The Exponential Cubic B-spline Algorithm for Burgers’ Equation

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Abstract

Purpose – The exponential cubic B-spline functions are used to set up the collocation method for finding solutions of the Burgers’ equation. The effect of the exponential cubic B-splines in the collocation method is sought by studying four text problems.

Design – The Burgers’ equation is fully-discretized using the Crank-Nicolson method for the time discretization and exponential cubic B-spline function for discretization of spatial variable.

Findings – The exponential cubic B-spline methods have produced acceptable solution of Burgers’ equation.

Originality/value – Burgers’ equation has never been solved by the collocation method using exponential cubic B-spline.

Keywords: collocation methods, exponential cubic B-spline, Burgers’ equation

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1 Introduction

This paper is concerned with adapting the exponential cubic B-spline function into the collocation method to develop a numerical method for finding numerical solutions of the Burgers’ equation of the form

\[ U_t + UU_x - \lambda U_{xx} = 0, \ a \leq x \leq b, \ t \geq 0 \]  

(1)

with the initial condition

\[ U(x, 0) = f(x), \ a \leq x \leq b \]  

(2)

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and the boundary conditions

\[ U(a, t) = \sigma_1, U(b, t) = \sigma_2 \]  

where subscripts \( x \) and \( t \) denote differentiation, \( \lambda = \frac{1}{\text{Re}} > 0 \) and \( \text{Re} \) is the Reynolds number characterizing the strength of viscosity, \( \sigma_1, \sigma_2 \) are the constants \( u = u(x,t) \) is a sufficiently differentiable unknown function and \( f(x) \) is a bounded function. Initial condition and boundary conditions will be defined in the later section depending on the test problems.

Burgers’ equation (1) has been widely used in modelling various problem in science and engineering with a broad range of applications, including gas dynamics, traffic flow, wave propagation in acoustics and hydrodynamics, etc. It is solved exactly for an arbitrary initial condition (1) so that it has been used as a test equation for the numerical methods. But some exact solutions contain series solution and these solutions are unpractical because of the slow convergence of the Fourier series when the small values of viscosity constant are used. Thus the numerical studies are necessary to find improving solutions of the Burgers’ equation. Besides the Burgers’ equation with small viscosity constant gives rise to appearance of steep front and shock waves in the solutions. Therefore one comes across complexity in writing the influential numerical method for finding solutions of the Burgers’ equation. Many numerical studies have been published on the numerical solution of the Burgers’ equation. Especially, attention has been given in deriving the numerical methods for the numerical solution of the Burgers’ equation with small values of viscosity constant.

Burgers’ equation was first introduced by (2). Solutions of the Burgers’ equation were presented by using some numerical methods with splines. A cubic spline collocation procedure has been developed for the numerical solution of the Burgers’ equation in the papers. A B-spline Galerkin Method is described to solve the Burgers’ equation over the both fixed and varied distribution of knots to define the B-splines in the studies of Davies [3, 6]. A Numerical method [7, 8, 9] is developed for solving the Burgers’ equation by using splitting method and cubic spline approximation method. In [10, 11, 15, 16, 17, 18, 20, 24, 23, 31], numerical solutions of the one-dimensional Burgers’ equation are obtained by a methods based on collocation of quadratic, cubic, quintic and septic B-splines over finite elements respectively, in which approximate functions in the collocation method for the Burgers’ equation are constructed by using the various degree B-splines. Galerkin methods based on various degree B-splines have been set up to find approximate solutions of the Burgers’ equation in the studies [6, 14, 18]. The least square method is combined with the B-splines to form numerical methods for solving the Burgers’ equation in the works [12, 30]. Numerical solutions of the Burgers’ equation is presented based on the cubic B-spline quasi-interpolation and the compact finite difference method in [26, 28]. Taylor-collocation and Taylor-Galerkin methods for the numerical solutions of the Burgers’ equation are formed by using both cubic and quadratic B-splines in the study [27] respectively. Differential quadrature methods based on cubic and quartic B-splines are set up to solve the Burgers equation in the works [29, 33, 34]. The hybrid spline difference method is developed to solve the Burgers’ equation by Chi-Chang Wang and
his colleagues[32]. The nonlinear polynomial splines method been proposed for solving the Burgers equation by M. Caglar and M. F. Ucar recently.

The exponential cubic B-spline function and its some properties are described in detail in the paper[35]. Since each exponential basis we use is twice continuously differentiable, we can form twice continuously approximate solution to the differential equations. There exist few articles which are used to form numerical methods to solve differential equations. The exponential cubic B-splines are used with the collocation method to find the numerical solution of the singular perturbation problem by Manabu Sakai and his colleague[36]. Another application of the collocation method using the cardinal exponential cubic B-splines was shown for finding the numerical solutions of the singularly perturbed boundary problem in the study of Desanka Radunovic[37]. The exponential cubic B-spline collocation method is set up to obtain the numerical solutions of the self-adjoint singularly perturbed boundary value problems in the work[38]. The only linear partial differential equation known as the convection-diffusion equation is solved by way of the exponential cubic B-spline collocation method in the study[39].

In this paper, we have compared results of the Burgers’ equation with those obtained with both the cubic B-spline collocation method and cubic B-spline Galerkin finite element method [15] [18] since the B-spline and exponential cubic B-spline functions have almost the same properties. In section 2, exponential cubic B-spline collocation method is described. In section 3, four classical test examples are studied to show the versatility of proposed algorithm and finally the conclusion is included to discuss the outcomes of the algorithm.

