On the non-minimal gravitational coupling to matter

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Received 22 September 2008, in final form 16 October 2008
Published 2 December 2008
Online at stacks.iop.org/CQG/25/245017

Abstract

The connection between $f(R)$ theories of gravity and scalar–tensor models with a ‘physical’ metric coupled to the scalar field is well known. In this work, we pursue the equivalence between a suitable scalar theory and a model that generalizes the $f(R)$ scenario, encompassing both a non-minimal scalar curvature term and a non-minimum coupling of the scalar curvature and matter. This equivalence allows for the calculation of the PPN parameters $\beta$ and $\gamma$ and, eventually, a solution to the debate concerning the weak-field limit of $f(R)$ theories.

PACS numbers: 04.20.Fy, 04.80.Cc, 11.10.Ef

1. Introduction

Contemporary cosmology is faced with the outstanding challenge of understanding the existence and nature of the so-called dark components of the universe: dark energy and dark matter. The former is required to explain the accelerated expansion of the universe, and accounts for about 74% of the energy content of the universe; the latter is hinted, for instance, by the flattening of galactic rotation curves and cluster dynamics [1], and constitutes about 22% of the universe’s energy budget. Several theories have been put forward to address these issues, usually resorting to the introduction of new fields; for dark energy, the so-called ‘quintessence’ models consider the slow-roll down of a scalar field, thus inducing the observed accelerated expansion [2, 3]. For dark matter, several weak-interacting particles (WIMPs) have been suggested, many arising from extensions to the standard model (e.g. axions, neutralinos). A scalar field can also account for a unified model of dark energy and dark matter [4]. Alternatively, one can implement this unification through an exotic equation of state, such as the generalized Chaplygin gas [5].

Other approaches consider that these observational challenges do not demand the inclusion of extra dark energy and dark matter states in the universe but, instead, they hint at an
incompleteness of the fundamental laws and tenets of general relativity (GR). Following this line of reasoning, one may resort e.g. to extensions of the Friedmann equation that include higher-order terms in the energy density $\rho$ (see [6] and references therein). Another approach considers changes on the fundamental action functional: a rather straightforward approach lies in replacing the linear scalar curvature term in the Einstein–Hilbert action by a function of the scalar curvature, $f(R)$; alternatively, one could resort to other scalar invariants of the theory [7] (see [8] and references therein for a discussion).

As with several other theories [9, 10], solar system tests could shed some light onto the possible form and behaviour of these $f(R)$ theories; amongst other considerations, this approach is based either on the more usual metric affine connection, or on the so-called Palatini approach [11], where both the metric and the affine connection are taken as independent variables. As an example of a phenomenological consequence of this extension of GR, it has been shown that $f(R) \propto R^n$ theories yield a gravitational potential which displays an increasing, repulsive contribution, besides the Newtonian term [12].

Another line of action lies in the comparison between present and future observational signatures and the parametrized post-Newtonian (PPN) metric coefficients arising from this extension of GR, taken in the weak field limit and when the added degree of freedom may be characterized by a light scalar field [10]. Regarding this, some disagreement exists in the community, some arguing that no significant changes are predicted at a post-Newtonian level (see e.g. [13] and references therein); others defending that $f(R)$ theories yield the PPN parameter $\gamma = 1/2$, which is clearly disallowed by the current experimental constraint $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$ [14]. This result first arose from the equivalence of the theory with a scalar field model [15], which led to criticism from several fronts [16]; however, a later study implied that the result $\gamma = 1/2$ could be obtained directly from the original $f(R)$ theory [17] (see [18] for a follow-up and criticism).

Despite the significant literature on these $f(R)$ models, another interesting possibility has been neglected until recent times: including not only a non-minimal scalar curvature term in the Einstein–Hilbert Lagrangian density, but also a non-minimum coupling between the scalar curvature and the matter Lagrangian density; indeed, these are only implicitly related in the action functional, since one expects that covariantly invariant terms in $\mathcal{L}$ should be constructed by contraction with the metric (e.g. the kinetic term of a real scalar field, $g^{\mu\nu} \dot{\chi}_\mu \chi_\nu$). In regions where the curvature is high (which, in GR, are related to regions of high energy density or pressure), the implications of such a theory could deviate considerably from those predicted by Einstein’s theory [19]. Related proposals have been put forward previously to address the problem of the accelerated expansion of the universe [20] and the existence of a cosmological constant [21]. Other works have studied the behaviour of matter, namely changes to geodetic motion [19], the possibility of modelling dark matter [22], the violation of the highly constrained equivalence principle [23] and the effect on the hydrostatic equilibrium of spherical bodies such as the Sun; also, a viability criterion for these generalized $f(R)$ theories has been obtained [24].

In this study, we focus on the equivalence of a theory displaying a non-minimal coupling of the scalar curvature with matter with a scalar–tensor theory; through a conformal transformation of the metric, this yields a purely scalar theory, that is, one in which the curvature term appears isolated from any scalar field contribution. For definitiveness, we recast the theory in a form that is as consistent as possible with the work of [25], and in close analogy with the available equivalence with $f(R)$ models [15].

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2 The connection between the equivalence principle and the interaction between dark energy and dark matter has been discussed in [30].
This work is divided into the following sections: first, we introduce the gravity model and discuss some of its features; then, we recast it as a scalar–tensor theory with a suitable dynamical identification of the scalar fields, and then as a scalar theory with a conformally related metric and redefined scalar fields. The latter prompts a computation of the PPN parameters $\beta$ and $\gamma$, which is followed by a discussion of our results.

2. The model

Following the discussion of the previous section, one considers the action

$$S = \int \left[ \kappa f_1(R) + f_2(R)L \right] \sqrt{-g} \, d^4x, \quad (1)$$

where $\kappa = c^4/16\pi G$, $f_i(R)$ (with $i = 1, 2$) are arbitrary functions of the scalar curvature $R$, $L$ is the Lagrangian density of matter and $g$ is the metric determinant; the metric signature is $(-, +, +, +)$. The standard Einstein–Hilbert action is recovered by taking $f_2 = 1$ and $f_1 = R - \lambda$, where $\lambda$ is the cosmological constant (from now on, one works in a unit system where $c = \kappa = 1$).

