On the dynamic consistency of hierarchical risk-averse decision problems

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Abstract In this paper, we consider a risk-averse decision problem for controlled-diffusion processes, with dynamic risk measures, in which there are two risk-averse decision makers (i.e., leader and follower) with different risk-averse related responsibilities and information. Moreover, we assume that there are two objectives that these decision makers are expected to achieve. That is, the first objective being of stochastic controllability type that describes an acceptable risk-exposure set vis-à-vis some uncertain future payoff, and while the second one is making sure the solution of a certain risk-related system equation has to stay always above a given continuous stochastic process, namely obstacle. In particular, we introduce multi-structure, time-consistent, dynamic risk measures induced from conditional $g$-expectations, where the latter are associated with the generator functionals of two backward-SDEs that implicitly take into account the above two objectives along with the given continuous obstacle process. Moreover, under certain conditions, we establish the existence of optimal hierarchical risk-averse solutions, in the sense of viscosity solutions, to the associated risk-averse dynamic programming equations that formalize the way in which both the leader and follower consistently choose their respective risk-averse decisions. Finally, we remark on the implication of our result in assessing the influence of the leader’s decisions on the risk-averseness of the follower in relation to the direction of leader-follower information flow.

Keywords Dynamic programming equation · forward-backward SDEs · hierarchical risk-averse decisions · value functions · viscosity solutions

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1 Introduction

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a probability space, and let \(\{B_t\}_{t \geq 0}\) be a \(d\)-dimensional standard Brownian motion, whose natural filtration, augmented by all \(\mathbb{P}\)-null sets, is denoted by \(\{\mathcal{F}_t\}_{t \geq 0}\), so that it satisfies the usual hypotheses (e.g., see [22] or [12]). We consider the following controlled-diffusion process over a given finite-time horizon \(T > 0\)

\[
dX^u_t = m(t, X_t^u, (u_t, v_t))dt + \sigma(t, X_t^u, (u_t, v_t))dB_t, \quad X^u_0 = x, \quad 0 \leq t \leq T, \tag{1}
\]

where

- \(X^u\) is an \(\mathbb{R}^d\)-valued controlled-diffusion process,
- \((u, v)\) is a pair of \((U \times V)\)-valued measurable decision processes such that for all \(t > s, (B_t - B_s)\) is independent of \((u_r, v_r)\) for \(r \leq s\) (nonanticipativity condition) and

\[
\mathbb{E}\int_s^t |u_\tau|^2 d\tau < \infty \quad \text{and} \quad \mathbb{E}\int_s^t |v_\tau|^2 d\tau < \infty, \quad \forall t \geq s,
\]

with \(U\) and \(V\) are open compact sets in \(\mathbb{R}^d\), with \(U \cap V = \emptyset\),
- \(m: [0, T] \times \mathbb{R}^d \times (U \times V) \rightarrow \mathbb{R}^d\) is uniformly Lipschitz, with bounded first derivative, and
- \(\sigma: [0, T] \times \mathbb{R}^d \times (U \times V) \rightarrow \mathbb{R}^{d \times d}\) is Lipschitz with the least eigenvalue of \(\sigma \sigma^T\) uniformly bounded away from zero for all \((x, (u, v)) \in \mathbb{R}^d \times (U \times V)\) and \(t \in [0, T]\), i.e.,

\[
\sigma(t, x, (u, v)) \sigma^T(t, x, (u, v)) \geq \lambda I_{d \times d}, \quad \forall (x, (u, v)) \in \mathbb{R}^d \times (U \times V), \quad \forall t \in [0, T],
\]

for some \(\lambda > 0\).

Notation: Let us introduce the following spaces that will be useful later in the paper.

- \(L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)\) is the set of \(\mathbb{R}^d\)-valued \(\mathcal{F}_T\)-measurable random variables \(\xi\) such that \(||\xi||^2 = \mathbb{E}\langle |\xi|^2 \rangle < \infty\);
- \(L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})\) is the set of \(\mathbb{R}\)-valued \(\mathcal{F}_T\)-measurable random variables \(\xi\) such that \(\|\xi\| = \text{ess inf}|\xi| < \infty\);
- \(S^2(t, T; \mathbb{R}^d)\) is the set of \(\mathbb{R}^d\)-valued adapted processes \((\varphi_s)_{t \leq s \leq T}\) on \(\Omega \times [t, T]\) such that \(||\varphi||^2_{[t, T]} = \mathbb{E}\{\sup_{t \leq s \leq T} |\varphi_s|^2\} < \infty\);
- \(H^2(t, T; \mathbb{R}^d)\) is the set of \(\mathbb{R}^d\)-valued progressively measurable processes \((\varphi_s)_{t \leq s \leq T}\) such that \(||\varphi||^2_{[t, T]} = \mathbb{E}\{\int_t^T |\varphi_s|^2 ds\} < \infty\).

In this paper, we consider a risk-averse decision problem for the above controlled-diffusion process, in which there are two hierarchical decision makers (i.e., leader and follower with differing risk-averse related responsibilities and information) choose their decisions from
progressively measurable strategy sets. That is, the leader’s decision \( u \) is a \( U \)-valued measurable control process from

\[
U_{[0,T]} \triangleq \left\{ u : [0,T] \times \Omega \to U \left| u \text{ is an } \{ \mathcal{F}_t \}_{t \geq 0} \text{- adapted} \right. \right. \\
\left. \left. \quad \quad \quad \quad \text{and } \mathbb{E} \int_0^T |u_t|^2 \, dt < \infty \right\}, \tag{2}
\]

and while the follower’s decision \( v \) is a \( V \)-valued measurable control process from

\[
V_{[0,T]} \triangleq \left\{ v : [0,T] \times \Omega \to V \left| v \text{ is an } \{ \mathcal{F}_t \}_{t \geq 0} \text{- adapted} \right. \right. \\
\left. \left. \quad \quad \quad \quad \text{and } \mathbb{E} \int_0^T |v_t|^2 \, dt < \infty \right\}. \tag{3}
\]

Furthermore, we consider the following two cost functionals that provide information about the accumulated risk-costs on the time interval \([0,T]\) w.r.t. the leader and follower, i.e.,

\[
\text{leader’s risk-cost: } \xi^l_{0,T}(u,v) = \int_0^T c_l(t,X^u,v_t,t,X^u,v_t,t,X^u,v_t) \, dt + \Psi_l(X_T), \tag{4}
\]

and

\[
\text{follower’s risk-cost: } \xi^f_{0,T}(u,v) = \int_0^T c_f(t,X^u,v_t,t,X^u,v_t,t,X^u,v_t) \, dt + \Psi_f(X_T), \tag{5}
\]

where \( c_l : [0,T] \times \mathbb{R}^d \times V \to \mathbb{R} \) and \( c_f : [0,T] \times \mathbb{R}^d \times W \to \mathbb{R} \) are measurable functions; and \( \Psi_l : \mathbb{R}^d \to \mathbb{R} \) and \( \Psi_f : \mathbb{R}^d \to \mathbb{R} \) are also assumed measurable functions.

Here, we remark that the corresponding solution \( X^u,v_t \) in (1) depends on the admissible risk-averse decision pairs \((u,v,) \) in \( U_{[0,T]} \otimes V_{[0,T]} \); and it also depends on the initial condition \( X^u,v_0 = x \). As a result of this, for any time-interval \([t,T]\), with \( t \in [0,T] \), the accumulated risk-costs \( \xi^l_{0,T} \) and \( \xi^f_{0,T} \) depend on the risk-averse decisions \((u,v,) \) in \( U_{[t,T]} \otimes V_{[t,T]} \).

Moreover, we also assume that \( f, \sigma, c_l, c_f, \Psi_l \) and \( \Psi_f \), for \( p \geq 1 \), satisfy the following growth conditions

\[
|m(t,x,(u,v))| + |\sigma(t,x,(u,v))| + |c_l(t,x,u)| + |\Psi_l(x)| \\
\leq K(1 + |x|^p + |u| + |v|) \tag{6}
\]

and

\[
|m(t,x,(u,v))| + |\sigma(t,x,(u,v))| + |c_f(t,x,v)| + |\Psi_f(x)| \\
\leq K(1 + |x|^p + |u| + |v|), \tag{7}
\]

for all \((t,x,(u,v)) \in [0,T] \times \mathbb{R}^d \times (U \times V)\) and for some constant \( K > 0 \).

On the same probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), we consider the following backward stochastic differential equation (BSDE)

\[
-dY_t = g(t,Y_t,Z_t) \, dt - Z_t \, dB_t, \quad Y_T = \xi, \tag{8}
\]

\footnote{For any \( t \in [0,T], U_{[t,T]} \) and \( V_{[t,T]} \) denote the sets of \( U \)- and \( V \)-valued \( \{ \mathcal{F}_s \}_{s \geq t} \)-adapted processes, respectively (see Definition\(\text{[3]}\)).}
where the terminal value $Y_T = \xi$ belongs to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ and the generator functional $g: \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, with property that $(g(t, y, z))_{0 \leq t \leq T}$ is progressively measurable for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. We also assume that $g$ satisfies the following assumption.

**Assumption 1**

1. $g$ is Lipschitz in $(y, z)$, i.e., there exists a constant $K > 0$ such that, $\mathbb{P}$-a.s., for any $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$

   \[
g(t, y_1, z_1) - g(t, y_2, z_2) \leq K (|y_1 - y_2| + \|z_1 - z_2\|).
\]

1. $g(t, 0, 0) \in \mathcal{H}^2(t, T; \mathbb{R})$.

1. $\mathbb{P}$-a.s., for all $t \in [0, T]$ and $y \in \mathbb{R}$, $g(t, y, 0) = 0$.

Then, we state the following lemma, which is used to establish the existence of a unique adapted solution (e.g., see [17] for additional discussions).

**Lemma 2** Suppose that Assumption 1 holds. Then, for any $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, the BSDE in (8), with terminal condition $Y_T = \xi$, i.e.,

\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T
\]

has a unique adapted solution

\[
(Y_t^{T, g, \xi}, Z_t^{T, g, \xi})_{0 \leq t \leq T} \in \mathcal{S}^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^d).
\]

Moreover, we recall the following comparison result that will be useful later (e.g., see [18]).

**Theorem 1 (Comparison Theorem)** Given two generators $g_1$ and $g_2$ satisfying Assumption 1 and two terminal conditions $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Let $(Y_t^{1, g_1}, Z_t^{1, g_1})$ and $(Y_t^{2, g_2}, Z_t^{2, g_2})$ be the solution pairs corresponding to $(\xi_1, g_1)$ and $(\xi_2, g_2)$, respectively. Then, we have

(i) Monotonicity: If $\xi_1 > \xi_2$ and $g_1 > g_2$, $\mathbb{P}$-a.s., then $Y_t^{1, g_1} > Y_t^{2, g_2}$, $\mathbb{P}$-a.s., for all $t \in [0, T]$;

(ii) Strictly Monotonicity: In addition to (i) above, if we assume that $\mathbb{P}(\xi_1 > \xi_2) > 0$, then $\mathbb{P}(Y_t^{1, g_1} > Y_t^{2, g_2}) > 0$, for all $t \in [0, T]$.

