CONTRACTIONS, HOPF ALGEBRA EXTENSIONS AND COVARIANT DIFFERENTIAL CALCULUS

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Abstract

We re-examine all the contractions related with the $\mathcal{U}_q(\mathfrak{su}(2))$ deformed algebra and study the consequences that the contraction process has for their structure. We also show using $\mathcal{U}_q(\mathfrak{su}(2)) \times \mathcal{U}(\mathfrak{u}(1))$ as an example that, as in the undeformed case, the contraction may generate Hopf algebra cohomology. We shall show that most of the different Hopf algebra deformations obtained have a bicrossproduct or a cocycle bicrossproduct structure, for which we shall also give their dual ‘group’ versions. The bicovariant differential calculi on the deformed spaces associated with the contracted algebras and the requirements for their existence are examined as well.

1. Introduction

As is well known, the standard Wigner-Înönü contraction [1] of simple Lie algebras with respect to a subalgebra leads to algebras which are the semidirect product of the preserved subalgebra and the resulting Abelian complement. Other types of contractions involving powers of the contraction parameter, first discussed in [2], may lead to a central extension structure. Due to the singular nature of the contraction process, (non-simple) groups/algebras which are the direct product/sum of two groups/algebras may not retain this direct product structure after the contraction limit if the contraction affects suitably the central trivial extension; one may refer to these groups as being pseudoextended [3,4] when the extension is trivial but behaves non-trivially under the contraction. A well known example is the direct product $P \times U(1)$, $P$ being the Poincaré group, for which a suitable limit leads to the centrally extended Galilei group [5,1,3,4].

One of the interests of non-commutative geometry is to provide a rationale for possible deformations of the spacetime manifold, which becomes a non-commutative
algebra. By extending standard Lie group arguments about quotient spaces, it is natural to associate these spacetime deformations with the deformation of inhomogeneous groups, which are non-simple. Since the standard deformation procedure [6, 7, 8] applies to the simple algebra/group case, the contraction of deformed simple algebras suggests itself as a possible way of obtaining deformed inhomogeneous algebras. This process usually requires involving the deformation parameter $q$ into the contraction [9, 10], and is rather complicated; in fact, the contraction of deformed algebras/groups is besieged by the appearance of divergences (the contraction is not always possible or the $R$-matrix diverges), and a complete theory is still lacking. Clearly, the difficulty lies in having a well defined contraction process in both the algebra and coalgebra sectors.

Contraction is not, however, the only way of finding deformations of inhomogeneous groups. Much in the same way we may construct Lie groups out of two by solving the corresponding group extension problem (which always has a solution for Abelian kernel, precisely the semidirect extension, see e.g. [4]), we may look for a similar direct construction for Hopf algebras without thinking of obtaining them by contraction. Such a construction already exists for certain cases, and leads to the bicrossproduct and cocycle bicrossproduct structure of Hopf algebras of Majid [11,12] (see also [13,14,15]; a summary of Majid’s theory is given in Appendix B). For instance, the $\kappa$-Poincaré algebra of Lukierski et al. [16], which is obtained from $U_q(so(2,3))$ by a contraction involving the deformation parameter $q$ written as $q = \exp 1/\kappa R$, where $R$ is the (de Sitter radius) contraction parameter so that $[\kappa] = L^{-1}$, has been shown [17] to possess such a bicrossproduct structure. In this paper we intend to re-examine in this new light the simplest contraction examples, including the earliest ones [10, 9] (several of them discussed from various points of view in [18,19,20,21,22,23,24,25,26]). We shall also look at the notion of central extension pseudocohomology for Hopf algebras, and find that the contraction process generates Hopf algebra extension cohomology as it does for its undeformed Lie counterpart. We shall discuss both the ‘algebra’ and ‘group’ aspects of the deformed Hopf algebras, and study whether they lead to a bicrossproduct or cocycle bicrossproduct structure.

In contrast, the problems associated with the contraction and the $R$-matrix behaviour will not be discussed here. In fact, the constructions presented in sec. 5 may be considered as a way of avoiding the search for an $R$-matrix. It would be interesting to perform a more general analysis of the consequences of the contraction process for the structure of the resulting deformed Hopf algebras. We hope to report on this elsewhere [27].

The analysis of the differential calculus on the ‘spaces’ associated with the inhomogeneous deformed groups is also of importance; this has been recently made for $\kappa$-spacetime algebras in [28, 29]. It was shown there that the demand of covariance for the differential calculus required to enlarge the spacetime algebra by an element related to a central extension of the Hopf algebra; this phenomenon will also appear
here for certain cases (sec. 6).

2. Contractions of \( U_q(su(2)) \)

The well known \( U_q(su(2)) \) deformed Hopf algebra is defined by \( (q = e^z) \)

\[
[J_3, J_1] = J_2, \quad [J_3, J_2] = -J_1, \quad [J_1, J_2] = \frac{1}{2}[2J_3]_q = \frac{\sinh(2zJ_3)}{2\sinh(z)}; \\
\Delta J_{1,2} = \exp(-zJ_3) \otimes J_{1,2} + J_{1,2} \otimes \exp(zJ_3), \quad \Delta J_3 = J_3 \otimes 1 + 1 \otimes J_3; \\
S(J_{1,2}) = -\exp(zJ_3) J_{1,2} \exp(-zJ_3), \quad S(J_3) = -J_3; \quad \epsilon(J_{1,2,3}) = 0. 
\]

Let us consider the different contractions of \( U_q(su(2)) \).

1. \( U_q(\mathcal{E}(2)) \). The standard contraction procedure with respect the Hopf subalgebra generated by \( J_3 \), implying the redefinitions \( J_1 = e^{-1}P_1, \; J_2 = e^{-1}P_2, \; J_3 = J \), leads ([26]; see also [19]) to

\[
[J, P_1] = P_2, \quad [J, P_2] = -P_1, \quad [P_1, P_2] = 0; \\
\Delta P_{1,2} = \exp(-zJ) \otimes P_{1,2} + P_{1,2} \otimes \exp(zJ), \quad \Delta J = J \otimes 1 + 1 \otimes J; \\
S(P_{1,2}) = -\exp(zJ) P_{1,2} \exp(-zJ), \quad S(J) = -J; \quad \epsilon(J, P_{1,2}) = 0. 
\]

This deformation of the Euclidean algebra is a Hopf algebra where the deformation only appears at the coalgebra level, and will be denoted \( U_q(\mathcal{E}(2)) \).

2. \( U_\omega(\mathcal{E}(2)) \). A second contraction, leading to another deformation of the Euclidean algebra, may be performed. This contraction [9,10] requires writing previously \( q = \exp(\epsilon/2) \) since it is not performed with respect to a Hopf subalgebra*: it is performed with respect \( J_2 \), which is a Hopf subalgebra only for \( q = 1 \). The redefinitions \( J_1 = e^{-1}P_2, \; J_2 = J, \; J_3 = e^{-1}P_1, \; z = \epsilon/2 \) in (2.1) lead to [9,10] the \( U_\omega(\mathcal{E}(2)) \) Euclidean Hopf algebra†

\[
[P_1, P_2] = 0, \quad [J, P_1] = P_2, \quad [J, P_2] = -\frac{\sinh(\omega P_1)}{\omega}; \\
\Delta P_1 = P_1 \otimes 1 + 1 \otimes P_1, \quad \Delta P_2 = \exp(-\omega P_1/2) \otimes P_2 + P_2 \otimes \exp(\omega P_1/2), \\
\Delta J = \exp(-\omega P_1/2) \otimes J + J \otimes \exp(\omega P_1/2); \quad S(P_{1,2}) = -P_{1,2}, \\
S(J) = -\exp(\omega P_1/2) J \exp(-\omega P_1/2) = -J + \frac{\omega}{2} P_2; \quad \epsilon(J, P_{1,2}) = 0. 
\]

Besides the above, we may consider two ‘non-standard’ contractions (i.e. involving

* It may be worth mentioning that contracting with respect a Hopf subalgebra (as in the case (1) above) is not a sufficient condition to define a contraction without involving the deformation parameter in it. The above is, in fact, a rather exceptional case.
† If we want to look at \( P_1, P_2 \) as deformed translation generators, \( [P_i] = L^{-1} \), it is sufficient to take \( [\epsilon] = L^{-1}, \; [\omega] = L \).
higher powers of the contraction parameter \( \epsilon \). They are obtained by extending to the deformed case the generalized contraction in [2].

(3) \( \mathcal{U}_G(\mathcal{G}(1+1)) \). A third contraction leads to a deformation of the Galilei algebra (the \((1+1)\) version of the \((1+3)\) deformed Galilei algebra in [30]). We make the redefinitions† \( z = \frac{\hat{\omega}}{\sigma} \), \( J_1 = \sigma^{-1} \epsilon^{-1} \tilde{V} \), \( J_2 = -\epsilon^{-2} \hat{X} \), \( J_3 = \epsilon^{-1} \sigma \hat{X}_t \). By taking the limit \( \epsilon \to 0 \), we get

\[
[X_t, \tilde{V}] = -\hat{X} \quad [X_t, \hat{X}] = 0 \quad [\hat{X}, \tilde{V}] = 0 ;
\]

\[
\Delta \hat{X}_t = \hat{X}_t \otimes 1 + 1 \otimes \hat{X}_t \quad \Delta \hat{X} = \exp(-\hat{\omega} \hat{X}_t) \otimes \hat{X} + \hat{X} \otimes \exp(\hat{\omega} \hat{X}_t) ,
\]

\[
\Delta \tilde{V} = \exp(-\hat{\omega} \hat{X}_t) \otimes \tilde{V} + \tilde{V} \otimes \exp(\hat{\omega} \hat{X}_t) ;
\]

\[
S(\hat{X}_t) = -\hat{X}_t , \quad S(\hat{X}) = -\hat{X} \quad S(\tilde{V}) = -\exp(\hat{\omega} \hat{X}_t) \tilde{V} \exp(-\hat{\omega} \hat{X}_t) = -\tilde{V} + \hat{\omega} \hat{X} ,
\]

\[
\epsilon(\hat{X}_t, \hat{X}, \tilde{V}) = 0 .
\]

We will denote this deformed Galilei algebra by \( \mathcal{U}_G(\mathcal{G}(1+1)) \). In the \( \hat{\omega} \to 0 \) limit, eq. (2.4) gives the Hopf structure of the enveloping algebra \( \mathcal{U}_G(\mathcal{G}(1+1)) \) of the Galilei Lie algebra.