Variation of this action with respect to the metric $g_{\mu\nu}$ yields the modified Einstein equations of motion, here arranged as

$$\left( F_1 + F_2 \right) \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 8\pi G f_2 T_{\mu\nu} + \frac{1}{2} \left[ f_1 - (F_1 + F_2) \right] g_{\mu\nu} + \left( \nabla_\mu - \nabla_\mu \right) (F_1 + F_2), \quad (2)$$

where one defines $\nabla_\mu \equiv \nabla_\mu \nabla_\nu$ for convenience, as well as $F_i(R) \equiv d f_i(R)/dR$, omitting the argument. The matter energy–momentum tensor is, as usual, defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \delta \left( \sqrt{-g} L \right) \delta g^{\mu\nu}. \quad (3)$$

The Bianchi identities, together with the identity $(\nabla_\mu - \nabla_\mu) F_i = R_{\mu\nu} \nabla^\mu F_i$, imply the non-(covariant) conservation law

$$\nabla^\mu T_{\mu\nu} = \frac{F_2}{f_2} (g_{\mu\nu} L - T_{\mu\nu}) \nabla^\mu R, \quad (4)$$

and, as expected, in the limit $f_2(R) = 1$, one recovers the conservation law $\nabla^\mu T_{\mu\nu} = 0$.

Since the energy–momentum tensor is not covariantly conserved, one concludes that the motion of matter distribution characterized by a Lagrangian density $L$ is non-geodesical. This, of course, may yield a violation of the equivalence principle, if the right-hand side of (4) differs between distinguishable matter distributions, which could be used to experimentally test the model and to obtain constraints on the function $f_2(R)$.

Finally, even if matter does not move according to geodesics given by the metric $g_{\mu\nu}$, one may still characterize the model via adequate PPN parameters, which indicate post-Newtonian deviations from the standard Schwarzschild form.

2.1. Equivalence with scalar–tensor theory

2.1.1. Survey. As in usual $f_1(R)$ models, one may rewrite the considered mixed curvature model as a scalar–tensor theory. One wishes to establish an equivalent action in the Jordan frame (where the scalar curvature appears coupled linearly to a function of scalar fields); this may be obtained in a similar fashion to usual $f(R)$ models (with $f_2(R) = 1$), so that the
equations of motion derived from the action functional coincide with those derived directly from action (1). However, one cannot simply consider an action of the form

$$S = \int [g(\Phi) R - V(\Phi) + h(\Phi) L] \sqrt{-g} \, d^4x.$$  \hspace{1cm} (5)

The equivalence with action (1) must stem from the equations of motion obtained through the variation of the action with respect to the non-dynamical scalar field $\Phi_1$; through a suitable definition of the functions $g(\Phi_1), h(\Phi_1)$ and the potential $V(\Phi_1)$, the variation of action (5) should yield the dynamic identification $\Phi_1 = \Phi_1(R)$, so that both actions are equal, that is

$$g(\Phi_1(R)) R - V(\Phi_1(R)) = f_1(R), \quad h(\Phi_1(R)) = f_2(R),$$  \hspace{1cm} (6)

or, conversely,

$$g(\Phi_1) R(\Phi_1) - V(\Phi_1) + h(\Phi_1) L = f_1(R(\Phi_1)), \quad h(\Phi_1) = f_2(\Phi_1).$$  \hspace{1cm} (7)

From action (5), the equation of motion for $\Phi_1$ reads

$$g'(\Phi_1) R - V'(\Phi_1) + h'(\Phi_1) L = 0,$$  \hspace{1cm} (8)

where the prime denotes differentiation with respect to $\Phi_1$. Integrating, one gets

$$g(\Phi_1) R - V(\Phi_1) + h(\Phi_1) L = f(R),$$  \hspace{1cm} (9)

which then yields an action without matter content,

$$S = \int f(R) \sqrt{-g} \, d^4x,$$  \hspace{1cm} (10)

indicating that one can resort to just one scalar field only when dealing with pure $f(R)$ models (that is, with $f_2(R) = 1$). This pathology has already been pointed out in [24], and in a following study [26] it is shown that one must resort to a second auxiliary real scalar field $\psi$.

In this approach, the authors chose an action of the form

$$S = \int \left[ f_1(\phi) + f_2(\phi) L + \psi(R - \phi) \right] \sqrt{-g} \, d^4x,$$  \hspace{1cm} (11)

so that the second scalar field enforces the identification $\phi = R$; however, variation of the above action with respect to $\phi$ yields the relation

$$\psi = F_1(\phi) + F_2(\phi) L,$$  \hspace{1cm} (12)

indicating that one cannot write one field solely in terms of the other, if $L \neq 0$ or $F_2 \neq 0$ at any point of spacetime (with $F_2 = 0$ falling back to the trivial case $f_2 = 1$ and $\psi = F_1(\phi)$).

Also, note that the GR limit $f_1 = R, f_2 = 1$ yields $\psi = 1$, so that this second scalar field becomes a Lagrangian multiplier enforcing the unnecessary relation $\phi = R$.

The on-shell action obtained by replacing (12) into action (11) is given by

$$S = \int \left[ f_1(\phi) + F_1(\phi)(R - \phi) + [f_2(\phi) + F_2(\phi)(R - \phi)] L \right] \sqrt{-g} \, d^4x.$$  \hspace{1cm} (13)

This differs from the unobtainable form of (5) in the dependence of $F(\phi, L) = F_1(\phi) + F_2(\phi) L$ on both the scalar field $\phi$ and the Lagrangian matter density.

It should be stressed that the GR limit $f_1 = R$ and $f_2 = 1$ disables the identification $\phi = R$: if one writes $f_1(\phi) = \phi + \epsilon \delta_1(\phi)$ and $f_2(\phi) = 1 + \lambda \delta_2(\phi)$, the equation of motion of $\phi$ becomes

$$(F_1' + F_2 L)(R - \phi) = (\epsilon \delta_1'' + \lambda \delta_2'')(R - \phi) = 0$$  \hspace{1cm} (14)

and taking the limit $\lambda, \epsilon \rightarrow 0$ gives a trivial identity.
Due to an analogous mechanism in the case of usual $f(R)$ theories (with $f_2 = 1$), it is misleading to consider the GR limit by inserting the expansion $f(R) = R + \epsilon \delta(R)$ into results that stem from the scalar field approach and then performing the limit $\epsilon \to 0$: the formalism itself breaks down at its inception. Hence, one cannot simply argue that the set of PPN parameters should depend on $\epsilon$, take the limit $\epsilon \to 0$ and state that, as a $f(R)$ theory collapses back to GR, so should the PPN parameter $\gamma$ approach unity—which does not happen if $\gamma = 1/2$ is constant and does not show a dependence on $\epsilon$.

As in usual $f(R)$ models, a suitable conformal transformation to the metric $g_{\mu\nu}^* = F(\phi, L)g_{\mu\nu}$ could be used to transform the above action (13) into a functional where the scalar curvature appears decoupled from other fields. However, the presence of the Lagrangian density in this transformation would render any comparison with standard scalar theories too complicated (including the extraction of the PPN parameters $\beta$ and $\gamma$); for this reason, the second scalar field $\psi$ will be kept in the remaining part of this study.

