On the kinetic equation in Zakharov’s wave turbulence theory for capillary waves

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Abstract

The wave turbulence equation is an effective kinetic equation that describes the dynamics of wave spectrum in weakly nonlinear and dispersive media. Such a kinetic model has been derived by physicists in the sixties, though the well-posedness theory remains open, due to the complexity of resonant interaction kernels. In this paper, we provide a global unique radial strong solution, the first such a result, to the wave turbulence equation for capillary waves.

Keyword: weak turbulence theory, capillary waves, water waves system, fluids mechanics

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1 Introduction

Over the last 60 years the theory of weak turbulence has been intensively developed. In weakly nonlinear and dispersive wave models, the weak turbulence kinetic equation can be formally derived, via the statistical approach, to describe the dynamics of resonant wave interactions. The model for slightly viscous capillary waves on the surface of a liquid reads as follows (cf. [30, 31, 41, 43])

$$\partial_t f + 2\nu|k|^2 f = Q[f]$$

(1.1)

in which $f(t, k)$ is the nonnegative wave density at wavenumber $k \in \mathbb{R}^d$, $d \geq 2$. Here, $\nu$ denotes the positive coefficient of fluid viscosity (strictly speaking, the model is derived under the assumption $\nu \sqrt{k} \ll 1$ so that the dispersion remains dominating the viscous dissipation; see [9] for more details on the addition of the viscous damping). The term $Q[f]$ denotes the integral collision operator, describing pure resonant three-wave interactions. The equation is a three-wave kinetic one, in which the collision operator is of the form

$$Q[f](k) = \int \int_{\mathbb{R}^{2d}} \left[ R_{k,k_1,k_2}[f] - R_{k_1,k,k_2}[f] - R_{k_2,k,k_1}[f] \right] dk_1 dk_2$$

(1.2)

with

$$R_{k,k_1,k_2}[f] := 4\pi |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2) \delta(\mathcal{E}_k - \mathcal{E}_{k_1} - \mathcal{E}_{k_2})(f_1 f_2 - f f_1 - f f_2)$$

with the short-hand notation $f = f(t, k)$ and $f_j = f(t, k_j)$. The Dirac delta function $\delta(\cdot)$ is to ensure the following resonant conditions for the wavenumbers:

$$k = k_1 + k_2, \quad \mathcal{E}_k = \mathcal{E}_{k_1} + \mathcal{E}_{k_2},$$

(1.3)

with $\mathcal{E}_k$ denoting the dispersion relation of the waves. The exact form of the collision kernel $V_{k,k_1,k_2}$ will be recalled below.
According to the weak turbulence theory (cf. [43, 44, 23]), equation (1.1) in the absence of viscosity admits nontrivial equilibria \( \rho_\infty \), called the Kolmogorov-Zakharov’s spectra:

\[
f_\infty(k) \approx C|k|^{-17/4}.
\]

Moreover, such a solution can be interpreted as a universal spectrum in the region of transparency. These solutions are the analogs of the familiar Kolmogorov energy spectrum prediction \( C|k|^{-5/3} \) of hydrodynamic turbulence.

The derivation of the equation dated back to the 60’s, starting with the pioneering work of Hasselmann, Zakharov and collaborators (cf. [18, 19, 40, 41, 43]). Since then, a lot works have been done, trying to understand the equation (see [40, 23, 17, 42, 41, 44, 7, 43, 28, 2, 30, 31, 12, 5, 15, 4, 16, 11, 34, 33, 25 and references therein). We refer to the books [27] for more discussions and references on the topic. Due to its complexity, the fundamental question on the global existence and uniqueness of solutions to the equation is still unsolved. In this paper, we give, for the first time, an answer to the fundamental question on the global existence and uniqueness of solutions to the equation, for the case where the solutions are radial.

In this paper, we develop new techniques, inspired by recent works on quantum kinetic theory. Let us mention that the kinetic wave equation (1.1) has a very similar structure with the quantum Boltzmann equation that describes the evolution of the excitations in a trapped Bose gas system, in which the temperature of the gas is below the Bose-Einstein condensate transition temperature (cf. [39, 45, 24, 21, 14, 22]). The collision operator that describes the interaction between excitations and condensates in the quantum Boltzmann equation reads

\[
C[f](k) = \int \int R_{k,k_1,k_2}[f] - R_{k_1,k,k_2}[f] - R_{k_2,k,k_1}[f] dk_1 dk_2
\]

with

\[
R_{k,k_1,k_2}[f] := |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2) \delta(\tilde{E}_k - \tilde{E}_{k_1} - \tilde{E}_{k_2})(f_1 f_2 - f f_1 - f f_2 - f)
\]

and \( |V_{k,k_1,k_2}|^2 = C^* |k||k_1||k_2| \), \( \tilde{E}_k = \sqrt{\kappa_1 |k|^2 + \kappa_2 |k|^4} \) for some positive constants \( C^*, \kappa_1, \kappa_2 \). Recent progresses on quantum Boltzmann equations (cf. [10, 23, 36, 35, 11, 20, 32]) have opened some opportunities to tackle this open problem, the existence and uniqueness of solutions to (1.1).

We note that, in the absence of the linear term in (1.5) or the viscous damping in (1.1), singularities are likely to form. Indeed, [37] constructed a self-similar blowup solution to the quantum Boltzmann equation, when the linear term is dropped.

1.1 Main result

Throughout the paper, we consider the following generalized version of (1.1)

\[
\partial_t f + 2\nu(|k|^2 + \varrho|k|^4) f = Q[f]
\]
for $\varrho \geq 0$. Solutions to the original model (1.1) will be obtained via the limit of $\varrho \to 0$.

The law of wave dispersion on the surface of infinitely deep liquid is of the form

$$\mathcal{E}_k = \sqrt{\sigma |k|^3}$$ (1.7)

for $\sigma$ the surface tension coefficient, and the collision kernel $V_{k,k_1,k_2}$ is defined by

$$V_{k,k_1,k_2} = \frac{1}{8\pi\sqrt{2\sigma}} \sqrt{\mathcal{E}_k \mathcal{E}_{k_1} \mathcal{E}_{k_2}} \left( \frac{L_{k_1,k_2}}{|k|\sqrt{|k_1||k_2|}} - \frac{L_{k,-k_1}}{|k_2|\sqrt{|k||k_1|}} - \frac{L_{k,-k_2}}{|k_1|\sqrt{|k||k_2|}} \right)$$ (1.8)

with $L_{k_1,k_2} = k_1 \cdot k_2 + |k_1||k_2|$; see [30, 31]. Without loss of generality, we assume the surface tension $\sigma = 1$. In the scope of our paper, we only consider the case $d = 2$ or 3, which are relevant dimensions in the physical applications.

We shall construct global unique radial solutions to (1.6) in weighted $L^1$ spaces. Precisely, for $N > 0$, let $L^1_N(\mathbb{R}^d)$ be the function space consisting of $f(k)$ so that the norm

$$\|f\|_{L^1_N} := \int_{\mathbb{R}^d} f(k) \mathcal{E}^N_k dk$$

is finite, with the dispersion relation $\mathcal{E}_k$ defined as in (1.7). In addition, for any $N > 0$ and $\vartheta_0 > 0$, we introduce

$$S_N := \left\{ f \in L^1_N(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) : f \geq 0, \quad f(k) = f(|k|), \quad \|f\|_{L^1} + \|f\|_{L^1_{N+3}} \leq \vartheta_0 \right\}.$$

We shall construct solutions of (1.6) that remain $S_N$, if initially so.

Our main result is as follows.

**Theorem 1.1** Let $N > 0$, and let $f_0(k) = f_0(|k|) \in S_N \cap L^2(\mathbb{R}^d)$, $d = 2$ or 3. Then for all $\varrho \geq 0$, the weak turbulence equation (1.6), with initial data $f(0,k) = f_0(k)$ and $\nu > 0$, has a unique global solution $f(t,k)$ so that

$$0 \leq f(t,k) = f(t,|k|) \in C([0,\infty); S_N \cap L^2(\mathbb{R}^d)) \cap C^1((0,\infty); L^1_N).$$ (1.9)

Moreover, there holds the propagation of moments: for any $n \geq 1$, if $f_0 \in L^1_n$, then there exists $C_n > 0$ such that

$$\sup_{t \geq 0} \|f(t,\cdot)\|_{L^1_k} \leq C_n.$$ (1.10)

**Remark 1.1** Notice that for the case where $\varrho > 0$, we have a stronger result: we can remove the $L^2$ dependence on the initial condition, and the solution exists in $C([0,\infty); S_N \cap L^1(\mathbb{R}^d)) \cap C^1((0,\infty); L^1_N)$.

Let us mention that the classical Boltzmann equation describes the evolution of the density function of a dilute classical gas. After the collision, two particles with velocities...
$k_1$ and $k_2$ change their velocities into $k_3$ and $k_4$. Since the energy of the particles is of the form $E_k = |k|^2$; the conservation of moment and energy then read

$$|k_1|^2 + |k_2|^2 = |k_3|^2 + |k_4|^2,$$

$$k_1 + k_2 = k_3 + k_4.$$ 

As a consequence, $k_1$, $k_2$, $k_3$, $k_4$ belong to the sphere centered at $\frac{k_1+k_2}{2}$ with radius $|k_1-k_2|$ and the classical Boltzmann collision operator can be expressed as a integration on a sphere (cf. [6, 38]).

Let us now turn to the collision operator of (1.6). As in the classical case, the collision operator involves surface integrals. Precisely, we introduce functions

$$H_0^k(w) := E_{k-w} + E_w - E_k,$$

$$H_1^k(w) := E_{k+w} - E_{k-w},$$  (1.11)

and the energy surfaces, dictated by the resonant conditions (1.3),

$$S_k := \{ w \in \mathbb{R}^d : H_0^k(w) = 0 \}$$

$$S'_k := \{ w \in \mathbb{R}^d : H_1^k(w) = 0 \}$$  (1.12)

with $E_k = \sqrt{\sigma}|k|^{3/2}$. The collision operator $Q[f]$ then reduces to

$$Q[f](k) = \int_{S_k} R_{k-k_2,k_2}[f] \frac{d\sigma(k_2)}{|\nabla H_0^k(k_2)|} - 2 \int_{S'_k} R_{k+k_2,k_2}[f] \frac{d\sigma(k_2)}{|\nabla H_1^k(k_2)|}. $$  (1.13)

Difficulties arise. First, surfaces $S_k$ and $S'_k$ are no longer a sphere as in the classical case, and the analysis on these surfaces can be tricky. More seriously, due to the lack of an integration over the whole space (compare with the classical Boltzmann equation), we are forced to bound surface integrals in term of (weighted) $L^1$ norms of solutions, a type of estimates that are in general false. In Section 2.2 we shall derive such estimates for radial functions.

By view of (1.3), the weak turbulence equation (1.6) conserves momentum and energy (in the absence of viscous damping), but does not conserve mass. As a consequence, one of the issues in dealing with (1.6) is that $L^1$ norms, say with weight $|k|^n$, of solutions do not close by itself, but are bounded by $L^1$ norms with a much higher-order weight. This is due to the high-order collision kernel. That is, roughly speaking, the kernel $|V_{k,k_1,k_2}|^2$ is of order $9/2$ in $|k|$, which is much higher than the order of classical Boltzmann collision kernel (typically, smaller than one). This apparent loss of weights in $|k|$ gives the impression that solutions could blow up in finite time, even in the presence of viscous damping: $2\nu|k|^2 f$, which gains precisely two order in $|k|$.

We end the introduction by giving the structure of the paper:

- We derive the momentum and energy identities and provide a careful study of the surface integrals on the energy manifolds.
- In Section 3 we provide an a priori estimate on the $L^1_N$ norm of the solution.
• An $L^2$ estimate on the solutions of (1.6) and the Hölder continuity of the collision
operator will be established in Sections 4 and 5 respectively.
• The proof of Theorem 1.1 is given in Section 6.

2 Conservation laws and energy surfaces

2.1 Momentum and energy identity
In this section, we obtain the basic properties of strong solutions of (1.6).

Lemma 2.1 There holds
\[
\int_{\mathbb{R}^d} Q[f](t,k)\varphi(k) \, dk = \iint_{\mathbb{R}^3} R_{k,k_1,k_2}[f]\left[\varphi(k) - \varphi(k_1) - \varphi(k_2)\right] \, dk \, dk_1 \, dk_2
\]
for any test functions $\varphi$ so that the integrals make sense.