In the following, we give the definition for a dynamic risk measure that is associated with the generator of BSDE in (8).

**Definition 1** For any $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, let $(Y_t^{T, g, \xi}, Z_t^{T, g, \xi})_{0 \leq t \leq T} \in \mathcal{S}^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^d)$ be the unique solution for the BSDE in (8) with terminal condition $Y_T = \xi$. Then, we define the dynamic risk measure $\rho_{t, T}^g$ of $\xi$ by

\[
\rho_{t, T}^g[\xi] \triangleq Y_t^{T, g, \xi}.
\]

\[\text{(11)}\]

\[\text{Here, we remark that, for any } t \in [0, T], \text{ the conditional } g\text{-expectation (denoted by } E_t[\xi|\mathcal{F}_t]\text{) is also defined by}
\]

\[
E_t[\xi|\mathcal{F}_t] \triangleq Y_t^{T, g, \xi}.
\]
Remark 1 Note that such a risk measure is widely used for evaluating the risk of uncertain future outcomes, and also assisting with stipulating minimum interventions required by financial institutions for risk management (e.g., see [2], [21], [8], [11] or [4] for related discussions). In Section 2, we use multi-structure, time-consistent, dynamic risk measures induced from conditional $g$-expectations, where the latter are associated with the generator functionals of two backward-SDEs that implicitly take into account the cost functionals of the leader and follower along with the given continuous obstacle process; and we provide a hierarchical framework for the risk-averse decision problem for the controlled-diffusion process.

Moreover, if the generator functional $g$ satisfies Assumption 1, then a family of time-consistent dynamic risk measures $\{\rho_{t,T}^{g}\}_{t\in[0,T]}$ has the following properties (see [21] for additional discussions).

Property 1

(p1) Convexity: If $g$ is convex for every fixed $(t,\omega) \in [0,T] \times \Omega$, then for all $\xi_1, \xi_2 \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R})$ and for all $\lambda \in L^\infty(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R})$ such that $0 \leq \lambda \leq 1$

$$\rho_{t,T}^g[\lambda\xi_1 + (1-\lambda)\xi_2] \leq \lambda\rho_{t,T}^g[\xi_1] + (1-\lambda)\rho_{t,T}^g[\xi_2];$$

(p2) Monotonicity: For $\xi_1, \xi_2 \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R})$ such that $\xi_1 > \xi_2 \mathbb{P}\text{-a.s.}$, then

$$\rho_{t,T}^g[\xi_1] > \rho_{t,T}^g[\xi_2], \quad \mathbb{P}\text{-a.s.};$$

(p3) Trans-invariance: For all $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R})$ and $\nu \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R})$

$$\rho_{t,T}^g[\xi + \nu] = \rho_{t,T}^g[\xi] + \nu;$$

(p4) Positive-homogeneity: For all $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R})$ and for all $\lambda \in L^\infty(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R})$ such that $\lambda > 0$

$$\rho_{t,T}^g[\lambda \xi] = \lambda\rho_{t,T}^g[\xi];$$

(p5) Normalization: $\rho_{t,T}^g[0] = 0$ for $t \in [0,T]$.

Remark 2 Note that, since the seminal work of Artzner et al. [2], there have been studies on axiomatic dynamic risk measures, coherency and consistency in the literature (e.g., see [6], [21], [24], [11] or [4]). Particularly relevant for us are, time-consistent, dynamic risk measures, induced from conditional $g$-expectations associated with generator functionals of BSDEs satisfying the above properties (p1)–(p5).

Here it is worth mentioning that some interesting studies on the dynamic risk measures, based on the conditional $g$-expectations, have been reported in the literature (e.g, see [21], [4] and [24] for establishing connection between the risk measures and the generator of BSDE; and see also [26] for characterizing the generator of BSDE according to different risk measures). Recently, the authors in [25] and [3] have provided interesting results on the risk-averse decision problem for Markov decision processes, in discrete-time setting, and, respectively, a hierarchical risk-averse framework for controlled-diffusion processes.
that the rationale behind our framework follows in some sense the settings of these papers. However, to our knowledge, the problem of dynamic consistent risk-aversion for controlled-diffusion processes has not been addressed in the context of hierarchical argument, and it is important because it provides a mathematical framework that shows how such a framework can be systematically used to obtain optimal risk-averse decisions.

The remainder of this paper is organized as follows. In Section 2, using the basic remarks made in Section 1, we state the hierarchical risk-averse decision problem for the controlled-diffusion process. In Section 3 we present our main results – where we introduce a framework under which the follower is required to respond optimally to the risk-averse decision of the leader so as to achieve an overall consistent risk-averseness. Moreover, we establish the existence of optimal risk-averse solutions, in the sense of viscosity solutions, to the associated risk-averse dynamic programming equations. Finally, Section 4 provides further remarks.

2 The hierarchical risk-averse decision problem formulation

In order to make our hierarchical formulation more precise, we further assume following.

Assumption 3

(3.1) \( g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \) is a measurable function that satisfies Assumption 1,

(3.2) \( \xi_{\text{Target}} \) is a real-valued random variable from \( L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}) \),

(3.3) \( h : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) is jointly continuous in \( t \) and \( x \); and satisfying

\[
  h(t, x) \leq K(1 + |x|^p), \quad (t, x) \in [0, T] \times \mathbb{R}^d, \tag{12}
\]

\[
  h(t, x) \leq \Psi_l(x) \text{ for } x \in \mathbb{R}^d \text{ and for some constant } K > 0,
\]

(3.4) an “obstacle” \( \{L_t, 0 \leq t \leq T\} \), which is continuous progressively real-valued process satisfying

\[
  \mathbb{E}\left\{ \sup_{0 \leq t \leq T} (L_t^+)\right\} < \infty. \tag{13}
\]

Then, for any \((t, x) \in [0, T] \times \mathbb{R}^d\), we consider the following forward-SDE with an initial condition \( X_t^{t, x; w} = x \)

\[
  dX_s^{t, x; w} = m(t, X_s^{t, x; w}, (u_s, v_s))ds + \sigma(s, X_s^{t, x; w}, (u_s, v_s))dB_s, \quad t \leq s \leq T, \tag{14}
\]

where \( w \triangleq (u, v) \) is a pair of \((U, V)\)-valued measurable decision processes. Furthermore, we also suppose that the data \( (\xi_{\text{Target}}, L) \) take the following forms

\[
  \xi_{\text{Target}} = \Psi_l(X_T^{t, x; w}) \quad \text{and} \quad L_s = h(s, X_s^{t, x; w}), \tag{15}
\]
Moreover, we introduce the following two risk-value functions w.r.t. the leader and follower, i.e.,

**leader:** \( V^l(t, x) = \rho^0_{t, T} [\xi^l_{t, T}(u, v)] \), such that, for a given \( \tilde{u} \in \mathcal{U}_{[t, T]} \),

\[
\exists \tilde{v} \in \mathcal{V}_{[t, T]} : \rho^0_{t, T} [\xi^l_{t, T}(\tilde{u}, \tilde{v})] \leq \rho^0_{t, T} [\xi^l_{t, T}(\tilde{u}, v)] \quad \forall v \in \mathcal{V}_{[t, T]},
\]

\[
\rho^0_{t, T} [\xi^l_{t, T}(\tilde{u}, v)] \geq L_t, \quad \mathbb{P} \text{-a.s.}
\]  

where

\[
\xi^l_{t, T}(v, w) = \int_t^T c_l(s, X^l_{t,x}; s, v_s) ds + \Psi_l(X^l_{t,x})
\]  

and similarly

**follower:** \( V^f(t, x) = \rho^0_{t, T} [\xi^f_{t, T}(u, v)] \),

where

\[
\xi^f_{t, T}(v, w) = \int_t^T c_f(s, X^f_{t,x}; s, w_s) ds + \Psi_f(X^f_{t,x}).
\]

Taking into account Assumption 3 (and with Markovian risk-averse decisions), we can express the above two risk-value functions using reflected- and standard-BSDE as follows

\[
V^l(t, x) \triangleq \tilde{Y}^l_{t,x} = \Psi_l(X^l_{t,x}) + \int_t^T g_l(s, X^l_{s,x}, \tilde{Y}^l_{s,x}, \tilde{Z}^l_{s,x}) ds
\]

\[
\quad + A^l_{t,x} - A^l_{s,x} - \int_t^T \tilde{Z}^l_{s,x} dB_s,
\]  

where \( \{A^l_{s,x} \} \) is increasing and continuous, and

\[
\int_t^T (\tilde{Y}^l_{s,x} - h(s, X^l_{s,x})) dA^l_{s,x} = 0,
\]

with \( L_s = h(s, X^l_{s,x}) \), and

\[
g_l(t, X^l_{s,x}, \tilde{Y}^l_{s,x}, \tilde{Z}^l_{s,x}) = c_l(s, X^l_{s,x}, u_s) + g(s, \tilde{Y}^l_{s,x}, \tilde{Z}^l_{s,x})
\]

and

\[
V^f(t, x) \triangleq \tilde{Y}^f_{t,x} = \Psi_f(X^f_{t,x}) + \int_t^T g_f(s, X^f_{s,x}, \tilde{Y}^f_{s,x}, \tilde{Z}^f_{s,x}) ds
\]

\[
\quad - \int_t^T \tilde{Z}^f_{s,x} dB_s,
\]  

where

\[
g_f(t, X^f_{s,x}, \tilde{Y}^f_{s,x}, \tilde{Z}^f_{s,x}) = c_f(s, X^f_{s,x}, v_s) + g(s, \tilde{Y}^f_{s,x}, \tilde{Z}^f_{s,x}).
\]

Noting the conditions in (6) and (7) (see also Remark 3), then \( \{\tilde{Y}^l_{s,x}, \tilde{Z}^l_{s,x}, A^l_{s,x} \}_{t \leq s \leq T} \) and \( \{\tilde{Y}^f_{s,x}, \tilde{Z}^f_{s,x} \}_{t \leq s \leq T} \) are adapted solutions on \([t, T] \times \Omega \) and belong to \( S^2(t, T; \mathbb{R}) \times \mathcal{H}^2(t, T; \mathbb{R}^d) \times S^2(t, T; \mathbb{R}) \).
Remark 3 Here, it is worth remarking that, for a given continuous progressively real-valued process \( \{ L_t, 0 \leq t \leq T \} \) satisfying (13) and for each \((t, x) \in [0, T] \times \mathbb{R}^d\), if there exists a unique triple \((\hat{Y}^{t,x,w}, \hat{Z}^{t,x,w}, A^{t,x,w})\) of \( \{ F_s^t \} \) progressively measurable processes, which solves the reflected-BSDE in (20). Then, this is equivalent to solve

\[
(\hat{Y}^{t,x,w}, \hat{Z}^{t,x,w}) = (\hat{Y}^{t,x,w} + A^{t,x,w}, \hat{Z}^{t,x,w}) \in S^2(t, T; \mathbb{R}) \times \mathcal{H}^2(t, T; \mathbb{R}^d)
\]

of the following standard-BSDE:

\[
\hat{Y}^{t,x,w}_s = \Psi_t(X^t_{T,w}) + \int_t^T g_l(s, X^t_{s,w}, \hat{Y}^{t,x,w}_s, \hat{Z}^{t,x,w}_s) ds - \int_t^T \hat{Z}^{t,x,w}_s dB_s.
\]

