On Varieties of Ordered Automata*

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Abstract. The Eilenberg correspondence relates varieties of regular languages to pseudovarieties of finite monoids. Various modifications of this correspondence have been found with more general classes of regular languages on one hand and classes of more complex algebraic structures on the other hand. It is also possible to consider classes of automata instead of algebraic structures as a natural counterpart of classes of languages. Here we deal with the correspondence relating positive $C$-varieties of languages to positive $C$-varieties of ordered automata and we present various specific instances of this correspondence. These bring certain well-known results from a new perspective and also some new observations. Moreover, complexity aspects of the membership problem are discussed both in the particular examples and in a general setting.

1 Introduction

Algebraic theory of regular languages is a well-established field in the theory of formal languages. The basic ambition of this theory is to obtain effective characterizations of various natural classes of regular languages. First examples of significant classes of languages, which were effectively characterized by properties of syntactic monoids, were the star-free languages by Schützenberger [22] and the piecewise testable languages by Simon [23]. A general framework for discovering relationships between properties of regular languages and properties of monoids was provided by Eilenberg [6], who established a one-to-one correspondence between the so-called varieties of regular languages and pseudovarieties of finite monoids. Here varieties of languages are classes closed for taking quotients, preimages under homomorphisms and Boolean operations. Thus a membership problem for a given variety of regular languages can be translated to a membership problem for the corresponding pseudovariety of finite monoids. An advantage of this approach is that pseudovarieties of monoids are exactly classes of finite monoids which have an equational description by pseudoidentities – see Reiterman [21]. For a thorough introduction to that theory we refer to surveys by Pin [17] and by Straubing and Weil [26].

Since not every natural class of languages is closed for taking all mentioned operations, various generalizations of the notion of varieties of languages have

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been studied. One possible generalization is the notion of positive varieties of languages introduced by Pin \[16\] – the classes need not be closed for taking complementation. Their equational characterization was given by Pin and Weil \[20\]. Another possibility is to weaken the closure property concerning preimages under homomorphisms – only homomorphisms from a certain fixed class \(C\) are used. In this way, one can consider \(C\)-varieties of regular languages which were introduced by Straubing \[25\] and whose equational description was presented by Kunc \[13\]. These two generalizations could be combined as suggested by Pin and Straubing \[19\].

In our contribution we do not use syntactic structures at all. We consider classes of automata as another natural counterpart to classes of regular languages. In fact, we deal with classes of semiautomata, which are exactly automata without the specification of initial nor final states. Characterizing of classes of languages by properties of minimal automata is quite natural, since usually we assume that an input of a membership problem for a fixed class of languages is given exactly by the minimal deterministic automaton. For example, if we want to test whether an input language is piecewise testable, we do not need to compute its syntactic monoid which could be quite large (see Brzozowski and Li \[2\]). Instead of that, we check a condition which must be satisfied by its minimal automaton and which was also established in \[23\]. This characterization was used in \[24\] and \[27\] to obtain a polynomial and quadratic algorithms, respectively, for testing piecewise testability. In \[11\], Simon’s condition was reformulated and the so-called confluent acyclic (semi)automata were defined. In this setting, this characterization can be viewed as an instance of Eilenberg type theorem between varieties of languages and varieties of semiautomata.

Moreover, each minimal automaton is implicitly equipped with an order in which the final states form an upward closed subset. This leads to a notion of ordered automata. Then positive \(C\)-varieties of ordered semiautomata can be defined as classes which are closed for taking certain natural closure operations. We recall here the general Eilenberg type theorem, namely Theorem 6.3, which states that positive \(C\)-varieties of ordered semiautomata correspond to positive \(C\)-varieties of languages.

Summarizing, there are three worlds:

- (L) classes of regular languages,
- (S) classes of finite monoids, sometimes enriched by an additional structure like the ordered monoids, monoids with distinguished generators, etc.,
- (A) classes of semiautomata, sometimes ordered semiautomata, etc.

Most variants of Eilenberg correspondence relate (L) and (S), the relationship between (A) and (S) was studied by Chaubard et al. \[4\], and finally the transitions between (L) and (A) were initiated by Říšik and Ito \[8\]. Here we continue in the last approach, to establish Theorem 6.3. In fact, this result is a combination of Theorem 5.1 of \[19\] (only some hints to a possible proof are given there) and the main result of \[4\] relating worlds (S) and (A). In contrary, in the present paper, one can find a self-contained proof which does not go through the classes of monoids.
The paper is structured as follows. In Sections 2 and 3 we recall the basic notions. In Sections 4 and 5 we study ordered semiautomata and some natural algebraic constructions on them. The next section is devoted to the detailed proof of Theorem 6.3. Section 7 explains how the unordered variant of this result can be obtained. Section 8 presents several instances of Theorem 6.3 and Section 9 discusses membership problem for $C$-varieties of semiautomata given by certain type of pseudoidentities.

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2 Positive $C$-Varieties of Languages

First of all, we recall basic definitions. Let $A^*$ be the set of all words over a finite alphabet $A$. We denote by $\lambda$ the empty word. The set $A^*$ equipped with the operation of concatenation forms a free monoid over $A$ with $\lambda$ being a neutral element. A language over alphabet $A$ is a subset of $A^*$. Note that all languages which are considered in the paper are regular. For a language $L \subseteq A^*$ and a pair of words $u, v \in A^*$, we denote by $u^{-1}L$ the quotient of $L$ by these words, i.e. the set $u^{-1}L = \{ w \in A^* | uwv \in L \}$. In particular, a left quotient is defined by $u^{-1}L = \{ w \in A^* | uw \in L \}$ and a right one is defined by $Lv^{-1} = \{ w \in A^* | wv \in L \}$.

For the propose of this paper, following Straubing \cite{25}, the category of homomorphisms $C$ is a category where objects are all free monoids over non-empty finite alphabets and morphisms are certain monoid homomorphisms among them. If the sets $A$ and $B$ are clear from the context, we write briefly $f \in C$ instead of $f \in C(A^*, B^*)$. This “categorical” definition means that $C$ satisfies the following properties:

– For each finite alphabet $A$, the identity mapping $id_A : A^* \rightarrow A^*$ belongs to $C$.
– If $f : B^* \rightarrow A^*$ and $g : C^* \rightarrow B^*$ belong to $C$, then their composition $gf : C^* \rightarrow A^*$ is also in $C$.

If $f : B^* \rightarrow A^*$ is a homomorphism and $L \subseteq A^*$, then by the preimage of $L$ in the homomorphism $f$ is meant the set $f^{-1}(L) = \{ v \in B^* | f(v) \in L \}$.

Definition 1. Let $C$ be a category of homomorphisms. A positive $C$-variety of languages $V$ associates to every non-empty finite alphabet $A$ a class $V(A)$ of regular languages over $A$ in such a way that

– $V(A)$ is closed under unions and intersections of finite families,
– $V(A)$ is closed under quotients, i.e.

$L \in V(A), u, v \in A^* \implies u^{-1}Lv^{-1} \in V(A),$

– $V$ is closed under preimages in morphisms of $C$, i.e.

$f : B^* \rightarrow A^*, f \in C, L \in V(A) \implies f^{-1}(L) \in V(B)$.
Note that the first condition in Definition 1 ensures that the languages \( \emptyset \) and \( A^* \) belong to \( \mathcal{V}(A) \) for every alphabet \( A \): \( \emptyset \) is the union of the empty system and \( A^* \) is the intersection of the empty system. In other words, the first condition can be equivalently formulated as \( \emptyset, A^* \in \mathcal{V}(A) \) and \( \mathcal{V}(A) \) is closed under binary unions and intersections. In particular, all \( \mathcal{V}(A) \)'s are nonempty.

If \( \mathcal{C} \) consists of all homomorphisms we get exactly the notion of the positive varieties of languages. When adding “each \( \mathcal{V}(A) \) is closed under complements”, we get exactly the notion of the \( \mathcal{C} \)-variety of languages.

3 The Canonical DFA

In this section we fix basic terminology concerning finite automata. First of all, note that all considered automata in the paper are deterministic, complete, finite and over finite alphabets. Moreover, we use the term semiautomaton when the initial and final states are not explicitly given.

A deterministic finite automaton (DFA) over the alphabet \( A \) is a five-tuple \( \mathcal{A} = (Q, A, \cdot, i, F) \), where \( Q \) is a non-empty set of states, \( \cdot : Q \times A \to Q \) is a complete transition function, \( i \in Q \) is the initial state and \( F \subseteq Q \) is the set of final states. The transition function can be extended to a mapping \( \cdot : Q \times A^* \to Q \) by \( q \cdot \lambda = q, \ q \cdot (ua) = (q \cdot u) \cdot a \), for every \( q \in Q, \ u \in A^*, \ a \in A \). The automaton \( \mathcal{A} \) accepts a word \( u \in A^* \) if and only if \( i \cdot u \in F \) and the language recognized by the automaton \( \mathcal{A} \) is \( \mathcal{L}_{\mathcal{A}} = \{ u \in A^* \mid i \cdot u \in F \} \). More generally, for \( q \in Q \), we denote \( \mathcal{L}_{\mathcal{A}, q} = \{ u \in A^* \mid q \cdot u \in F \} \). For a fixed \( \mathcal{A} \), we denote this language simply by \( \mathcal{L}_q \).

We recall the construction of the minimal automaton of a regular language which was introduced by Brzozowski. Since this automaton is uniquely determined and it plays a central role in our paper, we use the adjective “canonical” for it.

Definition 1. The canonical deterministic automaton of a regular language \( L \) is \( \mathcal{D}_L = (D_L, A, \cdot, L, F_L) \), where \( D_L = \{ u^{-1}L \mid u \in A^* \} \), \( q \cdot a = a^{-1}q \), for each \( q \in D_L, \ a \in A \), and \( F_L = \{ q \in D_L \mid \lambda \in q \} \).

A part of Brzozowski’s result is the correctness of the previous definition, because one needs to show that \( \mathcal{D}_L \) is really a finite deterministic automaton. The minimality of \( \mathcal{D}_L \) can be obtained as a consequence of the following lemma. Since the result will be modified later in the paper, the proof of the following lemma is also presented here.

Lemma 2 (\cite{1}). Let \( L \) be a regular language with the canonical automaton \( \mathcal{D}_L \) and let \( \mathcal{A} = (Q, A, \cdot, i, F) \) be an arbitrary DFA with \( \mathcal{L}_{\mathcal{A}} = L \). Then the following holds:

(i) For each \( q \in D_L \), we have that \( \mathcal{L}_{\mathcal{D}_L, q} = q \).

(ii) For each \( u \in A^* \), we have that \( \mathcal{L}_{\mathcal{A}, i \cdot u} = u^{-1}L \).

(iii) The rule \( \varphi : i \cdot u \mapsto u^{-1}L \), for every \( u \in A^* \), correctly defines a surjective mapping from \( Q^* = \{ i \cdot u \mid u \in A^* \} \) onto \( D_L \) satisfying \( \varphi((i \cdot u) \cdot a) = (\varphi(i \cdot u)) \cdot a \), for every \( u \in A^*, \ a \in A \).
Proof. (i) Let \( q \) be a state of \( D_L \), i.e. \( q = u^{-1}L \) for some \( u \in A^* \). Then \( v \in \mathcal{L}_{D_L,q} \) if and only if \( \lambda \in (u^{-1}L) \cdot v = (uv)^{-1}L \), which is equivalent to \( uv \in L \) and also to \( v \in u^{-1}L = q \).

(ii) Let \( u \in A^* \). Then, for every \( v \in A^* \), we have the following chain of equivalent formulas:

\[
v \in \mathcal{L}_{A,i,u}, \ (i \cdot u) \cdot v \in F, \ uv \in L, \text{ and } v \in u^{-1}L.
\]

(iii) The correctness of the definition of \( \varphi \) follows from (ii) and the surjectivity of \( \varphi \) is clear. Moreover, \( \varphi((i \cdot u) \cdot a) = \varphi(i \cdot ua) = (ua)^{-1}L = u^{-1}L \cdot a = \varphi(i \cdot u) \cdot a. \]

\( \square \)

4 Ordered Automata

First, we recall some basic terminology from the theory of ordered sets. By an ordered set we mean a set \( M \) equipped with an order \( \leq \), i.e. by a reflexive, antisymmetric and transitive relation. A subset \( X \) is called upward closed if, for every pair of elements \( x, y \in M \), the following property holds: \( x \leq y, x \in X \) implies \( y \in X \). For every subset \( X \), we denote by \( \uparrow X \) the smallest upward closed subset containing the subset \( X \), i.e. \( \uparrow X = \{ m \in M \mid \exists x \in X : x \leq m \} \). In particular, for \( x \in M \), we write \( \uparrow x \) instead of \( \uparrow \{x\} \). A mapping \( f : M \to N \) between two ordered sets \((M, \leq) \) and \((N, \leq)\) is called isotone if, for every pair of elements \( x, y \in M \), we have that \( x \leq y \) implies \( f(x) \leq f(y) \).

States of the canonical automaton \( D_L \) are languages, and therefore they are ordered naturally by the set-theoretical inclusion. The action by each letter \( a \in A \) is an isotone mapping: for each pair of states \( p, q \) such that \( p \subseteq q \), we have \( p \cdot a = a^{-1}p \subseteq a^{-1}q = q \cdot a \). Moreover, the set \( F_L \) of all final states is an upward closed subset with respect to \( \subseteq \). These observations motivate the following definition.

