On polynomial solutions to Fokker–Planck and sunked density evolution equations

Mathew Zuparic

Defence Science and Technology Organisation (DSTO), ACT 2600, Australia

E-mail: mathew.zuparic@dsto.defence.gov.au

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Abstract

We analytically solve for the time dependent solutions of various density evolution models. With specific forms of the diffusion, drift and sink coefficients, the eigenfunctions can be expressed in terms of hypergeometric functions. We obtain the relevant discrete and continuous spectra for the eigenfunctions. With non-zero sink terms the discrete spectra eigenfunctions are generalizations of well known orthogonal polynomials: the so-called associated-Laguerre, Bessel, Fisher–Snedecor and Romanovski functions. We use MacRobert’s proof to obtain closed form expressions for the continuous normalization of the Romanovski density function. Finally, we apply our results to obtain the analytical solutions associated with the Bertalanffy–Richards–Langevin equation.

Keywords: Fokker–Planck equation, density function, hypergeometric function, classical orthogonal polynomial, Bertalanffy–Richards–Langevin equation

1. Introduction

The Fokker–Planck equation has been the focus of many decades of study due to its relevance in physics, finance, probability and statistics [42, 46]. Reference [52] provides a particularly early example examining analytically tractable solutions to the Fokker–Planck equation, with more contemporary examples provided by [2, 5, 27–30].

Essentially, this work focuses on time dependent densities \( \mathcal{T}(x, t|y), t \geq 0 \), of a diffusion process described by range \( x \) given that it started at position \( y \), governed by

\[
\frac{\partial \rho(x, t)}{\partial t} = D \frac{\partial^2 \rho(x, t)}{\partial x^2} - \mu \frac{\partial \rho(x, t)}{\partial x} - s \rho(x, t)
\]
\[
\frac{\partial}{\partial t} T(x, t|y) = \left\{ \frac{\partial^2}{\partial x^2} s(x) - \frac{\partial}{\partial x} q(x) - r(x) \right\} T(x, t|y), \\
T(x, 0|y) = \delta(x - y), \quad s(x) > 0. \tag{1}
\]

Equation (1) is defined on some interval on \( \mathbb{R} \) with endpoints \((e_1, e_2)\), where \(-\infty \leq e_1 < e_2 \leq \infty\). The continuous functions \( s(x), q(x) \) and \( r(x) \) are referred to as the diffusion, drift and the sink coefficients respectively.

The applications of equation (1) are wide ranging. For \( r(x) = 0 \), equation (1) is commonly referred to as the Fokker–Planck equation and one can readily show that \( T(x, t|y) \) is conserved for \( t \geq 0 \). The corresponding solutions to the Fokker–Planck equation are the time dependent probability densities associated with the (Itō) stochastic Langevin process

\[
\dot{x}(t) = q(x(t)) + \sqrt{2s(x(t))} \eta(t), \quad x(0) = y, 
\]

where \( \eta(t) \) is a Gaussian white noise term with unit variance. The Langevin equation is ubiquitous in a wide range of applications, from its beginnings in Brownian motion (see chapter 1 of [46]), to finance [29], biological processes [19] and the synchronization of networked oscillators [48].

For \( r(x) \neq 0 \), the corresponding continuity equation is sinked, thus we expect the quantity being measured in equation (1) to seep away with time. It is conceptually important to develop analytically tractable solutions to equation (1) as they are Green’s functions, which are both inherently mathematically interesting, and highly applicable—for an account of their application in physics see chapter 7 of [36]. Following [17], Green’s/density functions appearing in this work also figure heavily in random matrix theory. We shall indicate some of these connections throughout this work. Additionally, many past results for the conserved Fokker–Planck equation (we offer [54] as a typical example) rely simply on the steady state density to gain insights. Sinked densities allow no such avenue for inquiry as their solutions decay with time. We shall highlight this behaviour in the proceeding sections.

The general strategy for solving equation (1) is as follows: we first obtain the weight function \( W(x) \), found by solving the corresponding Pearson equation

\[
\left\{ \frac{d}{dx} s(x) - q(x) \right\} W(x) = 0 \Rightarrow W(x) = \frac{\kappa}{s(x)} \exp\left\{ \int_{x'}^{x} \frac{q(\xi)}{s(\xi)} d\xi \right\}, \tag{2}
\]

for constants \( \kappa \) and \( x' \). Given the form of the diffusion, drift and sink coefficients, we categorise the spectrum of equation (1) as either discrete or mixed discrete/continuous. This gives us the general form of \( T(x, t|y) \) as

\[
T(x, t|y) = W(x) \sum_i e^{-i\lambda_i} \theta_i(x) \theta_i(y), \tag{3}
\]

where the sum will be an integral for the continuous parts of the spectrum. We then find the corresponding eigenvalues and eigenfunctions of the system through standard techniques [1, 24, 37]. The final step involves solving for the normalization constants which satisfy the initial condition. For the discrete spectrum eigenfunctions we utilize the orthogonal polynomial relation (chapter 3 of [24])

\[
\rho_n = \frac{1}{\int_{e_1}^{e_2} dx W(x) \theta_n^2(x)}, \tag{4}
\]

and for the continuous spectrum eigenfunctions we employ MacRobert’s inverse integral transform of the form
\[ \int_{e_1}^{e_2} \text{d}x W(x) \vartheta(\nu, x) \int_0^\infty \text{d}\mu \rho(\mu) \vartheta(\mu, x) = \rho(\nu) \Lambda(\nu). \] (5)

See [51] for an instance involving the Whittaker functions, and [23] and chapter 14 of [11] for examples involving Bessel/Hankel functions. These inverse integral transforms usually rely on some key results attributable to MacRobert [31] which we shall exploit when deriving the continuous normalization for the Romanovski case.

Given the pervasive nature of the Fokker–Planck equation, Green’s functions and orthogonal polynomials, most of the cases presented in this work have been fully solved in the literature without the sink term. What is new in this work is we present the full time dependent solutions for the sinked variants \( r(x) \neq 0 \), and apply these results to obtain new solutions to the stochastic Bertalanffy–Richards (B–R) equation [3, 41]. For an introduction to the application of the B–R equation in population modelling and biological processes see [19].

In the next section we detail the forms of the diffusion, drift and sink coefficients that lead to the orthogonal polynomial eigenfunctions considered in this work. In section 3 we give necessary information about Sturm–Liouville (S–L) operators, the Hilbert spaces their eigenfunctions span and how the sink terms in the S–L operators form associated variations of the orthogonal polynomials/eigenfunctions. In section 4 we detail how the form of the S–L operator determines the exact form of the spectra for the eigenfunctions, along with the corresponding solutions to equation (1). In section 5 we apply the results of this work to the stochastic B–R equation. Finally we offer implications of these results and flag future work.

2. Orthogonal polynomials

2.1. Negative eigenvalues

Applying the weight function, we decompose \( T(x, t|y) \) in equation (1) as

\[ T(x, t|y) = W(x) g(x, t|y). \] (6)

Hence the ensuing equation for \( g(x, t|y) \) is

\[ \frac{\partial}{\partial t} g(x, t|y) = \mathcal{H} g(x, t|y), \]

where \( \mathcal{H} \) is the S–L operator

\[ \mathcal{H} = s(x) \frac{d^2}{dx^2} + q(x) \frac{d}{dx} - r(x). \] (7)

We require that the operator \( \mathcal{H} \) be negative, i.e. all relevant eigenfunctions of \( \mathcal{H} \) have negative eigenvalues

\[ \mathcal{H} \vartheta_\lambda(x) = -\lambda \vartheta_\lambda(x), \quad \lambda \geq 0. \] (8)

The eigenvalues \( \lambda \) may be discrete or continuous depending on boundary conditions [29], to be specified explicitly in section 4.
2.2. Continuous classical orthogonal polynomials

For this work $s(x)$, $q(x)$ and $r(x)$ in equation (8) have the specific forms:

\[ s(x) \text{ is at most quadratic in } x, \]
\[ q(x) \text{ is at most linear in } x, \]
\[ r(x) \text{ is either 0, or } \lim_{x \to (0,\infty)} x r(x) = \text{constant}. \]  \tag{9}

Focusing on the $r(x) = 0$ case, theorem 4 of [24] states that there are six classes of discrete spectrum eigenfunctions to equation (8): Hermite, Laguerre, Jacobi, Bessel, Fisher–Snedecor (shifted-Jacobi), and Romanovski (pseudo-Jacobi) polynomials—the so-called continuous classical orthogonal polynomials. For $r(x) \neq 0$ the eigenfunctions generally have the same polynomial form, but are multiplied by the diffusion coefficient $s(x)$ raised to some power, $x$, specified in section 3 of this work. We do not formally consider the Hermite and Jacobi case in this work (we only refer to them) as the Hermite case offers nothing new, and the Jacobi case only has finite support in $x$. In [43], hypergeometric polynomial solutions to equation (8) for quite general forms of the diffusion and drift coefficients were studied.

