DECISION PROBLEMS IN THE SPACE OF DEHN FILLINGS

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Abstract. In this paper, we use normal surface theory to study Dehn filling on a knot-manifold. First, it is shown that there is a finite computable set of slopes on the boundary of a knot-manifold that bound normal and almost normal surfaces in a one-vertex triangulation of that knot-manifold. This is combined with existence theorems for normal and almost normal surfaces to construct algorithms to determine precisely which manifolds obtained by Dehn filling: 1) are reducible, 2) contain two-sided incompressible surfaces, 3) are Haken, 4) fiber over $S^1$, 5) are the 3–sphere, and 6) are a lens space. Each of these algorithms is a finite computation.

Moreover, in the case of essential surfaces, we show that the topology of the filled manifolds is strongly reflected in the triangulation of the knot-manifold. If a filled manifold contains an essential surface then the knot-manifold contains an essential vertex solution that caps off to an essential surface of the same type in the filled manifold. (Vertex solutions are the premier class of normal surface and are computable.)

1. Introduction

A compact, connected, orientable 3–manifold with connected boundary a torus is called a knot-manifold. Dehn filling is a method of obtaining closed 3–manifolds from a knot-manifold. It is a special case of a general construction from which one can obtain all closed, orientable, 3–manifolds \cite{22, 28}. Specifically, if $X$ is a knot-manifold, we call an isotopy class of a simple closed curve in $\partial X$ a slope. If $\alpha$ is a slope, the Dehn filling of $X$ along $\alpha$, denoted $X(\alpha)$, is the closed, orientable 3–manifold obtained from $X$ by attaching a solid torus $V_\alpha$ to $X$ via a homeomorphism from $\partial X$ to $\partial V_\alpha$ which takes a simple closed curve of slope $\alpha$ to the meridian of $V_\alpha$; i. e., to an essential curve in $\partial V_\alpha$ that bounds a disk in $V_\alpha$. The homeomorphism type of $X(\alpha)$ is completely determined by the identification of the slope $\alpha$ to a meridian of $V_\alpha$.

If $X$ is a knot-manifold and we select a homology basis, say $\mu$, $\lambda$, for $H_1(\partial X)$, then each slope $\alpha$ can be written $\alpha = p\mu + q\lambda$ where $p$ and $q$ are...
integers. Hence, if we include $\infty$ and forget orientation (sign) of a homology class, the slope $\alpha$ is uniquely associated with a rational number $p/q$ with $1/0$ associated with $\infty$. Hence, for a given knot-manifold $X$ we obtain a family of closed, orientable manifolds $X(\alpha), \alpha \in \mathbb{Q} \cup \{\infty\}$. The collection of such manifolds is called the space of Dehn fillings on $X$.

A great deal of work has been done to understand the manifolds in the space of Dehn fillings on a knot-manifold. In particular, for a hyperbolic knot-manifold $X$, a knot-manifold whose interior admits a complete Riemannian metric of constant sectional curvature $-1$, it has been shown [27] that $X(\alpha)$ is hyperbolic for all but finitely many slopes $\alpha$. For the past decade some of the most interesting work in low-dimensional topology has been toward understanding exceptions to $X(\alpha)$ being hyperbolic. In this sense, the exceptions include the possibilities that $X(\alpha)$ is reducible, toroidal, or a lens space.

If $\alpha$ and $\beta$ are slopes, we let $\Delta(\alpha, \beta)$ denote the absolute value of the homology intersection between $\alpha$ and $\beta$ and call $\Delta(\alpha, \beta)$ the distance between $\alpha$ and $\beta$. If for some homology basis of $H_1(\partial X)$ we have $\alpha = p/q$ and $\beta = r/s$, then $\Delta(\alpha, \beta) = |ps - qr|$. Now, if $X$ is a hyperbolic knot-manifold and $\alpha$ and $\beta$ are exceptional slopes, then in many situations bounds can be placed on $\Delta(\alpha, \beta)$ and thereby one obtains bounds on the numbers of exceptional Dehn fillings on $X$ [8]. The remarkable and very satisfactory consequence of these methods is that the bounds obtained are global; they do not depend on $X$. For example, it is conjectured that for $X$ a hyperbolic knot-manifold and $X$ not one of a finite number of exceptions formed by Dehn filling on a component of the Whitehead link in $S^3$, then $\Delta(\alpha, \beta) \leq 5$ if $\alpha$ and $\beta$ are exceptional slopes [8]. It is known that $\Delta(\alpha, \beta) \leq 1$ if $X(\alpha)$ and $X(\beta)$ are reducible [7, 1]; $\Delta(\alpha, \beta) \leq 5$ if $X(\alpha)$ and $X(\beta)$ have finite fundamental group [1]; and for all but the aforementioned exceptions on $X$, $\Delta(\alpha, \beta) \leq 5$ if $X(\alpha)$ and $X(\beta)$ are toroidal [5]. Results for mixed outcomes of exceptional Dehn fillings are given in [8]. For a knot-manifold embedded in $S^3$, a preferred basis for $H_1(\partial X)$ is the unique meridian and longitude pair, $\mu$ and $\lambda$, respectively.

Our work addresses most of these same issues about Dehn filling but from a different point of view. Namely, given a knot-manifold $X$, we are interested in determining precisely those slopes $\alpha$ on $\partial X$ for which Dehn filling leads to “interesting” phenomena for $X(\alpha)$. In particular, we consider for precisely what slopes $\alpha$ is $X(\alpha)$ reducible; is $X(\alpha)$ toroidal; does $X(\alpha)$ contain an embedded, incompressible, two-sided surface; is $X(\alpha)$ a Haken-manifold; does $X(\alpha)$ fiber over $S^1$; is $X(\alpha)$ homeomorphic with $S^3$; and is $X(\alpha)$ a lens space. Recall that given a 3–manifold $M$ there are algorithms to answer each of these questions regarding $M$. Namely, given a compact 3–manifold $M$, it can be decided if $M$ is reducible [23, 27, 18, 13]; it can be decided if $M$ is toroidal [4, 12]; it can be decided if $M$ contains an embedded, incompressible two-sided surface [4, 12, 18]; it can be decided if $M$ fibers over $S^1$ [11]; it can be decided if $M$ is homeomorphic with $S^3$.
and it can be decided if \( M \) is a lens space \([23]\). Now, one might think that the existence of these algorithms will solve our problem; however, for a given knot-manifold \( X \), the family of manifolds in the Dehn filling space of \( X \) is infinite. Hence, we have the situation that knowing that there is a manifold in the space of Dehn fillings of \( X \) that is of interest, then we can find one (our problem is recursively enumerable); however, without \textit{a priori} information, these algorithms, alone, will not necessarily determine if there is an interesting manifold, let alone determine all slopes for which such interesting phenomena occur. In this paper we provide the additional ingredients and algorithms to determine precisely the slopes, or manifolds in the space of Dehn fillings of \( X \), that exhibit the various “interesting” phenomena mentioned above.

We will assume 3-manifolds are given via triangulations or cell subdivisions. In most settings we use either one-vertex triangulations of the manifolds under considerations or at least a triangulation that restricts to a one-vertex triangulation on each torus component of the boundary. The existence of such triangulations is straight forward and discussion of these and other useful triangulation environments are given in \([16]\). We use normal and almost normal surface theory for these triangulations.

The study of Dehn fillings has exhibited strong relationships between the topology of \( X \) and those manifolds in the space of fillings of \( X \). Our methods re-enforce this relationship in a remarkable way. Given a knot-manifold \( X \) via a triangulation \( \mathcal{T} \) that restricts to a one-vertex triangulation on \( \partial X \), we use the methods of \([15, 16]\) to extend \( \mathcal{T} \) to a triangulation of \( X(\alpha) \); that is, for each slope \( \alpha \), we construct a triangulation \( \mathcal{T}(\alpha) \) of \( X(\alpha) \) that restricts to \( \mathcal{T} \) on \( X \). Furthermore, the triangulation \( \mathcal{T}(\alpha) \) restricts to a well understood one-vertex triangulation of \( V_\alpha \), the attached solid torus. Each of the problems we consider is to determine precisely the slopes \( \alpha \) for which a certain type of surface exists in the manifold \( X(\alpha) \). For example, reducibility is the existence of an embedded 2–sphere that does not bound a 3–cell; and to determine if \( X(\alpha) \) is \( S^3 \) or a lens space is to find a genus zero or genus one Heegaard surface, respectively. Normal and almost normal surface theory provide a parameterization of “interesting” surfaces by rational points in a computable, compact, convex, linear cell in \( \mathbb{R}^n \), the \textit{projective solution space}. If \( X \) is a knot-manifold and \( \mathcal{T} \) is a triangulation of \( X \), we denote the projective solution space of \( X \) with respect to \( \mathcal{T} \) by \( \mathcal{P}(X; \mathcal{T}) \). In this situation, if \( S \) is a properly embedded, normal surface in \( (X, \mathcal{T}) \), then either \( \partial S = \emptyset \) or \( \partial S \neq \emptyset \) and \( \partial S \) is a collection of pairwise disjoint, normal curves in \( \partial X \). If \( \partial S = \emptyset \) or is a collection of trivial and, hence, vertex-linking curves, then for any slope \( \alpha \), \( S \) determines a unique normal surface \( S(\alpha) \) in \( X(\alpha) \) (\( S(\alpha) \) is obtained from \( S \) by capping off \( \partial S \) with copies of the vertex-linking normal disks in the special triangulation of \( V_\alpha \) determined by \( \mathcal{T}(\alpha) \)). If \( \partial S \) contains a nontrivial component and determines a unique boundary slope \( \alpha \), then \( S \) determines a unique normal surface \( S(\alpha) \) in \( X(\alpha) \) just for the slope \( \alpha \) (\( S(\alpha) \) is obtained from \( S \) by capping off \( \partial S \) with copies
of the vertex-linking normal disks and meridional normal disks in the special triangulation of $V_\alpha$ determined by $T(\alpha)$. We show a Dehn filling $X(\alpha)$ will contain one of our “interesting” surfaces, listed above, if and only if there is a normal or almost normal surface $S$ in $(X, T)$ whose projective class is a vertex solution of $\mathcal{P}(X, T)$ and $S(\alpha)$ has the same interesting property in $X(\alpha)$. Hence, for any triangulation $T$ of $X$ that restricts to a one-vertex triangulation on $\partial X$, the normal and almost normal surfaces in $(X, T)$ whose projective classes in $\mathcal{P}(X, T)$ are vertex solutions completely determine for all slopes $\alpha$ whether the manifold $X(\alpha)$ is reducible, toroidal, contains an embedded, incompressible, two-sided surface, fibers over $S^1$, is $S^3$, or is a lens space. The vertex solutions of $\mathcal{P}(X, T)$ form a finite, computable set. It is this set which plays the fundamental role in most of our algorithms.

In Section 2, we recall material from normal and almost normal surface theory. We limit this to material that is directly relevant to this paper and assume the reader has some familiarity with this theory. More sweeping introductions from our point of view may be found in [14, 18, 13].

In Section 3, we introduce one of the fundamental features of using one-vertex triangulations: the relationship between interesting slopes and normal and almost normal surfaces. Also, we provide techniques to compute interesting slopes. We compute the projective solution space of normal curves in a one-vertex triangulation of a torus. It is represented as the standard 2–simplex, $\Delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_{i=1}^3 x_i = 1, x_i \geq 0\}$. The rational points of $\Delta$ represent projective classes of families of embedded, normal curves in the torus; the barycenter of $\Delta$ represents the trivial, vertex linking family and the edges (some $x_i = 0$) represent the various slopes of the families of embedded curves having nontrivial components. Now, given a triangulation $T$ of a knot-manifold $X$ that restricts to a one-vertex triangulation on $\partial X$, we call a slope $\alpha$ a boundary slope if there is a normal or almost normal surface $S$ properly embedded in $X$ and a component of $\partial S$ represents a curve of slope $\alpha$. We prove that for such a triangulation there are only finitely many slopes $\alpha$ which are boundary slopes; furthermore, we do this by showing that the boundary slopes are completely determined by the boundary slopes of normal or almost normal surfaces in $(X, T)$ whose projective class is a vertex solution of $\mathcal{P}(X, T)$, a finite set. This result generalizes the results of [10] (using similar techniques) and gives a new proof of the main theorem in [10] that there are only finitely many boundary slopes for embedded, incompressible and $\partial$-incompressible surfaces in $X$. However, a distinguishing feature of our work is a means to actually compute precisely the relevant boundary slopes from the triangulation $T$ of $X$.

In Section 4 we generalize the one-vertex triangulations of solid tori introduced in [15] to one-vertex triangulations called layered triangulations. We analyze the embedded, planar, normal surfaces in these layered triangulations, classifying such surfaces and obtaining lower bounds for their weights (the weight of a normal surface is the cardinality of its intersection with
the one-skeleton of the triangulation). These results lead to the special triangulations we use when studying Dehn fillings and enable us to relate the existence of interesting normal surfaces in the manifolds in the space of Dehn fillings to interesting surfaces in the original knot-manifold.

In Sections 5 and 6 we consider the central problems of this paper: given a knot-manifold $X$, to determine precisely those slopes $\alpha$ on $\partial X$ for which Dehn filling leads to “interesting” phenomena for $X(\alpha)$. We divide this work into two parts. In Section 5, we look at phenomena associated with embedded essential surfaces and in Section 6, we look at phenomena associated with Heegaard surfaces. We have organized the presentation so that following proofs of the existence of certain algorithms, we give step by step outlines of the algorithms.

In Section 5, Theorem 5.4 provides one of the major ingredients for our algorithms. It ties the topology of a knot-manifold $X$ quite tightly with that of the manifolds obtained by Dehn filling on $X$. For a triangulation $T$ that restricts to a one-vertex triangulation of $\partial X$, Theorem 5.4 gives that if $X(\alpha)$ is reducible, then a vertex-solution $S$ of $P(X, T)$ must be either an embedded, essential 2–sphere or planar surface and $S(\alpha)$ is an embedded, essential 2–sphere in $X(\alpha)$; if $X(\alpha)$ contains an embedded, incompressible, two-sided surface, then a vertex-solution $F$ of $P(X, T)$ must be an embedded, essential, non-planar surface and $F(\alpha)$ is an embedded, incompressible, two-sided surface in $X(\alpha)$. In the latter case, if $X(\alpha)$ contains an embedded, incompressible torus, then a vertex-solution $T$ of $P(X, T)$ must be an embedded, essential torus or punctured-torus and $T(\alpha)$ is an embedded, incompressible torus in $X(\alpha)$; and if $X(\alpha)$ fibers over $S^1$, then a vertex-solution $F$ of $P(X, T)$ has the property that $F(\alpha)$ is a fiber in a fibration of $X(\alpha)$ over $S^1$.

Theorem 5.7 and Theorem 5.8 show that there exists an algorithm to determine precisely those manifolds in the space of Dehn fillings of a knot-manifold $X$ that are reducible. In particular we have

**Algorithm R.** Given a knot-manifold $X$, determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is reducible.

Theorem 6.10 and Theorem 6.12 show that there exists an algorithm to determine precisely those manifolds which contain an embedded, incompressible, two-sided surface in the space of Dehn fillings of a knot-manifold $X$. Of particular importance in the proof of this theorem, and of independent interest, is Lemma 5.11 which provides an algorithm to determine for a given closed, two-sided, normal surface $S$ in $(X, T)$ precisely those slopes $\alpha$ for which $S$ is incompressible in $X(\alpha)$. The proof of this results investigates when one can determine if a 3–manifold contains an embedded, essential punctured disk. It uses a new estimate for curve length of the boundary of a normal surface discovered by Jaco and Rubinstein; we call this the ALE, average length estimate. It is used in [17] to give algorithms for the existence
of planar surfaces and their relationship to the Word Problem for 3–manifold groups. Our work here provides two algorithms: one that determines if a given closed surface is incompressible in a Dehn filling, Algorithm I, and one that determines precisely those slopes $\alpha$ for which the associated Dehn filling contains an embedded, incompressible, two-sided surface, Algorithm S.

**Algorithm I.** Suppose $X$ is a knot-manifold with a triangulation $T$ which restricts to a one-vertex triangulation on $\partial X$. Given an embedded, two-sided, closed, normal surface in $(X, T)$, determine precisely those slopes $\alpha$ for which the surface compresses in the Dehn filling $X(\alpha)$.

**Algorithm S.** Given a knot-manifold $X$, determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ contains an embedded, incompressible, two-sided surface.

We use Algorithm R and Algorithm S to give an algorithm to determine precisely those slopes for which the associated Dehn filling is a Haken-manifold, Algorithm H.

**Algorithm H.** Given a knot-manifold $X$, determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is a Haken-manifold.

At particular points in the application of Algorithm S, one may consider the alternative questions as to those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is either toroidal, the existence of an embedded, incompressible torus, or fibers over $S^1$, the existence of an embedded, incompressible surface that is a fiber in such a fibration.

Finally, in Section 6, we apply our techniques to similar considerations for Heegaard surfaces. We use almost normal surfaces introduced by H. Rubinstein [23] and thin position introduced by D. Gabai [4] as presented in the papers of Rubinstein [23] and A. Thompson [26]. The two main results of this section are given in Theorem 6.4 and Theorem 6.7 which provide algorithms to determine for a given knot-manifold $X$ precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is either $S^3$ or a lens space, respectively.

**Algorithm S.** Given a knot-manifold $X$, determine precisely those slopes $\alpha$ for which $X(\alpha)$ is the 3–sphere.

**Algorithm L.** Given a knot-manifold $X$, determine precisely those slopes $\alpha$ for which $X(\alpha)$ is a lens space.

The authors wish to thank J. Hyam Rubinstein, whose collaborations with the first author have lead to many useful ideas and tools used in this
work, as well as the members of the topology group at Oklahoma State University, who contributed to this paper through many useful formal and informal discussions. The first author also acknowledges, with thanks, the support of the Departments of Mathematics, and especially the topology groups, at University of California–Davis, California Institute of Technology, and University of Texas–Austin during his sabbatical leave from Oklahoma State University.

2. Normal curves and normal surfaces

Throughout this paper a 3–manifold will be given via a triangulation, where a triangulation $\mathcal{T}$ of a 3–manifold $M$ is a pairwise disjoint collection of tetrahedra, $\Delta = \{\Delta_1, \ldots, \Delta_t\}$, along with a family $\Phi$ of face identifications having $M$ the underlying point set of the identification space $\Delta/\Phi$. Under this definition the tetrahedra may not be embedded in $M$ and two distinct tetrahedra may meet in more than a face of each. Figure 1 shows a one tetrahedron triangulation of the 3–sphere, $S^3$, and Figure 2 shows the two tetrahedra triangulation of the familiar lens space presentation of real projective 3–space, $\mathbb{R}P^3$.

![Figure 1. A one tetrahedron triangulation of $S^3$.](image1)

![Figure 2. A two tetrahedra triangulation of $\mathbb{R}P^3$.](image2)

Triangulations of surfaces are considered in the same generality; that is, a triangulation $\mathcal{T}$ of a surface $S$ is a pairwise disjoint collection of triangles
$\Lambda = \{\lambda_1, \ldots, \lambda_s\}$, along with a family $\Psi$ of edge identifications having $S$ the underlying point set of the quotient space $\Lambda/\Psi$. Figure 4 shows a one-vertex triangulation of the torus $S^1 \times S^1$.

We shall assume the reader has a basic understanding of normal surface theory as well as the application of this theory to curves in 2–manifolds. The references [14] and [18] are sources to review normal surface theory. We also use the concept of an almost normal surface introduced by H. Rubinstein in [23]. If $\mathcal{T}$ is a triangulation of the 3–manifold $M$, a surface $F$ is almost normal (with respect to $\mathcal{T}$) if $F$ meets each tetrahedron of $\mathcal{T}$ in a collection of normal triangles and quadrilaterals and intersects one of the tetrahedra in one component that is either a normal octagon or a normal tube and possibly some normal triangles. See Figure 3. In Section 6 we prove the existence of almost normal surfaces using octagons only. Note however, that our restrictions on the slopes bounding almost normal surfaces developed in Section 3 also apply to almost normal surfaces possessing tubes.

![Figure 3. Exceptional pieces - an octagon and a tube.](image)

If $t$ is the number of tetrahedra in $\mathcal{T}$, then a normal isotopy class of a normal surface has a parameterization as an $n$–tuple of non-negative integers $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ ($n = 7t$), where $x_i$ is the number of elementary triangles and quadrilaterals of type $i$. Similarly, there is a parameterization of the normal isotopy classes of almost normal surfaces, but in this case $n$ is larger as there are 3 normal octagon types and 25 normal tube types in each tetrahedron.

Associated with the triangulation $\mathcal{T}$ is a system of linear equations. Non-negative integer solutions to this system give the parameterization of the normal isotopy classes of normal and almost normal surfaces. We add the equation $\sum_{i=1}^{n} x_i = 1$ along with the condition $x_i \geq 0, \forall i$ and obtain a compact, convex linear cell. The rational points in this cell correspond to projective classes of normal isotopy classes of normal and almost normal surfaces in $(M, \mathcal{T})$. We denote this compact, convex linear cell by $\mathcal{P}(M, \mathcal{T})$ and call it the projective solution space (of $(M, \mathcal{T})$).

If $S$ is a normal or almost normal surface in $M$ we do not distinguish and let $S$ denote the surface $S$, its normal isotopy class, and its representation as an $n$–tuple in $\mathbb{R}^n$. We denote the projective class of $S$ by $\bar{S} \in \mathcal{P}(M, \mathcal{T})$. 
The carrier of a normal surface $S$ is the unique minimal face of $\mathcal{P}(M, T)$ that contains $\bar{S}$. Two normal or almost normal surfaces $S$ and $S'$ are compatible if and only if each component of $S \cap S'$ is an embedded regular curve. Compatibility is equivalent to the normal sum $S + S'$ being defined. If $S$ and $S'$ are embedded normal or almost normal surfaces then $S$ and $S'$ are compatible if and only if they do meet in a tetrahedron in distinct normal quadrilateral types, a quadrilateral and an exceptional piece, or two exceptional pieces, where an exceptional piece is either an octagon or a tube. If $S$ is an embedded normal surface, then every normal surface with projective class in the carrier of $S$ is embedded and any two such normal surfaces are compatible.

W. Haken has observed [9] that there is a finite set of embedded normal surfaces $F_1, \ldots, F_N$ so that any normal surface $S$ can be written as a non-negative integer linear combination of the $F_i$’s; i.e.

$$S = \sum_{i=1}^{N} n_i F_i, \text{ each } n_i \text{ is a non-negative integer.}$$

There is a unique minimal such set, called the set of fundamental surfaces. A surface is fundamental if it cannot be written as a non-trivial sum of surfaces. Among these fundamental surfaces is an important set that have projective classes at the vertices of $\mathcal{P}(M, T)$. These latter surfaces are called vertex solutions. A surface is a vertex solution if no multiple of the surface can be written as a non-trivial sum of distinct surfaces. Note that the sum notation is used for both normal (or geometric) sum as well as coordinate-wise addition of $n$–tuples in $\mathbb{R}^n$.

We remind the reader that when normal surface theory is applied to curves in 2–manifolds; then every solution is realizable as an embedded family of properly embedded arcs and simple closed curves; i.e. there is always a unique embedded representative for a solution. This is not the situation for normal surfaces in 3–manifolds; and solutions that do not have embedded representatives (no realizable solutions) are not understood. In this paper we work only with embedded families of curves in 2–manifolds and with embedded surfaces in 3–manifolds.

