that $C_{L_1}$ is principal and monomial in $E_1$. Thus any local generator of the invertible $O_{X_1}$-submodule $C_{L_1}^{-1}L_1$ does not vanish anywhere.

### B Local normal forms of differentials and of the inner metric on singular surfaces

We complete the paper addressing the primary motivation of this work: describing locally, in a resolved manifold, the pull-back of the inner metric, by the resolution mapping, of an embedded real surface singularity. As a consequence of the previous sections we get a proof of the Hsiang and Pati property for real surfaces which is a bit different from the existing ones [2,15,19,24].
B.1 Hsiang and Pati property

Let $M$ be a smooth manifold. Two (Riemannian) metrics $g$ and $h$ on $M$ are quasi-isometric if there exists a positive constant $C$ such that $C^{-1}h \leq g \leq C h$. They are locally quasi-isometric if each point $a$ of $M$ admits an open neighbourhood $U$ such that the restricted metrics $g|_U$ and $h|_U$ are quasi-isometric.

The next result, of local nature, is the main tool used by Hsiang and Pati to get their result.

**Lemma 5** ([19, Section III]) Let $(X_0, 0)$ be a normal complex isolated surface singularity germ embedded in $(\mathbb{C}^N, 0)$. There exists a finite composition of points blowings-up $\sigma : (X, E) \to (X_0, 0)$ such that:

i) $X$ is a complex manifold of dimension two and $E := \sigma^{-1}(0)$, the exceptional divisor of this desingularization of $(X_0, 0)$, is a snc-divisor.

ii) Any regular point $a$ of $E$ admits local regular coordinates $(u, v)$, centered at $a$, such that in this chart $(E, a) = \{u = 0\}$ and the resolution mapping writes locally

$$(u, v) \to (x, y, z) = \sigma(a) + (u^{r+1}, u^{r+s+1}v; u^{r+1}g(u) + u^{r+s+1}Z(u, v)) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{N-2} \quad (6)$$

for non-negative integers $r, s$, for germs $f \in \mathbb{C}[u]$, $g \in \mathbb{C}[u]^{N-2}$, and $Z$ a regular map germ $(X, a) \to \mathbb{C}^{N-2}$.

iii) Any corner point $a$ of $E$ admits local regular coordinates $(u, v)$, centered at $a$, such that in this chart $(E, a) = \{uv = 0\}$ and the resolution mapping writes locally

$$(u, v) \to (x, y, z) = \sigma(a) + (u^m v^n, u^m v^n f(u, v) + u^n v^n g(u, v) + u^n v^n Z(u, v)) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{N-2} \quad (7)$$

for non-negative integers $p \geq m$ and $q \geq n$ such that $np - qm \neq 0$ and germs $f \in \mathcal{O}_a$, $g \in \mathcal{O}_a^{N-2}$ and $Z \in \mathcal{O}_a^{N-2}$ such that $df \land (u^m v^n) = dg \land (u^m v^n) = 0$.

Such local coordinates $(u, v)$, in either cases, are called Hsiang and Pati coordinates. The corollary of such systematic local presentation of the resolution mapping is Hsiang and Pati result of interest to us:

**Theorem 7** [19] Let $X_0$ be a normal complex surface singularity germ embedded in $\mathbb{CP}^N$. Let $g_{X_0}$ be the restriction to the regular part of $X_0$ of the Fubini-Study metric on $\mathbb{CP}^N$. There exists a finite composition of points blowings-up $\sigma : (X, E) \to X_0$ resolving the singularities of $X_0$ such that

i) Each point $a$ of $E$ admits Hsiang and Pati coordinates $(u, v)$ like in Lemma 5.

ii) When $a$ is a regular point of $E$, the (regular extension of the) pulled-back metric $\sigma^* g_{X_0}$ is quasi-isometric to the metric over $U$ given by

$$du^{r+1} \otimes du^{r+1} + du^{r+s+1}v \otimes du^{r+s+1}v.$$
iii) When \( q \) is a corner point of \( E \), the (regular extension of the) pulled-back metric \( \sigma^* g_{x_0} \) is quasi-isometric to the metric over \( U \) given by
\[
\sigma^* g_{x_0} = \sigma^* g_{x_0} = \sigma^* g_{x_0} = \sigma^* g_{x_0} = \sigma^* g_{x_0} = \sigma^* g_{x_0} = \sigma^* g_{x_0} = \sigma^* g_{x_0}.
\]

B.2 Preliminaries for local normal forms

Let \( M_0 \) be a regular connected manifold of dimension \( N \), equipped with a regular Riemannian metric \( g_0 \). Let \( X_0 \) be a sub-variety with no connected component of dimension other than 2.

Suppose given a Gauss regular resolution \( \sigma_1 : (X_1, E_1) \to X_0 \).

**Notation.** Let \( \Omega^1_{x_0} \) be the \( \mathcal{O}_{X_1} \)-dual to \( \Gamma_{X_1}(T^{\sigma_1}X_0) \). It is a locally free \( \mathcal{O}_{X_1} \)-module of rank 2.

A differential 1-form along \( \sigma_1 \) is a regular section \( X_1 \to (T^{\sigma_1}X_0)^\vee \).

Following Proposition 5 and then using Proposition 2, we can further assume that for any point \( a_1 \) in \( X_1 \) there exist a neighbourhood \( U_1 \) of \( a_1 \) and local regular sections \( \omega_1, \omega_2 \) of \( \Omega^1_{\sigma_1}|U_1 \), with orthogonal kernels (for the fiber-metric \( g^{\sigma_1} \) restriction of \( g_0 \circ \sigma_1 \to T^{\sigma_1}X_0 \)), such that over \( U_1 \) the following holds true:
\[
g^{\sigma_1} = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2. \tag{8}
\]

By definition \( g^{\sigma_1} \) is also the fiber metric onto \( T^{\sigma_1}X_0 \) which extends the restriction of the fiber-metric \( (g_0|_{X_0\times Y_0}) \circ \sigma_1 \to T^{\sigma_1}X_0|_{X_0 \times E_1} = T(X_0 \times Y_0)^{\sigma_1}|_{X_0 \times Y_0} \).

Suppose given a resolution of singularities \( \tilde{\pi} : (\tilde{X}, \tilde{E}) \to X_0 \) like in Theorem 6 factoring through \( \sigma_1 \), that is \( \tilde{\pi} = \sigma_1 \circ \tilde{\beta} \) for \( \tilde{\beta} : (\tilde{X}, \tilde{E}) \to (X_1, E_1) \) a locally finite sequence of point blowings-up. Thus Eq. (8) becomes
\[
g^{\tilde{\pi}} := \theta_1 \otimes \theta_1 + \theta_2 \otimes \theta_2, \tag{9}
\]

and where \( \theta_i := \omega_i \circ \tilde{\beta} \) is a differential 1-form along \( \tilde{\pi} \) for \( i = 1, 2 \). By definition \( g^{\tilde{\pi}} \) extends to the whole of \( T^{\tilde{\pi}}X_0 \) the restriction of the fiber-metric \( (g_0|_{X_0\times Y_0}) \circ \tilde{\pi} \to T^{\tilde{\pi}}X_0|_{\tilde{X} \times \tilde{E}} \).

