Sine-Gordon $\neq$ Massive Thirring,
and Related Heresies

Timothy R. Klassen$^1$ and Ezer Melzer$^2$

By viewing the Sine-Gordon and massive Thirring models as perturbed conformal field theories one sees that they are different (the difference being observable, for instance, in finite-volume energy levels). The UV limit of the former (SGM) is a gaussian model, that of the latter (MTM) a so-called fermionic gaussian model, the compactification radius of the boson underlying both theories depending on the SG/MT coupling. (These two families of conformal field theories are related by a “twist”.) Corresponding SG and MT models contain a subset of fields with identical correlation functions, but each model also has fields the other one does not, e.g. the fermion fields of MTM are not contained in SGM, and the bosonic soliton fields of SGM are not in MTM. Our results imply, in particular, that the SGM at the so-called “free-Dirac point” $\beta^2 = 4\pi$ is actually a theory of two interacting bosons with diagonal $S$-matrix $S = -1$, and that for arbitrary couplings the overall sign of the accepted SG $S$-matrix in the soliton sector should be reversed. More generally, we draw attention to the existence of new classes of quantum field theories, analogs of the (perturbed) fermionic gaussian models, whose partition functions are invariant only under a subgroup of the modular group. One such class comprises “fermionic versions” of the Virasoro minimal models.

---

$^1$ Newman Laboratory, Cornell University, Ithaca, NY 14853  klassen@strange.tn.cornell.edu

$^2$ Inst. for Theor. Physics, SUNY, Stony Brook, NY 11794  melzer@max.physics.sunysb.edu
1. Introduction

The sine-Gordon model (SGM) is a (1+1)-dimensional field theory of a pseudo-scalar field $\varphi$, defined classically by the lagrangian

$$ L_{SG} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{\alpha_0}{\beta^2} (1 - \cos \beta \varphi) . $$

(1.1)

Here $\alpha_0$ is a mass scale, $\beta$ a dimensionless coupling, and one identifies field configurations that differ by a period $\frac{2\pi}{\beta}$ of the potential. It has been shown rigorously \[1\] that one can make sense out of this theory also on the quantum level (at least for a certain range of $\beta$), the well-known classical (multi-)soliton solutions of (1.1) corresponding to nontrivial super-selection sectors in the quantum theory.

The massive Thirring model (MTM) is formally defined by the lagrangian

$$ L_{MTM} = i \bar{\Psi} \gamma^\mu \Psi - m_0 \bar{\Psi} \Psi - g^2 J^\mu J^\mu , $$

(1.2)

where $J^\mu = \bar{\Psi} \gamma^\mu \Psi$, in terms of a Dirac field $\Psi$. The quantum theory for the massless case $m_0 = 0$, the Thirring model, was proposed in \[2\], and discussed with increasing sophistication in \[3\] \[4\] \[5\], arbitrary Green’s functions of $\Psi$ finally being written down in \[3\]. The a priori ill-defined product of operators appearing in the current $J^\mu$ can be defined by requiring $J^\mu$ to obey appropriate Ward identities. There is (at least) a 1-parameter family of definitions of $J^\mu$ and one must be careful to specify which is used, otherwise the dimensionless coupling $g$ in (1.2) has no meaning.\[6\] It is advantageous to view the massless Thirring model as a perturbation of the massless one \[7\] \[8\] by the (suitably regularized) operator $\bar{\Psi} \Psi$, rather than a perturbation of a free massive Dirac theory by $J^\mu_\mu$, in the same way as one can attempt to define the SGM as a perturbation of its UV limit $\alpha_0 = 0$ by $\cos \beta \varphi$ \[8\] \[9\]. This general idea, defining a (1+1)-dimensional massive quantum field theory (QFT) as a perturbation of the conformal field theory (CFT) describing its UV limit, has been very successful in recent years. This approach is now known as conformal perturbation theory (CPT); it will be briefly discussed in sect. 4.2.

Since the work of Coleman \[8\] and Mandelstam \[9\] it is widely believed that the SGM and the MTM are equivalent, just being different lagrangian representations of the same

---

\[1\] Below and in sect. 2 we will see that the convention-independent way of defining the coupling in the Thirring model is in terms of the compactification radius $r$ of a free massless (pseudo-)scalar field.
underlying QFT, \textit{i.e.} that there is a 1−1 map between operators in the SGM and the MTM such that corresponding correlation functions are identical. This is not what is proved in \cite{8\cite{9}. Instead, Coleman basically showed that the correlation functions of the perturbing fields $\bar{\Psi}\Psi$ and $\cos \beta \phi$, respectively, are identical in the MT and SG models, provided their couplings are related by

$$1 + \frac{g}{\pi} = \frac{4\pi}{\beta^2}, \quad (1.3)$$

in Coleman’s conventions. The proof is given to all orders of CPT where it amounts to showing that all $N$-point functions of the perturbing fields are identical in the \textit{massless} theories. Mandelstam showed how to construct a fermion operator satisfying the MTM equation as a nonlocal functional of a pseudo-scalar field satisfying the SG equation. This is done directly in the massive theory.

Let us first address Coleman’s results. In the wake of \cite{10} many classes of CFTs have been understood in great detail (see \cite{11}[12] for reviews and references). There are numerous examples of distinct CFTs which nevertheless share a nontrivial subalgebra of operators. The most familiar examples are Virasoro minimal CFTs \cite{10} of the same central charge $c$ that belong to different series of modular invariants \cite{13}[14][15]. Perturbations of two such CFTs by a suitable common operator will lead to massive QFTs whose correlation functions are absolutely identical for certain fields, but different for others. A well-understood example, that will prove quite analogous to the case of MTM versus SGM, is that of the free Majorana fermion versus the Ising field theory, to be discussed in sect. 4.3. Other examples, involving local versus nonlocal realizations of supersymmetry, were recently discussed in \cite{16}[17].

So to investigate whether the SG and MT models really are equivalent it is useful to first study the CFTs describing their UV limits. Both models have an $O(2)\times O(2)$ internal symmetry in the massless limit, so their Hilbert spaces will split into super-selection sectors corresponding to the inequivalent representations of the $U(1)\times U(1)$ current algebra. These sectors are created by vertex operators, which can be expressed in terms of the left- and right-moving parts of a compactified massless scalar field. As we will discuss in detail in sect. 2, for a given compactification radius $r$ there are exactly two maximal, closed sets of mutually local vertex operators. One corresponds to the “standard” gaussian model, the other to the “fermionic gaussian model” \cite{18}. These two families of CFTs have central
charge $c=1$ for all $r$, and are related through a so-called $\mathbb{Z}_2$ twist [19] or GSO projection [20]. The vertex operators $V_{m,n}$ are labeled by “electric” and “magnetic” charges, $m,n$, respectively. In the standard, or bosonic gaussian model, $m,n \in \mathbb{Z}$, whereas in the fermionic one $m \in 2\mathbb{Z}, n \in \mathbb{Z}$ or $m \in 2\mathbb{Z}+1, n \in \mathbb{Z}+\frac{1}{2}$. With our conventions the Lorentz spin of $V_{m,n}$ is $mn$. All of this is the content of sect. 2.

The UV limit of the SGM is the bosonic gaussian model. The MTM, on the other hand, contains fermionic fields, i.e. fields of half-odd-integer Lorentz spin which anti-commute at spacelike separations. Therefore it can only correspond to the fermionic gaussian model in the UV limit.

We see that the bosonic and fermionic gaussian models have a common closed subset of sectors, labeled by $m \in 2\mathbb{Z}, n \in \mathbb{Z}$. In particular, both CFTs contain the field $(V_{0,1} + V_{0,-1}) \propto \cos \beta \varphi$. Perturbing by this field gives the SGM and MTM, respectively, as will be discussed in sect. 3. The fields corresponding to $m \in 2\mathbb{Z}, n \in \mathbb{Z}$ have identical correlation functions also in the massive theory (to all orders of CPT). The result of [8] is a special case of this; in our notation, Coleman showed that correlators involving the same number of massive analogs of $V_{0,1}$ and $V_{0,-1}$ are identical in SGM and MTM with the same compactification radius of the UV boson.

But as we will see in sect. 3, the solitons in SGM are created by massive analogs of $V_{\pm 1,0}$, whereas the bosonized components of the Dirac field $\Psi$ ($\Psi^\dagger$) creating the fermions of MTM are the massive versions of $V_{1,\pm \frac{1}{2}}, (V_{-1,\mp \frac{1}{2}})$. All these operators carry $\pm 1$ unit of the $U(1)$ charge that remains conserved in the massive theories, but have Lorentz spins $0$ and $\pm \frac{1}{2}$ in the respective models, showing that the SG solitons are bosons which can not be identified with the MT fermions!

Since in both the bosonic and fermionic gaussian models all operators can be expressed in terms of “the same” (pseudo-)scalar field $\varphi$, and its massive analog obeys the SG equation in both perturbed theories, it is not implausible that also the operators in the massive theories can be expressed in terms of the massive $\varphi$ [3]. Indeed, Mandelstam showed [4] that the Dirac field $\Psi$ of the MTM can be written (nonlocally) in terms of $\varphi$. However, this has no bearing on the question whether the SG and MT models are equivalent. Mandelstam’s work shows just as well (naively it is actually more obvious) that the bosonic fields which

\footnote{Note that $\varphi$ itself is not an operator in SGM or MTM, as we will see, but serves as “building block” of the theories, and under closer inspection will actually be seen to have different periodicity properties in SGM and MTM.}
we identify as creating the solitons of the SGM can be written in terms of \( \varphi \); the point is that the soliton and fermion fields are not relatively local and therefore cannot possibly be in the same theory.

So far we have not mentioned one of the most important features of the SGM and MTM — their (quantum) integrability — which allows one to obtain exact results for various quantities in these theories. In particular, the exact \( S \)-matrices of the two theories were first obtained within the “bootstrap approach” \([21][22]\), and later using the quantum inverse scattering method (for the MTM) \([23]\) and by exploiting the quantum group symmetry of the SGM \([24]\). A subtle issue in all these approaches is how to fix certain signs in the scattering amplitudes of the charged particles. This issue was never really resolved for the SGM; for the MTM it can be resolved by comparison with perturbation theory, for example.

Even though these signs can not be “detected” in scattering experiments in infinite volume, they can be observed in other circumstances. The discussion of “observables” distinguishing the SGM and MTM will be the general theme of sect. 4. In particular, in sect. 4.1 we explain that the \( S \)-matrix signs are related to the statistics of the (fields creating the) particles \([25][26]\). The fact that the solitons are bosons implies that the sign of the SG \( S \)-matrix in the soliton sector is opposite to that of the MTM in the fermion sector. As we have emphasized previously \([27][28]\), \( S \)-matrix signs can be directly observed in the finite-volume spectrum of a theory. In the remainder of sect. 4 we discuss the finite-volume partition function of the SGM, mainly at \( \beta^2 = 4\pi \), where the corresponding MTM is free. Starting with the exactly known partition function of the UV limit and using some other input, we derive what we believe is the exact partition function of the SGM at \( \beta^2 = 4\pi \). It provides an independent confirmation of our claim about the bosonic nature of the solitons and the signs of their scattering amplitudes.

In sect. 5 we discuss some extensions of our observations. One is about “massive orbifolds” of the SGM, defined as perturbations of \( c=1 \) orbifold CFTs. In particular, we propose exact \( S \)-matrices for one family of such theories. The other lies outside the SGM/MTM context, and concerns new classes of quantum field theories, conformal and otherwise, which are “essentially fermionic” (\( e.g. \) in that their partition functions are only invariant under the subgroup of the modular group generated by \( S \) and \( T^2 \)). As an example, we present fermionic versions of the Virasoro minimal models.

Our conventions are collected in appendix A, and some statistical mechanics consequences of our results are discussed in appendix B.
In the previous pages we have summarized the essential points of our paper. The impatient reader with some knowledge of the subject can just look up in the main text whatever might have caught her/his attention. For the remaining potentially desperate audience, *e.g.* readers familiar with the SG and MT models but not conformal field theory, or *vice versa*, we have provided a moderate amount of details in the following. Readers who are not familiar with either subject are referred to [21] for reviews of the former and [11][12] for the latter subject.

2. The UV CFTs

2.1. Gaussian CFTs

The gaussian CFTs, of central charge $c = 1$, are constructed using a compactified free massless scalar (or pseudo-scalar) field $\Phi(z, \bar{z})$ in two dimensions (see appendix A for our conventions). By definition, $\Phi$ takes values on a circle, *i.e.* $\Phi \sim \Phi + 2\pi r$, whose radius $r$ plays the role of a dimensionless coupling constant. Due to the decoupling of left- and right-moving modes (the field equation is $\partial \bar{\partial} \Phi(z, \bar{z}) = 0$), the theory is actually “enlarged” to include two (almost) independent real fields $\phi(z)$ and $\bar{\phi}(\bar{z})$, with $\Phi(z, \bar{z}) = \frac{1}{2}(\phi(z) + \bar{\phi}(\bar{z}))$. We will see later that both $\phi(z)$ and $\bar{\phi}(\bar{z})$ are compactified.

Because of IR divergences the fields $\phi(z)$ and $\bar{\phi}(\bar{z})$ do not exist as operators. They should be considered as “mathematical building blocks” of the gaussian models. The actual operators are composites of derivatives and certain exponentials of $\phi(z)$ and $\bar{\phi}(\bar{z})$ (the exponentials must be well-defined with respect to the compactification properties of $\phi$ and $\bar{\phi}$). The correlation functions of these operators can be obtained from the formal 2-point functions of $\phi(z)$ and $\bar{\phi}(\bar{z})$ (see appendix A). As will become clear in sects. 3 and 4, similar remarks apply to the massive field $\varphi$ “underlying” the SG and MT models.

To specify the gaussian CFTs we have to describe their Hilbert spaces, or equivalently their operator content. By definition, these theories contain the holomorphic and anti-holomorphic $U(1)$ currents $j(z) = i\partial \phi(z)$ and $\bar{j}(\bar{z}) = i\bar{\partial} \bar{\phi}(\bar{z})$. Consequently, the full Hilbert space $\mathcal{H}$ of a gaussian CFT can be written as $\mathcal{H} = \bigoplus_k \mathcal{H}_k \otimes \bar{\mathcal{H}}_k$, where the $\mathcal{H}_k$ ($\bar{\mathcal{H}}_k$) are $U(1)$-charged bosonic Fock spaces in which the modes of $j(z)$ ($\bar{j}(\bar{z})$) act. More explicitly (concentrating on the holomorphic side, the anti-holomorphic side is treated in the same manner), expanding $\phi(z) = q - i\alpha_0 \ln z + i \sum_{n \neq 0} \frac{1}{n} \alpha_n z^{-n}$, the Fock space $\mathcal{H}_k$ is generated by repeatedly applying modes $\alpha_{n<0}$ on the highest-weight state $|p_k\rangle$ ($p_k \in \mathbb{R}$) satisfying
\[\alpha_n|p_k\rangle = \delta_{n,0}p_k|p_k\rangle \text{ for } n \geq 0.\] Hence the Hilbert space \(\mathcal{H}\) is completely specified by the charges \((p_k, \bar{p}_k)\) labeling the different sectors.

To find the restrictions on the admissible sets of \((p_k, \bar{p}_k)\), it is simplest to consider the operator product algebra \([10]\) (OPA) consisting of the fields that create the states in \(\mathcal{H}\). The \(U(1) \times U(1)\) highest-weight state \(|p\rangle \otimes |\bar{p}\rangle\) is created by the vertex operator

\[
V_{m,n}(z, \bar{z}) = e^{ip\phi(z) + ip\bar{\phi}(\bar{z})} = e^{i(\frac{m}{2r} + nr)\phi(z) + i(\frac{m}{2r} - nr)\bar{\phi}(\bar{z})} = e^{i\frac{nr}{2} \Phi(z, \bar{z}) + 2inr\bar{\Phi}(z, \bar{z})},
\]

where \(\Phi \equiv \frac{1}{2}(\phi - \bar{\phi})\) and the quantum numbers \(m = r(p + \bar{p})\) and \(n = (p - \bar{p})/(2r)\) are referred to, respectively, as “momentum” and “winding” (motivated by string theory) or as “electric” and “magnetic” charges (as in the Coulomb gas). The normal ordering in (2.1) is defined in appendix A.

\(V_{m,n}\) is a primary field \([10]\) of conformal dimensions \((\Delta_{m,n}, \bar{\Delta}_{m,n}) = (\frac{1}{2}(\frac{m}{2r} + nr)^2, \frac{1}{2}(\frac{m}{2r} - nr)^2)\), so that its scaling dimension and (Lorentz) spin are \(d_{m,n} = \Delta_{m,n} + \bar{\Delta}_{m,n} = (\frac{m}{2r})^2 + (nr)^2\) and \(s_{m,n} = \Delta_{m,n} - \bar{\Delta}_{m,n} = mn\). Other states in the charge sector \((p, \bar{p})\), alternatively labeled by \((m, n)\), are created by the \(U(1) \times U(1)\)-descendants of \(V_{m,n}(z, \bar{z})\). These fields are generated by repeatedly taking the operator product expansion (OPE) of the currents \(j\) and \(\bar{j}\) with \(V_{m,n}\); for instance

\[
j(w)V_{m,n}(z, \bar{z}) = \frac{p}{w - z} V_{m,n}(z, \bar{z}) + : j(z) V_{m,n}(z, \bar{z}) : + \ldots
\]

\[
= \frac{p}{w - z} V_{m,n}(z, \bar{z}) + p^{-1} \partial_z V_{m,n}(z, \bar{z}) + \ldots,
\]

where the ellipsis stands for operators multiplied by \(c\)-functions that are less singular as \(w \to z\) than the terms shown. The normal ordered product of operators in the above, and in similar equations below, is defined by subtracting off the singular terms in the OPE of the operators in question (i.e. the first line of (2.2) defines the normal ordered product in this case). Descendant fields also appear in the OPE of two vertex operators,

\[
V_{m,n}(w, \bar{w})V_{m',n'}(z, \bar{z}) = (w - z)^{pp'}(\bar{w} - \bar{z})^{\bar{p}\bar{p}'} V_{m+m', n+n'}(z, \bar{z}) + \ldots
\]

\[
= (w - z)^{mn + m'n}|w - z|^{2(\frac{m}{2r} + nr)(\frac{m'}{2r} - n'r)} V_{m+m',n+n'}(z, \bar{z}) + \ldots,
\]

which in a gaussian CFT is a special case of the exact equation

\[
:e^{ip\phi(w)}: e^{ip'\phi(z)} = e^{-pp'\langle \phi(w)\phi(z) \rangle} : e^{ip\phi(w)} + ip'\phi(z) :.
\]

\[\text{(2.4)}\]
and its “complex conjugate”.