For normal curves and normal surfaces there is a notion of complexity analogous to geodesic curves and least arc surfaces. If $T$ is a triangulation of the surface $S$ and $C$ a family of normal curves in $S$ (with respect to $T$) then we define the length of $C$, written $L(C)$ to be the number of times $C$ meets the 1–skeleton of $T$;

$$L(C) = |C \cap T^{(1)}|.$$  

Similarly, if $T$ is a triangulation of the 3–manifold $M$ and $S$ is a normal surface or almost normal surface in $M$, then we define the weight of $S$, written $wt(S)$, to be the number of times $S$ meets the 1–skeleton of $T$;

$$wt(S) = |S \cap T^{(1)}|.$$
If $S$ and $S'$ are embedded compatible normal surfaces, then the normal sum $S+S'$ is defined and is a normal surface and we have:

1. If $S$ corresponds to the $n$–tuple $(x_1, \ldots, x_n)$ and $S'$ corresponds to the $n$–tuple $(x'_1, \ldots, x'_n)$; then $S+S'$ corresponds to the $n$–tuple $(x_1 + \cdots + x_n, x'_1 + \cdots + x'_n)$.
2. $\chi(S+S') = \chi(S) + \chi(S')$, where $\chi$ is the Euler characteristic.
3. $wt(S+S') = wt(S) + wt(S')$.
4. $L(\partial(S+S')) = L(\partial S) + L(\partial S')$.

The properties outlined in this section demonstrate that there is a nice theory of computation using normal (almost normal) surfaces. However, these computations are useful only if there exist interesting surfaces with normal or almost normal representatives. In most situations, this is the case. If $M$ is an irreducible 3–manifold then every essential surface has a normal representative in any triangulation of $M$, and every strongly irreducible Heegaard surface has an almost normal representative in any triangulation of $M$.

Unfortunately, when $M$ is a reducible manifold it may be necessary to alter an essential surface before finding a normal representative. Suppose $S$ is a surface properly embedded in the 3–manifold $M$ and $D'$ is a disk embedded in $M$ with $D' \cap S = \partial D'$. Furthermore, suppose $\partial D'$ bounds a disk $D \subset S$. Then $S' = (S \setminus D) \cup D'$ is a surface topologically equivalent to $S$. We say $S'$ is obtained from $S$ by a disk-swap. The two surfaces $S$ and $S'$ are said to be equivalent (in $M$) if and only if there is a sequence $S = S_1, \ldots, S_n = S'$ with $S = S_1$ and $S' = S_n$ where $S_{i+1}$ is obtained from $S_i$ by a disk swap and/or isotopy. Hence, if two surfaces $S$ and $S'$ are isotopic, then they are equivalent. Equivalent and isotopic are the same when the ambient manifold, $M$, is irreducible. The concept of “disk-swapping” applies to $\partial$-compressing disks as well and is a necessary extension of this concept in the case that the manifold $M$ has boundary and the surfaces in question are $\partial$–incompressible. Note that any surface that is equivalent to an incompressible and $\partial$–incompressible surface is also incompressible and $\partial$–incompressible.

Let $S$ be a normal (or almost normal) surface in $(M, T)$. Then $S$ is least weight if every normal (almost normal) surface $S'$ that is equivalent to $S$ in $M$, we have that $wt(S) \leq wt(S')$.

We now list the existence results mentioned above. The first is known from the work of H. Kneser [21].

2.1. Theorem. Let $M$ be a 3–manifold. If there is a 2–sphere embedded in $M$ that does not bound a 3–cell in $M$, then for any triangulation of $M$ there is a normal 2–sphere embedded in $M$ that does not bound a 3–cell in $M$.

The next theorem is from the work of W. Haken [9].

2.2. Theorem. Let $M$ be a 3–manifold. If there is an incompressible and $\partial$–incompressible surface $S'$ embedded in $M$, then for any triangulation of $M$ there is a normal surface $S'$ embedded in $M$ that is equivalent to $S$. 
Another reference where details can be found for the proofs of these results is [14]. Finally, we note M. Stocking’s result [25] for Heegaard surfaces.

2.3. Theorem. Let $M$ be an irreducible 3–manifold. If there is a non-trivial genus $g$ strongly irreducible Heegaard splitting of $M$, then for any triangulation of $M$ there is an almost normal genus $g$ surface isotopic to the Heegaard surface.

3. One-vertex triangulations and boundary slopes

3.1. One-vertex triangulations. Computations in normal curve and normal surface theory can often be simplified by selecting a special triangulation, in particular, by choosing a triangulation with a minimal number of top dimensional simplices. For surfaces with non-positive Euler Characteristic, a minimum triangulation (a triangulation with the minimal number of faces) requires a one-vertex triangulation, a triangulation of the surface having just one vertex; and while not so obvious, a minimum triangulation of a closed, orientable 3–manifold (triangulation with the minimal number of tetrahedra) requires a one-vertex triangulation, except for $S^3$, and the lens spaces $RP^3$ and $L(3,1)$. It turns out, however, that by using one-vertex triangulations we not only have the computational benefits but also can draw many topological conclusions from their nice combinatorial properties.

3.1. Theorem. Every closed surface with $\chi \leq 0$ admits a one-vertex triangulation.

For example, any closed, orientable surface with genus $g \geq 1$ is the quotient of a 4g-gon in the plane, formed by identifying edges in a way to give only one vertex. We can triangulate the 4g-gon by adding no additional vertices and $4g - 3$ edges. This induces a triangulation of the genus g surface with one vertex, $6g - 3$ edges and $4g - 2$ faces. The same construction also works for closed, non-orientable surfaces with $\chi \leq 0$.

For 3-manifolds, it is not as easy to show that they admit one-vertex triangulations and not as obvious (Euler characteristic arguments do not work) to show that with the exceptions noted above, a minimum triangulation must be a one-vertex triangulation. We have the following result from [16].

3.2. Theorem. Every closed, orientable 3-manifold admits a one-vertex triangulation. Furthermore, a compact, orientable 3-manifold with non-empty boundary, no component of which is a 2-sphere, admits a triangulation having all its vertices in the boundary and precisely one vertex in each boundary component.

There is a simpler version of this result that is satisfactory for most of our work.

3.3. Theorem. Given a triangulation $T$ of a compact, orientable 3-manifold with non-empty boundary, no component of which is a 2-sphere, then $T$ can
be modified to a triangulation $T'$ where $T'$ has precisely one vertex in each boundary component.

From the previous theorem, we see that each compact, orientable 3-manifold (no boundary component a 2-sphere) admits a triangulation that restricts to a one-vertex triangulation on each boundary component. We shall exploit this and especially use such triangulations for our study of knot-manifolds and Dehn fillings.

3.2. Normal curves in a one-vertex triangulation of a torus. In particular, we rely on some particularly nice properties of the space of normal curves in a one-vertex triangulation of a torus, $S^1 \times S^1$. We will assume that our knot-manifold has such a triangulation of its boundary and this will simplify computations involving properly embedded surfaces.

Pictured in Figure 4 is the one-vertex triangulation of a torus; the Euler characteristic of the torus determines that it has 2 triangles and three edges. Note that we refer to ‘the’ one-vertex triangulation; any other one-vertex triangulation of the torus is combinatorially equivalent to this one. The three edges are essential curves which meet in a single point. Any other triangulation also has three edges which meet in a single point and we can choose a homeomorphism of the torus mapping the edges of the new triangulation, hence the triangulation itself, to the triangulation of Figure 4.

Among the nice properties is that there is a 1-1 identification between normal curves and isotopy classes of curves on the torus.

3.4. Lemma. In the one-vertex triangulation of a torus every trivial normal curve is vertex-linking.

Proof. A trivial normal curve $C$ bounds a disk $D$ on the torus. Consider the intersection of this disk with the one-skeleton of the triangulation. If there is an arc of intersection which is not incident in $D$ to the vertex then it splits $D$ into two pieces, at least one of which does not contain the vertex. An outermost arc of intersection with this subdisk demonstrates that the trivial curve $C$ is not normal. Therefore the intersection is a collection of
arcs each of which is incident in $D$ to the vertex. This describes a vertex linking trivial curve.

### 3.5. Lemma

In a one-vertex triangulation of the torus two normal curves are normally isotopic if and only if they are isotopic.

**Proof.** It suffices to consider the case where the two normal curves $C_1$ and $C_2$ are connected, essential, in general position with respect to each other, and have been normally isotoped to intersect minimally. If $C_1$ and $C_2$ are disjoint then they cobound 2 annuli on the torus. One annulus, call it $A$, does not contain the vertex. As the boundary of $A$ consists of the normal curves $C_1$ and $C_2$, each edge of the triangulation must intersect $A$ in arcs running from $C_1$ to $C_2$. Thus, the edges meet $A$ in a parallel collection of such arcs and we may use $A$ to perform a normal isotopy of $C_1$ to $C_2$.

When the curves do intersect, there must be at least two bigons on the torus which are bounded by subarcs of $C_1$ and $C_2$. One of these bigons does not contain a vertex and an innermost such bounds a disk in which all edges of the triangulation intersect in arcs joining $C_1$ to $C_2$. We can use the bigon to construct a normal isotopy reducing the number of intersections between $C_1$ and $C_2$.

**Remark.** While Lemma 3.4 remains true for one-vertex triangulations of any surface, Lemma 3.5 is never true in a one-vertex triangulation of a surface of genus $\geq 2$. For example, each separating curve on such a surface possesses at least two distinct normal representatives, determined by the side of the curve to which the vertex lies.

Recall that the isotopy class of an essential simple closed curve on the torus is called a *slope* on the torus. If $C \subset S^1 \times S^1$ is a collection of pairwise disjoint curves with at least one non-trivial component, then the *slope* of $C$, denoted $\text{slope}(C)$, is the slope of one of the non-trivial components. By the preceding lemma, when using a one-vertex triangulation, it is equivalent to define slope as the normal isotopy class of an essential simple closed curve.

In the one-vertex triangulation $\mathcal{T}$ of a torus there are six normal arc types yielding variables, so the solution space and projective solution space are embedded in six-dimensional space, $\mathbb{R}^6$. However, in computing these spaces the system reduces to one with only three degrees of freedom and it becomes more natural to think of the solution space and the projective solution space as being embedded in $\mathbb{R}^3$. We will denote these representations of the solution and projective solution spaces by $\mathcal{S}_T \subset \mathbb{R}^3$, and $\mathcal{P}_T \subset \mathbb{R}^3$, respectively.

### 3.6. Theorem

Normal curves in a one-vertex triangulation $\mathcal{T}$ of a torus are projectively parameterized by $\mathcal{P}_T$, the set of rational points in the 2-simplex

$$\{(x_1, x_2, x_3) | x_1 + x_2 + x_3 = 1, x_i \geq 0\} \subset \mathbb{R}^3.$$

The vertices of this simplex represent the projective classes of the 3 edges of $\mathcal{T}$. 
Figure 5. Normal arcs in the one-vertex triangulation of the torus $S^1 \times S^1$.

Proof. Any normal curve in the one-vertex triangulation of the torus will meet the two simplices of $\mathcal{T}$ in a collection of normal arcs from the six types labeled $a_i$ in Figure 3.

Therefore, a normal curve can be identified by a point $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6, x_i \geq 0$, where $x_i$ denotes the number of arcs of the given type $a_i$. Furthermore, for each of the three edges, each of the two triangles must have the same number arcs which intersect that edge. This yields three matching equations (see Figure 5)

\begin{align*}
    x_1 + x_2 &= x_4 + x_5, \text{ along edge } e_3, \\
    x_1 + x_3 &= x_4 + x_6, \text{ along edge } e_2, \\
    x_2 + x_3 &= x_5 + x_6, \text{ along edge } e_1,
\end{align*}

which reduce to

\begin{align*}
    x_1 &= x_4 \\
    x_2 &= x_5 \\
    x_3 &= x_6.
\end{align*}

The solution space

$$S(S^1 \times S^1, \mathcal{T})$$

is the set of points with non-negative integer coordinates in the cone

\begin{equation*}
\{(x_1, x_2, x_3, x_4, x_5, x_6) | x_i \geq 0, x_1 = x_4, x_2 = x_5, x_3 = x_6 \} \subset \mathbb{R}^6.
\end{equation*}

However, it is more natural to forget about the coordinates $x_4, x_5$ and $x_6$ and represent the solution space by $S_\mathcal{T}$ the set of points with non-negative integer coordinates

\begin{equation*}
\{(x_1, x_2, x_3) | x_i \geq 0 \} \subset \mathbb{R}^3.
\end{equation*}

It can be seen from Figure 5 that the solutions $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ are the normal coordinates of the isotopy classes of the three edges, $e_1, e_2, e_3$, respectively, of the triangulation. Moreover, every solution to the normal equations can be written as a linear combination of these three solutions.
using non-negative integer coefficients, so they are the set of fundamental solutions of $S_T$.

We projectivize the solution space $S_T$ by adding the normalizing equation

$$x_1 + x_2 + x_3 = 1.$$ 

Any solution to the normal equations will have a unique projective representation as a triple of non-negative rational numbers. The resulting projective space $P_T$ is the set of points in

$$\{(x_1, x_2, x_3) | x_i \geq 0, x_1 + x_2 + x_3 = 1\} \subset \mathbb{R}^3.$$ 

See Figure 6.

Thus, the normal curves in $(S^1 \times S^1, T)$ are projectively parameterized by the set of rational points in the 2-simplex in $\mathbb{R}^3$ spanned by $(1,0,0), (0,1,0)$ and $(0,0,1)$. The vertices are the projective classes of each of the fundamental solutions, the three edges of the triangulation.

In the remaining discussion we will refer to a normal curve by its representation in $S_T$ or projective representation in $P_T$. A normal curve will be called Type I if its $x_1$ coordinate is less than or equal to each of its $x_2$ and $x_3$ coordinates. Type II, and Type III are defined analogously. See Figure 6.

Note that a curve may be of more than one type. For example, a collection of trivial curves is simultaneously all three types, and normal representatives of the edges $e_1, e_2, e_3$ are two types. If $C$ is a family of normal curves then we will let $\tau(C)$ denote the number of trivial curves in $C$. Two slopes, $\alpha$ and $\beta$, will be said to be complementary if $\alpha + \beta$ is a collection of trivial curves.

We now state without proof some useful, elementary facts about normal curves in a one-vertex triangulation of a torus.

1. The set of slopes on the torus is projectively represented by the points in the boundary of the projective space $P_T \subset \mathbb{R}^3$.
2. If the normal curve $C$ has representation the triple $(x_1, x_2, x_3) \in S_T$ then $\tau(C) = \min\{x_1, x_2, x_3\}$. 

\[
\begin{array}{c}
(0,1,0) \\
I \\
(0,0,1) \\
II \\
(1,0,0) \\
III
\end{array}
\]

\text{Figure 6. The projective solution space $P_T$.}
3. The projective class of a collection of trivial curves is the barycenter 
\((1/3,1/3,1/3) \in \mathcal{P}_T\).
4. If \(C_1\) and \(C_2\) are normal curves which are not the same type then 
\(\tau(C_1 + C_2) > \tau(C_1) + \tau(C_2)\). If \(C_1\) and \(C_2\) are normal curves which
are the same type then \(\tau(C_1 + C_2) = \tau(C_1) + \tau(C_2)\).
5. If \(C\) is a normal curve with projective class \(\bar{C}\) then the slope of \(C\) is
 determined by projecting the point \(\bar{C}\) from the barycenter to the 
boundary of \(\mathcal{P}_T\). (Figure 7.)
6. The slopes \(\alpha\) and \(\beta\) are complementary if and only if for any curve \(C_\alpha\) 
with slope \(\alpha\) and any curve \(C_\beta\) with slope \(\beta\), the line segment in \(\mathcal{P}_T\) 
connecting \(\bar{C}_\alpha\) and \(\bar{C}_\beta\) passes through the barycenter \((1/3,1/3,1/3)\).
Thus each slope has a unique complement.
7. Suppose \(C\) is a normal curve with parameterization \((x_1,x_2,x_3) \in S_T\).
If \(\mu(C) = \max\{x_1,x_2,x_3\}\) and \(\tau(C) = \min\{x_1,x_2,x_3\}\), then \(\text{slope}(C)\)
has projective class that of \((x_1 - \tau(C),x_2 - \tau(C),x_3 - \tau(C))\) and the
slope complementary to \(\text{slope}(C)\) has projective class that of \((\mu(C) - x_1,\mu(C) - x_2,\mu(C) - x_3)\).

3.3. Boundary slopes. In [10] Hatcher used the theory of incompressible 
branched surfaces developed by Floyd and Oertel [3] to show that the slopes
bounding incompressible and \(\partial\)-incompressible surfaces in a knot-manifold
are finite in number. Here we adapt Hatcher’s argument to normal surfaces
in a one-vertex triangulation and show that the result holds more generally
for the slopes bounding normal and almost normal surfaces; hence, our
results imply Hatcher’s result for incompressible and \(\partial\)-incompressible surfaces
as well.

3.7. Proposition. Let \(M\) be an orientable 3-manifold having a boundary
component a torus, \(T\), and let \(T\) be a triangulation of \(M\) that restricts
to a one-vertex triangulation of \(T\). Suppose that \(S_1\) and \(S_2\) are embedded
normal or almost normal surfaces and \(\partial S_1 \subset T\). If \(S_1\) and \(S_2\) are compatible
and both meet \(T\) in non-trivial slopes, then these slopes are either equal or
complementary.
Proof. Let $T'$ denote the induced one-vertex triangulation of the boundary torus $T$. We proceed in two steps. First, we show that if the slopes of the surfaces $S_1$ and $S_2$ are the same type in the one-vertex triangulation $T'$ of the boundary torus $T$ then they have the same slope in $T$. Next, we show that if they have different types in the triangulated torus $T'$ then their slopes in $T$ are complementary in $T'$.

So we first assume that $S_1$ and $S_2$ are compatible surfaces which intersect non-trivially and $\partial S_1 \cap T$ and $\partial S_2 \cap T$ are of the same type in $T'$, say type II. First perform a normal isotopy of $S_1$ and $S_2$ so that all of the trivial curves of $\partial S_1 \cap T$ and $\partial S_2 \cap T$ are disjoint from all other curves. All remaining intersections on $T$ between $S_1$ and $S_2$ lie on the non-trivial components of $\partial S_1 \cap T$ and $\partial S_2 \cap T$; denote these by $C_1$ and $C_2$, respectively, and let $(x_1, 0, x_3)$ and $(y_1, 0, y_3)$ denote their respective normal coordinates in $T'$.

The normal sum $S_1 + S_2$ restricts to a normal sum of the boundary curves $\partial S_1 + \partial S_2$, hence to a sum of the essential boundary curves in $T$, $C_1 + C_2$. The normal curves $C_1, C_2$ and $C_1 + C_2$ can be given an orientation by orienting the normal arcs $a_1, a_3, a_4$ and $a_6$ as indicated in Figure 8. (Here we are using that $x_2 = y_2 = 0$.)

Consider an intersection between the normal curves $C_1$ and $C_2$ that lies along an intersection of normal arcs of type $a_1 \cap a_1$, $a_1 \cap a_3$, or $a_3 \cap a_3$. The regular switch performed at such an intersection is the switch that follows the given orientation of the normal arcs, see Figure 4. It is easily verified that this is also true for intersections of type, $a_4 \cap a_4$, $a_4 \cap a_6$ and $a_6 \cap a_6$, i.e. for all intersections between $S_1$ and $S_2$ on the boundary component $T$.

If $a$ is an arc of intersection between the surfaces $S_1$ and $S_2$, both of its endpoints are in $C_1 \cap C_2$ ($\partial S_1 \subset T$). At each endpoint we know that the regular switch along $a$ follows the given orientation on the boundary curves. In Figure 4, we follow the regular switch along $a$ through the interior of the orientable manifold $M$ and see that the two endpoints of $a$ are intersections between $C_1$ and $C_2$ with opposite algebraic sign. If we consider all of the arcs of intersection, hence all points in $C_1 \cap C_2$, we see that the algebraic
intersection between $C_1$ and $C_2$ sums to 0. Since $T$ is a torus the surfaces $S_1$ and $S_2$ have the same slope on $T$.

We now assume that $S_1$ and $S_2$ are compatible, intersect non-trivially on $T$ and do not have the same type in the triangulated torus $T'$. As $S_1$ and $S_2$ are compatible each member of the collection $\{n_1S_1+n_2S_2 : n_1, n_2 \geq 0\}$ is an embedded normal surface contained in the same compatibility class and with non-empty boundary in $T$. Representing this collection of surfaces by their normal boundaries in $T'$ and projectivizing, we obtain the set of rational points on the segment joining $\partial S_1$ and $\partial S_2$ in $P_{T'}$, where the endpoints have different types. We can choose some surface $S'_1 = n'_1S_1 + n'_2S_2$ so that $\partial S'_1 \cap T$ has the same type as $\partial S_1 \cap T$. See Figure 11.

By the first step of the proof, the slopes of $S_1$ and $S'_1$ on $T$ are identical. This implies that the segment joining $S_1$ and $S_2$ passes through the point $(1/3, 1/3, 1/3) \subset P_{T'}$ and in particular means that the slopes of $S_1$ and $S_2$ on $T$ are complementary. (Recall the elementary facts 5 and 6). \qed
Remark. In Section 4 we will give an example of compatible normal surfaces with complementary slopes; hence it is necessary to include the two possibilities unlike the situation of [10]. Our methods include more general surfaces, normal and almost normal as opposed to incompressible and $\partial$-incompressible surfaces, and hence the corresponding branched surfaces may have monogons and bigons in their boundaries.

Since a normal or almost normal surface $S$ is compatible with all surfaces in its carrier we obtain the following.

3.8. Corollary. Let $X$ be a knot-manifold with a triangulation $\mathcal{T}$ that restricts to a one-vertex triangulation on $\partial X$. Suppose $S$ is an embedded normal or almost normal surface and $\partial S \neq \emptyset$. There are at most two slopes (complementary ones) for all surfaces in the carrier of $S$, $\mathcal{C}(S) \subset \mathcal{P}(X, \mathcal{T})$.

We note that if $S$ has no non-trivial boundary components, then every surface in the carrier of $S$ has the same slope as $S$, this is the case for $S$ incompressible. Now, if $S$ is a normal or almost normal surface then some multiple of $S$ can be written as a sum of embedded surfaces represented at the vertices in the carrier of $S$,

$$kS = \sum k_i V_i.$$  

Hence, in a knot-manifold, there can be at most two distinct boundary slopes for these summands, from which the slope of $S$ is inherited.

3.9. Corollary. Let $X$ be a knot-manifold with a triangulation $\mathcal{T}$ that restricts to a one-vertex triangulation on $\partial X$. All possible slopes for the boundaries of embedded normal or almost normal surfaces in $X$ are realized by the slopes of embedded surfaces represented at the vertices of $\mathcal{P}(X, \mathcal{T})$.

3.10. Corollary. Let $X$ be a knot-manifold with a triangulation $\mathcal{T}$ that restricts to a one-vertex triangulation on $\partial X$. Then there are only a finite number of slopes realized as the slopes of normal and almost normal surfaces in $(X, \mathcal{T})$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure11.png}
\caption{Compatible surfaces of different types are complementary.}
\end{figure}
The number of tetrahedra, \( t \), in the triangulation \( \mathcal{T} \) yields a very rough upper bound on the number of slopes bounding normal and almost normal surfaces. Let \( \{S_1, \ldots, S_n\} \) be a maximal collection of normal surfaces with distinct slopes. There is a sub-collection with at least \( n/2 \) surfaces, no two of which are compatible. For each pair of these \( n/2 \) surfaces there is a tetrahedron in which they possess distinct quadrilateral types. In the worst case possible, each surface in the sub-collection possess a quadrilateral in each tetrahedron, implying that \( n/2 \leq 3^t \). Thus, \( 2(3^t) \) is an upper bound on the number of slopes bounding normal surfaces.

A similar computation works for almost normal surfaces. Let \( \{S_1, \ldots, S_n\} \) be a maximal collection of almost normal surfaces with distinct slopes. If \( S_i \) possesses a tube in some tetrahedron, then we may compress the tube to obtain a normal surface with the same slope. There is a sub-collection of at least \( n/2 \) surfaces, no pair of which are compatible. In the worst case each surface possesses an octagon in one tetrahedron and quadrilaterals in all others. There are \( 3t \) choices for the octagon and in each of the remaining \( t - 1 \) tetrahedra we choose a quadrilateral. Therefore, \( n/2 \leq 3^t - t^{-1} \) and there are at most \( 2t^3 \) slopes of almost normal surfaces.