From Theorem 6, each point \( \tilde{a} \) of \( \tilde{X} \) admits a neighbourhood \( \tilde{U} \) such that for \( i = 1, 2 \), there exists a 1-form \( \mu_i \) of \( \Omega^1_{U_i} \), only with singularities adapted to \( \tilde{E} \), such that
\[
\tilde{\beta}^\ast(\theta_i \circ \Delta \sigma_1) = \mathcal{M}_i \mu_i
\]

with \( \mathcal{M}_i \) a monomial in \( \tilde{E} \), and \( \Delta \tilde{\pi} : T\tilde{X} \to T^{\tilde{\pi}}X_0 \subset (TM_0)^{\tilde{\pi}} \). Let \( \chi_1, \chi_2 \) be local regular sections \( \tilde{X} \to T^{\tilde{\pi}}X_0 \), which are orthogonal for the fiber-metric \( g^{\tilde{\pi}} \) on \( T^{\tilde{\pi}}X_0 \), and such that \( \theta_i(\chi_j) = \delta_{i,j} \) for \( i, j \in \{1, 2\} \). Suppose that \( \tilde{U} \) is small enough such that we can choose regular coordinates \((u, v)\), centered at \( \tilde{a} \) and adapted to \( \tilde{E} \), i.e. \((E, \tilde{E}) \subset (uv = 0)\).

Let \( Q \) be the matrix of the mapping \((\Delta \sigma_1) \circ \partial \tilde{\beta} \) in the basis \((\partial u, \partial v)\) and \((\chi_1, \chi_2)\). Let \( \text{adj}(Q) \) be the adjugate matrix of \( Q \) so that \( \text{adj}(Q) \cdot Q = Q \cdot \text{adj}(Q) = \psi \mathcal{M} \cdot Id \), where \( \mathcal{M} \) is a monomial in \( E \) and \( \psi \) an analytic unit over \( \tilde{U} \). Note that \( \psi \mathcal{M} \) is the (oriented) volume of the image of \( \tilde{\pi} \) and by Theorem 6, we have \( \mathcal{M} = \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_{1,2} \).

We obtain two regular vector fields on \( \tilde{X} \), namely \( \xi_i := \text{adj}(Q) \chi_i \), for \( i = 1, 2 \). They may vanish only on \( \tilde{E} \). We deduce that \( \mathcal{M}_i \mu_i(\xi_j) = \theta_i(\chi_j) = \psi \mathcal{M} \delta_{i,j} \) for \( i, j = 1, 2 \). Writing \( \mu_i = a_i du + b_i dv \) and \( \xi_i = \alpha_i \partial u + \beta_i \partial v \) we observe that
\[
a_i a_j + b_i \beta_j = \psi \mathcal{M}_k \mathcal{M}_{1,2} \delta_{i,j} \quad \text{with} \quad i \neq k \quad \text{and} \quad i, j, k \in \{1, 2\} \tag{10}
\]
\[
a_1 b_2 - a_2 b_1 = \psi \mathcal{M}_{1,2}. \tag{11}
\]
Since $\mu_1$ and $\mu_2$ may only vanish at isolated points, such that

$$(\alpha_i, \beta_i) = f_j \cdot (b_j, -a_j) \quad \text{with} \ i \neq j$$

In other words, in the basis above, the mapping $(\Delta \sigma_1) \circ D \tilde{\beta}$ over $\tilde{U}$ writes as

$$(\Delta \sigma_1) \circ D \tilde{\beta} = (f_1 \mu_1, f_2 \mu_2) : T_b \tilde{X} \ni \xi \to (f_1 \mu_1(\xi), f_2 \mu_2(\xi))$$

$$= (\theta_1, \theta_2)((\Delta \sigma_1) \circ D \tilde{\beta}) \cdot \xi \in T_b \tilde{\pi} \cdot X_0.$$  

Note also that along the way, we have proved the following expected

**Lemma 6** The $O_{\tilde{\pi}}$-module $(\Omega^1_{\tilde{\pi}}) \circ \Delta \tilde{\pi}$ is generated by $M_1 \mu_1$ and $M_2 \mu_2$.

**B.3 Local normal form of differentials**

Let $\tilde{a}$ be a point of $\tilde{E}$. Let $(u, v)$ be local coordinates centered at $\tilde{a}$ adapted to $\tilde{E}$, so that

$$\{u = 0\} \subset (\tilde{E}, \tilde{a}) \subset \{uv = 0\}.$$

The $O_{\tilde{X}}$-module $(\Omega^1_{M_0})_{\tilde{\pi}}$ is locally free of rank $n$ and, by definition, $\tilde{\pi}^* \Omega^1_{M_0} = (\Omega^1_{M_0})_{\tilde{\pi}} \circ \Delta \tilde{\pi}$. We recall that $T_{\tilde{\pi}}X_0$ is a vector sub-bundle of $T M_{0}^\perp |_{\tilde{X}}$. Let $a_0 := \tilde{\pi}(\tilde{a})$ be the image of $\tilde{a}$. For a germ of differential form $\theta \in \Omega^1_{M_0,a_0}$, the local section $\theta |_{T \tilde{\pi}X_0}$ nearby $\tilde{a}$ of $T_{\tilde{\pi}}X_0$ is defined as the restriction $\theta \circ \tilde{\pi}|_{T \tilde{\pi}X_0}$. Let

$$\Lambda_{\tilde{\pi}} := (\Omega^1_{M_0})_{\tilde{\pi}} \big|_{T_{\tilde{\pi}}X_0}$$

be the $O_{\tilde{X}}$-sub-module of $\Omega^1_{\tilde{\pi}}$ generated by the restrictions to $T_{\tilde{\pi}}X_0$.

**Claim 1** $\tilde{\pi}^* (\Omega^1_{M_0})_{\tilde{\pi}} = \Lambda_{\tilde{\pi}} \circ \Delta \tilde{\pi}$.

**Proof of Claim 1** Any germ at $\tilde{a}$ of a vector field $\xi$ along $\tilde{X}$ induces the germ at $\tilde{a}$ of the local section $D \tilde{\pi} \cdot \xi : (\tilde{X}, \tilde{a}) \to T_{\tilde{\pi}}X_0$. For $\theta \in \Omega^1_{M_0,a_0}$, we get that $\tilde{\pi}^* \theta$ belongs to $\Omega^1_{\tilde{X}, \tilde{a}}$ and for every $\tilde{b}$ nearby $\tilde{a}$, the linear form $(\tilde{\pi}^* \theta)(\tilde{b})$ is defined as

$$T_{\tilde{b}} \tilde{X} \ni \xi \to ((\theta(\tilde{\pi}(\tilde{b})))((D \tilde{\pi})(\tilde{b}) \cdot \xi)),$$

while the linear form $(\theta \circ \Delta \tilde{\pi})(\tilde{b})$ is defined as

$$T_{\tilde{b}} \tilde{X} \ni \xi \to ((\theta(\tilde{\pi}(\tilde{b})))((D \tilde{\pi})(\tilde{b}) \cdot \xi),$$

so that they coincide since $(D \tilde{\pi})(\tilde{b}) \cdot \xi$ lies in $(T_{\tilde{\pi}}X_0)_{\tilde{a}}$ which is contained in $T_{\tilde{\pi}(\tilde{a})}M_0$.

