ASYMPTOTICS OF UNITARY MULTIMATRIX MODELS: THE SCHWINGER-DYSON LATTICE AND TOPOLOGICAL RECURSION

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Abstract. We prove the existence of a \(1/N\) expansion in unitary multimatrix models which are Gibbs perturbations of the Haar measure, and express the expansion coefficients recursively in terms of the unique solution of a non-commutative initial value problem. The recursion obtained is closely related to the “topological recursion” which underlies the asymptotics of many random matrix ensembles and appears in diverse enumerative geometry problems. Our approach consists of two main ingredients: an asymptotic study of the Schwinger-Dyson lattice over noncommutative Laurent polynomials, and uniform control on the cumulants of Gibbs measures on product unitary groups. The required cumulant bounds are obtained by concentration of measure arguments and change of variables techniques.

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1. Introduction

1.1. A noncommutative initial value problem.

1.1.1. Given a unital \( \ast \)-algebra \( B \) defined over \( \mathbb{C} \), let

\[
L = B(u_1^{\pm 1}, \ldots, u_m^{\pm 1})
\]

denote the algebra of Laurent polynomials in \( m \) noncommutative variables \( u_1, \ldots, u_m \), with noncommutative coefficients in \( B \). That is,

\[
L = B \ast \mathbb{C}(u_1^{\pm 1}, \ldots, u_m^{\pm 1}),
\]

the free product of \( B \) and the group algebra of a free group of rank \( m \).

Assuming that the dimension of \( B \) is countable, select a basis

\[
1 = b_0, b_1, b_2, \ldots
\]

in \( B \). The set of reduced words of finite length in the letters

\[
u_1^{\pm 1}, \ldots, u_m^{\pm 1}, b_1, b_2, \ldots
\]

forms a basis in \( L \). We will reserve the term *monomial* for elements of this particular basis. For a norm on \( L \), we take the \( \ell^1 \)-norm relative to the monomial basis.

1.1.2. In his study of noncommutative analogues of entropy and Fisher information \cite{Voiculescu}, Voiculescu introduced linear maps, \( \partial_1, \ldots, \partial_m \), which act on monomials \( p \in L \) according to the formula

\[
\partial_i p = \sum_{p = p_1 u_i p_2} p_1 u_i \otimes p_2 - \sum_{p = p_1 u_i^{-1} p_2} p_1 \otimes u_i^{-1} p_2.
\]

In words, \( \partial_i p \) is the sum of all simple tensors obtained from \( p \) by tensoring on the right of a \( u_i \), less the sum of all simple tensors obtained by tensoring on the left of a \( u_i^{-1} \).

Viewing the variables \( u_1, \ldots, u_m \) as coordinates on a “noncommutative \( m \)-torus”, the maps \( \partial_i \) play the role of classical partial derivatives on \( U(1)^m \), see \cite{Voiculescu}. In particular, they annihilate constants,
\[ B \subseteq \text{Ker} \partial_i, \]

are \( B \)-bilinear

\[ \partial_i(bp'b') = b(\partial_i p)b' \]

when \( L \otimes L \) is given the natural \( B \)-bimodule structure, and satisfy the product rule

\[ \partial_i(pq) = (\partial_i p)(1 \otimes q) + (p \otimes 1)(\partial_i q) \]

when \( L \otimes L \) is given the natural algebra structure. Linear maps from an algebra into its tensor square with these properties are known as \textit{derivation-comultiplications} in free probability [24], and as \textit{double derivations} in noncommutative geometry [6].

1.1.3. Consider the noncommutative initial value problem

\[
\begin{aligned}
\tau \otimes \tau(\partial_i p) &= 0 \\
\tau|_B &= \sigma
\end{aligned}
\]

where \( \tau \) is an unknown unital trace on \( L \) and \( \sigma \) is a given unital trace on \( B \). It is straightforward to establish existence and uniqueness for (1.1) — indeed, in view of the Liebniz rule, (1.1) amounts to a recurrence reducing the computation of \( \tau \) on \( L \) to the computation of \( \sigma \) on \( B \). As a simple example, the reader is invited to check that

\[ \tau(b_1u_1b_2u_1^{-1}) = \sigma(b_1)\sigma(b_2). \]

Let \( \tau_\sigma \) denote the unique solution of (1.1). Then, the \( * \)-subalgebras

\[ B, \mathbb{C}\langle u_1^{\pm 1} \rangle, \ldots, \mathbb{C}\langle u_m^{\pm 1} \rangle \]

are \( * \)-free in the noncommutative probability space \( (L, \tau_\sigma) \), see [23, Proposition 5.17]. Since free independence has a very concrete combinatorial description [19], this amounts to a combinatorial rule allowing the efficient computation of \( \tau_\sigma(p) \) for any monomial \( p \in L \).

1.1.4. It is a fundamental result of Voiculescu that, if \( \sigma \) is the limit of a sequence of matrix traces, then \( \tau_\sigma \) is the limit of a sequence of \textit{random} matrix traces [22, 23].

Let

\[ \rho_N : B \to \text{Mat}_N(\mathbb{C}) \]

be a sequence of \( * \)-representations of \( B \) whose normalized characters approximate \( \sigma \), in the sense that

\[ \lim_{N \to \infty} \frac{1}{N} \text{Tr} \rho_N(b) = \sigma(b) \]

for each \( b \in B \). Note that, since any homomorphism from a normed \( * \)-algebra into a \( C^* \)-algebra is contractive [8, §1.3.7], the image of any \( b \in B \) under \( \rho_N \) satisfies

\[ \| \rho_N(b) \| \leq \| b \|_1, \]

where \( \| \cdot \| \) is the operator norm on \( \text{Mat}_N(\mathbb{C}) \). For each \( N \geq 1 \), let

\[ U_N = (U_1, \ldots, U_m) \]
be an \( m \)-tuple of \( N \times N \) random unitary matrices drawn independently from Haar measure on the unitary group \( U(N) \). For each \( p \in L \), denote by \( \rho_N(p)(U_N) \) the \( N \times N \) random matrix obtained by replacing the constants in \( p \) according to the representation \( \rho_N \), and replacing the variables \( u_1, \ldots, u_m \) with the random matrices \( U_1, \ldots, U_m \). Then, as shown in \([23]\), one has
\[
\lim_{N \to \infty} \frac{1}{N} \mathbb{E} \, \text{Tr} \, \rho_N(p)(U_N) = \tau_\sigma(p)
\]
for each \( p \in L \).

A unital trace \( \tau \) on a \( \ast \)-algebra \( A \) is called a character of \( A \) if it is positive, i.e. if \( \tau(a^*a) \geq 0 \) for all \( a \in A \). The approximation of \( \tau_\sigma \) by random matrix traces clearly implies its positivity. Thus, from an algebraic point of view, Voiculescu’s initial value problem (1.1) provides a means to induce characters of \( L \) from characters of the constant subalgebra \( B \). From a probabilistic perspective, one has an algebraic formalism — asymptotic freeness — describing the large \( N \) asymptotic behaviour of the trace of polynomial functions of the \( m \)-tuple \( U_N \) and the deterministic contractions \( \rho_N(b_i) \).

1.2. Initial value problem with potential.

1.2.1. Collins, Guionnet and Maurel-Segala \([5]\) considered a noncommutative initial value problem which generalizes (1.1), namely
\[
\tau \otimes \tau(\partial_i p) + \tau((D_i V)p) = 0
\]
\[
\tau|_B = \sigma
\]
Here \( V \in L \) is a fixed polynomial (the “potential”), and \( D_i \) is the Laurent version of the cyclic derivative of Rota, Sagan and Stein \([20]\), i.e. the endomorphism of \( L \) which acts on monomials according to
\[
D_i p = \sum_{p=p_1 u_i p_2} p_2 p_1 u_i - \sum_{p=p_1 u_i^{-1} p_2} u_i^{-1} p_2 p_1.
\]
In words, \( D_i p \) is the sum of the cyclic shifts of \( p \) ending in \( u_i \) less the sum of the cyclic shifts of \( p \) beginning with \( u_i^{-1} \). More functorially, \( D_i = m^{op} \circ \partial_i \), where \( m^{op} \in \text{Hom}(L \otimes L, L) \) is the map which reverses multiplication in \( L \). Note that \( D_i \) is not a derivation of \( L \). However, it does annihilate \( B \), so that (1.2) degenerates to (1.1) if \( V \in B \).

Although (1.2) is no longer recursive, the authors of \([5]\) established existence and uniqueness of continuous solutions provided that the potential \( V \) is sufficiently “close” to the constant subalgebra \( B \), in an appropriate sense. This was done by first proving that (1.2) admits at most one solution via a perturbative argument, and subsequently constructing a solution \( \tau_\sigma^V \) as a limit of traces on interacting random unitary matrices whose joint distribution is a Gibbs law on \( U(N)^m \).

1.2.2. As above, let \( \rho_N \) be a sequence of matrix representations of \( B \) whose characters approximate \( \sigma \). Consider the unit-mass measure Borel measure \( \mu_N^V \) on \( U(N)^m \) defined by
\[
\mu_N^V(dU) = \frac{1}{Z_N^m} e^{N \text{Tr} \, \rho_N(V)(U)} \mu_N(dU),
\]
where $\mu_N$ is Haar measure and $Z_N^V$ is a normalization constant (the “partition function”). Note that $\mu_N^V$ is invariant under translations of $V$ by elements of $B$. In particular, if $V \in B$ is a constant potential, $\mu_N^V$ degenerates to $\mu_N$. We refer to the sequence of measures $\mu_N^V$ as the Gibbs ensemble generated by $\rho_N(V)$. Note that $\mu_N^V$ is, in general, a complex measure. However, it is a genuine probability measure if the function

$$U \mapsto \text{Tr} \rho_N(V)(U)$$

is real-valued $\mu_N$-almost everywhere on $U(N)^m$. If this condition holds, we say that $\rho_N(V)$ generates a real Gibbs ensemble. In particular, $\rho_N(V)$ generates a real Gibbs ensemble if $V$ is selfadjoint up to cyclic symmetry, i.e. if each monomial in $V^*$ is a cyclic shift of a monomial in $V$.

Let $U_N^V = (U_1, \ldots, U_m)$ be an $m$-tuple of $N \times N$ random unitary matrices whose joint distribution is the real Gibbs law $\mu_N^V$. We then have a family of scalar valued random variables given by

$$\text{Tr} \rho_N(p)(U_N^V), \quad p \in L, \quad N \geq 1.$$ 

The mean and covariance statistics of this family induce two sequences of functionals on $L$: \begin{align*}
\mathcal{W}_{1N}^V(p) &= \mathbb{E} \text{Tr} \rho_N(p)(U_N^V) \quad (1.4) \\
\mathcal{W}_{2N}^V(p_1, p_2) &= \mathbb{E} \text{Tr} \rho_N(p_1)(U_N^V) \text{Tr} \rho_N(p_2)(U_N^V) - \mathbb{E} \text{Tr} \rho_N(p_1)(U_N^V) \mathbb{E} \text{Tr} \rho_N(p_2)(U_N^V).
\end{align*}

Using a change of variables argument, it was shown in [5] that these functionals satisfy the Schwinger-Dyson equation,

$$(1.4) \quad \mathcal{W}_{1N}^V \otimes \mathcal{W}_{1N}^V(\partial_p) + N \mathcal{W}_{1N}^V((D_V^p)p) = -\mathcal{W}_{2N}^V(\partial_p).$$

1.2.3. The existence of the functional equation (1.4), which holds at finite $N$, explains why solutions of (1.2) are limits of random matrix traces. Indeed, a straightforward compactness argument shows that the sequence of linear functionals $(N^{-1} \mathcal{W}_{1N}^V)_{N=1}^\infty$ admits a limit point. Furthermore, concentration of measure techniques may be used to demonstrate that

$$(1.5) \quad \sup_N |\mathcal{W}_{2N}^V(p_1, p_2)| < \infty$$

for any $p_1, p_2 \in L$, see [3] (Corollary 4.4.31) or Corollary [32]. It follows that any limit point $\tau$ of $(N^{-1} \mathcal{W}_{1N}^V)_{N=1}^\infty$ is a solution of (1.2). Given the existence of a unique solution $\tau^V$, one thus obtains the pointwise convergence of $N^{-1} \mathcal{W}_{1N}^V$ to $\tau^V$ on $L$.

1.3. Main result: higher cumulants and topological recursion.

1.3.1. In this article, we go beyond the framework of [5] and consider the higher cumulants of several interacting random unitary matrices distributed according to a Gibbs law.

Let $\rho_N(V)$ generate a real Gibbs ensemble $\mu_N^V$, and let $U_N^V$ be an $m$-tuple of $N \times N$ random unitary matrices whose distribution in $U(N)^m$ is $\mu_N^V$. Consider the mixed moment functionals on $L$ defined by
\[
M_{kN}(p_1, \ldots, p_k) = \mathbb{E} \prod_{j=1}^{k} \text{Tr} \rho_N(p_j)(U_N^{kN}),
\]
and the corresponding mixed cumulant functionals defined recursively by
\[
M_{kN}(p_1, \ldots, p_k) = \sum_{\pi \in \text{Par}(k)} \prod_{R \in \pi} W_{|R|N}(p_r : r \in R),
\]
where \(\text{Par}(k)\) is the lattice of partitions of \(\{1, \ldots, k\}\), the internal product being over the blocks of a given partition \(\pi\). Mixed moments and mixed cumulants contain the same probabilistic information, but cumulants are easier to work with. For example, for \(k \geq 2\), the cumulant \(W_{kN}^{V}\) vanishes whenever one of its arguments lies in \(B\) — we will refer to this property as \(B\)-connectedness, or simply connectedness. We use this term because the relation between moments and cumulants can equivalently be expressed using the exponential formula of enumerative combinatorics, which is frequently used to pass between possibly disconnected and connected combinatorial structures \([21]\) Chapter 5.

Our goal in this paper is to show that, when \(\|V\|_1\) is sufficiently small, the rescaled cumulants
\[
(1.6) \quad \tilde{W}_{kN}^{V} = N^{k-2} W_{kN}^{V}
\]
admit an \(N \to \infty\) asymptotic expansion on the asymptotic scale \(N^{-2}\). Furthermore, we will give a recurrence relation completely determining all the expansion coefficients in terms of the limit of \(\tilde{W}_{1N}^{V}\).

1.3.2. As in \([5]\), our approach is based on the method of Schwinger-Dyson equations in random matrix theory. Going beyond \([5]\), we consider an entire hierarchy of noncommutative partial differential equations obtained recursively from (1.4) which encodes the asymptotics of the higher cumulants \(W_{kN}^{V}\). To solve this hierarchy, one has to invert a certain partial differential operator acting on \(B^\perp\), the space of noncommutative Laurent polynomials with no constant term. We will prove the following quantitative result.

**Theorem 1.** Let \(V \in L\) be selfadjoint up to cyclic symmetry, and suppose there exists \(K \geq 1\)
\[
\|V\|_1 < \frac{7}{66} \frac{1}{\text{deg}(V)2^{(K-1)} \text{deg} V^{12 \text{deg}(V)}}.
\]
Let \(\rho_N : B \to \text{Mat}_N(\mathbb{C})\) be a sequence of matrix representations whose normalized characters admit an \(N \to \infty\) asymptotic expansion to \(h\) terms:
\[
N^{-1} \text{Tr} \rho_N(b) = \sum_{g=0}^{h} \sigma_g(b) \frac{1}{N^{2g}} + o\left(\frac{1}{N^{2h}}\right), \quad b \in B.
\]
For each \(k \in [1, K]\) and all \(p_1, \ldots, p_k \in B^\perp\), the renormalized \(k\)th cumulant \(\tilde{W}_{kN}^{V}\) of the real Gibbs ensemble generated by \(\rho_N(V)\) admits an \(N \to \infty\) asymptotic expansion to \(h \leq K - 1\) terms:
\[
\hat{W}^V_{kN}(p_1, \ldots, p_k) = \sum_{g=0}^{h} \frac{\tau^V_{kg}(p_1, \ldots, p_k)}{N^{2g}} + o\left(\frac{1}{N^{2h}}\right).
\]

The expansion coefficients \(\tau^V_{kg}\) may be described as follows:

1. \(\tau^V_{10}\) is the unique solution of the noncommutative initial value problem \((1.2)\) with \(\sigma = \sigma_0\);
2. For \(k = 1\) and \(g > 0\),

\[
\tau^V_{1g}(p) = -\sum_{\ell=1}^{g-1} \tau^V_{1\ell} \otimes \tau^V_{1(g-\ell)}(\Delta(\Xi^V_{10})^{-1} p) - \tau^V_{2(g-1)}(\Delta(\Xi^V_{10})^{-1} p);
\]
3. For \(k > 1\) and \(g > 0\), we have

\[
\tau^V_{kg}(p_1, \ldots, p_k) = -\sum_{f=1}^{g} \tau^V_{k(g-f)}(\Xi^V_{10})^{-1} p_1, \ldots, p_k \\
- \sum_{f=0}^{g} \sum_{l=1}^{g-f} \tau^V_{l(k-l)+f} \otimes \tau^V_{(g-f)}(\Xi^V_{10})^{-1} p_1 \# p_l \otimes p_l \\
- \tau^V_{(k-1)g}(\Xi^V_{10})^{-1} p_1, \ldots, \hat{p}_j, \ldots, p_k \\
- \tau^V_{(k+1)(g-1)}(\Delta(\Xi^V_{10})^{-1} p_1, \ldots, p_k),
\]

where the second sum on the right is over all proper nonempty subsets \(I\) of \(\{2, \ldots, k\}\).

These recurrences are given in terms of certain linear transformations \(\Xi \in \text{Hom}(B^\perp, L \otimes 2)\) and \(\Xi^V, T^V, \Xi^V \in \text{End} \ B^\perp\) which will be described in the next section.

**Remark 2.** Note that Theorem 1 is stated only for polynomials \(p_1, \ldots, p_k\) which have no constant term. It is always possible to reduce to this case. Indeed, if \(p = q + r\) with \(q \in B^\perp\) and \(r \in B\), we have

\[
\hat{W}^V_{1N}(p) = \hat{W}^V_{1N}(q + r) = \hat{W}^V_{1N}(q) + \hat{W}^V_{1N}(r),
\]

by linearity. Since the asymptotics of \(\hat{W}^V_{1N}(r)\) are given by \((1.7)\), one need only determine the asymptotics of \(\hat{W}^V_{1N}(q)\). For \(k \geq 2\), if \(p_i = q_i + r_i\) with \(q_i \in B^\perp\) and \(r_i \in B\), we have

\[
\hat{W}^V_{kN}(p_1, \ldots, p_k) = \hat{W}^V_{kN}(q_1 + r_1, \ldots, q_k + r_k) = \hat{W}^V_{kN}(q_1, \ldots, q_k),
\]

by multilinearity and connectedness.

