GLOBAL SHALLOW WATER MAGNETOHYDRODYNAMIC WAVES IN THE SOLAR TACHOCLINE

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1. INTRODUCTION

The tachocline, which is believed to exist near the base of the solar convection zone (Spiegel & Zahn 1992), has become a subject of rapidly growing scientific interest in the last years (Gough 2007). The existence of the tachocline is important for two main reasons. First, it may prevent the spread of the solar angular momentum from the convection zone to the interior (Spiegel & Zahn 1992; Gough & McIntyre 1998; Garaud 2007, p 3), thus retaining the inferred solid body rotation in the radiative zone; second, the processes of solar magnetic field generation and global magnetohydrodynamic (MHD) instabilities, which are crucial for solar activity, may take place there (Dikpati & Gilman 2005; Gilman & Cally 2007, p 243).

MHD waves and oscillations in the tachocline may play a significant role in both cases: they may redistribute the angular momentum in the horizontal direction, and some of them may become unstable due to the differential rotation, leading to magnetic flux emergence at the solar surface. Therefore, to study the complete spectrum of possible wave modes in this system is of vital importance. The tachocline is very thin compared to the solar radius; therefore, the ordinary shallow water approximation modified by the presence of a horizontal large-scale magnetic field can be easily applied (Gilman 2000).

The spectrum of various shorter-scale shallow water MHD waves has been recently studied in Cartesian coordinates by Schecter et al. (2001). However, global wave modes, those with a wavelength comparable to the solar radius, must be considered in spherical coordinates. Recently, Zaqarashvili et al. (2007) have studied spherical shallow water MHD waves in the simplest case, in which the high-frequency branch, i.e. magnetic Poincaré waves (or magneto-gravity waves) and the influence of the tachocline thickness on the wave dynamics have been ignored (that is, when the surface gravity speed is much higher than the surface rotation speed). This approximation has enabled us to study the dynamics of global magnetic Rossby waves in the lower, strongly stable part of the tachocline, but it fails in the upper, weakly stable overshoot region, where a negative buoyancy due to the subadiabatic stratification strongly reduces the gravity speed (Gilman 2000).

In this Letter, we present the analytical spectrum of global linear shallow water MHD waves for both parts of the tachocline in the absence of differential rotation.

2. GLOBAL SHALLOW WATER MHD WAVES

We use the linearized shallow water MHD equations in the rotating spherical coordinate system \((r, \theta, \phi)\), where \(r\) is the radial coordinate, \(\theta\) is the co-latitude, and \(\phi\) is the longitude (Zaqarashvili et al. 2007). We also assume the unperturbed toroidal magnetic field to be \(B_\theta = B(\theta) \sin \theta\), where \(B(\theta)\) may have an arbitrary profile with latitude, which can be chosen later. Then a normal mode analysis of the form \(\exp(-i\omega t + is\phi)\) gives the equation (Zaqarashvili et al. 2007)

\[
\left(\lambda D + s \mu \right) \left\{ \frac{1}{s^2 - \epsilon \lambda^2 \left(1 - \mu^2\right)} \left[ \lambda D - s \mu - \frac{\alpha^2}{\lambda^2} s^2 \lambda \left(D + 2 \mu\right) \right] \right\} 
\times \hat{u}_\theta + \left(\lambda^2 - \mu^2\right) \hat{u}_\theta + s^2 \alpha^2 \hat{u}_\phi + 2 \alpha^2 \mu D \hat{u}_\theta 
+ \mu s \frac{\alpha^2}{\lambda} \left(D + 2 \mu\right) \hat{u}_\theta + \mu D \left(\alpha^2\right) \hat{u}_\theta = 0, \tag{1}
\]

with

\[
\lambda = \frac{\omega}{2Q_0}, \quad \epsilon = \frac{4Q_0^2 R_0^2}{g H_0}, \quad \alpha^2 = \frac{v_A^2}{4Q_0^2 R_0^2}, \quad \nu_A = \frac{B(\mu)}{\sqrt{4\pi \rho}},
\mu = \cos \theta, \quad D = (1 - \mu^2) \frac{\partial}{\partial \mu} - \sin \theta \frac{\partial}{\partial \theta}, \quad \hat{u}_\theta = \sin \theta \hat{u}_\phi,
\]

