THE INVERSE SCATTERING PROBLEM
FOR THE MATRIX SCHRÖDINGER EQUATION

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Abstract: The matrix Schrödinger equation is considered on the half line with the general selfadjoint boundary condition at the origin described by two boundary matrices satisfying certain appropriate conditions. It is assumed that the matrix potential is integrable, is selfadjoint, and has a finite first moment. The corresponding scattering data set is constructed, and such scattering data sets are characterized by providing a set of necessary and sufficient conditions assuring the existence and uniqueness of the correspondence between the scattering data set and the input data set containing the potential and boundary matrices. The work presented here provides a generalization of the classical result by Agranovich and Marchenko from the Dirichlet boundary condition to the general selfadjoint boundary condition. The theory presented is illustrated with various explicit examples.

Mathematics Subject Classification (2010): 34L25 34L40 81U05 81Uxx
Keywords: matrix Schrödinger equation on the half line, selfadjoint boundary condition, Marchenko method, Jost matrix, scattering matrix, bound states, matrix Marchenko method, inverse scattering, characterization
Short title: Half-line matrix Schrödinger equation

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1. INTRODUCTION

In the direct scattering problem for the half-line Schrödinger equation, given the potential and the boundary condition we determine the spectrum of the corresponding Schrödinger operator, namely we determine the scattering matrix and the bound-state information consisting of the bound-state energies and the bound-state normalization constants. In the associated inverse problem, given the scattering matrix and the bound-state information, we determine the corresponding potential and the boundary condition. We refer to the data set consisting of the potential and the boundary condition as the input data set and denote it by \( D \). We refer to the data set consisting of the scattering matrix and the bound-state information as the scattering data set and denote it by \( S \). Thus, we can view the direct scattering problem as the mapping \( D : D \mapsto S \) and view the inverse scattering problem as the mapping \( D^{-1} : S \mapsto D \).

There are four aspects related to the direct and inverse problems. These are the existence, uniqueness, construction, and characterization. In the existence aspect in the direct problem, given \( D \) in a specified class we determine whether a corresponding \( S \) exists in some specific class. Thus, the domain of \( D \) as well as its range must be determined. The uniqueness aspect is concerned with whether there exists a unique \( S \) corresponding to a given \( D \) in the domain of \( D \), or two or more distinct sets \( S \) may correspond to the same \( D \). The construction deals with the recovery of \( S \) from \( D \). For the inverse problem to be well defined, one needs to show that the range of the direct scattering map \( D \) coincides with the domain of the inverse scattering map \( D^{-1} \). In the inverse problem the existence problem deals with the existence of some \( D \) corresponding to \( S \) specified in a particular class, which must coincide with the range of the direct scattering map. The uniqueness deals with the question whether \( D \) corresponding to a given \( S \) in the range of \( D \) is unique, and the construction consists of the recovery of \( D \) from \( S \). After the existence and uniqueness aspects in the direct and inverse problems are settled, one now turns the attention to the characterization problem, which consists of the identification of the class to which \( D \)
belongs and the identification of the class to which \( S \) belongs so that there is a one-to-one correspondence between \( D \) and \( S \).

The characterization aspect in the direct and inverse problems is usually the most difficult to establish. This is not surprising because the establishment of the characterization includes the establishment of the existence and the uniqueness in both the direct and inverse problems. Our goal in this monograph is to establish a characterization for the matrix Schrödinger operator on the half line with the general selfadjoint boundary condition. We do so in such a way that our characterization result naturally also holds in the scalar case, it holds for any selfadjoint boundary condition, it yields the construction in the corresponding direct and inverse problems, and it reveals how the individual conditions in the characterization affect the existence, uniqueness, and construction. We also provide some comments and explicit examples to clarify various issues so that our approach can be useful in establishing characterizations for other direct and inverse problems.

The only viable characterization in the literature for the matrix Schrödinger operator on the half line can be found in the seminal work by Agranovich and Marchenko [2]. However, the analysis in [2] is restricted to the Dirichlet boundary condition, and hence our study can be viewed as a generalization of the characterization in [2]. It is ironic that a characterization in the scalar case valid for a general selfadjoint boundary condition does not exist and cannot exist in the way a scattering matrix is defined in the existing literature. As indicated in Section 4 of [7], as a result of defining [8,14,37,38] the scattering matrix in one way with the Dirichlet boundary condition and in a different way with a non-Dirichlet boundary condition, it is impossible to have the uniqueness aspect unless separate characterizations are developed in the Dirichlet case and in the non-Dirichlet case, respectively. In the scalar case, it is known [7] that, in the absence of bound states, an input data set consisting of a real-valued potential with the Dirichlet boundary condition and another input data set consisting of a real-valued potential with the Neumann boundary condition may correspond to the same scattering matrix. The reader is referred to Section 4 of and Example 6.3 of [7] for further details. In our solution to the characterization problem given
in this monograph we do not encounter such a nonuniqueness issue because we define the scattering matrix in a unique way, without defining it in one way in the Dirichlet case and in another way in the non-Dirichlet case. Actually, we define the scattering matrix in such a way that the associated Schrödinger operator for the unperturbed problem has the Neumann boundary condition. This definition is motivated by the theory of quantum graphs [29,30,42]. In fact, in the matrix case, a boundary condition could be partly Dirichlet and partly non-Dirichlet, and the Dirichlet boundary condition in the matrix case is really a very special case and can be referred to as the purely Dirichlet case [9]. For further details we refer the reader to Section 4 of [9].

We analyze the existence, uniqueness, reconstruction, and characterization issues related to the relevant direct and inverse problems under the assumption that $D$ belongs to the Faddeev class and $S$ belongs to the Marchenko class. The Faddeev class consists of input data sets $D$ as in (4.1), where the potential $V$ and the boundary matrices $A$ and $B$ are as specified in Definition 4.1. The Marchenko class consists of scattering data sets $S$ as in (4.2), where the scattering matrix $S$ and the bound-state data $\{\kappa_j, M_j\}_{j=1}^N$ are as specified in Definition 4.5. In [2] the inverse problem for (2.1) is studied in the special case when $A = 0$ and $B = I$, with 0 being the $n \times n$ zero matrix and $I$ denoting the $n \times n$ identity matrix, and when the potential $V$ is not necessarily integrable, i.e. when the potential $V$ appearing in (2.1) satisfies (2.2), is Lebesgue measurable, and satisfies $\int_0^\infty dx \, x |V(x)| < +\infty$ instead of satisfying (2.3). A characterization of the corresponding scattering data was presented in [2]. Our work provides a generalization of the characterization of [2] to the case with the general selfadjoint boundary condition. When a non-Dirichlet boundary condition is used at $x = 0$, the integrability of the potential is necessary, and that is why the integrability of $V(x)$ in (2.3) is crucial. In particular, to be able to define the regular solution $\varphi(k, x)$ appearing in (9.5), it is necessary that the potential is integrable at $x = 0$. For further details on this issue we refer the reader to Theorem 1.2.1 of [2] and also [45]. In Chapter 26 we illustrate this issue with an explicit example.
Let us mention also the relevant references [23-25], where the direct and inverse problems for (2.1) are formally studied with the general selfadjoint boundary condition, not as in (2.4)-(2.6) but in a form equivalent to (2.4)-(2.6). However, the study in [23-25] lacks the large-$k$ analysis beyond the leading term and also lacks the small-$k$ analysis of the scattering data, which are both essential for the analysis of the relevant inverse problem. Thus, our study can also be considered as a complement to the work by Harmer [23-25]. In our monograph we use results from previous work [2,5,6,9,42-44], in particular [2,5,9,42].

The matrix Schrödinger equation on the half line with the general selfadjoint boundary condition has many applications in quantum mechanics, especially in the scattering of particles with internal structures such as spin [14,40] and in the scattering on quantum graphs [11-13,18,19,21,22,26,28-36]. A particular phenomenon governed by a matrix Schrödinger equation is a star graph, i.e. a one-vertex graph with a finite number of semi-infinite edges. In this case, a homogeneous boundary condition linear in the wavefunction and its derivative is imposed at the vertex, and the dynamics on each edge is governed by the Schrödinger equation. Physically, a star graph represents a finite number of very thin quantum wires connected at the vertex. The study of quantum wires has physical relevance to the design of elementary gates in quantum computing and in nanotubes for microscopic electronic devices, where string of atoms may form a star graph. The consideration of the general boundary condition at the vertex, rather than just the Dirichlet boundary condition, is relevant. For quantum graphs it is crucial that the boundary conditions at the vertices link the values of the wavefunction and its derivative arriving from different edges. An important case is the Kirchhoff boundary condition, which amounts to the continuity of the wavefunction at the vertex and also that at the vertex the sum of the derivatives of the wavefunction from all the edges is zero, which expresses the conservation of current at the vertex. Actually, a quantum graph is an idealization of quantum wires with very small cross sectional areas, where such wires meet at the vertices. The quantum graph is obtained in the zero limit for the cross sectional area. The boundary conditions at the vertices of the graph depends on how the limit is taken. From this point of view,
it is relevant to consider all possible selfadjoint boundary conditions since such conditions may result in various limiting procedures.

Our monograph is organized as follows. In Chapter 2 we introduce the matrix Schrödinger equation on the half line, introduce the related $n \times n$ matrix potential $V(x)$, and list the properties of the potential so that we can study the corresponding direct and inverse problems. The matrix potential is required to be hermitian as stated in (2.2) and also required to satisfy the so-called $L^1$-condition given in (2.3). The existence of the first moment for the potential in (2.3) ensures that the number of bound states can at most be finite, which makes the analysis of the corresponding direct and inverse problems manageable. In this chapter we also introduce the general selfadjoint boundary condition at $x = 0$ in terms of a pair of constant $n \times n$ matrices denoted by $A$ and $B$. The boundary condition is given in (2.4), where the boundary matrices satisfy (2.5) and (2.6). The boundary condition is unchanged if the boundary matrices are post multiplied by an invertible $n \times n$ matrix $T$. In Proposition 2.1 we show that the boundary matrices $A$ and $B$ are uniquely determined modulo post multiplication by $T$.

Chapter 3 contains a summary of various mathematical definitions, notations, and results used in our monograph. In our monograph we deal with vector-valued functions as well as matrix-valued functions. Our vectors may be row vectors or column vectors with $n$ components. In Section 3.a we review the standard vector norm, the standard scalar product, and the standard matrix norm in $\mathbb{C}^n$. In our monograph we use the standard matrix norm defined in (3.11) instead of another matrix norm used in [2]. Since all matrix norms are equivalent in a finite dimensional vector space, we could use any other matrix norm than that defined in (3.11). Our choice of the standard matrix norm given in (3.11) is motivated by the fact that we would like to have the inequalities (3.12)-(3.14) as well as the equalities given in (3.15). With any other matrix norm some constants depending on $n$ may need to be used in (3.12)-(3.14) to retain the inequalities, and similarly some constants depending on $n$ may be needed to retain the equalities in (3.15). In Section 3.b we review the basic facts about Banach spaces and Hilbert spaces, as our vector-valued and matrix-valued
functions belong to various Banach spaces or various Hilbert spaces. In our monograph we especially use the Banach space $L^1(\mathbb{R}^+)$, consisting of complex-valued integrable functions of a real-valued independent variable $x \in \mathbb{R}^+$. Similarly, we use $L^2(\mathbb{R}^+)$, the Hilbert space of square-integrable functions, and we also use $L^\infty(\mathbb{R}^+)$, the Banach space of bounded functions. Without explicitly mentioning we assume that all our functions are Lebesgue measurable and the integrals are assumed to be Lebesgue integrals. In Section 3.c we list various inequalities in various Banach spaces and Hilbert spaces, including some inequalities involving convolutions. In Section 3.4 we summarize certain basic facts on Hardy spaces that we use later on. The pointwise bounds stated in Propositions 3.1 and 3.2 turn out to be very useful later on, especially in solving some Riemann-Hilbert problems related to the characterization of scattering data sets. In Section 3.e we introduce various Banach spaces that become useful in solving various Riemann-Hilbert problems related to the characterization of scattering data sets. In Section 3.f we summarize basic facts on integral operators whose kernels depend on a sum. Our key integral equation, namely, the Marchenko integral equation has such a kernel as well as various other integral equations used in our characterization of scattering data sets. In Section 3.g we provide a summary of certain results in Banach and Hilbert spaces. Such results are used in the characterization presented in Sections 8 and 23.

In Chapter 4 we introduce the Faddeev class of input data sets $D$ and the Marchenko class of scattering data sets, in Definition 4.1 and Definition 4.5, respectively. These two classes conveniently allow us to summarize one of our main characterization results, Theorem 5.1 in Chapter 5, by saying that there is a one-to-one correspondence between the Faddeev class and the Marchenko class. In order to formulate various characterization results in an efficient manner, we list a set of properties for a scattering data set $S$ in Definition 4.2, and these properties are identified by using Arabic numerals, namely (1), (2), (3), and (4). Actually, there are two versions of (3), denoted by (3a) and (3b). There are five versions of (4), denoted by (4a), (4b), (4c), (4d), (4e). We also provide a second set of properties for a scattering data set $S$ in Definition 4.3, and those properties are
identified by using Roman numerals, namely (I), (III), (V), and (VI). Actually, there are three versions of (III), denoted by (IIIa), (IIIb), and (IIIc). There are eight versions of (V), denoted by (Va), (Vb), (Vc), (Vd), (Ve), (Vf), (Vg), (Vh).

In Chapter 5 we present one of our main characterization results. A stated in Theorem 5.1 we show that for each input data set D in the Faddeev class there exists and uniquely exists a corresponding scattering data set S in the Marchenko class. We also remark that by replacing the $L_1$-condition by the $L_p$-condition, where $p$ is an integer greater than one, the characterization result of Theorem 5.1 remains valid in a subclass. In other words, let us modify the definition of the Faddeev class so that the potential $V(x)$ satisfies (2.3) with $(1 + x)$ replaced by $(1 + x)^p$, and let us also modify the definition of the Marchenko class so that so that $F'_s(y)$ satisfies (4.8) of (2) with $(1 + y)$ replaced by $(1 + y)^p$. Then, the characterization result stated in Theorem 5.1 remains valid. Informally speaking, we then obtain the characterization in the $L_p$-class with $p = 2, 3, 4, \ldots$.

In Chapter 6 we analyze the interconnections among the properties listed in Definitions 4.2 and 4.3. We show that the two versions of (3), namely (3a) and (3b), are equivalent. We also show that the five versions of (4), namely (4a), (4b), (4c), (4d), and (4e), are all equivalent. We show that the three versions of (III), namely (IIIa), (IIIb), (IIIc), are also equivalent. We further show that all the eight versions of (V), namely (Va), (Vb), (Vc), (Vd), (Ve), (Vf), (Vg), (Vh), are equivalent. We then show that (3a) is equivalent to the combination of (IIIa) and (Va). This last equivalence also yields other equivalences between the two versions of (3) and all possible 24 combinations of (III) and (V). Informally speaking, we then have the equivalence between (1, 2, 3, 4) and (1, 2, III + V, 4).

With the help of various equivalences established in Chapter 6, in Chapter 7 we are able to present various equivalent versions of the characterization result of Theorem 5.1. The result in Theorem 5.1 is stated when the scattering data set S belongs to the Marchenko class, i.e. when S satisfies (1, 2, 3a, 4a). In Chapter 7 we present var-
ious characterization results, where the scattering data $S$ satisfies some six versions of $(1, 2, \text{III} + \text{V}, 4)$. Although the six characterization results presented in Theorems 7.1-7.6 are equivalent, some of these characterization versions may have certain advantages over others. For example, the characterizations stated in Theorems 7.1 and 7.2 allows us to verify the characterization conditions without having to solve the Marchenko integral equation first. Having so many equivalent versions of the characterization allows us to have many different options for methods in the analysis of the corresponding inverse scattering problem, and also it allows us to understand how various different methods are connected to each other. For example, in some cases it may be advantageous to solve an integral equation rather than a corresponding Riemann-Hilbert problem or vice versa. It may also be more convenient to look for a solution to an integral equation in the class of bounded and integrable functions rather than in the class of bounded and square-integrable functions. It may be more convenient to look for a solution to a homogenous Riemann-Hilbert problem in a more restricted class rather than in a Hardy class. In Chapter 7, in Theorems 7.9 and 7.10 we present two equivalent versions of yet another characterization of the scattering data, where this new characterization is based on using Levinson’s theorem. The details of this characterization are provided in Chapter 21, where it is also shown that the two versions of this characterization are equivalent to the previous characterization of Theorem 5.1 and all its equivalents.

In Chapter 8 we provide another characterization for the scattering data so that it uniquely corresponds to an input data set in the Faddeev class. This characterization is summarized in Theorem 8.1 and is different from the previous characterizations and its details are developed in Chapter 23. In our monograph we do not show the equivalence of this new characterization with the previous characterizations because that equivalence is still an open problem. It is our feeling that showing such an equivalence will reveal interconnections among various different areas of mathematical analysis, by not only contributing to the analysis of inverse scattering problems but perhaps by contributing to the field of analysis at large.
In Chapter 9 we provide an outline of the solution to the direct problem starting from a potential and a pair of boundary matrices. Namely, starting with an input data set \( \mathbf{D} \) in the Faddeev class we indicate how all the relevant quantities are constructed, among which are various solutions to the Schrödinger equation such as the Jost solution \( f(k, x) \), the regular solution \( \varphi(k, x) \), the physical solution \( \Psi(k, x) \), and the normalized bound-state matrix solutions \( \Psi_j(x) \). Other relevant quantities constructed from the input data set include the Jost matrix \( J(k) \), the scattering matrix \( S(k) \), the bound-state energies \(-\kappa_j^2\), the normalization matrices \( M_j \). The construction is outlined in Chapter 9.

In Chapter 10, we summarize the properties of various quantities constructed from the input data set in the Faddeev class. In particular, we present certain properties of the quantity \( K(x, y) \) constructed from the input data set \( \mathbf{D} \). Such properties later help us to establish that the constructed scattering data set \( \mathbf{S} \) belongs to the Marchenko class. The quantity \( K(x, y) \) plays a key role also in the analysis of the inverse scattering problem because as we see later it corresponds to the unique solution to the Marchenko integral equation (13.1).

In Chapter 11 various properties related to the bound states are established. In other words, starting with an input data set \( \mathbf{D} \), various quantities are constructed related to the discrete spectrum of the corresponding Schrödinger operator and their properties are established in Chapter 11.

In Chapter 12 certain properties of the constructed scattering matrix are established, especially related to the quantity \( F_s(y) \), which is related to the constructed scattering matrix \( S(k) \) through the relationship (4.7). In other words, starting from an input data set \( \mathbf{D} \) in the Faddeev class we construct the corresponding scattering data set \( \mathbf{S} \) and in Chapter 12 we analyze the properties of \( F_s(y) \), which plays a key role in the solution to the inverse problem.

In Chapter 13 the Marchenko integral equation is derived and it is shown that \( K(x, y) \) constructed from the input scattering data set \( \mathbf{D} \) actually satisfies the Marchenko equation.
Further, it is shown that $K_x(x, y)$ satisfies the derivative Marchenko integral equation given in (13.7). Let us mention that the derivative Marchenko equation (13.7) has a prominent role as the Marchenko integral equation (13.1) in the inverse scattering problem with the general selfadjoint boundary condition (2.4). This is because both $\psi(0)$ and $\psi'(0)$ appear in the boundary condition. The presence of $\psi'(0)$ in the boundary condition (2.4) makes the derivative Marchenko equation (13.7) as important as the Marchenko equation (13.1). In the Dirichlet case studied in [2], the boundary condition (2.4) reduces to having $\psi(0) = 0$, and hence the absence of $\psi'(0)$ in (2.4) in the Dirichlet case also diminishes the prominence of the derivative Marchenko integral equation (13.7) compared to the Marchenko equation (13.1).

In Chapter 14 we show that the boundary matrices $A$ and $B$ in the input data set $D$ appear in the large-$k$ asymptotics of the constructed scattering matrix $S(k)$. The fundamental result given in (14.2) shows how $A$ and $B$ are related to the large-$k$ limit of $S(k)$ and the matrix quantity $K(0, 0)$, where $K(x, y)$ is the solution to the Marchenko equation (13.1).

In Chapter 15 we establish various results related to the properties listed in Definition 4.2 and Definition 4.3. One consequence of the results in Chapter 15 is that if we construct the scattering data set $S$ from the input data set $D$, then the constructed $S$ satisfies the properties ($1, 2, 3, 4$) listed in Definition 4.2 and the properties ($I, III, V, VI$) listed in Definition 4.3.

In Chapter 16 we analyze the inverse problem of the construction of $D$ from a scattering data set $S$ satisfying one or more of the properties listed in Definitions 4.2 and 4.3. In particular, we show that the Marchenko integral equation (13.1) is uniquely solvable and its solution $K(x, y)$ has certain important properties. In this chapter we also show that the derivative Marchenko integral equation (13.7) is uniquely solvable and its unique solution is given by $K_x(x, y)$, the $x$-partial derivative of the solution $K(x, y)$ to the Marchenko equation. In Chapter 16 we also show how, starting from a scattering data set $S$, we can
construct the boundary matrices $A$ and $B$ uniquely modulo a postmultiplication by an invertible matrix $T$. This is achieved by solving the system given in (16.70) and we show that (16.70) is solvable when the scattering data set $S$ satisfies the properties (1) and (4) of Definition 4.2. In Chapter 16 we provide various other results to be used later on to obtain the characterization based on the method given in Chapter 23.

In Chapter 18 we obtain various results to show how some of the properties listed in Definitions 4.2 and 4.3 are related to each other. In other words, the results presented in Chapter 18 allow us to prove the equivalencies stated in Chapter 6. In particular, it shows how (3) is equivalent to the two combined properties (III) and (V), and it also shows how all the eight versions of (V) are all equivalent.

In Chapter 19 we analyze the inverse scattering problem when the bound-state data is missing in the scattering data set $S$. It shows that under certain circumstances we can supplement a scattering matrix with some appropriate bound-state data set so that the resulting scattering data set $S$ corresponds to an input data set $D$ in the Faddeev class.

Parseval’s equality is a fundamental equation equivalent to the completeness relation for the physical solution $\Psi(k, x)$ and the normalized bound-state matrix solutions $\Psi_j(x)$ of the Schrödinger equation with the boundary condition (2.4). In Chapter 20 we show that Parseval’s equality holds when the corresponding scattering data set $S$ satisfies certain minimal conditions. We also show how Parseval’s equality is related to the Marchenko equation by showing that the two are equivalent under some mild conditions on the scattering data set $S$.

Levinson’s theorem is a fundamental result in the scattering theory and it shows how the scattering matrix is intrinsically related to the bound states. For the matrix Schrödinger equation (2.1) with the boundary condition (2.4), the corresponding statement of Levinson’s theorem is summarized in (21.5). In Chapter 21 the details are provided for the two equivalent characterizations presented in Theorems 7.9 and 7.10, utilizing Levinson’s theorem. In Chapter 21 also the equivalence is shown between the characterization
using Levinson’s theorem and the previous characterization given in Theorem 5.1 and all its equivalents based on the results of Chapter 6.

In Chapter 22 we introduce the generalized Fourier map $F$ and establish its various relevant properties. In particular, we show that the generalized Fourier map is unitary, and we also establish various properties of its adjoint map $F^\dagger$. Such properties are needed to establish another characterization result, developed in Chapter 23, whose two equivalent versions are summarized in Theorems 8.1 and 8.2. In Section 22 we also show the orthonormality relations among the physical solution $\Psi(k, x)$ and the bound-state matrix solutions $\Psi_j(x)$, where the result is given in Proposition 22.4.

In Chapter 23 we present the alternate method to solve the inverse problem of constructing the input data set $D$ from a scattering data set $S$. A summary of the method is provided in the beginning of Chapter 22. Based on this alternate method, a characterization of the scattering data set $S$ is obtained so that there is a unique corresponding input data set $D$ in the Faddeev class. As already indicated, the resulting characterization is summarized in Theorem 8.1. As mentioned earlier, it is an open problem to show that this new characterization is equivalent to the characterization provided in Theorem 5.1.

We briefly mention two applications of our results in the field of quantum graphs. In Chapter 24 we show that a matrix Schrödinger equation with a diagonal potential matrix is equivalent to having a star graph. In Chapter 25 we show that a $2 \times 2$ matrix Schrödinger equation is unitarily equivalent to a Schrödinger equation on the full line with a point interaction at $x = 0$.

In Chapter 26 various explicitly solved examples are provided for the inverse problem when the scattering matrix $S(k)$ is a rational function of $k$. Broadly speaking we have developed three versions of the characterization of the scattering data. The first version is based on Theorem 5.1 and its various equivalent versions based on the results of Section 6 and summarized in Theorems 7.1-7.6. The second version utilizes Levinson’s theorem and the corresponding characterization is stated in Theorem 7.9 and its equivalence in
Theorem 7.10. The third version is provided in Theorem 8.1. In various examples provided in Chapter 26, we illustrate how a scattering data set may comply with these three versions of the characterization and how the failure of any one of our characterization conditions affects the solution to the inverse problem.


2. THE MATRIX SCHRÖDINGER EQUATION

In this chapter we introduce the matrix Schrödinger equation (2.1), the potential $V$, and the boundary matrices $A$ and $B$ used to describe the selfadjoint boundary condition.

Consider the matrix Schrödinger equation on the half line

$$-\psi'' + V(x) \psi = k^2 \psi, \quad x \in \mathbb{R}^+, \quad (2.1)$$

where $\mathbb{R}^+ := (0, +\infty)$, the prime denotes the derivative with respect to the spatial coordinate $x$, $k^2$ is the complex-valued spectral parameter, the potential $V$ is an $n \times n$ selfadjoint matrix-valued function of $x$ and belongs to class $L_1^1(\mathbb{R}^+)$, and $n$ is any positive integer. We assume that $n$ is fixed and is known. The selfadjointness of $V$ is expressed as

$$V(x) = V(x)^\dagger, \quad x \in \mathbb{R}^+, \quad (2.2)$$

where the dagger denotes the matrix adjoint (complex conjugate and matrix transpose). We equivalently say hermitian to describe a selfadjoint matrix. We remark that, unless we are in the scalar case, i.e. unless $n = 1$, the potential in not necessarily real valued. The condition $V \in L_1^1(\mathbb{R}^+)$ means that each entry of the matrix $V$ is Lebesgue measurable on $\mathbb{R}^+$ and

$$\int_0^\infty dx \, (1 + x) |V(x)| < +\infty, \quad (2.3)$$

where $|V(x)|$ denotes the operator matrix norm. Clearly, a matrix-valued function belongs to $L_1^1(\mathbb{R}^+)$ if and only if each entry of that matrix belongs to $L_1^1(\mathbb{R}^+)$. The wavefunction $\psi(k, x)$ appearing in (2.1) may be either an $n \times n$ matrix-valued function or it may be a column vector with $n$ components. We use $\mathbb{C}$ for the complex plane, $\mathbb{R}$ for the real line $(-\infty, +\infty)$, $\mathbb{R}^-$ for the left-half line $(-\infty, 0)$, $\mathbb{C}^+$ for the open upper-half complex plane, $\overline{\mathbb{C}^+}$ for $\mathbb{C}^+ \cup \mathbb{R}$, $\mathbb{C}^-$ for the open lower-half complex plane, and $\overline{\mathbb{C}^-}$ for $\mathbb{C}^- \cup \mathbb{R}$.

We are interested in studying (2.1) with an $n \times n$ selfadjoint matrix potential $V$ in $L_1^1(\mathbb{R}^+)$ under the general selfadjoint boundary condition at $x = 0$. There are various
equivalent formulations [5,9,23-25,29,30] of the general selfadjoint boundary condition at 
\( x = 0 \), and we find it convenient to state it [5,9] in terms of two constant \( n \times n \) matrices 
\( A \) and \( B \) as
\[
-B^\dagger \psi(0) + A^\dagger \psi'(0) = 0,
\]
where \( A \) and \( B \) satisfy
\[
-B^\dagger A + A^\dagger B = 0, \tag{2.5}
\]
\[
A^\dagger A + B^\dagger B > 0. \tag{2.6}
\]

The condition in (2.6) means that the \( n \times n \) matrix \( (A^\dagger A + B^\dagger B) \) is positive definite. 
One can easily verify that (2.4) remains invariant if the boundary matrices \( A \) and \( B \) are
replaced with \( AT \) and \( BT \), respectively, where \( T \) is an arbitrary \( n \times n \) invertible matrix.
The details of this invariance is provided in Proposition 2.1. We express this fact by saying
that the selfadjoint boundary condition in (2.4) is uniquely determined by the matrix pair 
\( (A,B) \) modulo an invertible matrix \( T \), and we equivalently state that (2.4) is equivalent
to the knowledge of \( (A,B) \) modulo \( T \).

The matrix \( (A^\dagger A + B^\dagger B) \) appearing in (2.6) is selfadjoint, and thus from (2.6) it follows that there exists a unique positive definite matrix \( E \) defined as
\[
E := (A^\dagger A + B^\dagger B)^{1/2}, \tag{2.7}
\]
in such a way that \( E \) is selfadjoint and invertible, and hence
\[
E = E^\dagger, \quad (E^\dagger)^{-1}(A^\dagger A + B^\dagger B)E^{-1} = I. \tag{2.8}
\]
From (1.16) of [5] we have
\[
AE^{-2}A^\dagger + BE^{-2}B^\dagger = I, \quad BE^{-2}A^\dagger - AE^{-2}B^\dagger = 0. \tag{2.9}
\]

The following proposition shows that the boundary matrices \( A \) and \( B \) appearing in
(2.4)-(2.6) are uniquely defined modulo post multiplication by an invertible matrix \( T \).
Proposition 2.1 Let $A_j$ and $B_j$ for $j = 1, 2$ be $n \times n$ matrices. Assume that

$$A_1^\dagger A_1 + B_1^\dagger B_1 > 0. \quad (2.10)$$

Let

$$L_j := \{(Z_1, Z_2) \in \mathbb{C}^{2n} : -B_j^\dagger Z_1 + A_j^\dagger Z_2 = 0, \ j = 1, 2\}. \quad (2.11)$$

Then, $L_1 = L_2$ if and only if there is an invertible matrix $T$ such that

$$A_2 = A_1 T, \quad B_2 = B_1 T. \quad (2.12)$$

Proof: It is immediate that if (2.12) holds, then $L_1 = L_2$. On the hand, assume that $L_1 = L_2$. Let $D_j$ for $j = 1, 2$ be the operators from $\mathbb{C}^{2n}$ into $\mathbb{C}^n$ defined as

$$D_j(Z_1, Z_2) := -B_j^\dagger Z_1 + A_j^\dagger Z_2, \quad Z_1, Z_2 \in \mathbb{C}^n, \ j = 1, 2. \quad (2.13)$$

We have $L_1 = L_2$ if and only if

$$\text{Ker}[D_1] = \text{Ker}[D_2]. \quad (2.14)$$

We will prove that (2.12) is satisfied. For that purpose, we first prove that the range of $D_1$ is equal to $\mathbb{C}^n$. Suppose that for some $W \in \mathbb{C}^n$, we have

$$\langle W, -B_1^\dagger Z_1 + A_1^\dagger Z_2 \rangle = 0, \quad Z_1, Z_2 \in \mathbb{C}^n, \quad (2.15)$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{C}^n$. In (2.15) choosing $Z_1 = -B Z$ and $Z_2 = A Z$ with $Z \in \mathbb{C}^n$, we obtain

$$\langle (A^\dagger A + B^\dagger B)W, Z \rangle = 0, \quad Z \in \mathbb{C}^n. \quad (2.16)$$

Choosing $Z$ as $Z = (A^\dagger A + B^\dagger B)W$, from (2.16) we get

$$\langle (A^\dagger A + B^\dagger B)W, (A^\dagger A + B^\dagger B)W \rangle = 0, \quad (2.17)$$

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and then, by (2.10), we obtain \( W = 0 \). Since the only vector in \( \mathbb{C}^n \) that is orthogonal to the range of \( D_1 \) is the zero vector, the range of \( D_1 \) is equal to \( \mathbb{C}^n \). Since the range of \( D_1 \) is \( \mathbb{C}^n \), for any \( U \in \mathbb{C}^n \) there exist \( Z_1 \) and \( Z_2 \) in \( \mathbb{C}^n \) such that

\[
U = -B_1^\dagger Z_1 + A_1^\dagger Z_2.
\]  

(2.18)

We denote by \( E \) the operator from \( \mathbb{C}^n \) into \( \mathbb{C}^n \) defined as

\[
E U = -B_2^\dagger Z_1 + A_2^\dagger Z_2.
\]  

(2.19)

Let us check that \( E U \) is well defined, i.e. that it is independent of the particular \( (Z_1, Z_2) \) that we use in (2.18) to represent \( U \). So, suppose that for some other \( \tilde{Z}_1 \) and \( \tilde{Z}_2 \) in \( \mathbb{C}^n \) we have

\[
U = -B_1^\dagger \tilde{Z}_1 + A_1^\dagger \tilde{Z}_2.
\]  

(2.20)

It follows from (2.18) and (2.20) that

\[
(Z_1 - \tilde{Z}_1, Z_2 - \tilde{Z}_2) \in \text{Ker} D_1.
\]  

(2.21)

But by (2.14)

\[
(Z_1 - \tilde{Z}_1, Z_2 - \tilde{Z}_2) \in \text{Ker}[D_2],
\]  

(2.22)

and then

\[
-B_2^\dagger Z_1 + A_2^\dagger Z_2 = -B_2^\dagger \tilde{Z}_1 + A_2^\dagger \tilde{Z}_2,
\]  

(2.23)

which proves that \( E U \) is well defined. We will also denote by \( E \) the \( n \times n \) matrix associated to the operator (2.19). Let us prove that \( E \) is invertible. Hence, suppose that for some \( U \in \mathbb{C}^n \) we have \( E U = 0 \). Then, choosing \( Z_1 \) and \( Z_2 \) as in (2.18), we have

\[
E U = -B_2^\dagger Z_1 + A_2^\dagger Z_2 = 0.
\]  

(2.24)

This means that \( (Z_1, Z_2) \in \text{Ker}[D_2] \). But by (2.14), \( (Z_1, Z_2) \in \text{Ker}[D_1] \), and consequently we obtain

\[
U = -B_1^\dagger Z_1 + A_1^\dagger Z_2 = D_1(Z_1, Z_2) = 0,
\]  

(2.25)
and, hence, $E$ is invertible. Taking $Z_2 = 0$ in (2.18) we get

$$EU = E ( -B_1^\dagger Z_1 ) = -B_2^\dagger Z_1, \quad Z_1 \in \mathbb{C}^n. \quad (2.26)$$

Hence,

$$EB_1^\dagger = B_2^\dagger. \quad (2.27)$$

In the same way, taking in (2.18) $Z_1 = 0$ we prove that

$$EA_1 = A_2. \quad (2.28)$$

Denoting $T = E^\dagger$ and taking the adjoint of (2.27) and (2.28) we obtain (2.12).
3. SOME MATHEMATICAL PRELIMINARIES

In this chapter we present certain mathematical results needed in later chapters.

3.a Vectors in $\mathbb{C}^n$ and matrices acting on $\mathbb{C}^n$

For a scalar-valued function $f(y)$, we have the absolute value given by

$$ |f(y)| := \sqrt{|f(y)|^2} = \sqrt{f(y)^* f(y)}, \quad (3.1) $$

where the asterisk denotes the complex conjugation. For vector-valued functions with $n$ components, we use the absolute-value notation to denote the length of the vector with $n$ components, whether it is a row vector or a column vector. For a vector-valued function $f(y)$, which is a column vector with $n$ components given by

$$ f(y) = \begin{bmatrix} f_1(y) \\ \vdots \\ f_n(y) \end{bmatrix}, \quad (3.2) $$

we have

$$ |f(y)| := \sqrt{|f_1(y)|^2 + \cdots + |f_n(y)|^2} = \sqrt{f(y)^\dagger f(y)}, \quad (3.3) $$

where the dagger denotes the matrix adjoint. For a vector-valued function $f(y)$, which is a row vector with $n$ components given by

$$ f(y) = [ f_1(y) \cdots f_n(y) ], \quad (3.4) $$

we have

$$ |f(y)| := \sqrt{|f_1(y)|^2 + \cdots + |f_n(y)|^2} = \sqrt{f(y)^* f(y)^T} = \sqrt{f(y) f(y)^\dagger}, \quad (3.5) $$

where the superscript $T$ denotes the matrix transpose. Thus, for any vector-valued function with $n$ components, we have

$$ |f(y)| = |f(y)^\dagger| = |f(y)^T| = |f(y)^*|. \quad (3.6) $$
For the standard scalar product in $\mathbb{C}^n$, we have

$$\langle f(y), g(y) \rangle := f_1(y)^* g_1(y) + \cdots + f_n(y)^* g_n(y). \quad (3.7)$$

Thus, for two column vectors $f(y)$ and $g(y)$ we have

$$\langle f(y), g(y) \rangle = f(y)^\dagger g(y), \quad (3.8)$$

and for two row vectors $f(y)$ and $g(y)$ we have

$$\langle f(y), g(y) \rangle = f(y)^* g(y)^T. \quad (3.9)$$

Thus, for the vector-valued function $f(y)$, whether it is a column vector or a row vector, we have

$$|f(y)| = \sqrt{\langle f(y), f(y) \rangle}. \quad (3.10)$$

If $A(y)$ is an $n \times n$ matrix-valued function and if $f(y)$ is a column vector with $n$ components, then $A(y) f(y)$ is a column vector with $n$ components. We define the matrix norm of $A(y)$, denoted by $|A(y)|$, as

$$|A(y)| := \sup_{|f(y)|=1} |A(y) f(y)|. \quad (3.11)$$

We then get the standard inequality

$$|A(y) f(y)| \leq |A(y)| |f(y)|. \quad (3.12)$$

If $A(y)$ and $B(y)$ are two $n \times n$ matrix-valued functions, we get

$$|A(y) B(y)| \leq |A(y)| |B(y)|. \quad (3.13)$$

If $g(y)$ is a row vector with $n$ components, we then get

$$|g(y) A(y)| \leq |g(y)| |A(y)|. \quad (3.14)$$

We also get

$$|A(y)^\dagger| = |A(y)^*| = |A(y)^T| = |A(y)|. \quad (3.15)$$
Since we use $|A(y)|$ to denote the matrix norm of the $n \times n$ matrix $A(y)$ and we also use $|f(y)|$ to denote the standard norm in $\mathbb{C}^n$ for the vector $f(y)$ with $n$ components, the notation $| \cdot |$ becomes clearer by checking whether it applies to a matrix or to a vector.

3.b Banach and Hilbert spaces

We introduce some elementary concepts from linear operator theory in Banach and in Hilbert spaces over the field of complex numbers. For a thorough presentation of this subject see [27].

Recall that a Banach space $B$ over the complex numbers is a vector space that has a norm $\| \cdot \|_B$ and that is complete. The norm satisfies the following conditions:

(a) We have $\|Y\|_B \geq 0$ for all $Y \in B$, and the equality holds if and only if $Y$ is the zero vector in $B$.

(b) For any complex number $\alpha$ and for any vector $Y \in B$ we have $\|\alpha Y\|_B = |\alpha| \|Y\|_B$.

(c) For any pair of vectors $Y$ and $Z$ in $B$, we have $\|Y + Z\|_B \leq \|Y\|_B + \|Z\|_B$.

The completeness of $B$ means that every Cauchy sequence in $B$ is convergent, i.e. for any sequence $\{Y^{(l)}\}_{l=1}^{\infty}$ with elements in $B$ that has the property $\|Y^{(l)} - Y^{(m)}\|_B \to 0$ as $l, m \to +\infty$ there exists an element $Y \in B$ such that $\|Y^{(l)} - Y\|_B \to 0$ as $n \to +\infty$.

Let us now consider the particular case of operators between Hilbert spaces. Recall that a Hilbert space, $\mathcal{H}$, is a Banach space that has a scalar product, $(\cdot, \cdot)_\mathcal{H}$ such that the norm is derived from the scalar product as, $\|Y\|_\mathcal{H} = ((Y, Y)_\mathcal{H})^{1/2}$. We take the scalar product antilinear-linear and it satisfies the following properties:

(a) For any pair of vectors $Y$ and $Z$ in $\mathcal{H}$ we have $(Y, Z)_\mathcal{H} = (Z, Y)^*_\mathcal{H}$, where the asterisk denotes complex conjugation.

(b) For any pair of vectors $Y$ and $Z$ in $\mathcal{H}$ and any complex number $\alpha$ we have $(Y, \alpha Z)_\mathcal{H} = \alpha (Y, Z)_\mathcal{H}$. By the antilinearity, we mean $(\alpha Y, Z)_\mathcal{H} = \alpha^* (Y, Z)_\mathcal{H}$.
(c) For any three vectors \(X, Y,\) and \(Z\) in \(\mathcal{H}\) we have \((X, Y + Z)_{\mathcal{H}} = (X, Y)_{\mathcal{H}} + (X, Z)_{\mathcal{H}}\).

(d) We have \((Y, Y)_{\mathcal{H}} \geq 0\) for all \(Y \in \mathcal{B}\), and the equality holds if and only if \(Y\) is the zero vector in \(\mathcal{H}\).

For a vector-valued function \(f(y)\), viewed as the vector \(f\) in the Banach space \(L^1(\mathbb{R}^+)\), we have the length of \(f\) defined as

\[
\|f\|_1 := \int_0^\infty dy |f(y)|. \tag{3.16}
\]

Note that (3.16) holds whether \(f(y)\) is a column vector or a row vector. Assume that the \(n \times n\) matrix-valued operator \(O\) acts on \(L^1(\mathbb{R}^+)\). Then, for a vector-valued function \(f(y)\), which is a column-vector with \(n\) components in \(L^1(\mathbb{R}^+)\), we have \((O f)(y)\) also a column vector with \(n\) components. Then, the operator norm \(\|O\|_1\) is given by

\[
\|O\|_1 := \sup_{\|f\|_1 = 1} \|O f\|_1 = \sup_{\|f\|_1 = 1} \int_0^\infty dy |(O f)(y)|. \tag{3.17}
\]

We have the standard inequality

\[
\|O f\|_1 \leq \|O\|_1 \|f\|_1. \tag{3.18}
\]

Since we use \(\|O\|_1\) to denote the operator norm of \(O\) on \(L^1(\mathbb{R}^+)\) and we also use \(\|f\|_1\) to denote the standard norm in \(L^1(\mathbb{R}^+)\) for the vector \(f\) with \(n\) components, the notation \(\| \cdot \|_1\) becomes clearer by checking whether it applies to a matrix or to a vector.

In the Hilbert space \(L^2(\mathbb{R}^+)\) we have the standard scalar product

\[
(f, g) := \int_0^\infty dy \left[ f_1(y)^* g_1(y) + \cdots + f_n(y)^* g_n(y) \right] = \int_0^\infty dy \langle f(y), g(y) \rangle. \tag{3.19}
\]

For a vector-valued function \(f(y)\), viewed as the vector \(f\) in the Hilbert space \(L^2(\mathbb{R}^+)\), we have the norm of \(f\) defined as

\[
\|f\|_2 := \left[ \int_0^\infty dy |f(y)|^2 \right]^{1/2}. \tag{3.20}
\]
Note that (3.20) holds whether \( f(y) \) is a column vector or a row vector. Comparing (3.19) and (3.20) we see that the standard norm \( || \cdot ||_2 \) in \( L^2(\mathbb{R}^+) \) given in (3.20) is induced by the standard scalar product in (3.19) via

\[
||f||_2 = \sqrt{(f,f)}_2. \tag{3.21}
\]

Assume that the \( n \times n \) matrix-valued operator \( O \) acts on \( L^2(\mathbb{R}^+) \). Then, for a vector-valued function \( f(y) \), which is a column-vector with \( n \) components in \( L^2(\mathbb{R}^+) \), we have \((Of)(y)\) also a column vector with \( n \) components. Then, the operator norm \( ||O||_2 \) is given by

\[
||O||_2 := \sup_{||f||_2=1} ||Of||_2 = \sup_{||f||_2=1} \left[ \int_0^\infty dy |(Of)(y)|^2 \right]^{1/2}. \tag{3.22}
\]

We have the standard inequality

\[
||Of||_2 \leq ||O||_2 ||f||_2. \tag{3.23}
\]

Since we use \( ||O||_2 \) to denote the operator norm of \( O \) on \( L^2(\mathbb{R}^+) \) and we also use \( ||f||_2 \) to denote the standard norm in \( L^2(\mathbb{R}^+) \) for the vector \( f \) with \( n \) components, the notation \( || \cdot ||_2 \) becomes clearer by checking whether it applies to a matrix or to a vector. Further results on the operator norm are given in Section 3.g.

For a vector-valued function \( f(y) \), viewed as the vector \( f \) in the Banach space \( L^\infty(\mathbb{R}^+) \), we have the norm of \( f \) defined as

\[
||f||_\infty := \text{ess sup}_{y \in \mathbb{R}^+} |f(y)|. \tag{3.24}
\]

Note that (3.24) holds whether \( f(y) \) is a column vector or a row vector.

The closure of a set is obtained by adding all the limit points to the set. A subspace of \( L^2(\mathbb{R}^+) \) is dense if its closure is equal to \( L^2(\mathbb{R}^+) \). We recall that there are various dense subspaces of \( L^2(\mathbb{R}^+) \). For example, \( C_0(\mathbb{R}^+) \), the subspace of continuous functions with compact support in \( \mathbb{R}^+ \) is dense in \( L^2(\mathbb{R}^+) \). Another dense subspace of \( L^2(\mathbb{R}^+) \) is
$C_0^\infty(\mathbb{R}^+)$, the subspace of infinitely differentiable functions with compact support in $\mathbb{R}^+$. Yet another dense subspace of $L^2(\mathbb{R}^+)$ is $L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$. A subspace of $L^2(\mathbb{R}^+)$ is dense if and only if its orthogonal complement is $\{0\}$.

3.c Useful inequalities

The Cauchy-Schwarz inequality in $\mathbb{C}^n$ is given by

$$|\langle f(y), g(y) \rangle| \leq |f(y)||g(y)|.$$  \hspace{1cm} (3.25)

The Cauchy-Schwarz inequality in $L^2(\mathbb{R}^+)$ is given by

$$|(f, g)| \leq ||f||_2 ||g||_2.$$  \hspace{1cm} (3.26)

We have the H"older inequality

$$||fg||_1 \leq ||f||_2 ||g||_2;$$  \hspace{1cm} (3.27)

which can informally be stated as the product of two square-integrable quantities is integrable. We also have

$$||fg||_1 \leq ||f||_1 ||g||_\infty,$$  \hspace{1cm} (3.28)

which can informally be stated as the product of an integrable quantity and a bounded quantity is integrable. We have Young’s inequality for products given by

$$|f(y)g(y)| \leq \frac{1}{2} \left[|f(y)|^2 + |g(y)|^2 \right],$$  \hspace{1cm} (3.29)

$$||fg||_1 \leq \frac{1}{2} \left[||f||_2^2 + ||g||_2^2 \right],$$  \hspace{1cm} (3.30)

the latter of which can informally be stated as the product of two square-integrable quantities is integrable. The properties in (3.25)-(3.30) hold for scalar, vector-valued, matrix-valued, and matrix-valued operator quantities.

We have the standard triangle inequalities

$$|f(y) + g(y)| \leq |f(y)| + |g(y)|,$$  \hspace{1cm} (3.31)
\[\|f + g\|_1 \leq \|f\|_1 + \|g\|_1,\]  
(3.32)

\[\|f + g\|_2 \leq \|f\|_2 + \|g\|_2,\]  
(3.33)

\[\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty,\]  
(3.34)

all holding for scalar, vector-valued, matrix-valued, and matrix-valued operator quantities.

Recall that the convolution \(f \ast g\) is defined as

\[(f \ast g)(y) := \int_{-\infty}^{\infty} \text{dz} \cdot f(y - z) g(z).\]  
(3.35)

We have the symmetry

\[(g \ast f)(y) = (f \ast g)(y),\]  
(3.36)

as well as Young’s inequalities for convolutions

\[\|f \ast g\|_1 \leq \|f\|_1 \|g\|_1,\]  
(3.37)

\[\|f \ast g\|_2 \leq \|f\|_1 \|g\|_2,\]  
(3.38)

\[\|f \ast g\|_\infty \leq \|f\|_1 \|g\|_\infty,\]  
(3.39)

all holding for scalar, vector-valued, matrix-valued, and matrix-valued operator quantities.

3.d Hardy spaces

We use \(H^2(\mathbb{C}^+\!\to\!)\) to denote the Hardy space of all complex-valued functions \(f(k)\) that are analytic in \(k \in \mathbb{C}^+\) with a finite norm defined as

\[\|f\|_{H^2(\mathbb{C}^+\!\to\!)} := \sup_{\rho > 0} \left[ \int_{-\infty}^{\infty} d\alpha |f(\alpha + i\rho)|^2 \right]^{1/2}.\]  
(3.40)

Thus, \(f(k)\) is square integrable along all lines in \(\mathbb{C}^+\) that are parallel to the real axis, i.e.

\[\int_{-\infty}^{\infty} d\alpha |f(\alpha + i\rho)|^2 \leq \left(\|f\|_{H^2(\mathbb{C}^+)\!\to\!}\right)^2, \quad \rho \in \mathbb{R}^+.\]  
(3.41)
The value of \( f(k) \) for \( k \in \mathbb{R} \) is defined \([41]\) to be the non-tangential limit of \( f(k + i\rho) \) as \( \rho \to 0^+ \), and in particular

\[
 f(k) := \lim_{\rho \to 0^+} f(k + i\rho), \quad k \in \mathbb{R}. \tag{3.42}
\]

It is known \([41]\) that such a non-tangential limit exists a.e. in \( k \in \mathbb{R} \) and belongs to \( L^2(\mathbb{R}) \) in the sense that

\[
 \lim_{\rho \to 0^+} \int_{-\infty}^{\infty} \, dk \, |f(k + i\rho) - f(k)|^2 = 0. \tag{3.43}
\]

It is known that \( f(k) \) belongs to \( \mathbf{H}^2(\mathbf{C}^+) \) if and only if there exists a corresponding function \( g(x) \) belonging to \( L^2(\mathbb{R}^+) \) in such a way that

\[
 f(k) = \int_{0}^{\infty} \, dx \, g(x) \, e^{ikx}, \tag{3.44}
\]

and that

\[
 ||f||_{\mathbf{H}^2(\mathbf{C}^+)} = \sqrt{2\pi} ||g||_2. \tag{3.45}
\]

The following pointwise estimate is useful for functions belonging to a Hardy space \( \mathbf{H}^2(\mathbf{C}^+) \).

**Proposition 3.1** Assume that \( f(k) \) belongs to the Hardy space \( \mathbf{H}^2(\mathbf{C}^+) \). Then, we have

\[
 |f(k)| \leq \frac{||f||_{\mathbf{H}^2(\mathbf{C}^+)}}{\sqrt{2\pi} \sqrt{2|k| \sin \theta}}, \quad k \in \mathbf{C}^+, \tag{3.46}
\]

where \( ||f||_{\mathbf{H}^2(\mathbf{C}^+)} \) is the norm of \( f \) defined in (3.40), and \( |k| \) and \( \theta \) are the absolute value and the argument of the point \( k \in \mathbf{C}^+ \), i.e. \( k = |k| e^{i\theta} \) is the polar representation of the complex number \( k \) with \( \theta \in (0, \pi) \). Thus, \( f(k) \) satisfies the pointwise estimate

\[
 |f(k)| \leq \frac{C}{\sqrt{|k| \sin \theta}}, \quad k \in \mathbf{C}^+, \tag{3.47}
\]

where \( C \) is a generic constant.

**PROOF:** In terms of the real and imaginary part of \( k \in \mathbf{C}^+ \), denoted by \( k_R \) and \( k_I \), respectively, we have \( k = k_R + i k_I \). Thus, from (3.44) we obtain

\[
 f(k) = \int_{0}^{\infty} \, dx \, g(x) \, e^{ik_R x - k_I x}. \tag{3.48}
\]
Applying the Cauchy-Schwarz inequality on the integrand in (3.48) we get
\[ |f(k)| \leq \int_0^\infty dx |g(x)| e^{-k_1 x} \leq \sqrt{\int_0^\infty dx |g(x)|^2} \sqrt{\int_0^\infty dx e^{-2k_1 x}}, \] (3.49)
yielding
\[ |f(k)| \leq ||g||_2 \frac{1}{\sqrt{2k_1}}. \] (3.50)
Using \( k_1 = |k| \sin \theta \) and (3.45) in (3.50) we obtain (3.46). When \( f(k) \) belongs to the Hardy space, its norm defined in (3.40) is finite, and hence (3.46) yields (3.47).

In a similar way, we use \( H^2(C^-) \) to denote the Hardy space of all complex-valued functions \( f(k) \) that are analytic in \( k \in C^- \) with a finite norm defined as
\[
||f||_{H^2(C^-)} := \sup_{\rho > 0} \left[ \int_{-\infty}^{\infty} d\alpha |f(\alpha - i\rho)|^2 \right]^{1/2}.
\] (3.51)
Thus, \( f(k) \) is square integrable along all lines in \( C^- \) that are parallel to the real axis, i.e.
\[
\int_{-\infty}^{\infty} d\alpha |f(\alpha - i\rho)|^2 \leq (||f||_{H^2(C^-)})^2, \quad \rho \in \mathbb{R}^+.
\] (3.52)
The value of \( f(k) \) for \( k \in \mathbb{R} \) is defined [40] to be the non-tangential limit of \( f(k - i\rho) \) as \( \rho \to 0^+ \), and in particular
\[
f(k) := \lim_{\rho \to 0^+} f(k - i\rho), \quad k \in \mathbb{R}.
\] (3.53)
Such a non-tangential limit exists a.e. in \( k \in \mathbb{R} \) and belongs to \( L^2(\mathbb{R}) \) in the sense that
\[
\lim_{\rho \to 0^+} \int_{-\infty}^{\infty} dk |f(k - i\rho) - f(k)|^2 = 0.
\] (3.54)
It is known that \( f(k) \) belongs to \( H^2(C^-) \) if and only if there exists a corresponding function \( g(x) \) belonging to \( L^2(\mathbb{R}^-) \) in such a way that
\[
f(k) = \int_{-\infty}^{0} dx g(x) e^{ikx},
\] (3.55)
and that
\[
||f||_{H^2(C^-)} = \sqrt{2\pi} ||g||_2.
\] (3.56)
The following pointwise estimate is useful for functions belonging to the Hardy space $H^2(C^-)$. Its proof is similar to the proof of Proposition 3.1 and hence is omitted.

**Proposition 3.2** Assume that $f(k)$ belongs to the Hardy space $H^2(C^-)$. Then, we have

$$|f(k)| \leq \frac{||f||_{H^2(C^-)}}{\sqrt{2\pi} \sqrt{2|k| \sin \theta}}, \quad k \in C^-,$$

where $||f||_{H^2(C^-)}$ is the norm of $f$ defined in (3.51), and $|k|$ and $\theta$ are the absolute value and the argument of the point $k \in C^+$, i.e. $k = |k|e^{i\theta}$ is the polar representation of the complex number $k$ with $\theta \in (-\pi, 0)$. Thus, $f(k)$ satisfies the pointwise estimate

$$|f(k)| \leq \frac{C}{\sqrt{|k| \sin \theta}}, \quad k \in C^-,$$

where $C$ is a generic constant.

3.e Other useful Banach spaces

We use $\hat{L}^1(C^+)$ to denote the Banach space of all complex-valued functions $f(k)$ that are analytic in $k \in C^+$ with the finite norm $||f||_{\hat{L}^1(C^+)}$ in such a way that there exists a corresponding function $g(x)$ belonging to $L^1(R^+)$ satisfying

$$f(k) = \int_{0}^{\infty} dx \, g(x) \, e^{ikx}.$$  \hspace{1cm} (3.59)

We define the norm $||f||_{\hat{L}^1(C^+)}$ as

$$||f||_{\hat{L}^1(C^+)} := \sqrt{2\pi} ||g||_1.$$  \hspace{1cm} (3.60)

Let us remark that if $f(k)$ belongs to $\hat{L}^1(C^+)$ then $f(k)$ is continuous in $k \in R$ and we have $f(k) = o(1)$ as $k \to \infty$ in $C^+$.

We use $\hat{L}^1(C^-)$ to denote the Banach space of all complex-valued functions $f(k)$ that are analytic in $k \in C^-$ with the finite norm $||f||_{\hat{L}^1(C^-)}$ in such a way that there exists a corresponding function $g(x)$ belonging to $L^1(R^-)$ satisfying

$$f(k) = \int_{-\infty}^{0} dx \, g(x) \, e^{ikx}.$$  \hspace{1cm} (3.61)
We define the norm $\|f\|_{\hat{L}^1(C^-)}$ as
\[
\|f\|_{\hat{L}^1(C^-)} := \sqrt{2\pi} \|g\|_1.
\] (3.62)

Note that if $f(k)$ belongs to $\hat{L}^1(C^-)$ then $f(k)$ is continuous in $k \in \mathbb{R}$ and we have $f(k) = o(1)$ as $k \to \infty$ in $\overline{C^-}$.

We use $\hat{L}^1_{\infty}(C^+)$ to denote the Banach space of all complex-valued functions $f(k)$ that are analytic in $k \in C^+$ with the finite norm $\|f\|_{\hat{L}^1_{\infty}(C^+)}$ in such a way that there exists a corresponding function $g(x)$ belonging to $L^1(R^+) \cap L^\infty(R^+)$ satisfying
\[
f(k) = \int_0^\infty dx \, g(x) e^{ikx}.
\] (3.63)
We define the norm $\|f\|_{\hat{L}^1_{\infty}(C^+)}$ as
\[
\|f\|_{\hat{L}^1_{\infty}(C^+)} := \sqrt{2\pi} (\|g\|_1 + \|g\|_\infty).
\] (3.64)

Note that if $f(k)$ belongs to $\hat{L}^1_{\infty}(C^+)$ then $f(k)$ must also belong to $H^2(C^+)$. In other words, we have $\hat{L}^1_{\infty}(C^+) \subset H^2(C^+)$. In a similar way, we use $\hat{L}^1_{\infty}(C^-)$ to denote the Banach space of all complex-valued functions $f(k)$ that are analytic in $k \in C^-$ with the finite norm $\|f\|_{\hat{L}^1_{\infty}(C^-)}$ in such a way that there exists a corresponding function $g(x)$ belonging to $L^1(R^-) \cap L^\infty(R^-)$ satisfying
\[
f(k) = \int_{-\infty}^0 dx \, g(x) e^{ikx}.
\] (3.65)
We define the norm $\|f\|_{\hat{L}^1_{\infty}(C^-)}$ as
\[
\|f\|_{\hat{L}^1_{\infty}(C^-)} := \sqrt{2\pi} (\|g\|_1 + \|g\|_\infty).
\] (3.66)
We remark that $\hat{L}^1_{\infty}(C^-) \subset H^2(C^-)$.

The definitions for $H^2(C^+), H^2(C^-), \hat{L}^1(C^+), \hat{L}^1(C^-), \hat{L}^1_{\infty}(C^+), \hat{L}^1_{\infty}(C^-)$ given above can naturally be extended to vector-valued or matrix-valued functions.
achieved by requiring that each entry of such functions belongs to the appropriate space and by replacing the absolute value used in the scalar case by the operator matrix norm.

For example, the quantity \( X(y) \) appearing in (4c) of Definition 4.3 is a row vector with \( n \) components. It belongs to \( L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \), and hence we can assume that it is bounded and integrable in \( y \in \mathbb{R} \) with the understanding that \( X(y) = 0 \) for \( y \in \mathbb{R}^- \). The quantity \( \hat{X}(k) \) appearing in (4d) of Definition 4.3 is also a row vector with \( n \) components, and it belongs to \( \hat{L}^1_\infty(\mathbb{C}^+) \). In fact, \( X(y) \) is related to \( \hat{X}(k) \) via a Fourier transform as

\[
X(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{X}(k) e^{-iky}, \quad y \in \mathbb{R}.
\] (3.67)

The quantity \( \hat{X}(k) \) belongs to \( \hat{L}^1_\infty(\mathbb{C}^+) \), and hence it is analytic in \( k \in \mathbb{C}^+ \), continuous in \( k \in \overline{\mathbb{C}^+} \), uniformly \( o(1) \) as \( k \to \infty \) in \( \mathbb{C}^+ \), and given by

\[
\hat{X}(k) = \int_{0}^{\infty} dy X(y) e^{iky}, \quad k \in \mathbb{R}.
\] (3.68)

We are also interested in those vector-valued \( \hat{X}(k) \) for which the corresponding \( X(y) \) belong to \( L^2(\mathbb{R}^-) \) and vanish for \( y \in \mathbb{R}^+ \). For such functions \( \hat{X}(k) \) belonging to \( H^2(\mathbb{C}^-) \) we still have (3.67) holding but (3.68) must be replaced with

\[
\hat{X}(k) = \int_{-\infty}^{0} dy X(y) e^{iky}, \quad k \in \mathbb{R}.
\] (3.69)

Such functions \( X(y) \) and \( \hat{X}(k) \) are related to \((\text{III}_a)\) and \((\text{III}_b)\) in Definition 4.3.

3.f Integral operators with kernels depending on a sum

Some general results pertaining to integral operators whose kernels depend on a sum are listed in the following three propositions for easy referencing. These results will be used to analyze various integral equations arising in the analysis of the inverse problem, including the Marchenko integral equation (13.1).

**Proposition 3.3** Let \( \epsilon \) be some fixed number in the interval \([0, +\infty)\). Assume that \( A(y) \) is an \( n \times n \) matrix-valued function integrable in \( y \in (\epsilon, +\infty) \). Then:
(a) The operator $A : X(y) \mapsto \int_{\epsilon}^{\infty} dz X(z) A(z+y)$ is compact on $L^1(\epsilon, +\infty)$, where $X(y)$ is a row vector with $n$ components.

(b) The linear integral equation given by

$$X(y) + A(y) + \int_{\epsilon}^{\infty} dz X(z) A(z+y) = 0,$$  \hspace{1cm} (3.70)

has a unique solution in $L^1(\epsilon, +\infty)$ if and only if the only solution in $L^1(\epsilon, +\infty)$ to the corresponding homogeneous equation

$$X(y) + \int_{\epsilon}^{\infty} dz X(z) A(z+y) = 0,$$  \hspace{1cm} (3.71)

is the trivial solution.

(c) If further $A(y)$ is assumed to be bounded in $y \in (\epsilon, +\infty)$, then any solution $X(y)$ in $L^1(\epsilon, +\infty)$ to (3.71) must also belong to $L^\infty(\epsilon, +\infty)$. In particular, any solution to (3.71) in $L^1(\epsilon, +\infty)$ must also belong to $L^2(\epsilon, +\infty)$.

(d) If further $A(y)$ is assumed to be bounded in $y \in (\epsilon, +\infty)$, then any solution $X(y)$ in $L^1(\epsilon, +\infty)$ to (3.70) must also belong to $L^\infty(\epsilon, +\infty)$. In particular, any solution to (3.70) in $L^1(\epsilon, +\infty)$ must also belong to $L^2(\epsilon, +\infty)$.

PROOF: The compactness in (a) is established in Lemma 3.3.1 of [2]. The result in (b) is a result of the compactness established in (a). The result in (c) is obtained with the help of (3.39), as follows. When $A(y)$ is bounded and $X(y)$ is integrable, from (3.71) we obtain

$$|X(y)| \leq \int_{\epsilon}^{\infty} dz |X(z)| |A(z+y)| \leq C \int_{\epsilon}^{\infty} dz |X(z)| < +\infty,$$ \hspace{1cm} (3.72)

for some constant $C$. Thus, $X(y)$ is bounded. Then, being in $L^1(\epsilon, +\infty)$ and $L^\infty(\epsilon, +\infty)$, the quantity $X(y)$ belongs to $L^p(\epsilon, +\infty)$ for all $p$ with $1 \leq p \leq +\infty$. In particular, it belongs to $L^2(\epsilon, +\infty)$. Thus, the proof of (c) is complete. From (3.70) we obtain

$$|X(y)| \leq |A(y)| + \int_{\epsilon}^{\infty} dz |X(z)| |A(z+y)| \leq C + C \int_{\epsilon}^{\infty} dz |X(z)| < +\infty,$$ \hspace{1cm} (3.73)

and hence any solution $X(y)$ in $L^1(\epsilon, +\infty)$ to (3.70) also belongs to $L^\infty(\epsilon, +\infty)$, and hence in particular to $L^2(\epsilon, +\infty)$. ■
The next result is the analog of Proposition 3.3, but for operators acting on $L^2(\epsilon, +\infty)$.

**Proposition 3.4** Let $\epsilon$ be some fixed number in the interval $[0, +\infty)$. Assume that $A(y)$ is an $n \times n$ matrix-valued function square integrable in $y \in (\epsilon, +\infty)$. Then:

(a) The operator $A : X(y) \mapsto \int_\epsilon^\infty dz X(z) A(z+y)$ is compact on $L^2(\epsilon, +\infty)$, where $X(y)$ is a row vector with $n$ components.

(b) The linear integral equation given by (3.70) has a unique solution in $L^2(\epsilon, +\infty)$ if and only if the only solution in $L^2(\epsilon, +\infty)$ to the corresponding homogeneous equation (3.71) is the trivial solution.

(c) Any solution $X(y)$ in $L^2(\epsilon, +\infty)$ to (3.71) must also belong to $L^\infty(\epsilon, +\infty)$.

(d) If further $A(y)$ is assumed to be bounded in $y \in (\epsilon, +\infty)$, then any solution $X(y)$ in $L^2(\epsilon, +\infty)$ to (3.70) must also belong to $L^\infty(\epsilon, +\infty)$.

**PROOF:** For the proof of (a), we proceed as follows. Let us first prove that the operator $A$ is bounded on $L^2(\epsilon, +\infty)$. Since $X(y)$ and $A(y)$ belong to $L^2(\epsilon, +\infty)$, we have their respective $L^2$-Fourier transforms

$$\hat{X}(k) := \int_\epsilon^\infty dy X(y) e^{iky},$$

$$\hat{A}(k) := \int_\epsilon^\infty dy A(y) e^{-iky},$$

with the understanding that $X(y) = 0$ and $A(y) = 0$ for $y \in (-\infty, \epsilon)$. Thus, we can write (3.74) and (3.75) as

$$\hat{X}(k) = \int_{-\infty}^\infty dy X(y) e^{iky},$$

$$\hat{A}(k) = \int_{-\infty}^\infty dy A(y) e^{-iky},$$

with $\hat{X}$ and $\hat{A}$ belonging to $L^2(\mathbb{R})$. Since the Fourier transform is a bijection on $L^2(\mathbb{R})$, from (3.76) and (3.77) yield

$$X(y) = \frac{1}{2\pi} \int_{-\infty}^\infty dk \hat{X}(k) e^{-iky},$$

(3.78)
\[ \mathcal{A}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{A}(k) e^{iky}. \] (3.79)

From (3.78) and (3.79), with the help of (11.37), we obtain
\[ \int_{\epsilon}^{\infty} dz \mathcal{A}(z + y) = \int_{-\infty}^{\infty} dz \mathcal{A}(z + y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{X}(k) \hat{A}(k) e^{iky}. \] (3.80)

From the second equality in (3.80) we obtain
\[ ||X \ast \mathcal{A}||_2 = \frac{1}{2\pi} ||\hat{X}||_2 ||\hat{A}||_2 = ||X||_2 ||\mathcal{A}||_2. \] (3.81)

From (3.82) we conclude that the operator \( \mathcal{A} \) is bounded on \( L^2(\epsilon, +\infty) \). Let \( \{\mathcal{A}^{(l)}\}_{l=1}^{\infty} \) be a sequence of \( n \times n \) matrix-valued functions belonging to \( C_0^\infty(\epsilon, +\infty) \) converging to \( \mathcal{A}(y) \in L^2(\epsilon, +\infty) \), i.e.
\[ \lim_{l \to +\infty} ||\mathcal{A}^{(l)} - \mathcal{A}||_2 = 0. \] (3.83)

Let \( \mathcal{A}^{(l)} \) be the operator on \( L^2(\epsilon, +\infty) \) for the mapping \( X(y) \mapsto \int_{\epsilon}^{\infty} dz X(z) \mathcal{A}^{(l)}(z + y) \).

From (3.83) we then conclude that the sequence of operators \( \{\mathcal{A}^{(l)}\}_{l=1}^{\infty} \) converge in norm to the operator \( \mathcal{A} \). Consequently, to prove that the operator \( \mathcal{A} \) is compact, it is enough to prove that the operator \( \mathcal{A}^{(l)} \) is compact. It is enough to prove that \( \mathcal{A}^{(l)} \) is a Hilbert-Schmidt operator on \( L^2(\epsilon, +\infty) \). We have
\[ \int_{\epsilon}^{\infty} dy \int_{\epsilon}^{\infty} dz |\mathcal{A}^{(l)}(y + z)|^2 \leq \int_{\epsilon}^{\infty} dy \left( \int_{\epsilon+y}^{\infty} dz |\mathcal{A}^{(l)}(z)|^2 \right) \leq \int_{\epsilon}^{\infty} dy \frac{1}{(1+y)^2} \int_{\epsilon+y}^{\infty} dz (1+z)^2 |\mathcal{A}^{(l)}(z)|^2. \] (3.84)

Because \( \mathcal{A}^{(l)}(y) \) is compactly supported in \( L^2(\epsilon, +\infty) \), the last integral in (3.84) is finite and hence \( \mathcal{A}^{(l)} \) is Hilbert-Schmidt and hence also compact on \( L^2(\epsilon, +\infty) \). Thus, the proof of (a) is complete. Let us now turn to the proof of (b). The result in (b) is a result of the compactness established in (a). The result in (c) is obtained without needing the boundedness of \( \mathcal{A}(y) \), as follows. Using (3.29) in (3.71) we obtain
\[ |X(y)| \leq \int_{\epsilon}^{\infty} dz |X(z)||\mathcal{A}(z + y)| \leq \frac{1}{2} \int_{\epsilon}^{\infty} dz \left( |X(z)|^2 + |\mathcal{A}(z + y)|^2 \right) < +\infty, \] (3.85)
for some constant \( C \). Thus, \( X(y) \) is bounded, establishing (c). Let us now turn to the proof of (d). From (3.70) we obtain

\[
|X(y)| \leq |A(y)| + \int_{\varepsilon}^{\infty} dz |X(z)| |A(z + y)|
\]

\[
\leq |A(y)| + \frac{1}{2} \int_{\varepsilon}^{\infty} dz \left( |X(z)|^2 + |A(z + y)|^2 \right),
\]

and hence from (3.86) we conclude (d). \( \blacksquare \)

The following result is the analog of Proposition 3.4 for \( y \in \mathbb{R}^- \).

**Proposition 3.5** Let \( A(y) \) be an \( n \times n \) matrix-valued function square integrable in \( y \in \mathbb{R}^- \). Then:

(a) The operator \( X(y) \mapsto \int_{-\infty}^{0} dz X(z) A(z + y) \) is compact on \( L^2(\mathbb{R}^-) \), where \( X(y) \) is a row vector with \( n \) components.

(b) Any solution \( X(y) \) in \( L^2(\mathbb{R}^-) \) to the linear homogeneous integral equation

\[
-X(y) + \int_{-\infty}^{0} dz X(z) A(z + y) = 0,
\]

must also belong to \( L^\infty(\mathbb{R}^-) \).

**PROOF:** The proof is similar to the proofs of (a) and (c) of Proposition 3.4. \( \blacksquare \)

3.g Further results in Banach and Hilbert spaces

Let \( B_1 \) and \( B_2 \) be Banach spaces over the complex numbers. We consider linear operators (operators in short) from \( B_1 \) into \( B_2 \) that are only defined in a linear subspace in \( B_1 \).

**Definition 3.6** An operator \( L \) from \( B_1 \) into \( B_2 \) is a linear function from a linear subspace \( \text{Dom}[L] \) of \( B_1 \) into \( B_2 \). We call \( \text{Dom}[L] \) the domain of \( L \). Then, for all complex numbers \( \alpha_1 \) and \( \alpha_2 \) we have

\[
L(\alpha_1 Y_1 + \alpha_2 Y_2) = \alpha_1 LY_1 + \alpha_2 LY_2, \quad Y_1, Y_2 \in \text{Dom}[L].
\]

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In our monograph we only consider operators that are densely defined in $B_1$. That is to say, we assume that the closure of the domain of $L$, denoted by $\text{Dom}[L]$, is equal to $B_1$.

An important class of operators is the class of closed operators defined below.

**Definition 3.7** An operator $L$ from $B_1$ into $B_2$ is said to be a closed operator if for every sequence $\{Y^{(l)}\}_{l=1}^{\infty}$ with elements in $\text{Dom}[L]$ converging to an element $Y \in B_1$, i.e. $Y := \lim_{l \to +\infty} Y^{(l)}$, and such that $\{LY^{(l)}\}_{l=1}^{\infty}$ converging to an element $W \in B_2$, i.e. $W := \lim_{l \to +\infty} LY^{(l)}$, the following is true: $Y$ belongs to $\text{Dom}[L]$ and $W = LY$.

Next, we introduce the class of closable operators.

**Definition 3.8** An operator $L$ from $B_1$ into $B_2$ is said to be closable, if for every sequence $\{Y^{(l)}\}_{l=1}^{\infty}$ with elements in $\text{Dom}[L]$ converging to zero, i.e. $\lim_{l \to +\infty} Y^{(l)} = 0$, and such that $\{LY^{(l)}\}_{l=1}^{\infty}$ converging to an element $W \in B_2$, i.e. $W := \lim_{l \to +\infty} LY^{(l)}$, we then have $W = 0$.

Next we define an order relation between two operators.

**Definition 3.9** Let $L_1$ and $L_2$ be operators from $B_1$ into $B_2$. We say that $L_1 \subset L_2$ if $\text{Dom}[L_1] \subset \text{Dom}[L_2]$ and if $L_1Y = L_2Y$ for all $Y \in \text{Dom}[L_1]$.

When $L_1 \subset L_2$ we say that $L_1$ is a restriction of $L_2$ or that $L_2$ is an extension of $L_1$. Observe that $L_1 \subset L_2$ means intuitively that $L_1$ acts in the same way as $L_2$ but with a smaller domain.

A closable operator $L$ has always a closed extension that we call the closure of $L$, which we denote by $\overline{L}$. The domain of $\overline{L}$, denoted by $\text{Dom}[\overline{L}]$ consists of all vectors $Y \in B_1$ that are the limits of sequences $\{Y^{(l)}\}_{l=1}^{\infty}$ with elements in the domain of $L$ in such a way that each corresponding sequence $\{LY^{(l)}\}_{l=1}^{\infty}$ converges to an element $W \in B_2$ and we have

$$\overline{L}Y = W, \quad Y \in \text{Dom}[\overline{L}].$$

(3.89)

Note that since $L$ is closable, if $Y = 0$, then $W = 0$ and then, we do not reach the contradiction $\overline{L}0 \neq 0$. Observe that if $L$ is closable, then, $\overline{L}$ is the smallest closed extension
of $L$, i.e. if $D$ is any closed extension of $L$ then $L \subset D$.

**Definition 3.10** An operator $L$ from $B_1$, into $B_2$ is bounded if for some constant $C$ we have

$$||LY||_{B_2} \leq C||Y||_{B_1}, \quad Y \in \text{Dom}[L], (3.90)$$

and we define the operator norm of $L$ from $B_1$ into $B_2$, denoted by $||L||_{B_2B_1}$ as

$$||L||_{B_2B_1} = \sup_{||Y||_{B_1} = 1} ||LY||_{B_2}. (3.91)$$

Since we assume that the domain of $L$ is dense, $L$ can be uniquely extended (taking the closure) to a bounded operator with domain $B_1$. Hence, we will always assume that the bounded operators have as their domain the whole Banach space.

**Definition 3.11** Let $\mathcal{H}$ be a Hilbert space. The adjoint of an operator $L$ from $\mathcal{H}$ into $\mathcal{H}$, which we denote by $L^\dagger$, is defined as the map $L^\dagger : Y \mapsto W$ with the domain given by

$$\text{Dom}[L^\dagger] := \{Y \in \mathcal{H} : (Y, LV)_\mathcal{H} = (W, V)_\mathcal{H} \text{ for all } V \in \text{Dom}[L]\}. (3.92)$$

Since $\text{Dom}[L]$ is dense in $\mathcal{H}$, the operator $L^\dagger$ is well defined in the sense that there is at most only one $W \in \mathcal{H}$ such that $(Y, LV)_\mathcal{H} = (W, V)_\mathcal{H}$ for any $V \in \text{Dom}[L]$.

**Definition 3.12** Let $\mathcal{H}$ be a Hilbert space, and let $L$ be an operator from $\mathcal{H}$ into $\mathcal{H}$. Then, we say that $L$ is symmetric if $L \subset L^\dagger$. We say that $L$ is selfadjoint if $L = L^\dagger$.

In applications in differential operators it is relatively simple to verify that an operator $L$ is symmetric. However, it is usually a rather delicate issue to verify that it is also selfadjoint, i.e. to verify that $\text{Dom}[L^\dagger] = \text{Dom}[L]$.

**Definition 3.13** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. An operator $U$ from $\mathcal{H}_1$ into $\mathcal{H}_2$ is said to be isometric if

$$||UY||_{\mathcal{H}_2} = ||Y||_{\mathcal{H}_1}, \quad Y \in \mathcal{H}_1. (3.93)$$
It is known [27] that \( U \) is isometric if and only if we have

\[
(UY_1, UY_2)_{\mathcal{H}_2} = (Y_1, Y_2)_{\mathcal{H}_1}, \quad Y_1, Y_2 \in \mathcal{H}_1.
\]  

Moreover, \( U \) is isometric if and only if \( U^\dagger U = I_{\mathcal{H}_1}, \) where by \( I_{\mathcal{H}_1} \) we denote the identity operator in \( \mathcal{H}_1. \)

**Definition 3.14** Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two Hilbert spaces. An operator \( U \) from \( \mathcal{H}_1 \) into \( \mathcal{H}_2 \) is unitary if it is isometric and onto, where the onto property is expressed as \( \text{Ran}[U] = \mathcal{H}_2. \)

It is known [27] that \( U \) is unitary if and only if \( U^\dagger U = I_{\mathcal{H}_1} \) and \( UU^\dagger = I_{\mathcal{H}_2}, \) with \( I_{\mathcal{H}_1} \) and \( I_{\mathcal{H}_2} \) denoting the identity operators on \( \mathcal{H}_1 \) and \( \mathcal{H}_2, \) respectively.

### 3.h Other miscellaneous results

We recall [1] that the Sobolev space \( \mathbf{H}^1(\mathbb{R}^+) \) consists of all vectors in \( L^2(\mathbb{R}^+) \) with also their first derivatives in \( L^2(\mathbb{R}^+) \). In other words, it consists of all square-integrable functions \( X(y) \) defined for \( y \in \mathbb{R}^+ \) where the derivative \( X'(y) \) exists in the distribution sense and belongs to \( L^2(\mathbb{R}^+) \). We remark that \( \mathbf{H}^1(\mathbb{R}^+) \) is a Hilbert space with the scalar product

\[
(X, Y)_{\mathbf{H}^1(\mathbb{R}^+)} := (X, Y)_2 + (X', Y')_2.
\]  

We recall that the Riemann-Lebesgue lemma states that the Fourier transform of an integrable function vanishes at infinity. For example, if \( g(x) \) appearing in (3.59) belongs to \( L^1(\mathbb{R}^+) \), then \( f(k) \) appearing in (3.59) vanishes as \( k \to \pm \infty \) according to the Riemann-Lebesgue lemma.

Let us introduce

\[
\sigma(x) := \int_x^\infty dz |V(z)|, \quad \sigma_1(x) := \int_x^\infty dz z |V(z)|, \quad x \geq 0.
\]  

Note that \( \sigma(x) \) and \( \sigma_1(x) \) are nonincreasing functions of \( x \in [0, +\infty) \), and it is readily seen that \( \sigma(0) \) and \( \sigma_1(0) \) are both finite when the potential satisfies (2.3). As also shown on p.
68 of [2], we have
\[ x \sigma(x) \leq \sigma_1(x), \quad \int_x^\infty dz \sigma(z) \leq \sigma_1(x). \quad (3.97) \]

With the help of (3.96) we get
\[ \int_0^\infty dz \sigma'(z)^2 \leq \sigma(0) \int_0^\infty dz \sigma(z) = \sigma(0) \sigma_1(0) < +\infty, \quad (3.98) \]
\[ \int_0^\infty dz z \sigma'(z)^2 \leq \sigma_1(0) \int_0^\infty dz z \sigma(z) = [\sigma_1(0)]^2 < +\infty. \quad (3.99) \]

The inequalities in (3.97)-(3.99) are used later on in some proofs.
4. THE FADDEEV CLASS AND THE MARCHENKO CLASS

In this chapter, in preparation for a characterization of the scattering data, we introduce the Marchenko class of scattering data sets and the Faddeev class of input data sets. Our characterization basically consists of showing that there is a one-to-one correspondence between the Marchenko class and the Faddeev class.

