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On norm equivalence between the displacement and velocity vectors for free linear dynamical systems

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Abstract: As the main new result, under certain hypotheses, for free vibration problems, the norm equivalence of the displacement vector \( y(t) \) and the velocity vector \( \dot{y}(t) \) is proven. The pertinent inequalities are applied to derive some two-sided bounds on \( y(t) \) and \( \dot{y}(t) \) that are known so far only for the state vector \( x(t) = [y^T(t), \dot{y}^T(t)]^T \). Sufficient algebraic conditions are given such that norm equivalence between \( y(t) \) and \( \dot{y}(t) \) holds, respectively, does not hold, as the case may be.

Numerical examples illustrate the results for vibration problems of \( n \) degrees of freedom with \( n \in \{1, 2, 3, 4, 5\} \) by computing the mentioned algebraic conditions and by plotting the graphs of \( y(t) \) and \( \dot{y}(t) \). Some notations and definitions of References Kohaupt (2008b, 2011) are necessary and are therefore recapitulated. The paper is of interest to Mathematicians and Engineers.

Subjects: Applied Mathematics; Computer Mathematics; Engineering Technology; Engineering Mathematics; Science; Technology

Keywords: initial value problem; free vibration problem; state-space description; two-sided bounds; sufficient algebraic conditions; norm equivalence between displacement and velocity

ABOUT THE AUTHOR

Ludwig Kohaupt received the equivalent to the master's degree (Diplom-Mathematiker) in Mathematics in 1971 and the equivalent to the PhD (Dr.phil.nat.) in 1973 from the University of Frankfurt/Main. From 1974 until 1979, Kohaupt was a teacher in Mathematics and Physics at a Secondary School. During that time (from 1977 until 1979), he was also an auditor at the Technical University of Darmstadt in Engineering Subjects, such as Mechanics, and especially Dynamics. From 1979 until 1990, he joined the Mercedes-Benz car company in Stuttgart as a computational engineer, where he worked in areas such as Dynamics (vibration of car models), Cam Design, Gearring, and Engine Design. Then, in 1990, Kohaupt combined his preceding experiences by taking over a professorship at the Beuth University of Technology Berlin (formerly known as TFH Berlin). He retired on 01 April 2014.

PUBLIC INTEREST STATEMENT

Under certain conditions, the norm equivalence of the displacement vector \( y(t) \) and the velocity vector \( \dot{y}(t) \) for free linear dynamical systems is derived. Hereby, a relation of the form \( c_0 \|y(t)\| \leq \|\dot{y}(t)\| \leq c_1 \|y(t)\|, \ t \geq t_1, \) is understood with positive constants \( c_0 \) and \( c_1 \). As a consequence, under norm equivalence, one has that \( \lim_{t \to 0} y(t) = 0 \) is equivalent to \( \lim_{t \to 0} \dot{y}(t) = 0 \) and that boundedness of \( y(t) \) is equivalent to boundedness of \( \dot{y}(t) \).
1. Introduction

In free vibration problems with one degree of freedom and mild damping, the displacement as well as the velocity have zeros in any sufficiently large interval. But, with increasing dimension, it is likely that not all components of the displacement or of the velocity are zero simultaneously, in other words, it is increasingly likely with increasing dimension \( n \) that \( y(t) \neq 0, t \geq t_1 \), and \( y(t) \neq 0, t \geq t_1 \), for sufficiently large \( t_1 \). Then, the question arises as to whether an inequality of the form \( c_1 \| y(t) \| \leq \| y(t) \| \leq c_2 \| y(t) \|, t \geq t_1 \), can be proven for sufficiently large \( t_1 \); this property, if it is valid, will be called norm equivalence between \( y(t) \) and \( \dot{y}(t) \).

Now, the contents of this paper will be outlined.

In Section 2, the state-space description \( x = Ax, x(t_0) = x_0 \), of the dynamical problem \( M\ddot{y} + B\dot{y} + Ky = 0 \), \( y(t_0) = y_0, \dot{y}(t_0) = \dot{y}_0 \) is given where \( M, B, \) and \( K \) are the mass, damping, and stiffness matrices, as the case may be; \( y_0 \) is the initial displacement and \( \dot{y}_0 \) is the initial velocity.

In Section 3, under certain hypotheses, the above-mentioned norm equivalence between \( y(t) \) and \( \dot{y} \) is proven for diagonalizable system matrix \( A \), and in Section 4, the same is done for general system matrix \( A \).

The pertinent norm inequalities between \( y(t) \) and \( \dot{y} \) are applied in Section 5 to improve Kohaupt (2011, Theorems 12 and 16). In both sections, also sufficient algebraic conditions are established that guarantee the validity, respectively, invalidity of norm equivalence of \( y(t) \) and \( \dot{y} \). In Section 6, numerical examples illustrate the results for vibration problems of \( n \) degrees of freedom for \( n \in \{1, 2, 3, \ldots, 9\} \). More precisely, in Section 6.1, the mentioned algebraic conditions are computed and the graphs of \( ||y(t)|| \) and \( ||\dot{y}(t)|| \) are plotted showing that, in the examples, for \( n \in \{3, 4, 5\} \) there is no norm equivalence; in Section 6.2, for \( n = 4 \), a case with \( \nu_2[A] < v[A] \) is illustrated by graphs, where \( \nu_2[A] \) is the spectral abscissa of matrix \( A \) with respect to \( x_0 \) and \( v[A] \) is the spectral abscissa of \( A \); in Section 6.3, for \( n = 2 \), a model with non-diagonalizable matrix \( A \) is constructed and analyzed in detail. Section 7 contains computational aspects, and in Section 8, conclusions are drawn. The non-cited references [1], [2], [6], and [7] are given because they may be useful to the reader.

2. The state-space description of \( M\ddot{y} + B\dot{y} + Ky = 0 \), \( y(t_0) = y_0, \dot{y}(t_0) = \dot{y}_0 \)

Let \( M, B, K \in \mathbb{R}^{nxn} \) and \( y_0, \dot{y}_0 \in \mathbb{R}^n \). Further, let \( M \) be regular. The matrices \( M, B, \) and \( K \) are the mass, damping, and stiffness matrices, as the case may be; \( y_0 \) is the initial displacement and \( \dot{y}_0 \) is the initial velocity. We study the initial value problem

\[
M\ddot{y} + B\dot{y} + Ky = 0, \; y(t_0) = y_0, \; \dot{y}(t_0) = \dot{y}_0,
\]

where \( y(t) \) is the sought displacement and \( z(t) = y(t) \) is the associated velocity.

2.1. State-space description

Let

\[
x_1 = \begin{bmatrix} y \\ z \end{bmatrix}, \quad x_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}
\]

and

\[
A = \begin{bmatrix} 0 & E \\ -M^{-1}K & -M^{-1}B \end{bmatrix}.
\]

\( x \) is called state vector and \( A \) is called system matrix. Herewith, the above second-order initial value problem is equivalent to the first-order initial value problem of double size,

\[
x = Ax, \; x(t_0) = x_0.
\]

(1)
In the sequel, we need only the special form of $x(t)$.

3. Norm equivalence inequalities for diagonalizable matrix $A$
To show the norm equivalence between $y(t)$ and $\dot{y}(t)$ for diagonalizable matrix $A$, is a very simple task.

First, we formulate some hypotheses and conditions.

3.1. Hypotheses and conditions for diagonalizable matrix $A$

(H1) $m = 2n$ and $A \in \mathbb{R}^{m \times m}$,
(H2) $T^{-1}AT = J$ = diag($\lambda_k$)$_{k=1,...,m}$ where $\lambda_k = \lambda_k(A)$, $k = 1, \ldots, m$ are the eigenvalues of $A$,
(H3) $\lambda_i = \lambda_i(A) \neq 0$, $i = 1, \ldots, m$,
(HS) the eigenvectors $p_1, \ldots, p_n, \bar{p}_1, \ldots, \bar{p}_n$ form a basis of $\mathbb{E}^{m \times m}$.

Remark Let (H1) be fulfilled and let $Ap = \lambda p$. Then, we have $A\bar{p} = \bar{\lambda} \bar{p}$, where the bar denotes the complex conjugate. So, together with $(\lambda, p)$, also $(\bar{\lambda}, \bar{p})$ is a solution of the eigenvalue problem $Ap = \lambda p$. But, if $\lambda$ and $p$ would be real, then $p$ and $\bar{p}$ would not be linearly independent. This situation cannot happen when hypothesis (HS) is supposed.

Remark In the sequel, when the special hypothesis (HS) is chosen, we do this in order to be specific in the construction of a solution basis, used already in Kohaupt (2008b, 2011). Other cases such as $A \in \mathbb{R}^{1 \times 1}$ can be handled in a similar manner, however.

Hypothesis (H4) defined in Kohaupt (2008b, 2011) is not need here.

As a preparation to the derivation of the norm equivalence of the displacement and velocity vectors, we collect some definitions, respectively, notations and representations for the solution vector $x(t)$ from Kohaupt (2011).

3.2. Representation of the basis $x^{(i)}_k(t), x^{(r)}_k(t), k = 1, \ldots, n$
Under the hypotheses (H1), (H2), and (HS), from Kohaupt (2011), we obtain the following real basis functions for the ODE $\dot{x} = Ax$:

\[
\begin{align*}
\dot{x}^{(i)}_k(t) &= e^{\bar{x}^t(t-t_0)} \left[ \cos \lambda_k(t-t_0) p_k^{(i)} - \sin \lambda_k(t-t_0) p_k^{(r)} \right], \\
\dot{x}^{(r)}_k(t) &= e^{\bar{x}^t(t-t_0)} \left[ \sin \lambda_k(t-t_0) p_k^{(i)} + \cos \lambda_k(t-t_0) p_k^{(r)} \right],
\end{align*}
\]

$\dot{x}^{(i)}_k(t), \dot{x}^{(r)}_k(t)$, $k = 1, \ldots, n$, where

$\dot{x}^{(i)}_k(t)$ = $\lambda_k^{(i)} + i\lambda_k^{(r)} = \text{Re} \lambda_k + i\text{Im} \lambda_k$,

$\dot{x}^{(r)}_k(t)$ = $\lambda_k^{(r)} + ip_k^{(r)} = \text{Re} p_k + i\text{Im} p_k$,

$k = 1, \ldots, m = 2n$ are the decompositions of $\lambda_k$ and $p_k$ into their real and imaginary parts. As in Kohaupt (2011), the indices are chosen such that $x_{n+k} = \lambda_k, x_{n+k} = \bar{p}_k, k = 1, \ldots, n$.