Using the Liouville transformation (equation (14) of this work) to transform equation (8) into a corresponding Schrödinger equation, we note that the Laguerre, Bessel and Romanovski potentials correspond to the Coulomb (chapter 6 of [34]), Morse [35] and trigonometric Scarf [44] potentials, albeit with an additional parameter corresponding to the sink coefficient.

More recently the Laguerre polynomials were applied as solutions to the nonlinear Madelung fluid equation [7] and Burger’s equation with a time-dependent forcing term [53]. Both [8] and [18] highlight the connection between the Laguerre polynomials and the algebra $su(1, 1)$, which is then exploited to construct the coherent Laguerre function, and explore squeezed states in the Calogero–Sutherland model. [14] also highlights connections between Lie algebras and the associated Laguerre functions. Additionally, the non-sinked variant of equation (3) for the Laguerre case features in the financial Cox–Ingersoll–Ross model [10, 12], amongst other applications.

With regards to the Bessel polynomials, in [15] the ladder operators of the associated Bessel functions were explored. The non-sinked variant of equation (3) for the Bessel case was derived as the Fokker–Planck equation of an ergodic diffusion with reciprocal gamma invariant distribution in [27], and features in many financial models (see section 6.5 of [29]).

Following chapter 4 of [24], the Fisher–Snedecor polynomials are a variant of the Jacobi polynomials under a simple linear transformation such that the corresponding weight function’s support is extended to the positive real line. In [2] the non-sinked variant of equation (3) for the Fisher–Snedecor case was derived as the Fokker–Planck equation for an ergodic diffusion with Fisher–Snedecor invariant distribution. Refer to [52] for the corresponding Jacobi expression of equation (3) which has finite support in $x$ and an entirely discrete spectrum.

The Romanovski polynomials have received a fair amount of attention lately due to their application in supersymmetric quantum mechanics [38], quantum chromodynamics [9, 39], and connections with Yang–Mills integrals [49]. The non-sinked variant of equation (3) for the Romanovski case (without a closed form expression for the continuous spectrum normalization) was first given in [28] as the Fokker–Planck equation for an ergodic diffusion with the symmetric scaled Student invariant distribution.

Multidimensional generalizations of the classical polynomials of course exist (see [25], amongst other works) and are a current active field of study. Of particular relevance to this work we see in chapters 2 and 3 of [17] that the probability density functions of the
The eigenvalues of the chiral, Laguerre, Jacobi and Cauchy ensembles of random matrices give the multidimensional generalizations of the Hermite, Laguerre, Jacobi and Romanovski weights respectively. Additionally, chapter 11 of the aforementioned work considers potentials which correspond to various classes of quantum Calogero–Sutherland models. In particular we see that propositions 11.3.1 and 11.3.2 give multidimensional generalizations of the corresponding Schrödinger Hermite, Laguerre and Jacobi potentials (amongst other more general cases), with the (restricted) Green functions of these three cases constructed in chapter 11.6.

3. One-dimensional S–L operators

3.1. Hilbert Space and finite orthogonality

Due to $H$ in equation (8) being a non-positive, self-adjoint S–L operator, the full set of solutions to equation (8)—present in equation (3)—necessarily form a (weighted) square-integrable Hilbert space $L^2((e_2, e_1), W(x))$ with respect to the weighted inner product [26, 33]

$$\langle \theta_j(x) | \theta_k(x) \rangle \equiv \int_{e_1}^{e_2} dx W(x) \theta_j(x) \theta_k(x) < \infty. \tag{10}$$

The emergence of the continuous spectrum in equation (3) for certain classes of eigenfunctions is due to the discrete spectrum eigenfunctions possessing so-called finite orthogonality (see chapters 3 and 4 of [24]): only a finite subset of the Bessel, Fisher–Snedecor and Romanovski polynomials obey equation (10). Thus the continuous spectrum eigenfunctions are required to construct the Hilbert space. Following chapter 22 of [21] and chapters 7 and 8 of [40], if the spectrum of $H$ is mixed, the corresponding Hilbert space is separable into the following orthogonal subspaces

$$L^2_{pp} ((e_2, e_1), W(x)) \oplus L^2_{ac} ((e_2, e_1), W(x)), \tag{11}$$

where $L^2_{pp}$ denotes the subspace of the Hilbert space containing pure point (discrete) spectrum, and $L^2_{ac}$ denotes the subspace of the Hilbert space containing absolutely continuous spectrum.

3.2. Associated orthogonal functions

Recently in [14–16] the associated variants of the Laguerre, Bessel and Romanovski polynomials, respectively, were considered (the associated Fisher–Snedecor functions are a simple variation on the Romanovski case). The new results in this paper involve applying the aforementioned results and constructing the associated sinked densities. We list the canonical

| Case       | $s(x)$ | $q(x)$ | $r(x)$ |
|------------|--------|--------|--------|
| Laguerre (L) | $x$    | $\sigma + 1 - x$ | $\frac{x^2}{\sigma + 1}$ |
| Bessel (B)  | $x^2$  | $(\sigma + 2)x + 1$ | $\frac{x}{\sigma}$ |
| Fisher–Snedecor (F–S) | $x^2 + x$ | $2(\sigma_1 + 1)x + \sigma_1 + \sigma_2 + 1$ | $\frac{(x + \sigma_1 + \sigma_2 + 1)(1 + 2x)}{4(x + 1)}$ |
| Romanovski (R) | $x^2 + 1$ | $2(\sigma_1 + 1)x + \sigma_2$ | $\frac{(x + \sigma_1 + \sigma_2)(1 + 2x)}{x^2 + 1}$ |

Table 1. Canonical forms of $s(x)$, $q(x)$ and $r(x)$ considered in this work.
forms of the diffusion, drift and sink coefficients of the four relevant cases in table 1, and give the weight functions and the corresponding support of $x$ for each case in table 2.

We note that the inclusion of the sink expression adds a new parameter, $\gamma$, to each density equation. Past studies [13, 50] have assured the negativity of the $S$–$L$ operator by assuming $s(x) > 0$ and $r(x) \geq 0$ for $x \in (e_1, e_2)$. In this work, due to the particular forms of $r(x)$ not following this restriction, we rely on the equivalent requirement: decomposing the eigenfunction

$$\theta_j(x) = s^x(x)\varphi_j(x), \quad x \in \mathbb{R},$$

the corresponding $S$–$L$ operator for $\varphi_j(x)$ becomes

$$\tilde{H}\varphi_j(x) \equiv \left\{ s(x) \frac{d^2}{dx^2} + \tilde{q}(x) \frac{d}{dx} - \tilde{r} \right\}\varphi_j(x) = -i\omega\varphi_j(x), \quad (12)$$

where

$$\tilde{q}(x) = 2xs'(x) + q(x),$$

$$\tilde{r} = r(x) - x\left\{ s''(x) + \frac{(x - 1)(s'(x))^2 + q(x)s'(x)}{s(x)} \right\}. \quad (13)$$

We require that $\tilde{q}(x)$ is a also a linear function in $x$ and $\tilde{r}$ is a positive constant. This guarantees that the original $S$–$L$ operator $\mathcal{H}$ in equation (7) is negative. In table 3 we give $\tilde{x}$, and the ensuing expressions of $\tilde{q}(x)$ and $\tilde{r}$ for each case. We reiterate that in order for $\tilde{H}$ to be negative, each $\tilde{r}$ given in table 3 needs to be positive.

The solutions to equation (12) in the form of hypergeometric functions are standard in the mathematical literature. For many technical aspects of the details in the proceeding sections of this work we refer to [24] for discrete spectrum eigenfunctions and [1, 37] for the corresponding continuous spectrum eigenfunctions. We shall consider the eigen-spectra for each case explicitly in section 4.
4. Spectral categories and solutions

The eigenvalue spectrum \( \lambda \) of the S–L operator given in equation (7) is determined by the behaviour of the operator at the boundaries. Two types of behaviour of \( \phi \) at the boundaries are relevant—designated non-oscillatory and oscillatory.