where \(\hat{u}_\theta\) is the poloidal velocity component, \(g\) is the reduced gravity in the tachocline, \(\rho\) is the medium density, \(Q_0\) is the angular velocity, \(R_0\) is the distance from the solar center to the tachocline, and \(H_0\) is the tachocline thickness. This is the general equation for the linear dynamics of the shallow water MHD system, which contains various kinds of waves (magnetic Poincaré and magnetic Rossby waves). When \(\epsilon = 0\) and \(B = \text{const}\), this equation transforms into the associated Legendre equation and governs the dynamics of fast and slow magnetic Rossby waves (Zaqarashvili et al. 2007). This approximation neglects magnetic Poincaré waves and the influence of the tachocline thickness on the dynamics of magnetic Rossby
waves. Therefore, Equation (1) must be solved with $\epsilon \neq 0$ in order to give the complete spectrum of waves. The subadiabatic stratification in the tachocline provides negative buoyancy, which leads to a reduced gravity and consequently an increased value of $\epsilon$. In the lower strongly stable part of the tachocline, $\epsilon$ is still much smaller than unity, but in the upper overshoot layer, $\epsilon$ becomes much greater than unity. Therefore, the solution of Equation (1) for the two extreme cases $\epsilon \ll 1$ and $\epsilon \gg 1$ covers both parts of the tachocline. Here, we use a magnetic field profile typical of the Sun, namely

$$B(\mu) = B_0 \mu,$$

which has opposite signs in the northern and southern hemispheres (Gilman & Fox 1997). The magnetic field profile (2) leads to $\alpha = \alpha_0 \mu$, where $\alpha_0 = B_0/(2\Omega_0 R_0 \sqrt{4\pi \rho})$. In the following subsections, we will give analytical solutions and dispersion relations of global shallow water MHD waves for each case separately.

### 2.1 $\epsilon \ll 1$ (Strongly Stable Stratification; Valid for the Radiative Part of Tachocline)

Reduced gravity in the strongly stable radiative part of the tachocline can be estimated as $500–1.5 \times 10^4 \text{ cm s}^{-2}$ (Schecter et al. 2001). Then, for the layer thickness $H_0 = 10^9 \text{ cm}$ we get $\epsilon = 4.5 \times 10^{-3}–0.13$ (where $\Omega_0 = 2.6 \times 10^{-6} \text{ s}^{-1}$, $R_0 = 5 \times 10^{10} \text{ cm}$ have been used). Therefore, $\epsilon \ll 1$ is a good approximation in the radiative part of the tachocline.

In this approximation and using the weak magnetic field limit, i.e. $\alpha_0^2 \ll 1$, Equation (1) leads to the spheroidal wave equation (second-order terms with $\epsilon$ and $\alpha_0^2$ are neglected), whose typical solutions are the spheroidal wave functions $S_m^{\lambda}$ (Abramowitz & Stegun 1964), where $n$ plays the role of the poloidal wavenumber. The approximate solution can be written as

$$\hat{u}_\alpha \propto u_0 S_m(\epsilon_1, \mu) \sqrt{s^2 - \epsilon \lambda^2 (1 - \mu^2)} / \sqrt{\lambda^2 - s^2} \alpha_0^2 \mu^2,$$

where $u_0$ is the amplitude and

$$\epsilon_1 = \sqrt{\epsilon \left( 1 + 3 \frac{\lambda^2}{s^2} - 2 \frac{\lambda}{s} + \frac{\alpha_0^2 s^2}{\lambda^2} \left( 7 + \frac{s}{2\lambda} \right) \right)}.$$

Note that $n$ is the integer for the solutions, which vanish at the poles ($\mu = \pm 1$). The dispersion relation is found to be

$$\epsilon(s^2 + 1) \left( \frac{\lambda}{s} \right)^4 - n(n + 1) \frac{\lambda^2}{s^2} - \left( \frac{\lambda}{s} \right)^2 + \alpha_0^2 = 0.$$

The dispersion relation can be split into high (magnetic Poincaré) and low (magnetic Rossby) frequency branches. For the magnetic Poincaré waves, we have

$$\frac{\lambda_m}{s} \approx \pm \frac{n(n + 1)}{\epsilon(s^2 + 1)}.$$

This is the dispersion relation of ordinary surface gravity waves, and we can recover it for the high $s$ harmonics (Longuet-Higgins 1965).