We have already noted that every knot-manifold possesses a one-vertex triangulation. The slope of every incompressible and \( \partial \)-incompressible surface is realized by the slope of a normal surface, surface (see [18] and Section 5). This implies Hatcher’s theorem [11].

3.11. Corollary. Let \( X \) be a knot-manifold. Then there are a finite number of slopes bounding incompressible and \( \partial \)-incompressible surfaces in \( X \).

4. Layered triangulations of the solid torus

In this section we give a method, also used in [15] and [16], for extending a one-vertex triangulation of a knot-manifold \( X \) to that of a manifold \( X(\alpha) \) obtained by Dehn filling. This is accomplished by showing that a one-vertex triangulation on the boundary of a solid torus can be extended to a special one-vertex triangulation of the solid torus (Theorem 4.1).

The special one-vertex triangulations referred to above are layered triangulations of solid tori. We are able to give a classification of the embedded, planar, normal surfaces in a layered triangulation of a solid torus (Proposition 4.2), which will be of use in Section 5. The classification is given in terms of the three types of normal surfaces, defined as follows:

1. \( D_\mu \) will designate any normal disk with essential boundary, i.e., a meridional disk for the solid torus.
2. \( D_\tau \) will designate any normal disk with trivial boundary, i.e., a disk which is parallel into the boundary of the solid torus.
3. \( A_\alpha \) will designate any annulus with essential boundary which is parallel into an annulus in the boundary, so that the parallel annulus in the boundary contains the vertex of the triangulation.
Consider the tetrahedron $\Delta$ pictured in Figure 12. Glue the back two faces of the tetrahedron together by making the ordered identification $\langle a, b, c \rangle \sim \langle b, c, d \rangle$. All four vertices are identified to a single vertex and the induced edge identifications are indicated in the figure. It is easy to check that $M = \Delta / \sim$ is a manifold with a single torus boundary component, where the boundary torus has a one-vertex triangulation consisting of the two front faces of the tetrahedron. The normal surface consisting of two triangles cutting of the vertices $\langle a \rangle$ and $\langle d \rangle$ along with the quadrilateral which separates the edges $\langle a, b \rangle$ and $\langle c, d \rangle$ is a properly embedded disk, $D$. See Figure 13(2). Moreover, after cutting along the disk $D$ the resulting manifold $M - N(D)$ is a ball. We have therefore constructed a triangulation of a solid torus with one tetrahedron, three faces, three edges (each contained in the boundary), and a single vertex. This triangulation of the solid torus will be referred to as the one-tetrahedron solid torus. Any other triangulation of a solid torus with a one-tetrahedron is combinatorially equivalent to this one: such a triangulation must be obtained by gluing two (adjacent) faces of a tetrahedron together with an orientation-reversing identification; the ordered identification $\langle a, b, c \rangle \sim \langle c, d, b \rangle$ is equivalent by relabelling $\Delta$, $\langle b \rangle \leftrightarrow \langle d \rangle$; and the ordered identification $\langle a, b, c \rangle \sim \langle d, b, c \rangle$ forms a triangulation of the ball.

The connected normal surfaces contained within the one-tetrahedron solid torus are determined by their quadrilateral type (or lack thereof). Pictured in Figure 13 are all of the connected normal surfaces which can be properly embedded in the one-tetrahedron solid torus. They are:

1. A disk of type $D_\tau$ with boundary the trivial curve $(1, 1, 1)$.
2. A disk of type $D_\mu$ with boundary $(2, 0, 1)$.
3. An annulus of type $A_\alpha$ with boundary $(0, 2, 0)$.
4. A Möbius band with boundary $(0, 0, 1)$.
5. An annulus of type $A_\alpha$ which is the double of the Möbius band. It has boundary $(0, 0, 2)$.

If $\mathcal{T}$ is a one-vertex triangulation of a solid torus then the boundary torus has a one-vertex triangulation. Any one of the three edges $e$ in this triangulation may be thought of as the diagonal of the rectangle bounded

---

**Figure 12.** The one-tetrahedron solid torus.
by the other two edges. We can change the boundary triangulation to a new one by exchanging $e$ for $e'$, the other diagonal of the rectangle, this is known as a Type I Pachner move. Fortunately, we can realize the Type I Pachner move by gluing an additional tetrahedron $\Delta$ to the boundary of $\mathcal{T}$. See Figure 14 with $e = e_2$.

Glue the edge $e$ in the boundary torus to an edge $e'$ of a disjoint tetrahedron $\Delta$. In addition glue the two faces on the boundary torus that are adjacent to $e$ to the faces adjacent to $e$ on $\Delta$. The result is a one-vertex triangulation of a solid torus with the boundary changed by a Type I Pachner move.

This move on a triangulation of a solid torus will be called layering at the edge $e$ and we denote the new triangulation by $\mathcal{T}' = \mathcal{T} \cup_e \Delta$. Inductively, define a layered triangulation of a solid torus with $t$ layers, $\mathcal{T}_t$, to be any triangulation of a solid torus so that,

1. $\mathcal{T}_1 = \mathcal{T}$, a one-tetrahedron solid torus,
2. $\mathcal{T}_t = \mathcal{T}_{t-1} \cup_e \Delta_t, t \geq 2$, a layering at $e$ of a layered triangulation with $t - 1$ layers.

Note that layering a solid torus has the effect of covering the boundary edge $e$ and adding a new boundary edge $e'$. Thus $\mathcal{T}_t$ will possess one vertex in the boundary torus and $t + 2$ edges, 3 contained in the boundary torus.

We now give a theorem of [16], that layered triangulations are general enough to perform an arbitrary Dehn filling.

Figure 13. The normal surfaces in a one-tetrahedron solid torus.
4.1. Theorem. Suppose $\mathcal{T}$ is a one-vertex triangulation of the torus $S^1 \times S^1$. For any slope $\mu$ on $\mathcal{T}$ there is an algorithm to extend $\mathcal{T}$ to a layered triangulation of a solid torus in which $\mu$ bounds a meridional disk. Furthermore, for any positive integer $N$ there is such a layered triangulation with greater than $N$ tetrahedra.

Proof. We construct the layered triangulation in reverse order, layering tetrahedra on the prescribed boundary, altering the normal representative of $\mu$ until it is a $(2, 0, 1)$ curve. This curve (perhaps after relabelling) bounds a meridional disk in the one tetrahedron solid torus, so we may glue our layers to the one tetrahedron solid torus in reverse order and obtain the desired layered triangulation.

We will keep track of $\mu$ by its *intersection numbers*, a triple which indicates the number of intersections between $\mu$ and each of the three edges of the triangulation of the boundary torus,

$$[y_1, y_2, y_3] = [\#(\mu \cap e_1), \#(\mu \cap e_2), \#(\mu \cap e_3)].$$

Note that these are *not* the normal coordinates of $\mu$ in the solution space $\mathcal{S}_\mathcal{T}$ (see Section 3). We may convert from normal coordinates to intersection numbers as follows

$$[y_1, y_2, y_3] = [x_2 + x_3, x_1 + x_3, x_1 + x_2].$$

The length of a normal curve, $L(\mu)$, is the sum of its intersection numbers $L(\mu) = y_1 + y_2 + y_3 = 2x_1 + 2x_2 + 2x_3$, an even number.

The slope $\mu$ is uniquely represented by a normal curve in the triangulation of the torus. Attaching a layer at the edge $e_2$, is equivalent to a Type $I$
Pachner move, replacing the set of edges \{e_1, e_2, e_3\} by the edges \{e_1', e_2', e_3\}. Choose an orientation on the torus and \(\mu\) and orient \(e_1\) and \(e_2\) so that the oriented intersection numbers \(<\mu, e_1> = y_1\) and \(<\mu, e_3> = y_3\). The edges \(e_1\) and \(e_3\) are a basis for the homology of the boundary torus and the edge \(e_2\) intersects each once, so we may orient \(e_2\) so that either \(e_2' = e_1 + e_3\) or \(e_2 = e_1 - e_3\) with respect to homology. Thus, \(y_2 = y_1 + y_3\) or \(y_2 = |y_1 - y_3|\). Then \(e_2'\) can be oriented so that \(e_2' = e_1 - e_3\) or \(e_2' = e_1 + e_3\), respectively, and then \(y_2' = |y_1 - y_3|\) or \(y_2' = y_1 + y_3\), respectively. Thus, layering a tetrahedron on the boundary at \(e_2\) changes the intersection numbers of \(\mu\),

\[
[y_1, y_1 + y_3, y_3] \leftrightarrow [y_1, |y_1 - y_3|, y_3],
\]

with the direction of the map determined by whether \(y_2 = y_1 + y_3\) or \(y_2 = |y_1 - y_3|\).

Layering tetrahedra at the other edges alters the corresponding intersection coordinate in precisely the same manner. By attaching to the edge with highest intersection coordinate, \(L(\mu)\) will strictly decrease, unless with respect to some ordering of the edges, \(y_1 + y_3 = |y_1 - y_3|\), i.e. one intersection coordinate is zero. This means that \(\mu\) is in fact disjoint from some edge and is therefore the normal representative of that edge. With respect to some ordering of the edges, \(\mu\) has intersection numbers \([0, 1, 1]\) and \(L(\mu) = 2\).

If we ever have that \(L(\mu) = 6\) then the intersection triple of \(\mu\) is \([1, 3, 2]\), up to ordering. In normal coordinates this is \((2, 0, 1)\), the curve that bounds a meridional disk in the one-tetrahedron solid torus. We may choose an ordering of the edges so that we are able to glue the layered tetrahedra, in reverse order, to the one-tetrahedron solid torus and obtain a layered triangulation of the solid torus in which \(\mu\) bounds a meridional disk.

If the original length is \(L(\mu) = 4\) then the intersection triple for \(\mu\) is \([2, 1, 1]\) after a choice of edges. By layering a tetrahedron at either \(e_2\) or \(e_3\), assuming this ordering) we obtain the triple \([2, 3, 1]\). As noted in the previous paragraph we may then attach the one-tetrahedron solid torus. (Assuming this ordering, layering at \(e_1\) lowers \(L(\mu)\).) If the original length is \(L(\mu) = 2\) then its intersection triple is \([0, 1, 1]\), up to ordering. In this ordering, by layering at the first edge we change the intersection triple \([0, 1, 1]\) to \([2, 1, 1]\). (Layering at the edges with intersection values \(1\) only changes the ordering.) We then add one more layer as in the previous case.

If the original length \(L(\mu) > 6\) then we may layer a sequence of tetrahedra from the boundary, always attached to the edge with highest intersection coordinate. Continue this process, strictly decreasing \(L(\mu)\) until \(L(\mu) \leq 6\). In fact, this process must terminate with \(L(\mu) = 6\). Because \(L(\mu)\) is strictly decreasing, and by the remarks of the previous paragraph, if the process terminates at \(L(\mu) = 2\) or \(L(\mu) = 4\) then \(L(\mu) = 6\) was a previous step.

It is easy to obtain such a triangulation with an arbitrary number of tetrahedra. First layer \(N\) tetrahedra on the boundary torus in any fashion (keeping track of \(\mu\)). Then, as specified above, layer tetrahedra which
reduce the length of $\mu$ until the boundary can be capped off with the one-tetrahedron torus. A one-vertex triangulation of a solid torus with at least $N + 1$ tetrahedra is obtained.

In Section 5 it will be necessary for us to understand the normal surfaces that can be embedded in a layered triangulation of a solid torus $T_t = T_{t-1} \cup e \Delta_t$. Layering identifies two faces of $\Delta_t$ with the boundary of $T_{t-1}$ and leaves the other two faces of $\Delta_t$ as the new boundary torus. The one-tetrahedron solid torus has no closed normal surfaces and each elementary disk type in $\Delta_t$ meets both the old boundary and the new boundary. It follows that in a layered triangulation of a solid torus, there can be no closed normal surfaces and each normal surface intersects each tetrahedron.

So if $P_t \subset T_t$ is a normal surface, then it was obtained by attaching a non-empty collection of elementary disks in $\Delta_t$ to a normal surface $P_{t-1} \subset T_{t-1}$. Call the elementary quadrilateral type in $\Delta_t$ which separates the attaching edge $e$ and the new edge $e'$ the banding quad in $\Delta_t$. See Figure 16. The boundary of $P_{t-1}$ and the number of banding quads attached in $\Delta_t$ completely determine the number of each of the other disk types that are used in $\Delta_t$. In particular, if no banding quads are attached then the surface $P_t$ is homeomorphic to the surface $P_{t-1}$, and we say that $P_t$ was obtained by pushing $P_{t-1}$ through $\Delta_t$. See Figure 15.

We are particularly interested in the planar normal surfaces embedded in $T_t$. The one-tetrahedron solid torus $T_1$ contains a unique surface of type $D_\mu$, a unique surface of type $D_\tau$ and two distinct surfaces of type $A_\alpha$. How can new normal planar surfaces be obtained by the process of layering? We list some (all, by Proposition 4.2) ways to construct new planar surfaces $P_t \subset T_t$ from planar surfaces $P_{t-1} \subset T_{t-1}$:

1. In any layer, each surface of type $D_\mu, D_\tau$ or $A_\alpha$ in $T_{t-1}$ may be pushed through $\Delta_t$ to obtain a surface of the same type in $T_t$. See Figure 15.

2. In any layer $\Delta_t$ we may attach a banding quad and two elementary triangles to a surface of type $D_\tau \subset T_{t-1}$ to produce a surface of type
$A_\alpha \subset T_t$, see Figure 16. The band was attached along the edge $e'$ so this annulus is parallel to a neighborhood of the edge $e'$, an annulus in the boundary containing the vertex.

**Figure 16.** Banding a trivial disk $D_\tau$ to create an annulus $A_\alpha$.

3. If the attaching edge $e$ happens to be the slope of the meridional disk $D_\mu \subset T_{t-1}$, then attach a single banding quad and two triangles in $\Delta_t$ to 2 copies of $D_\mu$ to obtain a surface of type $D_\tau \subset T_t$. See Figure 17. (Banding two meridional disks in a solid torus produces a trivial disk.)

**Figure 17.** Banding 2 meridional disks $D_\mu$ to create a trivial disk $D_\tau$.

4. If the attaching edge $e$ happens to be the slope of the meridional disk $D_\mu \subset T_{t-1}$, then attach two banding quads to two copies of $D_\mu$ to obtain a surface of type $A_\alpha \subset T_t$. See Figure 18. This annulus is also parallel to a neighborhood of the edge $e'$, an annulus in the boundary containing the vertex.

In the proof of the following proposition we show that the above moves are sufficient to generate all normal planar surfaces in a layered solid torus.
4.2. Proposition. Let $\mathcal{T}_t$ be a layered triangulation of a solid torus $T$ with $t$ layers. Then the only connected planar normal surfaces which can be properly embedded in $(T, \mathcal{T}_t)$ are of type $D_\mu$, $D_\tau$, and $A_\alpha$. Furthermore, there is unique (up to normal isotopy) surface of type $D_\mu$. The weight of any surface of these types is bounded below by:

1. $\text{wt}(D_\mu) \geq t + 4$,
2. $\text{wt}(D_\tau) \geq 2(t + 2)$, and
3. $\text{wt}(A_\alpha) \geq 2(t + 1)$.

Proof. We first give an inductive proof that all normal planar surfaces in $\mathcal{T}_t$ are of type $D_\mu$, $D_\tau$ or $A_\alpha$. When $t = 1$ the triangulation $\mathcal{T}_1$ is the one-tetrahedron triangulation of the solid torus. The normal planar surfaces were shown in Figure 13, each is of type $D_\mu$, $D_\tau$ or $A_\alpha$. Assume that the result holds for any layered triangulation $\mathcal{T}_{t-1}$ with $t - 1$ tetrahedra, $t \geq 2$. We now show that the result also holds for any layered triangulation with $t$ tetrahedra, $\mathcal{T}_t = \mathcal{T}_{t-1} \cup e \Delta_t$. With no loss of generality we may assume that $e = e_2$.

Let $P_t$ be a connected normal planar surface in $\mathcal{T}_t$. Then $P_{t-1} = P_t \cap \mathcal{T}_{t-1}$ is a (possibly disconnected) planar normal surface in $\mathcal{T}_{t-1}$.

Claim. We may assume that every component of $P_{t-1}$ has a banding quad in $\Delta_t$ attached to it.

If any component of $P_{t-1}$ does not have a banding quad attached to it, then it is merely pushed through the tetrahedron $\Delta_t$ to a surface which is of precisely the same type in $\mathcal{T}_t$. This component satisfies the conclusion of the theorem and in particular is homeomorphic to $P_t$ ($P_t$ is connected). We therefore assume that every component of $P_{t-1}$ has a banding quad in $T$ attached to it.

A banding quad is a band along the edge $e'_2$ and joins a normal arc of $\partial P_{t-1}$ of type $a_2$ to a normal arc of type $a_5$. The number of arcs of type $a_2$
is equal to the number of type $a_5$ for any normal curve (recall that $x_2 = x_5$). Number the arcs of type $a_2$ with the numbers $1, \ldots, x_2$ counting from the vertex to the edge $e_2$, and number the arcs labeled $a_5$ from $1, \ldots, x_2$ also counting from the vertex to the edge $e_2$, see Figure 19. If a banding quad is attached to an arc of type $a_2$ labeled $i$ then all arcs of type $a_2$ with greater labels must also have banding quads attached, it is impossible to attach a normal triangle, see Figure 17. The same holds true for arcs of type $a_5$, and it follows that each banding quad joins an arc of type $a_2$ to an arc of type $a_5$ with the same label.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure19.png}
\caption{Numbering the arcs of type $a_2$ and $a_5$.}
\end{figure}

**Claim.** If some component of $P_{t-1}$ is of type $D_\tau$ then $P_t$ is an annulus $A_\alpha$.

We are assuming that there is a banding quad attached to the component of $P_{t-1}$ of type $D_\tau$. The boundary of this component consists of a single trivial curve, so it possesses one $a_2$ arc and one $a_5$ arc. Moreover they must have the same label as there can only be trivial curves between $\partial D_\tau$ and the vertex, and each trivial curve possesses an equal number of arcs of type $a_2$ and $a_5$. So the banding quad attached to $D_\tau$ joins $D_\tau$ to itself. This is move (2) described before the proposition, which creates an annulus $A_\alpha \subset T$, see Figure 16. Since $P_t$ is connected, $D_\tau$ was the only component of $P_{t-1}$.

**Claim.** If no component of $P_{t-1}$ is of type $D_\tau$ then $\partial P_{t-1}$ consists of essential curves parallel to the edge $e_2$.

We are assuming that $\partial P_{t-1}$ has a banding quad attached so it must possess a normal arc of type $a_2$ hence the coordinate $x_2 > 0$. Then $\partial P_{t-1}$ has no trivial curves: by the inductive hypothesis a trivial curve implies that $P_{t-1}$ contains a trivial disk $D_\tau$ and by the conclusions of the previous claim $P_{t-1}$ must itself be a trivial disk $D_\tau$. Therefore one of the two coordinates $x_1, x_3$ must be 0. No normal arc of $\partial P_{t-1}$ of type $a_1$ can be connected across $e_2$ to an arc of type $a_4$ (nor $a_3$ to $a_6$), for this would force a quad which is not the banding quad to be attached, which prohibits any banding quads from being attached and means that the surface was pushed through $\Delta_t$, see Figure 13. Therefore all arcs of type $a_1$ are connected across $e_2$ to those of type $a_6$ and $x_1 = x_6 (= x_3)$. Then both $x_1 = x_3 = 0$ and only $x_2 > 0$. The normal curve $\partial P_{t-1}$ consists entirely of curves parallel to the edge $e_2$. 
Claim. No component of $P_{t-1}$ is an annulus $A_\alpha$ (or $P_t$ has positive genus).

By our previous claim, any annulus $A_\alpha \subset P_{t-1}$ has boundary disjoint from $e_2$ and is parallel into the boundary annulus containing the vertex, i.e. parallel to a neighborhood of the edge $e_2$. Thus any collection of such annuli is nested, and we may choose an innermost annulus $A_\alpha$ with respect to the edge $e_2$. No component of $P_{t-1}$ is of type $D_\tau$ and any component of type $D_\mu$ cannot have boundary contained in the annulus in the boundary to which the $A_\alpha$’s are parallel. So the boundary of the innermost $A_\alpha$ is adjacent to the edge $e_2$, i.e. it has arcs of type $a_2$ and $a_5$ with label $x_2$. A banding quad is attached from $\partial A_\alpha$ to itself along these arcs yielding a once punctured torus. We may either attach two elementary triangles or a banding quad to the remaining boundary arcs of type $a_2$ and $a_5$ (Figure 17 or Figure 18). However, in either case the surface $P_t$ has positive genus, a contradiction.

We are left with the case that $P_{t-1}$ is a collection of meridional disks $D_\mu$.

Claim. $P_{t-1}$ is not a single copy of $D_\mu$ (for then $P_t$ would be a Möbius band).

If $P_{t-1}$ is a single copy of $D_\mu$ then the banding quad is glued from the single normal arc of $\partial P_{t-1}$ of type $a_2$ to that of type $a_5$. The surface produced has a single boundary component and $\chi = 0$, hence it is a Möbius band (with boundary $e'_2$).

Claim. If $P_{t-1}$ is 2 copies of $D_\mu$ and a single banding quad is attached then $P_t$ has type $D_\tau$. If $P_{t-1}$ is 2 copies of $D_\mu$ and 2 banding quads are attached then $P_t$ has type $A_\alpha$.

These cases are the moves (3) and (4) listed preceding the theorem.

Claim. $P_{t-1}$ does not contain more than 2 meridional disks $D_\mu$ (for then $P_t$ would be disconnected).

Let $D_1, \ldots, D_{x_2}, x_2 \geq 2$ be a collection of meridional disks $D_\mu$ numbered to induce our previous labeling of the arcs of type $a_2$. See Figure 19 and Figure 21. The boundary of $D_i$ is parallel to $e_2$ and consists of an arc of type $a_2$ labeled $i$ along with an arc of type $a_5$ labeled $x_2 - i + 1$. Since at least one banding quad is attached, there is necessarily a banding quad attached to the two arcs labeled $x_2$, this quad bands $D_{x_2}$ to $D_1$. There is either a banding quad attached to the two arcs labeled 1 or elementary triangles are added to each. In either event, $D_1$ and $D_{x_2}$ are attached to each other and to no other disk. Then there must be no other disks, for then $P_t$ would be disconnected.

Note that a surface of type $D_\mu$ was created only by pushing through each layer $\Delta_1$, a unique process. For a given layered solid torus $T_t$, there is a unique surface of type $D_\mu$.

So we have that a normal planar surface $P_t \subset T_t$ is one of the three types, $D_\mu, D_\tau$ and $A_\alpha$. We now obtain lower bounds on their weights. Let $P_t = P_t \cap T_t$. Typically $P_t$ meets each of the $t + 2$ edges of the triangulation.
However, there are three ways that a planar surface $P_t$ can miss an edge of the triangulation $T_t$:

1. $P_t$ does not intersect some edge $e$ in the core triangulation $T_1$. This happens only when the surface $P_1$ was one of the two annuli of type $A_\alpha \subset T_1$. It follows that $P_t$ is of type $A_\alpha$ and was obtained by pushing through every subsequent layer.

2. In some layer the surface $P_i$ is obtained by attaching a banding quad to the surface $P_{i-1}$ and $P_i$ misses the new edge. Then $P_i$ has the same slope as the new edge $e'$, and is therefore an annulus $A_\alpha$. (A trivial disk $D_\tau$ has trivial boundary which intersects the new edge, and the type $D_\mu$ cannot be created through banding.)

