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EXISTENCE AND CLASSIFICATION OF CHARACTERISTIC POINTS AT BLOW-UP FOR A SEMILINEAR WAVE EQUATION IN ONE SPACE DIMENSION

By FRANK MERLE and HATEM ZAAG

Abstract. We consider the semilinear wave equation with power nonlinearity in one space dimension. We first show the existence of a blow-up solution with a characteristic point. Then, we consider an arbitrary blow-up solution $u(x,t)$, the graph $x \mapsto T(x)$ of its blow-up points and $S \subset \mathbb{R}$ the set of all characteristic points and show that $S$ has an empty interior. Finally, given $x_0 \in S$, we show that in selfsimilar variables, the solution decomposes into a decoupled sum of (at least two) solitons, with alternate signs and that $T(x)$ forms a corner of angle $\frac{\pi}{2}$ at $x_0$.

1. Introduction.

1.1. Known results and the case of non-characteristic points. We consider the one dimensional semilinear wave equation

\begin{equation}
\tag{1.1}
\begin{cases}
\partial_{tt} u = \partial_{xx} u + |u|^{p-1} u, \\
u(0) = u_0 \text{ and } u_t(0) = u_1,
\end{cases}
\end{equation}

where $u(t) : x \in \mathbb{R} \rightarrow u(x,t) \in \mathbb{R}$, $p > 1$, $u_0 \in H^1_{\text{loc},u}$ and $u_1 \in L^2_{\text{loc},u}$ with $\|v\|_{L^2_{\text{loc},u}} = \sup_{a \in \mathbb{R}} \int_{|x-a|<1} |v(x)|^2 \, dx$ and $\|v\|_{H^1_{\text{loc},u}}^2 = \|v\|_{L^2_{\text{loc},u}}^2 + \|\nabla v\|_{L^2_{\text{loc},u}}^2$ (we will also consider below a twin equation (4.16)). The Cauchy problem for equation (1.1) in the space $H^1_{\text{loc},u} \times L^2_{\text{loc},u}$ follows from the finite speed of propagation and the well-posedness in $H^1 \times L^2$ (see Ginibre, Soffer and Velo [6]). The existence of blow-up solutions for equation (1.1) follows from Levine [9]. More blow-up results can be found in Caffarelli and Friedman [4, 5], Alinhac [1, 2] and Kichenassamy and Littman [7, 8].

If $u$ is a blow-up solution of (1.1), we define (see for example Alinhac [1]) a 1-Lipschitz curve $\Gamma = \{(x,T(x))\}$ such that the maximal influence domain $D$ of $u$ (or the domain of definition $D$ of $u$) is written as:

\begin{equation}
\tag{1.2}
D = \{(x,t) \mid t < T(x)\}.
\end{equation}
\( \bar{T} = \inf_{x \in \mathbb{R}} T(x) \) and \( \Gamma \) are called the blow-up time and the blow-up graph of \( u \). An important notion for the blow-up graph is the notion of characteristic point (even though the existence of characteristic points remained unknown before this paper).

A point \( x_0 \) is a non-characteristic point (or a regular point) if

\[
\delta_0 \in (0, 1) \text{ and } t_0 < T(x_0) \text{ such that } u \text{ is defined on } C_{x_0,T(x_0),\delta_0} \cap \{ t \geq t_0 \}
\]

where \( C_{x_0,T(x_0),\delta_0} = \{ (x,t) \mid t < \bar{t} - \delta |x - \bar{x}| \} \). We denote by \( \mathcal{R} \) (resp. \( \mathcal{S} \)) the set of non-characteristic (resp. characteristic) points.

Following our earlier work \([10, 11]\), we aim at describing the blow-up behavior for any blow-up solution, especially \( \Gamma \) and the solution near \( \Gamma \).

Given some \( (x_0, T_0) \) such that \( 0 < T_0 \leq T(x_0) \), a natural tool is to introduce the following self-similar change of variables:

\[
w_{x_0,T_0}(y, s) = (T_0 - t)^{\frac{2}{p-1}} u(x, t), \quad y = \frac{x - x_0}{T_0 - t}, \quad s = -\log \left( T_0 - t \right).
\]

If \( T_0 = T(x_0) \), then we simply write \( w_{x_0} \) instead of \( w_{x_0,T(x_0)} \). The function \( w = w_{x_0,T_0} \) satisfies the following equation for all \( y \in B = B(0,1) \) and \( s \geq -\log T_0 \):

\[
\partial_{ss}^2 w = L w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial_{y,s}^2 w
\]

where

\[
L w = \frac{1}{\rho} \partial_y \left( \rho (1 - y^2) \partial_y w \right) \text{ and } \rho(y) = (1 - y^2)^{\frac{2}{p-1}}.
\]

The Lyapunov functional for equation (1.5)

\[
E(w(s)) = \int_{-1}^{1} \left( \frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1 - y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy
\]

is defined for \((w, \partial_s w) \in \mathcal{H}\) where

\[
\mathcal{H} = \left\{ q \mid \|q\|_{\mathcal{H}}^2 = \int_{-1}^{1} \left( q_1^2 + (q'_1)^2 (1 - y^2) + q_2^2 \right) \rho dy < +\infty \right\}.
\]

We will note

\[
\mathcal{H}_0 = \left\{ r \in H^1_{\text{loc}}(-1,1) \mid \|r\|_{\mathcal{H}_0}^2 = \int_{-1}^{1} (r^2 (1 - y^2) + r^2) \rho dy < +\infty \right\}.
\]
We also introduce for all $|d| < 1$ the following stationary solutions of (1.5) defined by

\begin{equation}
\kappa(d, y) = \kappa_0 \left( \frac{1 - d^2}{(1 + dy)^{\frac{p - 1}{p}}} \right) \text{ where } \kappa_0 = \left( \frac{2(p + 1)}{(p - 1)^2} \right)^{\frac{1}{p - 1}} \text{ and } |y| < 1.
\end{equation}

When $x_0$ is non-characteristic, we have a good understanding of the solution’s behavior near $x_0$. More precisely, we established the following results in [11, 12]:

(Blow-up behavior for $x_0 \in \mathcal{R}$, see Corollary 4 in [11], Theorem 1 (and the following remark) and Lemma 2.2 in [12]).

(i) The set of non-characteristic points $\mathcal{R}$ is non empty and open.

(ii) (Selfsimilar blow-up profile for $x_0 \in \mathcal{R}$) There exist positive $\mu_0$ and $C_0$ such that if $x_0 \in \mathcal{R}$, then there exist $\delta_0(x_0) > 0$, $d(x_0) \in (-1, 1)$, $|\theta(x_0)| = 1$, $s_0(x_0) \geq -\log T(x_0)$ such that for all $s \geq s_0$:

\begin{align}
\| \left( w_{x_0}(s) \right) - \theta(x_0) \left( \kappa(d(x_0), \cdot) \right) \|_{\mathcal{H}} &\leq C_0 e^{-\mu_0(s - s_0)}, \\
\| \left( w_{x_0}(s) \right) - \theta(x_0) \left( \kappa(d(x_0), \cdot) \right) \|_{H^1 \times L^2(|y| < 1 + \delta_0)} &\to 0 \text{ as } s \to \infty.
\end{align}

Moreover, $E(w_{x_0}(s)) \to E(\kappa_0)$ as $s \to \infty$.

(iii) The function $T(x)$ is $C^1$ on $\mathcal{R}$ and for all $x_0 \in \mathcal{R}$, $T'(x_0) = d(x_0) \in (-1, 1)$. Moreover, $\theta(x_0)$ is constant on connected components of $\mathcal{R}$.

1.2. Existence of characteristic points. For characteristic points, the only available result about existence or non-existence is due to Caffarelli and Friedman [4, 5] who proved (using the maximum principle) the non-existence of characteristic points for equation (1.1):

• under conditions on initial data that ensure that for all $x \in \mathbb{R}$ and $t \geq 0, u \geq 0$ and $\partial_t u \geq (1 + \delta_0)|\partial_x u|$ for some $\delta_0 > 0$,

• for $p \geq 3$ with $u_0 \geq 0, u_1 \geq 0$ and $(u_0, u_1) \in C^4 \times C^3(\mathbb{R})$.

From this example, it was generally conjectured by most people that there were no blow-up solutions for equation (1.1) with characteristic points: for all $(u_0, u_1)$ which lead to blow-up, $\mathcal{R} = \mathbb{R}$.

Our first result is to disprove this fact. Existence of characteristic points is seen as a consequence of two facts:

• on the one hand, the study of the blow-up profile at a non-characteristic point,

• on the other hand, connectedness arguments related to the sign of the blow-up profile.

To state our results, let us consider $u(x, t)$ a blow-up solution of equation (1.1) (take for example initial data $(u_0, u_1) \in H^1 \times L^2(\mathbb{R})$ satisfying

\[ \int_{\mathbb{R}} \left( \frac{1}{2} |\partial_x u_0|^2 + \frac{1}{2} u_1^2 - \frac{1}{p + 1} |u_0|^{p + 1} \right) dx < 0, \]
which gives blow-up by Levine [9]). The first result follows from the study near a regular point, that ensures the existence of an explicit signed profile.

**Proposition 1.** If the initial data \((u_0, u_1)\) is odd and \(u(x, t)\) blows up in finite time, then \(0 \in S\).

The second one follows from the continuity of the profile on the connected components of \(R\) (see [12, Theorem 1]).

**Theorem 2.** (Existence and generic stability of characteristic points)
(i) (Existence) Let \(a_1 < a_2\) be two non-characteristic points such that \(w_{a_i}(s) \to \theta(a_i)\kappa(d_{a_i}, \cdot)\) as \(s \to \infty\) with \(\theta(a_1)\theta(a_2) = -1\) for some \(d_{a_i}\) in \((-1, 1)\), in the sense (1.11). Then, there exists a characteristic point \(c \in (a_1, a_2)\).

(ii) (Stability) There exists \(\epsilon_0 > 0\) such that if \(\|(\tilde{u}_0, \tilde{u}_1) - (u_0, u_1)\|_{H_{loc}^1 \times L_{loc}^2} \leq \epsilon_0\), then, \(\tilde{u}(x, t)\) the solution of equation (1.1) with initial data \((\tilde{u}_0, \tilde{u}_1)\) blows up and has a characteristic point \(\tilde{c} \in [a_1, a_2]\).

**Remark.** It is enough to take \((u_0, u_1)\) with large plateaus of opposite signs to guarantee that \(u(x, t)\) blows up satisfying the hypotheses of this theorem.

Since a solution in one space dimension is also a solution in higher dimensions, we get from the finite speed of propagation the following existence result in \(N\) dimensions:

**Corollary 3.** (Existence of characteristic points in higher dimensions) Consider \(\tilde{u}(x_1, t)\) a blow-up solution of (1.1) in one space dimension with a characteristic point. Then, for \(R\) large enough, initial data \((u_0, u_1)\) such that \(u_i(x) = \tilde{u}_i(x_1)\) for \(|x| < R\), the solution \(u(x, t)\) of equation (1.1) with initial data \((u_0, u_1)\) blows up and has a characteristic point.

### 1.3. Non-existence results for characteristic points
In this section, we give sufficient conditions under which no characteristic point can occur. Our analysis in fact relates the fact that \(x_0\) is a characteristic point to sign changes of the solution in a neighborhood of \((x_0, T(x_0))\). We claim the following:

**Theorem 4.** Consider \(u(x, t)\) a blow-up solution of (1.1) such that \(u(x, t) \geq 0\) for all \(x \in (a_0, b_0)\) and \(t_0 \leq t < T(x)\) for some real \(a_0, b_0\) and \(t_0 \geq 0\). Then, \((a_0, b_0) \subset R\).

**Remark.** This result can be seen as a generalization of the result of Caffarelli and Friedman, with no restriction on initial data. Indeed, from our result, taking nonnegative initial data suffices to exclude the occurrence of characteristic points.
1.4. Shape of the blow-up set near characteristic points and properties of $S$. We have the following proposition:

**Proposition 5.** ($S$ has an empty interior) Consider $u(x,t)$ a blow-up solution of (1.1). The set of characteristic points $S$ has an empty interior.

**Remark.** This implies in particular that $S$ has zero measure. Direct arguments give no more than the fact that $S \neq \mathbb{R}$ (a point $x_0$ such that $T(x_0)$ is the blow-up time is non-characteristic; see in Section 1.1, point (i) of the result of [11, 12]). The proof of Proposition 5 uses the description of the solution in the $w$ variable as a non trivial decoupled sum of (at least 2) solitons $\pm \kappa(d_i(s))$ (see Theorem 6 below).

Now, we have the following theorem, which is the main result of our analysis, where for a given $x_0 \in S$, we are able to give the precise behavior of the solution near $(x_0, T(x_0))$:

**Theorem 6.** (Description of the behavior of $w_{x_0}$ where $x_0$ is characteristic) Consider $u(x,t)$ a blow-up solution of (1.1) and $x_0 \in S$. Then, it holds that

\[
\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \left( \sum_{i=1}^{k(x_0)} e_i^* \kappa(d_i(s), \cdot) \right) \right\|_{\mathcal{H}} \to 0 \text{ and } E(w_{x_0}(s)) \to k(x_0) E(\kappa_0)
\]

as $s \to \infty$, for some

\[k(x_0) \geq 2, \quad e_i^* = e_1^*(-1)^{i+1}, \quad e_1^* = \pm 1\]

and continuous $d_i(s) = -\tanh \zeta_i(s) \in (-1,1)$ for $i = 1, \ldots, k(x_0)$. Moreover, for some $C_0 > 0$, for all $i = 1, \ldots, k(x_0)$ and $s$ large enough,

\[
\left| \zeta_i(s) - \left( i - \frac{k(x_0) + 1}{2} \right) \frac{(p-1)}{2} \log s \right| \leq C_0.
\]

**Remark.** In [11], we proved a much weaker version of this result, with (1.13) valid just with $k(x_0) \geq 0$ and no information on the signs of $e_i^*$, or the position of $\zeta_i(s)$. Note that eliminating the case

\[k(x_0) = 0\]

is the most difficult part in our analysis. In some sense, we put in relation the notion of characteristic point at $x_0$ and the notion of decomposition of $w_{x_0}$ in a decoupled sum of (at least 2) solitons $\pm \kappa(d_i(s))$. This result can be seen as a result of decomposition up to dispersion into a sum of decoupling solitons in dispersive problems. According to the value of $k(x_0)$, this sum appears to have a multipole nature (dipole if $k(x_0) = 2$, tripole if $k(x_0) = 3, \ldots$). In some sense, it says that the
space $H$ is a critical space to measure dispersion and blow-up in the cone (i.e. if $E(w_{x_0},T_0) < E(\kappa_0)$, then:

- $T(x_0) > T_0$ and the solution can be extended in a strictly greater cone;
- $(w_{x_0, T_0}, \partial_s w_{x_0, T_0})$ converges as $s \to \infty$ to zero in $H^1 \times L^2((-1,1))$.

**Remark.** Estimate (1.14) comes from the fact that the centers’ positions $\zeta_i(s)$ satisfy the finite-dimensional system given in (3.3). From (1.14), we see that the distance between two solitons $\zeta_{i+1}(s) - \zeta_i(s) \sim \frac{(p-1)}{2} \log s = \frac{(p-1)}{2} \log |\log(T-t)|$ as $t \to T(x_0)$. If $r = E(\frac{k(x_0)}{2})$ then we see from (1.14) that $r$ solitons go to $-\infty$ as $s \to \infty$ (for $i = 1, \ldots, r$) and $r$ solitons go to $+\infty$ (for $i = k(x_0) + 1 - r, \ldots, k(x_0)$). If $k(x_0) = 2r + 1$, then the central soliton (for $i = r + 1 = \frac{k(x_0)+1}{2}$) stays bounded for $s$ large.

Extending the definition of $k(x_0)$ defined for $x_0 \in S$ in Theorem 6 by setting $k(x_0) = 1$ for all $x_0 \in \mathcal{R}$ and using the monotonicity of the Lyapunov functional $E(w)$, we have the following consequence from the blow-up behavior in the characteristic case (Theorem 6) and in the non-characteristic case (the result of [11] cited here in Section 1.1):

**Corollary 7.** (A criterion for non-characteristic points)

(i) For all $x_0 \in \mathbb{R}$ and $s_0 \geq -\log T(x_0)$, we have

$$E(w_{x_0}(s_0)) \geq k(x_0) E(\kappa_0).$$

(ii) If for some $x_0$ and $s_0 \geq -\log T(x_0)$, we have

$$E(w_{x_0}(s_0)) < 2E(\kappa_0),$$

then $x_0 \in \mathcal{R}$.

We also have the following consequences in the original variables:

**Proposition 8.** (Description of $T(x)$ for $x$ near $x_0$)

(i) If $x_0 \in S$ and $0 < |x - x_0| \leq \delta_0$, then

$$0 < T(x) - T(x_0) + |x - x_0| \leq \frac{C_0 |x - x_0|}{\log (x - x_0)^{(k(x_0)-1)(p-1)/2}}$$

for some $\delta_0 > 0$ and $C_0$, where $k(x_0)$ is defined in Theorem 6.

(ii) If $x_0 \in S$, then $T(x)$ is right and left differentiable at $x_0$, with

$$T'_l(x_0) = 1 \text{ and } T'_r(x_0) = -1.$$
(iii) For all $t \in [T(x_0) - \tau_0, T(x_0))$ for some $\tau_0 > 0$, there exist $z_1(t) < \cdots < z_k(t)$ continuous in $t$ such that

$$e_1^* (-1)^{i+1} u(z_i(t), t) > 0$$

and $z_i(t) \to x_0$ as $t \to T(x_0)$.

**Remark.** From (iii), we have the existence of zero lines $x_1(t) < \cdots < x_{k-1}(t)$ (not necessarily continuous in $t$) such that $u(x_i(t), t) = 0$ and $x_i(t) \to x_0$ as $t \to T(x_0)$.

**Remark.** In a forthcoming paper [13], we improve (1.15) by finding a lower bound of the same type as the upper bound.

The paper is organized as follows. Section 2 is devoted to the proofs of Proposition 1 and Theorem 2 (note that Corollary 3 follows straightforwardly from Theorem 2 and the finite speed of propagation). In Section 3, we consider a characteristic point and study the equation in selfsimilar variables. As for Section 4, it is devoted to the proof of Theorems 6, 4 and 4', as well as Propositions 5 and 8 (note that Corollary 7 is a direct consequence of Theorem 6 and the result of [11] cited here in Section 1.1).

### 2. Existence and stability of characteristic points.

Here in this section, we consider $u(x, t)$ a blow-up solution of equation (1.1). As mentioned in the introduction, we prove in this section the existence of characteristic points (Proposition 1 and Theorem 2).

**Proof of Proposition 1.** Assuming that $(u_0, u_1)$ is odd, we would like to prove that $0 \in S$. Arguing by contradiction, we assume that $0 \in R$.

On the one hand, using the result of [11] stated in (1.12), we see that for some $d(0) \in (-1, 1)$ and $\theta = \pm 1$,

$$\| w_0(s) - \theta \kappa(d(0), \cdot) \|_{L^\infty(-1, 1)} \leq C \| w_0(s) - \theta \kappa(d(0), \cdot) \|_{H^1(-1, 1)} \longrightarrow 0 \quad \text{as} \quad s \longrightarrow \infty.$$

In particular,

$$| w_0(0, s) | \longrightarrow \kappa(d(0), 0) > 0 \quad \text{as} \quad s \longrightarrow \infty. \quad (2.1)$$

On the other hand, since the initial data is odd, the same holds for the solution, in particular, $u(0, t) = 0$ for all $t \in [0, T(0))$, hence $w_0(0, s) = 0$ for all $s \geq - \log T(0)$, which contradicts (2.1). This concludes the proof of Proposition 1. \qed

**Remark.** We do not need to know that for $x_0 \in R$, $w_{x_0}$ converges to a particular profile to derive this result. It is enough to know that $w_{x_0}$ approaches the set \{ $\theta(x_0) \kappa(d, \cdot) \mid |d| < 1 - \eta$ \} for some $\eta > 0$, which is a much weaker result.
We now turn to the proof of Theorem 2. It is a consequence of three results from our earlier work:

- the continuity with respect to initial data of the blow-up time at \( x_0 \in \mathcal{R} \).

**PROPOSITION 2.1.** (Continuity with respect to initial data at \( x_0 \in \mathcal{R} \)) There exists \( A_0 > 0 \) such that if \( \tilde{T}(x_0) \to T(x_0) \) as \((\tilde{u}_0, \tilde{u}_1) \to (u_0, u_1)\) in \( H^1 \times L^2(|x| < A_0) \), where \( \tilde{T}(x_0) \) is the blow-up time of \( \tilde{u}(x, t) \) at \( x = x_0 \), the solution of equation (1.1) with initial data \((\tilde{u}_0, \tilde{u}_1)\).

**Proof.** This is a direct consequence of the Liouville Theorem and its applications given in [12]. See Appendix A for a sketch of the proof. \(\Box\)

- the continuity of the blow-up profile on \( \mathcal{R} \) proved in Theorem 1 in [12] (in particular, the fact that \( \theta(x_0) \) given in (1.11) is constant on the connected components of \( \mathcal{R} \).

- the following trapping result from [11]:

**PROPOSITION 2.2.** (See Theorem 3 in [11] and its proof) There exists \( \epsilon_0 > 0 \) such that if \( w \in C([s^*, \infty), \mathcal{H}) \) for some \( s^* \in \mathbb{R} \) is a solution of equation (1.5) such that

\[
\forall s \geq s^*, \quad E(w(s)) \geq E(\kappa_0) \quad \text{and} \quad \| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d^*, \cdot) \\ 0 \end{pmatrix} \|_{\mathcal{H}} \leq \epsilon^*
\]

for some \( d^* = -\tanh \xi^* \), \( \omega^* = \pm 1 \) and \( \epsilon^* \in (0, \epsilon_0] \), then there exists \( d_\infty = -\tanh \xi_\infty \) such that

\[
|\xi_\infty - \xi^*| \leq C_0 \epsilon^* \quad \text{and} \quad \| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d_\infty, \cdot) \\ 0 \end{pmatrix} \|_{\mathcal{H}} \rightarrow 0.
\]

Let us use these results to prove Theorem 2.

**Proof of Theorem 2.** We consider \( a_1 < a_2 \) two non-characteristic points such that \( w_{a_i}(s) \to \theta(a_i) \kappa(d_{a_i}, \cdot) \) with \( \theta(a_1) \theta(a_2) = -1 \) for some \( d_{a_i} \) in \((-1, 1)\), in the sense (1.11). Up to changing \( u \) in \(-u\), we can assume that \( \theta(a_1) = 1 \) and \( \theta(a_2) = -1 \). We aim at proving that \( (a_1, a_2) \cap \mathcal{S} \neq \emptyset \) and the stability of such a property with respect to initial data.

(i) If we assume by contradiction that \([a_1, a_2] \subset \mathcal{R} \), then the continuity of \( \theta(x_0) \) where \( x_0 \in [a_1, a_2] \) implies that \( \theta(x_0) \) is constant on \([a_1, a_2] \). This is a contradiction, since \( \theta(a_1) = 1 \) and \( \theta(a_2) = -1 \).

(ii) By hypothesis and estimate (1.12), there is \( \delta_0 > 0 \) and \( s_0 \in \mathbb{R} \) such that

\[
\left\| \begin{pmatrix} w_{a_i}(s_0) \\ \partial_s w_{a_i}(s_0) \end{pmatrix} - \theta(a_i) \begin{pmatrix} \kappa(d_{a_i}, \cdot) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(|y| < 1 + \delta_0)} \leq \frac{\epsilon_0}{2}
\]
where ε₀ is defined in Proposition 2.2. From the continuity with respect to initial data for equation (1.1) at the fixed time $T(a_i) - e^{-s_0}$ and the continuity of the blow-up time with respect to initial data (Proposition 2.1), we see there exists $η(ε₀) > 0$ such that if
\[
\| (\tilde{u}_0, \tilde{u}_1) - (u_0, u_1) \|_{H^1_{loc,u} \times L^2_{loc,u}} \leq η,
\]
then $\tilde{u}(x, t)$ the solution of equation (1.1) with initial data $(\tilde{u}_0, \tilde{u}_1)$ is such that $\tilde{w}_{α_i}(y, s_0)$ is defined for all $|y| < 1 + δ_0/2$
\[
\begin{align*}
\| \left( \frac{\tilde{w}_{α_i}(s_0)}{∂_s \tilde{w}_{α_i}(s_0)} \right) - \theta(a_i) \left( k(d_{α_i}, \cdot) \right) \|_{H^1 \times L^2(|y| < 1 + δ_0/2)} &\leq \frac{3}{4} ε₀, \text{ and} \\
\| \left( \frac{\tilde{w}_{α_i}(s_0)}{∂_s \tilde{w}_{α_i}(s_0)} \right) - \theta(a_i) \left( k(d_{α_i}, \cdot) \right) \|_H &\leq ε₀.
\end{align*}
\]
where $\tilde{w}_{α_i}$ is the selfsimilar version defined from $\tilde{u}(x, t)$ by (1.4).

Two cases then arise (by the way, we will prove later in Theorem 6 that the Lyapunov functional stays above $2E(κ_0)$ at a characteristic point, which means by (2.4) that $a_1$ and $a_2$ are non-characteristic points for $η$ small enough, but we cannot use Theorem 6 for the moment):

- If $a_1$ or $a_2$ is a characteristic point of $\tilde{u}(t)$, then the proof is finished.
- Otherwise, (1.11) holds for $\tilde{w}_{α_i}$ from the fact that the point is non-characteristic. Thus, from the monotonicity of $E(\tilde{w}_{α_i}(s))$, (2.2) holds with $ω^* = \theta(a_i)$. Applying Proposition 2.2, we see that $\tilde{w}_{α_i}(s) \to \theta(a_i)k(\tilde{d}_{α_i}, \cdot)$ as $s \to ∞$, for some $\tilde{d}_{α_i} \in (-1, 1)$. Noting that $θ(a_1) = 1$ and $θ(a_2) = -1$, we apply (i) to get the result. This concludes the proof of Theorem 2. ∎

3. Refined behavior for $w_{x_0}$ where $x_0$ is characteristic. In this section, we consider $x_0 \in S$. We know from [12] that
\[
\begin{align*}
(3.1) \quad &\left\| \left( \frac{w_{x_0}(s)}{∂_s w_{x_0}} \right)^{k(x_0)} - ∑_{i=1}^{k(x_0)} e_i \left( k(d_i(s), \cdot) \right) \right\|_H \to 0 \quad \text{as} \quad s \to ∞
\end{align*}
\]
for some $k(x_0) ≥ 0$, $e_i = ±1$ and continuous $d_i(s) = -\tanh ζ_i(s) \in (-1, 1)$ for $i = 1, \ldots, k(x_0)$ with
\[
(3.2) \quad ζ_1(s) < \cdots < ζ_{k(x_0)}(s) \text{ and } ζ_{i+1}(s) - ζ_i(s) \to ∞ \quad \text{for all} \quad i = 1, \ldots, k - 1.
\]
Since $w_{x_0}(s)$ is convergent when $k(x_0) ≤ 1$ (to 0 when $k(x_0) = 0$ and to some $κ(d_∞)$ by Proposition 2.2), we focus throughout this section on the case
\[
k(x_0) \geq 2.
\]
For simplicity in the notations, we forget the dependence of $w_{x_0}$ and $k(x_0)$ on $x_0$. 

