THE LINK OF \( \{ f(x, y) + z^n = 0 \} \) AND ZARISKI’S CONJECTURE

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Abstract. We consider suspension hypersurface singularities of type \( g = f(x, y) + z^n \), where \( f \) is an irreducible plane curve singularity. For such germs, we prove that the link of \( g \) determines completely the Newton pairs of \( f \) and the integer \( n \) except for two pathological cases, which can be completely described. Even in the pathological cases, the link and the Milnor number of \( g \) determine uniquely the Newton pairs of \( f \) and \( n \). In particular, for such \( g \), we verify Zariski’s conjecture about the multiplicity. The result also supports the following conjecture formulated in the paper. If the link of an isolated hypersurface singularity is a rational homology 3-sphere then it determines the embedded topological type, the equivariant Hodge numbers and the multiplicity of the singularity. The conjecture is verified for weighted homogeneous singularities too.

1. Introduction.

In the last decades an intense research effort has been concentrated on the following problem: what kind of analytic invariants or smoothing invariants (if they exist) can be determined from the topology of a normal surface singularity \((X, x)\).

Some of the results became already classical. E.g., Mumford’s result which states that \((X, x)\) is smooth if and only if the fundamental group of the link \(L_X\) is trivial \cite{21}. Or its generalization by Neumann \cite{30} which claims that the oriented homeomorphism type of the link contains the same information as the resolution graph of \((X, x)\). Or Artin’s computations of the multiplicity and the embedded dimension of rational singularities \cite{3, 4}, and their generalizations by Laufer \cite{17} for minimally elliptic singularities, and by S. S.-T. Yau \cite{47} for some elliptic singularities.

In general, these questions are very difficult, even if we restrict ourselves to some special families, e.g. to complete intersections or hypersurface singularities; and even if we permit ourselves to use, instead of the topology of \((X, x)\), a richer topological information, e.g. in the case of hypersurfaces the embedded topological type. For example, Zariski conjectured three decades ago that the embedded topological type of an isolated hypersurface singularity determines its multiplicity \cite{49}. This was verified till now only for quasi-homogeneous singularities \cite{24, 32, 33, 48} (and some other sporadic cases).

In a different direction, the conjecture of Neumann and Wahl \cite{33} about a possible connection between the Casson invariant of the link (provided that it is an integral homology sphere) and the signature of the Milnor fiber opened new windows for the theory.

Recently, the subject revives even with a larger intensity, see e.g. \cite{24, 34, 35, 27, 28, 29}, their introductions and listed references. Basically, these articles claim that if the link of a \(\mathbb{Q}\)-Gorenstein singularity is a rational homology sphere, then it codifies an extremely rich analytic information about the singularity.

1991 Mathematics Subject Classification. [].

Key words and phrases. suspension singularities, plane curve singularities, Newton pairs, resolution graphs, cyclic covers, Zariski’s conjecture, multiplicity, Milnor number, links of singularities.

The second author is partially supported by NSF grant DMS-0071820.
The present article is in the spirit of the above efforts. We will consider the family of suspension hypersurface singularities of type \( f(x,y) + z^n \), where \( f \) is an irreducible plane curve singularity.

For this family, we not only answer positively both main conjectures, (namely the Zariski’s conjecture, and also the possibility to recover the main analytic and smoothing invariants from the link), but we succeed to obtain much sharper statements.

The main result of the paper is the following (cf. 6.4):

**Theorem 1.** Let \( f : (\mathbb{C}^2,0) \to (\mathbb{C},0) \) be an irreducible plane curve singularity with Newton pairs \( \{(p_i,q_i)\}_{i=1}^{s} \) and let \( n \geq 2 \) be an integer. Let \( L_X \) be the link of the hypersurface singularity \( (X,0) = (\{f(x,y) + z^n = 0\}, 0) \). Then, except for two pathological cases \( S1 \) and \( S2 \) (which are described completely in 6.1 and 6.3, and can be characterized perfectly in terms of \( L_X \)), from the link \( L_X \) one can recover completely the Newton pairs of \( f \) and the integer \( n \) (provided that we disregard the “z-axis ambiguity”, cf. 6.2.) In both exceptional cases the links have non-trivial first Betti numbers. In particular, the above statement holds without any exception provided that the link is a rational homology sphere.

On the other hand, in the cases \( S1 \) and \( S2 \), the link together with the Milnor number of the hypersurface singularity \( f + z^n \) determine completely the Newton pairs of \( f \) and the integer \( n \) (cf. the two addendums in 6.1 and 6.3).

Here some remarks are in order.

(1) \( L_X \) determines the number \( s \) of Newton pairs of \( f \) in all the cases.

The exceptional case \( S1 \) appears when \( s = 1 \), and the corresponding singularities have the equisingular type of some special Brieskorn singularities. This case can be easily classified.

The exceptional case \( S2 \) appears for \( s = 2 \) with some other strong additional restrictions. In this case any link \( L_X \) can be realized by at most two possible pairs \((f,n)\). This case again is completely clarified.

In all other cases, e.g. when \( s \geq 3 \), the theorem assures uniqueness. This is slightly surprising. At the beginning of our study, we expected here more and more complicated special families providing interesting coincidences for their links. But, it turns out that this is not the case: if the plumbing graph of the link (or equivalently, the resolution graph of \((X,x)\)) has more and more complicated structure, then it becomes more “over-determined”, and it leaves no room for any ambiguity for \( f \) and \( n \).

In fact, in order to reach our goal, it was sufficient to consider a rather limited information about this graph: the determinants of its maximal strings and the determinants of some subgraphs with only one rupture vertex.

Except for the two pathological cases (when the graphs are really very simple), in all other cases already these determinants determine all the Newton pairs and \( n \).

(2) In general, it is very difficult to characterize those resolution graphs (or links) which can be realized by, say, hypersurface singularities, or complete intersections, or by any family of germs defined by some analytic property.

Our proof gives a complete characterization of those graphs which can be realized as resolution graphs of some \( \{f + z^n = 0\} \) for some irreducible \( f \). Indeed, the proof is a precise recipe how one can recover the Newton pairs of \( f \) and \( n \). If one runs this algorithm (the steps of the proof of 6.4) for an arbitrary minimal resolution graph, and at some point it fails, then the graph definitely is not of this type. If the algorithm goes through and provides some candidates for the Newton pairs of \( f \) and for \( n \), then one has to compute the minimal resolution graph of \( f + z^n = 0 \) (using e.g. 3.3) and compare it with the initial graph.
they are the same, then the answer is yes. But it can happen that these two graphs are not the same (since our algorithm is based on a very limited number of determinants: these determinants can be the same even if the graphs are not).

(3) For the case when \( f \) is arbitrary (i.e. reduced), but \( n = 2 \), and the link is rational homology sphere, Laufer established the uniqueness in [10].

In [40, 41], A. Pichon proved by a different method that any link can be realized in a \textit{finitely many} ways as the link of \( f + z^n = 0 \) (in her case \( f \) is reduced too). We do not see at this moment how one can show the above uniqueness result by her algorithm.

Some of our statements (after some identifications) can be compared with some results of A. Pichon. E.g. our last formula from [4, 5] can be compared with Proposition 3 [40].

Now we return back to our main theorem and its corollaries.

**Corollary 1.** Assume that \( g(x,y,z) = f(x,y) + z^n \) is a suspension hypersurface singularity with \( f \) irreducible, and not of type described in the pathological cases S1 and S2 (cf. [6, 8] and [23]). Then the link \( L_X \) of \((X,0) = \{g = 0\}, 0\) determines completely the following data:

1. The embedded topological type of \((X,0)\) (i.e. the embedding \( L_X \subset S^5 \)), in particular, the Milnor fibration and all the homological package derived from it.
2. All the equivariant Hodge numbers associated with the vanishing cohomology of \( g \), in particular, the geometric genus of \((X,0)\).
3. The multiplicity of \( g \).

In particular, if \( L_X \) is a rational homology sphere, then \( L_X \) determines (1) (2) and (3). If \( g \) is a pathological case listed in S1 or S2, then \( L_X \) together with the Milnor number of \( g \) determines completely (1) (2) and (3).

Indeed, once we have the Newton pairs of \( f \) and the integer \( n \), then the proof involves the description of the corresponding invariants for plane curve singularities and different “Sebastiani-Thom type” formulae, for more details see e.g. [1, 4, 22, 23, 29]. In fact, recently in [29], the geometric genus (together with the Milnor number and the signature of the Milnor fiber) was computed in terms of the Seiberg-Witten invariant of the link, provided that the link is a rational homology sphere.

It is well-known that the Milnor number of \( g \) can be determined from the embedded topological type of \( g \) (a fact firstly noticed by Teissier, see also [48]). Therefore, we get:

**Corollary 2.** [Zariski’s conjecture for this family] The multiplicity of \( g = f(x,y) + z^n \) (\( f \) irreducible) is determined by the embedded topological type of \( g \).

Corollary 1 (see also Theorem 2 below) motivates the following conjecture.

**Conjecture.** Let \( g : (\mathbb{C}^3,0) \to (\mathbb{C},0) \) be an isolated hypersurface singularity whose link \( L_X \) is a rational homology sphere. Then the fundamental group of the link characterizes completely the embedded topological type, the equivariant Hodge numbers and the multiplicity of \( g \). More generally, if the link of a \( \mathbb{Q} \)-Gorenstein singularity \((X,0)\) is a rational homology sphere, then the multiplicity of \((X,0)\) is determined by the (oriented) homeomorphism type of the link.

Notice that the last general property is true for rational [3, 4] and elliptic [17, 17, 24] singularities. The first part can be verified in the following cases. If \( g = f + z^n \) with \( f \) irreducible, then Conjecture is true by Theorem 1 above (see also [30] for the relation between \( L_X \) and its fundamental group). If \( f \) is arbitrary but \( n = 2 \), then it is true by [19].
For weighted homogeneous singularities the next theorem answers positively (since its proof is very short, we decided to put the whole statement at the end of this introduction).

**Theorem 2.** The above conjecture is true for any weighted homogeneous hypersurface singularity.

Indeed, the Poincaré polynomial of the singularity can be determined from the link (see also [31]). Then, by a recent result of Ebeling [9] (Theorem 1) follows that the characteristic polynomial of the algebraic monodromy can be recovered from the link $L_X$. (For this notice that $\psi_A$ used by Ebeling is also link-invariant.) Then, by [46] (cf. also with [38]), we get that the weights, multiplicity and the embedded topological type is determined by $L_X$. The statement about the Hodge data follows from [13].

2. Resolution graphs associated with analytic germs

2.1. **The embedded resolution.** Let $(X,x)$ be a normal surface singularity and let $f : (X,x) \to (\mathbb{C},0)$ be the germ of an analytic function.

An embedded resolution $\phi : (\mathcal{Y},D) \to (U,f^{-1}(0))$ of $(f^{-1}(0),x) \subset (X,x)$ is characterized by the following properties. There is a sufficiently small neighborhood $U$ of $x$ in $X$, smooth analytic manifold $\mathcal{Y}$, and analytic proper map $\phi : \mathcal{Y} \to U$ such that:

1) if $E = \phi^{-1}(x)$, then the restriction $\phi|_{\mathcal{Y}\setminus E} : \mathcal{Y}\setminus E \to U\setminus \{x\}$ is biholomorphic, and $\mathcal{Y}\setminus E$ is dense in $\mathcal{Y}$;

2) $D = \phi^{-1}(f^{-1}(0))$ is a divisor with only normal crossing singularities.

$E$ is called the exceptional divisor of $\phi$. Let $\cup_{w \in \mathcal{W}} E_w$ be its decomposition into irreducible divisors. The closure $S$ of $\phi^{-1}(f^{-1}(0) \setminus \{0\})$ is called the strict transform of $f^{-1}(0)$. Let $\cup_{a \in \mathcal{A}} S_a$ be its irreducible decomposition. Obviously, $D = E \cup S$.

In the sequel we will assume that $\mathcal{W} \neq \emptyset$, any two irreducible components of $E$ have at most one intersection point, and no irreducible exceptional divisor has a self–intersection points. This can always be realized by some additional blow ups.

2.2. **The embedded resolution graph.** We construct the embedded resolution graph $\Gamma(X,f)$ of the pair $(X,f)$, associated with a fixed resolution $\phi$, as follows. Its vertices $\mathcal{V} = \mathcal{W} \coprod \mathcal{A}$ consist of the nonarrowhead vertices $W$ corresponding to the irreducible exceptional divisors, and arrowhead vertices $\mathcal{A}$ corresponding to the irreducible divisors of the strict transform $S$. If two irreducible divisors corresponding to $v_1, v_2 \in \mathcal{V}$ have an intersection point then $(v_1,v_2)$ ($=(v_2,v_1)$) is an edge of $\Gamma(X,f)$.

The graph $\Gamma(X,f)$ is decorated as follows. Any $w \in \mathcal{W}$ is decorated by the self–intersection $e_w := E_w \cdot E_w$ and genus $g_w$ of $E_w$; and any $v \in \mathcal{V}$ by the multiplicity $m_v$ of $f$. More precisely, for any $v \in \mathcal{V}$, let $m_v$ be the vanishing order of $f \circ \phi$ along the irreducible divisor corresponding to $v$. For example, if $f$ defines an isolated singularity, then $m_a = 1$ for any $a \in \mathcal{A}$.

In all our graph–diagrams, we put the multiplicities in parentheses (e.g.: (3)) and the genera in brackets (e.g.: [3]), with the convention that we omit [0].

2.3. **The resolution of $(X,x)$.** We say that $\phi : \mathcal{Y} \to U$ is a resolution of $(X,x)$ if $\mathcal{Y}$ is a smooth analytic manifold, $U$ a neighbourhood of $x$ in $X$, $\phi$ is a proper analytic map, such that $\mathcal{Y}\setminus E$ (where $E = \phi^{-1}(x)$) is dense in $\mathcal{Y}$ and the restriction $\phi|_{\mathcal{Y}\setminus E} : \mathcal{Y}\setminus E \to U \setminus \{x\}$ is biholomorphic.

