Infinite sequence of new conserved quantities for $N = 1$ SKdV and the supersymmetric cohomology

S. Andrea*, A. Restuccia**, A. Sotomayor**

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*Departamento de Matemáticas,
**Departamento de Física
Universidad Simón Bolívar
***Departamento de Ciencias Básicas
Unexpo, Luis Caballero Mejías

e-mail: sandrea@usb.ve, arestu@usb.ve, sotomayo81@yahoo.es

Abstract

An infinite sequence of new non-local conserved quantities for $N = 1$ Super KdV (SKdV) equation is obtained. The sequence is constructed, via a Gardner transformation, from a new conserved quantity of the Super Gardner equation. The SUSY generator defines a nilpotent operation from the space of all conserved quantities into itself. On the ring of $C_\infty$ superfields the local conserved quantities are closed but not exact. However on the ring of $C_{NL,1}^{\infty}$ superfields, an extension of the $C_\infty$ ring, they become exact and equal to the SUSY transformed of the subset of odd non-local conserved quantities of the appropriate weight. The remaining odd non-local ones generate closed geometrical objects which become exact when the ring is extended to the $C_{NL,2}^{\infty}$ superfields and equal to the SUSY transformed of the new even non-local conserved quantities we have obtained. These ones fit exactly in the SUSY cohomology of the already known conserved quantities.

1 Introduction

Supersymmetric integrable systems are an interesting scenario to analyze ADS/CFT correspondence [1], in particular in relation to $N = 4$ Super Yang-Mills models. Super conformal algebras are also realized in Super KdV equations(SKdV) [2, 3, 4]. SKdV equations are also directly related to supersymmetric quantum mechanics. In fact, the whole SKdV hierarchy arises from the asymptotic expansion of the Green’s function of the Super heat operator [5].
One of the main properties of the integrable systems is the presence of an infinite sequence of conserved quantities.

For SUSY integrable systems the complete structure of conserved quantities has not been understood.

For $N = 1$ SKdV an infinite sequence of local conserved quantities was found in [2]. It was then observed, by analyzing the symmetries of SKdV, the existence of odd non-local conserved quantities [3, 7]. In [8] they were obtained from a Lax formulation of the Super KdV hierarchy and generated from the super residue of a fractional power of the Lax operator.

In [9] all these odd non-local conserved quantities were obtained from a single conserved quantity of the SUSY Gardner equation (SG), which was introduced in [2], see also [10]. So far only two conserved quantities of the SG equation are known, the one generating all the local conserved quantities of SKdV, it is of even parity and of dimension 1, and the above mentioned one of odd parity and of dimension $\frac{1}{2}$.

In this paper we introduce a new infinite sequence of non-local conserved quantities of $N = 1$ SKdV. We construct them via a Gardner map [11], from a new conserved quantity of SG. It is non-local and it has even parity and dimension 1. We then introduce the SUSY cohomology in the space of conserved quantities and obtain the relation between all the conserved quantities local, odd non-local and even non-local. In this sense it is natural the introduction of the new even conserved quantities. The SUSY cohomology we introduce here is used for the $N = 1$ KdV system, however several of the arguments are general, beyond SKdV, and we expect they will be useful in the analysis of other integrable systems.

In section 2 we present basic definitions, in sections 3 and 4 the new conserved quantities and in section 5 the SUSY cohomology.

2 Basic facts

A first step in order to analyze $N = 1$ SKdV and its known local and odd non-local conserved quantities is to consider the ring of polynomials with one odd generator $a_1$ and a superderivation $D$ defined by $Da_n = a_{n+1}$ ($n \in \mathbb{N}$). The elements $a_n$ have the same parity as the positive integer $n$ and satisfy $a_{n_1}a_{n_2} = \pm a_{n_2}a_{n_1}$ with a minus sign only in the case when $n_1$ and $n_2$ are odd. On the products $D$ acts following the rule $D(a_{n_1}a_{n_2}) = (Da_{n_1})a_{n_2} + (-1)^{n_1}a_{n_1}Da_{n_2}$. The explicit algebraic presentation is given by the ring $A$ of elements of the form $b(a_1, a_2, \ldots)$, with $b$ any polynomial, and $D = a_2 \frac{\partial}{\partial a_1} + a_3 \frac{\partial}{\partial a_2} + \cdots$