3.3. The spectral abscissa of $A$ with respect to the initial vector $x_0 \in \mathbb{R}^n$
Let $x^{(i)}_k, k = 1, \ldots, m = 2n$ be the eigenvectors of $A^*$ corresponding to the eigenvalues $\lambda_k, k = 1, \ldots, m = 2n$. Under (H1), (H2), and (HS), the solution $x(t)$ of (1) has the form

\[
x(t) = \sum_{k=1}^{m=2n} c_{ik} x_k^{(i)} e^{\lambda_k^{(i)}(t-t_0)} + \sum_{k=1}^{n} c_{ik} e^{\lambda_k^{(i)}(t-t_0)} + c_{kn} \bar{p}_k^{(r)} e^{\lambda_k^{(r)}(t-t_0)}
\]

with uniquely determined coefficients $c_{ik}, k = 1, \ldots, m = 2n$. Using the relations
\[ c_{2k} = c_{1,1+k} = \overline{c}_{1k}, \quad k = 1, \ldots, n, \quad (4) \]

(see Kohaupt, 2008b, Section 3.1 for the last relation), then according to Kohaupt (2008a), the spectral abscissa of \( A \) with respect to the initial vector \( x_0 \in \mathbb{R}^n \) is given by

\[ \nu_0 = \nu_x[A] = \max_{k=1,\ldots,m=2n} \{ \lambda^{(r)}(A) | x_0 \perp u_k^r \} \]
\[ = \max_{k=1,\ldots,m=2n} \{ \lambda^{(r)}(A) | c_{1k} \neq 0 \} \]
\[ = \max_{k=1,\ldots,n} \{ \lambda^{(r)}(A) | c_{1k} \neq 0 \} \]
\[ = \max_{k=1,\ldots,n} \{ \lambda^{(r)}(A) | x_0 \perp u_k^r \} \quad (5) \]

3.4. **Index sets**

In the sequel, we need the following index sets:

\[ J_\nu = \{ k_0 \in \mathbb{N} | 1 \leq k_0 \leq n \ \text{and} \ \lambda^{(r)}_{k_0}(A) = \nu_0 \} \quad (6) \]

and

\[ J_{\nu}^- = \{ 1, \ldots, n \} \setminus J_\nu \]
\[ = \{ k_0 \in \mathbb{N} | 1 \leq k_0 \leq n \ \text{and} \ \lambda^{(r)}_{k_0}(A) < \nu_0 \}. \quad (7) \]

3.5. **Appropriate representation of \( x(t) \)**

We have

\[ x(t) = \sum_{k=1}^{n} [c^{(r)}_k x^{(r)}_k(t) + c^{(i)}_k x^{(i)}_k(t)] \]

with

\[ c^{(r)}_k = 2 \Re c_{1k}, \quad c^{(i)}_k = -2 \Im c_{1k}, \quad k = 1, \ldots, n \]

(cf. Kohaupt, 2008b). Thus, due to (2),

\[ x(t) = \sum_{k=1}^{n} e^{\lambda^{(r)}(t-t_0)} f_k(t) \quad (8) \]

with

\[ f_k(t) = c^{(r)}_k [\cos \lambda^{(i)}_k(t-t_0)p^{(r)}_k - \sin \lambda^{(i)}_k(t-t_0)p^{(i)}_k] \]
\[ + c^{(i)}_k [\sin \lambda^{(i)}_k(t-t_0)p^{(r)}_k + \cos \lambda^{(i)}_k(t-t_0)p^{(i)}_k], \quad k = 1, \ldots, n \quad (9) \]

3.6. **Appropriate representation of \( bdy(t) \) and \( \dot{y}(t) \)**

Let

\[ p_k = \begin{bmatrix} q_k \\ r_k \end{bmatrix}, \quad p^{(r)}_k = \begin{bmatrix} q^{(r)}_k \\ r^{(r)}_k \end{bmatrix}, \quad p^{(i)}_k = \begin{bmatrix} q^{(i)}_k \\ r^{(i)}_k \end{bmatrix}, \quad k = 1, \ldots, m = 2n. \]

Then, from (8), (9),

\[ p_k = \begin{bmatrix} q_k \\ r_k \end{bmatrix}, \quad p^{(r)}_k = \begin{bmatrix} q^{(r)}_k \\ r^{(r)}_k \end{bmatrix}, \quad p^{(i)}_k = \begin{bmatrix} q^{(i)}_k \\ r^{(i)}_k \end{bmatrix}, \quad k = 1, \ldots, m = 2n. \]

with \( q_k, r_k \in \mathbb{C}^m, \quad q^{(r)}_k, r^{(r)}_k, q^{(i)}_k, r^{(i)}_k \in \mathbb{R}^n, \quad k = 1, \ldots, m = 2n. \)
After these preparations, for the quantities $J_{q_k}, g_k(t), h_k(t)$, we formulate the following conditions:

There exists a $t_1 \geq t_0$ such that

$$\mathcal{C}_g(t \geq t_1) \quad \sum_{k \in J_{q_k}} g_k(t) \neq 0, \quad t \geq t_1 \geq t_0,$$

$$\mathcal{C}_h(t \geq t_1) \quad \sum_{k \in J_{h_k}} h_k(t) \neq 0, \quad t \geq t_1 \geq t_0.$$

For these conditions, there are sufficient algebraic conditions, as the case may be:

(A) $q_k^{(i)}, q_k^{(j)}, k \in J_{q_k}$ are linearly independent,

(B) $r_k^{(i)}, r_k^{(j)}, k \in J_{h_k}$ are linearly independent.

In the examples of Section 6, also the case occurs that the above conditions are not fulfilled. For this, we formulate the following conditions:

For every $t_1 \geq t_0$ there exists a $t \geq t_1$ such that

$$\mathcal{C}_g(t \geq t_1) \quad \sum_{k \in J_{q_k}} g_k(t) = 0,$$

$$\mathcal{C}_h(t \geq t_1) \quad \sum_{k \in J_{h_k}} h_k(t) = 0.$$

For these conditions, there are sufficient algebraic conditions, as the case may be (see Kohaupt, 2011):

(A) $J_{q_k} = \{ k_0 \}$ and $q_k^{(i)}, q_k^{(j)}$ are linearly dependent,

(B) $J_{h_k} = \{ k_0 \}$ and $r_k^{(i)}, r_k^{(j)}$ are linearly dependent.
Further, here and in the sequel, we denote by \( \| \cdot \| \) any vector norm.

**Theorem 1** (Norm equivalence of \( y(t) \) and \( y(t) \) for diagonalizable matrix \( A \))

Let the hypotheses \((H1), (H2), \) and \((HS)\) as well as the conditions \((C_0, t \geq t_1)\), \((C_0, t \geq t_1)\) or the sufficient algebraic conditions \((A_{b1}, (A_{b1}, b_1))\) be fulfilled. Then, there exist constants \(c_0 > 0\) and \(c_1 > 0\) such that \(c_0\|y(t)\| \leq \|y(t)\| \leq c_1\|y(t)\|, \ t \geq t_2,\)

for sufficiently large \(t_2 \geq t_1 \geq t_0\).

**Proof** The proof follows immediately from Kohaupt (2011, Theorems 7 and 13) or Kohaupt (2011, Theorems 8 and 14).

4. Norm equivalence inequalities for general matrix \( A \)

In this section, we prove the same statement for a general square matrix \( A \) as in Theorem 1 for a diagonalizable matrix \( A \). This cannot be deduced in a similar way as for Theorem 1, that is, it cannot be done by Kohaupt (2011, Theorems 9 and 15) since they contain the factor \( e^{(\mathbf{t} - \mathbf{t}_0)} \) on the right-hand side. Neither can it be done by Kohaupt (2011, Theorems 12 and 16) since they contain the factor \( e^{(\mathbf{t} - \mathbf{t}_0)} \) on the right-hand side and the factor \( e^{(\mathbf{t} - \mathbf{t}_0)} \) on the left-hand side. Nevertheless, the same equivalence inequalities as in Theorem 1 hold true in the general case. The proof, however, is much more involved.

Again, first we formulate some hypotheses and conditions.

4.1. Hypotheses and conditions for general square matrix \( A \)

\((H1')\) \( m = 2n \) and \( A \in \mathbb{R}^{mxm}, \)
\((H2')\) \( T^{-1}AT = J = \text{diag}(J_i(\lambda_i))_{i=1,...,r}, \) where \( J_i(\lambda_i) \in \mathbb{C}^{m_i \times m_i} \) are the canonical Jordan forms,
\((H3')\) \( \lambda_i \neq 0, \ i = 1, ..., r, \)
\((HS')\) \( r = 2\rho, \) and the principal vectors \( p_{11}^{(1)} , ..., p_{1\rho}^{(1)} , ..., p_{\rho 1}^{(1)}, ..., p_{\rho \rho}^{(1)} \) form a basis of \( \mathbb{C}^{mxm}. \)

We mention that for the special hypothesis \((HS')\) similar remarks hold as for \((HS)\) in the case of diagonalizable matrices \( A. \)

Let \((H1'), (H2'), \) and \((HS')\) be fulfilled and \( Ap_k^{(l)} = \lambda_k p_k^{(l)}, k = 1, ..., m_i, l = 1, ..., r, \) where the indices are chosen such that \( \lambda_k = \hat{\lambda}_l, l = 1, ..., \rho \) and \( p_k^{(l)} = \hat{p}_k^{(l)}, k = 1, ..., m_i, l = 1, ..., \rho. \) The vectors \( p_k^{(l)} \) are the principal vectors of stage \( k \) corresponding to the eigenvalue \( \lambda_k \) of \( A. \)

Hypothesis \((H4')\) defined in Kohaupt (2008b, 2011) is not needed here.

In the case of a general square matrix \( A, \) we also have to collect some definitions, respectively, notations and representations of \( x(t) \) from Kohaupt (2008b, 2011).

4.2. Representation of the basis \( x_k^{(r)}(t), x_k^{(l)}(t), k = 1, ..., m_i, l = 1, ..., \rho \)

Under the hypotheses \((H1'), (H2'), \) and \((HS')\), from Kohaupt (2008b) we obtain the following real basis functions for the ODE \( x = Ax: \)
\[ x_k^{(i)}(t) = e^{i\lambda_k t} \left\{ \cos \lambda_k^i(t - t_0) \left[ p_1^{(l)} \frac{(t - t_0)^{k-1}}{(k-1)!} + \ldots + p_{k-1}^{(l)} (t - t_0) + p_k^{(l)} \right] 
- \sin \lambda_k^i(t - t_0) \left[ p_1^{(r)} \frac{(t - t_0)^{k-1}}{(k-1)!} + \ldots + p_{k-1}^{(r)} (t - t_0) + p_k^{(r)} \right] \right\}, \]

\[ x_k^{(l)}(t) = e^{i\lambda_k t} \left\{ \sin \lambda_k^i(t - t_0) \left[ p_1^{(l)} \frac{(t - t_0)^{k-1}}{(k-1)!} + \ldots + p_{k-1}^{(l)} (t - t_0) + p_k^{(l)} \right] 
+ \cos \lambda_k^i(t - t_0) \left[ p_1^{(r)} \frac{(t - t_0)^{k-1}}{(k-1)!} + \ldots + p_{k-1}^{(r)} (t - t_0) + p_k^{(r)} \right] \right\} \]

\[ k = 1, \ldots, m, \ l = 1, \ldots, \rho, \]

where

\[ p_k^{(l)} = p_k^{(r)} + i p_k^{(i)} \]

is the decomposition of \( p_k^{(i)} \) into its real and imaginary part.