2 Exponential Cubic B-spline Collocation Method

Knots are equally distributed over the problem domain as

$$\pi : a = x_0 < x_1 < \ldots < x_N = b$$

with mesh spacing $h = (b - a)/N$. The exponential cubic B-splines, $B_i(x)$, at the points of $\pi$, can be defined as

$$B_i(x) = \begin{cases} 
  b_2 \left( (x_{i-2} - x) - \frac{1}{p} \sinh (p (x_{i-2} - x)) \right) & \text{for } [x_{i-2}, x_{i-1}], \\
  a_1 + b_1 (x_i - x) + c_1 \exp (p (x_i - x)) + d_1 \exp (-p (x_i - x)) & \text{for } [x_{i-1}, x_i], \\
  a_1 + b_1 (x - x_i) + c_1 \exp (p (x - x_i)) + d_1 \exp (-p (x - x_i)) & \text{for } [x_i, x_{i+1}], \\
  b_2 \left( (x - x_{i+2}) - \frac{1}{p} \sinh (p (x - x_{i+2})) \right) & \text{for } [x_{i+1}, x_{i+2}], \\
  0 & \text{otherwise.}
\end{cases}$$

(4)
where

\[ a_1 = \frac{phc}{phc-s}, \quad b_1 = \frac{p}{2} \left( \frac{(phc-s)(1-c)}{c(c-1)+s^2} \right), \quad b_2 = \frac{p}{2(phc-s)}, \]

\[ c_1 = \frac{1}{4} \left( \frac{\exp(-ph)(1-c)+s(\exp(-ph)-1)}{(phc-s)(1-c)} \right) \]

\[ d_1 = \frac{1}{4} \left( \frac{\exp(ph)(c-1)+s(\exp(ph)-1)}{(phc-s)(1-c)} \right) \]

and \( c = \cosh(ph), \ s = \sinh(ph), \ p \) is a free parameter. When \( p = 1 \), graph of the exponential cubic B-splines over the interval \([0,1]\) is depicted in Fig. 1.

Table 1: Values of \( B_i(x) \) and its principle two derivatives at the knot points

| \( x \) | \( x_{i-2} \) | \( x_{i-1} \) | \( x_i \) | \( x_{i+1} \) | \( x_{i+2} \) |
| --- | --- | --- | --- | --- | --- |
| \( B_i \) | 0 | \( \frac{s-ph}{2(phc-s)} \) | 1 | \( \frac{s-ph}{p(c-1)} \) | 0 |
| \( B_i' \) | 0 | \( \frac{p(c-1)}{2(phc-s)} \) | 0 | \( \frac{2(phc-s)}{p(c-1)} \) | 0 |
| \( B_i'' \) | 0 | \( \frac{p^2s}{2(phc-s)} \) | \( \frac{p^2s}{phc-s} \) | \( \frac{2(phc-s)}{p^2s} \) | 0 |

Now suppose that an approximate solution \( U_N \) to the unknown \( U \) is given by

\[ U_N(x,t) = \sum_{i=-1}^{N+1} \delta_i B_i(x) \]  

where \( \delta_i \) are time dependent parameters to be determined from the collocation points \( x_i, i = 0, ..., N \), the boundary and initial conditions. Evaluation of Eq. (5), its first and
second derivatives at knots \( x_i \) using the Table 1 yields the nodal values \( U_i \) in terms of parameters

\[
U_i = U(x_i, t) = \frac{s - ph}{2(phc - s)} \delta_{i-1} + \delta_i + \frac{s - ph}{2(phc - s)} \delta_{i+1},
\]

\[
U'_i = U'(x_i, t) = \frac{p(1 - c)}{2(phc - s)} \delta_{i-1} + \frac{p(c - 1)}{2(phc - s)} \delta_{i+1}
\]

\[
U''_i = U''(x_i, t) = \frac{p^2 s}{2(phc - s)} \delta_{i-1} - \frac{p^2 s}{phc - s} \delta_i + \frac{p^2 s}{2(phc - s)} \delta_{i+1}.
\]

The Crank–Nicolson scheme is used to discretize time variables of the unknown \( U \) in the Burgers’ equation so that one obtain the time discretized form of the equation as

\[
\frac{U^{n+1} - U^n}{\Delta t} + \frac{(UU_x)^{n+1} + (UU_x)^n}{2} - \lambda \frac{U_{xx}^{n+1} + U_{xx}^n}{2} = 0
\]

where \( U^{n+1} = U(x, t) \) is the solution of the equation at the \((n + 1)\)th time level. Here \( t^{n+1} = t^n + \Delta t \), and \( \Delta t \) is the time step, superscripts denote \( n \) th time level, \( t^n = n \Delta t \).

