EXTREMAL POLARIZATION CONFIGURATIONS FOR INTEGRABLE KERNELS

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Abstract. Our main result shows that if a lower-semicontinuous kernel $K$ satisfies some mild additional hypotheses, then asymptotically polarization optimal configurations are precisely those that are asymptotically distributed according to the equilibrium measure for the corresponding minimum energy problem.

1. Background and Results

Suppose $\mathcal{A}$ is a compact set that is embedded in $\mathbb{R}^t$. Let $K(x, y) : \mathcal{A} \times \mathcal{A} \to [0, \infty]$ be a kernel given by $K(x, y) = f(|x - y|)$, where $f : [0, \infty) \to [0, \infty]$ is a lower semi-continuous function and $|\cdot|$ represents the Euclidean distance in $\mathbb{R}^t$. We will let $\mathcal{M}(\mathcal{A})$ denote the set of positive probability measures with support in $\mathcal{A}$. For any $\mu \in \mathcal{M}(\mathcal{A})$, the kernel generates a potential $U^\mu$ by

$$U^\mu(x) = \int_{\mathcal{A}} K(x, y) d\mu(y), \quad x \in \mathbb{R}^t,$$

which is also non-negative and lower semi-continuous (by Fatou’s Lemma; see [10, Section 1.2]). For any configuration $\omega_N = (a_1, \ldots, a_N)$ of $N$ (possibly not distinct) points in $\mathcal{A}$, we define its polarization by

$$P(\omega_N) := \min_{x \in \mathcal{A}} \frac{1}{N} \sum_{y \in \omega_N} K(x, y).$$

Equivalently, $P(\omega_N)$ is the minimum in $\mathcal{A}$ of the potential generated by the probability measure $\nu_N$ that assigns weight $N^{-1}$ to each point in $\omega_N$ (counting multiplicities). If we associate such $N$-point configurations with the space $\mathcal{A}^N$, then the extremal $N$-point polarization problem is to find

$$\mathcal{P}(\mathcal{A}, N) := \sup_{\omega_N \in \mathcal{A}^N} P(\omega_N).$$

If $\mathcal{M}_N(\mathcal{A})$ denotes the set of all probability measures $\nu$ of the form

$$\nu = \frac{1}{N} \sum_{j=1}^N \delta_{a_j}, \quad a_j \in \mathcal{A}, \quad j = 1, \ldots, N,$$

then $\mathcal{P}(\mathcal{A}, N)$ can be defined as

$$\mathcal{P}(\mathcal{A}, N) = \sup_{\nu \in \mathcal{M}_N(\mathcal{A})} \min_{x \in \mathcal{A}} U^\nu(x).$$

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Polarization problems have a lengthy history, with many substantial results appearing in [1, 3, 5, 6, 8, 11]. One of the most fundamental results is [11, Theorem 2], which asserts that
\[
\lim_{N \to \infty} P(A, N) = \sup_{\mu \in \mathcal{M}(A)} \min_{x \in A} U^\mu(x).
\]  
(1)
Any measure \( \mu \) that achieves the supremum on the right-hand side of (1) is called a polarization extremal measure. One consequence of our results will be a demonstration of the uniqueness of the polarization extremal measure for a large class of kernels \( K \) and compact sets \( A \) and a proof that these extremal measures are also extremal for the minimum energy problem.

The minimum energy problem for the kernel \( K \) and the set \( A \) is to find a configuration \( \omega_N = (a_1, \ldots, a_N) \in A^N \) that minimizes then the energy functional
\[
E(\omega_N) := \sum_{i,j=1 \atop i \neq j}^N K(a_i, a_j).
\]

It is well-known that if there is \( \mu \in \mathcal{M}(A) \) so that
\[
I[\mu] := \int_A \int_A K(x, y) d\mu(x) d\mu(y) < \infty,
\]
then there is a measure \( \mu_{eq} \in \mathcal{M}(A) \) so that
\[
\lim_{N \to \infty} \frac{\min_{\omega_N \in A^N} E(\omega_N)}{N^2} = I[\mu_{eq}].
\]
The quantity \( I[\mu] \) is known as the \( K \)-energy of \( \mu \) and the measure \( \mu_{eq} \) is known as a \( K \)-equilibrium measure. The set of \( K \)-equilibrium measures is given by
\[
\left\{ \mu \in \mathcal{M}(A) : I[\mu] = \inf_{\nu \in \mathcal{M}(A)} I[\nu] \right\},
\]
(see [9]).

