Coherent dissipative structures in chains of coupled $\chi^{(3)}$ resonators

Aleksandr K. Tusnin$^1$,† Alexey M. Tikan$^1$,‡ Kenichi Komagata$^{1,2}$, and Tobias J. Kippenberg$^{1,*}$

$^1$Institute of Physics, Swiss Federal Institute of Technology Lausanne (EPFL), Lausanne, Switzerland
$^2$Present address: Laboratoire Temps-Fréquence, Avenue de Bellevaux 51, 2000 Neuchâtel, Switzerland

(Dated: April 26, 2021)

We theoretically investigate pattern formation and nonlinear dynamics in an array of equally-coupled, optically driven, Kerr nonlinear microresonators. We show that the nonlinear dynamics of the system can be associated with an effective two-dimensional space due to the multimode structure of each resonator. As a result, two fundamentally different dynamical regimes (elliptic and hyperbolic) arise at different regions of the hybridized dispersion surface. We demonstrate the formation of global nonlinear optical patterns in both regimes which correspond to coherent optical frequency combs on the individual resonator level. In addition we show that the presence of an additional dimension leads to the observation of wave collapse.

Introduction.—Solitons are localized structures whose existence relies on the balance between dispersion and nonlinearity. Since the first discovery of solitary waves by Scott Russell in 1834 [1], the development of the soliton theory has undergone several important stages. First described for the Korteweg-de Vries equation [2], it has been found in a variety of nonlinear dispersive models including the Sine-Gordon and the Nonlinear Schrödinger equations (NLSE) [3], exactly integrable by the Inverse Scattering Transform method [4]. These universal models describe various physical systems. The NLSE governs light propagation in optical fibers [5, 6], surface gravity water waves under slowly varying envelope approximation [7, 8] and many more [9, 10]. Later, solitons have been discovered in open and driven systems with additional dissipation and gain such as mode-locked lasers [11] or high-Q resonators possessing Kerr nonlinearity [12]. In these systems, an additional balance between losses and (parametric) gain is required to maintain such dissipative solitons.

To leading order, light propagation in a single optical resonator with a Kerr nonlinearity is described by the Lugiato-Lefever equation (LLE) [13] - driven dissipative modification of the NLSE. Bright and dark dissipative Kerr solitons (DKSs) of the one-dimensional LLE are stable and have been extensively studied in the last decade as the key effect underpinning coherent and broadband frequency combs [12, 14] [see Fig. 1 (a)], widely applied from distance measurements and telecommunication to neuromorphic optical computing [15–19]. Remarkably, the fact that the core of the LLE is the NLSE equation has been used to introduce a new way of dissipative soliton characterization with a non-preserved Inverse Scattering Transform spectrum [20–22], which discrete eigenvalues identify the solitonic content of the complex field and provide with the profound link between these systems.

The LLE is a complex model exhibiting various dynamical features including (besides DKS) Arnold tongues [23], cnoidal waves [24], breathers [25], soliton crystals [26], and chaotic structures of various types [27, 28]. DKS are found in the bi-stable region of the parameter space and usually accessed by tuning over the cavity resonance [12]. However, up to now, studies of the complex dynamics have been mostly limited by the single-resonator case.

Historically, two-dimensional version of the LLE has been first derived in the context of transverse pattern formation in nonlinear cavities [29] and recently applied to a novel type of cylindrical high-Q cavities [30]. The extension of the LLE to the two-dimensional case makes inherent dissipative solitons stable (contrary to its conservative counterpart [31]) only in a narrow region of parameters in the monostable part of the tilted resonance [32]. The bi-stable part, on the contrary, features only hexagonal patterns [33].

The generation of DKSs has been recently experimentally observed in hybridized modes of a photonic dimer [34], which nonlinear dynamics is described by two linearly-coupled LLEs [35]. The development of this field is directed towards the study of the DKS generation in chains and lattices of resonators including topological arrangements [36, 37]. Despite the fact that the single mode arrangements have been extensively studied before in a system of coupled resonators known as CROW (coupled resonator optical waveguide) [38, 39], the description of the multimode case is missing.