The nonlinear term \((UU_x)^{n+1}\) in Eq. (7) is linearized by using the following form [3]:

\[
(UU_x)^{n+1} = U^{n+1}U_x^n + U^nU_x^{n+1} - U^nU_x^n
\]

So Equation (7) is discretized in time as

\[
U^{n+1} - U^n + \frac{\Delta t}{2}(U^{n+1}U_x^n + U^nU_x^{n+1}) - \lambda \frac{\Delta t}{2}(U_{xx}^{n+1} - U_{xx}^n) = 0
\]

Substitution of (6) into (9) leads to the fully-discretized equation:

\[
\left( \alpha_1 + \frac{\Delta t}{2}(\alpha_1 L_2 + \beta_1 L_1 - \lambda \gamma_1) \right) \delta_{m-1}^{n+1} + \left( \alpha_2 + \frac{\Delta t}{2}(\alpha_2 L_2 - \lambda \gamma_2) \right) \delta_{m}^{n+1} + \\
\left( \alpha_3 + \frac{\Delta t}{2}(\alpha_3 L_2 + \beta_2 L_1 - \lambda \gamma_3) \right) \delta_{m+1}^{n+1} = (\alpha_1 + \lambda \frac{\Delta t}{2} \gamma_1) \delta_{m-1}^{n} + \\
(\alpha_2 + \lambda \frac{\Delta t}{2} \gamma_2) \delta_{m}^{n} + (\alpha_3 + \lambda \frac{\Delta t}{2} \gamma_3) \delta_{m+1}^{n}
\]

where

\[
L_1 = \alpha_1 \delta_{i-1} + \alpha_2 \delta_i + \alpha_3 \delta_{i+1}, \\
L_2 = \beta_1 \delta_{i-1} + \beta_2 \delta_{i+1},
\]

\[
\alpha_1 = \frac{s - ph}{2(phc - s)}, \quad \alpha_2 = 1, \quad \alpha_3 = \frac{s - ph}{2(phc - s)}
\]

\[
\beta_1 = \frac{p(1 - c)}{2(phc - s)}, \quad \beta_2 = \frac{p(c - 1)}{2(phc - s)}
\]

\[
\gamma_1 = \frac{p^2 s}{2(phc - s)}, \quad \gamma_2 = -\frac{p^2 s}{phc - s}, \quad \gamma_3 = \frac{p^2 s}{2(phc - s)}
\]
The system consist of \( N + 1 \) linear equation in \( N + 3 \) unknown parameters \( d^{n+1} = (\delta_{-1}^{n+1}, \delta_{0}^{n+1}, \ldots, \delta_{N+1}^{n+1}) \). To make solvable the system, boundary conditions \( \sigma_1 = U_0, \sigma_2 = U_N \) are used to find two additional linear equations:

\[
\begin{align*}
\delta_{-1} &= \frac{1}{\alpha_1}(U_0 - \alpha_2 \delta_0 - \alpha_3 \delta_1), \\
\delta_{N+1} &= \frac{1}{\alpha_3}(U_N - \alpha_1 \delta_{N-1} - \alpha_2 \delta_N).
\end{align*}
\]

(11) can be used to eliminate \( \delta_{-1}, \delta_{N+1} \) from the system (10) which then becomes the solvable matrix equation for the unknown \( \delta_0^{n+1}, \ldots, \delta_{N}^{n+1} \). A variant of Thomas algorithm is used to solve the system.

Initial parameters \( \delta_0^0, \delta_0^0, \ldots, \delta_0^{N+1} \) can be determined from the initial condition and first space derivative of the initial conditions at the boundaries as the following:

1. \( U_N(x_i, 0) = U(x_i, 0), \ i = 0, ..., N \)
2. \( (U_x)_N(x_0, 0) = U'(x_0) \)
3. \( (U_x)_N(x_N, 0) = U'(x_N) \).

3 Numerical tests

Numerical method described in the previous section will be tested on three text problems for getting solutions of the Burgers’ equation. Four kinds of examples are presented in order to demonstrate the versatility and the accuracy of the proposed method. The discrete \( L_2 \) and \( L_\infty \) error norm

\[
\begin{align*}
L_2 &= |U - U_N| = \sqrt{h \sum_{j=0}^{N} |(U_j - (U_N)_j)^n|^2}, \\
L_\infty &= |U - U_N|_\infty = \max_j |U_j - (U_N)_j^n|,
\end{align*}
\]

are used to measure error between the analytical and numerical solutions.

(a) The Burger’s equation, with the sine wave initial condition \( U(x, 0) = \sin(\pi x) \) and boundary conditions \( U(0, t) = U(1, t) = 0 \), has analytic solution in the form of the infinite series defined by [21] as

\[
U(x, t) = \frac{4\pi \lambda \sum_{j=1}^{\infty} j \mathbf{I}_j \left( \frac{1}{2\pi \lambda} \right) \sin(j\pi x) \exp(-j^2\pi^2\lambda t)}{\mathbf{I}_0 \left( \frac{1}{2\pi \lambda} \right) + 2 \sum_{j=1}^{\infty} \mathbf{I}_j \left( \frac{1}{2\pi \lambda} \right) \cos(j\pi x) \exp(-j^2\pi^2\lambda t)}
\]

(12)

where \( \mathbf{I}_j \) are the modified Bessel functions. This problem gives the decay of sinusoidal disturbance. Numerical solutions at different times are depicted in the Figs 2-5 for the
parameters $N = 40$, $\Delta t = 0.0001$, $\lambda = 1, 0.1, 0.01, 0.001$ from the figures we see that the smaller viscosities $\lambda$ cause to develop the sharp front thorough the right boundary and amplitude of the sharp front starts to decay as time progress. These properties of solutions are in very good agreement with findings of B Saka and I Dag [22, 24].

