IMRO: A PROXIMAL QUASI-NEWTON METHOD FOR SOLVING \( \ell_1 \)-REGULARIZED LEAST SQUARE PROBLEM

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Abstract. We present a proximal quasi-Newton method in which the approximation of the Hessian has the special format of “identity minus rank one” (IMRO) in each iteration. The proposed structure enables us to effectively recover the proximal point. The algorithm is applied to \( \ell_1 \)-regularized least square problem arising in many applications including sparse recovery in compressive sensing, machine learning and statistics. Our numerical experiment suggests that the proposed technique competes favourably with other state-of-the-art solvers for this class of problems. We also provide a complexity analysis for variants of IMRO, showing that it matches known best bounds.

Key words. Proximal Methods, Quasi-Newton Methods, Sparse Recovery, Basis Pursuit Denoising Problem, \( \ell_1 \)-regularized Least Square Problem, Convex Optimization, Minimization of Composite functions

1. Introduction. Compressive sensing (CS) \cite{Candes2006,Donoho2006,Donoho2005} refers to the idea of encoding a large sparse signal through a relatively small number of linear measurements. This approach is essentially applying a linear operator \( A \in \mathbb{R}^{m \times n} \) to a signal \( x \in \mathbb{R}^n \) and storing \( \hat{b} = Ax \) instead. Naturally we want \( \hat{b} \in \mathbb{R}^m \) to be of a smaller dimension than \( x \); hence in practice \( m \ll n \). The main question is how to decode \( \hat{b} \) to recover signal \( x \), i.e. finding the solution to the underdetermined system of linear equations

\[
Ax = \hat{b}.
\]

Sparse recovery particularly aims at finding the sparsest solution to (1.1). The sparsest solution might be obtained by solving

\[
\begin{align*}
\min \quad & \|x\|_0 \\
\text{s.t.} \quad & Ax = \hat{b},
\end{align*}
\]

where \( \|x\|_0 \) corresponds to the number of nonzero entries of \( x \). Problem (1.2) is, however, NP-hard and difficult to solve in practice. Therefore the following linear programming relaxation was suggested for recovering the sparse solution:

\[
\begin{align*}
\min \quad & \|x\|_1 \\
\text{s.t.} \quad & Ax = \hat{b}.
\end{align*}
\]

The theory of compressive sensing has been well established. Candès, Tao, Donoho, and Romberg are among the pioneers of compressive sensing theory; see \cite{Candes2006,Donoho2006,Donoho2005} and references therein. In fact, they have shown that under some conditions (1.3) can recover the solution to (1.2).

In the presence of the noise in computing and storing \( \hat{b} \), the measurement is often \( b = \hat{b} + \hat{\epsilon} \); hence it is customary to replace \( Ax = \hat{b} \) with \( \|Ax - b\| \leq \epsilon \) in (1.3), where

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\end{itemize}
\( \epsilon \) is an estimated upper bound on the noise. The resulting problem is

\[
\min \|x\|_1 \quad \text{s.t.} \quad \|Ax - b\| \leq \epsilon \quad (BP_\epsilon).
\]

Problem (1.3) is usually referred to as “Basis Pursuit” (BP) problem, while \( BP_\epsilon \) refers to its least square constrained variant, i.e. (1.4).

Other common problems in sparse recovery are

\[
\min \frac{1}{2}\|Ax - b\|^2 + \lambda \|x\|_1, \quad (BPDN)
\]

and

\[
\min \|Ax - b\| \quad \text{s.t.} \quad \|x\|_1 \leq \tau. \quad (LASSO)
\]

In the literature of compressive sensing, (1.5) is often called “Basis Pursuit Denoising Problem” (BPDN) or \( l_1 \)-regularized least square problem, and (1.6) goes by the name of “LASSO” (Least Absolute Shrinkage and Selection Operator). It is possible to show that formulations (1.4), (1.5) and (1.6) attain the same optimizer provided that certain relationship holds between \( \epsilon \), \( \lambda \) and \( \tau \). However, there is no simple manner to compute this relationship without already knowing the optimal solution. The algorithms proposed in this work are tailored for solving the BPDN problem, i.e. (1.5).

In what follows we first review some of the notation used in this paper followed by a brief review on some of the related techniques to our method. IMRO is presented in Section 4. The convergence of IMRO is established in Section 5. In Section 6 we present the accelerated variant of IMRO. Our computational experiment is presented in Section 7. Finally we conclude our discussion in Section 8.

2. Notation. We work with real Euclidean vector space \( \mathbb{R}^n \) equipped with the inner product \( \langle x, y \rangle = \sum_{i=1}^n x_i y_i \). A linear operator (matrix) is defined as \( A: \mathbb{R}^n \to \mathbb{R}^m \). Its adjoint is denoted by \( A' \), and we have \( \langle Ax, y \rangle = \langle x, A'y \rangle \). Matrices, vectors and constants are denoted by upper-case, lower-case and Greek alphabet, respectively. The identity matrix is denoted by \( I \). We reserve the notation of \( \| \cdot \| \) for Euclidean norm; all other norms are denoted with the proper index.

The notation \( C_L \) stands for the class of continuously differentiable convex functions with Lipschitz continuous gradient, where \( L \) is the Lipschitz constant. We refer to functions of the form

\[ F(x) := f(x) + p(x) \]

as composite functions, where \( f(x) \in C_L \) and \( p(x) \) is a convex, possibly nonsmooth, function. Note that BPDN problem is an example of minimization of a composite function.

The proximal operator refers to

\[
\text{Prox}_p(y) = \arg\min_x \left\{ \frac{1}{2}\|x - y\|^2 + p(x) \right\}.
\]

The (soft) shrinkage (also called thresholding) operator is denoted by \( S \), and defined as

\[
S_\lambda(y) = \arg\min_x \left\{ \frac{1}{2}\|x - y\|^2 + \lambda\|x\|_1 \right\}.
\]
Thus, $S_\lambda \equiv \text{Prox}_p$ for the choice $p(x) = \lambda \|x\|_1$. One may easily check that

$$S_\lambda(y) = \begin{cases} 
  y_i - \lambda & \text{if } y_i \geq \lambda, \\
  0 & \text{if } |y_i| \leq \lambda, \\
  y_i + \lambda & \text{if } y_i \leq -\lambda,
\end{cases} \tag{2.1}$$

which is equivalent to

$$S_\lambda(y) = \text{sgn}(y) \odot \max\{|y| - \lambda, 0\}, \tag{2.2}$$

where $\odot$ denotes the entry-wise or Hadamard product.

The scaled norm associated with a positive definite matrix $H \succ 0$ is defined as

$$\|x\|_H = \sqrt{x^t H x}. \tag{2.3}$$

It is not too difficult to see that the scaled norm satisfies all the axioms of a norm. Moreover, let the scaled proximal mapping (or operator) associated with positive definite matrix $H$ be defined as

$$\text{Prox}_H^p(y) = \arg \min_x \left\{ \frac{1}{2} \|x - y\|_H^2 + p(x) \right\}. \tag{2.4}$$

3. Related Work. Algorithms that rely solely on the function value and the gradient of each iterate are referred to as first-order methods. Due to the large size of the problems arising in compressive sensing, first-order methods are more desirable in sparse recovery. There are numerous gradient-based first-order algorithms proposed for sparse recovery, see for example [31, 14, 36, 11, 52, 28, 48, 5].

In [11, 10] an efficient root finding procedure has been employed for finding the solution of $BP_\epsilon$ through solving a sequence of LASSO problems. In other words, a sequence of LASSO problems for different values of $\tau$ is solved using a spectral projected gradient method [15]; and as $\tau \rightarrow \tau^*$, the solution of the LASSO problem coincides with the solution of $BP$. In [52], the solution of $BP$ problem is recovered through solving a sequence of LASSO problems with an updated observation vector $b$. GPSR [31] is a gradient projection technique for solving the bound constrained QP reformulation of $BPDN$.

Many other state-of-the-art algorithms in compressive sensing are inspired by iterative thresholding/shrinkage idea [24, 25, 23]. ISTA (iterative shrinkage thresholding algorithm) is an extension of the steepest descent idea to composite functions using the thresholding operator. Recall that in the steepest descent method, the general format of the generated sequence is

$$x^{k+1} = x - \alpha \nabla f(x^k), \tag{3.1}$$

which might be considered as the solution to the following quadratic approximation of $f$:

$$x^{k+1} = \arg \min_x \left\{ f(x^k) + \langle x - x^k, \nabla f(x^k) \rangle + \frac{1}{2\alpha} \|x - x^k\|^2 \right\}. \tag{3.2}$$

Consider the problem of minimizing a composite function, i.e.,

$$\min \ F(x) = f(x) + p(x). \tag{3.3}$$
The idea of steepest descent method has been extended to minimizing a composite function by simply using the same approximation model as in (3.2) for \( f(x) \). As a result we get the following iterative scheme:

\[
x^{k+1} = \arg \min_x \left\{ f(x^k) + (x - x^k, \nabla f(x^k)) + \frac{1}{2\alpha} \|x - x^k\|^2 + p(x) \right\}.
\] (3.4)

Shuffling the linear and quadratic terms and ignoring the constants in (3.4), it can equivalently be written as

\[
x^{k+1} = \arg \min_x \left\{ \frac{1}{2\alpha} \|x - (x^k - \alpha \nabla f(x^k))\|^2 + p(x) \right\}.
\] (3.5)

Using the notion of Prox operator we conclude that

\[
x^{k+1} = \text{Prox}_{\alpha p}(x^k - \alpha \nabla f(x^k)).
\] (3.6)

The iterative scheme of (3.6) is the “generalized gradient method” or “proximal gradient method”. Note that it actually coincides with steepest descent method in the absence of \( p(x) \). It is also sometimes called “Forward-backward Splitting Method” [23, 49]. It gets its name from the two separate stages during each iteration while minimizing (3.3); the first stage is taking a forward step \( x^k - \alpha \nabla f(x^k) \) involving only \( f \), and the second stage is a backward step \( \text{Prox}_{\alpha p}(x^k - \alpha \nabla f(x^k)) \) which involves only \( p \). Finding the proximal point may not be a trivial task in general, but for solving BPDN it can be computed efficiently because the \( l_1 \)-norm is separable.

The algorithm that goes by the name of ISTA in the literature of sparse recovery refers to a proximal gradient method for composite functions in which \( p(x) = \lambda \|x\|_1 \). The general form of ISTA is

\[
x^{k+1} = S_{\lambda \alpha} (x^k - \alpha \nabla f(x^k)).
\] (3.7)

Each iteration of ISTA can be computed efficiently; it, however, could suffer from slow rate of convergence. In general, it has sublinear (i.e. \( O(k^{-1}) \)) rate of convergence [17, 43]. In [17], it has been shown that generalized gradient algorithms achieve linear convergence (i.e., \( \|x^k - x^*\| \leq q \lambda^k \) for some constants \( q \) and \( \lambda \in (0, 1) \)) provided that \( F(x^{k+1}) \leq F(x^k) - \delta \left( g(x^k) - g(x^{k+1}) + \langle \nabla f(x^k), x^k - x^{k+1} \rangle \right) \) and \( \|x^k - x^*\|^2 \leq \sigma \left( F(x^k) - F^* \right) \) for some constants \( \delta, \sigma \geq 0 \).

FISTA (fast ISTA) [3] is the accelerated variant of ISTA that was built upon the Nesterov’s idea [42, 44]. Each iteration of FISTA has the following format:

\[
x^{k+1} = S_{\lambda \alpha} (y^k - \alpha \nabla f(y^k)),
\] (3.8)

\[
t^{k+1} = \frac{1 + \sqrt{1 + 4t^k}}{2},
\] (3.9)

\[
y^{k+1} = x^k + \left( \frac{t^k - 1}{t^{k+1}} \right) (x^{k+1} - x^k).
\] (3.10)

FISTA is not restricted to BPDN problem, and it was originally proposed for minimizing a general composite function. Replacing (3.8) with

\[
x^{k+1} = \text{Prox}_{\lambda \alpha} (y^k - \alpha \nabla f(y^k))
\] (3.11)
would generalize FISTA to an algorithm well-suited for minimizing any composite function. The Nesterov’s accelerated proximal gradient algorithm has been adopted for solving $BP_\epsilon$ in [5], and for solving the LASSO problem in [33].

The alternating direction method (ADM) is also a technique that can be applied to $BP\epsilon$, see [50, 51] and references therein. It is suited for minimizing the summation of (separable) convex functions, say $f(x) + p(y)$, over a linear set of constraints. The augmented Lagrangian technique then solves for $x$ and $y$ alternately while fixing the other variable. The alternating linearization method (ALM) [32] also applies to minimizing composite functions. In (3.4), we linearize $f$ at every iteration to build the quadratic approximation model; in ALM a similar model based on $p$ is also minimized at every iteration. Nesterov’s accelerated technique has also been adopted, and the resulting algorithm is called FALM, for fast ALM.

In order to incorporate more information about the function without trading off the efficiency of the algorithms, Newton/quasi-Newton proximal methods [8, 39] have attracted researchers quite recently. Most of previous extensions on quasi-Newton methods are suited either for nonsmooth problems [40, 53], or for constrained problems with simple enough constraints [18, 45, 26].