In this work, we extend previous studies [39] by examining a chain of multimode (i.e. possessing the synthetic frequency dimension in addition to the spatial dimension) Kerr nonlinear microresonators coupled in a ring [see Fig. 1 (d)]. Due to the similarity of the inherent dispersion profile [see Fig. 1 (e)], the considered system exhibits analogy with topologically nontrivial photonic lattices possessing edge states [37, 40, 41]. Using Bloch mode representation, we derive an effective two-dimensional model which captures the emerging dynamics. Pumping different Bloch modes, we can access two fundamentally different regimes: elliptic and hyperbolic. We examine dissipative patterns formed in different...
As normalized, the system of equations can be written described by the set of linearly coupled LLEs [13]. In the bus waveguides –

Intrinsic linewidth is denoted as \( \kappa \) modulation instability gain lobes (depicted in red) in elliptic (anomalous spatial dispersion band structure are depicted by greed and blue, respectively. (c,f) Hybridized integrated dispersion of multimode CROWs and with a rate \( S \) configuration. Each resonator has an individual bus waveguide which supplies the pump. Thus \( J \) resonators are coupled with a rate \( J \); (e) corresponding integrated dispersion profile which includes second-order dispersion \( D \) range) and \( k \) band structure in the single-mode case with respect to \( k_0 = N/2 \). Normal \( (j_2 < 0) \) and anomalous \( (j_2 > 0) \) region of the band structure are depicted by greed and blue, respectively. (c,f) Hybridized integrated dispersion of multimode CROWs and modulation instability gain lobes (depicted in red) in elliptic (anomalous spatial dispersion \( j_2 > 0 \) at \( k_0 = 0 \)) and hyperbolic (normal spatial dispersion \( j_2 < 0 \) at \( k_0 = N/2 \)) regions, respectively.

FIG. 1. Hybridized dispersion in coupled resonator optical waveguides (CROWs). (a) Single optical resonator possessing \( \chi^{(3)} \) nonlinearity. Dissipative soliton generation in the resonator results into a coherent frequency comb at the output; (b) corresponding integrated dispersion profile which includes second-order dispersion \( d_2 \). (d) CROWs in the ring configuration. Each resonator has an individual bus waveguide which supplies the pump. Thus \( J \) resonators are coupled with a rate \( J \); (e) corresponding integrated dispersion profile which includes second-order dispersion \( D \) region of the band structure are depicted by greed and blue, respectively. (c,f) Hybridized integrated dispersion of multimode CROWs and modulation instability gain lobes (depicted in red) in elliptic (anomalous spatial dispersion \( j_2 > 0 \) at \( k_0 = 0 \)) and hyperbolic (normal spatial dispersion \( j_2 < 0 \) at \( k_0 = N/2 \)) regions, respectively.

regimes and show mechanisms of photon transfer between Bloch modes employing the notion of nonlinear dispersion relation (NDR) introduced for NLSE in Ref. [42] and further applied for characterization of optical pulses in optical microresonators in Refs. [34, 35, 43].

Model.—We consider a chain of \( N \) identical optical resonators with periodic boundary conditions. Modes of each resonator can be found as \( \omega_\mu = \omega_0 + \mu D_1 + \mu^2 D_2/2 + \ldots \), where \( \mu \) is comb index, \( D_1 = 2\pi \text{FSR} \) (free spectral range) and \( D_2 \) is the second-order dispersion coefficient. Integrated dispersion is defined as \( D_{\text{int}} = \omega_\mu - \omega_0 - \mu D_1 \). Intrinsic linewidth is denoted as \( \kappa_0 \) and coupling to the bus waveguides \( \kappa_{\text{ex}} \). Resonators are coupled with a rate \( J \). In the meanfield approximation, this system can be described by the set of linearly coupled LLEs [13]. In the normalized form the system of equations can be written as

\[
\frac{\partial \Psi_\ell}{\partial \tau} = -(1 + i\zeta_0)\Psi_\ell + i d_2 \frac{\partial^2 \Psi_\ell}{\partial \varphi^2} + i j \left( \Psi_{\ell-1} + \Psi_{\ell+1} \right) + i |\Psi_\ell|^2 \Psi_\ell + f_\ell, \tag{1}
\]

