Hessian Equations of Krylov Type on Kähler Manifolds

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Abstract
In this paper, we consider a Hessian equation with its structure as a combination of elementary symmetric functions on closed Kähler manifolds. We provide a sufficient and necessary condition for the solvability of this equation, which generalizes the results of Hessian equation and Hessian quotient equation. As a consequence, we can solve a complex Monge–Ampère type equation proposed by Chen in the case that one of the coefficients is negative. In this case, the equation is related the deformed Hermitian Yang-Mills equation. The key to our argument is a novel use of the special properties of the Hessian quotient operator $\sigma_k / \sigma_{k-1}$.

Keywords Complex Hessian equations · Cone condition · Kähler manifolds

Mathematics Subject Classification 32W50 · 53C55

1 Introduction

Let $(M, \omega)$ be a closed Kähler manifold of complex dimension $n \geq 2$ and fix a real smooth closed $(1, 1)$-form $\chi_0$ on $M$. For any $C^2$ function $u : M \to \mathbb{R}$, we can obtain a new real $(1, 1)$-form

$$\chi_u = \chi_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u.$$ 

We consider the following Hessian type of equation on $(M, \omega)$

$$\chi_u^k \wedge \omega^{n-k} = \sum_{i=0}^{k-1} \alpha_i(z) \chi_u^i \wedge \omega^{n-i}, \quad 2 \leq k \leq n,$$  

(1.1)
where $\alpha_0(z), \ldots, \alpha_{k-1}(z)$ are real smooth functions on $M$. This equation includes some of the most important partial differential equations in complex geometry and analysis.

- For $k = n$, $\alpha_1 = \cdots = \alpha_{n-1} = 0$, (1.1) is the complex Monge–Ampère equation $\chi^n = \alpha_0(z)\omega^n$ which was famously solved on compact Kähler manifolds by Yau in his resolution of Calabi conjecture [56], and on compact Hermitian manifolds by Tosatti-Weinkove [50] with some earlier work by Cherrier [14], Hanani [27], Guan–Li [23] and Zhang [58].

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All previous results on Eq. (1.1) require all coefficients $\alpha_i \geq 0$ for $0 \leq i \leq k-1$. Thus, it is interesting to ask:

**Question 1.1** Can we solve Eq. (1.1) for some $\alpha_i(z)$ which change sign or are negative? In particular, can we solve Chen’s problem for some $\alpha_i < 0$?

**Question 1.1** is related to the deformed Hermitian Yang–Mills equation which was introduced by Jacob and Yau [30]:

$$\text{Im}((\omega - \Theta)^n) = \tan(\hat{\theta})\text{Re}((\omega - \Theta)^n),$$

(1.2)

where $\Theta$ is the curvature of the Chern connection of a metric $h$ on a holomorphic line bundle $L$ over a compact Kähler manifold $(M, \omega)$ of complex dimension $n$, $\hat{\theta}$ is a constant parameter (called the “phase angle”). If $(M, \omega)$ is a compact Kähler three-fold, we can rewrite Eq. (1.2) in the following form (see [37] for details)

$$\Omega_\phi^3 = 3 \sec^2(\hat{\theta})\Omega_\phi \wedge \omega^2 + 2 \tan(\hat{\theta}) \sec^2(\hat{\theta})\omega^3,$$

(1.3)
where $\Omega_{\varphi} = \Omega + \sqrt{-1} \partial \bar{\partial} \varphi$. Equation (1.3) is a special case of (1.1) and has one coefficient $2 \tan(\hat{\theta}) \sec^2(\hat{\theta}) < 0$ when the phase angle $\hat{\theta} \in (\frac{\pi}{2}, \pi)$. Thus, it is necessary to consider Question 1.1.

In this paper, we make some progress on this direction. To state our main theorem, we first recall the

Then, we need to introduce the cone condition.

Definition 1.1 Following [19, 25, 43, 46], we set

$$C_k(\omega) := \{ [\chi] : \exists \chi' \in [\chi] \cap \Gamma_{k-1}(M), \text{ such that }$$

$$k(\chi')^{k-1} \wedge \omega^{n-k} > \sum_{i=1}^{k-1} i\alpha_i(\chi')^{i-1} \wedge \omega^{n-i} \}.$$  (1.4)

where $\alpha_i(z)$ with $1 \leq i \leq k-1$ are real smooth functions on $M, [\chi] = \{ \chi + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u : u \in C^2(M) \}$ and the Gårding’s cone $\Gamma_{k-1}(M)$ is defined by (2.10). If $[\chi] \in C_k(\omega)$, we say that $\chi$ satisfies the cone condition for (1.1).

Next, we make the following assumption on the coefficient functions $\alpha_i(z)$ ($0 \leq i \leq k-1$).

Assumption 1.1 Suppose that $\alpha_0(z), \alpha_1(z), \ldots, \alpha_{k-1}(z)$ are real smooth functions on $M$ satisfying

(i) either $\alpha_l(z) > 0$ or $\alpha_l(z) \equiv 0$ for $0 \leq l \leq k-2$ and all $z \in M$;
(ii) $\sum_{l=0}^{k-2} \alpha_l(z) > 0$ for all $z \in M$;
(iii) $\alpha_i(z) \geq c_{k,i}$ for $0 \leq i \leq k-1$ and

$$\int_M \chi_0^k \wedge \omega^{n-k} \leq \sum_{i=0}^{k-1} c_{k,i} \int_M \chi_0^i \wedge \omega^{n-i},$$  (1.5)

where $c_{k,0}, \ldots, c_{k,k-1}$ are constants.

Remark 1.1 We make no sign requirement for $\alpha_{k-1}(z)$ in Assumption 1.1.

Based on Proposition 2.1 (7), it is easy to see the condition $[\chi_0] \in \hat{C}_k(\omega)$ is necessary for the solvability of (1.1). We show such condition is also sufficient.

Theorem 1.1 Let $(M, \omega)$ be a closed Kähler manifold of complex dimension $n \geq 2$ and $\chi_0$ be a smooth closed real $(1,1)$-form on $M$. Assume that $\alpha_0(z), \alpha_1(z), \ldots, \alpha_{k-1}(z)$ satisfy Assumption 1.1. Then, there exist a smooth function $u$ and $a \in \mathbb{R}$ satisfies $\chi_u \in \Gamma_{k-1}(M)$ and

$$\chi_u^k \wedge \omega^{n-k} = \sum_{l=0}^{k-2} \alpha_l(z) \chi_u^l \wedge \omega^{n-l} + [\alpha_{k-1}(z) + a] \chi_u^{k-1} \wedge \omega^{n-k+1}$$

provided that $[\chi_0] \in \hat{C}_k(\omega)$. 

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In particular, if \( \alpha_0, \alpha_1, \ldots, \alpha_{k-1} \) are constants, Eq. (1.1) was proposed by Chen (see Conjecture/Question 4 in [11]) for \( k = n \). Assume \( \alpha_i \geq 0 \) for all \( 0 \leq i \leq k - 1 \), it was solved by Collins-Székelyhidi [15] for \( k = n \) and by Phong-Tô [40] for the general \( k \). As a corollary of Theorem 1.1, we can solve Chen’s problem for \( \alpha_{k-1} < 0 \).

**Theorem 1.2** Let \((M, \omega)\) be a closed Kähler manifold of complex dimension \( n \geq 2 \) and \( \chi_0 \) be a smooth closed real \((1, 1)\)-form on \( M \). Assume \( \alpha_{k-1} \in \mathbb{R} \) and \( \alpha_0, \alpha_1, \ldots, \alpha_{k-2} \) are nonnegative real numbers satisfying

\[
\sum_{i=0}^{k-2} \alpha_i > 0 \quad \text{and} \quad \int_M \chi_0^k \wedge \omega^{n-k} = k^{-1} \sum_{i=0}^{k-1} \alpha_i \int_M \chi_0^i \wedge \omega^{n-i}.
\]

Then, there exists a smooth function \( u \) with \( \chi_u \in \Gamma_{k-1}(M) \) satisfies (1.1) if and only if \( [\chi_0] \in C_k(\omega) \).

**Remark 1.2** Integrating both sides of (1.1) on \( M \), it is easy to see the equality (1.6) is a necessary condition for the solvability of (1.1) for \( \alpha_i \in \mathbb{R} \) \((0 \leq i \leq k - 1)\).

**Remark 1.3** After I finished my paper and posted it on arXiv:2107.12035, Dr. Tsai sent me their manuscript [52]. In [52], Tsai considered the parabolic flow of the equation (1.1) and gave a parabolic proof of Theorem 1.2. More interesting, Tsai applied Theorem 1.2 to give an alternative way to solve the deformed Hermitian Yang-Mills equation on threefold from [37]. Thus, we look forward to finding more applications of Theorem 1.2 in the future.

Equation (1.1) is just a Hessian type of equation with its structure as a combination of elementary symmetric functions (see Sect. 2.1 for details):

\[
\sigma_k(\chi_u) = \sum_{i=0}^{k-1} \beta_i(z_\alpha(\chi_u)),
\]

where \( \beta_i(z) = c_k^{C_n} \alpha_i(z) \) for \( 0 \leq i \leq k - 1 \). Such type of equation has been firstly considered by Krylov about twenty years ago. In [32], he considered the Dirichlet problem of the following degenerate equation in a \((k - 1)\)-convex domain \( D \) in \( \mathbb{R}^n \):

\[
\sigma_k(D^2u) = \sum_{i=0}^{k-1} \beta_i(x) \sigma_i(D^2u)
\]

with all coefficient \( \beta_i(x) \geq 0 \) for \( 0 \leq i \leq k - 1 \). Recently, Krylov’s equation was extended by Guan–Zhang [26] to the case without the sign requirement for the coefficient function \( \beta_{k-1}(x) \) and the corresponding Neumann problems of Krylov’s equation in real and complex spaces were studied by the author with his collaborators in [7, 8]. Moreover, such type of equations with its structure as a combination of elementary symmetric functions arise naturally from many important geometric problems,
such as the so-called Fu–Yau equation arising from the study of the Hull–Strominger system in theoretical physics, see Fu–Yau [20, 21] and Phong–Picard–Zhang [38, 39, 41]. Furthermore, the special Lagrangian equations introduced by Harvey and Lawson [28] can also be written as the alternative combinations of elementary symmetric functions. Lastly, the Krylov type of equations in prescribed curvature problems were also studied by the author with his collaborators in [9, 10].

Compared with Hessian equations or Hessian quotient equations, the ellipticity and concavity of (1.7) is not so easy to see. Moreover, we even have several ways to make (1.7) elliptic and concave. In [15, 47], the authors rewrite Eq. (1.7) in the form

\[-\sum_{i=0}^{k-1} \beta_i(z) \frac{\sigma_i(\chi_u)}{\sigma_k(\chi_u)} = -1. \quad (1.8)\]

The disadvantage of the form (1.8) is that the ellipticity and concavity of (1.8) require all $\beta_i \geq 0$ for $0 \leq i \leq k - 1$. In this paper, we follow the viewpoint of Guan–Zhang [26] to rewrite Eq. (1.7) in the form

\[\frac{\sigma_k(\chi_u)}{\sigma_{k-1}(\chi_u)} - \sum_{l=0}^{k-2} \beta_l(z) \frac{\sigma_l(\chi_u)}{\sigma_{k-1}(\chi_u)} = \beta_{k-1}(z). \quad (1.9)\]

Then, we have the following very important fact (see Lemma 2.3 for details) which was first observed by Guan–Zhang [26] for Krylov’s equation.

**Fact 1.1** Assume $\beta_l(z) \geq 0$ for $0 \leq l \leq k - 2$ and $\chi_u \in \Gamma_{k-1}(M)$, then Eq. (1.8) is elliptic and concave. Compared with the form (1.8), The advantages of the form (1.9) lie in two aspects: (i) we make no sign requirement for $\beta_{k-1}(z)$ and (ii) the proper admissible set of solutions is $\Gamma_{k-1}$ which is larger than $\Gamma_k$.

**Fact 1.1** is based on the special properties of the Hessian quotient operator $\frac{\sigma_k}{\sigma_{k-1}}$ which are key to our argument. In details, Proposition 2.1(4)(5)(7) declare that if $l = k - 1$, some properties of Hessian quotient operator $\frac{\sigma_k}{\sigma_l}$ hold true in a larger cone $\Gamma_{k-1}$. The root of these special properties goes back to the Newton inequality (2.2).