The above theorem remains true if the expansion \((1.7)\) is unknown, provided \(\sigma_0\) is replaced in the inductive relations by \(N^{-1} \text{Tr} \rho_N(b), b \in B\).
1.3.3. This paper is part of a broad program in random matrix theory, with contributions from many authors, which seeks to determine the asymptotics of both microscopic and macroscopic statistics of various classes of random matrices by leveraging information from an appropriate manifestation of the Schwinger-Dyson equations. For an overview of the microscopic side of the story, the reader is referred to [9], while the macroscopic side is surveyed in [7]. In particular, the recursion for the expansion coefficients \( \gamma^V_{kg} \) given in Theorem 1 is closely related to the topological recursion of mathematical physics [1, 3], and its modern re-imagining [10].

Theorem 1 is the unitary analogue of the theorems of Guionnet and Maurel-Segala [13, 14] and Maurel-Segala [18] on the asymptotics of the trace of polynomial functions in several interacting random Hermitian matrices whose joint distribution is a perturbation of the \( m \)-fold product GUE measure. The present paper complements these results by adapting the SD equations technology to the setting of perturbations of the \( m \)-fold product CUE measure, hence generalizing [5] to all order expansions. An additional feature of the present work is the inclusion of the background algebra \( B \), whose basis elements act as “external sources” from the random matrix viewpoint.

1.4. Asymptotic expansion of the free energy. Let \( V \in L \) be a potential such that \( \rho_N(V) \) generates a real Gibbs ensemble \( \mu^V_N \), and consider the partition function

\[
Z^V_N = \int_{U(N)^m} e^{N \text{Tr} \rho_N(V)(U)} \mu_N(dU)
\]

of \( \mu^V_N \). It was proved in [5] that

\[
Z^V_N \sim e^{N^2 F^0_V},
\]

with \( F^0_V \) a quantity independent of \( N \). To make this precise, we introduce the free energy of \( \mu^V_N \), which is by definition the quantity

\[
F_N^V := \frac{1}{N^2} \log Z^V_N.
\]

As a corollary of Theorem 1 we obtain the following refinement of [5] to all orders:

**Corollary 3.** Under the hypotheses of Theorem 1 the free energy \( F_N^V \) admits an \( N \to \infty \) expansion to \( h \leq K - 1 \) terms:

\[
\frac{1}{N^2} \log Z^V_N = \sum_{g=0}^{h} \frac{F^V_g}{N^{2g}} + o \left( \frac{1}{N^{2h}} \right).
\]

The coefficients \( F^V_g \) in this expansion depend only on \( V \) and the functionals \( \sigma_0, \ldots, \sigma_h \).

In fact, it may be shown that each \( F^V_g \) is an analytic function of the coefficients of \( V \) whose Taylor expansion serves as a generating function enumerating certain graphs drawn on a compact orientable surface of genus \( g \). These embedded graphs, or maps as they are known, are similar to the maps enumerated by the expansion of free energy of perturbations of the product GUE measure, except that they possess additional edge data. The \( g = 0 \) case of this expansion was developed in [5]. As this graphical description is rather involved, we shall not pursue the detailed development of its extension to higher genus in the present paper. We want to
stress however that the $F_V^g$’s are absolutely summable series whose coefficients are determined by the restriction of the normalized trace to the $*$-subalgebra $\rho_N(B) \subseteq \text{Mat}_N(\mathbb{C})$.

1.5. Central Limit Theorem. A corollary of the above large $N$ expansion is the following central limit theorem:

**Corollary 4.** Under the hypotheses of Theorem 1 with $K \geq 1$, for any selfadjoint polynomial $p$ in $L$, for any $\lambda \in \mathbb{R}$,

$$
\lim_{N \to \infty} \int_{U(N)^m} e^{\lambda(\text{Tr}(\rho_N(p)U) - N\tau_V^g(p))} \mu_N(dU) = e^{\frac{\lambda^2}{2} \gamma_V^g(p)}
$$

where

$$\gamma_V^g(p) = -\tau_V^g(P^g(\Xi_V^g)^{-1}p).$$

Corollary 4 should be compared with the analogous central limit theorem for traces of polynomial functions in several random Hermitian matrices whose joint law is a deformation of the product GUE measure, see [14, Theorem 4.7].

1.6. Topological combinatorics. In the present article, we content ourselves with the derivation of Theorem 1 and postpone the study of a general combinatorial/topological interpretation of the functionals $\tau_V^g$ and the affiliated coefficients $F_V^g$ to a future work. We do mention, however, the relation of Theorem 1 to the study of one particularly interesting unitary matrix model, namely the Harish-Chandra-Itzykson-Zuber model [15, 16, 17, 25].

Let $B = \mathbb{C}\langle x, y \rangle$ be the algebra of polynomials in two selfadjoint noncommutative variables $x, y$, and set $V_t = xuyu^{-1}$, with $t \in \mathbb{R}$ a real parameter. Let $\rho_N$ satisfying (1.7). The partition function of the corresponding real Gibbs ensemble $\mu_N^V$ is the HCIZ integral

$$Z_N^V = \int_{U(N)} e^{t\text{Tr}(\rho_N(x)U\rho_N(y)U^{-1})} dU.$$

Theorem 1 when combined with the results of [5] and [11], establishes the following topological expansion of the HCIZ free energy

$$F_N^V = \frac{1}{N^2} \log Z_N^V.$$

**Theorem 5.** For each $t \in \left( -\frac{7}{2^{1/2} \alpha - \frac{\alpha}{10000}}, \frac{7}{2^{1/2} \alpha - \frac{\alpha}{10000}} \right)$, the HCIZ free energy admits an $N \to \infty$ asymptotic expansion to $h \leq K - 1$ terms:

$$F_N^V = \sum_{g=0}^h \frac{F_g(t)}{N^2g} + o \left( \frac{1}{N^{2h}} \right),$$

The coefficients $F_g(t)$ in this expansion are analytic in a neighbourhood of $t = 0$, with Maclaurin series given by

$$F_g(t) = \sum_{d=1}^\infty \frac{t^d}{d!} \sum_{\alpha, \beta \vdash d} (-1)^{\ell(\alpha) + \ell(\beta)} \sigma_g(\alpha)\sigma_g(\beta) \bar{H}_g(\alpha, \beta),$$

where the internal sum is over all pairs of partitions $\alpha, \beta \vdash d$. 
\[ \sigma_g(x^\alpha) = \prod_{i=1}^{\ell(\alpha)} \sigma_g(x^{\alpha_i}), \quad \sigma_g(y^\beta) = \prod_{i=1}^{\ell(\beta)} \sigma_g(y^{\beta_i}), \]

and the \( \vec{H}_g(\alpha, \beta) \)'s are the monotone double Hurwitz numbers.

The monotone double Hurwitz number \( \vec{H}_g(\alpha, \beta) \) with \( \alpha, \beta \vdash d \) counts a combinatorially restricted subclass of the set of degree \( d \) branched covers of the Riemann sphere by a compact, connected Riemann surface of genus \( g \) such that the covering map has profile \( \alpha \) over \( \infty \), \( \beta \) over \( 0 \), and simple branching over the \( r \)th roots of unity, where \( r = 2g - 2 + \ell(\alpha) + \ell(\beta) \) by the Riemann-Hurwitz formula. For more on monotone double Hurwitz numbers, see [11]. Theorem 5 is the perturbative version of an asymptotic expansion of the HCIZ free energy conjectured to hold by Matytsin in [17].

1.7. Organization. The paper is organized as follows.

In Section 2, we cover necessary preliminaries. Most importantly, we introduce a deformation of the \( \ell^1 \)-norm on \( L \) which will play a crucial role in our analysis.

Section 3 treats the noncommutative initial value problem (1.2). We prove uniqueness of continuous solutions in a perturbative regime via an argument which is more conceptual than that employed in [5]. In particular, we introduce a pair of partial differential operators acting on \( B^\perp \) and deduce uniqueness from the invertibility of these operators.

Section 4 introduces the Schwinger-Dyson lattice over \( L \). In particular, we give all equations of this hierarchy in explicit form. We then present a secondary form of the SD lattice equations which describes them completely in terms of the first row of the lattice and the fundamental operators introduced in Section 3. This is somewhat similar in spirit to the description of classical integrable systems by means of Lax pairs. Finally, we introduce the notion of uniformly bounded solutions of the SD equations. Uniformly bounded solutions lead to a renormalized form of the SD lattice which is well-poised for asymptotic analysis.

Section 5 carries out the asymptotic analysis of an abstractly given uniformly bounded solution of the SD lattice. Our treatment is perturbative: we work exclusively in the regime where the potential \( V \) is “close” to the constant subalgebra \( B \). In this regime, the fundamental operators which describe the SD lattice are automorphisms of the completion of \( B^\perp \) in an appropriate norm. We obtain an abstract version of Theorem 1 listed as Theorem 24 below, which shows how the recursion relations of Theorem 1 arise intrinsically from the structure of the SD lattice, without any reference to random matrices.

Section 6 makes the connection with matrix models. Almost by definition, the cumulants of the Gibbs ensemble generated by \( \rho_N(V) \) form a solution of the SD lattice equations — however, this solution may not be uniformly bounded, so that Theorem 24 is not a priori applicable. When \( \rho_N(V) \) generates a real Gibbs ensemble, probabilistic tools such as concentration of measure can be brought in to verify uniform boundedness. These probabilistic arguments are carried out in Section 6. The upshot of this analysis is that Theorem 1 ultimately emerges as a corollary of its more abstract version, Theorem 24.

In Section 7, we derive a central limit theorem for the trace of polynomial functions of the \( m \)-tuple \( U_N^V = (U_1, \ldots, U_m) \) of \( N \times N \) random unitary matrices whose
joint distribution in $U(N)^m$ is the Gibbs measure $\mu_N^{\otimes m}$. We then establish the asymptotic expansion of the free energy of $\mu_N^{\otimes m}$. Finally, we combine Theorem 1 with results from [11] to obtain a proof of Theorem 5.

2. Preliminaries

2.1. Algebras and characters. All algebras in this article are normed unital $\ast$-algebras defined over $\mathbb{C}$. Homomorphisms respect $\ast$-structure.

A character of an algebra $A$ is a linear functional which is normalized, tracial, and nonnegative:

$$\tau(1) = 1, \quad \tau(ab) = \tau(ba), \quad \tau(a^*a) \geq 0.$$ 

Characters are also known as tracial states. Characters play the role of expectation functionals in noncommutative probability theory. The collection of all characters of $A$ forms a convex set, denoted $\text{Char} A$.

2.2. Constants, scalars, and correlators. Following the convention of [20], the term “constant” refers to elements of the algebra $B$, while “scalar” is reserved for elements of the one-dimensional subalgebra $\mathbb{C} 1 \subseteq B$.

For $k \geq 2$, a connected $k$-correlator is a symmetric $k$-linear functional on $L$ which is tracial in each argument, and vanishes whenever one of its arguments is a constant.

2.3. Degree filtration. Given a monomial $p \in L$, we define $\deg^+_i(p)$ to be the number of occurrences of the variable $u_i$ in $p$. Similarly, we denote by $\deg^-_i(p)$ the number of occurrences of $u_i^{-1}$ in $p$. We set

$$\deg_i(p) = \deg^+_i(p) + \deg^-_i(p),$$ 

the number of occurrences of $u_i^{\pm 1}$ in $p$, and

$$\deg(p) = \sum_{i=1}^m \deg_i(p),$$ 

the number of occurrences of $u_1^{\pm 1}, \ldots, u_m^{\pm 1}$ in $p$. Note that the degree function is not a valuation — we have

$$\deg(p_1 p_2) \leq \deg(p_1) + \deg(p_2)$$

for all monomials $p_1, p_2 \in L$, but this is not in general an equality due to the possibility of cancellations.

Let $L_d$ denote the vector subspace of $L$ spanned by the monomials of degree at most $d$. Thus $L_0 = B$, and

$$L_0 \subseteq L_1 \subseteq \ldots L_d \subseteq \ldots, \quad L_k L_d \subseteq L_{k+l}, \quad \bigcup_{d=0}^{\infty} L_d = L,$$

so that we have a filtration of $L$. We extend the domain of the degree function by declaring $\deg(p) = d$ for any $p \in L_d$. The degree filtration does not see the difference between constants and scalars.

In Section 6 we will need the notion of balanced polynomials.
Definition 6. A monomial $p \in L$ is said to be balanced if

$$\sum_{i=1}^{m} \text{deg}_{i}^{+}(p) = \sum_{i=1}^{m} \text{deg}_{i}^{-}(p).$$

A polynomial is balanced if it is the sum of balanced monomials.

2.4. Inner product. We equip $L$ with the inner product in which the monomials form an orthonormal basis. We then have

$$B_{\perp} = \bigcup_{d=1}^{\infty} L_{d},$$

the space of Laurent polynomials with no constant term.

Any $p \in L$ decomposes as

$$p = \sum_{q} \langle q, p \rangle q,$$

where the sum is over the monomial basis in $L$. Our convention is that inner products are linear in the second argument.

2.5. Parametric norm. Let $\xi$ be a positive parameter, and for each $p \in L$ set

$$\| p \|_{\xi} = \sum_{q} |\langle q, p \rangle| \xi^{\text{deg}(q)},$$

where the summation is over the monomial basis in $L$. Our proposition is that inner products are linear in the second argument.

Proposition 7. For any $\xi \geq 1$, $(L, \| \cdot \|_{\xi})$ is a normed $\ast$-algebra.

Proof. We leave it to the reader to check that $\| \cdot \|_{\xi}$ is a vector space norm with respect to which the involution in $L$ is an isometry. We prove that $\| \cdot \|_{\xi}$ is an algebra norm. This is where the condition $\xi \geq 1$ is required. Indeed

$$\| p_{1}p_{2} \|_{\xi} = \left\| \left( \sum_{q_{1}} \langle q_{1}, p_{1} \rangle q_{1} \right) \left( \sum_{q_{2}} \langle q_{2}, p_{2} \rangle q_{2} \right) \right\|_{\xi}$$

$$\leq \sum_{q_{1}, q_{2}} |\langle q_{1}, p_{1} \rangle \langle q_{2}, p_{2} \rangle| \| q_{1}q_{2} \|_{\xi},$$

since $\| \cdot \|_{\xi}$ is a vector space norm,

$$= \sum_{q_{1}, q_{2}} |\langle q_{1}, p_{1} \rangle \langle q_{2}, p_{2} \rangle| \xi^{\text{deg}(q_{1}) + \text{deg}(q_{2})} \xi^{\text{deg}(q_{1}) + \text{deg}(q_{2})},$$

since $\xi \geq 1$,

$$= \| p_{1} \|_{\xi} \| p_{2} \|_{\xi}.$$ 

The norm $\| \cdot \|_{\xi}$ is a deformation of the usual $\ell^{1}$-norm, which is the case $\xi = 1$. Note that, while all monomials are unit vectors in the $\ell^{1}$-norm, this is not the case for $\xi > 1$. In the range $\xi > 1$, the $\xi$-norm favours monomials of high degree and penalizes monomials of low degree.
Proposition 8. For any \( p \in L \) and any \( 1 \leq \xi_1 \leq \xi_2 \), we have
\[
\|p\|_{\xi_1} \leq \|p\|_{\xi_2} \leq \|p\|_{\xi_1} \left( \frac{\xi_2}{\xi_1} \right)^{\deg(p)}.
\]

Proof. The first inequality is obvious. For the second, we argue as follows:
\[
\|p\|_{\xi_2} = \sum_q |\langle q, p \rangle|_{\xi_2}^{\deg(q)} = \sum_q |\langle q, p \rangle|_{\xi_1}^{\deg(q)} \left( \frac{\xi_2}{\xi_1} \right)^{\deg(q)}
\leq \left( \frac{\xi_2}{\xi_1} \right)^{\deg(p)} \sum_q |\langle q, p \rangle|_{\xi_1}^{\deg(q)} = \left( \frac{\xi_2}{\xi_1} \right)^{\deg(p)} \|p\|_{\xi_1},
\]
where to obtain the inequality we used the fact that, by definition, \( \deg(q) \leq \deg(p) \) for any monomial \( q \) appearing in \( p \). \( \square \)

2.6. Continuous functionals and operators. A linear functional \( f \in \text{Hom}(L, \mathbb{C}) \) is \( \xi \)-continuous if and only if there exists a constant \( C \) such that
\[
|f(p)| \leq C \|p\|_{\xi}
\]
for all \( p \in L \). We denote by \( \text{Hom}_{\xi}(L, \mathbb{C}) \) the set of all \( \xi \)-continuous linear functionals on \( L \); it is a vector subspace of \( \text{Hom}(L, \mathbb{C}) \). The \( \xi \)-norm of \( f \in \text{Hom}_{\xi}(L, \mathbb{C}) \), denoted \( \|f\|_{\xi} \), is the infimum over all \( C \) such that the above Lipschitz inequality holds. We have
\[
\xi_1 \leq \xi_2 \implies \text{Hom}_{\xi_1}(L, \mathbb{C}) \subseteq \text{Hom}_{\xi_2}(L, \mathbb{C}).
\]
A linear operator \( T \in \text{End} L \) is \((\xi_1, \xi_2)\)-continuous if and only if there exists a constant \( C \) such that
\[
\|Tp\|_{\xi_2} \leq C \|p\|_{\xi_1}
\]
for all \( p \in L \), and \( \|T\|_{\xi_1, \xi_2} \) is the infimum over all \( C \) such that this inequality holds. The set of \((\xi_1, \xi_2)\)-continuous linear operators on \( L \) is a unital \(*\)-subalgebra of \( \text{End}(L) \) denoted \( \text{End}_{\xi_1, \xi_2}(L) \). For any \( \xi_0 \geq 1 \),
\[
\xi_1 \leq \xi_2 \implies \text{End}_{\xi_1, \xi_0}(L) \subseteq \text{End}_{\xi_2, \xi_0}(L).
\]
We will shorten \( \text{End}_{\xi, \xi}(L) \) to \( \text{End}_{\xi}(L) \), and refer to elements of this algebra as \( \xi \)-continuous rather than \((\xi, \xi)\)-continuous.

