THE LAW OF LARGE NUMBERS FOR QUANTUM STOCHASTIC FILTERING AND CONTROL OF MANY-PARTICLE SYSTEMS

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There is an extensive literature on the dynamic law of large numbers for systems of quantum particles, that is, on the derivation of an equation describing the limiting individual behavior of particles in a large ensemble of identical interacting particles. The resulting equations are generally referred to as nonlinear Schrödinger equations or Hartree equations, or Gross–Pitaevskii equations. In this paper, we extend some of these convergence results to a stochastic framework. Specifically, we work with the Belavkin stochastic filtering of many-particle quantum systems. The resulting limiting equation is an equation of a new type, which can be regarded as a complex-valued infinite-dimensional nonlinear diffusion of McKean–Vlasov type. This result is the key ingredient for the theory of quantum mean-field games developed by the author in a previous paper.

Keywords: quantum dynamic law of large numbers, quantum filtering, homodyne detection, Belavkin equation, nonlinear stochastic Schrödinger equation, quantum interacting particles, quantum control, quantum mean-field games, infinite-dimensional McKean–Vlasov diffusion on manifold

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1. Introduction

There is an extensive literature on the dynamic law of large numbers for systems of quantum particles, i.e., on the derivation of an equation describing the limiting individual behavior of particles in a large ensemble of identical interacting particles. The resulting equations are generally referred to as nonlinear Schrödinger equations or Hartree equations, or Gross–Pitaevskii equations. The first result of this kind appeared in [1]. Various ingenious approaches were employed afterwards in order to extend the conditions of applicability of the convergence results (say, for various classes of unbounded interacting potentials), exploit various scaling regimes, and establish the proper rates of convergence, see reviews in [2], [3]. In this paper, following [4], we for the first time extend some of these convergence results to a stochastic framework. Specifically, we work with the Belavkin stochastic filtering of many-particle quantum systems. The resulting limiting equation is an equation of a new type, which can be seen as a complex-valued infinite-dimensional nonlinear diffusion of McKean–Vlasov type on a manifold. An optimal control of these equations is beyond the scope of this paper. It leads to slightly more general Hamilton–Jacobi–Bellman equations on manifolds, such as those analyzed in [5].

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In [4], the author merged two recently developed branches of game theory, quantum games and mean-field games (MFGs), creating quantum MFGs. MFGs represent a very popular recent development in game theory. It was initiated in [6], [7]. For recent developments, we refer the reader to [8]–[11] and the numerous references therein. The main ingredient in the rigorous treatment of quantum MFGs was the quantum stochastic law of large numbers mentioned above, stating that the continuously observed (via nondemolition measurement and quantum filtering) and controlled quantum system of many particles can be described, in the limit of a large number of particles, by a decoupled system of identical nonlinear stochastic evolutions of individual particles evolving in accordance with some new kind of nonlinear stochastic Schrödinger equation or nonlinear quantum filtering equation. However, this fact was proved only under a very restrictive assumption of the so-called conservative homodyne measurement. In this paper, we obtain this result in its full generality, namely, for arbitrary coupling operators (not necessarily anti-Hermitian as in [4]) with a homodyne detection optical device. We also obtain a similar result for jump-type stochastic Schrödinger equations describing the continuous observations and filtering of counting type.

These results accentuate the difference between classical and quantum mean-field control. The average used in the classical case is the measure obtained by averaging the Dirac δ-measures arising from the initial conditions of individual particles, but in the quantum case the average of the initial conditions themselves is used.

The generality chosen requires working with the nonlinear quantum filtering equation, which leads to a more general limiting equation than the one in [4], namely, to the equation that can be most naturally written in terms of the density matrix \( \gamma \) as

\[
\begin{align*}
\frac{d\gamma_t}{dt} &= -i[H + u(t, \gamma_t)\hat{H}, \gamma_t] dt - i[A^n, \gamma_t] dt + \left(L\gamma_t L^* - \frac{1}{2} L^* L\gamma_t - \frac{1}{2} \gamma_t L^* L\right) dt + \\
&\quad + (\gamma_t L^* + L\gamma_t - \gamma_t \text{tr}(\gamma_t(L + L^*))) dB_t, \quad \eta_t(y, z) = E\gamma_t(y, z).
\end{align*}
\]

Here \( H, \hat{H}, \) and \( L \) are linear operators in some Hilbert space \( L^2(X) \), \( H \) and \( \hat{H} \) are self-adjoint operators, \( X \) is some Borel space with a given Borel measure denoted by \( dx \), \( A^n \) is the integral operator in \( L^2(X) \) with the integral kernel

\[
A^n(x, x') = \int_{X^2} A(x, y; x', y') \eta_t(y, y') dy dy',
\]

\( B_t \) is a standard Brownian motion, \( E \) denotes the corresponding expectation, and \( u(t, \gamma) \) is a continuous function with values in an interval \([-U, U]\) with some positive \( U \); in what follows, \( u(t, \gamma) \) is interpreted as certain control over the behavior of the system.

This equation can be regarded as infinite-dimensional complex-valued nonlinear diffusion of McKean–Vlasov type on the manifold of quantum mechanical states defining a nonlinear Markov process on this manifold in the sense of [12].

We show that Eq. (1) describes the limiting individual behavior of a quantum particle that is a part of an ensemble of interacting particles subject to continuous observation of diffusive type (see Sec. 2 for the theory of such observations) with the identical coupling operator \( L \) applied to each particle.

In terms of the state vectors \( \psi \in L^2(X) \), the limiting equation acquires a more complicated form

\[
\begin{align*}
\frac{d\psi_t(x)}{dt} &= -i[H + u(t, \psi_t)\hat{H} + A^E(\psi_t \otimes \psi_t)] - (\text{Re } L)\psi_t(x) dt - \\
&\quad - \frac{1}{2}(L - (\text{Re } L)\psi_t)^* (L - (\text{Re } L)\psi_t) \psi_t(x) dt + (L - (\text{Re } L)\psi_t)\eta_t dB_t,
\end{align*}
\]

where, for an operator \( L \),

\[
\text{Re } L = \frac{L + L^*}{2}, \quad \text{Im } L = \frac{L - L^*}{2i}, \quad \langle L \rangle_v = \frac{(v, L v)}{(v, v)}.
\]
In the particular case of a self-adjoint operator $L$, this equation is simplified to the equation

$$d\psi_t(x) = -i[H + u(t, \psi_t)]\psi_t(x) dt - \frac{1}{2}(L - \langle L \psi_t \rangle)^2 \psi_t(x) dt + (L - \langle L \psi_t \rangle) \psi_t dB_t. \quad (4)$$

These equations are different from the nonlinear Schrödinger equations that are usually discussed in the current literature (see, e.g., [13]–[16]). In our equation, the nonlinearity depends on the expectations of the correlations calculated with respect to the solution and thus bears analogy to the classical McKean–Vlasov nonlinear diffusions.

The content of this paper is as follows. In Sec. 2, we recall the basic theory of quantum continuous measurement and filtering. In the main Section 3, our new nonlinear equations are introduced in the case of homodyne (or diffusive) detection and the main result on the convergence of $N$-particle observed quantum evolutions to the decoupled system of these equations is obtained, together with explicit rates of convergence. The convergence of solutions is proved here under the assumption that solutions exist. To repair this drawback, in Sec. 4 we prove the well-posedness of our nonlinear Schrödinger equation (2), including a continuous dependence on the initial conditions with explicit estimates for errors. Because a direct check with Ito’s rule shows that for $\psi$ satisfying (2), the density matrix $\gamma_t = \psi_t \otimes \psi_t$ satisfies (1), this implies the existence of decomposable solutions of (1), which are used in Sec. 3. In Sec. 5, the case of a counting observation is analyzed, the limiting stochastic equation of jump type is introduced, and the corresponding convergence results are proved. In this case, however, we have to reduce the generality to the case of unitary coupling operators, because of incompatible intensities of jumps, as explained there.

Appendix A contains a derivation of a quite remarkable technical estimate that is central in our proof of the main result. Appendix B presents certain results on stochastic differential equations (SDEs) in Hilbert spaces, including McKean–Vlasov type diffusions, given in the form that is most handy for the application to the well-posedness result in Sec. 4.

2. Nondemolition observation and quantum filtering

The general theory of quantum nondemolition observation, filtering, and resulting feedback control was built essentially in [17]–[19]. For alternative simplified derivations of the main filtering equations given below (bypassing the heavy theory of quantum filtering), we refer to [20]–[23] and the references therein. For the technical side of organizing feedback quantum control in real time, see, e.g. [24]–[26].

We briefly describe the main result of this theory.

The nondemolition measurement of quantum systems can be organized in two versions: photon counting and homodyne or diffusive type detection. Roughly speaking, counting measurements measure the number of photons reaching a certain detector and therefore are described mathematically by Poisson-type random noises, and diffusive type observations measure the tracks of particles in certain media and therefore are described mathematically by diffusive processes generated by Brownian motions.

We start with the homodyne (mathematically speaking, diffusive type) detection. Under this type of measurement, the output process $Y_t$ is a usual Brownian motion (under an appropriate probability distribution). The quantum filtering equation can be written in two equivalent ways.