A second step to complete the construction is to extend the ring $A$ with a new set of generators of the form $\lambda_n = D^{-1}h_n$, with $h_n$ being the non trivial integrands of the known even local conserved quantities. An element of the new ring $\tilde{A}$ is a polynomial of the form $b(a_1, a_2, \ldots, \lambda_1, \lambda_3, \ldots)$ and the corresponding superderivation is given by $\tilde{D} = D + P$ with $P = h_1 \frac{\partial}{\partial \lambda_1} + h_3 \frac{\partial}{\partial \lambda_3} + h_5 \frac{\partial}{\partial \lambda_5} + \cdots$
This algebraic presentation is directly connected to the analytical one by the substitution $a_1 \leftrightarrow \Phi$, where $\Phi$ is a superfield, that is, a function $\Phi : \mathbb{R} \rightarrow \Lambda$, with $\Lambda$ a Grassmann algebra one of whose generators $\theta$ has been singled out. In this setting $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$ and $\Phi(x, \theta) = \xi(x) + \theta u(x)$, with $t$ an implicit variable. The parity of the generator $a_1$ requires $\xi$ to be odd and $u$ to be even for each $x \in \mathbb{R}$.

The covariant derivative $D$ has the property $D^2 = \frac{\partial}{\partial \theta}$ acting on differentiable superfields, and the generator of SUSY transformations $Q = -\frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$ satisfies $DQ + QD = 0$.

In order to realize “integration” the ring of infinitely differentiable superfields $C^\infty(\mathbb{R}, \Lambda)$ must be restricted. We then introduce the ring of Schwartz superfields, that is, $\Theta$ algebra one of whose generators $\theta$ served quantities. We introduce then the ring of non-local superfields, ideal of $\int \Phi dxd\theta$, where $\Phi$ is a superfield, that is, a function $\Phi : \mathbb{R} \rightarrow \Lambda$, with $\Lambda$ a Grassmann algebra one of whose generators $\theta$ has been singled out. In this setting $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$ and $\Phi(x, \theta) = \xi(x) + \theta u(x)$, with $t$ an implicit variable. The parity of the generator $a_1$ requires $\xi$ to be odd and $u$ to be even for each $x \in \mathbb{R}$.

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In order to realize “integration” the ring of infinitely differentiable superfields $C^\infty(\mathbb{R}, \Lambda)$ must be restricted. We then introduce the ring of Schwartz superfields, that is,

$$C^\infty_1(\mathbb{R}, \Lambda) = \left\{ \Phi \in C^\infty(\mathbb{R}, \Lambda) / \lim_{x \rightarrow \pm \infty} x^p \frac{\partial^q}{\partial x^q} \Phi = 0 \right\},$$

for every $p, q \geq 0$. The “superintegration” in this space is well defined and we write $\int \Phi dxd\theta = \int_{-\infty}^{\infty} udx$. It is natural to introduce the space of integrable superfields $C^\infty_{NL,1}(\mathbb{R}, \Lambda) = \left\{ \Phi \in C^\infty(\mathbb{R}, \Lambda) / \frac{\partial}{\partial \theta} \Phi \in C^\infty_{NL,1}(\mathbb{R}, \Lambda) \right\}$ and clearly $C^\infty_{NL,1}(\mathbb{R}, \Lambda) \subset C^\infty_{NL,1}(\mathbb{R}, \Lambda)$.

The next ring to be considered allows us to deal with the known odd non-local conserved quantities. We introduce then the ring of non-local superfields $C^\infty_{NL,1}(\mathbb{R}, \Lambda) = \left\{ \Phi \in C^\infty(\mathbb{R}, \Lambda) / \frac{\partial}{\partial \theta} \Phi \in C^\infty_{NL,1}(\mathbb{R}, \Lambda) \right\}$. We have $C^\infty_{NL,1}(\mathbb{R}, \Lambda) \subset C^\infty_{NL,1}(\mathbb{R}, \Lambda) \subset C^\infty_{NL,1}(\mathbb{R}, \Lambda)$. If $\Psi = U + \theta V \in C^\infty_{NL,1}(\mathbb{R}, \Lambda)$, we take, as in the case of integrable superfields $\int \Psi dxd\theta = \int_{-\infty}^{\infty} Vdx$. In conclusion, for a given superfield to be integrable it is sufficient that the application of $\frac{\partial}{\partial \theta} = D|_{\theta=0}$ to it gives a Schwartz superfield.

