The applications of the partial Hamiltonian approach to mechanics and other areas

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Abstract

The partial Hamiltonian systems of the form \( \dot{q}^i = \frac{\partial H}{\partial p^i}, \dot{p}^i = -\frac{\partial H}{\partial q^i} + \Gamma^i(t, q^i, p_i) \) arise widely in different fields of the applied mathematics. The partial Hamiltonian systems appear for a mechanical system with non-holonomic nonlinear constraints and non-potential generalized forces. In dynamic optimization problems of economic growth theory involving a non-zero discount factor the partial Hamiltonian systems arise and are known as a current value Hamiltonian systems. It is shown that the partial Hamiltonian approach proposed earlier for the current value Hamiltonian systems arising in economic growth theory Naz et al [1] is applicable to mechanics and other areas as well. The partial Hamiltonian approach is utilized to construct first integrals and closed form solutions of optimal growth model with environmental asset, equations of motion for a mechanical system with non-potential forces, the force-free Duffing Van Der Pol Oscillator and Lotka-Volterra models.

Key words: Partial Hamiltonian system, Economic growth theory, Mechanics, partial Hamiltonian function, First integrals

1 Introduction

The classical mechanics was reformulated as Lagrangian mechanics by Joseph Louis Lagrange in 1788 [2]. In 1833, William Rowan Hamilton [3] formulated the Hamiltonian mechanics by utilizing the Legendre transformation [4,5]. Later on, the notions of Lagrangian and Hamiltonian became popular in other fields as well e.g. continuum mechanics, fluid mechanics, quantum mechanics, plasma physics, engineering, mathematical biology, economic growth theory and many other fields dealing with dynamic optimization problems. A dynamic optimization problem involves the determination of the extremal of the
functional involving time, dependent, independent variables and their derivative up to finite order. There are three major approaches to deal with dynamic optimization problems: calculus of variations, dynamic programming and optimal control theory. The calculus of variation utilizes the notion of a standard Lagrangian and provides a set of equations known as Euler-Lagrange equations \[2,6\]. The dynamic programming was introduced by Richard Ernest Bellman \[7\]. The optimal control theory is an extension of calculus of variation and is developed by Lev Semyonovich Pontryagin \[8\].

The first integrals or conservation laws for differential equations are essential in constructing exact solutions (see e.g.\[9\]-\[11\] and references therein). The first integrals/conservation laws for the Euler-Lagrange differential equations can be established with the help of celebrated Noether’s theorem \[12\] provided a standard Lagrangian exists. Most of the differential equations that describe the real world phenomena do not admit standard Lagrangian and thus Noether’s theorem \[12\] cannot be applied to construct the first integrals and conservation laws. The partial Lagrangian approach \[13,14\] was developed to construct first integrals/conservation laws for differential equations which do not have standard Lagrangian. The partial or discount free Lagrangian approach \[15\] is developed to derive the first integrals and closed-form solutions for the calculus of variation problems involving partial or discount free Lagrangian in economic growth theory. Naeem and Mahomed \[16\] provided notions of approximate partial Lagrangian and approximate Euler-Lagrange equations for perturbed ODEs. There are methods to obtain first integrals which do not rely on the knowledge of a Lagrangian function. The characteristic method \[17\] and direct method \[18,19\] have been successfully applied to establish first integrals of several differential equations. The most effective and systematic Maple based computer package GeM developed by Cheviakov \[20]-\[21\] works in an excellent way to derive first integrals. A review of all different approaches to construct first integrals for differential equations is presented in \[22,23\].

The Legendre transformation provides the equivalence of the Euler-Lagrange and Hamiltonian equations \[5\]. In 2010, Dorodnitsyn and Kozlov \[24\] established the relation between symmetries and first integrals for both continuous canonical Hamiltonian equations and discrete Hamiltonian equations by utilizing the Legendre transformation. Thus well-known Noether’s theorem was formulated in terms of the Hamiltonian function and symmetry operators. A current value Hamiltonian approach was proposed by Naz et al \[1,25\] to derive the first integrals and closed-form solutions for systems of first-order ODEs arising from the optimal control problems involving current value Hamiltonian in economic growth theory. Mahomed and Roberts \[26\] focused on the characterization of Hamiltonian symmetries and their first integrals. This is applicable to standard Hamiltonian systems.

