Incommensurate Phase on a Disordered Surface: Instability Against the Formation of Overhangs and Finite Loops

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Abstract

The stability of the quenched incommensurate phase in two dimensions against the creation of overhangs and finite loops (OH/FL) in the replica space is investigated for a model of domain walls with \( N \) colors. Introducing a chemical potential \( \epsilon \) for OH/FL, the probability for the formation of these objects is studied for \( \epsilon \to 0 \). In the pure limit this probability vanishes with \( \sqrt{\epsilon} \), whereas the fluctuations of this probability are long-range correlated in the quenched system. This indicates an instability related to the symmetry in replica space. It is accompanied by the creation of a massless boson. The latter leads to a power law decay with exponent \( \propto 1/N \) for the product of the correlation functions along the domain walls.

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The model for the commensurate-incommensurate transition (CIT) in $d = 1 + 1$ dimensions is the prototype for a class of systems characterized by directed domain walls, directed random walks, directed polymers or flux lines. The common feature is that of interacting random walks where the walkers choose randomly steps forward, left or right but not backward. The interaction is due to the condition that walks are not allowed to cross. In the presence of quenched disorder these systems should exhibit a new phase which is related to a glass or frozen phase. The latter can be regarded as freezing of the domain walls (directed random walks etc.) in a random potential. From a conceptual point of view the CIT model has attracted attention because it is soluble without disorder [1], and the effect of disorder can be regarded as a perturbation. The disordered case has been studied by various methods like the Bethe ansatz [2], scaling arguments [3], variational approaches [4] and with perturbation theory [5]. Although these works agree on the fact that the critical exponent of the density changes from $1/2$ in the pure system to 1 in the disordered system they came to different conclusions concerning the decay of the correlation function in the quenched system. In a recent article by Tsvelik [6] it was argued that quenched disorder is irrelevant with respect to the asymptotic decay of the density density function. This is surprising since the correlation function should be more sensitive to additional fluctuations than the density. Therefore, at least near the CIT one should expect that disorder changes the qualitative properties of the model. For the corresponding model with disorder correlated along the direction of the domain walls was found that the long-range correlation is destroyed [7]. In the following we will present a calculation which shows that there is indeed a new feature due to disorder which affects the correlations. It is characterized by the formation of overhangs and finite loops (OH/FL). A crucial point in the various approaches to the CIT in the presence of disorder is the replica trick. It has been argued that there is no replica symmetry breaking (however, see [8]). In contrast to this we will start from a broken replica symmetry given by an external field (chemical potential) which creates OH/FL.

The grand canonical statistics of domain walls in $d = 1 + 1$ dimensions can be described by a fermion Lagrangian density in imaginary time representation [6].
\[
L = c^+ \partial_\tau c - \partial_x c^+ \partial_x c - v(x, \tau) c^+ c + \mu c^+ c
\]  

(1)

and the partition function

\[
Z = \int \mathcal{D}[c^+, c] \exp(-\int_0^\infty d\tau \int dx L).
\]  

(2)

The imaginary time \(\tau\) is along the direction of the domain walls and \(x\) is perpendicular to \(\tau\). Disorder is introduced by the random potential \(v(x, \tau)\) which is Gaussian distributed with zero mean and \(\langle v(x, \tau)v(x', \tau') \rangle = \sigma^2 \delta(x-x')\delta(\tau-\tau')\).

The partition function is invariant under a time-reversal transformation. Only the time differential operator is changed as \(\partial_\tau \rightarrow \partial_T^\tau = -\partial_\tau\) by this transformation. The integration of the non-interacting fermions in \(Z\) gives the space-time determinant \(\det(\partial_\tau + \partial_x^2 + \mu - v)\). Then the time-reversal invariance of \(Z\) is obvious because the transposition of the matrix leaves the determinant invariant.

Averaging of the free energy or of an expectation value can be performed either by using a supersymmetric generalization of the fermion Lagrangian [3] or by using the replica trick \(\ln Z = \lim_{n \to 0} (Z^n - 1)/n\). Since both representations are equivalent if we choose the correct structure, we will use the replica trick here for convenience. Introducing an even number of replicas, say \(2n\), we take one sector of \(n\) replicas with \(\partial_\tau\) and the other sector of \(n\) replicas with \(\partial_T^\tau\). This can be written in a spinor representation for the Lagrangian density as

\[
L = \begin{pmatrix}
  c^+ \\
  d^+
\end{pmatrix} \cdot \begin{pmatrix}
  0 & \partial_\tau + \partial_x^2 - v + \mu \\
  \partial_T^\tau + \partial_x^2 - v + \mu & 0
\end{pmatrix} \begin{pmatrix}
  d \\
  c
\end{pmatrix},
\]  