### 4.3. The spectral abscissa of \( A \) with respect to the initial vector \( x_0 \in \mathbb{R}^n \)

Let \( u_k^{(i)}, \ k = 1, \ldots, m, \) be the principal vectors of stage \( k \) of \( A^* \) corresponding to the eigenvalue \( \lambda_k, \ l = 1, \ldots, r = 2\rho. \) Under \((H1'), (H2'),\) and \((HS'),\) the solution \( x(t) \) of (1) has the form

\[ x(t) = \sum_{l=1}^{r=2} \sum_{k=1}^{m} c_{1k}^{(l)} x_k^{(l)}(t) = \sum_{l=1}^{r=2} \sum_{k=1}^{m} [c_{1k}^{(l)} x_k^{(l)}(t) + c_{2k}^{(l)} x_k^{(l)}(t)]. \]

with uniquely determined coefficients \( c_{1k}^{(l)}, \ k = 1, \ldots, m, \ l = 1, \ldots, r = 2\rho. \) Using the relations

\[ c_{1k}^{(l)} = (x_0, u_k^{(l)*}), \ k = 1, \ldots, m, \ l = 1, \ldots, \rho \]

\[ c_{2k}^{(l)} = c_{1k}^{(l)*}, \ l = 1, \ldots, \rho \]

(see [Section 3.2] Kohaupt, 2008b for the last relation), then the spectral abscissa of \( A \) with respect to the initial vector \( x_0 \in \mathbb{R}^n \) is

\[ \nu_{\alpha} = \nu_{\alpha}(A) = \max_{l=1, \ldots, r=2\rho} \left\{ \lambda_l^{(i)}(A) \mid x_0 \perp M_{\alpha(A)} = [u_1^{(i)*}, \ldots, u_m^{(i)*}] \right\} \]

\[ \nu_{\alpha} = \max_{l=1, \ldots, r=2\rho} \left\{ \lambda_l^{(i)}(A) \mid c_{1k}^{(l)} \neq 0 \text{ for at least one } k \in \{1, \ldots, m\} \right\} \]

\[ \nu_{\alpha} = \max_{l=1, \ldots, r=2\rho} \left\{ \lambda_l^{(i)}(A) \mid c_{1k}^{(l)} \neq 0 \text{ for at least one } k \in \{1, \ldots, m\} \right\} \]

\[ \nu_{\alpha} = \max_{l=1, \ldots, r=2\rho} \left\{ \lambda_l^{(i)}(A) \mid x_0 \perp M_{\alpha(A)} = [u_1^{(i)*}, \ldots, u_m^{(i)*}] \right\} \]

\[ \nu_{\alpha} = \max_{l=1, \ldots, r=2\rho} \left\{ \lambda_l^{(i)}(A) \mid c_{1k}^{(l)} \neq 0 \text{ for at least one } k \in \{1, \ldots, m\} \right\} \]

\[ \nu_{\alpha} = \max_{l=1, \ldots, r=2\rho} \left\{ \lambda_l^{(i)}(A) \mid x_0 \perp M_{\alpha(A)} = [u_1^{(i)*}, \ldots, u_m^{(i)*}] \right\} \]

### 4.4. Index sets

For the sequel, we need the following index sets:

\[ J_{\alpha} = \{ l_0 \in \mathbb{N} \mid 1 \leq l_0 \leq \rho \text{ and } \lambda_l^{(i)}(A) = \nu_{\alpha} \} \]

(19)

and

\[ J_{\alpha} = \{ 1, \ldots, \rho \} \setminus J_{\alpha} \]

\[ \nu_{\alpha} = \{ l_0 \in \mathbb{N} \mid 1 \leq l_0 \leq \rho \text{ and } \lambda_l^{(i)}(A) < \nu_{\alpha} \}. \]

(20)
4.5. Appropriate representation of \( x(t) \)

We have

\[
x(t) = \sum_{i=1}^{\rho} \sum_{k=1}^{m_i} [c_k^{(i)} x_k^{(i)}(t) + c_k^{(i)} x_k^{(i)}(t)]
\]

with

\[
c_k^{(i)} = 2 \text{Re } c_k^{(i)}, \quad c_k^{(i)} = -2 \text{Im } c_k^{(i)}, \quad k = 1, \ldots, m, \quad l = 1, \ldots, \rho
\]

(cf. Kohaupt, 2008b). Thus, due (18),

\[
x(t) = \sum_{i=1}^{\rho} e^{i\omega(t-t_0)} \sum_{k=1}^{m_i} f_k^{(i)}(t)
\]

with

\[
f_k^{(i)}(t) = c_k^{(i)} \left\{ \cos \lambda_k^{(i)}(t-t_0) \left[ p_1^{(i)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + p_k^{(i)}(t-t_0) + p_k^{(i)} \right] 
- \sin \lambda_k^{(i)}(t-t_0) \left[ \frac{p_1^{(i)}(t-t_0)^{k-1}}{(k-1)!} + \ldots + p_k^{(i)}(t-t_0) + p_k^{(i)} \right] \right\}
+ c_k^{(i)} \left\{ \sin \lambda_k^{(i)}(t-t_0) \left[ p_1^{(i)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + p_k^{(i)}(t-t_0) + p_k^{(i)} \right] 
+ \cos \lambda_k^{(i)}(t-t_0) \left[ \frac{p_1^{(i)}(t-t_0)^{k-1}}{(k-1)!} + \ldots + p_k^{(i)}(t-t_0) + p_k^{(i)} \right] \right\}
\]

\( k = 1, \ldots, m, \quad l = 1, \ldots, \rho \)

4.6. Appropriate representation of \( y(t) \) and \( y'(t) \)

Set

\[
p_k^{(i)} = \begin{bmatrix} q_k^{(i)} \\ r_k^{(i)} \end{bmatrix}, \quad p_k^{(i)} = \begin{bmatrix} q_k^{(i)} \\ r_k^{(i)} \end{bmatrix}, \quad p_k^{(i)}(t) = \begin{bmatrix} q_k^{(i)} \\ r_k^{(i)} \end{bmatrix}.
\]

with \( q_k^{(i)}, r_k^{(i)} \in \mathbb{C}^n, q_k^{(i)}, r_k^{(i)} \in \mathbb{R}^n, k = 1, \ldots, m, \quad l = 1, \ldots, \rho. \)

Then, from (24), (25)

\[
y(t) = \sum_{i=1}^{\rho} e^{i\omega(t-t_0)} \sum_{k=1}^{m_i} g_k^{(i)}(t)
\]

with

\[
g_k^{(i)}(t) = c_k^{(i)} \left\{ \cos \lambda_k^{(i)}(t-t_0) \left[ q_k^{(i)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + q_k^{(i)}(t-t_0) + q_k^{(i)} \right] 
- \sin \lambda_k^{(i)}(t-t_0) \left[ q_1^{(i)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + q_k^{(i)}(t-t_0) + q_k^{(i)} \right] \right\}
+ c_k^{(i)} \left\{ \sin \lambda_k^{(i)}(t-t_0) \left[ q_k^{(i)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + q_k^{(i)}(t-t_0) + q_k^{(i)} \right] 
+ \cos \lambda_k^{(i)}(t-t_0) \left[ q_k^{(i)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + q_k^{(i)}(t-t_0) + q_k^{(i)} \right] \right\}
\]

\( k = 1, \ldots, m, \quad l = 1, \ldots, \rho \)
After these preparations, for the quantities $J_{v_1}$, $J_{v_2}$, $g_{\lambda}^{(i)}(t)$, $h_{\lambda}^{(i)}(t)$, we formulate the following conditions:

There exists a $t_1 \geq t_0$ such that

\[
(C'_g(t \geq t_1)) \sum_{k \in J_{v_1}}^{m_1} \sum_{k=1}^{m_1} g_{\lambda}^{(i)}(t) \neq 0, \quad t \geq t_1 \geq t_0,
\]

\[
(C'_h(t \geq t_1)) \sum_{k \in J_{v_2}}^{m_1} \sum_{k=1}^{m_1} h_{\lambda}^{(i)}(t) \neq 0, \quad t \geq t_1 \geq t_0.
\]

For these conditions, there are sufficient algebraic conditions, as the case may be (see Kohaupt, 2011). However, the following sufficient conditions are not so stringent in that only conditions on components of eigenvectors are used and not on the set of components of all principal vectors. The sufficient algebraic conditions read:

\[
(A'_{g_1}) \quad q_{\lambda}^{(i)}, \quad r_{\lambda}^{(i)}, \quad l \in J_{v_1}, \quad \text{are linearly independent,}
\]

\[
(A'_{h_1}) \quad r_{\lambda}^{(i)}, \quad r_{\lambda}^{(i)}, \quad l \in J_{v_2}, \quad \text{are linearly independent.}
\]

The above conditions are not always fulfilled. For this, we formulate the following conditions:

For every $t_1 \geq t_0$ there exists a $t \geq t_1$ such that

\[
(C_g(t \geq t_1)) \sum_{k \in J_{v_1}}^{m_1} \sum_{k=1}^{m_1} g_{\lambda}^{(i)}(t) = 0,
\]

\[
(C_h(t \geq t_1)) \sum_{k \in J_{v_2}}^{m_1} \sum_{k=1}^{m_1} h_{\lambda}^{(i)}(t) = 0.
\]
For these conditions, there are sufficient algebraic conditions, as the case may be, (see Kohaupt, 2011). However, the following sufficient conditions are not so stringent in that we do not suppose on the algebraic multiplicity that \( m_0 = 1 \). The sufficient algebraic conditions read:

\[
\begin{align*}
(A_{h,1}^T) & \quad J_{y_0} = \{ l_0 \} \quad \text{and} \quad q_{1}^{(l_0)} r_{1}^{(l_0)} \quad \text{are linearly dependent}, \\
(A_{h,1}^T) & \quad J_{y_0} = \{ l_0 \} \quad \text{and} \quad r_{1}^{(l_0)} r_{1}^{(l_0)} \quad \text{are linearly dependent}.
\end{align*}
\]

For the next lemma, we set:

\[
\begin{align*}
Y(t) & = \left\| \sum_{i \in J_y} \sum_{k=1}^{m_i} g_i^{(l)}(t) \right\|, \\
Z(t) & = \left\| \sum_{i \in J_y} \sum_{k=1}^{m_i} h_i^{(l)}(t) \right\|.
\end{align*}
\]

The definition of the spectral abscissa \( \nu_0 = \nu_{k,[A]} \) of matrix \( A \) with respect to the initial vector \( x_0 \) can also be found in Kohaupt (2011).

**Lemma 2** Let hypotheses \((H1^*), (H2^*), \) and \((HS^*)\) be fulfilled. Then,

\[
\begin{align*}
\frac{1}{2} Y(t) e^{\nu_0(t-t_0)} & \leq \|y(t)\| \leq 2 Y(t) e^{\nu_0(t-t_0)}, \ t \geq t_1, \\
\frac{1}{2} Z(t) e^{\nu_0(t-t_0)} & \leq \|z(t)\| \leq 2 Z(t) e^{\nu_0(t-t_0)}, \ t \geq t_1,
\end{align*}
\]

for sufficiently large \( t_1 \geq t_0 \).

