Di-electron and two-photon widths in charmonium

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The vector and pseudoscalar decay constants are calculated in the framework of the Field Correlator Method. Di-electron widths: $\Gamma_{ee}(J/\psi) = 5.41$ keV, $\Gamma_{ee}(\psi(3686)) = 2.47$ keV, $\Gamma_{ee}(\psi(3770)) = 0.248$ keV, in good agreement with experiment, are obtained with the same coupling, $\alpha_s = 0.165$, in QCD radiative corrections. We show that the larger $\alpha_s = 0.191 \pm 0.004$ is needed to reach agreement with experiment for $\Gamma_{\gamma\gamma}(J/\psi) = 7.22$ keV, $\Gamma_{\gamma\gamma}(\chi_c^0 P_0) = 3.3$ keV, $\Gamma_{\gamma\gamma}(\chi_c^0 P_2) = 0.54$ keV, and also for $\Gamma(J/\psi \to 3g) = 59.5$ keV, $\Gamma(\psi \to \gamma 2g) = 5.7$ keV. Meanwhile even larger $\alpha_s = 0.238$ gives rise to good description of $\Gamma(\psi \to 3g) = 52.7$ keV, $\Gamma(\psi \to \gamma 2g) = 3.5$ keV, and provides correct ratio of the branching fractions: $B(J/\psi \to \text{light hadrons})/B(\psi \to \text{light hadrons}) = 0.24$.

I. INTRODUCTION

Low-lying states of heavy quarkonia have been an important laboratory to study both perturbative and nonperturbative phenomena in QCD. However, recent discoveries of higher resonances, in particular $X(3872)$ [1] and $Y(4260)$ [2] have shown that these and some other new resonances cannot be interpreted as conventional $Q\bar{Q}$ mesons. To understand the nature of new resonances, evidently, two- (or many-) channel consideration is needed. However, in strict sense it cannot be done now because nonperturbative theory of strong decays is not still well developed in QCD. Therefore for identification of new resonances with $J^{PC} = 1^{--}$, observed in $e^+e^-$ via the initial state radiation [3], [4], a special role belongs to di-electron widths and also two-photon widths for $C$-even resonances, which are reasonably well described by existing QCD formulas. At this point it is worthwhile to remind that the di-electron width of a $Q\bar{Q}$ meson is by two orders (may be even more) larger than di-electron width of a compact four-quark system [5].

In our paper, firstly, we calculate the decay constants of vector ($V$) and pseudoscalar ($P$) mesons in charmonium using the Field Correlator Method (FCM), which has been successfully applied to heavy-light mesons [6]. Due to relativistic corrections di-electron widths and their ratios, calculated here, agree with experiment with high accuracy. Therefore from the absolute values of di-electron widths some important factors, containing the squared wave functions at the origin, can be extracted and then used in different annihilation decays.

We pay a special attention to the influence of radiative corrections on different annihilation rates. The absolute values of di-electron widths are shown to agree with experimental numbers only if the QCD radiative corrections are taken into account. Unfortunately, at present there is no consensus about the true value of the strong coupling in them. These corrections, known in first (one-loop) approximation [7], [8], enter the di-electron and two-photon widths as separate factors: $\beta_V = 1 - \frac{16\alpha_s}{\pi}$, $\beta_P = 1 - \frac{3.7\alpha_s}{\pi}$. In [9] these factors are put equal unity: $\beta_V = \beta_P = 1.0$, i.e. the QCD correction is neglected, while in [10]-[13] their values are almost two times smaller, $\beta_V = 0.52 \pm 0.06$. The reason of this uncertainty partly occurs because the contribution of higher corrections remains unknown. Therefore, although by derivation the coupling $\alpha_s$ in different annihilation widths is defined at the standard scale $\mu = 2m_Q$ or $\mu = M_{V(P)}$ [14] (in the $\overline{MS}$ scheme), factually, this strong coupling appears to be an effective one and can differ in different annihilation decays, since for them higher order perturbative corrections can be different.

In our paper we show that in the $\psi$- family the di-electron widths are described with the same coupling, which turns out to be relatively small: $\alpha_s = 0.165$ or $\beta_V = 0.72$ (the same ”universality” is observed in bottomonium [12]). Meanwhile to describe two-photon widths of $\eta_c$, $\chi_c(1^3P_0)$, $\chi_c(1^3P_2)$, and also three-gluon annihilation rate of $J/\psi$ only the choice of larger coupling, $\alpha_s = 0.191 \pm 0.004$, gives rise to agreement with experiment. Even larger $\alpha_s = 0.25(2)$ provides correct number for the $\psi'$ width $\Gamma(\psi' \to 3g)$. Thus our analysis shows that low-lying charmonium states have no an universal scale for different annihilation decays and therefore any ratio of their widths cannot be used to extract characteristic strong coupling (for the discussion see [10], [17]); in particular, they are different for the $J/\psi$ and $\psi'$ three-gluon annihilation rates.