In the direct scattering problem related to (2.1) and (2.4), our input data set $D$ is given by

$$D := \{V, A, B\}, \quad (4.1)$$

with the understanding that $V$ is equivalent to the knowledge of the $n \times n$ matrix $V(x)$ for $x \in \mathbb{R}^+$, the boundary matrices $A$ and $B$ are two constant $n \times n$ matrices and they are defined up to a postmultiplication by an invertible $n \times n$ matrix $T$. In the inverse scattering problem our scattering data set $S$ is given by

$$S := \{S, \{\kappa_j, M_j\}_{j=1}^N\}, \quad (4.2)$$

with the understanding that $S$ is equivalent to the knowledge of the $n \times n$ scattering matrix $S(k)$ specified for $k \in \mathbb{R}$, the $\kappa_j$ are $N$ distinct positive numbers related to the bound-state energies $-\kappa_j^2$, the $M_j$ are $N$ constant $n \times n$ matrices related to the normalizations of matrix-valued bound-state wavefunctions, and each $M_j$ is nonnegative and hermitian and has rank $m_j$ for some integer between 1 and $n$. Thus, we use $N$ to denote the number of bound states without counting the multiplicities. We remark that the possibility $N = 0$ is included in our consideration, in which case the scattering data set consists of the scattering matrix alone. The finiteness of $N$ is guaranteed [9] by (2.3). The integer $m_j$ corresponds to the multiplicity of the bound state associated with $k = i\kappa_j$, and hence the integer $\mathcal{N}$ defined as the sum of the ranks of $N$ matrices $M_j$, i.e.

$$\mathcal{N} := \sum_{j=1}^N m_j, \quad (4.3)$$

corresponds the total number of bound states including the multiplicities.
In our analysis of the direct problem related to (2.1) and (2.4), we assume that our input data set $D$ belongs to the Faddeev class defined below.

**Definition 4.1** The input data set $D$ given in (4.1) is said to belong to the Faddeev class $\mathcal{F}$ if the potential $V$ satisfies (2.2) and (2.3) and the matrices $A$ and $B$ satisfy (2.5) and (2.6). In other words, $D$ belongs to the Faddeev class if the $n \times n$ matrix-valued potential $V$ appearing in (2.1) is hermitian and belongs to class $L^1_1(\mathbb{R}^+)$ and the constant $n \times n$ matrices $A$ and $B$ appearing in (2.4) satisfy (2.5) and (2.6).

In our monograph, we provide various equivalent formulations of the characterization of the scattering data set $S$ given in (4.2) when the corresponding $D$ in (4.1) belongs to the Faddeev class. In order to state those various conditions in an efficient manner, we first introduce a set of properties that are all indicated with an Arabic numeral.

**Definition 4.2** The properties $(1)$, $(2)$, $(3_a)$, $(3_b)$, $(4_a)$, $(4_b)$, $(4_c)$, $(4_d)$, $(4_e)$ for the scattering data set $S$ in (4.2) are defined as follows:

1. The scattering matrix $S(k)$ satisfies
   \[ S(-k) = S(k)^\dagger = S(k)^{-1}, \quad k \in \mathbb{R}, \]  
   \[ \text{and there exist constant } n \times n \text{ matrices } S_\infty \text{ and } G_1 \text{ in such a way that} \]
   \[ S(k) = S_\infty + \frac{G_1}{ik} + o \left( \frac{1}{k} \right), \quad k \to \pm \infty. \]  
   \[ \text{The quantity } S(k) - S_\infty \text{ is the Fourier transform of an } n \times n \text{ matrix } F_s(y) \text{ in such } \]
   \[ \text{a way that } F_s(y) \text{ is bounded in } y \in \mathbb{R} \text{ and integrable in } y \in \mathbb{R}^+. \text{ Thus, the constant matrix } S_\infty \text{ is obtained from the scattering matrix } S(k) \text{ via} \]
   \[ S_\infty := \lim_{k \to \pm \infty} S(k), \]
   \[ \text{and the quantity } F_s(y) \text{ is related to } S(k) \text{ as} \]
   \[ F_s(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left[ S(k) - S_\infty \right] e^{iky}, \quad y \in \mathbb{R}. \]
(2) For the matrix $F_s(y)$ defined in (4.7), the derivative $F'_s(y)$ exists a.e. for $y \in \mathbb{R}^+$ and it satisfies
\[ \int_0^\infty dy (1 + y) |F'_s(y)| < +\infty, \] (4.8)
where we recall that the norm in (4.8) is the operator norm of a matrix.

(3a) The physical solution $\Psi(k,x)$ satisfies the boundary condition (2.4), i.e.
\[ -B^\dagger \Psi(k,0) + A^\dagger \Psi'(k,0) = 0, \quad k \in \mathbb{R}. \] (4.9)
We clarify this property as follows: The scattering matrix appearing in $S$ yields a particular $n \times n$ matrix-valued solution $\Psi(k,x)$ to (2.1) known as the physical solution given in (9.4) and also yields a pair of matrices $A$ and $B$ (modulo an invertible matrix) satisfying (2.5) and (2.6). Our statement (3a) is equivalent to saying that (2.4) is satisfied if we use in (2.4) the quantities $\Psi(k,x)$, $A$, and $B$ constructed from $S(k)$ appearing in $S$.

(3b) The Jost matrix $J(k)$ satisfies
\[ J(-k) + S(k) J(k) = 0, \quad k \in \mathbb{R}. \] (4.10)
We clarify this property as follows: The scattering matrix $S(k)$ given in $S$ yields a Jost matrix $J(k)$ constructed as in (9.2), unique up to a post multiplication by an invertible matrix. Using the scattering matrix $S(k)$ given in $S$ and the Jost matrix constructed from $S(k)$, we find that (4.10) is satisfied.

(4a) The Marchenko equation (13.1) at $x = 0$ given by
\[ K(0,y) + F(y) + \int_0^\infty dz K(0,z) F(z + y) = 0, \quad y \in \mathbb{R}^+, \] (4.11)
has a unique solution $K(0,y)$ in $L^1(\mathbb{R}^+)$. Here, $F(y)$ is the $n \times n$ matrix related to $F_s(y)$ given in (4.7) as
\[ F(y) := F_s(y) + \sum_{j=1}^N M_j^2 e^{-\kappa_j y}, \quad y \in \mathbb{R}^+. \] (4.12)
(4b) The only solution in $L^1(\mathbb{R}^+)$ to the homogeneous Marchenko integral equation at $x = 0$ given by

$$K(0, y) + \int_0^\infty dz K(0, z) F(z + y) = 0, \quad y \in \mathbb{R}^+,$$  

(4.13)

which is the homogeneous version of the Marchenko equation at $x = 0$ given by (4.11), is the trivial solution $K(0, y) \equiv 0$. Here, $F(y)$ is the quantity defined in (4.12).

(4c) The only integrable solution $X(y)$, which is a row vector with $n$ integrable components in $y \in \mathbb{R}^+$, to the linear homogeneous integral equation

$$X(y) + \int_0^\infty dz X(z) F(z + y) = 0, \quad y \in \mathbb{R}^+,$$  

(4.14)

is the trivial solution $X(y) \equiv 0$. Here, $F(y)$ is the quantity defined in (4.12).

(4d) The only solution $\hat{X}(k)$ to the system

$$\begin{cases}
\hat{X}(i\kappa_j) M_j = 0, & j = 1, \ldots, N, \\
\hat{X}(-k) + \hat{X}(k) S(k) = 0, & k \in \mathbb{R},
\end{cases}$$  

(4.15)

where $\hat{X}(k)$ is a row vector with $n$ components belonging to the class $\hat{L}^1(\mathbb{C}^+)$, is the trivial solution $\hat{X}(k) \equiv 0$.

(4e) The only solution $h(k)$ to the system

$$\begin{cases}
M_j h(i\kappa_j) = 0, & j = 1, \ldots, N, \\
h(-k) + S(k) h(k) = 0, & k \in \mathbb{R},
\end{cases}$$  

(4.16)

where $h(k)$ is a column vector with $n$ components belonging to the class $\hat{L}^1(\mathbb{C}^+)$, is the trivial solution $h(k) \equiv 0$.

Let us make some comments on the condition (1) of Definition 4.2 above. The property (4.5) implies that $S(k) - S_\infty$ is square integrable in $k \in \mathbb{R}$ and hence also implies $F_s(y) \in L^2(\mathbb{R}^+)$, and in fact (4.5) contains even more information. The quantity $G_1$ and hence (4.5) itself are used to construct the boundary matrices $A$ and $B$ appearing in (2.4)-(2.6). On the other hand, for the construction of the potential and various other quantities the square integrability of $S(k) - S_\infty$ is sufficient. So that we can use such a weaker condition
to obtain various results, we specify the property (I) in the following definition. We also introduce various other properties, which are denoted by using a Roman numeral.

**Definition 4.3** The properties (I), (III<sub>a</sub>), (III<sub>b</sub>), (III<sub>c</sub>), (V<sub>a</sub>), (V<sub>b</sub>), (V<sub>c</sub>), (V<sub>d</sub>), (V<sub>e</sub>), (V<sub>f</sub>), (V<sub>g</sub>), (V<sub>h</sub>), and (VI) for the scattering data set S in (4.2) are defined as follows:

(I) The scattering matrix S<sub>k</sub> satisfies (4.4), the quantity S<sub>∞</sub> defined in (4.6) exists, the quantity S<sub>k</sub> − S<sub>∞</sub> is square integrable in k ∈ R, and the quantity F<sub>s</sub>(y) defined in (4.7) is bounded in y ∈ R and integrable in y ∈ R<sup>+</sup>.

(III<sub>a</sub>) For the function F<sub>s</sub>(y) given in (4.7), the derivative F<sub>s</sub>'(y) for y ∈ R<sup>−</sup> can be written as the sum of two functions, one of which is integrable and the other is square integrable in y ∈ R<sup>−</sup>. Furthermore, the only solution X(y), which is a row vector with n square-integrable components in y ∈ R<sup>−</sup>, to the linear homogeneous integral equation

\[-X(y) + \int_{-\infty}^{0} dz X(z) F_s(z + y) = 0, \quad y \in R^-,\]

is the trivial solution X(y) ≡ 0.

(III<sub>b</sub>) The only solution \(\hat{X}(k)\) to the homogeneous Riemann-Hilbert problem

\[-\hat{X}(−k) + \hat{X}(k) S(k) = 0, \quad k \in R,\]

where \(\hat{X}(k)\) is a row vector with n components belonging to the class \(H^2(C^-)\), is the trivial solution \(\hat{X}(k) \equiv 0\).

(III<sub>c</sub>) The only solution \(h(k)\) to the homogeneous Riemann-Hilbert problem

\[-h(−k) + S(k) h(k) = 0, \quad k \in R,\]

where \(h(k)\) is a column vector with n components belonging to the class \(H^2(C^-)\), is the trivial solution \(h(k) \equiv 0\).

(V<sub>a</sub>) Each of the N normalized bound-state matrix solutions \(Ψ_j(x)\) constructed as in (9.8) satisfies the boundary condition (2.4), i.e.

\[-B^dΨ_j(0) + A^dΨ'_j(0) = 0, \quad j = 1, \ldots, N.\]
We clarify this statement as follows. The scattering matrix \( S(k) \) and the bound-state data \( \{ \kappa_j, M_j \}_{j=1}^N \) given in \( S \) yield \( n \times n \) matrices \( \Psi_j(x) \) as in (9.8), where each \( \Psi_j(x) \) is a solution to (2.1) at \( k = i\kappa_j \). As stated in (3b) of Definition 4.2, the scattering matrix given in \( S \) yields a pair of matrices \( A \) and \( B \) (modulo an invertible matrix) satisfying (2.5) and (2.6). The statement (V\(_a\)) is equivalent to saying that (2.4) is satisfied if we use in (2.4) the quantities \( \Psi_j(x), A, \) and \( B \) constructed from the quantities appearing in \( S \).

(V\(_b\)) The normalization matrices \( M_j \) appearing in \( S \) satisfy

\[
J(i\kappa_j)\hat{M}_j = 0, \quad j = 1, \ldots, N.
\]  

(4.21)

We clarify this condition as follows: As indicated in (3b) in Definition 4.2, the scattering matrix \( S(k) \) given in \( S \) yields a Jost matrix \( J(k) \). Using in (4.21) the matrix \( M_j \) given in \( S \) and the Jost matrix constructed from \( S(k) \), at each \( \kappa_j \)-value listed in \( S \) the matrix equation (4.21) holds.

(V\(_c\)) The linear homogeneous integral equation

\[
X(y) + \int_0^\infty dz X(z) F_s(z + y) = 0, \quad y \in \mathbb{R}^+,
\]  

(4.22)

has precisely \( N \) linearly independent row vector solutions with \( n \) components which are integrable in \( y \in \mathbb{R}^+ \). Here \( F_s(y) \) is the matrix defined in (4.7).

(V\(_d\)) The homogeneous Riemann-Hilbert problem given by

\[
\hat{X}(-k) + \hat{X}(k) S(k) = 0, \quad k \in \mathbb{R},
\]  

(4.23)

has precisely \( N \) linearly independent row vector solutions whose \( n \) components belong to the class \( \hat{L}^1(C^+) \).

(V\(_e\)) The homogeneous Riemann-Hilbert problem given by

\[
h(-k) + S(k) h(k) = 0, \quad k \in \mathbb{R},
\]  

(4.24)
has precisely $N$ linearly independent column vector solutions whose $n$ components belong to the class $\hat{L}^1(C^+)$.

(Vf) The number of linearly independent square-integrable solutions $X(y)$ to (4.22) in $y \in R^+$ is equal to the nonnegative integer $N$ given in (4.3). The matrix $F_s(y)$ appearing in the kernel of (4.22) is defined in (4.7).

(Vg) The homogeneous Riemann-Hilbert problem given in (4.23) has precisely $N$ linearly independent row vector solutions whose $n$ components belong to the class $H^2(C^+)$, where $N$ is the nonnegative integer specified in (4.3).

(Vh) The homogeneous Riemann-Hilbert problem given in (4.24) has precisely $N$ linearly independent row vector solutions whose $n$ components belong to the class $H^2(C^+)$, where $N$ is the nonnegative integer specified in (4.3).

(VI) The scattering matrix $S(k)$ is continuous in $k \in R$.

As a result of the square integrability of $S(k) - S_\infty$ stated in (I) of Definition 4.3 it follows that the Fourier transform $F_s(y)$ given in (4.7) is square integrable in $y \in R$. For easy referencing we state the result in the following proposition.

Proposition 4.4 Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in R$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. If $S$ satisfies either (I) of Definition 4.3 or (1) of Definition 4.2, then $F_s(y)$ defined in (4.7) is bounded in $y \in R$, integrable in $y \in R^+$, and square integrable in $y \in R$.

Having defined various properties for the scattering data set $S$, we are able to formulate various characterizations for $S$ so that it corresponds to a unique input data set $D$ in the Faddeev class specified in Definition 4.1. Such a typical characterization has the form (1, 2, 3, 4), by which we mean $S$ satisfies the properties (1), (2), either of the two properties (3a) or (3b), and any one of the five properties listed as (4a), (4b), (4c), (4d), (4e). Another
typical characterization has the form \((1, 2, \text{III} + V, 4)\), by which we mean \(S\) satisfies \((1, 2, 3, 4)\), except for the fact that instead of satisfying either of the two properties \((3_a)\) or \((3_b)\), it instead satisfies two other properties, the first being one of the three properties \((\text{III}_c), (\text{III}_c), (\text{III}_c)\) and the second being one of the eight properties \((V_a), (V_b), (V_c), (V_d), (V_e), (V_f), (V_g), (V_h)\). The notation used, although maybe awkward, enables us to streamline all the characterizations of \(S\). One such characterization, namely that \(S\) satisfying \((1, 2, 3_a, 4_a)\), enables us to define the Marchenko class of scattering data sets as follows.

**Definition 4.5** Consider a scattering data set \(S\) as in (4.2), which consists of an \(n \times n\) scattering matrix \(S(k)\) for \(k \in \mathbb{R}\), a set of \(N\) distinct positive constants \(\kappa_j\), and a set of \(N\) constant \(n \times n\) hermitian and nonnegative matrices \(M_j\) with respective positive ranks \(m_j\), where \(N\) is a nonnegative integer. We say that \(S\) belongs to the Marchenko class \(\mathcal{M}\) if \(S\) satisfies \((1, 2, 3_a, 4_a)\), i.e. if \(S\) satisfies the four properties \((1), (2), (3_a), (4_a)\) specified in Definition 4.2.

As stated in Theorem 5.1 in the next chapter we prove that if \(D\) belongs to the Faddeev class then there exists and uniquely exists a corresponding scattering data set \(S\) in the Marchenko class. We provide the steps to construct \(S\) when \(D\) is given. Furthermore, we show that, for each scattering data set \(S\) in the Marchenko class, there exists and uniquely exists a corresponding input data set \(D\) in the Faddeev class. We provide the steps to construct \(D\) when \(S\) is given.

Let us comment on the condition \((3_a)\) of Definition 4.2, which states that the physical solution \(\Psi(k, x)\) constructed from the scattering data set \(S\) must satisfy the boundary condition (2.4) where \(A\) and \(B\) are the boundary matrices constructed from \(S\). One may then question why we do not include in Definition 4.5 a separate condition that the bound-state solutions \(\Psi_j(x)\) constructed as in (9.8) are required to satisfy the boundary condition (2.4). The answer is given in the following proposition, which indicates that the fulfilment of \((3_a)\) actually implies that the constructed bound-state solutions indeed satisfy the boundary
Proposition 4.6 Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ belongs to the Marchenko class, i.e. it satisfies the conditions (1), (2), (3a), (4a) stated in Definition 4.5. Then, we also have $(V_a)$ holding. We clarify this statement as follows. The scattering matrix $S(k)$ and the bound-state data $\{\kappa_j, M_j\}_{j=1}^N$ given in $S$ yield $n \times n$ matrices $\Psi_j(x)$ as in (9.8), where each $\Psi_j(x)$ is a solution to (2.1) at $k = i\kappa_j$. As stated in (3a) of Definition 4.2, the scattering matrix given in $S$ yields a pair of matrices $A$ and $B$ (modulo an invertible matrix) satisfying (2.5) and (2.6). Our statement $(V_a)$ is equivalent to saying that (2.4) is satisfied if we use in (2.4) the quantities $\Psi_j(x)$, $A$, and $B$ constructed from the quantities appearing in $S$.

PROOF: The result follows from Proposition 18.2(a). □

Let us comment on Proposition 4.6. It states that, if the scattering data set $S$ belongs to the Marchenko class, then (3a) in Definition 4.2 implies $(V_a)$ of Definition 4.3. As we show in Example 26.5, if $S$ does not satisfy (3a), and hence if $S$ does not belong to the Marchenko class, it may still be possible that $(V_a)$ holds even though (3a) does not hold. We remark that, in the absence of bound states, the condition $(V_a)$ stated in Proposition 4.6 becomes redundant.
5. THE CHARACTERIZATION OF THE SCATTERING DATA

Next we present one of our main results by showing that the four conditions given in Definition 4.5 for the Marchenko class form a characterization of the scattering data sets $S$ that have one-to-one correspondence with the input data sets $D$ in the Faddeev class specified in Definition 4.1.

**Theorem 5.1** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Consider also an input data set $D$ as in (4.1) consisting of an $n \times n$ matrix potential $V$ satisfying (2.2) and (2.3) and a pair of constant $n \times n$ matrices $A$ and $B$ satisfying (2.5) and (2.6). Let $F$ be the Faddeev class of input data sets $D$, as specified in Definition 4.1. Let $M$ be the Marchenko class of scattering data sets $S$, as specified in Definition 4.5. Then, we have the following:

(a) For each $D \in F$, there exists and uniquely exists a scattering data set $S \in M$.

(b) Conversely, for each $S \in M$, there exists and uniquely exists an input data set $D \in F$, where $A$ and $B$ are uniquely determined up to a postmultiplication by an invertible $n \times n$ matrix $T$.

(c) Let $\tilde{S}$ be the scattering data set corresponding to $D$ given in (b), where $D$ is constructed from the scattering data set $S$. Then, we must have $\tilde{S} = S$, i.e. the scattering data set constructed from $D$ must be equal to the scattering data set used to construct $D$.

(d) The characterization outlined in (a)-(c) can equivalently be stated as follows. A set $S$ as in (4.2) is the scattering data set corresponding to an input data set $D$ in the Faddeev class if and only if $S$ satisfies $\text{(1)}, \text{(2)}, \text{(3)}_a$, and $\text{(4)}_a$ stated in Definition 4.5.

**PROOF:** First, when $D$ belongs to the Faddeev class, the corresponding $S$ belongs to the Marchenko class, as proved in Theorem 15.10. Let us now turn to the inverse problem. The unique construction of $D$ from $S$ is outlined in Chapter 16, and in particular the uniqueness
of the construction is proved in Proposition 16.2. As indicated in Proposition 16.1(a), when (I) is satisfied, the Marchenko equation (13.1) is uniquely solvable and hence its solution $K(x, y)$ is uniquely constructed for each $x \in \mathbb{R}^+$. The condition (4a) assures that the Marchenko equation (13.1) is uniquely solvable also at $x = 0$. The potential $V(x)$ is then uniquely constructed from $K(x, y)$ as in (10.4). As indicated in Proposition 16.11(a) the condition (1) ensures that the constructed potential $V(x)$ satisfies (2.2), and as indicated in Proposition 16.11(a) the condition (2) ensures that the constructed potential $V(x)$ satisfies (2.3). The boundary matrices $A$ and $B$ are uniquely constructed (modulo $T$), as indicated in Proposition 16.9, and they satisfy (2.5) and (2.6). The physical solution $\Psi(k, x)$ is constructed as in (9.4). As indicated in Proposition 16.11(b), the constructed $\Psi(k, x)$ satisfies (2.1), and the condition (3a) assures that $\Psi(k, x)$ satisfies the boundary condition (2.4). We still need to show that, in the direct scattering problem the constructed $D$ yields the same $S$ that is used as input to the inverse scattering problem. Mathematically speaking, we have $D = \mathcal{D}^{-1}(S)$ and we need to show that $D(D) = S$. Let us use a tilde to denote any quantity constructed by using $D$ as input in the direct scattering map. Thus, we have $\tilde{S} := \mathcal{D}(D)$ and we would like to show that $\tilde{S} = S$. The proof is obtained as follows.

We claim that the Jost solution $f(k, x)$ constructed from $S$ as in (10.6) is the same as the Jost solution $\tilde{f}(k, x)$ obtained from $V$ as in [2,5,24]

$$\tilde{f}(k, x) = e^{ikx}I + \frac{1}{k} \int_x^\infty dy \left[ \sin k(y - x) \right] V(y) \tilde{f}(k, y). \tag{5.1}$$

This is true because, as indicated in Proposition 16.11(a), $f(k, x)$ satisfies (2.1) and (9.1) when $V$ is used as the potential in (2.1). It is already known [2,5,24] that $\tilde{f}(k, x)$ given in (5.1) satisfies (2.1) and (9.1) when $V$ is used as the potential in (2.1). Thus, as the solution $\tilde{f}(k, x)$ to (5.1) is unique, we have $\tilde{f}(k, x) \equiv f(k, x)$. We then have the equivalence of the Jost matrices, i.e. $\tilde{J}(k) \equiv J(k)$ because both $J(k)$ and $\tilde{J}(k)$ are constructed as in (9.2) using the same $f(k, x)$ and the same $A$ and $B$ as input. As (9.3) implies, we must also have the equivalence of the scattering matrices, i.e. $\tilde{S}(k) \equiv S(k)$. Then, from (4.6) it follows that we have the equivalence of the large $k$-limits of the scattering matrix, i.e. $\tilde{S}_\infty = S_\infty$. Because of (4.7), we get the equivalence $\tilde{F}_s(y) \equiv F_s(y)$. We already have
\( \tilde{K}(x, y) \equiv K(x, y) \), where \( \tilde{K}(x, y) \) constructed from \( \tilde{f}(k, x) \), or equivalently from \( f(k, x) \), as in (10.1). For each \( x \in [0, +\infty) \), as indicated in Proposition 16.1(b), the quantity \( K(x, y) \) is integrable in \( y \in \mathbb{R}^+ \). Thus, we can view the Marchenko integral equation (13.1) with \( K(x, y) \) as input and \( F(y) \) as the unknown. As indicated in the proof of Theorem 13.2, the Marchenko equation then yields \( F(y) \) uniquely with the properties stated in Proposition 16.7. Consequently, we have \( \tilde{F}(y) \equiv F(y) \). Then, from (4.7) and (4.12) we obtain

\[
\sum_{j=1}^{N} \tilde{M}_j^2 e^{-\tilde{\kappa}_j y} = \sum_{j=1}^{N} M_j^2 e^{-\kappa_j y}, \quad y \in \mathbb{R}^+.
\] (5.2)

Since (5.2) holds for all \( y \in \mathbb{R}^+ \), we can use a recursive argument to prove that \( \tilde{N} = N \) and also \( \tilde{\kappa}_j = \kappa_j \) for \( j = 1, \ldots, N \). Because \( \kappa_j \)-values are distinct, the functions \( e^{-\kappa_j y} \) are linearly independent in \( y \in \mathbb{R}^+ \), and hence (5.2) also yields \( \tilde{M}_j^2 = M_j^2 \). Then, from the nonnegativity of \( M_j \) it follows that we must have \( \tilde{M}_j = M_j \). Thus, we have proved that \( \tilde{S} = S \). 

Let us now provide some remarks on the four characterization conditions appearing in Definition 4.5. First, let us make some remarks concerning (1) and (4a) of Definition 4.2. As shown in Example 26.5, even if the unitarity of \( S(k) \) in (1) does not hold, corresponding to \( S \) we may be able to construct a unique \( D \) in the Faddeev class. However, the scattering data set \( \tilde{S} \) corresponding to \( D \) may not agree with \( S \), a point emphasized in the proof of Theorem 5.1. Concerning the symmetry relation of (1), namely \( S(-k) = S(k)^\dagger \) for \( k \in \mathbb{R} \) stated in the first equality in (4.4), we have the following remark. If that symmetry relation does not hold, the hermitian property of various constructed quantities fails, and the constructed potential may not be hermitian and the constructed Schrödinger operator may no longer be selfadjoint. The condition that \( F_s(y) \) is bounded and integrable in \( y \in \mathbb{R} \) appearing in (1) assures that the Marchenko integral operator related to (13.1) is compact and the Marchenko integral equation (13.1), for each \( x \in \mathbb{R}^+ \), has a unique solution \( K(x, y) \) for \( y \in (x, +\infty) \) and the potential \( V(x) \) can be constructed. However, without the additional assumption (4a), it is possible that, for \( x = 0 \), the Marchenko integral equation
may not have a solution and hence the constructed potential may be singular at \( x = 0 \), as illustrated in Examples 26.7 and 26.14. The property (4.5) of (1) enables us to construct the boundary matrices \( A \) and \( B \).

Let us now comment on (2) of Definition 4.2. The condition (2) assures us that the constructed potential \( V \) belongs to class \( L^1_1(\mathbb{R}^+) \). Let us remark that the characterization result stated in Theorem 5.1 involving the Faddeev class and the Marchenko class still holds if we modify the definitions of the Faddeev class and the Marchenko class in such a way that we replace the \( L^1_1(\mathbb{R}^+) \) condition given in (2.3) by

\[
\int_0^\infty dx (1 + x)^p |V(x)| < +\infty, \tag{5.3}
\]

and replace (4.8) by

\[
\int_0^\infty dy (1 + y)^p |F'(y)| < +\infty, \tag{5.4}
\]

where \( p \) is any positive integer. We can summarize this by stating that the \( L^1_1(\mathbb{R}^+) \) characterization provided in our monograph extends to the \( L^1_p(\mathbb{R}^+) \) characterization for any positive integer \( p \). In fact, the idea behind the \( L^1_2(\mathbb{R}) \) characterization given by Deift and Trubowitz in [16] for the full-line scalar Schrödinger equation was to modify the \( L^1_1(\mathbb{R}) \) characterization given by Faddeev in [20].
6. EQUIVALENTS FOR SOME CHARACTERIZATION CONDITIONS

The following result shows that the properties \((4_a), (4_b), (4_c), (4_d), (4_e)\) presented in Definition 4.2 are all equivalent.

**Proposition 6.1** Consider a scattering data set \(S\) as in (4.2), which consists of an \(n \times n\) scattering matrix \(S(k)\) for \(k \in \mathbb{R}\), a set of \(N\) distinct positive constants \(\kappa_j\), and a set of \(N\) constant \(n \times n\) hermitian and nonnegative matrices \(M_j\) with respective positive ranks \(m_j\), where \(N\) is a nonnegative integer. Assume that \(S\) satisfies (I) of Definition 4.3. Then, the property \((4_a)\) is equivalent to any of the four properties \((4_c), (4_d), (4_e), (4_b)\).

**PROOF:** The equivalences among \((4_c), (4_d),\) and \((4_e)\) are established in (c) and (d) of Proposition 15.3. Note that \((4_c)\) and \((4_b)\) are equivalent because, comparing \((4.14)\) and \((4.13)\) we see that each row of the matrix solution \(K(0,y)\) to \((4.13)\) is a row vector solution \(X(y)\) to \((4.14)\) and, conversely, each row vector solution \(X(y)\) is a row of the matrix solution \(K(0,y)\) to \((4.13)\). Finally, the equivalence of \((4_a)\) and \((4_b)\) is established in Proposition 16.1(c). [1]

In the next proposition we show the equivalences among \((III_a), (III_b),\) and \((III_c)\) of Definition 4.3.

**Proposition 6.2** Consider a scattering data set \(S\) as in (4.2), which consists of an \(n \times n\) scattering matrix \(S(k)\) for \(k \in \mathbb{R}\), a set of \(N\) distinct positive constants \(\kappa_j\), and a set of \(N\) constant \(n \times n\) hermitian and nonnegative matrices \(M_j\) with respective positive ranks \(m_j\), where \(N\) is a nonnegative integer. Assume that \(S\) satisfies (I) of Definition 4.3. Then, the properties \((III_a), (III_b),\) and \((III_c)\) are equivalent.

**PROOF:** The result is a direct consequence of Proposition 15.11. [1]

Next we show that, when the scattering data set \(S\) belongs to the Marchenko class, it is possible to replace \((3_a)\) and \((4_a)\) by various equivalent conditions.

**Proposition 6.3** Consider a scattering data set \(S\) as in (4.2), which consists of an \(n \times n\) scattering matrix \(S(k)\) for \(k \in \mathbb{R}\), a set of \(N\) distinct positive constants \(\kappa_j\), and a set of
$N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ satisfies the three conditions (1), (2), (4a) stated in Definition 4.2. Then, the condition (3a) for the Marchenko class is equivalent to (3b) of Definition 4.2.

PROOF: Note that (3a) in Definition 4.2 is given in (4.9), where where $A$ and $B$ are the boundary matrices constructed from $S$ as described in Proposition 16.9 and $\Psi(k,x)$ is the physical solution constructed as in (9.4). Using (9.4) in (4.9) we obtain
\[- \left[ B^\dagger f(-k,0) - A^\dagger f'(k,0) \right] - \left[ B^\dagger f(-k,0) - A^\dagger f'(k,0) \right] S(k) = 0. \tag{6.1}\]

With the help of (9.2) we can construct the Jost matrix $J(k)$, and then we can write (6.1) in terms of the constructed $J(k)$ constructed as
\[- J(k)^\dagger - J(-k)^\dagger S(k) = 0, \quad k \in \mathbb{R}. \tag{6.2}\]

As a result of the first equality in (4.4), we can replace $S(k)$ by $S(-k)^\dagger$ and hence rewrite (4.23) as
\[- \left[ J(-k) + S(k) J(k) \right]^\dagger = 0, \quad k \in \mathbb{R}, \tag{6.3}\]
which is equivalent to (4.10).

In the next proposition we show that the property (3a) is equivalent to the combination of the two properties (I(a)) and (V(a)).

**Proposition 6.4** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ satisfies (1), (2), and (4a) of Definition 4.2. Then, the property (3a) is equivalent to the combination of the two properties (I(a)) and (V(a)).

PROOF: The result is a direct consequence of Proposition 18.2. □
Next we show that the properties \((V_a), (V_b), (V_c), (V_d), (V_e), (V_f), (V_g), (V_h)\) listed in Definition 4.3 are all equivalent.

**Proposition 6.5** Consider a scattering data set \(S\) as in (4.2), which consists of an \(n \times n\) scattering matrix \(S(k)\) for \(k \in \mathbb{R}\), a set of \(N\) distinct positive constants \(\kappa_j\), and a set of \(N\) constant \(n \times n\) hermitian and nonnegative matrices \(M_j\) with respective positive ranks \(m_j\), where \(N\) is a nonnegative integer. Assume that \(S\) satisfies (1), (2), and (4a) of Definition 4.2 as well as (IIIa). Then, the property \((V_a)\) appearing in Proposition 4.6 is equivalent to any of the seven properties \((V_b), (V_c), (V_d), (V_e), (V_f), (V_g), (V_h)\).

**PROOF:** The equivalences among \((V_c), (V_d), (V_e)\) are established in (a) and (b) of Proposition 15.7. The equivalences among \((V_f), (V_g), (V_h)\) are given in (a) and (b) of Proposition 15.6. By (e) and (f) of Proposition 18.3 we know that \((V_a)\) and \((V_b)\) are equivalent. Since (IIIa) is also satisfied, by Proposition 6.4 we know that the combination of (IIIa) and \((V_a)\) is equivalent to (3a). However, then \(S\) satisfies all the four conditions (1), (2), (3a), (4a), in which case we know that Proposition 18.8 implies that \((V_c)\) and \((V_f)\) are equivalent and we also know that Proposition 15.8 implies that \((V_c)\) is satisfied. We also know from Proposition 18.7 that \((V_f)\) implies that \((V_b)\) is satisfied. Thus, the proof is complete. \(\blacksquare\)

The following result provides various other properties equivalent to the combination of (3a) and (4a) on \(S\) stated in Definition 4.2.

**Proposition 6.6** Consider a scattering data set \(S\) as in (4.2), which consists of an \(n \times n\) scattering matrix \(S(k)\) for \(k \in \mathbb{R}\), a set of \(N\) distinct positive constants \(\kappa_j\), and a set of \(N\) constant \(n \times n\) hermitian and nonnegative matrices \(M_j\) with respective positive ranks \(m_j\), where \(N\) is a nonnegative integer. Assume that \(S\) satisfies the four conditions (1), (2), (3a), (4a) of Definition 4.5, i.e. that \(S\) belongs to the Marchenko class, and hence \(S\) correspond to a unique input data set \(D\) in the Faddeev class. Then, it is possible to make any combination of the following changes in (3a) and (4a) in such a way that each modified scattering data set set still uniquely corresponds to the original set \(D\).
(a) One can replace \((3_a)\) by \((3_b)\).

(b) One can replace \((4_a)\) by any one of the four conditions \((4_b), (4_c), (4_d), (4_e)\).

(c) One can replace \((3_a)\) by two conditions, the first of which is any of the three conditions \((III_a), (III_b), (III_c)\), and the second of which is any of the eight conditions \((V_a), (V_b), (V_c), (V_d), (V_e), (V_f), (V_g), (V_h)\).

PROOF: We note that (a) follows from Proposition 6.3, (b) follows from Proposition 6.1, and (c) follows from Propositions 6.2, 6.4, and 6.5. ■
7. ALTERNATE CHARACTERIZATIONS OF THE SCATTERING DATA

In this chapter we provide two alternate sets of characterizations of the scattering data set $S$ in the Marchenko class in Definition 4.5 so that it corresponds to the input data set $D$ belonging to the Faddeev class introduced in Definition 4.1.

First, in Theorems 7.1-7.6 we present six different versions of a characterization equivalent to the characterization given in Theorem 5.1.

Next, in Theorems 7.9 and 7.10 we present two different versions of a different characterization based on the use of Levinson’s theorem, where the details of this characterization are developed in Chapter 21.

A slight drawback of using $(3_a)$ of Definition 4.2 as a characterization condition is that, from the given scattering data set $S$, one first needs to construct the boundary matrices $A$ and $B$ as well as the physical solution $\Psi(k, x)$. Using the equivalent statements presented in Proposition 6.6 for the characterization conditions $(3_a)$ and $(4_a)$, it is possible to arrange various equivalent formulations of the characterization of $S$ belonging to the Marchenko class. For example, we can assemble a set of five conditions so that they will form the characterization in the general selfadjoint case, generalizing the characterization presented in the Dirichlet case in the seminal work [2]. In this formulation, the emphasis on the conditions is on the Fourier transform of the scattering matrix. These five conditions can directly be checked without first having to construct the boundary matrices $A$ and $B$ and the physical solution $\Psi(k, x)$. The proof is omitted because the result directly follows from Proposition 6.6.

**Theorem 7.1** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. The set $S$ is the scattering data set corresponding to a unique input data set $D$ as in (4.1) in the Faddeev class specified in Definition 4.1 if and
only if \( S \) satisfies the five conditions \((1), (2), (III_a), (4_c), (V_c)\).

It is possible to have another characterization of the scattering data by modifying the condition \((V_c)\) of Theorem 7.1 and by replacing it with \((V_f)\) and this is done in the next theorem. Again the proof is omitted because the result directly follows from Proposition 6.6.

**Theorem 7.2** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. The set \( S \) is the scattering data set corresponding to a unique input data set \( D \) as in (4.1) in the Faddeev class specified in Definition 4.1 if and only if \( S \) satisfies the five conditions consisting of \((1), (2), (III_a), (4_c), (V_f)\).

In the next theorem, again with the help of Propositions 6.6, we present an alternate set of five characterization conditions on the scattering data set \( S \), where the emphasis on the conditions is on the scattering matrix itself. We again omit the proof because the theorem is a direct consequence of Propositions 6.6.

**Theorem 7.3** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. The set \( S \) is the scattering data set corresponding to a unique input data set \( D \) as in (4.1) in the Faddeev class specified in Definition 4.1 if and only if \( S \) satisfies the five conditions consisting of \((1), (2), (III_b), (4_d), (V_d)\).

It is possible to have still another characterization of the scattering data by modifying the condition \((V_d)\) of Theorem 7.3 and by replacing it with \((V_g)\) and this is done in the next theorem. The proof is again omitted because the result is actually a corollary of Proposition 6.6.

**Theorem 7.4** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \)
scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. The set $S$ is the scattering data set corresponding to a unique input data set $D$ as in (4.1) in the Faddeev class specified in Definition 4.1 if and only if $S$ satisfies the five conditions consisting of if and only if $S$ satisfies the five conditions consisting of (1), (2), (III), (4), and (V).

We next present another characterizations of the scattering data, which is a direct consequence of Proposition 6.6, and hence by omitting the proof.

**Theorem 7.5** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. The set $S$ is the scattering data set corresponding to a unique input data set $D$ as in (4.1) in the Faddeev class specified in Definition 4.1 if and only if $S$ satisfies the following five conditions consisting of (1), (2), (III), (4), and (V).

It is possible to have still another characterization of the scattering data by modifying the condition (V) of Theorem 7.4 and by replacing it with (V) and this is done in the next theorem. Again the proof is omitted because the result directly follows from Proposition 6.6.

**Theorem 7.6** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. The set $S$ is the scattering data set corresponding to a unique input data set $D$ as in (4.1) in the Faddeev class specified in Definition 4.1 if and only if $S$ satisfies the following four conditions consisting of (1), (2), (III), (4), and (V).

Next, we present a different characterization based on the use of Levinson’s theorem. Levinson’s theorem is given in Theorem 21.1 and the details of the derivation of this
characterization are presented in Chapter 21. Before we state the characterization and one of its equivalent, we introduce the property \((L)\) in the following definition.

**Definition 7.7** The set of conditions \((L)\) for the scattering data set \(S\) in (4.2) is defined as follows:

\((L)\) We say that \(S\) satisfies the property \((L)\) if the scattering matrix \(S(k)\) in \(S\) is continuous for \(k \in \mathbb{R}\) and Levinson’s theorem (21.5) is satisfied with \(\mu, n_D,\) and \(N\) coming from \(S\). Here, \(\mu\) is the algebraic (and geometric) multiplicity of the eigenvalue +1 of the zero-energy scattering matrix \(S(0)\), \(n_D\) is the algebraic (and geometric) multiplicity of the eigenvalue –1 of the hermitian matrix \(S_\infty\) defined in (4.6), and \(N\) is the nonnegative integer equal to the sum of the ranks \(m_j\) of the matrices \(M_j\) appearing in \(S\).

We also introduce three new properties, namely, \((4c,2)\), \((4d,2)\), and \((4e,2)\) that we need in the characterizations using Levinson’s theorem. Actually, they resemble the respective properties \((4c)\), \((4d)\), and \((4e)\) of Definition 4.2, but involving \(L^2(\mathbb{R}^+)\) and \(H^2(\mathbb{C}^+)\), respectively, instead of \(L^1(\mathbb{R}^+)\) and \(\hat{L}^1(\mathbb{C}^+)\).

**Definition 7.8** The properties \((4c,2)\), \((4d,2)\), and \((4e,2)\) for the scattering data set \(S\) in (4.2) are defined as follows:

\((4c,2)\) The only square-integrable solution \(X(y)\), which is a row vector with \(n\) square-integrable components in \(y \in \mathbb{R}^+\), to the linear homogeneous integral equation

\[
X(y) + \int_0^\infty dz X(z) F(z + y) = 0, \quad y \in \mathbb{R}^+, \quad (7.1)
\]

is the trivial solution \(X(y) \equiv 0\). Here, \(F(y)\) is the quantity defined in (4.12).

\((4d,2)\) The only solution \(\hat{X}(k)\) to the system

\[
\begin{align*}
\hat{X}(ik_j) M_j &= 0, \quad j = 1, \ldots, N, \\
\hat{X}(-k) + \hat{X}(k) S(k) &= 0, \quad k \in \mathbb{R},
\end{align*}
(7.2)
\]

where \(\hat{X}(k)\) is a row vector with \(n\) components belonging to the Hardy space \(H^2(\mathbb{C}^+)\), is the trivial solution \(\hat{X}(k) \equiv 0\).
The only solution \( h(k) \) to the system
\[
\begin{cases}
M_j h(i\kappa_j) = 0, & j = 1, \ldots, N, \\
h(-k) + S(k) h(k) = 0, & k \in \mathbb{R},
\end{cases}
\]
where \( h(k) \) is a column vector with \( n \) components belonging to the Hardy space \( H^2(\mathbb{C}^+) \), is the trivial solution \( h(k) \equiv 0 \).

Next we present the characterization of the scattering data utilizing Levinson’s theorem.

**Theorem 7.9** Consider a scattering data set \( \mathbf{S} \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) nonnegative, hermitian matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. The set \( \mathbf{S} \) is the scattering data set corresponding to a unique input data set \( \mathbf{D} \) as in (4.1) in the Faddeev class specified in Definition 4.1 if and only if \( \mathbf{S} \) satisfies the four properties consisting of (1) and (2) of Definition 4.2, \((4e,2)\) of Definition 7.8, and \((L)\) of Definition 7.7.

**PROOF:** The proof is given at the end of Chapter 21.

A characterization equivalent to the one given in Theorem 7.9 is presented next. In this equivalent characterization, the condition \((4e,2)\) of Theorem 7.9 is replaced with either of \((4e,2)\) and \((4e,2)\) of Definition 7.8.

**Theorem 7.10** Consider a scattering data set \( \mathbf{S} \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. The set \( \mathbf{S} \) is the scattering data set corresponding to a unique input data set \( \mathbf{D} \) as in (4.1) in the Faddeev class specified in Definition 4.1 if and only if \( \mathbf{S} \) satisfies the following conditions consisting of (1), (2) of Definition 4.2, \((L)\) of Definition 7.7, and either condition \((4e,2)\) or condition \((4d,2)\) of Definition 7.8.

**PROOF:** The proof follows from Proposition 21.2 and Theorem 7.9.

Let us comment on Levinson’s theorem and the property \((L)\) used in Theorem 7.7.
We know from Theorem 21.1, which was proved in Theorem 9.3 of [9], that if the scattering data set \( S \) containing \( S(k) \) belongs to the Marchenko class, then the change in the argument of the determinant of \( S(k) \) as \( k \) moves from \( k = +\infty \) to \( k = 0^+ \) is related to the nonnegative integer \( N \) given in (4.3). This result, mathematically related to an argument principle, is known as Levinson’s theorem. The proof of (21.5) given in [9] essentially uses the property stated in (3b), and it has an important consequence; namely, in the Marchenko class of scattering data set, the scattering matrix \( S(k) \) itself reveals \( N \) even though in general it does not reveal any further information on the bound-states. In other words, in general, neither \( \kappa_j \)-values nor \( M_j \) appearing in (4.2), and in fact not even \( N \) itself, can be extracted from \( S(k) \). As seen from (21.5), given \( S(k) \), one can extract the nonnegative integer \( \mu \) as the multiplicity of the eigenvalue +1 of \( S(0) \), extract the nonnegative integer \( n_D \) as the multiplicity of the eigenvalue \(-1\) of \( S_\infty \), extract the positive integer \( n \) from the size of the \( n \times n \) matrix \( S(k) \), and also evaluate the left-hand side of (21.5) directly from \( S(k) \). Thus, using (21.5) we obtain the value of \( N \) associated with the given \( S(k) \). In case the value of \( N \) predicted by (21.5) turns out to be negative, we know that the given \( S(k) \) cannot be a part of the scattering data set \( S \) belonging to the Marchenko class no matter how we choose \( \{\kappa_j, M_j\}_{j=1}^N \) in \( S \) given in (4.2). If the value of \( N \) predicted by (21.5) turns out to be zero, then we know that we cannot have any bound states in \( S \) if it must belong to the Marchenko class. If the value of \( N \) predicted by (21.5) turns out to be one, then we know that we must have exactly one bound state of multiplicity one if \( S(k) \) is to be a part of the scattering data \( S \) in the Marchenko class. In other words, we must have \( N = 1 \) and \( M_1 \) must have rank one. There may be certain restrictions on the positive constant \( \kappa_1 \) and the nonnegative, hermitian, rank-one matrix \( M_1 \) so that \( S \) belongs to the Marchenko class. If the value of \( N \) predicted by (21.5) turns out to be two, then we may be able to choose \( \{\kappa_j, M_j\}_{j=1}^N \) in \( S \) given in (4.2) in two different ways. In the first possibility we could have \( N = 1 \), the value of \( \kappa_1 \) could be chosen as a positive constant, and \( M_1 \) could be chosen as a nonnegative, hermitian matrix of rank two. In the second possibility we could have \( N = 2 \), the values of \( \kappa_1 \) and \( \kappa_2 \) could be chosen as two distinct positive constants, and and \( M_1 \)
and $M_2$ could be chosen as two nonnegative, hermitian matrices of rank one. As shown in some of the examples in Section 26, not all these choices may be viable and the condition of being in the Marchenko class may restrict some of these choices. Clearly, the same argument applies in how many different ways we might be able to choose $\{\kappa_j, M_j\}_{j=1}^N$ in $S$ when the value of $N$ predicted by (21.5) is three or higher, but again the constraints for belonging to the Marchenko class may restrict some of the available choices. In Section 26 various illustrative examples are provided to indicate how $S(k)$ predicts $N$ via (21.5) and how such a restriction and other restrictions play a role on the bound-state information for the scattering data set $S$ to belong to the Marchenko class.
8. ANOTHER CHARACTERIZATION OF THE SCATTERING DATA

In this chapter we provide yet another characterization of the scattering data set \( S \) in the Marchenko class in Definition 4.5 so that it corresponds to the input data set \( D \) belonging to the Faddeev class introduced in Definition 4.1.

This new characterization has some similarities and differences compared to the characterization given in Section 5 and the alternate characterizations given in Section 7. Related to this new characterization, the construction of the potential in the solution to the inverse problem is the same as in the previous characterizations; namely, one constructs the potential by solving the Marchenko equation. Hence, the conditions (1), (2), (4\(_a\)) of Definition 4.5 in the first characterization, the conditions (1), (2), (4\(_c\)) in the alternate characterizations of Section 7, and the conditions (I), (2), (4\(_c\)) in this new characterization are essentially used to construct the potential. This new characterization differs from the earlier ones in regard to the satisfaction of the boundary condition by the physical solution \( \Psi(k, x) \) and by the bound-state matrix solutions \( \Psi_j(x) \). It is based on the alternate solution to the inverse problem as summarized in Section 23. It uses six conditions, where five of the conditions are already listed in Definitions 4.2 and 4.3; namely, (I), (2), (4\(_c\)), either of \((V_e)\) or \((V_h)\), and (VI). It also uses the condition (A), which is not listed in Definitions 4.2 and 4.3. The condition (A), stated in the following theorem, somehow resembles (III\(_c\)) of Proposition 4.3, but there are also some major differences. In (III\(_c\)) a solution is sought to the homogeneous Riemann-Hilbert problem (4.19) as a column vector with \( n \) components where each of those components belongs to \( H^2(C^-) \), and the only solution is expected to be the trivial solution \( h(k) \equiv 0 \). On the other hand, in (A) one solves a nonhomogeneous Riemann-Hilbert problem and the solution is sought as a column vector where each of the \( n \) components belongs the Hardy space \( H^2(C^+) \), and certainly the corresponding solution is in general nontrivial and such a solution is not required to be unique. The condition (VI), which is the continuity of the scattering matrix \( S(k) \), is mainly needed to prove that the physical solution satisfies the boundary condition.
Theorem 8.1 Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. The set $S$ is the scattering data set corresponding to a unique input data set $D$ as in (4.1) in the Faddeev class specified in Definition 4.1 if and only if $S$ satisfies the following six conditions: (2) and (4c) of Definition 4.2, (I), (VI), and either one of (Ve) or (Vh) of Definition 4.3, and the following condition named (A):

(A) Consider the nonhomogeneous Riemann-Hilbert problem given by

$$h(k) + S(-k) h(-k) = g(k), \quad k \in \mathbb{R},$$

where the nonhomogeneous term $g(k)$ belongs to a dense subset $\tilde{\mathcal{Y}}$ of the vector space $\mathcal{Y}$ of column vectors with $n$ square-integrable components and satisfying $g(-k) = S(k) g(k)$ for $k \in \mathbb{R}$. Then, for each such given $g(k)$, the equation (8.1) has a solution $h(k)$ as a column vector with $n$ components belonging to the Hardy space $H^2(\mathbb{C}^+)$. 

PROOF: The proof is given in Section 23, after the proof of Proposition 23.6. ■
9. THE SOLUTION TO THE DIRECT PROBLEM

In this chapter we summarize the solution to the direct scattering problem of obtaining the scattering data set $S$ given in (4.2) from the input data set $D$ given in (4.1). We assume that $D$ belongs to the Faddeev class specified in Definition 4.1. The proofs and details will be provided in later chapters. We postpone the proof that $S$ belongs to the Marchenko class specified in Definition 4.5 when $D$ belongs to the Faddeev class, because that proof will be given in Theorem 15.10.

The relevant existence and uniqueness in the construction of $S$ from $D$ are implicit in each step described below.

(a) When our input data set $D$ belongs to the Faddeev class, regardless of the boundary matrices $A$ and $B$, the matrix Schrödinger equation (2.1) possesses the $n \times n$ matrix-valued Jost solution $f(k, x)$ satisfying the asymptotic condition

$$f(k, x) = e^{ikx}[I + o(1)], \quad x \to +\infty. \quad (9.1)$$

The existence and uniqueness of $f(k, x)$ as well as its relevant properties are summarized in Proposition 10.1.

(b) In terms of the boundary matrices $A$ and $B$ in $D$ and the Jost solution $f(k, x)$ obtained in (a), we construct the Jost matrix $J(k)$, an $n \times n$ matrix-valued function of $k$, as

$$J(k) := f(-k^*, 0)^\dagger B - f'(-k^*, 0)^\dagger A, \quad k \in \mathbb{R}, \quad (9.2)$$

where the asterisk denotes complex conjugation. When $D$ belongs to the Faddeev class, the relevant properties of the Jost matrix are summarized in Proposition 10.2. The redundant appearance of $k^*$ instead of $k$ in (9.2) when $k \in \mathbb{R}$ is useful in extending the Jost matrix analytically from $k \in \mathbb{R}$ to $k \in \mathbb{C}^+$.

(c) In terms of the Jost matrix $J(k)$, uniquely obtained from $D$ as indicated in (9.2), we construct the scattering matrix $S(k)$, an $n \times n$ matrix-valued function of $k$, as

$$S(k) := -J(-k) J(k)^{-1}, \quad k \in \mathbb{R}. \quad (9.3)$$
When $D$ belongs to the Faddeev class, the relevant properties of $S(k)$ are summarized in Proposition 10.3.

(d) In terms of the Jost solution $f(k, x)$ obtained in (a) and the scattering matrix $S(k)$ obtained in (c), we construct the physical solution $\Psi(k, x)$ as

$$\Psi(k, x) := f(-k, x) + f(k, x) S(k), \quad k \in \mathbb{R}. \quad (9.4)$$

In Proposition 10.5, we show that the $n \times n$ matrix-valued $\Psi(k, x)$ is a solution to (2.1) and satisfies the boundary condition (2.4) and we also summarize the relevant properties of $\Psi(k, x)$.

(e) Instead of constructing the physical solution via (9.4), one can alternatively construct it in an equivalent way as follows: When our input data set $D$ belongs to the Faddeev class, as indicated in Proposition 10.4, the matrix Schrödinger equation (2.1) possesses a unique $n \times n$ matrix-valued solution $\varphi(k, x)$ satisfying the initial conditions [5]

$$\varphi(k, 0) = A, \quad \varphi'(k, 0) = B, \quad k \in \mathbb{R}. \quad (9.5)$$

where $A$ and $B$ are the matrices appearing in (2.4)-(2.6) and (4.1). The solution $\varphi(k, x)$ is known as the regular solution because it is entire in $k$ for each fixed $x \in \mathbb{R}^+$.

In terms of the regular solution $\varphi(k, x)$ and the Jost matrix $J(k)$ appearing in (9.2) we can introduce the physical solution as

$$\Psi(k, x) = -2ik \varphi(k, x) J(k)^{-1}. \quad (9.6)$$

One can show that the expressions given in (9.4) and (9.6) are equivalent, and this can be shown by using the relationship given in (3.5) of [5], i.e.

$$\varphi(k, x) = \frac{1}{2ik} f(k, x) J(-k) - \frac{1}{2ik} f(-k, x) J(k), \quad (9.7)$$

where we recall that $f(k, x)$ is the Jost solution appearing in (9.1).

(f) As indicated in Proposition 10.2, the determinant of the Jost matrix $J(k)$ has an analytic extension from $k \in \mathbb{R}$ to $k \in \mathbb{C}^+$ and that determinant is nonzero in $\mathbb{C}^+$.
except perhaps at a finite number of $k$-values on the positive imaginary axis. We assume that there are $N$ such $k$-values occurring at $k = i\kappa_j$ and use $m_j$ to denote the multiplicity of the zero of $\det[J(k)]$ at $k = i\kappa_j$. Thus, the set of $N$ distinct values of positive $\kappa_j$ appearing in $S$ is uniquely constructed from $D$. Furthermore, the positive integers $m_j$ are also uniquely determined by $D$ as a result of the unique construction of $J(k)$. It is possible that the determinant of $J(k)$ never vanishes in $k \in \mathbb{C}^+$, in which case we have $N = 0$. When $N = 0$, the scattering data set $S$ given in (4.2) consists of $S(k)$ alone, and the summation term appearing in (4.12) is absent.

(g) At $k = i\kappa_j$, as shown in Proposition 11.4, (2.1) has $m_j$ linearly independent column vector-valued solutions, where we recall that the values of $m_j$ are already uniquely determined from $D$, as indicated in the previous step. It is possible to rearrange those linearly independent column vector solutions into an $n \times n$ matrix $\Psi_j(x)$, in such a way that $\Psi_j(x)$ can be uniquely constructed as

$$\Psi_j(x) := f(i\kappa_j, x) M_j, \quad j = 1, \ldots, N,$$

(9.8)

where $M_j$ is an $n \times n$ nonnegative hermitian matrix of rank $m_j$. The unique construction of $M_j$ is given in (11.22) in terms of the projection matrix $P_j$ appearing in (11.1) and the matrix $B_j^{-1/2}$, where $B_j$ is defined in (11.3). The relevant properties of $P_j$ are given in (11.1) and those of $B_j^{-1/2}$ are summarized in Proposition 11.2.

(h) As a part of the direct problem, when $D$ belongs to the Faddeev class, we show that the collection of the set of $N$ matrices $\Psi_j(x)$ given in (9.8) and the physical solution $\Psi(k, x)$ given in (9.4) satisfy the orthonormalization condition and the completeness condition (Parseval’s equality), which are summarized in Proposition 22.2 and Proposition 20.2, respectively.

(i) As a part of the direct problem, when $D$ belongs to the Faddeev class, we verify that $S$ belongs to the Marchenko class, by showing that the four conditions listed in Definition 4.5 are satisfied by $S$. This is done in Theorem 15.10.
(j) We show that the equivalence statements given in Proposition 6.6, hold when the input data set $\mathbf{D}$ belongs to the Faddeev class.
10. SOME RELEVANT RESULTS RELATED TO THE DIRECT PROBLEM

In this chapter we elaborate and justify some of the steps outlined in Section 3 for the unique construction of the scattering data set \( S \) when our input data set \( D \) belongs to the Faddeev class. The remaining steps outlined in Section 3 will be proved in later chapters.

Proposition 10.1 Assume that the input data set \( D \) appearing in (4.1) belongs to the Faddeev class specified in Definition 4.1. Then, regardless of the boundary matrices \( A \) and \( B \), (2.1) has a unique \( n \times n \) matrix-valued solution \( f(k, x) \), known as the Jost solution, satisfying the asymptotics (9.1). Furthermore, we have:

(a) For each fixed \( x \in [0, +\infty) \), the quantity \( f(k, x) \) has an analytic extension from \( k \in \mathbb{R} \) to \( k \in \mathbb{C}^+ \) and that extension is continuous in \( k \in \mathbb{C}^+ \).

(b) The quantity \( K(x, y) \) defined as

\[
K(x, y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left[ f(k, x) - e^{ikx} I \right] e^{-iky},
\]

vanishes when \( y < x \), i.e.

\[
K(x, y) = 0, \quad x > y,
\]

and it is related to the potential via

\[
K(x, x) = \frac{1}{2} \int_x^{\infty} dz \, V(z), \quad x \geq 0,
\]

\[
V(x) = -2 \frac{dK(x, x)}{dx}, \quad x \in \mathbb{R}^+.
\]

The constant \( n \times n \) matrix \( K(0, 0) \) is well defined, hermitian, and related to the potential as

\[
K(0, 0) = \frac{1}{2} \int_0^{\infty} dz \, V(z).
\]

We remark that we use \( K(x, x) \) to denote \( K(x, x^+) \) and use \( K(0, 0) \) to denote \( K(0, 0^+) \).

(c) The Jost solution \( f(k, x) \) has the representation

\[
f(k, x) = e^{ikx} I + \int_x^{\infty} dy \, K(x, y) e^{iky},
\]
where $K(x, y)$ is the quantity given in (10.1).

(d) The quantity $K(x, y)$ appearing in (10.1) satisfies

$$|K(x, y)| \leq \frac{1}{2} e^{\sigma_1(x)} \sigma \left( \frac{x + y}{2} \right), \quad y \geq x \geq 0,$$

(10.7)

where $\sigma(x)$ and $\sigma_1(x)$ are the scalar quantities defined in (3.96). Hence, the quantity $K(x, y)$ is integrable in $y \in [x, +\infty)$ for each fixed $x \geq 0$.

(e) The quantity $K(x, y)$ appearing in (10.1) is continuous in $(x, y)$ in the region $0 \leq x \leq y$.

(f) The quantity $K_x(x, y)$, i.e. the $x$-derivative of $K(x, y)$, exists a.e. and satisfies

$$K_x(x, y) = 0, \quad x > y,$$

(10.8)

$$|K_x(x, y)| \leq \frac{1}{4} |V \left( \frac{x + y}{2} \right)| + \frac{1}{2} e^{\sigma_1(x)} \sigma \left( \frac{x + y}{2} \right) \sigma(x), \quad 0 \leq x \leq y.$$  

(10.9)

Hence, the quantity $K_x(x, y)$ is integrable in $y \in [x, +\infty)$ for each fixed $x \geq 0$.

PROOF: We refer the reader to Theorem 1.3.1 of [2] for (a)-(d), in particular to (1.3.11) of [2] for (10.3). and to Lemma 1.3.1 of [2] for (f). We have the following remarks to complete the proof. The hermitian property of $K(0, 0)$ in (10.5) is directly obtained by using in (10.5) the hermitian property of the potential $V(x)$. The properties that, for each fixed $x \geq 0$, the quantities $K(x, y)$ and $K_x(x, y)$ are each integrable in $y \in [x, +\infty)$ is proved as follows. From (2.3) and (3.96) we know that $|V(y)|$ and $\sigma(y)$ are both in $L^1(\mathbb{R}^+)$. Furthermore, as a result of (2.3) we know that $\sigma(x)$ and $\sigma_1(x)$ are both finite for each $x \geq 0$. Then, (10.7) and (10.9) imply that $K(x, y)$ and $K_x(x, y)$, respectively, are integrable in $y \in [x, +\infty)$ for each fixed $x \geq 0$. The continuity stated in (e) can be proved as follows. The matrix $K(x, y)$ satisfies the integral equation (1.3.6) of [2]. That integral equation can be solved by the method of successive approximation, by representing $K(x, y)$ as a uniformly convergent infinite series, where each term is continuous in the region $0 \leq x \leq y$. As a result of the uniform convergence, $K(x, y)$ is continuous in the same region, proving (e). The property (10.8) follows from (10.2). The estimate (10.9) can be found in (1.3.9) of [2] and can
be obtained by solving the integral equation (1.3.8) of [2] iteratively. In general, in the Faddeev class, the potential \( V(x) \) is not continuous in \( x \in \mathbb{R}^+ \) and hence \( V(x) \) exists a.e. in \( x \in \mathbb{R}^+ \). Then, applying the method of successive approximation to (1.3.8) of [2], it follows that the quantity \( K_x(x, y) \) in general exists only a.e.

We remark that the Jost solution \( f(k, x) \) appearing in (9.1) is only affected by the potential \( V \) in the input data set \( D \) and not by the boundary matrices \( A \) and \( B \) appearing in (2.4)-(2.6).

Next we present the relevant properties of the Jost matrix \( J(k) \) introduced in (9.2).

**Proposition 10.2** Assume that the input data set \( D \) appearing in (4.1) belongs to the Faddeev class specified in Definition 4.1. Let \( J(k) \) be the corresponding Jost matrix constructed as in (9.2). Then:

(a) The Jost matrix \( J(k) \) is invertible for \( k \in \mathbb{R} \setminus \{0\} \). It is either invertible at \( k = 0 \) or it has a simple zero at \( k = 0 \). The matrix \( J(k)^{-1} \) has at most a simple pole at \( k = 0 \), and hence \( kJ(k)^{-1} \) is continuous in \( k \in \mathbb{R} \).

(b) The Jost matrix \( J(k) \) has an analytic extension from \( k \in \mathbb{R} \) to \( k \in \mathbb{C}^+ \) in such a way that \( J(k) \) is continuous in \( k \in \overline{\mathbb{C}^+} \). Furthermore, we have

\[
J(k) = -ikA + B + K(0, 0)A + o(1), \quad k \to \infty \text{ in } \overline{\mathbb{C}^+},
\]

where \( A \) and \( B \) are the boundary matrices in the input data set \( D \) and \( K(0, 0) \) is the constant \( n \times n \) matrix given in (10.5).

(c) The determinant of \( J(k) \) in \( \mathbb{C}^+ \) is nonzero, except at a finite number of distinct \( k \)-values on the positive imaginary axis, say at \( k = ik_j \) for \( j = 1, \ldots, N \), where the multiplicity of the zero at \( k = ik_j \) is denoted by \( m_j \). The matrix \( J(k)^{-1} \) is meromorphic in \( k \in \mathbb{C}^+ \) with simple poles at \( k = ik_j \) for \( j = 1, \ldots, N \). If \( N = 0 \), then the determinant of \( J(k) \) does not vanish in \( k \in \overline{\mathbb{C}^+} \setminus \{0\} \).

(d) The multiplicity \( m_j \) of the zero \( k = ik_j \) for the determinant of \( J(k) \) is equal to the nullity of the matrix \( J(ik_j)^\dagger \).
(e) We have
\[
J(k) = J(ik_j) + (k - ik_j) \dot{J}(ik_j) + O\left((k - ik_j)^2\right), \quad k \to ik_j \text{ in } \mathbb{C}^+, \quad (10.11)
\]
\[
J(k)^{-1} = \frac{N_j}{k - ik_j} + L_j + O(k - ik_j), \quad k \to ik_j \text{ in } \mathbb{C}^+, \quad (10.12)
\]
for some constant \( n \times n \) matrices \( N_j \) and \( L_j \), where an overdot indicates the \( k \)-derivative.

(f) The constant \( n \times n \) matrices \( N_j \) and \( L_j \), appearing in (10.12) satisfy
\[
N_j J(ik_j) = 0, \quad L_j J(ik_j) + N_j \dot{J}(ik_j) = I. \quad (10.13)
\]

(g) We have \( J(k)^{-1} = O(k) \) as \( k \to \infty \) in \( k \in \overline{\mathbb{C}^+} \).

PROOF: A proof of (a) can be found in Theorems 5.1 and 6.3 of [5]. For the first statement
in (b) we refer the reader to Theorem 3.1(a) of [9]. The large-\( k \) asymptotics of the Jost
matrix \( J(k) \) in (10.10) is known from (7.11) of [9]. Let us now prove (c). For this, let us
first argue that the number of zeros of \( \det[J(k)] \) in \( \overline{\mathbb{C}^+} \) must be finite, and the argument is
as follows. By (3.1) of [9] we know that any zero of \( \det[J(k)] \) in \( \mathbb{C}^+ \) yields a bound state
for (2.1) and (2.4), and by the first paragraph of Section 8 of [9] we know that such a zero
can only occur on the positive imaginary axis in \( \mathbb{C}^+ \). From (a) we know that \( \det[J(k)] \)
is nonzero for \( k \in \mathbb{R} \setminus \{0\} \). By (6.5) of [9] we know that \( \det[J(k)] \) either does not vanish
at \( k = 0 \) or its zero at \( k = 0 \) has a finite multiplicity. Furthermore, by (7.17) of [9] we
know that \( \det[J(k)] \) does not vanish as \( k \to \infty \) in \( \overline{\mathbb{C}^+} \). From (b) we know that \( \det[J(k)] \)
is analytic in \( k \in \mathbb{C}^+ \) and continuous in \( k \in \overline{\mathbb{C}^+} \). Hence, there cannot be any accumulation
points for the zeros of \( \det[J(k)] \) in \( \overline{\mathbb{C}^+} \) and the number of zeros of \( \det[J(k)] \) in \( \overline{\mathbb{C}^+} \) must be
finite. Having proved that the number of zeros of \( \det[J(k)] \) in \( \overline{\mathbb{C}^+} \) is finite, the rest of (c)
follows from Theorems 8.4 and 8.4 of [9]. Similarly, (d) follows from Theorems 8.4 and 8.4
of [9]. As for the proof of (e), we remark that the results in (10.11) and (10.12) follow from
(8.26) and (8.32), respectively, of [9]. Concerning (f), we obtain (10.13) by using (10.10)
and (10.12) in the identity \( J(k)^{-1}J(k) = I \). Finally, we note that (g) follows from (5.8)
and (7.13) of [9]. Thus, the proof is complete. \( \blacksquare \)
Next we present some relevant properties of the scattering matrix $S(k)$ introduced in (9.3).

**Proposition 10.3** Assume that the input data set $D$ appearing in (4.1) belongs to the Faddeev class specified in Definition 4.1. Then:

(a) The scattering matrix $S(k)$ defined in (9.3) is continuous for $k \in \mathbb{R}$, and it satisfies (4.4).

(b) The large-$k$ asymptotics of $S(k)$ is given by

$$
S(k) = S_\infty + \frac{G(k)}{ik} + O \left( \frac{1}{k^2} \right), \quad k \to \pm \infty,
$$

where $S_\infty$ is the constant $n \times n$ matrix defined in (4.6) and uniquely determined by the boundary matrices $A$ and $B$ as

$$
S_\infty = \lim_{k \to \pm \infty} \left[ -(B + ikA)(B - ikA)^{-1} \right],
$$

and the matrix $G(k)$ is continuous and uniformly bounded for $k \in \mathbb{R}$. In fact, we have

$$
G(k) = G_1 + G_2(k),
$$

where $G_1$ is the constant $n \times n$ matrix appearing in (4.5) and defined as

$$
G_1 := W_1 + \frac{1}{2} \int_0^\infty dz V(z) S_\infty + \frac{1}{2} \int_0^\infty dz S_\infty V(z),
$$

with $W_1$ being some constant $n \times n$ hermitian matrix, $S_\infty$ the constant matrix appearing in (4.6) and (10.14), and $G_2(k)$ the $n \times n$ matrix defined as

$$
G_2(k) := \frac{1}{2} \int_0^\infty dz V(z) e^{-2ikz} + \frac{1}{2} \int_0^\infty dz S_\infty V(z) S_\infty e^{2ikz},
$$

with the property that $G_2(k) = o(1)$ as $k \to \pm \infty$.

(c) The constant matrices $S_\infty$ and $G_1$, defined in (4.6) and (10.17), respectively, are both hermitian.
Each eigenvalue of the constant matrix $S(0)$ is either $+1$ or $-1$.

PROOF: The continuity of $S(k)$ is stated in Proposition 3.3 of [9]. We quote the property in (4.4) directly from (3.11) of [9]. The large-$k$ asymptotics in (10.14) is given in Theorem 7.6 of [9], where the properties of $G(k)$ is obtained from (10.17) and (10.18) by using the fact that $V(x)$ is integrable in $x \in \mathbb{R}^+$ and hence the Riemann-Lebesgue lemma on the right-hand side of (10.18) yields $G_2(k) = o(1)$ as $k \to \pm \infty$. The hermitian property of $S_\infty$ directly follows from (4.4) and (10.14). Since $W_1, S_\infty$, and $V(x)$ are all hermitian, from (10.17) we see that $G_1$ is also hermitian. The fact that the eigenvalues of $S(0)$ can only be $+1$ or $-1$ is proved in Proposition 6.3 of [9].

The relevant properties of the regular solution $\varphi(k, x)$ appearing in (9.5) are given in the following proposition.

**Proposition 10.4** Assume that the input data set $D$ appearing in (4.1) belongs to the Faddeev class specified in Definition 4.1. Then, (2.1) has a unique solution $\varphi(k, x)$, known as the regular solution, satisfying the initial conditions given in (9.5). For each fixed $x \in [0, +\infty)$, the regular solution $\varphi(k, x)$ is entire in $k$ and it satisfies as $k \to \infty$ in $C^+$

$$
-2ik \varphi(k, x) e^{ikx} = -ik \left(1 + e^{2ikx}\right) A + \left(1 - e^{2ikx}\right) B \nonumber
$$
$$
+ \frac{1}{2} \int_0^x dz \left(1 + e^{2ikz}\right) \left(1 - e^{2ik(x-z)}\right) V(z) A + O\left(\frac{1}{k}\right).
$$

(10.19)

PROOF: The regular solution $\varphi(k, x)$ satisfies the integral relation given in (3.7) of [5], which is

$$
\varphi(k, x) = A \cos kx + B \frac{\sin kx}{k} + \frac{1}{k} \int_0^x dz \left[\sin k(x-z)\right] V(z) \varphi(k, z).
$$

(10.20)

By iterating (10.20) one can establish [5] the existence and uniqueness of the regular solution and also prove that $\varphi(k, x)$ is entire in $k$ for each fixed $x \in [0, +\infty)$. Multiplying both sides of (10.20) with $e^{ikx}$, after some simplification, we obtain

$$
e^{ikx} \varphi(k, x) = \frac{1}{2} \left(1 + e^{2ikx}\right) A - \frac{1}{2ik} \left(1 - e^{2ikx}\right) B \nonumber
$$
$$
+ \frac{1}{2ik} \int_0^x dz \left[e^{2ik(x-z)} - 1\right] V(z) \left[e^{ikz} \varphi(k, z)\right].
$$

(10.21)
We obtain (10.19) from (10.21) via iteration. ■

Next, we establish some relevant properties of the physical solution \( \Psi(k, x) \) appearing in (9.4) and in (9.6).

**Proposition 10.5** Assume that the input data set \( D \) appearing in (4.1) belongs to the Faddeev class specified in Definition 4.1. Let \( \Psi(k, x) \) be the physical solution defined in (9.4). Then:

(a) The representation (9.6) for the physical solution \( \Psi(k, x) \) is equivalent to the representation given in (9.4).

(b) For each fixed \( x \in [0, +\infty) \), the quantity \( \Psi(k, x) \) is continuous in \( k \in \mathbb{R} \) and meromorphic in \( k \in \mathbb{C}^+ \) with simple poles coinciding with the poles of \( J(k)^{-1} \) as indicated in (10.12), i.e. simple poles at \( k = i\kappa_j \) for \( j = 1, \ldots, N \).

(c) The physical solution \( \Psi(k, x) \) satisfies (2.1) and the boundary condition (2.4).

(d) For each fixed \( x \in [0, +\infty) \), we have

\[
e^{ikx} \Psi(k, x) = W_2 + W_3 e^{2ikx} + O\left(\frac{1}{k}\right), \quad k \to \infty \text{ in } \mathbb{C}^+,
\]

for some constant \( n \times n \) matrices \( W_2 \) and \( W_3 \).

**Proof:** We establish (a) in a straightforward manner with the help of (9.3) and (9.7). Let us now turn to the proof of (b). As indicated in Theorem 3.1 of [5], \( J(k)^{-1} \) is continuous in \( k \in \mathbb{R} \setminus \{0\} \) with a possible simple pole at \( k = 0 \). We know from Proposition 10.4 that \( \varphi(k, x) \) is entire in \( k \) for each fixed \( x \in [0, +\infty) \). Furthermore, as indicated in Proposition 10.2(c) the matrix \( J(k)^{-1} \) is meromorphic in \( \mathbb{C}^+ \) with simple poles at \( k = i\kappa_j \) for \( j = 1, \ldots, N \).

With the help of (9.6), (10.12), and (10.19) we conclude (b). The proof of (c) is obtained as follows. From Proposition 10.1 we know that each column of the Jost solution \( f(k, x) \) satisfies (2.1). Since \( k \) appears as \( k^2 \) in (2.1), each column of \( f(-k, x) \) is also a solution to (2.1). As seen from (9.4), each column of \( \Psi(k, x) \) is a linear combination of columns of \( f(k, x) \) and \( f(-k, x) \). Hence, the \( n \times n \) matrix \( \Psi(k, x) \) is a solution to (2.1). From (2.5) and
(9.5) we see that the regular solution \( \varphi(k, x) \) satisfies the selfadjoint boundary condition given in (2.4), and from (9.6) we see that each column of the physical solution \( \Psi(k, x) \) is a linear combination of columns of \( \varphi(k, x) \). Thus, the physical solution also satisfies the boundary condition (2.4). Hence, the proof of (c) is complete. Let us now turn to the proof of (d). From (10.8), (10.9), and (7.12) of [9], we respectively have

\[
J_0(k)^{-1} = W_4 + O \left( \frac{1}{k} \right), \quad k \to \infty \text{ in } \mathbb{C}^+,
\]

(10.23)

\[
A J_0(k)^{-1} = -\frac{1}{ik} W_5 + O \left( \frac{1}{k^2} \right), \quad k \to \infty \text{ in } \mathbb{C}^+,
\]

(10.24)

\[
J(k)^{-1} = J_0(k)^{-1} \left[ I + O \left( \frac{1}{k} \right) \right], \quad k \to \infty \text{ in } \mathbb{C}^+,
\]

(10.25)

where \( W_4 \) and \( W_5 \) are some constant \( n \times n \) matrices and \( J_0(k) \) is the Jost matrix corresponding to the input data set \( \mathbf{D} \) when \( V \) is the zero potential. Using (10.19) and (10.25) on the right-hand side of (9.6), with the help of (10.23) and (10.24) and after some simplification, we obtain (10.22).
11. BOUND STATES

In this chapter, we continue our analysis of the direct scattering problem and continue the justification of the steps outlined in Section 3 when our input data set $D$ given in (4.1) belongs to the Faddeev class specified in Definition 4.1. We prove various relevant properties of the corresponding scattering data set $S$ given in (4.2), especially those properties related to the bound-state information in $S$.

Concerning the bound states of the Schrödinger operator associated with (2.1) and (2.4), we have the following basic facts. By definition, a bound state is a column vector solution to (2.1) which also satisfies the boundary condition (2.4). Because of the self-adjointness of the Schrödinger operator, a bound state, if it exists, must occur when the spectral parameter $k^2$ is real. When $k^2 > 0$ or $k^2 = 0$, none of $2n$ linearly independent column vector solutions to (2.1) are square integrable, as argued in the first paragraph of Section 8 of [9]. When $k^2 < 0$, i.e. when $k$ is on the positive imaginary axis in $\mathbb{C}^+$, we have the following argument. Among the $(2n)$ linearly independent column-vector solutions to (2.1), only $n$ of them are square integrable in $x \in \mathbb{R}^+$, and the columns of $f(k,x)$ are such solutions. Among the $(2n)$ linearly-independent column-vector solutions to (2.1), only $n$ of them satisfy (2.4), and the columns of the regular solution $\varphi(k,x)$ are such solutions. Thus, a particular $k$-value on the positive imaginary axis in $\mathbb{C}$ corresponds to a bound state provided a column vector at that $k$-value can be expressed as a linear combination of the columns of $f(k,x)$ and also of the columns of $\varphi(k,x)$. It turns out that such $k$-values correspond to the zeros of the determinant of the Jost matrix $J(k)$ given in (9.2), which occur at $k = i\kappa_j$ for $j = 1, \ldots, N$, as indicated in Proposition 10.2(c). Furthermore, at $k = i\kappa_j$ there are exactly $m_j$ linearly-independent column vectors satisfying both (2.1) and (2.4). This is elaborated in Propositions 11.2 and 11.3. Below we provide a summary of the basic facts on the bound states for the relevant Schrödinger operator, and for the proof and further details on the bound states we refer the reader to [9].

**Proposition 11.1** Assume that the input data set $D$ appearing in (4.1) belongs to the
Faddeev class. Then:

(a) The bound states corresponding to the Schrödinger operator related to (2.1) and (2.4)-(2.6) occur only at the \( k \)-values on the positive imaginary axis in \( \mathbb{C} \) where the determinant of Jost matrix \( J(k) \) given in (9.2) vanishes. Such \( k \)-values are denoted by \( k = i\kappa_j \) for \( j = 1, \ldots, N \), where the \( \kappa_j \) are distinct positive numbers and \( N \) is a nonnegative integer. If \( N \) is zero, then there are no bound states.

(b) For each \( k = i\kappa_j \) there are exactly \( m_j \) linearly independent square-integrable column-vector solutions to (2.1) that also satisfy (2.4). Here, \( m_j \) is the positive integer equal to the dimension of the kernel of \( J(i\kappa_j) \). The positive integer \( m_j \).

Let \( \text{Ker}[J(i\kappa_j) \dagger] \) denote the kernel of the matrix \( J(i\kappa_j) \). We use \( P_j \) to denote the orthogonal projection matrix onto \( \text{Ker}[J(i\kappa_j) \dagger] \). Then, \( P_j \) is an \( n \times n \) hermitian, idempotent matrix, i.e. we have

\[
P_j^\dagger = P_j, \quad P_j^2 = P_j, \quad j = 1, \ldots, N. \tag{11.1}
\]

Having defined the orthogonal projections \( P_j \), let us now define the normalization matrices \( M_j \) at each bound state with \( k = i\kappa_j \). Proceeding as on pp. 60–61 of [2], we define the constant \( n \times n \) matrices \( A_j \) and \( B_j \) as

\[
A_j := \int_0^\infty dx f(i\kappa_j, x) \dagger f(i\kappa_j, x), \quad j = 1, \ldots, N, \tag{11.2}
\]

\[
B_j := (I - P_j) + P_j A_j P_j, \quad j = 1, \ldots, N, \tag{11.3}
\]

where \( f(k, x) \) is the Jost solution appearing in (9.1) and \( P_j \) is the hermitian projection matrix appearing in (11.1). We remark that the definitions of \( A_j \) and \( B_j \) in (11.2) and (11.3), respectively, are the same as the corresponding definitions appearing on pp. 61–62 of [2]. Because [2] uses the Dirichlet boundary condition instead of (2.4), the matrix \( B_j \) appearing in (11.3) is a generalization of the corresponding matrix in [2]. The properties of \( B_j \) are similar to those in [2] and are listed in the following proposition.

**Proposition 11.2** Assume that the input data set \( D \) appearing in (4.1) belongs to the Faddeev class. Then:

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(a) The matrix $B_j$ defined in (11.3) is hermitian and positive definite.

(b) There exists a unique $n \times n$ matrix $B_j^{1/2}$ such that $B_j^{1/2}B_j^{1/2} = B_j$. In fact, $B_j^{1/2}$ is also hermitian and positive definite.

(c) The matrices $B_j$ and $B_j^{1/2}$ are both invertible. The inverse of $B_j^{1/2}$, denoted by $B_j^{-1/2}$, is also hermitian and positive definite.

(d) Each of the matrices $B_j$, $B_j^{-1}$, $B_j^{1/2}$, and $B_j^{-1/2}$ commutes with the projection matrix $P_j$ given in (11.1).