For each of the four instances, the range of \( x \) is given either by \( \pm \infty \) (Laguerre, Bessel and Fisher–Snedecor) or \( \pm 0 \) (Romanovski). Hence the four cases have three possible boundaries: \( \pm 0, \pm \infty \). The three S–L operators \( \mathcal{H} \) which have a boundary at 0 are classed as non-oscillatory at that boundary and require no special treatment. In the Feller boundary classification scheme (see chapter 1 of [4] for instance), the remaining boundaries at \( \pm \infty \) are classed as natural boundaries\(^1\) and require closer examination.

4.1. The Liouville transformation

Transforming the variable \( x \) and the eigenfunction \( \psi_\lambda(x) \) via the following

\[
    z(x) = \int_{x'}^x \frac{dz}{\sqrt{s(z)}} \quad \phi_\lambda(z) = \psi_\lambda(x) \sqrt{W(x) s(x)} \sqrt{s(x)} ,
\]

for constant \( x' \), the corresponding S–L equation for \( \phi_\lambda(z) \) becomes the Schrödinger equation

\[
    \begin{cases}
        \frac{d^2}{dz^2} - \mathcal{V}(z) \phi_\lambda(z) = -\lambda \phi_\lambda(z),
    \end{cases}
\]

with the potential

\[
    \mathcal{V}(x) = \frac{d}{dx} \left( \frac{1}{2} \frac{\sqrt{s(x)}}{\sqrt{s(x)}} \right)^2 - \frac{\sqrt{s(x)}}{2} \frac{d^2}{dx^2} \sqrt{s(x)} + \frac{q^2(x)}{4 s(x)} + \frac{d}{dx} \frac{q(x)}{2}
\]

\[
    = \frac{q(x)}{2} \frac{d}{dx} \sqrt{s(x)} + r(x),
\]

which includes the sink term \( r(x) \). In table 4 we list the relevant expressions regarding the Liouville transformation for our four cases.

\(^1\) Not to be confused with the term natural boundary regarding the analytic continuation of functions (see chapter 14.3 of [47]), amongst other uses of the term.
Given the Liouville transformations listed in table 4, the classification of the spectrum of $H$ can now be given.

4.2. Spectral classification

Following theorems 1–3 of [29], the spectral properties depend on the behaviours of the diffusion $s(x)$ and the potential $V(x)$ through the following specifications:

Classification at natural boundaries

- If $\lim_{x \to \pm \infty} s(x) \neq \pm \infty$, then $H$ is classed as non-oscillatory at that boundary.
- If $\lim_{x \to \pm \infty} s(x) = \pm \infty$, and $\lim_{z \to \pm \infty} V(z) = \infty$, then $H$ is classed as non-oscillatory at that boundary.
- If $\lim_{z \to \pm \infty} V(z) = A_+ < \infty$, and/or $\lim_{z \to \pm \infty} V(z) = A_- < \infty$, and $\lim_{x \to \pm \infty} s^2(x)$ ($V(x) - A_+ > -\frac{1}{4}$), then $H$ at the corresponding boundary ($\pm \infty$) is classed as non-oscillatory for $\lambda \in [0, A_+]$ and oscillatory for $\lambda > A_+$.

Hence we have three possible spectral categories, which are detailed below.

4.3. Spectral category I

If $H$ at both boundaries exhibits no oscillatory behaviour, then the spectrum is purely discrete and equation (3) is given by

$$T(x, t|y) = W(x) \sum_{n=0}^{\infty} e^{-\lambda_nt} \rho_n \hat{\theta}_n(x) \hat{\theta}_n(y),$$  \hspace{1cm} (17)$$

where the normalization coefficients are given by equation (4). We notice from table 4 that the associated Laguerre functions ($\hat{\theta}_n(x) \equiv x^\gamma L_n^{(2\gamma+\sigma)}(x)$) fall under this category. The hypergeometric form of the Laguerre polynomials are given by

$$L_n^{(2\gamma+\sigma)}(x) = \frac{(2\gamma+\sigma)_n}{n!} \text{I}_n \left( \frac{-n}{2\gamma+\sigma+1}, \lambda \right).$$

Following chapter 9.12 of [24] the eigenvalues and normalization constants are given by

$$\lambda_n = n + \gamma, \quad \rho_n = \frac{n!}{\Gamma(n + \sigma + 2\gamma + 1)}, \quad \gamma \geq 0, \quad \sigma + 2\gamma > -1.$$ 

Hence the expression of the density for the Laguerre case is

$$T(x, t|y) = x^{\gamma+\sigma} y^{\gamma} e^{-x-y} \sum_{n=0}^{\infty} \frac{\Gamma(n + \sigma + 2\gamma)(x)L_n^{(\sigma+2\gamma)}(y)}{\Gamma(n + \sigma + 2\gamma + 1)}$$

$$\cdot \exp \left( \frac{\sigma t - x + ye^{-t}}{1 - e^{-t}} \right) L_{\sigma+2\gamma} \left( \frac{2\sqrt{xye^{-t}}}{1 - e^{-t}} \right).$$  \hspace{1cm} (18)$$

where $L_\nu(x)$ is the modified Bessel function of the first kind of order $\nu$ (chapter 10 of [37]). The product form of the density is obtained through the application of the Hille–Hardy formula (chapter 18 of [37]).
4.4. Spectral category II

If $H$ exhibits oscillatory behaviour at only one of the boundaries for $\lambda > \Lambda$, then equation (3) is given by

$$T(x, t|y) = W(x) \left\{ \sum_{n=0}^{N} e^{-i t \rho_n} \vartheta_n(x) \vartheta_n(y) + \int_{0}^{\infty} d\mu e^{-(\lambda + \mu)^{\frac{1}{2}}} \vartheta(\mu, x) \vartheta(\mu, y) \right\}, \quad \lambda_N < \Lambda, \quad (19)$$

where (following lemmas 41 and 42 of [13]) the eigenfunction with the continuous eigenvalue, $\vartheta(\mu, x)$, is the non-trivial solution to equation (8) which is square-integrable with $W(x)$ and valid in the neighbourhood of the boundary (in this case $\infty$) for which $H$ exhibits the oscillatory behaviour. For this particular category we note that [21, 40]

$$\vartheta_n(x) \in L^2_{ep}(e_2, e_1, W(x)), \quad n \in \{0, 1, ..., N\},$$

$$\vartheta(\mu, x) \in L^2_{en}(e_2, e_1, W(x)), \quad \mu > 0.$$  

Since each respective subspace of the Hilbert space is orthogonal to the other, we are assured that any weighted inner product of a discrete and continuous eigenfunction is zero

$$\langle \vartheta_n(x) | \vartheta(\mu, x) \rangle \equiv \int_{e_1}^{e_2} dW(x) \vartheta_n(x) \vartheta(\mu, x) = 0.$$  

The discrete normalization constants in equation (19) are given by equation (4) and the continuous normalization $\rho(\mu)$ can be obtained through the application of the MacRobert inverse integral transform in equation (5).

From the forms of $\varphi(z)$ in table 4 and the corresponding support of $x$, we see that the Bessel and Fisher–Snedecor cases fall under this particular mixed spectral category. Additionally we see from table 4 that the highest discrete eigenvalue for each case is constrained by

$$\lambda_N < \begin{cases} 
\left(\frac{\sigma + 1}{2}\right)^2 & \text{Bessel,} \\
\left(\frac{\sigma_1 + 1}{2}\right)^2 & \text{Fisher – Snedecor.} 
\end{cases}$$

Addressing the discrete spectrum eigenfunctions, the associated Bessel function ($\vartheta_n(x) \equiv x^\gamma B^{(2\gamma+\sigma)}_n(x)$) and associated Fisher–Snedecor functions ($\vartheta_n(x) \equiv (x^2 + x)^{\frac{1}{2}} F^{(2\gamma+\sigma)}_n(x)$) have the following hypergeometric forms

$$B^{(2\gamma+\sigma)}_n(x) = {}_2F_0\left(\begin{array}{c} -n, \ 2\gamma + \sigma + n + 1 \end{array} \bigg| -x \right),$$

$$F^{(2\gamma+\sigma)}_n(x) = \left(\gamma + \sigma_2 + 1\right) {}_2F_1\left(\begin{array}{c} -n, \ 2\gamma + n + 1 \\ \gamma + \sigma_2 + 1 \end{array} \bigg| -x \right).$$

Following [2] and chapter 9.12 of [24], the eigenvalues and normalization constants for each case is given in table 5.

