CLASSICAL $r$-MATRICES

AND CONSTRUCTION OF QUANTUM GROUPS

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ABSTRACT

A problem of constructing quantum groups from classical $r$-matrices is discussed.

May 1994.

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1. Classical \( r \)-matrices \([1, 2]\) play a very important role in a theory of quantum groups \([3, 4]\). They are closely related to a structure of Poisson-Lie group which appears as a classical limit of a quantum group. Poisson-Lie group is a Lie group \( G \) together with a Poisson bracket defined on it in such a way that the group action becomes a Hamiltonian map. It is convenient to define the Poisson bracket by means of a bivector field \( \Lambda \)

\[ \{ \phi, \psi \} = \Lambda(d\phi, d\psi) \]  

or rather to consider instead of \( \Lambda \) an object \( \eta : G \to G \otimes G \) \((G \) is a Lie algebra of \( G \))

\[ \eta(x) = (T_{\rho_x^{-1}} \otimes T_{\rho_x^{-1}})x \Lambda_x \]  

where \( \rho \) denotes a right group action. The fact that the action of the group is a Hamiltonian map can be expressed as a cocycle condition on \( \eta \)

\[ \eta(g_1 g_2) = \eta(g_1) + \text{Ad} \otimes (g_1) \eta(g_2) \]  

Possible solutions of this equation are in the form of coboundary

\[ \eta(g) = \text{Ad} \otimes r - r \]  

where \( r \in G \otimes G \) is called a classical \( r \)-matrix. A requirement that the Poisson bracket defined by \( \Lambda \) via \( r \) satisfies the Jacobi identities imposes some conditions on \( r \). Namely \( r \) has to satisfy so called modified classical Yang-Baxter equation (MCYBE) which means that the following object constructed out of \( r \) \((\text{Schouten bracket})\)

\[ <r, r> = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \]  

is an invariant tensor \((\text{under the adjoint action})\) in \( G \otimes G \otimes G \).

As it is relatively easy to find solutions of \((\text{5})\) one can construct a lot of examples of Poisson-Lie groups. Moreover for many interesting groups every \( \eta(g) \) is a coboundary. This is e.g. the situation of complex simple Lie groups \([1]\), Lorentz group \([5]\) and the group of symmetries of a space-time of arbitrary metric signature in dimensions \( \geq 3 \) \([6]\). Two dimensions are exceptional in this sense as for the euclidean group in \( D = 2 \) it is possible to construct \( \eta(g) \) which is not a coboundary. One should have in mind that also other examples are known in which the structure of Poisson-Lie group is not given by a classical \( r \) matrix \([7]\).
2. It is clear that by studying classical $r$-matrices one gets some insight into a possible structure of quantum groups. For example it is known that classifications of $r$-matrices for D=4 Lorentz and Poincaré groups gives similar patterns to that of the analogous classification of quantum deformations of those groups [3], [3], [5], [4].

In this paper the starting point is a $r$-matrix satisfying MCYBE. We shall try to investigate what can be said about (possible) existing quantum groups having a Poisson-Lie group defined by this $r$ as its classical limit. In other terms we shall try to construct a quantum group out of $r$ only. This problem has been discussed by several authors [10]. Our discussion will be on much more elementary level.

One possible way of reasoning is the following. With the help of $r$ one can construct Poisson bracket in the space of functions on $G$ (we shall always have in mind elements of some suitable chosen matrix representation of $G$). Then one can try to replace Poisson brackets by commutators. However naive it looks like this is a method to construct some new examples of quantum groups: $\kappa$-Poincaré [11], ISL(2,$C$) group [12] and supergroups: $\kappa$-superPoincaré [13]. In general it is clear that the described method is ambiguous due to appearance of operator ordering problems. Only in some particular cases these ambiguities are not present or can be relatively easy solved. In the case of $sl(2)$ ($[H,X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = H$, group element is $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$) with $r$-matrix $r = X \wedge Y$ we obtain a typical Poisson bracket $\{a,b\} = ab$ and it is a priori difficult to guess that an ordering leading to a "good" quantum group is $[a,b] = \lambda ab + (1 - \lambda)ba$ with $q = \frac{2 - \lambda}{1 - \lambda}$ with similar or even more complicated tricks for other commutators. Very similar is the situation with Poisson brackets coming from $r$ matrix $r = X \wedge H$.