We codify the topology and the combinatorics of the pair $(\mathcal{Y}, E)$ in the dual resolution graph $\Gamma(X)$ of $(X,x)$ associated with $\phi$. If the divisor $E$ is not a normal crossing divisor,
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then this codification can be slightly complicated, so in the sequel we will assume that the irreducible components of \( E \) are smooth and intersect each other transversally, the irreducible components have no self-intersection points, and there is no intersection point which is contained in more than two components. In this case, similarly as in the situation of the embedded resolution, the vertices of the dual graph correspond to the irreducible components of \( E \), the edges to the intersections of these components, and each vertex \( w \) carries two decorations: the genus \( g_w \) of \( E_w \) and the self-intersection \( E_w \cdot E_w \).

Actually, one can obtain a possible graph \( \Gamma(X) \) from any \( \Gamma(X, f) \) by deleting all the arrows and multiplicities of the graph \( \Gamma(X, f) \).

2.4. Definitions. Let \( \Gamma \) be a decorated graph (as above, with or without arrowheads). For any \( w \in \mathcal{W} \), we denote by \( V_w \) the set of vertices \( v \in V \) adjacent to \( w \). Its cardinality \( \#V_w \) is called the degree \( \delta_w \) of \( w \). A vertex \( w \in \mathcal{W} \) is called a rupture vertex if either \( g_w > 0 \) or \( \delta_w \geq 3 \). (Notice that it can happen that \( w \) is a rupture vertex in some \( \Gamma(X, f) \), but not in the graph obtained from \( \Gamma(X, f) \) after one deletes its arrows.) A vertex is called leaf vertex if \( \delta_w = 1 \).

2.5. Some properties of resolution graphs.

(1) Notice that the combinatorics of the graph and the self-intersections codify completely the intersection matrix \( I := (E_w \cdot E_v)_{(w, v) \in \mathcal{W} \times \mathcal{W}} \) of the irreducible components of \( E \). This matrix is negative definite, see e.g. [21] page 230; [18] page 49, or [11]. We write \( \det(\Gamma) := \det(-I) > 0 \). By convention, the determinant of the empty graph is 1.

(2) The graphs \( \Gamma(X, f) \) and \( \Gamma(X) \) are connected (see [18], or Zariski’s Main Theorem, e.g. in [13]).

(3) The next identities connect the self-intersections and multiplicities:

\[
e_w m_w + \sum_{v \in V_w} m_v = 0 \text{ for any } w \in \mathcal{W}.
\]

Obviously, \( m_v > 0 \) for any \( v \in \mathcal{V} \), hence the set of multiplicities determine the self-intersections completely. Similarly, since the intersection matrix \( I \) is negative definite, these relations determine the multiplicities \( \{m_w\}_{w \in \mathcal{W}} \) in terms of the self-intersections \( \{e_w\}_{w \in \mathcal{W}} \) and the multiplicities \( \{m_a\}_{a \in \mathcal{A}} \).

Using matrices, the above set of relations can be written as follows. Fix a total ordering of the set \( \mathcal{W} \). Let \( \mathbf{m}_\mathcal{W} \) be the column vector with \( |\mathcal{W}| \) entries \( \{m_w\}_{w \in \mathcal{W}} \). Similarly, define the column vector \( \mathbf{m}_\mathcal{A} \) with \( |\mathcal{W}| \) entries whose \( w^{th} \) entry is \( \sum_{a \in \mathcal{A} \cap V_w} m_a \). Then

\[
I \cdot \mathbf{m}_\mathcal{W} + \mathbf{m}_\mathcal{A} = 0.
\]

If \( \Gamma = \Gamma(X) \) is a tree, then the inverse matrix \( I^{-1} \) can be computed in terms of determinants of some subgraphs as follows. Consider two vertices \( w_1, w_2 \in \mathcal{W} \) and the shortest path which connects them. Let \( \Gamma_{w_1 w_2} \) be the maximal (in general non-connected) subgraph of \( \Gamma \) which has no vertices on this path. Then the \( (w_1, w_2) \)-entry of \( I^{-1} \) is given by

\[
I_{w_1 w_2}^{-1} = -\frac{\det(\Gamma_{w_1 w_2})}{\det(\Gamma)}.
\]

(4) There are many (embedded) resolutions, hence many (embedded) resolution graphs too. Nevertheless, they are all connected by quadratic modifications (i.e. blow up and/or down of \(-1\)-curves with \( g = 0 \)). Notice that by the above conventions, we can blow down a \(-1\)-curve \( E_w \) with \( g_w = 0 \) if and only if \( \delta_w \leq 2 \). If the (embedded) resolution graph has
no rational $-1$-curve $E_w$ with $\delta_w \leq 2$, then we say that it is minimal. There is a unique minimal (embedded) resolution graph denoted by $\Gamma^{\text{min}}(X)$ (resp. $\Gamma^{\text{min}}(X,f)$). (Obviously, deleting the arrows from $\Gamma^{\text{min}}(X,f)$ it can happen that we obtain a non-minimal resolution graph of $(X,x)$.)

2.6. The link of $(X,x)$. Fix an embedding $(X,x) \subset (\mathbb{C}^N,0)$ for some $N$. Then, for sufficiently small $\epsilon > 0$, $L_X := \{ z \in X : |z| = \epsilon \}$ is a connected oriented differentiable manifold (independent of the different choices). It is called the link of $(X,x)$.

From topological point of view, $L_X$ characterizes completely $(X,x)$. Moreover, in the presence of a resolution $\phi$, for $U$ sufficiently small, the inclusion $E \hookrightarrow Y$ admits a strong deformation retract and the restriction of $\phi$ identifies $\partial Y$ with $L_X$. This shows that $L_X$ is the plumbed manifold associated with $\Gamma(X)$ (for details, see [10]), i.e. $\Gamma(X)$ determines completely $L_X$. The converse is also true: Neumann in [10] proved that the (minimal) resolution graph $\Gamma(X)$ is determined by the oriented homeomorphism type of $L_X$.

2.7. Fact. The homology of $L_X$. [13, 13, 2] $H_1(L_X,\mathbb{Z}) \approx \text{coker}(I) \oplus \mathbb{Z}^{g+c_1}$, where $g := \sum_{w \in \mathcal{W}} g_w$ and $c_1$ denotes the number of independent cycles in $\Gamma = \Gamma(X)$ (e.g. $c_1 = 0$ if and only if $\Gamma$ is a tree).

In particular, $L_X$ is an integral (resp. rational) homology sphere if and only if $g = c_1 = 0$ and $\det(\Gamma) = 1$ (resp. $g = c_1 = 0$).

2.8. Example. Irreducible plane curve singularities. (see e.g. [9] or [14]) If $(X,x)$ is smooth, hence $\approx (\mathbb{C}^2,0)$, then $f$ is called plane singularity. In this case, the graph $\Gamma(\mathbb{C}^2,f)$ is a tree, and $g_w = 0$ for any $w \in \mathcal{W}$.

In this article we are mainly interested in irreducible plane curve singularities (i.e. when $|\mathcal{A}| = 1$). Their equisingular type (and link also) is completely characterized by the set of Newton pairs $\{(p_k,q_k)\}_{k=1}^l$ (see e.g. [14], page 49). Here $(p_k,q_k) = 1$, $p_k \geq 2$, $q_k \geq 1$ and $q_1 > p_1$.

The minimal embedded resolution graph can be reconstructed from the Newton pairs as follows (see e.g. in [9], [14] or [23]). First determine $u_i^1$ and $v_i^1$ ($u_i^0, v_i^0 \geq 1$, and $u_i^1, v_i^1 \geq 2$ for $l > 0$) from the continued fractions:

$$\frac{p_i}{q_i} = u_i^0 - \frac{1}{u_i^1 - \frac{1}{\ddots - \frac{1}{u_i^s}}}, \quad \frac{q_i}{p_i} = v_i^0 - \frac{1}{v_i^1 - \frac{1}{\ddots - \frac{1}{v_i^s}}}.$$ 

Then $\Gamma^{\text{min}}(\mathbb{C}^2,f)$ is given by

$$\begin{array}{cccccccc}
-u_1^1 & -u_1^0 & -u_2^0 & -u_2^1 & -u_3^0 & -u_3^1 & -u_4^0 & -u_4^1 \\
-v_1^1 & -v_1^0 & -v_2^0 & -v_2^1 & -v_3^0 & -v_3^1 & -v_4^0 & -v_4^1 \\
& -v_1^2 & -v_1^1 & \ddots & -v_1^2 & -v_1^1 & \ddots & -v_1^2 \\
& & -v_2^2 & -v_2^1 & \ddots & -v_2^2 & \ddots & -v_2^1 \\
& & & -v_3^2 & -v_3^1 & \ddots & \ddots & -v_3^1 \\
& & & & -v_4^2 & -v_4^1 & \ddots & -v_4^1 \\
& & & & & -v_5^2 & \ddots & -v_5^1 \\
& & & & & & -v_6^2 & \ddots \\
& & & & & & & -v_7^2 \\
& & & & & & & \vdots \\
& & & & & & & -v_s^2 \\
& & & & & & & \vdots \\
& & & & & & & -v_s^1 \\
& & & & & & & \vdots \\
& & & & & & & -v_1^1 \\
\end{array}$$

This has the following schematic form:
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- \( \bar{v}_0 \)
- \( v_1 \)
- \( v_2 \)
- \( \cdots \)
- \( v_{s-1} \)
- \( v_s \)

Here we emphasized only those vertices \( \{ \bar{v}_k \}_{k=0}^s \) and \( \{ v_k \}_{k=1}^s \) which have degree \( \delta \neq 2 \). We denote the set of these vertices by \( \mathcal{W}^s \). The dash-line between two such vertices replaces a string \( \cdots \). In our discussions below, the corresponding self-intersections will be less important, but the multiplicities of the vertices \( v \in \mathcal{W}^s \) will be crucial. They can be easily described in terms of the integers \( \{ a_k \}_{k=1}^s \):

\[
a_1 = q_1 \quad \text{and} \quad a_{k+1} = q_{k+1} + pk_{k+1}p_{ka_k} \quad \text{if} \quad s - 1 \geq k \geq 1.
\]

Then again, \( (p_k, a_k) = 1 \) for any \( k \). Clearly, the two sets of pairs \( \{(p_k, q_k)\}_{k=1}^s \) and \( \{(p_n, a_n)\}_{k=1}^s \) determine each other completely. In fact, the set of pairs \( \{(p_k, a_k)\}_{k=1}^s \) constitutes the set of decoration of the so called \( \text{splice} \), or \( \text{Eisenbud-Neumann} \) diagram of \( f \), cf. \ref{10}, page 51. Then by \ref{10}, section 10, one has:

\[
m_{v_k} = a_kp_kp_{k+1} \cdots p_s \quad \text{for} \quad 1 \leq k \leq s;
\]
\[
m_{\bar{v}_0} = p_1p_2 \cdots p_s;
\]
\[
m_{\bar{v}_k} = a_kp_{k+1} \cdots p_s \quad \text{for} \quad 1 \leq k \leq s - 1;
\]
\[
m_{\bar{v}_s} = a_s.
\]

2.9. Example. Hirzebruch–Jung singularities. (See \ref{14}, \ref{15}, \ref{5}). For a normal surface singularity, the following conditions are equivalent. If \( (X, x) \) satisfies (one of) them, then it is called Hirzebruch–Jung singularity (where we prefer to include the smooth germ too).

(a) \( (X, x) \) has a resolution graph \( \Gamma(X) \) which is a string with \( g_w = 0 \) for any \( w \in \mathcal{W} \). (If the graph is minimal then it is either empty or \( e_w \leq -2 \) for any \( w \).

(b) There is a finite proper map \( \rho : (X, x) \to (\mathbb{C}^2, 0) \) such that reduced discriminant locus of \( \rho \), in some local coordinates \( (u, v) \) of \( (\mathbb{C}^2, 0) \), is a subset of \( \{ uv = 0 \} \).

(c) \( (X, x) \) is either smooth or is isomorphic with exactly one of the “model spaces” \( \{ A_{n, q} \}_{n/q} \), where \( A_{n, q} \) is the normalization of \( \{ xy^{n-q} + z^n = 0 \} \), where \( 0 < q < n \), \( (n, q) = 1 \).

For such \( (X, x) \), \( \pi_1(L_X) = \mathbb{Z}_n \), where \( n = 1 \) if and only if \( (X, x) \) is smooth (see e.g. \ref{5}), otherwise \( n \) is the number from (c). Then, by \ref{7}, we also have \( n = \det(\Gamma(X)) \) (fact which is valid for non-minimal graphs as well).

In some of our applications we will need to recover this integer \( n \) from the geometry of the map \( \rho \) from (b). Consider the induced regular covering \( \rho_{reg} : \rho^{-1}(\{ uv \neq 0 \}) \to \{ uv \neq 0 \} \). Let \( \pi_1 = \mathbb{Z}^2 \) be the fundamental group of \( \{ uv \neq 0 \} \) generated by \( e_1 \) and \( e_2 \) representing two elementary loops around the axes \( u \) and \( v \). Let \( \rho_* : \pi_1 \to G \) be the monodromy representation of \( \rho_{reg} \), and \( \rho_*|\mathbb{Z}(e_i) \) be its restriction to \( \mathbb{Z}(e_i) \), the subgroup generated by \( e_i \) (\( i = 1, 2 \)).