2.7. Tensor powers. We will need the tensor powers of \( L \),
\[
L^{\otimes k} = L \otimes \cdots \otimes L \overset{k\text{ times}}{\longrightarrow}.
\]
We equip \( L^{\otimes k} \) with the natural algebra structure in which simple tensors are multiplied according to the rule
\[
(p_1 \otimes \cdots \otimes p_k)(q_1 \otimes \cdots \otimes q_k) = p_1 q_1 \otimes \cdots \otimes p_k q_k.
\]
Simple tensors all of whose factors are monomials form a basis of this algebra which, by abuse of language, we will refer to as the monomial basis of \( L^{\otimes k} \). We equip \( L^{\otimes k} \) with the inner product in which the monomial basis is orthonormal.
All of the above constructions for $L$ go through for $L^\otimes k$. We have the degree function defined by

$$\deg(p_1 \otimes \cdots \otimes p_k) = \deg(p_1) + \cdots + \deg(p_k),$$

and the corresponding degree filtration in $L^\otimes k$. We also have the corresponding $\xi$-norm on $L^\otimes k$, which is defined by

$$\|T\|_\xi = \sum_{q_1 \otimes \cdots \otimes q_k} |\langle q_1 \otimes \cdots \otimes q_k, T \rangle|^{\xi \deg(q_1 \otimes \cdots \otimes q_k)}$$

for all $T \in L^\otimes k$, the summation being over the monomial basis of $L^\otimes k$. We have the identity

$$\|p_1 \otimes \cdots \otimes p_k\|_\xi = \|p_1\|_\xi \cdots \|p_k\|_\xi$$

for simple tensors in $L^\otimes k$.

By convention, $L^\otimes 0$ is the line $\mathbb{C}1$ in $L$ spanned by the unit element. Note that there is a unique algebra isomorphism $\mathbb{C}1 \to \mathbb{C}$ given by $1 \mapsto 1$, and under this identification the $\xi$-norm identifies with the usual norm on $\mathbb{C}$ for any value of $\xi$.

We will consider linear transformations

$$T : (L^\otimes k_1, \| \cdot \|_{\xi_1}) \to (L^\otimes k_2, \| \cdot \|_{\xi_2})$$

mapping between the various tensor powers of $L$. A linear transformation $T \in \text{Hom}(L^\otimes k_1, L^\otimes k_2)$ is $(\xi_1, \xi_2)$-continuous if and only if there exists a constant $C$ such that

$$\|Tp_1 \otimes \cdots \otimes p_k\|_{\xi_2} \leq C\|p_1 \otimes \cdots \otimes p_k\|_{\xi_1}$$

for all monomials $p_1 \otimes \cdots \otimes p_k \in L^\otimes k_1$. The operator norm of $T$, denoted $\|T\|_{\xi_1, \xi_2}$, can be calculated by infimizing $C$ over the monomial basis.

Allowing different instances of the $\xi$-norm on the source and target of our linear maps is useful for the following reason. Certain linear transformations which we will need to deal with are not $(\xi, \xi)$-continuous for any $\xi \geq 1$, but are $(\xi_1, \xi_2)$-continuous, and even contractive, if the ratio $\xi_1/\xi_2$ is large enough.

2.8. Completion. We denote by $L_\xi$ the completion of $L$ with respect to the $\xi$-norm. Viewing $L$ as the algebra of polynomial functions $p(u_1, \ldots, u_m)$ on a noncommutative $m$-torus, $L_\xi$ may be viewed as the algebra of functions $f(u_1, \ldots, u_m)$ whose Fourier coefficients $\langle q, f \rangle$ decay faster than $\xi^{\deg(q)}$. In particular, $\xi_1 \leq \xi_2$ implies $L_{\xi_1} \supseteq L_{\xi_2}$.

3. The initial value problem revisited

In this section we consider the noncommutative initial value problem (1.2) and prove uniqueness of solutions in a perturbative regime.

**Theorem 9.** Let $\sigma \in \text{Hom}_1(B, C)$ be a unital trace. If $V \in L$ satisfies

$$\|\Pi V\|_1 \leq \frac{7}{66} \cdot \frac{1}{\deg(V)12^{2\deg(V)}},$$

where $\Pi$ is the orthogonal projection of $L$ onto $B^\perp$, then there is at most one unital trace $\tau \in \text{Hom}_1(L, C)$ which satisfies for all $p \in L$. 

\[ \tau \otimes \tau (\partial_i p) + \tau ((D_i V)p) = 0 \]
\[ \tau|_\mathcal{B} = \sigma \].

A non-quantitative version of the same result was obtained in [5, Theorem 3.1]. Here we give a new, more conceptual argument based on the inversion of a certain differential operator acting on noncommutative Laurent polynomials. The methods developed in this section will be repeatedly applied in the remainder of the paper, and their introduction at an early stage clarifies the exposition.

3.1. The cyclic gradient trick. Our approach to the initial value problem (1.2) is based on considering its implications for the coordinates of the cyclic gradient of a monomial \( p \),

\[ Dp = (D_1 p, \ldots, D_m p). \]

Any solution \( \tau \) of (1.2) must satisfy

\[ \tau \otimes \tau (\partial_i D_i p) + \tau ((D_i V)(D_i p)) = 0, \quad 1 \leq i \leq m. \]  

**Proposition 10.** For any unital trace \( \tau \) on \( L \), we have

\[ \tau \otimes \tau (\partial_i D_i p) = \tau (D_i p) + \tau \otimes (\Delta_i p), \]

where \( D_i \in \text{End} \ L \) acts on monomials according to

\[ D_i p = \text{deg}_i(p)p, \]

and \( \Delta_i \in \text{Hom}(L, L \otimes L) \) acts on monomials according to

\[ \Delta_i p = \sum_{p=\sum u_{i}^{-1}u_{2}u_{1}} q_{1}u_{i} \otimes q_{2}u_{i} - \sum_{p=\sum u_{i}^{-1}u_{2}u_{1}} q_{1}u_{i}^{-1}q_{2}u_{i} \]
\[ - \sum_{p=\sum u_{i}^{-1}u_{2}u_{1}} q_{1}u_{i}^{-1}q_{1} - \sum_{p=\sum u_{i}^{-1}u_{2}u_{1}} q_{2}u_{i}^{-1}q_{2} \]

**Proof.** Let \( p \in L \) be a monomial. We will expand the tensor \( \partial_i D_i p \) into simple tensors. We have

\[ D_i p = \sum_{p=p_1 u_{i}^{-1}p_2} p_2 p_1 u_{i} - \sum_{p=p_1 u_{i}^{-1}p_2} u_{i}^{-1} p_2 p_1, \]

the sum of the cyclic shifts of \( p \) ending in \( u_{i} \) less the sum of the cyclic shifts of \( p \) beginning with \( u_{i}^{-1} \). Applying \( \partial_i \), this becomes
\[ \partial_i D_i p = \sum_{p=p_1 u_i p_2} \partial_i p_2 p_1 u_i - \sum_{p=p_1 u_i^{-1} p_2} \partial_i u_i^{-1} p_2 p_1 \]
\[ = \sum_{p=p_1 u_i p_2} \left( p_2 p_1 u_i \otimes 1 + \sum_{p_2 p_1 u_i = q_1 u_i q_2 u_i} q_1 u_i \otimes q_2 u_i - \sum_{p_2 p_1 u_i = q_1 u_i^{-1} q_2 u_i} q_1 \otimes u_i^{-1} q_2 u_i \right) \]
\[ - \sum_{p=p_1 u_i^{-1} p_2} \left( \sum_{u_i^{-1} p_2 p_1 = q_1 u_i q_2} u_i^{-1} q_1 u_i \otimes q_2 - \sum_{u_i^{-1} p_2 p_1 = q_1 u_i^{-1} q_2} u_i^{-1} q_1 \otimes u_i^{-1} q_2 - 1 \otimes u_i^{-1} p_2 p_1 \right) \]

Applying \( \tau \otimes \tau \) to this tensor and using the fact that \( \tau \) is a unital trace, the result follows.

By Proposition II, equation 3.1 may be rewritten

\[ (3.3) \]
\[ \tau \left( (D_i + \frac{1}{2} (1 \otimes \tau + \tau \otimes 1)d) \Delta_i + P^V_i \right)p = 0, \]

where \( P^V_i p = (D_i V)(D_i p) \). Summing over \( 1 \leq i \leq m \), we have

\[ \tau \left( (D + \frac{1}{2} T_\tau + P^V) \right)p = 0, \]

where

\[ D = \sum_{i=1}^m D_i, \quad T_\tau = (1 \otimes \tau + \tau \otimes 1) \sum_{i=1}^m \Delta_i, \quad P^V = \sum_{i=1}^m P^V_i. \]

**Remark 11.** The characteristic property of the operators \( D = \sum_{i=1}^m D_i \) and \( \Delta = \sum_{i=1}^m \Delta_i \) is that

\[ \tau \otimes (\partial (\Delta_i p)) = \tau (Dp) + \tau \otimes (\tau \Delta p) \]

for any unital trace \( \tau \) on \( L \). The transformation \( \sum_{i=1}^m \partial_i D_i \) is a natural noncommutative analogue of the Laplacian on an \( m \)-torus. The operator \( D \) is called the **number operator**, and the transformation \( \Delta \) is called the **reduced Laplacian**.

The summands \( \Delta_i \) of the reduced Laplacian \( \Delta \in \text{Hom}(L, L \otimes ^2) \) act on monomials according to the formula 3.2. If \( p \) is a monomial of degree zero, then the outer sums in this formula are empty, and \( \Delta \circ \tau = 0 \otimes 0 \). If \( p \) is a monomial of degree one, then the inner sums in this formula are empty, and \( \Delta \circ \tau = 0 \otimes 0 \). If \( p \) is a monomial of degree \( d \geq 2 \) which factors as \( p = p_1 u_i p_2 \), and if the cyclic shift \( p_2 p_1 u_i \) factors as \( p_2 p_1 u_i = q_1 u_i q_2 u_i \), then the tensor \( q_1 u_i \otimes q_2 u_i \) has degree at most \( d \), but neither of its factors has degree zero. If \( p_2 p_1 u_i \) factors as \( p_2 p_1 u_i = q_1 u_i^{-1} q_2 u_i \), then the tensor \( q_1 \otimes q_2 \) has degree at most \( d - 2 \). Similarly, if \( p \) factors as \( p = p_1 u_i^{-1} p_2 \) and the cyclic shift \( u_i^{-1} p_2 p_1 \) factors as \( u_i^{-1} q_1 u_i q_2 \), then the tensor \( q_1 \otimes q_2 \) has degree at most \( d - 2 \). If \( u_i^{-1} p_2 p_1 \) factors as \( u_i^{-1} q_1 u_i^{-1} q_2 \), then the tensor \( u_i^{-1} q_1 \otimes u_i^{-1} q_2 \) has degree at most \( d \), but neither of its factors has degree zero. From these considerations, we conclude that
\[ L_d \xrightarrow{\Delta} \bigvee_{k=1}^{d-1} L_k \otimes L_{d-k}. \]

Since \( T_\tau \in \text{End } L \) is the contraction of \( \Delta \) by \( \text{Id} \otimes \tau + \tau \otimes \text{Id} \), we conclude from the above that it is strictly upper triangular with respect to the degree filtration in \( L \):

\[ \ldots \xrightarrow{T_\tau} L_3 \xrightarrow{T_\tau} L_2 \xrightarrow{T_\tau} L_1 \xrightarrow{T_\tau} \{0\}. \]

The number operator \( D \) acts diagonally in \( L \) with spectrum 0, 1, 2, \ldots and corresponding eigenspaces

\[ B = L_0, L_1/L_0, L_2/L_1, \ldots. \]

Obviously, the kernel of \( D \) is \( B \).

The operator \( P^V \) is the dot product, \( D^V \cdot D^p \), of the cyclic gradient of \( V \) with the cyclic gradient of \( p \):

\[ P^V p = \sum_{i=1}^{m} (D^i V)(D^i p). \]

Unlike \( D \) and \( T_\tau \), the operator \( P^V \) does not respect the degree filtration in \( L \). When \( V \) is “small,” in an appropriate sense, \( P^V \) will be a perturbation of the upper triangular operator \( D + T_\tau \), hence our notation.

Since the operators \( D, T_\tau, P^* \) all annihilate \( B \), equation (3.3) contains no information concerning the behaviour of \( \tau \) on \( B \). This is an artifact of the cyclic gradient trick, but it results in no loss of information since our initial value problem stipulates \( \tau|_B = \sigma \). Now, the operator \( D \) is an automorphism of \( B^\perp \), the space of polynomials with no constant term, and hence we may regularize by the inverse of this operator. As we will see in a moment, this regularization has the effect of making the operators we have introduced in order to describe the SD equations \( \xi \)-continuous in the range \( \xi > 1 \).

**Definition 12.** For any linear transformation \( T \) with domain \( L \), we define its *degree regularization* by

\[ \overline{T} := TD^{-1}. \]

It is understood that the domain of the regularized operator \( \overline{T} \) is restricted to \( B^\perp \).

We now regularize equation (3.3), obtaining

\[ \tau \left( (\text{Id} + \frac{1}{2} \overline{T_\tau} + P^V)p \right) = 0. \]

A demerit of the operator

\[ \text{Id} + \frac{1}{2} \overline{T_\tau} + P^V \]

is that, since \( B^\perp \) is not invariant under the action of the strictly upper triangular operator \( \overline{T_\tau} \), it is not an endomorphism of \( B^\perp \). To rectify this, let \( \Pi \) be the orthogonal projection of \( L \) onto \( B^\perp \), and let \( \Pi^\prime \) be the complementary projection of \( L \) on \( B \).
Definition 13. Let \(\tau\) be a unital trace on \(L\), and let \(V \in L\) be a polynomial. The first fundamental operator associated to the data \(\tau, V\) is the endomorphism of \(B^\perp\) defined by

\[ \Psi^V_\tau = \text{Id} + \frac{1}{2} \Pi T_\tau + \Pi^V. \]

The second fundamental operator associated to \(\tau, V\) is the endomorphism of \(B^\perp\) defined by

\[ \Xi^V_\tau = \text{Id} + \Pi T_\tau + \Pi^V. \]

Note that the first and second fundamental operators associated to a given unital trace \(\tau\) are essentially the same; the precise relation between them is

\[ \Xi^V_\tau = \Psi^V_\tau + \frac{1}{2} \Pi T_\tau. \]

In the next section, we will study a lattice of noncommutative partial differential equations, the Schwinger-Dyson lattice, whose rows are described by these operators. The first fundamental operator governs the first row of the lattice, while the higher rows are controlled by the second fundamental operator.

The following property of the fundamental operators follows immediately from their definition.

Proposition 14. For any linear functionals \(\tau_0, \ldots, \tau_h\), we have

\[ \Psi^V_{\sum_{g=0}^h \tau_g} = \Psi^V_{\tau_0} + \sum_{g=1}^h \frac{1}{2} \Pi T_{\tau_g} \]

and

\[ \Xi^V_{\sum_{g=0}^h \tau_g} = \Xi^V_{\tau_0} + \sum_{g=1}^h \Pi T_{\tau_g}. \]

In terms of the first fundamental operator, equation (3.4) becomes

(3.5) \[ \tau(\Psi^V_{\tau} p) = -\frac{1}{2} \sigma(\Pi T_\tau p). \]

Suppose that \(\tau, \tau'\) are two solutions of the initial value problem (1.2), and set \(\delta = \tau' - \tau\).

Proposition 15. We have the quadratic constraint

\[ \delta(\Xi^V_\tau p) = -\delta \otimes (\delta p), \quad p \in B^\perp. \]

Proof. Since \(\tau, \tau'\) are solutions of (1.2), we have

\[ [\tau' \otimes \tau' - \tau \otimes \tau] (\partial_t p) + [\tau' - \tau] ((D_t V)p) = 0. \]

Using the identity

\[ \tau' \otimes \tau' - \tau \otimes \tau = \delta \otimes \tau + \tau \otimes \delta + \delta \otimes \delta, \]

this can be rewritten
The continuity properties
\[\delta((D,V)p) = -\delta(D)p.\]
Now use the cyclic gradient trick: replace \(p\) with \(D_ip\), and sum over \(1 \leq i \leq m\).

3.2. Operator norm estimates. In this subsection, we establish basic continuity properties of the regularized upper triangular operator \(\mathcal{T}_\bullet\), the regularized perturbation \(\tilde{\mathcal{T}}\), and the regularized reduced Laplacian \(\tilde{\Delta}\). These continuity properties will be essential in the analysis to follow.