Note also that one could also opt for the equivalence with a theory with just one scalar field, by taking the action

$$S = \int \left[ f_1(\phi) + F_1(\phi)(R - \phi) + f_2(R) L \right] \sqrt{-g} \, d^4x. \quad (15)$$

However, the latter may not be written in the Jordan frame, if the scalar curvature appears in a nonlinear fashion in $f_2(R)$.

2.1.2. Two-field scalar–tensor model. In this study, one establishes the equivalence between action (1) and a scalar–tensor theory through (11), here rewritten as a Jordan–Brans–Dicke theory with a potential,

$$S = \int \left[ \psi R - V(\phi, \psi) + f_2(\phi) L \right] \sqrt{-g} \, d^4x, \quad (16)$$

where $\phi$ and $\psi$ are scalar fields and one defines

$$V(\phi, \psi) = \phi \psi - f_1(\phi). \quad (17)$$

As discussed in the previous section, variation of the action with respect to both scalar fields yields the dynamical equivalence $\phi = R$ and $\psi = F_1(\phi) + F_2(\phi) L$. Substituting into (16), one recovers the action for the mixed curvature model (1).

With the above considerations taken into account, one should note that the second scalar field $\psi$ is not a function of the curvature (or $\phi$) alone, but also of the matter Lagrangian $L$ (which is itself a scalar). The degrees of freedom of the matter fields $\chi$ which appear in the Lagrangian $L(g_{\mu\nu}, \chi)$ (including any kinetic terms) are displayed in the Einstein field equations, through the corresponding energy–momentum tensor $T_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8 \pi G f_2(\phi) \frac{T_{\mu\nu}}{\psi} - \frac{1}{2} g_{\mu\nu} \frac{V(\phi, \psi)}{\psi} + \frac{1}{\psi} \left[ g_{\mu\nu} - g_{\mu\nu} \square \right] \psi, \quad (18)$$

which, introducing $\phi = R$ and $\psi = F_1(\phi) + F_2(\phi) L$, recovers (2).

Using the Bianchi identities and the previous relation, one obtains

$$\nabla^\mu T_{\mu\nu} = \frac{1}{f_2(\phi)} [\nabla_\nu V(\phi, \psi) - R \nabla_\nu \psi - F_2(\phi) T_{\mu\nu} \nabla^\mu \phi]. \quad (19)$$

Since

$$\nabla_\nu V(\phi, \psi) = [\psi - F_1(\phi)] \nabla_\nu \phi + \phi \nabla_\nu \psi \quad (20)$$

we get

$$\nabla^\mu T_{\mu\nu} = \frac{1}{f_2(\phi)} [(\phi - R) \nabla_\nu \psi + [(\psi - F_1(\phi)) g_{\mu\nu} - F_2(\phi) T_{\mu\nu}] \nabla^\mu \phi], \quad (21)$$

which, upon the substitution, $\psi = F_1(\phi) + F_2(\phi) L$ and $\phi = R$, collapses back to (4).
3. Equivalence with a scalar theory

One may now perform a conformal transformation (see e.g. [27]), so that the curvature appears decoupled from the scalar fields $\phi, \psi$ (yielding the action in the so-called Einstein frame): writing $g^{\mu\nu} = \psi g^{\mu\nu} = A^{-2}(\psi) g^{\mu\nu}$, with $A(\psi) = \psi^{-1/2}$, one obtains

$$\sqrt{-g} = \psi^{-2}/\sqrt{-g^*}, \quad R = \psi \left[ R^* - 6\sqrt{\psi} \nabla^2 \left( \frac{1}{\sqrt{\psi}} \right) \right],$$

(22)

where $\nabla^2$ denotes the D’Alembertian operator, defined from the metric $g^{\mu\nu}$. From the definition of the energy–momentum tensor, this implies that $T^\mu_\nu = A^2(\psi) T^{\star\mu}_\nu$.

Introducing the above into action (16) yields

$$S = \int \left[ R^* - 6\sqrt{\psi} \nabla^2 \left( \frac{1}{\sqrt{\psi}} \right) - 4U + f_2 A^4 \mathcal{L}(A^2 g^{\mu\nu}_{\star}, \chi) \right] \sqrt{-g^*} \, d^4x,$$

(23)

where one defines

$$U(\phi, \psi) = \frac{1}{4} A^4(\psi) V(\phi, \psi) = \frac{\phi}{4\psi} - \frac{f_1(\phi)}{4\psi^2}.$$

(24)

Note that there are two couplings between the scalar fields $\phi, \psi$ and matter: the explicit coupling given by the factor $f_2(\phi) A^2(\psi)$, and an additional coupling due to the rewriting of the metric (in the Jordan frame) $g_{\mu\nu}$ in terms of the new metric $g^*_{\mu\nu}$.

One now attempts to recast the action in terms of two other scalar fields, endowed with a canonical kinetic term. For this, one first integrates the covariant derivative term by parts and uses the metric compatibility relations, obtaining

$$-6 \int \sqrt{\psi} \nabla^2 \left( \frac{1}{\sqrt{\psi}} \right) \sqrt{-g^*} \, d^4x = -3 \int g^{*\mu\nu} \frac{\psi_\mu \psi_\nu}{\psi^2} \sqrt{-g^*} \, d^4x.$$

(25)

By resorting to the divergence theorem, the first integral may be dropped, yielding

$$-6 \int \sqrt{\psi} \nabla^2 \left( \frac{1}{\sqrt{\psi}} \right) \sqrt{-g^*} \, d^4x = -3 \int g^{*\mu\nu} \frac{\psi_\mu \psi_\nu}{\psi^2} \sqrt{-g^*} \, d^4x.$$

(26)

One obtains the action

$$S = \int \left[ R^* - \frac{3}{2} g^{*\mu\nu} \frac{\psi_\mu \psi_\nu}{\psi^2} - 4U(\phi, \psi) + f_2 A^4 \mathcal{L}(A^2 g^{*\mu\nu}, \chi) \right] \sqrt{-g^*} \, d^4x.$$  

(27)

One may redefine the two scalar fields, so that their kinetic terms may be recast in the canonical way. Specifically, one aims at writing (see [25])

$$\frac{3}{2} \frac{\psi_\mu \psi_\nu}{\psi^2} = \frac{3}{2} (\log \psi)_\mu (\log \psi)_\nu \equiv 2 \sigma_{ij} \phi_i^j \phi_i^j,$$

(28)

with $i, j = 1, 2$; $\sigma_{ij}$ is the metric of the two-dimensional space of scalar fields (field-space metric, for short), and $\phi^1, \phi^2$ the two new scalar fields. Clearly, this prompts for the identification

$$\phi^1 = \frac{\sqrt{3}}{2} \log \psi, \quad \phi^2 = \phi,$$

(29)

and the field metric

$$\sigma_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$  

(30)
which indicates that only $\psi^1$ has a kinetic term. Despite this, it is clear that $\psi^2 = \psi$ is a distinct degree of freedom, since one cannot rewrite the potential $U(\phi, \psi)$ in terms of one scalar field alone:

$$U(\psi^1, \psi^2) = \frac{1}{4} \exp \left(-\frac{2\sqrt{3}}{3} \psi^1 \right) \left[ \psi^2 - f_1(\psi^2) \exp \left(-\frac{2\sqrt{3}}{3} \psi^1 \right) \right].$$  \hspace{1cm} (31)

In the trivial case $f_2(R) = 1$ or $L = 0$, one gets $\psi = F_1(\phi)$, that is, $\psi^1 \propto \log F_1(\psi^2)$: one degree of freedom is lost and this potential may be written as a function of just one of the fields.