In this paper, I focus on the partial Hamiltonian systems of the form \[\dot{q}^i = \]
\[ \frac{\partial H}{\partial p_i} \dot{q}^i - \frac{\partial H}{\partial q_i} + \Gamma_i(t, q^i, p_i) \] which arises widely in economic growth theory, physics, mechanics, biology and in some other fields of applied mathematics. The partial Hamiltonian systems appear for the mechanical systems with non-holonomic nonlinear constraints and non-potential generalized forces. In dynamic optimization problems of economic growth theory involving a non-zero discount factor the partial Hamiltonian systems arise and are known as current value Hamiltonian systems. These type of systems arise also in classical field theories almost every conservative fluid and plasma theory has this form. The partial Hamiltonian systems have a real physical structure.

The layout of the paper is as follows. Preliminaries on known forms of partial Hamiltonian systems are presented in Section 2. In Section 3, real world applications are presented to show effectiveness of partial Hamiltonian approach. The first integrals of the optimal growth model with environmental asset, equations of motion for a mechanical system, the force-free duffing Van Der Pol oscillator and Lotka-Volterra system are established. Finally, conclusions are presented in Section 4.

2 Preliminaries

Let \( t \) be the independent variable which is usually time and \((q, p) = (q^1, ..., q^n, p_1, ..., p_n)\) the phase space coordinates. The following results are adopted from [24,1,25]:

The Euler operator \( \delta/\delta q^i \) and the variational operator \( \delta/\delta p_i \) are defined as

\[
\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - D \frac{\partial}{\partial \dot{q}^i}, \quad i = 1, 2, \ldots, n,
\]

and

\[
\frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i} - D \frac{\partial}{\partial \dot{p}_i}, \quad i = 1, 2, \ldots, n,
\]

where

\[
D_t = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} + \cdots
\]

is the total derivative operator with respect to the time \( t \). The summation convention applies for repeated indices here and in the sequel.

The variables \( t, q^i, p_i \) are independent and connected by the differential relations

\[
\dot{p}_i = D_t(p_i), \quad \dot{q}^i = D_t(q^i), \quad i = 1, 2, \ldots, n.
\]
\[ \dot{q}^i = \frac{\partial H}{\partial p_i}, \]
\[ \dot{p}_i = -\frac{\partial H}{\partial q_i} + \Gamma^i(t, q^i, p_i), \quad i = 1, \ldots, n, \] (5)

where \( \Gamma^i \) are in general non-zero functions of \( t, q^i, p^i \).

The generators of point symmetries in the space \( t, q, p \) are operators of the form [24,1,25]
\[ X = \xi(t, q, p) \frac{\partial}{\partial t} + \eta^i(t, q, p) \frac{\partial}{\partial q^i} + \zeta_i(t, q, p) \frac{\partial}{\partial p_i}. \] (6)

Naz et al [1,25] provided following criteria to derive partial Hamiltonian operators and associated first integrals:

An operator \( X \) of the form (6) is said to be a partial Hamiltonian operator corresponding to a current value Hamiltonian \( H(t, q, p) \), if there exists a function \( B(t, q, p) \) such that
\[ \zeta_i \frac{\partial H}{\partial p_i} + p_i D_t(\eta^i) - X(H) - HD_t(\xi) = D_t(B) + (\eta^i - \xi \frac{\partial H}{\partial p_i})(-\Gamma^i) \] (7)
holds on the system (5) then system has a first-integral
\[ I = p_i \eta^i - \xi H - B. \] (8)

The Hamiltonian system of form (5) arises in economic growth theory, mechanics, physics, biology and in some other fields of applied mathematics.

**Remark 1:** The function \( H \) which results in a Hamiltonian system of form (5) is defined in different ways. In Economic growth theory \( H \) is a current value Hamiltonian function see e.g. [1,25,27]. In a first-order mechanical system with non-holonomic constraints or non-potential generalized forces or external forces the function \( H \) which gives rise to a standard Hamiltonian but it yields a Hamiltonian system of form (5) see e.g. [28,29]. One can also find these structures in different fields of applied Mathematics. These systems have a real physical structure.