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This section is organized as follows. In Subsection 3.1, assuming an ODE on the solitons’ center, we find their behavior. Then, in Subsection 3.2, we study equation (1.5) around the solitons’ sum and derive in Subsection 3.3 the ODE satisfied by the solitons’ center. Finally, we prove in Subsection 3.4 the corner property near characteristic points.

### 3.1. Time behavior of the solitons’ centers.

We will prove the following:

**Proposition 3.1.** (Refined behavior of \( w_{x_0} \) where \( x_0 \in S \)) Assuming that \( k \geq 2 \), there exists another set of parameters (still denoted by \( \zeta_1(s), \ldots, \zeta_k(s) \)) such that (3.1) and (3.2) hold and:

1. For all \( i = 1, \ldots, k \), \( e_i = (-1)^{i+1} e_1 \).
2. For some \( C_0 > 0 \), for all \( i = 1, \ldots, k \) and \( s \) large enough, we have:
   \[
   \left( i - \frac{(k+1)}{2} \right) \frac{p-1}{2} \log s - C_0 \leq \zeta_i(s) \leq \left( i - \frac{(k+1)}{2} \right) \frac{p-1}{2} \log s + C_0.
   \]

The following finite-dimensional system satisfied by \( (\zeta_i(s)) \) is crucial in our proof:

**Proposition 3.2.** (Equations satisfied by the solitons’ centers) Assuming that \( k \geq 2 \), there exists another set of parameters (still denoted by \( \zeta_1(s), \ldots, \zeta_k(s) \)) such that (3.1) and (3.2) hold and for all \( i = 1, \ldots, k \) and \( s \) large enough:

\[
\frac{1}{c_1} \zeta'_i = -e_{i-1} e_i e^{-\frac{2}{p-1} (\zeta_i - \zeta_{i-1})} + e_i e_{i+1} e^{-\frac{2}{p-1} (\zeta_{i+1} - \zeta_i)} + R_i
\]

where

\[
|R_i| \leq C J^{1+\delta_0}, \quad J(s) = \sum_{j=1}^{k-1} e^{-\frac{2}{p-1} (\zeta_{j+1} - \zeta_j(s))},
\]

\( e_0 = e_{k+1} = 0 \), for some \( c_1 > 0 \) and \( \delta_0 > 0 \).

**Remark.** Note that system (3.3) looks like the finite Toda lattice system which has a second derivative in time instead of the first derivative (see Toda [14]).

**Proof.** See subsection 3.3. \( \square \)

Let us now give the proof of Proposition 3.1.

**Proof of Proposition 3.1.** We proceed in two parts, proving (i) and then (ii).

**Part 1: Proof of (i).** Given some \( s_0 \in \mathbb{R} \), we first define for all \( s \geq s_0 \), \( J_0(s) \) and \( j_0(s) \in \{1, \ldots, k-1\} \),

\[
J_0(s) \equiv \max_{i=1, \ldots, k-1} \int_{s_0}^{s} e^{-\beta (\zeta_{i+1}(s') - \zeta_i(s'))} ds' = \int_{s_0}^{s} e^{-\beta (\zeta_{j_0(s)+1}(s') - \zeta_{j_0(s)}(s'))} ds'
\]

\[
(3.5)
\]
where $\beta = \frac{2}{p-1}$. Then, we claim that

\[(3.6) \quad J_0(s) \to \infty \quad \text{as} \quad s \to \infty.\]

Indeed, we write from (3.3) and (3.5)

\[
\left| \zeta_i(s) - \zeta_i(s_0) \right| \leq C \sum_{i=1}^{k-1} \int_{s_0}^s e^{-\beta(\zeta_{i+1}(s') - \zeta_i(s'))} ds' \leq C J_0(s)
\]

and (3.2) implies (3.6).

Integrating equation (3.3), this yields as $s \to \infty$ for all $i = 1, \ldots, k$:

\[
\frac{\zeta_1(s) + \cdots + \zeta_i(s)}{k - i + 1} = e_i e_{i+1} \frac{c_1}{i} \int_{s_0}^s e^{-\beta(\zeta_{i+1}(s') - \zeta_i(s'))} ds' + o(J_0(s)),
\]

\[(3.7)\]

\[
\frac{\zeta_1(s) + \cdots + \zeta_k(s)}{k - j_0(s)} = -e_i e_{i+1} \frac{c_1}{k - i} \int_{s_0}^s e^{-\beta(\zeta_{i+1}(s') - \zeta_i(s'))} ds' + o(J_0(s)).
\]

\[
(3.8)
\]

Using (3.2), we write for $s$ large,

\[
\frac{\zeta_1(s) + \cdots + \zeta_{j_0(s)}(s)}{j_0(s)} < \frac{\zeta_{j_0(s)+1}(s) + \cdots + \zeta_k(s)}{k - j_0(s)},
\]

if $i < j_0(s)$,

\[
\frac{\zeta_1(s) + \cdots + \zeta_i(s)}{j_0(s)} < \frac{\zeta_1(s) + \cdots + \zeta_{j_0(s)}(s)}{j_0(s)},
\]

if $i > j_0(s)$,

\[
\frac{\zeta_{j_0(s)+1}(s) + \cdots + \zeta_k(s)}{k - j_0(s)} < \frac{\zeta_{i+1}(s) + \cdots + \zeta_k(s)}{k - i}.
\]

Then, using (3.7), (3.8) and (3.5), we write for $s$ large,

\[
e_{j_0(s)} e_{j_0(s)+1} e_{j_0(s)+1} \frac{c_1}{j_0(s)} J_0(s) \leq -e_{j_0(s)} e_{j_0(s)+1} e_{j_0(s)+1} \frac{c_1}{k - j_0(s)} J_0(s) + o(J_0(s)),
\]

if $i < j_0(s)$,

\[
(3.9) \quad e_i e_{i+1} \frac{c_1}{i} \int_{s_0}^s e^{-\beta(\zeta_{i+1}(s') - \zeta_i(s'))} ds' \leq e_{j_0(s)} e_{j_0(s)+1} e_{j_0(s)+1} \frac{c_1}{k - j_0(s)} J_0(s) + o(J_0(s)),
\]

if $i > j_0(s)$,

\[
(3.10) \quad -\frac{c_1 e_{j_0(s)} e_{j_0(s)+1}}{k - j_0(s)} J_0(s) + o(J_0(s)) \leq -e_i e_{i+1} \frac{c_1}{k - i} \int_{s_0}^s e^{-\beta(\zeta_{i+1}(s') - \zeta_i(s'))} ds'.
\]

Therefore, for $s$ large, $J_0(s) \left( e_{j_0(s)} e_{j_0(s)+1} \left( e_{j_0(s)+1} \frac{c_1}{j_0(s)} + \frac{c_1}{k - j_0(s)} \right) + o(1) \right) \leq 0$, hence,

\[
e_{j_0(s)} e_{j_0(s)+1} = -1.
\]
Then, (3.9) and (3.10) write together with (3.5)

\[ \forall i, \quad \frac{1}{2k} J_0(s) \leq -e_i e_{i+1} \int_{s_0}^{s} e^{-\beta(\zeta_{i+1}(s') - \zeta_i(s'))} ds' \leq J_0(s), \]

which gives for all \( i \) and \( s \) large,

\[ (3.11) \quad e_i e_{i+1} = -1 \quad \text{and} \quad \frac{J_0(s)}{C_0} \leq \int_{s_0}^{s} e^{-\beta(\zeta_{j+1}(s') - \zeta_j(s'))} ds' \leq J_0(s) \rightarrow \infty. \]

Using a finite induction, we get \( e_i = (-1)^{i+1} e_1 \). This closes the proof of (i) of Proposition 3.1.

**Part 2: Proof of (ii).** Using (i), we rewrite system (3.3) in the following:

**Corollary 3.3.** (Equations satisfied by the solitons’ centers) Assuming that \( k \geq 2 \), it holds that

\[ (3.12) \quad \frac{1}{c_1} \zeta'_i = -e^{-\frac{2}{p r}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p r}(\zeta_{i+1} - \zeta_i)} + R_i \]

with \( |R_i| \leq C J^{1+\delta_0} \) for \( s \) large enough.

Introducing

\[ (3.13) \quad L_i(s) = \zeta_{i+1}(s) - \zeta_i(s), \quad \text{where} \quad i = 1, \ldots, k - 1 \]

and

\[ (3.14) \quad \sigma(s) = \sum_{j=1}^{r} \epsilon_0^{j-1} S_j(s) \quad \text{where} \quad S_j(s) = \sum_{i=j}^{k-j} L_i(s), \quad \epsilon_0 = \frac{1}{1000} \]

and

\[ (3.15) \quad r = E\left(\frac{k}{2}\right) \in \mathbb{N}^* \quad \text{(note that either} \quad k = 2r + 1 \quad \text{or} \quad k = 2r), \]

we claim that (ii) follows from the following:

**Lemma 3.4.** Assuming that \( k \geq 2 \), the following holds for \( s \) large enough:

(i) For some \( C_0 > 0 \) and for all \( i = 1, \ldots, k - 1 \),

\[ |L_i(s) - L_1(s)| \leq C_0. \]
Let us first derive (ii) of Proposition 3.1 from Lemma 3.4 and then prove Lemma 3.4.

Using (i) of Lemma 3.4, we see that (3.14) and (ii) of Lemma 3.4 give for some $C_0 > 0$ and $s$ large enough,

$$L_1(s) - C_0 \leq \frac{\sigma(s)}{\gamma} \leq L_1(s) + C_0 \quad \text{with} \quad \gamma = \sum_{j=1}^{r} e^{j-1}(k+1-2j) > 0,$$

(3.16)

$$\frac{1}{C_0} e^{-\frac{2}{p-1} \sigma(s)} \leq \sigma'(s) \leq C_0 e^{-\frac{2}{p-1} \sigma(s)}.$$

Therefore, for $s$ large enough, we have

$$\frac{1}{C_0} e^{-\frac{2}{p-1} \sigma(s)} \leq \sigma'(s) \leq C_0 e^{-\frac{2}{p-1} \sigma(s)} \quad \text{and} \quad \frac{1}{C_0} \leq \left(e^{\frac{2}{p-1} \sigma(s)}\right)' \leq C_0.$$

Integrating this and using (3.16), we see that for $s$ large, we have

$$\frac{s}{C_0} - C_0 \leq e^{\frac{2}{p-1} \frac{\sigma(s)}{\gamma}} \leq C_0 s + C_0,$$

$$\log s - C_0 \leq \frac{2}{p-1} \frac{\sigma(s)}{\gamma} \leq \log s + C_0,$$

$$\log s - C_0 \leq \frac{2}{p-1} L_1(s) \leq \log s + C_0.$$

Using (i) of Lemma 3.4, we see that for all $i = 1, \ldots, k-1$ and $s$ large enough, we have

$$\frac{(p-1)}{2} \log s - C_0 \leq L_i(s) \leq \frac{(p-1)}{2} \log s + C_0.$$

(3.17)

Therefore, we write from Corollary 3.3

$$\left(\zeta_1(s) + \cdots + \zeta_k(s)\right)' \leq \frac{C_1}{k} \sum_{i=1}^{k} R_i(s) \leq C J(s)^{1+\delta_0} \leq \frac{C_0}{s^{1+\delta_0}}$$

for $s$ large enough. Hence,

$$\zeta(s) \equiv \frac{\zeta_1(s) + \cdots + \zeta_k(s)}{k} \quad \text{converges to some} \quad \bar{\zeta}_0 \quad \text{as} \quad s \longrightarrow \infty.$$  

(3.18)

Now, according to (3.15), we consider two cases: $k = 2r$ and $k = 2r + 1.$
Since we have from (3.18),

\[ (3.19) \quad -C_0 \leq \left( \frac{\zeta_r(s) + \zeta_{r+1}(s)}{2} \right) - \left( \frac{\zeta_r(s) + \zeta_{r+1}(s)}{2} - \zeta_{r+1-j}(s) \right) \leq C_0. \]

Since we have from (3.18),

\[
\sum_{j=1}^{r} \left( \zeta_{r+j}(s) - \frac{\zeta_r(s) + \zeta_{r+1}(s)}{2} \right) - \left( \frac{\zeta_r(s) + \zeta_{r+1}(s)}{2} - \zeta_{r+1-j}(s) \right) \\
= \sum_{j=1}^{r} \left( \zeta_{r+j}(s) + \zeta_{r+1-j}(s) \right) - 2r \left( \frac{\zeta_r(s) + \zeta_{r+1}(s)}{2} \right) = k \left( \bar{\zeta}(s) - \frac{\zeta_r(s) + \zeta_{r+1}(s)}{2} \right),
\]

we see from (3.19) that for \( s \) large enough,

\[
-C_0 \leq \bar{\zeta}(s) - \frac{\zeta_r(s) + \zeta_{r+1}(s)}{2} \leq C_0,
\]

and from (3.18), we see that

\[ (3.20) \quad -C_0 \leq \frac{\zeta_r(s) + \zeta_{r+1}(s)}{2} \leq C_0. \]

Now, since we have from the definition (3.13) of \( L_i \) and (3.17) for all \( i = 1, \ldots, k \) and \( s \) large enough,

\[
\left( i - \left( r + \frac{1}{2} \right) \right) \frac{(p-1)}{2} \log s - C_0 \leq \zeta_i(s) - \frac{\zeta_r(s) + \zeta_{r+1}(s)}{2} \leq \left( i - \left( r + \frac{1}{2} \right) \right) \frac{(p-1)}{2} \log s + C_0
\]

and \( r + \frac{1}{2} = \frac{k+1}{2} \), the conclusion of (ii) follows from (3.20), when \( k = 2r \).

Case \( k = 2r + 1 \). We omit the proof since it is quite similar to the case \( k = 2r \) (one has just to handle \( \zeta_{r+1} \) instead of \( \frac{\zeta_r + \zeta_{r+1}}{2} \)).

It remains to prove Lemma 3.4 in order to finish the proof of Proposition 3.1.

Proof of Lemma 3.4. We first derive from Corollary 3.3 the following ODE system satisfied by the \( L_i(s) \):

Claim 3.5. Assuming that \( k \geq 2 \), it holds that for all \( i = 1, \ldots, k-1 \) and \( s \) large enough,

\[
\frac{1}{c_1} L_i'(s) = -e^{-\frac{2}{r+1}L_i-1(s)} + 2e^{-\frac{2}{r+1}L_i(s)} - e^{-\frac{2}{r+1}L_{i+1}(s)} + \tilde{R}_i,
\]

where \( |\tilde{R}_i| \leq C J^{1+\delta_0} \) with the convention \( L_0(s) \equiv +\infty \) and \( L_k(s) \equiv +\infty \).
The proof of (i) in Lemma 3.4 is much longer than the proof of (ii). Therefore, we first derive (ii) assuming that (i) is true, then we prove (i).

(ii) Using Claim 3.5, the fact that $\tilde{R}_t = o(J)$ and (3.14), we see that for $j = 2, \ldots, r$, we have

$$\frac{1}{c_1} S_j^s(s) = e^{-\frac{2}{p-r} L_j(s)} + e^{-\frac{2}{p-r} L_{k-j}(s)} - e^{-\frac{2}{p-r} L_{j-1}(s)} - e^{-\frac{2}{p-r} L_{k-j+1}(s)} + o(J(s)),$$

$$\frac{1}{c_1} S_1^s(s) = e^{-\frac{2}{p-r} L_1(s)} + e^{-\frac{2}{p-r} L_{k-1}(s)} + o(J(s))$$

as $s \to +\infty$. Therefore, using (3.14), we write

$$\sigma'(s) = \sum_{j=1}^{r-1} \frac{\epsilon_0^{-1}}{r} S_j^s(s) = \sum_{j=1}^{r-1} \frac{\epsilon_0^{-1}}{r} (1 - \epsilon_0) \left( e^{-\frac{2}{p-r} L_j(s)} + e^{-\frac{2}{p-r} L_{k-j}(s)} \right)$$

$$+ \epsilon_0^{-1} \left( e^{-\frac{2}{p-r} L_r(s)} + e^{-\frac{2}{p-r} L_{k-r}(s)} \right) + o(J(s))$$

as $s \to \infty$. Since for all $j = 1, \ldots, r - 1$, $\frac{\epsilon_0^{-1}}{2} < \epsilon_0^{-1} (1 - \epsilon_0) < 1$ and $\epsilon_0^{-1} \leq 1$, the conclusion of (ii) of Lemma 3.4 follows by taking $s$ large enough.

(i) Proceeding by contradiction, we consider a sequence $\varphi_n \to \infty$ and find two sequences $s_n < s'_n$ both going to $+\infty$ as $n \to +\infty$ such that

$$\tilde{L}(s_n) = \varphi_n, \quad \tilde{L}(s'_n) = 1 + \varphi_n \text{ and } \forall s \in (s_n, s'_n), \quad \tilde{L}(s) > \varphi_n,$$

where

$$\tilde{L}(s) = \max_{i=1, \ldots, k-1} L_i(s) - \min_{i=1, \ldots, k-1} L_i(s).$$

Introducing

$$\bar{m}_n = \min_{j=1, \ldots, k-1} L_i(s_n) \text{ and } \bar{M}_n = \max_{j=1, \ldots, k-1} L_i(s_n),$$

we see from (3.21) that

$$\bar{M}_n - \bar{m}_n = \tilde{L}(s_n) = \varphi_n \to \infty \quad \text{as } n \to \infty.$$

Up to extracting a subsequence, we assume that for all $i = 1, \ldots, k-1$, we have

$$L_i(s_n) - \bar{m}_n \to l_i \in [0, +\infty] \quad \text{as } n \to \infty.$$

Introducing

$$I_0 = \{i = 1, \ldots, k-1 \mid l_i = 0\}, \quad I_+ = \{i = 1, \ldots, k-1 \mid l_i \in (0, +\infty)\},$$

$$I_\infty = \{i = 1, \ldots, k-1 \mid l_i = +\infty\},$$

we see that $\{1, \ldots, k-1\} = I_0 \cup I_+ \cup I_\infty$ and that $I_0 \neq \emptyset$. 

Let us now introduce the following change of variables for all \( i = 1, \ldots, k - 1 \) and \( s \in [s_n, s'_n] \),

\[
a_{i,n}(\theta) = L_i(s) - \bar{m}_n \quad \text{where} \quad s = s_n + \theta \frac{2}{e^{p-1} \bar{m}_n} c_1
\]

where \( c_1 \) appears in Claim 3.5. Using Claim 3.5 and the fact that \( \tilde{R}_i = o(J) \) as \( s \to \infty \), we see that for all \( i = 1, \ldots, k - 1 \) and \( \theta \geq 0 \),

\[
a'_{i,n}(\theta) = -e^{-\frac{2}{p-1} a_{i-1,n}(\theta)} + 2e^{-\frac{2}{p-1} a_{i,n}(\theta)} - e^{-\frac{2}{p-1} a_{i+1,n}(\theta)} + o\left( \sum_{j=1}^{k-1} e^{-\frac{2}{p-1} a_{j,n}(\theta)} \right)
\]

as \( n \to \infty \), where we take by convention \( a_{0,n}(\theta) \equiv +\infty \) and \( a_{k,n}(\theta) \equiv +\infty \). Now, fixing

\[
\theta_0 = e^{-\frac{8}{p-1}},
\]

we claim the following:

**Claim 3.6.** For all \( \eta_0 > 0 \), there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), \( i = 1, \ldots, k - 1 \) and \( \theta \in [0, \theta_0] \), we have

\[
-4 \leq a_{i,n}(\theta) \leq \varphi_n + \eta_0.
\]

**Proof.** See below. \( \square \)

Using equation (3.28) and Claim 3.6 (with \( \eta_0 = 1 \)), we see that for some \( C_0 \), for all \( n \geq n_0 \), \( i = 1, \ldots, k - 1 \) and \( \theta \in [0, \theta_0] \), we have

\[
|a'_{i,n}(\theta)| \leq C_0.
\]

Using Ascoli’s theorem, (3.25) and the definition (3.27) of \( a_{i,n} \), we see that for all \( i = 1, \ldots, k - 1 \),

\[
a_{i,n}(\theta) \to a_i(\theta) \quad \text{as} \quad n \to \infty, \quad \text{uniformly for} \quad \theta \in [0, \theta_0],
\]

where \( a_i(\theta) \equiv +\infty \) if \( i \in I_\infty \), and \( a_i \in C([0, \theta_0]) \) if not. Moreover, getting to the limit in equation (3.28) and introducing \( a_0(\theta) \equiv +\infty \) and \( a_{k-1}(\theta) \equiv +\infty \), we see that the vector \( (a_i(\theta))_i \) satisfies the following system (first in an integral sense, then in a strong sense):

\[
a'_i(\theta) = -e^{-\frac{2}{p-1} a_{i-1}(\theta)} + 2e^{-\frac{2}{p-1} a_i(\theta)} - e^{-\frac{2}{p-1} a_{i+1}(\theta)}
\]

for all \( \theta \in [0, \theta_0] \), with initial data

\[
a_i(0) = l_i \in [0, +\infty].
\]
Using (3.30), we see that

\[(3.34) \quad \forall i \in I_0 \cup I_+, \forall \theta \in [0, \theta_0], \quad |a'_i(\theta)| \leq C_0.\]

Now, we claim that for all \(i = 1, \ldots, k - 1\) and for some \(\theta_i \in (0, \theta_0]\), we have

\[(3.35) \quad a_i(\theta) > 0 \quad \text{for all } \theta \in (0, \theta_i].\]

Indeed, if \(i \in I_\infty\), this is clear from the convergence (3.31).

If \(i \in I_+\), then this is clear from (3.34).

Now, if \(i \in I_0\), then we define \(r_1\) and \(r_2\) in \([0, k] \cup I_\infty \cup I_+\) such that \(r_1 < i < r_2\) and for all \(j = r_1 + 1, \ldots, r_2 - 1, j \in I_0\). Therefore, we have from (3.26) and (3.33),

\[(3.36) \quad a_{r_1}(0) > 0, \quad a_{r_2}(0) > 0 \quad \text{and} \quad a_j(\theta) = 0 \quad \text{for all } \theta \in (0, \hat{\theta}).\]

The following Claim allows us to conclude:

**Claim 3.7.** For some \(\delta_0 > 0\), we have

\[(3.37) \quad \forall j = r_1, \ldots, r_2, \forall \theta \in (0, \delta_0], \quad a_j(\theta) > 0.\]

**Proof.** See below. \(
\)

Applying Claim 3.7, we see that (3.35) holds.

Using (3.35) and introducing \(\hat{\theta} = \inf_{i=1,\ldots,k-1} \theta_i \in (0, \theta_0]\), we see that for some \(\hat{\alpha} > 0\), we have

\[
\forall i = 1, \ldots, k - 1, \quad a_i(\hat{\theta}) \geq 4\hat{\alpha} > 0 \quad \text{and} \quad a_i(\theta) \geq 0 \quad \text{for all } \theta \in [0, \hat{\theta}].
\]

From the convergence (3.31), we see that for \(n\) large enough, we have

\[
\forall i = 1, \ldots, k - 1, \quad a_{i,n}(\hat{\theta}) \geq 3\hat{\alpha} > 0 \quad \text{and} \quad a_i(\theta) \geq -\frac{1}{4} \quad \text{for all } \theta \in [0, \hat{\theta}].
\]

Using Claim 3.6 (with \(\eta_0 = \min(\hat{\alpha}, \frac{1}{4})\)), we see from the definitions (3.27) and (3.22) of \(a_{i,n}\) and \(\hat{L}(s)\) that for

\[
\hat{s}_n = s_n + \theta \frac{e^{\frac{\hat{s}_n}{c_1}} m_n}{c_1},
\]

we have

\[(3.38) \quad \hat{L}(\hat{s}_n) \leq \hat{m}_n + \varphi_n + \eta_0 - (\hat{m}_n + 3\hat{\alpha}) = \varphi_n + \eta_0 - 3\hat{\alpha} \leq \varphi_n - 2\hat{\alpha},\]

\[(3.39) \quad \forall s \in [s_n, \hat{s}_n], \quad \hat{L}(s) \leq \hat{m}_n + \varphi_n + \eta_0 - \left(\hat{m}_n - \frac{1}{4}\right) = \varphi_n + \eta_0 + \frac{1}{4} \leq \varphi_n + \frac{1}{2}.
\]

Since we have from (3.21) and (3.39) that \(\hat{s}_n \leq s'_n\), we see that (3.38) is in contradiction with (3.21). It remains to prove Claims 3.6 and 3.7 in order to finish the proof of Lemma 3.4.
Proof of Claim 3.6. Note first from the definitions (3.27) and (3.23) of \( a_{i,n} \), \( \bar{m}_n \) and \( \bar{M}_n \), and from (3.24), that we have for all \( n \in \mathbb{N} \) and \( i = 1, \ldots, k \),

\[
0 \leq a_{i,n}(0) \leq \varphi_n.
\]

Consider \( \eta_0 > 0 \). Proceeding by contradiction to prove Claim 3.6, we define a sub-sequence (still denoted by \( (a_{i,n})_n \)) such that for all \( n \in \mathbb{N} \), there is \( \theta^* = \theta^*(n) \in (0, \theta_0) \) such that (3.29) holds for all \( i = 1, \ldots, k-1 \) and \( \theta \in [0, \theta^*] \), and for some \( i^* = i^*(n) \), we have

\[
a_{i^*,n}(\theta^*) = -4 \text{ or } a_{i^*,n}(\theta^*) = \varphi_n + \eta_0.
\]

From (3.40), we can define \( \bar{\theta} = \bar{\theta}(n) \in [0, \theta^*] \) such that

\[
a_{i^*,n}(\bar{\theta}) = \varphi_n \text{ and } \forall \theta \in [\bar{\theta}, \theta^*], \quad a_{i^*,n}(\theta) \geq \varphi_n.
\]

Since (3.29) holds for all \( i = 1, \ldots, k-1 \) and \( \theta \in [0, \theta^*] \), we derive from equation (3.28) that for \( n \) large enough and \( \theta \in [0, \theta^*] \),

\[
a'_{i^*,n}(\theta) \geq -e^{-2 \frac{1}{p-1} a_{i^*,n-1,\theta}(\theta)} + 2e^{-2 \frac{1}{p-1} a_{i^*,n,\theta}(\theta)} - e^{-2 \frac{1}{p-1} a_{i^*,n+1,\theta}(\theta)} - \frac{1}{k-1} \sum_{j=1}^{k-1} e^{-2 \frac{1}{p-1} a_{j,n}(\theta)}.
\]

Integrating this for \( \theta \in [0, \theta^*] \) and using (3.40), we get

\[
a_{i^*,n}(\theta^*) \geq a_{i^*,n}(0) - 3 \theta^* e^{8 \frac{1}{p-1}} \geq -3 \theta_0 e^{8 \frac{1}{p-1}} = -3,
\]

on the one hand.