3. The surface $P_{i-1}$ is pushed through some layer $\Delta_i$ and the new surface $P_i$ misses the new edge $e'$. Then both $P_{i-1}$ and $P_i$ have slope $e'$ and are either copies of meridional disks $D_\mu$ or an annulus $A_\alpha$. Note that the edge $e$ which was covered by $\Delta_i$ intersects the slope of the new edge $e'$ twice, hence every boundary component of the surface $P_i$ intersects $e$ twice. Moreover, each edge $e'$ missed in this fashion determines a distinct edge $e$ that is covered. So although, the edge $e'$ is missed, an earlier edge $e$ makes up for the deficit and we may count the edge $e'$ as if it was intersected by each boundary component of $P_i$.

If $P_t$ is a meridional disk $D_\mu$ then it was obtained by pushing through each layer $\Delta_i$; every surface $P_i$ is also of type $D_\mu$. Every edge in $T_1$ is intersected by the original disk $P_1$. See Figure 13. If any layered edge $e'$ is missed then by (3) above, some earlier edge is intersected twice. We can therefore count 1 intersection for each edge. We may also count an extra two intersections because $P_1 = D_\mu$ hit edge $e_2$ three times, and we have only counted 1 (being hit three times means that it can not correspond to an edge buried by reason (3) above). We have, $wt(D_\mu) \geq t + 4$.

If $P_t$ is a trivial disk $D_\tau$, then each intermediate surface $P_i$ is either of type $D_\tau$ or 2 copies of $D_\mu$. In any event, any edge that is met is met twice. Both $D_\tau$ and $2D_\mu$ meet each edge of $T_1$. If any subsequent edge is missed, it is due to reason (3) listed above, and the surface $P_i$ is 2 copies of $D_\mu$. Each boundary curve of $2D_\mu$ intersects some earlier edge twice and we count 2 intersections for each edge of the triangulation, $wt(D_\tau) \geq 2(t + 2)$. 

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**Figure 20.** A collection of meridional disks.
Suppose that $P_t$ is type $A_\alpha$. If $P_1$ was an annulus then $P_t$ was obtained by pushing through each layer and each $P_i$ is of type $A_\alpha$. Then $P_1$ misses one of the initial edges of $\mathcal{T}_1$, and by reason (3) above if any subsequent edge is missed then an earlier edge was met twice. For each edge except one we count 2 intersections, one for each boundary component of $A_\alpha$, $\text{wt}(P_t) \geq 2(t+1)$. If some $P_i$ is a surface of type $D_\tau$ then its weight was computed in the previous paragraph. A subsequent edge can be missed only when the banding quad is attached, or, after the banding quad is attached and due to reason (3) above. Thus, we can count 2 for all but one of the subsequent edges, $\text{wt}(P_\tau) \geq 2(t+1)$. The final case is that $A_\alpha$ was obtained by attaching two bands to $2D_\mu$ in a single layer. In this case, the three initial edges are met twice each. Using (3) above, we count all subsequent edges for two intersections except for the edge corresponding to the bands attached, $\text{wt}(P_t) \geq 2(t+1)$. Regardless, of the construction we have the bound $\text{wt}(A_\alpha) \geq 2(t+1)$.

Our understanding of layered triangulations allows us to construct an example of compatible surface with complementary slopes.

4.3. Example. Consider the annulus of type $A_\alpha$ contained in $\mathcal{T}_1$ pictured in Figure 13(3); it is disjoint from the edge $e_2$ of the triangulation. Attach a new layer $\Delta_2$ at the edge $e_2$ and use triangles to push the annulus through $\Delta_2$ to obtain an annulus $A_1 \subset \mathcal{T}_2$, see Figure 21.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure21.png}
\caption{Annuli $A_1$ and $A_2$ in $\mathcal{T}_2$.}
\end{figure}

Construct another annulus, $A_2 \subset \mathcal{T}_2$, by taking $D_\tau \subset \mathcal{P}_1$ and attaching a banding quad, and two triangles in $\Delta_2$. These surfaces have distinct slopes, $\partial A_1 = (2,0,2)$ and $\partial A_2 = (0,2,0)$ in normal coordinates with respect to $e_1, e_2', e_3$. Yet, their quads are in different tetrahedra and the surfaces are thus compatible. Indeed, their slopes are complementary, $\partial A_1 + \partial A_2 = \partial (A_1 + A_2) = (2,2,2)$ is two trivial curves. We can also see this by constructing $A_1 + A_2$ by recombining the same pieces, see Figure 22.
Band the annulus to itself by attaching the quad and two triangles and push $D_2$ through $\Delta_2$ by using all 4 triangles types. The former surface is a once punctured torus and the latter a vertex linking disk. These normal surfaces have trivial boundary and are disjoint. The normal surface $A_1 + A_2$ is the disjoint union of a vertex linking disk and a once punctured torus.

5. Decision Problems in the Space of Dehn Fillings: Essential Surfaces.

In this section we consider the existence of certain interesting surfaces in Dehn fillings of a knot-manifold $X$. Recall that a surface $S$ properly embedded in a 3-manifold $M$ is compressible if there is an embedded disk $D \subset M$ so that $\partial D \subset S$ is a non-trivial curve in $S$. If $S \neq S^2$ is not compressible, we say $S$ is incompressible. A properly embedded surface in $M$ is $\partial$-compressible if there is an embedded disk $D \subset M$ so that $\partial D = a \cup b$, where $a$ and $b$ are arcs in $\partial D$, $a \cap b = \partial a = \partial b$, $a \subset \partial M$, and $b \subset S$ is not parallel into $\partial S$. If $S$ is not a disk and $S$ is not $\partial$-compressible, we say $S$ is $\partial$-incompressible. A properly embedded surface is essential if it is either a 2-sphere not bounding a 3-cell in $M$, a disk not equivalent to a disk parallel into $\partial M$, or it is two-sided, incompressible, $\partial$-incompressible and not equivalent to a surface which is parallel into $\partial M$. If $M$ contains an essential 2-sphere, then $M$ is said to be reducible; otherwise $M$ is irreducible. The 3-manifold is toroidal if it contains an essential, embedded torus; otherwise, it is atoroidal. Finally, a 3-manifold is said to be a Haken-manifold if it is irreducible and contains an embedded, incompressible surface. An irreducible 3-manifold with nonempty boundary is a Haken-manifold.

If a knot-manifold $X$ is given, we provide an algorithm to determine precisely those slopes for which a Dehn filling is reducible or those slopes for
which a Dehn filling contains an embedded, incompressible, two-sided surface. Putting these results together, we determine precisely those slopes for which a Dehn filling is a Haken-manifold. In the case of the incompressible two-sided surface, our algorithm also may be used to distinguish those slopes for which a Dehn filling is toroidal, and those slopes for which a Dehn filling is fibered over $S^1$.

In [6], it is shown for $X$ an irreducible knot-manifold, there are at most 3 reducible Dehn fillings of $X$; also, bounds are given in [6] for toroidal Dehn fillings when $X$ is atoroidal. In [29], it is shown that when $X$ contains an embedded, essential surface, and when there is no embedded annulus having one boundary a non-trivial curve in this surface and the other a curve in $\partial X$, then there are at most 3 Dehn fillings in which this surface compresses. Again, we comment that we do not get such a priori global bounds; however, our methods do give new proofs that for a given manifold bounds do exist and for a given knot-manifold we give a method to compute precisely the slopes for which these interesting phenomena happen. The output of these algorithms will be a set of slopes described by a finite set of points and/or by a line in the Dehn filling space. If $\alpha$ is a slope on $\partial X$ then the line of slopes determined by $\alpha$, $L_\alpha$, is the infinite set of slopes which intersect $\alpha$ precisely once, i.e., $L_\alpha = \{\beta | \Delta(\alpha, \beta)\}$.

We begin this section by recalling results from normal surface theory on deciding if a given manifold contains an essential 2–sphere or if it contains an embedded, incompressible, two-sided surface.

5.1. Theorem. [12, 18] Let $T$ be a triangulation of the irreducible 3–manifold $X$. Suppose $S$ is a normal surface in $(X,T)$ that is least weight in its isotopy class. If $S$ is two-sided, incompressible and $\partial$-incompressible, then every rational point in the carrier of $S$ in $\mathcal{P}(X,T)$, is the projective class of an embedded, incompressible and $\partial$-incompressible, two-sided, normal surface in $(X,T)$.

The preceding theorem, in the case for embedded closed, incompressible, two-sided surfaces, is one of the main results of [12]. The theorem was extended to include embedded incompressible and $\partial$-incompressible surfaces (extended to the bounded case) in [18]. We need analogous results for embedded, essential, normal 2–spheres and for embedded, incompressible, two-sided, closed, normal surfaces when the 3-manifold may not be irreducible. The desired result for 2-spheres follows from recent work of W. Jaco and L. Reeves [13] where the assumption on the 2-sphere is that it is an absolute least weight, embedded, essential, normal 2–sphere, Theorem 5.2. Similar results appear in [14]. The latter case, involving incompressible surfaces, requires modification of the proof in [12] and consideration of a possibly larger equivalence class of embedded, incompressible, two-sided, least weight, normal surfaces. The result we need is given in Theorem 5.3. below. The proof of Theorem 5.3, including the case with nonempty boundary and embedded,
incompressible and \(\partial\)-incompressible surfaces, can be obtained from straightforward modification of the proof in [12].

5.2. Theorem. \([13]\) Let \(T\) be a triangulation of the 3–manifold \(M\). If \(\Sigma\) is a least weight, embedded, normal 2–sphere in \((M,T)\), then every rational point in the carrier of \(\Sigma\) in \(\mathcal{P}(M,T)\), is the projective class of a normal surface each component of which is an embedded, essential, normal 2–sphere in \((M,T)\).

We have given the conclusion of the preceding theorem to allow for the possibility that some projective class in the carrier of \(\Sigma\) in \(\mathcal{P}(M,T)\) may have no representative that is connected. By using projective classes we also have the possibility that some representative may be an embedded projective plane; however, its double, also a representative of the same projective class, will then be a 2–sphere.

In what follows, we use disk swapping, which was defined in Section 2, for equivalence between surfaces. Hence, if two surfaces \(S\) and \(S'\) are isotopic, they are equivalent. Being equivalent and isotopic are the same when the ambient manifold is irreducible. The concept of “disk swapping” applies to “\(\partial\)–compressing disks” as well and is a necessary extension of this concept in the case that the manifold \(X\) has boundary and the surfaces in question are \(\partial\)-incompressible. Furthermore, any two embedded 2-spheres are equivalent via disk-swapping and so an embedded, essential, normal 2-sphere that is least weight in its equivalence class is a least weight, embedded, essential, normal 2-sphere. Note that the word essential is crucial, as a least weight normal 2-sphere may not be essential and a least weight 2–sphere is not normal and has zero weight.

5.3. Theorem. Let \(T\) be a triangulation of the 3–manifold \(M\). Suppose \(S\) is an embedded normal surface in \((M,T)\) that is least weight in its equivalence class. If \(S\) is two-sided, incompressible, and \(\partial\)-incompressible, then every rational point in the carrier of \(S\) in \(\mathcal{P}(M,T)\), is the projective class of an embedded, incompressible, \(\partial\)-incompressible, two-sided, normal surface in \(M\).

The following theorem is the primary tool for many of the results of this section. We obtained the results of this section prior to discovering this theorem. While it simplifies our earlier proofs, its major appeal, however, is that of greatly simplifying the algorithms and exhibiting the fundamental roll of the topology of \(X\) to that of \(X(\alpha)\). Specifically, using special one-vertex triangulations for Dehn fillings, as in [14], which fix a triangulation \(T\) of \(X\) for all the Dehn fillings of \(X\), we show that \(X(\alpha)\) contains an essential surface if and only if one of the vertex-solutions of \(\mathcal{P}(X,T)\) is an embedded, essential surface in \(X\) and is either closed or “caps off” to a surface which is essential in \(X(\alpha)\). It follows that there are a finite number of surfaces in \(X\) (all computable) which determine the existence (or nonexistence) of an essential surface in all Dehn fillings of \(X\).
5.4. Theorem. Suppose $X$ is a knot-manifold and $T$ is a triangulation of $X$ that restricts to a one-vertex triangulation of $\partial X$. If $X(\alpha)$ contains an embedded, essential surface, then there is an embedded, essential, normal surface $G$ in $(X,T)$ such that the projective class of $G$ is a vertex-solution of $\mathcal{P}(X,T)$, the boundary slope of $G$ is $\alpha$ (if $\partial G \neq \emptyset$), and $G(\alpha)$ is an embedded, essential, normal surface in $(X(\alpha),T(\alpha))$.

In fact, if $X(\alpha)$ is reducible, then a vertex-solution $S$ of $\mathcal{P}(X,T)$ must be either an embedded, essential 2–sphere or planar surface and $S(\alpha)$ is an embedded, essential 2–sphere in $X(\alpha)$; if $X(\alpha)$ contains an embedded, incompressible, two-sided surface, then a vertex-solution $F$ of $\mathcal{P}(X,T)$ must be an embedded, essential, non-planar surface and $F(\alpha)$ is an embedded, incompressible, two-sided surface in $X(\alpha)$; and, in the latter case, if $X(\alpha)$ contains an embedded, incompressible torus, then a vertex-solution $T$ of $\mathcal{P}(X,T)$ must be an embedded, essential torus or punctured-torus and $T(\alpha)$ is an embedded, incompressible torus in $X(\alpha)$, and if $X(\alpha)$ fibers over $S^1$, then a vertex-solution $F$ of $\mathcal{P}(X,T)$ must be an embedded, essential, two-sided surface and $F(\alpha)$ is a fiber in a fibration of $X(\alpha)$ over $S^1$.

Proof. We are given that $X(\alpha)$ contains an embedded, essential surface. Hence, $X(\alpha)$ contains an embedded, essential 2–sphere or an embedded, incompressible, two-sided surface or both. We have organized the proof to handle the general situation; however, we indicate the specific considerations and give the details needed to arrive at the special conclusions given in the second part of the statement of the theorem.

Suppose $\Gamma$ is an embedded, essential surface in $X(\alpha)$. Among all essential surfaces in $X(\alpha)$ that are equivalent with $\Gamma$ (recall that equivalence means equivalent via disk-swapping and isotopy) consider those that meet $V_\alpha$ in the smallest number of components. We can find such a surface that meets $V_\alpha$ in a collection of pairwise disjoint copies of the meridional disk or not at all. Furthermore, assuming notation has been chosen so that $\Gamma$ is itself such a surface, then for $G = X \cap \Gamma$, $G$ is an embedded, essential surface with boundary slope $\alpha$ ( $\partial G \neq \emptyset$) or a closed essential surface in $X$. There is no loss in generality to assume that $G$ is also normal in $(X,T)$.

Having made these observations, it follows that there is an embedded, essential, normal, surface $G$ with boundary slope $\alpha$ (if $\partial G \neq \emptyset$) in $(X,T)$ such that:

i. $G(\alpha)$ is defined and is equivalent to $\Gamma$ in $X(\alpha)$,
ii. $G(\alpha)$ meets $V_\alpha$ in the minimal number of components among all embedded, essential surfaces in $X(\alpha)$ that are equivalent to $\Gamma$, and
iii. if $G'$ is an embedded, essential, normal surface that is either closed or has boundary slope $\alpha$ in $(X,T)$ and $G'(\alpha)$ satisfies i and ii, then $wt(G) \leq wt(G')$; i.e., $G$ is least weight in $(X,T)$ with respect to conditions i and ii.

It follows from Theorem 5.3 above that every surface with projective class in the carrier of $G$ in $\mathcal{P}(X,T)$, is an embedded, essential, normal surface
in \((X, T)\); furthermore, such a surface, if it has boundary, has essential boundary and, therefore, by Corollary 3.8, each boundary component has slope \(\alpha\). Hence, all the surfaces with projective class in the carrier of \(G\) are either closed in \(X\) or cap off with meridional disks in \(V_\alpha\) to give closed surfaces in \(X(\alpha)\). In particular, the surfaces with projective classes at the vertices of the carrier of \(G\) cap off to closed surfaces in \(X(\alpha)\). What we need to show is that surfaces in the carrier of \(G\) in \(\mathcal{P}(X, T)\) cap off to essential surfaces in \(X(\alpha)\). In addition, to achieve the specific conclusions of the second part of the theorem, we need to show that if \(G\) is a \(2\)-sphere or is planar, then the surfaces with projective classes at the vertices of the carrier of \(G\) are \(2\)-spheres or are planar and cap off to essential \(2\)-spheres; if \(G\) is non-planar, the surfaces with projective classes at the vertices of the carrier of \(G\) are non-planar and cap off to incompressible surfaces; and if \(G\) is a torus or punctured torus, then the surfaces with projective classes at the vertices of the carrier of \(G\) are either tori or punctured tori and cap off to incompressible surfaces; and, finally, if \(G(\alpha)\) is a fiber in a fibration over \(S^1\), then the surfaces with projective classes at the vertices of the carrier of \(G\) cap off to fibers in fibrations over \(S^1\).

The triangulation \(T\) induces a one-vertex triangulation on \(\partial X\) and so, a one-vertex triangulation on \(\partial V_\alpha\). By Theorem 4.1 and Proposition 4.2 there is a layered, one-vertex triangulation of \(V_\alpha\), extending this triangulation on \(\partial V_\alpha\) so that any planar, normal surface in \(V_\alpha\) has weight \(\geq wt(G)\). We can extend the triangulation \(T\) to a triangulation, say \(T(\alpha)\), of \(X(\alpha)\) using such a layered, one-vertex triangulation of \(V_\alpha\). If \(F\) is a normal surface whose projective class is in the carrier of \(G\) then we may write \(kG = F + F'\), where \(F'\) is some other normal surface whose projective class is in the carrier of \(G\). Then \(F\) and \(F'\) both have slope \(\alpha\) and cap off to normal surfaces \(F(\alpha)\) and \(F'(\alpha)\) in \((X(\alpha), T(\alpha))\). Furthermore, we may write \(kG(\alpha) = F(\alpha) + F'(\alpha)\) so it follows that \(F(\alpha)\) and \(F'(\alpha)\) are surfaces whose projective classes are in the carrier of \(G(\alpha)\) in \(\mathcal{P}(X(\alpha), T(\alpha))\). We want to show that the surfaces in the carrier of \(G\) in \(\mathcal{P}(X, T)\) cap off to essential surfaces in \(X(\alpha)\). Furthermore, we will arrive at such a conclusion using Theorem 5.3 above in \(X(\alpha)\) and results from 13 (generalizing 12, 18) which will give us the special conclusions of the second part of the theorem.

We claim \(G(\alpha)\) is least weight in its equivalence class in \((X(\alpha), T(\alpha))\). For suppose \(\Gamma'\) is a normal surface equivalent to \(G(\alpha)\) in \((X(\alpha), T(\alpha))\) and \(wt(\Gamma') < G(\alpha)\). It was observed in 15 that each component of \(\Gamma' \cap V_\alpha\) must be a (normal) planar surface in \(V_\alpha\). Now, by Proposition 4.2, each component of \(\Gamma' \cap V_\alpha\) is either a normal annulus or a normal disk (\(V_\alpha\) has a layered triangulation) and therefore by the choice of the layered triangulation of \(V_\alpha\) each component of \(\Gamma' \cap V_\alpha\) has weight \(\geq wt(G)\).

By our choice of \(G\), it follows that \(\Gamma'\) meets \(V_\alpha\) in at least as many meridional disks as \(G(\alpha)\). If the number of components in \(\Gamma' \cap V_\alpha\) were more, then by the choice of \(T(\alpha)\), \(wt(\Gamma') \geq wt(G(\alpha))\). Hence, we must have that \(\Gamma' \cap V_\alpha\) has precisely the same number of components as \(G(\alpha) \cap V_\alpha\) and each
component of intersection is a meridional disk; for otherwise the number of components of \( \Gamma' \cap V_{\alpha} \) could be reduced, contradicting our choice of \( G' \). Let \( G' = \Gamma' \cap X \). Thus \( G' \) is an embedded, essential, normal surface in \((X, T)\) with boundary slope \( \alpha \) and \( G'(\alpha) \) satisfies i and ii above. So, \( wt(G) \leq wt(G') \) and, therefore, \( wt(G(\alpha)) \leq wt(G'(\alpha)) = wt(\Gamma') \). Hence, \( G(\alpha) \) is a least weight, embedded, essential, normal surface in \((X(\alpha), T(\alpha))\).

If \( G(\alpha) \) is an essential 2–sphere, then by [13], Theorem 5.2 above, every rational point in the carrier of \( G(\alpha) \) in \( \mathcal{P}(X(\alpha), T(\alpha)) \), is the projective class of a normal surface each component of which is an embedded, essential, normal 2–sphere in \((X(\alpha), T(\alpha))\). However, any normal surface in \((X, T)\) with projective class in the carrier of \( G \) in \( \mathcal{P}(X, T) \) can be capped off to a normal surface in \((X(\alpha), T(\alpha))\) whose projective class is in the carrier of \( G(\alpha) \) in \( \mathcal{P}(X(\alpha), T(\alpha)) \). It follows that any normal surface in \((X, T)\) with projective class in the carrier of \( G \) in \( \mathcal{P}(X, T) \) is the projective class of a normal surface each component of which is an embedded, essential, 2–sphere or planar surface in \((X, T)\) and caps off to an embedded, essential, 2–sphere in \( X(\alpha) \); in particular, this is true for any normal surface whose projective class is a vertex-solution of the carrier of \( G \) in \( \mathcal{P}(X, T) \).

If \( G(\alpha) \) is an embedded, incompressible, two-sided surface, then by Theorem 5.3 above every rational point in the carrier of \( G(\alpha) \) in \( \mathcal{P}(X(\alpha), T(\alpha)) \), is the projective class of an embedded, incompressible, two-sided, normal surface in \( X(\alpha) \). It follows that any normal surface in \((X, T)\) with projective class in the carrier of \( G \) in \( \mathcal{P}(X, T) \) is the projective class of an embedded, essential, non-planar surface in \((X, T)\) and caps off to an embedded, incompressible, two-sided, normal surface in \( X(\alpha) \); in particular, this is true for any normal surface whose projective class is a vertex-solution of the carrier of \( G \) in \( \mathcal{P}(X, T) \). If \( G(\alpha) \) is a torus, then every surface in the carrier of \( G(\alpha) \) in \( \mathcal{P}(X(\alpha), T(\alpha)) \), is the projective class of an embedded, incompressible, two-sided, normal torus. Hence, any normal surface in \((X, T)\) with projective class in the carrier of \( G \) in \( \mathcal{P}(X, T) \) is the projective class of an embedded, punctured torus or torus and caps off to an embedded, incompressible, two-sided, normal torus in \( X(\alpha) \). Finally, if \( G(\alpha) \) is a fiber in a fibration of \( X(\alpha) \) over \( S^1 \), then every surface in the carrier of \( G(\alpha) \) in \( \mathcal{P}(X(\alpha), T(\alpha)) \) is the projective class of a fiber in a fibration of \( X(\alpha) \) over \( S^1 \) [14]. So, any normal surface in \((X, T)\) with projective class in the carrier of \( G \) in \( \mathcal{P}(X, T) \) caps off to a fiber in a fibration of \( X(\alpha) \) over \( S^1 \). This completes the proof.

5.1. Reducible Manifolds in Dehn Surgery Space. Given a knot-manifold \( X \), we consider the problem of determining precisely those slopes \( \alpha \) for which the Dehn filling \( X(\alpha) \) is reducible. We consider two distinct situations. The first is when the knot-manifold \( X \) is irreducible. In this situation most (all but a finite number) Dehn fillings are irreducible. If the knot-manifold \( X \) is reducible, then we show that only in very special situations does one get an irreducible Dehn filling.
From above we have that a Dehn filling $X(\alpha)$ is reducible if and only if at least one of a finite number of constructable planar surfaces in $X$ leads to an essential 2–sphere in $X(\alpha)$ or $X$ itself contains a constructable essential 2-sphere that remains essential in $X(\alpha)$. However, for an algorithm to decide these issues, we need a result of H. Rubinstein, which provides a method to recognize if a given normal 2–sphere is essential ([23] and [26], a solution to the 3–sphere recognition problem).