(4) \( \mathcal{U}_G(HW) \). Finally, there is another contraction of \( \mathcal{U}_q(su(2)) \). It is obtained by making in (2.1) the redefinitions \( J_1 = \epsilon^{-1} \hat{X}_q \), \( J_2 = \epsilon^{-1} \hat{X}_p \), \( J_3 = \epsilon^{-2} \Xi \) and \( z = \hat{\omega} \epsilon^2 / 2 \). The result is the \( \hat{\omega} \)-deformed Heisenberg-Weyl \( \mathcal{U}_G(HW) \) Hopf algebra

\[
[\Xi, \hat{X}_q] = 0 \quad [\Xi, \hat{X}_p] = 0 \quad [\hat{X}_q, \hat{X}_p] = \frac{\sinh(\hat{\omega} \Xi)}{\hat{\omega}} ;
\]

\[
\Delta \hat{X}_{q,p} = \exp(-\hat{\omega} \Xi / 2) \otimes \hat{X}_{q,p} + \hat{X}_{q,p} \otimes \exp(\hat{\omega} \Xi / 2) ;
\]

\[
S(\hat{X}_{q,p}) = -\hat{X}_{q,p} \quad S(\Xi) = -\Xi \quad \epsilon(\hat{X}_{q,p}, \Xi) = 0,
\]

(2.5)

(denoted Heisenberg quantum group \( H(1)_q \) in [9,10]). By making the change of basis \( X_{q,p} = \exp(-\hat{\omega} \Xi / 2) \hat{X}_{q,p} \) the \( \mathcal{U}_G(HW) \) algebra takes the form

\[
[\Xi, X_{p,q}] = 0 \quad [X_q, X_p] = \frac{1 - \exp(-2\hat{\omega} \Xi)}{2\hat{\omega}} ;
\]

\[
\Delta X_{q,p} = X_{q,p} \otimes 1 + \exp(-\hat{\omega} \Xi) \otimes X_{q,p} ;
\]

\[
\Delta \Xi = 1 \otimes \Xi + \Xi \otimes 1 \quad S(X_{q,p}) = -X_{q,p} \exp(\hat{\omega} \Xi) \quad \epsilon(X_{q,p}, \Xi) = 0 ;
\]

(2.6)

in the undeformed limit \( \hat{\omega} \to 0 \), the standard expressions for the Hopf structure of \( \mathcal{U}(HW) \) are recovered.

† The parameter \( \sigma, [\sigma] = TL^{-1/2} \), is introduced to give standard dimensions to the generators of the Galilei algebra ([\( e \] = \( L^{-1/2} \), [\( \hat{\omega} \] = \( T \)), but it disappears after the contraction.

‡ If one wishes to have \( q \) and \( p \) with dimensions of length and momentum (and \( [X_q] = L^{-1}, [X_p] = \epsilon^{-1} \lambda \Xi \), \( J_3 = \epsilon^{-2} \lambda \Xi \), \( z = \hat{\omega} \epsilon^2 / 2 \lambda \), with \( [e] = L^{-1}, [\lambda] = \text{[momentum]}L^{-1} \), \( [\hat{\omega}] = \text{[action]} \), \( [\Xi] = \text{[action]}^{-1} \); \( \lambda \) disappears in the final expressions (2.5).
3. Structure of the $U_q(su(2))$ contractions

As mentioned, the bicrossproduct [11,12] of Hopf algebras (see Appendix B) may be used as an alternative construction of deformed Hopf algebras when the undeformed ones are not simple. Non-simple algebras may arise from contraction, a process which for ordinary Lie algebras leads to a semidirect product algebra. Thus, it is worth exploring whether the above deformed Hopf algebras are the (right-left) bicrossproduct $H △ ▽ A$ of two Hopf algebras $H$ and $A$ or have a cocycle bicrossproduct structure. The notation $H △ ▽ A$, for instance, indicates that $A$ is a right $H$-module algebra for the right action $\alpha : A \otimes H \to A$, $\alpha(a,h) \equiv a \triangleright h$, and that $H$ is a left $A$-comodule coalgebra for the left coaction $\beta : H \to A \otimes H$ ($H$ is a left quantum space); $\alpha$ and $\beta$ must also satisfy certain compatibility conditions [11,12].

(1) Let us first consider $U_q(\mathcal{E}(2))$, eqs. (2.2). At the algebra level it has a semidirect structure. However, if we take $A$ as the undeformed Hopf algebra generated by $P_1$, $P_2$ and $H$ as that generated by $J$ we see that with independence of $\beta$, we cannot reproduce $\Delta(P_1,J)$ in $U_q(\mathcal{E}(2))$; in fact, $P_1$, $P_2$ in (2.2) do not generate a Hopf subalgebra of $U_q(\mathcal{E}(2))$. Thus, $U_q(\mathcal{E}(2))$ has not a bicrossproduct structure.

(2) Let us now look at $U_\omega(\mathcal{E}(2))$, eqs. (2.3). The redefinitions

$$P_x = P_1 \quad , \quad P_y = \exp(-\omega P_1/2)P_2 \quad , \quad J' = \exp(-\omega P_1/2)J \quad ,$$

allow us to write $U_\omega(\mathcal{E}(2))$ in terms of $(P_x, P_y, J')$ in the form

$$[P_x, P_y] = 0 \quad , \quad [J', P_x] = P_y \quad , \quad [J', P_y] = -\frac{1}{2\omega}(1 - \exp(-2\omega P_x)) - \frac{\omega}{2}P_y^2 \ ;$$

$$\Delta P_x = P_x \otimes 1 + 1 \otimes P_x \quad , \quad \Delta P_y = P_y \otimes 1 + \exp(-\omega P_x) \otimes P_y \ ;$$

$$\Delta J' = J' \otimes 1 + \exp(-\omega P_x) \otimes J' \quad ; \quad \epsilon(J', P_{x,y}) = 0 \ ;$$

$$S(P_x) = -P_x \quad , \quad S(P_y) = -\exp(\omega P_x)P_y \quad , \quad S(J') = -\exp(\omega P_x)J' \ .$$

(3.2)

If we now take for $A$ the commutative non-cocommutative Hopf translation subalgebra $U_\omega(\mathcal{T r}(2))$ of $(P_x, P_y)$ contained in (3.2) and $H$ is the commutative and cocommutative algebra generated by $J'$, the bicrossproduct structure $H △ ▽ A$ of (3.2) is exhibited if

$$\alpha(P_{x,y}, J') \equiv P_{x,y} \triangleright J' := [P_{x,y}, J'] \quad , \quad \beta(J') := \exp(-\omega P_x) \otimes J' \quad ,$$

since it may be seen that the compatibility axioms (B.10), (B.11), (B.12), (B.13) and (B.14) are satisfied and that (B.16), (B.17), (B.18), define the coproducts, antipodes and counits in (3.2). This shows that $U_\omega(\mathcal{E}(2)) = U(u(1)) △ ▽ U_\omega(\mathcal{T r}(2))$. 

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¶ The formulae (B.-) refer to the corresponding (B'.-) ones given in Appendix B; they may be found in the original papers [11] or in the Appendix of [29] (there with the same numbering).
Consider now the deformed \((1 + 1)\) Galilei Hopf algebra \(\mathcal{U}_G(G(1 + 1))\) of (2.4). It was found in [29] (for the \((1 + 3)\) case) that it is also endowed with a bicrossproduct structure. To show this, we make the redefinitions

\[ X_t = \tilde{X}_t \quad , \quad X = \exp(-\tilde{\omega}\tilde{X}_t)\tilde{X} \quad , \quad V = \exp(-\tilde{\omega}\tilde{X}_t)\tilde{V} \quad . \]  

With them, the \(\mathcal{U}_G(G(1 + 1))\) Hopf algebra takes the form

\[
\begin{align*}
[X_t, V] &= -X \quad , \quad [X, V] = \tilde{\omega}X^2 \quad , \quad [X, X_t] = 0 \quad ; \\
\Delta X_t &= X_t \otimes 1 + 1 \otimes X_t \quad , \quad \Delta X = X \otimes 1 + \exp(-2\tilde{\omega}X_t) \otimes X \quad , \\
\Delta V &= V \otimes 1 + \exp(-2\tilde{\omega}X_t) \otimes V \quad ; \quad \epsilon(V, X, X_t) = 0 \quad ; \\
S(X_t) &= -X_t \quad , \quad S(X) = -\exp(2\tilde{\omega}X_t)X \quad , \quad S(V) = -\exp(2\tilde{\omega}X_t)V \quad ,
\end{align*}
\]  