The proximal quasi-Newton method is obtained by replacing the diagonal matrix $\frac{1}{\sigma}I$ in the quadratic term of (3.4) with a suitable positive definite matrix. In other words, define $m_H(x, x^k)$ as

$$m_H(x, x^k) = f(x^k) + \langle x - x^k, \nabla f(x^k) \rangle + \frac{1}{2} (x - x^k)^t H (x - x^k), \quad (3.12)$$

where $H \succ 0$, and solve

$$\min m_H(x, x^k) + p(x), \quad (3.13)$$

at each iteration. Ignoring the constant terms and using the definition of scaled norm, we can rewrite (3.12) as

$$m_H(x, x^k) = \|x - (x^k - H^{-1}\nabla f(x^k))\|^2_H. \quad (3.14)$$

By definition of scaled proximal mapping, we can now define the proximal quasi-Newton algorithm as

$$x^{k+1} = \text{Prox}_p^H (x^k - H^{-1}\nabla f(x^k)). \quad (3.15)$$

4. IMRO Algorithm. We present a practical variant of proximal quasi-Newton methods for solving BPDN problem in this section. Recall that the BPDN problem is

$$\min_x F(x) := \frac{1}{2}\|Ax - b\|^2 + \lambda \|x\|_1. \quad (4.1)$$

Let us denote the quadratic part of $F(x)$ with $f(x)$, and the $l_1$-regularization term with $p(x)$. We note that $f \in C_L$, with $L = \|A\|^2$. Applying the proximal quasi-Newton scheme of (3.13) and (3.14) to BPDN we get that

$$x^{k+1} = \arg \min_x \|x - (x^k - H^{-1}\nabla f(x^k))\|^2_H + \lambda \|x\|_1, \quad (4.2)$$

In our proposed proximal quasi-Newton scheme $H$ has the following format:

$$H = \sigma I - uu^t. \quad (4.3)$$
Note that $H > 0$ provided $\sigma > ||u||^2$. In fact the term “IMRO” stands for “identity minus rank one,” which is the proposed format for matrix $H$. We will see shortly that one of the advantages of IMRO is the efficiency in computing $x^{k+1}$. In [8], Becker and Fadili suggest a proximal quasi-Newton method in which $H$ is an identity plus rank one matrix. The methodology that we develop for selecting $\sigma$ and $u$ presented in Sections 4.2 and 4.3 does not seem to extend to the case of identity plus rank one according to our analysis, but this question may need future investigation.

In the remainder of this section we first describe how we may find $x^{k+1}$ using the special structure of $H$ in IMRO. Our discussion is then followed by two different variants of IMRO and their properties.

4.1. Computing $x^{k+1}$ in IMRO. In this section we explain how we can attain the solution of (4.2), $x^{k+1}$, in linearithmic time, i.e. $O(n \log n)$. Note that optimality conditions for (4.2) imply that

$$H \left( x - (x^k - H^{-1}\nabla f^k) \right) + \lambda \xi = 0,$$

(4.4)

where $\xi \in \partial(||x^{k+1}||_1)$. Let us denote $x^k - H^{-1}\nabla f^k$ by $x^c$. Since in IMRO $H = \sigma I - uu^t$, $H^{-1}$ might be computed in closed form:

$$(\sigma I - uu^t)^{-1} = \frac{1}{\sigma} I - \frac{1}{\sigma(||u||^2 - \sigma)} uu^t,$$

(4.5)

so we are able to calculate $x^c$ easily. Condition (4.4) may now be restated as

$$(\sigma I - uu^t)(x - x^c) + \lambda \xi = 0,$$

(4.6)

Recall that $\xi_i = 1$ if $x_i > 0$, $\xi_i = -1$ if $x_i < 0$, and $\xi_i \in [-1, 1]$ if $x_i = 0$. In the latter case, we have the freedom to select $\xi_i$ as any point in $[-1, 1]$ in order to make (4.6) hold.

Let $\mu$ (to be found) be a scalar equal to $u^t(x - x^c) / \sigma$. Then equation (4.6) reduces to

$$x - x^c - u\mu + \frac{\lambda}{\sigma} \xi = 0.$$

(4.7)

By (4.7), we conclude that $i$th entry of $x$ is either 0 or $x_i^c + u_i\mu - \lambda / \sigma$ (for $x_i > 0$) or $x_i^c + u_i\mu + \lambda / \sigma$ (for $x_i < 0$). Using this and sign of $u_i$, we may now find the proper interval for $\mu$ so that the mentioned equations for $x_i$ holds true; in other words:

$$x_i > 0 \Rightarrow x_i^c + u_i\mu - \frac{\lambda}{\sigma} > 0 \Rightarrow \mu > \frac{\lambda / \sigma - x_i^c}{u_i} \text{ if } u_i > 0,$$

$$x_i < 0 \Rightarrow x_i^c + u_i\mu + \frac{\lambda}{\sigma} < 0 \Rightarrow \mu < \frac{-\lambda / \sigma - x_i^c}{u_i} \text{ if } u_i < 0,$$

(4.8)

$$x_i = 0 \Rightarrow \xi_i = \frac{(x_i^c + u_i\mu)\sigma}{\lambda}.$$

(4.9)

Note that by definition of $\mu$, we have

$$u^tx - \mu \sigma = u^tx^c.$$

(4.10)
Searching over all the breakpoints mentioned in (4.8) and (4.9) (i.e. $\frac{\lambda}{\sigma - x^c_i}$ and $\frac{-\lambda}{\sigma - x^c_i}$), enables us to find the proper value of $\mu$ for which (4.11) holds. By taking the inner product of both sides of (4.7) with $u$, we obtain

$$u^t x = u^t \left( x^c - \frac{\lambda \xi}{\sigma} \right) + u^t u \mu, \quad (4.12)$$

hence equation (4.11) has the equivalent form of

$$(\text{lhs}) \quad u^t \left( x^c - \frac{\lambda \xi}{\sigma} \right) + (u^t u - \sigma) \mu = u^t x^c \quad (\text{rhs}). \quad (4.13)$$

Note that both terms on the left-hand side are functions of $\mu$, the first term via the implicit dependence of $\xi_i$ on $\mu$ whenever $x_i = 0$, while the second term explicitly depends on $\mu$. In fact, when $x_i = 0$, the $i$th contribution dependence of the first term is exactly $-u_i^2 \mu$, in effect cancelling the $i$th contribution from the second term. Thus, we see that the left-hand side of (4.13) is a piecewise linear continuous function of $\mu$, where the pieces are given by intervals between the above-mentioned breakpoints. Furthermore, the slope is always nonpositive because the second term contributes $u^t u - \sigma$ to the slope, a negative number, while the $i$th contribution from the first term is either 0 (when $x_i \neq 0$) or $-u_i^2$ (when $x_i = 0$).

This monotonicity allows us to find the correct $\mu$ solving (4.13). To find $\mu$, we sort all the breakpoints (a vector of size $2n$); we start with an initial value of $\mu$ small enough such that $\text{lhs} > \text{rhs}$; we then increment the value of $\mu$ over the sorted breakpoints until we reach the desired interval $[\mu_l, \mu_u]$ such that $\text{lhs} \mu_l > \text{rhs}$ and $\text{lhs} \mu_u < \text{rhs}$, or the value of $\mu^*$ for which $\text{lhs}_{\mu^*} = \text{rhs}_{\mu^*}$. In the case that we reach the interval, a simple interpolation solves (4.13). Note that we may efficiently update the lhs when reaching a breakpoint, since only one of $x_i$’s changes sign for each breakpoint. The following chart visualizes how the search process is actually carried out:

\[
\begin{align*}
\quad u_i > 0 : & \quad - - - - - \quad -\frac{\lambda \sigma - x^c_i}{u_i} \quad - - - - - \quad \frac{\lambda}{\sigma} - \frac{x^c_i}{u_i} \quad - - - - - \quad \frac{\lambda}{\sigma} - \frac{x^c_i}{u_i} \quad - - - - - \\
\quad x_i < 0 : & \quad \frac{\lambda}{\sigma} - x^c_i \quad \frac{\lambda}{\sigma} - x^c_i \quad \frac{\lambda}{\sigma} - x^c_i \quad \frac{\lambda}{\sigma} - x^c_i
\end{align*}
\]

\[
\begin{align*}
\quad u_i < 0 : & \quad - - - - - \quad \frac{\lambda}{\sigma} - x^c_i \quad - - - - - \quad \frac{\lambda}{\sigma} - x^c_i \quad - - - - - \quad \frac{\lambda}{\sigma} - x^c_i \quad - - - - - \\
\quad x_i > 0 : & \quad - - - - - \quad \frac{\lambda}{\sigma} - x^c_i \quad - - - - - \quad \frac{\lambda}{\sigma} - x^c_i \quad - - - - - \quad \frac{\lambda}{\sigma} - x^c_i
\end{align*}
\]

The algorithm below summarized all we said above for finding $x^{k+1}$. The presented pseudocode is in MATLAB notation. “slp” in the following algorithm stands for the slope of lhs in (4.13) of the current piece (i.e., the derivative with respect to $\mu$).

**Algorithm 1.**

*Input:* $\sigma$, $u$, $x^c$, and $\lambda$

**slp-Update Subroutine:**

Let $i = |a(j, 2)|$

if $a(j, 2) < 0$

if $u_i < 0$

$\text{slp} = \text{slp} + u_i^2$

7
else
  \( \text{slp} = \text{slp} - u_i \)
else
  if \( u_i < 0 \)
    \( \text{slp} = \text{slp} - u_i \)
  else
    \( \text{slp} = \text{slp} + u_i \)

\textbf{Main Procedure}

Let \( \mathcal{I} = \{ i : u_i \neq 0 \} \)

Form \( a \in \mathbb{R}^{2|\mathcal{I}| \times 2} \) such that \( a(i,:) = \left[ \frac{\hat{\sigma} - x_i^0}{u_i}, +i \right] \) and \( a(|\mathcal{I}| + i,:) = \left[ -\frac{\hat{\sigma} - x_i^0}{u_i}, -i \right] \)

Let \( \bar{a} := \text{sorted } \text{"a" on first column} \)

Let \( \text{rhs} := ut \) \( x^c \)

Choose \( \mu < \bar{a}(1,1) \) such that \( \text{lhs} := u^\tau x^\mu - \mu \sigma > \text{rhs} \), where \( x^\mu \) is derived by (4.7)

\( \text{slp} = ut \) \( x^\mu - \sigma \), where \( \mathcal{I}' = \{ i : x_i^\mu \neq 0 \} \)

for \( j = 1, 2, \ldots, 2|\mathcal{I}| \)

Let \( \mu^+ = \bar{a}(j,1) \)

\( \text{lhs}^+ = \text{lhs} + \text{slp}(\mu^+ - \mu) \)

Update \( \text{slp} \) using \( \text{slp-Update Subroutine } (\bar{a}(j,:), \text{slp}) \)

if \( \text{lhs}^+ \leq \text{rhs} \)

\( \mu^* = \frac{\text{rhs} - \text{lhs}^+}{\text{lhs} - \text{lhs}^+} \mu + \frac{(\text{lhs} - \text{rhs})\mu^+}{\text{lhs} - \text{lhs}^+} \)

Find \( x_\mu^* \) by (4.7)

return \( \mu = \mu^+ \) and \( \text{lhs} = \text{lhs}^+ \)

The computation of \( x^{k+1} \) can actually be done in linear time, i.e., \( O(n) \) (rather than \( O(n \log n) \)). The linear-time algorithm for finding \( \mu \) is based on the fact that there is an algorithm to find the median of an unsorted array of size \( n \) in \( O(n) \). So after computing the \( 2n \) breakpoints, we can find the median of the break points and calculate the \( \text{lhs} \) and \( \text{rhs} \) of (4.13) in \( O(n) \). If the \( \text{lhs} \geq \text{rhs} \), then we can discard all the breakpoints below the median. Likewise, if \( \text{lhs} \leq \text{rhs} \) we can drop all the values above the median. This step can also be done in \( O(n) \), and reduces the size of the problem to \( \frac{n}{2} \). The same procedure can be applied to the remaining breakpoints until we reach the desired interval for \( \mu \) (an interval \( [\mu_l, \mu_u] \) such that \( \text{lhs}_{u_1} \geq \text{rhs} \) and \( \text{lhs}_{\mu_u} \leq \text{rhs} \)). Thus, the total running time is of the form \( O(n) + O(\frac{n}{2}) + O(\frac{n}{4}) + \cdots \), which is \( O(n) \).

Two variants of IMRO are proposed in this paper. The difference between these two variants lies in the derivation of \( \sigma^k \) and \( u^k \). We refer to these variants as IMRO-1D for IMRO on a one-dimensional subspace, and IMRO-2D for IMRO on a two-dimensional subspace.

\textbf{4.2. IMRO-1D.} In IMRO-1D, we find \( \sigma \) and \( u \) such that the approximation model \( m_H(x, x^k) \) equals \( f(x) \) on a one-dimensional affine space \( x^k + \alpha v \). Moreover, we require \( m_H(x, x^k) \) to be an upper approximation for \( f(x) \). The latter property has some theoretical benefits in the convergence of the algorithm as we shall see in Section 5. The formal statement of these imposed constraints is

\[ m_H(x, x^k) = f(x) \text{ whenever } x \in x^k + \text{Span } \{ v \}, \]  
\[ m_H(x, x^k) \geq f(x) \text{ } \forall x, \]  

(4.14)  

(4.15)
for some nonzero vector $v$ to be determined later. Using (3.12), we deduce that (4.14) is equal to

$$\frac{1}{2} v^t H v = \frac{1}{2} v^t A^t A v,$$

and condition (4.15) implies that

$$\frac{1}{2} (x - x^k)^t H (x - x^k) \geq \frac{1}{2} (x - x^k)^t A^t A (x - x^k).$$

(4.17)

Obviously (4.17) holds if and only if $H \succeq A^t A$. By (4.16) and (4.17), the required conditions on $H$ boils down to

$$v^t (H - A^t A) v = 0,$$

(4.18)

$$H \succeq A^t A.$$  

(4.19)

In the rest of this subsection we show how we can compute $\sigma$ and $u$ such that the above conditions are satisfied.