In what follows, we introduce a hierarchical framework that requires a certain level of risk-averseness be achieved for the leader as a priority over that of the follower. For example, suppose that the risk-averse decision for the leader \( \hat{u} \in \mathcal{U}_{[t,T]} \) is given. Then, the problem of finding an optimal risk-averse decision for the follower, i.e., \( \hat{v} \in \mathcal{V}_{[t,T]} \), which minimizes the accumulated risk-cost under \( v \) is then reduced to finding an optimal risk-averse solution for

\[
\inf_{v \in \mathcal{V}_{[t,T]}} J_f[(\hat{u}, v)],
\]

where

\[
J_f[(\hat{u}, v)] = \rho^{\mathcal{V}, \mathcal{U}}_{t,T}[\xi^{f}_{t,T}(\hat{u}, v)].
\]

(22)

Note that, for a given \( \hat{u} \in \mathcal{U}_{[t,T]} \), if the forward-backward stochastic differential equations (FBSDEs) in (14), (21) and the reflected-BSDE in (20) admit unique solutions, then we have

\[
\hat{v} \triangleq F(\hat{u}) \in \mathcal{V}_{[t,T]} \quad \text{satisfying} \quad \rho^{\mathcal{V}, \mathcal{U}}_{t,T}[\xi^{f}_{t,T}(\hat{u}, \hat{v})] \leq \rho^{\mathcal{V}, \mathcal{U}}_{t,T}[\xi^{f}_{t,T}(\hat{u}, v)], \quad \forall v \in \mathcal{V}_{[t,T]},
\]

(23)

for some measurable mapping \( F: \mathcal{U}_{[t,T]} \rightarrow \mathcal{V}_{[t,T]} \). Moreover, if we substitute \( \hat{w} = (\hat{u}, F(\hat{u})) \) into (14), then the corresponding solution \( X^{t,x,\hat{w}}_s \) depends uniformly on \( \hat{u} \) for \( s \in [t, T] \). Further, the risk-averse decision problem (which minimizes the accumulated risk-cost under \( u \) w.r.t. the follower) is then reduced to finding an optimal risk-averse solution for

\[
\inf_{u \in \mathcal{U}_{[t,T]}} J_l[(u, F(u))],
\]

(25)

where

\[
J_l[(u, F(u))] = \rho^{\mathcal{V}, \mathcal{U}}_{t,T}[\xi^{f}_{t,T}(u, F(u))].
\]

(26)

Remark 4 Note that the generator functionals \( g_l \) and \( g_f \) contain a common term \( g \) that acts on different processes (see also equation (20) and (21)). Moreover, due to differing cost functionals w.r.t. the leader and follower, \( \rho^{\mathcal{V}, \mathcal{U}}_{t,T}[\cdot] \) and \( \rho^{\mathcal{V}, \mathcal{U}}_{t,T}[\cdot] \) provide multi-structure dynamic risk measures.
Problem (P). can state the risk-averse decision problem as follow.

Next, we introduce the definition of admissible hierarchical risk-averse decision system \( \Sigma_{[t,T]} \), with time-consistent, dynamic risk measures, which provides a logical construct for our main results (e.g., see also [15]).

**Definition 2** For a given finite-time horizon \( T > 0 \), we call \( \Sigma_{[t,T]} \) an admissible hierarchical risk-averse decision system, if it satisfies the following conditions:

1. \( \{ \Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P} \} \) is a complete probability space;
2. \( \{ B_s \}_{s \geq t} \) is a \( d \)-dimensional standard Brownian motion defined on \( \{ \Omega, \mathcal{F}, \mathbb{P} \} \) over \([t, T]\) and \( \mathcal{F}^t \triangleq \{ \mathcal{F}^t_s \}_{s \in [t,T]} \), where \( \mathcal{F}^t_s = \sigma \{ (B_s; \ t \leq s \leq T) \} \) is augmented by all \( \mathbb{P} \)-null sets in \( \mathcal{F} \);
3. \( u : \Omega \times [s,T] \to U \) and \( v : \Omega \times [s,T] \to V \) are \( \{ \mathcal{F}^t_s \}_{s \geq t} \)-adapted processes on \( \Omega \) with
   \[
   \mathbb{E} \int_s^T |u_t|^2 dt < \infty \quad \text{and} \quad \mathbb{E} \int_s^T |v_t|^2 dt < \infty, \quad s \in [t,T];
   \]
4. There exists at least one measurable mapping \( F : u. \in \mathcal{U}_{[t,T]} \rightsquigarrow v. \in \mathcal{V}_{[t,T]} \) with \( v. = F(u.) \) whenever \( u. \in \mathcal{U}_{[t,T]} \) satisfies (22);
5. For any \( x \in \mathbb{R}^d \), the FBSDEs in (23), (21) and the reflected-BSDE in (20) admit a unique solution set \( \{ X^{s,x,u}_{\cdot}, Y^{s,x,u}_{\cdot}, Z^{s,x,u}_{\cdot}, \Psi^{s,x,u}_{\cdot} \} \) on \( \{ \Omega, \mathcal{F}, \mathcal{F}^t, \mathbb{P} \} \) with \( w. = (u., F(u.)) \).

Then, with restriction to the above admissible hierarchical risk-averse decision system, we can state the risk-averse decision problem as follow.

**Problem (P).** Find a pair of risk-averse strategies \((u^*, v^*) \in \mathcal{U}_{[0,T]} \otimes \mathcal{V}_{[0,T]}\) w.r.t. the leader and that of the follower, with \( \xi^{\text{Target}} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) \), such that

\[
u^* \in \left\{ \arg \inf_{J_f} \left[ (u, v) \right] \left| v = F(u) \right. \wedge (u, F(u)) \text{ restricted to } \Sigma_{[0,T]} \right\},
\]

and

\[
\nu^* \in \left\{ \arg \inf_{J_f} \left[ (u, v) \right] \left| v^* = F(u^*) \right. \wedge (u^*, F(u^*)) \text{ restricted to } \Sigma_{[0,T]} \right\},
\]

where \( F \) is a measurable mapping the set \( \mathcal{U}_{[0,T]} \) onto \( \mathcal{V}_{[0,T]} \) and, furthermore, the accumulated risk-costs \( J_l \) and \( J_f \) over the time-interval \([0, T]\) are given

\[
J_l \left[ (v, w) \right] = \int_0^T c_l(s, X^{0,x,v}_s, u_s) ds + \Psi_l(X^{0,x,v}_T), \quad \Psi_l(X^{0,x,v}_T) = \xi^{\text{Target}}
\]

and

\[
J_f \left[ (v, w) \right] = \int_0^T c_f(s, X^{0,x,v}_s, v_s) ds + \Psi_f(X^{0,x,v}_T),
\]

where \( c_l, c_f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) and \( \Psi_l, \Psi_f : \mathbb{R}^d \to \mathbb{R} \).
where $X^0_{0,x,w} = x$ and $w = (u,v)$.

In the following section, we establish the existence of optimal risk-averse solutions, in the sense of viscosity, for the risk-averse optimization problems in (27) and (28) with restriction to $\Sigma_{[0,T]}$. Note that, for a given $u \in U_{[0,T]}$, the risk-averse optimization problem in (28) has a unique solution on $V_{[0,T]}$. Moreover, as we will see later on, the problem in (27) makes sense if the follower is involved not only in minimizing his own accumulated risk-cost (in response to the risk-averse decision of the leader) but also in minimizing that of the leader.

3 Main results

In this section, we present our main results, where we introduce a hierarchical framework under which the follower is required to respond optimally to the risk-averse decision of the leader so as to achieve an overall risk-averseness. Moreover, such a framework allows us to establish the existence of optimal risk-averse solutions, in the sense of viscosity solutions, to the associated risk-averse dynamic programming equations.

We now state the following propositions that will be useful for proving our main results in Subsections 3.1 and 3.3.

Proposition 1 Suppose that the generator functional $g$ satisfies Assumption 7. Further, let the statements in (6), (7) and Assumption 3 along with (15) hold true. Then, for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and for every $w = (u,v) \in U_{[t,T]} \otimes V_{[t,T]}$, the FBSDEs in (14), (21) and the reflected-BSDE in (20) admit unique adapted solutions

$$
\begin{align*}
\left\{ & (\tilde{Y}^{t,x,w}, \tilde{Z}^{t,x,w}, A^{t,x,w}) \in S^2(t,T;\mathbb{R}^d) \times \mathcal{H}^2(t,T;\mathbb{R}^d) \times S^2(t,T;\mathbb{R}) \\
& (\tilde{Y}^{t,x,w}, \tilde{Z}^{t,x,w}) \in S^2(t,T;\mathbb{R}) \times \mathcal{H}^2(t,T;\mathbb{R}^d) \right\} 
\end{align*}
$$

Furthermore, the risk-values w.r.t. the leader and follower, i.e., $V^u_{[t,T]}(t,x)$ and $V^v_{[t,T]}(t,x)$, are deterministic.

Proof Notice that $m$ and $\sigma$ are bounded and Lipschitz continuous w.r.t. $(t,x) \in [0,T] \times \mathbb{R}^d$ and uniformly for $(u,v) \in U \times V$. Then, for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $w = (u,v)$ are progressively measurable processes, there always exists a unique path-wise solution $X^{t,x,w} \in S^2(t,T;\mathbb{R}^d)$ for the forward SDE in (14). On the other hand, consider the following BSDEs

$$
-d\tilde{Y}^{t,x,w}_s = g(s, X^{t,x,w}_s, \tilde{Y}^{t,x,w}_s, \tilde{Z}^{t,x,w}_s) ds - \tilde{Z}^{t,x,w}_s dB_s, \quad (32)
$$

where

$$
\tilde{Y}^{t,x,w}_{T} = A^{t,x,w}_T + \int_t^T c_l(\tau, X^{t,x,w}_\tau, u_\tau) d\tau + \Psi_l(X^{t,x,w}_\tau)
$$

Note that

$$
v^* \in \left\{ \arg \inf_{u} J_f \left[ (u,v) \right] \right\}_{w = F^{-1}(v) \& \left( F^{-1}(v), v \right) \text{restricted to } \Sigma_{[0,T]}}.
$$

where $F^{-1}(v) : V_{[1,T]} \mapsto u \in U_{[t,T]}$. 

and

\[-d\tilde{Y}_s^{t,x;u} = g_f(s, X_s^{t,x;u}, \tilde{Y}_s^{t,x;u}, \tilde{Z}_s^{t,x;u}) ds - \tilde{Z}_s^{t,x;u} dB_s, \tag{33}\]

where

\[\tilde{Y}_T^{t,x;u} = \int_t^T c_f(\tau, X_\tau^{t,x;u}, u_\tau) d\tau + \Psi_f(X_T^{t,x;u}).\]