On the other hand, from equation (3.28), (3.29) (valid for \( \theta \in [0, \theta^*] \)) and (3.42), we have for \( n \) large enough and for all \( \theta \in [\bar{\theta}, \theta^*] \),

\[
a'_{i^*,n}(\theta) \leq -e^{-\frac{1}{p-1} a_{i^*,n-1,\theta}(\theta)} + 2e^{-\frac{1}{p-1} a_{i^*,n,\theta}(\theta)} - e^{-\frac{1}{p-1} a_{i^*,n+1,\theta}(\theta)}
\]

\[
+ \frac{\eta_0}{4(k-1)} \sum_{j=1}^{k-1} e^{-\frac{2}{p-1} a_{j,n}(\theta)}
\]

\[
\leq 0 + 2e^{-\frac{1}{p-1}} + 0 + \frac{\eta_0}{4(k-1)} (k-1) e^{8 \frac{1}{p-1}} \leq \frac{\eta_0}{2} e^{8 \frac{1}{p-1}}.
\]

Integrating this for \( \theta \in [\bar{\theta}, \theta^*] \), using (3.42) and the fact that \( \theta^* - \bar{\theta} \leq \theta_0 = e^{-\frac{8}{p-1}} \), we get

\[
a_{i^*,n}(\theta^*) \leq a_{i^*,n}(\bar{\theta}) + \frac{\eta_0}{2} e^{8 \frac{1}{p-1}} (\theta^* - \bar{\theta}) \leq \varphi_n + \frac{\eta_0}{2}.
\]

Since (3.43) and (3.44) are in contradiction with (3.41), this concludes the proof of Claim 3.6. \( \square \)
Proof of Claim 3.7. In order to conclude, it is enough to prove that for all \( l = 0, \ldots, l^* \equiv E(\frac{r_2 - r_1}{2}) \), we have

\[
(3.45) \quad a_{r_1+1}^{(l)}(0) > 0, \quad a_{r_2-1}^{(l)}(0) > 0 \quad \text{and if } r_1 + l + 1 \leq j \leq r_2 - l - 1, \quad \text{then } a_j^{(l)}(0) = 0,
\]

where we use the notation

\[
f^{(0)} = f \quad \text{and } f^{(l)} \quad \text{is the } l\text{th derivative of } f.
\]

Indeed, if (3.45) holds, then it is easy to see that for any \( j = r_1, \ldots, r_2 \), we have

\[
a_j^{(l)}(0) > 0 \quad \text{and if } 0 \leq j' \leq l - 1, \quad \text{then } a_i^{(j')}(0) = 0
\]

where \( l = j - r_1 \) if \( j \leq l^* \) and \( l = r_2 - j \) if \( j \geq l^* + 1 \), hence

\[
a_j(\theta) \sim \frac{a_j^{(l)}(0)}{l!} \theta^l \quad \text{as } \theta \to 0 \quad \text{with } a_j^{(l)}(0) > 0,
\]

and (3.37) holds. It remains then to prove (3.45) in order to conclude the proof of Claim 3.7.

If \( l = 0 \), then (3.45) is true by (3.36).

In order to prove (3.45) for all \( l = 1, \ldots, l^* \), we proceed by induction.

- If \( l = 1 \), then using (3.32) and (3.45) (with \( l = 0 \)), we write

\[
a_{r_1+1}^{(l)}(0) = -e^{-\frac{2}{p-1}a_{r_1}(0)} + 2 - e^{-\frac{2}{p-1}a_{r_1+2}(0)} \geq 1 - e^{-\frac{2}{p-1}a_{r_1}(0)} > 0,
\]

\[
a_{r_2-1}^{(l)}(0) = -e^{-\frac{2}{p-1}a_{r_2-2}(0)} + 2 - e^{-\frac{2}{p-1}a_{r_2}(0)} \geq 1 - e^{-\frac{2}{p-1}a_{r_2}(0)} > 0,
\]

if \( r_1 + 2 \leq j \leq r_2 - 2 \), \( a_j^{(l)}(0) = -1 + 2 - 1 = 0 \).

- Take \( l = 2, \ldots, l^* \) and assume that (3.45) holds for any \( l' = 0, \ldots, l - 1 \). Therefore, it is easy to see that

\[
a_{r_1+1}^{(l-1)}(0) > 0 \quad \text{and if } 0 \leq j' \leq l - 2, \quad a_j^{(j')}(0) = 0, \quad \text{for all } j' = 0, \ldots, l - 1,
\]

\[
a_{r_2-1}^{(l-1)}(0) > 0 \quad \text{and if } 0 \leq j' \leq l - 2, \quad a_j^{(j')}(0) = 0.
\]

In the following, we prove that (3.45) holds for \( l \). Starting from equation (3.32), we prove thanks to a straightforward induction that

\[
\frac{(p-1)}{2} a_j^{(l)} = \left[ a_{j-1}^{(l-1)} + P_l(\sum_{j=1}^{j-1} a_{j-1}^{(l-2)}) \right] e^{-\frac{2}{p-1}a_j} - 2 \left[ a_j^{(l-1)} + P_l(\sum_{j=1}^{j-1} a_{j-1}^{(l-2)}) \right] e^{-\frac{2}{p-1}a_j} + \left[ a_{j+1}^{(l-1)} + P_l(\sum_{j=1}^{j+1} a_{j+1}^{(l-2)}) \right] e^{-\frac{2}{p-1}a_j},
\]

(3.47)
where \( P_2 \equiv 0 \) and for \( l \geq 3 \), \( P_1 \) is a polynomial of \( l - 2 \) variables satisfying
\[
(3.48) \quad P_1(0, \ldots, 0) = 0.
\]

Using (3.47), (3.46) and (3.48), we write
\[
a^{(l)}_{r_1+l}(0) = a^{(l-1)}_{r_1+l-1}(0) + a^{(l-1)}_{r_1+l+1}(0) \geq a^{(l-1)}_{r_1+l-1}(0) > 0,
\]
\[
a^{(l)}_{r_2-l}(0) = a^{(l-1)}_{r_2-l-1}(0) + a^{(l-1)}_{r_2-l+1}(0) \geq a^{(l-1)}_{r_2-l-1}(0) > 0,
\]

if \( r_1 + l + 1 \leq j \leq r_2 - l - 1 \), \( a^{(l-1)}_j(0) = 0 \).

which is the conclusion of (3.45) with the index \( l \). Thus, (3.45) holds. This concludes the proof of Claim 3.7 and Lemma 3.4 too. Thus, Proposition 3.1 holds. □

3.2. Refinement of (3.1) for \( k \geq 2 \). Note that the case \( k = 1 \) has already been treated in [11] giving rise to estimate (1.11).

In the 1-soliton case, we have a decomposition of the solution of the form
\[
\left( \begin{array}{c}
w(y, s) \\
\partial_s w(y, s)
\end{array} \right) = \left( \begin{array}{c}
\kappa(d(s), y) \\
0
\end{array} \right) + q(y, s)
\]

where \( q = (q_1, q_2) \) is small in \( \mathcal{H} \). Using a Lyapunov functional on \( q \) in \( \mathcal{H} \), we are able to establish that in some sense for \( s \) large,
\[
\frac{d}{ds} \| q(s) \|_{\mathcal{H}} \leq -\alpha_0 \| q(s) \|_{\mathcal{H}} \quad \text{where} \quad \alpha_0 > 0,
\]
and thus have exponential decay
\[
\| q(s) \|_{\mathcal{H}} \leq e^{-\alpha_0 s}.
\]

Here, as announced in the beginning of the section, we assume that \( k \geq 2 \) and we also decompose the solution as
\[
(3.49) \quad \left( \begin{array}{c}
w(y, s) \\
\partial_s w(y, s)
\end{array} \right) = \sum_{i=1}^{k} e_i \left( \begin{array}{c}
\kappa(d_i(s), y) \\
0
\end{array} \right) + q(y, s) \quad \text{where} \quad q = \left( \begin{array}{c}
q_1 \\
q_2
\end{array} \right)
\]
is small in \( \mathcal{H} \). Note then that for any \( \delta_1, \ldots, \delta_k \) in \( \mathbb{R}^k \), \( \sum_{i=1}^{k} e_i \kappa(\delta_i, y) \) is never a solution of equation (1.5) when \( k \geq 2 \). Nevertheless, with \( \delta_i = d_i(s) \) given in (3.1), we see from the decoupling estimate (3.2) that \( \sum_{i=1}^{k} e_i \kappa(d_i(s), y) \) is an approximate solution of (1.5), up to some error term of the order
\[
H_0(s) = \sum_{i=1}^{k-1} h(\zeta_{i+1}(s) - \zeta_i(s))
\]
where

\[(3.50) \quad h(\zeta) = e^{-\frac{p}{p - 1} \zeta} \text{ if } p < 2, \quad h(\zeta) = e^{-2\zeta \sqrt{\zeta}} \text{ if } p = 2 \text{ and } h(\zeta) = e^{-\frac{2}{p - 1} \zeta} \text{ if } p > 2.\]

Again, since \( q \) can be assumed to be orthogonal to the zero direction of the linearized operator of (1.5) around each \( e^{i\kappa(d_i(s), y)} \) (by a modulation technique as for \( k = 1 \)), we will prove in this subsection that in some sense, for \( s \) large,

\[(3.51) \quad \frac{d}{ds} \|q(s)\|_H \leq -\alpha_0 \|q(s)\|_H + CH_0(s)\]

(in fact, in order to get this result, one has to control the negative part of the spectrum of the linearized operator of equation (1.5) around each soliton \( e^{i\kappa(d_i(s), y)} \); this is done thanks to a linearized version of the Lyapunov function \( E(w) \) (1.7)). Then, by a direct computation, we will derive that for all \( \epsilon > 0 \) there exists \( s_0(\epsilon) > 0 \) such that for all \( s \geq s_0(\epsilon) \),

\[(3.52) \quad \left| \frac{d}{ds} H_0(s) \right| \leq \epsilon H_0(s),\]

which implies formally a slower decay in time. Gathering (3.51) and (3.52), we will prove that for \( s \) large,

\[\|q(s)\|_H \leq CH_0(s).\]

In other words, this means that the size of the defect \( q \) is controlled by the size of the soliton interaction \( H_0 \). More precisely, we claim the following:

**Proposition 3.8. (Size of \( q \) in terms of the distance between solitons)** There exists another set of parameters (still denoted by \( \zeta_1(s), \ldots, \zeta_k(s) \)) such that (3.1) and (3.2) hold and for some \( s^* \in \mathbb{R} \) and for all \( s \geq s^* \),

\[(3.53) \quad \|q(s)\|_H \leq C \sum_{i=1}^{k-1} h(\zeta_{i+1}(s) - \zeta_i(s)),\]

where \( q \) and \( h \) are defined in (3.49) and (3.50).

Before proving the estimate, we need to use a modulation technique to slightly change the \( \zeta_i(s) \) in order to guarantee some orthogonality conditions. In order to do so, we need to introduce for \( \lambda = 0 \) or 1, for any \( d \in (-1, 1) \) and \( r \in \mathcal{H} \),

\[(3.54) \quad \pi(\lambda)(r) = \phi(W_\lambda(d), r)\]
where:

\[
\phi(q, r) = \int_{-1}^{1} \left( q_1 r_1 + q_1^2 r_1^2 (1 - y^2) + q_2 r_2 \right) dy
\]

(3.55)

\[
\phi(q, r) = \int_{-1}^{1} \left( q_1 (-L r_1 + r_2) + q_2 r_2 \right) dy,
\]

\[
W_\lambda(d, y) = (W_{\lambda,1}(d, y), W_{\lambda,2}(d, y)),
\]

(3.56)  \[ W_{1,2}(d, y)(y) = c_1(d) \frac{1 - y^2}{(1 + dy)^{\frac{2}{p-1} + 1}}, \quad W_{0,2}(d, y) = c_0 \frac{y + d}{1 + dy} \kappa(d, y), \]

with \(0 < c_1(d) \leq C(1 - d^2)^{\frac{1}{p-1}}\), \(c_0 > 0\), and \(W_{\lambda,1}(d, y) \in \mathcal{H}_0\) is uniquely determined as the solution of

(3.57)  \[ -L r + r = \left( \lambda - \frac{p + 3}{p - 1} \right) W_{\lambda,2}(d) - 2 y \partial_y W_{\lambda,2}(d) + \frac{8}{p - 1} \frac{W_{\lambda,2}(d)}{1 - y^2} \]

(in [11], we defined \(W_{0,2}(d, y)\) by \(c_0(d)(y + d)^{\frac{1}{p} - \frac{1}{p-1}}\) with

\[
1 = c_0(d)(1 - d^2)^{\frac{1}{p} - \frac{1}{p-1}} \int_{-1}^{1} \frac{4}{p - 1} \frac{(y + d)^2}{1 + dy} \frac{\rho}{\sqrt[1-\frac{1}{2}]{1 - y^2}} dy.
\]

Setting \(y = \tanh \xi\), we compute the integral and get \(c_0(d) = c'_0(1 - d^2)^{\frac{1}{p} - \frac{1}{p-1}}\). Using (1.10), we get (3.56)). Recall from [11, Lemma 4.4, page 85] that

(3.58)  \[ \forall d \in (-1, 1), \quad \|W_\lambda(d)\|_{\mathcal{H}} \leq C. \]

We now have the following:

**Lemma 3.9.** (Modulation technique) Assume that \(k \geq 2\).

(i) (Choice of the modulation parameters) There exist other values of the parameters (still denoted by \(d_i(s)\)) of class \(C^1\), such that \(\zeta_{i+1}(s) - \zeta_i(s) \to \infty\) as \(s \to \infty\) where \(d_i(s) = -\tanh \zeta_i(s)\),

(3.59)  \[ \|q(s)\|_{\mathcal{H}} \to 0 \quad \text{and} \quad \pi_0^{d_i(s)}(q) = 0 \quad \text{for all} \quad i = 1, \ldots, k, \]

where \(\pi_0^d\) and \(q\) are defined in (3.54) and (3.53) respectively.

(ii) (Equation on \(q\)) For \(s\) large, we have

(3.60)  \[ \frac{\partial}{\partial s} \left( \begin{array}{c} q_1 \\ q_2 \end{array} \right) = L \left( \begin{array}{c} q_1 \\ q_2 \end{array} \right) - \sum_{j=1}^{k} e_j d_j'(s) \left( \begin{array}{c} \partial_d \kappa(d_j(s), y) \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ \text{R} \end{array} \right) + \left( \begin{array}{c} 0 \\ f(q_1) \end{array} \right) \]

\(d_j(s) = -\tanh \zeta_j(s)\).
where

\[
L \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_2 \\ L q_1 + \psi q_1 - \frac{p+3}{p-1} q_2 - 2y \partial_y q_2 \end{pmatrix},
\]

\[
\psi(y, s) = p |K(y, s)|^{p-1} - \frac{2(p+1)}{(p-1)^2}, \quad K(y, s) = \sum_{j=1}^k \epsilon_j \kappa(d_j(s), y),
\]

\[
f(q_1) = |K + q_1|^{p-1}(K + q_1) - |K|^{p-1}K - p|K|^{p-1}q_1,
\]

\[
R = |K|^{p-1}K - \sum_{j=1}^k \epsilon_j \kappa(d_j)^p.
\]

**Remark.** From the modulation technique, it is clear that the distance between old and new parameter \( \zeta_i(s) \) goes to zero as \( s \to \infty \).

**Proof.** See the proof of [11, Proposition 5.1] where the case \( k = 1 \) is treated. There is no difficulty in adapting the proof to \( k \geq 2 \). □

In the following, we will show that Proposition 3.8 holds with the set of parameters \( \zeta_1(s), \ldots, \zeta_k(s) \) given by the modulation technique of Lemma 3.9. Before giving the proof, we start by reformulating the problem.

Let us first remark that equation (3.60) can be localized near each soliton’s center which allows us to view it locally as a perturbation of the case of one soliton already treated in [11]. For this, given \( i = 1, \ldots, k \), we need to expand the linear operator of equation (3.60) as

\[
L(d_i(s)) = L(d_i(s)) + (0, V_i(y, s)q_1)
\]

with

\[
L_d \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_2 \\ \left( L q_1 + (p \kappa(d_i(s), y))^{p-1} - \frac{2(p+1)}{(p-1)^2} q_1 - \frac{p+3}{p-1} q_2 - 2y q_2 \right) \end{pmatrix},
\]

\[
V_i(y, s) = p |K(y, s)|^{p-1} - p \kappa(d_i(s), y)^{p-1}.
\]

Since the solitons’ sum is decoupled (remember from (i) of Lemma 3.9 that

\[
\zeta_{i+1} - \zeta_i \to \infty \quad \text{as} \quad s \to \infty,
\]

the properties of \( L_{d_i(s)} \) will be essential in our analysis.

From [11, Section 4], we know that for any \( d \in (-1, 1) \), the operator \( L_d \) has 1 and 0 as eigenvalues, the rest of the eigenvalues are negative. More precisely,
introducing

\[ F_1(d, y) = (1 - d^2)^{\frac{p}{p-1}} \left( \frac{(1 + dy)^{-\frac{2}{p-1}} - 1}{(1 + dy)^{-\frac{2}{p-1}} - 1} \right), \]

\[ F_0(d, y) = (1 - d^2)^{\frac{1}{p-1}} \left( \frac{y + d}{(1 + dy)^{\frac{2}{p-1}} + 1} \right), \]

we have

\[ L_d (F_\lambda(d)) = \lambda F_\lambda(d) \text{ and } \|F_1(d)d\|_H + \|F_0(d)d\|_H \leq C. \]

The projection on \( F_\lambda(d) \) is defined in (3.54) by \( \pi^d_\lambda(r) = \phi(W_\lambda(d), r) \). Of course,

\[ L^*_d W_\lambda(d) = \lambda W_\lambda(d) \]

where \( L^*_d \) is the conjugate of \( L_d \) with respect to the inner product \( \phi \), and the choice of the constants \( c_1(d) \) and \( c_0 \) guarantees the orthogonality condition

\[ \pi^d_\lambda(F_\mu(d)) = \phi(W_\lambda(d), F_\mu(d)) = \delta_{\lambda, \mu}. \]

In the following, we give a decomposition of the solution which is well adapted to the proof:

**Lemma 3.10.** (Decomposition of \( q \)) If we introduce for all \( r \) and \( r \) in \( H \) the operator \( \pi_-(r) \equiv r_-(y, s) \) defined by

\[ r(y, s) = \sum_{i=1}^{k} \left( \pi^d_1(r) F_1(d_i(s), y) + \pi^d_0(r) F_0(d_i(s), y) \right) + \pi_-(r) \]

and the bilinear form

\[ \varphi(r, r) = \int_{-1}^{1} \left( r_1'(1 - y^2) - \psi r_1 + r_2 r_2 \right) \rho dy \]

where \( \psi(y, s) \) is defined in (3.60), then:

(i) for \( s \) large enough and for all \( r \) and \( r \) in \( H \), we have

\[ |\varphi(r, r)| \leq C\|r\|_H \|r\|_H, \]
(ii) for some \( C_0 > 0 \) and for all \( s \) large enough, we have:

\[
q(y,s) = \sum_{i=1}^{k} \alpha_i^*(s) F_i(d_i(s), y) + q_-(y,s),
\]

\[
\frac{1}{C_0} \left\| q_-(s) \right\|_H^2 - C_0 J(s)^2 \left\| q(s) \right\|_H^2 \leq A_-(s) \leq C_0 \left\| q_-(s) \right\|_H^2,
\]

\[
\frac{1}{C_0} \left\| q(s) \right\|_H^2 \leq \sum_{i=1}^{k} (\alpha_i^*(s))^2 + A_-(s) \leq C_0 \left\| q(s) \right\|_H^2
\]

where

\[
\bar{J}(s) = \sum_{j=1}^{k-1} (\zeta_{j+1} - \zeta_j) e^{-\frac{2}{p-1}(\zeta_{j+1} - \zeta_j)},
\]

\[
\alpha_i^*(s) = \pi_\lambda^d_i(s) (q(s)) \text{ and } A_-(s) = \varphi(q_-(s), q_-(s)).
\]

**Remark.** Note that the choice of \( d_i(s) \) made in (3.59) guarantees that for \( s \) large enough,

\[
\forall i = 1, \ldots, k, \quad \alpha_i^0(s) \equiv \alpha_i^{i'}(s) \equiv 0.
\]

Moreover, we see from (3.71) that \( A_-(s) \) is nearly positive and nearly equivalent to \( \| q_-(s) \|_H^2 \). Note that from (B.25) (proved in the proof of Claim 3.10), (3.70), (3.64) and (3.71) we have for \( s \) large,

\[
|\alpha_i^*(s)| \leq C \left\| q(s) \right\|_H;
\]

\[
\left\| q_-(s) \right\|_H \leq C \min \left( \left\| q(s) \right\|_H, \sqrt{|A_-(s)| + J(s) \left\| q(s) \right\|_H} \right).
\]

**Remark.** The operator \( \pi_- \) depends on the time variable \( s \). In [11], we had only one soliton, and we decomposed \( q \) as follows:

\[
q(y,s) = \pi_1^d(q) F_1(d,y) + \pi_0^d(q) F_0(d,y) + \pi^d(q),
\]

where we had only one \( d(s) \) (note that this decomposition is in fact a definition of the operator \( \pi_\lambda^d \)). Here, due to (3.62), we have a decoupling effect, in the sense that \( \pi_\lambda^d_j(q) \) for \( j \neq i \) cannot be “seen” when \( y \) is close to \(-d_i(s)\), the “center” of the soliton \( \kappa(d_i(s), y) \). Hence, \( \pi_-(q) \) is more or less \( \pi_\lambda^d_i(q) \) and we are reduced to the situation of one soliton already treated in [11]. This idea will be essential in our proof since given some \( i = 1, \ldots, k \), we have two types of terms in equation (3.60):

- terms involving the soliton \( \kappa(d_i(s), y) \) for which we refer the reader to [11],
- interaction terms involving a different soliton \( \kappa(d_j(s), y) \) which we treat in details.

*Proof of Lemma 3.10.* See Appendix B. \( \square \)
In order to prove Proposition 3.8, we project equation (3.60) according to the decomposition (3.67). More precisely, we have the following:

**Lemma 3.11.** For $s$ large enough, the following holds:

(i) (Control of the positive modes and the modulation parameters)

\[
\forall i = 1, \ldots, k, \quad |\alpha_i^{1'}(s) - \alpha_i^1(s)| + |\zeta_i'(s)| \leq C\|q(s)\|^2_H + CJ(s)
\]

where $J(s)$ is defined in (3.4).

(ii) (Control of the negative part)

\[
\left( R_- + \frac{1}{2} A_- \right)' \leq -\frac{3}{p-1} \int_{-1}^1 q_{-2}^2 \frac{\rho}{1-y^2} dy + o\left(\|q(s)\|^2_H\right) + C \sum_{m=1}^{k-1} \left( h(\zeta_{m+1} - \zeta_m) \right)^2 + CJ(s) \sqrt{|A_-(s)|}
\]

for some $R_-(s)$ satisfying

\[
|R_-| \leq C\|q(s)\|_{\mathcal{H}}^{\bar{p}+1}
\]

where $\bar{p} = \min(p, 2)$ and $h(s)$ is defined in (3.50).

(iii) (An additional relation)

\[
\frac{d}{ds} \int_{-1}^1 q_1 q_2 \rho \leq \frac{4}{5} A_- + C \sum_{m=1}^{k-1} h(\zeta_{m+1} - \zeta_m)^2
\]

\[
+ C \int_{-1}^1 q_{-2}^2 \frac{\rho}{1-y^2} + C \sum_{i=1}^{k} (\alpha_i^1)^2.
\]

**Proof.** See Appendix C. \qed

With Lemma 3.11, we are ready to prove Proposition 3.8.

**Proof of Proposition 3.8.** Let us first explain the idea of the proof. From (3.70), we see that it is enough to bound $\alpha_i^1(s)$, $d_i(s)$ (in fact $\zeta_i(s)$) and $\|q_-(s)\|_{\mathcal{H}}$ by

\[
C \sum_{m=1}^{k-1} h(\zeta_{m+1} - \zeta_m)^2
\]

in order to conclude.

Clearly, the control of $\alpha_i^1(s)$ and $\zeta_i(s)$ follows from (i) of Lemma 3.11.

As for the control of $\|q_-(s)\|_{\mathcal{H}}$, it is enough from (3.71) to bound $A_-(s)$ which is controlled through a combination between (ii) and (iii) of Lemma 3.11.