Fig. 2: Solutions at different times for $\lambda = 1$, $N = 40$, $\Delta t = 0.0001$.

Fig. 3: Solutions at different times for $\lambda = 0.1$, $N = 40$, $\Delta t = 0.0001$.

Fig. 4: Solutions at different times for $\lambda = 0.01$, $N = 40$, $\Delta t = 0.0001$.

Fig. 5: Solutions at different times for $\lambda = 0.001$, $N = 40$, $\Delta t = 0.0001$.

Two dimensional solutions are depicted from time $t = 0$ to $t = 1$ with time increment $\Delta t = 0.01$ for space increment $h = 0.25$ and various $\lambda$ in Fig. 6-9. When the smaller $\lambda = 0.001$ is taken, the solutions starts to decay after about time $t = 0.6$ when $N = 40$ is used. So to have acceptable solution with $\lambda = 0.001$, we decrease the space step to $h = 0.02$ and graph of the solution is shown in Fig 9.
A comparison has been made between the present collocation method and alternative approaches including the cubic B-spline collocation method and Cubic B-spline Galerkin finite element method for parameters of $\Delta t = 0.0001, N = 80, \lambda = 1, 0.1, 0.01$. Exact solutions for $\lambda > 10^{-2}$ are not practical because of the low convergence of the infinite series so that these results are not compared with the exact solutions. It can be seen from Table 2 a, 2 b, 2 c and 3 a, 3 b that accuracy of the presented solutions is much the same with both the cubic B-spline collocation method and cubic B-spline Galerkin finite element method. When the size of the space variable is reduced, the error becomes less than that of the cubic B-spline collocation methods and is almost close to the cubic B-spline Galerkin finite element method and solution values are documented in Table 3.
at time $t = 0.1$

Table 2 a: Comparison of the numerical solutions of Problem 1 for $\lambda = 1$ and $N = 80$, $\Delta t = 0.0001$ at different times with the exact solutions

| $x$  | $t$  | Present $p = 1$ | Ref. [15] | Ref. [18] | Exact |
|------|------|-----------------|-----------|-----------|-------|
| 0.25 | 0.4  | 0.01356         | 0.01357   | 0.01357   | 0.01357 |
|      |      |                 | 0.00189   | 0.00189   | 0.00189 |
|      |      |                 | 0.00026   | 0.00026   | 0.00026 |
|      |      |                 | 0.00004   | 0.00004   | 0.00004 |
|      |      |                 | 0.00000   | 0.00000   | 0.00000 |
| 0.50 | 0.4  | 0.01922         | 0.01923   | 0.01924   | 0.01924 |
|      |      |                 | 0.00267   | 0.00267   | 0.00267 |
|      |      |                 | 0.00037   | 0.00037   | 0.00037 |
|      |      |                 | 0.00005   | 0.00005   | 0.00005 |
|      |      |                 | 0.00000   | 0.00000   | 0.00000 |
| 0.75 | 0.4  | 0.01362         | 0.01362   | 0.01363   | 0.01363 |
|      |      |                 | 0.00189   | 0.00189   | 0.00189 |
|      |      |                 | 0.00026   | 0.00026   | 0.00026 |
|      |      |                 | 0.00004   | 0.00004   | 0.00004 |
|      |      |                 | 0.00000   | 0.00000   | 0.00000 |
Table 2 b: Comparison of the numerical solutions of Problem 1 for $\lambda = 0.1$ and $N = 40$, $\Delta t = 0.0001$ at different times with the exact solutions

| $x$ | $t$ | Present | Ref. [15] | Ref. [20] | Ref. [18] | Exact |
|-----|-----|---------|-----------|-----------|-----------|-------|
| 0.25 | 0.4 | 0.30890 | 0.30891 | 0.30890 | 0.30890 | 0.3089 | 0.3089 |
| 0.6 | 0.24075 | 0.24075 | 0.24074 | 0.24074 | 0.24074 | 0.24074 |
| 0.8 | 0.19569 | 0.19568 | 0.19568 | 0.19568 | 0.19568 | 0.19568 |
| 1.0 | 0.16257 | 0.16257 | 0.16257 | 0.16257 | 0.16257 | 0.16257 |
| 3.0 | 0.02720 | 0.02720 | 0.02720 | 0.02720 | 0.02720 | 0.02720 |
| 0.50 | 0.4 | 0.56965 | 0.56966 | 0.56964 | 0.56964 | 0.56963 |
| 0.6 | 0.44722 | 0.44723 | 0.44721 | 0.44721 | 0.44721 | 0.44721 |
| 0.8 | 0.35925 | 0.35926 | 0.35924 | 0.35924 | 0.35924 | 0.35924 |
| 1.0 | 0.29192 | 0.29193 | 0.29191 | 0.29191 | 0.29191 | 0.29191 |
| 3.0 | 0.04019 | 0.04020 | 0.04020 | 0.04020 | 0.04020 | 0.04020 |
| 0.75 | 0.4 | 0.62537 | 0.62538 | 0.62543 | 0.62541 | 0.62544 |
| 0.6 | 0.48714 | 0.48715 | 0.48719 | 0.48719 | 0.48719 | 0.48719 |
| 0.8 | 0.37385 | 0.37385 | 0.37390 | 0.37390 | 0.37390 | 0.37392 |
| 1.0 | 0.28741 | 0.28741 | 0.28746 | 0.28746 | 0.28746 | 0.28747 |
| 3.0 | 0.02976 | 0.02978 | 0.02977 | 0.02977 | 0.02977 | 0.02977 |

