Universality of Replica-Symmetry Breaking in the Transverse Field Sherrington–Kirkpatrick Model

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Abstract
The existence theorem for replica-symmetry breaking (RSB) in the transverse field Sherrington–Kirkpatrick (SK) model is extended to the model with a general random exchange interactions. The relation between the expectation value of the exchange interaction energy and the Duhamel correlation function of spin operators can be obtained by an approximate integration by parts for general random interactions. In addition to the Falk–Bruch inequality, these explicit evaluations enable us to prove that the variance of overlap between two replica spin operators does not vanish under sufficiently weak transverse field in sufficiently low temperature. The absence of the ferromagnetic long range order is also shown to distinguish RSB from the $\mathbb{Z}_2$-symmetry breaking.

Keywords Spin glass · Quantum spin systems · Spontaneous symmetry breaking · Long-range order · Replica symmetry breaking

1 Introduction

Replica-symmetry breaking (RSB) is studied extensively as a spontaneous breaking of the replica-symmetry in spin systems with random interactions. This phenomenon has been studied deeply in mean field classical spin glass models, since Talagrand proved the Parisi formula [9] for the Sherrington–Kirkpatrick (SK) model [12] rigorously [13]. It is known that the replica-symmetry breaking phase includes the spin glass phase and a part of the ferromagnetic phase in the SK model. Quite recently, Leschke, Manai, Ruder and Warzel have proven a remarkable theorem that replica-symmetry breaking phase exists in the transverse field SK model with centered Gaussian random interactions [8]. It is shown that the variance of the longitudinal spin overlap does not vanish in the system under sufficiently weak transverse...
field in sufficiently low temperatures. This is a first rigorous result for replica-symmetry breaking in quantum disordered systems.

In the present paper, we extend their theorem to non-Gaussian random exchange interactions satisfying an arbitrary symmetric distribution. It is pointed out that the ferromagnetic long range order gives a non-zero variance of longitudinal spin overlap. This paper is organized as follows: In Sect. 2, the Hamiltonian and physical quantities are defined and the main theorem for RSB existence in the transverse field SK model is described. In Sect. 3, the main theorem is proven by several lemmas. Finally, several possible extension of the main theorem is discussed.

2 Definitions and Main Result

We study disordered quantum spin systems. A sequence of spin operators \((\sigma^w_i)_{w=x,y,z,i=1,2,...,N}\) on a Hilbert space \(\mathcal{H} := \bigotimes_{i=1}^N \mathcal{H}_i\) is defined by a tensor product of the Pauli matrix \(\sigma^w\) acting on \(\mathcal{H}_i \simeq \mathbb{C}^2\) and unities. These operators are self-adjoint and satisfy the commutation relations

\[
\begin{align*}
[\sigma^y_k, \sigma^z_j] &= 2i \delta_{k,j} \sigma^x_j, \\
[\sigma^z_k, \sigma^x_j] &= 2i \delta_{k,j} \sigma^y_j, \\
[\sigma^x_k, \sigma^y_j] &= 2i \delta_{k,j} \sigma^z_j,
\end{align*}
\]

and each spin operator satisfies

\[
(\sigma^w_j)^2 = 1.
\]

The following Hamiltonian with coupling constants \(h, J \in \mathbb{R}\)

\[
H_N (\sigma, h, J, \gamma) := J U_N (\sigma^z, \gamma) - h \sum_{j=1}^N \sigma^x_j,
\]

consists of exchange interactions \(U_N\) defined by

\[
U_N (\sigma^z, \gamma) := -\frac{1}{\sqrt{N}} \sum_{1 \leq i < j \leq N} \gamma_{i,j} \sigma^z_i \sigma^z_j
\]

where \(\gamma := (\gamma_{i,j})_{1 \leq i < j \leq N}\) is a sequence of independent identically distributed random variables satisfying a probability density function

\[
P(\gamma) = \prod_{1 \leq i < j \leq N} p(\gamma_{i,j}).
\]