**PROOF:** A condensed proof in the Dirichlet case can be found on pp. 60–61 of [2]. For the sake of establishing the notation and clarity, we provide a short proof. Using (11.1) in (11.3), we see that $B_j = B_j^\dagger$, and hence $B_j$ is hermitian. For any column vector $v \in \mathbb{C}^n$, let us use $||v||_1$ to denote the standard length of $v$, i.e. let $||v||_1 := \sqrt{v^\dagger v}$. Note that $B_j$ is positive definite if $v^\dagger B_j v > 0$ for any nonzero vector $v \in \mathbb{C}^n$. From (11.1) and (11.3) it follows that

$$v^\dagger B_j v = |(I - P_j)v|^2 + \int_0^\infty dx |f(i\kappa_j, x) P_j v|^2. \quad (11.4)$$

From (11.4) we see that $v^\dagger B_j v \geq 0$ and that we have $v^\dagger B_j v = 0$ if and only if $v = P_j v$ and $f(i\kappa_j, x) v \equiv 0$. On the other hand, as seen from (9.1), we have $f(i\kappa_j, x) v = e^{-\kappa_j x} v + o(1)$ as $x \to +\infty$, and hence $f(i\kappa_j, x) v \equiv 0$ if and only if $v = 0$. Thus, we have completed the proof of (a). The existence of $B_j^{1/2}$ stated in (b) directly follows from (a). Since $B_j$ is hermitian and positive definite, all the eigenvalues of $B_j$ are real and in fact positive. Thus, $B_j$ can be diagonalized, with the help of a unitary matrix $U$, into a diagonal matrix $D_j$, where $D_j = U^\dagger B_j U$. It is clear that there exists a unique matrix $D_j^{1/2}$ such that $D_j = D_j^{1/2}D_j^{1/2}$ and that $D_j^{1/2}$ is invertible with the inverse denoted by $D_j^{-1/2}$. Consequently, we have $B_j^{1/2} = UD_j^{1/2}U^\dagger$ and that $B_j^{1/2}$ is hermitian and positive definite. Thus, we have proved (b). The invertibility of $B_j$ and $B_j^{1/2}$ directly follows from the invertibility of $D_j$ and $D_j^{1/2}$, respectively, and in fact we have $B_j^{-1} = UD_j^{-1}U^\dagger$ and $B_j^{-1/2} = UD_j^{-1/2}U^\dagger$. Since $D_j^{-1/2}$ is a diagonal matrix with positive entries, it follows that $B_j^{-1/2}$ is hermitian and positive definite. Thus, the proof of (c) is complete. From (11.1) and (11.3) it directly
follows that $P_j B_j = B_j P_j$. Multiplying the latter equation by $B_j^{-1}$ on the left and on the right, we establish $P_j B_j^{-1} = B_j^{-1} P_j$. Since $B_j = U D_j U^\dagger$, in $P_j B_j = B_j P_j$ let us replace $B_j$ with $U D_j U^\dagger$. With some minor algebra, this yields $(U P_j U)^D_j = D_j (U P_j U)$. Thus, $U P_j U$ commutes with the diagonal matrix $D_j$. One can directly verify that $U P_j U$ must also commute with each of the diagonal matrices $D_j^{1/2}$ and $D_j^{-1/2}$. From $(U P_j U)^D_j^{1/2} = D_j^{1/2} (U P_j U)$, using $B_j^{1/2} = U D_j^{1/2} U^\dagger$ and some minor algebra we get $P_j B_j^{1/2} = B_j^{1/2} P_j$. Multiplying $P_j B_j^{1/2} = B_j^{1/2} P_j$ by $B_j^{-1/2}$ on the left and on the right, we also establish $P_j B_j^{-1/2} = B_j^{-1/2} P_j$. \[\Box\]

Let us now clarify the relationship between the Jost solution $f(k, x)$ and the regular solution $\varphi(k, x)$ at a bound-state value $k = i\kappa_j$.

**Proposition 11.3** Assume that the input data set $D$ appearing in (4.1) belongs to the Faddeev class. Let $P_j$ be the projection matrix appearing in (11.1), where the columns of $P_j$ belong to $\text{Ker}[J(i\kappa_j)^\dagger]$. Let $f(k, x)$ and $\varphi(k, x)$ be the Jost solution and the regular solution appearing in (9.1) and (9.5), respectively. Then:

(a) Corresponding to $P_j$, there exists a unique $n \times n$ matrix $Q_j$ whose columns belonging to $\text{Ker}[J(i\kappa_j)]$ in such a way that

$$f(i\kappa_j, x) P_j = \varphi(i\kappa_j, x) Q_j.$$  \hspace{1cm} (11.5)

(b) The matrix $Q_j$ can be expressed explicitly in various equivalent forms such as

$$Q_j = E^{-2} (A^\dagger + iB^\dagger) \left[ f(i\kappa_j, 0) - i f'(i\kappa_j, 0) \right] P_j,$$  \hspace{1cm} (11.6)

$$Q_j = E^{-2} \left[ A^\dagger f(i\kappa_j, 0) + B^\dagger f'(i\kappa_j, 0) \right] P_j,$$  \hspace{1cm} (11.7)

where $A$, $B$, and $E$ are the constant $n \times n$ matrices appearing in (2.4)-(2.9).

(c) The matrix $Q_j$ can also be expressed explicitly as

$$Q_j = -2i\kappa_j N_j A_j P_j,$$  \hspace{1cm} (11.8)

where $N_j$ and $A_j$ are the matrices appearing in (10.12) and (11.2), respectively.

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PROOF: The existence of $Q_j$ and the fact that the columns of $Q_j$ belong to the kernel of $J(i\kappa_j)$ directly follow from Theorem 10.1(d) of [9]. To prove that $Q_j$ appearing in (11.5) has the form given in (11.6), it is enough to prove that each side of (11.5) satisfies (2.1) at $k = i\kappa_j$ and that both sides agree at $x = 0$ and that the $x$-derivatives of both sides agree at $x = 0$, due to the fact that the relevant initial-value problem has a unique solution. Since every column of $f(i\kappa_j, x)$ and of $\varphi(i\kappa_j, x)$ satisfies (2.1) at $k = i\kappa_j$, it is clear that each side of (11.5) satisfies (2.1) at $k = i\kappa_j$. Recall that $P_j$ is the orthogonal projection onto $\text{Ker}[J(i\kappa_j)\dagger]$, and hence we have $J(i\kappa_j)\dagger P_j = 0$. With the help of (2.9), (9.2), (9.5), and the fact that $J(i\kappa_j)\dagger P_j = 0$, one can directly verify that the right-hand side of (11.5) at $x = 0$ with $Q_j$ as in (11.6) has the value equal to $f(i\kappa_j, 0) P_j$. Similarly, one can directly verify that the $x$-derivative of the right-hand side of (11.5) at $x = 0$ with $Q_j$ as in (11.6) has the value equal to $f'(i\kappa_j, 0) P_j$. Thus, (11.6) is established. With the help of (9.2) and $J(i\kappa_j)\dagger P_j = 0$, one can simplify (11.6) to (11.7). Let us now turn to the proof of (c). From the first equality in (10.13) it follows that $J(i\kappa_j)\dagger N_j\dagger = 0$, and hence the columns of $N_j\dagger$ belong to $\text{Ker}[J(i\kappa_j)\dagger]$. Thus, we have $P_j N_j\dagger = N_j\dagger$, yielding

$$N_j = N_j P_j,$$

where we have used the first equality in (11.1). With the help of (11.9), we see that the right-hand side of (11.8) satisfies

$$-2i\kappa_j N_j A_j P_j = -2i\kappa_j N_j P_j A_j P_j.$$  \hspace{1cm} (11.10)

From (11.2) and (11.5) we get

$$P_j A_j P_j = \int_0^\infty dx \left[ \varphi(i\kappa_j, x) Q_j\right]\dagger \left[ \varphi(i\kappa_j, x) Q_j\right].$$  \hspace{1cm} (11.11)

Using (8.22) of [9] at $\kappa = \kappa_j$, multiplying the resulting equation on the left and on the right by $P_j$, and integrating the resulting equation over $x \in \mathbb{R}^+$, we obtain

$$-2i\kappa_j P_j A_j P_j = P_j \left[ -f(i\kappa_j, 0)\dagger \dot{f}(i\kappa_j, 0) + f'(i\kappa_j, 0)\dagger \dot{f}(i\kappa_j, 0) \right] P_j,$$  \hspace{1cm} (11.12)
where we recall that \(A_j\) is the matrix in (11.2). The adjoint of (11.12) yields

\[
2i\kappa_j P_j A_j P_j = P_j \left[ -\hat{f}'(i\kappa_j, 0)^\dagger f(i\kappa_j, 0) + \hat{f}(i\kappa_j, 0)^\dagger f'(i\kappa_j, 0) \right] P_j, \tag{11.13}
\]

where we have used \(P_j^\dagger = P_j\) and \(A_j^\dagger = A_j\), as seen from (11.1) and (11.2), respectively.

From (9.5) and (11.5) we see that

\[
f(i\kappa_j, 0) P_j = AQ_j, \quad f'(i\kappa_j, 0) P_j = BQ_j, \tag{11.14}
\]

where \(A\) and \(B\) are the boundary matrices appearing in (2.4)-(2.6). Using (11.14) in (11.13) we get

\[
2i\kappa_j P_j A_j P_j = P_j \left[ -\hat{f}'(i\kappa_j, 0)^\dagger A + \hat{f}(i\kappa_j, 0)^\dagger B \right] Q_j. \tag{11.15}
\]

From the second equality of (8.11) of [9] and from the first equality of (8.12) of [9], we see that

\[
\hat{f}(i\kappa_j, 0)^\dagger = \left. -\frac{df(-k^*, 0)^\dagger}{dk} \right|_{k = i\kappa_j}, \quad \hat{f}'(i\kappa_j, 0)^\dagger = \left. -\frac{df'(-k^*, 0)^\dagger}{dk} \right|_{k = i\kappa_j}. \tag{11.16}
\]

From (9.2) and (11.16) it follows that

\[
\hat{f}'(i\kappa_j, 0)^\dagger A - \hat{f}(i\kappa_j, 0)^\dagger B = \hat{J}(i\kappa_j), \tag{11.17}
\]

where we recall that an overdot indicates the \(k\)-derivative. Thus, using (11.17) in (11.15) we obtain

\[
-2i\kappa_j P_j A_j P_j = P_j \hat{J}(i\kappa_j) Q_j. \tag{11.18}
\]

Using (11.18) on the right-hand side of (11.10) we obtain

\[
-2i\kappa_j N_j A_j P_j = N_j P_j \hat{J}(i\kappa_j) Q_j. \tag{11.19}
\]

With the help of (11.9), we write (11.19) as

\[
-2i\kappa_j N_j A_j P_j = N_j \hat{J}(i\kappa_j) Q_j. \tag{11.20}
\]

Using the second equality of (10.13) on the right-hand side of (11.20) we obtain

\[
-2i\kappa_j N_j A_j P_j = [I - L_j \hat{J}(i\kappa_j)] Q_j. \tag{11.21}
\]
By (a) we know that the columns of \( Q_j \) belong to \( \text{Ker}[J(i\kappa_j)] \), and hence we have \( J(i\kappa_j)Q_j = 0 \). Thus, (11.21) simplifies to (11.8).

As on p. 61 of [9], we define the normalization matrices \( M_j \) as

\[
M_j := B_j^{-1/2}P_j, \quad j = 1, \ldots, N, \tag{11.22}
\]

where \( P_j \) and \( B_j \) are the matrices appearing in (11.1) and (11.3), respectively. Recall that the existence of \( B_j^{-1/2} \) and its basic properties are stated in Proposition 11.2. Let us now consider the \( n \times n \) matrix-valued function \( \Psi_j(x) \) defined in (9.8). Some relevant properties of \( \Psi_j(x) \) are indicated in the following proposition.

**Proposition 11.4** Assume that the input data set \( D \) in (4.1) belongs to the Faddeev class specified in Definition 4.1. Then:

(a) The normalization matrix \( M_j \) defined in (11.22) is hermitian and nonnegative and has rank equal to \( m_j \), which is the dimension of \( \text{Ker}[J(i\kappa_j)^\dagger] \).

(b) At each bound state \( k = i\kappa_j \), the matrix \( \Psi_j(x) \) defined in (9.8) satisfies (2.1) and the boundary condition (2.4).

(c) The matrix \( \Psi_j(x) \) is normalized in the sense that

\[
\int_0^\infty dx \Psi_j(x)^\dagger \Psi_j(x) = P_j, \quad j = 1, \ldots, N, \tag{11.23}
\]

where \( P_j \) is the projection matrix appearing in (11.1). Furthermore, the matrices \( \Psi_j(x) \) for \( j = 1, \ldots, N \) are orthogonal in the sense that

\[
\int_0^\infty dx \Psi_l(x)^\dagger \Psi_j(x) = 0, \quad l \neq j. \tag{11.24}
\]

(d) For each \( j = 1, \ldots, N \), the normalized bound-state matrix solution \( \Psi_j(x) \) defined in (9.8) yields \( m_j \) linearly-independent column-vector solutions to (2.1) at \( k = i\kappa_j \), where \( m_j \) is equal to the rank of the normalization matrix \( M_j \) given in (11.22).

(e) Any column-vector solution to (2.1) at \( k = i\kappa_j \) satisfying (2.4) can be written as \( \Psi_j(x)v \) for some column vector \( v \) in \( \mathbb{C}^n \). Equivalently stated, any eigenfunction of
the Schrödinger operator associated with (2.1) and (2.4) can be written as $\Psi_j(x) v$ for some $v \in \mathbb{C}^n$ and some positive integer $j$ with $1 \leq j \leq N$.

PROOF: The projection matrix $P_j$ appearing in (11.1) has $m_j$ linearly independent columns, where $m_j$ is the dimension of $\text{Ker}[J(i\kappa_j)^\dagger]$. Thus, the rank of $P_j$ is $m_j$. From (11.1) we know that $P_j$ is hermitian. By Proposition 11.2(c) we know that $B_j^{-1/2}$ is hermitian and invertible. Thus, from (11.22) we conclude that $M_j$ is hermitian and has rank equal to $m_j$. The nonnegativity of $M_j$ follows from (11.1), (11.22), the positive definiteness of $B_j^{-1/2}$, and $B_j^{-1/2}P_j = P_j B_j^{-1/2}$, where the last two properties are assured by Proposition 11.2.

Let us now prove (b). With the help of (11.5), (11.22), and Proposition 11.2(d), we can write (9.8) as

$$\Psi_j(x) = \varphi(i\kappa_j, x) Q_j P_j B_j^{-1/2}, \quad j = 1, \ldots, N. \tag{11.25}$$

Since $\varphi(k, x)$ is an $n \times n$ matrix-valued solution to (2.1), each column of $\varphi(i\kappa_j, x)$ satisfies (2.1) at $k = i\kappa_j$. Hence, the right-hand side of (11.25) satisfies the corresponding matrix Schrödinger equation at $k = i\kappa_j$. Using (9.5) in (2.4), with the help of (2.5) we conclude that (2.4) is satisfied by the right-hand side of (11.25) and hence also by $\Psi_j(x)$. Let us now turn to the proof of (c). Using (11.25) we write the left-hand side of (11.23) as

$$\int_0^\infty dx \, \Psi_j(x)^\dagger \Psi_j(x) = B_j^{-1/2} P_j \int_0^\infty dx \, [\varphi(i\kappa_j, x) Q_j]^\dagger [\varphi(i\kappa_j, x) Q_j] P_j B_j^{-1/2}, \tag{11.26}$$

where we have used the fact that $P_j$ and $B_j^{-1/2}$ are hermitian. Using (11.11) on the right-hand side of (11.26) we obtain

$$\int_0^\infty dx \, \Psi_j(x)^\dagger \Psi_j(x) = B_j^{-1/2} \left[ P_j A_j P_j \right] B_j^{-1/2}. \tag{11.27}$$

With the help of (11.3) we replace $P_j A_j P_j$ on the right-hand side of (11.27) by $B_j - (I - P_j)$ and hence obtain

$$\int_0^\infty dx \, \Psi_j(x)^\dagger \Psi_j(x) = B_j^{-1/2} \left[ P_j B_j - P_j \right] P_j B_j^{-1/2}. \tag{11.28}$$

Next, with the help of Proposition 11.2(d) and the second equality in (11.1), we simplify the right-hand side of (11.28) and obtain (11.23). The proof of (11.24) will be given in the
proof of Proposition 22.4(a). Let us now prove (d). Each of the \(n\) columns of \(f(i\kappa_j,x)\) satisfies (2.1) at \(k = i\kappa_j\). From (9.1) we get

\[
f(i\kappa_j,x) = e^{-\kappa_j x} [I + o(1)], \quad x \to +\infty,
\]

and hence from (11.29) we conclude that \(f(i\kappa_j,x)\) yields \(n\) linearly-independent column-vector solutions to (2.1) at \(k = i\kappa_j\). Thus, from (9.8) we conclude that \(\Psi_j(x)\) yields as many linearly-independent column-vector solutions as the rank of the matrix \(M_j\). By (a) we know that the rank of \(M_j\) is \(m_j\), and hence we conclude that \(\Psi_j(x)\) yields \(m_j\) linearly-independent column-vector solutions to (2.1) at \(k = i\kappa_j\). Hence, the proof of (d) is complete. Let us now prove (e). From Theorem 8.1(c) of [9] it follows that any bound-state column-vector solution to (2.1) at \(k = i\kappa_j\) must be a linear combination of the columns of \(\Psi_j(x)\). Thus, any column-vector solution to (2.1) at \(k = i\kappa_j\) satisfying (2.4) must be a linear combination of the columns of \(\Psi_j(x)\), indicating that such a column-vector solution must be of the form \(\Psi_j(x)v\) for some constant column vector in \(\mathbb{C}^n\). By definition, an eigenvector of the Schrödinger operator is a square-integrable column vector solution to (2.1) satisfying the boundary condition (2.4). Thus, an eigenvector of the Schrödinger operator must have the form \(\Psi_j(x)v\) for some \(v \in \mathbb{C}^n\) and some positive integer \(j\) with \(1 \leq j \leq N\). This completes the proof of (e). 

The following result is needed later on in the derivation of the Marchenko integral equation (13.1).

**Proposition 11.5** Assume that the input data set \(D\) appearing in (4.1) belongs to the Faddeev class specified in Definition 4.1. Then, at each bound state with \(k = i\kappa_j\), the Jost solution \(f(k,x)\) appearing in (9.1) and the regular solution \(\varphi(k,x)\) appearing in (9.5) are related to each other as

\[
f(i\kappa_j,x) M_j^2 = -2i\kappa_j \varphi(i\kappa_j,x) N_j, \quad j = 1, \ldots, N,
\]

where \(M_j\) is the normalization matrix defined in (11.22) and \(N_j\) is the residue matrix appearing in (10.12).
PROOF: Using (9.8) and (11.25), after multiplication with $M_j$ on the right, we obtain

$$f(i\kappa_j, x) M_j^2 = \varphi(i\kappa_j, x) Q_j P_j B_j^{-1/2} M_j. \quad (11.31)$$

Using (11.3), (11.8)-(11.10), (11.22), and Proposition 11.2(d), we simplify the right-hand side of (11.31) and obtain (11.30).

The following result is also later needed in the derivation of the Marchenko integral equation (13.1).

**Proposition 11.6** Assume that the input data set $\mathbf{D}$ appearing in (4.1) belongs to the Faddeev class specified in Definition 4.1. Let $\Psi(k, x)$ be the physical solution appearing in (9.4) and (9.6), $f(k, x)$ be the Jost solution appearing in (9.1), and $M_j$ be the normalization matrix given in (11.22). Then, for $y > x \geq 0$ we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \Psi(k, x) e^{iky} = -\sum_{j=1}^{N} f(i\kappa_j, x) M_j^2 e^{-\kappa_j y}. \quad (11.32)$$

PROOF: For each fixed $x \in [0, +\infty)$, let us consider the quantity $e^{ikx} \Psi(k, x) - (W_2 + W_3 e^{2ikx})$ appearing in (10.22). By Proposition 10.5, that quantity is continuous in $k \in \mathbb{R}$, behaves like $O(1/k)$ as $k \to \infty$ in $\mathbb{C}^+$, and is meromorphic in $\mathbb{C}^+$ with simple poles at $k = i\kappa_j$ for $j = 1, \ldots, N$. Thus, for $y > x \geq 0$, with the help of residues we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left[ e^{ikx} \Psi(k, x) - (W_2 + W_3 e^{2ikx}) \right] e^{ik(y-x)} = i \sum_{j=1}^{N} \text{Res} \left[ e^{ikx} \Psi(k, x) e^{ik(y-x)}, i\kappa_j \right], \quad (11.33)$$

where the notation $\text{Res}[g(k), i\kappa_j]$ is used to denote the residue of a function $g(k)$ at $k = i\kappa_j$. With the help of (9.6), (10.12), and Proposition 10.5(b), we evaluate each residue on the right-hand side of (11.33) as

$$\text{Res} \left[ e^{ikx} \Psi(k, x) e^{ik(y-x)}, i\kappa_j \right] = 2i\kappa_j e^{-\kappa_j y} \varphi(i\kappa_j, x) N_j. \quad (11.34)$$

Using (11.30) we can write (11.34) as

$$\text{Res} \left[ e^{ikx} \Psi(k, x) e^{ik(y-x)}, i\kappa_j \right] = -e^{-\kappa_j y} f(i\kappa_j, x) M_j^2. \quad (11.35)$$
We have

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left( W_2 + W_3 e^{2ikx} \right) e^{ik(y-x)} = W_2 \delta(y-x) + W_3 \delta(y+x),
\]

(11.36)

where \( \delta(x) \) is the Dirac delta distribution given by

\[
\delta(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx}, \quad x \in \mathbb{R}.
\]

(11.37)

When \( y > x \geq 0 \), the right-hand side of (11.36) vanishes. Thus, for \( y > x \geq 0 \), using (11.35) and (11.36) in (11.33) we obtain (11.32).
12. FURTHER PROPERTIES OF THE SCATTERING DATA

In this chapter we continue to elaborate on the steps outlined in Section 3 to solve the direct scattering problem, by establishing various properties of the scattering data set \( S \) corresponding to an input data set \( D \) in the Faddeev class. Toward that goal, we obtain some relevant properties of \( F_s(y) \) and \( F(y) \) defined in (4.7) and (4.12), respectively.

**Theorem 12.1** Let the scattering data set \( S \) in (4.2) correspond to the input data set \( D \) in (4.1) that belongs to the Faddeev class specified in Definition 4.1, where \( S(k) \) is the corresponding scattering matrix defined in (9.3), the \( \kappa_j \) are the distinct positive constants appearing in (9.8) related to the bound states, and \( M_j \) are the \( n \times n \) normalization matrices appearing in (4.12) and (11.22). Let \( F_s(y) \) and \( F(y) \) be the \( n \times n \) matrix-valued functions defined in (4.7) and (4.12), respectively. We then have the following:

(a) The matrices \( F_s(y) \) and \( F(y) \) are both hermitian, i.e. \( F_s(y)^\dagger = F_s(y) \) for each \( y \in \mathbb{R}^+ \cup \mathbb{R}^- \) and \( F(y)^\dagger = F(y) \) for each \( y \in \mathbb{R}^+ \).

(b) The derivative matrix \( F_s'(y) \) is hermitian, i.e. \( F_s'(y)^\dagger = F_s'(y) \) for each \( y \in \mathbb{R}^+ \cup \mathbb{R}^- \).

(c) The matrices \( F_s(y) \) and \( F(y) \) are continuous and bounded in \( y \in \mathbb{R}^+ \), and they both vanish as \( y \to +\infty \).

(d) The matrix \( F_s(y) \) is continuous and bounded in \( y \in \mathbb{R}^- \), and it vanishes as \( y \to -\infty \).

(e) The matrices \( F_s(y) \) and \( F(y) \) are each bounded and integrable in \( y \in \mathbb{R}^+ \).

(f) For \( y \in \mathbb{R}^- \) the matrix \( F_s(y) \) can be written as a sum as

\[
F_s(y) = F_s^{(1)}(y) + F_s^{(2)}(y), \quad y \in \mathbb{R}^-,
\]

where \( F_s^{(1)}(y) \) is bounded and integrable for \( y \in \mathbb{R}^- \) and \( F_s^{(2)}(y) \) is bounded and square integrable for \( y \in \mathbb{R}^- \). Consequently, \( F_s(y) \) itself is bounded and square integrable in \( y \in \mathbb{R}^- \).

(g) The matrix \( F_s(y) \) has a jump discontinuity at \( y = 0 \), which is given by

\[
F_s(0^+) - F_s(0^-) = G_1
\]
where $G_1$ is the constant matrix appearing in (10.17). Hence, $F_s'(y)$ contains a delta-function term at $y = 0$, which is given by $G_1 \delta(y)$.

(h) For $y \in \mathbb{R}^-$, the matrix $F_s'(y)$ can be written as a sum of two matrix-valued functions, one of which is integrable and the other is square integrable.

PROOF: Note that $S_\infty$ appearing in (4.6) and (4.7) corresponds to the scattering matrix when the potential is zero. Hence, from Proposition 10.3(a) it follows that both $S(k)$ and $S_\infty$ satisfy (4.4). With the help of the first equality in (4.4) for $S(k)$ and $S_\infty$, from (4.7) we conclude that $F_s(y)$ is hermitian. The matrix $M_j$ given in (11.22) is hermitian because the matrix $B_j^{-1/2}$ is hermitian as stated Proposition 11.2(c) and the matrix $P_j$ is hermitian as indicated by the first equality in (11.1). Since $\kappa_j$ appearing in (4.12) is positive for each $j = 1, \ldots, N$, we also conclude that $F(y)$ is hermitian. Thus, (a) holds. From (4.7) we have

$$F_s'(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{iG(k)}{k+i} e^{iky}. \quad (12.3)$$

Using the first equality in (4.4) for $S(k)$ and $S_\infty$, from (12.3) we conclude that $F_s(y)$ is hermitian, and hence (b) is proved. For $y \in \mathbb{R}^+$, it is readily seen that the summation part in (4.12) is continuous, bounded, and integrable on $\mathbb{R}^+$ and it vanishes as $y \to +\infty$. Thus, it is sufficient to prove (c)-(f) only for $F_s(y)$. Let us decompose $S(k) - S_\infty$ as

$$S(k) - S_\infty = \left[ S(k) - S_\infty + \frac{iG(k)}{k+i} \right] - \frac{iG(k)}{k+i}, \quad (12.4)$$

where $G(k)$ is the matrix appearing in (10.14) and (10.16). Using (12.4) in (4.7) we see that

$$F_s(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left[ S(k) - S_\infty + \frac{iG(k)}{k+i} \right] e^{iky} - \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{iG(k)}{k+i} e^{iky}. \quad (12.5)$$

With the help of Proposition 10.3, we conclude that the term in the brackets on the right-hand side in (12.4) is continuous in $k \in \mathbb{R}$ and is $O(1/k^2)$ as $k \to \pm\infty$. Hence, that term is both integrable and square integrable in $k \in \mathbb{R}$. Thus, the first integral in (12.5) is continuous in $y \in \mathbb{R}$, vanishes as $y \to \pm\infty$, and is square integrable in $y \in \mathbb{R}$. It is also
bounded in $y \in \mathbb{R}$ as a result of the continuity in $y \in \mathbb{R}$ and the zero asymptotics as $y \to \pm \infty$. Thus, we only need to show that (c)-(f) hold for the second integral in (12.5). From Proposition 10.3(b) we know that $G(k)$ is the sum of $G_1$ and $G_2(k)$ given in (10.17) and (10.18), respectively. Hence, with the help of (10.17) and (10.18), we are able to evaluate the second integral in (12.5) explicitly by using the residues, and we get

$$
-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{k + i} e^{iky} \begin{cases} 
  g_+(y), & y > 0, \\
  g_-(y), & y < 0,
\end{cases}
$$

(12.6)

where we have defined

$$
g_+(y) := -\frac{1}{2} \int_{y/2}^{\infty} dz V(z) e^{-(2z-y)},
$$

(12.7)

and

$$
g_-(y) := -G_1 e^y - \frac{1}{2} \int_0^{y/2} dz V(z) e^{-(2z-y)} - \frac{1}{2} \int_{-y/2}^{y/2} dz S_\infty V(z) S_\infty e^{2z+y}.
$$

(12.8)

Our proof for (c)-(f) will be complete if we can show that $g_+(y)$ given in (12.7) is continuous and bounded in $y \in \mathbb{R}^+$, vanishes as $y \to +\infty$, and is integrable on $y \in \mathbb{R}^+$ and also that $g_-(y)$ given in (12.8) is continuous and bounded in $y \in \mathbb{R}^-$, vanishes as $y \to -\infty$, and is integrable on $y \in \mathbb{R}^-$. We will show that (2.3) guarantees the aforementioned properties.

In terms of $\sigma(x)$ and $\sigma_1(x)$ defined in (3.96), from (12.7) we obtain

$$
||g_+(y)|| \leq \frac{1}{2} \int_{y/2}^{\infty} dz |V(z)| \leq \frac{1}{2} \sigma \left( \frac{y}{2} \right).
$$

(12.9)

From (2.3), (3.96), and (12.9) it follows that $g_+(y)$ is bounded and integrable in $y \in \mathbb{R}^+$ and vanishes as $y \to +\infty$. Let us now prove the continuity of $g_+(y)$ in $\mathbb{R}^+$. From (12.7) we see that

$$
e^{-y}g_+(y) = -\frac{1}{2} \int_{y/2}^{\infty} dz V(z) e^{-2z},
$$

(12.10)

where the integrand is integrable as a result of (2.3). Thus, by the Lebesgue differentiation theorem, the right-hand side is a continuous function of $y$. Since $e^y$ is continuous, the function $g_+(y)$, being a product of two continuous functions, is also continuous in $y \in \mathbb{R}^+$. In a similar manner, one can prove that $g_-(y)$ is continuous and bounded in $y \in \mathbb{R}^-$, vanishes as $y \to -\infty$, and is integrable on $y \in \mathbb{R}^-$. Note that the first term, $-G_1 e^y$, on the right-hand side of (12.8) is readily seen to satisfy these four properties. For the second
term on the right-hand side of (12.8), by taking $e^y$ outside the integral, we see that the second integral is the product of $e^y$ with a constant $n \times n$ matrix, and hence it readily satisfies all the four properties. As for the third term on the right-hand side of (12.8), let us break it into two terms as

$$-\frac{1}{2} \int_{0}^{-y/2} dz S_{\infty} V(z) S_{\infty} e^{2z+y} = -\frac{1}{2} e^{y/2} \int_{0}^{-y/4} dz S_{\infty} V(z) S_{\infty} e^{2z+y/2}$$

$$-\frac{1}{2} \int_{-y/4}^{-y/2} dz S_{\infty} V(z) S_{\infty} e^{2z+y}.$$  \hspace{1cm} (12.11)

For the first term on the right-hand side of (12.11) we have

$$\left| -\frac{1}{2} e^{y/2} \int_{0}^{-y/4} dz S_{\infty} V(z) S_{\infty} e^{2z+y/2} \right| \leq \frac{1}{2} e^{y/2} |s_{\infty}|^2 \int_{0}^{-y/4} dz |V(z)|$$

$$\leq \frac{1}{2} e^{y/2} |s_{\infty}|^2 \sigma(0).$$ \hspace{1cm} (12.12)

For the second term on the right-hand side of (12.11) we get

$$\left| -\frac{1}{2} \int_{-y/4}^{-y/2} dz S_{\infty} V(z) S_{\infty} e^{2z+y} \right| \leq \frac{1}{2} |s_{\infty}|^2 \int_{-y/4}^{-y/2} dz |V(z)|$$

$$= \frac{1}{2} |s_{\infty}|^2 \left[ \sigma \left( -\frac{y}{4} \right) - \sigma \left( -\frac{y}{2} \right) \right].$$ \hspace{1cm} (12.13)

The left-hand side of (12.12) is bounded and integrable in $y \in \mathbb{R}^-$ and it vanishes as $y \to -\infty$ because that left-hand side is bounded by a constant multiple of $e^{y/2}$ on $\mathbb{R}^-$. From (12.13) we see that its left-hand side is bounded by a constant multiple of $\sigma(-y/4)$, and we already know from (2.3) and the first equality in (3.96) that $\sigma(-y/4)$ is bounded on $\mathbb{R}^-$ and vanishes as $y \to -\infty$ and we also know from (2.3) and the second equality in (3.96) that $\sigma(-y/4)$ is integrable in $\mathbb{R}^-$. Hence, the left-hand side of (12.13) is bounded and integrable on $\mathbb{R}^-$ and it vanishes as $y \to -\infty$. In order to complete the proof of our theorem, we only need to prove that the left-hand side of (12.11) is continuous in $y \in \mathbb{R}^-$. For a given $y \in \mathbb{R}^-$, we can find a constant $a < 0$ so that $a < y$. We can write the left-hand side of (12.11) as

$$-\frac{1}{2} \int_{0}^{-y/2} dz S_{\infty} V(z) S_{\infty} e^{2z+y} = -\frac{1}{2} e^{y-a} \int_{0}^{-y/2} dz S_{\infty} V(z) S_{\infty} e^{2z+a}.$$ \hspace{1cm} (12.14)
By the Lebesgue differentiation theorem, the integral on the right-hand side of (12.14) is a continuous function of \( y \) in \( y \in [a,0) \) because the integrand is integrable as a result of (2.3). Since \( e^{y-a} \) is also continuous, we conclude that the right-hand side of (12.14) is continuous in \( y \in [a,0) \) for every \( a < 0 \). Thus, we have completed the proof that the left-hand side of (12.11) is continuous in \( y \in \mathbb{R}^{-} \). Having completed the proof of (c)-(f), let us now prove (g). Earlier in the proof, we have already indicated that the first integral in (12.5) is continuous in \( y \in \mathbb{R} \). Thus, from (12.5) and (12.6), it follows that the left-hand side of (12.2) is given by

\[
F_s(0^+) - F_s(0^-) = g_+(0) - g_-(0).
\]

(12.15)

Using (12.7) and (12.8) on the right-hand side of (12.15), we get (12.2). Thus, the proof of (g) is complete. Finally, let us prove (h). From (10.14)-(10.18) it follows that

\[
\{ ik \left( S(k) - S_\infty \right) - G_1 - G_2(k) \} e^{iky} = \frac{1}{2} \left( \frac{1}{k} \right), \quad k \to \pm \infty.
\]

(12.16)

By Proposition 10.3(a) we know that the scattering matrix \( S(k) \) is continuous in \( k \in \mathbb{R} \). Hence, from (12.15) we conclude that the left-hand side of (12.16) is square integrable in \( k \in \mathbb{R} \). Using (10.17), (10.18), (11.37), and (12.16) we conclude that

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \{ ik \left( S(k) - S_\infty \right) - G_1 - G_2(k) \} e^{iky} = F'_s(y) - G_1 \delta(y) - \frac{1}{2} V \left( \frac{y}{2} \right) - \frac{1}{2} S_\infty V \left( \frac{-y}{2} \right) S_\infty,
\]

(12.17)

with the understanding that \( V(x) = 0 \) for \( x \in \mathbb{R}^- \). When \( y \in \mathbb{R}^- \), from (2.3) we know that the last term on the right-hand side of (12.17) is integrable. The left-hand side of (12.17) is square integrable in \( y \in \mathbb{R} \) because it is the Fourier transform of a square-integrable function of \( k \in \mathbb{R} \). Thus, from (12.17) we conclude (h).
13. THE MARCHENKO INTEGRAL EQUATION

In this chapter, when the input data set $D$ given in (4.1) belongs to the Faddeev class, we derive the matrix Marchenko integral equation and provide the basic properties of its kernel. We also introduce the derivative Marchenko integral equation (13.7), whose kernel coincides with the kernel of (1.1) but whose nonhomogeneous term differs from the nonhomogeneous term of (1.1). We remark that the kernel of (13.1) coincides with the kernel of (13.7) and the two equations only differ by their nonhomogeneous terms. In the Dirichlet case, the boundary matrix appearing $A$ in (2.4) is the zero matrix and the boundary matrix $B$ then can be chosen as the identity matrix $I$. Thus, in the Dirichlet case, as it is studied in [2], only the Marchenko equation (13.1) plays a relevant role in the inverse problem, and the derivative Marchenko equation (13.7) is hardly relevant to the inverse problem. On the other hand, in the case of the general selfadjoint boundary condition, which we study in this monograph, the roles of (13.1) and (13.7) are equally important in the analysis of the inverse problem.

**Theorem 13.1** Let the scattering data set $S$ in (4.2) correspond to the input data set $D$ in (4.1) that belongs to the Faddeev class specified in Definition 4.1, and let $K(x, y)$ be the quantity appearing in (10.1), and $F_s(y)$ and $F(y)$ be the quantities defined in (4.7) and (4.12), respectively. Then, $K(x, y)$ satisfies the Marchenko integral equation given by

$$K(x, y) + F(x + y) + \int_x^\infty dz K(x, z) F(z + y) = 0, \quad 0 \leq x < y. \quad (13.1)$$

**PROOF:** When $D$ belongs to the Faddeev class, the existence of $K(x, y)$ and its properties are assured by Proposition 10.1. The existence of $F_s(y)$ and $F(y)$ and their properties are assured by Theorem 12.1. Let us write (9.4) as

$$[f(-k, x) - e^{-ikx}I] + [S(k) - S^\in\infty]e^{ikx} + [f(k, x) - e^{ikx}I][S(k) - S^\in\infty]$$

$$= \Psi(k, x) - e^{-ikx}I - S^\in\infty e^{ikx} - [f(k, x) - e^{ikx}I]S^\in\infty. \quad (13.2)$$
Taking the Fourier transform of both sides of (13.2) and using (10.1) and (11.37), we obtain

\[
K(x, y) + F_s(x + y) + \int_x^\infty dz K(x, z) F_s(z + y) = \frac{1}{2\pi} \int_{-\infty}^\infty dk \Psi(k, x) e^{iky} - I\delta(y - x) - S\delta(y + x) - K(x, -y) S\delta.
\]

(13.3)

When \( y > x \geq 0 \), with the help of (10.2) we see that only the first term on the right-hand side of (13.3) is nonzero and in fact that term is explicitly evaluated in (11.32). Thus, using (11.32) in (13.3) we get

\[
K(x, y) + F_s(x + y) + \int_x^\infty dz K(x, z) F_s(z + y) = -\sum_{j=1}^N f(ik_j, x) M_j^2 e^{-k_jy}. \quad 0 \leq x < y.
\]

(13.4)

From (10.6) we obtain

\[
f(ik_j, x) M_j^2 e^{-k_jy} = e^{-k_j(x+y)} M_j^2 + \int_x^\infty dz K(x, z) e^{-k_j(z+y)} M_j^2.
\]

(13.5)

Using (13.5) on the right-hand side of (13.4), with the help of (4.12), we write (13.4) as (13.1).

By taking the \( x \)-derivative of the Marchenko integral equation (13.1) we obtain the integral equation

\[
K_x(x, y) + F'(x + y) - K(x, x) F(x + y) + \int_x^\infty dz K_x(x, z) F(z + y) = 0, \quad 0 \leq x < y,
\]

(13.6)

where we recall that the subscript \( x \) is used to denote the \( x \)-derivative. We call the integral equation associated with (13.6), i.e.

\[
L(x, y) + F'(x + y) - K(x, x) F(x + y) + \int_x^\infty dz L(x, z) F(z + y) = 0, \quad 0 \leq x < y,
\]

(13.7)

the derivative Marchenko integral equation. We remark that (13.7) along with the Marchenko equation (13.1) plays a key role in the analysis of the inverse scattering problem related to (2.1) with the general selfadjoint boundary condition (2.4). Its solvability in the context of the inverse problem is analyzed in Proposition 16.5.
We have the following further comment on comparing the Marchenko equation (13.1) and the derivative Marchenko equation (13.7) when the input data set $D$ belongs to the Faddeev class. Concerning the Marchenko equation (13.1), as pointed out in Theorem 12.1(c) the nonhomogeneous term $F(x + y)$ in (13.1) is continuous when $y > x \geq 0$, and as pointed out in Theorem 12.1(c) Proposition 10.1(e) the solution $K(x, y)$ to (13.1) is continuous when $y > x \geq 0$. On the other hand, concerning the derivative Marchenko equation (13.7), the nonhomogeneous term contains $F'(x + y)$ and that term is in general not continuous when $y > x \geq 0$, and as indicated in Proposition 10.1(f) the quantity $K_x(x, y)$ is in general not continuous and exists a.e. but for each $x \geq 0$ it is integrable in $y \in [x, +\infty)$.

In the next theorem, certain relevant properties of the kernel of the Marchenko equation (13.1) is presented when the input data set $D$ belongs to the Faddeev class.

**Theorem 13.2** Let the scattering data set $S$ in (4.2) correspond to the input data set $D$ in (4.1) that belongs to the Faddeev class specified in Definition 4.1, and let $F_s(y)$ and $F(y)$ be the quantities defined in (4.7) and (4.12), respectively. We then have the following:

(a) The matrix $F(y)$ satisfies

$$|F(y)| \leq C \sigma \left(\frac{y}{2}\right), \quad y \in \mathbb{R}^+, \quad (13.8)$$

where $C$ is a generic constant and $\sigma(x)$ is the quantity defined in (3.96). Furthermore, we have

$$\int_0^{\infty} dy |F(y)| < +\infty, \quad (13.9)$$
$$\int_0^{\infty} dy (1 + y) |F(y)|^2 < +\infty. \quad (13.10)$$

(b) The derivative $F'(y)$ exists a.e. for $y \in \mathbb{R}^+$ and satisfies

$$\left|F'(y) - \frac{1}{4} V \left(\frac{y}{2}\right)\right| \leq C \left[\sigma \left(\frac{y}{2}\right)\right]^2, \quad y \in \mathbb{R}^+, \quad (13.11)$$

where $C$ is a generic constant and $V(x)$ is the potential appearing in the data set $D$. 98
(c) The derivative $F'(y)$ satisfies

$$\int_0^\infty dy (1 + y) |F'(y)| < +\infty.$$  \hfill (13.12)

PROOF: The inequality in (13.8) can be found in (3.2.4) of [2], where the main idea behind the proof in [2] is to view the Marchenko equation (13.1) with $K(x, y)$ as being the input and $F(x + y)$ as being the unknown quantity, to use (10.7), and to get the corresponding property of $F(x + y)$. The inequality in (13.11) is given in (3.2.7) of [2], where the basic idea behind the proof is to get the appropriate property of $F'(y)$ from (13.6) with the help of (13.8) and (10.9). Thus, it is enough to establish (13.9), (13.10), and (13.12). To prove (13.9), we integrate both sides of (13.8) over $y \in \mathbb{R}^+$ and use the second definition in (3.97) and obtain

$$\int_0^\infty dy |F(y)| \leq C \int_0^\infty dy \sigma\left(\frac{y}{2}\right) = 2C \sigma_1(0) < +\infty,$$  \hfill (13.13)

which establishes (13.9). Let us now prove (13.10). Squaring both sides of (13.8) and then multiplying by $(1 + y)$ and integrating over $y \in \mathbb{R}^+$, we obtain

$$\int_0^\infty dy (1 + y) |F'(y)|^2 \leq C^2 \int_0^\infty dy (1 + y) \sigma\left(\frac{y}{2}\right) \leq C^2 [2\sigma(0) \sigma_1(0) + 4[\sigma_1(0)]^2] < +\infty,$$  \hfill (13.14)

where we have used (3.98) and (3.99). Thus, (13.10) is established. Let us finally prove (13.12). Since we have

$$|F'(y)| \leq \frac{1}{4} \left| V\left(\frac{y}{2}\right) \right| + \left| F'(y) - \frac{1}{4} V\left(\frac{y}{2}\right) \right|,$$  \hfill (13.15)

after using (13.11) in (13.15), we can multiply both sides of the resulting inequality by $(1 + y)$ and integrate over $y \in \mathbb{R}^+$ in order to obtain

$$\int_0^\infty dy (1 + y) |F'(y)| \leq \frac{1}{4} \int_0^\infty dy (1 + y) \left| V\left(\frac{y}{2}\right) \right| + C \int_0^\infty dy (1 + y) \left[ \sigma\left(\frac{y}{2}\right) \right]^2.$$  \hfill (13.16)

By (2.3) the first integral on the right-hand side in (13.16) is finite. The finiteness of the second integral follows from (3.98) and (3.99). Thus, (13.11) is established. \hfill \blacksquare
14. THE BOUNDARY MATRICES

In this chapter, when the input data set $D$ belongs to the Faddeev class, we show that the boundary matrices $A$ and $B$ appearing in (2.4)-(2.6) are related to the large-$k$ limit of the scattering matrix $S(k)$.

**Proposition 14.1** Let the input data set $D$ in (4.1) belong to the Faddeev class specified in Definition 4.1. Let $S$ in (4.2) be the scattering data set corresponding to $D$, i.e. $S(k)$ be the scattering matrix defined as in (9.3), where the Jost matrix $J(k)$ is defined as in (9.2). Let $S_\infty$ be the constant $n \times n$ matrix appearing in (4.6), $G_1$ be the constant $n \times n$ matrix appearing in (10.14) and (10.17) and equivalently given as

$$G_1 = \lim_{k \to \pm\infty} ik[S(k) - S_\infty],$$  \hspace{1cm} (14.1)

and $K(0,0)$ be the constant $n \times n$ matrix obtained as in (10.5) from $K(x,y)$ appearing in (10.1) and (13.1). Then, the boundary matrices $A$ and $B$ appearing in (2.4)-(2.6) and (4.1) satisfy the linear homogeneous matrix system

$$\begin{cases}
(I - S_\infty) A = 0, \\
(I + S_\infty) B = [G_1 - S_\infty K(0,0) - K(0,0) S_\infty] A.
\end{cases}$$ \hspace{1cm} (14.2)

**PROOF:** When $D$ belongs to the Faddeev class, the scattering matrix is constructed as in the steps of (a)-(c) of Chapter 9 leading to (9.3). From (9.3) we see that

$$-J(-k) = S(k) J(k), \hspace{1cm} k \in \mathbb{R}. \hspace{1cm} (14.3)$$

The large-$k$ asymptotics of the Jost matrix $J(k)$ is given in (10.8), from which we get

$$-J(-k) = -ikA - B - K(0,0) A + o(1), \hspace{1cm} k \to \pm\infty. \hspace{1cm} (14.4)$$

From Proposition 10.3(b) we have

$$S(k) = S_\infty + G_1 \frac{1}{ik} + o\left(\frac{1}{k}\right), \hspace{1cm} k \to \pm\infty. \hspace{1cm} (14.5)$$
Using (14.4) and (14.5) in (14.3), we obtain the expansion

\[-ikA - B - K(0, 0) A + o(1) = -ik S_\infty A + S_\infty B + S_\infty K(0, 0) A - G_1 A + o(1), \quad k \to \pm \infty.\]

(14.6)

By equating the coefficients of \(ik\) in (14.6) we obtain the first line of (14.2) and by equating the next order terms, i.e. the constant terms, in (14.6), we obtain

\[(I + S_\infty) B = [G_1 - S_\infty K(0, 0) - K(0, 0)] A.\]

(14.7)

Since we already have \(S_\infty A = A\) from the first line in (14.2), we can use that identity in (14.7) to obtain the second line of (14.2). □

The following result is useful in the analysis of the selfadjoint boundary condition given in (2.4).

**Proposition 14.2** Let \(\psi(x)\) and \(\phi(x)\) be two \(n \times n\) matrices satisfying the boundary condition (2.4), where the boundary matrices \(A\) and \(B\) satisfy (2.5) and (2.6). Then, we have

\[\psi'(0) \dagger \phi(0) - \psi(0) \dagger \phi'(0) = 0.\]

(14.8)

The result remains valid when \(\psi(x)\) and \(\phi(x)\) are column vectors with \(n\) entries.

**PROOF:** Since \(\psi(x)\) and \(\phi(x)\) satisfy (2.4) we have

\[-B \dagger \psi(0) + A \dagger \psi'(0) = 0, \quad -B \dagger \phi(0) + A \dagger \phi'(0) = 0,\]

(14.9)

The boundary matrices \(A\) and \(B\) appearing in (2.4) and satisfying (2.5) and (2.6) also satisfy (2.9), where \(E\) is the invertible matrix defined in (2.7). The left-hand side in (14.8) can be evaluated with the help of the first equality in (2.9) as

\[\psi'(0) \dagger \phi(0) - \psi(0) \dagger \phi'(0) = \psi'(0) \dagger \left[AE^{-2} A \dagger + BE^{-2} B \dagger\right] \phi(0) - \psi(0) \dagger \left[AE^{-2} A \dagger + BE^{-2} B \dagger\right] \phi'(0),\]

(14.10)

which can be written as

\[\psi'(0) \dagger I \phi(0) - \psi(0) \dagger I \phi'(0) = \left[A \dagger \psi'(0) \right] \dagger E^{-2} [A \dagger \phi(0)] + \left[B \dagger \psi'(0) \right] \dagger E^{-2} [B \dagger \phi(0)] - \left[A \dagger \psi(0) \right] \dagger E^{-2} [A \dagger \phi'(0)] - \left[B \dagger \psi(0) \right] \dagger E^{-2} [B \dagger \phi'(0)].\]

(14.11)
Using (14.9) on the right-hand side of (14.11), we obtain

\[
\psi'(0) \dagger \phi(0) - \psi(0) \dagger \phi'(0) \\
= [B^\dagger \psi(0)]^\dagger E^{-2} [A^\dagger \phi(0)] + [B^\dagger \psi'(0)]^\dagger E^{-2} [A^\dagger \phi'(0)] \\
- [A^\dagger \psi(0)]^\dagger E^{-2} [B^\dagger \phi(0)] - [A^\dagger \psi'(0)]^\dagger E^{-2} [B^\dagger \phi'(0)].
\]

(14.12)

We can rewrite (14.12) by rearranging its right-hand side and we get

\[
\psi'(0) \dagger \phi(0) - \psi(0) \dagger \phi'(0) \\
= \psi(0)^\dagger B E^{-2} A^\dagger \phi(0) + \psi'(0)^\dagger B E^{-2} A^\dagger \phi'(0) \\
- \psi(0)^\dagger A E^{-2} B^\dagger \phi(0) - \psi'(0)^\dagger A E^{-2} B^\dagger \phi'(0),
\]

(14.13)

or equivalently we get

\[
\psi'(0) \dagger \phi(0) - \psi(0) \dagger \phi'(0) \\
= \psi'(0)^\dagger [B E^{-2} A^\dagger - A E^{-2} B^\dagger] \phi(0) + \psi(0)^\dagger [B E^{-2} A^\dagger - A E^{-2} B^\dagger] \phi'(0),
\]

(14.14)

Using the second equality in (2.9) on the right-hand side of (14.14), we see that the right-hand side vanishes and hence (14.14) yields (14.8). We remark that the result in (14.8) also remains valid if \( \psi(x) \) and \( \phi(x) \) are column vectors with \( n \) components because the left-hand side in (14.8) is well defined in that case and is a scalar.
15. THE EXISTENCE AND UNIQUENESS IN THE DIRECT PROBLEM

In this chapter we provide various results related to the solution of the direct problem when the input data set \( \mathbf{D} \) belongs to the Faddeev class specified in Definition 4.1. In particular, we provide various results related to the solvability of the three key integral equations given in (4.22), (4.14), and (4.17), respectively, as well as various functional equations in \( k \in \mathbb{R} \) appearing in Definitions 4.2 and 4.3. Such results are used to prove that if the input data \( \mathbf{D} \) belongs to the Faddeev class specified in Definition 4.1, then a corresponding scattering data set \( \mathbf{S} \) exists, is unique, and belongs to the Marchenko class specified in Definition 4.5. We also provide a proof of the alternate formulations of the characterization condition \((4_0)\) stated in Proposition 4.3, which is established in Proposition 15.4.

In the following proposition we apply Propositions 3.3 to the specific operators related to (4.22) and (4.14).

**Proposition 15.1** Consider a scattering data set \( \mathbf{S} \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Assume that \( \mathbf{S} \) satisfies \((1)\) of Definition 4.3. Let \( F_s(y) \) and \( F(y) \) be the matrices defined in (4.7) and (4.12), respectively. Then we have the following:

(a) The integral operator associated with (4.22) is compact on \( L^1(\mathbb{R}^+) \).

(b) The integral operator associated with (4.14) is compact on \( L^1(\mathbb{R}^+) \).

(c) Any solution \( X(y) \) in \( L^1(\mathbb{R}^+) \) to (4.22) must actually belong to \( L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \), and in particular to \( L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \).

(d) Any solution \( X(y) \) in \( L^1(\mathbb{R}^+) \) to (4.14) must actually belong to \( L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \), and in particular to \( L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \).

**PROOF:** Since the \( \kappa_j\)-values are all positive, we see that \( F(y) \) defined in (4.12) is also
bounded and integrable in \( y \in \mathbb{R}^+ \) when \( F_s(y) \) is bounded and integrable in \( y \in \mathbb{R}^+ \), which is assured by (I). Thus, from Proposition 3.3 we conclude that (a)-(d) hold.

Next, we apply Propositions 3.2 to the specific operator related to (4.17).

**Proposition 15.2** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Assume that \( S \) satisfies (I) of Definition 4.3. Then:

(a) The integral operator associated with (4.17) is compact on \( L^2(\mathbb{R}^-) \).

(b) Any solution \( X(y) \) in \( L^2(\mathbb{R}^-) \) to (4.17) must actually belong to \( L^2(\mathbb{R}^-) \cap L^\infty(\mathbb{R}^-) \).

PROOF: By (I) we already know that \( F_s(y) \) is bounded and square integrable in \( y \in \mathbb{R} \). From Proposition 3.5 we then conclude that (a) and (b) hold.

One consequence of the following result is the equivalence among (4.1c), (4.1d), and (4.1e) in Proposition 4.3.

**Proposition 15.3** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Assume that \( S \) satisfies (I) of Definition 4.3. Let \( F_s(y) \) and \( F(y) \) be the quantities defined in (4.7) and (4.12), respectively, where \( S_\infty \) is the constant \( n \times n \) matrix defined as in (4.6). Then, we have the following:

(a) Any solution \( \hat{X}(k) \) in \( \hat{L}^1(\mathbb{C}^+) \) to (4.15) must actually belong to \( \hat{L}^1(\mathbb{C}^+) \).

(b) Any solution \( h(k) \) in \( \hat{L}^1(\mathbb{C}^+) \) to (4.16) must actually belong to \( \hat{L}^1(\mathbb{C}^+) \).

(c) The row vector \( \hat{X}(k) \) with \( n \) components belonging to \( \hat{L}^1(\mathbb{C}^+) \) satisfies (4.15) if and only if the column vector \( h(k) \) with \( n \) components belonging to \( \hat{L}^1(\mathbb{C}^+) \) satisfies (4.16), where \( \hat{X}(k) \) and \( h(k) \) are related to each other as \( h(k) = \hat{X}(-k^*)^\dagger \).

(d) The row vector \( X(y) \) whose \( n \) components belonging to \( L^1(\mathbb{R}^+) \) is a solution to (4.14)
if and only if \( \hat{X}(k) \in \hat{L}^1(C^+) \) satisfies (4.15), where \( X(y) \) and \( \hat{X}(k) \) are related to each other as in (3.67) and (3.68).

**PROOF:** From Proposition 15.1(d) we know that any solution \( X(y) \) in \( L^1(R^+) \) to (4.14) must actually belong to \( \hat{L}^1(R^+) \cap L^\infty(R^+) \). Since \( \hat{X}(k) \) is related to \( X(y) \) as in (3.67) and (3.68), it follows that we must have \( \hat{X}(k) \in \hat{L}^1(C^+) \) and in fact \( \hat{X}(k) \in \hat{L}^1_\infty(C^+) \). Thus (a) is proved. Actually, (b) is a direct consequence of (c) because \( h(k) = \hat{X}(-k^*)^\dagger \). On the other hand, the proof of (c) is obtained by taking the matrix adjoint of both sides (4.15) and by using \( S(k)^\dagger = S(-k) \), which follows from (4.4). Thus, it only remains to prove (d).

As already mentioned, \( X(y) \) belongs to both \( \hat{L}^1(R^+) \) and \( L^\infty(R^+) \), and hence \( X(y) \) must in particular belong to \( L^2(R^+) \). Therefore, it is sufficient to give the proof of (d) by only assuming that \( \hat{X}(k) \in H^2(C^+) \). Let us first show that if \( \hat{X}(k) \in H^2(C^+) \) satisfies (4.15), then \( X(y) \in L^2(R^+) \) given in (3.67) satisfies (4.14). From the second line of (4.15) we obtain

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{X}(-k) e^{iky} + \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{X}(k) S(k) e^{iky} = 0, \tag{15.1}
\]

Using (3.67) and (4.7) in (15.1), with the help of (11.36) we obtain

\[
X(y) + X(-y) S_\infty + \int_{-\infty}^{\infty} dz X(z) F_s(z + y), \quad y \in R. \tag{15.2}
\]

Using \( X(y) = 0 \) for \( y \in R^- \), from (15.2) we get

\[
X(y) + \int_{0}^{\infty} dz X(z) F_s(z + y) = 0, \quad y \in R^+. \tag{15.3}
\]

Using (4.12) in (15.3) we get

\[
X(y) + \int_{0}^{\infty} dz X(z) \left[ F(z + y) - \sum_{j=1}^{N} M_j^2 e^{-\kappa_j(z+y)} \right] = 0, \quad y \in R^+. \tag{15.4}
\]

With the help of (3.68) we can write (15.4) as

\[
X(y) + \int_{0}^{\infty} dz X(z) F(z + y) - \sum_{j=1}^{N} \hat{X}(i\kappa_j) M_j M_j e^{-\kappa_j y} = 0, \quad y \in R^+. \tag{15.5}
\]
Each term in the summation is zero because of the first line of (4.15), yielding (4.14). Thus, we have proved that (4.15) implies (4.14). Let us now prove the converse, namely, show that if $X(y) \in L^2(\mathbb{R}^+)$ satisfies (4.14), then $\hat{X}(k)$ satisfies (4.15). It is clear that if $X(y) \equiv 0$, then the assertion clearly holds. We can then proceed by assuming that $X(y)$ is a nontrivial solution to (4.14). Let us multiply both sides of (4.14) with $X(y)^\dagger$ and integrate over $y \in \mathbb{R}$ with the understanding that $X(y) = 0$ for $y \in \mathbb{R}^-$. Thus, if $X(y)$ satisfies (4.14) then we have

$$\int_{-\infty}^{\infty} dy \left[ X(y) + \int_{-\infty}^{\infty} dz \, X(z) \, F(z+y) \right] X(y)^\dagger = 0. \quad (15.6)$$

Using (3.67), (4.7), and (4.12) in (15.6), with the help of (11.37) we simplify the resulting equation and obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left( \hat{X}(-k) + \hat{X}(k) \, [S(k) - S_\infty] \right) \hat{X}(-k)^\dagger + \sum_{j=1}^{N} \hat{X}(i\kappa_j) \, M_j^2 \hat{X}(i\kappa_j)^\dagger = 0. \quad (15.7)$$

Since the matrices $M_j$ are hermitian, we can write the summation term in (15.7) as

$$\sum_{j=1}^{N} \hat{X}(i\kappa_j) \, M_j^2 \hat{X}(i\kappa_j)^\dagger = \sum_{j=1}^{N} [\hat{X}(i\kappa_j) \, M_j] \, [\hat{X}(i\kappa_j)^\dagger \, M_j] = \sum_{j=1}^{N} |\hat{X}(i\kappa_j) \, M_j|^2, \quad (15.8)$$

which indicates that the right-hand side in (15.8) is nonnegative and that it is zero if and only if we have each vector $\hat{X}(i\kappa_j) \, M_j$ is equal to zero for $j = 1, \ldots, N$. We can simplify the integral part of (15.7) further by using

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \hat{X}(k) \, S_\infty \hat{X}(-k)^\dagger = \int_{-\infty}^{\infty} dy \, X(y) \, S_\infty \, X(-y)^\dagger = 0, \quad (15.9)$$

which follows from (3.68) and the fact that $X(y) = 0$ for $y \in \mathbb{R}^-$. Thus, (15.7) is equivalent to

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left( \hat{X}(-k) + \hat{X}(k) \, S(k) \right) \hat{X}(-k)^\dagger + \sum_{j=1}^{N} |\hat{X}(i\kappa_j) \, M_j|^2 = 0. \quad (15.10)$$

Note that (15.10) is equivalent to

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left( \hat{X}(k) + \hat{X}(-k) \, S(-k) \right) \hat{X}(k)^\dagger + \sum_{j=1}^{N} |\hat{X}(i\kappa_j) \, M_j|^2 = 0. \quad (15.11)$$
By letting $\hat{X}_1(k) := \hat{X}(-k) S(-k)$, we notice that

$$||\hat{X}_1||_2 = ||\hat{X}||_2.$$  \hspace{1cm} (15.12)

We remark that (15.12) is a consequence of the unitarity of $S(k)$ and is seen from

\begin{align*}
\left(\hat{X}_1, \hat{X}_1\right) &= \int_0^\infty dk \left[\hat{X}(-k) S(-k)[\hat{X}(-k) S(-k)]^\dagger\right] \\
&= \int_0^\infty dk \hat{X}(-k) S(-k) S(-k)^\dagger \hat{X}(-k)^\dagger \\
&= \int_0^\infty dk \hat{X}(-k) \hat{X}(-k)^\dagger = \left(\hat{X}, \hat{X}\right) = ||\hat{X}||^2,
\end{align*}

which is equivalent to

$$||\hat{X}_1||_2^2 = ||\hat{X}||_2^2,$$  \hspace{1cm} (15.14)

which in turn is equivalent to (15.12). Writing the integral term in (15.11) in terms of the scalar product on $L^2(\mathbb{R})$, from (15.11) we obtain

\begin{align*}
\left(\hat{X}, \hat{X}\right) + \left(\hat{X}_1, \hat{X}\right) + 2\pi \sum_{j=1}^N |\hat{X}(i\kappa_j) M_j|^2 = 0. \hspace{1cm} (15.15)
\end{align*}

The first and third terms in (15.15) are real and in fact nonnegative. Thus, the second term in (15.15) must be real. Applying the Schwarz inequality on that second term we get

$$|\left(\hat{X}_1, \hat{X}\right)| \leq ||\hat{X}_1||_2 ||\hat{X}||_2.$$  \hspace{1cm} (15.16)

Using (15.12) in (15.16) we get

$$|\left(\hat{X}_1, \hat{X}\right)| \leq ||\hat{X}||_2^2.$$  \hspace{1cm} (15.17)

As indicated earlier, $\left(\hat{X}_1, \hat{X}\right)$ is real valued, and hence (15.17) yields

$$-||\hat{X}||_2^2 \leq \left(\hat{X}_1, \hat{X}\right) \leq ||\hat{X}||_2^2.$$  \hspace{1cm} (15.18)

The first inequality in (15.18) yields

$$\left(\hat{X}, \hat{X}\right) + \left(\hat{X}_1, \hat{X}\right) \geq 0.$$  \hspace{1cm} (15.19)
Then, from (15.15) and (15.19) we get
\[
\begin{align*}
\sum_{j=1}^{N} |\hat{X}(i\kappa_j) M_j|^2 &= 0, \\
(\hat{X}, \hat{X}) + (\hat{X}_1, \hat{X}) &= 0.
\end{align*}
\] (15.20)

The first line in (15.20) holds if and only if the first line of (5.15) holds. Let us now show that the second line of (15.20) implies the second line of (4.15). The second line of (15.20) implies that
\[-(\hat{X}_1, \hat{X}) = (\hat{X}, \hat{X}).\] (15.21)

Comparing (15.21) with (15.17) we see that the Schwarz inequality yields an equality, which happens if \(\hat{X}_1(k) = c \hat{X}(k)\). Then, (15.21) yields
\[-c (\hat{X}, \hat{X}) = (\hat{X}, \hat{X}),\] (15.22)
which is possible only if \(c = -1\), as we assume that \(\hat{X}(k)\) is nonzero. Thus, we must have \(\hat{X}_1 = -\hat{X}(k)\), or equivalently \(\hat{X}(-k) S(k) = -\hat{X}(k)\), which yields the second line of (4.15). Thus, the proof is complete.

The following proposition shows that if the scattering data set \(S\) given in (4.2) belongs to the Marchenko class, then the \(S\) satisfies the properties \((4_c), (4_d), (4_e)\) of Definition 4.2. Among other implications, it also indicates that if the input data set \(D\) belongs to the Faddeev class, then the only solution to each of (4.14), (4.15), and (4.16) is the trivial solution.

**Proposition 15.4** Consider a scattering data set \(S\) as in (4.2), which consists of an \(n \times n\) scattering matrix \(S(k)\) for \(k \in \mathbb{R}\), a set of \(N\) distinct positive constants \(\kappa_j\), and a set of \(N\) constant \(n \times n\) hermitian and nonnegative matrices \(M_j\) with respective positive ranks \(m_j\), where \(N\) is a nonnegative integer. Assume that \(S\) satisfies \((1), (2), (3_a), \text{and} (4_a)\) of Definition 4.5. Let \(F(y)\) be the quantity constructed from \(S\) as in (4.12). Then:

(a) The only solution \(\hat{X}(k)\) to (4.15), as a row vector with \(n\) components belonging to the class \(\hat{L}^1(\mathbb{C}^+)\), is the trivial solution \(\hat{X}(k) \equiv 0\).

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(b) The only solution \( X(y) \), which is a row vector with \( n \) integrable components in \( y \in \mathbb{R}^+ \) to the integral equation given in (4.14) is the trivial solution \( X(y) \equiv 0 \).

(c) The only solution \( h(k) \) to (4.16), as a column vector with \( n \) components belonging to the class \( \hat{L}^1(\mathbb{C}^+) \), is the trivial solution \( h(k) \equiv 0 \).

PROOF: It is enough to prove (a) because (b) and (c) directly follow from (a), as indicated in Proposition 15.3(c) and Proposition 15.3(d). From (4.4) we see that we have

\[
S(k) = [S(k)^\dagger]^{-1}, \quad k \in \mathbb{R}.
\]

As a result of Proposition 5.1(c), the Jost matrix \( J(k) \) constructed from \( S \) satisfies the properties listed in Proposition 10.2, and the results in Chapter 11 remain valid, and in particular Proposition 11.2 holds and \( M_j \) appearing in \( S \) satisfies (11.22). Since the constructed \( J(k) \) satisfies (4.10), using (4.10) and (15.23) we obtain

\[
S(k) = -[J(-k^*)^\dagger]^{-1} J(k^*)^\dagger, \quad k \in \mathbb{R},
\]

where for convenience we have written \( k \) as \( k^* \) for \( k \in \mathbb{R} \) in (15.24). The reason for this is that \( [J(-k^*)^\dagger]^{-1} \) can be extended from \( k \in \mathbb{R} \) to \( k \in \mathbb{C}^+ \) meromorphically with simple poles at \( k = i\kappa_j \) for \( j = 1, \ldots, N \), as a result of the fact that \( J(k) \) has a similar extension from \( k \in \mathbb{R} \) to \( k \in \mathbb{C}^+ \), as stated in Proposition 10.2(c). Consequently, \( [J(k^*)^\dagger]^{-1} \) can be extended from \( k \in \mathbb{R} \) to \( k \in \mathbb{C}^- \) meromorphically with simple poles at \( k = -i\kappa_j \) for \( j = 1, \ldots, N \). Using (15.24) in the second line of (4.15) we obtain

\[
\hat{X}(k) [J(-k^*)^\dagger]^{-1} = \hat{X}(-k) [J(k^*)^\dagger]^{-1}, \quad k \in \mathbb{R}.
\]

Since \( \hat{X}(k) \in \hat{L}^1(\mathbb{C}^+) \), it follows that \( \hat{X}(k) \) has an analytic extension from \( k \in \mathbb{R} \) to \( k \in \mathbb{C}^+ \), it is continuous in \( k \in \overline{\mathbb{C}^+} \), and vanishes uniformly as \( k \to \infty \) in \( \overline{\mathbb{C}^+} \). On the other hand, it follows from Proposition 10.2 that \( [J(-k^*)^\dagger]^{-1} \) is meromorphic in \( k \in \mathbb{C}^+ \) with simple poles at \( k = i\kappa_j \) for \( j = 1, \ldots, N \) and that it is continuous for \( k \in \mathbb{R} \) except for a possible simple pole at \( k = 0 \). From (10.12) it follows that

\[
[J(-k^*)^\dagger]^{-1} = -\frac{N_j^\dagger}{k - i\kappa_j} + O(1), \quad k \to i\kappa_j \text{ in } \mathbb{C}^+.
\]

(15.26)
Let us define
\[
\Xi(k) := \begin{cases} 
  k \hat{X}(k) [J(-k^*)]^{-1} + k \sum_{j=1}^{N} \frac{2i\kappa_j \hat{X}(i\kappa_j) N_j^\dagger}{k^2 + \kappa_j^2}, & k \in \mathbb{C}^+, \\
  k \hat{X}(-k) [J(k^*)]^{-1} + k \sum_{j=1}^{N} \frac{2i\kappa_j \hat{X}(i\kappa_j) N_j^\dagger}{k^2 + \kappa_j^2}, & k \in \mathbb{C}^-.
\end{cases}
\]

(15.27)

With the help of (15.25) we observe that \(\Xi(k)\) is analytic in \(k \in \mathbb{C}^+\) and continuous in \(k \in \mathbb{C}^-\). Furthermore, \(\Xi(k) = o(k)\) as \(k \to \infty\) in \(k \in \mathbb{C}^+\) because we know that \(\hat{X}(k) = o(1)\) because \(\hat{X}(k) \in \hat{L}^1(\mathbb{C}^+)\) we know that \([J(-k^*)]^{-1} = O(1)\) as a result of Proposition 10.2(g). Similarly, from the second line of (15.27), with the help of (15.26) we observe that \(\Xi(k)\) is analytic in \(k \in \mathbb{C}^-\), continuous in \(k \in \mathbb{C}^-\), and \(o(k)\) as \(k \to \infty\) in \(\mathbb{C}^-\). Thus, \(\Xi(k)\) must be entire and in fact a constant row vector. Then, from the first line of (15.27) we obtain
\[
k \hat{X}(k) [J(-k^*)]^{-1} + k \sum_{j=1}^{N} \frac{2i\kappa_j \hat{X}(i\kappa_j) N_j^\dagger}{k^2 + \kappa_j^2} = c,
\]
where \(c\) is a constant row vector with \(n\) components. Note that (15.28) yields
\[
\hat{X}(k) = \frac{c J(-k^*)}{k} - \sum_{j=1}^{N} \left( \frac{2i\kappa_j \hat{X}(i\kappa_j) N_j^\dagger J(-k^*)}{k^2 + \kappa_j^2} \right), \quad k \in \mathbb{C}^+.
\]

(15.29)

Similarly, from the second line of (15.27) we obtain
\[
k \hat{X}(-k) [J(k^*)]^{-1} + k \sum_{j=1}^{N} \frac{2i\kappa_j \hat{X}(i\kappa_j) N_j^\dagger}{k^2 + \kappa_j^2} = c,
\]
which yields
\[
\hat{X}(-k) = \frac{c J(k^*)}{k} - \sum_{j=1}^{N} \left( \frac{2i\kappa_j \hat{X}(i\kappa_j) N_j^\dagger J(k^*)}{k^2 + \kappa_j^2} \right), \quad k \in \mathbb{C}^-.
\]

(15.31)

or equivalently
\[
\hat{X}(k) = -\frac{c J(-k^*)}{k} - \sum_{j=1}^{N} \left( \frac{2i\kappa_j \hat{X}(i\kappa_j) N_j^\dagger J(-k^*)}{k^2 + \kappa_j^2} \right), \quad k \in \mathbb{C}^+.
\]

(15.32)
Comparing (15.29) and (15.32) we see that $c = 0$ and

$$\hat{X}(k) = -\sum_{j=1}^{N} \left( \frac{2i\kappa_j \hat{X}(i\kappa_j) N_j^\dagger J(-k^*)^\dagger}{k^2 + \kappa_j^2} \right), \quad k \in \mathbb{C}^+. \tag{15.33}$$

We will now show that $\hat{X}(i\kappa_j) N_j^\dagger = 0$. From the first line of (4.15) we know that $\hat{X}(i\kappa_j) M_j = 0$. From (11.22) and the fact that $P_j$ and $B_j^{-1/2}$ commute, as assured by Proposition 11.2(d), it follows that the first line of (4.15) can be written as $\hat{X}(i\kappa_j) P_j B_j^{-1/2} = 0$. Since $B_j^{-1/2}$ is invertible, as assured by Proposition 11.2(c), we have $\hat{X}(i\kappa_j) P_j = 0$. From (11.9) we have $N_j^\dagger = P_j N_j^\dagger$, where we have used the first equality in (11.1). Thus, we obtain

$$\hat{X}(i\kappa_j) N_j^\dagger = \hat{X}(i\kappa_j) P_j N_j^\dagger = 0, \quad j = 1, \ldots, N, \tag{15.34}$$

and hence, using (15.34) in (15.33) we conclude that $\hat{X}(k) \equiv 0$. Thus, the proof is complete.

The following proposition shows that if the scattering data set $S$ given in (4.2) belongs to the Marchenko class, then the Jost matrix $J(k)$ constructed from $S$ as in (9.2) satisfies a certain useful property. Among other implications, it also indicates that if the input data set $D$ belongs to the Faddeev class, then the corresponding Jost matrix given in (9.2) constructed from $D$ satisfies that useful property.

**Proposition 15.5** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ satisfies (1), (2), (3a), and (4a) of Definition 4.5. Let $J(k)$ be the Jost matrix constructed from $S$ via (9.2). Let $\kappa_j$ for $j = 1, \ldots, N$ be the set of distinct positive numbers related to the zeros of $\det[J(k)]$, as indicated in Proposition 10.2(c). Let $N_j$ for $j = 1, \ldots, N$ be the set of $n \times n$ matrices as in the first equality in (10.13), i.e. each $N_j^\dagger$ belongs to the kernel of $J(i\kappa_j)^\dagger$ and has rank $m_j$, as indicated in in Proposition 10.2. Then, we have the following:

(a) For each $j = 1, \ldots, N$, the matrix $J(k) N_j/(k^2 + \kappa_j^2)$ is analytic in $k \in \mathbb{C}^+$, continuous in $k \in \overline{\mathbb{C}^+}$, and $O(1/k)$ as $k \to \infty$ as $k \to \infty$ in $\overline{\mathbb{C}^+}$.  

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(b) For each \( j = 1, \ldots, N \) the matrix \( J(k) N_j/(k^2 + \kappa_j^2) \) belongs to \( \tilde{L}^1(\mathbb{C}^+) \) and in fact to \( \tilde{L}^1(\mathbb{C}^+) \). Hence, \( J(k) N_j/(k^2 + \kappa_j^2) \) also belongs to the Hardy space \( \mathbf{H}^2(\mathbb{C}^+) \).

(c) For each \( j = 1, \ldots, N \) we have

\[
\frac{J(k) N_j}{k^2 + \kappa_j^2} = \int_0^\infty dy e^{iky} J(y), \tag{15.35}
\]

where \( J(y) \) belongs to \( L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \) and is given by

\[
J(y) := e^{-\kappa_j y} \left[ B + K(0,0) A - \frac{A}{2} \right] N_j + \frac{1}{2\kappa_j} \int_0^\infty dz e^{-\kappa_j |y-z|} \left[ K(0,z)^\dagger B - K_x(0,z)^\dagger B \right] N_j, \quad y \in \mathbb{R}^+. \tag{15.36}
\]

Here, \( A \) and \( B \) are the boundary matrices constructed from \( S \) as in Proposition 14.1, and \( K(x,y) \) is the unique solution to the Marchenko equation (13.1).

PROOF: From Definition 4.5 we know that \( S \) belongs to the Marchenko class and hence from Theorem 5.1 we know that \( J(k) \) constructed from \( S \) satisfies Proposition 10.2. Thus, \( J(k) \) is analytic in \( k \in \mathbb{C}^+ \) and continuous in \( k \in \overline{\mathbb{C}^+} \). Thus, the matrix \( J(k)/(k^2 + \kappa_j^2) \) is meromorphic in \( k \in \mathbb{C}^+ \) with a simple pole at \( k = i\kappa_j \). On the other hand, with the help of (10.12) we conclude that the matrix \( J(k) N_j/(k^2 + \kappa_j^2) \) is analytic in \( k \in \mathbb{C}^+ \) and continuous in \( k \in \overline{\mathbb{C}^+} \). Furthermore, from (10.10) it follows that \( J(k) N_j/(k^2 + \kappa_j^2) \) is \( O(1/k) \) as \( k \to \infty \) in \( \overline{\mathbb{C}^+} \). Hence, the proof of (a) is complete. From (a) it follows that \( J(k) N_j/(k^2 + \kappa_j^2) \) belongs to the Hardy space \( \mathbf{H}^2(\mathbb{C}^+) \), and hence we have established the existence of \( J(y) \) as in (15.35) belonging to \( L^2(\mathbb{R}^+) \) with \( J(y) = 0 \) for \( y \in \mathbb{R}^- \). From the \( x \)-derivative of (10.6) we obtain

\[
f'(k,x) = ik e^{ikx} I - K(x,x) e^{ikx} + \int_x^\infty dy K_x(x,y) e^{iky}, \quad k \in \mathbb{R}, \quad x \geq 0, \tag{15.37}
\]

We remark that the integral term in (15.37) is well defined because \( K_x(x,y) \) for each \( x \geq 0 \) is integrable in \( y \in \mathbb{R}^+ \), as assured by Proposition 10.1(f). From (10.6) and (15.37) we respectively obtain

\[
f(k,0) = I + \int_{-\infty}^\infty dy K(0,y) e^{iky}, \tag{15.38}
\]
\[ f'(k, 0) = i k I - K(0, 0) + \int_{-\infty}^{\infty} dy \, K_x(0, y) e^{iky}. \]   \hspace{1cm} (15.39)

where we have used (10.2) and (10.8). Then, using (15.38) and (15.39) in (9.2) we write the Jost matrix \( J(k) \) as

\[ J(k) = B - ik A + K(0, 0) A + \int_{0}^{\infty} dy \, [K(0, y)^\dagger B - K_x(0, y)^\dagger A] e^{iky}, \]   \hspace{1cm} (15.40)

where we have used the fact that \( K(0, 0)^\dagger = K(0, 0) \), which follows from (2.2) and (10.5).

From (d) and (f) of Proposition 10.1 we know that \( (K(0, y)^\dagger B - K_x(0, y)^\dagger A) \) belongs to \( L^1(\mathbb{R}^+) \). Let us replace \( ik A \) in (15.40) with \( [-\kappa_j A + i(k - i\kappa_j) A] \) so that we have

\[ B - ik A + K(0, 0) A = [B + \kappa_j A + K(0, 0) A] - i(k - i\kappa_j) A, \]   \hspace{1cm} (15.41)

and hence

\[ \frac{B - ik A + K(0, 0) A}{k^2 + \kappa_j^2} = \frac{B + \kappa_j A + K(0, 0) A}{k^2 + \kappa_j^2} - \frac{i A}{(k + i\kappa_j)}. \]   \hspace{1cm} (15.42)

Note that

\[ \frac{1}{k^2 + \kappa_j^2} = \frac{1}{2\kappa_j} \left[ \frac{i}{k + i\kappa_j} - \frac{i}{k - i\kappa_j} \right]. \]   \hspace{1cm} (15.43)

On the other hand, we have the explicit expressions

\[ \frac{i}{k + i\kappa_j} = \int_{0}^{\infty} dy \, e^{-\kappa_j y + iky}, \quad \frac{i}{k - i\kappa_j} = -\int_{-\infty}^{0} dy \, e^{\kappa_j y + iky}. \]   \hspace{1cm} (15.44)

From (15.43) and (15.44) we get

\[ \frac{1}{k^2 + \kappa_j^2} = \frac{1}{2\kappa_j} \int_{-\infty}^{\infty} dy \, e^{-\kappa_j |y|} e^{iky}. \]   \hspace{1cm} (15.45)

Using (15.40)-(15.45) we obtain

\[ \frac{J(k)}{k^2 + \kappa_j^2} = \frac{1}{2\kappa_j} [B + \kappa_j A + K(0, 0) A] \int_{-\infty}^{\infty} dy \, e^{-\kappa_j |y|} e^{iky} - A \int_{0}^{\infty} dy \, e^{-\kappa_j y} e^{iky} \\
+ \frac{1}{2\kappa_j} \int_{-\infty}^{\infty} dy \, e^{iky} \int_{0}^{\infty} dz \, e^{-\kappa_j |y - z|} [K(0, z)^\dagger B - K_x(0, z)^\dagger A]. \]   \hspace{1cm} (15.46)

Postmultiplying both sides of (15.46) with \( N_j \), the resulting left-hand side satisfies the properties listed in (a) and (b), and hence the integral in the resulting right-hand side
vanishes over $y \in \mathbb{R}^{-}$. This yields

$$\frac{J(k) N_j}{k^2 + \kappa_j^2} = \frac{1}{2\kappa_j} \left[ B + \kappa_j A + K(0, 0) A \right] N_j \int_0^\infty dye^{-\kappa_j y} e^{iky} - A N_j \int_0^\infty dye^{-\kappa_j y} e^{iky}$$

$$+ \frac{1}{2\kappa_j} \int_0^\infty dye^{iky} \int_0^\infty dz e^{-\kappa_j |y-z|} \left[ K(0, z)^\dagger B - K_x(0, z)^\dagger A \right].$$

(15.47)

Combining the first two integrals in (15.47) into one, we obtain (15.35) and (15.36). In order to complete the proof, we need to show that $J(y)$ is integrable and bounded in $y \in \mathbb{R}^+$. Since $e^{-\kappa_j y}$ is bounded and integrable in $y \in \mathbb{R}^+$, we conclude that the first terms on the right-hand side is belongs to $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$. Thus, we only need to prove that the integral term in (15.36) belongs to $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$. Note that, that integral term is essentially the convolution of $e^{-\kappa_j |y|}$, which is bounded and integrable in $y \in \mathbb{R}$, with the matrix-valued function $\left[ K(0, y)^\dagger B - K_x(0, y)^\dagger A \right]$. With the help of (3.37) and (3.39) we conclude that that integral term belongs to $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$. Thus, the proof is complete. □

One consequence of the following proposition is the equivalence among $(V_f)$, $(V_g)$, and $(V_h)$ in Proposition 6.5.