(3)

where \(c, d\) are \(n\)-component fermions, and the matrix in (3) is diagonal w.r.t. to the \(n\) replica components. Now we introduce a chemical potential \(\epsilon\) which allows the formation of OH/FL by combining the two different replica sectors. This means that the chemical potential creates a particle-hole pair \(c^+ d\) which can be annihilated at another space-time point by \(d^+ c\). Including this new chemical potential in the Lagrangian density (3) we replace the zeros in the diagonal elements by \(i\epsilon\). The imaginary unit \(i\) guarantees a positive weight for the OH/FL in the partition function because OH/FL contain always a sequence of reversed Grassmann variables which contribute a minus sign. The density of OH/FL can be measured
by varying locally the chemical potential: $\rho_\epsilon(x, \tau) \propto \langle c^+(x, \tau)d(x, \tau) + d^+(x, \tau)c(x, \tau) \rangle$. In the pure limit $v = 0$ this density can be evaluated by a simple calculation as $\rho_\epsilon \sim \text{const.} \sqrt{\epsilon}$; i.e., it vanishes continuously for vanishing chemical potential $\epsilon$. Thus, the pure system is stable against the formation of OH/FL. Another interesting quantity is related to the fact that OH/FL are given by a combination of local edges (s. Fig.1) which are separated by a distance in $x$ and $\tau$. The correlation of an edge at $(x, \tau)$ and another one at $(0, 0)$ is

$$\langle c^+(x, \tau)d(x, \tau)d^+(0, 0)c(0, 0) \rangle.$$  \hfill (4)

Because this is a correlation function of non-interacting fermions it will factorize into

$$\langle c^+(x, \tau)d(x, \tau) \rangle \langle d^+(0, 0)c(0, 0) \rangle - \langle c^+(x, \tau)c(0, 0) \rangle \langle d^+(0, 0)d(x, \tau) \rangle.$$  \hfill (4)

The first term measures the density of OH/FL whereas the second term is the product of correlations along the domain walls from $(0, 0)$ to $(x, \tau)$.

Our calculation will be based on the generalization of the domain wall model to one which has walls with $N$ different colors. Without quenched disorder the new model separates into $N$ independent models. However, the disorder potential is chosen as $v_{\alpha\beta}(x, \tau)$ with $\alpha, \beta = 1, ..., N$ such that the domain walls may change statistically the color from $\alpha$ to $\beta$. This mechanism reduces the hard-core of the domain walls (i.e., Pauli’s principle of the fermions) because the domain walls can avoid each other by changing the color. The complex matrix elements $v_{\alpha\beta}$ are statistically independent except for the symmetry $v_{\alpha\beta} = v^*_{\beta\alpha}$. After averaging we obtain an effective interaction for the fermions ($\hat{c} = (d, c)$) in the replica model

$$\frac{\sigma^2}{2N} \sum_{i,j=1,\alpha,\beta=1}^{2n} \hat{c}_{i,\alpha}^+ \hat{c}_{i,\beta} \hat{c}_{j,\beta}^+ \hat{c}_{j,\alpha}.$$  \hfill (5)

The limit $N \rightarrow \infty$ can be solved by a saddle approximation. This becomes obvious if one decouples the effective fermion-fermion interaction of the quenched model by a new random matrix field $Q$ which includes fluctuations of both chemical potentials $\mu$ and $\epsilon$: we replace

$$\sigma^2 N^{-1} \hat{c}_{i,\alpha}^+ \hat{c}_{i,\beta} \hat{c}_{j,\beta}^+ \hat{c}_{j,\alpha} = -\sigma^2 N^{-1} \hat{c}_{i,\alpha}^+ \hat{c}_{j,\beta} \hat{c}_{j,\alpha}^+ \hat{c}_{i,\beta} \hat{c}_{j,\alpha} + 2iQ_{ji} \sum_\alpha \hat{c}_{i,\alpha}^+ \hat{c}_{j,\alpha}.$$  \hfill (5)

