Distributive laws for actions of monoidal categories

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Abstract

Given a monoidal category $\mathcal{C}$, an ordinary category $\mathcal{M}$, and a monad $T$ in $\mathcal{M}$, the lifts in a strict sense of a fixed action of $\mathcal{C}$ on $\mathcal{M}$ to an action of $\mathcal{C}$ on the Eilenberg-Moore category $\mathcal{M}^T$ of $T$-modules are in a bijective correspondence with certain families of natural transformations, indexed by the objects in the monoidal category $\mathcal{C}$. These families are analogues of distributive laws between monads.

1 Actions of monoidal categories on other ordinary categories appear in diverse setups. Anyway, apart from explaining the main result of this paper, we will limit ourselves just to our original motivation, and other applications will be left to the reader’s imagination and future.

2 Given some group-like (symmetry) object, e.g. a Hopf algebra $H$ (e.g. $H = \mathcal{O}(G)$ for an algebraic group $G$), and an auxiliary category $\mathcal{M}$ viewed as a category $\mathcal{D}coh_X$ of (i.e. “quasicoherent”) sheaves on a (noncommutative) space $X$, what is the appropriate notion of the action of $G$ on $X$ and how do we represent it practically in such a situation? Standard observations:

2a Affine case. If $\mathcal{M}$ is the category of (say left) modules over an algebra $A$, then in DRINFELD’s quantum group philosophy one replaces group action by Hopf (co)actions of some Hopf algebra on $A$.

2b (Co)monads from (co)actions. 1) Affine case. Every Hopf coaction of bialgebra $H$ on an algebra $A$ induces a comonad, say $T$, on $A - \text{Mod}$.

2) Globalizing. In some cases one may globalization this picture by introducing a global (co)monad $T'$ and having some device which will locally compare/reduce its action to the action induced from “affine Hopf-coaction” (as in a)). In the commutative situation, and if $H = \mathcal{O}(G)$ is (a Hopf algebra of functions on a group), one can localize the space with action only to the invariant open sets, otherwise there is no induced action. Although we have used generalizations of such methods to noncommutative setup elsewhere, a priori there is no sufficient supply of affine open sets which are action-invariant.
One occasionally replaces the Hopf algebra (or more general symmetry objects) by a monoidal category $\tilde{C} = (C, \otimes)$ of its (co)modules. Say, in the case of modules, the Hopf algebra itself is a distinguished comonoid in $\tilde{C}$. Its image under the (action) functor $L : C \to \text{End} \mathcal{M}$ is the comonad $T$ we mentioned before. The category $\mathcal{M}^T$ of modules over this monad is a version of the category of $G$-equivariant sheaves on $X$ in this setup.

The crucial point missing from 2a-c, and from the bulk of recorded “noncommutative” literature is failing to adhere to the old Grothendieck’s advice that geometrical study of morphisms (hence actions in particular) should be carried in relative setup. In particular, group schemes should live over some fixed scheme $V$ and the objects they act on should live over that same scheme $V$. Similarly, Hopf algebras are defined in some (symmetric, but generalizations apply) monoidal category $V$. The space $X = \text{Spec } M'$ should live over some “Spec $V$” what means that we have direct image (forgetful) functor from $M$ to $V$, satisfying some properties (e.g. having an inverse image functor) needed to call $X$ a space or a scheme (e.g. a non-commutative scheme over $V$ (20)). The action has to respect the forgetful functor in the sense that if one applies the action and then forgets down to $V$ then we (should) obtain exactly the natural action of the Hopf algebra (i.e. of its category of (co)modules) on $V$. In this paper, we show a general nonsense result on what additional data are necessary to lift known actions of monoidal category on $V$ (e.g. the natural action of $H - \text{Comod}$ on $\text{Vec}_k$ where $H$ is a bialgebra over a field $k$) to new actions on a simple, but important class of categories over $V$: namely the categories of modules over monads in $V$. Fundamental importance of such categories in geometric context (as local models) has been enlightened in [20].

Correct choice of $V$ enables unified correct treatment of diverse flavours of symmetries, e.g. relative group schemes, Hopf algebras in braided categories, (finite) group algebras in monoidal categories different than (super)vector spaces, accounting for (multi)graded rings etc. Useful sources of actions in quantum vector bundle theory, e.g. “entwining structures” ([7,4]) naturally appear in our perspective. Natural generalizations work, primarily for bialgebroids. The relative viewpoint is also inherent in replacing Hopf algebras by tensor categories in view of Tannakian theory which does require a fibre functor. For sensible consideration of actions one should not detach a monoidal category from its origin.

Some relevant references for the geometrical motivation for this work are [7,14,15,21,23].