Table 2 c: Comparison of the numerical solutions of Problem 1 for $\lambda = 0.01$ and $N = 40$, $\Delta t = 0.0001$ at different times with the exact solutions

| $x$ | $t$ | Present | Ref. [15] | Ref. [20] | Ref. [18] | Exact |
|-----|-----|---------|-----------|-----------|-----------|-------|
| 0.25 | 0.4 | 0.34192 | 0.34192 | 0.34192 | 0.34192 | 0.34191 |
| 0.6 | 0.26897 | 0.26897 | 0.26894 | 0.26897 | 0.26897 | 0.26896 |
| 0.8 | 0.22148 | 0.22144 | 0.22148 | 0.22148 | 0.22148 | 0.22148 |
| 1.0 | 0.18819 | 0.18816 | 0.18819 | 0.18819 | 0.18819 | 0.18819 |
| 3.0 | 0.07511 | 0.07511 | 0.07509 | 0.07511 | 0.07511 | 0.07511 |
| 0.50 | 0.4 | 0.66071 | 0.66071 | 0.66071 | 0.66071 | 0.66071 |
| 0.6 | 0.52942 | 0.52942 | 0.52942 | 0.52942 | 0.52942 | 0.52942 |
| 0.8 | 0.43914 | 0.43914 | 0.43914 | 0.43914 | 0.43914 | 0.43914 |
| 1.0 | 0.37442 | 0.37442 | 0.37442 | 0.37442 | 0.37442 | 0.37442 |
| 3.0 | 0.15018 | 0.15018 | 0.15018 | 0.15018 | 0.15018 | 0.15018 |
| 0.75 | 0.4 | 0.91027 | 0.91027 | 0.91023 | 0.91027 | 0.91026 |
| 0.6 | 0.76725 | 0.76725 | 0.76728 | 0.76724 | 0.76724 | 0.76724 |
| 0.8 | 0.64740 | 0.64740 | 0.64740 | 0.64740 | 0.64740 | 0.64740 |
| 1.0 | 0.55605 | 0.55605 | 0.55609 | 0.55605 | 0.55605 | 0.55605 |
| 3.0 | 0.22483 | 0.22483 | 0.22481 | 0.22481 | 0.22481 | 0.22481 |
Table 3a: Problem 1 for $\lambda = 1$, $t = 0.1$, $\Delta t = 0.0001$ at different size with the exact solutions

|       | Present | Ref. [15] | Ref. [18] | Present | Ref. [15] | Ref. [18] | Exact |
|-------|---------|-----------|-----------|---------|-----------|-----------|-------|
| $p = 1$ | $h = 0.05$ |           |           | $h = 0.025$ |           |           |       |
| $0.10936$ | $0.10937$ | $0.10953$ | $0.10949$ | $0.10949$ | $0.10954$ | $0.10954$ |       |
| $0.20943$ | $0.20945$ | $0.20978$ | $0.20970$ | $0.20949$ | $0.20979$ | $0.20979$ |       |
| $0.29136$ | $0.29138$ | $0.29188$ | $0.29176$ | $0.29175$ | $0.29189$ | $0.29190$ |       |
| $0.34723$ | $0.34726$ | $0.34791$ | $0.34775$ | $0.34773$ | $0.34792$ | $0.34792$ |       |
| $0.37076$ | $0.37080$ | $0.37156$ | $0.37137$ | $0.37136$ | $0.37157$ | $0.37158$ |       |
| $0.35819$ | $0.35823$ | $0.35902$ | $0.35883$ | $0.35881$ | $0.35904$ | $0.35905$ |       |
| $0.30911$ | $0.30914$ | $0.30988$ | $0.30971$ | $0.30969$ | $0.30990$ | $0.30991$ |       |
| $0.22719$ | $0.22722$ | $0.22779$ | $0.22766$ | $0.22765$ | $0.22781$ | $0.22782$ |       |
| $0.12034$ | $0.12036$ | $0.12066$ | $0.12060$ | $0.12060$ | $0.12068$ | $0.12069$ |       |

Table 3b: Problem 1 for $\lambda = 1$, $t = 0.1$, $\Delta t = 0.0001$ at different size with the exact solutions

|       | Present | Ref. [15] | Ref. [18] | Present | Ref. [15] | Ref. [18] | Exact |
|-------|---------|-----------|-----------|---------|-----------|-----------|-------|
| $p = 1$ | $h = 0.0125$ |           |           | $h = 0.00625$ |           |           |       |
| $0.10953$ | $0.10952$ | $0.10954$ | $0.10954$ | $0.10953$ | $0.10954$ | $0.10954$ |       |
| $0.20977$ | $0.20975$ | $0.20979$ | $0.20979$ | $0.20977$ | $0.20979$ | $0.20979$ |       |
| $0.29186$ | $0.29184$ | $0.29189$ | $0.29189$ | $0.29186$ | $0.29190$ | $0.29190$ |       |
| $0.34788$ | $0.34788$ | $0.34792$ | $0.34792$ | $0.34788$ | $0.34792$ | $0.34792$ |       |
| $0.37153$ | $0.37153$ | $0.37158$ | $0.37156$ | $0.37153$ | $0.37158$ | $0.37158$ |       |
| $0.35899$ | $0.35896$ | $0.35904$ | $0.35903$ | $0.35900$ | $0.35904$ | $0.35905$ |       |
| $0.30986$ | $0.30983$ | $0.30990$ | $0.30989$ | $0.30986$ | $0.30990$ | $0.30991$ |       |
| $0.22778$ | $0.22776$ | $0.22782$ | $0.22781$ | $0.22778$ | $0.22782$ | $0.22782$ |       |
| $0.12067$ | $0.12065$ | $0.12069$ | $0.12068$ | $0.12067$ | $0.12069$ | $0.12069$ |       |