(which is eq. (6.1) in [29] for \(\mathcal{G}_R\) with \(1/2\kappa = \tilde{\omega}\)). The bicrossproduct structure is summarized in the definitions of the action \(\alpha\) and the coaction \(\beta\) (\(\mathcal{A}\) is the Abelian, non-cocommutative Hopf subalgebra \(\mathcal{U}_G(Tr(2))\) generated by \(X\) and \(X_t\), and \(\mathcal{H}\) is given by the commutative and cocommutative Hopf algebra generated by \(V\))

\[
\begin{align*}
\alpha(X, V) &\equiv X \triangleright V := [X, V] = \tilde{\omega}X^2 \quad , \quad \alpha(X_t, V) &\equiv X_t \triangleright V := [X_t, V] = -X \quad , \\
\beta(V) &:= \exp(-2\tilde{\omega}X_t) \otimes V \quad .
\end{align*}
\]  

It may be shown that the bicrossproduct conditions are verified and hence that \(\mathcal{U}_G(G(1 + 1)) = \mathcal{U}(u(1)) \triangleright \mathcal{U}_G(Tr(2))\).*

Finally, we now show that \(\mathcal{U}_G(HW)\) [(2.6)] has both a bicrossproduct and a cocycle bicrossproduct structure. This parallels the fact that the Heisenberg-Weyl (HW) Lie group, \((q', p', \theta')(q, p, \theta) = (q + q', p + p', \theta + \theta + q'p)\), may be considered as the semidirect extension of \(\mathbb{R}\) (coordinate \(q\)) by the invariant subgroup \(\mathbb{R}^2\) (\(id. p, \theta\)) (the action of \(\mathbb{R}\) on \(\mathbb{R}^2\) being given by \(q : (p, \theta) \mapsto (p, qp + \theta)\)), or as a central extension of \(\mathbb{R}^2\) (coordinates \(p, q\)) by \(\mathbb{R}\) (\(id. \theta\)) (the \(\mathbb{R}\)-valued two-cocycle being given in its ‘asymmetric’ form \(\xi(p', q'; p, q) = q'p\)).

4a) The bicrossproduct structure follows taking for \(\mathcal{A}\) the Abelian \(\tilde{\omega}\)-deformed Hopf subalgebra generated by \(X_q\) and \(\Xi\) in (2.6), for \(\mathcal{H}\) the undeformed algebra generated by \(X_p\) (\(\Delta X_p = X_p \otimes 1 + 1 \otimes X_p\), \(S(X_p) = -X_p\), \(\epsilon(X_p) = 0\)) and for \(\alpha\) and \(\beta\)

\[
X_q \triangleright X_p = [X_q, X_p] \quad , \quad \Xi \triangleright X_p = 0 \quad ; \quad \beta(X_p) = \exp(-\tilde{\omega}\Xi) \otimes X_p \quad .
\]  

This induces the appropriate coproduct for \(X_p\) (identified as \(X_p \otimes 1\) in \(\mathcal{H} \otimes \mathcal{A}\)) and antipode (eq. (2.6)) from (B.16) and (B.18), respectively; the commutators in (2.6) follow from (B.15).

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* Note that the expressions (2.4), (3.5) and (3.6) may be obtained from standard contraction of their analogous ones in the Euclidean case (2.3), (3.2) and (3.3).
4b) The **cocycle bicrossproduct structure** is constructed from the **undeformed** Hopf algebras \( A = U(\Xi) \) (generated by \( \Xi \)) and \( H = (Tr(2)) \) (\( id. \ X_q, X_p \)). Since we wish to obtain a deformation of a central extension algebra the action \( \alpha \) must be trivial. We take

\[
\alpha(\Xi, X_{q,p}) = [\Xi, X_{q,p}] = 0 \quad , \quad \beta(X_{q,p}) = \exp(-\hat{\omega}\Xi) \otimes X_{q,p} \quad (3.8)
\]

and \( \xi : H \otimes H \to A \) (antisymmetric cocycle), \( \psi : H \to A \otimes A \) given by

\[
\xi(X_q, X_p) = -\xi(X_p, X_q) = \frac{1}{2} \exp(-\hat{\omega}\Xi) \frac{\sinh(\hat{\omega}\Xi)}{\hat{\omega}} \quad , \quad \psi(X_{q,p}) = 0 \quad (3.9)
\]

(\( i.e. \ \psi \) trivial, \( \psi(h) = 1 \otimes 1 \epsilon(h) \)) plus (B.19) and (B.28). Thus, the deformed character of the resulting algebra (and hence \( \hat{\omega} \)) enters in this case through \( \beta \) and \( \xi \) only. Since \( \alpha \) and \( \psi \) are trivial and \( A \) and \( H \) have the cocommutative Hopf algebra structure associated with the Abelian enveloping algebras \( U(\Xi), U(Tr(2)) \), it is not difficult to check that the compatibility conditions (B.14), (B.25), (B.26) and (B.27) are fulfilled. Moreover, (B.29) and (B.30) reduce using (B.19) to

\[
(h \otimes a)(g \otimes b) = hg \otimes ab + 1 \otimes \xi(h, g)ab \quad , \quad h, g \in H \quad (3.10)
\]

\[
\Delta(h \otimes 1) = h \otimes 1 \otimes 1 \otimes 1 + 1 \otimes h^{(1)} \otimes h^{(2)} \otimes 1 \quad , \quad h, g \in H \quad . \quad (3.11)
\]

Denoting the elements \( (h \otimes 1) \) and \( (g \otimes 1) \) in \( H \otimes A \) by \( \tilde{h} \) and \( \tilde{g} \), eq. (3.10) leads to

\[
[h, g] = 2\xi(h, g) \quad ([h, g] = 0 \text{ in } H) \quad \text{so that the commutators in (2.6) are recovered for} \quad \xi \text{ given by (3.9). Similarly,}
\]

\[
\Delta(X_{q,p}) = X_{q,p} \otimes 1 \otimes 1 \otimes 1 + 1 \otimes \exp(-\hat{\omega}\Xi) \otimes X_{q,p} \otimes 1 \equiv X_{q,p} \otimes 1 + \exp(-\hat{\omega}\Xi) \otimes X_{q,p} \quad (3.12)
\]

plus \( \Delta(\Xi) = \Xi \otimes 1 + 1 \otimes \Xi \). With \( \epsilon(X_{q,p}) = 0 = \epsilon(\Xi) \), the Hopf algebra structure of \( U_{\hat{\omega}}(HW) \) is obtained by adding the antipode as defined by (2.6).
4. The cocycle extended Euclidean Hopf algebras $\mathcal{U}_\rho(\tilde{E}(2))$, $\mathcal{U}_\rho(\bar{E}(2))$

Consider $\mathcal{U}_\omega(\mathcal{E}(2)) \times \mathcal{U}(u(1))$. This Hopf algebra has a trivial central factor and, as such, it might have been obtained from $\mathcal{U}_q(su(2)) \times \mathcal{U}(u(1))$ by contraction, since the redefinitions given in sec. 2 (2) (and (3.1)) do not affect the $\mathcal{U}(u(1))$ part ($\Xi$). However a generalization of the pseudocohomology mechanism [3,4] mentioned in the introduction may also be used here to obtain non-trivial extensions of Hopf algebras by contracting trivial products (see [29] for the case of the deformed extended (1+3) Galilei Hopf algebra). We now find two deformations of the centrally extended Euclidean algebra using this procedure.

(a) $\mathcal{U}_\rho(\tilde{E}(2))$. Consider the $\mathcal{U}_q(su(2)) \times \mathcal{U}(u(1))$ Hopf algebra generated by $(J_1, J_2, J_3, \Xi')$ given by eqs. (2.1) plus the $\mathcal{U}(u(1))$ relations

$$\Delta \Xi' = \Xi' \otimes 1 + 1 \otimes \Xi' \text{ , } S(\Xi') = -\Xi' \text{ , } \epsilon(\Xi') = 0 \text{ ; } [\Xi', \text{all}] = 0 \text{ .}$$

(4.1)

The redefinition $J_3 = J_3' + \Xi'$ [9] leaves (2.1) and (4.1) unchanged but for

$$[J_1, J_2] = \frac{\sinh(2z(J_3' + \Xi'))}{2\sinh(z)} \text{ ; }$$

$$\Delta J_{1,2} = \exp(-z(J_3' + \Xi')) \otimes J_{1,2} + J_{1,2} \otimes \exp(z(J_3' + \Xi')) \text{ ,}$$

$$\Delta J_3' = J_3' \otimes 1 + 1 \otimes J_3' \; ;$$

$$S(J_{1,2}) = -\exp(z(J_3' + \Xi')) J_{1,2} \exp(-z(J_3' + \Xi')) \text{ , } S(J_3') = -J_3' \; .$$

(4.2)

Because $[J_1, J_2]$ involves $\Xi'$, we refer to $\mathcal{U}_q(su(2)) \times \mathcal{U}(u(1))$ in the form (4.2) as a pseudoextension (the trivial direct product structure is disguised beneath the election of the generators).

