ANISOTROPIC MOSER-TRUDINGER INEQUALITY INVOLVING 
$L^n$ NORM

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ABSTRACT. The paper is concerned about a sharp form of Anisotropic Moser-Trudinger inequality which involves $L^n$ norm. Let

$$\lambda_1(\Omega) = \inf_{u \in W^{1,n}_0(\Omega), u \not\equiv 0} \frac{||F(\nabla u)\|_{L^n(\Omega)}}{||u\|_{L^n(\Omega)}}$$

be the first eigenvalue associated with $n$-Finsler-Laplacian. Using blowing up analysis, we obtain that

$$\sup_{u \in W^{1,n}_0(\Omega), ||F(\nabla u)\|_{L^n(\Omega)}} \int_\Omega e^{\lambda_n(1+\alpha)||u\|_{L^n(\Omega)}} \frac{1}{n-1} \frac{||u||_{L^n(\Omega)}^{n-1}}{dx}$$

is finite for any $0 \leq \alpha < \lambda_1(\Omega)$, and the supremum is infinite for any $\alpha \geq \lambda_1(\Omega)$, where $\lambda_n = n \frac{\kappa_n^{-1}}{n-1}$ ($\kappa_n$ is the volume of the unit wulff ball) and the function $F$ is positive, convex and homogeneous of degree 1, and its polar $F^o$ represents a Finsler metric on $\mathbb{R}^n$. Furthermore, the supremum is attained for any $0 \leq \alpha < \lambda_1(\Omega)$.

Keywords: Moser-Trudinger inequality, $n$-Finsler-Laplacian, Blow-up analysis

MSC: 35A05, 35J65

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. The Sobolev embedding theorem states that $W^{1,n}_0(\Omega)$ is embedded in $L^p(\Omega)$ for any $p > 1$, or equivalently, using the Dirichlet norm $\|u\|_{W^{1,n}_0(\Omega)} = (\int_\Omega \|
abla u\|^n dx)^{\frac{1}{n}}$ on $W^{1,n}_0(\Omega)$,

$$\sup_{u \in W^{1,n}_0(\Omega), \|
abla u\|_{L^n(\Omega)}} \int_\Omega \|u\|^p dx < +\infty.$$ 

But it is well known that $W^{1,n}_0(\Omega)$ is not embedded in $L^\infty(\Omega)$. Hence, one is led to look for a function $g(s) : \mathbb{R} \rightarrow \mathbb{R}^+$ with maximal growth such that

$$\sup_{u \in W^{1,n}_0(\Omega), \|
abla u\|_{L^n(\Omega)}} \int_\Omega g(u) dx < +\infty.$$ 

The Moser-Trudinger inequality states that the maximal growth is of exponential type, which was shown by Pohozhaev [28], Trudinger [29] and Moser [22]. This

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inequality says that
\[
\sup_{u \in W_0^{1,n}(\Omega), ||\nabla u||_{L^n(\Omega)} \leq 1} \int_{\Omega} e^{\alpha|u|^\frac{n}{n-1}} \, dx < +\infty
\]
for any \( \alpha \leq \alpha_n \), where \( \alpha_n = n\omega_{n-1} \) and \( \omega_{n-1} \) is the surface area of the unit ball in \( \mathbb{R}^n \). The inequality is optimal, i.e. for any \( \alpha > \alpha_n \) there exists a sequence of \( \{u_\epsilon\} \) in \( W_0^{1,n}(\Omega) \) with \( ||\nabla u_\epsilon||_{L^n(\Omega)} \leq 1 \) such that
\[
\int_{\Omega} e^{\alpha u_\epsilon} \, dx \to \infty \quad \text{as} \quad \epsilon \to 0.
\]
On the other hand, for any fixed \( u \in W_0^{1,n}(\Omega) \), it is also known that
\[
\int_{\Omega} e^{\alpha|u|^\frac{n}{n-1}} \, dx < +\infty
\]
for any \( \alpha > 0 \).

Another interesting question about Moser-Trudinger inequalities is whether extremal functions exist or not. The first result in this direction is due to Carleson and Chang \cite{6}, who proved that the supremum is attained when \( \Omega \) is a unit ball in \( \mathbb{R}^n \). Then Struwe \cite{24} got the existence of extremals for \( \Omega \) close to a ball. Struwe’s technique was then used and extended by Flucher \cite{11} to \( \Omega \) which is the more general bounded smooth domain in \( \mathbb{R}^2 \). Later, Lin \cite{19} generalized the existence result to a bounded smooth domain in dimension-\( n \). Recently Mancini and Martinazzi \cite{23} reproved the Carleson and Chang’s result by using a new method based on the Dirichlet energy, also allowing for perturbations of the functional. Even Thizy \cite{30} also gave examples in which slightly perturbed Moser-Trudinger inequalities do not admit extremals (as conjectured in \cite{23}), hence shown that the existence of extremal for the Moser-Trudinger inequality should not be taken for granted and holds only under some rigid conditions. Actually, the inequality (1) is viewed as a \( n \)-dimensional analog of the Sobolev inequality, and it plays an important role in analytic problems and in geometric problems. Now there are many generalizations of the classical Moser-Trudinger inequality (1), see for instance \cite{1, 18, 21, 10, 12, 58, 59, 50, 40, 56} and the references therein.

In 2012, Wang and Xia \cite{34} proved the following Moser-Trudinger type inequality
\[
\int_{\Omega} e^{\lambda|u|^\frac{n}{n-1}} \, dx \leq C(n)||\Omega||
\]
for all \( u \in W_0^{1,n}(\Omega) \) and \( \int_{\Omega} F^n(\nabla u) \, dx \leq 1 \). Here \( \lambda \leq \lambda_n = n\pi \kappa_{n-1}^{\frac{1}{n-1}} \), \( \lambda_n \) is optimal in the sense that if \( \lambda > \lambda_n \) we can find a sequence \( \{u_k\} \) such that \( \int_{\Omega} e^{\lambda|u_k|^\frac{n}{n-1}} \, dx \) diverges. Later, in \cite{43} and \cite{44} there have shown that the supremum is attained when \( \Omega \) is bounded domain in \( \mathbb{R}^n \).

Adimurthi and Druet \cite{11}, Y.Y Yang \cite{38} have proved that, when \( \lambda_1(\Omega) > 0 \) be the first eigenvalue of the \( n \)-Laplacian with Dirichlet boundary condition in \( \Omega \), then (1) For any \( 0 \leq \alpha < \lambda_1(\Omega) \),
\[
\sup_{u \in W_0^{1,n}(\Omega), ||\nabla u||_{L^n(\Omega)} = 1} \int_{\Omega} e^{\alpha|u|^\frac{n}{n-1}(1+||u||_{L^n(\Omega)}^\frac{n}{n-1})} \, dx < +\infty.
\]
(2) For any $\alpha \geq \lambda_1(\Omega)$,

$$
\sup_{u \in W^{1,n}_0(\Omega), \|\nabla u\|_{L^n(\Omega)} = 1} \int_\Omega e^{\alpha \|u\|_{L^n(\Omega)}^n} (1 + \alpha \|u\|_{L^n(\Omega)})^{\frac{n}{n-1}} dx = +\infty.
$$

Furthermore, when $0 \leq \alpha < \lambda_1(\Omega)$, the supremum is also attained.

For simplicity, we introduce the notions

$$
J_{\lambda}^\alpha(u) = \int_\Omega e^{\lambda (1 + \alpha \|u\|_{L^n(\Omega)}^n)} \|u\|_{L^n(\Omega)}^{\frac{n}{n-1}} |u|^{\frac{n}{n-1}} dx, \mathcal{H} = \{ u \in W^{1,n}_0(\Omega) : \|F(\nabla u)\|_{L^n(\Omega)} = 1 \}.
$$

Let $\lambda_1(\Omega)$ be the first eigenvalue of Finsler-n-Laplacian with Dirichlet boundary condition in $\Omega$. It is defined by

$$
\lambda_1(\Omega) = \inf_{u \in W^{1,n}_0(\Omega), u \neq 0} \frac{\|F(\nabla u)\|_{L^n(\Omega)}^n}{\|u\|_{L^n(\Omega)}^n},
$$

from [4], we can know that $\lambda_1(\Omega) > 0$ and it is also achieved uniquely by a positive function $\varphi$ satisfying

$$
\begin{align*}
-\Delta \varphi &= \lambda_1(\Omega) |\varphi|^{n-2} \varphi, & \text{in } \Omega \\
\varphi &= 0, & \text{on } \partial \Omega,
\end{align*}
$$

where $-\Delta$ is Finsler-Laplacian operator that can be found in section 2.

In this paper we state the following:

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain and let $\lambda_1(\Omega) > 0$ be the first eigenvalue of the Finsler-n-Laplacian with Dirichlet boundary condition in $\Omega$. Then we have

1. For any $0 \leq \alpha < \lambda_1(\Omega)$, $\sup_{u \in \mathcal{H}} J_{\lambda}^\alpha(u) < +\infty$.
2. For any $\alpha \geq \lambda_1(\Omega)$, $\sup_{u \in \mathcal{H}} J_{\lambda}^\alpha(u) = +\infty$.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain. For any $0 \leq \alpha < \lambda_1(\Omega)$, $\sup_{u \in \mathcal{H}} J_{\lambda}^\alpha(u) = \infty$ is attained by some $C^1_{\text{loc}}$ maximizer. In other words, there exists $u_\alpha \in \mathcal{H} \cap C^1(\Omega)$ such that $J_{\lambda}^\alpha(u_\alpha) = \sup_{u \in \mathcal{H}} J_{\lambda}^\alpha(u)$.

When $F(x) = |x|$, Adimurthi and Druet [1], Y.Y Yang [38] have proved the above theorem. But the $F(x) \neq |x|$, it is more different, need much more delicate work. Now we describe the main idea to prove Theorem 1.1 and Theorem 1.2. The proof of point (2) of Theorem 1.1 is base on test functions computations which are present in Section 2. The point of point (1) of Theorem 1.1 is based on the blow up analysis. The proof of Theorem 1.2 is based on two facts: an upper bound of $J_{\lambda}^\alpha$ on $\mathcal{H}$ can be derived under the assumption that blowing up occur; a sequence of functions $\phi_\epsilon \in \mathcal{H}$ can be constructed to show that the above upper bound is not an upper bound. This contradiction implies that no blowing up occur, and then Theorem 1.2 holds. Though the method we carry out blowing up analysis is routine, we will encounter new difficulties when $0 < \alpha < \lambda_1(\Omega)$.

We organize this paper as follows. In Section 2, we gives some notes about anisotropic function $F(x)$ and the properties of the function $F(x)$, moreover, we prove point (2) of Theorem 1.1 we use blowing up analysis to prove point of (1) of Theorem 1.1 in section 3 and section 4. An upper bound of $J_{\lambda}^\alpha$ is derived ,moreover a sequence of functions is constructed to reach a contradiction in section
5, which completes the proof of Theorem 1.2. In section 6, we show the asymptotic representation of certain Green function which has been used in Section 5.

2. Notations and preliminaries

In this section, we will give the notations and preliminaries.

Throughout this paper, let $F : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative convex function of class $C^2(\mathbb{R}^n \setminus \{0\})$ which is even and positively homogenous of degree 1, so that

$$F(t\xi) = |t|F(\xi) \quad \text{for any} \quad t \in \mathbb{R}, \xi \in \mathbb{R}^n.$$  

A typical example is $F = C(x_1^2 + \cdots + x_n^2)$, which completes the proof of Theorem 1.2. In section 6, we show the asymptotic representation of certain Green function which has been used in Section 5.

One can see easily that $\phi_F$. Let us see how to define the minimization problem

$$\min_{u \in W^{1,n}_0(\Omega)} \int_\Omega F^n(\nabla u)dx,$$

its Euler equation contains an operator of the form

$$Q_n u := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (F^{n-1}(\nabla u) F_{\xi_i}(\nabla u)).$$

Note that these operators are not linear unless $F$ is the Euclidean norm in dimension two. We call this nonlinear operator as $n$-Finsler-Laplacian. This operator $Q_n$ was studied by many mathematicians, see [34, 13, 34, 2, 4, 37] and the references therein.

Consider the map

$$\phi : S^{n-1} \to \mathbb{R}^n, \quad \phi(\xi) = F_2(\xi).$$

its image $\phi(S^{n-1})$ is a smooth, convex hypersurface in $\mathbb{R}^n$, which is called Wulff shape of $F$. Let $F^o$ be the support function of $K := \{x \in \mathbb{R}^n : F(x) \leq 1\}$, which is defined by

$$F^o(x) := \sup_{\xi \in K} \langle x, \xi \rangle.$$

It is easy to verify that $F^o : \mathbb{R}^n \to [0, +\infty)$ is also a convex, homogeneous function of class of $C^2(\mathbb{R}^n \setminus \{0\})$. Actually $F^o$ is dual to $F$ in the sense that

$$F^o(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F(\xi)}, \quad F(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F^{\circ}(\xi)}.$$  

One can see easily that $\phi(S^{n-1}) = \{x \in \mathbb{R}^n \mid F^o(x) = 1\}$. We denote $\mathcal{W}_F := \{x \in \mathbb{R}^n \mid F^o(x) \leq 1\}$ and $\kappa_n := |\mathcal{W}_F|$, the Lebesgue measure of $\mathcal{W}_F$. We also use the notion $\mathcal{W}_r(0) := \{x \in \mathbb{R}^n \mid F^o(x) \leq r\}$. We call $\mathcal{W}_r(0)$ a Wulff ball of radius $r$ with center at 0. For later use, we give some simple properties of the function $F$, which follows directly from the assumption on $F$, also see [34, 13, 3].

**Lemma 2.1.** We have

(i) $|F(x) - F(y)| \leq F(x + y) \leq F(x) + F(y)$;

(ii) $\frac{1}{C} \leq |\nabla F(x)| \leq C$, and $\frac{1}{C} \leq |\nabla F^o(x)| \leq C$ for some $C > 0$ and any $x \neq 0$;

(iii) $\langle x, \nabla F^o(x) \rangle = F^o(x), \langle x, \nabla F^o(x) \rangle = F^o(x)$ for any $x \neq 0$;
(iv) $F(∇F^o(x)) = 1$, $F^o(∇F(x)) = 1$ for any $x \neq 0$;
(v) $F^o(x)F_t(∇F^o(x)) = x$ for any $x \neq 0$;
(vi) $F_t(\xi) = \text{sgn}(t)F(\xi)$ for any $\xi \neq 0$ and $t \neq 0$.