To complete the list of basic properties of the vertex operators let us write down their \( N \)-point function:

\[
\langle V_{m_1,n_1}(z_1,\bar{z}_1) \ldots V_{m_N,n_N}(z_N,\bar{z}_N) \rangle = \delta_{\Sigma m_i,0} \delta_{\Sigma n_i,0} \prod_{i<j}^N (z_i - z_j)^{p_i \bar{p}_j} (\bar{z}_i - \bar{z}_j)^{\bar{p}_i \bar{p}_j}.
\] (2.5)

The OPA of a consistent CFT has to contain a single copy of the identity operator, be closed and associative under the OPE, with each field \( A \) contain its conjugate field\(^3\) \( A^* \), and consist of fields that are all \textit{mutually local} \(^4\). The latter is the requirement that the \( c \)-function coefficients appearing in the OPE of any two operators in the OPA are single-valued, so that correlators are single-valued as well.

We will now show that for CFTs with a \( U(1) \times U(1) \) current algebra these assumptions allow basically only two classes of OPAs. Defining \( L = \{(m, n) \in \mathbb{R}^2 \mid V_{m,n} \in \text{OPA}\} \), the above requirements immediately imply that \( L \) is an additive subgroup of \( \mathbb{R}^2 \), satisfying \( (m, n), (m', n') \in L \Rightarrow mn' + m'n \in \mathbb{Z} \), which follows from mutual locality. Note that this latter constraint implies\(^5\) that \( L \) is a \textit{discrete} subgroup of \( \mathbb{R}^2 \), and that the spin of any vertex operator \( V_{m,n} \) is half-integer, \( s_{m,n} = mn \in \frac{1}{2} \mathbb{Z} \).

Recall that \( \Phi = \frac{1}{2}(\phi + \bar{\phi}) \) lives on a circle of radius \( r \), so that \( V_{m,0} \) is well-defined only if \( m \in \mathbb{Z} \). Assume, then, that \( L \) contains some \((m, 0)\) with \( m \in \mathbb{Z}_0 \) and another \((m', n')\) with \( n' \neq 0 \). More precisely, let \( m \) be the smallest positive integer such that \((m, 0) \in L \), and \( n' \) the smallest positive real number such that \((m'', n') \in L \) for some \( m'' \in \mathbb{R} \). Then there exists a unique \( m' \in \mathbb{R} \) such that \( 0 < m' \leq m \) and \( m' - m'' \in m\mathbb{Z} \). It follows that \( L = \{(mk + m'l, n'l) \mid k, l \in \mathbb{Z}\} \), where \( a \equiv mn' \) and \( b \equiv 2m'n' \) must be positive integers. We now use our freedom to rescale the radius \( r \to r \rho \) and at the same time transform \( L = \{(m,n) \to ((m\rho, n/\rho))\} \), which has no effect on the physics.\(^6\) For \( b \) even we rescale \( r \to n'r \) so that the lattice becomes \( L_e(a,b) = \{(ak + \frac{1}{2}bl, l) \mid k, l \in \mathbb{Z}\} \). For

---

3 The conjugate \( A^* \) is a field of the same conformal dimensions as \( A \), satisfying \( A^*(w,\bar{w})A(z,\bar{z}) = (w-z)^{-2\Delta A} (\bar{w}-\bar{z})^{-2\Delta A} + \ldots \).  

4 We ignore the less interesting case of theories with only purely electric or magnetic charges. In these cases the OPA is (a closed subalgebra of) that of an \textit{un}compactified free massless boson, obtained in the limit \( r \to \infty \) or \( r \to 0 \), respectively.

5 Note that \textit{a priori} \( r \) is not a physical parameter, since it enters the “observable” charges \((p,\bar{p})\) in combination with \((m,n)\). Only with the canonical lattices for the \((m,n)\), given below in eq. (2.6), can \( r \) be considered a meaningful parameter characterizing a class of theories.
b odd, \( r \to 2n'r \), and \( L_o(a, b) = \{(2ak + bl, \frac{l}{2}) \mid k, l \in \mathbb{Z}\} \). In both cases, the complete set of allowed lattices is labeled by the integers \( a \) and \( b \) satisfying \( 0 < b \leq 2a \). Note that always \( L_e(a, b) \subseteq L_e(1, 2) \) and \( L_o(a, b) \subseteq L_o(1, 1) \), so that for given \( r \) there are exactly two “maximal” lattices of electric and magnetic charges leading to an acceptable OPA of vertex operators, namely,

\[
L_b = L_e(1, 2) = \{(m, n) \mid m, n \in \mathbb{Z}\}, \quad L_f = L_o(1, 1) = \{(m, n) \mid m \in 2\mathbb{Z}, n \in \mathbb{Z} \text{ or } m \in 2\mathbb{Z} + 1, n \in \mathbb{Z} + \frac{1}{2}\}.
\] (2.6)

We emphasize that \( L_b \) and \( L_f \) are not equivalent; an obvious difference, that explains our notation, is that the OPA corresponding to \( L_f \) contains operators with half-odd-integer spin, whereas \( L_b \) corresponds to only integer spin. We will refer to the CFTs whose OPA is specified by \( L_b \) and \( L_f \) as the **bosonic** and **fermionic gaussian** CFTs, respectively.\(^6\) Note that the OPAs of the two theories are generated by a quartet of “fundamental” operators, which can be chosen to be \( V_{0, \pm 1} \) and \( V_{\pm 1, 0} \) in the bosonic case and \( V_{\pm 1, \pm 1/2} \) in the fermionic case; these quartets are of course not mutually local with respect to one another.

Concerning the theories based on proper sublattices of \( L_b \) and \( L_f \) we should say the following. Their correlation functions satisfy all the standard axioms. However, we will see below (footnote 9) that the partition functions of these theories are not invariant under exchange of “space” and “time”. In other words, the euclidean covariant correlation functions of these models cannot be obtained from a path integral with a euclidean invariant action. This may sound strange, but just seems to show that a path integral is not the only way to get a consistent set of correlation functions. If for a **massive** theory, \( e.g. \) a perturbation of the above models, a set of Green’s functions satisfying the Wightman axioms is sufficient to lead to a sensible particle interpretation is quite a different question. Be that as it may, the theories based on sublattices of \( L_b \) and \( L_f \) will play no role in the rest of this paper.

We have to digress for a moment to discuss the statistics of the fields in the bosonic and fermionic gaussian models. The commutation relations \( V_{m,n}(w, \bar{w})V_{m',n'}(z, \bar{z}) = (-1)^{mn' + m'n}V_{m',n'}(z, \bar{z})V_{m,n}(w, \bar{w}) \) for \( w \neq z \), which follow from (2.3), are not the “standard” ones for generic \((m, n) \neq (m', n')\), neither in the bosonic nor the fermionic gaussian CFT. To obtain the standard ones, namely with “fermionic” operators (defined by

\(^6\) The bosonic theories have been extensively discussed in the literature; the fermionic ones were considered by Friedan and Shenker [18].
s ∈ \( \mathbb{Z} + \frac{1}{2} \) anti-commuting among themselves and “bosonic” operators \( s ∈ \mathbb{Z} \) commuting with all others, one has to multiply the \( V_{m,n} \) (and their descendants, of course) by appropriate “Klein factors”. These are unitary operators that commute with all observables, i.e. just multiply states by phases that are constant on sectors of given global charge. In our case the global charges are \( (m,n) \), and the Klein-transformed vertex operators \( \hat{V}_{m,n} \equiv K_{m,n}V_{m,n} \) creating the charged sectors should satisfy

\[
\hat{V}_{m,n}(w,\bar{w})\hat{V}_{m',n'}(z,\bar{z}) = (-1)^{2mn'}\hat{V}_{m',n'}(z,\bar{z})\hat{V}_{m,n}(w,\bar{w}) \quad \text{for} \; w \neq z \tag{2.7}
\]
in both the bosonic and the fermionic CFTs. The Klein operators \( K_{m,n} \) are defined by

\[
K_{m,n} |m_0,n_0\rangle = K_{m,n}(m_0,n_0) |m_0,n_0\rangle , \tag{2.8}
\]
and have to obey \( K_{m,n}(m_1,n_1)K_{m,n}(m_2,n_2) = K_{m,n}(m_1 + m_2,n_1 + n_2) \). Eq. (2.7) then leads to \( K_{m,n}(m',n')/K_{m',n'}(m,n) = (-1)^{mn'-m'n} \), which together with the other requirements on the \( K_{m,n} \) has as general solution

\[
K_{m,n}(m',n') = e^{\pi i(\alpha mn' + (\alpha - 1)m'n)} , \quad \alpha ∈ \mathbb{R} . \tag{2.9}
\]

Note that \( K_{m,n} K_{m',n'} = K_{m+m',n+n'} \), \( K_{0,0} = 1 \), and \( K_{m,n} V_{m',n'} = K_{m,n}(m',n') V_{m',n'} \). For the conjugate of a vertex operator we have \( \hat{V}_{m,n}^* = V_{m,n}^* K_{m,n} = V_{-m,-n} K_{-m,-n} = K_{m,n}^{-1}(m,n) \hat{V}_{-m,-n} \). The OPE of Klein-transformed vertex operators is identical to (2.3), except that the rhs should be multiplied by \( K_{m',n'}^{-1}(m,n) \).

The choice of \( \alpha \) does affect certain correlation functions and the parity properties of the fields. In a CFT this \( \alpha \)-dependence would presumably not be considered “observable”. In a perturbed CFT, however, some correlation functions of the perturbing field \( V \) certainly are observable (they determine the finite-volume energy levels, see sect. 4), and one should choose \( \alpha \) so that \( \hat{V} = V \). We will be interested in \( V_{k,0} \) and \( V_{0,k} \) perturbations, and therefore choose \( \alpha = 0 \) in the first and \( \alpha = 1 \) in the second case.

Returning to the bosonic and fermionic gaussian models, it is illuminating to reinterpret their operator content as follows. Looking at the OPE of \( \phi(w), \bar{\phi}(\bar{w}) \) with \( V_{m,n}(z,\bar{z}) \) and taking \( w, \bar{w} \) around \( z, \bar{z} \) one sees that \( V_{m,n}(z,\bar{z}) \) creates a “jump” of size \( 2\pi nr \) in \( \Phi \) at \( (z,\bar{z}) \), and one of size \( \pi m/r \) in \( \tilde{\Phi} \). Therefore \( \footnote{This implies, in particular, that Klein factors cannot change the commutation relations of a field with itself. And indeed, these commutation relations are already correct for the \( V_{m,n} \).} \) one should identify

\[
\phi(w) = \Phi(z) , \quad \tilde{\phi}(\bar{w}) = \tilde{\Phi}(\bar{z}) . \tag{18}
\]
\((\Phi, \bar{\Phi}) \sim (\Phi, \bar{\Phi}) + (2\pi nr, \pi m/r)\) for all \((m, n) \in L_{b,f}\) for the bosonic/fermionic model.

In other words, \((\Phi, \bar{\Phi})\) lives on a torus,

\[
(\Phi, \bar{\Phi}) \in \mathbb{R}^2/\Lambda_{b,f}(r), \quad \Lambda_{b,f}(r) = \{(2\pi nr, \pi m/r) | (m, n) \in L_{b,f}\}.
\] (2.10)

Given these target spaces for \((\Phi, \bar{\Phi})\) in the bosonic/fermionic model, their OPA consists exactly of all well-defined vertex operators \(V_{m,n} = \exp\left(i\frac{m}{r} \Phi + 2i\pi r \bar{\Phi}\right)\).

An interesting consequence of the above way of defining the bosonic and fermionic gaussian CFTs is that it makes the observation of duality, i.e. the equivalence of certain pairs of theories with different \(r\), clear from the start. Namely, note that

\[
\Lambda_{b}\left(\frac{1}{2r}\right) = (\Lambda_{b}(r))^t, \quad \Lambda_{f}\left(\frac{1}{r}\right) = (\Lambda_{f}(r))^t,
\] (2.11)

where ‘\(^t\)’ denotes reflection with respect to the \(\Phi = \bar{\Phi}\) line. This means that the bosonic (fermionic) theory at \(\frac{1}{2r}(\frac{1}{r})\) can be obtained from that at \(r\) by the simple field redefinition \((\Phi, \bar{\Phi}) \rightarrow (\bar{\Phi}, \Phi)\) or equivalently \((\phi, \bar{\phi}) \rightarrow (\phi, -\bar{\phi})\). The effect of such a field redefinition on the vertex operators is \(V_{m,n} \rightarrow V_{n,m} (V_{2n,m/2})\) for the bosonic (fermionic) theory. Note that the self-dual radii of the bosonic and fermionic theories are different, being \(r = \frac{1}{\sqrt{2}}\) in the bosonic case (level one SU(2)-WZW model with the unique modular invariant partition function) and \(r = 1\) in the fermionic case (free Dirac point, see below).

Duality is more commonly demonstrated at the level of partition functions. The well-known calculation of the torus partition function within the operator formalism leads to

\[
Z_{b,f}(q, r) = \text{Tr}_{\mathcal{H}_{b,f}(r)} q^{L_0-1/24} \bar{q}^{\bar{L}_0-1/24} = |\eta(q)|^{-2} \sum_{(m,n) \in L_{b,f}} q^{\frac{1}{2}(\frac{m^2}{r} + nr)} \bar{q}^{\frac{1}{2}(\frac{m^2}{r} - nr)},
\] (2.12)

where \(\eta(q) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k)\) is the Dedekind eta function, \(q = e^{2\pi i \tau}\) (\(\tau\) is the modulus of the torus), and duality

\[
Z_b\left(q, \frac{1}{2r}\right) = Z_b(q, r), \quad Z_f\left(q, \frac{1}{r}\right) = Z_f(q, r)
\] (2.13)

---

\(^8\) Note that \(\Phi\) and \(\bar{\Phi}\) have opposite parities under space (and time) reflection, one being a scalar, the other a pseudo-scalar. However, all correlation functions are independent of the parity of \(\Phi\), so the change of parity accompanying a duality transformation is unobservable. As we will see later, in a perturbed gaussian CFT the parity of \(\Phi\) is not arbitrary, but rather determined dynamically.
is manifest. Alternatively, $Z_b$ can be obtained within the euclidean path-integral formalism as the partition function of a compactified free massless scalar field with periodic boundary conditions along the two cycles of the torus. This derivation (which employs zeta function regularization) makes clear the full modular invariance of $Z_b(q, r)$, namely $Z_b(q, r) = Z_b(e^{2\pi i} q, r) = Z_b(\bar{q}, r)$, where $\bar{q} = e^{-2\pi i/\tau}$ $^9$ $Z_f$, on the other hand, can easily be seen not to be fully modular invariant but rather to satisfy only $Z_f(q, r) = Z_f(e^{4\pi i} q, r) = Z_f(\tilde{q}, r)$. This invariance under $\Gamma'$, the subgroup of the modular group $\Gamma$ generated by $T^2$ and $S$ ($\Gamma$ itself is generated by $T$: $\tau \rightarrow \tau + 1$ and $S$: $\tau \rightarrow -1/\tau$) suggests that $Z_f$ corresponds to a euclidean path integral with anti-periodic boundary conditions along both cycles of the torus, appropriate for fermions. This is is indeed the case $^{30}$, as we will discuss in subsect. 2.2.

To conclude this subsection we mention the internal symmetries of the gaussian CFTs that will be relevant later. For generic $r$ both the bosonic and fermionic theories have an $O(2) \times O(2)$ symmetry, the symmetry group of the toroidal target space in which $(\Phi, \tilde{\Phi})$ lives. We will denote the two $O(2)$’s by $O(2)$ and $\tilde{O}(2)$. In a similar notation, they decompose into $\mathbb{Z}_2 \times U(1)$ and $\tilde{\mathbb{Z}}_2 \times \tilde{U}(1)$. The $U(1)$, $\tilde{U}(1)$ act as shifts on $\Phi$, $\tilde{\Phi}$, while the $\mathbb{Z}_2$, $\tilde{\mathbb{Z}}_2$ are generated by $R: (\Phi, \tilde{\Phi}) \rightarrow (-\Phi, \tilde{\Phi})$ and $\tilde{R}: (\Phi, \tilde{\Phi}) \rightarrow (\Phi, -\tilde{\Phi})$, respectively.

2.2. The Thirring model and its bosonization

The (massless) Thirring model $^2$ is a (1+1)-dimensional QFT of a massless Dirac fermion $\Psi(x)$ with a four-fermion self-interaction,

$$L_{TM} = i\bar{\Psi} \gamma^\mu \partial_\mu \Psi + \frac{\lambda}{2} J_\mu J^\mu$$

(2.14)

(here $\bar{\Psi} = \Psi^\dagger \gamma^0$). The (formal) field equation is

$$i\partial_\mu \Psi(x) = -\lambda J(x) \Psi(x) \ .$$

(2.15)

$^9$ Modular invariance can of course be verified directly from (2.12); in particular, generalizing (2.12) and denoting by $Z_L(q, r)$ the partition function corresponding to an electric/magnetic charge lattice $L$, Poisson resummation (see e.g. $^{12}$) leads to $Z_L(q, r) = Z_{\hat{L}}(\hat{q}, r)$ where $\hat{L}$ = $\{(m, \hat{n}) \in \mathbb{R}^2 \ | \ m\hat{n} + n\hat{m} \in \mathbb{Z} \ \forall (m, n) \in L\}$. Hence $Z_L(q, r) = Z_L(\hat{q}, r)$ iff $L = \hat{L}$, which holds for $L = L_{b,f}$ but not for any proper sublattices of $L_{b,f}$. This last observation explains our earlier remark concerning the theories based on such sublattices. Note that for $c=1$ CFTs with a $U(1) \times U(1)$ symmetry, this result, together with our classification of such theories given above, extends those of $^{29}$, where rationality of the CFT was assumed.
To make this equation meaningful, in particular to give absolute meaning to the dimensionless real coupling $\lambda$, the normalization of $\Psi$ has to be specified and the rhs, as well as the current $J^\mu(x)$ itself (formally $\bar{\Psi}(x)\gamma^\mu\Psi(x)$), have to be carefully regularized.\footnote{Our conventions below are equivalent to those of Johnson \cite{4}, which can be reproduced from the parametrization of \cite{3} by setting $g = \lambda$ and $\sigma = -\lambda/2$, or from that of \cite{31} by taking $g = -\lambda$ and $c = (\pi[1 - (\lambda/2\pi)^2])^{-1}$. Coleman’s $g$ \cite{8} is related to $\lambda$ via $\lambda = -g/(1 + g^2/2\pi)$, and Mandelstam’s $g$ \cite{9} differs from Coleman’s just by sign.}

The lagrangian (2.14) has an $O(2) \times O(2)$ symmetry, generated by $(\Psi_1 \Psi_2) \rightarrow (e^{i\theta_1} \Psi_1 e^{i\theta_2} \Psi_2)$, $\theta_{1,2} \in \mathbb{R}$, $(\Psi_1 \Psi_2) \rightarrow (\Psi_2 \Psi_1)$, and $(\Psi_1 \Psi_2) \rightarrow (\Psi_1^* \Psi_2^*)$, in the representation (A.1) of the Dirac matrices. This symmetry also exists at the quantum level, as the exact correlation functions show. The next thing to note is that the theories at $\lambda$ and $-\lambda$ are equivalent, as they are related by a simple field redefinition $\Psi = (\Psi_1 \Psi_2) \rightarrow (\Psi_1^\dagger \Psi_2^\dagger)$ (having the effect of exchanging $J^\mu$ and the axial current $\tilde{J}^\mu = \epsilon^{\mu\nu}J_\nu$). Again, less formally this Thirring duality can be seen in the exact correlation functions (this was also noticed in \cite{32}). These correlation functions also show that the allowed region of the coupling is $-2\pi < \lambda < 2\pi$.