1. As a linear equation for a non-normalized state,

$$d\chi_t = -\left[iH\chi_t + \frac{1}{2}L^*L\chi_t\right] dt + L\chi_t dY_t, \quad (5)$$

where the unknown vector $\chi_t$ is from the Hilbert space of the observed quantum system, which we sometimes refer to as the atom, the self-adjoint operator $H$ is the Hamiltonian of the corresponding initial (not observed) quantum evolution, and the operator $L$ is the coupling operator of the atom to the optical measurement device specifying the chosen version of the homodyne detection.
2. As a nonlinear equation for the normalized state \( \phi = \chi/\|\chi\| \),

\[
\begin{align*}
\nonumber d\phi_t &= - \left[ i (H - \langle Re L \rangle_{\phi_t}) \operatorname{Im} L + \frac{1}{2} (L - \langle Re L \rangle_{\phi_t})^\ast (L - \langle Re L \rangle_{\phi_t}) \right] \phi_t \, dt + \\
&\quad + (L - \langle Re L \rangle_{\phi_t}) \phi_t \, dB_t,
\end{align*}
\]

where

\[
dB_t = dY_t - \langle L + L^\ast \rangle_{\chi_t} \, dt = dY_t - \langle L + L^\ast \rangle_{\phi_t} \, dt
\]

defines the so-called innovation process \( B_t \). It can be verified directly via Ito’s rule that (5) implies (6) for \( \phi = \chi/\|\chi\| \).

In the most important case of a self-adjoint \( L \), Eqs. (5) and (6) are simplified to the equations

\[
\begin{align*}
d\chi_t &= - \left[ i H \chi_t + \frac{1}{2} L^2 \chi_t \right] \, dt + L \chi_t \, dY_t
\end{align*}
\]

and

\[
\begin{align*}
d\phi_t &= - \left[ i H + \frac{1}{2} (L - \langle L \rangle_{\phi_t})^2 \right] \phi_t \, dt + (L - \langle L \rangle_{\phi_t}) \phi_t \, dB_t.
\end{align*}
\]

The innovation process is the most natural driving noise to deal with because it turns out to be the standard Brownian motion (or the Wiener process) with respect to a fixed (initial vacuum) state of the homodyne detector. This fact is one of the key statements of the filtering theory.

In [4], we worked with the special case of anti-Hermitian operators \( L \), i.e., \( L^* = -L \), for which nonlinear filtering equation (9) coincides with the linear one, the output process coincides with the innovation process, and the linear equation preserves the norms. This case is therefore referred to as the conservative one, and is regarded as the less relevant for concrete observations (see the discussion in [27]).

The theory extends naturally to the case of several, say \( N \), coupling operators \( \{L_j\} \). For example, the nonlinear quantum filtering equation can then be written as

\[
\begin{align*}
\begin{align*}
\nonumber d\phi_t &= -i H \phi_t \, dt + \sum_j \left[ i \langle Re L_j \rangle_{\phi_t} \operatorname{Im} L_j - \frac{1}{2} (L_j - \langle Re L_j \rangle_{\phi_t})^\ast (L_j - \langle Re L_j \rangle_{\phi_t}) \right] \phi_t \, dt + \\
&\quad + \sum_j (L_j - \langle Re L_j \rangle_{\phi_t}) \phi_t \, dB^j_t,
\end{align*}
\end{align*}
\]

where the \( N \)-dimensional innovation process \( B_t = \{B^j_t\} \) is the standard \( N \)-dimensional Wiener process connected with the output process \( Y_t = \{Y^j_t\} \) by the equations

\[
dB^j_t = dY^j_t - \langle L_j + L^*_j \rangle_{\chi_t} \, dt.
\]

We recall that the density matrix or density operator \( \gamma \) corresponding to a unit vector \( \chi \in L^2(X) \) is defined as the orthogonal projection operator on \( \chi \). This operator is usually expressed either as the tensor product \( \gamma = \chi \otimes \chi \) or in Dirac’s bra–ket notation, most common in physics, as \( \gamma = |\chi\rangle \langle \chi| \). Of course, in the tensor notation \( \gamma \) is formally an element of the tensor product \( L^2(X^2) \). However, considered as an integral operator, it is identified with the corresponding integral operator.

As can be verified by direct application of Ito’s formula, Eq. (6) can be rewritten in terms of the density matrix \( \gamma \) as

\[
\begin{align*}
\nonumber d\gamma_t &= -i [H, \gamma_t] \, dt + \left( L \gamma_t L^* - \frac{1}{2} L^* L \gamma_t - \frac{1}{2} \gamma_t L^* L \right) \, dt + \\
&\quad + \left[ \gamma_t L^* + L \gamma_t - \gamma_t \operatorname{tr}(\gamma_t (L^* + L)) \right] \, dB_t,
\end{align*}
\]

which is the equation we work with to prove the convergence result.
We make some comments about the evolution of traces. Equation (11) is best suited for dealing with matrices $\gamma$ with unit trace. A convenient extension to arbitrary traces can be written as

$$d\gamma_t = -i[H, \gamma_t] dt + \left( L\gamma_t L^* - \frac{1}{2} L^* L \gamma_t - \frac{1}{2} \gamma_t L^* L \right) dt +$$

$$+ \left[ \gamma_t L^* + L\gamma_t - \frac{\gamma_t}{\text{tr}\gamma_t} \text{tr}(\gamma_t (L^* + L)) \right] dB_t.$$

(12)

It can be seen directly from this equation that $d \text{tr}\gamma_t = 0$ and hence the trace is preserved by evolution (12) (as long as it is well-posed, of course). For Eq. (11), we obtain

$$d \text{tr}\gamma_t = \text{tr}(\gamma_t (L^* + L))(1 - \text{tr}\gamma_t) dB_t.$$

(13)

Hence, evolution (11) does not preserve traces in general. But $\text{tr}\gamma_t = 1$ is a solution of (13). Therefore, if the initial condition has $\text{tr}\gamma_0 = 1$ and Eq. (13) is well posed, then the solutions of (11) do preserve the trace.

Similar remarks on the preservation of norms of Eq. (6) are worth mentioning. Calculating $d(\phi, \phi)$ from this equation, we see first that the terms with $H$ and $\text{Im} L$ disappear and then, by Ito’s rule, all terms with the differential $dt$ vanish. Therefore,

$$d(\phi_t, \phi_t) = ((L - \langle \text{Re} L \rangle \phi), \phi_t) dB_t + (\phi_t, (L - \langle \text{Re} L \rangle \phi_t) \phi_t) dB_t =$$

$$= ((L + L^* - 2\langle \text{Re} L \rangle \phi), \phi_t) dB_t =$$

$$= 2((\langle \text{Re} L \rangle \phi_t), \phi_t) dB_t = 0,$$

(14)

because $\langle \text{Re} L \rangle \phi_t = (\phi_t, (\text{Re} L) \phi_t)/(\phi_t, \phi_t)$. Hence, solutions of (6) preserve the norm almost surely.

If we used the non-normalized expression $(\phi, (\text{Re} L) \phi)$ instead of $\langle \text{Re} L \rangle \phi$ in Eq. (6), as some authors do, we would obtain

$$d(\phi_t, \phi_t) = (\phi_t, (\text{Re} L) \phi_t)(1 - (\phi_t, \phi_t)) dB_t,$$

whence for the initial condition with $\|\phi_0\| = 1$ we would conclude that $\|\phi_t\| = 1$ by uniqueness, as in the density matrix case.

Apart from the diffusive-type continuous observation, one can organize continuous observation of counting type and corresponding counting measurements. This type of measurement is described by a different kind of Belavkin quantum filtering equations, namely, by the equation

$$d\gamma_t = -i[H, \gamma_t] dt + \sum_j \left( -\frac{1}{2} \{L_j^* L_j, \gamma_t\} + \text{tr}(L_j \gamma_t L_j^*) \gamma_t \right) dt +$$

$$+ \sum_j \left( \frac{L_j \gamma_t L_j^*}{\text{tr}(L_j \gamma_t L_j)} - \gamma_t \right) dN^j_t,$$

(15)

where $H$ is again the Hamiltonian of the free (not observed) motion of the quantum system, the operators $\{L_j\}$ define the coupling of the system to the measurement devices, the counting (observed) processes $N^j_t$ are independent and have the intensities $\text{tr}(L_j^* L_j \gamma_t)$, and therefore the compensated processes

$$M^j_t = N^j_t - \int_0^t \text{tr}(L_j^* L_j \gamma_s) ds.$$
are martingales. In terms of the compensated processes $M^f_j$, Eq. (15) becomes

$$d\gamma_t = -i[H, \gamma_t] dt + \sum_j \left( L_j \gamma_t L_j^* - \frac{1}{2} \{ L_j^* L_j, \gamma_t \} \right) dt + \sum_j \left( \frac{L_j \gamma_t L_j^*}{\text{tr}(L_j \gamma_t L_j)} - \gamma_t \right) dM^f_j. \quad (16)$$

As in the case of diffusive measurements, this dynamics preserves the set of pure states. Namely, if $\phi$ satisfies the equation

$$d\phi_t = - \left( iH + \frac{1}{2} \sum_j (\{ L_j^* - 1 \} L_j - (L_j - \|L_j\phi_t\|^2)) \right) \phi_t dt + \sum_j \left( \frac{L_j \phi_t}{\|L_j\phi_t\|} - \phi_t \right) dM^f_j, \quad (17)$$

then $\gamma_t = \phi_t \otimes \bar{\phi}_t$ satisfies Eq. (16).