This is a proper place to comment on the notion of a "good" quantum group. At the beginning there is some matrix group. Quantum group is a Hopf algebra with elements polynomials constructed out of noncommutative matrix elements modulo some relations among them. Usually these relations are introduced by $R$ matrix (the notation is explained in [3])

$$RT_1T_2 = T_2T_1R \quad (6)$$

but in general it need not to be a case. The comultiplication defined on the algebra generators is the same way as in the classical (undeformed) case (an advantage of defining quantum relations by means of (5) is that comultiplication becomes then the Hopf algebra homomorphism). Important is that both deformed and undeformed (i.e. commutative) algebras of polynomials should be of the same size i.e. that the dimensionality of the space
of "deformed" polynomials in matrix elements should be the same as in the undeformed case. Moreover there should exist a deformation parameter such that in the limit when it goes to zero both algebras become identical. Usually it is taken as granted that the statement about the dimensionalities is satisfied if relations are introduced by a $R$ matrix satisfying the quantum Yang Baxter equation (QYBE):

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$  \hspace{1cm} (7)

But it turns out to be nontrivial to prove it even in simple cases [14]. Similarly one can demand also that

$$\tilde{R}R = \lambda 1 \otimes 1$$  \hspace{1cm} (8)

(here $\tilde{R} = PRP$ where $P(x \otimes y) = y \otimes x$). We should however remember that the famous $R$ for $sl(n)$ discussed in [3] satisfies (7) but not (8). We shall come back to this point later.

Let us also notice that when we discuss different choices of normal ordering in the case of $sl(2)$ what is wrong about almost all orderings is that they give rise to too restrictive quadratic relations.

3. Going beyond the naive quantization scheme it is known that in some cases one can construct out of $r$ an element $R = e^{hr}$ satisfying QYBE. As $R = 1 \otimes 1 + hr + ...$ it is clear that $r$ has to satisfy CYBE, and not only the MCYBE (it means that the expression given in (8) equals zero). Among examples that can be treated in this way let us mention $sl(2)$ with $r$-matrix $r = X \wedge H$ in the matrix representation $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ [15]. In this way the nonstandard deformation of $sl(2)$ has been obtained. Another example is provided by a Poincaré group. Here group elements are described by 5-dimensional matrices $g = \begin{pmatrix} \Lambda & x \\ 0 & 1 \end{pmatrix}$. Poincaré algebra elements are represented by matrices $P_\mu = e_\mu^A$, $M_j = \epsilon_{jkl} e^k_l$, $L_j = e_j^0 + e_0^j$ and $r$ matrix is $r = M_3 \wedge L_3 + \alpha P_1 \wedge P_2 + \beta P_0 \wedge P_3$. This $r$-matrix is taken from the list given in [3]. It satisfies CYBE but that does not imply that $R = e^{hr}$ has to satisfy QYBE. It can however be calculated that

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} = 1 \otimes 1 \otimes 1 +$$

$$+ \alpha (P_1 \otimes P_2 \otimes 1 - P_2 \otimes P_1 \otimes 1 + P_1 \otimes 1 \otimes P_2 - P_2 \otimes 1 \otimes P_1 + 1 \otimes P_1 \otimes P_2 - 1 \otimes P_2 \otimes P_1) +$$