where \( d_2 = D_2/\kappa \), \( \kappa = \kappa_0 + \kappa_{\text{ex}} \) is the total linewidth, \( \zeta_0 = 2 k_0 / \kappa \) normalized detuning and \( \delta \omega = \omega_0 - \omega_\mu \) is the laser cavity detuning, \( j = 2 J / \kappa \), \( f_\ell = \sqrt{8 \kappa_{\text{ex}} g_0 / \kappa^3} \text{Re} \langle \Psi_\ell \rangle \), \( g_0 \) is the single-photon Kerr frequency shift, \( s_{\ell}^n \) is the pump rate of the \( \ell \)-th resonator with corresponig phase \( \phi_\ell \), \( \Psi_\ell = \sqrt{2 g_0 / \kappa A} \) and \( |A| \) is photon density inside the \( \ell \)-th resonator. The linear part can be diagonalized by the Fourier transform

\[
\psi_{\mu k} = \frac{1}{2\pi \sqrt{N}} \int \sum_{\ell=1}^{N} \Psi_\ell e^{2 i \ell (k N + \mu \varphi)} d\varphi, \tag{2}
\]

where \( k \) is the supermode index and \( \mu \) is the comb line index. With the Kerr term Eq. (1) transforms to

\[
\frac{\partial \psi_{\mu k}}{\partial \tau} = -(1 + i\zeta_0) \psi_{\mu k} - i \left[ d_2 \mu^2 - 2 j \cos \frac{2 \pi k}{N} \right] \psi_{\mu k} + \frac{i}{N} \sum_{k_1, k_2, k_3} \psi_{\mu_1 k_1} \psi_{\mu_2 k_2} \psi_{\mu_3 k_3} \delta_{\mu_1+\mu_2-\mu_3} \delta_{k_1+k_2-k_3-k} + \delta_{k, k_0} \delta_{\mu, 0} \delta_{\mu k_0}, \tag{3}
\]

where \( k_1, k_2, k_3 \) are the supermode indices and \( \mu_1, \mu_2, \mu_3 \) are the comb line indices.
FIG. 2. **Coherent dissipative structures in a driven nonlinear photonic ring lattice.** Panels (a-d) correspond to the elliptic region \((k_0 = 0, d_2 > 0, j_2 > 0)\), (e-h) to the hyperbolic \((k_0 = N/2, d_2 > 0, j_2 < 0)\). (a,e) represent global spatio-temporal mode-locked structures along the lattice; (b,f) field profile in the 1st resonator. The 2D spectral profiles of the states (a) and (e) obtained via Eq. (3) are presented in (c) and (g) respectively. The spectral profile in elliptic regime (c) is formed on a disk, whereas the spectrum of the pattern in hyperbolic regime (g) tends to align one of the asymptotes of the hyperbola depicting modulation instability gain in Eq. (4). The Fourier spectra of the states (b) and (f) are presented in (d) and (h).

where we assumed that we pump the mode with indexes \(\mu = 0, k = k_0\), and \(\hat{f}_{k_0}\) is Fourier transform of \(f_t\). From Eq. (3) one can see that the nonlinear term incorporates a conservation law on the supermode and comb indexes, revealing the two-dimensional nature of the system. The term in square brackets in Eq. (3) depicts the two-dimensional dispersion surface that has a parabolic profile in \(\mu\) axis and cosine in \(k\). Such surfaces are presented in Fig. 1(e,f). This surface is periodic in \(k\), and we shift the origin of \(k\) axis for clarity in two different pump schemes which we describe below.

Let us consider two cases: excitation of the supermodes \(k_0 = 0\) and \(k_0 = N/2\). Considering second order Taylor series expansion of the cosine, one can see that these two supermodes have different dispersion: \(k_0 = 0\) corresponds to the anomalous group velocity dispersion, \(k_0 = N/2\) to the normal. Thus, the local dispersion surface has either parabolic or saddle shape, respectively. Performing the inverse Fourier transform of Eq. (3) with quadratic approximation of the cosine term, one can readily obtain that Eq. (1) is approximated by two dimensional (2D) version of the LLE with the dispersion operator of elliptic or hyperbolic form in the long wavelength limit [44]. The elliptic LLE has been investigated in transverse nonlinear optics [29], and the existence of stable hexagonal patterns and solitons has been reported [32, 45]. We expect to observe similar states in the considered system when we pump the mode \(\psi_{\mu=0,k=0}\). In contrast to the elliptic LLE, the hyperbolic form has not been extensively investigated in context of nonlinear optics, to the best of our knowledge. It is important to note that pumping any other mode \(k_0 \neq 0, N/2\) leads to the decrease of local quadratic supermode dispersion (coefficient at \(k^2\)) and appearance of a linear term resulting into additional group velocity (coefficient at \(k\)) of the formed structures along the array of the resonators. Thus, we restrict ourselves to pumping the supermodes \(k_0 = 0, N/2\).