From Lemma 2 (and see also Remark 3), the equations in (32) and (33) admit unique solutions \((\tilde{Y}^{t,x;u}, \tilde{Z}^{t,x;u})\) and \((\tilde{Y}^{t,x;u}, \tilde{Z}^{t,x;u})\) in \(S^2(t, T; \mathbb{R}) \times \mathcal{H}^2(t, T; \mathbb{R}^d)\). Furthermore, if we introduce the following

\[\tilde{Y}_s^{t,x;u} = Y_s^{t,x;u} - A_s^{t,x;u} - \int_t^s c_f(\tau, X_\tau^{t,x;u}, u_\tau) d\tau, \quad s \in [t, T]\]

and

\[\tilde{Y}_s^{t,x;u} = \tilde{Y}_s^{t,x;u} - \int_t^s c_f(\tau, X_\tau^{t,x;u}, v_\tau) d\tau, \quad s \in [t, T].\]

Then, the forward of the reflected BSDE in (20) and that of the BSDE in (21) hold, respectively, with \((\tilde{Y}^{t,x;u}, \tilde{Z}^{t,x;u}, A^{t,x;u})\) and \((\tilde{Y}^{t,x;u}, \tilde{Z}^{t,x;u}, A^{t,x;u})\). Moreover, we also observe that \(\tilde{Y}_t^{t,x;u}\) and \(\tilde{Y}_t^{t,x;u}\) are deterministic. This completes the proof of Proposition 1. \(\square\)

**Proposition 2** Let \((t, x) \in [0, T] \times \mathbb{R}^d\) and \(u = (u, v) \in \mathcal{U}[t, T] \otimes \mathcal{V}[t, T]\) be restricted to \(\Sigma_{[t,T]}\) (cf. Definition 2). Then, for any \(r \in [t, T]\) and \(\mathbb{R}^d\)-valued \(\mathcal{F}_r\)-measurable random variable \(\eta\), we have

\[V_r^u(r, \eta) = \tilde{Y}_r^{t,x;u}\]
\[\triangleq \tilde{Y}_r^{q_r, \tilde{Y}_r^{t,x;u}} \left[ \int_r^T c_f(s, X_s^{r,\eta;u}, u_s) ds + \Psi_f(X_T^{r,\eta;u}) \right], \quad \mathbb{P}\text{-a.s.} \tag{34}\]

and

\[V_r^v(r, \eta) = \tilde{Y}_r^{t,x;v}\]
\[\triangleq \tilde{Y}_r^{q_r, \tilde{Y}_r^{t,x;v}} \left[ \int_r^T c_f(s, X_s^{r,\eta;u}, v_s) ds + \Psi_f(X_T^{r,\eta;v}) \right], \quad \mathbb{P}\text{-a.s.} \tag{35}\]

**Proof** For any \(r \in [t, T]\), with \(t \in [0, T]\), we consider the following probability space \((\Omega, \mathcal{F}, \mathbb{P}(\cdot|\mathcal{F}_r), \{\mathcal{F}_r\}_{r \in [t, T]})\) and notice that \(\eta\) is deterministic under this probability space. Then, for any \(s \geq r\), there exist progressively measurable processes \(\psi_1\) and \(\psi_2\) such that

\[(u_s(\Omega), v_s(\Omega)) = (\psi_1(\Omega, B_{\wedge s}(\Omega)), \psi_2(\Omega, B_{\wedge s}(\Omega))), \tag{36}\]

\[= (\psi_1(s, \tilde{B}_{\wedge s}(\Omega) + B_r(\Omega)), \psi_2(s, \tilde{B}_{\wedge s}(\Omega) + B_r(\Omega))), \tag{37}\]

where \(\tilde{B}_s = B_s - B_r\) is a standard \(d\)-dimensional brownian motion. Note that the pairs \((u, v)\) are \(\mathcal{F}_r\)-adapted processes, then we have the following restriction w.r.t. \(\Sigma_{[t,T]}\)

\[\Omega, \mathcal{F}, \{\mathcal{F}_r\}, \mathbb{P}(\cdot|\mathcal{F}_r)(\omega'), B, (u, v) \in \Sigma_{[t,T]}, \tag{38}\]

where \(\omega' \in \Omega'\) such that \(\Omega' \in \mathcal{F}_r\) with \(\mathbb{P}(\Omega') = 1\). Furthermore, noting Lemma 2 if we work under the probability space \((\Omega', \mathcal{F}, \mathbb{P}(\cdot|\mathcal{F}_r), \{\mathcal{F}_r\})\), then both statements in (34) and (35) hold \(\mathbb{P}\text{-almost surely. This completes the proof of Proposition 2}. \(\square\)
In what follows, we restrict our discussion w.r.t. the generator functional $g_f$ which is associated with the 
follower. Moreover, for $w = (u, v) \in U \times V$ and any $\phi(x) \in C_0^\infty(\mathbb{R}^d)$, we introduce a 
family of second-order linear operators, associated with (1), as follow

$$
L_t^{(u,v)} \phi(x) = \frac{1}{2} \text{tr} \left\{ a(t, x, w) D^2_x \phi(x) \right\} + m(t, x, w) D_x \phi(x), \quad t \in [0, T],
$$

(39)

where $a(t, x, w) = \sigma(t, x, w) \sigma^T(t, x, w)$, $D_x$ and $D^2_x$ (with $D^2_x = (\partial^2 / \partial x_i \partial x_j)$) 
are the gradient and the Hessian (w.r.t. the variable $x$), respectively. Further, on the space $C_b^{1,2}([t, T] \times \mathbb{R}^d)$, 
for any $(t, x) \in [0, T] \times \mathbb{R}^d$, we consider the following Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE)

$$
\frac{\partial \varphi(t, x)}{\partial t} + \inf_{u \in V} \left\{ L_t^{(u,v)} \varphi(t, x) + g_f(t, \varphi(t, x), D_x \varphi(t, x) \cdot \sigma(t, x, (u,v))) \right\} = 0
$$

(40)

with the following boundary condition

$$
\varphi(T, x) = \Psi_f(T, x), \quad x \in \mathbb{R}^d.
$$

(41)

**Remark 5** Here, we remark that the above equation in (40) together with (41), is associated with 
the risk-averse decision problem for the 
follower, restricted to $\Sigma_{[t, T]}$ (cf. Definition 2). Moreover, it represents a generalized HJB 
equation with additional terms $g_f$. Note that the problem of FBSDEs and reflected BSDEs (cf. equations (14), 
(20) and (21)), and the solvability of the related HJB partial differential equations (PDEs) 
have been well studied in literature (e.g., see [1], [9], [13], [15], [16], [18], [19] and [20]).

Next, we recall the definition of viscosity solutions for (40) together with (41) (e.g., see [5], [10] or [14] 
for additional discussions on the notion of viscosity solutions).

**Definition 3** The functions $\varphi: [0, T] \times \mathbb{R}^d$ is viscosity solutions for (40) together with the 
boundary conditions in (41), if the following conditions hold

(i) for every $\psi \in C_b^{1,2}([0, T], \times \mathbb{R}^d)$ such that $\varphi \geq \psi$ on $[0, T] \times \mathbb{R}^d$,

$$
\sup_{(t,x)} \left\{ \varphi(t, x) - \psi(t, x) \right\} = 0,
$$

(42)

and for $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ such that $\psi(t_0, x_0) = \varphi(t_0, x_0)$ (i.e., a local maximum 
at $(t_0, x_0)$), then we have

$$
\frac{\partial \psi(t_0, x_0)}{\partial t} + \inf_{u \in V} \left\{ L_t^{(u,v)} \psi(t_0, x_0) + g_f(t_0, x_0, \psi(t_0, x_0), D_x \psi(t_0, x_0) \cdot \sigma(t_0, x_0, (u,v))) \right\} \geq 0
$$

(43)

(ii) for every $\psi \in C_b^{1,2}([0, T], \times \mathbb{R}^d)$ such that $\psi \leq \varphi$ on $[0, T] \times \mathbb{R}^d$,

$$
\inf_{(t,x)} \left\{ \varphi(t, x) - \psi(t, x) \right\} = 0,
$$

(44)
and for \((t_0, x_0) \in [0, T] \times \mathbb{R}^d\) such that \(\psi(t_0, x_0) = \varphi(t_0, x_0)\) (i.e., a local minimum at \((t_0, x_0)\)), then we have
\[
\frac{\partial \psi(t_0, x_0)}{\partial t} + \inf_{v \in V} \left\{ c_t^{(u,v)}(t_0, x_0) + g_f(t_0, x_0, \psi(t_0, x_0), D_x \psi(t_0, x_0) \cdot \sigma(t_0, x_0, (u, v)) \right\} \leq 0.
\] (45)

3.1 On the risk-averse optimality condition for the follower

Suppose that, for a given leader’s risk-averse decision \(\hat{u} \in \mathcal{U}_{[t,T]}\), the decision for the follower is an optimal solution to \((13)\). Then, with restriction to \(\Sigma_{[t,T]}\), such a solution is characterized by the following propositions (i.e., Propositions \(3\) and \(4\)).

**Proposition 3** Suppose that the generator functional \(g\) satisfies Assumption \(7\). Further, let the statements in \((6), (7)\) and Assumption \(4\) along with \((15)\) hold true. Let \(\hat{u} \in \mathcal{U}_{[t,T]}\) be given, then the risk-value function \(v (t, x)\) of the follower is given by
\[
V^f_v(t, x) = \inf_{v \in V_{[t,T]} \cup \Sigma_{[t,T]}} \left\{ \int_t^T c_f(s, X^f_{s:t}, \tilde{v}_s) ds + V^f_v(r, X^f_{r:T}) \right\}
\] (46)
for any \((t, x) \in [0, T] \times \mathbb{R}^d\) and \(r \in [t,T]\), with \(w = (\hat{u}, v)\).

**Proof** Notice that \(\hat{u} \in \mathcal{U}_{[t,T]}\) is given. Then, for any \(\epsilon > 0\), there exists \(\bar{v} \in V_{[t,T]}\) such that \(V^f_v(t, x) + \epsilon \geq V^f_v(t, x)\). Further, if we apply the properties of time-consistency and translation to \(V^f_v(t, x)\), then we have
\[
V^f_v(t, x) + \epsilon \geq V^f_v(t, x)
\]
\[= \rho^q_{t', r} \left[ \int_t^{t'} c_f(s, X^f_{s:t'}, \bar{v}_s) ds + \Psi_f(X^f_{t':t}) \right]
\[= \rho^q_{t', r} \left[ \int_t^T c_f(s, X^f_{s:t'}, \bar{v}_s) ds + \rho^q_{r', T} \left[ \int_r^T c_f(s, X^f_{s:t'}, \bar{v}_s) ds + \Psi_f(X^f_{t':t}) \right] \right],
\] (47)
where \(\bar{\bar{v}} = (\hat{u}, \bar{v})\) is restricted to \(\Sigma_{[t,T]}\). Moreover, if we apply Proposition \(2\) then we have
\[
V^f_v(t, x) + \epsilon \geq \rho^q_{t', r} \left[ \int_t^T c_f(s, X^f_{s:t'}, \bar{v}_s) ds + V^f_v(r, X^f_{t':t}) \right]
\[\geq \rho^q_{t', r} \left[ \int_t^T c_f(s, X^f_{s:t'}, \bar{v}_s) ds + \inf_{v \in V_{[t',T]} \cup \Sigma_{[t',T]}} \left[ \int_t^{t'} c_f(s, X^f_{s:t'}, \bar{v}_s) ds + V^f_v(r, X^f_{t':t}) \right] \right].
\] (48)
Since $\epsilon$ is arbitrary, we obtain (46). On the other hand, to show the reverse inequality $\leq$, let $v$ (which is restricted to $\Sigma_{[t,T]}$) be an $\epsilon$-optimal solution, for a fixed $\epsilon > 0$, to the the problem on the right-hand side of (46). That is,

$$
\inf_{v \in V_{[t,T]} | \Sigma_{[t,T]}} \rho_{t,T}^\rho \left[ \int_t^T c_f(s, X^{t,x;\bar{w}}_s, \bar{v}_s)ds + V^\rho_f(r, X^{t,x;\bar{w}}_T) \right] + \epsilon
\geq \rho_{t,T}^\rho \left[ \int_t^T c_f(s, X^{t,x;\bar{w}}_s, \bar{v}_s)ds + V^\rho_f(r, X^{t,x;\bar{w}}_T) \right].