Since the inverse field metric $\sigma^{ij}$ will be required to raise Latin indexes throughout the text, one still has to deal with the particular form of $\sigma_{ij}$, which is non-invertible. In order to cope with this caveat, one is free to add an antisymmetric part, rewriting it as

$$\sigma_{ij} = \begin{pmatrix} 1 & a \\ -a & 0 \end{pmatrix}. \hspace{1cm} (32)$$

with inverse

$$\sigma^{ij} = a^{-1} \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}. \hspace{1cm} (33)$$

Clearly, no physical results can depend on the value of $a$ that shows in the off-diagonal part of $\sigma_{ij}$—in particular, the values of the PPN parameters $\beta$ and $\gamma$, as shall be shown.

With this choice, the action now reads

$$S = \int \left[R^* - 2g^{\mu\nu} \alpha^{ij} \phi^i \phi^j_{,\mu} - 4U(\phi^1, \phi^2) + f_2(\phi^2) L^* \right] \sqrt{-g^*} \, d^4x,$$  \hspace{1cm} (34)

where one defines $L^* = A^*(\phi^1) L$, with $A(\phi^1) = \exp(-(\sqrt{3}/3)\psi^1)$. Variation of the action with respect to the metric $g^{\mu\nu}$ yields

$$R^*_{\mu\nu} - \frac{1}{2} g^*_{\mu\nu} R^* = 8\pi G f_2 T^*_{\mu\nu} + \sigma_{ij} (2\phi^i_{,\mu} \phi^j_{,\nu} - g^*_{\mu\nu} g^{\sigma\delta} \phi^i_{,\sigma} \phi^j_{,\delta}) - 2g^*_{\mu\nu} U.$$  \hspace{1cm} (35)

Defining $T^* = g^{\mu\nu} T^*_{\mu\nu}$ and $\alpha_1 = A^{-1}(\partial A/\partial \phi^i)$, so that

$$\alpha_1 = -\sqrt{3}, \hspace{1cm} \alpha_2 = 0,$$  \hspace{1cm} (36)

one obtains, after some algebra

$$\frac{\delta(\sqrt{-g^*} f_2 L^*)}{\sqrt{-g^*}} = \frac{\sqrt{3}}{3} f_2 T^* \delta \phi^1 + F_2 L^* \delta \phi^2 = (-\alpha_1 f_2 T^* + \delta_2 F_2 L^*) \delta \phi^i.$$  \hspace{1cm} (37)

One defines $B_i = \partial U/\partial \phi_i$ and uses the field-space metric $\alpha_{ij}$ to raise and lower Latin indexes, so that $\alpha^i = \alpha^{ij} \alpha_{j}$ and $B^i = \sigma^{ij} B_j$; the Euler–Lagrange equation for each scalar field reads

$$0 = -4\alpha_1 \Box \phi^i + 4B_i - 16\pi G (-\alpha_1 f_2 T^* + \delta_2 F_2 L^*) \delta \phi^i \rightarrow \alpha_1 \Box \phi^i = B_i + 4\pi G (\alpha_1 f_2 T^* - \delta_2 F_2 L^*) \rightarrow$$  \hspace{1cm} (38)

$$\Box \phi^i = B^i + 4\pi G (\alpha^i f_2 T^* - \sigma^{ij} F_2 L^*).$$

This form enables a prompt comparison with the $f_2 = 1$ limit, yielding the usual expression for just one scalar field, $\Box \phi = B + 4\pi G a T^*$ (with $\psi = \phi^1$ and $\alpha = \alpha_1$) \cite{13}. One may also explicitly write the equation of motion for each field,

$$a \Box \phi^1 = -B_2 + 4\pi G F_2 L^*, \hspace{1cm} (39)$$

$$a \Box \phi^2 = B_1 + \frac{B_2}{a} - \frac{4\pi G \sqrt{3}}{3a} f_2 T^* - \frac{4\pi G}{a} F_2 L^*.$$
Using the Bianchi identities, the expression for the non-covariant conservation of the energy–
momentum tensor in the Einstein frame is also attained:

$$\nabla^\ast \mu T_\mu^\ast = \frac{\sqrt{3}}{3} T^\ast \nabla^\ast \varphi^1 + \frac{f_2}{f_2^2} \left( g^\ast_\mu_\nu \mathcal{L}^\ast - T^\ast_\mu_\nu \right) \nabla^\ast \varphi^2. \quad (40)$$

If, for consistency, one rewrites this in the Jordan frame, the conformal transformation
properties of the contravariant derivative eliminate the first term in the rhs, and the substitution
$$\varphi^2 = \phi = R$$ yields (4). Again, taking $$f_2(R) = 1$$ one recovers
$$\nabla^\ast \mu T_\mu^\ast = -\alpha T_\mu^\ast \nabla^\ast \phi.$$ (41)

Note that the sign of the rhs of (41) is opposite to the result found in [25], due to the different
sign conventions of $$T_{\mu \nu}$$ and $$R_{\mu \nu}$$.

4. Parametrized post-Newtonian formalism

Assuming that the effect of the non-minimum coupling of curvature to matter is perturbative,
one may write $$f_2(R) = 1 + \lambda \delta_2(R)$$, with $$\lambda \delta_2 \ll 1$$, so that the current bounds on the equivalence
principle are respected. Substituting into (40) one gets, at zeroth-order in $$\lambda$$,

$$\nabla^\ast \mu T_\mu^\ast \simeq -\alpha \nabla^\ast \phi, \nu,$$ (41)

which amounts to ignoring the $$f_2(\varphi^2)$$ factor in action (34), so that $$f_2$$ manifests itself only
through the coupling $$A^2(\varphi^1)$$ present in $$T^\ast$$, and the derivative of $$\varphi^1$$ (since $$\varphi^1 \propto \log \psi$$ and
$$\psi = F_1 + F_2 \mathcal{L}$$). In this case, the matter action reads

$$S_m = \int A^4(\psi) \mathcal{L}(A^2(\psi) g^\ast_{\mu \nu}, \chi) \sqrt{-g^\ast} \, d^4x.$$ (42)

Note that the sign of the rhs of (41) is opposite to the result found in [25], due to the different
sign conventions of $$T_{\mu \nu}$$ and $$R_{\mu \nu}$$.

