Computing Height-Optimal Tangles Faster

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Abstract. We study the following combinatorial problem. Given a set of \( n \) \( y \)-monotone wires, a tangle determines the order of the wires on a number of horizontal layers such that the orders of the wires on any two consecutive layers differ only in swaps of neighboring wires. Given a multiset \( L \) of swaps (that is, unordered pairs of numbers between 1 and \( n \)) and an initial order of the wires, a tangle realizes \( L \) if each pair of wires changes its order exactly as many times as specified by \( L \). The aim is to find a tangle that realizes \( L \) using the smallest number of layers. We show that this problem is NP-hard, and we give an algorithm that computes an optimal tangle for \( n \) wires and a given list \( L \) of swaps in \( O((2|L|/n^2 + 1)^{n^2/2} \cdot \varphi \cdot n) \) time, where \( \varphi \approx 1.618 \) is the golden ratio. We can treat lists where every swap occurs at most once in \( O(n!\varphi^n) \) time. We implemented the algorithm for the general case and compared it to an existing algorithm. Finally, we discuss feasibility for lists with a simple structure.

1 Introduction

The subject of this paper is the visualization of so-called chaotic attractors, which occur in chaotic dynamic systems. Such systems are considered in physics, celestial mechanics, electronics, fractals theory, chemistry, biology, genetics, and population dynamics. Birman and Williams [3] were the first to mention tangles as a way to describe the topological structure of chaotic attractors. They investigated how the orbits of attractors are knotted. Later Mindlin et al. [6] showed how to characterize attractors using integer matrices that contain numbers of swaps between the orbits. Our research is based on a recent paper of Olszewski et al. [7]. In the framework of their paper, one is given a set of wires that hang off a horizontal line in a fixed order, and a multiset of swaps between the wires; a tangle then is a visualization of these swaps, i.e., an order in which the swaps are performed, where only adjacent wires can be swapped and disjoint swaps can be done simultaneously. For an example of a list of swaps (described by an \((n \times n)\)-matrix) and a tangle that realizes this list, see Fig. 1. Olszewski et al. gave an algorithm for minimizing the height of a tangle. They didn’t analyze the asymptotic running time of their algorithm (which we estimate below), but tested it on a benchmark set.
Wang \cite{8} used the same optimization criterion for tangles, given only the final permutation. She showed that there is always a height-optimal tangle where no swap occurs more than once. She used odd-even sort, a parallel variant of bubble sort, to compute tangles with at most one layer more than the minimum. Bereg et al. \cite{12} considered a similar problem. Given a final permutation, they showed how to minimize the number of bends or moves (which are maximal “diagonal” segments of the wires).

**Framework, Terminology, and Notation.** We modify the terminology of Olszewski et al. \cite{7} in order to introduce a formal algebraic framework for the problem. Given \( n \) wires, a (swap) \( L = (l_{ij}) \) of order \( n \) is a symmetric \( n \times n \) matrix with non-negative entries and zero diagonal. The length of \( L \) is \( |L| = \sum_{i<j} l_{ij} \). A list \( L' = (l'_{ij}) \) is a sublist of \( L \) if \( l'_{ij} \leq l_{ij} \) for each \( i, j \in [n] \). A list is simple if all its entries are zeros or ones.

A permutation is a bijection of the set \([n] = \{1, \ldots, n\}\) onto itself. The set \( S_n \) of all permutations of the set \([n]\) is a group whose multiplication is a composition of maps (i.e., \((\pi \sigma)(i) = \pi(\sigma(i))\) for each pair of permutations \( \pi, \sigma \in S_n \) and each \( i \in [n] \)). The identity of the group \( S_n \) is the identity permutation \( \text{id}_n \). We write a permutation \( \pi \in S_n \) as the sequence of numbers \( \pi^{-1}(1)\pi^{-1}(2)\ldots\pi^{-1}(n) \). For instance, the permutation \( \pi \) of \([4]\) with \( \pi(1) = 3 \), \( \pi(2) = 4 \), \( \pi(3) = 2 \), and \( \pi(4) = 1 \) is written as \( 4312 \). We denote the set of all permutations of order 2 in \( S_n \) by \( S_{n,2} \), that is, \( \pi \in S_{n,2} \) if and only if \( \pi\pi = \text{id}_n \) and \( \pi \neq \text{id}_n \). For example, \( 2143 \in S_{4,2} \).

For \( i, j \in [n] \) with \( i \neq j \), the swap \( ij \) is the permutation that exchanges \( i \) and \( j \), whereas the other elements of \([n]\) remain fixed. A set \( S \) of swaps is disjoint if each element of \([n]\) participates in at most one swap of \( S \). Therefore, the product \( \prod S \) of all elements of a disjoint set \( S \) of swaps does not depend on the order of factors and belongs to \( S_{n,2} \). Conversely, for each permutation \( \varepsilon \in S_{n,2} \) there exists a unique disjoint set \( S(\varepsilon) \) of swaps such that \( \varepsilon = \prod S(\varepsilon) \).

A permutation \( \pi \in S_n \) supports a permutation \( \varepsilon \in S_{n,2} \) if, for each swap \( ij \in S(\varepsilon) \), \( i \) and \( j \) are neighbors in the sequence \( \pi \). By induction with respect to \( n \), we can easily show that any permutation \( \pi \in S_n \) supports exactly \( F_{n+1} - 1 \) permutations of order 2, where \( F_n \) is the \( n \)-th number in the Fibonacci sequence.

Permutations \( \pi \) and \( \sigma \) are adjacent if there exists a permutation \( \varepsilon \in S_{n,2} \) such that \( \pi \) supports \( \varepsilon \) and \( \sigma = \pi \varepsilon \). In this case, \( \sigma \varepsilon = \pi \varepsilon \varepsilon = \pi \) and \( \sigma \) supports \( \varepsilon \), too. A tangle \( T \) of height \( h \) is a sequence \( \langle \pi_1, \pi_2, \ldots, \pi_h \rangle \) of permutations in which every two consecutive permutations are adjacent. A tangle can also be viewed as a sequence of \( h - 1 \) layers, each of which is a set of disjoint swaps. A subtangle of \( T \) is a sequence \( \langle \pi_{k}, \pi_{k+1}, \ldots, \pi_{\ell} \rangle \) of consecutive permutations of \( T \). For a tangle \( T \), we define \( L(T) = (l_{ij}) \) as the symmetric \( n \times n \) matrix with zero diagonal, where \( l_{ij} \) is the number of occurrences of swap \( ij \) in \( T \). We say that \( T \) realizes the list \( L(T) \). To make the reader familiar with our formalism, we observe the following.