**Proposition 16.** Let \(f \in \text{Hom}_1(L, \mathbb{C})\). Then \(\tilde{\mathcal{T}}_f \in \text{End}_\xi(B^\perp)\) for any \(\xi > 1\), and
\[
\|\tilde{\mathcal{T}}_f\|_\xi < 4\|f\|_1 \frac{\xi + 1}{\xi(\xi - 1)}.
\]

**Proof.** By definition, the unregularized operator \(\mathcal{T}_f\) acts on monomials \(p\) according to
\[
\mathcal{T}_f p = \sum_{i=1}^m \sum_{p = p_1 u_i p_2} \left( \sum_{p_2 p_1 u_i = q_1 u_i q_2 u_i} (q_1 u_i f(q_2 u_i) + f(q_1 u_i)q_2 u_i) 
- \sum_{p_2 p_1 u_i = q_1 u_i^{-1} q_2 u_i} (q_1 f(q_2) + f(q_1)q_2) 
- \sum_{i=1}^m \sum_{p = p_1 u_i^{-1} p_2} \sum_{u_i^{-1} p_2 p_1 = u_i^{-1} q_1 u_i q_2} (q_1 f(q_2) + f(q_1)q_2) 
- \sum_{u_i^{-1} p_2 p_1 = u_i^{-1} q_1 u_i^{-1} q_2} (u_i^{-1} q_1 f(u_i^{-1} q_2) + f(u_i^{-1} q_1)u_i^{-1} q_2) \right).
\]

Using the triangle inequality in \((L, \|\cdot\|_\xi)\) and that \(|f(p)| \leq \|f\|_1 \|p\|_1 = \|f\|_1\) for all monomials \(p \in L\), we obtain
\[
\|\mathcal{T}_f p\|_\xi \leq \|f\|_1 \sum_{i=1}^m \sum_{p = p_1 u_i p_2} \left( \sum_{p_2 p_1 u_i = q_1 u_i q_2 u_i} (\|q_1 u_i\|_\xi + \|q_2 u_i\|_\xi) + \sum_{p_2 p_1 u_i = q_1 u_i^{-1} q_2 u_i} (\|q_1\|_\xi + \|q_2\|_\xi) \right) 
+ \|f\|_1 \sum_{i=1}^m \sum_{p = p_1 u_i^{-1} p_2} \left( \sum_{u_i^{-1} p_2 p_1 = u_i^{-1} q_1 u_i q_2} (\|q_1\|_\xi + \|q_2\|_\xi) + \sum_{u_i^{-1} p_2 p_1 = u_i^{-1} q_1 u_i^{-1} q_2} (\|u_i^{-1} q_1\|_\xi + \|u_i^{-1} q_2\|_\xi) \right).
\]

Since \(\xi > 1\), we have, for \(\deg(p) = d\),
\[ \sum_{p_2 p_1 u_i = q_1 u_i q_2 u_i} \|q_1 u_i\| \leq \xi + \cdots + \xi^{d-1} < \frac{1}{\xi - 1} \|p\| \]
\[ \sum_{p_2 p_1 u_i = q_1 u_i^{-1} q_2 u_i} \|q_1\| \leq 1 + \cdots + \xi^{d-2} < \frac{1}{\xi(\xi - 1)} \|p\| \]
\[ \sum_{u_i^{-1} p_2 p_1 = u_i^{-1} q_1 u_i q_2} \|q_1\| \leq 1 + \cdots + \xi^{d-2} < \frac{1}{\xi(\xi - 1)} \|p\| \]
\[ \sum_{u_i^{-1} p_2 p_1 = u_i^{-1} q_1 u_i^{-1} q_2} \|u_i^{-1} q_1\| \leq \xi + \cdots + \xi^{d-1} < \frac{1}{\xi - 1} \|p\|, \]
and similarly for the four additional sums involving the symbol \( q_2 \). Thus, we have
\[ \| T f \| \leq 4 \| f \|_1 \xi^d \| p \|, \]
from which the claim follows. \( \square \)

**Proposition 17.** For any \( V \in L \) and any \( \xi \geq 1 \), we have \( P^V \in \text{End}_\xi(B^\perp) \) and
\[ \| P^V \| \leq \| IV \|_1 \text{deg}(V) \xi^{\text{deg}(V)}. \]

**Proof.** The operator \( P^V \) acts on monomials \( p \in B^\perp \) according to
\[ P^V p = \frac{1}{\text{deg} p} \sum_{i=1}^m (D_i V)(D_i p). \]
Thus
\[ \| P^V \| \leq \frac{1}{\text{deg} p} \sum_{i=1}^m \| (D_i V)(D_i p) \| \leq \frac{1}{\text{deg} p} \sum_{i=1}^m \| D_i V \| \| D_i p \| \xi \]
\[ \leq \frac{1}{\text{deg} p} \left( \sum_{i=1}^m \| D_i V \| \xi \right) \left( \sum_{i=1}^m \| D_i p \| \xi \right). \]
Since \( p \) is a monomial, we have
\[ \| D_i p \| \xi = \left\| \sum_{p=p_1 u_i, p_2} p_2 p_1 u_i - \sum_{p=p_1 u_i^{-1}, p_2} u_i^{-1} p_2 p_1 \right\| \xi \]
\[ \leq \sum_{p=p_1 u_i, p_2} \| p_2 p_1 u_i \| + \sum_{p=p_1 u_i^{-1}, p_2} \| u_i^{-1} p_2 p_1 \| \xi \]
\[ \leq (\text{deg}_1^+(p) + \text{deg}_1^-(p))\|p\|_1 \text{deg}_1(p)\|p\|_1 \xi, \]
so that
\[ \sum_{i=1}^m \| D_i p \| \xi \leq \text{deg}(p)\|p\|_1 \xi. \]
To estimate the factor depending on \( V \), we proceed as follows:
\[
\sum_{i=1}^{m} \|D_i V\|_{\xi} = \sum_{i=1}^{m} \left\| D_i \sum_{q} \langle q, V \rangle q \right\|_{\xi} \\
\leq \sum_{i=1}^{m} \sum_{q \in B^\perp} |\langle q, V \rangle| \|D_i q\|_{\xi} \\
= \sum_{q \in B^\perp} |\langle q, V \rangle| \|q\|_{\xi} \deg(q) \\
\leq \deg(V) \sum_{q \in B^\perp} |\langle q, V \rangle| \|q\|_{\xi} \\
= \deg(V) \|IV\|_{\xi} \\
\leq \deg(V) \xi^{\deg(V)} \|IV\|_{1},
\]
where the last inequality follows from Proposition 8. Thus we have proved

\[
(3.6) \quad \|P^VD^{-1}p\|_{\xi} \leq \|IV\|_{1} \deg(V) \xi^{\deg(V)} \|p\|_{\xi},
\]
from which the claim follows.

\[
\square
\]

**Proposition 18.** For any \(\xi_1, \xi_2 \geq 1\) such that \(\xi_1 \geq 2\xi_2\), the regularized reduced Laplacian \(\Xi\) is a contractive mapping of \((B^\perp, \| \cdot \|_{\xi_1})\) into \((L^2, \| \cdot \|_{\xi_2})\).

**Proof.** Let \(p \in B^\perp\) be a monomial of degree \(d\). We have

\[
\Xi p = \frac{1}{d} \sum_{i=1}^{m} \Delta_i p.
\]

Now,

\[
\|\Delta_i p\|_{\xi_2} \leq \sum_{p=p_1 u_i p_2} \left( \sum_{p_1 p_2 u_i = q_1 u_i q_2 u_i} \|q_1 u_i\|_{\xi_2} \|q_2 u_i\|_{\xi_2} + \sum_{p_1 p_2 u_i = q_1^{-1} u_i q_2^{-1} u_i} \|q_1\|_{\xi_2} \|q_2\|_{\xi_2} \right) \\
+ \sum_{p=p_1 u_i^{-1} p_2} \left( \sum_{u_i^{-1} p_1 p_2 u_i = q_1 u_i q_2} \|q_1\|_{\xi_2} \|q_2\|_{\xi_2} + \sum_{u_i^{-1} p_1 p_2 = q_1^{-1} u_i q_2^{-1} q_2} \|q_1\|_{\xi_2} \|q_2\|_{\xi_2} \right) \\
= \sum_{p=p_1 u_i p_2} \left( (\deg_i^+(p) - 1)\xi_2^d + \deg_i^-(p)\xi_2^{d-2} \right) + \sum_{p=p_1 u_i^{-1} p_2} \left( (\deg_i^+(p)\xi_2^{d-2} + (\deg_i^-(p) - 1)\xi_2^d \right) \\
\leq \deg_i(p)^2 \xi_2^d.
\]

We thus have, since \(d \leq 2^d \leq (\xi_2)\xi_2^d\) for all \(d \in \mathbb{N}\),

\[
\|\Xi p\|_{\xi_2} \leq \frac{\xi_2^d}{d} \sum_{i=1}^{m} (\deg_i(p))^2 \leq d\xi_2^d \leq \left(\frac{\xi_1}{\xi_2}\right)^d \xi_2^d = \xi_1^d = \|p\|_{\xi_1}.
\]

\[
\square
\]
3.3. Uniqueness. Let \( \tau \) be a unital trace on \( L \) such that \( \| \tau \|_1 \leq 1 \), and let \( V \in L \) be a potential. From Propositions 16 and 17 we conclude that the fundamental operators associated to \( \tau, V \) are \( \xi \)-continuous endomorphisms of \( B^\perp \) whose norms satisfy

\[
\| \Psi^V_\tau - \text{Id} \|_\xi < \frac{\xi + 1}{\xi(\xi - 1)} + \| \Pi V \|_1 \deg(V) \xi^{\deg(V)}
\]

\[
\| \Xi^V_\tau - \text{Id} \|_\xi < 4 \frac{\xi + 1}{\xi(\xi - 1)} + \| \Pi V \|_1 \deg(V) \xi^{\deg(V)}.
\]

Consequently, \( \Psi^V_\tau \) and \( \Xi^V_\tau \) extend uniquely to continuous endomorphisms of \( B^\perp_\xi \), the completion of \( B^\perp \) in the norm \( \| \cdot \|_\xi \). Now, since \( B^\perp_\xi \) is complete, \( C(B^\perp_\xi) \) is a Banach algebra. Thus, if

\[
K(\xi, V) := 4 \frac{\xi + 1}{\xi(\xi - 1)} + \| \Pi V \|_1 \deg(V) \xi^{\deg(V)} < 1,
\]

then \( \Psi^V_\tau \) and \( \Xi^V_\tau \) are continuous automorphisms of \( B^\perp_\xi \) with inverses

\[
(\Psi^V_\tau)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\frac{1}{2} T_\tau + \overline{P}^V)^n
\]

\[
(\Xi^V_\tau)^{-1} = \sum_{n=0}^{\infty} (-1)^n (T_\tau + \overline{P}^V)^n
\]

with norms bounded by

\[
\| (\Psi^V_\tau)^{-1} \|_\xi \leq \frac{1}{1 - K(\xi, V)}; \quad \| (\Xi^V_\tau)^{-1} \|_\xi \leq \frac{1}{1 - K(\xi, V)}.
\]

We next show that the condition \( K(\xi, V) \) small enough implies uniqueness. Let \( \tau, \tau' \) be solutions of (1.2) such that \( \| \tau \|_1, \| \tau' \|_1 \leq 1 \), and set \( \delta = \tau' - \tau \). Then, by Proposition 15 and the invertibility of \( \Xi^V_\tau \), we have the identity

\[
\delta = -(\delta \otimes \delta) \overline{\Delta}(\Xi^V_\tau)^{-1}
\]

in \( \text{Hom}(B^\perp_\xi, \mathbb{C}) \). Taking operator norms, we obtain the inequality

\[
\| \delta \|_\xi \leq \| \delta \|_\xi \| (\text{Id} \otimes \delta) \overline{\Delta} \|_\xi \| (\Xi^V_\tau)^{-1} \|_\xi.
\]

If \( \| \delta \|_\xi \neq 0 \), we may cancel it from both sides of this inequality to obtain

\[
1 \leq \| (\text{Id} \otimes \delta) \overline{\Delta} \|_\xi \| (\Xi^V_\tau)^{-1} \|_\xi.
\]

Using the fact that \( \| \delta \|_1 \leq \| \tau' \|_1 + \| \tau \|_1 \leq 2 \), we proceed as in the proof of Proposition 16 and find that

\[
\| (\text{Id} \otimes \delta) \overline{\Delta} \|_\xi \leq 4 \frac{\xi + 1}{\xi(\xi - 1)}.
\]

Combining this with (3.8), we obtain the inequality

\[
1 < 4 \frac{\xi + 1}{\xi(\xi - 1)} \frac{1}{1 - K(\xi, V)}.
\]
Let us choose a particular value \( \xi_0 \) of \( \xi \), large enough so that \( 4 \frac{\xi_0 + 1}{\xi_0 (\xi_0 - 1)} < 1 \). For example, choosing \( \xi_0 = 12 \), we have

\[
4 \frac{\xi_0 + 1}{\xi_0 (\xi_0 - 1)} = \frac{13}{33} < \frac{1}{2},
\]

and

\[
K(\xi_0, V) = K(12, V) = \frac{13}{33} + \|IV\|_1 \deg(V) 12^{\deg(V)}.
\]

Thus if

\[
\|IV\|_1 \deg(V) 12^{\deg(V)} < \frac{1}{2} - \frac{13}{33} = \frac{7}{66},
\]

we obtain the fallacious inequality \( 1 < 1 \). This proves Theorem 9.

4. The Schwinger-Dyson lattice

In this section, we introduce the Schwinger-Dyson lattice over \( L \). The Schwinger-Dyson lattice with potential \( V \) is a countable set of noncommutative partial differential equations. The equations \( SD(k, N) \) in this hierarchy are indexed by two discrete parameters, the order, \( k \), and the rank, \( N \). A solution of the Schwinger-Dyson lattice with potential \( V \) is an array

\[
W_{11}^V \ W_{12}^V \ldots \ W_{1N}^V \ldots \\
W_{21}^V \ W_{22}^V \ldots \ W_{2N}^V \ldots \\
\vdots \quad \vdots \quad \ddots \quad \vdots \\
W_{k1}^V \ W_{k2}^V \ldots \ W_{kN}^V \ldots \\
\vdots \quad \vdots \quad \ddots \quad \vdots
\]

whose elements are symmetric multilinear functionals

\[
W_{kN}^V : L \times \cdots \times L \rightarrow \mathbb{C}.
\]

In order to qualify as a solution of the SD lattice, we insist that

\[
\tilde{W}_{1N}^V := N^{-1} W_{1N}^V
\]

is a unital trace, and that \( W_{kN}^V \) is a connected \( k \)-correlator for \( k \geq 2 \).

By definition, the first equation in the SD lattice, \( SD(1, N) \), is

\[
W_{1N}^V \otimes W_{1N}^V (\partial_i p) + NW_{1N}^V ((D_i V)p) = - W_{2N}^V (\partial_i p).
\]

Note that this equation is invariant under translations of \( V \) by elements of \( B \). The subsequent equations in the hierarchy are obtained by repeated application of the Gibbs rule,

\[
\frac{d}{d\xi} W_{kN}^V (\partial_{p_k+1} (p_1, \ldots, p_k)) \mid_{\xi=0} = W_{(k+1)N}^V (p_1, \ldots, p_k, p_{k+1}).
\]

In this section we give the equations of the SD hierarchy in explicit form. First, we present the SD equations in their primary form, obtained directly from the first equation and iteration of the Gibbs rule. We then obtain the secondary form of the
SD lattice equations by applying the primary form to the coordinates of a cyclic gradient,

$$\mathcal{D} p = (\mathcal{D}_1 p, \ldots, \mathcal{D}_m p).$$

Strictly speaking, the secondary form is a specialization of the primary form since not every \(m\)-tuple of noncommutative polynomials occurs as a cyclic gradient. However, the secondary form of the SD equations has the advantage that it is concisely described by the fundamental operators \(\Psi_{V_{1N}}^V\) and \(\Xi_{V_{1N}}^V\).

Finally, we introduce the notion of uniformly bounded solutions of the SD lattice. Quite simply, a uniformly bounded solution is one whose elements \(W_{1N}^V(z_1 \otimes \cdots \otimes z_N)\) are multilinear functionals whose norms are bounded independently of \(N\). We will see that, as soon as \(\Xi_{V_{1N}}^V\) is invertible, uniformly bounded solutions in fact exhibit polynomial decay in \(N\).

4.1. Primary form of the SD equations. Select \(p_1, \ldots, p_k \in L\), and consider the perturbed first order equation

$$W_{1N}^V \otimes W_{1N}^V(\partial_i p_1) + N W_{1N}^V((\mathcal{D}_1 V) p_1) = -W_{2N}^V(\partial_i p_1),$$

where

$$V_z = V + \sum_{j=2}^k \frac{z_j}{N} p_j$$

and \(k \geq 2\). Equivalently, the perturbed equation is

$$W_{1N}^V \otimes W_{1N}^V(\partial_i p_1) + N W_{1N}^V((\mathcal{D}_1 V) p_1) = -\sum_{j=2}^k z_j W_{1N}^V((\mathcal{D}_j p_j) p_1) - W_{2N}^V(\partial_i p_1).$$

Applying the Gibbs rule to the perturbed first order equation \(k - 1\) times, once for each of the variables \(z_2, \ldots, z_k\), will yield the \(k\)th order SD equation at rank \(N\).

To write down SD\((k, N)\) explicitly, we start by differentiating the term \(W_{1N}^V \otimes W_{1N}^V(\partial_i p_1)\). Let \(q_1 \otimes q_2\) be a simple tensor in \(L \otimes 2\), and consider

$$W_{1N}^V \otimes W_{1N}^V(q_1 \otimes q_2) = W_{1N}^V(q_1) W_{1N}^V(q_2).$$

Differentiating with respect to the parameters \(z_2, \ldots, z_k\) and applying the Gibbs rule yields the sum

$$\sum_{r=1}^k \sum_{I \subseteq \{2, \ldots, k\}} W_{1N}^V(q_1 \otimes p_I) W_{1N}^V(q_2 \otimes p_{T}) = \sum_{I \subseteq \{2, \ldots, k\}} W_{1N}^V(q_1 \otimes p_I) W_{1N}^V(q_2 \otimes p_{I^C})$$

where the sum on the right is over all subsets of \(\{2, \ldots, k\}\), including the empty set, \(I^C = \{2, \ldots, k\} \setminus I\), and

$$p_I = \bigotimes_{i \in I} p_i, \quad p_T = \bigotimes_{i \in T} p_i.$$

This may be equivalently written
\[ \sum_{I \subseteq \{2, \ldots, k\}} \mathcal{W}_{(\{I\}+1)N}^V \otimes \mathcal{W}_{(\{I\}+1)N}^V (q_1 \otimes q_2 \# p_1 \otimes p_1^c), \]

where we are using the notation \( q_1 \otimes q_2 \# T = q_1 \otimes T \otimes q_2 \) for any tensor \( T \).

Application of the Gibbs rule to the next term, \( \mathcal{N} \mathcal{W}_{kN}^V ((D_1V)p_1) \), yields the contribution

\[ \mathcal{N} \mathcal{W}_{kN}^V ((D_1V)p_1, p_2, \ldots, p_k). \]

We now move to the right hand side of the perturbed equation. Application of the Gibbs rule yields the contributions

\[ \sum_{j=2}^{k} \mathcal{W}_{(\{j\}-1)N}^V ((D_jp_j)p_1, p_2, \ldots, \hat{p}_j, \ldots, p_k), \quad \mathcal{W}_{(k+1)N}^V (\partial_p p_1, p_2, \ldots, p_k), \]

where, in the first contribution, the hat denotes an omitted argument.