$$+ \beta (P_0 \otimes P_3 \otimes 1 - P_3 \otimes P_0 \otimes 1 + P_0 \otimes 1 \otimes P_3 - P_3 \otimes 1 \otimes P_0 + 1 \otimes P_0 \otimes P_3 - 1 \otimes P_3 \otimes P_0)$$
\[ + \sin(h) (M_3 \otimes L_3 \otimes 1 - L_3 \otimes M_3 \otimes 1 + M_3 \otimes 1 \otimes L_3 - L_3 \otimes 1 \otimes M_3) + \\
+ 1 \otimes M_3 \otimes L_3 - 1 \otimes L_3 \otimes M_3) + (1 - \cos(h))(M_3^2 \otimes L_3^2 \otimes 1 + L_3^2 \otimes M_3^2 \otimes 1 + \\
+ M_3^2 \otimes 1 \otimes L_3^2 + L_3^2 \otimes 1 \otimes M_3^2 + 1 \otimes M_3^2 \otimes L_3^2 + 1 \otimes L_3^2 \otimes M_3^2) + \\
+ \alpha \sin(h)(P_2 \otimes L_3 \otimes P_2 + P_1 \otimes L_3 \otimes P_1)+ \\
+ \beta \sin(h)(P_0 \otimes M_3 \otimes P_0 - P_3 \otimes M_3 \otimes P_3)+ \\
+ \alpha(1 - \cos(h))(P_2 \otimes L_3^2 \otimes P_1 - P_1 \otimes L_3^2 \otimes P_2)+ \\
+ \beta(1 - \cos(h))(P_0 \otimes M_3^2 \otimes P_3 - P_3 \otimes M_3^2 \otimes P_0)+ \\
+ \sin^2(h)(M_3^2 \otimes L_3 \otimes L_3 - M_3 \otimes L_3^2 \otimes M_3 + L_3^2 \otimes M_3 \otimes M_3+ \\
- L_3 \otimes M_3^2 \otimes L_3 + M_3 \otimes M_3 \otimes L_3^2 + L_3 \otimes L_3 \otimes M_3^2)+ \\
+(1 - \cos(h))^2(-M_3^2 \otimes L_3^2 \otimes 14 - L_3^2 \otimes 14 \otimes M_3^2 - 14 \otimes M_3^2 \otimes L_3^2)+ \\
+ \sin(h)(1 - \cos(h))(-M_3 \otimes L_3 \otimes 14 + L_3 \otimes M_3 \otimes 14 + \\
- M_3 \otimes 14 \otimes L_3 + 14 \otimes L_3 \otimes M_3 + L_3 \otimes 14 \otimes M_3 - 14 \otimes M_3 \otimes L_3) \tag{9} \]

In the above formula \(14\) denotes a diagonal matrix with 1’s in the first four rows.

4. Interesting enough there are some examples in which \( R = e^{h r} \) does not satisfy QYBE however when put into (3) gives rise to a good (established by means of other methods) quantum group \( G \). This is the situation of the mentioned above \( \kappa \)-Poincare group with \( r = \sum_j L_j \wedge P_j \) [11]. Similar is the situation of \( D = 2 \) euclidean group in three-dimensional representation \( (X = (0 0 1) \quad J = (0 1 0) \quad h) \) with the \( r \)-matrix \( r = J \wedge X \) [16]. Still another example is provided by the Heisenberg group with its Lie algebra generators represented as \( A = (0 1 0) \quad A^+ = (0 0 1) \), 

\[ H = (0 0 1) \quad \text{and} \quad r = A \wedge A^+ \] [17].

As this point can give rise to some confusion let us comment on it shortly. The fact that \( R \) satisfies or not QYBE has nothing to do with whether the resulting Hopf algebra is associative or not. It is always assumed to be associative. QYBE is just a
constraint on $R$ that is supposed to ensure that the space of cubic (and then higher orders) polynomials in noncommuting matrix elements variables can be large enough to be of the same dimensionality as in the undeformed case (it eliminates a possibility of extra unwanted cubic relations). Of course QYBE is not a necessary condition to be so. Suppose for a moment that $R$ does not satisfy the QYBE. Then it is possible to define an invertible element $B$ by means of

$$R_{12}R_{13}R_{23} = BR_{23}R_{13}R_{12}. \quad (10)$$

$B$ has necessarily the properties (they follow from the fact that the algebra is associative)

$$t_{ij}t_{kl}t_{mn}B_{ns \ ir \ jp} = B_{mn \ kl \ ijp\ lr \ ns} \quad (11)$$

As we work in the particular representation in which $T$ is given $B$ is a 3-fold tensor product of numerical matrices. $t_{ij}$ are matrix elements which commutation relations are given in (6). $B$ can be viewed as a generalization of the element $<r,r>$ in (5) to the case of noncommuting $t_{ij}$. If we analyze its structure as a power series in the deformation parameter $h$ we get

$$B = 1 \otimes 1 \otimes 1 + h^2 <r,r> + ... \quad (12)$$

In a very similar way if the relations are introduced by a $R$ matrix which does not satisfy the condition (8) we automatically obtain an invertible element $C = \tilde{R}R$ with the properties