The crucial fact we use in the formulation for next sections is that $C^\infty_{1}(\mathbb{R}, \Lambda)$ is an ideal of $C^\infty_{NL,1}(\mathbb{R}, \Lambda)$.

We remind that a candidate for a conserved quantity (for example in the $C^\infty_{1}(\mathbb{R}, \Lambda)$ ring) associated to a given partial differential equation (PDE)

$$\Phi_t = k(\Phi, D\Phi, D^2\Phi, \ldots) \quad (1)$$

may be presented by $H = \int h(\Phi, D\Phi, D^2\Phi, \ldots) dxd\theta$ ($k, h \in C^\infty_{1}(\mathbb{R}, \Lambda)$). Then, $H$ is a conserved quantity if $H_t = 0$, which is equivalent to have $\frac{d}{ds}{\big|}_{s=0} H(\Phi + s\Psi) = 0$ whenever $\Psi(x, t) = \frac{d}{ds}\Phi(x, t) = k(\Phi(x, t), D\Phi(x, t), \ldots)$. In terms of functional derivatives a necessary and sufficient condition for $H$ to be a conserved quantity of (1) is to have $\delta h = Dg$ for some $g \in C^\infty_{1}(\mathbb{R}, \Lambda)$. This condition is used in section 4.

To describe the known local conserved quantities the last setting is sufficient and the ring $\tilde{\mathcal{A}}$ may be reduced by considering only polynomials of the form $b(a_1, a_2, \ldots)$.

For the study of the known odd non-local conserved quantities we must consider the complete ring $\tilde{\mathcal{A}}$. For example, the fourth odd non-local known conserved quantity is given by

$$\int \left\{ \frac{1}{24} (D^{-1}\Phi)^4 - \frac{1}{2} (D\Phi)^2 + (D^{-1}\Phi)D^{-1}(\Phi D\Phi) \right\} dxd\theta, \quad (2)$$
and the terms $D^{-1}\Phi$ and $D^{-1}(\Phi D\Phi)$ clearly belongs to the ring extension $C_{NL,1}^\infty$ mentioned before, where $\Phi = h_1$ and $\Phi D\Phi = h_3$. We put $a_0 \equiv \lambda_1 = D^{-1}\Phi$ in accordance with notation of [9]. The formal analysis of this type of conserved quantities is similar to the local ones but in this case the ring $C_{NL,1}^\infty(\mathbb{R}, \Lambda)$ plays a fundamental role.

To obtain the known local and non-local conserved quantities we start with the pair of SuperKdV and Super Gardner equations given by

$$\Phi_t = D^6\Phi + 3D^2(\Phi D\Phi),$$  

(3)

and

$$\chi_t = D^6\chi + 3D^2(\chi D\chi) - 3\epsilon^2(D\chi D^2(\chi D\chi),$$  

(4)

with $\epsilon$ a given parameter and $\chi = \sigma + \theta w$ another odd superfield [2].

(3) and (4) are connected by the Super Gardner map given by

$$\Phi = \chi + \epsilon D^2\chi - \epsilon^2(\chi D\chi).$$  

(5)

It holds

$$\Phi_t - D^6\Phi - 3D^2(\Phi D\Phi) = [1 + \epsilon D^2 - \epsilon^2(D\chi + \chi D)] 
\{\chi_t - D^6\chi - 3D^2(\chi D\chi) + 3\epsilon^2(D\chi) D^2(\chi D\chi)\},$$  

(6)

whenever $\Phi$ and $\chi$ are related by (5). We recall that (5) maps solutions of (4) into (3). The inverse is also true if we restrict the space of possible solutions of (4) to formal series

$$\chi = \sum_{n=0}^{\infty} a_n[\Phi]\epsilon^n.$$  

$\int \chi dx\theta$ is a conserved quantity of (4), it induces the infinite known local conserved quantities for (3). In the language of [9] the right members of (3), (4), (5) are denoted by $g, f, r$ respectively, with $g \in A$ and $f, r \in A[\epsilon]$. Condition (6) is equivalent to $g \circ r = r' f$ and is necessary and sufficient for (5) to map solutions of Super Gardner to SuperKdV.