To solve Eq. (1.9), we used ideas of Sun [46] and Székelyhidi [42]. In [42], Székelyhidi considered a general class of fully nonlinear equations on compact Hermitian manifolds which take the form equation

\[f(\lambda_1, \ldots, \lambda_n) = h,\]

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $\chi_u$ with respect to $\omega$ and $h$ is a given function on $M$, and $f$ is defined in open symmetric cone $\Gamma$ with vertex at the origin and satisfies

(i) $\frac{\partial f}{\partial \lambda_i} > 0$ for all $i$, and $f$ is concave,

(ii) $\sup_{\Gamma} f < \inf_M h$,

(iii) For any $\sigma < \sup_{\Gamma} f$ and $\lambda \in \Gamma$, we have $\lim_{t \to \infty} f(t\lambda) > \sigma$.

Since Eq. (1.9) satisfies (i) and (ii), but not (iii), this prevents us from applying the general machinery of Székelyhidi [42] directly. In [52], the author explains how
to avoid the use of condition (iii) by using some nice properties from our equations. Indeed, condition (iii) is mainly used in two places. The first place is for the gradient estimate. Székelyhidi develops a general Liouville-type theorem and utilizes the third condition in his blow up argument. Since the equation considered in [42] is very general, the whole argument is based heavily on the geometric property of admissible cone and concavity of $f$. In particular, condition (iii) gives additional geometric information for the level set of $f$. But now, we have an exact equation, so we can apply the blow up argument from Dinew–Kolodziej [16] to the equation directly without studying these geometric properties. The second place is for the $C^2$ estimate. Same as before, the geometric properties provided by condition (iii) was used in a crucial way to bound $\sum_i \frac{\partial f}{\partial \lambda_i}$. For our case, we can compute it directly, see Lemma 2.6. Moreover, the $C^2$ estimate in our paper involves differentiating the $\beta_i(z)$, since the $\beta_i(z)$ may vary with $z$, while Székelyhidi’s formalism $f(\lambda)$ does not involve extra $z$-dependence. We use Lemma 4.1 to handle there extra terms.

2 Preliminaries

In this section, we give some basic properties of elementary symmetric functions, which could be found in [33, 44], and establish some key lemmas.

2.1 Basic Properties of Elementary Symmetric Functions

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, the $k$-th elementary symmetric function is defined by

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.\]

We also set $\sigma_0 = 1$ and denote by $\sigma_k(\lambda|i)$ the $k$-th symmetric function with $\lambda_i = 0$. Recall that the Gårding’s cone is defined as

$$\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \ \forall \ 1 \leq i \leq k \}.$$

Proposition 2.1 (1) Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $k = 1, \cdots, n$, then

$$\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i), \ \forall \ 1 \leq i \leq n,$n

$$\sum_{i=1}^n \lambda_i \sigma_{k-1}(\lambda|i) = k \sigma_k(\lambda), \ \sum_{i=1}^n \sigma_k(\lambda|i) = (n-k) \sigma_k(\lambda). \quad (2.1)$$

(2) For $1 \leq k \leq n-1$ and $\lambda \in \mathbb{R}^n$, we have

$$(n-k+1)(k+1)\sigma_{k-1}(\lambda)\sigma_{k+1}(\lambda) \leq k(n-k)\sigma_k^2(\lambda). \quad (2.2)$$

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In particular, we have
\[ \sigma_{k-1}(\lambda)\sigma_{k+1}(\lambda) \leq \sigma_k^2(\lambda). \]  
(2.3)

(3) For \( \lambda \in \Gamma_k \) and \( n \geq k > l \geq 0, r > s \geq 0, k \geq r, l \geq s, \) we have
\[
\left[ \frac{\sigma_k(\lambda)/C_n^k}{\sigma_l(\lambda)/C_n^l} \right]^{\frac{1}{l-r}} \leq \left[ \frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s} \right]^{\frac{1}{r-s}}.
\]  
(2.4)

(4) For \( \lambda \in \Gamma_k \) and \( 0 \leq l < k \leq n, \) we have
\[
\frac{\partial [\sigma_k/\sigma_l](\lambda)}{\partial \lambda_i} > 0, \quad \forall \ 1 \leq i \leq n.
\]  
Moreover,
\[
\frac{\partial [\sigma_{k-1}/\sigma_k](\lambda)}{\partial \lambda_i} \geq 0, \quad \forall \ 1 \leq i \leq n,
\]  
holds true for \( \lambda \in \Gamma_{k-1}. \)

(5) For any \( n \geq k > l \geq 0, \)
\[
\left[ \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)} \right]^{\frac{1}{l-r}}
\]  
is a concave function in \( \Gamma_k. \) Moreover,
\[
\frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)}
\]  
is a concave function in \( \Gamma_{k-1}. \)

(6) For \( 1 \leq k \leq n \) and \( \lambda, \mu \in \Gamma_k, \) then we have
\[
\sum_{i=1}^{n} \mu_i \sigma_{k-1}(\lambda|i) \geq k[\sigma_k(\mu)]^{\frac{1}{k}}[\sigma_k(\lambda)]^{1-\frac{1}{k}}.
\]  
In particular,
\[
\sum_{i=1}^{n} \mu_i \sigma_{k-1}(\lambda|i) > 0.
\]  
(2.5)

(7) Assume \( \lambda \in \Gamma_k \) and \( 1 \leq l < k \leq n, \) then for any \( i \in \{1, 2, \ldots, n\} \) we have
\[
\frac{\sigma_{k-1}(\lambda|i)}{\sigma_l-1(\lambda|i)} > \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}.
\]  
(2.6)
Moreover,

\[
\frac{\sigma_{k-1}(\lambda|i)}{\sigma_{k-2}(\lambda|i)} \geq \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)}
\]  

(2.7)

holds true for \( \lambda \in \Gamma_{k-1} \).

**Proof** The proofs of (1)–(5) could be found in [33, 44]. The inequality (6) is the Gårding’s inequality (see [6]). (7) follows directly as in (4).

\( \square \)

Then, we list the following well-known result (See [1]).

**Lemma 2.1** If \( A = (a_{ij}) \) is a Hermitian matrix, \( \lambda_i = \lambda_i(A) \) is one of its eigenvalues \((i = 1, \ldots, n)\) and \( F = F(A) = f(\lambda(A)) \) is a symmetric function of \( \lambda_1, \ldots, \lambda_n \), then for any Hermitian matrix \( B = (b_{ij}) \), we have the following formulas:

\[
\frac{\partial^2 F}{\partial a_{ij} \partial a_{st}} b_{ij} b_{st} = \frac{\partial^2 f}{\partial \lambda_p \partial \lambda_q} b_{pp} b_{qq} + 2 \sum_{p<q} \frac{\partial f}{\partial \lambda_p} - \frac{\partial f}{\partial \lambda_q} \frac{\lambda_p - \lambda_q}{\lambda_p - \lambda_q} b_{pq}^2.
\]  

(2.8)

In addition, if \( f \) is a concave function and

\[ \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n, \]

then we have

\[ f_1 \geq f_2 \geq \cdots \geq f_n, \]  

(2.9)

where \( f_i = \frac{\partial f}{\partial \lambda_i} \).

Let \( \lambda(a_{ij}) \) denote the eigenvalues of a Hermitian matrix \( (a_{ij}) \). Define \( \sigma_k(a_{ij}) = \sigma_k(\lambda(a_{ij})) \). This definition can be naturally extended to Kähler manifolds (and more generally, Hermitian manifolds). Let \( A^{1,1}(M) \) be the space of real smooth \((1, 1)\)-forms on \((M, \omega)\). For any \( \chi \in A^{1,1}(M) \), we write in a local coordinate chart \((z^1, \ldots, z^n)\)

\[
\omega = \sqrt{-1} \frac{1}{2} g_{i\bar{j}} d z^i \wedge d \bar{z}^j
\]

and

\[
\chi = \sqrt{-1} \frac{1}{2} \chi_{i\bar{j}} d z^i \wedge d \bar{z}^j.
\]

In particular, in a local normal coordinate system \( g_{i\bar{j}} = \delta_{i\bar{j}} \), the matrix \( \chi_{i\bar{j}} \) is a Hermitian matrix. We denote \( \lambda(\chi_{i\bar{j}}) \) by the eigenvalues of the matrix \( \chi_{i\bar{j}} \).
Definition 2.1 We define $\sigma_k(\chi)$ with respect to $\omega$ as

$$\sigma_k(\chi) = \sigma_k(\lambda(\chi_{ij})).$$

The definition of $\sigma_k$ is independent of the choice of local normal coordinate system. In fact, $\sigma_k$ can be defined without the use of local normal coordinate by

$$\sigma_k(\chi) = C_k^n \chi^k \wedge \omega^{n-k}. $$

where $C_k^n = \frac{n!}{(n-k)!k!}$. We also define the Gårding’s cone on $M$ by

$$\Gamma_k(M) = \{ \chi \in A^{1,1}(M) : \sigma_i(\chi) > 0, \ \forall \ 1 \leq i \leq k \}. \quad (2.10)$$

Using the above notation, we can rewrite Eq. (1.1) in the following local form:

$$\sigma_k(\chi_{iu}) = \sum_{l=0}^{k-1} \beta_l(z) \sigma_l(\chi_{iu}), \quad (2.11)$$

where $\beta_l(z) = \frac{C_k^n}{C_l^n} \alpha_l(z)$.

Moreover, we also have a local version of the cone condition (1.4).

Lemma 2.2 $\chi \in C_k(\omega)$ is equivalent to

$$\sigma_{k-1}(\chi|i) - \sum_{l=1}^{k-1} \beta_l \sigma_{l-1}(\chi|i) > 0 \quad (2.12)$$

for any $i \in \{1, 2, \ldots, n\}$, where $(\chi|i)$ denotes the matrix obtained by deleting the $i$-th row and $i$-th column of $\chi$, and $\beta_l(z) = \frac{C_k^n}{C_l^n} \alpha_l(z)$ is denoted as before.

Proof The equality (2.12) follows directly if we observe that coefficient of $(n-1, n-1)$ form $\prod_{j=1, j \neq i}^{n} dz^j d\bar{z}^j$ in $\chi^{l-1} \wedge \omega^{n-l}$ is

$$(l-1)!(n-1)! \sigma_{l-1}(\chi|i) = \frac{1}{l} \frac{n!}{C_l^n} \sigma_{l-1}(\chi|i).$$

$\square$

2.2 The Ellipticity and Concavity

To make Eq. (2.11) elliptic and concave, we follow an important observation by Guan–Zhang [26] to rewrite (2.11) in the form:

$$F(u_{ij}, z) := \frac{\sigma_k(\chi_{iu})}{\sigma_{k-1}(\chi_{iu})} - \sum_{l=0}^{k-2} \beta_l(z) \frac{\sigma_l(\chi_{iu})}{\sigma_{l-1}(\chi_{iu})} = \beta_{k-1}(z). \quad (2.13)$$
Then, as a corollary of Propositions 2.1 (4)(5), we obtain (see also Proposition 2.2 in [26]):

**Lemma 2.3** If \( u \) is a \( C^2 \) function with \( \chi_u \in \Gamma_{k-1}(M) \), and \( \beta_l(z) \) (\( 0 \leq l \leq k - 2 \)) are nonnegative, then the operator \( F \) is elliptic and concave. Moreover, if \( \sum_{l=0}^{k-2} \beta_l(z) > 0 \) for all \( z \in M \), the operator \( F \) is strictly elliptic.

### 2.3 Some Key Lemmas

In this subsection, we prove some inequalities and lemmas which will play an important role in the establishment of the a priori estimates. In the following, for \( \lambda(z) = (\lambda_1(z), \ldots, \lambda_n(z)) \in C^0(M, \mathbb{R}^n) \), we set

\[
f(\lambda(z), z) := \frac{\sigma_k(\lambda(z))}{\sigma_{k-1}(\lambda(z))} - \sum_{l=0}^{k-2} \beta_l(z) \frac{\sigma_l(\lambda(z))}{\sigma_{k-1}(\lambda(z))} = \beta_{k-1}(z) \tag{2.14}
\]

and denote by \( f_i(\lambda) = \frac{\partial f}{\partial \lambda_i}(\lambda) \).