2.10. Lemma. \( \mathbb{Z}_n \cong \ker(\rho_*) / \ker(\rho_*|\mathbb{Z}(e_1)) \times \ker(\rho_*|\mathbb{Z}(e_2)) \).

Proof. This follows from the classification of the subgroups of \( \mathbb{Z}^2 \), see e.g. \ref{8}, III.5. Another proof goes as follows. It is clear that \( \pi_1(\rho^{-1}(\{ uv \neq 0 \})) = \ker(\rho_*) \), and \( \mathbb{Z}_n = \pi_1(L_X) \) is the quotient group of the previous group by the subgroup generated by all the loops staying above \( e_1 \) and \( e_2 \).
2.11. The “model” $X(a, b, N)$. From our point of view, it is more convenient to consider a bigger class of “models” instead of $\{A_{n,q}\}_{n,q}$. More precisely, for any three strictly positive integers $a, b$ and $N$, with $(a, b, N) = 1$, we define $(X, x) = (X(a, b, N), x)$ as the unique singularity lying over the origin in the normalization of $\{(a^i b^j + \gamma N = 0), 0\}$. Let the germ $\gamma : (X(a, b, N), x) \to (\mathbb{C}, 0)$ be induced by $(\alpha, \beta, \gamma) \mapsto \gamma$. In the sequel, we give the embedded resolution graph $\Gamma(X, \gamma)$ of the germ $\gamma$ (for details, see [24]).

First, consider the unique $0 \leq \lambda < N/(a, N)$ and $m_1 \in \mathbb{N}$ with:

$$b + \lambda \cdot \frac{a}{(a, N)} = m_1 \cdot \frac{N}{(a, N)}.$$ 

If $\lambda \neq 0$, then consider the continued fraction:

$$\frac{N/(a, N)}{\lambda} = k_1 - \frac{1}{k_2 - \frac{1}{\ddots - \frac{1}{k_s}}}.$$ 

Then the embedded resolution graph $\Gamma(X, \gamma)$ of $\gamma$ is the following string:

$$\text{St}(a, b, N) : \left(\frac{a}{(a,N)}\right) \xrightarrow{-k_1} (m_1) \xrightarrow{-k_2} (m_2) \cdots \xrightarrow{-k_s} (m_s) \left(\frac{b}{(b,N)}\right).$$

The arrow at the left (resp. right) hand side codifies the strict transform of $\{\alpha = 0\}$ (resp. of $\{\beta = 0\}$). All vertices have genus $g_w = 0$. The first vertex has multiplicity $m_1$ given by the above congruence. Hence $m_2, \ldots, m_s$ can be easily computed using 2.14(3), namely:

$$-k_1m_1 + \frac{a}{(a,N)} + m_2 = 0, \quad -k_im_i + m_{i-1} + m_{i+1} = 0 \quad \text{for} \quad i \geq 2.$$ 

This resolution resolves also the germs $\alpha$ and $\beta$ (induced by the projection $(\alpha, \beta, \gamma) \mapsto \alpha$, resp. $(\alpha, \beta, \gamma) \mapsto \beta$). Indeed, as we already mentioned, the strict transform of $\{\alpha = 0\}$ (resp. $\{\beta = 0\}$) is irreducible and it is exactly that strict transform component of $\gamma$ which is codified by the left (resp. right) arrowhead of $\text{St}(a, b, N)$. The multiplicity of $\alpha$ (resp $\beta$) along this strict transform component, or arrowhead, is $N/(a, N)$ (resp. $N/(b, N)$). Obviously, the multiplicity of $\alpha$ (resp $\beta$) on the right (resp. left) arrowhead is zero.

Therefore, the embedded resolution graph $\Gamma(X, \alpha^i \beta^j \gamma^k)$ ($k > 0$) of the germ $\alpha^i \beta^j \gamma^k$ defined on $X$ can be deduced easily from the above resolution graphs. It has the same shape, the same self–intersections and genera. Its multiplicity on the left arrowhead is $(iN + ka)/(a, N)$, and on the right arrowhead $(jN + kb)/(b, N)$.

Finally, if $\lambda = 0$, then the string $\text{St}(a, b, N)$ has no vertices. In this case, $(X(a, b, N), x)$ is smooth and the zero set of $\gamma$ (on $X$) has only a normal crossing singularity: in some local coordinates $(u, v)$ of $(X, x)$, it can be represented as $\gamma = u^{a/(a,N)}v^{b/(b,N)}$.

2.12. Remark. Form the point of view of the above classification 2.14(c), $X(a, b, N)$ is an $A_{n,q}$–singularity, where $n = N/(a, N)(b, N)$ and $q = \lambda/(b, N)$ (cf. e.g. [8], page 83-84). In fact, $n$ can be deduced from 2.14 as well. Indeed, in this case $G = \mathbb{Z}_N$, $\rho_s$ is onto, hence $\ker(\rho_s)$ has index $N$ in $\mathbb{Z}^2$. On the other hand the index of $\ker(\rho_s|\mathbb{Z}(e_i))$ in $\mathbb{Z}$ is $N/(a, N)$ for $i = 1$ and $N/(b, N)$ for $i = 2$. Hence their product has index $N^2/(a, N)(b, N)$ in $\mathbb{Z}^2$. In particular, $\det(\Gamma(X))$ is $n = N/(a, N)(b, N)$. 
Similarly, if $e$ is a loop in $\Gamma(X,f)$, then we write $m_e := (m_{w_1}, m_{w_2})$.

3.1. **Cyclic coverings.** Let $(X, x)$ be a normal surface singularity and $f : (X, x) \to (\mathbb{C}, 0)$ the germ of an analytic function. For simplicity we assume that $f$ defines an isolated singularity at $x$. For any integer $n \geq 1$, take $b : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ given by $z \mapsto z^n$, and let $X_{f,n}$ be the normalization of the fiber product $\{(x', z) \in (X \times \mathbb{C}, x, 0) : f(x') = z^n\}$. The second projection $(x', z) \in X \times \mathbb{C} \mapsto z \in \mathbb{C}$ induces an analytic map $X_{f,n} \to \mathbb{C}$, still denoted by $z$. The first projection $(x', z) \mapsto x'$ gives rise to a ramified cyclic $n$-covering $pr : X_{f,n} \to X$, branched along $f^{-1}(0)$. Since $f^{-1}(0)$ has an isolated singular point at $x$, one can verify that there is only one (singular) point of $X_{f,n}$ lying above $x \in X$.

We regard $\mathbb{Z}/n$ as the group of $n^{th}$-roots of unity $\{e^{2\pi ik/n} : 0 \leq k \leq n - 1\}$. Then $(x', z) \mapsto (x', e^{2\pi ik/n}z)$ induces a $\mathbb{Z}/n$-Galois action of $X_{f,n}$ over $X$.

3.2. **The (embedded) resolution graph of cyclic coverings.** It is known (see e.g. [26]) that in general the graphs $\Gamma(X_{f,n}, z)$ and $\Gamma(X_f, n)$ cannot be determined from the embedded resolution graph $\Gamma(X, f)$ of $f$ and the integer $n$. Nevertheless, if $L_X$ is a rational homology sphere, this is possible. The algorithm for a plane curve singularity $f$ is given in [25] (based on idea of the construction used in the book of Laufer [18]). For other particular cases, see [2, 37, 38, 14]. The general case is discussed in [21]. This algorithm can also be compared with some results of E. Hironaka about global cyclic coverings.

3.3. **Theorem.** The graph $\Gamma(X_{f,n}, z)$. [25, 26] Let $f : (X, x) \to (\mathbb{C}, 0)$ be as above and assume that $L_X$ is a rational homology sphere. Fix the an embedded resolution graph $\Gamma(X, f)$. Let $\{m_v\}_{v \in V}$ denote the corresponding multiplicities of $f$. For any $w \in W(\Gamma(X, f))$, it is convenient to write

$$M_w := \gcd(m_v \mid v \in \mathcal{V}_w \cup \{w\}).$$

Similarly, if $e = (w_1, w_2)$ is an edge of $\Gamma(X, f)$, then we write $m_e := (m_{w_1}, m_{w_2})$. 

In this section we review some properties of cyclic coverings and their resolution graphs. The reader is invited to consult [27] for more details.
Next, fix an integer \( n \), and consider the cyclic covering \((X_{f,n}, z)\) as above. In order to eliminate any confusion between the decorations of the graph \( \Gamma(X, f) \) and \( \Gamma(X_{f,n}, z) \), we denote the multiplicities (respectively the genera) of \( \Gamma(X_{f,n}, z) \) by \( m_{v'} \) (respectively \( g_{v'} \) for any \( v' \in \mathcal{V}(\Gamma(X_{f,n}, z)) \)).

Then a possible embedded resolution graph \( \Gamma(X_{f,n}, z) \) can be constructed as follows.

The graph \( \Gamma(X_{f,n}, z) \) can be considered as a “covering” \( q : \Gamma(X_{f,n}, z) \to \Gamma(X, f) \).

(a) Above \( w \in \mathcal{W}(\Gamma(X, f)) \) there are \((M_w, n)\) vertices \( v' \in q^{-1}(w) \) of \( \Gamma(X_{f,n}, z) \), each with multiplicity \( m_{v'} = m_w/(m_w, n) \) and genus \( g_{v'} \), where:

\[
2 - 2g_{v'} = \frac{(2 - \delta_w)(m_w, n) + \sum_{v \in \mathcal{V}_w} (m_w, m_v, n)}{(M_w, n)}.
\]

(In fact, by 2.5(3), if \( \delta_w \leq 2 \) in \( \Gamma(X, f) \), then \( g_{v'} = 0 \).) The vertices in \( q^{-1}(w) \) can be indexed by the group \( \mathbb{Z}_{(M_w, n)} \).

(b) An edge \( e = (w_1, w_2) \) of \( \Gamma(X, f) \) (where \( w_1, w_2 \in \mathcal{W}(\Gamma(X, f)) \))

\[
\begin{array}{c}
(m_{w_1}) \\
\end{array} \quad \begin{array}{c}
(m_{w_2}) \\
\end{array}
\]

is covered by \((m_e, n)\) copies of strings in \( \Gamma(X_{f,n}, z) \), each of type (cf. 2.11):

\[
St(m_{w_1}/(m_e, n), m_{w_2}/(m_e, n), n/(m_e, n)).
\]

These strings can be indexed by the group \( \mathbb{Z}_{(m_e, n)} \). The arrowheads of the strings are identified with the vertices \( q^{-1}(w_1) \), respectively \( q^{-1}(w_2) \), via the natural morphisms \( \mathbb{Z}_{(m_e, n)} \to \mathbb{Z}_{(M_{w_1}, n)} \), respectively \( \mathbb{Z}_{(m_e, n)} \to \mathbb{Z}_{(M_{w_2}, n)} \). (If this string is empty, i.e. if in 2.11 \( \lambda = 0 \), then above the edge \( e \) we insert \((m_e, n)\) edges by the same procedure.)

(c) An arrowhead of \( \Gamma(X, f) \)

\[
\begin{array}{c}
(m_{w}) \\
\end{array}
\]

is covered by one string of type \( St(m_w, 1, n) \) (cf. 2.11), whose right arrowhead will remain an arrowhead of \( \Gamma(X_{f,n}, z) \) with multiplicity 1, and its left arrowhead is identified with the unique vertex above \( w \). (Similarly as above, if this string is empty, then above this arrow we insert a unique arrow supported by the unique vertex staying above \( w \).)

(d) In this way, we obtain all the vertices, edges and arrowheads of \( \Gamma(X_{f,n}, z) \), and all the multiplicities of \( z \). Moreover, by the description of the strings (cf. 2.11), one has all the self–intersections of the vertices which are situated on the new strings. The self–intersections of the vertices \( q^{-1}(w) \) \( (w \in \mathcal{W}(\Gamma(X, f))) \) can be computed using 2.7(3) (from the multiplicities of \( z \)).

The isomorphism type of the above “covering” graph is independent of the choice a different identification (of the corresponding sets with cyclic groups), cf. 21. If we drop the arrowheads and multiplicities of \( \Gamma(X_{f,n}, z) \), we obtain \( \Gamma(X_{f,n}) \). The graphs \( \Gamma(X_{f,n}, z) \) and \( \Gamma(X_{f,n}) \), in general, are not minimal.

3.4. Definition. Let \( (X, f) \) and \( n \) be as in 3.3. Assume that in the above algorithm, we start with the minimal (good) embedded resolution graph \( \Gamma(X, f) \) of \( (X, f) \). Then the output graph of the algorithm (without any modification by any blow up or down) will be called the canonical embedded resolution graph of \( (X_{f,n}, z) \). In the sequel, we denote
On the link of \( \{ f(x, y) + z^n = 0 \} \) it by \( \Gamma^{can}(X_{f,n}, z) \). The name is motivated by \[8\], where Laufer proved that the above algorithm for a plane curve singularity \( f \) provides exactly the canonical resolution of \( X_{f,n} \) in the sense of Zariski, provided that \( n = 2 \).

4. The Resolution Graph of \( \{ f(x, y) + z^n = 0 \} \).

In this section we make the algorithm \[8\] very explicit in the case when \( (X, x) = (\mathbb{C}^2, 0) \) and \( f \) is an irreducible plane curve singularity. Clearly, in this case, \( X_{f,n} \) can be identified with the hypersurface singularity \( \{ f(x, y) + z^n = 0 \} \) and \( z \) with the natural map induced by the \( z \)-projection. We will assume that \( n \geq 2 \).

In the sequel we will use the following notations as well. Recall that \( \{(p_k, q_k)\}_{k=1}^s \) denotes the set of Newton pairs of \( f \), and the integers \( \{a_k\}_{k=1}^s \) are defined in \[2.8(1)\]. Then we define

- \( d_k := (n, p_{k+1}p_{k+2} \cdots p_s) \) for \( 0 \leq k \leq s - 1 \), and \( d_s := 1 \);
- \( h_k := d_{k-1}/d_k = (p_k, n/d_k) \) and \( p'_k := p_k/h_k \) for \( 1 \leq k \leq s \);
- \( \tilde{h}_k := (a_k, n/d_k) \) and \( a'_k := a_k/\tilde{h}_k \) for \( 1 \leq k \leq s \).