A monoidal category is a 6-tuple $\tilde{C} = (C, \otimes, 1, a, r, l)$ where $\otimes : C \times C \to C$ is a bifunctor (monoidal or tensor product),
1 \in \text{Ob}\mathcal{C} is a distinguished object (unit object),

\[ a : \_ \otimes (\_ \otimes \_ ) \Rightarrow (\_ \otimes \_ ) \otimes \_ \] is a trinatural equivalence of trifunctors (associativity constraint); (i.e. a family of isomorphisms \( \{a_{X,Y,Z} : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z\}_{X,Y,Z \in \text{Ob}\mathcal{C}} \), natural in \( X,Y,Z \)).

\[ \rho : \text{Id}_\mathcal{C} \rightarrow \text{Id}_\mathcal{C} \otimes 1 \] (right unit coherence) and \( \lambda : \text{Id}_\mathcal{C} \rightarrow 1 \otimes \text{Id}_\mathcal{C} \) (left unit coherence) are natural equivalences of functors; the compositions \( X \otimes (Y \otimes (Z \otimes W)) \xrightarrow{a_{X,Y,Z} \otimes 1} (X \otimes Y) \otimes (Z \otimes W) \xrightarrow{a_{X,Y,Z,W}} ((X \otimes Y) \otimes Z) \otimes W \) and \( X \otimes (Y \otimes (Z \otimes W)) \xrightarrow{1 \otimes a_{Y,Z,W}} X \otimes ((Y \otimes Z) \otimes W) \xrightarrow{a_{X,Y,Z,W}} (X \otimes (Y \otimes Z)) \otimes W \) should agree \( \forall X,Y,Z,W \in \text{Ob}\mathcal{C} \) and the triangle coherence identity \( a_{X,1,Y} \circ (\rho_X \otimes Y) = X \otimes \lambda_Y \) holds. The monoidal category is strict if all coherence isomorphisms \( a_{X,Y,Z}, \rho_X, \lambda_Y \) are identity maps and hence may be omitted from the data.

4a A monoidal functor from a monoidal category \((\mathcal{A}, \otimes, 1, a, \rho, \lambda)\) to \((\mathcal{B}, \otimes', 1', a', \rho', \lambda')\) is a triple \((F, \chi, \xi)\) where \( F : \mathcal{A} \rightarrow \mathcal{B} \) is a functor and \( \chi : F(\_ ) \otimes F(\_ ) \Rightarrow F(\_ \otimes \_ ) \) and \( \xi : F(\_ ) \otimes 1' \rightarrow F(\_ \otimes 1) \) natural transformations satisfying some natural coherence relations \([16, 2]\).

5 The category \( \text{End}\mathcal{A} \) of endofunctors in a given category \( \mathcal{A} \) is a strict monoidal category with respect to the tensor product given on objects (endofunctors) as the composition, and on morphisms (natural endotransformations) as the Godement’s “vertical product” \( \ast : (F : f \Rightarrow f', G : g \Rightarrow g') \mapsto G \ast F = g \circ f \Rightarrow g' \circ f' \), where \( (G \ast F)_M := g' (F_M) \circ g(F_M) \).

A semigroup \( T = (T, \mu) \) in \( \text{End}\mathcal{A} \) called a nonunital monad. In other words, \( T \) is an endofunctor in \( \mathcal{A} \) and \( \mu : T \circ T \rightarrow T \) is a natural transformation of functors satisfying the associativity \( (\text{Id} \ast T) \circ T = (T \ast \text{Id}) \circ T \). If \( (T, \mu) \) is a monoid in \( \text{End}\mathcal{A} \), i.e. in addition \( \exists \) (automatically unique) natural transformation \( \eta : \text{Id} \Rightarrow T \) satisfying \( \mu \circ \eta_T = \text{Id} = \mu \circ T(\eta) \), then we say that \( (T, \mu, \eta) \) is a monad \([16]\). A comonad is a monad in \( \mathcal{A}^{\text{op}} \).

6 A right action of a monoidal category is a bifunctor
\[ \Diamond : \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{M}, \quad (M, Q) \mapsto M \Diamond Q, \]
and a family of morphisms \( \Psi_{M,X,Y}^Z : M \Diamond (X \otimes Y) \rightarrow (M \Diamond X) \Diamond Y \) in \( \mathcal{M} \) natural in \( M, X \) and \( Y \) and such that the “action associativity” pentagon

\[ \begin{array}{ccc}
M \Diamond (X \otimes (Y \otimes Z)) & \xrightarrow{\Psi_{M,X,Y,Z}^Y \otimes Z} & (M \Diamond X) \Diamond (Y \otimes Z) \\
\downarrow M \Diamond a_{X,Y,Z}^{-1} & & \downarrow \Psi_{M \Diamond X,Y,Z}^Y \\
M \Diamond ((X \otimes Y) \otimes Z) & \xrightarrow{\Psi_{M,X \otimes Y,Z}^{X \otimes Y,Z}} & ((M \Diamond X) \Diamond Y) \Diamond Z
\end{array} \]
commutes. For applications we have in mind, we do not require that the unit object \(1\) acts strictly trivially, i.e. \(M \otimes 1 \neq M\) in general, but rather we demand a natural equivalence of functors \(u = u^\otimes : \text{Id} \to \text{Id} \otimes 1\), compatible with coherence isomorphisms in \(\tilde{C}\). More precisely, the diagram

\[
\begin{array}{c}
(M \otimes 1) \otimes Q \\ \downarrow \psi_M^{1,Q} \\
M \otimes (1 \otimes Q) \\
\end{array}
\begin{array}{c}
M \otimes Q \\
\downarrow (u_M) \otimes Q \\
M \otimes Q \\
\end{array}
\begin{array}{c}
(M \otimes Q) \otimes 1 \\
\downarrow \psi_M^{Q,1} \\
M \otimes (Q \otimes 1) \\
\end{array}
\]

commutes for all \(M \in \text{Ob} \mathcal{M}\) and all \(Q \in \text{Ob} \mathcal{C}\).