Here, \(\mathbb{E}\) denotes sample expectation of a function \(f(\gamma)\) over the sequence \(\gamma\)

\[
\mathbb{E} f(\gamma) := \int d\gamma P(\gamma) f(\gamma).
\]

Assume that the probability function \(p(\gamma_{i,j})\) of each \(\gamma_{i,j}\) \((1 \leq i < j \leq N)\) is even function and each moment is given by

\[
\mathbb{E} \gamma_{i,j} = 0, \quad \mathbb{E} \gamma_{i,j}^2 = 1, \quad \mathbb{E} |\gamma_{i,j}|^3 < \infty.
\]

Note that the Hamiltonian is invariant under \(\mathbb{Z}_2\)-symmetry \(U \sigma^z_i U^\dagger = -\sigma^z_i\) for the discrete unitary transformation \(U := \exp \left( i \pi / 2 \sum_{i=1}^N \sigma^x_i \right)\).
Here, we define Gibbs state for the Hamiltonian. For a positive $\beta$, the partition function is defined by

$$Z_N(\beta, h, J, \gamma) := \text{Tr} e^{-\beta H_N(\sigma, h, J, \gamma)}$$

where the trace is taken over the Hilbert space $\mathcal{H}$.

Let $f$ be an arbitrary function of a sequence of spin operators $\sigma = (\sigma_i^w)_{w=x,y,z,i=1,2,3,\ldots,N}$. The expectation of $f$ in the Gibbs state is given by

$$\langle f(\sigma) \rangle = \frac{1}{Z_N(\beta, h, J, \gamma)} \text{Tr} f(\sigma) e^{-\beta H_N(\sigma, h, J, \gamma)}.$$  \hspace{1cm} (5)

Here, we introduce a fictitious time $t \in [0, 1]$ and define a time evolution of operators with the Hamiltonian. Let $O$ be an arbitrary linear operator on the Hilbert space $\mathcal{H}$, and we define an operator valued function $O(t)$ of $t \in [0, 1]$ by

$$O(t) := e^{-\beta t H_N} O(e^{\beta t H_N}).$$

The Duhamel function by

$$(O_1, O_2, \ldots, O_k) := \int_{[0,1]^k} dt_1 \ldots dt_k \langle T[O_1(t_1) O_2(t_2) \ldots O_k(t_k)] \rangle,$$

where $O_1(t_1), \ldots, O_k(t_k)$ are time dependent operators, and the symbol $T$ is a multilinear mapping of the chronological ordering. If we define a partition function with arbitrary linear operators $O_0, O_1, \ldots, O_k$ on the Hilbert space $\mathcal{H}$ and real numbers $x_1, \ldots, x_k$

$$Z(x_1, \ldots, x_k) := \text{Tr} \exp \left[ O_0 + \sum_{i=1}^k x_i O_i \right],$$

the Duhamel function of $k$ operators represents the $k$-th order derivative of the partition function [3, 7, 11]

$$\beta^k(O_1, \ldots, O_k) = \frac{1}{Z} \frac{\partial^k Z}{\partial x_1 \ldots \partial x_k}.$$

Also, the connected Duhamel function is defined by

$$\beta^k(O_1; \ldots; O_k) = \frac{\partial^k}{\partial x_1 \ldots \partial x_k} \log Z.$$

To study the spontaneous $\mathbb{Z}_2$-symmetry breaking, define order operator

$$m = \frac{1}{N} \sum_{i=1}^N \sigma_i^z.$$  \hspace{1cm} (7)

Note that $\langle \sigma_i^z \rangle = 0$ and thus $\langle m \rangle = 0$ in the $\mathbb{Z}_2$-symmetric Gibbs state for any temperature.