**Observation 1** The tangle in Fig. 7 realizes the list \( L_n \) specified there; all tangles that realize \( L_n \) have the same order of swaps along each wire.
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Our Contribution

A list is $\pi$-feasible if it can be realized by a tangle starting from a permutation $\pi$. An id$_n$-feasible list is feasible. For example, the list defined by the two swaps 13 and 24 is $\pi$-feasible. By $E$, we denote the (simple) list $E = (e_{ij})$ with $e_{ij} = 1$ if $i \neq j$, and $e_{ij} = 0$ otherwise. This list is feasible for any permutation; a tangle realizing $E$ is commonly known as a pseudo-line arrangement. So tangles can be thought of as generalizations of pseudo-line arrangements where the numbers of swaps are prescribed and even feasibility becomes a difficult question.

A list $L = (l_{ij})$ can also be considered a multiset of swaps, where $l_{ij}$ is the multiplicity of swap $ij$. By $ij \in L$ we mean $l_{ij} > 0$. A tangle is simple if all its permutations are distinct. Note that the height of a simple tangle is at most $nl$.

The height $h(L)$ of a feasible list $L$ is the minimum height of a tangle that realizes $L$. A tangle $T$ is optimal if $h(T) = h(L(T))$. In the TANGLE-HEIGHT MINIMIZATION problem, we are given a swap list $L$ and the goal is to compute an optimal tangle $T$ realizing $L$. As initial wire order, we always assume id$_n$.

Our Contribution. We show that TANGLE-HEIGHT MINIMIZATION is NP-hard (see Section 2). We give an exact algorithm for simple lists running in $O(n!\varphi^n)$ time, where $\varphi = \frac{\sqrt{5}+1}{2} \approx 1.618$ is the golden ratio, and an exact algorithm for general lists running in $O((2|L|/n^2+1)^{n^2/2}\varphi^n n)$ time, which is polynomial in $|L|$ for fixed $n \geq 2$ (see Section 3). We implemented the algorithm for general lists.

Proof. For $i, j \in [n - 2]$ with $i \neq j$, the wires $i$ and $j$ swap exactly once, so their order reverses. Additionally, each wire $i \in [n - 2]$ swaps twice with the wire $k \in \{n-1, n\}$ that has the same parity as $i$. Observe that wire $i \in [n - 2]$ must first swap with each $j \in [n - 2]$ with $j > i$, then twice with the correct $k \in \{n-1, n\}$, say $k = n$, and finally with each $j' \in [n - 2]$ with $j' < i$. Otherwise, if $i$ swaps with $i - 1$ before swapping with $n$, then $i$ cannot reach $n$ because $i - 1$ swaps only with $n - 1$ among the two wires $\{n - 1, n\}$ and thus separates $i$ from $n$. This establishes the unique order of swaps along each wire.

Fig. 1. A list $L_n$ for $n$ wires and a tangle of minimum height $h = 3n - 4$ realizing $L_n$ for id$_n$. Here, $n = 7$. The tangle is not simple because $\pi_2 = \pi_4$. $L_n = \left(\begin{array}{cccccc} 0 & 1 & 1 & \ldots & 1 & 0 & 2 \\ 1 & 0 & 1 & \ldots & 1 & 2 & 0 \\ 1 & 1 & 0 & \ldots & 1 & 0 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & 0 & 0 & 2 \\ 0 & 2 & 0 & \ldots & 0 & 0 & n - 1 \\ 2 & 0 & 2 & \ldots & 2 & n - 1 & 0 \end{array}\right)$ (The bold zeros and twos must be swapped if $n$ is even.)
and compared it to the algorithm of Olszewski et al. \cite{7} using their benchmark set (see Section \[4\]). We show that the asymptotic runtimes of the algorithms of Olszewski et al. \cite{7} for simple and for general lists are $O(\varphi^{|L|/5} - 1/n^2)$ and $2^{O(n^2)}$, respectively. In Section \[5\] we discuss feasibility for lists with simple structure.

2 Complexity

We show the NP-hardness of TANGLE-HEIGHT MINIMIZATION by reduction from 3-PARTITION. An instance of 3-PARTITION is a multiset $A$ of $3n$ positive integers $n_1, \ldots, n_{3m}$, and the task is to decide whether $A$ can be partitioned into $m$ groups of three elements each that all sum up to the same value $B = \sum_{i=1}^{3m} n_i/m$. 3-PARTITION remains NP-hard if restricted to instances where $B$ is polynomial in $m$, and $B/4 < n_1 < B/2$ for each $i \in [3m]$ \cite{4}. We reduce from this version.

**Theorem 1.** The decision version of TANGLE-HEIGHT MINIMIZATION is NP-hard.

**Proof.** Given an instance $A$ of 3-PARTITION, we construct in polynomial time a list $L$ of swaps such that there is a tangle $T$ realizing $L$ with height at most $H = 2m^2B + 7m^2$ if and only if $A$ is a yes-instance of 3-PARTITION.

In $L$, we use two inner wires $\omega$ and $\omega'$ with $\omega = \omega + 1$ that swap $2m$ times. Thus, in a tangle realizing $L$, $\omega$ and $\omega'$ provide a twisted structure with $m + 1$ "loops" of $\omega$ and $\omega'$ ($\omega$ on the left side and $\omega'$ on the right side) and $m$ "loops" of $\omega'$ and $\omega$ ($\omega'$ on the left side and $\omega$ on the right side). We call them $\omega-\omega'$ loops and $\omega'-\omega$ loops, respectively. The first $\omega-\omega'$ loop is open, that is, it is bounded by the start permutation and the first $\omega-\omega'$ swap. Symmetrically, the last $\omega-\omega'$ loop is open. All other $\omega-\omega'$ loops and all $\omega'-\omega$ loops are closed, that is, they are bounded by two consecutive $\omega-\omega'$ swaps. Apart from $\omega$ and $\omega'$, the list $L$ uses three different types of wires. Refer to Fig. 2 for an illustration.