(b) As the second example, we consider particular solution of Burgers’ equation with initial condition

$$U(x, 1) = \exp\left(\frac{1}{8\lambda}\right), 0 \leq x \leq 1,$$

and boundary conditions $U(0, t) = 0$ and $U(1, t) = 0$.

This problem has the following analytical solution

$$U(x, t) = \frac{2}{1 + \sqrt{\frac{1}{t_0} \exp\left(\frac{2^2}{4\lambda t}\right)}}, \quad t \geq 1, \quad 0 \leq x \leq 1,$$

This solution represents the propagation of the shock and the selection of the smaller $\lambda$ result in steep shock solution. So the success of the numerical method depends on dealing with the steep shock efficiently.
The propagation of the shock is studied with parameters $\lambda = 0.005, 0.0005$. Numerical solutions obtained by exponential collocation method can be favorably compared with results reported in the papers [15, 18] at some times in the same Table 4-5. Figs. 10 and 11 show propagation of shock. As time advances, the initial steep shock becomes smoother when the larger viscosity is used but for the small viscosity it is steeper. These observations are in complete agreement with those reported in the papers [9].
Table 5: Comparison of results at different times. $\lambda = 0.005$ $[a, b] = [0.1]$ with $h = 0.005$ and $\Delta t = 0.01$

| $x$ | $t = 1.7$ | $t = 1.7$ | $t = 2.4$ | $t = 2.4$ | $t = 3.1$ | $t = 3.1$ | $t = 3.1$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
|     | Present   | Ref. (CBGM) | Exact     | Present   | Ref. (CBGM) | Exact     | Present   | Ref. (CBGM) | Exact     |
| 0.1 | 0.058823  | 0.058823  | 0.058823  | 0.041666  | 0.041666  | 0.032258  | 0.032258  |
| 0.2 | 0.117645  | 0.117644  | 0.117645  | 0.083332  | 0.083332  | 0.0644515 | 0.0644515 |
| 0.3 | 0.176458  | 0.176458  | 0.176458  | 0.124995  | 0.124995  | 0.096771  | 0.096771  |
| 0.4 | 0.235169  | 0.235170  | 0.235168  | 0.166640  | 0.166640  | 0.129021  | 0.129021  |
| 0.5 | 0.291919  | 0.291909  | 0.291904  | 0.208115  | 0.208115  | 0.161231  | 0.161231  |
| 0.6 | 0.295900  | 0.294962  | 0.294910  | 0.247431  | 0.247402  | 0.193130  | 0.193127  |
| 0.7 | 0.041931  | 0.042942  | 0.041929  | 0.251663  | 0.251672  | 0.221836  | 0.221867  |
| 0.8 | 0.000639  | 0.000669  | 0.000646  | 0.073049  | 0.073025  | 0.214751  | 0.215135  |
| 0.9 | 0.000005  | 0.000005  | 0.000005  | 0.003008  | 0.003023  | 0.071252  | 0.070874  |

$L_2 \times 10^3$ 0.00459 0.35126 0.00457 0.24448 0.02295 0.235

$L_\infty \times 10^3$ 0.06494 1.20726 0.04639 0.80176 0.42187 4.79061
Table 4: Comparison of results at different times. \( \lambda = 0.0005 \) with \( h = 0.005 \) and \( \Delta t = 0.01 \)

| \( x \) | \( t = 1.7 \) | \( t = 1.7 \) | \( t = 1.7 \) | \( t = 2.5 \) | \( t = 2.5 \) | \( t = 3.25 \) | \( t = 3.25 \) | \( t = 3.25 \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Present | Ref. [15] | Exact | Present | Ref. [15] | Exact | Present | Ref. [15] | Exact |
| 0.1 | 0.05882 | 0.05883 | 0.05882 | 0.04000 | 0.04000 | 0.03077 | 0.03077 | 0.03077 |
| 0.2 | 0.11765 | 0.11765 | 0.11765 | 0.08000 | 0.08000 | 0.06154 | 0.06154 | 0.06154 |
| 0.3 | 0.17647 | 0.17648 | 0.17647 | 0.12000 | 0.12000 | 0.09231 | 0.09231 | 0.09231 |
| 0.4 | 0.23529 | 0.23531 | 0.23529 | 0.16000 | 0.16000 | 0.12308 | 0.12308 | 0.12308 |
| 0.5 | 0.29412 | 0.29414 | 0.29412 | 0.20000 | 0.20000 | 0.15385 | 0.15385 | 0.15385 |
| 0.6 | 0.35294 | 0.35296 | 0.35294 | 0.24000 | 0.24000 | 0.18462 | 0.18462 | 0.18462 |
| 0.7 | 0.00000 | 0.00000 | 0.00000 | 0.28000 | 0.28000 | 0.21538 | 0.21538 | 0.21538 |
| 0.8 | 0.00000 | 0.00000 | 0.00000 | 0.00828 | 0.00811 | 0.00977 | 0.24615 | 0.24616 | 0.24615 |
| 0.9 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.12394 | 0.12358 | 0.12435 |