If there is no characteristic length ruling the added gravitational interaction (that is, both
scalar fields are light, leading to long-range interactions), this manifestation of a ‘physical
metric’ in the matter action allows one to resort to calculate the PPN parameters
$$\beta$$ and $$\gamma$$ [25]:

$$\beta - 1 = - \frac{1}{2} \left[ \frac{\alpha^i \alpha^j \alpha_{j,i}}{(1 + \alpha^2)^2} \right]_0,$$  $$\gamma - 1 = -2 \left[ \frac{\alpha^2}{1 + \alpha^2} \right]_0,$$  (43)

where $$\alpha_{j,i} = \partial \alpha_j / \partial \varphi^i$$ and $$\alpha^2 = \alpha^i \alpha^i = \sigma_{ij} \alpha^i \alpha^j$$; the subscript ‘0’ indicates that the quantities
should be evaluated at their asymptotic values $$\varphi^0$$, related with the cosmological values of the
curvature and matter Lagrangian density (as shall be further discussed in the following section).

Since the $$\alpha_i$$’s are constant for $$i = 1, 2$$, one gets that $$\alpha_{j,i} = 0$$; hence, the PPN parameter
$$\beta$$ is unitary. Moreover, since $$\alpha_1 = (\alpha_2, 0)$$ is orthogonal to $$\alpha' = (0, \alpha_1 / \alpha)$$, one gets $$\alpha^2 = 0$$. Hence,
the PPN parameter $$\gamma$$ is also unitary. As argued previously, these results do not reflect
the particular value of $$a$$ chosen in the antisymmetric part of $$\sigma_{ij}$$. In light of the overall discussion presented, it is clear that the two degrees of freedom
embodied in the two independent scalar fields $$\varphi^1$$ and $$\varphi^2$$ stem not only from the non-minimal
curvature term in action (1), but also of the non-minimal coupling of $$R$$ to the matter Lagrangian
density. Given this, it is clear that a ‘natural’ choice for the two scalar fields (in the Jordan
frame) would be $$\phi = R$$ and $$\psi = \mathcal{L}$$: this more physical interpretation \textit{ab initio} comes at the
cost of more evolved calculations, since both redefined scalar fields in the Einstein frame will
deepend on $$\phi$$ and $$\psi$$. This less pedagogical approach is deferred to appendix A.

5. Discussion and conclusions

The result $$\gamma = 1$$ is key to our study: it is clear that, in the standard $$f_2(R) = 1$$ theories,
$$\alpha^2$$ does not vanish (it is a purely algebraic, not matricial result and $$\alpha^2 = 1/3$$), and the
resulting PPN parameter $\gamma = 1/2$, which violates well-known observational bounds! A more thorough discussion on the ongoing debate concerning the value of the PPN parameter $\gamma$ for $f(R)$ theories is deferred to appendix B—with special focus to what is believed to be a misconception in the identification of the equivalence with a scalar–tensor theory.

In the $f_2(R) \neq 1$ case, the added degree of freedom that a non-minimal coupling of curvature to matter implies yields not one, but two scalar fields: as a result, a two-dimensional field-space metric $\sigma_{ij}$ arises; from the redefinition of the fields necessary to absorb non-canonical kinetic terms after the conformal transformation to the Einstein frame, it follows that this enables a vanishing $\alpha^2 = \alpha_i \alpha^j \sigma_{ij}$ term, yielding no post-Newtonian observational signature that discriminates these models from general relativity. However, this conclusion is valid only in zeroth-order in $\lambda$: if more terms are allowed, the non-covariant conservation law for the energy–momentum tensor is no longer of the form treated in [25], and more elaborate calculations would have to be performed in order to extract the PPN parameters $\beta$ and $\gamma$.

It is important to highlight that a naive analysis of the model under scrutiny might predict no difference between the PPN parameters arising from the trivial $f_2(R) = 1$ and the non-minimal $f_2(R) \neq 1$ cases: indeed, since this function is coupled to the matter Lagrangian density one might expect that, outside of the matter distribution ($\mathcal{L} = 0$), the theory would collapse back to the usual $f(R)$ scenario. However, this misinterpretation is resolved by (43), which is evaluated at the cosmological values $\phi_0$: for this reason, the relevant value for $\mathcal{L}$ is given by the Lagrangian density of the overall cosmological fluid, not of the local environment. The issue of suitably identifying this contribution is discussed in [28].

Finally, note that the result $\beta = \gamma \simeq 1$ is approximate, since it corresponds to dropping the term $f_2(\phi^2)$ in action (34); a future work should consider a perturbative approach, which could perhaps yield a small, $\lambda$-dependent deviation from unity, thus marking a clear (even if small) departure between the model studied here and general relativity.

Acknowledgments

The authors would like to thank S Capozziello for fruitful and elucidating discussions. The work of JP is sponsored by the FCT under the grant BPD 23287/2005. OB acknowledges the partial support of the FCT project POCI/FIS/56093/2004.

Appendix A. Alternative formulation of the two-field equivalent scalar–tensor theory

We present here an alternative formulation where instead one chooses to express action (1) through the equivalent expression

$$S = \int \left[ F(\phi, \psi) R - V(\phi, \psi) + f_2(\phi) \mathcal{L}(g_{\mu\nu}, \chi) \right] \sqrt{-g} \, d^4x,$$

where one defines

$$F(\phi, \psi) = F_1(\phi) + F_2(\phi) \psi, \quad V(\phi, \psi) = \phi F(\phi, \psi) - f_1(\phi).$$

One obtains the equivalence with the model under scrutiny by writing the equations of motion for the scalar fields,

$$F_2(\phi) (L - \psi) + [F'_1(\phi) + F'_2(\phi) \psi] (R - \phi) = 0,$$

$$F_2(\phi) (R - \phi) = 0,$$

implying that $\phi = R$ (or $F_2(\phi) = 0 \rightarrow f_2 = 1$, the trivial result) and, therefore, $\psi = \mathcal{L}$. Substituting into (A.1), one recovers the action for the mixed curvature model (1).
As discussed before, the identification of the $f_1 = R$ and $f_2 = 1$ case with GR must be approached with caution, as it disables the identification $f = R$: one may write $f_1(\phi) = \phi + \epsilon \delta_1(\phi)$ and $f_2(\phi) = 1 + \lambda \delta_2(\phi)$, so that equations (A.3) become
\[
\lambda \delta_2(\phi) (\mathcal{L} - \psi) + [\epsilon \delta_1(\phi) + \lambda \delta_2(\phi) \psi](R - \phi) = 0, \tag{A.4}
\]
and taking the limit $\epsilon, \lambda \to 0$ gives a trivial identity.