Calculated here $\Gamma_{ee}$ for $J/\psi$, $\psi' = \psi(3686)$, and $\psi'' = \psi(3770)$ (with the mixing angle $\theta = 11^0$) agree with experiment with accuracy $\leq 5\%$ and this allows us to extract some important factors from the di-electron widths. The essential fact is that correct ratio of the branching fractions, $R_{L,H} = B(J/\psi \to \text{light hadrons})/B(\psi' \to \text{light hadrons}) = 0.24$, appears
to be two times larger then in the "12% rule" mostly because different $\alpha_s$ describe corresponding annihilation rates.

The unclear situation still remains with two-photon width of $\eta'_c$, because its value can depend on possible influence of virtual decay channel $DD^*$ and possibly other channels [18]. In closed-channel approximation $\Gamma(\eta'_c \to gg) = 3$ keV is obtained if the same $\alpha_s = 0.24$, which provides correct number for $\Gamma(\eta_c \to KK\pi)$, the value $\Gamma(\eta'_c) = 1.3 \pm 0.6$ keV has been reported. However, if via the $DD^*$ channel the mixing of the $3^1S_0$ and $2^1S_0$ states occurs, then even with small $3^1S_0$ contribution to the w.f. of $\eta'_c$ (4% to the norm) its two-photon width is becoming essentially smaller, $\Gamma_{\gamma\gamma} \leq 1.9$ keV.

II. VECTOR AND PSEUDOSCALAR DECAY CONSTANTS

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The decay constants $f_V$ and $f_P$ are calculated here with the use of the analytic expressions, derived in [6]. To obtain these expressions the functional integral representation for the correlator of the currents (in V and P channels) is used and on the final stage this correlator is expanded in the complete set of the eigenfunctions (e.f.) of the relativistic string Hamiltonian (RSH) [20, 21]. As the first step we use here this RSH to calculate charmonium spectrum and define relativistic corrections to the decay constants:

$$f_V^2(nS) = 12 \frac{|\phi_n(0)|^2}{M_V(nS)} \xi_V = \frac{3 |R_n(0)|^2}{\pi M_V(nS)} \xi_V, \quad (1)$$

$$f_P^2(nS) = 12 \frac{|\phi_n(0)|^2}{M_P(nS)} \xi_P = \frac{3 |R_n(0)|^2}{\pi M_P(nS)} \xi_P. \quad (2)$$

Here the relativistic factors $\xi_P, \xi_V$, referring to the P and V channels, are different and given by the expressions:

$$\xi_V = \frac{m^2 + \omega^2 + \frac{1}{3} \sqrt{<\mathbf{p}^2>}}{2\omega^2}, \quad \xi_P = \frac{m^2 + \omega^2 - \sqrt{<\mathbf{p}^2>}}{2\omega^2}. \quad (3)$$

The values of $\omega$ and the wave functions (w.f.) at the origin are given in Appendix. In [1] and [2] $R_n(0)$ (n=1,2) refers to the physical radial w.f. at the origin for $J/\psi$, $\psi' = \psi(3686)$, and $R_D(0)$ is the w.f. of $\psi''(3770)$, i.e. the S-D mixing (with the mixing angle $\theta = 11^\circ$) is taken into account. In Appendix for pure $2S$ and $1D$ states their w.f. at the origin are denoted as $R_{2S}(0), R_{1D}(0)$.

The characteristic feature of RSH is that it contains a minimal number of fundamental parameters: the string tension $\sigma$, $\Lambda_{QCD}$ for $n_f = 4$, and the pole quark mass $m_c$. The string tension is taken from the analysis of the Regge trajectories of light-light mesons [21] and the spectra of heavy-light mesons [6], where the preferable value is $\sigma = 0.180$ GeV$^2$. The pole mass of c quark is now known with rather good accuracy, $m_c = (1.40 \pm 0.05)$ GeV [22] and for $n_f = 4$ the QCD constant $\Lambda_{\overline{MS}} = 255(5)$ MeV is taken.