**Proposition 19** (Primary SD equations). The \( k \)th order Schwinger-Dyson equation at rank \( N \), \( \text{SD}(k,N) \), is

\[ \sum_{I \subseteq \{2, \ldots, k\}} \mathcal{W}_{(\{I\}+1)N}^V \otimes \mathcal{W}_{(\{I\}+1)N}^V (\partial_p p_1 \# p_1 \otimes p_1^c) + \mathcal{N} \mathcal{W}_{kN}^V ((D_1V)p_1, \ldots, p_k) = \]

\[- \sum_{j=2}^{k} \mathcal{W}_{(\{j\}-1)N}^V ((D_jp_j)p_1, p_2, \ldots, \hat{p}_j, \ldots, p_k) - \mathcal{W}_{(k+1)N}^V (\partial_p p_1, p_2, \ldots, p_k). \]

4.2. **Secondary form of the SD equations.** Using the fact that \( \mathcal{W}_{1N}^V \) is a unital map, we use the cyclic gradient trick and argue as in Section 3.1 to obtain the secondary form of the SD equations. They are expressed in terms of the first and second fundamental operators associated to \( \mathcal{W}_{1N}^V \).

**Proposition 20** (Secondary SD equations). For any \( p \in B^\perp \), we have

\[ \mathcal{W}_{1N}^V (\Psi_{\mathcal{W}_{1N}^V}^V p) = - \frac{1}{2N} \mathcal{W}_{1N}^V (\Pi \mathcal{T} \mathcal{W}_{1N}^V p) - \frac{1}{N} \mathcal{W}_{2N}^V (\Delta p). \]

For any \( k \geq 2 \) and \( p_1, \ldots, p_k \in B^\perp \), we have

\[ \mathcal{W}_{kN}^V (\Xi_{\mathcal{W}_{1N}^V}^V p_1, \ldots, p_k) = - \frac{1}{N} \sum_{I} \mathcal{W}_{(\{I\}+1)N}^V \otimes \mathcal{W}_{(\{I\}+1)N}^V (\Delta p_1 \# p_1 \otimes p_T) \]

\[- \frac{1}{N} \sum_{j=2}^{k} \mathcal{W}_{(\{j\}-1)N}^V (\mathcal{P}_{T_j} p_1, \ldots, \hat{p}_j, \ldots, p_k) - \frac{1}{N} \mathcal{W}_{(k+1)N}^V (\Delta p_1, \ldots, p_k), \]

where the first sum on the right is over all proper nonempty subsets \( I \) of \( \{2, \ldots, k\} \).

In other words, the sum over \( I \) above is taken over all subsets of \( \{2, \ldots, k\} \) which are neither the full set \( \{2, \ldots, k\} \), nor the empty set \( \emptyset \).
4.3. Uniform boundedness and renormalization. So far, we have considered the SD lattice equations from a purely algebraic perspective. We now inject a modicum of analytic structure by introducing the notion of uniformly bounded solutions of the SD lattice.

**Definition 21.** A solution \((W^{V}_{k,N})_{k,N=1}^\infty\) of the SD lattice with potential \(V\) is said to be \(\xi\)-uniformly bounded if the following conditions hold:

1. \(\sup_{N} \|W^{V}_{1N}\|_1 \leq 1\);
2. For each \(k \geq 2\), \(\sup_{N} \|W^{V}_{kN}\|_\xi < \infty\).

Given a \(\xi\)-uniformly bounded solution as above, we define its renormalization by

\[\tilde{W}^{V}_{kN} = N^{k-2}W^{V}_{kN}, \quad k, N \geq 1.\]

In terms of the renormalized functionals \(\tilde{W}^{V}_{kN}\), uniform boundedness means that we have a sequence \((C_{k})_{j=2}^\infty\) of positive constants (i.e. numbers independent of \(N\)) such that

\[\|\tilde{W}^{V}_{kN}\|_\xi \leq C_{k}N^{k-2}, \quad k \geq 2.\]

In fact, the SD equations can be used to substantially improve upon this sequence of inequalities at the cost of geometrically dilating the \(\xi\)-norm.

**Theorem 22.** Let \((W^{V}_{k,N})_{k,N=1}^\infty\) be a \(\xi\)-uniformly bounded solution of the SD lattice with potential \(V\), and suppose that the corresponding fundamental operators are continuous automorphisms of \(B_\xi^k\). Set

\[\xi_l := 2^{l-2}\xi, \quad l \geq 2.\]

There exists an array \((C_{kl})_{k,l=2}^\infty\) of constants such that

\[\|\tilde{W}^{V}_{kN}\|_{\xi_l} \leq C_{kl}N^{\max(0,k-l)}, \quad k, l \geq 2.\]

**Proof.** Schematically, the theorem statement can be represented as an entrywise inequality between two \(\frac{N}{2} \times \frac{N}{2}\) matrices,

\[
\begin{bmatrix}
\|\tilde{W}^{V}_{1N}\|_{\xi_2} & \|\tilde{W}^{V}_{1N}\|_{\xi_3} & \|\tilde{W}^{V}_{1N}\|_{\xi_4} & \|\tilde{W}^{V}_{1N}\|_{\xi_5} & \cdots \\
\|\tilde{W}^{V}_{2N}\|_{\xi_2} & \|\tilde{W}^{V}_{2N}\|_{\xi_3} & \|\tilde{W}^{V}_{2N}\|_{\xi_4} & \|\tilde{W}^{V}_{2N}\|_{\xi_5} & \cdots \\
\|\tilde{W}^{V}_{3N}\|_{\xi_2} & \|\tilde{W}^{V}_{3N}\|_{\xi_3} & \|\tilde{W}^{V}_{3N}\|_{\xi_4} & \|\tilde{W}^{V}_{3N}\|_{\xi_5} & \cdots \\
\|\tilde{W}^{V}_{4N}\|_{\xi_2} & \|\tilde{W}^{V}_{4N}\|_{\xi_3} & \|\tilde{W}^{V}_{4N}\|_{\xi_4} & \|\tilde{W}^{V}_{4N}\|_{\xi_5} & \cdots \\
\|\tilde{W}^{V}_{5N}\|_{\xi_2} & \|\tilde{W}^{V}_{5N}\|_{\xi_3} & \|\tilde{W}^{V}_{5N}\|_{\xi_4} & \|\tilde{W}^{V}_{5N}\|_{\xi_5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\leq
\begin{bmatrix}
C_{22}N^0 & C_{23}N^0 & C_{24}N^0 & C_{25}N^0 & \cdots \\
C_{32}N^1 & C_{33}N^0 & C_{34}N^0 & C_{35}N^0 & \cdots \\
C_{42}N^2 & C_{43}N^1 & C_{44}N^0 & C_{45}N^0 & \cdots \\
C_{52}N^3 & C_{53}N^2 & C_{54}N^1 & C_{55}N^0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

We present a proof of this inequality by induction on the column parameter, \(l\).

For \(l = 2\), the desired statement coincides with the definition of \(\xi\)-uniform boundedness, and the invertibility of the fundamental operators is not required.

For the induction step, fix \(m \geq 2\) and suppose that there exists an array of constants...
such that

\[ \| \tilde{W}_{kN}^V \|_{\xi_l} \leq C_{kl} N^{\max(0,k-l)}, \quad k \geq 2, \quad l = 2, \ldots, m. \]

We will extend this to an array

\[
\begin{array}{ccc}
C_{22} & \ldots & C_{2m} \\
C_{32} & \ldots & C_{3m} \\
\vdots & & \vdots \\
C_{k2} & \ldots & C_{km} \\
\vdots & & \vdots \\
\end{array}
\]

such that

\[ \| \tilde{W}_{kN}^V \|_{\xi_{m+1}} \leq C_{k(m+1)} N^{\max(0,k-(m+1))}, \quad k \geq 2. \]

Let us return to the secondary form of the SD equations, which in terms of the renormalized functionals \( \tilde{W}_{kN}^V \) becomes

\[
\tilde{W}_{kN}^V(\Xi_{W_{kN}}^V p_1, \ldots, p_k) = - \sum_I \tilde{W}_{(|I|+1)N}^V \otimes \tilde{W}_{(k-|I|)N}^V (\Xi_{p_I} \otimes p_I)
\]

\[
- \sum_{j=2}^{k} \tilde{W}_{(k-1)N}^V (p_{|I|} \otimes p_I) - \frac{1}{N^2} \tilde{W}_{(k+1)N}^V (\Xi_{p_1} \otimes p_k)
\]

\[=: S_{kN}^{(1)}(p_1, \ldots, p_k) + S_{kN}^{(2)}(p_1, \ldots, p_k) + S_{kN}^{(3)}(p_1, \ldots, p_k), \]

valid for all \( k \geq 2 \). We will use the induction hypothesis to estimate the \( \xi_{m+1} \)-norm of the three contributions \( S_{kN}^{(1)}, S_{kN}^{(2)}, S_{kN}^{(3)} \).

We begin with \( S_{kN}^{(1)} \). Recall that the summation in this group of terms is over proper nonempty subsets \( I \) of \( \{2, \ldots, k\} \). We have

\[
\| S_{kN}^{(1)} \|_{\xi_{m+1}} \leq \sum_I \| (\tilde{W}_{(|I|+1)N}^V \otimes \tilde{W}_{(k-|I|)N}^V) \Xi \|_{\xi_{m+1}}
\]

\[
= \sum_{r=1}^{k-2} \binom{k-2}{r} \| (\tilde{W}_{(r+1)N}^V \otimes \tilde{W}_{(k-r)N}^V) \Xi \|_{\xi_{m+1}}
\]

\[
\leq \sum_{r=1}^{k-2} \binom{k-2}{r} \| \tilde{W}_{(r+1)N}^V \|_{\xi_m} \| \tilde{W}_{(k-r)N}^V \|_{\xi_m} \| \Xi \|_{\xi_{m+1}, \xi_m}
\]

\[
\leq \sum_{r=1}^{k-2} \binom{k-2}{r} \| \tilde{W}_{(r+1)N}^V \|_{\xi_m} \| \tilde{W}_{(k-r)N}^V \|_{\xi_m},
\]
The second to last inequality follows from the diagram

\[
((B^\perp)^{\otimes k}, \| \cdot \|_{\xi_{m+1}}) \xrightarrow{\bigotimes \text{Id}^{\otimes (k-1)}} ((B^\perp)^{\otimes (k+1)}, \| \cdot \|_{\xi_{m}}) \xrightarrow{\hat{V}_V^{(r+1)N} \otimes \hat{W}_V^{(k-r)N}} \mathbb{C},
\]

and the final inequality is Proposition 18. We now invoke the induction hypothesis, obtaining

\[
\|S_{kN}^{(1)}\|_{\xi_{m+1}} \leq \sum_{r=1}^{k-2} \binom{k-2}{r} \|\hat{V}_V^{(r+1)N}\|_{\xi_{m}} \|\hat{W}_V^{(k-r)N}\|_{\xi_{m}} \leq \sum_{r=1}^{k-2} \binom{k-2}{r} C_{(r+1)m} N^{\max(0,r+1-m)} C_{(k-r)m} N^{\max(0,k-r-m)} \leq C_{k(m+1)}^{(1)} N^{\max(0,k-(m+1))},
\]

where \(C_{k(m+1)}^{(1)} = \sum_{r=1}^{k-2} \binom{k-2}{r} C_{(r+1)m} C_{(k-r)m}\).

Next, we estimate the contribution \(S_{kN}^{(2)}\). We have

\[
\|S_{kN}^{(2)}\|_{\xi_{m+1}} \leq \sum_{j=2}^{k} \|\hat{V}_V^{(k-j)N}\|_{\xi_{m}} \|X_j\|_{\xi_{m+1}} \leq \sum_{j=2}^{k} \|\hat{V}_V^{(k-j)N}\|_{\xi_{m}} \|X_j\|_{\xi_{m+1}, \xi_m},
\]

where \(X_j \in \text{Hom}((B^\perp)^{\otimes k}, (B^\perp)^{\otimes (k-1)})\) is the map which operators on simple tensors \(p_1 \otimes \cdots \otimes p_k\) according to

\[
X_j p_1 \otimes \cdots \otimes p_k = (\hat{P}_V p_1) \otimes \cdots \otimes \hat{p}_j \otimes \cdots \otimes p_k,
\]

and the second inequality is the diagram

\[
((B^\perp)^{\otimes k}, \| \cdot \|_{\xi_{m+1}}) \xrightarrow{X_j} ((B^\perp)^{\otimes (k-1)}, \| \cdot \|_{\xi_{m}}) \xrightarrow{\hat{V}_V^{(k-j)N}} \mathbb{C}.
\]

Now, we claim that the first arrow in this diagram is a contractive mapping. Indeed, for any monomial \(p_1 \otimes \cdots \otimes p_k \in (B^\perp)^{\otimes k}\), we have

\[
\|X_j p_1 \otimes \cdots \otimes p_k\|_{\xi_m} = \|\hat{P}_V p_1\|_{\xi_m} \|\hat{p}_j \otimes \cdots \otimes p_k\|_{\xi_m} \leq \|p_j\|_{1} \deg(p_1) C^{\deg(p_1)} \|\hat{p}_j \otimes \cdots \otimes p_k\|_{\xi_m} \leq \|p_1\|_{\xi_{m+1}} \|p_2 \otimes \cdots \otimes \hat{p}_j \otimes \cdots \otimes p_k\|_{\xi_m} \leq \|p_1\|_{\xi_{m+1}} \|p_2 \otimes \cdots \otimes \hat{p}_j \otimes \cdots \otimes p_k\|_{\xi_{m+1}},
\]
where Proposition 17 was applied to obtain the first inequality. Thus, returning to our estimate on the $\xi_{m+1}$-norm and applying the induction hypothesis, we have

$$\|S_{kN}^{(2)}\|_{\xi_{m+1}} \leq \sum_{j=2}^{k} \|\tilde{W}_{V}^{(k-1)N}\|_{\xi_{m+1}}$$

$$\leq \sum_{j=2}^{k} C_{(k-1)m} N_{\max(0,k-1-m)} \leq C_{(k(m+1))m}^{(2)} N_{\max(0,k-(m+1))},$$

where $C_{(k(m+1))m}^{(2)} = \sum_{j=2}^{k} C_{(k-1)m}$. Finally, we estimate the contribution $S_{kN}^{(3)}$ in $\xi_{m+1}$-norm. From the diagram

$$((B_{\perp}^{\otimes k} \parallel \cdot \parallel_{\xi_{m+1}}) \xrightarrow{\Sigma} ((B_{\perp}^{\otimes (k+1)} \parallel \cdot \parallel_{\xi_{m}}) \xrightarrow{\tilde{W}_{V}^{(k+1)N}_{1}} C,$$

Proposition 18, and the induction hypothesis, we have the estimate

$$\|S_{kN}^{(3)}\|_{\xi_{m+1}} \leq \frac{1}{N^2} \|\tilde{W}_{V}^{(k+1)N}\|_{\xi_{m+1}}$$

$$\leq \frac{1}{N^2} \|\tilde{W}_{V}^{(k+1)N}\|_{\xi_{m}} \|\Sigma\|_{\xi_{m}, \xi_{m+1}}$$

$$\leq \frac{1}{N^2} \|\tilde{W}_{V}^{(k+1)N}\|_{\xi_{m}}$$

$$\leq \frac{1}{N^2} C_{(k+1)m}^{(3)} N_{\max(0,k+1-m)}$$

$$= C_{(k+1)m}^{(3)} N_{\max(0,k-1-m)}.$$

We thus have

$$\|S_{kN}^{(3)}\|_{\xi_{m+1}} \leq C_{(k+1)m}^{(3)} N_{\max(0,k-(m+1))},$$

with $C_{(k+1)m}^{(3)} = C_{(k+1)m}$. We have now shown that

$$\|\tilde{W}_{V}^{(k)N}_{1}\|_{\xi_{m+1}} \leq C_{k(m+1)} N_{\max(0,k-(m+1))}, \quad k \geq 2, N \geq 1,$$

where $C_{k(m+1)} = C_{k(m+1)}^{(1)} + C_{k(m+1)}^{(2)} + C_{k(m+1)}^{(3)}$, provided that the domain of $\tilde{W}_{V}^{(k)N}$ is restricted to

$$\Xi_{\tilde{W}_{V}^{(k)N}}(B_{\perp}^{\otimes k}) \otimes (B_{\perp}^{\otimes (k-1)}).$$

Since our hypotheses dictate that the fundamental operator $\Xi_{\tilde{W}_{V}^{(k)N}}$ is invertible and bounded in $B_{\perp}^{\otimes k}$, the proof is complete.

**Corollary 23.** Under the hypotheses of Theorem 22, the sequence $(\tilde{W}_{V}^{(k)N})_{N=1}^{\infty}$ has a nonempty set of limit points as a linear form from $L_{\xi_{k}}$ into $C$ for any $k \in [1, K]$.

**Proof.** For $k = 1$, we have that $\|\tilde{W}_{V}^{(1)}\|_{1} \leq 1$ for all $N \geq 1$, so that

$$|\tilde{W}_{V}^{(1)}(p)| \leq \|p\|_{1} = 1$$

for any monomial $p \in L$. Thus, the sequence $(\tilde{W}_{V}^{(1)}(p))_{N=1}^{\infty}$ is a bounded sequence of complex numbers, and hence admits a limit point. Using the countability of the
monomial basis in $L$ together with a diagonalization argument yields the existence of a limit point of $(\tilde{W}_{kN}^V)_{N=1}^\infty$ with respect to the topology of pointwise convergence.