In [9] it was shown that $\int \exp(\epsilon D^{-1}\chi) dx d\theta$ is also a conserved quantity for (4). This induces the known infinite sequence of odd non-local conserved quantities of SKdV.

3 Infinite sequence of new non-local conserved quantities of SKdV

We will introduce in this section a new non-local conserved quantity of the Super Gardner equation. From it, following the previous section, we may obtain an infinite set of new non-local conserved quantities of SKdV. There are two already known conserved quantities of the Super Gardner equation. The first one [2] provides the infinite set of local conserved quantities of SKdV equation. The other one [9] give rise to the infinite set of odd non-local conserved quantities of SKdV, originally found in [6] and also obtained from the Lax operator in [8]. This two conserved quantities of Super Gardner equation exhausts all the known conserved quantities of SKdV equation. We will now introduce a new infinite set of even non-local conserved quantities of SKdV.
The quantity $H_G$,

$$
H_G = \int \left\{ \frac{1}{2} D^{-1} \left[ \exp(\epsilon D^{-1} \chi) - \exp(-\epsilon D^{-1} \chi) - 2 \right] \right. \\
+ \frac{1}{\epsilon} \left[ \exp(\epsilon D^{-1} \chi) - 2 \right] D^{-1} \left[ \exp(-\epsilon D^{-1} \chi) - 2 \right] \right\} dxd\theta
$$

(7)

where $\chi \in C_1^\infty$, exists and is conserved by every solution of the Super Gardner equation. It has the same parity as the local conserved quantities and opposite to the already known non-local conserved quantities of SKdV. When we apply the inverse Gardner transformation we obtain an infinite set of well defined even non-local conserved quantities of SKdV.

The terms

$$
D^{-1} \left[ \exp(\epsilon D^{-1} \chi) + \exp(-\epsilon D^{-1} \chi) \right]
$$

and

$$
D^{-1} \left[ \frac{\exp(-\epsilon D^{-1} \chi)}{\epsilon} - 1 \right]
$$

do not belong to $C_{NL,1}^\infty$, but to $C_{NL,2}^\infty = \{ \Phi \in C^\infty(\mathbb{R}, \Lambda) / D^2 \Phi \in C_1^\infty(\mathbb{R}, \Lambda) \}$.

Although each term is not integrable the complete integrand belongs to $C_1^\infty$.

We notice that (7) may be rewritten as

$$
H_G = \int \frac{1}{2} D^{-1} \left\{ D \left[ \frac{\exp(\epsilon D^{-1} \chi) - 1}{\epsilon} \right] D^{-1} \left[ \frac{\exp(-\epsilon D^{-1} \chi) - 1}{\epsilon} \right] \right\} dxd\theta.
$$

(8)

We may then perform the $\theta$ integration and obtain

$$
H_G = \int_{-\infty}^{\infty} \frac{1}{2} \left\{ D \left[ \frac{\exp(\epsilon D^{-1} \chi) - 1}{\epsilon} \right] D^{-1} \left[ \frac{\exp(-\epsilon D^{-1} \chi) - 1}{\epsilon} \right] \right\} dx
$$

(9)

which under the assumption $\chi \in C_1^\infty$ is a well defined integral. The first two new conserved quantities of Super KdV which may be obtained from the inverse Gardner transformation are

$$
H_1^{NL} = \int \frac{1}{2} D^{-1}(\Phi D^2 \Phi) dxd\theta,
$$

(10)

$$
H_3^{NL} = \int \left[ -\frac{1}{2} D^{-1}(\Phi D^2 \Phi) + D^{-1}(D\Phi \cdot \Phi \cdot D^2 \Phi) + \frac{1}{24} D^{-1}(D^{-1} \Phi)^4 - \frac{1}{6} D^{-2} \Phi (D^{-1} \Phi)^3 + \frac{1}{8} (D^{-1} \Phi)^2 D^{-1}(D^{-1} \Phi)^2 \right] dxd\theta
$$

(11)

which exactly agree with the two non-local conserved quantities obtained in [12] by the supersymmetric recursive gradient procedure. The proofs of existence in [12] for the recursive gradient procedure were only given for local conserved quantities, there are no proofs in the literature for the recursive gradient procedure involving non-local quantities.
Hence the existence of the infinite set of conserved quantities is not guaranteed from that approach.

