Inverse problem solution and spectral data characterization for the matrix Sturm–Liouville operator with singular potential

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Abstract
The self-adjoint matrix Sturm–Liouville operator on a finite interval with singular potential of class $W^{-1/2}$ and the general self-adjoint boundary conditions is studied. This operator generalizes the Sturm–Liouville operators on geometrical graphs. We investigate the inverse problem that consists in recovering the considered operator from the spectral data (eigenvalues and weight matrices). The inverse problem is reduced to a linear equation in a suitable Banach space, and a constructive algorithm for the inverse problem solution is developed. Moreover, we obtain the spectral data characterization for the studied operator. In addition, the main results are applied to the Sturm–Liouville operator on a graph of arbitrary geometrical structure.

Keywords Inverse spectral problems · Matrix Sturm–Liouville operator · Singular potential · Method of spectral mappings · Spectral data characterization

Mathematics Subject Classification 34A55 · 34B09 · 34B24 · 34L40

1 Introduction

This paper is devoted to an inverse spectral problem for the matrix Sturm–Liouville operator $-Y'' + Q(x)Y$, where $Q(x)$ is a Hermitian $(m \times m)$-matrix function called the potential.
Inverse problems of spectral analysis consist in reconstruction of operators from their spectral information. The greatest success in inverse problem theory has been achieved for the scalar Sturm–Liouville operators (for $m = 1$), see the classical monographs [1–4] and references therein. Matrix Sturm–Liouville operators have been intensively studied in connection with various applications. In particular, inverse problems for such operators are used in quantum mechanics [5], in elasticity theory [6], for description of electromagnetic waves [7] and nuclear structure [8], for solving matrix nonlinear evolution equations by inverse spectral transform [9].

For the matrix Sturm–Liouville operators on a finite interval, this issue has been independently solved by Chelkak [8], Malamud [11], Yurko [12], and Shieh [13]. Yurko [14] proposed a constructive method, based on spectral mappings, for solving such inverse problems. Further, this method has been developed by Bondarenko [15] for working with multiple eigenvalues. The most difficult and, at the same time, the most important issue of inverse problem theory is the spectral data characterization. For the matrix Sturm–Liouville operators from various spectral characteristics has been proved by Carlson [10], Chabanov [11], Malamud [11], Yurko [12], and Shieh [13]. Yurko [14] proposed a constructive method, based on spectral mappings, for solving such inverse problems. Further, this method has been developed by Bondarenko [15] for working with multiple eigenvalues. The most difficult and, at the same time, the most important issue of inverse problem theory is the spectral data characterization. For the matrix Sturm–Liouville operators on a finite interval, this issue has been independently solved by Chelkak and Korotyaev [16], by Mykytyuk and Trush [17], and by Bondarenko [15,18]. The latter approach was also generalized for a certain class of non-self-adjoint matrix Sturm–Liouville operators [19].

The present paper deals with the matrix Sturm–Liouville operator with the self-adjoint boundary conditions in the general form defined below. Denote by $S_0$ and $S_m$ the spaces of complex $m$-vectors and $(m \times m)$-matrices, respectively. For an interval $I$ and a class $A(I)$ of functions defined on $I$ (e.g., $A = L_2, C, \ldots$), we denote by $A(I; \mathbb{C}^m)$ and $A(I; \mathbb{C}^{m\times m})$ the classes of complex-valued $m$-vector functions and $(m \times m)$-matrix functions, respectively, with entries from $A(I)$.

Consider the matrix Sturm–Liouville problem $L = L(\sigma, T_1, T_2, H_2)$:

$$\ell Y := -(Y^{[1]}_1)' - \sigma(x)Y^{[1]} + \sigma^2(x)Y = \lambda Y, \quad x \in (0, \pi), \quad (1.1)$$

$$V_1(Y) := T_1 Y^{[1]}(0) - T^{\perp}_1 Y(0) = 0,$$

$$V_2(Y) := T_2 Y^{[1]}(\pi) - H_2 Y(\pi) - T^{\perp}_2 Y(\pi) = 0. \quad (1.2)$$

where $Y = [y_j(x)]_{j=1}^m$ is a vector function, $\sigma \in L_2((0, \pi); \mathbb{C}^{m\times m})$, $\sigma(x) = (\sigma(x))^*$ a.e. on $(0, \pi)$, $Y^{[1]}(x) := Y'(x) - \sigma(x)Y(x)$ is the quasi-derivative, $\lambda$ is the spectral parameter, for $j = 1, 2$, $T_j \in \mathbb{C}^{m\times m}$, $T_j$ is an orthogonal projection matrix, $T^{\perp}_j = I - T_j$, $H_2 \in \mathbb{C}^{m\times m}$, $H_2 = H_2^* = T_2 H_2 T_2$, $I$ is the $(m \times m)$-unit matrix, the symbol $*$ denotes the conjugate transpose. Under these assumptions, the problem $L$ is
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self-adjoint. We suppose that $Y$ belongs to the domain

$$D(L) := \{Y : Y, Y^{[1]} \in AC([0, \pi]; \mathbb{C}^m), \ell Y \in L_2((0, \pi); \mathbb{C}^m)\}.$$

Equation (1.1) can be rewritten in the equivalent form

$$-Y'' + Q(x)Y = \lambda Y, \quad x \in (0, \pi),$$

with the singular potential $Q(x) = \sigma'(x)$ of class $W_2^{-1}((0, \pi); \mathbb{C}^{m \times m})$. The derivative of $L_2$-function is understood in the sense of distributions. However, it is more convenient to use the form (1.1).

Relations (1.2) describe the general self-adjoint form of separated boundary conditions. The matrix Sturm–Liouville operator given by (1.1)–(1.2) causes interest because it generalizes Sturm–Liouville operators on geometrical graphs. The latter operators are used for modeling wave propagation in graph-like structures consisting of thin tubes, strings, beams, etc. Differential operators on graphs attract much attention of mathematicians and physicists in recent years in connection with applications in nanotechnology, organic chemistry, mechanics, and other branches of science and engineering (see [20–23] and references therein). The general self-adjoint boundary conditions in the form

$$TY'(v) + HY(v) = 0, \quad T^\perp Y(v) = 0,$$

where $T$ and $T^\perp$ are complimentary projection matrices, $H = H^* = THT$, have been introduced by Kuchment [24]. In the literature (see, e.g., [25]), the other equivalent forms of parametrization also appear:

$$AY(v) + BY'(v) = 0,$$

where the $(m \times 2m)$-matrix $[A, B]$ has the maximal rank $m$ and the matrix $AB^*$ is Hermitian, and

$$-i(U + I)Y(v) + (U - I)Y'(v) = 0,$$

where $U$ is a unitary matrix.

Inverse problems for the matrix Sturm–Liouville operator on a finite interval with general self-adjoint boundary conditions and regular potential of class $L_2$ have been recently studied by Xu [26]. However, paper [26] is only concerned with uniqueness theorems. The issues of constructive solution and spectral data characterization for this operator appeared to be more difficult for investigation because of complex asymptotic behavior of the spectrum and structural properties of the problem. In [27,28], properties of the spectral data have been investigated for the matrix Sturm–Liouville operator with boundary condition in the general self-adjoint form at $x = \pi$ and with Dirichlet boundary condition at $x = 0$. Further, a constructive solution procedure has been developed for the corresponding inverse spectral problem (see [29]). Those results have
been applied for obtaining the spectral data characterization for the Sturm–Liouville operator on the star-shaped graph (see [30]).

In addition, it is worth mentioning that inverse scattering problems have been studied for the matrix Sturm–Liouville operators on the half-line and on the line (see, e.g., [5,9,31–35]). In particular, Harmer [31,32] and Aktosun and Weder [35] investigated inverse scattering on the half-line with boundary condition in the general self-adjoint form at the origin. Harmer [31] also applied those results to the inverse scattering problem on the star-shaped graph consisting of infinite rays. However, the matrix Sturm–Liouville operators on infinite domains usually have a bounded set of eigenvalues, so the inverse problems for them are in some sense easier for investigation than analogous problems on a finite interval.

The majority of mentioned results deal with the case of regular (square summable or summable) potentials. For the Sturm–Liouville operators with singular (distributional) potentials, there is an extensive literature concerning the scalar case. Inverse problems for the scalar operators in the form 

\[-(y^{(1)})' - \sigma(x)y^{(1)} - \sigma^2(x)y, \quad \sigma \in L^2(0, \pi),\]

were studied by Hryniv and Mykytyuk [36,37], Savchuk and Shkalikov [38], Djakov and Mityagin [39], and by other authors.

Mykytyuk and Trush [17] investigated inverse problems for the matrix Sturm–Liouville operators with potential of class $W^{-1}_2$ on a finite interval in a special form, which can be easily reduced to a Dirac-type operator. Analogous reduction was applied by Eckhardt and co-authors [40,41] to the matrix Sturm–Liouville operators on the half-line and on the line. It is also worth mentioning that the spectral data characterization for the matrix Dirac-type operators with the separated and non-separated boundary conditions was obtained by Mykytyuk and Puyda [42,43]. However, we find it to be inconvenient to apply the method of reduction to the Dirac-type system to the problem (1.1)–(1.2) because of the two reasons. First, such reduction is non-unique, and so the spectral data of the second-order problem do not uniquely specify the coefficients of the first-order problem. Second, applying the reduction similar to the one in [17] to (1.1)–(1.2), we arrive at the Dirac-type system with boundary conditions depending on the spectral parameter. The latter system has not been studied so far and requires a separate investigation. Therefore, we choose another method to deal with the inverse matrix Sturm–Liouville problem.

In this paper, we solve the inverse spectral problem for the matrix Sturm–Liouville operator (1.1)–(1.2) with singular potential and with general self-adjoint boundary conditions at the both ends of the interval. We obtain an algorithm for reconstruction of the operator by its spectral data and provide the spectral data characterization. On the one hand, our approach is based on the spectral properties of the operator (1.1)–(1.2) obtained in our previous study [44]. On the other hand, we rely on the method of spectral mappings for constructive solution of the inverse problem. This method has been initially developed by Yurko for operators with regular coefficients (see [4]). This method allows one to reduce a nonlinear inverse problem to a linear equation in a suitable Banach space. Such reduction leads to a constructive procedure for solving an inverse problem and also can be used for investigating global solvability, local solvability, stability, and other issues of inverse problem theory. Yurko’s method has been modified for the Sturm–Liouville operators with singular potentials by Freiling, Ignatiev, and Yurko [45] and by Bondarenko [46]. An approach to inverse problems
for the matrix Sturm–Liouville operators has been developed in [15,19,29,30]. In the present paper, we combine the ideas of the mentioned studies to solve the inverse problem for the operator (1.1)–(1.2).