**Definition 1.** An ordered automaton over the alphabet \( A \) is a six-tuple \( \mathcal{O} = (Q, A, \cdot, \leq, i, F) \), where

- \( (Q, A, \cdot, i, F) \) is a usual DFA;
- \( \leq \) is an order on the set \( Q \);
- the action by every letter \( a \in A \) is an isotone mapping from the ordered set \( (Q, \leq) \) to itself;
- \( F \) is an upward closed subset of \( Q \) with respect to \( \leq \).

The definitions of the acceptance and the recognition are the same as in the case of DFA’s. Since a composition of isotone mappings is isotone, it follows from Definition 1 that the action by every word \( u \in A^* \) is an isotone mapping from the ordered set of states into itself.

Moreover, the ordered semiautomaton \( Q = (Q, A, \cdot, \leq) \) accepts the language \( L \subseteq A^* \) if we can complete \( Q \) to an ordered automaton \( \mathcal{O} = (Q, A, \cdot, \leq, i, F) \) such that \( L = \mathcal{L}_\mathcal{O} \).

The following result states that Brzozowski’s construction gives the minimal ordered automata \( \mathcal{O}_L = (D_L, A, \cdot, \subseteq, L, F_L) \).
Lemma 2. Let \( O = (Q, A, \cdot, \leq, i, F) \) be an ordered automaton recognizing the language \( L = L_O \). Then, for states \( p \leq q \), we have \( L_p \subseteq L_q \). Moreover, the mapping \( \varphi \) from Lemma 3 is an isotope one onto the canonical ordered automaton \( O_c = (D_L, A, \cdot, \leq, L, F_L) \).

Proof. Let \( p \leq q \) hold in the given ordered automaton \( O \). If \( w \in L_p \), then \( p \cdot w \in F \). Now \( p \cdot w \leq q \cdot w \) implies \( q \cdot w \in F \) and therefore \( w \in L_q \). Moreover, having \( u, v \in A^* \) such that \( p = i \cdot u \), \( q = i \cdot v \), we get \( L_{i \cdot u} \subseteq L_{i \cdot v} \). By Lemma 2, it means that \( u^{-1}L \subseteq v^{-1}L \), i.e. \( \varphi(i \cdot u) \subseteq \varphi(i \cdot v) \).

When relating languages with algebraic structures (not our task here), the following property of the minimal/canonical ordered automaton is a crucial one.

Lemma 3 (Pin [17, Section 3]). The transition monoid of the minimal automaton of a regular language \( L \) is isomorphic to the syntactic monoid of \( L \).

Similarly, the ordered transition monoid of the minimal ordered automaton of \( L \) is isomorphic to the syntactic ordered monoid of \( L \).

The next lemma clarifies how the quotients of a language can be obtained changing the initial and the final states appropriately.

Lemma 4. Let \( O = (Q, A, \cdot, \leq, i, F) \) be an ordered automaton recognizing the language \( L = L_O \). Let \( u, v \in A^* \). Then
(i) \( u^{-1}L = L_B \) where \( B = (Q, A, \cdot, \leq, i \cdot u, F) \),
(ii) \( Lv^{-1} = L_C \) where \( C = (Q, A, \cdot, \leq, i, F_v) \) and \( F_v = \{ q \in Q \mid q \cdot v \in F \} \).

Proof. (i) It follows from Lemma 2 (ii).
(ii) To show that \( C \) is an ordered semiautomaton, we need to prove that \( F_v \) is upward closed. Let \( p \in F_v \) and \( p \leq q \in Q \). From \( p \in F_v \) we have \( p \cdot v \in F \) and from \( p \leq q \) we obtain \( p \cdot v \leq q \cdot v \). Since \( F \) is upward closed we get \( q \cdot v \in F \), which implies \( q \in F_v \).

Now, for every \( w \in A^* \), the following is a chain of equivalent statements:

\[ w \in Lv^{-1}, \quad wv \in L, \quad (i \cdot w) \cdot v \in F, \quad i \cdot w \in F_v, \quad \text{and} \quad w \in L_C. \]

Thus we proved the equality \( Lv^{-1} = L_C \).

The next result characterizes languages which are recognized by changing the final states in the canonical ordered automaton.

Lemma 5. Let \( H \subseteq D_L \) be upward closed. Then \( (D_L, A, \cdot, \leq, L, H) \) recognizes a language which can be expressed as a finite union of finite intersections of languages of the form \( Lv^{-1} \).

Proof. For an arbitrary upward closed subset \( X \subseteq D_L \), we define \( \text{Past}(X) = \{ w \in A^* \mid L \cdot w \in X \} \). Since \( \text{Past}(X) = \bigcup_{p \in X} \text{Past}(\langle p \rangle) \), it is enough to prove that, for each \( p \in D_L \), the set \( \text{Past}(\langle p \rangle) \) can be expressed as a finite intersection of right quotients of the language \( L \).
Let $p$ be an arbitrary state in $D_L$. For each $q \in D_L$ such that $p \nsubseteq q$, we have $p = L_p \nsubseteq L_q = q$. This means that there is $v_q \in A^*$ with the property $v_q \in L_p \setminus L_q$. Equivalently, $p \cdot v_q \notin F_L$ and $q \cdot v_q \notin F_L$. Now if $w \in \text{Past}(\gamma)$ then $L \cdot w \supseteq p$. Therefore $(L \cdot w) \cdot v_q \supseteq p \cdot v_q \in F_L$, from which we get $wv_q \in L$, i.e. $w \in Lv_q^{-1}$. We have showed that $\text{Past}(\gamma) \subseteq Lv_q^{-1}$.

Now we claim that

$$\text{Past}(\gamma) = \bigcap_{p \notin q} Lv_q^{-1}.$$  

We already saw the $\subseteq$-part. To prove the opposite inclusion, let $w$ be an arbitrary word from $\in \bigcap_{p \notin q} Lv_q^{-1}$. Fixing $q$ for a moment, we see that $wv_q \in L$, i.e. $L \cdot wv_q \in F_L$. In the case of $q = L \cdot w$ we would have $q \cdot v_q = (L \cdot w) \cdot v_q \in F_L$ – a contradiction. Hence $L \cdot w \neq q$, and this holds for each $q \supseteq p$. Therefore $L \cdot w \supseteq p$ and we deduce that $w \in \text{Past}(\gamma)$. \hfill \Box

There is a natural question how the minimal ordered (semi)automaton can be computed from a given automaton.

**Proposition 6.** There exists an algorithm which computes, for a given automaton $A = (Q, A, \cdot, i, F)$, the minimal ordered automaton of the language $\mathcal{L}(A)$.

**Proof.** Our construction is based on Hopcroft minimization algorithm for DFA’s. We may assume that all states of $A$ are reachable from the initial state $i$. Let $R = (Q \times F) \cup ((Q \setminus F) \times Q)$. Then we construct the relation $R$ from $R$ by removing unsuitable pairs of states step by step. At first, we put $R_1 = R$. Then for each integer $k$, if we find $(p, q) \in R_k$ and a letter $a \in A$ such that $(p \cdot a, q \cdot a) \notin R_k$, then we remove $(p, q)$ from the current relation $R_k$, that is, we put $R_{k+1} = R_k \setminus \{(p, q)\}$. This construction stops after, say, $m$ steps. So, $R_{m+1} = R$ satisfies $(p, q) \in R \implies (p \cdot a, q \cdot a) \in R$, for every $p, q \in Q$ and $a \in A$. Now, we observe that, $(p, q) \in R$ if and only if, for every $u \in A^*$, $(p \cdot u, q \cdot u) \in R$. The condition can be equivalently written as

$$(p, q) \in R \text{ if and only if } (\forall u \in A^*: p \cdot u \in F \implies q \cdot u \in F). \quad (1)$$

It follows that $R$ is a quasiorder on $Q$ and we can consider the corresponding equivalence relation $\rho = \{(p, q) \mid (p, q) \in R, (q, p) \in R\}$ on the set $Q$. Then the quotient set $Q/\rho = \{[q]_\rho \mid q \in Q\}$ has a structure of the automaton: the rule $[q]_\rho \cdot a = [q \cdot a]_\rho$, for each $q \in Q$ and $a \in A$, defines correctly actions by letters using (1). Furthermore, the relation $\leq$ on $Q/\rho$ defined by the rule $[p]_\rho \leq [q]_\rho$ if $(p, q) \in R$, is an order on $Q/\rho$ compatible with actions by letters. So, $A_\rho = (Q/\rho, A_\cdot, \leq, [i]_\rho, F_\rho)$, where $F_\rho = \{|f \in F\}$, is an ordered automaton recognizing $\mathcal{L}(A)$. Moreover, if there are two states $[p]_\rho, [q]_\rho \in Q/\rho$ such that $\mathcal{L}(A_\rho, p) = \mathcal{L}(A_\rho, q)$, then $(p, q) \in \rho$. Thus, the ordered automaton $A_\rho$ is isomorphic to the minimal ordered automaton of the language $\mathcal{L}(A)$. \hfill \Box

Note also that the classical power-set construction makes from a non-deterministic automaton an ordered deterministic automaton which is ordered by the set-theoretical inclusion. Thus, for the purpose of a construction of the minimal ordered automaton, one may also use Brzozowski’s minimization algorithm using power-set construction for the reverse of the given language.
5 Algebraic Constructions on Ordered Semiautomata

To get an Eilenberg correspondence between classes of languages and the classes of semiautomata we need an appropriate definition of a variety of semiautomata. The notion of variety of semiautomata would be given in terms of closure properties with respect to certain constructions on semiautomata.

Positive \( C \)-varieties of languages are closed under quotients, therefore the choice of an initial state and final states in ordered automata can be left free due to Lemma 4.

If an ordered automaton \( O = (Q, A, \cdot, \leq, i, F) \) is given, then we denote by \( O \) the corresponding ordered semiautomaton \( (Q, A, \cdot, \leq) \). In particular, for the canonical ordered automaton \( D_L = (D_L, A, \cdot, \subseteq, L, F) \) of the language \( L \), we have \( D_L = (D_L, A, \cdot, \subseteq) \).

Since positive \( C \)-varieties of languages are closed under taking finite unions and intersections, we include the closedness with respect to direct products of ordered semiautomata.

**Definition 1.** Let \( n \geq 1 \) be a natural number. Let \( O_j = (Q_j, A, \cdot_j, \leq_j) \) be an ordered semiautomaton for \( j = 1, \ldots, n \). We define the ordered semiautomaton \( O_1 \times \cdots \times O_n = (Q_1 \times \cdots \times Q_n, A, \cdot, \leq) \) as follows:

- for each \( a \in A \), we put \( (q_1, \ldots, q_n) \cdot a = (q_1 \cdot_1 a, \ldots, q_n \cdot_n a) \) and
- we have \( (p_1, \ldots, p_n) \leq (q_1, \ldots, q_n) \) if and only if, for each \( j = 1, \ldots, n \), the inequality \( p_j \leq_j q_j \) is valid.

The ordered semiautomaton \( O_1 \times \cdots \times O_n \) is called a product of the ordered semiautomata \( O_1, \ldots, O_n \).

We would like to know, which languages are recognized by a product of ordered semiautomata.

**Lemma 2.** Let the ordered semiautomaton \( O \) be the product of the ordered semiautomata \( O_1, \ldots, O_n \). Then the following holds:

(i) If, for each \( j = 1, \ldots, n \), the language \( L_j \) is recognized by \( O_j \), then both \( L_1 \cap \cdots \cap L_n \) and \( L_1 \cup \cdots \cup L_n \) are recognized by \( O \).

(ii) If the language \( L \) is recognized by \( O \), then \( L \) is a finite union of finite intersections of languages recognized by \( O_1, \ldots, O_n \).

**Proof.** Let \( O_j = (Q_j, A, \cdot_j, \leq_j) \), \( j = 1, \ldots, n \). Denote \( Q = Q_1 \times \cdots \times Q_n \) and \( O = (Q, A, \cdot, \leq) \).

(i) Let \( F_1, \ldots, F_n \) be sets of final states used for recognition of the languages \( L_1, \ldots, L_n \). Put \( F = F_1 \times \cdots \times F_n \) for the intersection \( L_1 \cap \cdots \cap L_n \) and

\[
F = \{ (q_1, \ldots, q_n) \mid \text{there exists } j \in \{1, \ldots, n\} \text{ such that } q_j \in F_j \}
\]

for the union \( L_1 \cup \cdots \cup L_n \). It is not hard to see that, in the both cases, \( F \) is indeed an upward closed subset.

(ii) Let \( L \) be a language recognized by \( (Q, A, \cdot, \leq) \), i.e let \( F \) be an upward closed subset of \( Q \), and \( i \in Q \) such that \( L \) is recognized by \( (Q, A, \cdot, \leq, i, F) \). Since
$F = \bigcup_{p \in F} \uparrow p$, we see that $L = \bigcup_{p \in F} L_p$, where $L_p$ is recognized by the ordered automaton $(Q, A, \leq, i, \uparrow p)$. Furthermore, for such $p$, we have $p = (p_1, \ldots, p_n)$ and we can write $\uparrow p = \uparrow p_1 \times \cdots \times \uparrow p_n$. Let $i = (i_1, \ldots, i_n)$ and let $L_{(p, i)}$ be a language recognized by the ordered automaton $(Q_j, A, \leq, i_j, \uparrow p_j)$. Then one can check that $L_p = L_{(p, i)} \cap \cdots \cap L_{(p, n)}$. □

Also the following construction is useful.