6a A right action of a monoidal category \((\mathcal{C}, \otimes, a, r, l)\) on a (non-monoidal) category \(\mathcal{M}\) may be also given by a contravariant monoidal functor from \(\mathcal{C}\) to \(\text{End} \mathcal{M}\).

Given an action \((\otimes, \Psi, u)\), the associated coherent monoidal functor \(\mathcal{L} : \mathcal{C} \to \text{End} \mathcal{M}\), is then given by \(\mathcal{L}(C)(M) := \mathcal{L}_C(M) := M \otimes C\) (the rest of the coherence structure left to the reader).

6b Example. Every monoidal category \(\tilde{C} = (\mathcal{C}, \otimes, 1, a, r, l)\) acts on its underlying category \(\mathcal{C}\) from the right by the action \((\otimes, \Psi, u) = (\otimes, a, r^{-1})\).

7 A distributive law from a monoidal action \(\otimes\) of \(\mathcal{C}\) on \(\mathcal{M}\) to a monad \(T = (T, \mu, \eta)\) is a binatural transformation \(l : T(\otimes \_ \_) \Rightarrow (T \_ \_ \_ \otimes)\) of bifunctors \(\mathcal{M} \times \mathcal{C} \to \mathcal{M}\), satisfying the list of axioms (D1-4) below. Denote by \((l^Q)_M = l^Q_M : T(M \otimes Q) \to TM \otimes Q\) the morphisms in \(\mathcal{M}\) forming \(l\). Clearly, each \(l^Q : T(\otimes Q) \Rightarrow T(\_ \_ \_ \otimes Q)\) is a natural transformation, hence \(l\) may be viewed as a family \(\{l^Q\}_{Q \in \text{Ob} \mathcal{C}}\). The diagrams (D1-4) are required to commute:

\[
\begin{array}{c}
TT(M \otimes Q) \\
\downarrow (\mu_M) \otimes Q \\
T(M \otimes Q) \\
\downarrow \mu_M \otimes Q \\
T(M \otimes Q) \\
\end{array}
\begin{array}{c}
T(T(M \otimes Q)) \\
\downarrow T(\mu_M) \otimes Q \\
T(M \otimes Q) \\
\end{array}
\begin{array}{c}
T(Q \otimes Q') \\
\downarrow (\Psi_M^{Q,Q'}) \\
T(M \otimes Q) \otimes Q' \\
\downarrow \Psi_M^{Q,Q'} \\
T((M \otimes Q) \otimes Q') \\
\end{array}
\]

\[
\begin{array}{c}
T(M \otimes Q) \\
\downarrow T(\Psi_M^{Q,Q'}) \\
T(M \otimes Q) \otimes Q' \\
\downarrow (\Psi_M^{Q,Q'}) \\
T((M \otimes Q) \otimes Q') \\
\end{array}
\begin{array}{c}
T(M \otimes Q) \\
\downarrow T(\Psi_M^{Q,Q'}) \\
T(M \otimes Q) \otimes Q' \\
\downarrow \Psi_M^{Q,Q'} \\
T((M \otimes Q) \otimes Q') \\
\end{array}
\]

\[
T(M \otimes Q) \\
\downarrow T(\Psi_M^{Q,Q'}) \\
T(M \otimes Q) \otimes Q' \\
\downarrow \Psi_M^{Q,Q'} \\
T((M \otimes Q) \otimes Q') \\
\end{array}
\]

4
8 Given two $\mathcal{C}$-categories $(\mathcal{M}, \hat{\varnothing}, \Psi, u), (\hat{\mathcal{M}}, \hat{\varnothing}, \hat{\Psi}, \hat{u})$, a pair $(F, \zeta)$ where $F : \mathcal{M} \rightarrow \mathcal{M}$ is a functor, and $\zeta : F(\hat{\varnothing}) \Rightarrow F(\varnothing)$ a binatural equivalence of bifunctors $\hat{\mathcal{M}} \times \mathcal{C} \rightarrow \mathcal{M}$, is called a $\mathcal{C}$-functor if the following diagrams commute

\[
\begin{array}{ccc}
T(M) & \xrightarrow{\zeta_{M, 1}} & F(M) \\
\downarrow T(\hat{u}) & & \downarrow u_F(M) \\
T(M \diamond 1) & \xrightarrow{\hat{\zeta}} & TM \diamond 1
\end{array}
\]

We say that $\mathcal{C}$-functor $(F, \zeta)$ is strict if $\zeta_{M, Q} = \text{Id}_{F(M \diamond Q)}$ for all $M$ and $Q$. Then we naturally omit $\zeta$ from the notation.