### 4.2.1. Finding $\sigma$ and $u$ in IMRO-1D.

Conditions (4.18) and (4.19) imply that $v \in N(H - A^t A)$. (4.20)

Without loss of generality, we assume that $v$ is normalized, i.e. $\|v\| = 1$. The following lemma gives us the formula for $\sigma$ and $u$ in IMRO-1D.

**Lemma 1.** (4.18) and (4.19) are satisfied for

$$\sigma = \|A\|^2,$$

(4.21)

and

$$u = \begin{cases} \frac{\sigma v - A^t A v}{\sqrt{\sigma - \|A v\|^2}} & \text{if } v \text{ is not a dominant singular vector of } A, \\ 0 & \text{otherwise}. \end{cases}$$

(4.22)

**Proof.** Note that $\|A\|^2 = \lambda_{\text{max}}(A^t A) = \sigma_{\text{max}}^2(A)$, where $\lambda_{\text{max}}$ and $\sigma_{\text{max}}$ stand for the maximum eigenvalue and maximum singular value, respectively. Let us first consider the case where $v$ is a dominant singular vector of $A$. In this case $H = \sigma I = \|A\|^2 I \succeq A^t A$ and $(\sigma I - A^t A) v = 0$, so both requirements hold.

Suppose $v$ is not a dominant singular vector of $A$. Then the denominator in the formula for $u$ is positive and $u$ is defined. We, therefore, have

$$(H - A^t A) v = \sigma v - (u^t v) u - A^t A v = \sigma v - (\sqrt{\sigma - \|A v\|^2}) \frac{\sigma v - A^t A v}{\sqrt{\sigma - \|A v\|^2}} - A^t A v = 0,$$

which concludes equality (4.18). It remains to show (4.19), that is $x^t (\sigma I - u u^t) x \geq x^t A^t A x$ for all $x \in \mathbb{R}^n$. Equivalently, we will show that for all $x \in \mathbb{R}^n$ such that $\|x\| = 1$ we have $\sigma \geq x^t A^t A x + (u^t x)^2$, i.e.,

$$\sigma \geq \sup_{\|x\| = 1} \left\| \begin{pmatrix} A \\ u \end{pmatrix} x \right\|^2 = \left\| \begin{pmatrix} A \\ u \end{pmatrix} \right\|^2.$$
In fact, we prove that \( \sigma = \left\| \begin{pmatrix} A \\ u^t \end{pmatrix} \right\|^2 \). Clearly
\[
\left\| \begin{pmatrix} A \\ u^t \end{pmatrix} \right\| \geq \|A\|,
\]
because
\[
\left\| \begin{pmatrix} A \\ u^t \end{pmatrix} \right\| = \left\| \begin{pmatrix} Ax \\ u^tx \end{pmatrix} \right\| \geq \|Ax\| \quad \forall x \in \mathbb{R}^n.
\]

It remains to show that \( \left\| \begin{pmatrix} A \\ u^t \end{pmatrix} \right\| \leq \|A\| \). By the value of \( \sigma \), we have \( \sigma I - A^tA \succeq 0 \), so we can define \( B \) such that \( BB^t = \sigma I - A^tA \). Note that
\[
x^t \begin{pmatrix} A^t & u \end{pmatrix} \begin{pmatrix} A \\ u^t \end{pmatrix} x = x^tA^tAx + (u^tx)^2 = x^tA^tAx + \frac{(x^t(\sigma I - A^tA)v)^2}{\sigma - \|Av\|^2}
\]
\[
= x^tA^tAx + \frac{(x^t(\sigma I - A^tA)v)^2}{v^t(\sigma - A^tA)v}
\]
\[
= x^tA^tAx + \frac{(x^tBB^tv)^2}{v^tBB^tv}
\]
\[
\leq x^tA^tAx + x^tBB^tx = \sigma x^tx, \quad (4.23)
\]
where the last inequality is ensured by Cauchy-Schwarz inequality, i.e.,
\[
(x^tBB^tv)^2 \leq \|B^tx\|^2\|B^tv\|^2 = (x^tBB^t)(v^tBB^tv).
\]

Combining the definition of induced matrix norms and the result obtained in (4.23), we get
\[
\left\| \begin{pmatrix} A \\ u^t \end{pmatrix} \right\| = \sup_{x: \|x\|=1} x^t \begin{pmatrix} A^t & u \end{pmatrix} \begin{pmatrix} A \\ u^t \end{pmatrix} x \leq \sup_{x: \|x\|=1} \sigma x^tx = \sigma = \|A\|
\]
which yields the result we wanted to show. \( \Box \)

4.3. IMRO-2D. IMRO-2D is a variant of IMRO algorithm in which the quadratic model \( m_H(x, x^k) \) matches the function on the two-dimensional space of \( x^k + \text{Span} \{ \nabla f^k, d^k \} \), where \( d^k = x^k - x^{k-1} \). Without loss of generality, let us assume that \( \nabla f^k \) and \( d^k \) are normalized.

The imposed condition for IMRO-2D requires
\[
f(x^k) + (\nabla f^k, x - x^k) + \frac{1}{2}(x - x^k)^tH(x - x^k) = \frac{1}{2}\|A(x^k + (x - x^k))\|^2, \quad (4.24)
\]
for all \( x \in \{ x^k + \text{Span} \{ \nabla f^k, d^k \} \} \), that is when \( x - x^k = \alpha \nabla f^k + \beta d^k \).

Condition (4.24), therefore, reduces to
\[
\frac{1}{2}(x - x^k)^tH(x - x^k) = \frac{1}{2}(x - x^k)^tA^tA(x - x^k), \quad \text{i.e.,}
\]
\[
\frac{1}{2}(\alpha \nabla f^k + \beta d^k)^tH(\alpha \nabla f^k + \beta d^k) = \frac{1}{2}(\alpha \nabla f^k + \beta d^k)^tA^tA(\alpha \nabla f^k + \beta d^k) \quad (4.25)
\]
for all \( \alpha, \beta \in \mathbb{R} \). The fact that \( H = \sigma I - uu^t \) in IMRO enables us to find \( \sigma, \tau, \) and \( \rho \) such that (4.25) is satisfied for \( \sigma \) and \( u = \tau \nabla f^k + \rho d^k \). This is the topic covered in the remainder of this subsection.
4.3.1. Finding $\sigma$ and $u$ in IMRO-2D. By (4.25), we need to solve

$$(\alpha \nabla f^k + \beta d^k)^t A^t A (\alpha \nabla f^k + \beta d^k) = (\alpha \nabla f^k + \beta d^k)^t (\sigma I - uu^t)(\alpha \nabla f^k + \beta d^k),$$  (4.26)

for $\sigma$ and $u$. We first derive $\sigma$, then using $\sigma$ we will compute vector $u$.

Let $S$ be the following matrix

$$S = (\nabla f^k \ d^k)^t A^t A (\nabla f^k \ d^k) = \begin{pmatrix}
(\nabla f^k)^t A^t A \nabla f^k \\
(d^k)^t A^t A \nabla f^k \\
(d^k)^t A^t A d^k
\end{pmatrix}.  \quad (4.27)$$

Then (4.26) imposes the following equations on $\sigma$ and $u$:

$$
\begin{align*}
S_{11} &= \sigma - (\nabla f^k)^t uu^t \nabla f^k, \\
S_{12} &= \sigma (\nabla f^k)^t d^k - (\nabla f^k)^t uu^t d^k, \\
S_{22} &= \sigma - (d^k)^t uu^t d^k.
\end{align*}  \quad (4.28)
$$

Let $\epsilon$ denote $(\nabla f^k)^t d^k$, an easily computable scalar. Then

$$
\det(S) = S_{11}S_{22} - S_{12}^2 = \sigma^2 (1 - \epsilon^2) + \sigma (- (\nabla f^k)^t uu^t \nabla f^k - (d^k)^t uu^t d^k + 2\epsilon (\nabla f^k)^t uu^t d^k)
$$

$$= \sigma^2 (1 - \epsilon^2) + \sigma (S_{11} - \sigma + S_{22} - \sigma + 2\epsilon (\sigma - S_{12})),
$$

using the set of equations in (4.28). Hence $\sigma$ can be calculated by solving the following quadratic equation

$$\sigma^2 (1 - \epsilon^2) + \sigma (S_{11} - S_{22} + 2\epsilon S_{12}) + \det(S) = 0.  \quad (4.29)$$

Suppose $u = \tau \nabla f^k + \rho d^k$; using (4.28) we get

$$
\begin{align*}
\sigma - S_{11} &= \tau^2 + 2\epsilon \tau \rho + \epsilon^2 \rho^2 = (\tau + \epsilon \rho)^2, \\
\epsilon \sigma - S_{12} &= \epsilon \tau^2 + \tau \rho + \epsilon^2 \tau \rho + \epsilon \rho^2 = (\tau + \epsilon \rho)(\epsilon \tau + \rho), \\
\sigma - S_{22} &= \epsilon^2 \tau^2 + 2\epsilon \tau \rho + \rho^2 = (\epsilon \tau + \rho)^2,
\end{align*}  \quad (4.30)$$

so $(\tau + \epsilon \rho) = \sqrt{\sigma - S_{11}}$ and $(\epsilon \tau + \rho) = \sqrt{\sigma - S_{22}} \text{sgn}(\epsilon \sigma - S_{12})$. Therefore $\tau$ and $\rho$ are the solutions of the following linear system:

$$
\begin{pmatrix}
1 & \epsilon \\
\epsilon & 1
\end{pmatrix}
\begin{pmatrix}
\tau \\
\rho
\end{pmatrix}
= \begin{pmatrix}
\sqrt{\sigma - S_{11}} \\
\sqrt{\sigma - S_{22}} \text{sgn}(\epsilon \sigma - S_{12})
\end{pmatrix}.  \quad (4.31)
$$

4.3.2. Validity of IMRO-2D. In what follows, we show that IMRO-2D is a valid algorithm, namely $\exists \sigma, \tau, \rho \in \mathbb{R}$ that solve (4.29) and (4.31).

PROPERTY 1. Let $\eta_1$, $\eta_2$, and $\eta_3$ be defined as in (4.29). Then

$$
\begin{align*}
\eta_1 &\geq 0,  \quad (4.32) \\
\eta_2 &\leq 0,  \quad (4.33) \\
\eta_3 &\geq 0.  \quad (4.34)
\end{align*}
$$

Proof:

- Note that $\epsilon^2 \leq \epsilon \leq 1$ because $\epsilon = (\nabla f^k)^t d^k$ and $\|\nabla f^k\| = \|d^k\| = 1$, therefore $\eta_1 = 1 - \epsilon^2 \geq 0$. 

11
• $\eta_3 = \det(S)$ and $S \geq 0 \Rightarrow \eta_3 \geq 0$.
• $(S_{11} - S_{22})^2 \geq 0 \iff S_{11}^2 + S_{22}^2 + 2S_{11}S_{22} \geq 4S_{11}S_{22} \geq 4S_{12}^2 \geq 4\epsilon^2 S_{12}^2$, where the last two inequalities hold by $S \geq 0$ and $\epsilon^2 \leq 1$, respectively; therefore
  $$(S_{11} + S_{22})^2 \geq 4\epsilon^2 S_{12}^2 \Rightarrow S_{11} + S_{22} \geq 2\epsilon S_{12} \Rightarrow \eta_2 \leq 0.$$

☐

Claim 1. Equation (4.29) has a real solution, i.e., $\sigma$ exists.

Proof. We want to show that $\eta_2^2 - 4\eta_1 \eta_3 \geq 0$. Note that

$$\eta_2^2 - 4\eta_1 \eta_3 = (S_{11}^2 + S_{22}^2 + 4\epsilon^2 S_{12}^2 + 2S_{11}S_{22} - 4\epsilon S_{11}S_{12} - 4\epsilon S_{22}S_{12})$$
$$+ (-4S_{11}S_{22} + 4S_{12}^2 + 4\epsilon^2 S_{11}S_{22} - 4\epsilon^2 S_{12}^2), \quad (4.35)$$

Now let us make the following substitutions in (4.35)

$$S_{11}^2 = (1 - \epsilon^2)S_{11}^2 + \epsilon^2 S_{11}^2, \quad (4.36)$$
$$S_{22}^2 = (1 - \epsilon^2)S_{22}^2 + \epsilon^2 S_{22}^2, \quad (4.37)$$
$$4\epsilon^2 S_{11}S_{22} = 2\epsilon^2 S_{11}S_{22} + 2\epsilon^2 S_{11}S_{22}, \quad (4.38)$$

to get

$$\eta_2^2 - 4\eta_1 \eta_3 = (1 - \epsilon^2)S_{11}^2 + \epsilon^2 S_{11}^2 + (1 - \epsilon^2)S_{22}^2 + \epsilon^2 S_{22}^2 - 4\epsilon S_{11}S_{12} - 4\epsilon S_{22}S_{12}$$
$$- 2S_{11}S_{22} + 4S_{12}^2 + 2\epsilon^2 S_{11}S_{22} + 2\epsilon^2 S_{11}S_{22}$$
$$= (\epsilon S_{11} + \epsilon S_{22} - 2S_{12})^2 + (1 - \epsilon^2)(S_{11} - S_{22})^2 \geq 0.$$

☐

Claim 2. $\sigma \geq S_{11}$ and $\sigma \geq S_{22}$; therefore $u$ exists.