Similarly to the results of Chelkak and Korotyaev [16] and of Mykytyuk and Trush [17], our main theorem on the spectral data characterization (Theorem 2.6) contains asymptotic formulas for the eigenvalues and the weigh matrices, some structural properties of the spectral data, and the completeness of a special vector-functional system. Furthermore, it can be shown that Theorem 1.2 from [17] is a special case of our Theorem 2.6 for the Dirichlet boundary conditions. However, the problem with the Neumann boundary conditions considered in [17] concerns the Dirac-type operator and is different from ours.

The constructive algorithm for solving the inverse problem, obtained in this paper, can also be applied to the matrix Sturm–Liouville operators with regular potentials and general self-adjoint boundary conditions, and it is a novel result for this class of operators. Nevertheless, obtaining necessary and sufficient conditions of inverse problem solvability is a separate problem for each class of operators. In fact, this problem consists in construction of a one-to-one correspondence between a class of operators and a class of spectral data. Therefore, our results for singular potentials do not directly imply the spectral data characterization for the potentials \( Q(x) \) of classes \( L_2((0, \pi); \mathbb{C}^{m \times m}) \) and \( L_1((0, \pi); \mathbb{C}^{m \times m}) \). In order to obtain the characterization for the latter classes, one has to describe additional properties of the spectral data that make the potential to belong to an appropriate class. For \( Q \in L_2((0, \pi); \mathbb{C}^{m \times m}) \), the Dirichlet boundary condition at \( x = 0 \), and the general self-adjoint boundary condition at \( x = \pi \), the spectral data characterization has been obtained in [30]. For the both boundary conditions of the general form, the problem is more technically difficult and has not been studied yet.

The paper is organized as follows. In Sect. 2, we describe asymptotical and structural properties of the spectral data, formulate the inverse problem, the corresponding uniqueness theorem, and our main theorem (Theorem 2.6) on the characterization of the spectral data. The proof of Theorem 2.6 is contained in Sects. 3–6. In Sect. 3, we reduce the nonlinear inverse problem to a linear equation in a special Banach space. That equation is called the main equation of the inverse problem. In Sect. 4, we obtain auxiliary estimates concerning the operator participating in the main equation and some other characteristics. In Sect. 5, it is proved that, under the conditions of Theorem 2.6, the main equation is uniquely solvable. In Sect. 6, the proof of Theorem 2.6 is finished. By using the solution of the main equation, we construct \( \sigma \) and \( H_2 \), and finally arrive at Algorithm 6.8 for constructive solution of the inverse problem. In Sect. 6, we apply our main results to the Sturm–Liouville operator on a graph.

We overcome the following difficulties specific for our problem.

1. The problem \( L \) can have an infinite number of groups of multiple and/or asymptotically multiple eigenvalues. Therefore, in the construction of the main equation in Sect. 3, we use the special grouping (3.5) of the eigenvalues with respect to their asymptotics.

2. Because of the singular potential, we need to obtain some precise estimates related with the operator participating in the main equation (see Lemmas 4.4–4.5). These
estimates play an important role in the proofs of the main equation solvability. Such estimates are not needed in the case of regular potential.

3. When the matrix function $\sigma(x)$ is constructed by using the spectral data, we cannot directly substitute this function into Eq. (1.1) and so have to approximate it by smooth matrix functions $\sigma^N$.

2 Preliminaries and main results

In this section, we define the spectral data and provide their properties obtained in [44]. Further, we formulate the inverse problem (Inverse Problem 2.4), the corresponding uniqueness theorem (Proposition 2.5), and our main result (Theorem 2.6). The latter theorem gives necessary and sufficient conditions for the inverse problem solvability, or, in other words, the spectral data characterization.

Let us start with the notations:

1. Denote $\rho := \sqrt{\lambda}$, $\arg \rho \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (unless stated otherwise).
2. We use the Euclidean norm in $\mathbb{C}^m$:

   $$\|a\| = \left( \sum_{j=1}^{m} |a_j|^2 \right)^{1/2}, \quad a = [a_j]_{j=1}^{m},$$

   and the corresponding matrix norm $\|A\|$ equal to the maximal singular value of $A$.
3. The scalar product in the Hilbert space $L_2(\mathbb{I}; \mathbb{C}^m)$ is defined as follows:

   $$(Y, Z) = \int_{\mathbb{I}}(Y(x))^\ast Z(x) \, dx = \sum_{j=1}^{m} \int_{\mathbb{I}} y_j(x) z_j(x) \, dx,$$

   $$Y = [y_j(x)]_{j=1}^{m}, \quad Z = [z_j(x)]_{j=1}^{m} \in L_2(\mathbb{I}; \mathbb{C}^m).$$
4. The same symbol $C$ is used for various positive constants independent of $n, x, \lambda$, etc.

Let $\varphi(x, \lambda)$ be the matrix solution of Eq. (1.1) satisfying the initial conditions $\varphi(0, \lambda) = T_1, \varphi^{(1)}(0, \lambda) = T_1^\perp$. Clearly, the matrix functions $\varphi(x, \lambda)$ and $\varphi^{(1)}(x, \lambda)$ are entire in $\lambda$ for each fixed $x \in [0, \pi]$. The eigenvalues of the problem $L$ coincide with the zeros of the entire characteristic function $\Delta(\lambda) := \det V_2(\varphi(x, \lambda))$ with their multiplicities.

The matrix function $\varphi(x, \lambda)$ can be represented in the form

$$\varphi(x, \lambda) = (\cos \rho x \, T_1 + \sin \rho x \, T_1^\perp + K_\lambda(\rho))(T_1 + \rho^{-1} T_1^\perp), \quad (2.1)$$

where

$$K_\lambda(\rho) = \int_{-\pi}^{\pi} \mathcal{K}(x, t) \exp(i \rho t) \, dt,$$
the kernel $\mathcal{K}(x, .)$ belongs to $L_2((-x, x); \mathbb{C}^m \times m)$ for each fixed $x \in [0, \pi]$ and the norm \(\|\mathcal{K}(x, .)\|_{L_2((-x, x); \mathbb{C}^m \times m)}\) is bounded uniformly by $x \in (0, \pi]$. Using (2.1) and the analogous relation for $\varphi^{(1)}(x, \lambda)$, we have proved the following proposition in [44].

**Proposition 2.1** The spectrum of $L$ is a countable set of real eigenvalues \(\{\lambda_{nk}\}_{(n, k) \in J}\), counted with their multiplicities and numbered in non-decreasing order: $\lambda_{nk1} \leq \lambda_{nk2}$ if $(n_1, k_1) < (n_2, k_2)$. The following asymptotic relation holds:

\[
\rho_{nk} := \sqrt{\lambda_{nk}} = n + r_k + \varepsilon_{nk}, \quad (n, k) \in J, \quad \{\varepsilon_{nk}\} \in l_2, \quad (2.2)
\]

where

\[
J := \{(n, k) : n \in \mathbb{N}, k = 1, m\}
\]

\[\cup \{ (0, k) : k = p^\perp + 1, m \}, \quad p^\perp := \dim(\text{Ker} T_1 \cap \text{Ker} T_2), \quad (2.3)\]

\[\{r_k\}_{k=1}^m \text{ are the zeros of the function } \det(W^0(\rho)) \text{ on } [0, 1), 0 \leq r_1 \leq r_2 \leq \cdots \leq r_m < 1, \quad \text{W}^0(\rho) := (T_2 T_1 + T_2^\perp T_1^\perp) \sin \rho \pi + (T_2^\perp T_1 - T_2 T_1^\perp) \cos \rho \pi. \quad (2.4)\]

The Weyl solution of $L$ is the matrix solution $\Phi(x, \lambda)$ of Eq. (1.1) satisfying the boundary conditions $V_1(\Phi) = I$, $V_2(\Phi) = 0$. The matrix function $M(\lambda) := T_1 \Phi(0, \lambda) + T_2^\perp \Phi^{(1)}(0, \lambda)$ is called the Weyl matrix of $L$. The matrix functions $M(\lambda)$ and $\Phi(x, \lambda)$ for each fixed $x \in [0, \pi]$ are meromorphic in $\lambda$. All their singularities are the simple poles at $\lambda = \lambda_{nk}$, $(n, k) \in J$. Denote

\[
\alpha_{nk} := \text{Res}_{\lambda=\lambda_{nk}} M(\lambda), \quad (n, k) \in J.
\]

The matrices \(\{\alpha_{nk}\}_{(n, k) \in J}\) are called the weight matrices and the collection \(\{\lambda_{nk}, \alpha_{nk}\}_{(n, k) \in J}\) is called the spectral data of $L$.

Let $\lambda_{nk1} = \lambda_{nk2} = \cdots = \lambda_{nk_j}$ be a group of multiple eigenvalues maximal by inclusion, $(n_1, k_1) < (n_2, k_2) < \cdots < (n_r, k_r)$. Clearly, $\alpha_{nk1} = \alpha_{nk2} = \cdots = \alpha_{nk_r}$. Define $\alpha'_{nk1} := \alpha_{nk1}$, $\alpha'_{nkj} := 0, j = 2, r$. We obtain the sequence of matrices \(\{\alpha'_{nk}\}_{(n, k) \in J}\).

