Using Periodicity Theorems for Computations in Higher Dimensional Clifford Algebras

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Abstract

We present different methods for symbolic computer algebra computations in higher dimensional ($\geq 9$) Clifford algebras using the CLIFFORD and Bgebra packages for Maple. This is achieved using graded tensor decompositions, periodicity theorems and matrix spinor representations over Clifford numbers. We show how to code the graded algebra isomorphisms and the main involutions, and we provide some benchmarks.

1 Introduction

Clifford algebras are used in several areas of mathematics, physics, and engineering. Since computing power has increased tremendously, in terms of available memory and actual processing speed, practical symbolic computations, say on a laptop finishing within minutes, are by now possible for Clifford algebras over vector spaces of dimensions higher than 8. When CLIFFORD was designed, way back in the early 90's [1], such computations were impossible. However, the design of CLIFFORD, restricting the vector space dimension to less than or equal to 9, incorporated from the beginning the idea that using mod-8 and other periodicity isomorphisms of real Clifford algebras allows nevertheless computations in higher dimensional algebras as well. Using periodicity theorems also 'groups' Clifford algebras as in a periodic table, the spinorial chess board [10]. Hence, implementing the periodicity in symbolic computations—as described in this paper—makes use of these intrinsic algebra features. Recent applications in engineering, for example when modeling geomet-
ric transformations in robotics, rely on real Clifford algebras like $\mathcal{C}_{8,2}$ and thus enforce the need for using the periodicity in computations with CLIFFORD.

In this note, we will describe three ways showing how CLIFFORD deals with (real) Clifford algebras of higher dimensions ($\geq 9$). One way is to use Bigebra, an extension of CLIFFORD, to utilize graded tensor products of Clifford algebras. Another method utilizes ungraded tensor products and periodicity isomorphisms without a need for introducing spinorial bases. The third method uses, on top of the periodicity theorems, a spinor representation of one factor of the tensor decomposition, hence computing in Clifford algebra valued matrix rings. For example, from $\mathcal{C}_{p+1,q+1} \simeq \mathcal{C}_{p,q} \otimes \mathcal{C}_{1,1} \simeq \text{Mat}(2, \mathcal{C}_{p,q})$ we see that computing say in $\mathcal{C}_{8,2}$ can be done with the $2 \times 2$ matrices over $\mathcal{C}_{7,1}$. CLIFFORD supports computing with matrices having Clifford entries. Similar computations still make sense, e.g., for conformal symmetries using Vahlen matrices $\text{Mat}(2, \mathcal{C}_{3,1}) \simeq \mathcal{C}_{4,2}$.

After recalling our basic notations and the periodicity theorems, we start explaining how we use Bigebra to compute with graded and ungraded tensor products of Clifford algebras. Then we proceed to the third matrix-based method indicated above. As this method relies on spinor representations of real Clifford algebras, one encounters not only real, but also complex, quaternionic, double real, and double quaternionic spinor representations. For the sake of simplicity and space, we will just deal with real representations of simple Clifford algebras. The periodicity theorems single out the signature cases where a graded algebra isomorphism is available onto an ungraded tensor decomposition and hence, introducing spinor bases, matrix tensor products can be employed.

## 2 Tensor product decompositions and periodicity for $\mathcal{C}_\ell$-algebras

We study Clifford algebras over finite (real) vector spaces. While these algebras can be defined by a universal property, for actual computations in a CAS (Computer Algebra System) we use generators and relations.

### 2.1 Basic notations, quadratic and bilinear forms

Let $V$ be a finite dimensional real (or complex) vector space with scalar multiplication $\mathbb{R} \times V \to V :: (\lambda, v) \mapsto \lambda v$. A quadratic form is a map $Q : V \to \mathbb{R}$ such that $Q(\lambda v) = \lambda^2 Q(v)$, with associated polar bilinear form $B : V \times V \to \mathbb{R}$ derived from $Q$ and defined as $2B(v, w) = Q(v + w) - Q(v) - Q(w)$ (hence $Q(v) = B(v, v)$). $Q$ is called non-degenerate if $Q(v) = 0$ implies $v = 0$ ($v \in V, \forall w \in V, B(w, v) = 0$ implies $v = 0$). An isomorphism $V \simeq \mathbb{R}^n = \oplus \mathbb{R}$ defines a set of generators for $V$ from the injections of the direct sum $i_k : \mathbb{R} \to \mathbb{R}^n :: 1 \mapsto e_k \simeq v_k$, that is a basis for $V$. Sylvester’s theorem states that there exists a basis for $V$ such that the quadratic form $Q$ is diagonal with entries $\pm 1, 0$ in the real case ($0$ only when $Q$ is degenerate; just $+1$’s
in the non-degenerate complex case). Under these isomorphisms, the real quadratic space \((Q, V)\) is isomorphic to a space \(\mathbb{R}^{p,q} = (\mathbb{R}^n, \delta_{p,q})\) with a diagonal quadratic (polarized) form \(\delta_{p,q}\) with \(p\) ones and \(q\) minus ones. \((p,q)\) is called the signature of \(Q\) and \(p+q = n = \text{dim}V\). Furthermore we have generators \(e_k\) of \(\mathbb{R}^{p,q}\) such that \(Q(e_k) = +1\) for \(1 \leq k \leq p\) and \(Q(e_k) = -1\) for \(p < k \leq p+q\).