Addressing the continuous spectrum eigenfunctions, the Bessel ($\vartheta(\mu, x) \equiv x^\gamma \psi_b(\mu, x)$) (see [27] and chapter 13 of [1, 37]) and Fisher–Snedecor ($\vartheta(\mu, x) \equiv (x^2 + x)^{\frac{1}{2}} \psi_f(\mu, x)$) (see [2] and chapter 15 of [1, 37]) cases have the following hypergeometric forms
\[\psi_{B}(\mu, x) = \left\{ \begin{array}{ll}
2F_0 \left( \gamma + \frac{\sigma + 1}{2} - i\mu, \ y + \frac{\sigma + 1}{2} + i\mu \bigg| -x \right) & |x| \leq 0, \\
\frac{\Gamma(-2\mu)}{\Gamma(\gamma + \frac{\sigma + 1}{2})} \psi_{B}(\mu, x) + \frac{\Gamma(2\mu)}{\Gamma(y + \frac{\sigma + 1}{2} + i\mu)} \psi_{B}(-\mu, x) & |x| > 0,
\end{array} \right. \]
\[\psi_{F}(\mu, x) = \left\{ \begin{array}{ll}
2F_1 \left( \gamma + \frac{1}{2} + i\mu, \ y + \frac{1}{2} - i\mu \bigg| -1 \right) & |x| \leq 1, \\
\Pi(\mu) \psi_{F}(\mu, x) + \Pi(-\mu) \psi_{F}(-\mu, x) & |x| > 1,
\end{array} \right. \]

where
\[\psi_{B}(\mu, x) = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{x}} e^{\frac{-\sigma + 1}{2}} I_{1} \left( \gamma + \frac{\sigma + 1}{2} + i\mu \bigg| \frac{1}{x} \right), \\
\psi_{F}(\mu, x) = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{x}} e^{\frac{+1}{2} + i\mu} \frac{1}{2} \left( \gamma + \frac{1}{2} + i\mu, \ y + \frac{1}{2} + \sigma + i\mu \bigg| -1 \right), \\
\Pi(\mu) = \frac{\Gamma(y + 1 + \sigma) \Gamma(-2\mu)}{\Gamma(\gamma + 1 + \sigma + i\mu) \Gamma(\gamma + 1 + \sigma - i\mu)},
\end{array} \right. \]

and the corresponding eigenvalues and normalizations (\cite{2, 27, 51}) are given in table 6. Hence, the expression for the density of the Bessel case is
\[
T(x, t | y) = x^{\gamma + \mu} e^{t} \left\{ \sum_{n=0}^{-\sigma + 1} \frac{(-2\gamma - \sigma - 2n - 1)}{n! \Gamma(-2\gamma - \sigma - 2n)} + \int_0^\infty dy \frac{\Gamma(y + \frac{1}{2} + i\mu) e^{-\frac{1}{2} \left( \frac{y+1}{2} + i\mu \right)} \psi_{B}(\mu, x) \psi_{B}(\mu, y)}{2\pi \Gamma(2\mu)} \psi_{B}(\mu, -\mu) \psi_{B}(\mu, y) \right\}. \tag{20}
\]

| Case | \( \lambda_n \) | \( \beta_n \) | Restrictions |
|------|-----------------|-----------------|-------------|
| B    | \(-\gamma + n)(\gamma + \sigma + n + 1)\) | \(-\gamma + n)\) | \(\gamma(\gamma + \sigma + 1) \leq 0\) |
|      | \(\sigma + 2\gamma < -1\) | \(n < -\gamma - \frac{\sigma + 1}{2}\) | |
| F–S  | \((\sigma - \gamma - n)(\gamma + \sigma + n + 1)\) | \((\sigma - \gamma + n)\) | \(\gamma(\gamma + \sigma + 1) \geq 0\) |
|      | \(2\gamma < -1\) | \(n < -\gamma - \frac{1}{2}\) | |
and the corresponding expression for the Fisher–Snedecor case is

\[
\mathcal{T}(x, t | y) = x^{\frac{\mu}{2} + \sigma_2} (x + 1)^{\frac{\mu}{2} - \sigma_1} \left( y^2 + y \right)^{\frac{\gamma - 1}{2}} \\
\times \left\{ \sum_{n=0}^{\gamma - 1} \frac{(-2n - 1 - 2\mu) \Gamma(-n - \gamma + \sigma_2) e^{(\gamma + n - \sigma_1)(\gamma + \sigma_1 + n + 1)t}}{n! \Gamma(-n - 2\mu) \Gamma(1 + n + \gamma + \sigma_2)} \right\} \\
\times F_n^{(\sigma_2)}(x) F_n^{(\sigma_1)}(y) + \int_0^\infty d\mu \frac{e^{- x^{\frac{\mu}{2} + \sigma_2}}}{2\pi \Gamma(\mu)} \psi_F(\mu, x) \psi_F(\mu, y) \right\}, \quad (21)
\]

### 4.5. Spectral category III

If \( \mathcal{H} \) exhibits oscillatory behaviour at both boundaries for \( \lambda > \Lambda_\pm \), and \( \Lambda_+ = \Lambda_- \), then equation (3) is given by

\[
\mathcal{T}(x, t | y) = W(x) \left\{ \sum_{n=0}^N e^{-\lambda_n} \rho_n(x) \theta_n(y) \\
+ \sum_{i,j=1}^2 \int_0^\infty d\mu e^{-(\Lambda + \mu)t} \rho_{ij}(\mu) \theta_i(\mu, x) \theta_j(\mu, y) \right\}, \quad \lambda_N < A, \quad (22)
\]

where (following section 5.3 of [29]) the eigenfunctions with the continuous eigenvalues, \( \theta_i(\mu, x), i = \{1, 2\} \), are the linearly independent solutions to equation (8) which are square-integrable with \( W(x) \), and are valid in the neighbourhood of the natural boundaries (in this case, \( \pm \infty \)) for which \( \mathcal{H} \) exhibits oscillatory behaviour. Similar to spectral category II, we note that [21, 40]

\[
\theta_n(x) \in L^2_{pp}\left( (e_2, e_1), W(x) \right), \quad n \in \{0, 1, ..., N\}, \\
\theta_i(\mu, x) \in L^2_{ac}\left( (e_2, e_1), W(x) \right), \quad i \in \{1, 2\}, \quad \mu > 0.
\]

The discrete normalization constants in equation (22) are given by equation (4), and in the appendix we explicitly derive the continuous normalizations \( \rho_{ij}(\mu) \), using the aforementioned MacRobert style proof.

From the forms of \( \Psi(z) \) in table 4 and the corresponding support of \( x \), we see that the Romanovski case fall under this particular mixed spectral category, and its highest discrete eigenvalue satisfies
Concerning the discrete spectrum eigenfunctions, the associated Romanovski functions 
\( R_n^{(\nu, \sigma)}(x) \equiv (x^2 + 1)^{-\nu} R_n^{(\nu, \sigma)}(x) \) have the following hypergeometric form

\[
R_n^{(\nu, \sigma)}(x) = (-2i)^n \frac{\left( \gamma + i \frac{\sigma_2}{2} + 1 \right)}{(n + 2\gamma + 1)_n} \binom{2\nu}{n} \left( \nu - \gamma + n + \frac{1}{2} \right) \exp \left( -2i \frac{n}{2} \right).
\]

Following chapter 9.9 of [24], the eigenvalues and normalization constants for this case are given by

\[
\lambda_n = (\sigma_1 - \sigma - n)(\gamma + \sigma_1 + n + 1),
\]

\[
\beta_n = \frac{\Gamma(-n - 2\gamma) \Gamma\left( -\gamma - n + i \frac{\sigma_2}{2} \right)}{2^{2n+\nu+1} \nu! \nu! \Gamma(-2n - 2\gamma) - \Gamma(-2n - 2\gamma)}.
\]

under the restrictions

\[(\sigma_1 - \gamma)(\gamma + \sigma_1 + 1) > 0, \quad 2\gamma < -1, \quad n < -\gamma - \frac{1}{2}.
\]