In order to give the actual proof, we proceed as in section 5.3 page 113 in [11], though the situation is a bit different because of the presence of the forcing terms $J(s)$ and $\sum_{i=1}^{k} h(\zeta_{i+1} - \zeta_i)^2$ in the differential inequalities in Lemma 3.11.
If we introduce

\[ a(s) = \sum_{i=1}^{k} \alpha_i^2(s), \quad b(s) = A_-(s) + 2R_-(s) \quad \text{and} \quad H(s) = \sum_{m=1}^{k-1} h(\zeta_{m+1} - \zeta_m)^2 \]

where \( h \) is defined in (3.50), then we see from (i) of Lemma 3.9 and (3.79) that

\[ a(s) + b(s) + H(s) \rightarrow 0 \quad \text{as} \quad s \rightarrow \infty. \]

Moreover, we see from (3.79) and (3.72) that

\[ |b - A_-| \leq \frac{1}{1000} (A_- + \sum_{i=1}^{k} (\alpha_i^2)^2) \]

for \( s \) large enough, hence

\[ \frac{99}{100} A_-- \frac{1}{100} a \leq b \leq \frac{101}{100} A_- + \frac{1}{100} a. \]

Therefore, since \( J(s) \leq H(s) \) by (3.4) and (3.50), we have for \( s \) large,

\[ \forall \epsilon > 0, \quad CJ \sqrt{|A_-|} \leq \epsilon (a + b) + \frac{C'}{\epsilon} H(s). \]

Using (3.81), (3.83) and (3.82), we rewrite estimates (3.72) and Lemma 3.11 with the new variables, in the following:

**COROLLARY 3.12.** (Equations in the new framework) There exists \( K_0 \geq 1 \) such that for all \( \epsilon > 0 \), there exists \( s_0(\epsilon) \in \mathbb{R} \) such that for all \( s \geq s_0(\epsilon) \), the following holds:

(i) (Size of the solution)

\[ \frac{1}{K_0} (a + b) \leq \|q\|_{\mathcal{H}}^2 \leq K_0 (a + b), \]

\[ \left| \int_{-1}^{1} q_1 q_2 \rho \, dy \right| \leq K_0 (a + b). \]

(ii) (Equations)

\[ \frac{3}{2} a - \epsilon b - K_0 H \leq a' \leq \frac{5}{2} a + \epsilon b + K_0 H, \]

\[ b' \leq \frac{6}{p-1} \int_{-1}^{1} q_2^2 \frac{\rho}{1-y^2} \, dy + \epsilon (a + b) + \frac{K_0}{\epsilon} H, \]

\[ \frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho \, dy \leq -\frac{3}{5} b + K_0 \int_{-1}^{1} q_2^2 \frac{\rho}{1-y^2} \, dy + K_0 a + K_0 H, \]

\[ |H'| \leq \epsilon H. \]

We proceed in 2 steps:

- In Step 1, we show that \( a \) is controlled by \( b + H \).
- In Step 2, we show that \( b \) is controlled by \( H \) and conclude the proof using (3.84).
**Step 1:** \( a \) is controlled by \( b + H \). We claim that for \( \epsilon \) small enough, we have:

\[
\forall s \geq s_0(\epsilon), \quad a(s) \leq \epsilon b(s) + \frac{K_0}{\epsilon} H(s).
\]

Indeed, from Corollary 3.12, we see that for all \( s \geq s_0(\epsilon) \), we have

\[
a' \geq \frac{3}{2} a - \left( \epsilon b + \frac{K_0}{\epsilon} H \right),
\]

\[
\left( \epsilon b + \frac{K_0}{\epsilon} H \right)' \leq 2\epsilon \left( \epsilon b + \frac{K_0}{\epsilon} H \right) + \epsilon^2 a.
\]

Introducing \( \gamma_1(s) = a(s) - (\epsilon b(s) + \frac{K_0}{\epsilon} H(s)) \), we see that for all \( s \geq s_0(\epsilon) \),

\[
\gamma_1' = a' - \left( \epsilon b' + \frac{K_0}{\epsilon} H' \right) \geq \frac{3}{2} a - \left( \epsilon b + \frac{K_0}{\epsilon} H \right) - 2\epsilon \left( \epsilon b + \frac{K_0}{\epsilon} H \right) - \epsilon^2 a
\]

\[
= \left( \frac{3}{2} - \epsilon^2 - 1 - 2\epsilon \right) a + (1 + 2\epsilon) \gamma_1 \geq \gamma_1
\]

if \( \epsilon \) is small enough. Since \( \gamma_1(s) \to 0 \) as \( s \to \infty \) (see (3.82)), this implies \( \gamma_1(s) \leq 0 \), hence (3.87) follows.

**Step 2:** \( b \) is controlled by \( H \). We claim that in order to conclude, it is enough to prove for some \( K_1 > 0 \) that

\[
\forall s \geq s_0(\epsilon), \quad f(s) \leq K_1 H(s)
\]

where

\[
f = b + \eta \int_{-1}^{1} q_1 q_2 \rho \, dy \quad \text{and} \quad \eta = \frac{1}{2} \min \left( \frac{1}{2K_0}, \frac{6}{(p-1)K_0} \right).
\]

Indeed, using (3.85) and (3.87), and taking \( \epsilon \) small enough, we get for all \( s \geq s_0(\epsilon) \),

\[
\left| \int_{-1}^{1} q_1 q_2 \rho \, dy \right| \leq 2K_0 b + \frac{K_0^2}{\epsilon} H \quad \text{and} \quad |f - b| \leq 2K_0 \eta b + \eta \frac{K_0^2}{\epsilon} H \leq \frac{b}{2} + \eta \frac{K_0^2}{\epsilon} H,
\]

hence

\[
\frac{b}{2} - \eta \frac{K_0^2}{\epsilon} H \leq f \leq 2b + \eta \frac{K_0^2}{\epsilon} H.
\]

Therefore, if (3.88) holds, then using (3.84), (3.90) and (3.87), we see that for some \( K_2 > 0 \) and for all \( s \geq s_0(\epsilon) \), \( \|q(s)\|_{\mathcal{H}}^2 \leq K_0(a(s) + b(s)) \leq K_2 H(s) \) which is the desired conclusion of Proposition 3.8. It remains to prove (3.88).
Using Corollary 3.12, (3.87), (3.89) and the fact that $K_0 \geq 1$, and taking $\epsilon$ small enough, we get for all $s \geq s_0(\epsilon)$:

\begin{align*}
(3.91) \quad b' &\leq -\frac{6}{p-1} \int_{-1}^{1} q_{.-2} \frac{\rho}{1-y^2} dy + 2\epsilon b + 2\frac{K_0}{\epsilon} H, \\
(3.92) \quad \frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho dy &\leq -\frac{2}{5} b + K_0 \int_{-1}^{1} q_{.-2} \frac{\rho}{1-y^2} dy + 2\frac{K_0^2}{\epsilon} H, \\
(3.93) \quad f' &\leq -\left(\frac{2}{5} \eta - 2\epsilon\right) b - \left(\frac{6}{p-1} - K_0 \eta\right) \int_{-1}^{1} q_{.-2} \frac{\rho}{1-y^2} dy + \left(2\frac{K_0}{\epsilon} + 2\frac{K_0^2}{\epsilon} \eta\right) H \\
&\leq -\frac{\eta}{4} b + 3\frac{K_0}{\epsilon} H \leq -\frac{\eta}{8} f + 4\frac{K_0}{\epsilon} H.
\end{align*}

If $\gamma_2(s) = f(s) - \frac{64K_0}{\eta \epsilon} H(s)$, then we write from (3.86) and (3.93), for all $s \geq s_0(\epsilon)$,

\[\gamma_2' = f' - \frac{64K_0}{\eta \epsilon} H' \leq -\frac{\eta}{8} f + 4\frac{K_0}{\epsilon} H + \frac{64K_0}{\eta \epsilon} \epsilon H = -\frac{\eta}{8} \gamma_2 + \frac{K_3}{\epsilon} H \leq -\frac{\eta}{8} \gamma_2\]

because $K_3 = -\frac{64K_0}{\eta} + \frac{64K_0}{\eta} \epsilon + 4K_0 = -4K_0 + \frac{64K_0}{\eta} \epsilon \leq 0$ if $\epsilon$ is small enough. Therefore, for all $s \geq s_0(\epsilon)$, $\gamma_2(s) \leq e^{-\frac{\eta}{8}(s-s_0)} \gamma_2(s_0)$, hence

\[f(s) \leq \frac{64K_0}{\eta \epsilon} H(s) + e^{-\frac{\eta}{8}(s-s_0)} |\gamma_2(s_0)|.
\]

Since we have from (3.86) and (3.81),

\[H(s) \geq e^{-\epsilon(s-s_0)} H(s_0) \quad \text{and} \quad H(s_0) > 0,
\]

taking $\epsilon \leq \frac{\eta}{8}$, we see that (3.88) follows from (3.94) and (3.95). This concludes the proof of Proposition 3.8.

\[\square\]

### 3.3. An ODE system satisfied by the solitons’ centers.

With Proposition 3.8, we are ready to prove Proposition 3.2 now. The proof consists in refining the projection of equation (3.60) with the projector $\pi_0^d$ (3.54), already performed in the proof of (i) of Lemma 3.11 (see Part 1 page 632).

**Proof of Proposition 3.2.** Using (C.1), (C.2), (C.7), (C.8), (C.11), the differential inequality (3.77) on $\zeta_i$ and the fact that $\alpha_0^i(s) \equiv \alpha_0^i(s) \equiv 0$ (see (3.74)), we write for some $\delta_2 > 0$ and for $s$ large enough,

\[e^{-\frac{2K_0}{p-1}} \psi(s) + \pi_0^d(s) \begin{pmatrix} 0 \\ R \end{pmatrix} \leq C \|q(s)\|^2_{H^1} + CJ(s)^{1+\delta_2}.
\]
Since we have from Proposition 3.8,

\[(3.97) \quad \|q(s)\|^2_H \leq C \sum_{i=1}^{k-1} (h(\zeta_{i+1} - \zeta_i))^2 \leq C J(s)^{1+\delta_3},\]

for some \(\delta_3 > 0\) and for \(s\) large enough, where \(h(\zeta)\) is defined in (3.50), it is clear that if one proves that for some \(c'_1 > 0\), \(\delta_4 > 0\) and for \(s\) large enough,

\[(3.98) \quad \frac{1}{c'_1} \pi_0^{d_i} \begin{pmatrix} 0 \\ R \end{pmatrix} = -e_{i-1} e^{-\frac{2}{p-1}|\zeta_i(s) - \zeta_{i-1}(s)|} + e_{i+1} e^{-\frac{2}{p-1}|\zeta_{i+1}(s) - \zeta_{i}(s)|} + C J(s)^{1+\delta_4}
\]

(with the convention \(\zeta_0(s) \equiv -\infty\) and \(\zeta_{k+1}(s) \equiv +\infty\)), then, Proposition 3.2 immediately follows from (3.96) and (3.97) (with \(\delta_0 = \min(\delta_2, \delta_3, \delta_4)\)). It remains to prove (3.98) in order to conclude the proof of Proposition 3.2.

**Proof of (3.98).** We claim first that

\[(3.99) \quad \left| R - \sum_{j=1}^{k} p\kappa(d_j(s))^{p-1} 1_{\{y_j-1(s) < y < y_j(s)\}} \sum_{l \neq j} e_l \kappa(d_l(s)) \right| \leq C \sum_{j=1}^{k} \kappa(d_j(s))^{p-\bar{p}} 1_{\{y_j-1(s) < y < y_j(s)\}} \sum_{l \neq j} \kappa(d_l(s))^{\bar{p}}
\]

where \(y_i\) are the solitons’ separators defined in (E.1).

Indeed, let us take \(y \in (y_j-1(s), y_j(s))\) and set \(X = (\sum_{l \neq j} e_l \kappa(d_l(s)))/e_j \kappa(d_j(s))\).
From the fact that \(\zeta_{j+1}(s) - \zeta_j(s) \to \infty\) and (E.1), we have \(|X| \leq 3\) hence

\[\|1 + X\|^{p-1}(1 + X) - 1 - pX \leq CX^2\]

and for \(y \in (y_{j-1}(s), y_j(s))\) and \(s\) large,

\[\left| K^{p-1} K - e_j \kappa(d_j(s))^p - p\kappa(d_j(s))^{p-1} \sum_{l \neq j} e_l \kappa(d_l(s)) \right| \leq C \kappa(d_j(s))^{p-2} \sum_{l \neq j} \kappa(d_l(s))^2.
\]

Since for \(y \in (y_{j-1}(s), y_j(s))\), \(|\sum_{l \neq j} e_l \kappa(d_l(s))^p| \leq \sum_{l \neq j} \kappa(d_l(s))^p\) and \(\kappa(d_j(s)) \geq \kappa(d_l(s))\) if \(l \neq j\), this concludes the proof of (3.99).

Now, we prove (3.98). Using (3.99), (3.54), (3.56) and the notations of Lemma E.1, we write

\[\left| \pi_0^{d_i} \begin{pmatrix} 0 \\ R \end{pmatrix} - p c_0 \sum_{j=1}^{k} \sum_{l \neq j} e_l A_{i,j,l} \right| \leq C \sum_{j=1}^{k} \sum_{l \neq j} B_{i,j,l}.
\]
Since we have from (iii) and (iv) of Lemma E.1, $|A_{i,j,l}| + |B_{i,j,l}| \leq CJ^{1+\delta_1}$, except for $A_{i,i,l}$ with $l = i \pm 1$ where we have $|A_{i,i,l} - c'' \operatorname{sgn}(l - i)e^{-\frac{1}{1-\gamma}(|x - \xi_{i,l}|)}| \leq CJ^{1+\delta_1}$ where $\delta_4 = \min(\delta_5, \delta_6) > 0$, we get (3.98). Since Proposition 3.2 follows from (3.98) and (3.96), this concludes the proof of Proposition 3.2 too.

3.4. The blow-up set has the corner property near $x_0 \in \mathcal{S}$ when $k(x_0) \geq 2$.

We derive here the following consequence of Proposition 3.1:

**Proposition 3.13.** (Existence of signed lines and the corner property near $x_0 \in \mathcal{S}$) If $x_0 \in \mathcal{S}$ with $k(x_0) \geq 2$, then:

(i) For all $j = 1, \ldots, k$,

$$u(z_j(t), t) \sim c_j \kappa_0 \cosh \frac{1}{\kappa_0} \zeta_j(s)(T(x_0) - t)^{-\frac{1}{\kappa_0}} \text{ as } t \to T(x_0),$$

where $t \mapsto z_j(t)$ is continuous and defined by

$$z_j(t) = x_0 + (T(x_0) - t) \tanh \zeta_j(s) \text{ with } s = -\log (T(x_0) - t).$$

(ii) We have for some $\delta_0 > 0$ and $C_0 > 0$,

$$\text{if } \frac{x - x_0}{T(x_0) - x - x_0} \leq T(x) \leq T(x_0)$$

$$- |x - x_0| + \frac{C_0 |x - x_0|}{\log (x - x_0)} \leq \frac{x_2 - x_1}{T(x_1) - t},$$

where $z_j(t)$ corresponds in the original variables to the center of the $j$th soliton in the description (3.1).

**Remark.** The point $z_j(t)$ corresponds in the original variables to the center of the $j$th soliton in the description (3.1).

**Remark.** In the next section, we prove that for all $x_0 \in \mathcal{S}$, we have $k(x_0) \geq 2$. Thus, the result will hold for all $x_0 \in \mathcal{S}$.

**Proof.** It follows from Proposition 3.1 and the following:

**Lemma 3.14.** (Upper blow-up bound for equation (1.1)) For all real $x_1, x_2$ and $t$ satisfying

$$\left(1 - e^{-3}\right)T(x_1) \leq t < T(x_1) \text{ and } |y| \leq 1 \text{ where } y = \frac{x_2 - x_1}{T(x_1) - t},$$

it holds that

$$|u(x_2, t)| \leq K(x_1)(T(x_1) - t)^{-\frac{2}{p-1}(1 - y^2)} - \frac{1}{p-1},$$

where $K(x_1)$ depends only on $p$ and an upper bound on $T(x_1)$ and $1/T(x_1)$.

**Proof.** Take $x_1 \in \mathbb{R}$. Using [11, Proposition 3.5, page 66], we see that for all $s \geq -\log T(x_1) + 3$, we have $\|w_{x_1}(s)\|_{\mathcal{H}_0} \leq K(x_1)$, where $K(x_1)$ depends only
on $p$ and an upper bound on $T(x_1)$ and $1/T(x_1)$. Using (i) of Lemma B.1, we see that

$$\text{(3.103)} \quad \text{if } |y| < 1, \text{ then } \left| w_{x_1}(y,s) \right| \leq K(x_1)(1-y^2)^{-\frac{1}{p-1}}. $$

Now, taking $x_1$, $x_2$ and $t$ satisfying (3.102) and introducing $s = -\log(T(x_1) - t)$, we see that $s \geq -\log T(x_1) + 3$ and the conclusion follows from (3.103) thanks to the selfsimilar transformation (1.4).

**Proof of Proposition 3.13.** (i) Using Proposition 3.1 and (i) of Lemma B.1, we see that we have

$$\text{(3.104)} \quad \text{sup}_{|y|<1} \left( (1-y^2)^{\frac{1}{p-1}} w_{x_0}(y,s) - \sum_{i=1}^{k(x_0)} c_i \left( 1-y^2 \right)^{\frac{1}{p-1}} \kappa(d_i,y) \right) \longrightarrow 0 \text{ as } s \longrightarrow \infty.$$

Since we have

$$\text{(3.105)} \quad \kappa(d_i(s),y) \left( 1-y^2 \right)^{\frac{1}{p-1}} = \kappa_0 \cosh^{-\frac{2}{p-1}} \left( \xi - \zeta_i(s) \right) \text{ if } y = \tanh \xi$$

and $\zeta_{i+1}(s) - \zeta_i(s) \rightarrow \infty$ as $s \rightarrow \infty$, we apply (3.104) with $y = -d_j(s) = \tanh \zeta_j(s)$ to get

$$\text{(3.106)} \quad \left( 1-d_j(s)^2 \right)^{\frac{1}{p-1}} w_{x_0}(-d_j(s),s) \longrightarrow e_j^* \kappa_0 \text{ as } s \longrightarrow \infty.$$

Since $\left( 1-d_j(s)^2 \right)^{\frac{1}{p-1}} = \cosh^{\frac{2}{p-1}} \zeta_j(s)$, $e_j^* = e_1^j(-1)^{j+1}$ and $s \mapsto d_j(s) \in (-1,1)$ is continuous, using the selfsimilar transformation (1.4), we see that (i) follows.

(ii) Let us introduce $B(x)$ by

$$\text{(3.107)} \quad \frac{T(x_0) - T(x)}{|x_0 - x|} = 1 - B(x).$$

From the fact that $x \mapsto T(x)$ is 1-Lipschitz, we see that

$$\text{(3.108)} \quad \left| \frac{T(x_0) - T(x)}{|x_0 - x|} \right| \leq 1 \text{ hence } 0 \leq B(x) \leq 2,$$

and the left-hand inequality of (3.101) follows. Using (3.107), we see that in order to prove the right-hand inequality, it is enough to prove that for $x_0 - x$ small,

$$\text{(3.109)} \quad B(x) \leq C \left( \frac{(k-1)(p-1)}{2^p} \right)^{\frac{1}{2}} \text{ where } l = \left| \log (x_0 - x) \right|.$$
bounds for $|u(z_1(\tilde{t}), \tilde{t})|$ where $z_1(\tilde{t})$ defined in (3.100) is the “center” of the first soliton and

$$\tilde{t} = \bar{t}(x) = T(x_0) - 2(x_0 - x).$$

The following claim allows us to conclude:

**Claim 3.15.** It holds that:

(i) $|u(z_k(\tilde{t}), \tilde{t})| \geq \frac{1}{C} (x_0 - x)^{-\frac{2}{p-1}} l^{\frac{k-1}{2}}.$

(ii) $|u(z_k(\tilde{t}), \tilde{t})| \leq C (x_0 - x)^{-\frac{2}{p-1}} B(x)^{-\frac{1}{p-1}}.$

Indeed, if Claim 3.15 holds, then we see that

$$\frac{1}{C} (x_0 - x)^{-\frac{2}{p-1}} l^{\frac{k-1}{2}} \leq |u(z_k(\tilde{t}), \tilde{t})| \leq C (x_0 - x)^{-\frac{2}{p-1}} B(x)^{-\frac{1}{p-1}}$$

and (3.109) follows. It remains to prove Claim 3.15 to conclude the proof of Proposition 3.13.

**Proof of Claim 3.15.** (i) Using (i) of Proposition 3.13 with $j = 1$, we get for $x_0 - x$ small enough,

$$|u(z_k(\tilde{t}), \tilde{t})| \geq \frac{\kappa_0}{2} \cosh^{\frac{2}{p-1}} \zeta_1(\tilde{s})(T(x_0) - \tilde{t})^{-\frac{2}{p-1}}$$

where

$$\tilde{s} = -\log(T(x_0) - \tilde{t}) = -\log 2 - \log(x_0 - x).$$

Recalling from (ii) of Proposition 3.1 that

$$\left| \zeta_1(\tilde{s}) + \frac{(k-1)(p-1)}{4} \log(\tilde{s}) \right| \leq C_0,$$

hence, from (3.112)

$$\cosh \zeta_1(\tilde{s}) \geq \frac{e^{-\zeta_1(\tilde{s})}}{2} \geq C \tilde{s}^{\frac{(k-1)(p-1)}{4}} \geq C l^{\frac{(k-1)(p-1)}{4}},$$

the conclusion follows from (3.111) and (3.110).

(ii) The idea is to apply Lemma 3.14 for some well-chosen $x_1, x_2$ and $t$. We claim that condition (3.102) holds with $x_1 = x$, $x_2 = z_1(\tilde{t})$ and $t = \tilde{t}$. Indeed:

- using (3.108) and (3.110), we write for $x_0 - x$ small enough,

$$T(x) \geq T(x_0) - (x_0 - x) > T(x_0) - 2(x_0 - x) = \bar{t}(x) \geq T(x_0)(1 - e^{-3});$$
• using (3.100), (3.110), (3.113) and (3.112), we write

\[
\begin{align*}
    z_1\left(\tilde{t}\right) - x &= x_0 + 2\left(x_0 - x\right)\left(-1 + 2e^{2\xi_1(\tilde{s})} + O\left(e^{4\xi_1(\tilde{s})}\right)\right) - x \\
    &= \left(x_0 - x\right)\left(-1 + e^{O(1)\frac{-(k-1)(p-1)}{2}}\right) \\
    &= \left(x_0 - x\right)\left(-1 + e^{O(1)l^{-\frac{(k-1)(p-1)}{2}}}\right).
\end{align*}
\] (3.115)

Since we have from (3.108) and the definitions (3.107) and (3.110) of \(B(x)\) and \(\tilde{t}\),

\[
T(x) - \tilde{t} = T(x) - T(x_0) + 2\left(x_0 - x\right)
\]

\[
= \left(x_0 - x\right)\left(1 + B(x)\right) \in [x_0 - x, 3(x_0 - x)],
\] (3.116)

we deduce that \(y = \frac{z_1\left(\tilde{t}\right) - x}{T(x) - \tilde{t}}\) satisfies

\[
y = \frac{-1 + e^{O(1)}l^{-\frac{(k-1)(p-1)}{2}}}{1 + B(x)} \in \left[-1 + \frac{l^{-\frac{(k-1)(p-1)}{2}}}{C}, -\frac{1}{3} + Cl^{-\frac{(k-1)(p-1)}{2}}\right].
\] (3.117)

Thus, from (3.114) and (3.117), condition (3.102) holds and Lemma 3.14 applies and gives

\[
|u(z_1(\tilde{t}), \tilde{t})| \leq K(x)(T(x) - \tilde{t})^{-\frac{1}{p-1}}\left(1 - y^2\right)^{-\frac{1}{p-1}}
\] (3.118)

where \(K(x) \leq K_0\) uniformly for \(x\) close to \(x_0\). Since we have from (3.117) and (3.108),

\[
\left(1 - y^2\right) \geq \frac{1}{C}(1 + y) = \frac{1}{C} \frac{B(x) + e^{O(1)}l^{-\frac{(k-1)(p-1)}{2}}}{1 + B(x)} \geq \frac{B(x)}{C},
\] (3.119)

the conclusion follows from (3.119), (3.116) and (3.118). This concludes the proof of Claim 3.15 and Proposition 3.13 too. \(\Box\)

4. Properties of \(S\). We proceed in 3 subsections. We first prove that the interior of \(S\) is empty which is the desired conclusion of Proposition 5. Then, we give the proofs of Theorems 4, 4’ and 6 as well as Proposition 8.

4.1. Soliton characterization on \(S\). This subsection is devoted to the proof of the following result:

**Proposition 4.1.** (i) The interior of \(S\) is empty.

(ii) For all \(x_0 \in S\), \(k(x_0) \geq 2\).

Before proving this proposition, let us first state the following Lemmas:

**Lemma 4.2.** (Characterization of the interior of \(S\)) For any \(x_1 < x_2\), the following statements are equivalent:
(a) \((x_1, x_2) \in \mathcal{S}\).
(b) There exists \(x^* \in [x_1, x_2]\) such that for all \(x \in [x_1, x_2]\), \(T(x) = T(x^*) - |x - x^*|\).

**Lemma 4.3.** Consider \(x_1 < x_2\) such that \(e \equiv \frac{T(x_2) - T(x_1)}{x_2 - x_1} = \pm 1\). Then,
(i) for all \(x \in [x_1, x_2]\), \(T(x) = T(x_1) + e(x - x_1)\),
(ii) \((x_1, x_2) \in \mathcal{S}\).

**Lemma 4.4.** (Boundary properties of \(\mathcal{S}\))
(i) For all \(x_0 \in \partial \mathcal{S}\), \(k(x_0) \neq 0\).
(ii) Consider \(x_0 \in \partial \mathcal{S}\) with \(k(x_0) = 1\). If there exists a sequence \(x_n \in \mathbb{R}\) converging from the left (resp. the right) to \(x_0\), then \(x_0\) is left-non-characteristic (resp. right-non-characteristic).

**Remark.** We mean by \(x_0\) is left-non-characteristic (resp. right-non-characteristic) that it satisfies condition (1.3) only for \(x < x_0\) (resp. for \(x > x_0\)).

**Remark.** It is not possible to prove by a direct argument that \(k(x_0) \geq 1\) when \(x_0\) is arbitrary in \(\mathcal{S}\). We need to prove it first for \(x_0 \in \partial \mathcal{S}\) and then prove that the interior is empty. See the derivation of Proposition 4.1 from Lemma 4.4.

We now give the proofs of Lemmas 4.2, 4.3 and 4.4.

**Proof of Lemma 4.2.** (a)⇒(b): Let us introduce \(x^* \in [x_1, x_2]\) such that
\[
T(x^*) = \max_{x_1 \leq x \leq x_2} T(x).
\]
We claim that \(T(x)\) is nondecreasing on \([x_1, x^*]\), and nonincreasing on \([x^*, x_2]\).
Indeed, let us prove the first fact, the second being similar. If for some \(x' \leq x''\) in \([x_1, x^*]\), we have \(T(x') > T(x'')\), then \(\min_{x' \leq x \leq x''} T(x) \leq T(x'') < T(x') \leq T(x^*)\). Therefore, this minimum is achieved at a point \(x^*\) different from \(x'\) and \(x^*\), hence
\[
\tilde{x} \in (x', x^*) \subset (x_1, x_2).
\]
In other words, \(\tilde{x}\) is a local minimum, hence non-characteristic, which is in contradiction with (a).

The result clearly follows if we prove that
\[
\forall x \in (x_1, x^*), \quad T(x) = T(x_1) + (x - x_1), \tag{4.1}
\]
\[
\forall x \in (x^*, x_2), \quad T(x) = T(x_2) - (x - x_2). \tag{4.2}
\]
We only prove (4.1) since (4.2) follows similarly.

Assume by contradiction that for some \(x' \in (x_1, x^*)\), we have
\[
\frac{T(x') - T(x_1)}{x' - x_1} = m_0 \not\in \{-1, 1\}. \tag{4.3}
\]
Then, since \( x \mapsto T(x) \) is 1-Lipschitz and nondecreasing, it follows that \( 0 \leq m_0 < 1 \). Considering a family of lines of slope \( \frac{1+m_0}{2} \) growing from below, we find \( \lambda_0 \in \mathbb{R} \) and \( x_0 \in [x_1, x'] \) such that

\[
\forall x \in [x_1, x'], \quad T(x) \geq \frac{1+m_0}{2} (x - x_1) + \lambda_0 \quad \text{and} \quad T(x_0) = \frac{1+m_0}{2} (x_0 - x_1) + \lambda_0.
\]

(4.4)

If \( x_0 \in (x_1, x') \), then for all \( x \in [x_1, x'] \), \( T(x) \geq \frac{1+m_0}{2} (x - x_0) + T(x_0) \), hence \( x_0 \) is non-characteristic (the cone of slope \( \frac{1+m_0}{2} \) is convenient).