(49)
$$

Then, for every $y \in \mathbb{R}^d$, let $\tilde{v}(y) \in V_{[t,T]}$ be such that $V^\rho_f(r, y) + \epsilon \geq V^\tilde{v}(y)(t, x)$ and restricted to $\Sigma_{[t,T]}$. Due to the measurable selection theorem, we may assume that the function $y \to \tilde{v}(y)$ is Borel measurable. Further, suppose that a control function $v^0$ is defined as following

$$
v^0 = \begin{cases} 
\tilde{v}_s, & s \in [t, T) \\
\tilde{v}_T(X^{t,x;\tilde{w}}_T), & s \in [t, T].
\end{cases}
$$

(50)

Note that, from the above definition, $v^0$ is restricted to $\Sigma_{[t,T]}$. Then, using the properties of the monotonicity, translation and time-consistency, we obtain the following

$$
\rho_{t,T}^\rho \left[ \int_t^T c_f(s, X^{t,x;\bar{w}}_s, \bar{v}_s)ds + V^\rho_f(r, X^{t,x;\bar{w}}_T) \right]
\geq \rho_{t,T}^\rho \left[ \int_t^T c_f(s, X^{t,x;\bar{w}}_s, \bar{v}_s)ds + V^\rho_f(X^{t,x;\bar{w}}_T)(r, X^{t,x;\bar{w}}_T) - \epsilon \right], \text{ with } \bar{w} = (\tilde{u}, \tilde{v})
\geq \rho_{t,T}^\rho \left[ \int_t^T c_f(s, X^{t,x;w^0}_s, v^0_s)ds + \Psi_f(X^{t,x;w^0}_T) \right] - \epsilon, \text{ with } w^0 = (\tilde{u}, v^0)
\geq V^{v^0}_f(t, x) - \epsilon.

(51)
$$

If we combine the inequalities from (49) and (51), then we have

$$
\inf_{v \in V_{[t,T]} | \Sigma_{[t,T]}} \rho_{t,T}^\rho \left[ \int_t^T c_f(s, X^{t,x;\bar{w}}_s, \bar{v}_s)ds + V^\rho_f(r, X^{t,x;\bar{w}}_T) \right] + \epsilon \geq V^{v^0}_f(t, x) - \epsilon
\geq V^{v^0}_f(t, x) - \epsilon.
$$

(52)

Note that, since $\epsilon$ is arbitrary, we obtain (46). This completes the proof of Proposition 3. □

Then, we have the following results (i.e., Propositions 4 and 5) that characterize the measurable mapping $F$ in (13).

**Proposition 4** Suppose that the generator functional $g$ satisfies Assumption 7. Let $V$ be a compact set in $\mathbb{R}^d$ and $\hat{u}, \hat{v} \in U_{[t,T]}$ be given. Then, the risk-value function $V^\rho_f(\cdot, \cdot)$ is the viscosity solution of (10) with boundary condition $\Psi_f(T, x)$ for $x \in \mathbb{R}^d$ and with $w = (\hat{u}, \hat{v})$.

**Proof** Suppose that $\varphi \in C^{1,2}_b([0, T] \times \mathbb{R}^d)$ and assume that $\varphi \geq V^\rho_f$ on $[0, T] \times \mathbb{R}^d$ and $\max_{(t,x)}[V^\rho_f(t, x) - \varphi(t, x)] = 0$. We consider a point $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ so that
$\varphi(t_0, x_0) = V^*_f(t_0, x_0)$ (i.e., a local maximum at $(t_0, x_0)$). Further, for a small $\delta t > 0$, we consider a constant control $v_s = \alpha$ for $s \in [t_0, t_0 + \delta t]$. Then, from (46), we have

$$
\varphi(t_0, x_0) = V^*_f(t_0, x_0)
$$

$$
\leq \rho^g_{t_0, t_0 + \delta t} \left[ \int_{t_0}^{t_0 + \delta t} c_f(s, X_s^{t_0, x_0}, \alpha) ds + V^*_f(t_0 + \delta t, X_{t_0 + \delta t}^{t_0, x_0}) \right]
$$

$$
\leq \rho^g_{t_0, t_0 + \delta t} \left[ \int_{t_0}^{t_0 + \delta t} c_f(s, X_s^{t_0, x_0}, \alpha) ds + \varphi(t_0 + \delta t, X_{t_0 + \delta t}^{t_0, x_0}) \right], \text{ with } w = (\hat{u}, \alpha).
$$

(53)

Using the translation property of $\rho^g_{t_0, t_0 + \delta t}[\cdot]$, we obtain the following inequality

$$
\rho^g_{t_0, t_0 + \delta t} \left[ \int_{t_0}^{t_0 + \delta t} c_f(s, X_s^{t_0, x_0}, \alpha) ds + \varphi(t_0 + \delta t, X_{t_0 + \delta t}^{t_0, x_0}) - \varphi(t_0, x_0) \right] \geq 0.
$$

(54)

Notice that $\varphi \in C^{1,2}_b([0, T] \times \mathbb{R}^d)$, then, using the Itô formula, we can evaluate the difference between $\varphi(t_0 + \delta t, X_{t_0 + \delta t}^{t_0, x_0})$ and $\varphi(t_0, x_0)$ as follow

$$
\varphi(t_0 + \delta t, X_{t_0 + \delta t}^{t_0, x_0}) - \varphi(t_0, x_0) = \int_{t_0}^{t_0 + \delta t} \left[ \frac{\partial}{\partial t} \varphi(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha)) + L^t(\hat{u}_s, \alpha) \varphi(s, X_s^{t_0, x_0}) \right] ds
$$

$$
+ \int_{t_0}^{t_0 + \delta t} D_x \varphi(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha)) \cdot \sigma(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha)) dB_s.
$$

(55)

Moreover, if we substitute the above equation into (53), then we obtain

$$
\rho^g_{t_0, t_0 + \delta t} \left[ \int_{t_0}^{t_0 + \delta t} \left[ c_f(s, X_s^{t_0, x_0}, \alpha) + \frac{\partial}{\partial t} \varphi(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha)) + L^t(\hat{u}_s, \alpha) \varphi(s, X_s^{t_0, x_0}) \right] ds
$$

$$
+ \int_{t_0}^{t_0 + \delta t} D_x \varphi(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha)) \cdot \sigma(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha)) dB_s \right] \geq 0,
$$

(56)

which amounts to solving the following BSDE

$$
\hat{Y}_{t_0}^{t_0, x_0} = \int_{t_0}^{t_0 + \delta t} \left[ \frac{\partial}{\partial t} \varphi(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha)) + L^t(\hat{u}_s, \alpha) \varphi(s, X_s^{t_0, x_0}) \right] ds
$$

$$
+ \int_{t_0}^{t_0 + \delta t} D_x \varphi(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha)) \cdot \sigma(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha)) dB_s
$$

$$
+ \int_{t_0}^{t_0 + \delta t} g_f(s, X_s^{t_0, x_0}, \varphi(s, X_s^{t_0, x_0}), D_x \varphi(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha)) \cdot \sigma(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha))) ds
$$

$$
- \int_{t_0}^{t_0 + \delta t} \hat{Z}_{s}^{t_0, x_0} dB_s.
$$

(57)

From Lemma 2, the above BSDE admits unique solutions, i.e.,

$$
\hat{Z}_{s}^{t_0, x_0} = D_x \varphi(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha)) \cdot \sigma(s, X_s^{t_0, x_0}, (\hat{u}_s, \alpha)), \quad t_0 \leq s \leq t_0 + \delta t
$$
Further, if we substitute the above results in (56), we obtain

\[
Y_{t_0}^{x_0, w} = \int_{t_0}^{t_0 + \delta t} \left[ \frac{\partial}{\partial t} \varphi(s, X_s^{t_0, x_0}), W_t^{(\bar{u}, \alpha)} \varphi(s, X_s^{t_0, x_0}), \right. \\
\left. + g_f(s, X_s^{t_0, x_0}), \varphi(s, X_s^{t_0, x_0}), D_x \varphi(s, X_s^{t_0, x_0}), \sigma(s, X_s^{t_0, x_0}, (\bar{u}_s, \alpha)) \right] ds.
\]

Further, if we substitute the above results in (56), we obtain

\[
\int_{t_0}^{t_0 + \delta t} \left[ \frac{\partial}{\partial t} \varphi(s, X_s^{t_0, x_0}), W_t^{(\bar{u}, \alpha)} \varphi(s, X_s^{t_0, x_0}), \right. \\
\left. + g_f(s, X_s^{t_0, x_0}), \varphi(s, X_s^{t_0, x_0}), D_x \varphi(s, X_s^{t_0, x_0}), \sigma(s, X_s^{t_0, x_0}, (\bar{u}_s, \alpha)) \right] ds \geq 0.
\]

Then, dividing the above equation by \( \delta t \) and letting \( \delta t \to 0 \), we obtain

\[
\frac{\partial}{\partial t} \varphi(t_0, x_0) + L_t^{(\bar{u}, \alpha)} \varphi(t_0, x_0) + g_f(t_0, x_0, \varphi(t_0, x_0), D_x \varphi(t_0, x_0), \sigma(t_0, x_0, (\bar{u}_t, \alpha))) \geq 0.
\]

Note that, since \( \alpha \in V \) is arbitrary, we can rewrite the above condition as follow

\[
\frac{\partial}{\partial t} \varphi(t_0, x_0) + \min_{\alpha \in V} \left\{ L_t^{(\bar{u}, \alpha)} \varphi(t_0, x_0), \right. \\
\left. + g_f(t_0, x_0, \varphi(t_0, x_0), D_x \varphi(t_0, x_0), \sigma(t_0, x_0, (\bar{u}_t, \alpha))) \right\} \geq 0.
\]

which attains its minimum in \( V \) (which is a compact set in \( \mathbb{R}^d \)). Thus, \( V_f^V (\cdot, \cdot) \) is a viscosity subsolution of (53), with boundary condition \( \varphi(T, x) = \Psi_f(T, x) \).