**Remark 1.** Equation (15) is slightly nonstandard in that the driving noises $N^j$ depend on the solutions. However, there is a natural way to rewrite it in terms of independent driving noises. Namely, with standard independent Poisson random measure processes $N^j(dx \, dt)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ (with the Lebesgue measure as intensity), we can rewrite Eq. (15) in the equivalent form

$$d\gamma_t = - i[H, \gamma_t] dt + \sum_j \left( - \frac{1}{2} \{ L_j^* L_j, \gamma_t \} + \text{tr}(L_j \gamma_t L_j^*) \gamma_t \right) dt + \sum_j \left( \frac{L_j \gamma_t L_j^*}{\text{tr}(L_j \gamma_t L_j)} - \gamma_t \right) 1(\text{tr}(L_j \gamma_t L_j) \leq x) N^j(dx \, dt)$$

(see the details of this construction in [28]). Alternatively, one can make sense of (15) in terms of the general theory of weak SDEs from [29]. Anyway, this problem is not relevant to our paper, where we deal only with the simplest case of counting measurements where operators $L_j$ are unitary. In this case, $\text{tr}(L_j \gamma_t L_j^*) = \text{tr}(\gamma_t) = 1$ and $N_j$ are just the standard Poisson processes.

The theory of quantum filtering reduces the analysis of quantum dynamic control and games to the controlled version of evolutions (11). The simplest situation concerns the case where the homodyne device is fixed, i.e., the operators $L_j$ are fixed, and the players can control the Hamiltonian $H$, say, by applying appropriate electric or magnetic fields to the atom. Thus Eqs. (11) or (15) become modified by allowing $H$ to depend on one or several control parameters.

### 3. The limiting equation: diffusive measurement

Let $X$ be a Borel space with a fixed Borel measure denoted by $dx$. For a linear operator $O$ in $L^2(X)$, we let $O_j$ denote the operator in $L^2(X^N)$ that acts on functions $f(x_1, \ldots, x_N)$ as $O$ acting on the variable $x_j$.

Let $H$ be a self-adjoint operator in $L^2(X)$ and $A$ a self-adjoint integral operator in $L^2(X^2)$ with the kernel $A(x, y; x', y')$ that acts on functions of two variables as

$$A \psi(x, y) = \int_{X^2} A(x, y; x', y') \psi(x', y') \, dx' \, dy'.$$

It is assumed that $A$ is symmetric in the sense that it takes symmetric functions $\psi(x, y)$ (symmetric with respect to permutation of $x$ and $y$) to symmetric functions. In terms of the kernel, this means the equation

$$A(x, y; x', y') = A(y, x; y', x').$$
We consider the quantum evolution of \( N \) particles driven by the standard interaction Hamiltonian

\[
H(N) f(x_1, \ldots, x_N) = \sum_{j=1}^{N} H_j f(x_1, \ldots, x_N) + \frac{1}{N} \sum_{i<j \leq N} A_{ij} f(x_1, \ldots, x_N),
\]

(18)

with \( A_{ij} \) denoting the action of \( A \) on the variables \( x_i \) and \( x_j \).

**Remark 2.** Alternatively, we can consider the operator of multiplication by a symmetric function \( V(x_i, x_j) \) rather than the integral operator \( A \) in (18).

We assume further that this quantum system is observed via coupling to a collection of identical one-particle operators \( L \). That is, we consider filtering equation (10) of the type

\[
d\Psi_{N,t} = \sum_j \left[ i (\Re L_j) \phi \Im L_j - \frac{1}{2} (L_j - \langle \Re L_j \rangle_{\Psi_{N,t}})^* (L_j - \langle \Re L_j \rangle_{\Psi_{N,t}}) \right] \Psi_{N,t} \, dt - i H(N) \Psi_{N,t} \, dt + \sum_{j=1}^{N} (L_j - \langle \Re L_j \rangle_{\Psi_{N,t}}) \Psi_{N,t} \, dB_j^t.
\]

(19)

We show that as \( N \to \infty \), the solution of this equation and the corresponding density matrices \( \Gamma_{N,t} = \Psi_{N,t} \otimes \overline{\Psi_{N,t}} \) are close to the product of the solutions of certain one-particle nonlinear stochastic equations, or more precisely, that the partial traces \( \Gamma^{(j)}_{N,t} \) of \( \Gamma_{N,t} \) with respect to all variables except the \( j \)th are close to the density matrices \( \gamma_{j,t} \) of the individual equations.

However, we need a more general setting for the study of control. We assume that the individual Hamiltonian \( H \) has a control component, i.e., it can be written as \( H + u \tilde{H} \) with two self-adjoint operators \( H \) and \( \tilde{H} \), and \( u \) is a real control parameter taken from a bounded interval \([-U, U]\). We suppose that for the idealized limiting evolution, \( u \) is chosen as a certain function of an observed density matrix \( \gamma_{j,t} \), i.e., \( u = u(t, \gamma_{j,t}) \). In the original \( N \)-particle evolution, \( u \) is then chosen based on the approximation \( \Gamma^{(j)}_{N,t} \) to \( \gamma_{j,t} \), i.e., as \( u = u(t, \Gamma^{(j)}_{N,t}) \). Thus the controlled and observed \( N \)-particle evolution is given by Eq. (19) with the nonlinear controlled Hamiltonian \( H_u(N) \) instead of \( H \):

\[
H_u(N) f(x_1, \ldots, x_N) = \sum_{j=1}^{N} (H_j + u(t, \Gamma^{(j)}_{N,t}) \tilde{H}_j) f(x_1, \ldots, x_N) + \frac{1}{N} \sum_{i<j \leq N} A_{ij} f(x_1, \ldots, x_N),
\]

(20)

where \( u(t, \gamma) \) is some continuous function.

The corresponding density matrix \( \Gamma_{N,t} = \Psi_{N,t} \otimes \overline{\Psi_{N,t}} \) satisfies an equation of type (11):

\[
d\Gamma_{N,t} = -i[H_u(N), \Gamma_{N,t}] \, dt - i \sum_{i<j \leq N} [A_{ij}, \Gamma_{N,t}] \, dt + \sum_j \left( L_j \Gamma_{N,t} L_j^* - \frac{1}{2} L_j^* L_j \Gamma_{N,t} - \frac{1}{2} \Gamma_{N,t} L_j^* L_j \right) \, dt + \sum_j \left( \Gamma_{N,t} L_j^* + L_j \Gamma_{N,t} - \Gamma_{N,t} \, \text{tr}(\Gamma_{N,t}(L_j^* + L_j)) \right) \, dB_j^t.
\]

(21)
Our main objective in this paper is to prove that as \( N \to \infty \), the limiting evolution of each particle is described by the nonlinear stochastic equation

\[
d\psi_{j,t}(x) = -i[H + u(t, \gamma_{j,t})\hat{H} + A^{\phi}_t - \langle \text{Re}\, L \rangle \phi \text{Im}\, L]\psi_{j,t}(x)\, dt - \frac{1}{2}(L - \langle \text{Re}\, L \rangle \psi_{j,t})^*(L - \langle \text{Re}\, L \rangle \psi_{j,t})\psi_{j,t}(x)\, dt + (L - \langle \text{Re}\, L \rangle \psi_{j,t})\psi_{j,t} dB_t^j,
\]

where \( A^{\phi}_t \) is the integral operator in \( L^2(X) \) with the integral kernel

\[
A^{\phi}(x; y) = \int_{X^2} A(x, y; x', y')\eta_t(y, y')\, dy\, dy'
\]

and

\[
\eta_t(y, z) = \mathbb{E}(\psi_{j,t}(y)\bar{\psi}_{j,t}(z)).
\]

The equation for the corresponding density matrix \( \gamma_{j,t} = \psi_{j,t} \otimes \bar{\psi}_{j,t} \) is

\[
d\gamma_{j,t} = -i[H + u(t, \gamma_{j,t})\hat{H}, \gamma_{j,t}]\, dt - i[A^{\phi}_t, \gamma_{j,t}]\, dt + (L\gamma_{j,t}L^* - \frac{1}{2}L^*L\gamma_{j,t} - \frac{1}{2}\gamma_{j,t}L^*L)\, dt +
\]

\[
+ (\gamma_{j,t}L^* + L\gamma_{j,t} - \gamma_{j,t}\text{tr}(\gamma_{j,t}(L + L^*)))\, dB_t^j,
\]

\[
\eta_t(y, z) = \mathbb{E}(\psi_{j,t}(y)\bar{\psi}_{j,t}(z)) = \mathbb{E}\gamma_{j,t}(y, z).
\]

The result extends directly to the case where the measurement related to each particle is multidimensional, i.e., the operator \( L \) is vector-valued, \( L = (L^1, \ldots, L^k) \), in which case each noise \( dB_t^j \) is also \( k \)-dimensional, and hence the last term in (22) should be understood as the inner product,

\[
(L - \langle L \rangle \psi_{j,t})\psi_{j,t} dB_t^j = \sum_{l=1}^{k} (L_l^i - \langle L_l^i \rangle \psi_{j,t}) dB_{j,t}^i,
\]

with all other terms containing \( L \) understood in the same way.

