MAXIMAL LINEABILITY OF THE SET OF CONTINUOUS SURJECTIONS

NACIB GURGEL ALBUQUERQUE

Abstract. Let $m, n$ be positive integers. In this short note we prove that the set of all continuous and surjective functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ contains (excluding the 0 function) a $c$-dimensional vector space. This result is optimal in terms of dimension.

1. Preliminaries

Lately the study of the linear structure of certain subsets of surjective functions in $\mathbb{R}^R$ (such as everywhere surjective functions, perfectly everywhere surjective functions, or Jones functions) has attracted the attention of several authors working on Real Analysis and Set Theory (see, e.g. [1, 2, 4, 6, 7]). The previously mentioned functions are, indeed, very “pathological”: for instance an everywhere surjective function $f$ in $\mathbb{R}^R$ verifies that $f(I) = \mathbb{R}$ for every interval $I \subset \mathbb{R}$ and the other classes (perfectly everywhere surjective functions and Jones functions) are particular cases of everywhere surjective functions and, thus, with even “worse” behavior. It has been shown [5] that there exists a $2^c$-dimensional vector space every non-zero element of which is a Jones function and, thus, everywhere surjective (here, $c$ stands for the cardinality of $\mathbb{R}$). Of course, this previous result is optimal in terms of dimension since $\dim(\mathbb{R}^R) = 2^c$. However, all the previous classes are nowhere continuous, thus, it is natural to ask about the set of continuous surjections. The aim of this short note is to prove, in a more general framework that of $\mathbb{R}^R$, that (for every $m, n \in \mathbb{N}$) the set of continuous surjections from $\mathbb{R}^m$ onto $\mathbb{R}^n$ is $c$-lineable [1] (that is, it contains a $c$-dimensional vector space every non-zero element of which is a continuous surjective function from $\mathbb{R}^m$ onto $\mathbb{R}^n$). Since $\dim C(\mathbb{R}^m, \mathbb{R}^n) = c$ we have that this result would be the best possible in terms of dimension, that is, the set of continuous surjections from $\mathbb{R}^m$ onto $\mathbb{R}^n$ is maximal lineable [3].

While there are many trivial examples of surjective continuous functions in $\mathbb{R}^R$, coming up with a concrete example of a continuous surjective function from $\mathbb{R}$ onto $\mathbb{R}^2$ is a totally different story. The existence of a continuous surjection from $\mathbb{R}$ onto $\mathbb{R}^2$ (a Peano type function) can be found in [8, p. 42] or [9, p. 274]. Both references use the existence of a continuous surjection from $[0, 1]$ onto $[0, 1]^2$ (a Peano curve in $[0, 1]^2$ or a space filling curve). The existence of this curve is proved, for instance, in [8] invoking a result due to A. D. Alexandrov: there is a continuous surjection from the Cantor space $K$ onto any arbitrary nonempty compact metric space (see [8, p. 40]); in [9, section 44] the construction of the Peano curve is done geometrically, and is a consequence of the completeness of the space $C(X, M)$ of all continuous functions from a topological space $X$ to a complete metric space $M$, considering $C(X, M)$ with the uniform metric.

2010 Mathematics Subject Classification. 15A03.
Key words and phrases. lineability; spaceability; algebrability; Peano type function.
The author is supported by Capes.
2. The lineability of the set of continuous surjections from $\mathbb{R}^m$ to $\mathbb{R}^n$

Let $m$ and $n$ be positive integers. Throughout this note we shall denote

$$S_{m,n} = \{ f : \mathbb{R}^m \to \mathbb{R}^n : f \text{ is continuous and surjective} \}.$$

The following result shows that $S_{m,n} \neq \emptyset$, and uses the fact that $S_{1,2} \neq \emptyset$ (p. 42).

**Proposition 2.1.** Let $m, n \in \mathbb{N}$. There exists a continuous surjection $f : \mathbb{R}^m \to \mathbb{R}^n$.