If \( x_0 = x' \), then since \( T(x) \) is non-decreasing on \( (x_1, x^*) \), it follows that \( x_0 \) is again non-characteristic. In these two cases, we have a contradiction with the fact that \( (x_1, x_2) \in \mathcal{S} \).

If \( x_0 = x_1 \), then we have from (4.4), \( T(x_1) = \lambda_0 \) and \( T(x') \geq \frac{1+m_0}{2} (x' - x_1) + T(x_1) \), in contradiction with (4.3).

Thus, (4.1) holds. Since (4.2) follows similarly, (b) follows too.

(b)⇒(a): For any \( x \in (x_1, x_2) \), the left-slope of \( x \mapsto T(x) \) is 1 or \(-1\), hence, by definition, \( x \in \mathcal{S} \) and (a) follows. This concludes the proof of Lemma 4.2. □

We now give the proof of Lemma 4.3:

**Proof of Lemma 4.3.** Up to replacing \( u(x,t) \) by \( u(-x,t) \), we can assume that \( x_1 < x_2 \) and

\[
eq \frac{T(x_2) - T(x_1)}{x_2 - x_1} = 1.
\]

(4.5)

(i) If \( x \in (x_1, x_2) \), we use the fact that \( x \mapsto T(x) \) is \( l \)-Lipschitz together with (4.5) to write:

\[
T(x) \leq T(x_1) + (x - x_1),
\]

\[
T(x) \geq T(x_2) - (x_2 - x) = T(x_1) + (x - x_1)
\]

and (i) follows.

(ii) It follows from (i), just by applying the fact that (b) implies (a) in Lemma 4.2 (take \( x^* = x_2 \)). This concludes the proof of Lemma 4.3. □

We now give the proof of Lemma 4.4.

**Proof of Lemma 4.4.** Consider \( x_0 \in \partial \mathcal{S} \). Up to replacing \( u(x,t) \) by \( u(-x,t) \), we can assume that for some sequence

\[
x_n \in \mathcal{R}, \quad \text{we have } x_n < x_0 \text{ and } x_n \to x_0 \text{ as } n \to \infty.
\]

(4.6)
Therefore, we have
\begin{equation}
\forall x < x_0, \quad T(x) > T(x_0) - (x_0 - x).
\end{equation}

Indeed, note first form the fact that \( x \mapsto T(x) \) is 1-Lipschitz that for all \( x < x_0 \),
\( T(x) \geq T(x_0) - (x_0 - x) \). By contradiction, if for some \( \hat{x} < x_0 \), we have \( T(\hat{x}) = T(x_0) - (x_0 - \hat{x}) \), then we see from Lemma 4.3 that \((\hat{x}, x_0) \subset \mathcal{S}\), in contradiction with (4.6). Thus, (4.7) holds.

To prove (i) and (ii), we proceed by contradiction. Assume then that

\( k(x_0) = 0 \) (case 1),
\( \text{or } k(x_0) = 1 \text{ and } x_0 \text{ is not left-non-characteristic} \) (case 2).

Using (3.1) when \( k(x_0) = 0 \) and Proposition 2.2 when \( k(x_0) = 1 \), we see that
\begin{equation}
\left\| \left( \frac{w_{x_0}(s)}{\partial_s w_{x_0}(s)} \right) - \left( \frac{w_\infty}{0} \right) \right\|_\mathcal{H} \rightarrow 0 \text{ as } s \rightarrow \infty,
\end{equation}

where
\begin{equation}
w_\infty(y) = 0 \text{ if } k(x_0) = 0 \text{ and } w_\infty(y) = e^* \kappa(d(x_0), y) \text{ if } k(x_0) = 1,
\end{equation}

for some \( e^* = \pm 1 \) and \( d(x_0) \in (0, 1) \). Now, we claim the following continuity result:

**Claim 4.5.** For all \( \epsilon_0 > 0 \), there exists \( \tilde{t} < T(x_0) \) and \( \tilde{x} < x_0 \) such that for all \( x' \in (\tilde{x}, x_0) \),
\begin{equation}
\left\| \left( \frac{w_{x'}(\tilde{s}_0(x'))}{\partial_s w_{x'}(\tilde{s}_0(x'))} \right) - \left( \frac{w_\infty}{0} \right) \right\|_\mathcal{H} \leq \epsilon_0
\end{equation}

where \( \tilde{s}_0(x') = -\log(T(x') - \tilde{t}) \).

**Proof.** See Appendix D.

Let us first use this lemma to find a contradiction.

**Case 1:** \( k(x_0) = 0 \). Consider some \( \epsilon_0 > 0 \) (to be fixed small enough later). Using this claim, (4.9) and (4.6), we see that for some \( \tilde{t} < T(x_0) \) and for \( n \) large enough, we have
\[
x_n \in \mathcal{R} \text{ and } \left\| \left( \frac{w_{x_n}(\tilde{s}_0(x'))}{\partial_s w_{x_n}(\tilde{s}_0(x'))} \right) \right\|_\mathcal{H} \leq \epsilon_0.
\]

Using the continuity of \( E(w) \) in \( \mathcal{H} \) (which is a consequence of Lemma B.1), we see that
\[
E\left( w_{x_n}(\tilde{s}_0(x')) \right) \leq C \epsilon_0 \leq \frac{1}{2} E(\kappa_0)
\]
The use of the geometrical interpretation of $E$ known from the limit and the monotonicity of $E(w_{x_n}(s))$ stated in Section 1.1 that $E(w_{x_n}(s)) \geq E(\kappa_0) > 0$, which is a contradiction.

Case 2: $k(x_0) = 1$ and $x_0$ is not left-non-characteristic. Since $x_0$ is not left-non-characteristic, we see from (4.7) that there exists a sequence $\hat{x}_n$ such that

$$\hat{x}_n < x_0, \quad \hat{x}_n \rightarrow x_0 \text{ and } \hat{m}_n = \frac{T(\hat{x}_n) - T(x_0)}{\hat{x}_n - x_0} \in \left[1 - \frac{1}{n}, 1\right].$$

Considering a family of lines of slope $\frac{1 + \hat{m}_n}{2}$, we can select one such that

$$\forall x \in [\hat{x}_n, x_0],$$

(4.12) $T(x) \geq \frac{(1 + \hat{m}_n)}{2}(x - \hat{x}_n) + \lambda_n$ and $T(\hat{x}_n) = \frac{(1 + \hat{m}_n)}{2}(\hat{x}_n - \hat{x}_n) + \lambda_n$

for some $\lambda_n \in \mathbb{R}$ and $\hat{x}_n \in [\hat{x}_n, x_0]$.

If $\hat{x}_n = x_0$, then for all $x \in [\hat{x}_n, x_0]$, $T(x) \geq \frac{(1 + \hat{m}_n)}{2}(x - x_0) + T(x_0)$, which is in contradiction with the fact that $x_0$ is left-non-characteristic.

If $\hat{x}_n = \hat{x}_n$, then we have from (4.12), $T(\hat{x}_n) = \lambda_n$ and $T(x_0) \geq \frac{1 + \hat{m}_n}{2}(x_0 - \hat{x}_n) + T(\hat{x}_n)$, in contradiction with (4.11).

If $\hat{x}_n \in (\hat{x}_n, x_0)$, then $\hat{x}_n \in \mathcal{R}$ (the cone of slope $\frac{1 + \hat{m}_n}{2}$ is convenient). Since $\hat{x}_n \rightarrow x_0$ and $\hat{x}_n \in \mathcal{R}$, we see from Claim 4.5 that for some $\hat{t} < T(x_0)$ and $n$ large enough, we have

(4.13) $\left\| \left( \begin{array}{c} w_{\hat{x}_n}(\hat{s}_0(\hat{x}_n)) \\ \partial_s w_{\hat{x}_n}(\hat{s}_0(\hat{x}_n)) \end{array} \right) - \epsilon^*(\kappa(d(x_0), \cdot), 0) \right\|_{\mathcal{H}} \leq \epsilon^*$

where $\epsilon^*$ is introduced in Proposition 2.2. Since the energy barrier follows from the fact that $\hat{x}_n \in \mathcal{R}$, Proposition 2.2 applies and we have for $n$ large enough,

$$\left( w_{\hat{x}_n}(s), \partial_s w_{\hat{x}_n}(s) \right) \longrightarrow \epsilon^*(\kappa(d_n), 0) \text{ in } \mathcal{H} \text{ as } s \longrightarrow \infty,$$

where $|d_n - d(x_0)| < \eta_0$ for some $\eta_0 > 0$ small enough so that $|d(x_0) \pm \eta_0| < 1$. The use of the geometrical interpretation of $d_n$ is crucial for the conclusion. Indeed, from (4.13) and the regularity result of [11] cited in Section 1.1, we see that $x \mapsto T(x)$ is differentiable at $x = \hat{x}_n$ and that

$$T'(\hat{x}_n) = d_n \leq d(x_0) + \eta_0 < 1$$

on the one hand. On the other hand, using (4.12) and (4.11), we see that

$$T'(\hat{x}_n) = \frac{1 + \hat{m}_n}{2} \longrightarrow 1 \text{ as } n \longrightarrow \infty$$

which is a contradiction. This concludes the proof of Lemma 4.4. \qed

Now, we are ready to prove Proposition 4.1.
Proof of Proposition 4.1. (i) Let us assume by contradiction that $S$ contains some non empty interval $(a',b')$. Since $S \neq \mathbb{R}$ by the result of [11] cited in Section 1.1, by maximizing this interval and up to replacing $u(x,t)$ by $u(-x,t)$, we can assume that:

$$(a,b) \subset S \text{ with } a \in \partial S, \quad b > a$$

and, either $b \in \partial S$ or $b = +\infty$. If $b$ is finite, then up to replacing $u(x,t)$ by $u(-x,t)$, we can assume that $T(b) \geq T(a)$. Using Lemma 4.2 and the fact that $T(x) \geq 0$, we see that for some $\tilde{b} < b$, we have

$$(4.14) \quad \forall x \in (a,\tilde{b}), \quad T(x) = T(a) + (x - a).$$

We consider three cases and find a contradiction in each case.

- If $k(a) = 0$, then a contradiction occurs from (i) of Lemma 4.4.
- If $k(a) = 1$, then from the fact that $a \in \partial S$, there exists a sequence $x_n \in \mathcal{R}$ such that $x_n \to a$ as $n \to \infty$. Since $(a,b) \subset S$, it follows that $x_n < a$ for $n$ large enough. Therefore, applying (ii) of Lemma 4.4, we see that $a$ is left-non-characteristic. Since it is clearly right-non-characteristic by (4.14), $a$ is in fact non-characteristic, which contradicts the fact that $a \in \partial S \subset S$ (note that $S$ is closed since its complementary set $\mathcal{R}$ is open by the result of [12] cited in Section 1.1).
- If $k(a) \geq 2$, then the corner property stated in Proposition 3.13 is in contradiction with (4.14).

Thus, (i) of Proposition 4.1 follows.

(ii) Consider $x_0 \in S$. From (i), we have $x_0 \in \partial S$. Using (i) of Lemma 4.4, we see that $k(x_0) \geq 1$. The result follows if we rule out the case $k(x_0) = 1$.

Assume by contradiction that $k(x_0) = 1$. Since the interior of $S$ is empty, we can construct 2 sequences $x_n$ and $y_n$ in $\mathcal{R}$, such that $x_n \to x_0$ from the left, and $y_n \to x_0$ from the right. Applying (ii) of Lemma 4.4, we see that $x_0$ is in fact left-non-characteristic and right-non-characteristic, hence non-characteristic. This contradicts the fact that $x_0 \in S$. Thus, (ii) follows. This concludes the proof of Proposition 4.1.

4.2. On characteristic points for equation (1.1). We prove Proposition 8, as well as Theorems 6 and 4 here.

Proof of Theorem 6. Consider $u(x,t)$ a solution of equation (1.1) and $x_0 \in S$. Using (ii) of Proposition 4.1, we see that $k(x_0) \geq 2$. Therefore, Proposition 3.1 applies and directly gives the conclusion of Theorem 6.

Proof of Proposition 8. Consider $u(x,t)$ a solution of equation (1.1) and $x_0 \in S$. Using (ii) of Proposition 4.1, we see that $k(x_0) \geq 2$. Therefore, Proposition 3.1 and Proposition 3.13 apply and give the conclusion of Proposition 8, except for the strict inequality in (1.15), which we prove now.
Assume by contradiction that for some $x_1 < x_0$, we have equality in the left-hand inequality of (1.15). Then, we see from Lemma 4.3 that $(x_1, x_0) \subset S$, which contradicts the fact that the interior of $S$ is empty (see (i) of Proposition 4.1). Thus, (1.15) follows and Proposition 8 follows too.

**Proof of Theorem 4.** Consider $u(x, t)$ a solution of equation (1.1) that blows up on a graph $x \mapsto T(x)$ such that for some $a_0 < b_0$ and some $t_0 \geq 0$, we have

\[ \forall x \in (a_0, b_0) \text{ and } t \in [t_0, T(x)], \quad u(x, t) \geq 0. \]  

(4.15)

We would like to prove that $(a_0, b_0) \subset \mathcal{R}$. Proceeding by contradiction, we assume that there exists $x_0 \in (a_0, b_0) \cap S$. Using Proposition 8, we see that for some $e_1 = \pm 1$ and $t_1 \in [t_0, T(x_0)]$, there are continuous $t \mapsto z_i(t)$ where $i = 1$ and 2 such that $z_i(t) \to x_0$ as $t \to T(x_0)$ and

\[ \forall t \in [t_1, T(x_0)], \quad e_1 u(z_1(t), t) > 0 \text{ and } e_1 u(z_2(t), t) < 0. \]

Therefore, $u$ changes sign in $(a_0, b_0) \times [t_1, T(x_0))$, which is in contradiction with (4.15). Thus, $(a_0, b_0) \subset \mathcal{R}$ and Theorem 4 follows.

**4.3. Non-existence of characteristic points for equation (4.16).** This subsection is dedicated to the following twin of equation (1.1):

\[
\begin{aligned}
\frac{\partial^2}{\partial t^2} u &= \frac{\partial^2}{\partial x^2} u + |u|^p, \\
u(0) &= u_0 \text{ and } u_t(0) = u_1.
\end{aligned}
\]  

(4.16)

Taking $p > 1$ and $(u_0, u_1) \in H^1_{\text{loc}, u} \times L^2_{\text{loc}, u}$, we claim the following twin of Theorem 4:

**THEOREM 4'.** The set of characteristic points $S$ is empty for any blow-up solution of equation (4.16).

**Proof.** To prove this result, one needs to perform for equation (4.16), an almost identical analysis to what we did for equation (1.1), in our previous papers, including the present paper. The adaptation is straightforward and the results are similar, until the characterization of stationary solutions in self-similar variables which makes the splitting point between the two equations (1.1) and (4.16):

\[
\text{(4.17) We now have only nonnegative solutions: 0 and } \kappa(d, y) \text{ for any } d \in (-1, 1).
\]

Taking into account this fact, we proceed by contradiction to prove Theorem 4' and consider $x_0 \in S$. Arguing as for equation (1.1), we show that $w_{x_0}(y, s)$ (defined as in (1.4)) decomposes into a finite sum of $k(x_0) \in \mathbb{N}$ solitons (which are all nonnegative due to (4.17)), in the sense that

\[
w_{x_0}(y, s) = \sum_{i=1}^{k(x_0)} \kappa(d_i(s), y) + q(y, s) \text{ with } \|q(s)\|_\mathcal{H} \to 0 \text{ as } s \to \infty.
\]  

(4.18)
where (when $k(x_0) \geq 1$) $d_i(s) = -\tanh \zeta_i(s)$ and (when $k(x_0) \geq 2$)

\begin{equation}
(4.19) \quad \zeta_{i+1}(s) - \zeta_i(s) \longrightarrow \infty \quad \text{as} \quad s \longrightarrow \infty \quad \text{for} \quad i = 1, \ldots, k - 1.
\end{equation}

Ruling out the cases $k(x_0) = 0$ and $k(x_0) = 1$ is similar to the case of equation (1.1) (we do it first for $x_0 \in \partial S$, then we show that $S = \partial S$).

Ruling out the case $k(x_0) \geq 2$ is the new feature, for which we give details here. When $k(x_0) \geq 2$, we see from the decomposition (4.18) that the solitons’ centers satisfy the same ODE system (3.3) as in the case of equation (1.1), with

\begin{equation}
(4.20) \quad e_i = 1 \quad \text{for all} \quad i = 1, \ldots, k,
\end{equation}

Because of (4.20), the interaction between the solitons has to be attractive, which is incompatible with (4.19). This concludes the proof of Theorem 4'.

**Appendix A. Continuity with respect to initial data of the blow-up time at a non-characteristic point.** This section is devoted to the proof of Proposition 2.1. The proof is more or less included in the arguments of the proof of [12, Lemma 2.2]. We only give here a sketch of the proof (see [12] for more details).

**Sketch of the proof of Proposition 2.1.** We will prove the continuity in the norm $H^1 \times L^2(\mathbb{R})$ since the result with the norm $H^1 \times L^2(|x| < A_0)$ follows from the finite speed of propagation.

Using the continuity of $u(t_0)$ for $t_0 < T(x_0)$ with respect to initial data, it follows that $T(x_0)$ is lower semi-continuous as a function of initial data.

For the upper continuity, we consider $T_0 > T(x_0)$ to be taken close enough to $T(x_0)$ and aim at proving that $\tilde{u}(x, t)$ blows up in finite time $\tilde{T}(x_0) < T_0$, where $\tilde{u}(x, t)$ is the solution of equation (1.1) with initial data $(\tilde{u}_0, \tilde{u}_1)$ close enough to $(u_0, u_1)$.

Up to changing $u$ in $-u$, we know from (1.12) that for some $d_0 \in (-1, 1)$ and $\delta_0 > 0$,

\begin{equation}
(A.1) \quad \left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \begin{pmatrix} \kappa(d_0, \cdot) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(|y| < 1 + \delta_0)} \longrightarrow 0 \quad \text{as} \quad s \longrightarrow \infty.
\end{equation}

Consider $s_0 < 0$ to be fixed later and introduce $t_0 < T(x_0)$ such that $\frac{T(x_0) - t_0}{T_0 - t_0} = 1 - e^{s_0}$. Using the self-similar transformation (1.4) and (A.1), we see that

\begin{equation}
(A.2) \quad \left\| \begin{pmatrix} w_{x_0}(y, - \log(T_0 - t_0)) \\ \partial_s w_{x_0}(y, - \log(T_0 - t_0)) \end{pmatrix} - \begin{pmatrix} w_-(s_0) \\ \partial_s w_-(s_0) \end{pmatrix} \right\|_{H^1 \times L^2(-1, 1)} \longrightarrow 0.
\end{equation}
as \( T_0 \to T(x_0) \), where \( w_-(y,s) = \kappa_0 \frac{(1-d^2) \frac{1}{p-1}}{(1-e^{s0} + d_0y)^{p-1}} \) is a particular solution of equation (1.5).

Since \( E(w_-(s_0)) < 0 \) for some \( s_0 < 0 \) from Appendix B in [12], we see from (A.2) and the continuity of \( u(t_0) \) with respect to initial data that for \( T_0 \) close enough to \( T(x_0) \) and \((\tilde{u}_0, \tilde{u}_1)\) close enough to \((u_0, u_1)\), we have

(A.3) \[ E\left(\tilde{w}_{x_0,T_0}(-\log(T_0 - t_0))\right) < 0. \]

Using the blow-up criterion of Antonini and Merle (see [3, Theorem 2]), we see that \( \tilde{w}_{x_0,T_0} \) cannot be defined for all \((y,s) \in (-1,1) \times [-\log T_0, \infty)\), which means that \( \tilde{u} \) blows up in finite time and that \( \tilde{T}(x_0) < T_0 \). This yields the upper semi-continuity and concludes the proof of Proposition 2.1. \( \Box \)

Appendix B. Estimates on the quadratic form \( \varphi \). This section is devoted to the proof of Lemma 3.10. The heart of the proof is the case of one soliton already treated in [11]. Since the sum of the solitons is decoupled (see (3.62)), going from the 1-soliton case to the multi-soliton case is done by localization of the function near each soliton’s center. Thus, our proof reduces to the control of the interaction between the different truncations of the function. We proceed in 3 subsections:

- in the first subsection, we give some preliminaries and recall the 1-soliton case from [11].
- in the second subsection, we first change the problem to the \( \xi \) variable, where \( y = \tanh \xi \), then, we apply the 1-soliton case to truncations of the function, locally near each soliton’s center, and sum-up the local information to get a global estimate (in the \( \xi \) variable).
- in the third subsection, we give the proof of Lemma 3.10.

B.1. The 1-soliton case. We first recall the following result from [11].

Claim B.1. (i) (A Hardy-Sobolev type identity) For all \( h \in H_0 \), it holds that

\[ \| h \|_{L^2_{\frac{1}{1-y^2}}} + \| h \|_{L^{p+1}_0} + \left\| h \left(1 - y^2\right)^{\frac{1}{p-1}} \right\|_{L^{p}(-1,1)} \leq C \| h \|_{H_0}. \]

(ii) (Boundedness of \( \kappa(d,y) \) in several norms) For all \( d \in (-1,1) \), it holds that

\[ \| \kappa(d,y) \|_{L^{p+1}_0} + \left\| \kappa(d,y) \left(1 - y^2\right)^{\frac{1}{p-1}} \right\|_{L^{p}(-1,1)} \leq C \| \kappa(d,y) \|_{H_0} \leq CE(\kappa_0). \]

(iii) (Same energy level for \( \kappa(d,y) \)) For all \( d \in (-1,1) \), it holds that

\[ E(\kappa(d,y)) = E(- \kappa(d,y)) = E(\kappa_0). \]
(iv) (Continuity of the Lyapunov functional) If \((w_i(y,s), \partial_s w_i(y,s)) \in \mathcal{H}\) for \(i = 1\) and \(2\) and for some \(s \in \mathbb{R}\), then

\[
|E(w_1(s)) - E(w_2(s))| \leq C \left\| \left( \frac{w_1(s)}{\partial_s w_1(s)} \right) - \left( \frac{w_2(s)}{\partial_s w_2(s)} \right) \right\|_{\mathcal{H}}^p + \left\| \left( \frac{w_2(s)}{\partial_s w_2(s)} \right) \right\|_{\mathcal{H}}^p.
\]

**Proof.** For (i), see [11, Lemma 2.2, page 51]. For (ii), use (i) and [11, identity (49), page 59]. For (iii), see [11, Proposition 1(ii), page 47]. The proof of (iv) is straightforward from the definition (1.7) of \(E(w)\). □

To prove estimates about \(\varphi\), we take advantage of the decoupling in the solitons’ sum (see (3.62)) and use information we proved in [11] for the 1-soliton version of \(\varphi\) defined for all \(d \in (-1,1), r \) and \(r \) in \(\mathcal{H}\) by

\[
\varphi_d(r, r) = \int_{-1}^{1} \left( r'_1 r'_1 (1 - y^2) - \left( p\kappa(d)^{p-1} - \frac{2(p+1)}{(p-1)^2} \right) r_1 r_1 + r_2 r_2 \right) \rho dy.
\]

In [11], we proved the following

**Lemma B.2.** (Estimates on the bilinear form \(\varphi_d\)) There exists \(\epsilon_0 > 0\) such that for all \(d \in (-1,1), r \) and \(r \) in \(\mathcal{H}\), we have

\[
|\varphi_d(r, r)| \leq C \|r\|_{\mathcal{H}} \|r\|_{\mathcal{H}},
\]

\[
\varphi_d(r, r) \geq \epsilon_0 \|r\|_{\mathcal{H}}^2 - \frac{1}{\epsilon_0} \sum_{\lambda=0}^{1} |\pi^1_{\lambda}(r)|^2.
\]

**Proof.** For the first line, see [11, estimate (138), page 91].

For the second line, we recall from [11, Proposition 4.7, page 90] that for some \(\epsilon_1 > 0\), for any \(d \in (-1,1)\) and \(r \in \mathcal{H}\), we have

\[
\varphi_d(r_{-d}, r_{-d}) \geq 2\epsilon_1 \|r\|_{\mathcal{H}}^2 - \frac{1}{\epsilon_1} \sum_{\lambda=0}^{1} |\pi^1_{\lambda}(r)|^2
\]

where \(r^d = \pi^d_{\lambda}(r)\) is defined in (3.76). Using (B.2) and (3.64), we write

\[
\varphi_d(r_{-d}, r_{-d}) \leq \varphi_d(r, r) + C \sum_{\lambda=0}^{1} |\pi^1_{\lambda}(r)|^2 + C \|r\|_{\mathcal{H}} \sum_{\lambda=0}^{1} |\pi^1_{\lambda}(r)|^2
\]

\[
\leq \varphi_d(r, r) + C \sum_{\lambda=0}^{1} |\pi^1_{\lambda}(r)|^2 + \epsilon_1 \|r\|_{\mathcal{H}}.
\]

Using (B.3), the conclusion follows. This concludes the proof of Lemma B.2. □
B.2. The $\xi$ variable and truncation to go from the local to the global. It happens that the proof is clearer in the $\xi$ variable where

$$y = \tanh \xi.$$  

More precisely, let us introduce the transformations

$$r(y) \mapsto \tilde{T}r(\xi) = \bar{r}(\xi) = r(y)(1 - y^2)^{1/(p-1)},$$  

and for $r = (r_1, r_2)$, the notation

$$\tilde{T}(r) = \bar{r} = \left( \frac{\bar{r}_1}{\bar{r}_2} \right) = \left( \tilde{T}(r_1) \tilde{T}(r_2) \right).$$  

In the following claim, we transform $\varphi$ and $\varphi_d$ in the new set of variables. Let us first introduce the quadratic forms (where $d \in (-1, 1)$):

$$\varphi_d(q, q) = \int_{\mathbb{R}} (q'_1 q'_1 + \beta_d(\xi) q_1 q_1 + q_2 q_2) d\xi$$

and

$$\varphi(q, q) = \int_{\mathbb{R}} (q'_1 q'_1 + \beta(\xi, s) q_1 q_1 + q_2 q_2) d\xi$$

where (using (1.10))

$$\beta_d(\xi) = \frac{4}{(p-1)^2} - p(\bar{\kappa}(d, y))^{p-1} = \frac{4}{(p-1)^2} - p\bar{\kappa}_0(\xi - \zeta)^{p-1} \text{ with } d = -\tanh \zeta,$$

$$\bar{\kappa}_0(\xi) = \kappa_0 \cosh^{-2} \frac{\xi}{p-1},$$

and

$$\beta(\xi, s) = \frac{4}{(p-1)^2} - p|\bar{K}(\xi, s)|^{p-1} = \frac{4}{(p-1)^2} - p \left| \sum_{i=1}^{k} e_i \bar{\kappa}_0(\xi - \zeta_i(s)) \right|^{p-1}.$$  

In the following claim, we give the effect of the new transformation:

**Claim B.3.** (Translation in the $\xi$ variable)

(i) There exists $C_0 > 0$ such that for all $r \in \mathcal{H}$, we have

$$\frac{1}{C_0} \|r\|_{\mathcal{H}} \leq \|\bar{r}\|_{H^1 \times L^2(\mathbb{R})} \leq C_0 \|r\|_{\mathcal{H}}.$$  

(ii) If $r_1 \in \mathcal{H}_0$, then $(1 - y^2)\tilde{T} \left( Lr_1 - \frac{2(p+1)}{(p-1)^2} r_1 \right) = (\partial^2_{\xi} \bar{r}_1 - \frac{4}{(p-1)^2} \bar{r}_1).$

(iii) For all $r, \bar{r}$ in $\mathcal{H}$ and $d \in (-1, 1)$, we have

$$\varphi(r, \bar{r}) = \varphi(\bar{r}, \bar{r}) \text{ and } \varphi_d(r, \bar{r}) = \varphi_d(\bar{r}, \bar{r})$$

where $\varphi$ and $\varphi_d$ are introduced in (B.6) and (B.5).
Proof. (i) Consider \( r = (r_1, r_2) \in \mathcal{H} \). Using (B.4), we first write

\[
\int_{-1}^{1} r_1(\xi)^2 d\xi = \int_{-1}^{1} r_1(y)^2 \frac{\rho(y)}{1 - y^2} dy \quad \text{and} \quad \int_{-1}^{1} r_2(\xi)^2 d\xi = \int_{-1}^{1} r_2(y)^2 \rho(y) dy.
\]

Using Lemma B.1, we obtain

\[
\|r_1\|_{L^2_{\rho}}^2 \leq \int_{-1}^{1} \tilde{r}_1(\xi)^2 d\xi \leq \|r\|^2_{\mathcal{H}}.
\]

Now, using again (B.4), we write

\[
\partial_\xi \tilde{r}_1(\xi) = \partial_y r_1(y)(1 - y^2)^{\frac{p+1}{2}} - \frac{2y}{p-1}(1 - y^2)^{\frac{p}{2}} r_1(y),
\]

therefore,

\[
|\partial_\xi \tilde{r}_1|^2 \leq 2|\partial_y r_1|^2 \rho(1 - y^2)^2 + C|r_1|^2 \rho,
\]

\[
|\partial_y r_1|^2 \rho(1 - y^2)^2 \leq 2|\partial_\xi \tilde{r}_1|^2 + C|\tilde{r}_1|^2.
\]

Integrating this and using Lemma B.1, we write

\[
\int_{-1}^{1} |\partial_\xi \tilde{r}_1|^2 d\xi \leq 2 \int_{-1}^{1} |\partial_y r_1|^2 \rho(1 - y^2) dy + C \int_{-1}^{1} r_1^2 \frac{\rho}{1 - y^2} dy \leq C\|r\|^2_{\mathcal{H}},
\]

\[
\int_{-1}^{1} |\partial_y r_1|^2 \rho(1 - y^2) dy \leq 2 \int_{-1}^{1} |\partial_\xi \tilde{r}_1|^2 d\xi + C \int_{-1}^{1} |\tilde{r}_1|^2 d\xi.
\]

Gathering (B.9), (B.10) and (B.11), we conclude the proof of (i).