**Modulation instability.**—First, we locally investigate the optical parametric oscillations. Similarly to Ref. [46], we consider only four-wave mixing processes between the pump mode \((\mu_0 = 0, k_0 = 0, N/2)\) and the first sidebands \((\mu, k)\). Assuming the pump mode at a steady state, we linearize the system for sideband amplitudes \(\psi_{\mu k}\) and investigate their temporal dynamics. Corresponding characteristic equation provides with parametric gain, which at the threshold (gain equals to loss) reveals the position of the first sidebands

\[
d_{2}\mu^2 \pm j_{2}k^2 = 4|\psi_{00}|^4 + \sqrt{|\psi_{00}|^4 - 1} - \zeta_{0}^\mp, \mu, k \neq 0, \tag{4}
\]

\[
d_{2}\mu^2 = 2|\psi_{00}|^4 + \sqrt{|\psi_{00}|^4 - 1} - \zeta_{0}^\mp, k = 0, \tag{5}
\]

\[
j_{2}k^2 = 2|\psi_{00}|^4 + \sqrt{|\psi_{00}|^4 - 1} - \zeta_{0}, \mu = 0, k_0 = 0 \tag{6}
\]

where \(j_2 = (2\pi/N)^2 j\) is the amplitude of the local supermode dispersion, \(\pm\) corresponds to excitation of \(k_0 = 0\)
duce the minimum \( k \) respectively. From Eq. (6), similarly to Ref. [46], one can therefore, we call these regions elliptic and hyperbolic respectively. From Eq. (6), similarly to Ref. [46], one can deduce the minimum \( k_{th} = \sqrt{1/j_2} \) at threshold condition \( (\varphi_0 = 1) \), meaning that the first sidebands along \( k \) axis can appear if \( j_2 \leq 1 \). On the other hand, Eq. (4) regime yields \( d_2\mu_{th}^2 \pm j_2\kappa_{th}^2 = 3 \), mitigating the above condition in the elliptic regime \( j_2 < 3 \) and indicating that in the hyperbolic regime the two dimensional dynamics is always present no matter the value of \( j_2 \). An example of calculated modulation instability gain lobes (Eq. (4)) is presented in Fig. 1(c,f) for both regions in case of \( d_2 = 0.04 \) and \( j_2 = 1 \).

**Coherent dissipative structures.**—We continue our analysis with numerical simulation of the set of equations (1), considering 20 optical resonators with \( d_2 = 0.04 \) and \( j = 10.13 \) \( (j_2 = 1) \). In order to find a numerical solution, we employ step-adaptive Dormand-Prince Runge-Kutta method of Order 8(5,3) [47] and approximate the dispersion operator by the second order finite difference scheme. We fix the pump amplitude \( |f_0| \) and choose the supermode to pump by fixing relative phases \( \phi_0 \). When all the resonators pumped in phase (opposite phases), we excite the system in elliptic \( k_0 = 0 \) (hyperbolic \( k_0 = N/2 \)) regime. In order to observe coherent structures, we scan the resonance by changing the normalized laser detuning \( \zeta_0 \) and bring the system into an unstable state. Having stimulated the pattern formation, we further tune towards the monostable region \( (\zeta_0^3 < \sqrt{3}) \) and obtain stable coherent structures in both regimes (Fig. 2). One can see that in the elliptic regime at \( |f_0| = 1.05 \) and \( \zeta_0 = 20.5 \), we observe formation of a hexagonal pattern [Fig. 2(a)]. On a single resonator level this corresponds to locked pulses [Fig. 2(b)] with a typical comb spectrum shown in Fig. 2(d). The corresponding 2D \( k-\mu \) spectral profile in Fig. 2(c) shows that the sidebands form a disk, occupying the supermodes from both anomalous \( (|k-k_0| < 5) \) and normal dispersion regimes \( (|k-k_0| \geq 5) \). This can signify that 2D solitons in this system will be substantially different from the 2D LLE counterpart, because solitons are well localized in the direct space and have spectrum wider than the spectrum the hexagonal pattern shown in Fig. 2(c). Even though such localized structures are of the particular interest, their investigation is beyond the scope of this letter. In the hyperbolic regime at \( |f_0| = 2.35 \) and \( \zeta_0 = -20.3 \), we observe a train of pulses in each resonator locked to each other [Fig. 2(e,f)]. The corresponding 2D spectral profile [Fig. 2(g)] follows several lines in \( k-\mu \) space, that qualitatively follow one of the asymptotes of the hyperbola depicting modulation instability gain lobes in Fig. 1(f). Comparing the comb spectra at the 1st resonator [Fig. 2(d)] with the elliptic case [Fig. 2(h)], one can notice that the state at the hyperbolic regime has a wider comb spectrum.