Proof By definition, we compute
\[
\int_{\mathbb{R}^d} Q[f](t,k)\varphi(k) \, dk = \iint_{\mathbb{R}^3} R_{k,k_1,k_2} - R_{k_1,k,k_2} - R_{k_2,k,k_1}\varphi\, dk \, dk_1 \, dk_2.
\]
By switching the variables $k \leftrightarrow k_1$, $k \leftrightarrow k_2$ in the first integral on the right, the lemma follows at once.

As a direct consequence, we obtain the following corollary.

Corollary 2.1 (Momentum and energy identities) Smooth solutions $f(t,k)$ of (1.6) satisfy
\[
\frac{d}{dt} \int_{\mathbb{R}^d} f(t,k)k \, dk + 2\nu \int_{\mathbb{R}^d} f(t,k)k(|k|^2 + \varrho|k|^4) \, dk = 0 \tag{2.1}
\]
and
\[
\frac{d}{dt} \int_{\mathbb{R}^d} f(t,k)\mathcal{E}_k \, dk + 2\nu \int_{\mathbb{R}^d} f(t,k)\mathcal{E}_k(|k|^2 + \varrho|k|^4) \, dk = 0 \tag{2.2}
\]
for all $t \geq 0$.

Proof This follows from Lemma 2.1 by taking $\varphi(k) = k$ and $\varphi(k) = \mathcal{E}_k$, and using the resonant conditions (1.3).
2.2 Energy surfaces

Our first step is to study the surface integrals. For sake of generality, we consider in this section the following power-law energy function

$$\mathcal{E}_k = \mathcal{E}(k) = |k|^{\gamma}, \quad 1 < \gamma \leq 2.$$  (2.3)

In the case when $\gamma = 1$, the surface $S_k$ degenerates into a straight line $S_k = \{\alpha k\}_{\alpha \in [0,1]}$, and the surface integral reduces to a line integral. Such an energy corresponds to the dispersion law of phonons, and has been studied in [1, 8, 10].

**Lemma 2.2 (Surface $S_p$)** Let $\gamma \in (1, 2]$ and $S_p$ be defined as in (1.12)-(2.3). Then, for each $p$, $S_p = |p|S_{e_p}$, with $e_p = p/|p|$. In addition, there hold the following properties:

i. $\{0, p\} \subset S_p \subset B(p, |p|^{\gamma})$.

ii. The surface $S_p$ is invariant under the rotation around $p$, and can be parametrized by

$$S_p = \left\{ w(\alpha, e_q) = \alpha p + s(\alpha)e_q : \alpha \in [0, 1], \quad |e_q| = 1, \quad e_q \cdot p = 0 \right\}$$

for some function $s(\alpha)$ that is smooth in $(0,1)$; see Figure 1. In the two dimensional case: $d = 2$, $S_p$ is a curve parametrized by $\alpha \in [0, 1]$.

iii. $s(\alpha) = s(1 - \alpha)$, $s(0) = 0$, and $s(\alpha)$ is strictly increasing and invertible on $(0, \frac{1}{2})$.

iv. There are universal constants $c_0, C_0$ so that the surface area satisfies

$$c_0|p|^{d-\gamma} \leq \int_{S_p} \frac{d\sigma(w)}{|\nabla_w H_{p}^p (w)|} \leq C_0|p|^{d-\gamma},$$

uniformly in $p \in \mathbb{R}^d$. 

Figure 1: Sketched is the surface $S_p$, centered at $\frac{p}{2}$ and having 0 and $p$ as its south and north poles, respectively.
v. There holds

\[ \int_{S_p} F(|w|) \frac{d\sigma(w)}{|\nabla_w H_0^p(w)|} \leq C|p|^{d-\gamma-2} \int_{|w| \leq |p|} F(|w|)|w|^{2-d} dw. \]

**Proof** Let \( p \in \mathbb{R}^d \setminus \{0\} \). It is clear that \( S_p = |p|S_{e_p} \), with \( e_p = p/|p| \). Thus, it suffices to study the case when \( |p| = 1 \). As for (i), it is clear that \( \{0, p\} \subset S_p \). Next, since \( a\gamma + b\gamma \leq (a+b)\gamma \), we have for \( w \in S_p \)

\[ 1 = \left( |p - w|^\gamma + |w|^\gamma \right)^{2/\gamma} \geq |p - w|^2 + |w|^2. \]

This proves that \( w \in B(\frac{5}{2}, \frac{1}{2}) \). (i) follows.

As for (ii), we write \( w = \alpha p + q \) for \( q \) orthogonal to \( p \). By orthogonality, \( |w| \) and \( |w-p| \) do not depend on the direction of \( q \), and neither does \( S_p \). That is, \( S_p \) is invariant under the rotation around \( p \). We set

\[ w(\alpha, s) = \alpha p + se_q. \]

We shall prove the existence of a function \( s = s(\alpha) \) for \( \alpha \in (0, 1) \), so that \( w(\alpha, s(\alpha)) \in S_p \).

To this end, let

\[ H_0^p(\alpha, s) := \mathcal{E}(p - w(\alpha, s)) + \mathcal{E}(w(\alpha, s)) - \mathcal{E}(p), \]

as in [1.11]. Clearly, \( H_0^p(\alpha, s) = 0 \) if and only if \( w(\alpha, s) \in S_p \). Observe that \( H_0^p(\alpha, 0) < 0 \) (by convexity of \( \mathcal{E}(p) \)) and \( H_0^p(\alpha, s) > 0 \) for sufficiently large \( s \) (and hence large \( |w(\alpha, s)| \)). The existence of a such \( s(\alpha) \) follows. In addition, a direct computation yields

\[ \nabla_w H_0^p = \frac{w - p}{|p - w|} \mathcal{E}'(p - w) + \frac{w}{|w|} \mathcal{E}'(w) \]  

(2.4)

and hence \( \partial_s H_0^p(\alpha, s) = e_q \cdot \nabla_w H_0^p \) is positive, since \( \mathcal{E}'(w) > 0 \) (for \( w \neq 0 \)). That is, \( H_0^p(\alpha, s) \) is increasing in \( s \) for each \( \alpha \) and \( s(\alpha) \) is uniquely determined, so that \( H_0^p(\alpha, s(\alpha)) = 0 \). The smoothness of \( s(\alpha) \) follows from that of \( \mathcal{E}(\cdot) \). This proves (ii).

Next, the symmetry stated in (iii) is clear from the definition of \( S_p \), and it suffices to study \( s(\alpha) \) for \( \alpha \in (0, \frac{1}{2}] \). Observe that for \( w_\alpha = w(\alpha, s(\alpha)) \in S_p \), we have \( 0 = F(\alpha, s(\alpha)) \) and

\[ 0 = \partial_s H_0^p + s'(\alpha) \partial_s H_0^p = p \cdot \nabla_w H_0^p + s'(\alpha)e_q \cdot \nabla_w H_0^p \]

\[ = -\frac{\mathcal{E}'(p - w_\alpha)}{|p - w_\alpha|} + (s(\alpha)s'(\alpha) + \alpha) \left[ \frac{\mathcal{E}'(p - w_\alpha)}{|p - w_\alpha|} + \frac{\mathcal{E}'(w_\alpha)}{|w_\alpha|} \right]. \]

(2.5)

Setting \( \mathcal{E}_1(w) := |w|^{\gamma-2} \), we have

\[ s(\alpha)s'(\alpha) = \frac{(1 - \alpha)\mathcal{E}_1(p - w_\alpha) - \alpha\mathcal{E}_1(w_\alpha)}{\mathcal{E}_1(p - w_\alpha) + \mathcal{E}_1(w_\alpha)}. \]

(2.6)

Observe that the function in the numerator in (2.6) is decreasing in \( \alpha \), and vanishes at \( \alpha = \frac{1}{2} \). Hence, for \( \alpha \in (0, \frac{1}{2}] \), we have \( s'(\alpha) > 0 \), and hence \( s(\alpha) \) is invertible, on \( (0, \frac{1}{2}) \). This yields (iii).
Next, we compute the surface area of $S_p$. Let us consider the case when $d \geq 3$; the case when $d = 2$ is simpler, as the surface is parametrized solely by $\alpha \in [0, 1]$. Writing $w(\alpha, \theta) = \alpha p + s(\alpha)e_\theta$, we have
\[
\frac{d\sigma(w)}{\sqrt{|\nabla w|}} = \left|\partial_\alpha w \times \partial_\theta w\right| d\alpha d\theta = \left|(p + s'(\alpha)e_\theta) \times s(\alpha)\partial_\theta e_\theta\right| d\alpha d\theta
\]
which, in companion with (2.9), implies
\[
0 = \partial_\alpha w_\alpha \cdot \nabla_w H^p_0
\]
\[
= \frac{1}{2} \partial_\alpha |s(\alpha)|^2 \left\{ \frac{\mathcal{E}'(|p - w_\alpha|)}{|p - w_\alpha|} + \frac{\mathcal{E}'(|w_\alpha|)}{|w_\alpha|} \right\} + \alpha |p|^2 \frac{\mathcal{E}'(|p - w_\alpha|)}{|p - w_\alpha|} - (1 - \alpha) |p|^2 \frac{\mathcal{E}'(|p - w_\alpha|)}{|p - w_\alpha|}.
\]
It is straightforward that
\[
\partial_\alpha |s(\alpha)|^2 = 2|p|^2 \left\{ \frac{\mathcal{E}'(|w_\alpha|)}{|w_\alpha|} + (\alpha - 1) \frac{\mathcal{E}'(|p - w_\alpha|)}{|p - w_\alpha|} \right\}.
\]
Now, let us compute $|\nabla H^p_0|$ under the new parametrization
\[
|\nabla H^p_0|^2 = |p|^2 \left[ \frac{\mathcal{E}'(|w_\alpha|)}{|w_\alpha|} + (\alpha - 1) \frac{\mathcal{E}'(|p - w_\alpha|)}{|p - w_\alpha|} \right]^2
\]
\[
+ |s(\alpha)|^2 \left[ \frac{\mathcal{E}'(|p - w_\alpha|)}{|p - w_\alpha|} + \frac{\mathcal{E}'(|w_\alpha|)}{|w_\alpha|} \right]^2,
\]
which, in companion with (2.9), implies
\[
|\nabla H^p_0|^2 = \frac{\partial_\alpha |s(\alpha)|^2}{4|p|^2} \left[ \frac{\mathcal{E}'(|p - w_\alpha|)}{|p - w_\alpha|} + \frac{\mathcal{E}'(|w_\alpha|)}{|w_\alpha|} \right]^2
\]
\[
+ |s(\alpha)|^2 \left[ \frac{\mathcal{E}'(|p - w_\alpha|)}{|p - w_\alpha|} + \frac{\mathcal{E}'(|w_\alpha|)}{|w_\alpha|} \right]^2.
\]
We get the following representation of $|\nabla H^p_0|
\[
|\nabla H^p_0| = \sqrt{\frac{|\partial_\alpha |s(\alpha)|^2|^2}{4} + |\alpha|^2 |p|^2} \left[ \frac{\mathcal{E}'(|p - w_\alpha|)}{|p - w_\alpha|} + \frac{\mathcal{E}'(|w_\alpha|)}{|w_\alpha|} \right].
\]
As for the surface integral of a radial function $G(|w|)$, we introduce the radial variable $u = |W_\alpha| = \sqrt{\alpha^2 |p|^2 + |q_\alpha|^2}$. We compute $2udu = \partial_\alpha |W_\alpha|^2 d\alpha$ and hence
\[
\frac{d\sigma(w)}{|\nabla H^p_0|} = \frac{|p|}{2} \left[ \frac{\mathcal{E}'(|p - w_\alpha|)}{|p - w_\alpha|} + \frac{\mathcal{E}'(|w_\alpha|)}{|w_\alpha|} \right] \partial_\alpha |W_\alpha|^2 d\alpha d\theta.
\]
Figure 2: Sketched is the trace of $S'_p$ on any two dimensional plane containing $p$.

which in combination with (2.8) yields

$$\frac{d\sigma(w)}{|\nabla H^p_0|} = \frac{|w|}{c'(\|p-w\|)} \frac{d\theta}{|p|} = \frac{|p-w\|^{2-\gamma}}{|p|(|\gamma-1|)} d\theta,$$

Since

$$\frac{u\|p\| + u^{2-\gamma}}{|p|(|\gamma-1|)} \leq \frac{u\|p-w\|^{2-\gamma}}{|p|(|\gamma-1|)} \leq \frac{u\|p\| + u^{2-\gamma}}{|p|(|\gamma-1|)},$$

upon noting that $d\sigma(S_p) = \|p\|^{d-1}d\sigma(S_{ep})$ and defining $v = \frac{u}{|p|}$, we obtain

$$\int_{S_p} \frac{d\sigma(w)}{|\nabla H^p_0|} \geq c_1 \|p\|^{d-\gamma} \int_0^1 v|1-v|^{2-\gamma} dv \geq c_0 \|p\|^{d-\gamma},$$

and

$$\int_{S_p} \frac{d\sigma(w)}{|\nabla H^p_0|} \leq C_1 \|p\|^{d-\gamma} \int_0^1 v|1+v|^{2-\gamma} dv \leq C_0 \|p\|^{d-\gamma},$$

for some $c_0, c_1, C_0, C_1$, depending only on $\gamma$ (in particular, independent of $p$). This proves (iv).