(11) may be rewritten using (8) as

\[ H_{NL}^3 = \int D^{-1} \left[ \Phi \left( -\frac{1}{2} D^2 \Phi - D\Phi \cdot D^{-2} \Phi - \frac{1}{2} D^{-2} \Phi \cdot (D^{-1} \Phi)^2 + \frac{1}{4} D^{-1} \Phi \cdot D^{-1} (D^{-1} \Phi)^2 \right) \right] dxd\theta, \]

where the integrand is manifestly in \( C_\infty^I \).

In the next section we prove the claims of existence and conservation of \( H_G \) and consequently of the new set of infinite conserved quantities of SKdV.

## 4 Conservation of \( H_G \) under the Super Gardner flow

The proposed non-local quantity \( H_G \) for Super Gardner, written in terms of an appropriate analytical extension of the ring \( \mathcal{A} \) (we make use of an abuse of notation taking \( D^{-1} \chi \leftrightarrow a_0, \chi \leftrightarrow a_1 \) an so on), is defined by

\[ h_G = (\exp^{\epsilon a_0} - 1) D^{-1} (\exp^{-\epsilon a_0} - 1) + D^{-1} (\exp^{\epsilon a_0} + \exp^{-\epsilon a_0} - 2). \]

For \( h_G \) to be integrable, \( h_G \in C_\infty^I \), it must satisfy, \( \frac{\partial}{\partial \theta} h_G \in C_\infty^I \). From \( a_1 \leftrightarrow \chi = \sigma + \theta w \) and \( D^{-1} a_1 = a_0 = \int_{-\infty}^x w + \theta \sigma \) it follows that \( e^{\epsilon a_0} - 1 \in C_\infty^I \). The inclusion

\[ D^{-1} \frac{\partial}{\partial \theta} C_\infty^I \subset C_\infty^I \]

gives \( D^{-1} \frac{\partial}{\partial \theta} (e^{\epsilon a_0} - 1) \in C_\infty^I \).

Using \( D^{-1} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta} D^{-1} = \) the identity operator, we compute

\[
\frac{\partial h_G}{\partial \theta} = C_\infty^I + (e^{\epsilon a_0} - 1) \left( I - D^{-1} \frac{\partial}{\partial \theta} \right) (e^{-\epsilon a_0} - 1) + \left( I - D^{-1} \frac{\partial}{\partial \theta} \right) (e^{\epsilon a_0} + e^{-\epsilon a_0} - 2)
\]

\[ = C_\infty^I + (e^{\epsilon a_0} - 1) (e^{-\epsilon a_0} - 1) + (e^{\epsilon a_0} + e^{-\epsilon a_0} - 2)
\]

\[ = C_\infty^I. \]

This proves that the proposed non-local conserved quantity is integrable.

Next it must be shown that when \( a_1 \) is replaced by \( a_1 + sf \) and \( a_0 \) by \( a_0 + sg \) we have

\[
\frac{d}{ds} \bigg|_{s=0} h_G (a_0 + sg, a_1 + sf, a_2 + sDf, \ldots) \in DC_\infty^I.
\]

Using \( \delta \Omega(a_0, a_1, \ldots) \) to denote \( \frac{d}{ds} \bigg|_{s=0} \Omega (a_0 + sg, a_1 + sf, a_2 + sDf, \ldots) \) for any \( \Omega(a_0, a_1, \ldots) \), we have

\[
\frac{1}{\epsilon} \delta e^{\epsilon a_0} = (e^{\epsilon a_0}) g.
\]
Thus

\[
\frac{1}{\epsilon} \delta h_G = e^{\epsilon a_0} g D^{-1} (e^{-\epsilon a_0} - 1) + (e^{\epsilon a_0} - 1) D^{-1} (-e^{-\epsilon a_0} g)
\]

\[
+ D^{-1} ((e^{\epsilon a_0} - e^{-\epsilon a_0}) g).
\]