Next we describe the isoperimetric inequality and co-area formula with respect to $F$. For a domain $Ω \subset \mathbb{R}^n$, a subset $E \subset Ω$ and a function of bounded variation $u \in BV(Ω)$, we define the anisotropic bounded variation of $u$ with respect to $F$ is

$$\int_Ω |∇u|_F = \sup \{ \int_Ω u \text{div} σ dx, σ \in C^1_c(Ω; \mathbb{R}^n), F^o(σ) \leq 1 \}.$$ 

We set anisotropic perimeter of $E$ with respect to $F$ is

$$P_F(E) := \int_Ω |∇X_E|_F,$$

where $X_E$ is the characteristic function of the set $E$. It is well known (also see [12]) that the co-area formula

$$\int_Ω |∇u|_F = \int_0^∞ P_F(|u| > t)dt$$

and the isoperimetric inequality

$$P_F(E) \geq nκ_n^2 |E|^{1 - \frac{1}{n}}$$

hold. Moreover, the equality in (6) holds if and only if $E$ is a Wulff ball.

In the sequel, we will use the convex symmetrization with respect to $F$. The convex symmetrization generalizes the Schwarz symmetrization (see [33]). It was defined in [2] and will be an essential tool for establishing the Lions type concentration-compactness alternative under the anisotropic Dirichlet norm. Let us consider a measurable function $u$ on $Ω \subset \mathbb{R}^n$. The one dimensional decreasing rearrangement of $u$ is

$$u^* = \sup \{ s \geq 0 : |\{ x ∈ Ω : |u(x)| > s \}| > t \}, \quad \text{for} \; t ∈ \mathbb{R}.$$ 

The convex symmetrization of $u$ with respect to $F$ is defined as

$$u^*(x) = u^*(κ_n F^o(x)^n), \quad \text{for} \; x ∈ Ω^*.$$ 

Here $κ_n F^o(x)^n$ is just Lebesgue measure of a homothetic Wulff ball with radius $F^o(x)$ and $Ω^*$ is the homothetic Wulff ball centered at the origin having the same measure as $Ω$. Throughout this paper, we assume that $Ω$ is bounded smooth domain in $\mathbb{R}^n$ with $n ≥ 2$.

The following properties about Finsler-Laplacian can be found in [44]:

**Lemma 2.2.** Assume that $u ∈ W_0^{1,n}(Ω)$ is a solution to the equation

$$-Q_n(u) = f.$$ 

If $f ∈ L^q(Ω)$ for some $q > 1$, then $||u||_L^∞(Ω) ≤ C||f||_{L^q(Ω)}^{1 - \frac{1}{n}},$ where $C$ is only depends on $a, b, n, Ω, q$.

Set

$$P = \begin{cases} \infty & \int_Ω F^n(∇u)dx < 1, \\ \int_Ω F^n(∇u)dx = 1. \end{cases}$$
Lemma 2.3. Let \( u \in W^{1,n}_0(\Omega) \), \( u \neq 0 \). Assume that \( \{u_k\} \) is a sequence of functions \( W^{1,n}_0(\Omega) \) such that \( \int_\Omega F^n(\nabla u_k)dx \leq 1 \), and
\[
u_k \rightharpoonup u \quad \text{weakly in} \quad W^{1,n}_0(\Omega), \quad u_k \rightarrow u \quad \text{a.e. in} \ \Omega.
\]
Then for every \( p < P \), there exists a constant \( C = C(n, p) \) such that for each \( k \)
\[
\int_\Omega \exp(n \frac{\nabla}{\nabla u_k} \frac{1}{\kappa_n}) p|u_k|^\frac{n}{n-1} dx \leq C.
\]
Moreover, this conclusion fails if \( p \geq P \).

Test functions computations

In this subsection we will prove the conclusion (2) of Theorem 1.1. we will build explicit test functions to show the unboundedness of Moser-Trudinger function under large parameter.

Since the Moser-Trudinger inequality is invariant under translation, we may assume that \( 0 \in \Omega \) and \( \Omega_1 \subset \Omega \). We now fix some \( x_\delta \in \Omega_1 \) such that \( F^n(x_\delta) = \delta \). Choosing \( t_\delta \) such that \( t_\delta^n \log \frac{1}{\epsilon} \rightarrow \infty \) and \( t_\delta^{n+1} \log \frac{1}{\epsilon} \rightarrow 0 \). Set
\[
\varphi_\epsilon(x) = \begin{cases} \frac{n}{\lambda_n} \log \frac{1}{\epsilon} \frac{\nabla}{\nabla u_k} \frac{1}{\kappa_n}, & F^n(x) \leq \delta, \\ \left( \frac{n}{\lambda_n} \log \frac{1}{\epsilon} \frac{\nabla}{\nabla u_k} \frac{1}{\kappa_n} \right) \left( \log \delta - \log F^n(x) - t_\delta \varphi(x_\delta) \right)(\log F^n(x)), & \epsilon < F^n(x) \leq \delta, \\ t_\delta \varphi(x_\delta) + \theta(x)(\varphi(x_\delta) - \varphi(x_\delta)), & F^n(x) > \delta. \end{cases}
\]

In above definition of \( \varphi_\epsilon(x) \), \( \varphi \) is the eigenvalue function. \( \theta(x) \in C^2(\Omega) \) is a cut-off function satisty \( |\nabla \theta(x)| \leq \frac{C}{\delta} \) and
\[
\theta(x) = \begin{cases} 0, & F^n(x) \leq \delta \\ \theta \in (0, 1), \delta < F^n(x) < 2\delta \\ 1, & F^n(x) \geq 2\delta. \end{cases}
\]

Let \( \delta = \frac{1}{t_\delta^n} \frac{1}{\log \frac{1}{\epsilon}} \), it is easy to see \( \epsilon < \delta \) if \( \epsilon \) is small enough. We obtain that
\[
\int_{c < F^n(x) \leq \delta} F^n(\nabla \varphi_\epsilon(x)) dx = \int_{c < F^n(x) \leq \delta} \left| \frac{n}{\lambda_n} \log \frac{1}{\epsilon} \frac{\nabla}{\nabla u_k} \frac{1}{\kappa_n} + t_\delta \varphi(x_\delta) \right|^n dx
\]
\[
= \frac{n\kappa_n}{\lambda_n} - \left( \frac{n}{\lambda_n} \log \frac{1}{\epsilon} \frac{\nabla}{\nabla u_k} \frac{1}{\kappa_n} + t_\delta \varphi(x_\delta) \right)^n
\]
\[
= 1 - n \frac{n+1}{\kappa_n} \frac{1}{\epsilon} \frac{1}{\nabla u_k} \frac{1}{\kappa_n} t_\delta \varphi(x_\delta)(1 + o_\epsilon(1)),
\]
where \( o_\epsilon(1) \rightarrow 0 \) as \( \epsilon \rightarrow 0 \). We also have
\[
\int_{\delta F^n(x) \leq 2\delta} F^n(\nabla \varphi_\epsilon(x)) dx = t_\delta^n \int_{\delta F^n(x) \leq 2\delta} \left| \theta(x) \nabla \varphi_\epsilon + \nabla \theta(x)(\varphi(x) + \varphi(x_\delta)) \right|^n dx
\]
\[
= t_\delta^n \Omega(\delta^n)
\]
and
\[
\int_{F^n(x) \geq 2\delta} F^n(\nabla \varphi_\epsilon(x)) dx = t_\delta^n \int_{F^n(x) \geq 2\delta} F^n(\nabla \varphi(x)) dx
\]
\[
= t_\delta^n (1 + O(\delta^n)).
\]

Summing the above integral estimates for \( F^n(\nabla \varphi_\epsilon) \) up, we have
\[
\int_\Omega F^n(\nabla \varphi_\epsilon(x)) dx = 1 - n \frac{n+1}{\kappa_n} \frac{1}{\epsilon} \frac{1}{\nabla u_k} \frac{1}{\kappa_n} t_\delta \varphi(x_\delta)(1 + o_\epsilon(1)) + t_\delta^n (1 + O(\delta^n)).
\]
Then

\[ \|F(\nabla \varphi(x))\|_{L^n(\Omega)} \leq 1 + \frac{n^{a+1}}{n-1} \kappa_n \left( \frac{\lambda}{\epsilon} \right)^{\frac{n}{n-1}} \frac{1}{\epsilon} t_{\epsilon} \varphi(x) (1 + o_{\epsilon}(1)) - \frac{1}{n-1} t_{\epsilon} (1 + O(\delta^n)). \]

Set \( v_\epsilon(x) = \frac{\varphi(x)}{\|F(\nabla \varphi(x))\|_{L^n(\Omega)}} \), then \( \|F(v_\epsilon(x))\|_{L^n(\Omega)} = 1 \). Furthermore,

\[ \lambda_1(\Omega) \|v_\epsilon(x)\|^n_{L^n(\Omega)} \geq \frac{\lambda_1(\Omega) t_n^n}{\|F(\nabla \varphi(x))\|_{L^n(\Omega)}^n} \int_{|\varphi(x)| \geq \epsilon} |\varphi(x)|^n dx \]

\[ \geq \lambda_1(\Omega) t_n^n \|\varphi(x)\|^n_{L^n(\Omega)} + O(\delta^n) \left[ 1 + n^{a+1} \kappa_n \left( \frac{\lambda}{\epsilon} \right)^{\frac{n}{n-1}} t_{\epsilon} \varphi(x) (1 + o_{\epsilon}(1)) \right] - t_n^n (1 + O(\delta^n)) \]

\[ = t_n^n \lambda_1(\Omega) \|\varphi(x)\|^n_{L^n(\Omega)} + O(\delta^n) (1 + O(t_n^n)) \]

\[ = t_n^n (1 + O(t_n^n) + O(\delta^n)), \]

where we have used \( \lambda_1(\Omega) \|\varphi(x)\|^n_{L^n(\Omega)} = 1 \).

Next we establish the integral estimates on the domain of \( \{ x \in \Omega : F^n(x) < \epsilon \} \). We have

\[ \lambda_n (1 + \lambda_1(\Omega)) \|v_\epsilon\|^n_{L^n(\Omega)} \frac{\lambda}{\epsilon} |v_\epsilon| \frac{\lambda}{\epsilon} \]

\[ \geq n \log \frac{1}{\epsilon} \left( 1 + \lambda_1(\Omega) \|v_\epsilon\|^n_{L^n(\Omega)} \frac{\lambda}{\epsilon} \right) \|F(\nabla \varphi(x))\|^n_{L^n(\Omega)} \]

\[ = n \log \frac{1}{\epsilon} \left( 1 + t_n^n (1 + O(t_n^n) + O(\delta^n)) \right) \frac{\lambda}{\epsilon} \]

\[ \cdot \left( 1 + n^{a+1} \kappa_n \left( \frac{\lambda}{\epsilon} \right)^{\frac{n}{n-1}} t_{\epsilon} \varphi(x) (1 + o_{\epsilon}(1)) \right) \]

\[ - \frac{n}{n-1} t_n^n (1 + O(\delta^n)) \]

\[ = n \log \frac{1}{\epsilon} + \frac{n^{a+1}}{n-1} \kappa_n \left( \frac{\lambda}{\epsilon} \right)^{\frac{n}{n-1}} t_{\epsilon} \varphi(x) (1 + o_{\epsilon}(1)) \]

\[ - \frac{n}{n-1} \log \frac{1}{\epsilon} t_n^n (1 + O(\delta^n)) + \frac{n}{n-1} \log \frac{1}{\epsilon} t_n^n (1 + O(t_n^n)) \]

\[ + O(\delta^n) + o_{\epsilon}(1) \]

\[ = n \log \frac{1}{\epsilon} + \frac{n^{a+1}}{n-1} \kappa_n \left( \frac{\lambda}{\epsilon} \right)^{\frac{n}{n-1}} t_{\epsilon} \varphi(0) (1 + o_{\epsilon}(1)), \]

where the fact that \( \varphi(x) = \varphi(0) + o_{\epsilon}(1) \) is applied. Note that \( \log \frac{1}{\epsilon} t_n^n O(\delta^n) = o_{\epsilon}(1) \).

Considering the above estimates, we deduce that

\[ \int_{\Omega} \epsilon \exp \left( \lambda_n (1 + \lambda_1(\Omega)) \|v_\epsilon\|^n_{L^n(\Omega)} \frac{\lambda}{\epsilon} |v_\epsilon| \frac{\lambda}{\epsilon} \right) dx \geq C \exp \left[ n \frac{a+1}{n-1} \kappa_n \left( \frac{\lambda}{\epsilon} \right)^{\frac{n}{n-1}} t_{\epsilon} \varphi(0) (1 + o_{\epsilon}(1)) \right] \]

\[ \to + \infty \]

as \( \epsilon \to 0 \), since \( \varphi(0) > 0 \) and \( (\log \frac{1}{\epsilon})^{\frac{n}{n-1}} t_{\epsilon} \to + \infty \). Here \( C \) is a positive constant independent of \( \epsilon \). The conclusion (2) in Theorem [11] holds.