Finally, we check the surface integral of a radial function $f(|w|)$. It is clear that

$$\int_{S_p} f(|w|) \frac{d\sigma(w)}{|\nabla H^p_0|} \leq C \|p\|^{d-\gamma-2} \int_0^1 f(u) u du.$$

The lemma follows by the spherical coordinates $dw = |w|^{d-1}d(|w|)d\sigma(S^d)$.  

\textbf{Lemma 2.3 (Surface $S'_p$)} Let $S'_p$ be defined as in (1.12)-(2.3). Then, $S'_p = |p|S'_{ep}$, with $e_p = p/|p|$. In addition, there is a positive constant $C_0$ so that

$$\int_{S'_p} F(|w|) \frac{d\sigma(w)}{|\nabla H^p_1(w)|} \leq C_0 \|p\|^{d-2-\gamma} \int_0^\infty F(|w|)(1 + |w|/|p|)^{2-\gamma}|w|^{2-d} dw \quad (2.12)$$

uniformly in $p \in \mathbb{R}^d$. 

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Proof It is clear that \( S'_p = |p|S'_{e_p} \), in which the surface

\[
S'_{e_p} = \left\{ w(\alpha, \theta) = \alpha e_p + s(\alpha)e_\theta : \alpha \in [0,\alpha_0), \ \theta \in [0,2\pi] \right\}
\]

(2.13)

for some monotonic function \( s(\alpha) \) and some positive constant \( \alpha_0 \); see Figure 2. We stress that the parametrization \( \alpha, s(\alpha) \) and \( \theta \) are independent of \( p \). As a consequence, the surface integral on \( S'_p \) is independent of \( p \).

As in (2.7), we have

\[
d\sigma(w) = \sqrt{|p|^2|s(\alpha)|^2 + \frac{1}{4}|\partial_\alpha(|s(\alpha)|^2)|^2} \, d\alpha d\theta
\]

and hence, the surface is estimated by

\[
\frac{d\sigma(w)}{|\nabla H^p_1(w)|} = \frac{\sqrt{|p|^2|s(\alpha)|^2 + \frac{1}{4}|\partial_\alpha(|s(\alpha)|^2)|^2}}{2\partial_\alpha|w_\alpha|^2|\nabla H^p_1(|\alpha p + s(\alpha)|)|} d\alpha d\theta.
\]

Let us introduce the variable \( u = |w| = \sqrt{\alpha^2|p|^2 + |s(\alpha)|^2} \). We compute

\[
2udu = \partial_\alpha|w_\alpha|^2 d\alpha
\]

and hence

\[
\frac{d\sigma(w)}{|\nabla H^p_1(w_\alpha)|} = C\sqrt{|p|^2|s(\alpha)|^2 + \frac{1}{4}|\partial_\alpha(|s(\alpha)|^2)|^2} \left| \frac{u}{2\partial_\alpha|w_\alpha|^2|\nabla H^p_1(|\alpha p + s(\alpha)|)|} \right| u dud\theta.
\]

(2.14)

We recall that \( H^p_1(w_\alpha) = 0 \) and hence

\[
0 = \partial_\alpha w_\alpha \cdot \nabla w H^p_1 = \frac{1}{2} \partial_\alpha|w_\alpha|^2 \left[ \frac{\mathcal{E}'(p + w_\alpha)}{|w_\alpha|} - \frac{\mathcal{E}'(w_\alpha)}{|w_\alpha|} \right] + |p|^2 \mathcal{E}'(p + w_\alpha)
\]

which leads to

\[
|p|^2 \frac{\mathcal{E}'(p + w_\alpha)}{|p + w_\alpha|} = \frac{1}{2} \partial_\alpha|w_\alpha|^2 \left[ \frac{\mathcal{E}'(w_\alpha)}{|w_\alpha|} - \frac{\mathcal{E}'(p + w_\alpha)}{|p + w_\alpha|} \right],
\]

(2.15)

and

\[
\partial_\alpha|s(\alpha)|^2 = 2 \frac{-\alpha|p|^2 \mathcal{E}'(w_\alpha) + (1 + \alpha)|p|^2 \mathcal{E}'(p + w_\alpha)}{\left[ \mathcal{E}'(w_\alpha) - \mathcal{E}'(p + w_\alpha) \right]}.
\]

(2.16)

We deduce from (2.16) that

\[
|\nabla H^p_1|^2 = |p|^2 \left[ -\alpha \frac{\mathcal{E}'(w_\alpha)}{|w_\alpha|} + (\alpha + 1) \frac{\mathcal{E}'(p + w_\alpha)}{|p + w_\alpha|} \right]^2 + |s(\alpha)|^2 \left[ \frac{\mathcal{E}'(p + w_\alpha)}{|p + w_\alpha|} - \frac{\mathcal{E}'(w_\alpha)}{|w_\alpha|} \right]^2
\]

\[
= \frac{\partial_\alpha|s(\alpha)|^2}{4|p|^2} \left[ \mathcal{E}'(p + w_\alpha) - \mathcal{E}'(w_\alpha) \right]^2 + |s(\alpha)|^2 \left[ \frac{\mathcal{E}'(p + w_\alpha)}{|p + w_\alpha|} - \frac{\mathcal{E}'(w_\alpha)}{|w_\alpha|} \right]^2,
\]

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which implies
\[ \frac{d\sigma(w)}{|\nabla H_{1}(w)|} = C \frac{|p|}{\partial_{\alpha}|w_{\alpha}|^{2}} \left[ \frac{\mathcal{E}'(|w_{\alpha}|)}{|w_{\alpha}|} - \frac{\mathcal{E}'(|p+w_{\alpha}|)}{|p+w_{\alpha}|} \right] udud\theta. \]

The above and (2.15) yield
\[ \frac{d\sigma(w)}{|\nabla H_{1}(w)|} = C \frac{udu}{2|p|\mathcal{E}'(|p+w|)}. \]

We obtain
\[ \int_{S} F(|w|) \frac{d\sigma(w)}{|\nabla H_{1}(w)|} = C|p|^{d-2-\gamma} \int_{0}^{\infty} F(u)(1+u/|p|)^{2-\gamma}du. \]

The lemma follows by the spherical coordinates \( dw = |w|^{d-1}d(|w|)d\sigma(S^{d}). \)

\section{3 Weighted \( L^{1} \) estimates}

In this section, we shall derive uniform estimates on the weighted \( L^{1} \) norm, where weights are \( n^{th} \)-order monomials of \( \mathcal{E}_{k} \), which are defined by
\[ \mathcal{M}_{n}[g] = \int_{\mathbb{R}^{d}} \mathcal{E}^{n}_{k}g(k)dk \]

in which we recall the energy function \( \mathcal{E}_{k} = |k|^{3/2} \). We stress that our estimates might depend on the positive coefficient of viscosity, but is independent of \( \varrho \), in the equation; see \[1.6\].

\subsection{3.1 Estimate of the collision operator}

We first obtain the following estimate on the collision operator \( Q[g] \).

\textbf{Lemma 3.1} \textit{Let} \( N > 1 \). \textit{For any positive and radial function} \( g(p) = g(|p|) \), \textit{there exists a constant} \( C_{N} \), \textit{depending on} \( N \), \textit{such that the following holds}
\[ \int_{\mathbb{R}^{d}} Q[g](k)\mathcal{E}_{k}^{N} dk \leq C_{N} \sum_{n=[N/2]}^{N-1} \left( \mathcal{M}_{n+\frac{7-2d}{2}}[g]\mathcal{M}_{N-n+\frac{5}{2}}[g] + \mathcal{M}_{n+\frac{11-2d}{2}}[g]\mathcal{M}_{N-n+\frac{5}{2}}[g] \right). \]

\textbf{Proof} \textit{It is sufficient to prove the lemma for} \( N \) \textit{to be natural numbers. By Lemma [2.1], we have}
\[ \int_{\mathbb{R}^{d}} Q[g](k)\mathcal{E}_{k}^{N} dk = \int\int_{\mathbb{R}^{2d}} R_{k,k_{1},k_{2}}[g] \left[ \mathcal{E}_{k}^{N} - \mathcal{E}_{k_{1}}^{N} - \mathcal{E}_{k_{2}}^{N} \right] dkdk_{1}dk_{2}. \]
Using the resonant conditions $k = k_1 + k_2$ and $\mathcal{E}_k = \mathcal{E}_{k_1} + \mathcal{E}_{k_2}$, dictated by the Dirac delta functions in $R_{k,k_1,k_2}[g]$, we can write

$$\mathcal{E}_k^N - \mathcal{E}_{k_1}^N - \mathcal{E}_{k_2}^N = (\mathcal{E}_{k_1} + \mathcal{E}_{k_2})^N - \mathcal{E}_{k_1}^N - \mathcal{E}_{k_2}^N = \sum_{n=1}^{N-1} \binom{N}{n} \mathcal{E}_{k_1}^n \mathcal{E}_{k_2}^{N-n}. $$

Thus, we obtain

$$\int_{\mathbb{R}^d} Q[g](k) \mathcal{E}_k^N dk = \int_{\mathbb{R}^2d} R_{k_1+k_2,k_1,k_2}[g] \sum_{n=1}^{N-1} \binom{N}{n} \mathcal{E}_{k_1}^n \mathcal{E}_{k_2}^{N-n} dk_1 dk_2$$

$$= \int_{\mathbb{R}^2d} R_{k_1+k_2,k_1,k_2}[g] \sum_{n=1}^{[N/2]-1} \binom{N}{n} \mathcal{E}_{k_1}^n \mathcal{E}_{k_2}^{N-n} dk_1 dk_2$$

$$+ \int_{\mathbb{R}^2d} R_{k_1+k_2,k_1,k_2}[g] \sum_{n=[N/2]}^{N-1} \binom{N}{n} \mathcal{E}_{k_1}^n \mathcal{E}_{k_2}^{N-n} dk_1 dk_2$$

$$=: I_1 + I_2. $$

Clearly, due to the symmetry of $k_1$ and $k_2$, it suffices to give estimates on $I_2$. Indeed, we write

$$I_1 = \int_{\mathbb{R}^2d} R_{k_1+k_2,k_1,k_2}[g] \sum_{n=N-[N/2]+1}^{N} \binom{N}{n} \mathcal{E}_{k_1}^{N-n} \mathcal{E}_{k_2}^{n} dk_1 dk_2,$n

which is in fact $I_2$. We now estimate $I_2$. Recall that

$$R_{k,k_1,k_2}[g] = 4\pi |V_{k_1,k_2}|^2 \delta(k-k_1-k_2) \delta(\mathcal{E}_k - \mathcal{E}_{k_1} - \mathcal{E}_{k_2})(g_1 g_2 - g_1 g_2 - g g_2)$$

and the energy surface $S'_k$ is defined as in (1.12). Thus, using the nonnegativity of $g(k)$, we can drop the last two terms $g_1 g_2 + gg_2$ in (3.4), yielding

$$I_2 \leq 4\pi \int_{\mathbb{R}^d} \int_{S'_k} |V_{k_1+k_2,k_1,k_2}|^2 g_1 g_2 \sum_{n=[N/2]}^{N-1} \binom{N}{n} \mathcal{E}_{k_1}^n \mathcal{E}_{k_2}^{N-n} \frac{d\sigma(k_1)}{\sqrt{H'_k(k_1)}},$$

in which $H'_k$ is defined as in (1.11). Let us now estimate the collision kernel $V_{k_1+k_2,k_1,k_2}$, defined as in (1.8). We recall

$$V_{k,k_1,k_2} = \frac{1}{8\pi \sqrt{2\sigma}} \sqrt{\mathcal{E}_k \mathcal{E}_{k_1} \mathcal{E}_{k_2}} \left( \frac{L_{k_1,k_2}}{|k||k_1||k_2|} - \frac{L_{k,-k_1}}{|k_2||k_1|} - \frac{L_{k,-k_2}}{|k_1||k||k_2|} \right)$$

with $L_{k_1,k_2} = k_1 \cdot k_2 + |k_1||k_2|$ and $\mathcal{E}_k = |k|^{3/2}$. It is clear that $|L_{k_1,k_2}| \leq 2|k_1||k_2|$. In addition, the energy identity $\mathcal{E}_k = \mathcal{E}_{k_1} + \mathcal{E}_{k_2}$ in particular implies that $|k_1| \leq |k|$ and $|k_2| \leq |k|$, due to the monotonicity of $\mathcal{E}_k$. Hence, for $k = k_1 + k_2$, we compute

$$0 \leq L_{k,-k_1} = |k||k_1| - k \cdot k_1 = |k_1|(|k| - |k_1|) - k_1 \cdot k_2 \leq 2|k_1||k_2|. $$