We use the first type of wires of $L$ to represent the numbers in $A$. To this end, we introduce wires $\alpha_1, \alpha_2, \ldots, \alpha_{3m}$, which we call $\alpha$-wires, and wires $\alpha'_1, \alpha'_2, \ldots, \alpha'_{3m}$, which we call $\alpha'$-wires. Initially, these wires are ordered $\alpha_{3m} < \cdots < \alpha_1 < \omega < \omega' < \alpha'_1 < \cdots < \alpha'_{3m}$. For each $i \in [3m]$, we have $2m^3n_i$ swaps $\alpha_i-\alpha'_i$. We use the factor 2 in the number of $\alpha_i-\alpha'_i$ swaps to make the initial permutation and the final permutation of this part the same. The factor $m^3$ helps us to prove the correctness because it dominates the number of intermediate swaps, which are swaps that cannot occur on the same layer as any $\alpha_i-\alpha'_i$ swap. The intermediate swaps together will require a total height of only $O(m^2)$. Clearly, all $\omega-\omega'$ swaps are intermediate swaps, but we will identify more below.

We now argue why no two $\alpha_i-\alpha'_i$ swaps can appear on the same layer. Clearly, the same swap cannot appear multiple times on the same layer. Also, there cannot be two swaps $\alpha_i-\alpha'_i$ and $\alpha_j-\alpha'_j$ with $i \neq j$ on the same layer because $L$ does not contain any swap $\alpha_i-\alpha'_j$ or $\alpha_j-\alpha'_i$. For the $\alpha$-wires and the $\alpha'$-wires to swap with each other, for each $i \in [3m]$, $L$ has two $\alpha_i-\omega'$ swaps and two $\alpha'_i-\omega$ swaps, but no $\alpha_i-\omega$ swaps and no $\alpha'_i-\omega'$ swaps. Therefore, $\alpha_i-\alpha'_i$ swaps can only
Fig. 2. Example of our reduction from 3-Partition to Tangle-Height Minimization with $A_1 = \{n_1, n_5, n_7\}$, $A_2 = \{n_2, n_4, n_9\}$, $A_3 = \{n_3, n_6, n_8\}$, $m = 3$, $B = \sum_{i=1}^{3m} n_i/m$, and $M = 2m^3$. 
occur within $\omega' - \omega$ loops. Every pair of $\alpha$-wires swaps twice, and so does every pair of $\alpha'$-wires. This allows each $\alpha$-wire to once pass all $\alpha$-wires to its right in order to reach an $\omega' - \omega$ loop, and then to go back. Observe that the order in which the $\alpha$-wires do this is not fixed. Note that some of the $\alpha - \omega'$ and $\alpha' - \omega$ swaps are intermediate swaps that are needed for the $\alpha$- and $\alpha'$-wires to enter and to leave the $\omega' - \omega$ loops.

Using the second type of wires, we now build a rigid structure around the $\omega - \omega'$ loops. We use the construction of Fig. 1 on both sides of the wires $\omega$ and $\omega'$, as follows. For each $i \in [m]$, we introduce wires $\beta_i, \delta_i$ and $\beta'_i, \delta'_i$ such that $\delta_i < \beta_i < \cdots < \delta_1 < \beta_1 < \alpha_{3m}$ and $\alpha'_{3m} < \delta'_i < \beta'_i < \cdots < \delta'_m < \beta'_m$.

On each side, every pair of wires of the second type swaps exactly once – as the green wires in Fig. 1. Hence, in the final permutation, their order is reversed on both sides. For every $i \in [m]$, each of the wires $\beta_i$ and $\delta_i'$ has two swaps with $\omega$ and each of the wires $\delta_i$ and $\beta'_i$ has two swaps with $\omega'$. To allow them to pass the $\alpha$-wires, each $\beta$- and each $\delta$-wire swaps twice with each $\alpha$-wire. The same holds on the right-hand side for the $\alpha'$-, $\beta'$- and $\delta'$-wires. Note that this does not restrict the choice of the $\omega' - \omega$ loops where the $\alpha_i - \alpha_i'$ swaps take place. This is important for the correctness of our reduction.

Further note that some of the swaps of the $\beta$- and $\delta$-wires with the wires $\omega$, $\omega'$, and the $\alpha$-wires are intermediate swaps. For example, $\beta_1$ has to swap with all $\alpha$-wires and twice with the wire $\omega$ before any swap of an $\alpha$- and an $\alpha'$-wire can occur. Accordingly, some of the swaps of the $\beta'$- and $\delta'$-wires with $\omega$, $\omega'$, and the $\alpha'$-wires are intermediate swaps as well. Still, it is obvious that the number of layers needed to accommodate all intermediate swaps is $O(m^2)$.

We denote the third type of wires by $\gamma_i, \gamma'_i$ for $i \in [m]$. On the left side, the $\gamma$-wires are initially on the far left, that is, we set $\gamma_1 < \cdots < \gamma_m < \delta_m$. In the final permutation $\pi$, these $\gamma$-wires end up in between the $\beta$- and $\delta$-wires in the order $\pi(\gamma_1) < \pi(\beta_1) < \cdots < \pi(\gamma_m) < \pi(\beta_m) < \pi(\delta_m)$. On the right side, the $\gamma'$-wires start in a similarly interwoven configuration: $\delta'_1 < \beta'_1 < \gamma'_1 < \cdots < \delta'_m < \beta'_m < \gamma'_m$. The $\gamma'$-wires end up in order on the far right; see Fig. 2.

To ensure that each $\omega' - \omega$ loop has a fixed minimum height, we introduce many swaps between the $\gamma$- and $\beta$-wires, and between the $\gamma'$- and $\beta'$-wires: For $i \in [m]$, every $\gamma_i$ has $(m - i + 1) \cdot 2m^3B$ swaps with $\beta_i$, and every $\gamma'_i$ has $i \cdot 2m^3B$ swaps with $\beta'_i$. Additionally, every $\gamma_i$ has one swap with every $\beta_j$ and $\delta_j$ with $j < i$, and every $\gamma'_i$ has one swap with every $\beta'_j$ and $\delta'_j$ with $j > i$. Recall that the subinstance of $L$ induced by $\delta_m, \beta_m, \ldots, \delta_1, \beta_1, \omega, \omega'$ is the same as the instance $L_n$ with wires 1, 2, $\ldots$, $n$ in Fig. 1. Observe that, for any realization of the list $L_n$, the order of the swaps along each wire is the same as in the tangle on the right side. Therefore, by Observation 1 no $\gamma_i - \beta_i$ swap is above the $i$-th $\omega - \omega'$ loop; see Fig. 2. (Recall that we start counting from the first open $\omega - \omega'$ loop.) Accordingly, no $\gamma'_i - \beta'_i$ swap is below the $(i + 1)$-th $\omega - \omega'$ loop. Since there are $(m - i + 1) \cdot 2m^3B$ swaps of $\gamma_i - \beta_i$, occurring on different layers, the subtangle below and including the $i$-th $\omega - \omega'$ loop has height at least $(m - i + 1) \cdot 2m^3B$.
the subtangle above and including the \((i + 1)\)-th \(\omega^i-\omega^i\) loop has height at least \(i \cdot 2m^3B\). Thus, the whole tangle has height at least \(2m^4B\).