We start our discussion with the computation of the numbers \( M_w \) (\( w \in \mathcal{W}^* \)) and the decorations \( m_{w'} \) and \( g_{w'} \) from \[3.3(a)\] for any \( v' \in q^{-1}(\mathcal{W}^*) \). Notice that because of the Galois action, for a fixed \( w \in \mathcal{W}^* \), the integers \( m_{w'} \) and \( g_{w'} \) do not depend on the choice of \( v' \in q^{-1}(w) \), but only on \( w \). Therefore, sometimes we prefer to denote them simply by \( m_w \) and \( g_w \).

4.1. Lemma. Assume that \( \Gamma^{min}(\mathbb{C}^2, f) \) is the minimal embedded resolution graph of the irreducible plane curve singularity \( f \). Then the following facts hold:

- \( M_{v_k} = m_{v_k} \) \((0 \leq k \leq s)\) \((\text{see } 2.3(2) \text{ for } m_{v_k})\)
- \( M_{v_k} = p_{k+1} \cdots p_s \) \((1 \leq k \leq s - 1)\)
- \( M_{v_1} = 1 \).

For any \( 1 \leq k \leq s \), fix two integers \( i_k \) and \( j_k \) with \( a_ki_k + p_kj_k = 1 \). Then the multiplicities of the three vertices from the set \( V_{v_k} \), modulo \( m_{v_k} = a_kp_{k+1} \cdots p_s \), are:

- \(-i_ka_kp_{k+1} \cdots p_{s}; -j_kp_{k+1} \cdots p_s \), and \( p_{k+1} \cdots p_s \) \((k \leq s - 1)\)
- \(-i_ka_k; -j_kp_s \) \(\text{ and } 1 \) \((k = s)\).

Proof. The first identity follows from \[2.5(3)\]. For the other statement see \[32\] or the proof of \((3.2)\) in \[22\].

Using this, the graph \( \Gamma^{can}(X_{f,n}, z) \) has the following data:

4.2. Corollary. (cf. also with \[28\].) Let \( q : \Gamma^{can}(X_{f,n}, z) \to \Gamma^{min}(\mathbb{C}^2, f) \) be the “graph projection” considered in the algorithm \[7.3\]. Then
(a) For any \( f \) and \( n \), \( \Gamma^{can}(X_{f,n}, z) \) is a tree with

- \( \#q^{-1}(v_s) = 1 \)
- \( \#q^{-1}(\bar{v}_s) = h_{s+1} \cdots h_s \) \((1 \leq k \leq s - 1)\)
- \( \#q^{-1}(\bar{v}_0) = h_s \).

(b)

- \( m_{v_0} = p'_1p'_2 \cdots p'_s \)
- \( m_{v_k} = a'_{k}p'_k \cdots p'_s \) \((1 \leq k \leq s - 1)\)
- \( m_{v_{s}} = a'_s \)
- \( m_{v_k} = a'_k p_k p_{k+1} \cdots p'_s \) \((1 \leq k \leq s)\).
$$g_{vk} = 0 \quad (0 \leq k \leq s)$$
$$g_{vk} = (h_k - 1)(\tilde{h}_k - 1)/2 \quad (1 \leq k \leq s).$$

In particular, the link of $X_{f,n}$ is a rational homology sphere if and only if $(h_k - 1)(\tilde{h}_k - 1) = 0$ for any $1 \leq k \leq s$ (cf. 2.7).

Proof. Use 3.3 and 4.1. For the last case in (c) notice that $2 - 2g_{v'}$ (with $g_{v'} = g_{vk}$) equals $-(a_kp_k, n/d_k) + (a_k, n/d_k) + (p_k, n/d_k) + 1$, hence the identity follows from the definition of $h_k$ and $\tilde{h}_k$ and $(a_k, p_k) = 1$. Notice that the fact that $\Gamma_{\text{can}}(X_{f,n}, z)$ is a tree follows also from Durfee’s theorem [8], since the algebraic monodromy of $f$ has finite order [20].

By the above discussions, the graph $\Gamma_{\text{can}}(X_{f,n}, z)$ has the following schematic form (where the dash-lines replace strings as above, and we omit the genera and the self-intersections):

4.3. Example. Assume that $s = 1$ and write $p = p_1$ and $a = a_1$. Take $n$ such that $h = (p, n)$ and $\tilde{h} = (a, n) = 1$. Then $X_{f,n}$ can be identified with the Brieskorn hypersurface singularity $\{(x, y, z) \in \mathbb{C}^3 : x^a + y^p + z^n = 0\}$. Then the link is a Seifert 3-manifold with Seifert invariants: $a, a, \cdots, a, p/h, n/h$ ($a$ appearing $h$ times, hence all together there are $h + 2$ special fibers corresponding to the $h + 2$ arms, cf. the above graph-diagram). These numbers also give (up to a sign) the determinants of the corresponding arms of the graph $\Gamma(X_{f,n})$. (For more details about Seifert manifolds and their plumbings, see e.g. [16, 36], or [27], section 6 for this special case; see also 6.1 below).

4.4. The maximal strings of $\Gamma_{\text{can}}(X_{f,n}, z)$. The next goal is to compute the determinants of the maximal strings of $\Gamma_{\text{can}}(X_{f,n}, z)$. For this, fix a vertex $w \in \mathcal{W}^*(\Gamma_{\text{min}}(\mathbb{C}^2, f))$
On the link of \( \{ f(x, y) + z^n = 0 \} \)
and \( v' \in q^{-1}(w) \). Consider the shortest path in \( \Gamma^{\text{can}}(X_{f, n}, z) \) which connects \( v' \) and the arrowhead.

If \( w \neq v_s \), then on this path there is at least one rupture vertex of \( \Gamma^{\text{can}}(X_{f, n}, z) \). Let \( v'' \) (\( v'' \neq v' \)) be the closest one to \( v' \). If \( w = v_k \) (\( 1 \leq k \leq s - 1 \)), then let \( \Gamma(v') \) be the string which contains all the vertices between \( v' \) and \( v'' \) (excluding \( v' \) and \( v'' \)), and all the edges connecting them. If \( w = \hat{v}_k \) (\( 1 \leq k \leq k \)), then \( \Gamma(v') \) is the string constructed similarly, but at this time we include \( v' \) and its connecting edge as well. If \( w = v_s \), then the above path is already a string. Let \( \Gamma(v') \) be the string which contains all the vertices between \( v' \) and the arrowhead (excluding \( v' \)), and all the edges connecting them.

In this way we have a codification of all the maximal strings of \( \Gamma^{\text{can}}(X_{f, n}, z) \). Notice also that the isomorphism type of the string \( \Gamma(v') \) does not depend on the choice of \( v' \in q^{-1}(w) \), but only on \( w \). Therefore, sometimes it is preferable to denote this type by \( \Gamma(w) \). Denote by \( D(v') \) (or by \( D(w) \)) the determinant \( \det(\Gamma(v')) \). If \( \Gamma(w) = \emptyset \), then by definition \( D(w) = 1 \) (cf. 2.5(1), see also 4.6).

4.5. Proposition. Consider the vertices \( w \in W^*(\Gamma^{\text{min}}(\Gamma^2, f)) \) as above. Then \( D(w) \) has the following values:

\[
\begin{align*}
D(\tilde{v}_0) &= a_1^1 \\
D(\tilde{v}_k) &= p_k^k & (1 \leq k \leq s) \\
D(v_k) &= n/(h_1\tilde{h}_s) \\
D(v_{k+1}) &= n \cdot q_{k+1}^k/(d_{k-1}\tilde{h}_k\tilde{h}_{k+1}) & (1 \leq k \leq s - 1)
\end{align*}
\]

Proof. We start with the ("difficult") case \( D(v_k) \) (\( 1 \leq k \leq s - 1 \)). Using the notations of 2.8, the maximal string in \( \Gamma^{\text{min}}(\Gamma^2, f) \) between \( v_k \) and \( v_{k+1} \) has the following form:

\[
(m_{v_k}) = \begin{pmatrix}
-u_{k+1}^1 & -u_{k+1}^2 & \cdots & -u_{k+1}^{l_{k+1}} \\
\end{pmatrix} \quad (m_{v_{k+1}})
\]

where \( m_{v_k} = a_kp_k \cdots p_s \) and \( m_{v_{k+1}} = a_{k+1}p_{k+1} \cdots p_s \), cf. 2.8(2). Moreover, \( p_{k+1}/q_{k+1} = u_k^0 - \lambda/q_k \), and the quotient \( q_k \) gives the continued fraction \([u_k^1, \cdots, u_{k+1}^{l_{k+1}}] \). This can be identified with the string \( St(a, b, N) \) in the description of the model \( X(a, b, N) \) in 2.11. By this identification \( a = 1 \), \( b = p_{k+1} \), and \( \lambda = q_k \). Therefore, the above string (without arrowheads) is the graph of the normalization of \( \{ z^{p_{k+1}}y \}^{p_{k+1} \cdots p_s} \). In this model, reading the multiplicities of the arrowheads associated with the coordinate functions, one gets for them 1 and \( p_{k+1} \) in the case \( z \), and 0 and \( q_k \) in the case \( y \). Therefore, \( m_{v_k} \) and \( m_{v_{k+1}} \) are the arrow-multiplicities of \( \{ z^{p_{k+1}}y \}^{p_{k+1} \cdots p_s} \). In particular, the collection of graphs \( \{ \Gamma(v') \}_{v' \in q^{-1}(v_k)} \) is the (non-connected) graph of the normalization of

\[
X = \{ (x, y, z, w) : z^{q_{k+1}} = x y^{p_{k+1}} ; \ w^n = (z^{p_k}a_k y)^{p_{k+1} \cdots p_s} \}.
\]

\( X \) has \( d_k = (n, p_{k+1} \cdots p_s) \) (isomorphic) irreducible components, number which agrees exactly with \( \# q^{-1}(v_k) \). Hence, \( D(v') \) is the graph of the normalization of

\[
X_1 = \{ z^{q_{k+1}} = x y^{p_{k+1}} ; \ w^n/d_k = (z^{p_k}a_k y)^{p_{k+1} \cdots p_s}/d_k \}.
\]

Then apply 2.13 for \( q = q_{k+1}, \ p = p_{k+1}, \ N = n/d_k, \ \tau = p_k a_k, \ P = p_{k+1} \cdots p_s/d_k \) and \( a = a_{k+1} \).

The other identities can be computed by a similar argument. But also notice that in all the other cases the corresponding maximal string contains a leaf vertex of \( \Gamma^{\text{can}}(X_{f, n}) \).
Therefore, $D(v')$ can be identified with the corresponding Seifert invariant, similarly as in \[4.3\] Then the first three identities also follow from \[4.3\] \[\square\]

4.6. **Remark.** In $\Gamma^{can}(X_{f,n}, z)$ the following hold:

1. If $w = \tilde{v}_k$ ($0 \leq k \leq s$) then $\Gamma(w) \neq \emptyset$. Indeed, $\Gamma(w)$ contains at least as many vertices as the corresponding arm in $\Gamma^{min}(\mathbb{C}^2, f)$, which is clearly not empty.

2. The same argument is valid for any $\Gamma(v_k)$ ($1 \leq k \leq s - 1$) provided that $q_{k+1} > 1$. In fact, for such $w = v_k$, $\Gamma(v_k) = \emptyset$ if and only if $q_{k+1} = 1$ and $n = d_k h_k h_{k+1}$.

3. $\Gamma(v_s) = \emptyset$ if and only if $n = h_s h_s$.

Here appears a natural question: is it possible to distinguish the arms $\Gamma(v')$ ($v' \notin q^{-1}(\tilde{v}_0)$) from the arms of type $\Gamma(v')$ ($v' \in q^{-1}(\tilde{v}_0)$)? The next corollary says that if $g_{v_1} = 0$ then already their determinants are different:

4.7. **Corollary.** (a) If $D(\tilde{v}_0) = D(\tilde{v}_1)$ then $D(\tilde{v}_0) = D(\tilde{v}_1) = 1$ and $g_{v_1} \neq 0$.

(b) If $D(\tilde{v}_s) = D(\tilde{v}_a)$ then $D(\tilde{v}_s) = D(\tilde{v}_a) = 1$.

*Proof.* (a) If $D(\tilde{v}_0) = D(\tilde{v}_1)$, then $a_1/h_1 = p_1/h_1$ by \[4.3\]. Since $(a_1, p_1) = 1$, one gets $a_1/h_1 = p_1/h_1 = 1$. But then $h_1 \geq 2$ and $h_1 \geq 2$ since $a_1 = q_1 > p_1 \geq 2$. (b) Similarly by \[4.5\] one has $p_s/h_s = n/h_s h_s$. But this two numbers are also relatively prime. \[\square\]

4.8. **The subgraphs** $\Gamma_{\pm}(v_k)$. Above we discussed the case of maximal strings of $\Gamma^{can}(X_{f,n}, z)$. Obviously, one can consider the determinants of much bigger subgraphs delimited by different rupture vertices. In this way one obtains a large number of rather subtle invariants of this graph. Nevertheless, in order to recover the Newton pairs of $f$ and the integer $n$ from this graph, it is enough to consider only a restrictive sub-family of them.

Let us fix an integer $k$ ($1 \leq k \leq s$). Consider the maximal subgraph of $\Gamma^{can}(X_{f,n}, z)$ which does not contain any vertex from the set $q^{-1}(v_k)$. It has many connected components. The component which supports the arrowhead of $\Gamma^{can}(X_{f,n}, z)$ is denoted by $\Gamma_+(v_k)$. There are $h_k h_{k+1} \cdots h_s$ more components (isomorphic to each other), which contain vertices above $\tilde{v}_k$. They are strings of type $\Gamma(\tilde{v}_k)$ (cf. \[4.2\]a) and \[4.4\]). Finally, there are $h_k \cdots h_s$ isomorphic components containing vertices above $\tilde{v}_k$. We denote such a component by $\Gamma_-(v_k)$. $D_{\pm}(v_k)$ denotes $\det(\Gamma_{\pm}(v_k))$.