**Claim 2** $\Lambda_{\tilde{\pi}} = \Omega^1_{\tilde{\pi}}$.

**Proof of Claim 2** We just need to show that $\Omega^1_{\tilde{\pi}} \circ \Delta \tilde{\pi}$ is a subset of $\tilde{\pi}^* (\Omega^1_{M_0})_{\tilde{\pi}}$. Let $\theta_1, \theta_2$ as in Eq. (9) and let $\chi_1, \chi_2$ be the dual basis (for the fiber metric $g^{\tilde{E}}$). Let $\omega_1, \omega_2$ be two 1-forms of $\Omega^1_{M_0,a_0}$ such that $\omega_i(a_0)(\chi_j(\tilde{a})) = \delta_{i,j}$. Thus the sections $\omega^1_{\tilde{\pi}}$ and $\omega^2_{\tilde{\pi}}$ are linearly independent nearby $\tilde{a}$, and the claim is proved.

Combining Claim 1 and Claim 2 with Lemma 6 yields the following important

**Proposition 6** The $O_{\tilde{X}}$-module $\tilde{\pi}^* \Omega^1_{M_0}$ is locally generated at $\tilde{a}$ by $M_1 \mu_1$ and $M_2 \mu_2$. 

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Let \((x, y, z_3, \ldots, z_n)\) be local regular coordinates centered at the image point \(a_0 := \pi(a)\). Obviously \(\pi^*(\Omega^1_{\mathcal{O}_a})\) is \(\mathcal{O}_a\)-generated by \(\pi^*dx, \pi^*dy, \pi^*dz_3, \ldots, \pi^*dz_n\). Since it is of local rank 2, we can assume that the coordinates at \(a_0\) were such that it is generated by \(\pi^*dx, \pi^*dy\). Proposition 6 implies, up to a linear change in \(x\) and \(y\), that nearby \(\tilde{a}\) the following relations hold:

\[
\begin{align*}
\text{Unit} \cdot \mathcal{M}_1\mu_1 &= \pi^*dx + A\pi^*dy \\
\text{Unit} \cdot \mathcal{M}_2\mu_2 &= B\pi^*dx + \pi^*dy
\end{align*}
\]  

for \(A, B \in \mathfrak{m}_\tilde{a}\mathcal{O}_\tilde{a}\).

- Assume that \(\tilde{a}\) is a regular point of \(\tilde{E}\).

We start with the following obvious

**Lemma 7** Since \(\mu_1 \land \mu_2 = \text{Unit} \cdot \mathcal{M}(du \land dv)\), if \(\mu_i(\tilde{a}) = 0\) and \(\mu_j(\tilde{a}) \neq 0\) when \((i, j) = (1, 2)\) or \((i, j) = (2, 1)\), then \(\mu_j = \pi u + \pi v\) and \(\mathcal{M} = \pi^{1+r}\) for some non-negative integer \(r\).

When \(\tilde{a}\) is a regular point of \(\tilde{E}\), we find \(\mathcal{M}_1 = u^r\) and \(\mathcal{M}_2 = u^s\) with \(s \geq r \geq 0\). Let us write \(\mu_i = a_i du + b_i dv\), and \(v_w\) for \(\partial_w(\pi^*x)\) and \(y_w\) for \(\partial_w(\pi^*y)\), where \(w\) is either \(u\) or \(v\). From Eqs. (12) and (13) we find the following relations:

\[
\begin{align*}
x_u + Ay_u &= u^r \psi_1 a_1 \\
Bx_u + y_u &= u^r \psi_2(a^s - r a_2) \\
x_v + Ay_v &= u^r \psi_1 b_1 \\
Bx_v + y_v &= u^r \psi_2(a^s - r b_2)
\end{align*}
\]  

where each \(\psi_i\) is a local unit.

We deduce that \(x = a_0 + u^{1+r}X(u, v)\) and \(y = b_0 + u^{1+r}Y(u, v)\) for constants \(a_0, b_0\). So we can write

\[
\begin{align*}
X &= x_0 + vx_1(v) + ux_2(u) + uvx_3(u, v) \\
Y &= y_0 + vy_1(v) + uy_2(u) + uvy_3(u, v).
\end{align*}
\]

We are using this local description of the blowing-up mapping to obtain the following possible local forms.

**Proposition 7** Assume the point \(\tilde{a}\) is a regular point of \(\tilde{E}\).

1) If \(\mu_1(\tilde{a}) \neq 0\), then we find \(\mu_1 = du + u(\cdots)dv\).

2) Suppose \(\mu_1(\tilde{a}) = 0\) and write

\[
\mu_1 = [v^k \phi(v) + uc_1(u, v)]du + uDdv,
\]

with \(k = 1\) and \(\phi(0) \neq 0\) if \(D(\tilde{a}) = 0\). We are in one of the situations listed below:

i) Suppose \(k = 1\) and \(D(\tilde{a}) \neq 0\). We can choose the local regular coordinates \((u, v)\) centered at \(\tilde{a}\) and adapted to \(\tilde{E}\) such that \(x = a_0 + u^{r+1}v\) and \(y = b_0 + v^r + 1\) and thus \(r = s\) and \(t = 1\). Moreover we find out that \(\mu_2 = du + u(\cdots)dv\).

ii) If \(k = 1\) and \(D(\tilde{a}) = 0\), then \(r = s\) and \(\mu_2 = du + u(\cdots)dv\).

iii) If \(D(\tilde{a}) \neq 0\) and \(k \geq 2\), then the conclusion of point i) holds true.

**Proof** 1) Since \(u^{r+1}(X_v + AY_v) = u^r \psi_1 b_1\), we get \(b_1 = uc_1\) for some \(c_1 \in \mathcal{O}_\tilde{a}\).