**Proposition 2.2** The weight matrices are Hermitian non-negative definite: $\alpha_{nk} = \alpha_{nk}^* \geq 0$, $(n, k) \in J$. For each $(n, k) \in J$, rank($\alpha_{nk}$) equals the multiplicity of the eigenvalue $\lambda_{nk}$. Furthermore, the asymptotic relation holds:

\[
\alpha_n^{(k)} := \sum_{r_s \in J_k} \alpha'_{nk} = \frac{2}{\pi} (T_1 + n T_1^\perp)(A_k + K_{nk})(T_1 + n T_1^\perp), \quad n \geq 1, \quad k \in J, \quad (2.5)
\]

where

\[
J := \{1\} \cup \{k = \overline{2, m} : r_k \neq r_{k-1}\}, \quad J_k := \{s = \overline{1, m} : r_s = r_k\}, \quad \{\|K_{nk}\|\} \in l_2,
\]

\[
A_k := \pi \text{Res}_{\rho=r_k} (W^0(\rho))^{-1} U^0(\rho),
\]

\[
U^0(\rho) := (T_2 T_1 + T_2^\perp T_1^\perp) \cos \rho \pi + (T_2 T_1^\perp - T_2^\perp T_1) \sin \rho \pi,
\]
The matrices \( \{A_k\}_{k \in J} \) are orthogonal projection matrices having the following properties:

\[
\text{rank}(A_k) = |J_k|, \quad A_k A_s = 0, \; k \neq s, \quad \sum_{k \in J} A_k = I.
\]

Consider a group of multiple eigenvalues \( \lambda_{n_1 k_1} = \lambda_{n_2 k_2} = \cdots = \lambda_{n_r k_r} \), maximal by inclusion. By Proposition 2.2, we have \( \text{rank}(\alpha_{n_1 k_1}) = r \), so \( \text{Ran}\alpha_{n_1 k_1} \) is an \( r \)-dimensional subspace in \( \mathbb{C}^m \). Choose a basis \( \{\chi_{n j k j}\}_{j=1}^r \) in this subspace. This choice is non-unique. Proposition 2.3 is valid for any choice of the basis. Thus, we have defined the vector sequence \( \{\chi_{nk}\}_{(n,k) \in J} \). Consider the sequence of vector functions

\[
\mathcal{X} := \{X_{nk}\}_{(n,k) \in J}, \quad X_{nk}(x) := \left( \cos(\rho_{nk} x) T_1 + \frac{\sin(\rho_{nk} x)}{\rho_{nk}} T_1^\perp \right) \chi_{nk}.
\]  

(2.6)

**Proposition 2.3** *The sequence \( \mathcal{X} \) is complete in \( L_2((0, \pi); \mathbb{C}^m) \).*

Proposition 2.3 immediately follows from [44, Theorem 5.1], which asserts the completeness of the following sequence \( \mathcal{Y} \). Put

\[
T_{nk} := \begin{cases} 
T_1 + \rho_{nk} T_1^\perp, & \rho_{nk} \neq 0, \\
I, & \rho_{nk} = 0,
\end{cases} \quad B_{nk} := \frac{\pi}{2} T_{nk}^{-1} \alpha_{nk} T_{nk}^{-1}, \quad (n, k) \in J.
\]

Clearly, \( \text{rank}(B_{nk}) \) equals the multiplicity of the eigenvalue \( \lambda_{nk} \).

For any group of multiple eigenvalues \( \lambda_{n_1 k_1} = \lambda_{n_2 k_2} = \cdots = \lambda_{n_r k_r} \), considered above, choose an orthonormal basis \( \{\varepsilon_{n j k j}\}_{j=1}^r \) in the \( r \)-dimensional subspace \( \text{Ran} \alpha_{n_1 k_1} \). Thus, we have defined the vector sequence \( \{\varepsilon_{nk}\}_{(n,k) \in J} \). Define

\[
\mathcal{Y} := \{Y_{nk}\}_{(n,k) \in J}, \quad Y_{nk}(x) := \begin{cases} 
(\cos(\rho_{nk} x) T_1 + \sin(\rho_{nk} x) T_1^\perp) \varepsilon_{nk}, & \rho_{nk} \neq 0, \\
(T_1 + x T_1^\perp) \varepsilon_{nk}, & \rho_{nk} = 0.
\end{cases}
\]  

(2.7)

Clearly, the completeness of \( \mathcal{X} \) is equivalent to the completeness of \( \mathcal{Y} \) independently of the choice of the bases in the corresponding subspaces.

Now we turn to discuss the following inverse spectral problem.

**Inverse Problem 2.4** *Given the spectral data \( \{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J} \), find \( \sigma, T_1, T_2, H_2 \).*

Along with the problem \( L \), we consider the problem \( \tilde{L} = L(\tilde{\sigma}, \tilde{T}_1, \tilde{T}_2, \tilde{H}_2) \) of the same form but with different coefficients. We agree that if a symbol \( \gamma \) denotes an object related to \( L \), then the symbol \( \tilde{\gamma} \) with tilde denotes the similar object related to \( \tilde{L} \). Note that the quasi-derivatives for these two problems are supposed to be different: \( Y^{[1]} = Y' - \sigma Y \) for \( L \) and \( Y^{[1]} = Y' - \tilde{\sigma} Y \) for \( \tilde{L} \). In [44], the following uniqueness theorem has been obtained.

**Proposition 2.5** *If \( \lambda_{nk} = \tilde{\lambda}_{nk}, \alpha_{nk} = \tilde{\alpha}_{nk}, (n, k) \in J, J = \tilde{J}, \) then

\[
\sigma(x) = \tilde{\sigma}(x) + H_1^\circ \quad \text{a.e. on} \; (0, \pi), \quad T_1 = \tilde{T}_1, \quad T_2 = \tilde{T}_2, \quad H_2 = \tilde{H}_2 - T_2 H_1^\circ T_2,
\]

(2.8)
where
\[ H_1^\Diamond = (H_1^\Diamond)^* = T_1^\perp H_1^\Diamond T_1^\perp. \] (2.9)

Thus, the spectral data \( \{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J} \) uniquely specify the problem \( L \) up to a transform (2.8) given by an arbitrary matrix \( H_1^\Diamond \) satisfying (2.9). Conversely, the transform (2.8) does not change the spectral data.

The main result of this paper is the following theorem, which provides the characterization of the spectral data \( \{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J} \) of the problem \( L \).

**Theorem 2.6** Let \( T_1, T_2 \in \mathbb{C}^{m \times m} \) be arbitrary fixed orthogonal projection matrices. Then, for a collection \( \{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J} \) to be the spectral data of a problem \( L = L(\sigma, T_1, T_2, H_2) \) in the form (1.1)–(1.2), the following conditions are necessary and sufficient:

(i) \( \lambda_{nk} \in \mathbb{R}, \alpha_{nk} \in \mathbb{C}^{m \times m}, \alpha_{nk} = \alpha_{nk}^* \geq 0, \text{rank}(\alpha_{nk}) \) is equal to the multiplicity of the corresponding value \( \lambda_{nk} \) (i.e., to the number of times \( \lambda_{nk} \) occurs in the sequence), for all \((n, k) \in J \), and \( \alpha_{nk} = \alpha_{ls} \) if \( \lambda_{nk} = \lambda_{ls} \).

(ii) The asymptotic relations (2.2) and (2.5) hold, where \( \{r_k\}_{k=1}^m \) and \( \{A_k\}_{k \in J} \) are defined as in Propositions 2.1 and 2.2, respectively, by using the fixed \( T_1 \) and \( T_2 \).

(iii) The sequence \( X \) defined by (2.6) is complete in \( L_2((0, \pi); \mathbb{C}^m) \).

In Theorem 2.6, the index set \( J \) is defined by the fixed matrices \( T_1 \) and \( T_2 \) via (2.3). We suppose that the matrices \( T_1 \) and \( T_2 \) are initially given, but this is done only for convenience of formulation. By necessity, \( T_1 \) and \( T_2 \) can be uniquely recovered from the spectral data \( \{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J} \) by [44, Algorithm 6.7]. Note that, in condition (iii), the sequence \( X \) depends on the choice of \( \{\chi_{nk}\} \). Obviously, if condition (iii) holds for some choice of \( \{\chi_{nk}\} \), then it holds for any possible choice of \( \{\chi_{nk}\} \).

The necessity part of Theorem 2.6 readily follows from Propositions 2.1–2.3. Therefore, our goal is to prove the sufficiency part. For this purpose, we need one more proposition proved in [44].

**Proposition 2.7** Suppose that the data \( \{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J} \) satisfy the conditions (i)–(iii) of Theorem 2.6. Then the sequence \( Y \) constructed by (2.7) is complete in \( L_2((0, \pi); \mathbb{C}^m) \).

The proof of Theorem 2.6 is based on several auxiliary theorems and lemmas provided in Sects. 3–6. This proof is constructive and yields Algorithm 6.8 for solving Inverse Problem 2.4.

### 3 Main equation

The goal of this section is to reduce the nonlinear Inverse Problem 2.4 to the linear **main equation** in a special Banach space. For construction of this Banach space, we group the eigenvalues with respect to their asymptotics (2.2). In the next sections, the main equation is used for the proof of Theorem 2.6 and for constructive solution of the inverse problem.
Consider the problem $L = L(\sigma, T_1, T_2, H_2)$ with the spectral data $[\lambda_{nk}, \alpha_{nk}]_{(n,k) \in J}$. Without loss of generality, we may assume that $\lambda_{nk} \geq 0$ and $\rho_{nk} = \sqrt{\lambda_{nk}} \geq 0$, $(n, k) \in J$. One can easily achieve this condition by a shift:

$$\sigma(x) := \sigma(x) + cx, \quad H_2 := H_2 - c\pi T_2, \quad \lambda_{nk} := \lambda_{nk} + c, \quad c > 0.$$ 

Fix the model problem $\tilde{L} := L(0, T_1, T_2, 0)$. We have

$$\tilde{\varphi}(x, \lambda) = \cos \rho x T_1 + \frac{\sin \rho x}{\rho} T_1^{\perp}, \quad \tilde{\rho}_{nk} = n + r_k, \quad (3.1)$$

$$\tilde{\alpha}_{nk} = \begin{cases} \frac{2}{\pi} (T_1 + \tilde{\rho}_{nk} T_1) A_k (T_1 + \tilde{\rho}_{nk} T_1), & \tilde{\rho}_{nk} \neq 0, \\ \frac{1}{\pi} T_1 A_k T_1, & \tilde{\rho}_{nk} = 0. \end{cases} \quad (3.2)$$

Denote $(Z, Y) := ZY^{[1]} - Z^{[1]} Y$. Introduce the notations

$$\tilde{D}(x, \lambda, \mu) := \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\phi}(x, \mu) \rangle}{\lambda - \mu} = \int_0^x \tilde{\varphi}(t, \lambda) \tilde{\phi}(x, \mu) d\mu,$$

$$\lambda_{nk0} := \lambda_{nk}, \quad \lambda_{nk1} := \tilde{\lambda}_{nk}, \quad \rho_{nk0} := \rho_{nk}, \quad \rho_{nk1} := \tilde{\rho}_{nk},$$

$$\alpha_{nk0} := \alpha_{nk}, \quad \alpha_{nk1} := \tilde{\alpha}_{nk}, \quad \alpha'_{nk0} := \alpha'_{nk}, \quad \alpha'_{nk1} := \tilde{\alpha}'_{nk}. \quad (3.3)$$

Using the contour integration in the $\lambda$-plane (see [46]), we prove the following lemma.

**Lemma 3.1** The following relation holds

$$\tilde{\varphi}(x, \lambda_{nki}) = \varphi(x, \lambda_{nki}) + \sum_{(l, s) \in J} (\varphi(x, \lambda_{ls0}) \alpha'_{ls0} \tilde{D}(x, \lambda_{ls0}, \lambda_{nki})$$

$$- \varphi(x, \lambda_{ls1}) \alpha'_{ls1} \tilde{D}(x, \lambda_{ls1}, \lambda_{nki})), \quad (3.4)$$

for $(n, k) \in J$, $i = 0, 1$. The series converges in the sense $\lim_{N \to \infty} \sum_{l \leq N} (\ldots)$ absolutely and uniformly by $x \in [0, \pi]$.