**Proof** We prove only the first relation. The second one is proven in a similar way. One has

\[
y(t) = \sum_{i=1}^{\rho} e^{i\nu_0(t-t_0)} \sum_{k=1}^{m_i} g_i^{(l)}(t)
\]

\[
= e^{i\nu_0(t-t_0)} \sum_{i \in J_y} \sum_{k=1}^{m_i} g_i^{(l)}(t) + \sum_{i \in J_y} e^{i\nu_0(t-t_0)} \sum_{k=1}^{m_i} g_i^{(l)}(t).
\]

This implies

\[
\|y(t)\| \geq e^{i\nu_0(t-t_0)} \left\| \sum_{i \in J_y} \sum_{k=1}^{m_i} g_i^{(l)}(t) \right\| - \left\| \sum_{i \in J_y} e^{i\nu_0(t-t_0)} \sum_{k=1}^{m_i} g_i^{(l)}(t) \right\|
\]

\[
= e^{i\nu_0(t-t_0)} \left\| \sum_{i \in J_y} \sum_{k=1}^{m_i} g_i^{(l)}(t) \right\| \left[ 1 - \frac{\left\| \sum_{i \in J_y} e^{i\nu_0(t-t_0)} \sum_{k=1}^{m_i} g_i^{(l)}(t) \right\|}{e^{i\nu_0(t-t_0)}} \right]
\]

\[
\geq e^{i\nu_0(t-t_0)} \left\| \sum_{i \in J_y} \sum_{k=1}^{m_i} g_i^{(l)}(t) \right\| \cdot \frac{1}{2}
\]

for sufficiently large \( t_1 \geq t_0 \) since the fraction in the bracket tends to zero. Further,
\[ \|y(t)\| \leq e^{\|\mathcal{A}t\|} \left\| \sum_{i \in I_0} \sum_{k=1}^{m_i} g_i^{(k)}(t) \right\| + \left\| \sum_{i \in I_0} \sum_{k=1}^{m_i} e^{\|\mathcal{A}(t-t_0)\|} g_i^{(k)}(t) \right\| \\
= e^{\|\mathcal{A}t\|} \left\| \sum_{i \in I_0} \sum_{k=1}^{m_i} g_i^{(k)}(t) \right\| \left[ 1 + \frac{\left\| \sum_{i \in I_0} \sum_{k=1}^{m_i} g_i^{(k)}(t) \right\|}{\sum_{i \in I_0} \sum_{k=1}^{m_i} g_i^{(k)}(t)} \right] \\
\leq e^{\|\mathcal{A}t\|} \left\| \sum_{i \in I_0} \sum_{k=1}^{m_i} g_i^{(k)}(t) \right\| \cdot 2 \\
\]

for sufficiently large \( t_1 \geq t_0 \) again since the fraction tends to zero.

For the formulation of the next lemma, we introduce some abbreviations. So, we define

\[ m_i' := \max_{k=1, \ldots, m_i} \left\{ k \mid |c_k^{(l)}|^2 + |c_k^{(i)}|^2 > 0 \right\} = \max_{k=1, \ldots, m_i} \left\{ k \mid c_k^{(l)} \neq 0 \right\}, l \in J_i, \]

where the quantities \( c_k^{(l)} \), \( c_k^{(i)} \), and \( c_k^{(l)} \) are contained in each of the quantities \( g_k^{(l)}(t), h_k^{(i)}(t), \) and \( f_k^{(i)}(t) \).

This clearly implies

\[ c_{l,m_i'}^{(j)} \neq 0, \quad l \in J_i. \]

Further, define

\[ m' := \max_{i \in I_0} m_i', \]

and

\[ J_i' := \{ l \in J_i \mid m_i' = m' \} \subset J_i. \]

as well as

\[ u(t) := \sum_{i \in I_0} \sum_{l \in J_i} c_{m_i'}^{(l)} [\cos x_i^{(l)}(t - t_0)q_1^{(l)} - \sin x_i^{(l)}(t - t_0)q_1^{(l)}] \]

\[ + \frac{c_{m_i'}^{(l)}}{m_i'} [\sin x_i^{(l)}(t - t_0)q_1^{(l)} + \cos x_i^{(l)}(t - t_0)q_1^{(l)}], \]

\[ v(t) := \sum_{i \in I_0} \sum_{l \in J_i} c_{m_i'}^{(l)} [\cos x_i^{(l)}(t - t_0)r_1^{(l)} - \sin x_i^{(l)}(t - t_0)r_1^{(l)}] \]

\[ + \frac{c_{m_i'}^{(l)}}{m_i'} [\sin x_i^{(l)}(t - t_0)r_1^{(l)} + \cos x_i^{(l)}(t - t_0)r_1^{(l)}], \]

\[ \eta(t) := \|u(t)\|, \]

\[ \zeta(t) := \|v(t)\|, \]

\[ p(t) \leq \frac{(t - t_0)^{m' - 1}}{(m' - 1)!}. \]

After these preparations, we are now in a position to state the following lemma.

**Lemma 3** Let hypotheses \((H1'), (H2'), \) and \((H3')\) be fulfilled. Then,

\[ \frac{1}{2} p(t) \eta(t) \leq Y(t) \leq 2 p(t) \eta(t), t \geq t_1, \]

\[ \frac{1}{2} p(t) \zeta(t) \leq Z(t) \leq 2 p(t) \zeta(t), t \geq t_1. \]
for sufficiently large \( t_1 \geq t_0 \).

**Proof**  We prove only the first relation. The second one is proven in a similar way. One has

\[
\sum_{l_0} \sum_{k=1}^{m} g_k(t) = \sum_{l_0} \sum_{k=1}^{m'} c_k(t) \left\{ \cos \lambda_k(t-t_0) \left[ q_{k}^{(1)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + q_{k}^{(1)} (t-t_0) + q_{k}^{(1)} \right] 
- \sin \lambda_k(t-t_0) \left[ q_{k}^{(2)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + q_{k}^{(2)} (t-t_0) + q_{k}^{(2)} \right] 
+ c_k \left\{ \sin \lambda_k(t-t_0) \left[ q_{k}^{(3)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + q_{k}^{(3)} (t-t_0) + q_{k}^{(3)} \right] 
+ \cos \lambda_k(t-t_0) \left[ q_{k}^{(4)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + q_{k}^{(4)} (t-t_0) + q_{k}^{(4)} \right] \right\} \right].
\]

This delivers

\[
\sum_{l_0} \sum_{k=1}^{m} g_k(t) = \sum_{l_0} \sum_{k=1}^{m'} c_k(t) \left\{ \cos \lambda_k(t-t_0) \left[ q_{k}^{(1)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + q_{k}^{(1)} (t-t_0) + q_{k}^{(1)} \right] 
- \sin \lambda_k(t-t_0) \left[ q_{k}^{(2)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + q_{k}^{(2)} (t-t_0) + q_{k}^{(2)} \right] 
+ c_k \left\{ \sin \lambda_k(t-t_0) \left[ q_{k}^{(3)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + q_{k}^{(3)} (t-t_0) + q_{k}^{(3)} \right] 
+ \cos \lambda_k(t-t_0) \left[ q_{k}^{(4)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \ldots + q_{k}^{(4)} (t-t_0) + q_{k}^{(4)} \right] \right\} \right]\]

+ \ldots

+ c_{m_1} \left\{ \cos \lambda_{m_1}(t-t_0) \left[ q_{m_1}^{(1)} \frac{(t-t_0)^{m_1-1}}{(m_1-1)!} + \ldots + q_{m_1}^{(1)} (t-t_0) + q_{m_1}^{(1)} \right] 
- \sin \lambda_{m_1}(t-t_0) \left[ q_{m_1}^{(2)} \frac{(t-t_0)^{m_1-1}}{(m_1-1)!} + \ldots + q_{m_1}^{(2)} (t-t_0) + q_{m_1}^{(2)} \right] \right\} \right].
\]

We note that the terms containing the vectors \( q_{m_1}^{(1)} \) and \( q_{m_1}^{(2)} \) with \( c_{m_1}^{(1)} = c_{m_1}^{(2)} \) and \( c_{m_1}^{(1)} = c_{m_1}^{(2)} \) give us the function \( \eta(t) \), and we mention that it has the factor \( p(t) \), both defined above. The rest of the sum is denoted by \( R(t) \), it can be estimated from above by polynomials of degree less than \( m_1 \). So, we obtain

\[
Y(t) = \left\| \sum_{l_0} \sum_{k=1}^{m} g_k(t) \right\| = \left\| \sum_{l_0} \sum_{k=1}^{m'} c_k(t) \right\| = \left\| p(t) \eta(t) + R(t) \right\|.
\]

This entails, taking into account the definition of the function \( \eta(t) \),

\[
Y(t) = \left\| p(t) \eta(t) (1 + R(t)/p(t) \eta(t)) - \right\| \leq p(t) \eta(t)(1 - ||R(t)||/p(t) \eta(t)) \geq \frac{1}{2} p(t) \eta(t),
\]

for sufficiently large \( t_1 \geq t_0 \) since the last fraction tends to zero as \( t \) tends to infinity. Similarly,
\[ Y(t) = \| p(t)u(t) | 1 + R(t)/p(t)u(t) | \| \leq p(t)\eta(t)(1 + \| R(t) \| /p(t)u(t)) \leq 2p(t)\eta(t), \]

for sufficiently large \( t_1 \geq t_\varphi \)

The next lemma is also important for results in the sequel.

**Lemma 4.** Let hypotheses \((H1^{'1}), (H2^{'1})\) and \((HS^{'})\) be fulfilled. If additionally the sufficient algebraic condition \((A_{g,1}^{'})\), respectively, \((A_{h,1}^{'})\) is satisfied, then

\[ \nu(t) \neq 0, \quad \tau \geq t_1, \]

\[ \zeta(t) \neq 0, \quad \tau \geq t_1. \]

as the case may be, for sufficiently large \( t_1 \geq t_\varphi \)

On the other hand, if additionally the sufficient algebraic condition \((A_{g,1}^{'})\), respectively, \((A_{h,1}^{'})\) is satisfied, then for every \( t_1 \geq t_\varphi \) there exists a \( t \geq t_1 \) such that

\[ \nu(t) = 0, \]

\[ \zeta(t) = 0, \]

as the case may be, meaning correspondingly,

\[ Y(t) = 0, \]

\[ Z(t) = 0. \]

**Proof.** We prove only the first relation. The second one is proven in a similar way. Assume that for all \( t_1 \geq t_\varphi \) there exists a \( t \geq t_1 \) such that \( \nu(t) = 0 \). Then, \( u(t) = 0 \) so that

\[
\sum_{i \in J_{\gamma}} \left\{ \left[ c_{m_i}^{(r)} \cos \lambda_i(t - t_\varphi) + c_{m_i}^{(u)} \sin \lambda_i(t - t_\varphi) \right] q_{1,1}^{(i)} \right\} + \left[ -c_{m_i}^{(r)} \sin \lambda_i(t - t_\varphi) + c_{m_i}^{(u)} \cos \lambda_i(t - t_\varphi) \right] q_{2,1}^{(i)} = 0.
\]

Now, due to \((A_{g,1}^{'})\), the vectors \( q_{1,1}^{(i)}, q_{2,1}^{(i)}, l \in J_{\gamma} \) are linearly independent. Therefore,

\[
c_{m_i}^{(r)} \cos \lambda_i(t - t_\varphi) + c_{m_i}^{(u)} \sin \lambda_i(t - t_\varphi) = 0,
\]

\[
-c_{m_i}^{(r)} \sin \lambda_i(t - t_\varphi) + c_{m_i}^{(u)} \cos \lambda_i(t - t_\varphi) = 0,
\]

\( l \in J_{\gamma} \). From this, we conclude that

\[
c_{m_i}^{(r)} = c_{m_i}^{(u)} = 0, \quad l \in J_{\gamma}.
\]

or,

\[
c_{m_i}^{(r)} = c_{m_i}^{(u)} = 0, \quad l \in J_{\gamma}.
\]

This delivers a contradiction since we have seen above that \( c_{m_i}^{(i)} \neq 0 \), \( l \in J_{\gamma} \).