To obtain a non-trivial Hopf algebra extension from it, we now make a rescaling involving $\Xi'$,

$$J_1 = \epsilon^{-1} X_1 \text{ , } J_2 = \epsilon^{-1} X_2 \text{ , } J_3' = N \text{ , } \Xi' = \Xi / \epsilon^2 \; ,$$

(4.3)

redefine $z$ as $z = \rho \epsilon^2$ and take the limit $\epsilon \rightarrow 0$. The resulting Hopf algebra is given by

$$[N, X_1] = X_2 \text{ , } [N, X_2] = -X_1 \text{ , } [X_1, X_2] = \frac{\sinh 2\rho \Xi}{2\rho} \text{ , } [\Xi, \text{all}] = 0 \; ;$$

$$\Delta X_i = \exp(-\rho \Xi) \otimes X_i + X_i \otimes \exp(\rho \Xi) \; (i = 1, 2) \text{ , } \Delta N = N \otimes 1 + 1 \otimes N \; ;$$

$$S(X_{1,2}) = -X_{1,2} \text{ , } S(N) = -N \text{ , } S(\Xi) = -\Xi \text{ ; } \epsilon(X_{1,2}, N, \Xi) = 0 \; .$$

(4.4)

This Hopf algebra will be denoted by $\mathcal{U}_\rho(\tilde{E}(2))$. 

8
It is convenient to make in (4.4) the change
\[ Y_i = \exp(-\rho\Xi)X_i \quad . \]  
(4.5)

This modifies only
\[ [Y_1, Y_2] = \frac{1}{4\rho}(1 - \exp(-4\rho\Xi)) \quad , \]
\[ \Delta Y_i = \exp(-2\rho\Xi) \otimes Y_i + Y_i \otimes 1 \quad , \quad S(Y_i) = -\exp(2\rho\Xi)Y_i \quad , \]
(4.6)

which reproduces the Heisenberg-Weyl \( U_\rho(HW) \) Hopf algebra of (2.6) with \( 2\rho = \hat{\omega} \).

a1) \( U_\rho(\hat{\mathcal{E}}(2)) \) has the bicrossproduct structure \( \mathcal{H} \triangleright \triangleleft \mathcal{A} \), in which \( \mathcal{A} \) is the deformed Heisenberg-Weyl \( U_\rho(HW) \) Hopf subalgebra in \( U_\rho(\hat{\mathcal{E}}(2)) \) generated by \( (Y_1, Y_2, \Xi) \) with primitive coproduct for \( \Xi \) and \( \Delta(Y_i) \) given in (4.6), and \( \mathcal{H} \) is the commutative and cocommutative algebra generated by \( N \). The right action \( \triangleright \) of \( N \) on \( \mathcal{A} \) is then designed to reproduce the commutators in \( U_\rho(\hat{\mathcal{E}}(2)) \)
\[ \alpha(Y_1, N) = [Y_1, N] = -Y_2 \quad , \quad \alpha(Y_2, N) = [Y_2, N] = Y_1 \quad , \quad \Xi \triangleright N = 0 \quad , \]
(4.7)

and the coaction \( \beta \) is taken to be trivial, \( \beta(N) = 1 \otimes N \), since the coproducts in both \( \mathcal{H} \) and \( \mathcal{A} \) are already those in \( U_\rho(\hat{\mathcal{E}}(2)) \).

a2) The cocycle extension structure of \( U_\rho(\hat{\mathcal{E}}(2)) \) is achieved by taking \( \mathcal{A} \) generated by \( \Xi \) [eq. (4.1)] and \( \mathcal{H} \) as the undeformed Euclidean algebra \( U(\mathcal{E}(2)) \). The action \( \alpha \) of \( \mathcal{H} \) on \( \mathcal{A} \) is trivial (we want \( \Xi \) to be central), and so is the map \( \psi \) ((B.22), (B.23)); the antisymmetric cocycle \( \xi \) and coaction \( \beta \) are given by (cf. (4.6))
\[ \xi(Y_1, Y_2) = \frac{1}{8\rho}(1 - \exp(-4\rho\Xi)) \quad , \quad \beta(Y_i) = \exp(-2\rho\Xi) \otimes Y_i \quad , \quad \beta(N) = 1 \otimes N \quad (4.8) \]

(the coaction on \( N \) is trivial). We may check that all relations (B.19)-(B.26), (B.27)*, (B.28) are fulfilled and that (B.29)-(B.30) then reproduce (4.6); thus, \( U_\rho(\hat{\mathcal{E}}(2)) \) has a cocycle bicrossproduct structure.

* Since \( \psi \) and \( \triangleright \) are trivial, this formula reduces to \( \Delta \xi(h \otimes g) = \xi(h_{(1)} \otimes g_{(1)})h_{(2)}^{(1)} \otimes \xi(h_{(2)} \otimes g_{(2)}) \) .
(b) \( \mathcal{U}_\rho(\tilde{\mathcal{E}}(2)) \). Consider again the algebra \( \mathcal{U}_q(su(2)) \times \mathcal{U}(u(1)) \) given by eqs. (2.1) plus the relations (4.1) for the central \( u(1) \) generator, now denoted \( \hat{\Xi}' \). The redefinition \( J_1 = J'_1 + \hat{\Xi}' \) leaves (4.1) and (2.1) unchanged but for

\[
\begin{align*}
[J_3, J_2] &= -(J'_1 + \hat{\Xi}') , \\
\Delta J'_1 &= \exp(-zJ_3) \otimes J'_1 + J'_1 \otimes \exp(zJ_3) \\
&+ (\exp(-zJ_3) - 1) \otimes \hat{\Xi}' + \hat{\Xi}' \otimes (\exp(zJ_3) - 1) .
\end{align*}
\] (4.9)

If we now make the rescaling

\[
\begin{align*}
J_3 &= \epsilon^{-1}P_x ,  \\
J_2 &= -\epsilon^{-1}P_y ,  \\
J'_1 &= J' ,  \\
\hat{\Xi}' &= \epsilon^{-2}\hat{\Xi} ,
\end{align*}
\] (4.10)

and set \( z = \epsilon^3 \hat{\rho} \), in the limit \( \epsilon \to 0 \) we obtain the Hopf algebra \( \mathcal{U}_\rho(\tilde{\mathcal{E}}(2)) \) given by

\[
\begin{align*}
[J', P_x] &= P_y ,  \\
[J', P_y] &= -P_x ,  \\
[P_x, P_y] &= \hat{\Xi} ,  \\
[\hat{\Xi}, \text{all}] &= 0 ;  \\
\Delta J' &= J' \otimes 1 + 1 \otimes J' + \hat{\rho} (\hat{\Xi} \otimes P_x - P_x \otimes \hat{\Xi}) ;  \\
\Delta P_{(x,y)} &= P_{(x,y)} \otimes 1 + 1 \otimes P_{(x,y)} ,  \\
\Delta \hat{\Xi} &= \hat{\Xi} \otimes 1 + 1 \otimes \hat{\Xi} ;  \\
S[(J', \hat{\Xi}, P_x, P_y)] &= -(J', \hat{\Xi}, P_x, P_y) ,  \\
\epsilon[(J', \hat{\Xi}, P_x, P_y)] &= 0 .
\end{align*}
\] (4.11)

This algebra has a cocycle extension structure. To show this, we make the non-linear change

\[
J = J' + \hat{\rho} \Xi P_x
\] (4.12)

This modifies only

\[
\begin{align*}
[J, P_y] &= -P_x + \hat{\rho} \hat{\Xi}^2 ,  \\
\Delta J &= J \otimes 1 + 1 \otimes J + 2\hat{\rho} \hat{\Xi} \otimes P_x ,  \\
S(J) &= -J + 2\hat{\rho} \hat{\Xi} P_x .
\end{align*}
\] (4.13)

If \( \mathcal{A} \) is taken as the Hopf subalgebra generated by \( \hat{\Xi} \) and \( \mathcal{H} \) is the undeformed Euclidean Hopf algebra \( \mathcal{U}(\mathcal{E}(2)) \), the algebra (4.11), (4.13) is obtained as the right-left cocycle bicrossproduct with \( \alpha \) and \( \psi \) trivial and \( \beta \) and \( \xi \) defined by

\[
\begin{align*}
\beta(J) &= 1 \otimes J + 2\hat{\rho} \hat{\Xi} \otimes P_x ,  \\
\xi(P_x, P_y) &= \frac{1}{2} \hat{\Xi} ,  \\
\xi(J, P_y) &= \frac{\hat{\rho}}{2} \hat{\Xi}^2 .
\end{align*}
\] (4.14)

(a') Let us go back to the case (a) above. If we make the redefinitions \( \tilde{N} = iN \), \( A = i\frac{(X_1-iX_2)}{\sqrt{2}} \), \( A^+ = i\frac{(X_1+iX_2)}{\sqrt{2}} \), the \( \mathcal{U}_\rho(\tilde{\mathcal{E}}(2)) \) algebra in the basis (4.4) takes the
\[
\begin{align*}
[\bar{N}, A] &= -A , \quad [\bar{N}, A^+] = A^+ , \quad [A, A^+] = -\frac{i\sinh 2\rho\Xi}{2\rho} , \quad [\Xi, \text{all}] = 0 ; \\
\Delta \bar{N} &= \bar{N} \otimes 1 + 1 \otimes \bar{N} , \quad \Delta A = \exp(-\rho\Xi) \otimes A + A \otimes \exp(\rho\Xi) , \\
\Delta A^+ &= \exp(-\rho\Xi) \otimes A^+ + A^+ \otimes \exp(\rho\Xi) , \quad \Delta \Xi = \Xi \otimes 1 + 1 \otimes \Xi ; \\
S(\bar{N}) &= -\bar{N} , \quad S(A) = -A , \quad S(A^+) = -A^+ , \quad \epsilon(\bar{N}, A, A^+, \Xi) = 0 .
\end{align*}
\] (4.15)