For $k \geq 2$, we have that $\|\tilde{W}_{kN}^V\|_{\xi_k} \leq C_{kk}$ for all $N \geq 1$. Thus, for any monomial $p_1 \otimes \cdots \otimes p_k \in L^{\otimes k}$, we have

$$|\tilde{W}_{kN}^V(p_1, \ldots, p_k)| \leq C_{kk}^{\deg(p_1) + \cdots + \deg(p_k)}$$

for all $N \geq 1$, so that $(\tilde{W}_{kN}^V(p_1, \ldots, p_k))_{N=1}^\infty$ is a bounded sequence of complex numbers. The same countability/diagonalization argument now applies to deduce the existence of a limit point of $(\tilde{W}_{kN}^V(p_1, \ldots, p_k))_{N=1}^\infty$.

\[\square\]

5. **Asymptotic analysis of the SD equations**

Let $V \in L$ be a polynomial verifying

(5.1) $$\|\Pi V\|_1 \leq \frac{7}{66} \frac{1}{\deg(V)(2^{K-1}12)^{\deg(V)}}.$$ 

Set $\xi = 12$ and $\xi_l = 2^{l-2}\xi$ for $l \geq 2$. Let

$$\begin{align*}
\tilde{W}_{11}^V & | \tilde{W}_{12}^V | \cdots | \tilde{W}_{1N}^V |
\tilde{W}_{21}^V & | \tilde{W}_{22}^V | \cdots | \tilde{W}_{2N}^V |
\vdots & & \vdots & & \vdots
\tilde{W}_{k1}^V & | \tilde{W}_{k2}^V | \cdots | \tilde{W}_{kN}^V |
\vdots & & \vdots & & \vdots
\end{align*}$$

be a $\xi$-uniformly bounded solution of the SD lattice with potential $V$, and let

$$\begin{align*}
\tilde{W}_{11}^V & | \tilde{W}_{12}^V | \cdots | \tilde{W}_{1N}^V |
\tilde{W}_{21}^V & | \tilde{W}_{22}^V | \cdots | \tilde{W}_{2N}^V |
\vdots & & \vdots & & \vdots
\tilde{W}_{k1}^V & | \tilde{W}_{k2}^V | \cdots | \tilde{W}_{kN}^V |
\vdots & & \vdots & & \vdots
\end{align*}$$

be its renormalization. Suppose that $\tilde{W}_{1N}^V$ restricted to $B$ admits an $N \to \infty$ asymptotic expansion to $h$ terms on the asymptotic scale $N^{-2}$:

$$\tilde{W}_{1N}^V(b) = \sum_{g=0}^h \frac{\sigma_g(b)}{N^{2g}} + o\left(\frac{1}{N^{2h}}\right), \quad b \in B.$$ 

Under these hypotheses, we prove the following abstract version of Theorem 1.

**Theorem 24.** For each $k \in [1, K]$, each $h \leq K - 1$, and all $p_1, \ldots, p_k \in B^\perp$, the functional $\tilde{W}_{kN}^V$ admits an $N \to \infty$ asymptotic expansion to $h$ terms:

$$\tilde{W}_{kN}^V(p_1, \ldots, p_k) = \sum_{g=0}^h \frac{\tau_{kg}^V(p_1, \ldots, p_k)}{N^{2g}} + o\left(\frac{1}{N^{2h}}\right)$$

The expansion coefficients $\tau_{kg}^V$ may be described as follows:
Remark 25. We reiterate that the asymptotics of $\hat{\mathcal{W}}_{1N}^V$ when restricted to $B$ are part of our hypotheses, while for $k \geq 2$ we need only consider $\hat{\mathcal{W}}_{kN}^V(p_1, \ldots, p_k)$ for arguments with no constant terms, by multilinearity and connectedness. This is why Theorem 24 is stated for $p_1, \ldots, p_k \in B^\perp$.

Theorem 24 is proved using Theorem 9 together with Theorem 22 and further analysis of the renormalized SD equations

$$
\hat{\mathcal{W}}_{kN}^V(\Xi_{1N}^V p_1, \ldots, p_k) = -\sum_{l} \hat{\mathcal{W}}_{(l|+1)N}^V \otimes \hat{\mathcal{W}}_{(l|-1)N}^V (\Xi_{1N}^V \# p_l \otimes p_T)
$$

(5.2)

Using equation (5.2), we establish theorem 24 through a double induction on the parameters $k$ and $g$. As will be clear from the proof, the error terms are uniform on potentials $V$ satisfying (5.2). In particular our argument allows that the potential $V$ may itself depend on $N$, for example be given by

$$
V = V_0 + \frac{1}{N} V_1
$$

for each $N \geq 1$, with $V_0, V_1 \in L$ fixed. In this case, $\tau_{kg}^V$ depends on $N$ through $V$, and can be again expanded recursively. We will make use of this fact in Section 6.

5.1. External base step: $g = 0$. Here we will prove that

$$
\lim_{N \to \infty} \hat{\mathcal{W}}_{kN}^V (p_1, \ldots, p_k) = \tau_{k0}^V (p_1, \ldots, p_k)
$$

for all $k$ such that $K(\xi_{k+2}, V) < 1$ and $p_1, \ldots, p_k \in B^\perp$, with $\tau_{k0}^V$ as given in the statement of Theorem 24. The proof is by induction on $k$. (1) $\tau_{10}^V$ is the unique solution of the noncommutative initial value problem (1.2) with $\sigma = \sigma_0$;

(2) For $k = 1$ and $g > 0$,

$$
\tau_{1g}^V (p) = -\sum_{\ell=1}^{g-1} \tau_{1\ell} \otimes \tau_{1(g-\ell)} (\Xi_{10}^{-1} p) - \tau_{2(g-1)} (\Xi_{10}^{-1})
$$

(3) For $k > 1$ and $g > 0$,

$$
\tau_{kg}^V(p_1, \ldots, p_k) = -\sum_{f=1}^{g} \tau_{k(g-f)} (\Xi_{10}^{-1} p_1, \ldots, p_k)
$$

$$
-\sum_{f=0}^{g} \sum_{I} \tau_{(|I|+1)f} \otimes \tau_{(|I|-1)(g-f)} (\Xi_{10}^{-1} p_1 \# p_I \otimes p_T)
$$

$$
-\sum_{j=2}^{k} \tau_{(k-1)g} (\Xi_{10}^{-1} p_1, \ldots, p_{j-1}, p_j) - \tau_{(k+1)(g-1)} (\Xi_{10}^{-1} p_1, \ldots, p_k),
$$

where the first sum on the right is over all proper nonempty subsets $I$ of $\{2, \ldots, k\}$. For (2)

For $p_1 \in B^\perp$, we have

$$
\tau_{1g}^V (p_1) = -\sum_{\ell=1}^{g-1} \tau_{1\ell} \otimes \tau_{1(g-\ell)} (\Xi_{10}^{-1} p_1) - \tau_{2(g-1)} (\Xi_{10}^{-1})
$$

and can be again expanded recursively. We will make use of this fact in Section 6.
5.1.1. **Internal base step:** \( k = 1 \). Let \( \hat{W}^V_{1N} \) denote the set of limit points of the sequence \( (\hat{W}^V_{1N})_{n=1}^\infty \). By Corollary 23 this is a nonempty set.

Let \( \tau \in \lim \hat{W}^V_{1N} \). Then, there is a subsequence \( (\hat{W}^V_{1N,n})_{n=1}^\infty \) of \( (\hat{W}^V_{1N})_{N=1}^\infty \) converging pointwise to \( \tau \) on \( L \). From the initial form of \( SD(1, N) \), we have that

\[
\hat{W}^V_{1N} \otimes \hat{W}^V_{1N}((\partial_1 p) + \hat{W}^V_{1N,n}((\partial_1 V)p) = -\frac{1}{N^2} \hat{W}^V_{2N}((\partial_1 p)
\]

for all \( n \geq 1 \). By Theorem 22 we have

\[
|\hat{W}^V_{2N}((\partial_1 p)| \leq \|\hat{W}^V_{2N}\|_\xi_2 \|\partial_1 p||_\xi_2 \leq C_{22} N^{\max(0,2-N_2)} \deg(p)||p||_\xi_2
\]

for all \( n \geq 1 \). Taking the \( n \to \infty \) limit we obtain that for \( p \in L_{\xi_2} \),

\[
\tau \otimes \tau(\partial_1 p) + \tau((\partial_1 V)p) = 0.
\]

Moreover, \( |\tau| = \sigma_0 \), by hypothesis. Thus \( \tau \) is a solution of the initial value problem (1.2) with \( \sigma = \sigma_0 \). By Theorem 0 this initial value problem admits at most one solution if \( K(\xi, V) < 1 \). Consequently, \( \lim \hat{W}^V_{1N} \) consists of a single point, and this is \( \tau_{10}^V \).

5.1.2. **Internal induction step:** \( k > 1 \). Fix \( k \geq 2 \), and suppose that \( \hat{W}^V_{kN} \) converges to \( \tau_{10}^V \) as a linear form on \( L_{\xi_k}^r \) for all \( 1 \leq r < k \), with \( \tau_{r0}^V \) as specified in Theorem 24. Assume also \( K(\xi_{k+2}, V) < 1 \) so that \( \hat{W}^V_{kN} \) is invertible in \( L_{\xi_{k+2}} \).

By Corollary 23, the sequence \( (\hat{W}^V_{kN})_{n=1}^\infty \) has a nonempty set of limit points. Let \( \tau \in \lim \hat{W}^V_{kN} \) be a limit point. To prove that \( \tau = \tau_{k0}^V \), we return to the renormalized form of \( SD(k, N) \) given in equation 6.2. Let \( (\hat{W}^V_{kN})_{n=1}^\infty \) be a subsequence converging to \( \tau \). We then have that

\[
\hat{W}^V_{kN,n}(\overline{\Xi}^V_{\hat{W}^V_{kN}}p_1, \ldots, p_k) = \sum_{I} \hat{W}^V_{(|I|+1)N_n} \otimes \hat{W}^V_{(k-|I|)N_n}(\overline{\Xi}_p p_1 \# p_I \otimes p_T)
\]

\[
- \sum_{j=2}^{k} \hat{W}^V_{(k-1)N_n}(p_j, \ldots, p_k) - \frac{1}{N^2} \hat{W}^V_{(k+1)N_n}(\overline{\Xi}p_1, \ldots, p_k)
\]

for all \( n \geq 1 \). By Proposition 18 and Theorem 22, we have that

\[
|\hat{W}^V_{(k+1)N_n}(\overline{\Xi}p_1, \ldots, p_k)| \leq \|\hat{W}^V_{(k+1)N_n}\|_{\xi_{k+2}} \|\overline{\Xi}\|_{\xi_{k+2}, \xi_{k+1}} \|p_1 \otimes \cdots \otimes p_k\|_{\xi_{k+2}} \leq C_{(k+1)(k+1)}\|p_1 \otimes \cdots \otimes p_k\|_{\xi_{k+2}}.
\]

Hence, by the induction hypothesis, we have for \( p_i \in L_{\xi_{k+2}} \),

\[
\tau(\overline{\Xi}^V_{\tau_{10}^V}p_1, \ldots, p_k) = \sum_{I} \tau_{(|I|+1)0}^V \otimes \tau_{(k-|I|)0}^V(\overline{\Xi}_p p_1 \# p_I \otimes p_T)
\]

\[
- \sum_{j=2}^{k} \tau_{(k-1)0}^V(p_j, \ldots, p_k).
\]

Since \( \overline{\Xi}^V_{\tau_{10}^V} \) is invertible in \( L_{\xi_{k+2}} \), \( \tau \) is uniquely defined and we obtain \( \tau = \tau_{k0}^V \), as required.
5.2. The second column: \( g = 1 \). In this section we obtain the second term in the asymptotics of \( \mathcal{W}_{kN}^V \), i.e. the limit of the error functional

\[
\delta_1 \mathcal{W}_{kN}^V(p_1, \ldots, p_k) = \mathcal{W}_{kN}^V(p_1, \ldots, p_k) - \tau_{k0}^V(p_1, \ldots, p_k).
\]

As in the previous section, our argument is inductive in \( k \).

5.2.1. Internal base step: \( k = 0 \). Our starting point is the following quadratic constraint, which is the analogue of Proposition 15 for the error functional.

**Proposition 26.** For any \( p \in B^1 \), we have

\[
\delta_1 \mathcal{W}_{1N}^V(\Xi_{\tau_{10}}^V p) = -\delta_1 \mathcal{W}_{1N}^V \otimes \delta_1 \mathcal{W}_{1N}^V(\Delta p) - \frac{1}{N^2} \mathcal{W}_{2N}^V(\Delta p).
\]

**Proof.** The proof is analogous to the proof of Proposition 15. We start with the equations

\[
\begin{align*}
\delta_1 \mathcal{W}_{1N}^V \otimes \delta_1 \mathcal{W}_{1N}^V(\partial_i p) + \delta_1 \mathcal{W}_{1N}^V((\mathcal{D}_i V)p) &= -\frac{1}{N^2} \mathcal{W}_{2N}^V(\partial_i p) \\
\tau_{10}^\nu \otimes \tau_{10}^\nu(\partial_i p) + \tau_{10}^\nu((\mathcal{D}_i V)p) &= 0.
\end{align*}
\]

Subtracting the second equation from the first yields the identity

\[
[\delta_1 \mathcal{W}_{1N}^V \otimes \delta_1 \mathcal{W}_{1N}^V - \tau_{10}^\nu \otimes \tau_{10}^\nu](\partial_i p) + [\delta_1 \mathcal{W}_{1N}^V - \tau_{10}^\nu]((\mathcal{D}_i V)p) = -\frac{1}{N^2} \mathcal{W}_{2N}^V(\partial_i p).
\]

which is equivalent to

\[
\delta_1 \mathcal{W}_{1N}^V \left( (\mathcal{D}_i \otimes \tau_{10}^\nu + \tau_{10}^\nu \otimes \mathcal{D}_i) \partial_i p \right) + \delta_1 \mathcal{W}_{1N}^V((\mathcal{D}_i V)p) = -\delta_1 \mathcal{W}_{1N}^V \otimes \delta_1 \mathcal{W}_{1N}^V(\partial_i p) - \frac{1}{N^2} \mathcal{W}_{2N}^V(\partial_i p).
\]

Now use the cyclic gradient trick. \( \Box \)

We now use Proposition 26 to obtain an upper bound on the error functional, showing in particular that it is bounded on the correct asymptotic scale.

**Proposition 27.** For any \( N \geq 1 \), we have

\[
\|\delta_1 \mathcal{W}_{1N}^V\|_{\xi_3} \leq \frac{C_{22}\|\Xi_{\tau_{10}}^V\|^{-1}||\Xi_{\tau_{10}}^V\|_{\xi_3}^{-1}}{1 - 4 \frac{\xi_3 + 1}{\xi_3 (\xi_3 - 1)} \|\Xi_{\tau_{10}}^V\|^{-1}} \cdot \frac{1}{N^2}
\]

which is finite as soon as \( K(\xi_3, V) < 1 \) by \( (3.8) \).

**Proof.** Since \( \Xi_{\tau_{10}}^V \) is continuous and invertible, the error constraint implies the identity

\[
\begin{align*}
\delta_1 \mathcal{W}_{1N}^V &= -\delta_1 \mathcal{W}_{1N}^V \otimes \delta_1 \mathcal{W}_{1N}^V \Xi_{\tau_{10}}^V(\Xi_{\tau_{10}}^V)^{-1} - \frac{1}{N^2} \mathcal{W}_{2N}^V \Xi_{\tau_{10}}^V(\Xi_{\tau_{10}}^V)^{-1} \\
&= -\delta_1 \mathcal{W}_{1N}^V(\mathcal{D}_i \otimes \delta_1 \mathcal{W}_{1N}^V) \Xi_{\tau_{10}}^V(\Xi_{\tau_{10}}^V)^{-1} - \frac{1}{N^2} \mathcal{W}_{2N}^V \Xi_{\tau_{10}}^V(\Xi_{\tau_{10}}^V)^{-1}
\end{align*}
\]

in \( \text{Hom}(B^1_\xi, \mathbb{C}) \). Since \( |\delta_1 \mathcal{W}_{1N}^V(p)| \leq 2 \) for all monomials \( p \), the same argument as in the proof of Proposition 13 yields the bound
\[ \| (\text{Id} \otimes \delta_1 \hat{W}_1^V) \Xi \|_{\xi_3} \leq 4 \frac{\xi_3 + 1}{\xi_3 (\xi_3 - 1)}. \]

Furthermore, we have
\[ \| \hat{W}_2^V \|_{\xi_3} \leq \| \hat{W}_2^V \|_{\xi_2} \| \Xi \|_{\xi_3, \xi_2} \leq C_{22}, \]
by Proposition 18 and Theorem 22. We thus have the inequality
\[ \| \delta_1 \hat{W}_1^V \|_{\xi_3} \leq \| \delta_1 \hat{W}_1^V \|_{\xi_4} \left( \frac{\xi_3 + 1}{\xi_3 (\xi_3 - 1)} \right) \| (\Xi_{10}^V)^{-1} \|_{\xi_3} + \frac{C_{22}}{N^2} \| (\Xi_{10}^V)^{-1} \|_{\xi_3}, \]
from which the result follows. \( \square \)

We can now complete the proof of the base step. Consider the set \( \lim_{N \to \infty} N^2 \delta_1 \hat{W}_1^V \) of limit points of the sequence \( (N^2 \delta_1 \hat{W}_1^V)_{N=1}^\infty \). By Proposition 27 this set is nonempty, just as in the proof of Corollary 23. Moreover, by Proposition 26 any limit point \( \tau \) must satisfy the equation
\[ \tau(\Xi_{10}^V) = -\tau(\Delta p). \]
Since \( \Xi_{10}^V \) is invertible, we conclude that \( N^2 \delta_1 \hat{W}_1^V \) converges to \( \tau_{11}^V \) given by
\[ \tau_{11}^V(p) = -\tau(\Delta(\Xi_{10}^V)^{-1}p), \]
as required.

5.2.2. Internal induction step: \( k \geq 2 \). Fix \( k \geq 2 \), and suppose that
\[ \lim_{N \to \infty} N^2 \delta_1 \hat{W}_1^V = \tau_{r_1}^V \]
for all \( 1 \leq r < k \), with \( \tau_{r_0}^V \) and \( \tau_{r_1}^V \) as in Theorem 24. Assume \( K(\xi_{k+2}, V) < 1 \).

As in the induction step for the first column (i.e. \( g = 0 \)), we return to the renormalized equation \( \text{SD}(k, N) \):
\[ \hat{W}_{kN}^V(\Xi_{10}^V, p_1, \ldots, p_k) = -\sum_{l} \hat{W}_{l+1N}^V \otimes \hat{W}_{k-lN}^V(\Delta p_l \# p_l \otimes p_r) \]
\[ -\sum_{j=2}^k \hat{W}_{(k-j)N}^V(p_j, \ldots, p_k) - \frac{1}{N^2} \hat{W}_{(k+1)N}^V(\Delta p_k, \ldots, p_k). \]
Expanding \( \hat{W}_{kN}^V \) to two terms and applying Proposition 14, the left hand side is
\[ \text{LHS} = \hat{W}_{kN}^V(\Xi_{10}^V, \tau_{r_0}^V, \tau_{r_1}^V, p_1, \ldots, p_k) + o \left( \frac{1}{N^2} \right) \]
\[ = \hat{W}_{kN}^V(\Xi_{10}^V, p_1, \ldots, p_k) + \frac{1}{N^2} \hat{W}_{kN}^V(\tau_{r_0}^V, p_1, \ldots, p_k) + o \left( \frac{1}{N^2} \right) \]
\[ = \hat{W}_{kN}^V(\Xi_{10}^V, p_1, \ldots, p_k) + \frac{1}{N^2} \tau_{r_0}^V(\tau_{r_0}^V, p_1, \ldots, p_k) + o \left( \frac{1}{N^2} \right), \]
where we used the external base step in the last line. Similarly, applying the induction hypothesis and the external base step, the right hand side becomes...
RHS = \tau_k V(p_1, \ldots, p_k) \\
- \frac{1}{N^2} \sum_I [\tau_I V(\tau_{(I|-)} + \tau_{(I|-)} V(\tau_{(I|-)} [\Delta p_I \# p_I \otimes p_T]) \\
- \frac{1}{N^2} \sum_{j=2}^k (\mathcal{P}^I V(p_1, \ldots, \hat{p}_j, \ldots, p_k) - \frac{1}{N^2} \tau_{(I|-)} V(p_1, \ldots, p_k) + o\left(\frac{1}{N^2}\right).