To study replica-symmetry breaking, we consider $n$ replicated spin model defined by the following Hamiltonian

$$\sum_{a=1}^n H_N(\sigma^a, h, J, \gamma).$$

$$\text{(8)}$$

$$\text{Springer}$$
The overlap operator $R_{a,b}$ between different replicated spins is defined by

$$R_{a,b} = \frac{1}{N} \sum_{i=1}^{N} \sigma_{i}^{a} \sigma_{i}^{b},$$

for $a, b = 1, 2, \ldots, n$, and $a \neq b$.

**Theorem 1** Consider the transverse field SK model defined by the Hamiltonian (1) and its replicated model (8). In the infinite volume limit, the variance of the overlap operator calculated in the replica symmetric and $\mathbb{Z}_{2}$-symmetric Gibbs state does not vanish

$$\lim_{N \to \infty} \text{inf} \mathbb{E}( (R_{1,2} - \mathbb{E}(R_{1,2}) )^2 ) > 0,$$

for sufficiently weak coupling constant $h$ in sufficiently low temperature.

In addition, the expectation value of the ferromagnetic order operator vanishes, and there is no ferromagnetic long range order

$$\mathbb{E}(m) = 0 = \lim_{N \to \infty} \mathbb{E}(m^2) = \lim_{N \to \infty} \mathbb{E}(m^4),$$

in the entire region of coupling constant space.

Theorem 1 shows that the overlap operator $R_{1,2}$ is not self-averaging in the replica symmetric and $\mathbb{Z}_{2}$-symmetric Gibbs state which has no ferromagnetic long range order. The Falk–Bruch inequality [6, 10] enables us to prove Theorem 1.

### 3 Proof

The following lemma proven by Carmona and Hu [1] is useful to study the model with random interactions satisfying a general distribution. See also Ref. [2] by Chen.

**Lemma 1** (An approximate integration by parts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function with a bounded continuous third-order partial derivative. Define the following difference between two expectation values over a random variable $\gamma$ obeying $p(\gamma)$

$$\Delta(f) := \mathbb{E}(f(\gamma)) - \mathbb{E}(f'(\gamma)).$$

The absolute value of $\Delta(f)$ is bounded by

$$|\Delta(f)| \leq \frac{3}{2} \mathbb{E}|\gamma|^3 \sup_x |f''(x)|.$$

**Proof** In the Taylor series of $f(\gamma)$ and $f'(\gamma)$, for any $\gamma \in \mathbb{R}$, there exist $c, c' \in (0, 1)$, such that

$$f(\gamma) = f(0) + \gamma f'(0) + \frac{\gamma^2}{2} f''(c\gamma),$$

$$f'(\gamma) = f'(0) + \gamma f''(c'\gamma).$$
\[ E\gamma = 0 \text{ and } E\gamma^2 = 1 \text{ give} \]
\[ \Delta(f) = E[\gamma f(\gamma) - f'(\gamma)] = E\left[ \frac{\gamma^3}{2} f''(c\gamma) - \gamma f''(c'\gamma) \right] \]
\[ \leq E\left( \frac{|\gamma|^3}{2} + |\gamma| \right) \sup_x |f''(x)| \]
\[ \leq \frac{3}{2} E|\gamma|^3 \sup_x |f''(x)|, \]

since Jensen’s inequality gives
\[ E|\gamma| = E|\gamma|E|\gamma|^2 = E(|\gamma|^3)^{\frac{1}{3}}E(|\gamma|^3)^{\frac{2}{3}} \leq E|\gamma|^3. \]

This completes the proof. \(\square\)

First, Lemma 2 also for the model defined in the Hamiltonian (1) is proven as in [8].