It remains to prove that there is a tangle \(T\) realizing \(L\) with height at most \(H = 2m^4B + 7m^2\) if and only if \(A\) is a yes-instance of \(3\)-Partition.

First, assume that \(A\) is a yes-instance. Let \(L\) be a tangle constructed in the same way as the example given in Fig. 2. Then it is clear that \(T\) realizes \(L\).

We now estimate the height of \(T\). For each partition of three elements \(n_i, n_j, n_k\) of a solution of \(A\), we assign exactly one \(\omega^i-\omega^i\) loop, in which we let the swaps of the pairs \((\alpha_i, \alpha'_i), (\alpha_j, \alpha'_j), (\alpha_k, \alpha'_k)\) occur. Therefore, every \(\omega^i-\omega^i\) loop has height \(2m^3B + c\), where \(c\) is a small constant for the involved wires to enter and leave the loop. Observe that the additional height for the intermediate swaps we need at the beginning, at the end, and between each two consecutive \(\omega^i-\omega^i\) loops is always at most \(6m + k\) for some small constant \(k\). So in total, the height of the constructed tangle is \(m \cdot (2m^3B + c) + (m + 1) \cdot (6m + k) = 2m^4B + 6m^2 + (c + k + 6)m + k\). This is at most \(H\) for \(m > c + 2k + 6\).

Now, assume that \(A\) is a no-instance. This means that any tangle realizing \(L\) has an \(\omega^i-\omega^i\) loop of height at least \(2m^3(B + 1)\) because there is no 3-Partition of \(A\) and, for each unit of an item in \(A\), there are \(2m^3\) swaps. Assume that the \(i\)-th \(\omega^i-\omega^i\) loop has height at least \(2m^3(B + 1)\). We know that the subtangle from the very beginning to the end of the \(i\)-th \(\omega^i-\omega^i\) loop has height at least \((i - 1) \cdot 2m^3B\) and the subtangle from the beginning of the \((i + 1)\)-th \(\omega^i-\omega^i\) loop to the very end has height at least \((m - i) \cdot 2m^3B\). In between, there is the \(i\)-th \(\omega^i-\omega^i\) loop with height \(2m^3(B + 1)\). Summing these three values up, we have a total height of at least \(2m^3B + 2m^3\). Since this is greater than \(H\) for \(m > 3.5\), we conclude that \(L\) cannot be realized by a tangle of height at most \(H\), and thus our reduction is complete.

\[\square\]

3 Exact Algorithms

The two algorithms that we describe in this section test whether a given list is feasible and, if yes, construct an optimal tangle realizing the list.

For a permutation \(\pi \in S_n\) and a list \(L = (l_{ij})\), we define a map \(\pi L: [n] \rightarrow [n], i \mapsto \pi(i)+|\{j : \pi(i) < \pi(j) \leq n \text{ and } l_{ij} \text{ odd}\}| - |\{j : 1 \leq \pi(j) < \pi(i) \text{ and } l_{ij} \text{ odd}\}|\). For each wire \(i \in [n]\), \(\pi L(i)\) is the position of the wire after all swaps in \(L\) have been applied to \(\pi\). A list \(L\) is called \(\pi\)-consistent if \(\pi L \in S_n\), or, more rigorously, if \(\pi L\) induces a permutation of \([n]\). An \(\text{id}_n\)-consistent list is consistent. For example, the list \(\{12, 23, 13\}\) is consistent, whereas the list \(\{13\}\) is not. If \(L\) is not consistent, then it is clearly not feasible. However, not all consistent lists are feasible e.g., the list \(\{13, 13\}\) is consistent but not feasible. For a list \(L = (l_{ij})\), we define \(1(L) = (l_{ij} \mod 2)\). Since \(\text{id}_n L = \text{id}_n 1(L)\), the list \(L\) is consistent if and only if \(1(L)\) is consistent. We can compute \(1(L)\) and check its consistency in \(O(n + 1|L|) = O(n^2)\) time. Hence, in the sequel we assume that all lists are consistent. For any permutation \(\pi \in S_n\), we define the simple list \(L(\pi) = (l_{ij})\) such that for \(0 \leq i < j \leq n\), \(l_{ij} = 0\) if \(\pi(i) < \pi(j)\), and \(l_{ij} = 1\) otherwise.

We use the following two lemmas which are proved in Appendix A.
Lemma 1. For every permutation \( \pi \in S_n \), \( L(\pi) \) is the unique simple list with \( \text{id}_n L(\pi) = \pi \).

Lemma 2. For every tangle \( T = \langle \pi_1, \pi_2, \ldots, \pi_h \rangle \), we have \( \pi_1 L(T) = \pi_h \).

Simple lists. Let \( L \) be a consistent simple list. Wang’s algorithm [5] creates a simple tangle from \( \text{id}_n L \), so \( L \) is feasible. Let \( T = (\text{id}_n = \pi_1, \pi_2, \ldots, \pi_h = \text{id}_n L) \) be any tangle such that \( L(T) \) is simple. Then, by Lemma 2, \( \text{id}_n L(T) = \pi_h \).

By Lemma 1, \( L(\pi_h) \) is the unique simple list with \( \text{id}_n L(\pi_h) = \pi_h = \text{id}_n L \), so \( L(T) = L(\pi_h) = L \) and thus \( T \) is a realization of \( L \).

We compute an optimal tangle realizing \( L = (l_{ij}) \) as follows. Consider the graph \( G_L \) whose vertex set \( V(G_L) \) consists of all permutations \( \pi \in S_n \) with \( L(\pi) \leq L \) (componentwise). A directed edge \((\pi, \sigma)\) between vertices \( \pi, \sigma \in V(G_L) \) exists if and only if \( \pi \) and \( \sigma \) are adjacent as permutations and \( L(\pi) \cap L(\sigma^{-1}) = \emptyset \); the latter means that the set of (disjoint) swaps whose product transforms \( \pi \) to \( \sigma \) cannot contain swaps from the set whose product transforms \( \text{id}_n \) to \( \pi \). The graph \( G_L \) has at most \( n! \) vertices and maximum degree \( F_{n+1} - 1 \), see introduction (page 2). Notice, that \( F_n = (\varphi^n - (-\varphi)^{-n})/\sqrt{5} \in O(\varphi^n) \). Furthermore, for each \( h \geq 0 \), there is a natural bijection between tangles of height \( h + 1 \) realizing \( L \) and paths of length \( h \) in the graph \( G_L \) from the initial permutation \( \text{id}_n \) to the permutation \( \text{id}_n L \). A shortest such path can be found by BFS in \( O(E(G_L)) = O(n!\varphi^n) \) time.