Putting these together and using the fact that \Xi_{t_0}^{V_0} is invertible, we conclude that

\tilde{\mathcal{W}}_{kN} V(p_1, \ldots, p_k) = \tau_{k0} V(p_1, \ldots, p_k) + \frac{\tau_{k1} V(p_1, \ldots, p_k)}{N^2} + o\left(\frac{1}{N^2}\right),

where

\tau_{k1}(p_1, \ldots, p_r) = -\tau_{k0}(\xi_{t_0} V(\tau_{t_0})^{-1} p_1, \ldots, p_k)
- \frac{1}{N^2} \sum_I [\tau_I V(\tau_{(I|-)} + \tau_{(I|-)} V(\tau_{(I|-)} [\Delta(\xi_{t_0} V(\tau_{t_0})^{-1} p_1 \# p_I \otimes p_T])

- \frac{1}{N^2} \sum_{j=2}^k (\mathcal{P}^I V(\tilde{\xi}_{t_0} V_{t_0})^{-1} p_1, \ldots, \hat{p}_j, \ldots, p_k) - \tau_{(I|-)} V(\tilde{\xi}_{t_0} V_{t_0})^{-1} p_1, \ldots, p_k),

as required.

5.3. External induction step: \( g \geq 2 \). For each \( g \) in the range \( 0 \leq g \leq h \), define

\delta_g \tilde{\mathcal{W}}_{kN} = \tilde{\mathcal{W}}_{kN} V - \left(\tau_{k0} V + \frac{\tau_{k1} V(p_1, \ldots, p_k)}{N^2(g-1)}\right), \quad k \geq 1,

where by convention \( \delta_0 \tilde{\mathcal{W}}_{kN} = \tilde{\mathcal{W}}_{kN} V \). Our goal is to prove that

\lim_{N \to \infty} N^{2g} \delta_g \tilde{\mathcal{W}}_{kN} V = \tau_{kg}

for each \( k \geq 1 \) so that \( K(\xi_{k+2}, V) < 1 \) and \( g \leq K = \max\{k : K(\xi_{k+2}, V) < 1\} - 1 \), with \( \tau_{kg} \) as in Theorem 21. So far we have shown this for \( g = 0, 1 \).

We now fix \( 2 \leq g \leq h \), and suppose that

\lim_{N \to \infty} N^{2f} \delta_f \tilde{\mathcal{W}}_{kN} V = \tau_{kf}

for each \( 0 \leq f < g \leq K - 1 \) and all \( k \in [1, K] \).

5.3.1. Internal base step: \( k = 1 \). Using Proposition 20, writing for \( i = 1, 2 \)

\delta_1 \tilde{W}_{kN} V = \sum_{f=1}^{g-1} \frac{1}{N^2} \tau_{1f} + \frac{1}{N^2(g-1)} \delta_g \tilde{W}_{kN} V

with \( \delta_g \tilde{W}_{kN} V \) going to zero as \( N \) goes to infinity, and identifying each orders in \( N^{-2f}, 1 \leq f \leq g - 1 \), we arrive at

\begin{align*}
N^{2g} \delta_g \tilde{W}_{kN} V(\xi_{t_0} V) & = -\sum_{f=1}^{g-1} \tau_{1f} \otimes \tau_{1(g-f)}(\Delta p) - \tau_{2(g-1)}(\Delta p) + o(1)
\end{align*}
from which we deduce that the sequence \( N^2 \delta_g \hat{W}_{1N}^V \) converges to \( \tau_{1g}^V \) as claimed in Theorem \([24]\). Note here that the \( g \)th term in the expansion of \( \hat{W}_{1N}^V \) depends on the \( g - 1 \) first terms in the expansion of \( \hat{W}_{2N}^V \). We will soon see that the latter itself will depend on the \( g - 2 \) first terms in the expansion of \( \hat{W}_{3N}^V \), so that ultimately the \( g \)th term in the expansion of \( \hat{W}_{1N}^V \) will depend on the first term in the expansion of \( \hat{W}_{(g+1)N}^V \). This is the reason why we can obtain the expansion only up to order \( K - 1 \).

5.3.2. Internal induction step: \( k \geq 2 \). Let \( k \geq 2 \), and suppose that \( \hat{W}_{1N}^V \) admits the expansion

\[
\hat{W}_{rN}^V(p_1, \ldots, p_k) = \sum_{f=0}^{g} \frac{\tau_{r,f}^V(p_1, \ldots, p_r)}{N^{2f}} + o\left(\frac{1}{N^{2g}}\right)
\]

for all \( 1 \leq r < k \), with the expansion coefficients \( \tau_{r,f}^V \) as given in Theorem \([24]\). To complete the induction, we must prove the claimed expansion for \( \hat{W}_{rN}^V \). As above, we return to the renormalized SD equation,

\[
\hat{W}_{kN}^V(\Xi_{\hat{W}_{kN}^V}, p_1, \ldots, p_k) = - \sum_{l} \hat{W}_{l|l+1|N}^V \otimes \hat{W}_{(k-l)N}^V (\Sigma p_1 \# p_l \otimes p_T)
\]

\[
- \sum_{j=2}^{k} \hat{W}_{(k-1)N}^V (\Pi_{f} p_1, \ldots, \tilde{p}_j, \ldots, p_k) - \frac{1}{N^2} \hat{W}_{(k+1)N}^V (\Sigma p_1, \ldots, p_k),
\]

By the induction hypothesis and Proposition \([14]\) the left hand side of the SD equation expands as

\[
LHS = \hat{W}_{kN}^V(\Xi_{\hat{W}_{kN}^V}, p_1, \ldots, p_k)
\]

\[
= \hat{W}_{kN}^V\left(\Xi_{\hat{W}_{kN}^V}, \sum_{f=1}^{g} \frac{1}{N^{2f}} T_{\tau_{1f}^V} p_1, \ldots, p_k\right) + o\left(\frac{1}{N^{2g}}\right)
\]

\[
= \hat{W}_{kN}^V(\Xi_{\hat{W}_{1N}^V}, p_1, \ldots, p_k) + \sum_{f=1}^{g} \frac{1}{N^{2f}} \hat{W}_{kN}^V (T_{\tau_{1f}^V} p_1, \ldots, p_k) + o\left(\frac{1}{N^{2g}}\right)
\]

\[
= \hat{W}_{kN}^V(\Xi_{\hat{W}_{1N}^V}, p_1, \ldots, p_k) + \sum_{f=1}^{g} \frac{1}{N^{2f}} \left(\sum_{e=0}^{g-1} \frac{1}{N^{2e}} T_{\tau_{1f}^V} p_1, \ldots, p_k\right) + o\left(\frac{1}{N^{2g}}\right)
\]

\[
= \hat{W}_{kN}^V(\Xi_{\hat{W}_{1N}^V}, p_1, \ldots, p_k) + \sum_{e=0}^{g-1} \sum_{f=1}^{g} \frac{1}{N^{2(e+f)}} T_{\tau_{1f}^V} p_1, \ldots, p_k + o\left(\frac{1}{N^{2g}}\right),
\]

so that the term of order \( N^{-2g} \) is

\[
\sum_{f=1}^{g} T_{k(g-f)} (T_{\tau_{1f}^V} p_1, \ldots, p_k).
\]

Similarly, the term of order \( N^{-2g} \) on the right hand side is
\[- \sum_{I} \sum_{f=0}^{g} \tau_I^f \tau_{(I|-I)(g-f)} (\Delta \Xi^{-1}_{\tau_{10}} p_1 \# p_f \otimes \#) \]
\[- \sum_{j=2}^{k} \tau_{(k-1)g} (\tau_{k-j} \Xi^{-1}_{\tau_{10}} p_1, \ldots, \hat{p}_j, \ldots, p_k) - \tau_{(k+1)(g-1)} (\Delta \Xi^{-1}_{\tau_{10}} p_1, \ldots, p_k). \]

Putting these two together, we obtain the expansion
\[
\tilde{W}_{V_k^N}(p_1, \ldots, p_k) = \tau_{V_0^k} (p_1, \ldots, p_k) + \cdots + \tau_{V_k^g} (p_1, \ldots, p_k) \frac{1}{N^{2g}} + o \left( \frac{1}{N^{2g}} \right),
\]
with the expansion coefficients as claimed in Theorem 24.

6. Matrix model solutions of the SD lattice

In this section, we treat matrix model solutions of the SD lattice. Select $V_0, V_1, \ldots, V_k \in L$ and set

\[ V = \sum_{\ell=0}^{k} \frac{1}{N^\ell} V_\ell, \quad N \geq 1. \]

Let

\[ \rho_N : B \to \text{Mat}_N(\mathbb{C}), \quad N \geq 1, \]

be a sequence of matrix representations of $B$, and consider the Gibbs ensemble generated by $\rho_N(V)$. We recall that this is the sequence of complex, unit-mass Borel measures $\mu_N^V$ defined by the density

\[ \mu_N^V (dU) = \frac{1}{Z_N^V} e^{N \text{Tr} \rho_N(V)} \mu_N (dU), \]

where $\mu_N$ is Haar measure on the compact group $U(N)^m$.

For each $p \in L$, set

\[ W^V_{kN}(p_1, \ldots, p_k) = \int \text{Tr} \rho_N(p)(U) \mu_N^V (dU), \quad N \geq 1. \]

It is immediate from the form of the density and the moment-cumulant formula that the higher cumulants may be obtained by iterating the Gibbs rule,

\[ \frac{d}{dz} W^V_{kN}(p_1, \ldots, p_k) |_{z=0} = W^V_{(k+1)N}(p_1, \ldots, p_k). \]

Thus, by construction, the cumulants

\[
\begin{array}{cccccccc}
W_{11}^V & W_{12}^V & \cdots & W_{1N}^V & \cdots \\
W_{21}^V & W_{22}^V & \cdots & W_{2N}^V & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
W_{k1}^V & W_{k2}^V & \cdots & W_{kN}^V & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots 
\end{array}
\]

form a solution of the SD lattice with potential $V$.

The condition
for all monomials \( p \in L \) is immediate: the matrix \( \rho_N(p)(U) \) is a product of unitary matrices and contractions. We thus have \( \| \tilde{W}_N \|_1 \leq 1 \). However, the existence of \( \xi \geq 1 \) such that this solution is \( \xi \)-uniformly bounded not automatic. In this section, we use concentration of measure techniques or change of variables tricks to verify that uniform boundedness holds for the cumulants of real Gibbs ensembles.

6.1. Concentration of measure. Suppose that \( \rho_N(V) \) generates a real Gibbs ensemble \( \mu_N^V \). Let \( U_N^V = (U_1, \ldots, U_m) \) be an \( m \)-tuple of \( N \times N \) random unitary matrices whose joint distribution in \( U(N)^m \) is \( \mu_N^V \).

In this subsection, we assume that \( V \) is both selfadjoint and balanced, as in Definition 5. Then, for any \( \theta_1, \ldots, \theta_k \in [0, 2\pi) \) and any \( U_1, \ldots, U_m \in U(N)^m \), we have

\[
\text{Tr} \rho_N(V)(e^{i\theta_1}U_1, \ldots, e^{i\theta_m}U_m) = \text{Tr} \rho_N(V)(U_1, \ldots, U_m).
\]

**Lemma 28.** Suppose that \( V = \sum \beta_i q_i \) with \( \sum |\beta_i| \deg(q_i)^2 \) small enough. Then, there exist constants \( c, C > 0 \) such that for any monomial \( q \in L \), for any \( \delta \geq 0 \)

\[
\mu_N^V \left\{ U \in U(N)^m : |\text{Tr} \rho_N(q)(U) - E\text{Tr} \rho_N(q)(U_N^V)| \geq \delta \deg(q) \right\} \leq Ce^{-cd^2}
\]

**Proof.** Let us first remark that, since \( V \) is balanced, for any balanced monomial \( q \) the law of

\[
X_N^q := \frac{1}{N} \text{Tr} \rho_N(q)(U_N^V)
\]

under \( \mu_N^V \) is the same law as under \( \mu_N^V \), the law defined under the Haar measure on \( SU(N) \) and with the same density. If \( q \) is not balanced, notice that its expectation under \( \mu_N^V \) vanishes as it has the same law as \( e^{i\theta_k}X_N^q \) with some \( \theta_k \) independent of \( X_N^q \) and taken uniformly on \([0, 2\pi)\) (see details in the proof of [2, Corollary 4.4.31]). Moreover, the law of \( |X_N^q| \) under under \( \mu_N^V \) is the same as its law under \( \mu_N^V \). Hence, it is enough to prove concentration inequalities under \( \mu_N^V \). But we know, by a result of Gromov, see [2, Theorem 4.27], that the Ricci curvature of \( SU(N) \) is bounded below by \( 2^{-1}(N + 2) - 1 \). On the other hand, the Hessian of the function

\[
U \mapsto \text{Tr} \rho_N(V)(U)
\]

is bounded above by \( \sum |\beta_i| \deg(q_i)^2 \). Hence, by the Bakry-Emery criterion, see [2, Corollary 4.4.25], we know that if \( 0 < c = 2^{-1} - \sum |\beta_i| \deg(q_i)^2 \), we have

\[
(6.1) \quad \tilde{\mu}_N^V \left( |G - \int G\tilde{\mu}_N^V| \geq \delta \right) \leq 2e^{-\frac{(c(N+1))^2}{4|\Gamma_1(G)|_\infty^2}},
\]

for all measurable functions \( G \) on \( SU(N) \), where \( \Gamma_1 \) is the carré du champ. On the other hand it is well known (see e.g. [12, p. 75] for a similar argument on \( SO(N) \)) that the metric on \( SU(N) \) can be lower bounded by the Euclidean metric on the full set of matrices. In particular, if \( G = \frac{1}{N} \text{Tr} \rho_N(q)(U) \), we can bound \( \| \Gamma_1(G) \|_\infty \) from above by noticing that
\[ \left| \text{Tr} \rho_N(q)(U) - \text{Tr} \rho_N(q)(U) \right| \leq \sum_{i=1}^{m} \sum_{1 \leq j \leq m} \left| \text{Tr} \left( \rho_N(p_1)(U)((U_j)^* - (\tilde{U_j})^*)\rho_N(p_2)(U) \right) \right| \]

\[ \leq \sum_{j=1}^{m} \text{deg}_j(q) \left| U_j - \tilde{U}_j \right| \]

\[ \leq \sqrt{N} \left( \sum_{j=1}^{m} \text{deg}_j(q)^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^{m} \text{Tr} (U_j - \tilde{U}_j)(U_j - \tilde{U}_j)^* \right)^{\frac{1}{2}} \]

Hence,

\[ \| \Gamma_1(\text{Tr} \rho_N(q)(U)) \|_\infty \leq N \sum_{j=1}^{m} \text{deg}_j(q)^2 . \]

Plugging this estimate back into (6.1) completes the proof, as \( \sum_{j=1}^{m} \text{deg}_j(q)^2 \leq \text{deg} q^2 . \)

6.2. Uniform boundedness via change of variables. In this section we get estimates on correlators by using change of variables and the good controls we had on the operator \( \Xi^* . \) This elaborates on the change of variable approach introduced in this context in [5]. We assume throughout this section that \( V \) is self-adjoint (but not necessarily balanced). We shall first prove that

**Lemma 29.** For any monomials \( p_1, \ldots, p_m \) such that \( p_i = -p_i^* , \) \( 1 \leq i \leq m \) , let \( Z_{p_1, \ldots, p_m} : U(N)^m \to \mathbb{C} \) be the function

\[ Z_{p_1, \ldots, p_m}(U) = \sum_{i=1}^{m} \left( \frac{1}{N} \text{Tr} \otimes \frac{1}{N} \text{Tr} \rho_N(\partial_i p_i)(U) + \frac{1}{N} \text{Tr} \rho_N(\partial_i V p_i)(U) \right) . \]

Then there exists a universal constant \( C \) depending only on \( V \) such that for any \( \lambda \in \mathbb{R} , \)

\[ \int e^{\lambda N Z_{p_1, \ldots, p_m}(U)} \mu_N(dU) \leq e^{C \lambda^2 (\text{max}_i \text{deg}(p_i))^2} . \]

**Proof.** We start by noting that \( Z_{p_1, \ldots, p_m} \) is real when \( p_i = -p_i^* . \) Indeed we have if \( p = -p^* \) and for all \( i \)

\[ D_i p = \sum (p, q) D_i q \]

\[ = \sum (p, q) [ \sum_{p=q_1 U_i q_2} q_2 q_1 U_i - \sum_{p=q_1 U_i^* q_2} U_i^* q_2 q_1 ] \]

\[ D_i p^* = \sum (p, q) [ - \sum_{p=q_1 U_i q_2} U_i^* q_2 q_1 + \sum_{q=q_1 U_i^* q_2} q_2 q_1 U_i ] \]

\[ = -(D_i p)^* \] (6.2)
so that $\mathcal{D}_p = (\mathcal{D}_p)^*$ if $p^* = -p$, which implies that $\text{Tr}_{\mathbb{P}}(\mathcal{D}_p(U))$ is real if $V$ is self-adjoint and $p_i = -p_i^*$. Similarly

$$\partial_i p = \sum (p, q) \sum_{\epsilon = \pm 1} \epsilon \sum_{p = q_1 u_1^{\epsilon = 1} q_2} q_1 u_1^{\epsilon = 1} \otimes u_1^{\epsilon = -1} q_2$$

$$\partial_i p^* = -\sum (p, q) \sum_{\epsilon = \pm 1} \epsilon \sum_{p = q_2 u_1^{\epsilon = -1} q_2} q_2 u_1^{\epsilon = -1} \otimes u_1^{\epsilon = 1} q_1^*$$

implies that

$$\frac{1}{N} \text{Tr} \otimes \frac{1}{N} \text{Tr}_{\mathbb{P}}(\partial_i p)(U) = -\frac{1}{N} \text{Tr} \otimes \frac{1}{N} \text{Tr}_{\mathbb{P}}(\partial_i p^*)(U)$$

which shows that $(\frac{1}{N} \text{Tr} \otimes \frac{1}{N} \text{Tr}_{\mathbb{P}}(\partial_i p)(U))$ is real if $p = -p^*$. Hence $Z_{p_1, \ldots, p_m}$ is real when $p_i = -p_i^*$ for all $i \in \{1, \ldots, m\}$.