It is inconvenient to use (3.4) as the main equation of the inverse problem, since the series in (3.4) only converges “with brackets”. Below we transform (3.4) into a linear equation in a specially constructed Banach space.

Let $J := \{ j_k \}_{k=1}^{|J|}$. Divide the square roots $\{ \rho_{nki} \}$ of the eigenvalues into collections (multisets) as follows:

$$G_1 := \{ \rho_{nki} : (n, k) \in J, n \leq n_0, i = 0, 1 \},$$

$$G_{|J|+q+s+1} := \{ \rho_{nki} : n = n_0 + q + 1, r_k = r_j, i = 0, 1 \}, \quad q \geq 0, \quad s = 1, |J|, \quad (3.5)$$

In view of asymptotics (2.2), we can choose and fix $n_0$ such that $G_n \cap G_k = \emptyset$ for $n \neq k$. 
For any multiset \( G \) of real numbers, let \( B(G) \) be the finite-dimensional space of matrix functions \( f : G \to \mathbb{C}^{m \times m} \) such that \( f(\rho) = f(\theta) \) if \( \rho = \theta \). The norm in \( B(G) \) is defined as follows:

\[
\|f\|_{B(G)} = \max\left\{ \max_{\rho \in G} \|f(\rho)\|, \max_{\rho \neq \theta} |\rho - \theta|^{-1} \|f(\rho) - f(\theta)\| \right\}.
\] (3.6)

Introduce the Banach space \( \mathcal{B} \) of infinite sequences:

\[
\mathcal{B} := \{ f = \{ f_n \}_{n \geq 1} : f_n \in B(G_n), n \geq 1, \|f\|_{\mathcal{B}} := \sup_{n \geq 1} \|f_n\|_{B(G_n)} < \infty \}. \quad (3.7)
\]

For \((n, k) \in J, i = 0, 1\), denote

\[
T_{nki} := \begin{cases}
T_1 + T_1^\perp \rho_{nki}, & \rho_{nki} \neq 0, \\
I, & \rho_{nki} = 0,
\end{cases}
\phi_{nki}(x) := \phi(x, \lambda_{nki}) T_{nki}.
\]

Put \( \phi(x) := [\phi_n(x)]_{n=1}^{\infty}, \phi_n(x)(\rho_{l_{s_j}}) := \phi_{l_{s_j}}(x) \) for \( \rho_{l_{s_j}} \in G_n \). Analogously, define \( \tilde{\phi}(x) \) replacing \( \phi \) by \( \tilde{\phi} \). Using relation (2.1), we obtain the estimates

\[
\|\phi_{nki}(x)\| \leq C, \quad \|\phi_{nki}(x) - \phi_{l_{s_j}}(x)\| \leq C |\rho_{nki} - \rho_{l_{s_j}}|, \quad \rho_{nki}, \rho_{l_{s_j}} \in G_q,
\]

for \( q \geq 1, x \in [0, \pi] \). Hence, \( \phi(x) \in \mathcal{B} \) and, similarly, \( \tilde{\phi}(x) \in \mathcal{B} \) for each fixed \( x \in [0, \pi] \). In addition, \( \phi(x) \) and \( \tilde{\phi}(x) \) are uniformly bounded in \( \mathcal{B} \) with respect to \( x \in [0, \pi] \).

For each fixed \( x \in [0, \pi] \), define the linear operator \( \tilde{R}(x) : \mathcal{B} \to \mathcal{B}, \tilde{R}(x) = [\tilde{R}_{k,n}(x)]_{k,n=1}^{\infty} \), acting on an element \( f = \{ f_n \}_{n=1}^{\infty} \) of \( \mathcal{B} \) by the following rule:

\[
(f \tilde{R}(x))_n = \sum_{k=1}^{\infty} f_k \tilde{R}_{k,n}(x), \quad \tilde{R}_{k,n}(x) : B(G_k) \to B(G_n), \quad (3.8)
\]

\[
(f_k \tilde{R}_{k,n}(x))(\rho_{\eta_{q_i}}) = \sum_{\rho_{l_{s_j}} \in G_k} (-1)^j f_k(\rho_{l_{s_j}}) T_{l_{s_j}}^{-1} \alpha_{l_{s_j}}' D(x, \lambda_{l_{s_j}}, \lambda_{\eta_{q_i}}) T_{\eta_{q_i}} \rho_{\eta_{q_i}} \in G_n. \quad (3.9)
\]

Here we put operators to the right of operands to show the order of matrix multiplication.

Theorem 3.2 The series (3.8) converges in the \( B(G_n) \)-norm. For each fixed \( x \in [0, \pi] \), the operator \( \tilde{R}(x) \) is bounded and can be approximated by finite-dimensional operators in the norm \( \|\cdot\|_{\mathcal{B} \to \mathcal{B}} \).

Theorem 3.2 is proved in Sect. 4. Taking the above definitions into account, we rewrite relation (3.4) in the form

\[
\phi(x)(I + \tilde{R}(x)) = \tilde{\phi}(x), \quad x \in [0, \pi]. \quad (3.10)
\]
where $I$ is the unit operator in $\mathcal{B}$. For each fixed $x \in [0, \pi]$, relation (3.10) is a linear equation with respect to $\phi(x)$ in the Banach space $\mathcal{B}$. Note that $\tilde{\phi}(x)$ and $\tilde{R}(x)$ are constructed by the spectral data $\{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J}$ and by the model problem $\tilde{L}$, while the unknown element $\phi(x)$ is related to the problem $L$. Further, the solution of the main Eq. (3.10) is used for constructive solution of Inverse Problem 2.4. Therefore, we call (3.10) the main equation of the inverse problem.

### 4 Estimates

In this section, we investigate properties of the operator $\tilde{R}(x)$ and obtain the estimates needed in further proofs. It is supposed that $\tilde{R}(x)$ is the operator constructed by the collection $\{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J}$ satisfying the asymptotics (2.2) and (2.5) and by the problem $\tilde{L} = L(0, T_1, T_2, 0)$. We emphasize that $\{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J}$ are not assumed to be the spectral data of some problem $L$. This allows us to use the results of this section in the proof of the sufficiency in Theorem 2.6.

For $n \geq 1$, denote

$$\hat{\alpha}(G_n) := \sum_{\rho \sim j \in G_n} (-1)^j T_{\bar{s}j}^{-1} \alpha_{\bar{s}j} T_{\bar{s}j}^{-1},$$

$$\xi_n := \sum_{\rho, \theta \in G_n} |\rho - \theta| + \|\hat{\alpha}(G_n)\|.$$  

(4.1)

(4.2)

It follows from the asymptotic formulas (2.2) and (2.5) that

$$\Xi := \left( \sum_{n=1}^{\infty} \xi_n^2 \right)^{1/2} < \infty.$$  

(4.3)

Put

$$\tilde{D}_T(x, \rho, \theta) := (T_1 + \rho T_1^\perp) \tilde{D}(x, \rho^2, \theta^2)(T_1 + \theta T_1^\perp).$$

Substituting (3.1) into (3.3) and using (2.2), (4.2), we obtain the following lemma.

**Lemma 4.1** For $x \in [0, \pi]$, $n, k \geq 1$, $\rho, \zeta \in G_n$, $\theta, \chi \in G_k$, the following estimates hold

$$\|\tilde{D}_T(x, \rho, \theta)\| \leq \frac{C}{|n - k| + 1},$$

$$\|\tilde{D}_T(x, \rho, \theta) - \tilde{D}_T(x, \rho, \chi)\| \leq \frac{C \xi_k}{|n - k| + 1},$$

$$\|\tilde{D}_T(x, \rho, \theta) - \tilde{D}_T(x, \rho, \chi) - \tilde{D}_T(x, \zeta, \theta) + \tilde{D}_T(x, \zeta, \chi)\| \leq \frac{C \xi_n \xi_k}{|n - k| + 1}.$$
For \( x \in [0, \pi] \), \( n \geq 1 \), \( \rho, \zeta \in G_n \), \( \theta \in \mathbb{C} \), we have

\[
\| \tilde{D}(x, \rho, \theta) \| \leq \frac{C \exp(|\text{Im} \theta| x)}{\theta - m_n + 1}, \quad \| \tilde{D}(x, \rho, \theta) - \tilde{D}(x, \zeta, \theta) \| \leq \frac{C \xi_n \exp(|\text{Im} \theta| x)}{|\theta - m_n| + 1},
\]

where \( m_n = l + r_s \), \((l, s) : \rho_{ls} \in G_n\). In all the estimates, the constant \( C \) does not depend on \( n, k, x \), etc.

**Lemma 4.2** For \( x \in [0, \pi] \), the following estimates hold

\[
\| \tilde{R}_{k,n}(x) \|_{B(G_k) \to B(G_n)} \leq \frac{C \xi_k}{|n - k| + 1}, \quad n, k \geq 1, \tag{4.4}
\]

\[
\| \tilde{R}(x) \|_{\mathfrak{B} \to \mathfrak{B}} \leq C \Xi, \tag{4.5}
\]

where the constant \( C \) does not depend on \( n, k \) and \( x \).

**Proof** Estimate (4.4) is proved by using (3.6), (3.9), and the summation rule

\[
\sum_{u=1}^{v} a_u b_u c_u = \sum_{u=1}^{v} (a_u - a_1) b_u c_u + a_1 \sum_{u=1}^{v} b_u (c_u - c_1) + a_1 \sum_{u=1}^{v} b_u c_1. \tag{4.6}
\]

We put

\[
a_u = f_k(\rho_{lsj}), \quad b_u = (-1)^j T^{-1}_{lsj} \omega_{lsj} T^{-1}_{lsj}, \quad c_u = T_{lsj} \tilde{D}(x, \lambda_{lsj}, \lambda_{qi}) T_{qi},
\]

apply the estimates \( \| f_k \|_{B(G_k)} \leq C \), (4.2) and Lemma 4.1, and so arrive at (4.4). Relations (3.7) and (3.8) yield

\[
\| \tilde{R}(x) \|_{\mathfrak{B} \to \mathfrak{B}} \leq \sup_{n \geq 1} \sum_{k=1}^{\infty} \| \tilde{R}_{k,n}(x) \|_{B(G_k) \to B(G_n)}.
\]

Using (4.3) and (4.4), we arrive at (4.5). \( \square \)

**Proof of Theorem 3.2** It readily follows from (4.3), (4.4), and (4.5) that the series (3.8) converges in \( B(G_n) \)-norm and the operator \( \tilde{R}(x) \) is bounded. Define the finite-dimensional operators

\[
\tilde{R}^N(x) = [\tilde{R}_{k,n}(x)]_{k=1}^{N}, \quad \tilde{R}^N_{k,n} = \begin{cases} \tilde{R}_{k,n}(x), & k \leq N, \\ 0, & k > N, \end{cases}, \quad N \geq 1.
\]

Using (4.4), it is easy to show that the sequence \( \{ \tilde{R}^N(x) \} \) converges to \( \tilde{R}(x) \) in the norm \( \| . \|_{\mathfrak{B} \to \mathfrak{B}} \) for each fixed \( x \in [0, \pi] \). \( \square \)
Further we need the following auxiliary proposition, which easily follows from asymptotics (2.2) and the Riesz-basicity of the sequences \( \{\cos(n + \zeta_n)x\}_{n=0}^{\infty}, \{\sin(n + \zeta_n)x\}_{n=1}^{\infty} \) in \( L_2(0, \pi) \), \( \{\zeta_n\} \in l_2 \), \( n + \zeta_n \neq k + \zeta_k \) for \( n \neq k \) (see [47]).