5.5. Theorem. [23, 26] Suppose $T$ is a triangulation of the 3–manifold $M$. Given a normal 2–sphere $\Sigma$ in $(M, T)$ it can be decided if $\Sigma$ bounds a 3–cell in $M$.

5.6. Theorem. [23, 26] Given a compact 3–manifold $M$, it can be decided if $M$ is irreducible; furthermore, [13, 13] if $M$ is not irreducible, there is an algorithm to construct an irreducible (a minimal irreducible or prime) decomposition of $M$.

It is known for $X$ an irreducible knot-manifold there are only finitely many slopes $\alpha$ for which $X(\alpha)$ is reducible; [24] showed that if $\alpha$ and $\beta$ are both slopes for which $X(\alpha)$ and $X(\beta)$ are reducible then $\Delta(\alpha, \beta) \leq 2$. Later in [6], it was shown that $\Delta(\alpha, \beta) \leq 1$ holds. Hence, there is a global finite bound; namely, $X(\alpha)$ is reducible for at most 3 slopes. We do not get a global bound but do get a new proof that the number is finite for any knot-manifold $X$ and show that there is an algorithm to determine precisely those slopes $\alpha$ for which $X(\alpha)$ is reducible.

5.7. Theorem. Given an irreducible knot-manifold $X$, there is an algorithm to determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is reducible; in particular, it follows that there are only finitely many slopes $\alpha$ for which $X(\alpha)$ is reducible.

**Proof.** We assume $X$ is given via a triangulation $T$ that restricts to a one-vertex triangulation on $\partial X$. By Theorem 5.4, $X(\alpha)$ is reducible if and only if there is a vertex-solution $S$ of $P(X, T)$ that is planar ($X$ is assumed to be irreducible) and $S(\alpha)$ is an embedded, essential 2–sphere in $X(\alpha)$. Let $A = \{\alpha_1, \ldots, \alpha_n\}$ be the set of boundary slopes of embedded, planar, normal surfaces with projective classes at a vertex of $P(X, T)$. If $A = \emptyset$, then $X(\alpha)$ is irreducible for all $\alpha$. If $A \neq \emptyset$, then $X(\alpha)$ can only be reducible for $\alpha = \alpha_i$ for some $i, 1 \leq i \leq m$. So, suppose $P_{i_1}, \ldots, P_{i_m}$ is the set of all embedded, connected planar, normal surfaces with projective classes at a vertex of $P(X, T)$ having slope $\alpha_i$; $X(\alpha_i)$ will be reducible if and only if some $P_{i_j}$ caps off to an essential 2-sphere in $X(\alpha_i)$. This can be checked by the algorithm of Theorem 5.4, stated above.

It follows that there are at most a finite number of slopes $\alpha$ such that $X(\alpha)$ is reducible; and these slopes are among the boundary slopes of embedded, planar, normal surfaces with projective classes at a vertex of $P(X, T)$. 

In the hypothesis of Theorem 5.7, it is assumed that it is known that the knot-manifold $X$ is irreducible. Of course, Theorem 5.6 tells us that it
can be decided if a 3–manifold is irreducible; so, the issue is, in the case \( X \) is reducible, can we decide those slopes \( \alpha \) for which \( X(\alpha) \) is reducible (or, more accurately, those slopes \( \alpha \) for which \( X(\alpha) \) is irreducible; since the generic case when \( X \) is reducible, is for \( X(\alpha) \) to be reducible). We can do this; however, we need the results of Section 6 for the complete proof; in particular, we need Theorem 6.4 which provides an algorithm to decide precisely those slopes for which a Dehn filling gives the 3-sphere.

5.8. Theorem. Given a reducible knot-manifold \( X \), there is an algorithm to determine precisely those slopes \( \alpha \) for which the Dehn filling \( X(\alpha) \) is irreducible.

Proof. We are assuming the knot-manifold \( X \) is reducible. By Theorem 5.7 there is an algorithm to construct an irreducible decomposition of \( X \). If \( X \) contains a non-separating, embedded 2–sphere the algorithm will find one and it follows that \( X(\alpha) \) will be reducible for all \( \alpha \). So, we may assume every 2–sphere embedded in \( X \) separates \( X \). However, if \( X \) contains two independent, separating 2–spheres (i.e., \( X \) contains disjoint, essential 2–spheres \( S_1 \) and \( S_2 \) where \( S_1 \cup S_2 \) does not bound a product \( S^2 \times [0,1] \)), then, again, the algorithm will construct such a pair and it follows that \( X(\alpha) \) is reducible for all slopes \( \alpha \). Thus, the only possibility for \( X(\alpha) \) to be irreducible when \( X \) is reducible is that \( X \) has a separating, essential 2–sphere \( S \), each component of \( \tilde{X}_S \) (the manifold obtained from \( X \) by splitting at \( S \) and capping off each 2–sphere boundary component with a 3–cell) is irreducible, and, for notation chosen so that \( M \) is the component of \( \tilde{X}_S \) containing \( \partial X \), \( M \) is a knot-manifold in \( S^3 \), i.e., \( M \) embeds in \( S^3 \).

Again by Theorem 5.7, the algorithm will find a separating, essential 2–sphere \( S \) in \( X \) and thus determine the knot-manifold \( M \), as above. Now by Theorem 6.4, we can decide if the knot-manifold \( M \) embeds in \( S^3 \) and determine precisely those slopes \( \alpha \) for which \( M(\alpha) \) is homeomorphic with \( S^3 \). If \( M \) does not embed in \( S^3 \), then for all slopes \( \alpha \), \( X(\alpha) \) is reducible. If the knot-manifold \( M \) embeds in \( S^3 \) and it is not a solid torus, there is only one slope \( \alpha \) for which the Dehn filling \( M(\alpha) \) is homeomorphic with \( S^3 \), and the algorithm finds this slope. If the knot-manifold \( M \) is a solid torus and \( \mu \) is the meridional slope (the algorithm of Theorem 5.9, for example, finds the meridional slope), then for every slope \( \alpha \) with \( \Delta(\alpha, \mu) \leq 1 \), the line \( L_\mu, X(\alpha) \) is irreducible.

It follows from the proof of the previous theorem that whenever the knot-manifold \( X \) is reducible, one of the following holds: \( X(\alpha) \) is reducible for every slope \( \alpha \); or \( X \) is a connected sum of a non-trivial knot-manifold in \( S^3 \) and an irreducible manifold and there is precisely one computable slope \( \alpha \) for which \( X(\alpha) \) is irreducible; or \( X \) is a connected sum of a solid torus and an irreducible manifold and there is a computable line of slopes for which \( X(\alpha) \) is irreducible.
Algorithm R. Given a knot-manifold \( X \), determine precisely those slopes \( \alpha \) for which the Dehn filling \( X(\alpha) \) is reducible.

**Step 1.** We assume the knot-manifold \( X \) is given via a triangulation. Endow \( X \) with a triangulation \( T \) that restricts to a one-vertex triangulation on \( \partial X \). (An algorithm to do this is given in [15].)

**Step 2.** Compute the vertices of \( \mathcal{P}(X,T) \); i.e. find all embedded normal surfaces whose projective class is a vertex of \( \mathcal{P}(X,T) \).

**Step 3.** Determine if \( X \) is reducible (if \( X \) has an essential, embedded 2–sphere). (Recall that \( X \) has an essential, embedded 2–sphere if and only if there is an essential, embedded normal 2–sphere in \((X,T)\) whose projective class is at a vertex of \( \mathcal{P}(X,T) \) [18, 13]; furthermore, it can be decided if a given embedded, normal 2–sphere is essential [23] and if a finite, pairwise disjoint collection of normal 2–spheres is independent [23, 13].) Begin the algorithm to construct a minimal irreducible decomposition of \( X \) [18, 13].

If an embedded, non-separating 2–sphere is found, then for every slope \( \alpha \), the manifold \( X(\alpha) \) will be reducible and the algorithm terminates.

If a pair of independent, embedded, normal 2-spheres is found, then for every slope \( \alpha \), the manifold \( X(\alpha) \) will be reducible and the algorithm terminates.

So, the only possibility left, if \( X \) is reducible (the irreducible decomposition is not empty), is that there is one essential (separating) 2–sphere in the irreducible decomposition of \( X \). If this is the situation and we let \( S \) denote such a normal 2–sphere and let \( M \) denote the component of \( \hat{X}_S \) that contains \( \partial X \), then \( M \) is a knot manifold with \( \partial M = \partial X \). We wish to determine precisely those slopes \( \alpha \) for which \( M(\alpha) \) is the 3–sphere. The algorithm in Section 6 (Theorem 6.4), which includes the possibility that \( M \) is a solid torus, can be used to determine such slopes \( \alpha \), either precisely one slope or a line of slopes and the algorithm terminates.

If the irreducible decomposition is empty, then go to the next step.

**Step 4.** List the vertices of \( \mathcal{P}(X,T) \) that correspond to the projective classes of planar, normal surfaces in \((X,T)\). (Recall that if the knot-manifold \( X \) is irreducible and \( X(\alpha) \) is reducible, then a vertex-solution \( S \) of \( \mathcal{P}(X,T) \) must be planar with \( S(\alpha) \) being an embedded, essential 2–sphere in \( X(\alpha) \).) If there are none, then for every slope \( \alpha \), \( X(\alpha) \) is irreducible. Otherwise, let \( \{S_1, \ldots, S_k\} \) be all the planar normal surfaces whose projective class is a vertex of \( \mathcal{P}(X,T) \). Calculate the boundary slope of each \( S_i \), \( 1 \leq i \leq k \); let \( \{\alpha_1, \ldots, \alpha_k\} \) be the set of slopes where \( \alpha_i \) is the boundary slope of \( S_i \), \( 1 \leq i \leq k \).

**Step 5.** Determine if \( S_i(\alpha_i) \) is essential in \( X(\alpha_i) \), \( 1 \leq i \leq k \), using the algorithm of Theorem 5.5. If \( S_i(\alpha_i) \) is essential in \( X(\alpha_i) \), then \( X(\alpha_i) \) is reducible. If \( X \) is irreducible, the finite list of slopes \( (\alpha_i) \) for which \( X(\alpha_i) \) is reducible is precisely the set of slopes \( \alpha \) for which \( X(\alpha) \) is reducible.

This completes Algorithm R.
5.2. Haken-Manifolds in Dehn Surgery Space. In this section we provide an algorithm to determine precisely those manifolds in the space of Dehn fillings that are Haken-manifolds. The main problem, after the previous section, is given a knot-manifold $X$ to determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ contains an embedded, closed, incompressible, two-sided surface. The problem splits in a manner similar to that in the last section. If the knot-manifold $X$ does not contain an embedded, closed, essential, two-sided surface, then the generic Dehn filling is not expected to contain an embedded, closed, incompressible, two-sided surface. We give an algorithm to determine precisely those slopes $\alpha$ for which $X(\alpha)$ contains an embedded, incompressible, two-sided surface. We obtain a new proof that there are only finitely many slopes $\alpha$ for which $X(\alpha)$ contains such a surface.

On the other hand, if $X$ contains an embedded, closed, essential, two-sided surface, then the generic Dehn filling is expected to contain an embedded, closed, incompressible, two-sided surface. We give, in this case, an algorithm to determine precisely those slopes $\alpha$ for which $X(\alpha)$ does not contain an embedded, closed, incompressible, two-sided surface. Of independent interest in this proof is an algorithm to show that given an embedded, closed, two-sided surface $S$, we can find precisely those slopes $\alpha$ for which $S$ compresses in $X(\alpha)$. From this and the results in the previous section it is easy to determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is a Haken-manifold.

First, we recall a result due to Haken [9]. A proof appears in [12] for handle-decompositions; the proof in the case of triangulations requires some modification to the proof for handle decompositions. Proofs for triangulations and refinements in the algorithm appear in [18, 13].

5.9. Proposition. [9] Let $M$ be a 3-manifold with triangulation $T$. Given a two-sided, normal surface $F$ in $(M; T)$, there is an algorithm to decide if $F$ is incompressible in $M$.

5.10. Theorem. Given a knot-manifold $X$ which does not contain an embedded, closed, essential, two-sided surface there is an algorithm to determine precisely those slopes $\alpha$ for which $X(\alpha)$ contains an embedded, closed, incompressible, two-sided surface; in particular, it follows that there are at most finitely many slopes $\alpha$ for which $X(\alpha)$ contains such a surface.

Proof. Suppose for the slope $\alpha$, the Dehn filling $X(\alpha)$ contains an embedded, incompressible, two-sided surface. It follows from Theorem 5.4 that there is an embedded, essential, two-sided surface $S$ of $(X, T)$ whose projective class in $\mathcal{P}(X, T)$ is a vertex, the boundary slope of $S$ is $\alpha$ ($\partial S \neq \emptyset$, since by assumption $X$ does not contain an embedded, closed, essential, two-sided surface), and $S(\alpha)$ is an embedded, incompressible, two-sided surface in $X(\alpha)$. So, the only slopes $\alpha$ for which it is possible that $X(\alpha)$ contain an embedded, closed, incompressible, two-sided surface are among a subset of boundary slopes coming from embedded, essential, non-planar, normal
surfaces in \((X, T)\) whose projective classes are at vertices of \(P(X, T)\). This is a finite set.

Let \(\{S_1, \ldots, S_n\}\) be the set of embedded, non-planar, normal surfaces in \((X, T)\) whose boundary consists of only non-trivial curves in \(\partial X\) and whose projective class in \(P(X, T)\) is a vertex. Let \(\alpha_i\) be the slope of \(X_i, 1 \leq i \leq n\). We check if \(S_i(\alpha_i)\) is incompressible in \(X(\alpha_i)\). It is precisely those slopes \(\alpha_{ij}\) for which \(S_i(\alpha_{ij})\) is incompressible in \(X(\alpha_{ij})\) that satisfy the conclusion of the theorem.

We now consider the situation when the knot-manifold \(X\) contains an embedded, closed, essential, two-sided surface. First, we make some notational conventions regarding a planar surface. If \(D\) is a disk and \(p_1, \ldots, p_n\) are points in the interior of \(D\), we call the planar surface obtained from \(D\) by removing the interior of a small regular neighborhood about each point \(p_i, i \leq i \leq n\), a punctured-disk. In this situation, if \(P = D \setminus \bigcup_{1}^{n} \text{Int} N(p_i)\), where \(N(p_i)\) is a small regular neighborhood of \(p_i\) in \(D\), we call the boundary component \(\partial D\) of \(P\) the boundary of \(P\), written \(\text{bdry}(P)\), and the boundary components \(\partial N(p_i)\) the punctures of \(P\). Similarly, if \(A\) is an annulus and the \(p_i\)'s and \(N(P_i)\)'s are defined the same, then we call \(Q = A \setminus \bigcup_{1}^{n} \text{Int} N(p_i)\) a punctured annulus and call the boundary components of \(A\) the boundary of \(Q\), denoted \(\text{bdry}(Q)\), and the boundary components of \(\partial N(p_i)\) the punctures of \(Q\). In this way we distinguish boundary components of such planar surfaces.

We find that the generic case, when the knot-manifold \(X\) contains an embedded, closed, essential, twosided surface, is for \(X(\alpha)\) to contain an embedded, closed, incompressible, two-sided surface. In particular, Wu has shown \([29]\) that if \(X\) contains an embedded, essential, closed, two-sided surface \(S\) and there is no annulus from \(S\) to \(\partial X\), then \(S\) will compress in the Dehn filling \(X(\alpha)\) for at most 3 slopes \(\alpha\). We are able to show that given an embedded, closed, two-sided, normal surface \(S\), there is an algorithm to determine precisely those slopes \(\alpha\) for which \(S\) compresses in \(X(\alpha)\). Again, our techniques give finiteness in the case considered by Wu but do not give similar global bounds; and we obtain complete answers when there is an annulus embedded in \(X\) having one boundary a non-trivial curve in \(S\) and the other in \(\partial X\) (see \([2]\)).

First, we have the following lemma which has independent interest.

5.11. Lemma. Let \(T\) be a triangulation of the knot-manifold \(X\) that restricts to a one-vertex triangulation of \(\partial X\). Given a closed, two-sided, normal surface \(S\) in \((X, T)\), there is an algorithm to decide precisely those slopes \(\alpha\) for which \(S\) is incompressible in \(X(\alpha)\).

Proof. Given \(S\) normal in \((X, T)\), there are algorithms to decide if \(S\) is incompressible in \(X\) (\([4]\), see Proposition 5.9) and if \(S\) is equivalent to a boundary parallel surface \([18]\). If \(S\) compresses in \(X\) or \(S\) is equivalent to a boundary-parallel surface, then \(S\) will compress in \(X(\alpha)\) for every \(\alpha\). Hence,
we may assume $S$ is essential in $X$ (and $S$ is not $S^2$). The surface $S$ will be normal in $(X(\alpha), T(\alpha))$ for all $\alpha$.

We consider two possibilities:

i. there is no annulus in $X$ having one boundary component a non-trivial curve in $S$ and the other a curve in $\partial X$, or

ii. there is an annulus in $X$ having one boundary component a non-trivial curve in $S$ and the other a curve in $\partial X$.

Split $X$ at $S$ and let $X_S$ denote the component of the 3-manifold which contains $\partial X$. Then $\partial X_S$ consists of one ($S$ separates $X$) or two ($S$ does not separate $X$) copies of $S$ along with the torus, $\partial X$. Note that if $S$ does not separate $X$, then for every $\alpha$ we have that either $X(\alpha)$ is reducible or $X(\alpha)$ contains an embedded, closed, incompressible, two-sided surface; however, we do not need to make a distinction as to $S$ separating or not separating $X$.

Observe that if $S$ compresses in $X(\alpha)$ for some $\alpha$, there is a punctured disk $P$ embedded in $X_S$ with $bdr(P)$ in a copy of $S$ in $\partial X_S$ and punctures in the torus $\partial X$. In this case, there is no loss in generality to assume that $P$ is essential. Hence, we have that the punctures form a non-empty, pairwise disjoint collection of simple closed curves in $\partial X$, each having slope $\alpha$. In particular, in situation ii above, the existence of such an annulus gives that $S$ compresses (the annulus must also meet $\partial X$ in a non-trivial curve) in $X(\alpha)$ where $\alpha$ is the slope of the boundary curve of the annulus in $\partial X$.

Also, we observe in situation ii that there is a unique slope on $\partial X$ for an annulus which joins $S$ to $\partial X$; for otherwise, the characteristic Seifert-Pair Theorem [19, 20] gives a contradiction to our assumption that $S$ is essential (not equivalent to a surface parallel to $\partial X$). Finally, if $S$ compresses in $X(\alpha)$ and $\alpha$ is not a boundary slope in $\partial X$, then $S$ completely compresses in $X(\alpha)$.

Now, in situation i, where there is no annulus in $X_S$ having one boundary component a non-trivial curve in $S$ and the other in $\partial X$, we shall show that there is a finite and computable set of slopes $\alpha$ for which $S$ compresses in $X(\alpha)$. (In [29] this set is shown to have no more than 3 slopes.)

Let $T_S$ be a triangulation of $X_S$ having precisely one vertex in the component $\partial X$ of $\partial X_S$ ([15]).

If $P$ is an essential punctured disk as above, then we may assume that $P$ is normal in $(X_S, T_S)$ and $P$ is least weight in its equivalence class. We have,

$$P = \sum_i k_i F_i + \sum_{i'} l_{i'} K_{i'} + \sum_j m_j A_j + \sum_{j'} n_{j'} A'_{j'}$$

where all of the summands are essential, normal, fundamental surfaces in $(X_S, T_S)$ ([14]), and notation has been chosen so that $\chi(F_i) < 0$, each $K_{i'}$ is either a torus or Klein bottle, each $A_j$ is an annulus or Möbius band with its boundary in $S$, and each $A'_{j'}$ is an annulus or Möbius band with its
boundary in $\partial X$. Of course, it is possible that there are no factors $K_{i'}, A_{j}$ and $A'_{j'}$. We have written the most general sum in this situation and in fact, each $l_{i'}, m_{j}$ and $n_{j'}$ might be zero.

Suppose some $n_{j'} \neq 0$. Then we may write $P = F + A'_{j'}$ for some normal surface $F$ in $(X_S, \mathcal{T}_S)$. Hence, either $\partial A'_{j'} \cap \partial F = \emptyset$ and $\partial A'_{j'}$ has slope $\alpha$ or all intersections between $A'_{j'}$ and $F$ run from $\partial X$ to $\partial X$ (we can assume there are no trivial curves of intersection). So, by Proposition 3.7 and in this latter case, $\partial A'_{j'}$ has slope $\alpha$ and $\alpha$ is a boundary slope for $X$. In fact, there is an essential normal annulus or Möbius band in $(X, \mathcal{T})$ (possibly $A'_{j'}$ itself) with boundary slope $\alpha$ and whose projective class is a vertex of $\mathcal{P}(X_S, \mathcal{T}_S)$. So, if $n_{j'} \neq 0$ we arrive at the conclusion that $\alpha$ is a computable boundary slope of $X$ and we can check if $S$ compresses in $X(\alpha)$. Hence, we check if $S$ is incompressible in $X(\alpha)$ for all boundary slopes $\alpha$ corresponding to embedded, normal annuli having projective class at a vertex of $\mathcal{P}(X_S, \mathcal{T}_S)$, a finite, computable set. Note that in this situation there can be at most one slope bounding an essential annulus with boundary in $\partial X$. Otherwise, $X$ would have to be a twisted I-bundle over a Klein bottle [19] which contains no two-sided essential surface. If we find more than one slope realized by vertex annuli, we have the option of first checking which of these annuli are essential.

So, we may suppose that $n_{j'} = 0, \forall j'$; for otherwise, $P$ would have boundary slope the same as $A'_{j'}$. Let $L(\partial G)$ denote the length of the boundary of the normal surface, $G$, in $(X_S, \mathcal{T}_S)$, we have:

$$L(\partial P) = \sum_i k_i L(\partial F_i) + \sum_j m_j L(\partial A_j).$$

Also,

$$-\chi(P) = \sum_i k_i (-\chi(F_i)).$$

Let

$$C = \max \left\{ \frac{L(\partial F_i)}{-\chi(F_i)} \right\}.$$ 

Notice that $C$ is computable for $F_i$ ranging over the embedded, normal, fundamental surfaces in $(X_S, \mathcal{T}_S)$ with $\chi(F_i) < 0$ and that $L(\partial F_i) < -\chi(F_i)C$ for all such $F_i$. Let $\gamma$ be the length of $\text{bdry}(P)$ and let $\gamma_\alpha$ denote the length of the slope $\alpha$. If $P$ has $p$ punctures, then $-\chi(P) = p - 1$ and
L(\partial P) = \gamma + p\gamma_\alpha. Thus,
\[
\gamma + p\gamma_\alpha = \sum_i k_i L(\partial F_i) + \sum_j m_j L(\partial A_j) \\
\leq \sum_i (-\chi(F_i))C + \sum_j m_j L(\partial A_j) \\
= -\chi(P)C + \sum_j m_j L(\partial A_j).
\]

From this and the fact that \(\gamma - \sum_j m_j L(\partial A_j) \geq 0\), we have
\[
\gamma_\alpha \leq C.
\]
So, in situation i and if \(S\) compresses in \(X(\alpha)\), it either compresses at a boundary slope \(\alpha\) corresponding to the boundary slope of an essential normal annulus or Möbius band in \((X, T)\) whose projective class is a vertex of \(P(X, T)\) or it compresses in \(X(\alpha)\) where \(\gamma_\alpha\), the length of the slope \(\alpha\), satisfies \(\gamma_\alpha \leq C\), where \(C\) is computable from certain fundamental solutions in \((X_S, T_S)\). In either case, there are at most finitely many slopes \(\alpha\) for which \(S\) compresses and we can determine precisely those slopes \(\alpha\) where \(S\) compresses in \(X(\alpha)\).