Theorem 2. For a simple list of order \( n \), TANGE-HEIGHT MINIMIZATION can be solved in \( O(n!\varphi^n) \) time.

General lists. W.l.o.g., assume that \( |L| \geq n/2 \); otherwise, there is a wire \( k \in [n] \) that doesn’t belong to any swap. This wire splits \( L \) into smaller lists with independent realizations. (If there is a swap \( ij \) with \( i < k < j \), then \( L \) is infeasible.)

Let \( L = (l_{ij}) \) be the given list. We compute an optimal tangle realizing \( L \) (if it exists) as follows. Let \( \lambda \) be the number of distinct sublists of \( L \). We consider them ordered non-decreasingly by their length. Let \( L' \) be the next list to consider. We first check its consistency by computing the map \( \text{id}_n L' \). If \( L' \) is consistent, we compute an optimal realization \( T(L') \) of \( L' \) (if it exists), adding a permutation \( \text{id}_n L' \) to the end of a shortest tangle \( T(L'') = \langle \pi_1, \ldots, \pi_h \rangle \) with \( \pi_h \) adjacent to \( \text{id}_n L' \) and \( L'' + L(\langle \pi_h, \text{id}_n L' \rangle) = L' \). This search also checks the feasibility of \( L' \) because such a tangle \( T(L') \) exists if and only if the list \( L' \) is feasible. Since there are \( F_{n+1} - 1 \) permutations adjacent to \( \text{id}_n L' \), we have to check at most \( F_{n+1} - 1 \) lists \( L'' \). Hence, in total we spend \( O(\lambda(F_{n+1} - 1)n) \) time for \( L \). Assuming that \( n \geq 2 \), we bound \( \lambda \) as follows, where we obtain the first inequality from the inequality between arithmetic and geometric means, the second one from Bernoulli’s inequality, and the third one from \( 1 + x \leq e^x \).

\[
\lambda = \prod_{i<j} (l_{ij} + 1) \leq \left( \frac{\sum_{i<j} (l_{ij} + 1)}{n^2} \right) \left( \frac{|L|}{2} + 1 \right) \leq \left( \frac{2|L|}{n^2} + 1 \right)^{\frac{n^2}{2}} \leq e^{|L|}.
\]
**Theorem 3.** For a list $L$ of order $n$, **Tangle-Height Minimization** can be solved in $O((2|L|/n^2 + 1)^{n^2/2} \cdot \phi^n \cdot n)$ time.

### 4 Theoretical and Experimental Comparison

In order to be able to compare the algorithm of Olszewski et al. [7] to ours, we first analyze the asymptotic runtime behavior of the algorithm of Olszewski et al. Their algorithm constructs a search tree whose height is bounded by the height $h(L)$ of an optimal tangle for the given list $L$. The tree has $1 + d + d^2 + \cdots + d^{h(L)-1} = (d^{h(L)} - 1)/(d - 1)$ vertices, where $d = F_{n+1} - 1$ is a bound on the number of edges leaving a vertex. Neglecting the time it takes to deal with each vertex, the total running time is $\Omega((\phi^{(n+1)(h(L)-1)} \cdot 5^{-(h(L)-1)/2} \cdot |L|)$, where $2|L|/n \leq h(L) - 1 \leq |L|$, this is at least $\Omega(\phi^{2|L|} \cdot 5^{-|L|/n} \cdot n)$, which is exponential in $|L|$ for fixed $n \geq 2$ and, hence, slower than our algorithm for the general case if we assume that $|L| \geq n/2$ (see Theorem 3).

It is known (see, e.g., Wang [8]) that, for any simple list $L$, $h(L) \leq n + 1$. This implies that, on simple lists, the algorithm of Olszewski et al. runs in $O((\phi^{(n+1)n} \cdot 5^{-n} \cdot n) = 2^{O(n^2)}$ time, whereas our algorithm for simple lists runs in $O(n! \cdot \phi^n) = 2^{O(n \log n)}$ time.

We implemented the algorithm for general lists (see Theorem 3) and compared the running time of our implementation with the one of Olszewski et al. [7]. Their code and a database of all possible elementary linking matrices (most of them non-simple) of 5 wires (14 instances), 6 wires (38 instances), and 7 wires (115 instances) are available at [https://gitlab.uni.lu/PCOG]. We used their code and their benchmarks to compare our implementations. Both their and our code is implemented in Python3.

The matrices in the benchmark are quite small: the largest instance for 5 wires has 8 swaps, the largest instance for 6 wires has 15 swaps, and the largest instance for 7 wires has 27 swaps. Further, the algorithm of Olszewski et al. could not solve any of the six instances with 7 wires and $\geq 22$ swaps within two hours (while our algorithm solved four of them within 10 seconds and the other two within 50 seconds), so we removed them from the data set. For better comparisons, we additionally created 10 random matrices each for $n = 5$ and $|L| = 9, \ldots, 49$, for $n = 6$ and $|L| = 16, \ldots, 49$, and for $n = 7$ and $|L| = 22, \ldots, 49$. To this end, we randomly and uniformly generated vectors of length $n(n+1)/2$ and sum $|L|$ by drawing samples from a multinomial distribution and rejecting them if the corresponding swap list is not feasible. This gave us 414 instances for 5 wires, 358 instances for 6 wires, and 379 instances for 7 wires in total. Our source code, the benchmarks, and the experimental data are available at [https://github.com/PhKindermann/chaotic-attractors].

We ran our experiments on a single compute node of the High Performance Computing Cluster of the University of Würzburg. This node consists of two Intel Xeon Gold 6134 processors, both with eight cores of 3.20 GHz. The node
Fig. 3. Comparison of our algorithm (blue circles) with the algorithm of Olszewski et al. [7] (red triangles). The means are plotted as a trend curve. The elapsed time is plotted on a log-scale. The shaded regions correspond to randomly generated instances.