Our analysis is carried out via the extension of the method suggested by Pickl in a deterministic case (see [30], [31]) to the present stochastic framework. In Pickl’s approach, the main measures of the deviation of the solutions \( \Psi_{N,t} \) of \( N \)-particle systems from the product of the solutions \( \psi_t \) of the Hartree equations are the following positive numbers from the interval \([0, 1]\):

\[
\alpha_N(t) = 1 - \langle \psi_t, \Gamma_{N,t}\psi_t \rangle.
\]

In the present stochastic case, these quantities depend not only on the number of particles in the product but also on the concrete choice of these particles. The proper stochastic analog of \( \alpha_N(t) \) is the collection of random variables

\[
\alpha_{N,j}(t) = 1 - \langle \psi_{j,t}, \Gamma_{N,t}\psi_{j,t} \rangle = 1 - \text{tr}(\gamma_{j,t}\Gamma_{N,t}^{(j)}) = 1 - \text{tr}(\gamma_{j,t}\Gamma_{N,t}^{(j)}),
\]

where the last equality holds by the definition of the partial trace. Here, \( \gamma_{j,t} \) is identified with the operator in \( L^2(X^N) \) acting on the \( j \)th variable and \( \Gamma_{N,t}^{(j)} \) denotes the partial trace of \( \Gamma_{N,t} \) with respect to all variables except the \( j \)th.
The key property of Eqs. (21) and (23) is that their solutions preserve the set of operators with unit trace (see the discussion around Eq. (13) above). Hence, Eq. (24) can be rewritten as

\[ \alpha_{N,j}(t) = \text{tr}((1 - \gamma_{j,t})\Gamma_{N,t}) = \text{tr}((1 - \gamma_{j,t})\Gamma_{N,t}^{(j)}). \]  

Due to the i.i.d. property of the solutions of (22), the expectations \( \mathbf{E}\alpha_{N}(t) = \mathbf{E}\alpha_{N,j}(t) \) are well defined (independent of a particular choice of particles).

The expressions \( \alpha_{N,j} \) can be linked with the traces by the following inequalities, due to Knowles and Pickl:

\[ \alpha_{N,j}(t) \leq \text{tr}|\Gamma_{N,t}^{(j)} - \gamma_{j,t}| \leq 2\sqrt{2\alpha_{N,j}(t)}, \]

see Lemma 2.3 in [31].

**Theorem 1.** Let \( H, \hat{H}, \) and \( L \) be operators in \( L^2(X) \), with \( H \) and \( \hat{H} \) self-adjoint, and \( \hat{H} \) and \( L \) bounded. Let \( A \) be a symmetric self-adjoint integral operator \( A \) in \( L^2(X^2) \) with a Hilbert–Schmidt kernel, i.e., a kernel \( A(x, y; x', y') \) such that

\[ \|A\|_{HS}^2 = \int_{X^4} |A(x, y; x', y')|^2 dx\,dy\,dx'\,dy' < \infty, \]

\[ A(x, y; x', y') = A(y, x; y', x'), \quad A(x, y; x', y') = A(x', y'; x, y). \]

Let the function \( u(t, \gamma) \) with values in a bounded interval \([-U, U]\) be Lipschitz in the sense that

\[ |u(t, \gamma) - u(t, \tilde{\gamma})| \leq \kappa |\gamma - \tilde{\gamma}|. \]

Let \( \psi_{j,t} \) be solutions of Eqs. (22) with i.i.d. initial conditions \( \psi_{j,0}, \|\psi_{j,0}\| = 1 \). Let \( \Psi_{N,t} \) be a solution of the \( N \)-particle equation (19) with \( H(N) \) of type (20), with some initial condition \( \Psi_{N,0}, \|\Psi_{N,0}\|_2 = 1 \) such that

\[ \alpha_N(0) = \alpha_{N,j}(0) = 1 - \mathbf{E}\text{tr}(\gamma_{j,0}\Gamma_{N,0}) = 1 - \mathbf{E}\text{tr}(\gamma_{j,0}\Gamma_{N,0}^{(j)}) \]

are equal for all \( j \). The main example of such an initial condition is of course the product

\[ \Psi_{N,0} = \prod \psi_{j,0}(x_j), \]

where \( \alpha_{N,j}(0) = 0 \) for all \( j \). Then

\[ \mathbf{E}\alpha_{N}(t) \leq e^{(12\|A\|_{HS} + 6\kappa\|\hat{H}\| + 28\|L\|^2)t} \mathbf{E}\alpha_{N}(0) + \]

\[ + (e^{(12\|A\|_{HS} + 6\kappa\|\hat{H}\| + 28\|L\|^2)t} - 1) \frac{1}{\sqrt{N}}. \]

By (25) it follows that if \( \alpha_{N}(0) \to 0 \) as \( N \to \infty \) (for instance, if \( \alpha_{N}(0) = 0 \)), then \( \mathbf{E}\text{tr}|\Gamma_{N,t}^{(j)} - \gamma_{j,t}| \to 0 \) as \( N \to \infty \). Of course, everything remains unchanged for a vector-valued \( L \).

**Proof.** Using definition (24) and Ito’s product rule, we derive that

\[ d\alpha_{N,j}(t) = -\text{tr}(d\Gamma_{N,t}\gamma_{j,t}) - \text{tr}(\Gamma_{N,t}d\gamma_{j,t}) - \text{tr}(d\Gamma_{N,t}d\gamma_{j,t}) = \]

\[ = (C_j + D_j)\,dt + \sum_j F_j\,dB^j_t, \]

where we let \( C_j \) denote the part of the differentials that is independent of \( L_j \) and \( D_j \) denote the part that depends on \( L_j \). These parts clearly separate, and hence \( D_j \) does not contain \( H, \hat{H}, \) and \( A_{jk} \).
We now estimate $\mathbb{E}|\tilde{\alpha}_{N,t}|$, and in doing so we are interested only in the estimates for $|C_j|$ and $|D_j|$. The part $C_j$ was calculated and estimated in [4] (proofs of Theorems 3.1 and 3.2 there), yielding the estimate

$$
|C_j| \leq 12\|A\|_{\text{HS}} \left( 2\mathbb{E}_{\alpha_N}(t) + \frac{1}{\sqrt{N}} \right) + 4\sqrt{2\xi}\|\tilde{H}\|\mathbb{E}_{\alpha_N}(t).
$$

(32)

As was shown in [4], $D_j$ vanishes in the case $L_j = -L_j^*$. In the general case, we derive from (31), (23), and (21) that

$$
D_j = \text{tr} \left[ \sum_k \left( \frac{1}{2} L_k^* L_k \Gamma_N + \frac{1}{2} \Gamma_N L_k^* L_k - L_k \gamma_N L_k^* \right) \gamma_j + \right.
$$

$$
+ \Gamma_N \left( \frac{1}{2} L_j^* L_j \gamma_j + \frac{1}{2} \gamma_j L_j L_j^* - L_j \gamma_j L_j^* \right) -
$$

$$
- \left( \Gamma_N L_j^* + L_j \Gamma_N - \Gamma_N \text{tr}(\Gamma_N(L_j^* + L_j))(\gamma_j L_j^* + L_j \gamma_j - \gamma_j \text{tr}(\gamma_j(L_j^* + L_j))) \right),
$$

where we omitted the index $t$ for brevity. The terms with $k \neq j$ vanish because

$$
\text{tr}(L_k^* L_k \Gamma_N \gamma_j) = \text{tr}(\Gamma_N L_k^* L_k \gamma_j) = \text{tr}(L_k \Gamma_N L_k^* \gamma_j) = \text{tr}(\Gamma_N \gamma_j L_k^* L_k).
$$

We are left with

$$
D_j = \text{tr} \left[ \frac{1}{2} L_j^* L_j \Gamma_N \gamma_j + \frac{1}{2} \Gamma_N L_j^* L_j \gamma_j - L_j \Gamma_N L_j^* \gamma_j + \right.
$$

$$
+ \frac{1}{2} \Gamma_N L_j^* L_j \gamma_j + \frac{1}{2} \Gamma_N L_j \gamma_j L_j^* - \Gamma_N \Gamma_j \gamma_j L_j^* \gamma_j -
$$

$$
- \Gamma_N L_j^* \gamma_j L_j^* - \Gamma_N L_j \gamma_j L_j - L_j \Gamma_N \gamma_j L_j - L_j \Gamma_N L_j \gamma_j +
$$

$$
+ \Gamma_N (L_j^* + L_j \Gamma_N) \gamma_j \text{tr}(\gamma_j(L_j^* + L_j)) +
$$

$$
+ \Gamma_N \text{tr}(\Gamma_N(L_j^* + L_j))(\gamma_j L_j^* + L_j \gamma_j) -
$$

$$
- \Gamma_N \gamma_j \text{tr}(\Gamma_N(L_j^* + L_j)) \text{tr}(\gamma_j(L_j^* + L_j)) \right].
$$

More cancelations yield the following rather awkwardly looking expression:

$$
D_j = -\text{tr}(\gamma_j L_j \Gamma_N L_j^* + \gamma_j L_j^* \Gamma_N L_j + \gamma_j L_j \Gamma_N L_j^* + \gamma_j L_j \Gamma_N L_j) +
$$

$$
+ \text{tr}(\gamma_j \Gamma_N L_j^* + \gamma_j L_j \Gamma_N) \text{tr}(\gamma_j(L_j^* + L_j)) +
$$

$$
+ \text{tr}(\gamma_j \Gamma_N L_j + \gamma_j L_j \Gamma_N) \text{tr}(\Gamma_N(L_j^* + L_j)) -
$$

$$
- \text{tr}(\Gamma_N \gamma_j) \text{tr}(\Gamma_N(L_j^* + L_j)) \text{tr}(\gamma_j(L_j^* + L_j)).
$$

However, by Lemma 1 in Appendix A,

$$
|D_j| \leq 28\|L\|^2 \text{tr}((1 - \gamma_j)\Gamma_N).
$$

Combining this estimate with (32), we have

$$
|\mathbb{E}_{\alpha_N,t}| \leq 12\|A\|_{\text{HS}} \left( 2\mathbb{E}_{\alpha_N}(t) + \frac{1}{\sqrt{N}} \right) + 4\sqrt{2\xi}\|\tilde{H}\| + 28\|L\|^2 \mathbb{E}_{\alpha_N}(t).
$$

Applying Gronwall’s lemma yields (30).
4. A well-posedness result

This section is devoted to the well-posedness of Eq. (2):

\[ d\psi_t(x) = -i[H\psi_t(x) + u(t, \psi_t)\hat{H} + A^E(\hat{\psi} \otimes \psi_t)](x) - \langle \text{Re } L \rangle \psi_t(x) \, dt + \]
\[ + \frac{1}{2}(L - \langle \text{Re } L \rangle \psi_t)^*(L - \langle \text{Re } L \rangle \psi_t)\psi_t(x) \, dt + (L - \langle \text{Re } L \rangle \psi_t)\psi_t \, dB_t, \]  
\[ \text{Eq. (33)} \]

In this analysis, we use the results in Appendix B and the notation for spaces and processes introduced there. Equation (33) is of type (47) (see Appendix B), with the Hilbert space \( \mathcal{H} = L^2(X) \) and with \( A = -iH \).