**Lemma 2** There exists a function \( \Delta_N \) of the sequence \( \gamma \), such that the expectation of the square of spin overlap is represented by
\[ E\langle R^2_{1,2} \rangle = \frac{N - 1}{N} E(A, A) + \frac{2}{\beta J N} E(U_N) - \frac{2}{\beta J} \Delta_N + \frac{1}{N}, \]
where
\[ A := \sigma_1^z \sigma_2^z. \]

and
\[ \lim_{N \to \infty} \Delta_N = 0. \]

**Proof** The left hand side is represented as
\[ E\langle R^2_{1,2} \rangle = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E\left( \sigma_i^{z,1} \sigma_j^{z,2} \sigma_j^{z,1} \sigma_i^{z,2} \right) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E\left( \sigma_i^z \sigma_j^z \right)^2 \]
\[ = \frac{1}{N} + \frac{N - 1}{N} E(\sigma_1^z \sigma_2^z)^2 = \frac{1}{N} + \frac{N - 1}{N} E(A)^2. \]

The expectation value of the exchange energy is
\[ \frac{1}{N} E(U_N) = -\frac{1}{N^2} \sum_{1 \leq i < j \leq N} E\gamma_{i,j} \langle \sigma_i^z \sigma_j^z \rangle \]
\[ = -\frac{1}{N^2} \sum_{1 \leq i < j \leq N} \left[ \frac{\beta J}{\sqrt{N}} E(\sigma_i^z \sigma_j^z; \sigma_i^z \sigma_j^z) + \Delta(\langle \sigma_i^z \sigma_j^z \rangle) \right] \]
\[ = -\frac{\beta J (N - 1)}{2N} E(\sigma_1^z \sigma_2^z; \sigma_1^z \sigma_2^z) - \frac{1}{N^2} \sum_{1 \leq i < j \leq N} \Delta(\langle \sigma_i^z \sigma_j^z \rangle) \]
\[ = -\frac{\beta J (N - 1)}{2N} E[(A, A) - \langle A \rangle^2] + \Delta_N, \]
where
\[ \Delta_N := -\frac{1}{N^2} \sum_{1 \leq i < j \leq N} \Delta(\langle \sigma_i^z \sigma_j^z \rangle). \]
The identity (15) is obtained from above identities (17) and (18). Lemma 1 gives the upper bound on $|\Delta(\langle \sigma_i^z \sigma_j^z \rangle)|$

$$|\Delta(\langle \sigma_i^z \sigma_j^z \rangle)| \leq \frac{3}{2} \mathbb{E}|\gamma_{i,j}|^3 \sup \left| \frac{\partial^2}{\partial \gamma_{i,j}^2} \langle \sigma_i^z \sigma_j^z \rangle \right|$$

$$= \frac{3}{2} \mathbb{E}|\gamma_{i,j}|^3 \frac{2 \beta^2 J^2}{N} \sup |(\sigma_i^z \sigma_j^z; \sigma_i^z \sigma_j^z)|,$$

$$= \frac{3 \beta^2 J^2 c_3}{2N} \mathbb{E}|\gamma_{i,j}|^3.$$

(20)

where $c_3 := \sup |(A; A)| = \sup |(A, A, A) - 3(A, A)\langle A \rangle + 2\langle A \rangle^3 | \leq 6$ for $A := \sigma_i^z \sigma_2^z$.

Therefore

$$|\Delta_N| \leq \frac{1}{N^3} \sum_{1 \leq i < j \leq N} \frac{3 \beta^2 J^2 c_3}{2N} \mathbb{E}|\gamma_{i,j}|^3 \leq \frac{3 \beta^2 J^2 c_3}{4 \sqrt{N}} \left(1 - \frac{1}{N}\right) \mathbb{E}|\gamma_{i,j}|^3.$$

(21)

Therefore

$$\lim_{N \to \infty} \Delta_N = 0.$$

This completes the proof.

The following lemma is one key which Leschke, Manai, Ruder and Warzel have used to prove the non-zero variance of the overlap operator [8].