For our computations, we will make the following additional assumptions on the kernel \( K \) and the set \( A \):

(A1) There is a \( \mu \in \mathcal{M}(A) \) so that \( I[\mu] < \infty \).

(A2) The kernel \( K \) has a unique equilibrium measure, which we denote by \( \mu_{eq} \).

(A3) The support of \( \mu_{eq} \) is all of \( A \).

(A4) The potential function
\[
U^e(x) := \int_A K(x, y) d\mu_{eq}(y)
\]
is equal to a positive constant on all of \( A \), which we denote by \( W_K \).

The conditions (A1-A4), while far from being generic, are satisfied by a very large collection of compact sets \( A \) and kernels \( K \) and we will explore some examples in Section 2. Note that condition (A3) is not heavily restrictive in the sense that if supp(\( \mu_{eq} \)) \( \neq A \), then we can redefine \( A \) to be the support of \( \mu_{eq} \) so that (A3) is then satisfied. All four conditions are satisfied when \( A = S^d \subset \mathbb{R}^d \) and \( K(x, y) = |x - y|^{-s} \) for any \( s \in (0, d) \). In this case, the \( K \)-equilibrium measure is normalized surface-area measure on \( S^d \) (see [7]).
Now we are ready to state our main result.

**Theorem 1.1.** Let the compact set $A$ and lower semi-continuous kernel $K$ satisfy conditions (A1-A4). For each $N \geq 2$, choose some $\omega_N \in A^N$ and let $\nu_N$ be the probability measure that assigns mass $N^{-1}$ to each point in $\omega_N$ (counting multiplicities). The following are equivalent:

(a) The measures $\{\nu_N\}_{N \geq 2}$ converge weakly to $\mu_{eq}$.
(b) It holds that $\lim_{N \to \infty} P(\omega_N) = W_K$.
(c) It holds that $\lim_{N \to \infty} \left( \frac{1}{N} \sum_{y \in \omega_N} K(x, y) - P(\omega_N) \right) = 0$ in $L^1(\mu_{eq})$.

**Remark.** In [2], Borodachov and Bosuwan showed that if $K(x, y) = |x - y|^{-d}$ and $A$ is a $d$-dimensional manifold, then any sequence of polarization optimal configurations is asymptotically equidistributed on $A$ as $n \to \infty$. This is distinct from our results because the kernel does not satisfy (A1).

**Proof.** Assume that (a) is true. For every $n \geq 2$, define

$$U_n(x) := \int K(x, y) d\nu_n(y) = \frac{1}{n} \sum_{y \in \omega_n} K(x, y).$$

It is clear (by Fubini’s Theorem) that

$$P(\omega_N) = \min_{x \in A} U_n(x) \leq \int U_n(x) d\mu_{eq}(x) = W_K.$$  \hspace{1cm} (2)

Let $\{f_\delta\}_{\delta > 0}$ be a collection of non-negative continuous functions on $[0, \text{diam}(A)]$ converging pointwise to $f$ from below as $\delta \to 0^+$. Let $x_n$ be a point in $A$ where $U_n$ attains its minimum. By passing to a subsequence if necessary, we may assume that $x_n$ converges to some $x_\infty$ and $U_n(x_n)$ converges to $\lim inf U_m(x_m)$ as $n \to \infty$.