Now, we state the following corollary.

**Corollary 5.** Let hypotheses \((H1^{'1}), (H2^{'1}), \) and \((HS^{'})\) be fulfilled.
(i) If, further, the conditions ($C'_y(t \geq t_1)$) and ($C'_z(t \geq t_1)$) are satisfied, then
\[
\frac{1}{4} \frac{Z(t)}{Y(t)} \leq \frac{\|Z(t)\|}{\|Y(t)\|} \leq 4 \frac{Z(t)}{Y(t)}, \quad t \geq t_2,
\]
and $Y(t) > 0$ and $Z(t) > 0$ for sufficiently large $t_2 \geq t_1 \geq t_o$.

(ii) If, instead, the sufficient algebraic conditions ($A'_g$) and ($A'_h$) are fulfilled, then,
\[
\frac{1}{16} \frac{\zeta(t)}{\eta(t)} \leq 1 \frac{Z(t)}{Y(t)} \leq \frac{\|Z(t)\|}{\|Y(t)\|} \leq 4 \frac{Z(t)}{Y(t)} \leq 16 \frac{\zeta(t)}{\eta(t)}, \quad t \geq t_1,
\]
and $\eta(t) > 0$ and $\zeta(t) > 0$ for sufficiently large $t_1 \geq t_o$.

Proof

This follows from Lemmas 2 to 4.

The last corollary is the basis for the derivation of the following theorem that is the main theoretical result of this paper.

THEOREM 6  (Norm equivalence of $y(t)$ and $\dot{y}(t)$; general square matrix $A$)

Let hypotheses ($H_1^*$), ($H_2^*$), and ($H^*_S$) be fulfilled.

(i) If, further, the conditions ($C'_y(t \geq t_1)$) and ($C'_z(t \geq t_1)$) are satisfied, then there exist constants $c_0 > 0$ and $c_1 > 0$ such that
\[
c_0 \|y(t)\| \leq \|\dot{y}(t)\| = \|z(t)\| \leq c_1 \|y(t)\|, \quad t \geq t_2,
\]
for sufficiently large $t_2 \geq t_1 \geq t_o$.

(ii) If, instead, ($A'_g$) and ($A'_h$) hold, then the above equivalence inequalities are valid for $t_1 = t_o$.

Proof

(i) Due to Corollary 5, we have
\[
\frac{1}{4} \frac{Z(t)}{Y(t)} \leq \frac{\|Z(t)\|}{\|Y(t)\|} \leq 4 \frac{Z(t)}{Y(t)}, \quad t \geq t_2,
\]
for sufficiently large $t_2 \geq t_1 \geq t_o$. Now, $Y(t)$ and $Z(t)$ are positive for $t \geq t_2$. Thus, due to the periodicity and continuity of $Y(t)$ and $Z(t)$ the extreme values
\[
Y_{\min} := \min_{t \geq t_2} Y(t),
\]
\[
Z_{\min} := \min_{t \geq t_2} Z(t),
\]
\[
Y_{\max} := \max_{t \geq t_2} Y(t),
\]
\[
Z_{\max} := \max_{t \geq t_2} Z(t)
\]
exist and are positive. Therefore,
\[
\frac{1}{4} \frac{Z_{\min}}{Y_{\max}} \leq \frac{\|Z(t)\|}{\|Y(t)\|} \leq 4 \frac{Z_{\max}}{Y_{\min}}, \quad t \geq t_2,
\]
so that Theorem 6 follows with $c_0 = \frac{1}{4} Z_{\min}/Y_{\max}$ and $c_1 = 4 Z_{\max}/Y_{\min}$.

(ii) Due to Corollary 5, we have
\[
\frac{1}{16} \frac{\zeta(t)}{\eta(t)} \leq \frac{\|Z(t)\|}{\|Y(t)\|} \leq 16 \frac{\zeta(t)}{\eta(t)}, \quad t \geq t_1.
\]
Now, $\eta(t)$ and $\zeta(t)$ are periodic and continuous as well as positive for sufficiently large $t_1 \geq t_o$. Thus, the extreme values
$$\eta_{\min} := \min_{t \in [t_1]} \eta(t),$$

$$\zeta_{\min} := \min_{t \in [t_1]} \zeta(t),$$

$$\eta_{\max} := \max_{t \in [t_1]} \eta(t),$$

$$\zeta_{\max} := \max_{t \in [t_1]} \zeta(t)$$

exist and are positive. Therefore,

$$\frac{1}{16} \frac{\zeta_{\min}}{\eta_{\max}} \leq \frac{\|x(t)\|}{\|y(t)\|} \leq 16 \frac{\zeta_{\max}}{\eta_{\min}}, \quad t \geq t_1,$$

so that Theorem 6 follows with $c_0 = \frac{1}{16} \frac{\zeta_{\min}}{\eta_{\max}}$ and $c_1 = 16 \frac{\zeta_{\max}}{\eta_{\min}}$. \hfill \square

5. Applications

As applications, we improve Kohaupt (2011, Theorems 12 and 16). The corresponding results are known so far only for $x(t)$ (cf. Kohaupt, 2011).

**Theorem 7** (Improvement of Kohaupt, 2011, Theorem 12)

Let hypotheses (H1'), (H2'), and (H3') be fulfilled. Moreover, let $w(t)$ be defined by Kohaupt (2011, (41)). If the conditions ($C'_\gamma(t \geq t_1)$) and ($C'_\gamma(t \geq t_1)$) are satisfied, then there exist constants $\eta_0 > 0$ and $\eta_1 > 0$ such that

$$\eta_0 \|w(t)\| \leq \|y(t)\| \leq \eta_1 \|w(t)\|, \quad t \geq t_1,$$

for sufficiently large $t_1 \geq t_0$. The same holds true if the sufficient algebraic conditions ($A_{\eta_0}$) and ($A'_{\eta_1}$) hold.

**Proof** From Theorem 6 and the equivalence of norms in finite-dimensional spaces, it follows that

$$c_{0,\infty} \|y(t)\|_{\infty} \leq \|y(t)\|_{\infty} \leq c_{1,\infty} \|y(t)\|_{\infty}, \quad t \geq t_1.$$  

(28)

Further,

$$\|y(t)\|_2 \leq \left( \|y(t)\|_2^2 + \|y'(t)\|_2^2 \right)^{1/2} = \|x(t)\|_2, \quad t \geq t_0.$$  

(29)

Moreover, using (28), we get

$$\|y(t)\|_{\infty} = \frac{1}{2} \|y(t)\|_{\infty} + \frac{1}{2} \|y(t)\|_{\infty} = \frac{1}{2} \|y(t)\|_{\infty} + \frac{1}{2} \|y'(t)\|_{\infty}$$

$$\geq \min \left\{ \frac{1}{2}, \frac{1}{2} \right\} (\|y(t)\|_{\infty} + \|y'(t)\|_{\infty})$$

$$\geq \min \left\{ \frac{1}{2}, \frac{1}{2} \right\} \max \{\|y(t)\|_{\infty}, \|y'(t)\|_{\infty}\}$$

$$= \min \left\{ \frac{1}{2}, \frac{1}{2} \right\} \|x(t)\|_{\infty}, \quad t \geq t_1.$$  

(30)

Due to the equivalence of norms in finite-dimensional spaces, from (29) and (30), we infer that there exist constants $\gamma_0 > 0$ and $\gamma_1 > 0$ such that

$$\gamma_0 \|x(t)\| \leq \|y(t)\| \leq \gamma_1 \|x(t)\|,$$

(31)

$t \geq t_0$, for sufficiently large $t_1$. By (31) and Kohaupt (2011, Theorem 6), the proof follows. \hfill \square

Further, we have  

...
Theorem 8  (Improvement of Kohaupt, 2011, Theorem 16) Let hypotheses (H1'), (H2'), and (H5') be fulfilled. Moreover, let \( w(t) \) be defined by Kohaupt (2011, (41)). If the conditions \( \langle C'_g(t \geq t_1) \rangle \) and \( \langle C'_n(t \geq t_1) \rangle \) are satisfied, then there exist constants \( \zeta_0 > 0 \) and \( \zeta_1 > 0 \) such that

\[
\zeta_0 \| w(t) \| \leq \| z(t) \| = \| \dot{y}(t) \| \leq \zeta \| w(t) \|, \quad t \geq t_1,
\]

for sufficiently large \( t_1 \geq t_0 \). The same holds true if the sufficient algebraic conditions \( (A'_{g,0}) \) and \( (A'_{n,0}) \) hold.

Proof  The proof is similar to that of Theorem 7 and is therefore omitted.

6. Numerical examples

In this section, we illustrate the obtained results by examples.

We consider the multi-mass vibration model in Figure 1 for \( n \in \{1, 2, 3, 4, 5\} \).

The associated initial value problem is given by

\[
M \ddot{y} + B \dot{y} + Ky = 0, \quad y(0) = y_0, \quad \dot{y}(0) = \dot{y}_0
\]

where \( y = [y_1, \ldots, y_n]^T \) and

\[
M = \begin{bmatrix}
m_1 & m_2 & m_3 & \cdots & m_n \\
m_2 & m_3 & m_4 & \cdots & m_5 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
m_n & m_1 & m_2 & \cdots & m_2 \\
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
b_1 + b_2 & -b_2 & -b_3 & \cdots & -b_n \\
-b_2 & b_2 + b_3 & b_3 + b_4 & \cdots & -b_n \\
-\cdots & \cdots & \cdots & \cdots & \cdots \\
-b_{n-1} & b_{n-1} + b_n & -b_n & b_n + b_{n+1} \\
\end{bmatrix},
\]

\[
K = \begin{bmatrix}
k_1 + k_2 & -k_2 & -k_3 & \cdots & -k_n \\
-k_2 & k_2 + k_3 & k_3 + k_4 & \cdots & -k_n \\
-\cdots & \cdots & \cdots & \cdots & \cdots \\
-k_{n-1} & k_{n-1} + k_n & -k_n & k_n + k_{n+1} \\
\end{bmatrix}
\]

or, in the state-space description

\[
\dot{x}(t) = A x(t), \quad x(0) = x_0,
\]
where the state vector $x$ is given by $x = [y', z']'$, $z = y$, and

where the system matrix $A$ has the form

$$A = \begin{bmatrix}
0 & E \\
-E^{-1}K & -E^{-1}B
\end{bmatrix}.$$  

As in Kohaupt (2011), we specify the values as

$m_j = 1, \quad j = 1, \ldots, n$

$k_j = 1, \quad j = 1, \ldots, n + 1$

and

$$b_j = \begin{cases} 
1/2, & j \text{ even} \\
1/4, & j \text{ odd.}
\end{cases}$$