However \( D(e^{\epsilon a_0})(F_0 + \epsilon F_1 + \epsilon^2 F_2) = e^{\epsilon a_0} g. \)

Therefore, with \( F'(\epsilon) = F_0 + \epsilon F_1 + \epsilon^2 F_2, \)

\[
\frac{1}{\epsilon} \delta h_G = (De^{\epsilon a_0} F(\epsilon)) D^{-1} (e^{-\epsilon a_0} - 1) (e^{\epsilon a_0} - 1) e^{-\epsilon a_0} F(-\epsilon)
\]

\[
+ e^{\epsilon a_0} F'(\epsilon) - e^{-\epsilon a_0} F(-\epsilon).
\]

At this point it is crucial that \( F'(\epsilon) \in C^\infty_1, \) as is \( e^{\epsilon a_0} F'(\epsilon) D^{-1} (e^{-\epsilon a_0} - 1). \) Therefore, except for a term in \( DC^\infty_1, \)

\[
\frac{1}{\epsilon} \delta h_G = e^{\epsilon a_0} F'(\epsilon) (e^{-\epsilon a_0} - 1) - (1 - e^{-\epsilon a_0}) F(-\epsilon)
\]

\[
+ e^{\epsilon a_0} F'(\epsilon) - e^{-\epsilon a_0} F(-\epsilon)
\]

\[
\frac{1}{\epsilon} \delta h_G = F(\epsilon) - e^{\epsilon a_0} F(\epsilon) - F(-\epsilon) + e^{-\epsilon a_0} F(-\epsilon)
\]

\[
+ e^{\epsilon a_0} F'(\epsilon) - e^{-\epsilon a_0} F(-\epsilon).
\]

Having arrived at

\[
\frac{1}{\epsilon} \delta h_G = F(\epsilon) - F(-\epsilon) = 2\epsilon F_1,
\]

with \( F_1 = a_1 a_4 - a_2 a_3 = -D(a_1 a_3). \)

This verifies that \( \delta h_G \in DC^\infty_1, \) showing in last term that

\[
\int h_G dxd\theta = \int \{(e^{\epsilon a_0} - 1) D^{-1} (e^{-\epsilon a_0} - 1) + D^{-1} (e^{\epsilon a_0} + e^{-\epsilon a_0} - 2)\} a_0 = D^{-1} \chi dxd\theta \tag{14}
\]

is indeed a non-local conserved quantity for the Super Gardner equation

\[
\frac{\partial}{\partial t} \chi(x, t) = f(\chi, D\chi, D^2\chi, \ldots),
\]

with \( f \) having the particular expression given by

\[
f = (a_7 + 3a_1 a_4 + 3a_2 a_3) - 3\epsilon^2 (a_1 a_2 a_4 + a_2 a_3).
\]
5 The SUSY cohomology on the space of conserved quantities

The invariance under supersymmetry of SKdV equations implies that the SUSY transformations of conserved quantities are also conserved quantities. That is, if \( H = \int h, h \in C^{\infty}_I \), is conserved under the SKdV flow then

\[
\delta_Q H := \int Qh
\]

is also a conserved quantity.

The operation \( \delta_Q \) acting on functionals of the above form is well defined since under the change, leaving \( H \) invariant,

\[
h \rightarrow h + Dg
\]

with \( g \in C^{\infty}_I \), we have

\[
Qh \rightarrow Qh + QDg = Qh + D(-Qg)
\]

where \( Qg \in C^{\infty}_I \).

\( \delta_Q \) is a superderivation satisfying \( \delta_Q \delta_Q = 0 \). In fact,

\[
\delta_Q \delta_Q H = \int Q^2 h = -\partial_\theta h|_{-\infty}^\infty = 0
\]

since \( h \in C^{\infty}_I \).