A natural question arises what is significance of functions \( \Gamma^i(t, q^i, p_i) \)?

**Remark 2:** In each field the functions \( \Gamma^i(t, q^i, p_i) \) are interpreted in different ways but are always some physical quantities. In correspondence to the economic growth theory \( \Gamma^i \) is associated with discount factor. In mechanics the systems of form (5) arise and the functions \( \Gamma^i \) contain non-potential generalized forces. In some other mechanical systems the functions \( \Gamma^i \) are related to the
generalized constrained forces. In other fields as well the functions $\Gamma^i(t, q', p_i)$ have structural properties and describe some physical phenomena.

3 Applications

In this Section, the first-integrals and closed-form solutions for some real world models are derived to show effectiveness of partial Hamiltonian approach developed by Naz et al [1]. The first integrals and closed-form solutions of optimal growth model with environmental asset are established. The first integrals of the equations of motion for a mechanical system are established. Both of these models have real Hamiltonian structure. The force-free Duffing Van Der Pol Oscillator and Lotka-Volterra model have no real Hamiltonian structure but these can be expressed as a partial Hamiltonian system.

3.1 Optimal growth model with environmental asset

The partial Hamiltonian approach [1] is applied to the optimal growth model with an environmental assets investigated by Le Kama and Schubert [32]. The social planner seeks to maximize

$$\max \int_0^\infty \frac{(cs^\phi)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, \quad \phi \geq 0, \sigma > 0, \sigma \neq 1, \rho > 0,$$

subject to the constraint

$$\dot{s} = ms - c,$$

where $s(t)$ is the stock of environmental asset, $c(t)$ is the consumption, $\sigma$ is the inverse of intertemporal elasticity of substitution, $\phi$ is the relative preference for environment, $m > 0$ is the regeneration rate for environmental asset and $\rho$ is the discount factor.

3.1.1 First integrals

The current value Hamiltonian for this model is

$$H(t, c, s, p) = \frac{(cs^\phi)^{1-\sigma}}{1-\sigma} + p(ms - c),$$

where $p(t)$ is the costate variable. The necessary first order conditions for optimal control yield

$$p = c^{-\sigma} s^\phi(1-\sigma),$$

$$\dot{s} = ms - c,$$
\[
\dot{p} = (\rho - m - \phi \frac{c}{s})p,
\]
and the growth rate of consumption, with the aid of equations (12)-(14), is given by
\[
\frac{\dot{c}}{c} = \phi(\frac{1}{\sigma} - 1)m + \phi \frac{c}{s} + \frac{m - \rho}{\sigma}.
\]
(15)
The initial and transversality conditions are of the following form:
\[
c(0) = c_0, \quad s(0) = s_0,
\]
(16)
and
\[
\lim_{t \to \infty} e^{-\rho t} p(t)s(t) = 0.
\]
(17)
The partial Hamiltonian operator determining equation with the aid of equations (11)-(14) yields
\[
c^{-\sigma} s^{\phi(1-\sigma)}[\eta_t + (ms - c)\eta_s] - \eta[c^{1-\sigma} \phi s^{\phi(1-\sigma)-1} + c^{-\sigma} s^{\phi(1-\sigma)}m]
- \left[\frac{(cs)^{1-\sigma}}{1-\sigma} + c^{-\sigma} s^{\phi(1-\sigma)}(ms - c)\right]\xi_t + (ms - c)\xi_s
= B_t + (ms - c)B_s - \left[\eta - \xi (ms - c)\right]\rho c^{-\sigma} s^{\phi(1-\sigma)},
\]
in which \(\xi(t, s), \eta(t, s)\) and \(B(t, s)\). Separating equation (18) with respect to powers of \(c\) yields following overdetermined system for \(\xi(t, s), \eta(t, s)\) and \(B(t, s)\):
\[
\begin{align*}
c^{2-\sigma} : \quad & \xi_s = 0, \\
c^{1-\sigma} : \quad & \eta_s + \phi \frac{\eta}{s} + \sigma \xi_t - \rho \xi = 0, \\
c^{-\sigma} : \quad & \eta_t + ms \eta_s - m\eta - ms \xi_t + \rho(\eta - ms \xi) = 0, \\
c^0 : \quad & B_s = 0, \\
& B_t = 0.
\end{align*}
\]
(19)
The solution of system (19) provides following partial Hamiltonian operators \(\xi(t, s), \eta(t, s)\) and gauge term \(B(t, s)\):
\[
\eta = \frac{a_1 \rho}{(1-\sigma)(\phi + 1)} e^{-\rho t} + a_2 c^{\frac{(\rho - m \phi - m)(1-\sigma)}{\sigma}} t + a_3 s^{-\phi} e^{-(\rho - m \phi - m)t},
\]
(20)
The first integrals corresponding to operators and gauge terms given in equation (20) can be computed from formula and are given as follows:
\[ I_1 = \frac{\rho \rho e^{-\rho t}}{(\phi + 1)(1 - \sigma)} - e^{-\rho t} \left[ \frac{(cs\phi)^{1-\sigma}}{1 - \sigma} + p(ms - c) \right] \]