(3.5)
The same bound holds for \( L_{k,-k_2} \). This proves that

\[
|V_{k,k_1,k_2}| \leq C_0 \sqrt{\mathcal{E}_k \mathcal{E}_{k_1,k_2}} \left( \frac{\sqrt{|k_1||k_2|}}{|k|} + \frac{\sqrt{|k_1|}}{\sqrt{|k|}} + \frac{\sqrt{|k_2|}}{\sqrt{|k|}} \right)
\]

for some universal constant \( C_0 \). Using again \( |k| \geq \max\{|k_1|, |k_2|\} \), we obtain

\[
|V_{k,k_1,k_2}| \leq C_0 \sqrt{\mathcal{E}_k \mathcal{E}_{k_1,k_2}}
\]

for all \((k,k_1,k_2)\) satisfying the resonant conditions \( k = k_1 + k_2 \) and \( \mathcal{E}_k = \mathcal{E}_{k_1} + \mathcal{E}_{k_2} \). Hence, we have

\[
I_2 \leq C_0 \int_{\mathbb{R}^d} \int_{S'_{k_2}} \mathcal{E}_{k_1} \mathcal{E}_{k_2}(\mathcal{E}_{k_1} + \mathcal{E}_{k_2}) g_1 g_2 \sum_{n=\lfloor N/2 \rfloor}^{N-1} \binom{N}{n} \mathcal{E}_{k_1}^{n} \mathcal{E}_{k_2}^{N-n} \frac{d\sigma(k_1)}{|\nabla H_{k_2}^k(k_1)|} dk_2
\]

\[
\leq C_0 \sum_{n=\lfloor N/2 \rfloor}^{N-1} \binom{N}{n} \int_{\mathbb{R}^d} g_2 \mathcal{E}_{k_2}^{N-n+1} \left( \int_{S'_{k_2}} \mathcal{E}_{k_2}^{n+1}(\mathcal{E}_{k_1} + \mathcal{E}_{k_2}) g_1 \frac{d\sigma(k_1)}{|\nabla H_{k_2}^k(k_1)|} \right) dk_2.
\]

Next, applying Lemma \( \text{[2,3]} \) with \( \gamma = 3/2 \), to the surface integral on \( S'_{k_2} \) and recalling that \( g_1 = g(|k_1|) \), we obtain

\[
\int_{S'_{k_2}} \frac{g_2 \mathcal{E}_{k_2}^{n+1}(\mathcal{E}_{k_1} + \mathcal{E}_{k_2})}{|\nabla H_{k_2}^k(k_1)|} \frac{d\sigma(k_1)}{|\nabla H_{k_2}^k(k_1)|}
\]

\[
\leq C_0 \int_{\mathbb{R}^d} g(k) \mathcal{E}_{k_1}^{n+1}(\mathcal{E}_{k_1} + \mathcal{E}_{k_2}) \mathcal{E}_{k_2}^{2(3-d)/3}(\mathcal{E}_{k_1}^{3/2} + \mathcal{E}_{k_2}^{3/2}) \mathcal{E}_{k_1}^{2(3-d)/3} dk_1.
\]

This proves

\[
I_2 \leq C_N \sum_{n=\lfloor N/2 \rfloor}^{N-1} \left( \mathcal{M}_{n+1+\frac{4-2d}{3}}[g] \mathcal{M}_{N-n+1+\frac{2d-4}{3}}[g] + \mathcal{M}_{n+1+\frac{8-2d}{3}}[g] \mathcal{M}_{N-n+1+\frac{4d-8}{3}}[g] \right).
\]

This proves the lemma. \( \Box \)

**Remark 3.1** We note that by writing \( L_{k,-k_1} \) and \( L_{k,-k_2} \) as in \( (3.5) \), the kernel \( |V_{k,k_1,k_2}| \) is radial in \( k \).

### 3.2 Weighted \( L^r_N \) \((N > 1)\) estimates

**Proposition 3.1** Let \( N > 1 \). Suppose that \( f_0(k) = f_0(|k|) \) is a nonnegative radial initial data satisfying

\[
\int_{\mathbb{R}^d} f_0(k)(\mathcal{E}_k + \mathcal{E}^N_k) \ dk < \infty.
\]
Lemma 3.2

For some finite constant $C$, we obtain

$$\sup_{t \geq 0} \int_{\mathbb{R}^d} f(t, k) \mathcal{E}_k^N \, dk \leq C_N$$

(3.8)

for some finite constant $C_N$ depending on the initial data and the viscosity.

We need the following simple lemma.

**Lemma 3.2** For $M > n > p$, there holds

$$\mathcal{M}_n[g] \leq \mathcal{M}_{\frac{M-n}{p}}^M[g] \mathcal{M}_{n-p}^M[g].$$

(3.9)

**Proof**

The lemma follows from the definition of $\mathcal{M}_n$ and the following Hölder inequality

$$\int_{\mathbb{R}^d} g(k) \mathcal{E}_k^p \, dk \leq \left( \int_{\mathbb{R}^d} g(k) \mathcal{E}_k^p \, dk \right)^{\frac{M-n}{M-p}} \left( \int_{\mathbb{R}^d} g(k) \mathcal{E}_k^M \, dk \right)^{\frac{n-p}{M-p}}.$$

**Proof** [Proof of Proposition 3.1]

Using $\varphi = \mathcal{E}_k^N$ as a test function in (1.6), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(t, k) \mathcal{E}_k^N \, dk + 2\nu \int_{\mathbb{R}^d} (|k|^2 + g|k|^4) f(t, k) \mathcal{E}_k^N \, dk = \int_{\mathbb{R}^d} Q[f](k) \mathcal{E}_k^N \, dk.$$

By using Proposition 3.1 and recalling the definition of $\mathcal{M}_M$, the above yields

$$\frac{d}{dt} \mathcal{M}_N[f(t)] + 2\nu \mathcal{M}_{\frac{N+1}{3}}[f(t)] \leq C_N \sum_{n=[N/2]}^{N-1} \left( \mathcal{M}_{n+\frac{7-2d}{3}}[f(t)] \mathcal{M}_{-n+\frac{2d+1}{3}}[f(t)] + \mathcal{M}_{n+\frac{11-2d}{3}}[f(t)] \mathcal{M}_{-n+\frac{2d+1}{3}}[f(t)] \right).$$

Now using Lemma 3.2, with $p = 7/3$ and $M = N$, we get

$$\sum_{n=[N/2]}^{N-1} \mathcal{M}_{n+\frac{7-2d}{3}}[f(t)] \mathcal{M}_{-n+\frac{2d+1}{3}}[f(t)] \leq \sum_{n=[N/2]}^{N-1} \mathcal{M}_2[f(t)] \mathcal{M}_N[f(t)]$$

$$\leq C_N \mathcal{M}_2[f(t)] \mathcal{M}_N[f(t)]$$

and

$$\sum_{n=[N/2]}^{N-2} \mathcal{M}_{n+\frac{7-2d}{3}}[f(t)] \mathcal{M}_{-n+\frac{2d+1}{3}}[f(t)] \leq \sum_{n=[N/2]}^{N-2} \mathcal{M}_2[f(t)] \mathcal{M}_N[f(t)]$$

$$\leq C_N \mathcal{M}_2[f(t)] \mathcal{M}_N[f(t)].$$

We obtain

$$\frac{d}{dt} \mathcal{M}_N[f(t)] + \nu \mathcal{M}_{\frac{N+1}{3}}[f(t)] \leq C_N \mathcal{M}_2[f(t)] \mathcal{M}_N[f(t)].$$

(3.10)
The uniform boundedness of $\mathcal{M}_N[f(t)]$ follows from the standard Gronwall’s lemma, upon using the following energy inequality (see (2.2)):

$$\mathcal{M}_1[f(t)] + \int_0^t \mathcal{M}_2[f(s)] \, ds \leq \mathcal{M}_1[f_0].$$

The proposition is proved.

3.3 Weighted $L_{\frac{1}{3}}$ estimates

Proposition 3.2 Let $f_0(k) = f_0(|k|)$ be nonnegative and satisfy

$$\int_{\mathbb{R}^d} f_0(k) \mathcal{E}_k^{1/3}(1 + \mathcal{E}_k^{5/3}) \, dk < \infty.$$  

Then, corresponding nonnegative radial solutions $f(t, k) = f(t, |k|)$ of (1.6), with $f(0, k) = f_0(k)$, satisfy

$$\int_{\mathbb{R}^d} f(t, k) \mathcal{E}_k^{\frac{1}{3}} \, dk \leq c_0 e^{c_1 t}, \quad \forall t \geq 0,$$

(3.11)

for some universal constants $c_0, c_1$ depending on the initial data and the viscosity.

Proof By Proposition 3.1, the $L_{\frac{1}{3}}$-norm of $f$ is bounded for $s \in [1, \frac{7}{3}]$. Using $\mathcal{E}_k^{1/3}$ as a test function in (1.6), we obtain

$$\frac{d}{dt} \mathcal{M}_{\frac{1}{3}}[f(t)] + 2\nu \mathcal{M}_{\frac{1}{3}}[f(t)] \leq \int_{\mathbb{R}^d} \mathcal{Q}[f](k) \mathcal{E}_k^{\frac{1}{3}} \, dk.$$  

(3.12)

We now divide the proof into two steps.