$$C_{ij \ kl}t_{jm}t_{ln} = t_{ij}t_{kl}C_{jm \ ln} \quad (13)$$

Both conditions: (8) and (7) are desirable for construction of good quantum groups but certainly neither is a necessary one (strictly speaking it is even not obvious if they are sufficient; one should have in mind that for general matrix groups the resulting relations are not purely quadratic – e.g. for the Poincaré group, another complication comes from extra relations given e.g. by the determinant). It would be interesting to find some weaker conditions on $R$. Equations (11) and (13) are however not suitable for that purpose as they are identities in the algebra defined by (6) and so carry no information about the space of polynomials the size of which we would like to be able to estimate.

5. It turns out to be interesting to analyze in more detail the case of $R = e^{hr}$ where $h$ is a deformation parameter and $r$ is a classical antisymmetric $r$ matrix. Such $R$ have two interesting general properties. First of all they satisfy (8). Moreover they give rise to
Poisson-Lie group structure on $G$ in the limit as $h \to 0$. What is missing is an estimate of the dimensionality of the space of elements of the resulting Hopf algebra. We do not know if the requirement of the proper dimensionality can be expressed as some condition on $R$. We have checked only that in some simple examples one quite surprisingly obtains correct results.

The first such example is given by $sl(2)$ with $r$-matrix $r = X \wedge Y$. In the given above two-dimensional representation for $X$ and $Y$ one calculates that $R = e^{hr}$ equals

\[
R = 1 \otimes 1 + \sin(h)(X \otimes Y - Y \otimes X) + (\cos(h) - 1)(U \otimes D + D \otimes U)
\]  

(14)

where $U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This $R$ does not satisfy QYBE. When however put into (6) it gives rise to the standard deformation of $sl(2)$ with $q = \frac{\cos(h)}{1 + \sin(h)}$.

As we explained above as a byproduct of this construction of a standard deformation of $sl(2)$ we are able to calculate an element $B$ with the interesting property (11). In the explicite way it is given as

\[
B = 1 \otimes 1 \otimes 1 + (-\sin^4(h) + \sin^5(h))U \otimes U \otimes D + (-\sin^4(h) - \sin^6(h))U \otimes D \otimes U + (-\sin^4(h) + \sin^5(h))U \otimes D \otimes D + (-\sin^4(h) - \sin^5(h))D \otimes U \otimes U + (-\sin^4(h) - \sin^6(h))D \otimes U \otimes D + (-\sin^4(h) - \sin^5(h))D \otimes D \otimes U
\]

\[
+ \cos(h)(\sin^2(h) + \sin^3(h) + \sin^4(h))X \otimes Y \otimes U
\]

\[
+ \cos(h)(-\sin^2(h) + \sin^3(h) + \sin^5(h))X \otimes Y \otimes D
\]

\[
+ \cos(h)(-\sin^2(h) - \sin^3(h) - \sin^5(h))Y \otimes X \otimes U
\]

\[
+ \cos(h)(\sin^2(h) - \sin^3(h) + \sin^4(h))Y \otimes X \otimes D
\]

\[
+ (-\sin^2(h) + \sin^4(h))X \otimes U \otimes Y + (\sin^2(h) - \sin^6(h))X \otimes D \otimes Y
\]

\[
+ (\sin^2(h) - \sin^6(h))Y \otimes U \otimes X + (-\sin^2(h) + \sin^4(h))Y \otimes D \otimes X
\]

\[
+ \cos(h)(\sin^2(h) - \sin^3(h) + \sin^4(h))U \otimes X \otimes Y
\]

\[
+ \cos(h)(-\sin^2(h) - \sin^3(h) - \sin^5(h))D \otimes X \otimes Y
\]

\[
+ \cos(h)(-\sin^2(h) + \sin^3(h) + \sin^5(h))U \otimes Y \otimes X
\]

\[
+ \cos(h)(\sin^2(h) + \sin^3(h) + \sin^4(h))D \otimes Y \otimes X
\]  