**Lemma 3** The following Duhamel function of $A := \sigma_1^z \sigma_2^z$ is bounded from the below

$$(A, A) \geq \frac{1}{2h} \left(1 - e^{-2 \beta h}\right).$$

(23)

**Proof** The Falk–Bruch inequality [6, 10] gives the following lower bound on Duhamel function of the operator $A$ defined by (16)

$$(A, A) \geq (A^2) \Phi \left( \frac{\beta}{4 \langle A^2 \rangle} [A, [H_N, A]] \right),$$

(24)

where the function $\Phi : [0, \infty) \to [0, \infty)$ is defined by $\Phi(r \tanh r) := \frac{\tanh r}{r}$. Using $A^2 = 1$ and $[A, [H_N, A]] = 4h(\sigma_1^x + \sigma_2^x)$, the Falk-Bruch inequality implies

$$(A, A) \geq \Phi(\beta h (\sigma_1^x + \sigma_2^x)).$$

(25)

Since the function $\Phi$ is monotonically decreasing and $\langle \sigma_i^x \rangle \leq 1$,

$$(A, A) \geq \Phi(2 \beta h).$$

(26)

The following lower bound is given by Dyson et al. [5]

$$\Phi(t) \geq \frac{1}{t}(1 - e^{-t})$$

for $t \geq 0$. This and the bound (26) for $(A, A)$ complete the proof.

The following lemma is obtained by Carmona and Hu [1]
Lemma 4  The expectation value of the exchange energy has the following lower bound
\[
\lim_{N \to \infty} \frac{1}{N} \mathbb{E} \langle U_N \rangle \geq -\kappa,
\]  
(27)
where \(\kappa \simeq 0.763\) is given by the ground state energy in the SK model with the standard Gaussian r.v.s \(g\) [4, 9].

Proof  The expectation of exchange energy is bounded by its ground state energy
\[
\mathbb{E} \langle U_N \rangle \geq \mathbb{E} \inf_{\langle \sigma | \sigma \rangle = 1} \left\langle \sigma \left( -\sum_{i<j} \frac{\gamma_{i,j}}{\sqrt{N}} \sigma_i \sigma_j \right) \right\rangle,
\]
where the normalized ground state \(|\sigma\rangle\) in the SK model with r.v.s \(\gamma\) is given by an eigenstate of each \(\sigma_i^z\), such that \(\sigma_i^z |\sigma\rangle = \sigma_i |\sigma\rangle\). Carmona and Hu give the following identity [1]
\[
\lim_{N \to \infty} \frac{1}{N} \mathbb{E} \inf_{\langle \sigma | \sigma \rangle = 1} \left\langle \sum_{i<j} \frac{\gamma_{i,j}}{\sqrt{N}} \sigma_i \sigma_j \right\rangle = -\kappa.
\]
The right hand side is originally given by Parisi [9].

Proof of Theorem 1  Lemma 2, inequalities (23) and (27) imply that the square of the spin overlap has the following expectation bounded from the below in the infinite volume limit
\[
\lim_{N \to \infty} \mathbb{E} \langle R_{1,2}^2 \rangle \geq \frac{1}{2\beta h} \left( 1 - e^{-2\beta h} \right) - \frac{2\kappa}{\beta J},
\]  
(28)
where \(\kappa \simeq 0.763\) has been evaluated already [4, 9]. The \(\mathbb{Z}_2\)-symmetry gives \(\langle \sigma_i^z \rangle = 0\) and therefore
\[
\langle R_{1,2} \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i^z \rangle^2 = 0.
\]
This and the bound (28) yield
\[
\lim_{N \to \infty} \left[ \mathbb{E} \langle R_{1,2}^2 \rangle - \left( \mathbb{E} \langle R_{1,2} \rangle \right)^2 \right] > 0,
\]  
(29)
for sufficiently weak \(h\) and sufficiently low temperature.