**Lemma 2.4** Let \( \beta_0(z), \ldots, \beta_{k-2}(z) \) be nonnegative functions on \( M \). Assume that \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_{k-1} \), then we have for any \( \mu = (\mu_1, \ldots, \mu_n) \in \Gamma_{k-1} \)

\[
\sum_{i=1}^{n} f_i(\lambda) \mu_i \geq f(\mu) + (k - l) \sum_{l=0}^{k-2} \beta_l \frac{\sigma_l(\lambda)}{\sigma_{k-1}(\lambda)}. \tag{2.15}
\]

**Proof** Since \( f \) is concave in \( \Gamma_{k-1} \) (see Lemma 2.3), we have

\[
f(\mu) \leq \sum_i f_i(\lambda)(\mu_i - \lambda_i) + f(\lambda).
\]

Using the homogeneousness of the Hessian quotient operators, we have

\[
\sum_i f_i(\lambda) \lambda_i = \frac{\sigma_k(\lambda(z))}{\sigma_{k-1}(\lambda(z))} - \sum_{l=0}^{k-2} (l - k + 1) \beta_l(z) \frac{\sigma_l(\lambda(z))}{\sigma_{k-1}(\lambda(z))}
\]

\[
= f(\lambda) + \sum_{l=0}^{k-2} (k - l) \beta_l \frac{\sigma_l(\lambda)}{\sigma_{k-1}(\lambda)}.
\]

Thus,

\[
\sum_i f_i(\lambda) \mu_i \geq f(\mu) + (k - l) \sum_{l=0}^{k-2} \beta_l \frac{\sigma_l(\lambda)}{\sigma_{k-1}(\lambda)}.
\]

So, the proof is complete. \( \square \)
Lemma 2.5 Let $\beta_0(z), \ldots, \beta_{k-2}(z)$ be nonnegative continuous functions and $\beta_{k-1}(z)$ be a continuous function on $M$. Assume $\mu(z) = (\mu_1(z), \ldots, \mu_n(z)) \in \Gamma_{k-1}$ for any $z \in M$ and $\mu$ satisfies
\[
\frac{\sigma_{k-1}(\mu|\mu)}{\sigma_{k-2}(\mu|\mu)} - \sum_{l=1}^{k-2} \beta_l(z) \frac{\sigma_{l-1}(\mu|\mu)}{\sigma_{k-2}(\mu|\mu)} > \beta_{k-1}(z)
\] (2.16)
for arbitrary $z \in M$ and $i \in \{1, 2, \ldots, n\}$. Assume also that $\lambda(z) \in \Gamma_{k-1}$ for all $z \in M$ and satisfies (2.14). Then there are constants $N, \theta > 0$ depending on $|\mu|_{C^0(M)}$ and $|\beta|_{C^0(M)}$ with $0 \leq i \leq k - 1$ such that for any $z \in M$, if
\[
\max_{1 \leq i \leq n} \{\lambda_i(z)\} \geq N,
\]
we have at $z$
\[
\sum_i f_i(\lambda)(\lambda_i - \mu_i) \leq -\theta - \theta \sum_i f_i(\lambda)
\] (2.17)
or
\[
f_1(\lambda)\lambda_1 \geq \theta.
\] (2.18)

Proof For any $z \in M$, without loss of generality, we may assume $\lambda_1(z) = \max_{1 \leq i \leq n} \{\lambda_i(z)\}$. If $\lambda_1 \gg -\mu_1 + \epsilon$, we have
\[
\sum_i f_i(\lambda)(\lambda_i - \mu_i)
\] 
\[
= \sum_{i \geq 2} f_i(\lambda)(\lambda_i - (\mu_i - \epsilon)) - \epsilon \sum_i f_i(\lambda) + f_1(\lambda)(\lambda_1 - \mu_1 + \epsilon)
\] 
\[
\leq \sum_{i \geq 2} f_i(\lambda)(\lambda_i - (\mu_i - \epsilon)) - \epsilon \sum_i f_i(\lambda) + 2f_1(\lambda)\lambda_1.
\] (2.19)
Choosing $\epsilon$ small enough which is independent of $z$ such that $(\mu_1 - \epsilon, \ldots, \mu_n - \epsilon) \in \Gamma_{k-1}$ satisfies (2.16). So we can choose $\lambda_1$ large enough which is independent of $z$ such that $\tilde{\mu} = (\lambda_1, \mu_2 - \epsilon, \ldots, \mu_n - \epsilon) \in \Gamma_{k-1}$ and
\[
f(\tilde{\mu}) \geq \beta_{k-1} + \tilde{\epsilon}
\] (2.20)
in view that
\[
\lim_{\lambda_1 \to +\infty} f(\tilde{\mu}) = \frac{\sigma_{k-1}(\mu')}{\sigma_{k-2}(\mu')} - \sum_{l=1}^{k-2} \beta_l \frac{\sigma_{l-1}(\mu')}{\sigma_{k-2}(\mu')} > \beta_{k-1},
\]
where $\mu' = (\mu_2 - \epsilon, \ldots, \mu_n - \epsilon)$ and we use the inequality (2.16). Now, we rewrite (2.19)

$$
\sum_i f_i(\lambda)(\lambda_i - \mu_i) \\
\leq \sum_i f_i(\lambda)(\lambda_i - \tilde{\mu}_i) - \epsilon \sum_i f_i(\lambda) + 2f_1(\lambda)\lambda_1.
$$

(2.21)

Since $f$ is concave in $\Gamma_{k-1}$ (see Lemma 2.3), there is

$$
f(\tilde{\mu}) \leq \sum_i f_i(\lambda)(\tilde{\mu}_i - \lambda_i) + f(\lambda),
$$

which implies together with (2.21)

$$
\sum_i f_i(\lambda)(\lambda_i - \mu_i) \leq f(\lambda) - f(\tilde{\mu}) - \epsilon \sum_i f_i(\lambda) + 2f_1(\lambda)\lambda_1.
$$

Thus, when $\lambda_1$ is large enough, we have by (2.20)

$$
\sum_i f_i(\lambda)(\lambda_i - \mu_i) \leq \tilde{\epsilon} - \epsilon \sum_i f_i(\lambda) + 2f_1(\lambda)\lambda_1.
$$

If $f_1(\lambda)\lambda_1 \leq \tilde{\epsilon}/4$, we obtain

$$
\sum_i f_i(\lambda)(\lambda_i - \mu_i) \leq -\tilde{\epsilon}/2 - \epsilon \sum_i f_i(\lambda).
$$

Then, we complete the proof if we choose $\theta = \min\{\tilde{\epsilon}/4, \epsilon\}$. \qed

**Lemma 2.6** Assume $\beta_l(z) > 0$ ($0 \leq l \leq k - 2$) and $\lambda \in \Gamma_{k-1}$ satisfies (2.14), then we have

$$
0 < \frac{\sigma_l(\lambda)}{\sigma_{k-1}(\lambda)} \leq C, \quad 0 \leq l \leq k - 2,
$$

(2.22)

where the constant $C$ depends on $n, k$, $\frac{1}{\inf M \beta_l}$, $\sup_M |\beta_{k-1}|$. Moreover, we have

$$
-\sup_M |\beta_{k-1}| \leq \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)} \leq C,
$$

(2.23)

where the constant $C$ depends on $n, k$, $\sup_M |\beta_i|$ ($0 \leq i \leq k - 1$).

**Proof** On the one hand, if $\frac{\sigma_k}{\sigma_{k-1}} \leq 1$, then we get from the equation (2.14)

$$
\beta_l \frac{\sigma_l}{\sigma_{k-1}} \leq \frac{\sigma_k}{\sigma_{k-1}} - \beta_{k-1} \leq 1 + \sup |\beta_{k-1}|, \quad 0 \leq l \leq k - 2.
$$

(2.24)
On the other hand, if \( \frac{\sigma_k}{\sigma_{k-1}} > 1 \), i.e. \( \frac{\sigma_{k-1}}{\sigma_k} < 1 \). We can get for \( 0 \leq l \leq k - 2 \) by the Newton-MacLaurin inequality (2.4),

\[
\frac{\sigma_l}{\sigma_{k-1}} \leq \frac{(C_n^k)^{k-1-l}C_n^l}{(C_{n-1}^{k-1})^{k-l}} \leq \frac{(C_n^k)^{k-1-l}C_n^l}{(C_{n-1}^{k-1})^{k-l}} \leq C(n, k).
\] (2.25)

So, we get (2.22). Moreover, combining (2.24) and (2.25) results in

\[
\beta_l \frac{\sigma_l}{\sigma_{k-1}} \leq 1 + \sup_l |\beta_{k-1}| + C(n, k) \sup_l \beta_l, \quad 0 \leq l \leq k - 2.
\]

Then, it follows that

\[
- \sup_l |\beta_{k-1}| \leq \frac{\sigma_k}{\sigma_{k-1}} = \sum_{l=0}^{k-2} \beta_l \frac{\sigma_l}{\sigma_{k-1}} + \beta_{k-1}
\]

\[
\leq C(n, k) \left( 1 + \sum_{l=0}^{k-2} \sup_l \beta_l + \sup_l |\beta_{k-1}| \right).
\]

Thus, the proof is complete. \( \square \)

**Lemma 2.7** Assume \( \beta_l(z) \geq 0 \) (\( 0 \leq l \leq k - 2 \)) and \( \lambda \in \Gamma_{k-1} \), then

\[
\sum_{i=1}^{n} f_i \geq \frac{n-k+1}{k}.
\] (2.26)

**Proof** By direct computations, we can get by Proposition 2.1(4)(3)(1)

\[
\sum_{i=1}^{n} f_i \geq \sum_{i=1}^{n} \partial \left( \frac{\sigma_k}{\sigma_{k-1}} \right) \frac{\partial}{\partial \lambda_i} = \sum_{i=1}^{n} \frac{\sigma_{k-1}(\lambda_i)\sigma_{k-1} - \sigma_k \sigma_{k-2}(\lambda_i)}{\sigma_{k-1}^2}
\]

\[
= \frac{(n-k+1)\sigma_{k-1}^2 - (n-k+2)\sigma_k \sigma_{k-2}}{\sigma_{k-1}^2}
\]

\[
\geq \frac{n-k+1}{k},
\]

where we get the last inequality from Newton’ inequality (2.2). Hence (2.26) holds. \( \square \)

**3 C^0 Estimate**

In this section, we follow the idea of Sun [46] to derive \( C^0 \) estimate directly from the cone condition. Maybe the same estimate can be also obtained by the use of the Alexandrov-Bakelman-Pucci argument, first employed in the context of complex Monge–Ampère equation by Błocki [3, 4] (see also Székelyhidi [42] for a general class of fully non-linear equations).
3.1 Some Lemmas

Since \( \chi_0 \in \Gamma_{k-1}(M) \), we may assume that there is a uniform constant \( \tau > 0 \) such that
\[
\chi_0 - \tau \omega \in \Gamma_{k-1}(M) \quad \text{and} \quad \omega - \tau \chi_0 \in \Gamma_{k-1}(M).
\]
(3.1)

Then, we can succeed in extending Lemma 2.3 in [46] to our case.