**Definition 3.** Let $I = \{1, \ldots, n\}$ be a non-empty finite set and, for each $j \in I$, let $Q_j = (Q_j, A, \leq, j)$ be an ordered semiautomaton. We define the disjoint union $Q = (Q, A, \leq)$ of ordered semiautomata $Q_1, \ldots, Q_n$ in the following way:

- $Q = \{ (q, j) \mid j \in I, q \in Q_j \}$.
- for each $a \in A$ and $(p, j), (q, k) \in Q$, we put $(q, j) \cdot a = (q \cdot j, a, j)$ and
- we put $(q, j) \leq (p, k)$ if and only if $j = k$ and $q_j \leq p_j$.

Clearly, $L$ is recognized by a disjoint union of ordered semiautomata if and only if it is recognized by some of them. A further useful notion is a homomorphism of ordered semiautomata.

**Definition 4.** Let $(Q, A, \leq)$ and $(P, A, \circ, \preceq)$ be ordered semiautomata and $\varphi : Q \rightarrow P$ be a mapping. Then $\varphi$ is called a homomorphism of ordered semiautomata if it is isotone and $\varphi(q \cdot a) = \varphi(q) \circ a$ for all $a \in A, q \in Q$. If there exists a surjective homomorphism of ordered semiautomata from $(Q, A, \leq)$ to $(P, A, \circ, \preceq)$, then we say that $(P, A, \circ, \preceq)$ is a homomorphic image of $(Q, A, \leq)$.

We say that $\varphi$ is backward order preserving if, for every $p, q \in Q$, the inequality $\varphi(p) \preceq \varphi(q)$ implies $p \preceq q$. If the homomorphism $\varphi$ is surjective and backward order preserving, then we say that $(Q, A, \leq)$ is isomorphic to $(P, A, \circ, \preceq)$.

In what follows, we use often simply $(P, A, \preceq)$ instead of $(P, A, \circ, \preceq)$. Note that every backward order preserving mapping is injective.

In the setting of the previous definition, one can prove by induction with respect to the length of words that for an arbitrary homomorphism $\varphi$ of semiautomata that the equality $\varphi(q \cdot u) = \varphi(q) \circ u$ holds for every state $q \in Q$ and every word $u \in A^*$."

**Lemma 5.** Let an ordered semiautomaton $(P, A, \preceq)$ be a homomorphic image of an ordered semiautomaton $(Q, A, \leq)$ and $L$ be recognized by $(P, A, \preceq)$. Then $L$ is also recognized by $(Q, A, \leq)$.

**Proof.** If $L$ is recognized by an ordered automaton $P = (P, A, \leq, i, F)$, with $F$ being an upward closed subset, and $\varphi$ is a surjective homomorphism of a semiautomaton $(Q, A, \leq)$ onto the semiautomaton $\overline{P} = (P, A, \preceq)$, then we can choose some $i' \in Q$ such that $\varphi(i') = i$ and we can consider $F' = \{ q \in Q \mid \varphi(q) \in F \}$. Now $F'$ is an upward closed subset in $(Q, \leq)$, because $\varphi$ is an isotone mapping and $F$ is upward closed.
Moreover, for an arbitrary \( u \in A^* \), the following is a chain of equivalent statements:

\[
i' \cdot u \in F', \quad \varphi(i' \cdot u) \in F, \quad \varphi(i') \cdot u \in F, \quad i \cdot u \in F, \quad \text{and } u \in L.
\]

The statement of the lemma follows.

**Definition 6.** An ordered semiautomaton \((Q, A, \cdot, \leq)\) is trivial if \( q \cdot a = q \) for all \( q \in Q \) and \( a \in A \), and \( \leq \) is the equality relation on \( Q \). In particular, for a natural number \( n \), we define the ordered semiautomaton \( T_n(A) = (I_n, A, \cdot, =) \), where \( I_n = \{1, \ldots, n\} \), the transition function \( \cdot \) is defined by the rule \( j \cdot a = j \) for all \( j \in I_n \) and \( a \in A \).

It follows directly from the definition that every trivial ordered semiautomaton is isomorphic to some \( T_n(A) = (I_n, A, \cdot, =) \).

**Lemma 7.** The disjoint union of ordered semiautomata \( Q_j = (Q_j, A, j, \leq_j), j \in \{1, \ldots, n\} \), is a homomorphic image of the product \( Q_1 \times \cdots \times Q_n \times T_n(A) \).

**Proof.** Clearly, the mapping defined by the rule

\[
\varphi : (q_1, \ldots, q_n, j) \mapsto (q_j, j), \quad \text{for every } q_1 \in Q_1, \ldots, q_n \in Q_n, j \in \{1, \ldots, n\},
\]

is a surjective homomorphism of the considered semiautomata.

**Definition 8.** Let \((Q, A, \cdot, \leq)\) be an ordered semiautomaton and \( P \subseteq Q \) be a non-empty subset. If \( p \cdot a \in P \) for every \( p \in P, a \in A \), then \((P, A, \cdot, \leq)\) is called a subsemiautomaton of \((Q, A, \cdot, \leq)\).

In the previous definition, the transition function and order on \( P \) are restrictions of the corresponding data on the set \( Q \) and so they are denoted by the same symbols.

Using the notions of a subsemiautomaton and a homomorphic image, we can formulate the minimality of the canonical ordered semiautomaton in a bit precise way.

**Lemma 9.** Let \((Q, A, \cdot, \leq)\) be an ordered semiautomaton recognizing the language \( L \). Then the canonical ordered semiautomaton \( D_L \) is a homomorphic image of some subsemiautomaton of \((Q, A, \cdot, \leq)\).

**Proof.** Let \( L \) be recognized by the ordered automaton \( A = (Q, A, \cdot, \leq, i, F) \). It is easy to see that the subset \( Q' = \{i \cdot u \mid u \in A^*\} \) constructed in Lemma 2 forms a subsemiautomaton of \((Q, A, \cdot, \leq)\). Furthermore, we defined there the mapping \( \varphi : Q' \to D_L \) by the rule \( \varphi(q) = u^{-1}L \), where \( q = i \cdot u \). This mapping \( \varphi \) is a surjective homomorphism of ordered semiautomata.

We say that an ordered semiautomaton \((Q, A, \cdot, \leq)\) is 1-generated if there exists a state \( i \in Q \) such that \( Q = \{i \cdot u \mid u \in A^*\} \).
Lemma 10. Let \((Q, A, \leq)\) be a 1-generated ordered semiautomaton. Then this semiautomaton is isomorphic to a subsemiautomaton of a product of the canonical ordered semiautomata of languages recognized by the ordered semiautomaton \((Q, A, \leq)\).

Proof. Let \(A = (Q, A, \leq)\) be a 1-generated ordered semiautomaton, i.e. \(Q = \{ i \cdot u \mid u \in A^* \}\) for some \(i \in Q\). For each \(q \in Q\), we consider the ordered automaton \(A_q = (Q, A, \leq, i, q)\). This automaton recognizes the language \(L_{A_q}\), which we denote by \(L(q)\). Using Lemma 8 there is a surjective homomorphism \(\varphi_q : A \to D_{L(q)}\) of ordered semiautomata, because \(A\) is 1-generated and thus \(Q' = Q\) here.

Assume that \(A\) has \(n\) states and denote them by \(q_1, \ldots, q_n\). We can consider the product of the canonical ordered semiautomata \(D_{L(q_k)} = (D_{L(q_1)} \times \cdots \times D_{L(q_n)})\) for all \(k = 1, \ldots, n\), i.e. \(D_{L(q)} = D_{L(q_1)} \times \cdots \times D_{L(q_n)}\). Moreover, since we have the mapping \(\varphi_q\) for each \(q \in Q\), we can define a mapping \(\varphi : Q \to D_{L(q_1)} \times \cdots \times D_{L(q_n)}\) by the rule \(\varphi(p) = (\varphi_{q_1}(p), \ldots, \varphi_{q_n}(p))\), for \(p \in Q\). To prove the statement of the lemma it is enough to show that this mapping \(\varphi\) is an backward order preserving homomorphism of ordered semiautomata.

Let \(p \leq q\) hold in \(Q\). For each \(r \in Q\) the homomorphism \(\varphi_r\) is isotone and hence \(\varphi_r(p) \leq \varphi_r(q)\). Thus we get \(\varphi(p) \leq \varphi(q)\) and we see that \(\varphi\) is an isotone mapping. In the similar way one can check that \(\varphi(p \cdot a) = \varphi(p) \cdot a\) for every \(p \in Q\) and \(a \in A\). These facts mean that \(\varphi\) is a homomorphism of ordered semiautomata.

Now assume that \(p\) and \(q\) are two states in \(Q\) such that \(\varphi(p) \leq \varphi(q)\). Then we have \(\varphi(p) \subseteq L(p)\). By the definition of the mapping \(\varphi_p\) given in Lemma 2 for each \(r \in Q\), we have \(\varphi_p(r) \subseteq L_{A_{r,p}}q\). In particular, we can write \(L_{A_{r,p}}q \subseteq L_{A_{r,q}}q\), for all \(p \leq q\) in \(Q\). Thus the mapping \(\varphi\) is order preserving.

\(\square\)

Lemma 11. Let \(A = (Q, A, \leq)\) be an ordered semiautomaton. Then the semiautomaton \(A\) is a homomorphic image of a disjoint union of its 1-generated subsemiautomata.

Proof. For every \(q \in Q\), we consider the subset of \(Q\) given by \(Q_q = \{ q \cdot u \mid u \in A^* \}\) consisting from all states reachable from \(q\). Clearly, \(Q_q\) form a 1-generated subsemiautomaton of \(A\). We consider disjoint union of all these semiautomata. The set of all states of this ordered semiautomaton is \(P = \{ (p, q) \mid \text{there exists } u \in A^* \text{ such that } q \cdot u = p \}\). We show that the mapping \(\varphi : P \to Q\) given by the rule \(\varphi((p, q)) = p\) is a surjective homomorphism of ordered semiautomata. Indeed, for \(a \in A\), we have \((p, q) \cdot a = (p \cdot a, q)\), and hence \(\varphi((p, q) \cdot a) = \varphi((p \cdot a, q)) = p \cdot a = \varphi((p, q)) \cdot a\). Moreover, \((p, q) \leq (p', q')\) in \(P\) implies \(q = q'\) and \(p \leq p'\), which means that \(\varphi((p, q)) \leq \varphi((p', q'))\). Finally, the surjectivity follows from the fact \(\{ (q, q) \mid q \in Q \}\) is closed under \(\varphi\).

\(\square\)

Since positive \(C\)-varieties of languages are closed under taking preimages in morphisms from the category \(C\) we need a construction on ordered semiautomata which enables the recognition of such languages.
Definition 12. Let \( f : B^* \rightarrow A^* \) be a homomorphism and \( A = (Q, A, \cdot, \leq) \) be an ordered semiautomaton. By \( A^f \) we denote the semiautomaton \( (Q, B, \cdot, f) \) where \( q \cdot f(b) = q \cdot f(b) \) for every \( q \in Q \) and \( b \in B \). We speak about \( f \)-renaming of \( A \) and we say that \( (P, B, \circ, \leq) \) is an \( f \)-subsemiautomaton of \( (Q, A, \cdot, \leq) \) if it is a subsemiautomaton of \( A^f \). In other words, if \( P \subseteq Q \), the order \( \leq \) is the restriction of \( \leq \), and \( \circ \) is a restriction of \( \cdot \).

If we consider \( f = \text{id}_A : A^* \rightarrow A^* \), we can see that \((Q, A, \cdot, \leq)^{\text{id}} = (Q, A, \cdot, \leq)\) and that id-subsemiautomata of \((Q, A, \cdot, \leq)\) are exactly its subsemiautomata.

Lemma 13. Consider a homomorphism \( f : B^* \rightarrow A^* \) of monoids.

(i) Let \( L \) be a regular language which is recognized by an ordered automaton \((Q, A, \cdot, \leq, i, F)\). Then the automaton \( B = (Q, B, \circ, \leq, i, F) \), where \( q \circ b = q \cdot f(b) \) for every \( q \in Q \), \( b \in B \), recognizes the language \( f^{-1}(L) \).

(ii) Let \( A = (Q, A, \cdot, \leq) \) be an ordered semiautomaton. If \( K \subseteq B^* \) is recognized by the ordered semiautomaton \( A^f \), then there exists a language \( L \subseteq A^* \) recognized by \( A \) such that \( K = f^{-1}(L) \).

Proof. (i) For every \( u \in B^* \), we have the following chain of equivalent formulas:

\[
    u \in f^{-1}(L), \quad f(u) \in L, \quad i \cdot f(u) \in F, \quad i \circ u \in F, \quad \text{and} \quad u \in L_B.
\]

(ii) If \( K \subseteq B^* \) is recognized by the ordered semiautomaton \( A^f \), then there is a state \( i \in Q \), and an upward closed subset \( F \subseteq Q \) such that \( K = L_B \), where \( B = (Q, B, \cdot, \leq, i, F) \). Now we consider \( L = L_{A'} \), where \( A' = (Q, A, \cdot, \leq, i, F) \). Now the equality \( K = f^{-1}(L) \) follows from Lemma 13 (i).

\[ \Box \]

Lemma 14. Let \( f \) be an arbitrary homomorphism \( f : B^* \rightarrow A^* \).