Rescaling the continuous spectrum eigenfunctions

\[\theta_1(\mu, x) \equiv \left( x^2 + 1 \right)^{\frac{\nu}{2}} \chi_1(\mu, x), \quad \theta_2(\mu, x) \equiv \left( x^2 + 1 \right)^{\frac{\nu}{2}} \chi_2(\mu, x),\]

(\text{where } \chi^* \text{ is the complex conjugate of } \chi), \text{ the eigenvalues are parameterized by}

\[\lambda(\mu) = \left( \sigma_1 + \frac{1}{2} \right)^2 + \mu^2, \quad \mu > 0\]

and hypergeometric forms of \( \chi \) are (see [28] and chapter 15 of [1, 37])

\[\chi_1(\mu, x) = \left\{ \begin{array}{ll}
\binom{2\nu}{n} \left( \gamma + \frac{1}{2} + i\mu, \gamma + \frac{1}{2} - i\mu \right), & |x| \leq \sqrt{3}, \\
\tilde{\Gamma}(\mu)\tilde{\chi}(\mu, x) + \tilde{\Gamma}(-\mu)\tilde{\chi}(-\mu, x), & |x| > \sqrt{3},
\end{array} \right. \]

where

\[
\tilde{\chi}(\mu, x) = \left( \frac{i\mu - 1}{2} \right)^{\gamma - \frac{i\mu}{2}} \binom{2\nu}{n} \left( \gamma + \frac{1}{2} + i\mu, \gamma + \frac{1}{2} - i\mu \right),
\]

\[
\tilde{\Gamma}(\mu) = \frac{\Gamma\left( \gamma + 1 + i\frac{\sigma_2}{2} \right) \Gamma\left( -2i\mu \right)}{\Gamma\left( \gamma + 1 + i\mu \right) \Gamma\left( \frac{1}{2} + i\frac{\sigma_2}{2} - i\mu \right)}.
\]
The continuous orthogonality relations are given by
\[ \int_{-\infty}^{\infty} d\nu \left( x^2 + 1 \right)^{\nu} e^{\nu \arctan(x)} \chi_{\nu} (\nu, x) \int_{0}^{\infty} d\mu \rho_{\nu, \mu}(\mu) \chi_{\mu} (\mu, x) = \rho_{\nu, \mu}(\nu) A_{\nu, \mu}(\nu), \]
where \( \nu \in \mathbb{R}_+ \) and
\[ A_{\nu, \mu}(\nu) = A_{\nu, \mu}^2(\mu) = 2^{\nu + \frac{3}{2}} \pi \tilde{F}(\mu) \tilde{F}(-\mu) \cosh \left( \frac{\pi}{2} \left( \sigma_2 - i(2\gamma + 1) \right) \right), \]
\[ A_{\nu, \mu}(\mu) = A_{\nu, \mu}^2(\mu) = 2^{\nu + \frac{3}{2}} \pi \left\{ \left[ \tilde{F}(\mu) \right]^2 \cosh \left( \frac{\pi}{2} (\sigma_2 + 2\mu) \right) \right. \]
\[ + \left. \left[ \tilde{F}(-\mu) \right]^2 \cosh \left( \frac{\pi}{2} (\sigma_2 - 2\mu) \right) \right\}. \] (26)

Thus the continuous normalizations are given by
\[ \rho_{\nu, \mu}(\nu) = \rho_{\nu, \mu}^2(\mu) = \frac{A_{\nu, \mu}^2(\nu)}{|A_{\nu, \mu}(\nu)|^2 - A_{\nu, \mu}^2(\mu)}, \]
\[ \rho_{\nu, \mu}(\mu) = \rho_{\nu, \mu}^2(\mu) = \frac{A_{\nu, \mu}(\mu)}{|A_{\nu, \mu}(\mu)|^2 - |A_{\nu, \mu}(\mu)|^2}. \] (27)

We provide the detailed derivation of equation (26) in the appendix using the aforementioned MacRobert method.

Hence the complete density function for the Romanovski case is
\[ T(x, t|y) = \left( x^2 + 1 \right)^{\gamma y + \frac{\nu_1}{2}} e^{\nu_1 \arctan(x)} (y^2 + 1)^{\gamma y + \frac{\nu_2}{2}} \left\{ \sum_{n=0}^{\gamma y + \nu_1 + \gamma y + \nu_2 + 1} \frac{\Gamma(-n - 2\gamma)}{2^{2n+\gamma+1} n!} \right\} \times \frac{\Gamma\left(-n - \gamma + \frac{\nu_1}{2}\right)}{\Gamma(-2n - 2\gamma)} \frac{\Gamma\left(-n - \gamma + \frac{\nu_2}{2}\right)}{\Gamma(-2n - 2\gamma - 1)} \tilde{R}_n^{(\nu_2, \gamma_2)}(x) \tilde{R}_n^{(\nu_2, \gamma_2)}(y) \]
\[\int \sum \mu \rho \mu \chi \mu \chi \mu \mu \rho + \sigma \mu = \infty - + + \{()\}(()) \{(),\}(1,2,2) \quad (28)\]

We give an example of equation (28) in figure 1. Notice that as time increases the total area of the density (which begins at unity) decreases. For \( t > 0.5 \) we notice that the density is barely distinguishable visually. We compare this to the stationary density of the non-sunked case—the left most density—where area is conserved for all \( t \).

5. Application—B–R Langevin equation

We now present an application of this work—time dependent distributions corresponding to various instances of the B–R Langevin equation. The B–R equation is a deterministic system given by

\[\dot{x}(t) = ax(t) - bx^\zeta(t), \quad \zeta > 1, \quad x(0) = y,\]

where \([a, b] \in \mathbb{R} \cup \{0\}\). We note that when \([a, b] > 0\), the \(ax\) and \(bx^\zeta\) terms act as growth and decay terms respectively; the greater the value of \(\zeta\), the more pronounced the decay. The choice \(\zeta = 2\) gives the famous logistic equation.

5.1. Stochastic perturbations and the Fokker–Planck equation

To proceed we consider the two uncorrelated noise terms \(\eta_1\) and \(\eta_2\) with variance \(\Omega\),

\[\{\eta_i(t)\} = 0, \quad \{\eta_i(t_1)\eta_j(t_2)\} = \delta_{ij}\Omega \delta(t_1 - t_2),\]

where \(\langle...\rangle\) means an ensemble average over the noise. We perturb the growth and decay coefficients by \(\eta_1\) and \(\eta_2\) respectively to obtain the following (Itô) stochastic Langevin equation

\[\dot{x}(t) = \left\{a + \alpha \eta_1(t)\right\}x(t) - \left\{b + \beta \eta_2(t)\right\}x^\zeta(t), \quad (29)\]

where \([\alpha, \beta] \in \mathbb{R}\). Equation (29) can be solved exactly through the transformation

\[x^{1-\zeta} = \xi, \quad y^{1-\zeta} = \xi^*, \quad (30)\]

leading to the linear Langevin equation

\[\dot{\xi}(t) = (\zeta - 1)\left\{b + \beta \eta_2(t) - \left\{a + \alpha \eta_1(t)\right\}\xi(t)\right\},\]

and the formal solution

\[\xi(t) = \left\{\frac{(\zeta - 1) \int_0^t \mathrm{d} \tau_1 \left\{b + \beta \eta_2(\tau_1)\right\} e^{(\zeta - 1) \int_0^\tau \mathrm{d} \tau (a + \alpha \eta_1(\tau))} + \xi^*}{e^{(\zeta - 1) \int_0^t \mathrm{d} \tau (a + \alpha \eta_1(\tau))}}\right\}.\]

We refer to the above solution for \(\xi(t)\) as formal as it contains integrals of specific instances of the noise terms (meaning that each solution will be different for different noise instances). In order to make general statements about the above system, we shall construct its probability
density function. Following chapter 4.5 of [46], the stochastic process in equation (29) obeys the following Fokker–Planck equation
\[
\frac{\partial}{\partial t} T(x, t|y) = \left\{ \frac{\partial^2}{\partial x^2} s(x) - \frac{\partial}{\partial x} q(x) \right\} T(x, t|y), \quad T(x, 0|y) = \delta(x - y),
\]
where \( s(x) = \frac{\Omega}{2} (\alpha^2 x^2 + \beta^2 x^2), \quad q(x) = ax - bx^2. \) (31)

As mentioned in earlier sections, since the above equation contains no sink term \( T(x, t|y) \) is conserved, and its most natural interpretation is density of probability, where \( x \) and \( y \) are the population of a species.