It is well known that the solution of the Thirring model can be expressed in terms of a free massless (pseudo-)scalar field, namely the correlation functions of $\Psi$ are identical to correlators of certain vertex operators of the form (2.1). This is the so-called bosonization of the model, whose history goes back (at least) to \cite{3}. Less known, apparently, is the precise gaussian CFT that is equivalent to the Thirring model at a given $\lambda$. Since the components of the Thirring field $\Psi$ are of Lorentz spin $\pm \frac{1}{2}$, this gaussian CFT must be fermionic, the components corresponding to $V_{\pm \frac{1}{2}}$. Up to $O(2) \times O(2)$ transformations there are two inequivalent choices of bosonization, related by duality. They are

\begin{align}
(\text{i}) & \quad \sqrt{2\pi} \Psi_1(x) \leftrightarrow \hat{V}_{1,\frac{1}{2}}(z, \bar{z}), \quad \sqrt{2\pi} \Psi_2(x) \leftrightarrow \hat{V}_{1,-\frac{1}{2}}(z, \bar{z}), \\
(\text{ii}) & \quad \sqrt{2\pi} \Psi_1(x) \leftrightarrow \hat{V}_{1,\frac{1}{2}}(z, \bar{z}), \quad \sqrt{2\pi} \Psi_2(x) \leftrightarrow \hat{V}_{-1,\frac{1}{2}}(z, \bar{z}),
\end{align}

the factors of $\sqrt{2\pi}$ arising from the different but standard normalizations of the kinetic term in Minkowski space QFT and euclidean CFT.

Furthermore, the scaling dimension of $\Psi$ is $d_\Psi = \frac{1}{2}[1 + (\lambda/2\pi)^2]/[1 - (\lambda/2\pi)^2]$, which can be read off from the 2-point function \cite{4} (see also \cite{3} \cite{31}). Consequently, the compactification radius in the corresponding fermionic gaussian CFT is identified as

\begin{equation}
r(\lambda) = \sqrt{\frac{1 \pm \frac{\lambda}{2\pi}}{1 \mp \frac{\lambda}{2\pi}}},
\end{equation}

\footnote{Our conventions below are equivalent to those of Johnson \cite{4}, which can be reproduced from the parametrization of \cite{3} by setting $g = \lambda$ and $\sigma = -\lambda/2$, or from that of \cite{31} by taking $g = -\lambda$ and $c = (\pi[1 - (\lambda/2\pi)^2])^{-1}$. Coleman’s $g$ \cite{8} is related to $\lambda$ via $\lambda = -g/(1 + g^2/2\pi)$, and Mandelstam’s $g$ \cite{9} differs from Coleman’s just by sign.}
The two possibilities, related by the Thirring duality $\lambda \leftrightarrow -\lambda$, are equivalent in view of the duality of the fermionic gaussian CFT $r \leftrightarrow 1/r$.\footnote{The fact that bosonization of the Thirring model with a given coupling $g$, in Coleman’s notation, leads to the dual (fermionic) gaussian CFTs at $r = (1 + \frac{g}{\pi})^{\pm 1/2}$ was already noticed in \cite{30}. We emphasize that it simply reflects the above-mentioned duality of the Thirring models at $g$ and $-g/(1 - \frac{g}{\pi})$.} We see that the self-dual $r = 1$ fermionic gaussian CFT is the bosonized Thirring model at $\lambda = 0$, \textit{i.e.} the free massless Dirac theory. Note that in this case the bosonized $\Psi_1 (\Psi_2)$ is of conformal weights ($\frac{1}{2}, 0$) ($\lambda, 0$)) according to (2.16), hence holomorphic (anti-holomorphic), in agreement with our conventions in appendix A.

We can also derive (2.17) directly in CFT, by writing down the CFT analog of (2.15). This will explicitly exhibit the definition of the current $J_\mu$ we use, and show how the two options in (2.17) are correlated with those in (2.16). We have (see (2.2))

$$\partial \hat{V}_{1, -\frac{1}{2}}(z, \bar{z}) = -\pi (1 - r) : \partial \phi(z) \hat{V}_{1, -\frac{1}{2}}(z, \bar{z}) : , \quad (2.18)$$

and from (2.4) and the properties of the Klein factors

$$\hat{V}_{1, \frac{1}{2}}^*(w, \bar{w}) \hat{V}_{1, \frac{1}{2}}(z, \bar{z}) = |w - z|^{-\frac{1}{2} (\frac{1}{r} - r)^2} (w - z)^{-1} \times \left[ 1 - \frac{i}{2} \left( \frac{1}{r} + r \right) (w - z) \partial \phi(z) - \frac{i}{2} \left( \frac{1}{r} - r \right) (\bar{w} - \bar{z}) \partial \bar{\phi}(\bar{z}) + \ldots \right]. \quad (2.19)$$

Now define

$$J(z) = \lim_{w \to z} A \left[ |w - z|^\frac{1}{2} (\frac{1}{r} - r)^2 \hat{V}_{1, \frac{1}{2}}^*(w, \bar{w}) \hat{V}_{1, \frac{1}{2}}(z, \bar{z}) - \frac{1}{w - z} \right] = -\frac{i}{2} \left( \frac{1}{r} + r \right) \partial \phi(z) \quad (2.20)$$

where $A$ denotes a suitable average, \textit{e.g.} $A[\ldots] \equiv \oint_z \frac{dw}{2\pi i} \frac{1}{w - z} \ldots$ (the contour integral being around $z$), which gets rid of the “non-covariant” term proportional to $\frac{\bar{w} - \bar{z}}{w - z}$ in (2.19). The above, together with the analogous equations for the fields of opposite chirality, then give

$$\partial \hat{V}_{1, -\frac{1}{2}}(z, \bar{z}) = \frac{1 - r^2}{1 + r^2} : J(z) \hat{V}_{1, -\frac{1}{2}}(z, \bar{z}) : , \quad (2.21)$$

This is equivalent to (2.13) with choice (i) in (2.16) and the upper case in (2.17), $\lambda = -2\pi \frac{1 - r^2}{1 + r^2}$, with $(J(z), \bar{J}(\bar{z})) \leftrightarrow 2\pi (J_0(x) \pm J_1(x))$. The dual bosonization choice, (ii) of (2.16), is treated similarly. It corresponds to $\lambda = 2\pi \frac{1 - r^2}{1 + r^2}$ and $(J(z), \bar{J}(\bar{z})) \leftrightarrow$
$2\pi(\pm J_0(x) + J_1(x))$. [In both cases $(J(z), \bar{J}(\bar{z}))$ are defined so that they equal $-\frac{i}{2}(\frac{1}{r} + \bar{r})(\partial \phi(z), \bar{\partial} \phi(\bar{z}))$.]

As alluded to earlier, a direct euclidean path-integral calculation \[30\] of the partition function of the Thirring model, with anti-periodic boundary conditions on $\Psi$ along both cycles of the torus, yields the result (2.12) for $Z_f(q, r(\lambda))$. The bosonic partition function $Z_b(q, r(\lambda))$ is obtained \[30\] by summing up all four possible boundary conditions, demonstrating that the bosonic gaussian CFT is \[33\] a GSO-projected Thirring model. At the level of the OPA, the bosonic theory is obtained from the Thirring model by a “twist” \[19\] with respect to the total fermion number $(-1)^F = (-1)^{F_1 + F_2 + \bar{F}_1 + \bar{F}_2} (= (-1)^{2n}$ in the fermionic gaussian language) followed by a projection onto the $(-1)^F = 1$ sector. In this framework the (bosonic) vertex operators $V_{0,\pm 1}, V_{\pm 1,0}$ play the role of spin fields \[34\] for the Thirring fermion, in analogy with $\sigma$ of the Ising CFT being the spin field for the free Majorana fermion. It is also possible to construct the Thirring model by twisting the bosonic gaussian CFT, or, in a more unified picture, to view both CFTs as two different mutually-local projections of one nonlocal “theory” based on $L = \{(m,n) \mid m \in \mathbb{Z}, n \in \frac{1}{2}\mathbb{Z}\}$.

It is amusing to note the interplay between duality and the above $\mathbb{Z}_2$ twist. For instance, the equivalent bosonic theories at $r = 1$ and $r = \frac{1}{2}$ can be obtained by twisting either the free Thirring (self-dual) point $\lambda = 0$, or of the interacting Thirring model at $\lambda = \pm \frac{6\pi}{5}$. There is no contradiction, though, as we would like to think of the resulting bosonic theory as interacting anyhow (cf. sect. 4), in the sense that it contains a nontrivial interacting sector — the $m \in 2\mathbb{Z} + 1$ sector in the $r = 1$ representation, or $n \in 2\mathbb{Z} + 1$ in the dual $r = \frac{1}{2}$ one. When twisting the free Thirring model this sector arises as the twisted sector that survives the $(-1)^F = 1$ projection, the “non-interacting” sector of the bosonic theory coming from the untwisted sector of the free Thirring model. When twisting the $\lambda = \pm \frac{6\pi}{5}$ Thirring model the situation is more intricate; the interacting and non-interacting sectors of the bosonic theory are built of $(-1)^F = 1$ operators in both the twisted and untwisted Thirring sectors.

3. Identification of the Sine-Gordon and Massive Thirring Models

We are now ready to discuss the massive QFTs whose UV limits are the bosonic and fermionic gaussian CFTs described in the previous section. It has proven very fruitful, both from a conceptual as well as a practical point of view (see e.g. \[8\] \[33\] \[36\] \[37\]), to view such theories as relevant perturbations of their UV CFTs.
We will consider only perturbations by a single relevant \( (d<2) \), spinless \( (s=0) \), real operator, i.e. either by \( V_{m,0}^{(\pm)} \), \( m=1,2,\ldots < 2\sqrt{2}/r \) (with \( m \) only even in the fermionic case), or \( V_{0,n}^{(\pm)} \), \( n=1,2,\ldots < \sqrt{2}/r \). [Here for any operator \( A \), \( A^{(+)} \equiv \frac{1}{\sqrt{2}}(A + A^*) \) and \( A^{(-)} \equiv \frac{1}{\sqrt{2}}(A - A^*) \), where the conjugate operator \( A^* \) was defined in sect. 2.1.] Due to the \( O(2) \times O(2) \) symmetry of the unperturbed theory, perturbations by any linear combination of the purely electric operators \( V_{m,0}^{(+)} \) and \( V_{m,0}^{(-)} \) lead to the same massive theory, and similarly for the magnetic operators \( V_{0,n}^{(+)} \) and \( V_{0,n}^{(-)} \). We can therefore restrict attention to the \( V_{m,0}^{(+)} \) and \( V_{0,n}^{(+)} \) perturbations with positive couplings. Using the “counting argument” it is not difficult to see that any such perturbation is integrable (see e.g. [38]).

The euclidean action of the perturbed theories can therefore be written as

\[
A_{b,f}(r, V) = A_{b,f}(r) + \mu \int V, \quad (3.1)
\]

where \( A_{b,f}(r) \) is the action of the bosonic/fermionic gaussian CFT with \( \Phi \) living on a circle of radius \( r \), \( V \) is either \( V_{m,0}^{(+)} = \sqrt{2}\cos(m\Phi/r) \) or \( V_{0,n}^{(+)} = \sqrt{2}\cos(2nr\tilde{\Phi}) \), and \( \mu > 0 \) is of mass dimension \( y = 2 - d_V > 0 \). By duality the theories described by \( A_{b}(r, V_{m,0}^{(+)}(1/2r)) \) and \( A_{b}(V_{0,m}^{(+)}(1/2r)) \) are identical, and the same is true for \( A_{f}(r, V_{0,n}^{(+)}(1/2r)) \) and \( A_{f}(V_{2n,0}^{(+)}(1/2r)) \). Except for these identifications, the theories defined above are distinct.

Since both \( \Phi \) and \( \tilde{\Phi} \) are free fields in the massless theory, it is clear that “underlying” all of the massive theories \( (3.1) \) is a field \( \varphi (\propto \Phi \) or \( \tilde{\Phi} \) that obeys a SG equation. This demonstrates one of the weaknesses of a lagrangian approach on the non-perturbative level, where often there is no direct relation between the particle spectrum and the fields in the lagrangian: Distinct QFTs can “contain” fields that obey the same equations of motion (even when all renormalization effects are taken into account). In other words, a lagrangian does not in general define a unique QFT.

The best one can hope for in a lagrangian approach is that the equations of motion for the “fundamental fields” of the theory follow from a lagrangian. (By fundamental fields we mean operators whose derivatives and OPEs generate the full operator algebra of the theory.) As we will discuss below, this is the case for the MTM where the components of \( \Psi \) and \( \Psi^\dagger \), i.e. the massive analogs of \( V_{\pm 1,\pm 1/2} \), are the fundamental fields.

The perturbations of the bosonic gaussian CFTs, on the other hand, illustrate that the fundamental fields do not necessarily obey lagrangian equations of motion. Here the

\[\text{12} \text{ This is a statement on the quantum level; further non-uniqueness might be associated with passing from the classical to the quantum lagrangian.}\]
fundamental fields are presumably the massive analogs of \( V_{\pm 1,0} \) and \( V_{0,\pm 1} \). But in the UV limit these fields obey first order equations of motion (derived similarly to (2.21)) and there is no first order Lorentz-invariant lagrangian for fields of Lorentz spin 0. Since there is already no lagrangian for the fundamental fields in the massless limit, there is no hope for one describing the fundamental fields in the theory perturbed by a relevant operator.

Sticking to a lagrangian approach, in cases like the perturbed bosonic CFTs one must be content with a lagrangian for a non-fundamental field, e.g. the SG field \( \varphi \). To discover the full operator content of the theory one should look for soliton solutions, first on the classical level, and then try to construct the corresponding super-selection sectors on the quantum level. This is not easy, in general, and it is one of the many advantages of describing the massive theories by (3.1) that the super-selection sectors are manifest — each super-selection sector corresponds to the union of sectors of the CFT that have the same global charge with respect to the symmetry that remains unbroken in the perturbed theory.

Let us now complete the lagrangian description of the theories specified by (3.1), starting with the bosonic theories. This equation and the form of the allowed \( V \) implies that in terms of the SG lagrangian (1.1),

\[
A_b(r,V_{k,0}^{(+)}(\varphi)) \leftrightarrow \mathcal{L}_{SG}(\beta = \sqrt{\pi k}/r), \quad \varphi \equiv \frac{\Phi}{\sqrt{\pi}} \sim \varphi + \frac{2\pi}{\beta} k
\]

\[
A_b(r,V_{0,k}^{(+)}(\varphi)) \leftrightarrow \mathcal{L}_{SG}(\beta = \sqrt{4\pi kr}), \quad \varphi \equiv \frac{\Phi}{\sqrt{\pi}} \sim \varphi + \frac{2\pi}{\beta} k,
\]

(3.2)

where we used the fact that the SG coupling does not renormalize. In contrast, the mass parameter \( \alpha_0 \) in (1.1) does; its relation to \( \mu \) in (3.1) therefore depends on the scale at which \( \cos \beta \varphi \) is normal ordered [8], and is irrelevant here. Much more interesting is the dimensionless coefficient \( \kappa = \mu m_1^{-\nu} \) relating the “bare coupling” \( \mu \) to a physical mass scale \( m_1 \), e.g. the mass of the lightest particle in the perturbed theory. (In sect. 4.3 we will find the exact value of \( \kappa \) in a special case.)

Note that for the \( V_{k,0}^{(+)} \) and \( V_{0,k}^{(+)} \) perturbations in (3.2) exactly \( k \) periods of the cosine potential fit on the circle on which \( \varphi \) lives. These theories were denoted by \( \text{SG}(\beta, k) \) in [28]. Intuitively one expects that \( \text{SG}(\beta, k) \) will have a \( k \)-fold degenerate vacuum, and that for \( k > 1 \) these QFTs will be “kink theories”, i.e. have nontrivial restrictions on their multi-particle Hilbert spaces. For example, in finite volume with periodic boundary conditions the only allowed soliton states in \( \text{SG}(\beta, k) \) are such that \( (# \text{solitons} - # \text{antisolitons}) \in k\mathbb{Z} \).

We will discuss these theories in more detail in [39].
The ordinary SG model corresponds to the $k=1$ case, i.e. we identify $\text{SG}(\beta) = \text{SG}(\beta,1) = A_b(r = \sqrt{\pi/\beta}, V_{1,0}^{(+)}(r)) = A_b(r = \beta/\sqrt{4\pi}, V_{0,1}^{(+)}(r))$. This is in accordance with the common belief that the UV limit of the SG models are bosonic gaussian CFTs. As in other cases, arguments for this would presumably be given in the context of lattice models, either by a study of partition functions in the thermodynamic limit, or by exhibiting a field which obeys the SG equation in the continuum limit. Neither of these methods allows one to see the difference between the bosonic and fermionic gaussian models. Knowledge of the finite-temperature infinite-volume partition function is equivalent to that of the finite-volume ground state energy, which is identical in SGM and MTM (cf. sect. 4), and a field obeying the SG equation can also be written down for perturbed fermionic gaussian CFTs. One might therefore in fact have some doubts about our above identification and wonder if the SGM is perhaps a perturbed fermionic CFT.

However, there is a simple argument to dispel such doubts, based on the periodicity of the field $\varphi$. Namely, exactly as in (3.2) one sees that the theories $A_f(r = \sqrt{\pi k/\beta}, V_{k,0}^{(+)}(r)) = A_f(r = \beta/\sqrt{4\pi k}, V_{0,k}^{(+)}(r))$, $k \in 2\mathbb{N}$, involve an “underlying” field $\varphi$ obeying the SG equation with coupling $\beta$; but now the number $k$ of periods of the potential that fit on the “target space” of $\varphi$, cf. (2.10), is always even. Therefore none of these theories can correspond to the standard SGM, where exactly one period fits on the target space of $\varphi$. We should also add that for the above perturbed fermionic CFTs $k/2$, not $k$, is the degeneracy of the vacuum, as we will discuss in 39. (In the special case $k = 2$ this just corresponds to the fact that the vacuum of the MTM is non-degenerate, see below.) This is possible, and not in contradiction to our above remarks about $\text{SG}(\beta,k)$, because the SG “field” $\varphi$ is compactified and therefore not really a well-defined field, neither in the massless nor the perturbed gaussian models; its periodicity just encodes the operator content of the theory (at least partially). The conventional claim that $\varphi$ creates the lowest bound state in SGM should be replaced by the statement that $\sin \beta \varphi$ creates this particle, cf. sect. 4.4.