Among the benchmark instances of Olszewski et al., our algorithm could solve almost all in less than 1 second, and the maximum running time was 8 seconds for one instance. The benchmark instances that could not be solved within 2 hours by the algorithm of Olszewski et al. could also all be solved in less than 1 minute by our algorithm. We solved all 414 instances for 5 wires within 2 hours. Within the 12-hour time, we solved 303 instances with 6 wires and 333 instances with 7 wires. The algorithm by Olszewski et al. solved 163 instances with 5 wires, 97 instances with 6 wires, and 120 instances with 8 wires, within 12 hours each. Our algorithm used at most 2 GB memory, whereas for the algorithm of Olszewski et al. the 384 GB RAM did not suffice for many instances.
5 Deciding Feasibility

Since computing a tangle of minimum height realizing a given list turned out to be NP-hard, the question arises whether it is already NP-hard to decide if a given list is feasible. As we could not answer this question in its full generality, we are investigating the feasibility for special classes of lists in this section.

Recall that for a list \( L = (l_{ij}) \), we defined \( 1(L) = (l_{ij} \mod 2) \), see Section 3. Now we also define \( 2(L) = (l'_{ij}) \), where \( l'_{ij} = 0 \) if \( l_{ij} = 0 \), \( l'_{ij} = 1 \) if \( l_{ij} \) is odd, and \( l'_{ij} = 2 \) otherwise. Clearly, \( \pi 1(L) = 2(L) = \pi L \) for each \( \pi \in S_n \). A list \( (l_{ij}) \) is even if all \( l_{ij} \) are even, and odd if all non-zero \( l_{ij} \) are odd. A list \( L \) is even if and only if the list \( 1(L) \) is the zero list. A list \( L \) is odd if and only if \( 1(L) = 2(L) \).

Simple Lists. If we restrict our study to simple lists, we can easily decide feasibility. We use the following lemma, which is well-known (see, e.g., Wang [8]).

Lemma 3 (Wang [8]). For any \( n \geq 2 \) and permutations \( \pi, \sigma \in S_n \), there is a tangle \( T \) of height at most \( n + 1 \) that starts from \( \pi \), ends at \( \sigma \), and the list \( L(T) \) is simple.

Proposition 1. A simple list \( L \) is feasible if and only if \( L \) is consistent. Thus, we can check the feasibility of \( L \) in \( O(n + |L|) \) time.

Proof. Clearly, if \( L \) is feasible, then \( L \) is also consistent. If \( L \) is consistent, then \( \text{id}_n \ L \) is a permutation. By Lemma 3 there exists a tangle \( T \) which starts from \( \text{id}_n \), ends at \( \text{id}_n \ L \), and the list \( L(T) \) is simple. By Lemma 2 \( \pi L(T) = \pi L \). By Lemma 1 \( L(T) = L \). So \( L \) is also feasible. As discussed in the beginning of Section 3, we can check the consistency of \( L \) in \( O(n + |L|) \) time, which is equivalent to checking the feasibility of \( L \). \( \square \)

Odd Lists. For odd lists, feasibility reduces to that of simple lists. For \( A \subseteq [n] \), let \( L_A \) be the list that consists of all swaps \( ij \) of \( L \) such that \( i, j \in A \). We prove the following Proposition 2 in Appendix B.

Proposition 2. For \( n \geq 3 \) and an odd list \( L \), the following statements are equivalent:

1. The list \( L \) is feasible.
2. The list \( 1(L) \) is feasible.
3. For each triple \( A \subseteq [n] \), the list \( L_A \) is feasible.
4. For each triple \( A \subseteq [n] \), the list \( 1(L_A) \) is feasible.
5. The list \( L \) is consistent.
6. The list \( 1(L) \) is consistent.
7. For each triple \( A \subseteq [n] \), the list \( L_A \) is consistent.
8. For each triple \( A \subseteq [n] \), the list \( 1(L_A) \) is consistent.

Note that, for any feasible list \( L \), it does not necessarily hold that \( 2(L) \) is feasible; see, e.g. list \( L_n \) from Observation 1.
Even Lists. For even lists, it is not as clear as for odd lists whether we can
decide feasibility efficiently. An even list is always consistent, since it does not
contain odd swaps and the final permutation is the same as the initial one.
We conjecture that the following characterization is true, and we give some
alternative formulations (see Proposition 3).

We say that a list \((l_{ij})\) is non-separable if, for every \(1 \leq i < k < j \leq n\),
\(l_{ik} = l_{kj} = 0\) implies \(l_{ij} = 0\). Clearly, non-separability is a necessary condition
for a list to be feasible. For even lists, we conjecture that this is also sufficient.
Note that any triple \(A \subseteq [n]\) of an even list is feasible if and only if it is non-
separable (which is not true for general lists, e.g., \(L = \{12, 23\}\) is not feasible).

Conjecture 1. Every non-separable even list \(L\) is feasible.

We have verified the correctness of Conjecture 1 for \(n \leq 8\) by testing all
lists using a computer. Moreover, Conjecture 1 is true for sufficiently “rich” lists
according to the following lemma, which we prove in Appendix B.

Lemma 4. Every even non-separable list \(L = (l_{ij})\) with \(l_{ij} \geq n\) or \(l_{ij} = 0\) for
every \(1 \leq i, j \leq n\) is feasible.

We now give some alternative formulations of Conjecture 1. To this end, we
define a minimal feasible (even) list to be a(n even) list where we cannot remove
swaps to obtain another feasible (even) list without creating new zero-entries.
We say that a list is 0–2 if all its entries are either 0 or 2.

Proposition 3. The following claims are equivalent:

1. Every non-separable even list \(L\) is feasible. (Conjecture 1)
2. Every non-separable 0–2 list \(L\) is feasible.
3. For each feasible even list \(L\), the list \(2(L)\) is feasible.
4. Every minimal feasible even list \(L\) is a 0–2 list.