On the other hand, suppose that \( \varphi \in C^{1,2}_b([0, T] \times \mathbb{R}^d) \) and assume that \( \varphi \leq V^f \) on \([0, T] \times \mathbb{R}^d \) and \( \min_{(t, x)} [V^f(t, x) - \varphi(t, x)] = 0 \). Then, we consider a point \((t_0, x_0) \in [0, T] \times \mathbb{R}^d \) so that \( \varphi(t_0, x_0) = V_f^V (t_0, x_0) \) (i.e., a local minimum at \((t_0, x_0)\)). Further, for a small \( \delta t > 0 \), let \( \tilde{\nu}_s \), which is restricted to \( \Sigma_{t_0, t_0 + \delta t} \), be an \( \epsilon \delta t \)-optimal control for (56) at \((t_0, x_0)\). Then, proceeding in this way as (58), with \( w = (\bar{u}_s, \tilde{\nu}_s) \), we obtain the following

\[
\int_{t_0}^{t_0 + \delta t} \left[ \frac{\partial}{\partial t} \varphi(s, X_s^{t_0, x_0}), W_t^{(\bar{u}, \alpha)} \varphi(s, X_s^{t_0, x_0}), \right. \\
\left. + g_f(s, X_s^{t_0, x_0}), \varphi(s, X_s^{t_0, x_0}), D_x \varphi(s, X_s^{t_0, x_0}), \sigma(s, X_s^{t_0, x_0}, (\bar{u}_s, \tilde{\nu}_s)) \right] ds \leq \epsilon \delta t.
\]

As a result of this, we also obtain the following

\[
\int_{t_0}^{t_0 + \delta t} \min_{\alpha \in V} \left\{ \frac{\partial}{\partial t} \varphi(s, X_s^{t_0, x_0}), W_t^{(\bar{u}, \alpha)} \varphi(s, X_s^{t_0, x_0}), \right. \\
\left. + g_f(s, X_s^{t_0, x_0}), \varphi(s, X_s^{t_0, x_0}), D_x \varphi(s, X_s^{t_0, x_0}), \sigma(s, X_s^{t_0, x_0}, (\bar{u}_s, \alpha)) \right\} ds \leq \epsilon \delta t.
\]
Note that the mapping
\[
(s, x, \alpha) \rightarrow \left[ \frac{\partial}{\partial t} \varphi(t, x) + \mathcal{L}^{(\hat{u}_t, \alpha)}_t \varphi(t, x) + g_f(t, x, \varphi(t, x), D_x \varphi(t, x) \cdot \sigma(t, x, (\hat{u}_t, \alpha))) \right]
\]
is continuous and, since \( V \) is compact, then \( s \rightarrow X^{s, x_0, w}_t \) is also continuous. As a result, the expression under the integral in (61) is continuous. Further, if we divide both sides of (61) by \( \delta t \) and letting \( \delta t \to 0 \), then we obtain the following
\[
\frac{\partial}{\partial t} \varphi(t_0, x_0) + \min_{\alpha \in V} \mathcal{L}^{(\hat{u}_t, \alpha)}_t \varphi(t_0, x_0) + g_f(t_0, x_0, \varphi(t_0, x_0), D_x \varphi(t_0, x_0) \cdot \sigma(t_0, x_0, (\hat{u}_t, \alpha))) \leq \epsilon. \tag{62}
\]

Notice that, since \( \epsilon \) is arbitrary, we conclude that \( V^*_f(t, x) \) is a viscosity supersolution of (45), with boundary condition \( \varphi(T, x) = \Psi_f(T, x) \). This completes the proof of Proposition 4. \( \square \)

**Remark 6**  Note that if \( V^*_f \in C^{1,2}_{\bar{b}}([0, T] \times \mathbb{R}^d) \), then such a solution also satisfies (45) with boundary condition \( V^*_f(T, x) = \Psi_f(T, x) \). Furthermore, using the verification theorem, one can also identify \( V^*_f \) as the optimal value function.

**Proposition 5**  Suppose that Proposition 4 holds and let \( \varphi \in C^{1,2}_{\bar{b}}([0, T] \times \mathbb{R}^d) \) satisfy (45) with \( \varphi(T, x) = \Psi_f(T, x) \) for \( x \in \mathbb{R}^d \). Then, \( \varphi(t, x) \leq V^*_f(t, x) \) for any control \( v \in \mathcal{V}_{[t, T]} \) with restriction to \( \Sigma_{[t, T]} \) and for all \((t, x) \in [0, T] \times \mathbb{R}^d \). Furthermore, if an admissible control process \( \hat{v} \in \mathcal{V}_{[t, T]} \) exists, for almost all \((s, \omega) \in [0, T] \times \Omega\), together with the corresponding solution \( X^{s, x_0, w}_t \), with \( \hat{w}_s = (u_s, \hat{v}_s) \), and satisfies
\[
\dot{\hat{v}}_s \in \arg \inf_{v \in \mathcal{V}_{[t, T]} \mid \Sigma_{[t, T]}} \left\{ \mathcal{L}^{(u_s, v)}_s \varphi(s, X^{s, x_0, w}_s) + g_f(s, X^{s, x_0, w}_s, \varphi(s, X^{s, x_0, w}_s), D_x \varphi(s, X^{s, x_0, w}_s) \cdot \sigma(s, X^{s, x_0, w}_s, (\hat{u}_s, v_s))) \right\}
\]
\[
\hat{v}_s \in \mathbb{F}(\hat{u}) \text{ with } \mathbb{F}: \hat{u} \in \mathcal{U}_{[t, T]} \rightarrow \hat{v} \in \mathcal{V}_{[t, T]}
\]

Then, \( \varphi(t, x) = V^*_f(t, x) \) for all \((t, x) \in [0, T] \times \mathbb{R}^d \).

**Proof**  Assume that \((t, x) \in [0, T] \times \mathbb{R}^d \) is fixed. For any \( v \in \mathcal{V}_{[t, T]} \), restricted to \( \Sigma_{[t, T]} \), we consider a process \( \kappa(s, X^{s, x_0, w}_t) \), with \( w = (\hat{u}, v) \), for \( s \in [t, T] \). Then, using Itô integral formula, we can evaluate the difference between \( \kappa(T, X^{T, x_0, w}_T) \) and \( \kappa(t, x) \) as follow:
\[
\kappa(T, X^{T, x_0, w}_T) - \kappa(t, x) = \int_t^T \left[ \frac{\partial}{\partial t} \kappa(s, X^{s, x_0, w}_s) + \mathcal{L}^{(\hat{u}_s, v_s)}_s \kappa(s, X^{s, x_0, w}_s) \right] ds + \int_t^T D_x \kappa(s, X^{s, x_0, w}_s) \cdot \sigma(s, X^{s, x_0, w}_s, (\hat{u}_s, v_s)) dB_s. \tag{64}
\]

\* Notice that \( \kappa(t, x) \in C^{1,2}_{\bar{b}}([0, T] \times \mathbb{R}^d) \).
Using (60), we further obtain the following
\[ \frac{\partial}{\partial t} \kappa(s, X^{t,x,w}_s) + L_t(\hat{u}_s, v_s) \kappa(s, X^{t,x,w}_s) 
+ g_f(s, X^{t,x,w}_s, \kappa(s, X^{t,x,w}_s), D_x \kappa(s, X^{t,x,w}_s) \cdot \sigma(s, X^{t,x,w}_s, (\hat{u}_s, v_s))) \geq 0. \tag{65} \]

Furthermore, if we combine (64) and (65), then we obtain
\[ \kappa(t, x) \leq \Psi_f(T, X^{t,x,w}_T) 
+ \int_t^T g_f(s, X^{t,x,w}_s, \kappa(s, X^{t,x,w}_s), D_x \kappa(s, X^{t,x,w}_s) \cdot \sigma(s, X^{t,x,w}_s, (\hat{u}_s, v_s))) ds 
- \int_t^T D_x \kappa(s, X^{t,x,w}_s) \cdot \sigma(s, X^{t,x,w}_s, (\hat{u}_s, v_s)) dB_s. \tag{66} \]

Define \( \tilde{Z}^{t,x,w}_s = D_s \kappa(s, X^{t,x,w}_s) \cdot \sigma(s, X^{t,x,w}_s, (\hat{u}_s, v_s)), \) for \( s \in [t, T], \) then \( \kappa(t, x) \leq V_f(t, x). \)

Moreover, if there exists at least one \( \hat{v} \) satisfying (63). Then, for \( v = \hat{v}, \) the inequality in (65) becomes an equality (i.e., \( \kappa(t, x) = V_f(t, x) \)). Note that the corresponding pathwise solution \( X^{t,x,w}_t \), with \( \hat{v} = (\hat{a}, \hat{v}) \) and \( \hat{v} = F(\hat{a}), \) is progressively measurable, since \( \hat{v} \in V_{[t,T]} \) is restricted to \( \Sigma_{[t,T]} \). This completes the proof of Proposition 5. \( \square \)

### 3.2 On the stochastic controllability

As we have already mentioned in the previous sections (i.e., Section 2 and Subsection 3.1), for a given leader’s risk-averse decision \( \hat{u} \in U_{[t,T]}, \) the risk-averse optimization in (43) (or equation (21)) admits a unique solution \( \hat{v} = F(\hat{u}), \) which is restricted to \( \Sigma_{[t,T]} \). However, the situation is more involved for the risk-averse optimization in (43) (or equation (20)). Notice that it is not even clear that, for every \( \xi_{\text{Target}} \in L^2(\Omega, F_T, \mathbb{P}, \mathbb{R}), \) there exist decision processes \( u \in U_{[t,T]} \) and \( A^{t,x,w}_t \in S^2(t, T; \mathbb{R}), \) with \( w = (u, F(u)), \) such that
\[ \tilde{Y}^{t,x,w}_T \geq h(T, X^{t,x,w}_T), \tag{67} \]
\[ \int_t^T (\tilde{Y}^{t,x,w}_s - h(s, X^{t,x,w}_s)) dA^{t,x,w}_s = 0, \quad \text{and} \quad L_T = h(T, X^{t,x,w}_T), \tag{68} \]
for all \( t \in [0, T]. \)

Moreover, verifying the above conditions amounted to solving the stochastic controllability type problem, which is indeed useful to describe the set of all acceptable risk-exposures, when \( t = 0, \) vis-à-vis \( \xi_{\text{Target}} \in L^2(\Omega, F_T, \mathbb{P}, \mathbb{R}), \) i.e.,
\[ A_0 = \left\{ \rho_{0,T} \mathbb{E}^{\xi_{\text{Target}}} \geq L_T \right\} = \left\{ \int_0^T (\tilde{Y}^{0,x,w}_s - h(s, X^{0,x,w}_s)) dA^{0,x,w}_s = 0, \right\} \tag{69} \]
\[ L_T = h(T, X^{0,x,w}_T), \quad t \in [0, T]. \]
In the following subsection, we provide additional results that provide conditions under which the problem in (27) makes sense, if the follower is involved not only in minimizing his own accumulated risk-cost (in response to the decision of the leader) but also in minimizing that of the leader.

### 3.3 On the risk-averse optimality condition for the leader

In this subsection, we provide conditions under which the leader chooses its optimality risk-averse decision, whenever the follower responds optimally to the leader’s decision, i.e., \( v = F(u) \), with restriction to \( \Sigma_{[0,T]} \). Therefore, we suppose here that Proposition 5 holds and, further, we will establish a two-way connection between the reflected-BSDE in (20) and a probabilistic representation for the solution of related parabolic obstacle PDE problem.