Let $\gamma > 0$ be fixed. Since $f_\delta$ is uniformly continuous, when $n$ is sufficiently large we have

$$U_n(x_n) = \int K(x_n, y) d\nu_n(y) \geq \int f_\delta(|x_n - y|) d\nu_n(y) \geq \int f_\delta(|x_\infty - y|) d\nu_n(y) - \gamma$$

$$\to \int f_\delta(|x_\infty - y|) d\mu_{eq}(y) - \gamma,$$  \hspace{1cm} (3)

as $n \to \infty$. Taking the supremum over all $\delta > 0$ shows

$$\liminf_{n \to \infty} U_n(x_n) \geq \int K(x_\infty, y) d\mu_{eq}(y) - \gamma = W_K - \gamma,$$

where we used assumption (A4) in this last equality. Since $\gamma > 0$ was arbitrary, this proves part (b).

Now let us assume (b) is true. We know from [2] that

$$\int U_n(x) d\mu_{eq}(x) = W_K.$$  \hspace{1cm} (4)
However, our assumption (b) implies \( \min U_n(x) \to W_K \) as \( n \to \infty \). Therefore, the functions \( \{U_n\}_{n \geq 2} \) are such that the minima converge to the average, which is \( n \)-independent. We then calculate

\[
\int_A \left| U_n(x) - \min_{z \in A} U_n(z) \right| d\mu_{eq}(x) = \int_A \left( U_n(x) - \min_{z \in A} U_n(z) \right) d\mu_{eq}(x) \to 0, \quad n \to \infty,
\]

which proves (c).

Now, let us assume that (c) is true. By appealing to (4), we can write

\[
W_K - P(\omega_N) = \int \left( U_n(x) - \min_{z \in A} U_n(z) \right) d\mu_{eq}(x) \to 0,
\]
as \( n \to \infty \), which proves (b).

Finally, assume (b) is true and let \( N \subseteq \mathbb{N} \) be a subsequence through which \( \nu_n \) converges weakly to \( \nu_\infty \) as \( n \to \infty \) through \( N \). We have already seen that (b) implies (c), so \( U_n - W_K \) converges to 0 in probability as \( n \to \infty \) through \( N \). We may therefore take a further subsequence \( N_1 \subseteq N \) so that \( U_n \) converges to \( W_K \) \( \mu_{eq} \)-almost everywhere as \( n \to \infty \) through \( N_1 \) (see [14, page 169]). If we use the functions \( \{f_\delta\}_{\delta > 0} \) defined above, then we calculate for \( \mu_{eq} \)-almost every \( x \):

\[
W_K = \lim_{\delta \to 0, n \to \infty} U_n(x) = \lim_{\delta \to 0, n \to \infty} \int K(x, y) d\nu_n(y) \geq \limsup_{\delta \to 0, n \to \infty} \int f_\delta(|x - y|) d\nu_n(y)
\]

\[
= \int f_\delta(|x - y|) d\nu_\infty(y).
\]

Taking the supremum over all \( \delta > 0 \) shows

\[
U^{\nu_\infty}(x) \leq W_K \tag{5}
\]

\( \mu_{eq} \)-almost everywhere, in particular at all isolated points of \( A \) (by (A3)). Finally, we note that the potential on the left-hand side of (5) is lower-semicontinuous as a function of \( x \). Therefore (5) holds for all \( x \in A \) that are not isolated points of \( A \), and hence for all \( x \in A \). From this, it follows that \( \nu_\infty \) has the same \( K \)-energy as \( \mu_{eq} \), and the uniqueness of the \( K \)-equilibrium measure implies that \( \nu_\infty \) must be \( \mu_{eq} \), which proves (a).

**Remark.** Notice that the implication (b)\( \Rightarrow \) (c) in Theorem 1.1 does not make use of assumption (A3).

**Corollary 1.2.** Assume the hypotheses of Theorem 1.1 on \( A \) and \( K \).

(i) \( \lim_{N \to \infty} P(A, N) = W_K \)

(ii) For any sequence \( \{\omega_N^*\}_{N \geq 2} \) of polarization optimal configurations having corresponding counting measure \( \{\nu_N^*\}_{N \geq 2} \), it holds that \( \nu_N^* \) converges weakly to \( \mu_{eq} \) as \( N \to \infty \)

(iii) \( \mu_{eq} \) is the unique polarization extremal measure.