As an auxiliary tool, we study the more standard equations

\[ d\psi_t(x) = -i[H\psi_t(x) + u(t, \psi_t)\hat{H} + A^\xi \psi_t(x) - \langle \text{Re } L \rangle \psi_t(x) \, dt + \]
\[ + \frac{1}{2}(L - \langle \text{Re } L \rangle \psi_t)^*(L - \langle \text{Re } L \rangle \psi_t)\psi_t(x) \, dt + (L - \langle \text{Re } L \rangle \psi_t)\psi_t \, dB_t, \]  
\[ \text{Eq. (34)} \]

with \( \xi(y, z) \) a given continuous function in \( \mathcal{H} \otimes \mathcal{H} \) and

\[ d\psi_t(x) = -i[H\psi_t(x) + u_t\hat{H} + A^\xi \psi_t(x) - \langle \text{Re } L \rangle \psi_t(x) \, dt + \]
\[ + \frac{1}{2}(L - \langle \text{Re } L \rangle \psi_t)^*(L - \langle \text{Re } L \rangle \psi_t)\psi_t(x) \, dt + (L - \langle \text{Re } L \rangle \psi_t)\psi_t \, dB_t, \]  
\[ \text{Eq. (35)} \]

with \( u_t \) a given continuous function. These equations are of type (48) of Appendix B.

The results in Appendix B are not directly applicable to Eqs. (33) and (34) because the Lipschitz continuity is only local for terms containing \( u \) and \( \xi \). However, as was proved in [27], the terms containing \( L \) are uniformly Lipschitz for a bounded \( L \).

**Remark 3.** The proof of this fact was obtained in [27] by elementary direct estimates, which were rather lengthy and not very intuitive, however. To see this fact more directly, we can note that the derivatives of all coefficients involving \( L \) in (34) are bounded. For instance, the last coefficient in (34) has the form

\[ f(\psi) = \frac{(\psi, M\psi)}{(\psi, \psi)} \psi, \]

with a self-adjoint \( M \). Taking a simpler situation with a real Hilbert space, we find

\[ \frac{\partial f}{\partial \psi} = \frac{(\psi, M\psi)}{(\psi, \psi)} + 2 \frac{M\psi \otimes \psi}{(\psi, \psi)} - 2 \frac{(M\psi, \psi)\psi \otimes \psi}{(\psi, \psi)^2}, \]

\[ \left\| \frac{\partial f}{\partial \psi} \right\| \leq 5\|M\|. \]

**Theorem 2.** Let \( H, \hat{H}, \) and \( L \) be self-adjoint operators in \( L^2(X) \), with \( \hat{H} \) and \( L \) bounded, and \( A \) be an integral operator in \( L^2(X^2) \) with a kernel \( A(x, y; x', y') \) satisfying (27) and (28). Let the function \( u(t, \gamma) \) with values in a bounded interval \([-U, U]\) be Lipschitz either as a function of \( \psi \) or as a function of \( \psi \otimes \hat{\psi} \), i.e., either

\[ |u(t, \psi) - u(t, \hat{\psi})| \leq \kappa \|\psi - \hat{\psi}\|, \]  
\[ \text{Eq. (36)} \]

or

\[ |u(t, \psi) - u(t, \hat{\psi})| \leq \kappa(\|\psi\| + \|\hat{\psi}\|)\|\psi - \hat{\psi}\|. \]  
\[ \text{Eq. (37)} \]

Then
1. equation (33) is globally well-posed in the sense of mild solutions;

2. for any continuous curve $\xi_t$ in $(L^2(X))^\otimes 2$, Eq. (34) is globally well-posed in the sense of mild solutions;

3. for any continuous curve $\xi_t$ in $(L^2(X))^\otimes 2$ and a continuous function $u_t$, Eq. (35) is globally well-posed in the sense of mild solutions;

4. solutions $\psi_t$ of Eq. (34) depend continuously on the initial conditions in the following precise sense: for two solutions with the initial conditions $Y_1$ and $Y_2$, we have the estimate

$$
\|\psi^1 - \psi^2\|_{ad,T} \leq 2e^{aT+bT^2} \|Y_1 - Y_2\| \tag{38}
$$

(see (50) for the definition of the norm), where $a$ and $b$ are constants depending on $\kappa$, $U$, $\|\hat{H}\|$, $\|L\|$, and $\|A\|_{HS}$;

5. for all these equations to hold in the strong sense, it is sufficient to assume that there exists an invariant core $D$ of the group $e^{tH}$ and a norm $\| \cdot \|_D$ on it such that $\| \cdot \|_D \geq \| \cdot \|$, $D$ is a Banach space under this norm, and the operators $\hat{H}$, $L$, and $A^5$ are bounded operators $L^2(X) \to D$, uniformly with respect to bounded sets of curves $\xi_t$ for the last operator, and the initial condition is taken from $D$.

**Proof.** By the remark on the Lipschitz continuity of the coefficients involving $L$, Eq. (35) satisfies the conditions of Proposition 1 (see Appendix B), and hence statement 3 is a consequence of this proposition. Moreover, by Proposition 3, statement 5 follows from statements 1–3. To show statements 1 and 3, we note that solutions of (35) preserve norms almost surely. This is a standard fact shown by calculations (14).

**Remark 4.** Calculations (14) were performed assuming the strong (not just mild) form of the equation, i.e., strictly speaking, only under the assumptions of statement 4. But by approximating the operators $\hat{H}$, $L$, and $A^5$ by operators satisfying the assumptions of statement 4, we obtain this result in the general case.

In particular, in seeking solutions of Eqs. (33) and (34), we can reduce our attention to curves $\xi_t$ such that $\|\xi_t\| \leq 1$. Therefore, comparing two mild solutions $\psi^1_t$ and $\psi^2_t$ of Eq. (35) with the same initial condition $\psi_0$, $\|\psi_0\| = 1$, but with different pairs $(\xi^1_t, u^1_t)$, $(\xi^2_t, u^2_t)$ such that $\|\xi^j_t\| \leq 1$, we find, similarly to calculations performed in the proof of Proposition 1, that

$$
E\|\psi^1_t - \psi^2_t\|^2 \leq (a + bt)E \int_0^t \|\psi^1_s - \psi^2_s\|^2 ds + \|\hat{H}\| \left( \int_0^t |u^1_s - u^2_s|^2 ds \right)^2 + \|A\|_{HS}^2 \left( \int_0^t \|\xi^1_s - \xi^2_s\| ds \right)^2 \leq (a + bt)E \int_0^t \|\psi^1_s - \psi^2_s\|^2 ds + \|\hat{H}\|^2 t \|u^1_t - u^2_t\|^2 + \|A\|_{HS}^2 \|\xi^1_t - \xi^2_t\|^2,
$$

where $a$ and $b$ are constants depending on $U$, $\|\hat{H}\|$, $\|L\|$, and $\|A\|_{HS}$, and where we let $\| \cdot \|_t$ denote the maximum of the corresponding norms of the functions at all times $s \in [0, t]$.

Consequently, by Gronwall’s lemma and because $\psi^1_0 = \psi^2_0$,

$$
E\|\psi^1_t - \psi^2_t\|^2 \leq e^{aT+bT^2} t(\|\hat{H}\|^2 \|u^1_t - u^2_t\|^2 + \|A\|_{HS}^2 \|\xi^1_t - \xi^2_t\|^2),
$$

and hence

$$
\|\psi^1_t - \psi^2_t\|_{ad,T} \leq e^{(a+bT^2)/2} \sqrt{t}(\|\hat{H}\| \|u^1_t - u^2_t\|_t + \|A\|_{HS} \|\xi^1_t - \xi^2_t\|_t).
$$
Thus, by choosing \( t \) small enough, we can make the Lipschitz constant of the map \((\xi, u) \rightarrow X\) arbitrarily small. Because \( u(t, \gamma) \) is Lipschitz and the map \( X \rightarrow E(\bar{X} \otimes X) \) is Lipschitz (uniformly for \( X \) with \( \|X\| = 1 \)), we find that the composition map

\[ (\xi, u) \rightarrow \psi, \rightarrow \mathbf{E}(\bar{\psi} \otimes \psi) \]

is a contraction for small times. Hence, it has a unique fixed point. By iteration, we build a unique solution of Eq. (33), thus proving 1.

Statement 2 is proved analogously.