**FIG. 3.** Numerical reconstruction of the nonlinear dispersion relation in the elliptic and hyperbolic regions in the unstable regime. Panels (a-c) correspond to the elliptic region \((k_0 = 0, d_2 > 0, j_2 > 0)\), (b-f) to the hyperbolic \((k_0 = N/2, d_2 > 0, j_2 < 0)\). Spatiotemporal diagrams of unstable states in 0-th comb mode are shown in (a) and (d); The corresponding nonlinear dispersion relation (NDR) in elliptic region (b) demonstrates excitation of all the optical and spatial modes, whereas the NDR in the hyperbolic region (e) reveals that photon transfer between the spatial supermodes is suppressed in the vicinity of the pump mode \( \mu = 0 \); The panes (c) and (f) represent the nonlinear supermode dispersion relation [Eq. (7)] of 0-th comb line for the state in (a) and 25-th comb line for the state in (d).
Wave collapse.—Since LLE is NLSE with external force and dissipation, it can possess similar features, and in particular, the effect called wave collapse [48, 49]. Even though the thorough derivation of the wave collapse relies on conservation laws that are not presented in our driven-dissipative system, we show that the qualitative behavior is defined by the dispersion surface topography, namely its curvature. It has been shown for 2D elliptic focusing NLSE that a pulse of finite width can explosively shrink to an infinitely small area concentrating there finite amount of energy [31, 50] becoming ultrabroad in the spectral domain. We observe a similar effect in our system. Exciting incoherent dynamics by pumping the elliptic region at $|f| = 2.35$ and $\zeta_0 = 22.1$, we observe rapid formation and dissipation of narrow pulses in each cavity. A typical spatio-temporal diagram at single resonator level is shown in Fig. 3(a). We observe occasional appearance of the pulses in different parts of the cavity and further their rapid compression, during which the peak amplitude significantly exceeds the background level. However, investigating the pulse width dynamics, we find that it does not completely shrink. In order to find what limits the minimum pulse width, we computed the nonlinear dispersion relation [42] [Fig. 3(b)] which is nothing more, but 2D Fourier transform of the field dynamics in Fig. 3(a). We observe the high photon occupancy of the region beneath the parabolas which indicates the presence of 2D dissipative nonlinear structures. Furthermore, all the hybridized parabolas are populated by the photons, meaning that supermodes from both dispersion regions are excited. We continue by reconstructing the supermode NDR for 0-th comb line ($\mu_0 = 0$) for all resonators in the following way

$$NDR(\Omega, \mu_0, k) = \frac{1}{\sqrt{N_\ell N_\ell}} \sum_{\ell,n} \psi_{\mu_0}(t) e^{i(2\pi k \ell/N-\Omega n)}, \quad (7)$$

where $t_n = \Delta t n$ with $\Delta t = T/N_\ell$ time-step, $T$ is simulation time with $N_\ell$ number of discretization points. The result is shown in Fig. 3(c). It reveals that the whole cosine band structure is populated, including the region of the normal dispersion, which prevents the full wave collapse.

We continue analysis by exciting the hyperbolic region under the same conditions (same pump power and relative detuning $\zeta_0 = -17.0$). As mentioned earlier, the local dispersion topography has an opposite sign of the supermode dispersion with respect to the elliptic region. In the conservative long wavelength limit, this corresponds to the hyperbolic NLSE which does not possess the wave collapse mechanism [50]. Indeed, we observe that the spatio-temporal diagram [Fig. 3(d)] does not demonstrate any extreme events, showing slow (with respect to the elliptic case) incoherent dynamics. Further, comparing the NDR [Fig. 3(e)] with the elliptic case, we show how the mode occupancy differs. Indeed, the nonlinear structures [horizontal line in Fig. 3(e)] form on the upper parabola. In the vicinity of $\mu = 0$ the normal supermode group velocity dispersion suppresses the parametric oscillations and prevents photon transfer along the $k$ axis. Nevertheless, the photon transfer to other supermodes is stimulated in the area where the line crosses the lower parabolas, resulting in appearance of dispersive waves [34, 35]. Reconstructing the supermode NDR [Fig. 3(f)] for $\mu = 25$ comb line [the average crossing position in Fig. 3(e)], we observe the predominant population of the center of the band. In this region the Bloch supermodes have maximal propagation velocity, allowing the dispersive waves with frequencies in the vicinity of $\pm25$-th comb line to propagate along the circumference of the CROW.