9 If $U : \hat{\mathcal{M}} \rightarrow \mathcal{M}$ is some (usually “forgetful”) functor, then a $\mathcal{C}$-category structure $(\hat{\mathcal{M}}, \hat{\Psi}, \hat{u})$ is called a strict lift of a $\mathcal{C}$-category structure $(\mathcal{M}, \Psi, u)$ to $\mathcal{M}$ along $U$ if $U$ is a strict $\mathcal{C}$-functor from $(\hat{\mathcal{M}}, \hat{\Psi}, \hat{u})$ to $(\mathcal{M}, \Psi, u)$. In other words, $U(M \diamond \hat{\varnothing}) = U(M) \diamond \varnothing$, $u_{U(M)} = U(\hat{u}_M)$ and $U(\hat{\Psi}_M^{Q, Q'}) = \Psi_{U_M}^{Q, Q'}$.

10 In this article we are primarily interested in the case when $\hat{\mathcal{M}} = \mathcal{M}^T$ is the Eilenberg-Moore category of a monad $T$ in $\mathcal{M}$. Its objects are $T$-modules, that is pairs of the form $(M, \nu)$, where $\nu : TM \rightarrow M$ is a $T$-action, i.e. a morphism satisfying $\nu \circ T(\nu) = \nu \circ \mu_M$ and the unitality $\nu \circ \eta = \text{id}_M$; the morphisms $f : (M, \nu) \rightarrow (N, \xi)$ are those morphisms $f : M \rightarrow N$ which intertwine the action in the sense $\xi \circ T(f) = f \circ \nu$. There is a forgetful functor $U = U^T : \mathcal{M}^T \rightarrow \mathcal{M}$ given by $U^T : (M, \nu) \mapsto M$. This forgetful functor will be in place of functor $U : \hat{\mathcal{M}} \rightarrow \mathcal{M}$ in previous discussion. $U^T$ has a left adjoint $F^T : M \mapsto (TM, \mu_M)$ whose image is so-called Kleisli category of a monad $T$ and its objects are called free $T$-modules. Note that $U^TF^T = T$. 

\[
\begin{array}{ccc}
\xi_{M, Q} & \searrow & \eta_{M, Q} \\
\downarrow T(M) & & \downarrow T(M) \\
T(M \diamond Q) & \xrightarrow{\hat{\zeta}_{M, Q}} & TM \diamond Q
\end{array}
\]
Suppose some action \((\hat{\odot}, \hat{\Psi}, \hat{\mu})\) strictly lifts \((\odot, \Psi, \mu)\) to \(\mathcal{M}^T\). Then we may write \((M, \nu)\hat{\odot}Q = (M\odot Q, (\hat{M}, \nu))\) where \((\hat{M}, \nu)^Q : T(M\odot Q) \to (M\odot Q)\) is a (canonical) \(T\)-action (depending on \(M, \nu, Q\)). Rule \((M, \nu) \mapsto (\hat{M}, \nu)^Q\) defines a natural transformation \((\hat{\odot})^Q : T(U(\odot)\odot Q) \to U(\odot)\odot Q)\) of functors from \(\mathcal{M}^T\) to \(\mathcal{M}\). We will sometimes abuse the notation abbreviating \(\nu^Q\) for \((\hat{M}, \nu)^Q\). The counit of the adjunction \(\epsilon : F^T U T \Rightarrow \text{Id}_{\mathcal{M}^T}\) is given by \(\epsilon_{(M, \nu)} = \nu : F^T M = (TM, \mu_M) \to (M, \nu)\). Then \(U^T(\epsilon_{(M, \nu)}) = \nu : TM \to M\).

Thus \((\hat{M}, \nu)^Q = U^T(\epsilon_{(M, \nu)}\hat{\odot}Q), \) i.e. \((\hat{\odot})^Q = U^T(\epsilon_{\hat{\odot}Q})\). We may also write

\[
(\hat{\odot})^Q F^T = U^T(\epsilon_{F^T(\hat{\odot}Q)}) : M \mapsto \left( (\hat{M}, \mu_M)^Q : T(TM\odot Q) \to TM\odot Q \right)
\]

what is a natural transformation of functors \(\mathcal{M} \to \mathcal{M}\).

**Theorem.** Distributive laws from an action \((\odot, \Psi, \mu)\) of a monoidal category \(\mathcal{C} = (\mathcal{C}, \odot, 1, \alpha, \rho, l)\) on a category \(\mathcal{M}\) to a monad \(T = (T, \mu, \eta)\) in \(\mathcal{M}\) are in a natural bijective correspondence \(l \mapsto (\hat{\odot}, \hat{\Psi}, \hat{\mu})\) with the monoidal actions of \(\mathcal{C}\) on \(\mathcal{M}^T\) strictly lifting the action \((\odot, \Psi, \mu)\) of \(\mathcal{C}\) on \(\mathcal{M}\) along the forgetful functor \(U = U^T : \mathcal{M}^T \to \mathcal{M}\), \(U^T : (M, \nu) \mapsto M\).