**Lemma 3.1** Let \((M, \omega)\) be a Kähler manifold of complex dimension \( n \geq 2 \) and \( \alpha_0(z), \ldots, \alpha_{k-1}(z) \) be continuous functions on \( M \) which satisfy \( \alpha_l(z) \geq 0 \) for all \( z \in M \) and \( 0 \leq l \leq k-2 \). Suppose that \( \chi_0 \in \Gamma_{k-1}(M) \) satisfies (3.1). If \( u \in C^2(M) \) satisfies \( \chi_u \in \Gamma_{k-1}(M) \), then we have the following pointwise inequalities:

1. for \( 0 \leq t \leq 1 \) and \( 1 \leq l \leq k-1 \)
\[
\chi_{tu}^{l-1} \wedge \omega^{n-l} \geq (1-t)^{l-1} t^{l-1} \omega^{n-1}.
\]
(3.2)

2. for \( 0 < t \leq 1 \) and \( 1 \leq l \leq k-1 \)
\[
\chi_u^l \wedge \omega^{n-l} \leq \frac{1}{t} \chi_{tu}^l \wedge \omega^{n-l}.
\]
(3.3)

Moreover, if \( u \) is a solution to the equation (1.1) and \( \chi_0 \) satisfies the cone condition (1.4), then there exists a uniform constant \( C > 0 \) such that for \( 0 \leq t \leq 1 \)
\[
k \chi_{tu}^{k-1} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} l \alpha_l \chi_{tu}^{l-1} \wedge \omega^{n-l}
\]
\[
> C(1-t) \chi_{tu}^{k-2} \wedge \omega^{n-k+1}.
\]
(3.4)

and consequently
\[
k \chi_{tu}^{k-1} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} l \alpha_l \chi_{tu}^{l-1} \wedge \omega^{n-l}
\]
\[
> C t^{k-2} (1-t)^{k-1} \omega^{n-1}.
\]
(3.5)

**Proof** For \( 1 \leq l \leq k-1 \) and \( 1 \leq i \leq n \), \( \sigma_i^l (\lambda) \) and \( \sigma_{i-1}^{l-1} (\lambda | i) \) are concave in \( \Gamma_l \) and \( \Gamma_{l-1} \) respectively. Thus,
\[
\sigma_i^l (\chi_{tu}) \geq (1-t) \sigma_i^l (\chi_0) + t \sigma_i^l (\chi_u),
\]
(3.6)
\[
\sigma_{i-1}^{l-1} (\chi_{tu} | i) \geq (1-t) \sigma_{i-1}^{l-1} (\chi_0 | i) + t \sigma_{i-1}^{l-1} (\chi_u | i)
\]
\[
\geq (1-t) \sigma_{i-1}^{l-1} (\chi_0 | i),
\]
(3.7)
and
\[
\sigma_{l-1}^1(\chi_0|i) \geq \sigma_{l-1}^1(\chi_0 - \tau \omega|i) + \tau \sigma_{l-1}^1(\omega|i) \geq \tau \sigma_{l-1}^1(\omega|i).
\] (3.8)

Combining (3.7) and (3.8) gives
\[
\sigma_{l-1}(\chi_{tu}|i) \geq (1 - t)^{l-1} \tau^{l-1} \sigma_{l-1}(\omega|i),
\]
which is just the local form of (3.2). Similarly, (3.3) is a consequence of (3.6). Now we only need to prove (3.4) and (3.5). Since
\[
f(\chi|i) := \frac{\sigma_{k-1}(\chi|i)}{\sigma_{k-2}(\chi|i)} - \sum_{l=1}^{k-2} \beta_l \frac{\sigma_{l-1}(\chi|i)}{\sigma_{k-2}(\chi|i)}
\]
is concave in \( \Gamma_{k-2} \) (see Lemma 2.3), we can obtain
\[
f(\chi_{tu}|i) \geq (1 - t) f(\chi_0|i) + t f(\chi_u|i).
\] (3.9)

Moreover, we have by Proposition 2.1(7)
\[
f(\chi_u|i) = \frac{\sigma_{k-1}(\chi_u|i)}{\sigma_{k-2}(\chi_u|i)} - \sum_{l=1}^{k-2} \beta_l \frac{\sigma_{l-1}(\chi_u|i)}{\sigma_{k-2}(\chi_u|i)} \geq \frac{\sigma_k(\chi_u)}{\sigma_{k-1}(\chi_u)} - \sum_{l=1}^{k-2} \beta_l \frac{\sigma_l(\chi_u)}{\sigma_{k-1}(\chi_u)} = \frac{\beta_0}{\sigma_{k-1}(\chi_u)} + \beta_{k-1} \geq \beta_{k-1}.
\] (3.10)

where \( \beta_0, \ldots, \beta_{k-1} \) are defined in the equation (1.7). In addition, since \( \chi_0 \) satisfies the cone condition (1.4), there exists some uniform constant \( \delta > 0 \) which is independent of \( z \) such that
\[
f(\chi_0|i) > \beta_{k-1}(z) + \delta,
\] (3.11)
where we write the cone condition in a local version as Lemma 2.2. Substituting (3.10) and (3.11) into (3.9), we get
\[
\frac{\sigma_{k-1}(\chi_{tu}|i)}{\sigma_{k-2}(\chi_{tu}|i)} - \sum_{l=1}^{k-2} \beta_l \frac{\sigma_{l-1}(\chi_{tu}|i)}{\sigma_{k-2}(\chi_{tu}|i)} > (1 - t)(\beta_{k-1} + \delta) + t \beta_{k-1},
\]
this is to say
\[\sigma_{k-1}(\chi_{tu}|i) - \sum_{l=1}^{k-1} \beta_l \sigma_{l-1}(\chi_{tu}|i) > \delta (1 - t) \sigma_{k-2}(\chi_{tu}|i),\]

which is just the local form of (3.4). Then, (3.5) follows by substituting (3.2) into it. \(\square\)

Next, we derive the following important inequality by iterating the inequality (3.15) in [46].

**Lemma 3.2** Let \((M, \omega)\) be a Kähler manifold of complex dimension \(n \geq 2\). Suppose that \(\chi_0 \in \Gamma_{k-1}(M)\) satisfies (3.1). If \(u \in C^2(M)\) satisfies \(\chi_u \in \Gamma_{k-1}(M)\), then we have the following inequalities for \(l < k\):

\[
k - 1 \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{tu}^{l-2} \wedge \omega^{n-l} + \frac{1}{l-1} \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{tu}^{l-1} \wedge \omega^{n-l-1} \geq \tau \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{tu}^{l-2} \wedge \omega^{n-l+1}.
\]

(3.12)

**Proof** Using integration by parts and Garding’s inequality (2.5), it yields

\[
\int_0^{\frac{1}{2}} dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{tu}^{l-1} \wedge \omega^{n-l} \geq \tau \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{tu}^{l-2} \wedge \omega^{n-l+1} + \frac{1}{l-1} \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{tu}^{l-1} \wedge \omega^{n-l} \geq \tau \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{tu}^{l-2} \wedge \omega^{n-l+1}.
\]

Thus,

\[
\frac{l}{l-1} \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{tu}^{l-1} \wedge \omega^{n-l} \geq \tau \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{tu}^{l-2} \wedge \omega^{n-l+1}.
\]

So, we obtain by iteration
\[
\frac{k-1}{l-1} \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{k-2} \wedge \omega^{n-k+1} \\
\geq \tau^{k-l} \int_0^{\frac{1}{2}} dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{l-2} \wedge \omega^{n-l+1},
\]
which completes the proof. \qed

Once we have established the following inequality (3.13), we can use it to derive \(C^0\) estimate (3.14) by the standard argument in [49, 50] without using the equation (1.1).

Lemma 3.3 Let \((M, \omega)\) be a closed Kähler manifold of complex dimension \(n \geq 2\) and \(\alpha_0(z), \ldots, \alpha_{k-1}(z)\) be \(C^2\) functions on \(M\) which satisfy \(\alpha_i(z) \geq 0\) for all \(z \in M\) and \(0 \leq l \leq k - 2\). Suppose that \(\chi_0 \in \Gamma_{k-1}(M)\) is closed, and satisfies (3.1) and the cone condition (1.4). If \(u \in C^2(M)\) is a solution to the equation (1.1) and satisfies \(\chi_u \in \Gamma_{k-1}(M)\), then there exists uniform constants \(C\) and \(p_0\) such that for \(p \geq p_0\) the following inequality holds

\[
\int_M |\partial e^{-\frac{p}{2}u}|^2 g \omega^n \leq Cp \int_M e^{-pu} \omega^n. \tag{3.13}
\]

Thus, there exists a uniform constant \(C\) depending on the given data \(M, \omega, \chi_0,\) and \(|\alpha_i|_{C^0(M)}(0 \leq i \leq k - 1)\) such that

\[
|u|_{C^0(M)} \leq C \text{ with } \sup_M u = 0. \tag{3.14}
\]

Proof The fundamental theorem of Calculus tells us that

\[
(\chi_u^k \wedge \omega^{n-k} - \chi_0^k \wedge \omega^{n-k}) - \sum_{l=0}^{k-1} \alpha_l (\chi_u^l \wedge \omega^{n-l} - \chi_0^l \wedge \omega^{n-l})
\]

\[
= \int_0^1 \sqrt{-1} \partial \bar{\partial} u \wedge (k \chi_{tu}^{k-1} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} l \alpha_l \chi_{tu}^{l-1} \wedge \omega^{n-l}) dt.
\]

Multiplying the above equality by term \(e^{-pu}\) and integrating on \(M\) give

\[
\int_M e^{-pu} \left[(\chi_u^k \wedge \omega^{n-k} - \chi_0^k \wedge \omega^{n-k}) - \sum_{l=0}^{k-1} \alpha_l (\chi_u^l \wedge \omega^{n-l} - \chi_0^l \wedge \omega^{n-l})\right]
\]

\[
= p \int_0^1 dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \left(k \chi_{tu}^{k-1} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} l \alpha_l \chi_{tu}^{l-1} \wedge \omega^{n-l}\right)
\]

\[
- \sum_{l=1}^{k-1} l \int_0^1 dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} \alpha_l \wedge \chi_{tu}^{l-1} \wedge \omega^{n-l}.
\]
\[ \geq p \int_0^1 dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \left( k \chi_{tu}^{k-1} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} l \alpha_l \chi_{tu}^{l-1} \wedge \omega^{n-l} \right) \]
\[ - \sum_{l=1}^{k-1} C_l \frac{1}{p} \int_0^1 dt \int_M e^{-pu} \chi_{tu}^{l-1} \wedge \omega^{n-l+1}. \] (3.15)

Using the equation (1.1), we have
\[ \int_M e^{-pu} \left[ (\chi^k_0 \wedge \omega^{n-k} - \chi^k_0 \wedge \omega^{n-k}) - \sum_{l=0}^{k-1} \alpha_l (\chi^l_0 \wedge \omega^{n-l} - \chi^l_0 \wedge \omega^{n-l}) \right] \]
\[ = \int_M e^{-pu} \left( - \chi^k_0 \wedge \omega^{n-k} + \sum_{l=0}^{k-1} \alpha_l \chi^l_0 \wedge \omega^{n-l} \right) \]
\[ \leq C \int_M e^{-pu} \omega^n. \] (3.16)

Using the inequality (3.3), we obtain
\[ \int_0^1 dt \int_M e^{-pu} \chi_{tu}^{l-1} \wedge \omega^{n-l+1} \]
\[ \leq 2^{l-1} \int_0^1 dt \int_M e^{-pu} \chi_{tu}^{l-1} \wedge \omega^{n-l+1} \]
\[ \leq 2^l \int_0^1 dt \int_M e^{-pu} \chi_{tu}^{l-1} \wedge \omega^{n-l+1}. \] (3.17)

Plugging the inequalities (3.16) and (3.17) into (3.15), it yields
\[ p \int_0^1 dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \left( k \chi_{tu}^{k-1} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} l \alpha_l \chi_{tu}^{l-1} \wedge \omega^{n-l} \right) \]
\[ \leq \sum_{l=1}^{k-1} 2^l \frac{C_l}{p} \int_0^1 dt \int_M e^{-pu} \chi_{tu}^{l-1} \wedge \omega^{n-l+1} + C \int_M e^{-pu} \omega^n. \] (3.18)