In this letter, we considered nonlinear dynamics in coupled high-Q optical resonators with $\chi^{(3)}$ nonlinearity operating in multimode regime. We have shown that this system possesses a 2D dispersion surface whose different parts correspond to two fundamentally different regimes of operation: elliptic and hyperbolic. Simulating the full set of coupled LLEs, we have demonstrated a variety of nonlinear effects in elliptic regime, such as hexagonal patterns formation and wave collapses in the chaotic state inherent to known 2D systems, along with unconventional coherent and incoherent states in hyperbolic regime. Our theory sheds light on nonlinear interactions in integrated photonic lattices and will be helpful for future investigations of multimode systems with complex band structures and different topological properties. Even though, it is still a challenging task to obtain a smooth dispersion profile experimentally [36], recent progress in the integrated nonlinear photonics suggests that such system as investigated here can be realized in the nearest future.

ACKNOWLEDGMENTS

This publication was supported by Contract18AC00032 (DRINQS) from the Defense Advanced Research Projects Agency (DARPA), Defense Sciences Office (DSO). This material is based upon work supported by the Air Force Office of Scientific Research under award number FA9550-19-1-0250. This work was further supported by the European Union’s Horizon 2020 Research and Innovation Program under the Marie Skłodowska-Curie grant agreement 812818 (MICRO-COMB), and by the Swiss National Science Foundation under grant agreements 192293 and 176563.

[1] J. S. Russell, Report on Waves: Made to the Meetings of the British Association in 1842-43 (1845).

[2] N. J. Zabusky and M. D. Kruskal, Interaction of “solitons”...
tions” in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. 15, 240 (1965).

[3] M. J. Ablowitz and H. Segur, Solitons and the inverse scattering transform (SIAM, 1981).

[4] S. Novikov, S. Manakov, L. Pitaevskii, and V. E. Zakharov, Theory of solitons: the inverse scattering method (Springer Science & Business Media, 1984).

[5] G. P. Agrawal, Nonlinear fiber optics, in Nonlinear Science at the Dawn of the 21st Century (Springer, 2000) pp. 195–211.

[6] F. Copie, S. Randoux, and P. Suret, The physics of the one-dimensional nonlinear schrödinger equation in fiber optics: Rogue waves, modulation instability and self-focusing phenomena, Reviews in Physics 8, 100037 (2020).

[7] V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, J. Appl. Mech. Tech. Phys. 9, 190 (1968).

[8] C. Kharif, E. Pelinovsky, and A. Shunyaev, Rogue waves in the ocean (Springer Science & Business Media, 2008).

[9] M. Onorato, S. Residori, U. Bortolozzo, A. Montina, and F. T. Arecchi, Rogue waves and their generating mechanisms in different physical contexts, Phys. Rep. 528, 47 (2013).

[10] T. F. Biﬁch, The nonlinear Schrödinger equation (Springer, 2015).

[11] P. Grelu and N. Akhmediev, Dissipative solitons for massively parallel coherent optical communications, Nature Physics, 887 (2018).

[12] T. Herr, V. Brasch, J. D. Jost, C. Y. Wang, N. M. Kondratiev, M. L. Gorodetsky, and T. J. Kippenberg, Temporal solitons in optical microresonators, Nat. Photonics 8, 145 (2014).

[13] Y. K. Chembo and C. R. Menyuk, Spatiotemporal luguto-lefever formalism for kerr-cob generation in whispering-gallery-mode resonators, Phys. Rev. A 87, 053852 (2013).

[14] T. J. Kippenberg, A. L. Gaeta, M. Lipson, and M. L. Gorodetsky, Dissipative kerr solitons in optical microresonators, Science 361, eaam8083 (2018).