Step 1: Estimating the collision integral. We can estimate the right hand side of (3.12) as

$$\int_{\mathbb{R}^d} \mathcal{E}_k^{\frac{1}{3}} \mathcal{Q}[f](k) \, dk \leq \iint_{\mathbb{R}^d} |R_{k,k_1,k_2}[f]| \left( \mathcal{E}_k^{\frac{1}{3}} + \mathcal{E}_{k_1}^{\frac{1}{3}} + \mathcal{E}_{k_2}^{\frac{1}{3}} \right) \, dk \, dk_1 \, dk_2,$$

in which, recall

$$R_{k,k_1,k_2}[g] = 4\pi |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2) \delta(\mathcal{E}_k - \mathcal{E}_{k_1} - \mathcal{E}_{k_2})(f_1 f_2 - f f_1 - f f_2).$$

By the resonant conditions $k = k_1 + k_2$ and $\mathcal{E}_k = \mathcal{E}_{k_1} + \mathcal{E}_{k_2}$, we the integrals can be re-expressed in terms of the surface integrals over $\mathbb{R}^d \times S_{k}$ and $\mathbb{R}^d \times S'_{k}$, as follows

$$\int_{\mathbb{R}^d} \mathcal{E}_k^{\frac{1}{3}} \mathcal{Q}[f](k) \, dk \leq C \int_{\mathbb{R}^d} \int_{S'_{k_1}} |V_{k_1+k_2,k_1,k_2}|^2 |f_1 f_2| \left( \mathcal{E}_k^{\frac{1}{3}} + \mathcal{E}_{k_1}^{\frac{1}{3}} + \mathcal{E}_{k_2}^{\frac{1}{3}} \right) \, d\sigma(k_1) \frac{d\sigma(k_2)}{|\nabla H_{\mathcal{E}_{k_1}}(k_2)|} \, dk_1$$

$$+ C \int_{\mathbb{R}^d} \int_{S_{k}} |V_{k_1+k_2,k_1,k_2}|^2 |f_1 f_2| \left( \mathcal{E}_k^{\frac{1}{3}} + \mathcal{E}_{k_1}^{\frac{1}{3}} + \mathcal{E}_{k_2}^{\frac{1}{3}} \right) \, d\sigma(k_1) \frac{d\sigma(k_2)}{|\nabla H_{\mathcal{E}_{k_1}}(k_2)|} \, dk_1$$

$$=: I_1 + I_2.$$
Estimate on $I_1$. By the Cauchy-Schwarz inequality and the conservation law $\mathcal{E}_{k_1+k_2} = \mathcal{E}_{k_1} + \mathcal{E}_{k_2}$,
\[
\frac{\mathcal{E}_{k_1+k_2}}{2} \leq C_N (\mathcal{E}_{k_1}^{\frac{1}{2}} + \mathcal{E}_{k_2}^{\frac{1}{2}})
\]
which then leads to
\[
I_1 \leq C \int_{\mathbb{R}^d} \int_{S_{k_2}'} |V_{k_1+k_2,k_1,k_2}|^2 |f_1||f_2| (\mathcal{E}_{k_1}^{\frac{1}{2}} + \mathcal{E}_{k_2}^{\frac{1}{2}}) \frac{d\sigma(k_1)}{\|\nabla H_{k_2}(k_1)\|} \, dk_2.
\]
Let us note from (3.6) that $|V_{k_1,k_2}|^2 \leq C_0 \mathcal{E}_{k_1} \mathcal{E}_{k_2}$. By Lemma 2.3 and the same argument used for (3.7), $I_1$ can be bounded the following way
\[
I_1 \leq C \int_{\mathbb{R}^d} \int_{S_{k_2}'} |V_{k_1+k_2,k_1,k_2}|^2 |f_1||f_2| (\mathcal{E}_{k_1}^{\frac{1}{2}} + \mathcal{E}_{k_2}^{\frac{1}{2}}) (\mathcal{E}_{k_1}^{\frac{1}{2}} + \mathcal{E}_{k_2}^{\frac{1}{2}}) \frac{d\sigma(k_1)}{\|\nabla H_{k_2}(k_1)\|} \, dk_2
\]
\[
\leq C \int_{\mathbb{R}^d} \int_{S_{k_2}'} \mathcal{E}_{k_1} \mathcal{E}_{k_2} |f_1||f_2| (\mathcal{E}_{k_1}^{\frac{1}{2}} + \mathcal{E}_{k_2}^{\frac{1}{2}}) \frac{d\sigma(k_1)}{\|\nabla H_{k_2}(k_1)\|} \, dk_2
\]
\[
\leq C \int_{\mathbb{R}^d} \int_{S_{k_2}'} \mathcal{E}_{k_1} \mathcal{E}_{k_2} |f_1||f_2| (\mathcal{E}_{k_1}^{\frac{1}{2}} + \mathcal{E}_{k_2}^{\frac{1}{2}}) (\mathcal{E}_{k_1}^{\frac{2d-4}{2d}} \mathcal{E}_{k_2}^{\frac{2d-4}{2d}} + \mathcal{E}_{k_1}^{\frac{8-2d}{2d}} \mathcal{E}_{k_2}^{\frac{8-2d}{2d}}) \, dk_1 \, dk_2
\]
\[
\leq C \int_{\mathbb{R}^d} \int_{S_{k_2}'} |f_1||f_2| (\mathcal{E}_{k_1}^{\frac{1}{2}} + \mathcal{E}_{k_2}^{\frac{1}{2}}) \frac{d\sigma(k_2)}{\|\nabla H_{k_2}(k_1)\|} \, dk
\]

Estimate on $I_2$. We turn to estimate $I_2$. Again, recalling $\mathcal{E}_k = \mathcal{E}_{k-k_2} + \mathcal{E}_{k_2} \geq \max\{\mathcal{E}_{k-k_2}, \mathcal{E}_{k_2}\}$, we bound
\[
I_2 \leq C \int_{\mathbb{R}^d} \int_{S_{k_2}} |V_{k-k_2,k_2}|^2 |f_1||f_2| (\mathcal{E}_{k-k_2}^{\frac{1}{2}} + \mathcal{E}_{k_2}^{\frac{1}{2}}) \frac{d\sigma(k_2)}{\|\nabla H_{k_2}(k_1)\|} \, dk
\]
Notice that $|V_{k-k_2,k_2}|^2 \leq C_0 \mathcal{E}_{k-k_2} \mathcal{E}_{k_2} \leq C_0 \mathcal{E}_{k_2}^2 \mathcal{E}_{k_2}$. By Lemma 2.2, the following holds true
\[
\int_{\mathbb{R}^d} \int_{S_{k_2}} |V_{k-k_2,k_2}|^2 |f_1||f_2| \mathcal{E}_{k}^{\frac{1}{2}} \frac{d\sigma(k_2)}{\|\nabla H_{k_2}(k_1)\|} \, dk
\]
\[
\leq C_0 \int_{\mathbb{R}^d} \int_{S_{k_2}} \mathcal{E}_{k-k_2} \mathcal{E}_{k_2} |f_1||f_2| \mathcal{E}_{k}^{\frac{1}{2}} \frac{d\sigma(k_2)}{\|\nabla H_{k_2}(k_1)\|} \, dk
\]
\[
\leq C_0 \int_{\mathbb{R}^d} \int_{S_{k_2}} |\mathcal{E}_{k-k_2}|^{2-d} |f_1||f_2| \mathcal{E}_{k}^{\frac{1}{2}} |k|^{d-2} \, dk \, dk_2
\]
\[
\leq C_M \|f\|_{L^\frac{1}{2}} \|f\|_{L^\frac{1}{2}} \leq C_M \|f\|_{L^\frac{1}{2}} \|f\|_{L^\frac{1}{2}}
\]
Combining (3.2) and (3.14) and using the fact that the $L^s_1$-norm of $f$ is bounded for $s \in [1, 2]$, we obtain
\[
\int_{\mathbb{R}^d} \mathcal{E}_{k}^{\frac{1}{2}} |Q[f](k)| \, dk \leq C^* \|f\|_{L^\frac{1}{2}}.
\]
Step 2: Estimating the $L^1_3$-norm. Putting together the two estimates (3.12) and (3.15) yields

$$\frac{d}{dt}M_1^3[f(t)] + 2\nu M_2^3[f(t)] \leq C^* M_1^3[f(t)],$$

(3.16)

which implies the bound on $M_1^3[f(t)]$.

The proof of the lemma is complete.

4 $L^2$ estimates

Proposition 4.1 Suppose that $f_0(k) = f_0(|k|)$ is a nonnegative radial initial data with

$$\int_{\mathbb{R}^d} f_0(k)\mathcal{E}_k(1 + \mathcal{E}_k^{2d-4}) \, dk < \infty$$

and

$$\int_{\mathbb{R}^d} |f_0(k)|^2 \, dk < \infty.$$

Then, corresponding nonnegative radial solutions $f(t, k) = f(t, |k|)$ of (1.6), with $f(0,k) = f_0(k)$, satisfy

$$\int_{\mathbb{R}^d} |f(t,k)|^2 \, dk \leq c_0 e^{c_1 t}.$$  

(4.1)

for some universal constants $c_0, c_1$ depending on the initial data and the viscosity.

Proof Using $f$ as a test function in (1.6), we obtain the following identity

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} f^2 \, dk + \nu \int_{\mathbb{R}^d} (|k|^2 + \rho |k|^4) f^2 \, dk = \int_{\mathbb{R}^d} Q[f] f \, dk.$$  

(4.2)

As an application of Lemma 2.1 the right hand side of (4.2) could be expressed as

$$\int_{\mathbb{R}^d} Q[f] f \, dk = 4\pi \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2) \delta(\mathcal{E}_k - \mathcal{E}_{k_1} - \mathcal{E}_{k_2})$$

$$\times (f_1 f_2 - f f_1 - f f_2)(f - f_1 - f_2) \, dk_1 dk_2 dk.$$  

(4.3)

By taking into account the positivity of $f$, the term inside the integral of (4.3) can be bounded by removing all the terms containing the negative sign, giving

$$(f_1 f_2 - f f_1 - f f_2)(f - f_1 - f_2) \leq 3 f f_1 f_2 + f f_1^2 + f f_2^2$$

$$\leq \frac{5}{2} f (f_1^2 + f_2^2).$$
Inserting the above inequality into \(4.3\) and using the symmetry in \(k_1\) and \(k_2\), we find
\[
\int_{\mathbb{R}^d} Q[f]f dk \leq C \int \int \int_{\mathbb{R}^d} |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2)\delta(E_k - E_{k_1} - E_{k_2})f(f_1^2 + f_2^2)dk_1dk_2dk
\leq C \int \int \int_{\mathbb{R}^d} |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2)\delta(E_k - E_{k_1} - E_{k_2})f f_2^2 dk_1dk_2dk.
\]

Then, again using the definition of the Dirac functions \(\delta(k - k_1 - k_2)\) and \(\delta(E_k - E_{k_1} - E_{k_2})\), we obtain
\[
\int_{\mathbb{R}^d} Q[f]f dk \leq C \int \int \int_{\mathbb{R}^d} |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2)\delta(E_k - E_{k_1} - E_{k_2})f f_2^2 dk_1dk_2dk.
\]

Recall that \(|V_{k,k_1,k_2}|^2\) is bounded by \(C E_k E_{k_1} E_{k_2}\), and on the surface \(S_k\), \(E_{k_2} \leq E_k\) and \(E_{k_2} \leq E_k\). This together with Lemma \(2.2\) yields
\[
\int_{\mathbb{R}^d} Q[f]f dk \leq C \int \int \int_{\mathbb{R}^d} |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2)\delta(E_k - E_{k_1} - E_{k_2})f f_2^2 dk_1dk_2dk.
\]

Now, by interpolating the results of Proposition \(3.1\), the \(L_{1.5}^1\) norm of \(f\) is bounded. Hence,\[
\int_{\mathbb{R}^d} Q[f]f dk \leq C \int \int |k|^{1/2} f_2^2 dk.
\]

Putting this into \(4.2\) yields
\[
\frac{d}{dt} \int_{\mathbb{R}^d} f^2 dk \leq \int_{\mathbb{R}^d} \left(C |k|^{7/3 - d} - \nu |k|^2 \right) f^2 dk.
\]

Let us note that the function \(\rho(x) = C x^{7/3 - 2d} - \nu x^2\), \(d = 2, 3, x \in \mathbb{R}_+\) is bounded from above by some positive constant \(C_1\) (depending on \(\nu\)). This proves
\[
\frac{d}{dt} \int_{\mathbb{R}^d} f^2 dk \leq C_1 \int_{\mathbb{R}^d} f^2 dk,
\]
which yields the proposition.

\section{Holder estimates for \(Q[f]\)}

In this section, we study the Hölder continuity of the collision operator \(Q[f]\) with respect to weighted \(L_N^1\) norm:
\[
\|f\|_{L_N^1} = \int_{\mathbb{R}^d} f(k) E_k^N dk.
\]
Proposition 5.1 Let $M, N \geq 1$, and let $\mathcal{S}_M$ be any bounded subset of $L^1_1(\mathbb{R}^d) \cap L^1_{1+3}(\mathbb{R}^d)$, with $L^1_1$ and $L^1_{1+3}$ norms bounded by $M$. Then, there exists a constant $C_{M,N}$, depending on $M, N$, so that

$$
\|Q[g] - Q[h]\|_{L^1_N} + \|Q[g] - Q[h]\|_{L^1_{\frac{1}{3}}} \leq C_{M,N}\left(\|g - h\|_{L^1_N} + \|g - h\|_{L^1_{\frac{1}{3}}}\right)^{\frac{1}{3}}
$$

(5.1)

for all $g, h \in \mathcal{S}_M$.

We first prove the following lemma.

Lemma 5.1 Let $M, N > 0$, and let $\mathcal{S}_M$ be any bounded subset of $L^1_1(\mathbb{R}^d) \cap L^1_{1+2}(\mathbb{R}^d)$, with $L^1_1$ and $L^1_{1+2}$ norms bounded by $M$. Then, there exists a constant $C_{M,N}$, depending on $M, N$, so that

$$
\|Q[g] - Q[h]\|_{L^1_N} \leq C_{M,N}\left(\|g - h\|_{L^1_N} + \|g - h\|_{L^1_{1+2}}\right)
$$

(5.2)

for all $g, h \in \mathcal{S}_M$.