### 2.2 Grassmann algebra, \(\mathbb{Z}\)- and \(\mathbb{Z}_2\)-gradings, main involutions

We can associate functorially to any (say finite, real) vector space \(V\) the Grassmann algebra of antisymmetric tensors, \(\wedge : V \rightarrow \bigwedge V = \bigoplus_{i=0}^n \wedge^i V\). Using the linearly ordered basis \(\{e_k\}\) of \(V\) \((e_k < e_j\) if \(k < l\)), we get a basis

\[
\{e_l\} = \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l} \mid I = (i_1, i_2, \ldots, i_l), \ i_1 < i_2 < \cdots < i_l, i_j \in \{1, \ldots, n\}\}
\]

for the space \(\bigwedge V\) of antisymmetric tensors. We can extend the order on \(V\) to the inverse lexicographic order on \(\bigwedge V\). We associate to \(e_l\) a degree \(|l|\) \((\text{number of generators } e_k \text{ in } e_l)\) and a grade \((\text{or parity})\) as \(|l| \mod 2\). The Grassmann algebra is \(\mathbb{Z}\)-graded \(w.r.t.\) the degree, that is \(\bigwedge^p V \bigwedge^q V \subseteq \bigwedge^{p+q} V\). This can be restricted to a \(\mathbb{Z}_2\)-grading \(\bigwedge V = \bigwedge^0 V \oplus \bigwedge^1 V\) \(w.r.t.\) the grade, hence decomposing into even and odd parts \((\bigwedge^0 V\) is a subalgebra, \(\bigwedge^1 V\) is a \(\bigwedge^0 V\)-module). The subspaces \(\bigwedge^i V\) have \(\binom{n}{i}\) \(\text{many basis elements and hence } \bigwedge V\) \(\text{has dimension } 2^n = \sum \binom{n}{i}\). Homogeneous elements \(v_1 \wedge \cdots \wedge v_l\) of \(\bigwedge^i V\) are called extensors \((\text{or blades})\). A general element \(X \in \bigwedge V\) is an aggregate \(X = \sum x_i e_i\) with real coefficients \(x_i\). A presentation using generators and relations is given as

\[
\bigwedge V := \langle e_k \mid e_k e_l = -e_l e_k, i, k \in \{1, \ldots, n\} \rangle.
\]

The Grassmann algebra comes with two main involutions. The \emph{grade involution} extends the map \(- : V \rightarrow V\) \(v \mapsto -v\) \((\text{additive inverse in } V)\) to \((-)\) on \(\bigwedge V\). On generators this reads as \(\hat{e}_1 = (-1)^{|l|} e_l\) \(\text{and it extends by linearity. For example, } \hat{e}_1 = -e_1, \hat{e}_{1,2,3} = \hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3 = (-1)^3 e_{1,2,3} = -e_{1,2,3}\) while \(\hat{e}_{1,2} := \hat{e}_1 \wedge \hat{e}_2 = e_{1,2}\).

A second involution makes use of the opposite algebra. Let \((A, m_A)\) \(w.r.t.\) \(m_A : A \times A \rightarrow A \ (a,b) \mapsto ab\) be an algebra. The \emph{opposite algebra} \(A^{op} = (A, m_A^{op})\) is the same vector space \(A\) \(w.r.t.\) the multiplication \(m_A^{op} : A \times A \rightarrow A\) \(w.r.t.\) \((a,b) \mapsto ba\). The opposite wedge product \(\wedge^{op}\) \(\text{is given as } u \wedge^{op} v = v \wedge u\) \((\text{no signs) producing the opposite Grassmann algebra } \bigwedge^{op} V\). The \emph{reversion involution} on \(\bigwedge V\) extends the identity map \(1 : V \rightarrow V\) \(w.r.t.\) \(\bigwedge V\) \(\text{in such a way that } \text{rev} \circ \wedge^{op} = \wedge \circ (\text{rev} \otimes \text{rev})\). In other words, \(\text{rev} : \bigwedge^{op} V \rightarrow \bigwedge V\) is a Grassmann algebra isomorphism. As \(\text{rev}\) is invertible, this can be be used to define \(\wedge^{op}\). On basis elements we have \(\text{rev}(e_l) = (-1)^{|l|(|l|-1)/2} e_l\) with the usual notation \(\text{rev}(u) = \bar{u}\).

Let \(V^* = [V, \mathbb{R}]\) be the dual vector space \((\text{linear forms})\). One finds that \(\bigwedge V^* \cong (\bigwedge V)^*\) is the dual Grassmann algebra, and we can define a pairing.
\[ \langle - | - \rangle : \bigwedge^* V \times \bigwedge V \to \mathbb{R} : (e_i^*, e_k) \mapsto \begin{cases} \det(e_i^*(e_k)), & \text{when } |i| = |k|; \\ 0, & \text{when } |i| \neq |k|. \end{cases} \]

The interior product \( \mathcal{J} \) is defined as the adjoint w.r.t. the duality pairing of multiplication in \( \bigwedge V^* \), that is \( \langle u \mid \mathcal{J} w \rangle := \langle v^\vee \wedge u \mid w \rangle \). Let \( x, y \in V \) and \( u, v, w \in \bigwedge V \) then \( \mathcal{J} \) is defined operationally by the rules (Chevalley construction)

\[
x \mathcal{J} y = \langle x \mid y \rangle, \quad x \mathcal{J} (u \wedge v) = (x \mathcal{J} u) \wedge v + u \wedge (x \mathcal{J} v), \quad (u \wedge v) \mathcal{J} w = u \mathcal{J} (v \mathcal{J} w). \tag{1}
\]

Using a nondegenerate bilinear form \( B \) as duality, one obtains \( x \mathcal{J} y = B(x, y) \).