Proof. 1 \(\Rightarrow\) 2. By definition.
2 \(\Rightarrow\) 3. Since the list \(L\) is feasible, it is non-separable and, thus, also the
list \(2(L)\) is non-separable. Since \(2(L)\) is non-separable and 0–2 (because \(L\) is
even), \(2(L)\) is feasible.
3 \(\Rightarrow\) 4. Clearly, a list \(L\) never has fewer swaps than \(2(L)\). Therefore, all
minimal feasible lists are 0–2.
4 \(\Rightarrow\) 1. Let \(L = (l_{ij})\) be an even non-separable list. By Lemma 4, the list
\(nL := (n \cdot l_{ij})\) is feasible. Let \(L'\) be a minimal feasible even list that we obtain from
\(nL\) by removing swaps without creating new zero-entries. Since every minimal
feasible even list \(L'\) is 0–2 by assumption, we have \(L' = 2(L)\). Hence, any tangle
realizing \(L'\) can be extended to a tangle realizing \(L\) using the same procedure as
in the proof of Proposition 2 (2 \(\Rightarrow\) 1), so \(L\) is feasible. \(\square\)
6 Conclusions and Open Problems

Inspired by the practical research of Olszewski et al. [7], we have considered tangle-height minimization. We have shown that the problem is NP-hard, but we note that membership in NP is not obvious because the minimum height can be exponential in the size of the input. We leave open the complexity of the feasibility problem for general lists. Even if feasibility turns out to be NP-hard, can we decide it faster than finding optimal tangles?

For the special case of simple lists, we have a faster algorithm, but its running time of $O(n! \phi^n)$ is still depressing given that odd-even sort [8] can compute a solution of height at most one more than the optimum in $O(n^2)$ time. This leads to the question whether height-minimization is NP-hard for simple lists.

Our most tantalizing open problem, however, is whether Conjecture 1 holds.

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Appendix

A Omitted proofs of Section 3

Lemma 1. For every permutation \( \pi \in S_n \), \( L(\pi) \) is the unique simple list with \( \text{id}_n L(\pi) = \pi \).

Proof. By definition, \( \text{id}_n L(\pi) \) is a map from \([n]\) to \( \mathbb{Z} \),

\[
i \mapsto i + |\{j : i < j \leq n \text{ and } \pi(i) > \pi(j)\}| - |\{j : 1 \leq j < i \text{ and } \pi(i) < \pi(j)\}|
= i + |\{j : i < j \leq n \text{ and } \pi(i) > \pi(j)\}| + |\{j : 1 \leq j < i \text{ and } \pi(i) > \pi(j)\}|
- |\{j : 1 \leq j < i \text{ and } \pi(i) < \pi(j)\}|
= i + |\{j : 1 < j \leq n \text{ and } \pi(i) > \pi(j)\}| - |\{j : 1 \leq j < i\}|
= i + (\pi(i) - 1) - (i - 1) = \pi(i).
\]

Assume that \( L = (l_{ij}) \) is a simple list such that \( \text{id}_n L = \pi \). That is, for each \( i \in [n] \), we have
\[
\pi(i) = i + |\{j : i < j \leq n \text{ and } l_{ij} = 1\}| - |\{j : 1 \leq j < i \text{ and } l_{ij} = 1\}|
\]

We show that the list \( L \) is uniquely determined by the permutation \( \pi \) by induction with respect on \( n \). For \( n = 2 \), there exist only two simple lists \((0, 0)\) and \((0, 1)\). Since \( \text{id}_2 (0, 0) = \text{id}_2 (0, 1) = 21 \), we have uniqueness. Now assume that \( n \geq 3 \) and we have already proved the induction hypothesis for \( n - 1 \). Then, for \( k = \pi^{-1}(n) \), we have
\[
n = \pi(\pi^{-1}(n)) = \pi(k) = k + |\{j : k < j \leq n \text{ and } l_{kj} = 1\}|
- |\{j : 1 \leq j < k \text{ and } l_{kj} = 1\}|.
\]

Since
\[
|\{j : k < j \leq n \text{ and } l_{kj} = 1\}| \leq |\{j : k < j \leq n\}| = n - k
\]
and
\[
|\{j : 1 \leq j < k \text{ and } l_{kj} = 1\}| \geq 0,
\]
the equality holds if and only if \( l_{kj} = 1 \) for each \( k < j \leq n \) and \( l_{kj} = 0 \) for each \( 1 \leq j < k \). These conditions determine the \( k \)-th row (and column) of the matrix \( L \).

It is easy to see that a map \( \pi' : [n - 1] \to \mathbb{Z} \) such that \( \pi'(i) = \pi(i) \) for \( i < k \) and \( \pi'(i) = \pi(i + 1) \) for \( i \geq k \) is a permutation.

Let \( L' = (l'_{ij}) \) be a simple list of order \( n - 1 \) obtained from \( L \) by removing the \( k \)-th row and column. For each \( i \in [n - 1] \), we have
\[
\text{id}_n L'(i) = i + |\{j : i < j \leq n - 1 \text{ and } l'_{ij} = 1\}| - |\{j : 1 \leq j < i \text{ and } l'_{ij} = 1\}|
\]

If \( i < k \) then \( l_{ik} = 0 \). Thus,
\[
\text{id}_n L'(i) = i + |\{j : i < j \leq n \text{ and } l_{ij} = 1\}| - |\{j : 1 \leq j < i \text{ and } l_{ij} = 1\}|
= \pi(i) = \pi(i).
\]
If on the other hand \( i \geq k \), then \( l_{i+1,k} = 1 \). Hence,

\[
id_n L'(i) = \begin{align*}
i + |\{ j : i < j \leq n - 1 \text{ and } l_{i+1,j+1} = 1 \}| \\
- |\{ j : 1 \leq j < i + 1 \text{ and } l_{i+1,j} = 1 \}| + 1 \\
= i + 1 + |\{ j : i + 1 < j \leq n \text{ and } l_{i+1,j+1} = 1 \}| \\
- |\{ j : 1 \leq j < i + 1 \text{ and } l_{i+1,j} = 1 \}| \\
= \pi(i + 1) = \pi'(i).
\end{align*}
\]

Thus, \( id_n L' = \pi' \). By the inductive hypothesis, the list \( L' \) is uniquely determined by the permutation \( \pi' \), so the list \( L \) is uniquely determined by the permutation \( \pi \).

\[ \square \]

**Lemma 2.** For every tangle \( T = \langle \pi_1, \pi_2, \ldots, \pi_h \rangle \), we have \( \pi_1 L(T) = \pi_h \).

*Proof.* We have \( L(T) = (l_{ij}) \), where

\[
l_{ij} = |\{ t : 1 \leq t < h, \pi^{-1}_t \pi_{t+1}(i) = j \text{ and } \pi^{-1}_t \pi_{t+1}(j) = i \}|
\]

for each distinct \( i,j \in [n] \). If \( \pi_1(i) < \pi_1(j) \), then it is easy to see that \( \pi_h(i) < \pi_h(j) \) if and only if \( l_{ij} \) is even. On the other hand, by the definition for each \( i \in [n] \),

\[
\pi_1 L(T)(i) = \begin{align*}
\pi_1(i) + |\{ j : \pi_1(i) < \pi_1(j) \leq n \text{ and } l_{ij} \text{ is odd} \}| \\
- |\{ j : 1 \leq \pi_1(j) < \pi_1(i) \text{ and } l_{ij} \text{ is odd} \}|
\end{align*}
\]

\[
= \begin{align*}
\pi_1(i) + |\{ j : \pi_1(i) < \pi_1(j) \leq n \text{ and } \pi_h(i) > \pi_h(j) \}| \\
- |\{ j : 1 \leq \pi_1(j) < \pi_1(i) \text{ and } \pi_h(i) < \pi_h(j) \}|
\end{align*}
\]