(i) If an ordered semiautomaton \( B = (P, A, \circ, \leq) \) is a homomorphic image of an ordered semiautomaton \( A = (Q, A, \cdot, \leq) \), then \( B^f \) is a homomorphic image of the ordered semiautomaton \( A^f \).

(ii) If an ordered semiautomaton \( B = (P, A, \circ, \leq) \) is a subsemiautomaton of an ordered semiautomaton \( A = (Q, A, \cdot, \leq) \) then \( B^f \) is a subsemiautomaton of \( A^f \).

(iii) If an ordered semiautomaton \( B = A_1 \times \cdots \times A_n \) is a product of a family of ordered semiautomata \( A_1, \ldots, A_n \), then \( B^f = A_1^f \times \cdots \times A_n^f \).

Proof. (i) Let \( \varphi \) be a surjective homomorphism from an ordered semiautomaton \((Q, A, \cdot, \leq)\) onto \((P, A, \circ, \leq)\). Then \( \varphi \) is a surjective mapping from \((Q, \leq)\) to \((P, \leq)\). The states and the order in the semiautomaton \((Q, A, \cdot, \leq)^f\) resp. \((P, A, \circ, \leq)^f\) are unchanged and hence \( \varphi \) is an isomorphism mapping from the ordered semiautomaton \((Q, A, \cdot, \leq)\) onto \((P, A, \circ, \leq)\). Now let \( b \in B \) be an arbitrary letter and \( q \in Q \) be an arbitrary state. Then \( \varphi(q \cdot f(b)) = \varphi(q) \cdot f(b) = \varphi(q) \circ b. \)

Therefore \( \varphi \) is a surjective homomorphism of ordered semiautomata.

(ii) This is clear.

(iii) Let \( A_j = (Q_j, A_j, \cdot, \leq_j) \) be an ordered semiautomaton for every \( j = 1, \ldots, n \). Let \( (P, A, \circ, \leq) \) be the product \( A_1 \times \cdots \times A_n \) and \((R, B, \circ, \leq)\) be the
product of ordered semiautomata $A^f_1 \times \cdots \times A^f_n$. Directly from the definitions we have that $P = R = Q_1 \times \cdots \times Q_n$ and $\preceq = \subseteq$. Furthermore, for an arbitrary element $(q_1, \ldots, q_n)$ from the set $P = R$, we have

$$(q_1, \ldots, q_n) \circ b = (q_1, \ldots, q_n) \circ f(b) = (q_1 \cdot f_1(b), \ldots, q_n \cdot f_n(b)) = (q_1 \cdot 1 b, \ldots, q_n \cdot n b)$$

which is equal to $(q_1, \ldots, q_n) \circ b$. This means that the action by each letter $b$ is defined in the ordered semiautomaton $(P,A,\circ,\preceq)^f$ in the same way as in the ordered semiautomaton $(R,B,\cdot,\subseteq)$.

6 Eilenberg Type Correspondence for Positive $\mathcal{C}$-Varieties of Ordered Semiautomata

Definition 1. Let $\mathcal{C}$ be a category of homomorphisms. A positive $\mathcal{C}$-variety of ordered semiautomata $\mathcal{V}$ associates to every non-empty finite alphabet $A$ a class $\mathcal{V}(A)$ of ordered semiautomata over $A$ in such a way that

$\begin{align*}
- \mathcal{V}(A) \neq \emptyset & \text{ is closed under disjoint unions and direct products of non-empty finite families, and homomorphic images,} \\
- \mathcal{V} & \text{ is closed under } f\text{-subsemiautomata for all } (f : B^* \to A^*) \in \mathcal{C}.
\end{align*}$

Remark 2. We define $\mathcal{T}(A)$ as a class of all trivial ordered semiautomata over an alphabet $A$, i.e. $\mathcal{T}(A)$ contains all semiautomata $\mathcal{T}_n(A)$ and all their isomorphic copies. By Lemma 7 the first condition in the definition of positive $\mathcal{C}$-variety of ordered semiautomata can be written equivalently in the following way: $\mathcal{T}(A) \subseteq \mathcal{V}(A)$ and $\mathcal{V}(A)$ is closed under direct products of non-empty finite families and homomorphic images. In particular, the class of all trivial ordered semiautomata $\mathcal{T}$ forms the smallest positive $\mathcal{C}$-variety of ordered semiautomata whenever the considered category $\mathcal{C}$ contains all isomorphisms.

As mentioned in the introduction, Theorem 5.3 has already been proved in special cases. The technical difference is that Ésik and Ito in [8] used disjoint union of automata and Chaubard, Pin and Straubing [4] did not use this construction because they used trivial automata instead of them.

Now we are ready to state the Eilenberg type correspondence for positive $\mathcal{C}$-varieties of ordered semiautomata.

For each positive $\mathcal{C}$-variety of ordered semiautomata $\mathcal{V}$, we denote by $\alpha(\mathcal{V})$ the class of regular languages given by the following formula

$$(\alpha(\mathcal{V}))(A) = \{ L \subseteq A^* \mid \overline{D_L} \in \mathcal{V}(A) \}.$$
Theorem 6.3. Let \( C \) be a category of homomorphisms. The mappings \( \alpha \) and \( \beta \) are mutually inverse isomorphisms between the lattice of all positive \( C \)-varieties of ordered semiautomata and the lattice of all positive \( C \)-varieties of regular languages.

Proof. First of all, we fix a category of homomorphism \( C \) for the whole proof. The proof will be done when we show the following statements:

1. \( \alpha \) is correctly defined, i.e. for every positive \( C \)-variety of ordered semiautomata \( V \), the class \( \alpha(V) \) is a positive \( C \)-variety of languages.
2. \( \beta \) is correctly defined, i.e. for every positive \( C \)-variety of regular languages \( L \), the class \( \beta(L) \) is a positive \( C \)-variety of ordered semiautomata.
3. \( \beta \circ \alpha = \text{id} \), i.e. for each positive \( C \)-variety of ordered semiautomata \( V \) we have \( \beta(\alpha(V)) = V \).
4. \( \alpha \circ \beta = \text{id} \), i.e. for each positive \( C \)-variety of languages \( L \), we have \( \alpha(\beta(L)) = L \).

We prove these facts in separate lemmas. The exception is the second item which trivially follows from the definition of the mapping \( \beta \). Before the formulation of these lemmas we prove some technicalities.

Lemma 4. For each positive \( C \)-variety of ordered semiautomata \( V \) and an alphabet \( A \) we have

\[
(\alpha(V))(A) = \{ L \subseteq A^* | \exists A = (Q, A, ·, \leq, i, F) : L = L_A \text{ and } A \in V(A) \}.
\]

Proof. The inclusion “\( \subseteq \)” is trivial, because one can take for the ordered automaton \( A \) the canonical automaton \( D_L \). To prove the opposite inclusion, let \( L = L_A \), where \( A = (Q, A, ·, \leq, i, F) \) with \( A \in V(A) \). By Lemma 9 and the assumption that \( V \) is closed under taking subsemiautomata and homomorphic images, we have that \( D_L \in V(A) \). Therefore \( L \in (\alpha(V))(A) \). \( \square \)

Lemma 5. If \( V \) is a positive \( C \)-variety of ordered semiautomata, then \( \alpha(V) \) is a positive \( C \)-variety of regular languages.

Proof. We need to prove that \( (\alpha(V))(A) \) is closed under taking intersections, unions and quotients. Secondly, we must show the closure property with respect to taking preimages in morphisms from the category \( C \).

For each \( A \), the class \( (\alpha(V))(A) \) given by the formula from Lemma 4 is closed under unions and intersections of finite families, since \( V(A) \) is closed under products of finite families (see Lemma 2). The class \( (\alpha(V))(A) \) is also closed under quotients, since we can change initial and final states freely (see Lemma 4).

Furthermore, by Lemma 13 we see the following observation. Since \( V \) is closed under taking \( f \)-subsemiautomata for each homomorphism \( f \) from \( C \), the class \( \alpha(V) \) is closed under preimages in the same homomorphisms. \( \square \)
All three constructions – direct product, homomorphic image and subsemiautomaton – are standard constructions of universal algebra. From the general theory (see e.g. Burris and Sankappanavar [3]) we want to use only the fact that if one needs to generate the smallest class closed with respect to all three constructions together and containing a class $X$, then it is enough to consider a homomorphic images of subalgebras in products of algebras form $X$. Note that from this point of view, an alphabet $A$ is fixed, and $A$ serves as a set of unary function symbols. Then a semiautomaton over $A$ is a unary algebra.

For a class of ordered semiautomata $X$ over a fixed alphabet $A$ we denote by

- $HX$ the class of all homomorphic images of ordered semiautomata from $X$,
- $IX$ the class of all isomorphic copies of ordered semiautomata from $X$,
- $SX$ the class of all subsemiautomata of ordered semiautomata from $X$,
- $PX$ the class of all products of non-empty finite families of ordered semiautomata from $X$,
- $DX$ the class of all disjoint unions of non-empty finite families of ordered semiautomata from $X$.

It is clear that the operators $H$, $I$ and $S$ are idempotent, i.e. for each class of ordered semiautomata $X$ we have $HHX = HX$ etc. Furthermore, $IPP = IP$ and $IDD = ID$.

**Lemma 6.** For each class $X$ of ordered semiautomata over a fixed alphabet $A$, we have:

$$DX \subseteq HP(X \cup T(A))$$

and

$$PHX \subseteq HPX, \quad SHX \subseteq HSX, \quad PSX \subseteq SPX.$$  

**Proof.** The first property follows from Lemma 7. The other properties are well-known facts from universal algebra, see e.g. [3, Chapter II, Section 9] – a modification for the ordered case is straightforward. \qed

**Lemma 7.** For each positive $C$-variety of regular languages $L$ we have

$$(\beta(L))(A) = HSP(\{D_L \mid L \in L(A)\} \cup T(A)).$$

**Proof.** For every alphabet $A$, we denote $X = \{D_L \mid L \in L(A)\} \cup T(A)$ and we denote by $L(A)$ the right-hand side of the formula in the statement, i.e. $L(A) = HSPX$.

Since $\beta(L)$ is a positive $C$-variety of ordered semiautomata, we have $T(A) \subseteq (\beta(L))(A)$ by Remark 2. Therefore $X \subseteq (\beta(L))(A)$ and the inclusion $L(A) \subseteq (\beta(L))(A)$ follows from the fact that $\beta(L)(A)$ is closed under operators $H, S$ and $P$.

To prove the opposite inclusion $(\beta(L))(A) \subseteq L(A)$, we first prove that $L$ is a positive $C$-variety of ordered semiautomata.

By the first property in Lemma 6 we get

$$D(L(A)) = DHSPX \subseteq HP(HSPX).$$
By the other properties of Lemma 8 and idempotency of the operators we get $\text{HPSP}_X \subseteq \text{HSP}_X = L(A)$. Thus $D(L(A)) \subseteq L(A)$. In the same way one can prove another inclusions $H(L(A)) \subseteq L(A)$, $S(L(A)) \subseteq L(A)$, $P(L(A)) \subseteq L(A)$. Therefore $L(A)$ is closed under all four operators $H$, $S$, $P$ and $D$.

It remains to prove that $L$ is closed under $f$-renaming. So, let $f : B^* \to A^*$ belong to $C$. We need to show that $(Q, A, \cdot, \leq)_f$ belongs to $L(B)$ whenever $(Q, A, \cdot, \leq)$ is from $L(A)$. For an arbitrary set $Y$ of ordered semiautomata over the alphabet $A$, we denote $Y^f = \{ (Q, A, \cdot, \leq)_f | (Q, A, \cdot, \leq) \in Y \}$. Using this notation, we need to show that $(L(A))^f \subseteq L(B)$.

At first, we show a weaker inclusion $X^f \subseteq L(B)$. Trivially $(\text{T}(A))^f = (\text{T}(B))$. Now let $L$ be an arbitrary language from $L(A)$. We consider $(D_L, A, \cdot, \subseteq)_f$ which is an ordered semiautomaton over $B$. By Lemma 14 the set $Z$ of all regular languages which are recognized by the ordered semiautomaton $(D_L, A, \cdot, \subseteq)_f$ contains only languages of the form $f^{-1}(K)$, where $K$ is recognized by $(D_L, A, \cdot, \subseteq)$. Since $L(A)$ is closed under unions, intersections and quotients, every such language $K$ belongs to $L(A)$ by Lemmas 4 and 5. This means that the set $Z$ is a subset of $L(B)$ because $L$ is closed under preimages in the homomorphism $f$. Therefore $(D_L, A, \cdot, \subseteq)_f$ can be reconstructed from these canonical ordered semiautomata of languages from $Z$. Finally, $(D_L, A, \cdot, \subseteq)_f$ is a homomorphic image of a disjoint union of certain subsemiautomata which are isomorphic, by Lemma 11 to subsemiautomata of products of the canonical ordered semiautomata of languages from $Z$. Hence $(D_L, A, \cdot, \subseteq)_f$ belongs to $L(B)$ which is closed under homomorphic images, subsemiautomata, products and disjoint unions as we proved above. So, we proved $X^f \subseteq L(B)$.