5.2. Restricting the Heun equation

Our goal of this section is to analytically solve for various cases of equation (31) using our polynomial solutions for density functions given in section 4. Applying the standard decomposition in equation (6), and the nonlinear transformation in equation (30), the resulting expression for \( g(\xi, t|\xi') \) is
\[
\frac{\partial}{\partial t} g(\xi, t|\xi') = (\xi' - 1) \left\{ \frac{\Omega(\xi - 1)}{2} \left( \alpha^2 \xi^2 + \beta^2 \right) \frac{\partial^2}{\partial \xi^2} + \left( \frac{\Omega\xi^2 - 2a}{2} \right) \frac{\partial}{\partial \xi} \right\} g(\xi, t|\xi'). \tag{32}
\]

Since the B–R equation is used extensively in population modelling, where the variable \( x \) represents the number of living members of a species, only eigenfunctions in the range \( \mathbb{R}_+ \) will be considered, hence leaving out the Romanovski example. This leaves three relevant cases, Laguerre, Bessel and Fisher–Snedecor.

The \( S–L \) operator on the right hand side of equation (32) leads to the Heun differential equation (see chapter 31 of [37]). Due to the Heun equation possessing four distinct singular points, there is no equivalent hypergeometric closed form expression for the Heun functions [22]. Nevertheless, we find the following mapping between the Heun system and hypergeometric solutions:

- \( \alpha = b = 0 \) leads to the Laguerre case
- \( \beta = 0 \) leads to the Bessel case
- \( b \neq 0 \) leads to the Fisher–Snedecor case.

We shall only detail the Laguerre and Fisher–Snedecor cases in this work as the Bessel case was first solved in [45] and along with [52] is one of the earlier results involving analytical expressions of densities with mixed spectra. The case \( \alpha = 0 \) in equation (32) leads to the biconfluent Heun equation, whose solution suffers the same non-closed properties as the Heun equation (see chapter 31 of [37]). Additionally, the case \( \alpha = a = 0 \) leads to the Bessel process with constant drift [30], which is a peculiar hypergeometric case (beyond the scope of this work) where the spectrum is mixed but the discrete part contains an infinite number of eigenvalues.
5.3. Laguerre case

Setting \( a = b = 0 \), the weight function \( W(\xi) \) for this case is

\[
W(\xi) = \xi^\alpha e^{\frac{\xi^2}{2}}, \quad \omega_L = \frac{\Omega(\zeta - 1)\beta^2}{a}.
\]

Applying the following change in variables

\[
\xi = \sqrt{\omega_L} \xi, \quad \xi' = \sqrt{\omega_L} \xi',
\]

equation (32) becomes

\[
\frac{\partial}{\partial \tau} g(\xi, \xi'; \tau) = \left( \frac{\partial^2}{\partial \xi^2} + (1 + \sigma - \zeta) \frac{\partial}{\partial \xi} \right) g(\xi, \xi'; \tau),
\]

\[
\tau = \frac{\alpha t}{\sigma}, \quad \sigma = \frac{1}{2(\zeta - 1)},
\]

where equation (33) is the standard Laguerre–Fokker–Planck equation. Hence, due to the initial condition, the time dependent solution for the density in this section is

\[
T(x, t | y) = \exp \left\{ - \frac{\left(e^{-(\xi + \xi')} + e^{-\xi - \xi'}\right)}{a \xi (1 - e^{-2\xi})} \right\} \phi_{\omega_L, \xi \xi'}^{2, 2} \left(1 - e^{-2\xi'}\right).
\]

To the best of our knowledge, equation (34) is a new result of a specific example of a B–R Fokker–Planck equation.

Making the connection with the Langevin equation this density is generated from,

\[
\dot{x}(t) = ax(t) - \beta x(t) \eta_2(t),
\]

since for \( a > 0 \) the deterministic system is divergent, but the density is normalizable, this particular case is an example of multiplicative noise stabilizing the system [32].

5.4. Fisher–Snedecor case

Setting \( b = 0 \) the weight function \( W(\xi) \) for this case is

\[
W(\xi) = \xi^\alpha e^{\xi^2 + \alpha_F} \xi^2 e^{\xi^2},
\]

\[
\omega_F = \frac{\beta^2}{a^2}, \quad \sigma_1 = \frac{\Omega(\zeta - 1)\alpha_F}{2\Omega(\zeta - 1)a^2} - 1, \quad \sigma_2 = \frac{\Omega(\zeta - 1)\alpha_F}{2\Omega(\zeta - 1)a^2}.
\]

Applying the following change in variables

\[
\xi = \sqrt{\omega_F} \xi, \quad \xi' = \sqrt{\omega_F} \xi',
\]

equation (32) becomes

\[
\frac{\partial}{\partial \tau} g(\xi, \xi'; \tau) = \left( \frac{\partial^2}{\partial \xi^2} + 2(\sigma_1 + 1)\zeta + \sigma_1 + \sigma_2 + 1 \right) \frac{\partial}{\partial \xi} g(\xi, \xi'; \tau),
\]

\[
\tau = \frac{2(\zeta - 1)\alpha_F}{\sigma_2 - \sigma_1 - 1},
\]

\[ (35) \]
where equation (35) is the standard Fisher–Snedecor–Fokker–Planck equation. Hence the time dependent solution for the density is

$$\mathcal{T}(x, t) = \frac{2(\zeta - 1)}{\omega_F^{2n+1}} \sum_{n=0}^{\infty} \Gamma(-n, -2n - 1 - 2\sigma_1)
\times \left\{ \left( \frac{2(1-\zeta)}{\omega_F} \right)^{2n(\sigma_1+\sigma_2)} \psi_F\left( \mu, \frac{x^{2(1-\zeta)}}{\omega_F} \right) \right\}.
$$

As with the Laguerre case, to the best of our knowledge, equation (36) is a new result of a specific instance of a B–R Fokker–Planck equation. In figure 2, we give a specific example of equation (36) at various times. It is elementary to show that the weight function, which is proportional to the steady state density, peaks at the value

$$\Omega = \Omega_0 = 17 = 1.54.
$$

6. Conclusions and future work

In this work we have given closed form expressions of sinked densities associated with (at most) quadratic diffusion and linear drift. The eigenfunctions relating to the discrete part of the spectrum are associated variants of classical orthogonal polynomials. We have given a MacRobert style proof to obtain a new closed form expression for the continuous spectrum.
normalization associated with the Romanovski density. This technique is sufficiently generalizable, given one knows enough about the analytic continuation properties of the hypergeometric function under consideration. We then applied these results to obtain the time dependent Fokker–Planck solutions associated with various cases of the B–R Langevin equation.

Given the pervasive nature of Langevin equations (and the densities and Green’s functions associated with them) in the physical sciences, we anticipate that these results are a stepping stone to a richer understanding of a variety of processes, both conserved and non-conserved. Specifically, we hope that processes involving mixed spectra eigenfunctions become increasingly commonplace, as more analytic examples of solution appear which increase our mathematical understanding and our ability to apply such results in novel ways. Paraphrasing the relevant introduction of [6]: in a world of ever increasing computing power, we must never overlook the benefits provided from analytic solutions in terms of special functions. They provide insight for understanding non-trivial relationships among physical variables with unsurpassed economy of effort, and are an invaluable tool for the validation of more complicated models which require computational treatment.