Notice that, for each \( t \in [0,T] \), the natural filtration of the Brownian motion \( \{B_s - B_t, t \leq s \leq T\} \), augmented by the \( \mathbb{P} \)-null sets of \( \mathcal{F}_s \), is denoted by \( \{\mathcal{F}_t\}_{t \leq s < T} \).

Then, for each \( (t,x) \in [0,T] \times \mathbb{R}^d \), with \( w = (u,F(u)) \), there exists a unique triple \( \{\hat{Y}_t,x;w\}, \{\hat{Z}_t,x;w\}, \{A_t,x;w\} \) of \( \{\mathcal{F}_t\} \) progressively measurable processes, which solves the following reflected-BSDE:

\[
\begin{align*}
(i) & \quad \mathbb{E} \int_t^T \left( |\hat{Y}^{t,x;w}_r|^2 + |\hat{Z}^{t,x;w}_r|^2 \right) dr < \infty \\
(ii) & \quad \hat{Y}^{t,x;w}_s = \Psi_l(X^{t,x;w}_s) + \int_t^s g(t,r,X^{t,x;w}_r,\hat{Y}^{t,x;w}_r,\hat{Z}^{t,x;w}_r) dr \\
& \quad \quad \quad + A^{t,x;w}_s - A^{t,x;w}_t - \int_t^s \hat{Z}^{t,x;w}_r dB_r, \quad t < s \leq T, \quad (70) \\
(iii) & \quad \hat{Y}^{t,x;w}_t \geq h(s,X^{t,x;w}_s), \quad t < s \leq T \\
(iv) & \quad \{A^{t,x;w}_t\} \text{ is increasing and continuous, and} \\
& \quad \int_t^T (\hat{Y}^{t,x;w}_s - h(s,X^{t,x;w}_s)) dA^{t,x;w}_s = 0.
\end{align*}
\]

More precisely, we consider the following related parabolic obstacle PDE problem. Then, roughly speaking, its solution is a function \( \varphi : [0,T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) which satisfies

\[
\min \left\{ \varphi(t,x) - h(t,x), - \frac{\partial \varphi}{\partial t}(t,x) - \inf_{u \in U} \left\{ \mathcal{L}^{(u,F(u))}_t \varphi(t,x) + g(t,x,\varphi(t,x),D_s \varphi(t,x) \cdot \sigma(s,x,u,F(u))) \right\} = 0, \right. \\
\left. (t,x) \in (0,T) \times \mathbb{R}^d, \right. \\
\varphi(T,x) = \Psi_l(x), \quad x \in \mathbb{R}^d. \quad (71)
\]

Hence, we consider such a solution for (71) in the sense of viscosity. For the sake of convenience, we also provide the definition of viscosity solutions for the above parabolic obstacle PDE problem (cf. Definition 3).
Definition 4

(a) \( \varphi \in C([0, T] \times \mathbb{R}^d) \) is said to be a viscosity subsolution of (71) if \( \varphi(T, x) \leq \Psi_t(x) \), \( x \in \mathbb{R}^d \), and at any point \( (t, x) \in (0, T) \times \mathbb{R}^d \),

\[
\min\left\{ \varphi(t, x) - h(t, x), -\frac{\partial \varphi}{\partial t}(t, x) - \inf_{u \in U} \left\{ \mathcal{L}_t^{(u, F(u))} \psi(t, x) + g(t, x, \varphi(t, x), D_x \psi(t, x) \cdot \sigma(s, x, (u, F(u)))) \right\} \right\} \leq 0.
\]

In other words, at any point \( (t, x) \), where \( \varphi(t, x) > h(t, x) \)

\[
-\frac{\partial \varphi}{\partial t}(t, x) - \inf_{u \in U} \left\{ \mathcal{L}_t^{(u, F(u))} \psi(t, x) + g(t, x, \varphi(t, x), D_x \psi(t, x) \cdot \sigma(s, x, (u, F(u)))) \right\} \leq 0.
\]

(b) \( \varphi \in C([0, T] \times \mathbb{R}^d) \) is said to be a viscosity supersolution of (71) if \( \varphi(T, x) \geq \Psi_t(x) \), \( x \in \mathbb{R}^d \), and at any point \( (t, x) \in (0, T) \times \mathbb{R}^d \),

\[
\min\left\{ \varphi(t, x) - h(t, x), -\frac{\partial \varphi}{\partial t}(t, x) - \inf_{u \in U} \left\{ \mathcal{L}_t^{(u, F(u))} \psi(t, x) + g(t, x, \varphi(t, x), D_x \psi(t, x) \cdot \sigma(s, x, (u, F(u)))) \right\} \right\} \geq 0.
\]

(c) \( \varphi \in C([0, T] \times \mathbb{R}^d) \) is said to be a viscosity solution of (71) if is both a viscosity sub-
and supersolution.

Lemma 4 For \( (t, x) \in [0, T] \times \mathbb{R}^d \), define

\[
\varphi(t, x) \triangleq \hat{Y}_t^{t,x,w}.
\]

Then, \( \varphi \in C([0, T] \times \mathbb{R}^d) \) is a deterministic quantity.

Proof We define \( \hat{Y}_s^{t,x,w} \) for all \( s \in [0, T] \) by choosing \( \hat{Y}_s^{t,x,w} = \hat{Y}_t^{t,x,w} \) for \( 0 \leq s \leq t \). It is suffices to show that whenever \( (t_n, x_n) \to (t, x) \)

\[
E\left\{ \sup_{0 \leq s \leq T} \left| \hat{Y}_{s_n}^{t_n,x_n,w} - \hat{Y}_s^{t,x,w} \right|^2 \right\} \to 0.
\]

Indeed, this will show that

\[
(s, t, x) \to \hat{Y}_s^{t,x,w}
\]

is mean-square continuous, and so is

\[
(t, x) \to \hat{Y}_t^{t,x,w}.
\]

But \( \hat{Y}_t^{t,x,w} \) is deterministic, since it is \( \mathcal{F}_t \)-measurable.
Furthermore, note that (72) is a consequence of Proposition 9 (see the Appendix section) and the following convergences as \( n \to \infty \):

\[
\mathbb{E} \left\{ \left| \psi_t(X^l_{T,x,w}) - \psi_t(X^l_{T,x,n,w}) \right|^2 \right\} \to 0
\]
\[
\mathbb{E} \left\{ \sup_{0 \leq s \leq T} \left| h(s, X^l_{s,x,w}) - h(s, X^l_{s,x,n,w}) \right|^2 \right\} \to 0
\]
\[
\mathbb{E} \left\{ \int_0^T \left| 1_{[t,T]} g_t(s, X^l_{s,x,w}, Y^l_{s,x,w}, Z^l_{s,x,w}) \right|^2 \, ds \right\} \to 0,
\]

that follow from Assumptions 1 and 3, and the growth conditions of \( g_t, \psi_t \) and \( h \). This completes the proof of Lemma 4. \( \square \)

**Proposition 6** Suppose that Lemma 4 holds, then \( \varphi \) is a viscosity solution of the parabolic obstacle PDE in (71).

**Proof** In order to prove the above proposition, we will use an approximation for the reflected-BSDE of (20) by penalizing. For each \( (t, x) \in [0, T] \times \mathbb{R}^d, n \in \mathbb{N} \), let \( \left\{ (Y^l_{n,s,x,w}, Z^l_{n,s,x,w}), t \leq s \leq T \right\} \) denote the solution of the following approximated reflected-BSDE

\[
\begin{align*}
\dot{Y}^l_{n,s,x,w} &= \psi_t(X^l_{s,x,w}) + \int_t^s g_t(r, X^l_{r,x,w}, Y^l_{r,x,w}, Z^l_{r,x,w}) \, dr \\
&\quad + n \int_t^s \left( Y^l_{n,r,x,w} - h(r, X^l_{r,x,w}) \right) \, dr - \int_t^s Z^l_{n,r,x,w} \, dB_r, \quad t < s \leq T.
\end{align*}
\]

Then, from (23), it is clear that

\[
\varphi_n(t, x) \equiv Y^l_{n,t,x,w}, \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^d,
\]

is also the viscosity solution of the following parabolic PDE

\[
\frac{\partial \varphi_n}{\partial t}(t, x) + \inf_{u \in U} \left\{ L^u(F(u)) \varphi_n(t, x) \right\} + \dot{g}(t, x, \varphi_n(t, x), D_x \varphi_n(t, x) \cdot \sigma(t, x, (u, F(u)))) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}
\]

\[
\varphi_n(T, x) = \psi_t(x), \quad x \in \mathbb{R}^d,
\]

where

\[
\dot{g}(t, x, \varphi(t, x), D_x \varphi(t, x) \cdot \sigma(t, x, (u, F(u)))) = g_t(t, x, \varphi(t, x), D_x \varphi(t, x) \cdot \sigma(t, x, (u, F(u)))) - (\varphi(t, x) - h(t, x))^-.
\]

Notice that, for each \( t \in [0, T], x \in \mathbb{R}^d \), we have

\[
\varphi_n(t, x) \uparrow \varphi(t, x) \quad \text{as} \quad n \to \infty.
\]

Since \( \varphi_n \) and \( \varphi \) are continuous, it follows from Dini’s theorem that the above convergence is uniform on compacts.
Next, we show that \( \varphi \) is a subsolution of (71). Let \((t, x)\) be a point at which \( \varphi(t, x) > h(t, x) \), and let

From Lemma 6.1 in [5], there exists sequences

\[ n_j \to +\infty \quad \text{and} \quad (t_j, x_j) \to (t, x) \]

such that

\[ \left( \frac{\partial \psi_j}{\partial t}, D_x \psi_j, D_x^2 \psi_j \right) \to \left( \frac{\partial \psi}{\partial t}, D_x \psi, D_x^2 \psi \right) \]

but for any \( j \)

\[ -\frac{\partial \psi_j}{\partial t}(t_j, x_j) - \inf_{u \in U} \left\{ L^{(u)}_{i,F(u)} \psi_j(t_j, x_j) \right. \]

\[ \left. - g_i(t_j, x_j, \varphi_{n_j}(t_j, x_j), D_x \psi_j(t_j, x_j) \cdot \sigma(t_j, x_j, (u, F(u)))) \right\} \]

\[ - n_j (\varphi_{n_j}(t_j, x_j) - h(t_j, x_j))^{\_} \leq 0. \]

From the assumption that \( \varphi(t, x) > h(t, x) \) and the uniform convergence of \( \varphi_n \), it follows that for \( j \) large enough \( \varphi_{n_j}(t_j, x_j) > h(t_j, x_j) \); hence taking the limit as \( n_j \to +\infty \) in the above inequality yields

\[ -\frac{\partial \psi}{\partial t}(t, x) - \inf_{u \in U} \left\{ L^{(u)}_{i,F(u)} \psi(t, x) \right. \]

\[ \left. - g_i(t, x, \varphi(t, x), D_x \psi(t, x) \cdot \sigma(t, x, (u, F(u)))) \right\} \leq 0 \]

and we have proved that \( \varphi \) is a subsolution of (71).