We deal with the first term on the right side of the inequality (3.18). First, using the fundamental theorem of Calculus, we rewrite it
\[ \frac{1}{p} \int_0^1 dt \int_M e^{-pu} \chi_{tu}^{l-1} \wedge \omega^{n-l+1} \]
\[ = (l - 1) \int_0^1 dt \int_0^t ds \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{su}^{l-2} \wedge \omega^{n-l+1} \]
\[ + \frac{1}{2p} \int_M e^{-pu} \chi_0^{l-1} \wedge \omega^{n-l+1}. \]
Then, we obtain

\[
\frac{1}{p} \int_0^1 dt \int_M e^{-pu} \chi_{iu}^{l-1} \wedge \omega^{n-l+1} \\
\leq \frac{l-1}{2} \int_0^1 dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{iu}^{l-2} \wedge \omega^{n-l+1} \\
+ \frac{1}{2p} \int_M e^{-pu} \chi_{0}^{l-1} \wedge \omega^{n-l+1} \\
\leq \frac{k-1}{\tau^{k-l}} \int_0^1 dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{iu}^{k-2} \wedge \omega^{n-k+1} \\
+ \frac{C}{2p} \int_M e^{-pu} \omega^n, 
\tag{3.19}
\]

where we use the inequality (3.12) to get the last inequality. To cancel the first term on the right side of (3.19), we will use a part of the left term in (3.18). In details, we can get the following positive term for \(0 \leq t \leq \frac{1}{2}\) from the inequality (3.4)

\[
p e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \left( k \chi_{iu}^{k-1} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} l \alpha_l \chi_{iu}^{l-1} \wedge \omega^{n-l} \right) \\
\geq C p e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{iu}^{k-2} \wedge \omega^{n-k+1}. \tag{3.20}
\]

Thus, if we choose \(p\) sufficiently large, the integral of the term (3.20) on \(M\) can kill the first term on the right side of (3.19). Then, (3.18) becomes

\[
\frac{P}{2} \int_0^1 dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \left( k \chi_{iu}^{k-1} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} l \alpha_l \chi_{iu}^{l-1} \wedge \omega^{n-l} \right) \\
\leq C \int_M e^{-pu} \omega^n,
\]

which implies in view of (3.5)

\[
p \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega^{n-1} \leq C \int_M e^{-pu} \omega^n.
\]

So, our proof is completed. \(\square\)

4 \(C^2\) Estimate

4.1 Notations and Some Lemmas

In local complex coordinates \((z^1, \ldots, z^n)\), the subscripts of a function \(u\) always denote the covariant derivatives of \(u\) with respect to \(\omega\) in the directions of the local frame
\[ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n} \]. Namely,
\[ u_i = \nabla_{\frac{\partial}{\partial z^i}} u, \quad u_{i\overline{j}} = \nabla_{\frac{\partial}{\partial z^j}} \nabla_{\frac{\partial}{\partial z^i}} u, \quad u_{i\overline{j}k} = \nabla_{\frac{\partial}{\partial z^k}} \nabla_{\frac{\partial}{\partial z^j}} \nabla_{\frac{\partial}{\partial z^i}} u. \]

But, the covariant derivatives of a \((1, 1)\)-form \(\chi\) with respect to \(\omega\) will be denoted by indices with semicolons, e.g.,
\[ \chi_{i\overline{j};k} = \nabla_{\frac{\partial}{\partial z^k}} \chi \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right), \quad \chi_{i\overline{j};kl} = \nabla_{\frac{\partial}{\partial z^l}} \nabla_{\frac{\partial}{\partial z^k}} \chi \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right). \]

We recall the following commutation formula on Kähler manifolds \((M, \omega)\) [23, 24, 29].

**Lemma 4.1** For \(u \in C^4(M)\), we have
\[ u_{i\overline{j}l} = u_{il\overline{j}} - u_{pj} R_{i\overline{m}j\overline{l}} g^{\overline{mp}}, \quad u_{i\overline{j}m} = u_{pm\overline{j}} - u_{i\overline{j}l} u_{p\overline{l}m}, \quad u_{i\overline{j}l} = u_{pi\overline{j}}, \]
\[ u_{i\overline{j}l\overline{m}} = u_{mj\overline{i}l} + u_{pj} R_{i\overline{m}q\overline{l}} g^{\overline{pq}} - u_{p\overline{m}} R_{i\overline{j}l\overline{q}} g^{\overline{pq}}, \]
where \(R\) is the curvature tensor of \((M, \omega)\).

For the convenience of notations, we will denote
\[ F_k(\chi) := \sigma_k(\chi) / \sigma_{k-1}(\chi), \quad F_l(\chi) := - \sigma_l(\chi) / \sigma_{k-1}(\chi), \quad 0 \leq l \leq k-2, \]
\[ F(\chi, z) := F_k(\chi) + \sum_{l=0}^{k-2} \beta_l(z) F_l(\chi), \]
and
\[ F^{i\overline{j}} := \frac{\partial F}{\partial \chi_{i\overline{j}}}, \quad F^{i\overline{j},rs} := \frac{\partial^2 F}{\partial \chi_{i\overline{j}} \partial \chi_{r\overline{s}}}, \quad F_{i\overline{j}} := \frac{\partial F_l}{\partial \chi_{i\overline{j}}}, \quad F_{i\overline{j},rs} := \frac{\partial^2 F_l}{\partial \chi_{i\overline{j}} \partial \chi_{r\overline{s}}}, \]
where \(1 \leq i, j, r, s \leq n\) and \(0 \leq l \leq k-2\).

**Lemma 4.2** Let \(\beta_0, \ldots, \beta_{k-2}\) are nonnegative \(C^2\) functions on \((M, \omega)\) and \(\beta_{k-1} \in C^2(M)\). Assume \(\chi \in \Gamma_{k-1}(M)\) satisfies \(F(\chi, z) = \beta_{k-1}(z)\). For any \(z \in M\), we choose a normal coordinate such that at this point \(\omega = \sqrt{-1} \delta_{ij} dz^i \wedge d\overline{z}^j\). Then, we have for any \(\delta < 1\) at \(z\)
\[ F^{i\overline{j}} \chi_{i\overline{j};p\overline{q}} \geq (1 - \delta^2) F^{i\overline{j},rs} \chi_{i\overline{j};p\overline{r}q\overline{s};p\overline{q}} + \nabla_{\overline{p}} \nabla_{\overline{q}} \beta_{k-1} \]
\[ + \sum_{l=0}^{k-2} \frac{1}{1 + \frac{1}{k+1-l}} \frac{\| \nabla_{\overline{p}} \beta_l \|^2}{\delta^2 \beta_l} F_l - \sum_{l=0}^{k-2} \nabla_{\overline{p}} \nabla_{\overline{q}} \beta_l \cdot F_l, \quad (4.1) \]
where \(\frac{\| \nabla_{\overline{p}} \beta_l \|^2}{\beta_l}\) should be understood as zero when \(\beta_l = 0\).
Proof Differentiating the equation (2.13) once, we have

$$\nabla_p \beta_{k-1} = F^{i\bar{j}} \chi_{i\bar{j};p} + \sum_{l=0}^{k-2} \nabla_p \beta_l \cdot F_l.$$

Differentiating the equation (2.13) again, we obtain

$$\nabla_{\bar{\nabla}} \nabla_p \beta_{k-1} = F^{i\bar{j},r\bar{s}} \chi_{i\bar{j};p} \chi_{r\bar{s};\bar{p}} + F^{i\bar{j}} \chi_{i\bar{j};p\bar{p}} + \sum_{l=0}^{k-2} \left( \nabla_{\bar{\nabla}} \beta_l \cdot F^{i\bar{j}}_l \chi_{i\bar{j};p} \right) + \sum_{l=0}^{k-2} \nabla_{\bar{\nabla}} \nabla_p \beta_l \cdot F_l.$$

Moreover, since the operator $\left( \frac{\sigma_k - 1}{\sigma_l} \right)^{\frac{1}{k-l}}$ is concave for $0 \leq l \leq k - 2$ (see Proposition 2.1(5)), we have

$$- F^{i\bar{j},r\bar{s}}_l \chi_{i\bar{j};p} \chi_{r\bar{s};\bar{p}} \geq -\left( 1 + \frac{1}{k - 1 - l} \right) F^{-1}_l F^{i\bar{j}}_l F^{r\bar{s}}_l \chi_{i\bar{j};p} \chi_{r\bar{s};\bar{p}}. \quad (4.2)$$

Since $F_k$ is concave in $\Gamma_{k-1}$, throwing away the nonpositive term

$$\delta^2 \frac{\partial^2 F_k}{\partial \chi_{i\bar{j}} \partial \chi_{r\bar{s}}} \chi_{i\bar{j};p} \chi_{r\bar{s};\bar{p}}$$

gives

$$\nabla_{\bar{\nabla}} \nabla_p \beta_{k-1} - (1 - \delta^2) F^{i\bar{j},r\bar{s}} \chi_{i\bar{j};p} \chi_{r\bar{s};\bar{p}}$$

$$\leq \sum_{l=0}^{k-2} \delta^2 \beta_l F^{i\bar{j},r\bar{s}}_l \chi_{i\bar{j};p} \chi_{r\bar{s};\bar{p}}$$

$$+ F^{i\bar{j}} \chi_{i\bar{j};p\bar{p}} + \sum_{l=0}^{k-2} \left( \nabla_{\bar{\nabla}} \beta_l \cdot F^{i\bar{j}}_l \chi_{i\bar{j};p} + \nabla_p \beta_l \cdot F^{i\bar{j}}_l \chi_{i\bar{j};p} \right) + \sum_{l=0}^{k-2} \nabla_{\bar{\nabla}} \nabla_p \beta_l \cdot F_l$$

$$\leq \delta^2 \sum_{l=0}^{k-2} \beta_l \left( 1 + \frac{1}{k - 1 - l} \right) F^{-1}_l F^{i\bar{j}}_l \chi_{i\bar{j};p}^2 + F^{i\bar{j}} \chi_{i\bar{j};p\bar{p}}$$

$$+ \sum_{l=0}^{k-2} \left( \nabla_{\bar{\nabla}} \beta_l \cdot F^{i\bar{j}}_l \chi_{i\bar{j};p} + \nabla_p \beta_l \cdot F^{i\bar{j}}_l \chi_{i\bar{j};p} \right) + \sum_{l=0}^{k-2} \nabla_{\bar{\nabla}} \nabla_p \beta_l \cdot F_l$$

$$= \frac{\delta^2 (k - l)}{k - 1 - l} \sum_{l=0}^{k-2} \beta_l F^{-1}_l \left| F^{i\bar{j}}_l \chi_{i\bar{j};p} + \frac{1}{1 + \frac{1}{k - 1 - l}} \frac{\nabla_p \beta_l}{\delta^2} F_l \right|^2.$$
\[-\sum_{l=0}^{k-2} \frac{1}{1 + \frac{1}{k-l-1}} \frac{\left|\nabla_{p} \beta_{l}\right|^{2}}{\delta^{2} \beta_{l}} F_{l} + F^{i\overline{j}} \chi_{i\overline{j};\, p\overline{p}} + \sum_{l=0}^{k-2} \nabla_{\overline{p}} \nabla_{p} \beta_{l} \cdot F_{l}\]

\[\leq -\sum_{l=0}^{k-2} \frac{1}{1 + \frac{1}{k-l-1}} \frac{\left|\nabla_{p} \beta_{l}\right|^{2}}{\delta^{2} \beta_{l}} F_{l} + F^{i\overline{j}} \chi_{i\overline{j};\, p\overline{p}} + \sum_{l=0}^{k-2} \nabla_{\overline{p}} \nabla_{p} \beta_{l} \cdot F_{l},\]

where we used the inequality (4.2) to get the second inequality. So, our proof is completed. \(\square\)

### 4.2 \(C^{2}\) Estimate

**Lemma 4.3** Let \((M, \omega)\) be a closed Kähler manifold of complex dimension \(n \geq 2\). Suppose that \(\chi_{0}\) satisfies the cone condition (1.4) and \(\alpha_{0}(z), \alpha_{1}(z), \ldots, \alpha_{k-1}(z)\) are real \(C^{2}\) functions on \(M\) satisfying (i) and (ii) in Assumption 1.1. If \(u \in C^{4}(M)\) is a solution to the equation (2.13) with \(\chi_{u} \in \Gamma_{k-1}(M)\), we have

\[\sup_{M} \left|\partial \overline{\partial} u\right| \leq C \left(\sup_{M} \left|\nabla u\right|^{2} + 1\right),\]

where the constant \(C\) depends on \(M, \omega, \chi_{0}, |u|_{C^{0}}, \inf_{M} \alpha_{l} (0 \leq l \leq k - 2)\) and \(|\alpha_{l}|_{C^{2}(M)} (0 \leq i \leq k - 1)\).