The RSH for a meson can be presented as in [23]:

$$H_0 = \frac{p^2 + m_c^2}{\omega} + \omega + V_0(r). \quad (4)$$

This Hamiltonian $H_0$ is unperturbed part of general Hamiltonian,

$$H = H_0 + \Delta H, \quad (5)$$

where $\Delta H = V_{SD}(r) + \Delta V_{str} + V_{SE}$ includes the spin-dependent part $V_{SD} = V_{SS} + V_{LS} + V_T$, the string correction $V_{str}$, and the self-energy term $V_{SE}$, which are considered as a perturbation. Notice that in heavy quarkonia the string correction (as well as the self-energy term) is always small, $|V_{str}| \leq 5$ MeV, and can be neglected. The Hamiltonian $H_0$ has an advantage as compared to the spinless Salpeter equation (SSE), since its w.f. at the origin for $L = 0$ is a regular function while the S-wave w.f. of SSE diverges at small $r$ [24] and has to be regularized.

In einbein approximation the spin-averaged mass $\overline{M}_{nL}$ can be presented as:

$$\overline{M}_{nL} = \omega_{nL} + \frac{m_c^2}{\omega_{nL}} + E_{nL}(\omega_c + \Delta_{SE}), \quad (6)$$

where the e.v. $E_{nL}$ are the solutions of the so-called einbein equation:

$$\left[ \frac{\omega_{nL}^2}{\omega_{nL}} + V_0(r) \right] \phi_{nL}(r) = E_{nL} \phi_{nL}. \quad (7)$$

Notice that the mass formula [6] does not contain an arbitrary (fitting) constant $C_0$. It is essential that in einbein approximation for a given state the values, $\overline{M}_{nL}$ and $\omega_{nL}$, are defined from the extremum condition: $\frac{\partial \overline{M}_{nL}}{\partial \omega_{nL}} = 0$, which provides the accuracy $\sim 5 - 7\%$ [23]. For the mass [6] this extremum condition gives rise to the relation:

$$\omega_{nL}^2 = m_c^2 - \frac{\partial E_{nL}}{\partial \omega_{nL}}. \quad (8)$$

Then for a given nL state the dynamical mass $\omega_{nL}$ and $\overline{M}_{nL}$ are calculated. Notice that $\omega_{nL}$, being the kinetic energy of a quark, plays the role of the constituent quark mass which slightly differs for the states with different quantum numbers.
The masses given in Table 1 are calculated with the static potential which contains perturbative gluon-exchange (GE) term and nonperturbative confining term:

\[
V_0(r) = -\frac{4}{3}\frac{\alpha_B(r)}{r} + \sigma r,
\]

(9)

where the vector coupling in coordinate space \(\alpha_B(r)\) is defined as in [25]:

\[
\alpha_B(r) = \frac{2}{\pi} \int_0^\infty dq \frac{\sin(qr)}{q}\alpha_B(q),
\]

\[
\alpha_B(q) = \frac{4\pi}{\beta_0 t_B} \left( 1 - \frac{\beta_1}{\beta_0} \frac{\ln t_B}{t_B} \right)
\]

(10)

with \(t_B = \ln \frac{\bar{\Delta}_M^2 + M^2}{\bar{\Lambda}_B^2}\). Here \(M_B(\sigma, \Lambda_B) = (1.00 \pm 0.05)\) GeV is so-called background mass [25], and \(\Lambda_B(n_f)\) can be expressed through \(\Lambda_{\gamma N}\) in 2-loop approximation. \(\Lambda_B(n_f = 4) = 0.360(10)\) MeV corresponds to the \(\Lambda_{\gamma N} = 0.254(7)\) GeV and in this case the freezing value \(\alpha_{\text{crit}} = 0.547\).

Although here we consider low-lying states, to represent gross features of the charmonium spectrum and the position of higher levels we take into account flattening of linear confining potential. This phenomenon occurs due to creation of virtual light-quark pairs [21] and it is becoming essential for higher levels, in particular, for the mass of the \(5^3S_1\) state. The flattening of linear potential is defined by the analytic function for which the form and parameters are taken just the same as in [21], where they have been extracted from the light meson radial Regge trajectories. The origin of flattening comes from the virtual \(dq\) pairs creation on the surface inside the Wilson loop \(\langle W(C)\rangle\), having large size, and due to these virtual loops the string tension (as well as the surface) is becoming smaller and dependent on the \(Q\bar{Q}\) separations \(r\). This potential provides a correlated mass shift down of all radial excitations with \(n \geq 3\).