\[ I_2 = c^{(\rho - m\phi - m - \phi + 1)(1 - \sigma)} \left[ p - \frac{(cs\phi)^{1-\sigma}}{1 - \sigma} \right] \]

\[ I_3 = ps - \phi e^{(m\phi + m - \rho) t}. \]

3.1.2 Closed-form solution

One can utilize either any one or any two of these first integrals to derive the closed-form solution of system (12)-(14). Setting \( I_3 = a \) yields

\[ c = a e^{-\frac{1}{\sigma}} s^{-\phi} e^{(m\phi + m - \rho) t}. \]

Equation (22) transforms equation (13) into Bernoulli's equation for variable \( s \) which gives

\[ s(t) = \left( \frac{\sigma a^{-\frac{1}{\sigma}} (\phi + 1) e^{m\phi + m - \rho} t}{\rho + m(\sigma - 1)(\phi + 1)} + be^{m(\phi + 1)t} \right)^{\frac{1}{\phi + 1}}, \]

and then equation (13) gives \( c(t) \). It is worthy to mention her that one can directly arrive at expression for \( s \) given in (23) by setting \( I_1 = b \) and then no integration is required. The transversality condition (17) takes following form

\[ \lim_{t \to \infty} e^{-\rho t} p(t) s(t) = as^{\phi + 1} e^{-m(\phi + 1)t}, \]

and it goes to zero provided \( \rho + m(\sigma - 1)(\phi + 1) > 0 \) and \( b = 0 \). The final form of closed-form solutions subject to initial conditions \( c(0) = c_0 \) and \( s(0) = s_0 \) is given as follows:

\[ s(t) = s_0 e^{m\phi + m - \rho t}, \]

\[ c(t) = c_0 e^{\sigma(\phi + 1) - \rho t}, \]

\[ p(t) = c_0 s_0^{-\sigma} \phi e^{(\rho - m - \phi s_0^\sigma) t}, \]

where \( \frac{\sigma}{s_0}(\phi + 1) = \rho + m(\sigma - 1)(\phi + 1) \). The growth rates of all variables of economy are

\[ \dot{s} = \frac{m(\phi + 1) - \rho}{\sigma(\phi + 1)}, \]

\[ \dot{c} = \frac{m(\phi + 1) - \rho}{\sigma(\phi + 1)}, \]

\[ \dot{p} = \rho - m - \phi \frac{c_0}{s_0}. \]
The variables $s$ and $c$ grow at the same constant rate which makes true economics sense. The variable $p$ grows at a constant rate $\rho - m - \phi_{s_0}$.

### 3.2 First integrals for equations of motion for a mechanical system

Next it is shown how to construct first integrals with the aid of partial Hamiltonian approach [1] for more variables case.