3. Maximizers of the Subcritical Case

In this section, we will show the existence of the maximizers for Moser-Trudinger functional in the subcritical case. We begin with the following existence of the maximizers of the subcritical Moser-Trudinger function.
Proposition 3.1. For any small $\epsilon$ and $0 \leq \alpha < \lambda_1$, there exists some $u_\epsilon \in C^1(\Omega) \cap H$ satisfying

$$J_{\lambda_\epsilon - \epsilon}^\alpha(u_\epsilon) = \sup_{u \in H} J_{\lambda_\epsilon - \epsilon}^\alpha(u).$$

**Proof.** For any fixed $\epsilon$, let $\{u_{\epsilon,j}\}$ be a sequence such that

$$\lim_{j \to +\infty} J_{\lambda_\epsilon - \epsilon}^\alpha(u_{\epsilon,j}) = \sup_{u \in H} J_{\lambda_\epsilon - \epsilon}^\alpha(u).$$

Since $u_{\epsilon,j}$ is bounded in $W_0^{1,n}(\Omega)$, there exists a subsequence of $u_{\epsilon,j}$ (We do not distinguish subsequence and sequence in the paper) such that

$$u_{\epsilon,j} \rightarrow u_\epsilon \text{ weakly in } W_0^{1,n}(\Omega),$$

$$u_{\epsilon,j} \rightarrow u_\epsilon \text{ strongly in } L^p(\Omega), \text{ for any } p \geq 1,$$

$$u_{\epsilon,j} \rightarrow u_\epsilon \text{ a.e. } \Omega$$

as $j \to +\infty$. Hence

$$g_j := \exp[(\lambda_\epsilon - \epsilon) (1 + \alpha\|u_{\epsilon,j}\|\|u_{\epsilon,j}\|^{-\alpha} - \epsilon)] \rightarrow g_j := \exp[(\lambda_\epsilon - \epsilon) (1 + \alpha\|u_\epsilon\|\|u_\epsilon\|^{-\alpha} - \epsilon)]$$

a.e. in $\Omega$. We claim that $u_\epsilon \neq 0$, suppose not, $1 + \alpha\|u_{\epsilon,j}\|\|u_{\epsilon,j}\|^{-\alpha} \rightarrow 1$, from which one can see that $g_j$ is bounded in $L^p(\Omega)$ for some $p > 1$ and $g_j \rightarrow 1$ in $L^1(\Omega)$. Hence $\sup_{u \in H} J_{\lambda_\epsilon - \epsilon}^\alpha(u) = |\Omega|$, which is impossible. Therefore $u_\epsilon \neq 0$. Thanks to Lemma 2.2, we have

$$\lim \sup_{j \to +\infty} \int_\Omega \exp[\lambda_\epsilon q\|u_{\epsilon,j}\|\|u_{\epsilon,j}\|^{-\alpha}] dx < +\infty.$$  

Due to (1), we get

$$1 + \alpha\|u_\epsilon\|_L^\alpha(\Omega) < \frac{1}{1 - \|F(\nabla u_\epsilon)\|_L^\alpha(\Omega)}$$

for $0 \leq \alpha < \lambda_1$. Thus, $g_j$ is bounded in $L^s(\Omega)$ for some $s > 1$. Since $g_j \rightarrow g_\epsilon$ a.e. in $\Omega$, we infer that $g_j \rightarrow g_\epsilon$ strongly in $L^1(\Omega)$ as $j \to +\infty$, Therefore, the extremal function is attained for the case of $\lambda_\epsilon - \epsilon$ and $\|\nabla u_\epsilon\|_{L^\alpha(\Omega)} = 1$. Clearly we can choose $u_\epsilon \geq 0$. It is not difficult to check that the Euler-Lagrange equation of $u_\epsilon$ is

$$
\begin{cases}
-Q_\alpha u_\epsilon = \beta_\epsilon \lambda_\epsilon^{-1} u_\epsilon^{-\alpha} e^{\alpha u_\epsilon^{-\alpha}} + \gamma_\epsilon u_\epsilon^{-1}, \\
u_\epsilon \in W_0^{1,n}(\Omega), \quad \|F(\nabla u_\epsilon)\|_{L^\alpha(\Omega)} = 1, \quad u_\epsilon \geq 0, \\
\alpha_\epsilon = (\lambda_\epsilon - \epsilon)(1 + \alpha\|u_\epsilon\|\|u_\epsilon\|^{-\alpha})^{-1}, \\
\beta_\epsilon = (1 + \alpha\|u_\epsilon\|\|u_\epsilon\|^{-\alpha})/(1 + 2\alpha\|u_\epsilon\|\|u_\epsilon\|^{-\alpha}), \\
\gamma_\epsilon = \alpha/(1 + 2\alpha\|u_\epsilon\|\|u_\epsilon\|^{-\alpha}), \\
\lambda_\epsilon = \int_\Omega u_\epsilon^{-\alpha} e^{\alpha u_\epsilon^{-\alpha}} dx.
\end{cases}
$$

Since $\beta_\epsilon \lambda_\epsilon^{-1} u_\epsilon^{-\alpha} e^{\alpha u_\epsilon^{-\alpha}}$ and $\gamma_\epsilon u_\epsilon^{-1}$ are bounded in $L^s(\Omega)$ for some $s > 1$, then by Lemma 2.2, we have $u_\epsilon \in L^\infty(\Omega)$. It implies $\beta_\epsilon \lambda_\epsilon^{-1} u_\epsilon^{-\alpha} e^{\alpha u_\epsilon^{-\alpha}} + \gamma_\epsilon u_\epsilon^{-1} \in L^\infty(\Omega)$. Then by Theorem 1 in [20], we easily get $u_\epsilon \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$, which implies that $u_\epsilon \in C^{1,\alpha}(\Omega)$.

The following observation is important.
Lemma 3.2. For any $\alpha \in [0, \lambda_1(\Omega))$, we have
\[
\lim_{\epsilon \to 0} J_{\lambda_{n-\epsilon}}^\alpha(u_\epsilon) = \sup_{u \in H} J_{\lambda_{n-\epsilon}}^\alpha(u).
\]

Proof. Obviously,
\[
\lim_{\epsilon \to 0} J_{\lambda_{n-\epsilon}}^\alpha(u_\epsilon) \leq \sup_{u \in H} J_{\lambda_{n-\epsilon}}^\alpha(u).
\]
On the other hand, for any $u \in W^{1,n}_0(\Omega)$ with $\|F(\nabla u)\|_{L^n(\Omega)} \leq 1$, from Fatou’s Lemma and Proposition 3.1, we have
\[
\int e^{\lambda_n|u|^\alpha |\nabla u|^2_{L^n(\Omega)}} \frac{dx}{|\Omega|} \leq \liminf_{\epsilon \to 0} \int e^{(\lambda_{n-\epsilon})|u|^\alpha |\nabla u|^2_{L^n(\Omega)}} \frac{dx}{|\Omega|}
\]
which implies
\[
\lim_{\epsilon \to 0} J_{\lambda_{n-\epsilon}}^\alpha(u_\epsilon) \geq \sup_{u \in H} J_{\lambda_{n-\epsilon}}^\alpha(u).
\]
Hence the result holds.

4. Blow-up Analysis

In this section, we consider the convergence of the maximizing sequence in section 3. There are two cases. The one case is that $M_\epsilon = \max_{\Omega} u_\epsilon(x_\epsilon)$ is bounded. In this case, it is clear that $u_\epsilon$ is bounded in $W^{1,n}_0(\Omega)$. Then we can assume without loss of generality
\[
\begin{align*}
&u_\epsilon \to u_0 \quad \text{weakly in } W^{1,n}_0(\Omega), \\
&u_\epsilon \to u_0 \quad \text{strongly in } L^q(\Omega), \forall q \geq 1, \\
&u_\epsilon \to u_0 \quad \text{a.e. in } \Omega.
\end{align*}
\]
Since, for any $u \in W^{1,n}_0(\Omega)$ with $\int_\Omega F^n(\nabla u)dx \leq 1$, by the Lebesgue dominated convergence theorem we have
\[
\int e^{\lambda_n|u|^\alpha |\nabla u|^2_{L^n(\Omega)}} \frac{dx}{|\Omega|} = \lim_{\epsilon \to 0} \int e^{(\lambda_{n-\epsilon})|u|^\alpha |\nabla u|^2_{L^n(\Omega)}} \frac{dx}{|\Omega|}
\]
Hence $u_0$ is the desired maximizer.

The other case is that $M_\epsilon = u_\epsilon(x_\epsilon) \to +\infty$ and $x_\epsilon \to x_0$ as $\epsilon \to 0$. In this case, the maximizing sequence $u_\epsilon$ blows up as $\epsilon \to 0$, where $x_0$ is called the blow-up point. In the sequel, we will analyze the blow-up behaviors of $u_\epsilon$.

First, by an inequality $e^t \leq 1 + te^t$, we have
\[
|\Omega| < \int e^{|\nabla u|^{\alpha n}} dx \leq |\Omega| + \alpha \epsilon \lambda.
\]
This leads to $\liminf_{\epsilon \to 0} \lambda_\epsilon > 0$.

Case 1. $x_0$ lies in the interior of $\Omega$. 

Lemma 4.1. There holds \( u_0 = 0 \) and \( F^n(\nabla u_0)dx \to \delta_{x_0} \) in the sense of measure as \( \epsilon \to 0 \), where \( \delta_{x_0} \) is the dirac measure at \( x_0 \).

Proof. Suppose \( u_0 \neq 0 \), then for any \( \alpha \in [0, \lambda_1(\Omega)] \), we have

\[
1 + \alpha ||u_\epsilon||_{L^n(\Omega)}^n \to 1 + \alpha ||u_0||_{L^n(\Omega)}^n \leq 1 + ||F(\nabla u_0)||_{L^n(\Omega)}^n < \frac{1}{1 - ||F(\nabla u_0)||_{L^n(\Omega)}^n}.
\]

Hence \( e^{\alpha \cdot |u_\epsilon|^\frac{n}{n-1}} \) is bounded in \( L^s(\Omega) \) for some \( s > 1 \) provided \( \epsilon \) is sufficiently small. Lemma 2.2 implies that \( u_\epsilon \) is uniformly bounded in \( \Omega \). It contradicts the assumption that \( M_\epsilon \to +\infty \).

Assume that \( F^n(\nabla u_\epsilon)dx \to \mu \) in the sense of measure as \( \epsilon \to 0 \), if \( \mu \neq \delta_{x_0} \), we claims that there exists a cut-off function \( \phi(x) \in C_0^1(\Omega) \), which is supported in \( W_r(x_0) \) for some \( r > 0 \) with \( 0 < \phi(x) < 1 \) in \( W_r(x_0) \setminus W_{r/2}(x_0) \) and \( \phi(x) = 1 \) in \( W_{r/2}(x_0) \) satisfying

\[
\int_{W_r(x_0)} \phi F^n(\nabla u_\epsilon)dx \leq 1 - \eta
\]

for some \( \eta > 0 \) and small enough \( \epsilon \). We prove the claim by contradiction. Suppose by the contradiction that there exist sequences of \( \eta_i \to 0 \) and \( r_i \to 0 \) as \( i \to +\infty \) such that

\[
\int_{W_{r_i}(x_0)} \phi_i F^n(\nabla u_\epsilon)dx > 1 - \eta_i.
\]

for every \( \phi_i \in C_0^1(W_{r_i}(x_0)) \) and \( \phi_i = 1 \) in \( W_{r_i/2}(x_0) \). Then

\[
\int_{W_{r_i}(x_0)} \phi_i F^n(\nabla u_\epsilon)dx > 1 - \eta_i. \tag{11}
\]

Taking \( i \to \infty \), the left hand side of (11) converges to 0. However, \( 1 - \eta_i \to 1 \), this contradiction leads to the claim. Since \( u_\epsilon \to 0 \) strongly in \( L^q(\Omega) \) for any \( q > 1 \), we may assume that

\[
\int_{W_r(x_0)} F^n(\nabla u_\epsilon)dx \leq 1 - \eta
\]

provided \( \epsilon \) is sufficient small. By (2), \( e^{\alpha \cdot |u_\epsilon|^\frac{n}{n-1}} \) is uniformly bounded in \( L^s(W_{r_0}(x_0)) \) for some \( s > 1 \) and \( 0 < r_0 < r \). Applying Lemma 2.2, \( u_\epsilon \) is uniformly bounded in \( W_{r_0/2}(x_0) \), which contradicts the fact that \( M_\epsilon \to +\infty \) again. Therefore, \( F^n(\nabla u_\epsilon)dx \to \delta_{x_0} \) as \( \epsilon \to 0 \).

Now we set

\[
r^n = \lambda_\epsilon \beta_\epsilon^{-1} M_\epsilon^{-\frac{n}{n-1}} e^{-\alpha_\epsilon M_\epsilon^{-\frac{n}{n-1}}}.
\]

Fixed any \( \delta \in (0, \frac{\alpha_\epsilon}{\beta_\epsilon}) \), by the expression of \( r_\epsilon \) in (12) and \( \lambda_\epsilon \) in (11), we have

\[
r^n \exp\{\delta M_\epsilon^{-\frac{n}{n-1}}\} = \lambda_\epsilon \beta_\epsilon^{-1} M_\epsilon^{-\frac{n}{n-1}} \exp\{-\alpha_\epsilon M_\epsilon^{-\frac{n}{n-1}}\} \exp\{\delta M_\epsilon^{-\frac{n}{n-1}}\}
\]

\[
= \beta_\epsilon^{-1} M_\epsilon^{-\frac{n}{n-1}} \exp\{(\delta - \alpha_\epsilon) M_\epsilon^{-\frac{n}{n-1}}\} \int_{\Omega} u_\epsilon^{-\frac{n}{n-1}} \exp(\alpha_\epsilon u_\epsilon^{-\frac{n}{n-1}})dx
\]

\[
\leq \beta_\epsilon^{-1} M_\epsilon^{-\frac{n}{n-1}} \exp\{(2\delta - \alpha_\epsilon) M_\epsilon^{-\frac{n}{n-1}}\} \int_{\Omega} u_\epsilon^{-\frac{n}{n-1}} \exp((\alpha_\epsilon - \delta) u_\epsilon^{-\frac{n}{n-1}})dx
\]

\[
\leq C \beta_\epsilon^{-1} M_\epsilon^{-\frac{n}{n-1}} \exp\{(2\delta - \alpha_\epsilon) M_\epsilon^{-\frac{n}{n-1}}\}
\]

\[
\to 0
\]

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as $\epsilon \to 0$. From above, we can easily get the fact that $\beta_\epsilon \to 1$, $\alpha_\epsilon \to \lambda_n$, $r_\epsilon \to 0$, $\gamma_\epsilon \to \alpha$ as $\epsilon \to 0$.

Define the rescaling functions
\[ v_\epsilon(x) = \frac{u_\epsilon(x_\epsilon + r_\epsilon x)}{M_\epsilon}, \]
\[ w_\epsilon(x) = \frac{1}{M_\epsilon^{\frac{1}{n}}} (u_\epsilon(x_\epsilon + r_\epsilon x) - M_\epsilon), \]
where $v_\epsilon(x)$ and $w_\epsilon(x)$ are defined on $\Omega_\epsilon = \{ x \in \mathbb{R}^n : x_\epsilon + r_\epsilon x \in \Omega \}$. By a direct calculation we obtain that
\[ -\text{div}(F^{n-1}(\nabla v_\epsilon)F_\xi(\nabla v_\epsilon)) = \frac{v_\epsilon^{1+\frac{1}{n}}}{M_\epsilon^\alpha} e^{\frac{1}{\alpha}(v_\epsilon^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - M_\epsilon^{\frac{n}{n-1}})} + r_\epsilon^{n} \gamma_\epsilon \epsilon^{n-1} \]
in $\Omega_\epsilon$.

Since $0 \leq v_\epsilon \leq 1$, $\frac{v_\epsilon^{1+\frac{1}{n}}}{M_\epsilon^\alpha} e^{\frac{1}{\alpha}(v_\epsilon^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - M_\epsilon^{\frac{n}{n-1}})} + r_\epsilon^{n} \gamma_\epsilon \epsilon^{n-1}$ is uniformly bounded in $L^\infty(B_r(0))$, by Theorem 1 in [32], $v_\epsilon$ is uniformly bounded in $C^{1,\alpha}(B_{\epsilon}(0))$. By Ascoli-Arzelà’s theorem, we can find a subsequence $\epsilon_j \to 0$ such that $v_{\epsilon_j} \to v$ in $C^1_{loc}(\mathbb{R}^n)$ and satisfies
\[ -\text{div}(F^{n-1}(\nabla v)F_\xi(\nabla v)) = 0 \]
in $\mathbb{R}^n$. Furthermore, we have $0 \leq v \leq 1$ and $v(0) = 1$. The Liouville theorem (see [15]) leads to $v \equiv 1$.