From Table 1 one can also see that in the potential with \(\sigma(r) = 0\) the masses \(M(3S), M(4S), M(5S)\) are shifted down by \(\sim 20\) MeV, \(\sim 60\) MeV, and \(\sim 100\) MeV, respectively, and turn out to be close to the experimental values with the exception of the \(\psi(4040)\), which is strongly affected by the nearby \(S\) wave threshold. Since the hyperfine splitting of the \(5^3S_1\) level is small, \(\leq 6\) MeV [13], its mass practically coincides with the centroid mass, \(M(5S) = 4.70\) GeV, calculated here, and lies close to the mass of the Belle resonance \(Y(4660)\) [4]. From Table 1 one can see that with exception of the \(1P\) state all other masses are in good agreement with experiment, even for the states above the \(D\bar{D}\) threshold. The reason, why only for the \(1P\) states the centroid mass has smaller value (in einbein approximation) needs an additional analysis.

The \(V\) and \(P\) decay constants, calculated with the use of [1] and [2], are given in Tables 2 and 3 together with those from some other papers.
TABLE II: The decay constants $f_{V}$ (in MeV) of the $J/\psi$, $\psi'$ = $\psi(3686)$, $\psi'' = \psi(3770)$ mesons.

| State | This paper | BL [9] EFG [11] | Wang[12] | Experiment |
|-------|------------|-----------------|-----------|------------|
|       | (Rel)      | $\beta_{V} = 1.0$ | $\beta_{V} = 0.72$ | $\beta_{V} = 1.0$ |
|       |            | $\alpha_{s} = 0$ | $\alpha_{s} = 0.165$ | $\alpha_{s} = 0$ |
| $J/\psi$ | 483 | 545 | 551 | 459(28) | 415(6) | 490(7) |
| $\psi'$ | 357 | 371 | 401 | 364(24) | 302(4) | 356(4) |
| $\psi''$ | 115 | 318 | 96(5) | 113(6) | 490(7) |

TABLE III: The decay constants $f_{P}$ (in MeV) of the $\eta_{c}$, $\eta'_{c}$, $\eta_{c2}$ mesons.

| State | This paper | BL [9] LP[10] | Experiment |
|-------|------------|---------------|------------|
|       | (Rel)      | $\beta_{P} = 1.0$ | $\alpha_{s}(\eta_{c}) = 0.195$ | $\alpha_{s}(\eta'_{c}) = 0.25$ |
|       |            | $\alpha_{s} = 0$ | $\alpha_{s}(\eta_{c2}) = 0.195$ | $\alpha_{s}(\eta'_{c2}) = 0.25$ |
| $\eta_{c}$ | 453 | 493 | 480 | 404(57) | 454(64) |
| $\eta'_{c}$ | 336$^{a}$, 267$^{b}$ | 260 | 303 | 189(40) | 213(47) |
| $\eta_{c2}$ | 41 | | | |

$^{a}$ The influence of virtual decay channels is neglected.
$^{b}$ Mixing of $2^{1}S_{0}$ and $3^{1}S_{0}$ states ($\theta = 11^{o}$) is taken into account.

It is worth pointing out that the ”experimental” $f_{V}$ and $f_{P}$, extracted from the experimental di-electron and two-photon widths, depend on chosen values of radiative corrections, $\beta_{V}$ and $\beta_{P}$. Therefore in Tables 1 and 2 we give two variants of ”experimental” decay constants, which correspond to $\beta_{V} = 0.72$ ($\alpha_{s} = 0.165$) and 1.0, and $\beta_{P} = 0.79$ ($\alpha_{s} = 0.195$) and 1.0 ($\alpha_{s} = 0$). It is also important that $\eta'_{c}(2^{1}S_{0})$ and $\eta_{c2}(1^{3}D_{2})$ cannot be mixed and therefore their w.f. at the origin are defined by the w.f. $R_{2S}(0)$ and $R_{D}(0)$ for pure $2S$ and $1D$ states (see Appendix). From Table 3 one can see that for $\beta_{V} = 0.72$ our decay constants $f_{V}$ are in good agreement with experiment. Note that for $\psi''$ calculated constant $f_{V}$ is almost three times smaller than in [9].

III. DI-ELECTRON WIDTHS

The leptonic width of a vector state in heavy quarkonia is expressed via the decay constant $\bar{c}$, $\bar{c}$:

$$\Gamma_{ee}(V) = \frac{4\pi e_{L}^{2}c^{2}}{3M_{V}^{2}} \beta_{V} \gamma_{V},$$

$$\beta_{V} = 1 - \frac{16}{3\pi} \alpha_{s},$$

Best description of di-electron widths is obtained here taking in $\alpha_{s} = 0.165$, or $\beta_{V} = 0.72$, for which $\Gamma_{ee}(J/\psi) = 5.14$ keV, $\Gamma_{ee}(\psi') = 2.47$ keV, and $\Gamma_{ee}(\psi'') = 0.248$ keV are obtained (see Table 4). In our calculations of the di-electron widths of $\psi'(3686)$ and $\psi''(3770)$ the S-D mixing with the mixing angle $\theta = 11^{o}$ is taken into account (In Appendix we give their physical w.f. at the origin and also several m.c.). In Table 4 calculated widths are presented together with some theoretical predictions and experimental data.