To show no ferromagnetic long range order, let us evaluate the expectation value of \(m^2\)
\[
\mathbb{E} \langle m^2 \rangle = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E} \left\langle \sigma_i^z \sigma_j^z \right\rangle = -\frac{1}{N} + \frac{N-1}{N} \mathbb{E} \langle \sigma_i^z \sigma_j^z \rangle.
\]  
(30)
For a unitary transformation \(U_1 := \exp(i \frac{\pi}{2} \sigma_i^x)\), the operator \(\sigma_i^z\) is transformed into \(U_1 \sigma_i^z U_1^\dagger = -\sigma_i^z\). Since \(p(\gamma_{1,j}) = p(-\gamma_{1,j})\) is assumed, \(\mathbb{E} \langle \sigma_i^z \sigma_j^z \rangle = \mathbb{E} \langle U_1 \sigma_i^z \sigma_j^z U_1^\dagger \rangle = -\mathbb{E} \langle \sigma_i^z \sigma_j^z \rangle\) which gives \(\mathbb{E} \langle \sigma_i^z \sigma_j^z \rangle = 0\) and thus
\[
\lim_{N \to \infty} \mathbb{E} \langle m^2 \rangle = 0.
\]
Also, \(\mathbb{E} \langle m^4 \rangle\) can be represented in
\[
\mathbb{E} \langle m^4 \rangle = \frac{N!}{N^4(N-4)!} \mathbb{E} \langle \sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z \rangle + \frac{3N-2}{N^3}.
\]  
(31)
\[ \mathbb{E}(\sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z) = 0 \] is proven in the same argument as for \( \mathbb{E}(\sigma_1^z \sigma_2^z) = 0 \). Then,
\[ \mathbb{E}(m^4) = 0. \]
This completes the proof of Theorem 1.

Theorem 1 implies that RSB occurs and the variance of \( \langle m^2 \rangle \) vanishes
\[ \lim_{N \to \infty} \left[ \mathbb{E}(m^2)^2 - (\mathbb{E}(m^2))^2 \right] = 0, \]
since \( \langle m^4 \rangle \geq \langle m^2 \rangle^2 \). Therefore, the Chebyshev inequality implies that \( \langle m^2 \rangle \) vanishes with probability 1 in the infinite volume limit, and there is no ferromagnetic long range order in each sample.

3.1 Discussions

Here, we discuss several extensions of our result to some other models, where there is a possibility of \( \mathbb{Z}_2 \)-symmetry breaking.

In the model with random exchange interactions satisfying a non-centered distribution \( \mathbb{E} \gamma_{i,j} \neq 0 \), spontaneous \( \mathbb{Z}_2 \)-symmetry breaking can appear. If there is ferromagnetic long range order, then \( \mathbb{E}(\sigma_1^z \sigma_2^z) \neq 0 \) gives a finite variance of the overlap
\[ \mathbb{E}(R_{1,2}^2) - (\mathbb{E}(R_{1,2}))^2 = \frac{N-1}{N} \mathbb{E}(\sigma_1^z \sigma_2^z)^2 \geq \frac{N-1}{N} \left( \mathbb{E}(\sigma_1^z \sigma_2^z) \right)^2 \neq 0, \]
in the \( \mathbb{Z}_2 \)-symmetric Gibbs state. If ferromagnetic long range order exists,
\[ \lim \inf_{N \to \infty} \left[ \mathbb{E}(R_{1,2}^2) - (\mathbb{E}(R_{1,2}))^2 \right] > 0, \]
should be proven in \( \mathbb{Z}_2 \)-symmetry breaking Gibbs state with spontaneous magnetization to show the existence of RSB. Since \( \mathbb{E}(R_{1,2}) \) does not vanish because of the \( \mathbb{Z}_2 \)-symmetry breaking \( \langle \sigma_1^z \rangle \neq 0 \), the proof becomes nontrivial.

In models defined by Hamiltonians with \( \mathbb{Z}_2 \)-symmetry breaking terms, such as longitudinal fields or \( p \)-spin interactions for an odd positive integer \( p \), the order parameter \( \mathbb{E}(R_{1,2}) \) does not vanish because of the \( \mathbb{Z}_2 \)-symmetry breaking \( \langle \sigma_1^z \rangle \neq 0 \), then the proof becomes nontrivial also.

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Data Availability  All data are provided in full in this paper.

Declarations

Conflict of interest  There is no conflict of interest.

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