We can now identify the soliton creation operators in the SGM. Note that for a $V_{0,k}^{(+)}(V_{k,0}^{(+)}(r))$ perturbation of a bosonic gaussian CFT the electric charge $m$ (magnetic charge $n$) is super-selected also in the massive theory. We identify $m$ ($n$) as the soliton number ($= \# \text{solitons} - \# \text{antisolitons}$). This is confirmed by the following fact, considering for

---

13 Elaborating on an earlier remark in this section, we now see that distinct QFTs can involve a field obeying the same equations of motion even when the periodicity of this field is taken into account.
example a \( V_{0,k}^{(+)} \) perturbation: \( V_{m,0} \) creates a discontinuity of \( \pi m/r \) in \( \Phi \) (cf. sect. 2.1), corresponding to a jump of \( \frac{\pi m}{\sqrt{\pi r}} = \frac{2\pi m}{\beta} \) in the SG field \( \varphi = \Phi/\sqrt{\pi} \), in accord with the well-known normalization of the topological charge in the SG model: \( \beta [\varphi(\infty) - \varphi(-\infty)] \in \mathbb{Z} \).

The massive theory will provide a length scale over which the discontinuity created by \( V_{m,0} \) is smeared out. The soliton/antisoliton creation operators are presumably the simplest operators of soliton number \( \pm 1 \), namely massive analogs of \( V_{\pm 1,0} \). Note that these operators have Lorentz-spin 0, i.e. are bosonic. The work of Mandelstam [9] essentially shows that the above considerations, phrased in the language of CFT, also hold in the massive theory.

Let us now discuss the perturbed fermionic theories \( A_f(r, V_{2,0}^{(+)}), A_f(r, V_{0,1}^{(+)} \) in more detail. Here we should be careful and use the Klein-transformed vertex operators \( \hat{V}_{m,n} \) throughout. Using (2.3) and the properties of the Klein factors we can then express the perturbing field in terms of the fundamental fields \( \hat{V}_{\pm 1, \pm \frac{1}{2}} \) as

\[
\frac{1}{\sqrt{2}} \lim_{w \to z} |w - z|^{\frac{1}{2}(r^2 - r^{-2})} \left[ \hat{V}_{1, \frac{1}{2}}^*(w, \bar{w}) \hat{V}_{\pm 1, \mp \frac{1}{2}}(z, \bar{z}) + \hat{V}_{\pm 1, \mp \frac{1}{2}}^*(w, \bar{w}) \hat{V}_{1, \frac{1}{2}}(z, \bar{z}) \right] = \begin{cases} \hat{V}_{0,1}^{(+)}(z, \bar{z}) \\ \hat{V}_{2,0}^{(+)}(z, \bar{z}) \end{cases},
\]

where we take \( \alpha = 1 \) (0) in (2.3) for the upper (lower) case. In terms of the fermion of the TM we therefore have

\[
\frac{1}{\sqrt{2}} \lim_{y \to x} |y - x|^{\frac{1}{2}(r^2 - r^{-2})} 2\pi \bar{\Psi}(y) \Psi(x) \leftrightarrow \begin{cases} \hat{V}_{0,1}^{(+)}(z, \bar{z}) \\ \hat{V}_{2,0}^{(+)}(z, \bar{z}) \end{cases},
\]

the upper/lower case corresponding to the bosonization choice (i)/(ii) in (2.16). The perturbations in question therefore really correspond to the addition of a properly regularized \([31] [8]\) mass term to the massless Thirring lagrangian.

To recapitulate, we identify \( A_f(r, V_{0,1}^{(+)} \) as the MTM at \( \lambda = -2\pi \frac{1-r^2}{1+r^2} \) (or \( g = \pi(1 - r^2)/r^2 \) in Coleman’s conventions), and \( A_f(r, V_{2,0}^{(+)} \) as the MTM at \( \lambda = 2\pi \frac{1-r^2}{1+r^2} \) (\( g = -\pi(1 - r^2) \)). The condition for the relevance of the perturbation, \( r < \sqrt{2} \) and \( r > 1/\sqrt{2} \), respectively, translates into \(-2\pi < \lambda < 2\pi/3 \) \( (g > -\pi/2) \) in both cases. Note that within the massive TM there is no duality between the attractive \((-2\pi < \lambda < 0)\) and repulsive \((0 < \lambda < 2\pi/3)\) regimes.

The perturbations by \( V_{0,k}^{(+)} \) and \( V_{k,0}^{(+)} \) with even \( k > 2 \) correspond to adding suitably regularized terms of \( k \)-th order in \( \Psi \) to the massless Thirring lagrangian. They describe
“fermionic kink theories”, and will be discussed in [39]. Let us here just note the obvious fact that the conserved global charge, \( m(2n) \) for a \( V^{(+)}_{0, \frac{1}{2}} \) perturbation of a fermionic CFT, is fermion number, \( \text{# fermions} - \text{# antifermions} \).

Finally, we should elaborate on our remarks in the introduction concerning Mandelstam’s results [9]. He works directly with the massive theories, where things are somewhat more involved than in the massless case, but we can illustrate the basic points of his analysis by translating it into CFT language. The starting point is to find operators that create a “jump” of one period in the SG field \( \varphi \), as candidates for soliton creation operators. Mandelstam notices that such operators can either commute or anti-commute, corresponding to, in our notation, massive analogs of \( V_{\pm 1, n} \) with \( n \in \mathbb{Z} \) or \( n \in \mathbb{Z} + \frac{1}{2} \), respectively (we choose \( \varphi \propto \tilde{\Phi} \) in this discussion). He pursues the study of the simplest anti-commuting operators, \( V_{\pm 1, \pm \frac{1}{2}} \), leading to the bosonized MTM. His analysis works just as well for \( V_{\pm 1, 0} \), which leads to the SGM.

The massive \( V_{m,n} \) are identical in structure to the CFT ones; the only differences are:

(i) Instead of a compactified free massless field one uses a compactified massive field \( \varphi \) obeying the SG equation to construct the \( V_{m,n} \).

(ii) Since for a massive field the left- and right-moving modes do not decouple, one should express the “dual” of \( \varphi \) as a nonlocal functional of \( \varphi \), i.e. use the analog of the CFT expression

\[
V_{m,n}(z, \bar{z}) = : \exp \left( \frac{imr}{r} \int_{\infty}^{(z, \bar{z})} (dw \partial_w \tilde{\Phi} - d\bar{w} \partial_{\bar{w}} \tilde{\Phi}) + 2i nr \tilde{\Phi}(z, \bar{z}) \right) :, \tag{3.5}
\]

ignoring Klein factors and other subtleties. As emphasized previously, the operators \( V_{\pm 1, 0} \) and \( V_{\pm 1, \pm \frac{1}{2}} \) cannot belong to the same theory since they are not mutually local. In the perturbed theory mutual locality of arbitrary vertex operators is equivalent to requiring that the “Mandelstam strings” in (3.5) be “invisible”; the constraint \( mn' + m'n \in \mathbb{Z} \) (see sect. 2) can then be interpreted as a Dirac-like quantization condition on the allowed electric and magnetic charges.

4. Observables in SGM and MTM

4.1. S-Matrices

In sect. 3 we have identified the SG and MT models as perturbed CFTs, which immediately gave us a detailed understanding of their UV limits, in particular how they differ.
The next step is to uncover the difference between the massive theories. Given certain smoothness assumptions about the UV limit, to be discussed in subsect. 4.2, we right away know that some correlation functions differ and some are identical in corresponding SG and MT models. Namely, correlators of fields whose UV limit is in one of the “even” sectors \((m, n) \in 2\mathbb{Z} \times \mathbb{Z}\) will be identical, whereas all other correlation functions will be distinct, since they contain some fields from the “odd” sectors where the UV limits of SGM and MTM have no fields in common.

It is of course very difficult (see however [40]) to calculate these correlation functions exactly, and in any case we are more interested in directly “observable” quantities, e.g. \(S\)-matrix elements and finite-volume energy levels. Considering the SG/MT models from now on as \(V^{(+)}_{0,1}\)-perturbed CFTs, we know that the electric charge \(m\) corresponds to soliton/fermion number. [In the notation of subsect. 2.1, the \(O(2)\) symmetry of the CFT remains unbroken; the parity \((-1)^m\) of the sectors in the massive theory refers then to the \(\mathbb{Z}_2\) subgroup of the unbroken \(U(1)\).] The bound states of the solitons, respectively, fermions correspond to \(m = 0\), i.e. one of the even sectors where the correlation functions in SGM and MTM are identical. Therefore the scattering amplitudes involving only bound states will be identical in both theories.

The \(S\)-matrix in the soliton/fermion sector is determined almost uniquely by the general requirements of factorizable \(S\)-matrix theory for an \(O(2)\)-doublet of scattering particles [21][41]. The only ambiguity are CDD factors. They have to be the same in SGM and MTM, up to a sign perhaps, to guarantee that the bound state amplitudes, which can be obtained using the bootstrap [21][22], agree. If one makes the very plausible assumption that there are no bound states in the repulsive regime of the MTM, the CDD factors must be trivial \((=\pm 1)\) [41], and the only ambiguity that remains is one overall sign for the \(S\)-matrix in the soliton/fermion sector.

For the MTM the \(S\)-matrix so obtained has been checked perturbatively to third order [42], confirming in particular the choice of sign made in [21][41]. However, for the SGM there is no lagrangian for the soliton fields which would allow for a perturbative check, and we in fact claim that the overall sign of the SG \(S\)-matrix in the soliton sector is opposite to that of the MTM!

There is a very simple argument for this: Let \(S_{aa}^a(\theta)\) denote the \(S\)-matrix element for the scattering of two particles of species \(a\) in a factorizable \(S\)-matrix theory; \(\theta\) is the relative rapidity and we are assuming that the scattering is purely elastic, i.e.

\[S_{aa}^{cd}(\theta_1 - \theta_2)\] denotes the amplitude for the process \(a(\theta_1) + b(\theta_2) \rightarrow c(\theta_2) + d(\theta_1)\).
\[S^{cd}_{aa}(\theta) = \delta^c_a \delta^d_a S^{aa}_{aa}(\theta)\] (as for the solitons in SGM and the fermions in MTM, by charge conservation). By considering multi-particle states one can then easily show [20] that an exclusion principle (in rapidity space) holds only if \(S^{aa}_{aa}(0) = (-1)^{F_a} \mp 1\) if particle \(a\) is a boson/fermion. Since presumably any interacting (1+1)-dimensional QFT satisfies an exclusion principle, our identification of the SG solitons as bosons then proves our claim.

A similar argument can be given using the relation between the parity of a bound state and the sign of the residue of a pole in an appropriate channel of the \(S\)-matrix. Such a relation follows from the following two statements:

(i) Poles of the \(S\)-matrix corresponding to bound states with symmetric (anti-symmetric) wave functions have positive (negative) imaginary residue, when the \(S\)-matrix is considered as a function of the relative rapidity \(\theta\).

(ii) Boson-antiboson (\(b\bar{b}\)) bound states with symmetric (anti-symmetric) wave function have positive (negative) \(C\) and \(P\) parity. For fermion-antifermion (\(f\bar{f}\)) bound states this relationship is reversed.

Statement (ii) is basically obvious for the corresponding unbound 2-particle states (see e.g. chapter 15 of [43]), and can therefore be expected to be true at least for weak bound states. (i) can be proved in potential scattering and is also believed to hold in QFT [22][25]. By combining (i) and (ii) we see that the relation between the \(C\) and \(P\) parity and the sign of the residue will be opposite for the case of a \(b\bar{b}\) and \(f\bar{f}\) bound state. If we can establish that the parities of the bound states in SGM and MTM are the same, our claim about the sign of the soliton \(S\)-matrix follows. But it is well known how to determine the parities of the bound states [21][22]. In a 2-particle basis of \(C=\pm 1\) eigenstates, \(|s(\theta_1)\bar{s}(\theta_2)|\pm|\bar{s}(\theta_1)s(\theta_2)|\) for SGM, say, the soliton-antisoliton (\(s\bar{s}\)) \(S\)-matrix is diagonal with amplitudes \(S_{\pm}(\theta)\) [21][22], respectively. One finds that a pole corresponding to the \(n\)-th bound state appears in \(S_+\) only for even \(n\), in \(S_-\) only for odd \(n\) (and this fact is obviously independent of the overall sign of the \(s\bar{s}\) \(S\)-matrix).

For completeness, we note that amplitudes involving one soliton/fermion and a bound state have the same sign in SGM/MTM. This can be shown using the bootstrap.

\[\] 15 Ignoring the special case of kink theories [28], the only QFT we know for which \(S^{aa}_{aa}(0) \neq -(-1)^{F_a}\) are free bosons, which indeed do not satisfy an exclusion principle.

\[\] 16 Recall that the number of bound states in SGM (and the corresponding MTM) is equal to the largest integer smaller than \(\lambda \equiv \frac{8\pi}{\beta^2} - 1\); their masses are \(m_n(\lambda) = 2m \sin \frac{2\pi n}{\lambda}n = 1, 2, \ldots < \lambda\) in terms of the soliton (fermion) mass \(m\).
So as to not always argue about the signs of $S$-matrix elements, we will in the following subsections look at a rather different kind of observable that distinguishes the SG and MT models, namely the finite-volume spectrum, as encoded in the partition function. The finite-volume partition function has proven to be a useful and illuminating probe of QFTs defined as perturbed CFTs. The reason is that it provides an interpolation between the UV CFT for small volume, which is usually understood completely, and the IR behaviour of the theory for large volume, which allows one to obtain some information about the $S$-matrix, in particular signs. The arguments we use in this analysis are unrelated to those employed above, and will, in particular, provide independent evidence for the claim that the solitons are bosons. The small-volume behaviour of a partition function can be determined in CPT, to which we now turn.

4.2. Conformal Perturbation Theory

The idea of defining a massive (1+1)-dimensional QFT as a (spinless) relevant perturbation of a CFT has been around for a long time. It has become particularly fruitful in the last few years, after many classes of CFTs were understood in great detail. In the case of integrable perturbations, this approach has led to exact non-perturbative results when used in conjunction with other techniques, like the bootstrap, exploiting quantum group symmetries, the thermodynamic Bethe Ansatz, and, more generally, the numerical and analytical study of the finite-volume spectrum of a non-scale-invariant QFT. (For all of this see [37][24][26][44][45][46][27][47][17][28] and references therein.)

The reason that makes this idea so powerful is that a relevant (i.e. super-renormalizable) perturbation of a CFT is a very “benign” perturbation. For example, there are good reasons [48][46] to believe that a perturbative expansion around the UV CFT will have a finite radius of convergence with appropriate IR and, if necessary (see below), UV cutoffs. Furthermore, the super-selection structure of the massive theory is manifest, since that of the CFT is usually known, and the massive sectors will simply be labeled by the charges that are conserved by the perturbation. Of course, the structure within a sector, e.g. if there are bound states in the vacuum sector, is not obvious in this approach either. Still, one can often ascertain facts which are not at all trivial in a standard lagrangian approach; for example, as mentioned in subsect. 4.1, that the properties of the bound states of the solitons/fermions in the SGM/MTM are absolutely identical.

The expected “smoothness” of relevant perturbations of CFTs is reflected in the finite-volume spectrum of the perturbed theory. (We will always put the theory on a cylinder,
i.e. use periodic boundary conditions for bosons and anti-periodic ones for fermions.) The finite-volume energy levels must be smooth functions of the “volume” of space, namely the circumference $L$ of the cylinder. For small volume the eigenstates are simply labeled by the states of the UV CFT, and the $i$-th energy gap $\hat{E}_i(L) = E_i(L) - E_0(L)$ above the ground state behaves like \cite{19} (for a perturbed unitary CFT at least)

$$\hat{E}_i(L) \to \frac{2\pi}{L} d_i \quad \text{as} \quad L \to 0,$$

where $d_i$ is the scaling dimension of the UV conformal state $|i\rangle$ created by $\phi_i$. The ground state energy itself behaves like $E_0(L) \to -\pi c/6L$, where $c$ is the central charge of the (unitary) UV CFT. [In the cases considered the ground state level presumably does not cross any other level for all $L \geq 0$, in other words, there is no phase transition for nonzero temperature.] The total momentum $P$ of a state is quantized in finite volume, and we have the exact relation $P_i(L) = \frac{2\pi}{L} s_i$ to the spin $s_i$ of the UV field $\phi_i$. The finite-volume spectrum then provides a smooth interpolation between the states of the UV CFT and those of the massive theory in the IR, where any $\hat{E}_i(L)$ simply approaches some sum of masses.

To emphasize that defining a QFT as a perturbed CFT is essentially a non-perturbative definition of the theory, we remark that using the “truncated conformal space approach” of Yurov and Zamolodchikov \cite{50} \cite{51} one can in principle calculate the finite-volume energy gaps of the perturbed theory to arbitrary accuracy; the method is non-perturbative, though numerical. We will not use this technique here, but instead consider the analytical calculation of the small-volume expansion coefficients for the energy levels, i.e. CPT. The action of the perturbed CFT is

$$A_\lambda = A_{\text{CFT}} + \lambda \int d^2 \xi \ V(\xi) \ ,$$

where the integral is over the cylinder, $\lambda = \kappa m^y$ for some dimensionless constant $\kappa$, $d = 2 - y$ is the scaling dimension of $V$, and $m$ a mass scale in the perturbed theory, the mass of the lightest particle in the cases we will consider. The dimensionless gap scaling functions $\hat{e}_i(\rho)$ can then be expanded as

$$\hat{e}_i(\rho) \equiv \frac{L}{2\pi} \hat{E}_i(L) = d_i + \sum_{n=1}^{\infty} \hat{a}_{i,n} \ \rho^{yn}, \quad \rho \equiv L m ,$$

23
where \( \hat{a}_{i,n} \equiv a_{i,n} - a_{0,n} \) with

\[
a_{i,n} = \frac{(2\pi)^{1-y} (-\kappa)^n}{n!} \langle i| i \rangle \int \prod_{j=1}^{n-1} \frac{d^2z_j}{(2\pi|z_j|)^y} \langle i| V(1, 1) \prod_{j=1}^{n-1} V(z_j, \bar{z}_j)|i\rangle_{\text{conn}} .
\]

(4.4)

The correlators here are connected (with respect to the “in- and out-states” created by \( \phi_i \), \( \phi_0 \) being the identity operator) critical \((n+2)\)-point functions on the \textit{plane}, with \( \langle i| \ldots |j \rangle \) denoting \( \langle \phi_i^*(\infty, \infty) \ldots \phi_j(0, 0) \rangle \). [To obtain (4.4) one has to transform the correlators from the cylinder to the plane; if \( \phi_i \) is not the “planar” state \(|i\rangle\) appearing in the above has to be interpreted suitably, taking into account the nontrivial transformation properties of \( \phi_i \).