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Appendix. MacRobert’s proof of equation (26)

We begin by conveniently labelling the double integral of equation (26)

$$I_{ij}(\nu) = \int_{-\infty}^{\infty} dx \left( x^2 + 1 \right)^{\nu/2} e^{\nu \arctan(x)} \chi_i(x) \int_{a}^{b} d\mu \rho_{ij}(\mu) \chi_j(\mu),$$

{\{a, b\} \in \mathbb{R}_+ \quad a < b, \quad i, j \in \{1, 2\}, \quad \nu \in \mathbb{R}_+ .}

Focusing on the case \(i = j = 1\), we apply equation (24) to split up \(\chi_i(\mu, x)\) for the region \(x > \sqrt{3}\) and deform the \(\mu\) integral onto the contours \(\phi(+)\) and \(\phi(-)\) as shown in figure A1 to obtain
\[ I_{1,1}(\nu) = \int_{-\infty}^{\infty} \mathrm{d}x \left( x^2 + 1 \right) e^{\sigma_1 \arctan(x)} \chi_1(\nu, x) \int_{\phi^{(+)}} \mathrm{d}\mu \hat{\Gamma}(\mu) \rho_{1,1}^{(\nu)}(\mu) \tilde{\chi}(\mu, x) \]
\[ \quad + \int_{-\infty}^{\infty} \mathrm{d}x \left( x^2 + 1 \right) e^{\sigma_2 \arctan(x)} \chi_1(\nu, x) \int_{\phi^{(-)}} \mathrm{d}\mu \hat{\Gamma}(\mu) \rho_{1,1}^{(\nu)}(-\mu) \tilde{\chi}(-\mu, x). \]  
(A.1)

Following chapter 14 of [11] and chapter 7 of [20], we may reverse the order of integration as each term in equation (A.1) falls off like \( x^{-1-\text{Im}(\mu)}(1 + O(x^{-1}) + ...) \), \( \text{Im}(\mu) \in \mathbb{R}_+ \), on the respective contours \( \phi^{(+)} \) and \( \phi^{(-)} \), as \( x \to \infty \). Hence \( I_{1,1}(\nu) \) becomes
\[ I_{1,1}(\nu) = \int_{\phi^{(+)}} \mathrm{d}\mu \hat{\Gamma}(\mu) \rho_{1,1}^{(\nu)}(\mu) \int_{-\infty}^{\infty} \mathrm{d}x \left( x^2 + 1 \right) e^{\sigma_1 \arctan(x)} \chi_1(\nu, x) \tilde{\chi}(\mu, x) \]
\[ \quad + \int_{\phi^{(-)}} \mathrm{d}\mu \hat{\Gamma}(\mu) \rho_{1,1}^{(\nu)}(\mu) \int_{-\infty}^{\infty} \mathrm{d}x \left( x^2 + 1 \right) e^{\sigma_2 \arctan(x)} \chi_1(\nu, x) \tilde{\chi}(\mu, x). \]  
(A.2)

To proceed we note that (chapter 15.5 of [1]) \( \chi(\pm \mu, x) \), and their complex conjugates, obey the same governing S–L equation as \( \chi(\mu, x) \) and \( \chi(-\mu, x) \), as they are the corresponding linearly independent solutions in the neighbourhood of the singular point \( \infty \). Thus decomposing either of the aforementioned eigenfunctions as
\[ \{ J_{\nu}(\mu, x), \tilde{J}_{\nu}(\mu, x) \} = \sqrt{\left( x^2 + 1 \right)^{\sigma_1} e^{\sigma_2 \arctan(x)}} \{ \chi_1(\mu, x), \tilde{\chi}(\mu, x) \}, \]
the resulting governing equation for the \( J \) is
\[ \left\{ \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left( \frac{\mu^2}{x^2 + 1} + \frac{x^2 - 4\nu\sigma_2 x + 4\nu^2 - \sigma_2^2 - 3}{4\left( x^2 + 1 \right)^2} \right) \right\} \times \{ J_{\nu}(\mu, x), \tilde{J}_{\nu}(\mu, x) \} = 0. \]  
(A.3)

We now recast equation (A.2) in terms of the \( J \). Through considering two copies of equation (A.3), one for \( J_{\nu}(\mu, x) \) and one for \( \tilde{J}_{\nu}(\pm \mu, x) \), we multiply the equation for \( J_{\nu}(\mu, x) \) by \( \tilde{J}_{\nu}(\pm \mu, x) \), and vice versa. Subtracting the two expressions awards us with
\[ \frac{J_{\nu}(\mu, x)\tilde{J}_{\nu}(\pm \mu, x)}{x^2 + 1} = \frac{J_{\nu}(\pm \mu, x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}J_{\nu}(\nu, x) - J_{\nu}(\nu, x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}J_{\nu}(\pm \mu, x)}{\mu^2 - \nu^2}. \]  
(A.4)

Integrating equation (A.4) over all of \( x \), and applying integration by parts we obtain
\[ \int_{-\infty}^{\infty} \mathrm{d}x \frac{J_{\nu}(\mu, x)J_{\nu}(\pm \mu, x)}{x^2 + 1} = \int_{-\infty}^{\infty} \mathrm{d}x \left( x^2 + 1 \right) e^{\sigma_2 \arctan(x)} \chi_1(\nu, x) \tilde{\chi}(\pm \mu, x) \]
\[ \quad = \left[ \frac{J_{\nu}(\pm \mu, x)\frac{\mathrm{d}}{\mathrm{d}x}J_{\nu}(\nu, x) - J_{\nu}(\nu, x)\frac{\mathrm{d}}{\mathrm{d}x}J_{\nu}(\pm \mu, x)}{\mu^2 - \nu^2} \right]_{x=-\infty}^{x=\infty}. \]

Using equation (24) the asymptotic forms of the desired limits are given by
\[ \lim_{x \to \pm \infty} \tilde{J}_{\nu}(\mu, x) \sim P_{\nu}(\mu) x^{-\nu} + O\left( x^{-\nu-1} \right), \]
\[ \lim_{x \to \pm \infty} \frac{1}{\mathrm{d}x} \tilde{J}_{\nu}(\mu, x) \sim \pm \left( \frac{1}{2} - \frac{i\mu}{2} \right) P_{\nu}(\mu) x^{-\nu} + O\left( x^{-\nu-1} \right). \]
\[
\lim_{x \to \pm \infty} J_1(\nu, x) \sim \hat{F}(\nu)F_\pm(\nu)\chi^{\pm -iu} + \hat{F}(-\nu)\chi^{\pm +iu} + O\left(x^{-\frac{1}{2}\pm\mp iu}\right),
\]
\[
\lim_{x \to \pm \infty} \frac{d}{dx} J_1(\nu, x) \sim \pm \left(\frac{1}{2} - iu\right)\hat{F}(\nu)F_\pm(\nu)\chi^{-\frac{1}{2} - iu} + \pm \left(\frac{1}{2} + iu\right)\hat{F}(-\nu)\chi^{-\frac{1}{2} + iu} + O\left(x^{-\frac{3}{2}\pm\mp iu}\right),
\]

where
\[
F_\pm(\mu) = 2^{\nu + 1 + \mp iu} e^{\pm \frac{\pi}{4} \left(\sigma_2 + 2\mu - i(2\gamma + 1)\right)}.\]

Using the above asymptotic forms, equation (A.2) becomes

\[
I_{1,1}(\nu) = \lim_{z \to \infty} \int_{\phi^{(+)}}^{\phi^{(-)}} d\mu \rho_{1,1}(\mu)K(\mu, \nu) \left(\frac{\sin(\mu + \nu)z + i\cos(\mu + \nu)z}{\mu + \nu}\right)
+ \lim_{z \to \infty} \int_{\phi^{(-)}}^{\phi^{(+)}} d\mu \rho_{1,1}(\mu)K(-\mu, -\nu) \left(\frac{\sin(\mu + \nu)z - i\cos(\mu + \nu)z}{\mu + \nu}\right)
+ \lim_{z \to \infty} \int_{\phi^{(-)}}^{\phi^{(+)}} d\mu \rho_{1,1}(\mu)K(\mu, -\nu) \left(\frac{\sin(\mu - \nu)z + i\cos(\mu - \nu)z}{\mu - \nu}\right)
+ \lim_{z \to \infty} \int_{\phi^{(+)}}^{\phi^{(-)}} d\mu \rho_{1,1}(\mu)K(-\mu, \nu) \left(\frac{\sin(\mu - \nu)z - i\cos(\mu - \nu)z}{\mu - \nu}\right),
\]

(A.5)

where \(z = \log x\) and
\[
K(\mu, \nu) = 2^{\nu + 2 + (\mu + \nu)} \hat{F}(\mu)\hat{F}(\nu) \cosh\frac{\pi}{2} (\sigma_2 - i(2\gamma + 1) + \mu + \nu).
\]

In the following we consider the Dirichlet integral expressions from chapter 1 of [31] and chapter 3 of [11]:

\[
\lim_{z \to \infty} \int_{-\alpha}^{\beta} d\xi M(\xi) \cos(\xi z) = 0, \quad \lim_{z \to \infty} \int_{-\alpha}^{\beta} d\xi M(\xi) \sin(\xi z) = 0,
\]
\[
\lim_{z \to \infty} \int_{-\alpha}^{\beta} d\xi M(\xi) \frac{\cos(\xi z)}{\xi} = 0,
\]
\[
\lim_{z \to \infty} \int_{-\alpha}^{\beta} d\xi M(\xi) \frac{\sin(\xi z)}{\xi} = \frac{\pi}{2} \left(M(0+) + M(0-)\right),
\]

where \(\alpha, \beta \in \mathbb{R}_+\) and the analytic function \(M(\xi)\) obeys Dirichlet’s conditions on the interval \((-\alpha, \beta)\). Thus we deform the contours \(\phi^{(+)}\) and \(\phi^{(-)}\) back to the real line segment \((a, b)\), and let \(a \to 0\) and \(b \to \infty\). Assuming that the function \(\rho_{1,1}(\mu)\) obeys Dirichlet’s conditions, we immediately obtain the following expression for \(I_{1,1}(\nu)\),
\[
I_{1,1}(\nu) = \pi \rho_{1,1}(\nu) \left[K(\nu, -\nu) + K(-\nu, \nu)\right],
\]
which is the required form given in equation (26). The expression for \(I_{2,2}(\nu)\) is simply the complex conjugate of the case just considered. The remaining cases can be verified in an equivalent method considered in this appendix.