Variation of action (A.1) with respect to the metric yield the Einstein equations,
\[
 F(\phi, \psi) \left(R_{\mu
u} - \frac{1}{2}g_{\mu
u}R\right) = 8\pi G f_2(\phi)T_{\mu\nu} - \frac{1}{2}g_{\mu
u}V(\phi, \psi) + (\Box_{\mu\nu} - g_{\mu\nu}\Box)F(\psi, \phi), \tag{A.5}
\]
which, introducing $f = R$ and $\psi = \mathcal{L}$, recovers (2).

Using the Bianchi identities and the previous relation, one obtains
\[
\nabla^\mu T_{\mu\nu} = \frac{1}{f_2(\phi)} [\nabla_\nu V(\phi, \psi) - R \nabla_\nu F(\phi, \psi) - F_2(\phi)T_{\mu\nu} \nabla^\mu \phi]. \tag{A.6}
\]
Since
\[
\nabla_\nu V(\phi, \psi) = [\psi F_2(\phi) + \phi (F_1'(\phi) + \psi F_2'(\phi))]\nabla_\nu \phi + \phi F_2(\phi)\nabla_\nu \psi \tag{A.7}
\]
and
\[
\nabla_\nu F(\phi, \psi) = (F_1'(\phi) + F_2'(\phi)\psi)\nabla_\nu \phi + F_2(\phi)\nabla_\nu \psi, \tag{A.8}
\]
we get
\[
\nabla^\mu T_{\mu\nu} = \frac{1}{f_2(\phi)} \times [F_2(\phi)(g_{\mu\nu} \psi - T_{\mu\nu})\nabla^\mu \phi
+ ([F_1'(\phi) + \psi F_2'(\phi)]\nabla_\nu \phi + F_2(\phi)\nabla_\nu \psi)(\phi - R)], \tag{A.9}
\]
which, upon the substitution $\psi = \mathcal{L}$ and $\phi = R$, collapses back into (4).

\section*{A.1. Equivalence with a scalar theory}

In this alternative formulation, the adequate conformal transformation to decouple the scalar curvature from the scalar fields $\phi, \psi$ is given by $g^*_{\mu\nu} = F(\phi, \psi)g_{\mu\nu} = A^{-2}(\phi, \psi)g_{\mu\nu}$, with $A(\phi, \psi) = F^{-1/2}(\phi, \psi)$. One obtains
\[
\sqrt{-g} = F^{-2}\sqrt{-g^*}, \quad R = F \left[R^* - 6\sqrt{F} \left(\frac{1}{\sqrt{F}}\right)\right]. \tag{A.10}
\]
Introducing the above into action (A.1) yields
\[
S = \int \left[R^* - 6\sqrt{F} \left(\frac{1}{\sqrt{F}}\right) - 4U + f_2A^4 L(A^2g^*_{\mu\nu}, \chi)\right]\sqrt{-g^*} d^4x, \tag{A.11}
\]
where one defines $U(\phi, \psi) = A^4(\phi, \psi)V(\phi, \psi)/4$.

One now attempts to recast the action in terms of two other scalar fields, endowed with canonical kinetic term. For this, one repeats the integration of the covariant derivative term by parts and uses the metric compatibility relations, finally obtaining the action
\[
S = \int \left[R^* - \frac{3}{2}g^*_{\mu\nu}F_\mu F_\nu F_\mu F_\nu + 4U + f_2A^4 L(A^2g^*_{\mu\nu}, \chi)\right]\sqrt{-g^*} d^4x. \tag{A.12}
\]
As one aims to allow for an immediate comparison with the $f_2 = 1$ scenario, it is interesting to isolate the contributions arising from the non-minimal scalar curvature term and from its coupling with matter as clearly as possible, namely
\[
\log F = \log(F_1 + F_2\psi) = \log F_1 + \log \left(1 + \frac{\psi F_2}{F_1}\right). \tag{A.13}
\]
assuming that $F_1 \neq 0$. If one defines

$$
\varphi^1 = \frac{\sqrt{3}}{2} \log F_1(\phi), \quad \varphi^2 = \frac{\sqrt{3}}{2} \log \left( 1 + \frac{\psi F_2(\phi)}{F_1(\phi)} \right), \tag{A.14}
$$

then the comparison is transparent: when $f_2 = 1$, the second scalar field vanishes, and the first scalar field coincides with the usual redefined scalar field $\varphi^1 = \log F_1$ arising in $f_1(R)$ models (although in many studies some terms are overlooked, see appendix B). Also, when $f_1 = R$, the first scalar field $\varphi^1$ vanishes.

For the particular choice of fields (equations (A.14)), one obtains cross-products between the derivatives of $\varphi^1$ and $\varphi^2$, so that the $\sigma_{ij}$ field-space metric displays non-vanishing off-diagonal elements; since one has absorbed the numerical factors in the redefinition, this field-space metric is trivially given by

$$
\sigma_{ij} = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix} + \begin{pmatrix}
0 & 1-a \\
1-a & 0
\end{pmatrix} = \begin{pmatrix}
1 & 2-a \\
a & 1
\end{pmatrix}, \tag{A.15}
$$

with inverse

$$
\sigma^{ij} = \frac{1}{(1-a)^2} \begin{pmatrix}
1 & a-2 \\
-a & 1
\end{pmatrix}. \tag{A.16}
$$

As already discussed, the physical results will not depend on the value of $a$. With this choice, the action now reads

$$
S = \int \left[ R^* - 2g^{\mu\nu} \sigma_{ij} \partial_{[\mu} \varphi^i \partial_{\nu]} \varphi^j - 4U(\varphi^1, \varphi^2) + f_2(\phi(\varphi^1))L^* \right] \sqrt{-g^*} \, d^4x. \tag{A.17}
$$

where one defines $L^* = A^4(\varphi^1, \varphi^2)L$ as before, but using the new redefined fields $\varphi^i$; note that the non-minimal coupling is written in terms of $\varphi^1$ alone. The Einstein field equations are

$$
R_{\mu\nu}^* - \frac{1}{2}g_{\mu\nu}^* R^* = 8\pi G f_2 T_{\mu\nu}^* + 2\sigma_{ij} \partial_{[\mu} \varphi^i \partial_{\nu]} - g_{\mu\nu}^* \sigma^{\alpha\beta} \sigma_{ij} \partial_{[\alpha} \varphi^i \partial_{\beta]} - 2g_{\mu\nu}^* U, \tag{A.18}
$$

and variation of the matter action with respect to each scalar field $\varphi^i$ yields, after some algebra

$$
\frac{\delta}{\delta \varphi^i} \left[ f_2 T^* + \frac{2F_1 F_2}{F_1^2} \delta_{ij} L^* \right] \alpha_i \delta \varphi^i = \left[ -f_2 \alpha_i T^* + F_2 \delta_{ij} \frac{\partial \phi}{\partial \varphi^j} L^* \right] \delta \varphi^i = \left[ 2f_2 T^* + \frac{2F_1 F_2}{F_1^2} \delta_{ij} L^* \right] \alpha_i \delta \varphi^i, \tag{A.19}
$$

where, as before, one uses

$$
\alpha_i = \frac{\partial \log A}{\partial \varphi^i} = -\frac{1}{2} \frac{\partial \log F}{\partial \varphi^i} = -\frac{\sqrt{3}}{3} = \alpha, \tag{A.20}
$$

and, given definitions equations (A.14),

$$
\frac{\partial \phi}{\partial \varphi^i} = \frac{2\sqrt{3} F_1}{3 F_1} = -2\alpha F_1. \tag{A.21}
$$