As an easy consequence of (4.4) note that all energy levels in the even sectors of MTM and SGM must be identical. In particular, the SG and MT models have exactly the same finite-volume ground state energy \( E_0(L) \). (In these type of arguments we are making the very plausible assumption, cf. above, that equality in CPT means exact equality.)

Due to UV divergences the \( a_{i,n} \) themselves are actually not well-defined when \( d = 2 - y \geq 1 \), but since these divergences do not depend on \( \phi_i \) they cancel in the \( \hat{a}_{i,n} \). See [52] for more details about the UV divergences and their nontrivial effect on the \( e_i(\rho) = (L/2\pi)E_i(L) \), as well as the regularization of possible IR divergences of the \( a_{i,n} \) (by analytic continuation in \( d_i \)) that is implicit in (4.4).

4.3. Free MTM versus SG\((\sqrt{4\pi})\)

We now discuss in more detail the free MTM and the SGM at its so-called “free Dirac point” \( \text{SG}(\sqrt{4\pi}) \), \textit{i.e.} the \( V_{0,1}^{(+)} \) perturbations of the fermionic, respectively, bosonic gaussian CFT of radius \( r = 1 \). The free MTM just describes a free massive fermion \( f \) and its antiparticle \( \bar{f} \). \( \text{SG}(\sqrt{4\pi}) \) contains only the soliton \( s \) and the antisoliton \( \bar{s} \). Our aim is to find the \textit{exact} finite-volume partition function of \( \text{SG}(\sqrt{4\pi}) \).

4.3.1 The Partition Functions

It is of course easy to write down the partition function of the free MTM, eq. (4.10) below. The only nontrivial ingredient — still easily obtained — is the finite-volume ground state energy; all energy gaps just follow from the dispersion relation of a free massive particle. In contrast, our derivation of the partition function \( Z_{\text{SG}} \) of \( \text{SG}(\sqrt{4\pi}) \) is not rigorous, and to motivate it we will first briefly discuss a different pair of closely related but distinct \((1+1)\)-dimensional QFTs whose partition functions are rigorously known. The
theories in question are the free massive Majorana fermion and the Ising field theory \[53\] \[54\] in its high-temperature phase (IFT), defined equivalently as a scaling limit of the lattice Ising model (without a magnetic field) from above the critical temperature, or as the leading thermal perturbation (with the appropriate sign of the coupling) of the Ising CFT. We have discussed these theories in detail before (see sect. 6 of \[16\] and sect. 3.1 of \[28\]) and therefore can be brief here.

The finite-volume partition function of the IFT, with periodic boundary conditions on the spin field, can be derived by taking a scaling limit of the Onsager solution of the Ising model on a finite lattice (cf. \[55\] \[46\]). It reads

\[
Z_{\text{IFT}}(\rho) = \frac{1}{2} q^{e_+(\rho)} \left\{ \prod_{n \in \mathbb{Z} + \frac{1}{2}} (1 + q^{e_n(\rho)}) + \prod_{n \in \mathbb{Z} + \frac{1}{2}} (1 - q^{e_n(\rho)}) \right. \\
+ \left. q^{\hat{e}_-(\rho)} \left[ \prod_{n \in \mathbb{Z}} (1 + q^{e_n(\rho)}) - \prod_{n \in \mathbb{Z}} (1 - q^{e_n(\rho)}) \right] \right\} .
\]

Here \( q = e^{-2\pi/T L}, \) \( L \) is the “volume” of space, \( T \) the temperature, \( \rho = L m \) where \( m \) is the (infinite-volume) mass of the particle,

\[
e_n(\rho) = \sqrt{\left( \frac{\rho}{2\pi} \right)^2 + n^2} , \quad (4.6)
\]

\( \hat{e}_-(\rho) = e_-(\rho) - e_+(\rho), \) and \[16\] \[52\]

\[
e_\pm(\rho) = -\frac{\rho}{4\pi^2} \int_{-\infty}^{\infty} d\theta \cosh \theta \ln(1 \pm e^{-\rho \cosh \theta}) . \quad (4.7)
\]

We have written everything in terms of dimensionless quantities. The dimensionful finite-volume energies \( E_i(L) \) are read off from \((4.5)\) by expanding it as \( \sum \) \( e^{-E_i(L)/T} \), for instance the ground state and first excitation energies are \( E_0(L) = (2\pi/L)e_+(\rho) \) and \( E_1(L) = (2\pi/L)e_-(\rho) + m, \) respectively; the \( (2\pi/L)e_n(\rho) \) are just energy gaps of a free particle of mass \( m \) and momentum \( 2\pi n/L. \)

Note that the first two terms in \((4.5)\) (together with the factor \( \frac{1}{2} \)) amount to a projection on states with an even number of particles, including the vacuum, the last two on odd-particle states. These even and odd sectors are distinguished by their parity under the \( \mathbb{Z}_2 \) symmetry of the theory, corresponding to the spin-reversal symmetry of the Ising model.
For comparison recall that the partition function of the free massive Majorana fermion (with anti-periodic boundary conditions) is

\[ Z_{\text{Maj}}(\rho) = q^{e_+(\rho)} \prod_{n \in \mathbb{Z} + 1/2} \left( 1 + q^{\epsilon_n(\rho)} \right). \]  

(4.8)

The spectrum in the even sector, here with respect to the total fermion number, is identical to that of the IFT, but the odd sectors of the two theories are different.

The main point of (4.5) and (4.8) is that their qualitative features are exactly the same as those of the partition functions of the corresponding UV CFT, obtained from the above by \( m \rightarrow 0 \). The building blocks of \( Z_{\text{IFT}} \) are the partition functions of a free massive Majorana fermion with the four possible boundary conditions in the time and space directions, and \( Z_{\text{IFT}} \) is obtained from \( Z_{\text{Maj}} \) by a GSO projection (cf. sect. 2.2). The only differences from the massless case are the ground state scaling functions, i.e. \( e_+(\tau) \) in both sectors of the Majorana theory and \( e_{\pm}(\rho) \) in the even/odd sector of IFT, and the replacement of \(|n|\) by the rescaled free particle energies \( \epsilon_n(\rho) \).

Because of a widespread confusion in the literature, we emphasize again that the IFT and the theory of a free Majorana fermion are not identical. The IFT describes an interacting boson created by the spin field \( \sigma \), with constant \( S \)-matrix \( S = -1 \) \([54]\), whereas the free massive Majorana theory describes, of course, a fermion with \( S \)-matrix \( S = +1 \). It “just so happens” that the partition function of the IFT with any allowed (see below) boundary conditions on \( \sigma \) is identical to some linear combination of “partition functions” of a free Majorana fermion with various boundary conditions. We say “partition functions” in quotes because for a true partition function, namely the generating function of the energy levels \( \text{Tr} e^{-\beta H} \), there is no choice for the boundary conditions in the time direction. In a path integral formalism, for example, one must choose anti-periodic boundary conditions in the time direction for fermions and periodic ones for bosons, otherwise one will get a different trace, e.g. \( \text{Tr} (-1)^F e^{-\beta H} \) for a fermion with periodic temporal boundary conditions. The “natural” boundary conditions in the spatial direction are the ones that

---

17 This gives the case of a torus with perpendicular cycles, \( q = \bar{q} \). Generalization of (4.5), (4.8) and other partition functions given below to an arbitrary torus amounts to the replacement \( q^{\epsilon_n(\rho)} \rightarrow |q|^{\epsilon_n(\rho)} (q/\bar{q})^{1/2} \), explicitly identifying \( n \) as the momentum quantum number.

18 If we denote by \((\alpha, \beta)\), \( \alpha, \beta \in \{P, A\} \), the boundary conditions (periodic or anti-periodic) in the time, \( \alpha \), and space, \( \beta \), directions, then the four terms in (4.5) correspond to \((A, A), (P, A), (A, P), (P, P)\), respectively.
are identical to those in the time direction — then the partition function will be invariant under exchange of space and time, and presumably (cf. sect. 5) reveal the true mutually local operator content of the theory — although one can also impose various other spatial boundary conditions (depending on the symmetry of the theory).

Some other features of (4.5) will be illuminated as we now proceed to the case of actual interest. The partition function $Z_f(q, r=1)$ of the free massless Thirring model, alias the free massless Dirac fermion, can be written as $q^{-\frac{1}{12}} \prod_{n \in \mathbb{Z} + \frac{1}{2}} (1 + q^{|n|})^2$, assuming $q = \bar{q}$ for simplicity again. We can also rewrite the partition function of the UV limit of SG$(\sqrt{4\pi})$,

$$Z_b(q, r = 1) = q^{-\frac{1}{12}} \left\{ \prod_{n \in \mathbb{Z} + \frac{1}{2}} (1 + q^{|n|})^2 \right\}_+ + q^{\frac{1}{4}} \left\{ \prod_{n \in \mathbb{Z}} (1 + q^{|n|})^2 \right\}_-$$. (4.9)

i.e. as a GSO projection of the massless Dirac fermion partition function. [Here and below $[ \prod \ldots ]_\pm$ denotes keeping only terms in the product with an even/odd number of factors $q^{|n|}$ or $q^{\epsilon_n}(\rho)$.] The IFT-Majorana analogy suggests that the GSO mechanism extends to the massive regime also in our case, namely that the partition function of SG$(\sqrt{4\pi})$ is related by a GSO projection to that of the free massive Dirac fermion. In this way $Z_{SG}$ will be automatically modular invariant, in particular invariant under the exchange $L \leftrightarrow 1/T$ (cf. [56] for the modular invariance of $Z_{IFT}$).

The building blocks of $Z_{SG}$ will therefore be “partition functions” of a free massive Dirac fermion with the four possible boundary conditions. These partition functions can be calculated either using $\zeta$-function regularization, cf. [56] for the Majorana case, or, in a way that is simpler and allows one to obtain more explicit expressions, by noting that their only nontrivial aspect is the Casimir energy which depends on the boundary conditions in the spatial direction. For a free theory the finite-volume Casimir energy can be easily calculated using the thermodynamic Bethe Ansatz (TBA), see e.g. [46] for details. For the “natural” boundary conditions of a free fermion, anti-periodic, each fermionic degree of freedom (1 for the Majorana, 2 for the Dirac case) contributes $\frac{2\pi}{L} e_+(\rho)$; for periodic boundary conditions it is $\frac{2\pi}{L} e_-(\rho)$, cf. (4.7).

Putting everything together, the partition function of the free MTM, i.e. the free Dirac fermion, is

$$Z_{Dirac}(\rho) = q^{2e_+(\rho)} \prod_{n \in \mathbb{Z} + \frac{1}{2}} (1 + q^{\epsilon_n}(\rho))^2$$. (4.10)

27
and that of $\text{SG}(\sqrt{4\pi})$ reads

$$Z_{\text{SG}}(\rho) = q^{2e_+ (\rho)} \left\{ \prod_{n \in \mathbb{Z}+1/2} (1 + q^{e_n (\rho)})^2 \right\}_+ + q^{2e_- (\rho)} \left\{ \prod_{n \in \mathbb{Z}} (1 + q^{e_n (\rho)})^2 \right\}_-. \quad (4.11)$$

The only assumption used to obtain the above is that $Z_{\text{SG}}$ is related by a GSO projection to $Z_{\text{Dirac}}$. One nontrivial point in implementing this projection in the massive case is the $[\ ]_-$ projector — as opposed to $[\ ]_+$ — that we used for the second product in (4.11). In the massless case the two choices give the same result, because the partition function of a free massless fermion with periodic boundary conditions in both directions vanishes identically. But in the massive case the two choices lead to different physics; in particular, if we use the $[\ ]_+$ projection the ground state is doubly degenerate and no odd-particle states exist with periodic boundary conditions. This choice is therefore not appropriate for the standard SGM; instead, we claim that it gives the partition function of $\text{SG}(\sqrt{4\pi}, 2)$, mentioned in sect. 3.19 The generalization to $\text{SG}(\sqrt{4\pi}, k)$ for any $k$ will be discussed in [39].

Though (4.11) is highly plausible, we should provide independent evidence for its correctness. We will do so below using CPT, which indicates that the small $\rho$ behaviour of $Z_{\text{SG}}$ is correct. First, however, we want to discuss the “IR interpretation” of $Z_{\text{SG}}$, which will also help us to see how conformal states in the UV CFT “evolve” into multiparticle states in the massive theory. Note that $Z_{\text{Dirac}}$ is just the square of $Z_{\text{Maj}}$, in accord with the fact that the free massive Dirac theory describes the scattering of two, not one, fermions with trivial diagonal $S$-matrix $S = 1$. Similarly, comparing $Z_{\text{SG}}$ and $Z_{\text{IFT}}$ one expects that $\text{SG}(\sqrt{4\pi})$ describes the scattering of two bosons with diagonal $S$-matrix $S = -1$. We will show that the large $\rho$ behaviour of $Z_{\text{SG}}$ is in perfect agreement with this.

4.3.2 IR Check of $Z_{\text{SG}}$

The first check of $Z_{\text{SG}}$ is provided by the ground state energy, whose rescaled form is $2e_+ (\rho)$. From the way $Z_{\text{SG}}$ was obtained we know that it is the Casimir energy of two free fermions of mass $m$ with anti-periodic boundary conditions. Fortunately — and this is trivial to see with the TBA, cf. [10] — a theory of two bosons of mass $m$ with diagonal $S$-matrix $S = -1$ has exactly the same ground state energy with periodic boundary

$\text{IFT}$

19 The corresponding replacement of the minus sign in front of the last term in (4.5) by a plus sign gives the partition function of the theory obtained by taking a scaling limit of the Ising model from below the critical temperature, see sect. 3.1 of [28] for details.
conditions, in agreement with the above picture of \( \text{SG}(\sqrt{4\pi}) \). (Of course, we know independently from CPT that corresponding MT and SG models have the same finite ground state energy for any \( \beta \).)

It is well known \cite{57} \cite{58} that the \( S \)-matrix of any (massive) QFT determines the large-volume behaviour of its energy gaps. The corrections to the infinite-volume gaps (= sums of masses) have contributions that are powerlike in \( 1/L \), and others that fall off exponentially with \( L \) (= the smallest extension of the spatial volume). These latter corrections are due to off-shell effects, basically “virtual particles traveling around the world” (and also tunneling if the QFT has a degenerate vacuum in infinite volume). For 1-particle states at zero momentum there are only exponential corrections and the leading ones can be calculated in terms of the \( S \)-matrix \cite{57} \cite{27}.

The powerlike dependence of energy gaps of multi-particle states in any integrable (1+1)-dimensional QFT can be determined exactly given its factorizable \( S \)-matrix \cite{58} \cite{50} \cite{59} \cite{28}. In the case of a diagonal and furthermore constant \( S \)-matrix this is in fact basically trivial: For an \( N \)-particle state in a theory with \( S = -1 \) in finite volume \( L \) with periodic boundary conditions, the allowed “single-particle momenta” \( p_j \) are determined by the quantization conditions

\[
e^{ip_j L} (-1)^{N-1} = 1, \quad j = 1, \ldots, N. \quad (4.12)
\]

The momenta are therefore of the form \( p_j = \frac{2\pi}{L} n_j \), where all \( n_j \in \mathbb{Z} \) for an odd- and all \( n_j \in \mathbb{Z} + \frac{1}{2} \) for an even-particle state, and in both cases the number of \( n_j \) with the same value must not exceed the number of particle species in the theory (because of the exclusion principle, cf. subsect. 4.1). Up to exponentially small contributions, the energy gap of a state in terms of the “momentum quantum numbers” \( n_j \) is

\[
\hat{E}(L) = \frac{2\pi}{L} \sum_{j=1}^{N} \epsilon_{n_j}(m_j L). \quad (4.13)
\]

A look at (4.11) now reveals that up to exponential corrections its energy levels, as well as their degeneracies, are exactly those of a theory of two bosons of mass \( m \) with \( S = -1 \).

Note that in the even sector of \( \text{SG}(\sqrt{4\pi}) \) (and IFT) there are no exponential corrections to (4.13), and in the odd sector the corrections are the same for every state, namely, for \( \text{SG}(\sqrt{4\pi}) \),

\[
\frac{\Delta E(L)}{m} = \frac{2\pi}{mL} 2\hat{e}_-(mL) = \frac{4}{\pi} K_1(mL) + \mathcal{O}(e^{-3mL}). \quad (4.14)
\]
This “universality” of the off-shell corrections is certainly not true in a generic interacting QFT, and is a sign of how “closely” SG($\sqrt{4\pi}$) and IFT are related to free theories. More precisely, the vanishing of the off-shell effects in the even sector of SG($\sqrt{4\pi}$) and IFT is clear from the fact that this sector is created by the same fields that create the even sector in the free MTM and the free Majorana theory, respectively.

It is not known how to calculate even the leading off-shell corrections for an arbitrary multi-particle state in terms of the $S$-matrix, except for 1-particle states at zero momentum, i.e. finite-volume masses. For a (1+1)-dimensional QFT with non-degenerate vacuum in infinite volume, containing only particles of the same mass $m$ and no poles in its scattering amplitudes, the finite-size mass shift of a particle $a$ is (see eq. (75) of [27])

$$\frac{\Delta m_a(L)}{m} = -\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-mL \cosh \theta} \cosh \theta \sum_b \left( S_{ab}(\theta + \frac{i\pi}{2}) - 1 \right) + O(e^{-\sigma L}),$$

with $\sigma \geq \sqrt{3}m$. Therefore, a particle that scatters with itself and exactly one other mass-degenerate particle with scattering amplitudes $S_{ab}(\theta) \equiv -1$ will have a finite-size mass shift of $\Delta m(L) = 4 \int d\theta e^{-mL \cosh \theta} \cosh \theta + O(e^{-\sigma L}) = \frac{4}{\pi} K_1(mL) + O(e^{-\sigma L})$, in agreement with (4.14).