**Proposition 15.6** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Let $F_s(y)$ be the quantity defined in (4.7). Assume that $S$ satisfies (I) of Definition 4.3. Then, we have the following:

(a) The row vector $X(y)$ whose $n$ components belonging to $L^2(\mathbb{R}^+)$ is a solution to (4.22) if and only if the row vector $\hat{X}(k)$ with $n$ components in $H^2(\mathbb{C}^+)$ is a solution to (4.23), where $\hat{X}(k)$ and $X(y)$ are related to each other as in (3.67) and (3.68). We remark that any solution $X(y)$ in $L^2(\mathbb{R}^+)$ to (4.22) actually belongs to $L^2(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$. 

(b) The row vector $\hat{X}(k)$ whose $n$ components belonging to $H^2(\mathbb{C}^+)$ is a solution to (4.23) if and only if the column vector $h(k)$ with $n$ components in $H^2(\mathbb{C}^+)$ is a solution to
(4.24), where \( \hat{X}(k) \) and \( h(k) \) are related to each other as \( h(k) = \hat{X}(-k^*)^\dagger \).

PROOF: Let us first remark that the fact that any solution \( X(y) \) in \( L^2(R^+) \) to (4.22) must belong to \( L^2(R^+) \cap L^\infty(R^+) \) directly follows from the analog of Proposition 3.2. We remark that the rest of the proof essentially follows by repeating the proof of Proposition 15.3, as the following argument indicates. If we replace \( F(y) \) appearing in (4.14) by \( F_s(y) \) then we get (4.22). Thus, the proof of Proposition 15.3 can be repeated by ignoring the portions in that proof related to the bound states. In fact, the proof of Proposition 15.3 is given when \( X(y) \) only belongs to \( L^2(R^+) \), and hence the results stated in (a) and (b) hold.

In the next proposition we present the result of Proposition 15.6 in a more restricted class. One of its consequences is the equivalences among \((V_c),(V_d),(V_e)\) in Proposition 6.5.

**Proposition 15.7** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbf{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Assume that \( S \) satisfies (I) of Definition 4.3. Let \( F_s(y) \) be the quantity defined in (4.7). Then, we have the following:

(a) The row vector \( X(y) \) whose \( n \) components belonging to \( L^1(R^+) \) is a solution to (4.22) if and only if the row vector \( \hat{X}(k) \) with \( n \) components in \( \hat{L}^1(C^+) \) is a solution to (4.23), where \( \hat{X}(k) \) and \( X(y) \) are related to each other as in (3.67) and (3.68).

(b) The row vector \( \hat{X}(k) \) whose \( n \) components belonging to \( \hat{L}^1(C^+) \) is a solution to (4.23) if and only if the column vector \( h(k) \) with \( n \) components in \( \hat{L}^1(C^+) \) is a solution to (4.24), where \( \hat{X}(k) \) and \( h(k) \) are related to each other as \( h(k) = \hat{X}(-k^*)^\dagger \).

(c) Any solution \( X(y) \) in \( L^1(R^+) \) to (4.22) must actually belong to \( L^1(R^+) \cap L^\infty(R^+) \).

(d) Any solution \( \hat{X}(k) \) in \( \hat{L}^1(C^+) \) to (4.23) must actually belong to \( \hat{L}^1_\infty(C^+) \).

(e) Any solution \( h(k) \) in \( \hat{L}^1(C^+) \) to (4.24) must actually belong to \( \hat{L}^1_\infty(C^+) \).
PROOF: We remark that if $X(y)$ belongs to $L^1(R^+)$, from (3.67) and (3.68) it follows that $\hat{X}(k)$ belongs to $\hat{L}^1(C^+)$. Furthermore, from $h(k) = \hat{X}(-k^*)^\dagger$ it follows that $h(k)$ also belongs to $\hat{L}^1(C^+)$. Let us also remark that (c) directly follows from Proposition 3.1(c), and (c) implies (d) and (e). Thus, we only need to prove (a) and (b). We note that (a) and (b) directly follows from (a) and (b) of Proposition 15.6 because $X(y) \in L^1(R^+) \cap L^\infty(R^+)$ implies that $X(y) \in L^2(R^+)$ and as a result $\hat{X}(k) \in \hat{L}^1(C^+)$ implies that $\hat{X}(k) \in H^2(C^+)$ and that $h(k) \in \hat{L}^1(C^+)$ implies that $h(k) \in H^2(C^+)$. Thus, the proof is complete.

One implication of the following result is that if the scattering data set $S$ belongs to the Marchenko class then $(V_f)$, $(V_g)$, and $(V_h)$ of Definition 4.3 are satisfied. Among its other implications, it also indicates that if the input data set $D$ belongs to the Faddeev class then the number of linearly independent solutions to each of (4.22), (4.23), (4.24) is equal to the nonnegative integer $N$ appearing in (4.3).

**Proposition 15.8** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in R$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ satisfies (1), (2), (3a), and (4a) of Definition 4.5. Let $F_s(y)$ be the quantity constructed from $S$ as in (4.7). Then:

(a) The number of linearly independent solutions $\hat{X}(k)$ to (4.23), as row vectors whose $n$ components belonging to $H^2(C^+)$, is equal to $N$, where $N$ is the nonnegative integer defined in (4.3).

(b) The number of linearly independent solutions $X(y)$ to (4.22), as a row vector whose $n$ components belonging to $L^2(R^+)$, is equal to $N$.

(c) The number of linearly independent solutions $h(k)$ to (4.24), as column vectors whose $n$ components belonging to $H^2(C^+)$, is equal to $N$.

PROOF: By Proposition 15.7, we know that the solution $X(y)$ to (4.22) and the solution $\hat{X}(k)$ to (4.23) are related to each other as in (3.67) and (3.68). Furthermore, the same
proposition indicated that the solution $h(k)$ to (4.24) and the solution $\hat{X}(k)$ to (4.23) are related to each other as $h(k) = \hat{X}(-k^*)^\dagger$. Thus, it is enough to prove (a), and the results in (b) and (c) follow from (a). For the proof of (a) we proceed as follows. We remark that (4.23) is identical to the second line of (4.15). Hence, we equivalently need to solve the second line of (4.15) and look for the general solution $\hat{X}(k)$ belonging to $H^2(C^+)$. Thus, we can repeat the beginning of the proof of Proposition 15.4 and show that if $\hat{X}(k) \in H^2(C^+)$ is a solution to (4.23), then the quantity $\Xi(k)$ defined in (15.27) is sectionally analytic in $k \in C$, i.e. it is analytic in $k \in C^+ \cup C^-$. However, for the rest of the proof we cannot use the argument given in the proof of Proposition 15.4 because $\hat{X}(k)$ cannot be assumed continuous in $k \in C^+$ and cannot be assumed to have the behavior $o(1)$ as $k \to \infty$ in $C^+$.

So, we proceed as follows. First, we prove that the domain of analyticity of $\Xi(k)$ given in (15.27) but with $\hat{X}(k) \in H^2(C^+)$ extends to the entire complex plane $C$. This is done as follows. Since $\hat{X}(k) \in H^2(C^+)$ we have $\hat{X}(k + i\epsilon) \to \hat{X}(k)$ as $\epsilon \to 0^+$ a.e. in $k \in R$ and strongly in $L^2(R)$. For $0 < \epsilon < 1$, we use $C_\epsilon$ to denote the positive boundary of the rectangle in $k \in C^+$ with respective corners located at $a + i\epsilon, b + i\epsilon, b + i, a + i$, where $a$ and $b$ are some positive parameters with $a < b$. Similarly, we use $C_{-\epsilon}$ to denote the positive boundary of the rectangle in $C^-$ with respective corners located at $b - i\epsilon, a - i\epsilon, a - i, b - i$. Since $\Xi(k)$ defined in (15.27) is analytic in $k \in C^+ \cup C^-$, it follows from the Cauchy integral formula that for any $k$ inside $C_\epsilon$ we have

$$\Xi(k) = \frac{1}{2\pi i} \int_{C_\epsilon} dt \frac{\Xi(t)}{t - k} + \frac{1}{2\pi i} \int_{C_{-\epsilon}} dt \frac{\Xi(t)}{t - k},$$

(15.48)

where the contribution by the second integral is zero. Let us choose $a$ and $b$ so that in the limit $\epsilon \to 0^+$ we have $\Xi(a \pm i\epsilon) \to \Xi(a)$ and $\Xi(b \pm i\epsilon) \to \Xi(b)$. Then, letting $\epsilon \to 0^+$ in (15.22) we get

$$\Xi(k) = \frac{1}{2\pi i} \int_{C_0} dt \frac{\Xi(t)}{t - k},$$

(15.49)

where $C_0$ is the positively oriented boundary of the rectangle with corners at $-a - ib, a - ib, a + ib,$ and $-a + ib$. From the representation in (15.49), we conclude that $\Xi(k)$ is analytic in the interior of the rectangle bounded by $C_0$, including the segment of the real
axis contained in that rectangle. Since we can let $a \to -\infty$ and $b \to +\infty$, we conclude that $\Xi(k)$ is in fact entire in $k$. With the help of (15.27) we conclude that $\Xi(k)$ is an odd function of $k$ in $C$ and we have

$$\Xi(-k) = -\Xi(k), \quad k \in C. \quad (15.50)$$

Since $\Xi(k)$ defined in (15.27) has an analytic extension to the entire complex plane, in its Maclaurin expansion of $\Xi(k)$ given by

$$\Xi(k) = \sum_{p=1}^{\infty} \frac{1}{p!} \left[ \Xi^{(p)}(0) \right] k^p, \quad k \in C, \quad (15.51)$$

where the coefficient $\Xi^{(p)}(0)$ can be evaluated, with the help of the generalized Cauchy integral formula as

$$\Xi^{(p)}(0) := \frac{d^p \Xi(k)}{dk^p} \bigg|_{k=0} = \frac{p!}{2\pi i} \int_{T_r} dt \frac{\Xi(t)}{t^{p+1}}, \quad p = 0, 1, 2, \ldots, \quad (15.52)$$

where $T_r$ is the circle of radius $r$ centered at $k = 0$ traversed in the positive direction, with $r := |k|$. Because of (15.50), from (15.52) we conclude that $\Xi^{(p)}(0) = 0$ for even values of $p$ in (15.51). We will now estimate the integral in (15.52). Using (15.50), we have

$$\left| \int_{T_r} dt \frac{\Xi(t)}{t^{p+1}} \right| \leq 2 \int_{T_r^+} |dt| \frac{|\Xi(t)|}{r^{p+1}}, \quad (15.53)$$

where we use $T_r^+$ to denote the upper semicircle of $T_r$. From Proposition 10.2(b) it follows that there exists some positive number $r_0$ such that

$$|J(k)| \leq C |k|, \quad |k| \geq r_0, \quad k \in \overline{C^+}, \quad (15.54)$$

for some generic constant $C$. On the other hand, since $\hat{X}(k) \in H^2(C^+)$, by (3.46) we have

$$|\hat{X}(k)| \leq \frac{C}{\sqrt{|k| \sin \theta}}, \quad k \in C^+. \quad (15.55)$$

Using (15.54) and (15.55) in the first line of (15.27), we have with $k = re^{i\theta}$

$$|\Xi(re^{i\theta})| \leq \frac{C \sqrt{r}}{\sqrt{\sin \theta}}, \quad r \geq r_0, \quad k \in T_r^+, \quad (15.56)$$
for some generic constant $C$. Using (15.56) in (15.53) we get the estimate

$$\left| \int_{T_r} dt \frac{\Xi(t)}{t^{p+1}} \right| \leq \frac{C}{r^{p+1/2}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}},$$

(15.57)

where we have used $|dt| = r d\theta$ and that $\sin \theta = \sin(\pi - \theta)$ for $\theta \in (0, \pi/2)$. The integral on the right-hand side of (15.57) is convergent because the singularity of the integrand at $\theta = 0$ is an integrable singularity, as we have $\sin \theta = \theta + O(\theta^3)$ as $\theta \to 0$. In fact, that integral is related to the complete elliptic integral of the first kind and we have

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}} = \sqrt{2} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \frac{1}{2} \sin^2 \theta}} = 2.62206,$$

(15.58)

where we us an overline on a digit to indicate a round off. Thus, using (15.52), (15.57), and (15.58), and letting $r \to +\infty$, we conclude that $\Xi^{(p)}(0) = 0$ for $p = 0, 1, \ldots$, and hence from (15.51) we conclude that $\Xi(k) \equiv 0$. Then, from (15.27) we obtain $\hat{X}(k)$ as in the adjoint of (15.45), i.e.

$$\hat{X}(k) = -\sum_{j=1}^N \left( \frac{2i\kappa_j \hat{X}(i\kappa_j) N_j^\dagger J(-k^*)^\dagger}{k^2 + \kappa_j^2} \right), \quad k \in \mathbb{C}^+. \quad (15.59)$$

Let us now investigate the general solution $\hat{X}(k)$ to (4.23) we have constructed in (15.59), and let us show that it contains precisely $N$ linearly independent row vector solutions. For the proof we proceed as follows. From the first equality in (10.13) we know that $J(i\kappa_j)^\dagger N_j^\dagger = 0$ and hence each column of $N_j^\dagger$ belongs to the kernel of $J(i\kappa_j)^\dagger$. From Proposition 10.2(d) we then conclude that the rank of $N_j^\dagger$ is equal to $m_j$, which is the nullity of the matrix $J(i\kappa_j)^\dagger$. Since (4.23) is a linear homogeneous system, we then conclude that the number of linearly independent solutions to (4.23) is equal to the sum of the ranks of $N_j^\dagger$ for all $j = 1, \ldots, N$, which is $N$ given in (4.3). Let us remark that we can directly infer from the explicit solution given in (15.59) that $\hat{X}(k)$ indeed belongs to the Hardy space $H^2(\mathbb{C}^+)$. This can be argued as follows. From Proposition 10(b), (10.11), and the first equality in (10.13) it follows that $\hat{X}(k)$ is analytic in $k \in \mathbb{C}^+$. Then, with the help of Proposition 10(b) we conclude that $\hat{X}(k)$ is continuous in $k \in \overline{\mathbb{C}^+}$.
we conclude that \( \hat{X}(k) = O(1/k) \) as \( k \to \infty \) in \( \mathbb{C}^+ \). Thus, we can conclude that
\[
\hat{X}(k) \in H^2(\mathbb{C}^+).
\]

The following result is similar to the one given in Proposition 15.8. One of its implications is that if the scattering data set \( S \) belongs to the Marchenko class then \( (V_e), (V_d) \), and \( (V_c) \) of Definition 4.3 are satisfied. Among its other implications, it also indicates that if the input data set \( D \) belongs to the Faddeev class then the number of linearly independent solutions to each of (4.22), (4.23), (4.24) is equal to the nonnegative integer \( N \) appearing in (4.3), where the solutions are sought in a more restricted class than that used in Proposition 15.8.

**Proposition 15.9** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Assume that \( S \) satisfies (1), (2), (3), and (4) of Definition 4.5. Let \( F_s(y) \) be the quantity constructed from \( S \) as in (4.7). Then:

(a) The number of linearly independent solutions \( \hat{X}(k) \) to (4.23), as row vectors whose \( n \) components belonging to \( \hat{L}^1(\mathbb{C}^+) \), is equal to \( N \), where \( N \) is the nonnegative integer defined in (4.3).

(b) The number of linearly independent solutions \( X(y) \) to (4.22), as a row vector whose \( n \) components belonging to \( L^1(\mathbb{R}^+) \), is equal to \( N \).

(c) The number of linearly independent solutions \( h(k) \) to (4.24), as column vectors whose \( n \) components belonging to \( \hat{L}^1(\mathbb{C}^+) \), is equal to \( N \).

**PROOF:** By Proposition 15.7, we know that the solution \( X(y) \) to (4.22) and the solution \( \hat{X}(k) \) to (4.23) are related to each other as in (3.67) and (3.68). Furthermore, the same proposition indicates that the solution \( h(k) \) to (4.24) and the solution \( \hat{X}(k) \) to (4.23) are related to each other as \( h(k) = \hat{X}(-k^*)^\dagger \). Thus, it is enough to prove (a), and the results in (b) and (c) follow from (a). For the proof of (a) we proceed as follows. We remark that
(4.23) is identical to the second line of (4.15). Hence, we equivalently need to solve the second line of (4.15) and look for the general solution \( \hat{X}(k) \) belonging to \( \hat{L}^1(C^+) \). Thus, we can repeat the proof of Proposition 15.4 from the beginning and obtain the solution \( \hat{X}(k) \) given as in the adjoint of (15.45), i.e.

\[
\hat{X}(k) = -\sum_{j=1}^{N} \left( \frac{2i\kappa_j \hat{X}(i\kappa_j) N_j^\dagger J(-k^*)^\dagger}{k^2 + \kappa_j^2} \right), \quad k \in \overline{C^+}.
\]  

(15.60)

Since (4.23) and (4.15) differ from each other in the sense that the solution to (4.23) does not need to satisfy the first line of (4.15), the solution \( \hat{X}(k) \) given in (15.60) is the general solution to (4.23). Let us show that the general solution \( \hat{X}(k) \) given in (15.60) contains precisely \( N \) linearly independent row vector solutions. For this we can use the same argument given in the proof of Proposition 15.6, namely, that the rank of \( N_j^\dagger \) is equal to \( m_j \), which is the nullity of the matrix \( J(i\kappa_j)^\dagger \). Since (4.23) is a linear homogeneous system, then from (15.60) we see that the number of linearly independent solutions to (4.23) is equal to the sum of the ranks of \( N_j^\dagger \) for all \( j = 1, \ldots, N \), which is \( N \) given in (4.3). Let us remark that the solution \( \hat{X}(k) \) in (15.59) and the solution \( \hat{X}(k) \) in (15.60) coincide. We know from Proposition 15.8 that \( \hat{X}(k) \) of (15.59) belongs to the Hardy space \( H^2(C^+) \). Let us now prove that \( \hat{X}(k) \) of (15.60) indeed belongs to \( \hat{L}^1(C^+) \). In other words, we would like to prove that \( \hat{X}(k) \) of (15.60) is analytic in \( k \in C^+ \) and that it is the Fourier transform of some function \( X(y) \), as in (3.67) and (3.68), where \( X(y) \in L^1(R^+) \) and \( X(y) = 0 \) for \( y \in R^- \). In fact, this directly follows using Proposition 15.5 in (15.60). }

In the next theorem, we show that if the input data set \( D \) given in (4.1) belongs to the Faddeev class specified in Definition 4.1, then a corresponding scattering data set \( S \) as in (4.2) exists, is unique, and belongs to the Marchenko class specified in Definition 4.5.

**Theorem 15.10** For any input data set \( D \) in the Faddeev class specified in Definition 4.1, there exists and uniquely exists a corresponding scattering data set \( S \) as in (4.2) belonging to the Marchenko class specified in Definition 4.5.

*PROOF:* The existence and uniqueness of \( S(k) \) are implicitly given in the construction steps (a)-(c) given in Chapter 9. The existence and uniqueness of the construction of
the Jost function $J(k)$ are implicitly given in the construction steps (a) and (b) given in Chapter 9. Then, the existence and uniqueness of the constants $\kappa_j$, their multiplicities $m_j$, and their number $N$ are assured because such quantities are related to $J(k)$ as described in the step (f) in Chapter 9. The existence and uniqueness of the normalization matrices $M_j$ are implicit in the construction summary given in the step (g) in Chapter 9. Having proved the existence and uniqueness of the corresponding $S$, let us now prove that $S$ belongs to the Marchenko class, i.e. that all the four conditions stated in Definition 4.5 are satisfied. Let us first prove that (1) in Definition 4.5 holds when $D$ belongs to the Faddeev class. By Proposition 10.3(a) we know that $S(k)$ satisfies (4.4). The property (4.5) follows from Proposition 10.3(b). The boundedness of $F_s(y)$ defined in $y \in \mathbb{R}$ and its integrability in $y \in \mathbb{R}^+$ are given in Theorem 12.1 Thus, (1) is satisfied. Note that (2) follows from Theorem 13.2(c) and the fact that the difference $F(y) - F_s(y)$, as seen from (4.12), is a linear combination of exponential functions with negative exponents in $y \in \mathbb{R}^+$ given by the right-hand side of (5.2). The property (3a) in Definition 4.5 follows from Proposition 10.5(c). Let us now prove (4a). By Proposition 15.1(b), the Marchenko integral operator for $x = 0$ is compact on $L^1(\mathbb{R}^+)$. Thus, (4.11) has a unique solution in $L^1(\mathbb{R}^+)$ if the only solution in $L^1(\mathbb{R}^+)$ to (4.13) is the trivial solution. By Proposition 15.4(d) we know that the only solution in $L^1(\mathbb{R}^+)$ to (4.14) is the trivial solution, and hence the only solution in $L^1(\mathbb{R}^+)$ to (4.13) is also the trivial solution. Hence, (4a) holds.

The following proposition is related to the equivalences among (IIIa), (IIIb), and (IIIc) of Definition 4.3.

**Proposition 15.11** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ satisfies (I) of Definition 4.3. Let $F_s(y)$ be the quantity defined in (4.7), where $S_\infty$ is the constant $n \times n$ matrix defined as in (4.6). Then, we have the following:
(a) The row vector \( X(y) \) whose \( n \) components belonging to \( L^2(\mathbb{R}^-) \) is a solution to (4.17) if and only if \( \hat{X}(k) \) related to \( X(y) \) as in (3.67) and (3.69) is a solution in the Hardy space \( H^2(\mathbb{C}^-) \) to (4.18).

(b) The row vector \( \hat{X}(k) \) whose \( n \) components belonging to the Hardy space \( H^2(\mathbb{C}^-) \) is a solution to (4.18) if and only if the column vector \( h(k) \) with \( n \) components belonging to \( H^2(\mathbb{C}^-) \) satisfies (4.19), where \( \hat{X}(k) \) and \( h(k) \) are related to each other as \( h(k) = \hat{X}(-k^*)^\dagger \).

PROOF: We remark that the proof of (b) is obtained by taking the matrix adjoint of both sides of (4.18) and by using \( S(k)^\dagger = S(-k) \) for \( k \in \mathbb{R} \), which follows from (4.4). So, we only need to prove (a). From (3.67) and (3.69) we see that \( X(y) \in L^2(\mathbb{R}^-) \) if and only if \( \hat{X}(k) \in H^2(\mathbb{C}^-) \). Thus, we first need to show that if \( \hat{X}(k) \in H^2(\mathbb{C}^-) \) satisfies (4.18) then \( X(y) \in L^2(\mathbb{R}^-) \) satisfies (3.70). After that proof, we need to prove the converse. First, let us show that (4.18) implies (4.17). From (4.18) we get

\[
-\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{X}(-k) e^{iky} + \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{X}(k) S(k) e^{iky} = 0, \quad (15.61)
\]

Using (3.67) and (4.7) in (15.61), with the help of (11.36) we obtain

\[
-X(y) + X(-y) S_{\infty} + \int_{-\infty}^{\infty} dz X(z) F_{s}(z + y), \quad y \in \mathbb{R}. \quad (15.62)
\]

Using \( X(y) = 0 \) for \( y \in \mathbb{R}^+ \), from (15.62) we get

\[
-X(y) + \int_{-\infty}^{0} dz X(z) F_{s}(z + y) = 0, \quad y \in \mathbb{R}^- \quad (15.63)
\]

and hence we have proved that (4.18) implies (4.17). Let us now prove the converse. By assuming that \( X(y) \) satisfies (4.17) we would like to show that \( \hat{X}(k) \) satisfies (4.18). We can extend \( X(y) \) to \( y \in \mathbb{R} \) by letting \( X(y) = 0 \) for \( y \in \mathbb{R}^+ \). Let us multiply both sides of (4.17) with \( X(y)^\dagger \) and integrate over \( y \in \mathbb{R} \) with the understanding that \( X(y) = 0 \) for \( y \in \mathbb{R}^+ \). We get

\[
\int_{-\infty}^{\infty} dy \left[ -X(y) + \int_{-\infty}^{\infty} dz X(z) F_{s}(z + y) \right] X(y)^\dagger = 0. \quad (15.64)
\]
Using (3.67) and (4.7) in (15.64), with the help of (11.37) we simplify the resulting equation and obtain
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left( -\hat{X}(-k) + \hat{X}(k) [S(k) - S_{\infty}] \right) \hat{X}(-k)^\dagger = 0. \quad (15.65)
\]
We can simplify the integral part of (15.65) further by using
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{X}(k) S_{\infty} \hat{X}(-k)^\dagger = \int_{-\infty}^{\infty} dy X(y) S_{\infty} X(-y)^\dagger = 0, \quad (15.66)
\]
which follows from (3.69) and the fact that \( X(y) = 0 \) for \( y \in \mathbb{R}^+ \). Thus, (15.65) is equivalent to
\[
\int_{-\infty}^{\infty} dk \left( -\hat{X}(-k) + \hat{X}(k) S(k) \right) \hat{X}(k)^\dagger = 0. \quad (15.67)
\]
Note that (15.67) can be written as
\[
\int_{-\infty}^{\infty} dk \left( -\hat{X}(-k) + \hat{X}(k) S(-k) \right) \hat{X}(k)^\dagger = 0. \quad (15.68)
\]
By letting \( \hat{X}_1(k) := \hat{X}(-k) S(-k) \), as indicated in (15.13) we know that (15.12) holds and we have
\[
||\hat{X}_1||_2 = ||\hat{X}||_2. \quad (15.69)
\]
We can write (15.68) in terms of the scalar product on \( L^2(\mathbb{R}) \) as
\[
-\left( \hat{X}, \hat{X} \right) + \left( \hat{X}_1, \hat{X} \right) = 0. \quad (15.70)
\]
or equivalently as
\[
\left( \hat{X}_1, \hat{X} \right) = ||\hat{X}||_2^2. \quad (15.71)
\]
Applying the Schwarz inequality on the left-hand side of (15.71) we get
\[
| \left( \hat{X}_1, \hat{X} \right) | \leq ||\hat{X}_1||_2 ||\hat{X}||_2, \quad (15.72)
\]
where the equality holds if and only if \( \hat{X}_1(k) = c \hat{X}(k) \) for some constant \( c \). Using (15.69) in (15.72) we get
\[
| \left( \hat{X}_1, \hat{X} \right) | \leq ||\hat{X}||_2^2. \quad (15.73)
\]
Comparing (15.71) and (15.73) we see that we must have the equality holding in the Schwarz inequality and hence
\[ \hat{X}_1(k) = c \hat{X}(k). \]  
(15.74)

Using (15.74) in (15.71) we determine that either \( \hat{X}(k) \equiv 0 \) or \( c = 1 \). In the former case \( \hat{X}(k) \) clearly satisfies (4.18), because (4.18) is a homogeneous Riemann-Hilbert problem. In the latter case (15.74) with \( c = 1 \) yields
\[ \hat{X}(-k) S(-k) = \hat{X}(k), \quad k \in \mathbb{R}, \]  
(15.75)

which is equivalent to (4.18). Hence, the proof is complete.

The following result shows that the only solution to the linear homogeneous integral equation appearing in (III\(_a\)) of Definition 4.3 is the trivial solution. The result is analogous to a part of Theorem 3.5.1 of [2]. However, we could not rely on the proof given in [2] because the proof in [2] assumes that the quantity \( \hat{X}(k) \) given in (3.68) is analytic in \( k \in \mathbb{C}^+ \), continuous in \( k \in \overline{\mathbb{C}}^+ \), and uniformly \( o(1) \) as \( k \to \infty \) in \( \overline{\mathbb{C}}^+ \) when the quantity \( X(y) \) in (3.68) is only square integrable in \( y \in \mathbb{R}^+ \). In Example 26.2 we illustrate that \( \hat{X}(k) \) may not have such a nice behavior unless \( X(y) \) is also integrable.

**Proposition 15.12** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Assume that \( S \) satisfies (1), (2), (3\(_a\)), and (4\(_a\)) of Definition 4.5. Let \( F_s(y) \) be the quantity constructed from \( S \) as in (4.7). Then:

(a) The only solution \( X(y) \) in \( L^2(\mathbb{R}^-) \) to (4.17) is the trivial solution \( X(y) \equiv 0 \).

(b) The only solution \( \hat{X}(k) \) in \( H^2(\mathbb{C}^-) \) to (4.18) is the trivial solution \( \hat{X}(k) \equiv 0 \).

(c) The only solution \( h(k) \) in \( H^2(\mathbb{C}^-) \) to (4.19) is the trivial solution \( h(k) \equiv 0 \).

**PROOF:** By Proposition 15.11 we know that \( X(y) \in L^2(\mathbb{R}^-) \) satisfies (4.17) if and only if \( \hat{X}(k) \in H^2(\mathbb{C}^-) \) satisfies (4.18), where \( X(y) \) and \( \hat{X}(k) \) are related to each other as in (3.67)
and (3.69). Furthermore, by the same proposition we know that \( h(k) \in H^2(\mathbb{C}^-) \) is a solution to (4.19) if and only if \( \hat{X}(k) \in H^2(\mathbb{C}^-) \) is a solution to (4.18), where \( h(k) = \hat{X}(-k^*)^\dagger \).

Hence, to prove our proposition, it is enough to prove that \( \hat{X}(k) \equiv 0 \). As also indicated in the proof of Proposition 15.4, the Jost matrix \( J(k) \) constructed from the scattering data set \( S \) satisfies (4.10) and possesses all the properties listed in Proposition 10.2. Thus, in (4.18) we can replace \( S(k) \) by \(-J(-k) J(k)^{-1} \) and we get

\[
\hat{X}(-k) + \hat{X}(k) J(-k) J(k)^{-1} = 0, \quad k \in \mathbb{R}, \tag{15.76}
\]
or equivalently

\[
\hat{X}(-k) J(k) + \hat{X}(k) J(-k) = 0, \quad k \in \mathbb{R}, \tag{15.77}
\]
where \( \hat{X}(k) \) is analytic in \( k \in \mathbb{C}^- \) and \( J(k) \) is analytic in \( k \in \mathbb{C}^+ \) and continuous in \( k \in \mathbb{C}^+ \). We then consider the sectionally analytic function \( \Xi(k) \) defined on the complex plane \( \mathbb{C} \) as

\[
\Xi(k) := \begin{cases} 
\hat{X}(-k) J(k), & k \in \mathbb{C}^+, \\
-\hat{X}(k) J(-k), & k \in \mathbb{C}^-.
\end{cases} \tag{15.78}
\]

Since \( \hat{X}(k) \) belongs to the Hardy space \( H^2(\mathbb{C}^-) \), it follows that for any \( k \in \mathbb{R} \) we have \( \hat{X}(k - i\epsilon) \to \hat{X}(k) \) as \( \epsilon \to 0^+ \) a.e. in \( k \in \mathbb{R} \) and strongly in \( L^2(\mathbb{R}) \). As in the proof of Proposition 15.8, for \( 0 < \epsilon < 1 \), we use \( C_\epsilon \) to denote the positive boundary of the rectangle in \( \mathbb{C}^+ \) with respective corners located at \( a + i\epsilon, b + i\epsilon, b + i, a + i \), where \( a \) and \( b \) are some positive parameters with \( a < b \). Similarly, we use \( C_{-\epsilon} \) to denote the positive boundary of the rectangle in \( \mathbb{C}^- \) with respective corners located at \( b - i\epsilon, a - i\epsilon, a - i, b - i \). Since \( \Xi(k) \) defined in (15.78) is analytic in \( k \in \mathbb{C}^+ \cup \mathbb{C}^- \), it follows from the Cauchy integral formula that for any \( k \) inside \( C_\epsilon \) we have

\[
\Xi(k) = \frac{1}{2\pi i} \int_{C_\epsilon} dt \frac{\Xi(t)}{t - k} + \frac{1}{2\pi i} \int_{C_{-\epsilon}} dt \frac{\Xi(t)}{t - k}, \tag{15.79}
\]

where the contribution by the second integral is zero. Let us choose \( a \) and \( b \) so that in the limit \( \epsilon \to 0^+ \) we have \( \Xi(a \pm i\epsilon) \to \Xi(a) \) and \( \Xi(b \pm i\epsilon) \to \Xi(b) \). Then, letting \( \epsilon \to 0^+ \) in (15.79) we get

\[
\Xi(k) = \frac{1}{2\pi i} \int_{C_0} dt \frac{\Xi(t)}{t - k}, \tag{15.80}
\]
where $C_0$ is the positively oriented boundary of the rectangle with corners at $-a - ib$, $a - ib$, $a + ib$, and $-a + ib$. From the representation in (15.80), we conclude that $\Xi(k)$ is analytic in the interior of the rectangle bounded by $C_0$, including the segment of the real axis contained in that rectangle. Since we can let $a \to -\infty$ and $b \to +\infty$, we conclude that $\Xi(k)$ is in fact entire in $k$. With the help of (15.78) we conclude that $\Xi(k)$ is an odd function of $k$ in $C$ and we have

$$\Xi(-k) = -\Xi(k), \quad k \in C. \quad (15.81)$$

Since $\Xi(k)$ defined in (15.78) has an analytic extension to the entire complex plane, in its Maclaurin expansion of $\Xi(k)$ given by

$$\Xi(k) = \sum_{p=1}^{\infty} \frac{1}{p!} \left[ \Xi^{(p)}(0) \right] k^p, \quad k \in C, \quad (15.82)$$

where the coefficient $\Xi^{(p)}(0)$ can be evaluated, with the help of the generalized Cauchy integral formula as

$$\Xi^{(p)}(0) := \frac{d^p \Xi(k)}{dk^p} \bigg|_{k=0} = \frac{p!}{2\pi i} \int_{T_r} dt \frac{\Xi(t)}{t^{p+1}}, \quad p = 0, 1, 2, \ldots, \quad (15.83)$$

where $T_r$ is the circle of radius $r$ centered at $k = 0$ traversed in the positive direction, with $r := |k|$. Because of (15.81), from (15.83) we conclude that $\Xi^{(p)}(0) = 0$ for even values of $p$ in (15.82). We will now estimate the integral in (15.83). Using (15.81), we have

$$\left| \int_{T_r} dt \frac{\Xi(t)}{t^{p+1}} \right| \leq 2 \int_{T_r^+} |dt| \left| \frac{\Xi(t)}{r^{p+1}} \right|, \quad (15.84)$$

where we use $T_r^+$ to denote the upper semicircle of $T_r$. From Proposition 10.2(b) it follows that there exists some positive number $r_0$ such that

$$|J(k)| \leq C |k|, \quad |k| \geq r_0, \quad k \in \overline{C^+}, \quad (15.85)$$

for some generic constant $C$. On the other hand, since $\hat{X}(k) \in H^2(C^-)$, we have $\hat{X}(-k) \in H^2(C^+)$, and hence by (3.46) we have

$$|\hat{X}(-k)| \leq \frac{C}{\sqrt{|k| \sin \theta}}, \quad k \in C^+. \quad (15.86)$$
Using (15.85) and (15.86) in the first line of (15.78), we have with \( k = r e^{i\theta} \)

\[
|\Xi(r e^{i\theta})| \leq \frac{C \sqrt{r}}{\sqrt[4]{\sin \theta}}, \quad r \geq r_0, \quad k \in T_r^+,
\]

(15.87)

for some generic constant \( C \). Using (15.87) in (15.84) we get the estimate

\[
\left| \int_{T_r} \frac{\Xi(t)}{t^{p+1}} \right| \leq \frac{C}{r^{p+1/2}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}},
\]

(15.88)

where we have used \(|dt| = r \, d\theta\) and that \( \sin \theta = \sin(\pi - \theta) \) for \( \theta \in (0, \pi/2) \). From (15.58) we know that the integral on the right-hand side of (15.88) is convergent. Thus, using (15.58), (15.83), (15.88), and letting \( r \to +\infty \), we conclude that \( \Xi^{(p)}(0) = 0 \) for \( p = 0, 1, \ldots \), and hence from (15.82) we conclude that \( \Xi(k) \equiv 0 \), yielding \( \hat{X}(k) \equiv 0 \). Thus, our proof is complete.

Among various implications of Proposition 15.12, one consequence is that if \( D \) belongs to the Faddeev class then each of (III\(_a\)), (III\(_b\)), and (III\(_c\)) of Definition 4.3 is satisfied.
16. THE SOLUTION TO THE INVERSE PROBLEM

In this chapter, given the scattering data set $S$ in (4.2) belonging to the Marchenko class specified in Definition 4.5, our primary goal is to show that there exists a unique data set $D$ belonging to the Faddeev class, with the understanding that the boundary matrices $A$ and $B$ are unique up to a multiplication from the right by an invertible matrix.

We summarize the construction of $D$ from $S$ as follows, where the existence and uniqueness are implicit at each step:

(a) From the large-$k$ asymptotics of the scattering matrix $S(k)$, with the help of (4.6), we determine the $n \times n$ constant matrix $S_\infty$. We then determine the constant $n \times n$ matrix $G_1$ specified in (14.1). As we show in Proposition 16.4, the matrices $S_\infty$ and $G_1$ are hermitian when $S$ satisfies the condition (1) of Definition 4.5.

(b) In terms of the quantities in $S$, we uniquely construct the $n \times n$ matrix $F_s(y)$ by using (4.7) and the $n \times n$ matrix $F(y)$ by using (4.12).

(c) One uses the matrix $F(y)$ as input to the Marchenko integral equation (13.1). When $F(y)$ is integrable in $y \in (x, +\infty)$ for each $x \in \mathbb{R}^+$, we show in Proposition 16.1 that, for each fixed $x \in \mathbb{R}^+$, there exists a solution $K(x, y)$ integrable in $y \in (x, +\infty)$ to (13.1) and such a solution is unique. The solution $K(x, y)$ can be constructed by iterating (13.1). Even though $K(x, y)$ is constructed for $y > x > 0$, one can extend $K(x, y)$ to $y \in \mathbb{R}^+$ by letting $K(x, y) = 0$ for $0 \leq y < x$.

(d) Having obtained $K(x, y)$ uniquely from $S$, one constructs the potential $V(x)$ via (10.4) and also constructs the Jost solution $f(k, x)$ via (10.6). Then, as indicated in Proposition 16.11, by using (I) of Definition 4.3 and (2) and (4a) of Definition 4.2, one proves that the constructed $V$ satisfies (2.2) and (2.3) and that the constructed $f(k, x)$ satisfies (2.1) with the constructed potential $V(x)$.

(e) Having obtained $K(x, y)$ uniquely, one also obtains the $n \times n$ constant matrix $K(0, 0)$ via (10.5) and proves that $K(0, 0)$ is hermitian.
With the help of the uniquely constructed $n \times n$ constant hermitian matrices $S_\infty$, $G_1$, and $K(0,0)$, one constructs the boundary matrices $A$ and $B$ by solving the linear homogeneous system (14.2) in such a way that the rank of the $2n \times n$ matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is equal to $n$. The details of this step are provided in Proposition 16.9. One proves that the solution pair of matrices $A$ and $B$ to (14.2) is unique up to a multiplication on the right by an invertible matrix. One also proves that any solution to (14.2) satisfies (2.5) and the full rank of the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ guarantees that (2.6) is satisfied. Note that, if $K(0,0)$ were not well defined, then, as seen from (10.5), the potential $V$ constructed via (10.4) would not be integrable, which cannot happen if the scattering data set belongs to the Marchenko class.

Having constructed the Jost solution $f(k, x)$, one then constructs the physical solution $\Psi(k, x)$ via (9.4) and the normalized bound-state matrices $\Psi_j(x)$ via (9.8). One then proves that the constructed matrix $\Psi(k, x)$ satisfies (2.1) and (2.4) and that the constructed $\Psi_j(x)$ satisfies (2.1) at $k = i\kappa_j$ and also (2.4), where $A$ and $B$ are the matrices constructed as explained in the previous step. In the proof that $\Psi(k, x)$ and $\Psi_j(x)$ each satisfy (2.4), one uses (3a) of Definition 4.5.

The following proposition discusses the unique solvability of the Marchenko equation (13.1), and it also indicates the equivalence of (4a) and (4b) of Definition 4.2.

**Proposition 16.1** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ satisfies (I) of Definition 4.3. Then:

(a) For each fixed $x > 0$, the Marchenko integral operator associated with (13.1) is compact on $L^1(x < y < +\infty)$, the corresponding homogeneous Marchenko equation has only the trivial solution in $L^1(x < y < +\infty)$, and the Marchenko equation (13.1) has a unique solution $K(x, y)$ in $y \in L^1(x < y < +\infty)$. Moreover, for each fixed $x > 0$, the solution $K(x, y)$ to the Marchenko equation actually belongs to $L^1(x < y < +\infty) \cap L^\infty(x < +\infty)$. 

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\( y < +\infty \). Hence, \( K(x, y) \) in particular belongs to \( L^2(x < y < +\infty) \).

(b) Without any further assumption, the result in (a) may not hold at \( x = 0 \). If the scattering data further satisfies (4a) of Definition 4.5, then the result in (a) also holds at \( x = 0 \). In other words, if the scattering data set \( S \) satisfies (I) and (4a), then for each fixed \( x \geq 0 \) the Marchenko integral equation (13.1) is uniquely solvable in \( L^1(x < y < +\infty) \), and the unique solution \( K(x, y) \) actually belongs to \( L^1(x < y < +\infty) \cap L^\infty(x < y < +\infty) \). Hence, \( K(x, y) \) in particular belongs to \( L^2(x < y < +\infty) \).

(c) The condition (4a) and (4b) of Proposition 4.2 are equivalent.

PROOF: Because the scattering data satisfies (I), it follows that \( F_s(y) \) given in (4.7) is bounded and integrable in \( y \in \mathbf{R}^+ \). Then, from (4.12) we see that \( F(y) \) given in (4.12) is also bounded and integrable in \( y \in \mathbf{R}^+ \). Hence, the result of Proposition 3.3 applies. Thus, for each fixed \( x > 0 \), the Marchenko integral operator is compact on \( L^1(x < y < +\infty) \). It is proved in Theorem 3.4.1 on p. 82 of [2] that the Marchenko equation (13.1) has a unique solution in \( L^1(x < y < +\infty) \). Then, from Proposition 3.3(c) it follows that the solution \( K(x, y) \) to the Marchenko equation also belongs to \( L^\infty(x < y < +\infty) \). The integrability and the boundedness implies the square integrability, and hence \( K(x, y) \) in particular belongs to \( L^2(x < y < +\infty) \). Thus, the proof of (a) is complete. As later shown in Example 26.14, where (I) is satisfied but (4a) is not satisfied, the result in (a) does not necessarily hold at \( x = 0 \). The further assumption (4a) assures that the result of (a) also holds at \( x = 0 \). Thus, the proof of (b) is completed. As stated in Proposition 3.3(b), the equivalence of (4a) and (4b) directly follows from the fact that the Marchenko integral operator is compact on \( L^1(\mathbf{R}^+) \), which is already established in (a). \( \blacksquare \)

As Proposition 16.1(a) indicates, the unique solvability of the Marchenko equation (13.1) on \( L^1(x, +\infty) \) for each \( x \in \mathbf{R}^+ \) is solely determined by \( S(k) \) satisfying (I) and that the unique solvability is unaffected by the bound-state data. In other words, for the unique solvability of the Marchenko equation, it does not matter what \( N \) is, what the \( \kappa_j \)-values are, and what \( M_j \) are as long as \( N \) is a finite nonnegative integer, the \( \kappa_j \) are distinct
and positive, and the $M_j$ are nonnegative hermitian $n \times n$ matrices of positive rank. On the other hand, if we want to relate our scattering data set $S$ in (4.2) to some data set $D$ in (4.1) belonging to the Faddeev class, where $V$ is the potential and \( \{\kappa_j, M_j\}_{j=1}^{N} \) is the bound-state data set, then $S$ must satisfy further restrictions. In other words, the potential $V(x)$ constructed from the unique solution $K(x, y)$ to the Marchenko equation (13.1) via (10.4) must satisfy (2.2) and (2.3) and that the physical solution $\Psi(k, x)$ and the bound-state solutions $\Psi_j(x)$ constructed via (9.4) and (9.8), respectively, must satisfy the boundary condition (2.4). We further impose (2) of Definition 4.2 on the scattering data set $S$ so that (2.3) is satisfied. We also impose (3a) of Definition 4.2 on the scattering data set $S$ so that $\Psi(k, x)$ and $\Psi_j(x)$ each satisfy (2.4).

The following result shows that when $S$ is a scattering data set that belongs to the Marchenko class, then the solution to the inverse problem $S \mapsto D$ must be unique, with the understanding that the boundary matrices $A$ and $B$ are uniquely determined only up to a postmultiplication by an invertible matrix.

**Proposition 16.2** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ belongs to the Marchenko class specified in Definition 4.5. Then, two distinct input data sets $D_1 := \{V_1, A_1, B_1\}$ and $D_2 := \{V_2, A_2, B_2\}$ in the Faddeev class corresponding to the same $S$ must be related to each other as

$$V_1(x) \equiv V_2(x), \quad A_1 = A_2 T, \quad B_1 = B_2 T,$$

(16.1)

where $T$ is an $n \times n$ invertible matrix.

**PROOF:** As shown in Proposition 6.2, the scattering data set $S$ yields a unique solution $K(x, y)$ to the Marchenko integral equation, and the potential $V(x)$ is uniquely constructed from $K(x, y)$ via (10.4). Hence, we must have $V_1(x) \equiv V_2(x)$. The Jost solution $f(k, x)$ is constructed from $K(x, y)$ as in (10.6), and hence the constructed $f(k, x)$ is unique. The
physical solution $\Psi(k, x)$ is constructed from $f(k, x)$ and $S(k)$ as in (9.4) and hence we must have

$$\Psi_1(k, x) \equiv \Psi_2(k, x), \quad \Psi'_1(k, x) \equiv \Psi'_2(k, x),$$

(16.2)

where $\Psi_1(k, x)$ and $\Psi_2(k, x)$ are the constructed physical solutions associated with $D_1$ and $D_2$, respectively. Let $\varphi_1(k, x)$ and $\varphi_2(k, x)$ be the constructed regular solutions associated with $D_1$ and $D_2$, respectively. We then see from (9.5) that

$$\varphi_1(k, 0) = A_1, \quad \varphi'_1(k, 0) = B_1, \quad \varphi_2(k, 0) = A_2, \quad \varphi_2(k, 0) = B_2.$$  

(16.3)

Let $J_1(k)$ and $J_2(k)$ be the constructed Jost matrices associated with $D_1$ and $D_2$, respectively. From (9.6) we see that

$$\Psi_1(k, x) = -2ik \varphi_1(k, x) J_1(k), \quad \Psi_2(k, x) = -2ik \varphi_2(k, x) J_2(k).$$

(16.4)

Using (16.3) and (16.4) in (16.2) we obtain

$$A_1 J_1(k) = A_2 J_2(k), \quad B_1 J_1(k) = B_2 J_2(k), \quad k \in \mathbb{R} \setminus \{0\}.$$  

(16.5)

By Proposition 10.2(a) we know that the constructed Jost matrix is invertible for $k \in \mathbb{R} \setminus \{0\}$, from (16.5) we obtain

$$A_1 = A_2 J_2(k) J_1(k)^{-1}, \quad B_1 = B_2 J_2(k) J_1(k)^{-1}, \quad k \in \mathbb{R} \setminus \{0\},$$

(16.6)

confirming the second and third equalities in (16.1) with $T$ being equal to $J_2(k) J_1(k)^{-1}$ for any real nonzero $k$-value.

Analogous to (3.96), let us define

$$\tau(x) := \int_x^\infty dz |F'(z)|, \quad \tau_1(x) := \int_x^\infty dz z |F'(z)|, \quad x \geq 0,$$

(16.7)

where $F'(y)$ is the derivative of the quantity $F(y)$ appearing in (4.12). Comparing (16.7) with (3.97)-(3.99), we see that

$$x \tau(x) \leq \tau_1(x), \quad \int_x^\infty dz \tau(z) \leq \tau_1(x),$$

(16.8)
\[ \int_0^\infty dz \, (1 + z) [\tau(z)]^2 \leq [\tau(0) + \tau_1(0)] \tau_1(0). \quad (16.9) \]

Comparing (4.8) and (16.7) we see that \( \tau(0) \) and \( \tau_1(0) \) are both finite when the condition (2) of Definition 4.2 holds.

In the next proposition, we continue to present certain properties of the solution \( K(x, y) \) to the Marchenko equation (13.1).

**Proposition 16.3** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Let \( F_s(y) \) and \( F(y) \) be the quantities defined in (4.7) and (4.12), respectively. Assume that \( S \) satisfies (I) of Definition 4.3 and that \( F'_s(y) \) is integrable in \( y \in \mathbb{R}^+ \). Then:

(a) The quantities \( F_s(y) \) and \( F(y) \) are continuous in \( y \in \mathbb{R}^+ \) and vanish as \( y \to +\infty \). Furthermore, they have finite limits, \( F_s(0^+) \) and \( F(0^+) \), respectively, as \( y \to 0^+ \). Consequently, they are uniformly bounded in \( y \in \mathbb{R}^+ \).

(b) The solution \( K(x, y) \) to the Marchenko equation satisfies

\[ |K(x, y)| \leq C \tau(x + y), \quad 0 < x \leq y, \quad (16.10) \]

where \( C \) is a generic constant and \( \tau(x) \) is the scalar quantity defined in (16.7). If \( S \) further satisfies (4a), then we have

\[ |K(x, y)| \leq C \tau(x + y), \quad 0 \leq x \leq y. \quad (16.11) \]

**PROOF:** Since \( F'_s(y) \) is assumed to be integrable in \( y \in \mathbb{R}^+ \), we have

\[ F_s(y) = -\int_y^\infty dz \, F'_s(z), \quad y \in \mathbb{R}^+. \quad (16.12) \]

Using Lebesgue’s dominated convergence theorem on (16.12) we conclude that \( F_s(y) \) is continuous in \( y \in \mathbb{R}^+ \), \( F_s(y) \) vanishes as \( y \to +\infty \), and \( F_s(0^+) \) is finite. These three
properties also hold for \(F(y)\) because \(F(y)\) is related to \(F_s(y)\) as in (4.12), where all \(\kappa_j\) are positive. Hence, the second sentence of (a) is valid. Let us now prove (b). From (16.12) we obtain
\[
|F(y)| \leq \int_y^\infty dz |F'(z)| = \tau(y), \quad y \geq 0.
\] (16.13)

From the Marchenko equation (13.1) we obtain
\[
|K(x, y)| \leq |F(x + y)| + \int_x^\infty dz |K(x, z)| |F(z + y)|, \quad 0 < x \leq y.
\] (16.14)

Using (16.13) in (16.14) we obtain
\[
|K(x, y)| \leq \tau(x + y) + \int_x^\infty dz |K(x, z)| \tau(z + y), \quad 0 < x \leq y.
\] (16.15)

Since \(\tau(x)\) defined in (16.7) is a nonincreasing function in \(x \in \mathbb{R}^+\), from (16.15) we get
\[
|K(x, y)| \leq \tau(x + y) + \tau(x + y) \int_x^\infty dz |K(x, z)|, \quad 0 < x \leq y.
\] (16.16)

As asserted in Proposition 16.1(a), \(K(x, y)\) is integrable in \(y \in (x, +\infty)\), and hence the integral in (16.16) is bounded, yielding
\[
|K(x, y)| \leq \tau(x + y) + C \tau(x + y), \quad 0 < x \leq y,
\] (16.17)

for some constant \(C\). From (16.17) we obtain (16.10) for some generic constant \(C\). Under the further assumption that \(S\) satisfies (4\(a\)), the results in (16.14)-(16.17) hold also at \(x = 0\), and hence we conclude (16.11). Thus, the proof is complete.

In the next proposition, we study the relevant properties of some quantities related to the scattering matrix \(S(k)\) and the solution \(K(x, y)\) to the Marchenko integral equation (13.1).

**Proposition 16.4** Consider a scattering data set \(S\) as in (4.2), which consists of an \(n \times n\) scattering matrix \(S(k)\) for \(k \in \mathbb{R}\), a set of \(N\) distinct positive constants \(\kappa_j\), and a set of \(N\) constant \(n \times n\) hermitian and nonnegative matrices \(M_j\) with respective positive ranks \(m_j\), where \(N\) is a nonnegative integer. Assume that \(S\) satisfies (I) of Definition 4.3. Let
$F_s(y)$ and $F(y)$ be the quantities defined in (4.7) and (4.12), respectively, where $S_\infty$ is the constant $n \times n$ matrix defined as in (4.6). Let $K(x,y)$ be the unique solution to (13.1), whose existence and uniqueness for each fixed $x > 0$ are assured by Proposition 16.1(a).

Then:

(a) The matrix $S_\infty$ satisfies

$$S_\infty = S_\infty^\dagger = S_\infty^{-1},$$

and hence $S_\infty$ is hermitian and each eigenvalue of $S_\infty$ is either $+1$ or $-1$.

(b) The matrices $F_s(y)$ and $F(y)$ are hermitian for every $y \in \mathbb{R}^+$.

(c) The $n \times n$ matrix $K(x,x)$ is hermitian for each fixed $x \in \mathbb{R}^+$. Under the further assumption of $(4_a)$ of Definition 4.2, the matrix $K(x,x)$ is hermitian for $x \geq 0$. In particular, the hermitian property of $K(0,0)$ is assured under the additional assumption of $(4_a)$.

(d) If the scattering data set $S$ satisfies (1) of Definition 4.2, then the $n \times n$ constant matrix $G_1$ given in (14.1) is hermitian.

PROOF: Note that (4.4) yields (16.18) and hence $S_\infty$ is hermitian and unitary. Thus, each eigenvalue of $S_\infty$ must be real and equal to either $+1$ or $-1$, establishing (a). Let us now turn to the proof of (b). The hermitian property of $F_s(y)$ is obtained by using the first equality of (4.4) as well as (a) in (4.7). Because the matrices $M_j$ are hermitian, the $\kappa_j$-values are real, and we have already proved that $F_s(y)$ is hermitian for every $y \in \mathbb{R}^+$, it follows from (4.12) that $F(y)$ is hermitian for every $y \in \mathbb{R}^+$. Thus, the proof of (b) is complete. The proof of (c) can be found on p. 122 of [2] and based on the fact that $F(y)$ is hermitian and that the Marchenko equation (13.1) is uniquely solvable for each $x \in \mathbb{R}^+$, as indicated in Proposition 16.1(b). Under the additional assumption of $(4_a)$, the Marchenko equation is uniquely solvable for each $x \geq 0$ and the proof of the hermitian property of $K(x,x)$ for each $x \geq 0$ again follows from p. 122 of [2]. Let us now turn to the proof of (d). The assumption (1) assures the existence of $G_1$, and the hermitian property of $G_1$
follows from the fact that $S(k)$ is hermitian as stated in the first equality in (4.4) and that $S_{\infty}$ is hermitian as stated in (a) and we have $ik$ appearing as a factor on the right-hand side of (14.1).

From Theorem 15.10 and Proposition 16.4(c) it follows that the matrix $K(x, x)$ is hermitian for each $x \geq 0$ when the input data set $D$ in (4.1) belongs to the Faddeev class. On the other hand, we remark that, in the Faddeev class, in general the solution $K(x, y)$ to the Marchenko integral equation (13.1) is not hermitian when $0 \leq x < y$.

As mentioned in Chapter 13, in the analysis of the inverse problem for (2.1) with the general selfadjoint boundary condition (2.4), the derivative Marchenko equation (13.7) plays an equally important role as the Marchenko equation (13.1). Let us further elaborate on this issue. In order to prove that the physical solution $\Psi(k, x)$ constructed as in (9.4) satisfies the boundary condition, one needs to construct both $f(k, x)$ and $f'(k, x)$, where $f(k, x)$ is the Jost solution to (2.1) constructed as in (10.6) and $K(x, y)$ is obtained by solving the Marchenko equation (13.1). Thus, unless the boundary condition (2.4) is the Dirichlet boundary, the construction of the physical solution also requires the construction of $f'(k, x)$, and this requires the analysis of the derivative Marchenko equation (13.7). In the special case with the Dirichlet boundary condition studied in [2], the analysis of the derivative Marchenko equation (13.7) is not needed for the satisfaction of the boundary condition. It turns out that, under appropriate restrictions on our scattering data, which are all satisfied when the scattering data belongs to the Marchenko class, the derivative Marchenko equation (13.7) is uniquely solvable and its unique solution is given by $K(x, y)$, where $K(x, y)$ is the unique solution to the Marchenko equation (13.1). A proof of this is given in Lemma 5.3.3 of [2] under the restriction that the derivative $F'(y)$ is continuous in $y \in \mathbb{R}^+$, where $F(y)$ is the quantity defined in (4.12). In general, when the input data set $D$ in (4.1) is in the Faddeev class, the constructed $F'(y)$ in the corresponding scattering data set $S$ is not necessarily continuous in $y \in \mathbb{R}^+$. We extend the proof in [2] without requiring the continuity of $F'(y)$ in $y \in \mathbb{R}^+$, by only using the integrability of $F'(y)$ in $y \in \mathbb{R}^+$. From Definition 4.2 we see that the integrability of $F'(y)$ in $y \in \mathbb{R}^+$ is assured if
S satisfies the property (2).

Proposition 16.5 Consider a scattering data set \( \mathbf{S} \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Assume that \( \mathbf{S} \) satisfies (I) of Definition 4.3. Assume also that \( F'_s(y) \) is integrable in \( y \in \mathbb{R}^+ \), where \( F_s(y) \) is the quantity defined in (4.7). Then:

(a) For each fixed \( x > 0 \), the derivative Marchenko integral operator associated with (13.7) is compact on \( L^1(x < y < +\infty) \), the corresponding homogeneous derivative Marchenko equation has only the trivial solution in \( L^1(x < y < +\infty) \), and the derivative Marchenko equation (13.7) has a unique solution \( L(x, y) \) in \( L^1(x < y < +\infty) \).

(b) For each fixed \( x > 0 \), the unique solution \( L(x, y) \) to (13.7) is equal to \( K_x(x, y) \), where \( K(x, y) \) is the unique solution to the Marchenko equation (13.1), whose existence and uniqueness are established in Proposition 16.1(a).

(c) Without any further assumption, the result in (a) may not hold at \( x = 0 \). If the scattering data further satisfies (4a) in Definition 4.5, then the result in (a) also holds at \( x = 0 \). In other words, if the scattering data set \( \mathbf{S} \) satisfies (I) and (4a) and we also have the integrability of \( F'_s(y) \) in \( y \in \mathbb{R}^+ \), then (13.7) is uniquely solvable with the solution \( L(x, y) \) which is integrable in \( L^1(x < y < +\infty) \) for each \( x \in [0, +\infty) \).

(d) When (4a) is also satisfied, the solution \( L(x, y) \) to (13.7) satisfies \( L(x, y) = K_x(x, y) \) for \( 0 \leq x \leq y \), where \( K(x, y) \) is the solution to (13.1), whose existence and uniqueness is established in Proposition 16.1(b).

PROOF: The proofs of (c) and (d) under the additional assumption (4a) are similar to the proofs of (a) and (b) under the assumption (I). Thus, we only present the proofs for (a) and (b). Let us compare the Marchenko equation (13.1) and the derivative Marchenko equation (13.7). They have the same kernel and they only differ in their nonhomogeneous terms. The nonhomogeneous term in (13.1) is \( F(x + y) \) and the nonhomogeneous term in (13.7)
is $F'(x + y) - K(x, x) F(x + y)$. We will show that both nonhomogeneous terms belong to $L^1(x < y < +\infty)$ for each fixed $x > 0$, and as a result, the unique solvability stated in (a) directly follows from Proposition 16.1(a). After that, it only remains to prove (b) by showing that the unique solution to (13.7) is given by $K_x(x, y)$. Let us first prove that the aforementioned nonhomogeneous terms are both integrable in $y \in (x, +\infty)$ for each fixed $x > 0$. The integrability of $F(x + y)$ for $y \in (x, +\infty)$ is assured by the integrability of $F_s(y)$ in $y \in \mathbb{R}^+$, which is ensured by (I), and the fact that $F(y)$ and $F_s(y)$ are related to each other as in (4.12) where the $\kappa_j$ are all positive. The integrability of $F'(x + y)$ is assured by the assumed integrability of $F_s'(y)$ in $y \in \mathbb{R}^+$ and again by the fact that $F(y)$ and $F_s(y)$ are related to each other as in (4.7) where the $\kappa_j$ are all positive. The integrability of $K(x, x) F(x + y)$ is assured by (3.28) and by the fact that $K(x, x)$ is bounded, as assured by Proposition 16.1(a), and that $F(x + y)$ is integrable when $y \in (x, +\infty)$. Thus, the nonhomogeneous term in (13.7) belongs to $L^1(x < y < +\infty)$ for each fixed $x > 0$, and the proof of (a) is complete. Hence, it only remains to prove (b) by showing that the unique solution $L(x, y)$ to (13.7) is given by $K_x(x, y)$. A proof of this is given in Lemma 5.3.3 of [2] by assuming that $F_s'(y)$ is continuous in $y \in \mathbb{R}^+$. We will extend that proof by assuming that $F_s'(y)$ is integrable in $y \in \mathbb{R}^+$ instead of assuming that $F_s'(y)$ is continuous in $y \in \mathbb{R}^+$. We proceed as follows. By Proposition 16.1(a) we know that (13.1) has a unique solution $K(x, y)$, which can be written as

$$K(x, y) = -F(x + y) (I + O_x)^{-1},$$

(16.19)

where $O_x$ is the integral operator on $L^1(x < y < +\infty)$ defined as

$$(X O_x)(y) := \int_x^\infty dz X(z) F(z + y), \quad 0 < x \leq y.$$

(16.20)

From Proposition 16.1(b) we know that for each $x \geq 0$ the Marchenko integral operator is compact and the homogeneous Marchenko integral equation has only the trivial solution. Consequently, the operator $(I + O_x)$ is invertible and the inverse $(I + O_x)^{-1}$ is bounded as an operator on $L^1(x < y < +\infty)$. By (a) we are assured that (13.7) has a unique solution
and we use $L(x, y)$ to denote that solution. With the help of (16.20) we get

$$L(x, y) = [F'(x + y) - K(x, x) F(x + y)] (I + O_x)^{-1}, \quad (16.21)$$

where $K(x, x)$ is obtained from (16.19) as $K(x, x^+)$. Let us now approximate $F(y)$ by an appropriate sequence $\{F^{(m)}(y)\}_{m=1}^\infty$ converging to $F(y)$. We will soon see how to choose the sequence. Let us use $K^{(m)}(x, y)$ to denote the solution to (13.1) when $F^{(m)}(y)$ is used there instead of $F(y)$, i.e. we would like $K^{(m)}(x, y)$ to satisfy

$$K^{(m)}(x, y) + F^{(m)}(x + y) + \int_x^\infty dz K^{(m)}(x, z) F^{(m)}(z + y) = 0, \quad 0 < x \leq y. \quad (16.22)$$

From (13.1) and (16.22) we obtain

$$|K^{(m)}(x, y) - K(x, y)| \leq |F^{(m)}(x + y) - F(x + y)|$$

$$+ \int_x^\infty dz |K^{(m)}(x, z) - K(x, z)| |F^{(m)}(x + y)| \quad (16.23)$$

$$+ \int_x^\infty dz |K(x, z)| |F^{(m)}(x + y) - F(x + y)|.$$ 

We also would like $K^{(m)}_x(x, y)$, the $x$-derivative of $K^{(m)}(x, y)$, to be the solution to (13.7), i.e.

$$K^{(m)}_x(x, y) + F^{(m)'}(x + y) - K^{(m)}(x, x) F^{(m)}(x + y)$$

$$+ \int_x^\infty dz K^{(m)}_x(x, z) F^{(m)}(z + y) = 0, \quad 0 < x \leq y, \quad (16.24)$$

where $F^{(m)'}(y)$ denotes the $y$-derivative of $F^{(m)}(y)$. From (13.7) and (16.24) we obtain the analog of (16.23) given by

$$|K^{(m)}_x(x, y) - L(x, y)| \leq |F^{(m)'}(x + y) - F'(x + y)|$$

$$+ |K^{(m)}(x, x) - K(x, x)| |F^{(m)}(x + y)|$$

$$+ |K(x, x)| |F^{(m)}(x + y) - F(x + y)|$$

$$+ \int_x^\infty dz |K^{(m)}(x, z) - K(x, z)| |F^{(m)}(x + y)|$$

$$+ \int_x^\infty dz |K(x, z)| |F^{(m)}(x + y) - F(x + y)|. \quad (16.25)$$
From (16.23) and (16.25) we see that it is appropriate to choose the sequence \( \{ F^{(m)}(y) \}_{m=1}^{\infty} \) in such a way that \( F^{(m)}(y) \) belongs to \( C_0^\infty(\mathbb{R}^+) \), is uniformly bounded, i.e. satisfying \( |F^{(m)}(y)| \leq c \) for some constant \( c \) for all \( m \geq 1 \) and \( y \in \mathbb{R}^+ \), and converging uniformly to \( F(y) \) in every compact subset of \( \mathbb{R}^+ \), and also satisfying

\[
\lim_{m \to +\infty} \int_0^\infty dy |F^{(m)}(y) - F(y)| = 0,
\]

\[
\lim_{m \to +\infty} \int_0^\infty dy |F^{(m)'}(y) - F'(y)| = 0.
\]

Our goal is now to prove that

\[
K_x(x, y) = L(x, y), \quad 0 < x \leq y,
\]

where we recall that \( K_x(x, y) \) is the \( x \)-derivative of the unique solution \( K(x, y) \) to (13.1). In analogy with (16.20) let us introduce the sequence of operators on \( L^1(x < y < +\infty) \) given by \( \{ O^{(m)}_x \}_{m=1}^{\infty} \) and converging to \( O_x \) as \( m \to +\infty \). We define

\[
\left( X O^{(m)}_x \right)(y) := \int_x^\infty dz X(z) F^{(m)}(z + y), \quad 0 < x \leq y.
\]

Since the constructed sequence \( \{ F^{(m)}(y) \}_{m=1}^{\infty} \) converges to \( F(y) \), we have the convergence \( ||O^{(m)}_x - O_x||_{L^1(x < y < +\infty)} \to 0 \) as \( m \to +\infty \) in the uniform norm of the bounded operators on \( L^1(x < y < +\infty) \). Then, for \( m \) large enough, the operator \( (I + O^{(m)}_x) \) is invertible and we have

\[
\lim_{m \to +\infty} \left\| \left( I + O^{(m)}_x \right)^{-1} - (I + O_x)^{-1} \right\|_{L^1(x < y < +\infty)} = 0,
\]

in the operator norm on \( L^1(x < y < +\infty) \). To see these, we proceed as follows. The invertibility of the operator \( (I + O_x) \) has already been established, as argued below (16.20). Furthermore, the result in (16.30) follows from the bounded invertibility theorem, i.e. Theorem 1.16 on p. 196 of [27]. From (16.30) we get

\[
\lim_{m \to +\infty} F^{(m)}(x + y) \left( I + O^{(m)}_x \right)^{-1} = F(x + y) (I + O_x)^{-1}.
\]

Having constructed the sequence \( \{ F^{(m)}(y) \}_{m=1}^{\infty} \) converging to \( F(y) \), let us now consider the Marchenko equation (13.1) but \( F(y) \) replaced with \( F^{(m)}(y) \) there. Since \( (I + O^{(m)}_x) \) is
invertible for large $m$, the unique solution to the corresponding Marchenko equation with input $F^{(m)}(y)$ for large $m$ is, analogous to (16.19), given by

$$K^{(m)}(x, y) = -F^{(m)}(x + y) \left( I + O^{(m)}_x \right)^{-1}. \quad (16.32)$$

Proceeding in a similar manner, for the derivative Marchenko integral equation (13.7) but $F(y)$ and $F'(y)$ replaced with $F^{(m)}(y)$ and $F^{(m)}(y)$, respectively, there, for large enough $m$ we obtain

$$K^{(m)}_x(x, y) = \left[ F^{(m)}(x + y) - K^{(m)}(x, x) F^{(m)}(x + y) \right] \left( I + O^{(m)}_x \right)^{-1}. \quad (16.33)$$

In order to obtain (16.33) we have used the invertibility of the operator $\left( I + O^{(m)}_x \right)$ for large $m$ as well as the fact that the unique solution to the derivative Marchenko equation (13.7) with input $F^{(m)}(y)$ and $F^{(m)}(y)$ for large $m$ is given by $K^{(m)}_x(x, y)$, which is established in Lemma 5.3.3 of [2]. From (16.30) we obtain

$$\lim_{m \to +\infty} \left[ F^{(m)}(x + y) - K^{(m)}(x, x) F^{(m)}(x + y) \right] \left( I + O^{(m)}_x \right)^{-1} = [F'(x + y) - K(x, x) F(x + y)] (I + O_x)^{-1}. \quad (16.34)$$

Then, we see that (16.31) and (16.32) yields

$$\lim_{m \to +\infty} K^{(m)}(x, y) = K(x, y), \quad (16.35)$$

and we also see that (16.21) and (16.34) yields

$$\lim_{m \to +\infty} K^{(m)}_x(x, y) = L(x, y). \quad (16.36)$$

For large $m$, we know that $K^{(m)}_x(x, y)$ for large $m$ is integrable in $y \in \mathbb{R}^+$ for each fixed $x \geq 0$, and hence we have

$$K^{(m)}(x, y) = K^{(m)}(0, y) + \int_0^x dz K^{(m)}_x(z, y). \quad (16.37)$$

Postmultiplying (16.37) by a test function $\varphi(y)$ belonging to $C^\infty_0(\mathbb{R}^+)$, and integrating over $y \in \mathbb{R}^+$ we obtain

$$\int_0^\infty dy K^{(m)}(x, y) \varphi(y) = \int_0^\infty dy K^{(m)}(0, y) \varphi(y) + \int_0^\infty dy \int_0^x dz K^{(m)}_x(z, y) \varphi(y). \quad (16.38)$$
Since $K_{x}^{(m)}(z,y)$ is integrable in $(z,y)$ in the domain of its integration in (16.38), we can change the order of integration there and obtain

$$
\int_{0}^{\infty} dy K^{(m)}(x,y) \varphi(y) = \int_{0}^{\infty} dy K^{(m)}(0,y) \varphi(y) + \int_{0}^{x} dz \int_{0}^{\infty} dy K_{x}^{(m)}(z,y) \varphi(y).
$$  

(16.39)

By letting $m \to +\infty$ in (16.39), with the help of (16.35) and (16.36), we obtain

$$
\int_{0}^{\infty} dy K(x,y) \varphi(y) = \int_{0}^{\infty} dy K(0,y) \varphi(y) + \int_{0}^{x} dz \int_{0}^{\infty} dy L(z,y) \varphi(y).
$$  

(16.40)

Because $L(x,y)$ is integrable in $(z,y)$ in its domain of integration in (16.40), we can change the order of integration there and obtain

$$
\int_{0}^{\infty} dy K(x,y) \varphi(y) = \int_{0}^{\infty} dy K(0,y) \varphi(y) + \int_{0}^{\infty} dy \int_{0}^{x} dz L(z,y) \varphi(y).
$$  

(16.41)

because of the arbitrariness of the test function $\varphi(y)$ in (16.41), we obtain

$$
K(x,y) = K(0,y) + \int_{0}^{x} dz L(z,y),
$$  

(16.42)

and by taking the $x$-derivative of both sides of (16.42) we obtain (16.28). Thus, the proof is complete.

In the next proposition we obtain a useful bound on the solution $K_{x}(x,y)$ to the derivative Marchenko equation (13.7).

**Proposition 16.6** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_{j}$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_{j}$ with respective positive ranks $m_{j}$, where $N$ is a nonnegative integer. Assume that $S$ satisfies (I) of Definition 4.3. Assume also that $F'_{s}(y)$ is integrable in $y \in \mathbb{R}^{+}$, where $F_{s}(y)$ is the quantity defined in (4.7). Then:

(a) For each fixed $x > 0$, the unique solution $K_{x}(x,y)$ to the derivative Marchenko integral equation (13.7) satisfies

$$
|K_{x}(x,y)| \leq |F'(x + y)| + C, \quad 0 < x \leq y,
$$  

(16.43)
where \( F(y) \) is the quantity defined in (4.12) and \( C \) is a generic constant.

(b) Without any further assumption, the result in (a) may not hold at \( x = 0 \). If the scattering data further satisfies (4_a) in Definition 4.2, then the result in (a) also holds at \( x = 0 \), i.e. we have

\[
|K_x(x,y)| \leq |F'(x+y)| + C, \quad 0 \leq x \leq y. \tag{16.44}
\]

(c) If the scattering data further satisfies (4_a) in Definition 4.2, then we have

\[
||K_x(x,\cdot)||_{L^1(x<y<\infty)} \leq C ||F'(x+\cdot) - K(x,x) F(x+\cdot)||_{L^1(x<y<\infty)}, \quad x \geq 0, \tag{16.45}
\]

for some generic constant \( C \).

PROOF: By Proposition 16.5(b) we know that \( K_x(x,y) \) exists and that \( K_x(x,y) \) is integrable in \( y \in (x, +\infty) \) for each fixed \( x > 0 \). From (13.6) we obtain

\[
|K_x(x,y)| \leq |F'(x+y)| + |K(x,x)| |F(x+y)| + \int_x^\infty dz |K_x(x,z)||F(z+y)|, \quad 0 < x \leq y. \tag{16.46}
\]

From Proposition 16.3(a) we know that \( F(y) \) is uniformly bounded in \( y \in \mathbb{R}^+ \). Hence, (16.46) yields

\[
|K_x(x,y)| \leq |F'(x+y)| + C |K(x,x)| + C \int_x^\infty dz |K_x(x,z)|, \quad 0 < x \leq y. \tag{16.47}
\]

From the aforementioned integrability of \( K_x(x,y) \) we conclude that the integral in (16.47) is finite. Furthermore, using (16.11) in the middle term in (16.47) we obtain

\[
|K_x(x,y)| \leq |F'(x+y)| + C \tau(x+y) + C, \quad 0 < x \leq y. \tag{16.48}
\]

where \( \tau(x) \) is the scalar quantity defined in (16.7). Since \( \tau(x) \) is a nonincreasing function in \( x \in \mathbb{R}^+ \), we have \( \tau(x+y) \leq \tau(0) < +\infty \), where the finiteness of \( \tau(0) \) follows from the first definition in (16.7) with the assumption that \( F'(y) \in L^1(\mathbb{R}^+) \). Thus, (16.48) yields (16.44). Hence, the proof of (a) is complete. If we further have (4_a), then from
Propositions 16.3(b), 16.5(c), and 16.5(d), it follows that (16.43) holds also at \( x = 0 \), and hence (16.44) is justified, and the proof of (b) is completed. Let us now turn to the proof of (c). As in (16.20) let us use \( \mathcal{O}_x \) to denote the Marchenko integral operator on \( L^1(x < y < +\infty) \). From Proposition 16.5 we know that the derivative Marchenko integral equation is uniquely solvable and the solution is given by \( K_x(x,y) \). Thus, from (16.21) we obtain
\[
K_x(x,y) = [F'(x+y) - K(x,x)F(x+y)](I + \mathcal{O}_x)^{-1}.
\]
(16.49)

In the proof of Proposition 16.5 we have seen that the term \( F'(x+y) - K(x,x)F(x+y) \) belongs to \( L^1(x < y < +\infty) \), holding for each \( x \geq 0 \) under the additional assumption of \( (4_a) \). Since \( (I + \mathcal{O}_x)^{-1} \) is bounded on \( L^1(x < y < +\infty) \), from (16.49) we obtain
\[
||K_x(x,\cdot)||_{L^1(x<y<+\infty)} \\
\leq ||F'(x+\cdot) - K(x,x)F(x+\cdot)||_{L^1(x<y<+\infty)} \cdot ||(I + \mathcal{O}_x)^{-1}||_{L^1(x<y<+\infty)} \\
\leq C \cdot ||F'(x+\cdot) - K(x,x)F(x+\cdot)||_{L^1(x<y<+\infty)},
\]
(16.50)
yielding (16.45). \( \blacksquare \)

The next proposition provides certain properties of \( F_s(y) \) defined in (4.7) and of its derivative \( F'_s(y) \) given in (12.3).

**Proposition 16.7** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Assume that \( S \) satisfies (I) of Definition 4.3. Further assume that \( F'_s(y) \) given in (12.3) is integrable in \( y \in \mathbb{R}^+ \) and is the sum of an integrable function and a square integrable function in \( y \in \mathbb{R}^- \). Then, we have the following:

(a) The matrix \( F_s(y) \) given in (4.7) is continuous in \( y \in \mathbb{R} \setminus \{0\} \).

(b) The limits \( F_s(0^+) \) and \( F_s(0^-) \) exist and we have (12.2) satisfied, where \( G_1 \) is the constant matrix appearing in (4.5).
(c) The matrix $F_s'(y)$ can be written as

$$F_s'(y) = G_1 \delta(y) + \tilde{F}_s'(y), \quad y \in \mathbb{R},$$

where the regular part $\tilde{F}_s'(y)$ of $F_s'(y)$ is defined as

$$\tilde{F}_s'(y) := \begin{cases} F_s'(y), & y \in \mathbb{R}^+, \\ F_s'(y), & y \in \mathbb{R}^- \end{cases}.$$ (16.52)

PROOF: Since $F_s'(y)$ is integrable in $y \in \mathbb{R}^+$, we can conclude that $F_s(y)$ is continuous in $y \in \mathbb{R}^+$ and that $F_s(0^+)$ exists. Since $F_s'(y)$ is the sum of an integrable function and a square-integrable function in $y \in \mathbb{R}^-$, it is locally integrable in $y \in \mathbb{R}^-$ and hence $F_s(y)$ is continuous in $y \in \mathbb{R}^-$ and that $F_s(0^-)$ exists. Thus, we have proved (a) and the first part of (b). Let us now prove the rest of the proposition. Since $F_s'(y)$ exists in $y \in \mathbb{R} \setminus \{0\}$ and the limits $F_s(0^\pm)$ exist, using integration by parts and interpreting the derivative in the distribution sense, we conclude that

$$F_s'(y) = \left[ F_s(0^+) - F_s(0^-) \right] \delta(y) + \tilde{F}_s'(y),$$

where $\tilde{F}_s'(y)$ is the quantity defined in (16.53). It only remains to prove that (12.2) holds, i.e. the coefficient of the delta-distribution in (16.53) is equal to $G_1$. Let us define

$$H(k) := \int_{-\infty}^{\infty} dy \tilde{F}_s'(y) e^{-iky}.$$ (16.54)

The quantity $H(k)$ exists because it corresponds to the Fourier transform of the sum of an integrable function and a square-integrable function in $y \in \mathbb{R}$. Let us use $H_\infty(k)$ to denote the Fourier transform of the integrable part of $\tilde{F}_s'(y)$ and use $H_2(k)$ to denote the Fourier transform of the square-integrable part of $\tilde{F}_s'(y)$. Thus, we have

$$H(k) = H_\infty(k) + H_2(k), \quad k \in \mathbb{R}.$$ (16.55)

We remark that $H_\infty(k)$ is bounded in $k \in \mathbb{R}$ because it is the Fourier transform of an integrable function, and furthermore, with the help of the Riemann-Lebesgue lemma, it follows that

$$H_\infty(k) = o(1), \quad k \to \pm \infty.$$ (16.56)
On the other hand, $H_2(k)$ itself is square integrable in $k \in \mathbb{R}$ because it is the Fourier transform of a square-integrable function. Let us use $\hat{F}'_s(k)$ to denote the Fourier transform of $F'_s(y)$, where we have defined

$$\hat{F}'_s(k) := \int_{-\infty}^{\infty} dy F'_s(y) e^{-iky}.$$  \hfill (16.57)

From (4.5) and (12.3) we see that $\hat{F}'_s(k)$ is given by

$$\hat{F}'_s(k) = G_1 + H_3(k), \quad k \in \mathbb{R},$$  \hfill (16.58)

where, as seen from (4.5), we have $H_3(k)$ has the behavior of the product of $ik$ and $o(1/k)$ as $k \to \pm \infty$. Thus, we obtain

$$H_3(k) = o(1), \quad k \to \pm \infty.$$  \hfill (16.59)

From (16.53)-(16.59) we obtain

$$F_s(0^+) - F_s(0^-) + H_\infty(k) + H_2(k) = G_1 + H_3(k), \quad k \in \mathbb{R}.$$  \hfill (16.60)

Let us write (16.60) as

$$F_s(0^+) - F_s(0^-) - G_1 + H_\infty(k) = H_3(k) - H_\infty(k), \quad k \in \mathbb{R}.$$  \hfill (16.61)

With the help of (16.56) and (16.59) we see that the right-hand side of (16.61) vanishes as $k \to \pm \infty$. In general, the Fourier transform of a square-integrable function does not vanish as $k \to \pm \infty$ and hence we cannot conclude that $H_2(k) = o(1)$ as $k \to \pm \infty$. On the other hand, since the right-hand side of (16.61) has the limit zero as $k \to \pm \infty$, we conclude that $H_2(k)$ must have a limit as $k \to \pm \infty$ and, since $H_2(k)$ itself is square integrable in $k \in \mathbb{R}$, that limit must be zero. Thus, the left-hand side of (16.61) in the limit as $k \to \pm \infty$ yields

$$F_s(0^+) - F_s(0^-) - G_1 = 0,$$  \hfill (16.62)

proving that (12.2) holds, and hence (16.62) asserts that (16.51) holds.  \hfill □
The following result is needed in the proof of Proposition 18.1.