Also we have in $\Omega_\epsilon$
\[ -\text{div}(F^{n-1}(\nabla w_\epsilon)F_\xi(\nabla w_\epsilon)) = \frac{v_\epsilon^{1+\frac{1}{n}}}{M_\epsilon^\alpha} e^{\frac{1}{\alpha}(v_\epsilon^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - M_\epsilon^{\frac{n}{n-1}})} + r_\epsilon^{n} M_\epsilon \gamma_\epsilon \epsilon^{n-1}. \]
(14)

For any $r > 0$, since $0 \leq u_\epsilon(x_\epsilon + r_\epsilon x) \leq M_\epsilon$, we have $-\text{div}(F^{n-1}(\nabla u_\epsilon)F_\xi(\nabla u_\epsilon)) = O(1)$ in $B_r(0)$ for small $\epsilon$. Then from Theorem 1 in [32] and Ascoli-Arzelà’s theorem, there exists $w \in C^{1}(\mathbb{R}^n)$ such that $w_\epsilon$ converges to $w$ in $C^1_{loc}(\mathbb{R}^n)$. Therefore we have
\[ |u_\epsilon^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - M_\epsilon^{\frac{n}{n-1}}| = M_\epsilon^{\frac{n}{n-1}}(u_\epsilon^{\frac{n}{n-1}}(x_\epsilon) - 1) = \frac{n}{n-1} w_\epsilon(x_\epsilon)(1 + O((u_\epsilon(x_\epsilon) - 1)^2)). \]
(15)

By taking $\epsilon \to 0$, we know that $w$ satisfies
\[ -\text{div}(F^{n-1}(\nabla w)F_\xi(\nabla w)) = e^{\frac{n}{n-1}\lambda_n} w \]
in the distributional sense. We also have the facts $w(0) = 0 = \max_{x \in \mathbb{R}^n} w(x)$. Moreover, for any $R > 0$, we have
\[ 1 \geq \lim_{\epsilon \to 0} \epsilon \int_{W_{\epsilon,R}(x_\epsilon)} \frac{u_\epsilon^{\frac{n}{n-1}}}{\lambda_n} e^{\alpha_\epsilon |u_\epsilon^{\frac{n}{n-1}}|} dx \]
\[ = \lim_{\epsilon \to 0} \epsilon \int_{W_{\epsilon}(0)} \frac{u_\epsilon^{\frac{n}{n-1}}}{\lambda_n} e^{\alpha_\epsilon (u_\epsilon^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - M_\epsilon^{\frac{n}{n-1}})} dx \]
\[ = \int_{W_{\epsilon}(0)} e^{\frac{n}{n-1}\lambda_n w} dx. \]
(17)

Taking $R \to +\infty$, we have $\int_{\mathbb{R}^n} e^{\frac{n}{n-1}\lambda_n w} dx \leq 1$.
On the other hand, we claim
\[
\int_{\mathbb{R}^n} e^{\frac{n}{n-1} \lambda_n w} dx \geq 1. \tag{18}
\]
Actually we can prove it by level-set-method. For \( t \in \mathbb{R} \), let \( \Omega_t = \{ x \in \Omega | w(x) > t \} \) and \( \mu(t) = |\Omega_t| \). By the divergence theorem,
\[
\int_{\Omega_t} -\text{div}(F^{n-1}(\nabla w) F_\xi(\nabla w)) dx = \int_{\partial\Omega_t} F^{n-1}(\nabla w) < F_\xi(\nabla w), \frac{\nabla w}{|\nabla w|} > ds
\]
\[
= \int_{\partial\Omega_t} F^{n}(\nabla w) ds.
\]
By using the isoperimetric inequality \( \text{(6)} \) and the co-area formula \( \text{(5)} \), it follows from Hölder inequality that
\[
n\kappa^n \mu(t)^{1-\frac{1}{n}} \leq P_F(\Omega_t) = \int_{\partial\Omega_t} F^{n}(\nabla w) ds
\]
\[
\leq \left( \int_{\partial\Omega_t} F^{n}(\nabla w) ds \right)^{\frac{1}{n}} \left( \int_{\partial\Omega_t} |\nabla w| ds \right)^{1-\frac{1}{n}}
\]
\[
= \left( \int_{\Omega_t} e^{\frac{n}{n-1} \lambda_n w} dx \right)^{\frac{1}{n}} (-\mu'(t))^{1-\frac{1}{n}}. \tag{19}
\]
Hence
\[
\int_{\mathbb{R}^n} e^{\frac{n}{n-1} \lambda_n w} dx = \frac{n}{n-1} \lambda_n \int_{-\infty}^{\max w} e^{\frac{n}{n-1} \lambda_n t} \mu(t) dt
\]
\[
\leq \frac{n}{n-1} \lambda_n \int_{-\infty}^{\max w} e^{\frac{n}{n-1} \lambda_n t} = \mu'(t) \left( \int_{\Omega_t} e^{\frac{n}{n-1} \lambda_n w} dx \right)^{\frac{1}{n}} dt
\]
\[
= \int_{-\infty}^{\max w} \frac{d}{dt} \left( \int_{\Omega_t} e^{\frac{n}{n-1} \lambda_n w} dx \right)^{\frac{1}{n}} dt = \left( \int_{\mathbb{R}^n} e^{\frac{n}{n-1} \lambda_n w} dx \right)^{\frac{n}{n-1}},
\]
which implies the claim.

Thus we get that \( \int_{\mathbb{R}^n} e^{\frac{n}{n-1} \lambda_n w} dx = 1 \), which implies that the equality holds in the above iso-perimeter inequality. Therefore \( \Omega_t \) must be a wulf ball. In other words, \( w \) is radial symmetric with respect to \( F^o(x) \). We can immediately get
\[
w(r) = -\frac{n-1}{\lambda_n} \log(1 + \kappa^n \frac{r^n}{\kappa^n-1}) \tag{20}
\]
where \( r = F^o(x) \).

**Lemma 4.2.** For any \( L > 1 \), we set \( u_{\epsilon, L} = \min \{ u_\epsilon, \frac{M_\epsilon}{L} \} \). Then we have
\[
\limsup_{\epsilon \to 0} \int_{\Omega} F^n(\nabla u_{\epsilon, L}) dx \leq \frac{1}{L}.
\]
**Proof.** We chose \( (u_\epsilon - \frac{M_\epsilon}{L})^+ \) as a test function of \( \text{(10)} \) to get
\[
-\int_{\Omega} (u_\epsilon - \frac{M_\epsilon}{L})^+ \text{div}(F^{n-1}(\nabla u_\epsilon) F_\xi(\nabla u_\epsilon)) dx
\]
\[
= \int_{\Omega} (u_\epsilon - \frac{M_\epsilon}{L})^+ \frac{\beta u_\epsilon}{\lambda_\epsilon} e^{\frac{n}{n-1} \lambda_n u_\epsilon} + \gamma_\epsilon u_\epsilon^{n-1} dx. \tag{21}
\]
Thus we have
\[ \int_{\Omega} (u_{e} - \frac{M_{e}}{L})^{+} [\frac{\beta_{e} u_{e}^{\frac{n}{n-1}}}{\lambda_{e}} e^{\alpha_{e} |u_{e}|^{\frac{n}{n-1}}} + \gamma_{e} u_{e}^{n-1}] \mathrm{d}x \]
\geq \int_{W_{e, R}(x_{e})} (u_{e} - \frac{M_{e}}{L})^{+} \frac{\beta_{e} u_{e}^{\frac{1}{1-n}}}{\lambda_{e}} e^{\alpha_{e} |u_{e}|^{\frac{n}{n-1}}} \mathrm{d}x + o_{e}(1)
\]
\[ = \int_{W_{e}(0)} (u_{e}(x_{e} + r_{e}x) - \frac{M_{e}}{L})^{+} \frac{\beta_{e} u_{e}^{\frac{n-1}{n}}} {\lambda_{e}} e^{\alpha_{e} |u_{e}|^{\frac{n}{n-1}}} (x_{e} + r_{e}x) \mathrm{d}x + o_{e}(1) \]
\[ = \int_{W_{e}(0)} (v_{e} - \frac{1}{L})^{+} \beta_{e} u_{e}^{\frac{1}{n-1}} e^{\alpha_{e} |u_{e}|^{\frac{n}{n-1}}} (x_{e} + r_{e}x - \frac{M_{e}}{L})^{+} \mathrm{d}x + o_{e}(1) \]
\[ \to \int_{W_{e}(0)} (1 - \frac{1}{L}) e^{\frac{1}{1-n} \lambda_{e} \omega} \mathrm{d}x. \quad (22) \]

In virtue of the divergence theorem and Lemma 2.1, the estimation of the left hand side of (21) is

\[ - \int_{\Omega} (u_{e} - \frac{M_{e}}{L})^{+} \mathrm{div}(F^{n-1}(\nabla u_{e}) F_{\xi}(\nabla u_{e})) \mathrm{d}x \]
\[ = - \int_{\Omega} (u_{e} - \frac{M_{e}}{L})^{+} \mathrm{div}(F^{n-1}(\nabla (u_{e} - \frac{M_{e}}{L})^{+})) \mathrm{d}x \]
\[ = \int_{\Omega} F^{n}(\nabla (u_{e} - \frac{M_{e}}{L})^{+}) \mathrm{d}x. \quad (23) \]

Putting (21), (22), (23) together, and taking \( R \to \infty \), we obtain
\[ \int_{\Omega} F^{n}(\nabla (u_{e} - \frac{M_{e}}{L})^{+}) \mathrm{d}x \geq 1 - \frac{1}{L}. \]

Noticing that
\[ \int_{\Omega} F^{n}(\nabla u_{e}) \mathrm{d}x = \int_{\Omega} F^{n}(\nabla u_{e,L}) \mathrm{d}x + \int_{\Omega} F^{n}(\nabla (u_{e} - \frac{M_{e}}{L})^{+}) \mathrm{d}x. \]

Thus the conclusion can be obtained due to the fact \( \int_{\Omega} F^{n}(\nabla u_{e}) \mathrm{d}x = 1 \). \hfill \square

**Remark 4.3.** From Lemma 4.2, applying \( L^{\frac{n-1}{n}} u_{e,L} \) to inequality (2), we get
\[ \int_{\Omega} e^{\frac{\lambda_{e,L} |u_{e,L}|^{\frac{n}{n-1}}}{\lambda_{e}}} \mathrm{d}x \leq C < +\infty. \quad (24) \]

For any \( L > 1 \), since \( e^{\alpha_{e,L} \frac{1}{n-1} |u_{e,L}|^{\frac{n}{n-1}}} \) is uniformly bounded in \( L^{1}(\Omega) \), then by (24) \( e^{\alpha_{e,L} |u_{e,L}|^{\frac{n}{n-1}}} \) is uniformly bounded in \( L^{q}(\Omega) \) for some \( q > 1 \). Due to \( u_{e,L} \) converges to 0 almost everywhere in \( \Omega \), it implies that \( e^{\alpha_{e,L} |u_{e,L}|^{\frac{n}{n-1}}} \) converges to 1 in \( L^{1}(\Omega) \). Thus we have
\[ \lim_{\epsilon \to 0} \int_{\{L u_{e} \leq M_{e}\}} e^{\alpha_{e,L} |u_{e,L}|^{\frac{n}{n-1}}} \mathrm{d}x = \lim_{\epsilon \to 0} \int_{\Omega} e^{\alpha_{e,L} |u_{e,L}|^{\frac{n}{n-1}}} \mathrm{d}x = |\Omega|. \]
Hence
\[
\lim_{\epsilon \to 0} \int_{\Omega} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx = \lim_{\epsilon \to 0} \int_{\{Lu_{\epsilon} \leq M_{\epsilon}\}} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx + \lim_{\epsilon \to 0} \int_{\{Lu_{\epsilon} > M_{\epsilon}\}} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx \\
\leq |\Omega| + \lim_{\epsilon \to 0} \frac{\lambda_{\epsilon} L {\frac{n}{n-1}}} {M_{\epsilon}} \int_{\{Lu_{\epsilon} > M_{\epsilon}\}} \frac{u_{\epsilon}^{\frac{n}{n-1}}} {\lambda_{\epsilon}} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx \\
\leq |\Omega| + \lim_{\epsilon \to 0} \frac{\lambda_{\epsilon} L {\frac{n}{n-1}}} {M_{\epsilon}}.
\]

Taking \(L \to 1\), we get
\[
\lim_{\epsilon \to 0} \int_{\Omega} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx \leq |\Omega| + \lim_{\epsilon \to 0} \frac{\lambda_{\epsilon}} {M_{\epsilon} {\frac{n}{n-1}}}. \tag{25}
\]

Noticing that \(\lim_{\epsilon \to 0} \int_{\Omega} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx > |\Omega|\), we have
\[
\lim_{\epsilon \to 0} \frac{M_{\epsilon}} {\lambda_{\epsilon}} = 0. \tag{26}
\]

The following Lemma will be used in Section 5.

