Efficient classical-communication-assisted local simulation of n-qubit GHZ correlations

Tracey E. Tessier, Ivan H. Deutsch, and Carlton M. Caves
Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131
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We present a local hidden-variable model supplemented by classical communication that reproduces the quantum-mechanical predictions for measurements of all products of Pauli operators on an n-qubit GHZ state (or “cat state”). The simulation is efficient since the required amount of communication scales linearly with the number of qubits, even though there are Bell-type inequalities for these states for which the amount of violation grows exponentially with n. The structure of our model yields insights into the Gottesman-Knill theorem by demonstrating that, at least in this limited case, the correlations in the set of nonlocal hidden variables represented by the stabilizer generators are captured by an appropriate set of local hidden variables augmented by n – 2 bits of classical communication.

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INTRODUCTION

Bell’s theorem codifies the observation that entangled quantum-mechanical systems exhibit stronger correlations than are achievable within any local hidden-variable (LHV) model. Beyond philosophical implications, the ability to operate outside the constraints imposed by local realism serves as a resource for many information processing tasks such as communication and cryptography.

The role of entanglement in quantum computation is less clear, for the issue is not one of comparing quantum predictions to a local realistic description, but rather one of comparing a quantum computation to the efficiency of a realistic simulation. Nevertheless, various results indicate some connection between entanglement and computational power. Entanglement is a necessity if a pure-state quantum computer is to have scalable physical resources. Moreover, systems with limited entanglement can often be efficiently simulated classically. Jozsa and Linden showed that if the entanglement in a quantum computer extends only to some fixed number of qubits, independent of problem size, then the computation can be simulated efficiently on a classical computer.

Despite these results, global entanglement is by no means sufficient for achieving an exponential quantum advantage in computational efficiency. The set of Clifford gates (Hadamard, Phase, and CNOT) acting on a collection of n qubits, each initialized to a fiducial state |0⟩, can generate globally entangled states, yet according to the Gottesman-Knill (GK) theorem, the outcomes of all measurements of products of Pauli operators can be simulated with O(n^2) resources on a classical computer. The GK theorem is an expression of properties of the n-qubit Pauli group P_n, which consists of all products of Pauli operators multiplied by ±1 or ±i: the allowed (Clifford) gates preserve P_n, and the allowed measurements are the Hermitian operators in P_n.

One approach to understanding the information processing capabilities of entangled states is to translate a quantum protocol involving entanglement into an equivalent protocol that utilizes only classical resources, e.g., the shared randomness of LHVs and ordinary classical communication. Toner and Bacon showed that the quantum correlations arising from local projective measurements on a maximally entangled state of two qubits can be simulated exactly using a LHV model augmented by just a single bit of classical communication. Pironio took this analysis a step further, showing that the amount of violation of a Bell inequality imposes a lower bound on the average communication needed to reproduce the quantum-mechanical correlations.

In this article we analyze the classical resources required to simulate measurements made on the n-qubit GHZ state (also called a “cat state”). We present a LHV model, augmented by classical communication, that simulates the quantum-mechanical predictions for measurements of arbitrary products of Pauli operators on this state. The simulation is efficient since the required amount of communication scales linearly with n. These results are somewhat surprising in light of the existence of Bell-type inequalities for n-qubit GHZ states where the amount of violation grows exponentially in the number of qubits.

Since the n-qubit GHZ state is generated by a circuit composed solely of Clifford gates, and since we consider only measurements of observables in P_n, our result yields an alternative perspective on the GK theorem. Whereas the GK simulation tracks the evolution of nonlocal hidden variables that specify the generators of the stabilizer, we simulate the circuit using local hidden variables that are supplemented by an efficient amount of classical communication to predict measurement outcomes. We conjecture that our result is general, i.e., that any GK circuit can be simulated with a LHV model plus an amount of communication that scales at most polynomially in the number of qubits. The existence of such an efficient classical model is currently under investigation.
Consider now the three-qubit GHZ state, $|\psi_3\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$, generated by the quantum circuit shown in Fig. 1. In the language of the GK theorem, the evolution of the state is tracked by the evolution of the stabilizer generators. The Hadamard gate $H$ transforms the Pauli operators $X, Y, Z$ according to $H X H^\dagger = Z$, $H Y H^\dagger = -Y$, and $H Z H^\dagger = X$. Similarly, under the action of CNOT, we have

$$
C (XI) C^\dagger = XX, \quad C (IX) C^\dagger = IX, \\
C (YI) C^\dagger = YX, \quad C (IY) C^\dagger =ZY, \\
C (ZI) C^\dagger = ZI, \quad C (IZ) C^\dagger = ZZ,
$$

(1)

where the first qubit is the control, the second is the target, and $I$ represents the identity operator.