It remains to show 4. For two solutions \( \psi^1_t \) and \( \psi^2_t \) of Eq. (34) with some unit initial condition \( Y_1, Y_2 \) and some \( \xi_t \) with \( \|\xi_t\| \leq 1 \), we find, similarly to the foregoing, that

\[ \mathbf{E}\|\psi^1_t - \psi^2_t\|^2 \leq 2\|Y_1 - Y_2\|^2 + (a + bt)\mathbf{E}\int_0^t \|\psi^1_s - \psi^2_s\|^2 \, ds. \]

By Gronwall’s lemma it hence follows that

\[ \mathbf{E}\|\psi^1_t - \psi^2_t\|^2 \leq 2e^{at+bt^2}\|Y_1 - Y_2\|^2, \]

implying (38). ■

5. The limiting equation: counting measurement

The analog of Eq. (21) describing the observation of a collection of identical quantum particles, arising from the general quantum filtering equation (16), or observation of counting type is the equation

\begin{align*}
\frac{d\Gamma_{N,t}}{dt} &= \left( -i \sum_j [H_j, \Gamma_{N,t}] - \frac{i}{N} \sum_{l<j \leq N} [A_{lj}, \Gamma_{N,t}] + 
\sum_k \left( L_k \Gamma_{N,t} L_k^* - \frac{1}{2} L_k^* L_k \Gamma_{N,t} - \frac{1}{2} \Gamma_{N,t} L_k^* L_k \right) \right) dt + 
\sum_k \left( \frac{L_k \Gamma_{N,t} L_k^*}{\text{tr}(\Gamma_{N,t} L_k^* L_k)} - \Gamma_{N,t} \right) dM^k_t,
\end{align*}

(39)

where \( M^k_t \) are martingales such that \( dM^k_t = dN^k_t + \text{tr}(\Gamma_{N,t} L_k^* L_k) \) and \( N^k_t \) are counting processes with the intensities \( \text{tr}(\Gamma_{N,t} L_k^* L_k) \). In terms of pure states, quantum filtering equation (17) takes the form

\begin{align*}
\frac{d\Psi_{N,t}}{dt} &= -iH(N)\Psi_{N,t} - \frac{1}{2} \sum_j ((L_j^* - 1)L_j - (L_j - \|L_j\Psi_{N,t}\|^2))\Psi_{N,t} \, dt + 
\sum_j \left( \frac{L_j\Psi_{N,t}}{\|L_j\Psi_{N,t}\|} - \Psi_{N,t} \right) dM^j_t.
\end{align*}

(40)

The corresponding analog of limiting equation (23) is

\begin{align*}
\frac{d\gamma_{j,t}}{dt} &= \left( -i[H_j, \gamma_{j,t}] - i[A_{\bar{\eta}_j}, \gamma_{j,t}] + \left( L_j \gamma_{j,t} L_j^* - \frac{1}{2} L_j^* L_j \gamma_{j,t} - \frac{1}{2} \gamma_{j,t} L_j^* L_j \right) \right) dt + 
\left( \frac{L_j \gamma_{j,t} L_j^*}{\text{tr}(\gamma_{j,t} L_j^* L_j)} - \gamma_{j,t} \right) dM^j_t,
\end{align*}

(41)
where \( M_t^j \) are martingales such that \( dM_t^j = dN_t^k + \text{tr}(\gamma_j, t L_j^* L_j) \) and \( N_t^k \) are counting processes with the intensities \( \text{tr}(\gamma_j, t L_j^* L_j) \). The corresponding equation for pure states is

\[
d\psi_{j,t} = -i(H_j + A^0)\psi_{j,t} \, dt - \frac{1}{2}(L_j^* - 1)L_j - (L_j - ||L_j\psi_{j,t}||^2)\psi_{j,t} \, dt + \left( \frac{L_j\psi_{j,t}}{||L_j\psi_{j,t}||} - \psi_{j,t} \right) dM_t^j.
\] (42)

We can clearly see the principle problem with this situation, which does not occur in the diffusive case. In that case, both the approximating and the limiting equations are written in terms of the same Brownian motions. In the present case, the driving noises are different (intensities of jumps depend on different objects). To avoid this problem, we deal only with a special conservative case where the operator \( L \) is unitary: \( L^* = L^{-1} \). Equation (39) then becomes linear, the intensities of all jump processes \( N_t^j \) become identical and equal to unity (and thus independent of a state of the process), and hence the noises \( M_t^j \) in (39) and (41) can be identified.

**Remark 5.** The natural idea for dealing with the general case would be by organizing certain coupling between the counting processes \( N_t^k \) in Eqs. (39) and (41). The natural coupling is the marching coupling where the intensity of the coupled part is given by the minimum of the intensities of individual noises. We did not manage to prove convergence under such a coupling. Our conjecture is that it does not hold. The reason is that convergence can be proved for the coupling that makes the intensity of the coupled processes equal to the maximum of the intensities of individual noises. Evidently, such coupling destroys the individual dynamics.

We therefore consider the dynamics of \( \alpha_{N,j} \),

\[
d\alpha_{N,j} = -\text{tr}(d\Gamma_{N,t} \gamma_j(t)) - \text{tr}(\Gamma_{N,t} \, d\gamma_j(t)) - \text{tr}(d\Gamma_{N,t} \, d\gamma_j(t)),
\]
assuming that \( L^{-1} = L^* \) and that the noises \( M_t^j \) in (39) and (41) are identified.

The part containing \( H \) and \( A \) is the same as for diffusion, and hence we are interested only in the part containing \( L_j \).

We recall the Ito multiplication rule for counting processes \( dN_t^j dN_t^j = dN_t^j \). Under the unitarity assumption for \( L \), it follows that

\[
dM_t^j dM_t^j = dN_t^j = dM_t^j + \text{tr}(\Gamma_{N} L_j^* L_j) \, dt = dM_t^j + dt.
\]

The part of the stochastic differential (at \( dM_t^j \)) in the expression for \( d\alpha_{N,j} \) is of no interest for us because we seek the expectation of \( d\alpha_{N,j} \), which is not affected by these martingale terms.

It turns out that under the unitarity assumption, the part at \( dt \) depending on \( L_j \) vanishes. In fact (omitting the index \( t \) for brevity), taking into account that \( \gamma_j \) and \( \Gamma_N \) have unit traces, the coefficient at \( dt \) depending on \( L_j \) is

\[
\text{tr}\left[ \frac{1}{2} \Gamma_N \gamma_j + \frac{1}{2} \Gamma_N L_j^* \gamma_j + \frac{1}{2} \Gamma_N L_j \gamma_j - \frac{1}{2} \Gamma_N \gamma_j \gamma_j - \frac{1}{2} \Gamma_N L_j^* L_j \right] - \text{tr}\left[ (L_j \gamma_j L_j^* - \gamma_j)(L_j \Gamma_N L_j^* - \Gamma_N) \right] =
\]

\[
= \text{tr}\left[ 2\gamma_j \Gamma_N - \gamma_j L_j \Gamma_N L_j^* - \gamma_j L_j^* \Gamma_N L_j - L_j \gamma_j L_j^* L_j \Gamma_N L_j^* + \gamma_j L_j \Gamma_N L_j^* + \gamma_j L_j^* \Gamma_N L_j - \gamma_j \Gamma_N \right] = 0.
\]

Thus, turning to the expectations of \( \alpha_{N,j} \), we have the same situation as in Theorem 1, but in the simpler version where \( L \) is absent in all estimates. Consequently, the following result holds.
Let \( H \) and \( \tilde{H} \) be self-adjoint operators in \( L^2(X) \), with \( \tilde{H} \) bounded, and \( L \) a unitary operator. Let \( A \) be a symmetric self-adjoint integral operator in \( L^2(X^2) \) with a Hilbert–Schmidt kernel \( A(x, y; x', y') \) satisfying (27) and (28). Let the function \( u(t, \gamma) \) with values in a bounded interval \([-U, U]\) be Lipschitz in the sense of \( \Gamma \).

By the positivity of \( \Gamma \), it follows that \( \tilde{E} \) is equal for all \( t \). Let \( \psi \) be solutions of Eqs. (42) with i.i.d. initial conditions \( \psi_{j,0}, \|\psi_{j,0}\| = 1 \). Let \( \Psi_{N,0} \) be a solution of the \( N \)-particle equation (40) with \( H(N) \) of type (20), with some initial condition \( \Psi_{N,0}, \|\Psi_{N,0}\|_2 = 1 \) such that

\[ \alpha_N(0) = \alpha_{N,j}(0) = 1 - E \text{tr}(\gamma_{j,0}^j) \Gamma_N(0) = 1 - E \text{tr}(\gamma_{j,0}^j) \Gamma_{N,j}(0) \]

are equal for all \( j \). Then

\[ E\alpha_N(t) \leq \exp\{(12\|A\|_\text{HS} + 6\|\tilde{H}\|)t\}E\alpha_N(0) + (\exp\{(12\|A\|_\text{HS} + 6\|\tilde{H}\|)t\} - 1) \frac{1}{\sqrt{N}} \]

\textbf{Appendix A: A technical estimate}

\textbf{Lemma 1.} Let \( \gamma \) be a one-dimensional projector in a Hilbert space, \( \Gamma \) be a density matrix (positive operator with unit trace) and \( L \) be a bounded operator in this Hilbert space. Then

\[ -4 \text{tr}(L\gamma L\Gamma) + 2 \text{tr}(\Gamma(L + \gamma L)) \text{tr}(\Gamma L) - 4 \text{tr}(\Gamma \gamma) \text{tr}(\Gamma L) \leq 20\|L\|^2 \text{tr}((1 - \gamma)\Gamma) \]

(43)

for a self-adjoint \( L \), and

\[ -\text{tr}(\gamma L\Gamma L^* + \gamma L^* \Gamma L + \gamma L^* \Gamma L^* + \gamma L \Gamma L) + \]

\[ + \text{tr}(\gamma \Gamma L^* + \gamma L \Gamma) \text{tr}(\gamma L^* + L) + \text{tr}(\gamma \Gamma L + \gamma L \Gamma) \text{tr}(\Gamma N(L^* + L)) - \]

\[ - \text{tr}(\Gamma \gamma) \text{tr}(\Gamma(L^* + L)) \text{tr}(\Gamma(L^* + L)) \leq 28\|L\|^2 \text{tr}((1 - \gamma)\Gamma) \]

(44)

for a general \( L \).