**Lemma 4.4.**
\[
\lim_{\epsilon \to 0} \int_{\Omega} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx = |\Omega| + \lim_{R \to +\infty} \lim_{\epsilon \to 0} \int_{W_{R, \epsilon}(x_{\epsilon})} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx.
\]

**Proof.** On one hand,
\[
\lim_{\epsilon \to 0} \sup_{\Omega} \int_{W_{R, \epsilon}(x_{\epsilon})} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx \\
\leq \lim_{\epsilon \to 0} \sup_{\Omega} \int e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx - \lim_{\epsilon \to 0} \inf_{\Omega} \int_{W_{R, \epsilon}(x_{\epsilon})} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx \\
\leq \lim_{\epsilon \to 0} \sup_{\Omega} \int e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx - |\Omega|. \tag{27}
\]

On the other hand,
\[
\int_{W_{R, \epsilon}(x_{\epsilon})} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx = \frac{\lambda_{\epsilon}} {M_{\epsilon}^{\frac{n}{n-1}}} \left( \int_{W_{R}(0)} e^{\frac{n}{n-1}} \lambda_{\epsilon} w \, dx + o_{\epsilon}(1) \right),
\]
which gives
\[
\lim_{R \to +\infty} \lim_{\epsilon \to 0} \int_{W_{R, \epsilon}(x_{\epsilon})} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx = \lim_{\epsilon \to 0} \frac{\lambda_{\epsilon}} {M_{\epsilon}^{\frac{n}{n-1}}}. \tag{28}
\]

Combining (25), (27), (28) and Lemma 3.2 we get the result. \[\square\]

Now we claim that
\[
\lim_{\epsilon \to 0} \int_{\Omega} \frac{M_{\epsilon}} {\lambda_{\epsilon} \beta_{\epsilon} u_{\epsilon}^{\frac{n}{n-1}}} e^{\alpha_{\epsilon} |u_{\epsilon}|^{\frac{n}{n-1}}} \, dx = 1. \tag{29}
\]
To this purpose, we denote \( \varphi_\epsilon = \frac{M_\epsilon}{\lambda_\epsilon} \beta_\epsilon u_\epsilon^\frac{1}{n} e^{\alpha_\epsilon |u_\epsilon|} \). Clearly
\[
\int_{\Omega} \varphi_\epsilon \, dx = \int_{\{u_\epsilon < M_\epsilon\}} \varphi_\epsilon \, dx + \int_{\{u_\epsilon \geq M_\epsilon\} \setminus W_{\epsilon R}(x_\epsilon)} \varphi_\epsilon \, dx + \int_{W_{\epsilon R}(x_\epsilon)} \varphi_\epsilon \, dx.
\]
We estimate the three integrals on the right hand side respectively. By \(26\) and Lemma \(4.1\) we have
\[
0 \leq \int_{\{u_\epsilon < M_\epsilon\}} \varphi_\epsilon \, dx \leq \frac{M_\epsilon \beta_\epsilon}{\lambda_\epsilon} \int_{\{u_\epsilon < M_\epsilon\}} \frac{u_\epsilon^\frac{1}{n}}{u_\epsilon} e^{\alpha_\epsilon |u_\epsilon|} \, dx
\]
\[
\leq \frac{M_\epsilon \beta_\epsilon}{\lambda_\epsilon} \int_{\Omega} \frac{u_\epsilon^\frac{1}{n}}{u_\epsilon} e^{\alpha_\epsilon |u_\epsilon|} \, dx = o_\epsilon(1)O\left(\frac{1}{L}\right).
\]  
(30)

Moreover for any \( R > 0 \), we have
\[
\int_{\{u_\epsilon \geq M_\epsilon\} \setminus W_{\epsilon R}(x_\epsilon)} \varphi_\epsilon \, dx \leq \int_{\{u_\epsilon \geq M_\epsilon\} \setminus W_{\epsilon R}(x_\epsilon)} \frac{L_\epsilon \beta_\epsilon}{\lambda_\epsilon} \frac{u_\epsilon^\frac{1}{n}}{u_\epsilon} e^{\alpha_\epsilon |u_\epsilon|} \, dx
\]
\[
\leq \frac{L_\epsilon \beta_\epsilon}{\lambda_\epsilon} \int_{\Omega} \frac{u_\epsilon^\frac{1}{n}}{u_\epsilon} e^{\alpha_\epsilon |u_\epsilon|} \, dx \to L \left(1 - \int_{W_R(0)} e^{\frac{1}{n} \lambda w} \, dx\right),
\]  
(31)

and
\[
\int_{W_{\epsilon R}(x_\epsilon)} \varphi_\epsilon \, dx = \int_{W_R(0)} \beta_\epsilon u_\epsilon^\frac{1}{n} e^{\alpha_\epsilon \left(|u_\epsilon| - M_\epsilon \frac{n}{\lambda_\epsilon}\right)} \, dx
\]
\[
\to \int_{W_R(0)} e^{\frac{n}{\lambda} \lambda w} \, dx.
\]  
(32)

Putting \(30\), \(31\), \(32\) together and taking \( \epsilon \to 0 \) first, then letting \( R \to \infty \), we conclude \(29\).

In the similar way, we also can obtain that
\[
\lim_{\epsilon \to 0} \int_{\Omega} \beta_\epsilon M_\epsilon \frac{u_\epsilon^\frac{1}{n}}{u_\epsilon} e^{\alpha_\epsilon |u_\epsilon|} \phi(x) \, dx = \phi(x_0)
\]  
(33)

for any \( \phi(x) \in C^0_c(\Omega) \).

The following phenomenon was first discovered by Brezis and Merle \(3\), developed later by Struwe \(25\). We deduce the new version involving n-Finsler-Laplacian.

**Lemma 4.5.** Let \( \{f_\epsilon\} \) be a uniformly bounded sequence of functions in \( L^1(\Omega) \), and \( \{\psi_\epsilon\} \subset C^1(\overline{\Omega}) \cap W^{1,n}_0(\Omega) \) satisfy
\[
- \text{div}(F^{n-1}(\nabla \psi_\epsilon)F_\xi(\nabla \psi_\epsilon)) = f_\epsilon + \alpha \psi_\epsilon |\psi_\epsilon|^{n-2} \quad \text{in } \Omega.
\]  
(34)

where \( 0 \leq \alpha < \lambda_1(\Omega) \) is a constant. Then for any \( 1 < q < n \), we have \( \|\nabla \psi_\epsilon\|_{L^q(\Omega)} \leq C \) for some constant \( C \) depending only on \( q, n, \Omega \) and the upper bound of \( \|f_\epsilon\|_{L^1(\Omega)} \).

**Proof.** When \( \alpha = 0 \). We use an argument of M. Struwe to prove that \( \|\nabla \psi_\epsilon\|_{L^1(\Omega)} \leq C\|f_\epsilon\|_{L^1(\Omega)} \) for some constant \( C \) depending only on \( q, n, \Omega \). Without loss of generality, we assume \( \|f_\epsilon\|_{L^1(\Omega)} = 1 \). For \( t \geq 1 \), denote \( \psi_\epsilon^t = \min\{\psi_\epsilon^+, t\} \), where \( \psi_\epsilon^+ \) is a positive part of \( \psi_\epsilon \). Testing Eq.(34) with \( \psi_\epsilon^t \), we have \( \int_{\Omega} F^n(\nabla \psi_\epsilon^t) \, dx \leq \int_{\Omega} |f_\epsilon| \psi_\epsilon^t \, dx \leq t \).

Assume \( |\Omega| = |W_d| \), where \( W_d = \{x \in \mathbb{R}^n : F^0(x) \leq d\} \).
Let $\psi^*_t$ be the nonincreasing rearrangement of $\psi^t$, and $|W_\rho| = |\{x \in W_d : \psi^*_t \geq t\}|$. It is known that $\|F(\nabla \psi^*_t)\|_{L^\infty(W_d)} \leq \|F(\nabla \psi^t)\|_{L^\infty(\Omega)}$, and we have
\[
\inf_{\phi \in W^{1,n}_0(W_d), \phi|_{W_\rho} = t} \int_{W_d} F^n(\nabla \phi) \, dx \leq \int_{W_d} F^n(\nabla \psi^*_t) \, dx \leq t. \tag{35}
\]
The above infimum can be attained by
\[
\phi_1(x) = \left\{ \begin{array}{ll}
t \log \frac{\delta(x)}{\rho} / \log \frac{\delta}{\rho} & \text{in } W_d \setminus W_\rho, \\
\quad & \text{in } W_\rho.
\end{array} \right.
\]
Calculating $\|F(\nabla \phi_1)\|_{L^\infty(W_d)}^n$, we have by (35), $\rho \leq de^{-C_1 t}$ for some constant $C_1 > 0$. Hence
\[
\{x \in \Omega : \psi_\varepsilon \geq t\} = |W_\rho| \leq \kappa_n d^n e^{-nC_1 t}.
\]
For any $0 < \delta < nC_1$,
\[
\int_{\Omega} e^{\delta \psi^*_t} \, dx \leq e^{\delta |\Omega|} + \sum_{m=1}^{\infty} e^{(m+1)\delta} |\{x \in \Omega : m \leq \psi_\varepsilon \leq m + 1\}|
\leq e^{\delta |\Omega|} + \kappa_n d^n e^{\delta} \sum_{m=1}^{\infty} e^{-(nC_1 - \delta)m} \leq C_2
\]
for some constant $C_2$. Testing Eq. (34) with $\log \frac{1 + 2\psi^*_t}{1 + \psi^*_t}$, we have
\[
\int_{\Omega} \frac{F^n(\nabla \psi^*_t)}{(1 + \psi^*_t)(1 + 2\psi^*_t)} \, dx \leq \log 2.
\]
By the Young inequality, we have for any $1 < q < n$,
\[
\int_{\Omega} F^q(\nabla \psi^*_t) \, dx \leq \int_{\Omega} \frac{F^n(\nabla \psi^*_t)}{(1 + \psi^*_t)(1 + 2\psi^*_t)} \, dx + \int_{\Omega} (1 + \psi^*_t)(1 + 2\psi^*_t)^{\frac{q}{n-q}} \, dx
\leq C_3(1 + \int_{\Omega} e^{\delta \psi^*_t} \, dx) \leq C_4,
\]
for some constants $C_3$ and $C_4$ depending only on $q$, $n$ and $\Omega$. Let $\psi^-_t$ be the negative part of $\psi_t$. Similarly, we have $\int_{\Omega} F^q(\nabla \psi^-_t) \, dx \leq C_5$ for some constant $C_5$ depending only on $q$, $n$ and $\Omega$. Then by Lemma 2.4, the lemma holds.

When $\alpha \in (0, \lambda_1(\Omega))$. Suppose $\psi_\varepsilon$ is unbounded in $L^{n-1}(\Omega)$. Then there exist a subsequence $\{\varepsilon_j\}$ such that $||\psi_{\varepsilon_j}||_{L^{n-1}(\Omega)} \to +\infty$ as $j \to +\infty$. Let $w_{\varepsilon_j} = \psi_{\varepsilon_j}/||\psi_{\varepsilon_j}||_{L^{n-1}(\Omega)}$. Then we have $||w_{\varepsilon_j}||_{L^{n-1}(\Omega)} = 1$, and $-Q_n(w_{\varepsilon_j})$ is bounded in $L^1(\Omega)$. Hence $w_{\varepsilon_j}$ is bounded in $W^{1,q}_0(\Omega)$ for any $0 < q < n$. Assume $w_{\varepsilon_j}$ converges to $w$ weakly in $W^{1,q}_0(\Omega)$ and strongly in $L^{n-1}(\Omega)$. It can be easily derived that $w$ is a weak solution of $-Q_n u = \alpha \lambda_1(\Omega) u$ in $\Omega$. Since $0 < \alpha < \lambda_1(\Omega)$, $w$ must be zero. On the other hand, $||w_{\varepsilon_j}||_{L^{n-1}(\Omega)} = 1$ leads to $||w||_{L^{n-1}(\Omega)} = 1$, contradiction. Therefore $\psi_\varepsilon$ must be bounded in $L^{n-1}(\Omega)$. It implies $f_\varepsilon + \alpha \psi_\varepsilon \psi^-_t \to 0$ in $L^1(\Omega)$. Then, for any $1 < q < n$, there exist a constant $C$ depending only on $q, n, \Omega$, and the upper bounded of $||f_\varepsilon||_{L^1(\Omega)}$ such that $||\nabla \psi^-_t||_{L^n(\Omega)} \leq C$. Thus the proof is finished. \hfill \Box

The following lemma reveals how $u_\varepsilon$ converges away from $x_0$.
Lemma 4.6. $M^\frac{1}{\alpha} u_\epsilon \rightharpoonup G_\alpha$ weakly in $W^{1,q}_0(\Omega)$ for any $1 < q < n$, where $G_\alpha$ is a Green function satisfying
\begin{equation}
\begin{cases}
-\text{div}(F^{n-1}(\nabla G_\alpha)F_\xi(\nabla G_\alpha)) = \delta_{x_0} + \alpha G^{n-1}_\alpha & \text{in } \Omega, \\
G_\alpha = 0 & \text{on } \partial\Omega.
\end{cases}
\end{equation}
(36)
Furthermore, $M^\frac{1}{\alpha} u_\epsilon \to G_\alpha$ in $C^1(\Omega')$ for any domain $\Omega' \subset \overline{\Omega}\setminus\{x_0\}$.

Proof. By Eq. (10), we have
\begin{equation}
-\int Q_n(M^\frac{1}{\alpha} u_\epsilon)dx = \frac{M_\alpha \beta_\epsilon u_\epsilon^{\frac{1}{\alpha}}}{\lambda_\epsilon} e^{\alpha_\epsilon|u_\epsilon|^{\frac{n}{n-1}}} + \gamma_\epsilon M_\epsilon u_\epsilon^{n-1}.
\end{equation}
(37)
Due to (29) and Lemma 4.5 we obtain that $M^\frac{1}{\alpha} u_\epsilon$ is uniformly bounded in $W^{1,q}_0(\Omega)$ for any $1 < q < n$. Assume $M^\frac{1}{\alpha} u_\epsilon \rightharpoonup G_\alpha$ weakly in $W^{1,q}_0(\Omega)$. Testing Eq. (37) with $\phi \in C^\infty_0(\Omega)$, we have by (33)
\begin{align*}
-\int \phi Q_n(M^\frac{1}{\alpha} u_\epsilon)dx &= \int \phi \frac{M_\alpha \beta_\epsilon u_\epsilon^{\frac{1}{\alpha}}}{\lambda_\epsilon} e^{\alpha_\epsilon|u_\epsilon|^{\frac{n}{n-1}}} dx + \gamma_\epsilon \int \phi M_\epsilon u_\epsilon^{n-1} dx \\
&\to \phi(x_0) + \alpha \int \phi G_\alpha^{n-1} dx.
\end{align*}
Hence
\begin{equation}
\int \nabla \phi F^{n-1}(\nabla G_\alpha)F_\xi(\nabla G_\alpha) dx = \phi(x_0) + \alpha G_\alpha^{n-1},
\end{equation}
in the sense of measure and whence (35) holds.

For any fixed small $\delta$, we choose a cut-off function $\xi(x) \in C^\infty_0(\Omega\setminus\mathcal{W}_3(x_0))$ such that $\xi(x) = 1$ on $\Omega\setminus\mathcal{W}_3(x_0)$. By Lemma 11 we get $\int \Omega F^n(\nabla(\xi u_\epsilon))dx \to 0$ as $\epsilon \to 0$. Then $e^{(\xi u_\epsilon)^{\frac{n}{n-1}}}$ is bounded in $L^s(\Omega\setminus\mathcal{W}_3(x_0))$ for any $s > 1$. In particular, $e^{u_\epsilon^{\frac{n}{n-1}}}$ is bounded in $L^s(\Omega\setminus\mathcal{W}_3(x_0))$. Since $M^\frac{1}{\alpha} u_\epsilon$ is bounded in $L^q(\Omega)$ for any $q > 1$, Hölder inequality implies that $\frac{M_\epsilon u_\epsilon^{\frac{1}{\alpha}}}{\lambda_\epsilon} e^{\alpha_\epsilon|u_\epsilon|^{\frac{n}{n-1}}}$ is uniformly bounded in $L^{\infty}(\Omega\setminus\mathcal{W}_3(x_0))$. From the proof of the Lemma 4.5 and $0 \leq \gamma_\epsilon < \lambda_1(\Omega)$, we have $\gamma_\epsilon M_\epsilon u_\epsilon^{n-1} \in L^1(\Omega\setminus\mathcal{W}_3(x_0))$. Then by Lemma 2.2 we have
\begin{equation}
||M^\frac{1}{\alpha} u_\epsilon||_{L^{\infty}(\Omega\setminus\mathcal{W}_3(x_0))} < C.
\end{equation}

Theorem 1 in [32] and Ascoli-Arzela’s theorem, we have $M^\frac{1}{\alpha} u_\epsilon$ converges to $G_\alpha$ in $C^1_{\text{loc}}(\Omega\setminus\mathcal{W}_3(x_0))$. \hfill \Box

Lemma 4.7. Asymptotic representation of Green function $G_\alpha$ is
\begin{equation}
G_\alpha = -\frac{1}{(n\kappa_\alpha)^{\frac{n}{n-1}}} \log F^\alpha(x - x_0) + C_G + \psi(x)
\end{equation}
(38)
where $C_G$ is a constant, $\psi(x_0) = 0$ and $\psi(x) \in C^0(\Omega) \cap C^1_{\text{loc}}(\Omega\setminus\{x_0\})$ such that $\lim_{x \to x_0} F^\alpha(x - x_0) \nabla \psi(x) = 0$.