### 2.3 (Real) Clifford algebras

Given a quadratic space \((V, Q)\) we can functorially associate to it the universal Clifford algebra \(\mathcal{C}D(V, Q)\) \[\text{[13]}\]. Given the isomorphism \((V, Q) \simeq \mathbb{R}^p\delta\) this provides the Clifford algebra \(\mathcal{C}D(\mathbb{R}^n, \delta_{p,q}) \ (n = p + q)\) also often denoted as \(\mathbb{R}^{p,q}\). As vector spaces one has \(\mathbb{R}^{p,q} = \bigwedge \mathbb{R}^n\), hence we have a Grassmann basis \(\{e_i\}\) spanning the vector space underlying the Clifford algebra. The Clifford product can be implemented in various ways. A Clifford algebra presentation reads

\[
\mathcal{C}D_{p,q} := \langle e_i \mid e_i e_j + e_j e_i = 0, \ i \neq j; \ e_i^2 = 1 \text{ if } 1 \leq i \leq p, \ \text{otherwise } e_i^2 = -1 \rangle
\]

where the Clifford multiplication is juxtaposition and \(i, j \in \{1, \ldots, n\}\). Another way to define the Clifford multiplication uses the interior multiplication and the Clifford map \(\gamma\). Let \(x \in V, \ u \in \bigwedge V\) then \(\gamma_x u = xu = x \mathcal{J} u + x \wedge u\) and extend by \(\{1\}\) with \(x \mathcal{J} y = B(x, y)\) and linearity. Defining \(\mathcal{C}D\) as a quotient of the tensor algebra it can be seen that it is not graded, but only \(\mathbb{Z}_2\)-graded, \(\mathcal{C}D = \mathcal{C}D_0 \oplus \mathcal{C}D_1\) with \(\mathcal{C}D_0\) the even sub-Clifford algebra and \(\mathcal{C}D_1\) a \(\mathcal{C}D_0\)-module. In terms of the Grassmann \(\mathbb{Z}\)-degrees, it is a filtered algebra \(\mathcal{C}D^{\mathbb{Z}} \mathcal{C}D^j \subseteq \bigoplus_{i+j \neq 0, j-i} \mathcal{C}D^i\). The quadratic space \((V, Q)\) decomposes as \((V, Q) = (V_1 + V_2, Q_1 \perp Q_2)\) with restricted quadratic forms \(Q_1 = Q|_{V_1}\) on \(V_1\) and \(Q_2 = Q|_{V_2}\) on \(V_2\). The Grassmann algebra functor is exponential, that is,

\[
\bigwedge (V_1 + V_2) = \bigwedge (V_1) \hat{\otimes} \bigwedge (V_2)
\]

(the graded tensor product \(\hat{\otimes}\) is defined below and we use equality for categorical isomorphisms). Similarly we get for Clifford algebras

\[
\mathcal{C}D(V_1 + V_2, Q_1 \perp Q_2) = \mathcal{C}D(V_1, Q_1) \hat{\otimes} \mathcal{C}D(V_2, Q_2),
\]

and it is this decomposition which will be used below to compute in CLIFFORD in dimensions \(\geq 9\). For Clifford algebras with non-symmetric bilinear forms such a decomposition is in general not direct, see \[\text{[13]}\].
2.4 Tensor products of (graded) algebras

Let \((A, m_A)\) and \((B, m_B)\) be \(\mathbb{K}\)-algebras. That is, we have right and left scalar multiplications \(\rho : A \times \mathbb{K} \to A\) and \(\lambda : \mathbb{K} \times B \to B\). This allows to define the tensor product \(A \otimes_{\mathbb{K}} B\) as a universal object via a co-equalizer of two arrows \(\rho \circ 1\) and \(1 \circ \lambda\):

\[
\begin{array}{ccc}
A \times \mathbb{K} \times B & \xrightarrow{\rho \circ 1} & A \times B \\
1 \circ \lambda & \xrightarrow{\sigma} & A \otimes_{\mathbb{K}} B,
\end{array}
\]

which produces the common relations such as \(a \lambda \otimes b = a \otimes \lambda b\), multilinearity etc. We have (vector space) injections \(i_A : A \to A \otimes B\) and \(i_B : B \to A \otimes B\), and want to transport the algebra structure too. We define the product algebra on \(A \otimes B\) from the algebra structures on \(A\) and \(B\) as follows: \(m_{A \otimes B} := (m_A \otimes m_B)(1 \otimes \sigma \otimes 1)\), where \(\sigma : A \otimes B \to B \otimes A\). The graded multiplication \(m_{A \otimes B}\) is then given by \(m_{A \otimes B} := (m_A \otimes m_B)(1 \otimes \hat{\sigma} \otimes 1)\). Using this setup, the injections above become algebra isomorphisms \(i_A : a \mapsto (a \otimes 1)\) and \(i_B : b \mapsto (1 \otimes b)\).

In the Grassman algebra case, splitting the space \(V = V_1 + V_2\) with \(n\) basis vectors \(e_i\) into two sets with, respectively, \(p\) \((1 \leq i \leq p)\) and \(q\) \((p < i \leq n)\) vectors, we get the maps \(e_i \mapsto e_i \hat{\otimes} 1\) \((1 \leq i \leq p)\) and \(e_j \mapsto 1 \hat{\otimes} e_j\) \((p < j \leq n)\). In the CAS computations below we will standardize the indices, that is, we will reindex \(j \mapsto j - p\) so that \(i \in \{1, \ldots, p\}\) and \(j - p \in \{1, \ldots, n - p\}\). The graded tensor product ensures that we still have the desired anti-commutation relations

\[
(e_i \hat{\otimes} 1)(1 \hat{\otimes} e_j) = (e_i \hat{\otimes} e_j) \quad \text{and} \quad (1 \hat{\otimes} e_j)(e_i \hat{\otimes} 1) = (-1)^{1 \cdot 1}(e_i \hat{\otimes} e_j).
\]