\textbf{Proof.} By the approximation argument, it suffices to prove the Lemma for a finite-dimensional Hilbert space \( \mathbb{C}^n \).

Let \( \alpha = \text{tr}((1 - \gamma)\Gamma) \). We choose an orthonormal basis such that \( \gamma \) is the projection on the first basis vector. By the positivity of \( \Gamma \), it follows that

\[ |\Gamma_{jk}| \leq \alpha \quad \text{for} \quad j, k \neq 1, \quad \text{and} \quad \max(|\Gamma_{j1}|, |\Gamma_{1j}|) \leq \sqrt{\alpha} \quad \text{for} \quad j \neq 1. \]

(45)

Let \( L \) be a self-adjoint matrix. Then the expression under the modulus sign in the left-hand side of (43) becomes

\[ -4(L\Gamma L)_{11} + 2((L\Gamma)_{11} + (\Gamma L)_{11})((\text{tr}(\Gamma L) + L_{11}) - 4\Gamma_{11}L_{11} \text{tr}(\Gamma L) = \]

\[ = -4 \sum_{j,k} L_{1j} \Gamma_{jk} L_{k1} + \]

\[ + 2 \left[ 2L_{11} \Gamma_{11} + \sum_{j \neq 1} (\Gamma_{1j} L_{j1} + L_{1j} \Gamma_{j1}) \right] (\text{tr}(\Gamma L) + L_{11}) - 4\Gamma_{11}L_{11} \text{tr}(\Gamma L) = \]

\[ = -4L_{11} \sum_{j \neq 1} (\Gamma_{1j} L_{j1} + L_{1j} \Gamma_{j1}) - 4 \sum_{j \neq 1, k \neq 1} L_{1j} \Gamma_{jk} L_{k1} + \]

\[ + 2 \sum_{j \neq 1} (\Gamma_{1j} L_{j1} + L_{1j} \Gamma_{j1}) (\text{tr}(\Gamma L) + L_{11}) = \]
Everything apart from the last term is already estimated by (43).

Moreover,

$$\left| \sum_{j \neq 1, k \neq 1} L_{kj} \Gamma_{jk} \right| = |\text{tr}[(1 - \gamma)L(1 - \gamma)\Gamma(1 - \gamma)]| \leq \|L\| |\text{tr}[(1 - \gamma)\Gamma]| \leq \|L\| \alpha. $$

and therefore

$$\left| \sum_{j \neq 1} (\Gamma_{1j} L_{jj}) \right|^2 \leq \sum_{j \neq 1} |\Gamma_{1j}|^2 \sum_{j \neq 1} |L_{jj}|^2 \leq \|L\|^2 \alpha,$$

$$\left| \sum_{j \neq 1} (\Gamma_{j1} L_{1j}) \right|^2 \leq \|L^T\|^2 \alpha = \|L\|^2 \alpha,$$

whence

$$\left| \sum_{j \neq 1} (\Gamma_{1j} L_{jj} \pm L_{1j} \Gamma_{j1}) \right| \leq 2\|L\| \sqrt{\alpha}. \tag{46}$$

Finally,

$$\left| \sum_{j \neq 1, k \neq 1} L_{1j} \Gamma_{jk} L_{k1} \right|^2 \leq \sum_{j, k} |L_{1j}|^2 |L_{k1}|^2 \sum_{j \neq 1, k \neq 1} |\Gamma_{jk}|^2 \leq \|L\|^4 \left( \sum_{j \neq 1} |\Gamma_{jj}| \right)^2 \leq \|L\|^4 \alpha^2,$$

where the estimate

$$|\Gamma_{jk}|^2 \leq \Gamma_{jj} \Gamma_{kk}$$

for all $j$ and $k$ was used (arising from the positivity of $\Gamma$).

Putting the estimates together, we obtain (43).

For a general $L$, we can write $L = L^* + L^a$, where $L^* = (L + L^*)/2$ is self-adjoint and $L^a = (L - L^*)/2$ is anti-Hermitian. Substituting this in the left-hand side of (44) leads to several cancelations, such that the expression under the modulus sign in the left-hand side becomes

$$-4 \text{tr}(L^a \gamma L^* \Gamma) + 2 \text{tr}(\Gamma(L^a \gamma + \gamma L^a)) \text{tr}(GL^a + \gamma L^*) - 4 \text{tr}(\Gamma \gamma) \text{tr}(GL^a) \text{tr}(\gamma L^*) +$$

$$2 \text{tr}(\gamma \Gamma, L^a) \text{tr}(\Gamma L_s) - \text{tr}(\gamma L^*)).$$

Everything apart from the last term is already estimated by (43).
By (46) (which is valid for arbitrary $L$),

$$|\text{tr}(\gamma[\Gamma, L^a])| = ||\Gamma, L^a||_11 = \left| \sum_{j \neq 1} (\Gamma_{ij} L_i^a - L_i^a \Gamma_{ij}) \right| \leq 2\|L\|\sqrt{\alpha}$$

and

$$|\text{tr}(\Gamma L_s) - \text{tr}(\gamma L^s)| = \left| (\Gamma_{11} - 1)L_{11} + \sum_{j \neq 1} (\Gamma_{ij} L_i^s - L_i^s \Gamma_{ij}) + \sum_{j, k \neq 1} (\Gamma_{jk} L_k^s) \right| \leq 4\|L\|\sqrt{\alpha},$$

which implies (44).

**Remark 6.** Our proof of Lemma 1 is based on some remarkable cancelation of terms in concrete calculations via coordinate representations. We do not see any intuitive reasons for its validity. Nor is it clear whether this can be extended to arbitrary density matrices $\gamma$, not just one-dimensional projectors.

**Appendix B: McKean–Vlasov diffusions in Hilbert spaces**

The McKean–Vlasov nonlinear diffusions are well-studied processes, due to a large variety of applications. However, there seem to be only very few publications pertaining to the infinite-dimensional case (see [32] and the references therein). Here, for completeness, we therefore provide some basic results for a class of McKean–Vlasov diffusions in Hilbert spaces, stressing explicit bounds and errors. Namely, we are interested in the Cauchy problems

$$dX_t = AX_t\, dt + b_t(X_t, E X_t^{\otimes K})\, dt + (\sigma(X_t), dB_t), \quad X_0 = Y,$$

in a complex Hilbert space $\mathcal{H}$ equipped with the scalar product $(\cdot, \cdot)$ and the corresponding norm $\| \cdot \|$, and (as an auxiliary tool) in more standard equations

$$dX_t = AX_t\, dt + b_t(X_t, \xi_t)\, dt + (\sigma(X_t), dB_t), \quad X_0 = Y.$$  

(47)

Here, $K = 1$ or $K = 2$ (which are the most important cases for applications), $B_t$ is the standard $n$-dimensional Wiener process defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\sigma(Y) = (\sigma_1(Y), \ldots, \sigma_n(Y))$ with each $\sigma_j$ a continuous map $\mathcal{H} \to \mathcal{H}$, $b$ a continuous map $\mathbb{R} \times \mathcal{H} \times \mathcal{H}^{\otimes K} \to \mathcal{H}$, $A$ a generator of a strongly continuous operator semigroup $e^{tA}$ of contractions in $\mathcal{H}$, and $\xi_t$ a given (deterministic) continuous curve in $\mathcal{H}^{\otimes K}$. With some abuse of notation we let $(\cdot, \cdot)$ and $\| \cdot \|$ also denote the scalar product and the norm in the tensor product Hilbert space $\mathcal{H}^{\otimes K}$. In (47), $E$ denotes the expectation with respect to the Wiener process. A solution process of (47) is called a nonlinear diffusion of the McKean–Vlasov type, because of the dependence of the coefficient on this expectation.