Proof. We will prove it for $S_{11}$; the proof for $S_{22}$ would be similar.

$$(\epsilon S_{11} - S_{12})^2 \geq 0 \Rightarrow S_{12}^2 \geq -\epsilon^2 S_{11}^2 + 2\epsilon S_{11}S_{12},$$
$$\Rightarrow S_{12}^2 - S_{11}S_{22} \geq -\epsilon^2 S_{11}^2 + 2\epsilon S_{11}S_{12} - S_{11}S_{22} + S_{11}^2 - S_{11}^2,$$
$$\Rightarrow -\eta_3 \geq S_{11}^2 \eta_1 + S_{11} \eta_2,$$
$$\Rightarrow -4\eta_1 \eta_3 \geq 4\eta_1 (S_{11}^2 \eta_1 + S_{11} \eta_2) = 4S_{11}^2 \eta_1^2 + 4S_{11} \eta_1 \eta_2,$$
$$\Rightarrow \eta_2^2 + 4\eta_1 \eta_3 \geq 4S_{11}^2 \eta_1^2 + 4S_{11} \eta_1 \eta_2 + \eta_2^2 = (2S_{11} \eta_1 + \eta_2)^2,$$
$$\Rightarrow \sqrt{\eta_2^2 + 4\eta_1 \eta_3} \geq 2S_{11} \eta_1 + \eta_2,$$
$$\Rightarrow \sigma = \frac{-\eta_2 + \sqrt{\eta_2^2 + 4\eta_1 \eta_3}}{2\eta_1} \geq S_{11}.$$

☐

Claim 3. Suppose $\sigma$ and $u$ are as defined in IMRO-2D by (4.29) and (4.31). Then $H = (\sigma I - u^t) \geq 0$.

Proof. We will prove that $\sigma \geq ||u||^2$, which implies that $\sigma ||x||^2 \geq ||u||^2 ||x||^2 \geq (u^t x)^2$ for all $x \in \mathbb{R}^n$; thus $H \geq 0$.

Recall that $u = \tau \nabla f^k + \rho d^k$, $||\nabla f^k|| = ||d^k|| = 1$ and $\epsilon = (\nabla f^k)^t d^k$ by definition, so

$$||u||^2 = \tau^2 + \rho^2 + 2\tau \rho \epsilon. \quad (4.39)$$
In addition, recall (4.30) in which we had
\[\begin{align*}
\sigma - S_{11} &= \tau^2 + 2\epsilon \rho + \epsilon^2 \rho^2, \\
\epsilon \sigma - S_{12} &= \epsilon \tau + \epsilon^2 \tau \rho + \epsilon^2 \rho^2, \\
\sigma - S_{22} &= \epsilon^2 \tau^2 + 2\epsilon \tau \rho + \rho^2.
\end{align*}\]

Let us multiply the second equation by \(-2\epsilon\) and add the result to the summation of the other two equations to get
\[\begin{align*}
lhs &= \sigma - S_{11} - 2\epsilon^2 \sigma + 2\epsilon S_{12} + \sigma - S_{22} = 2(1 - \epsilon^2)\sigma + \eta_2 = 2\eta_1 \sigma + \eta_2, \\
rhs &= \tau^2 + 2\epsilon \tau \rho + \epsilon^2 \rho^2 - 2\epsilon^2 \tau^2 - 2\epsilon \tau \rho - 2\epsilon^3 \tau \rho - 2\epsilon^2 \rho^2 + \epsilon^2 \rho^2 + 2\epsilon \tau \rho + \rho^2 \\
&= \tau^2(1 - \epsilon^2) + 2\epsilon \tau \rho(1 - \epsilon^2) + \rho^2(1 - \epsilon^2) = (1 - \epsilon^2)\|u\|^2 = \eta_1 ||u||^2, \\
\Rightarrow 2\eta_1 \sigma + \eta_2 &= \eta_1 ||u||^2, \\
\Rightarrow -\eta_2 + \sqrt{\eta_2^2 - 4\eta_1 \eta_3} + \eta_2 &= \eta_1 ||u||^2, \\
\Rightarrow ||u||^2 &= \sqrt{\eta_2^2 - 4\eta_1 \eta_3}.\end{align*}\]

By the value of \(\sigma\), we have
\[\sigma - ||u||^2 = -\frac{\eta_2 + \sqrt{\eta_2^2 - 4\eta_1 \eta_3} - \sqrt{\eta_2^2 - 4\eta_1 \eta_3}}{2\eta_1} = -\frac{\eta_2 - \sqrt{\eta_2^2 - 4\eta_1 \eta_3}}{2\eta_1} \geq 0,\]
where the final inequality holds by property $\|$1$, i.e.,
$$\eta_1 \geq 0, \; \eta_3 \geq 0 \Rightarrow -4\eta_1 \eta_3 \leq 0,$$

hence
$$\eta_2^2 - 4\eta_1 \eta_3 \leq \eta_2^2 \Rightarrow \sqrt{\eta_2^2 - 4\eta_1 \eta_3} \leq |\eta_2| = -\eta_2.$$

Note that \(\sigma > ||u||^2\) unless \(\eta_1 = 0\) (i.e. \(\epsilon = 0\)) or \(\eta_3 = 0\) (i.e. \(\det(S) = 0\)). Both cases happen only if \(\nabla f^k\) and \(d^k\) are parallel, otherwise \(H = \sigma I - uu^t > 0\).

Before we start the analysis on the convergence of IMRO, we would like to point out that IMRO-2D reduces to linear CG (LCG) in the absence of \(\lambda \|x\|_1\) term. The following theorem explains why IMRO-2D is essentially linear CG when the regularization term is missing.

**Theorem 1.** Suppose IMRO-2D is applied to minimizing the quadratic function
\[\frac{1}{2}\|Ax - b\|^2.\]
Then the sequence of iterates generated by IMRO-2D is the same as iterates generated in linear CG.

**Proof.** Notice that
\[f(x) = \frac{1}{2}\|Ax - b\|^2 = \frac{1}{2}x^tQAx - x^tA^tb + \frac{1}{2}b^tb.\]
The proof is by induction. Let \(x^0\) and \(r^0 = \nabla f(x^0) = Qx^0 - c\) be the starting point for both algorithms. The subscript CG distinguishes iterates for LCG from iterates obtained by IMRO-2D. For IMRO-2D at the first iteration we have
\[\frac{1}{2}(r^0)^t(\sigma I)r^0 = \frac{1}{2}(r^0)^tA^tAr^0 \Rightarrow \sigma = \frac{(r^0)^tA^tAr^0}{(r^0)^tr^0},\]
and

\[ x^1 = x^0 + \frac{1}{\sigma}(-r^0). \]

By the fact that for LCG, \( p^0_{CG} = -r^0 \) and \( \alpha^0 = \frac{(x^0) r^0}{\langle r^0, A r^0 \rangle} = \frac{1}{\sigma} \), we get \( x^1_{CG} = x^1 \).

Suppose this holds true for \( k \), i.e. \( x^k_{CG} = x^k \). To ensure that \( x^{k+1}_{CG} = x^{k+1} \) it suffices to show that \( x^{k+1} \in x^k + \text{Span} \{ r^k, p^k \} \). Because \( m_H(x, x^k) = f(x) \) on the space of \( x^k + \text{Span} \{ r^k, p^k \} \), hence \( x^{k+1} \) must be the minimizer of \( f(x) \) over the space \( x^k + \text{Span} \{ r^k, p^k \} \) which is \( x^{k+1}_{CG} \).

Using optimality condition for our model \( m_H(x, x^k) \) we get that

\[
\begin{align*}
x^{k+1} &= x^k - H^{-1}r^k = x^k - (\sigma I - uu^t)^{-1}r^k \\
&= x^k - \left( \frac{1}{\sigma} I - \frac{1}{\| u \|^2} uu^t \right) r^k \\
&= x^k - \frac{1}{\sigma} r^k + \frac{u^t r^k}{\sigma(\| u \|^2 - \sigma)}(\tau r^k + \rho p^k) \in \{ x^k + \text{Span} \{ r^k, p^k \} \}.
\end{align*}
\]

The general framework of IMRO is captured in the following algorithm.

**Algorithm 2.**

Let \( x^0 \in \mathbb{R}^n \) be an arbitrary starting point and \( x^1 = \text{Prox}_p(\mathbf{x}^0 - \nabla f^0 / \sigma) \).

For \( i = 1, 2, \ldots \)

Find \( \sigma \) and \( u \):
- **equations** [4.21] and [4.22] for IMRO-1D
- **equations** [4.29] and [4.31] for IMRO-2D

Find \( x^{k+1} \) using Algorithm 1

Update \( \nabla f^k \) and \( d^k \)

Note that we have not explained the choice of vector \( v \) (which determines \( u \)) in IMRO-1D. Our choice of \( v \) is discussed in Section 7.

### 5. Convergence of IMRO.

The difference between IMRO and other proximal quasi-Newton methods is the special structure of \( H \). The format of \( H \) in IMRO facilitates computation of the next iterate as mentioned earlier. The convergence properties of IMRO, however, can mostly be generalized to other variants of proximal quasi-Newton methods.

In the preceding sections, we established that \( H \succeq 0 \) for both IMRO-1D and IMRO-2D. Furthermore, the conditions under which \( H \) is singular are apparently unusual (that \( v^\circ \) is a dominant singular vector of \( A \) in the case of IMRO-1D; that \( \nabla f^k \) and \( d^k \) are parallel in the case of IMRO-2D) and never arose in our computational experiments. Therefore, for the remainder of this section, we assume \( H \succ 0 \). If one of these unusual cases arose in practice, we could simply modify the algorithm by replacing \( \sigma \) by \( \sigma + \epsilon \) for some small \( \epsilon > 0 \) to ensure that \( H \succ 0 \). Let \( p(x) \) be a convex function, possibly nonsmooth. Our notation is as follows.

\[
M_H(x, x^k) := m_H(x, x^k) + p(x),
\]

and

\[
\begin{align*}
M_H(x^k) &= \min M_H(x, x^k), \\
x_H^k &= \arg \min M_H(x, x^k), \\
g_H^k &= H(x^k - x_H^k).
\end{align*}
\]
Throughout this section, we use the compact notation of $M_k^H(x)$, $\bar{x}_k^H$, and $g_k^H$ for $M_H(x, x^k)$, $x_H^k(x^k)$ and $g_H^k(x^k)$, respectively.

Note that optimality conditions for (5.1) implies that
\[
g_k^H = \nabla f^k + \xi^k, \tag{5.3}
\]
where $\xi^k \in \partial (p(\bar{x}_H^k))$. We will see in this section that the notion of scaled gradient, $g_k^H$, mimics some of the properties of gradient. An important property of $g_k^H$ is captured below.

**Property 2.** $g_k^H = 0$ if and only if $x^k$ is the optimizer of the problem (3.3).

**Proof.** Note that if $g_k^H = 0$, then $x^k - \bar{x}_H^k = 0$ because $H > 0$ (thus invertible). This implies that $x^k = \bar{x}_H^k$. Therefore (5.3) reduces to optimality condition for (3.3). Likewise if $x^k$ is the optimal solution of (3.3), then $\nabla f^k + \xi^k = 0$ implies that $\bar{x}_H^k = x^k$; thus $g_k^H = 0$. \[\square\]

The following lemma (which is directly based on [39, Proposition 2.3]) shows that in fact direction $\bar{x}_H^k - x^k$ is a descent direction; in other words using this direction armed with a line search we attain the next iterate, $x^{k+1}$, for which we have $F(x^{k+1}) < F(x^k)$.

**Lemma 2.** Suppose the scheme of (5.2) (also in (5.15)) with some $H > 0$ has been applied to problem (3.3), where $x^*$ is not the optimizer. Let $x^{k+1} = x^k + \alpha(\bar{x}_H^k - x^k)$. Then $F(x^{k+1}) < F(x^k)$, for sufficiently small step size $\alpha > 0$.