Now Lemma 14 has the following consequences $(HY)^f \subseteq H(Y^f)$, $(SY)^f \subseteq S(Y^f)$ and $(PY)^f \subseteq P(Y^f)$ for an arbitrary set of ordered semiautomata $Y$ over the alphabet $A$. If we use all these properties we get $(L(A))^f = (\text{HSP}_X)^f \subseteq \text{HSP}(X^f) \subseteq \text{HSP}(L(B)) = L(B)$ because $L(B)$ is closed under all three operators. Hence $L$ is closed under taking $f$-subsemiautomata and therefore $L$ is a positive $C$-variety of ordered semiautomata.

The inclusion $(\beta(L))(A) \subseteq L(A)$ follows from the definition of $\beta(L)$ which is the smallest positive $C$-variety of ordered semiautomata containing $X$. Since the opposite inclusion is also proved we have finish the proof of the lemma.

**Lemma 8.** For each positive $C$-variety of ordered semiautomata $\forall$ we have $\beta(\alpha(\forall)) = \forall$.

**Proof.** Let $A$ be an arbitrary alphabet. By Lemma 11

$$(\beta(\alpha(\forall)))(A) = \text{HSP}(\{ D_L \mid L \in (\alpha(\forall))(A) \} \cup \text{T}(A)).$$

We denote $X = \{ D_L \mid L \in (\alpha(\forall))(A) \}$. If we use the definition of the mapping $\alpha$ then we see that $X = \{ (Q, A, \cdot, \leq) \in \forall(A) \mid \exists L \subseteq A^* : D_L = (Q, A, \cdot, \leq) \}$. In particular $X \subseteq \forall(A)$. Since we also have $\text{T}(A) \subseteq \forall(A)$ we see that $X \cup \text{T}(A) \subseteq \forall(A)$. Hence

$$(\beta(\alpha(\forall)))(A) = \text{HSP}(X \cup \text{T}(A)) \subseteq \forall(A)$$
because $\mathcal{V}(A)$ is closed under taking homomorphic images, subsemiautomata and products.

In the proof of Lemma 4 we already saw that every ordered semiautomaton $(Q, A, \cdot, \leq)$ can be reconstructed from the canonical ordered automata of languages which are recognized by $(Q, A, \cdot, \leq)$ by Lemma 10 and 11. Therefore $\mathcal{V}(A) \subseteq \text{HSP}(\mathcal{X} \cup \mathcal{T}(A))$ and we proved the equality $\mathcal{V}(A) = \text{HSP}(\mathcal{X} \cup \mathcal{T}(A))$, which means that $\beta(\alpha(\mathcal{V})) = \mathcal{V}$.

\begin{proof}
We want to prove that for every $A$ the equality $(\alpha(\beta(\mathcal{L}))) = (A = \mathcal{L}(A))$ holds. Let $L \in \mathcal{L}(A)$ be an arbitrary language. By the definition of the mapping $\beta$, we have $\overline{D}_L \in (\beta(\mathcal{L}))(A)$. Therefore, by definition of $\alpha$, we have $L \in \alpha(\beta(\mathcal{L}))(A)$ and we have proved the inclusion “$\subseteq$”.

To prove the opposite one, let $K \in \alpha(\beta(\mathcal{L}))(A)$ be an arbitrary language. Then there is an ordered automaton $\mathcal{A} = (Q, A, \cdot, \leq, i, F)$ such that $K = \mathcal{L}(\mathcal{A})$ and $\mathcal{A} \in (\beta(\mathcal{L}))(A) = \text{HSP}(\{ \overline{D}_L \mid L \in \mathcal{L}(A) \} \cup \mathcal{T}(A))$. If $K$ is recognized by $\mathcal{A}$, where $\mathcal{A} \in \mathcal{HX}$ for the class of ordered semiautomata $\mathcal{X} = \text{SP}(\{ \overline{D}_L \mid L \in \mathcal{L}(A) \} \cup \mathcal{T}(A))$, then there is an ordered automaton $\mathcal{B}$ such that $\mathcal{A}$ is a homomorphic image of $\mathcal{B}$ in $\mathcal{X}$. By Lemma 4 the language $K$ is recognized by $\mathcal{B}$. Thus we can assume that $\mathcal{A}$ belongs to $\mathcal{HX}$ for the class of ordered semiautomata $\mathcal{X} = \text{SP}(\{ \overline{D}_L \mid L \in \mathcal{L}(A) \} \cup \mathcal{T}(A))$. In the same way we can also assume that $\mathcal{A}$ belongs to $\mathcal{P}(\{ \overline{D}_L \mid L \in \mathcal{L}(A) \} \cup \mathcal{T}(A))$. By Lemma 2 we know that $K$ is a finite union of finite intersections of languages which are recognized by ordered semiautomata from the class $\{ \overline{D}_L \mid L \in \mathcal{L}(A) \} \cup \mathcal{T}(A)$. Furthermore, trivial semiautomata recognize only languages $\emptyset$ and $A^*$ which belong to every $\mathcal{L}(A)$, hence we may consider $\{ \overline{D}_L \mid L \in \mathcal{L}(A) \}$ instead of $\{ \overline{D}_L \mid L \in \mathcal{L}(A) \} \cup \mathcal{T}(A)$ in the previous sentence. Since the canonical automaton $\mathcal{D}_L$ recognizes only finite unions offinite intersections of quotients of the language $L$ (by Lemmas 4 and 5), and since $\mathcal{L}(A)$ is closed under taking quotients, unions and intersections, we see that $K$ belongs to $\mathcal{L}(A)$.

The previous lemma finishes the proof of Theorem 6.
\end{proof}

\section{$\mathcal{C}$-Varieties of Semiautomata}

For an ordered semiautomaton $(Q, A, \cdot, \leq)$ we define its dual $(Q, A, \cdot, \leq)^d = (Q, A, \cdot, \leq^d)$ where $\leq^d$ is the dual order to $\leq$, i.e. $p \leq^d q$ if and only if $q \leq p$. Instead of the symbol $\leq$ we usually use the symbol $\geq$. Trivially, the resulting structure $(Q, A, \cdot, \geq)$ is also an ordered semiautomaton. For a positive $\mathcal{C}$-variety of ordered semiautomata $\mathcal{V}$ we denote by $\mathcal{V}^d$ its dual, i.e. for every alphabet $A$ we consider $\mathcal{V}^d(A) = \{(Q, A, \cdot, \leq^d) \mid (Q, A, \cdot, \leq) \in \mathcal{V}(A) \}$. It is clear that $(\mathcal{V}^d)^d = \mathcal{V}$ and that $\mathcal{V}^d$ is a positive $\mathcal{C}$-variety of ordered semiautomata.

We say that $\mathcal{V}$ is selfdual if $\mathcal{V}^d = \mathcal{V}$. In other words, $\mathcal{V}$ is selfdual if and only if every $\mathcal{V}(A)$ is closed under taking duals of its members. An alternative characterization follows.

\begin{lemma}
Let $\mathcal{L}$ be a positive $\mathcal{C}$-variety of languages $\mathcal{L}$. Then $\alpha(\beta(\mathcal{L})) = \mathcal{L}$.
\end{lemma}

\begin{proof}
We want to prove that for every $A$ the equality $(\alpha(\beta(\mathcal{L}))) = (A = \mathcal{L}(A))$ holds. Let $L \in \mathcal{L}(A)$ be an arbitrary language. By the definition of the mapping $\beta$, we have $\overline{D}_L \in (\beta(\mathcal{L}))(A)$. Therefore, by definition of $\alpha$, we have $L \in \alpha(\beta(\mathcal{L}))(A)$ and we have proved the inclusion “$\subseteq$”.

To prove the opposite one, let $K \in \alpha(\beta(\mathcal{L}))(A)$ be an arbitrary language. Then there is an ordered automaton $\mathcal{A} = (Q, A, \cdot, \leq, i, F)$ such that $K = \mathcal{L}(\mathcal{A})$ and $\mathcal{A} \in (\beta(\mathcal{L}))(A) = \text{HSP}(\{ \overline{D}_L \mid L \in \mathcal{L}(A) \} \cup \mathcal{T}(A))$. If $K$ is recognized by $\mathcal{A}$, where $\mathcal{A} \in \mathcal{HX}$ for the class of ordered semiautomata $\mathcal{X} = \text{SP}(\{ \overline{D}_L \mid L \in \mathcal{L}(A) \} \cup \mathcal{T}(A))$, then there is an ordered automaton $\mathcal{B}$ such that $\mathcal{A}$ is a homomorphic image of $\mathcal{B}$ in $\mathcal{X}$. By Lemma 4 the language $K$ is recognized by $\mathcal{B}$. Thus we can assume that $\mathcal{A}$ belongs to $\mathcal{HX}$ for the class of ordered semiautomata $\mathcal{X} = \text{SP}(\{ \overline{D}_L \mid L \in \mathcal{L}(A) \} \cup \mathcal{T}(A))$. In the same way we can also assume that $\mathcal{A}$ belongs to $\mathcal{P}(\{ \overline{D}_L \mid L \in \mathcal{L}(A) \} \cup \mathcal{T}(A))$. By Lemma 2 we know that $K$ is a finite union of finite intersections of languages which are recognized by ordered semiautomata from the class $\{ \overline{D}_L \mid L \in \mathcal{L}(A) \} \cup \mathcal{T}(A)$. Furthermore, trivial semiautomata recognize only languages $\emptyset$ and $A^*$ which belong to every $\mathcal{L}(A)$, hence we may consider $\{ \overline{D}_L \mid L \in \mathcal{L}(A) \}$ instead of $\{ \overline{D}_L \mid L \in \mathcal{L}(A) \} \cup \mathcal{T}(A)$ in the previous sentence. Since the canonical automaton $\mathcal{D}_L$ recognizes only finite unions of finite intersections of quotients of the language $L$ (by Lemmas 4 and 5), and since $\mathcal{L}(A)$ is closed under taking quotients, unions and intersections, we see that $K$ belongs to $\mathcal{L}(A)$.

The previous lemma finishes the proof of Theorem 6.
\end{proof}
Lemma 1. Let $V$ be a positive $C$-variety of ordered semiautomata. Then $V$ is selfdual if and only if for each alphabet $A$, we have that:

$$(Q, A, \cdot, \leq) \in V(A) \implies (Q, A, \cdot, \geq) \in V(A).$$

Proof. If $V^d = V$ and $(Q, A, \cdot, \leq) \in V(A)$ then we also have $(Q, A, \cdot, \geq) \in V(A)$. Now the ordered semiautomaton $(Q, A, \cdot, \leq)$ is isomorphic to a subsemiautomaton of the product of the ordered semiautomata $(Q, A, \cdot, \leq)$ and $(Q, A, \cdot, \geq)$, namely the subsemiautomaton with the set of states $\{(q, q) \mid q \in Q\}$.

To prove the converse, it is enough to see that an arbitrary ordered semiautomaton $(Q, A, \cdot, \leq)$ is a homomorphic image of the semiautomaton $(Q, A, \cdot, \geq)$: the identity mapping is a homomorphism of the considered order semiautomata. $\square$

Recall that a $C$-variety of regular languages is a positive $C$-variety of languages which is closed under taking complements. The canonical ordered semiautomaton of the complement of a regular language $L$ is the dual of the canonical ordered semiautomaton of $L$, i.e. $D_L^c = (D_L)^d$. This easy observation helps to prove the following statement.

Proposition 2. There is one to one correspondence between $C$-varieties of regular languages and selfdual positive $C$-varieties of ordered semiautomata.

Proof. The mentioned correspondence is given by the pairs of the mappings $\alpha$ and $\beta$ from Theorem 6.3. For a selfdual positive $C$-variety of ordered semiautomata $V$, we know that $(\alpha(V))(A)$ is closed under complements. This means that $\alpha(V)$ is a $C$-variety of regular languages. Therefore, it remains to show that, for an arbitrary $C$-variety of regular languages $L$, the positive $C$-variety of ordered semiautomata $\beta(L)$ is selfdual. By Lemma 7, $(\beta(L))(A) = \text{HSP} (\{ D_L \mid L \in L(A) \} \cup T(A))$, where the set $\{ D_L \mid L \in L(A) \} \cup T(A)$ is selfdual. However, for every selfdual class of semiautomata $X$, the classes of ordered semiautomata $PX$, $SX$ and $HX$ are selfdual again. Hence $\beta(L)$ is selfdual. $\square$

Since every ordered semiautomaton $(Q, A, \cdot, \leq)$ is a homomorphic image of the ordered semiautomaton $(Q, A, \cdot, \geq)$ we can consider the notion of $C$-varieties of semiautomata instead of selfdual positive $C$-varieties of ordered semiautomata: $C$-varieties of semiautomata are classes of semiautomata which are closed under taking $f$-subsemiautomata, homomorphic images, disjoint unions and finite products.

Let $A(A)$ be the class of all ordered semiautomata over the alphabet $A$. Notice that $A$ forms the greatest positive $C$-variety of ordered semiautomata for each category $C$.