Since the coupling in [12] appears to be the same for $J/\psi$, $\psi'$, $\psi''$, it is of interest to compare their ratios (where the radiative corrections are cancelled): $R_{\psi'} = \Gamma_{ee}(\psi')/\Gamma_{ee}(J/\psi)$, $R_{\psi''} = \Gamma_{ee}(\psi'')/\Gamma_{ee}(J/\psi)$ with experimental numbers which turn out to be very close to each other:

$$R_{\psi'_{th}} = 0.46, \quad R_{\psi'_{exp}} = 0.45 \pm 0.02,$$
$$R_{\psi''_{th}} = 0.046, \quad R_{\psi''_{exp}} = 0.044 \pm 0.03.$$ (13)

Having such agreement with experiment we expect that our w.f. at the origin are defined with good accuracy, in particular, the following numbers are obtained for the factors:

$$\left| \frac{R_{J/\psi}(0)}{M_{J/\psi}^{2}} \right|^{2} = 0.085\text{GeV}, \quad \left| \frac{R_{\psi'}(0)}{M_{\psi'}^{2}} \right|^{2} = 0.040\text{GeV},$$ (14)
$$\left| \frac{R_{\psi''}(0)}{M_{\psi''}^{2}} \right|^{2} = 0.0040\text{GeV}, \quad \left| \frac{R_{\psi'}(0)}{M_{\psi'}^{2} \Delta_{ee}} \right|^{2} = 0.0027\text{GeV},$$ (15)

We estimate the accuracy for these factors as $\leq 10\%$ and later use them to define three- and two-gluon annihilation rates for $J/\psi$, $\psi'$, $\eta_{c}$, $\eta'_{c}$.

Notice that relativistic corrections decrease the decay constants $f_{V}$ and $f_{P}$ and provide better agreement with experiment. However, for hadronic decays this type of relativistic corrections is not still calculated in FCM, and therefore the accuracy of calculated hadronic widths is.
worse than for the decay constants \( f_P \) and \( f_P \). Concluding this Section, we would like to underline that our calculations give the ratio of the branching fractions for the \( e^+e^- \) annihilation:

\[
R_{ee} = \frac{B_{ee}(\gamma\gamma)}{B_{ee}(J/\psi)} = \frac{\Gamma_{\gamma\gamma}(J/\psi)}{\Gamma_{\gamma\gamma}(\psi')}, \quad \frac{M^2(J/\psi)}{M^2(\psi')} \cdot \frac{\xi_V(2S)}{\xi_V(1S)} = \left( 12.6 \pm 0.03 \right)\%,
\]

which is in good agreement with experimental number, or the so-called "12% rule", \( R_{ee}(exp) = (12.3 \pm 0.03)\% \).

### IV. TWO-PHOTON WIDTHS OF \( \eta_c, \eta'_c, \chi_c (1^3P_0), \chi_c (1^3P_2) \)

The two-photon widths of \( \eta_c (1S) \), \( \eta'_c (2S) \) can be expressed via the decay constants \( f_P \):

\[
\Gamma_{\gamma\gamma}(P) = \frac{4\pi a^2 e^4}{M_P} f_P^2 \beta_P,
\]

\[\beta_P = 1 - \frac{20 - \frac{\pi^2}{3}}{3\pi} \alpha_s.\]

The two-photon widths of the scalar \( \chi_{c0} \) and tensor \( \chi_{c2} \) mesons can be derived in FCM in the same manner as it has been done for the \( V, P \) decay constants in [3] with the following result:

\[
f_S^2(T) = \frac{3|R_P'(0)|^2}{4\pi M_S(T)\omega_P^2}.
\]