For the magnetic Rossby waves, we have approximately

$$n(n + 1) \left( \frac{\lambda}{s} \right)^2 + \lambda - \alpha_0^2 = 0.$$

It is clear from Equation (6) that the magnetic field causes the splitting of ordinary Rossby waves into fast and slow modes (Zaqarashvili et al. 2007). For $\alpha_0 \ll 1$, the dispersion relations for the fast and slow magnetic Rossby modes can be approximated as

$$\frac{\lambda_f}{s} \approx - \frac{1}{n(n + 1)} - \alpha_0^2, \quad \frac{\lambda_s}{s} \approx \alpha_0^2.$$

We see that the fast magnetic Rossby mode has a dispersion relation similar to that of the ordinary Rossby waves but slightly modified by the magnetic field. Slow magnetic Rossby waves, however, are new wave modes as their dispersion relation is different from Rossby and Alfvén wave dispersion relations (the dispersion relation of Alfvén waves should be $\lambda = \pm \epsilon \alpha_0$). There is a significant difference between the frequencies of fast and slow magnetic Rossby modes for sufficiently small $\alpha_0$.

The dispersion diagram for the $s = 1$ harmonics of shallow water MHD waves according to Equation (4) is shown in Figure 1. Here $\epsilon = 0.01$ and $\alpha_0 = 0.05$. The frequencies of magnetic Poincaré waves (solid lines) are much higher than the rotational frequency, $\Omega_0$, and they increase with increasing $n$. For example, the frequency of the $n = 3$ harmonics is $\sim 50 \Omega_0$, which corresponds to oscillations with period $\sim 13$ hr. On the other hand, the frequencies of magnetic Rossby waves are, in general, much lower than the rotational frequency. The absolute value of the frequency of fast magnetic Rossby waves (dotted line in the zoom) decreases with increasing $n$, such as happens with the HD Rossby wave.

Figure 2 shows the dependence of the $s = 1, n = 2$ and $s = 1, n = 3$ harmonics on $\epsilon$ and $\alpha_0$ (i.e. the magnetic field strength). The frequency (in absolute value) of fast (dotted) and slow (dashed) magnetic Rossby modes significantly increases when the magnetic field is increased. We also see that the frequency difference between the $n = 2,3$ harmonics of fast magnetic Rossby waves is almost independent of $\alpha_0$. In contrast, the $n = 2,3$ harmonics of slow magnetic Rossby waves diverge when the magnetic field is increased. It turns out that the frequencies of magnetic Rossby waves have almost no dependence on $\epsilon$; therefore, they are not shown in the upper panel. On the other hand, the frequency of magnetic Poincaré waves depends on $\epsilon$ (lower panel) and significantly decreases.
with increasing \( \epsilon \). However, these frequencies have almost no dependence on \( \alpha_0 \), therefore magnetic Poincaré waves are only slightly affected by the magnetic field.

2.2. \( \epsilon \gg 1 \) (Weakly Stable Stratification; Valid in the Upper Overshoot Part of Tachocline)

In the weakly stable overshoot region (or say upper tachocline) reduced gravity is much smaller, being 0.05–5 cm s\(^{-2} \) (Schechter et al. 2001). Then, for the thickness \( H_0 = 5 \times 10^8 \) cm, we get \( \epsilon = 27–2.7 \times 10^3 \). Therefore, \( \epsilon \gg 1 \) is a good approximation in the overshoot region.