Obviously, $\Gamma_-(v_1) = \Gamma(\tilde{v}_0)$, and $\Gamma_+(v_s) = \Gamma(v_s)$ whose determinants are computed in \[4.5\].

4.9. **Proposition.** Assume that $s \geq 2$.

(a) $D_-(v_2) = (a'_1)^{h_1-1} \cdot (p'_1)^{h_1-1} \cdot a'_2$.

(b) $D_+(v_{s-1}) = n \cdot D(v_{s-1})^{h_{s-1}} \cdot D(\tilde{v}_s)^{h_{s-1}/(h_s h_s h_{s-1} h_{s-1})}$.

*Proof.* (a) Fix a vertex $v' \in q^{-1}(v_2)$ and one of the graphs $\Gamma_-(v')$. Let $w_1$ be its unique rupture point, and $w_2$ denote that vertex which was connected by an edge with $v'$ in $\Gamma^{can}(X_{f,n}, z)$. (If $\Gamma(v_1) = \emptyset$ then $w_1 = w_2$, but the proof is valid in this case as well.) We put back on the vertices of $\Gamma_-(v')$ the multiplicities of $\Gamma^{can}(X_{f,n}, z)$. They will form a compatible set (i.e. will satisfy \[2.5\]3)) provided that we put on $w_2$ an arrow with multiplicity $m_{v'} = m_{w_2}$. This graph with arrowhead and multiplicities has the following schematic form:
On the link of \( \{ f(x, y) + z^n = 0 \} \)

Notice that \( \Gamma_-(v') \setminus \{ w_1 \} \) has \( h_1 + \tilde{h}_1 + 1 \) connected components, \( h_1 \) of type \( \Gamma(\tilde{v}_0) \), \( \tilde{h}_1 \) of type \( \Gamma(\tilde{v}_1) \), and one of type \( \Gamma(v_1) \). Therefore, by 2.3(3) one gets:

\[
\frac{m_{v_1}}{m_{v_2}} = -I_{w_1 w_2} = \frac{D(\tilde{v}_0)^{h_1} \cdot D(\tilde{v}_1)^{\tilde{h}_1}}{D_{-}(v_2)}.
\]

Now, use 4.2(b) and 4.3. For part (b) we proceed similarly, but now with the graph \( \Gamma_+(v_{s-1}) \). Its schematic form, together with the multiplicities of \( \Gamma_{\text{can}}(X_{f,n}, z) \), is

\[
\begin{align*}
(m_{v_{s-1}}) & \quad \vdots \quad (m_{v_s}) \\
(m_{v_{s+1}}) & \quad \vdots \quad (m_{v_s}) \\
\vdots & \quad \vdots \\
\end{align*}
\]

If from this graph we delete its rupture point (and the arrows and multiplicities) then we get the following connected components: \( h_s \) of type \( \Gamma(v_{s-1}) \), one of type \( \Gamma(v_s) \), and \( \tilde{h}_s \) of type \( \Gamma(\tilde{v}_s) \). Therefore, from 2.3(3), similarly as above, one gets:

\[
m_{v_s} = \frac{D(v_{s-1})^{h_s} \cdot D(\tilde{v}_s)^{\tilde{h}_s}}{D_{+}(v_{s-1})} + h_s \cdot \frac{D(v_{s-1})^{h_s-1} \cdot D(\tilde{v}_s)^{\tilde{h}_s} \cdot D(v_s)}{D_{+}(v_{s-1})} \cdot m_{v_{s+1}}.
\]

Then use again 4.2(b) and 4.3 (and \( a_s = q_s + p_s p_{s-1} a_{s-1} \)).

4.10. Remark. The above formulae and proofs can be easily generalized for the other subgraphs as well. For example, for \( \{ D_{-}(v_k) \}_{k \geq 2} \) one can prove (by computing \( m_{v_{k-1}} / m_{v_k} \) by the above method) the following inductive formula:

\[
\frac{D_{-}(v_k)}{a_k} = \left[ \frac{D_{-}(v_{k-1})}{a_{k-1}} \right]^{h_{k-1}} \cdot (a_{k-1})^{h_{k-1}-1} \cdot (p_{k-1})^{h_{k-1}-1}.
\]

5. From \( \Gamma_{\text{can}}(X_{f,n}, z) \) to \( \Gamma_{\text{min}}(X_{f,n}) \)

Let \( \Gamma_{\text{min}}(X_{f,n}, z) \) be the minimal embedded resolution graph of \( (X_{f,n}, z) \). This can be obtained from \( \Gamma_{\text{can}}(X_{f,n}, z) \) by a sequence of blow downs (and without any blow up).

5.1. Proposition. All the rupture vertices of \( \Gamma_{\text{can}}(X_{f,n}, z) \) survive in \( \Gamma_{\text{min}}(X_{f,n}, z) \) as rupture vertices (i.e. they are not blown down in the minimalization procedure, and in \( \Gamma_{\text{min}}(X_{f,n}, z) \) they still live as rupture vertices).

Proof. From 2.3 follows that a string of type \( \Gamma(\tilde{v}_k) \) \((0 \leq k \leq s)\) is completely collapsed in the minimalization procedure if and only if its determinant \( D(\tilde{v}_k) \) equals 1.

First we verify that all the rupture vertices above \( v_1 \) will survive (as rupture vertices). Let \( v' \) be one of them considered in \( \Gamma_{\text{can}} \). It supports \( h_1 \) strings of type \( \Gamma(\tilde{v}_0) \), \( \tilde{h}_1 \) strings of type \( \Gamma(\tilde{v}_1) \) and another edge, denoted by \( e \). Recall that \( D(\tilde{v}_0) = a'_1 \) and \( D(\tilde{v}_k) = p'_k \), cf.
By the 4.7, if both $D(\tilde{v}_0)$ and $D(\tilde{v}_1)$ equal one, then $g_{v'} \neq 0$. Hence $v'$ will be a rupture vertex in $\Gamma_{min}(X_{f,n}, z)$.

If $D(\tilde{v}_0) \neq 1$ but $D(\tilde{v}_1) = p_1/h_1 = 1$ then the strings of type $\Gamma(\tilde{v}_0)$ will survive. Their number is $h_1 = p_1 \geq 2$. Symmetrically, if $D(\tilde{v}_1) \neq 1$ but $D(\tilde{v}_0) = a_1/\tilde{h}_1 = 1$ then $\tilde{h}_1 = a_1 \geq 2$ strings of type $\Gamma(\tilde{v}_1)$ will survive. If both determinants are greater than one, then all the strings will survive with total number $h_1 + \tilde{h}_1 \geq 2$. Since the arrowhead survives, and $\Gamma_{min}(X_{f,n}, z)$ is connected, the edge $e$ will survive as well. Hence $v'$ has degree at least three in $\Gamma_{min}(X_{f,n}, z)$.

By induction, we assume that for a fixed $k$, all the rupture vertices above any $v_i \in W^s$ survive for any $i \leq k - 1$. We show that this is the case for $v_k$ as well. For this, fix an arbitrary $v' \in q^{-1}(v_k)$.

First notice that by the inductive step, the $h_k$ subgraphs $\Gamma_{-}(v')$ will survive (they cannot be completely contracted since they contain rupture points that survive). Similarly as above, since the arrowhead survives, the edge connecting $v'$ with $\Gamma_{+}(v_k)$ will also survive. If $D(\tilde{v}_k) = 1$, then $h_k = p_k \geq 2$. If $D(\tilde{v}_k) \neq 1$, then all the graphs $\Gamma(\tilde{v}_k)$ will survive. Hence, in any case $\delta_{v'} \geq 3$ in $\Gamma_{min}(X_{f,n}, z)$.

Now, recall that $\Gamma_{min}(X_{f,n})$ denotes the minimal (good) resolution graph of $(X_{f,n}, 0)$. It can be obtained from $\Gamma_{min}(X_{f,n}, z)$ by deleting its arrowhead (and all the multiplicities) and blowing down successively all the $(-1)$-curves with genus zero and new degree $\leq 2$. In fact, there is exactly one case when after deleting the arrowhead of $\Gamma_{min}(X_{f,n}, z)$ we do not obtain a minimal graph, and this is described completely in the next proposition. In the sequel we refer to this “pathological” situation as the “P-case”.

5.2. Proposition. Assume that by deleting the arrowhead of $\Gamma_{min}(X_{f,n}, z)$ we obtain a non-minimal graph. Then $\Gamma_{min}(X_{f,n}, z)$ has the following schematic form with the two left branches isomorphic and with $e \leq -3$ (we omit the multiplicities). The rational $(-1)$-curve is the unique vertex $v' = q^{-1}(v_s)$ (which survives in $\Gamma_{min}(X_{f,n}, z)$, cf. 5.4).

\begin{center}
\begin{tikzpicture}
  \node (e) at (-0.5, 0) {$e$};
  \node (v') at (0, 0) {$v'$};
  \node (v) at (0.5, 0) {$v_s$};
  \draw[->] (e) -- (v');
  \draw[->] (v') -- (v);
\end{tikzpicture}
\end{center}

This situation can happen if and only if $n = p_s = 2$.

In this case, $\Gamma_{min}(X_{f,n})$ is obtained from $\Gamma_{min}(X_{f,n}, z)$ by deleting its arrowhead and blowing down $v'$. No other blow downs are necessary.

Moreover, in this case, all the vertices of $\Gamma_{min}(X_{f,n}, z)$ have genus zero.

Proof. If the graph obtained from $\Gamma_{min}(X_{f,n}, z)$ by deleting its arrowhead is not minimal, then the vertex $v'$ in $\Gamma_{min}(X_{f,n}, z)$ which supports the arrowhead should be a $(-1)$ rational curve of degree 3 in $\Gamma_{min}(X_{f,n}, z)$. This can happen only if this vertex $v'$ is exactly the unique vertex $q^{-1}(v_s)$ (and $g_{v_s} = 0$). This also shows that $\Gamma(v_s)$ was collapsed in $\Gamma_{min}(X_{f,n}, z)$, hence $D(v_s) = n/\tilde{h}_s = 1$. Hence:

$$h_s\tilde{h}_s = n \geq 2 \quad \text{and} \quad (h_s - 1)(\tilde{h}_s - 1) = g_{v_s} = 0.$$ 

Assume first that $h_s = 1$ and $\tilde{h}_s > 1$. Since $D(\tilde{v}_s) = p_s > 1$, the $\tilde{h}_s$ strings $\Gamma(\tilde{v}_s)$ are present in $\Gamma_{min}(X_{f,n}, z)$. This can happen if and only if $\tilde{h}_s = 2$ and $\Gamma_{-}(v_s)$ is collapsed completely in $\Gamma_{min}(X_{f,n}, z)$. Since for $s \geq 2$ the rupture points $q^{-1}(v_1)$ survive in $\Gamma_{min}(X_{f,n}, z)$, this
On the link of \( \{ f(x, y) + z^n = 0 \} \)
can happen only if \( s = 1 \) and \( D(\tilde{v}_0) = a_1/\tilde{h}_1 = 1 \). This shows that \( a_1 = \tilde{h}_1 = 2 \), which
contradicts the inequality \( a_1 = q_1 > p_1 \geq 2 \).

Therefore \( \tilde{h}_s = 1 \) and \( h_s > 1 \). Since the degree of \( v' \) (in \( \Gamma_{min}(X_{f,n}, z) \)) is at least \( 1 + h_s \)
(hence \( 1 + h_s \leq 3 \)), one gets \( h_s = 2 \), and also the fact that the graphs of type \( \Gamma(\tilde{v}_s) \) are
collapsed in \( \Gamma_{min}(X_{f,n}, z) \), hence \( p_s/h_s = 1 \). Therefore, \( n = h_s = p_s = 2 \).

Then \( \tilde{h}_k = 1 \), hence \( g_{v_k} = 0 \) for any \( k \).

Finally, notice that \( e \leq -3 \) (cf. the diagram) since after we blow down \( v' \) we get a
subgraph of type \( e + 1 \) which must be negative definite.

On the other hand, using the algorithm described in sections 3 and 4, one can verify easily that if \( n = p_s = 2 \) then the above situation always occurs.

5.3. **Remark.** Assume that \( g_{v_k} = 0 \). Then if a family of strings supported by any fixed \( v' \in q^{-1}(v_k) \) is collapsed completely during the minimalization procedure, then the cardinality of this family (in spite of the fact that it is missing in \( \Gamma_{min}(X_{f,n}) \)) can be determined, and it is one. More precisely, if for \( k \geq 1 \), the \( \tilde{h}_k \) graphs of type \( \Gamma(\tilde{v}_k) \) are completely collapsed, then \( \tilde{h}_k = p_k \geq 2 \). Then by the genus-formula \( G_2(c) \) one gets \( \tilde{h}_k = 1 \). Similarly, if \( k = 1 \) and the \( h_1 \) graphs of type \( \Gamma(\tilde{v}_0) \) are collapsed, then \( h_1 = 1 \).

In order to recover the Newton pairs of \( f \) and the integer \( n \) from the graph \( \Gamma_{min}(X_{f,n}) \), we
need some information about some subgraphs \( \mathcal{G} \) of \( \Gamma_{min}(X_{f,n}) \) of the following type. Each
\( \mathcal{G} \) is a connected component of \( \Gamma_{min}(X_{f,n}) \setminus \{ v \} \) for some rupture vertex \( v \) of \( \Gamma_{min}(X_{f,n}) \), and it contains exactly one rupture vertex of \( \Gamma_{min}(X_{f,n}) \). In a general setting their precise definition is the following. [We recall that the determinant of a (decorated) graph is the
determinant of the negative of its intersection matrix (cf. [2:3].)]

5.4. **Definitions.** Let \( \Gamma \) be a decorated tree (with self-intersections and genera \( \{ g_v \}_v \), without
arrowheads and multiplicities). Assume that it has at least two rupture vertices.