**Proof.** I. From distributive laws to lifted actions: \(l \mapsto (\hat{\odot}, \hat{\Psi}, \hat{\mu})\).

Given a distributive law \(l\), the action of \(\mathcal{C}\) on \(\mathcal{M}\) is lifted to an action of \(\mathcal{C}\) on \(\mathcal{M}^T\) which is of the form \((M, \nu)\hat{\odot}Q := (M\odot Q, \nu^Q)\), where \(\nu^Q\) is the composition

\[
\begin{align*}
T(M\odot Q) &\xrightarrow{\iota_M^Q} TM\odot Q \xrightarrow{\nu^Q} M\odot Q.
\end{align*}
\]

Actually, to live up to our claims we extend these formulas to a bifunctor \(\hat{\odot}\), using the more obvious formulas on morphisms

\[
\begin{align*}
f \odot Q := f \odot Q &\quad \forall f : (M, \nu) \to (N, \xi), \\
f \odot Q g := f \odot g &\quad \forall g : Q \to Q'.
\end{align*}
\]

Pair \((M\odot Q, \nu^Q_M)\) is indeed a \(T\)-module by the commutativity of diagram

\[
\begin{array}{ccc}
TT(M\odot Q) &\xrightarrow{T(\iota_M^Q)}& T(TM\odot Q) &\xrightarrow{T(\mu_{M\odot Q})}& T(M\odot Q) \\
\mu_{M\odot Q} \downarrow & & \downarrow \iota_M^Q & & \downarrow \iota_M^Q \\
TTM\odot Q &\xrightarrow{T\nu^Q_M} & TM\odot Q \\
\mu_{M\odot Q} \downarrow & & \downarrow \nu^Q & & \downarrow \nu^Q \\
T(M\odot Q) &\xrightarrow{\iota_M^Q} & TM\odot Q &\xrightarrow{\nu^Q} & M\odot Q.
\end{array}
\]
The pentagon on the left is (D1), the upper-right square expresses the naturality of \( l^Q \) and the lower-right square is obtained from the action axiom for \( \nu \) by applying \( \Diamond Q \). The action \( \nu^Q : T(M \Diamond Q) \to M \Diamond Q \) is unital as \( \nu^Q \circ \eta_{M \Diamond Q} = (\nu \Diamond Q) \circ l^Q_M \circ \eta_{MOQ} = (\nu \Diamond Q) \circ (\eta_M \Diamond Q) = (\nu \circ \eta_M) \Diamond Q = \text{id}_M \Diamond Q = \text{id}_{M \Diamond Q} \).

If \( f : (M, \nu) \to (N, \xi) \) then \( f \Diamond Q \) in \( M \) is indeed a \( T \)-module map from \((M \Diamond Q, \nu^Q)\) to \((N \Diamond Q, \xi^Q)\), i.e. \((f \Diamond Q)\nu = T(f \Diamond Q)\xi^Q\) by the commutativity of the following diagram:

\[
\begin{array}{cccc}
T(M \Diamond Q) & \xrightarrow{T(f \Diamond Q)} & TM \Diamond Q & \xrightarrow{T(f \Diamond Q)} & M \Diamond Q \\
\downarrow \quad \nu^Q & & \downarrow \quad \nu^Q & & \downarrow \quad \nu^Q \\
T(N \Diamond Q) & \xrightarrow{T(f \Diamond Q)} & TN \Diamond Q & \xrightarrow{T(f \Diamond Q)} & N \Diamond Q.
\end{array}
\]

Here the left square is commutative by the naturality of \( l^Q \) and the right square by the functoriality of \( \Diamond \) in the first variable.

\( u_l : \text{Id} \Diamond l \Rightarrow \text{Id} \) is defined to be given simply by \((u_l)_{(M, \nu)} := u_M \). It is indeed a map of \( T \)-modules \((u_l)_{(M, \nu)} : (M, \nu) \Diamond l 1 = (M \Diamond 1, \nu^1) \to (M, \nu) \) by axiom (D4) and naturality of \( u \):

\[
\begin{array}{cccc}
TM & \xrightarrow{T(u_M)} & T(M \Diamond 1) & \\
\downarrow \quad \nu & & \downarrow \quad \nu^1 & \\
M & \xrightarrow{u_M} & M \Diamond 1.
\end{array}
\]