Now, similarly to the beginning of the proof of Lemma 1, we can show that \( \pi_1 L(T)(i) = \pi_h(i) \).

\[ \square \]

**B Omitted proofs of Section 5**

**Proposition 2.** For \( n \geq 3 \) and an odd list \( L \), the following statements are equivalent:

1. The list \( L \) is feasible.
2. The list \( 1(L) \) is feasible.
3. For each triple \( A \subseteq [n] \), the list \( L_A \) is feasible.
4. For each triple \( A \subseteq [n] \), the list \( 1(L_A) \) is feasible.
5. The list \( L \) is consistent.
6. The list \( 1(L) \) is consistent.
7. For each triple \( A \subseteq [n] \), the list \( L_A \) is consistent.
8. For each triple \( A \subseteq [n] \), the list \( 1(L_A) \) is consistent.
Lemma 4. Every even non-separable list $L = (l_{ij})$ with $l_{ij} \geq n$ or $l_{ij} = 0$ for every $1 \leq i, j \leq n$ is feasible.

Proof. We define a binary relation $\leq_L$ on the set of wires $[n]$ for each $i, j \in [n]$ as follows. We set $i \leq_L j$ if and only if $i \leq j$ and $l_{ij} = 0$. Since the list $L$ is non-separable, the relation $\leq_L$ is transitive, so it is a partial order. The dimension $d$ of a partial order on the set $[n]$ is at most $\left\lfloor \frac{n}{2} \right\rfloor$, so this is, there exist $d$ linear orders $\leq_1, \ldots, \leq_d$ on $[n]$ such that for each $i, j \in [n]$ we have $i \leq_L j$ if and only if $i \leq t_j$ for each $t \in [d]$. So $\leq_L$ can be seen as the intersection of $\leq_1, \ldots, \leq_d$. For each linear order $\leq_k$ with $k \in [d]$, let $\pi_k$ be the (unique) permutation of $[n]$ such that $\pi_k^{-1}(1) \leq \pi_k^{-1}(2) \leq \cdots \leq \pi_k^{-1}(n)$. Observe that $L(\pi) = 1(L)$, so the list $1(L)$ is feasible.

4. Follows from Proposition 1, because the list $1(L)$ is simple.
8. Follows from the equality $id_n 1(L_A) = id_n L_A$.
3. For every triple $A \subseteq [n]$, we can argue as in the proof (2 $\Rightarrow$ 1).
1. We decompose $L$ into $1(L)$ and $L' = (L - 1(L))$. Note that $L' = (l'_{ij})$ is an even list. Let $ij \in L'$. Then $ij \in 1(L)$ because $L$ is odd. Consider a tangle $T$ realizing $1(L)$. Let $\pi$ be the layer in $T$ where swap $ij$ occurs. Behind $\pi$, insert $l_{ij}$ new layers such that the difference between one such layer and its previous layer is only the swap $ij$. Observe that every second new layer equals $\pi$ – in particular the last one, which means that we can continue the tangle with the remainder of $T$. Applying this operation to all swaps in $L'$ yields a tangle realizing $L$.
2. We prove the proposition by proving three cycles of implications $1 \Rightarrow 5 \Rightarrow 6 \Rightarrow 2 \Rightarrow 1, 3 \Rightarrow 7 \Rightarrow 8 \Rightarrow 4 \Rightarrow 3, \text{ and } 1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$.

Proof. We prove the proposition by proving three cycles of implications $1 \Rightarrow 5 \Rightarrow 6 \Rightarrow 2 \Rightarrow 1, 3 \Rightarrow 7 \Rightarrow 8 \Rightarrow 4 \Rightarrow 3, \text{ and } 1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$.

1. Clearly, all feasible lists are consistent.
5. Consistency of $L$ means that $id_n L \in S_n$. Since $id_n 1(L) = id_n L$, the list $1(L)$ is consistent, too.
6. Follows from Proposition 1 because the list $1(L)$ is simple.
2. We decompose $L$ into $1(L)$ and $L' = (L - 1(L))$. Note that $L' = (l'_{ij})$ is an even list. Let $ij \in L'$. Then $ij \in 1(L)$ because $L$ is odd. Consider a tangle $T$ realizing $1(L)$. Let $\pi$ be the layer in $T$ where swap $ij$ occurs. Behind $\pi$, insert $l_{ij}$ new layers such that the difference between one such layer and its previous layer is only the swap $ij$. Observe that every second new layer equals $\pi$ – in particular the last one, which means that we can continue the tangle with the remainder of $T$. Applying this operation to all swaps in $L'$ yields a tangle realizing $L$.
3. Consistency of $L$ means that $id_n L \in S_n$. Since $id_n 1(L) = id_n L$, the list $1(L)$ is consistent, too.
7. Clearly, all feasible lists are consistent.
4. Follows from Proposition 1 because the list $1(L)$ is simple.
8. Follows from the equality $id_n 1(L_A) = id_n L_A$.
1. We decompose $L$ into $1(L)$ and $L' = (L - 1(L))$. Note that $L' = (l'_{ij})$ is an even list. Let $ij \in L'$. Then $ij \in 1(L)$ because $L$ is odd. Consider a tangle $T$ realizing $1(L)$. Let $\pi$ be the layer in $T$ where swap $ij$ occurs. Behind $\pi$, insert $l_{ij}$ new layers such that the difference between one such layer and its previous layer is only the swap $ij$. Observe that every second new layer equals $\pi$ – in particular the last one, which means that we can continue the tangle with the remainder of $T$. Applying this operation to all swaps in $L'$ yields a tangle realizing $L$.
the remaining (even) number of $l_{ij} - l'_{ij}$ swaps of the wires $i$ and $j$ for each non-zero entry of $L$ after an execution of an $i \rightarrow j$ swap in $T'$. Thus, the feasibility of $L$ follows from the feasibility of $L'$. \hfill \square