In this case, we follow the calculation of Longuet-Higgins (1968) (that paper considers spherical hydrodynamic shallow water waves in the Earth context), introduce a new function \( \eta = \epsilon^{1/4} \mu \) and also consider that \( \lambda = \epsilon^{-1/4} L \) and \( \alpha_0 = \epsilon^{-1/4} L \), where \( L \) is of order unity. Then, after some algebra and keeping only large terms, Equation (1) leads to the Weber (parabolic cylinder) equation, whose solutions finite at \( \eta \rightarrow \pm \infty \) can be expressed in terms of Hermite polynomials \( H_n(\eta) \) of order \( \nu \), where \( \nu \) satisfies

\[
\epsilon \lambda^2 - \frac{s}{\lambda} + \frac{a_0^2 s^2}{\lambda^2} = (2\nu + 1)\sqrt{\epsilon}, \quad (\nu = 0, 1, 2, \ldots),
\]

and the solution is proportional to

\[
\hat{u}_0 \propto e^{-\frac{1}{2}\eta^2} H_n(\eta).
\]

Therefore, the function \( \hat{u}_0 \) is exponentially small beyond the turning points \( \cos \theta = \epsilon^{-1/4} \sqrt{2\nu + 1} \). Due to the large \( \epsilon \), the co-latitude \( \theta \) satisfying this condition is close to \( 90^\circ \); therefore, the solution is confined to the neighborhood of the equator.

Equation (8) yields the wave dispersion relation as

\[
\lambda^4 - \frac{2\nu + 1}{\sqrt{\epsilon}} \lambda^2 - \frac{s}{\lambda} + \frac{a_0^2 s^2}{\epsilon} = 0. \tag{10}
\]

In the nonmagnetic case, this Equation transforms into Equation (8.11) of Longuet-Higgins (1968). This is a fourth-order equation, and it describes magnetic Poincaré and magnetic Rossby waves. For large values of \( \epsilon \), the solutions can be written separately for both types of waves.

The solutions for Poincaré waves can be written approximately as

\[
\lambda_{mp} \approx \pm \frac{(2\nu + 1)^{1/2}}{\epsilon^{1/4}} + \frac{s}{(4\nu + 2)\epsilon^{1/2}}. \tag{11}
\]

It turns out that the magnetic field has almost no influence on the dynamics of Poincaré waves, and the dispersion relation is the same as for the nonmagnetic case (Longuet-Higgins 1968).

The dispersion relation for the magnetic Rossby waves can be written as

\[
(2\nu + 1)\sqrt{\epsilon \lambda^2 + s\lambda} - a_0^2 s^2 = 0. \tag{12}
\]

The magnetic field causes the splitting of ordinary Rossby waves into fast and slow magnetic Rossby waves as in the \( \epsilon \ll 1 \) case. If \( \sqrt{\epsilon a_0^2} \ll 1 \), then we can write

\[
\frac{\lambda_F}{s} \approx -\frac{1}{(2\nu + 1)\epsilon^{1/2} - a_0^2}, \quad \frac{\lambda_s}{s} \approx a_0^2 \tag{13}
\]

Fast magnetic Rossby waves are similar to hydrodynamic Rossby waves modified by the magnetic field. Slow magnetic Rossby waves have a similar dispersion relation as in the \( \epsilon \ll 1 \) case.

The dispersion diagram for the \( s = 1 \) harmonics of magnetic Poincaré and magnetic Rossby waves according to Equation (10) is shown in Figure 3. Here \( \epsilon = 2700 \) and \( a_0 = 0.05 \). The behavior of fast magnetic Rossby waves (dotted line) is similar to the small \( \epsilon \) case: the absolute value of their frequency significantly decreases with increasing \( \nu \) (which now plays the role of the poloidal wavenumber). However, we immediately note that the frequency now is much smaller than that

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**Figure 2.** Upper panel: frequencies of the \( s = 1, n = 2 \) and \( s = 1, n = 3 \) harmonics of fast (dotted) and slow (dashed) magnetic Rossby modes vs. \( \alpha_0 \) for \( \epsilon = 0.0045 \). Magnetic Poincaré waves do not depend significantly on the magnetic field, therefore, they are not shown here. Lower panel: frequencies of the \( s = 1, n = 2 \) and \( s = 1, n = 3 \) harmonics of magnetic Poincaré waves vs. \( \epsilon \) for \( \alpha_0 = 0.05 \).