Here in [19] instead of the quark mass \( m_Q \) (in nonrelativistic limit) the kinetic energy \( \omega_P \) enters. Then with the QCD corrections the two-photon widths of the \( P \)-wave mesons are defined as [3]:

\[
\Gamma_{\gamma\gamma}(\chi_{c0}) = \frac{108a^2 e^4 |R_P'(0)|^2}{M_S^2 T^2 \omega_P^2} \left( 1 - \frac{28 - \frac{3\pi^2}{9}}{3\pi} \alpha_s \right),
\]

\[
\Gamma_{\gamma\gamma}(\chi_{c2}) = \frac{144a^2 e^4 |R_P'(0)|^2}{5M_S^2 \omega_P^2} \left( 1 - \frac{16 \alpha_s}{3\pi} \right).
\]

In our analysis we use \( \alpha_s = 0.195 \) for \( \eta_c, \chi_{c0} \), and \( \chi_{c2} \) to obtain good numbers for the following ratios of two-photon widths (in which relativistic factors \( \xi_P(\eta_c) = 0.785, \xi_P(\eta'_c) = \xi_P(\chi_{c2}) = 0.73 \) are used, see Appendix):

\[
\left( \frac{\Gamma_{\gamma\gamma}(\chi_{c0})}{\Gamma_{\gamma\gamma}(\eta_c)} \right)_{th} = 0.458, \quad \left( \frac{\Gamma_{\gamma\gamma}(\chi_{c2})}{\Gamma_{\gamma\gamma}(\eta_c)} \right)_{th} = 0.41 \pm 0.02.
\]

For \( \chi_{c0} \) the width is by 10\% larger than the experimental one, while for \( \chi_{c2} \) the absolute value and the ratios:

\[
\left( \frac{\Gamma_{\gamma\gamma}(\chi_{c0})}{\Gamma_{\gamma\gamma}(\eta_c)} \right)_{th} = 0.075, \quad \left( \frac{\Gamma_{\gamma\gamma}(\chi_{c2})}{\Gamma_{\gamma\gamma}(\eta_c)} \right)_{exp} = 0.076 \pm 0.002
\]

coincide with the experimental values. Different two-photon widths are given in Table 5.

### TABLE V: The two-photon widths (in keV) with \( \alpha_s = 0.195 \) for \( \eta_c, \chi_{c0}, \chi_{c2} \) and with \( \alpha_s = 0.24 \) for \( \eta'_c, \eta_{c2} \).

| State   | This paper | BL [9] | EFG [11] | KLW [12] | Experiment |
|---------|------------|--------|----------|----------|------------|
| \( \eta_c \) | 7.2201 ± 0.001 | 5.5 | 7.14(95) | 7.1 ± 2.7 | \[22\] |
| \( \eta_c \) | 3.0(02) ± 0.4(02) | 1.9 | 2.59 | 1.8 | 4.44(48) | 1.3 ± 0.6 | \[18\] |
| \( \eta_{c2} \) | 0.042 | 1.21 | - | | |
| \( \chi_{c0}(1^3P_0) \) | 3.31 | 3.28 | 2.9 | 3.78 | 2.9 ± 0.43 |
| \( \chi_{c2}(1^3P_2) \) | 0.54 | - | 0.52 | | 0.53 ± 0.06 |

\( a) \) See the footnote \( a) \) to Table 3.

\( b) \) See the footnote \( b) \) to Table 3.

We would like to stress here that calculated two-photon width of \( \eta'_c \) (with \( \alpha_s = 0.24 \)) is larger than in the CLEO experiment \[19\], nevertheless with the same \( \alpha_s \) we have obtained hadronic width \( \Gamma(\eta'_c \to \eta_{c2}) = 11.0 \text{ MeV} \) in agreement with \( \Gamma_{tot} = 14 \pm 7 \text{ MeV} \) \[22\].

One cannot exclude that the w.f. at the origin of \( \eta'_c \) is affected by the virtual \( DD^* \) decay channel which lies only by 130 MeV higher than \( \eta'_c \). Then via this channel the w.f. of \( \eta'_c \) can be mixed with \( 3^P_S \) state (which is now often identified with \( X(3940) \) \[20\]). For example, the 20\% admixture in the w.f. of \( \eta'_c \) (or the 4\% contribution to the norm) gives rise to the two-photon \( \Gamma_{\gamma\gamma}(\eta'_c) = 1.9 \text{ keV} \).