**Proof** For convenience, in the following argument, we write \(\chi = \chi_{u}\) for short (suppressing the subscript \(u\)). Following the work of Hou–Ma–Wu [29], we define for \(z \in M\) and a unit vector \(\xi \in T_{z}^{1,0} M\)

\[W(z, \xi) = \ln(\chi^{i\overline{j}} \xi^{i} \xi^{j}) + \varphi(|\nabla u|^{2}) + \psi(u),\]

where

\[\varphi(s) = -\frac{1}{2} \log \left(1 - \frac{s}{2K}\right) \text{ for } 0 \leq s \leq K - 1\]

and

\[\psi(t) = -A \log \left(1 + \frac{t}{2L}\right) \text{ for } -L + 1 \leq t \leq 0.\]

Here, we set

\[K := \sup_{M} |\nabla u|^{2} + 1, \quad L := \sup_{M} |u| + 1, \quad A := 2L\Lambda,\]

and \(\Lambda\) is a large constant that we will choose later. Clearly, \(\varphi\) satisfies

\[\frac{1}{2K} \geq \varphi' \geq \frac{1}{4K}, \quad \varphi'' = 2(\varphi')^{2} > 0,\]
and $\psi$ satisfies the bounds

$$2\Lambda \geq -\psi' \geq \Lambda, \quad \psi'' \geq \frac{2\varepsilon}{1 - \varepsilon} (\psi')^2 \quad \text{for all} \quad \varepsilon \leq \frac{1}{2\Lambda + 1}. $$

The function $W$ must achieve its maximum at some point $z$ in some unit direction of $\eta$. Around $z$, we choose a normal chart such that $\chi$ is diagonal and

$$\chi_{\eta\bar{\eta}} = \chi_{1\bar{1}} \geq \cdots \geq \chi_{n\bar{n}}. \quad (4.3)$$

It follows from (2.9)

$$F^{1\bar{1}} \leq \cdots \leq F^{n\bar{n}}. \quad (4.4)$$

Furthermore, we arrive at

$$W_i = \frac{\chi_{1\bar{1}}}{\chi_{1\bar{1}}} + \varphi' \nabla_i (|\nabla u|^2) + \psi' u_i = 0 \quad (4.5)$$

and

$$W_{i\bar{i}} = \frac{\chi_{1\bar{1}}}{\chi_{1\bar{1}}} - \frac{|\chi_{1\bar{1}}|^2}{\chi_{1\bar{1}}} + \varphi'' |\nabla_i (|\nabla u|^2)|^2 + \varphi' \nabla_i (|\nabla u|^2) + \psi'' |u_i|^2 + \psi' u_{i\bar{i}} \leq 0. \quad (4.6)$$

Commuting derivatives, we have the identity from Lemma 4.1

$$\chi_{1\bar{1}} = \chi_{i\bar{i}};_{1\bar{1}} + \chi_{1\bar{1}} R_{1\bar{1}i\bar{i}} - \chi_{i\bar{i}} R_{i\bar{i}1\bar{1}} + \chi_{01} R_{1\bar{1}i\bar{i}} - \chi_{0i} R_{1\bar{1}i\bar{i}} - \sum m \chi_{0m} R_{1\bar{1}i\bar{i}} + \sum m \chi_{0m} R_{i\bar{i}1\bar{1}}.$$

Using the above equality and the equality

$$F^{i\bar{i}} \chi_{i\bar{i}} = F_k + \sum_{l=0}^{k-2} (l - k + 1) \beta_l F_l,$$

we have

$$F^{ij} \chi_{1\bar{1};ij} \geq F^{ij} \chi_{i\bar{i};1\bar{1}} - C \chi_{1\bar{1}} \sum_i F^{i\bar{i}} - C F_k - C \sum_{l=0}^{k-1} |F_l| - C \sum_i F^{i\bar{i}}. $$

Then, plugging the inequality (4.1) into the above inequality yields

$$F^{ij} \chi_{1\bar{1};ij} \geq -(1 - \delta^2) F^{ij} \chi_{i\bar{i};1\bar{1}} \chi_{r\bar{r}:1\bar{1}} - C (1 + \chi_{1\bar{1}}) \sum_i F^{i\bar{i}} - C F_k - C \sum_{l=0}^{k-1} |F_l|.$$
\[ + \nabla_T \nabla_1 \beta_{k-1} - \sum_{l=0}^{k-2} \frac{1}{1 + \frac{1}{k+1-l}} \frac{1}{\delta^4 \beta_l} F_l - \sum_{l=0}^{k-2} \nabla_T \nabla_1 \beta_l \cdot F_l \]

\[ \geq -(1 - \delta^2) F^{i\tilde{i}, r\tilde{r}} \chi_{i\tilde{r};1} x_{r\tilde{r};T} - C (1 + x_{1\tilde{T}}) \sum_i F^{i\tilde{i}} - C, \]

where we used the fact that \( F_k \) and \( F_l \) \((0 \leq l \leq k - 2)\) are bounded (see (2.22) and (2.23)) to get the last inequality.

Multiplying (4.6) by \( F^{i\tilde{i}} \) and summing it over index \( i \), we can infer from the above inequality

\[ 0 \geq -\frac{F^{i\tilde{i}} |x_{1\tilde{T};i}|^2}{x_{1\tilde{T}}^2} - (1 - \delta^2) \frac{1}{x_{1\tilde{T}}} F^{i\tilde{i}, r\tilde{r}} \chi_{i\tilde{r};1} x_{r\tilde{r};T} \]

\[ + \varphi'' F^{i\tilde{i}} |\nabla_i(|\nabla u|)^2| + \varphi' F^{i\tilde{i}} \nabla_i \nabla_i (|\nabla u|^2) + \psi'' F^{i\tilde{i}} |u_i|^2 + \psi' F^{i\tilde{i}} u_{i\tilde{i}} - C \sum_i F^{i\tilde{i}} - \frac{C}{x_{1\tilde{T}}}. \]  

(4.7)

To proceed, we need the following calculation

\[ \nabla_i (|\nabla u|^2) = \sum_j (-u_j x_{0i\tilde{j}} + u_{ji} u_{\tilde{j}}) + u_i x_{ii} \]

and

\[ \nabla_T \nabla_i (|\nabla u|^2) = \sum_j (x_{0i\tilde{j}} x_{0i\tilde{j}} + u_{ji} u_{i\tilde{j}} - x_{0i\tilde{j}} x_{0i\tilde{j}} - u_j x_{0i\tilde{j}}) + \sum_{j,k} R_{i\tilde{j}j\tilde{k}} u_k u_{\tilde{j}} \]

\[ + x_{i\tilde{i}}^2 - 2 x_{0i\tilde{i}} x_{i\tilde{i}} + 2 \sum_j \text{Re} \{x_{i\tilde{i};j} u_{\tilde{j}}\} \]

\[ \geq -2 \sum_j \text{Re} \{x_{0i\tilde{j}} u_{\tilde{j}}\} + \sum_{j,k} R_{i\tilde{j}j\tilde{k}} u_k u_{\tilde{j}} + \frac{1}{2} x_{i\tilde{i}}^2 - 2 x_{0i\tilde{i}} \]

\[ + 2 \sum_j \text{Re} \{x_{i\tilde{i};j} u_{\tilde{j}}\}, \]

where \( \text{Re} \{z\} \) denotes the real part of the complex number \( z \). Thus, multiplying the above inequality by \( \varphi' F^{i\tilde{i}} \), summing them from 1 to \( n \) and using the fact that \( \frac{1}{2K} \geq \varphi' \geq \frac{1}{4K} \) to deal with the first two terms, we have

\[ \varphi' \sum_i F^{i\tilde{i}} \nabla_T \nabla_i (|\nabla u|^2) \]

\[ \geq -C |\nabla u| + |\nabla u|^2 \sum_i F^{i\tilde{i}} + \frac{1}{2} \varphi' \sum_i F^{i\tilde{i}} x_{i\tilde{i}}^2 - 2 \varphi' \sum_i F^{i\tilde{i}} x_{0i\tilde{i}} \]
\[-2\varphi' \sum_j \Re \left\{ \sum_{l=0}^{k-1} (\nabla_j \beta_l \cdot F_{\bar{\imath}}) u_{\bar{\imath}} \right\} \geq -C \frac{1 + |\nabla u| + |\nabla u|^2}{K} \sum_i F^{i\bar{\imath}} + \frac{1}{2} \varphi' \sum_i F^{i\bar{\imath}} \chi_i^{2,\bar{\imath}}, \tag{4.8}\]

where we used the following equality

\[
F^{i\bar{\imath}} \chi_{i\bar{\imath};p} = - \sum_{l=0}^{k-1} \nabla_p \beta_l \cdot F_{\bar{\imath}} \tag{4.9}
\]

to get the first inequality and (4.9) can be directly deduced by (2.13).

Taking the inequality (4.8) into (4.7), it yields

\[
0 \geq - F^{i\bar{\imath}} |\chi_{1\bar{\imath};i}|^2 \chi_{1\bar{\imath}}^2 \chi_{i\bar{\imath};1} \chi_{r\bar{\tau};\bar{\imath}} + \varphi'' F^{i\bar{\imath}} |\nabla_i (|\nabla u|^2)|^2 + \frac{\varphi'}{2} F^{i\bar{\imath}} \chi_i^{2,\bar{\imath}} + \psi'' F^{i\bar{\imath}} |u_i|^2 + \psi' F^{i\bar{\imath}} u_{i\bar{\imath}} - C \sum_i F^{i\bar{\imath}} - C. \tag{4.10}
\]

Now, we divide our proof into two cases separately, depending on whether \( \chi_{n\bar{n}} < -\delta \chi_{1\bar{\imath}} \) or not, for a small \( \delta \) to be chosen later.

**Case 1.** \( \chi_{n\bar{n}} < -\delta \chi_{1\bar{\imath}} \). In this case, it follows that \( \chi_{1\bar{\imath}}^2 \leq \frac{1}{\delta^2} \chi_{n\bar{n}}^2 \). So, we only need to bound \( \chi_{n\bar{n}}^2 \). Note that

\[-(1 - \delta^2) \frac{1}{\chi_{1\bar{\imath}}} F^{i\bar{\imath}} F^{r\bar{\tau}} \chi_{i\bar{\imath};1} \chi_{r\bar{\tau};\bar{\imath}} \geq 0 \text{ and } \psi'' F^{i\bar{\imath}} |u_i|^2 \geq 0, \]

we can obtain from (4.10)

\[
0 \geq - \frac{F^{i\bar{\imath}} |\chi_{1\bar{\imath};i}|^2}{\chi_{1\bar{\imath}}^2 \chi_{1\bar{\imath}}} + \varphi'' F^{i\bar{\imath}} |\nabla_i (|\nabla u|^2)|^2 + \frac{\varphi'}{2} F^{i\bar{\imath}} \chi_i^{2,\bar{\imath}} + \psi' F^{i\bar{\imath}} u_{i\bar{\imath}} - C \sum_i F^{i\bar{\imath}} - C. \tag{4.11}
\]

Note that we get from (2.15)

\[
\psi' \sum_i F^{i\bar{\imath}} u_{i\bar{\imath}} = \psi' \sum_i F^{i\bar{\imath}} \left[ \chi_{i\bar{\imath}} - (\chi_{0i\bar{\imath}} - \tau) - \tau \right]
= \psi' \left[ F_k + \sum_{l=0}^{k-1} (l - k + 1) \beta_l F_{\bar{\imath}} - \sum_i F^{i\bar{\imath}} (\chi_{0i\bar{\imath}} - \tau) - \tau \sum_i F^{i\bar{\imath}} \right]\]
\[ \geq -C \Lambda + \tau \Lambda \sum_i F^{i\bar{r}}, \quad (4.12) \]

where we used the fact that \( 2\Lambda \geq -\psi' \geq \Lambda \) to get the last inequality. Plugging (4.12) into (4.11), we get by choosing \( \Lambda \) large enough

\[ \phi'' F^{i\bar{r}} |\nabla_i (|\nabla u|^2)|^2 + \frac{\phi'}{2} F^{i\bar{r}} \chi_{ij}^2 \leq \frac{F^{i\bar{r}} |x_{1i};j|^2}{\chi^2_{1i}} + C(1 + \Lambda). \quad (4.13) \]