**Proposition 16.8** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Let \( F_s(y) \) be the quantity defined in (4.7). Assume that \( S \) satisfies (I) of Definition 4.3. Furthermore, assume that \( F'_s(y) \) given in (12.3) is integrable in \( y \in \mathbb{R}^+ \) and is the sum of an integrable function and a square integrable function in \( y \in \mathbb{R}^- \). Then, we have the following:

\[
\int_{-\infty}^{\infty} dy F_s(y) F_s(y + x) = -F_s(-x) S_{\infty} - S_{\infty} F_s(x), \quad x \in \mathbb{R}, \tag{16.63}
\]

\[
\int_{-\infty}^{\infty} dy F_s(z + y) F_s(y + x) = -F_s(z - x) S_{\infty} - S_{\infty} F_s(x - z), \quad x, z \in \mathbb{R}, \tag{16.64}
\]

\[
\int_{-\infty}^{\infty} dy F'_s(y) F_s(y + x) = -F'_s(-x) S_{\infty} + S_{\infty} F'_s(x), \quad x \in \mathbb{R} \setminus \{0\}, \tag{16.65}
\]

\[
\int_{-\infty}^{\infty} dy \bar{F}'_s(y) F_s(y + x) = -\bar{F}'_s(-x) S_{\infty} + S_{\infty} \bar{F}'_s(x) - G_1 F_s(x), \quad x \in \mathbb{R}, \tag{16.66}
\]

where \( \bar{F}'_s(y) \) is the quantity related to \( F'_s(y) \) as in (16.52) and \( G_1 \) is the constant matrix appearing in (4.5).

**PROOF:** Note that (16.63) is obtained from (16.64) by setting \( z = 0 \) there and hence we can skip the proof of (16.63). Let us first prove (16.64). If (1) holds, then from Proposition 4.4 we can conclude that \( F_s(y) \) is square integrable in \( y \in \mathbb{R} \). Hence, the integrand in (16.64), being the product of two square-integrable functions, is integrable, and as a result the integral on the left-hand side of (16.64) exists. Using (4.7) we evaluate the left-hand side of (16.64), and with the help of (11.37) we get

\[
\int_{-\infty}^{\infty} dy F_s(z + y) F_s(y + x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left[ S(k) - S_{\infty} \right] \left[ S(-k) - S_{\infty} \right] e^{ik(z-x)}. \tag{16.67}
\]

From (4.4) and (16.60) we have

\[
\left[ S(k) - S_{\infty} \right] \left[ S(-k) - S_{\infty} \right] = - \left[ S(k) - S_{\infty} \right] S_{\infty} - S_{\infty} \left[ S(-k) - S_{\infty} \right], \quad k \in \mathbb{R}. \tag{16.68}
\]
Using (16.68) on the right-hand side of (16.67), with the help of (4.7) we obtain (16.64). Let us now turn to the proof of (16.65). With the help of (4.7), (11.37), and (12.3), we evaluate the left-hand side of (16.65) as
\[
\int_{-\infty}^{\infty} dy F'_s(y) F_s(y + x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk ik [S(k) - S_\infty] [S(-k) - S_\infty] e^{-ikx}. \tag{16.69}
\]
Using (16.68) on the right-hand side of (16.69), with the help of (12.3), we simplify the right-hand side of (16.69) and establish (16.65). As seen from (16.51), \(F'_s(x)\) contains a Dirac-delta distribution at \(x = 0\), and hence we need to exclude \(x = 0\) from (16.65). Finally, (16.66) is obtained from (16.66) with the help of (16.51) and the fact that the integrand in (16.66) is integrable. The integrability of the integrand follows from the fact that \(F'_s(y)\) is integrable in \(y \in \mathbb{R}^+\) and is the sum of an integrable function and a square-integrable function in \(y \in \mathbb{R}^-\) and that \(F_s(y)\) is bounded in \(y \in \mathbb{R}\) and integrable in \(y \in \mathbb{R}^+\) and is square integrable in \(y \in \mathbb{R}^-\), as asserted by Proposition 4.4.

The next result shows that, if the scattering data set \(S\) given in (3.97) satisfies (1) and (4a) of Definition 4.2, then the boundary matrices appearing in (2.4)-(2.6) can be constructed by solving (14.2).

**Proposition 16.9** Consider a scattering data set \(S\) as in (4.2), which consists of an \(n \times n\) scattering matrix \(S(k)\) for \(k \in \mathbb{R}\), a set of \(N\) distinct positive constants \(\kappa_j\), and a set of \(N\) constant \(n \times n\) hermitian and nonnegative matrices \(M_j\) with respective positive ranks \(m_j\), where \(N\) is a nonnegative integer. Assume \(S\) satisfies (1) and (4a) specified in Definition 4.2. Let \(A\) and \(B\) be any solution to (14.2) in such a way that the rank of the \(2n \times n\) matrix \([A\ B]\) is equal to \(n\). Then:

(a) Such a solution exists.

(b) If \(A\) and \(B\) make up such a solution, then \(AT\) and \(BT\) also constitute such a solution for any \(n \times n\) invertible matrix \(T\).

(c) Such a solution consisting of \(A\) and \(B\) satisfies (2.5) and (2.6).

(d) If \((A, B)\) and \((\tilde{A}, \tilde{B})\) are any two solutions to (14.2), then we must have \(\tilde{A} = AT\) and
\[ \tilde{B} = BT \] for some \( n \times n \) invertible matrix \( T \).

PROOF: Let us write (14.2) in the block matrix form as
\[
\begin{bmatrix}
I - S_{\infty} & 0 \\
S_{\infty}K(0,0) + K(0,0)S_{\infty} - G_1 & I + S_{\infty}
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\] (16.70)

Note that the coefficient matrix in (16.70) is a block lower-triangular matrix. By Proposition 16.4(a) we know that \( S_{\infty} \) is hermitian and hence it can be diagonalized with the help of a unitary matrix. The same unitary matrix diagonalizes the block diagonal matrices \((I - S_{\infty})\) and \((I + S_{\infty})\) in the coefficient matrix in (16.70). Furthermore, by Proposition 16.4(a) we know that each eigenvalue of \( S_{\infty} \) is equal to +1 or -1. Thus, with the help of the unitary matrix that diagonalizes \( S_{\infty} \), we can transform the coefficient matrix in (16.70) into a lower-triangular matrix where exactly \( n \) of the diagonal entries are zero and the remaining \( n \) diagonal entries are nonzero. Thus, the nullity of the coefficient matrix in (16.70) is exactly \( n \) and hence the general solution of (16.70) contains \( n \) linearly independent columns. Thus, (a) is proved. From (14.2) we see that (b) is valid for any invertible \( n \times n \) matrix \( T \). Let us now turn to the proof of (c). By Proposition 16.4 we know that the three constant matrices \( K(0,0), S_{\infty}, \) and \( G_1 \) appearing in (14.2) are all hermitian. From the first line of (14.2) we have \( S_{\infty}A = A \), and hence
\[ A^\dagger S_{\infty} = A^\dagger, \] (16.71)
where we have used \( S_{\infty}^\dagger = S_{\infty} \). Since \( S_{\infty}, G_1, \) and \( K(0,0) \) are all hermitian, from the second line of (14.2) we obtain
\[ B^\dagger(I + S_{\infty}) = A^\dagger[G_1 - K(0,0)S_{\infty} - S_{\infty}K(0,0)]. \] (16.72)
By multiplying (16.72) on the right by \( A \) we obtain
\[ B^\dagger(I + S_{\infty})A = A^\dagger[G_1 - K(0,0)S_{\infty} - S_{\infty}K(0,0)]A. \] (16.73)
Using (16.72) on the right-hand side of (16.73) we get
\[ B^\dagger(I + S_{\infty})A = A^\dagger(I + S_{\infty})B. \] (16.74)
Using $S_\infty A = A$ and $A^\dagger S_\infty = A^\dagger$, respectively, in (16.74) we simplify (16.74) and obtain

$$2B^\dagger A = 2A^\dagger B,$$

(16.75)

which is equivalent to (2.5). Next, let us show that (2.6) is satisfied. Note that we can write the left-hand side of (2.6) as

$$A^\dagger A + B^\dagger B = \begin{bmatrix} A^\dagger & B^\dagger \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}.$$  

(16.76)

We need to show that the matrix product appearing on the right-hand side of (16.76) is positive. From the left-hand side of (16.76) we see that that product is itself hermitian and hence it has $n$ linearly independent eigenvectors with all real eigenvalues. To prove the positivity it is enough to prove that none of the eigenvalues can be negative or zero. Let $v$ be an eigenvector of $(A^\dagger A + B^\dagger B)$ with the corresponding eigenvalue $\lambda$. Then, from (16.76) we get

$$\lambda \langle v, v \rangle = \langle v, (A^\dagger A + B^\dagger B) v \rangle = \langle \begin{bmatrix} A \\ B \end{bmatrix} v, \begin{bmatrix} A \\ B \end{bmatrix} v \rangle \geq 0,$$

(16.77)

and hence $\lambda \geq 0$. On the other hand, we remark that $\lambda$ cannot be zero because that would imply that $\begin{bmatrix} A \\ B \end{bmatrix} v = 0$ and hence the kernel of the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ would contain a nonzero vector. That would mean that the nullity of the matrix would be at least one. Since the nullity and the rank must add up to $n$, the rank of $\begin{bmatrix} A \\ B \end{bmatrix}$ would have to be strictly less than $n$, contradicting the fact that the rank of $\begin{bmatrix} A \\ B \end{bmatrix}$ is equal to $n$. Thus, the proof of (c) is complete. Let us now prove (d). Let us use $\begin{bmatrix} A^{(j)} \\ B^{(j)} \end{bmatrix}$ and $\begin{bmatrix} \tilde{A}^{(j)} \\ \tilde{B}^{(j)} \end{bmatrix}$ to denote the $j$th column of the $2n \times n$ matrices $\begin{bmatrix} A \\ B \end{bmatrix}$ and $\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}$, respectively. From the proof of (a) we know that the general solution to (16.70) contains exactly $n$ linearly independent column vector solutions. The $n$ columns of $\begin{bmatrix} A \\ B \end{bmatrix}$ must be linearly independent because the rank of $\begin{bmatrix} A \\ B \end{bmatrix}$ is $n$. Similarly, the $n$ columns of $\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}$ must be linearly independent because the rank of $\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}$ is $n$. Thus, we can express $\begin{bmatrix} \tilde{A}^{(j)} \\ \tilde{B}^{(j)} \end{bmatrix}$ as a linear combination of columns of $\begin{bmatrix} A \\ B \end{bmatrix}$ as

$$\begin{bmatrix} \tilde{A}^{(j)} \\ \tilde{B}^{(j)} \end{bmatrix} = \sum_{l=1}^{n} \begin{bmatrix} A^{(l)} \\ B^{(l)} \end{bmatrix} T_{lj}, \quad j = 1, \ldots, n,$$

(16.78)
for some coefficients $T_{lj}$. Thus, (16.78) implies that

$$
\begin{bmatrix}
\tilde{A} \\
\tilde{B}
\end{bmatrix}
= \begin{bmatrix} A \\ B \end{bmatrix} T,
$$

(16.79)

where $T$ is the $n \times n$ matrix whose $(l, j)$-entry is equal to $T_{lj}$. A similar argument implies that

$$
\begin{bmatrix}
A \\
B
\end{bmatrix}
= \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} \tilde{T}, \quad j = 1, \ldots, n,
$$

(16.80)

for some $n \times n$ matrix $\tilde{T}$. From (16.79) and (16.80) we obtain

$$
\begin{bmatrix}
\tilde{A} \\
\tilde{B}
\end{bmatrix}
= \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} \tilde{T} \tilde{T}.
$$

(16.81)

Let us multiply (16.81) by $[ \tilde{A}^\dagger \tilde{B}^\dagger ]$, which yields

$$
[ \tilde{A}^\dagger \tilde{B}^\dagger ] \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} (\tilde{T} \tilde{T} - I) = 0,
$$

(16.82)

or equivalently

$$
(\tilde{A}^\dagger \tilde{A} + \tilde{B}^\dagger \tilde{B})(\tilde{T} \tilde{T} - I) = 0,
$$

(16.83)

From the proof of (c), we know that the matrix $(\tilde{A}^\dagger \tilde{A} + \tilde{B}^\dagger \tilde{B})$ has rank $n$ and hence is invertible. Thus, (16.83) implies that $\tilde{T} \tilde{T} = I$, and hence $T$ and $\tilde{T}$ are both invertible and are inverses of each other. Then, from (16.79) we conclude that $\tilde{A} = AT$ and $\tilde{B} = BT$, completing the proof of (d). ■

As indicated in Proposition 10.1(b), $K(x, x)$ denotes $K(x, x^+)$, where $K(x, y)$ is the solution to the Marchenko equation (13.1). In the next proposition we analyze the $x$-derivative of $K(x, x)$.

**Proposition 16.10** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Let $F_s(y)$ and $F(y)$ be the quantities defined in (4.7) and (4.12), respectively, where $S_\infty$ is the constant $n \times n$ matrix defined as in (4.4). Let
$K(x, y)$ be the quantity defined in (10.1). Assume that $S$ satisfies (I) of Definition 4.3 as well as (2) and (4a) of Definition 4.2. Then, we have

$$\left| \frac{dK(x, x)}{dx} + 2 F'(2x) \right| \leq c |\tau(x)|^2, \quad x > 0,$$

(16.84)

where $c$ is some constant and $\tau(x)$ is the quantity defined in (16.7). Furthermore, we have

$$\int_0^\infty dx \ (1 + x) \left| \frac{dK(x, x)}{dx} \right| < +\infty.$$

(16.85)

PROOF: The proof of (16.84) can be found in the proof of Lemma 5.3.4 on p. 115 of [2] and in the remark on p. 116 in [2]. Let us now prove (16.85). Using the triangle inequality, we have

$$\left| \frac{dK(x, x)}{dx} \right| \leq \left| \frac{dK(x, x)}{dx} \right| + 2 |F'(2x)| + 2 |F'(2x)|.$$

(16.86)

Multiplying both sides of (16.86) with $(1 + x)$ and integrating over $x \in \mathbb{R}^+$, with the help of (16.84) we obtain

$$\int_0^\infty dx \ (1 + x) \left| \frac{dK(x, x)}{dx} \right| \leq c \int_0^\infty dx \ (1 + x) [\tau(x)]^2 + 2 \int_0^\infty dx \ (1 + x) |F'(2x)|.$$

(16.87)

Because we assume (2), from (4.9) we see that $\tau(0)$ and $\tau_1(0)$ are both finite, where $\tau(x)$ and $\tau_1(x)$ are the quantities defined in (16.7). Then, we see that (16.9) implies that the first integral on the right-hand side of (16.87) is finite. Furthermore, (4.8) implies that the second integral on the right-hand side of (16.87) is also finite. Thus, (16.85) holds.

In the next proposition, we study the relevant properties of the potential and the Jost solution constructed from the scattering data set $S$ given in (3.97).

**Proposition 16.11** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Let $F_s(y)$ and $F(y)$ be the quantities defined in (4.7) and (4.12), respectively, where $S_\infty$ is the constant $n \times n$ matrix defined as in (4.6). Assume that $S$ satisfies (I) of Definition 4.3 as well as (2) and (4a) of Definition 4.2. Let
K(x, y) be the unique solution to (13.1), whose existence and uniqueness are assured by Proposition 16.1(b). Then:

(a) The Jost solution \( f(k, x) \) constructed as in (10.6) satisfies (2.1) and (9.1) when the potential \( V(x) \) appearing in (2.1) is constructed as in (10.4). Furthermore, the potential \( V(x) \) constructed via (10.4) satisfies (2.2) and (2.3).

(b) The physical solution \( \Psi(k, x) \) constructed as in (9.4) satisfies (2.1) when the potential \( V(x) \) appearing in (2.1) is constructed as in (10.4).

(c) The normalized bound-state matrix solution \( \Psi_j(x) \) constructed as in (9.8) satisfies (2.1) with \( k = i\kappa_j \) when the potential \( V(x) \) appearing in (2.1) is constructed as in (10.4).

PROOF: The part of the proof of (a) related to the satisfaction of (2.1) can be found in Theorem 5.4.1 on p. 117 of [2] and in the Remark on p. 121 of [2], by recalling that the integrability of the constructed potential \( V(x) \) in our case is assured by the integrability of \( F'(y) \) stated in (2), as indicated in Proposition 16.10. The fact that the constructed \( f(k, x) \) satisfies (9.1) follows from (10.6) by using the fact that the solution \( K(x, y) \) to the Marchenko equation (13.1) is integrable in \( y \in [x, +\infty) \), as indicated in Proposition 16.1(b). Hence, the proof of (e) is complete. Note that (9.4) indicates that each column of the physical solution \( \Psi(k, x) \) is a linear combination of the columns of \( f(-k, x) \) and of \( f(k, x) \).

By (a) we know that each column of \( f(k, x) \) satisfies (2.1), but then we conclude that each column of \( f(-k, x) \) also satisfies (2.1) because \( k \) appears as \( k^2 \) in (2.1). Thus, from (9.4) it follows that (b) holds. From (9.8) we see that each column of \( \Psi_j(x) \) is a linear combination of the columns of \( f(i\kappa_j, x) \), and by (a) we know that \( f(k, x) \) satisfies (2.1). Thus, the result stated in (c) holds. \( \blacksquare \)
17. ADDITIONAL RESULTS RELATED TO THE INVERSE PROBLEM

In this chapter we present certain auxiliary results needed in later chapters.

The next proposition presents certain properties of the Jost solution $f(k, x)$, the physical solution $\Psi(k, x)$, and the bound-state matrix solutions $\Psi_j(x)$.

**Proposition 17.1** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ satisfies (I) of Definition 4.3 and (4a) of Definition 4.2. We then have the following:

(a) The Jost solution $f(k, x)$ constructed from the scattering data as in (10.6) is uniformly bounded, i.e.

$$|f(k, x)| \leq C, \quad k \in \mathbb{R}, \quad x \geq 0,$$

for some constant $C$ independent of $k$ and $x$.

(b) The physical solution $\Psi(k, x)$ constructed from the scattering data as in (9.4) is uniformly bounded, i.e.

$$|\Psi(k, x)| \leq C, \quad k \in \mathbb{R}, \quad x \geq 0,$$

for some constant $C$ independent of $k$ and $x$.

(c) The matrix $f(i\kappa_j, x)$ constructed from the scattering data as in (10.6) is uniformly bounded, integrable, and square integrable for $x \in [0, +\infty)$, and it satisfies

$$|f(i\kappa_j, x)| \leq Ce^{-\kappa_j x}, \quad x \geq 0, \quad j = 1, \ldots, N.$$

(d) Each bound-state matrix solution $\Psi_j(x)$ constructed from the scattering data as in (9.8) is uniformly bounded, integrable, and square integrable for $x \in [0, +\infty)$, and it satisfies

$$|\Psi_j(x)| \leq Ce^{-\kappa_j x}, \quad x \geq 0, \quad j = 1, \ldots, N.$$
PROOF: Let us use $C$ for a generic constant that may take different values in different appearances. From (10.6), we obtain
\[ |f(k, x)| \leq 1 + \int_{x}^{\infty} dy |K(x, y)|, \tag{17.5} \]
where $K(x, y)$ is the solution to the Marchenko equation. From Proposition 16.1(b) we know that $K(x, y)$ is integrable in $y \in (x, +\infty)$ for each fixed $x \geq 0$. Thus, the integral in (17.5) is finite, and hence it (17.5) yields (17.1). Thus, (a) holds. Let us now prove (b). From (9.4) we get
\[ |\Psi(k, x)| \leq |f(-k, x)| + |f(k, x)||S(k)|. \tag{17.6} \]
Because $S(k)$ is unitary, as assumed in (I), we have $|S(k)| = 1$ for $k \in \mathbb{R}$. Thus, using (17.1) in (17.6), we conclude (17.2), and hence the proof of (b) is complete. Let us now turn to the proof of (c). From (10.6) we get
\[ f(i\kappa_j, x) = e^{-\kappa_j x}I + \int_{x}^{\infty} dy K(x, y) e^{-\kappa_j y}. \tag{17.7} \]
From (17.7) we obtain
\[ |f(i\kappa_j, x)| \leq e^{-\kappa_j x} + e^{-\kappa_j x} \int_{x}^{\infty} dy |K(x, y)|. \tag{17.8} \]
Again, as a result of the integrability of $K(x, y)$ in $y \in (x, +\infty)$ for each fixed $x \geq 0$, the integral in (17.8) is finite and hence (17.8) yields (17.3).

Because each $\kappa_j$ is positive, the right-hand side in (17.3) is bounded, integrable, and square integrable in $x \in [0, +\infty)$. Thus, we conclude that $|f(i\kappa_j, x)|$ has the properties stated in (c). Let us now prove (d). From (9.8), since each $M_j$ is a constant $n \times n$ matrix, we see that (17.3) implies (17.4) and hence (c) implies (d). \]

The next proposition is the analog of Proposition 17.1 and presents certain properties of the Jost solution $f'(k, x)$, the physical solution $\Psi'(k, x)$, and the bound-state matrix solutions $\Psi'_j(x)$.

**Proposition 17.2** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of
constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ satisfies (I) of Definition 4.3 and (4a) of Definition 4.2. Assume also that $F'_s(y)$ is integrable in $y \in \mathbb{R}^+$, where $F_s(y)$ is the quantity defined in $(4.7)$. We then have the following:

(a) The $x$-derivative of the Jost solution $f(k, x)$ constructed as in $(10.6)$, i.e. the matrix $f'(k, x)$ satisfies

$$|f'(k, x)| \leq C (1 + |k|), \quad k \in \mathbb{R}, \quad x \geq 0,$$

for some constant $C$ independent of $k$ and $x$.

(b) The matrix $\Psi'(k, x)$, i.e. the $x$-derivative of the physical solution $\Psi(k, x)$ constructed from the scattering data as in $(9.4)$, satisfies

$$|\Psi'(k, x)| \leq C (1 + |k|), \quad k \in \mathbb{R}, \quad x \geq 0,$$

for some constant $C$ independent of $k$ and $x$.

(c) Each matrix $f'(ik_j, x)$, i.e. the $x$-derivative of the Jost solution $f(ik_j, x)$ constructed from the scattering data as in $(10.6)$, is uniformly bounded, integrable, and square integrable for $x \in [0, +\infty)$, and it satisfies

$$|f'(ik_j, x)| \leq Ce^{-\kappa_j x}, \quad x \geq 0, \quad j = 1, \ldots, N.$$

(d) Each matrix $\Psi'_j(x)$, i.e. the $x$-derivative of the bound-state matrix solution $\Psi_j(x)$ constructed from the scattering data as in $(9.8)$, is uniformly bounded, integrable, and square integrable for $x \in [0, +\infty)$, and it satisfies

$$|\Psi'_j(x)| \leq Ce^{-\kappa_j x}, \quad x \geq 0, \quad j = 1, \ldots, N.$$

PROOF: We again use $C$ to denote a generic constant. Recall that $f'(k, x)$ is constructed from the solution $K(x, y)$ to the Marchenko equation $(13.1)$ and its $x$-derivative given by $K_x(x, y)$. This is accomplished by using $K(x, y)$ and $K_x(x, y)$ in $(15.37)$. We obtain

$$|f'(k, x)| \leq |k| + |K(x, x)| + \int_x^\infty dy |K_x(x, y)|, \quad x \geq 0.$$  

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By Proposition 16.1(b) we know that $K(x, y)$ is bounded when $0 \leq x \leq y$ and hence $|K(x, x)| \leq C$ for $x \geq 0$. By Proposition 16.5(c) and 16.5(d) we know that $K_x(k, y)$ is integrable in $y \in (x, +\infty)$ for each $x \geq 0$, and hence the integral in (17.13) is finite. Thus, (17.13) yields (17.9), proving (a). Let us turn to the proof of (b). From (9.4), we have

$$
\Psi'(k, x) = f'(-k, x) + f'(k, x) S(k).
$$

(17.14)

and hence we obtain

$$
|\Psi'(k, x)| \leq |f'(-k, x)| + |f'(k, x)| |S(k)|,
$$

(17.15)

Because $S(k)$ is unitary, we have $|S(k)| = 1$ for $k \in \mathbb{R}$. Hence, using (17.9) in (17.15) we obtain (17.10), proving (b). Let us now turn to the proof of (c). From (15.37) we have

$$
f'(i\kappa_j, x) = -\kappa_j e^{-\kappa_j x} I - K(x, x) e^{-\kappa_j x} + \int_x^\infty K_x(x, y) e^{-\kappa_j y}, \quad x \geq 0.
$$

(17.16)

From (17.16) we get

$$
|f'(i\kappa_j, x)| \leq \kappa_j e^{-\kappa_j x} + |K(x, x)| e^{-\kappa_j x} + \int_x^\infty dy |K_x(x, y)| e^{-\kappa_j y}, \quad x \geq 0,
$$

(17.17)

which implies

$$
|f'(i\kappa_j, x)| \leq e^{-\kappa_j x} \left[ \kappa_j + |K(x, x)| + \int_x^\infty dy |K_x(x, y)| \right], \quad x \geq 0,
$$

(17.18)

As argued earlier, $|K(x, x)| \leq C$ for every $x \geq 0$ and $K_x(x, y)$ is integrable in $y \in (x, +\infty)$ for every $x \geq 0$. Thus, (17.18) yields (17.11). Since $\kappa_j$ is positive, from (17.11) we conclude the properties stated in (c). Thus, the proof of (c) is completed. We note that (d) directly follows from (c) because as seen from (9.8), the matrices $\Psi'_j(x)$ and $f'(i\kappa_j, x)$ are related to each other by the constant matrix $M_j$ via

$$
\Psi'_j(x) = f'(i\kappa_j, x) M_j.
$$

(17.19)

Thus, the proof is complete. \[ \Box \]
The properties established in Proposition 16.1(b) for the solution \( K(x, y) \) to the Marchenko integral equation enable us to define certain useful integral operators related to \( K(x, y) \). Toward that goal, we first need the following result.

**Proposition 17.3** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Assume that \( S \) satisfies (I) of Definition 4.3 as well as (2) and (4a) of Definition 4.2. Let \( K(x, y) \) be the solution to the Marchenko equation (13.1), and consider the integral equation

\[
M(x, y) + K(x, y) + \int_x^y dz \, M(x, z) K(z, y) = 0, \quad 0 \leq x < y,
\]

where \( M(x, y) \) is an \( n \times n \) matrix. Then:

(a) The integral equation (17.20) is uniquely solvable for \( M(x, y) \).

(b) The solution \( M(x, y) \) to (17.20) satisfies

\[
|M(x, y)| \leq C \tau(x + y), \quad 0 \leq x < y,
\]

where \( C \) is a generic constant and \( \tau(x) \) is the scalar quantity defined in (16.7).

(c) For each fixed \( x \geq 0 \), the entries of the matrix \( M(x, y) \) belong to \( L^1(x < y < +\infty) \) and \( L^\infty(x < y < +\infty) \), and hence they in particular belong to \( L^2(x < y < +\infty) \).

(d) The integral equation associated with (17.21), which is given by

\[
\tilde{M}(x, y) + K(x, y) + \int_x^y dz \, K(x, z) \tilde{M}(z, y) = 0, \quad 0 \leq x < y,
\]

is also uniquely solvable for \( \tilde{M}(x, y) \). In fact, the unique solution \( \tilde{M}(x, y) \) to (17.22) satisfies

\[
\tilde{M}(x, y) \equiv M(x, y),
\]

where \( M(x, y) \) is the unique solution to (17.20).
PROOF: The Volterra equation (17.20) can be solved by using the method of successive approximation as

\[ M(x, y) = \sum_{j=0}^{\infty} M^{(j)}(x, y), \]  
(17.24)

where we have defined

\[ M^{(0)}(x, y) := -K(x, y), \]  
(17.25)

\[ M^{(j)}(x, y) := -\int_{x}^{y} dz M^{(j-1)}(x, z) K(z, y), \quad j = 1, 2, \ldots. \]  
(17.26)

From the recurrence relations (17.25) and (17.26) we get

\[ M^{(j)}(x, y) = (-1)^{j+1} \int_{x}^{y} dz_{j} \int_{x}^{z_{j-1}} dz_{j-1} \cdots \int_{x}^{z_{2}} dz_{1} K(x, z_{1}) K(z_{1}, z_{2}) \cdots K(z_{j-1}, z_{j}) K(z_{j}, y). \]  

(17.27)

Since \( K(x, y) = 0 \) for \( x > y \), we can write (17.27) as

\[ M^{(j)}(x, y) = (-1)^{j+1} \int_{x}^{y} dz_{j} \int_{x}^{y} dz_{j-1} \cdots \int_{x}^{y} dz_{1} K(x, z_{1}) K(z_{1}, z_{2}) \cdots K(z_{j-1}, z_{j}) K(z_{j}, y). \]  

(17.28)

With the help of (16.10), from (17.25) and (17.26) we get

\[ |M^{(0)}(x, y)| \leq C \tau(x + y), \quad |M^{(1)}(x, y)| \leq C \tau(x + y) \int_{x}^{y} dz C \tau(x + z). \]  
(17.29)

Using induction, from (17.29) we obtain

\[ |M^{(j)}(x, y)| \leq \frac{1}{j!} C \tau(x + y) \left[ \int_{x}^{y} dz C \tau(x + z) \right]^{j}, \quad j = 1, 2, \ldots. \]  
(17.30)

Hence, the series in (17.24) converges uniformly and yields \( M(x, y) \) as the unique solution to (17.20). Thus, the proof of (a) is complete. Using (17.30) in (17.29), for \( 0 \leq x \leq y \) we obtain

\[ |M(x, y)| \leq C \tau(x + y) \exp \left( C \int_{x}^{y} dz \tau(x + z) \right) \leq C \tau(x + y) \exp \left( C \int_{x}^{\infty} dz \tau(x + z) \right) \leq C \tau(x + y) e^{C \tau_{1}(0)}. \]  
(17.31)
The property (2) implies that $\tau_1(0)$ is finite. Hence, (17.31) implies (17.21) with some generic constant $C$ not necessarily equal to $C$ appearing in (17.31). Thus, (b) is proved.

Let us now turn to the proof of (c). Since we assume that (2) of Definition 4.2 holds, (16.8) implies that $\tau(0)$ is also finite. Then, using the second inequality in (16.8) and the fact that $\tau(x)$ is a nonincreasing function of $x$, we see from (17.21) that $|M(x, y)|$ belongs to $L^1(x < y < +\infty)$ and $L^\infty(x < y < +\infty)$ for each fixed $x \geq 0$. Any function that belongs to $L^1$ and $L^\infty$ must also belong to $L^p(x < y < +\infty)$ for $1 < p < +\infty$, and hence $|M(x, y)|$ belongs to $L^2(x < y < +\infty)$ as well. These properties satisfied by the matrix norm $|M(x, y)|$ imply that each entry of the matrix also satisfy such properties related to $L^1$, $L^\infty$, and $L^2$. Thus, the proof of (c) is complete. Let us now turn to the proof of (d). We can solve (17.22) the same way we solve (17.20). As in (17.24)-(17.28), we solve (17.22) by the method of successive approximations as

$$
\tilde{M}(x, y) = \sum_{j=0}^{\infty} \tilde{M}^{(j)}(x, y),
$$

with

$$
\tilde{M}^{(0)}(x, y) := -K(x, y),
$$

$$
\tilde{M}^{(j)}(x, y) = -\int_x^y dz K(x, z) \tilde{M}^{(j-1)}(z, y), \quad j = 1, 2, \ldots.
$$

The recurrence relations (17.32) and (17.33) yield

$$
\tilde{M}^{(j)}(x, y) = (-1)^{j+1} \int_x^y dz_1 \int_{z_1}^y dz_2 \cdots \int_{z_{j-1}}^y dz_j K(x, z_1) K(z_1, z_2) \cdots K(z_{j-1}, z_j) K(z_j, y).
$$

Since $K(x, y) = 0$ for $x > y$, as in (17.27) and (17.28), from (17.35) we obtain

$$
\tilde{M}^{(j)}(x, y) = (-1)^{j+1} \int_x^y dz_1 \int_x^y dz_2 \cdots \int_x^y dz_j K(x, z_1) K(z_1, z_2) \cdots K(z_{j-1}, z_j) K(z_j, y).
$$

Comparing the pair of equations (17.25) and (17.28) with the pair of equations (17.33) and (17.36), we see that

$$
M^{(j)}(x, y) = \tilde{M}^{(j)}(x, y), \quad j = 0, 1, \ldots,
$$

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Since the iterates in (17.20) and (17.22) coincide, from the unique solvability of (17.20) proved in (a) we conclude the unique solvability of (17.22). Furthermore, by comparing (17.24) and (17.32), from (17.32) we also conclude (17.23).

Associated with the unique solution $K(x,y)$ to the Marchenko equation (13.1), let us define the operators $K, P$ as

$$ K : Y(x) \mapsto \int_x^\infty dy K(x,y) Y(y), \quad (17.38) $$

$$ P : Y(x) \mapsto \int_0^x dy K(y,x)^\dagger Y(y). \quad (17.39) $$

Associated with the unique solution $M(x,y)$ to (17.20) and also to (17.23), let us define the operators $M, Q$ as

$$ M : Y(x) \mapsto \int_x^\infty dy M(x,y) Y(y), \quad (17.40) $$

$$ Q : Y(x) \mapsto \int_0^x dy M(y,x)^\dagger Y(y). \quad (17.41) $$

The next proposition presents certain properties of the four operators in (17.39)-(17.41) and some relationships among them.

**Proposition 17.4** The four operators $K, M, P, Q$ defined in (17.38)-(17.41), respectively, are related to each other as

$$ (I + M)(I + K) = I, \quad (17.42) $$

$$ (I + K)(I + M) = I. \quad (17.43) $$

$$ (I + Q)(I + P) = I, \quad (17.44) $$

$$ (I + P)(I + Q) = I. \quad (17.45) $$

The operator equations (17.42)-(17.45) hold on $L^1(\mathbb{R}^+)$. They also hold on $L^2(\mathbb{R}^+)$. 

**PROOF:** By postmultiplying (17.20) by $Y(y)$ and integrating the resulting equation in $y \in (x, +\infty)$, we obtain

$$ \int_x^\infty dy M(x,y) Y(y) + \int_x^\infty dy K(x,y) Y(y) + \int_x^\infty dy \int_x^y dz M(x,z) K(z,y) Y(y) = 0. \quad (17.46) $$

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By interchanging the order of integration in the double integral in (17.46) we obtain
\[
\int_{x}^{\infty} dy M(x, y) Y(y) + \int_{x}^{\infty} dy K(x, y) Y(y) + \int_{x}^{\infty} dz \int_{z}^{\infty} dy M(x, z) K(z, y) Y(y) = 0,
\]
which can be written as
\[
(MY)(x) + (KY)(x) + (MKY)(x) = 0,
\]
yielding
\[
M + K + MK = 0,
\]
which is equivalent to (17.42). In a similar way, since \(M(x, y)\) solves (17.22), we postmultiply (17.22), with \(\tilde{M}(x, y)\) replaced with \(M(x, y)\) there, by \(Y(y)\) and integrate the resulting equation in \(y \in (x, +\infty)\). By changing the order of integration in the term involving the double integral, we get
\[
(MY)(x) + (KY)(x) + (MKY)(x) = 0,
\]
yielding
\[
QY + PY + PQY = 0,
\]
(17.54)
which is equivalent to (17.45). In a similar way, since $M(x, y)$ solves (17.22), we take the matrix adjoint of (17.22), with $\tilde{M}(x, y)$ replaced with $M(x, y)$ there, and obtain

$$M(x, y)^\dagger + K(x, y)^\dagger + \int_x^y dz M(z, y)^\dagger K(x, z)^\dagger = 0, \quad 0 \leq x < y. \quad (17.55)$$

By postmultiplying (17.55) with $Y(x)$ and integrating the resulting equation in $x \in (0, y)$, after changing the order of integration in the term containing the double integral, we obtain

$$\int_0^y dx M(x, y)^\dagger Y(x) + \int_0^y dx K(x, y)^\dagger Y(x) + \int_0^y dz \int_0^z dx M(z, y)^\dagger K(x, z)^\dagger Y(x) = 0. \quad (17.56)$$

We recognize that (17.56) is equivalent to

$$(QY)(y) + (PY)(y) + (QPY)(y) = 0, \quad (17.57)$$

which yields (17.44). We remark that the change of the order of integration in the double integrals is justified with the help of (16.11), (17.21), and the analogous inequalities for $K(x, y)^\dagger$ and $M(x, y)^\dagger$.

Some further properties of the four operators $K$, $P$, $M$, and $Q$ defined in (17.38)-(17.41), respectively, are presented in the next proposition.

**Proposition 17.5** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ satisfies (I) of Definition 4.3 as well as (2) and (4a) of Definition 4.2. Then, we have the following:

(a) Each of the four operators $K$, $P$, $M$, and $Q$ defined in (17.38)-(17.41), respectively, is a bounded operator from $L^1(\mathbb{R}^+) \rightarrow L^1(\mathbb{R}^+)$. 

(b) Each of the four operators $K$, $P$, $M$, and $Q$ is a bounded operator from $L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$. 

(c) Each of the four operators $(I + K)$, $(I + P)$, $(I + M)$, and $(I + Q)$ is invertible on $L^1(\mathbb{R}^+)$. The corresponding inverses $(I + K)^{-1}$, $(I + P)^{-1}$, $(I + M)^{-1}$, and $(I + Q)^{-1}$
are bounded on $L^1(\mathbb{R}^+)$, and we have

\[(I+K)^{-1} = I + M, \quad (I+M)^{-1} = I + K, \quad (I+P)^{-1} = I + Q, \quad (I+Q)^{-1} = I + P.\]

(17.58)

(d) Each of the four operators $(I+K)$, $(I+P)$, $(I+M)$, and $(I+Q)$ is invertible on $L^2(\mathbb{R}^+)$. The corresponding inverses $(I+K)^{-1}$, $(I+P)^{-1}$, $(I+M)^{-1}$, and $(I+Q)^{-1}$ are bounded on $L^2(\mathbb{R}^+)$, and we have

\[(I+K)^{-1} = I + M, \quad (I+M)^{-1} = I + K, \quad (I+P)^{-1} = I + Q, \quad (I+Q)^{-1} = I + P.\]

(17.59)

(e) The operator $P$ on $L^2(\mathbb{R}^+)$ corresponds to the adjoint operator for $K$ on $L^2(\mathbb{R}^+)$. Similarly, the operator $Q$ on $L^2(\mathbb{R}^+)$ corresponds to the adjoint operator for $M$ on $L^2(\mathbb{R}^+)$. 

PROOF: Let us first remark that (16.11) and (17.21) hold and they also imply

\[|K(y,x)^\dagger| \leq C\tau(x+y), \quad 0 \leq x \leq y,\]

(17.60)

\[|M(y,x)^\dagger| \leq C\tau(x+y), \quad 0 \leq x \leq y,\]

(17.61)

for some generic constants $C$, not necessarily having the same value in different appearances, and where $\tau(x)$ is the scalar function defined in (16.7). In proving (a) and (b), it is enough to give the proof for only one of the four operators because those four operators satisfy the same upper bound in the four inequalities (16.11), (17.21), (17.60), (17.61). Thus, let us start the proof of (a) for the operator $K$ appearing in (17.38). It is enough to prove that $||KY||_1 \leq C||Y||_1$, for some constant $C$ and $Y(x) \in L^1(\mathbb{R}^+)$, where we recall that $|| \cdot ||_1$ denotes the standard norm in $L^1(\mathbb{R}^+)$. From (17.38) we get

\[||KY||_1 := \int_0^\infty dx |(KY)(x)| \leq \int_0^\infty dx \int_x^\infty dy |K(x,y)||Y(y)|. \]

(17.62)
Using (16.11) in (17.62) we get
\[ ||KY||_1 \leq C \int_0^\infty dx \int_x^\infty dy \tau(x+y) |Y(y)| \]
\[ \leq C \int_0^\infty dx \int_0^\infty dy |Y(y)| \]
\[ = \frac{1}{2} C \tau_1(0) ||Y||_1. \] (17.63)

Since \( \tau_1(0) \) is finite when (2) holds, from (17.63) we conclude that the operator norm of \( K \) on \( L^1(\mathbb{R}) \) is bounded and hence (a) is proved for \( K \). Since the proof of (a) can be repeated for the other three operators in exactly the same way, the proof of (a) is complete. Let us now return to the proof of (b) for the operator \( K \). The operator norm \( |K|_2 \) on \( L^2(\mathbb{R}^+) \) satisfies
\[ |K|_2 := \sup_{||Y||_2=1} ||KY||_2 = \sup_{||Y||_2=1} \left( \sup_{||\tilde{Y}||_2} |(KY, \tilde{Y})_2| \right), \] (17.64)
and hence we will consider estimating \( (KY, \tilde{Y})_2 \) when \( Y(x) \) and \( \tilde{Y}(x) \) belong to \( L^2(\mathbb{R}^+) \).

From (17.38) we get
\[ |(KY, \tilde{Y})_2| \leq \int_0^\infty dx \int_x^\infty dy |Y(y)| \int_0^\infty K(x,y) \tilde{Y}(x) |Y(y)| \int_0^\infty \int_x^\infty dy |K(x,y) \tilde{Y}(x)|. \] (17.65)

Using Young’s inequality
\[ |Y(y)| \cdot |\tilde{Y}(x)| \leq \frac{1}{2} \left( |Y(y)|^2 + |\tilde{Y}(x)|^2 \right), \] (17.66)
from (17.65) we obtain
\[ |(KY, \tilde{Y})_2| \leq \frac{1}{2} \int_0^\infty dx \int_x^\infty dy |K(x,y)| |Y(y)|^2 + \frac{1}{2} \int_0^\infty dx \int_x^\infty dy |K(x,y) \tilde{Y}(x)|^2. \] (17.67)

Using (17.60) in (17.67) we get
\[ |(KY, \tilde{Y})_2| \leq \frac{C}{2} \int_0^\infty dx \int_x^\infty dy \tau(x+y) |Y(y)|^2 + \frac{C}{2} \int_0^\infty dx \int_x^\infty dy \tau(x+y) |\tilde{Y}(x)|^2, \] (17.68)
which yields

\[(KY, \tilde{Y})_2 \leq \frac{C}{2} \int_0^\infty dx \tau(2x) \int_x^\infty dy |Y(y)|^2 + \frac{C}{2} \int_0^\infty dx |\tilde{Y}(x)|^2 \int_x^\infty dy \tau(x+y). \quad (17.69)\]

From (17.69) we obtain

\[|(KY, \tilde{Y})_2| \leq \frac{C}{4} \tau_1(0) ||Y||^2_2 + \frac{C}{2} \tau_1(0) ||\tilde{Y}||^2_2. \quad (17.70)\]

Since \(\tau_1(0)\) is finite when (2) holds, from (17.64) and (17.70) we get \(|K|_2 \leq C\), and hence the operator \(K\) is bounded on \(L^2(R^+)\). Thus, the proof of (b) is complete for \(K\). The same proof can be repeated to prove that the remaining three operators on \(L^2(R^+)\) also have finite operator norms, and hence the proof of (b) is complete. Let us now turn to the proof of (c). From (a) it follows that the four operators \((I + K), (I + M), (I + P), (I + Q)\) are bounded on \(L^1(R^+)\). Then, from (11.11)-(11.13) we conclude that each of these four operators are invertible on \(L^1(R^+)\) and their inverses are also bounded on \(L^1(R^+)\) and (17.58) holds. The proof of (d) on \(L^2(R^+)\) is similar to the proof of (c) on \(L^1(R^+)\). As for the proof of (e), we observe from the kernels in (17.38) and (17.39) that the operators \(K\) and \(P\) on \(L^2(R^+)\) are adjoints of each. Similarly, from the kernels in (17.40) and (17.41) we observe that the operators \(M\) and \(Q\) on \(L^2(R^+)\) are adjoints of each. \[\blacksquare\]
18. FURTHER RESULTS RELATED TO THE INVERSE PROBLEM

In this chapter we provide some results related to the characterization conditions presented in Chapters 4-7.

Recall that there are two essential parts in solving the inverse scattering problem. Given the scattering data set \( \mathbf{S} \) as in (4.2), the first part involves the construction of the corresponding input data set \( \mathbf{D} \) as in (4.1), i.e. to construct the corresponding potential \( V(x) \) and the boundary matrices \( A \) and \( B \) appearing in (4.1). The second part involves proving that the physical solution \( \Psi(k,x) \) constructed as in (9.4) as well as the matrix bound-state solutions \( \Psi_j(x) \) constructed as in (9.8) satisfy the boundary conditions. Paraphrasing, the second part involves showing that we have \( \Delta(k) \equiv 0 \), where we have defined

\[
\Delta(k) := -B^\dagger \Psi(k,0) + A^\dagger \Psi'(k,0),
\]

and showing that \( \Delta_j = 0 \) for \( j = 1, \ldots, N \), where we have defined

\[
\Delta_j := -B^\dagger \Psi_j(0) + A^\dagger \Psi'_j(0), \quad j = 1, \ldots, N.
\]

We know that the property \((3a)\) of Definition 4.2 is equivalent to saying \( \Delta(k) \equiv 0 \) and the condition \((V_a)\) of Definition 4.3 is equivalent to having \( \Delta_j = 0 \) for \( j = 1, \ldots, N \).

We are interested in analyzing \( \Delta(k) \) defined in (18.1) carefully to understand it better. The following proposition provides an integral representation of \( \Delta(k) \) resembling a Fourier transform. It expresses \( \Delta(k) \) in terms of the solution \( K(x,y) \) to the Marchenko equation (13.1) and the solution \( K_x(x,y) \) to the derivative Marchenko equation (13.7) as well as \( F_s(y) \) in (4.7) and its derivative \( F'_s(y) \).

**Proposition 18.1** Let \( \mathbf{S} \) in (4.2) be the scattering data set consisting of a scattering matrix \( S(k) \), the constants \( \kappa_j \) as distinct positive numbers, and the matrices \( M_j \) as \( n \times n \) nonnegative hermitian matrices. Assume that \( \mathbf{S} \) satisfies the conditions \((1), (2), \) and \((4_a)\) stated in Definition 4.5. Let \( F_s(y) \) be the quantity given in (4.7) in such a way that \( F'_s(y) \) for \( y \in \mathbb{R}^- \) is the sum of an integrable function and a square-integrable function. Then, we have the following:
(a) We can represent the quantity $\Delta(k)$ defined in (18.1) as

$$\Delta(k) = \int_{-\infty}^{\infty} dy \hat{\Delta}(y) e^{-iky}, \quad (18.3)$$

where $\hat{\Delta}(y)$ is the quantity given by

$$\hat{\Delta}(y) := -B^\dagger \Gamma_{10}(y) + A^\dagger \Gamma_{11}(y), \quad (18.4)$$

with $\Gamma_{10}(y)$ and $\Gamma_{11}(y)$ given by

$$\Gamma_{10}(y) := K(0, y) + F_s(y) + K(0, -y) S_\infty + \int_{-\infty}^{\infty} dz K(0, z) F_s(z + y), \quad (18.5)$$

$$\Gamma_{11}(y) := K_x(0, y) + F'_s(y) + K_x(0, -y) S_\infty - K(0, 0) F_s(y) + \int_{-\infty}^{\infty} dz K_x(0, z) F_s(z + y). \quad (18.6)$$

Here $A$ and $B$ are the boundary matrices constructed as in Proposition 16.9, $S_\infty$ is the constant matrix defined as in (4.6), $K(x, y)$ is the unique solution to the Marchenko equation (13.1), and $F'_s(y)$ is the quantity related to $F'_s(y)$ as in (16.52). Because of (10.2) and (10.8), the lower integration limits $x = -\infty$ in (18.5) and (18.6) can be replaced with $x = 0$.

(b) The matrix $\hat{\Delta}(y)$ is integrable in $y \in \mathbb{R}^+$, and it is the sum of an integrable function and a square-integrable function in $\mathbb{R}^-$.

(c) For $y \in \mathbb{R}^+$, the matrix $\hat{\Delta}(y)$ can be expressed in terms of $K(0, y)$, $K_x(0, y)$, $F_s(y)$, and $F'_s(y)$ as

$$\hat{\Delta}(y) = -\Gamma_1(y)^\dagger B + \Gamma_2(y)^\dagger A, \quad y \in \mathbb{R}^+, \quad (18.7)$$

where $\Gamma_1(y)$ and $\Gamma_2(y)$ are obtained from (18.5) and (18.6), respectively, by using $K(0, -y) = 0$ and $K_x(0, -y) = 0$ and given by

$$\Gamma_1(y) := K(0, y) + F_s(y) + \int_{0}^{\infty} dz K(0, z) F_s(z + y), \quad (18.8)$$

$$\Gamma_2(y) := K_x(0, y) + F'_s(y) - K(0, 0) F_s(y) + \int_{0}^{\infty} dz K_x(0, z) F_s(z + y). \quad (18.9)$$
(d) For \( y \in \mathbb{R}^+ \), the matrix \( \hat{\Delta}(y) \) can be expressed in terms of the normalized bound-state matrices \( \Psi_j(x) \) constructed as in (9.8) as
\[
\hat{\Delta}(y) = \sum_{j=1}^{N} \left[ B_j \Psi_j(0) - A_j \Psi_j'(0) \right] M_j e^{-\kappa_j y}, \quad y \in \mathbb{R}^+,
\]
(18.10)
or equivalently expressed in terms of the Jost matrix \( J(k) \) constructed as in (9.2) as
\[
\hat{\Delta}(y) = \sum_{j=1}^{n} J(i\kappa_j) M_j^2 e^{-\kappa_j y}, \quad y \in \mathbb{R}^+,
\]
(18.11)
and hence \( \hat{\Delta}(y) \) is in fact continuous in \( y \in \mathbb{R}^+ \) and vanishes as \( y \to +\infty \).

(e) The matrix \( \hat{\Delta}(y) \) satisfies the integral equation
\[
\int_{-\infty}^{\infty} dz \hat{\Delta}(z) F_s(z + y) = \hat{\Delta}(y) - \hat{\Delta}(-y) S_\infty, \quad y \in \mathbb{R}.
\]
(18.12)

PROOF: When (1) and (4a) of Definition 4.5 are satisfied, by Proposition 16.1 we are assured the existence and uniqueness of \( K(x, y) \) as the solution to the Marchenko equation (13.1). We also know that \( K(x, y) \) is integrable in \( y \in (x, +\infty) \) for each fixed \( x \geq 0 \). Similarly, if (1), (2), and (4a) are satisfied, Proposition 16.5 implies that \( K_x(x, y) \) is the unique solution to (13.7). For each \( x \geq 0 \), the quantity \( K_x(x, y) \) is integrable in \( y \in (x, +\infty) \). By solving (13.1) and (13.7), we obtain \( K(0, y) \) and \( K_x(0, y) \). Then, with the help of (10.6), we construct \( f(k, 0) \) and \( f'(k, 0) \) in terms of \( K(0, y) \) and \( K_x(0, y) \) as in (15.38) and (15.39), respectively. From the physical solution \( \Psi(k, x) \) constructed as in (9.4) we get
\[
\Psi(k, 0) = f(-k, 0) + f(k, 0) S(k), \quad \Psi'(k, 0) = f'(-k, 0) + f'(k, 0) S(k).
\]
(18.13)
From (4.7) and (12.3), we respectively have
\[
S(k) = S_\infty + \int_{-\infty}^{\infty} dy F_s(y) e^{-iky},
\]
(18.14)
\[
\text{i}k S(k) = \text{i}k S_\infty + \int_{-\infty}^{\infty} dy F'_s(y) e^{-iky}.
\]
(18.15)
Using (15.38) and (15.39) in (18.13), with the help of (11.37), (18.14), (18.15), and Proposition 16.7 we can express $\Delta(k)$ given in (18.1) in terms of $K(0, y)$ and $K_x(0, y)$ as

$$\Delta(k) = -B^\dagger \Gamma_{12}(k) + A^\dagger \Gamma_{13}(k), \quad (18.16)$$

where we have defined

$$\Gamma_{12}(k) := I + S_\infty + \int_{-\infty}^{\infty} dy \Gamma_{10}(y) e^{-iky}, \quad (18.17)$$

$$\Gamma_{13} := -ik(I - S_\infty) - K(0, 0)(I + S_\infty) + \int_{-\infty}^{\infty} dy \left[ G_1 \delta(y) + \Gamma_{10}(y) \right] e^{-iky}, \quad (18.18)$$

with $\Gamma_{10}(y)$ and $\Gamma_{11}(y)$ are the quantities defined in (18.5) and (18.6), respectively. Using (18.17) and (18.18) in (18.16), we obtain

$$\Delta(k) = \Gamma_{14} + \int_{-\infty}^{\infty} dy \left[ -B^\dagger \Gamma_{10} + A^\dagger \Gamma_{11}(k) \right] e^{-iky}, \quad (18.19)$$

where we have defined the constant matrix $\Gamma_{14}$ as

$$\Gamma_{14} := -ik A^\dagger(I - S_\infty) - B^\dagger(I + S_\infty) + A^\dagger \left[ G_1 - K(0, 0) - K(0, 0) S_\infty \right]. \quad (18.20)$$

Since the boundary matrices $A$ and $B$ satisfy (16.70), the left-hand side of (18.20) vanishes and we get $\Gamma_{14} = 0$. Using $\Gamma_{14} = 0$ in (18.19), we see that with the help of (18.4) we can write (18.19) as (18.3), completing the proof of (a). Let us now turn to the proof of (b). For $y \in \mathbb{R}^+$, each term on the right-hand sides of (18.5) and (16.6) is integrable, where this property of $F_s(y)$ follows from (1), that of $K(0, y)$ follows from Proposition 16.1(a), $K(0, -y) = 0$ as a result of (10.2), $K_x(0, y)$ is integrable as indicated in Proposition 16.5, the integrability of $\bar{F}'_s(y)$ follows from (2) and (16.52), and each of the integrals in (18.5) and (18.6) is integrable in $y \in \mathbb{R}^+$ because they are essentially the convolution of two integrable functions. Thus, from (18.4) we conclude that $\hat{\Delta}(y)$ is integrable in $y \in \mathbb{R}^+$. Let us now show that for $y \in \mathbb{R}^-$ the quantity $\hat{\Delta}(y)$ is the sum of an integrable function and a square-integrable function. From (18.4) we see that it is enough to show that each term on the right-hand sides of (18.5) and (18.6) is either integrable or square integrable.
in \( y \in \mathbb{R}^- \). For \( y \in \mathbb{R}^- \), by Proposition 4.4 we know that \( F_s(y) \) is square integrable, by Proposition 16.1 we know that \( K(0, -y) \) is integrable, by (10.2) we have \( K(0, y) = 0 \), by Proposition 16.5 we know that \( K_x(0, -y) \) is integrable, by (10.8) we have \( K_x(0, y) = 0 \), as seen from (16.3) the quantity \( F_s'(y) \) coincides with \( F_s'(y) \) for \( y \in \mathbb{R}^- \), and as indicated in the statement of our proposition \( F_s'(y) \) is assumed to be the sum of an integrable function and a square-integrable function for \( y \in \mathbb{R}^- \). As for the integral in (18.5), \( K(0, y) \) is square integrable in \( y \in \mathbb{R} \) as result of (10.2) and Proposition 16.1(b), and moreover \( F_s(y) \) is square integrable in \( y \in \mathbb{R} \) as implied by Proposition 4.4. The integrals in (18.5) and (18.6) are each essentially a convolution of an integrable and a square integrable functions and hence from Young’s inequality for convolutions it follows that those two integrals are each square integrable in \( y \in \mathbb{R}^- \). Hence, the proof of (b) is complete. Let us now turn to the proof of (c). For \( y \in \mathbb{R}^+ \), from (10.2) and (10.8) it follows that we have

\[
\Gamma_1(y) = \Gamma_{10}(y), \quad \Gamma_2(y) = \Gamma_{11}(y), \quad y \in \mathbb{R}^+.
\]

Thus, using (18.21) in (18.4) we obtain (18.7), completing the proof of (c). Let us now turn to the proof of (d). Using (4.7) and (4.12), we can write the Marchenko equation at \( x = 0 \) given in (4.11) as

\[
K(0, y) + F_s(y) + \int_0^\infty dz K(0, z) F_s(z + y) = -\sum_{j=1}^N e^{-\kappa_j y} \left( I + \int_0^\infty dz K(0, z) e^{-\kappa_j z} \right) M_j^2, \quad y > 0.
\]

Using (18.22) we recognize the quantity in the parentheses on the right-hand side of (18.22) as \( f(i\kappa_j, 0) \), and hence with the help of (9.8) we obtain

\[
\left( I + \int_0^\infty dz K(0, z) e^{-\kappa_j z} \right) M_j^2 = \Psi_j(0) M_j, \quad j = 1, \ldots, N,
\]

where \( \Psi_j(x) \) is the bound-state matrix solution constructed as in (9.8). Using (18.23) in (18.22) we obtain

\[
K(0, y) + F_s(y) + \int_0^\infty dz K(0, z) F_s(z + y) = -\sum_{j=1}^N \Psi_j(0) M_j^2 e^{-\kappa_j y}, \quad y > 0.
\]
We remark that (18.24) could also be obtained from (13.4) at \(x = 0\) and by using (9.8). Similarly, using (4.7) and (4.12) in (13.6) at \(x = 0\), i.e. in the derivative Marchenko equation at \(x = 0\), we obtain

\[
K_x(0, y) + F_s'(y) - K(0, 0) F_s(y) + \int_0^\infty dz K(0, z) F_s(z + y) = -\sum_{j=1}^N e^{-\kappa_j y} \left(-\kappa_j I - K(0, 0) + \int_0^\infty dz K_z(0, z) e^{-\kappa_j z}\right) M_j^2, \quad y > 0. \tag{18.25}
\]

Using (15.39) we recognize the quantity in the parentheses on the right-hand side of (18.25) as \(f'(i\kappa_j, 0)\), and hence with the help of (9.8) we get

\[
\left(-\kappa_j I - K(0, 0) + \int_0^\infty dz K_z(0, z) e^{-\kappa_j z}\right) M_j^2 = \Psi_j'(0) M_j, \quad j = 1, \ldots, N, \tag{18.26}
\]

and hence, using (18.26) in (18.25), for \(y \in \mathbb{R}^-\) we obtain

\[
K_x(0, y) + F_s'(y) - K(0, 0) F_s(y) + \int_0^\infty dz K(0, z) F_s(z + y) = -\sum_{j=1}^N \Psi_j'(0) M_j e^{-\kappa_j y}. \tag{18.27}
\]

With the help of (18.8) and (18.9), we see that we can write (18.24) and (18.27), respectively, as

\[
\Gamma_1(y) = -\sum_{j=1}^N \Psi_j(0) M_j e^{-\kappa_j y}, \quad y \in \mathbb{R}^+, \tag{18.28}
\]

\[
\Gamma_2(y) = -\sum_{j=1}^N \Psi_j'(0) M_j e^{-\kappa_j y}, \quad y \in \mathbb{R}^+, \tag{18.29}
\]

where \(\Psi_j(x)\) is the bound-state matrix solution constructed as in (9.8). Premultiplying (18.28) by \(-B^\dagger\) and premultiplying (18.29) by \(A^\dagger\) and adding the resulting equations, we obtain

\[
-B^\dagger \Gamma_1(y) + A^\dagger \Gamma_2(y) = \sum_{j=1}^n \left[B^\dagger \Psi_j(0) - A^\dagger \Psi_j'(0)\right] M_j e^{-\kappa_j y}, \quad y \in \mathbb{R}^+. \tag{18.30}
\]

By taking the adjoint of both sides of (18.30) we obtain

\[
-G_1(y)^\dagger B + G_2(y)^\dagger A = \sum_{j=1}^n M_j \left[\Psi_j(0)^\dagger B - \Psi_j'(0)^\dagger A\right] e^{-\kappa_j y}, \quad y \in \mathbb{R}^+, \tag{18.31}
\]
where we have used the hermitian property of $M_j$. Comparing (18.7) with (18.31) we obtain (18.10). Let us now show that (18.10) and (18.11) are equivalent. Using (9.2) and (9.8) we see that

$$
\Psi_j(0)^\dagger B - \Psi_j'(0)^\dagger A = M_j J(i\kappa_j), \quad j = 1, \ldots, N,
$$

or equivalently, by taking the adjoint of (18.32) we have

$$
B^\dagger \Psi_j(0) - A^\dagger \Psi_j'(0)^\dagger = J(i\kappa_j)^\dagger M_j, \quad j = 1, \ldots, N.
$$

(18.33)

Hence, using (18.33) in (18.10) we get (18.11).

$$
-\Gamma_1(y)^\dagger B + \Gamma_2(y)^\dagger A = \sum_{j=1}^n M_j^2 J(i\kappa_j) e^{-\kappa_j y}, \quad y \in \mathbb{R}^+.
$$

(18.34)

From (18.10) or (18.11) we observe that the only $y$ dependence of $\hat{\Delta}(y)$ is through the exponential factors $e^{-\kappa_j}$ and hence $\hat{\Delta}(y)$ is continuous in $y \in \mathbb{R}^+$ and vanishes as $y \to +\infty$. Thus, the proof of (d) is complete. Let us now turn to the proof of (e). Using (18.5) and (18.6) in (18.4) and isolating the integral terms, we obtain

$$
\int_{-\infty}^\infty dz \left[ -B^\dagger K(0, z) + A^\dagger K_x(0, z) \right] F_s(z + y) = \hat{\Delta}(y) + B^\dagger \Gamma_{15}(y) - A^\dagger \Gamma_{16}(y), \quad y \in \mathbb{R},
$$

(18.35)

where we have defined

$$
\Gamma_{15}(y) := K(0, y) + F_s(y) + K(0, -y) S_\infty,
$$

(18.36)

$$
\Gamma_{16}(y) := K_x(0, y) + \frac{\partial}{\partial y} F_s(y) + K_x(0, -y) S_\infty - K(0, 0) F_s(y).
$$

(18.37)

From (18.35), for $x \in \mathbb{R}$ we obtain

$$
\int_{-\infty}^\infty dz \left[ -B^\dagger K(0, y) + A^\dagger K_x(0, y) \right] F_s(y + x) = \hat{\Delta}(x) + B^\dagger \Gamma_{15}(x) - A^\dagger \Gamma_{16}(x),
$$

(18.38)

$$
\int_{-\infty}^\infty dz \left[ -B^\dagger K(0, y) + A^\dagger K_x(0, y) \right] F_s(y - x) = \hat{\Delta}(-x) + B^\dagger \Gamma_{15}(-x) - A^\dagger \Gamma_{16}(-x).
$$

(18.39)
In order to prove (e), let us start with the left-hand side of (18.12). Postmultiplying (18.4) by \( F_s(y + x) \) and integrating the resulting equality over \( y \in \mathbb{R} \), we obtain
\[
\int_{-\infty}^{\infty} dy \hat{\Delta}(y) F_s(y + x) = \int_{-\infty}^{\infty} dy \left[ -B^\dagger \Gamma_{10}(y) + A^\dagger \Gamma_{11}(y) \right] F_s(y + x) \quad y \in \mathbb{R}. \tag{18.40}
\]

We can evaluate the right-hand side of (18.40) with the help of (16.63), (16.64), and (16.66), and we get
\[
\int_{-\infty}^{\infty} dy \hat{\Delta}(y) F_s(y + x) = \Gamma_{31}(x) + \Gamma_{32}(x) + \Gamma_{33}(x) + \Gamma_{34}(x) + \Gamma_{35}(x), \quad x \in \mathbb{R}, \tag{18.41}
\]
where we have defined
\[
\Gamma_{31}(x) := \int_{-\infty}^{\infty} dy \left[ -B^\dagger K(0, y) F_s(y + x) + A^\dagger K_x(0, y) F_s(y + x) \right], \tag{18.42}
\]
\[
\Gamma_{32}(x) := \int_{-\infty}^{\infty} dy \left[ -B^\dagger K(0, -y) S_\infty F_s(y + x) + A^\dagger K_x(0, -y) S_\infty F_s(y + x) \right], \tag{18.43}
\]
\[
\Gamma_{33}(x) := B^\dagger \left[ F_s(-x) S_\infty + S_\infty F_s(x) \right] + A^\dagger \left[ -\bar{\Phi}'(x) S_\infty + S_\infty \bar{\Phi}_s(x) - G_1 F_s(x) \right]
+ A^\dagger K(0, 0) \left[ F_s(-x) S_\infty + S_\infty F_s(x) \right], \tag{18.44}
\]
\[
\Gamma_{34}(x) := \int_{-\infty}^{\infty} dz \left[ B^\dagger K(0, z) F_s(z - x) S_\infty - A^\dagger K_x(0, z) F_s(z - x) S_\infty \right], \tag{18.45}
\]
\[
\Gamma_{35}(x) := \int_{-\infty}^{\infty} dz \left[ B^\dagger K(0, z) S_\infty F_s(x - z) - A^\dagger K_x(0, z) S_\infty F_s(x - z) \right]. \tag{18.46}
\]
We observe that
\[
\Gamma_{32}(x) = -\Gamma_{35}(x). \tag{18.47}
\]

Using (18.38) in (18.42), using (18.39) in (18.45), with the help of (18.47), we rewrite (18.41) as
\[
\int_{-\infty}^{\infty} dy \hat{\Delta}(y) F_s(y + x) = \hat{\Delta}(x) + B^\dagger \Gamma_{15}(x) - A^\dagger \Gamma_{16}(x) + \Gamma_{33}(x) - \hat{\Delta}(-x) S_\infty
- B^\dagger \Gamma_{15}(-x) + A^\dagger \Gamma_{16}(-x) S_\infty, \quad x \in \mathbb{R}, \tag{18.48}
\]
Since \( A \) and \( B \) satisfy (16.70), the adjoint of (16.70) yields (16.71) and (16.72). With the help of (16.71) and (16.72), we obtain
\[
B^\dagger \Gamma_{15}(x) - A^\dagger \Gamma_{16}(x) + \Gamma_{33}(x) - B^\dagger \Gamma_{15}(-x) + A^\dagger \Gamma_{16}(-x) S_\infty = 0, \quad x \in \mathbb{R}, \tag{18.49}
\]
and hence using (18.49) in (18.48) we obtain (18.12). Thus, the proof of (e) is complete. □

From Proposition 18.1, we obtain several important results. The following proposition indicates that (3a) of Definition 4.2 implies (V_a) of Definition 4.3. Furthermore, it indicates that (3a) is equivalent to the combination of two properties, namely (III_a) and (V_a) of Definition 4.3.

**Proposition 18.2** Let S in (4.2) be the scattering data set consisting of a scattering matrix S(k), the constants κ_j as distinct positive numbers, and the matrices M_j as n x n nonnegative hermitian matrices. Let F_s(y) be the quantity defined in (4.7), A and B be the boundary matrices constructed as in Proposition 16.9 and Ψ(k, x) be the physical solution constructed as in (9.4). Then:

(a) If (1), (2), (3a), and (4a) of Definition 4.2 hold, then (V_a) of Definition 4.3 holds.

(b) If (1), (2), (4a) of Definition 4.2 hold then (3a) of Definition 4.2 is equivalent to the combination of (III_a) and (V_a) of Definition 4.3.

**PROOF:** Let us first argue that Proposition 18.1 is applicable in both cases of (a) and (b) because in both cases we have F_s'(y) for y ∈ R− is the sum of an integrable function and a square-integrable function. In (a), since S satisfies (1), (2), (3a), and (4a) it belongs to the Marchenko class, and hence by Theorem 5.1(b) there exists a corresponding unique input data set D in the Faddeev class. Then, by Theorem 12.1(h) we see that F_s'(y) for y ∈ R− is the sum of an integrable function and a square-integrable function. On the other hand, in (b) it is assumed that (III_a) holds and hence, by the definition of (III_a), it follows that F_s'(y) for y ∈ R− is the sum of an integrable function and a square-integrable function.

Thus, we are able to apply Proposition 18.1. As seen from (18.3) of Proposition 18.1, Δ(k) and ˆΔ(y) are Fourier transforms of each other. Hence, if (3a) is satisfied, then we must have Δ(k) ≡ 0, yielding ˆΔ(y) ≡ 0. Then, we must have the right-hand-side of (18.10) vanishing for all y ∈ R+. Since the κ_j are distinct, from (18.10) we conclude that

\[ [-B^\dagger \Psi_j(0) + A^\dagger \Psi'_j(0)] M_j = 0, \quad j = 1, \ldots, N. \]  

(18.50)
Since $M_j$ is hermitian, (18.50) can also be written as

$$[-B^\dagger \Psi_j(0) + A^\dagger \Psi'_j(0)] M_j^\dagger = 0, \quad j = 1, \ldots, N. \quad (18.51)$$

From (18.51) we get

$$a_j [-B^\dagger \Psi_j(0) + A^\dagger \Psi'_j(0)] M_j^\dagger b_j^\dagger = 0, \quad j = 1, \ldots, N, \quad (18.52)$$

where $a_j$ and $b_j$ are arbitrary row vectors with $n$ components each. Let $f(k, x)$ be the Jost solution constructed from the solution $K(x, y)$ to the Marchenko equation as in (10.6).

Choosing $b_j$ in (18.52) as

$$b_j = a_j [-B^\dagger f(i\kappa_j, 0) + A^\dagger f'(i\kappa_j, 0)], \quad (18.53)$$

we see that (18.52) yields

$$a_j [-B^\dagger \Psi_j(0) + A^\dagger \Psi'_j(0)] M_j^\dagger [-B^\dagger f(i\kappa_j, 0) + A^\dagger f'(i\kappa_j, 0)]^\dagger a_j^\dagger = 0, \quad j = 1, \ldots, N, \quad (18.54)$$

or equivalently, after using (9.8), we have

$$a_j [-B^\dagger \Psi_j(0) + A^\dagger \Psi'_j(0)] [-B^\dagger \Psi_j(0) + A^\dagger \Psi'_j(0)]^\dagger a_j^\dagger = 0, \quad j = 1, \ldots, N. \quad (18.55)$$

From (18.55) we see that, for each $j = 1, \ldots, N$, the row vector $a_j [-B^\dagger \Psi_j(0) + A^\dagger \Psi'_j(0)]$ has zero length and hence it must be equal to the zero vector. On the other hand since $a_j$ is arbitrary, we conclude that the matrix $[-B^\dagger \Psi_j(0) + A^\dagger \Psi'_j(0)]$ must be zero, yielding (4.20). Hence, the proof of (a) is complete. Let us now turn to the proof of (b). Since $(V_a)$ is assumed to be satisfied, we have the right-hand side of (18.10) is zero and hence (18.10) implies that $\hat{\Delta}(y) = 0$ for $y \in \mathbb{R}^+$. Then, from (18.12) we get

$$\hat{\Delta}(y) + \int_{-\infty}^0 dz \hat{\Delta}(z) F_s(z + y) = 0, \quad y \in \mathbb{R}^- \quad (18.56)$$

Let us now prove that $\hat{\Delta}(y)$ belongs to $L^2(\mathbb{R}^-)$. This is justified as follows. From Proposition 18.1(b) we know that $\hat{\Delta}(y)$ is the sum of an integrable function and a square-integrable
function in \( y \in \mathbb{R}^- \). With the help of Propositions 3.3(c) and 3.4(c) we can conclude that if a solution to (18.56) is the sum of an integrable function and a square-integrable function then that solution must be bounded and hence also be square integrable. Since we assume that (III) holds, we know that the only solution in \( L^2(\mathbb{R}^-) \) to (18.56) is the trivial solution \( \hat{\Delta}(y) = 0 \) for \( y \in \mathbb{R}^- \). However, then we have \( \hat{\Delta}(y) \equiv 0 \) because we have already had \( \hat{\Delta}(y) = 0 \) for \( y \in \mathbb{R}^+ \) as a result of (V). Then, by (18.3) we see that \( \Delta(k) \equiv 0 \), yielding (3a) of Definition 4.2. Conversely, if (3a) holds, then \( \hat{\Delta}(y) \equiv 0 \), and as a result (V) holds because \( \hat{\Delta}(y) = 0 \) for \( y \in \mathbb{R}^+ \) as indicated by (18.10), and (III) holds because \( \hat{\Delta}(y) = 0 \) for \( y \in \mathbb{R}^- \) as the trivial solution to (18.56).

With the help of Proposition 18.1 and 18.2, we get the following result, which also indicates the equivalence of (V) and (Vb) in Proposition 6.5.

**Proposition 18.3** Let \( S \) in (4.2) be the scattering data set consisting of a scattering matrix \( S(k) \), the constants \( \kappa_j \) as distinct positive numbers, and the matrices \( M_j \) as \( n \times n \) nonnegative hermitian matrices. Assume that \( S \) satisfies the conditions (1), (2), and (4a) stated in Definition 4.2. Let \( A \) and \( B \) be the boundary matrices constructed as in Proposition 16.9, \( J(k) \) be the Jost matrix constructed as in (9.2), \( \Psi_j(x) \) be the bound-state matrix solutions constructed as in (9.8), \( \hat{\Delta}(y) \) be the quantity defined in (18.4) and represented as in (18.7) for \( y \in \mathbb{R}^+ \), and \( \Gamma_1(y) \) and \( \Gamma_2(y) \) be the matrices appearing in (18.8) and (18.9), respectively. Then, the following statements are equivalent:

(a) We have the matrix equality

\[ \hat{\Delta}(y) = 0, \quad y \in \mathbb{R}^+. \]  

(b) We have the matrix equality

\[ -\Gamma_1(y)^\dagger B + \Gamma_2(y)^\dagger A = 0, \quad y \in \mathbb{R}^+. \]  

(c) The bound-state matrix solutions \( \Psi_j(x) \) satisfy (18.50).
The Jost matrix $J(k)$ satisfies
\[ J(i\kappa_j)^\dagger M_j^2 = 0, \quad j = 1, \ldots, N. \] (18.59)

The bound-state matrix solutions $\Psi_j(x)$ satisfy the boundary condition (2.4). This is the same as saying that $(V_a)$ of Proposition 4.6 holds, i.e. (4.20) holds.

The Jost matrix $J(k)$ satisfies (4.21), i.e. the scattering data set $S$ satisfies $(V_b)$ of Definition 4.3.

PROOF: From (18.7) we know that (a) and (b) are equivalent. From (18.30) we see that (b) holds if and only if (c) holds, as a result of the functions $e^{-\kappa_j y}$ with $j = 1, \ldots, N$ being linearly independent in $y \in \mathbb{R}^+$. Then, from (18.33) we see that (e) and (f) are equivalent. From (18.33), after postmultiplying with $M_j$, we see that (c) and (d) are equivalent. Hence, it is enough to prove the equivalence of (c) and (e). Note that (e) implies (c), as seen by postmultiplying (4.20) with $M_j$. Let us now prove that (c) implies (e). Arguing as in (18.50)-(18.55) in the proof of of Proposition 18.2, we prove that $a_j \left[ -B^\dagger \Psi_j(0) + A^\dagger \Psi_j'(0) \right] = 0$ for any $a_j \in \mathbb{C}^n$ when $j = 1, \ldots, N$, which proves (4.20) and hence confirms (e).

In the next proposition we explore a key feature of the solution $X(y)$ to (4.22), which will be used later in Proposition 18.6.

**Proposition 18.4** Let $S$ in (4.2) be the scattering data set consisting of a scattering matrix $S(k)$, the constants $\kappa_j$ as distinct positive numbers, and the matrices $M_j$ as $n \times n$ nonnegative hermitian matrices. Assume that $S$ satisfies the conditions (1), (2), and (4_a) of Definition 4.2. Let $F_s(y)$ be the quantity given in (4.7) in such a way that $F_s'(y)$ for $y \in \mathbb{R}^-$ is the sum of an integrable function and a square-integrable function. We have the following:

(a) Any solution $X(y)$ in $L^1(\mathbb{R}^+)$ to (4.22) satisfies
\[ \int_0^\infty dy \ X(y) \hat{\Delta}(y) = 0, \] (18.60)
where \( \hat{\Delta}(y) \) is the quantity defined in (18.4) and whose value for \( y \in \mathbb{R}^+ \) is given by (18.7).

(b) Any solution \( \hat{X}(k) \) in \( \tilde{L}^1(\mathbb{C}^+) \) to (4.23) satisfies

\[
\sum_{j=1}^{N} \hat{X}(ik_j) M_j^2 J(ik_j) = 0, \tag{18.61}
\]

where \( J(k) \) is the Jost matrix constructed as in (9.2).

PROOF: Note that \( F_s(y) \) and \( F(y) \) are hermitian as stated in Proposition 16.4(b), and \( K(0,0) \) is hermitian as stated in Proposition 16.4(d). Thus, from (18.8) and (18.9), we respectively get

\[
\Gamma_1(y) = K(0,y) + F_s(y) + \int_0^\infty dz F_s(y + z) K(0,z), \tag{18.62}
\]

\[
\Gamma_2(y) = K_x(0,y) + F'_s(y) - F_s(y) K(0,0) + \int_0^\infty dz F_s(y + z) K_x(0,z). \tag{18.63}
\]

Let \( X(y) \) be the general solution in \( L^1(\mathbb{R}^+) \) to (4.22). Using (18.62) and (18.63) we obtain

\[
\int_0^\infty dy X(y) \left[ -\Gamma_1(y) B + \Gamma_2(y) A \right] = -\int_0^\infty dy X(y) F_s(y) B + \int_0^\infty dy X(y) F'_s(y) A - \int_0^\infty dy X(y) F_s(y) K(0,0) A - \Lambda_1 + \Lambda_2, \tag{18.64}
\]

where we have defined

\[
\Lambda_1 := \int_0^\infty dy \left[ X(y) + \int_0^\infty dz X(z) F_s(z + y) \right] K(0,y) B, \tag{18.65}
\]

\[
\Lambda_2 := \int_0^\infty dy \left[ X(y) + \int_0^\infty dz X(z) F_s(z + y) \right] K_x(0,y) A. \tag{18.66}
\]

Because \( X(y) \) satisfies (4.22), from (18.65) and (18.66) we see that

\[
\Lambda_1 = 0, \quad \Lambda_2 = 0, \tag{18.67}
\]

and hence the right-hand side of (18.64) only consists of the first three integrals there. Recall that \( X(y) = 0 \) for \( y \in \mathbb{R}^- \) and \( F'_s(y) \) and \( \tilde{F}'_s(y) \) coincide when \( y \in \mathbb{R}^+ \), where
\(\tilde{F}'_s(y)\) is the regular part of \(F'_s(y)\) defined in (16.52). Thus, with the help of (18.67), we can write (18.64) as

\[
\int_0^\infty dy X(y) \left[ -\Gamma_1(y) B + \Gamma_2(y) A \right] = \int_0^\infty dy X(y) \Lambda_3(y),
\]

(18.68)

where we have defined

\[
\Lambda_3(y) := -F_s(y) B + \tilde{F}'_s(y) A - F_s(y) K(0,0) A, \quad y \in \mathbb{R}.
\]

(18.69)

Note that we can replace the lower integration limit \(y = 0\) by \(y = -\infty\) in the integral on the right-hand side of (16.75) to have

\[
\int_0^\infty dy X(y) \Lambda_3(y) = \int_{-\infty}^\infty dy X(y) \Lambda_3(y).
\]

(18.70)

We justify (18.70) as follows. As asserted in Proposition 15.1(c), the solution \(X(y)\) in \(L^1(\mathbb{R}^+)\) to (4.22) is bounded in \(y \in \mathbb{R}^+\). Since \(X(y) = 0\) for \(y \in \mathbb{R}^-\), it follows that \(X(y)\) is bounded and integrable in \(y \in \mathbb{R}\). By (1) we know that \(F_s(y)\) is bounded in \(y \in \mathbb{R}\). Thus the product \(X(y) F_s(y)\) is integrable in \(y \in \mathbb{R}\) and vanishes for \(y \in \mathbb{R}^-\). By (2) we know that \(F'_s(y)\) is integrable in \(y \in \mathbb{R}^+\), and for \(y \in \mathbb{R}^-\) it is the sum of an integrable function and a square-integrable function. From (16.52) it follows that \(\tilde{F}'_s(y)\) is also integrable in \(y \in \mathbb{R}^+\), and for \(y \in \mathbb{R}^-\) it is the sum of an integrable function and a square-integrable function. Thus, the product \(X(y) \tilde{F}'_s(y)\) is integrable in \(y \in \mathbb{R}^+\), vanishes for \(y \in \mathbb{R}^-\), and is integrable in any finite interval containing \(y = 0\). Thus, (18.70) is justified. With the help of (3.67), (4.7), (11.37) we get

\[
\int_{-\infty}^\infty dy X(y) F_s(y) = \frac{1}{2\pi} \int_{-\infty}^\infty dk \hat{X}(k) \left[ S(k) - S_\infty \right],
\]

(18.71)

and with the help of (3.67), (12.3), (16.51) we obtain

\[
\int_{-\infty}^\infty dy X(y) \tilde{F}'_s(y) = \frac{1}{2\pi} \int_{-\infty}^\infty dk \hat{X}(k) \left[ ik (S(k) - S_\infty) - G_1 \right].
\]

(18.72)

Using (18.71) and (18.72) in (16.76), the integral on the right-hand side of (18.70) yields

\[
\int_{-\infty}^\infty dy X(y) \Lambda_3(y) = \frac{1}{2\pi} \int_{-\infty}^\infty dk \hat{X}(k) \Lambda_4(k),
\]

(18.73)
where we have defined

\[ \Lambda_4(k) := [-S(k) + S_\infty] B + (ik [S(k) - S_\infty] - G_1) A - [S(k) - S_\infty] K(0, 0) A. \]  

(18.74)

Since \( X(y) \) satisfies (4.22), from Proposition 15.7 it follows that \( \hat{X}(k) \) satisfies (4.23); hence, we can replace \( \hat{X}(k) S(k) \) on the right-hand side of (18.73) by \( -\hat{X}(-k) \). We then obtain

\[ \int_0^\infty dy X(y) \left[ -\Gamma_1(y) B + \Gamma_2(y) A \right] = \frac{1}{2\pi} \int_{-\infty}^\infty dk \left[ \hat{X}(k) \Lambda_5(k) + \hat{X}(k) \Lambda_6(k) \right], \]

(18.75)

where we have defined

\[ \Lambda_5(k) := B - ik A + K(0, 0) A, \]

(18.76)

\[ \Lambda_6(k) := S_\infty B - ik S_\infty A - G_1 A + S_\infty K(0, 0) A. \]

(18.77)

We can replace \( \hat{X}(-k) \Lambda_5(k) \) by \( \hat{X}(k) \Lambda_5(-k) \) in the integrand on the right-hand side of (18.75). Then, using (18.76) and (18.77) in (18.75) we obtain

\[ \int_0^\infty dy X(y) \left[ -\Gamma_1(y) B + \Gamma_2(y) A \right] = \frac{1}{2\pi} \int_{-\infty}^\infty dk \hat{X}(k) \Lambda_7(k), \]

(18.78)

where we have defined

\[ \Lambda_7(k) := (I + S_\infty) B + ik (I - S_\infty) A - G_1 A + K(0, 0) A + S_\infty K(0, 0) A. \]

(18.79)

When (1) and (4a) in Definition 4.2 hold, we have (16.70) satisfied, which makes the right-hand side of (18.79) vanish. Thus, using \( \Lambda_7(k) \equiv 0 \) on the right hand side of (18.78), we see that (18.78) yields (18.60). Thus, the proof of (a) is complete. Let us now turn to the proof of (b). Using (18.34) in (18.60), we obtain

\[ \sum_{j=1}^n \int_0^\infty dy X(y) e^{-\kappa_j y} M_j^2 J(i\kappa_j) = 0. \]

(18.80)

From (3.68) we see that

\[ \int_0^\infty dy X(y) e^{-\kappa_j y} = \hat{X}(i\kappa_j), \quad j = 1, \ldots, N. \]

(18.81)
and hence using (18.81) on the left-hand side of (18.80) we obtain (18.61). Hence, the proof of (b) is complete. □

In the next proposition we show that the results stated in Proposition 18.4 for the solutions $X(y) \in L^1(\mathbb{R}^+)$ to (4.22) and $\hat{X}(k) \in L^1(\mathbb{C}^+)$ to (4.23) actually hold for the solutions $X(y) \in L^2(\mathbb{R}^+)$ to (4.22) and $\hat{X}(k) \in L^2(\mathbb{C}^+)$ to (4.23).