In this way we can form graded tensor products of Clifford algebras \(\mathcal{C}_{p,q} \otimes \mathcal{C}_{r,s}\) too, and that is what we aim for. Tensor products for matrix algebras are usually called Kronecker products, and are taken without the grading. Let \(A \simeq \text{Mat}(n, \mathbb{R})\) and \(B \simeq \text{Mat}(m, \mathbb{R})\) be matrix algebras. Recall that matrices need a choice of basis giving matrices \([a_{ij}]\) and \([b_{ij}]\). The definition of the matrix tensor algebra \(A \otimes B \simeq \text{Mat}(n \cdot m, \mathbb{R})\) includes a choice of how to form a basis for \(A \otimes B\), which consists of elements \(E_{n,m} = e_{i,j} \otimes e_{k,l}\), that is, a reindexing function \([i, j, (k, l)] \mapsto (n, m)\). One way is to define a tensor \([a_{ij}] \otimes [b_{kl}] = [a_{ij} \cdot [b_{kl}]]\) by inserting the matrix \([b_{kl}]\) as blocks into the matrix \([a_{ij}]\); another obvious choice would exchange the role of \([a]\) and \([b]\). Category theory shows that the definition of the tensor algebra is unique up to a unique isomorphism depending on the particular choices. However, actual computations in a CAS need to be consistent and explicit in the choice of these isomorphisms.
2.5 Periodicity of Clifford algebras

We have seen in the previous section that we can tensor Clifford algebras of any signature provided we employ the graded tensor product.

**Theorem 1.** Let $\mathcal{C}_{p,q}$, $\mathcal{C}_{r,s}$ be two real Clifford algebras, then

$$\mathcal{C}_{p+r,q+s} \simeq \mathcal{C}_{p,q} \hat{\otimes} \mathcal{C}_{r,s} \quad (4)$$

(which does not even need nondegeneracy, or even symmetry, of the involved forms).

This result will be used in section 3 to describe a general method using Bigréa to do practical symbolic CA computations in higher dimensional real Clifford algebras of any signature.

Using matrices over Clifford numbers, like $\text{Mat}(2, \mathcal{C}_{p,q})$, needs considering ungraded tensors, as the matrix algebra tensor products are ungraded. Doing so employs (graded) algebra isomorphisms described on the generators of the factor Clifford algebras inside the ambient Clifford algebra. This leads to the well-known periodicity relations which are summarized in the following\footnote{For additional references on the periodicity theorems see [5, 12, 18].}

**Theorem 2.** For real Clifford algebras we have the following periodicity theorems and isomorphisms:

1) $\mathcal{C}_{q+1,p-1} \simeq \mathcal{C}_{p,q}$ if $p \geq 1$ (see [16]),
2) $\mathcal{C}_{p,q+1} \simeq \mathcal{C}_{q,p+1}$ (see [18]),
3) $\mathcal{C}_{p,q} \simeq \mathcal{C}_{p-4,q+4}$ if $p \geq 4$ (see [11, 16]),
4) $\mathcal{C}_{p+4,q} \simeq \mathcal{C}_{p,q} \otimes \mathcal{C}_{4,0} \simeq \mathcal{C}_{p,q} \otimes \text{Mat}(2, \mathbb{H})$ (see [18]),
5) $\mathcal{C}_{p,q+4} \simeq \mathcal{C}_{p,q} \otimes \mathcal{C}_{4,0} \simeq \mathcal{C}_{p,q} \otimes \text{Mat}(2, \mathbb{H})$ (see [18]),
6) $\mathcal{C}_{p+1,q+1} \simeq \mathcal{C}_{p,q} \otimes \mathcal{C}_{1,1} \simeq \mathcal{C}_{p,q} \otimes \text{Mat}(2, \mathbb{R}) \simeq \text{Mat}(2, \mathcal{C}_{p,q})$ (see [16]),
7) $\mathcal{C}_{p,q+8} \simeq \mathcal{C}_{p,q} \otimes \text{Mat}(16, \mathbb{R}) \simeq \text{Mat}(16, \mathcal{C}_{p,q})$ with $\text{Mat}(2, \mathbb{H}) \otimes \text{Mat}(2, \mathbb{H}) \simeq \text{Mat}(8, \mathbb{R})$ (see [11, 16, 18]),
8) $\mathcal{C}_{p+8,q} \simeq \mathcal{C}_{p,q} \otimes \text{Mat}(16, \mathbb{R}) \simeq \text{Mat}(16, \mathcal{C}_{p,q})$ with $\text{Mat}(2, \mathbb{H}) \otimes \text{Mat}(2, \mathbb{H}) \simeq \text{Mat}(8, \mathbb{R})$ (see [11, 16, 18]).

Note that here all tensor products are ungraded and these structure results characterize the cases (signatures) where a graded isomorphism from the graded case to the ungraded case exists. The isomorphisms with matrix rings need spinor representations and will be discussed briefly in subsection 2.6.

By way of example we show a typical isomorphism on generators for the isomorphism 6.

**Example 1.** Let $\{e_i\}$ be the set of orthonormal generators for $\mathcal{C}_{p,q}$ with $e_i^2 = 1$ for $i \in \{1, \ldots, p\}$ and $e_i^2 = -1$ for $i \in \{p+1, \ldots, p+q\}$, and let $f_i^2 = 1 = -f_j^2$ be the orthogonal generators for $\mathcal{C}_{1,1}$. Then, $e_i \otimes f_1 f_2, 1 \otimes f_1, 1 \otimes f_2$ form a set of generators for $\mathcal{C}_{p+1,q+1}$. Indeed, since $f_1 f_2 = -f_2 f_1$ and $(f_1 f_2)^2 = 1$, for $i, j \in \{1, \ldots, p+q\}$ and $i \neq j$, we find the following familiar relations in $\mathcal{C}_{p,q} \otimes \mathcal{C}_{1,1}$:
\[(e_i \otimes f_1 f_2)^2 = (e_i \otimes f_1 f_2)(e_i \otimes f_1 f_2) = e_i^2 \otimes (f_1 f_2)^2 = \begin{cases} 1 \otimes 1, & \text{if } 1 \leq i \leq p; \\ -1 \otimes 1, & \text{otherwise,} \end{cases}\]

\[(1 \otimes f_1)^2 = 1 \otimes f_1^2 = 1 \otimes 1 = -1 \otimes f_2^2 = -(1 \otimes f_2)^2,\]

\[(e_i \otimes f_1 f_2)(e_j \otimes f_1 f_2) + (e_j \otimes f_1 f_2)(e_i \otimes f_1 f_2) = (e_i e_j + e_j e_i) \otimes (f_1 f_2)^2 = 0,\]

\[(e_i \otimes f_1 f_2)(1 \otimes f_1) + (1 \otimes f_1)(e_i \otimes f_1 f_2) = e_i \otimes (-f_1^2 f_2) + e_i \otimes (f_1 f_2^2) = 0,\]

\[(1 \otimes f_1)(1 \otimes f_2) + (1 \otimes f_2)(1 \otimes f_1) = 1 \otimes (f_1 f_2 + f_2 f_1) = 0.\]

For further details see, e.g., [10, 17].