Dirichlet’s conditions for function \(M(\xi)\) on the interval \((-\alpha, \beta)\) entail: (I) \(M(\xi)\) contains only a finite number of discontinuities on the interval, (II) \(M(\xi)\) contains a finite number of turning points on the interval.
References

[1] Abramowitz M and Stegun I (ed) 1972 Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables (Applied Mathematical Series vol 55) (Washington DC: US Government Printing Office)

[2] Avram F, Leonenko N and Šuvak N 2013 Spectral representation of transition density of Fisher–Snedecor diffusion Stoch.—Int. J. Probab. Stoch. Process. 85 346–69

[3] von-Bertalanffy L 1949 Problems of organic growth Nature 163 156–8

[4] Borodin A and Salminen P 1996 Handbook of Brownian Motion (Cambridge, MA: Birkhauser Boston)

[5] Bric M, Kaupužis J and Mahnke R 2013 How to solve the Fokker–Planck equation treating mixed eigenvalue spectrum? Condens. Matter Phys. 16 13002

[6] Broadbridge P 1999 The forced Burgers equation, plant roots and Schrödinger’s eigenfunctions J. Eng. Math. 36 25–39

[7] Büyükaşik S and Pashaev O 2012 Exactly solvable Madelung fluid and complex Burgers equations: a quantum Sturm–Liouville connection J. Math. Chem. 50 2716–45

[8] Compean M and Kirchbach M 2007 Trigonometric quark confinement potential of QCD traits Eur. Phys. J. A 33 1–4

[9] Cox J, Ingersoll J and Ross S 1985 A theory of the term structure of interest rates Econometrica 53 385–407

[10] Davies B 2002 Integral Transforms and Their Applications (Texts in Applied Mathematics vol 41) 3rd edn (New York: Springer)

[11] Davydov D and Linetsky V 2003 Pricing options on scalar diffusions: an eigenfunction expansion approach Oper. Rev. 51 185–209

[12] Dunford N and Schwartz J 1988 Linear Operators: II. Spectral Theory, Self-Adjoint Operators in Hilbert Space (New Jersey: Wiley)

[13] Fakhri H and Chenaghlou A 2006 Ladder operators and recursion relations for the associated Bessel polynomials Phys. Lett. A 358 345–53

[14] Fakhri H and Mojaveri B 2011 The remarkable properties of the associated Romanovski functions J. Phys. A: Math. Theor. 44 195205

[15] Forrester P 2010 Log Gases and Random Matrices (The London Mathematical Society Monograph Series vol 34) (Princeton, NJ: Princeton University Press)

[16] Fu H and Sasaki R 1996 Exponential and Laguerre squeezed states for su(1, 1) algebra and the Calogero–Sutherland model Phys. Rev. A 53 3836–44

[17] García O 1983 A stochastic differential equation model for the height growth of forest stands Biometrics 39 1059–72

[18] Graf U 2010 Introduction to Hyperfunctions and Their Integral Transforms (An Applied and Computational Approach) (New York: Springer)

[19] Koekoek R, Leskey P and Swarttouw R 2010 Hypergeometric Orthogonal Polynomials and their q-Analogues (Berlin: Springer)

[20] Koornwinder T 1975 Two Variable Analogues of the Classical Orthogonal Polynomials (Theory and Application of Special Functions) ed R Askey (New York: Academic) pp 435–95

[21] Langer H and Schenki W 1990 Generalised second-order differential equations, corresponding gap diffusions and susupperharmonic transformations Math. Nachr. 148 7–45

[22] Leonenko N and Šuvak N 2010 Statistical inference for reciprocal gamma diffusion process J. Stat. Plan. Inference 140 30–51

[23] M Zuparic 2015 A new generalization of the Hankel integral transform J. Phys. A: Math. Theor. 48 (2015) 135202
[29] Linetsky V 2004 The spectral decomposition of the option value Int. J. Theor. Appl. Finance 7 337–84
[30] Linetsky V 2004 The spectral representation of Bessel processes with constant drift: applications in queueing and finance J. Appl. Probab. 41 327–44
[31] MacRobert T 1947 Spherical Harmonics: An Elementary Treatise on Harmonic Functions, with Applications 2nd edn (New York: Dover Publications)
[32] Mao X, Marion G and Renshaw E 2002 Environmental Brownian noise suppresses explosions in population dynamics Stoch. Process. Appl. 97 95–110
[33] McKean H 1956 Elementary solutions for certain parabolic partial differential equations Trans. Am. Math. Soc. 82 519–48
[34] Merzbacher E 1997 Quantum Mechanics 3rd edn (New Jersey: Wiley)
[35] Morse P 1929 Diatomic molecules according to the wave mechanics: II. Vibrational levels Phys. Rev. 34 57–64
[36] Morse P and Feshach H 1953 Methods of theoretical physics International Series in Pure and Applied Physics (New York: McGraw Hill)
[37] Olver F, Lozier D, Boisvert R and Clark C (ed) 2010 NIST Handbook of Mathematical Functions (New York: Cambridge University Press)
[38] Quesne C 2013 Extending Romanovski polynomials in quantum mechanics J. Math. Phys. 54 122103
[39] Raposo A, Weber H, Alvarez-Castillo D and Kirchbach M 2007 Romanovski polynomials in selected physics problems Cent. Eur. J. Phys. 5 253–84
[40] Reed M and Simon B 1981 Functional analysis Methods of Modern Mathematical Physics vol 1 (San Diego, CA: Academic)
[41] Richards F 1959 A flexible growth function for empirical use J. Exp. Bot. 10 290–300
[42] Risken H 1989 The Fokker–Planck Equation 2nd edn (Heidelberg: Springer)
[43] Saad N, Hall R and Ciftci H 2006 Criterion for polynomial solutions to a class of linear differential equations of second order J. Phys. A 39 13445–54
[44] Scarf F 1958 New soluble energy band problem Phys. Rev. 112 1137–41
[45] Schenzle A and Brand H 1979 Multiplicative stochastic processes in statistical physics Phys. Lett. 69A 313–5
[46] Schuss Z 2010 Theory and applications of stochastic processes Series in Applied Mathematical Sciences vol 170 (New York: Springer)
[47] Stewart I and Tall D 1999 Complex Analysis (Cambridge: Cambridge University Press)
[48] Strogatz S 2000 From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators Physica D 143 1–20
[49] Tierz M 2007 SL(2, R) matrix model and supersymmetric Yang–Mills integrals Phys. Rev. D 76 107701
[50] Weidmann J 1987 Spectral theory of ordinary linear operators Lecture Notes in Mathematics vol 1258 (Berlin: Springer)
[51] Wimp J 1964 A class of integral transforms Proc. Edinburgh Math. Soc. 14 33–40
[52] Wong E 1964 The Construction of a Class of Stationary Markoff Processes (Stochastic Processes in Mathematical Physics and Engineering vol 26) ed R Bellman and A Philips (Providence, R.I.: American Mathematical Society) pp 264–76
[53] Yadav M 2014 Solutions of a system of forced Burgers equation in terms of generalized Laguerre polynomials Acta Math. Sci. B 34 1461–72
[54] Zuparic M and Kalloniatis A 2013 Stochastic (in)stability of synchronization of oscillators on networks Physica D 255 35–51