Recently two-photon width of \( \eta_c \) has been calculated in lattice QCD \[30, 31\]. Such calculations from first principles are very important for the theory, however, in \[30\] rather small number, \( \Gamma_{\gamma\gamma}(\eta_c) = 2.65(26)(80)(53) \text{ keV} \) and corresponding \( f_P(\eta_c) = 373 \text{ MeV} \) are obtained in quenched approximation. Meanwhile in \[31\] the calculations in lattice QCD with exact chiral symmetry, where heavy quarks are treated as the Dirac fermions, the larger decay constant, \( f_P(\eta_c) = 438(11) \text{ MeV} \), is obtained and this number is in agreement with our result, \( f_P(\eta_c) = 453 \text{ MeV} \).
V. THREE-AND TWO-GLUON
ANNIHILATION RATES

The QCD corrections to di-electron widths have appeared to be ≤ 30%, but they are even more important for some hadronic decays. From [7], [8], [14] we know that the widths of the three-gluon annihilation and the decay into $\gamma g g$ for the vector mesons, as well as for two-gluon annihilation of P, S, T mesons:

$$\Gamma(V \to g g g) = \frac{40(\pi^2 - 9)\alpha_s^3}{81\pi M^2_V} |R_n(0)|^2 \left(1 - 3.7\frac{\alpha_s}{\pi}\right)$$

(24)

$$\Gamma(V \to \gamma g g) = \frac{32(\pi^2 - 9)\alpha_s^2\alpha}{9\pi M^2_V} |R_n(0)|^2 \left(1 - 6.7\frac{\alpha_s}{\pi}\right)$$

(25)

$$\Gamma(P \to g g) = \frac{8\alpha_s^2}{3M_P^2} |R_n(0)|^2 \left(1 + 4.8\frac{\alpha_s}{\pi}\right),$$

(26)

$$\Gamma(\chi_{c0} \to g g) = \frac{24\alpha_s^2}{M^2_{\chi_{c0}}|R_P(0)|^2} \left(1 + 9.5\frac{\alpha_s}{\pi}\right),$$

(27)

$$\Gamma(\chi_{c2} \to g g) = \frac{32\alpha_s^2}{5M^2_{\chi_{c2}}|R_P(0)|^2} \left(1 - 2.2\frac{\alpha_s}{\pi}\right).$$

(28)

It can be easily shown that with $\alpha_s = 0.165$, as for considered above di-electron widths, the three-gluon annihilation width of $J/\psi$ is smaller than in experiment being equal 42 keV. The best fit to this annihilation rate is obtained taking $\alpha_s = 0.187$ (practically the same as in two-photon width), for which

$$\Gamma(J/\psi \to g g g) = 59.5 \text{ keV},$$

$$\Gamma(J/\psi \to \gamma g g) = 5.7 \text{ keV}.$$

Then the sum of these annihilation widths, being equal the width $\Gamma(J/\psi \to \text{light hadrons}) = 65.2 \text{ keV}$, is in good agreement with the experimental number $B(J/\psi \to \text{light hadrons}) = (69 \pm 3)\%$ [16], [17], or $\Gamma(J/\psi \to \text{light hadrons})_{\text{exp}} = 64 \pm 3 \text{ keV}$.

On the other hand this value, $\alpha_s = 0.187$, is not sufficient to provide correct number for the three-gluon annihilation rate of $\psi'(3686)$ and in this case the best fit is obtained taking $\alpha_s = 0.238$:

$$\Gamma(\psi' \to g g g) = 52.7 \text{ keV},$$

$$\Gamma(\psi' \to \gamma g g) = 3.5 \text{ keV},$$

with their sum $\Gamma(\psi' \to \text{light hadrons}) = 56.2 \text{ keV}$, in good agreement with the experimental branching fraction $B(\psi' \to \text{light hadrons}) = (16.9 \pm 3)\%$ [16], [17], or $\Gamma(\psi' \to \text{light hadrons})_{\text{exp}} = 57 \pm 10 \text{ keV}$. This fact means that the ratio

$$R_{\gamma \psi} = \frac{\Gamma(\psi' \to \text{light hadrons})}{\Gamma(J/\psi \to \text{light hadrons})} = 0.24 \pm 0.01,$$

appears to be in good agreement with experimental number, $R_{\gamma \psi}(\exp) = 0.24 \pm 0.04$, being two times larger than the ratio $R_{\gamma \psi} (\text{exp})$. Nevertheless for the same $\alpha_s = 0.24$ calculated here two-photon width of $\eta_c', \Gamma_{\gamma \eta_c}(\eta_c') = 3.0 \text{ keV}$, turns out to be larger than in the CLEO experiment [18]. From our point of view the problem of this width can be solved taking into account the influence of the $DD^*$ decay channel.

Thus our analysis shows that there is no an universal effective strong coupling which allows to describe different annihilation processes for all low-lying charmonium states. Such universal description is possible only for di-electron widths.