**Theorem 3 ([10], Thm. 5.8).** With the notation as above, let \(V_2\) have dimension \(2k\) and let \(\omega\) be the volume element in \(\mathcal{C}(V_2, Q_2)\) with \(\omega^2 = \lambda \neq 0\). There exists a vector space isomorphism between the module \(\mathcal{C}(V_1 \oplus V_2, Q_1 \perp Q_2)\) and the module \(\mathcal{C}(V_1, \frac{1}{\lambda} Q_1) \otimes \mathcal{C}(V_2, Q_2)\) given on generators as \((x, y) \mapsto x \otimes \omega + 1 \otimes y\), and there is a graded algebra isomorphism

\[
\mathcal{C}(V_1 \oplus V_2, Q_1 \perp Q_2) \simeq \mathcal{C}(V_1, \frac{1}{\lambda} Q_1) \otimes \mathcal{C}(V_2, Q_2). \tag{5}
\]

The involutions extend as \((x \otimes y) \simeq \hat{x} \otimes \hat{y} \) and \(\text{rev}(x \otimes y) \simeq \text{rev}(x) \otimes \text{rev}(y)\) if \(|x| \equiv 0 \mod 2\) even and \(\text{rev}(x \otimes y) \simeq \text{rev}(x) \otimes \text{rev}(y)\) otherwise. Then all periodicity isomorphisms in theorem2 are special cases of this one4.

To exemplify this, let \((x, y)\) be any pair of generators with \(x \in V_1\) and \(y \in V_2\) which upon the embedding \(V_1 \oplus V_2 \hookrightarrow \mathcal{C}(V_1 \oplus V_2, Q_1 \perp Q_2)\) we write as the sum \(x + y\). Then,

\[(x + y)^2 = x^2 + (xy + yx) + y^2 = (Q_1(x) + Q_2(y))1 = (Q_1 \perp Q_2)(x, y) \tag{6}\]

due to the orthogonality of \(x\) and \(y\). On the other hand, in the (ungraded) tensor product algebra in the right-hand-side of (5) we find, as expected,

\[
(x \otimes \omega + 1 \otimes y)^2
= (x \otimes \omega)(x \otimes \omega) + (x \otimes \omega)(1 \otimes y) + (1 \otimes y)(x \otimes \omega) + (1 \otimes y)(1 \otimes y)
= x^2 \otimes \omega^2 + x \otimes \omega y + x \otimes y \omega + 1 \otimes y^2
= \frac{1}{\lambda} Q_1(x)1 \otimes \lambda 1 + 1 \otimes Q_2(y) = Q_1(x) \otimes 1 + 1 \otimes Q_2(y)
= (Q_1(x) + Q_2(y))(1 \otimes 1) \tag{7}\]

due the anti-commutativity \(y \omega = -\omega y\) for every \(y \in V_2\) assured by the even dimension of \(V_2\). Furthermore, this last computation shows that the assumption \(\omega^2 = \lambda \neq 0\) and the appearance of the factor \(\frac{1}{\lambda}\) modifying \(Q_1\) in \(\mathcal{C}(V_1, \frac{1}{\lambda} Q_1)\), are both necessary.

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2 The requirement \(\omega^2 = \lambda \neq 0\) is equivalent to \(Q_2\) being non-degenerate. See also [16] p. 218.
Since we are using Grassmann bases in all Clifford algebras, it is interesting to calculate the image of a Grassmann basis monomial, lets say of degree 2, in the product algebra on the right of (5). Let \( x_1, x_2 \) be orthogonal generators in \( V_1 \) and let \( y_1, y_2 \) be orthogonal generators in \( V_2 \). Then, the wedge product in the Clifford algebra \( \mathcal{C}(V_1 \oplus V_2, Q_1 \perp Q_2) \) is computed as expected

\[
(x_1 + y_1) \wedge (x_2 + y_2) = x_1 \wedge x_2 + x_1 \wedge y_2 + y_1 \wedge x_2 + y_1 \wedge y_2.
\]  

(8)

On the other hand, since \( (x_1 + y_1) \wedge (x_2 + y_2) = (x_1 + y_1)(x_2 + y_2) \), its image in \( \mathcal{C}(V_1, \frac{1}{2} Q_1) \otimes \mathcal{C}(V_2, Q_2) \) under the isomorphism (5) is the following rather complicated element:

\[
(x_1 x_2) \otimes \omega^2 + (x_1 \wedge 1) \otimes \omega y_2 + (1 \wedge x_2) \otimes y_1 \omega + (1 \wedge 1) \otimes (y_1 y_2) =
(x_1 \wedge x_2) \otimes 1 + x_1 \otimes \omega y_2 + x_2 \otimes y_1 \omega + 1 \otimes (y_1 \wedge y_2).
\]  

(9)

The isomorphism in (5) is given by the procedures \texttt{bas2Tbas} (from left to right) and its inverse \texttt{Tbas2bas} (from right to left). In the worksheets [6] we show both procedures as well as we verify the assertions regarding the involutions.