4.3.3 CPT Check of $Z_{SG}$

We now want to apply the CPT results (4.3)(4.4) to check that the small $\rho$ behaviour of $Z_{SG}$ is consistent with the formulation of SG($\sqrt{4\pi}$) as the $V_{0,1}^{(+)}$-perturbed bosonic gaussian model at $r = 1$. To keep things simple we will only consider levels in the zero momentum sector of the spectrum that in the UV limit are created by (spinless) vertex operators that do not mix under the perturbation with other operators of the same scaling dimension. Since spin and the electric charge $m$ are conserved by a $V_{0,1}^{(+)}$ perturbation, the UV operators $V_{m,0}$, $m \in \mathbb{Z}$, obviously satisfy these criteria. The only operators that $V_{0,\pm n}$, $n>0$, could couple to by a $V_{0,1}^{(+)}$ perturbation are spinless descendants of $V_{0,\pm(n-1)}$, but since $d_{0,n} - d_{0,n-1} = 2n - 1$ is odd, such operators cannot have the same scaling dimension as $V_{0,\pm n}$. The $V_{0,\pm n}$ are therefore part of a basis in the space of $s=m=0$, $d=n^2$ fields in which the perturbation is diagonal, so that results of standard (non-degenerate) perturbation theory apply.

The first order term $\hat{a}_{1}(A)$ vanishes for all operators $A$ in the set $\{V_{\pm m,0}, V_{0,\pm n}\}$, $m, n \in \mathbb{N}$. This means that the levels corresponding to the operators $V_{\pm(2m+1),0}$, which are

20 In the following we denote $\hat{a}_{i,n}$ of subsect. 4.1 by $\hat{a}_n(\phi_i)$. 30
the only ones in this set in the odd sector, must have exactly one term $\epsilon_0(\rho) = \frac{\rho}{2\pi}$ in their energy, to cancel the $O(\rho)$ term in $2\hat{\epsilon}_-(\rho) = \frac{1}{4} - \frac{2}{\pi^2} \rho^2 + O(\rho^4)$. We will see that this is indeed true after we have identified the “IR labels”, namely the momentum quantum numbers $n_i$ in (4.13), of the levels corresponding to the above operators. To identify these levels we proceed to $\hat{a}_2(A)$.

The correlators involved can be read off from (2.5), and the one complex integral that has to be performed can be done either directly in polar coordinates or using the “generalized beta function” of the complex number field (cf. [60])

$$\int d^2z \ z^{s+m/2} \bar{z}^{s-m/2} \ (1-z)^{t+n/2} \ (1-\bar{z})^{t-n/2} = \varepsilon\pi \ \frac{\Gamma(s+\frac{|m|}{2}+1) \Gamma(t+\frac{|n|}{2}+1) \Gamma(-s-t+\frac{|m+n|}{2}-1)}{\Gamma(-s+\frac{|m|}{2}) \Gamma(-t+\frac{|n|}{2}) \Gamma(s+t+\frac{|m+n|}{2}+2)} \ . \quad (4.16)$$

where $s, t \in \mathbb{C}$ (by analytic continuation from the region where the integral converges), $m, n \in \mathbb{Z}$, and $\varepsilon = 1$ if $mn \geq 0$ and $(-1)^{\min(|m|,|n|)}$ otherwise.

The results are

$$\hat{a}_2(V_{0,\pm n}) = \frac{\kappa^2}{2} \left[ \psi(n+\frac{1}{2}) - \psi(\frac{1}{2}) \right] = \kappa^2 \left( 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \right) \ , \quad (4.17)$$

and

$$\hat{a}_2(V_{\pm m,0}) = \frac{\kappa^2}{2} \left[ \psi(\frac{m}{2}+\frac{1}{2}) - \psi(\frac{1}{2}) \right] = \kappa^2 \left( \ln 2 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{m-1} \right) \quad m \text{ odd} \ , \quad m \text{ even} \ , \quad (4.18)$$

where $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$.

To identify the corresponding levels as (multi-)particle states let us label them $s(n_1, \ldots, n_k) \ s(n_{k+1}, \ldots, n_N)$ in terms of their momentum quantum numbers. Because of the exclusion principle we can restrict the quantum numbers to $n_1 > n_2 > \ldots > n_k$ and $n_{k+1} > n_{k+2} > \ldots > n_N$, say. Note that according to $Z_{SG}$ the above state has UV spin $s = \sum_{i=1}^{N} n_i$, and UV scaling dimension $d = \frac{1-(-1)^N}{8} + \sum_{i=1}^{N} |n_i|$.

$V_{\pm 1,0}$ are the lowest dimension operators above the vacuum and one might expect that they correspond to the lowest excited states also for finite $\rho$, namely $s(0)$ and $\bar{s}(0)$, with scaled energy $2\hat{\epsilon}_-(\rho) + \epsilon_0(\rho) = \frac{1}{4} + \frac{\ln 2}{\pi^2} \rho^2 + O(\rho^4)$. This identification is consistent with (4.13), which furthermore allows us to conclude that

$$\kappa = \frac{1}{\sqrt{2\pi}} \ . \quad (4.19)$$

31
The identification of $V_{\pm 1,0}$ as soliton/antisoliton creation operators in the UV is in agreement with our remarks in sect. 3, showing that the electric charge that is still super-selected in a $V_{0,k}^{(+)}$ perturbation should be identified as soliton number. Using $\epsilon_n(\rho) = |n| + \frac{\rho^2}{8\pi^2} |n| + O(\rho^4)$, for $n \neq 0$, this identification then allows us to conclude

$$V_{\pm m,0} \leftrightarrow \begin{cases} \sqrt{\frac{m-1}{2}} \rho^{m-3}, \ldots, 1, 0, -1, \ldots, -\frac{m-1}{2} \quad \text{for } m \text{ odd} \\ \sqrt{\frac{m-1}{2}} \rho^{m-3}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{m-1}{2} \quad \text{for } m \text{ even} \end{cases},$$

which are the only states in the charge $\pm m$ sector with the right UV scaling dimension $d_{\pm m,0} = (\frac{m}{2})^2 = \frac{1}{4} + 2(1 + 2 + \ldots + \frac{m-1}{2})$ for $m$ odd, and $d_{\pm m,0} = (\frac{m}{2})^2 = 2(\frac{1}{2} + \frac{3}{2} + \ldots + \frac{m-1}{2})$ for $m$ even. [In the above $s^\pm$ stands for $s, \bar{s}$, respectively.]

The degenerate pair of levels corresponding to $V_{0,\pm n}$ is naturally identified as the pair $s(n - \frac{1}{2}, -\frac{3}{2}, \ldots, -\frac{1}{2})$ and $s(-\frac{1}{2}, -\frac{3}{2}, \ldots, -n + \frac{1}{2})$. From previous experience [52, 28] it is also plausible to conjecture that, say, the pair $s(l + \frac{1}{2})\bar{s}(-l - \frac{1}{2})$ and $s(-l - \frac{1}{2})\bar{s}(l + \frac{1}{2})$ corresponds to certain spinless descendants of $V_{0,\pm 1}$ at (left and right) level $l$. Finally, turning to sectors with nonzero momentum, we conjecture that the special Virasoro primaries in the vacuum sector of $d = \pm s = n^2$, $n \in \mathbb{N}$ (which can be expressed in term of the derivatives of $\phi$ and $\bar{\phi}$, respectively, using Schur polynomials [61]) correspond to the states $s(\pm \frac{1}{2}, \pm \frac{3}{2}, \ldots, \pm n - \frac{1}{2})$.

4.4. Away from $\beta = \sqrt{4\pi}$

In the previous subsections we studied the SGM at $\beta = \sqrt{4\pi}$ in some detail. We will now briefly discuss what happens as $\beta$ moves below $\sqrt{4\pi}$, where bound states of the solitons, so-called breathers, appear in the spectrum. In particular, we will identify the field that creates the first breather by looking at the finite-volume spectrum.

Consider the zero momentum sector of the spectrum of SG($\sqrt{4\pi}$). Recall that the soliton/antisoliton, whose rescaled energy gap at rest is $\epsilon_0(\rho)$, are created by $V_{\pm 1,0}$ in the UV, and the lowest 2-particle states $s(\frac{1}{2})\bar{s}(\frac{1}{2}) \pm s(-\frac{1}{2})\bar{s}(\frac{1}{2})$ of energy $2\epsilon_{\frac{1}{2}}(\rho)$ by $V_{0,1}^{(+)}$. Now lower $\beta$ by an infinitesimal amount $\delta$. The “picture” of the finite-volume spectrum, i.e. the energy levels, can only change by an infinitesimal amount, but we must now accommodate the first breather, a weak bound state of mass $m_1 = 2m - O(\delta)$, in this picture. It is rather clear what will happen: The bound state should correspond to the lowest of the former 2-particle levels, the question only is which of $V_{0,1}^{(\pm)}$ creates it in the UV. Using (4.16) one sees that $\hat{a}_2(V_{0,1}^{(\pm)})$ is positive/negative, so that presumably $V_{0,1}^{(-)}$ is the lowest level for all $\rho$. We therefore identify $V_{0,1}^{(-)} \propto \sin \beta \phi$ as creating the first breather for $\beta^2 < 4\pi$. 

32
Initially this is justified only for small volume and $\beta$ just below $\sqrt{4\pi}$. But besides the standard smoothness arguments, there are various other reasons why this is very plausible. We note, for example, that the scaling dimension of $V_{0,1}^{(-)}$ drops below that of $V_{\pm 1,0}$ exactly when the mass of the first breather drops below that of the solitons. More convincingly, standard perturbation theory around the free massive theory $\beta = 0$ provides quantitative evidence $[65][66]$ that $\varphi$ creates the first breather, which is in any case the only plausible candidate for what it could create. Of course, from a non-perturbative point of view $\varphi$ is compactified, i.e. not really well-defined. The simplest well-defined field that creates the same asymptotic state as $\varphi$, that is, gives rise to the same $S$-matrix, is $\sin \beta \varphi$. (See chapter 2 of [67] for a discussion of when two different fields lead to the same $S$-matrix.)

Recall from sect. 4.1 that the first breather, and as we now know $\sin \beta \varphi$, are $C$ and $P$ odd. By just looking at the SG lagrangian it seems that one can simply choose $\sin \beta \varphi$ to have any $C$ and $P$ parity. On a non-perturbative level this is not the case, though, because $\sin \beta \varphi$ creates a particle that is a bound state of the underlying solitons, and its parity properties must be consistent with the dynamics of these solitons. Similarly, one can argue that for the other $V_{0,k}^{(+)}$ ($V_{k,0}^{(+)}$) perturbations the SG “field” $\varphi \propto \hat{\Phi}$ ($\Phi$) is a pseudo-scalar (cf. [38]). This is not just true for the perturbed bosonic gaussian CFTs, but also for the fermionic ones, where it is in fact obvious: There we have direct access to the “underlying” fermions and by bosonization the fermion current $J^{\mu} \equiv \bar{\Psi} \gamma^{\mu} \Psi \propto \epsilon^{\mu\nu} \partial_{\nu} \varphi$. The fact that $J^{\mu}$ is a current, not a pseudo-current, implies that $\varphi$ must be a pseudo-scalar.

We conjecture that the $n$-th breather is created by $V_{0,n}^{n^{(+)}}$ in the UV limit. Since the $n$-th breather can be interpreted as a bound state of $n$ lightest breathers $[66]$, this is in accord with the fact that a suitably defined $: (V_{0,1}^{(-)})^n :$ equals $V_{0,n}^{n^{(+)}}$.

\footnote{This happens at $r = 1/\sqrt{2}$, the self-dual point, where the bosonic gaussian model has a level one $\hat{su}(2) \times \hat{su}(2)$ Kac-Moody symmetry (see e.g. [22]). Under a $V_{0,1}^{(+)}$ perturbation a global $SU(2)$ survives, with respect to which the conformal fields $V_{\pm 1,0}$ and $V_{0,1}^{(-)}$ — and thus, according to our identifications, also the soliton/antisoliton and the first breather — form a triplet. This implies that their finite-volume levels must be exactly degenerate for all $L$, and that the SG amplitudes at $\beta^2 = 2\pi$ should be invariant under permutations of the three particles. This is indeed true for our sign of the SG $S$-matrix. It was argued on different grounds in [52] (cf. also [54]) that $SG(\sqrt{2\pi})$ has an $SU(2)$ “isospin” symmetry. The argument involves viewing $SG(\sqrt{2\pi})$ as a subtheory of the $SU(2)$ Schwinger model in a certain limit. The solitons turn out to be “quark-antiquark” bound states in terms of the Schwinger model fermions. Therefore they should be bosons! (This was not commented upon in [53].)
Finally we should comment on what looks perhaps a bit strange in the above sketched picture of the finite-volume spectrum as $\beta^2$ goes below $4\pi$. Namely, the energy level which at $\beta^2 = 4\pi$ corresponds to $s(\frac{1}{2})\bar{s}(-\frac{1}{2}) - s(-\frac{1}{2})\bar{s}(\frac{1}{2})$, “belongs” to the breather when $\beta$ is just infinitesimally smaller. But there is still a lowest anti-symmetric 2-particle state; what is its energy? The only possibility is that its energy gets “bumped up” to $2\epsilon_{3/2}(\rho)$. [Similarly, one of each of the two zero momentum 2-particle levels of energy $2\epsilon_{n+\frac{1}{2}}$, $n = 1, 2, \ldots$] Naively it looks as if the spectrum changes discontinuously. However, an infinitely weak bound state in finite volume is not distinguishable from an unbound 2-particle state, so that there is no observable discontinuity at $\beta^2 = 4\pi$; sets of energy levels of given symmetry properties change completely smoothly. Note also that there is nothing wrong with the fact that the (zero momentum) breather starts its existence with energy $\frac{2\pi}{L} 2\epsilon_{\frac{1}{2}} = \sqrt{(2m)^2 + (\frac{2\pi}{L})^2}$, which differs at $O(L^{-2})$ from $2m$, whereas finite-size mass corrections are supposedly exponentially small in $L$. The point is that for a weak bound state of mass $m_1 = 2m - \delta, \delta \ll 1$, all we know is that the finite-size mass corrections are $O(e^{-\delta m L})$ for $\delta m L \gg 1$; when $\delta$ is infinitesimal they can certainly be $O(L^{-2})$ for all $L$.

5. Discussion and Further Examples

In retrospect the inequivalence of the SG and MT models is rather obvious. Let us briefly summarize the main points and why, we think, this was not noticed much earlier. The equality of correlation functions in certain sectors of SGM and MTM appeared to be good evidence for SGM=MTM at a time when not many examples of distinct QFTs with nontrivial subsectors of identical operators were known. With the advent of “modern CFT” many such examples were discovered, but then the emphasis on full modular invariance (due to the influence of string theory) prevented an appreciation of the significance of the fermionic gaussian models. Furthermore, the fact that various QFTs can be constructed out of the same compactified boson was often believed to mean that these theories are “equivalent”; but, if they contain operators that are not mutually local, they must be distinct. It is also important to realize that in the (or at least one) right way of looking at the SG and MT models the SG “field” $\varphi$ is compactified. This is crucial for understanding the “higher harmonic” analogs of the SG and MT models, where several periods of the cosine potential fit on the circle on which $\varphi$ lives, so that these models are “kink theories” with a degenerate vacuum in infinite volume.
Our story of course also has a statistical mechanics version. Among lattice models whose critical points are described by \( c=1 \) CFTs one must not only distinguish between models in the orbifold and bosonic gaussian families of universality classes, but also the latter from the fermionic gaussian one. This is more subtle, since the bosonic and fermionic gaussians have the same generic \( O(2) \times O(2) \) symmetry, and even a quantitative measure like the free energy per site in the thermodynamic limit will not distinguish between these universality classes (the same situation arises in the case of IRF lattice models related by “orbifolding” \([9]\)). One needs critical exponents corresponding to fermionic operators or (subleading) eigenvalues of the transfer matrix corresponding to levels in the “odd sectors” to distinguish them. Some remarks on consequences of our results for various scaling limits of the spin \( \frac{1}{2} \) XYZ chain are given in appendix B.

The bosonic and fermionic gaussian CFTs are related by a twist (followed by a projection), and we discussed the various repercussions this has in the perturbed theories. Our analysis should be extended to other pairs of non-scale-invariant QFTs defined as “the same” perturbation of CFTs related by twists, alternatively a (generalized) orbifold construction. There are numerous such examples, and we will now briefly discuss several of them, leaving details for future work.

One class of examples was already encountered in sect. 3, namely the \( V_{0,k}^{(+)} \) perturbations, \( k \in \mathbb{N} \), of the gaussian CFTs. We note that the operator \( V_{0,k}^{(+)} = \sqrt{2} \cos(2kr\Phi) \) at \( r \) is identical — in terms of its behaviour in correlation functions — to \( V_{0,1}^{(+)} \) at radius \( r' = kr \). Moreover, the gaussian CFT (bosonic or fermionic) at \( r \) can be obtained from the one at \( r' \) by a \( \mathbb{Z}_k \)-twist \([19][2]\), the \( \mathbb{Z}_k \) being the discrete subgroup of the \( U(1) \) symmetry (see subsect. 2.1) of the model at \( r' \), generated by \((\Phi, \tilde{\Phi}) \rightarrow (\Phi + 2\pi r'/k, \tilde{\Phi})\). We may therefore think of the theories \( A_{b,f}(r, V_{0,k}^{(+)}) \) with \( k \geq 1 \) as \( \mathbb{Z}_k \)-“massive orbifolds” of \( A_{b,f}(kr, V_{0,1}^{(+)}) \) (or \emph{vice versa}; then the \( \mathbb{Z}_k \) is the subgroup of the \( \tilde{U}(1) \) that is not broken by a \( V_{0,k}^{(+)} \) perturbation). This point of view is rather useful in practice. For instance, it allows us to obtain \([39]\) the full exact finite-volume spectra and S-matrices of the kink theories \( A_{b,f}(1/k, V_{0,k}^{(+)}) \) by twisting those of \( A_{b,f}(1, V_{0,1}^{(+)}) \), the latter being the SGM at \( \beta^2 = 4\pi \) and the free MTM, respectively.

Another class of examples is provided by perturbations of the \( c=1 \) orbifold CFTs (see \emph{e.g.} \([2]\)). These CFTs are related to the bosonic gaussian ones through a \( \mathbb{Z}_2 \) twist, in this case with respect to \( RR: (\Phi, \tilde{\Phi}) \rightarrow (-\Phi, -\tilde{\Phi}) \). They are bosonic, \emph{i.e.} do not contain any fields of half-odd-integer Lorentz spin, and just have a discrete \( \mathbb{D}_4 \) global symmetry (the symmetry group of the square). Contrary to \( V_{m,0}^{(-)} \) and \( V_{0,n}^{(-)} \), which are
projected out, the operators $V_{m,0}^{(+)}$ and $V_{0,n}^{(+)}$ are still part of the orbifold operator algebra (OPA), and can be used to generate integrable perturbations. However, the sign of the perturbation now matters in contrast to the gaussian case, the two perturbations being related by the continuum analog of the Kramers-Wannier duality of the underlying Ashkin-Teller model. The corresponding theories $A^{(+)}_{\text{orb}}(r, V_{0,k}^{(+)})$, $k = 1, 2, \ldots \prec \sqrt{2}/r$, have to be studied separately, as they differ in their kink-structure and hence also their $S$-matrix and finite-volume spectrum [28].