A Hamiltonian function for a mechanical system with non-potential forces $\Gamma^1 = -p_2$ and $\Gamma^2 = 0$ is

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{q_2^2}{2}.$$  

The equations of motion for a mechanical system are

$$\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_1 = -p_2, \quad \dot{p}_2 = -q_2.$$  

The partial Hamiltonian operator determining equation (7) with the aid of equations (30)-(33) yields

$$p_1(\eta_1^1 + p_1\eta_1^1 + p_2\eta_1^1) + p_2(\eta_2^2 + p_1\eta_2^2 + p_2\eta_2^2)$$

$$-\eta_2^2 q_2 - (\frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{q_2^2}{2})(p_1 \xi_t + p_1 \xi_{q_1} + p_2 \eta_1^1)$$

$$= B_1 + p_1 B_{q_1} + p_2 B_{q_2} + (\eta_1^1 - \xi p_1)p_2$$

in which $\xi(t, q_1, q_2)$, $\eta_1^1(t, q_1, q_2)$, $\eta_2^2(t, q_1, q_2)$ and $B(t, q_1, q_2)$. Separating equation (34) with respect to powers of $p_1$ and $p_2$ as $\eta_1^1$, $\eta_2^2$, $\xi$, $B$ do not contain $p_1$, $p_2$, an over determined system for $\eta_1^1$, $\eta_2^2$, $\xi$, $B$ is obtained. After lengthy calculations, following expressions for $\eta_1^1$, $\eta_2^2$, $\xi$, $B$ are obtained:

$$\xi = c_1, \quad \eta_1^1 = -c_4 q_2 + c_2 t + c_3, \quad \eta_2^2 = c_4 \sin t + c_5 \cos t + c_2,$$

$$B = \frac{c_1}{2} q_2^2 + c_2 (q_1 - t q_2) - c_3 q_2 + c_4 q_2 \cos t - c_5 q_2 \sin t.$$  

After utilizing formula (8) the first integrals associated with partial Hamiltonian operators and gauge terms given in (35) can be expressed as following form:
\[ I_1 = q_2^2 + q_2 p_1 + \frac{p_1^2}{2} + \frac{p_2^2}{2}, \quad (36) \]
\[ I_2 = q_1 - tq_2 - tp_1 - p_2, \quad (37) \]
\[ I_3 = q_2 + p_1, \quad (38) \]
\[ I_4 = q_2 \cos t - p_2 \sin t, \quad (39) \]
\[ I_5 = q_2 \sin t + p_2 \cos t. \quad (40) \]

One can utilize these first integrals to construct closed-form solution for the partial Hamiltonian system (30)-(33).

### 3.3 Force-free Duffing Van Der Pol Oscillator

The standard form of force-free Duffing Van der pol oscillator equation is

\[ \ddot{q} + (\alpha + \beta q^2)\dot{q} - \gamma q + q^3 = 0, \quad (41) \]

where \( \alpha, \beta \) and \( \gamma \) are arbitrary parameters. Equation (41) arises in a model describing the propagation of voltage pulses along a neuronal axon \cite{33,34}. Chandrasekar et al \cite{33} derived only one first integral for the case when \( \alpha = \frac{4}{\beta} \) and \( \gamma = -\frac{3}{\beta^2} \). After making a series of variable transformations and applying the Preller-Singer method, one first integral was computed in \cite{34} and then inverse transformations provide the first integrals of the original equations satisfying parameter restriction \( \beta^2 \gamma + 3 \alpha \beta - 9 = 0 \). I derive here more first integrals of force-free Duffing Van der pol oscillator using the partial Hamiltonian approach and these are not reported before in literature. The partial Hamiltonian function for second-order ODE (41) is

\[ H = -\frac{p^2}{2} + \frac{\gamma}{2} q^2 - \frac{q^4}{4}, \quad (42) \]

and it satisfies

\[ \dot{q} = -p, \quad (43) \]
\[ \dot{p} = -\gamma q + q^3 + \Gamma \quad (44) \]

where \( \Gamma = -(\alpha + \beta q^2)p \). The partial Hamiltonian operator determining equation (7) with the aid of equations (42)-(44) yields

\[
p(\eta_t - p\eta_q) - \eta(\gamma q - q^3) - \left(-\frac{p^2}{2} + \frac{\gamma}{2} q^2 - \frac{q^4}{4}\right)(\xi_t - p\xi_q)
= B_t - pB_q + (\eta + p\xi)(\alpha + \beta q^2)p,
\quad (45)\]
in which \( \xi(t, q) \), \( \eta(t, q) \) and \( B(t, q) \). Separating equation (45) with respect to powers of \( p \) as \( \eta, \xi, B \) do not contain \( p \) yields