**Proof.** Let us denote $\bar{x}_H^k - x^k$ by $d^k$. Since $\bar{x}_H^k$ is the unique optimizer of (5.1) and $x_H^k \neq x^k$ by assumption, we have
\[
\frac{1}{2} \|d^k\|_H^2 + \langle \nabla f^k, d^k \rangle + f(x^k) + p(\bar{x}_H^k) < \frac{\alpha^2}{2} \|d^k\|_H^2 + \alpha \langle \nabla f^k, d^k \rangle + f(x^k) + p(x^{k+1})
\]
\[
\leq \frac{\alpha^2}{2} \|d^k\|_H^2 + \alpha \langle \nabla f^k, d^k \rangle + f(x^k) + \alpha p(\bar{x}_H^k) + (1 - \alpha)p(x^k),
\]
where the last inequality follows from convexity of $p(x)$. Rearranging the terms and dividing by $1 - \alpha$ we get
\[
\langle \nabla f^k, d^k \rangle + p(\bar{x}_H^k) - p(x^k) < -\frac{1}{2} \alpha \|d^k\|_H^2. \tag{5.4}
\]
Using the convexity of $p(x)$ and the Taylor expansion for $f(x)$, we derive
\[
F(x^{k+1}) - F(x^k) = f(x^{k+1}) - f(x^k) + p(x^{k+1}) - p(x^k)
\]
\[
\leq \alpha \langle \nabla f^k, d^k \rangle + O(\alpha^2) + \alpha p(\bar{x}_H^k) + (1 - \alpha)p(x^k) - p(x^k)
\]
\[
= \alpha \left[ \langle \nabla f^k, d^k \rangle + p(\bar{x}_H^k) - p(x^k) \right] + O(\alpha^2) < 0, \tag{5.5}
\]
by (5.4) for sufficiently small values of $\alpha$. \[\square\]

The above lemma indicates that both variants of IMRO generate a decreasing sequence when paired with a line search.

**Lemma 3.** Suppose the scheme of (5.2) with some $H > 0$ has been applied to problem (3.3), and
\[
F(x_H^k) \leq M_k^H(\bar{x}_H^k). \tag{5.6}
\]
Suppose $x^k$ is not the optimizer. Let $x^{k+1} = x^k + \alpha(\bar{x}_H^k - x^k)$, where $\alpha \in (0, 1]$. Then $F(x^{k+1}) < F(x^k)$.
Proof. By the hypothesis that $H > 0$, $M_H^k(x)$ is strongly convex and $\bar{x}_H^k$ is the unique minimizer of $M_H$. We, therefore, get

$$F(\bar{x}_H^k) \leq M_H^k(\bar{x}_H^k) < M_H^k(x^k) = F(x^k).$$

We now attain the desired result using the convexity of $F(x)$:

$$F(x^{k+1}) \leq \alpha F(\bar{x}_H^k) + (1 - \alpha)F(x^k) < F(x^k)$$

Note that for IMRO-1D condition (5.6) always hold because $H \succeq A^tA$. The following lemma which is based on [3, Lemma 2.3] is the essence of showing the convergence properties of IMRO.

**Lemma 4.** Suppose the scheme of (5.2) with some $H > 0$ has been applied to problem (3.3) and $x^{k+1} = \bar{x}_H^k$. In addition suppose that

$$F(x^{k+1}) \leq M_H^k(x^{k+1}).$$

Then for $\forall x \in \mathbb{R}^n$ we have

$$F(x) - F(x^{k+1}) \geq \frac{1}{2}\|x^{k+1} - x^k\|_H^2 + \langle g_H^k, x - x^k \rangle.$$

**Proof.** Recall that we have

$$H(x^{k+1} - x^k) + \nabla f(x^k) + \xi = 0,$$

where $\xi \in \partial (p(x^{k+1}))$. By hypothesis we have

$$F(x) - F(x^{k+1}) \geq F(x) - M_H^k(x^{k+1}),$$

and by convexity of $f(x)$ and $p(x)$ we derive

$$f(x) \geq f(x^k) + \langle \nabla f(x^k), x - x^k \rangle,$$

$$p(x) \geq p(x^{k+1}) + \langle \xi, x - x^{k+1} \rangle.$$ (5.10)

Summing the above inequalities, we get

$$F(x) \geq f(x^k) + \langle \nabla f(x^k), x - x^k \rangle + p(x^{k+1}) + \langle \xi, x - x^{k+1} \rangle.$$ (5.11)

Substituting (5.11) in (5.8) gives us

$$F(x) - F(x^{k+1}) \geq f(x^k) + \langle \nabla f(x^k), x - x^k \rangle + p(x^{k+1}) + \langle \xi, x - x^{k+1} \rangle$$

$$- \frac{1}{2}\|x^{k+1} - x^k\|_H^2 - \langle \nabla f(x^k), x^{k+1} - x^k \rangle - f(x^k) - p(x^{k+1})$$

$$= - \frac{1}{2}\|x^{k+1} - x^k\|_H^2 - \langle \nabla f(x^k) + \xi, x - x^{k+1} \rangle$$

$$= - \frac{1}{2}\|x^{k+1} - x^k\|_H^2 + \langle H(x^k - x^{k+1}), x - x^k + (x^k - x^{k+1}) \rangle$$

$$= \frac{1}{2}\|x^{k+1} - x^k\|_H^2 + \langle g_H^k, x - x^k \rangle.$$
As mentioned earlier, in IMRO we have $H \preceq \sigma I$. Thus $H^{-1} \succeq \frac{1}{\sigma} I$ and
\[
\|x\|_{H^{-1}}^2 \geq \frac{1}{\sigma} \|x\|^2. \tag{5.12}
\]
Also recall that $\bar{x}_H^k - x^k = H^{-1} g_H^k$. As a result we get the following corollary.

**Corollary 1.** Suppose $x^{k+1} = \bar{x}_H^k$, and $F(x^{k+1}) \leq M_H^k(x^{k+1})$. Then for $\forall x \in \mathbb{R}^n$ we have
\[
F(x) \geq F(x^{k+1}) + \langle g_H^k, x - x^k \rangle + \frac{1}{2} \|g_H^k\|_{H^{-1}}^2 \tag{5.13}
\]
\[
\geq F(x^{k+1}) + \langle g_H^k, x - x^k \rangle + \frac{1}{2\sigma} \|g_H^k\|^2. \tag{5.14}
\]

**Proof.** Immediately follows from Lemma 4 and inequality (5.12). When $x^{k+1} = \bar{x}_H^k$, and $F(x^{k+1}) \leq M_H^k(x^{k+1})$, we get
\[
F(x^{k+1}) \leq F(x) - \frac{1}{2\sigma} \|g_H^k\|^2, \tag{5.15}
\]
by applying corollary 1 at $x = x^k$. Inequality (5.15) clarifies more similarities between scaled gradient, $g_H$, and the notion of gradient in smooth unconstrained problems. One of the helpful properties of an algorithm for unconstrained smooth optimization is to have a sufficient reduction in the objective value at each iteration, i.e.,
\[
f(x^{k+1}) \leq f(x^k) - \frac{1}{2L} \|\nabla f^k\|^2. \tag{5.16}
\]
Assuming that $\sigma = L = \|A\|^2$ (as in IMRO-1D), inequality (5.15) implies
\[
F(x^{k+1}) \leq F(x) - \frac{1}{2L} \|g_H^k\|^2, \tag{5.17}
\]
which captures the sufficient reduction in objective value at each iteration.

**Lemma 5.** Suppose $x^{k+1} = \bar{x}_H^k$, and $F(x^{k+1}) \leq M_H^k(x^{k+1})$. Then
\[
F(x) - F(x^{k+1}) \geq \frac{(F(x^{k+1}) - F(x^*))^2}{2\sigma^2 \delta^2}, \tag{5.18}
\]
where $\delta$ is the diameter of the level set of $x^0$.

**Proof.** By Lemma 3 at $x^*$ we have
\[
F(x^*) - F(x^{k+1}) \geq \frac{1}{2} \|x^{k+1} - x^k\|^2_H + \langle H(x^k - x^{k+1}), x^* - x^k \rangle
\]
\[
= \frac{1}{2} \langle (H(x^k - x^{k+1}), x^k - x^{k+1} + 2x^* - 2x^k) \rangle
\]
\[
= \frac{1}{2} \langle (H(x^* - x^{k+1}) - H(x^* - x^k), (x^* - x^{k+1}) + (x^* - x^k)) \rangle
\]
\[
= \frac{1}{2} \|x^* - x^{k+1}\|_H^2 - \|x^* - x^k\|^2_H
\]
\[
= \frac{1}{2} (\|x^* - x^{k+1}\|_H - \|x^* - x^k\|_H) \|x^* - x^{k+1}\|_H + \|x^* - x^k\|_H
\]
\[
\geq \frac{1}{2} (-\|x^{k+1} - x^k\|_H) (\|x^* - x^{k+1}\|_H + \|x^* - x^k\|_H),
\]

where the last line is by triangle inequality. Therefore we have
\begin{equation}
\|x^{k+1} - x^k\|_H \geq \frac{2(F(x^*) - F(x^{k+1}))}{\|x^* - x^{k+1}\|_H + \|x^* - x^k\|_H} \geq \frac{F(x^{k+1}) - F(x^*)}{\sigma \delta},
\end{equation}
by the fact that \(\|x^k - x^*\|, \|x^{k+1} - x^*\| \leq \delta\) because IMRO is a descent method, and \(\|x\|_H \leq \sigma \|x\|\) by the choice of \(H\) in IMRO.
Moreover, by Lemma 4 at \(x^k\) we have
\begin{equation}
F(x^k) - F(x^{k+1}) \geq \frac{1}{2}\|x^{k+1} - x^k\|_H^2.
\end{equation}
Applying inequality (5.19) concludes the result we wanted to show. □

The sublinear convergence of IMRO-1D is established in the following lemma.

**Lemma 6.** Suppose \(\{\omega^k\} \to \omega^*\) is a decreasing sequence; \(\omega^k - \omega^{k+1} \geq (\omega^{k+1} - \omega^*)^2 / \mu\) for all \(k\); and \(\omega_1 - \omega^* \leq 4\mu\). Then for all \(k\) we have
\(\omega^k - \omega^* \leq \frac{4\mu}{k} \).

**Proof.** Proof is by induction. For \(k = 1\) the result holds by hypothesis. Let \(p_k = \frac{4\mu}{k}\), then
\[
\omega^{k+1} - \omega^* = \omega^k - \omega^* + \omega^{k+1} - \omega^k
\leq \omega^k - \omega^* - \frac{(\omega^{k+1} - \omega^*)^2}{\mu}
\leq p_k - \frac{(\omega^{k+1} - \omega^*)^2}{\mu}.
\]
Let \(\nu = \omega^{k+1} - \omega^*\). Then the above inequality is
\begin{equation}
\frac{\nu^2}{\mu} + \nu - p_k \leq 0,
\end{equation}
which has nonnegative solution given by
\begin{equation}
\nu \leq \frac{-\mu + \sqrt{\mu^2 + 4p_k \mu}}{2} = \frac{2p_k}{1 + \sqrt{1 + \frac{4p_k}{\mu}}}.\quad (5.22)
\end{equation}

Note that function \(f(x) = \frac{1}{1 + \sqrt{x + 1}}\) is convex; thus on any interval \([0, a]\), it is bounded above by its secant interpolant. We now consider two separate cases; when \(k = 1\) to show that lemma holds for \(k + 1 = 2\), and when \(k \geq 2\) to show that the lemma holds for \(k + 1 \geq 3\). For \(k = 1\) we have \(p_1 \leq 4\mu\), so \(\frac{4p_1}{\mu} \leq 16\), therefore
\begin{equation}
\nu \leq 2p_1 \left(\frac{1}{2} + \frac{1}{1 + \sqrt{1}} - \frac{1}{2} \frac{4p_1}{\mu}\right) \leq \frac{4\mu}{2},\quad (5.23)
\end{equation}
For \(k \geq 2\), \(p_k \leq \frac{4\mu}{k} \leq 2\mu\), so \(\frac{4p_k}{\mu} \leq 8\), hence
\begin{align}
\nu & \leq 2p_k \left(\frac{1}{2} + \frac{1}{1 + \sqrt{9}} - \frac{1}{2} \frac{4p_k}{\mu}\right) \\
& = 4\mu \left(\frac{1}{k} + 4 \left(\frac{1}{4} - \frac{1}{2}\right) \frac{1}{k^2}\right) \leq \frac{4\mu}{k + 1},\quad (5.24)
\end{align}
where the last inequality follows from the fact that \( \frac{1}{k} - \frac{1}{k^*} \leq \frac{1}{k+1} \). \( \square \)

Applying the previous lemma with \( \omega^k = F(x^k) \) and \( \mu = \max \left( \frac{f(x^k) - f(x)}{1} , 2\sigma^2 \delta^2 \right) \) along with Lemma 5 completes our proof for sublinear convergence of IMRO-1D.

### 6. FIMRO - Accelerated Variant of IMRO

In this section, we discuss how we may apply the accelerated technique of Nesterov [43, Chapter 2] to IMRO-1D. We assume in this section that \( \sigma \geq L = \|A\|^2 \).

As in Nesterov’s method, we have two sequences \( \{y^k\} \) and \( \{x^k\} \) in this section. The model \( M_H(x,y^k) \) is built using \( y^k \), while its solution generates \( \{x^k\} \). In other words we have the following:

\[
\begin{align*}
M^*_H(y^k) &= \min \ M_H(x,y^k), \\
x^*_H(y^k) &= \arg \min \ M_H(x,y^k), \\
g^*_H(y^k) &= H(y^k - x^*_H(y^k)) \equiv g_H.
\end{align*}
\]

**Definition 1.** [43, Definition 2.2.1] A pair of sequences \( \{\phi^k(x)\} \) and \( \{\lambda^k\} \), \( \lambda^k \geq 0 \) is called an estimate sequence of function \( f(x) \) if \( \lambda^k \to 0 \) and for any \( x \in \mathbb{R}^n \) and all \( k \geq 0 \) we have

\[
\phi^k(x) \leq (1 - \lambda^k) f(x) + \lambda^k \phi^0(x).
\]

The following two lemmas which are analogous to [43] Lemmas 2.2.2 and 2.2.3] summarize how we can construct an estimate sequence.