**Proposition 18.5** Let $S$ in (4.2) be the scattering data set consisting of a scattering matrix $S(k)$, the constants $\kappa_j$ as distinct positive numbers, and the matrices $M_j$ as $n \times n$ nonnegative hermitian matrices. Assume that $S$ satisfies the conditions (1), (2), and (4a) of Definition 4.2. Let $F_s(y)$ be the quantity given in (4.7) in such a way that $F_s'(y)$ for $y \in \mathbb{R}^+$ is the sum of an integrable function and a square-integrable function. We have the following:

(a) Any solution $X(y)$ in $L^2(\mathbb{R}^+)$ to (4.22) satisfies

$$\int_0^\infty dy \, X(y) \hat{\Delta}(y) = 0,$$

(18.82)

where $\hat{\Delta}(y)$ is the quantity defined in (18.4) and whose value for $y \in \mathbb{R}^+$ is given by (18.7).

(b) Any solution $\hat{X}(k)$ in $H^2(\mathbb{C}^+)$ to (4.23) satisfies

$$\sum_{j=1}^N \hat{X}(ik_j) M_j^2 J(ik_j) = 0,$$

(18.83)

where $J(k)$ is the Jost matrix constructed as in (9.2).

**PROOF:** In the proof of Proposition 18.4, let us replace the solution $X(y) \in L^1(\mathbb{R}^+)$ to (4.22) with $X(y) \in L^2(\mathbb{R}^+)$ to (4.22). Then, the proof remains valid provided we can prove that the integrability of $X(y) F_s(y)$ in $y \in \mathbb{R}^+$ as well as the integrability of $X(y) F_s'(y)$ in $y \in \mathbb{R}^+$ are unaffected. The following argument shows that those integrabilities are indeed unaffected. We know from Proposition 14.6(a) that any solution $X(y) \in L^2(\mathbb{R}^+)$ to (4.22) must actually belong to $L^2(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$. Thus, $X(y) F_s(y)$ still remains integrable in

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\( y \in \mathbb{R}^+ \) because \( F_s(y) \in L^1(\mathbb{R}^+) \) and \( X(y) F_s(y) \) is a product of a bounded quantity with an integrable quantity. Similarly, \( X(y) \hat{F}'_s(y) \) still remains integrable in \( y \in \mathbb{R}^+ \) because \( \hat{F}'_s(y) \in L^1(\mathbb{R}^+) \) and \( X(y) F_s(y) \) is a product of a bounded quantity with an integrable quantity. Then, with the minor replacement that we use Proposition 15.6 instead of Proposition 15.7 in the proof of Proposition 18.4, we know that the proof constitutes a proof for Proposition 18.5.

The next proposition shows that the property \((V_c)\) implies \((V_b)\) in Proposition 6.6.

**Proposition 18.6** Let \( S \) in (4.2) be the scattering data set consisting of a scattering matrix \( S(k) \), the constants \( \kappa_j \) as distinct positive numbers, and the matrices \( M_j \) as \( n \times n \) nonnegative hermitian matrices. Assume that \( S \) satisfies the conditions (1), (2), and \( (4_a) \) of Definition 4.2. Then, the condition \((V_c)\) of Proposition 4.1 implies \((V_b)\) of Proposition 6.6.

**PROOF:** Let \( N \) be the nonnegative integer defined in (4.3). We would like to show that, if (4.22) has \( N \) linearly independent solutions in \( L^1(\mathbb{R}^+) \), then the Jost matrix constructed as in (9.2) from the input scattering data satisfies (4.21). For the proof we proceed as follows. Let \( X^{(l)}(y) \) for \( l = 1, \ldots, N \) be linearly independent solutions in \( L^1(\mathbb{R}^+) \) to (4.22). By Proposition 15.7(a) we know that (4.23) has also \( N \) linearly independent solutions given by \( \hat{X}^{(l)}(k) \) with \( l = 1, \ldots, N \), where each \( \hat{X}^{(l)}(k) \) is related to \( X^{(l)}(y) \) as in (3.67) and (3.68). Thus, the general solution to (4.23) can be written as a linear combination as

\[
\hat{X}(k) = \sum_{l=1}^{N} \gamma_l \hat{X}^{(l)}(k), \tag{18.84}
\]

where the \( \gamma_l \) are arbitrary scalar coefficients. From Proposition 6.1 we know that \( (4_a) \) and \( (4_d) \) of Definition 4.2 are equivalent, and hence \( (4_d) \) holds. Note that (4.23) and the second line of (4.15) coincide. Thus, the quantity \( \hat{X}(k) \) given in (18.84) satisfies the second line of (4.15). Since \( (4_d) \) holds, this means that if \( \hat{X}(k) \) given in (18.84) also satisfies

\[
\hat{X}(ik_j) M_j = 0, \quad j = 1, \ldots, N, \tag{18.85}
\]

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then we must have \( \hat{X}(k) \equiv 0 \). Then, from (18.85) we can conclude that the linear homogeneous system given by

\[
\sum_{l=1}^{N} \gamma_l \hat{X}^{(l)}(i\kappa_j) M_j = 0, \quad j = 1, \ldots, N.
\]

(18.86)
can only have the trivial solution \( \gamma_l = 0 \) for \( l = 1, \ldots N \). We can write (18.86) in the matrix notation as

\[
\begin{bmatrix}
\gamma_1 & \cdots & \gamma_N
\end{bmatrix}
\begin{bmatrix}
\hat{X}^{(1)}(i\kappa_1) M_1 & \cdots & \hat{X}^{(1)}(i\kappa_N) M_N \\
\vdots & \ddots & \vdots \\
\hat{X}^{(N)}(i\kappa_1) M_1 & \cdots & \hat{X}^{(N)}(i\kappa_N) M_N
\end{bmatrix}
= [0 \cdots 0].
\]

(18.87)

We remark that in (18.87) the unknown is a row vector with \( N \) entries, the coefficient matrix has the matrix size \( N \times (nN) \), with \( \hat{X}^{(l)}(i\kappa_j) M_j \) being an \( 1 \times n \) matrix, and in the zero vector on the right-hand side of (18.87) each zero represents the zero row vector with \( n \) entries. Since the only solution to (18.87) must be the zero solution, the rank of the coefficient matrix in (18.87) must be the same as the number of unknowns, i.e. must be equal to \( N \). This implies that any nonhomogeneous system associated with (18.87) must have a unique solution. In particular, let us consider the nonhomogeneous system given by

\[
\sum_{l=1}^{N} \gamma_l \hat{X}^{(l)}(i\kappa_j) M_j = a^{(j)} M_j, \quad j = 1, \ldots, N,
\]

(18.88)

which we can write in the matrix notation as

\[
\begin{bmatrix}
\gamma_1 & \cdots & \gamma_N
\end{bmatrix}
\begin{bmatrix}
\hat{X}^{(1)}(i\kappa_1) M_1 & \cdots & \hat{X}^{(1)}(i\kappa_N) M_N \\
\vdots & \ddots & \vdots \\
\hat{X}^{(N)}(i\kappa_1) M_1 & \cdots & \hat{X}^{(N)}(i\kappa_N) M_N
\end{bmatrix}
= [a^{(1)} M_1 \cdots a^{(N)} M_N],
\]

(18.89)

where each \( a^{(j)} \) is a row vector with \( n \) components. Because of the full rank of the coefficient matrix in (18.89), the corresponding nonhomogeneous system has a unique solution \( \gamma_l \) for \( l = 1, \ldots, N \) for any choice of \( a^{(j)} \) with \( j = 1, \ldots, N \). With those values of \( \gamma_l \), the row vector \( \hat{X}(k) \) given in (15.59) is a solution to (4.23) and we have

\[
\hat{X}(i\kappa_j) M_j = a^{(j)} M_j, \quad j = 1, \ldots, N.
\]

(18.90)
By Proposition 18.4(b) we know that any solution to (4.23) must satisfy (18.61). Using (18.90) in (18.61) we get
\[
\sum_{l=1}^{N} a^{(j)} M_j M_j^\dagger J(ik_j) = 0,
\]
where we have used the fact that the matrix \( M_j \) is hermitian. Since the \( a^{(j)} \) can be chosen arbitrarily, we conclude that each term in the summation in (18.91) must vanish, i.e. we have
\[
a^{(j)} M_j M_j^\dagger J(ik_j) = 0, \quad j = 1, \ldots, N.
\]
(18.92)
Let us multiply (18.92) by an arbitrary column vector \( b_j^\dagger \) having \( n \) components. We get
\[
a^{(j)} M_j M_j^\dagger J(ik_j) b_j^\dagger = 0, \quad j = 1, \ldots, N.
\]
(18.93)
Choosing \( a^{(j)} \) as \( a^{(j)} = b_j J(ik_j)^\dagger \), from (18.93) we obtain
\[
\begin{bmatrix} b_j J(ik_j)^\dagger M_j \end{bmatrix} \begin{bmatrix} M_j^\dagger J(ik_j) b_j^\dagger \end{bmatrix} = 0, \quad j = 1, \ldots, N,
\]
(18.94)
or equivalently
\[
\begin{bmatrix} M_j^\dagger J(ik_j) b_j^\dagger \end{bmatrix} \begin{bmatrix} M_j^\dagger J(ik_j) b_j^\dagger \end{bmatrix} = 0, \quad j = 1, \ldots, N.
\]
(18.95)
The left-hand side of (18.95) is the length of the vector \( M_j^\dagger J(ik_j) b_j^\dagger \), and hence that vector must be the zero vector, yielding
\[
M_j^\dagger J(ik_j) b_j^\dagger = 0, \quad j = 1, \ldots, N.
\]
(18.96)
Since \( b_j^\dagger \) can be chosen arbitrarily, (18.96) implies that
\[
M_j^\dagger J(ik_j) = 0, \quad j = 1, \ldots, N.
\]
(18.97)
Comparing (18.97) with (4.21) we conclude that (\( V_b \)) of Definition 4.3 is satisfied.

The next proposition shows that the condition (\( V_f \)) of Proposition 6.6 implies (\( V_b \)) there.
Proposition 18.7 Let $S$ in (4.2) be the scattering data set consisting of a scattering matrix $S(k)$, the constants $\kappa_j$ as distinct positive numbers, and the matrices $M_j$ as $n \times n$ nonnegative hermitian matrices. Assume that $S$ satisfies the conditions (1), (2), and (4a) of Definition 4.2. Then, the condition $(V_f)$ implies $(V_b)$ in Proposition 6.6.

PROOF: The proof of Proposition 18.6 also constitutes a proof for Proposition 18.7 with the following minor modifications. In the proof of Proposition 18.6 we replace $X^{(l)}(y) \in L^1(\mathbb{R}^+)$ with $X^{(l)}(y) \in L^2(\mathbb{R}^+)$, replace the reference to Proposition 15.7 by the reference to Proposition 15.6, replace the reference to Proposition 18.4(b) by the reference to Proposition 18.5(b), and replace the mention of $\hat{X}(k) \in \hat{L}^1(\mathbb{C}^+)$ by the mention of $\hat{X}(k) \in H^2(\mathbb{C}^+)$. Hence, $(V_f)$ implies $(V_b)$. 

The following results shows the equivalence of $(V_c)$ and $(V_f)$ when the scattering data set $S$ satisfies all the four conditions (1), (2), (3a), and (4a) of Definition 4.5.

Proposition 18.8 Let $S$ in (4.2) be the scattering data set consisting of a scattering matrix $S(k)$, the constants $\kappa_j$ as distinct positive numbers, and the matrices $M_j$ as $n \times n$ nonnegative hermitian matrices. Assume that $S$ satisfies the conditions (1), (2), (3a), and (4a) of Definition 4.5. Then, the properties $(V_c)$ and $(V_f)$ of Definition 4.3 are equivalent.

PROOF: The general solution $\hat{X}(k) \in H^2(\mathbb{C}^+)$ to (4.23) given in (15.59) and the general solution $\hat{X}(k) \in L^1(\mathbb{C}^+)$ to (4.23) given in (15.60) are identical. As argued in Propositions 15.8 and 15.9 they each contain $N$ linearly independent row vector solutions. Thus, $(V_c)$ and $(V_f)$ are equivalent. 

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19. INVERSE PROBLEM BY USING ONLY THE SCATTERING MATRIX

In this chapter we assume that our scattering data set $S$ given in (4.2) does not contain any information on the bound states and it consists of the scattering matrix $S(k)$ alone. We then investigate whether we can supplement $S(k)$ with some appropriate bound-state data so that the resulting $S$ becomes a scattering data set for some input data set $D$ as in (4.1) belonging to the Faddeev class.

The following result shows that, if the scattering matrix $S(k)$ satisfies (I) of Definition 4.3, then we can always find some bound-state data set so that the resulting $S$ satisfies also (4c) and (Vf) of Theorem 7.2. The analogous result in the Dirichlet case is given in Corollary to Lemma 5.6.1 of [2].

**Proposition 19.1** If the scattering matrix $S(k)$ satisfies (I) of Definition 4.3, then there exists at least one bound-state data set $\{\kappa_l, M_l\}_{l=1}^p$ in such a way that the resulting scattering data set $\{S, \{\kappa_l, M_l\}_{l=1}^p\}$ satisfies (4c) and (Vf) of Theorem 7.2. Here $p$ is either zero, in which case the resulting scattering data set consists of $S(k)$ alone, or $p$ is a positive integer in such a way that the $\kappa_l$ are distinct positive numbers and the $n \times n$ matrices $M_l$ are each nonnegative, hermitian, and of rank one.

**PROOF:** Consider the vector space of row vectors with $n$ components as functions in $x$ belonging to $L^2(\mathbb{R}^+)$. For any arbitrary positive integer $p$, consider a $p$-dimensional subspace of the aforementioned vector space, which is spanned by $p$ linearly independent vector-valued functions in $x \in \mathbb{R}^+$. As indicated in Lemma 5.6.1 of [2], one can explicitly construct at least one set of $p$ nonnegative, hermitian matrices $M_l$, each of which is an $n \times n$ constant matrix of rank one, and $p$ distinct positive numbers $\kappa_l$ in such a way that if $X(y)$ is any vector in the aforementioned $p$-dimensional subspace and if we have

$$\int_0^\infty dy X(y) M_l e^{-\kappa_l y} = 0, \quad l = 1, \ldots, p,$$

(19.1)

then $X(y) \equiv 0$. Note that the sum of the ranks of $M_l$ is $p$ because each $M_l$ has rank one. Given $S(k)$ satisfying (I) of Definition 4.3, consider the integral equation (4.22), where
$F_s(y)$ is constructed as in (4.7). Since (I) holds, by Proposition 4.4 we know that $F_s(y)$ belongs to $L^2(\mathbb{R}^+)$. By Proposition 3.4(a), the operator associated with (4.22) is compact on $L^2(\mathbb{R}^+)$. Any solution to (4.22) in $L^2(\mathbb{R}^+)$ must be an eigenfunction of that operator with eigenvalue $-1$. From Theorem 6.26 on p. 185 of [27] it follows that any nonzero eigenvalue of a compact operator on a Hilbert space must have finite multiplicity. Thus, the number of linearly independent solutions in $L^2(\mathbb{R}^+)$ to (4.22) must be some finite nonzero integer $p$. Thus, the solution space for (4.22) is a $p$-dimensional subspace of row vectors with $n$ components that belong to $L^2(\mathbb{R}^+)$. If $p = 0$, then we assume that there are no bound states and hence (4.22) becomes the same as (4.14), as seen from (4.12). By Proposition 15.7(c) we know that any solution in $L^1(\mathbb{R}^+)$ to (4.14) must be bounded and hence must belong to $L^2(\mathbb{R}^+)$. Thus, if the only solution in $L^2(\mathbb{R}^+)$ to (4.14) is the trivial solution then the only solution in $L^1(\mathbb{R}^+)$ to (4.22) must also be the trivial solution. Then, (V') of Theorem 7.2 and (4.e) are both satisfied because the only solutions to (4.14) and (4.22) in $L^2(\mathbb{R}^+)$ are $X(y) \equiv 0$. If $p \geq 1$, then (4.22) has $p$ linearly independent solutions in $L^2(\mathbb{R}^+)$. Analogous to (4.12), let us set

$$F(y) = F_s(y) + \sum_{l=1}^{p} M_l^2 e^{-\kappa_l y}, \quad y \in \mathbb{R}^+, \quad (19.2)$$

where we recall that the sum of the ranks of $M_l$ is equal to $p$. Then, using (19.2) we see that the left-hand side of (4.14) satisfies

$$X(y) + \int_0^\infty dz X(z) F(z + y) = X(y) + \int_0^\infty dz X(z) F_s(z + y)$$

$$+ \sum_{l=1}^{p} \left[ \int_0^\infty dz X(z) M_l e^{-\kappa_l z} \right] M_l e^{-\kappa_l y}, \quad y \in \mathbb{R}^+. \quad (19.3)$$

From (19.3) and the conclusion associated with (19.1) we observe the following: Any solution in $L^2(\mathbb{R}^+)$ to (4.22) satisfying (19.1) must be a solution in $L^2(\mathbb{R}^+)$ to (4.14), and conversely any solution in $L^2(\mathbb{R}^+)$ to (4.14) can be expressed as a solution in $L^2(\mathbb{R}^+)$ to (4.22) satisfying (19.1). Note that, by Proposition 15.3(d), $X(y)$ satisfies (19.3) if any only
if its Fourier transform $\hat{X}(k)$ given in (3.68) satisfies both lines of (4.15). The first line of (4.15) implies that (19.1) is satisfied by $X(y)$. By Proposition 15.7(a), the second line of (4.15) implies, after using (3.67), that $X(y)$ satisfies (4.22). Then, $X(y)$ satisfies (19.1) and (4.22), and furthermore it belongs to $L^2(\mathbb{R}^+)$ because it is in $L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ by Proposition 15.1(c). By the construction of $\{\kappa_l, M_l\}_{l=1}^p$, we know that any solution $X(y)$ in $L^2(\mathbb{R}^+)$ to (4.22) satisfying (19.1) must be the trivial solution $X(y) \equiv 0$. Thus, the scattering data set $\{S, \{\kappa_l, M_l\}_{l=1}^p\}$ satisfies (4.22) of Theorem 7.2. By construction, the number of linearly independent solutions in $L^2(\mathbb{R}^+)$ to (4.22) is equal to $p$, and hence $(V_f)$ of Theorem 7.1 also holds.

With the help of Proposition 19.1 we obtain the following result.

**Proposition 19.2** The properties (1), (2), and (IIIa) are necessary and sufficient for an $n \times n$ scattering matrix $S(k)$ to be the scattering matrix for some input data set $D$ as in (4.1) belonging to the Faddeev class.

**PROOF:** By Theorem 7.1 we know that there exists a unique input data set $D$ in the Faddeev class if the scattering data set $S$ satisfies (1), (2), (IIIa), (4c), and $(V_c)$. If the scattering matrix $S(k)$ in $S$ satisfies (1), then (I) of Definition 4.3 is automatically satisfied. Then, as Proposition 19.1 implies we can always complete $S(k)$ to a scattering data set $S$ satisfying (4c) and $(V_c)$ by adding some bound-state data set $\{\kappa_l, M_l\}_{l=1}^p$ for some nonnegative integer $p$, and such a procedure certainly does not affect (1), (2), (IIIa). Thus, the resulting scattering data set $\{S, \{\kappa_l, M_l\}_{l=1}^p\}$ satisfies all of (1), (2), (IIIa), (4c), $(V_c)$ of Theorem 7.1, and hence it corresponds to an input data set $D$ in the Faddeev class.

The following result is related to the characterization stated in Theorem 7.9, and it states that unless a scattering matrix $S(k)$ satisfies Levinson’s theorem, it is impossible to supplement it with any bound-state data set so that the resulting scattering data set corresponds to an input data set $D$ in the Faddeev class.

**Corollary 19.3** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$
scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. If $N$ defined in (4.3) does not satisfy (21.5), i.e. if the scattering matrix does not satisfy Levinson’s theorem, then as stated in Theorem 7.9, the associated scattering data set $S$ cannot belong to the Marchenko class and hence at least one of the conditions in any of Theorems 5.1, 7.1-7.6 must be violated.

Based on an intrinsic property of the scattering matrix $S(k)$, in some cases we might be able to conclude that it is impossible that such a scattering matrix may correspond to an input data set $D$ in the Faddeev class. The following result directly follows from Corollary 19.3.

**Corollary 19.4** Consider an $n \times n$ scattering matrix $S(k)$ satisfying (I) of Definition 4.3 and (2) of Theorem 7.1. Consider (21.5) related to $S(k)$, where each term except $N$ is uniquely determined by $S(k)$. If the value of $N$ obtained by solving (21.5) algebraically is negative, then the scattering matrix $S(k)$ cannot be a scattering matrix for some input data set $D$ as in (4.1) belonging to the Faddeev class.
20. PARSEVAL’S EQUALITY

Parseval’s equality is the completeness relation for the physical solutions \( \Psi(k,x) \) appearing in (9.4) and the normalized bound-state solutions \( \Psi_j(x) \) appearing in (9.8), and it is formulated as

\[
\frac{1}{2\pi} \int_0^\infty dk \, \Psi(k,x) \Psi(k,y)^\dagger + \sum_{j=1}^N \Psi_j(x) \Psi_j(y)^\dagger = \delta(x-y) \, I, \quad x,y \in \mathbb{R}^+.
\]

(20.1)

In this chapter we prove that if the scattering data set \( S \) given in (4.2) satisfies (I) of Definition 4.3 and (2) of Definition 4.2 then Parseval’s equality holds. In this chapter, we also show that if Parseval’s equality holds then the Marchenko integral equation (13.1) is uniquely solvable.

The following proposition is needed in the proof of Parseval’s equality. Even though the physical solution \( \Psi(k,x) \) appearing in (9.4) and (9.6) is not an even function in \( k \in \mathbb{R} \), the following result shows that \( \Psi(k,x) \Psi(k,y)^\dagger \) is even in \( k \) for \( k \in \mathbb{R} \).

**Proposition 20.1** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Assume that \( S \) satisfies (I) of Definition 4.3. Then, for any \( x,y \in \mathbb{R}^+ \), the physical solution \( \Psi(k,x) \) constructed from \( S \) as in step (9.4) satisfies

\[
\Psi(-k,x) \Psi(-k,y)^\dagger = \Psi(k,x) \Psi(k,y)^\dagger, \quad k \in \mathbb{R}.
\]

(20.2)

**PROOF:** The physical solution is constructed from \( f(k,x) \) and \( S(k) \) via (9.4). In turn, \( f(k,x) \) is constructed from the solution \( K(x,y) \) to the Marchenko equation (13.1) as in (10.6). Proposition 16.1(a) assures the existence and the uniqueness of \( K(x,y) \) for each \( x \in \mathbb{R}^+ \) if (I) holds. Using (9.4) on the right-hand side of (20.2) and simplifying the result with the help of (4.4), we observe that the right-hand side of (20.2) is an even function of \( k \in \mathbb{R} \). \( \blacksquare \)
In the following theorem we show that Parseval’s equality holds if the Marchenko equation (13.1) is uniquely solvable.

**Proposition 20.2** Consider a scattering data set \( S \) as in (4.2), which consists of an \( n \times n \) scattering matrix \( S(k) \) for \( k \in \mathbb{R} \), a set of \( N \) distinct positive constants \( \kappa_j \), and a set of \( N \) constant \( n \times n \) Hermitian and nonnegative matrices \( M_j \) with respective positive ranks \( m_j \), where \( N \) is a nonnegative integer. Assume that \( S \) satisfies (I) of Definition 4.3. Let \( \Psi(k,x) \) be the corresponding physical solutions constructed from \( S \) as in (9.4) and \( \Psi_j(x) \) be the normalized bound-state solutions constructed as in (9.8). Then, Parseval’s relation (20.1) holds for \( x,y \in \mathbb{R}^+ \).

**PROOF:** Because of (20.2) the integral in (20.1) can be written in the equivalent form as

\[
\frac{1}{2\pi} \int_0^\infty dk \Psi(k,x) \Psi(k,y)^\dagger = \frac{1}{4\pi} \int_{-\infty}^\infty dk \Psi(k,x) \Psi(k,y)^\dagger. \tag{20.3}
\]

From (9.4) we obtain

\[
\Psi(k,x) \Psi(k,y)^\dagger = f(-k,x) f(-k,y)^\dagger + f(-k,x) S(k)^\dagger f(k,y)^\dagger + f(k,x) S(k) f(-k,y)^\dagger + f(k,x) f(k,y)^\dagger,
\]

where we have used the unitarity of \( S(k) \) expressed in the second equality in (4.4). With the help of (20.4) and the first equality in (4.4), we obtain

\[
\frac{1}{4\pi} \int_{-\infty}^\infty dk \Psi(k,x) \Psi(k,y)^\dagger = \frac{1}{2\pi} \int_{-\infty}^\infty dk \left[ f(k,x) f(k,y)^\dagger + f(k,x) S(k) f(-k,y)^\dagger \right]. \tag{20.5}
\]

Let us replace \( S(k) \) in (20.5) by the sum of \( S_\infty \) and \( [S(k) - S_\infty] \), where \( S_\infty \) is the constant \( n \times n \) matrix given in (4.6). Furthermore, let us use (4.7) and (10.6) so that we can write the right-hand side of (20.5) in terms of \( F_s(y) \) and \( K(x,y) \). As indicated in Proposition 16.1(a) the existence and uniqueness of \( K(x,y) \) for \( x,y \in \mathbb{R}^+ \) is assured by (I) of Definition 4.3.
With the help of

\[
\frac{1}{4\pi} \int_{-\infty}^{\infty} dk \, \Psi(k, x) \, \Psi(k, y)^\dagger \\
= \delta(x - y) \, I + K(y, x)^\dagger + K(x, y) + S_\infty \delta(x + y) \\
+ S_\infty K(y, -x)^\dagger + K(x, -y) S_\infty + F_s(x + y) \\
+ \int_{-\infty}^{\infty} d\xi \, K(x, \xi) \, K(y, \xi)^\dagger + \int_{-\infty}^{\infty} dz \, K(x, z) \, F_s(z + y) \\
+ \int_{-\infty}^{\infty} d\xi \, F_s(x + \xi) \, K(y, \xi)^\dagger + \int_{-\infty}^{\infty} d\xi \, K(x, \xi) \, S_\infty K(y, -\xi)^\dagger \\
+ \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} dz \, K(x, z) \, F_s(z + \xi) \, K(y, \xi)^\dagger,
\]

(20.6)

where we have written the integration limits over \(\mathbb{R}\) because, in the Marchenko equation (13.1), we know that \(K(x, y) = 0\) for \(y < x\). We will consider the case \(0 < x \leq y\) and and the case \(0 < y < x\) separately. When we have \(0 < x \leq y\), we see that the second, fourth, fifth, sixth, and eleventh terms on the right-hand side in (20.6) vanish. We can group the third, seventh, and ninth terms into one group and the eighth, tenth, and twelfth terms into another group. Then, from (20.6) we obtain

\[
\frac{1}{4\pi} \int_{-\infty}^{\infty} dk \, \Psi(k, x) \, \Psi(k, y)^\dagger - \delta(x - y) \, I \\
= \left[ K(x, y) + F_s(x + y) + \int_{x}^{\infty} dz \, K(x, z) \, F_s(z + y) \right] \\
+ \int_{y}^{\infty} d\xi \left[ K(x, \xi) + F_s(x + \xi) + \int_{x}^{\infty} dz \, K(x, z) \, F_s(z + \xi) \right] \, K(y, \xi)^\dagger,
\]

(20.7)

where we have used \(K(x, y) = 0\) for \(y < x\). Let us now consider the summation term in (20.1). With the help of (9.8) we get

\[
\Psi_j(x) \, \Psi_j(y)^\dagger = f(i\kappa_j, x) \, M_j^2 \, f(i\kappa_j, y)^\dagger,
\]

(20.8)
where we have used $M_j^\dagger = M_j$. Using (10.6) on the right-hand side of (20.8), we obtain

$$
\Psi_j(x) \Psi_j(y)^\dagger = M_j^2 e^{-\kappa_j(x+y)} + \int_x^\infty dz \ K(x, z) M_j^2 e^{-\kappa_j(z+y)}
+ \int_x^\infty d\xi \ M_j^2 K(y, \xi)^\dagger e^{-\kappa_j(x+\xi)}
\times \int_y^\infty d\xi \ \int_x^\infty dz \ K(x, z) M_j^2 K(y, \xi)^\dagger e^{-\kappa_j(z+\xi)}
\tag{20.9}
$$

Applying the summation $\sum_{j=1}^N$ on both sides of (20.9), we can add the resulting equation to (20.7). With the help of (4.12), we can combine the resulting four summation terms with the four terms on the right-hand side of (20.7). This yields

$$
\frac{1}{2\pi} \int_0^\infty dk \ \Psi(k, x) \Psi(k, y)^\dagger + \sum_{j=1}^N \Psi_j(x) \Psi_j(y)^\dagger - \delta(x - y) I
\tag{20.10}
$$

$$
= P(x, y) + \int_y^\infty d\xi \ P(x, \xi) K(y, \xi)^\dagger,
$$

where we have used (20.3) in the first term on the left-hand side and have defined

$$
P(x, y) := K(x, y) + F(x + y) + \int_x^\infty dz \ K(x, z) F(z + y). \tag{20.11}
$$

Since $K(x, y)$ satisfies the Marchenko equation (13.1) for $0 < x \leq y$, comparing (13.1) and (20.11) we see that $P(x, y) = 0$ for $0 < x \leq y$, and hence (20.11) yields (20.1). Let us now consider the case $0 < y < x$. In this case, we see that $K(x, y) = 0$ for $x > y$ implies that the third, fourth, fifth, sixth, and eleventh terms on the right-hand side of (20.6) vanish. We can group the second, seventh, and tenth terms into one group and the eighth, ninth, and twelfth terms into another group. Then, from (20.6), instead of (20.7), we get

$$
\frac{1}{4\pi} \int_{-\infty}^\infty dk \ \Psi(k, x) \Psi(k, y)^\dagger - \delta(x - y) I
\tag{20.12}
$$

$$
= \left[ K(y, x) + F_s(y + x) + \int_y^\infty d\xi \ K(y, \xi) F_s(\xi + x) \right]^\dagger
\times \int_y^\infty dz \ K(x, z) \left[ K(y, z) + F_s(y + z) + \int_y^\infty d\xi \ K(y, \xi) F_s(\xi + z) \right]^\dagger,
$$
where we have used $K(x, y) = 0$ for $x > y$ and also used $F_s(y)^\dagger = F_s(y)$, where the hermitian property of $F_s(y)$ follows from (I) as indicated in Proposition 16.4(b). Proceeding as in the previous case $0 < x \leq y$, after (20.8) and (20.9), instead of (20.10) we get

$$
\frac{1}{2\pi} \int_0^\infty dk \, \Psi(k, x) \Psi(k, y)^\dagger + \sum_{j=1}^N \Psi_j(x) \Psi_j(y)^\dagger - \delta(x - y) I
$$

$$
= P(y, x)^\dagger + \int_y^\infty dz \, K(x, z) P(y, z)^\dagger,
$$

(20.13)

where $P(x, y)$ is the quantity defined in (20.11). In this case we have $0 < y < x$, and hence the Marchenko equation (13.1) yields $P(y, x) = 0$ for $0 < y < x$. Thus, the right-hand side of (20.10) vanishes for $0 < y < x$. Hence, the proof is completed.

In the next proposition we show that the Marchenko equation (13.1) is satisfied if Parseval’s equality (20.1) holds.

**Proposition 20.3** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ satisfies (I) of Definition 4.3. Let $\Psi(k, x)$ be the corresponding physical solutions constructed from $S$ as in (9.4) and $\Psi_j(x)$ be the normalized bound-state solutions constructed as in (9.8). Assume further that Parseval’s equality (20.1) holds for $x, y \in \mathbb{R}^+$. Then, for each $x \in \mathbb{R}^+$, the Marchenko equation (13.1) is uniquely solvable for $y > x > 0$ where the solution $K(x, y)$ in $y$ belongs to $L^1(x, +\infty)$.

**PROOF:** Let us look for a solution $K(x, y)$ in $y$ in $L^1(x, +\infty)$ to (13.1) for $0 < x < y$. Thus, we can let $K(x, y) = 0$ for $y < x$.

If Parseval’s equality (20.1) holds for $x, y \in \mathbb{R}^+$, then the left-hand side of (20.10) vanishes for $0 < x < y$, yielding

$$
P(x, y) + \int_y^\infty d\xi \, P(x, \xi) K(y, \xi)^\dagger = 0, \quad 0 < x < y.
$$

(20.14)

Let us view (20.14) as a homogeneous integral equation where $P(x, y)$ is the unknown in $L^1(x < y < +\infty)$ for each $x > 0$ and $K(y, \xi)^\dagger$ is the kernel of the integral operator. Since
\( K(x, y) \) is considered in \( y \) to belong to \( L^1(x, +\infty) \), from Proposition 3.3(a), we conclude that the integral operator associated with (20.14) is compact on \( L^1(x, +\infty) \). We notice that (20.14) is a homogeneous Volterra integral equation. Thus, it is uniquely solvable and its solution is given by \( P(x, y) = 0 \) for \( 0 < x < y \). Then, from (20.11) we see that \( K(x, y) \) must satisfy the Marchenko equation (13.1). Since (I) of Definition 4.3 holds, from Proposition 16.1(a) it follows that (13.1) must have a unique solution, and hence we know that \( K(x, y) \) must be the unique solution to (13.1). Thus, the proof is completed. \( \blacksquare \)
21. ALTERNATE CHARACTERIZATION VIA LEVINSON’S THEOREM

In this chapter we give a new characterization where we use Levinson’s theorem. Let us first recall this theorem.

For \( \theta_j \in (0, \pi] \) with \( j = 1, 2, \ldots, n \), let us denote

\[
\tilde{A} := -\text{diag}\{ \sin \theta_1, \ldots, \sin \theta_n \}, \quad \tilde{B} := \{ \cos \theta_1, \ldots, \cos \theta_n \}.
\]  

(21.1)

The matrix pair \((\tilde{A}, \tilde{B})\) satisfy (2.5) and (2.6), and the boundary conditions (2.4) for these matrices is given by

\[
(\cos \theta_j) \psi_j(0) + (\sin \theta_j) \psi'_j(0) = 0, \quad j = 1, \ldots, n.
\]  

(21.2)

The particular case \( \theta_j = \pi \) corresponds to the Dirichlet boundary condition, and the case \( \theta_j = \pi/2 \) corresponds to the Neumann boundary condition. In general there will be \( n_D \) values with \( \theta_j = \pi \), \( n_N \) values with \( \theta_j = \pi/2 \), and then, \( n_M = n - n_D - n_N \) values with \( \theta_j \in (0, \pi/2) \cup (\pi/2, \pi) \), that correspond to mixed boundary conditions.

Let \((A, B)\) be any matrix pair satisfying (2.5) and (2.6). It is proven in Proposition 4.3 of [9] that there exists a matrix pair \((\tilde{A}, \tilde{B})\) as in (21.1), a unitary matrix \(U\), and invertible matrices \(T_1\) and \(T_2\) such that

\[
A = U \tilde{A} T_1 U^\dagger T_2, \quad B = U \tilde{A} T_1 U^\dagger T_2.
\]  

(21.3)

Furthermore, in Proposition 4.1 of [9] it is proven that under the transformation \((A, B) \mapsto (\tilde{A}, \tilde{B})\) with \((\tilde{A}, \tilde{B})\) as in (21.1) and \(U, T_1,\) and \(T_2\) as in (21.3), we have the Jost matrix and the scattering matrix transforming, respectively, as

\[
J(k) = U \tilde{J}(k) T_1 U^\dagger T_2, \quad S(k) = U \tilde{S}(k) U^\dagger,
\]  

(21.4)

where \(\tilde{J}(k)\) and \(\tilde{S}(k)\) denote the Jost matrix and the scattering matrix, respectively, for the potential \(\tilde{V}(x) := U^\dagger V(x) U\) and the boundary conditions (2.4) with the matrix pair \((\tilde{A}, \tilde{B})\) instead of the pair \((A, B)\). Note that \(\tilde{V}(x)\) satisfies (2.2) and (2.3) because \(U\) is unitary.
and $V(x)$ satisfies (2.2) and (2.3). This means that our problem with the general boundary condition specified by (2.4)-(2.6) with the matrix pair $(A, B)$ is actually equivalent to a problem with the boundary conditions (21.2) given by the diagonal matrix pair $(\tilde{A}, \tilde{B})$ and the potential $\tilde{V}(x) := U^\dagger V(x) U$. Note that the transformation $(A, B) \mapsto (AT, BT)$ with an invertible matrix $T$ is just a reparametrization of the boundary condition and the transformation $V(x) \mapsto UV(x) U^\dagger$ with a unitary matrix $U$ is a change of representation in the quantum mechanical sense.

We present Levinson’s theorem next.

**Theorem 21.1** Suppose that the input data set $D := \{V, A, B\}$ belongs to the Faddeev class. Then, the number $N$ of bound states (including multiplicities) is related to the argument of the determinant of the scattering matrix as

$$\arg[\det S(0^+)] - \arg[\det S(+\infty)] = \pi [2N + \mu - (n - n_D)],$$

(21.5)

where $\mu$ is the algebraic (and geometric) multiplicity of the eigenvalue $+1$ of the zero-energy scattering matrix $S(0)$, and $n_D$ is the number defined after (21.2), namely it is the number of Dirichlet boundary conditions in the representation where the boundary conditions are given as in (21.2) by matrices $\tilde{A}, \tilde{B}$ that satisfy (21.1) and (21.3). The quantity $n_D$ is also equal to the algebraic (and geometric) multiplicity of the eigenvalue $-1$ of the hermitian matrix $S_\infty$ defined in (4.6). Furthermore, the number $N$ of bound states (including multiplicities) is equal to the sum of the ranks $m_j$ of the matrices $M_j$ for $j = 1, \ldots, N$ that appear in the definition (4.2) of the unique data set $S$ that corresponds to the input data set $D$, according to Theorem 5.1, $N = \sum_{j=1}^N m_j$.

**PROOF:** This result is proven in Theorem 9.3 of [9].

Levinson’s theorem is a remarkable result, linking scattering information encoded in the scattering matrix to bound state information.

Next we show that the three properties $(4c,2)$, $(4d,2)$, and $(4e,2)$ of Definition 7.8 are equivalent.
Proposition 21.2 Consider a scattering data set $\mathbf{S}$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ nonnegative, hermitian matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $\mathbf{S}$ satisfies (I) of Definition 4.3. Then, the three conditions $(4_{c,2})$, $(4_{d,2})$, and $(4_{e,2})$ of Definition 7.8 are equivalent.

PROOF: The proposition is proved the same way as in (c) and (d) of Proposition 15.3. Note that the proof of Proposition 15.3(d) amounts to reduce the problem to the case of $L^2(\mathbb{R}^+)$ and of $H^2(\mathbb{C}^+)$, which we consider here. ■

Our main goal in this chapter is to prove the characterization results stated in Theorems 7.9 and 7.10. The strategy of the proof of Theorem 7.9 is to prove that all the conditions of Theorem 7.6 are satisfied. Then, the proof of Theorem 7.10 would follow as a result of the equivalence indicated in Proposition 21.2.

Before we prove Theorem 7.9 we first obtain certain preliminary results that we need. As in Lemma 1 on p. 265 of [2], the Corollary and Lemma 2 on page 268 of [2], we prove that if conditions (1) and (2) hold then there is a nonnegative integer $q$ such that the matrix $S_q(k)$ defined by

$$S_q(k) := S(k) \left( \frac{k+i}{k-i} \right)^{2q}, \quad (21.6)$$

satisfies the conditions (1), (2), and (III.a) of Proposition 19.2. Then, this proposition implies that there is a set of $N_q$ distinct positive constants $\kappa_{q,j}$ and a set of $N_q$ nonnegative, hermitian $n \times n$ matrices $M_{q,j}$ with respective positive ranks $m_{q,j}$, where $N_q$ is a nonnegative integer, such that $\mathbf{S}_q = \{S_q(k), \{\kappa_{q,j}, M_{q,j}\}_{j=1}^{N_q}\}$ is the scattering data set of a unique input data set $\mathcal{D} := \{V_q, A_q, B_q\}$ in the Faddeev class.

Remark 21.3 Note that by (21.6) $S_q(0) = S(0)$. Then, the algebraic (and geometric) multiplicity of the eigenvalue $+1$ of $S_q(0)$ is the number $\mu$ that appears in condition (L) in Definition 7.7. Furthermore, by (21.6),

$$S_{q,\infty} := \lim_{k \to \infty} S_q(k) = \lim_{k \to \infty} S(k) = S_\infty. \quad (21.7)$$
Then, the algebraic (and geometric) multiplicity of the eigenvalue $-1$ of the hermitian matrix $S_{q,\infty}$ is the number $n_D$ that appears in condition (L) in Definition 7.7.

Let us denote by $J_q(k)$ the Jost matrix for $S_q$. We have

$$S_q(k) = -J_q(-k) [J_q(k)]^{-1}, \quad k \in \mathbb{R}. \quad (21.8)$$

By (21.6) and (21.8), we have

$$S(k) = -\left(\frac{k - i}{k + i}\right)^{2q} J_q(-k) [J_q(k)]^{-1}, \quad k \in \mathbb{R}. \quad (21.9)$$

**Definition 21.4** Let $h(k)$ be a column vector with $n$ components, that is defined for $k \in \mathbb{C}^+$ in such a way that it is analytic in $k \in \mathbb{C}^+$. We say that $h(k)$ is of finite order if for some real number $\nu$ and some constant $C_\nu$ we have

$$|h(k)| \leq C_\nu (1 + |k|^\nu), \quad k \in \mathbb{C}^+. \quad (21.10)$$

The order of $h(k)$ is the infimum of the $\nu$-values such that (21.10) holds for some constant $C_\nu$. The definition for order also applies for functions defined for $k \in \mathbb{C}^-$, analytic in $\mathbb{C}^-$, and (21.10) holds in $k \in \mathbb{C}^-$.

We now consider the solutions to the following equation that appears in condition $(V_h)$ in Theorem 7.6:

$$h(-k) + S(k) h(k) = 0, \quad k \in \mathbb{R}, \quad (21.11)$$

where $h(k)$ is a column vector that is analytic in $\mathbb{C}^+$ and of finite order.

It follows from Proposition 10.2(a) that $k [J_q(k)]^{-1}$ is continuous in $k \in \mathbb{R}$ and

$$J_q(k) = B_q - i k A_q + O(1), \quad k \to \infty \text{ in } \mathbb{C}^+. \quad (21.12)$$

Then, using (21.9) and (21.12) we prove as in Lemma 3 in page 270 of [2] that every solution to (21.11) with finite order is of the form

$$h(k) = (k + i)^{2q} J_q(k) \left[ \sum_{j=1}^{N_q} \frac{2 i \kappa_{q,j}}{k^2 + \kappa_{q,j}^2} N_{q,j} d^{(j)} + p(k^2) \right], \quad (21.13)$$
where $i\kappa_{q,j}$ with $j = 1, \ldots, N_q$ are the poles of $[J_q(k)]^{-1}$ and $N_{q,j}$ is the residue of $[J_q(k)]^{-1}$ at $i\kappa_{q,j}$. Moreover, $p(k)$ is a column vector with $n$ components that are polynomials in $k$, and $d^{(j)}$ is the column vector with $n$ components given by

$$d^{(j)} := [(k + i)^{-2q} h_1(i\kappa_{q,j}) \cdots (k + i)^{-2q} h_n(i\kappa_{q,j})]^T$$

where we recall that the superscript $T$ denotes the matrix transpose. By (21.12) and (21.13), each component $h_l(k)$ with $l = 1, \ldots, n$, of a solution of finite order behaves as $C_l k^{\beta_l}$ as $|k| \to \infty$, where $C_l$ is a complex number different from zero, and $\beta_l$ is an integer. The order, $\beta$, of the solution $h(k)$ is equal to $\beta := \max\{\beta_l : l = 1, \ldots, n\}$. Note that from (21.13) it follows that $\beta$ can be an even or an odd integer.

We have the following result.

**Proposition 21.5** Suppose that conditions (1) and (2) of Definition 4.2 are satisfied. Then, there are $n$ solutions $h^{(j)}(k)$ of finite order for $j = 1, \ldots, n$ to (21.11) with the following properties:

(a) Each $h^{(j)}(k)$ is of order $-\beta_j$, with $-\beta_1 \geq -\beta_2 \geq \cdots \geq -\beta_n$. Moreover, $-\beta_1$ is the smallest possible order of all solutions to (21.11) with finite order. Note that from (21.13) it follows that $\beta_1$ is finite.

(b) For every $j = 1, \ldots, n$, the solutions $h^{(j)}(k)$ cannot be represented in the form

$$h^{(j)}(k) = \sum_{l=1}^{j-1} p_l(k) h^{(l)}(k),$$

where the $p_l(k)$ are polynomials.

(c) All the solutions $h(k)$ to (21.11) with finite order can be represented as

$$h(k) = \sum_{j=1}^{n} p_j(k^2) h^{(j)}(k),$$

for some polynomials $p_j$ with $j = 1, \ldots, n$. Furthermore, any solution $h(k)$ to (21.11) of order smaller than $-\beta_1$ can be represented as

$$h(k) = \sum_{j=1}^{l-1} p_j(k^2) h^{(j)}(k),$$

for some $l \geq 2$. Note that the constants $C_l$ may vanish for $l > 1$, and $\beta_l$ is an integer.

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for some polynomials \( p_j(k) \) with \( j = 1, \ldots, l-1 \).

**PROOF:** This proposition is proven as in Appendix I of [2] (see also pages 393–404 of [39]).  

By (21.13), each of the \( h^{(j)}(k) \) is of the form,

\[
  h^{(j)}(k) = J_q(k) r^{(j)}(k), \quad j = 1, \ldots, n,
\]

where \( r^{(j)}(k) \) is a column vector with \( n \) components that are rational functions of \( k \). Let

\[
  \omega^{(j)}(k) := k^{\beta_j} h^{(j)}(k), \quad j = 1, \ldots, n.
\]

Moreover, we define following \( n \times n \) matrices:

\[
  Z(k) := [z_{i,j}(k)], \quad z_{i,j}(k) := h_{i}^{(j)}(k), \quad 1 \leq i, j \leq n, \quad (21.20)
\]

\[
  \Omega(k) := [\omega_{i,j}(k)], \quad \omega_{i,j}(k) := \omega_{i}^{(j)}(k), \quad 1 \leq i, j \leq n, \quad (21.21)
\]

\[
  R(k) := [r_{i,j}(k)], \quad r_{i,j}(k) := r_{i}^{(j)}(k), \quad 1 \leq i, j \leq n.
\]

Then,

\[
  Z(-k) + S(k) Z(k) = 0, \quad k \in \mathbb{R}, \quad (21.23)
\]

and

\[
  Z(k) = J_q(k) R(k), \quad k \in \mathbb{C}^+. \quad (21.24)
\]

**Proposition 21.6** Suppose that conditions (1) and (2) of Definition 4.2 are satisfied. Then, the following statements hold:

(a) The determinant of \( Z(k) \) is different from zero for all \( k \in \mathbb{C}^+ \setminus \{0\} \).

(b) The following two limits exist:

\[
  \omega^{(j)}(\infty) := \lim_{k \to \infty} \omega^{(j)}(k), \quad (21.25)
\]

\[
  [\Omega(\infty)] = \lim_{k \to \infty} [\Omega(k)], \quad (21.26)
\]
and the determinant of $\Omega(\infty)$ is different from zero.

(c) The order of

$$\sum_{j=1}^{n} p_j(k) h^{(j)}(k),$$

(21.27)

where the $p_j(k)$ for $j = 1, \ldots, n$ are polynomials, is equal to the highest of the orders of the individual terms in the summation in (21.27).

(d) The determinant of $R(k)$ is different from zero for all $k \in \mathbb{C}^+$. 

PROOF: This proposition is proven as in Appendix I of [2] (see also pages 393–404 of [39]). We give some details for the reader’s convenience. Let us prove that (d) holds. Suppose that for some $k_0 \in \mathbb{C}^+$ the determinant of $R(k_0)$ is zero. Then, the columns of $R(k_0)$ are linearly dependent and there are some $\lambda_j$ with $j = 1, \ldots, n$ that are not all equal to zero, such that

$$\sum_{j=1}^{n} \lambda_j r^{(j)}(k_0) = 0.$$ 

(21.28)

Let

$$h(k) := \frac{1}{k^2 - k_0^2} J_q(k) \sum_{j=1}^{n} \lambda_j r^{(j)}(k) = \frac{1}{k^2 - k_0^2} \sum_{j=1}^{n} \lambda_j h^{(j)}(k).$$

(21.29)

By (21.13) and (21.18) we have

$$h(k) = \frac{1}{k^2 - k_0^2} (k + i)^2 q J_q(k) \left( (k^2 - k_0^2)Q + O((k^2 - k_0^2)^2) \right), \quad k \to k_0,$$

(21.30)

for some constant column vector $Q$. Then, $h(k)$ is a solution to (21.11). Let $\lambda_m$ be the last of the coefficients in (21.29) that is different from zero. Hence, $h(k)$ is a solution to (21.11) of order $(-\beta_m - 2)$, and by Proposition 21.5(c) it can be represented as

$$h(k) = \sum_{j=1}^{m-1} p_j(k^2) h^{(j)}(k).$$

(21.31)

By (21.29) and (21.31) we have

$$h^{(m)}(k) = \frac{1}{\lambda_m} \left( \sum_{j=1}^{m-1} (k^2 - k_0^2)p_j(k^2) h^{(j)}(k) - \sum_{j=1}^{m-1} \lambda_j h^{(j)}(k) \right).$$

(21.32)
However, by Proposition 21.5(b) this is not possible, and then the determinant of $R(k)$ never vanishes. We prove in the same way that the determinant of $Z(k)$ does not vanish for $k \in \mathbb{C}^+$ using that $Z(k)$ is analytic on $\mathbb{C}^+$. Moreover, by (21.24) the determinant of $Z(k)$ is different from zero on $\mathbb{R} \setminus \{0\}$, because the determinant of $R(k)$ and of $J_q(k)$ are nonzero on $\mathbb{R} \setminus \{0\}$.

**Remark 21.7** Let $\tilde{A}_q$ and $\tilde{B}_q$ be the matrices as in (21.1) related to $A_q$ and $B_q$ as in (21.3), i.e.

$$
\tilde{A}_q := -\text{diag}\{\sin \theta_{q,1}, \ldots, \sin \theta_{q,n}\}, \quad \tilde{B}_q := \text{diag}\{\cos \theta_{q,1}, \ldots, \cos \theta_{q,n}\},
$$

$$0 < \theta_{q,j} \leq \pi, \quad j = 1, \ldots, n, \quad (21.33)
$$

where $n_D$ of the $\theta_{q,j}$ are equal to $\pi$ and the $(n - n_D)$ remaining $\theta_{q,j}$ are different from $\pi$. We can always reorder the $\theta_{q,j}$ in such a way that the first $n_D$ of the $\theta_{q,j}$ are equal to $\pi$. By Theorem 7.6 of [9] the number of $\theta_{q,j} = \pi$ that appear in (21.33) is equal to the algebraic (and geometric) multiplicity of the eigenvalue equal to $-1$ of the matrix $S_{q,\infty}$, where $S_q(k)$ is the matrix defined in (21.8). By Remark 21.7 this is precisely the number, $n_D$, of eigenvalues $-1$ of the matrix $S_{\infty}$ that appears in condition (L) of Definition 7.7.

**Proposition 21.8** Suppose that conditions (1) and (2) of Definition 4.2 are satisfied. Then, among the solutions $h^{(j)}(k)$ with $j = 1, \ldots, n$, we have $n_D$ of them having even order and $n - n_D$ of them having odd order.

**PROOF:** Let $\tilde{A}_q$ and $\tilde{B}_q$ be the matrices that appear in Remark 21.7. Let $P_1$ and $P_2$ be the diagonal matrices defined as

$$
P_1 := \text{diag}\{1, \cdots, 1, 0, \cdots, 0\}, \quad (21.34)
$$

with the first $n_D$ diagonal entries equal to one and the remaining $(n - n_D)$ diagonal entries equal to zero, and

$$
P_2 := \text{diag}\{0, \cdots, 0, 1, \cdots, 1\}, \quad (21.35)
$$

with the first $n_D$ diagonal entries equal to zero and the remaining $n - n_D$ diagonal entries...
equal to one. Then,

\[ \tilde{B}_q - ik\tilde{A}_q = \text{diag} \{ -1, -1, \cdots, -1, \cos \theta_{q,n_{D}+1} + ik \sin \theta_{q,n_{D}+2}, \cdots, \cos \theta_{q,n} + ik \sin \theta_{q,n} \}, \]

with \(-1\) in the first \(n_D\) diagonal entries and the remaining ones different from zero. Hence,

\[ P_1(\tilde{B}_q - ik\tilde{A}_q) = \text{diag} \{ -1, \cdots, -1, 0, \cdots, 0 \}, \]

(21.37)

has \(-1\) in the first \(n_D\) diagonal entries and 0 in the remaining \(n - n_D\) diagonal entries. Furthermore,

\[ P_2(\tilde{B}_q - ik\tilde{A}_q) = \text{diag} \{ 0, \cdots, 0, \cos \theta_{q,n_{D}+1} + ik \sin \theta_{q,n_{D}+1}, \cdots, \cos \theta_{q,n} + ik \sin \theta_{q,n} \}, \]

(21.38)

has zeros in the first \(n_D\) diagonal entries and the remaining \(n - n_D\) diagonal entries are nonzero. By (21.13) we have

\[ j^\beta k^{(j)}(k) = k^{\beta_j + 2q} \left( 1 + \frac{i}{k} \right)^{2q} J_q(k) c^{(j)}(k^2), \]

(21.39)

for some column vector \(c^{(j)}(k)\) with components that are rational functions. By (21.3) for \(A_q, B_q\) and \(\tilde{A}_q, \tilde{B}_q\), and (21.12), we have

\[ U^\dagger J_q(k) = \left( \tilde{B}_q - ik\tilde{A}_q + O(1) \right) T_1 U^\dagger T_2, \quad k \to \infty \text{ in } \mathbb{C}^+. \]

(21.40)

Hence, by (21.19), (21.25), and (21.37)-(21.40) we get

\[ P_1 U^\dagger \omega^{(j)}(\infty) = \lim_{k \to \infty} \left[ \Theta_1 + O(1) \right] T_1 U^\dagger T_2 k^{\beta_j + 2q} \left( 1 + \frac{i}{k} \right)^{2q} c^{(j)}(k^2), \]

(21.41)

and,

\[ P_2 U^\dagger \omega^{(j)}(\infty) = \lim_{k \to \infty} \left[ \Theta(k) + O(1) \right] T_1 U^\dagger T_2 k^{\beta_j + 2q} \left( 1 + \frac{i}{k} \right)^{2q} c^{(j)}(k^2) \]

(21.42)

where

\[ \Theta_1 := \text{diag} \{ -1, \cdots, -1, 0, \cdots, 0 \}, \]
Θ(k) := diag \{0,\ldots,0,\cos \theta_{q,nD+1} + ik \sin \theta_{q,nD+1},\ldots,\cos \theta_{q,n} + ik \sin \theta_{q,n}\}.

Suppose that the order of \(h^{(j)}(k)\) is even. Then, the only way in which the limit on the right-hand side of (21.42) can be finite is if that limit is zero. It follows that

\[ P_2 U^\dagger \omega^{(j)}(\infty) = 0, \quad \text{if } \beta_j \text{ is even.} \quad (21.43) \]

Similarly if the order of \(h^{(j)}(k)\) is odd, then the only way in which the limit on the right-hand side of (21.41) can be finite is if it is zero. As a result, we have

\[ P_1 U^\dagger \omega^{(j)}(\infty) = 0, \quad \text{if } \beta_j \text{ is odd.} \quad (21.44) \]

Let us introduce the following subspaces of \(\mathbb{C}^n\):

\[ V_{\text{even}} := \left\{ v \in \mathbb{C}^n : v = \sum_{(j: \beta_j \text{ is even})} \lambda_j U^\dagger \omega^{(j)}(\infty), \quad \lambda_j \in \mathbb{C} \right\}, \quad (21.45) \]

\[ V_{\text{odd}} := \left\{ v \in \mathbb{C}^n : v = \sum_{(j: \beta_j \text{ is odd})} \lambda_j U^\dagger \omega^{(j)}(\infty), \quad \lambda_j \in \mathbb{C} \right\}. \quad (21.46) \]

The subspace \(V_{\text{even}}\) is the subspace of all linear combinations of the \(U^\dagger \omega^{(j)}(\infty)\) with \(h^{(j)}(k)\) of even order, and \(V_{\text{odd}}\) is the subspace of all linear combinations of the \(U^\dagger \omega^{(j)}(\infty)\) with \(h^{(j)}(k)\) of odd order. Let us denote by \(n_{\text{even}}\), the number of \(h^{(j)}(k)\) of even order and by \(n_{\text{odd}}\), the number of \(h^{(j)}(k)\) of odd order. As by Proposition 21.6 the determinant of \(\Omega(\infty)\) is different from zero, the \(\omega^{(j)}(\infty)\) for \(j = 1,\ldots,n\) are linearly independent. Then, from the unitarity of \(U\), (21.43), and (21.44) we get

\[ V_{\text{even}} \subset P_1 \mathbb{C}^n, \quad \dim [V_{\text{even}}] = n_{\text{even}} \leq \dim [P_1 \mathbb{C}^n] = n_D, \quad (21.47) \]

\[ V_{\text{odd}} \subset P_2 \mathbb{C}^n, \quad \dim [V_{\text{odd}}] = n_{\text{odd}} \leq \dim [P_2 \mathbb{C}^n] = n - n_D. \quad (21.48) \]

Then, \(V_{\text{even}} \cap V_{\text{odd}} = \{0\}\), and consequently we have

\[ \dim [V_{\text{even}} + V_{\text{odd}}] = \dim [V_{\text{even}}] + \dim [V_{\text{odd}}] = n_{\text{even}} + n_{\text{odd}}. \quad (21.49) \]
Suppose that \( n_{\text{even}} < n_D \). Then, by (21.47) and (21.48) we have
\[
\dim [V_{\text{even}} + V_{\text{odd}}] < n_D + n - n_D < n. \tag{21.50}
\]
This implies that the vectors \( U^\dagger \omega^{(j)}(\infty) \) with \( j = 1, \ldots, n \) generate a subspace of \( \mathbb{C}^n \) of dimension smaller than \( n \). This is impossible because as the \( \omega^{(j)}(\infty) \) with \( j = 1, \ldots, n \) are linearly independent, and \( U \) is unitary, the vectors \( U^\dagger \omega^{(j)}(\infty) \) with \( j = 1, \ldots, n \) are linearly independent. Consequently, we have \( n_{\text{even}} = n_D \) and \( n_{\text{odd}} = n - n_D \).

We can now compute the number of linearly independent solutions to (21.11) that have negative order. Let us assume that there are \( d \geq 0 \) orders \( \beta_j \) that are positive, that is to say
\[
\beta_1 \geq \beta_2 \geq \cdots \geq \beta_d > 0 \geq \beta_{d+1} \geq \cdots \geq \beta_n. \tag{21.51}
\]

**Proposition 21.9** Suppose that conditions (1) and (2) of Definition 4.2 are satisfied. Denote by \( n_{\{\text{odd}, +\}} \) the number of the \( \beta_j \)-values for \( j = 1, \ldots, d \), that are odd. Then, the number, \( N_+ \), of linearly independent solutions to (21.11) that are of negative order satisfies
\[
2N_+ = \sum_{j=1}^{d} \beta_j + n_{\{\text{odd}, +\}}. \tag{21.52}
\]

**PROOF:** It follows from Proposition 21.5(c) and Proposition 21.6(c) that all solutions to (21.11) of negative order, i.e. of order not exceeding \(-1\), are of the form
\[
h(k) = \sum_{j=1}^{d} p_j(k^2) h^{(j)}(k), \tag{21.53}
\]
where \( p_j(k) \) is a polynomial of degree \( \gamma_j \) so that \( 2\gamma_j < \beta_j \). If \( \beta_j \) is even this implies that \( \gamma_j \leq \frac{\beta_j}{2} - 1 \). Since a polynomial of order \( \frac{\beta_j}{2} - 1 \) has \( \frac{\beta_j}{2} \) independent coefficients, for each \( h^{(j)}(k) \) of even order, \( \beta_j \), there are \( \frac{\beta_j}{2} \) linearly independent solutions to (21.11) of negative order. If \( \beta_j \) is odd, the inequality \( 2\gamma_j < \beta_j \) implies that \( \gamma_j \leq \frac{\beta_j}{2} - \frac{1}{2} \). Then, for each \( h^{(j)}(k) \) of odd order \( \beta_j \), there are \( \frac{\beta_j}{2} + \frac{1}{2} \) linearly independent solutions to (21.11) of negative order. Then, the number, \( N_+ \), of linearly independent solutions of negative order is given by
\[
\sum_{(j: 1 \leq j \leq d, \ \beta_j \ is \ even)} \frac{\beta_j}{2} + \sum_{(j: 1 \leq j \leq d, \ \beta_j \ is \ odd)} \left( \frac{\beta_j}{2} + \frac{1}{2} \right) = \sum_{j=d+1}^{n} \frac{\beta_j}{2} + \frac{n_{\{\text{odd}, +\}}}{2}. \tag{21.54}
\]
Then, from (21.54) we obtain (21.52). \[\square\]

We now consider the equation that appears in condition (\(III_c\)) in Theorem 7.6, i.e. consider

\[-h(-k) + S(k) h(k) = 0, \quad k \in \mathbb{R}, \quad (21.55)\]

where \(h(k)\) is a column vector with \(n\) components in \(H^2(C^-)\). Taking the adjoint of (21.23), then taking the inverse, using \(S(k)^\dagger = S(k)^{-1}\) which appears in (4.4), and multiplying by \(k\) we obtain

\[k \left[ Z(-k)^\dagger \right]^{-1} + k S(k) \left[ Z(k)^\dagger \right]^{-1} = 0, \quad k \in \mathbb{R} \setminus \{0\}. \quad (21.56)\]

Let us define

\[Y(k) := k \left[ Z(k^*)^\dagger \right]^{-1}, \quad k \in \mathbb{C}^{-} \setminus \{0\}. \quad (21.57)\]

Then, by (21.56) we have

\[-Y(-k) + S(k) Y(k) S(k) = 0, \quad k \in \mathbb{R} \setminus \{0\}. \quad (21.58)\]

Letting

\[Y(k) := [y_{i,j}], \quad y_{i,j}(k) := y_{i}^{(j)}(k), \quad 1 \leq i, j \leq n, \quad (21.59)\]

we see that the columns \(y_{i}^{(j)}(k)\) of \(Y(k)\) are solutions of (21.55) analytic for \(k \in \mathbb{C}^-\). Recall that Proposition 10.2(a) indicates that \(kJ_q(k)^{-1}\) is continuous for \(k \in \mathbb{R}\). Moreover, from Proposition 21.6(d) and (21.24) we conclude that \(Y(k)\) is continuous at \(k = 0\). We prove as in Appendix I of [2] that \(y_{i}^{(j)}\) are solutions of order equal to \(\beta_{j} + 1\) with \(j = 1, \ldots, n\) that \(Y(k)\) satisfies the analogs of (a) and (b) of Proposition 21.5, and that any solution to (21.55) of finite order can be represented as

\[h(k) = \sum_{j=1}^{n} p_j(k^2) y^{(j)}(k), \quad (21.60)\]

for some polynomials \(p_j\) with \(j = 1, \ldots, n\), and the order of

\[\sum_{j=1}^{n} p_j(k) y^{(j)}(k), \quad (21.61)\]

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where the \( p_j(k) \) for \( j = 1, \ldots, n \) are polynomials, is equal to the highest of the orders of the individual terms in the sum in (21.61).

**Proposition 21.10** Suppose that conditions (1) and (2) of Definition 4.2 are satisfied. Denote by \( n_{\{\text{odd}, -}\} \) the number of the \( \beta_j \) with \( j = d + 1, \ldots, n \) that are odd. Then, the number, \( N_- \), of linearly independent solutions to (21.56) that are of negative order satisfies

\[
2N_- = \sum_{j=d+1}^{d} |\beta_j| - n_{\{\text{odd}, -\}}.
\]

(21.62)

**PROOF:** Since any solution to (21.55) of finite order can be represented as (21.60), and since the order of (21.61) is equal to the highest of the orders of the individual terms in the sum, it follows that any solution to (21.55) of negative order is of the form

\[
h(k) = \sum_{j=d+1}^{n} p_j(k^2) y^{(j)}(k),
\]

(21.63)

where \( p_j(k) \) is a polynomial of degree, \( \gamma_j \), so that, \( 2\gamma_j + \beta_j + 1 < 0 \). If \( \beta_j \) is even this implies that \( \gamma_j \leq |\beta_j|/2 - 1 \). Since a polynomial of order \( |\beta_j|/2 - 1 \) has \( |\beta_j|/2 \) independent coefficients, for each \( y^{(j)}(k) \) of even order, \( \beta_j \), there are \( |\beta_j|/2 \) linearly independent solutions to (21.55) of negative order. If \( \beta_j \) is odd, \( 2\gamma_j + \beta_j + 1 < 0 \) implies that \( \gamma_j < |\beta_j|/2 - 1/2 - 1 \). Then, for each \( h^{(j)}(k) \) of odd order, \( \beta_j \), there are \( |\beta_j|/2 - 1/2 \) linearly independent solutions to (21.55) of negative order. Hence, the number, \( N_- \), of linearly independent solutions of negative order is given by,

\[
\sum_{(j: \ d+1 \leq j \leq n, \ \beta_j \ is \ even)} \frac{|\beta_j|}{2} + \sum_{(j: \ d+1 \leq j \leq n, \ \beta_j \ is \ odd)} \left( \frac{|\beta_j|}{2} - \frac{1}{2} \right) = \sum_{j=1}^{d} \frac{|\beta_j|}{2} - \frac{n_{\{\text{odd}, -\}}}{2}.
\]

(21.64)

Equation (21.62) follows from (21.64). \( \blacksquare \)

Let us define the total index, \( \beta \), of \( S(k) \) as

\[
\beta := \sum_{j=1}^{n} \beta_j.
\]

(21.65)

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By Propositions 21.8, 21.9, and 21.10 and since \( n_{\{\text{odd, +}\}} + n_{\{\text{odd, -}\}} = n - n_D \), we have

\[
\beta = 2N_+ - 2N_- - (n - n_D). \tag{21.66}
\]

With the help of (21.19)-(21.21), (21.65), and Proposition 21.6(b), we obtain

\[
\det[Z(k)] = C k^{-\beta} [1 + o(1)], \quad k \to \infty \text{ in } \mathbb{C}^+.
\tag{21.67}
\]

**Proposition 21.11** Suppose that conditions (1) and (2) of Definition 4.2 are satisfied. Then:

(a) Every column vector \( h(k) \) that is a solution to (21.11) in \( \mathbf{H}^2(\mathbb{C}^+) \) is of negative order.

(b) Every column vector \( h(k) \) that is a solution to (21.55) in \( \mathbf{H}^2(\mathbb{C}^-) \) is of negative order.

**PROOF:** Suppose that \( h(k) \) is a solution to (21.11) in \( \mathbf{H}^2(\mathbb{C}^+) \). Let \( \hat{X}(k) := h(-k^*)^\dagger \).

Then \( \hat{X}(k) \) is a solution to (4.23) in \( \mathbf{H}^2(\mathbb{C}^+) \). Then, by (21.9) we have

\[
\hat{X}(-k) = \left( \frac{k - i}{k + i} \right)^{2q} \hat{X}(k) J_q(-k) J_q^{-1}(k), \quad k \in \mathbb{R},
\tag{21.68}
\]

and as in the proof of (15.25) we obtain

\[
(k - i)^{2q} \hat{X}(k) [J(-k^*)]^\dagger - 1 = (k + i)^{2q} \hat{X}(-k) [J(k^*)]^\dagger - 1, \quad k \in \mathbb{R} \setminus \{0\}.
\tag{21.69}
\]

As in (15.27) we let

\[
\Xi(k) := \begin{cases} 
(k - i)^{2q} \left( k \hat{X}(k) [J(-k^*)]^\dagger - 1 + k \sum_{j=1}^{N_q} \frac{2i\kappa_{q,j} \hat{X}(i\kappa_{q,j}) N_{q,j}^\dagger}{k^2 + \kappa_{q,j}^2} \right), & k \in \mathbb{C}^+, \\
(k + i)^{2q} \left( k \hat{X}(-k) [J(k^*)]^\dagger - 1 + k \sum_{j=1}^{N_q} \frac{2i\kappa_{q,j} \hat{X}(i\kappa_{q,j}) N_{q,j}^\dagger}{k^2 + \kappa_{q,j}^2} \right), & k \in \mathbb{C}^-.
\end{cases}
\tag{21.70}
\]

As in the proof of Proposition 15.8 we prove that \( \Xi(k) \) is an entire odd function and that \( \Xi^{(p)}(0) = 0 \) for \( p \) odd, \( p \geq 2q - 1 \). Then, \( \Xi(k) \) is a polynomial order smaller or equal to \( 2q - 1 \). Hence, \( \hat{X}(k) \) is of finite order, but since it is in \( \mathbf{H}^2(\mathbb{C}^+) \) the order has to be
negative. In consequence $\hat{X}(k)$ is of negative order, which proves that $h(k)$ is of negative order. Let us now prove \( b \). Suppose that $h(k)$ is a solution to (21.55) in $H^2(C^-)$. Then, by (21.6) and since $S_q(k) = S_q(-k)^\dagger$, which follows from (4.4), we have

\[-(k - i)^{2q} J_q(-k)^\dagger h(-k) = -(k - i)^{2q} J_q(k)^\dagger h(k), \quad k \in \mathbb{R}. \quad (21.71)\]

Let us define the sectionally analytic function

\[g(k) := \begin{cases} 
-(k + i)^{2q} J_q(k)^\dagger h(k), & k \in \mathbb{C}^+, \\
-(k - i)^{2q} J_q(k^*)^\dagger h(k), & k \in \mathbb{C}^-.
\] \quad (21.72)\]

As in the proof of Proposition 15.8 we prove that $g(k)$ is an odd entire function and that $g^{(p)}(0) = 0$ for $p$ odd, $p \geq 2q - 1$. Then, $g(k)$ is a polynomial order smaller or equal to $2q - 1$. Hence, $h(k)$ is of finite order, but since it is in $H^2(C^-)$ that order has to be negative. \qed

Next, we provide a proof of the characterization stated in Theorem 7.9.

**Proof of Theorem 7.9:** If the input data set $D := \{V, A, B\}$ belongs to the Faddeev class it is proven in Theorem 7.6 that conditions of (1) and (2) hold. The continuity of $S(k)$ in $k \in \mathbb{R}$ is proven in Proposition 10.3(a). The proof of Levinson’s theorem (21.5) is given in Theorem 9.3 of [9]. It remains to prove that (4e,2) holds. By Proposition 21.2 we can, equivalently, prove that (4d,2) is satisfied. By Theorem 5.1 we know that the conditions of Proposition 15.4 hold. As in the proof of Proposition 15.4(a) we prove that (15.25) holds and we define $\Xi(k)$ as in (15.27). Then, as in Proposition 15.8 we prove that $\Xi(k) \equiv 0$. We complete the proof that $\hat{X}(k) \equiv 0$ as in the proof of Proposition 15.4(a). We now prove that if conditions of (1), (2), (4e,2), and (L) are satisfied, then $\mathbf{S}$ is the scattering data of a unique input data $D := \{V, A, B\}$ in the Faddeev class. For this purpose we verify that all the conditions of Theorem 7.6 are satisfied. Conditions (1) and (2) hold by assumption. By Corollary 6.2 of [9], we have

\[
\det[J_q(k)] = C k^\mu (1 + o(1)), \quad k \to 0. \quad (21.73)
\]
By Proposition 21.6(d), (21.23), (21.24), (21.67), (21.73), and using contour integration we prove, as in Appendix I of [2], that

$$\arg[\det[S(0^+)]] - \arg[\det[S(\infty)]] = \pi (\beta + \mu).$$  \hfill (21.74)

Furthermore, since by assumption (L) Levinson’s theorem (21.5) holds, and using (21.65) we obtain

$$\mathcal{N} = N_+ - N_-.$$  \hfill (21.75)

Recall that $N_+$ is the number of linearly independent solutions to (21.11) of negative order. Then, arguing as in Appendix I of [2] we prove that $N_+$ cannot be larger than $\mathcal{N}$ because otherwise (4e,2) would be violated. Then, $N_+ = \mathcal{N}$ and $N_- = 0$. The order of any solution to (21.11) of negative order is an integer not exceeding $-1$. Then, it follows by a contour integration that the Fourier transform (3.5) of any solution to (21.11) of negative order is zero for $y \in \mathbb{R}^-$. Hence, any solution to (21.11) that is of negative order is in $H^2(C^+)$. Since by Proposition 21.8 any solution to (21.11) that is in $H^2(C^+)$ is of negative order, it follows that $N_+$ is the number of linearly independent solutions to (21.11) that are in $H^2(C^+)$. Hence, (Vh) of Theorem 7.6 holds. Finally $N_-$ is the number of linearly independent solutions to (21.54) that are of negative order. Again, by a contour integration we can show that these solutions are in $H^2(C^-)$ and since by Proposition 21.8 any solution to (21.55) that is in $H^2(C^-)$ is of negative order, $N_-$ is the number of linearly independent solutions to (21.55) that are in $H^2(C^-)$. Since we have proven that $N_- = 0$ it follows that (IIIc) holds. It only remains to prove that (4e) is satisfied. By Proposition 6.1 it is enough to prove that condition (4c) holds. On the contrary, suppose that (4.14) has a nontrivial integrable solution, $X(y)$. Since by condition (I) of Definition 4.3 the matrix $F_s(y)$ is bounded in $y \in \mathbb{R}$ and it is integrable for $y \in \mathbb{R}^+$, $X(y)$, is bounded, and then $X(y) \in L^2(\mathbb{R}^+)$. Then, condition (4e,2) will be violated, and by Proposition 21.2 also condition (4e,2) will be violated. In consequence, condition (4e) holds. 


22. THE GENERALIZED FOURIER MAP

The Fourier transform between square-integrable functions of $x$ and square-integrable functions of $k$ is an essential tool in the analysis of inverse scattering and spectral problems. In the case of the Schrödinger equation with a matrix-valued potential the corresponding Fourier transform acts between vectors with $n$ components that are square-integrable functions of $x$ and vectors with $n$ components that are square-integrable functions of $k$. It is possible to generalize such a Fourier transform and introduce a generalized Fourier map $F$ acting from a Hilbert space involving functions of $x$ into another Hilbert space involving functions of $k$.

We use $C_0(R^+)$ to denote the subspace of $L^2(R^+)$ consisting of column vectors with $n$ components that are continuous functions of $x$ with compact support in $R^+$. Thus, any function $Y(x)$ in $C_0(R^+)$ is continuous, bounded, integrable, and square integrable in $x \in R^+$. It is known that $C_0(R^+)$ is a dense subspace of $L^2(R^+)$.

We define the Hilbert space $\mathcal{R}$ as the direct sum given by

$$\mathcal{R} := \text{Ran}[M_1] \oplus \cdots \oplus \text{Ran}[M_N] \oplus L^2(R^+),$$

(22.1)

where $N$ is the number of bound states appearing in (4.2) and $\text{Ran}[M_j]$ denotes the range of the matrix $M_j$ appearing in (4.3) and (10.22). Let us use $(Z_1, \ldots, Z_N, Z)$ to denote an element in $\mathcal{R}$, where $Z_j$ is a constant column vector with $n$ components and has the form $M_j v_j$ for some vector $v_j$ in $C^n$, and $Z$ is a column vector with $n$ components that are square-integrable functions of $k \in R^+$. The scalar product in $\mathcal{R}$ is defined as

$$\left((Z_1, \ldots, Z_N, Z), (\tilde{Z}_1, \ldots, \tilde{Z}_N, \tilde{Z})\right)_\mathcal{R} := Z_1^\dagger \tilde{Z}_1 + \cdots + Z_N^\dagger \tilde{Z}_N + \int_0^\infty dk Z(k)^\dagger \tilde{Z}(k),$$

(22.2)

and the norm $|| \cdot ||_\mathcal{R}$ is defined to be the norm induced by the scalar product in (22.2).

Inspired by [42], we introduce the generalized Fourier map $F$ acting on the subspace $C_0(R^+)$ in terms of the components $F_1, \ldots, F_N, F_c$ as

$$FY = (F_1Y, \ldots, F_NY, F_cY),$$

(22.3)
where each component $F_j$ is associated with the bound state at $k = i\kappa_j$, and $F_c$ is associated with the continuous spectrum of the matrix Schrödinger operator. Here, the component $F_j$ is defined with the help of the bound-state matrix solution $\Psi_j(x)$ appearing in (9.8) as

$$Z_j := F_j Y := \int_0^\infty dx \, \Psi_j(x)^\dagger Y(x), \quad j = 1, \ldots, N, \quad (22.4)$$

and the component $F_c$ is defined with the help of the physical solution $\Psi(k, x)$ appearing in (9.4) as

$$Z(k) := (F_c Y)(k) := \frac{1}{\sqrt{2\pi}} \int_0^\infty dx \, \Psi(k, x)^\dagger Y(x), \quad k \in \mathbb{R}^+. \quad (22.5)$$

**Proposition 22.1** Consider a scattering data set $S$ as in (4.2), which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbb{R}$, a set of $N$ distinct positive constants $\kappa_j$, and a set of $N$ constant $n \times n$ hermitian and nonnegative matrices $M_j$ with respective positive ranks $m_j$, where $N$ is a nonnegative integer. Assume that $S$ satisfies (I) of Definition 4.3 and (4_a) of Definition 4.2. Then, we have the following:

(a) The map $F_j$ defined in (22.4) maps $C_0(\mathbb{R}^+)$ into $\text{Ran}[M_j]$ appearing in (22.1). It can be extended to a bounded map from $L^2(\mathbb{R}^+)$ into $\text{Ran}[M_j]$.

(b) The Fourier map $F$ defined in (22.3) maps the subspace $C_0(\mathbb{R}^+)$ into the Hilbert space $\mathcal{R}$ defined in (22.1). It is an isometry, i.e. $(FY, FY)_{\mathcal{R}} = (Y, Y)_2$ for any $Y(x) \in C_0(\mathbb{R}^+)$. It can be extended to an isometric map from $L^2(\mathbb{R}^+)$ into $\mathcal{R}$.

(c) The map $F_c$ defined in (22.5) maps $C_0(\mathbb{R}^+)$ into $L^2(\mathbb{R}^+)$. It can be extended to a bounded map from $L^2(\mathbb{R}^+)$ into $L^2(\mathbb{R}^+)$.