Similarly, the redefinitions \( \hat{J} = i J, \hat{A} = i \frac{P_x - i P_y}{\sqrt{2}}, \hat{A}^+ = i \frac{P_x + i P_y}{\sqrt{2}} \) take the algebra \( \mathcal{U}_\rho(\hat{\mathcal{E}}(2)) \) in the basis (4.13) to the form

\[
\begin{align*}
[\hat{J}, \hat{A}] &= -\hat{A} , \quad [\hat{J}, A^+] = \hat{A}^+ , \quad [A, A^+] = -i\hat{\Xi} , \quad [\Xi, \text{all}] = 0 ; \\
\Delta \hat{J} &= \hat{J} \otimes 1 + 1 \otimes \hat{J} + \sqrt{2}\rho\hat{\Xi} \otimes (\hat{A} + \hat{A}^+) , \\
\Delta \hat{A} &= 1 \otimes \hat{A} + \hat{A} \otimes 1 , \quad \Delta \hat{A}^+ = 1 \otimes \hat{A}^+ + \hat{A}^+ \otimes 1 , \\
S(\hat{J}) &= -\hat{J} + \sqrt{2}\rho\hat{\Xi}(\hat{A} + \hat{A}^+) , \quad S(\hat{A}) = -\hat{A} , \quad S(\hat{A}^+) = -\hat{A}^+ , \quad \epsilon(\hat{J}, \hat{A}, \hat{A}^+, \hat{\Xi}) = 0 .
\end{align*}
\] (4.16)

The algebras (4.15) [9,10] and (4.16) are a deformation of the four-generator oscillator algebra which is recovered in the limits \( \rho \to 0 \) [(4.15)], \( \rho \to 0 \) [(4.16)]. Eqs. (4.15) or (4.16) do not, however [10], define the algebra of the \( q \)-oscillator [31,32,33]. The oscillator algebra may be obtained by contraction using the finite-dimensional representations of \( su_q(2) \) [34]. To derive it directly, without resorting to the \( su_q(2) \) representations, consider the four generators algebra \( \mathcal{U}_q(\mathfrak{su}(2)) \times \mathcal{U}(\mathfrak{u}(1)) \) with \( [J, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J]_q \equiv \sinh 2zJ/\sinh z \), \( [\Xi, \text{all}] = 0 \). Now, we perform the redefinitions \( * J_+ = [2/\epsilon]q^{1/2} \bar{a}^+, \quad J_- = [2/\epsilon]q^{1/2} \bar{a}^+ \), \( J = N - \Xi/\epsilon \); this means that \( \bar{a}, \bar{a}^+ \) and \( N \) are independent generators. Assuming \( q \) real, \( z > 0 \), the contraction leads to \( [N, \bar{a}] = -\bar{a} , \quad [N, \bar{a}^+] = \bar{a}^+ \), \( [\bar{a}, \bar{a}^+] = q^{-2N} \); the familiar \( q \)-commutator relations \( [N, a] = -a , \quad [N, a^+] = a^+ , \quad [a, a^+]_q = q^{-N} \) follow for \( a = q^{N/2}\bar{a} \), \( a^+ = \bar{a}^+ q^{N/2} \). The coproduct in \( \mathcal{U}_q(\mathfrak{su}(2)) \times \mathcal{U}(\mathfrak{u}(1)) \), however, does not have a limit and this explains why the Hopf structure for the \( q \)-oscillator (as defined by these relations) is lost (for recent references on this point, see [35,36]).

* Notice that, were it not by the \( q \)-bracket \( [x]_q = (q^x - q^{-x})/(q - q^{-1}) \), these redefinitions would be equivalent to those in (4.3); this exhibits once more the non-commutative nature of many contraction/deformation diagrams.

† The above oscillator algebra, where \( N \) is treated as an independent generator, has a non-trivial central element, \( z = q^{-N+1}([N]_q - a^+ a) \) and many irreducible representations (for \( 0 < q < 1 \)) [34] inequivalent to the Fock space ones with vacuum state and number operator \( N, \quad [N]_q = a^+ a \), for which \( z = 0 \).
5. The dual case: structure of the deformed Hopf group algebras

The previous deformed algebras may be dualized making use of the bicrossproduct construction. The dual of a bicrossproduct Hopf algebra is also a bicrossproduct Hopf algebra; thus, if $\mathcal{H}$ and $\mathcal{A}$ are Hopf algebras from which the bicrossproduct $\mathcal{H}\bowtie\mathcal{A}$ is constructed, then their duals $\mathcal{H}$ and $\mathcal{A}$ lead to the dual bicrossproduct $\mathcal{H}\triangleright\triangleleft\mathcal{A}$. This dualization will exhibit the ‘group-like’ (rather than ‘algebra-like’) aspects of the deformation. In fact, this procedure of obtaining the duals of certain deformed Hopf algebras is quite an efficient one, since the construction often embeds the non-commuting properties in some of the $(\alpha, \beta, \xi, \psi)$ operations, while the original algebras $\mathcal{H}$ and $\mathcal{A}$ are often undeformed or easy to dualize. We may even follow a step by step procedure.

(1) The case of Fun$_q(E(2))$ has been discussed in [19], and will not be repeated here.

(2) Consider now $\mathcal{U}_\omega(E(2))$ [sec. 2(2)] which has a bicrossproduct structure according to sec. 3(2). We now show that the dual algebra Fun$_\omega(E(2))$ [20,37,38] is easily recovered by looking at its bicrossproduct structure. We take for $\mathcal{A}$ the dual algebra Fun$_\omega(E(2))$ of $\mathcal{U}_\omega(Tr(2))$ defined by

\[
\Delta x = x \otimes 1 + 1 \otimes x \quad , \quad \Delta y = y \otimes 1 + 1 \otimes y \quad , \quad [x,y] = -\omega y \quad ,
\]

\[
S(x,y) = (-x,-y) \quad , \quad \epsilon(x,y) = 0 \quad , \quad x,y \in A \quad ,
\]

and $H$ is generated by $\varphi$ with

\[
\Delta \varphi = \varphi \otimes 1 + 1 \otimes \varphi \quad .
\]

The duals $\bar{\beta}$ and $\bar{\alpha}$ of $\alpha$ and $\beta$ are found to be

\[
\bar{\beta}(x) = x \otimes \cos \varphi + y \otimes \sin \varphi \quad , \quad \bar{\beta}(y) = -x \otimes \sin \varphi + y \otimes \cos \varphi \quad ;
\]

\[
\bar{\alpha}(x \otimes \varphi) = x \bar{\varphi} \varphi = -\omega \sin \varphi \quad , \quad \bar{\alpha}(y \otimes \varphi) = y \bar{\varphi} \varphi = \omega(1 - \cos \varphi) \quad .
\]

The compatibility conditions (B'.10)-(B'.14) are satisfied, and (B'.15), (B'.16), (B'.17) and (B'.18) determine the Hopf structure of Fun$_\omega(E(2))$,

\[
[x,y] = -\omega y \quad , \quad [x,\varphi] = -\omega \sin \varphi \quad , \quad [y,\varphi] = \omega(1 - \cos \varphi) \quad ,
\]

\[
\Delta \varphi = \varphi \otimes 1 + 1 \otimes \varphi \quad , \quad \Delta x = 1 \otimes x + x \otimes \cos \varphi + y \otimes \sin \varphi \quad ,
\]

\[
\Delta y = 1 \otimes y - x \otimes \sin \varphi + y \otimes \cos \varphi \quad , \quad \epsilon(\varphi; x,y) = 0 \quad ,
\]

\[
S(x) = -\cos \varphi x + \sin \varphi y \quad , \quad S(y) = -\sin \varphi x - \cos \varphi y \quad .
\]

(3) A discussion of the Galilei case will be presented elsewhere.
Consider now the case of the deformed Heisenberg-Weyl ‘group’ \( \text{Fun}_\rho(\text{HW}) \), (see [39]) dual of the algebra \( \mathcal{U}_\rho(\text{HW}) \) as given in (4.6) (i.e., (2.6) for \( \hat{\omega} = 2\rho \)). It was shown in sec. 3 (4b) that \( \mathcal{U}_\rho(\text{HW}) \) could be obtained as the cocycle bicrossproduct \([11,12]\) (Appendix B) \[ \mathcal{U}_\rho(\text{HW}) = \mathcal{H} \triangleright \triangleleft \mathcal{A} \] of the undeformed algebras \( \mathcal{H} = \mathcal{U}(\text{Tr}(2)) \) and \( \mathcal{A} = \mathcal{U}(u(1)) \) by using the non-trivial \( \beta \) and \( \xi \) given (3.8), (3.9). Thus, the deformed Heisenberg-Weyl group algebra \( \text{Fun}_\rho(\text{HW}) \) may be found as the cocycle bicrossproduct of \( \mathcal{H} = \text{Tr}(2) \) and \( \mathcal{A} = \mathcal{U}(1) \) using the duals \( \bar{\alpha} : \mathcal{A} \otimes \mathcal{H} \rightarrow \mathcal{H} \) and \( \bar{\psi} : \mathcal{A} \rightarrow \mathcal{H} \otimes \mathcal{H} \) of \( \beta \) and \( \xi \) respectively. Using \((y_1, y_2; \chi)\) for the parameters of \( \text{Tr}(2) \) and \( \mathcal{U}(1) \), \(<Y_i, y_j>_i = \delta_{ij} \), \(<\Xi, \chi>_i = 1 \) the dualization of \( \beta \) immediately leads to

\[
\bar{\alpha}(\chi, y_i) \equiv \chi \bar{\psi} y_i = -2\rho y_i \quad \text{or} \quad [\chi, y_i] = -2\rho y_i \quad ,
\]