\[
\begin{align*}
p^3 : & \quad \xi_q = 0, \\
p^2 : & \quad -\eta_q + \frac{1}{2} \xi_t = \xi(\alpha + \beta q^2), \\
p : & \quad \eta_t + \xi_q(\frac{\gamma}{2} q^2 - \frac{q^4}{4}) = -B_q + \eta(\alpha + \beta q^2), \\
p^0 : & \quad B_t + \eta(\gamma q - q^3) + \xi_t(\frac{\gamma}{2} q^2 - \frac{q^4}{4}) = 0. \tag{46}
\end{align*}
\]

After lengthy calculations finally system (46) gives

\[
\begin{align*}
\xi &= c_1 e^{\frac{\beta t}{3}}, \quad \eta = -\frac{1}{\beta} q(\beta^2 q^2 + 3\alpha \beta - 9)c_1 e^{\frac{\beta t}{3}} + c_2 e^{\frac{\beta t}{3}}, \\
B &= -\frac{q^2}{18\beta^2} \left( \beta^4 q^4 + \left( 6\alpha \beta^3 - \frac{45}{2} \beta^2 \right) q^2 + 9\alpha^2 \beta^2 - 81 \alpha \beta + 162 \right) e^{\frac{\beta t}{3}} + \frac{q}{3\beta} (\beta^2 q^2 + 3\alpha \beta^2 - 9)c_2 e^{\frac{\beta t}{3}}. \tag{47}
\end{align*}
\]

provided \( \beta^2 \gamma + 3\alpha \beta - 9 = 0 \). The first integrals can be constructed from formula (8). The partial Hamiltonian operators and first integrals are given by

\[
\begin{align*}
X_1 &= e^{\frac{\beta t}{3}} \frac{\partial}{\partial t} - \frac{q}{\beta} (\beta^2 q^2 + 3\alpha \beta - 9)e^{\frac{\beta t}{3}} \frac{\partial}{\partial q}, \\
I_1 &= \frac{1}{2} \left[ p - (\alpha \beta - 3) \frac{q}{\beta} - \frac{\beta}{3} q^3 \right] e^{\frac{\beta t}{3}}, \\
X_2 &= e^{\frac{\beta t}{3}} \frac{\partial}{\partial q}, \\
I_2 &= \left[ p - (\alpha \beta - 3) \frac{q}{\beta} - \frac{\beta}{3} q^3 \right] e^{\frac{\beta t}{3}}. \tag{48}
\end{align*}
\]

provided \( \beta^2 \gamma + 3\alpha \beta - 9 = 0 \). Notice that \( I_1 = \frac{I_2^2}{2} \). Now I utilize \( I_2 \) to construct solution of (43) and (44). Set \( I_2 = a_1 \) gives

\[
p = (\alpha \beta - 3) \frac{q}{\beta} + \frac{\beta q^3}{3} + a_1 e^{-\frac{\beta t}{3}}. \tag{49}
\]

Equation (43) with the aid of equation (49) results in

\[
\dot{q}(t) + (\alpha \beta - 3) \frac{q}{\beta} + \frac{\beta q^3}{3} + a_1 e^{-\frac{\beta t}{3}} = 0, \tag{50}
\]
where \( a_1 \) is an arbitrary constant and one can choose it as zero without loss of generality. Equation (50) reduces to a Bernoulli’s equation if \( a_1 = 0 \) which yields

\[
q(t) = \pm \sqrt{9a_2(\alpha \beta - 3)^2e^{-\frac{2(\alpha \beta-3)}{\beta}t} - 3\beta^2(\alpha \beta - 3)e^{-\frac{4(\alpha \beta-3)}{\beta}t}}.
\]  

The exact solution for second order ODE (41) or its equivalent system (43)-(44) is given in (49) and (51). This solution holds provided \( \beta^2 \gamma + 3 \alpha \beta - 9 = 0 \). This solution is new in literature and partial Hamiltonian approach has made it possible to derive this solution.