**Proof.** Let us take $f \in S_{1,2}$. If $f_i := \pi_i \circ f$, $i = 1, 2$ denotes the $i$-coordinates functions of $f$ ($f = (f_1, f_2)$), then the map $id_\mathbb{R} \times f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $id_\mathbb{R} \times f(t, s) := (t, f_1(s), f_2(s))$ is a continuous surjection. Thus, $(id_\mathbb{R} \times f) \circ f$ is in $S_{1,3}$. Proceeding in an induction manner, we can assure the existence of a function $g$ belonging to $S_{1, n}$ for every $n \in \mathbb{N}$. Hence, defining $F : \mathbb{R}^m \to \mathbb{R}^n$ by $F := g \circ \pi_1$, i.e.,

$$F(x) = F(x_1, \ldots, x_m) = g(x_1), \quad \text{for all } x = (x_1, \ldots, x_m) \in \mathbb{R}^m$$

($\pi_1 : \mathbb{R}^m \to \mathbb{R}$ denotes the canonical projection over the first coordinate), we conclude that $F \in S_{m,n}$ ($F$ is composition of continuous surjective functions). \qed

Attempting maximal lineability of $S_{m,n}$ (that is, $\mathfrak{c}$-lineability) we make use of the following remark (inspired in a result from [1]), which indicates a method to obtain our main result.

**Remark 2.2.** Given a continuous surjection $f : \mathbb{R}^m \to \mathbb{R}^n$, suppose we have $X \subset C(\mathbb{R}^n; \mathbb{R}^n)$ a subset of $\mathfrak{c}$-many linearly independent functions such that every nonzero element of $\text{span}(X)$ is a continuous surjection. Then, we have that

$$\mathcal{Y} := \{ F \circ f \}_{F \in X} \subset C(\mathbb{R}^m; \mathbb{R}^n)$$

has cardinality $\mathfrak{c}$, is linearly independent and is formed just by continuous surjections. Moreover,

$$\text{span}(\mathcal{Y}) \subset S_{m,n} \cup \{0\},$$

obtaining the $\mathfrak{c}$-lineability of $S_{m,n}$.

In order to continue we shall need two lemmas and some notation. First, let us consider (for $r > 0$) the homeomorphism $\phi_r : \mathbb{R} \to \mathbb{R}$ given by

$$\phi_r(t) := e^{rt} - e^{-rt}.$$

**Lemma 2.3.** The subset $\mathfrak{A} := \{ \phi_r \}_{r \in \mathbb{R}^+}$ of $\mathbb{R}^\mathbb{R}$ is linearly independent, has cardinality $\mathfrak{c}$, and every nonzero element of $\text{span}(\mathfrak{A})$ is continuous and surjective.

**Proof.** First let us prove that every nonzero element $\phi = \sum_{i=1}^{k} \alpha_i \cdot \phi_{r_i} \in \text{span}(\mathfrak{A})$ is surjective. We may suppose that $r_1 > r_2 > \cdots > r_k$ and $\alpha_1 \neq 0$. Writing

$$\phi(t) = e^{rt} - e^{-rt},$$

we conclude that $\lim_{t \to +\infty} \phi(t) = \text{sign}(\alpha_1) \cdot \infty$ and $\lim_{t \to -\infty} \phi(t) = -\text{sign}(\alpha_1) \cdot \infty$. Thus, the continuity of $\phi$ assures its surjection. Now let us see that $\mathfrak{A}$ is linearly independent: suppose that $\psi = \sum_{i=1}^{n} \lambda_i \cdot \phi_{s_i} = 0$. If there is some $\lambda_j \neq 0$, we may suppose that $s_1 > \cdots > s_n$ and $\lambda_1 \neq 0$. Repeating the argument above, we obtain

$$\lim_{t \to +\infty} \psi(t) = \text{sign}(\lambda_1) \cdot \infty \quad \text{and} \quad \lim_{t \to -\infty} \psi(t) = -\text{sign}(\lambda_1) \cdot \infty,$$
which contradicts $\psi = 0$. This proves that $\mathfrak{A}$ is linearly independent. The other assertions are easy to prove.

For each $r = (r_1, \ldots, r_n) \in (\mathbb{R}^+)^n$, let $\varphi_r$ be the homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by $\varphi_r = (\phi_{r_1}, \ldots, \phi_{r_n})$, i.e.,

$$\varphi_r(x) := (\phi_{r_1}(x_1), \ldots, \phi_{r_n}(x_n)),$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Working on each coordinate, and using the previous lemma, we have the following.