If we have $C$-variety of semiautomata $V$ then we can consider all possible compatible orderings on these semiautomata and define the positive $C$-variety of ordered semiautomata $V^o$ in the following sense

$$V^o(A) = \{(Q, A, \cdot, \leq) \in A(A) \mid (Q, A, \cdot, \geq) \in V(A)\}.$$
Clearly, $\mathcal{V}^o$ is selfdual. Conversely, for a selfdual positive $\mathcal{C}$-variety of ordered semiautomata $\mathcal{V}$, we can consider

$$\mathcal{V}^u(A) = \{ (Q, A, \cdot) \mid \text{there is an order } \leq \text{ such that } (Q, A, \cdot, \leq) \in \mathcal{V}(A) \}.$$  

Now two mappings $\mathcal{V} \mapsto \mathcal{V}^o$ and $\mathcal{V} \mapsto \mathcal{V}^u$ are mutually inverse mappings between $\mathcal{C}$-varieties of semiautomata and selfdual positive $\mathcal{C}$-varieties of ordered semiautomata.

Using this easy correspondence, we obtain the following result as the consequence of Proposition 2.

**Theorem 7.3.** There is one to one correspondence between $\mathcal{C}$-varieties of regular languages and $\mathcal{C}$-varieties of semiautomata.

Note that this results can be also obtained by composing the results by Pin, Straubing [19] with those of Chaubard, Pin and Straubing [14].

### 8 Examples

In this section we present several instances of Eilenberg type correspondence. Some of them are just reformulations of examples already mentioned in existing literature. In particular, the first three subsections correspond to pseudovarieties of aperiodic, $\mathcal{R}$-trivial and $\mathcal{J}$-trivial monoids, respectively. Also Subsection 8.4 has a natural counterpart in pseudovarieties of ordered monoids satisfying the inequality $1 \leq x$. In all these cases, $\mathcal{C}$ is the category of all homomorphisms denoted by $\mathcal{C}_{all}$. Nevertheless, we believe that these correspondences viewed from the perspective of varieties of (ordered) semiautomata are of some interest. Another four subsections works with different categories $\mathcal{C}$ and Subsections 8.6 and 8.8 bring new examples of (positive) $\mathcal{C}$-varieties of (ordered) automata.

#### 8.1 Counter-Free Automata

The star free languages were characterized by Schützenberger [22] as languages having aperiodic syntactic monoids. Here we recall the subsequent characterization of McNaughton and Papert [14] by counter-free automata.

**Definition 1.** We say that a semiautomaton $(Q, A, \cdot)$ is counter-free if, for each $u \in A^*$, $q \in Q$ and $n \in \mathbb{N}$ such that $q \cdot u^n = q$, we have $q \cdot u = q$.

**Proposition 2.** The class of all counter-free semiautomata forms a variety of semiautomata.

**Proof.** It is easy to see that disjoint unions, subsemiautomata, products and $f$-renamings of a counter-free semiautomata are again counter-free.

Let $\varphi : (Q, A, \cdot) \to (P, A, \circ)$ be a surjective homomorphism of semiautomata and let $(Q, A, \cdot)$ be counter-free. We prove that also $(P, A, \circ)$ is a counter-free semiautomaton.
Take $p \in P, u \in A^*$ and $n \in \mathbb{N}$ such that $p \circ u^n = p$. Let $q \in Q$ be an arbitrary state such that $\varphi(q) = p$. Then, for each $j \in \mathbb{N}$, we have $\varphi(q \cdot u^j) = p = \varphi(u^n)$. Since the set $\{q, q \cdot u^n, q \cdot u^{2n}, \ldots\}$ is finite, there exist $k, \ell \in \mathbb{N}$ such that $q \cdot u^k = q \cdot u^{k+\ell}$. If we take $r = q \cdot u^k$, then $r \cdot u^\ell = r$. Since $(Q, A, \cdot)$ is counter-free, we get $r \cdot u = r$. Consequently, $p \circ u = \varphi(r) \circ u = \varphi(r \cdot u) = \varphi(r) = p$. 

The promised link between languages and automata follows.

**Proposition 3 (McNaughton, Papert [14]).** Star free languages are exactly the languages recognized by counter-free semiautomata.

Note that this characterization is effective, although testing whether a regular language given by a DFA is aperiodic is even PSPACE-complete problem by Cho and Huynh [5].

### 8.2 Acyclic Automata

The content $c(u)$ of a word $u \in A^*$ is the set of all letters occurring in $u$.

**Definition 4.** We say that a semiautomaton $(Q, A, \cdot)$ is acyclic if, for every $u \in A^+$ and $q \in Q$ such that $q \cdot u = q$, we have $q \cdot a = q$ for every $a \in c(u)$.

Note that one of the conditions in Simon’s characterization of piecewise testable languages is that a minimal DFA is acyclic – see [23].

One can prove the following proposition in a similar way as in the case of counter-free semiautomata.

**Proposition 5.** The class of all acyclic semiautomata forms a variety of semiautomata.

According to Pin [15, Chapter 4, Section 3], a semiautomaton $(Q, A, \cdot)$ is called extensive if there exists a linear order $\preceq$ on $Q$ such that $(\forall q \in Q, a \in A) q \preceq q \cdot a$. Note that such an order need not to be compatible with actions of letters. One can easily show that a semiautomaton is acyclic if and only if it is extensive. We prefer to use the term acyclic, since we consider extensive actions by letters (compatible with ordering of a semiautomaton) later in the paper. Anyway, testing whether a given semiautomaton is acyclic can be decided using the breadth-first search algorithm.

**Proposition 6 (Pin [15]).** The languages over the alphabet $A$ accepted by acyclic semiautomata are exactly disjoint unions of the languages of the form

$$A_0^* a_1 A_1^* a_2 A_2^* \ldots A_{n-1}^* a_n A_n^*$$

where $a_i \notin A_{i-1} \subseteq A$ for $i = 1, \ldots, n$.

Note that the languages above are exactly those having $\mathcal{R}$-trivial syntactic monoids.
8.3 Acyclic Confluent Automata

In our paper [11] concerning piecewise testable languages, we introduced a certain condition on automata being motivated by the terminology from the theory of rewriting systems.

**Definition 7.** We say that a semiautomaton $(Q, A, \cdot)$ is confluent, if for each state $q \in Q$ and every pair of words $u, v \in A^*$, there is a word $w \in A^*$ such that $c(w) \subseteq c(uv)$ and $(q \cdot u) \cdot w = (q \cdot v) \cdot w$.

In [11], this definition was studied in the context of acyclic (semi)automata, in which case several equivalent conditions were described. One of them can be rephrased in the following way.

**Lemma 8.** Let $(Q, A, \cdot)$ be an acyclic semiautomaton. Then $(Q, A, \cdot)$ is confluent if and only if, for each $q \in Q$, $u, v \in A^*$, we have $q \cdot u \cdot (uv)^{|Q|} = q \cdot v \cdot (uv)^{|Q|}$.

**Proof.** Assume that $(Q, A, \cdot)$ is a confluent acyclic semiautomaton and let $q \in Q$, $u, v \in A^*$ be arbitrary. We consider the sequence of states 

$$q \cdot u, q \cdot u \cdot (uv), q \cdot u \cdot (uv)^2, \ldots, q \cdot u \cdot (uv)^{|Q|}.$$ 

Since the sequence contains more members than $|Q|$, we have $p = q \cdot u \cdot (uv)^k = q \cdot a \cdot (uv)^k$ for some $0 \leq k < \ell \leq |Q|$. Since $(Q, A, \cdot)$ is acyclic, we have $p \cdot a = p$ for every $a \in c(uv)$. Therefore, $p = q \cdot u \cdot (uv)^k = q \cdot u \cdot (uv)^{k+1} = \ldots = q \cdot u \cdot (uv)^{|Q|}$ and we have $p \cdot w = p$ for every $w \in A^*$ such that $c(w) \subseteq c(uv)$. Similarly, for $r = q \cdot v \cdot (uv)^{|Q|}$, we obtain the same property $r \cdot w = r$ for the same words $w$. Taking into account that $(Q, A, \cdot)$ is confluent we obtain the existence of a word $w$ such that $c(w) \subseteq c(uv)$ and $p \cdot w = r \cdot w$. Hence $p = r$ and the first implication is proved. The second implication is evident. \qed

Using the condition from Lemma 8, one can prove that the class of all acyclic confluent semiautomata is a variety of semiautomata similarly as in Proposition 2. Finally, the main result from [11] can be formulated in the following way. It is mentioned in [11] that the defining condition is testable in a polynomial time.

**Proposition 9 (Klíma and Polák [11]).** The variety of all acyclic confluent semiautomata corresponds to the variety of all piecewise testable languages.

8.4 Ordered Automata with Extensive Actions

We say that an ordered semiautomaton $(Q, A, \cdot, \leq)$ has extensive actions if, for every $q \in Q, a \in A$, we have $q \leq q \cdot a$. Clearly, the defining condition is testable in a polynomial time. The transition ordered monoids of such ordered semiautomata are characterized by the inequality $1 \leq x$. It is known [17, Proposition 8.4] that the last inequality characterizes the positive variety of all finite unions of languages of the form

$$A^* a_1 A^* a_2 A^* \ldots A^* a_\ell A^*,$$

where $a_1, \ldots, a_\ell \in A$, $\ell \geq 0$. 

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Therefore we call them positive piecewise testable languages. In this way one can obtain the following statement, which we prove directly using the theory presented in this paper.

**Proposition 10.** The class of all ordered semiautomata with extensive actions is a positive variety of ordered semiautomata and corresponds to the positive variety of all positive piecewise testable languages.

**Proof.** It is a routine to check that the class of all ordered semiautomata with extensive actions is a positive variety of ordered semiautomata. Using Theorem 6.3, we need to show that a language \( L \) is positive piecewise testable if and only if its canonical ordered semiautomaton has extensive actions. To prove that the canonical semiautomaton of a positive piecewise testable language has extensive actions, it is enough to prove this fact for languages of the form \( A^*a_1A^*a_2A^*\ldots A^*a_\ell A^* \) with \( a_1,\ldots,a_\ell \in A, \ell \geq 0 \). This observation follows from the description of the canonical ordered (semi)automata of a language given in Section 4. Indeed, for every language \( K = A^*b_1A^*b_2A^*\ldots A^*b_kA^* \) we have \( K \subseteq b^{-1}K \), because \( b^{-1}K = K \) or \( b^{-1}K = A^*b_2A^*\ldots A^*b_kA^* \) depending on the fact whether \( b \neq b_1 \) or \( b = b_1 \).

Assume that the canonical automaton \( O_L = (D_L, A, \cdot, \subseteq, L, F_L) \) of a language \( L \) has extensive actions; consequently \( (D_L, A, \cdot) \) is an acyclic semiautomaton. Since \( F \) is upward closed, for every \( p \in F \) and \( a \in A \), we have \( p \cdot a \in F \). In other words, for every \( p \in F \), we have \( L_p = A^* \). However by Lemma 2 we have \( L_p = p \), so we get that \( F \) contains just one final state \( p = A^* \). Now we consider a simple path in \( O_L \) from \( i \) to \( p \) labeled by a word \( u = a_1a_2\ldots a_n \) with \( a_k \in A \), i.e. \( i \neq i \cdot a_1 \neq i \cdot a_1a_2 \neq \cdots \neq i \cdot u = p \). If we consider a word \( w \) such that \( w = w_0a_1w_1a_1\ldots a_nw_n \), where \( w_0, w_1,\ldots, w_n \in A^* \), then one can easily prove by an induction with respect to \( k \) that \( i \cdot a_1\ldots a_k \leq i \cdot w_0a_1w_1a_1\ldots a_kw_k \). For \( k = n \), we get \( p \leq i \cdot w \in F \), thus \( i \cdot w = p \). Hence we can conclude with \( A^*a_1A^*a_2\ldots a_nA^* \subseteq L \). We can consider the language \( K \), which is the union of such languages \( A^*a_1A^*a_2\ldots a_nA^* \) for all possible simple paths in \( O_L \) from \( i \) to \( p \). Now \( K \subseteq L \) follows from the previous argument and \( L \subseteq K \) is clear, because every \( w \in L \) describes a unique simple path from \( i \) to \( p \).

\( \square \)

Note that a usual characterization of the class of positive piecewise testable languages is given by a forbidden pattern for DFA (see e.g. [26] page 531). This pattern consists of two words \( v, w \in A^* \) and two states \( p \) and \( q = p \cdot v \) such that \( p \cdot w \in F \) and \( q \cdot w \notin F \). In view of [1] from Section 4 the presence of the pattern is equivalent to the existence of two states \( [p]_\rho \not\leq [q]_\rho \), such that \( [p]_\rho \cdot v = [q]_\rho \) in the minimal automaton of the language. The membership for the class of positive piecewise testable languages is decidable in polynomial time — see [17] Corollary 8.5 or [26] Theorem 2.20.

### 8.5 Autonomous Automata

We recall examples from the paper [8]. We call a semiautomaton \((Q, A, \cdot)\) autonomous if for each state \( q \in Q \) and every pair of letters \( a, b \in A \), we have
q \cdot a = q \cdot b. For a positive integer d, let \( V_d \) be the class of all autonomous semiautomata being disjoint unions of cycles whose lengths divide d. Clearly, the defining conditions are testable in a linear time.