Up to now, we have described the convergence behavior of $u_\epsilon$ near $x_0$ and away from $x_0$ when the concentration point $x_0$ in the interior of $\Omega$.

Case 2. $x_0$ lies on $\partial\Omega$. 


Lemma 5.1. Assume that $u_\epsilon$ is a normalized concentrating sequence in $W^{1,n}_0(W,\rho)$ with a blow up point at the origin, i.e.

1. $\int_{W_\rho} F^n(|\nabla u_\epsilon|)dx = 1$,
2. $u_\epsilon \rightharpoonup 0$ weakly in $W^{1,n}_0(W,\rho)$,
3. for any $0 < r < \rho$, $\int_{W_\rho \setminus W_r} F^n(\nabla u_\epsilon)dx \to 0$.

Proof. Suppose not, there exists $R > 0$ such that $d_\epsilon \leq Rr_\epsilon$. Take some $y_\epsilon \in \partial\Omega$ such that $d_\epsilon = |x_\epsilon - y_\epsilon|$. Let $\overline{\pi}_\epsilon = M_\epsilon^{-1}u_\epsilon(y_\epsilon + r, x)$. By a reflection argument, similar to the case 1, we have $\overline{\pi}_\epsilon \to 1$ in $C^1(B_R^*)$ for $||\overline{\pi}_\epsilon||_{L^\infty(B_R^*)} = 1$. This contradicts $\overline{\pi}_\epsilon(0) = 0$.

Set

$$\Omega_\epsilon = \{x \in \mathbb{R}^n | x_\epsilon + r, x \in \Omega\}.$$ 

By Lemma 4.8 we have $\lim_{\epsilon \to 0} \frac{dist(x_\epsilon, \partial\Omega)}{r_\epsilon} \to +\infty$, then $\Omega_\epsilon \to \mathbb{R}^n$. Let $w_\epsilon$ be define in (13) and $w$ be define in (20). Similar arguments to Case 1 imply that $w_\epsilon \to w$ in $C^1_{loc}(\mathbb{R}^n)$. We proceed as in Case 1, $M_\epsilon^{-1}u_\epsilon \to \overline{G}_\alpha$ weakly in $W^{1,n}_0(\Omega)$, and in $C^1(\Omega)$, where $\overline{G}$ satisfies the following equation:

$$\left\{ \begin{array}{ll}
-Q_\alpha \overline{G}_\alpha &= 0 & \text{in } \Omega,
\overline{G}_\alpha &= 0 & \text{in } \partial\Omega.
\end{array} \right.$$ 

The above equation has a unique solution $\overline{G} = 0$. Hence

$$M_\epsilon^{-1}u_\epsilon \to 0 \text{ weakly in } W^{1,n}_0(\Omega), \quad M_\epsilon^{-1}u_\epsilon \to 0 \text{ in } C^1(\Omega \setminus \{x_0\}). \quad (39)$$

This is all we need to know about the convergence behavior of $u_\epsilon$ when the concentration point $x_0$ lies on the boundary of $\Omega$.

The proof point (1) of Theorem 1.1. If $M_\epsilon$ is bounded, elliptic estimates implies that the Theorem holds. If $M_\epsilon \to +\infty$, then we have $||u_\epsilon||_{L^\infty(\Omega)} \to 0$. A straightforward calculation gives

$$J_{\lambda_n - \epsilon}^\alpha(u_\epsilon) = \int_{\Omega} e^{(\lambda_n - \epsilon)|u_\epsilon|^n} \left((1 + \alpha||u_\epsilon||_{L^n(\Omega)})^{\frac{n}{n-1}} - 1\right) e^{(\lambda_n - \epsilon)|u_\epsilon|^{\frac{n}{n-1}}} \frac{dx}{\omega_n} \leq e^{\lambda_n M_\epsilon^{-1}u_\epsilon ||u_\epsilon||_{L^n(\Omega)}^{\frac{n}{n-1}} - 1} \int_{\Omega} e^{\lambda_n |u_\epsilon|\frac{n}{n-1}} \frac{dx}{\omega_n} \leq e^{\lambda_n ||u_\epsilon||_{L^n(\Omega)}^{\frac{n}{n-1}} - 1} \int_{\Omega} e^{\lambda_n |u_\epsilon|\frac{n}{n-1}} \frac{dx}{\omega_n}.$$ 

Notice that $\alpha$ satisfies $0 \leq \alpha < \lambda_1(\Omega)$. When $x_0 \in \Omega$, we have $||M_\epsilon^{-1}u_\epsilon||_{L^n(\Omega)} \to ||G_\alpha||_{L^n(\Omega)}$, when $x_0 \in \partial\Omega$, $||M_\epsilon^{-1}u_\epsilon||_{L^n(\Omega)} \to 0$. Hence, together with Lemma 3.2 and (2) completes the proof of point (1) of Theorem 1.1.

5. PROOF OF THEOREM 1.2

In this section, we will prove our main Theorem. We first give a Lemma in 43.
Then
\[
\limsup_{\epsilon \to 0} \int_{\mathcal{W}_0} (e^{\lambda_\alpha|u_\epsilon|^{\frac{2}{n\kappa}}/\epsilon} - 1) dx \leq \kappa_n \rho^n e^{1+\frac{1}{2}+\cdots+\frac{1}{n\kappa}}. \tag{40}
\]

Motivated by the arguments in [6, 36, 38], we first compute an upper bound of \( T_0 \) if \( u_\epsilon \) blows up.

**Lemma 5.2.** If \( \limsup_{\epsilon \to 0} \|u_\epsilon\|_\infty = \infty \), then
\[
\sup_{u \in H_0^1} J_{\lambda_\alpha}^\alpha (u) \leq |\Omega| + \kappa_n e^{\lambda_\alpha C_G + 1 + \frac{1}{2} + \cdots + \frac{1}{n\kappa}}. \tag{41}
\]

**Proof.** **Case 1.** \( x_0 \) lies in the interior of \( \Omega \).

Note that \( x_0 \) is a blow-up point of \( u_\epsilon \). We chose \( \mathcal{W}_0(x_0) \subset \Omega \) for sufficient small \( \delta > 0 \). By [36] we have
\[
\int_{\Omega \setminus \mathcal{W}_0(x_0)} F^n(\nabla G_\alpha) dx = \int_{\partial(\Omega \setminus \mathcal{W}_0(x_0))} G_\alpha F^{n-1}(\nabla G_\alpha)(F_\xi(\nabla G_\alpha), \nu) ds - \int_{\Omega \setminus \mathcal{W}_0(x_0)} G_\alpha \text{div} (F^{n-1}(\nabla G_\alpha) F_\xi(\nabla G_\alpha)) dx \\
+ \alpha \int_{\Omega \setminus \mathcal{W}_0(x_0)} |G_\alpha|^n dx
\]
\[
= - \int_{\partial \mathcal{W}_0(x_0)} G_\alpha F^{n-1}(\nabla G_\alpha)(F_\xi(\nabla G_\alpha), \nu) ds + \alpha \int_{\Omega \setminus \mathcal{W}_0(x_0)} |G_\alpha|^n dx. \tag{42}
\]
Due to [38] and Lemma 2.1 we have on \( \partial \mathcal{W}_0(x_0) \)
\[
F(\nabla G_\alpha) = F(-\frac{1}{(n\kappa_n)^{\frac{1}{2}} - \frac{1}{2}} \nabla F^o(x - x_0) + o(\frac{1}{F^o(x - x_0)}))
\]
\[
= \frac{1}{(n\kappa_n)^{\frac{1}{2}} - \frac{1}{2}} + o(\frac{1}{\delta}). \tag{43}
\]
and
\[
(F_\xi(\nabla G_\alpha), \nu) = (F_\xi(\nabla G_\alpha), \frac{\nabla F^o(x - x_0)}{|\nabla F^o(x - x_0)|})
\]
\[
= (F_\xi(\nabla G_\alpha), (-\frac{1}{n\kappa_n})^{\frac{1}{2}} - \frac{1}{2} \nabla F^o(x - x_0) - \frac{1}{n\kappa_n}) \frac{\nabla G_\alpha - o(\frac{1}{F^o(x - x_0)})}{|\nabla F^o(x - x_0)|}
\]
\[
= -\frac{1}{n\kappa_n}^{\frac{1}{2}}\cdot \delta \frac{F(\nabla G_\alpha)}{|\nabla F^o(x - x_0)|} - \frac{o(\frac{1}{\delta})}{|\nabla F^o(x - x_0)|}
\]
\[
= -\delta(1 + o_\delta(1)) \frac{1}{|\nabla F^o(x - x_0)|}. \tag{44}
\]
where \( o_\delta(1) \to 0 \) as \( \delta \to 0 \). Putting [38], [43], [44] into [42], we obtain
\[
\int_{\Omega \setminus \mathcal{W}_0(x_0)} F^n(\nabla G_\alpha) dx = -\frac{1}{(n\kappa_n)^{\frac{1}{2}} - \frac{1}{2}} \log \delta + C_G + \alpha \int_{\Omega \setminus \mathcal{W}_0(x_0)} |G_\alpha|^n dx + o_\delta(1) \tag{45}
\]
Hence from Lemma 4.6 we have
\[
\int_{\Omega \setminus \mathcal{W}_0(x_0)} F^n(\nabla u_\epsilon) dx = \frac{1}{M_\epsilon^{\frac{1}{2}} - \frac{1}{2}} \log \delta + C_G + \alpha \int_{\Omega \setminus \mathcal{W}_0(x_0)} |G_\alpha|^n dx + o_\delta(1) + o_\epsilon(1), \tag{46}
\]
where \( o_\epsilon(1) \to 0 \) as \( \epsilon \to 0 \).
Next we let $b_\epsilon = \sup_{\partial \mathcal{W}_\delta(x_0)} u_\epsilon$ and \( \overline{u}_\epsilon = (u_\epsilon - b_\epsilon)^+ \). Then \( \overline{u}_\epsilon \in W^{1,n}_0(\mathcal{W}_\delta(x_0)) \).

From (46) and the fact that \( \int_{\mathcal{W}_\delta(x_0)} F^n(\nabla u_\epsilon) dx = 1 - \int_{\Omega \setminus \mathcal{W}_\delta(x_0)} F^n(\nabla u_\epsilon) dx \), we have
\[
\int_{\mathcal{W}_\delta(x_0)} F^n(\nabla u_\epsilon) dx = \tau_\epsilon \leq 1 - \frac{1}{M_\epsilon^{n-1}} \log \delta + C_\delta + \alpha \int_{\Omega \setminus \mathcal{W}_\delta(x_0)} |G_\alpha|^n dx + o_\delta(1) + o_\epsilon(1).
\]

By Lemma 5.1,
\[
\limsup_{\epsilon \to 0} \int_{\mathcal{W}_\delta(x_0)} \left( e^{\lambda_n \overline{u}_\epsilon^{1/n}} \overline{u}_\epsilon^{1-1/n} - 1 \right) dx \leq \kappa_n \delta^{n} e^{1 + \frac{1}{n} + \cdots + \frac{1}{n-1}}.
\]

Now we focus on the estimate in the bubbling domain \( \mathcal{W}_{R\epsilon}(x_\epsilon) \). According to the rescaling functions in Section 4, we can assume that \( w_\epsilon \to w \) in \( C^1_{loc}(\mathbb{R}^n) \), and whence \( u_\epsilon = M_\epsilon + o_\epsilon(R) \), where \( o_\epsilon(R) \to 0 \) as \( \epsilon \to 0 \) for any fixed \( R > 0 \). Then from Lemma 4.6 we have
\[
\alpha_n |u_\epsilon|^{\frac{n}{n-1}} \leq \lambda_n (1 + \alpha ||u_\epsilon||_{L^n(\Omega)})^{\frac{n}{n-1}} (\overline{u}_\epsilon + b_\epsilon)^{\frac{n}{n-1}} \\
\leq \lambda_n \overline{u}_\epsilon^{\frac{n}{n-1}} + \frac{\lambda_n \alpha}{n-1} ||G_\alpha||_{L^n(\Omega)}^{\frac{n}{n-1}} + \frac{n}{n-1} \alpha \epsilon_{b_\epsilon} \overline{u}_\epsilon^{\frac{n}{n-1}} + o_\epsilon(1),
\]
and
\[
b_\epsilon \overline{u}_\epsilon^{\frac{n}{n-1}} = - \frac{1}{(n \kappa_n)^{n-1}} \log \delta + C_\delta + o_\delta(1) + o_\epsilon(1).
\]

Notice that
\[
\lambda_n \overline{u}_\epsilon^{\frac{n}{n-1}} \leq \frac{\lambda_n \overline{u}_\epsilon^{\frac{n}{n-1}}}{\tau_\epsilon^{\frac{n}{n-1}}} - \frac{\lambda_n}{n-1} \left( \frac{1}{(n \kappa_n)^{n-1}} \log \delta + \alpha ||G_\alpha||_{L^n(\Omega)}^{\frac{n}{n-1}} + C_\delta + o_\delta(1) + o_\epsilon(1) \right) \\
= \frac{\lambda_n \overline{u}_\epsilon^{\frac{n}{n-1}}}{\tau_\epsilon^{\frac{n}{n-1}}} + \frac{n}{n-1} \log \delta - \frac{\alpha \lambda_n}{n-1} ||G_\alpha||_{L^n(\Omega)}^{\frac{n}{n-1}} - \frac{\lambda_n}{n-1} C_\delta + o_\delta(1) + o_\epsilon(1).
\]