1. Let \( \Gamma(\mathcal{R}) \) be the minimal connected subgraph of \( \Gamma \) which contains all the rupture
vertices \( \mathcal{R} \) of \( \Gamma \). Let \( \mathcal{L}(\Gamma(\mathcal{R})) \) be the set of leaf vertices of \( \Gamma(\mathcal{R}) \). For any \( v \in \mathcal{L}(\Gamma(\mathcal{R})) \)
let \( \mathcal{G}(v) \) be the maximal connected subgraph of \( \Gamma \) which contains \( v \) but contains no other
rupture vertex of \( \Gamma \). The determinant \( \text{det}(\mathcal{G}(v)) \) is denoted by \( D(v) \).

2. For any \( v \in \mathcal{L}(\Gamma(\mathcal{R})) \), let \( v_{root} \) be the unique rupture vertex of \( \Gamma \) with the property that
on the shortest path in \( \Gamma \) connecting \( v \) and \( v_{root} \) there are no other rupture vertices of \( \Gamma \). Then clearly \( v_{root} \) is adjacent with a certain vertex of \( \mathcal{G}(v) \) (in fact \( \mathcal{G}(v) \) is one of the
connected components of \( \Gamma \setminus \{ v_{root} \} \)).

3. For each rupture vertex \( v \in \mathcal{R} \), denote by \( St(v) \) the set of maximal strings of \( \Gamma \) which are
supported by \( v \) (on one end) and contain a leaf vertex of \( \Gamma \) (on the other end). More
precisely, these strings are those connected components of \( \Gamma \setminus \{ v_{root} \} \) which have an adjacent vertex with \( v \) (in \( \Gamma \)). We write \( St(v) \) as a disjoint union of its subsets \( \{ St_i(v) \}_{i \in I(v)} \) which are
the level sets of \( \text{det} : St(v) \to \mathbb{Z} \). We set \( D_i := \text{det}(St) \) for \( St \in St_i(v) \) and \( \#_i := \# St_i(v) \). Then we define

\[
D_{St}(v) := \left\{ \prod_{i \in I(v)} D_i^{#_i} \right\} \text{ if } St(v) \neq \emptyset \\
1 \quad \text{ if } St(v) = \emptyset,
\]

and \( \alpha(v) \in \mathbb{Q} \cup \{ \infty \} \) by

\[
\alpha(v) = \left\{ \prod_{i \in I(v)} \frac{#_i}{2g_v} + 1 \right\} \text{ if } St(v) = \emptyset \text{ and } g_v = 0 \\
\prod_{i \in I(v)} \frac{#_i}{2g_v} + 1 \quad \text{ if } St(v) = \emptyset \text{ and } g_v \neq 0.
\]
\[\alpha(v) = \infty \text{ if and only if the degree } \delta_v \text{ of } v \text{ in } \Gamma \text{ is } 2, \text{ St}(v) = \emptyset \text{ and } g_v \neq 0.\]

(4) For each \(v \in \mathcal{L}(\Gamma(\mathcal{R}))\) we define the \(\beta\)-invariant by

\[\beta(v) := \frac{D(v)}{D_{\text{St}}(v)} \cdot \frac{\alpha(v_{\text{root}})}{\alpha(v)}.\]

5.5. In the next paragraphs we apply these definitions for \(\Gamma = \Gamma^{\text{min}}(X_{f,n})\). Here, we prefer to regard \(\Gamma^{\text{min}}(X_{f,n})\) together with \(\Gamma^{\text{can}}(X_{f,n}, z)\), as a minimalization of \(\Gamma^{\text{can}}(X_{f,n}, z)\). In particular, we will define subsets, subgraphs, etc. in \(\Gamma^{\text{min}}(X_{f,n})\) as the images of well-defined subsets, subgraphs, etc. of \(\Gamma^{\text{can}}(X_{f,n}, z)\) by the minimalization procedure. (Of course, in the next section will be a crucial task to recover some of these sets only from the abstract graph \(\Gamma^{\text{min}}(X_{f,n})\). The key result for this is the next 5.7.)

In order to avoid any confusion, for any subset of vertices of \(\Gamma^{\text{can}}(X_{f,n}, z)\), we will denote by \(\pi(A)\) the image of \(A\) by the minimalization procedure. Hence, \(\pi(A)\) denotes those vertices of \(\Gamma^{\text{min}}(X_{f,n})\) which have ancestors in \(A\), and survive in \(\Gamma^{\text{min}}(X_{f,n})\); in some cases this set can be empty.

The following facts follow easily from the structure-results proved in section 4 and the above propositions 5.1 and 5.2.

5.6. Facts. Assume that \(\Gamma = \Gamma^{\text{min}}(X_{f,n})\) with \(s \geq 2\). Then the following hold:

(a) The set \(\mathcal{L}(\Gamma(\mathcal{R}))\) is the disjoint union of two sets \(\mathcal{R}_1\) and \(\mathcal{L}\mathcal{R}_s\), where

(i) \(\mathcal{R}_1 := \pi(q^{-1}(v_1))\);

(ii) \(\mathcal{L}\mathcal{R}_s := \emptyset\) if \(h_s > 1\). Otherwise \(\mathcal{L}\mathcal{R}_s := \pi(q^{-1}(v_s))\), the image by the minimalization procedure of the unique rupture vertex of \(\Gamma^{\text{can}}(X_{f,n}, z)\) sitting above \(v_s\). (In both cases, by 5.1 and 5.2, these sets are subsets of the rupture vertices of \(\Gamma^{\text{min}}(X_{f,n})\).)

(b) The subgraphs \(G(v)\) for \(v \in \mathcal{L}(\Gamma(\mathcal{R})) = \mathcal{R}_1 \cup \mathcal{L}\mathcal{R}_s\) (cf. part (a)) can be identified as follows:

(i) Assume that we are not in the “P-case” with \(s = 2\). For each \(v \in q^{-1}(v_1)\) consider the unique subgraph of type \(\Gamma_-(v_{\text{root}})\) in \(\Gamma^{\text{can}}(X_{f,n}, z)\), for some \(v_{\text{root}} \in q^{-1}(v_2)\), which contains \(v\). Then its image in \(\Gamma^{\text{min}}(X_{f,n})\) by the minimalization procedure is \(G(v)\).

(ii) Assume that \(h_s = 1\). For \(v = q^{-1}(v_s)\) consider \(\Gamma_+(v_{\text{root}})\) in \(\Gamma^{\text{can}}(X_{f,n}, z)\) with \(v_{\text{root}} := q^{-1}(v_{s-1})\). Then its image in \(\Gamma^{\text{min}}(X_{f,n})\) by the minimalization procedure is \(G(v)\).

(c) \(\alpha\) and \(\beta\) are constant on \(\mathcal{R}_1\).

(The motivation for the notation \(\mathcal{L}\mathcal{R}_s\) is the following: later we will use the symbol \(\mathcal{R}_s\) for \(\pi(q^{-1}(v_s))\); hence \(\mathcal{L}\mathcal{R}_s = \mathcal{R}_s\) if \(\pi(q^{-1}(v_s))\) is a “leaf rupture vertex”, otherwise it is empty.)

The main point is that in (a), the cases (i) and (ii) can be distinguished by the genus and \(\beta\)-invariant.

5.7. Proposition. Assume that \(\Gamma = \Gamma^{\text{min}}(X_{f,n})\) with \(s \geq 2\).

(a) If there exists at least one \(v \in \mathcal{L}(\Gamma(\mathcal{R}))\) with \(g_v \neq 0\), then \(\mathcal{R}_1 = \{v \in \mathcal{L}(\Gamma(\mathcal{R})) : g_v \neq 0\}\) and \(\mathcal{L}\mathcal{R}_s = \{v \in \mathcal{L}(\Gamma(\mathcal{R})) : g_v = 0\}\) (\(\mathcal{L}\mathcal{R}_s\) can be empty).

(b) If \(g_v = 0\) for any \(v \in \mathcal{L}(\Gamma(\mathcal{R}))\) and \(\mathcal{L}\mathcal{R}_s \neq \emptyset\), then \(\beta(v) \in (0, \infty)\) and

\[
\begin{align*}
\beta(v) &> 2 & \text{if } v \in \mathcal{R}_1, \\
\beta(v) &\leq 1/2 & \text{if } v \in \mathcal{L}\mathcal{R}_s.
\end{align*}
\]

Proof. (a) \(g_v\) is constant on \(\mathcal{R}_1\) and \(g_v = 0\) for (the unique) \(v \in \mathcal{L}\mathcal{R}_s\) provided that \(\mathcal{L}\mathcal{R}_s \neq \emptyset\), since in this case \(h_t = 1\) (cf. 4.2).
On the link of \( \{ f(x,y) + z^n = 0 \} \)

Now we prove (b). Since \( h_s = 1 \) we can exclude the “P-case”.

First assume that \( s \geq 3 \).

If \( v \in \mathcal{R}_1 \) then by \( 4.4 \) and \( 5.3 \) one has \( \alpha(v) = h_1 h_1 \). For \( v_{\text{root}} \), analyzing the three different cases from the definition of \( \alpha(v_{\text{root}}) \), and using \( 5.3 \) and the genus formula, we get \( \alpha(v_{\text{root}}) = \tilde{h}_2 \). On the other hand, \( \mathcal{D}(v) = (d'_1)^{h_1-1} (p'_1)^{h_1-1} \) (cf. \( 4.3(\text{b}) \)) and \( \mathcal{D}_{St}(v) = (d'_1)^{h_1} (p'_1)^{h_1} \) (use \( 4.3 \) and notice that if a string is collapsed completely then its determinant is one).

Therefore, \( \beta(v) = a_2/(a_1 p_1) > p_2 \geq 2 \), cf. \( 2.8(\text{I}) \).

If \( v = \pi(q^{-1}(v_s)) \) then \( \alpha(v) = \tilde{h}_s \) (use \( 5.3 \) and \( 4.7 \)). By similar argument as above, \( \alpha(v_{\text{root}}) = \tilde{h}_s \). By \( 4.3(\text{b}) \) and \( h_s = 1 \) one has \( \mathcal{D}(v) = \mathcal{D}(v_{s-1}) = n(p'_s)^{h_s-1}/(h_s h_s-1) \). By \( 4.3 \) \( \mathcal{D}_{St}(v) = (p'_s)^{h_s} \cdot n/(h_s h_s) \). Therefore, using again \( h_s = 1 \) one gets \( \beta(v) = 1/(h_s-1 p_s) \leq 1/2 \).

Assume that \( s = 2 \) and let \( v'_i = q^{-1}(v_i) \) \((i = 1,2)\). Then \( \alpha(v'_i) = h_i \tilde{h}_i \) \((i = 1,2)\). Hence the computation of \( \beta(v'_i) \) is the same as above, and it gives \( a_2/(a_1 p_1) > 2 \). For \( v'_2 \) we have an additional \( h_1 \) and we get \( \beta(v'_2) = 1/p_2 \leq 1/2 \).

\[ \Box \]

6. From \( \Gamma^{\text{min}}(X_{f,n}) \) Back to \( f \) and \( n \)

Our final goal is to recover the Newton pairs of \( f \) and the integer \( n \) from the graph \( \Gamma^{\text{min}}(X_{f,n}) \). In general, this is not possible. Nevertheless, by our main theorem, there are only two cases when such an ambiguity appears. They are presented in the next subsections.

6.1. Example. The S1-coincidence. Assume that \( (X,0) = (x^{p_1} + y^{p_1} + z^n = 0,0) \) is a Brieskorn singularity with \( (q_1,p_1) = 1 \). Let us first analyze how one can recover the set of integers \( \{q_1,p_1,n\} \) from the minimal resolution graph \( \Gamma \) of \( (X,0) \). In this case, the computation of the graph \( \Gamma \) from the integers \( \{q_1,p_1,n\} \) is a classical, well-known fact (cf. also with our algorithm). The graph is either a string (with all genera zero) or a star-shaped graph (where only the central vertex might have a non-zero genus). If \( \Gamma \) is a string, then \( (X,0) \) is a Hirzebruch-Jung hypersurface singularity. But there is only one family of such singularities, namely the \( A_{q_1-1} \)-singularities provided by the integers \( \{q_1,2,2\} \). In this case, \( q_1 \) is just the determinant of the string.

There is a rich literature of star-shaped graphs and Seifert 3-manifolds, and also of their subclass given by Brieskorn hypersurface singularities. The reader is invited to consult [28], section 3, case (I) (cf. also with [11, 38]).

If one wants to recover the integers \( \{q_1,p_1,n\} \), then one considers the set of strings \( St(v) \) of the central vertex \( v \). Recall \( 5.4(3) \) for the notations. Then \#1(\( v \)) \( \leq 3 \). If \#1(\( v \)) \( = 3 \) then \( \{q_1,p_1,n\} = \{D_1\#2\#3,D_2\#1\#3,D_3\#1\#2\} \). If one \( St_{t_0} \) is missing (empty) then \( D_{t_0} = 1 \) and \#t_0 can determined from the genus of the central vertex (see e.g. our genus formula \( 12(c) \) or \( 12(3.5) \)). Hence the previous procedure still works.

Similarly, in our special situation \( (q_1,p_1) = 1 \), one can show that if two subsets \( St_i \) are empty, then one can still recover \( \{q_1,p_1,n\} \) excepting only one case, namely when \( St(v) \) consists of only one string. In our terminology, this can happen only when the string which supports the arrowhead survives and all the others are contracted (i.e. \( p'_1 = a'_1 = 1 \)). Similar ambiguity appears when \( St(v) = St_\emptyset \).