Fig. 10: Shock propagation, \( \lambda = 0.005 \) Fig. 11: Shock propagation, \( \lambda = 0.0005 \)

(c) Travelling wave solution of the Burgers’ equation has the form:

\[
U(x, t) = \frac{\alpha + \mu + (\mu - \alpha) \exp \eta}{1 + \exp \eta}, \quad 0 \leq x \leq 1, \ t \geq 0,
\]

where

\[
\eta = \frac{\alpha(x - \mu t - \gamma)}{\lambda},
\]

and \( \alpha, \mu, \) and \( \gamma \) are arbitrary constants. The boundary conditions is

\[
U(0, t) = 1, \ U(1, t) = 0.2 \text{ or } U_x(0, t) = 0, \ U_x(1, t) = 0, \text{ for } t \geq 0
\]

and initial condition is obtained from the analytical solution \((14)\) when \( t = 0 \). Analytical solution takes values between 1 and 0.2 and the propagation of the wave front through the
right will be observed with varying $\lambda$. The smaller $\lambda$ we take for the Burgers’ equation, the steeper the wave front propagates. The robustness of the algorithm will be shown by monitoring the motion of the wave front with smaller $\lambda$. The algorithm has run for the values $\alpha = 0.4$, $\mu = 0.6$, $\gamma = 0.125$ and $\lambda = 0.01$, $h = 1/36$, $\Delta t = 0.001$. A comparison of the results obtained from the present method with those by I Dag and his coworkers is shown in table 5. $L_\infty$-norm of the methods given data in the table 6 has been found as $0.004, 0.004, 0.005, 0.006, 0.005$ respectively. Numerical solutions obtained with present method are in good agreement with results obtained by the methods documented in the Table 6. Visual motion of the wave front is depicted in the Figs. 12-13 for the $\lambda = 0.01, 0.05$. The numerical results demonstrate the formation of the steep front and very steeper front. Error graphs of the numerical solutions are also shown in the Figs. 14-15, from figures the maximum error occurs in the middle of the solution domain. Solutions from time $t = 0$ to $t = 1.2$ at some times are visualised in 3D graph to see the propagation of the sharp behaviours in Figs 16-17.

| Table 6: Comparison of results at time $t = 0.5$, $h = 1/36$, $\Delta t = 0.01$, $\lambda = 0.01$ |
|--------------------------------------------------|
| $x$  | Present $p = 1$ | Ref. [15] $\Delta t = 0.025$ | Ref. [18] (QBGM) | Ref. [18] (CBGM) | Exact |
| 0.000 | 1. | 1. | 1. | 1. | 1. |
| 0.056 | 1. | 1. | 1. | 1. | 1. |
| 0.111 | 1. | 1. | 1. | 1. | 1. |
| 0.167 | 1. | 1. | 1. | 1. | 1. |
| 0.222 | 1. | 1. | 1. | 1. | 1. |
| 0.278 | 0.998 | 0.999 | 0.998 | 0.998 | 0.998 |
| 0.333 | 0.980 | 0.986 | 0.980 | 0.980 | 0.980 |
| 0.389 | 0.847 | 0.850 | 0.841 | 0.842 | 0.847 |
| 0.444 | 0.452 | 0.448 | 0.458 | 0.457 | 0.452 |
| 0.500 | 0.238 | 0.236 | 0.240 | 0.241 | 0.238 |
| 0.556 | 0.204 | 0.204 | 0.205 | 0.205 | 0.204 |
| 0.611 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 |
| 0.667 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 |
| 0.722 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 |
| 0.778 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 |
| 0.833 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 |
| 0.889 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 |
| 0.944 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 |
| 1.000 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 |
Figure 12: Solutions at different times for $\lambda = 0.01$ $h = 1/36$, $\Delta t = 0.001$, $x \in [0, 1]$.

Figure 13: $L_2$ error norm for $\lambda = 0.005$ $h = 1/36$, $p = 1$, $\Delta t = 0.001$, $x \in [0, 1]$.

Fig. 14: $L_2$ error norm for $\lambda = 0.01$ and $h = 1/36$.

Fig. 15 $L_2$ error norm $\lambda = 0.005$ and $h = 1/36$. 
(d) The initial condition and boundary conditions are taken as

\[ U(x, 0) = \lambda [x + \tan(x/2)], \]

\[ U(0.5, t) = \frac{\lambda}{1 + \lambda t} \left[ 0.5 + \tan \left( \frac{1}{4(1 + \lambda t)} \right) \right], \quad t \geq 0, \]

\[ U(1.5, t) = \frac{\lambda}{1 + \lambda t} \left[ 1.5 + \tan \left( \frac{3}{4(1 + \lambda t)} \right) \right], \quad t \geq 0, \]

The exact solution of the problem is \[ \text{[8]} \]

\[ U(x, t) = \frac{\lambda}{1 + \lambda t} \left[ x + \tan \left( \frac{x}{2(1 + \lambda t)} \right) \right]. \]