To obtain effective bounds for growth and continuous dependence on the parameters of Eqs. (48), we follow the strategy developed systematically in [33] for deterministic Banach space-valued equations. Namely, we use the generalized fixed-point principle of Weissinger type in the following form (see, e.g., Propositions 9.1 and 9.3 in [33] for simple proofs). If $\Phi$ is a map from a complete metric space $(M, \rho)$ to itself such that

$$\rho(\Phi^k(x), \Phi^k(y)) \leq a_k \rho(x, y),$$

with $a = 1 + \sum_j \alpha_j < \infty$, then $\Phi$ has a unique fixed point $x^*$, and $\rho(x, x^*) \leq a \rho(x, \Phi(x))$ for any $x$. Moreover, for any two maps $\Phi_1$ and $\Phi_2$ satisfying these conditions and such that $\rho(\Phi_1(x), \Phi_2(x)) \leq \epsilon$ for all $x$, the corresponding fixed points allow the estimate

$$\rho(x_1^*, x_2^*) \leq \epsilon a.$$
We start with (48) and work with the so-called mild form of this equation:

\[ X_t = Y + \int_0^t e^{A(t-s)} [b_s(X_s, \xi_s) \, ds + (\sigma(X_s), dB_s)]. \tag{49} \]

For a Hilbert space \( B \) that is either \( \mathcal{H} \) or \( \mathcal{H}^{\otimes 2} \), let \( C_{ad}([0, T], B) \) denote the Banach space of adapted continuous \( B \)-valued processes, equipped with the norm

\[ \|X\|_{ad, T} = \sup_{t \in [0, T]} \sqrt{E\|X_t\|^2}. \tag{50} \]

We let \( C([0, T], B) \) denote its subspace of deterministic curves. For elements \( \xi \) of this subspace, the norm becomes the standard sup-norm

\[ \|\xi\|_{ad, T} = \sup_{t \in [0, T]} \|\xi_t\|. \]

We let \( C_Y, ad([0, T], B) \) and \( C_Y([0, T], B) \) denote the subsets of these spaces consisting of curves with \( X(0) = Y \).

In our estimates, we encounter the so-called Le Roy function of index 1/2

\[ R(z) = \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}} \tag{51} \]

which plays the same role for stochastic equations as the exponential and the Mittag-Leffler function play for deterministic equations.

**Proposition 1.** Let

\[ \|b_t(Z_1, \xi_1) - b_t(Z_2, \xi_2)\| \leq \kappa_1 \|Z_1 - Z_2\| + \kappa_2 \|\xi_1 - \xi_2\|, \]

\[ \|\sigma(Z_1) - \sigma(Z_2)\| \leq \kappa_3 \|Z_1 - Z_2\|. \tag{52} \]

Then for any \( T > 0 \), \( Y \in \mathcal{H} \), and \( \xi = \xi \in C([0, T], \mathcal{H}^{\otimes K}) \), Eq. (49) has the a unique global solution \( X \in C_{ad}([0, T], \mathcal{H}) \), and it satisfies the estimate

\[ \|X - Y\|_{ad, T}^2 \leq 2tM^2(t) \left[ \int_0^t |b_s(Y, \xi_s)|^2 \, ds + \sigma^2(Y) \right], \tag{53} \]

where

\[ M(t) = R\left( \sqrt{2(\kappa_3^2 + \kappa_2^2)} \max(\sqrt{t}, t) \right). \tag{54} \]

Moreover, for two initial conditions \( Y_1 \) and \( Y_2 \) and two curves \( \xi_1 \) and \( \xi_2 \), the corresponding solutions satisfy the estimate

\[ \|X_1 - X_2\|_{ad, T}^2 \leq 2M^2(t)(\|Y_1 - Y_2\|^2 + t^2\kappa_2^2\|\xi_1 - \xi_2\|^2_{ad, T}). \tag{55} \]

**Proof.** A solution of (49) is a fixed point of the map

\[ [\Phi_{Y, \xi}(X)](t) = Y + \int_0^t e^{A(t-s)} [b_s(X_s, \xi_s) \, ds + (\sigma(X_s), dW_s)]. \tag{56} \]
in $C_{Y, \text{ad}}([0, T])(\mathcal{H})$. For two curves $X^1, X^2 \in C_{Y, \text{ad}}([0, T])(\mathcal{H}^{\otimes K})$, by the contraction property of the semigroup $e^{At}$ and Ito’s isometry, we have

$$\mathbb{E}[[\Phi_{Y, \xi}(X^1)](t) - [\Phi_{Y, \xi}(X^2)](t)]^2 \leq$$

$$\leq 2\mathbb{E}\left[\int_0^t e^{A(t-s)}(b_s(X^1_s, \xi_s) - b_s(X^2_s, \xi_s)) \, ds\right]^2 +$$

$$+ 2\mathbb{E}\left[\int_0^t e^{A(t-s)}(\sigma(X^1_s) - \sigma(X^2_s)) \, dW_s\right]^2 \leq$$

$$\leq 2\mathbb{E}\left(\int_0^t \kappa_1 \|X^1_s - X^2_s\| \, ds\right)^2 + 2\kappa_3^2 \mathbb{E}\int_0^t \|X^1_s - X^2_s\|^2 \, ds.$$

Applying the Cauchy–Schwarz inequality to the first integral gives

$$\left(\int_0^t \kappa_1 \|X^1_s - X^2_s\| \, ds\right)^2 \leq \kappa_1^2 \int_0^t \|X^1_s - X^2_s\|^2 \, ds,$$

whence

$$\mathbb{E}[[\Phi_{Y, \xi}(X^1)](t) - [\Phi_{Y, \xi}(X^2)](t)]^2 \leq 2(\kappa_3^2 + \kappa_1^2) \mathbb{E}\int_0^t \|X^1_s - X^2_s\|^2 \, ds.$$

This estimate is easy to iterate. Namely, for $t \leq 1$, the $k$th power of $\Phi$ satisfies the estimate

$$\mathbb{E}[[\Phi^k_{Y, \xi}(X^1)](t) - [\Phi^k_{Y, \xi}(X^2)](t)]^2 \leq 2^k(\kappa_3^2 + \kappa_1^2)^k \mathbb{E}\int_0^t \|X^1_s - X^2_s\|^2_{\text{ad}, T} \, ds.$$

and therefore

$$\|\Phi^k_{Y, \xi}(X^1) - \Phi^k_{Y, \xi}(X^2)\|_{\text{ad}, T} \leq \frac{1}{\sqrt{k!}}[2(\kappa_3^2 + \kappa_1^2)t]^{k/2} \|X^1 - X^2\|_{\text{ad}, T}.$$

For $t \geq 1$, we have the estimate for the $k$th power of $\Phi$,

$$\mathbb{E}[[\Phi^k_{Y, \xi}(X^1)](t) - [\Phi^k_{Y, \xi}(X^2)](t)]^2 \leq 2^k(\kappa_3^2 + \kappa_1^2)^k \frac{t^{2k}}{(2k - 1)!!} \|X^1_s - X^2_s\|^2_{\text{ad}, T},$$

whence

$$\|\Phi^k_{Y, \xi}(X^1) - \Phi^k_{Y, \xi}(X^2)\|_{\text{ad}, T} \leq \frac{1}{\sqrt{k!}}[2(\kappa_3^2 + \kappa_1^2)t]^{k/2} \|X^1 - X^2\|_{\text{ad}, T}.$$

Therefore, the conditions of the generalized fixed-point theorem above is satisfied for $a = M(t)$, implying all statements of the proposition. \hfill \blacksquare

A solution of the mild form

$$X_t = Y + \int_0^t e^{A(t-s)}[b_s(X_s, \mathbb{E}X^{\otimes K} \, ds + (\sigma(X_s), dB_s)] \tag{57}$$

of Eq. (47) is a fixed point of the map $\Gamma: C_Y([0, T], H^{\otimes K}) \rightarrow C_Y([0, T], H^{\otimes K})$ that sends $\xi$ to $\mathbb{E}X^{\otimes K}$, where $X$ is the solution of (49). By (55), for two curves $\xi^1$ and $\xi^2$, we have estimates for the corresponding solutions $X^1$ and $X^2$:

$$\|\mathbb{E}(X^1_t) - \mathbb{E}(X^2_t)\| \leq 2M(t)t \kappa_2 \|\xi^1 - \xi^2\|_{\text{ad}, t}$$

for $K = 1$ and

$$\|\mathbb{E}(X^1_t)^{\otimes 2} - \mathbb{E}(X^2_t)^{\otimes 2}\| \leq \mathbb{E}[[X^1_t - X^2_t] \otimes (X^1_t - X^2_t)] + \mathbb{E}[[X^2_t] \otimes (X^1_t - X^2_t)] \leq$$

$$\leq 2M(t)t \kappa_2 \|\xi^1 - \xi^2\|_{\text{ad}, t} \max(\|X^1_t\|_{\text{ad}, t}, \|X^2_t\|_{\text{ad}, t})$$

for $K = 2$. 

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Therefore, for small times \( t \leq t_0 \), the map \( \Gamma \) is a contraction and hence has a unique fixed point. For \( K = 1 \), the time \( t_0 \) is independent of \( Y \) and we can therefore build a unique global solution by iterations. We have thus proved the following statement.

**Proposition 2.** Let the assumptions of Proposition 1 hold. If \( K = 1 \), Eq. (57) has a unique global solution for any initial \( Y \), and

\[
\| \mathbf{E}X - Y \|_{\text{ad}, T} \leq C(T),
\]

with a constant \( C(T) \) depending on \( \kappa_1, \kappa_2, \kappa_3 \), and \( \| Y \| \). If \( K = 2 \), Eq. (57) has a unique local solution for times of the order of \( \| Y \|^{-1} \).

We finally note a situation where solutions of mild equations also solve the SDEs.

**Proposition 3.** Let \( D \) be an invariant core for the semigroup \( e^{At} \), which is itself a Banach space with respect to some norm \( \| \cdot \|_D \). Let \( b \) and \( \sigma \) be continuous maps \( \mathbb{R} \times H \times H^{\otimes K} \to D \) and \( H \to D \). Then for any \( Y \in D \), the solutions of mild equations (57) and (49) solve the corresponding SDEs.

**Proof.** This follows from the direct application of Ito’s rule. The differentiability required here is a consequence of the assumptions made. ■

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