2) Suppose that \(\mu_1(\tilde{a}) = 0\).
i) Assume that \( \mu_1 = udv + (v\phi + uc_1)du \) with \( \phi \) a local analytic unit such that \( -\phi(0) \notin \mathbb{Q}_{>0} \). Since \( a_1(\tilde{a}) = 0 \) we find that \( x_0 = 0 \). From Eq. (16), we find \( X_v + AY_v = \psi_1 \) with \( A(\tilde{a}) = 0 \), and we see that \( x_1(0) \neq 0 \). Let \( \tilde{v} := X(v, u) \). Thus (\( u, v \) \( \rightarrow (u, X(u, v)) \)) is a regular change of coordinates so that \( x = a_0 + u^{r+1}\tilde{v} \). Since \( v = \tilde{v}z_1(\tilde{v}) + uz_2(u, \tilde{v}) \), with \( z_1(0) \neq 0 \), we deduce that \( \mu_1 = Unit[ud\tilde{v} + (\tilde{v}\phi(\tilde{v}) + u\tilde{c}_1)du] \) with \( \tilde{\phi}(0) \neq 0 \). Suppose the coordinates \((u, v)\), centered at \( \tilde{a} \) and adapted to \( \tilde{E} \), are also such that

\[
\mu_1 = udv + (v\phi(v) + uc_1)du, \quad \text{with} \quad -\phi(0) \notin \mathbb{Q}_{>0},
\]

and \( x = a_0 + u^{r+1}v \). Since \( \psi_1 \cdot u^r \mu_1 = dx + Ady \), we get

\[
\psi_1 \cdot [udv + (v\phi + uc_1)du] = [(r + 1)vdv + udv] + A[((r + 1)Y + uY_u)du + uY_ydv]
\]

with \( A(\tilde{a}) = 0 \) and \( \psi_1 \) a local analytic unit. Thus we find

\[
\psi_1 = 1 + uAY_v
\]

\[
\psi_1(v\phi + uc_1) = u + A[(r + 1)Y + uY_u]
\]

We deduce that \( y_0 \neq 0 \), so that \( Y \) is a local analytic unit, and thus \( y = b_0 + u^{r+1}Y \). Let \( \varepsilon \) be the sign of \( y_0 \). Let \( \tilde{u}(u, v) = u(\varepsilon Y)^{-\frac{1}{r+1}} \). The change of coordinates \((u, v) \rightarrow (\tilde{u}, \tilde{v})\) is regular, centered at \( \tilde{a} \) and adapted to \( \tilde{E} \), and we have \( y = b_0 + \varepsilon u^{r+1} \). Thus \( x = a_0 + \xi(\tilde{a}, v)\tilde{u}^{r+1}v \) for a local analytic unit \( \xi \). Thus taking \( \tilde{v} := \psi_v \), we have found local coordinates centered at \( \tilde{a} \) adapted to \( \tilde{E} \) and such that

\[
x = a_0 + \tilde{u}^{r+1}\tilde{v} \quad \text{and} \quad y = b_0 + \varepsilon \tilde{u}^{r+1}
\]

so that \( r = s \). This implies that \( t = 1 \), which can only occur if \( \mu_2(\tilde{a}) \neq 0 \) (otherwise \( t \geq 2 \)). Since \( r = s \) and \( \mu_2(\tilde{a}) \neq 0 \), we deduce from point 1) that \( \mu_2 = Unit(du + \tilde{u}\tilde{c}_2dv) \).

ii) Assume that \( \mu_1 = (v + uc_1)du + uDdv \) with \( D(\tilde{a}) = 0 \). Let us write \( A = vA_1(v) + u(\ldots) \). Equation (14) provides

\[
(1 + r)x_0 + v(x_1(v) + [y_0 + v\psi_1(v)]A_1(v)) + u(\ldots) = [v\psi_1(0, v) + u(\ldots)].
\]

(18)

We deduce that

\[
x_0 = 0 \quad \text{and} \quad (1 + r)v(x_1(v) + y_0A_1(v)) = v\psi_1(0, v).
\]

Thus \( x_1(v) + y_0A_1(v) \) is an analytic unit. Since \( D(\tilde{a}) = A(\tilde{a}) = 0 \) and by Eq. (16) we get

\[
X_v + AY_v = \psi_1 \cdot D.
\]

(19)

We deduce that \( x_1(0) = 0 \) so that \( y_0 \neq 0 \). Up to a change of coordinates as in i), we can assume that \( Y = b_0 \pm u^{r+1} \). Equation (15) reads

\[
B[(r + 1)X + uX_u] + (r + 1)Y + uY_u = \psi_2u^{s-r}a_2,
\]

(20)

and provides \( a_2(\tilde{a}) \neq 0 \) and \( r = s \). Tank to this latter condition we are back in point 1) by permuting \( \mu_1 \) and \( \mu_2 \) so that \( \mu_2 = du + u(\ldots)dv \).

iii) Suppose \( \mu_1 = (v^{k+2}\phi(v) + uc_1)du + udv \) with \( k \geq 0 \) so that \( D = 1 \). From Eq. (16), we deduce that \( x_1(0) = 1 \). Adapting Eq. (18) to our situation we get

\[
(x_1 + y_0A_1)(0) = (v^{k+1}\psi_1(0, v))(0) = 0,
\]
so that \( y_0 A_1(0) = -x_1(0) = 1 \). So we have \( x_1(0) \neq 0, y_0 \neq 0 \), thus we reach, after two changes of variables (one to change \( u \) and the next one to change \( v \), the same conclusion as \( i \)). \( \square \)

**Remark 4** An obvious, but unexpected, consequence of Proposition 7 is that any regular point of \( \widetilde{E} \) cannot be a simultaneous singular point of both (local) foliations \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \). Moreover according to our notations, at any regular point \( \tilde{a} \) of \( \widetilde{E} \) we can assume that we always have \( \mu_1(\tilde{a}) \neq 0 \).

We now use these pairs of normal forms to obtain the following Hsiang and Pati type result

**Proposition 8** Let \( \tilde{a} \) be a regular point of \( \widetilde{E} \).

There exist local coordinates \((u, v)\) centered at \( \tilde{a} \) and adapted to \( \widetilde{E} = \{ u = 0 \} \) such that

1) If \( \mu_2(\tilde{a}) \neq 0 \), then the module \( \pi^* \Omega^1_{M_0} \) is locally generated at \( \tilde{a} \) by \( d(u^{r+1}) \) and \( d(u^{s+1+m}v) \) for a non-negative integer \( m \).

2) If \( \mu_2 = a_2du + udv \) with \( a_2 \in \mathfrak{m}_{\tilde{a}} \), then \( t = 1 \) and the module \( \pi^* \Omega^1_{M_0} \) is locally generated at \( \tilde{a} \) by \( d(u^{r+1}) \) and \( d(u^{s+1}v) \).

3) If \( \mu_2 = (v + uc_2)du + u^2dv \) with \( e_2 \in \mathfrak{m}_{\tilde{a}} \), then \( t \geq 2 \) and the module \( \pi^* \Omega^1_{M_0} \) is locally generated at \( \tilde{a} \) by \( d(u^{r+1}) \) and \( d(u^{s+t}v) \).

**Proof** For simplicity let \( \Theta := \pi^* \Omega^1_{M_0} \). We recall that \( \mu_1 \cdot \mu_2 = \text{Unit} \cdot u^t du \wedge dv \).

By Proposition 7 we find \( \mu_1 = du + uc_1 dv \). Equation (14) gives \( x_0 \neq 0 \). Up to an adapted change of coordinates in \( u \), we assume that \( x = a_0 \pm u^{r+1} \). Up to replacing \( y \) by \( y \pm y_0x \), we assume that \( y_0 = 0 \).

1) Suppose \( \mu_2(\tilde{a}) \neq 0 \).