(ii) See page 60 in [11].

(iii) We only prove the estimate for \( \varphi \) since it is even easier for \( \varphi_d \). Using the definitions (3.68), (1.6) and (3.60) of \( \varphi, \mathcal{L} \) and \( \psi \), integration by parts and the change of variables (B.4), we write

\[
\varphi(r, \mathbf{r}) = \int_{-1}^{1} \left[ -\mathcal{L} r_1 \cdot \mathbf{r}_1 - \psi r_1 \mathbf{r}_1 + r_2 \mathbf{r}_2 \right] \rho dy
\]

\[
= \int_{-1}^{1} \left( -\mathcal{L} r_1 + \frac{2(p+1)}{(p-1)^2} r_1 \right) \mathbf{r}_1 \rho dy - p \int_{-1}^{1} r_1 \mathbf{r}_1 |K|^{p-1} \rho dy + \int_{-1}^{1} r_2 \mathbf{r}_2 \rho dy
\]

\[
= \int_{-1}^{1} (1 - y^2) \tilde{T} \left( -\mathcal{L} r_1 + \frac{2(p+1)}{(p-1)^2} r_1 \right) \tilde{r}_1 d\xi - p \int_{-1}^{1} \tilde{r}_1 \tilde{r}_1 |\tilde{K}|^{p-1} d\xi + \int_{-1}^{1} \tilde{r}_2 \tilde{r}_2 d\xi
\]

Using (ii) and integration by parts, we see that

\[
\varphi(r, \mathbf{r}) = - \int_{-1}^{1} \left( \partial_\xi \tilde{r}_1 - \frac{4}{(p-1)^2} \tilde{r}_1 \right) \tilde{r}_1 d\xi - p \int_{-1}^{1} \tilde{r}_1 \tilde{r}_1 |\tilde{K}|^{p-1} d\xi + \int_{-1}^{1} \tilde{r}_2 \tilde{r}_2 d\xi
\]

\[
= \varphi(\tilde{r}, \tilde{\mathbf{r}})
\]

where \( \varphi \) is introduced in (B.6). This concludes the proof of Claim B.3. \( \square \)
As we said earlier, we take advantage of the decoupling in the solitons' sum. In the following claim, we first translate to the \( \xi \) variable the 1-soliton estimate given for \( \varphi_d \) in Lemma B.2, then, in (ii), taking \( d = d_i(s) \) and applying (i) to a truncation of the function, we gather the local information to derive a global estimate for \( \bar{\varphi} \).

**Claim B.4.** (Truncation in the \( \xi \) variable)

(i) There exist \( \epsilon_0 > 0 \) and \( A_0 > 0 \) such that for all \( A > A_0, \ d \in (-1,1) \) and \( q \in H^1 \times L^2(\mathbb{R}) \), we have

\[
\bar{\varphi}_d(q \sqrt{X_{A,d}}, q \sqrt{X_{A,d}}) \geq \epsilon_0 \| q \sqrt{X_{A,d}} \|_{H^1 \times L^2}^2 - \frac{\epsilon_0}{\delta k} \| q \|_{H^1 \times L^2}^2 - \frac{1}{\epsilon_0} \sum_{\lambda=0}^{1} \| \pi_{\lambda}^d(\bar{T}^{-1}(q)) \|_{L^2}^2
\]

where \( X_{A,d}(\xi) = \chi_{1,0}(\frac{\xi - \zeta}{A}) \), \( \tanh \zeta = -d \) and \( \chi_{1,0} \in C^\infty(\mathbb{R}, [0,1]) \) is even, decreasing for \( \xi > 0 \) with \( \chi_{1,0}(\xi) = 1 \) if \( |\xi| < 1 \) and \( \chi_{1,0}(\xi) = 0 \) if \( |\xi| > 2 \).

(ii) There exists \( \epsilon_2 > 0 \) such that for \( s \) large enough and for all \( q \in H^1 \times L^2 \), we have

\[
\bar{\varphi}(q) \geq \epsilon_2 \| q \|_{H^1 \times L^2}^2 - \frac{1}{\epsilon_2} \sum_{i=1}^{k} \sum_{\lambda=1}^{2} \| \pi_{\lambda}^d(\bar{T}^{-1}(q)) \|_{L^2}^2.
\]

**Proof.** (i) Using the 1-soliton case recalled in Lemma B.2 and Claim B.3, we get for all \( q \in H^1 \times L^2(\mathbb{R}) \),

\[
(B.12) \quad \bar{\varphi}_d(q, q) \geq \epsilon_0 \| q \|_{H^1 \times L^2(\mathbb{R})}^2 - \frac{\epsilon_0}{\epsilon_0} \sum_{\lambda=0}^{1} \| \pi_{\lambda}^d(\bar{T}^{-1}(q)) \|_{L^2}^2.
\]

Now, we claim that (i) follows from the fact that for all \( d \in (-1,1) \) and \( \lambda = 0 \) or 1, we have

\[
(B.13) \quad |\pi_{\lambda}^d(\bar{T}^{-1}(u))| \leq C \int \bar{\kappa}_0(\xi - \zeta) \left( |u_1(\xi)| + |u_2(\xi)| \right) d\xi \ \text{where} \ \ d = -\tanh \zeta.
\]

Indeed, consider \( q \in H^1 \times L^2, \ d \in (-1,1), \ A > 0 \) and \( \lambda = 0 \) or 1. Taking

\[
u = \bar{T}^{-1}(q(1 - \sqrt{X_{A,d}}))\]

using the Cauchy-Schwartz inequality and performing the change of variables \( z = \xi - \zeta \), we see that

\[
|\pi_{\lambda}^d(\bar{T}^{-1}(q(1 - \sqrt{X_{A,d}})))| \leq C \int \bar{\kappa}_0(\xi - \zeta) (1 - \sqrt{X_{A,d}}) \left( |q_1(\xi)| + |q_2(\xi)| \right) d\xi
\]

\[
\leq C \left( \int \bar{\kappa}_0(z)^2 (1 - \sqrt{X_{A,d}})^2 dz \right)^{1/2} \| q \|_{H^1 \times L^2}.
\]
Using Lebesgue’s theorem, we find $A_0 > 0$ such that if $A \geq A_0$, then
\[
|\pi^d_A(T^{-1}(q(1 - \sqrt{\chi_A,d}))| \leq \frac{\epsilon_0}{4\sqrt{k}}\|q\|_{H^1 \times L^2}
\]
(uniformly in $d \in (-1,1)$ of course). Since $\pi^d_A$ is linear, this gives
\[
|\pi^d_A(T^{-1}(q\sqrt{\chi_A,d}))|^2 \leq 2|\pi^d_A(T^{-1}(q))|^2 + \frac{\epsilon_0^2}{8k}\|q\|_{H^1 \times L^2}^2.
\]
Using (B.12) with $q\sqrt{\chi_A,d}$, (i) follows. It remains to prove (B.13) to finish the proof of (i) of Claim B.4.

**Proof of (B.13).** Consider $d \in (-1,1)$, $\lambda = 0$ or $1$ and $u \in H^1 \times L^2$. If we introduce $r = T^{-1}(u)$ which is in $\mathcal{H}$ by (i) of Claim B.3, then we have from (3.54) and integration by parts
\[
\pi^d_A(r) = \int_{-1}^{1} \left[ (-\mathcal{L}W_{\lambda,1}(d) + W_{\lambda,1}(d))r_1 + W_{\lambda,2}(d)r_2 \right] \rho(y) dy.
\]
Since we have from (3.56), (3.57) and (1.10)
\[
W_{\lambda,2}(d,y) \leq C\kappa(d,y) \text{ and } | -\mathcal{L}W_{\lambda,1}(d,y) + W_{\lambda,1}(d,y)| \leq C\frac{\kappa(d,y)}{1 - y^2},
\]
we get from (B.14) and the transformation (B.4)
\[
|\pi^d_A(r)| \leq C \int_{-1}^{1} \kappa(d,y)|r_1(y)| \frac{\rho(y)}{1 - y^2} dy + C \int_{-1}^{1} \kappa(d,y)|r_2(y)|\rho(y) dy
\leq C \int \tilde{\kappa}(d,\xi)|u_1(\xi)|d\xi + \int \tilde{\kappa}(d,\xi)|u_2(\xi)|d\xi.
\]
Since we have from (B.4) and (B.7),
\[
\tilde{\kappa}(d,\xi) \leq \tilde{\kappa}(d,\xi) = \tilde{\kappa}_0 \cosh^{-\frac{2}{p-1}}(\xi - \zeta) = \kappa_0(\xi - \zeta) \text{ with } d = -\tanh \zeta,
\]
(B.13) follows. This concludes the proof of (i) in Claim B.4.

(ii) Introducing the notation
\[
\chi_i = \chi_{A,d_i}(\xi) = \chi_{1,0} \left( \frac{\xi - \zeta_i}{A} \right)
\]
and using (B.6), we write

\begin{equation}
\bar{\varphi}(q, q) = \int (\partial_\xi q_1)^2 + \frac{4}{(p-1)^2} \int q_1^2 + \int q_2^2 - p \int |\bar{K}|^{p-1} q_1^2 \\
= \sum_{j=1}^{k} \left[ \int (\partial_\xi q_1)^2 \chi_j + \frac{4}{(p-1)^2} \int q_1^2 \chi_j + \int q_2^2 \chi_j - p \int |\bar{K}|^{p-1} q_1^2 \chi_j \right] \\
+ \int (\partial_\xi q_1)^2 \left( 1 - \sum_{j=1}^{k} \chi_j \right) + \frac{4}{(p-1)^2} \int q_1^2 \left( 1 - \sum_{j=1}^{k} \chi_j \right) \\
+ \int q_2^2 \left( 1 - \sum_{j=1}^{k} \chi_j \right) - p \int |\bar{K}|^{p-1} q_1^2 \left( 1 - \sum_{j=1}^{k} \chi_j \right) \\
= \sum_{j=1}^{k} \bar{\varphi}(q_1, q_1) + \bar{\varphi} \left( q_1 \left( 1 - \sum_{j=1}^{k} \chi_j \right), q_1 \left( 1 - \sum_{j=1}^{k} \chi_j \right) \right) + I_1(s)
\end{equation}

where

\begin{equation}
I_1(s) = -\sum_{j=1}^{k} \left\{ \int q_1^2 \left( \partial_\xi \sqrt{\chi_j} \right)^2 - 2 \int q_1 \partial_\xi q_1 \sqrt{\chi_j} \partial_\xi \sqrt{\chi_j} \right\} \\
- \int q_1^2 \left( \partial_\xi \sqrt{1 - \sum_{j=1}^{k} \chi_j} \right)^2 - 2 \int q_1 \partial_\xi q_1 \sqrt{1 - \sum_{j=1}^{k} \chi_j} \partial_\xi \sqrt{1 - \sum_{j=1}^{k} \chi_j}.
\end{equation}

Using the definitions (B.6) and (B.5) of \( \bar{\varphi} \) and \( \bar{\varphi}_d \), we write

\begin{equation}
\bar{\varphi}(q_1, q_1) = \bar{\varphi}_d(s)(q_1, q_1) - I_2(s),
\end{equation}

\begin{equation}
\bar{\varphi} \left( q_1 \left( 1 - \sum_{j=1}^{k} \chi_j \right), q_1 \left( 1 - \sum_{j=1}^{k} \chi_j \right) \right) \geq c_0(p) \left\| q_1 \left( 1 - \sum_{j=1}^{k} \chi_j \right) \right\|_{H^1 \times L^2}^2 - I_3(s)
\end{equation}

where \( c_0(p) = \min \left( 1, \frac{4}{(p-1)^2} \right) \),

\[ I_2(s) = p \int \left( |\bar{K}(\xi, s)|^{p-1} - \bar{K}_0(\xi - \zeta_i(s))^{p-1} \right) q_1^2 \chi_j, \]

\[ I_3(s) = p \int |\bar{K}|^{p-1} q_1^2 \left( 1 - \sum_{j=1}^{k} \chi_j \right). \]
Since we have from Claim B.4 and (B.8)
\[
|\partial_x x_j| \leq C/A, \quad \left\| \left( |\tilde{K}(x, s)|^{p-1} - \tilde{K}_0(x - \xi(s))^p \right)^{\frac{1}{2}} \right\|_{L^\infty} \leq C(A)J(s),
\]
(B.19)
\[
\left\| |\tilde{K}|^{p-1} \left( 1 - \sum_{j=1}^k x_j \right) \right\|_{L^\infty} \leq Ce^{-2A}
\]
where \(J(s) \to 0\) is defined in (3.4), it follows that for \(A\) and \(s\) large enough,
\[
(B.20) \quad |I_1(s)| + |I_2(s)| + |I_3(s)| \leq \frac{C}{A} \|q\|_{H^1}^2.
\]
Therefore, using (B.15), (B.16), (B.17), (B.18), (B.20) and Claim B.4, we write for \(A\) and \(s\) large enough,
\[
\bar{\varphi}(q, q) \geq \epsilon_0 \sum_{j=1}^k \|q\sqrt{x_j}\|_{H^1 \times L^2}^2 + c_0(p) \left\| q \sqrt{1 - \sum_{j=1}^k x_j} \right\|_{H^1 \times L^2}^2
\]
(B.21)
\[-\frac{\epsilon_0}{4} \|q\|_{H^1 \times L^2}^2 - \frac{1}{\epsilon_0} \sum_{j=1}^k \sum_{\lambda=0}^1 |\pi_{\lambda} d_j(s)(T^{-1}(q))| \|q\|_{H^1 \times L^2}.
\]
Since (B.15) holds with \(\bar{\varphi}\) replaced by the canonical inner product of \(H^1 \times L^2\), we use (B.20) to write
\[
\|q\|_{H^1 \times L^2}^2 \leq \sum_{j=1}^k \|q\sqrt{x_j}\|_{H^1 \times L^2}^2 + \left\| q \sqrt{1 - \sum_{j=1}^k x_j} \right\|_{H^1 \times L^2}^2 + \frac{C}{A} \|q\|_{H^1 \times L^2}^2
\]
hence for \(A\) and \(s\) large enough,
\[
\|q\|_{H^1 \times L^2}^2 \leq 2 \sum_{j=1}^k \|q\sqrt{x_j}\|_{H^1 \times L^2}^2 + 2 \left\| q \sqrt{1 - \sum_{j=1}^k x_j} \right\|_{H^1 \times L^2}^2
\]
and (ii) follows from (B.21). This concludes the proof of Claim B.4. \(\Box\)

**B.3. Proof of Lemma 3.10.** Now we are ready to start the proof of Lemma 3.10.

*Proof of Lemma 3.10.* (i) Since \(\psi(y, s) = p|K(y, s)|^{p-1} - \frac{2(p+1)}{(p-1)^2} C_k(y, s)\) with \(K(y, s) = \sum_{j=1}^k e_j \kappa(d_j(s), y)\) by (3.60), we split \(\varphi(r, r')\) into 2 parts as follows:

- We first use the definition (1.8) of the norm in \(\mathcal{H}\) to write
  \[
  \left| \int_{-1}^1 \left( \partial_{y} r_1 \partial_{y} r_1' (1 - y^2) + \frac{2(p+1)}{(p-1)^2} r_1 r_1' + r_2 r_2' \right) \rho dy \right| \leq C \|r\|_{\mathcal{H}} \|r'\|_{\mathcal{H}}.
  \]
Then, using Claim B.1, we write
\[
\left| \int_{-1}^{1} |K(s)|^{p-1} r_1 r_1' \rho \, dy \right| \leq C \int_{-1}^{1} \frac{|r_1| |r_1'|}{1 - y^2} \rho \, dy \leq C \frac{\| r_1 \|_{L^2} \| r_1' \|_{L^2}}{1 - y^2}
\]
\[
\leq C \| r \|_{\mathcal{H}} \| r \|_{\mathcal{H}}.
\]
Using these two bounds gives the conclusion of (i).

(ii) **Proof of (3.70).** It immediately follows from (3.67), (3.73) and (3.74).

**Proof of (3.71).** The right inequality follows from (i). For the left inequality, we use Claim B.3 to translate (ii) of Claim B.4 back to the $y$ variable: for some $\epsilon_2 > 0$, for $s$ large enough and for all $r \in \mathcal{H}$,
\[
\varphi(r, r) \geq \epsilon_2 \| r \|_{\mathcal{H}}^2 - \frac{1}{\epsilon_2} \sum_{i=1}^{k} \sum_{\lambda=0}^{1} |\pi_{\lambda}^{d_i}(s) (r) |^2.
\]
Using (B.22) with $r(y) = q_-(y, s)$, we write
\[
\varphi(q_-, q_-) \geq \epsilon_2 \| q_- \|_{\mathcal{H}}^2 - \frac{1}{\epsilon_2} \sum_{i=1}^{k} \sum_{\lambda=0}^{1} |\pi_{\lambda}^{d_i}(q_-) |^2.
\]
Since $\pi_{\lambda}^{d_i}(F_{d_i}^{d_1}) = \delta_{\lambda, 1}$ by (3.66), we use (3.70) to write
\[
\pi_{\lambda}^{d_i}(q_-) = \pi_{\lambda}^{d_i}(q) - \sum_{j=1}^{k} \pi_{\lambda}^{d_j}(q) \pi_{\lambda}^{d_i}(F_{d_i}^{d_j}) = \sum_{j \neq i} \pi_{\lambda}^{d_j}(q) \pi_{\lambda}^{d_i}(F_{d_i}^{d_j}).
\]
Using (3.73), (3.54) and (3.58), we see that
\[
|\alpha_1^j| = |\pi_{1}^{d_j}(q)| = |\phi(W_{\lambda}(d_j), q)| \leq \| W_{\lambda}(d_j) \|_{\mathcal{H}} \| q \|_{\mathcal{H}} \leq C \| q \|_{\mathcal{H}}.
\]
Using (3.54), integration by parts and the definition (1.6) of $L$, we write
\[
\pi_{\lambda}^{d_i}(F_{\mu}(d_{j})) = \int_{-1}^{1} \left( W_{\lambda, 1}(d_i) F_{\mu, 1}(d_j) + \partial_y W_{\lambda, 1}(d_i) \partial_y F_{\mu, 1}(d_j) (1 - y^2) \right) \rho \, dy
\]
\[
+ \int_{-1}^{1} W_{\lambda, 2}(d_i) F_{\mu, 2}(d_j) \rho \, dy
\]
\[
= \int_{-1}^{1} \left( - L W_{\lambda, 1}(d_i) + W_{\lambda, 1}(d_i) \right) F_{\mu, 1}(d_j) \rho \, dy
\]
\[
+ \int_{-1}^{1} W_{\lambda, 2}(d_i) F_{\mu, 2}(d_j) \rho \, dy.
\]
Since we have from the definitions (1.10), (3.56), (3.57) and (3.63) of $\kappa(d,y)$, $W_\lambda(d,y)$ and $F_\mu(d,y)$, for all $(d,y) \in (-1,1)^2$,

$$\left| W_{\lambda,2}(d,y) \right| + \left| LW_{\lambda,1}(d,y) - W_{\lambda,1}(d,y) \right| \leq C\frac{\kappa(d,y)}{1 - y^2},$$
$$\left| F_{\mu,1}(d,y) \right| \leq C\kappa(d,y),$$

we use (i) of Lemma E.1 to write for $s$ large enough,

$$\left| \pi_d^{d_i}\left( F_\mu(d_i) \right) \right| \leq C \int_{-1}^{1} \kappa(d_i) \kappa(d_j) \frac{\rho}{1 - y^2} dy$$
$$\leq C|\zeta_i - \zeta_j| e^{-\frac{r}{2}|\zeta_i - \zeta_j|} \leq C\tilde{J}(s)$$

by definition (3.73) of $\tilde{J}$. Using (B.24) and (B.25), we see that for $s$ large enough,

$$\left| \pi_d^{d_i}\left( q_- \right) \right| \leq C\tilde{J}\|q\|_\mathcal{H}.$$ 

Using (B.23), we see that the left inequality in (3.71) follows.

**Proof of** (3.72). The right inequality follows from (B.25), (3.69) and (3.71). For the left inequality in (3.72), we write from the bilinearity of $\varphi$, (3.70), (3.69) and (3.64)

$$\varphi(q_-,q_-) \geq \varphi(q,q) - C \sum_{i=1}^{k} |\alpha_i|^2 - C\|q\|_\mathcal{H} \sum_{i=1}^{k} |\alpha_i|$$
$$\geq \varphi(q,q) - C\epsilon_1 \sum_{i=1}^{k} |\alpha_i|^2 - \frac{\epsilon_2}{2}\|q\|^2_\mathcal{H}$$

where $\epsilon_1 > 0$ is introduced in (B.22). Using (B.22) with $r = q$, we get the left inequality in (3.72). This concludes the proof of Lemma 3.10. \qed

**Appendix C. Projection of equation (3.60) on the different modes.** We prove Lemma 3.11 in subsection C.2 below. We need to project equation (3.60) on the different modes defined in (3.70). If we compare equation (3.60) to its 1-soliton version treated in [11], we see that the main difference is the presence of the soliton interaction in $(0,R)$ and $(0,\psi q_1)$. So, going from the 1-soliton case to the multi-soliton-case is almost straightforward, except for the two terms $(0,R)$ and $(0,\psi q_1)$, which need a careful handling. The projection of these two terms is responsible for the presence of $J(s)$ and $\sum_{i=1}^{k} h(\zeta_{i+1} - \zeta_i)^2$ terms in Lemma 3.11. Therefore, to make our proof more reader friendly, we first recall the 1-soliton case from [11], and then give the proof of Lemma 3.11.
C.1. The 1-soliton case. When \( k = 1 \), we simply write \( d(s) \) instead of \( d_i(s) \) and decompose \( q(y,s) \) as

\[
q(y,s) = \alpha_1(s)F_1(d(s),y) + q_-(y,s),
\]

where \( d(s) = -\tanh\zeta(s) \) is chosen by modulation so that \( \alpha_0(s) = \pi^{d_0(s)}_0(q) = 0. \) Since \( \pi^{d(s)}_{\lambda}(q_-) = 0 \) for \( \lambda = 0 \) or \( 1 \), we see from Lemma B.2 that \( \varphi_{d(s)}(q_-,q_-) \geq 0 \), hence \( \alpha_-(s) = \sqrt{\varphi_{d(s)}(q_-,q_-)} \) is well defined. In [11], we proved the following:

**Lemma C.1.** Under the hypotheses of Lemma 3.11, assuming \( k = 1 \), we have for \( s \) large enough,

(i) (Control of the positive mode and the modulation parameter)

\[
|\alpha'_1 - \alpha_1| + |\zeta'| \leq C\|q(s)\|^2_H.
\]

(ii) (Control of the negative part)

\[
\left( R_- + \frac{1}{2} \alpha_-^2 \right)' \leq -\frac{4}{p-1} \int_{-1}^{1} q_-^2 2 \frac{\rho}{1-y^2} dy + C\|q(s)\|^3
\]

for some \( R_- (s) \) satisfying \( |R_- (s)| \leq C_0\|q(s)\|^{1+\tilde{p}}_H \) where \( \tilde{p} = \min(p,2) > 1 \).

(iii) (Additional relation)

\[
\frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho \leq -\frac{4}{5} \alpha_-^2 + C \int_{-1}^{1} q_-^2 2 \frac{\rho}{1-y^2} + C\alpha_1^2.
\]

**Proof.** See [11, Proposition 5.2, page 103]. \( \square \)

C.2. The multi-soliton case. This subsection is devoted to the proof of Lemma 3.11. We proceed in 3 parts to prove (i), (ii), and finally (iii).