\( i = 1, 2. \) Let us now dualize \( \xi = \frac{1 - \exp(-4\rho \Xi)}{8\rho} \). What was really needed in (3.9) to compute \([Y_1, Y_2]\) was the difference \( \xi(Y_1, Y_2) - \xi(Y_2, Y_1) \); the ambiguity in \( \xi(Y_1, Y_2) \) is related to the coboundary ambiguity. A suitable election produces

\[
\bar{\psi}(\chi) = \frac{1}{2}(y_1 \otimes y_2 - y_2 \otimes y_1) \quad ,
\]

from which \( \Delta(\chi) \) is easily found using (B’.34) since \( \bar{\beta} \) is trivial (\( \alpha \) is trivial). In all, \( \text{Fun}_\rho(\text{HW}) \) is determined by

\[
[y_i, y_j] = 0 \quad , \quad [\chi, y_i] = -2\rho y_i \quad , \\
\Delta y_i = y_i \otimes 1 + 1 \otimes y_i \quad , \quad \Delta \chi = \chi \otimes 1 + 1 \otimes \chi + \frac{1}{2}(y_1 \otimes y_2 - y_2 \otimes y_1) \quad ; \\
S(y_i) = -y_i \quad , \quad S(\chi) = -\chi \quad , \quad \epsilon(y_i, \chi) = 0 \quad .
\]

The coproduct mimics the familiar HW group law, and the non-commutativity is just reflected in the non-zero \([\chi, y_i]\) commutator.

(a) Extended Euclidean group \( \text{Fun}_\rho(\tilde{E}(2)) \). The dual \( \text{Fun}_\rho(\tilde{E}(2)) \) of the algebra (a) given by eqs. (4.6) (and (4.4)) is generated by the elements \((y_1, y_2, \chi, \varphi) \) (<
\[ Y_i, y_j = \delta_{ij}, \ <N, \phi> = 1, \ <\Xi, \chi> = 1 \] for which

\[
[y_1, \phi] = [y_2, \phi] = [\chi, \phi] = [y_1, y_2] = 0 ,
\]
\[
[\chi, y_1] = -2\rho y_1 \ , \ [\chi, y_2] = -2\rho y_2 ;
\]
\[
\Delta \phi = \phi \otimes 1 + 1 \otimes \phi \ , \ \Delta y_1 = 1 \otimes y_1 + y_1 \otimes \cos \phi + y_2 \otimes \sin \phi ,
\]
\[
\Delta y_2 = 1 \otimes y_2 + y_2 \otimes \cos \phi - y_1 \otimes \sin \phi ,
\]
\[
\Delta \chi = 1 \otimes \chi + \chi \otimes 1 + \frac{1}{2} [y_1 \otimes \cos \phi y_2 + y_2 \otimes \sin \phi y_2 - y_2 \otimes \cos \phi y_1 + y_1 \otimes \sin \phi y_1] ;
\]
\[
S(\phi) = -\phi \ , \ S(y_1) = -\cos \phi y_1 + \sin \phi y_2 , \ S(y_2) = -\sin \phi y_1 - \cos \phi y_2 ,
\]
\[
S(\chi) = -\chi ; \ \epsilon(\phi, y_1, y_2, \chi) = 0 .
\]

(5.8)

It is not difficult to check directly that \( \text{Fun}_\rho(\tilde{E}(2)) \) is a Hopf algebra; we shall now obtain (5.8) by dualization in two different ways. For the dual in the basis (4.15), see [24].

a1) \( \text{Fun}_\rho(\tilde{E}(2)) \) is the bicrossproduct \( \text{Fun}U(1) \bigtriangleup \text{Fun}_\rho(\text{HW}) \), where \( U(1) \) is generated by \( \phi \) and \( \text{Fun}_\rho(\text{HW}) \) is given in (5.7). To see this, it is sufficient to dualize the right action \( \triangleleft \) (eq. (4.7)), \( Y_1 \triangleleft N = -Y_2, Y_2 \triangleleft N = Y_1, \Xi \triangleleft N = 0 \) to obtain

\[
\bar{\beta}(y_1) = y_1 \otimes \cos \phi + y_2 \otimes \sin \phi \ , \ \bar{\beta}(y_2) = -y_1 \otimes \sin \phi + y_2 \otimes \cos \phi ,
\]
\[
\bar{\beta}(\chi) = 1 \otimes \chi ,
\]

(5.9)

for which the coproducts and antipodes in (5.8) are obtained from (B’.16) and (B’.18). Clearly \( [y_i, \phi] = 0 = [\chi, \phi] \) since \( \bar{\alpha} (\bar{\beta}) \) is dual to \( \beta \), which is trivial.

a2) \( \text{Fun}_\rho(\tilde{E}(2)) \) has also a cocycle bicrossproduct structure. To see this, we take \( A \) as the Hopf algebra generated by \( \chi \) with primitive coproduct and \( H \) as the (undeformed) Euclidean group Hopf algebra of generators \( (y_1, y_2, \phi) \) with \( \Delta y_i, \Delta \phi, S(y_i, \phi) \) and \( \epsilon(y_i, \phi) \) as in (5.8). Then, since \( \alpha \) and \( \psi \) were trivial in sec. 4 a2), \( \bar{\beta} \) and \( \bar{\xi} \) are trivial \( (\bar{\beta}(a) = a \otimes 1_H, \bar{\xi}(a, b) = \epsilon(a)\epsilon(b)1_H) \) and \( \bar{\psi} : A \to H \otimes H , \bar{\alpha} : A \otimes H \to H \) may be found from (4.8) to be

\[
\bar{\psi}(\chi) = \frac{1}{2} [y_1 \otimes \cos \phi y_2 + y_2 \otimes \sin \phi y_2 - y_2 \otimes \cos \phi y_1 + y_1 \otimes \sin \phi y_1] ,
\]
\[
\chi\bar{\phi}y_1 = -2\rho y_1 \ , \ \chi\bar{\phi}y_2 = -2\rho y_2 \ , \ \chi\bar{\phi}\phi = 0 .
\]

(5.10)

The relations (B’.19)-(B’.28) are fulfilled* and the cocycle bicrossproduct structure of

\* The only non-trivial properties are (B’.23) and (B’.25). The first one is the dual cocycle condition, verified because the dual cocycle \( \bar{\psi} \) is the undeformed one, and the second one is due to the compatibility between the coproduct and the commutators.
Fun\(\hat{\rho}(\tilde{E}(2))\) follows from (B’.29) (which for \(\bar{\xi}\) trivial and \(a\) with primitive coproduct leads to \([a, h] = a \bar{\triangleleft} h\)) and (B’.34).

(b) Extended Euclidean group Fun\(\hat{\rho}(\bar{\tilde{E}}(2))\). This is the dual Fun\(\hat{\rho}(\tilde{E}(2))\) of the Hopf algebra \(U\hat{\rho}(\tilde{E}(2))\) (see eqs. (4.11) and (4.13)). It is generated by the elements \((x, y, \varphi, \bar{\chi})\) \(<P_x, x > = < P_y, y > = < J, \varphi > = < \bar{\Xi}, \bar{\chi} > = 1\) with relations

\[
\begin{align*}
[x, \varphi] &= [y, \varphi] = [\hat{x}, \varphi] = [x, y] = 0, \\
[\hat{x}, x] &= -2\hat{\rho}\sin \varphi, \quad [\hat{x}, y] = 2\hat{\rho}(1 - \cos \varphi), \\
\Delta \varphi &= \varphi \otimes 1 + 1 \otimes \varphi, \quad \Delta x = 1 \otimes x + x \otimes \cos \varphi + y \otimes \varphi, \\
\Delta y &= 1 \otimes y + y \otimes \cos \varphi - x \otimes \sin \varphi, \\
\Delta \hat{x} &= 1 \otimes \hat{x} + \hat{x} \otimes 1 + \frac{1}{2}[x \otimes \cos \varphi y + y \otimes \sin \varphi y - y \otimes \cos \varphi x + x \otimes \sin \varphi x], \\
S(\varphi) &= -\varphi, \quad S(x) = -\cos \varphi x + \sin \varphi y, \quad S(y) = -\cos \varphi y - \sin \varphi x, \\
S(\chi) &= -\chi; \quad \epsilon(\varphi, x, y, \chi) = 0,
\end{align*}
\]

which define a Hopf algebra as it may be checked. Now, we take \(A\) as the Hopf group algebra generated by \(\hat{\chi}\) and \(H\) as the dual undeformed Euclidean group Hopf algebra Fun\(E(2)\) (eqs. (5.4) for \(\omega = 0\)) of generators \((x, y, \varphi)\). If we now define \(\bar{\beta}, \bar{\xi}\) to be trivial plus

\[
\begin{align*}
\bar{\psi}(\hat{x}) &= \frac{1}{2}[x \otimes \cos \varphi y + y \otimes \sin \varphi y - y \otimes \cos \varphi x + x \otimes \sin \varphi x], \\
\hat{x} \bar{\triangleleft} x &= -2\hat{\rho}\sin \varphi, \quad \hat{x} \bar{\triangleleft} y = 2\hat{\rho}(1 - \cos \varphi), \quad \hat{x} \bar{\triangleleft} \varphi = 0,
\end{align*}
\]

from \(\xi\) and \(\beta\) in eq. (4.14), the Hopf algebra (5.11) is recovered using (B’.29) and (B’.34), which exhibits the cocycle bicrossproduct structure of (5.11). Due to the commutators \([\chi, x], [\chi, y]\), there is no Hopf Fun\(\hat{\rho}(\text{HW})\) subalgebra here and no bicrossproduct structure in contrast with the previous a1) case.
6. Differential calculus on the Euclidean and Galilean planes

We shall now introduce a covariant differential calculus [40] (see Appendix A) on the different homogeneous spaces which can be constructed. Clearly, to have a proper action on the ‘homogeneous’ part, a bicrossproduct structure is needed. Let us consider now a few different cases.