**Lemma 2.4.** The set $\mathfrak{B} = \{\varphi_r\}_{r \in (\mathbb{R}^+)^n}$ of $C(\mathbb{R}^n; \mathbb{R}^n)$ is linearly independent, has cardinality $c$, and every nonzero element of span($\mathfrak{B}$) is continuous and surjective.

Now it is time to state and prove our main result.

**Theorem 2.5.** $S_{m,n}$ is $c$-lineable.

**Proof.** Let $f \in S_{m,n}$. Using the notation of the previous lemma and the ideas of the Remark 2.2, we now prove that the set $\mathfrak{C} = \{F \circ f\}_{F \in \mathfrak{B}}$ is so that span($\mathfrak{C}$) is the space we are looking for.

The surjectivity of $f$ assures that $G \circ f = 0$ implies $G = 0$, for every function $G$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus, if $G_i \in \mathfrak{B}$, $i = 1, \ldots, k$ and

$$0 = \sum_{i=1}^{k} \alpha_i \cdot G_i \circ f = \left(\sum_{i=1}^{k} \alpha_i G_i\right) \circ f,$$

then $\sum_{i=1}^{k} \alpha_i \cdot G_i = 0$; so since $\mathfrak{B}$ is linearly independent, we conclude that $\alpha_i = 0$, $i = 1, \ldots, k$ and thus, $\mathfrak{C}$ is linearly independent. Thus, clearly, it has cardinality $c$. Furthermore, any nonzero function

$$\sum_{i=1}^{l} \lambda_i \cdot F_i \circ f = \left(\sum_{i=1}^{l} \lambda_i F_i\right) \circ f$$

of span($\mathfrak{C}$) is continuous and surjective, since it is the composition of continuous surjective functions (recall that, from Lemma 2.3, $\sum_{i=1}^{l} \lambda_i F_i$ is a continuous surjective function). Therefore, span($\mathfrak{C}$) only contains, except the zero function, continuous surjective functions.

**Remark 2.6.** As we mentioned in the Introduction, and since $\dim C(\mathbb{R}^m, \mathbb{R}^n) = c$, this result is the best possible in terms of dimension. The next step (in sense of trying a similar result in higher dimensions) could be related to the lineability of $S_{m,N}$ (the set of the continuous surjections from $\mathbb{R}^m$ onto $\mathbb{R}^N$ with the product topology). However this is not possible, since $S_{m,N} = \emptyset$ ([4], p. 275).

**References**

[1] R. Aron, V. I. Gurariy, and J. B. Seoane-Sepúlveda, Lineability and spaceability of sets of functions on $\mathbb{R}$, Proc. Amer. Math. Soc. **133** (2005), no. 3, 795–803 (electronic).

[2] R. M. Aron and J. B. Seoane-Sepúlveda, Algebrability of the set of everywhere surjective functions on $C$, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), no. 1, 25–31.

[3] L. Bernal-González, Algebraic genericity of strict-order integrability, Studia Math. **199** (2010), no. 3, 279–293.

[4] L. Bernal-González, D. M. Pellegrino, and J. B. Seoane-Sepúlveda, Linear subsets of nonlinear sets in topological vector spaces, Bulletin of American Mathematical Society, in press.

[5] J. L. Gámez-Merino, Large algebraic structures inside the set of surjective functions, Bull. Belg. Math. Soc. Simon Stevin **18** (2011), no. 2, 297–300.

[6] J. L. Gámez-Merino, G. A. Muñoz-Fernández, V. M. Sánchez, and J. B. Seoane-Sepúlveda, Sierpiński-Zygmund functions and other problems on lineability, Proc. Amer. Math. Soc. **138** (2010), no. 11, 3863–3876.
[7] J. L. Gámez-Merino, G. A. Muñoz-Fernández, and J. B. Seoane-Sepúlveda, *Lineability and additivity in $\mathbb{R}^n$*, J. Math. Anal. Appl. **369** (2010), no. 1, 265–272.

[8] A. B. Kharazishvili, *Strange functions in real analysis, 2nd ed.*, Pure and Applied Mathematics (Boca Raton), vol. 272, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[9] J. R. Munkres, *Topology, 2nd ed.*, Prentice Hall, Upper Saddle River, NJ, 2000.