### 2.6 Spinor representations, Clifford valued matrix representations

A Clifford algebra is an abstract algebra, but we may want to realize it as a concrete matrix algebra. It is, however, well known that matrix representations may be very inefficient for CAS purposes. The simplest representation is the (left) regular representation, sending \( a \in A \mapsto \lambda_a = m_A(a, -) \in \text{End}(A) \), the left multiplication operator by \( a \). This representation is usually highly reducible. The smallest faithful representations of a Clifford algebra are given by spinor representations. Algebraically, a spinor representation is given by a minimal left ideal which can be generated by left multiplication from a primitive idempotent \( f_i = f^2_i \) with \( \not\exists f_k, f_l \neq 0 \) idempotents such that \( f_i = f_k + f_l \) and \( f_k f_l = f_i f_k = 0 \). The vector space \( S := \mathcal{C}_{p,q} f_i \) is a spinor space, and it carries a faithful irreducible representation of \( \mathcal{C}_{p,q} \) for simple algebras.\(^3\) However, when \( \mathcal{C}_{p,q} \) is not simple, and in several signatures \( (p,q) \) this space is not really a vector space, but a module over \( \mathbb{K} = f_i \mathcal{C}_{p,q} f_i \) with \( \mathbb{K} \) isomorphic to \( \mathbb{R} \), \( \mathbb{C} \), \( 2\mathbb{R} = \mathbb{R} \oplus \mathbb{R} \), or \( 2\mathbb{H} = \mathbb{H} \oplus \mathbb{H} \) depending on the signature \( (p,q) \). The spinor bi-module \( \mathcal{C}_p S_{\mathbb{K}} \) carries a left \( \mathcal{C}_{p,q} \) and right \( \mathbb{K} \) action (scalar product). Looking up tables of spinor representations [16] [18] yields that \( \mathcal{C}_{p,q} \) is simple and has a real representation if \( p - q \equiv 0, 2 \mod 8 \), distinguished by the fact that a normalized volume element \( \omega \) squares to \( \omega^2 = +1 \) if \( p - q \equiv 0, 1 \mod 4 \) and \( \omega^2 = -1 \) if \( p - q \equiv 2, 3 \mod 4 \). Avoiding non real \( \mathbb{K} \)'s, we find the isomorphisms \( \mathcal{C}_{0,0} \simeq \mathbb{R} \simeq \text{Mat}(1, \mathbb{R}) \), \( \{ \mathcal{C}_{2,0}, \mathcal{C}_{1,1} \} \simeq \text{Mat}(2, \mathbb{R}) \), \( \{ \mathcal{C}_{3,1}, \mathcal{C}_{2,2} \} \simeq \text{Mat}(4, \mathbb{R}) \), \( \{ \mathcal{C}_{4,2}, \mathcal{C}_{3,3}, \mathcal{C}_{1,6} \} \simeq \text{Mat}(8, \mathbb{R}) \), and \( \{ \mathcal{C}_{8,0}, \mathcal{C}_{5,3}, \mathcal{C}_{4,4}, \mathcal{C}_{1,7}, \mathcal{C}_{0,8} \} \simeq \text{Mat}(16, \mathbb{R}) \).

\(^3\) For semi-simple Clifford algebras, we realize the spinor representation in \( S \oplus \hat{S} \) where \( \hat{S} = \mathcal{C}_{p,q} f_i \) (see, e.g., [16]).
Mat(16, R), which can be looked up using the command clidata in CLIFFORD. The main reason for using these isomorphisms is that \( R \otimes C_{p,q} \simeq C_{p,q} \), so we don’t have to worry about \( \mathbb{K} \) tensor products of spinor modules and use only \( S \otimes S' \simeq \text{Mat}(2^k, \mathbb{R}) \).

**Example 2.** A spinor basis for \( C_{1,1} \simeq \text{Mat}(2, \mathbb{R}) \) used in the isomorphism 6), given the orthogonal generators \( e_1^2 = 1 = -e_2^2 \) for \( C_{1,1} \) and a primitive idempotent \( f \), maybe chosen as

\[
S = C_{1,1} = \text{span}_R \{ f = \frac{1}{2} (1 + e_1 e_2), e_1 f = \frac{1}{2} (e_1 + e_2) \}. \tag{10}
\]

In this basis, the following matrices represent the basis elements in \( C_{1,1} \):

\[
[1] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [e_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [e_2] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad [e_1 e_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}
\]

In general, the images of the additional generators \( e_i \otimes e_j e_k \) needed to generate \( C_{p,q} \otimes C_{1,1} \simeq \text{Mat}(2, C_{p,q}) \) are given by \( e_i \otimes [e_j e_k] \) with entries from \( C_{p,q} \), and read \( e_i \otimes e_j e_k \simeq \begin{pmatrix} e_i & 0 \\ 0 & -e_j \end{pmatrix} \). The isomorphism 6) gives, via iteration, \( C_{p,q} \simeq C_{p-q,0} \otimes \text{Mat}(2^p, \mathbb{R}) \) if \( p \geq q \) and \( C_{p,q} \simeq C_{1,0} \otimes \text{Mat}(2^q, \mathbb{R}) \) if \( q \geq p \).

## 3 Computing with CLIFFORD and Bigebra in tensor algebras

We provide examples of Maple code how to set up tensor products of Clifford algebras in CLIFFORD and Bigebra. For the usage of these packages see [145], the help pages which come with the package, and the package website [3]. The Maple worksheets with code for the described methods will be posted at [6].