(1) Due to the lack of a bicrossproduct structure, the inhomogeneous part of the $U_q(\mathcal{E}(2))$ algebra $(P_1, P_2)$ does not constitute a Hopf subalgebra, and the construction of the space algebra as the dual of $(P_1, P_2)$ cannot be performed.

(2) The Euclidean plane $E^2_\omega$ is introduced as the dual ($\langle P_i, x_j \rangle = \delta_{ij}$) of the translation Hopf subalgebra $U_\omega(Tr(2))$ of $U_\omega(\mathcal{E}(2))$ generated by $P_i$ (eq. (3.2)). Since $U_\omega(Tr(2))$ is commutative but not cocommutative, we obtain (eqs. (5.1))

$$\Delta x = x \otimes 1 + 1 \otimes x \quad \Delta y = y \otimes 1 + 1 \otimes y \quad [x, y] = -\omega y$$

for the $E^2_\omega$-plane algebra associated with $U_\omega(\mathcal{E}(2))$. Let us construct a bicovariant differential calculus on $E^2_\omega$ which is consistent (i.e. covariant) under the action of $J$. The (left) action of $J$ on $E^2_\omega$ is defined by duality, $\langle P_xJ, x \rangle = \langle P_x, Jx \rangle$ etc., from which follows that

$$J \triangleright x = y \quad J \triangleright y = -x$$

To define a first order ($J$-)covariant differential calculus we have to determine all commutators $[x_i, dx_j]$ in a way which is consistent with the action (6.2) (which for instance, implies $J \triangleright x dy = (J \triangleright_1 x) d(J \triangleright_2 y)$) and with the Jacobi identity. Although it is not difficult to check that the set of covariance equations (like $J \triangleright (x_i dx_j) - J \triangleright (dx_j x_i) = J \triangleright [x_i, dx_j]$) has a unique solution given by

$$[x, y] = -\omega y \quad [x, dx] = 0 = [x, dy] \quad [y, dx] = \omega dy \quad [y, dy] = -\omega dx$$

the above commutators do not satisfy the Jacobi identity and thus fail to provide a consistent differential calculus. This situation is not new, and has already appeared for the differential calculus on other spacetime algebras [28,29]. We now show that the solution proposed there, and which involves an enlargement of the algebra which has been found to be associated with a Hopf algebra cocycle extension [29], also applies here. We stress that this problem is associated to the deformed character of (3.2) as expressed by $\omega$, being of course absent for the undeformed Euclidean Hopf algebra $U(\mathcal{E}_2)$. 

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Consider the trivial extension $U_\omega(E(2))\times U(u(1))$ mentioned in sec. 4, obtained by adding the primitive Hopf algebra generated by $\Xi$ to $U_\omega(E(2))$. The previous procedure applied to $(P_x, P_y, \Xi)$ leads now to an enlarged Euclidean algebra $\tilde{E}_\omega$ generated by $(x, y, \chi)$ (<$\Xi, \chi>$ = 1) and to the additional relations
\[
[x, x] = 0 = [x, y] \ , \quad \Delta \chi = \chi \otimes 1 + 1 \otimes \chi \ , \quad J \triangleright \chi = 0 \quad .
\]
(6.4)

Proceeding as before, we find that there is a unique solution for the rotation covariant differential calculus on the above enlarged Euclidean ‘space’ specified by (cf. (6.3))
\[
[y, x] = -\omega y \ , \quad [x, \chi] = 0 \ , \quad [x, dx] = \omega d\chi \ , \quad [x, dy] = 0 \ , \quad [x, d\chi] = \omega dx \ ,
\]
\[
[y, \chi] = 0 \ , \quad [y, dx] = \omega dy \ , \quad [y, dy] = -\omega(dx - d\chi) \ , \quad [y, d\chi] = \omega dy \ ,
\]
\[
[\chi, dx] = \omega dx \ , \quad [\chi, dy] = \omega dy \ , \quad [\chi, d\chi] = \omega d\chi \ ,
\]
and satisfying Jacobi identity.

(3) We define the two-dimensional Galilean plane $G^2_\omega$ as the dual ($<$ $X, x >$ = 1 = < $X_t, t >$, < $X, t >$ = 0 = < $X_t, x >$) of $U_\omega(Tr(2))$. The commutativity (non-cocommutativity) of $U_\tilde{\omega}(Tr(2))$ implies the relations
\[
[x, t] = 2\tilde{\omega} x \ ; \quad \Delta x = x \otimes 1 + 1 \otimes x \ , \quad \Delta t = t \otimes 1 + 1 \otimes t \quad ,
\]
(6.6)
for the $G^2_\omega$ algebra. Following the same pattern of case (2) we construct a bicovariant differential calculus (covariant under the action of the ‘boost’ $V$) that satisfies Leibniz’s rule and Jacobi identity. The (left) action of $V$ on $G^2_\omega$ is given by
\[
V \triangleright x = -t \quad , \quad V \triangleright t = 0 \quad .
\]
(6.7)

Using (6.7), we find that the covariance requirement implies the system of equations
\[
V \triangleright [x, dx] = -[t, dx] - [x, dt] \quad , \quad V \triangleright [x, dt] = -[t, dt] + 2\tilde{\omega} dt
\]
\[
V \triangleright [x, dt] = -[t, dt] \quad , \quad V \triangleright [t, dt] = 0 \quad .
\]
(6.8)

The unique solution linear in $dx, dt$ that satisfies (6.8), Leibniz’s rule and Jacobi identity is*

\[
[x, dx] = 0 \ , \quad [x, dt] = \tilde{\omega} dx \ , \quad [t, dx] = -\tilde{\omega} dx \ , \quad [t, dt] = \tilde{\omega} dt \quad .
\]
(6.9)

Thus, this case is different from the Euclidean case $E^2_\omega$. On $G^2_\omega$ there is a covariant differential calculus without any additional one-form.†

* Even if there is no deformation ($\tilde{\omega} = 0$) there exists a non-trivial solution (see [29]) given by $[x, dx] = \mu dt$ and all other commutators equal to zero.

† For the differential calculus on the deformed Newtonian spacetime associated with the (1 + 3) version of the deformed Galilei algebra $G_\kappa$ see [29].
(a) It was seen (eq. (6.5)) that to define a $J$-covariant differential calculus on $E^2_\omega$ it was necessary to enlarge it to $\tilde{E}_\omega$. Let us now show that two $N$-covariant calculi may be similarly constructed on $\text{Fun}_\rho(\text{HW})$ (eqs. (5.7)) as the dual of the $U_\rho(\tilde{E}(2))$ subalgebra of $U_\rho(\tilde{E}(2))$ (sec. 4(a)). The left action $\rhd$ of $N$ on $(y_1, y_2, \chi)$ is obtained from (4.7) and given by

\[ N \rhd y_1 = y_2 \quad , \quad N \rhd y_2 = -y_1 \quad , \quad N \rhd \chi = 0 \quad . \]  

(6.10)

Proceeding as before, we find the commutators

\[ [y_i, y_j] = 0 \quad , \quad [y_i, \chi] = 2\rho y_i \quad , \quad [y_i, dy_j] = 0 \quad , \]
\[ [\chi, dy_i] = \lambda dy_i \quad , \quad [y_i, d\chi] = (\lambda + 2\rho)dy_i \quad , \quad [\chi, d\chi] = \mu d\chi \quad . \]  

(6.11)

The Jacobi identity requires $\lambda = -2\rho$ or $\lambda - \mu = -2\rho$. The bicovariance requirement now determines two bicovariant differential calculi over $\text{Fun}_\rho(\text{HW})$ (on the $E^2_\omega$ plane the coproduct of the generators was primitive, hence the differentials are bi-invariant by (A.4) and the bicovariance is trivial). We first find, using (5.7) and (A.4),

\[ \Delta_L dy_i = 1 \otimes dy_i \quad , \quad \Delta_R dy_i = dy_i \otimes 1 \quad , \]
\[ \Delta_L d\chi = 1 \otimes d\chi + \frac{1}{2}(y_1 \otimes dy_2 - y_2 \otimes dy_1) \quad , \]
\[ \Delta_R d\chi = d\chi \otimes 1 + \frac{1}{2}(dy_1 \otimes y_2 - dy_2 \otimes y_1) \quad ; \]  

(6.12)

it is easy to show that the coactions (6.12) satisfy (A.2). If we use now (A.1) to calculate $\Delta_L[\chi, d\chi] = \mu \Delta_L d\chi$ we find $\mu = 2\lambda$; the same condition is obtained using $\Delta_R$. Then, (6.11) leads to ($\lambda = -2\rho, \mu = -4\rho$)

\[ [y_i, y_j] = 0 \quad , \quad [y_i, \chi] = 2\rho y_i \quad , \quad [y_i, dy_j] = 0 \quad , \]
\[ [\chi, dy_i] = -2\rho dy_i \quad , \quad [y_i, d\chi] = 0 \quad , \quad [\chi, d\chi] = -4\rho d\chi \]  

(6.13)

and ($\lambda = 2\rho, \mu = 4\rho$)

\[ [y_i, y_j] = 0 \quad , \quad [y_i, \chi] = 2\rho y_i \quad , \quad [y_i, dy_j] = 0 \quad , \]
\[ [\chi, dy_i] = 2\rho dy_i \quad , \quad [y_i, d\chi] = 4\rho dy_i \quad , \quad [\chi, d\chi] = 4\rho d\chi \]  

(6.14)

Since (A.3) is satisfied, eqs. (6.13), (6.14) determine two first order $N$-covariant differential calculi over $\tilde{E}_\rho$. 