The Euler–Lagrange equation for each field $\varphi^i$ reads

$$
\Box \varphi^i = B^i + 4\pi G \left[ f_2 T^* + \frac{2F_1 F_2}{F_1^2} \delta_{ij} L^* \right] \alpha^i. \tag{A.22}
$$

In the $f_2 = 1$ limit, one recovers $\Box \varphi = B + 4\pi G a T^*$, as before.
Using the Bianchi identities, the expression for the non-covariant conservation of the energy–momentum tensor is

$$\nabla^\mu T^\mu_{\nu} = - \left[ \frac{2 F_1}{F_2} \frac{F_2}{f_2} \delta_{i1} (g^*_{\mu\nu} \mathcal{L}^* - T^*_{\mu\nu}) + g^*_{\mu\nu} T^* \right] \alpha_i \nabla^\mu \phi^i. \quad (A.23)$$

As discussed regarding (40), the first term of the rhs is eliminated in the inverse conformal transformation back to the Jordan frame, and one obtains an expression that collapses to (4), after a suitable manipulation.

Also, note that, taking first the minimal coupling $f_2(\phi) = 1$, one gets $\psi^2 = 0$ and (A.23) collapses to the expected expression $\nabla^\mu T^\mu_{\nu} = - T^* \alpha \nabla^\nu \phi$ (with $\phi \equiv \phi^1 \propto \log F(\phi)$, following redefinition [13]), reflecting the presence of a single scalar degree of freedom $\phi^1$.

Conversely, taking the trivial case $f_1(R) = R$ yields $\psi^1 = 0$ and, for $f_2(R) \neq 1$, one keeps only the scalar degree of freedom $\psi^2$: the above expression is no longer valid, as the troublesome $f_1$ term results from the variation of the action with respect to $\phi^1$, which now vanishes. This pathology reflects the particular choice of fields considered, and does not occur in the main body of this study, as (21) shows.

**A.2. Parametrized post-Newtonian formalism**

Once again, assuming a perturbative effect of $f_2(R) = 1 + \lambda \delta_2(R)$, with $\lambda \delta_2 \ll 1$ yields, to zeroth-order in $\lambda$

$$\nabla^\nu T_{\mu\nu} \simeq - \alpha_i T^* \phi^i, \quad (A.24)$$

which amounts to ignoring the $f_2(\phi)$ factor in action (A.17), so that $f_2$ manifests itself only through the coupling $A^2(\phi, \psi)$; in this case, one may write

$$\mathcal{L}^* \simeq A^2(\phi^1, \phi^2) \mathcal{L}. \quad (A.25)$$

The computation of the PPN parameters $\beta$ and $\gamma$ is very similar to that already performed: since $\alpha_i$’s are constant for $i = 1, 2$, one gets that $\alpha_{i,j} = 0$ and $\beta = 1$. One also obtains

$$\alpha^2 = \sigma^{ab} \alpha_i \alpha_j = \frac{a^2}{(1-a)^2} \left[ \begin{array}{cc} 1 & 1 \\ a-2 & 1 \end{array} \right] \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] = 0, \quad (A.26)$$

independently of the value of $a$, as already discussed. Hence, one confirms that $\gamma = 1$. As required, the particular choice of field-metric $\sigma$ and scalar fields (initially in the Jordan frame) $\phi$ and $\psi$ do not change the obtained results $\beta = \gamma = 1$.

**Appendix B. Discussion of previous results**

In this appendix one addresses the issue of the $f_2(R) = 1$, $f_1(R) = f(R)$ case, very much discussed in the literature (see e.g. [7])—in particular, the claim that $f(R)$ models characterized by a light scalar field still allow for the PPN parameters $\beta$ and $\gamma$ to be close to unity. The action for this model is, in the Jordan frame, given by

$$S = \int \left[ f(R) + \mathcal{L}(g_{\mu\nu}, \chi) \right] \sqrt{-g} \, d^4x, \quad (B.1)$$

which is dynamically equivalent to

$$S = \int \left[ F(\phi) R - V(\phi) + \mathcal{L}(g_{\mu\nu}, \chi) \right] \sqrt{-g} \, d^4x, \quad (B.2)$$

with the usual identification $\phi = R$, and the definition $V(\phi) = \phi F(\phi) - f(\phi)$.
A conformal transformation $g^\ast_{\mu\nu} = Fg_{\mu\nu} = A^{-2}g_{\mu\nu}$, with $A(\phi) = F^{-1/2}(\phi)$, yields the action, in the Einstein frame,

$$S = \int \left[ R^* - \frac{3}{2} g^*_{\mu\nu} F,_{\mu}F,_{\nu} - 4U + A^4 L(A^2 g^*_{\mu\nu}, \chi) \right] \sqrt{-g^*} \, d^4x,$$

(B.3)

where $U(\phi) = A^4(\phi)V(\phi)/4$.

One may redefine the scalar field as

$$\varphi = \frac{\sqrt{3}}{2} \log F(\phi) = -\sqrt{3} \log A,$$

(B.4)

so that the action becomes

$$S = \int \left[ R^* - 2g^{*\mu\nu} \varphi,_{\mu}\varphi,_{\nu} - 4U + A^4 L(A^2 g^*_{\mu\nu}, \chi) \right] \sqrt{-g^*} \, d^4x,$$

(B.5)

and the matter action depends not on the Einstein metric $g^*_{\mu\nu}$, but on the original Jordan metric $g_{\mu\nu} = A^2 g^*_{\mu\nu}$:

$$S_m = \int A^4 L(A^2 g^*_{\mu\nu}, \chi) \sqrt{-g} \, d^4x = \int L(g_{\mu\nu}, \chi) \sqrt{-g} \, d^4x.$$

(B.6)

One obtains

$$\alpha = \frac{\partial \log A}{\partial \varphi} = -\frac{1}{2} \frac{\partial \log F}{\partial \varphi} = -\frac{\sqrt{3}}{3},$$