**Proof.** (i) As in [2], we have \( P(A, N) \leq W_K \). Now, if \( \{\omega_N\}_{N \geq 2} \) is such that \( \nu_N \) converges weakly to \( \mu_{eq} \) as \( N \to \infty \), then we have

\[
P(A, N) \geq P(\omega_N) \to W_K,
\]
as \( N \to \infty \) by Theorem 1.1.

(ii) This is immediate from the equivalence of (a) and (b) in Theorem 1.1.
(iii) Let $\mu_p$ be a polarization extremal measure and $U_p(x)$ the corresponding potential. Then by definition,

$$\min_{x \in A} U_p(x) = W_K.$$  

However, $\int U_p(x) d\mu_{eq}(x) = W_K$, so $U_p(x) = W_K$ $\mu_{eq}$-almost everywhere. Since (A3) implies $\text{supp}(\mu_{eq}) = A$ and $U_p(x)$ is lower-semicontinuous, this implies $U_p(x) \leq W_K$ on all of $A$ (as in the proof of Theorem 1.1). Therefore, $\mu_p$ has $K$-energy equal to $W_K$ and hence must be $\mu_{eq}$ by (A2). □

2. Examples

In this section we will explore some examples that highlight the utility and some subtleties of the results of Section 1. Throughout this section we will refer to the notion of asymptotic optimality, which we define as in [2]. A sequence of configurations $\{\omega_N\}_{N=2}^{\infty}$ (where each $\omega_N \in A^N$) is said to be asymptotically optimal for the polarization problem if

$$\lim_{N \to \infty} \frac{P(\omega_N)}{P(A,N)} = 1.$$  

With this terminology, Theorem 1.1 can be restated as a collection of statements that are equivalent to the condition of asymptotic optimality of the sequence of point configurations $\{\omega_N\}_{N \geq 2}$.

2.1. Example: Riesz potentials on the solid ball. Let us assume $t \geq 2$ and $A = \{x \in \mathbb{R}^t : |x| \leq 1\}$ and consider the Riesz kernel $K(x,y) = |x-y|^{-s}$ for some $0 < s \leq t-2$. It was shown in [5, Section 3] that the $N$-point configuration consisting of $N$ points at the origin is in fact optimal for the polarization problem on the solid ball with this choice of kernel. It is obvious that a point mass has infinite energy, so the counting measures for the optimal polarization configurations do not, in this case, converge weakly to the equilibrium measure. Thus we see that it is not clear how asymptotically optimal polarization configurations behave when the conditions (A1-A4) are not satisfied. This example shows that the equivalences stated in Theorem 1.1 need not hold in general.

2.2. Example: Random and greedy point configurations. Suppose that $A$ and $K$ are such that conditions (A1-A4) are satisfied. For each $N \geq 2$, let $\omega_N$ be a collection of $N$ points in $A$ chosen at random with distribution $\mu_{eq}$ and let $\nu_N$ be the probability measure assigning weight $N^{-1}$ to each point in $\omega_N$. The Strong Law of Large Numbers implies that as $N \to \infty$, the measures $\{\nu_N\}_{N \geq 2}$ almost surely converge weakly to $\mu_{eq}$. Theorem 1.1 implies that $P(\omega_N) \to W_K$ as $N \to \infty$. Therefore, randomly chosen points from the appropriate distribution are almost surely asymptotically optimal for the polarization problem.

In [9], López-García and Saff studied greedy energy points, which are sequences of $N$-point configurations $\{\omega_N\}_{N \geq 2}$ that are optimal for the energy problem subject to the constraint that $\omega_{N-1} \subseteq \omega_N$. More precisely, we define a sequence $\{a_n\}_{n=1}^{\infty}$ by choosing $a_1 \in A$ arbitrarily, and then for each $n > 1$ we choose $a_n \in A$ so that

$$\frac{1}{n-1} \sum_{i=1}^{n-1} K(a_n, a_i) = P((a_i)_{i=1}^{n-1}).$$
The set $\omega_N$ is then taken to be $(a_i)_{i=1}^N$. We recall [9, Theorem 2.1(iii)], which says that under the assumptions (A1-A4), it holds that
\[
\lim_{n \to \infty} \frac{1}{n-1} \sum_{i=1}^{n-1} K(a_n, a_i) = W_K.
\]
In other words, the sequence of configurations $\{\omega_N\}_{N \geq 2}$ is asymptotically optimal for the polarization problem. By Theorem 1.1 we conclude that the measures
\[
\frac{1}{N} \sum_{i=1}^N \delta_{a_i}
\]
converge weakly to $\mu_{eq}$, which is the same conclusion as [9, Theorem 2.1(iii)].