**Proposition 4.3** (i) Let \( \{\zeta_{nki}\} \) be an arbitrary sequence from \( l_2 \). Then the series

\[
F_c(x) := \sum_{n,k,i} \zeta_{nki} \cos(\rho_{nki}x), \quad F_s(x) := \sum_{n,k,i} \zeta_{nki} \sin(\rho_{nki}x)
\]

converge in \( L_2(0, \pi) \) and

\[
\|F_c\|_{L_2(0,\pi)}, \|F_s\|_{L_2(0,\pi)} \leq C \|\{\zeta_{nki}\}\|_{l_2},
\]

where the constant \( C \) depends only on \( \{\rho_{nki}\} \) and not on \( \{\zeta_{nki}\} \).

(ii) Let \( F(x) \) be arbitrary function from \( L_2(0, \pi) \). Put

\[
\zeta_{c,nki} = \int_0^\pi F(x) \cos(\rho_{nki}x) \, dx, \quad \zeta_{s,nki} = \int_0^\pi F(x) \sin(\rho_{nki}x) \, dx.
\]

Then the sequences \( \{\zeta_{c,nki}\} \) and \( \{\zeta_{s,nki}\} \) belong to \( l_2 \) and

\[
\|\{\zeta_{c,nki}\}\|_{l_2}, \|\{\zeta_{s,nki}\}\|_{l_2} \leq C \|F\|_{L_2(0,\pi)},
\]

where the constant \( C \) depends only on \( \{\rho_{nki}\} \) and not on \( F \).

In (i) and (ii), the indices \((n, k, i)\) run over the set: \((n, k) \in J, i = 0, 1\).

For convenience, for any sequence \( f = \{f_n\}_{n=1}^\infty \in \mathcal{B} \), we denote \( f_{lsj} := f_n(\rho_{lsj}) \), where \( n \) is such that \( \rho_{lsj} \in G_n \).

**Lemma 4.4** Suppose that \( f \in \mathcal{B} \) and \( g(x) = \tilde{R}(x)f \). Then the corresponding sequence \( \{\|g_{nki}(x)\|\} \) belongs to \( l_2 \) for each fixed \( x \in [0, \pi] \) and

\[
\|\{\|g_{nki}(x)\|\}\|_{l_2} \leq C \|f\|_{\mathcal{B}},
\]

uniformly by \( x \in [0, \pi] \).

**Proof** Substituting (3.1) into (3.3), we obtain

\[
\tilde{D}(x, \rho_{lsj}, \rho_{nki})
= T_{lsj}^{-1} \int_0^x (\cos(\rho_{lsj}t) \cos(\rho_{nki}t) T_1 + \sin(\rho_{lsj}t) \sin(\rho_{nki}t) T_1^\perp) \, dx \, T_{nki}^{-1}.
\]

(4.7)

For simplicity, throughout this proof we assume that \( \rho_{nki} \neq 0 \) and \( \rho_{lsj} \neq 0 \). The opposite case requires minor technical changes. Using (3.9) and (4.7), we derive

\[
\tilde{g}_{nki}(x)T_1 = \int_0^x F(t) \cos(\rho_{nki}t) \, dt,
\]

(4.8)

\[
F(t) := \sum_{l,s,j} (-1)^j f_{lsj} T_{lsj}^{-1} a'_{lsj} T_{lsj}^{-1} T_1 \cos(\rho_{lsj}t).
\]

(4.9)
Using the summation rule \((4.6)\) in \((4.9)\), relations \((2.2)\), \((4.2)\), and Proposition \(4.3(i)\), we prove that \(F \in L_2((0, \pi); \mathbb{C}^{m \times m})\) and \(\|F\|_{L_2} \leq C\|f\|_B\). Applying Proposition \(4.3(ii)\) to \((4.8)\), we show that \(\{\|g_{nki}(x)T_1\|\} \in l_2\) for each fixed \(x \in [0, \pi]\) and the \(l_2\)-norm of this sequence does not exceed \(C\|F\|_{L_2}\), where \(C\) does not depend on \(x \in [0, \pi]\). Similar arguments are valid for \(g_{nki}T_1^\perp\). This concludes to proof. \(\square\)

**Lemma 4.5** Suppose that \(f \in B\) and \(g(x) = \tilde{R}(x)f\). Let indices \(q \in J\), \(k, s \in J_q\), \(i, j \in \{0, 1\}\) be fixed. Then the sequence \(\{\|g_{nki} - g_{nsj}\|\}\) belongs to \(l_1\) for each fixed \(x \in [0, \pi]\) and

\[
\{\|g_{nki}(x) - g_{nsj}(x)\|\}_1 \leq C\|f\|_B
\]

uniformly by \(x \in [0, \pi]\).

**Proof** For fixed \(k, s, i, j\) satisfying the conditions of the lemma, we have

\[
\cos(\rho_{nki}t) - \cos(\rho_{nsj}t) = \gamma_n t \sin(n + r_k)t + \zeta_n(t),
\]

\[
\{\gamma_n\} \in l_2, \quad \sum_n \max_{t \in [0, \pi]} |\zeta_n(t)| \leq C, \tag{4.10}
\]

where \(\gamma_n\) does not depend on \(t, i \in [0, \pi]\). Using \((4.8), (4.10),\) and Proposition \(4.3,\) we obtain

\[
(g_{nki}(x) - g_{nsj}(x))T_1 = \int_0^x F(t)(\cos(\rho_{nki}t) - \cos(\rho_{nsj}t)) \, dt = \gamma_n K_n(x) + Z_n(x),
\]

where

\[
\{\|K_n(x)\|\}_{l_2} \leq C\|F\|_{L_2}, \quad \{\|Z_n(x)\|\}_{l_2} \leq C\|F\|_{L_2} \max_{t \in [0, \pi]} |\zeta_n(t)|
\]

uniformly by \(x \in [0, \pi]\). Taking the estimate \(\|F\|_{L_2} \leq C\|f\|_B\) into account, we obtain the assertion of the lemma for the sequence \(\{\|(g_{nki}(x) - g_{nsj}(x))T_1\|\}\). The sequence \(\{\|(g_{nki}(x) - g_{nsj}(x))T_1^\perp\|\}\) can be studied similarly. \(\square\)

## 5 Solvability of main equation

In this section, we suppose that the collection \(\{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J}\) satisfies the conditions of Theorem \(2.6\) and prove the unique solvability of the main equation \((3.10)\).

**Theorem 5.1** For each fixed \(x \in [0, \pi]\), the operator \((\mathcal{I} + \tilde{R}(x)):\mathcal{B} \rightarrow \mathcal{B}\) has a bounded inverse, so the main equation \((3.10)\) has a unique solution \(\phi(x) \in \mathcal{B}\).

**Proof** Fix \(x \in [0, \pi]\). By virtue of Theorem \(3.2\), the operator \(\tilde{R}(x)\) can be approximated by finite-dimensional operators. Therefore, in view of Fredholm’s Theorem, it suffices to prove that the homogeneous equation

\[
\beta(x)(\mathcal{I} + \tilde{R}(x)) = 0, \quad \beta(x) = \{\beta_n(x)\}_{n=1}^\infty \in \mathcal{B}, \tag{5.1}
\]
has the only solution $\beta(x) = 0$ in $\mathcal{B}$. Since $\beta(x) = -\tilde{R}(x)\beta(x)$, Lemmas 4.4 and 4.5 imply

$$\{||\beta_{nki}(x)||\} \in l_2, \quad \{||\beta_{nki}(x) - \beta_{nsj}(x)||\} \in l_1, \quad \text{(5.2)}$$

for fixed $k, s \in J_q, q \in J, i, j \in \{0, 1\}$. Introduce the matrix functions:

$$\gamma(x, \lambda) := -\sum_{l,s,j} (-1)^j \beta_{lsj}(x)T_{lsj}^{-1}\alpha_{lsj}' \tilde{D}(x, \lambda_{lsj}, \lambda), \quad \text{(5.3)}$$

$$\Gamma(x, \lambda) := -\sum_{l,s,j} (-1)^j \beta_{lsj}(x)T_{lsj}^{-1}\alpha_{lsj}' \tilde{E}(x, \lambda_{lsj}, \lambda), \quad \text{(5.4)}$$

$$\tilde{E}(x, \lambda, \mu) := \frac{\langle \tilde{\phi}(x, \lambda), \Phi(x, \mu) \rangle}{\lambda - \mu},$$

$$\mathcal{B}(x, \lambda) := \Gamma(x, \lambda)(\gamma(x, \lambda)). \quad \text{(5.5)}$$

In (5.3) and (5.4), the indices $(l, s, j)$ run over the set: $(l, s) \in J, j = 0, 1$. The matrix function $\gamma(x, \lambda)$ is entire in $\lambda$, while $\Gamma(x, \lambda)$ and $\mathcal{B}(x, \lambda)$ are meromorphic in $\lambda$ with the simple poles $\{\lambda_{nki}\}$. Relation (5.1) implies $\gamma(x, \lambda_{nki}) = \beta_{nki}(x)T_{nki}^{-1}$, $(n, k) \in J$, $i = 0, 1$. Calculations show that

$$\text{Res}_{\lambda = \lambda_{nki}} \mathcal{B}(x, \lambda) = \gamma(x, \lambda_{nki}) \alpha_{nki0}(\gamma(x, \lambda_{nki}))^\ast, \quad \text{Res}_{\lambda = \lambda_{nki1}} \mathcal{B}(x, \lambda) = 0 \quad \text{(5.6)}$$

if $\lambda_{nki0} \neq \lambda_{11}, (n, k) \in J$. The opposite case requires minor changes.