If \( t = 0 \), then we can assume that \( \mu_2 = dv \). Equation (17) provides \( Y_v = \psi_2 u^{s-r} \), so that up to changing \( v \) by \( \text{Unit} \cdot v \), we can assume \( y = b_0 + u^{r+2}y_1(u) + u^{s+1}v \). So we get result in this case.

If \( t \geq 1 \) and \( \mu_2(\tilde{a}) \neq 0 \) then we deduce \( \mu_2 = du + (uc_1 + \text{Unit} \cdot u^t)dv \). And we still have \( \mu_2 = udv + a_2du \) but only \( \mu_1 = du + u(\ldots)dv \). Writing \( Y = Yu_1(u) + u^p v Z(u, v) \) for a non-negative integer \( p \) and with \( Z \) such that \( Z(0, v) \neq 0 \), we deduce from Eqs. (16) and (17)

\[
AY_v = u^p A(Z + v Z_v) = \psi_1 c_1 \quad \text{and} \quad Y_v = \psi_2 u^{s-r}(c_1 + u^{t-1} \psi_3)
\]

where \( \psi_3 \) is a local unit. Since \( A(\tilde{a}) = 0 \), we deduce \( Z + v Z_v = \text{Unit} \cdot u^{s+t-r-1-p} \) and therefore \( p = s - r + t - 1 \). Thus up to changing \( v \) into \( \text{Unit} \cdot v \), we can assume \( y = b_0 + u^{r+2}y_1(u) + u^{s+t}v \). So we also get the desired result in this case.

2) Assume \( \mu_2 = udv + a_2 du \) with \( a_2 \in \mathfrak{m}_{\tilde{a}} \). Thus \( t = 1 \) and from Eq. (17) we get \( Y_v = \psi_2 u^{s-r} \). So that after a change of coordinates in \( v \) we can assume that \( y = b_0 + u^{r+2}y_1(u) + u^{s+1}v \). And we find the announced result.

3) Suppose \( \mu_2 = (v + uc_2)du + u^2dv \) with \( e_2 \in \mathfrak{m}_{\tilde{a}} \). Thus we deduce \( t \geq 2 \). Since \( \mu_1 = du + uc_1 dv \) we deduce that that \( \mu_2 = (\ldots) \mu_1 + u^t dv \). The module \( \Theta \) is generated by \( u^t \mu_1 \) and \( u^{s+t} dv \). We will use the following

**Lemma 8** Suppose \( \mu_1 = du + u^p Cdv \), with \( p \geq 1 \) so that \( C(0, v) \neq 0 \). There exists a change of coordinates \((w, v) \rightarrow (w + w^p \alpha(v), v)\) such that \( \alpha(0) = 0 \) and

\[
\mu_1 = \text{Unit}[dw + w^{p+1}(\cdots)dv].
\]
Proof Let \( C = c_0(v) + uC_1(u,v) \), so that \( c_0(v) \) is not identically 0. Let \( u = w + w^p \alpha(v) \). Since \( du = \left[ 1 + pw^{p-1} \alpha(v) \right] dw + w^p \alpha'(v) \, dv \), we get
\[
\mu_1 = \left[ 1 + pw^{p-1} \alpha(v) \right] dw + w^p \left( \alpha'(v) + c_0(v) \right) dv + w^{p+1} \beta(v, w) \, dv
\]
with \( \beta \in \mathcal{O}_\Xi^2 \). Taking \( \alpha \) the primitive of \(-c_0\) vanishing at \( v = 0 \) provides the result. \( \square \)

Remark 5 The equation we solve in \( W := 1 + w^{p-1} \alpha(v) \) in the proof of theLemma admits a formal solution (that is a solution in the real formal power series in two variables), so that up to a formal change of coordinates \( u = \tilde{u} W(u, v) \) for \( W \) a formal power series and a unit, we would find \( \mu_1 = d\tilde{u} \).

Thanks to the Lemma, used finitely many times, we deduce (despite these changes of coordinates) that \( \Theta \) is generated by \( u' \, du \) and \( u^{p+1} \, dv \) which is the result. \( \square \)

Using Hsiang and Pati proof (and caring with the fact that \(-1\) has no real square root), with a few elementary computations, we deduce from Proposition 8 the existence of Hsiang and Pati coordinates:

Corollary 4 (see also \([2]\)) There exist adapted coordinates \((u, v)\) at the regular point \( \tilde{a} \) such that the resolution mapping \( \tilde{\pi} \) locally writes
\[
(u, v) \rightarrow (x, y; z) = \tilde{\pi}(\tilde{a}) + (\pm u^{k+1}, u^{k+1} f(u) + u^{l+1} v; u^{k+1} z(u) + u^{l+1} v Z(u, v))
\]
in \( \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-2} \)
for non-negative integers \( k, l \), for germs \( f \in \mathbb{R}[u], \, z \in \mathbb{R}[u]^{N-2} \) and \( Z \) a regular map germ \((X, a) \rightarrow \mathbb{R}^{N-2} \).

• Assume now that \( \tilde{a} \) is a corner point of \( \tilde{E} \).

If \( \tilde{a} \) is a regular point of \( \mu_i \), for \( i = 1 \) or \( 2 \), then we can write \( \mu_i = dz + z(\cdots) \, dw \) where \((w, z) = (u, v) \) or \((v, u) \). In this case the logarithmic local generator of \( \mathcal{F}_i \) writes
\[
\mu_i^{\log} = z^{-1} \mu_i = d_{\log} z + w(\cdots) \, d_{\log} w.
\]

If \( \tilde{a} \) is a singular point of \( \mu_i \) (thus adapted to \( \tilde{E} \)), for \( i = 1 \) or \( 2 \), then \( \mu_i = wdz + z(\cdots) \, dw \) and where \((w, z) = (u, v) \) or \((v, u) \). The logarithmic local generator of \( \mathcal{F}_i \) writes in this case
\[
\mu_i^{\log} = w^{-1}z^{-1} \mu_i = d_{\log} z + (\cdots) \, d_{\log} w.
\]

By Proposition 6, the sub-module \( \tilde{\pi}^* \Omega^1_{\mathcal{M}_0, \tilde{\pi}(\tilde{a})} \) of \( \Omega^1_{\tilde{X}, \tilde{a}} \) is generated over \( \mathcal{O}_{\tilde{a}} \) by \( \mathcal{M}_1 \mu_1 \) and \( \mathcal{M}_2 \mu_2 \). Thus \( \tilde{\pi}^* \Omega^1_{\tilde{X}} \), as a \( \mathcal{O}_{\tilde{X}} \) sub-module of \( \Omega^1_{\tilde{X}}(\log \tilde{E}) \) is locally generated at \( \tilde{a} \) by \( \mathcal{M}_1^{\log} \mu_1^{\log} \) and \( \mathcal{M}_2^{\log} \mu_2^{\log} \).