Combining (47), (49), we obtain in \( \mathcal{W}_{R\epsilon}(x_\epsilon) \)
\[
\alpha_n |u_\epsilon|^{\frac{n}{n-1}} \leq \frac{\lambda_n \overline{u}_\epsilon^{\frac{n}{n-1}}}{\tau_\epsilon^{\frac{n}{n-1}}} - n \log \delta + \lambda_n C_\delta + o_\delta(1) + o_\epsilon(1).
\]

Therefore, we have
\[
\limsup_{\epsilon \to 0} \int_{\mathcal{W}_{R\epsilon}(x_\epsilon)} e^{\alpha_n |u_\epsilon|^{\frac{n}{n-1}}} dx \\
\leq \delta^{-n} e^{\lambda_n C_\delta + o_\delta(1)} \limsup_{\epsilon \to 0} \int_{\mathcal{W}_{R\epsilon}(x_\epsilon)} \left( e^{\lambda_n \overline{u}_\epsilon^{1/n}} \overline{u}_\epsilon^{1-1/n} - 1 \right) dx \\
\leq \delta^{-n} e^{\lambda_n C_\delta + o_\delta(1)} \limsup_{\epsilon \to 0} \int_{\mathcal{W}_{\delta}(0)} \left( e^{\lambda_n \overline{u}_\epsilon^{1/n}} \overline{u}_\epsilon^{1-1/n} - 1 \right) dx \\
\leq \delta^{-n} e^{\lambda_n C_\delta + o_\delta(1)} \kappa_n \delta^{n} e^{1 + \frac{1}{n} + \cdots + \frac{1}{n-1}}.
\]

Taking \( \delta \to 0 \), we have
\[
\limsup_{\epsilon \to 0} \int_{\mathcal{W}_{R\epsilon}(x_\epsilon)} e^{\alpha_n |u_\epsilon|^{\frac{n}{n-1}}} dx \leq \kappa_n e^{\lambda_n C_\delta + 1 + \frac{1}{n} + \cdots + \frac{1}{n-1}}.
\]

Then by the Lemma 4.4, we obtain
\[
\limsup_{\epsilon \to 0} \int_{\Omega} e^{\alpha_n |u_\epsilon|^{\frac{n}{n-1}}} dx \leq |\Omega| + \kappa_n e^{\lambda_n C_\delta + 1 + \frac{1}{n} + \cdots + \frac{1}{n-1}}.
\]
It follows Lemma 3.2 to get sup_{u \in \mathcal{H}} J_{\lambda_n}^\alpha (u) \leq |\Omega| + \kappa_n e^{\lambda_n C_G + 1 + \frac{1}{p} + \cdots + \frac{1}{n-1}}.

Case 2. $x_0$ lies on the boundary of $\Omega$.

We proceed and use the same notions as in case 1. By \cite{[39]}, $M_\epsilon \to u_\epsilon \rightharpoonup 0$ weakly in $W_{0,q}^1(\Omega)$ for any $1 < q < n$, and in $C^1(\overline{\Omega}\setminus \{x_0\})$. Hence

$$\int_{W_\delta(x_\epsilon)} F^n(\nabla u_\epsilon) dx \leq \tau_n = 1 - \frac{a(1)}{M_\epsilon^{\alpha/(n-1)}},$$

and we have in $\omega_{R\epsilon}, (x_\epsilon)$,

$$\alpha_n |u_\epsilon|^\frac{\alpha}{\alpha} \leq \lambda_n |u_\epsilon|^{\frac{\alpha}{\alpha}} + o_n(1).$$

Combining \eqref{30}, \eqref{31} and Lemma 4.1 we have

$$\limsup_{\epsilon \to 0} \int_{\Omega} e^{\alpha_n |u_\epsilon|^\frac{\alpha}{\alpha}} dx \leq |\Omega| + O(\delta^n) e^{1 + \frac{1}{p} + \cdots + \frac{1}{n-1}}.$$  

Letting $\delta \to 0$, \eqref{52} together with Lemma 3.2 gives sup_{u \in \mathcal{H}} J_{\lambda_n}^\alpha (u) \leq |\Omega|, which is impossible. Therefore we conclude that $x_0$ cannot lie on $\partial \Omega$.

In Lemma 5.2 we have got the upper bound of sup_{u \in \mathcal{H}} J_{\lambda_n}^\alpha (u) if $u_\epsilon$ blows up. Next we will construct an explicit test function to get the lower bound of sup_{u \in \mathcal{H}} J_{\lambda_n}^\alpha (u), which will contradict the the upper bound of sup_{u \in \mathcal{H}} J_{\lambda_n}^\alpha (u). Thus we get a contradiction and consequently we complete the proof of Theorem. The similar arguments can be seen in \cite{[40], [42].

Lemma 5.3. There holds

$$\sup_{u \in \mathcal{H}} J_{\lambda_n}^\alpha (u) > |\Omega| + \kappa_n e^{\lambda_n C_G + 1 + \frac{1}{p} + \cdots + \frac{1}{n-1}}.$$  

Proof. Define a sequence of functions in $\Omega$ by

$$\phi_\epsilon = \begin{cases} \frac{C + C^{-\frac{1}{n-1}} (-\frac{n}{\lambda_n} \log (1 + \kappa_n |\nabla u_\epsilon|)^{\frac{\alpha}{\alpha}} + b)}{(1 + \alpha C^{-\frac{1}{n-1}} |G_\epsilon|_{L^n(\Omega)})^{\frac{1}{n}}}, & x \in W_{R\epsilon}(x_0), \\ \frac{C^{-\frac{1}{n-1}} (G - \eta \psi)}{(1 + \alpha C^{-\frac{1}{n-1}} |G_\epsilon|_{L^n(\Omega)})^{\frac{1}{n}}}, & x \in W_{2R\epsilon}(x_0) \setminus W_{R\epsilon}(x_0), \\ \frac{C^{-\frac{1}{n-1}} G}{(1 + \alpha C^{-\frac{1}{n-1}} |G_\epsilon|_{L^n(\Omega)})^{\frac{1}{n}}}, & x \in \Omega \setminus W_{2R\epsilon}(x_0), \end{cases}$$

where $G$ and $\psi$ are functions given in \cite{[38]}. $R = -\log \epsilon$, $\eta \in C_0^1(W_{2R\epsilon}(x_0))$ satisfying that $\eta = 1$ on $W_{2R\epsilon}(x_0)$ and $|\nabla \eta| \leq \frac{\epsilon}{R^n}$, $b$ and $C$ are constants depending only on $\epsilon$ to be determined later. Clearly $W_{2R\epsilon}(x_0) \subset \Omega$ provided that $\epsilon$ is sufficiently small. In order to assure that $\phi_\epsilon \in W_{0,1,n}^1(\Omega)$, we set

$$C + C^{-\frac{1}{n-1}} (-\frac{n}{\lambda_n} \log (1 + \kappa_n |\nabla u_\epsilon|)^{\frac{\alpha}{\alpha}} + b) = C^{-\frac{1}{n-1}} (-\frac{1}{(n\kappa_n)^{\frac{1}{n-1}}} \log (R \epsilon) + C_G),$$

which gives

$$C^{-\frac{1}{n-1}} = -\frac{1}{(n\kappa_n)^{\frac{1}{n-1}}} \log \epsilon + \frac{1}{\lambda_n} \log \kappa_n - b + C_G + O(R^{-\frac{1}{n-1}}).$$

Next we make sure that $\int_\Omega F^n(\nabla \phi_\epsilon) dx = 1.$
Then together with (45), we have

\[ \mathcal{W}_{R_0}(x_0) \left( 1 + \kappa_n \left( \int_{R_0(x_0)} \phi \right)^n \right) \]

\[ \int_{\mathcal{W}_{R_0}(x_0)} \left( \frac{F^n(x - x_0)}{\epsilon} \right) \frac{1}{e^{n-1}} \, dx = n \kappa_n \int_0^{R_0} \frac{(\frac{1}{e} \frac{1}{e})}{e^{n-1}} \, ds \]

\[ = n - 1 \int_{\kappa_n}^{\frac{1}{L}} \frac{1}{e^{n-1}} \, dt. \]

Then it follows that

\[ \int_{\mathcal{W}_{R_0}(x_0)} F^n(\nabla \phi) \, dx = \frac{n - 1}{\lambda_n (C \pi \alpha + \alpha \|G\|_{L^n(\Omega)})} \int_0^{\frac{1}{\kappa_n}} R^{n\alpha - \tau} \left( \frac{t + 1 - 1}{(1 + t)^n} \right) \, dt \]

\[ = \frac{n - 1}{\lambda_n (C \pi \alpha + \alpha \|G\|_{L^n(\Omega)})} \int_0^{\frac{1}{\kappa_n}} R^{n\alpha - \tau} \left( \frac{1}{(1 + t)^n} \right) \, dt \]

\[ = \frac{n - 1}{\lambda_n (C \pi \alpha + \alpha \|G\|_{L^n(\Omega)})} \left[ - \frac{1}{2} + \cdots + \frac{1}{n - 1} \right] \]

\[ + \log(1 + \kappa_n) R^{n\alpha - \tau} + O(R^{-\frac{1}{1 - \alpha}}), \quad (55) \]

where we have used the fact that

\[ - \sum_{k=0}^{n-2} \frac{C_{n-1}^{k}}{n - k - 1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n - 1}. \]

Noting that \( \psi(x) \) satisfies that \( |\nabla \psi(x)| = o(\frac{1}{F^n(x - x_0)}) \) as \( x \to x_0 \), and using Lemma 2.1 and (35), we have

\[ \int_{\mathcal{W}_{2 R_0}(x_0) \setminus \mathcal{W}_{R_0}(x_0)} F^n(\nabla G - G - \eta \psi) \, dx = o_{R_0}(1). \]

Then together with (45), we have

\[ \int_{\Omega \setminus \mathcal{W}_{R_0}(x_0)} F^n(\nabla \phi) \, dx \]

\[ = \frac{1}{C \pi \alpha + \alpha \|G\|_{L^n(\Omega)}} \int_{\Omega \setminus \mathcal{W}_{R_0}(x_0)} F^n(\nabla G) \, dx \]

\[ - \int_{\mathcal{W}_{2 R_0}(x_0) \setminus \mathcal{W}_{R_0}(x_0)} F^n(\nabla G - G - \eta \psi) \, dx \]

\[ = \frac{1}{C \pi \alpha + \alpha \|G\|_{L^n(\Omega)}} \left( - \frac{1}{(n \kappa_n) \pi \alpha} \log(R) \right) \]

\[ + \frac{\alpha \lambda_n}{n - 1} \|G\|_{L^n(\Omega)} + C_G + o_{R_0}(1). \quad (56) \]
Putting (53), (56) together, we have

\[
\int_{\Omega} F^n(\nabla \phi_\epsilon) dx = \frac{n-1}{\lambda_n (C^{\frac{n}{n+1}} + \alpha ||G_\alpha||_{L^n(\Omega)})} \left(- \frac{n}{n-1} \log \epsilon + \frac{1}{n-1} \log \kappa_n + \frac{\lambda_n}{n-1} C_G \right.
\]

\[+ \frac{\alpha \lambda_n}{n-1} ||G_\alpha||^n_{L^n(\Omega)} - (1 + \frac{1}{2} + \cdots + \frac{1}{n-1}) + O(R^{-\frac{n}{n-1}}) + o_{Re}(1). \]

Since \( \int_{\Omega} F^n(\nabla \phi_\epsilon) dx = 1 \), we have

\[
C_{\pi^{\frac{n}{n-1}}} = \frac{n-1}{\lambda_n} \left(- \frac{n}{n-1} \log \epsilon + \frac{1}{n-1} \log \kappa_n + \frac{\lambda_n}{n-1} C_G \right.
\]

\[\left. - (1 + \frac{1}{2} + \cdots + \frac{1}{n-1}) + O(R^{-\frac{n}{n-1}}) + o_{Re}(1). \right) \quad (57)
\]

Consequently from (54), we have

\[
b = \frac{(n-1)}{\lambda_n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) + O(R^{-\frac{n}{n-1}}) + o_{Re}(1). \quad (58)
\]

Since

\[
||\phi_\epsilon||^n_{L^n(\Omega)} = \frac{||G_\alpha||^n_{L^n(\Omega)} + O(C^{\frac{n^2}{n-1}} R^n \epsilon^n) + O((R_\epsilon)^n (\log(Re))^n))}{C^{\frac{n}{n-1}} + \alpha ||G_\alpha||_{L^n(\Omega)}},
\]

using the inequality

\[
(1 + t)^{\frac{n}{n-1}} \geq 1 - \frac{t}{n-1}, \text{ for } t \text{ small}.
\]

In view of (54) and (58), there holds in \( \mathcal{W}_{Re}(x_0) \),

\[
\lambda_n |\phi_\epsilon(x)|^{\frac{n}{n-1}} (1 + \alpha ||\phi_\epsilon||_{L^n(\Omega)})^{\frac{n}{n-1}} \geq \lambda_n C^{\frac{n}{n-1}} - n \log(1 + \frac{1}{\kappa_n} |F^o(x - x_0)|) + \frac{n \lambda_n}{n-1} b - \frac{\alpha^2 \lambda_n ||G_\alpha||_{L^n(\Omega)}}{(n-1)C^{\frac{n}{n-1}}} 
\]

\[+ O(C^{-\frac{n}{n-1}}) + O(C^{\frac{n^2}{n-1}} (Re)^n) + O((R_\epsilon)^n (- \log(Re))^n) \]

\[\geq -n \log \epsilon + \log \kappa_n + \lambda_n C_G - (1 + \frac{1}{2} + \cdots + \frac{1}{n-1}) \]

\[\left. - n \log(1 + \frac{1}{\kappa_n} |F^o(x - x_0)|) \right) + \frac{\alpha^2 \lambda_n ||G_\alpha||_{L^n(\Omega)}}{(n-1)C^{\frac{n}{n-1}}} 
\]

\[+ O(C^{-\frac{n}{n-1}}) + O(R^{-\frac{n}{n-1}}) + o_{Re}(1). \]

where we have used the inequality \(|1 + t|^{\frac{n}{n-1}} \geq 1 + \frac{n}{n-1} t + O(t^3) \) for small \( t \). By using the fact

\[
\sum_{k=0}^{n-2} \frac{C_{n-2} (-1)^{n-k-2}}{n-k-1} = \frac{1}{n-1},
\]

\[
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\]
one can get

\[
\int_{W_R(x_0)} e^{-n \log \epsilon - n \log(1 + \kappa_n \left( \frac{F'(x-x_0)}{\epsilon} \right)^{\frac{n}{n-1}}} dx
\]

\[
= \frac{1}{\epsilon^n} \int_{W_R(x_0)} \frac{1}{(1 + \kappa_n \left( \frac{F'(x-x_0)}{\epsilon} \right)^{\frac{n}{n-1}}} dx
\]

\[
= (n - 1) \int_0^{R_n \frac{1}{\epsilon}} t^{n-2} \frac{dt}{(1 + t)^n}
\]

\[
= (n - 1) \int_0^{R_n \frac{1}{\epsilon}} (t + 1 - 1) t^{n-2} \frac{dt}{(1 + t)^n}
\]

\[
\geq (n - 1) \left( \frac{1}{n-4} + O(R^{-\frac{n}{n-1}}) \right) = 1 + O(R^{-\frac{n}{n-1}}).
\]