The required associativity maps of \( T \)-modules \((\Phi_l)_{Q,Q'}^{Q_1,Q_2} : ((M, \nu) \Diamond l (Q \otimes Q'), \nu^{Q \otimes Q'}) \to (((M, \nu) \Diamond l Q) \otimes l Q', (\nu Q)^Q) \) are defined to be identical to the maps of underlying objects in \( M \), \( \Phi_{Q,Q'}^{Q_1,Q_2} : M \Diamond (Q \otimes Q') \to (M \Diamond Q) \otimes Q', \) but one has to check that they are in fact maps of the \( T \)-modules:

\[
\begin{array}{cccc}
T(M \Diamond (Q \otimes Q')) & \xrightarrow{\nu^{Q \otimes Q'}} & TM \Diamond (Q \otimes Q') & \xrightarrow{\nu^{Q \otimes Q'}} & M \Diamond (Q \otimes Q') \\
\downarrow \quad T\psi_{QM}^{Q,Q'} & & \Downarrow \quad \psi_{QM}^{Q,Q'} & & \Downarrow \quad \psi_{QM}^{Q,Q'} \\
T((M \Diamond Q) \Diamond Q') & \xrightarrow{\nu^{Q \otimes Q'} & & (T \Diamond Q) \Diamond Q' & \xrightarrow{l^Q_M \circ Q} & (M \Diamond Q) \Diamond Q'.
\end{array}
\]

The left square is commutative by the definition of the distributive laws and the right by the naturality of \( \Psi_{Q,Q'}^{Q,Q} \). Hence the commutativity of the
external (round) square follows, expressing the fact that $Ψ_{M,Q,Q'}$ is a $T$-module map. Consequently, setting $\tilde{Ψ}_{(M,ν)} := Ψ_{M,Q,Q'}$ is meaningful; and the action pentagon for $\tilde{Ψ}$ is automatically commutative in category $\tilde{M}$; the same for compatibilities with $u_t$.

II. From lifted actions to distributive laws: $(\tilde{♦}, \tilde{Ψ}, \tilde{u}) \mapsto l$.

We will repeatedly use the notation and properties from $\text{11}$.

The action axiom ensures that any $T$-action $ν$ is a map of $T$-modules $ν : (TM, μ_M) \to (M, ν)$. As $\tilde{♦}$ is a bifunctor, the map $ν\tilde{♦}Q$ is also a map of $T$-modules, $ν\tilde{♦}Q : (TM, μ_M)\tilde{♦}Q \to (M, ν)\tilde{♦}Q$.

That means that the following diagram in $\mathcal{M}$ commutes:

$$
\begin{array}{ccc}
T(TM\diamond Q)\tilde{µ}_M^Q & \xrightarrow{T(ν\diamond Q)} & TM\diamond Q \\
\downarrow & & \downarrow ν\diamond Q \\
T(M\diamond Q) & \xrightarrow{ν\diamond Q} & M\diamond Q,
\end{array}
$$

where we also used that $U^T(ν\diamond Q) = (ν\diamond Q)U^T$. This implies that

$$(ν\diamond Q) \circ T(ν\diamond Q) = ν\diamond Q \circ T(ν\diamond Q) = ν\diamond Q \circ T((ν \circ η_M)\diamond Q) = ν\diamond Q \circ T(id_M\diamond Q) = ν\diamond Q \circ id_{(M\diamond Q)} = ν\diamond Q.$$ 

Therefore if we set

$$T(M\diamond Q) \xrightarrow{T(η_M\diamond Q)} T(TM\diamond Q) \xrightarrow{\tilde{µ}_M^Q = U(ε_{FM\diamond Q})} TM\diamond Q,$$

that will define a candidate for a distributive law $l = \{l^Q\}_{Q \in \mathcal{C}}$ for which $ν\diamond Q = (ν\diamond Q) \circ l^Q_M ≡ ν\diamond Q$ for all $T$-modules $(M, ν)$ (cf. the notation $ν\diamond Q$ from part I). This means, in effect, that the lift $\diamond l Q$ agrees with $\diamond Q$ (*).

We have to check that this family is really a distributive law. The fact that $l^Q_M$ form a natural transformation $l^Q$ follows as for every $f : M \to N$ in $\mathcal{M}$ the diagram

$$
\begin{array}{ccc}
T(M\diamond Q) & \xrightarrow{T(η_M\diamond Q)} & T(TM\diamond Q) \xrightarrow{\tilde{µ}_M^Q} TM\diamond Q \\
\downarrow T(f\diamond Q) & & \downarrow T(f) \\
T(N\diamond Q) & \xrightarrow{T(η_N\diamond Q)} & T(TN\diamond Q) \xrightarrow{\tilde{µ}_N^Q} TN\diamond Q
\end{array}
$$

8
commutes, where we identified $\tilde{\mu}_M$ with the corresponding map in $\mathcal{M}$.

Axiom (D3) reads $\eta_M \circ Q = l_Q^M \circ \eta_M \circ Q = U(\epsilon_{FM\hat{Q}})T(\eta_M \circ Q)\eta_M \circ Q$ hence it follows by the commutativity of the pentagon

\[
\begin{array}{c}
M \circ Q \xrightarrow{\eta_M \circ Q} TM \circ Q \xrightarrow{id} TM \circ Q \\
T(M \circ Q) \xrightarrow{T(\eta_M \circ Q)} T(TM \circ Q) \xrightarrow{U(\epsilon_{FM\hat{Q}})} T(TM \circ Q),
\end{array}
\]

where the right-hand square is commutative by naturality of $\eta$ and the triangle represents the adjunction identity $U(\epsilon_Z) \circ \eta_{UZ} = \text{id}_Z$ for $Z = FM\hat{Q}$.