Proof By definition of the collision operator, we compute

$$
Q[g] - Q[h] = \int_{\mathbb{R}^d} \left[ R_{k,k_1,k_2}[g] - R_{k,k_1,k_2}[h] - 2(R_{k_1,k,k_2}[g] - R_{k_1,k,k_2}[h]) \right] dk_1 dk_2
$$

and hence

$$
\|Q[g] - Q[h]\|_{L^1_N} = \int_{\mathbb{R}^d} \mathcal{E}_k^N |Q[g](k) - Q[h](k)| dk
$$

$$
\leq \int_{\mathbb{R}^d} \mathcal{E}_k^N |R_{k,k_1,k_2}[g] - R_{k,k_1,k_2}[h]| dk dk_1 dk_2
$$

$$
+ 2 \int_{\mathbb{R}^d} \mathcal{E}_k^N |R_{k_1,k,k_2}[g] - R_{k_1,k,k_2}[h]| dk dk_1 dk_2
$$

$$
= \int_{\mathbb{R}^d} |R_{k,k_1,k_2}[g] - R_{k,k_1,k_2}[h]| \left(\mathcal{E}_k^N + \mathcal{E}_k^N + \mathcal{E}_k^N\right) dk dk_1 dk_2.
$$

Recall that

$$
R_{k,k_1,k_2}[g] = C|V_{k_1,k,k_2}|^2 \delta(k - k_1 - k_2) \delta(E_k - E_{k_1} - E_{k_2})(g_{12}g_2 - g_{12}g_1 - gg_2).
$$

Using the resonant conditions $k = k_1 + k_2$ and $\mathcal{E}_k = \mathcal{E}_{k_1} + \mathcal{E}_{k_2}$, we write the triple integrals in term of the surface integrals over $\mathbb{R}^d \times S_k$ and $\mathbb{R}^d \times S'_{k_1}$. It follows at once that

$$
\|Q[g] - Q[h]\|_{L^1_N} \leq C \int_{\mathbb{R}^d} \int_{S_{k_1}} |V_{k_1+k_2,k_1,k_2}|^2 |g_{12}g_2 - h_{12}h_2| \left(\mathcal{E}_{k_1+k_2}^N + \mathcal{E}_{k_1}^N + \mathcal{E}_{k_2}^N\right) \frac{d\sigma(k_2)}{|\nabla H_{k_1}^N(k_2)|} dk_1
$$

$$
+ 8\pi \int_{\mathbb{R}^d} \int_{S_k} |V_{k,k_2,k_2}|^2 |g_{22}g_2 - h_{22}h_2| \left(\mathcal{E}_{k}^N + \mathcal{E}_{k-2k}^N + \mathcal{E}_{k_2}^N\right) \frac{d\sigma(k_2)}{|\nabla H_{k_2}^N(k_2)|} dk
$$

$$
=: J_1 + J_2,
$$

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in which $H_j^k$ are defined as in (1.11).

**Estimate on $J_1$.** Using the triangle inequality and the conservation law $\mathcal{E}_{k_1+k_2} = \mathcal{E}_{k_1} + \mathcal{E}_{k_2}$, we have

$$\mathcal{E}_{k_1+k_2}^N \leq C_N(\mathcal{E}_{k_1}^N + \mathcal{E}_{k_2}^N)$$

and

$$|g_1g_2 - h_1h_2| \leq |g_1 - h_1||g_2| + |h_1||g_2 - h_2|.$$  

Thus, we obtain

$$J_1 \leq C_N \int_{\mathbb{R}^d} \int_{S_{k_2}} |V_{k_1+k_2, k_1, k_2}|^2 |g_1 - h_1||g_2| (\mathcal{E}_{k_1}^N + \mathcal{E}_{k_2}^N) \frac{d\sigma(k_1)}{|\nabla H_{k_2}^1(k_1)|} \, dk_2$$

$$+ C_N \int_{\mathbb{R}^d} \int_{S_{k_1}'} |V_{k_1+k_2, k_1, k_2}|^2 |h_1||g_2 - h_2| (\mathcal{E}_{k_1}^N + \mathcal{E}_{k_2}^N) \frac{d\sigma(k_2)}{|\nabla H_{k_1}^1(k_2)|} \, dk_1.$$

(5.3)

Recall from (3.6) that $|V_{k_1, k_2}|^2 \leq C_0 \mathcal{E}_{k_1} \mathcal{E}_{k_2}$. Thus, together with Lemma 2.3 and the same argument used for (3.7), we estimate the first integral term in $J_1$, yielding

$$\int_{\mathbb{R}^d} \int_{S_{k_2}} |V_{k_1+k_2, k_1, k_2}|^2 |g_1 - h_1||g_2| (\mathcal{E}_{k_1}^N + \mathcal{E}_{k_2}^N) \frac{d\sigma(k_1)}{|\nabla H_{k_2}^1(k_1)|} \, dk_2$$

$$\leq C_0 \int_{\mathbb{R}^d} \int_{S_{k_2}} \mathcal{E}_{k_1} \mathcal{E}_{k_2} |g_1 - h_1||g_2| (\mathcal{E}_{k_1}^N + \mathcal{E}_{k_2}^N) (\mathcal{E}_{k_1} + \mathcal{E}_{k_2}) \frac{d\sigma(k_1)}{|\nabla H_{k_2}^1(k_1)|} \, dk_2$$

$$\leq C_0 \int_{\mathbb{R}^d} \int_{S_{k_2}} \mathcal{E}_{k_1} \mathcal{E}_{k_2} (\mathcal{E}_{k_1}^N \mathcal{E}_{k_2}^N + \mathcal{E}_{k_1}^{2d-4} \mathcal{E}_{k_2}^{2d-8} + \mathcal{E}_{k_1}^{2d-4} \mathcal{E}_{k_2}^{2d-8}) |g_1 - h_1||g_2| (\mathcal{E}_{k_1} + \mathcal{E}_{k_2}) \, dk_1 \, dk_2$$

$$\leq C_M \left( \|g - h\|_{L^1_{k_1+1}} + \|g - h\|_{L^1_{k_1+1}} \right),$$

in which we have used the boundedness of $g$ in $L^1_{k_1+1} \cap L^1_{k_1+1}$. By symmetry, the same estimate holds for the second integral in $J_1$.

**Estimate on $J_2$.** We turn to estimate $J_2$. Again, using

$$|gg_2 - hh_2| \leq |g - h||g_2| + |h||g_2 - h_2|,$$

and recalling $\mathcal{E}_k = \mathcal{E}_{k-k_2} + \mathcal{E}_{k_2} \geq \max\{\mathcal{E}_{k-k_2}, \mathcal{E}_{k_2}\}$, we estimate

$$J_2 = C \int_{\mathbb{R}^d} \int_{S_k} |V_{k-k_2, k_2}|^2 |gg_2 - hh_2| (\mathcal{E}_k^N + \mathcal{E}_{k-k_2}^N + \mathcal{E}_{k_2}^N) \frac{d\sigma(k)}{|\nabla H_0^k(k_2)|} \, dk$$

$$\leq C_N \int_{\mathbb{R}^d} \int_{S_k} |V_{k-k_2, k_2}|^2 |g - h||g_2| \mathcal{E}_k^N \frac{d\sigma(k)}{|\nabla H_0^k(k_2)|} \, dk$$

$$+ C_N \int_{\mathbb{R}^d} \int_{S_k} |V_{k-k_2, k_2}|^2 |h||g_2 - h_2| \mathcal{E}_k^N \frac{d\sigma(k)}{|\nabla H_0^k(k_2)|} \, dk.$$

(5.4)
Recall that $|V_{k,k-k_2,k_2}|^2 \leq C_0 \mathcal{E}_k \mathcal{E}_{k-k_2} \mathcal{E}_{k_2} \leq C_0 \mathcal{E}_k^2 \mathcal{E}_{k_2}$. Therefore, using Lemma 2.2 with $\gamma = 3/2$, we estimate

$$
\int_{\mathbb{R}^d} \int_{S_k} |V_{k,k-k_2,k_2}|^2 |g-h||g_2| \mathcal{E}_k^N \frac{d\sigma(k_2)}{|\nabla H_0^k(k_2)|} \, dk \\
\leq C_0 \int_{\mathbb{R}^d} \int_{S_k} \mathcal{E}_k^2 \mathcal{E}_{k-k_2} |g-h||g_2| \mathcal{E}_k^N \frac{d\sigma(k_2)}{|\nabla H_0^k(k_2)|} \, dk \\
\leq C_0 \int_{\mathbb{R}^d} \int_{S_k} \mathcal{E}_{k-k_2} |k_2|^{2-d} |g-h||g_2| |k|^{d-\frac{7}{2}} \mathcal{E}_k^{N+2} \, dk \, dk_2 \\
\leq C_M \|g-h\|_{L^1_{N+2d-\frac{3}{2}}}.
$$

in which we have again used the boundedness of $g$ with respect to $L^1_{\frac{7-2d}{3}}$ norm.

We now estimate the second integral in $J_2$.

$$
\int_{\mathbb{R}^d} \int_{S_k} |V_{k,k-k_2,k_2}|^2 |h||g_2| |h_2| \mathcal{E}_k^N \frac{d\sigma(k_2)}{|\nabla H_0^k(k_2)|} \, dk \\
\leq C_0 \int_{\mathbb{R}^d} \int_{S_k} \mathcal{E}_k^2 \mathcal{E}_{k-k_2} |h||g_2| |h_2| \mathcal{E}_k^N \frac{d\sigma(k_2)}{|\nabla H_0^k(k_2)|} \, dk \\
\leq C_N \|g-h\|_{L^1_{N+2d-\frac{3}{2}}}
$$

in which again the boundedness of $h$ in $L^1_{N+2d-\frac{3}{2}}$ was used.

Combining, we obtain

$$
\|Q[g] - Q[h]\|_{L^1_N} \leq C_{M,N} \left( \|g-h\|_{L^1_{N+2d-\frac{3}{2}}} + \|g-h\|_{L^1_{\frac{7-2d}{3}}} \right).
$$

Since $\mathcal{E}_k^{N+2d-\frac{3}{2}} + \mathcal{E}_k^{\frac{7-2d}{3}} \leq C(\mathcal{E}_k^3 + \mathcal{E}_k^{N+2})$, the above reduces to

$$
\|Q[g] - Q[h]\|_{L^1_N} \leq C_{M,N} \left( \|g-h\|_{L^1_{\frac{3}{2}}} + \|g-h\|_{L^1_{N+2}} \right), \quad (5.5)
$$

The proof of the lemma is complete.

\[\blacksquare\]

**Proof** [Proof of Proposition 5.1] The proposition now follows straightforwardly from the previous lemma. Indeed, we recall the interpolation inequality (see Lemma 3.2):

$$
\|g\|_{L^p} \leq \|g\|_{L^{\frac{q-n}{p-n}}}^{\frac{n}{q-n}} \|g\|_{L^{\frac{n-n}{p-n}}}^{\frac{n}{n-n}}
$$

for $q > n > p$. Together with the boundedness of $g, h$ in $L^1_1 \cap L^1_{N+3}$, we obtain

$$
\|g-h\|_{L^1_{N+2}} \leq \|g-h\|_{L^1_N}^{\frac{1}{2}} \|g-h\|_{L^1_{N+3}}^{\frac{1}{2}} \leq C_M \|g-h\|_{L^1_{N+3}}^{\frac{1}{2}}.
$$
Lemma 5.1 yields
\[ \|Q[g] - Q[h]\|_{L^N} \leq C_{M,N} \left( \|g - h\|_{L^N} + \|g - h\|_{L^1} \right)^{1/3} \]
which holds for all \( N > 0 \). In particular, the above holds for \( \|Q[g] - Q[h]\|_{L^1} \). The proposition follows.

\section{Proof of Theorem 1.1}

\subsection{Case 1: \( \varrho > 0 \)}

The proof of our main theorem, Theorem 1.1, for the case \( \varrho > 0 \) uses the following abstract theorem, introduced in \cite{1,36} inspired by the previous works of \cite{3,26}. For sake of completeness, the proof of the abstract theorem will be given in the Appendix.