**Proof:** The integrand in (22.4) is integrable when $Y(x)$ belongs to $C_0(\mathbb{R}^+)$ because $\Psi_j(x)$ is bounded in $x \in \mathbb{R}^+$, as stated in Proposition 17.1(d). That integrand remains integrable when $Y(x)$ belongs $L^2(\mathbb{R}^+)$ because the integrand now becomes a product of two square-integrable quantities, as a result of $\Psi_j(x)$ being square integrable in $x \in \mathbb{R}^+$, as stated in Proposition 17.1(d). Thus, the extension of $F_j$ from the domain of $C_0(\mathbb{R}^+)$ to the domain of $L^2(\mathbb{R}^+)$ is immediate. For any $Y(x) \in L^2(\mathbb{R}^+)$, from (9.8) and the fact
that $M_j$ is hermitian, it follows that $Z_j$ given in (22.4) has the form $Z_j = M_j v_j$, where $v_j$ is the constant vector in $C^n$ given by

$$v_j = \int_0^\infty dx f(i\kappa_j, x)^\dagger Y(x). \quad (22.6)$$

Thus, we have shown that $F_j$ maps $C_0(\mathbb{R}^+)$ into $\text{Ran}[M_j]$, and its extension maps $L^2(\mathbb{R}^+)$ into $\text{Ran}[M_j]$. Let us now show that the extended map is bounded from $L^2(\mathbb{R}^+)$, i.e. let us show that $|F_j Y| \leq C ||Y||_2$ for some constant $C$. Using the inequality

$$|\Psi_j(x)^\dagger Y(x)| \leq |\Psi_j(x)||Y(x)|, \quad x \in \mathbb{R}^+, \quad (22.7)$$

in (22.4) we obtain

$$|F_j Y| \leq \int_0^\infty dx |\Psi_j(x)^\dagger Y(x)| \leq \int_0^\infty dx |\Psi_j(x)||Y(x)|. \quad (22.8)$$

Applying the Schwarz inequality on the last integral term in (22.8) we get

$$|F_j Y|^2 \leq \int_0^\infty dy |\Psi_j(y)|^2 \int_0^\infty dz |Y(z)|^2. \quad (22.9)$$

As stated in Proposition 17.1(d), $\Psi_j(x)$ is square integrable in $x \in \mathbb{R}^+$, and hence the first integral in (22.9) is bounded by a constant. Thus, we get $|F_j Y| \leq C ||Y||_2$, completing the proof of (a). Let us now turn to the proof of (b). For $Y(x) \in C_0(\mathbb{R}^+)$, using (22.4) and (22.5) in (22.3) we obtain

$$(FY, FY)_\mathcal{R} = \sum_{j=1}^N \int_0^\infty dx Y(x)^\dagger \Psi_j(x) \int_0^\infty dy \Psi_j(y)^\dagger Y(y) \quad (22.10)$$

$$+ \frac{1}{2\pi} \int_0^\infty dk \left[ \int_0^\infty dx Y(x)^\dagger \Psi(k, x) \int_0^\infty dy \Psi(k, y)^\dagger Y(y) \right].$$

Since $Y(x)$ belongs to $C_0^\infty(\mathbb{R}^+)$, the order of the integrations in (22.10) can be interchanged, and hence the right-hand side in (22.10) can be rearranged and we get

$$(FY, FY)_\mathcal{R} = \int_0^\infty dx \int_0^\infty dy Y(x)^\dagger q(x, y) Y(y), \quad (22.11)$$

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where we have defined

\[ q(x, y) := \sum_{j=1}^{N} \Psi_j(x) \Psi_j(y) + \frac{1}{2\pi} \int_{0}^{\infty} dk \, \Psi(k, x) \Psi(k, y). \]  

(22.12)

From Parseval’s identity given in (20.1), we see that

\[ q(x, y) = \delta(x - y) \, I \]

and hence we get

\[ (FY, FY)_{\mathcal{R}} = (Y, Y)_2, \quad Y \in C_0(\mathbb{R}^+), \]  

(22.13)

which proves that the Fourier map \( F \) is an isometry from \( C_0(\mathbb{R}^+) \) into \( \mathcal{R} \). Since the subspace \( C_0(\mathbb{R}^+) \) is dense in \( L^2(\mathbb{R}^+) \), we can extend \( F \) to an isometry from \( L^2(\mathbb{R}^+) \) into \( \mathcal{R} \) so that (22.13) remains valid when \( Y(x) \in L^2(\mathbb{R}^+) \), i.e. we

\[ (FY, FY)_{\mathcal{R}} = (Y, Y)_2, \quad Y \in L^2(\mathbb{R}^+). \]  

(22.14)

Thus, we have completed the proof of (b). Let us now turn to the proof of (c). Using (22.2) in (22.14) we obtain

\[ Z_1^\dagger \tilde{Z} + \cdots + Z_N^\dagger \tilde{Z}_N + (Z, Z)_2 = (Y, Y)_2, \quad Y \in C_0(\mathbb{R}^+). \]  

(22.15)

Thus, (22.15) implies that

\[ (Z, Z)_2 \leq (Y, Y)_2, \quad Y \in C_0(\mathbb{R}^+), \]  

(22.16)

and hence \( F_c \) defined in (22.5) is a bounded operator from \( C_0(\mathbb{R}^+) \) into \( L^2(\mathbb{R}^+) \). Since \( C_0(\mathbb{R}^+) \) is a dense subspace of \( L^2(\mathbb{R}^+) \), from (22.16) we conclude that \( F_c \) uniquely extends to a bounded map from \( L^2(\mathbb{R}^+) \) into \( L^2(\mathbb{R}^+) \). Let us elaborate on such an extension. For any \( Y(x) \) in \( L^2(\mathbb{R}^+) \) there exists a sequence \( \{Y^{(j)}(x)\}_{j=1}^{\infty} \) converging to \( Y(x) \) in the \( L^2 \)-sense with \( Y^{(j)}(x) \) belonging to \( C_0(\mathbb{R}^+) \) for all \( j \geq 1 \). Using (22.5) we then define

\[ Z^{(j)}(k) := \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dx \, \Psi(k, x)^\dagger Y^{(j)}(x), \quad j = 1, 2, \ldots. \]  

(22.17)

From (22.16) we know that

\[ (Z^{(j)}, Z^{(j)})_2 \leq (Y^{(j)}, Y^{(j)})_2, \quad j = 1, 2, \ldots, \]  

(22.18)
and hence $Z^{(j)}(k)$ is square integrable in $k \in \mathbb{R}^+$. Since the sequence $\{Y^{(j)}(x)\}_{j=1}^\infty$ is convergent in the $L^2$-sense, it follows from (22.18) that also the sequence $\{Z^{(j)}(k)\}_{j=1}^\infty$ is convergent in the $L^2$-sense to some limit $Z(k) \in L^2(\mathbb{R}^+)$. Using that limit, we then let $(F_cY)(k) := Z(k)$. This approach allows us interpret (22.5) as the extension of $F_c$ as a bounded map on $L^2(\mathbb{R}^+)$ even though the integral on the right-hand side of (22.5) may not literally exist for $Y(x) \in L^2(\mathbb{R}^+)$. Thus, the proof of (c) is completed.

From Proposition 22.1 it follows that (22.3), (22.4), and (22.5) can also be viewed as the extensions of $F$, $F_j$, and $F_c$, respectively, where their domains are now $L^2(\mathbb{R}^+)$. In fact, unless otherwise stated, we will use $F$, $F_j$, and $F_c$, to denote such extensions.

Next, we analyze the component $F_c$ of the Fourier map $F$ further. Toward this goal we first introduce the point subspace $P$ associated with our Schrödinger operator related to (2.1) and (2.4). By definition, the point subspace $P$ is the span of all eigenfunctions of the Schrödinger operator associated with (2.1) and (2.4). By Proposition 11.4(e) we know that the eigenfunctions of the Schrödinger operator correspond to the eigenvalues $-\kappa_j^2$ for $j = 1, \ldots, N$, and hence we can write $P$ in the form of a direct sum as

$$P = P_1 \oplus \cdots \oplus P_N.$$  \hfill (22.19)

It follows from Proposition 11.4 that each $P_j$ is the column space of the bound-state matrix solution $\Psi_j(x)$ appearing in (9.8), the dimension of $P_j$ is equal to the rank $m_j$ of $M_j$, and $P_j$ is equal to the subspace of $L^2(\mathbb{R}^+)$ consisting of all column vectors of the form $\Psi_j(x)v$ for all $v \in \mathbb{C}^n$.

**Proposition 22.2** Assume that the input data set $D$ in (4.1) belongs to the Faddeev class specified in Definition 4.1. Let $F_c$ be the map from $L^2(\mathbb{R}^+)$ into $L^2(\mathbb{R}^+)$ appearing in (22.5).

(a) The kernel of $F_c$ is equal to the point space $P$ defined in (22.19), i.e.

$$\text{Ker}[F_c] = P.$$  \hfill (22.20)
Thus, Ker[$\mathbf{F}_c$] is consists of linear combinations of columns of all bound-state matrix solutions $\Psi_j(x)$ for $j = 1, \ldots, N$ defined in (10.7) and hence the dimension of Ker[$\mathbf{F}_c$] is equal to the nonnegative integer $N$ appearing in (4.3).

(b) The subspace Ker[$\mathbf{F}_c$] of $L^2(\mathbb{R}^+)$ is a subspace of $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$. 

(c) The map $\mathbf{F}_c$ is unitary from $(\text{Ker}[\mathbf{F}_c])^\perp$ onto $L^2(\mathbb{R}^+)$. 

(d) The generalized Fourier map $\mathbf{F}$ defined in (22.5) is a unitary map from $L^2(\mathbb{R}^+)$ into $\mathcal{R}$, i.e. it is an isometry and onto $\mathcal{R}$.

PROOF: For the proof of (22.20) and the unitarity property stated in (c), we refer the reader to [42], where the proof is given in Theorem 6.7, (6.37), and (6.38) there. Actually, the notation $F_{A,B}^-$ is used in [42] to denote our map $\mathbf{F}_c$. Hence, Ker[$\mathbf{F}_c$] consists of the columns of $\Psi_j(x)$ for all $j = 1, \ldots, N$ and since the number of such columns contain exactly $N$ linearly independent columns, the dimension of the kernel of $\mathbf{F}_c$ is also $N$. Thus, the proof of (a) is complete. From Proposition 17.1(d) we know that each column of $\Psi_j(x)$ is bounded and integrable in $x \in \mathbb{R}^+$ besides being square integrable there. Then, from (a) we conclude that any column vector in $L^2(\mathbb{R}^+)$ belonging to Ker[$\mathbf{F}_c$] must be bounded and integrable in $x \in \mathbb{R}^+$. Thus, the proof of (b) is complete. Let us now prove (d). To prove the unitarity of $\mathbf{F}$ from $L^2(\mathbb{R}^+)$ into $\mathcal{R}$ we need to prove that $\mathbf{F}$ is an isometry and is also onto $\mathcal{R}$. The former property is assured by Proposition 22.1(b) and hence we only need to prove that $\mathbf{F}$ is onto. Since the kernel of the map $\mathbf{F}$ is a subspace of $L^2(\mathbb{R}^+)$, we have the orthogonal decomposition

$$L^2(\mathbb{R}^+) = \text{Ker}[\mathbf{F}_c] \oplus (\text{Ker}[\mathbf{F}_c])^\perp.$$  \hfill (22.21)

From (22.1), (22.3), and (22.21) we see that $\mathbf{F}$ is onto if we can show that the map given by

$$(\mathbf{F}_1, \ldots, \mathbf{F}_N) : \text{Ker}[\mathbf{F}_c] \rightarrow \text{Ran}[M_1] \oplus \cdots \oplus \text{Ran}[M_N],$$  \hfill (22.22)

is onto and also the map $\mathbf{F}_c$ maps $L^2(\mathbb{R}^+)$ onto $L^2(\mathbb{R}^+)$. The latter map, i.e. $\mathbf{F}_c$ is already onto because it is unitary, as stated in (c). On the other hand, the map given in (22.22) is
onto if and only if the map $F_j$ given in (22.4) maps $\text{Ker}[F_c]$ onto $\text{Ran}[M_j]$ for $j = 1, \ldots, N$. On the other hand, from (22.19) and (22.20) we see that the map given in (22.22) is onto if and only if $\mathcal{P}_j$ is isomorphic to $\text{Ran}[M_j]$ for $j = 1, \ldots, N$. The latter property follows from Proposition 11.4(d) and Proposition 11.4(e). Thus, the proof of (d) is complete.

As shown in the next proposition, the adjoint of the Fourier map, denoted by $F^\dagger$, can be expressed explicitly in terms of the physical solution $\Psi(k, x)$ and the bound-state matrix solutions $\Psi_j(x)$.

**Proposition 22.3** Assume that the scattering data set $S$ appearing in (4.2) satisfies (1) of Definition 4.3 and (4a) of Definition 4.2. Further, assume that the generalized Fourier map $F$ defined in (22.3) is unitary from $L^2(\mathbb{R}^+)$ onto $\mathcal{R}$, the Hilbert space specified in (22.1). Then, we have the following:

(a) The adjoint of the map $F_j$ given in (22.4), denoted by $F_j^\dagger$, maps $\text{Ran}[M_j]$ into $L^2(\mathbb{R}^+)$ and is given by

$$F_j^\dagger Z_j := \Psi_j(x) Z_j, \quad j = 1, \ldots, N,$$

where $\Psi_j(x)$ is the bound-state matrix solution appearing in (9.8).

(b) The adjoint of the map $F_c$ given in (22.5), denoted by $F_c^\dagger$, maps $L^2(\mathbb{R}^+)$ onto $(\text{Ker}[F_c])^\perp$ and is given by

$$(F_c^\dagger Z)(x) := \frac{1}{\sqrt{2\pi}} \int_0^\infty dk \Psi(k, x) Z(k), \quad x \in \mathbb{R}^+,$$

where $\Psi(k, x)$ is the physical solution in (9.4).

(c) The adjoint of the Fourier map $F$ given in (22.3)-(22.5), denoted by $F^\dagger$, maps the Hilbert space $\mathcal{R}$ given in (22.1) onto $L^2(\mathbb{R}^+)$ and described as

$$F^\dagger(Z_1, \ldots, Z_N, Z) = F_1^\dagger Z_1 + \cdots + F_N^\dagger Z_N + F_c^\dagger Z,$$

where $F_j^\dagger$ and $F_c^\dagger$ are as in (22.23) and (22.24), respectively.

**PROOF:** Because $F$ is unitary, it is onto. Hence, its respective components $F_j$ are each onto $\text{Ran}[M_j]$ and $F_c$ is onto $L^2(\mathbb{R}^+)$. Thus, the domain of $F^\dagger$ is $\mathcal{R}$, the domain of $F_j^\dagger$
is $\text{Ran}[M_j]$, and the domain of $F_c^\dagger$ is $L^2(\mathbb{R}^+)$. Using the definition of the adjoint map 
\[ \langle Z_j, F_j \tilde{Y} \rangle = \left( F_j^\dagger Z_j, \tilde{Y} \right)_2, \]
where $Z_j \in \text{Ran}[M_j]$, $\tilde{Y}(x) \in L^2(\mathbb{R}^+)$, and $F_j$ as in (22.4), we obtain (22.23), and hence (a) holds. Furthermore, we have 
\[ \left( Z, F_c \tilde{Y} \right)_2 = \left( F_c^\dagger Z, \tilde{Y} \right)_2, \]
where $Z(k) \in L^2(\mathbb{R}^+)$, $\tilde{Y}(x) \in L^2(\mathbb{R}^+)$, and $F_c$ as in (22.5). Thus, we obtain (22.24), and hence (b) holds. Moreover, by using the definition of the operator adjoint given by 
\[ ((Z_1, \ldots, Z_N, Z), F\tilde{Y})_{\mathcal{R}} = (F^\dagger (Z_1, \ldots, Z_N, Z), \tilde{Y})_2, \quad (22.26) \]
where $(Z_1, \ldots, Z_N, Z)$ and $\tilde{Y}$ are some arbitrary vectors in $\mathcal{R}$ and $L^2(\mathbb{R}^+)$, respectively. With the help of (22.2)-(22.5), one can verify directly that (22.26) yields (22.25). We see from (22.23) that, for any $Z_j \in \text{Ran}[M_j]$ its image under $F_j^\dagger$ given by $\Psi_j Z_j$ belongs to $L^2(\mathbb{R}^+)$, as seen from (9.8) and Proposition 17.1(d). On the other hand, the integral in (22.24) does not exist in the usual sense when $Z(k) \in L^2(\mathbb{R}^+)$ and needs to be understood in the $L^2$-sense, analogous to a similar description given in the proof of Proposition 22.1.

We provide a brief elaboration. For any $Z(k) \in L^2(\mathbb{R}^+)$ we can find a sequence $\{Z^{(l)}(k)\}_{l=1}^\infty$ converging to $Z(k)$ in the $L^2$-sense with $Z^{(l)}(k)$ belonging to $L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ for all $l \geq 1$. Using (22.24) we let 
\[ Y^{(l)}(x) := (F_c^\dagger Z^{(l)})(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dk \Psi(k, x) Z^{(l)}(k), \quad l = 1, 2, \ldots. \quad (22.27) \]

By Proposition 22.2(c) we know that $F_c$ is a unitary map from $(\text{Ker}[F_c])^\perp$ onto $L^2(\mathbb{R}^+)$, and hence $F_c^\dagger$ is unitary from $L^2(\mathbb{R}^+)$ onto $(\text{Ker}[F_c])^\perp$. Thus, from (22.27) we obtain
\[ ||Y^{(l)}||_2 = ||F_c^\dagger Z^{(l)}||_2 = ||Z^{(l)}||_2. \quad (22.28) \]

Since $Z^{(l)}(k)$ converges to $Z(k)$ in the $L^2$-sense, it follows from (22.28) that $Y^{(l)}(x)$ must converge in the $L^2$-sense to some $Y(x) \in L^2(\mathbb{R}^+)$, where we have let
\[ Y(x) := \lim_{l \to +\infty} (F_c^\dagger Z^{(l)})(x) = \lim_{l \to +\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty dk \Psi(k, x) Z^{(l)}(k). \quad (22.29) \]

Thus, the proof is complete. □
The following result indicates that the physical solution $\Psi(k, x)$ appearing in (9.4) and (9.6) along with the normalized bound-state matrix solutions $\Psi_j(x)$ defined in (9.8) satisfy certain orthonormality properties.

**Proposition 22.4** Assume that the input data set $D$ appearing in (4.1) belongs to the Faddeev class specified in Definition 4.1. Then, the physical solution $\Psi(k, x)$ appearing in (9.4) and (9.6) and the normalized bound-state matrix solutions $\Psi_j(x)$ appearing in (9.8) satisfy

\[
\int_0^\infty dx \, \Psi_l(x)\dagger \Psi_j(x) = \delta_{lj} P_j, \quad l, j = 1, \ldots, N, \tag{22.30}
\]

\[
\int_0^\infty dx \, \Psi_l(x)\dagger \Psi(k, x) = 0, \quad l = 1, \ldots, N, \quad k \in \mathbb{R}^+, \tag{22.31}
\]

\[
\int_0^\infty dx \, \Psi(k, x)\dagger \Psi_l(x) = 0, \quad l = 1, \ldots, N, \quad k \in \mathbb{R}^+, \tag{22.32}
\]

\[
\frac{1}{2\pi} \int_0^\infty dx \, \Psi(k, x)\dagger \Psi(\ell, x) = \delta(k - \ell) I, \quad k, \ell \in \mathbb{R}^+, \tag{22.33}
\]

where $P_j$ is the projection matrix in (11.1), $\delta_{jl}$ is the Kronecker delta, and $\delta(k)$ is the Dirac delta distribution.

**PROOF:** The normality in (22.30), i.e. (22.30) when $j = l$, directly follows from (11.23).

Let us prove (22.30) when $j \neq l$. We know from Proposition 11.4(b) that $\Psi_j(x)$ satisfies (2.1) with $k = i\kappa_j$, i.e.

\[-\Psi_j''(x) + V(x) \Psi_j(x) = -\kappa_j^2 \Psi_j(x), \quad x \in \mathbb{R}^+, \tag{22.34}\]

Since the potential $V$ satisfies (2.2), (22.34) yields

\[-\Psi_l''(x)\dagger + \Psi_l(x)\dagger V(x) = -\kappa_l^2 \Psi_l(x)\dagger, \quad x \in \mathbb{R}^+, \tag{22.35}\]

Premultiplying (22.34) by $\Psi_l(x)\dagger$ and postmultiplying (22.35) by $\Psi_j(x)$ and subtracting the resulting matrix equations, we obtain

\[(\kappa_l^2 - \kappa_j^2) \Psi_l(x)\dagger \Psi_j(x) = \Psi_l''(x)\dagger \Psi_j(x) - \Psi_l(x)\dagger \Psi_j''(x), \quad x \in \mathbb{R}^+, \tag{22.36}\]
or equivalently we have

$$(\kappa^2_l - \kappa^2_j) \Psi_l(x)^\dagger \Psi_j(x) = \frac{d}{dx} \left[ \Psi_l'(x)^\dagger \Psi_j(x) - \Psi_l(x)^\dagger \Psi_j'(x) \right] , \quad x \in \mathbb{R}^+.$$  \hspace{1cm} (22.37)

By integrating over $x \in \mathbb{R}^+$, from (22.37) we obtain

$$(\kappa^2_l - \kappa^2_j) \int_0^\infty dx \Psi_l(x)^\dagger \Psi_j(x) = \Psi_l(0)^\dagger \Psi_j'(0) - \Psi_l'(0)^\dagger \Psi_j(0),$$  \hspace{1cm} (22.38)

where we have used the fact that $\Psi_j(x)$ and $\Psi_j'(x)$ vanishes as $x \to +\infty$ for $j = 1, \ldots, N$, as indicated by Proposition 17.1(d) and Proposition 17.2(d), respectively. By Proposition 14.2, the right-hand side of (22.38) vanishes because $\Psi_l(x)$ and $\Psi_j(x)$ each satisfy the boundary condition (2.4), as indicated in Proposition 11.4(b). Hence, we obtain

$$(\kappa^2_l - \kappa^2_j) \int_0^\infty dx \ Psi_l(x)^\dagger \Psi_j(x) = 0.$$

Since $\kappa_l \neq \kappa_j$, from (22.39) we obtain (22.30). Let us next prove (22.31) By Proposition 9.6(c), the physical solution $\Psi(k, x)$ satisfies (2.1), i.e. we have

$$-\Psi''(k, x) + V(x) \Psi(k, x) = k^2 \Psi(k, x), \quad x \in \mathbb{R}^+.$$  \hspace{1cm} (22.40)

Postmultiplying (22.35) with $\Psi(k, x)$ and premultiplying (22.40) with $\Psi_l(x)^\dagger$ and subtracting the resulting equations we obtain the analog of (22.37), i.e. we obtain

$$(\kappa^2_l + k^2) \Psi_l(x)^\dagger \Psi(k, x) = \frac{d}{dx} \left[ \Psi_l'(x)^\dagger \Psi(k, x) - \Psi_l(x)^\dagger \Psi'(k, x) \right] , \quad x \in \mathbb{R}^+.$$  \hspace{1cm} (22.41)

When $k \in \mathbb{R}$, from (17.2), (17.4), (17.10), and (17.12) it follows that, for each $k \in \mathbb{R}$, we have

$$\Psi_l'(x)^\dagger \Psi(k, x) - \Psi_l(x)^\dagger \Psi'(k, x) = o(1), \quad x \to +\infty,$$

and that $\Psi_l(x)^\dagger \Psi(k, x)$ is integrable in $x \in \mathbb{R}^+$. Thus, integrating over $x \in \mathbb{R}^+$, from (22.41) we obtain

$$(\kappa^2_l + k^2) \int_0^\infty dx \Psi_l(x)^\dagger \Psi(k, x) = \Psi_l(0)^\dagger \Psi'(k, 0) - \Psi_l'(0)^\dagger \Psi(k, 0), \quad k \in \mathbb{R}.$$  \hspace{1cm} (22.43)
By Proposition 9.6(b) we know that $\Psi(k, x)$ satisfies (2.4), and by Proposition 11.4(b) we know that $\Psi_l(x)$ satisfies (2.4). Hence, Proposition 14.2 indicates that the right-hand side of (22.43) must vanish. For $k \in \mathbb{R}$, we have $\kappa_l^2 + k^2 > 0$ and hence (22.43), with the right-hand side being zero, yields (22.31). Note that (22.32) can be obtained by taking the matrix adjoint of both sides of (22.31), and hence (22.32) is valid. It only remains to prove (22.33). As stated in Proposition 22.2(c) the map $F_c$ from $(\text{Ker}[F_c])^\perp$ onto $L^2(\mathbb{R}^+)$ given in (22.5) is unitary. This implies [27] that we have $F_c^{\dagger}F_c = I$ on $(\text{Ker}[F_c])^\perp$ and $F_cF_c^{\dagger} = I$ on $L^2(\mathbb{R}^+)$. The latter can be written as $F_cF_c^{\dagger}Z = Z$ for $Z \in L^2(\mathbb{R}^+)$, or equivalently as

$$
\frac{1}{2\pi} \int_0^\infty dx \Psi(k, x)^\dagger \int_0^\infty dl \Psi(l, x) Z(l) = Z(k), \quad Z(k) \in L^2(\mathbb{R}^+). \quad (22.44)
$$

Since $C_0(\mathbb{R}^+)$ is a subspace of $L^2(\mathbb{R}^+)$, we can use (22.44) with $Z(k) \in C_0(\mathbb{R}^+)$. Then, the integral on the left-hand side of (22.44) exists and we are allowed to interchange the order of integration and obtain

$$
\frac{1}{2\pi} \int_0^\infty dl \int_0^\infty dx \Psi(k, x)^\dagger \Psi(l, x) Z(l) = Z(k), \quad Z(k) \in C_0(\mathbb{R}^+). \quad (22.45)
$$

Then, from (22.45) we conclude the formal expression (22.33) understood in the distribution sense. □

In the following proposition we present some relations among the maps $F_j, F_c, F_j^{\dagger}, F_c^{\dagger}$ appearing in (22.4), (22.5), (22.23), (22.24), respectively.

**Proposition 22.5** Assume that the input data set $D$ appearing in (4.1) belongs to the Faddeev class. Then, the components $F_j$ and $F_c$ of the Fourier map $F$ defined in (22.3)-(22.5) satisfy

$$
\begin{align*}
F_lF_j^{\dagger}Z_j & = \delta_{lj}Z_j, \quad j, l = 1, \ldots, N, \quad Z_j \in \text{Ran}[M_j], \\
F_lF_c^{\dagger}Z & = 0, \quad l = 1, \ldots, N, \quad Z \in L^2(\mathbb{R}^+), \\
F_cF_l^{\dagger}Z_l & = 0, \quad l = 1, \ldots, N, \quad Z_l \in \text{Ran}[M_l], \\
F_cF_c^{\dagger}Z & = Z, \quad Z \in L^2(\mathbb{R}^+),
\end{align*}
\quad (22.46)
$$

where we recall that $\delta_{jl}$ is the Kronocker delta, $F_j^{\dagger}$ is the adjoint map in (22.23), $F_c^{\dagger}$ is the adjoint map in (22.24), and $Z$ is a column vector with $n$ components that are square-integrable functions of $k \in \mathbb{R}^+$.
PROOF: We know from Proposition 22.2(d) that the generalized Fourier map $F$ from $L^2(\mathbb{R}^+)$ into $\mathcal{R}$ is unitary. Consequently, we have $F^\dagger F = I$ on $L^2(\mathbb{R}^+)$ and we also have $FF^\dagger = I$ on $\mathcal{R}$. The latter property is equivalent to

$$FF^\dagger(Z_1, \ldots, Z_N, Z) = (Z_1, \ldots, Z_N, Z). \tag{22.47}$$

where $(Z_1, \ldots, Z_N, Z)$ denotes a typical element in $\mathcal{R}$. With the help of (22.25) we see that (22.47) is equivalent to

$$F(F^\dagger_1 Z_1 + \cdots + F^\dagger_N Z_N + F^\dagger_c Z) = (Z_1, \ldots, Z_N, Z), \tag{22.48}$$

From (22.3) we see that (22.48) is equivalent to

$$\begin{cases} F_l \left( \sum_{j=1}^N F^\dagger_j Z_j + F^\dagger_c Z \right) = Z_l, & l = 1, \ldots, N, \\ F_c \left( \sum_{j=1}^N F^\dagger_j Z_j + F^\dagger_c Z \right) = Z, \end{cases} \tag{22.49}$$

where $Z_l \in \text{Ran}[M_l]$ and $Z \in L^2(\mathbb{R}^+)$. Using (22.4) and (22.23), from (22.30) we obtain

$$F_l F^\dagger_j Z_j = \delta_{lj} P_j Z_j, \quad Z_j \in \text{Ran}[M_j]. \tag{22.50}$$

Since $Z_j = M_j v_j$ for some vector $v_j \in \mathbb{C}^n$, we have $P_j Z_j = P_j M_j v_j$. On the other hand, using (11.1), the definition of $M_j$ given in (11.22), and the fact that $B_j^{-1/2}$ commutes with $P_j$, as asserted by Proposition 11.2(d), we get

$$P_j M_j = M_j, \quad j = 1, \ldots, N, \tag{22.51}$$

yielding $P_j Z_j = M_j v_j$ or equivalently,

$$P_j Z_j = Z_j, \quad j = 1, \ldots, N. \tag{22.52}$$

From (22.50) and (22.52) we see that the first line of (22.46) is satisfied. Using the first line of (22.46) in the first line of (22.49), we get $F_l F^\dagger_c Z = 0$ and hence the second line of (22.30) is satisfied. By postmultiplying (22.32) by $Z_l$ in the range of $M_l$, we obtain the third line of (22.46). Then, using the third line of (22.46) in the second line of (22.49) we get the fourth line of (22.46). $\blacksquare$
In this chapter, we present an alternate solution to the inverse problem and this is related to the characterization given in Theorem 8.1.

The part of the solution to the inverse problem involving the construction of the potential is practically the same as the solution outlined in Chapter 16. However, the part of the solution related to the boundary condition is different than the procedure outlined in Chapter 16. We summarize the construction of $D$ from $S$ in this alternate method, where the existence and uniqueness are implicit at each step:

(a) From the large-$k$ asymptotics of the scattering matrix $S(k)$, with the help of (4.6), we determine the $n \times n$ constant matrix $S_\infty$. Contrary to the method of Chapter 16, we do not deal with the determination of the constant $n \times n$ matrix $G_1$ specified in via (14.1). It follows from (4.4) that the matrix $S_\infty$ is hermitian when $S$ satisfies the condition (I).

(b) In terms of the quantities in $S$, we uniquely construct the $n \times n$ matrix $F_s(y)$ by using (4.7) and the $n \times n$ matrix $F(y)$ by using (4.12). This step is the same as (b) of the summary of the method outlined in Chapter 16.

(c) If the condition $(4_c)$ is also satisfied, then one uses the matrix $F(y)$ as input to the Marchenko integral equation (13.1). When $F(y)$ is integrable in $y \in (x, +\infty)$ for each $x \geq 0$, as shown in Proposition 16.1, for each fixed $x \geq 0$, there exists a solution $K(x, y)$ integrable in $y \in (x, +\infty)$ to (13.1) and such a solution is unique. The solution $K(x, y)$ can be constructed by iterating (13.1). Even though $K(x, y)$ is constructed for $y > x \geq 0$, one can extend $K(x, y)$ to $y \in \mathbb{R}^+$ by letting $K(x, y) = 0$ for $0 \leq y < x$. This step is the same as (c) of the summary of the method outlined in Chapter 16.

(d) Having obtained $K(x, y)$ uniquely from $S$, one constructs the potential $V(x)$ via (10.4) and also constructs the Jost solution $f(k, x)$ via (10.6). Then, it follows from Proposition 16.11 that, by using (I), (2), and $(4_c)$ of Theorem 8.1, one proves that the
constructed $V(x)$ satisfies (2.2) and (2.3) and that the constructed $f(k, x)$ satisfies (2.1) with the constructed potential $V(x)$. This step is the same as (d) of the summary of the method outlined in Chapter 16.

(e) Having constructed the Jost solution $f(k, x)$, one then constructs the physical solution $Ψ(k, x)$ via (9.4) and the normalized bound-state matrices $Ψ_j(x)$ via (9.8). One then proves that the constructed matrix $Ψ(k, x)$ satisfies (2.1) and that the constructed $Ψ_j(x)$ satisfies (2.1) at $k = iκ_j$.

(f) Having constructed the potential $V(x)$, one forms a matrix-valued differential operator denoted by $L_{\text{min}}$, which acts as $(-D_x^2 I + V)$ with $D_x := d/dx$, with a domain that is a dense subset of $L^2(\mathbb{R}^+)$. More precisely, the domain of $L_{\text{min}}$ consists of the column vectors with $n$ components each of which is a function of $x$ belonging to a dense subset of $L^2(\mathbb{R}^+)$. The constructed operator $L_{\text{min}}$ is symmetric, i.e. it satisfies $L_{\text{min}} \subseteq L_{\text{min}}^\dagger$, but is not selfadjoint, i.e. it does satisfy $L_{\text{min}} = L_{\text{min}}^\dagger$. For the meaning of the operator inclusion, we refer the reader to Chapter 3.

(g) One then constructs a selfadjoint realization of $L_{\text{min}}$, namely an operator $L$ in such a way that $L_{\text{min}} \subseteq L$ and $L = L^\dagger$. The constructed operator $L$ is a restriction of $L_{\text{min}}^\dagger$, i.e. we have $L \subseteq L_{\text{min}}^\dagger$ but not $L = L_{\text{min}}^\dagger$.

(h) The construction of the operator $L$ is achieved by using the generalized Fourier map $F$ and its adjoint $F^\dagger$ introduced in Chapter 22, inspired by [42].

(i) Once the selfadjoint operator $L$ is constructed, it follows from the results in [45] that the domain of $L$ is a maximal isotropic subspace, which is sometimes also called a Lagrange plane. Once we know that the domain of $L$ is a maximal isotropic subspace, then it follows from Lemma 2.2 of [24] and Theorem 2.1 of [5] that the functions in the domain of $L$ must satisfy the boundary condition (2.4) for some boundary matrices $A$ and $B$ satisfying (2.5) and (2.6), where $A$ and $B$ are uniquely determined up to a postmultiplication by an invertible matrix $T$. 

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Finally, one proves that the constructed physical solution $\Psi(k, x)$ and the bound-state matrix solutions $\Psi_j(x)$ satisfy the boundary condition (2.4) with the boundary matrices $A$ and $B$ specified in the previous step; however, such a proof is different in nature than the proofs for any of the previous characterizations. For the constructed matrices $\Psi_j(x)$, it is immediate that they satisfy the boundary condition because they belong to the domain of $\mathcal{L}$. Thus, it remains to prove that the constructed $\Psi(k, x)$ satisfies the boundary condition. We note that the matrix $\Psi(k, x)$ does not belong to the domain of $\mathcal{L}$ because its entries do not belong to $L^2(\mathbb{R}^+)$. On the other hand, $\Psi(k, x)$ is locally square integrable in $x \in [0, +\infty)$, i.e. it is square integrable in every compact subset of $[0, +\infty)$. Hence, it is possible to use a simple limiting argument to prove that $\Psi(k, x)$ satisfies the boundary condition, and the condition (VI) is utilized in that limiting argument.

As in the previous characterization, we still need to prove that the input data set $D$ of (4.1) constructed from the scattering data set $S$ of (4.2) yields $S$. The proof of this step is the same as the proof given for the earlier characterizations and it is given in the proof of Theorem 5.1.

**Theorem 23.1** For any input data set $D$ in the Faddeev class specified in Definition 4.1, there exists and uniquely exists a corresponding scattering data set $S$ as in (4.2) satisfying the properties (I), (2), (A), (4c), either one of (Ve) or (Vh), and (VI), of Theorem 8.1.

**PROOF:** By Theorem 15.10 we know that for any input data set $D$ in the Faddeev class, there exists and uniquely exists a corresponding scattering data set $S$ in the Marchenko class, i.e. satisfying the properties (1), (2), (3a), (4a) of Definition 4.5. Hence, in order to prove our theorem, it is enough to prove that those four conditions imply (I), (4c), (5)', (5)''', and (VI). The property (1) implies (I), the confirmation of (VI) follows from Proposition 10.3(a), and Proposition 6.6 indicates that (4c), (Ve), and (Vh) hold. Thus, we only need to prove that (A) holds when $D$ belongs to the Faddeev class. In other words, we must prove that for any $g(k)$ belonging to a dense subset $\overset{\circ}{\Upsilon}$ of the vector space
Υ of column vectors with \( n \) components and satisfying \( g(-k) = S(k)g(k) \) for \( k \in \mathbb{R} \), the corresponding equation (8.1) has at least one solution \( h(k) \in \mathbf{H}^2(\mathbf{C}^+) \). Since \( \mathbf{Y} \) can be any dense subspace of \( \mathbf{Y} \), we can certainly choose \( \mathbf{Y} \) as \( \mathbf{Y} \). So, let us start with \( g(k) \in L^2(\mathbf{R}) \) satisfying \( g(-k) = S(k)g(k) \) for \( k \in \mathbf{R} \) and prove the existence of some \( h(k) \) satisfying (8.1). Because \( g(k) \in L^2(\mathbf{R}) \), there exists \( Y(x) \in (\text{Ker}[\mathbf{F}_c])^\perp \subset L^2(\mathbf{R}^+) \) such that

\[
(\mathbf{F}_cY)(k) = g(k), \quad k \in \mathbf{R}^+.
\] (23.1)

Here \( \mathbf{F}_c \) is the map defined in (22.5), and the existence of the corresponding \( Y(x) \) is guaranteed because \( \mathbf{F}_c \) onto \( L^2(\mathbf{R}^+) \) from \( (\text{Ker}[\mathbf{F}_c])^\perp \) as indicated by Proposition 22.2(c).

We will construct a solution \( h(k) \) to (8.1) with the help of \( Y(x) \). We proceed as follows. Using (22.5) we evaluate the left-hand side of (23.1) as

\[
(\mathbf{F}_cY)(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dx \, \Psi(k, x)^\dagger Y(x),
\] (23.2)

where \( \Psi(k, x) \) is the physical solution constructed from \( \mathbf{D} \) via the procedure outlined in Chapter 9, i.e. constructed as in (9.4). Thus, using (9.4) on the right-hand side of (23.2) we obtain

\[
(\mathbf{F}_cY)(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dx \, [f(-k, x)^\dagger + S(k)^\dagger f(k, x)^\dagger] Y(x).
\] (23.3)

Recall that \( f(k, x) \) is constructed from \( \mathbf{D} \) via (10.6) and in turn \( K(x, y) \) is obtained as the unique solution to the Marchenko equation. Thus, using (10.6) on the right-hand side of (23.3) we obtain

\[
(\mathbf{F}_cY)(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dx \, [e^{ikx} + S(-k) e^{-ikx}] Y(x)
\] + \[
\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dx \int_{x}^{\infty} dy \, [e^{iky} + S(-k) e^{-iky}] K(x, y)^\dagger Y(x),
\] (23.4)

where we have also replaced \( S(k)^\dagger \) by \( S(-k) \), which follows from (4.4). Changing the order of integrations in the second integral in (23.4) we obtain

\[
\int_{0}^{\infty} dx \int_{x}^{\infty} dy \, [e^{iky} + S(-k) e^{-iky}] K(x, y)^\dagger Y(x)
\] = \[
\int_{0}^{\infty} dy \int_{0}^{y} dx \, [e^{iky} + S(-k) e^{-iky}] K(x, y)^\dagger Y(x).
\] (23.5)
We remark that the change of the order of integrations in (23.24) can be justified by using an argument similar to the one given at the end of the proof of Proposition 22.1, i.e. by approximating \( Y(x) \in L^2(\mathbb{R}^+) \) appearing in (23.4) with a convergent sequence in \( C_0(\mathbb{R}^+) \).

In terms of the operator \( K^\dagger \) related to (17.38), we recognize the right-hand side of (23.4) and obtain

\[
\int_0^\infty dy \int_0^y dx \left[ e^{iky} + S(-k) e^{-iky} \right] K(x, y)^\dagger Y(x) = \int_0^\infty dy \left[ e^{iky} + S(-k) e^{-iky} \right] (K^\dagger Y)(y). 
\]  

(23.6)

Using (23.6) in (23.4), we get

\[
(F_c Y)(k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dy \left[ e^{iky} + S(-k) e^{-iky} \right] [(I + K^\dagger) Y(y)],
\]  

(23.7)

or equivalently we obtain

\[
(F_c Y)(k) = h(k) + S(-k) h(-k), \quad k \in \mathbb{R}^+, 
\]  

(23.8)

where we have defined

\[
h(k) := \frac{1}{\sqrt{2\pi}} \int_0^\infty dy e^{iky} \left[ (I + K^\dagger) Y(y) \right].
\]  

(23.9)

From Proposition 17.5(e) we know that \( (I + K^\dagger) \) is a bounded operator on \( L^2(\mathbb{R}^+) \) and hence \( (I + K^\dagger) Y \) belongs to \( L^2(\mathbb{R}^+) \) because \( Y \in L^2(\mathbb{R}^+) \). Because the integrand in (23.9) has support in \( y \in \mathbb{R}^+ \), we conclude that \( h(k) \) defined in (23.9) belongs to the Hardy space \( H^2(\mathbb{C}^+) \). Comparing (23.1) and (23.8) we see that for any \( g(k) \in \Upsilon \subset L^2(\mathbb{R}) \), there exists \( h(k) \in H^2(\mathbb{C}^+) \) such that

\[
h(k) + S(-k) h(-k) = g(k), \quad k \in \mathbb{R}^+.
\]  

(23.10)

By postmultiplying (23.10) with \( S(k) \) we get

\[
S(k) h(k) + S(k) S(-k) h(-k) = S(k) g(k), \quad k \in \mathbb{R}^+.
\]  

(23.11)

From (4.4) we know that \( S(k) S(-k) = I \) for \( k \in \mathbb{R} \), and because \( g(k) \in \Upsilon \) we have \( S(k) g(k) = g(-k) \) for \( k \in \mathbb{R} \). Thus, (23.11) is equivalent to

\[
S(-k) h(-k) + h(k) = g(k), \quad k \in \mathbb{R}^-.
\]  

(23.12)
From (23.11) and (23.12) we conclude that for any \( g(k) \in \Upsilon \subset L^2(\mathbb{R}) \), there exists \( h(k) \in H^2(\mathbb{C}^+) \) such that
\[
\begin{align*}
h(k) + S(-k) h(-k) &= g(k), \quad k \in \mathbb{R}, \\
\end{align*}
\]
which proves (A). Thus, we have shown that if \( D \) belongs to the Faddeev class, then there exists and uniquely exists a corresponding scattering data set \( S \) as in (4.2) satisfying the conditions (I), (2), (A), (4c), either one of (Ve) or (Vh), and (VI) of Theorem 8.1. □

**Proposition 23.2** Let \( \tilde{\Upsilon} \) be a dense set in \( \Upsilon \), where \( \Upsilon \) is defined in the statement of Theorem 8.1. Let us denote by \( \tilde{\Upsilon}_+ \) the set of all the restrictions of vectors in \( \tilde{\Upsilon} \) to \( \mathbb{R}^+ \), i.e.,
\[
\tilde{\Upsilon}_+ := \left\{ g(k) \in L^2(\mathbb{R}^+) : g(k) = f(k), k \in \mathbb{R}^+, \text{for some } f(k) \in \tilde{\Upsilon} \right\}.
\]
Then, \( \tilde{\Upsilon}_+ \) is dense in \( L^2(\mathbb{R}^+) \).

PROOF: Suppose that \( h(k) \in L^2(\mathbb{R}^+) \) and that,\[
(h(k), g(k))_2 = 0, \quad \forall g(k) \in \tilde{\Upsilon}_+.
\]
We will prove that \( h(k) \equiv 0 \), which implies that \( \tilde{\Upsilon}_+ \) is dense in \( L^2(\mathbb{R}^+) \). We extend \( h(k) \) to a function in \( \Upsilon \) defining \( h(k) := S(-k) h(-k) \) when \( k \leq 0 \). Since by (4.4) we have \( S(-k) S(k) = I \), it follows that \( h(k) \) is indeed a vector in \( \Upsilon \). Then,
\[
\begin{align*}
(h(k), g(k))_2 &= (h(k), g(k))_{L^2(\mathbb{R}^+)} + (h(k), g(k))_{L^2(\mathbb{R}^-)} \\
&= (h(k), g(k))_{L^2(\mathbb{R}^+)} + (S(k) h(k), S(k) g(k))_{L^2(\mathbb{R}^+)} \\
&= 2 (h(k), g(k))_{L^2(\mathbb{R}^+)} = 0,
\end{align*}
\]
where we used that by (4.4) \( S(k)^\dagger S(k) = I \). Hence, \( h(k) \) is orthogonal in \( \Upsilon \) to \( \tilde{\Upsilon} \), and since \( \tilde{\Upsilon} \) is dense in \( \Upsilon \) we have \( h(k) = 0 \) for \( k \in \mathbb{R} \). □

**Proposition 23.3** Suppose that the following conditions are satisfied: Conditions (2) and (4c) of Definition 4.2, (I), (VI), and either one of (Ve) or (Vh) of Definition 4.3 and
(A) of Theorem 8.1. Then, the generalized Fourier map, $F$, defined in (22.3) is a unitary operator from $L^2(\mathbb{R}^+) \rightarrow \mathcal{R}$, where $\mathcal{R}$ is the Hilbert space defined in (22.1).

**PROOF:** Under our assumptions Proposition 6.1 and Proposition 16.10 apply. Hence, we construct the potential $V(x)$ satisfying (2.2) and (2.3), the Jost solution $f(k,x)$, the physical solution, $\Psi(k,x)$, and the normalized bound-state matrix solutions $\Psi_j(x)$ for $j = 1, \cdots, N$. In consequence, the generalized Fourier map $F$ is well defined. By Proposition 22.1 (b), the Fourier map $F$ is isometric from $L^2(\mathbb{R}^+)$ into $\mathcal{R}$. Hence, to prove that it is unitary we only need to prove that $F$ is onto $\mathcal{R}$. But as the range of any isometric operator is closed, it is enough to prove that the range of $F$ is dense in $\mathcal{R}$. We first consider the case when condition $(V_h)$ of Definition 4.3 holds. Suppose that $Y(x) \in \text{Ker}[F_c]$, i.e. we have

$$(F_c Y)(k) = 0, \quad k \in \mathbb{R}^+. \quad (23.17)$$

Then, (23.1) holds with $g(k) = 0$. Furthermore, arguing exactly as in the proof of Theorem 23.1 we prove that equations (23.8) and (23.9) hold, i.e. we have

$$(F_c Y)(k) = h(k) + S(-k)h(k) = 0, \quad k \in \mathbb{R}^+, \quad (23.18)$$

$$h(k) := \frac{1}{\sqrt{2\pi}} \int_0^\infty dy e^{iky} \left[ (I + K\dagger) Y(y) \right]. \quad (23.19)$$

Then, exactly in the same way as in the proof of Theorem 23.1 we prove that (23.13) holds with $g(k) = 0$, i.e., that,

$$h(k) + S(-k)h(-k) = 0, \quad k \in \mathbb{R}. \quad (23.20)$$

By Proposition 17.5(d) and the equivalence of $(4_a)$ and $(4_c)$ given in Proposition 6.1, $(I + K\dagger)$ is a bijection on $L^2(\mathbb{R}^+)$ and since the Fourier transform is a bijection from $L^2(\mathbb{R}^+)$ onto $H^2(\mathbb{C}^+)$, we have that $h(k) \in H^2(\mathbb{C}^+)$. Hence, $Y(x) \in \text{Ker}[F_c]$ if and only if $h(k) \in H^2(\mathbb{C}^+)$ is a solution to (23.20). Then, the dimension of the kernel of $F_c$ is equal to the number of linearly independent solutions to (23.20) that are in $H^2(\mathbb{C}^+)$. By condition $(V_h)$ there are $N$ linearly independent solutions to (23.20) in $H^2(\mathbb{C}^+)$ and in consequence the kernel of $F_c$ has dimension $N$. 232
Let us define the following map from $L^2(\mathbb{R}^+)$ into $\text{Ran}[M_1] \oplus \cdots \oplus \text{Ran}[M_N]$,

$$F_p Y := (F_1 Y, \ldots, F_N Y), \quad Y(x) \in L^2(\mathbb{R}^+), \quad (23.21)$$

where the $F_j, j = 1, \cdots, N$, are defined in (22.4). It follows from (22.3) that,

$$FY = (F_p Y, F_c Y), \quad Y(x) \in L^2(\mathbb{R}^+). \quad (23.22)$$

By Proposition 22.1(b), the map $F_p$ is isometric from Ker[$F_c$] into $\text{Ran}[M_1] \oplus \cdots \oplus \text{Ran}[M_N]$ and since the dimension of Ker[$F_c$] and of $\text{Ran}[M_1] \oplus \cdots \oplus \text{Ran}[M_N]$ is $N$, we have that $F_p$ is unitary from Ker[$F_c$] onto $\text{Ran}[M_1] \oplus \cdots \oplus \text{Ran}[M_N]$. Furthermore,

$$L^2 = \text{Ker}[F_c] \oplus (\text{Ker}[F_c])^\perp. \quad (23.23)$$

Since $F_p$ is unitary from Ker$F_c$ onto $\text{Ran}[M_1] \oplus \cdots \text{Ran}[M_N]$, to prove that the range of $F$ is dense in $\mathcal{R}$ it is enough to prove that the range of $F_c$ is dense in $L^2(\mathbb{R}^+)$. By Proposition 23.2 it is sufficient to prove that for every $g(k) \in \tilde{\mathcal{Y}}_+$ there is a function $Y(x) \in (\text{Ker}[F_c])^\perp$ such that,

$$g(k) = (F_c Y)(k), \quad k > 0. \quad (23.24)$$

Since $g(k) \in \tilde{\mathcal{Y}}_+$, there is a $f(k) \in \tilde{\mathcal{Y}}$ such that $g(k) = f(k), k \in \mathbb{R}^+$. Furthermore, by condition (A) there is a $h \in \mathcal{H}_2^+(\mathbb{C}^+)$ such that,

$$f(k) = h(k) + S(-k) h(-k), \quad k \in \mathbb{R}. \quad (23.25)$$

Denote,

$$m(x) := (I + K^\dagger)^{-1} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} h(k) \, dk. \quad (23.26)$$

The vector $m(x)$ belongs to $L^2(\mathbb{R}^+)$, because the inverse Fourier transform is a bijection from $\mathcal{H}_2^+(\mathbb{C}^+)$ onto $L^2(\mathbb{R}^+)$ and $(I + K^\dagger)$ is a bijection on $L^2(\mathbb{R}^+)$. By (23.25) and (23.36) we have

$$f(k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dx \left( e^{ikx} + S(-k) e^{-ikx} \right) \left[ (I + K^\dagger) m(x) \right], \quad k \in \mathbb{R}. \quad (23.27)$$
As in the proof of (23.7) we prove that the right-hand side of (23.27) is equal to $(F_c m)(k)$, for $k \in \mathbb{R}^+$, and then,

$$f(k) = (F_c m)(k), \quad k \in \mathbb{R}^+. \quad (23.28)$$

Let us decompose $m(x)$ as,

$$m(x) = m_1(x) + m_2(x), \quad m_1(x) \in \text{Ker}[F_c], \quad m_2(x) \in (\text{Ker}[F_c])^\perp. \quad (23.29)$$

Hence, by (23.28) and (23.29)

$$f(k) = (F_c m_2)(k), k > 0, \quad (23.30)$$

and since $g(k) = f(k), k \in \mathbb{R}^+$, we obtain that,

$$g(k) = (F_c m_2)(k), k > 0. \quad (23.31)$$

Then, (23.24) holds with $Y(x) = m_2(x)$, what proves that the range of $F_c$ is dense in $L^2(\mathbb{R}^+)$. Suppose now that condition $(V_e)$ of Definition 4.3 holds. Then, there are $N$ linearly independent solutions to (23.20) that are in $\hat{L}^1(\mathbb{C}^+)$, but by Proposition 15.7(e) these solutions are in $\hat{L}_\infty^1(\mathbb{C}^+)$ and as $\hat{L}_\infty^1(\mathbb{C}^+) \subset H^2(\mathbb{C}^+)$, there are $N$ linearly independent solutions to (23.20) in $H^2(\mathbb{C}^+)$. We complete the proof as in the case when condition $(V_e)$ of Definition 4.3 is satisfied. \[\square\]

Let us define the operator $\hat{L}$ in the Hilbert space $\mathcal{R}$ defined in (22.1),

$$\hat{L}(Z_1, \cdots, Z_N, Z) := (-\kappa_1^2 Z_1, \cdots, -\kappa_N^2 Z_N, k^2 Z(k)), \quad (Z_1, \cdots, Z_N, Z) \in D(\hat{L}), \quad (23.32)$$

where the domain of $\hat{L}$, that we denote by $D(\hat{L})$, is defined as follows,

$$D(\hat{L}) := \{(Z_1, \cdots, Z_N, Z) \in \mathcal{R} : k^2 Z(k) \in L^2(\mathbb{R}^+)\}. \quad (23.33)$$

Since $\hat{L}$ is a multiplication operator defined on its maximal domain, it is selfadjoint, i.e. $\hat{L}^\dagger = \hat{L}$.
Using the generalized Fourier map $F$ defined in (22.3), we define operator $\mathcal{L}$ in $L^2(\mathbb{R}^+)$,

$$\mathcal{L} := F^\dagger \hat{\mathcal{L}} F. \quad (23.34)$$

Then, we have that,

$$(\mathcal{L}Y)(x) = (F^\dagger \hat{\mathcal{L}} FY)(x), Y(x) \in D(\mathcal{L}) := \left\{ Y(x) \in L^2(\mathbb{R}^+) : (FY)(k) \in D(\hat{\mathcal{L}}) \right\}, \quad (23.35)$$

where by $D(\mathcal{L})$ we denote the domain of $\mathcal{L}$.

**Proposition 23.4** Suppose that the following conditions are satisfied: Conditions (2) and (4c) of Definition 4.2, (I), (VI), and either one of (Ve) or (Vh) of Definition 4.3 and (A) of Theorem 8.1. Then, the operator $\mathcal{L}$ defined in (23.34) is selfadjoint and its domain, $D(\mathcal{L})$, is contained in $H^1(\mathbb{R}^+)$,

$$D(\mathcal{L}) \subset H^1(\mathbb{R}^+), \quad (23.36)$$

where by $H^1(\mathbb{R}^+)$ we denote the Sobolev space of all vectors in $L^2(\mathbb{R}^+)$ with first derivative in $L^2(\mathbb{R}^+)$.

**PROOF:** By Proposition 23.3 the generalized Fourier map $F$ is unitary from $L^2(\mathbb{R}^+)$ onto $\mathcal{R}$. Then, $\mathcal{L}$ is selfadjoint because by its definition in (23.34), $\mathcal{L}$ is unitarily equivalent to $\hat{\mathcal{L}}$ that is selfadjoint because it is a multiplication operator defined on its maximal domain.

Let us prove (23.36). By definition $D(\mathcal{L}) \subset L^2(\mathbb{R}^+)$. Hence, we only need to prove that for any $Y(x) \in D(\mathcal{L})$, we have that, $Y'(x) \in L^2(\mathbb{R}^+)$. Recall that by Proposition 6.1 conditions (4a), and (4c) are equivalent. Let $Y(x)$ belongs to $D(\mathcal{L})$. Then by (23.35), and since $F^\dagger F = I$, there is a $(Z_1, \ldots, Z_N, Z) \in D(\hat{\mathcal{L}})$ such that

$$Y'(x) = (F^\dagger (Z_1, \ldots, Z_N, Z))'(x) = \sum_{j=1}^N (F^\dagger_j Z_j)'(x) + (F^\dagger c Z)'(x). \quad (23.37)$$

By (17.12) and (22.4),

$$(F^\dagger_j Z_j)'(x) \in L^2(\mathbb{R}^+), j = 1, \ldots, N. \quad (23.38)$$

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By (9.4) and (22.24)
\[(\mathbf{F}_c^\dagger Z)'(x) = Y_1(x) + Y_2(x), \quad (23.39)\]
where
\[Y_1(x) := -\frac{1}{\sqrt{2\pi}} \int_0^\infty f'(-k, x) Z(k) \, dk, \quad (23.40)\]
and
\[Y_2(x) := \frac{1}{\sqrt{2\pi}} \int_0^\infty f'(k, x) S(k) Z(k) \, dk. \quad (23.41)\]

Note that the derivation under the integral sign in (23.40), (23.41) is justified by Lebesgue dominated convergence theorem, since the integrands in (23.40), (23.41) are integrable column vectors. This is proven using (17.12), that by (4.4), \[\|S(k)\| = 1, \text{ and that as } k^2 Z(k) \in L^2(\mathbb{R}^+), \text{ we have that } (1 + k) Z(k) \in L^1(\mathbb{R}^+),\]
\[
\int_0^\infty (1 + k) |Z(k)| \, dk \leq \left( \int_0^\infty \frac{1}{(1 + k)^2} \, dk \right)^{\frac{1}{2}} \left( \int_0^\infty (1 + k)^4 |Z(k)|^2 \, dk \right)^{\frac{1}{2}} < \infty. \quad (23.42)\]

Let us prove that \[Y_1(x) \in L^2(\mathbb{R}^+). \]
Recall that, as in (5.1), the Jost solution can be characterized as the unique solution of the integral equation,
\[f(k, x) = e^{ikx} I_n + \frac{1}{k} \int_x^\infty \sin(k(y - x)) V(y) f(k, y) \, dy. \quad (23.43)\]

Here, \(V(x)\) is the reconstructed potential obtained from (10.4).

We have that,
\[f'(-k, x) = -ik e^{ikx} I_n - \int_x^\infty \cos(k(y - x)) V(y) f(k, y) \, dy. \quad (23.44)\]

The differentiation under the integral sign is justified because by (17.11) and (2.3) the integrand in (23.44) is integrable.

By (23.44),
\[Y_1(x) = -\frac{1}{\sqrt{2\pi}} \int_0^\infty f'(-k, x) Z(k) \, dk = g_1(x) + g_2(x), \quad (23.45)\]
where,
\[g_1(x) := \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ikx} ik Z(k) \, dk \in L^2(\mathbb{R}^+), \text{ since } k Z(k) \in L^2(\mathbb{R}^+), \quad (23.46)\]
and
\[ g_2(x) := \frac{1}{\sqrt{2\pi}} \int_0^\infty dk \ Z(k) \int_x^\infty \cos k(y-x) \ V(y) \ f(k,y) \ dy. \]  

(23.47)

By Proposition 6.1, (10.4), (16.78), (17.1) and (23.42),
\[ |g_2(x)| \leq C \left( \int_0^\infty |Z(k)| \ dk \right) \int_x^\infty |V(y)| \ dy \]
\[ \leq C(1 + x)^{-1} \left( \int_0^\infty (1 + y) |V(y)| \ dy \right) \in L^2(\mathbb{R}^+). \]  

(23.48)

Then, by (2.3), Proposition 7.1, (17.1), and (23.42), we have
\[ Y_1(x) \in L^2(\mathbb{R}^+). \]  

(23.49)

Recall that by (4.4) \( \|S(k)\| = 1 \). This allows us to prove, as above,
\[ Y_2(x) \in L^2(\mathbb{R}^+). \]  

(23.50)

Hence, by (23.39) and (23.49) we get
\[ (F_c^\dagger Z)'(x) \in L^2(\mathbb{R}^+). \]  

(23.51)

By (23.37), (23.38), and (23.51), we conclude that \( Y'(x) \in L^2(\mathbb{R}^+) \). □

We now introduce some concepts from the spectral theory of differential operators [4]. We denote by \( \mathcal{L}_{\text{max}} \) the maximal operator associated to \(-D_x^2 I + V\), where \( D_x \) is the derivative operator \( d/dx \), namely
\[ \mathcal{L}_{\text{max}} Y(x) := (-D_x^2 + V(x)) Y(x), \quad Y(x) \in D(\mathcal{L}_{\text{max}}), \]  

(23.52)

where the domain of \( \mathcal{L}_{\text{max}} \), that we denote by \( D(\mathcal{L}_{\text{max}}) \), is defined as follows,
\[ D(\mathcal{L}_{\text{max}}) = \{ Y(x) \in L^2(\mathbb{R}^+) : Y(x), Y'(x) \text{ are absolutely continuous on } (0, \infty) \] and \((-D_x^2 + V(x)) Y(x) \in L^2(\mathbb{R}^+) \}. \]  

(23.53)

We designate by \( \mathcal{L}_{\text{min}} \) the minimal operator associated to \(-D_x^2 I + V\),
\[ \mathcal{L}_{\text{min}} Y(x) := (-D_x^2 + V(x)) Y(x), \quad Y(x) \in D(\mathcal{L}_{\text{min}}), \]  

(23.54)
where the domain of $\mathcal{L}_{\text{min}}$, that we denote by $D(\mathcal{L}_{\text{min}})$, is defined as follows,

$$D(\mathcal{L}_{\text{min}}) = \{Y(x) \in D(\mathcal{L}_{\text{max}}) : Y(x), \text{ has compact support in } (0, \infty)\}. \quad (23.55)$$

Clearly, $\mathcal{L}_{\text{min}} \subset \mathcal{L}_{\text{max}}$. By Theorems 3.7 in page 47 and Theorem 3.9 in page 49 of [42] $D(\mathcal{L}_{\text{min}})$ is dense in $L^2(\mathbb{R}^+)$, $\mathcal{L}_{\text{min}}$ is symmetric and $\mathcal{L}_{\text{min}} \subset \mathcal{L}\dagger_{\text{min}} = \mathcal{L}_{\text{max}}$. We denote by $\overline{\mathcal{L}_{\text{min}}}$ the closure of $\mathcal{L}_{\text{min}}$. Note the difference in notation. In [42] $\mathcal{L}_{\text{min}}$ is called $T'_0$, $\mathcal{L}_{\text{max}}$ is called $T_0$ and $\mathcal{L}_{\text{max}}$ is called $T$.

**Proposition 23.5** Suppose that the following conditions are satisfied: Conditions (2) and (4c) of Definition 4.2, (I), (VI), and either one of (Vc) or (Vh) of Definition 4.3 and (A) of Theorem 8.1. Then,

$$\overline{\mathcal{L}_{\text{min}}} \subset \mathcal{L} \subset \mathcal{L}_{\text{max}}, \quad (23.56)$$

where $\overline{\mathcal{L}_{\text{min}}}$ is the closure of the operator $\mathcal{L}_{\text{min}}$ defined in (23.54), $\mathcal{L}$ is the selfadjoint operator defined in (23.34) and $\mathcal{L}_{\text{max}}$ is the operator defined in (23.52).

**PROOF:** We first prove that $\mathcal{L} \subset \mathcal{L}_{\text{max}}$. For this purpose we define $\mathcal{L}_0$, a restriction of $\mathcal{L}$, as

$$(\mathcal{L}_0Y)(x) := (\mathcal{L}Y)(x), \quad Y(x) \in \text{Dom}[\mathcal{L}_0], \quad (23.57)$$

where the domain of $\mathcal{L}_0$ is defined as

$$\text{Dom}[\mathcal{L}_0] := \{Y(x) \in L^2(\mathbb{R}^+) : Y(x) = F\dagger(Z_1, \cdots, Z_N, Z) \text{ with }$$

$$(Z_1, \cdots, Z_N, Z) \in \mathcal{R}, \text{ where } Z(k) \in L^2(\mathbb{R}^+) \text{ and } Z(k) \text{ has compact support}\} \quad (23.58)$$

As $Z(k)$ has compact support, $k^2Z(k) \in L^2(\mathbb{R}^+)$ and then, $\mathcal{L}_0 \subset \mathcal{L}$. Moreover, $\overline{\mathcal{L}_0} = \mathcal{L}$. Furthermore, as $\mathcal{L}$ is closed because it is selfadjoint and since $\mathcal{L}_{\text{max}}$ is closed because it is the adjoint of $\mathcal{L}_{\text{min}}$, it is enough to prove that $\mathcal{L}_0 \subset \mathcal{L}_{\text{max}}$. Suppose that $Y(x) \in \text{Dom}[\mathcal{L}_0]$. Then,

$$\mathcal{L}_0Y(x) = F\dagger(-\kappa_1^2Z_1, \cdots, -\kappa_N^2Z_N, k^2Z(k)) = \sum_{j=1}^N F\dagger_j(-\kappa_j^2)Z_j + F\dagger c k^2Z(k). \quad (23.59)$$

But,

$$F\dagger_j(-\kappa_j^2)Z_j = \Psi_j(x)(-\kappa_j^2)Z_j = (-D_x^2 + V(x)) F\dagger_j Z_j, \quad j = 1, \ldots, N, \quad (23.60)$$
because $\Psi_j(x)$ is a solution to (2.1) with $k = i\kappa_j$ for $j = 1, \ldots, N$. Moreover,

$$
(F_c^* k^2 Z(k))(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \Psi(k, x) k^2 Z(k) \, dk
= (-D_x^2 + V(x)) \frac{1}{\sqrt{2\pi}} \int_0^\infty \Psi(k, x) Z(k) \, dk
= (-D_x^2 + V(x)) (F_c^* Z(k))(x),
$$

(23.61)

where we used that $\Psi(k, x)$ is a solution of (2.1) and that $Z(k)$ has compact support. By (23.35), (23.59), (23.60), (23.61)

$$
\mathcal{L}_0 Y(x) = (-D_x^2 + V(x)) Y(x) \in L^2(\mathbb{R}^+), \quad Y(x) \in \text{Dom}[\mathcal{L}_0].
$$

(23.62)

Furthermore,

$$
D_x^2 Y(x) = V(x) Y(x) - \mathcal{L}_0 Y(x).
$$

(23.63)

By Proposition 23.4 $Y(x)$ and $Y'(x)$ belong to $L^2(\mathbb{R}^+)$. Then,

$$
|Y(x)|^2 = - \int_x^\infty (|Y(x')|^2)' \, dx = - \int_x^\infty ((Y^\dagger)'(x)Y(x) + Y^\dagger(x)Y'(x)) \, dx,
$$

(23.64)

and then, by Schwarz inequality,

$$
|Y(x)|^2 \leq 2 \|Y(x)\|_2 \|Y'(x)\|_2.
$$

(23.65)

Since $V(x)$ satisfies (2.3), $Y(x)$ is bounded by (23.65), and $\mathcal{L}_0 Y(x) \in L^2(\mathbb{R}^+)$, we have that $D_x^2 Y(x) \in L^1(0, R)$ for any $R > 0$. In consequence $Y(x)$ and $Y'(x)$ are absolutely continuous and since by (23.62) $(-D_x^2 + V(x)) Y(x) \in L^2(\mathbb{R}^+)$ it follows that $Y(x)$ is in the domain of $\mathcal{L}_{\text{max}}$, and we have proved that,

$$
\text{Dom}[\mathcal{L}_0] \subset \text{Dom}[\mathcal{L}_{\text{max}}].
$$

(23.66)

Equation (23.62) and (23.66) imply that $\mathcal{L}_0 \subset \mathcal{L}_{\text{max}}$. Let us now prove that $\overline{\mathcal{L}_{\text{min}}} \subset \mathcal{L}$. Since $\mathcal{L}$ is closed, it is enough to prove that $\mathcal{L}_{\text{min}} \subset \mathcal{L}$. Suppose that $Y(x) \in \text{Dom}[\mathcal{L}_{\text{min}}]$. Then, denote

$$
Z_j := F_j Y, \quad j = 1, \ldots, N, \quad Z(k) := (F_c Y)(k).
$$

(23.67)
We have that,
\[ \mathbf{F} \mathbf{Y} = (Z_1, \ldots, Z_N, Z(k)). \] (23.68)

Moreover,
\[ \mathbf{F}_j \mathcal{L}_{\min} \mathbf{Y}(x) = \mathbf{F}_j(-D_x^2 + V(x))Y(x) \]
\[ = \int_0^\infty \Psi_j^\dagger(x)(-D_x^2 + V(x))Y(x) dx \] (23.69)
\[ = -\kappa_j^2 Z_j, \quad j = 1, \ldots, N, \]
where we used that \( \Psi_j(x) \) satisfies (2.1) and that \( Y(x) \) has compact support in \( \mathbb{R}^+ \) in order to integrate by parts. In a similar way we prove that
\[ (\mathbf{F}_c \mathcal{L}_{\min} \mathbf{Y})(k) = (\mathbf{F}_c(-D_x^2 + V(x))Y(x))(k) \]
\[ = \frac{1}{\sqrt{2\pi}} \int_0^\infty \Psi(k,x)^\dagger (-D_x^2 + V(x))Y(x) dx \] (23.70)
\[ = k^2 Z(k). \]

By (23.69), (23.70)
\[ \mathbf{F}(\mathcal{L}_{\min} \mathbf{Y})(x) = (\mathbf{F}(-D_x^2 + V(x))Y(x)) \]
\[ = (-\kappa_1^2 Z_1, \ldots, -\kappa_N^2 Z_N, k^2 Z(k)) \] (23.71)
\[ = \hat{\mathcal{L}}(Z_1, \ldots, Z_N, Z(k)) \in \mathcal{R}, \]
where we used that \( \mathcal{L}_{\min} \mathbf{Y}(x) = (-D_x^2 + V(x))Y(x) \in L^2(\mathbb{R}^+) \) and that \( \mathbf{F} \) is unitary from \( L^2(\mathbb{R}^+) \) onto \( \mathcal{R} \). In particular, this implies that \( k^2 Z(k) \in L^2(\mathbb{R}^+) \). This means that \( \mathbf{F} \mathbf{Y} \in \text{Dom}[\hat{\mathcal{L}}] \), and then, \( Y(x) \in \text{Dom}[\mathcal{L}] \). It follows that,
\[ \text{Dom}[\mathcal{L}_0] \subset \text{Dom}[\mathcal{L}]. \] (23.72)

Multiplying by \( \mathbf{F}^\dagger \) in both sides of (23.71), using \( \mathbf{F}^\dagger \mathbf{F} = I \) and (23.68) we obtain that,
\[ \mathcal{L}_{\min} \mathbf{Y}(x) = \left( \mathbf{F}^\dagger \hat{\mathcal{L}} \mathbf{F} \mathbf{Y} \right)(x) = \mathcal{L} \mathbf{Y}(x), \] (23.73)
where we used (23.34). Equations (23.72) and (23.73) imply that \( \mathcal{L}_{\min} \subset \mathcal{L} \).}

**Proposition 23.6** Suppose that the following conditions are satisfied: Conditions (2) and (4c) of Definition 4.2, (I),(VI), and either one of (V_e) or (V_h) of Definition 4.3 and (A)
of Theorem 8.1. Then, there is a pair of matrices \((A, B)\), unique up to post multiplication by an invertible matrix \(T\), that satisfy (2.6), (2.7), and such that, all vectors \(Y(x)\) in the domain of \(L\) satisfy

\[-B^\dagger Y(0) + A^\dagger Y'(0) = 0, \quad Y(x) \in \text{Dom}[\mathcal{L}].\]  

(23.74)

PROOF: As in the proof of Proposition 23.5 we prove that for any \(Y(x) \in \text{Dom}[\mathcal{L}]\), \(D_x^2 Y(x) \in L^1(0, R)\) for any \(R > 0\). Then, \(Y(x)\) and \(Y'(x)\) are absolutely continuo on \([0, \infty)\) and in consequence, \(Y(0)\) and \(Y'(0)\) are well defined so that (23.74) makes sense.

Let us define the quadratic form,

\[ [Y, G]_x := \sum_{j=1}^n (Y_j(x)^* G'_j(x) - Y'_j(x)^* G_j(x)), \quad Y, G \in \text{Dom}[\mathcal{L}_{\text{max}}]. \]  

(23.75)

Furthermore (see page 28 of [2] and Section II of [9]), the matrix Schrödinger equation (2.1) has the \(n \times n\) matrix solution \(g(k, x)\) that satisfies for each \(k \in \mathbb{C}^+ \setminus \{0\}\) the asymptotics

\[
\begin{cases}
  g(k, x) = e^{-ikx} \left( I + o \left( \frac{1}{x} \right) \right), & x \to +\infty \\
  g'(k, x) = -ike^{-ikx} \left( I + o \left( \frac{1}{x} \right) \right), & x \to +\infty.
\end{cases}
\]  

(23.76)

Moreover, the combined \(2n\) columns of \(f(k, x)\) and \(g(k, x)\) form a fundamental system of solutions to (2.1). Then, any column vector solution \(Y(k, x)\) to (2.1) can be written as follows,

\[ Y(k, x) = f(k, x) Q_1 + g(k, x) Q_2, \]  

(23.77)

for some constant column vectors \(Q_1, Q_2\). It follows from (23.76) that each of the \(n\) columns of \(g(k, x)\) grows exponentially as \(x \to +\infty\) for each fixed \(k \in \mathbb{C}^+\). Then, all solutions to (2.1) with \(k \in \mathbb{C}^+\) (or equivalently with \(\text{Im} \ k^2 \neq 0\)) that belong to \(L^2(R < x < \infty)\) for \(R \geq 0\) must have \(Q_2 = 0\). Note that it follows from (10.6) that the \(n\) columns of \(f(k, x)\) are linearly independent solution to (2.1) and that they decay exponentially as \(x \to +\infty\) for each fixed \(k \in \mathbb{C}^+\). Hence, for each fixed \(k^2\) with \(\text{Im}[k^2] \neq 0\) there are exactly \(n\) linearly
independent solution to (2.1) that are in \( L^2(R < x < \infty) \) for \( R \geq 0 \). Then, it follows from Proposition 23.5 and Theorem 4.8 in page 61 of [42] that,

\[
[Y, G]_{\infty} := \lim_{x \to +\infty} [Y, G]_x = 0, \quad \forall Y(x), G(x) \in \text{Dom}[L_{\text{max}}]. \tag{23.78}
\]

Hence, by Theorem 4.1 on page 53 of [42], we have that for \( Y(x), G(x) \in \text{Dom}[L] \),

\[
[Y, G]_0 = (LY, G) - (Y, LG) = 0, \tag{23.79}
\]

( what is obvious in our case because \( L \) is selfadjoint), and moreover, if \( Y \in \text{Dom}[L_{\text{max}}] \) and \( [Y, G]_0 = 0 \) for all \( G \in \text{Dom}[L] \), then, \( f \in \text{Dom}[L] \). This means that \( D(L) \) is a maximal isotropic subspace for \([\cdot, \cdot]_0\). In consequence, by Lemma 2.2 of [29] and Theorem 2.1 of [5] there exist matrices \( A, B \) that satisfy (2.6), (2.7) such that all the vectors on the domain of \( L \) satisfy the boundary conditions (23.74). □

**Proof of Theorem 8.1** By Theorem 23.1 if the input data set \( D \) belongs to the Faddeev class the conditions (2) and (4c) of Definition 4.2, (I), (VI), and either one of (Ve) or (Vh) of Definition 4.3 and (A), are satisfied. Let us assume that the scattering data set \( S \) satisfies the conditions: (2) and (4c) of Definition 4.2, (I), (VI), and either one of (Ve) or (Vh) of Definition 4.3 and (A). Using Proposition 6.1 and Proposition 16.10 we construct the potential \( V(x) \), that satisfies (2.2), (2.3), the Jost solution, \( f(k, x) \), the physical solution, \( \Psi(k, x) \), and the normalized bound state matrix solutions \( \Psi_j(x) \) for \( j = 1, \ldots, N \). We have to prove that the \( \Psi_j(x) \) for \( j = 1, \ldots, N \), and the physical solution \( \Psi(k, x) \) satisfy the boundary condition (2.4). By Proposition 23.4 the operator, \( L \) defined in (23.34), (23.35), is selfadjoint and by Proposition 23.6 the vectors in the domain of \( L \) satisfy the boundary condition (23.74). For any constant column vector \( Z_j \in \text{Ran} M_j \) with \( j = 1, \ldots, N \), let us define

\[
(0, \cdots, 0, Z_j, 0 \cdots, 0) \in \mathcal{R}, \tag{23.80}
\]

where the component at the position \( j \) is equal to \( Z_j \) and all the others are equal to zero. Then,

\[
(0, \cdots, 0, Z_j, 0, \cdots, 0) \in \text{Dom}[\hat{L}], \tag{23.81}
\]

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where the operator $\hat{L}$ is defined in (23.32), (23.33). Then,

$$
\Psi_j(x)Z_j = F_j^\dagger Z_j = F(0, \ldots, Z_j, 0 \ldots, 0) \in \text{Dom}[\mathcal{L}],
$$

(23.82)

where we used (23.35). Then, we have

$$
-B^\dagger \Psi_j(0)Z_j + A^\dagger \Psi_j'(0)Z_j = 0, \quad \forall Z_j \in \text{Ran}[M_j], \quad j = 1, \ldots, N.
$$

(23.83)

Given any constant column vector $Q \in \mathbb{C}^n$ let us decompose it as follows,

$$
Q = Z_j + \tilde{Q}, \quad Z_j \in \text{Ran}[M_j], \quad \tilde{Q} \in \text{Ker}[M_j], \quad j = 1, \ldots, N.
$$

(23.84)

Hence, by the definition of $\Psi_j$ in (9.8) and (23.82)

$$
-B^\dagger \Psi_j(0)Q + A^\dagger \Psi_j'(0)Q = -B^\dagger \Psi_j(0)Z_j + A^\dagger \Psi_j'(0)Z_j
$$

$$
= 0, \quad j = 1, \ldots, N, \quad \forall Q \in \mathbb{C}^n.
$$

(23.85)

But then,

$$
-B^\dagger \Psi_j(0) + A^\dagger \Psi_j'(0) = 0, \quad j = 1, \ldots, N.
$$

(23.86)

Let us now prove that the physical solution $\Psi(k, x)$ satisfies the boundary condition. For any $k_0 \in (0, \infty)$, $\varepsilon > 0$, we denote by $\chi_{[k_0-\varepsilon, k_0+\varepsilon]}(k)$ the characteristic function of $[k_0 - \varepsilon, k_0 + \varepsilon]$, i.e., $\chi_{[k_0-\varepsilon, k_0+\varepsilon]}(k) = 1$ for $k \in [k_0 - \varepsilon, k_0 + \varepsilon]$ and $\chi_{[k_0-\varepsilon, k_0+\varepsilon]}(k) = 0$ if $k \in [0, \infty] \setminus [k_0 - \varepsilon, k_0 + \varepsilon]$. Let us define,

$$
(0, \ldots, 0, \frac{1}{2\varepsilon} \chi_{[k_0-\varepsilon, k_0+\varepsilon]}(k)) \in \mathbb{R}.
$$

(23.87)

Since $k^2 \chi_{[k_0-\varepsilon, k_0+\varepsilon]}(k) \in L^2(\mathbb{R}^+)$, we have

$$
(0, \ldots, 0, \frac{1}{2\varepsilon} \chi_{[k_0-\varepsilon, k_0+\varepsilon]}(k)) \in \text{Dom}[\hat{\mathcal{L}}].
$$

(23.88)

Then,

$$
\psi_\varepsilon(k_0, x) : = \frac{1}{\sqrt{2\pi}} \int_{k_0-\varepsilon}^{k_0+\varepsilon} \psi(k, x) \frac{1}{2\varepsilon} dk
$$

$$
= F^\dagger(0, \ldots, 0, \frac{1}{2\varepsilon} \chi_{[k_0-\varepsilon, k_0+\varepsilon]}(k)) \in \text{Dom}[\mathcal{L}].
$$

(23.89)
In consequence,

\[-B^\dagger \psi \varepsilon (k_0, 0) + A^\dagger \psi_\varepsilon'(k_0, 0) = 0. \tag{23.90}\]

By the definition (9.4) of the physical solution and since \(f(k, x)\) is continuous in \(k\) for each fixed \(x\) and \(S(k)\) is continuous in \(k\), it follows from (23.89) and the mean value theorem that,

\[-B^\dagger \psi(k_0, 0) + A^\dagger \psi'(k_0, 0) = \lim_{\varepsilon \to 0} \left( -B^\dagger \psi \varepsilon (k_0, 0) + A^\dagger \psi_\varepsilon'(k_0, 0) \right) = 0, \tag{23.91}\]

which proves that the physical solution \(\Psi(k, x)\) satisfies the boundary condition (2.4) for \(k > 0\), and by continuity also for \(k = 0\). We prove that, in the direct problem, the reconstructed input data set, \(D := \{V, A, B\}\) yields the same scattering data set \(S\) used as input of the inverse scattering problem as in the proof of Theorem 5.1. The fact \(V\) is unique and that \((A, B)\) are unique up to the transformation \((A, B) \mapsto (AT, BT)\) for any invertible matrix \(T\) follows from Proposition 16.2. □
24. A STAR GRAPH

In this chapter we show that a matrix Schrödinger equation with a diagonal potential matrix is unitarily equivalent to a star graph. A star graph is a quantum graph with only one vertex and a finite number, \( n \), of semi infinite edges. The Hilbert space is given by

\[
\mathcal{H} := \bigoplus_{j=1}^{n} L^2(\mathbb{R}^+, \mathbb{C}),
\]

where by \( L^2(\mathbb{R}^+, \mathbb{C}) \) we denote the Hilbert space of square-integrable functions defined in \( \mathbb{R}^+ \) and with values in \( \mathbb{C} \). An element \( \mathbf{Y} \) of \( \mathcal{H} \) is a finite sequence

\[
\mathbf{Y} := \{Y_1(x), \ldots, Y_n(x)\},
\]

and the scalar product of \( \mathbf{Y} \) and \( \mathbf{W} \) in \( \mathcal{H} \) is described as

\[
(\mathbf{Y}, \mathbf{W}) := \sum_{j=1}^{n} (Y_j(x)W_j(x))_{L^2(\mathbb{R}^+, \mathbb{C})}.
\]

The Schrödinger equation on the star graph is given by

\[
\mathbf{L Y} = k^2 \mathbf{Y},
\]

where

\[
\mathbf{L Y} := (-Y_1''(x) + V_1(x)Y_1(x), \ldots, -Y_n''(x) + V_n(x)Y_n(x)),
\]

with the potentials \( V_j(x) \) for \( j = 1, \ldots, n \) being real-valued functions satisfying (2.3). The boundary conditions are given by

\[
-B^\dagger Y(0) + A^\dagger Y'(0) = 0,
\]

where the \( n \times n \) matrices \( A \) and \( B \) satisfy (2.5) and (2.6).