The stabilizer generators evolve through the circuit in Fig. 1 as $|\langle ZII, IZI, IIZ \rangle H_1 \langle XII, IZI, IIZ \rangle CNOT_{12} \langle XXI, ZZI, IIZ \rangle CNOT_{13} \langle XXY, ZZI, ZIZ \rangle \rangle$. The full final stabilizer, consisting of all products of the generators, includes $-XYX$, $-YXY$, $-YYX$, and $XXX$. This means that $|\psi_3\rangle$ is a $+1$ eigenstate of these four operators, which implies a deterministic violation of the assumptions of local realism.

The GK description provides an efficient method for simulating the outcome of a measurement of any product of Pauli operators on the globally entangled state $|\psi_3\rangle$, but it does so via the nonlocal stabilizer generators. We replace this nonlocal resource with a local description, augmented by classical communication, by constructing a LHV table where each row represents a qubit and each column represents a measurement. Locality is enforced by only allowing changes in rows corresponding to qubits that participate in an interaction.

For the initial state $|000\rangle$, a measurement of $Z$ on any qubit yields $+1$ with certainty, and a measurement of $X$ or $Y$ yields $\pm 1$ with equal probabilities. The first table in Fig. 2 gives corresponding LHVs for this state, with $R_j$ denoting a classical random variable that returns $\pm 1$ with equal probability and $j$ labeling the qubit to which the random variable refers. The table is read by choosing a measurement and multiplying the corresponding entries. The resulting product, with $i$ discarded whenever it appears, is the outcome predicted by the LHV model. The LHV table yields the correct quantum-mechanical predictions for measurements of the $4^3 = 64$ products of Pauli operators on the state $|000\rangle$. The use of $i$ in the model, apparently just a curiosity, actually plays a crucial role. It simulates some of the conflicting predictions of commuting LHVs and anticommuting quantum operators, which are the basis of Mermin’s GHZ argument.[10]

In addition, modeling the CNOT gates relies on the transformations of the Pauli operators, which suggest that to simulate $H$, we should swap the $X$ and $Z$ entries, i.e., $X^a = Z^b$ and $Z^a = X^b$, and flip the sign of the $Y$ entry, i.e., $Y^a = -Y^b$, where $b$ and $a$ denote LHV values before and after a gate. Applying these rules to the first row leads to the second table in Fig. 2, which corrects quantum-mechanical predictions for all measurements of Pauli operators on the state $|000 + |111\rangle |00\rangle /\sqrt{2}$. This is not surprising since the state remains a product state, and it is well known that a LHV model can be constructed for a single qubit.[11] The usefulness of our model only becomes apparent when we apply it to entangled states.

Applying the first CNOT gate in Fig. 1 yields the Bell entangled state $|000 + |111\rangle |00\rangle /\sqrt{2}$. To update the LHV table entries under a CNOT, we use the following rules for the control $c$ and the target $t$:

$$
X^a_c = X^b_c X^a_t, \quad Y^a_c = Y^b_c X^a_t, \quad Z^a_c = Z^b_c, \\
X^a_t = X^b_t, \quad Y^a_t = Z^b_c Y^a_t, \quad Z^a_t = Z^b_c Z^a_t.
$$

(2)

The update rules for $H$ and CNOT keep the $X$ and $Z$ entries real and the $Y$ entry imaginary, and the CNOT rule preserves the correlation $XYZ = i$ that holds for each qubit after the operation of the Hadamard gate. Using the rules (2) in the first two rows gives the third table in Fig. 2 to represent the Bell state.

The LHV rules (2) must be consistent with the fifteen transformations of nontrivial Pauli products under CNOT. For example, the transformation $C (XI) C^\dagger = XX$ requires that $X^b_c = X^a_c X^a_t$, which does follow from the rules (2). The CNOT rules (2) are derived from the six transformations (1), and because $C = C^\dagger$, these rules are consistent with five other transformations. Consis-

FIG. 1: Circuit to generate the three-qubit GHZ state.

FIG. 2: Evolution of the LHV model during creation of the three-qubit GHZ state. The rules for updating the LHV tables are suggested by the equations for transforming Pauli operators.
tency with the remaining four transformations, those being $C(XY)C^\dagger = YZ$, $C(XZ)C^\dagger = -YY$, and the inverse transformations, requires that $X^bY^b = Y^aZ^a_i = Y^bZ^b_cZ^b_iX^b$ and $X^bZ^b_i = -Y^aY^a_i = -Y^bZ^b_cX^b$. These relations do not hold generally, but they are satisfied if the initial entries for both the control and target are correlated according to $XYZ = i$, with $X$ and $Z$ real and $Y$ imaginary. In all our applications of CNOT, these conditions hold. That they are not generally true is the chief obstacle to extending our results to arbitrary GK circuits and the entangled states they produce.