**Proposition 11 (Ésik and Ito [8]).** (i) All autonomous semiautomata form a \( C_l \)-variety of semiautomata and the corresponding \( C_l \)-variety of languages consists of regular languages \( L \) such that, for all \( u, v \in A^* \), if \( u \in L, |u| = |v| \) then \( v \in L \).

(ii) The class \( V_d \) forms a \( C_l \)-variety of semiautomata and the corresponding \( C_l \)-variety of languages consists of all unions of \( (A^d)^*A^i, i \in \{0, \ldots, d-1\} \).

### 8.6 Synchronizing and Weakly Confluent Automata

Synchronizing automata are intensively studied in the literature. A semiautomaton \((Q, A, \cdot)\) is *synchronizing* if there is a word \( w \in A^* \) such that the set \( Q \cdot w \) is a one-element set. We use an equivalent condition, namely, for each pair of states \( p, q \in Q \), there exists a word \( w \in A^* \) such that \( p \cdot w = q \cdot w \) (see e.g. Volkov [28, Proposition 1]). In this paper we consider the classes of semiautomata which are closed for taking disjoint unions. So, we need to study disjoint unions of synchronizing semiautomata. Those automata can be equivalently characterized by the following weaker version of confluence. We say that a semiautomaton \((Q, A, \cdot)\) is *weakly confluent* if, for each state \( q \in Q \) and every pair of words \( u, v \in A^* \), there is a word \( w \in A^* \) such that \((q \cdot u) \cdot w = (q \cdot v) \cdot w\).

**Proposition 12.** A semiautomaton is weakly confluent if and only if it is a disjoint union of synchronizing semiautomata.

**Proof.** It is clear that a disjoint union of synchronizing semiautomata is weakly confluent.

To prove the opposite implication, assume that \((Q, A, \cdot)\) is a weakly confluent semiautomaton. We consider one connected component and an arbitrary pair \( p, q \) of its states. Then there exist states \( p_1, p_2, \ldots, p_n \) and letters \( a_1, \ldots, a_{n-1} \) such that \( p_1 = p, p_n = q \) and for each \( i \in \{1, \ldots, n-1\} \) we have \( p_i \cdot a_i = p_{i+1} \) or \( p_{i+1} \cdot a_i = p_i \). We claim, for each \( i \in \{1, \ldots, n\} \), the existence of a word \( w_i \) such that \( p_{i+1} \cdot w_i = p_i \cdot w_i \). This claim gives, in the case \( i = n \), that \( p \cdot w_n = q \cdot w_n \), which concludes the proof. In the rest of the proof we show the claim by the induction on \( i \). For \( i = 1 \) one can take any word for \( w_1 \). Now, assume that the claim is true for \( i \), i.e. there is a word \( w_i \) and state \( r_1 \) such that \( r_1 = p_{i+1} \cdot w_i = p_i \cdot w_i \). Furthermore, we denote \( r_2 = p_{i+1} \cdot w_i \). In the case \( p_{i+1} \cdot a_i = p_i \), we denote \( r_0 = p_{i+1} \) and we have \( r_0 \cdot w_i = r_1 \) and \( r_0 \cdot a_i w_i = r_2 \). In the case \( p_{i+1} \cdot a_i = p_i \), we denote \( r_0 = p_i \) and we have \( r_0 \cdot a_i w_i = r_1 \) and \( r_0 \cdot w_i = r_2 \). In both cases, since the semiautomaton is weakly confluent there exists \( u \in A^* \) such that \( r_1 \cdot u = r_2 \cdot u \). Now for \( w_{i+1} = w_i \) we have \( p_{i+1} \cdot w_{i+1} = (p_{i+1} \cdot w_i) \cdot u = r_1 \cdot u = r_2 \cdot u = (p_{i+1} \cdot w_i) \cdot u = p_{i+1} \cdot w_{i+1} \). \( \square \)

Since the synchronization property can be tested in the polynomial time (see [28]), Proposition 12 implies that the weak confluence of a semiautomaton can be tested in the polynomial time, as well.

In the next result we use the category \( C_s \) of all surjective homomorphisms. Note that \( f : B^* \to A^* \) is a surjective homomorphism if and only if \( A \subseteq f(B) \).
Proposition 13. The class of all weakly confluent semiautomata is a $\mathcal{C}_s$-variety of semiautomata.

Proof. Clearly, the class of all weakly confluent semiautomata $\mathcal{V}$ is closed under disjoint unions, subsemiautomata and homomorphic images. We need to check that $\mathcal{V}$ is closed under direct products of non-empty finite families.

Let $Q = (Q, A, \cdot, \leq)$ and $P = (P, A, \circ, \preceq)$ be a pair of weakly confluent semiautomata. Take a state $(q, p)$ in the product $Q \times P$ and let $u, v \in A^*$ be words. Since $(Q, A, \cdot, \leq)$ is weakly confluent, there is $w \in A^*$ such that $q \cdot u \cdot w = q \cdot v \cdot w$. Now we consider the words $uw$ and $vw$. Since $P$ is weakly confluent, there is $z \in A^*$ such that $p \cdot uw \cdot z = p \cdot vw \cdot z$. Hence $(q, p) \cdot u \cdot wz = (q, p) \cdot v \cdot wz$ and we proved that $Q \times P$ is weakly confluent. The general case for a direct product of a non-empty finite family of ordered semiautomata can be proved in the same way.

To finish the proof, assume that $f \in \mathcal{C}_s(B^*, A^*)$ is a surjective homomorphism. Let $A = (Q, A, \cdot)$ be a weakly confluent semiautomaton and $A' = (Q, B, \cdot')$ is its $f$-renaming. Taking $q \in Q$ and $u, v \in B^*$, we have $q \cdot f(u) = q \cdot f(v)$ and $q \cdot f(v) = q \cdot f(v')$. Since $A = (Q, A, \cdot)$ is weakly confluent, there is $w \in A^*$ such that $q \cdot f(u) \cdot w = q \cdot f(v) \cdot w$. Now we can consider a preimage $w' \in B^*$ of the word $w$ in the surjective homomorphism $f$. Finally, we can conclude that $(q \cdot f(u)) \cdot f(w') = (q \cdot f(v)) \cdot f(w')$. $\Box$

8.7 Automata for Finite Languages

Finite languages do not form a variety, because their complements, the so-called cofinite languages, are not finite. Moreover, the class of all finite languages is not closed for taking preimages under all homomorphisms. However, one can restrict the category of homomorphisms to the so-called non-erasing ones: we say that a homomorphism $f : B^* \rightarrow A^*$ is non-erasing if $f^{-1}(\lambda) = \{ \lambda \}$. The class of all non-erasing homomorphisms is denoted by $\mathcal{C}_{ne}$. Note that $\mathcal{C}_{ne}$-varieties of languages correspond to $+\mathcal{V}$-varieties of languages (see [25]).

We use certain technical terminology for states of a given semiautomaton $(Q, A, \cdot)$: we say that a state $q \in Q$ has a cycle, if there is a word $u \in A^+$ such that $q \cdot u = q$ and we say that the state $q$ is absorbing if for each letter $a \in A$ we have $q \cdot a = q$.

Definition 14. We call a semiautomaton $(Q, A, \cdot)$ strongly acyclic, if each state which has a cycle is absorbing.

It is evident that every strongly acyclic semiautomaton is acyclic.

Proposition 15. (i) The class of all strongly acyclic semiautomata forms a $\mathcal{C}_{ne}$-variety.

(ii) The class of all strongly acyclic confluent semiautomata forms a $\mathcal{C}_{ne}$-variety.
Proof. (i) It is easy to see that the class $V$ of all strongly acyclic semiautomata is closed under finite products, disjoint unions and subsemiautomata. Also the property of $f$-renaming is clear whenever we consider a non-erasing homomorphism $f : B^* \rightarrow A^*$. Finally, one can prove that the class $V$ is closed under homomorphisms in a similar way as in the case of counter-free semiautomata.

(ii) By the first part we know that all strongly acyclic semiautomata form a $C_{ne}$-variety. We also know that all acyclic confluent semiautomata form a variety of semiautomata, and hence they form also a $C_{ne}$-variety of semiautomata. Therefore all strongly acyclic confluent semiautomata, as an intersection of two $C_{ne}$-varieties, form a $C_{ne}$-variety again. 

Proposition 16. The $C_{ne}$-variety of all finite and all cofinite languages corresponds to the $C_{ne}$-variety of all strongly acyclic confluent semiautomata.

Proof. At first, consider an arbitrary finite language $L \subseteq A^*$ and its canonical automaton $(D_L, A, \cdot, L, F_L)$. Since $L$ is finite, there is only one state in $D_L$ which has a cycle, namely the state $\emptyset$. Moreover, this state is absorbing and it is reachable from all other states, because quotients of finite languages are finite. Therefore the semiautomaton $(D_L, A, \cdot)$ is strongly acyclic and confluent at the same time. Of course, if we start with the complement of a finite language $L$, the canonical semiautomaton is the same as for $L$.

Conversely, let $A = (Q, A, \cdot)$ be a strongly acyclic confluent semiautomaton. For an arbitrary state $q \in Q$, we take some path starting in $q$ of length $|Q|$. On that path there is a state $q'$ which has a cycle, i.e. $q \cdot u = q' = q' \cdot v$ for some $u \in A^*$, $v \in A^+$. Since $A$ is strongly acyclic, $q'$ is an absorbing state. Since $A$ is confluent, there is at most one such absorbing state $q'$ reachable from $q$. Now we choose $i \in Q$ and $F \subseteq Q$ arbitrarily and we consider the automaton $(Q, A, \cdot, i, F)$. By the previous considerations there is just one state reachable from $i$ which has a cycle. We denote it by $f$. Note that it is an absorbing state. One can see that, for each state $q \neq f$, the set $\{u \mid i \cdot u = q\}$ is finite and therefore $\{u \mid i \cdot u = f\}$ is a complement of the finite language. Thus depending on the fact $f \in F$, the language recognized by $(Q, A, \cdot, i, F)$ is cofinite or finite.

Proposition 17. The $C_{ne}$-variety of all prefix-testable languages corresponds to the $C_{ne}$-variety of all strongly acyclic confluent semiautomata.

The characterization from Proposition 16 can be modified for a positive $C_{ne}$-variety of finite languages $F$; where $F(A)$ consists from $A^*$ and all finite languages over $A$. To make the characterizing condition more readable, for a given strongly acyclic confluent semiautomaton and its state $q$, we call the uniquely
determined state $q'$, mentioned in the proof of Proposition 16, as a main follower of the state $q$.

**Proposition 18.** The positive $C_{ne}$-variety of all finite languages corresponds to the positive $C_{ne}$-variety of all strongly acyclic confluent ordered semiautomata satisfying $q' \leq q$ for each state $q$ and its main follower $q'$.

**Proof.** By the first paragraph of the proof of Proposition 16, every canonical ordered automaton of a finite language satisfies the additional condition $q' \leq q$ for each state $q$, because the main follower of $q$ is $\emptyset$.

Similarly, in the second part of the proof: Let $f$ be the considered main follower of $i$. Since it is also main follower of all reachable states from the initial state $i$, we see that $f$ is the minimal state among all reachable states from $i$. Now if $f$ is final, then all states are final, because the final states form upward closed subset. Consequently the language accepted by the ordered automaton is $A^*$ in this case. If $f$ is not final, then the language accepted by the ordered automaton is finite.

Note that, all considered conditions on semiautomata discussed in this subsection can be checked in polynomial time.

### 8.8 Automata for Languages Closed under Inserting Segments

We know that a language $L \subseteq A^*$ is positive piecewise testable if, for every pair of words $u, w \in A^*$ such that $uw \in L$ and for a letter $a \in A$, we have $uaw \in L$. So, we can add an arbitrary letter into each word from the language (at an arbitrary position) and the resulting word stays in the language. Now we consider an analogue, where we put into the word not only a letter but a word of a given fixed length. The length of a word $v \in A^*$ is denoted by $|v|$ as usually.

For each positive integer $n$, we consider the following property of a given regular language $L \subseteq A^*$:

$$(\text{for every } u, v, w \in A^*, \text{ if } uw \in L \text{ and } |v| = n, \text{ then } uvw \in L)$$

We say that $L$ is closed under $n$-insertions whenever $L$ satisfies this property. We show that the class of all regular languages closed under $n$-insertions form a positive $C$-variety of languages by describing the corresponding positive $C$-variety of ordered semiautomata.

At first, we need to describe an appropriate category of homomorphisms. Let $C_{lm}$ be the category consisting of the so-called length-multiplying (see [25]) homomorphisms: $f \in C_{lm}(B^*, A^*)$ if there exists a positive integer $k$ such that $|f(b)| = k$ for every $b \in B$.

**Definition 19.** Let $n$ be a positive integer and $Q = (Q, A, \cdot, \leq)$ be an ordered semiautomaton. We say that $Q$ has $n$-extensive actions if, for every $q \in Q$ and $u \in A^*$ such that $|u| = n$, we have $q \leq q \cdot u$. 
Note that ordered semiautomata from Subsection 8.4 are ordered semiautomata which have 1-extensive actions. Of course, these ordered semiautomata have $n$-extensive actions for every $n$. More generally, if $n$ divides $m$ and an ordered semiautomaton $Q$ has $n$-extensive actions, then $Q$ has $m$-extensive actions.