Axiom (D1) now follows by the commutativity of

\[
\begin{array}{c}
TT(M \circ Q) \xrightarrow{T(\eta_M \circ Q)} TT(TM \circ Q) \xrightarrow{T(\mu_M \circ Q)} T(TM \circ Q) \xrightarrow{T(TT \circ Q)} TTT \circ Q \\
T(M \circ Q) \xrightarrow{T(\eta_M \circ Q)} T(TM \circ Q) \xrightarrow{T(TM \circ Q)} T(TM \circ Q) \xrightarrow{T(\mu_M \circ Q)} T(TM \circ Q),
\end{array}
\]

Here the right-hand square is clearly just following from a naturality of $\mu$, and the little triangle in the middle from $\mu_M \circ \eta_{TM} = \text{id}_M$. The middle pentagon follows after translating all 4 nontrivial maps in terms of counit $\epsilon$ of the adjunction. Namely, using $TM \circ Q = U(FM\hat{Q})$, $\mu_X = U(\epsilon_{UX})$ and $\tilde{\mu}_X^Q = U(\epsilon_{FX\hat{Q}})$, we can write the pentagon (after erasing id-leg) as a square

\[
\begin{array}{c}
UFU \circ (FM\hat{Q}) \xrightarrow{UFU(\epsilon_{FM\hat{Q}})} UFU \circ (FM\hat{Q}) \\
UFU \circ (FM\hat{Q}) \xrightarrow{UFU(\epsilon_{FM\hat{Q}})} UFU \circ (FM\hat{Q}) \xrightarrow{UFU(\epsilon_{FM\hat{Q}})} UFU \circ (FM\hat{Q})
\end{array}
\]

which is commutative by the naturality of $\epsilon$ (without $\hat{Q}$ everywhere this is nothing but the associativity diagram for $\mu$). It is similar with the right-hand most rectangle which follows from

\[
\begin{array}{c}
FU(FUF \circ (FM\hat{Q}) \xrightarrow{FU(\epsilon_{FM\hat{Q}})} FU(FM\hat{Q}) \\
FU(FM\hat{Q}) \xrightarrow{FU(\epsilon_{FM\hat{Q}})} FU(FM\hat{Q}),
\end{array}
\]
after applying functor $U$ to every piece of the diagram and straightforward renamings; in this diagram $\mu_M$ is understood as a map $\epsilon_{FM} : FUFM \to FM$ rather than $U(\epsilon_{FM})$.

To show (D2), one inspects in a similar manner this diagram:

$$
\begin{array}{c}
T(M \diamond (Q \otimes Q')) \\
\downarrow \gamma \Psi_{Q'}^Q \quad \downarrow T(\eta_{MQ})^Q \quad \downarrow \gamma T(M \diamond Q) \\
T((M \diamond Q) \diamond Q') \\
\downarrow T(\eta_{M} \diamond Q) \\
T(T(M \diamond Q) \diamond Q') \\
\downarrow T((T(M \diamond Q) \diamond Q') \\
\downarrow U(\epsilon_{FM} Q \otimes Q') \\
T(M \diamond Q) \diamond Q' \\
\end{array}
$$

Finally, to show (D4), it is enough to consider the diagram

$$
\begin{array}{c}
T(M) \\
\downarrow T(\eta_M) \\
T(M \diamond 1) \\
\downarrow T(\eta_M \diamond 1) \\
T(T(M \diamond 1) \\
\downarrow T(u_{M}) \\
T(M \diamond 1). \\
\end{array}
$$

Finally, the fact that the correspondences $\hat{\diamond} \mapsto l$ and $l \mapsto \diamond_l$ are inverses at one side has been shown above (see (*)); and the other composition is left to the reader. Then the proof of the theorem is finished.

13 Given a commutative unital ring $k$ and a $k$-coalgebra $C$ with coproduct $\Delta = \Delta^C : c \mapsto \sum c_{(1)} \otimes_k c_{(2)}$ (Sweedler notation, \[17\]) we can consider the category $C = C^C$ of right $C$-comodules.