[15] P. Trocha, M. Karpov, D. Ganin, M. H. P. Pfeiffer, A. Kordts, S. Wolf, J. Kreckenberger, P. Marin-Palomo, C. Weimann, S. Randel, W. Freude, T. J. Kippenberg, and C. Koos, Ultrafast optical ranging using microresonator soliton frequency combs, Science 359, 887 (2018).

[16] J. Riemenhsberger, A. Lukashchuk, M. Karpov, W. Weng, E. Lucas, J. Liu, and T. J. Kippenberg, Massively parallel coherent laser ranging using a soliton microcomb, Nature 581, 164 (2020).

[17] P. Marin-Palomo, J. N. Kemal, M. Karpov, A. Kordts, J. Pfeuille, M. H. P. Pfeiffer, P. Trocha, S. Wolf, V. Brasch, M. H. Anderson, R. Rosenberger, K. Vijayan, W. Freude, T. J. Kippenberg, and C. Koos, Microresonator-based solitons for massively parallel coherent optical communications, Nature 546, 274 (2017).

[18] D. T. Spencer, T. Drake, T. C. Briles, J. Stone, L. C. Sinclair, C. Fredrick, Q. Li, D. Westley, B. R. Ilic, A. Bluestone, et al., An optical-frequency synthesizer using integrated photonics, Nature 557, 81 (2018).

[19] J. Feldmann, N. Youngblood, M. Karpov, H. Gehring, X. Li, M. Stappers, M. Le Gallo, X. Fu, A. Lukashchuk, A. S. Raja, J. Liu, C. D. Wright, A. Sebastian, T. J. Kippenberg, W. H. P. Pernice, and H. Bhaskaran, Parallel convolutional processing using an integrated photonic tensor core, Nature 589, 52 (2021).

[20] I. S. Chekhovskov, O. V. Shtyrina, M. P. Fedoruk, S. B. Medvedev, and S. K. Turitsyn, Nonlinear Fourier Transform for Analysis of Coherent Structures in Dissipative Systems, Phys. Rev. Lett. 122, 153901 (2019).

[21] S. K. Turitsyn, I. S. Chekhovskov, and M. P. Fedoruk, Nonlinear fourier transform for characterization of the coherent structures in optical microresonators, Optics Letters 45, 3059 (2020).

[22] S. K. Turitsyn, I. S. Chekhovskov, and M. P. Fedoruk, Nonlinear fourier transform for analysis of optical spectral combs, Phys. Rev. E 103, L020202 (2021).

[23] D. V. Skryabin, Z. Fan, A. Villois, and D. N. Puzyrev, Threshold of complexity and Arnold tongues in Kerr-ring microresonators, Phys. Rev. A 103, L011502 (2021).

[24] Z. Qi, S. Wang, J. Jaramillo-Villegas, M. Qi, A. M. Weiner, G. D’Aguanno, T. F. Carruthers, and C. R. Menyuk, Dissipative cnoidal waves (turing rolls) and the soliton limit in microring resonators, Optica 6, 1220 (2019).

[25] E. Lucas, M. Karpov, H. Guo, M. L. Gorodetsky, and T. J. Kippenberg, Breathing dissipative solitons in optical microresonators, Nat. Commun. 8, 1 (2017), 1611.6567.

[26] M. Karpov, M. H. Pfeiffer, H. Guo, W. Weng, J. Liu, and T. J. Kippenberg, Dynamics of soliton crystals in optical microresonators, Nature Physics 15, 1071 (2019).

[27] M. Anderson, F. Leo, S. Coen, M. Erkintalo, and S. G. Murdoch, Observations of spatiotemporal instabilities of temporal cavity solitons, Optica 3, 1071 (2016).

[28] S. Coulibaly, M. Taki, A. Bendahmane, G. Millot, B. Kibler, and M. G. Clerc, Turbulence-induced rogue waves in Kerr resonators, Physical Review X 9, 011054 (2019).

[29] A. Scroggie, W. Firth, G. McDonald, M. Tlidi, R. Lefever, and L. Lugato, Pattern formation in a passive kerr cavity, Chaos, Solitons & Fractals 4, 1323 (1994), special Issue: Nonlinear Optical Structures, Patterns, Chaos.

[30] S. B. Ivars, Y. V. Kartashov, L. Torner, J. A. Conejero, and C. Milián, Reversible self-replication of spatiotemporal kerr cavity patterns, Phys. Rev. Lett. 126, 063903 (2021).