We would like to notice also that in our calculations the ratio,

$$R_{\gamma \psi} = \frac{\Gamma(J/\psi \to \text{light hadrons})}{\Gamma_{ee}(J/\psi)} = 11.0 \pm 0.05$$

agrees with experimental number, $R_{\gamma \psi}(\exp) = 11.6 \pm 0.03$, and this fact justifies our choice of $\alpha_s = 0.187$ in radiative correction for $J/\psi$.

VI. CONCLUSIONS

We have shown that di-electron widths of $J/\psi$, $\psi'$, $\psi''$ describe experimental data with the accuracy better 5%, if the same $\alpha_s = 0.165$ is taken in the QCD radiative corrections. This fact can be considered as a test of the method used here.

For $\eta_c(1S)$, $\chi_{c0}$, $\chi_{c2}$ the larger effective coupling, $\alpha_s = 0.195$, is needed to fit experimental numbers for two-photon widths. Also close value of $\alpha_s = 0.187$ provides good description of the annihilation widths for the $J/\psi \to g g g$ and $J/\psi \to \gamma g g$ processes, so that $B(J/\psi \to \text{light hadrons}) = 70\%$ is obtained (its value in experiment is $(69 \pm 3)\%$ [10], [17]).

However, to describe experimental branching fraction, $B(\psi' \to \text{light hadrons}) = 16.7 \pm 3.0\%$ [10], [17], larger $\alpha_s = 0.238$ is needed, for which we obtain $\Gamma(\psi' \to g g g) = 52.7 \text{ keV}$ and $\Gamma(\psi' \to \gamma g g) = 3.5 \text{ keV}$, so that their sum is equal $\Gamma(\psi' \to \text{light hadrons}) = 56.2 \text{ keV}$, which corresponds to experimental branching fraction. Then the ratio of the branching fractions:

$$R_{\gamma \psi} \left(\frac{\Gamma_{ee}(J/\psi)}{\Gamma(J/\psi \to \text{light hadrons})}\right) = 0.24 \pm 0.01,$$

Acknowledgements

This work is supported by the Grant NSh-843.2006.2.
Here we use RSH to calculate the m.e., like $\omega(nL) < p^2 >$, and also the w. f. at the origin. For pure $2S$ state we denote its w. f. as $\tilde{R}_{2S}(0) = 0.767 \text{ GeV}^{3/2}$; for the $P$-wave states $R_P(0) = R'_P(0)/\omega_P = 0.183 \text{ GeV}^{3/2}$ with $R'_P(0) = 0.297 \text{ GeV}^{5/2}$, and for the $1D$ state $R_D(0) = \frac{5R''_D(0)}{2\sqrt{2}\omega_D} = 0.0942 \text{ GeV}^{3/2}$ with $R''_D(0) = 0.145 \text{ GeV}^{7/2}$. The values of $\omega(nL)$ are given in Table 6.

The physical w. f. of $\psi'$ and $\psi''$, calculated here, take into account the mixing angle $\theta = 11^\circ$, and these w. f. at the origin are given in Table 6 (all needed parameters are given in Section 2).

For $\psi'$ and $\psi''$, the S-D mixing can occur due to tensor forces and coupling to open $D\bar{D}$ channel. Then the w. f. at the origin of $\psi'$ and $\psi''$ are defined as in [28]:
\[
R_{\psi'}(0) = \cos \theta \tilde{R}_{2S}(0) - \frac{5}{2\sqrt{2}\omega} \sin \theta R''_D(0),
\]
(31)

Our calculations give $R_P(0) = \frac{|R_P(0)|^2}{\omega_P} = 0.183 \text{ GeV}^{3/2}$; the $\psi''$ w. f. at the origin, $R_D(0) = 0.238 \text{ GeV}^{3/2}$ ($\omega(1D) = 1.65 \text{ GeV}$), and for the $\psi'$ meson $R_{\psi'}(0) = 0.735$ (see Table 6).

**Table VI:** The dynamical masses $\omega_{nL}$, the radial wave functions at the origin for $J/\psi$, $\psi'$, $\psi''$, and $\chi_{cJ}$ (in GeV).

| State $\omega_{nL}$, GeV | $<p^2>$ | $\xi_V$ | $\xi_P$ |
|------------------------|---------|--------|--------|
| 1S                     | 1.59    | 0.905  | 0.541  | 0.929  | 0.785 |
| 2S                     | 1.65    | 0.735  | 0.722  | 0.910  | 0.733 |
| 1P                     | 1.62    | 0.183  | 0.619  | 1.0    | 1.0   |
| 1D                     | 1.65    | 0.238  | 0.721  | 0.911  | 0.733 |

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