But all these ambiguity cases can be classified very precisely. Consider an identity of type \( (h_i - 1)(h_i - 1) = 2g > 0 \) and an arbitrary positive integer \( l \). Then the triplet \( \{q_1,p_1,n\} = \{h_1,h_1,h_1l\} \) provides the following graph \( \Gamma \) (with \( l - 1 (-2) \)-vertices):
Now fix $l > 0$ and $g > 0$. Then, different triplets $\{h_1, h_1, h_1 h_1 l\}$ with $(h_1 - 1)(h_1 - 1) = 2g$, $h_1 > 1$ and $h_1 > 1$ provide the same graph. [For example, $(3, 7, 21)$ and $(4, 5, 20)$ provide the same graph consisting of a vertex with $g = 6$ and self-intersection $-1$.]

This is the only coincidence in the case of Brieskorn singularities with $(q_1, p_1) = 1$. Obviously, this cannot happen if $g = 0$.

**Addendum. Relation with the Milnor number.** Notice that in those cases when $\Gamma$ fails to determine the integers $\{q_1, p_1, n\}$, $\Gamma$ together with the Milnor number $\mu$ of the Brieskorn singularity do determine $\{q_1, p_1, n\}$. Indeed, in the “ambiguity cases” one has $(q_1, p_1, n) = (h_1, h_1, h_1 h_1 l)$, where $2g = (h_1 - 1)(h_1 - 1) > 0$ and $l$ are readable from the graph. But $\mu = (h_1 - 1)(h_1 - 1)(h_1 h_1 l - 1) = 2g(h_1 h_1 l - 1)$. This determines $h_1 h_1$, and finally $h_1$ and $\tilde{h}_1$ (using the genus formula).

6.2. **Remark. The “$z$-axis ambiguity”**. Recall that by our general aim, we have to recover the Newton pairs of $f$ and the integer $n$. In the Brieskorn case, after we recover the set $(q_1, p_1, n)$ we have to make a choice for the $z$-axis. Recall that $(p_1, q_1) = 1$. If $k$ integers among of $(p_1, q_1)$, $(p_1, n)$ and $(q_1, n)$ equal 1, then there are $k$ possibilities for the choice of the $z$-axis.

6.3. **Example. The $S_2$-coincidence.** The next coincidence appears when

$$s = 2, \quad a_1' = p_1' = \tilde{h}_2 = 1 \text{ (or equivalently, } s = 2, \quad (n, a_2) = 1 \text{ and } (n, p_2)a_1 p_1 | n \text{).}$$

In this case clearly $q_1 = \tilde{h}_1$ and $p_1 = h_1$, but the $\tilde{h}_1$ strings of type $\Gamma(\tilde{v}_0)$ and the $h_1$ strings of type $\Gamma(\tilde{v}_1)$ are not visible on the minimal graph since their determinants are one, hence they are contracted. The graph $\Gamma^{\min}(X_{f,n})$ has the following schematic form, where $g_{v_1} > 0$ and we omit the self-intersections:

\[
\begin{array}{c}
\text{[}g_{v_1} \text{]}
\end{array}
\]

\[
\begin{array}{c}
\vdots
\end{array}
\]

\[
\begin{array}{c}
\text{[}g_{v_2} \text{]}
\end{array}
\]

\[
\begin{array}{c}
\vdots
\end{array}
\]

\[
\begin{array}{c}
\text{[}g_{v_3} \text{]}
\end{array}
\]

The strings that appear on the right correspond to $\Gamma(v_2)$ and $\Gamma(\tilde{v}_2)$, but in general, we cannot decide which one is which. From the graph we can read $\tilde{h}_2$ and the genus $g_{v_1} = (h_1 - 1)(\tilde{h}_1 - 1)/2$, and of course, a lot of determinants.

Using $\tilde{h}_2$, $\tilde{h}_2$, and $D(v_1)$, $D_-(v_2)$, $D_+(v_1)$ and the set $\{D(v_2), D(\tilde{v}_2)\}$, we can also recover $a_2$, $q_2$, $n/(h_1 h_1)$, $h_1 h_1 p_2$ and the set $\{p_2, n\}$, where we cannot distinguish $p_2$ from $n$.

Notice that once we know $h_1 \tilde{h}_1$, then using the genus formula and $\tilde{h}_1 = q_1 > p_1 = h_1$, we obtain $h_1$ and $\tilde{h}_1$ without any ambiguity, hence (by the above equations) all the data. But for the three “variables” $h_1 \tilde{h}_1$, $p_2$, $n$ we know only the values $n/(h_1 \tilde{h}_1)$, $h_1 h_1 p_2$ and the set $\{p_2, n\}$. This, in general, has two possible solutions (which correspond by a permutation of $p_2$ and $n$). If this is the case, then it might happen that there are two different realizations of the same graph for two different pairs $(f, n)$. But for this, both solutions should provide
On the link of \{f(x, y) + z^n = 0\}

positive integers as candidates for the Newton pairs and \(n\). If this is not happening then the graph is uniquely realized (see Example 3 below).

The complete discussion of all the cases when the above equations which involve \(D(v_1)\), \(D_+(v_2)\), \(D_-(v_1)\) and the set \(\{ D(v_2), D(\bar{v}_2) \}\) associated with the graph provide exactly two “good” solutions for \((f, n)\) is long and tedious, so we decided not to give it here (nevertheless we think that Example 3 illuminates completely the problem). What is important is the fact that any graph (in this family) can be realized by at most two possible pairs \((f, n)\), and this coincidence in some cases really occurs. (Moreover, given a pair \((f, n)\), or the graph of \(X_{f,n}\), one can write down easily the possible candidate for the numerical data of \((f', n')\), the possible pair of \((f, n)\), with the same graph.)

In the next examples we will write \{\(p_1, q_1\), \(p_2, q_2\); \(n\}\} for the Newton pairs of \(f\) and the integer \(n\). Recall that \((\ast)\) implies \(p_1 = h_1\) and \(q_1 = h_1\).

**Example 1.** The two different solutions \((3, 7)\) and \((4, 5)\) for the genus formula \((h_1 - 1)(\tilde{h}_1 - 1) = 2 \cdot 6\) can be completed to the following two sets of invariants: \{\(3, 7\), \(20, 1\); \(21\}\} and \{\(4, 5\), \(21, 1\); \(20\}\}. For them the corresponding two graphs are the same:

\[
\begin{array}{cccccc}
-421 & -1 & -2 & -2 & -2 \\
& & & & & \\
[6] & & & & & \\
& & -21 & & & \\
\end{array}
\]

where the number of \((-2)\)-curves is 19. Here \(h_2 = 1\).

**Example 2.** If one wants examples with arbitrary \(h_2\), then one of the possibilities is the following: one multiplies in the above data (of Example 1) \(p_2\) and \(n\) by the wanted \(h_2\). E.g. the data \{\(3, 7\), \(40, 1\); \(42\}\} and \{\(4, 5\), \(42, 1\); \(40\}\} provide the same \(h_2 = 2\) and the same graph:

\[
\begin{array}{cccccc}
-841 & -1 & -2 & -2 & -2 \\
& & & & & \\
[6] & & & & & \\
& & -21 & & & \\
\end{array}
\]

where again, the number of \((-2)\)-curves is 19.

**Example 3.** Assume the data \{\(p_1, q_1\), \(p_2, q_2\); \(n\}\} of \((f, n)\) satisfies \((\ast)\), hence \(p_1 = h_1\) and \(q_1 = \tilde{h}_1\). If \((f, n)\) has “a pair” \((f_2, n_2)\) (with the same graph) then the data of \((f_2, n_2)\) has the form (cf. the above discussion) \{\(x, y\), \((n, q_2\), \(p_2)\}\}, where \(x\) and \(y\) can be determined by the equations: \(xy/p_2 = p_1q_1/n\) and \((x - 1)(y - 1) = (p_1 - 1)(q_1 - 1)\). It is easy to write down cases when this has no integral solutions.

E.g., the data \{\(2, 3\), \(5, 1\); \(6\}\} satisfies \((\ast)\), but it has no “pair”. Its minimal resolution graph can be realized in a unique way in the form \(f + z^n\) \((f\ irreducible)\) (cf. 3.4).

**Addendum. Relation with the Milnor number.** Even if the same graph is realized by two different pairs \((f_1, n_1)\) and \((f_2, n_2)\), the corresponding Milnor numbers \(\mu_i\) associated with the hypersurface singularities \(f_i + z^n\) \((i = 1, 2)\) distinguish the two cases. This follows from the formula \(\mu = 2g_{v_1}p_2 + (p_2 - 1)(a_2 - 1)(n - 1)\). Since \(g_{v_1} > 0\), \(a_2, np_2\) and \(n + p_2\) are readable from the graph, this relation determines \(p_2\), hence all the numerical data.

Now we are ready to formulate and prove the main result of the article.
6.4. **Theorem.** Let \( f : (\mathbb{C}^2,0) \to (\mathbb{C},0) \) be an irreducible plane curve singularity with Newton pairs \( \{(p_i,q_i)\}_{i=1}^s \) and let \( n \) be an integer \( \geq 2 \). Let \( \Gamma_{\min}(X_{f,n}) \) be the minimal (good) resolution graph of the hypersurface singularity \( (X_{f,n},0) := (\{f(x,y) + z^n = 0\},0) \). Then the following facts hold:

(a) The integer \( s \) is uniquely determined by \( \Gamma_{\min}(X_{f,n}) \).

(b) \( s = 1 \) if and only if \( \Gamma_{\min}(X_{f,n}) \) is either a string (with all the genera zero), or a star-shaped graph (where only the central vertex might have genus \( g \) non-zero). Moreover, \( f(x,y) + z^n \) has the same equisingularity type as the Brieskorn singularity \( x^{q_1} + y^{p_1} + z^n \).

(c) If \( s = 2 \) then it can happen that two pairs \( (f_1,n_1) \) and \( (f_2,n_2) \) (but not more) provide identical graphs \( \Gamma_{\min}(X_{f,n}) \). If this is the case then both of them should satisfy the numerical restrictions:

\[
(n,a_2) = 1, \quad \text{and} \quad (n,p_2)a_1p_1|n. \tag{*}
\]

(which can be recognized from the graph as well), and (automatically) at least one of the vertices has genus \( g > 0 \). This case is described in 6.3.

(d) In all other cases (i.e. for any \( s \geq 3 \) or for \( s = 2 \) excluding the exceptional case \((*)\)), \( \Gamma_{\min}(X_{f,n}) \) determines uniquely the Newton pairs of \( f \) and the integer \( n \) (by a precise algorithm which basically constitutes the next proof).

(e) In particular, except for the two cases \( S1 \) and \( S2 \) (cf. 6.1 and 6.3), from the link one can recover completely the Newton pairs of \( f \) and the integer \( n \) (provided that we disregard the \( z \)-axis ambiguity, cf. 6.2.) In particular, this is true without any exception provided that the link is a rational homology sphere.

On the other hand, in the cases \( S1 \) and \( S2 \), the link together with the Milnor number of the hypersurface singularity \( f + z^n \) determines completely the Newton pairs of \( f \) and the integer \( n \) (cf. the two addendums in 6.1 and 6.3).

**Proof.** We denote \( \Gamma_{\min}(X_{f,n}) \) by \( \Gamma \) and its rupture vertices by \( R \). It is convenient to separate those cases when \#\( R \) is small.

**Case A** Assume that \#\( R = 0 \). By 5.1 the set of rupture vertices of \( \Gamma_{\min}(X_{f,n},z) \) is never empty. Hence, by 5.2 \( R = \emptyset \) if and only if in the “P-case” we contract \( v' \), and \( v' \) is the unique rupture vertex of \( \Gamma_{\min}(X_{f,n},z) \). But \( \Gamma_{\min}(X_{f,n},z) \) has a unique rupture point if and only if \( s = 1 \). Therefore (cf. 5.2) this situation occurs if and only if \( s = 1, p_1 = n = 2 \) and \( (q_1,2) = 1 \); i.e. \( f(x,y) + z^n \) has the equisingularity type of \( x^{q_1} + y^2 + z^2 \). Clearly, \( q_1 \) can be recovered from the graph: it is its determinant. Cf. also with 5.1.

**Case B** Assume that \#\( R = 1 \). From 5.2 is clear that in the “P-case” the number of rupture vertices of \( \Gamma \) is even. Hence, this case is excluded, and the number of rupture vertices of \( \Gamma_{\min}(X_{f,n},z) \) is also 1. By 5.1 this can happen only of \( s = 1 \). In particular, \( f + z^n \) is of Brieskorn type: \( x^{q_1} + y^{p_1} + z^n \), with \( q_1 > p_1 \geq 2 \), \( (p_1,q_1) = 1 \) and \( n \geq 2 \) (where the case \( p_1 = n = 2 \) is excluded, see above). This case is completely covered by 6.7.

**Case C** Assume that \#\( R > 1 \). By 5.1 and 5.2 \#\( R > 1 \) if and only if \( s \geq 2 \). The proof (algorithm) consists of several steps, each step recovers some data.

**1** The set \( R_1 \) can be determined from 5.6 and 5.7. Indeed, we start with the set \( \mathcal{L}(\Gamma(R)) \) (where \( \Gamma = \Gamma_{\min}(X_{f,n}) \)). Then, if there exists at least one \( v \in \mathcal{L}(\Gamma(R)) \) with \( g_v \neq 0 \), then \( R_1 = \{v \in \mathcal{L}(\Gamma(R)) : g_v \neq 0\} \) (cf. 5.7(a)). If \( g_v = 0 \) for all \( v \), then we consider their
On the link of \( \{ f(x, y) + z^n = 0 \} \) \( \beta \)-invariants \( \beta(v) \). If they are all equal, then by \( \square \) one gets \( \mathcal{R}_1 = \mathcal{L}(\Gamma(\mathcal{R})) \). If they are not all equal, then only one can be \( \leq 1/2 \) (corresponding to \( \{ v \} = \mathcal{R}_a \)), and all the others are \( > 2 \) (and equal to each other) corresponding to \( \mathcal{R}_1 \) (cf. \( \square \)).