The third table in Fig. 2 gives the correct quantum-mechanical predictions for all measurements of Pauli products on the Bell state $\langle(00) + (11)\rangle/\sqrt{2}$. What is new are the correlations between the first two qubits in each column. For example, the single-qubit measurements $ZII$ and $IZI$ both return the random result $R_1$; the product of these outcomes always equals +1, the same as the outcome of a joint measurement of $ZZI$ on the first two qubits. In this context, the $i$'s in the correlated $Y$ entries now lead to a problem: the single-qubit measurements $YII$ and $IYI$ both give the random result $R_2$, with product +1, inconsistent with the outcome $(iR_1R_2)(iR_1R_2) = -1$ of a joint measurement of $YYI$ [13]. This problem persists throughout our analysis, occurring for joint measurements involving $Y$'s on some qubits and having outcomes that are certain (i.e., measurements of stabilizer elements). This is the reason our LHV model must be supplemented by classical communication.

At this point the problem is restricted to the joint measurements $YYI$ and $YYZ$ and the corresponding local measurements and thus can be corrected by flipping the sign of the outcome whenever a local measurement of $Y$ is made on the first qubit; i.e., the model returns the random result $-R_1R_2$ for a measurement of $YII$. This sign flip fixes the required correlations and is irrelevant to other joint measurements that involve $Y$ on the first qubit, all of which have random results. Since the sign flip depends only on the measurement on the first qubit, it requires no communication between the qubits. Thus at this stage, with Bell-state entanglement, the LHV model gives correct quantum-mechanical predictions for all observables in $P_3$ and their correlations.

We complete the simulation of the creation of the GHZ state by performing the CNOT between the first and third qubits, resulting in the last table in Fig. 2. This table yields correct quantum-mechanical predictions for all of the observables in $P_3$, including those that form the basis of Mermin's GHZ argument [14], i.e., $XXX = 1$ and $XXY = YXY = YYX = -1$. As promised, the imaginary $Y$ entries make this agreement possible.

Consider now the scheme for ensuring consistency with local measurement predictions for the three-qubit GHZ state. The only local measurements that yield inconsistent results are those associated with stabilizer elements that contain $Y$'s, i.e., the joint measurements $XYX$, $YXY$, and $YYX$. Let Alice, Bob, and Carol each possess one of the qubits. If we put Alice in charge of ensuring compatibility, she should flip the sign of her outcome whenever she and/or Bob measures $Y$ locally. This sign flip fixes the local correlations associated with $XYX$, $YXY$, and $YYX$ and is irrelevant to other possible joint measurements that involve $Y$'s on the first two qubits, all of which have random outcomes. To implement this scheme, Bob must communicate to Alice one bit denoting whether or not he measured $Y$. For the three-qubit GHZ state, we thus have a LHV model, assisted by one bit of classical communication, that duplicates the quantum-mechanical predictions for all measurements in $P_3$ and their correlations.

The circuit that creates the general $n$-qubit GHZ state, $|\psi_n\rangle = (|00\ldots0\rangle + |11\ldots1\rangle)/\sqrt{2}$, has the same topology as in Fig. 1 a Hadamard on the first qubit is followed by $n-1$ CNOT gates, with the leading qubit as the control and the remaining qubits serving successively as targets. The operator transformations show that $|\psi_n\rangle$ is specified by the $n$ stabilizer generators $(X^{\otimes n}, Z{ZI}^{\otimes(n-2)}, ZIZ{I}^{\otimes(n-3)}, \ldots, Z{I}^{\otimes(n-2)}Z)$. The full stabilizer consists of the $2^n$ observables in $P_n$ that yield +1 with certainty. It contains Pauli products that have (i) only $Y$'s and an even number of $Z$'s and (ii) only $X$'s and an even number of $Y$'s, with an overall minus sign if the number of $Y$'s is not a multiple of 4. Of the remaining $2 \times 4^n$ observables in $P_n$, $2^n$ are negatives of the stabilizer elements, thus yielding $-1$, while the rest return $\pm 1$ with equal probability [4].