### 3.4 Lotka-Volterra system

The notion of partial Hamiltonian function for a dynamical system of two first-order ODEs is explained with the help of Lotka-Volterra system [35,36,37]. Moreover, the first integrals and closed-form solutions are derived by newly developed partial Hamiltonian approach [1]. The two species Lotka-Volterra model for predator-prey interaction is governed by following two first-order ODEs:

\[
\dot{q} = aq - bpq,
\]

\[
\dot{p} = -mp + npq,
\]

where \( q \) is the number of prey, \( p \) is the number of predator, \( a \) and \( m \) are their per-capita rates of change in the absence of each other and \( b \) and \( n \) their respective rates of change due to interaction. A partial Hamiltonian function for system (52)-(53)

\[
H = apq - \frac{b}{2}p^2 q
\]

satisfies

\[
\dot{q} = \frac{\partial H}{\partial p},
\]

\[
\dot{p} = -\frac{\partial H}{\partial q} + \Gamma,
\]

with function \( \Gamma \)

\[
\Gamma = -\frac{bp^2}{2} + (a - m)p + npq.
\]

The partial Hamiltonian operators determining equation (7) with the aid of equations (54)-(57) yields
\[ p[\eta_t + (aq - bpq)\eta_q] - \eta(ap - \frac{bp^2}{2}) - (apq - \frac{bp^2q}{2})[\xi_t + (aq - bpq)\xi_q] \]
\[ = B_t + (aq - bpq)B_q - [\eta - \xi(aq - bpq)][-\frac{bp^2}{2} + (a - m)p + npq], \tag{58} \]

in which \( \xi(t, q), \eta(t, q) \) and \( B(t, q) \). Separating equation (58) with respect to powers of \( q \) as \( \eta, \xi, B \) do not contain \( q \), provides

\[ p^3 : q\xi_q + \xi = 0, \]
\[ p^2 : \eta_q - \frac{1}{2}\xi_t - \frac{3aq}{2}\xi_q - \frac{3a}{2} - m + nq\xi = 0, \tag{59} \]
\[ p : \eta_t + aq\eta_q + (nq - m)\eta - aq\xi_t - a^2q^2\xi_q \]
\[ + bqB_q - (a^2q - amq + aq^2n)\xi = 0, \]
\[ p^0 : B_t + qB_q = 0. \]

The solution of system (59) yields

\[ \xi = \frac{F_1(t)}{q}, \]
\[ \eta = \left(\frac{1}{2}F_1'(t) - mF_1(t)\right)\ln(q) + nqF_1(t) + F_2(t), \]
\[ B = \left( -\frac{1}{4b}F''_1(t) + \frac{3m}{4b}F_1'(t) - \frac{m^2}{2b}F_1(t) \right)(\ln(q))^2 \]
\[ + \frac{1}{b}\left( - F'_2(t) + mF_2(t) + \frac{(a - nq)}{2}F'_1(t) + mnqF_1(t) \right)\ln(q) \]
\[ - \frac{n^2q^2}{2b}F_1(t) - \frac{n}{b}qF_2(t) - \frac{n}{2b}F_1(t)q + F_3(t), \]

with \( F_1(t), F_2(t) \) and \( F_3(t) \) satisfying

\[ \frac{1}{4}F_1(t)'' - \frac{3m}{4}F''_1(t) + \frac{m^2}{2}F_1'(t) = 0, \]
\[ F_2''(t) - mF_2'(t) - \frac{3am}{2}F_1'(t) + am^2F_1(t) = 0, \]
\[ \frac{1}{2}F''_1(t) - (m - \frac{a}{2})F'_1(t) - amF_1(t) = 0, \tag{60} \]
\[ F'_1(t) + 2aF_1(t) = 0, \]
\[ F'_2(t) + 2aF_2(t) + F''_1(t) + 2aF'_1(t) - 2amF_1(t) = 0, \]
\[ F'_3(t) + \frac{a}{b}\left( \frac{a}{2}F'_1(t) - F'_2(t) + mF_2(t) \right) = 0. \]
System (60) yields

\[
F_1(t) = C_1 e^{-2at}, \quad F_2(t) = aC_1 e^{-2at} - \frac{1}{a} C_2 e^{-at}, \quad F_3(t) = C_3,
\]