**Lemma 7.** Suppose \( \{y^k\} \) is an arbitrary sequence, \( \{\alpha^k\} \) is a sequence such that \( \alpha^k \in (0,1) \) and \( \sum_{k=0}^{\infty} \alpha^k = \infty \), and \( \lambda^0 = 1 \). Moreover assume that

\[
F(x^*_H(y^k)) \leq M^*_H(x^*_H(y^k)).
\]

Then the pair of sequences \( \{\lambda^k\} \) and \( \{\phi^k(x)\} \) generated as

\[
\lambda^{k+1} = (1 - \alpha^k) \lambda^k,
\]

\[
\phi^{k+1}(x) = (1 - \alpha^k) \phi^k(x) + \alpha^k \left[ F(x^*_H(y^k)) + \langle g^*_H, x - y^k \rangle + \frac{1}{2\sigma} \|g^*_H\|^2 \right]
\]

is an estimate sequence for \( f(x) \).

**Proof.** Note that by Corollary 4 \( F(x) \geq \psi(x) \) for \( \forall x \in \mathbb{R}^n \). Our proof is by induction. The base case holds true for \( k = 0 \). Suppose it holds true for \( k \), then for \( k+1 \) we have

\[
\begin{align*}
\phi^{k+1}(x) &= (1 - \alpha^k) \phi^k(x) + \alpha^k \psi(x) \leq (1 - \alpha^k) \phi^k(x) + \alpha^k F(x) \\
&= (1 - (1 - \alpha^k) \lambda^k) F(x) + (1 - \alpha^k) \left( \phi^k(x) - (1 - \lambda^k) F(x) \right) \\
&\leq (1 - (1 - \lambda^k) \lambda^k) F(x) + (1 - \alpha^k) \lambda^k \phi^0(x) \\
&= (1 - \lambda^{k+1}) F(x) + \lambda^{k+1} \phi^0(x).
\end{align*}
\]

\( \square \)
In the following lemma we show how we may write $\phi^{k+1}(x)$ in closed form.

**Lemma 8.** Suppose $\phi^0(x) = F(x^0) + \frac{\sigma}{2} \| x - z^0 \|^2$. Then $\phi^{k+1}(x)$ generated by the recursive formulation of the previous lemma is

$$
\phi^{k+1}(x) = \phi^k + \frac{\gamma^k}{2} \| x - z^k \|^2,
$$

where

$$
\gamma^{k+1} = (1 - \alpha^k) \gamma^k,
$$

$$
z^{k+1} = z^k - \frac{\alpha^k}{\gamma^k+1} g_H,
$$

$$
\bar{\phi}^{k+1} = (1 - \alpha^k) \bar{\phi}^k + \alpha^k F(x_H^*(y^k)) + \frac{(\alpha^k)^2}{2\gamma^k} \| g_H \|^2 + \alpha^k \langle g_H, z^k - y^k \rangle.
$$

**Proof.** The proof is by induction. The base case for $k = 0$ holds. Suppose for $k$ we have

$$
\phi^k(x) = \bar{\phi}^k + \frac{\gamma^k}{2} \| x - z^k \|^2,
$$

then by the previous lemma we have

$$
\phi^{k+1}(x) = (1 - \alpha^k) \left[ \bar{\phi}^k + \frac{\gamma^k}{2} \| x - z^k \|^2 \right]
+ \alpha^k \left[ F(x_H^*(y^k)) + \langle g_H, x - y^k \rangle + \frac{1}{2\sigma} \| g_H \|^2 \right].
$$

Using the fact that $\phi$ is a quadratic function we get

$$
\nabla^2 \phi^{k+1}(x) = \gamma^{k+1} = (1 - \alpha^k) \gamma^k.
$$

By $\nabla \phi^{k+1}(x) = 0$, we get the minimizer of $\phi^{k+1}$ which is $\frac{1}{\gamma^k+1} z^{k+1}$, so

$$
\nabla \phi^{k+1}(x) = (1 - \alpha^k) \gamma^k (x - z^k) + \alpha^k g_H,
$$

$$
x = z^{k+1} = \arg \min \phi^{k+1}(x) = z^k - \frac{\alpha^k}{\gamma^{k+1}} g_H.
$$

To find $\bar{\phi}^{k+1}$ we set equal the $\phi^{k+1}(y^k)$ in both formulations of $\phi^{k+1}$. We, therefore, have

$$
\bar{\phi}^{k+1} + \frac{\gamma^{k+1}}{2} \| y^k - z^{k+1} \|^2 = \bar{\phi}^k + \frac{\gamma^k}{2} \| z^k - y^k \|^2 - \alpha^k \langle g_H, z^k - y^k \rangle + \frac{(\alpha^k)^2}{2\gamma^{k+1}} \| g_H \|^2
$$

$$
= (1 - \alpha^k) \bar{\phi}^k + (1 - \alpha^k) \gamma^k \| y^k - z^k \|^2 + \alpha^k F(x_H^*(y^k)) + \alpha^k \langle g_H, z^k - y^k \rangle.
$$

i.e.,

$$
\bar{\phi}^{k+1} = (1 - \alpha^k) \bar{\phi}^k + \alpha^k F(x_H^*(y^k)) + \frac{(\alpha^k)^2}{2\gamma^{k+1}} \| g_H \|^2 + \alpha^k \langle g_H, z^k - y^k \rangle. \quad (6.4)
$$
We would like to construct $y^k$ such that $\tilde{\phi}^{k+1} \geq F(x_H^*(y^k)) = F(x^{k+1})$. The benefit of this condition will be clear in Theorem 2. Note that for $k = 0$, $\phi^0 = F(x^0)$ and the condition holds. Let $x^{k+1} = x_H^*(y^k)$, and suppose the required condition is satisfied for $k$, i.e. $\tilde{\phi}^k \geq F(x^k)$. Using Corollary 1, at $x^k = y^k$ and $x = x^k$ we derive

$$\tilde{\phi}^k \geq F(x^k) \geq F(x_H^*(y^k)) + \langle g_H^k x^k - y^k \rangle + \frac{1}{2\sigma} \| g_H^k \|^2. \tag{6.5}$$

Substituting inequality (6.5) in the equation (6.4) we get

$$\tilde{\phi}^{k+1} \geq F(x_H^*(y^k)) + \left( \frac{1}{2\sigma} - \frac{(\alpha^k)^2}{2\gamma^k + 1} \right) \| g_H^k \|^2 + \langle g_H^k, \alpha^k(z^k - y^k) \rangle + (1 - \alpha^k)(x^k - y^k). \tag{6.6}$$

To make sure that $\tilde{\phi}^{k+1} \geq F(x_H^*(y^k))$, we need to set

$$\sigma(\alpha^k)^2 = (1 - \alpha^k)\gamma^k \equiv \gamma^{k+1}, \tag{6.7}$$

$$y^k = \alpha^k z^k + (1 - \alpha^k)x^k. \tag{6.8}$$

Equation (6.7) ensures that the coefficient of $\| g_H^k \|^2$ is zero, and (6.8) makes the linear term vanish. The proposed accelerated scheme is summarized as follows.

**Algorithm 3.**

Let $z^0 = x^0$ be arbitrary initial points, $\gamma^0 \geq L$.

for $i=0,1,\ldots$.

Compute $\alpha^k$ as $\sigma(\alpha^k)^2 = (1 - \alpha^k)\gamma^k$.

Let $\gamma^{k+1} = \sigma(\alpha^k)^2$.

Let $y^k = \alpha^k z^k + (1 - \alpha^k)x^k$.

Compute $f(y^k)$ and $\nabla f(y^k)$.

Find $\sigma^k$ and $u^k$:

- equations (4.21) and (4.22) (as in IMRO-1D).

Find $x^{k+1} = x_H^*(y^k)$ using algorithm 7.

(Note that for this choice of $x$ we get $F(x^{k+1}) \leq F(y^k) - \frac{1}{2\sigma} \| g_H^k \|^2$).

Let $z^{k+1} = z^k - \frac{\alpha^k}{\sigma^k + \tau} g_H^k$.

The following theorem from [13] Lemma 2.2.1 reveals the importance of condition $\tilde{\phi}^{k+1} \geq F(x_H^*(y^k))$.

**Theorem 2.** Suppose $F(x^k) \leq \tilde{\phi}^k$ holds true for a sequence $\{x^k\}$. Then

$$F(x^k) - F(x^*) \leq \lambda^k \left[ F(x^0) + \frac{\gamma^0}{2} \| x^0 - x^* \|^2 - F(x^*) \right].$$

**Proof.** By definition of an estimate sequence we get

$$F(x^k) \leq \tilde{\phi}^k \leq \min_x (1 - \lambda^k)F(x) + \lambda^k \phi^0(x) \leq (1 - \lambda^k)F(x^*) + \lambda^k \phi^0(x^*),$$

so

$$F(x^k) - F(x^*) \leq \lambda^k [\phi^0(x^*) - F(x^*)] = \lambda^k \left[ F(x^0) + \frac{\gamma^0}{2} \| x^0 - x^* \|^2 - F(x^*) \right].$$

☐
The beauty of the above theorem lies in the fact that the convergence of \( \{x^k\} \) follows the convergence rate of \( \{\lambda^k\} \). It remains to find the convergence rate of \( \{\lambda^k\} \).

**Lemma 9.** [43, Lemma 2.2.4] Suppose \( \gamma^0 \geq L \). Then

\[
\lambda^k \leq \frac{4\sigma}{(2\sqrt{\sigma} + k\sqrt{\gamma^0})^2}.
\]

**Proof.** Inductively we show that \( \gamma^0 \lambda^k \leq \gamma^k \). It holds by our assumption that \( \lambda^0 = 1 \) for 0. Suppose it holds for \( k \), then for \( k+1 \) we get

\[
\gamma^0 \lambda^{k+1} = \gamma^0 \lambda^k (1 - \alpha^k) \leq \gamma^k (1 - \alpha^k) = \gamma^{k+1} = \sigma (\alpha^k)^2 \quad \Rightarrow \quad \alpha^k \geq \sqrt{\frac{\gamma^0 \lambda^{k+1}}{\sigma}}.
\]

Let \( \tau^k = \frac{1}{\sqrt{\lambda^k}} \), be an increasing sequence; then we have

\[
\tau^{k+1} - \tau^k = \frac{\sqrt{\lambda^k} - \sqrt{\lambda^{k+1}}}{\sqrt{\lambda^k \lambda^{k+1}}} = \frac{\lambda^k - \lambda^{k+1}}{\sqrt{\lambda^k \lambda^{k+1}}} = \frac{\lambda^k - (1 - \alpha^k) \lambda^k}{2\lambda^k \sqrt{\lambda^{k+1}}} = \frac{\alpha^k}{2\sqrt{\lambda^{k+1}}} \geq \frac{1}{2} \sqrt{\frac{\gamma^0}{\sigma}}.
\]

Hence,

\[
\tau^k \geq 1 + \frac{k}{2} \sqrt{\frac{\gamma^0}{\sigma}} \quad \Rightarrow \quad \lambda \leq \frac{4\sigma}{(2\sqrt{\sigma} + k\sqrt{\gamma^0})^2}.
\]

The corollary below immediately follows from Theorem 2, Lemma 9, and the fact that \( F(x^0) - F(x^*) \leq \frac{L}{2} \|x^0 - x^*\|^2 \).

**Corollary 2.** Suppose \( \gamma^0 \geq L \). The generated sequence by Algorithm 3 satisfies

\[
F(x^k) - F(x^*) \leq \frac{\gamma^0 + L}{2} \left( \frac{4\sigma}{(2\sqrt{\sigma} + k\sqrt{\gamma^0})^2} \right) \|x^0 - x^*\|^2.
\]

Note that in IMRO-1D, \( \sigma \geq L \). Under the assumption that \( \sigma = \gamma^0 = L \), we have

\[
F(x^k) - F(x^*) \leq \frac{4L}{(2 + k)^2} \|x^0 - x^*\|^2.
\]

This bound of \( O(1/k^2) \) residual after \( k \) iterations is the best known for the class of first-order methods for BPDN.

7. Numerical Result. We compare IMRO in terms of both speed and accuracy with other available solvers listed below.

- **GPSR (gradient projection for Sparse Reconstruction)** [31] This gradient projection based algorithm first reformulates BPDN problem (1.5) into a bound constrained quadratic program (BCQP). The BCQP formulation is then solved through a projected gradient technique. In other words

\[
z^{k+1} = z^k + \lambda^k \left( \text{Proj}_+ \left( z^k - \alpha^k \nabla F(z^k) \right) - z^k \right)
\]
where $\lambda^k$ and $\alpha^k$ are step sizes and $\text{Proj}_{+}$ is the projection on nonnegative orthant. Two variants of the algorithm have been proposed. In the basic variant $\lambda^k = 1$ and $\alpha^k$ is determined through a backtracking line search. The BB version finds $\lambda^k \in [0,1]$ using the technique due to Barzilai and Borwein [2], then updates $\alpha^{k+1}$ accordingly. GPSR software is available at [29]. In our experiment we have used the BB version.