$Q$ is a Hermitean $2n \times 2n$ matrix field. Going back to the replicated partition function $Z^{2n}$ we integrate out the fermion field $\hat{c}_{i,\alpha}$. This leads to an effective action which depends only on the decoupling field as
\[
S_{\text{eff}} = N \int_0^\infty d\tau \int dx \frac{2}{\sigma^2} Tr_{2n}(Q(x, \tau)^2) \\
- N \ln \det \begin{pmatrix}
  i\epsilon + 2iQ_{21}(x, \tau) & \partial_x + \partial^2_x + \mu + 2iQ_{11}(x, \tau) \\
  \partial^T_x + \partial^2_x + \mu + 2iQ_{22}(x, \tau) & i\epsilon + 2iQ_{12}(x, \tau)
\end{pmatrix}.
\]

\(Tr_{2n}\) denotes the trace w.r.t. the \(2n\) replica and \(\det\) the determinant w.r.t. the \(2n\) replica and \(x, \tau\). This is the 1+1-dimensional replica version of the 2+1-dimensional ‘supersymmetric’ action found for the flux lines in a random potential \[9\].

\(S_{\text{eff}}\) depends on the number of colors only through the prefactor \(N\). Therefore, the effective field theory for large \(N\) enables us to do the functional integration in saddle point approximation. This approximation can be interpreted as the replacement of the chemical potentials \(\mu\) and \(\epsilon\) in the free fermion propagator by a self-energy term which is determined by the saddle point equation \(\delta S_{\text{eff}} = 0\). There are two contributions if we assume that \(Q\) provides only additive corrections to the two chemical potentials. One is a shift of the chemical potential \(\mu\) in the limit \(\epsilon \to 0\)

\[
\mu_s \equiv 2iQ_{jj} = -\sigma^2 \int dk d\omega \frac{-k^2 + \mu + \mu_s}{D(\epsilon')} (j = 1, ..., 2n)
\]

with \(D(\epsilon') = \epsilon'^2 + \omega^2 + (-k^2 + \mu + \mu_s)^2\), and the other is a shift of the chemical potential for OH/FL

\[
\epsilon' \equiv 2Q_{jj+n} = 2Q_{nj+n} = \epsilon'\sigma^2 \int dk d\omega / D(\epsilon') (j = 1, ..., n).
\]

\(\epsilon' = 0\) is always a solution of \(8\). This is a replica-symmetric (RS) solution. A replica-symmetry breaking (RSB) solution \(\epsilon' \neq 0\) can be found if

\[
\int dk d\omega / D(0) > 1/\sigma^2.
\]

Since the denominator \(D(\epsilon')\) is increasing with increasing \(\epsilon'^2\) there is an \(\epsilon'\) which satisfies

\[
\int dk d\omega / D(\epsilon') = 1/\sigma^2.
\]

This can be rewritten as
\[ \epsilon'^{1/2} = \sigma^2 \int d\bar{k} d\bar{\omega} \frac{1}{1 + \bar{\omega}^2 + (-k^2 + \mu'/\epsilon')^2}. \]  

Equation 11

In this case we obtain from (7) a renormalized chemical potential

\[ \mu' \equiv \mu + \mu_s = \mu - \epsilon'^{1/2} \sigma^2 \int d\bar{k} d\bar{\omega} \frac{-k^2 + \mu'/\epsilon'}{1 + \bar{\omega}^2 + (-k^2 + \mu'/\epsilon')^2}. \]

Equation 12

\( D(\epsilon') \) is increasing with decreasing \( \mu' < 0 \). This implies that there is a \( \mu_c < 0 \) (which depends on \( \sigma \)) such that \( \int d\bar{k} d\bar{\omega}/D(0) < 1/\sigma^2 \) for \( \mu' < \mu_c \). Consequently, there is only a RS solution \( \epsilon' = 0 \) for \( \mu' < \mu_c \). This behavior describes a transition from a RS solution to a RSB solution if we change \( \mu \) or the disorder \( \sigma \).

The evaluation of the density of domain walls in saddle point approximation requires a cut-off, because only a finite number of domain walls per unit length is reasonable. With the cut-off \( k^2 = 1 \) we obtain after integration over \( \omega \)

\[ \rho \propto 1 - \frac{i}{\sigma^2} (Q_{11} + Q_{22}) = 1 + \int_0^1 dk \frac{-k^2 + \mu'}{\sqrt{\epsilon'^2 + (-k^2 + \mu')^2}}. \]

Equation 13

\( \mu' < 0 \) and \( \epsilon' \to 0 \) implies \( \rho \to 0 \). The density vanishes linearly. For instance, the density of the RS solution is \( \rho \propto \sigma^2 + \mu \). Thus the density of the RS as well as the RSB solution are in agreement with the linear behavior of the Bethe ansatz calculation of Kardar [2].