We know that
\[
\mathcal{M}_1^{\log} = u^m v^n \quad \text{and} \quad \mathcal{M}_2^{\log} = u^p v^q \quad \text{with} \quad \mu_1^{\log} \wedge \mu_2^{\log} = \text{Unit} \cdot u^r v^s (d_{\log} u \wedge d_{\log} v).
\]

We can assume by Theorem 6 that \( p \geq m \geq 0, \, q \geq n \geq 0 \) and \( m + n \geq 1 \). When \( \mu_1 \) or \( \mu_2 \) vanishes at \( \tilde{a} \), we deduce \( \max(r, s) \geq 1 \). Since the local coordinates \((u, v)\) are centered at \( \tilde{a} \) and adapted to \( \tilde{E} \), for \( i = 1, 2 \) we can write \( \mu_i^{\log} = a_i d_{\log} u + b_i d_{\log} v \) (and \( a_i \) or \( b_i \) is a local unit), so that, up to permuting \( u \) and \( v \) we can always assume that \( a_1(\tilde{a}) \neq 0 \), thus \( m \geq 1 \).

Using again Eqs. (12) and (13) we obtain the following relations:
\[
ux_u + uAy_u = u^m v^n \psi_1 a_1 \tag{21}
\]
\[
ux_v + vAy_u = u^m v^n \psi_2 (u^{p-m}v^{q-n}a_2) \tag{22}
\]
\[
vx_u + vAy_v = u^m v^n \psi_1 b_1 \tag{23}
\]
\[
vx_v + vAy_v = u^m v^n \psi_2 (u^{p-m}v^{q-n}b_2) \tag{24}
\]
We deduce, up to changing $v$ into $Unit \cdot v$, that

$$x = X_0(v) \pm u^m v^n$$ and $$y = Y_0(v) + u^m v^n Y(u, v).$$

Since $m \geq 1$, Eqs. (23) and (24) imply that $X_0 = a_0$ and $Y_0 = b_0$ are real numbers. So we can write

$$Y = y_0 + vy_1(v) + uy_2(u) + uvy_3(u, v).$$

From Eq. (23), we deduce that necessarily $b_1(0) \neq 0$, thus we have deduced

**Lemma 9** For any local coordinates $(u, v)$ centered at the corner point $\tilde{\alpha}$ of $\tilde{E}$ and adapted to $\tilde{E}$ we find $\mu_1 = vdu + Unit \cdot udv$.

**Proposition 9** There exist local coordinates $(u, v)$ centered at $\tilde{\alpha}$ adapted to $\tilde{E}$ such that the resolution mapping $\tilde{\pi}^* \Omega^1_{M_0}$ is locally generated at $\tilde{\alpha}$ as an $\mathcal{O}_{\tilde{X}}$-submodule of $\Omega^1_{\tilde{X}}(\log \tilde{E})$ by

$$u^m v^n \log(u^m v^n) \text{ and } u^r v^s + q \log(u^r v^s + q).$$

Equivalently $\tilde{\pi}^* \Omega^1_{M_0}$ is $\mathcal{O}_{\tilde{X}}$-generated, as a submodule of $\Omega^1_{\tilde{X},\tilde{\alpha}}$ nearby $\tilde{\alpha}$ by $d(u^m v^n)$ and $d(u^r v^s + q)$. Therefore the plane vectors $(m, n)$ and $(p + r, q + s)$ are linearly independent.

**Proof** We already have that $x = x_0 \pm u^m v^n$, the sign “$\pm$” may be “$-$” only if $m, n$ are both even.

We can write $y - b_0 = u^m v^n [f(u, v) + z(u, v)]$ for regular germ $f$, $z \in \mathcal{O}_{\tilde{X}}$ such that each monomial $u^k v^l$ appearing in $f$ is such that $ml - kn = 0$, and each monomial $u^k v^l$ appearing in $z$ is such that $ml - kn \neq 0$. Thus

$$\frac{dx \wedge dy}{dx} = u^m v^n \frac{dx \wedge dz}{d} = Unit \cdot u^{r + p + m - 1} v^{s + q + n - 1} du \wedge dv.$$

Let $u^k v^l$ be a monomial of $z$, thus $dx \wedge d(u^k v^l) = (ml - nk)u^k v^l v^{s + q + n - 1} du \wedge dv$. Necessarily we deduce that $z = u^r v^s + q \alpha$ for a local analytic unit $\alpha$. Thus the plane vectors $(m, n)$ and $(p + r, q + s)$ are linearly independent. We are looking, if possible, for a change of local coordinates of the form $u = \tilde{u} U$ and $v = \tilde{v} V$ for local units $U, V$ such that $\tilde{u} = \pm u^m v^n$ and $\tilde{v} = \pm v^r v^s + q \alpha = \pm \tilde{v}^r v^s + q$. Let $\varepsilon$ be the sign of $\alpha(0)$. So we need $U^m V^n = 1$ and $U^{r + p} V^{s + q} = \varepsilon \alpha$ knowing that $m(s + q) - n(r + p) \neq 0$, this is equivalent to $V^{(s + q)n - (r + p)m} = (\varepsilon \alpha)^m$, which can be solved. Thus we can re-write $x = a_0 \pm \tilde{u}^m \tilde{v}^n$ and $y = a_0 + \tilde{u}^m \tilde{v}^n [f(\tilde{u}, \tilde{v}) \pm \tilde{v}^r v^s + q]$ where $h$ has only monomials $\tilde{u}^k \tilde{v}^l$ such that $ml - kn = 0$. These coordinates satisfy the announced result.  

As in the case of a regular point, using Hsiang and Pati proof, with a few elementary computations, we deduce from Proposition 9 the existence of Hsiang and Pati coordinates:

**Corollary 5** (see also [2]) There exist adapted coordinates $(u, v)$ at the corner point $\tilde{\alpha}$ such that the resolution mapping $\tilde{\pi}$ locally writes

$$(u, v) \rightarrow (x, y, z) = \tilde{\pi}(\tilde{\alpha}) + (\pm u^m v^n, u^m v^n f(u, v) \pm u^k v^l, u^m v^n z(u, v) + u^k v^l Z(u, v))$$

$$\in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$$

for non-negative integers $k \geq m$ and $l \geq n$ such that $nk - lm \neq 0$ and germs $f \in \mathcal{O}_{\tilde{\alpha}}$ and $z, Z \in \mathcal{O}_{\tilde{\alpha}}^{N-2}$ such that $df \wedge d(u^m v^n) = dz \wedge d(u^m v^n) \equiv 0.$
B.4 Local normal form for the induced metric

Following on the material presented in the previous subsection, we will give local normal forms of the metric \( g_0 \big|_{X_0} \) pulled-back onto the resolved surface \( \widetilde{X} \) at any regular point \( \widetilde{a} \) of the exceptional divisor \( E \) and add a few precisions at corner points.