**Figure 3.** Dispersion diagram of spherical shallow water waves in the presence of a toroidal magnetic field for large \( \epsilon \). The two extreme upper and lower solutions (solid lines) correspond to magnetic Poincaré waves, whereas the low frequency modes (also shown in the inset) are the fast (dotted) and slow (dashed) magnetic Rossby waves. Symbols denote solutions obtained with the approximate formulas (11) and (13). The parameters used to obtain the dispersion diagram are \( \epsilon = 2700, s = 1 \), and \( \alpha_0 = 0.05 \).

In the nonmagnetic case, this Equation transforms into Equation (8.11) of Longuet-Higgins (1968). This is a fourth-order equation, and it describes magnetic Poincaré and magnetic Rossby waves. For large values of \( \epsilon \), the solutions can be written separately for both types of waves.
wave harmonic with $\nu$ (Equation (10)) shows that the frequency of the magnetic Rossby to a decrease of the wave frequencies.

of the $\epsilon \ll 1$ case. This is also true for magnetic Poincaré waves (solid lines): their frequency is significantly small compared to the rotational frequency. The frequency of slow magnetic Rossby waves only slightly depends on $\nu$, but now it decreases with increasing $\nu$, contrary to $\epsilon \ll 1$ case.

Figure 4 shows the dependence of the $s = 1, \nu = 2$ and $s = 1, \nu = 3$ harmonics on $\epsilon$ and $\alpha_0$. The frequency of magnetic Rossby mode harmonics significantly depends on $\alpha_0$; the frequencies of both fast and slow magnetic Rossby modes increase with increasing $\alpha_0$. On the other hand, it turns out that the frequencies of magnetic Poincaré waves have almost no dependence on $\alpha_0$. However, they are significantly reduced with increasing $\epsilon$, becoming smaller than the rotational frequency for $\epsilon > 1000$. Fast magnetic Rossby waves also depend on $\epsilon$, their frequency significantly decreasing when this parameter is increased. Slow magnetic Rossby waves have almost no dependence on $\epsilon$, as is suggested from Equation (13). Thus, increasing $\epsilon$, which is equivalent to reducing $g$, generally leads to a decrease of the wave frequencies.

The numerical solution of the dispersion relation (Equation (10)) shows that the frequency of the magnetic Rossby wave harmonic with $\nu = 0$ becomes complex for sufficiently large $\alpha_0$, i.e. large magnetic field. This may point to some kinds of instabilities similar to the polar kink instability found by Cally (2003). However, the value of $\alpha_0$ for which the frequency becomes complex is significantly outside the range for which Equation (10) is valid (note that $\alpha_0$ should be proportional to $\epsilon^{-1/4}$). Therefore, the complex solution may not reflect a real instability.

3. CONCLUSIONS

We have derived analytical solutions and dispersion relations of global shallow water MHD waves for the solar tachocline in a rotating spherical coordinate system. The solutions and dispersion relations are obtained for weakly and strongly stable stratifications separately. The weakly stable stratification is valid in the upper overshoot part of tachocline, while the strongly stable stratification is valid for the radiative part of tachocline. The solutions include the Poincaré and Rossby waves modified by the presence of the magnetic field.

Here, we use a realistic latitudinal profile of the toroidal magnetic field $B_\phi \sim \cos \theta \sin \theta$ which changes sign at the equator. Similar to the $\epsilon = 0$ case (Zaqarashvili et al. 2007), the magnetic field leads to the splitting of hydrodynamic Rossby waves into fast and slow magnetic Rossby modes.

In the case of strongly stable stratification, the dispersion relation of magnetic Poincaré waves is similar to that for the hydrodynamic case (Longuet-Higgins 1965). The dispersion relation of magnetic Rossby waves (Zaqarashvili et al. 2007) is slightly modified by the thickness of the layer. In the case of weakly stable stratification, both magnetic Poincaré and magnetic Rossby waves are concentrated near the equator, thus leading to equatorially trapped waves.

Another remarkable feature is that the frequencies of all waves are significantly reduced in the large $\epsilon$ case compared to the $\epsilon \ll 1$ case. This means that the upper overshoot part of the tachocline supports oscillations with smaller frequency than the lower radiative part of the tachocline.

The solutions have been obtained without taking into account the differential rotation which is present in the tachocline. The future inclusion of differential rotation probably will lead to the instability of some modes, which was found by simulations in the inertial frame (Gilman & Cally 2007, p 243, and references therein).

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