(15)
Also the two-parameter deformation of the $gl(2)$ can be obtained in this way. Here the starting point is $r = \alpha X \wedge Y + \beta U \wedge D$ (of course $U = \frac{H+1}{2}$ and $D = \frac{1-H}{2}$). It is possible to calculate that with $s = h\sqrt{\beta^2 - \alpha^2}$

$$R = 1 \otimes 1 + (ch(s) - 1)(U \otimes D + D \otimes U) + \frac{sh(s)}{\sqrt{\beta^2 - \alpha^2}}(\alpha X \wedge Y + \beta U \wedge D)$$

Using this $R$ (which does not satisfy QYBE) one obtains the standard relations: $ab = qba$, $ac = pca$, $cd = qdc$, $bd = pdb$, $qbc = pcb$, $[a, d] = (p - \frac{1}{q})cb = (q - \frac{1}{p})bc$ with

$$q = \frac{\sqrt{\beta^2 - \alpha^2}ch(s) + \beta sh(s)}{\sqrt{\beta^2 - \alpha^2} + \alpha sh(s)}, \quad p = \frac{\sqrt{\beta^2 - \alpha^2} - \alpha sh(s)}{\sqrt{\beta^2 - \alpha^2}ch(s) + \beta sh(s)}$$

In the limit $\beta \rightarrow 0$ one gets $p \rightarrow q$.

It would be in our opinion interesting to continue the study of properties of $R$ matrices of the form $R = e^{hr}$. That includes both a discussion of new examples and investigation a general question what are the sufficient conditions on $r$ to give rise to a good quantum group.

ACKNOWLEDGMENTS.

I wish to thank dr. P.Maślanka for interesting discussions on sl(2). I thank dr. S.Zakrzewski for pointing out errors in the first version of this paper and for many stimulating comments about the notion of a good (being of correct size) quantum group. Finally I would like to thank prof. J. de Azcárraga for a warm hospitality at Valencia where this work was done. The author was supported by EEC grant number ERBCIPACT920488.
References

[1] V.G.Drinfeld, Soviet. Math. Dokl. 27 (1983) 68
[2] J.-H.Lu and A.Weinstein, J. Diff. Geometry 31 (1990) 501; M.A.Semenov-Tian-Shansky, Publ. RIMS, Kyoto University, 21 (1985) 1237
[3] N.Yu.Reshetikhin, L.A.Takhtadzyan and L.D.Faddeev, Leningrad Math. J., 1 (1990) 193
[4] V.G.Drinfeld, Quantum groups, Proc. ICM, Berkeley, 1986, vol.1, 789
[5] S.Zakrzewski, Poisson structures on the Lorentz group, to appear in Lett. Math. Phys.
[6] S.Zakrzewski, Talk at the Karpacz Winter School of Theoretical Physics 1994, unpublished
[7] I.Szymczak and S.Zakrzewski, J.Geom.Phys., 7 (1990) 553
[8] S.L.Woronowicz and S.Zakrzewski, Compositio Mathematica 90 (1994) 211
[9] P.Podleś, Talk at the Karpacz Winter School of Theoretical Physics 1994, unpublished
[10] see e.g. D.Gurewicz, V.Rubtsov and N.Zobin, J.Geom.Phys. 9 (1992) 25, S.Zakrzewski, Geometric quantization of Poisson groups – diagonal and soft deformations, Contemp. Math. 1993
[11] S.Zakrzewski, Quantum Poincaré group related to κ-Poincaré algebra, J. Phys. A27 (1994) 2075
[12] P.Maślanka, The quantum ISL(2,C) group, Łódź University preprint, IMUL 9/93
[13] P.Kosiński, J.Lukierski, P.Maślanka and J.Sobczyk, Quantum deformation of the Poincarè supergroup and κ-deformed superspace, Wrocław University preprint IFTWr 868/94
[14] S.Woronowicz, Rep. Math. Phys. 30 (1991) 259
[15] S.Zakrzewski, Lett. Math. Phys., 22, (1991) 287
[16] A.Ballesteros, E.Celeghini, R.Giachetti, E.Sorace and M.Tarlini, An R-matrix approach to the quantization of the euclidean group E(2), to be published in Journal of Physics A; P.Maślanka, J. Math. Phys., 35 (1994) 1976
[17] E.Celeghini, R.Giachetti, E.Sorace and M.Tarlini, J. Math. Phys., 32 (1991) 1155