Loading the package using 

\>

with(CLIFFORD); with(Bigebra); 

exposes the exported functions. To set up a Clifford algebra, say \( C_{2,2} \), one needs to define the dimension \( \text{dim} := 2^{2+2} \); and the bilinear form \( B := \text{linalg[diagonal]}([1,2,3,2]); \)

Basis elements \( e_i \) are written as strings, e.g., \( e_1 e_4 \) stands for \( e_1 \wedge e_4 \), etc., whereas \( \text{Id} \) stands for the identity of the Clifford algebra. Bigebra exports also the (graded) tensor product \( \&\text{t} \), which is multilinear and associative. Then, a tensor product \( e_1 e_2 \otimes e_1 \) reads \( \&\text{t}(e_1 e_2, e_1) \), and permutations of tensors are implemented by maps \( \text{switch}(\&\text{t}(e_1,e_2),1) = \&\text{t}(e_1,e_1) \) (the ungraded switch) or, in the graded case, \( \text{gswitch}(\&\text{t}(e_1,e_2),1) = -\&\text{t}(e_1,e_1) \) (the graded switch). The extra index \( i \) in either switch (here we have used 1 in each) tells \( \text{gswitch} \) to swap the \( i \)-th and the \( (i+1) \)-st elements. Again, you get help by typing \( \text{>gswitch} \) and \( \text{gswitch} \) at the Maple prompt.

---

\[ The \text{default name of the bilinear form in CLIFFORD and Bigebra is } B, \text{ however other names can also be used. So, when the bilinear form } B \text{ is left undefined (unassigned), computations are performed in a Clifford algebra } C(B) \text{ for an arbitrary bilinear form } B \text{ (see, e.g., [5813]). } \]
The Clifford product \texttt{cmul} by default implicitly uses the bilinear form \( B \) as in, for example, \( >\texttt{cmul}(e_1,e_2)=e_1we_2+B[1,2] \cdot \text{Id} \). However, it can also use \( B \) or any other Maple name explicitly as an optional argument, e.g., \( >\texttt{cmul}[K](e_1,e_2) \).

Let \( B_1, B_2 \) hold the bilinear forms for \( \mathcal{C}_{p+r+q+s} \), \( \mathcal{C}_{p,q} \) and \( \mathcal{C}_{r,s} \), and let \( \text{bas2GTbas} \) be the graded algebra isomorphism given by \( e_I \in \mathcal{C}_{p,q} \mapsto \&t (e_I, \text{Id}) \) (\( I \subseteq \{1,...,p+q\} \)) and \( e_J \in \mathcal{C}_{r,s} \mapsto \&t (\text{Id}, e_J) \) (\( J \subseteq \{1,...,r+s\} \)), then the following procedure \( \text{cmulGTensor} := \text{proc} (x,y,B1,B2) \)

\[
\begin{align*}
f4:=(a,b,x,y) &\rightarrow \text{cmul}[B1](a,b), \text{cmul}[B2](x,y): \\
eval (\text{subs}(`&t`=f4, gswitch (&t(x,y),2))) # ordinary switch
\end{align*}
\]

\end{proc}

implements the graded tensor product algebra as explained in section 2.4. We prove in the worksheet \texttt{cmulGTensor.mw} that \( \text{bas2TGbas} (\text{cmulGTensor}(X,Y,B1,B2)) = \text{cmul}[B] (\text{bas2TBas}(X), \text{bas2TGbas}(Y)) \) is the graded algebra isomorphism with the inverse \( \text{GTbas2bas} \). The grade and reversion involutions work as expected. The limit \( B \rightarrow 0 \) implements the wedge product on \( \bigwedge V = \bigwedge V_1 \otimes \bigwedge V_2 \).

We discuss in more detail the ungraded tensor product for \( \mathcal{C}_{p+1,q+1} \simeq \mathcal{C}_{p,q} \otimes \mathcal{C}_{1,1} \), the isomorphism 6) of Thm. 2, which we call \( \text{bas2TBas} \) while its inverse is \( \text{Tbas2bas} \). We have in \( \mathcal{C}_{1,1} \), with generators \( e_1^2 = 1 = -e_2^2 \), the volume element \( \omega = e_1e_2 \) with \( \omega^2 = \lambda = 1 \), as we have to use the bilinear form \( \frac{1}{\lambda} Q_1 \) of Thm. 3 so we can still use \( \frac{1}{\lambda} B_1 = B_1 \). The isomorphism 6) reads \( e_I \in \mathcal{C}_{p,q} \mapsto \&t (e_I, \omega) \) (\( I \subseteq \{1,...,p+q\} \)) and \( e_J \in \mathcal{C}_{1,1} \mapsto \&t (\text{Id}, e_J) \) (\( J \subseteq \{1,2\} \)). The tensor Clifford product is graded isomorphic to the Clifford product on \( \mathcal{C}_{p+1,q+1} \). For a proof see the worksheet \texttt{cmulTensor11.mw}. The following procedure implements the ungraded tensor product algebra as explained in section 2.5.

\[
\begin{align*}
\text{cmulGTensor} := \text{proc} (x,y,B1,B2) \\
\text{local f4; } \\
f4:=(a,b,x,y) &\rightarrow \text{cmul}[B1](a,b), \text{cmul}[B2](x,y): \\
eval (\text{subs}(`&t`=f4, gswitch (&t(x,y),2))) # ordinary switch \\
\end{align*}
\]

\end{proc}

The isomorphism \( \text{bas2TBas} \) is this time more involved, as are the grade and reversion involutions. We still get the isomorphism \( \text{bas2TBas} (\text{cmulTensor}(X,Y,B1,B2)) = \text{cmul}[B] (\text{bas2TBas}(X), \text{bas2TBas}(Y)) \) proved by direct computation explicitly. Further details are provided in the worksheet. Note that we do not have the limit \( B \rightarrow 0 \), as \( Q_2 \simeq B_2 \) needs to be nondegenerate, and the naive limit replacing \texttt{cmul} by wedge gives false results. In the

\[3\text{Computations in the worksheet \texttt{cmulGTensor.mw} are performed for arbitrary not necessarily symmetric or diagonal bilinear forms.}\]

\[6\text{Here, } \omega \text{ in } \&t (e_I, \omega) \text{ stands for the volume element } \omega. \text{ Then, in the code of \texttt{cmulTensor}, the form } \frac{1}{\lambda} B_1 \text{ is denoted as } 1B1.\]
worksheet we show how to produce the Grassmann basis for the tensor algebra, and how the isomorphism and the involutions work.