Using (5.3), the summation rule (4.6), (5.2), (4.2), and Lemma 4.1, we obtain

$$\|\gamma(x, \lambda)(T_1 + \rho T_1^\dagger)\| \leq C(x) \exp(|\text{Im} \rho|x) \sum_{n=1}^{\infty} \frac{\theta_n}{|\rho - m_n| + 1}. \quad \text{(5.7)}$$

Here and below, $\rho = \sqrt{\lambda}$, $\arg \rho \in [-\frac{\pi}{2}, \frac{\pi}{2})$, the notation $\{\theta_n\}$ stands for various $l_1$-sequences of non-negative numbers. Analogously to (5.7), we get

$$\|\Gamma(x, \lambda)(\rho T_1 + T_1^\dagger)\| \leq C(x) \exp(-|\text{Im} \rho|x) \sum_{n=1}^{\infty} \frac{\theta_n}{|\rho - m_n| + 1}, \quad \rho \in G_{\delta}, \quad |\rho| \geq \rho^\ast, \quad \text{(5.8)}$$

where

$$G_{\delta} := \{\rho \in \mathbb{C}: |\rho - (n + r_k)| \geq \delta, n \in \mathbb{Z}, k = \overline{1, m}\},$$

$\delta$ and $\rho^\ast$ are some positive reals. Suppose that $\lambda \in Y_{N+r}, Y_{N+r} := \{\lambda \in \mathbb{C}: |\lambda| = (N + r)^2\}$, where $N \in \mathbb{N}, r$ is fixed, $r \neq r_k, k = \overline{1, m}$. Using (5.5), (5.7), and (5.8), we obtain

$$\|\mathcal{B}(x, \lambda)\| \leq \frac{C(x)}{N} \left(\sum_{n=1}^{\infty} \frac{\theta_n}{|N - n| + 1}\right)^2.$$
Consequently,

\[ \| B(x, \lambda) \| \leq \frac{C(x)}{N} f_N, \quad f_N := \sum_{n=1}^{\infty} \frac{\theta_n}{(N - n + 1/2)^2}, \quad \lambda \in \Upsilon_{N+r}. \]

Obviously, \( \{ f_N \} \in l_1. \) This implies

\[ \lim_{N \to \infty} \frac{f_N}{1/N} = 0. \]

Hence, there exists a sequence \( \{ N_k \} \) such that

\[ \max_{\lambda \in \Upsilon_{N_k}} B(x, \lambda) = o(N_k^{-2}), \quad k \to \infty. \]

Therefore,

\[ \lim_{k \to \infty} \int_{\Upsilon_{N_k+r}} B(x, \lambda) d\lambda = 0. \]

Using the Residue Theorem and (5.6), we show that

\[ \sum_{(n,k) \in J} \gamma(x, \lambda_{nk}) \alpha'_{nk}(\gamma(x, \lambda_{nk}))^* = 0. \]

Since \( \alpha'_{nk} = (\alpha'_{nk})^* \geq 0, \) we get

\[ \gamma(x, \lambda_{nk}) \alpha_{nk} = 0, \quad (n,k) \in J. \quad (5.9) \]

It is easy to see that the matrix function \( \gamma(x, \rho^2)T_1 \) is even and \( \gamma(x, \rho^2)\rho T_1^\perp \) is odd. It follows from (5.7), (5.3), (5.2), (4.2), (4.7), and Proposition 4.3 that these matrix functions are \( O(\exp(|\Im \rho|)) \) and belong to \( L_2(\mathbb{R}; \mathbb{C}^{m \times m}). \) Applying the Paley–Wiener Theorem, we obtain the representation

\[ \gamma(x, \lambda) = \int_0^\pi (h(x, t))^* \left( \cos \rho t T_1 + \frac{\sin \rho t}{\rho} T_1^\perp \right) dt, \quad h(x, .) \in L_2((0, \pi); \mathbb{C}^m). \quad (5.10) \]

Combining (5.9) and (5.10), we get \( (h, X_{nk}) = 0 \) for each fixed \( x \in [0, \pi] \) and for all \( (n,k) \in J. \) Since the sequence \( X' = \{ X_{nk} \}_{(n,k) \in J} \) is complete in \( L_2((0, \pi); \mathbb{C}^m), \) it follows that \( h = 0 \) in \( L_2((0, \pi); \mathbb{C}^m) \) for each fixed \( x \in [0, \pi]. \) Consequently, \( \gamma(x, \lambda) \equiv 0 \) and \( \beta(x) = 0, \) so the homogeneous equation (5.1) has the unique solution in \( \mathcal{B}. \) This yields the claim. \( \square \)
6 Proof of sufficiency

In this section, we prove the sufficiency part of Theorem 2.6. Let \( \{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J} \) be a collection satisfying the conditions of Theorem 2.6. Suppose that the Banach space \( \mathcal{B} \), the element \( \hat{\phi}(x) \in \mathcal{B} \), and the operator \( \hat{R}(x) : \mathcal{B} \rightarrow \mathcal{B} \) for each fixed \( x \in [0, \pi] \) are constructed in accordance with Sect. 3. By virtue of Theorem 5.1, the main equation (3.10) has the unique solution \( \phi(x) \in \mathcal{B} \) for each fixed \( x \in [0, \pi] \). Similarly to [46, Lemma 5.3], we obtain the following result.

**Lemma 6.1** The elements \( \{\phi_{nki}(x)\} \) of \( \phi(x) \) can be represented in the form

\[
\phi_{nki}(x) = \cos(n + r_k)x T_1 + \sin(n + r_k)x T_1^\perp + \psi_{nki}(x), \quad (n, k) \in J, \quad i = 0, 1,
\]

where the matrix functions \( \psi_{nki} \) are continuous on \([0, \pi]\), the sequence \( \{\|\psi_{nki}(x)\|\} \) belongs to \( l_2 \) for each fixed \( x \in [0, \pi] \), and the \( l_2 \)-norm of this sequence is uniformly bounded by \( x \in [0, \pi] \).

Construct the matrix function \( \sigma(x) \) and the matrix \( H_2 \) as follows:

\[
\sigma(x) := -2 \sum_{n=1}^{\infty} \left( \sum_{\rho_{lsj} \in G_n} (-1)^j \phi_{lsj}(x) T_{lsj}^{-1} \alpha_{lsj} T_{lsj}^{-1} \hat{\phi}_{lsj}(x) - \frac{1}{2} (T_1 \hat{\alpha}(G_n) T_1 + T_1^\perp \hat{\alpha}(G_n) T_1) \right)
\]

(6.1)

\[
H_2 := 2 \sum_{n=1}^{\infty} \left( \sum_{\rho_{lsj} \in G_n} (-1)^j \phi_{lsj}(\pi) T_{lsj}^{-1} \alpha_{lsj} T_{lsj}^{-1} \hat{\phi}_{lsj}(\pi) - (T_1 \hat{\alpha}(G_n) T_1 + T_1^\perp \hat{\alpha}(G_n) T_1) \right) T_2,
\]

(6.2)

where \( \hat{\alpha}(G_n) \) is defined by (4.1).

Relying on Lemmas 4.5 and 6.1, Proposition 4.3, and relations (4.2), (4.3), we prove the following lemma.

**Lemma 6.2** The series (6.1) and (6.2) converge in \( L_2((0, \pi); \mathbb{C}^{m \times m}) \) and \( \mathbb{C}^{m \times m} \), respectively.

Thus, we have constructed \( \sigma(x) \) and \( H_2 \) by formulas (6.1) and (6.2), respectively. Consider the corresponding boundary value problem \( L = L(\sigma, T_1, T_2, H_2) \) of the form (1.1)–(1.2). It remains to prove the following theorem.

**Theorem 6.3** The values \( \{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J} \) are the spectral data of \( L \).

In order to prove Theorem 6.3, consider the data \( \{\lambda_{nk}^N, \alpha_{nk}^N\}_{(n,k) \in J} \) defined as follows:

\[
\lambda_{nk}^N = \begin{cases} 
\lambda_{nk}, & n \leq N, \\
\tilde{\lambda}_{nk}, & n > N,
\end{cases} \quad \alpha_{nk}^N = \begin{cases} 
\alpha_{nk}, & n \leq N, \\
\tilde{\alpha}_{nk}, & n > N,
\end{cases} \quad N \in \mathbb{N}.
\]

(6.3)

**Lemma 6.4** The collection \( \{\lambda_{nk}^N, \alpha_{nk}^N\}_{(n,k) \in J} \) satisfies conditions (i)–(iii) of Theorem 2.6 for all sufficiently large \( N \).
Proof Conditions (i)–(ii) are obvious, so we focus on the proof of (iii). We have to show that the sequence

\[ \mathcal{Y}^N := \{ Y^N_{nk} \}_{(n,k) \in J}, \quad Y^N_{nk} := \begin{cases} Y_{nk}, & n \leq N, \\ \tilde{Y}_{nk}, & n > N, \end{cases} \]

is complete in \( L_2((0, \pi); \mathbb{C}^m) \) for each sufficiently large \( N \). In view of condition (iii) of Theorem 2.6, the sequence \( \mathcal{X} \) is complete, so \( \mathcal{Y} \) is also complete. By virtue of Proposition 2.7, \( \mathcal{Y} \) is a Riesz basis. Consider the sequence

\[ \mathcal{Y}^N \bullet := \{ Y^N_{nk} \}_{(n,k) \in J}, \quad Y^N_{nk} := \begin{cases} Y_{nk}, & n \leq N, \\ Y^*_{nk}, & n > N, \end{cases} \]

where \( Y^*_{nk} \) is defined similarly to \( Y_{nk} \) (see (2.7)), but with \( \mathcal{E}_{nk} \) replaced by \( \mathcal{E}^*_nk := A_k \mathcal{E}_{nk} \). It is easy to show that

\[ \lim_{N \to \infty} \sum_{(n,k) \in J, n > N} \| Y_{nk} - Y^*_{nk} \|^2_{L_2((0,\pi);\mathbb{C}^m)} = 0. \]

Consequently, the sequence \( \mathcal{Y}^N \bullet \) is a Riesz basis for sufficiently large \( N \), so \( \mathcal{Y}^N \bullet \) is complete in \( L_2((0, \pi); \mathbb{C}^m) \) for such \( N \). It is easy to check that, for each fixed sufficiently large \( n \) and each fixed \( k \in J \), the vector functions \( \{ Y^N_{nk} \}_{s \in J_k} \) are linear combinations of \( \{ \tilde{Y}_{ns} \}_{s \in J_k} \). This implies that \( \mathcal{Y}^N \) is also complete in \( L_2((0, \pi); \mathbb{C}^m) \).