provided \( m = -a \). The partial Hamiltonian operators, gauge term and first integrals are given by

\[
\xi = \frac{e^{-2at}}{q}, \quad \eta = (a + nq)e^{-2at}, \quad B = -\frac{n^2 q^2}{2b} e^{-2at},
\]

\[
I_1 = \frac{1}{2b} e^{-2at}(bp + nq)^2,
\]

\[
\xi = 0, \quad \eta = -\frac{1}{a} e^{-at}, \quad B = -\frac{nq}{a2b} e^{-at},
\]

\[
I_2 = -\frac{1}{ab} e^{-at}(bp + nq),
\]

provided \( a = -m \). Notice that \( I_1 = \frac{1}{2} a^2 b I_2^2 \) and thus the first integrals are functionally dependent. One can construct solution of system (52)-(53) with the help of one of these first integrals.

Setting \( I_2 = \alpha_1 \) results in

\[
-\frac{1}{ab} e^{-at}(bp + nq) = \alpha_1
\]

and this yields

\[
p = -(a\alpha_1 e^{at} + \frac{nq}{b}),
\]

where \( \alpha_1 \) is arbitrary constant. Using value of \( p \) from equation (64) in equation (52) yields

\[
\dot{q} - a(1 + bc_1 e^{at})q = nq^2,
\]

and this provides

\[
q(t) = -\frac{ab\alpha_1 e^{at}}{n - ab\alpha_1 a_2 e^{-b\alpha_1 e^{at}}},
\]

Equation (64) and (66) gives following final form of exact solution for variable \( p(t) \):

\[
p = -a\alpha_1 e^{at} + \frac{an\alpha_1 e^{at}}{n - ab\alpha_1 a_2 e^{-b\alpha_1 e^{at}}},
\]

The exact solutions for \( q(t) \) and \( p(t) \) given in equations (66)-(67) are valid for \( a = -m \) case only. The assumption \( a = -m \) shows that the per-capita rates of change are same for the predator and prey in the absence of each other. The solution derived here is not reported before in literature and is established due to partial Hamiltonian approach.
4 Conclusions

The partial Hamiltonian systems arise widely in different fields of the applied mathematics. The partial Hamiltonian systems appear for the mechanical systems with non-holonomic nonlinear constraints and non-potential generalized forces. In dynamic optimization problems of economic growth theory involving a non-zero discount factor the partial Hamiltonian systems arise and are known as the current value Hamiltonian systems. These systems have a real physical structure. The partial Hamiltonian approach proposed earlier for a current value Hamiltonian systems arising in economic growth theory is applicable to mechanics and other areas.

In order to show effectiveness of approach the method is applied to four models: optimal growth model with environmental asset, the equations of motion for a mechanical system, the force-free duffing Van Der Pol oscillator and Lotka-Volterra model. The first integrals and closed-from solutions of the optimal growth model with environmental asset are derived. This model yields a current value Hamiltonian system. The partial Hamiltonian approach provided three first integrals for this model and then closed-from solutions are derived with the aid of these first integrals. The first integrals of the equations of motion for a mechanical system with non-potential forces are derived. A standard Hamiltonian exists for this model and I obtained five first integrals. The system of second-order ODEs, describing the force-free duffing Van Der Pol oscillator, is expressed as a partial Hamiltonian system of first-order ODEs. The partial Hamiltonian approach provided two first integrals. Only one of these first integrals is linearly independent and then this first integral is utilized to derive closed-form solution. Finally, I derived the first-integrals and closed-form solutions for Lotka-Volterra system described by two first-order ODEs. According to best of my knowledge all these solutions are new in literature and are obtained with the help of partial Hamiltonian approach.

It is worthy to mention here that the partial Hamiltonian systems arise in different fields of applied mathematics. In each field the functions $\Gamma^i(t, q^i, p_i)$ are interpreted in different ways but are always some physical quantities. The partial Hamiltonian systems arise widely in non-linear mechanics. In mechanics the functions $\Gamma^i$ contains non-potential generalized forces or generalized constrained forces or both. The functions $\Gamma^i$ are associated with discount factor for economic growth theory. In other fields as well the functions $\Gamma^i(t, q^i, p_i)$ have structural properties.
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