(B.7)

identical to the previous result for $\alpha_a$. However, in this case the field-space metric $\sigma$ is one dimensional, and simply given by $\sigma_{11} = \sigma^{11} = 1$, so that $\alpha^2 = 1/3$. One obtains $\alpha,\varphi = 0$, so that

$$\beta - 1 = \frac{1}{2} \left[ \frac{\alpha^2 \alpha,_{\varphi}}{(1 + \alpha^2)^2} \right]_0 = 0 \rightarrow \beta = 1,$$

$$\gamma - 1 = -2 \left[ \frac{\alpha^2}{1 + \alpha^2} \right]_0 = -\frac{1}{2} \rightarrow \gamma = \frac{1}{2}.$$

(B.8)

This indicates that general scalar–tensor theories with no a priori kinetic term for the long-range scalar field (in the Jordan frame) are incompatible with observations. As the above example shows, the equivalence between $f(R)$ theories and such models falls within this category, and is therefore observationally ruled out. Furthermore, the result $\alpha^2 = 1/3$ enables, for the case of a single scalar field (see [25]), the identification of the Brans–Dicke coupling parameter

$$2\omega + 3 = \frac{1}{\alpha^2} \rightarrow \omega = 0,$$

(B.9)

which shows that $f(R)$ models may also be recast as a (sometimes used) generalized Jordan–Brans–Dicke model with no kinetic term,

$$S = \int \left[ \phi R - V(\phi) + L \right] \sqrt{-g} \, d^4x,$$

(B.10)

with the dynamical identification $\phi = F(R)$ and a suitable potential $V(\phi) = R(\phi)F(R(\phi)) - f(R(\phi))$.

In several papers in the literature this equivalence with a scalar–tensor theory is given by the action

$$S = \int \left[ F(\phi) R - Z(\phi) g^{\mu\nu} \varphi,_{\mu}\varphi,_{\nu} - V(\phi) + L(g_{\mu\nu}, \chi) \right] \sqrt{-g} \, d^4x,$$

(B.11)
with $Z(\phi) = 1$; for later convenience, one retains the kinetic function $Z(\phi)$. As discussed above, one can opt by an equivalent Jordan–Brans–Dicke theory with a scalar field dynamically identified through $\phi = F(R)$, and no kinetic term, that is, $\omega = 0$.

It is easy to verify that variation of the action with respect to the scalar field $\phi$ will yield terms involving $Z'(\phi)$ (which vanishes, in the usual approach $Z(\phi) = 1$ and the four-dimensional D’Alembertian operator, similarly to the classical Klein–Gordon equation. For this reason, the presence of a kinetic term in the above action implies that the dynamical identification $\phi = R$ (arising from the equation of motion of the scalar field $\phi$) fails.

Moreover, the above conformal transformation $g^*_{\mu\nu} = F(\phi)g_{\mu\nu}$ yields

$$\int \sqrt{-g^*} d^4x \times \left[ R^* - \frac{3}{2} g^{*\mu\nu} F_{,\mu} F_{,\nu} - g^{*\mu\nu} Z(\phi) \frac{\phi_{,\mu} \phi_{,\nu}}{F(\phi)} \right.
\left. - 4U(\phi) + A^4(\phi) \mathcal{L}(A^2(\phi)g^{*}_{\mu\nu}, \chi) \right],$$

(B.12)

using the previous result relating $R^*$ and $R$, as well as $g^{*\mu\nu} = F(\phi)g^{\mu\nu}$ and $\sqrt{-g} = F^{-2}(\phi)\sqrt{-g^*}$. The usual redefinition of the scalar field follows (erroneously),

$$\left( \frac{\partial \phi}{\partial \phi} \right)^2 = \frac{3}{4} \left( \frac{\partial \log F(\phi)}{\partial \phi} \right)^2 + \frac{Z(\phi)}{2F(\phi)}.$$

(B.13)

The usual redefinition [15] is often presented as

$$\left( \frac{\partial \phi}{\partial \phi} \right)^2 = \frac{3}{4} \left( \frac{\partial \log F(\phi)}{\partial \phi} \right)^2 + \frac{1}{2F(\phi)},$$

(B.14)

which clearly corresponds to $Z(\phi) = 1$; this yields the canonical action

$$\int \sqrt{-g^*} d^4x \times \left[ R^* - 2g^{*,\mu\nu} \phi_{,\mu} \phi_{,\nu} - 4U(\phi) + A^4(\phi) \mathcal{L}(A^2(\phi)g^*_{\mu\nu}, \chi) \right].$$

(B.15)

which can be matched with action (B.12) through the relation

$$-2\phi_{,\mu} \phi_{,\nu} = -2 \frac{\partial \phi}{\partial \phi} \phi_{,\mu} \phi_{,\nu} = - \left[ \frac{3}{2} \left( \frac{\partial \log F(\phi)}{\partial \phi} \right)^2 \right] \phi_{,\mu} \phi_{,\nu}.$$

(B.16)

However, the above shows that a proper treatment should use $Z(\phi) = 0$, as there is no intrinsic kinetic term in the original, Jordan frame theory embodied in (B.11). This redefinition of the scalar field will affect the calculation of $\alpha = \partial \log A / \partial \phi$ and, as a consequence, yield incorrect predictions for the PPN parameters $\beta$ and $\gamma$; in particular, one obtains a dependence on $F(\phi)$ which would otherwise be missing. This can be seen from the following expressions [29],

$$\gamma - 1 = - \frac{F'(\phi)^2}{Z(\phi)F(\phi) + 2F'(\phi)^2}, \quad \beta - 1 = \frac{1}{4} \frac{F(\phi)F'(\phi)}{2Z(\phi)F(\phi) + 3F'(\phi)^2} \frac{d\gamma}{d\phi},$$

(B.17)

which, for $Z(\phi) = 1$, yields the PPN parameters $\beta$ and $\gamma$ used in [13].

Hence, it appears that the PPN coefficients calculated for a wide variety of $f(R)$ models, and obtained by the dynamical identification $\phi = R$, are inaccurate: by reinstating the correct factor $Z(\phi) = 0$ into (B.13), one recovers

$$\left( \frac{\partial \phi}{\partial \phi} \right)^2 = \frac{3}{4} \left( \frac{\partial \log F(\phi)}{\partial \phi} \right)^2 \rightarrow \phi = \frac{\sqrt{3}}{2} \log F(\phi),$$

(B.18)

so that the calculations for the PPN parameters $\gamma = 1/2$ and $\beta = 1$ follow as argued previously—and equations (B.17) clearly show. Moreover, note that (as already discussed)
one cannot simply identify GR with the limit $\epsilon \to 0$ of a model with $f(R) = R + \epsilon \delta(R)$, since the corresponding limit of equations (B.17) (taking the correct factor $Z(\phi) = 0$) is ill-defined for $F'(\phi) = \epsilon \delta''(\phi) \to 0$.

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