The proof of the lemma goes through the change of variables

$$U = (U_1, \ldots, U_m) \rightarrow \Psi(U) = (\Psi_1(U_1), \ldots, \Psi_m(U_m))$$

with

$$\Psi_j(U) = U_j e^\frac{\lambda}{N} p_j(U_j)$$

It was shown in [5, section 2] that for $N$ large enough $\Psi$ is a local diffeomorphism of $U(N)^m$. We thus have

$$1 = \int |\det J_\Psi(U)| e^{N \text{Tr}_{\mathbb{P}}(V)(\Psi(U)) - N \text{Tr}_{\mathbb{P}}(V)(U)} \mu_N(dU)$$

The Jacobian of $\Psi$ can be computed by

$$|\det J_\Psi(U)| = \exp(-\sum_{p \geq 1} \frac{(-\lambda/N)^p}{p} \text{Tr}(\Phi(U)^p))$$.

Here $\Phi$ is the linear map acting on $A_N = \{A \in M_N : A = -A^*\}$

$$\Phi_{ij}(U).A = \sum_{k=0}^{\infty} \frac{(\text{Ad}_{\frac{\lambda}{N} p_j(U)}(U))^k}{(k+1)!} (\rho_N(\partial_i p_j) \# A) e^\frac{\lambda}{N} p_j(U)$$

where $\text{Ad}_M H = MH - H M$. Moreover it was shown that

$$\text{Tr}(\Phi(U)) = \sum_{i=1}^{m} \text{Tr} \otimes \text{Tr}_{\mathbb{P}}(\partial_i p_i) + \sum_{k=1}^{\infty} \text{Tr} \otimes \text{Tr}\left(\frac{(\text{Ad}_{\frac{\lambda}{N} p_j(U)}(U))^k}{(k+1)!} \rho_{\mathbb{P}}(\partial_i p_j) e^\frac{\lambda}{N} p_j(U)\right)$$

$$+ \sum_{k=0}^{\infty} \text{Tr} \otimes \text{Tr}\left(\frac{(\text{Ad}_{\frac{\lambda}{N} p_j(U)}(U))^k}{(k+1)!} (\partial_i p_j)(e^\frac{\lambda}{N} p_j(U) - 1)\right)$$

We can estimate the other terms as $\|\text{Ad}_M H\|_\infty \leq 2\|M\|_\infty\|H\|_\infty$ and

$$\|\frac{\lambda}{N} \rho_{\mathbb{P}}(p_j(U))\|_\infty \leq 2\lambda$$

Therefore, for all $k \geq 0$

$$\|\text{Tr} \otimes \text{Tr}\left(\frac{(\text{Ad}_{\frac{\lambda}{N} p_j(U)}(U))^k}{(k+1)!} (\partial_i p_j(U) e^\frac{\lambda}{N} p_j(U)\right)\|_\infty \leq N^2 \frac{(\frac{\lambda}{N})^k}{(k+1)!} \text{deg}(p_i)$$

$$\|\text{Tr} \otimes \text{Tr}\left(\frac{(\text{Ad}_{\frac{\lambda}{N} p_j(U)}(U))^k}{(k+1)!} (\partial_i p_j(U) e^\frac{\lambda}{N} p_j(U) - 1)\right)\|_\infty \leq N^2 \frac{(\frac{\lambda}{N})^{k+1}}{(k+1)!} \text{deg}(p_i)$$
where there exists a universal finite constant $C$

**Corollary 30.**

Let $p$ be a polynomial such that $p = p^*$ and set

$$
Z_p = N\sum_{i=1}^{m} \frac{1}{N^2} \operatorname{Tr} \otimes \operatorname{Tr}(\partial_i \rho_N(U)) + \frac{1}{N} \operatorname{Tr}(\rho_N(\partial_i D_i V)(U)) + \varepsilon^N_p
$$

where $|\varepsilon^N_p| \leq C\Lambda^2$. Hence, (6.3) proves the claim. 

As a corollary we have

**Corollary 30.**

Let $p$ be a polynomial such that $p = p^*$ and set

$$
Z_p = N\sum_{i=1}^{m} \frac{1}{N^2} \operatorname{Tr} \otimes \operatorname{Tr}(\partial_i \rho_N(U)) + \frac{1}{N} \operatorname{Tr}(\rho_N(\partial_i D_i V)(U)) + \varepsilon^N_p
$$

Then there exists a universal finite constant $C$ depending only on $V$ such that for any $\lambda \in \mathbb{R}$,

$$
\int e^{\lambda Z_p(U)} \mu_N^V(dU) \leq e^{C\lambda^2 \sum |(p,q)| \deg(q)^3}.
$$

**Proof.** By (6.2), $Z_p$ is real. Moreover, by Hölder inequality, we have

$$
E[e^{\lambda Z_p}] = E[e^{\lambda \sum_q (p,q) Z_q}] \\
\leq \prod_q E[e^{\lambda \operatorname{sign}(p,q) Z_q}] \\
\leq \prod_{q,q'} E[e^{\lambda \operatorname{sign}(p,q) \operatorname{sign}(\partial_q \rho_N(U)^{q'} - (q')^*)} Z_{q_0,...,q_{i-1},q_i,q_{i+1},...,q_{i',1},a_0,...}] \langle p,q \rangle \langle D_i q, q' - (q')^* \rangle \\
\leq e^{C\lambda^2 \sum_q |(p,q)| \deg(q)^2} \\
\leq e^{C\lambda^2 \sum_q |(p,q)| \deg(q)^3}
$$

where we used the previous lemma.
We next use the previous lemma as well as the Schwinger-Dyson equations to prove concentration estimates.

**Lemma 31.** Let \( \delta_N(p) = \frac{1}{N} \text{Tr} \rho_N(p)(U) - \tau_{r_{10}}(p) \) and assume that \( \xi \geq 12 \) so that \( \Xi_{r_{10}}^{V} \) is invertible in \( L_\xi \). Assume moreover that \( \|V\|_1 \) is small enough so that we can choose \( \xi \) so that additionally

\[
\max_{\mu \in \text{Char}(L), \|\mu\| \leq 1} \|\Pi T_\mu D^{-1}\|_\xi \|(\Xi_{r_{10}}^{V})^{-1}\|_\xi < 2.
\]

Then, for all \( r \geq 1 \), there exists a finite constant \( C_r \) such that for all polynomials such that \( \|p\|_\xi \leq 1 \), we have

\[
E[|\delta_N(p)|^r]^{\frac{1}{r}} \leq \frac{C_r}{N}
\]

**Proof.** If we denote by \( L_N = \frac{1}{N} \text{Tr} \circ \rho_N \), we can rewrite the definition of \( Z_p \) as

\[
Z_p = L_N \otimes L_N \left( \sum \partial_1 D_1 p \right) + L_N \left( \sum \partial_1 V D_1 p \right) = L_N \left( \Psi_{L_N} D_1 p \right)
\]

so that

\[
\delta_N(\Xi_{r_{10}}^{V}) = -\frac{1}{2} \delta_N((\Pi T_\delta D^{-1}) p) + \frac{1}{N} Z_{D^{-1}p}
\]

so that taking the \( \ell^r \)-norm on both sides we deduce that for any polynomial \( p = p^* \) we have

\[
\|\delta_N(p)\|_r \leq \frac{1}{2} \|\delta_N((\Pi T_\delta D^{-1}(\Xi_{r_{10}}^{V})^{-1} p))\|_r + \frac{1}{N} \|Z_{D^{-1}p}\|_r
\]

We set \( \|\delta_N\|_r = \max_{\|\mu\| \leq 1, \|\xi\| \leq 1} \|\delta_N(p)\|_r \) and deduce that

\[
\|\delta_N\|_r \leq \frac{1}{2} \max_{\mu \in \text{Char}(L), \|\mu\| \leq 1} \|\Pi T_\mu D^{-1}\|_\xi \|(\Xi_{r_{10}}^{V})^{-1}\|_\xi \|\delta_N\|_r + \frac{1}{N} \max_{\|\mu\| \leq 1} \|Z_{D^{-1}p}\|_r
\]

Remembering that we chose the norm so that

\[
\max_{\mu \in \text{Char}(L), \|\mu\| \leq 1} \|\Pi T_\mu D^{-1}\|_\xi \|(\Xi_{r_{10}}^{V})^{-1}\|_\xi < 2
\]

the conclusion follows from the fact that the previous lemma yields the existence of a finite constant so that since \( \xi \) is large enough so that \( n^3 \leq \xi^n \) for all \( n \in \mathbb{N} \),

\[
\|Z_{D^{-1}(\Xi_{r_{10}}^{V})^{-1} p}\|_r \leq C_r^3 \sum_q \|D^{-1}(\Xi_{r_{10}}^{V})^{-1} p, q\| \|\delta_N\|_r \leq C_r^3 \|D^{-1}(\Xi_{r_{10}}^{V})^{-1}\|_\xi \|\delta_N\|_r < \infty
\]

\( \square \)

### 6.3. Uniform boundedness.

**Corollary 32.** Under the assumptions of Lemma 28 or section 6.2, for any \( k \geq 2 \), any monomials \( p_1, \ldots, p_k \) we have

\[
|\mathcal{W}_k^{V}(p_1, \ldots, p_k)| \leq C_k \prod_{i} \|\text{deg}(p_i)\|_\xi
\]

for a finite constant \( C_k \). In particular, \( (\mathcal{W}_k^{V})_{k,N=1}^{\infty} \) is \( \xi \)-uniformly bounded for any \( \xi \geq 12 \).
**Proof.** By induction we easily check that

\[ W^V_{kN}(p_1, \cdots, p_k) = \int \prod_{i=1}^k \left( \text{Tr} \, \rho_N(p_i)(U) - \mathbb{E} \text{Tr} \, \rho_N(p_i)(U_N) \right) d\mu_N^V(dU). \]

Hence the result follows by Hölder’s inequality. This is trivial in the setting when Lemma 31 applies but also when Lemma 28 does since it yields

\[
\int (\text{Tr} \, \rho_N(p)(U) - \mathbb{E} \text{Tr} \, \rho_N(p)(U))^2 \mu_N^V(dU)
\leq 2k C \int_0^\infty x^{2k-1} e^{-c x^2} dx.
\]

\[ \square \]

### 7. Consequences of the main result

#### 7.1. Expansion of the free energy: Proof of Corollary 3

The expansion of the free energy is a direct consequence of Theorem 1 as we can write

\[
\frac{1}{N^2} \log Z^N_V = \int_0^1 \frac{dt}{N^2} \log Z^N_{tV} dt = \int_0^1 \frac{1}{N} W^{tV}_{1N}(V) dt
\]

where for all \( t \in [0, 1] \) \( tV \) satisfies the hypothesis of Theorem 1 as soon as \( V \) does. Hence, the asymptotic expansion for the free energy is a direct consequence of the asymptotic expansion of \( W^{tV}_{1N}, t \in [0, 1] \) which is uniform in \( t \in [0, 1] \).

#### 7.2. Central limit theorem: Proof of Corollary 4

We write that

\[
\log \mu_N^V(\exp^{\lambda(\text{Tr} P)}) = \int_0^\lambda \frac{\mu_N^V(\text{Tr} P e^{\alpha \text{Tr} P})}{\mu_N^V(e^{\alpha \text{Tr} P})} d\alpha = \int_0^\lambda W^{V+\Phi P}_{1N}(P) d\alpha
\]

Moreover, by Theorem 24 we know that

\[
W^{V+\Phi P}_{1N}(P) = N \tau^{V+\Phi P}_{10}(P) + O\left(\frac{1}{N}\right)
\]

where it is not hard to check that the error is uniform in \( \alpha \in [0, \lambda] \). Therefore we need to compute the expansion of \( \tau^{V+N^{-1}V_i}_{10} \). It is not difficult to see that \( \tau^{V+\epsilon V_i}_{10} \) is smooth in \( \epsilon \) so that we can write

\[
\tau^{V+\epsilon V_i}_{10} = \tau^{V}_{10} + \tau^{V}_{10} \epsilon V_i + o(\epsilon)
\]

Plugging back this expansion into the Schwinger-Dyson equation shows that \( \tau^{V}_{10} \) is solution of

\[
\tau^{V}_{10} \exp^{V_{10}} = -\tau^{V}_{10} (\mathbb{P}^{V_{10}} P) \Rightarrow \tau^{V}_{10}(P) = -\tau^{V}_{10} (\mathbb{P}^{V_{10}} (\exp^{V_{10}})^{-1} p).
\]

Since \( \mathbb{P}^{V} \) is linear in \( V \) we find by taking \( \epsilon = N^{-1} \) that

\[
W^{V+\Phi P}(P) = N \tau^{V}_{10}(P) - \frac{\alpha}{N} \tau^{V}_{10} (\mathbb{P}^{V} (\exp^{V})^{-1} p) + o\left(\frac{1}{N}\right)
\]
from which we conclude that
\[
\lim_{N \to \infty} \log \mu_N^V(e^{A(\text{Tr}p - N\tau_{10}^V(p))}) = -\int_0^\lambda \alpha \tau_{10}^V(\mathbb{P}^V(\Xi_{10}^V)^{-1}p) d\alpha = -\frac{\lambda^2}{2} \tau_{10}^V(\mathbb{P}^V(\Xi_{10}^V)^{-1}p).
\]

7.3. **Proof of Theorem 5.** We now give the proof of Theorem 5 as stated in the introduction. Since the monomial \(xuyu^{-1}\) is selfadjoint up to cyclic symmetry, for any \(t \in \mathbb{R}\), the quadratic potential \(V_t = txuyu^{-1}\) generates a real Gibbs ensemble, i.e. the Borel measure \(\mu_{V_t}^N\) on \(U(N)\) with density
\[
\frac{1}{Z_{V_t}^N} e^{N \text{Tr}_N(V_t)} = \frac{1}{Z_{V_t}^N} e^{tN \text{Tr}_N(x)U\rho_N(y)U^{-1}}
\]
against the Haar measure \(\mu_N\) is a real probability measure. Thus, by Corollary 3, for real \(t\) satisfying
\[
|t| < \frac{7}{66} \cdot \frac{1}{22^k - 1} \cdot \frac{1}{12^2} = \frac{7}{19008},
\]
the free energy
\[
F_N(t) = F_{V_t}^N = \frac{1}{N^2} \log Z_{V_t}^N
\]
admits the asymptotic expansion for \(h \leq K - 1\)
\[
(7.1) \quad F_N(t) = \sum_{g=0}^h \frac{F_g(t)}{N^{2g}} + o\left(\frac{1}{N^{2h}}\right).
\]
That is, for any \(t \in (-\frac{7}{19008} - \frac{7}{19008}, \frac{7}{19008} + \frac{7}{19008})\), we have
\[
F_N(t) = \sum_{g=0}^{K-1} \frac{F_g(t)}{N^{2g}} + r_N(t),
\]
where
\[
\lim_{N \to \infty} \frac{r_N(t)}{N^{2(K-1)}} = 0.
\]
Moreover, there exists \(\varepsilon > 0\) such that the coefficients \(F_g(t)\) extend to analytic functions on the complex disc \(|t| < \varepsilon\).

Since \(U(N)\) is compact, the partition function \(Z_{V_t}^N\) is an entire function of \(t \in \mathbb{C}\). Thus \(F_N(t)\), being the principal branch of the logarithm of \(Z_N(t)\), is analytic in a complex neighbourhood of \(t = 0\). Thus the error
\[
r_N(t) = F_N(t) - \sum_{g=0}^h \frac{F_g(t)}{N^{2g}}
\]
is a difference of analytic functions, and hence is also analytic in a neighbourhood of \(t = 0\). Decomposing all relevant functions in Maclaurin series,
\[
F_N(t) = \sum_{d=1}^\infty F_N^{(d)}(0) \frac{t^d}{d!}, \quad F_g(t) = \sum_{d=1}^\infty F_g^{(d)}(0) \frac{t^d}{d!}, \quad r_N(t) = \sum_{d=1}^\infty r_N^{(d)}(0) \frac{t^d}{d!},
\]
we thus have

\[ F_N^{(d)}(0) = \sum_{g=0}^{\infty} \frac{F_g^{(d)}(0)}{N^{2g}} + r_N^{(d)}(0) \]

for all \( d \geq 1 \), whence

\[ F_N^{(d)}(0) = \sum_{g=0}^{\infty} \frac{F_g^{(d)}(0)}{N^{2g}} + o\left( \frac{1}{N^{2\ell}} \right) \]

as \( N \to \infty \), for each fixed \( d \geq 1 \).

The asymptotics of the Maclaurin coefficients of \( F_N(t) \) were obtained by a different method in [11], where a representation-theoretic argument was used to show that

\[ F_N^{(d)}(0) = \sum_{\alpha, \beta \vdash d} (-1)^{\ell(\alpha) + \ell(\beta)} \sigma_g(x^{\alpha}) \sigma_g(y^{\beta}) \vec{H}_g(\alpha, \beta) + o\left( \frac{1}{N^{2\ell}} \right), \]

with the \( \vec{H}_g(\alpha, \beta) \)'s the monotone double Hurwitz numbers. Thus, from the uniqueness of asymptotic expansions on a given asymptotic scale (the scale here being \( N^{-2} \)), it follows that

\[ F_g^{(d)}(0) = \sum_{\alpha, \beta \vdash d} (-1)^{\ell(\alpha) + \ell(\beta)} \sigma_g(x^{\alpha}) \sigma_g(y^{\beta}) \vec{H}_g(\alpha, \beta), \]

as required.

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