In addition, we have from (4.5)

\[ \sum_i \frac{F^{i\bar{r}} |x_{1i};j|^2}{\chi^2_{1i}} = \sum_i F^{i\bar{r}} |\psi' \nabla_i (|\nabla u|^2) + \psi' u_i|^2 \leq 2(\phi')^2 \sum_i F^{i\bar{r}} |\nabla_i (|\nabla u|^2)|^2 + 8\Lambda^2 K \sum_i F^{i\bar{r}}. \quad (4.14) \]

Moreover, we obtain in view of (4.3) and (4.4)

\[ \sum_i F^{i\bar{r}} \chi_{ij}^2 \geq F^{n\bar{n}} \chi_{n\bar{n}}^2 \geq \frac{1}{n} \chi_{n\bar{n}}^2 \sum_i F^{i\bar{r}}. \quad (4.15) \]

Substituting (4.14) and (4.15) into (4.13), we get

\[ \frac{1}{8nK} \chi_{n\bar{n}}^2 \sum_i F^{i\bar{r}} \leq 8\Lambda^2 K \sum_i F^{i\bar{r}} + C(1 + \Lambda). \]

Then, it follows that by (2.26)

\[ \chi_{11} \leq C K. \]

**Case 2.** \( \chi_{n\bar{n}} \geq -\delta \chi_{11} \). Define the set

\[ I = \{ i \in \{1, 2, \ldots, n\} : F^{i\bar{r}} > \delta^{-1} F^{1\bar{1}} \}. \]

Since each term in (2.8) is non-positive, it follows that

\[ \frac{1}{\chi_{11}} F^{i\bar{r}, r\bar{s}} \chi_{ij} \chi_{i\bar{r};j,1} \chi_{r\bar{s};1} \geq \frac{1 - \delta}{1 + \delta} \frac{1}{\chi^2_{1i}} \sum_{i \in I} F^{i\bar{r}} |x_{1i};1|^2. \]

Using Lemma 4.1, we have

\[ \chi_{i\bar{1};1} = \chi_{1\bar{i};1} + e_i, \]
where $e_i = \chi_{0i;1} - \chi_{01;:i}$. Then,

$$-\frac{1}{\chi_{1T}} F^{\bar{j},r\bar{\tau}}_{i j} \chi_{i j;1} \chi_{r\bar{\tau};1} \geq \frac{1 - \delta}{1 + \delta} \frac{1}{\chi_{1T}^2} \sum_{i \in I} F^{ii} |\chi_{1T;i} + e_i|^2$$

$$\geq \frac{1 - \delta}{1 + \delta} \frac{1}{\chi_{1T}^2} \sum_{i \in I} F^{ii} \left(|\chi_{1T;i}|^2 + 2 \text{Re}(\chi_{1T;i} e_i)\right), \quad (4.16)$$

where we abandon the positive term containing $|e_i|^2$ to get the last inequality. Using (4.5), we have in view of the fact $\varphi'' = 2(\varphi')^2$

$$\varphi'' \sum_{i \in I} F^{ii} |\nabla_i(|\nabla u|)^2|^2$$

$$\geq 2 \sum_{i \in I} F^{ii} \left(\delta |\chi_{1T;i}|^2 - \frac{\delta}{1 - \delta} |\psi' u_i|^2\right). \quad (4.17)$$

Choosing $\delta \leq \frac{1}{2A + 1}$, we have $\varphi'' \geq \frac{2\delta}{1 - \delta} (\psi')^2$. Then, we get by combining (4.16) and (4.17)

$$-\sum_{i \in I} \frac{F^{ii} |\chi_{1T;i}|^2}{\chi_{1T}^2} - \left(1 - \delta^2\right) \frac{1}{\chi_{1T}^2} F^{\bar{j},r\bar{\tau}}_{i j} \chi_{i j;1} \chi_{r\bar{\tau};1}$$

$$+ \varphi'' \sum_{i \in I} F^{ii} |\nabla_i(|\nabla u|)^2|^2 + \psi'' \sum_{i \in I} F^{ii} |u_i|^2$$

$$\geq \delta^2 \sum_{i \in I} \frac{F^{ii} |\chi_{1T;i}|^2}{\chi_{1T}^2} + \frac{2(1 - \delta^2)(1 - \delta)}{1 + \delta} \frac{1}{\chi_{1T}^2} \sum_{i \in I} F^{ii} \text{Re}(\chi_{1T;i} e_i)$$

$$\geq \delta^2 \sum_{i \in I} \frac{F^{ii} |\chi_{1T;i}|^2}{\chi_{1T}^2} + 2(1 - \delta)^2 \frac{1}{\chi_{1T}^2} \sum_{i \in I} F^{ii} \text{Re}(\chi_{1T;i} e_i)$$

$$\geq \delta^2 \sum_{i \in I} \frac{F^{ii} |\chi_{1T;i}|^2}{\chi_{1T}^2} - C \frac{4(1 - \delta)^4}{\delta^2} \frac{1}{\chi_{1T}^2} \sum_{i \in I} F^{ii}$$

$$\geq \delta^2 \sum_{i \in I} \frac{F^{ii} |\chi_{1T;i}|^2}{\chi_{1T}^2} - C \sum_{i \in I} F^{ii}, \quad (4.18)$$

where we choose $\chi_{1T}$ large enough to get the last inequality.

For the terms without an index in $I$, we have by (4.14)
\begin{equation}
- \sum_{i \not\in I} \frac{F^{i\tilde{i}}(\chi_{1\tilde{1}}; i)^2}{\chi_{1\tilde{1}}^2} + \varphi'' \sum_{i \not\in I} F^{i\tilde{i}} |\nabla_i (|\nabla u|^2)|^2 \\
\geq -8 \Lambda^2 K \sum_{i \not\in I} F^{i\tilde{i}} \\
\geq -\frac{8n \Lambda^2 K}{\delta} F^{1\tilde{1}},
\end{equation}

(4.19)

where we used the fact \( i \not\in I \) to get the last inequality.

Substituting (4.18) and (4.19) into (4.10)

\begin{equation}
C + C \sum_i F^{i\tilde{i}} + 8n \Lambda^2 K \frac{F^{1\tilde{1}}}{\delta} \\
\geq \frac{1}{8K} \sum_i F^{i\tilde{i}} \chi_{i\tilde{i}}^2 + \psi' \sum_i F^{i\tilde{i}} u_{i\tilde{i}}.
\end{equation}

(4.20)

Suppose that \( \chi_{1\tilde{1}} \geq N \). Since the cone condition (1.4) is equivalent to the condition (2.12) for \( \mu \) being the eigenvalues of \( \chi_0 \), Lemma 2.5 works. Thus we will divide the following argument into two cases:

**Case A:** If (2.17) holds, we have

\begin{equation}
\sum_i F^{i\tilde{i}} u_{i\tilde{i}} \leq -\theta - \theta \sum_i F^{i\tilde{i}},
\end{equation}

which implies

\begin{equation}
\psi' \sum_i F^{i\tilde{i}} u_{i\tilde{i}} \geq \Lambda \theta (1 + \sum_i F^{i\tilde{i}}).
\end{equation}

(4.21)

Substituting (4.21) into (4.20)

\begin{equation}
C + C \sum_i F^{i\tilde{i}} + 8n \Lambda^2 K \frac{F^{1\tilde{1}}}{\delta} \\
\geq \frac{1}{8K} \sum_i F^{i\tilde{i}} \chi_{i\tilde{i}}^2 + \Lambda \theta (1 + \sum_i F^{i\tilde{i}}),
\end{equation}

which implies if we choose \( \Lambda \) large enough

\begin{equation}
\frac{8n \Lambda^2 K}{\delta} F^{1\tilde{1}} \geq \frac{1}{8K} \sum_i F^{i\tilde{i}} \chi_{i\tilde{i}}^2 \geq \frac{1}{8K} F^{1\tilde{1}} \chi_{1\tilde{1}}^2.
\end{equation}

Thus,

\begin{equation}
\chi_{1\tilde{1}}^2 \leq \frac{64n \Lambda^2 K^2}{\delta}.
\end{equation}
**Case B:** Otherwise, we have by (2.18)

\[ F^{1\bar{1}} \chi_{1\bar{1}} \geq \theta. \]  

(4.22)

Thus, we get from (4.12)

\[
C(1 + \Lambda) + C \sum_i F^{i\bar{i}} + \frac{8n\Lambda^2 K}{\delta} F^{1\bar{1}} \\
\geq \frac{1}{8K} F^{1\bar{1}} \chi_{1\bar{1}}^2 + \tau \Lambda \sum_i F^{i\bar{i}}.
\]

(4.23)

Choosing \( \Lambda \) large enough, it follows from (4.23)

\[
C(1 + \Lambda) + \frac{8n\Lambda^2 K}{\delta} F^{1\bar{1}} \geq \frac{1}{8K} F^{1\bar{1}} \chi_{1\bar{1}}^2.
\]

(4.24)

Then, using (4.22), we choose \( \chi_{1\bar{1}} \) large enough such that

\[
\frac{1}{16K} F^{1\bar{1}} \chi_{1\bar{1}}^2 \geq \frac{1}{16K} \theta \chi_{1\bar{1}} \geq C(1 + \Lambda).
\]

Taking the above inequality into (4.24), we arrive

\[
\chi_{1\bar{1}} \leq \sqrt{\frac{128n}{\delta}} \Lambda K.
\]

So, we complete the proof. \( \Box \)

**5 The Gradient Estimate**

We will derive the gradient estimate using a blow-up argument and Liouville-type theorem due to Dinew-Kolodziej [16], see also [46]. Suppose the gradient estimate fails. Then, there exists a sequence solutions \( u_m \in C^4(M) \) satisfying

\[
N_m = \sup_M |\nabla u_m| \to +\infty, \quad \text{as} \quad m \to +\infty.
\]

and

\[
\chi_{n-k}^k \wedge \omega^{n-k} = \sum_{l=0}^{k-1} \alpha_{ml}(z) \chi_{um}^l \wedge \omega^{n-l}, \quad \sup_M u_m = 0.
\]

(5.1)
with a function $\alpha_{m,k-1}$ and some strictly positive functions $\alpha_{ml}$ for $0 \leq l \leq k-2$ satisfying

$$k \chi^{k-1} \wedge \omega^{n-k} > \sum_{i=1}^{k-2} i \alpha_{mi} \chi^{i-1} \wedge \omega^{n-i} + (k - 1)(\alpha_{m,k-1} + \epsilon) \chi^{k-2} \wedge \omega^{n-k+1}$$

for a small $\epsilon$. For each $m$, we assume that $|\nabla u_m|$ attains its maximum value at $z_m \in M$. Then, after passing a subsequence, $z_m$ converges some point $z \in M$. Choosing a normal coordinate chart around $z$, which we identify with an open set in $\mathbb{C}^n$ with coordinates $(z^1, \ldots, z^n)$, and such that $\omega(0) = \omega_0 := \sqrt{-1} \delta_{ij} dz^i \wedge d\bar{z}^j$. Without loss of generality, we may assume that the open set contains $B_1(0)$. Set

$$v_m(z) = u_m\left(\frac{z}{N_m}\right), \quad z \in B_{N_m}(0).$$

Then, we know from $C^2$ estimate

$$|v_m|_{C^2(B_{N_m}(0))} \leq C.$$ 

By passing to a subsequence again, we can assume $v_m$ is $C^{1,\alpha}$ convergent to a limit function $v \in C^{1,\alpha}$ with $\nabla v(0) = 1$. Then, we have from (5.1)

$$\left[\chi\left(\frac{z}{N_m}\right)\right]^k \wedge \left[\omega\left(\frac{z}{N_m}\right)\right]^{n-k} = \sum_{l=0}^{k-1} \alpha_{ml}\left(\frac{z}{N_m}\right)^l \wedge \left[\omega\left(\frac{z}{N_m}\right)\right]^{n-l},$$

which results in

$$\left[\chi_0\left(\frac{z}{N_m}\right) + N_m^2 \frac{\sqrt{-1}}{2} \partial \bar{\partial} v_m\right]^k \wedge \left[\omega\left(\frac{z}{N_m}\right)\right]^{n-k} = \sum_{l=0}^{k-1} \alpha_{ml}\left(\frac{z}{N_m}\right)^l \wedge \left[\omega\left(\frac{z}{N_m}\right)\right]^{n-l}.$$

Thus,

$$N_m^{2k}\left[O\left(\frac{1}{N_m^2}\right) \omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} v_m\right]^k \wedge \left[\omega_0 + O\left(\frac{|z|^2}{N_m^2}\right) \omega_0\right]^{n-k} = \sum_{l=0}^{k-1} \alpha_{ml}\left(\frac{z}{N_m}\right)^l \wedge \left[\omega_0 + O\left(\frac{|z|^2}{C_m^2}\right) \omega_0\right]^{n-l}.$$

Therefore, we have

$$\left(\sqrt{-1} \partial \bar{\partial} v\right)^k \wedge \omega_0^{n-k} = 0,$$
which is in the pluripotential sense. Moreover, we have for any $1 \leq l \leq k - 1$ by a similar reasoning

$$(\sqrt{-1} \partial \overline{\partial} v)^l \wedge \omega_0^{n-l} \geq 0.$$ 

Thus, the limiting function $v$ is $k$-subharmonic. A result of Błocki [2] tell us that $v$ is a maximal $k$-subharmonic function in $\mathbb{C}^n$. Then the Liouville theorem in [16] implies that $v$ is a constant, which contradicts the fact $\nabla v(0) = 1$.