**Proof of (i): Projection of equation (3.60) on \( F_1(d_i(s),\cdot) \) and \( F_0(d_i(s),\cdot) \).**

We prove (i) of Lemma 3.11 here. Fixing some \( i = 1, \ldots, k \) and projecting equation (3.60) with the projector \( \pi^d_\lambda (3.54) \) (where \( \lambda = 0 \) or \( 1 \)), we write (putting on top the main terms)

\[
\begin{align*}
\pi^d_\lambda (\partial_s q) &= \pi^d_\lambda (L_{d_i(s)}(q)) - e_i d'_i(s) \pi^d_\lambda \left( \partial_d \kappa (d_i(s),y) \right)_0 \\
&+ \pi^d_\lambda \left( 0 \right) + \pi^d_\lambda \left( f(q_1) \right) + \pi^d_\lambda \left( V_i(y,s) q_1 \right) \\
&- \sum_{j \neq i} e_j d'_j(s) \pi^d_\lambda \left( \partial_d \kappa (d_j(s),y) \right)_0.
\end{align*}
\]

(C.1)

Note that we expand the operator \( L(q) \) according to (3.61). In the following, we handle each term of (C.1) in order to finish the proof of (3.77).
Using the analysis performed in Claim 5.3 page 104 and Step 1 page 105 in [11] for the case of one soliton \((k = 1)\), we immediately get the following estimates:

\[
\left| \pi^{d_i(s)}_\lambda \left( \partial_s q - \alpha^{i'}(s) \right) \right| \leq \frac{C_0}{1 - (d_i(s))^2} \left| d_i'(s) \right| \left| q(s) \right|_\mathcal{H} \leq C_0 \left| \alpha_i'(s) \right| \left| q(s) \right|_\mathcal{H},
\]

\[(C.2)\]

\[
d_i'(s) \pi^{d_i(s)}_\lambda \left( L_{d_i(s)}(q) \right) = \lambda \alpha_i(s),
\]

\[
d_i'(s) \pi^{d_i(s)}_\lambda \left( \partial_d \kappa(d_i(s), y) \right) = - \frac{2 \kappa_0}{(p-1) (1 - (d_i(s))^2)} \delta_{\lambda,0} = \frac{2 \kappa_0}{(p-1)} \zeta_i'(s) \delta_{\lambda,0},
\]

\[
\left| f(q_1) \right| \leq C \delta_{p \geq 2} \left| q_1 \right|^p + C \left| K \right|^{p-2} \left| q_1 \right|^2
\]

(recall that

\[(C.3)\]

\[
d_i(s) = - \tanh \zeta_i(s), \quad \text{hence} \quad \zeta_i'(s) = - \frac{d_i'(s)}{1 - (d_i(s))^2}).
\]

Since we have from the definitions (3.60), (1.10) and (3.56) of \(R, \kappa(d, y)\) and \(W_{\lambda,2}(d, y)\)

\[
\left| R(y, s) \right| \leq C \sum_{j \neq i} \kappa(d_j, y)^p + \kappa(d_i, y)^{p-1} \kappa(d_j, y),
\]

\[
(C.4)
\]

\[
\left| W_{\lambda,2}(d_i, y) \right| \leq C \kappa(d_i, y),
\]

we use (3.54), (i) of Lemma E.1 and the definition (3.4) of \(J(s)\) to write for \(s\) large enough,

\[
\left| \pi^{d_i}_\lambda \left( 0 \right) \right| = \left| \int_{-1}^{1} W_{\lambda,2}(d_i) R \rho dy \right| \leq C \sum_{j \neq i} \int_{-1}^{1} \kappa(d_i) \kappa(d_j)^p \rho dy
\]

\[
+ \int_{-1}^{1} \kappa(d_i)^p \kappa(d_j) \rho dy \leq \sum_{j \neq i} e^{-\frac{2}{p-1} \left| \zeta_i - \zeta_j \right|} \leq C J.
\]

Using (3.54), (C.4), (C.2) and the Hölder inequality, we write

\[
\left| \pi^{d_i}_\lambda \left( f(q_1) \right) \right| \leq C \int_{-1}^{1} \kappa(d_i) \left| f(q_1) \right| \rho dy
\]

\[
\leq C \delta_{p \geq 2} \int_{-1}^{1} \kappa(d_i) \left| q_1 \right|^p \rho dy + C \int_{-1}^{1} \kappa(d_i) \left| K \right|^{p-2} \left| q_1 \right|^2 \rho dy
\]

\[
\leq C \delta_{p \geq 2} \left\| \kappa(d_i) \right\|_{L^{p+1}} \left\| q_1 \right\|^p_{L^{p+1}} + C J_i \left( 1 - y^2 \right)^{\frac{p-1}{2}} \left\| L^\infty \right\|^2
\]

where

\[
J_i = \int_{-1}^{1} \kappa(d_i) \left| K \right|^{p-2} dy.
\]

(C.6)
Using (v) of Lemma E.1 and Claim B.1, we see that
\begin{align}
\left| \pi_{\lambda}^{d_i} \left( \begin{array}{c} 0 \\ f(q_1) \end{array} \right) \right| & \leq C \int_{-1}^{1} \kappa(d_i) |f(q_1)| \rho \, dy \\
& \leq C \delta_{p \geq 2} \|q\|_{\mathcal{H}}^p + C \|q\|_{\mathcal{H}}^2 \leq C \|q\|_{\mathcal{H}}^2
\end{align}
(C.7)
where we use (3.59) in the last step.

- We claim that for some $\delta_1 > 0$ and for $s$ large enough,
\begin{align}
\left| \pi_{\lambda}^{d_i} \left( \begin{array}{c} 0 \\ V_i q_1 \end{array} \right) \right| & \leq C \|q\|_{\mathcal{H}}^2 + C J^{1+\delta_1}
\end{align}
(C.8)
and
\begin{align}
|V_i(y, s)| & \leq C 1_{\{y_{i-1} < y < y_i\}} \sum_{l \neq i} \kappa(d_i, y)^{p-2} \kappa(d_i, y) \\
& \quad + C \sum_{l \neq i} \kappa(d_i, y)^{p-1} 1_{\{y_{i-1} < y < y_i\}}
\end{align}
(C.9)
where $y_0 = -1$, $y_j = \tanh\left(\frac{\zeta_j + \zeta_{j+1}}{2}\right)$ if $j = 1, \ldots, k - 1$ and $y_k = 1$. In particular, we have,
\[-1 = y_0 < -d_1 < y_1 < -d_2 < \cdots < y_j < -d_j < y_{j+1} < \cdots < -d_k < y_k = 1\]
and $\kappa(d_j(s), y_{j+1}(s)) = \kappa(d_{j+1}(s), y_{j+1}(s))$ for $j = 1, \ldots, k - 1$ (use (3.105) to see this).

We first prove (C.9) and then (C.8). To prove (C.9), using (3.62), we see that:
- if $y \in (y_{i-1}(s), y_i(s))$, then $|\sum_{l \neq i} e^l \kappa(d_l(s), y)| \leq 3 \kappa(d_i, s) y$; hence from (3.61), $|V_i(y, s)| \leq C \sum_{l \neq i} \kappa(d_i, s, y)^{p-2} \kappa(d_i, s, y)$;
- if $y \in (y_{i-1}(s), y_i(s))$ for some $l \neq i$, then for all $j = 1, \ldots, k$, $\kappa(d_j(s), y) \leq \kappa(d_i(s), y)$, hence, from (3.61), $|V_i(y, s)| \leq C \sum_{j=1}^{k} \kappa(d_j(s), y)^{p-1} \leq C \kappa(d_i(s), y)^{p-1}$.

Thus, (C.9) follows. Now, we prove (C.8). Using (3.54), (C.4), Claim B.1 and (C.9), we write
\begin{align}
\left| \pi_{\lambda}^{d_i} \left( \begin{array}{c} 0 \\ V_i q_1 \end{array} \right) \right| & \leq C \int_{-1}^{1} \kappa(d_i) |V_i q_1| \rho \, dy \\
& \leq C \|q_1(1 - y^2)^{\frac{1}{p-1}}\|_{L^2}^2 + C \left( \int_{-1}^{1} \kappa(d_i) |V_i|(1 - y^2)^{\frac{1}{p-1}} \, dy \right)^2 \\
& \leq C \|q\|_{\mathcal{H}}^2 + C \sum_{l \neq i} \left( \int_{y_{i-1}}^{y_i} \kappa(d_i)^{p-1} \kappa(d_i)(1 - y^2)^{\frac{1}{p-1}} \, dy \right)^2 \\
& \quad + \left( \int_{y_{i-1}}^{y_i} \kappa(d_i)^{p-1} \kappa(d_i)(1 - y^2)^{\frac{1}{p-1}} \, dy \right)^2.
\end{align}
Using (ii) of Lemma E.1, (C.8) follows.
Consider $j \neq i$. Since we have from the definitions (1.10) and (3.63) of $\kappa(d,y)$ and $F_0(d,y)$,

\[(C.10) \quad \left( \frac{\partial_d \kappa(d,y)}{0} \right) = -\frac{2\kappa_0}{(p-1)(1-d^2)} F_0(d,y),\]

we use (B.27) and (C.3) to write

\[(C.11) \quad \left| d' \pi^d_i \left( \frac{\partial_d \kappa(d_j)}{0} \right) \right| \leq C \left| d' \pi^d_i \left( F_0(d_j) \right) \right| \leq C \bar{J} |\zeta'_j|\]

where $\bar{J}(s)$ is defined in (3.73). Using (C.1), (C.2), (C.5), (C.7), (C.8), (C.11) and (3.74), we write for all $i = 1, \ldots, k$ (starting with $\lambda = 0$ and then $\lambda = 1$),

\[(C.12) \quad \left| \zeta'_i \right| \leq C \left| \zeta'_i \right| ||q||_H + CJ + C ||q||^2_H + CJ \sum_{j \neq i} |\zeta'_j|,\]

\[(C.13) \quad \left| \alpha'_i - \alpha'_i \right| \leq C \left| \zeta'_i \right| ||q||_H + C \bar{J} + C ||q||^2_H + CJ \sum_{j \neq i} |\zeta'_j|,\]

Since $||q||_H + \bar{J} \to 0$ (see (i) of Lemma 3.9 and (3.73)), summing up (C.12) in $i$, we get,

\[\sum_{i=1}^{k} |\zeta'_i| \leq CJ + C ||q||^2_H.\]

Plugging this in (C.13), we get

\[\left| \alpha'_i - \alpha'_i \right| \leq CJ + C ||q||^2_H,\]

which closes the proof of (3.77). This concludes the proof of (i) of Lemma 3.11.

Proof of (ii): Differential inequality satisfied by $A_-(s)$. We proceed in 2 steps: we first project equation (3.60) with the projector $\pi_-$ defined in (3.67), and then use that equation to write a differential inequality for $A_- = \varphi(q_-,q_-)$.

Step 2.1 : Projection of equation (3.60) with $\pi_-$. In this claim, we project equation (3.60) with the projector $\pi_-$ defined in (3.67):

CLAIM C.2. (A partial differential inequality for $q_-$) For $s$ large enough, we have

\[\left\| \partial_s q_- - Lq_- - \sum_{i=1}^{k} \pi_1^{d_i}(q) \left( 0 \begin{array}{c} V_i F_{1,1}(d_i) \end{array} \right) - \left( \begin{array}{c} 0 \end{array} \right) \right\|_H \leq CJ + C ||q||^2_H\]

where $\bar{J}(s)$ is defined in (3.4).
Proof. Applying the projector \( \pi_- \) defined in (3.67) to equation (3.60), we write

\[
\pi_-(\partial_s q) = \pi_-(Lq) - \sum_{i=1}^{k} \pi_1^{d_i(s)}(q) LF_1(d_i(s), \cdot) + \pi_-(Lq) - \sum_{i=1}^{k} \pi_1^{d_i(s)}(q) LF_1(d_i(s), \cdot) - \pi_1^{d_i(s)}(Lq) F_1(d_i(s), \cdot) + \pi_-(Lq).
\]

In the following, we will estimate each term appearing in this identity.

- Proceeding as for estimate (213) in [11] in the case of one soliton, one can straightforwardly control the left-hand term as follows:

\[
\| \pi_-(\partial_s q) - \partial_s q \|_{\mathcal{H}} \leq C J\|q\|_{\mathcal{H}} + C\|q\|^3_{\mathcal{H}}.
\]

- We claim that for \( s \) large enough,

\[
\| \pi_-(Lq) - Lq - \sum_{i=1}^{k} \pi_1^{d_i(s)}(q) LF_1(d_i(s), \cdot) \|_{\mathcal{H}} \leq C\|q\|^2_{\mathcal{H}} + C J^{1+\delta_1},
\]

where \( \delta_1 > 0 \) is introduced in (C.8). Indeed, applying the operator \( L \) to (3.70) on the one hand, and using (3.67) with \( r = Lq \) on the other hand, we write

\[
Lq = \sum_{i=1}^{k} \pi_1^{d_i(s)}(q) LF_1(d_i(s), \cdot) + Lq - \sum_{i=1}^{k} \pi_1^{d_i(s)}(q) LF_1(d_i(s), \cdot) + \sum_{i=1}^{k} \pi_0^{d_i(s)}(Lq) F_0(d_i(s), \cdot) + \pi_-(Lq).
\]

Therefore,

\[
\pi_-(Lq) - Lq = \sum_{i=1}^{k} \pi_1^{d_i(s)}(q) LF_1(d_i(s), \cdot) - \pi_1^{d_i(s)}(Lq) F_1(d_i(s), \cdot)
\]

\[
- \sum_{i=1}^{k} \pi_0^{d_i(s)}(Lq) F_0(d_i(s), \cdot).
\]

Since we have from (3.54) and (3.65), \( \pi_\lambda^{d}(Lq) = \phi(W_\lambda(d, \cdot), Lq) = \phi(L_\lambda W_\lambda(d, \cdot), r) = \lambda \pi_\lambda^{d}(r) \), using this with (3.61) and (3.64) gives for \( \lambda = 0 \) or 1,

\[
LF_\lambda(d_i(s), \cdot) = L_{d_i(s)} F_\lambda(d_i(s), \cdot) + \left( V_i F_{\lambda,1}(d_i(s), \cdot) \right)
\]

\[
= LF_\lambda(d_i(s), \cdot) + \left( V_i F_{\lambda,1}(d_i(s), \cdot) \right)
\]

\[
\pi_\lambda^{d_i(s)}(Lq) = \pi_\lambda^{d_i(s)}(L_{d_i(s)} q) + \pi_\lambda^{d_i(s)}(0 V_i q_1)
\]

Using (C.17), (C.18) and (C.19) together with (C.8) and (3.64), we get (C.16).
• Using the definition (3.67) of the operator $\pi_-$, we see that

$$
\pi_-(\partial_d \kappa(d_i, \cdot) \begin{pmatrix} 0 \\ 0 \end{pmatrix}) = \left( \partial_d \kappa(d_i, \cdot) \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) - \sum_{j=1}^{k} \pi_d (\partial_d \kappa(d_j, \cdot) \begin{pmatrix} 0 \\ 0 \end{pmatrix}) F_1(d_j, \cdot)
$$

(C.20)

$$
- \sum_{j=1}^{k} \pi_d (\partial_d \kappa(d_j, \cdot) \begin{pmatrix} 0 \\ 0 \end{pmatrix}) F_0(d_j, \cdot).
$$

Using (C.10), it follows from the orthogonality relation (3.66) that for $\lambda = 0$ or 1,

$$
\pi_d (\partial_d \kappa(d_i, \cdot) \begin{pmatrix} 0 \\ 0 \end{pmatrix}) = - \frac{2\kappa_0}{(p-1)(1-d^2)} \sum_{j=1}^{k} \pi_d (\partial_d \kappa(d_j, \cdot) \begin{pmatrix} 0 \\ 0 \end{pmatrix}) F_1(d_j, \cdot).
$$

Therefore, it follows from (C.20) that

$$
\pi_-(\partial_d \kappa(d_i, \cdot) \begin{pmatrix} 0 \\ 0 \end{pmatrix}) = - \sum_{j \neq i} \pi_d (\partial_d \kappa(d_j, \cdot) \begin{pmatrix} 0 \\ 0 \end{pmatrix}) F_1(d_j, \cdot)
$$

$$
- \sum_{j \neq i} \pi_d (\partial_d \kappa(d_j, \cdot) \begin{pmatrix} 0 \\ 0 \end{pmatrix}) F_0(d_j, \cdot).
$$

Using (C.11), (3.64) and (3.77), we see that

$$
\left\| d_i'(s) \pi_-(\partial_d \kappa(d_i(s), \cdot) \begin{pmatrix} 0 \\ 0 \end{pmatrix}) \right\| \leq C J(s) |\zeta'(s)| \leq C J(s) \left( \| q(s) \|^2 + J(s) \right).
$$

(C.21)

\[ \| \pi_-(f(q_1)) - (0, f(q_1)) \| \leq C \sum_{\lambda=1,2; \ i=1}^{k} \left| \pi_{d_i}(s) \right| \leq C \| q(s) \|^2, \]

(C.22)

\[ \left\| \pi_-(R) - (0, R) \right\| \leq C \sum_{\lambda=1,2; \ i=1}^{k} \left| \pi_{d_i}(s) \right| \leq C J(s). \]

(C.23)

Using (C.14), (C.15), (C.16), (C.21), (C.22) and (C.23) closes the proof of Claim C.2.

Step 2.2: A differential inequality on $A_-(s)$. By definition (3.73) of $\alpha_-(s)$, it holds that

$$
\frac{1}{2} A'_-(s) = \varphi(\partial_s q_-, q_-) - \frac{p(p-1)}{2} \sum_{i=1}^{k} e_i d_i' I_i
$$

(C.24)
with
\[ I_i = \int_{-1}^{1} \partial_d \kappa(d_i) |K|^{p-2} (q_{-1})^2 \rho \, dy \text{ and } K = \sum_{j=1}^{k} e_j \kappa(d_j). \]

Since we have from (C.10) and (B.26), \(|\partial_d \kappa(d_i)| \leq C \frac{\kappa(d_i)}{1-d_i^2}\), using (C.3), (v) of Lemma E.1, (i) of Claim B.1 and (3.75), we see that

\[ |d'_i | I_i | \leq C \frac{d'_i}{1-d_i^2} \int_{-1}^{1} \kappa(d_i) |K|^{p-2} dy \| q_{-1} (1-y^2)^{\frac{1}{p-1}} \|_{L^\infty}^2 \]
\[ \leq C |\zeta'_i(s)||q_{-1}||_{H^1}^2 \leq C |\zeta'_i(s)||q||_{H^1}^2. \]

Using (C.24), (C.25) and (3.77), we get

\[ \left| \frac{1}{2} A_-(s) - \varphi(\partial_s q_-, q_-) \right| \leq C \|q||_{H}^2 (\|q||_{H^1}^2 + J). \]

Using (3.75), (3.69) and Claim C.2, we estimate \( \varphi(\partial_s q_-, q_-) \) in the following:
\[ |\varphi(\partial_s q_-, q_-) - \varphi(L q_-, q_-) - \int_{-1}^{1} q_{-2} f(q_1) \rho \, dy - \int_{-1}^{1} q_{-2} G \rho(y) \, dy| \]
\[ \leq C \|q||_{H} (J + \|q||_{H^1}^3) \leq C J \sqrt{|A_-|} + C J \|q||_{H} + C \|q||_{H}^3 \]
\[ \leq C J \sqrt{|A_-|} + C \|q||_{H}^3 + C \sum_{m=1}^{k-1} (h(\zeta_{m+1} - \zeta_m))^2 \]

where \( h \) is defined in (3.50) and

\[ G(y, s) = \sum_{i=1}^{k} \alpha_i^i(s) V_i(y, s) F_{1,1}(d_i(s), y) + R(y, s). \]

In the following, we estimate every term of (C.27) in order to finish the proof of (3.78).

- Arguing as in page 107 of [11], we write
\[ \varphi(L q_-, q_-) = -4 \int_{-1}^{1} q_{-2} \frac{\rho}{1-y^2} \, dy. \]

- Since we have from the definitions (1.10) and (3.63) of \( \kappa(d, y) \) and \( F_{1}(d, y) \),

\[ F_{1,1}(d, y) = F_{1,2}(d, y) \leq C \kappa(d, y), \]

using (3.70) and (C.7), we write

\[ \int_{-1}^{1} q_{-2} f(q_1) \rho \, dy - \int_{-1}^{1} q_{-2} f(q_1) \rho \, dy \leq C \sum_{i=1}^{k} |\alpha_i^i| \int_{-1}^{1} \kappa(d_i) |f(q_1)| \rho \, dy \leq C \|q||_{H^1}^3. \]
If we introduce
\[ F(q_1) = \int_0^{q_1} f(\xi) d\xi = \frac{|K + q_1|^{p+1}}{p + 1} - \frac{|K|^{p+1}}{p + 1} - |K|^{p-1} q_1 - \frac{p}{2} |K|^{p-1} q_1^2, \]
then it is easy to see that
\[ |F(q_1)| \leq C |q_1|^{p+1} + C \delta_{\{p \geq 2\}} |K|^{p-2} |q_1|^3. \]

(C.32)

Introducing \( R_- = -\int_1^{-1} F(q_1) \rho dy \) and using equation (3.60), we write
\[ R'_- + \int_{-1}^{1} q_2 f(q_1) \rho dy \]
\[ = R'_- + \int_{-1}^{1} \partial_s q_1 f(q_1) \rho dy + \sum_{i=1}^{k} e_i d'_i \int_{-1}^{1} \partial_d \kappa(d_i) f(q_1) \rho dy \]
\[ = \sum_{i=1}^{k} d'_i \int_{-1}^{1} (e_i \partial_d \kappa(d_i) f(q_1) - \partial_d \kappa F(q_1)) \rho dy \]
\[ = \frac{p(p-1)}{2} \sum_{i=1}^{k} e_i d'_i \int_{-1}^{1} \partial_d \kappa(d_i) |K|^{p-3} K q_1^2 \rho dy. \]

(C.33)

Therefore, using (C.31) and (C.33), arguing as for (C.25), using (v) of Lemma E.1 and (3.77), we write
\[ \left| \int_{-1}^{1} q_2 f(q_1) \rho dy + R'_- \right| \leq C \sum_{i=1}^{k} \frac{|d'_i|}{1 - d'_i^2} J_i \| q \|^2_{\mathcal{H}} \]
\[ \leq C (\| q \|^4_{\mathcal{H}} + J \| q \|^2_{\mathcal{H}}). \]

(C.34)

Note that from (C.32), the Hölder inequality and Claim B.1, we have
\[ \left| \int_{-1}^{1} F(q_1) \rho dy \right| \]
\[ \leq C \int_{-1}^{1} |q_1|^{p+1} \rho dy + C \delta_{\{p \geq 2\}} \int_{-1}^{1} |K|^{p-2} |q_1|^3 \rho dy \]
\[ \leq C \| q \|^{p+1}_{\mathcal{H}} + C \delta_{\{p \geq 2\}} \left( \int_{-1}^{1} |q_1|^{p+1} \rho dy \right)^{\frac{1}{p+1}} \left( \int_{-1}^{1} |K|^{p+1} \rho dy \right)^{\frac{p}{p+1}} \]
\[ \leq C \| q \|^{p+1}_{\mathcal{H}} + C \delta_{\{p \geq 2\}} \| q \|^3_{\mathcal{H}} \leq C \| q \|^\bar{p+1}_{\mathcal{H}} \]
\[ \text{where } \bar{p} = \min(p, 2). \]
Using the Cauchy-Schwartz inequality, we write

\[
\left| \int_{-1}^{1} q_{-2} G \rho \, dy \right| \leq \frac{1}{p - 1} \int_{-1}^{1} q_{-2} y \, dy + C \int_{-1}^{1} G^2 \rho (1 - y^2) \, dy.
\]

From the definition (C.28) of \( G \), we need to handle \( R \) and \( V_1 F_{1,1} \). We start by \( R \) first.

We claim that

\[
|R| \leq C \sum_{j=1}^{k} \kappa \left( d_j(s), y \right)^{p-1} 1_{\{y_{j-1}(s) < y < y_j(s)\}} \sum_{l \neq j} \kappa \left( d_l(s), y \right)
\]

where

\[
y_0 = -1, \quad y_j = \tanh \left( \frac{\zeta_j + \zeta_{j+1}}{2} \right) \text{ if } j = 1, \ldots, k-1 \text{ and } y_k = 1.
\]

In particular, we have

\[-1 = y_0 < -d_1 < y_1 < -d_2 < \cdots < y_j < -d_j < y_{j+1} < \cdots < -d_k < y_k = 1\]

and \( \kappa \left( d_j(s), y_{j+1}(s) \right) = \kappa \left( d_{j+1}(s), y_j(s) \right) \) for \( j = 1, \ldots, k-1 \) (to see this, just use the fact that \( \kappa(d, y)(1 - y^2)^{p-1} = \kappa_0 \cosh \frac{y^2}{2}(\xi - \zeta_i) \) if \( y = \tanh \xi \).

To prove (C.37), we take \( y \in (y_{j-1}(s), y_j(s)) \) and set \( X = (\sum_{l \neq j} e_l \kappa \left( d_l(s), y \right)) / e_j \kappa \left( d_j(s), y \right) \). From the fact that \( \zeta_{j+1}(s) - \zeta_j(s) \to \infty \), we have \( |X| \leq 2 \) hence

\[
\left| |1 + X|^{p-1} (1 + X) - 1 \right| \leq C |X|
\]

and for \( y \in (y_{j-1}(s), y_j(s)) \) and \( s \) large,

\[
\left| K^{p-1} K - e_j \kappa \left( d_j(s), y \right)^{p} \right| \leq C \kappa \left( d_j(s), y \right)^{p-1} \sum_{l \neq j} \kappa \left( d_l(s), y \right).
\]

Since for all \( y \in (y_{j-1}(s), y_j(s)) \), \( \kappa \left( d_j(s), y \right) \geq \kappa \left( d_l(s), y \right) \) if \( l \neq j \), this concludes the proof of (C.37).