- **l1-ls** [37] l1-ls solves the same reformulation of BPDN problem as in GPSR through a truncated Newton interior point method. In l1-ls, preconditioned conjugate gradient (PCG) has been adopted for finding the search direction. Although forming the Newton system explicitly requires $A^t A$, the computational cost of each iteration of PCG is reduced to a matrix vector multiplication by the proper choice of preconditioner. The MATLAB code of this solver is available at [38].

- **FPC and FPC-AS** [35, 36] Recall that first order optimality conditions imply that $x^*$ is the minimizer of a composite function if and only if
  \[ x^* = \text{Prox}_{\alpha p} \left( x^* - \alpha \nabla f(x^*) \right). \]  
  Equation (7.1) is called “fixed point equation”. “Fixed Point Continuation” (FPC) aims to solve equation (7.1) through a proximal gradient method. The resulting algorithm has the following general scheme
  \[ x^{k+1} = \text{Prox}_{\alpha p} \left( x^k - \alpha \nabla f(x^k) \right). \]  
  The developed theory on the convergence of this method suggests that the algorithm converges faster for larger values of $\lambda$. “Continuation” strategy is essentially solving BPDN problem for a decreasing values of $\bar{\lambda} \to \lambda$ and warm starting the algorithm from the terminating solution corresponding to the previous value of $\bar{\lambda}$. FPC has later been extended to FPC-AS [47]. For each continuation interval, FPC-AS first solves the problem through FPC, then hard-thresholds the solution for nonzero entries. The $\|x\|_1$ is replaced with $\text{sgn}(x)^t x$, and the smaller smooth problem is minimized to attain the final solution. FPC and FPC-AS software are available at [34, 46]. We have used FPC-AS in our experiment.

- **SpaRSA (Sparse Reconstruction by Separable Approximation)** [48] SpaRSA is a proximal gradient framework for composite functions in which the nonsmooth part, $p(x)$, is separable. When $p(x) = \|x\|_1$ as is BPDN problem, SpaRSA reduces to an ISTA algorithm. The Barzilai and Borwein [2] and a continuation scheme has been applied to enhance the performance of the algorithm. The MATLAB code is available at [30].

- **NestA** [5] NestA is an algorithm built upon the Nesterov’s accelerated technique for for minimizing convex functions with Lipschitz continuous gradients over simple convex sets [43]. NestA is tailored for solving problem BP, i.e. (1.4), with orthonormal matrix $A$. A continuation scheme has been adapted to improve the performance of NestA. The MATLAB package for NestA might be reached at [4].

- **SPGL1** [11] This method solves BP formulation, (1.4), through solving a sequence of LASSO problems, (1.6). Each LASSO problem is solved using a spectral projected gradient method [10]. In this technique, a single parameter function for LASSO problem is defined as $\phi(\tau) = \|Ax^* - b\|$. Using the dual information of the LASSO problem, one may recover derivative of $\phi$;
In case that \( \nabla \) to converge. We also include the residual of the solution at termination for each

\[ \xi \]

believed that instances with higher dynamic range are harder to solve.

\[ 3 \]

problem. "dynamic 3" gives random entries with a dynamic range of 10

\[ \text{Ent. type} \]

stands for the type of entries in matrix \( A \) or vector \( x^* \), the optimal solution of the

\[ \text{model, i.e., no line search has been employed. Our experiment (not reported here) } \]

extra computational cost.

\[ \theta \]

used. In our experiment, we have used the previous direction for

\[ v \]

questions regarding the implementation of IMRO-1D is what direction

\[ \text{[2], however, might be advantageous and needs further investigation. One of the key } \]

advantage in applying the line search. Other techniques such as Barzilai and Borwein

\[ \text{[2], however, might be advantageous and needs further investigation. One of the key } \]

questions regarding the implementation of IMRO-1D is what direction \( v \) should be

\[ \text{used. In our experiment, we have used the previous direction for } v, \text{ i.e. } v^k = x^k - x^{k-1}. \]

The choice of \( v \), however, needs further study. Our measurement on the computational
cost of each algorithm is the number of matrix-vector multiplications, i.e., the number
of calls to \( A \) or \( A^T \).

As mentioned earlier, NestA only runs on instance with orthonormal \( A \). In the
first part of our experiment we compare IMRO with NestA. For increasing the accuracy of IMRO, we can simply use a smaller tolerance on the norm of subgradient.
For NestA, parameter \( \mu \) controls the accuracy of the solution; the smaller parameter \( \mu \) gets, the more accurate the solution becomes. We have used the default value of \( \mu = 0.02 \) suggested by the authors, and \( \mu = 0.002 \) in our experiment. All other settings are set to their default value. The number of continuation steps in the default setting is five. The information on test cases with orthonormal \( A \) is summarized in Table [7.3]. These instances are generated by L1TestPack package [41]. "Ent. type" stands for the type of entries in matrix \( A \) or vector \( x^* \), the optimal solution of the problem. "dynamic 3" gives random entries with a dynamic range of \( 10^5 \). It is often believed that instances with higher dynamic range are harder to solve.

In Table [7.2] we have presented the number of \( A \) or \( A^T \) calls each algorithm takes
to converge. We also include the residual of the solution at termination for each

\[ \theta \]

\[ \text{FISTA} \]

We presented FISTA in section [1]. The algorithm has proposed by

Beck and Teboulle in [3] and by Nesterov in [42]. The algorithm is part of

the TFOCS package [7] that is available at [6].

The termination criterion used for IMRO is the measurement on the norm of subgradient of function \( F(x) \), \( \xi \), which at iteration \( k \) is

\[ \xi_i = \lambda + \nabla f_i^k \quad \text{if } x_i > 0, \]

\[ \xi_i = -\lambda + \nabla f_i^k \quad \text{if } x_i < 0, \]

\[ \xi_i = -\lambda \alpha + \nabla f_i^k \quad \text{if } x_i = 0 \quad \text{for some } \alpha \in [-1, 1]. \]

Note that for zero entries, if \( |\nabla f_i^k| \leq \lambda \), then there exists an \( \alpha \) such that \( \xi_i = 0 \). In case that \( |\nabla f_i^k| > \lambda \), then \( \xi_i \neq 0 \); so we take \( \alpha \) such that \( \xi_i \) is minimized, i.e. \( \xi_i = |\nabla f_i^k| - \lambda \). Therefore, the norm of subgradient at \( x^k \) is easily calculated with no extra computational cost.

In both variants of IMRO, we took \( x^{k+1} \) as the minimizer of the approximation model, i.e., no line search has been employed. Our experiment (not reported here) with IMRO coupled with a bisection line search suggested that there is no significant advantage in applying the line search. Other techniques such as Barzilai and Borwein [2], however, might be advantageous and needs further investigation. One of the key questions regarding the implementation of IMRO-1D is what direction \( v \) should be used. In our experiment, we have used the previous direction for \( v \), i.e. \( v^k = x^k - x^{k-1} \). The choice of \( v \), however, needs further study. Our measurement on the computational cost of each algorithm is the number of matrix-vector multiplications, i.e., the number of calls to \( A \) or \( A^T \).
Table 7.1
Test Cases with Orthonormal $A$

|      | $m$ | $n$ | Ent. type of $A$ | Ent. type of $x$ | $\lambda$ |
|------|-----|-----|------------------|------------------|-----------|
| Ins 1 | 2500 | 10000 | Gaussian         | Gaussian          | 0.5       |
| Ins 2 | 2500 | 10000 | Gaussian         | Gaussian          | 0.05      |
| Ins 3 | 2500 | 10000 | Gaussian         | dynamic 3         | 0.5       |
| Ins 4 | 2500 | 10000 | Gaussian         | dynamic 3         | 0.1       |

Table 7.2
Numerical Results (number of $A$ or $A^t$ calls) for Comparison of IMRO and NestA

|      | IMRO-2D | IMRO-1D | NestA |
|------|---------|---------|-------|
|      | tol it. $\|x_t - x^*\|$ | it. $\|x_t - x^*\|$ | it. $\|x_t - x^*\|$ | $\mu$ |
| Ins 1 | 1e-2 | 51 0.047 | 54 0.057 | 684 1.539 | 0.02 |
|      | 1e-6 | 138 7.119e-6 | 214 9.102e-06 | 1010 0.474 | 0.002 |
| Ins 2 | 1e-2 | 60 0.048 | 60 0.058 | 424 0.173 | 0.02 |
|      | 1e-6 | 120 6.755e-6 | 220 8.759e-06 | 572 0.026 | 0.002 |
| Ins 3 | 1e-2 | 198 0.051 | 208 0.061 | 532 1.488 | 0.02 |
|      | 1e-6 | 267 7.169e-6 | 347 7.515e-06 | 580 0.182 | 0.002 |
| Ins 4 | 1e-2 | 393 0.055 | 630 0.06 | 504 1.487 | 0.02 |
|      | 1e-6 | 474 7.194e-6 | 796 1.160e-05 | 552 0.646 | 0.002 |

The number of calls to $A$ or $A^t$, and the residual of the solution (i.e., $\|x_t - x^*\|$) are presented in Tables 7.4 and 7.5 respectively. In almost all cases IMRO-2D outperforms other algorithms, especially when a highly accurate solution is desired. The performance of some of the techniques like GPSR or SpaRSA has been influenced considerably by the increase in the dynamic range. Moreover, some of the solvers failed to converge when $A$ is ill-conditioned, or failed to reach high accuracy as in FPC-AS. IMRO-2D on the other hand, performs consistently well in all our test cases. Graph 7.2 visualizes the residual of the solution with respect to the number of $A$ or $A^t$ calls.

$$\frac{1}{2}\|A\hat{x} - b\|^2 + \lambda \|\hat{x}\|_1 \leq \frac{1}{2}\|A\hat{x} - b\|^2 + \lambda \|\hat{x}\|_1.$$ (7.3)
**Table 7.4**  
Numerical Results (number of A/AH calls) for BPDN solvers

|          | tol | IMRO-2D | IMRO-1D | FISTA | GPSR-BB | SpaRSA | TwiST | FPC-AS | L1ls (pcs) |
|----------|-----|---------|---------|-------|---------|--------|-------|--------|------------|
| Ins 1    | 1e-2| 37      | 52      | 32    | 24      | 24     | 60    | 23     | 368        |
|          | 1e-6| 103     | 232     | 106   | 64      | 48     | 153   | DNRT†  | DNC‡       |
| Ins 2    | 1e-2| 175     | 190     | 168   | 230     | 180    | 187   | 142    | 1311       |
|          | 1e-6| 253     | 392     | 258   | 264     | 203    | 277   | DNRT†  | DNC‡       |
| Ins 3    | 1e-2| 52      | 54      | 36    | 26      | 30     | 53    | 25     | 504        |
|          | 1e-6| 136     | 242     | 118   | 56      | 53     | 136   | DNRT†  | DNC‡       |
| Ins 4    | 1e-2| 322     | 416     | 382   | 1208    | 880    | 313   | 160    | 2201       |
|          | 1e-6| 394     | 616     | 480   | 1242    | 904    | 387   | DNRT†  | DNC‡       |
| Ins 5    | 1e-2| 136     | DNC     | 225   | 380     | DNC    | 356   | DNC    | 502        |
|          | 1e-6| 277     | DNC     | 997   | 884     | DNC    | 883   | DNC    | 502        |
| Ins 6    | 1e-2| 202     | 540     | 399   | 820     | 554    | DNRT† | DNRT†  | 2550       |
|          | 1e-6| 538     | 1292    | 1741  | 1416    | DNC    | 1033  | DNRT‡  | DNC‡       |

† Did Not Reach the Tolerance (DNRT): The solver terminates before reaching the desired tolerance.
‡ Did Not Converge (DNC): The solver reaches the maximum iteration count of 5 × k2D, where k2D is the number of iterations IMRO-2D takes to converge.
Table 7.3

Information on Test Cases

| L1TestPack | Ins 1 | O(1) | Gaussian | Ent. type of $x$ | $\lambda$ |
|------------|------|------|----------|-----------------|---------|
| $A \in \mathbb{R}^{2500 \times 1000}$ | 5    | 300  | 2048     | Gaussian, DCT   | 0.1     |
|           | 9    | 128  | 128      | Heaviside       | 0.1     |
| Sparco     | 10   | 1024 | 1024     | Heaviside       | 0.1     |
|           | 903  | 1024 | 1024     | 1D Convolution  | 0.1     |

Table 7.5

Accuracy of the Solution, i.e. $\|x_t - \hat{x}\|$, in IMRO and Other BPDN Solvers

| tol   | IMRO-2D | IMRO-1D | FISTA | GPSR-BB | SpaRSA | TwIST | FPC-AS |
|-------|---------|---------|-------|---------|--------|-------|--------|
| Ins 1 | 1e-2    | 1.4e-01 | 1.2e-01| 2.5e-01 | 1.1e-01| 1.6e-02| 1.3e-01| 5.7e-02| 1e-02|
|       | 1e-6    | 2.9e-05 | 2e-05 | 2.2e-05 | 2.7e-05| 1.8e-05| 4.5e-05| 2.3e-03| 1.9e-05|
| Ins 2 | 1e-2    | 1.1e-01 | 1.5e-01| 1.5e-01 | 1.1e-01| 3.7e-02| 1e-01  | 1.4e-02| 7.3e-03|
|       | 1e-6    | 2.6e-05 | 2.5e-05| 3.7e-05| 4.4e-05| 2.5e-05| 6.3e-05| 1.3e-04| 2.9e-03|
| Ins 3 | 1e-2    | 8.8e-02 | 1.3e-01| 1.2e-01 | 8.4e-02| 2.2e-02| 8.5e-02| 3.9e-02| 5e-03 |
|       | 1e-6    | 2.3e-05 | 2.5e-05| 3.1e-05| 3.2e-05| 2.1e-05| 3e-05  | 4.4e-03| 1.8e-05|
| Ins 4 | 1e-2    | 9.3e-02 | 1.6e-01| 1.2e-01 | 1.2e-01| 2.7e-02| 8.1e-02| 3.9e-02| 7.6e-03|
|       | 1e-6    | 3.4e-05 | 2.1e-05| 4.2e-05| 5.2e-05| 2.6e-05| 1.8e-04| 1.4e-04| 2.7e-05|
| Ins 5 | 1e-2    | 1e-01   | 1e-01  | 1e-01  | 1.1e-01| 9.2e+00| 8.8e-02| 1.7e+03| 5.9e-03|
|       | 1e-6    | 2.4e-05 | 2.2e-05| 4.3e-05| 2.7e-05| 9.2e+00| 2.6e-05| 1.7e+03| 1.7e-05|
| Ins 6 | 1e-2    | 1e+01   | 7.6e+00| 9.8e+00| 1e+01  | 1.8e+03| 1e+01  | 6.4e+03| 5.2e-01|
|       | 1e-6    | 2.2e-03 | 1.9e-03| 1.8e-03| 2.2e-03| 1.8e-03| 2.1e-03| 6.4e-03| 1.5e-03|

† For instance with DNRT or DNC in Table 7.4, we report the error of the final iterate.