Now, we consider situation ii, where there is an annulus in \(X\) having one boundary component a non-trivial curve in \(S\) and the other a curve in \(\partial X\). Note in this case, since \(S\) is assumed to be essential, it follows that the boundary curve of the annulus in \(\partial X\) is non-trivial.

First we observe that if there is such an annulus \(A\), then there is a normal one in \((X_S, T_S)\). We claim that if \(A\) is least weight (in its equivalence class) among all such annuli, then \(A\) is fundamental in \((X_S, T_S)\). To see this suppose \(A\) is not fundamental; then \(A = A' + A''\) where we may write such a sum with both \(A'\) and \(A''\) incompressible, \(\partial\)-incompressible, and not parallel into \(\partial X_S\) and \(A' \cap A''\) has the smallest number of components under these conditions [12]. But \(\chi(A') = \chi(A'') = \chi(A) = 0\). It follows that (with choice of notation) the possibilities are: both \(A'\) and \(A''\) are annuli each having one boundary component in \(S\) and one in \(\partial X\) (but this contradicts the choice of \(A\) being least weight); \(A'\) is an annulus having one boundary component a non-trivial curve in \(S\) and the other boundary a curve in \(\partial X\) and \(A''\) is a Möbius band, a torus, or a Klein bottle (but again this contradicts the choice of \(A\) being least weight); or both \(A'\) and \(A''\) are Möbius bands, one having its boundary in \(S\) and the other having its boundary in \(\partial X\). In this last possibility there is no loss in generality to assume that \(A' \cap A''\) has exactly one component and it is the non-separating (orientation-reversing) simple closed curve in each. Thus, a regular exchange along the intersection gives the normal annulus \(A\); but then an irregular exchange gives an annulus \(B\) having the same boundary as \(A\) but containing a fold (see Figure 23). So \(\text{wt}(B) < \text{wt}(A)\). But this also contradicts our choice of \(A\). So, as claimed, a least weight normal annulus in \((X_S, T_S)\) running from \(S\) to \(\partial X\) is fundamental.
Notice from this analysis, it is possible to have an annulus \( A \) with one boundary component a non-trivial curve in \( S \) and the other a curve in \( \partial X \) and have \( A = M_1 + M_2 \) where \( M_i \) is a Möbius band; but \( A \) cannot be least weight, the irregular switch at \( M_1 \cap M_2 \) gives a similar annulus with lower weight. Furthermore, there is a unique (up to isotopy) slope in \( \partial X \) for such an annulus \( A \). It follows that we can find such an \( A \) and the slope \( \alpha \), where \( \alpha \) is the slope of \( \partial A \) on \( \partial X \). This completes our claim.

Now, as we noted above, \( S \) compresses in \( X(\alpha) \). If \( S \) compresses in \( X(\beta) \), where \( \beta \neq \alpha \), then, as in situation i, there is a planar surface \( P \) embedded in \( X \) having \( \text{bdry}(P) \) in \( S \) and punctures in \( \partial X \). There is no loss to assume that \( P \) is essential in \( X_S \). It follows from [2] that \( \Delta(\alpha, \beta) = 1 \) and that \( S \) compresses in \( X(\beta) \) for all \( \beta \) where \( \Delta(\alpha, \beta) = 1 \), i.e. the “line” \( L_\alpha \).

So, by considering the fundamental surfaces in \( (X_S, T_S) \) we can determine if there is an annulus embedded in \( X_S \) having one boundary a non-trivial curve in \( S \) and the other in \( \partial X \). If there is, we can find its boundary slope, say \( \alpha \) in \( \partial X \). The surface \( S \) compresses in \( X(\alpha) \) but does not compress for any slope distinct from \( \alpha \) or \( S \) compresses in \( X(\beta) \) precisely for all those slopes \( \beta \in \{\alpha\} \cup L_\alpha \). We can use the algorithm given in [4], see Proposition 5.9, to determine which is the case; namely, check if \( S \) compresses in \( X(\beta) \) for some \( \beta \in L_\alpha \) (\( \beta \neq \alpha \)). This completes the proof of the lemma and provides an algorithm to decide precisely those slopes \( \alpha \) for which \( S \) compresses in \( X(\alpha) \).

**5.12. Theorem.** Given a knot-manifold \( X \) that contains an embedded, essential, closed, two-sided surface distinct from \( S^2 \), then there is an algorithm to determine precisely those slopes \( \alpha \) for which \( X(\alpha) \) does not contain an embedded, incompressible, closed, two-sided surface; in particular, the set of slopes \( \alpha \) for which \( X(\alpha) \) does not contain an embedded, incompressible, closed, two-sided surface is either a finite set of slopes or all but possibly finitely many slopes in the set \( \{\alpha_0\} \cup L_{\alpha_0} \) for some slope \( \alpha_0 \).

**Proof.** Suppose the knot-manifold \( X \) is given by a triangulation \( T \) that restricts to a one-vertex triangulation on \( \partial X \) (recall there is an algorithm to modify any triangulation of \( X \) to such a triangulation).
If the Dehn filling \( X(\alpha) \) contains an embedded, incompressible, two-sided surface, then by Theorem 5.2 there is an embedded, essential, normal surface \( S \) in \((X, \mathcal{T})\) such that the projective class of \( S \) is a vertex-solution of \( \mathcal{P}(X, \mathcal{T}) \), the boundary slope of \( S \) is \( \alpha \) (if \( \partial S \neq \emptyset \)), and \( S(\alpha) \) is an embedded, essential, normal surface in \((X(\alpha), \mathcal{T}(\alpha))\). These surfaces are constructable. The closed vertex solutions provide candidate surfaces to which we can apply the algorithms of Lemma 5.11. Of course, it is also possible that Dehn fillings along boundary slopes \( \alpha \) of \( X \) may create embedded, incompressible, two-sided surfaces in \( X(\alpha) \); hence, those vertex solutions that are bounded will also need to be taken into consideration.

Let \( \{S_1, \ldots, S_J\} \) denote, the embedded, essential, two-sided, connected, closed normal surfaces in \((X, \mathcal{T})\) that are not 2–spheres and whose projective class is a vertex of \( \mathcal{P}(X, \mathcal{T}) \); and let \( \{F_1, \ldots, F_K\} \) denote the embedded, essential, two-sided, connected, bounded, normal surfaces in \((X, \mathcal{T})\) that are not planar and whose projective class is a vertex of \( \mathcal{P}(X, \mathcal{T}) \). By hypothesis and \([12]\), the set \( \{S_1, \ldots, S_J\} \neq \emptyset \).

For each surface \( S_j, 1 \leq j \leq J \), use the algorithm of Lemma 5.11 to determine those slopes \( \alpha \) for which \( S_j \) compresses in \( X(\alpha) \). For each \( S_j \) we have that \( S_j \) compresses in at most a finite number of computable slopes or for the set of slopes \( \{\alpha_j\} \cup L_{\alpha_j} \) where \( \alpha_j \) (and hence, \( L_{\alpha_j} \)) is computable. Hence, we conclude that there is an embedded, incompressible, two-sided surface in \( X(\alpha) \) for all but a finite number of computable slopes \( \alpha \) (this includes the possibility \( \{\alpha_j\} \cup L_{\alpha_j} \cap (\{\alpha_{j'}\} \cup L_{\alpha_{j'}}) \), where \( j \neq j' \)) or for all but those slopes in the set \( \{\alpha_j\} \cup L_{\alpha_j} \). If any of the slopes where one of the closed surfaces in \( \{S_1, \ldots, S_J\} \) does not remain incompressible in \( X(\alpha) \) is a boundary slope of some \( F_k, 1 \leq k \leq K \), say the slope \( \beta_k \) which is the boundary slope of \( F_k \), then we check, \([4]\) (Theorem 5.9 above) to determine if \( F_k(\beta_k) \) is incompressible in \( X(\beta_k) \). This can only add slopes where \( X(\alpha) \) contains an embedded, incompressible, two-sided surface.

We now have the main theorem of this section.

5.13. Theorem. Given a knot-manifold \( X \) there is an algorithm to determine precisely those slopes \( \alpha \) for which \( X(\alpha) \) is a Haken-manifold; in particular, if \( X \) is irreducible and does not contain an embedded, essential, two-sided, closed surface, the set of slopes \( \alpha \) for which \( X(\alpha) \) is a Haken-manifold is finite; if \( X \) is irreducible and does contain an embedded, essential, two-sided, closed surface, the set of slopes \( \alpha \) for which \( X(\alpha) \) is not a Haken-manifold is either a finite set of slopes or all but possibly a finite number of slopes on the line \( \{\alpha_0\} \cup L_{\alpha_0} \) for some slope \( \alpha_0 \).

Proof. From the preceding subsection, we have an algorithm to determine precisely those slopes \( \beta \) for which the Dehn filling \( X(\beta) \) is reducible and from the above Theorems, we have algorithms to determine precisely those slopes \( \gamma \) for which the Dehn filling \( X(\gamma) \) contains an embedded, incompressible, two-sided surface. The combination of these algorithms will give us precisely those slopes \( \alpha \) for which \( X(\alpha) \) is a Haken-manifold. \( \square \)
We next outline the steps for an algorithm to decide for a given knot-manifold $X$ precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is a Haken-manifold, Algorithm H. We have organized the algorithms so that one can determine precisely those slopes $\alpha$ for which a Dehn filling $X(\alpha)$ contains an embedded, incompressible, two-sided surface, Algorithm S; and then we can apply our earlier algorithm to eliminate those slopes where the manifold is reducible, Algorithm R. A fundamental step in Algorithm S is to decide for any given embedded, two-sided, closed surface $S$ in $X$, precisely those slopes $\alpha$ for which the surface $S$ is incompressible in the Dehn filling $X(\alpha)$; we give this as an independent algorithm, Algorithm I.

**Algorithm I.** Suppose $X$ is a knot-manifold with a triangulation $T$ which restricts to a one-vertex triangulation on $\partial X$. Given an embedded, two-sided, closed, normal surface in $(X,T)$, determine precisely those slopes $\alpha$ for which the surface compresses in the Dehn filling $X(\alpha)$.

**Step 1.** Let $S$ be the given embedded, two-sided, closed, normal surface in $(X,T)$. Split $X$ at $S$ and let $X_S$ denote the component containing $\partial X$. The manifold $X_S$ has either one or two copies of $S$ in $\partial X_S$; and, if $S$ separates $X$, the manifold $X'_S$ has a single copy of $S$ in $\partial X'_S$. Endow $X_S$ and $X'_S$ with triangulations $T_S$ and $T'_S$, respectively, so that $T_S$ restricts to the triangulation $T$ on the boundary component $\partial X$ of $X_S$.

**Step 2.** Compute the fundamental solutions of $(X_S, T_S)$ and (in the case $S$ separates $X$) of $(X'_S, T'_S)$. We look for the existence of disks and annuli among these fundamental solutions.

**Step 3.** If a fundamental solution is an embedded disk with boundary a non-trivial curve in a copy of $S$, then the surface $S$ compresses in $X$ and therefore will compress in $X(\alpha)$ for every slope $\alpha$. If this is not the case but a fundamental solution is an embedded disk with boundary a non-trivial curve in $\partial X$, then the knot-manifold $X$ is reducible and $S$ is incompressible in $X(\alpha)$ for every slope $\alpha$. (Notice that if $\partial X$ compresses and $X$ is irreducible then $X$ is a solid torus, $S$ would necessarily compress and we would have a fundamental solution that is an embedded disk with boundary a non-trivial curve in a copy of $S$, i.e., we would have found such a disk in the first part of this step.) If either type of disk is found, then the algorithm is complete and we have either $S$ compresses for every slope or $S$ compresses for no slope.

**Step 4.** We have that no fundamental solution found in Step 2 is a disk with non-trivial boundary in either a copy of $S$ or in $\partial X$. Now, look for fundamental solutions that are embedded annuli having one boundary a non-trivial curve in a copy of $S$ and the other in $\partial S$. If there are two such annuli having distinct slopes in $\partial X$, then $S$ is equivalent to a peripheral torus and compresses in $X(\alpha)$ for every slope $\alpha$. If there is only one slope
for all such annuli, then go to Step 6. If there are no such annuli, then go to Step 5.

**Step 5.** We have that no fundamental solution found in Step 2 is an embedded disk or an embedded annulus having one boundary a non-trivial curve in \( S \) and the other in \( \partial X \). However, there may be fundamental solutions found in Step 2 that are embedded annuli or Möbius bands having their boundary non-trivial curves in \( \partial X \). Let \( \{A_1, \ldots, A_m\} \) denote such fundamental solutions and let \( \{F_1, \ldots, F_n\} \) be the set of all embedded fundamental solutions of \( (X_S, T_S) \) with \( \chi(F_i) < 0, \forall i \). Set

\[
C = \max \left\{ \frac{L(\partial F_i)}{-\chi(F_i)} \right\}.
\]

Let \( \{\alpha_1, \ldots, \alpha_K\} \) denote the slopes in \( \partial X \) that either have length \( \lambda_{\alpha_i} \leq C \) or are a boundary slope for some \( A_j, 1 \leq j \leq m \). Recall, in this situation, the surface \( S \) will compress in \( X(\alpha) \) if and only if there is an \( i, 1 \leq i \leq K, \alpha = \alpha_i \) and \( S \) compresses in \( X(\alpha_i) \). (Also recall that there is at most one slope bounding an essential annulus \( A_j \); it may be advantageous to check whether each vertex annulus is inessential before listing the slope.)

For \( \alpha_i \in \{\alpha_1, \ldots, \alpha_K\} \), build \( X(\alpha_i) \) via a layered triangulation and check if \( S \) compresses in \( X(\alpha_i) \). Let \( \{\alpha_{i_1}, \ldots, \alpha_{i_j}\} \) be the set of slopes in \( \{\alpha_1, \ldots, \alpha_K\} \) for which \( S \) compresses in \( X(\alpha_i) \). The algorithm terminates having found this finite set of slopes as precisely the set of slopes \( \alpha \) for which the surface \( S \) compresses in \( X(\alpha) \).

**Step 6.** Let \( A \) denote an embedded annulus in \( (X_S, T_S) \) having one boundary a non-trivial curve in a copy of \( S \) and the other in \( \partial S \) (If there is such an \( A \), then one may be constructed.) The component of \( \partial A \) in \( \partial S \) is a non-trivial curve, say, with slope \( \alpha_0 \). Choose any slope \( \beta \in L_{\alpha_0} \). Determine if \( S \) compresses in \( X(\beta) \). If \( S \) does not compress in \( X(\beta) \), then \( S \) is incompressible in all Dehn fillings \( X(\alpha), \alpha \neq \alpha_0 \), and so, \( S \) compresses in \( X(\alpha) \) for precisely one slope, the slope \( \alpha_0 \). If \( S \) compresses in \( X(\beta) \), then \( S \) compresses in all Dehn fillings \( X(\alpha) \) where \( \alpha \in \{\alpha_0\} \cup L_{\alpha_0} \). This completes Algorithm I.

We now consider an algorithm to decide for a given knot-manifold \( X \) the set of slopes \( \alpha \) for which the Dehn filling \( X(\alpha) \) contains an embedded, incompressible, two-sided surface.

**Algorithm S.** Given a knot-manifold \( X \), determine precisely those slopes \( \alpha \) for which the Dehn filling \( X(\alpha) \) contains an embedded, incompressible, two-sided surface.

**Step 1.** We have the knot-manifold \( X \) given via a triangulation. Endow \( X \) with a triangulation \( T \) that has precisely one vertex in \( \partial X \). (An algorithm is given in [13] to modify a given triangulation of \( X \) to such a triangulation.)
Step 2. Compute the vertices of $P(X, \mathcal{T})$.

Step 3. Make two lists: $\mathcal{G} = \{G_1, \ldots , G_J\}$, those vertices of $P(X, \mathcal{T})$ that are projective classes of embedded, closed, normal surfaces that are not 2–spheres in $(X, \mathcal{T})$; and $\mathcal{B} = \{B_1, \ldots , B_K\}$, those vertices $P(X, \mathcal{T})$ which are projective classes of embedded, non-planar, normal surfaces in $(X, \mathcal{T})$ and have nonempty boundary consisting entirely of non-trivial simple closed curves in $\partial X$. For each surface in $\mathcal{B}$ compute its boundary slope. Denote the boundary slope of $B_k$, $1 \leq k \leq K$, by $\beta_k$; these are a subset of the boundary slopes of $X$.

Step 4. If $\mathcal{G} = \emptyset$, it follows that $\partial X$ compresses and $X$ is a solid torus or a non-trivial connected sum of a solid torus and a 3–manifold $M'$. Furthermore, $M'$ does not contain any embedded, incompressible, two-sided, closed surfaces. So, in either case, $X(\alpha)$ does not contain an embedded, incompressible, two-sided closed surface for any slope $\alpha$ and the algorithm terminates.

Step 5. We have $\mathcal{G} = \{G_1, \ldots , G_J\} \neq \emptyset$. For each $G_i \in \mathcal{G}$, $1 \leq i \leq J$, apply Algorithm S to decide precisely those slopes $\alpha$ for which the surface $G_i$ compresses in the Dehn filling $X(\alpha)$. For the surface $G_i$, let this set of slopes be denoted $\mathcal{A}_i$. Recall the possibilities are: a finite set of slopes, or all the slopes on a “line” $\{\alpha_{i0}\} \cup L_{\alpha_{i0}}$ for some slope $\alpha_{i0}$, or every slope (i. e., $G_i$ either compresses in $X$ or is peripheral). Let

$$\mathcal{A} = \bigcap_{i=1}^{J} \mathcal{A}_i.$$  

Step 6. For each slope $\beta_k$ found in Step 3, construct $X(\beta_k)$ via a layered triangulation. It can be determined if the surface $B_k(\beta_k)$, $B_k$ also found in Step 3, compresses in $X(\beta_k)$. Let $\{\beta_{k_1}, \ldots , \beta_{k_n}\}$ be the set of slopes for which $B_{k_j}$, $1 \leq j \leq n$, does NOT compress in $X(\beta_{k_j})$.

Step 7. If $\mathcal{A}$ is finite (i. e., there are only finitely many slopes $\alpha$ for which all the surfaces in $\mathcal{G}$ compress in $X(\alpha$), then the set of slopes $\mathcal{A} \setminus \{\beta_{k_1}, \ldots , \beta_{k_n}\}$ is precisely the set of slopes $\alpha$ for which $X(\alpha)$ does NOT contain an embedded, incompressible, two-sided surface. If $\mathcal{A}$ is infinite, there are two possibilities: $\mathcal{A} = \{\alpha_{i0}\} \cup L_{\alpha_{i0}}$ for some slope $\alpha_{i0}$ or $\mathcal{A}$ is the set of all slopes. In the first case, the set of slopes $\mathcal{A} \setminus \{\beta_{k_1}, \ldots , \beta_{k_n}\}$ is precisely the set of slopes $\alpha$ for which $X(\alpha)$ does NOT contain an embedded, incompressible, two-sided surface. In the second case, $\{\beta_{k_1}, \ldots , \beta_{k_n}\}$ is precisely the set of slopes $\alpha$ for which $X(\alpha)$ does contain an embedded, incompressible, two-sided surface. This terminates Algorithm S.

Finally, we are prepared to give an algorithm to determine the manifolds in the space of Dehn fillings which are Haken-manifolds.

Algorithm H. Given a knot-manifold $X$, determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is a Haken-manifold.
Step 1. We have the knot-manifold $X$ given via a triangulation. Endow $X$ with a triangulation $T$ that has precisely one vertex in $\partial X$.

Step 2. Employ Algorithm S to determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ contains an embedded, incompressible, two-sided surface.

Step 3. Employ Algorithm R to determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is irreducible (those for which it is not reducible).

The slopes common to those found in Steps 2 and 3 are precisely the set of slopes $\alpha$ for which $X(\alpha)$ is a Haken-manifold.

5.3. Fibered manifolds in Dehn Surgery Space. In this section we provide an algorithm to determine for a given knot-manifold $X$ precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ fibers as a surface bundle over a circle. We wish to thank Robert Myers who suggested that our methods should solve this problem. Our proof is based on material from lectures of the first author given at University of Melbourne a decade ago. Revision of this work appears in [11]; we state the results we need below without proof.

5.14. Theorem. [11, 18] Suppose $T$ is a triangulation of the 3–manifold $M$. Given a properly embedded normal surface $F$ in $(M, T)$, there is an algorithm to determine if $F$ is a fiber in a fibration of $M$ over $S^1$.

5.15. Theorem. [11] Given a 3–manifold $M$, there is an algorithm to determine if $M$ is a fibration over $S^1$.

5.16. Theorem. Given a knot-manifold $X$ there is an algorithm to determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is a fibration over $S^1$.

Proof. We assume $X$ is given via a triangulation $T$ that restricts to a one-vertex triangulation on $\partial X$. We separate the argument into two cases depending on $X$ reducible or $X$ irreducible.

If $X$ is reducible, then by Theorem 5.8 we have:

- $X$ contains an embedded, non-separating (hence, essential) 2–sphere and $X(\alpha)$ is reducible for all $\alpha$,
- $X$ contains two separating, embedded, disjoint, inequivalent, essential 2–spheres and $X(\alpha)$ is reducible for all $\alpha$,
- $X$ is a connected sum of a nontrivial knot-manifold in $S^3$ and an irreducible 3–manifold and there is precisely one computable slope for which a Dehn filling is irreducible, or
- $X$ is a connected sum of a solid torus and an irreducible 3–manifold and there is a computable line of slopes for which $X(\alpha)$ is irreducible.

If $X$ contains an embedded, non-separating 2–sphere $S$, it may be possible that $X(\alpha)$ fibers over $S^1$ with fiber the surface $S$. There is an algorithm to
determine this; again, we call upon Theorem 6.4 of the next section. Hence, if \(X\) contains an embedded, non-separating 2–sphere, by Theorem 5.6, there is an algorithm to find one, say \(S\) is such a 2–sphere. Split the knot-manifold \(X\) at \(S\) to form the 3–manifold \(X_S\). The manifold \(X_S\) has two copies of \(S\) in its boundary, along with \(\partial X\). We can fill the two copies of \(S\) with 3–cells to get a new knot-manifold \(\tilde{X}_S\). The slopes \(\alpha\) for which \(X'(\alpha)\) is \(S^3\) are precisely the slopes for which \(X(\alpha)\) fibers over \(S^1\) with fiber the 2–sphere \(S\).

Hence, we have \(X(\alpha)\) does not fiber for all \(\alpha\), or \(X(\alpha)\) fibers for a unique, and computable, slope \(\alpha\), or \(X(\alpha)\) fibers for a computable line of slopes \(L_\alpha\).

If \(X\) contains two separating, embedded, disjoint, independent, essential 2–spheres, it is not possible for \(X(\alpha)\) to fiber for any \(\alpha\).

If \(X\) is a connected sum of a nontrivial knot-manifold in \(S^3\) and an irreducible 3–manifold, we have that there is precisely one computable slope for which a Dehn filling is irreducible; say for \(\alpha_0\), we have \(X(\alpha_0)\) irreducible. If we denote the irreducible 3–manifold by \(N\), then \(X(\alpha)\) will fiber over \(S^1\) with fiber a surface only for the slope \(\alpha_0\) and then if and only if we have \(N\) fibers over \(S^1\) with fiber a surface. By [11], Theorem 5.15 above, there is an algorithm to determine if \(N\) fibers over \(S^1\) with fiber a surface.

If \(X\) is a connected sum of a solid torus, say \(M\), and an irreducible 3–manifold, say \(N\), there is a computable line of slopes \(L_\mu\), where \(\mu\) is the (computable) slope of the meridian of the solid torus \(M\), for which Dehn fillings on \(X\) are irreducible. Hence, \(X(\alpha)\) can fiber over \(S^1\) with fiber a surface only for those slopes \(\alpha \in L_\mu\) and then if and only if we have \(N\) fibers over \(S^1\) with fiber a surface. Again, this can be determined by Theorem 5.15 above.