\textbf{Theorem 6.1} Let \([0,T]\) be a time interval, \( E := (E, \| \cdot \|) \) be a Banach space, \( S \) be a bounded, convex and closed subset of \( E \), and \( Q : S \rightarrow E \) be an operator satisfying the following properties:

(A) \( \| \cdot \| \) be a different norm of \( E \), satisfying \( \| \cdot \| \leq C_E \| \cdot \| \) for some universal constant \( C_E \), and the function

\[ |\cdot|_* : E \rightarrow \mathbb{R} \]

\[ u \rightarrow |u|_* \]

satisfying

\[ |u + v|_* \leq |u|_* + |v|_* \]

and

\[ |\alpha u|_* = |u|_* \]

for all \( u, v \) in \( E \) and \( \alpha \in \mathbb{R}_+ \).

Moreover,

\[ |u|_* = \|u\|_* \]

\[ \forall u \in S, \]

\[ |u|_* \leq \|u\|_* \leq C_E \|u\|, \forall u \in E, \]

and

\[ |Q[u]|_* \leq C_* (1 + |u|_*), \forall u \in S, \]

then

\[ S \subset B_* \left( O, (2R_* + 1) e^{(C_* + 1)T} \right) := \left\{ u \in E : \|u\|_* \leq (2R_* + 1) e^{(C_* + 1)T} \right\}, \]

for some positive constant \( R_* \geq 1 \).
(B) Sub-tangent condition

\[ \liminf_{h \to 0^+} h^{-1} \text{dist}(u + hQ[u], S) = 0, \quad \forall u \in S \cap B_\epsilon(O, (2R_* + 1)e^{(C_*+1)T}) , \]

(C) Hölder continuity condition

\[ ||Q[u] - Q[v]|| \leq C||u - v||^\beta, \quad \beta \in (0,1), \quad \forall u, v \in S , \]

(D) One-side Lipschitz condition

\[ [Q[u] - Q[v], u - v] \leq C||u - v||, \quad \forall u, v \in S , \]

where

\[ [\varphi, \phi] := \lim_{h \to 0^-} h^{-1}(||\phi + h\varphi|| - ||\phi||). \]

Then the equation

\[ \partial_t u = Q[u] \text{ on } [0,T] \times E, \quad u(0) = u_0 \in S \cap B_\epsilon(O, R_*) \]  \hspace{1cm} (6.1)

has a unique solution in \( C^1((0,T), E) \cap C([0,T], S) \).

We shall apply Theorem 6.1 for (1.6), which reads

\[ \partial_t f = \bar{Q}[f], \quad \bar{Q}[f] := Q[f] - 2\nu(||k||^2 + \varrho||k||^4)f, \]

in which \( \varrho > 0 \).

Fix an \( N > 1 \). We choose the Banach space \( E = L^1_\frac{1}{3}(\mathbb{R}^d) \cap L^1_N(\mathbb{R}^3) \), endowed with the following norm

\[ ||f||_E := ||f||_{L^1_\frac{1}{3}} + ||f||_{L^1_N}. \]

We define the function \( |\cdot|_* \) to be

\[ ||f||_* = \int_{\mathbb{R}^d} f(p)\xi_p^\frac{1}{3} dk. \]

Set

\[ ||f||_* = ||f||_{L^1_\frac{1}{3}}. \]

By (3.13), it is clear that for all \( f \geq 0, f \in E \), the following inequality holds true

\[ |Q[f]|_* \leq C_* (1 + ||f||_*). \]  \hspace{1cm} (6.2)

We then choose \( C_* \) in Theorem 6.1 as \( C^* \).

In addition, we take \( S_\varrho \) to be consisting of radial functions \( f \in L^1_\frac{1}{3}(\mathbb{R}^d) \cap L^1_{N+3}(\mathbb{R}^3) \) so that
(S1) $f \geq 0$;
(S2) $\|f\|_{L^1_3} \leq c_0$;
(S3) $\|f\|_{L^1_1} \leq c_1$;
(S4) $\|f\|_{L^1_{N+3}} \leq c_2$;

where

\[ c_0 := (2R + 1)e^{(C^* + 1)T}, \quad (6.3) \]

$R$, $c_1$ are some positive constant and

\[ c_2 = \frac{3\rho_*}{2}, \quad (6.4) \]

with $\rho_*$ defined below in (6.6). Note that from (3.15), $C^*$ depends on $c_1$ and $c_2$. Clearly, $S_\varrho$ is a bounded, convex and closed subset of $(E, \|\cdot\|_E)$. Moreover for all $f$ in $S_\varrho$, it is straightforward that $|f|_* = \|f\|_*$. By Proposition 3.1 and Remark 3.1, for $f_0 \in S_\varrho$, solutions to (1.6) are radial and remain in $S_\varrho$. Thus, it suffices to verify the four conditions ($\mathfrak{A}$), ($\mathfrak{B}$), ($\mathfrak{C}$) and ($\mathfrak{D}$) of Theorem 6.1.

6.1.1 Condition ($\mathfrak{A}$)

We choose the constant $R_*$ to be $R$, then for all $u$ in $S$, $\|u\|_* \leq (2R_* + 1)e^{(C^* + 1)T}$. Condition ($\mathfrak{A}$) is satisfied.

6.1.2 Condition ($\mathfrak{B}$)

For the sake of simplicity, we denote $N + 3$ by $N_*$. By using Proposition 3.1 and recalling the definition of $\mathfrak{M}_M$, for any $g$ that makes the integrals well-defined, we have

\[ \tilde{Q}[g] \leq -2\nu g \mathfrak{M}_{N_* + 2}[g] + C \sum_{n=[N_*/2]}^{N_* - 1} \left( \mathfrak{M}_{n + \frac{7 - 2d}{2}}[g] \mathfrak{M}_{N_* - n + \frac{2d - 1}{2}}[g] + \mathfrak{M}_{n + \frac{11 - 2d}{2}}[g] \mathfrak{M}_{N_* - n + \frac{2d - 2}{2}}[g] \right). \]

Now using Lemma 3.2 with $p = 1$ and $M = N_* + 1$, we get

\[ \sum_{n=[N_*/2]}^{N_* - 1} \left( \mathfrak{M}_{n + \frac{7 - 2d}{2}}[g] \mathfrak{M}_{N_* - n + \frac{2d - 1}{2}}[g] + \mathfrak{M}_{n + \frac{11 - 2d}{2}}[g] \mathfrak{M}_{N_* - n + \frac{2d - 2}{2}}[g] \right) \leq 2\nu g \mathfrak{M}_1[g] \mathfrak{M}_{N_* + 1}[g]. \]

By assuming that $\mathfrak{M}_1[g]$ is bounded by $c_1$, we find

\[ \int_{\mathbb{R}^d} \tilde{Q}[f](k) \mathcal{E}_k^N dk \leq C \mathfrak{M}_{N_* + 1}[g] - 2\nu g \mathfrak{M}_{N_* + 2}[g]. \]
Now, since $C|k|^{2} - \nu g|k|^{2}$ is bounded for all $k$ by some positive constant $c$, we deduce that $C\mathfrak{M}_{N,1}[g] - \nu g\mathfrak{M}_{N,2}[g]$ is also bounded by $C\mathfrak{M}_{N,1}[g]$. We then obtain the following estimate on $Q$

\[
\int_{\mathbb{R}^{d}} \bar{Q}[f](k)E_{k}^{N}dk \leq C\mathfrak{M}_{N,1}[g] - \nu g\mathfrak{M}_{N,2}[g].
\]

Applying again the Holder’s inequality (6.1.3), we end up with

\[
\mathfrak{M}_{N,2}[g] \leq \mathfrak{M}_{N,1}^{\frac{N}{N+1}}[g]\mathfrak{M}_{N,1}^{\frac{N}{N+2}}[g] \leq C\mathfrak{M}_{N,1}^{\frac{N}{N+2}}[g].
\]

Combining the above two estimates yields

\[
\int_{\mathbb{R}^{d}} \bar{Q}[f](k)E_{k}^{N}dk \leq \mathcal{P}[\mathfrak{M}_{N,1}[g]] := C_{1}\mathfrak{M}_{N,1}[g]\left(1 - C_{2}\mathfrak{M}_{N,1}^{\frac{2}{N+1}}[g]\right) \quad (6.5)
\]

where $C_{1}, C_{2}$ are positive constants depending on $c_{1}$. We set

\[
\rho_{*} = C_{2}^{-\frac{N+1}{2}}. \quad (6.6)
\]

Note that the function $\mathcal{P}(\cdot)$ in (6.5) satisfies $\mathcal{P}(x) < 0$ for $0 < x < \rho_{*}$ and $\mathcal{P}(x) > 0$ for $x > \rho_{*}$. In addition, we may take $C_{2}$ in (6.5) smaller, if needed, which allows $\rho_{*}$ and so $c_{2}$ in (6.4) to be arbitrarily large (but fixed).

Let $f$ be an arbitrary element of the set $S_{0} \cap B_{\rho}(O, (2R_{*} + 1)e^{(C_{*}+1)T})$. It suffices to prove the following claim: for all $\epsilon > 0$, there exists $h_{*}$ depending on $f$ and $\epsilon$ such that

\[
B_{E}(f + h\bar{Q}[f], \epsilon) \cap S_{0} \neq \emptyset, \quad 0 < h < h_{*}, \quad (6.7)
\]

in which $B_{E}(f, R)$ denotes the ball in $(E, \| \cdot \|_{E})$ centered at $f$ and having radius $R$. For $R > 0$, let $\chi_{R}(k)$ to be the characteristic function of the ball $B_{E}(0, R)$, and set

\[
w_{R} := f + h\bar{Q}[f_{R}], \quad f_{R}(k) = \chi_{R}(k)f(k), \quad (6.8)
\]

recalling $\bar{Q}[g] = Q[g] - 2\nu(|k|^{2} + g|k|^{4})g$. We shall prove that for all $R > 0$, there exists an $h_{R}$ so that $w_{R}$ belongs to $S_{0}$, for all $0 < h \leq h_{R}$. In view of (5.3), it is clear that $w_{R} \in L^{1}_{1} \cap L^{1}_{N}(\mathbb{R}^{d})$. We now check the conditions (S1)-(S3).

**Condition (S1).** Note that one can write $Q[f] = Q^{\text{gain}}[f] - Q^{\text{loss}}[f]$, with $Q^{\text{gain}}[f] \geq 0$ and $Q^{\text{loss}}[f] = fQ_{-}[f]$. Since $f_{R}$ is compactly supported, it is clear that $Q_{-}[f_{R}]$ is bounded by a positive constant $C_{f}$, depending on $f, R, c_{1}$, and $c_{2}$. Hence,

\[
w_{R} = f + h\left(Q[f_{R}] - 2\nu(|k|^{2} + g|k|^{4})f_{R}\right)
\geq f - hf_{R}\left(C_{f} + 2\nu R^{2} + gR^{4}\right)
\]

which is nonnegative, for sufficiently small $h$; precisely, $h \leq h_{R} := (C_{f} + 2\nu R^{2} + gR^{4})^{-1}$. 

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Condition (S2). Since
\[ \|f\|_* < (2R_* + 1)e^{(C_*+1)T}, \]
and
\[ \lim_{h \to 0} \|f - w_R\|_* = 0, \]
we can choose \( h_* \) small enough such that
\[ \|w_R\|_* < (2R_* + 1)e^{(C_*+1)T}. \]

Condition (S3). Using Lemma 2.1 with \( \varphi(k) = E_k \), we have
\[ \int_{\mathbb{R}^d} \tilde{Q}[f_R] E_k \, dk \leq -2\nu\|f_R\|_{L^1_\varphi} \leq 0. \] (6.9)
Hence, since \( f \in S_\varphi \),
\[ \int_{\mathbb{R}^d} w_R E_k \, dk = \int_{\mathbb{R}^d} (f + h\tilde{Q}[f_R]) E_k \, dk \leq \int_{\mathbb{R}^d} f E_k \, dk \leq c_1. \]

Condition (S4). Now, we claim that \( R \) and \( h_* \) can be chosen, such that
\[ \int_{\mathbb{R}^d} w_R E_k \, dk < \frac{3\rho_*}{2} \] (6.10)
with \( \rho_* \) defined as in (6.6). In order to see this, we consider two cases. First, if
\[ \int_{\mathbb{R}^d} f E_k \, dk < \frac{3\rho_*}{2}, \]
we deduce from the fact
\[ \lim_{h \to 0} \int_{\mathbb{R}^d} |w_R - f| E_k^{N_*} \, dk = \lim_{h \to 0} h \int_{\mathbb{R}^d} \tilde{Q}[f_R] E_k \, dk = 0, \]
that we can choose \( h_* \) small enough such that (6.10) holds. On the other hand, if we have
\[ \int_{\mathbb{R}^d} f E_k \, dk = \frac{3\rho_*}{2}, \]
we can then choose \( R \) large enough such that
\[ \int_{\mathbb{R}^d} f_R E_k \, dk > \rho_*, \]
which implies, by (6.5) and (6.6), that
\[ \int_{\mathbb{R}^d} \tilde{Q}[f_R] E_k \, dk < 0. \]
The estimate (6.10) follows by definition of \( w_R \).