For Sparco test cases, we plot the error of the solution for fixed number of calls to $A$ or $A^t$. The result is depicted in Figure 7.3. As the plot shows, IMRO-2D performs consistently well in these test cases as well.

8. Conclusion. We presented a proximal quasi-Newton method for solving the $l_1$-regularized least square problem. The approximation Hessian matrix in the suggested scheme has the format of identity minus rank one (IMRO) which allows us to compute the proximal point effectively. Two variants of this technique are proposed; in IMRO-1D the approximation model matches the function on a one-dimensional space, and in IMRO-2D it matches the function on a two-dimensional space. Our computational experiments have shown promising results. IMRO-2D, in particular,
outperformed other state-of-the-art solvers in almost all our test cases. An accelerated variant of IMRO, named FIMRO, was also proposed. Despite theoretical advantages of FIMRO, we did not observe significant practical improvement in our experiment not reported in this paper. The performance of FIMRO, however, needs further study as well as the possible directions (one dimensional space) in IMRO-1D. Other possible directions to pursue are to adapt a suitable line search for IMRO, and apply a continuation scheme to IMRO. Although in theory the convergence rate of IMRO does not depend on the regularization parameter, a continuation scheme may enhance the performance of IMRO in practice.
Fig. 7.2. Accuracy of the Solution for BPDN Solvers

(a) Ins 1
(b) Ins 2
(c) Ins 3
(d) Ins 4
(e) Ins 5
(f) Ins 6
Fig. 7.3. Accuracy of the Solution for Sparco Test Cases

(a) Sparco(5)

(b) Sparco(9)

(c) Sparco(10)

(d) Sparco(903)
REFERENCES

[1] R. Baraniuk. Compressive sensing. *IEEE Signal Processing Magazine*, 24(4):118–121, 2007.
[2] J. Barzilai and J. Borwein. Two point step size gradient method. *IMA J. Numer. Anal.*, 8:141–148, 1988.
[3] A. Beck and M. Teboulle. Fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM J. Imaging Sci.*, 2:183–202, 2009.
[4] S. Becker, J. Bobin, and E. Candès. NestA: a fast and accurate first-order method for sparse recovery. Available from: \(\text{http://www-stat.stanford.edu/~candes/nesta/#code}\).
[5] S. Becker, J. Bobin, and E. Candès. NestA: a fast and accurate first-order method for sparse recovery. *SIAM J. Imaging Sci.*, 4(1):1–39, 2011.
[6] S. Becker, E. J. Candès, and M. Grant. Software: Templates for convex cone problems with applications to sparse signal recovery (TFOCS), 2011. Available from: \(\text{http://cvxr.com/tfocs/paper/}\).
[7] S. Becker, E. J. Candès, and M. Grant. Templates for convex cone problems with applications to sparse signal recovery. *Mathematical Programming Computation*, 3(3), 2011.
[8] S. Becker and M. J. Fadili. A quasi-Newton proximal splitting method. *arXiv.org, math.OC*, June 2012. Available from: \(\text{http://arxiv.org/abs/1206.1156v1} [\text{arxiv:1206.1156v1}]\).
[9] E. van den Berg and M. P. Friedlander. SPGL1: A solver for large-scale sparse reconstruction. Available from: \(\text{http://www.cs.ubc.ca/~mpf/spgl1/}\).
[10] E. van den Berg and M. P. Friedlander. In pursuit of a root. Tech. Rep. TR-2007-19, Department of Computer Science, University of British Columbia, Vancouver, June 2007. Available from: \(\text{http://www.optimization-online.org/DB_HTML/2007/06/1708.html}\).
[11] E. van den Berg and M. P. Friedlander. Probing the pareto frontier for basis pursuit solutions. *SIAM J. Sci. Comput.*, 31(2):890–912, 2008.
[12] E. van den Berg, M. P. Friedlander, G. Hennenfent, F. Herrmann, R. Saab, and Ö. Yılmaz. Sparco: A testing framework for sparse reconstruction. Technical Report TR-2007-20, Dept. Computer Science, University of British Columbia, Vancouver, October 2007. Available from: \(\text{http://www.cs.ubc.ca/labs/scl/sparco/}\).
[13] J. M. Bioucas-Dias and M. A. T. Figueiredo. TwIST: Two-step iterative shrinkage/thresholding algorithms for linear inverse problems. Available from: \(\text{http://www.lx.it.pt/~bioucas/TwIST/TwIST.htm}\).
[14] J. M. Bioucas-Dias and M. A. T. Figueiredo. A new TwIST: Two-step iterative shrinkage/thresholding algorithms for image restoration. *IEEE Trans. Image Process.*, 16:2992–3004, 2007.
[15] E. G. Birgin, J. M. Martínez, and M. Raydan. Nonmonotone spectral projected gradient methods on convex sets. *SIAM J. on Optim.*, 10(4):1196–1211, 2000.
[16] E. G. Birgin, J. M. Martínez, and M. Raydan. Inexact spectral projected gradient methods on convex sets. *IMA J. Numer. Anal.*, 23(4):539–559, 2003.
[17] K. Bredies and D. A. Lorenz. Linear convergence of iterative soft-thresholding. *Journal of Fourier Analysis and Applications*, 14:813–837, 2008.
[18] R. H. Byrd, P. Lu, J. Nocedal, and C. Zhu. A limited memory algorithm for bound constrained optimization. *AISTATS*, 2009.
[19] E. Candès. Compressive sampling. *International Congress of Mathematics*, 3:1433–1452, 2006.
[20] E. Candès, J. Romberg, and T. Tao. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. on Information Theory*, 52(2):489–509, 2006.
[21] E. Candès and T. Tao. Near optimal signal recovery from random projections: Universal encoding strategies. *IEEE Trans. on Information Theory*, 52(12):5406–5425, 2006.
[22] E. Candès and M. Wakin. An introduction to compressive sampling. *IEEE Signal Processing Magazine*, 25(2):21–30, 2008.
[23] P. Combettes and V. Wajs. Signal recovery by proximal forward-backward splitting. *Multiscale Modeling and Simulation*, 4(4):1168–1200, 2005.
[24] P. L. Combettes and J. C. Pesquet. Proximal thresholding algorithm for minimization over orthonormal bases. *SIAM J. Optim.*, 18:1351–1376, 2007.
[25] P. L. Combettes and J. C. Pesquet. Proximal splitting methods in signal processing. *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, pages 185–212, 2011.
[26] I. Dhillon, D. Kim, and S. Sra. Tackling box-constrained optimization via a new projected quasi-Newton approach. *SIAM J. Sci. Comput.*, 32(0):3548–3563, 2010.
[27] D. Donoho. Compressed sensing. *IEEE Trans. on Information Theory*, 52(4):1289 – 1306, 2006.
[28] M. A. T. Figueiredo and R. D. Nowak. An EM algorithm for wavelet-based image restoration.
IEEE Trans. Image Process., 12:906–916, 2003.

[29] M. A. T. Figueiredo, R. D. Nowak, and S. J. Wright. Software: GPSR (gradient projection for sparse reconstruction). Available from: http://www.lx.it.pt/~mtf/GPSR/

[30] M. A. T. Figueiredo, R. D. Nowak, and S. J. Wright. Software: Sparse reconstruction by separable approximation. Available from: http://www.lx.it.pt/~mtf/SpaRSA/

[31] M. A. T. Figueiredo, R. D. Nowak, and S. J. Wright. Gradient projection for sparse reconstruction: Application to compressed sensing and other inverse problems. IEEE Journal of Selected Topics in Signal Processing, 1:586–597, 2007.

[32] D. Goldfarb, S. Ma, and K. Scheinberg. Fast alternating linearization methods for minimizing the sum of two convex functions. Mathematical Programming, 141(1-2):349–382, 2013.

[33] M. Gu, L. Lim, and C. Wu. ParNes: a rapidly convergent algorithm for accurate recovery of sparse and approximately sparse signals. Numerical Algorithms, 64(2):321–347, 2013.

[34] E. T. Hale, W. Yin, and Y. Zhang. Software: Fixed point continuation (FPC). Available from: http://www.caam.rice.edu/~optimization/L1/fpc/#soft

[35] E. T. Hale, W. Yin, and Y. Zhang. A fixed-point continuation method for $l_1$-regularized minimization with applications to compressed sensing. Technical report, Rice University, 2007.

[36] E. T. Hale, W. Yin, and Y. Zhang. Fixed-point continuation for $l_1$-minimization: Methodology and convergence. SIAM J. Optim., 19:1107–1130, 2008.

[37] S. Kim, K. Koh, M. Lustig, S. Boyd, and D. Gorinevsky. An interior-point method for large-scale $l_1$-regularized least squares. IEEE Journal of Selected Topics in Signal Processing, 1(4), 2007.

[38] K. Koh, S. Kim, S. Boyd, and Y. Lin. $l_1$-ls: Simple MATLAB solver for $l_1$-regularized least squares problems. Available from: http://www.stanford.edu/~boyd/l1_ls/

[39] J. D. Lee, Yuekai Sun, and Michael A. Saunders. Proximal Newton-type methods for minimizing composite functions, 2013. Available from: arXiv:1206.1623v11

[40] A. S. Lewis and M. L. Overton. Nonsmooth optimization via quasi-Newton methods. math. program., 141:135–163, 2013.

[41] D. A. Lorenz. LiTestPack: A software to generate test instances for $l_1$ minimization problems., 2011. http://www.tubraunschweig.de/iaa/personal/lorenz/littestpack.

[42] Y. E. Nesterov. A method of solving a convex programming problem with convergence rate $o(n^2)$. Soviet mathematics, Doklady, 27(2):372–376, 1983.

[43] Y. E. Nesterov. Introductory Lectures on Convex Optimization: A Basic Course. Springer, 2004.

[44] Y. E. Nesterov. Smooth minimization of nonsmooth functions. Math. Programming, 103:127–152, 2005.

[45] M. Schmidt, E. van den Berg, M. Friedlander, and K. Murphy. Optimizing costly functions with simple constraints: A limited-memory projected quasi-Newton algorithm. SIAM J. Sci. Comput., 16(5):1190–1208, 1995.

[46] Z. Wen, W. Yin, D. Goldfarb, and Y. Zhang. FPC AS: A MATLAB solver for $l_1$-regularization problems. Available from: http://www.caam.rice.edu/~optimization/L1/fpc/AS/

[47] Z. Wen, W. Yin, D. Goldfarb, and Y. Zhang. A fast algorithm for sparse reconstruction based on shrinkage, subspace optimization, and continuation. SIAM J. Sci. Comput., 32:1832–1857, 2010.

[48] S. J. Wright, R. D. Nowak, and M. A. T. Figueiredo. Sparse reconstruction by separable approximation. IEEE Transactions on Signal Processing, 57(7):2479–2493, 2009.

[49] T. Yamamoto, M. Yamagishi, and I. Yamada. Adaptive proximal forward-backward splitting for sparse system identification under impulsive noise. 20th European Signal Processing Conference (EUSIPCO), pages 2620 – 2624, 2012.

[50] J. Yang and Y. Zhang. Alternating direction algorithms for $l_1$-problems in compressive sensing. Technical report, Rice University, 2009.

[51] J. Yang, Y. Zhang, and W. Yin. A fast alternating direction method for tv1-2l signal reconstruction from partial Fourier data. IEEE J. Sel. Top. Signal Process., 4:228–297, 2010.

[52] W. Yin, S. Osher, D. Goldfarb, and J. Darbon. Bregman iterative algorithms for $l_1$-minimization with applications to compressed sensing. SIAM J. Imaging Sciences, 1(1):143–168, 2008.

[53] J. Yu, S. V. N. Vishwanathan, S. Günter, and N. N. Schraudolph. A quasi-Newton approach to nonsmooth convex optimization problems in machine learning. J. Mach. Learn. Res., 11:1145–1200, 2010.