[31] E. A. Kuznetsov, Wave collapse in nonlinear optics, in Self-focusing: Past and Present: Fundamentals and Prospects, edited by R. W. Boyd, S. G. Lukishova, and Y. Shen (Springer New York, New York, NY, 2009) pp. 175–190.

[32] W. J. Firth, G. K. Harkness, A. Lord, J. M. McSloy, D. Gomila, and P. Colet, Dynamical properties of two-dimensional Kerr cavity solitons, J. Opt. Soc. Am. B 19, 747 (2002).

[33] W. J. Firth, A. J. Scroggie, G. S. McDonald, and L. A. Lugato, Hexagonal patterns in optical bistability, Phys. Rev. A 46, R3609 (1992).

[34] A. Tikan, J. Riemensberger, K. Komagata, S. Hönl, M. Churava, C. Skehan, H. Guo, R. N. Wang, J. Liu, P. Seidler, et al., Emergent nonlinear phenomena in a driven dissipative photonic dimer, Nat. Phys. 1, 1 (2021).

[35] K. Komagata, A. Tikan, A. Tusnin, J. Riemensberger, M. Churava, H. Guo, and T. J. Kippenberg, Dissipative kerr solitons in a photonic dimer on both sides of exceptional point, arXiv preprint arXiv:2101.09237 (2021).

[36] A. Tikan, A. Tusnin, J. Riemensberger, M. Churava, K. Komagata, X. Ji, R. N. Wang, J. Liu, and T. J. Kippenberg, Symmetry protection of topological states
in multimode photonic resonator chains, arXiv preprint arXiv:2011.07976 (2020).

[37] S. Mittal, G. Moille, K. Srinivasan, Y. K. Chembo, and M. Hafezi, Topological frequency combs and nested temporal solitons, arXiv preprint arXiv:2101.02229 (2021).

[38] F. Morichetti, C. Ferrari, A. Canciamilla, and A. Melloni, The first decade of coupled resonator optical waveguides: bringing slow light to applications, Laser & Photonics Reviews 6, 74 (2012).

[39] L. Marti, J. Vasco, and V. Savona, Slow-light enhanced frequency combs and dissipative kerr solitons in silicon coupled-ring microresonators in the telecom band, arXiv preprint arXiv:2012.11439 (2020).

[40] M. Hafezi, S. Mittal, J. Fan, A. Migdall, and J. Taylor, Imaging topological edge states in silicon photonics, Nature Photonics 7, 1001 (2013).

[41] T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. C. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, and I. Carusotto, Topological photonics, Rev. Mod. Phys. 91, 015006 (2019).

[42] K. P. Leisman, D. Zhou, J. W. Banks, G. Kovačič, and D. Cai, Effective dispersion in the focusing nonlinear schrödinger equation, Phys. Rev. E 100, 022215 (2019).

[43] A. K. Tusnin, A. M. Tikan, and T. J. Kippenberg, Nonlinear states and dynamics in a synthetic frequency dimension, Phys. Rev. A 102, 023518 (2020).

[44] D. N. Christodoulides and R. I. Joseph, Discrete self-focusing in nonlinear arrays of coupled waveguides, Opt. Lett. 13, 794 (1988).

[45] G. D’Alessandro and W. J. Firth, Spontaneous hexagon formation in a nonlinear optical medium with feedback mirror, Phys. Rev. Lett. 66, 2597 (1991).

[46] T. Herr, K. Hartinger, J. Riemonsberger, C. Wang, E. Gavartin, R. Holzwarth, M. Gorodetsky, and T. Kippenberg, Universal formation dynamics and noise of kerr-frequency combs in microresonators, Nature photonics 6, 480 (2012).

[47] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, Numerical recipes 3rd edition: The art of scientific computing (Cambridge university press, 2007).

[48] V. E. Zakharov, Wave collapse, Soviet Physics Uspekhi 31, 672 (1988).

[49] Y. V. Kartashov, M. L. Gorodetsky, A. Kudlinski, and D. V. Skryabin, Two-dimensional nonlinear modes and frequency combs in bottle microresonators, Opt. Lett. 43, 2680 (2018).

[50] J. J. Rasmussen and K. Rypdal, Blow-up in nonlinear schroedinger equations-i a general review, Physica Scripta 33, 481 (1986).