Table 7 gives results of the problem (d) among exponential cubic B-spline collocation method, cubic B-spline collocation method and exact solutions for \( \lambda = 1/1000 \) at time \( t = 2.25 \). The program is run for a set of parameters \( p \), the best results is found for the value \( p = 1 \) seeing in the Table 7.
Table 7: Comparison of results at time $t = 2.25$, $\lambda = 1/1000$ with $h = 0.00625$, $\Delta t = 0.015$

| $x$ | Present ($p = 0.01$) | Present ($p = 0.1$) | Present ($p = 0.5$) | Present ($p = 1$) | cubic B-spline [15] | Exact |
|-----|----------------------|---------------------|---------------------|------------------|------------------|-------|
| 0.5 | 0.007329977          | 0.007329977         | 0.000753049         | 0.000753049      | 0.000753049      | 0.000753049 |
| 0.6 | 0.008817897          | 0.008817897         | 0.000905597         | 0.000905597      | 0.000906558      | 0.000906558 |
| 0.7 | 0.010323604          | 0.010323604         | 0.001061678         | 0.001061678      | 0.001061749      | 0.001061749 |
| 0.8 | 0.011851020          | 0.011851020         | 0.001218899         | 0.001218991      | 0.001218992      | 0.001218992 |
| 0.9 | 0.013402144          | 0.013402144         | 0.001378707         | 0.001378707      | 0.001378707      | 0.001378707 |
| 1.0 | 0.014978854          | 0.014978854         | 0.001541377         | 0.001541377      | 0.001541377      | 0.001541377 |
| 1.1 | 0.016584692          | 0.016584692         | 0.001707565         | 0.001707565      | 0.001707565      | 0.001707565 |
| 1.2 | 0.018262688          | 0.018262688         | 0.001877935         | 0.001877935      | 0.001877935      | 0.001877935 |
| 1.3 | 0.019914461          | 0.019914461         | 0.002053284         | 0.002053284      | 0.002053284      | 0.002053284 |
| 1.4 | 0.021662973          | 0.021662973         | 0.002231955         | 0.002231955      | 0.002234577      | 0.002234577 |
| 1.5 | 0.023483948          | 0.023483948         | 0.002423004         | 0.002423004      | 0.002423004      | 0.002423004 |

Visual representation of the absolute error over space interval $0.5 \leq x \leq 1.5$ is given at time $t = 2.25$ in Figs18-19.

Figure 18: Absolute error for $h = 1/36$, $p = 1$, $\Delta t = 0.001$, $t = 2.25$, $\lambda = 0.1$

Figure 19: Absolute error for $h = 1/36$, $p = 1$, $\Delta t = 0.001$, $t = 2.25$, $\lambda = 0.01$

In Table 8 the accuracy of the present method via $L_\infty$ norm is can be examined. As
number of $N$ and $\lambda$ parameter increase, according to Table 8, the error decrease.

| Parameters $p = 0.1$, $\Delta t = 0.01$ | $N = 16$          | $N = 32$          | $N = 64$          | $N = 128$         |
|----------------------------------------|------------------|------------------|------------------|------------------|
| $\lambda$                              | $1.19 \times 10^{-7}$ | $5.26 \times 10^{-8}$ | $3.94 \times 10^{-8}$ | $3.73 \times 10^{-8}$ |
| $1/2$                                   | $1.03 \times 10^{-7}$ | $4.69 \times 10^{-8}$ | $2.15 \times 10^{-8}$ | $1.48 \times 10^{-8}$ |
| $1/4$                                   | $7.19 \times 10^{-6}$ | $2.74 \times 10^{-6}$ | $1.53 \times 10^{-6}$ | $1.22 \times 10^{-6}$ |
| $1/8$                                   | $6.57 \times 10^{-5}$ | $2.06 \times 10^{-5}$ | $9.36 \times 10^{-6}$ | $6.52 \times 10^{-6}$ |
| $1/16$                                  | $1.45 \times 10^{-4}$ | $4.09 \times 10^{-5}$ | $1.51 \times 10^{-5}$ | $8.65 \times 10^{-6}$ |
| $1/32$                                  | $1.58 \times 10^{-4}$ | $4.18 \times 10^{-5}$ | $1.31 \times 10^{-5}$ | $5.98 \times 10^{-6}$ |
| $1/100$                                 | $1.32 \times 10^{-4}$ | $3.40 \times 10^{-5}$ | $9.2 \times 10^{-6}$  | $3.04 \times 10^{-6}$ |

4 Conclusion

The Exponential cubic B-Spline collocation method for the numerical solutions of the Burges’ equation is presented over the finite elements so that the continuity of the dependent variable and its first two derivatives is satisfied for the approximate solution throughout the solution range. The equation has been integrated into a system of the linearized iterative algebraic equations. The system of the iterative at each time step in which it has got a three-banded coefficients matrix is solved with the Thomas algorithm. Generally, Comparative results show that results of our finding is better than the that of the cubic B-spline collocation method and is much the same with that of the cubic B-spline Galerkin finite element method. Since cost of the cubic B-spline Galerkin method is higher than the suggested method, that is advantages of the Exponential cubic B-Spline collocation method over the cubic B-spline Galerkin method. During all runs of the algorithm, the best result are found for the free parameter $p = 1$ for the Exponential cubic B-Spline functions.

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