With the above numerical values, we have

$$M = E,$$

$$B = \begin{bmatrix}
\frac{3}{4}, & -\frac{1}{3} \\
-\frac{1}{2}, & \frac{1}{5} \\
-\frac{1}{2}, & \frac{1}{5} \\
\vdots & \ddots \\
-\frac{1}{2}, & \frac{1}{5} \\
\frac{3}{4}, & -\frac{1}{3} \\
\frac{3}{4}, & -\frac{1}{3} \\
\vdots & \ddots
\end{bmatrix}$$

(if $n$ is even), and

$$K = \begin{bmatrix}
2 & -1 & \vdots \\
-1 & 2 & -1 & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
-1 & 2 & -1 & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
-1 & 2 & -1 & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
2 & -1 & \vdots & \vdots
\end{bmatrix}.$$  

Remark We mention that, in all examples, condition (H3) resp. (H3') is fulfilled, i.e., that all eigenvalues are different from zero. Therefore, the sufficient algebraic conditions $(A_{g,1})$ and $(A_{h,1})$ are equivalent (since then $r_{k}^{(n)} = \lambda_{j}^{(n)}, k \in J$ (see Kohaupt, 2011)). The same holds true for $(A'_{g,1})$ and $(A'_{h,1})$, for $(A_{g,1})$ and $(A_{h,1})$, and for $(A'_{g,1})$ and $(A'_{h,1})$. So, we need only the first sufficient algebraic condition with index $g$, in each case. The stepsize in all figures is $\Delta t = 0.01$.

### 6.1. Illustration of the sufficient algebraic conditions

In this subsection, we illustrate the sufficient algebraic conditions that guarantee the validity, respectively, invalidity of the equivalence inequalities, as the case may be.

Remark In the following Examples 1-5, we have to consider the quantities $\bar{u}_j = (U_j(x_j)) = (v_0, v_j')$, $j = 1, \ldots, m = 2n$ because they play a role in the definition of $\nu_0 = \nu_k(A)$ (see Kohaupt, 2011). Due to the numbering $\lambda_{j}^{(n)}(A) = \lambda_{j}^{(A)} = \lambda_{j}(A')$, $j = 1, \ldots, n$, we have to study only the quantities $\bar{u}_j$ for $j = 1, \ldots, n$ (and not for $j = 1, \ldots, m = 2n$), see also the definition of $\nu_0$ on this.

Example 1: $n = 1$. We choose
\[ y_0 = -1, \ y_0 = 0. \]

Here, \( \tilde{u}_j = (UX_0)_j \neq 0, \ j = 1 \text{ and } 2 \)

\[
\lambda_1(A) = -0.37500000000000 + 1.36358901432946i, \\
\lambda_2(A) = -0.37500000000000 - 1.36358901432946i = \overline{\lambda_1(A)} = \lambda_1(A^*). \tag{32}
\]

Thus,

**Sufficient algebraic condition** (\( A_{5,1} \)):

\[ \nu_0 = v_{x_0} [A] = v[A] = Re \lambda_1(A) = -0.375. \]

We have

\[ q_1^{(1,1)} = 0.55668288399531, \quad q_1^{(1,2)} = -0.15309310892395. \]

Since \( q_1^{(1,1)}, q_1^{(1,2)} \) are linearly dependent, the equivalence inequalities between \( y(t) \) and \( \dot{y}(t) \) do not hold. This is consistent with the fact that \( |y(t)| \) and \( |\dot{y}(t)| \) have zeros (see Figures 2 and 3).

**Example 2:** \( n = 2 \). We choose

\[ y_0 = [-1, 1]^T, \ y_0 = [0, 0]^T. \]

Here, \( \tilde{u}_j = (UX_0)_j \neq 0, \ j = 1, 2 \)

\[
\lambda_1(A) = -0.62500000000000 + 1.61535599791501i, \\
\lambda_2(A) = -0.62500000000000 + 0.99215674164922i, \\
\lambda_3(A) = -0.62500000000000 - 1.61535599791501i = \overline{\lambda_1(A)} = \lambda_1(A^*), \\
\lambda_4(A) = -0.62500000000000 - 0.99215674164922i = \overline{\lambda_2(A)} = \lambda_2(A^*). \tag{33}
\]

Thus,

\[ y_0 = -1, \ y_0 = 0. \]

Figure 2. \( y = |y(t)| \) for \( n = 1 \).
Sufficient algebraic condition \((\mathbf{A})_{g,1}\):

We have

\[
q^{(2,r)}_1 = \begin{bmatrix} 0.46392985110715 \\ 0.46392985110715 \end{bmatrix}, \quad q^{(2,i)}_1 = \begin{bmatrix} 0.18646472388014 \\ 0.18646472388014 \end{bmatrix}.
\]

Since \(q^{(2,r)}_1, q^{(2,i)}_1\) are linearly dependent, the equivalence inequalities between \(y(t)\) and \(\dot{y}(t)\) do not hold. This is consistent with the fact that \(\|y(t)\|_2\) and \(\|\dot{y}(t)\|_2\) have zeros (see Figures 4 and 5).

**Example 3**: \(n = 3\). We choose

\[
\nu_0 = \nu_{x_0} = [A] = Re \lambda_2(A) = \max_{j=1,2} Re \lambda_j(A) = -0.125.
\]

**Figure 3.** \(y = |\dot{y}(t)|\) for \(n = 1\).

**Figure 4.** \(y = \|y(t)\|_2\) for \(n = 2\).
Here, \( \tilde{u}_j = (Ux_0) \neq 0, j = 1, 2, 3 \) and

\[
\begin{align*}
\lambda_1(A) &= -0.64018170840517 + 1.72238614772272i, \\
\lambda_2(A) &= -0.37500000000000 + 1.36358901432946i, \\
\lambda_3(A) &= -0.10981829159483 + 0.76176022151217i, \\
\lambda_4(A) &= -0.64018170840517 - 1.72238614772272i = \lambda_1(A^*), \\
\lambda_5(A) &= -0.37500000000000 - 1.36358901432946i = \lambda_2(A^*), \\
\lambda_6(A) &= -0.10981829159483 - 0.76176022151217i = \lambda_3(A^*).
\end{align*}
\]

Thus,

\[
\nu_0 = \nu_0[A] = v[A] = \text{Re} \lambda_j(A) = \max_{j=1,2,3} \text{Re} \lambda_j(A) = -0.10981829159483.
\]

**Sufficient algebraic condition \( (A_{5,1}) \):**

We have

\[
q_1^{(3,1)} = \begin{bmatrix} 0.24265406168356 \\ 0.29298741809281 \\ 0.17263029735493 \end{bmatrix}, \quad q_1^{(3,2)} = \begin{bmatrix} -0.32420386726516 \\ -0.47468024545176 \\ -0.35244526929766 \end{bmatrix}.
\]

Since \( q_1^{(3,1)}, q_1^{(3,2)} \) are *linearly independent*, the equivalence inequalities between \( y(t) \) and \( \dot{y}(t) \) hold. This is consistent with the fact that \( \|y(t)\|_2 \) and \( \|\dot{y}(t)\|_2 \) do not have zeros for \( t > 0 \) (see Figures 6 and 7).

**Example 4:** \( n = 4 \). We choose

\[
y_0 = [-1, 1, -1, 1]^T, \quad \dot{y}_0 = [0, 0, 0, 0]^T.
\]
Here, $\tilde{u}_j = (Ux_0)_j \neq 0$, $j = 1, \ldots, 4$ is not true since $\tilde{u}_4 = (Ux_0)_4 = (x_0, u_4^*) = 0$. We have

- $\lambda_1(A) = -0.68970367573270 + 1.76753133055006i$,
- $\lambda_2(A) = -0.56419829475905 + 1.51175297583756i$,
- $\lambda_3(A) = -0.18529632426730 + 1.16387666169148i$,
- $\lambda_4(A) = -0.06080170524095 + 0.61674103415901i$,
- $\lambda_5(A) = -0.68970367573270 - 1.76753133055006i = \overline{\lambda_1(A)} = \lambda_1(A^*)$,
- $\lambda_6(A) = -0.56419829475905 - 1.51175297583756i = \overline{\lambda_2(A)} = \lambda_2(A^*)$,
- $\lambda_7(A) = -0.18529632426730 - 1.16387666169148i = \overline{\lambda_3(A)} = \lambda_3(A^*)$,
- $\lambda_8(A) = -0.06080170524095 - 0.61674103415901i = \overline{\lambda_4(A)} = \lambda_4(A^*)$.
Thus,

\[ \nu_0 = \nu_{x_0} [A] \neq \nu [A] = \Re \lambda_4 (A) = \max_{j=1, \ldots, 4} \Re \lambda_j (A) = -0.06080170524095. \]

Since

\[ \tilde{u}_3 = (UX_0)_3 = (x_0, u^*_3) \neq 0, \]

it follows

\[ \nu_0 = \nu_{x_0} [A] = \Re \lambda_3 (A) = \max_{j=1, \ldots, 4} \{ \Re \lambda_j (A^*) \mid x_0 \not\perp u^*_j \} = -0.18529632426730. \]

**Sufficient algebraic condition (A_{g,1}):**

We have

\[
q^{(3,1)}_1 = \begin{bmatrix}
0.02911851244640 \\
0.04706992473723 \\
-0.04706992473723 \\
-0.02911851244640
\end{bmatrix},
q^{(3,1)}_1 = \begin{bmatrix}
0.38288215237183 \\
-0.24420344590968 \\
-0.24420344590968 \\
0.38288215237183
\end{bmatrix}.
\]

Since \(q^{(3,1)}_1, q^{(3,1)}_1\) are linearly independent, the equivalence inequalities between \(y(t)\) and \(\dot{y}(t)\) hold. This is consistent with the fact that \(\|y(t)\|_2\) and \(\|\dot{y}(t)\|_2\) do not have zeros for \(t > 0\) (see Figures 8 and 9).

**Remark** Here, we have a nontrivial example of a case with \(\nu_{x_0} [A] < \nu [A]\).

**Example 5:** \(n = 5\). This model was often used before by the author. We choose

\[ y_0 = [-1, 1, -1, 1, -1]' \], \(\dot{y}_0 = [0, 0, 0, 0, 0]' \).