(2) The sets \( \pi(q^{-1}(v_k)) \) \( (1 \leq k \leq s) \). Define a distance on the set \( \mathcal{R} \). If \( w_{1}, w_{2} \in \mathcal{R} \), and on the shortest path in \( \Gamma \) connecting them there are exactly \( l \) rupture vertices of \( \Gamma \) (including \( w_{1} \) and \( w_{2} \)), then we say that \( d(w_{1}, w_{2}) := l - 1 \geq 0 \). Moreover, for any subset \( \mathcal{R}' \subset \mathcal{R} \) and \( w \in \mathcal{R} \) we define \( d(\mathcal{R}', w) \) as usual by \( \min \{ d(w', w) : w' \in \mathcal{R}' \} \).

Then, for any \( k \geq 1 \), we write \( \mathcal{R}_k := \{ v \in \mathcal{R} : d(\mathcal{R}_1, v) = k - 1 \} \). Let \( s' := \max \{ k : \mathcal{R}_k \neq \emptyset \} \). We distinguish two cases:

(a) \( \#\mathcal{R}_{s'} = 1 \). Then \( s = s' \) and \( \pi(q^{-1}(v_k)) = \mathcal{R}_k \) for \( 1 \leq k \leq s \).

(b) \( \#\mathcal{R}_{s'} > 1 \). This can happen exactly in the “P-case” (cf. \( \square \)). In this situation, \( s = s' + 1, \pi(q^{-1}(v_k)) = \mathcal{R}_k \) for \( 1 \leq k \leq s - 1 \), and the (unique) vertex \( v' = q^{-1}(v_s) \) of \( \Gamma^{\min}(X_{f,n}, z) \) is “missing” in \( \Gamma \), i.e. \( \pi(\{v'\}) = \emptyset \) (cf. \( \square \)).

For a moment we postpone the “P-case”, and we assume (a). We will come back to the “P-case” in (11).

(3) The integers \( \{ h_k \}_{k=2}^{s} \) are determined by the identities \( h_k = \#\mathcal{R}_{k-1}/\#\mathcal{R}_k \) \( (h_1 \) will be determined later.)

(4) The sets \( \{ \pi(q^{-1}(\tilde{v}_k)) \}_{k=2}^{s} \) and the integers \( \{ \tilde{h}_k \}_{k=2}^{s} \) \( \{ p_k \}_{k=2}^{s} \) \( (s \geq 3) \). Fix \( 2 \leq k \leq s - 1 \) and some \( v \in \mathcal{R}_k \). Consider the set of strings \( St(v) \) supported by \( v \) (cf. \( \square \)).

If \( St(v) \neq \emptyset \), then \( \#St(v) = h_k \) and \( det : St(v) \to \mathbb{Z} \) is constant with value \( p_k' \). Then \( p_k = p_k' \cdot h_k \). Then this is happening for any choice of \( v \), and \( \pi(q^{-1}(\tilde{v}_k)) \) is the set of leaf vertices of \( \Gamma \) situated on the strings of type \( \cup \varepsilon St(v), v \in \mathcal{R}_k \).

If \( St(v) = \emptyset \) then all the strings of type \( \Gamma(\tilde{v}_k) \) are collapsed in \( \Gamma \), in particular \( \pi(q^{-1}(\tilde{v}_k)) = \emptyset \). Hence their determinants \( p_k' = p_k/h_k = 1 \). In particular, \( p_k = h_k \geq 2 \). Then \( \tilde{h}_k \) is given by the genus formula \( \tilde{h}_k - 1 = 2g_v/(h_k - 1) \).

For the “ends” \( k = 1 \) and \( k = s \) we need more special computations (since we have to separate the two different types of strings (which may be, or may not be “missing” from \( \Gamma \)).

In (5) we recover \( \tilde{h}_s \), in (6) and (7) \( n \) and \( p_s \) and the arrowhead of \( \Gamma^{\min}(X_{f,n}, z) \) (excepting the case (*)). In (8) we treat the invariants with index \( k = 1 \).

(5) The integer \( \tilde{h}_s \). If \( h_s > 1 \) then the genus formula for \( g_{vs} \) gives \( \tilde{h}_s \). If \( h_s = 1 \), the strings of type \( \Gamma(\tilde{v}_s) \) cannot be collapsed, hence \( St(v) \neq \emptyset \) for \( \{ v \} = \mathcal{R}_s \). Then \( \tilde{h}_s = \alpha(v) \), cf. \( \square \) \( 3 \) and \( \square \) \( 7 \) \( (b) \).

(6) \( p_s \) and \( n \) and the arrowhead of \( \Gamma^{\min}(X_{f,n}, z) \) in the following cases:

(i) \( \text{either } s \geq 3, \text{ or} \)

(ii) \( s = 2 \) but \( \{ a_1', p_1' \} \neq \{ 1 \} \)

First we show that in both cases we can compute the product \( h_{s-1} \tilde{h}_{s-1} \) \( \tilde{h}_{s-1} \). Indeed, in the case (i) this follows from (3) and (4). If \( s = 2 \) then we proceed as follows. Since \( a_1' \) and \( p_1' \) are not both 1, \( St(v) \neq \emptyset \) for \( v \in \mathcal{R}_1 \). If the determinant has two values on this set, then \( \tilde{h}_1 = \alpha(v) \), cf. \( \square \) \( 5 \) \( 3 \) and \( \square \) \( 7 \) \( (a) \). If all the determinants are the same, then either \( \Gamma(\tilde{v}_0) \) or \( \Gamma(\tilde{v}_1) \) is collapsed. If \( \Gamma(\tilde{v}_0) \) is collapsed then \( a_1' = 1 \) hence \( \tilde{h}_1 \geq 2 \). In the second case \( h_1 \geq 2 \). Hence in both cases the following procedure works: take \( c_1 := \#St(v) \) (which automatically is \( \geq 2 \)), compute \( c_2 \) by the genus formula \( 2g_{v_1} = (c_1 - 1)(c_2 - 1) \) and set \( \tilde{h}_1 \tilde{h}_1 = c_1 c_2 \).
Now, we go back to \( p_s \) and \( n \) and the position of the arrowhead.

Notice that by \( 5.3 \) and part (2), the determinants of type \( D(v_{s-1}) \) and \( D_+(v_{s-1}) \) are well-defined in \( \Gamma \), and their values do not change by the minimalization procedure. E.g. \( D_+(v_{s-1}) \) can be computed from \( 1.3(b) \). Notice also that \( D_{St}(v_s) = n(p_s'h_s\tilde{h}_s \text{ cf. } 1.5) \). Therefore

\[
\frac{D(v_{s-1})^{h_s-1}D_{St}(v_s)}{D_+(v_{s-1})} = \frac{h_{s-1}\tilde{h}_s-1p_s}{h_sh_s}.
\]

Hence this value can be determined from the graph, a fact which is true for \( h_{s-1}\tilde{h}_s-1 \) (see above) and \( h_s \) (cf. (3)) and \( \tilde{h}_s \) (cf. (5)) as well. Hence we get \( p_s \). In particular, we can compute the string determinants \( D(\tilde{v}_s) = p'_s \) and (using \( D_{St}(v_s) \)) \( D(v_s) = n/\{h_s\tilde{h}_s\} \) as well. This gives \( n \) too. If \( D(v_s) \neq 1 \) then we put the arrow on the string with this determinant (cf. \( 1.7 \)); if \( D(v_s) = 1 \) then we put the arrowhead on \( \pi(q^{-1}(v_s)) \). In this way we recover the arrow of \( \Gamma^\text{min}(X_{f,n}, z) \).

(7) \( p_s \) and \( n \) and the arrowhead of \( \Gamma^\text{min}(X_{f,n}, z) \) if

\( \text{(iii)} \quad \tilde{h}_s \neq 1. \)

Consider \( St(v) \) for \( \{v\} = R_s \), cf. \( 5.4(3) \). Then by a verification \( D^\text{red}_{St}(v) = (p'_s\tilde{h}_s-1) \). Since \( \tilde{h}_s \) is determined in (5), and it is \( \neq 1 \), one gets \( p'_s \). Then we repeat the arguments of (6).

(8) The integers \( a_1, p_1, h_1 \) and \( \tilde{h}_1 \) in the cases when one of the conditions (i) or (ii) or (iii) is valid. Fix a vertex \( v \in R_1 \) and consider \( St(v) \) as in \( 5.4(3) \).

If \( St(v) \neq \emptyset \), then \( \#I(v) \leq 2 \). If \( \#I(v) = 2 \), then compute the two numbers \( D_1 \cdot \#_2 \) and \( D_2 \cdot \#_1 \). They are the candidates for \( a_1 = q_1 \) and \( p_1 \), cf. \( 1.5 \). Since \( q_1 > p_1 \), these two numbers cannot be the same. If, say, \( D_1 \cdot \#_2 > D_2 \cdot \#_1 \), then \( St_1 \) is the index set of \( \pi(q^{-1}(v_0)) \) and \( St_2 \) of \( \pi(q^{-1}(v_1)) \). Hence \( h_1 = \#_1, h_1 = \#_2, q_1 = D_1 \cdot \#_2 \) and \( p_1 = D_2 \cdot \#_1 \).

If there is only one level set with data \( D_1 \) and \( \#_1 \), then in the above argument we write \( D_2 = 1 \) and we determine \( \#_2 \) using the genus formula \( 2g_{v_1} = (\#_1-1)(\#_2-1) \) (which is possible since \( D_2\#_1 = \#_1 \geq 2 \)). And we repeat the above argument.

Now we assume that \( St(v) = \emptyset \). This can happen only if \( a'_1 = p'_1 = 1 \), hence \( q_1 = \tilde{h}_1 \) and \( p_1 = h_1 \). First we determine \( H := h_1\tilde{h}_1 \).

\( D(v_1) \) gives an equation of type \( q_2 = H \cdot A \), where \( A \) is a positive number which can be determined from the graph by the previous steps. Moreover, \( D_-(v_2) = a_2 \), hence \( a_2 = D_-(v_2)\tilde{h}_2 \) is known from the graph. Finally, \( a_1p_1p_2 = Hp_2 \), where \( p_2 \) too is known from the graph. Then the identity \( a_2 = q_2 + a_1p_1p_2 \) gives a non-trivial linear equation for \( H \).

Then \( h_1\tilde{h}_1 = H \) and \( (h_1-1)(\tilde{h}_1-1) = 2g_{v_1} \) provides \( h_1 \) and \( \tilde{h}_1 \) modulo their permutation. But \( \tilde{h}_1 = q_1 > a_1 = h_1 \), hence we get \( h_1 \) and \( \tilde{h}_1 \).

(9) The integers \( \{a_k\}_{k=2} \) when one of the conditions (i) or (ii) or (iii) is valid. Once we have the position of the arrow, we have all the multiplicities \( \{m_{v_k}\}_{k} \) (cf. \( 2.3 \)), hence \( 1.2(b) \) gives all the integers \( a'_k \). An alternative way is to use inductively \( 1.10 \).

(10) Assume that the conditions (i), (ii) and (iii) are not valid. This means that \( s = 2 \) and \( a'_1 = p'_1 = \tilde{h}_2 = 1 \). This is exactly the case of S2-coincidence treated in \( 6.3 \).

(11) The “\( P \)-case”. Now we go back to step (2), case (b). In this case \#\( R_s \) = 2, so write \( R_s' = \{w_1,w_2\} \). Take the shortest path in \( \Gamma \) connecting \( w_1 \) and \( w_2 \). Take the edge “at the middle of the path”, blow it up, and put an arrow on it. This new graph is exactly
On the link of \( \{ f(x, y) + z^n = 0 \} \)
\( \Gamma_{\text{min}}(X_{f,n}, z) \). Set \( R_s := \{ v' \} \), where \( v' \) is the new vertex. Then we can repeat all the above arguments.

Notice that step (6) works, since if \( s = 2 \) and \( a'_1 = p'_1 = 1 \), then \( h_1 = p_1 \geq 2 \) and \( \tilde{h}_1 = a_1 \geq 2 \) hence \( g_{v_1} > 0 \). But in the “P-case” all the genera are zero. (Hence step (7) is not needed.) (In fact, since in this case we already have the position of the arrow, we can compute some of the invariants much faster using the multiplicities and [2(b)].)

6.5. In the above proof we were rather meticulous in separating the possible sets \( \pi(q^{-1}(v)) \).

The fruit of this is the following corollary (whose proof is left to the reader, and basically it is incorporated in the previous proof of the main theorem).

First recall that the cyclic covering \( X_{f,n} \to X \) has a \( \mathbb{Z}_n \) Galois action. This lifts to the level of the resolution, hence \( \Gamma_{\text{min}}(X_{f,n}) \) inherits a natural \( \mathbb{Z}_n \)-action as well. The question is: has the graph \( \Gamma_{\text{min}}(X_{f,n}) \) any extra symmetry?

Take for example the Brieskorn case 4.3. Then the Galois action permutes cyclically (via its image \( \mathbb{Z}_h \)) the \( h \) arms with Seifert invariants \( a \). On the other hand, the total symmetry group of the graph is the total permutation group of these arms. So, in this sense, the symmetry group of the graph is definitely larger than the (image) of the Galois action. On the other hand, their orbits are the same. This fact is valid in general.

6.6. Corollary. Assume that \( \sigma \) is a (decorated graph-) automorphism of \( \Gamma_{\text{min}}(X_{f,n}) \) which identifies two vertices, say, \( v_1 \) and \( v_2 \). Then \( v_1 \) and \( v_2 \) are in the same orbit with respect to the Galois action.

This result definitely cannot be extended to the general case when \( f \) is not irreducible. E.g., \( \Gamma_{\text{min}}(\mathbb{C}^2, 0) \) can have a symmetry (take e.g. \( f = (x^2 + y^3)(x^3 + y^2) \)) which lifts to an automorphism of \( \Gamma_{\text{min}}(X_{f,n}) \), which does not come from the Galois covering.

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