Suppose now $B = C$ is a bialgebra. Then the category $C^B$ is monoidal. Namely, if the coproduct on object $Q$ in $C$ is given, in Sweedler-like notation, by $x \mapsto \sum q_{(0)} \otimes q_{(1)}$ then the tensor product in $C$ is the tensor product $Q \otimes_k Q'$ of $k$-modules with the coaction $q \otimes q' \mapsto \sum q_{(0)} \otimes_k q_{(0)}' \otimes_k q_{(1)} q_{(1)}'$. Bialgebra $B$ is a monoid in $C^B$, hence it induces a monad in any category on which $C^B$ acts. Category $C^B$ acts on the category $V := k - \text{Mod}$ of $k$-modules via the "forgetful" right action, namely $V \diamond Q$ is $V \otimes_k Q$ as a $k$-module. In particular, $B$ induces a monoid with the underlying endofunctor $V \mapsto V \otimes_k B$ in $V$ as well.
A left $B$-module algebra $A = (A, \mu^A, \eta^A)$ is an associative unital algebra with a left $B$-action $\triangleright_A$ for which $b \triangleright_A (aa') = \sum (b_{(1)} \triangleright_A a)(b_{(1)} \triangleright_A a')$. Category $\hat{\mathcal{V}} = A - \text{Mod}$ of left $A$-modules is, in general, not monoidal (as $A$ is not commutative), but it is equipped with the forgetful functor $F : \hat{\mathcal{V}} \to \mathcal{V}$.

Algebra $A$ induces a natural monad $\mathbf{T} = (T, \mu, \eta)$ in $\mathcal{V}$ given by $T(V) = A \otimes_k V$ with $\mu_V = \mu^A \otimes_k V$. Clearly, $\mathbf{T}$-modules are the same thing as the left $A$-modules, as $\nu : TM = A \otimes M \to M$ is a $\mathbf{T}$-action iff it is a left $A$-action.

We may consider the problem of lifting the forgetful action of $\mathcal{C}^B$ on $\mathcal{V}$ to $\mathbf{T} - \text{Mod} \equiv (\mathcal{C}^B)^\mathbf{T}$. There is a natural distributive law $l$ from $\triangleleft$ to $\mathbf{T}$ compatible with $F$. Namely given a module $M$ with action $\triangleright_M : A \otimes M \to M$, $l^Q_M : T(M \triangleleft Q) = A \otimes (M \triangleleft Q) \to (A \otimes M \triangleleft Q) = TM \triangleleft Q$ is given by $a \otimes (m \triangleleft q) \mapsto \sum (q_{(1)} \triangleright_A a \otimes m) \otimes q_{(0)}$. Hence the induced action is $(M, \nu) \triangleleft l Q = (M \otimes Q, \nu^Q)$ where $a \triangleright_{TM} (m \triangleleft q) \equiv \nu^Q(a \otimes (m \triangleleft q)) = \sum (q_{(1)} \triangleright_A a) \triangleright_{TM} m \otimes q_{(0)}$.

Consider the associated monoidal functor $\mathcal{L} : \mathcal{C} \to \text{End } \hat{\mathcal{V}}$ for the lifted action. As the bialgebra itself is a monoid in $\mathcal{C}^B$, its image is also a monoid in End $\hat{\mathcal{V}}$, that is a monad in $\hat{\mathcal{V}}$. Call this new monad $\mathcal{L}_B$. Then $\mathcal{L}_B(M, \nu) = (M \otimes B, \nu^B)$. Its multiplication $\mu' = \mu^{\mathcal{L}_B} : M \otimes B \otimes B \to M \otimes B$ is simply $M \otimes \mu^B$. We leave as an exercise for the reader to analyse that the Eilenberg-Moore category $\hat{\mathcal{V}}^{\mathcal{L}_B}$ of this monad is equivalent to a certain category of $k$-modules equipped with two additional actions (the left action of $A$ inherited from $\mathcal{V}$ and a new right action of $B$) with the compatibility condition $a \triangleright (n \triangleright h) = [(h_{(2)} \triangleright_A a) \triangleright n] \triangleright h_{(1)}$. $\hat{\mathcal{V}}$ is not monoidal, and even if it is (if $A$ is commutative, say), monad $\mathcal{L}_B$ is not induced by tensoring by an algebra in $\hat{\mathcal{V}}$. Instead, $B$ is an algebra in another monoidal category – namely in $\mathcal{C}^B$; but also in the underlying category $\mathcal{V}$.

This simple example has a number of useful generalizations and applications which will be addressed elsewhere.

14 There are other flavours of compatibility between tensor structures and monads. Instead of $\mathcal{C}$-categories in the sense of actions, one can consider the different notion of categories enriched over $\mathcal{C}$ (i.e. the hom-sets consist of $\mathcal{C}$-objects, and carry tensor products with some natural properties), those are also called $\mathcal{C}$-categories; monads in such categories were explored before (18 [29]). One can also consider the monads in $\mathcal{C}$-itself, cf. [18 [19]. There are also various dual (left-hand, opmonoidal, comonadic etc.) versions.

14a Such a “dual” version which may be less obvious is as follows. Given $\mathcal{C}$-category $\mathcal{M}$ in our sense (i.e. with action $\triangleleft$), one considers the category $\mathcal{M}^\triangleright$ of $\mathcal{C}$-modules in $\mathcal{M}$, i.e. objects $M$ in $\mathcal{M}$ with families of maps $\nu_Q : M \triangleleft Q \to M$ satisfying a list of required identities making those families “actions”. Such modules also make a category in a natural way and one can consider liftings of a monad $\mathbf{T}$ in $\mathcal{M}$ to that category. The liftings will then
be given by another flavour of distributive laws.

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