A particular case of the above perturbed orbifold CFTs that has already been analyzed in some detail [14] [16] is that of $A^{(+)}_{\text{orb}}(\sqrt{N/2}, V_{2,0}^{(+)})$, $N = 2, 3, \ldots$ [The sign choice for the direction of the perturbation is a convention. We take the plus sign to correspond to the case where the vacuum of the theory is non-degenerate in infinite volume; in the opposite direction it is doubly degenerate.] These theories have been argued [14] to be described by the so-called $D_N^{(1)}$ diagonal $S$-matrix theories, which can be thought of as massive orbifolds of the SGM at the points $\beta^2 = 8\pi/N$ where soliton-antisoliton scattering is reflectionless. We pointed out that there are sign differences between the scattering amplitudes of the two “fundamental particles” (which are bosons) in the $D_N^{(1)}$ theory and those of the solitons in the corresponding SGM. For instance, in the $N = 2$ case the theory contains only the two self-conjugate fundamental particles, with the nonzero amplitudes $S_{11} = S_{22} = 1 = -S_{12} = -S_{21}$. Thus $A^{(+)}_{\text{orb}}(r = 1, V_{2,0}^{(+)})$ clearly describes two decoupled copies of the Ising field theory, and the partition function is $Z_{D_2^{(1)}}(\rho) = Z_{\text{IFT}}^2(\rho)$, cf. (4.5).

It is natural to look for the full one-parameter family of factorizable $S$-matrices describing $A^{(+)}_{\text{orb}}(r, V_{2,0}^{(+)})$ for arbitrary $r$, and see if and how they are related to the $S$-matrix of SG($\beta = \sqrt{4\pi/r}$), namely $A_b(\sqrt{2}, V_{1,0}^{(+)})$. The $S$-matrix should be $D_4$-symmetric for any $r$, since this symmetry of the UV CFT is not broken by the perturbation. A two-parameter family of $D_4$-symmetric “elliptic $S$-matrices”, formally describing the scattering of a doublet of particles, was in fact constructed by Zamolodchikov [58]. It is now believed that elliptic solutions to the bootstrap constraints are not relevant for QFT, i.e. only their degenerate trigonometric limits may describe relativistic QFTs. Zamolodchikov noticed that in one trigonometric limit, $l \rightarrow 0$ in his notation, the $D_4$ symmetry is enlarged to $O(2)$ and the $S$-matrix reduces to that of the SGM (or MTM, depending on the overall sign). But

\[^{22}\text{However, when writing [14] we still had the wrong impression that the latter are identical to the amplitudes of the fermions of the MTM.}\]
there is another limit, \( l \to 1 \), which is not \( O(2) \)-symmetric! In this limit the nonvanishing \( S \)-matrix elements for the doublet of self-conjugate particles are

\[
S_{11}^{11}(\theta) = S_{22}^{22}(\theta) = \frac{1}{2} (+s + t + r)(\theta),
\]

\[
S_{11}^{22}(\theta) = S_{22}^{11}(\theta) = \frac{1}{2} (-s + t + r)(\theta),
\]

\[
S_{12}^{21}(\theta) = S_{21}^{12}(\theta) = \frac{1}{2} (-s - t + r)(\theta),
\]

\[
S_{12}^{12}(\theta) = S_{21}^{21}(\theta) = \frac{1}{2} (+s - t + r)(\theta),
\]

in terms of the SG \( ss \) and \( s\bar{s} \) transmission and reflection amplitudes \( s, t \) and \( r \), respectively (with the correct sign). This is our conjecture for the exact \( S \)-matrix of the fundamental particles of the theory \( A_{\text{orb}}^{(+)}(r = \sqrt{4\pi/\beta}, V_{2,0}^{(+)} \) ). It seems to be true in general, cf. \cite{[69],[70]}, that the \( S \)-matrix elements of an orbifold theory are linear combinations of those in the “original” theory. Note that when \( \beta = \sqrt{8\pi/N}, N = 2, 3, \ldots \), the amplitudes (5.1) coincide with those of the \( D_N^{(1)} \) scattering theory (in a 1-particle basis where the fundamental particles are self-conjugate even for \( N \) odd, cf. \cite{[14]}). The bound state amplitudes can be obtained via the bootstrap; the number and masses of the bound states are exactly as in the corresponding SGM, but their interpretation as composites of the fundamental particles is different.

The other “thermal” perturbations of the orbifold CFTs deserve further investigation. In addition, there are “magnetic” perturbations \cite{[38],[71]} which have no analog in the gaussian case, being induced by twist operators that cannot be expressed as exponentials of a free boson.

Finally, we would like to discuss theories that can be obtained by “fermionic twists” from bosonic models. To gain some insight about such theories, consider first the CFTs at the UV limits of the SGM and its fermionic partner, the MTM. As discussed in sect. 2, the corresponding partition functions \( Z_{b,t}(q,r) \) seem to cover all \( \Gamma' \)-invariant partition functions of \( c=1 \) unitary CFTs having (at least) \( U(1) \) symmetry. [By this statement we mean the following. Let \( \chi_\Delta(q) \) be the character of the Virasoro irrep of central charge \( c=1 \) and highest weight \( \Delta \geq 0 \). We conjecture that the only \( Z(q) = \sum_{(\Delta,\bar{\Delta})} N_{\Delta,\bar{\Delta}} \chi_\Delta(q)\bar{\chi}_\bar{\Delta}(\bar{q}) \) with \( N_{\Delta,\bar{\Delta}} \in \mathbb{Z}_{\geq 0}, N_{1,0} \geq N_{0,0} = 1 \) satisfying \( Z(q) = Z(e^{4\pi i}q) = Z(\bar{q}) \), are \( Z_{b,t}(q,r) \) for some \( r \).] We are not aware of a proof of this statement (partial support, related to the particular case of full \( \Gamma \)-invariance, can be found in \cite{[23],[72]}). In any case, our discussion (sect. 2) of consistent OPAs for \( U(1)\times U(1) \)-symmetric CFTs suggests a deep general connection.
between (sub)modular invariance of the partition function and local properties of the corresponding OPA. Namely, given a $\Gamma'$-invariant partition function, there appears to exist a (not necessarily unique) consistent OPA — in particular satisfying mutual locality in a “maximal” way — giving rise to that partition function.

Such a connection is alluded to in many places in the CFT literature (see e.g. [15][73]) though always, it seems, restricted to the case of full modular invariance. We here draw attention to the more general case of $\Gamma'$-invariance whose consequences do not seem to have been systematically explored so far.

In some cases the construction of $\Gamma'$-invariant fermionic CFTs is rather trivial. For example, the Neveu-Schwarz sector (including the fermionic components of the super-fields) of any super-CFT fits the bill. A bit more generally, one can take the “Neveu-Schwarz sectors” of CFTs invariant under (not necessarily supersymmetric) fermionic $W$-algebras [74], namely chiral algebras containing chiral fields of half-odd-integer spin. However, this class of theories does not automatically cover CFTs that contain fermionic fields which are not chiral. To discover such CFTs using “chiral techniques” one apparently has to consider twisted bosonic $W$-algebras (see footnote 24 below).

Alternatively, putting aside for a moment the question of full consistency of the CFT, one might attempt to simply classify $\Gamma'$-invariant partition functions built of characters of the Virasoro or any other extended chiral algebra in the same way as full modular invariance has been analyzed. For Virasoro minimal models this constraint is highly restrictive. Although we do not have a complete proof at this point, we believe that there exist only two series of “fermionic” solutions, one of them exceptional (to be contrasted with five series in the bosonic case [14], three of which are exceptional). Explicitly, in the notation of [14], the $\Gamma'$- (but not $\Gamma$-) invariant partition functions we find read

\[
p' = 4\rho \quad (\rho \geq 1)
\]

\[
p' = 4\rho + 2 \quad (\rho \geq 1)
\]

\[
p' = 12
\]

We will refer to the partition functions on the first two lines as members of the fermionic $D$-series, the ones on the last line as the fermionic $E_6$-series. As in the bosonic case, the
unitary fermionic series of central charge \( c = 1 - \frac{6}{m(m+1)} \) corresponds to \( p = m, p' = m + 1 \) or \( p = m + 1, p' = m \), with \( m = 3, 4, \ldots \). There is one model in the unitary fermionic \( D \)-series for each \( m = 3, 4, \ldots \), and two more unitary fermionic models are found in the \( E_6 \)-series, i.e. at \( m = 11, 12 \).

Let us make some brief comments on (5.2):

(i) As discussed above, given a \( \Gamma' \)-invariant as in (5.2) does not quite guarantee the existence of a consistent CFT having that invariant as a partition function. In particular, one has to check that an OPA whose “content” only is encoded in the invariant satisfies all required properties. First note that invariance under the modular transformation \( T^2 \) implies \( s \in \frac{1}{2} \mathbb{Z} \) for all fields in the model. Next, closure of all the OPAs encoded in (5.2) is easily verified using the (chiral) fusion rules of the minimal models [10], amended with conservation of \((-1)^F (= 1 \text{ for fields of spin } s \in \mathbb{Z}, -1 \text{ for } s \in \mathbb{Z} + \frac{1}{2})\). Conservation of \((-1)^F \) also ensures mutual locality. However, associativity is much more difficult to prove, requiring essentially the calculation of all OPE coefficients. This problem in the fermionic case, which by definition involves a non-diagonal submodular invariant partition function, is similar to that of computing OPEs in non-diagonal bosonic CFTs. A satisfactory method to solve this problem in both cases still remains to be found (cf. [75] and references therein). We are nevertheless rather confident that all the partition functions listed in (5.2) do correspond to consistent CFTs.

(ii) The fields of \((-1)^F = 1\) in the model labeled by \((p, p')\) in the fermionic \( D \)-series coincide with the \( \mathbb{Z}_2 \)-even fields in the corresponding bosonic \( A \)- and \( D \)-model, and the same relation holds between corresponding fermionic and bosonic \( E_6 \)-models.\(^{23}\) This suggests that the series involved are related by \( \mathbb{Z}_2 \)-twists.

(iii) The fermionic \( D \)-series model \((p, p' = 4\rho)\) is minimal and diagonal with respect to the fermionic \( W(2, \Delta_{1,p-1}) \) chiral algebra, in the notation of [14], where \( \Delta_{1,p-1} = (p - 2)(p' - 2)/4 \in \mathbb{N} - \frac{1}{2} \). Similarly the fermionic models in the \( E_6 \)-series are minimal and diagonal with respect to \( W(2, \frac{p}{2} - 2, p - 3, \frac{5(p-2)}{2}) \). (In fact, some of the corresponding \( \Gamma' \)-invariants have been presented in [14].) The sums of Virasoro characters \( \chi_{rs} + \chi_{r,p-s} \)

\(^{23}\) Cf. [14] [15] for the \( \mathbb{Z}_2 \) symmetry in the bosonic models, present in the \( A, D \), and \( E_6 \)-models when one of \( p, p' \) is even. The fact that \( p' \) must be even in the first two lines of (5.2), allowing for the \( \mathbb{Z}_2 \) symmetry that any fermionic model must obviously have, motivated us to call the corresponding series of theories the \( D \)-series; the bosonic \( A \)-series contains also models with \( p, p' \) both odd, which have no fermionic counterparts.
and \( \chi_{1s} + \chi_{5s} + \chi_{7s} + \chi_{11s} \) which appear on the first and third line of (5.2), respectively, are characters of the above chiral algebras.

(iv) The unitary fermionic model \( m=3 \) is just the free massless Majorana fermion, and the \( m=4 \) model is the first model in the \( N=1 \) superconformal unitary series (containing only the Neveu-Schwarz sector), from which the tricritical Ising CFT is obtained by a fermionic twist. These two models are special in that their global symmetry is \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), whereas all other models in the fermionic unitary series are only \( \mathbb{Z}_2 \)-symmetric. This is due to the fact that for \( m=3, 4 \) all fields have left and right Kac labels of the form \((n,1)\) or \((1,n)\), so that one can define separate left and right charges \((-1)^F\) and \((-1)^\bar{F}\), not just their product \((-1)^F = (-1)^{2s}\). The action of each one of them on the bosonic fields in the models has the same effect as that of the Kramers-Wannier duality transformation in the corresponding bosonic CFTs. But while duality maps the full local operator algebra of the bosonic CFTs onto a different algebra, exchanging order and disorder operators (cf. e.g. appendix E of \([10]\)), it is promoted to a global \( \mathbb{Z}_2 \) symmetry in the fermionic \( m=3, 4 \) models. These observations become important when considering perturbations of these CFTs, see below.

(v) The unitary fermionic models with \( m \geq 5 \) are apparently new. We now discuss in some detail the \( m=5 \) case, the fermionic partner of the bosonic tetracritical Ising (\( A \)-series) and 3-state Potts (\( D \)-series) CFTs. Note that the partition function of the fermionic \( m=5 \) model can be rewritten as

\[
Z^D_{f} = Z^A_{b} - (|\chi_{1,2} - \chi_{4,2}|^2 + |\chi_{2,2} - \chi_{3,2}|^2) ,
\]

where \( Z^A_{b} \) is the partition function of the tetracritical Ising CFT, in a self-explanatory notation. (Formulas similar to (5.3) exist for all the partition functions on the second line of (5.2).) The expression in parenthesis in (5.3), evidently \( \Gamma' \)-invariant by itself, is \([13]\) the “partition function” \( Z^D_{b}(T,T) \) of the 3-state Potts CFT with \( \mathbb{Z}_2 \)-twisted boundary conditions on the complex spin field \( \sigma \) of the model (namely, \( \sigma(z + \omega_i, \bar{z} + \omega_i^*) = \sigma(z, \bar{z})^* \) for \( i = 1, 2 \), where \( \omega_i \) are the two periods of the torus). In this notation, the partition function of the 3-state Potts CFT is \( Z^D_{b} = Z^D_{b}(P,P) \), where \( P = \) periodic. Moreover, we note that

\[
\frac{1}{2} \left[ Z^D_{b}(P,P) + Z^D_{b}(P,T) + Z^D_{b}(T,P) \pm Z^D_{b}(T,T) \right] = \begin{cases} 
Z^A_{b} \\
Z^D_{f} 
\end{cases} \quad (5.4)
\]
(cf. [13] for the upper case), showing how the tetracritical Ising and fermionic \( m=5 \) CFTs are obtained by twisting the 3-state Potts CFT.\[^{23}\] In terms of the \( m=5 \) nonlocal “theory” containing the union of fields in the bosonic \( A_5, D_5 \) and fermionic models, the \( \mathbb{Z}_2 \)-odd \( D_5/A_5 \) fields can be thought of as order/disorder operators, their (double-valued) OPEs generating the fermionic \((-1)^F = -1\) fields. This is similar to the \( m=3 \) case, but in the case at hand the fermions are not free, having conformal dimensions \((\frac{1}{40}, \frac{21}{40}), (\frac{21}{40}, \frac{1}{40}), (\frac{1}{8}, \frac{13}{8}), (\frac{13}{8}, \frac{1}{8})\).

Given some new fermionic CFTs, the next step is to consider their relevant perturbations. Since we restrict ourselves to spinless perturbations, where the perturbing field is in the \((-1)^{F}=1\) sector of the theory, such fermionic perturbed CFTs necessarily have bosonic partners where “the same” perturbing field is in the even sector of the unperturbed bosonic CFT. The difference between the fermionic and bosonic theories is expected to be in general more drastic than in the case of the MT vs. SG models, where even though the operator algebras of the two theories are different, the particle spectrum is identical and there are only sign differences in certain \( S \)-matrix elements.

One particularly interesting family of theories is the \( \phi_{1,3} \)-perturbed models in the unitary series. Perturbations in the massive direction of the bosonic \( A \)-series models, known to lead to the restricted sine-Gordon (RSG) models \[^{77}\], describe the scattering of a multiplet of (bosonic) kinks \[^{70}\]. (The simplest such kink theory, the \( m=3 \) case, is just the IFT in the low-temperature phase \[^{28}\].) Correspondingly, one might expect the \( \phi_{1,3} \)-perturbed fermionic models to describe theories of fermionic kinks.

For \( m > 3 \) this is presumably true. For \( m=3 \), however, the enlarged symmetry of the theory (see (iv) above) shows that the sign of the perturbation is irrelevant — the sign of the fermion mass term in the lagrangian is just a convention — so there is no kink phase. For higher models in the fermionic series the direction of the perturbation does matter, presumably leading to a kink phase in one direction. It is rather clear that in this phase the degeneracy of the vacuum and therefore the kink structure will differ from that of the corresponding RSG models (implying, in particular, that the \( S \)-matrices of the perturbed

---

\[^{24}\] Recall that the 3-state Potts CFT, \( i.e. \) the bosonic \( D_5 \) model, is minimal and diagonal with respect to the \( W_3 \) algebra. It was noted in \[^{76}\] that the combinations of Virasoro characters \( \chi_{1,2} - \chi_{4,2} \) and \( \chi_{2,2} - \chi_{3,2} \) appearing in \[^{53}\] are essentially \( T \)-transformed characters of the twisted \( W_3 \) algebra, suggesting that this twisted chiral algebra plays a role in both the bosonic \( A_5 \) and fermionic \( D_5 \) models.
bosonic and fermionic theories will differ by more than just signs). In fact, since the
$m - 1$ degenerate vacua in the $m$-th RSG model are presumably created by the spinless
fields $\phi_{rr}$ ($r = 1, 2, \ldots, m - 1$) in the bosonic UV CFT, there are only \(\left\lfloor \frac{m}{2} \right\rfloor\) vacua
in the perturbed fermionic CFT, created by the even fields $\phi_{rr}$ ($r = 1, 3, \ldots, 2\left\lfloor \frac{m}{2} \right\rfloor - 1$)
that survive the twist. The relation between these massive theories is therefore similar
to that between $SG(\beta, k)$ with $k \in 2\mathbb{N}$ and its fermionic partner (see sect. 3 and [39]),
and requires further investigation.\(\footnote{In the \textit{massless} direction of the $\phi_{1,3}$ perturbation we expect RG flows [33]-[36] between the fermionic unitary CFTs $m$ and $m - 1$ ($m = 4, 5, \ldots$ in the fermionic $D$-series, $m = 12$ in the $E_6$-series), the reasoning being along the same lines as in [73].}

\textbf{Acknowledgements}

We would like to thank T. Banks, M. Douglas, P. Fendley, D. Friedan, V. Korepin,
A. LeClair, B. McCoy, M. Roček, K. Schoutens, R. Shrock, D. Spector, H. Tye, E. Verlinde,
and S. Yankielowicz for discussions. The work of T.R.K. is supported by NSERC and the
NSF, and that of E.M. by the NSF, grant no. 91-08054.