In each worksheet we have benchmarked computations using the generic cmul routine for the Clifford product from CLIFFORD (in dim $V \leq 9$) versus the Biegebra tensor routines cmul[G]Tensor. For orthonormal bases we roughly get equal run-times. The more complicated data structures of the tensor algebras is compensated by not computing some of the off-diagonal terms.

We add to this brief discussion that it is easily possible to iterate this morphism, and to provide CLIFFORD code for computations in Clifford algebras

$$C\ell_{p+k,q+k} \simeq C\ell_{p,q} \otimes C\ell_{1,1} \otimes \cdots \otimes C\ell_{1,1},$$

or use the mod 8 periodicity.

4 Computations using matrix algebras over Clifford numbers

The isomorphism 6) from Thm. 2 was explicitly defined by Lounesto in [16 Sect. 16.3]. We will use this matrix approach to perform computations in $\mathbf{Cl}_{8,2} \simeq \text{Mat}(2, \mathbf{Cl}_{7,1})$ [7]. Let $\{e_1, \ldots, e_8\}$ be an orthonormal basis of $\mathbb{R}^{7,1}$ generating the Clifford algebra $\mathbf{Cl}_{7,1}$ such that $e_i^2 = 1$ for $1 \leq i \leq 7$, $e_8^2 = -1$, and $e_ie_j = -e_je_i$ for $\leq i, j \leq 8$ and $i \neq j$. The following $2 \times 2$ matrices (compare with (11))

$$E_i = \begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix} \quad \text{for} \quad i = 1, \ldots, 8, \quad E_9 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_{10} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (12)$$

anti-commute and generate $\mathbf{Cl}_{8,2}$. In order to effectively compute in $\mathbf{Cl}_{8,2}$, we define the isomorphism $\phi : \text{Mat}(2, \mathbf{Cl}_{7,1}) \rightarrow \mathbf{Cl}_{8,2}$ as a Maple procedure phi from the asvd package. Its inverse is given by the Maple procedure evalm. In the first step, we compute 1,024 $2 \times 2$ matrices $E_I$ with entries in $\mathbf{Cl}_{7,1}$ which represent the basis monomials $e_I = e_{i_1}e_{i_2}\cdots e_{i_k}, i_1 < i_2 < \cdots < i_k, 0 \leq k \leq 8.$ We store these matrices in a list $\mathcal{B}$.

For example, for the basis $\mathcal{B}_k$ of $k$-vectors in CLIFFORD we compute all $\binom{8}{k}$ products $E_{i_1} \land \cdots \land E_{i_k}$ $\land \cdots \land \land E_{i_8}$ where $\land \cdots \land \cdots \land$ is a CLIFFORD procedure to compute a product of Clifford algebra-valued matrices with the Clifford product applied to the matrix entries. Once the matrix representation (12) has been chosen the list $\mathcal{B} = \bigcup_k \mathcal{B}_k$ can be saved and read into a next Maple session thus avoiding the need to repeat this step.

7 Notice that $E_2^2 = E_3^2 = 1$ for $1 \leq i \leq 7$ and $E_8^2 = E_{10}^2 = -1$ in this explicit representation. Also, $\mathbf{Cl}_{1,1} \simeq \text{Mat}(2, \mathbb{R})$ is such that the matrices $e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate $\mathbf{Cl}_{1,1}$.

8 The asvd package is part of the CLIFFORD library. It was introduced in [2].

9 Note that other representations are possible. See [16 Sect. 16.3].
Having computed the basis matrices $B$, we are now ready to compute in $Cl_{8,2}$.

For example, let $x = 2I^D + 4E_{1,2,3} - 10E_{1,5,7,8,10}$ and $y = -I^D + 4E_{1,2,3,7} + E_{1,5,6,8} - 3E_{1,4,5,7}$. We can find the Clifford product $x \&_{CM} y$ of $x$ and $y$ in $Cl_{8,2}$ using the following procedure:

```maple
sCM:=proc(x::algebraic,y::algebraic)
local xy;
global phi, BBB;
if not type(evalm(x),climatrix) then error
'evalm(x) must be of type climatrix'
end if:
if not type(evalm(y),climatrix) then error
'evalm(y) must be of type climatrix'
end if:
x y := display id(evalm(x) &cm evalm(y));
return phi(x y, BBB);
end proc:
```

Once the basis $B$ is stored in Maple as a list $BBB$, the procedure $\&_{CM}$ treats it as a global variable and it returns

$$
-40E_{2,3,5,8,10} + 2E_{1,5,6,8} - 16E_7 - 4E_{1,2,3} + 10E_{1,5,7,8,10} - 10E_{6,7,10}
$$

$$
+ 8E_{1,2,3,7} + 4E_{2,3,5,6,8} - 12E_{2,3,4,6,7} - 30E_{4,5,6,8,10} - 6E_{1,4,6,7} - 2I^D.
$$

(13)

Details of the above computations can be found in the worksheet $G82.mws$. In the worksheet we also define the respective grade and reversion involutions, and the graded algebra isomorphisms $Cl_{8,2} \leftrightarrow \text{Mat}(2, Cl_{7,1})$ are given by $\phi$ and $\text{evalm}$.

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10 In the following, we let $I^D$ denote the identity matrix in $\text{Mat}(2, Cl_{7,1})$ namely

$$
\begin{pmatrix}
1d & 0 \\
0 & 1d
\end{pmatrix}
$$

where $1d$, as before, denotes the identity element in $Cl_{7,1}$.

11 On a laptop running Intel(R) Core(TM) 2 Duo CPU T6670 @ 2.20 GHz it takes 6.5 sec to obtain this result. The computation time can be shortened using parallel processing available in Maple 15 and later.
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