Using (C.37), we see that

\[
\int_{-1}^{1} R^2 \rho (1 - y^2) \, dy \leq C \sum_{j=1}^{k} \sum_{l \neq j} \int_{y_{j-1}}^{y_j} \kappa \left( d_j(s), y \right)^{2(p-1)} \kappa \left( d_l(s), y \right)^2 \rho (1 - y^2) \, dy
\]

\[
\leq C \sum_{m=1}^{k-1} h \left( \zeta_{m+1} - \zeta_m \right)^2
\]

where \( h \) is defined in (3.50).
Now, we handle $V_i F_{1,1}$. Using (C.9), (C.30) and (i) of Lemma E.1, we see that
\[
\int_{-1}^{1} (V_i F_{1,1}(d_i))^2 \rho (1 - y^2) \, dy \\
\leq C \sum_{j \neq i} \int_{-1}^{1} \kappa(d_i)^2 \kappa(d_j)^2 (p-1) \rho (1 - y^2) \, dy \\
+ C \sum_{j \neq i} \int_{-1}^{1} \kappa(d_i)^2(p-1) \kappa(d_j)^2 (1 - y^2) \rho (1 - y^2) \, dy \to 0 \text{ as } s \to \infty.
\]

Hence, using (3.72), we see that
\[
(\alpha_i^1)^2 \int_{-1}^{1} (V_i F_{1,1}(d_i))^2 \rho (1 - y^2) \, dy = o(\|q\|_2^2).
\]

Gathering (C.26), (C.27), (C.29), (C.34), (C.36), (C.28), (C.39) and (C.40), we get to the conclusion of (3.78). Note that the estimate for $R_-(s)$ is given in (C.35).

**Proof of (iii): An additional estimate.** We prove estimate (3.80) here. The proof is the same as in the case of one soliton treated in [11], except for the term involving the interaction term $R(y, s)$ (3.60). Therefore, arguing exactly as in [11, pages 110 and 112], we write
\[
\frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho \, dy \leq \frac{9}{10} A_- + C J^2 + C \int_{-1}^{1} q_1^2 \frac{\rho}{1 - y^2} \, dy + C \sum_{i=1}^{k} |\alpha_i^1|^2 + \int_{-1}^{1} q_1 R \rho \, dy.
\]

Since we have from the Cauchy-Schwartz inequality, (i) of Claim B.1, (3.70) and (C.39)
\[
\left| \int_{-1}^{1} q_1 R \rho \, dy \right| \leq \left( \int_{-1}^{1} q_1^2 \frac{\rho}{1 - y^2} \, dy \right)^{\frac{1}{2}} \left( \int_{-1}^{1} R^2 (1 - y^2) \rho \, dy \right)^{\frac{1}{2}} \\
\leq C \|q\|_H \left( \int_{-1}^{1} R^2 (1 - y^2) \rho \, dy \right)^{\frac{1}{2}} \leq \frac{1}{10} \left( A_- + \sum_{i=1}^{k} (\alpha_i^1)^2 \right) \\
+ C \sum_{i=1}^{k-1} h(\zeta_{i+1} - \zeta_i)^2
\]

where $h$ is defined in (3.50), this concludes the proof of (3.80) and the proof of Lemma 3.11. □

**Appendix D. A continuity result in the selfsimilar variable.** We prove Claim 4.5 here. Consider $\epsilon_0 > 0$ and from (4.8), fix $\tilde{t}$ close enough to $T(x_0)$ so that
\[
\|w_{x_0}(s_0) - w_\infty\|_{L^2_\rho} \leq \epsilon_0 \text{ where } s_0 = -\log \left( T(x_0) - \tilde{t} \right).
\]
Note from (4.7) and the continuity of \( x \mapsto T(x) \) that \( u(x, \bar{t}) \) is well defined for all \( x \in [\bar{x}, x_0 + (T(x_0) - \bar{t})] \) for some \( \bar{x} < x_0 - (T(x_0) - \bar{t}) \). Therefore, using the selfsimilar transformation \((1.4)\), we see that

\[
\begin{aligned}
\text{(D.2)} \quad w(\cdot, s_0) \in L^2(\bar{y}, 0) \text{ where } \bar{y} = \frac{\bar{x} - x_0}{T(x_0) - \bar{t}} < -1.
\end{aligned}
\]

We aim at proving that for \( x' \) close enough to \( x_0 \), we have

\[
\begin{aligned}
\text{(D.3)} \quad \left\| \left( \begin{array}{c}
w_{x'}(\bar{s}_0(x')) \\
\partial_s w_{x'}(\bar{s}_0(x'))
\end{array} \right) - \left( \begin{array}{c}
w_{\infty} \\
0
\end{array} \right) \right\|_{H} \leq 6 \epsilon_0 \text{ where } \bar{s}_0(x') = -\log \left( T(x') - \bar{t} \right).
\end{aligned}
\]

For simplicity, we will only prove that

\[
\begin{aligned}
\text{(D.4)} \quad \left\| w_{x'}(\bar{s}_0(x')) - w_{\infty} \right\|_{L^2_{\rho}} \leq 2 \epsilon_0,
\end{aligned}
\]

provided that \( x_0 - x' \) is small. The estimates involving \( \partial_y w_{x'}(\bar{s}_0(x')) \) and \( \partial_s w_{x'}(\bar{s}_0(x')) \) follow in the same way.

Using the selfsimilar transformation \((1.5)\), we write

\[
\begin{aligned}
\text{(D.5)} \quad \forall \bar{y} \in (-1, 1), \quad w_{x'}(\bar{y}, \bar{s}_0(x')) = \theta \tau^{2 \bar{t}} w_{x_0}(y, s_0) \text{ where } y = \bar{y}\theta + \xi, \\
\theta = \frac{1}{1 + e^{\bar{s}_0(x')} (T(x_0) - T(x'))} \to 1, \\
\xi = (x' - x_0) e^{\bar{s}_0(x')} \theta \to 0 \text{ as } x' \to x_0.
\end{aligned}
\]

Therefore, performing a change of variables, we write for \( x_0 - x' \) small enough,

\[
\begin{aligned}
\left\| w_{x'}(\bar{s}_0(x')) - w_{\infty} \right\|_{L^2_{\rho}}^2 &= \int_{-1}^{1} \left| w_{x'}(\bar{y}, \bar{s}_0(x')) - w_{\infty}(\bar{y}) \right|^2 \rho(\bar{y}) d\bar{y} \\
&= \int_{-\theta + \xi}^{\theta + \xi} \left| \theta^{2 \bar{t}} w_{x_0}(y, s_0) - w_{\infty} \left( \frac{y - \xi}{\theta} \right) \right|^2 \rho \left( \frac{y - \xi}{\theta} \right) dy.
\end{aligned}
\]

Since we have from \((D.6), (D.3)\) and the fact that \( x \mapsto T(x) \) is \( 1 \)-Lipschitz,

\[
\theta + \xi = \frac{1 + (x' - x_0) e^{\bar{s}_0(x')}}{1 + e^{\bar{s}_0(x')} (T(x_0) - T(x'))} = \frac{T(x') - \bar{t} + x' - x_0}{T(x_0) - \bar{t}} \leq 1,
\]

it follows that

\[
\left\| w_{x'}(\bar{s}) - w_{\infty} \right\|_{L^2_{\rho}}^2 = \int_{\bar{y}}^{1} g(\theta, \xi, y) dy
\]

where \( \bar{y} < -1 \) is defined in \((D.2)\) and

\[
\text{(D.7)} \quad g(\theta, \xi, y) = \frac{1}{\theta} \left| \theta^{2 \bar{t}} w_{x_0}(y, s_0) - w_{\infty} \left( \frac{y - \xi}{\theta} \right) \right|^2 \rho \left( \frac{y - \xi}{\theta} \right).
\]
We claim that in order to conclude, it is enough to prove that for $x_0 - x'$ small enough,

\[
\forall y \in (-\theta + \xi, \theta + \xi), \quad g(\theta, \xi, y) \leq \bar{g}(y) \text{ for some } \bar{g} \in L^1(\bar{g}, 1).
\]

Indeed, since we have from (D.6) that

\[
\forall y \in (\bar{g}, 1), \quad g(\theta, \xi, y) \longrightarrow g(1, 0, y) \text{ as } x' \longrightarrow x_0,
\]

we use (D.8) to apply the Lebesgue Theorem and obtain that

\[
\|w_{x'}(\tilde{s}_0(x')) - w_{x_0}\|_{L_{\rho}^2}^2 = \int_{\bar{g}}^1 g(\theta, \xi, y) \, dy \longrightarrow \int_{-1}^1 g(1, 0, y) \, dy = \|w_{x_0}(s_0) - w_{x_0}\|_{L_{\rho}^2}^2
\]

as $x' \rightarrow x_0$. Using (D.1), we see that for $x_0 - x'$ small enough, (D.4) holds. It remains to prove (D.8) in order to conclude.

If $-\theta + \xi \leq y \leq 0$, then we have $\rho(\frac{y - \xi}{\theta}) \leq 1$. Using (D.7), (D.6), (D.2) and the definition (4.9) of $w_{x_0}$, we write for $x_0 - x'$ small enough:

\[
g(\theta, \xi, y) \leq C\left(\|w_{x_0}(y, s_0)\|_{L^\infty(-1, 1)}^2 + \|w_{x_0}\|_{L^\infty(-1, 1)}^2\right) \in L^1(\bar{g}, 0).
\]

If $0 \leq y \leq \theta + \xi$, then we have from (D.6), $\rho(\frac{y - \xi}{\theta}) \leq C(1 - \frac{y - \xi}{\theta})^{\frac{\theta^2}{\theta^2 - 1}} = C(1 - \frac{y + \xi}{\theta})^{\frac{\theta^2}{\theta^2 - 1}} \leq C(1 - y)^{\frac{\theta^2}{\theta^2 - 1}} \leq C(\theta(y))$. Therefore, using (D.7) and (D.1), we write

\[
g(\theta, \xi, y) \leq C\left(\|w_{x_0}(y, s_0)\|_{L^\infty(-1, 1)}^2 + \|w_{x_0}\|_{L^\infty(-1, 1)}^2\right) \rho(y) \in L^1(0, 1).
\]

Thus, (D.8) holds and so does (D.4).

Since the same technique works for $\|\partial_y w_{x'}(\tilde{s}_0(x')) - \frac{dw_{x_0}}{dy}\|_{L_{\rho}^2(1 - y^2)}^2$ and $\|\partial_y w_{x'}(\tilde{s}_0(x'))\|_{L_{\rho}^2}$, estimate (D.3) follows in the same way. This concludes the proof of Claim 4.5. □

**Appendix E. Computations in the $\xi$ variable.** In the following, we compute integrals involving the solitons $\kappa(d, y)$ (1.10).

Recalling that $y = -d_1(s) = \tanh \zeta_1(s)$ is the center of the $i$th soliton $\kappa(d_i(s), y)$, we introduce the following “separators” between the solitons:

\[
y_0 = -1, \quad y_j = \tanh \left(\frac{\zeta_j + \zeta_{j+1}}{2}\right) \text{ if } j = 1, \ldots, k - 1 \text{ and } y_k = 1
\]

Note in particular that we have

\[-1 = y_0 < -d_1 < y_1 < -d_2 < \cdots < y_j < -d_j < y_{j+1} < \cdots < -d_k < y_k = 1\]

and $\kappa(d_j(s), y_{j+1}(s)) = \kappa(d_{j+1}(s), y_{j+1}(s))$ for $j = 1, \ldots, k - 1$ (to see this, just use the fact that $\kappa(d, y)(1 - y^2)^{-\frac{1}{\rho^2}} = \kappa_0 \cosh^{-2}(\frac{\rho^2}{\rho^2 - 1}(\xi - \zeta_i)$ if $y = \tanh \xi$).
In the following lemma, we estimate various integrals involving the solitons $\kappa(d_i(s), y)$:

**LEMMA E.1.** (A table of integrals involving the solitons) We have the following estimates as $s \to \infty$:

(i) If $i \neq j$, $\alpha > 0$, $\beta > 0$ and $I_1 = \int_{-\infty}^{\infty} \kappa(d_j)^{\alpha} \kappa(d_i)^{\beta} (1 - y^2)^{\frac{\alpha + \beta}{2}} dy$, then:

- for $\alpha = \beta$, $I_1 \sim C_0 |\zeta_i - \zeta_j| e^{-\frac{2\beta}{p-1}|\zeta_i - \zeta_j|}$;
- for $\alpha \neq \beta$, $I_1 \sim C_0 e^{\frac{\alpha + \beta}{2} \min(\alpha, \beta)|\zeta_i - \zeta_j|}$ for some $C_0 = C_0(\alpha, \beta) > 0$.

(ii) If $i \neq j$, $\alpha > 0$, $\beta > 0$ and $I_2 = \int_{y_j-1}^{y_j} \kappa(d_j)^{\alpha} \kappa(d_i)^{\beta} (1 - y^2)^{\frac{\alpha + \beta}{2}} dy$, then:

- for $\alpha = \beta$, $I_2 \leq C |\zeta_{j+1} - \zeta_j| e^{-\frac{2\beta}{p-1}|\zeta_{j+1} - \zeta_j|} + C |\zeta_{j-1} - \zeta_j| e^{-\frac{2\beta}{p-1}|\zeta_{j-1} - \zeta_j|}$;
- for $\alpha > \beta$, $I_2 \leq C e^{-\frac{2\alpha}{p-1}|\zeta_{j+1} - \zeta_j|} + C e^{-\frac{2\beta}{p-1}|\zeta_{j-1} - \zeta_j|}$;
- for $\beta > \alpha$, $I_2 \leq C e^{-\frac{\alpha + \beta}{p-1}|\zeta_{j+1} - \zeta_j|} + C e^{-\frac{\alpha + \beta}{p-1}|\zeta_{j-1} - \zeta_j|}$.

(iii) Let $A_{i,j,l} = \int_{y_j-1}^{y_j} \frac{\kappa(d_j)\kappa(d_i)^p}{1 + y_j^2} dy$ with $l \neq j$. Then, for some $\epsilon_1'' > 0$ and $\delta_5(p) > 0$, we have:

- if $i = j$ and $l = i \pm 1$, then $|A_{i,i,l} - \text{sgn}(l-j)e^{\epsilon_1'' |\zeta_{j+1} - \zeta_j|} | \leq C J^{1+\delta_5}$,
- otherwise, $A_{i,j,l} \leq C J^{1+\delta_5}$, where $J$ is defined in (3.4).

(iv) If $l \neq j$, then $B_{i,j,l} = \int_{y_j-1}^{y_j} \kappa(d_i)\kappa(d_j)^p dy \leq C J^{1+\delta_6}$ for some $\delta_6(p) > 0$ (with $\bar{p} = \min(p, 2)$).

(v) For any $i = 1, \ldots, k$, it holds that $J_i = \int_{-\infty}^{\infty} \kappa(d_i)^2 |J|^{p-2} dy \leq C$ where $K(y,s)$ is defined in (3.60).

**Proof.** (i) With the change of variables $y = \tanh\xi$, we write

$$I_1 = \kappa_0^{\alpha + \beta} \int_{\mathbb{R}} \cosh \frac{2\alpha}{p-1} (\xi - \zeta_j) \cosh \frac{2\beta}{p-1} (\xi - \zeta_i) d\xi.$$ 

From symmetry, we can assume that $\alpha \geq \beta$ and $\zeta_i > \zeta_j$. Using the change of variables $z = \xi - \zeta_j$, we write

$$I_1 = \kappa_0^{\alpha + \beta} \int_{\mathbb{R}} \cosh \frac{2\alpha}{p-1} (z) \cosh \frac{2\beta}{p-1} (z + \zeta_j - \zeta_i) dz.$$ 

When $\alpha > \beta$, we get from Lebesgue’s Theorem $I_1 \sim C e^{-\frac{2\beta}{p-1}|\zeta_i - \zeta_j|}$.

When $\alpha = \beta$, we write from symmetry and Lebesgue’s Theorem

$$I_1 = 2\kappa_0^{\alpha + \beta} \int_{-\infty}^{\zeta_j - \zeta_i} \cosh \frac{2\alpha}{p-1} (z) \cosh \frac{2\beta}{p-1} (z + \zeta_j - \zeta_i) dz \sim C (\zeta_i - \zeta_j) e^{-\frac{2\beta}{p-1}(\zeta_i - \zeta_j)}.$$ 

(ii) Since $I_2 \leq I_1$ and $|\zeta_j - \zeta_i| \geq \min(|\zeta_{j+1} - \zeta_j|, |\zeta_j - \zeta_{j-1}|)$, the result follows from (i) if $\alpha \geq \beta$. When $\alpha < \beta$, we assume that $\zeta_i > \zeta_j$, the other case being parallel.
Using the change of variables \( y = \tanh \xi \) then \( z = \xi - \zeta_j \), we write

\[
I_2 = \kappa_0^{\alpha + \beta} \int \frac{(\zeta_{j+1} - \zeta_j)}{2} \cosh \frac{-2\alpha}{p-1} (z) \cosh \frac{-2\beta}{p-1} (z + \zeta_j - \zeta_i) \, dz \\
\sim e^{-\frac{2\alpha}{p-1} (\zeta_i - \zeta_j)} \int \frac{(\zeta_{j+1} - \zeta_j)}{2} \cosh \frac{-2\alpha}{p-1} (z)e^{\frac{2\beta}{p-1}} \, dz \\
\sim Ce^{-\frac{2\beta}{p-1} (\zeta_i - \zeta_j)} e^{\frac{2(\beta - \alpha)}{p-1} \frac{\zeta_{j+1} - \zeta_j}{2}} \leq Ce^{\frac{(\alpha + \beta)}{p-1} (\zeta_{j+1} - \zeta_j)}
\]

since \( \zeta_i - \zeta_j \geq \zeta_{j+1} - \zeta_j \), which yields the result.

(iii) If \( i = j \), we assume that \( \zeta_i > \zeta_i \), since the other case follows by replacing \( \xi \) by \(-\xi\) (generating a minus sign in the formula). Therefore, it holds that

\[
(\text{E.2}) \quad \zeta_i \geq \zeta_{i+1}.
\]

Using the change of variables \( y = \tanh \xi \), we write

\[
A_{i,i,l} = \kappa_0^{\beta+1} \int \frac{\zeta_i + \zeta_{i+1}}{2} \cosh \frac{-2\alpha}{p-1} (\xi - \zeta_i) \tanh (\xi - \zeta_i) \cosh \frac{-2\beta}{p-1} (\xi - \zeta_i) \, d\xi \\
= \kappa_0^{\beta+1} \int \frac{\zeta_{i+1} - \zeta_i}{2} \cosh \frac{-2\alpha}{p-1} (z) \tanh (z) \cosh \frac{-2\beta}{p-1} (z + \zeta_i - \zeta_l) \, dz.
\]

Since we see from (E.2) that when \( z \leq \frac{(\zeta_{i+1} - \zeta_i)}{2} \), it holds that \( z + \zeta_i - \zeta_l \leq \frac{(\zeta_{i+1} - \zeta_i)}{2} + \zeta_i - \zeta_{i+1} = -\frac{(\zeta_{i+1} - \zeta_i)}{2} \rightarrow -\infty \) as \( s \rightarrow \infty \), we deduce that

\[
\left| \cosh \frac{-2}{p-1} (z + \zeta_i - \zeta_l) - 2 \frac{2}{p-1} e^{\frac{2(z + \zeta_i - \zeta_l)}{p-1}} \right| \leq Ce^{\frac{2(z + \zeta_i - \zeta_l)}{p-1}} e^{(\zeta_{i+1} - \zeta_i)}.
\]

Therefore, using (E.3) and the definition (3.4) of \( J \), we see that

\[
|A_{i,i,l} - c''_1 e^{-\frac{2}{p-1}(\zeta_i - \zeta_l)}| \leq Ce^{-\frac{2}{p-1}(\zeta_i - \zeta_l)} e^{(\zeta_{i+1} - \zeta_i)} \leq C J^{1 + \frac{p-1}{2}}
\]

where

\[
c''_1 = 2 \frac{2}{p-1} \kappa_0^{\beta+1} \int_{\mathbb{R}} \cosh \frac{-2\alpha}{p-1} (z) \tanh (z) e^{\frac{2z}{p-1}} \, dz \\
= 2 \frac{2}{p-1} \kappa_0^{\beta+1} \int_0^{\infty} \cosh \frac{-2\alpha}{p-1} (z) \tanh (z) (e^{\frac{2z}{p-1}} - e^{-\frac{2z}{p-1}}) \, dz > 0,
\]

which gives the result when \( l = i + 1 \).

If \( l \geq i + 2 \), then \( e^{-\frac{2}{p-1}(\zeta_i - \zeta_l)} \leq e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i+1})} e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_l)} \leq J^2 \) and the result follows as well.
Now, if \( j \neq i \), then we have from the Cauchy-Schwartz inequality,

\[
A_{i,j,l} \leq \left( \int_{y_{j-1}}^{y_j} \kappa(d_j)^{p-1} \kappa(d_i)^2 \rho \, dy \right)^{1/2} \left( \int_{y_{j-1}}^{y_j} \kappa(d_j)^{p-1} \kappa(d_i)^2 \rho \, dy \right)^{1/2},
\]

and the conclusion follows from (ii) and the definition (3.4) of \( J(s) \).

(iv) If \( i = j \), then the result follows from (ii). If \( i \neq j \), using the Hölder inequality with \( P = \bar{p} + 1 \) and \( Q = \frac{\bar{p}+1}{\bar{p}} \), we write

\[
B_{i,j,l} \leq \left( \int_{y_{j-1}}^{y_j} \kappa(d_i)^{\bar{p}+1} \kappa(d_j)^{p-\bar{p}} \rho \, dy \right)^{\frac{1}{\bar{p}+1}} \left( \int_{y_{j-1}}^{y_j} \kappa(d_i)^{\bar{p}+1} \kappa(d_j)^{p-\bar{p}} \rho \, dy \right)^{\frac{\bar{p}}{\bar{p}+1}},
\]

and the result follows from (ii) and the definition (3.4) of \( J(s) \).

(v) Using the change of variables \( y = \tanh \xi \), we write

\[
J_i = \kappa_0^{p-1} \int_{\mathbb{R}} \cosh^{-\frac{2}{p-1}} (\xi - \zeta_i) |\bar{K}(\xi, s)|^{p-2} \, d\xi
\]

(E.4)

where \( \bar{K}(\xi, s) = \sum_{j=1}^{k} e_j \cosh^{-\frac{2}{p-1}} (\xi - \zeta_j) \).

If \( p \geq 2 \), then \( |\bar{K}(\xi, s)| \leq C \) and \( |J_i(s)| \leq C \).

If \( p < 2 \) and the \( e_j \) are the same, then \( |\bar{K}(\xi, s)| \geq \cosh^{-\frac{2}{p-1}} (\xi - \zeta_i) \) and \( |J_i(s)| \leq \int_{\mathbb{R}} \cosh^{-2}(\xi - \zeta_i) \, d\xi \leq C \).

It remains to treat the delicate case where \( p < 2 \) with the \( e_j \) not all the same. Taking advantage of the decoupling in the sum of the solitons (see (3.62)), we write

\[
J_i = \kappa_0^{p-1} \sum_{j=1}^{k} \int_{\theta_{j-1}+A}^{\theta_j+A} \cosh^{-\frac{2}{p-1}} (\xi - \zeta_i) |\bar{K}(\xi, s)|^{p-2} \, d\xi
\]

(E.5)

where \( \theta_0 = -\infty, \theta_j = \frac{\zeta_{j-1} + \zeta_{j+1}}{2} \) if \( j = 1, \ldots, k-1, \theta_k = \infty \) and \( A = A(p) \) is fixed such that

\[
e^{\frac{2A}{p-1}} \geq 2e^{\frac{2A}{p-1}}.
\]

This partition isolates each soliton in the definition of \( \bar{K}(\xi, s) \). It is shifted by \( A \) since \( \bar{K}(\xi, s) \) may be zero for some \( z_j(s) \sim \theta_j(s) \) if \( e_j e_{j+1} = -1 \), giving rise to a singularity in \( |\bar{K}(\xi, s)|^{p-2} \), integrable though delicate to control.

Consider some \( j = 1, \ldots, k-1 \).

If \( e_j = e_{j+1} \), then we have from (3.62) and (E.6) for all \( \xi \in (\theta_{j-1}+A, \theta_j+A) \),

\[
|\bar{K}(\xi, s)| \geq C(A) \cosh^{-\frac{2}{p-1}} (\xi - \zeta_j) \text{ and } \cosh^{-\frac{2}{p-1}} (\xi - \zeta_i) \leq C(A) \cosh^{-\frac{2}{p-1}} (\xi - \zeta_j),
\]

and cosh

\[
\cosh^{-2}(\xi - \zeta_i) \leq C(A) \cosh^{-\frac{2}{p-1}} (\xi - \zeta_j),
\]

where \( \cosh^{-2}(\xi - \zeta_i) \) is bounded.
Therefore, since for all $C$ into two parts, below and above $\xi_e$ as in the case $z\in\mathbb{R}$, because $\int_1^2 \ldots$, hence $E-mail: \text{merle@math.u-cergy.fr}$

If $e_j = -e_{j+1}$, then $\tilde{K}(z_j(s), s) = 0$ with $z_j(s) \sim \theta_j(s)$, which makes $|\tilde{K}(\xi, s)|^{p-2}$ singular at $\xi = z_j(s)$. For this we split the integral over the interval $(\theta_{j-1} + A, \theta_{j} + A)$ into two parts, below and above $\theta_j - A$:
- the part on the interval $(\theta_{j-1} + A, \theta_j - A)$ is bounded by the same argument as in the case $e_j = e_{j+1}$;
- the part on the interval $(\theta_j - A, \theta_j + A)$. Since we have from the definition (E.4) of $\tilde{K}(\xi, s)$

$$\partial_\xi \tilde{K}(\xi, s) = -\frac{2}{p-1} \sum_{l=1}^k c_l \sinh (\xi - \zeta_l) \cosh^{2/p-1} (\xi - \zeta_l),$$

it follows that for all $\xi \in (\theta_j - A, \theta_j + A)$, $|\partial_\xi \tilde{K}(\xi, s)| \geq C(A) \cosh^{2/p-1}(\theta_j - \zeta_j) = C(A)e^{-\frac{\xi_{j+1} - \xi_j}{p-1}}$ for some $C(A) > 0$, hence

$$|\tilde{K}(\xi, s)|^{p-2} = |\tilde{K}(\xi, s) - \tilde{K}(z_j(s), s)|^{p-2} \leq C(A)|\xi - z_j(s)|^{p-2} e^{-\frac{2}{p-1}(\xi - \zeta_j)}.$$

Therefore, since for all $\xi \in (\theta_j - A, \theta_j + A)$, $\cosh^{2/p-1}(\xi - \zeta_j) \leq C(A) \cosh^{2/p-1}(\theta_j - \zeta_j) \leq C(A)e^{-\frac{\xi_{j+1} - \xi_j}{p-1}}$, it follows that

$$\int_{\theta_j - A}^{\theta_j + A} \cosh^{2/p-1}(\xi - \zeta_j)|\tilde{K}(\xi, s)|^{p-2} d\xi \leq C(A)e^{-(\xi_{j+1} - \xi_j)} \int_{\theta_j - A}^{\theta_j + A} |\xi - z_j(s)|^{p-2} d\xi \leq C(A)e^{-(\xi_{j+1} - \xi_j)}$$

because $z_j(s) \sim \theta_j(s)$ as $s \to \infty$. Therefore, (v) follows from (E.5), (E.7) and (E.8). □

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