Then, we conclude the proof by showing that \( \varphi \) is a supersolution of (71). Let \((t, x)\) be an arbitrary point in \([0, T] \times \mathbb{R}^d \) and \( T \). We already know that \( \varphi(t, x) \geq h(t, x) \). By the same argument as above, there exist sequences:

\[ n_j \to +\infty \quad \text{and} \quad (t_j, x_j) \to (t, x) \]

such that

\[ \left( \frac{\partial \psi_j}{\partial t}, D_x \psi_j, D_x^2 \psi_j \right) \to \left( \frac{\partial \psi}{\partial t}, D_x \psi, D_x^2 \psi \right) \]

but for any \( j \)

\[ -\frac{\partial \psi_j}{\partial t}(t_j, x_j) - \inf_{u \in U} \left\{ L^{(u)}_{i,F(u)} \psi_j(t_j, x_j) \right. \]

\[ \left. - g_i(t_j, x_j, \varphi_{n_j}(t_j, x_j), D_x \psi_j(t_j, x_j) \cdot \sigma(t_j, x_j, (u, F(u)))) \right\} \]

\[ - n_j (\varphi_{n_j}(t_j, x_j) - h(t_j, x_j))^{\_} \geq 0. \]

Hence, we have

\[ -\frac{\partial \psi}{\partial t}(t, x) - \inf_{u \in U} \left\{ L^{(u)}_{i,F(u)} \psi(t, x) \right. \]

\[ \left. - g_i(t, x, \varphi(t, x), D_x \psi(t, x) \cdot \sigma(t, x, (u, F(u)))) \right\} \]
and taking the limit as \( n_j \to +\infty \), then we conclude that

\[
\begin{align*}
- \frac{\partial \psi}{\partial t}(t, x) &- \inf_{u \in U} \left\{ L^2(u, F(u)) \psi(t, x) \\
&- g_2(t, x, \varphi(t, x), D_x \psi(t, x) \cdot \sigma(t, x, (u, F(u)))) \right\} \geq 0.
\end{align*}
\]

This completes the proof of Proposition 7. \( \square \)

We conclude this subsection with the following proposition, which provides a condition for the leader to have an optimal risk-averse decision.

**Proposition 7** Let \( \varphi \in C([0, T] \times \mathbb{R}^d) \) be a viscosity solution for the parabolic obstacle PDE in (71), with boundary condition \( \varphi(T, x) = \Psi(T, x) \), for \( x \in \mathbb{R}^d \). Then, \( \varphi(t, x) \leq V^*_{u}(t, x) \) for some \( u \in U_{[t, T]} \), with restriction to \( \Sigma_{[t, T]} \), and for all \( (t, x) \in [0, T] \times \mathbb{R}^d \).

Furthermore, if an admissible decision process \( u^* \in U_{[t, T]} \) exists, for almost all \( (s, \Omega) \in [0, T] \times \Omega \), together with the corresponding solution \( X^*_{s, x, w} \), with \( w^* = (u^*, F(u^*)) \), and satisfies

\[
\begin{align*}
u^*_{s} \in \arg \inf_{u \in U_{[s, T]} \mid \Sigma_{[s, T]}} & \left\{ L^2(u, F(u)) \varphi(s, X^*_{s, x, w}) \\
&+ g_1(s, X^*_{s, x, w}, \varphi(s, X^*_{s, x, w}), D_x \varphi(s, X^*_{s, x, w}) \cdot \sigma(s, X^*_{s, x, w}, (u, F(u)))) \right\},
\end{align*}
\]

Then, \( \varphi(t, x) = V^*_{u}(t, x) \) for all \( (t, x) \in [0, T] \times \mathbb{R}^d \).

**Proof** The proof is similar to that of Proposition 5 except that we require a unique solution set \( \{ X^{t, x, w}, (Y^{t, x, w}, Z^{t, x, w}, A^{t, x, w}, \{ Y_{s, x, w}, Z^{t, x, w} \}) \} \) for the FBSDEs in (14) and the reflected-BSDE in (20) on \( (\Omega, F, P, F^1) \) for every initial condition \( (t, x) \in [0, T] \times \mathbb{R}^d \).

4 Further remarks

In this section, we further comment on the implication of our result in assessing the influence of the leader’s decisions on the risk-averseness of the follower in relation to the direction of leader-follower information flow. Note that the statement of Proposition 4 is implicitly accounted in Proposition 7. That is, for \( s \in [t, T] \), the risk-averseness of the follower, with restriction to \( \Sigma_{[t, T]} \),

\[
\begin{align*}
u^*_{s} \in \arg \inf_{u \in V_{[s, T]} \mid \Sigma_{[s, T]}} & \left\{ L^2_{s} (F^{-1}(v), v) \varphi(s, X^*_{s, x, w}) \\
&+ g_1(s, X^*_{s, x, w}, \varphi(s, X^*_{s, x, w}), D_x \varphi(s, X^*_{s, x, w}) \cdot \sigma(s, X^*_{s, x, w}, (F^{-1}(v), v))) \right\},
\end{align*}
\]

with \( w^* = (F^{-1}(v), v) \) and \( F^{-1} : v \in V_{[t, T]} \rightsquigarrow u \in U_{[t, T]} \), is a subproblem in (76).
On the other hand, the risk-averse decision of the leader

\[
u^*_s \in \arg \inf_{u \in U(s, \tau)} \left\{ \mathcal{L}_s(u, F(u)) \varphi(s, X^s_{t, x}) + g_l(s, X^s_{t, x}, \varphi(s, X^s_{t, x}), D_x \varphi(s, X^s_{t, x}), \sigma(s, X^s_{t, x}, (u_s, F(u_s)))) \right\},
\]

that is implicitly conditioned by the leader’s decision \( u \) and that of the follower’s decision \( v = F(u) \). As a result of this, the follower is involved not only in minimizing his own accumulated risk-cost (in response to the risk-averse decision of the leader) but also in minimizing that of the leader’s accumulated risk-cost. Hence, such an inherent interaction, due to the nature of the problem, constitutes a constrained information flow between the leader and that of the follower, in which the follower is required to respond optimally, in the sense of best-response correspondence to the risk-averse decision of the leader.

**Appendix**

In this section, we provide additional results (whose proofs are adaptations of [9]) that are related to the solutions of the reflected-BSDE in (70).

**Proposition 8** For \((t, x) \in [0, T] \times \mathbb{R}^d\) let \((\hat{Y}_{t, x}^s, \hat{Z}_{t, x}^s, \hat{A}_{t, x}^s)_{t \leq s \leq T}\) be the solution of the reflected BSDE satisfying (iii) and (iv) in (70). Then, for each \( t \in [0, T] \),

\[
\begin{aligned}
\hat{Y}_{t, x}^s &= \text{ess sup}_{\tau \in \mathcal{T}_t} \mathbb{E} \left\{ L_{\tau} 1_{\tau < T} + \xi_{\text{Target}}^s 1_{\tau = T} \right\} \\
&+ \int_t^T g_l(s, X^s_{t, x}, \varphi(s, X^s_{t, x}), D_x \varphi(s, X^s_{t, x}), \sigma(s, X^s_{t, x}, (u_s, F(u_s)))) ds \bigg| \mathcal{F}_t \right\},
\end{aligned}
\]

where \( \mathcal{T} \) is the set of all stopping times dominated by \( T \), and

\( \mathcal{T}_t = \{ \tau \in \mathcal{T} | t \leq \tau \leq T \} \).

Then, we have the following proposition, whose proof depends on the above proposition and use of the Gronwall’s lemma and Burkholder-Davis-Gundy’s inequality.

**Proposition 9** Suppose that \((g_l, \xi_{\text{Target}}, L)\) and \((\bar{g}_l, \bar{\xi}_{\text{Target}}, \bar{L})\) satisfies Assumption 3. Let \((\hat{Y}^0_{0, x}, \hat{Z}^0_{0, x}, \hat{A}^0_{0, x})\) and \((\bar{Y}^0_{0, x}, \bar{Z}^0_{0, x}, \bar{A}^0_{0, x})\) be solutions of reflected-BSDEs associated with \((g_l, \xi_{\text{Target}}, L)\) and \((\bar{g}_l, \bar{\xi}_{\text{Target}}, \bar{L})\), respectively. Define

\[
\begin{align*}
\Delta \xi^s_{\text{Target}} &= \xi^s_{\text{Target}} - \xi^s_{\text{Target}}, \\
\Delta L &= L - \bar{L}, \\
\Delta Y^0_{0, x} &= Y^0_{0, x} - \bar{Y}^0_{0, x}, \\
\Delta Z^0_{0, x} &= Z^0_{0, x} - \bar{Z}^0_{0, x}.
\end{align*}
\]

\[
\begin{align*}
\Delta g_l &= g_l - \bar{g}_l, \\
\Delta A^0_{0, x} &= A^0_{0, x} - \bar{A}^0_{0, x}.
\end{align*}
\]
Then, there exists a constant $\gamma$ such that
\[
\mathbb{E}\left\{ \sup_{0 \leq t \leq T} \left[ |\Delta Y_t^{0,x,w}|^2 + \int_0^T |\Delta Z_t^{0,x,w}|^2 dt + \Delta (A_t^{0,x,w})^2 \right] \right\} \\
\leq \gamma \mathbb{E}\left\{ \left[ |\Delta \xi_{t,T}|^2 + \int_0^T |\Delta g(t, X_t^{0,x,w}, Y_t^{0,x,w}, Z_t^{0,x,w})|^2 dt \right] \right\} + \gamma \left[ \mathbb{E}\left\{ \sup_{0 \leq t \leq T} (A_t^{0,x,w})^2 \right\} \right]^{\frac{1}{2}} \left[ \Gamma_T \right]^{\frac{1}{2}},
\]
where
\[
\Gamma_T = \mathbb{E}\left\{ (\xi_{t,T})^2 + \int_0^T g(t, x, 0, 0) dt + \sup(L_t^+)^2 \right\} + \mathbb{E}\left\{ (\xi_{t,T})^2 + \int_0^T g(t, x, 0, 0) dt + \sup(L_t^+)^2 \right\}.
\]

Then, we have the following uniqueness result from the Proposition\([\text{9}]\) with $g_t = \bar{g}_t$, $L = L$ and $\xi_{T,T} = \xi_{T,T}$.

**Corollary 1** Under Assumption\([\text{3}]\) there exists at most one progressively measurable triple
\[
(Y_s^{t,x,w}, Z_s^{t,x,w}, A_s^{t,x,w})_{t \leq s \leq T}
\]
satisfying (iii) and (v) in (70).

**Remark 7** Note that, for $(t,x) \in [0,T] \times \mathbb{R}^d$, if $(Y_s^{t,x,w}, Z_s^{t,x,w}, A_s^{t,x,w})_{t \leq s \leq T}$ satisfying (iii) and (v) in (70). Then, we have
\[
A_{T,T}^{t,x,w} - A_t^{t,x,w} = \sup_{t \leq s \leq T} \left\{ \xi_{s,T}^{T,T} + \int_t^s g(t, X_r^{t,x,w}, Y_r^{t,x,w}, Z_r^{t,x,w}) dr \right\} - \int_s^T Z_r^{t,x,w} dB_r - L_r \}
\]
for each $s \in [t,T]$.

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