For the local conserved quantities of SKdV, which we denote \( H_{2n+1}(\Phi), n = 0, 1, \ldots \), we have

\[
\delta_Q H_{2n+1}(\Phi) = 0, n = 0, 1, \ldots, \tag{15}
\]

where the index \( 2n + 1 \) denotes the dimension of \( H_{2n+1} \).

If we consider the ring \( C^{\infty}_I \) of superfields, \( H_{2n+1} \) is closed but not exact. However if we extend the ring to the superfields \( C^{\infty}_{NL,1}, C^{\infty}_I \subset C^{\infty}_{NL,1} \), then \( H_{2n+1} \) becomes exact and it is expressed in terms of \( \delta_Q H_{2n+1/2}, n = 0, 1, \ldots \) where \( H_{n+1/2}, n = 0, 1, \ldots \) denote the odd non-local conserved quantities of SKdV \( [6, 8, 9] \), they have dimension \( n + \frac{1}{2} \). The remaining \( H^{NL}_{2n+1}, n = 0, 1, \ldots \) plus a polynomial of lower dimensional conserved quantities is closed but not exact in \( C^{\infty}_{NL,1} \), however if we extend the ring of superfields to \( C^{\infty}_{NL,2} \) they become exact and equal to \( \delta_Q H^{NL}_{2n+1} \), where \( H^{NL}_{2n+1} \) are the even non-local conserved quantities we have introduced in the previous section. They have dimension \( 2n + 1 \). To obtain the exact relation between them we use the conserved quantities of the Super Gardner equation.

We denote them \( G_1, G^{NL}_{1/2} \) and \( G^{NL}_1 \). We have

\[
G_1 = \int \chi = \sum_{n=0}^{\infty} \epsilon^{2n} H_{2n+1} \tag{16}
\]
\[ G_{1/2}^{NL} = \int \frac{\exp(\epsilon D^{-1} \chi) - 1}{\epsilon} = \sum_{n=0}^{\infty} \epsilon^n H_{n+1/2}^{NL} \]  

(17)

\[ H_G \equiv G_1^{NL} = \int \left\{ D^{-1} \left[ \frac{\exp(\epsilon D^{-1} \chi) + \exp(-\epsilon D^{-1} \chi) - 2}{2 \epsilon^2} \right] + \right. \\
\left. + \frac{1}{2} \left[ \frac{\exp(\epsilon D^{-1} \chi) - 1}{\epsilon} \right] D^{-1} \left[ \frac{\exp(-\epsilon D^{-1} \chi) - 1}{\epsilon} \right] \right\} = \sum_{n=0}^{\infty} \epsilon^n H_{n+1}^{NL} \]  

(18)

where \( \chi \in C^\infty_{1/1} \).

The odd powers of \(\epsilon\), in (18), do not provide new conserved quantities of SKdV. For example

\[ H_2^{NL} = \frac{1}{2} H_{1/2}^{NL} H_{3/2}^{NL}. \]  

(19)

We then have

\[ \delta_Q G_1 = \int Q \chi = \chi|_{-\infty}^{+\infty} = 0 \]

hence we obtain (15).

We also have

\[ \delta_Q G_{1/2}^{NL} = \frac{\exp(\epsilon G_1) - 1}{\epsilon} = G_1 + \frac{1}{2} \epsilon G_1^2 + \cdots = \sum_{n=0}^{\infty} \epsilon^{2n} H_{2n+1} + \frac{1}{2} \epsilon (\sum_{n=0}^{\infty} \epsilon^{2n} H_{2n+1})^2 + \cdots \]  

(20)

from which we obtain the relation between the odd non-local and local conserved quantities. In particular we get

\[ \delta_Q H_{1/2}^{NL} = H_1, \]  

(21)

and

\[ \delta_Q H_{3/2}^{NL} = \frac{1}{2} H_1^2 = \delta_Q \left( \frac{1}{2} H_1 H_{1/2}^{NL} \right) \]

that is

\[ \delta_Q \left( H_{3/2}^{NL} - \frac{1}{2} H_1 H_{1/2}^{NL} \right) = 0. \]  

(22)