**Proposition 20.** Let $n$ be a positive integer. The class of all ordered semiautomata which have $n$-extensive actions form a positive $C_{lm}$-variety of ordered semiautomata. The corresponding positive $C_{lm}$-variety of languages consists of all regular languages closed under $n$-insertions.

**Proof.** The first part of the statement is easy to show. To establish the second part, let $L$ be a regular language over $A$ closed under $n$-insertions. For $u \in A^*$, we consider the state $K = u^{-1}L$ in the canonical ordered semiautomaton of $L$. Now we show that for every $v \in A^*$ such that $|v| = n$, we have $K \subseteq v^{-1}K$. Indeed, if $w \in K = u^{-1}L$ then $uw \in L$ and since $L$ is closed under $n$-insertions we get $uvw \in L$. Hence $vw \in K = u^{-1}L$, which implies $w \in v^{-1}K$. Therefore the canonical ordered semiautomaton of $L$ has $n$-extensive actions.

On contrary, let $L$ be recognized by $Q = (Q, A, \cdot, \leq, i, F)$ with $n$-extensive actions. For every $u, v, w \in A^*$ such that $uw \in L$ and $|v| = n$, we can consider the state $q = i \cdot u$ in $Q$. Since $Q$ has $n$-extensive actions we have $q \cdot v \geq q$. Hence $i \cdot uvw = q \cdot vw \geq q \cdot w = i \cdot uw \in F$ and we can conclude that $uvw \in L$. Thus $L$ is closed under $n$-insertions. \qed

For a fixed $n$, it is decidable in polynomial time whether a given ordered semiautomaton has $n$-extensive actions, because the relation $q \leq q \cdot u$ has to be checked only for polynomially many words $u$.

### 9 Membership Problem for $C$-Varieties of Semiautomata

In the previous section, the membership problem for (positive) $C$-varieties of semiautomata was always solved by an ad hoc argument. Here we discuss whether it is possible to give a general result in this direction. For that purpose, recall that $\omega$-identity is a pair of $\omega$-terms, which are constructed from variables by (repeated) successive application of concatenation and the unary operation $u \mapsto u^{\omega}$. In a particular monoid, the interpretation of this unary operation assigns to each element $s$ its uniquely determined power which is idempotent.

In the case of $C_{all}$ consisting of all homomorphisms, we mention Theorem 2.19 from [26] which states the following result: if the corresponding pseudovariety of monoids is defined by a finite set of $\omega$-identities then the membership problem of the corresponding variety of languages is decidable by a polynomial space algorithm in the size of the input automaton. Thus, Theorem 2.19 slightly extends the case when the pseudovariety of monoids is defined by a finite set of identities. The algorithm checks the defining $\omega$-identities in the syntactic monoid $M_L$ of a language $L$ and uses the basic fact that $M_L$ is the transition monoid of the minimal automaton of $L$. This extension is possible, because the unary operation $(\cdot)^{\omega}$ can be effectively computed from the input automaton.
We should mention that checking a fixed identity in an input semiautomaton can be done in a better way. Such a (NL) algorithm (a folklore algorithm in the theory) guesses a pair of finite sequences of states for two sides of a given identity $u = v$ which are visited during reading the word $u$ (and $v$ respectively) letter by letter. These sequences have the same first states and distinct last states. Then the algorithm checks whether for each variable, there is a transition of the automaton given by a word, which transforms all states in the sequence in the right way, when every occurrence of the variable is considered. If, for every used variable, there is such a word, we obtained a counterexample disproving the identity $u = v$.

Whichever algorithm is used, we can immediately get the generalization to the case of positive varieties of languages, because checking inequalities can be done in the same manner as checking identities. However, we want to use the mentioned algorithms to obtain a corresponding result for positive $\mathcal{C}$-varieties of ordered semiautomata for the categories used in this paper. For such a result we need the following formal definition. An $\omega$-inequality $u \leq v$ holds in an ordered semiautomaton $\mathcal{O} = (Q, A, \cdot, \leq)$ with respect to a category $\mathcal{C}$ if, for every $f \in \mathcal{C}(X^*, A^*)$ with $X$ being the set of variables occurring in $uv$, and for every $p \in Q$, we have $p \cdot f(u) \leq p \cdot f(v)$. Here $f(u)$ is equal to $f(u')$, where $u'$ is a word obtained from $u$ if all occurrences of $\omega$ are replaced by an exponent $n$ satisfying the equality $s^n = s^m$ in the transition monoid of $\mathcal{O}$ for its arbitrary element $s$.

**Theorem 9.1.** Let $\mathcal{O} = (Q, A, \cdot, \leq)$ be an ordered semiautomaton, let $u \leq v$ be an $\omega$-inequality and $\mathcal{C}$ be one of the categories $\mathcal{C}_{ne}, \mathcal{C}_l, \mathcal{C}_s$ and $\mathcal{C}_{lm}$. The problem whether $u \leq v$ holds in $\mathcal{O}$ with respect to $\mathcal{C}$ is decidable.

**Proof.** The result is a consequence of the following propositions.

**Proposition 2.** Let $\mathcal{O} = (Q, A, \cdot, \leq)$ be an ordered semiautomaton, $u \leq v$ be an $\omega$-inequality and $\mathcal{C}$ be one of the categories $\mathcal{C}_{ne}, \mathcal{C}_l$ and $\mathcal{C}_s$. The problem of deciding whether $u \leq v$ holds in $\mathcal{O}$ with respect to $\mathcal{C}$ can be solved by a polynomial space algorithm.

**Proof.** First of all, we prove the statement formally for $\mathcal{C} = \mathcal{C}_{all}$. We start with the case when $\omega$ operation is not used. It is mentioned in Section 9 that such an algorithm is a folklore in the theory.

Let $y_1 \ldots y_s \leq z_1 \ldots z_t$ be an inequality, where $y_1, \ldots, y_s, z_1, \ldots, z_t$ are variables from $X$. Recall, that this inequality holds in the transition ordered monoid of $\mathcal{A}$ if and only if for every homomorphism $f : X^* \rightarrow A^*$, the inequality of transformations $f(y_1) \circ \cdots \circ f(y_s) \leq f(z_1) \circ \cdots \circ f(z_t)$ is satisfied. This requirement can be reformulated as the inequality $q \cdot f(y_1) \cdots f(y_s) \leq q \cdot f(z_1) \cdots f(z_t)$ of states of $\mathcal{A}$, for every state $q \in Q$ and every homomorphism $f : X^* \rightarrow A^*$. This means that the inequality is not valid if and only if there exist such $f$ and states $p_0,p_1,\ldots,p_s,q_0,q_1,\ldots,q_t \in Q$, with $p_0 = q_0$ and $p_s \not\leq q_t$, which satisfy $p_{i-1} \cdot f(y_i) = p_i$ and $q_{j-1} \cdot f(z_j) = q_j$ for every $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$. Since the numbers $s$ and $t$ are constants, one can non-deterministically choose all
these states, and then decide whether for this choice of states the required homomorphism $f$ exists. For every variable $x$, denote by $I_x$ the set of all $i \in \{1, \ldots, s\}$ such that $y_i = x$, and by $J_x$ the set of all $j \in \{1, \ldots, t\}$ such that $z_j = x$. In order to decide existence of $f$, one has to check whether for every variable $x$ there exists a word $f(x) \in A^*$ such that $p_{i-1} \cdot f(x) = p_i$ for every $i \in I_x$ and $q_{j-1} \cdot f(x) = q_j$ for every $j \in J_x$. However, the existence of such a word $f(x)$ can be expressed as a condition on the product automaton of $|I_x| + |J_x|$ copies of the automaton $A$; namely, it is equivalent to reachability of the state with components $p_i$, for $i \in I_x$, and $q_j$, for $j \in J_x$, from the state with components $p_{i-1}$, for $i \in I_x$, and $q_{j-1}$, for $j \in J_x$. Recall that $|I_x| + |J_x|$ is a constant.

Now assume that $u$ and $v$ are $\omega$-terms. We are guessing the states as in the previous simple case, but we do this inductively with respect to the structure of the $\omega$-terms $u$ and $v$ from top to down. In this way we obtain a more complicated system of states comparing the sequences in the case of (linear) words. To explain the inductive construction, assume that we have guessed states $p$ and $q$ for a certain $\omega$-subterm $w$ assuming that $p \cdot f(w) = q$. If $w = w_1 w_2$ for $\omega$-terms $w_1$ and $w_2$, then we simply guess a state $r$ and assume that $p \cdot f(w_1) = r$ and $r \cdot f(w_2) = p$. The case $w = z^\omega$, with a subterm $z$, is more complicated. It is well known that, for every element $s$ in the transition monoid of the given automaton, the element $s^\omega$ is equal to $s^n$ for some $n \leq |Q|$. In particular, $s^n \cdot s^n = s^n$ holds for this $n$. So, we guess $n \leq |Q|$ and states $r_0, r_1, r_2, \ldots, r_{2n}$ such that $r_0 = p$, $r_n = r_{2n} = q$ and we assume that $r_{i-1} \cdot f(w) = r_i$ for every $i = 1, \ldots, 2n$. In this way, when we decompose all subterms, we obtain a system of states equipped with assumptions of the form $p \cdot f(x) = q$, where $p$ and $q$ are states and $x$ is a variable. Since the $\omega$-terms $u$ and $v$ are not part of the input, there are only constantly many steps of the algorithm decomposing the terms $u$ and $v$. Thus, at the end, the number of conditions is polynomial with respect the size of the input automaton. (In fact, the number of the conditions can be bounded by the number of all states in $Q$, which is linear.) The final part of the algorithm is the same: we just check, for each variable $x$, whether it is possible to satisfy all the conditions concerning $f(x)$ at the same time. Point out, that the number of conditions was constant in the case of identity $u = v$ in the first part, which gives log space algorithm in the that case.

Now we are ready to discuss another categories, where we search for $f \in \mathcal{C}(X^*, A^*)$. The case $\mathcal{C} = \mathcal{C}_{se}$ is trivial. When we test reachability in the product of certain number of copies of $\mathcal{O}$, we are looking for a non-empty path in the graph. The case $\mathcal{C} = \mathcal{C}_t$ is even easier, because we test reachability in one step. Seeing this case from another point of view, this case is easy, because there are only polynomially many homomorphisms in $\mathcal{C}_t(X^*, A^*)$ for fixed $X$ and $A$ where only $A$ is a part of the input. The case $\mathcal{C} = \mathcal{C}_s$ is also easy. We are looking for $f \in \mathcal{C}(X^*, A^*)$ such that $A \subseteq f(X)$. So, we can additionally guess, for each letter $a \in A$, a variable $x \in X$ such that $f(x) = a$.

We could conclude with the remark, that is well known that nondeterministic polynomial space is equivalent to deterministic polynomial space. $\Box$
Proposition 3. Let $\mathcal{O} = (Q, A, \cdot, \leq)$ be an ordered semiautomaton, $u \leq v$ be an $\omega$-inequality. The problem whether $u \leq v$ holds in $\mathcal{O}$ with respect to $C_{lm}$ is decidable.

Proof. We proceed as in the general case up to the place where the existence of $f(x)$ is discussed. We do not decide whether there is $f(x) \in A^\ast$ satisfying all conditions before we first complete the conditions in such a way that, for every $q \in Q$, the condition on $q \cdot f(x)$ is present. This is made by guessing missing pairs $q \cdot f(x)$ for all $q$ and $x$. Just now we test whether there are words $f(x)$ satisfying the conditions.

Only if there are such words, we continue. Next we try to describe all of them. It is possible, because, for every $x$, we know how $f(x)$ transform the semiautomaton $\mathcal{O}$. So, the language of all words which are considered as a potential words $f(x)$ is a regular language which is recognized by the transition monoid of the semiautomaton $\mathcal{O}$. We denote it as $L_x$. Furthermore, we are able to compute a regular expression $r_x$ describing $L_x$. We need to decide whether for each variable $x$ there is a word $w_x \in L_x$ such that all words’s $w_x$ have the same length. Thus, we need to know all possible lengths of words in $L_x$. For this purpose we consider the unique literal mapping $\psi : A^\ast \rightarrow \{a\}^\ast$, $\psi(A) = \{a\}$. Clearly, the language $\psi(L_x)$ is regular, because it is described by a regular expression $r_\psi$, which can be obtained from $r_x$, if we replace every letter from the alphabet $A$ by the letter $a$. Moreover, there is a word $w_x \in L_x$ of length $k$ if and only if $a^k \in \psi(L_x)$. The existence of an integer $k$ such that $L_x \cap A^k \neq \emptyset$ holds for every $x$, is equivalent to the fact $\bigcap_{x \in X} \psi(L_x) \neq \emptyset$. The later inequality is equivalent to non-emptiness of the language given by the generalized regular expression $\bigcap_{x \in X} \overline{r_x}$. So, one can deciding this question.

We did not discuss the complexity of the algorithm, because we do not see how to effectively construct the regular expression $\bigcap_{x \in X} \overline{r_x}$.

10 Further Remarks

At the end we could mention that one can extend the construction in at least two natural directions. First, the theory of tree languages is a field where many fundamental ideas from the theory of deterministic automata were successfully generalized. Another recent notion of biautomata (see [10] and [9]) is based on considering both-sided quotients instead of left quotients only. In both cases one can try to apply the previous constructions and consider varieties of (semi)automata. Some papers in this direction already exist [7].

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