By using \( \{ \lambda^N_n, \alpha^N_n \}_{(n,k) \in J} \) and the model problem \( \tilde{L} = L(0, T_1, T_2, 0) \), construct the element \( \tilde{\phi}^N(x) \) and the operator \( \tilde{R}^N(x) \) similarly to \( \tilde{\phi}(x) \) and \( \tilde{R}(x) \), respectively. Let \( \phi^N(x) \) be the solution of the main equation

\[ \phi^N(x)(\mathcal{I} + \tilde{R}^N(x)) = \tilde{\phi}^N(x), \quad x \in [0, \pi], \quad (6.4) \]

analogous to (3.10). By virtue of Theorem 5.1, the solution of (6.4) exists and is unique. Obviously, for the matrix sequences \( \{ \phi^N_{nk1}(x) \} \) and \( \{ \tilde{\phi}^N_{nk1}(x) \} \) corresponding to \( \phi(x) \) and \( \tilde{\phi}(x) \), respectively, the following relations hold: \( \tilde{\phi}^N_{nk1}(x) = \tilde{\phi}^N_{nk1}(x) \) for \( n \leq N, \phi^N_{nk0}(x) = \phi^N_{nk1}(x), \tilde{\phi}^N_{nk0}(x) = \tilde{\phi}^N_{nk1}(x) \) for \( n > N \). Taking these relations into account, similarly to (6.1) and (6.2), we define

\[ \sigma^N(x) := -2 \sum_{n=1}^{g(N)} \left( \sum_{\rho \in G_n} (-1)^j \phi^N_{ljsj}(\rho) T_{ljsj}^{-1} \alpha_{ljsj} T_{ljsj}^{-1} \phi^N_{ljsj}(\rho) - \frac{1}{2} (T_1 \tilde{\alpha}(G_n) T_1 + T_1 \tilde{\alpha}(G_n) T_1) \right) \]

\[ H^N_2 := T_2 \sum_{n=1}^{g(N)} \left( \sum_{\rho \in G_n} (-1)^j \phi^N_{ljsj}(\rho) T_{ljsj}^{-1} \alpha_{ljsj} T_{ljsj}^{-1} \phi^N_{ljsj}(\rho) - (T_1 \tilde{\alpha}(G_n) T_1 + T_1 \tilde{\alpha}(G_n) T_1) \right) T_2, \]

\[ (6.5) \]

\[ (6.6) \]
where \( g(N) \) is such that
\[
\bigcup_{n=1}^{g(N)} G_n = \{ \rho[l,s] : (l,s) \in J, l \leq N, j = 0, 1 \}.
\]

Here and above, we assume that \( N \) is large enough.

Let us show that \( \{\lambda_{nk}^N, \alpha_{nk}^N\}_{(n,k)\in J} \) are the spectral data of the problem \( L^N := L(\sigma^N, T_1, T_2, H_2^N) \), i.e., prove Theorem 6.3 for \( \{\lambda_{nk}^N, \alpha_{nk}^N\}_{(n,k)\in J} \). This special case is much easier for investigation than the general case, since the main equation (6.4) in the element-wise form contains a finite sum:
\[
\tilde\phi_{nki}^N(x) = \phi_{nki}^N(x) + \sum_{l,s,j : l \leq N} (-1)^j \phi_{lsj}^N(x) T_{lsj}^{-1} \alpha_{lsj}^j \tilde{D}(x, \lambda_{lsj}, \lambda_{nki}) T_{nki}.
\]

Therefore, one can show that \( \tilde{R}^N(x) \) is twice continuously differentiable with respect to \( x \in [0, \pi] \), and so does \( \phi^N(x) \) (see the proof of Lemma 1.6.9 from [4] for details). Moreover, the sums (6.5) and (6.6) are finite, so we do not need to care of their convergence. Define the matrix functions
\[
\phi^N(x, \lambda) := \tilde{\phi}(x, \lambda) - \sum_{l,s,j : l \leq N} (-1)^j \phi_{lsj}^N(x) T_{lsj}^{-1} \alpha_{lsj}^j \tilde{D}(x, \lambda_{lsj}, \lambda),
\]
\[
\Phi^N(x, \lambda) := \tilde{\Phi}(x, \lambda) - \sum_{l,s,j : l \leq N} (-1)^j \phi_{lsj}^N(x) T_{lsj}^{-1} \alpha_{lsj}^j \tilde{E}(x, \lambda_{lsj}, \lambda),
\]
\[
\sigma_*^N(x) := \sigma^N(x) - C^N, \quad H_{2,*}^N := H_2^N + T_2 C^N T_2, \quad C^N := T_1^\perp \sum_{n=1}^{g(N)} \hat{a}(G_n) T_1^\perp.
\]

Calculations yield the following lemma.

**Lemma 6.5** \( \phi^N(., \lambda) \in C^2([0, \pi]; \mathbb{C}^{m \times m}) \) for each fixed \( \lambda \in \mathbb{C} \), \( \Phi^N(., \lambda) \in C^2([0, \pi]; \mathbb{C}^{m \times m}) \) for each fixed \( \lambda \neq \lambda_{nki} \), and \( \sigma_*^N \in C^1([0, \pi]; \mathbb{C}^{m \times m}) \). Moreover, the following relations hold:
\[
-\frac{d^2}{dx^2} \phi^N(x, \lambda) + \frac{d}{dx} \sigma_*^N(x) \phi^N(x, \lambda) = \lambda \phi^N(x, \lambda), \quad x \in (0, \pi),
\]
\[
\phi^N(0, \lambda) = T_1, \quad \frac{d}{dx} \phi^N(0, \lambda) - \sigma_*^N(0) \phi^N(0, \lambda) = T_1^\perp,
\]
\[
-\frac{d^2}{dx^2} \Phi^N(x, \lambda) + \frac{d}{dx} \sigma_*^N(x) \Phi^N(x, \lambda) = \lambda \Phi^N(x, \lambda), \quad x \in (0, \pi),
\]
\[
T_1 \left( \frac{d}{dx} \Phi^N(0, \lambda) - \sigma_*^N(0) \Phi^N(0, \lambda) \right) - T_1^\perp \Phi^N(0, \lambda) = 0,
\]
\[
T_2 \left( \frac{d}{dx} \Phi^N(\pi, \lambda) - (\sigma_*^N(\pi) + H_{2,*}^N) \Phi^N(\pi, \lambda) \right) - T_2^\perp \Phi^N(\pi, \lambda) = 0.
\]
Lemma 6.5 implies that \( \varphi^N(x, \lambda) \) is the \( \varphi \)-type solution and \( \Phi^N(x, \lambda) \) is the Weyl solution of the boundary value problem \( L^N_* := L(\sigma^N_*, T_1, T_2, H^N_2) \). Hence, the Weyl matrix of \( L^N_* \) has the form
\[
M^N(\lambda) := T_1 \Phi^N(0, \lambda) + T_1^L \left( \frac{d}{dx} \Phi^N(0, \lambda) - \sigma^N_*(0) \Phi(0, \lambda) \right).
\]
Using (6.7), we derive
\[
M^N(\lambda) = \tilde{M}(\lambda) + \sum_{l,s,j: l \leq N} \frac{(-1)^j \alpha'_{lsj}}{\lambda - \lambda_{lsj}}. \tag{6.8}
\]
Recall that the Weyl matrix \( \tilde{M}(\lambda) \) has the poles \( \{\tilde{\lambda}_{nk}(n,k) \}_{(n,k) \in J} \) and the corresponding residues \( \{\tilde{\alpha}_{nk}(n,k) \}_{(n,k) \in J} \). Consequently, it follows from (6.3) and (6.8) that the Weyl matrix \( M^N(\lambda) \) has the poles \( \{\lambda^N_n(n,k) \}_{(n,k) \in J} \) and the corresponding residues \( \{\alpha^N_n(n,k) \}_{(n,k) \in J} \). Thus, \( \{\lambda^N_n, \alpha^N_n(n,k) \}_{(n,k) \in J} \) are the spectral data of the problem \( L^N_* = L(\sigma^N_*, T_1, T_2, H^N_2) \). Since the transform (2.8) with \( H^0 = -C^N \) does not change the spectral data, we conclude that \( \{\lambda^N_n, \alpha^N_n(n,k) \}_{(n,k) \in J} \) are also the spectral data of \( L^N = L(\sigma^N, T_1, T_2, H^N_2) \). Since \( \lambda^N_n \in \mathbb{R} \) and \( \alpha^N_n = (\alpha^N_n)^* \) for \( (n,k) \in J \), one can easily show that the matrices \( \sigma^N(x) \) for a.e. \( x \in (0, \pi) \) and \( H^N_2 \) are Hermitian.

The following two lemmas can be proved similarly to Lemmas 5.6 and 5.7 from [44].

**Lemma 6.6** \( \sigma^N \to \sigma \) in \( L_2((0, \pi); \mathbb{C}^{m \times m}) \) and \( H^N_2 \to H_2 \) as \( N \to \infty \), where \( \sigma, H_2, \sigma^N, H^N_2 \) are defined by (6.1), (6.2), (6.5), (6.6), respectively.

**Lemma 6.7** Suppose that \( \sigma \) and \( \sigma^N, N \in \mathbb{N} \), are arbitrary Hermitian matrix functions from \( L_2((0, \pi); \mathbb{C}^{m \times m}) \) such that \( \sigma^N \to \sigma \) in \( L_2((0, \pi); \mathbb{C}^{m \times m}) \) as \( N \to \infty \) and \( H_2, H^N_2, N \in \mathbb{N} \), are arbitrary Hermitian matrices from \( \mathbb{C}^{m \times m} \) such that \( H^N \to H \) as \( N \to \infty \). Let \( \{\lambda_{nk}, \alpha_{nk}(n,k) \}_{(n,k) \in J} \) and \( \{\lambda^N_n, \alpha^N_n(n,k) \}_{(n,k) \in J} \) be the spectral data of the problems \( L(\sigma, T_1, T_2, H_2) \) and \( L(\sigma^N, T_1, T_2, H^N_2) \), respectively. Then, for each fixed \( (n,k) \in J \),
\[
\lim_{N \to \infty} \lambda^N_n = \lambda_{nk}.
\]

Let \( \lambda_{n_1 k_1} = \lambda_{n_2 k_2} = \cdots = \lambda_{n_r k_r} \) be a group of multiple eigenvalues of \( L \), maximal by inclusion. Then
\[
\lim_{N \to \infty} \sum_{j=1}^r \alpha^N_{n_j k_j} = \alpha_{n_1 k_1}.
\]

Lemmas 6.6 and 6.7 together with (6.3) prove Theorem 6.3 for \( \{\lambda_{nk}, \alpha_{nk}(n,k) \}_{(n,k) \in J} \). Theorems 5.1,6.3 and Lemmas 6.1, 6.2 yield the sufficiency part of Theorem 2.6. Our proof of sufficiency in Theorem 2.6 is constructive and provides the following algorithm for solving Inverse Problem 2.4.
Algorithm 6.8 Suppose that the orthogonal projection matrices $T_1$, $T_2$ and the data \{\(\lambda_{nk}, \alpha_{nk}\)\}_{(n,k) \in J}$ satisfying conditions (i)–(iii) of Theorem 2.6 be given. We have to find $\sigma$ and $H_2$.