This is the generic situation, from (20), \( H_{2n+1}, n = 0, 1, \ldots \) is expressed as an exact quantity in terms of \( \delta_Q [H_{2n+1/2}^{NL} + \Sigma \text{products of lower dimensional conserved quantities}] \) while \( [H_{2n+1/2}^{NL} + \Sigma \text{products of lower dimensional conserved quantities}] \) is closed in the ring \( C_{NL,1}^\infty \). If we extend the ring of superfields to \( C_{NL,2}^\infty \), then the closed quantity becomes exact and expressed in terms of \( H_{2n+1}^{NL}, n = 0, 1, \ldots \). The integrand of \( H_{2n+1}^{NL} \) is expressed in terms of superfields in \( C_{NL,2}^\infty \) with the property that the whole integrand belongs to \( C_{1}^\infty \). In the case of \( H_{n+1/2}^{NL} \) the integrand is expressed in terms of superfields in \( C_{NL,1}^\infty \subset C_{1}^\infty \), hence each term is integrable. For example, \( H_{1/2}^{NL} \) (see (18)) may be expressed as

\[ H_1^{NL} = \int \left[ D^{-1} \left( \frac{1}{2} (D^{-1} \Phi)^2 \right) - \frac{1}{2} D^{-1} \Phi \cdot D^{-1} (D^{-1} \Phi) \right] \]  

(23)
each term in the integrand belongs to $C_{N L}^\infty$, it is not integrable but the combination is in $C^\infty$. This expression is in terms of $D^{-1}h$ where $h$ are the integrands of previously known conserved quantities $H_{n+\frac{1}{2}}, H_{2n+1}$. In this particular case

$$H_{\frac{3}{2}}^{NL} = \int \frac{1}{2}(D^{-1}\Phi)^2,$$

$$H_1 = \int \Phi,$$

$$H_{\frac{3}{2}}^{NL} = \int D^{-1}\Phi.$$

This is also a generic property of $H_{2n+1}^{NL}, n = 0, 1, \ldots$ and as we already knew of $H_{n+\frac{1}{2}}, n = 0, 1, \ldots$ whose integrands may be expressed in terms of polynomials in $D^{-1}h$ where $h$ are the integrands of the local conserved quantities $H_{2n+1}$.

From (23) we have

$$\delta_Q H_{1}^{NL} = H_{\frac{3}{2}}^{NL} - \frac{1}{2} H_1 H_{\frac{3}{2}}^{NL},$$

that is, the closed quantity becomes exact in $C_{N L, 2}^\infty$. Similar relations are obtained from (18) for higher dimensional conserved quantities. The general formula is

$$\sum_{n=0}^{\infty} \epsilon^n \delta_Q H_{n+1}^{NL} = \sum_{n=0}^{\infty} \epsilon^{2n} H_{2n+\frac{3}{2}}^{NL} - \frac{1}{2\epsilon} \left[ \exp \left( \sum_{n=0}^{\infty} \epsilon^{2n+1} H_{2n+1} \right) - 1 \right] \left[ \sum_{n=0}^{\infty} (-\epsilon)^n H_{n+\frac{1}{2}} \right]$$

We then have the following relations between the conserved quantities of SKdV equation:

\[
\begin{array}{ccccccc}
H_1 & H_3 & H_5 & H_7 & \ldots \\
H_{\frac{3}{2}}^{NL} & H_{\frac{3}{2}}^{NL} & H_{\frac{3}{2}}^{NL} & H_{\frac{3}{2}}^{NL} & \ldots \\
H_{1}^{NL} & H_{3}^{NL} & H_{5}^{NL} & H_{7}^{NL} & \ldots \\
\end{array}
\]

where the arrow denotes the action of $\delta_Q$, up to lower dimensional conserved quantities as explained previously.

The new conserved quantities $H_1^{NL}, H_3^{NL}, \ldots$ fit then exactly in the SUSY cohomology of the previously known conserved quantities.
6 Conclusions

We found a new infinite sequence of non-local conserved quantities of $N = 1$ SKdV equations. They have even parity and dimension $2n + 1$, $n = 0, 1, \ldots$. We introduced the SUSY cohomology in the space of conserved quantities: local, odd non-local and even non-local. We found all the cohomological relations between them.

Although we consider $N = 1$ SKdV, we expect the SUSY cohomological arguments to be valid in general for SUSY integrable systems.

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