Here, \(\tilde{u}_j = (UX_0)_j \neq 0, j = 1, \ldots, 5\) and

**Figure 8.** \(y = \|y(t)\|_2\) for \(n = 4\).
Thus, sufficient algebraic condition ($A_{g,1}$): We have

\[
\begin{align*}
q_1^{(5,1)} &= 0.17257779639348, \quad 0.31669429022046, \quad 0.31669429022046, \quad 0.19340987915336, \\
q_1^{(5,2)} &= 0.36598767554684, \quad 0.31669429022046, \quad 0.19340987915336.
\end{align*}
\]

Since $q_1^{(5,1)}$, $q_1^{(5,2)}$ are linearly independent, the equivalence inequalities between $y(t)$ and $\dot{y}(t)$ hold. This is consistent with the fact that $\|y(t)\|_2$ and $\|\dot{y}(t)\|_2$ do not have zeros for $t > 0$ (see Figures 10 and 11).
6.2. Illustration of a case with \( \nu_{x_0}[A] < \nu[A] \)

In most of the Examples 1–5 of Section 6.1, one has \( \nu_{x_0}[A] = \nu[A] \). However, in Example 4 of Section 6.1, we have seen that \( \nu_{x_0}[A] < \nu[A] \). To illustrate this result, we employ Kohaupt (2011, Theorems 7 and 13), where we restrict ourselves to the upper bounds

\[
\|y(t)\|_2 \leq Y_{1,2} e^{\nu_{x_0}(t-t_0)} \quad \text{and} \quad \|z(t)\|_2 \leq Z_{1,2} e^{\nu_{x_0}(t-t_0)},
\]

with the abbreviation \( \nu_0 = \nu_{x_0[A]} \). For comparison reasons, however, first we plot the upper bounds \( \|y(t)\|_2 \leq Y_{1,2} e^{\nu(A)(t-t_0)} \) and \( \|z(t)\|_2 \leq Z_{1,2} e^{\nu(A)(t-t_0)} \).

We have

\[
\nu[A] = \text{Re} \lambda_0(A) \doteq -0.060801,
\]

\[
\nu_{x_0}[A] = \text{Re} \lambda_3(A) \doteq -0.185296.
\]
In what follows, we give the point of contact $t_{s,u,2}$ between curve and upper bound as well as the optimal constants $Y_{1,2}$ and $Z_{1,2}$ computed by the differential calculus of norms.

For the upper bound $y = Y_{1,2} e^{(A)(t-t_0)}$, we obtain

$$t_{s,u,2} = 0.017579,$$
$$Y_{1,2} = 2.001064.$$  

The curve $y = \|y(t)\|_2$ and its upper bound can be seen in Figure 12.

For the upper bound $y = Z_{1,2} e^{(A)(t-t_0)}$, we obtain

$$t_{s,u,2} = 0.694699,$$
$$Z_{1,2} = 2.426929.$$  

The curve $y = \|z(t)\|_2$ and its upper bound can be seen in Figure 13.

For the upper bound $y = Y_{1,2} e^{(t-t_0)}$, we obtain

$$t_{s,u,2} = 0.0054810,$$
$$Y_{1,2} = 2.010068.$$  

The curve $y = \|y(t)\|_2$ and its upper bound can be seen in Figure 14.

For the upper bound $y = Z_{1,2} e^{(t-t_0)}$, we obtain

$$t_{s,u,2} = 0.730963,$$
$$Z_{1,2} = 2.652107.$$  

The curve $y = \|z(t)\|_2$ and its upper bound can be seen in Figure 15.

---

**Figure 12.** $y = \|y(t)\|_2$ and upper bound $y = Y_{1,2} e^{(A)(t-t_0)}$.  

![Graph showing the relationship between $y$, $\|y(t)\|_2$, and upper bounds $Y_{1,2}$ and $Z_{1,2}$]

---
Comparing the corresponding figures, it is evident that the spectral abscissa with respect to the initial vector $x_0$, i.e. $\nu_0 = \nu_{x_0}[A]$, has not only theoretical meaning, but sometimes also practical significance.

### 6.3. Illustration of a case with non-diagonalizable matrix $A$

In this subsection, we first construct an example with $n = 2$ degrees of freedom so that $A \in \mathbb{R}^{4 \times 4}$ is not diagonalizable. The aim is then to apply Theorems 7 and 8, where we restrict ourselves to the upper bounds $\|y(t)\|_2 \leq \eta_{1,2} \|\psi(t)\|_2$ and $\|z(t)\|_2 \leq \xi_{1,2} \|\psi(t)\|_2$.

(i) Construction of a non-diagonalizable matrix
A In the case \( n = 2 \), we have

\[
M = \begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
b_1 + b_2 & -b_2 \\
-b_2 & b_2 + b_3
\end{bmatrix},
\]
\[
K = \begin{bmatrix}
k_1 + k_2 & -k_2 \\
-k_2 & k_2 + k_3
\end{bmatrix},
\]

so that the pertinent characteristic equation reads

\[
|\lambda^2 M + \lambda B + K| = \det \begin{bmatrix}
\lambda^2 m_1 + \lambda (b_1 + b_2) + (k_1 + k_2) & \frac{\lambda (-b_2) - k_2}{\lambda (-b_2) - k_2} \\
\frac{\lambda (-b_2) - k_2}{\lambda (-b_2) - k_2} & \lambda^2 m_2 + \lambda (b_2 + b_3) + (k_2 + k_3)
\end{bmatrix} = 0.
\]

For the construction of a case with non-diagonalizable matrix \( A \), we choose

\[
b_2 = 0, \quad m_2 = m_1 = 1, \quad b_3 = b_1, \quad k_3 = k_1.
\]

Then,

\[
\lambda^2 m_1 + \lambda b_1 + (k_1 + k_2) = sk_2 \quad \text{with} \quad s \in \{+1, -1\}.
\]

Hence, with \( m_1 = 1 \),

\[
\lambda = -\frac{b_1}{2} \pm \sqrt{\left(\frac{b_1}{2}\right)^2 - k_1 - k_2 + sk_2}.
\]

Now, in order to get one real solution, we set

\[
k_1 = \left(\frac{b_1}{2}\right)^2.
\]
This implies
\[
\lambda = \begin{cases} 
-\frac{b_1}{2}, & s = +1, \\
-\frac{b_1}{2} \pm i \sqrt{2k_2}, & s = -1.
\end{cases}
\]

As numerical values for the quantities are not yet specified, we choose \(b_1 = 1/4, k_2 = 1/2\). On the whole, this delivers the following data:

\[
m_1 = m_2 = 1; b_1 = 1/4, b_2 = 0, b_3 = 1/4; k_1 = 1/64 = 1/2^4, k_2 = 1/2, k_3 = 1/64 = 1/2^4,
\]

which leads to

\[
M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

\[
B = \begin{bmatrix} b_1 + b_2 & -b_2 \\ -b_2 & b_2 + b_3 \end{bmatrix} = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}.
\]

\[
K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 1/64 + 1/2 & -1/2 \\ -1/2 & 1/2 + 1/64 \end{bmatrix} = \begin{bmatrix} 0.515625 & -0.5 \\ -0.5 & 0.515625 \end{bmatrix}.
\]

Now,

\[
A = \begin{bmatrix} E \\ -M^{-1}K \end{bmatrix} \begin{bmatrix} M^{-1}B \end{bmatrix}.
\]

The \textit{jordan} routine of MATLAB gives \([V, J] = \text{jordan}(A)\) with

\[
J = \begin{bmatrix} J_1(\lambda_1) & & \\
& J_2(\lambda_2) & \\
& & J_3(\lambda_3) \end{bmatrix} = \begin{bmatrix} -0.125 + i & & \\
& -0.125 - i & \\
& & -0.125 \\
& & 1 \end{bmatrix}.
\]

and

\[
V = [p_1, p_2, p_1^{(3)}, p_2^{(3)}] = \begin{bmatrix} 0.25 - 0.03125i & 0.25 + 0.03125i & 0.0625 + 0.5 \\ -0.25 - 0.03125i & -0.25 + 0.03125i & 0.0625 - 0.5 \\ 0.25390625i & -0.25390625i & -0.0078125 \\ -0.25390625i & 0.25390625i & -0.0078125 \end{bmatrix}.
\]

Further, the Matlab command \([V, J] = \text{jordan}(A^*)\) delivers \(J_s = J\). After rearranging the eigenvalues of \(A^*\) such that \(\lambda_k(A^*) = \overline{\lambda_k(A)}\), \(k = 1, 2, 3\), and calling the rearranged \(J_s\) now \(J_{A^*}\), and the rearranged \(V_s\) now \(U^*\), we obtain

\[
J_{A^*} = \begin{bmatrix} J_1(\overline{\lambda_1}) & & \\
& J_2(\overline{\lambda_2}) & \\
& & J_3(\overline{\lambda_3}) \end{bmatrix} = \begin{bmatrix} -0.125 - i & & \\
& -0.125 + i & \\
& & -0.125 \\
& & 1 \end{bmatrix},
\]

and,
We have

\[ U^* = [u^*_1, u^*_2, u_1^{(3)*}, u_2^{(3)*}] = \begin{bmatrix} 0.25 + 0.03125i & 0.25 - 0.03125i & 0.0625 & 0.5 \\ -0.25 - 0.03125i & -0.25 + 0.03125i & 0.5 & 0 \\ 0.25i & -0.25i & 0.5 & 0 \\ -0.25i & 0.25i & 0.5 & 0 \end{bmatrix}. \]

The next step is to replace the principal vector of stage 2, \( p_2^{(3)} \), by a principal vector of stage 2, \( w_2^{(3)} \), with

\[(w_2^{(3)}, u_1^{(3)}) = 0.\]

Following the method of 2007, for this, we seek \( w_2^{(3)} \) in the form

\[ w_2^{(3)} = p_2^{(3)} + a_1 p_1^{(3)}. \]

From \( (w_2^{(3)}, u_2^{(3)}) = 0 \), we obtain \( a_1^{(3)} = -(p_2^{(3)}, u_2^{(3)})/(p_1^{(3)}, u_1^{(3)}) \). Moreover, we normalize according to

\[ \tilde{p}_1 = p_1 / (p_1, u_1^{(3)}), \tilde{p}_2 = p_2 / (p_2, u_2^{(3)}), \tilde{p}_1^{(3)} = p_1^{(3)} / (p_1^{(3)}, u_1^{(3)}), \tilde{p}_2^{(3)} = w_2^{(3)} / (w_2^{(3)}, u_2^{(3)}), \]

and rename \( \tilde{p}_1, \tilde{p}_2, \tilde{p}_1^{(3)} \) to \( p_1, p_2, p_1^{(3)}, p_2^{(3)} \), as the case may be. Then,

\[
\begin{array}{c|c|c|c|c|}
(p_1, u_1^{(3)}) & 1 & 1 & 1 & 0 \\
(p_2, u_2^{(3)}) & 1 & 1 & 1 & 0 \\
(p_1^{(3)}, u_1^{(3)}) & 1 & 1 & 1 & 0 \\
(p_2^{(3)}, u_2^{(3)}) & 1 & 1 & 1 & 0 \\
\end{array}
\]

The numerical values of the new \( p_1, p_2, p_1^{(3)}, p_2^{(3)} \) are:

\[ [p_1, p_2, p_1^{(3)}, p_2^{(3)}] = \begin{bmatrix} 1 \\ -1 \\ -0.125 + i \\ -0.125 - i \\ 1 \\ 1 \end{bmatrix} =: P. \]

(ii) Complex basis functions

Similar as in Kohaupt (2008b), we obtain as complex basis functions

\[ x_1(t) = p_1 e^{i(t-t_0)}, \]
\[ x_2(t) = p_2 e^{i(t-t_0)} = \overline{x_1(t)}, \]
\[ x_1^{(3)}(t) = p_1^{(3)} e^{i(t-t_0)}, \]
\[ x_2^{(3)}(t) = [p_1^{(3)}(t - t_0) + p_2^{(3)}] e^{i(t-t_0)}. \]