## 6 The Proof of Main Theorem

To get higher-order estimates, we need to show Eq. (2.13) is uniformly elliptic. First, we recall the following lemma.

**Lemma 6.1** Under the assumption in Theorem 1.1, we have

$$\sigma_{k-1}(\chi_u) \geq C > 0. \quad (6.1)$$

**Proof** Without loss of generality, we assume there exists $0 \leq l_0 \leq k - 2$ such that $\beta_{l_0}(z) > 0$. Then, we get from the inequality (2.22)

$$0 < \frac{\sigma_{l_0}(\chi_u)}{\sigma_{k-1}(\chi_u)} \leq C.$$

Thus, we complete the proof by using the generalized Newton-MacLaurin inequality (2.4).

Thus, using (6.1) and the uniform estimates up to second order by Lemmas in the previous sections, we can deduce that the equation (2.13) is uniformly elliptic. Then, higher-order estimates follow from the Evans–Krylov theorem [18, 31] and the Schauder estimate (see also [51]). Next, we shall give a proof of Theorem 1.1 by the continuity method. The obstacle is that we have to find a uniform cone condition for the solution flow of the continuity method.

We define $\gamma(z)$ by

$$\chi_0^k \wedge \omega^{n-k} - \sum_{l=1}^{k-2} \alpha_l \chi_0^l \wedge \omega^{n-l} = \gamma(z) \chi_0^{k-1} \wedge \omega^{n-k+1},$$

which is equivalent to

$$\frac{\sigma_k(\chi_0)}{\sigma_{k-1}(\chi_0)} - \sum_{l=0}^{k-2} \beta_l(z) \frac{\sigma_l(\chi_0)}{\sigma_{k-1}(\chi_0)} = \frac{n - k + 1}{k} \gamma(z).$$
Using (2.6) and (2.7), we have

\[ \frac{\sigma_k - 1}{\sigma_k - 2} (\chi_{0|\tilde{i}}) \sigma_k - 1 \sigma_{k-1} (\chi_{0|\tilde{i}}) \]  
\[ > \frac{\sigma_k (\chi_0)}{\sigma_k - 1 (\chi_0)} - \sum_{l=0}^{k-2} \frac{\beta_l(z)}{\sigma_k - 1 (\chi_0)} \sigma_l (\chi_0) \]  
\[ = \frac{n - k + 1}{k} \gamma(z), \]

which is equivalent to

\[ k \chi_0^{k-1} \wedge \omega^{n-k} > \sum_{l=1}^{k-2} l \alpha_l (\chi_0^{l-1} \wedge \omega^{n-l}) \]  
\[ + (k - 1) \gamma \chi_0^{k-2} \wedge \omega^{n-k+1}. \quad (6.2) \]

Let

\[ \tilde{a}_{k-1}(z) = \max \{ \alpha_{k-1}(z), \gamma(z) \}. \]

Then, \( \chi_0 \) satisfies the cone condition by combining (6.2) and (1.4)

\[ k \chi_0^{k-1} \wedge \omega^{n-k} > \sum_{l=0}^{k-2} l \alpha_l (\chi_0^{l-1} \wedge \omega^{n-l}) \]  
\[ + (k - 1) \tilde{a}_{k-1} \chi_0^{k-2} \wedge \omega^{n-k+1}. \quad (6.3) \]

We will apply the continuity method introduced in [45, 46, 49] to complete my proof. First, we consider the family of the equations for \( t \in [0, 1] \)

\[ \chi_{u_t}^k \wedge \omega^{n-k} = \sum_{l=0}^{k-2} l \alpha_l (\chi_{u_t}^l \wedge \omega^{n-l}) \]  
\[ + \left[ (1 - t) \gamma(z) + t \tilde{a}_{k-1}(z) + a_t \right] \chi_{u_t}^{k-1} \wedge \omega^{n-k+1}, \quad (6.4) \]

where \( \chi_{u_t} \in \Gamma_{k-1}(M) \) and \( a_t \) is a constant for each \( t \). We consider

\[ T_1 = \{ s \in [0, 1] : \exists u_t \in C^2,\alpha(M) \text{ and } a_t \text{ solving (6.4) for } t \in [0, s] \} \]

Clearly, \( 0 \in T_1 \) and \( a_0 = 0 \).

**Lemma 6.2** \( T_1 \) is open.

**Proof** Rewriting the equations (6.4) in the form

\[ \log H(u_t, z, t) = \frac{\sigma_k (\chi_{u_t})}{\sigma_k - 1 (\chi_{u_t})} - \sum_{l=0}^{k-2} \frac{\beta_l(z)}{\sigma_k - 1 (\chi_{u_t})} \sigma_l (\chi_{u_t}) \]
\[
\frac{n-k+1}{k} \left[ (1-t)\gamma(z) + t\overline{\alpha}_{k-1}(z) + \alpha_t \right].
\]

Then we can follow the standard argument in [45, 49] to show \( T_1 \) is open, so we omit the details here.

**Lemma 6.3** \( T_1 \) is closed.

**Proof** It suffices to show (i) the uniform bound for \( \alpha_t \) and (ii) the uniform \( C^\infty \) estimates for all \( u_t \). In fact, at the maximum point of \( u_t \), using the ellipticity of \( \left( \frac{\partial}{\partial \gamma} \right)^{1/k} \) (see Proposition 2.1(5)), we have

\[
\chi_0^k \wedge \omega^{n-k} \geq \chi_{u_t}^k \wedge \omega^{n-k} = \sum_{l=0}^{k-2} \alpha_l(z) \chi_0^l \wedge \omega^{n-l} + \left[ (1-t)\gamma(z) + t\overline{\alpha}_{k-1}(z) + \alpha_t \right] \chi_0^{k-1} \wedge \omega^{n-k+1},
\]

Thus, we get from (6.4)

\[
\chi_0^k \wedge \omega^{n-k} \geq \sum_{l=0}^{k-2} \alpha_l(z) \chi_0^l \wedge \omega^{n-l} + \left[ (1-t)\gamma(z) + t\overline{\alpha}_{k-1}(z) + \alpha_t \right] \chi_0^{k-1} \wedge \omega^{n-k+1}.
\]

Thus, \( \alpha_t \leq 0 \). Similarly, at the minimum point of \( u_t \), we have

\[
\chi_0^k \wedge \omega^{n-k} \leq \sum_{l=0}^{k-2} \alpha_l(z) \chi_0^l \wedge \omega^{n-l} + \left[ (1-t)\gamma(z) + t\overline{\alpha}_{k-1}(z) + \alpha_t \right] \chi_0^{k-1} \wedge \omega^{n-k+1},
\]

Thus, \( \alpha_t \leq 0 \). Similarly, at the minimum point of \( u_t \), we have

\[
\chi_0^k \wedge \omega^{n-k} \leq \sum_{l=0}^{k-2} \alpha_l(z) \chi_0^l \wedge \omega^{n-l} + \left[ (1-t)\gamma(z) + t\overline{\alpha}_{k-1}(z) + \alpha_t \right] \chi_0^{k-1} \wedge \omega^{n-k+1}.
\]

Therefore, \( \alpha_t \) is uniformly bounded from below. Thus, the requirement (i) is met. For the requirement (ii), it is sufficient to show the cone condition

\[
k \chi_0^{k-1} \wedge \omega^{n-k} > \sum_{l=1}^{k-2} l\alpha_l \chi_0^{l-1} \wedge \omega^{n-l} + (k-1) \left[ (1-t)\gamma(z) + t\overline{\alpha}_{k-1}(z) + \alpha_t \right] \chi_0^{k-2} \wedge \omega^{n-k+1}.
\]

is uniform for the equations (6.4). In fact, multiplying (6.2) and (6.3) by \( 1-t \) and \( t \) respectively, and then summing them result in

\[
k \chi_0^{k-1} \wedge \omega^{n-k} > \sum_{l=1}^{k-2} l\alpha_l \chi_0^{l-1} \wedge \omega^{n-l}
\]
\[ + (k - 1) \left[ (1 - t) \gamma (z) + t \tilde{a}_{k-1}(z) \right] x_0^{k-2} \wedge \omega^{n-k+1} \]

\[ \geq \sum_{l=1}^{k-2} l \omega x_0^{l-1} \wedge \omega^{n-l} \]

\[ + (k - 1) \left[ (1 - t) \gamma (z) + t \tilde{a}_{k-1}(z) + a_t \right] x_0^{k-2} \wedge \omega^{n-k+1}, \]

where we used \( a_t \leq 0 \) to get the last inequality. \( \square \)

Therefore, there exists a solution \( v \) to the equation

\[ \chi_v^k \wedge \omega^{n-k} = \sum_{l=0}^{k-2} \alpha_l x_v^l \wedge \omega^{n-l} + \left[ \widetilde{\alpha}_{k-1}(z) + a_1 \right] x_0^{k-1} \wedge \omega^{n-k+1}. \]

Second, we consider the family of equations

\[ \chi_{u_t}^k \wedge \omega^{n-k} = \sum_{l=0}^{k-2} \alpha_l x_{u_t}^l \wedge \omega^{n-l} + \left[ (1 - t) \widetilde{\alpha}_{k-1}(z) + t \alpha_{k-1}(z) + b_t \right] x_0^{k-1} \wedge \omega^{n-k+1}. \quad (6.5) \]

where \( x_{u_t} \in \Gamma_{k-1}(M) \) and \( b_t \) is a constant for each \( t \). We consider

\[ \mathcal{T}_2 = \{ s \in [0, 1] : \exists u_t \in C^{2, \alpha}(M) \text{ and } b_t \text{ solving } (6.5) \text{ for } t \in [0, s] \} \]

Clearly, \( 0 \in \mathcal{T}_2 \) with \( u_0 = v \) and \( b_0 = a_1 \). The openness of \( \mathcal{T}_2 \) and uniform bound for \( b_t \) can be similarly deduced by the previous argument for (6.4). Moreover, from the argument for \( \mathcal{T}_1 \), we know it suffices to show the cone condition holds for the equations (6.5) in order to guarantee that the continuity method works.

Integrating (6.5) on \( M \), we get from the condition (1.5)

\[ \int_M x_0^k \wedge \omega^{n-k} = \int_M x_{u_t}^k \wedge \omega^{n-k} \]

\[ = \sum_{l=0}^{k-2} \alpha_l \int_M x_{u_t}^l \wedge \omega^{n-l} + \int_M \left[ (1 - t) \widetilde{\alpha}_{k-1} + t \alpha_{k-1} + b_t \right] x_0^{k-1} \wedge \omega^{n-k+1} \]

\[ \geq \sum_{l=0}^{k-2} c_k,l \int_M x_0^l \wedge \omega^{n-l} + (c_k,k-1 + b_t) \int_M x_0^{k-1} \wedge \omega^{n-k+1} \]

\[ \geq \int_M x_0^k \wedge \omega^{n-k} + b_t \int_M x_0^{k-1} \wedge \omega^{n-k+1}. \]
Thus, $b_t \leq 0$. So the cone condition holds for the equations (6.5). Therefore, we complete the proof of Theorem 1.1.

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