From Theorem 6, Proposition 6, Proposition 8 and Proposition 9 there exist
\[
U := \mathcal{M}_1^{\text{log}} \quad \text{and} \quad T := \mathcal{M}_2^{\text{log}} \quad \text{and} \quad V := T \cdot \mathcal{M}_{1,2}^{\text{log}},
\]
ordered monomials in \( \widetilde{E} \) with \( T = U \cdot (\cdots) \) and \( V = T \cdot (\cdots) \) such that the pulled-back metric on \( \widetilde{X} \) writes nearby \( \widetilde{a} \)
\[
\widetilde{\pi}^* g_0|_{X_0} = \lambda_1 (\mathcal{M}_1 \mu_1) \otimes (\mathcal{M}_1 \mu_1) + \lambda_2 (\mathcal{M}_2 \mu_2) \otimes (\mathcal{M}_2 \mu_2)
\]
for positive analytic units \( \lambda_1, \lambda_2 \). Moreover we know that \( \{ U \mu_1^{\text{log}}, T \mu_2^{\text{log}} \} \) and \( \{ U \mu_1^{\text{log}}, V \mu_1^{\text{log}}, V \mu_2^{\text{log}} \} \) are both \( O_{\widetilde{a}} \)-basis of the sub-module \( \widetilde{\pi}^* \Omega^1_{M_0} \) of \( \Omega^1_{\widetilde{X}}(\log E) \) nearby \( \widetilde{a} \). Thus we can write,
\[
\begin{align*}
U \mu_1^{\text{log}} &= C_1 U \mu_1^{\text{log}} + D_1 V \mu_1^{\text{log}} \\
T \mu_2^{\text{log}} &= C_2 U \mu_2^{\text{log}} + D_2 V \mu_2^{\text{log}}
\end{align*}
\]
with \( C_1 D_2 - C_2 D_1 = \text{Unit} \). Let us write
\[
\eta_1 := \mu_1^{\text{log}} \quad \text{and} \quad \eta_2 := \mu_2^{\text{log}}.
\]
If \( T \neq U \) in Eq. (26), then \( C_1 D_2 = \text{Unit} \). Thus we can write
\[
\begin{align*}
\widetilde{\pi}^* g_0|_{X_0} &= (\lambda_1 C_1^2 + \lambda_2 C_2^2) \eta_1 \otimes \eta_1 + (\lambda_1 D_1^2 + \lambda_2 D_2^2) \eta_2 \otimes \eta_2 \\
&\quad + (\lambda_1 C_1 D_1 + \lambda_2 C_2 D_2) (\eta_1 \otimes \eta_2 + \eta_2 \otimes \eta_1)
\end{align*}
\]
for positive analytic units \( \lambda_1, \lambda_2 \). Since \( C_1 D_2 - C_2 D_1 \) is a unit, as a quadratic form in the “variables” \( \eta_1, \eta_2 \), the pulled-back metric \( \widetilde{\pi}^* \kappa \) is positive definite nearby \( \widetilde{a} \), thus we deduce the following looked for and expected

**Proposition 10** At the point \( \widetilde{a} \) of \( \widetilde{E} \) the pulled-back metric \( \widetilde{\pi}^* g_0|_{X_0} \) is locally quasi-isometric to the following metric:
\[
U^2 \eta_1 \otimes \eta_1 + V^2 \eta_2 \otimes \eta_2 = dU \otimes dU + dV \otimes dV
\]
\[
= dM_1^{\text{log}} \otimes dM_1^{\text{log}} + d(M_2^{\text{log}} \mathcal{M}_{1,2}^{\text{log}}) \otimes (d(M_2^{\text{log}} \mathcal{M}_{1,2}^{\text{log}})).
\]

- When \( \widetilde{a} \) is a regular point of \( \widetilde{E} \), we can be a little more specific. We recall that by Lemma 8, we can assume that for any given \( p \geq s - r + t + 2 \), the coordinates \((u, v)\) are such that \( \mu_1 = du + u^p c_1 dv \). As a consequence of Proposition 7 and Proposition 8 and of elementary computations we find the following normal forms:

**Proposition 11** Let \( \widetilde{a} \) be a regular point of \( \widetilde{E} \) we obtain. For each integer number \( \rho \geq 2s + 2t + 1 \), there exists an local adapted coordinates \((u, v)\) such that
\[
\widetilde{\pi}^* g_0 = \lambda_1 u^{2r} du \otimes du + \lambda_2 u^{2s+2t} dv \otimes dv + u^\rho (\cdots)(du \otimes dv + dv \otimes du)
\]
for positive analytic units \( \lambda_1, \lambda_2 \).
Proof Let $\rho$ be given. We can already assume that $p$ is chosen so that $2r + p \geq \rho$. We write $\mu_2$ as $B_{\mu_1} + u^t dv$ for $B \in m_{\tilde{\lambda}}$ and $t \geq 1$. We an write $$\tilde{\pi}^* g_0 = [Unit^2 \cdot u^{2r} + Unit^2 \cdot u^{2s+2t} B^2][\mu_1 \otimes \mu_1 + Unit^2 \cdot u^{2s+t} B(\mu_1 \otimes dv + dv \otimes \mu_1)] + Unit^2 \cdot u^{2s+2t} dv \otimes dv.$$ Let us consider a change of variable of the form $u = w(1 + w^q A(v))$ with $q$ a positive integer. Then we deduce that
\[
\begin{align*}
du &= [1 + (q + 1)w^q A]dw + w^{q+1} A' dv \\
du \otimes dv &= [1 + (q + 1)w^q A]dw \otimes dv + w^q A' dv \otimes dv \\
du \otimes du &= [1 + (q + 1)w^q A]^2 dw \otimes dw + w^{q+1}[1 + (q + 1)w^q A]A'(dw \otimes dv + dv \otimes dw) + w^{q+2}(A')^2 dv \otimes dv.
\end{align*}
\]
Observe that $\mu_1 \otimes \mu_1 = (du)^2 + u^p(\cdots)$. In the new coordinates we find
\[
\tilde{\pi}^* g_0|_{x_0} = Unit^2 \cdot w^{2r} dw \otimes dw + w^r C(dv \otimes dw + dw \otimes dv) + w^{2r}(\cdots)dv \otimes dv
\]
where
\[
\begin{align*}
w^r C &= (\lambda_1 w^{2r+q+1}[1 + (q + 1)w^q A]^{2r+1} A' \\
&+ \lambda_2 w^{2s+t}[1 + (q + 1)w^q A]^{2s+t}[B + w'(\cdots)].
\end{align*}
\]
Since $B(0, v) = v^j b_0(v)$ for a positive integer $j$ and a local analytic unit $b_0(v)$, taking $q = 2s + 2r + t - 1$, we find $A(v) = v^j t_0(v)$ for a local analytic unit $t_0(v)$, resolving a differential equation in $v$ of the form $A'(\psi(A)) = v^j f(v)$ where $\psi$ and $f$ are local analytic unit in $v$, such that $w^r C = w^{2r+1}(\cdots)$. The metric then writes
\[
\tilde{\pi}^* g_0|_{x_0} = Unit^2 \cdot w^{2r} dw \otimes dw + Unit^2 \cdot w^{2s+2t}(u^{-1} dv + G dw) \otimes (u^{-1} dv + G dw)
\]
with $G \in m_{\tilde{\lambda}}$. 