Hence, if \(X\) is reducible, we can determine precisely those slopes \(\alpha\) for which the Dehn filling \(X(\alpha)\) fibers over \(S^1\).

So, suppose \(X\) is irreducible. The argument in this case is very similar to the combination of arguments used in Theorem 5.10, Lemma 5.11, and Theorem 5.12. By Theorem 5.4, if \(X(\alpha)\) fibers over \(S^1\), there is a vertex solution \(F\) of \(P(X,T)\) that is an embedded, essential, two-sided surface and \(F(\alpha)\) is a fiber in a fibration over \(S^1\). We may also assume that \(F\) does not separate \(X\) for otherwise \(F(\alpha)\) could not be a fiber in a fibration of \(X(\alpha)\) over \(S^1\).

Suppose \(\partial F \neq \emptyset\). By Theorem 5.14 there is an algorithm to determine if \(F(\alpha)\) is a fiber in a fibration of \(X(\alpha)\) over \(S^1\). There are only finitely many such surfaces we need to check.

Suppose \(\partial F = \emptyset\). We have the embedded, essential, closed normal surface \(F\) in \(X\) and we wish to determine precisely those slopes \(\alpha\) for which \(F = F(\alpha)\) is a fiber in a fibration of \(X(\alpha)\) over \(S^1\).

As in the proof of Lemma 5.11, we consider two possibilities:

i. there is no annulus in \(X\) having one boundary component a non-trivial curve in \(F\) and the other a curve in \(\partial X\), or

ii. there is an annulus in \(X\) having one boundary component a non-trivial curve in \(F\) and the other a curve in \(\partial X\).
Suppose we are in situation i where there is no annulus in \( X \) having one boundary component a non-trivial curve in \( F \) and the other a curve in \( \partial X \). Split \( X \) at \( F \) to get the 3–manifold \( X_F \). The manifold \( X_F \) has two copies (\( F \) does not separate \( X \)) of \( F \), say \( F_0 \) and \( F_1 \), along with \( \partial X \) as its boundary.

We extend our notion of a Dehn filling to this situation where the manifold \( X_F \) has components of the boundary other than the torus \( \partial X \). A slope \( \alpha \) will be an isotopy class of a simple closed curve in \( \partial X \) and a Dehn filling of \( X_F \) along \( \alpha \) is the 3–manifold obtained by attaching a solid torus \( V_\alpha \) to \( X_F \) via a homeomorphism of \( \partial X \) to \( \partial V_\alpha \) taking the slope \( \alpha \) to a meridian of \( V_\alpha \). We denote the Dehn filling of \( X_F \) along \( \alpha \) by \( X_F(\alpha) \). With this notation, we have \( X(\alpha) \) will fiber over \( S^1 \) with fiber \( F = F(\alpha) \) if and only if the Dehn filling \( X_F(\alpha) \) is homeomorphic to the product \( F \times [0,1] \). Hence, we wish to determine precisely those slopes \( \alpha \) for which the Dehn filling \( X_F(\alpha) \) is a product.

Give \( X_F \) a triangulation \( T_F \) that restricts to a one-vertex triangulation on the component of \( \partial X_F \) corresponding to \( \partial X \). If \( X_F(\alpha) \) is a product, then either there is an embedded, essential, punctured annulus \( Q \) in \( X_F \) with one component of \( \text{bdry} \( Q \) \) a nontrivial curve in \( F_0 \) and the other a nontrivial curve in \( F_1 \) and punctures in \( \partial X \) with slope \( \alpha \) or we have \( V_\alpha \), the attached solid torus, is contained in a 3–cell in \( X(\alpha) \). However, the latter situation could only happen if \( X \) were reducible. We conclude that for any Dehn filling with \( X(\alpha) \) a product, there is such a punctured annulus \( Q \) having punctures in \( \partial X \) with slope \( \alpha \).

We now use an average length estimate similar to that in the proof of Lemma 5.11 to give an algorithm to find such a punctured annulus.

If \( Q \) is an essential punctured annulus as above, then we may assume that \( Q \) is normal in \((X_F, T_F)\) and \( Q \) is least weight in its equivalence class. We have,

\[
Q = \sum_i k_i G_i + \sum_{j'} l'_j K_{j'} + \sum_j p_j A_{j'}^0 + \sum_{j'} q_{j'} A_{j'}^1 + \sum_{j''} r_{j''} A_{j''}^{0,1} + \sum_k s_k A_k^0
\]

where all of the summands are essential, normal, fundamental surfaces in \((X_S, T_S)\) (\([14]\)), and notation has been chosen so that \( \chi(G_i) < 0 \), each \( K_{j'} \) is either a torus or Klein bottle, \( A_{j'}^0 \) and \( A_{j'}^1 \) are annuli or Möbius bands with their boundaries in \( F_0 \) or \( F_1 \), respectively, each \( A_{j''}^{0,1} \) is an annulus with one boundary component in \( F_0 \) and the other in \( F_1 \), and each \( A_k^0 \) is an annulus or Möbius band with its boundary in the copy of \( \partial X \). Of course, it is possible that there are no factors \( K_{j'}, A_{j'}^0, A_{j'}^1, A_{j''}^{0,1}, \) and \( A_k^0 \). We have written the most general sum in this situation. Also, by assumption for this case, there are no annuli in \( X \) (and hence in \( X_F \)) having one boundary a nontrivial curve in \( F \) and the other boundary in \( \partial X \).

As in the proof of Lemma 5.11 if there are any annuli or Möbius bands of type \( A_k^0 \), then the slope \( \alpha \) is the same as the boundary slope of \( A_k^0 \), which is a computable slope of a fundamental surface of \((X_F, T_F)\). (Actually, if there
is an embedded, essential annulus $A_k$, then there is one whose projective class is also a vertex solution of $\mathcal{P}(X, T)$. So, we may assume that each $s_k = 0$.

Again we let $L(\partial G)$ denote the length of the boundary of a normal surface, $G$, in $(X, T)$, we have:

$$L(\partial Q) = \sum_i k_i L(\partial G_i) + \sum_j p_j L(\partial A_j^0) + \sum_{j'} q_{j'} L(\partial A_{j'}^1) + \sum_{j''} r_{j''} L(\partial A_{j''}^{0,1}).$$

Also,

$$-\chi(Q) = \sum_i k_i (-\chi(G_i)).$$

Let

$$C = \max \left\{ \frac{L(\partial G_i)}{-\chi(G_i)} \right\}.$$

Notice that $C$ is computable for $G_i$ ranging over the embedded, normal, fundamental surfaces in $(X, T)$ with $\chi(G_i) < 0$ and $L(\partial G_i) < -\chi(G_i) C$ for all such $G_i$. Let $\gamma_0$ and $\gamma_1$ be the length of the components of $\partial(Q)$ in $F_0$ and $F_1$, respectively, and let $\gamma_\alpha$ denote the length of the slope $\alpha$. If $Q$ has $q$ punctures, then $-\chi(Q) = q$ and $L(\partial Q) = \gamma_0 + \gamma_1 + q\gamma_\alpha$. Thus, if we set $L' = \sum_j p_j L(\partial A_j^0) + \sum_{j'} q_{j'} L(\partial A_{j'}^1) + \sum_{j''} r_{j''} L(\partial A_{j''}^{0,1})$, we have

$$\gamma_0 + \gamma_1 + q\gamma_\alpha = \sum_i k_i L(\partial G_i) + L'$$

$$\leq \sum_i (-\chi(G_i)) C + L'$$

$$= -\chi(Q) C + L'.$$

From this and the fact that $\gamma_0 + \gamma_1 - L' \geq 0$, we have

$$\gamma_\alpha \leq C.$$

So, in situation i and if $X_F(\alpha)$ is a product, $\alpha$ is either the boundary slope of an essential normal annulus or Möbius band in $(X, T)$ whose projective class is a vertex of $\mathcal{P}(X, T)$ or $\gamma_\alpha$, the length of the slope $\alpha$, satisfies $\gamma_\alpha \leq C$, where $C$ is computable from certain fundamental solutions in $(X, T)$. In either case, there are at most finitely many computable slopes $\alpha$ for which a Dehn filling of $X_F$ can be homeomorphic to a product $F \times [0, 1]$ and the Dehn filling $X(\alpha)$ can be a fibration over $S^1$ with fiber $F$.

Now, we consider situation ii where there is an annulus having one boundary component a nontrivial curve in $F_i$ and the other in $\partial X$, for either $i = 0, i = 1$ or both. Note that if there is any combination of such annuli, then by [13, 20] and the fact that $F$ is not peripheral, there is a unique slope for the components of all such annuli in $\partial X$. Furthermore, by the same argument as that in Lemma 5.11, if there is such an annulus, then there is one that is a fundamental solution of $(X_F, T_F)$; hence, the boundary slope on $\partial X$ of such an annulus, say $\alpha_0$, can be computed.
Notice that our analogy with Lemma 5.11 diverges at this point, as the existence of such an annulus gives that $F$ compresses in $X(\alpha_0)$ and so could not be a fiber in a fibration of $X(\alpha_0)$ over $S^1$. However, if for some Dehn filling along a slope $\alpha$, we do have that $X(\alpha)$ fibers over $S^1$ with fiber $F$, then, as above, there is an embedded, essential, punctured annulus in $X_F$ having one boundary component in $F_0$ and the other in $F_1$ and punctures in $\partial X$ having slope $\alpha$. It follows from [2] that $\Delta(\alpha, \alpha_0) \leq 1$; hence, it is only possible for $X(\alpha)$ to fiber over $S^1$ with fiber $F$ for $\alpha \in L_{\alpha_0}$. Furthermore, one such Dehn filling will fiber over $S^1$ with fiber $F$ if and only if all do. By Theorem 5.15, there is an algorithm to check if any one does fiber over $S^1$. This completes the proof. 

**Algorithm F.** Given a knot-manifold $X$, determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is a fibration over $S^1$.

**Step 1.** $X$ is given via a triangulation. Endow $X$ with a triangulation $T$ that restricts to a one-vertex triangulation on $\partial X$.

**Step 2.** Compute $\mathcal{P}(X, T)$.

**Step 3.** Construct the irreducible decomposition of $X$.

- If $X$ has a non-separating 2-sphere, say $S$, then split $X$ at $S$ to form $X_S$ and then fill the resulting 2-sphere boundary components with 3-cells to get the knot-manifold $\hat{X}_S$. Use the algorithm of Theorem 6.4 to determine those slopes $\alpha$ for which $\hat{X}_S(\alpha)$ is $S^3$. It is precisely these slopes $\alpha$ for which the Dehn filling $X(\alpha)$ fibers over $S^1$ with fiber the 2-sphere $S$; and the algorithm terminates.

- If $X$ has two separating, independent, essential 2-spheres, then $X(\alpha)$ does not fiber over $S^1$ for any Dehn filling $\alpha$; and the algorithm terminates.

- If $X$ has precisely one, separating, essential 2-sphere, say $S$, then split $X$ at $S$ and fill the resulting 2-sphere boundary components with 3-cells to get the two 3-manifolds $M$ and $N$, where we choose notation so that $M$ contains the copy of $\partial X$. Determine if $N$ is a fibration over $S^1$. If $N$ does not fiber over $S^1$, then $X(\alpha)$ will not fiber over $S^1$ for any $\alpha$; and the algorithm terminates. If $N$ does fiber over $S^1$, use Theorem 6.4 to determine those slopes $\alpha$ for which Dehn filling on $M$ along $\alpha$ gives $S^3$; it is precisely these slopes for which $X(\alpha)$ is a fibration over $S^1$ and the algorithm terminates.

- If $X$ is irreducible, go to the next step.

**Step 4.** Let $\mathcal{F} = \{F_1, \ldots, F_J\}$ denote the collection of all embedded, closed, non-separating, two-sided, normal surfaces whose projective class is a vertex-solution of $\mathcal{P}(X, T)$. Let $\mathcal{B} = \{B_1, \ldots, B_K\}$ denote the collection of all embedded, non-separating, two-sided, normal surfaces with boundary consisting of nontrivial curves in $\partial X$ whose projective class is a vertex-solution of $\mathcal{P}(X, T)$. Compute the boundary slopes of the surfaces in $\mathcal{B}$, let $\beta_k$ denote the boundary slope of $B_k$. 


Step 5. Check if $B_k(\beta_k)$ is a fiber in a fibration of $X(\beta_k)$ over $S^1$. In this way we get a possible finite number of slopes $\beta_{ki}$ for which $X(\beta_{ki})$ fibers over $S^1$.

Step 6. For each $F_j \in \mathcal{F}$, split $X$ at $F_j$ and form $X_{F_j}$. Triangulate $X_{F_j}$ with a triangulation that restricts to a one-vertex triangulation on $\partial X$, say $T_{F_j}$. Compute the fundamental solutions of $(X_{F_j}, T_{F_j})$.

Step 7. Consider those $j, 1 \leq j \leq J$, for which a fundamental solution is an embedded annulus with one boundary component in $F_j$ and the other in $\partial X$, compute the slope of the component of the boundary in $\partial X$, say $\alpha_j$. Compute the line $L_{\alpha_j} = \{ \beta : \Delta(\alpha_j, \beta) \leq 1 \}$. For some $\beta_0 \in L_{\alpha_j}$ check if $F_j$ is a fiber in a fibration of $X(\beta_0)$ over $S^1$. If yes, then $F_j$ is a fiber in a fibration over $S^1$ for all $\beta \in L_{\alpha_j}$. If no, then $X(\alpha)$ does not fiber over $S^1$ with $F_j$ a fiber for any $\alpha$.

Step 8. Consider those $j, 1 \leq j \leq J$, for which no fundamental solution is an embedded annulus with one boundary component in $F_j$ and the other in $\partial X$. If some fundamental solution is an embedded annulus with both its boundary components in $\partial X$, then compute the boundary slope of such an annulus in $\partial X$, say, $\alpha_{j0}$. Note there is only one such slope for such embedded essential annuli and it may be necessary to determine if the annulus is essential.

Let $\{G_{j1}, \ldots, G_{jNj}\}$ denote the fundamental solutions of $(X_{F_j}, T_{F_j})$ for which $\chi(G_{ji}) < 0$. Compute

$$C_j = \left\{ \frac{L(\partial G_{ji})}{-\chi(G_{ji})} \right\}.$$  

For each $j, 1 \leq j \leq J$, compute all slopes in $\partial X$ having length less than $C_j$. Let $\{\alpha_{j0}, \alpha_{j1}, \ldots, \alpha_{jK_j}\}$ be this set of slopes along with the slope $\alpha_{j0}$, if found above. Check if $F_j$ is a fiber in a fibration of $X(\alpha_{ji})$ for each of these slopes. It is precisely these slopes $\alpha$ for which the surface $F_j$ is a fiber in a fibration over $S^1$.

Step 9. The union of the slopes found in Step 5, in Step 7, and in Step 8 determine all slopes $\alpha$ for which $X(\alpha)$ fibers over $S^1$; and the algorithm terminates.

6. Decision Problems in the Space of Dehn Fillings: Heegaard Surfaces

In the last section we used normal surface theory to determine precisely those slopes for which Dehn fillings had “interesting” essential surfaces. In this section we use almost normal surface theory in order to add Heegaard surfaces to our list of interesting surfaces. We employ the ideas of J. H. Rubinstein (almost normal surfaces and sweep outs) and of D. Gabai (thin position), along with the work of A. Thompson [26] and M. Stocking [25].
We are able to give algorithms to determine for a given knot-manifold $X$ precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is either the 3–sphere or a lens space.

We will use two important solutions to the homeomorphism problem for 3–manifolds. The first is a restatement of Theorem 5.5, which is directly applicable to this section.

**6.1. Theorem.** [23, 26] Given a compact 3–manifold $M$, it can be decided if $M$ is homeomorphic to $S^3$.

**6.2. Theorem.** [23] Given a compact 3–manifold $M$ it can be decided if $M$ is homeomorphic to a lens space.

Both of these algorithms are based on the fact that given a triangulation of $S^3$ or of a lens space, a strongly irreducible Heegaard surface (a 2–sphere in $S^3$ or a torus in a lens space) is isotopic to an almost normal surface. (See [25] for the general case.) From these algorithms, if we are given a knot-manifold $X$ and a slope $\alpha$, we are able to determine if the Dehn filling $X(\alpha)$ is $S^3$ or a lens space; however, these algorithms are not sufficient (do not provide finite algorithms) to answer the general questions as to precisely which slopes $\alpha$ the Dehn filling $X(\alpha)$ is either $S^3$ or a lens space or whether there is a Dehn filling of $X$ that is either $S^3$ or a lens space.

The generic model [4] is that there are only a finite number of slopes along which a knot-manifold $X$ can be filled to produce $S^3$ or a lens space. We obtain a finiteness result and more by showing that the slopes giving Dehn fillings that are either $S^3$ or a lens space arise as the slopes of embedded normal or almost normal surfaces or as the slope of an edge in $\partial X$ of the triangulation, a so-called boundary edge. This is a finite computable set of slopes. We identify and analyze a few exceptional cases that arise when the core of the solid torus that is attached to $\partial X$ is isotopic into the minimal genus Heegaard splitting of the Dehn filling. In this event, thin position does not provide the desired conclusion. For example, for fillings giving $S^3$, this occurs when the core of the attached solid torus is isotopic into a 2–sphere; so, the core is an unknot and its exterior is a solid torus.

Fortunately, we are able to identify and analyze these exceptions. A knot-manifold is a solid torus if and only if it has compressible boundary and is irreducible. We can determine when a manifold has compressible boundary [1, 12] and when it is irreducible [23, 24] (see Theorem 5.6). Note that Haken’s original algorithm to recognize the unknot [8] consisted of finding a compressing disk for the boundary of the knot-manifold combined with the advance knowledge that the knot-manifold was contained in $S^3$ (was irreducible).

We make the following convention. If $X$ is embedded in $M$ as a knot-manifold, the exterior of a knot in $M$, then there is a unique slope in $\partial X$ that we call the meridional slope or a meridian. In $S^3$ there is a unique slope that has distance 1 from the meridian and bounds a properly embedded,
orientable surface in \( X \); its slope is called a longitude. However, in general, there is no such unique curve in \( \partial X \); so, we shall refer to any slope in the line of slopes having distance 1 from the meridian as a \textit{generalized longitude}.

We have the following lemma.

\textbf{6.3. Lemma.} \textit{Let \( X \) be the exterior of a non-trivial knot in \( S^3 \) and \( \mathcal{T} \) a one-vertex triangulation of \( X \). Then \( (X, \mathcal{T}) \) contains a normal or almost normal planar surface with an essential boundary curve that has slope a meridian.}

\textit{Remark.} For the purposes of this lemma an almost normal surface possesses a single octagon (no tubes).

\textit{Proof.} The proof of this lemma is adapted from Thompson’s proof of the existence of an almost normal sphere in a triangulation of \( S^3 \) \cite{Th}. It differs in that we guarantee that there is a level surface which intersects the boundary torus \( \partial X \) in a collection of curves that includes an essential curve with meridional slope. Both are applications of Gabai’s notion of \textit{thin position} \cite{Ga} to an embedding of the 1-skeleton of a triangulation, and we assume that the reader has a familiarity with the basic concepts. For more detailed information on thin position for graphs the reader is directed to \cite{Ja13}.

By assumption, \( X \) is the exterior of a non-trivial knot in \( S^3 \) and it is endowed with a one-vertex triangulation, \( \mathcal{T} \). Note however, that \( \mathcal{T} \) is \textit{not} a triangulation of \( S^3 \); the exterior of the 2-skeleton of \( \mathcal{T} \) is a collection of tetrahedra and a single solid torus (the neighborhood of the knot).

Consider the singular foliation of \( S^3 \) induced by its genus 0 Heegaard splitting. Each leaf of the foliation, \( S_t, 0 < t < 1 \), is a 2-sphere except for \( S_0 \) and \( S_1 \) which are single points. We think of this foliation in terms of the height function that it induces: \( h : S^3 \rightarrow [0, 1] \). Arrange \( \mathcal{T} \) to be in general position with respect to this foliation and so that the boundary vertex is held fixed at \( S_1 \). We define the width of the one-skeleton \( \mathcal{T}^{(1)} \) to be

\begin{equation}
 w(\mathcal{T}^{(1)}) = \sum |\mathcal{T}^{(1)} \cap S_t|,
\end{equation}

where the sum is taken over level surfaces, \( S_t \), where one level surface is chosen between each pair of successive critical values of \( h : \mathcal{T}^{(1)} \rightarrow [0, 1] \). Among such generic embeddings of \( \mathcal{T} \) choose one which minimizes the width of the 1-skeleton, \( w(\mathcal{T}^{(1)}) \). This is called a \textit{thin position} for \( \mathcal{T}^{(1)} \).

\textit{Claim.} Suppose that a sphere \( S \) intersects \( \partial X \) in a non-empty collection of curves, at least one of which is essential in \( \partial X \). Then the essential curves of intersection have slope a meridian on \( \partial X \).

We have a 2–sphere \( S \) which intersects \( \partial X \) in a non-empty collection of curves, at least one of which is essential in \( \partial X \). There is no loss in generality to assume that among all 2–spheres meeting \( \partial X \) in the same slope as \( S \), \( S \) has the minimal number of curves that are inessential in \( \partial X \). Let \( c \) be a curve of intersection which is innermost on the sphere \( S \). If \( c \) is inessential on \( \partial X \), then we may perform an isotopy of \( S \) that removes \( c \) (and perhaps
some other inessential curves of intersection), a contradiction. We conclude that \(c\) is an essential curve in \(\partial X\) bounding an embedded disk whose interior is disjoint from \(\partial X\). But, \(X\) is the exterior of a non-trivial knot in \(S^3\), so \(c\) must be the meridional slope on \(\partial X\). Any other essential curve in the intersection is parallel to \(c\), and is therefore also a meridian curve. This completes the proof of the claim.

Each of the boundary edges of the triangulation is a loop (\(T\) is a one-vertex triangulation on \(\partial X\)) and, therefore, defines a knot in \(S^3\). The bridge number of a knot \(K\) relative to a height function \(h\), is its minimum number of maxima, taken over all generic embeddings of knots \(K'\) that are ambient isotopic to \(K\).

**Claim.** If a boundary edge \(e\) of \(T\) has bridge number 1, then \(e\) is a meridian or a generalized longitude.

We assume that the boundary edge \(e\) of \(T\) has bridge number 1. Then there is an ambient isotopy of \(S^3\) so that with respect to the given genus zero Heegaard decomposition, \(e\) has only one maximum and one minimum. We may take this as the original embedding of \(X\) and \(\partial X\). Choose a level surface that intersects \(\partial X\) in a collection of curves that contains at least two essential components in \(\partial X\) (a Heegaard surface is separating). By the previous claim, each of these essential curves is a meridian. Moreover, there are at most 2 intersections between \(e\) and these curves, hence at most 1 intersection between \(e\) and each of these curves. For otherwise, \(e\) would necessarily contain more than one maximum and one minimum. As \(e\) intersects one of these meridians at most once, it is itself either a meridian (does not intersect) or a curve that meets a meridian exactly once, a generalized longitude. This completes the proof of the claim.

The three edges of \(T\) in \(\partial X\) meet pairwise in exactly one point. It is possible that one of these edges is a meridian and has bridge number 1. However, there must be at least two edges which are not bridge number 1. If two edges are bridge number 1, then at most one is a meridian, and so one is necessarily a generalized longitude. But, the fact that a generalized longitude has bridge number 1 implies that \(X\) is the exterior of an unknot, a contradiction. It follows that at least two boundary edges have bridge number at least 2, and in particular possess a maximum that is not the vertex of the triangulation.

Consider the height function as restricted to the 1-skeleton of the triangulation, \(h : \mathcal{T}^{(1)} \rightarrow [0, 1]\). A thick region for a set of edges \(E\) is a sub-interval \((a, b) \subset [0, 1]\) which consists only of regular values of \(h : \mathcal{T}^{(1)} \rightarrow [0, 1]\) and so that \(a\) is a critical value corresponding to a minimum of some edge \(e \in E\) and \(b\) is a critical value corresponding to a maximum of some edge \(e' \in E\) and this maximum is not the vertex.