Finally we evaluate the Gaussian fluctuations around the saddle point solution which are the \( 1/N \) corrections. The stability matrix can be taken from Ref. [9]. The fluctuations of \( Q_{11}, Q_{22} \) are massive with the eigenvalues \( \lambda_{\pm} \)

\[ \lambda_+ = \lambda_- + 4 \int d\bar{k} d\bar{\omega} \omega^2 / D(\epsilon')^2. \]

Equation 14

For \( \epsilon' \neq 0 \) \( \lambda_- \) is

\[ \lambda_- = 4\epsilon'^2 \int d\bar{k} d\bar{\omega} / D(\epsilon')^2; \]

Equation 15

i.e., the eigenvalues of the RSB solution are always positive (stable). On the other hand, for \( \epsilon' = 0 \) the eigenvalue \( \lambda_- \) is

\[ \lambda_- = 2/\sigma^2 - 2 \int d\bar{k} d\bar{\omega} / D(0). \]

Equation 16
Since inequality (9) holds if there are two saddle point solutions, $\lambda_-$ of the RS solution (16) is negative (unstable) when the RSB solution exists. The other two eigenvalues of the RSB solution are $\lambda_3 = 2(\lambda_+ - \lambda_-) > 0$ (related to $Q_{12} - Q_{21}$) and the massless one $\lambda_4 = 0$ (related to $Q_{12} + Q_{21}$). The massless fluctuations are a consequence of a global symmetry of the model [9]. They can be expressed as a non-linear sigma model for $Q$ with the constraint $Q^2 = 1$, $Tr_{2n}Q = 0$ and $Q\sigma_2Q = -\sigma_2$. The constraint can be satisfied if we parametrize $Q$ by an $n$-component field $\varphi$ as $Q = \sigma_3(\cos^2\varphi - \sin^2\varphi) + 2\sigma_1 \cos\varphi \sin\varphi$. This parametrization neglects a unitary transformation inside of each replica sector which is not expected to affect the properties of the $\varphi$-dependent part of the model. With the field $\varphi$ the action of the non-linear sigma model reads

$$S_{eff} = bN \int dx d\tau \varphi(\partial_x^2 + \partial_\tau^2)\varphi$$

with a positive constant $b$, depending on $\gamma$ but not on $N$ or $\epsilon$, which are given in Ref. [9].

From this bilinear action we can evaluate the correlation function of the density of the edges:

$$\langle Q_{12}(x, \tau)Q_{21}(0, 0) \rangle \sim \epsilon'const.\left(x^2 + \tau^2\right)^{-1/2\pi bN}$$

for large distances. Thus there is no replica symmetry breaking because the correlation of the order parameter decays. According to our previous remark this implies also a power law decay for the product of correlation functions along the domain walls. In contrast, a single averaged correlation function decays exponentially according to the finite length scale created by $\epsilon'^{-1}$. On a pure surface the correlation function decays like $(x^2 + \tau^2)^{-1/2}$.

In conclusion, we have found that the statistics of domain walls in the CIT model with quenched disorder is unstable against the formation of OH/FL in the replica space. This result was based on a saddle point calculation for an effective self-energy in the replica model with $2n$ replica and $N$ colors. The properties of the model are similar to those of the Gross Neveu model [10] and other 1+1-dimensional fermion models with attractive interaction [11]. The creation of OH/FL in the quenched CIT model is plausible because the random potential favors OH/FL which can easily circumvent an unfavorable potential
or freeze into the local potential wells. Since there are no OH/FL by definition, the replica model creates them spontaneously by combining wall elements coming from different replica sectors. However, the effect of the OH/FL is only marginal in our calculation because they appear with vanishing density. Nevertheless, the instability related to the OH/FL means a drastic change in the quenched state compared with the pure CIT problem.

A similar instability was also discussed in the context of the Gross Neveu model by Witten [12] who evaluated the correlation function of the massless bosons. Those bosons also lead to a power law decay with an exponent proportional to $1/N$.

It should be emphasized that the formation of OH/FL is a special type of instability regarding the freezing into local minima of the random potential. We cannot exclude other instabilities which may be even more favorable. Therefore, the purpose of this investigation was only to provide an example for an instability in the quenched CIT model. It indicates that the model has probably a rich structure which deserves further investigations based on effects related to replica symmetry.
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FIGURES

FIG. 1. A finite loop (a) and an overhang (b) created for the domain wall segments from different sectors of the replica space $j$ and $k$. 