Up to factoring further powers of $u$ from $G$, we may assume that $G(0, v) \neq 0$, which is the worse case scenario. Replacing $\mu_1$ by $dw$ and $\mu_2$ be $H du + u^{t-1} dv$, we check that after at most $t - 1$ consecutive changes of coordinates of the exceptional variable $u_{exc}$, of the form
\[
u_{exc, odd} := u_{exc, new} [1 + u_{exc, new}^{q_{exc, new}} A_{new}(v)],
\]
we find adapted coordinates $(x, v)$, with $(\tilde{E}, \tilde{\lambda}) = \{x = 0\}$, such that
\[
\tilde{\pi}^* g_0|_{x_0} = Unit^2 \cdot w^{2r} du \otimes du + Unit^2 \cdot u^{2s+2t} dv \otimes dv + x^{2s+2t+1}(\cdots)(du \otimes dv + dv \otimes du).
\]
From here, finitely many (iterated) changes of variables of the form $v = y + x^m J(y)$, for a positive integer $m$ and regular function germ $J$ to find, will provide the announced result. □

- When $\overline{a}$ is a corner point of $\tilde{E}$, we know that the form $\mu_1$ (attached to the “smallest” monomial in $\tilde{E}$, writes
\[
\mu_1 = uv[d_{\log} u + (\lambda + G)d_{\log} v], \text{ with } -\lambda \notin \mathbb{Q}_{\geq 0} \text{ and } G \in m_{\tilde{\lambda}}
\]
for local adapted coordinates $(u, v)$ at $\overline{a}$. Since our result is general and we do not have explicit equations of the foliations we are dealing with, the desingularization of the foliation locally given by $\mu_1$ will tell us very little about $\lambda$. Remarkably the very special context we are working in gives the value of $\lambda$, for free, namely

\[\tilde{\lambda}\] Springer


Corollary 6 Since $U \mu_1^\log = U d_{\log} U + HV d_{\log} V$ with $H \in O_{\tilde{a}}$, and $U = u^m v^n$, for positive integer numbers $m$, $n$, we find that $\lambda = \frac{n}{m}$, in other words we can choose $\mu_1$ such that

$$
\mu_1^\log = d_{\log} U + F d_{\log} v, \quad \text{with} \quad F \in m_{\tilde{a}}.
$$

Proof Let $\xi := nu \partial_u - mv \partial_v$ so that $d_{\log} U (\xi) \equiv 0$. Using Proposition 10 and evaluating both quasi-isometric metrics along the vector field $\xi$ provides the result. □

References

1. Aroca, J.M., Hironaka, H., Vincente, J.L.: Desingularization Theorems. Memorias de Matemática del Instituto “Jorge Juan”, vol. 30. Consejo Superior de Investigaciones Científicas, Madrid (1977)
2. Belotto, A., Bierstone, E., Grandjean, V., Milman, P.: Resolution of singularities of the cotangent sheaf of a singular variety. Adv. Math. 307, 780–832 (2017)
3. Bierstone, E., Milman, P.: Semianalytic and subanalytic sets. Inst. Hautes Études Sci. Publ. Math. 67, 5–42 (1988)
4. Bierstone, E., Milman, P.: Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. Invent. Math. 128(2), 207–302 (1997)
5. Cano, F.: Desingularization of plane vector fields. Trans. Amer. Math. Soc. 296(1), 83–93 (1986)
6. Cano, F.: Reduction of the singularities of codimension one singular foliations in dimension three. Ann. Math. 160(3), 907–1011 (2004)
7. Cano, F., Cerveau, D., Deserti, J.: Théorie élémentaire des feuilletages holomorphes singuliers, p. 207. Belin, Collection Échelles (2013)
8. van den Essen, A.: Reduction of Singularities of the Differential Equation $Ady = Bdx$, Équations Différentielles et Systèmes de Pfaff dans le Champ Complexe, Lecture Notes in Math. vol. 712, pp. 44–49. Springer-Verlag, Berlin (1979)
9. Dumortier, F.: Singularities of vector fields on the plane. J. Differ. Equ. 23(1), 53–106 (1977)
10. Grandjean, V.: Gradient trajectories for plane singular metrics I: oscillating trajectories. Demonstr. Math. 47(1), 69–78 (2014)
11. Grandjean, V.: Re-parameterizing and reducing families of normal operators. Israel J. Math. 230(2), 715–744 (2019)
12. Grandjean, V., Grieser, D.: The exponential map at a cusp singularity. J. Reine Angew. Math. 736, 33–67 (2018)
13. Grandjean, V., Sanz, F.: On restricted analytic gradients on analytic isolated surface singularities. J. Differ. Equ. 255(7), 1684–1708 (2013)
14. Grauert, H.: On Levi’s problem and the imbedding of real-analytic manifolds. Ann. Math. 68, 460–472 (1958)
15. Grieser, D.: Hsiang & Pati coordinates for real analytic isolated surface singularities, Notes (2000) 4 pages
16. Guaraldo, F., Macri, P., Tancredi, A.: Topics On Real Analytic Spaces. Friedr. Vieweg & Sohn, Braunschweig (1986)
17. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. Ann. Math. 79, 326–356 (1964)
18. Hironaka, H.: Subanalytic Sets, Number Theory, Algebraic Geometry and Commutative Algebra, in Honor of Yasuo Akizuki, pp. 453–493. Kinokuniya, Tokyo (1973)
19. Hsiang, W.C., Pati, V.: $L^2$-cohomology of normal algebraic surfaces. Invent. Math. 81, 395–412 (1985)
20. Ilyashenko, Y., Yakovenko, S.: Lectures on Differential Equations, AMS, (2007)
21. Kurdyka, K., Paunescu, L.: Hyperbolic polynomials and multi-parameter perturbation theory. Duke Math. J. 141(1), 123–149 (2008)
22. Morrey, C.B.: The analytic embedding of abstract real-analytic manifolds. Ann. Math. 68, 159–201 (1958)
23. Parusinski, A.: Subanalytic functions. Trans. Amer. Math. Soc. 344(2), 583–595 (1994)
24. Pardon, W., Stern, M.: Pure Hodge structure on the $L^2$-cohomology of varieties with isolated singularities. J. Reine Angew. Math. 533, 55–80 (2001)
25. Saito, K.: Theory of Logarithmic differential forms and logarithmic vector fields. J. Fac. Sci. Univ. Tokyo, Sect. IA Math. 27, 265–291 (1980)
26. Seidenberg, A.: Reduction of the singularities of the differential equation $Ady = Bdx$. Amer. J. Math. 90, 248–269 (1968)
27. Whitney, H.: Local Properties of Analytic Varieties, 1965 Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), pp. 205–244. Princeton Univ. Press, Princeton (1965)
28. Youssin, B.: Abstract of his talk, titled Monomial resolution of singularities, at the conference Geometric Analysis and Singular Space, Oberwolfach, 21-27 June 1998. See page 8 of https://www.mfo.de/document/9826/Report_25_98.ps

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