We may choose \(e\), a boundary edge that has bridge number at least 2. There is necessarily a thick region for the edge \(e\). Identify all of the thick
regions for \( e \) and within each of these choose a thick region that is a thick region for all edges of the triangulation. This yields a collection of thick regions \( \{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\} \). Within each of the thick regions \((a_i, b_i)\) we apply the four claims of Thompson, each of which follows from thin position.

**Claim.** For some \( t_i \in (a_i, b_i) \) there is a level 2-sphere \( S_i = S_{t_i} \) which intersects the boundary of each tetrahedron in normal curves or curves disjoint from the 1-skeleton.

We can assume (see [24]) that at the top of the thick region \((a_i, b_i)\), just below \( b_i \), there is a high disk for the 1-skeleton which is contained in the 2-skeleton. A high disk is a boundary compression for \( S_{t_i} \) in the exterior of the 1-skeleton that starts above \( S_{t_i} \). We may also assume that there is a low disk contained in the 2-skeleton at the bottom of this thick level, just above \( a_i \). Thin position guarantees that for some value of \( t_i \) in this thick region there is a level surface \( S_i = S_{t_i} \) for which there is no high or low disk contained in the 2-skeleton. For otherwise, at some level between there would be a pair of cancelling high and low disks. In particular, the intersection of \( S_i \) with the boundary of each tetrahedron does not contain any curves which intersect the 1-skeleton but are not normal. Such a curve implies an innermost arc joining an edge to itself which defines a bigon in the 2-skeleton that is either a high or low disk. This completes the proof of the claim.

**Claim.** \( S_i \) does not intersect any tetrahedron \( \Delta \) in a normal curve of length greater than 8.

If there is a normal curve \( c \subset \partial \Delta \) of length greater than 8, then this curve must intersect some edge \( e \) at least three times [24]. Following \( e \) through three consecutive intersections with \( c \) we note that they cobound two bigons on \( \partial \Delta \), one above \( c \) and one below \( c \). Moreover these bigons may be chosen to be disjoint except for a single point of intersection on \( c \). They may contain portions of other edges (including \( e \)), but, by pushing them slightly into \( \Delta \), see Figure [24], they become a cancelling pair of high and low disks for the 1-skeleton. In particular, they can be used to guide an isotopy of \( e \) that reduces the width of the 1-skeleton. (It is possible that there are other curves of intersection \( c' \) that also intersect the portion of \( e \) that bounds the bigons. In this case the isotopy is even more beneficial in reducing width.) This completes the proof of the claim.

**Claim.** The sphere \( S_i \) does not intersect any tetrahedron \( \Delta \) in parallel curves of length 8.

A normal curve of length 8 on \( \partial \Delta \) intersects two distinct edges twice [26]. If \( c \) and \( c' \) are an outermost pair of parallel curves of length 8 then some edge \( e \) hits each twice and there are bigons bounded by both \( c \) and \( e \) and \( c' \) and \( e \). As \( c \) and \( c' \) are parallel and outermost, one bigon is a subdisk of the other and when the larger one is pushed slightly into \( \Delta \), it acts simultaneously
as a high disk and low disk that can be used to reduce the width. This completes the proof of the claim.

Claim. The sphere $S_i$ does not intersect distinct tetrahedra, $\Delta$ and $\Delta'$, in curves of length 8.

If $c$ is a curve of length 8 in $\partial \Delta$, it intersects two edges $e$ and $e'$ exactly twice. This defines two bigons, when pushed into $\Delta$ one is a high disk for $e$ and the other a low disk for $e'$. These disks are not disjoint when pushed into $\Delta$ and do not by themselves contradict thin position. However, we have the same situation in $\Delta'$ so we may choose a high disk in $\Delta$ and a low disk in $\Delta'$ which reduce width and contradict thin position. This completes the proof of the claim.

Claim. For some $i$ the intersection $S_i \cap \partial X$ contains a meridional curve in $\partial X$.

Consider the collection of level surfaces $\{S_1, S_2, \ldots, S_n\}$ one chosen for each of the thick regions $\{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}$. The surface $S_i$ was chosen within a thick region for the boundary edge $e$ and necessarily intersects $e$. By the first claim, if for some $i$, the intersection $S_i \cap \partial X$ contains an essential curve then that curve is meridional and we are done. The alternative is that each of these intersections $S_i \cap \partial X$ consists entirely of trivial curves, the normal ones are vertex linking and the others disjoint from the boundary edges. Choose the outermost vertex linking curve $c$. The curve $c$ bounds a disk $D$ in $\partial X$. The boundary edge $e$ intersects $D$ in two arcs that are joined to the vertex, call the union of these arcs $e'$. The remainder of $e$ is a single arc in $\partial X - D$ which is connected to the endpoints of $e'$, call this arc $e''$.

Now $e''$ can only possess a single maximum or minimum. For otherwise there would be a thick region for $e$ between some maximum and minimum of $e''$, and we have chosen thick regions $(a_i, b_i)$ and a level surface $S_i$ within each such thick region. The level surface $S_i$ would intersect the interior of the edge $e''$.

Then $e'$ is parallel in $D$ to a subarc of $c = \partial D$. Perform this isotopy, see Figure 25. Then the boundary edge $e$ can be isotoped so that it has only a single minimum and maximum. This contradicts our choice of an edge $e$ with bridge number at least 2, and we conclude that there must be
a curve of intersection that is essential in \( \partial X \) and by the above it must be meridional. This completes the proof of the claim.

Let \( S \) be one of the level spheres \( S_i \) which possesses a meridional curve of intersection with \( \partial X \). The arguments above guarantee that \( S \) intersects the 2-skeleton of \( T \) in normal curves and curves disjoint from the 1-skeleton. However, the intersection of \( S \) with each tetrahedron of \( T \) may not consist entirely of disks, there may be planar surfaces with more than one boundary component (tubes). So, within each tetrahedron \( \Delta \) compress \( S \) to a collection of disks. Throw away any component (a disk or a sphere) which does not intersect the 1-skeleton. Any component of the resulting surface is a normal or almost normal sphere or planar surface in \((X, T)\). At least one of these components \( S' \) is planar and has non-empty boundary containing at least two meridional curves.

If \( X \) is not the solid torus, then Lemma 6.3 and Corollary 3.10 imply that there is a finite computable set of slopes \( \alpha \) so that \( X(\alpha) = S^3 \). In fact, by the work of Gordon and Luecke [6] there is at most one filling on \( X \) which can produce \( S^3 \). We complement this result by giving an algorithm that either computes this slope or demonstrates that it does not exist.

**6.4. Theorem.** Given a knot-manifold \( X \), there is an algorithm to determine precisely those slopes \( \alpha \) for which \( X(\alpha) = S^3 \). In particular, this gives an algorithm to determine whether \( X \) embeds in \( S^3 \).

**Proof.** We are given \( X \) via a triangulation, we may assume that this is a one-vertex triangulation \( T \).

First, we determine whether \( X \) has compressible boundary [4, 12] and/or whether \( X \) is irreducible (Theorem 5.6). If \( X \) has compressible boundary and is irreducible, then \( X \) is the exterior of the unknot in \( S^3 \). Dehn filling along any slope \( \alpha \) which intersects the slope of the compressing disk once will produce \( S^3 \). This set is computable as the slope of the compressing disk is a by-product of these computations. If \( X \) is reducible, then \( X \) is not the exterior of a knot in \( S^3 \), no filling produces \( S^3 \).
We may therefore assume that $X$ is irreducible and not the exterior of an unknot ($\partial X$ is incompressible). If for any $\alpha$ the manifold $X(\alpha)$ is $S^3$, then by Lemma 3.3, $X$ possesses a normal or almost normal surface with slope $\alpha$. By Corollary 3.9, the slope $\alpha$ is the slope of a normal or almost normal surface (using a single octagon) whose projective class is at a vertex of $\mathcal{P}(X, T)$. There are only finitely many such slopes.

For each slope $\alpha$ bounding an embedded normal or almost normal surface whose projective class is at a vertex of $\mathcal{P}(X, T)$, use the filling described in Section 4 to construct the manifold $X(\alpha)$. Use the 3-sphere recognition, Theorem 6.1 (see [23, 26]) to determine whether $X(\alpha)$ is $S^3$. If any $X(\alpha)$ is $S^3$ then this is the sole filling producing $S^3$. If after checking all of this finite number of fillings, none is $S^3$, then no Dehn filling gives the 3–sphere and $X$ does not embed in $S^3$.

**Algorithm S.** *Given a knot-manifold $X$, determine precisely those slopes $\alpha$ for which $X(\alpha)$ is the 3–sphere.*

**Step 1.** Endow $X$ with a one-vertex triangulation and compute the vertices of $\mathcal{P}(X, T)$. (Using both normal surfaces and almost normal surfaces with only octagons.)

**Step 2.** Determine whether $X$ is reducible. If so, no filling can produce the 3–sphere and the algorithm terminates.

**Step 3.** Determine whether $X$ has compressible boundary, i.e., whether there is a normal disk $D$ with its boundary an essential curve $\mu$ in $\partial X$. If so, then $X$ is a solid torus and Dehn filling along any slope on the line $L_\mu$ produces the 3–sphere.

**Step 4.** List the slopes $\{\alpha_1, \ldots, \alpha_n\}$ that correspond to embedded vertex surfaces of $\mathcal{P}(X, T)$. For each $\alpha_i$ construct $X(\alpha_i)$ via a layered triangulation and determine whether it is the 3–sphere using the algorithm given by Theorem 6.1. If any such filling is found, terminate the algorithm, it is the only filling producing the 3–sphere.

The slopes from Step 3 or slope from Step 4 are the only Dehn fillings yielding the 3–sphere.

Suppose $X$ is the exterior of a knot $K$ in a lens space. If $K$ is isotopic into a Heegaard torus and $X$ has incompressible boundary then we say that $K$ is a *generalized torus knot* (in a lens space). When the knot-manifold $X$ has compressible boundary or the exterior of a generalized torus knot we get special cases for lens space fillings. Note that the exterior of a generalized torus knot is the union of two solid tori glued along an incompressible and $\partial$-incompressible annulus $A$, i.e. a Seifert fibered space over the disk with 2 exceptional fibers. If $\alpha$ is the slope of the annulus $A$ on $\partial X$ and $\beta \in L_\alpha$ then $X(\beta)$ will possess a genus one Heegaard splitting, i.e. is either $S^3$ or a lens space. Thus, $X$ possesses an infinite number of slopes yielding lens spaces (at most one is $S^3$). We must be able to recognize this situation.
6.5. Lemma. Let $X$ be a knot-manifold. There is an algorithm to determine whether $X$ is a generalized torus knot exterior.

Proof. We may assume $X$ is given via a triangulation $\mathcal{T}$ that restricts to a one-vertex triangulation on $\partial X$.

Recall that the annulus $A$ characterizing a generalized torus knot exterior (an embedded, essential annulus separating the manifold into two solid tori) is vertical (composed entirely of regular fibers) with respect to the Seifert fibering of the manifold. Moreover, it is the unique essential annulus with boundary a regular fiber.

Hence, if $X$ is a generalized torus knot exterior it contains an essential annulus of the above type and if $\mathcal{T}$ is a triangulation of $X$ that restricts to a one-vertex triangulation on $\partial X$, then there is a normal annulus having these same properties and whose projective class is a vertex in $\mathcal{P}(X, \mathcal{T})$. To see this let $A$ be such an annulus and suppose that some multiple of $A$ can be written as a sum

$$kA = \sum_i k_i V_i,$$

where each $V_i$ is an incompressible and $\partial$-incompressible vertex surface (Theorem 6.3) and $\chi(V_i) \leq 0$ ($X$ has incompressible boundary and no summand can be a 2–sphere or $\mathbb{RP}^2$). Then $\chi(V_i) = 0, \forall i$ and some $V_i$, say $V_1$, has non-empty boundary with the same slope as $A$ (Proposition 6.7). So $V_1$ is either an annulus or a Möbius band and either $V_1$ or $2V_1$, respectively, is an essential annulus with the same boundary slope. This implies that $V_1$ or $2V_1$ is isotopic to $A$. Hence, there is such an annulus that has its projective class a vertex of $\mathcal{P}(X, \mathcal{T})$.

Now, to determine whether $X$ is a generalized torus knot exterior, first enumerate the vertices of $\mathcal{P}(X, \mathcal{T})$ that correspond to separating annuli. For each of these annuli $A$, split $X$ at $A$, retriangulate the components, and determine whether each is a solid torus (has compressible boundary and is irreducible). The knot-manifold $X$ is a generalized torus knot exterior if and only if we find such a decomposition. \hfill \Box

6.6. Lemma. Suppose $X$ is a knot-manifold with incompressible boundary which is not a generalized torus knot exterior, and that for some slope $\alpha$ the Dehn filling $X(\alpha)$ is a lens space. For any one-vertex triangulation $\mathcal{T}$ of $X$, either

1. $(X, \mathcal{T})$ contains a normal or almost normal surface (a punctured sphere or torus) with slope $\alpha$, or
2. $\alpha$ is the slope of an edge of the triangulation $\mathcal{T}$ in $\partial X$.

Remark. For the purposes of this lemma an almost normal surface possesses a single octagon (no tubes).

Proof. This lemma is an adaptation of Lemma 6.3, above, which was the case for non-trivial knots in $S^3$. We have that $X(\alpha)$ is a lens space and we
proceed as before, putting the 1-skeleton of $T$ in thin position. This time using a foliation of the lens space by level Heegaard tori $H_t$.

The first adjustment is a variation of the first claim of Lemma 6.3 for Heegaard tori in lens spaces.

Claim. Suppose that a Heegaard torus $H$ intersects $\partial X$ in a non-empty collection of curves, at least one of which is essential in $\partial X$. Then that curve has meridional slope on $\partial X$.

We have a Heegaard torus $H$ which intersects $\partial X$ in a non-empty collection of curves, at least one of which is essential in $\partial X$. There is no loss in generality to assume that among all Heegaard tori meeting $\partial X$ in the same slope as $H$, that $H$, itself, has the minimal number of curves. Suppose $c$ is a curve of intersection between $H$ and $\partial X$ which is inessential and inessential on the torus $H$. If $c$ is inessential on $\partial X$, then we may perform an isotopy of $H$ that removes $c$ (and perhaps some other inessential curves of intersection), a contradiction. So $c$ is an essential curve in $\partial X$ bounding an embedded disk whose interior is disjoint from $\partial X$. But $X$ has incompressible boundary, so $c$ must be the meridional slope on $\partial X$.

The alternative is that there is no curve $c$ which is inessential in the Heegaard torus $H$; hence, $H$ is cut into a collection of annuli by its intersection with $\partial X$. If any intersection curve is inessential in $\partial X$ then at least one of these annuli, call it $A$, joins an essential curve in $\partial X$ to an inessential curve in $\partial X$. This also shows that the intersection is meridional; perform a surgery on the annulus at the inessential end to produce a disk bounding the essential curve.

We are left in the case that every curve of intersection is essential in both $H$ and $\partial X$; thus cutting each into a collection of annuli. If any annulus is compressible in one of the solid tori bounded by $H$ then $\partial X$ is compressible or the slope is meridional. We are left assuming that each annulus is boundary parallel in the solid tori bounded by $H$. We can reduce the number of intersections (a contradiction) by pushing an outermost annulus out of one solid torus and into the other unless the surfaces intersect in exactly 2 curves, cutting each surface into 2 annuli. Each of the annuli from $\partial X$ are then isotopic to one of the annuli in $H$. If the two annuli are isotopic to distinct annuli, then $\partial X$ is isotopic to $H$, a contradiction. $X$ is not a solid torus. So they are both isotopic to the same annulus. This implies that the core of the attached solid torus is isotopic into the Heegaard torus and is either a generalized torus knot or $X$ has compressible boundary, a contradiction. This completes the proof of the claim.

The second claim follows exactly as before, an edge with bridge number 1 is either a meridian or a generalized longitude.

We now need to show that there is a boundary edge that has bridge number at least 2. If all three edges have bridge number 1, then two are generalized longitudes and the other a meridian. At this point, there is a notable difference with the $S^3$ case. In a lens space it is distinctly possible
for the longitude of a knot, hence the knot itself, to have bridge number 1, yet not be trivial (the boundary of its exterior is not compressible). In this case, we have the second conclusion of the theorem: one of the boundary edges is the slope of the meridian.

With this exception noted, we continue as before. Choose a boundary edge with bridge number at least 2, and identify its thick regions. Within each of these regions we choose a thick region for all edges of the triangulation. This produces a list of thick regions \{(a_1,b_1),(a_2,b_2), \ldots,(a_n,b_n)\}. In each thick region \((a_i,b_i)\) a level Heegaard torus \(H_i\) is found which intersects the boundary of each tetrahedron in normal curves and curves disjoint from the 1-skeleton. Furthermore, no normal curve is longer than 8, and there is at most one of length 8. One of these level surfaces \(H = H_i\) must intersect \(\partial X\) in a curve that is essential in \(\partial X\). These curves are meridional by the first claim. We compress \(H \cap X\) inside each tetrahedron and choose a normal or almost normal component \(H' \subset (X,T)\) with meridional slope. Compressing may have lowered genus so \(H'\) is either a punctured torus or a punctured sphere.

If the knot-manifold \(X\) has incompressible boundary and is not a generalized torus knot then Lemma 6.6 and Corollary 3.10 imply that there is a finite computable set of slopes \(\alpha\) so that \(X(\alpha)\) is a lens space. In fact, it is well known [2] that any pair of such slopes have distance at most 1, for a total of at most 3 slopes. Again, we complement this result by supplying an algorithm which either determines precisely these slopes or demonstrates that they do not exist.

6.7. Theorem. Given a knot-manifold \(X\) there is an algorithm to determine precisely those slopes \(\alpha\) for which the Dehn filling \(X(\alpha)\) is a lens space.

Proof. We may assume \(X\) is given via a one-vertex triangulation \(T\).

First, one determines whether \(X\) has compressible boundary, in which case a Dehn filling on \(X\) will produce a lens space only if \(X\) is a trivial knot in \(S^3\) or in a lens space. If \(X\) does have compressible boundary then we can find a normal disk \(D\) with essential boundary in \(\partial X\), call its slope \(\mu\). Cut \(X\) along \(D\) and cap off the resulting 2–sphere boundary with a ball to obtain a closed manifold \(\widetilde{X}_D\). Next, using the algorithm from [23], Theorem 6.1 above, determine whether \(\widetilde{X}_D\) is the 3–sphere, if so, \(X\) is a solid torus and every filling on \(L_\alpha\) produces the 3–sphere and every other filling produces a lens space. If \(\widetilde{X}_D\) is not the 3–sphere, using the algorithm from [23], Theorem 6.2 above, determine whether it is a lens space. If so, then \(X\) is the exterior of a trivial knot in a lens space and \(X(\beta)\) is a lens space precisely for \(\beta \in L_\mu\). If not, then no filling can produce a lens space.

If \(X\) is the exterior of a generalized torus knot, then we may determine so by Lemma 6.3. Moreover, that algorithm will produce the slope \(\alpha\) of the essential annulus. In this case \(X(\beta)\) is a lens space or \(S^3\) for precisely the slopes \(\beta \in L_\alpha\). By Theorem 7.4 we may identify which slope, if any, to fill along to obtain \(S^3\).
The remaining case is that $X$ has incompressible boundary and is not the exterior of a generalized torus knot. If for any $\alpha$ the manifold $X(\alpha)$ is a lens space, then by Lemma 6.6, $X$ possesses a normal or almost normal surface (using a single octagon) with slope $\alpha$ or $\alpha$ is the slope of a boundary edge. In the former case, Corollary 3.8 implies that $\alpha$ is the slope of a normal or almost normal surface whose projective class is at a vertex of $P(X, T)$. The lens space fillings can then be identified by performing the following steps. For each slope $\alpha$ which is either the slope of an embedded vertex normal or almost normal surface or the slope of one of the three boundary edges, use the filling described in Section 4 to construct the manifold $X(\alpha)$ and use the lens space recognition Theorem 23 to determine whether $X(\alpha)$ is a lens space.

Algorithm L. Given a knot-manifold $X$, determine precisely those slopes $\alpha$ for which $X(\alpha)$ is a lens space.

**Step 1.** Endow $X$ with a one-vertex triangulation and compute the vertex solutions of $P(X, T)$. (Again, we are considering both normal and almost normal surfaces with octagons.)

**Step 2.** If a vertex solution of $P(X, T)$ is a disk $D$ whose boundary is an essential curve in $\partial X$, then $\partial X$ is compressible. Determine the slope of $\partial D$, say $\mu$.

In this case, cut $X$ along $D$ and cap off the remaining 2–sphere boundary component with a ball. This yields a closed manifold $\hat{X}_D$. Determine whether $\hat{X}_D$ is the 3–sphere or a lens space. If $\hat{X}_D$ is the 3–sphere, then filling along every slope $\beta \in L_\mu$ yields $S^3$ and every other filling yields a lens space. If $\hat{X}_D$ is a lens space, then filling along every slope $\beta \in L_\mu$ yields that same lens space. In no other case is a lens space filling obtained; and the algorithm terminates.

**Step 3.** For each vertex solution of $P(X, T)$ that is a separating annulus, split $X$ along the annulus and determine if each component is a solid torus; i.e. if $X$ is a generalized torus knot exterior. If an annulus, say $A$, is found so that $X$ split at $A$ yields two solid tori, then compute $\alpha$, the boundary slope of $A$ (there is a unique such boundary slope). Then $X(\beta)$ is a lens space or $S^3$ for precisely those $\beta \in L_\alpha$. Algorithm $S$ can identify which slope, if any, produces $S^3$; and, the algorithm terminates.

If $X$ is not a generalized torus knot exterior (and $\partial X$ is incompressible), go to the next step.

**Step 4.** Enumerate the slopes $\{\alpha_1, \ldots, \alpha_n\}$ of vertex normal and almost normal surfaces. For each slope $\alpha_i$ construct $X(\alpha_i)$ via a layered triangulation of a solid torus and determine whether it is a lens space. If at any time three such slopes are found, terminate the algorithm, this is the maximum number of slopes yielding lens space fillings. This completes the algorithm.
7. Summary comments

The preceding considerations are well adapted to normal (and almost normal) surface theory. In each, our algorithms were based on finding interesting surfaces and are rather comprehensive in their application to exceptional and Haken Dehn fillings. However, there are some notable exclusions. We have not considered Dehn fillings that are Seifert fibered or have finite fundamental group (except for $S^3$ and lens spaces).

Our methods can be used to determine for a given knot-manifold $X$ those Dehn fillings that are Haken-manifolds and are Seifert fibered. The proof uses the methods of Section 6 and, while quite tedious, does not require new ideas. However, there is a major gap for applying our methods to determine small Seifert fibered manifolds, Seifert fibered manifolds that are not Haken-manifolds. The major problems here are probably associated to the lack of understanding of immersed (not embedded) normal surfaces. We give the following remaining open problems.

7.1. Problem. Given a 3–manifold $M$ that is known to be irreducible and not a Haken-manifold, is there an algorithm to determine if $M$ is a small Seifert fibered space?

7.2. Problem. Given a knot-manifold $X$ is there an algorithm to determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ is Seifert fibered?

While similar, the next problem probably calls for even a wider range of new ideas.

7.3. Problem. Given a knot-manifold $X$, is there an algorithm to determine precisely those slopes $\alpha$ for which the Dehn filling $X(\alpha)$ has finite fundamental group?

Finally, our objective has been to determine interesting phenomena in the space of Dehn fillings on a given knot-manifold $X$. One very interesting open problem is the homeomorphism problem for manifolds in the space of Dehn fillings on $X$.

7.4. Problem. Given the knot-manifold $X$ and slopes $\alpha$ and $\beta$ is there an algorithm to determine if $X(\alpha)$ and $X(\beta)$ are homeomorphic?

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