\textbf{Appendix A. Conventions}

When working in Minkowski space $x^\mu$, $\mu = 0, 1$, our signature is $(+, -)$. For the Dirac
matrices, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, we use the representation
\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (A.1)

Wick rotation to euclidean space corresponds to $(x^0, x^1) \rightarrow (x_1 = x^1, x_2 = ix^0)$, and
complex euclidean coordinates are defined by $(z, \bar{z}) = \frac{1}{2}(x_2 + ix_1, x_2 - ix_1)$.

For the free massless scalar field of the gaussian CFT we adopt the normalization (and
choice of regulating mass) used in [12], except that we denote the field by $\Phi(z, \bar{z})$ instead
of $X(z, \bar{z})$. Explicitly, the 2-point function is
\[
\langle \Phi(w, \bar{w})\Phi(z, \bar{z}) \rangle = -\frac{1}{2} \ln |w - z|,
\] (A.2)

\footnote{A first attempt to find the $S$-matrix for the $m=4$ case was made in [16].}
so that in Minkowski space $\varphi = \Phi/\sqrt{\pi}$ is described by the usual lagrangian $\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi$.

The correlation functions of the chiral fields $\phi(z)$ and $\bar{\phi}(\bar{z})$ in the decomposition $\Phi(z, \bar{z}) = \frac{1}{2}(\phi(z) + \bar{\phi}(\bar{z}))$ are $\langle \phi(w)\phi(z) \rangle = -\ln(w - z)$ and $\langle \bar{\phi}(\bar{w})\bar{\phi}(\bar{z}) \rangle = -\ln(\bar{w} - \bar{z})$.

The chiral fields have the mode expansion

$$
\phi(z) = q - i\alpha_0 \ln z + i \sum_{n \neq 0} \frac{1}{n} \alpha_n z^{-n}, \quad \bar{\phi}(\bar{z}) = \bar{q} - i\bar{\alpha}_0 \ln \bar{z} + i \sum_{n \neq 0} \frac{1}{n} \bar{\alpha}_n \bar{z}^{-n},
$$

where the nonvanishing commutators of the operators $q$, $\alpha_n$ and $\bar{q}$, $\bar{\alpha}_n$ are

$$[q, \alpha_0] = [\bar{q}, \bar{\alpha}_0] = i, \quad [\alpha_n, \alpha_m] = [\bar{\alpha}_n, \bar{\alpha}_m] = n\delta_{n+m,0}. \quad (A.4)$$

Since all left- and right-moving fields commute, the normal ordering of the $V_{m,n}$ in (2.1) follows from that of a chiral “half”

$$: e^{\phi(z)} : \equiv e^{\phi_- (z)} e^{q z^{-i\alpha_0}} e^{\phi_+ (z)}, \quad (A.5)$$

where $\phi_{\pm}(z)$ is the part of $\phi(z)$ that contains only modes with $n > 0$, respectively, $n < 0$.

**Appendix B. The Statistical Mechanics Connection**

Taking the scaling limit of certain types of one-dimensional quantum spin chains (or various two-dimensional lattice models) is often regarded as the way to rigorously define and study (1+1)-dimensional relativistic QFTs. In particular, the spin $\frac{1}{2}$ XYZ Heisenberg chain is often mentioned as an appropriate regularized system whose scaling limit defines the SGM. We will here examine this issue slightly more carefully, though still at a rather heuristic level.

Consider the hamiltonian

$$
H(\gamma, \varepsilon, \vec{h}) = \sum_{n=1}^{N} [(1 + \varepsilon)\sigma_n^x \sigma_{n+1}^x + (1 - \varepsilon)\sigma_n^y \sigma_{n+1}^y + \cos \gamma \sigma_n^z \sigma_{n+1}^z + \vec{h} \cdot \vec{\sigma}_n], \quad (B.1)
$$

where the $\vec{\sigma}_n$ are Pauli matrices at each site of the chain. Throughout the discussion below $N$ is taken to be even, $0 \leq \gamma < \pi$, and periodic boundary conditions $\vec{\sigma}_{N+1} = \vec{\sigma}_1$ are imposed. $H(\gamma, 0, 0)$ is then the (antiferromagnetic) XXZ chain, whose scaling limit is known [8] to be described by the bosonic gaussian CFT at $r = r(\gamma) = [2(1 - \gamma/\pi)]^{-1/2}$ (or its dual). The electric charge $m$ of states in the CFT is just the conserved total spin $S^z = \frac{1}{2} \sum_n \sigma_n^z$ in the $z$-direction (which takes integer values when $N$ is even). The
additional $U(1)$ symmetry of the CFT, corresponding to the magnetic charge $n$, is not manifest in the XXZ hamiltonian.

Now, based on the known large-distance behaviour of certain correlation functions in the XXZ antiferromagnet one identifies the CFT operators $V_{\pm 1,0}$ as the continuum versions of $\sigma_n^x \equiv \sigma_n^x \pm i \sigma_n^y$, respectively (up to symmetry transformations and possibly subleading irrelevant operators). We are therefore led to identify the scaling limit of the XXZ chain in a transverse field, $H(\gamma, 0, h\hat{n}_\perp)$ with $h \to 0$ (where $\hat{n}_\perp \cdot \hat{n}_z = 0$), as being described by $A_b(r(\gamma), V^{(+)}_{1,0}) = SG(\beta = \sqrt{2/(\pi - \gamma)})$.

This claim may sound surprising at first, since it is believed (see e.g. the first reference in [78]) that the XXZ chain in a transverse field is not integrable (contrary to the longitudinal case). However, such a situation would not be new: The Ising model in a magnetic field is apparently non-integrable on the lattice, but its scaling limit (at $T = T_c$) is conjectured to be an integrable QFT. Evidence supporting the claim in the Ising case were provided [80] by numerical simulations of the corresponding chain and lattice models, and we think it is worthwhile to perform similar checks of our prediction in the case of the XXZ chain in a transverse field.

Next, consider the scaling limit of the XYZ chain in the absence of a magnetic field, namely $H(\gamma, \varepsilon, 0)$ with $\varepsilon \to 0$, which differs from the XXZ hamiltonian by a $\sum_n (\sigma_n^x \sigma_{n+1}^x + \sigma_n^- \sigma_{n+1}^-)$ “perturbation”. It is now natural to conjecture that it is described by the perturbed CFT $A_b(r(\gamma), V^{(+)}_{2,0})$, when $0 < \gamma < \pi$ (cf. [38]). This theory is identified (see [28] and sect. 3.1) as $SG(\beta = \sqrt{8(\pi - \gamma)}, 2)$, namely a “massive orbifold” of $SG(\beta = \sqrt{8(\pi - \gamma)})$ in which the vacuum is doubly-degenerate. This latter fact is consistent with the asymptotic double degeneracy of the XYZ chain ground state energy in the thermodynamic limit [81].

Finally, let us comment about the relation between the XYZ hamiltonian and the lattice MTM. The MTM lattice hamiltonian can be obtained [32] from that of the XYZ chain by a Jordan-Wigner transformation. However, the boundary conditions on the resulting fermion operators have to be handled with care. Exactly like in the case of the Ising model versus the lattice free Majorana fermion, in order to preserve the original boundary conditions that were imposed on the bosonic variables of the XYZ chain one has to consider “simultaneously” different boundary conditions on the fermions [82]. This is precisely the lattice analog of the GSO projection, or “twist” with respect to the fermion number (see sect. 2), that goes back to [83] in the Ising case. The continuum MTM is obtained by taking the scaling limit of the Jordan-Wigner transformed XYZ chain with one choice
of boundary conditions on the fermions (imposing anti-periodic boundary conditions one
probes the full mutually-local operator algebra of the theory). In the perturbed CFT lan-
guage, this procedure of “undoing” the GSO projection takes us from $A_b(r(\gamma), V^{(+)}_{2,0})$ to
$A_f(r(\gamma), V^{(+)}_{2,0})$; the latter being the MTM is in agreement with the discussion of sect. 3.
References

[1] J. Fröhlich, in: *Renormalization Theory* (Erice 1975), ed. G. Velo and A.S. Wightman (Reidel, Dordrecht, 1976); J. Fröhlich and E. Seiler, Helv. Phys. Acta 49 (1976) 889
[2] W. Thirring, Ann. Phys. (NY) 3 (1958) 91
[3] V. Glaser, Nuovo Cimento 9 (1958) 990
[4] K. Johnson, Nuovo Cimento 20 (1961) 773
[5] A.S. Wightman, in: *Cargèse Lectures in Theoretical Physics*, 1964, ed. M. Levy (Gordon and Breach, New York, 1966)
[6] B. Klaiber, in: *Lectures in Theoretical Physics* (Boulder 1967), ed. A. Barut and W. Brittin (Gordon and Breach, New York, 1968), Vol. X, part A
[7] G.F. Dell’Antonio, Acta Phys. Austriaca 43 (1975) 43
[8] S. Coleman, Phys. Rev. D11 (1975) 2088
[9] S. Mandelstam, Phys. Rev. D11 (1975) 3026
[10] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333
[11] J.L. Cardy, in: *Fields, Strings, and Critical Phenomena*, Les Houches 1988, ed. E. Brézin and J. Zinn-Justin, (North Holland, Amsterdam, 1989)
[12] P. Ginsparg, in: *Fields, Strings, and Critical Phenomena*, Les Houches 1988, ed. E. Brézin and J. Zinn-Justin, (North Holland, Amsterdam, 1989)
[13] J.L. Cardy, Nucl. Phys. B270 (1986) 186; *ibid.* B275 (1986) 200
[14] A. Cappelli, C. Itzykson and J.-B. Zuber, Nucl. Phys. B280 (1987) 445
[15] D. Gepner, Nucl. Phys. B287 (1987) 111
[16] P. Fendley, Phys. Lett. 250B (1990) 96
[17] P. Fendley and K. Intriligator, Nucl. Phys. B372 (1992) 533, “Scattering and Thermodynamics in Integrable $N=2$ Theories”, BUHEP-92-5/HUTP-91-A067 (1992)
[18] D. Friedan and S. Shenker, in: *Conformal Invariance and Applications to Statistical Mechanics*, ed. C. Itzykson et al (World Scientific, Singapore, 1988), p. 578
[19] L. Dixon, D. Friedan, E. Martinec and S.H. Shenker, Nucl. Phys. B282 (1987) 13
[20] F. Gliozzi, J. Scherk and D. Olive, Nucl. Phys. B122 (1977) 253
[21] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. (NY) 120 (1980) 253
[22] M. Karowski and H.J. Thun, Nucl. Phys. B130 (1977) 295
[23] V.E. Korepin, Teor. Mat. Fiz. 41 (1979) 169
[24] D. Bernard and A. LeClair, Commun. Math. Phys. 142 (1991) 99
[25] M. Karowski, Nucl. Phys. B153 (1979) 244
[26] Al.B. Zamolodchikov, Nucl. Phys. B342 (1990) 695
[27] T.R. Klassen and E. Melzer, Nucl. Phys. B362 (1991) 329
[28] T.R. Klassen and E. Melzer, “Kinks in Finite Volume”, Cornell/Stony Brook preprint CLNS-92-1130/ITP-SB-92-01, to appear in Nucl. Phys. B
[29] E.B. Kiritsis, Phys. Lett. 217B (1989) 427

46
[30] D.Z. Freedman and K. Pilch, Phys. Lett. 213B (1988) 331, and Ann. Phys. (NY) 192
(1989) 331.
[31] G.F. Dell’Antonio, Y. Frishman and D. Zwanziger, Phys. Rev. D6 (1972) 988.
[32] A. Luther, Phys. Rev. B14 (1976) 2153.
[33] H. Neuberger, A.J. Niemi and G.W. Semenoff, Phys. Lett. 181B (1986) 244; H. Kawai, D.C. Lewellen and S.-H. Tye, Nucl. Phys. B288 (1987) 1; J. Bagger, D. Nemeschansky, N. Seiberg and S. Yankielowicz, Nucl. Phys. B289 (1987) 53.
[34] D. Friedan, Z. Qiu and S.H. Shenker, Phys. Lett. 151B (1985) 37.
[35] A.B. Zamolodchikov, Sov. J. Nucl. Phys. 46 (1987) 1090.
[36] A.W.W. Ludwig and J.L. Cardy, Nucl. Phys. B285 (1987) 687.
[37] A.B. Zamolodchikov, Adv. Stud. Pure Math. 19 (1989) 1.
[38] M. Henkel and H. Saleur, J. Phys. A23 (1990) 791.
[39] T.R. Klassen and E. Melzer, in preparation.
[40] F.A. Smirnov, Form Factors in Completely Integrable Models of Quantum Field Theory (World Scientific, Singapore, 1992).
[41] M. Karowski, H.J. Thun, T.T. Truong, P. Weisz, Phys. Lett. 67B (1977) 321.
[42] P. Weisz, Nucl. Phys. B122 (1977) 1.
[43] J. Bjorken and S. Drell, Relativistic Quantum Fields (McGraw-Hill, New York, 1965).
[44] T.R. Klassen and E. Melzer, Nucl. Phys. B338 (1990) 485.
[45] M. Lässig, G. Mussardo and J.L. Cardy, Nucl. Phys. B348 (1991) 591.
[46] T.R. Klassen and E. Melzer, Nucl. Phys. B350 (1991) 635.
[47] P. Fendley, Nucl. Phys. B374 (1992) 667.
[48] D.A. Kastor, E.J. Martinec and S.H. Shenker, Nucl. Phys. B316 (1989) 590.
[49] J.L. Cardy, J. Phys. A17 (1984) L385; H.W.J. Blöte, J.L. Cardy and M.P. Nightingale, Phys. Rev. Lett. 56 (1986) 742; I. Affleck, Phys. Rev. Lett. 56 (1986) 746.
[50] V.P. Yurov and A.B. Zamolodchikov, Int. J. Mod. Phys. A5 (1990) 3221, and Paris preprint ENS-LPS-273 (1990).
[51] M. Lässig and G. Mussardo, Computer Phys. Comm. 66 (1991) 71.
[52] T.R. Klassen and E. Melzer, Nucl. Phys. B370 (1992) 511.
[53] T.T. Wu, B.M. McCoy, C.A. Tracy and E. Baruch, Phys. Rev. B13 (1976) 316; B. Schroer and T.T. Truong, Nucl. Phys. B144 (1978) 80.
[54] M. Sato, T. Miwa and M. Jimbo, Proc. Japan Acad. 53A (1977) 6, 147, 153, 183, 219.
[55] A.E. Ferdinand and M.E. Fisher, Phys. Rev. 185 (1969) 832.
[56] H. Saleur and C. Itzykson, J. Stat. Phys. 48 (1987) 449.
[57] M. Lüscher, in: Progress in Gauge Field Theory (Cargèse 1983), ed. G. ’t Hooft et al (Plenum, New York, 1984), and Commun. Math. Phys. 104 (1986) 177.
[58] M. Lüscher, Commun. Math. Phys. 105 (1986) 153, and Nucl. Phys. B354 (1991) 531; M. Lüscher and U. Wolff, Nucl. Phys. B339 (1990) 222.
[59] M. Lässig and M.J. Martins, Nucl. Phys. B354 (1991) 666; M.J. Martins, Phys. Lett. 262B (1991) 39
[60] I.M. Gel’fand, M.I. Graev and I.I. Pyatetskii-Shapiro, Representation Theory and Automorphic Functions (Saunders, Philadelphia, 1966); E. Melzer, Int. J. Mod. Phys. A4 (1989) 4877
[61] M. Wakimoto and H. Yamada, Lett. Math. Phys. 7 (1983) 513
[62] E. Witten, Commun. Math. Phys. 92 (1984) 455; A.B. Zamolodchikov and V.A. Fateev, Sov. J. Nucl. Phys. 43 (1986) 657
[63] S. Coleman, Ann. Phys. (NY) 101 (1976) 239
[64] S. Meyer and P. Weisz, Phys. Lett. 68B (1977) 471
[65] I.Ya. Aref’eva and V.E. Korepin, JETP Lett. 20 (1974) 312
[66] R.F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D10 (1974) 4114, 4130, D11 (1975) 3424, 2443
[67] S. Coleman, Aspects of Symmetry (Cambridge University Press, Cambridge, 1985)
[68] A.B. Zamolodchikov, Commun. Math. Phys. 69 (1979) 155
[69] P. Fendley and P. Ginsparg, Nucl. Phys. B324 (1989) 549
[70] A.B. Zamolodchikov, Landau Institute preprint (1989)
[71] M. Henkel and A.W.W. Ludwig, Mass Spectrum of the 2D Ashkin-Teller Model in an External Magnetic Field, Geneva preprint UGVA/DPT 1990/05-672
[72] R. Dijkgraaf, E. Verlinde and H. Verlinde, in: Perspectives in String Theory, ed. P. Di Vecchia and J.L. Petersen (World Scientific, Singapore, 1988)
[73] T. Banks, in: The Santa Fe TASI-87, ed. R. Slansky and G. West (World Scientific, Singapore, 1988)
[74] W. Eholzer, M. Flohr, A. Honecker, R. Hübel, W. Nahm, and R. Varnhagen, Bonn preprint BONN-HE-91-22, and references therein
[75] T.R. Klassen and E. Melzer, “RG Flows in the D-Series of Minimal CFTs”, Cornell/Stony Brook preprint CLNS-91-1111/ITP-SB-91-57 (1991)
[76] Q. Ho-Kim and H.B. Zheng, Phys. Lett. 212B (1988) 71
[77] A. LeClair, Phys. Lett. 230B (1989) 103; D. Bernard and A. LeClair, Nucl. Phys. B340 (1990) 721; N.Yu. Reshetikhin and F.A. Smirnov, Commun. Math. Phys. 131 (1990) 157
[78] S.V. Pokrovski and A.M. Tsvelik, Sov. Phys. JETP 66 (1987) 1275; F.C. Alcaraz, M. Baake, U. Grimm and V. Rittenberg, J. Phys. A21 (1988) L117
[79] A. Luther and I. Peschel, Phys. Rev. B12 (1975) 3908; V.E. Korepin and A.G. Izergin, JETP Lett. 42 (1985) 512
[80] M. Henkel and H. Saleur, J. Phys. A22 (1989) L513; I.R. Sagdeev and A.B. Zamolodchikov, Mod. Phys. Lett. B3 (1989) 1375; P.G. Lauwers and V. Rittenberg, Phys. Lett. 233B (1989) 197; M. Henkel, J. Phys. A24 (1991) L133

48
[81] R.J. Baxter, *Exactly Solved Models in Statistical mechanics* (Academic Press, London, 1982)

[82] M. Lüscher, Nucl. Phys. B117 (1976) 475

[83] B. Kaufman, Phys. Rev. 76 (1949) 1232