To conclude, \( w_R \) defined as in (6.8) belongs to \( S_\varrho \), for \( 0 < h \leq h_R \) for sufficiently large \( R \). In addition, by definition, we compute

\[
\lim_{R \to \infty} \frac{1}{h} \| w_R - f - h\bar{Q}[f] \|_E = \lim_{R \to \infty} \| \bar{Q}[f] - \bar{Q}[f_R] \|_E = 0,
\]

thanks to the Holder property of \( \bar{Q}[f] \) with respect to \( \| \cdot \|_E \). This proves that for all \( \epsilon > 0 \), there is a large \( R_\epsilon \) so that \( w_{R_\epsilon} \in B_E(f + h\bar{Q}[f], h\epsilon) \), for all \( 0 < h \leq h_{R_\epsilon} \). This proves the claim (6.7), and hence condition (B) is verified.

6.1.3 Condition (C)

Condition (C) follows from Proposition 5.1.

6.1.4 Condition (D)

By the Lebesgue’s dominated convergence theorem, we have that

\[
\left[ \varphi, \phi \right] = \lim_{h \to 0^-} h^{-1} (\| \phi + h\varphi \|_E - \| \phi \|_E)
\]

\[
= \lim_{h \to 0^-} h^{-1} \int_{\mathbb{R}^d} (|\phi + h\varphi| - |\phi|)(\mathcal{E}_k + \mathcal{E}_k^N) \, dk
\]

\[
\leq \int_{\mathbb{R}^d} \varphi(k) \text{sign}(\phi(k))(\mathcal{E}_k + \mathcal{E}_k^N) \, dk.
\]

Hence, recalling \( \tilde{Q}[f] = Q[f] - 2\nu(|k|^2 + \varrho|k|^4)f \), we estimate

\[
\left[ \tilde{Q}[f] - \tilde{Q}[g], f - g \right] \leq \int_{\mathbb{R}^d} [\tilde{Q}[f](k) - \tilde{Q}[g](k)] \text{sign}((f - g)(k))(\mathcal{E}_k^{3/4} + \mathcal{E}_k^N) \, dk
\]

\[
\leq \|Q[f] - Q[g]\|_E - 2\nu\|(|k|^2 + \varrho|k|^4)(f - g)\|_E.
\]

Using Lemma 5.1 and recalling \( \| \cdot \|_E = \| \cdot \|_{L_1^{1/4}} + \| \cdot \|_{L_1^N} \), we have

\[
\|Q[f] - Q[g]\|_E \leq C_N \left( \|f - g\|_{L_1^{1/4}} + \|f - g\|_{L_1^N} \right).
\]

Since \( C(|k|^3 - \varrho|k|^4) \) is always bounded for \( \varrho > 0 \), we obtain

\[
\left[ \tilde{Q}[f] - \tilde{Q}[g], f - g \right] \leq C_N \|f - g\|_E.
\]

The condition (C) follows. The proof of Theorem 1.1 is complete for \( \varrho > 0 \).
6.2 Case 2: \( \varrho = 0 \)

Denote \( f_n \) to be the unique solution to (1.6) for \( \varrho = \frac{1}{n} \), starting with the same initial condition \( f_0 \). Proposition 3.1 asserts that \( f_n \) is uniformly bounded in \( L^\infty(0, \infty, L^1_N(\mathbb{R}^d)) \) for all \( n \). Moreover, according to Proposition 4.1, \( f_n \) is uniformly bounded in \( L^\infty(0, T, L^2(\mathbb{R}^d)) \) for all \( n \). By the Dunford-Pettis theorem and Smulian’s theorem, the sequence \( f_n \) is equicontinuous in \( t \) and it converges up to a subsequence to a nonnegative to a function \( f \geq 0 \) in the weak \( L^1 \) sense. Recalling from (5.2) that \( Q[f] \) is Lipschitz from \( L^1 \cap L^1_N + 2 \) to \( L^1_N \), and \( f_n \) converges weakly to \( f \) in \( L^1_N(\mathbb{R}^d) \) for all \( s \in [1, N + 3] \). This implies that \( Q[f_n] \) also converges to \( Q[f] \) in the the weak \( L^1 \) sense. As a consequence, \( f \) is a solution of (1.1).

A Appendix: Proof of Theorem 6.1

We recall below the proof of Theorem 6.1, which is Theorem 1.3 of [36], for the sake of completeness. The proof is divided into four parts.

Part 1: Fix an element \( v \) of \( S \), due to the Hölder continuity property of \( Q[u] \), we have
\[
\|Q[u]\| \leq \|Q[v]\| + C\|u - v\|^\beta, \quad \forall u \in S.
\]
According to our assumption, \( S \) is bounded by a constant \( C_S \). We deduce from the above inequality that
\[
\|Q[u]\| \leq \|Q[v]\| + C(\|u\| + \|v\|)^\beta \leq \|Q[v]\| + C(C_S + \|v\|)^\beta = : C_Q, \quad \forall u \in S.
\]
For an element \( u \) be in \( S \), there exists \( \xi_u > 0 \) such that for \( 0 < \xi < \xi_u \), \( u + \xi Q[u] \in S \), which implies
\[
B(u + \xi Q[u], \delta) \cap S\{u + \xi Q[u]\} \neq \emptyset,
\]
for \( \delta \) small enough. Choose \( \epsilon = 2C((C_Q + 1)\xi)^\beta \), then \( \|Q[u] - Q[v]\| \leq \frac{\epsilon}{2} \) if \( \|u - v\| \leq (C_Q + 1)\xi \), by the Hölder continuity of \( Q \). Let \( z \in B \left(u + \xi Q[u], \frac{\epsilon}{2}\right) \cap S\{u + \xi Q[u]\} \) and define
\[
t \mapsto \vartheta(t) = u + \frac{t(z - u)}{\xi}, \quad t \in [0, \xi].
\]
Since \( S \) is convex, \( \vartheta \) maps \([0, \xi]\) into \( S \). It is straightforward that
\[
\|\vartheta(t) - u\| \leq \xi\|Q[u]\| + \frac{\epsilon}{2} < (C_Q + 1)\xi,
\]
which implies
\[
\|Q(\vartheta(t)) - Q[u]\| \leq \frac{\epsilon}{2}, \quad \forall t \in [0, \xi].
\]
The above inequality and the fact that
\[
\|\dot{\vartheta}(t) - Q[u]\| \leq \frac{\epsilon}{2},
\]
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leads to
\[ \| \dot{\vartheta}(t) - Q(\vartheta(t)) \| \leq \epsilon, \quad \forall t \in [0, \xi]. \]  
\hfill (A.1)

**Part 2:** Let \( \vartheta \) be a solution to (A.1) on \([0, \tau]\). Inequality (A.1) leads to
\[ \left| \frac{\vartheta(\tau) - \vartheta(0)}{\tau} - Q(\vartheta(0)) \right|_* \leq C_E \epsilon, \]
which yields
\[ |\vartheta(\tau)|_* \leq |\vartheta(0)|_* + \tau C_s (|\vartheta(0)|_* + 1) + \tau C_E \epsilon. \]
Since we can assume that \( C_E \epsilon < 1 \), we obtain
\[ |\bar{\vartheta}(\tau)|_* \leq (|\bar{\vartheta}(0)|_* + 1)e^{(C_s + 1)\tau} \]
\hfill (A.2)
Using the procedure of Part 1, we assume that \( \bar{\vartheta} \) can be extended to the interval \([\tau, \tau + \tau']\).

The same arguments that lead to (A.2) imply
\[ |\bar{\vartheta}(\tau + \tau')|_* \leq \left( (|\bar{\vartheta}(\tau)|_* + 1)e^{(C_s + 1)\tau'} - 1 \right). \]
Combining the above inequality with (A.2) yields
\[ \| \bar{\vartheta}(\tau + \tau') \|_* = |\vartheta(\tau + \tau')|_* \leq \left( (|\bar{\vartheta}(0)|_* + 1) \left( e^{(C_s + 1)(\tau + \tau')} - 1 \right) \right) \leq (2R_* + 1)e^{(C_s + 1)(\tau + \tau')}, \]
\hfill (A.3)
where the last inequality follows from the fact that \( R_* \geq 1 \).

**Part 3:** From Part 1, there exists a solution \( \vartheta \) to the equation (A.1) on an interval \([0, h]\).
Now, we have the following procedure.

- **Step 1:** Suppose that we can construct the solution \( \vartheta \) of (A.1) on \([0, \tau]\) \((\tau < T)\). Since \( \vartheta(\tau) \in S \), by the same process as in Part 1 and by (A.2) and (A.3), the solution \( \vartheta \) could be extended to \([\tau, \tau + h_\tau]\) where \( \tau + h_\tau \leq T, h_\tau \leq \tau \).

- **Step 2:** Suppose that we can construct the solution \( \vartheta \) of (A.1) on a series of intervals \([0, \tau_1], [\tau_1, \tau_2], \ldots, [\tau_n, \tau_{n+1}], \ldots \). Observe that the increasing sequence \( \{\tau_n\} \) is bounded by \( T \), the sequence has a limit, defined by \( \tau \). Recall that \( Q(\vartheta) \) is bounded by \( C_Q \) on \([\tau_n, \tau_{n+1}]\) for all \( n \in \mathbb{N} \), then \( \dot{\vartheta} \) is bounded by \( \epsilon + C_Q \) on \([0, \tau]\). As a consequence \( \vartheta(\tau) \) can be defined as
\[ \vartheta(\tau) = \lim_{n \to \infty} \vartheta(\tau_n), \dot{\vartheta}(\tau) = \lim_{n \to \infty} \dot{\vartheta}(\tau_n), \]
which, together with the fact that \( S \) is closed, implies that \( \vartheta \) is a solution of (A.1) on \([0, \tau]\).
By Step 2, if the solution $\vartheta$ can be defined on $[0, T_0)$, $T_0 < T$, it could be extended to $[0, T_0]$. Now, we suppose that $[0, T_0]$ is the maximal closed interval that $\vartheta$ could be defined, by Step 1, $\vartheta$ could be extended to a larger interval $[T_0, T_0 + T_h]$, which means that $T = T_0$ and $\vartheta$ is defined on the whole interval $[0, T]$.

**Part 4:** Finally, let us consider a sequence of solution $\{u^\epsilon\}$ to (A.1) on $[0, T]$. We will prove that this is a Cauchy sequence. Let $\{u^\epsilon\}$ and $\{v^\epsilon\}$ be two sequences of solutions to (A.1) on $[0, T]$. We note that $u^\epsilon$ and $v^\epsilon$ are affine functions on $[0, T]$. Moreover by the one-side Lipschitz condition

$$\frac{d}{dt} \|u^\epsilon(t) - v^\epsilon(t)\| = \left[ u^\epsilon(t) - v^\epsilon(t), \dot{u}^\epsilon(t) - \dot{v}^\epsilon(t) \right]$$

$$\leq \left[ u^\epsilon(t) - v^\epsilon(t), Q[u^\epsilon(t)] - Q[v^\epsilon(t)] \right] + 2\epsilon$$

$$\leq C\|u^\epsilon(t) - v^\epsilon(t)\| + 2\epsilon,$$

for a.e. $t \in [0, T]$, which leads to

$$\|u^\epsilon(t) - v^\epsilon(t)\| \leq 2\epsilon \frac{e^{LT}}{L}.$$ 

By letting $\epsilon$ tend to 0, $u^\epsilon \rightarrow u$ uniformly on $[0, T]$. It is straightforward that $u$ is a solution to (6.1).

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