Then we obtain

\[
\int_{W_R(x_0)} e^{\lambda_n |\phi_\alpha(x)|^\frac{n}{n-1}} (1 + \alpha |\phi_\alpha(x)|_{L^\alpha(\Omega)}) \frac{dx}{(1 + \alpha |\phi_\alpha(x)|_{L^\alpha(\Omega)})^{\frac{n}{n-1}}}
\]

\[
\geq \kappa_n e^{\lambda_n C_G + (1 + \frac{1}{n} + \ldots + \frac{1}{n-1}) \left( 1 - \frac{\alpha^2 \lambda_n \|G_\alpha\|_{L^\alpha(\Omega)}}{(n-1) C^{-\frac{n-2}{n-1}}} \right)} + O(C^{-\frac{2n}{n-1}}) + O(R^{-\frac{n}{n-1}}) + o_{R\epsilon}(1).
\]

On the other hand, since

\[
\int_{W_{2R}(x_0)} |G_\alpha|^\frac{n}{n-1} dx = O((R\epsilon)^n \log^{\frac{n}{n-1}}(R\epsilon)) = o_{R\epsilon}(1),
\]

we obtain

\[
\int_{\Omega \setminus W_{R\epsilon}(x_0)} e^{\lambda_n |\phi_\alpha(x)|^\frac{n}{n-1}} (1 + \alpha |\phi_\alpha(x)|_{L^\alpha(\Omega)}) \frac{dx}{(1 + \alpha |\phi_\alpha(x)|_{L^\alpha(\Omega)})^{\frac{n}{n-1}}}
\]

\[
\geq \int_{\Omega \setminus W_{2R\epsilon}(x_0)} (1 + \lambda_n |\phi_\alpha(x)|_{L^\alpha(\Omega)}) \frac{dx}{(1 + \lambda_n |\phi_\alpha(x)|_{L^\alpha(\Omega)})^{\frac{n}{n-1}}}
\]

\[
\geq |\Omega| + \frac{\lambda_n \|G_\alpha\|\frac{n}{n-1}}{C^{-\frac{n-2}{n-1}}} + O(C^{-\frac{2n}{n-1}}) + o_{R\epsilon}(1)
\]

Together with the above integral estimates on $W_{R\epsilon}$, and the fact that

\[
C^{-\frac{2n}{n-1}} \rightarrow 0 \text{ and } R^{-\frac{n}{n-1}} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.
\]

Then we have

\[
\int_{\Omega} e^{\lambda_n |\phi_\alpha(x)|^\frac{n}{n-1}} (1 + \alpha |\phi_\alpha(x)|_{L^\alpha(\Omega)}) \frac{dx}{(1 + \alpha |\phi_\alpha(x)|_{L^\alpha(\Omega)})^{\frac{n}{n-1}}}
\]

\[
> |\Omega| + \kappa_n e^{\lambda_n C_G + (1 + \frac{1}{n} + \ldots + \frac{1}{n-1})},
\]

provided that $\epsilon > 0$ is chosen sufficiently small. Thus we get the conclusion of Lemma.

\[\Box\]

6. Asymptotic representation of $G_\alpha$

In this section we will give the asymptotic representation of Green function $G_\alpha$, similarly to [17 35 38].
The proof of Lemma 4.7: Since \( c_k^\alpha \rightarrow u_k \geq 0 \) in \( \Omega \setminus \{0\} \), we have \( G_\alpha \geq 0 \) in \( \Omega \setminus \{0\} \). Theorem 1 in [26] gives
\[
\frac{1}{K} \leq \frac{G_\alpha}{-\log r} \leq K \quad \text{in} \quad \Omega \setminus \{0\}
\]
for some constant \( k > 0 \). Assume \( \Gamma(r) = -c(n) \log r, \ c(n) = (n \kappa n)^{\frac{1}{n-1}}. \) Let \( G_k = \frac{G_{\alpha}(\gamma_x^k)}{F(\gamma_x^k)} \), which is defined in \( \{x \in \mathbb{R}^n \setminus \{0\}, \ r_k x \in \mathcal{W}_k\} \) for some small \( \delta > 0 \). Here \( r_k \to 0 \) as \( k \to +\infty \). Then \( G_k \) satisfy the equation
\[
-\sum_{i=1}^n \frac{\partial}{\partial x_i} (F^{n-1}(\nabla G_k) \frac{F_i(\nabla G_k)}{F(\nabla G_k)}) = \alpha r_k^n G_k^{n-1}.
\]
By theorem 1 in [32], when \( r_k \to 0 \), \( G_k \) converges to \( G^* \) in \( C^1_{loc}(\mathbb{R}^n \setminus \{0\}) \) and \( G^* \) is bounded, where \( G^* \) satisfying
\[
-\sum_{i=1}^n \frac{\partial}{\partial x_i} (F^{n-1}(\nabla G^*) \frac{F_i(\nabla G^*)}{F(\nabla G^*)}) = 0.
\]
From serrin’s result (see [25] and [37]), 0 is a removable singularity and \( G^* \) can be extended to \( \hat{G} \in C^1(\mathbb{R}^n) \). Consequently, form Liouville type theorem (see [15]), \( \hat{G} \) must be a constant. Let \( \gamma_k = \sup_{\mathcal{W}_k \setminus \mathcal{W}_{r_k}} \frac{G_\alpha(x)}{F(\gamma_x)} \), and \( \gamma = \lim_{k \to +\infty} \gamma_k, (\gamma > 0) \). This means the constant function \( \hat{G} = \gamma \).

Set
\[
G_\eta^0(x) = (\gamma + \eta)(\Gamma(x) - \Gamma(\delta)) - c(n)(\gamma + \eta)(F^\eta(x) - \delta) + \sup_{\partial \mathcal{W}_k} G_\alpha,
\]
\[
G_\eta^n(x) = (\gamma - \eta)(\Gamma(x) - \Gamma(\delta)) - c(n)(\gamma - \eta)(F^\eta(x) - \delta) + \inf_{\partial \mathcal{W}_k} G_\alpha.
\]
A straightforward calculation shows
\[
-Q_n G_\eta^0(x) = c^n - (n)(\gamma + \eta)^{n-1} \frac{\eta - 1}{F^\eta(x)} + 1)^{n-2},
\]
\[
-Q_n G_\eta^n(x) = c^n - (n)(\gamma - \eta)^{n-1} \frac{\eta - 1}{F^\eta(x)} - 1)^{n-2}.
\]
By, for any fixed \( 0 < \eta < \gamma \), we have
\[
-Q_n G_\eta^0(x) \geq -Q_n G \quad \text{in} \quad \mathcal{W}_\delta \setminus \mathcal{W}_{r_k},
\]
\[
G_\eta^0|_{\partial \mathcal{W}_k} \geq G_\alpha|_{\partial \mathcal{W}_k}, \quad G_\eta^0|_{\partial \mathcal{W}_{r_k}} \geq G_\alpha|_{\partial \mathcal{W}_{r_k}},
\]
provided that \( \delta \) are sufficiently small and \( r_k < \delta \). By the comparison principle (see [27]), we have
\[
G_\alpha \leq (\gamma + \eta) \Gamma(x) + C_\delta \quad \text{in} \quad \mathcal{W}_\delta \setminus \mathcal{W}_{r_k}
\]
for some constant \( C_\delta \). Letting \( \eta \to 0 \) first, then \( k \to \infty \), one has
\[
G_\alpha \leq \gamma \Gamma(x) + C_\delta \quad \text{in} \quad \mathcal{W}_\delta \setminus \{0\}.
\]
A similar argument gives \( G_\alpha \geq \gamma \Gamma(x) + C'_\delta \) in \( \mathcal{W}_\delta \setminus \{0\} \) for some constant \( C'_\delta \). Hence \( G_\alpha - \gamma \Gamma(x) \) is bounded in \( L^\infty(\mathcal{W}_\delta) \).

Next we prove the continuity of \( G_\alpha - \gamma \Gamma(x) \) at 0. We look at the points where the bounded function \( G_\alpha - \gamma \Gamma(x) \) achieves its supremum in \( \mathcal{W}_\delta \). Set \( \lambda = \sup_{\mathcal{W}_\delta}(G_\alpha - \gamma \Gamma(x)) \).
\( \lambda \) achieves at some point in \( \mathcal{W}_3 \setminus \{0\} \), then \( G_\alpha - \gamma \Gamma(x) - \gamma c(n) F^n(x) \) also achieves at some point in \( \mathcal{W}_3 \setminus \{0\} \). It follows from comparison principle (see [3]) that \( G_\alpha - \gamma \Gamma(x) - \gamma c(n) F^n(x) \) is a constant, hence we have done.

\( \lambda \) achieves at 0. Set

\[
  w_r(x) = G_\alpha(rx) - \gamma \Gamma(r) \quad \text{in} \quad \mathcal{W}_4 \setminus \{0\}.
\]

The function \( w_r \) satisfies \( -Q_\alpha w_r(x) + \alpha r^{n-1} G_\alpha r^{-1}(rx) = 0 \). We also have \( r^n G_\alpha r^{-1}(rx) \in L^\infty(\mathcal{W}_3) \) and \( |w_r - \gamma \Gamma(x)| \leq C_0 \) for \( C_0 = \sup_{\mathcal{W}_3 \setminus \{0\}} |G_\alpha(x) - \gamma \Gamma(x)| \). By Theorem 1 in [32], when \( r \to 0 \), \( w_r \to w \) in \( C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \), where \( w \in C^1(\mathbb{R}^n \setminus \{0\}) \) satisfies \( -Q_\alpha(w) = 0 \). For the sequence \( \xi_j = \frac{\delta}{r_j} F^n(\xi_j) = 1 \), which maybe assumed to converge to \( \xi^0 \in \partial \mathcal{W}_1 \), we have

\[
  w_r(\xi_j) - \gamma \Gamma(\xi_j) = G_\alpha(x_{r_j}) - \gamma \Gamma(x_{r_j}) \to \lambda.
\]

Hence

\[
  w(x) \leq \gamma \Gamma(x) + \lambda \quad \text{and} \quad w(\xi^0) = \gamma \Gamma(\xi^0) + \lambda.
\]

By comparison principle (see [37]), \( w(x) = \gamma \Gamma(x) + \lambda \) and hence \( w_r \to \gamma \Gamma(x) + \lambda \) in \( C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \). This implies

\[
  \lim_{r \to 0} (G_\alpha(rx) - \gamma \Gamma(rx)) = \lambda, \quad \lim_{r \to 0} \nabla_x (G_\alpha(rx) - \Gamma(rx)) = 0. \tag{65}
\]

The above equalities lead to the continuity of \( G_\alpha - \gamma \Gamma \) and \( \lim_{x \to 0} F^n(x) \nabla(G_\alpha - \gamma \Gamma) = 0 \).

We assume \( \sup_{x \in \mathcal{W}_3}(G_\alpha - \gamma \Gamma) = \sup_{F^n(x) = \delta}(G_\alpha - \gamma \Gamma) \), we define \( w_r \) as the above, then \( w_r \to w \) in \( C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) and \( |w - \gamma \Gamma| \leq C_0 \). We now look at the points where \( w - \gamma \Gamma \) achieves its supremum in \( \mathbb{R}^n \). Set \( \lambda = \sup_{x \neq 0}(w - \gamma \Gamma) \).

If \( \tilde{\lambda} \) is achieved at some point in \( \mathbb{R}^n \setminus \{0\} \), then \( w - \gamma \Gamma \) equals to some constant by strong maximum principle [14], which implies \( G_\alpha(rx) - \gamma \Gamma(rx) \to \lambda \in C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) as \( r \to 0 \). For any fixed \( \epsilon > 0 \), there exists \( n_0 \) such that \( n \geq n_0 \) and \( x \in \partial \mathcal{W}_1 \), we have

\[
  \gamma \Gamma(r_n x) + \tilde{\lambda} - \epsilon \leq G_\alpha(r_n x) \leq \gamma \Gamma(r_n x) + \tilde{\lambda} + \epsilon.
\]

Applying maximum principle [14] in \( \mathcal{W}_{r_{n_0}} \setminus \mathcal{W}_{r_n} \), we obtain

\[
  \gamma \Gamma(x) + \tilde{\lambda} - \epsilon \leq G_\alpha(x) \leq \gamma \Gamma(x) + \tilde{\lambda} + \epsilon,
\]

which leads to with \( \lambda \) replaced by \( \tilde{\lambda} \).

If \( \tilde{\lambda} \) is achieved at 0, we simply argue as in the above to deduce

\[
  \lim_{x \to 0} (w - \gamma \Gamma) = \tilde{\lambda} \quad \text{and} \quad \lim_{x \to 0} \lim_{r_n \to 0} (G_\alpha(r_n x) - \gamma \Gamma(r_n x)) = \tilde{\lambda}. \tag{66}
\]

If \( \tilde{\lambda} \) is achieved at \( \infty \), the same idea as in case can be applied when we defined \( \lambda(R) = \max_{\delta \leq F^n(x) \leq R} (w - \gamma \Gamma) = \max_{\partial \mathcal{W}_3}(w - \gamma \Gamma) \) and let \( R \) tend to \( \infty \). We obtain

\[
  \lim_{x \to \infty} (w - \gamma \Gamma) = \tilde{\lambda}, \quad \lim_{x \to \infty} \lim_{r_n \to 0} (G_\alpha(r_n x) - \gamma \Gamma(r_n x)) = \tilde{\lambda}. \tag{67}
\]

As long as we have [66] and [67], we can have use maximum principle [14] again to conclude [65] as before.

Integrating by parts on both sides of Eq. [36] over \( \mathcal{W}_3 \), we have

\[
  - \int_{\mathcal{W}_3} \text{div}(F^n(x) F^n_\xi \nabla G_\alpha) dx = 1 + \alpha \int_{\mathcal{W}_3} G_\alpha^{n-1} dx. \tag{68}
\]
Because $G_\alpha(x) = \gamma \Gamma(x) + o(1)$ and $\nabla G_\alpha(x) = \gamma \nabla \Gamma(x) + o(\frac{1}{\Gamma'(x)})$ as $x \to 0$. Inserting the above two equalities into (68), then letting $\delta \to 0$, we obtain $\gamma = 1$.

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