1. Find $r_k$ and $A_k$ by the formulas
\[
 r_k = \lim_{n \to \infty} (\sqrt{\lambda_{nk}} - n),
\]
\[
 A_k = \frac{\pi}{2} \lim_{n \to \infty} (T_1 + n^{-1}T_1^\perp)\alpha^{(k)}(T_1 + n^{-1}T_1^\perp), \quad k = 1, m.
\]

2. Fix the model problem $\tilde{L} := L(0, T_1, T_2, 0)$ and find \{\(\tilde{\lambda}_{nk}, \tilde{\alpha}_{nk}\)\}_{(n,k) \in J},
\[
\{\tilde{\phi}(x, \tilde{\lambda}_{nki})\}_{(n,k) \in J, i = 0, 1}, \text{ by using (3.1), (3.2) and (3.3) for } (l, s), (n, k) \in J, i, j = 0, 1.
\]

3. Divide the values \{\(\tilde{\rho}_{nki}\)\} into the groups \{\(G_n\)\}_{n = 1} according to (3.5).

4. Construct the Banach space $\mathcal{B}$, the sequence $\tilde{\phi}(x) \in \mathcal{B}$, and the operator $\tilde{R}(x) : \mathcal{B} \to \mathcal{B}$ for each fixed $x \in [0, \pi]$ as it is described in Sect. 3.

5. Find the solution $\phi(x)$ of the main equation (3.10).

6. Using the elements \{\(\phi_{lsj}(x)\)\} of $\phi(x)$, construct $\sigma$ and $H_2$ by formulas (6.1) and (6.2), respectively.

In view of Proposition 2.5, the solution constructed by Algorithm 6.8 is not the only solution of Inverse Problem 2.4. All the other solutions can be obtained by applying transform (2.8).

### 7 Application to the Sturm–Liouville operator on a graph

In this section, we apply our results to the Sturm–Liouville operator on a graph of arbitrary geometrical structure.

Let $\mathcal{G}$ be a geometrical graph with edges \{\(e_j\)\}_{j = 1} of the corresponding lengths \{\(l_j\)\}_{j = 1}.

The graph may contain cycles, loops, and multiple edges. For each edge $e_j$, $j = 1, m$, introduce the parameter $x_j \in [0, l_j]$. Denote the ends of $e_j$ by $w_{j-1}$ and $w_j$ so that the values $x_j = 0$ and $x_j = l_j$ correspond to the ends $w_{j-1}$ and $w_j$, respectively. Every vertex $v$ of the graph $\mathcal{G}$ can be considered as the equivalence class of all the ends $w_j$ incident to this vertex: $v = \{w_j, w_{j+1}, \ldots, w_{j+k}\}$. The number of elements in this class is called the degree of the vertex. The vertices of degree 1 are called the boundary vertices and the others are called the internal vertices. Denote by $\partial\mathcal{G}$, $\text{int} \mathcal{G}$, and $\mathcal{V} = \partial\mathcal{G} \cup \text{int} \mathcal{G}$ the sets of the boundary vertices, the internal vertices, and all the vertices of $\mathcal{G}$, respectively.

For $j = 1, m$, consider functions $y_j(x_j)$ and $\sigma_j(x_j)$, $x_j \in [0, l_j]$, such that $y_{j,1} \in AC[0, l_j]$, $(y_{j,1}') \in L_2(0, l_j)$, $\sigma_j \in L_2(0, l_j)$ is real-valued, $y_{j,1} = y_j' - \sigma_j y_j$.

\[
 y_j|_{w_{j-1}} = y_j(0), \quad y_j|_{w_j} = y_j(l_j), \\
 y_{j,1}|_{w_{j-1}} = -y_{j,1}(0), \quad y_{j,1}|_{w_j} = y_{j,1}(l_j), \quad j = 1, m.
\]
Consider the Sturm–Liouville eigenvalue problem on the graph $G$ given by the equations

$$-(y_j^{[1]}(x_j))' - \sigma_j(x_j)y_j^{[1]}(x_j) - \sigma_j^2(x_j)y_j(x_j) = \lambda y_j(x_j), \quad x_j \in (0, l_j), \quad j = 1, m,$$

(7.1)
on the edges and the following matching conditions in the vertices:

$$y|_{w_j} = y|_{w_k}, \quad w_j, w_k \in v, \quad \sum_{w_j \in v} (y_j^{[1]} - h_v y|_{w_j}) = 0, \quad v \in V^K, \quad (7.2)$$

$$y|_{w_j} = 0, \quad w_j \in v, \quad v \in V^D, \quad (7.3)$$

where $V^D \subseteq \partial G$, $V^K := V \setminus V^D$, and $\{h_v\}_{v \in V^K}$ are reals. The conditions (7.2) in the internal vertices $v \in \text{int} \ G$ generalize the standard matching conditions, which express Kirchoff’s law in electrical circuits, balance of tension in elastic string network, etc. (see [21,24]). We impose the Dirichlet boundary conditions (7.3) in some part $V^D$ of the boundary vertices. For $v \in V^K \cap \partial G$, formulas (7.2) turn into the Robin boundary conditions:

$$y_j^{[1]} - h_v y|_{w_j} = 0, \quad w_j \in v.$$

Let us assume that $l_j = \pi / j = \frac{1}{m}$. If the edge lengths are different, but rationally dependent: $l_j = n_j \Delta, n_j \in \mathbb{N}, \Delta \in \mathbb{R}$, one can split the edges by additional vertices in internal points and obtain a graph with equal edge lengths. Without loss of generality we may assume that the graph $G$ is bipartite, that is, the set $V$ can be divided into two disjoint subsets $V_1$ and $V_2$ so that every edge connects two vertices from different subsets. In order to make the graph bipartite, one can add auxiliary vertices in the middle points of the edges. Then the vertices of the initial graph form the subset $V_1$, and the new auxiliary vertices, the subset $V_2$. Suppose that the vertices of $V_1$ and $V_2$ correspond to $x_j = 0$ and $x_j = \pi$, respectively. In other words, if $w_{2j-1} \in v$, then $v \in V_1$ and if $w_{2j} \in v$, then $v \in V_2$.

Consider a vertex $v \in V_1 \cap V^K, s = 1$ or 2. Let $e_{j_1}, e_{j_2}, \ldots, e_{j_r}$ be the edges incident to $v$. Construct the matrices

$$T^v_s = [T^v_s]_{j,k=1}^m, \quad H^v_s = h_v T^v_s, \quad T^v_s := \begin{cases} \frac{1}{r}, & j, k \in \{j_i\}_{i=1}^r, \\ 0, & \text{otherwise}, \end{cases} \quad (7.4)$$

$$T_s := \sum_{v \in V_1 \cap V^K} T^v_s, \quad H_s := \sum_{v \in V_1 \cap V^K} H^v_s. \quad (7.5)$$

It is easy to check that $T_s$ is an orthogonal projector and $H_s = H^*_s = T_s H_s T_s$ for $s = 1, 2$. Put $\sigma(x) := \text{diag}(\sigma_j(x))_{j=1}^m$. Then the eigenvalue problem (7.1)–(7.3) on the graph $G$ is equivalent to the matrix Sturm–Liouville problem for Eq. (1.1) with the boundary conditions

$$T_1(Y^{[1]}(0) - H_1 Y(0)) - T_1^Y(0) = 0, \quad T_2(Y^{[1]}(\pi) - H_2 Y(\pi)) - T_2^Y(\pi) = 0.$$
One can get rid of the matrix $H_1$ applying the transform
\[
\sigma(x) := \sigma(x) + H_1, \quad H_1 := 0, \quad H_2 := H_2 - T_2 H_1 T_2, \quad (7.6)
\]
and so arrive at the problem (1.1)–(1.2). Clearly, transform (7.6) does not change the spectral data. Let $\{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J}$ be the spectral data of this problem. The spectral data characterization of the Sturm–Liouville problem (7.1)–(7.3) on the graph $G$ is given by Theorem 2.6 with the following additional condition:

There exists a matrix $H^\diamond \in \mathbb{C}^{m \times m}$ such that
\[
H^\diamond = (H^\diamond)^* = T_1 H^\diamond T_1 + T_1^\perp H^\diamond T_1^\perp,
\]
for which the matrix function $\sigma^\diamond(x) := \sigma(x) + H^\diamond$ is a.e. diagonal and the matrices
\[
H_1^\diamond := -T_1 H^\diamond T_1, \quad H_2^\diamond := H_2 - T_2 H^\diamond T_2
\]
have the form (7.4)–(7.5) with some reals $\{h_v\}_{v \in V_K}$, where $\sigma(x)$ and $H_2$ are constructed by formulas (6.1)–(6.2).

It is supposed that the matrices $T_1$ and $T_2$ in Theorem 2.6 have the form (7.4)–(7.5) corresponding to the structure of some bipartite graph. For the star-shaped graph, the provided additional condition can be replaced by the diagonality condition, which can be checked immediately after solving the main equation (see [30] for details).

In the literature, the Sturm–Liouville inverse problems on graphs were considered in various statements. In particular, Yurko [48] has proved the uniqueness theorems and obtained constructive procedures for solving the inverse problems which consist in recovering the Sturm–Liouville potentials on a tree (i.e. graph without cycles) from the Weyl vector $[M_j(\lambda)]_{j=1}^p$ and from the spectral data $\{\lambda_{nk}, \alpha_{nkj}\}$, $\alpha_{nkj} := \text{Res}_{\lambda=\lambda_{nk}} M_j(\lambda)$. Here $p$ is the number of the boundary vertices excluding the root-vertex. Without loss of generality we may assume that all the boundary vertices belong $\mathcal{V}_1$ and are incident to the edges $\{e_j\}_{j=1}^p$. Then the elements $M_j(\lambda)$ of the Weyl–Yurko vector equal to the corresponding diagonal elements $M_{jj}(\lambda)$ in our problem statement, and the weight numbers $\alpha_{nkj}$ equal to the diagonal elements of our weight matrices $\alpha_{nk, jj}$ ($j = 1, p$). Thus, our spectral data $\{\lambda_{nk}, \alpha_{nk}\}$ contain Yurko’s data as a subset and so our inverse problem for the operator on graph is overdetermined. Nevertheless, in this paper, the linear relations on the spectral data are found (conditions (i)–(iii) of Theorem 2.6) which provide the complete characterization.

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