Following the same procedure as in the three-qubit case, one finds that the LHV table representing the $n$-qubit GHZ state is given by

$$
\begin{array}{cccc}
X & Y & Z \\
\text{qubit 1} & R_2R_3\cdots R_n & iR_1R_2\cdots R_n & R_1 \\
\text{qubit 2} & R_2 & iR_1R_2 & R_1 \\
\text{qubit 3} & R_3 & iR_1R_3 & R_1 \\
\vdots & \vdots & \vdots & \vdots \\
\text{qubit n} & R_n & iR_1R_n & R_1 \\
\end{array}
$$

That this table gives the correct quantum-mechanical predictions for all measurements of Pauli products follows from the consistency of our LHV update rules, but it is nevertheless useful to check this directly. Suppose a Pauli product contains no $X$’s or $Y$’s, but consists solely of $I$’s and $Z$’s. Then it is clear from the table in Eq. (3) that the outcome is certain if and only if the number of $Z$’s in the product is even. Suppose now that the product has an $X$ or a $Y$ in the first position. Then it is apparent that to avoid a random variable in the overall product, all the other elements in the product must be $X$’s or $Y$’s and the number of $Y$’s must be even; the outcome is +1 if the number of $Y$’s is a multiple of 4 and $-1$ otherwise. Finally, suppose the Pauli product has an $X$ or a $Y$ in a
position other than the first. Then the only way to avoid a random variable in the overall product is to have an $X$ or a $Y$ in the first position, and we proceed as before. This argument shows that the LHV table for the $n$-qubit GHZ state gives correct quantum-mechanical predictions for measurements of all Pauli products.

It remains to ensure that the products of the LHV predictions for local measurements are consistent with the corresponding joint measurement results. As before, the source of the inconsistency is the $i$ in the $Y$ table entries, the very thing that allows us to get all the Pauli products correct. Stationing Alice at the first qubit and putting her in charge of ensuring consistency, we see that what she needs to know is the number of $i$'s in the product for the corresponding joint measurement. In particular, letting $q_j = i$ if $Y$ is measured on the $j$th qubit and $q_j = 1$ otherwise, Alice can ensure consistency by changing the sign of her local outcome if the product $p_n = q_1 \cdots q_n$ is $-1$ or $-i$ and leaving her local outcome unchanged if $p_n$ is $+1$ or $i$. This scheme requires $n - 1$ bits of communication as each of the other parties communicates to Alice whether or not he measured $Y$, but we can do a bit better. Alice's action is only important when $p_n$ is $+1$ or $-1$; when $p_n$ is $i$ or $-i$, the sign flip or lack thereof is irrelevant because the joint measurement outcome is random. As a result, Alice can get by with the truncated product $p_{n-1} = q_1 \cdots q_{n-1}$; she flips the sign of her local outcome if $p_{n-1}$ is $i$ or $-1$ and leaves the local outcome unchanged if $p_{n-1}$ is $-i$ or $1$. The scheme works because whether $q_n$ is $1$ or $i$, Alice flips when $p_n = -1$ and doesn't flip when $p_n = +1$, as required. This improved scheme requires $n - 2$ bits of classical communication; it generalizes our previous results for the Bell state and the three-qubit GHZ state.

The consistency scheme generalizes trivially to the case of measurements made on $l$ disjoint sets of qubits. For each set $k$ chosen from the $l$ sets, the table yields a measurement product that is the predicted outcome multiplied by $q_k = i$ or $q_k = 1$. Letting Alice be in charge of the first set, all but the last of the other sets communicates $q_k$ to Alice, who computes the product $q_1 \cdots q_{k-1}$ and decides whether to flip her set's outcome just as before. Consistency with the corresponding joint measurement is thus ensured at the price of $l - 2$ bits of communication.

Using local hidden variables and an efficient amount of classical communication, we have shown that it is possible to simulate the correlations that arise when measuring arbitrary products of Pauli operators on an $n$-qubit GHZ state. Though the $n$-qubit GHZ state is highly entangled, the probability distributions for the allowed measurements of Pauli products are essentially trivial, being either certainty or binary randomness. This property is shared by all states produced by GK circuits, leading us to conjecture that our results can be extended to measurements of Pauli products on any state produced by a GK circuit. In contrast, allowing just one additional nontrivial measurement, say of $(X + Z)/\sqrt{2}$, leads to correlations for which our simple simulation will no longer work. We anticipate that under this more general measurement scheme, the amount of classical communication required to make a LHV model work grows exponentially in the number of qubits.

Our model provides weak evidence that the power of quantum computation arises not directly from entanglement, but rather from the lack of an efficient, local realistic description assisted by an efficient amount of nonlocal, but classical communication. An efficient communication-assisted LHV model for all GK circuits would provide powerful additional evidence for this idea.

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* Electronic address: tessiert@info.phys.unm.edu

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