2.3. Example: Logarithmic potentials on curves in the plane. Consider the case when $\mathcal{A}$ is a union of $M \geq 1$ disjoint and mutually exterior Jordan curves in $\mathbb{R}^2$ and $K(x, y) = -\log(c|x - y|)$, where $c > 0$ is a constant chosen to ensure that $K(x, y) > 0$ when $x, y \in \mathcal{A}$. In this case, it is easily seen that condition (A1) is satisfied and [13, Theorem I.1.3] assures us that (A2) is satisfied. By [13, Theorem IV.1.3] and an application of Mergelyan’s Theorem (see [12, Theorem 20.5]), one can check that condition (A3) is satisfied as well.

The only condition that remains to verify before we can apply our results is (A4). There are several criteria that imply continuity of the logarithmic equilibrium potential. For example, [13, Theorem I.5.1] tells us that if $z_0 \in \mathcal{A}$ and we define (for some $\lambda \in (0, 1)$)
\[
A_n(z_0) := \{ z : z \in \mathcal{A}, \quad \lambda^{n+1} \leq |z - z_0| < \lambda^n \},
\]
then
\[
\sum_{n=1}^{\infty} \frac{-n}{\log(\text{cap}(A_n(z_0)))} = \infty
\]
implies continuity of the logarithmic equilibrium potential at $z_0$. The criterion that we will use is [13, Theorem I.4.8ii], which applies to every point of $\mathcal{A}$ because every point of $\mathcal{A}$ is on the boundary of two components of $\mathbb{R}^2 \setminus \mathcal{A}$, one of which is bounded and one of which is unbounded. Applying this result shows condition (A4) is satisfied, and hence Theorem 1.1 applies in this setting.

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References

[1] G. Ambrus, K. Ball, and T. Erdélyi, Chebyshev constants on the unit ball, Bull. London Math. Soc. 45(2) (2013), 236–248.
[2] S. Borodachov and N. Bosuwan, Asymptotics of discrete Riesz d-polarization on subsets of d-dimensional manifolds, Potential Anal. 41 (2014), no. 1, 35–49.
[3] S. Borodachov, D. Hardin, and E. B. Saff, Minimal Discrete Energy on the Sphere and Other Manifolds, Springer, to appear.
[4] K. L. Chung, A Course in Probability Theory, Academic Press, 2001.
[5] T. Erdélyi and E. B. Saff, Riesz polarization inequalities in higher dimensions, J. Approx. Theory 171 (2013), 128–147.
[6] B. Farkas and S. G. Révész, *Potential theoretic approach to Rendevous numbers*, Monatsh. Math. 148 (2006), no. 4, 309–331.

[7] D. Hardin and E. Saff, *Discretizing manifolds via minimum energy points*, Notices Amer. Math. Soc. 51 (2004), no. 10, 1186–1194.

[8] D. Hardin, A. Kendall, and E. Saff, *Polarization optimality of equally spaced points on the circle for discrete potentials*, Discrete Comput. Geom. 50 (2013), no. 1, 236–243.

[9] A. López-García and E. B. Saff, *Asymptotics of greedy energy points*, Math. Comp. 79 (2010), 2287–2316.

[10] J. Malý and W. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, American Mathematical Society, 1997.

[11] M. Ohtsuka, *On various definitions of capacity and related notions*, Nagoya Math. J. 30 (1967) 121–127.

[12] W. Rudin, *Real and Complex Analysis*, Third Edition, McGraw-Hill, Madison, WI, 1987.

[13] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer, 1997.

[14] J. Taylor, *An Introduction to Measure and Probability*, Springer, 1997.