Let us prove that the star graph is unitarily equivalent to a matrix Schrödinger equation with the diagonal matrix potential given by

\[
V(x) := \text{diag}\{V_1(x), \ldots, V_n(x)\},
\]
and with the boundary condition (2.4) given by the same matrices $A$ and $B$ as in (24.6).

We define the unitary operator $U$ from $\mathcal{H}$ onto $L^2(\mathbb{R}^+)$ as

$$
\psi(x) = U \mathbf{Y} := \begin{bmatrix}
Y_1(x) \\
\vdots \\
Y_n(x)
\end{bmatrix}.
$$

(24.8)

Clearly we have

$$
||U \mathbf{Y}||_2 = ||\mathbf{Y}||_\mathcal{H}, \quad Y(x) \in \mathcal{H},
$$

(24.9)

and $U$ is onto $\mathcal{H}$. Moreover, $\mathbf{Y}(x)$ is a solution to the system consisting of (24.4) and (24.5) if and only if $\Psi(x) := U \mathbf{Y}(x)$ is a solution to (2.1) and $Y(x)$ satisfies the boundary condition (24.6) if and only if $\Psi(x) := U \mathbf{Y}(x)$ satisfies the boundary condition (2.4) with the same matrices $A$ and $B$. Then $U$ establishes a unitary equivalence between both problems.
25. THE SCHröDINGER EQUATION ON THE FULL LINE

A 2 × 2 matrix Schrödinger equation is unitarily equivalent to a Schrödinger equation on the full line with a point interaction at \( x = 0 \). The Hilbert space for the Schrödinger equation on the line is \( L^2(\mathbb{R}, \mathbb{C}) \), where by \( L^2(\mathbb{R}, \mathbb{C}) \) we denote the Hilbert space of square integrable functions defined on \( \mathbb{R} \) and with values in \( \mathbb{C} \). We define the unitary operator \( U \) from \( L^2(\mathbb{R}^+) \) onto \( L^2(\mathbb{R}, \mathbb{C}) \) as

\[
Y(x) = U\psi := \begin{cases} 
\psi_1(x), & x \geq 0, \\
\psi_2(-x), & x < 0.
\end{cases}
\] (25.1)

Clearly we have

\[
\|U\psi\|_{L^2(\mathbb{R}, \mathbb{C})} = \|\psi\|_{L^2(\mathbb{R}^+)}, \quad \psi(x) \in L^2(\mathbb{R}^+),
\] (25.2)

and \( U \) is onto \( L^2(\mathbb{R}, \mathbb{C}) \). Suppose that the potential matrix \( V(x) \) is diagonal, i.e.

\[
V(x) := \text{diag}\{V_1(x), V_2(x)\}. \quad (25.3)
\]

We conclude that \( \psi(x) \) satisfies the Schrödinger equation (2.1) if and only if \( Y(x) := U\psi(x) \) satisfies the Schrödinger equation on the line given by

\[
-Y''(x) + Q(x)Y(x) = k^2 Y(x),
\] (25.4)

with the potential,

\[
Q(x) := \begin{cases} 
V_1(x), & x \geq 0, \\
V_2(-x), & x < 0.
\end{cases}
\] (25.5)

Moreover, \( \psi(x) \) satisfies the boundary condition (2.4) if and only if \( Y(x) \) satisfies the point-interaction condition

\[
\begin{cases} 
-(B^\dagger)_{11} Y(0^+) - (B^\dagger)_{12} Y(0^-) + (A^\dagger)_{11} Y'(0^+) - (A^\dagger)_{12} Y'(0^-) = 0, \\
-(B^\dagger)_{21} Y(0^+) - (B^\dagger)_{22} Y(0^-) + (A^\dagger)_{21} Y'(0^+) - (A^\dagger)_{22} Y'(0^-) = 0,
\end{cases}
\] (25.6)

where \((A^\dagger)_{ij}\) and \((B^\dagger)_{ij}\) denote the \((i, j)\)-th entry of the matrices \(A^\dagger\) and \(B^\dagger\), respectively. For example, suppose that \( \psi(x) \) satisfies the δ-type boundary condition \( \psi_1(0) = \psi_2(0) \) and \( \psi'_1(0) + \psi'_2(0) = \lambda \psi_1(0) \), where \( \lambda \) is a real number, for which the special case \( \lambda = 0 \)
corresponds to the Kirchhoff boundary condition. In this case, the matrices appearing in (25.6) are given by

\[ A := \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \]  

(25.7)

\[ B := \begin{bmatrix} -1 & \lambda \\ 1 & 0 \end{bmatrix}. \]  

(25.8)

The boundary conditions (25.6) corresponding to the matrices \( A \) and \( B \) given in (25.7) and (25.8) are \( Y(0^-) = Y(0^+) \) and \( Y'(0^+) - Y'(0^-) = \lambda Y(0^+) \), which is related to the Schrödinger equation on the line with a \( \delta \)-point interaction.
26. SOME EXPLICIT EXAMPLES

In this chapter, through some explicit examples, we illustrate our theoretical results obtained in previous chapters.

The explicit examples for the direct and inverse scattering problems mainly involve scattering matrices $S(k)$ that are rational functions of $k$ in the complex plane. In this case, the quantity $F(y)$ defined in (4.12) yields a separable kernel for the Marchenko integral equation (13.1). Then, it is possible to solve (13.1) explicitly by using methods of linear algebra.

In the first example, we show that, by using a special case of the method of [3,4,10], one can easily produce explicit examples for the input data set $D$ and the corresponding scattering data set $S$ for any values of $n$ when the scattering matrix $S(k)$ is a rational function of $k$. This amounts to [3,4,10] choosing $F(y)$ as a matrix-valued function containing a matrix exponential.

**Example 26.1** For any constant $n \times n$ hermitian matrix $a$ with positive eigenvalues, let us choose $F(y)$ given in (4.12) as

$$F(y) = c \, e^{-ay} \, c, \quad y \in \mathbb{R}^+, \quad (26.1)$$

where $c$ is a constant $n \times n$ hermitian matrix. We remark that we use a matrix exponential in (26.1) because $a$ is a matrix. The corresponding Marchenko equation has a separable kernel and hence can be solved explicitly by algebraic means. Let $m$ be the solution to the linear system

$$am + ma = c^2, \quad (26.2)$$

where the existence and uniqueness of $m$ is assured [17] and in fact it can be evaluated explicitly by using

$$m = \int_0^\infty dy \, e^{-ay} \, c^2 \, e^{-ay}. \quad (26.3)$$

Using $F(y)$ given in (26.1) as input to (13.1), one obtains the solution to (13.1) explicitly
as

\[ K(x, y) = -c (m + e^{2ax})^{-1} e^{a(x-y)} c, \]  

(26.4)
yielding

\[ K(x, x) = -c (m + e^{2ax})^{-1} c, \quad K(0, 0) = -c (m + I)^{-1} c. \]  

(26.5)

Using (26.4) in (10.6) and using the first equality of (26.5) in (10.4), we obtain the corresponding Jost solution \( f(k, x) \) and the potential \( V(x) \) as

\[ f(k, x) = e^{ikx} \left[ I - c (m + e^{2ax})^{-1} (a - ikI)^{-1} c \right], \]  

(26.6)

\[ V(x) = -4c (m + e^{2ax})^{-1} a e^{2ax} (m + e^{2ax})^{-1} c. \]  

(26.7)

From (26.6) we get

\[ f(k, 0) = I - c (m + I)^{-1} (a - ikI)^{-1} c, \]  

(26.8)

\[ f'(k, 0) = ikI + c (m + I)^{-1} \left[ 2a - ikI + 2a (m + I)^{-1} \right] (a - ikI)^{-1} c, \]  

(26.9)

Along with any pair of constant \( n \times n \) matrices \( A \) and \( B \) satisfying (2.5) and (2.6), using (26.8) and (26.9) in (9.2) we obtain the corresponding Jost matrix \( J(k) \) and then obtain the corresponding scattering matrix \( S(k) \) by using (9.3). The corresponding physical solution \( \Psi(k, x) \) can be obtained via (9.4). One can certainly enhance this method by choosing \( F_s(y) \) for \( y \in \mathbb{R}^- \) in a form similar to (26.1) by replacing \( a \) with some constant \( n \times n \) hermitian matrix with negative eigenvalues. The use of matrix exponentials allows us to write the explicit solutions in the direct and inverse problems in a compact form. For instance, by choosing

\[ a = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \]  

(26.10)

we get an example where \( a \) has eigenvalues 4, 2, and 2 and the matrix \( m \) is given by

\[ m = \begin{bmatrix} 11/48 & 3/16 & 1/8 \\ 3/16 & 43/48 & 5/8 \\ 1/8 & 5/8 & 1/2 \end{bmatrix}, \quad e^{-ay} = \frac{1}{2} \begin{bmatrix} e^{-2y} + e^{-4y} & e^{-2y} - e^{-4y} & 0 \\ e^{-2y} - e^{-4y} & e^{-2y} + e^{-4y} & 0 \\ 0 & 0 & 2e^{-2y} \end{bmatrix}, \]  

(26.11)
\[
(a - ikI)^{-1} = \frac{1}{(k + 2i)^2(k + 4i)} \begin{bmatrix}
  i(k + 2i)(k + 3i) & -(k + 2i) & 0 \\
  -(k + 2i) & i(k + 2i)(k + 3i) & 0 \\
  0 & 0 & i(k + 2i)(k + 4i)
\end{bmatrix}.
\]

(26.12)

The purpose of the next example is to emphasize the fact that, in seeking a solution to an integral equation such as (4.17), (4.22), and (7.1), it is important for us to state whether we look for a square-integrable solution or an integrable solution. For example, the class \(L^1(\mathbb{R}^+)\) is different from \(L^2(\mathbb{R}^+)\), and the class \(L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)\) is a proper subspace of the class \(L^2(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)\).

**Example 26.2** Assume that \(n = 1\) and that we have

\[
X(y) = \begin{cases}
  \frac{1}{1 + y}, & y \in \mathbb{R}^+, \\
  0, & y \in \mathbb{R}^-.
\end{cases}
\]

(26.13)

We notice that \(X(y)\) is square integrable in \(y \in \mathbb{R}^+\) but not integrable there. Using (3.68) we can evaluate \(\hat{X}(k)\) explicitly as

\[
\hat{X}(k) = -ie^{-ik} \left[ \frac{\pi}{2} - \text{Si}(k) - i \text{Ci}(k) \right],
\]

(26.14)

in terms of the sine integral function \(\text{Si}(k)\) and the cosine integral function \(\text{Ci}(k)\) defined as

\[
\text{Si}(k) := \int_0^k dt \frac{\sin t}{t}, \quad \text{Ci}(k) := \int_k^\infty dt \frac{\cos t}{t}.
\]

(26.15)

Even though \(\text{Si}(k)\) is well behaved at \(k = 0\), the function \(\text{Ci}(k)\) has a logarithmic singularity at \(k = 0\), which is seen from its representation

\[
\text{Ci}(k) = \gamma + \log k + \sum_{j=1}^\infty \frac{(-k^2)^j}{2j(2j)!},
\]

(26.16)

where \(\gamma\) is the Euler-Mascheroni constant. Thus, \(\hat{X}(k)\) given in (2.16b) is not continuous at 0. This example indicates that the proof given in [2] for Theorem 3.5.1 of [2] needs to be improved. For example, one can use a procedure such as that given in our own Propositions 15.12 to prove Theorem 3.5.1 of [2].
The purpose of the next example is to emphasize the fact that, in seeking a solution to a Riemann-Hilbert problem such as (4.19), (4.24), and (7.2), it is important for us to state whether we look for a solution in the Hardy space $H^2(\mathbb{C}^+)$ or in $\hat{L}^1(\mathbb{C}^+)$. In the next example we present a function that belongs to the Hardy space $H^2(\mathbb{C}^+)$ but not continuous on $\mathbb{R}$ and hence not in $\hat{L}^1(\mathbb{C}^+)$. 

Example 26.3 Let

$$\hat{X}(k) = \frac{\log k}{k + i}, \quad k \in \mathbb{C}^+, \quad (26.17)$$

where $\log$ denotes the principal branch of the logarithm function with the argument of $k$ limited to the interval $(-\pi, \pi)$. From (26.17), we see that $\hat{X}(k)$ is analytic in $\mathbb{C}^+$ but it is not continuous at $k = 0$ and hence it is not continuous in $k \in \mathbb{C}^+$. Letting $k = |k| e^{i\theta}$ with $\theta \in (-\pi, \pi)$, we can write $\hat{X}(k)$ in $\mathbb{C}^+$ as

$$\hat{X}(k) = \frac{\ln |k| + i\theta}{|k| \cos \theta + i(1 + |k| \sin \theta)}, \quad |k| > 0, \quad \theta \in (-\pi, \pi). \quad (26.18)$$

Let us use $k_R$ and $k_I$ for the real and imaginary parts of $k$. Because $\hat{X}(k)$ is continuous on the line $k = k_R + ik_I$ with $k_R \in \mathbb{R}$ for any fixed positive $k_I$, in order to prove the integrability of $|\hat{X}(k_R + ik_I)|^2$ in $k_R \in \mathbb{R}$ for each fixed $k_I > 0$, it is enough to check the integrability of $|\hat{X}(k_R + ik_I)|^2$ as $k_R \to \pm \infty$. From (26.18), we have

$$|\hat{X}(k_R + ik_I)|^2 = \frac{(\ln |k|)^2 + \theta^2}{|k|^2 + 2 |k| \sin \theta + 1}, \quad (26.19)$$

and hence we can find some constant $c > 1$ such that

$$|\hat{X}(k_R + ik_I)|^2 \leq C \frac{(\ln |k|)^2}{|k|^2}, \quad |k_R| \geq c, \quad (26.20)$$

for some generic constant $C$. Then, with the help of (26.20) we get

$$\int_{-\infty}^{\infty} dk_R |\hat{X}(k_R + ik_I)|^2 \leq C + C \int_{1}^{\infty} d|k| \frac{(\ln |k|)^2}{|k|^2} \leq C + C \int_{0}^{\infty} d\alpha \frac{\alpha^2}{e^{\alpha}} < +\infty, \quad (26.21)$$

where we have used the substitution $\alpha = \ln |k|$. Thus, $\hat{X}(k)$ given in (26.17) belongs to the Hardy space $H^2(\mathbb{C}^+)$. On the other hand, $\hat{X}(k)$ cannot be in $\hat{L}^1(\mathbb{C}^+)$ because the
Riemann-Lebesgue lemma requires that any function in $\hat{L}^1(C^+)\) must be continuous in $k \in \mathbb{R}$, whereas from (26.17) we know that $\hat{X}(k)$ is not continuous at $k = 0$ on the real axis.

In the next example, we present a scattering data set $S$ for which all the characterization conditions given in Theorem 5.1 are satisfied.

**Example 26.4** Assume that $n = 1$ and that the scattering matrix $S(k)$ is given by

$$S(k) = -\frac{k + i}{k - i}, \quad k \in \mathbb{R},$$

and that there are no bound states. Thus, from (4.3) we get $\mathcal{N} = 0$. This is compatible with (21.5) because $S(0) = 1$ and hence $\mu = 1$, $S_\infty = -1$ and hence $n_D = 1$, and the left-hand side of (21.5) is equal to $\pi$. By using the construction process outlined in the beginning of Chapter 16, we uniquely obtain

$$S_\infty = -1, \quad G_1 = 2,$$

$$F_s(y) = \begin{cases} 2e^{-y}, & y \in \mathbb{R}^+, \\ 0, & y \in \mathbb{R}^-, \end{cases}$$

$$F'_s(y) - G_1 \delta(y) = \begin{cases} -2e^{-y}, & y \in \mathbb{R}^+, \\ 0, & y \in \mathbb{R}^-, \end{cases}$$

$$K(x, y) = \begin{cases} -e^{-y} \text{sech} x, & y > x \geq 0, \\ 0, & y < x, \end{cases}$$

$$K_x(x, y) = \begin{cases} e^{-y} (\tanh x)(\text{sech} x), & y > x \geq 0, \\ 0, & y < x, \end{cases}$$

$$K(0, 0) = -1, \quad f(k, x) = e^{ikx} \left[ 1 - \frac{i}{k + i} \frac{e^{-x}}{\cosh x} \right],$$

$$V(x) = -2 \text{sech}^2 x, \quad \Psi(k, x) = -\frac{2ik}{k - i} \sin kx - \frac{1}{k - i} (\cos kx)(\tanh x),$$

$$A = 0, \quad J(k) = \frac{k}{k + i} B, \quad \Psi(k, 0) = 0, \quad \Psi'(k, 0) = -2i(k + i),$$

where $B$ is an arbitrary nonzero constant. One can directly verify that each of the four conditions (1), (2), (3a), (4a) of Theorem 5.1 is satisfied.
In the following example, we present a scattering data set $S$, for which, except for the second equality in (4.4), all the remaining characterization conditions in Theorem 7.1 are satisfied.

**Example 26.5** Assume that $n = 1$ and that the scattering matrix $S(k)$ is given by

$$S(k) = \frac{k}{k + i}, \quad k \in \mathbb{R},$$

and that there are no bound states. From (26.31) we get $S(0) = 0$, which indicates that $S(k)$ cannot be unitary for $k \in \mathbb{R}$. Nevertheless, using the construction process outlined in Chapter 16 we obtain

$$S_\infty = 1, \quad G_1 = 1,$$

$$F_s(y) = \begin{cases} 0, & y \in \mathbb{R}^+, \\ -e^y, & y \in \mathbb{R}^- \end{cases},$$

$$F'_s(y) - G_1 \delta(y) = \begin{cases} 0, & y \in \mathbb{R}^+, \\ -e^y, & y \in \mathbb{R}^- \end{cases},$$

$$F(y) = 0, \quad y \in \mathbb{R}^+.$$

Using the second line of (26.33) we determine that the only solution in $L^2(\mathbb{R}^-)$ to (4.17) is the trivial solution, and by using (26.35) we see that the only solution in $L^1(\mathbb{R}^+)$ to (4.14) is also the trivial solution. Since it is assumed that there are no bound states, (4.22) and (4.14) coincide and hence the only solution in $L^1(\mathbb{R}^+)$ to (4.22) is also the trivial solution. Then, we conclude that, in Theorem 7.1, all the conditions are satisfied, except for the unitarity of $S(k)$ in (1) stated in the second equality in (4.4). Equivalently stated, we have (2), (IIIc), (4c), and (Vc) are all satisfied and only (1) is not satisfied. Nevertheless, if we continue using the method of Chapter 16 to construct the corresponding input data set $D$, we find that

$$K(x, y) \equiv 0, \quad V(x) \equiv 0, \quad f(k, x) = e^{ikx}, \quad B = \frac{1}{2} A,$$

where $A$ is an arbitrary nonzero constant. For the input data set $D$ given in (26.36), by using the method of Chapter 9 we find that the corresponding Jost matrix $J(k)$ and the
scattering matrix $S(k)$ are given by

$$J(k) = \left( \frac{1}{2} - ik \right) A, \quad S(k) = \frac{k - i/2}{k + i/2}. \quad (26.37)$$

However, $S(k)$ given in (26.37) is not compatible with Levinson’s theorem. This is because for the scattering matrix $S(k)$ in (26.37), the left-hand side of (21.5) is $-\pi, \mu = 1$ because $S(0) = 1, n_D = 0$ because $S_\infty = 1$, and hence (21.5) yields $N = -1/2$, which is not a nonnegative integer.

In the following example, we present a scattering data set $S$, for which, except for the first equality in (4.4), all the remaining characterization conditions in Theorem 7.1 are satisfied.

**Example 26.6** Assume that $n = 1$ and that the scattering matrix $S(k)$ is given by

$$S(k) = i \frac{k - i}{k + i}, \quad k \in \mathbb{R}, \quad (26.38)$$

and that there are no bound states. We observe that the property $S(-k) = S(k)^\dagger$ in (4.4) is not satisfied by $S(k)$ given in (26.38). On the other hand, $S(k)$ given in (26.38) satisfies the second equality in (4.4) and hence it is unitary. Nevertheless, let us show that, in Theorem 7.1, all the remaining characterization conditions are satisfied. Using the construction process outlined in Chapter 16 we obtain

$$S_\infty = i, \quad G_1 = 2i, \quad (26.39)$$

$$F_s(y) = \begin{cases} 0, & y \in \mathbb{R}^+, \\ -2i e^y, & y \in \mathbb{R}^- \end{cases}, \quad (26.40)$$

$$F'_s(y) - G_1 \delta(y) = \begin{cases} 0, & y \in \mathbb{R}^+, \\ -2i e^y, & y \in \mathbb{R}^- \end{cases}, \quad (26.41)$$

$$F(y) = 0, \quad y \in \mathbb{R}^+. \quad (26.42)$$

Using the second line of (26.40) we determine that the only solution in $L^2(\mathbb{R}^-)$ to (4.17) is the trivial solution, and by using (26.42) we see that the only solution in $L^1(\mathbb{R}^+)$ to (4.14)
is also the trivial solution. Since it is assumed that there are no bound states, (4.22) and (4.14) coincide and hence the only solution in $L^1(\mathbb{R}^+)$ to (4.22) is also the trivial solution. Then, we conclude that, in Theorem 7.1, all the conditions are satisfied, except for the symmetry of $S(k)$ in (1) stated in the first equality in (4.4). Equivalently stated, we have (2), (III), (4c), and (V) are all satisfied and only (1) is not satisfied. We can continue to use the method of Chapter 16 to construct the corresponding input data set $D$, and we find that

$$K(x, y) \equiv 0, \quad V(x) \equiv 0, \quad f(k, x) = e^{ikx}, \quad A = B = 0. \quad (26.43)$$

Because the constructed $A$ and $B$ does not satisfy (2.6), we do not have a selfadjoint boundary condition as in (2.4). Nevertheless, let us also explicitly construct the corresponding physical solution $\Psi(k, x)$ via (9.4) as

$$\Psi(k, x) = e^{-ikx} + i e^{ikx} \frac{k - i}{k + i}. \quad (26.44)$$

From (26.7) we obtain

$$\Psi(k, 0) = \frac{(1 + i)(k + 1)}{k + i}, \quad \Psi'(k, 0) = -\frac{(1 + i)k(k - 1)}{k + i}, \quad (26.45)$$

and hence from (2.4) we see that (2.4) cannot be satisfied unless $A = B = 0$. One can directly verify that (2) and (4a) in Definition 4.2 are satisfied but neither of (1) and (3a) there are satisfied. This example indicates that the first equality of (4.4) given in (1) is necessary in the characterization stated in Theorem 7.1.

In the following example, we present a scattering data set $S$ for which some of the characterization conditions are not satisfied.

**Example 26.7** Assume that $n = 1$ and that the scattering matrix $S(k)$ is given by

$$S(k) = \frac{k + i}{k - i}, \quad k \in \mathbb{R}, \quad (26.46)$$

and that there are no bound states. We remark that the scattering matrix in (26.46) differs from that given in (26.22) only in the sign. By using the construction process outlined in the beginning of Chapter 16, we uniquely obtain

$$S_\infty = 1, \quad G_1 = -2, \quad (26.47)$$
\[
F_s(y) = \begin{cases} 
-2e^{-y}, & y \in \mathbb{R}^+, \\
0, & y \in \mathbb{R}^-,
\end{cases} \tag{26.48}
\]

\[
F_s'(y) - G_1 \delta(y) = \begin{cases} 
2e^{-y}, & y \in \mathbb{R}^+, \\
0, & y \in \mathbb{R}^-,
\end{cases} \tag{26.49}
\]

\[
F(y) = -2e^{-y}, & y \in \mathbb{R}^+. \tag{26.50}
\]

Using the second line of (26.48) in (4.17) we see that the only solution in \(L^2(\mathbb{R}^-)\) to (4.17) is the trivial solution. On the other hand, using (26.50) in (4.14) we observe that (4.14) has a one-parameter family of solutions given by \(X(y) = \alpha e^{-y}\) for \(y \in \mathbb{R}^+\), where \(\alpha\) is an arbitrary parameter. Since there are no bound states, we also obtain that (4.22) has a one-parameter family of solutions given by \(X(y) = \alpha e^{-y}\) for \(y \in \mathbb{R}^+\), where \(\alpha\) is an arbitrary parameter. We conclude that our scattering data set satisfies (1), (2), and (III\(_c\)) in Theorem 7.1, but (4\(_c\)) and (V\(_c\)) there both fail. Using the method of Chapter 16, we construct

\[
K(x, y) = \begin{cases} 
e^{-y} / \sinh x, & y > x > 0, \\
0, & y < x,
\end{cases} \tag{26.51}
\]

\[
K_x(x, y) = \begin{cases} 
-\cosh x e^{-y} / \sinh^2 x, & y > x > 0, \\
0, & y < x,
\end{cases} \tag{26.52}
\]

\[
K(0, 0) = -\infty, \quad f(k, x) = e^{ikx} \left[ 1 + \frac{i}{k} \frac{e^{-x}}{\sinh x} \right], \tag{26.53}
\]

\[
V(x) = \frac{8e^{2x}}{(e^{2x} - 1)^2}, \quad \Psi(k, x) = \frac{2k}{k - i} \cos kx - \frac{1}{k - i} (\sin kx)(\coth x), \tag{26.54}
\]

\[
V(x) = \frac{2}{x^2} - \frac{2}{3} + \frac{2x^2}{15} + O(x^4), \quad x \to 0. \tag{26.55}
\]

In this example, the boundary parameters \(A\) and \(B\) do not exist because \(K(0, 0)\) is not finite. Both \(\Psi(k, 0)\) and \(\Psi'(k, 0)\) blow up at \(x = 0\) and hence \(\Psi(k, x)\) does not satisfy a boundary condition like (2.4). In the characterization conditions specified in Theorem 5.1, we find that (1) and (2) are satisfied but (3\(_a\)) and (4\(_a\)) are not satisfied. Let us now check the equivalences indicated in Chapter 6. We cannot have (3\(_b\)) of Definition 4.2 or (V\(_b\)) of Definition 4.3 satisfied because we cannot construct a Jost matrix \(J(k)\) in this example as
we cannot construct the boundary matrices $A$ and $B$. In the absence of bound states, none of $(4_d), (4_e), (V_d), (V_e), (V_g), (V_h)$ are satisfied because each of $(4.15), (4.16), (4.23), (4.24)$ has the one-parameter family of solutions of the form $c/(k + i)$ for $k \in \mathbb{C}^+$ with $c$ being the arbitrary parameter. Note that neither $(4_c)$ nor $(V_c)$ is satisfied because each of $(4.14)$ and $(4.22)$ has the one-parameter family of solutions given by $c e^{-y}$ for $y \in \mathbb{R}^+$, with $c$ being the arbitrary parameter. Ironically, even though $(3_a)$ of Definition 4.2 fails, each of $(III_a), (III_b), (III_c)$ in Definition 4.3 is satisfied because $(4.17), (4.18), (4.19)$ each have only the trivial solution. In this example, the potential $V(x)$ constructed, although hermitian, does not satisfy (2.3). In summary, in Theorem 7.1, $(1), (2)$, and $(III_a)$ are satisfied, but $(4_c)$ and $(V_c)$ are not satisfied. In Theorem 5.1, $(1)$ and $(2)$ are satisfied, but $(3_a)$ and $(4_a)$ each fail.

In Example 26.7 presented above, we have observed that each of $(4.22), (4.23), (4.24)$ has one linearly independent solution. Thus, in order for $S(k)$ given in (26.46) to satisfy the characterization conditions, there has to be exactly one bound state associated with the corresponding scattering data. In the following example, we supplement the scattering matrix of (26.46) with one bound state and obtain an example where all the characterization conditions are met.

**Example 26.8** Assume that $n = 1$ and that the scattering matrix $S(k)$ is given by (26.46) and that there is exactly one bound state at $k = i\kappa_1$ with $\kappa_1 = 1$ and the Marchenko normalization constant $M_1 = \sqrt{2}$. The number of bound states is consistent with Levinson’s theorem. This is because from (26.8) we see that the left-hand side of (21.5) is $\pi, \mu = 0$ because $S(0) = -1$, $n_D = 0$ because $S_\infty = 1$ and hence the value of $\mathcal{N}$ predicted by Levinson’s theorem is equal to one. Since (26.46)-(26.49) still hold, and we also have

$$F(y) = 0, \quad y \in \mathbb{R}^+,$$

we see that (4.14) now has only the trivial solution and hence $(4_c)$ is satisfied. Using the method of Chapter 16, we construct

$$K(x, y) \equiv 0, \quad K(0, 0) = 0, \quad V(x) \equiv 0, \quad f(k, x) = e^{ikx},$$

where
\[
\Psi(k, x) = \frac{2(k \cos kx - \sin kx)}{k - i}, \quad \Psi_1(x) = \sqrt{2} e^{-x}, \quad J(k) = -i A(k + i),
\]
(26.58)
\[
\Psi(k, 0) = \frac{2k}{k - i}, \quad \Psi'(k, 0) = -\frac{2k}{k - i}, \quad \Psi_1(0) = \sqrt{2}, \quad \Psi_1'(0) = -\sqrt{2},
\]
(26.59)
with the boundary parameters \( B = A \), where \( A \) is an arbitrary nonzero constant. One can directly verify that each of the four conditions stated in Definition 4.5 is satisfied and that each of the five conditions in Theorem 7.1 is satisfied.

Next, we present an example of the scattering data set \( S \) failing only (\( III_a \)) in Theorem 7.1, failing only (\( 3_a \)) of Theorem 5.1, and failing only (\( L \)) of Theorem 7.9, whereas all the remaining conditions in those three theorems are satisfied.

**Example 26.9** Assume that \( n = 1 \) and \( S \) consists of the scattering matrix
\[
S(k) = \left( \frac{k - i}{k + i} \right)^2, \quad k \in \mathbb{R},
\]
(26.60)
and that there are no bound states. Using (26.60) in (10.14) we see that the constants \( S_\infty \) and \( G_1 \) appearing in (4.5) are given by
\[
S_\infty = 1, \quad G_1 = 4.
\]
(26.61)
Using (26.60) and (26.61) in (4.7), through some residue computations, we obtain
\[
F_s(y) = \begin{cases} 
0, & y > 0, \\
-4(1 + y) e^y, & y < 0.
\end{cases}
\]
(26.62)
Since there are no bound states, using (26.62) in (4.12) we obtain
\[
F(y) = 0, \quad y > 0.
\]
(26.63)
Using (26.60) and (26.62) we find that (1) and (2) in Theorem 7.1 are satisfied. Using (26.63) in (4.14) we see that the only solution in \( L^1(\mathbb{R}^+) \) to (4.14) is the trivial solution and hence (4c) in Theorem 7.1 is satisfied. Similarly, using the first line of (26.62) in (4.22) we see that the only solution in \( L^1(\mathbb{R}^+) \) to (4.22) is the trivial solution and hence (Vc) in Theorem 7.1 is satisfied. Let us use the second line of (26.62) in (4.17), which yields
\[
-X(y) + \int_{-\infty}^{0} dz X(z) \left[ -4(1 + z + y) e^{z+y} \right] = 0, \quad y \in \mathbb{R}^-.
\]
(26.64)
From (26.64) we see that its solution must have the form

\[ X(y) = \alpha e^y + \beta ye^y, \]  
(26.65)

for some constants \( \alpha \) and \( \beta \). Using (26.65) in (26.64) we find that \( \alpha = 0 \) and \( \beta \) is arbitrary. Thus, (26.64) has the nontrivial solution in \( L^2(\mathbb{R}^-) \) given by \( X(y) = \beta ye^y \), where \( \beta \) is arbitrary. Thus, (III) in Theorem 7.1 is violated, although (1), (2), (4c), and (Vc) are satisfied. In this example, the construction outlined in Chapter 16 by starting with (26.63) yields

\[ K(x, y) = 0, \quad K(0, 0) = 0, \quad V(x) \equiv 0, \quad f(k, x) = e^{ikx}, \]  
(26.66)

\[ \Psi(k, 0) = \frac{2k^2 - 2}{(k + i)^2}, \quad \Psi'(k, 0) = \frac{4k^2}{(k + i)^2}. \]  
(26.67)

Using \( S_\infty = 1, G_1 = 4, K(0, 0) = 0 \) in (14.2) we obtain \( B = 2A \) with \( A \) being an arbitrary nonzero constant. Thus, the boundary condition (2.4) is given by

\[ \psi'(0) - 2\psi(0) = 0. \]  
(26.68)

However, using (26.67) in (26.68) we see that the physical solution \( \Psi(k, x) \) does not satisfy the boundary condition, and hence (3a) in Theorem 5.1 does not hold. On the other hand, the remaining characterization conditions (1), (2), and (4a) in Theorem 5.1 all hold. Let us now check the compatibility of \( S(k) \) given in (26.60) with Levinson’s theorem. In other words, let us check if (L) of Theorem 7.9 is satisfied. Using (26.60), we find that the left-hand side of (21.5) is \(-2\pi, \mu = 1\) because \( S(0) = 1, n_D = 0 \) because \( S_\infty = 1, \) and hence (21.5) predicts \( N = -1, \) which is not a nonnegative integer. Thus, (L) in Theorem 7.9 does not hold. One can directly verify that the only solution in \( H^2(\mathbb{C}^+) \) to (7.3) is the trivial solution and hence all the four conditions in Theorem 7.9 are satisfied with the exception of (L). One can use the procedure described in Chapter 9 to solve the direct problem and show that the input data set \( D \) consisting of the zero potential and the boundary condition (26.68) does not correspond to the scattering matrix \( S(k) \) given in (26.60) with no bound states. In fact, with the help of (9.1)-(9.3) we get

\[ f(k, x) = e^{ikx}, \quad J(k) = -i(k + 2i)A, \quad S(k) = \frac{k - 2i}{k + 2i}. \]  
(26.69)
and there are no bound states because \( J(k) \) does not vanish on the positive imaginary axis and it vanishes only at \( k = -2i \).

As we have seen in Example 26.9, for \( S(k) \) given in (26.60) Levinson’s theorem predicts \( \mathcal{N} = -1 \). Hence, as indicated in Corollary 19.4, it is impossible to supplement \( S(k) \) with any bound-state data set so that the resulting scattering data set \( S \) can belong to the Marchenko class.

Next, we present an example satisfying all the characterization conditions in Theorem 7.1 except (4\(c\)) and (V\(c\)).

**Example 26.10** Assume that \( n = 1 \) and \( S \) consists of the scattering matrix

\[
S(k) = \left( \frac{k + i}{k - i} \right)^2, \quad k \in \mathbb{R},
\]

and that there are no bound states. Let us use the construction outlined in Chapter 16. Using (26.60) in (10.14) we see that the constants \( S_\infty \) and \( G_1 \) appearing in (4.5) are given by

\[
S_\infty = 1, \quad G_1 = -4,
\]

\[
F_s(y) = \begin{cases} 
4(-1 + y) e^{-y}, & y \in \mathbb{R}^+; \\
0, & y \in \mathbb{R}^-;
\end{cases}
\]

\[
F(y) = 4(-1 + y) e^{-y}, \quad y \in \mathbb{R}^+.
\]

One can directly verify that (1) and (2) in Theorem 7.1 are satisfied. Using the second line of (26.72) in (4.17) we see that the only solution in \( L^2(\mathbb{R}^-) \) to (4.17) is the trivial solution and hence (III\(a\)) of Theorem 7.1 is satisfied. Using (26.69) in (4.14) we see that the general solution in \( L^1(\mathbb{R}^+) \) to (4.14) is \( X(y) = \beta(-1 + y) e^{-y} \), where \( \beta \) is arbitrary, and hence (4\(c\)) of Theorem 7.1 is not satisfied. In the absence of bound states, (4.22) and (4.14) coincide and their general solutions must also coincide. Thus, (V\(c\)) of Theorem 7.1 is not satisfied because (4.22) has one linearly independent solution although there are no bound states. Therefore, unless the scattering matrix of (26.70) is accompanied with exactly one bound state, the corresponding scattering data set cannot correspond to an
input data set $D$ in the Faddeev class. Continuing with the method of Chapter 16, we obtain

$$K(x, y) = \frac{4 e^{x-y} \left[1 + x + e^{2x} - x e^{2x} + y (1 + e^{2x})\right]}{-1 + 4 x e^{2x} + e^{4x}},$$  \hspace{1cm} (26.74)

$$K(x, x) = \frac{4 + (4 - 8x) e^{2x}}{-1 + 4 x e^{2x} + e^{4x}},$$  \hspace{1cm} (26.75)

$$V(x) = -\frac{32 e^{2x} (1 + e^{2x}) (-1 - x + (-1 + x) e^{2x})}{(-1 + 4 x e^{2x} + e^{4x})^2},$$  \hspace{1cm} (26.76)

$$f(k, x) = e^{ikx} \frac{2 (k^2 x + ik + x) + 2ik \cosh(2x) + (k^2 - 1) \sinh(2x)}{(k + i)^2 (2x + \sinh(2x))}. \hspace{1cm} (26.77)$$

We have

$$K(x, x) = \frac{1}{x} - 2 + O(x), \quad V(x) = \frac{2}{x^2} - \frac{4}{3} + O(x^2), \quad x \to 0,$$  \hspace{1cm} (26.78)

$$\Psi(k, x) = O\left(\frac{1}{x}\right), \quad \Psi'(k, x) = O\left(\frac{1}{x}\right), \quad x \to 0.$$  \hspace{1cm} (26.79)

Because $K(0,0)$ is not finite, the boundary matrices $A$ and $B$ do not exist. Hence, $(3_a)$ of Theorem 5.1 does not hold. By Proposition 6.1 we have the equivalence of $(4_a)$ and $(4_c)$, and hence $(4_a)$ fails because $(4_c)$. In summary, in Theorem 5.1 we have $(1)$ and $(2)$ satisfied and we have $(\text{III}_c), (4_c)$, and $(\text{V}_c)$ not satisfied. In Theorem 5.1, we have $(1)$ and $(2)$ satisfied and we have $(3_a)$ and $(4_a)$ not satisfied. In this example, the number of bound states predicted by Levinson’s theorem is not consistent with the absence of bound states. This is because from (26.70) we see that the left-hand side of (21.5) is $2\pi$, $\mu = 1$ because $S(0) = 1$, $n_D = 0$ because $S_\infty = 1$ and hence the value of $N$ predicted by Levinson’s theorem is equal to one, inconsistent with the absence of bound states in this example.

In the next example we use the scattering matrix of Example 26.10 and we choose the number of bound states compatible with Levinson’s theorem. We illustrate that we can supplement the scattering data set of Example 26.10 with one bound state at $k = i\kappa_1$ for any $\kappa_1 > 0$ and with the normalization matrix $M_1$ for any positive constant $M_1$. The resulting scattering data set belongs to the Marchenko class.
Example 26.11 Assume that \( n = 1 \) and \( S \) consists of the scattering matrix \( S(k) \) given in (26.70) and that there is one bound state at \( k = i\kappa_1 \) for some \( \kappa_1 > 0 \) and \( M_1 > 0 \). Then, (26.71) and (26.72) still hold, but instead of (26.73) we get

\[
F(y) = 4(-1 + y) e^{-y} + M_1^2 e^{-\kappa_1 y}, \quad y \in \mathbb{R}^+.
\] (26.80)

Using (26.79) as input to (4.14) we obtain the fact that the only integrable solution to (4.14) is the trivial solution and hence (4c) is satisfied. We already know from Example 26.10 that (4.22) has the one-parameter family of solutions \( X(y) = \beta (-1 + y) e^{-y} \) for \( y \in \mathbb{R}^+ \) and hence in this example (Vc) is satisfied. Thus, all the five characterization conditions given in Theorem 7.1 hold. Then, based on the equivalence results of Chapter 6, all the four characterization conditions in Theorem 5.1 are satisfied, and hence the corresponding scattering data set \( S \) belongs to the Marchenko class for any choice of positive \( \kappa_1 \) and positive \( M_1 \). Since the explicit expressions for the constructed quantities are fairly lengthy, we display below the constructed quantities in the simplest choice of \( \kappa_1 = 1 \) and \( M_1 = 2 \). In this case we have

\[
F(y) = 4y e^{-y}, \quad y \in \mathbb{R}^+,
\] (26.81)

\[
K(x, y) = \frac{2 e^{-x-y} [1 + x - y - (x + y) e^{2x}]}{1 + 2x + \sinh(2x)},
\] (26.82)

\[
K(0, 0) = 2, \quad V(x) = \frac{4 \cosh x [2 \cosh x - (1 + 2x) \sinh x]}{[1 + 2x + \sinh(2x)]^2},
\] (26.83)

\[
f(k, x) = e^{ikx} \left[ 1 + \frac{2}{(k + i)^2} \frac{ik(e^{-2x} - 2x) + 1 + 2x}{1 + 2x + \sinh(2x)} \right],
\] (26.84)

\[
\Psi(k, 0) = 2 \frac{k + i}{k - i}, \quad \Psi'(k, 0) = -8 \frac{k + i}{k - i}, \quad \Psi_1(0) = 2, \quad \Psi_1'(0) = -8, \quad B = -4A,
\] (26.85)

where \( A \) is an arbitrary nonzero constant.

In the next example, we elaborate on the compatibility with Levinson’s theorem.

Example 26.12 Let the scattering matrix be given by

\[
S(k) = \left( \frac{k + i}{k - i} \right)^p, \quad k \in \mathbb{R},
\] (26.86)
where \( p \) is an integer. The left-hand side of (21.5) is then equal to \( p\pi \). We have \( S_\infty = 1 \) and hence \( n_D = 0 \). We have \( S(0) = (-1)^p \) and hence \( \mu = 1 \) if \( p \) is even and \( \mu = 0 \) if \( p \) is odd. Thus, Levinson’s theorem predicts that

\[
\mathcal{N} = \begin{cases} 
\frac{p}{2}, & p \text{ even integer}, \\
\frac{p-1}{2}, & p \text{ odd integer}.
\end{cases}
\] (26.87)

Thus, \( \mathcal{N} \) predicted by Levinson’s theorem is always an integer. However, if \( p \) is negative then \( \mathcal{N} \) in (26.87) is negative, and hence the scattering matrix cannot be a part of a scattering data set \( S \) in the Marchenko class. In Example 26.5 the predicted \( \mathcal{N} \) is \(-1/2\), but in that example the scattering matrix given in (26.31) is not unitary.

In the next example we elaborate on the choices of supplementing a scattering matrix with an appropriate bound-state data set so that the corresponding scattering data set can belong to the Marchenko class. The elaborations can be found in Chapter 19. This also provides an example where all the characterization conditions except (4c) may be satisfied in Theorem 7.1. It also illustrates a case where in Theorem 7.9 we have all the properties are satisfied except (4c,2) there.

**Example 26.13** Consider the scattering matrix given by

\[
S(k) = \begin{bmatrix} \left( \frac{k+i}{k-i} \right)^4 & 0 \\ 0 & 1 \end{bmatrix}, \quad k \in \mathbb{R}.
\] (26.88)

Note that associated with \( S(k) \) in (26.88), using the procedure outlined in Chapter 16, we obtain

\[
F_s(y) = \begin{cases} 
\begin{bmatrix} \left( -8 + 24y - 16y^2 + \frac{8}{3}y^3 \right) e^{-y} & 0 \\ 0 & 0 \end{bmatrix}, & y \in \mathbb{R}^+, \\
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & y \in \mathbb{R}^-.
\end{cases}
\] (26.89)

Using (26.89) in (4.17), as a solution to (4.17) we obtain \( X(y) = [0 \ 0] \) for \( y \in \mathbb{R}^+ \). Thus, (IIIc) of Theorem 7.1 is satisfied. Using (26.89) in (4.22) we get the two parameter family
of solutions given by

\[ X(y) = \begin{bmatrix} \alpha \left(1 - 3y + \frac{1}{3} y^3\right) e^{-y} + \beta \left(y^2 - \frac{1}{3} y^3\right) e^{-y} & 0 \end{bmatrix}, \quad y \in \mathbb{R}^+, \quad (26.90) \]

where \( \alpha \) and \( \beta \) are arbitrary parameters. The fact that \( X(y) \) in (26.90) contains two arbitrary parameters suggests that we must have \( N = 2 \). Thus, the property \((V_c)\) of Theorem 7.1 is satisfied if and only if we have \( N = 2 \). We can also get the same conclusion from Levinson’s theorem stated in (21.5). We see this as follows. We have \( S_\infty = I \) and \( S(0) = I \), and hence \( \mu = 2 \) and \( n_D = 0 \) in (21.5). Note that \( I \) here denotes the \( 2 \times 2 \) unit matrix. The left-hand side of (21.5) is equal to \( 4\pi \). Thus, Levinson’s theorem predicts that we must have \( N = 2 \). In other words, if \( S(k) \) given in (26.88) is a part of a scattering data \( S \) as in (4.2), then (4.2) must be compatible with (4.3). Note that we can achieve having \( N = 2 \) in essentially two ways. The first way is to have one bound state at some \( k = i\kappa_1 \) of multiplicity two, in which case the normalization matrix \( M_1 \) must have rank two. The second way is to have two simple bound states, say at \( k = i\kappa_1 \) and \( k = i\kappa_2 \) with \( \kappa_1 \neq \kappa_2 \), in which case the corresponding normalization matrices \( M_1 \) and \( M_2 \) each must have rank one. In this specific example, we will show that the first way does not necessarily always lead to a scattering data set in the Marchenko class. We see this as follows. Let us add one bound state at \( k = i \) with \( M_1 = \sqrt{8} I \). Since \( M_1 \) has rank two, this ensures that the property \((L)\) of Theorem 7.9 is satisfies. Using (26.89) in (4.7) we then have

\[ F(y) = \begin{bmatrix} \left(24 y - 16 y^2 + \frac{8}{3} y^3\right) e^{-y} & 0 \\ 0 & 8 e^{-y} \end{bmatrix}, \quad y \in \mathbb{R}^+. \quad (26.91) \]

Using (26.91) as input to (4.14) we obtain the general solution \( X(y) \) to (4.14) as

\[ X(y) = \begin{bmatrix} \gamma \left(1 - 3y + y^2\right) e^{-y} & 0 \end{bmatrix}, \quad y \in \mathbb{R}^+, \quad (26.92) \]

where \( \gamma \) is an arbitrary parameter. Thus, using the scattering data set \( S \) consisting of \( S(k) \) in (26.88), \( N = 1, \kappa_1 = 1, M_1 = \sqrt{8} I \), we see that the properties \((1), (2), (III_c), (V_c)\) of Theorem 7.1 are satisfied but \((4_c)\) is not satisfied. We also conclude that in Theorem 7.9
the conditions (1), (2), (L) are satisfied but not (4e,2). Similarly, in Theorem 7.10 the conditions (1), (2), (L) are satisfied but not (4c,2) or (4d,2).

Next, we present an example that does not satisfy all the characterization conditions because the number of bound states predicted by the solution to (4.22) is inconsistent with the number of bound states in the scattering data.

**Example 26.14** Let \( S(k) \) be the scattering matrix

\[
S(k) = \frac{1}{(k - i) \left( k - \frac{i}{3} \right)} \begin{bmatrix}
    k(k + i) & i/3(k + i) \\
    i/3(k + i) & k(k + i)
\end{bmatrix},
\]

and assume that there are no bound states. From (26.93) we get

\[
S_\infty = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} -\frac{7}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{7}{3} \end{bmatrix},
\]

The determinant of \( S(k) \) given in (26.93) is given by

\[
\det[S(k)] = \frac{(k + i)^2(k + i/3)}{(k - i)^2(k - i/3)}, \quad k \in \mathbb{R}.
\]

Using the first two equalities of (26.94) and also using (26.95) in (21.5), we see that the left-hand side of (21.5) is equal to \( 3\pi, \mu = 1 \) because \( S(0) \) has eigenvalues 1 with multiplicity one, \( n_D = 0 \) because \( S_\infty = I \) with \( I \) denoting the \( 2 \times 2 \) identity matrix. Thus, Levinson’s theorem predicts \( N = 2 \), which is not compatible with the assumption of no bound states. Using the construction process described in Chapter 16, from (26.93) we obtain

\[
F_s(y) = \begin{cases}
    \begin{bmatrix}
        -3e^{-y} + \frac{2}{3}e^{-y/3} & -e^{-y} + \frac{2}{3}e^{-y/3} \\
        -e^{-y} + \frac{2}{3}e^{-y/3} & -3e^{-y} + \frac{2}{3}e^{-y/3}
    \end{bmatrix}, & y \in \mathbb{R}^+, \\
    \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & y \in \mathbb{R}^-.
\end{cases}
\]

\[
F(y) = \begin{cases}
    \begin{bmatrix}
        -3e^{-y} + \frac{2}{3}e^{-y/3} & -e^{-y} + \frac{2}{3}e^{-y/3} \\
        -e^{-y} + \frac{2}{3}e^{-y/3} & -3e^{-y} + \frac{2}{3}e^{-y/3}
    \end{bmatrix}, & y \in \mathbb{R}^+,
\end{cases}
\]
Using (26.96) in (4.22) we obtain a two-parameter family of solutions to (4.22) given by

\[
X(y) = \left[ -\left( \alpha_1 + 6 \alpha_2 \right) e^{-y} + \alpha_2 e^{-y/3} \quad \alpha_1 e^{-y} + \alpha_2 e^{-y/3} \right],
\]

where \( \alpha_1 \) and \( \alpha_2 \) are arbitrary constants. Hence the number of bound states including the multiplicities must be two. In the absence of bound states, (4.22) and (4.14) coincide and hence (4.22) has the general solution given in (26.99), which is not the trivial solution.

Thus, if we assume that there are no bound states, even though (1), (2), and (III\(_a\)) in Theorem 7.1 are satisfied, neither (4\(_c\)) nor (V\(_c\)) are satisfied. Using the method of Chapter 16, we construct

\[
K(x, y) = \begin{bmatrix} \beta_1(x) & \beta_2(x) \\ \beta_2(x) & \beta_1(x) \end{bmatrix} e^{-y} + \beta_7(x) \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-y/3}, \quad y > x,
\]

\[
K(x, x) = \begin{bmatrix} \beta_3(x) & \beta_4(x) \\ \beta_4(x) & \beta_3(x) \end{bmatrix}, \quad V(x) = \begin{bmatrix} \beta_5(x) & \beta_6(x) \\ \beta_6(x) & \beta_5(x) \end{bmatrix},
\]

\[
f(k, x) = e^{ikx} \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{i e^{-x}}{k + i} \begin{bmatrix} \beta_1(x) & \beta_2(x) \\ \beta_2(x) & \beta_1(x) \end{bmatrix} + \frac{i e^{-x/3}}{k + i/3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right),
\]

where we have defined

\[
\beta_1(x) := \frac{3 e^x + 4 e^{5x/3} + 3 e^{7x/3}}{(e^{2x} - 1)(e^{2x/3} + 1)^2}, \quad \beta_2(x) := \frac{7 + 12 e^{2x/3} + 7 e^{4x/3} - 2 e^{8x/3}}{(e^{2x} - 1)(e^{2x/3} + 1)^2},
\]

\[
\beta_3(x) := \frac{1 + e^{4x/3} - 2 e^{8x/3}}{3(e^{2x} - 1)(e^{2x/3} + 1)^2}, \quad \beta_4(x) := \frac{e^x + e^{7x/3}}{3(e^{2x} - 1)(e^{2x/3} + 1)^2},
\]

\[
\beta_5(x) := \frac{-8 e^{2x/3} + 16 e^{4x/3} + 68 e^{2x} + 136 e^{8x/3} + 68 e^{10x/3} + 16 e^{4x} - 8 e^{14x/3}}{9(e^{2x} - 1)^2(e^{2x/3} + 1)^2},
\]

\[
\beta_6(x) := \frac{-8 e^{2x/3} + 4 e^{2x} - 8 e^{10x/3}}{9(1 + 2 e^{2x/3} + 2 e^{4x/3} + e^{2x})^2}, \quad \beta_7(x) := \frac{-2 e^{x/3} + 2 e^x - 2 e^{5x/3}}{3(e^{2x} - 1)(e^{2x/3} + 1)^2}.
\]
We have
\[ K(x, x) = \frac{1}{x} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix} + O(x), \quad x \to 0, \quad (26.107) \]
\[ V(x) = \frac{2}{x^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{27} \begin{bmatrix} 19 & 1 \\ 1 & 7 \end{bmatrix} + O(x^2), \quad x \to 0, \quad (26.108) \]
\[ \Psi(k, x) = -\frac{x^2}{9} \begin{bmatrix} 6k^2 + 7ik - 1 & i(k + i) \\ i(k + i) & 6k^2 + 7ik - 1 \end{bmatrix} + O(x^3), \quad x \to 0, \quad (26.109) \]
\[ \Psi'(k, x) = -\frac{2x}{9} \begin{bmatrix} 6k^2 + 7ik - 1 & i(k + i) \\ i(k + i) & 6k^2 + 7ik - 1 \end{bmatrix} + O(x^3), \quad x \to 0. \quad (26.110) \]

Even though \( \Psi(k, 0) = 0 \) and \( \Psi'(k, 0) = 0 \), there is no selfadjoint boundary condition of the form (2.4). This is because \( K(0, 0) \) is not finite and hence the boundary matrices \( A \) and \( B \) satisfying (16.70) with the further property (2.6) do not exist. As a result, (1) and (2) in Theorem 5.1 are satisfied, but (3a) and (4a) there are violated. In Theorem 7.9, the condition (L) is not satisfied but the remaining conditions (1), (2), (4e, 2) are satisfied.

Next, we use the scattering matrix in Example 26.14 with one bound state of multiplicity two in order to have an example where all the five characterization conditions in Theorem 7.1 are satisfied.

**Example 26.15** Let \( S(k) \) be the scattering matrix given in (26.93) and assume that we have one bound state of multiplicity two at \( k = i\kappa_1 \) with \( \kappa_1 = 1 \) and the rank-two normalization matrix \( M_1 \) given by
\[ M_1 = \begin{bmatrix} 1 + \frac{1}{\sqrt{2}} & 1 - \frac{1}{\sqrt{2}} \\ 1 - \frac{1}{\sqrt{2}} & 1 + \frac{1}{\sqrt{2}} \end{bmatrix}, \quad M_1^2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}. \quad (26.111) \]

Thus, using (26.70) and (26.94), from (4.12) we obtain
\[ F(y) = \frac{2}{3} e^{-y/3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad y \in \mathbb{R}^+. \quad (26.112) \]

Then, all the characterization conditions are met and we obtain
\[ K(x, y) = -\frac{2}{3} \frac{e^{-(x+y)/3}}{1 + 2e^{-2x/3}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad V(x) = -\frac{8e^{2x/3}}{9(2 + e^{2x/3})^2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad (26.113) \]
\[ f(k, x) = e^{ikx} \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2i}{(3k+i)(2+e^{2x/3})} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right), \quad (26.114) \]

\[ K(0, 0) = -\frac{2}{9} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \frac{1}{18} \begin{bmatrix} -17 & 1 \\ 1 & -17 \end{bmatrix} A, \quad (26.115) \]

\[ \Psi(0, 0) = \frac{1}{3(k-i)} \begin{bmatrix} 6k+i & i \\ i & 6k+i \end{bmatrix}, \quad \Psi'(0, 0) = \frac{1}{27(k-i)} \begin{bmatrix} -51k-8i & 3k-8i \\ 3k-8i & -51k-8i \end{bmatrix}, \quad (26.116) \]

\[ \Psi(0) = \frac{2}{3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \Psi'(0) = -\frac{16}{27} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad (26.117) \]

with the boundary matrix \( A \) being any invertible \( 2 \times 2 \) matrix. One can directly verify that all the five conditions in Theorem 7.1 and all the four conditions in Theorem 5.1 are all satisfied.

Next, we use the scattering matrix in Example 26.14 with two bound states, each with multiplicity one in order to have an example where all five the characterization conditions in Theorem 7.1 are satisfied.

**Example 26.16** Let \( S(k) \) be the scattering matrix given in (26.93) and assume that we have two bound states at \( k = i\kappa_1 \) and \( k = i\kappa_2 \) with \( \kappa_1 = 1 \) and \( \kappa_2 = 1/3 \) and with respective rank-one normalization matrices \( M_1 \) and \( M_2 \) given by

\[ M_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (26.118) \]

With the help of (4.12), (26.96), and (26.118) we get

\[ F(y) = \left( -2e^{-y} + \frac{4}{3} e^{-y/3} \right) I, \quad y \in \mathbb{R}^+, \quad (26.119) \]

where \( I \) is the \( 2 \times 2 \) identity matrix. The construction procedure described in Chapter 16 yields

\[ K(x, y) = \left( \alpha(x) e^{-(x+y)} + \beta(x) e^{-(x+y)/3} \right) I, \quad K(0, 0) = \frac{4}{3} I, \quad (26.120) \]

\[ f(k, x) = e^{ikx} \left[ 1 + \frac{i \alpha(x)}{k+i} + \frac{i \beta(x)}{k+i/3} \right] I, \quad B = -\frac{1}{6} \begin{bmatrix} 15 & 1 \\ 1 & 15 \end{bmatrix} A, \quad (26.121) \]
\[ V(x) = \left( \frac{16e^{2x/3} \left[ 18e^{4x/3} + 32e^{2x} + 18e^{8x/3} - 4e^{-4x} - 1 \right]}{9 \left[ 4e^{2x} + 2e^{8x/3} - 2e^{2x/3} - 1 \right]^2} \right) I, \tag{26.122} \]

where \( A \) is any invertible \( 2 \times 2 \) matrix, and the quantities \( \alpha(x) \) and \( \beta(x) \) are given by

\[
\alpha(x) := \frac{2e^{-x} + 2e^{-5x/3}}{1 + 2e^{-2x/3} - e^{-2x} - \frac{1}{2} e^{-8x/3}}, \quad \beta(x) := \frac{-4}{3} e^{-x/3} - \frac{2}{3} e^{-7x/3}.
\tag{26.123} \]

One can directly verify that all the five conditions in Theorem 7.1, all the four conditions in Theorem 5.1, and all the four conditions in Theorem 7.9 are all satisfied.

In the next example we present a scattering matrix, which does not satisfy (1) in Theorem 7.1 because it is not unitary.

**Example 26.17** Let \( S(k) \) be the scattering matrix given

\[
S(k) = \frac{1}{(k + i) \left( k + \frac{i}{3} \right)} \begin{bmatrix} k(k - i) & \frac{i}{3} (k - i) \\ \frac{i}{3} (k - i) & -k(k - i) \end{bmatrix}, \tag{26.124} \]

and assume that there are no bound states. The matrix in (26.124) is not unitary although it satisfies \( S(-k) = S(k)^\dagger \) for \( k \in \mathbb{R} \). Going through the construction procedure of Chapter 16, we obtain

\[
F_s(y) = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & y \in \mathbb{R}^+, \\
\begin{bmatrix} -3e^y + \frac{2}{3} e^{y/3} & e^y - \frac{2}{3} e^{y/3} \\ e^y - \frac{2}{3} e^{y/3} & 3e^y - \frac{2}{3} e^{y/3} \end{bmatrix}, & y \in \mathbb{R}^- \end{cases}, \tag{26.125} \]

\[
F(y) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & y \in \mathbb{R}^+, \tag{26.126} \]

\[
S_\infty = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 7 & -1 \\ 3 & -3 \\ -3 & 1 \\ -3 & 7 \end{bmatrix}, \quad K(x, y) \equiv 0, \quad K(0, 0) = 0, \tag{26.127} \]

\[
V(x) \equiv 0, \quad A = 0, \quad B = 0. \tag{26.128} \]

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In this example, it is impossible to construct the two boundary matrices $A$ and $B$ with the rank of $\begin{bmatrix} A \\ B \end{bmatrix}$ being two. Using the second line of (26.125) in (4.17), one determines that (4.17) has only the trivial solution and hence $(\text{III}_a)$ is satisfied. In summary, in Theorem 5.1, the property (1) is not satisfied because $S(k)^\dagger \neq S(k)^{-1}$ for $k \in \mathbb{R}$, (2) is satisfied, $(3_a)$ is violated because there does not exist a corresponding selfadjoint boundary condition. From the first line of (29.96) we conclude that (4.22) has only the trivial solution and hence $(V_c)$ is satisfied. From (26.126) we conclude that (4.14) has only the trivial solution and hence $(4_c)$ is satisfied. In Theorem 7.1, only (1) is violated because $S(k)$ is not unitary, while the remaining conditions (2), $(\text{III}_a)$, $(4_c)$, and $(V_c)$ are all satisfied.

In the next example we present a unitary scattering matrix, which does not satisfy (1) in Theorem 7.1 because the symmetry property $S(-k) = S(k)^\dagger$ for $k \in \mathbb{R}$ does not hold.

**Example 26.18** Let $S(k)$ be the scattering matrix given

$$
S(k) = \frac{1}{(k + i) \left( k + \frac{i}{3} \right)} \begin{bmatrix} k(k - i) & \frac{i}{3}(k - i) \\ -\frac{i}{3}(k - i) & -k(k - i) \end{bmatrix},
$$

and assume that there are no bound states. Note that $S(k)$ of (26.129) differs from $S(k)$ of (26.124) only in the sign of the $(2,1)$-entry. The matrix in (26.129) is unitary but it does not satisfy $S(-k) = S(k)^\dagger$ for $k \in \mathbb{R}$.

From (26.129) we get

$$
S_\infty = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S(0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
$$

The determinant of $S(k)$ given in (26.129) is given by

$$
\det[S(k)] = -\frac{(k - i)^2(k - i/3)}{(k + i)^2(k + i/3)}, \quad k \in \mathbb{R}.
$$

Using (26.130) and (26.131) in (21.5), we see that the left-hand side of (21.5) is equal to $-3\pi$, $\mu = 0$ because $S(0)$ has eigenvalues $i$ and $-i$, $n_D = 1$. Thus, Levinson’s theorem
predicts $N = -1$, which contradicts the expectation that $N$ is a nonnegative integer.

Going through the construction procedure of Chapter 16, we obtain

$$F_s(y) = \begin{cases} 
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & y \in \mathbb{R}^+, \\
\begin{bmatrix} -3e^y + \frac{2}{3}e^{y/3} & e^y - \frac{2}{3}e^{y/3} \\ -e^y + \frac{2}{3}e^{y/3} & 3e^y - \frac{2}{3}e^{y/3} \end{bmatrix}, & y \in \mathbb{R}^-.
\end{cases}$$  \tag{26.132}

$$F(y) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad y \in \mathbb{R}^+,$$  \tag{26.133}

$$G_1 = \begin{bmatrix} \frac{7}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{7}{3} \end{bmatrix}, \quad K(x,y) \equiv 0, \quad K(0,0) = 0,$$  \tag{26.134}

$$V(x) \equiv 0, \quad A = 0, \quad B = 0.$$  \tag{26.135}

In this example, $F_s(y)$ is not hermitian for $y \in \mathbb{R}^-$ and the matrix $G_1$ is not hermitian. One cannot construct the two boundary matrices $A$ and $B$ with the rank of $\begin{bmatrix} A \\ B \end{bmatrix}$ being two. Using the second line of (26.132) in (4.17), one determines that (4.17) has only the trivial solution and hence $\text{(III}_a\text{)}$ is satisfied. In summary, in Theorem 5.1, (1) is violated because $S(-k) \neq S(k)^\dagger$ for $k \in \mathbb{R}$, (2) is satisfied, $\text{(3}_a\text{)}$ is violated because there does not exist a corresponding selfadjoint boundary condition. From the first line of (29.100) we conclude that (4.22) has only the trivial solution and hence $\text{(V}_c\text{)}$ is satisfied. From (26.133) we conclude that (4.14) has only the trivial solution and hence $\text{(4}_c\text{)}$ is satisfied.

In Theorem 7.1, only (1) is violated because $S(-k) \neq S(k)^\dagger$ for $k \in \mathbb{R}$, while the remaining conditions (2), $\text{(III}_a\text{)}$, (4$_c$), and $\text{(V}_c\text{)}$ are all satisfied. In Theorem 7.9, (1) and (L) are not satisfied while the remaining two properties (2) and (4$_{c,2}$) are satisfied.

In the next example, we present a scattering data set that satisfies all the five conditions in Theorem 7.1, except for the unitarity of the scattering matrix.

**Example 26.19** Let $S(k)$ be the scattering matrix given

$$S(k) = \begin{bmatrix} k - i & 1 \\ k + i & k^2 + 1 \\ 1 & k - 2i \\ k^2 + 1 & k + 2i \end{bmatrix},$$  \tag{26.136}
and assume that there are no bound states. The scattering matrix given in (26.136) satisfies the first equality in (4.4), but it is not unitary and hence it does not satisfy the second equality in (4.4). Going through the construction procedure of Chapter 16, we obtain

\[
F_s(y) = \begin{cases}
\left[ \begin{array}{cc}
0 & \frac{1}{2} e^{-y} \\
\frac{1}{2} e^{-y} & 0 \\
\end{array} \right], & y \in \mathbb{R}^+,
\end{cases}
\]

(26.137)

\[
F(y) = \left[ \begin{array}{cc}
0 & \frac{1}{2} e^{-y} \\
\frac{1}{2} e^{-y} & 0 \\
\end{array} \right], & y \in \mathbb{R}^-.
\]

(26.138)

\[
S_\infty = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad K(0,0) = \begin{bmatrix} 2/15 & -8/15 \\ -8/15 & 2/15 \end{bmatrix},
\]

(26.139)

\[
K(x,y) = \frac{e^{x-y}}{16e^{4x}-1} \begin{bmatrix} 2 & -8e^{2x} \\ -8e^{2x} & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 13/15 & 8/15 \\ 8/15 & 28/15 \end{bmatrix} A,
\]

(26.140)

\[
V(x) = \frac{1}{(16e^{4x}-1)^2} \begin{bmatrix} 256e^{4x} & -32e^{2x} - 512e^{6x} \\ -32e^{2x} - 512e^{6x} & 256e^{4x} \end{bmatrix},
\]

(26.141)

\[
f(k,x) = e^{ikx} \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{i}{(k+i)(16e^{4x}-1)} \begin{bmatrix} 2 & -8e^{2x} \\ -8e^{2x} & 2 \end{bmatrix} \right),
\]

(26.142)

where \( A \) is any invertible \( 2 \times 2 \) constant matrix. One can directly verify that all the five conditions in Theorem 7.1 are satisfied except for the unitarity of \( S(k) \) in (1). This is because from the first line of (26.137) it is seen that (2) is satisfied. Using (26.138) in (4.14) one determines that (4.14) has only the trivial solution and hence (4c) is satisfied. Using the second line of (26.137) in (4.17) one determines that (4.17) has only the trivial solution and hence (IIIc) is satisfied. Using the first line of (26.137) in (4.22) we determine that (4.22) has only the trivial solution and hence (Vc) is satisfied. Similarly, one can determine that (2) and (4a) in Definition 4.5 are satisfied but neither (1) nor (3a) are satisfied. As
a result, the scattering data in this example does not belong to the Marchenko class. One can evaluate the scattering data set corresponding to the potential \( V(x) \) is (26.141) and the boundary matrices \( A \) and \( B \) specified in the second equality in (26.140). Using (26.142) in (9.2), with the help of (9.3) one can construct the corresponding Jost matrix and the scattering matrix and finds that

\[
J(k) = \frac{1}{225(k+i)} \begin{bmatrix}
-\i(225k^2 - 510ik + 931) & 16(15k + 34i) \\
16(15k + 34i) & -\i(225k^2 - 510ik + 931)
\end{bmatrix} A, \tag{26.143}
\]

\[
S(k) = \begin{bmatrix}
(k+i)(k-i)^2(225k^2 + 2537) & 480ik(k-i)(k^2 - 1) \\
480ik(k-i)(k^2 - 1) & (k+i)(k-i)^2(225k^2 + 2537)
\end{bmatrix} \frac{(k+i)(9k^2 - 30ik + 59)(25k^2 - 30ik + 43)}{(k+i)(9k^2 - 30ik + 59)(25k^2 - 30ik + 43)}. \tag{26.144}
\]

One can verify that \( S(k) \) given in (26.144) unitary. The determinant of \( J(k) \) given in (26.143) is given by

\[
\det[J(k)] = -\frac{(9k^2 - 30ik + 59)(25k^2 - 30ik + 43)}{225(k+i)^2}, \tag{26.145}
\]

and hence it vanishes in \( \mathbb{C}^+ \) at the two points \( k = i\kappa_1 \) and \( k = i\kappa_2 \), where

\[
\kappa_1 = \frac{3 + 2\sqrt{13}}{5}, \quad \kappa_2 = \frac{5 + 2\sqrt{21}}{3}. \tag{26.146}
\]

Using (11.1)-(11.3) we obtain

\[
A_1 = \begin{bmatrix}
\frac{5(583 - 145\sqrt{13})}{1161} & \frac{1}{3(4 + \sqrt{13})^2} \\
-\frac{1}{3(4 + \sqrt{13})^2} & \frac{5(583 - 145\sqrt{13})}{1161}
\end{bmatrix}, \tag{26.147}
\]

\[
B_1 = \begin{bmatrix}
\frac{905\sqrt{13} - 2798}{774} & \frac{3572 - 905\sqrt{13}}{774} \\
\frac{3572 - 905\sqrt{13}}{774} & \frac{905\sqrt{13} - 2798}{774}
\end{bmatrix}, \tag{26.148}
\]

\[
A_2 = \begin{bmatrix}
\frac{3(779 - 111\sqrt{21})}{7375} & \frac{6}{5(4 + \sqrt{21})^2} \\
-\frac{6}{5(4 + \sqrt{21})^2} & \frac{3(779 - 111\sqrt{21})}{7375}
\end{bmatrix}, \tag{26.149}
\]

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\[ B_2 = \begin{bmatrix} \frac{4562 - 633\sqrt{21}}{2950} & \frac{633\sqrt{21} - 1612}{2950} \\ \frac{633\sqrt{21} - 1612}{2950} & \frac{4562 - 633\sqrt{21}}{2950} \end{bmatrix}, \quad (26.150) \]

and via (11.22) we explicitly evaluate the normalization matrices \( M_1 \) and \( M_2 \) as

\[ M_1 = \frac{1}{4} \sqrt{\frac{49 + \frac{181}{\sqrt{13}}}{} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}, \quad M_2 = \frac{1}{12} \sqrt{147 + 211 \sqrt{\frac{3}{7}}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (26.151) \]

The scattering data set \( S \) consisting of \( S(k) \) in (26.144) with the corresponding two bound states specified in (26.146) and (26.151) now belongs to the Marchenko class, and it corresponds to the input data set \( D \) consisting of \( \{V, A, B\} \) appearing in (26.141) and the second equality in (26.140).

In the following example, we check the characterization conditions of Theorem 8.1

**Example 26.20** Assume that \( n = 1 \) and that the scattering matrix \( S(k) \) is given by

\[ S(k) = \frac{(k + i)(k + 2i)}{(k - i)(k - 2i)}, \quad k \in \mathbb{R}, \quad (26.152) \]

and that we have one bound state at \( k = i\kappa_1 \) with \( \kappa_1 = 1 \) and the normalization constant \( M_1 = \sqrt{6} \). We will prove that the conditions (I), (2), (III), (4c), (V), and (VI) of Theorem 8.1 are satisfied. From (26.152) we get

\[ S_\infty := \lim_{k \to \pm \infty} S(k) = -1. \quad (26.153) \]

Using (26.152) in (4.7), with the help of a contour integration, we get

\[ F_s(y) = \begin{cases} -6e^{-y} + 12e^{-2y}, & y > 0, \\ 0, & y < 0. \end{cases} \quad (26.154) \]

Then, from (4.12) we obtain

\[ F(y) = 12e^{-2y}, \quad y > 0. \quad (26.155) \]

From (26.152), (26.153), and (26.154) we conclude that (I) and (2) are satisfied. Note that (4.14) is given by

\[ X(y) + \int_0^\infty dz X(z) 12e^{-2z-2y} = 0, \quad y \in \mathbb{R}^+, \quad (26.156) \]
and hence any solution to (26.156) must be of the form

\[ X(y) = \alpha e^{-2y}, \quad y \in \mathbb{R}^+, \quad (26.157) \]

where \( \alpha \) is a constant to be determined. We see that the solution given in (26.157) belongs to \( L^1(\mathbb{R}^+) \). Using (26.157) in (26.156), we obtain

\[ 4\alpha e^{-2y} = 0, \quad y \in \mathbb{R}^+, \quad (26.158) \]

and hence we must have \( \alpha = 0 \), yielding \( X(y) = 0 \) for \( y \in \mathbb{R}^+ \). Thus, (4c) is also satisfied.

We now prove that conditions (III\(_e\)) and (V\(_h\)) hold. We take as \( \breve{\Upsilon} \) the set given by

\[ \breve{\Upsilon} := \left\{ X(k) : X(k) = \frac{k - 2i}{k + i} (U(k) - U(-k), U(k) \in H^2(C^+)) \right\}. \quad (26.159) \]

Note that

\[ X(-k) = S(k) X(k), \quad k \in \mathbb{R}, \quad X(k) \in \breve{\Upsilon}, \quad (26.160) \]

and then we have \( \breve{\Upsilon} \subset \Upsilon \). Let us prove that \( \breve{\Upsilon} \) is a dense set in \( \Upsilon \). Suppose that some \( W(k) \in \Upsilon \) is orthogonal to \( \breve{\Upsilon} \), i.e. for any \( X(k) \in \breve{\Upsilon} \) we have

\[ (W(k), X(k))_2 = 0, \quad X(k) \in \breve{\Upsilon}. \quad (26.161) \]

However, by (26.159), for any \( U(k) \) in the Hardy space \( H^2(C^+) \), we have the scalar product in \( L^2(\mathbb{R}) \) given by

\[ \left( \frac{k + 2i}{k - i} W(k) - \frac{k - 2i}{k + i} W(-k), U(k) \right)_2 = 0, \quad U(k) \in H^2(C^+). \quad (26.162) \]

Since \( H^2(C^+) \) is dense in \( L^2(\mathbb{R}) \), from (26.162) we obtain

\[ \frac{k + 2i}{k - i} W(k) - \frac{k - 2i}{k + i} W(-k) = 0. \quad (26.163) \]

On the other hand, since \( W(-k) = S(k) W(k) \) it follows that (26.163) implies that \( W(k) = -W(k) \) and then, \( W(k) = 0 \) Since the only vector in \( \Upsilon \) that is orthogonal to \( \breve{\Upsilon} \) is the zero
vector, $\mathcal{Y}$ is dense in $\mathcal{Y}$. For each $X(k) = \frac{k - 2i}{k + i}(U(k) - U(-k)) \in \mathcal{Y}$ we need to find a column vector $h(k) \in H^2(C^+)$ that solves the equation,

$$h(k) + S(-k) h(-k) = \frac{k - 2i}{k + i} (U(k) - U(-k)), \quad k \in \mathbb{R}.$$  

(26.164)

The solution is given by

$$h(k) = \frac{k - 2i}{k + i} U(k) \in H^2(C^+).$$  

(26.165)

This prove that the property (III$_e$) is satisfied. Let us prove that the property (V$_h$) is satisfied. Note that

$$S(k) = -J(-k) J^{-1}(k), \quad \text{where} \quad J(k) = \frac{k - i}{k + 2i}.$$  

(26.166)

Then, by (21.13) with $q = 0$ and Proposition 21.11, every solution to (4.24) that is in $H^2(C^+)$ is of the form given by

$$h(k) = \alpha J(k) \frac{1}{k^2 + 1}, \quad \text{for some} \ \alpha \in \mathbb{C}.$$  

(26.167)

Hence, there is a one-parameter family of solutions and, as there is only one bound state with multiplicity one, the property (V$_h$) holds. The property (VI) is also satisfied by the definition of $S(k)$ in (26.152).

In the following example, we elaborate on the necessity of the integrability of the potential when non-Dirichlet boundary conditions are used.

**Example 26.21** From Theorem 1.2.1 of [2], we know that if the potential $V(x)$ satisfies $\int_0^\infty dx \ x |V(x)| < +\infty$, then (2.1) has two linearly independent $n \times n$ matrix-valued solutions $G(k, x)$ and $H(k, x)$ satisfying the initial conditions

$$G(k, 0) = x[I + o(1)], \quad G'(k, 0) = I + o(1), \quad x \to 0^+,$$  

(26.168)

$$H(k, 0) = I + o(1), \quad H'(k, 0) = o\left(\frac{1}{x}\right), \quad x \to 0^+.$$  

(26.169)
Thus, any solution to (2.1) can be expressed as a linear combination of $G(k,x)$ and $H(k,x)$. Let us express the regular solution $\varphi(k,x)$ appearing in (9.5) as a linear combination of $G(k,x)$ and $H(k,x)$ as
\[
\varphi(k,x) = G(k,x) \alpha + H(k,x) \beta,
\] (26.170)
where $\alpha$ and $\beta$ are two constant $n \times n$ matrices to be determined by the boundary condition at $x = 0$. From the $x$-derivative of (26.170) we obtain
\[
\varphi'(k,x) = G'(k,x) \alpha + H'(k,x) \beta,
\] (26.171)
If the boundary condition (2.4) is the Dirichlet condition, which is the case considered in [2], then we have $A = 0$ and $B = I$. From (26.168)-(26.171) we see that by choosing $\alpha = I$ and $\beta = 0$ in (26.170) and (26.171), the regular solution $\varphi(k,x)$ satisfies the Dirichlet boundary condition. Then, as a result of (9.6), the physical solution $\Psi(k,x)$ also satisfies the Dirichlet boundary condition. On the other hand, if the boundary condition (2.4) is non-Dirichlet, then the regular solution $\varphi(k,x)$ and hence also the physical solution $\Psi(k,x)$ cannot be obtained from only $G(k,x)$ appearing in (168.171). In the non-Dirichlet case, the involvement of $H(k,x)$ with the behavior given in (26.169) makes it impossible to define a selfadjoint boundary condition at $x = 0$ if the potential $V(x)$ only satisfies $\int_0^\infty dx x |V(x)| < +\infty$ but not (2.3). Such potentials are nevertheless important in physical applications. For example, in the scalar case the truncated Coulomb potential given by
\[
V(x) = \begin{cases} 
\frac{1}{x}, & 0 < x < 1, \\
0, & x > 1,
\end{cases}
\] (26.172)
although satisfying $\int_0^\infty dx x |V(x)| < +\infty$, is a potential not integrable at $x = 0$ and hence we cannot use a selfadjoint non-Dirichlet boundary condition of the form (2.4)-(2.6) if (26.172) is used in the Schrödinger equation (2.1). In fact, the Schrödinger equation (2.1) when $k = 0$ with $V(x)$ given in (26.172) has two linearly independent solutions, one of which is regular at $x = 0$ and can be expressed in terms of the Bessel function of the first kind $I_1(\sqrt{2}x)$ and the other has a singular derivative at $x = 0$ and can be expressed in
terms of the modified Bessel function of the second kind $K_1(\sqrt{2}x)$. So, the general solution to (2.1) at $k = 0$ has the behavior as $x \to 0^+$ given by

$$
\psi(0, x) = \alpha \left[ -x - \frac{x^2}{2} + O(x^{5/2}) \right] + \beta \left[ 1 + (-1 + 2\gamma + \log x) x + \frac{x^2}{2} \log x + O(x^2) \right], \quad x \to 0^+,
$$

(26.173)

where $\alpha$ and $\beta$ are arbitrary constants and $\gamma$ is the Euler-Mascheroni constant appearing in (26.16). The derivative $\psi'(0, x)$ has the behavior as $x \to 0^+$ given by

$$
\psi'(0, x) = \alpha \left[ -1 - x - \frac{x^2}{4} + O(x^{5/2}) \right] + \beta \left[ 2\gamma + \log x + (-2 + 2\gamma + \log x) x + +O(x^2 \log x) \right], \quad x \to 0^+.
$$

(26.174)

When $k \neq 0$, the Schrödinger equation (2.1) with $V(x)$ given in (26.172) has two linearly independent solutions, one of which is regular at $x = 0$ and can be expressed in terms of the Kummer confluent hypergeometric function $\, _1F_1(a, b, z)$, with some appropriate $a, b, z$ expressed in terms of $k$ and $x$, and the other has a singular derivative at $x = 0$ and can be expressed in terms of the Tricomi confluent hypergeometric function $U(a, b, z)$. So, the general solution to (2.1) with $k \neq 0$ has the behavior as $x \to 0^+$ given by

$$
\psi(k, x) = \alpha \left[ x + \frac{x^2}{2} + \left( \frac{1}{12} - \frac{k^2}{6} \right) x^3 + O(x^4) \right] + \beta \left[ \frac{1}{\Gamma(1/(2ik))} + O(x) \right], \quad x \to 0^+,
$$

(26.175)

where $\Gamma(1/(2ik))$ is the gamma function evaluated at $1/(2ik)$. The derivative $\psi'(k, x)$ has the behavior as $x \to 0^+$ given by

$$
\psi'(k, x) = \alpha \left[ 1 + x + \left( \frac{1}{4} - \frac{k^2}{2} \right) x^2 + O(x^3) \right] + \beta \left[ \frac{\log x}{\Gamma(1/(2ik))} + O(1) \right], \quad x \to 0^+.
$$

(26.176)

As argued earlier, in the example of the truncated Coulomb potential (26.172), only in the Dirichlet case with $\beta = 0$ we can have the regular and physical solutions to the Schrödinger equation satisfy the self adjoint boundary condition at $x = 0$. In the non-Dirichlet case we must have $\beta \neq 0$, and hence there are neither regular nor physical nontrivial solutions to (2.1) with $V(x)$ as in (26.172) satisfying a non-Dirichlet boundary condition of the form (2.4)-(2.6).
Acknowledgments. The research leading to this article was supported in part by CONACYT under project CB2015, 254062, and by Project PAPIIT-DGAPA-UNAM IN102215.
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