The general solution of \( x = Ax \) is given by

\[ x(t) = c_{11} x_1(t) + \overline{c_{11}} \overline{x_1(t)} + c_{13} x_1^{(3)}(t) + c_{23} e^{i(t-t_0)} x_2^{(3)}(t), \]

where we prefer the usage of the double index for the first coefficient (\( c_{1k} \) with \( k = 1 \) in the notation of Kohaupt (2008b)). The boundary condition \( x(t_0) = x_0 \) is met for \( t = t_x \), delivering

\[ x_0 = c_{11} p_1 + \overline{c_{11}} p_1 + c_{13} p_1^{(3)} + c_{23} p_2^{(3)}. \]
Scalar multiplication by the columns of $U^*$ leads to

$$c_{11} = (x_0, u_{11}^*), \quad c_1^{(3)} = (x_0, u_2^{(3r)}), \quad c_2^{(3)} = (x_0, u_1^{(3r)}).$$

(iii) **Real basis functions**

As in Kohaupt (2008b), for the splitting in the real and imaginary parts, we set

$$p_k^{(i)} = p_k^{(r)} + ip_k^{(i)},$$

$$e^{-i(t-t_0)} = e^{|i(t-t_0)|} = e^{-i\lambda_k^{(i)}(t-t_0)},$$

$k = 1, \ldots, m_1; l = 1, \ldots, r$ where $r = 3$ and $m_1 = 1$, $m_2 = 1$, and $m_3 = 2$, and where we have set $p_k^{(1)} = p_k$, $k = 1, 2$, and so on. Then the solution with real basis is given by

$$x(t) = c_1^{(r)} x_1^{(r)}(t) + c_1^{(i)} x_1^{(i)}(t) + c_2^{(3r)} x_2^{(3r)}(t) + c_2^{(3r)} x_2^{(3r)}(t)$$

with

$$c_1^{(r)} = 2 \text{Re}(c_{11}),$$

$$c_1^{(i)} = -2 \text{Im}(c_{11}),$$

$$c_1^{(3r)} = c_1^{(3)},$$

$$c_1^{(3r)} = c_2^{(3)},$$

and with the real basis functions

$$x_1^{(r)}(t) = e^{-i\lambda_1^{(r)}(t-t_0)} \{ \cos \lambda_1^{(r)}(t-t_0) \, p_1^{(r)} - \sin \lambda_1^{(r)}(t-t_0) \, p_1^{(i)} \},$$

$$x_1^{(i)}(t) = e^{-i\lambda_1^{(r)}(t-t_0)} \{ \sin \lambda_1^{(r)}(t-t_0) \, p_1^{(r)} + \cos \lambda_1^{(r)}(t-t_0) \, p_1^{(i)} \},$$

$$x_2^{(3r)}(t) = e^{-i\lambda_2^{(r)}(t-t_0)} \, p_1^{(3r)},$$

$$x_2^{(3r)}(t) = e^{-i\lambda_2^{(r)}(t-t_0)} \{ p_1^{(3r)}(t-t_0) + p_2^{(3r)} \}.$$

---

**Figure 16.** $y = \|y(t)\|_2$ and upper bound $y = \eta_{12} \|\psi(t)\|_2$.
(iv) Vector $\psi(t)$

From Kohaupt (2011), we have

$$
\psi_1(t) = (x_0, p_1) e^{\psi_1(t-t_0)},
$$
$$
\psi_2(t) = (x_0, p_2) e^{\psi_2(t-t_0)},
$$
$$
\psi_3(t) = (x_0, p_3^{(3)}) e^{\psi_3(t-t_0)},
$$
$$
\psi_4(t) = (x_0, p_3^{(3)}(t-t_0) + p_2^{(3)}) e^{\psi_4(t-t_0)},
$$

and thus

$$
\psi(t) = [\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t)]^T.
$$

Remark  Since $\psi_2(t) = \overline{\psi}_1(t)$, we could also use the vector $\psi(t)$ without component $\psi_2(t)$. Then merely the constants in the upper bounds would change.

(v) Optimal upper bounds on $y(t)$ and $z(t) = y(t)$

The displacement $y(t)$ and the velocity $\dot{y}(t)$ can be computed from $x(t) = [y(t), \dot{y}(t)]^T$. For comparison reasons, we have determined $x(t)$ also by $x(t) = e^{A(t-t_0)}x_0$ and obtained numerically identical results. The advantage of the representation of $x(t)$ by the real basis functions is that we get more insight into its vibration behavior than without it.

We restrict ourselves to the upper bounds $\|y(t)\|_2 \leq \eta_{1,2} \|\psi(t)\|_2$ and $\|z(t)\|_2 \leq \zeta_{1,2} \|\psi(t)\|_2$. As initial condition, we choose

$$
y_0 = [-1, 1]^T; \; \dot{y}_0 = z_0 = [0, 0]^T.
$$

In the sequel, we denote by $t_{s,0,2}$ the point of contact between the considered curve and optimal upper bound.
For the coefficients of the solution, we obtain
\[ c = [c_1, c_2, c_3] = [-0.5 + 0.0625i, -0.5 - 0.0625i, 0, 0]^T \]
and thus
\[ c_1^{(r)} = 2 \Re c_1 = -1, \quad c_1^{(i)} = -2 \Im c_1 = -0.125, \quad c_2^{(3)} = c_1^{(3)} = 0, \quad c_2^{(3)} = c_2^{(3)} = 0. \]
The curve \( y = \|y(t)\|_2 \) and its optimal upper bound \( y = n_{1,2} \|\psi(t)\| \) can be seen in Figure 16. One has
\[ t_{s,u,2} = 0.124354, \quad n_{1,2} = 0.503891. \]
The curve \( y = \|z(t)\|_2 = \|\dot{y}(t)\| \) and its optimal upper bound \( y = \zeta_{1,2} \|\psi(t)\| \) are drawn in Figure 17. One gets
\[ t_{s,u,2} = 1.570796, \quad n_{1,2} = 0.507812. \]
Since \( c_1^{(3)} = (x_0, u_2^{(3)}) = 0 \) and \( c_2^{(3)} = (x_0, u_1^{(3)}) = 0 \),
it follows that the solution part corresponding to \( \lambda_j(A) \) is suppressed. Thus, the solution behaves like a one-mass model with eigenvalues \( \lambda_j(A) \) and \( \bar{\lambda}_j(A) \). It is evident that the representation of the solution by the real basis functions offers much more insight into the vibration behavior than the representation \( x(t) = e^{A(t-t_0)^i}x_0 \) that does not allow such an interpretation. We mention that here the upper bounds \( y = n_{1,2} \|\psi(t)\| \) and \( y = \zeta_{1,2} \|\psi(t)\| \) are numerically identical with those of \( y = Y_{1,2} e^{(A(t-t_0) \} \text{ and } y = Z_{1,2} e^{(A(t-t_0) \}, as the case may be. The values \( t_{s,u,2} \) and best constants \( n_{1,2} \) are obtained by the differential calculus of norms.

7. Computational aspects
In this subsection, we say something about the used computer equipment and the computation time.

(i) As to the computer equipment, the following hardware was available: a Pentium D 940 (3.2 GHz), and 64 GB mass storage facility, and a 2048 MB DDR2-SDRAM 533 MHz (2x1024 MB) high-speed memory. As software package for the computations, we used 368-Matlab, Version 4.2.c; for the generation of the figures, Version 6.5, in order to be able to caption them; and for the jordan routine, likewise Version 6.5.

(ii) The computation time \( t \) of an operation was determined by the command sequence \( t1 = \text{clock; operation}; t = \text{etime(clock,t1)} \); it is put out in seconds rounded to two decimal places, by MATLAB. For example, to compute the points of contact and to generate the table of values \( t, y(t), y(t), y(t), t = 0(0.01)25 \) for Figure 16, we obtained \( t_{16} = 2.69 \) s.

8. Conclusion
In a one-mass vibration model with no or mild damping, the displacement \( y(t) \) and the velocity \( \dot{y}(t) \) cannot satisfy the equivalence relation \( c_1 \|y(t)\| \leq \|\dot{y}(t)\| \leq c_1 \|y(t)\| \), \( t \geq t_1 \), for sufficiently large \( t \), since \( y(t) = 0 \), respectively, \( \dot{y}(t) = 0 \) for some \( t \) occurs in any sufficiently large interval, which is not a disadvantage because then the lower bound for both functions is simply the time axis. On the other hand, for multi-mass vibration models, one can imagine that the case \( y(t) \neq 0 \), \( \dot{y}(t) \neq 0 \), \( t \geq t_1 \), occurs; the probability for this to happen increases intuitively with increasing dimension since then it will be unlikely that all components of \( y(t) \) or \( \dot{y}(t) \) will be zero simultaneously at any time. In the case \( y(t) \neq 0 \), \( \dot{y}(t) \neq 0 \), \( t \geq t_1 \), naturally the question of norm equivalence between the
quantities \( y(t) \) and \( \dot{y}(t) \) arises. In this paper, sufficient conditions are given under which the norm equivalence of \( y(t) \) and \( \dot{y}(t) \) can be proven. Whereas the case of diagonalizable matrices \( A \) is simple to treat, the case of general square matrices needs much more effort. As application, improvements of some theorems of Kohaupt (2011) are presented. Moreover, the algebraic conditions for norm equivalence are illustrated for several examples of diagonalizable and non-diagonalizable matrices \( A \), and their validity is underpinned by the graphs of \( y = \|y(t)\|_2 \) and \( \dot{y} = \|\dot{y}(t)\|_2 \).

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References
Coppel, W. A. (1965). Stability and asymptotic behavior of differential equations. Boston, MA: D.C. Heath.
Kohaupt, L. (2007). Construction of a biorthogonal system of principal vectors for the matrices \( A \) and \( A^* \) with applications. \( \dot{x} = Ax, \ x(0) = x_0 \). Journal of Computational Mathematics and Optimization, 3, 163–192.
Kohaupt, L. (2008a). Solution of the matrix eigenvalue problem \( VA + AV^* = \mu V \) with applications to the study of free linear systems. Journal of Computational and Applied Mathematics, 213, 142–165.
Kohaupt, L. (2008b). Solution of the vibration problem \( MY' + BY + Ky = 0, \ y(0) = y_0, \ \dot{y}(0) = \dot{y}_0 \) without the hypothesis \( BM^{-1}K = KM^{-1}B \) or \( B = \alpha M = \beta K \). Applied Mathematical Sciences, 2, 561–574.
Kohaupt, L. (2010). Two-sided bounds on the displacement \( y(t) \) and the velocity \( \dot{y}(t) \) of the vibration system \( MY' + BY + Ky = 0, \ y(0) = y_0, \ \dot{y}(0) = \dot{y}_0 \) with application of the differential calculus of norms. The Open Applied Mathematics Journal, 5, 1–18.
Müller, P. C., & Schiehlen, W. O. (1985). Linear vibrations. Dordrecht: Martinus Nijhoff.
Thomson, W. T. (1971). Theory of vibration with applications. Prentice-Hall, NJ: Englewood Cliffs.