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Appendix A: Bicovariant differential calculus

Let $A$ be a Hopf algebra and let $\Delta$ and $\epsilon$ be its coproduct and counit. A first order bicovariant differential calculus over $A$ is defined [40] by a pair $(\Gamma, d)$ where $d : A \rightarrow \Gamma$ is a linear mapping satisfying Leibniz’s rule and $\Gamma$ is a bicovariant $A$-bimodule $(\Gamma, \Delta_L, \Delta_R)$ i.e., the linear mappings $\Delta_L : \Gamma \rightarrow A \otimes \Gamma$, $\Delta_R : \Gamma \rightarrow \Gamma \otimes A$ and the exterior derivative $d$ satisfy

$$
\Delta_L(a\omega) = \Delta(a)\Delta_L(\omega) , \quad \Delta_R(a\omega) = \Delta(a)\Delta_R(\omega) , \quad \Delta_L(\omega a) = \Delta_L(\omega)\Delta(a) , \quad \Delta_R(\omega a) = \Delta_R(\omega)\Delta(a) ,
$$

(A.1)

$$(\Delta \otimes id)\Delta_L = (id \otimes \Delta_L)\Delta_L , \quad (id \otimes \Delta)\Delta_R = (\Delta_R \otimes id)\Delta_R , \quad (\epsilon \otimes id)\Delta_L = id , \quad (id \otimes \epsilon)\Delta_R = id ,$$

(A.2)

$$(id \otimes \Delta_R)\Delta_L = (\Delta_L \otimes id)\Delta_R ,$$

(A.3)

$$
\Delta_L d = (id \otimes d)\Delta , \quad \Delta_R d = (d \otimes id)\Delta ,
$$

(A.4)

where the left (right) equations in (A.2) express that $\Gamma$ is a left (right) $A$-comodule, (A.3) is the result of bicovariance (commutation of the left and right coactions), and (A.4) expresses the compatibility of the exterior derivative $d$ with $\Delta$ and $\Delta_{L,R}$. Eqs. (A.1), (A.2) and (A.3) characterize $(\Gamma, \Delta_L, \Delta_R)$ as a bicovariant bimodule over $A$; the addition of (A.4) determines a first order bicovariant differential calculus $(\Gamma, d)$. An element $\omega \in \Gamma$ is called left (right) invariant if $\Delta_L(\omega) = 1 \otimes \omega$ $\Delta_R(\omega) = \omega \otimes 1$.

As in the undeformed (Lie) case, the basis elements of the vector space $\Gamma_{\text{inv}} \subset \Gamma$ of the left-invariant elements generate $\Gamma$ as a left free module.

Appendix B: Bicrossproduct of Hopf algebras and cocycles

We list here for convenience the basic formulae of Majid’s bicrossproduct and cocycle bicrossproduct constructions and refer to [11,12] (see also [14]) for details. The expressions which characterize $H\triangleright\rhd A$ (used in secs. 3,4) involve the mappings $\alpha : A \otimes H \rightarrow A$ (right $H$-module action), $\beta : H \rightarrow A \otimes H$ (left $A$-comodule coaction), $\xi : H \otimes H \rightarrow A$ (two-cocycle) and $\psi : H \rightarrow A \otimes A$ (hence the more detailed notation $H^{\beta,\psi}\triangleright\lhd_{\alpha,\xi} A$, see [11]). Those of the dual case ($H_{\bar{\alpha},\bar{\xi}} \lhd\triangleright\bar{\beta},\bar{\psi} A$ when all ingredients
are indicated) involve the respective dual operations; they were used in sec. 5. We may think of \( H \triangleright A \) as emphasizing the ‘algebra-like’ aspects and of \( H \triangleright A \) as giving the ‘group-like’ ones* (in the undeformed case they correspond, respectively, to the cocommutative Hopf algebra constructed on the enveloping algebra \( \mathcal{U}(G) \) of \( G \), and to the Abelian Hopf algebra of functions \( \text{Fun}(G) \) over a Lie group \( G \) with coproduct given by the group law). Both sets of formulae are in correspondence once \( H, A \) are replaced by their respective duals \( H, A \), \( \bar{\beta}, \bar{\alpha}, \bar{\psi}, \bar{\xi} \); thus we shall only reproduce here those for the second case. Those useful for \( H \triangleright A \) may be found in the original papers [11,12] (or in the Appendix of [29] with the same numbers they are referred to in the main text, also corresponding to the dual formulae for \( H \triangleright A \) below).

Let \( H \) and \( A \) be Hopf algebras and let

a) \( H \) be a left \( A \)-module algebra (\( H \triangleright A \))

b) \( A \) be a right \( H \)-comodule coalgebra (\( H \triangleright A \)) i.e., there exist linear mappings

\[
\bar{\alpha} : A \otimes H \rightarrow H \quad , \quad \bar{\alpha}(a \otimes h) \equiv a \triangleright h \quad , \quad a \in A, \; h \in H \quad ; \quad (B'1)
\]

\[
\bar{\beta} : A \rightarrow A \otimes H \quad , \quad \bar{\beta}(a) = a^{(1)} \otimes a^{(2)} \quad , \quad a^{(1)} \in A, \; a^{(2)} \in H \quad (B'2)
\]

such that the properties of

a1) \( \bar{\alpha} \) being a left \( A \)-module action \( \bar{\delta} \):

\[
1_A \triangleright h = h \quad , \quad (B'3)
\]

\[
a' \triangleright (a \triangleright h) = a' \triangleright a \triangleright h \quad ; \quad (B'4)
\]

a2) \( H \) being a left \( A \)-module algebra:

\[
a \triangleright 1_H = 1_H \epsilon(a) \quad , \quad a \triangleright (hg) = (a^{(1)} \triangleright h)(a^{(2)} \triangleright g) \quad ; \quad (B'5)
\]

b1) \( \bar{\beta} \) being a right \( H \)-comodule coaction:

\[
a^{(1)} \otimes \epsilon_H(a^{(2)}) = a \otimes 1_H \equiv a' \quad [(id \otimes \epsilon) \circ \beta = id] \quad ; \quad (B'6)
\]

* We use the bicrossproduct notation \( \triangleright \triangleleft \) or \( \triangleright \trianglecirc \) rather than the (right, left) crossproduct \( \triangleright <, > \triangleright \) or the (left, right) cross coproduct \( \triangleright <, > \triangleright \) even if the coactions \( \beta, \bar{\beta} \) or the actions \( \alpha, \bar{\alpha} \) are trivial, and omit explicit reference to them (or to \( \xi, \psi \) etc.)
\[ a^{(1)(1)} \otimes a^{(1)(2)} \otimes a^{(2)} = a^{(1)} \otimes a^{(2)} \otimes a^{(2)} \] (B').7

b2) A being a right \( H \)-comodule coalgebra:

\[ \epsilon_A(a^{(1)})a^{(2)} = 1_H \epsilon_A(a) \] , \quad \epsilon \otimes \bar{\beta} = \epsilon \] , \quad \text{(B').8}

\[ a^{(1)} \otimes a^{(1)} \otimes a^{(2)} = a^{(1)} \otimes a^{(2)} \otimes a^{(2)} \] ((\Delta \otimes \epsilon) \otimes \bar{\beta} = (id \otimes \Delta) \otimes \bar{\beta}) \] , \quad \text{(B').9}

where \( m_H \) is the multiplication in \( H \) and \( \tau \) is the twist mapping, are fulfilled.

Then, if the compatibility conditions

\[ \epsilon_H(\bar{\beta} h) = \epsilon_A(a)\epsilon_H(h) \] , \quad \text{(B').10}

\[ \Delta(\bar{\alpha} h) \equiv (\bar{\alpha} h)_{(1)} \otimes (\bar{\alpha} h)_{(2)} = (a^{(1)} \otimes a^{(2)})_{(1)} (a^{(1)} \otimes a^{(2)})_{(2)} \] , \quad \text{(B').11}

\[ \bar{\beta}(1_A) \equiv 1^{(1)}_A \otimes 1^{(2)}_A = 1_A \otimes 1_H \] , \quad \text{(B').12}

\[ \bar{\beta}(ab) \equiv (ab)_{(1)} \otimes (ab)_{(2)} = a^{(1)}_{(1)} b_{(1)} \otimes a^{(2)}_{(1)} (a^{(2)}_{(2)} \otimes b_{(2)}) \] , \quad \text{(B').13}

\[ a^{(1)}(2) \otimes (a^{(1)} \otimes \bar{\beta} h) a^{(2)} = a^{(1)}(1) \otimes a^{(2)}(1) (a^{(2)} \otimes \bar{\beta} h) \] , \quad \text{(B').14}

are satisfied* , there is a Hopf algebra structure on \( [11] \) \( K \equiv H \otimes A \) called the (left-right) bicrossproduct \( H \ll A \) (\( H \ll A \)) for short) defined by

\[ (h \otimes a)(g \otimes b) = h(a^{(1)} \otimes g) \otimes a^{(2)} b , \quad h, g \in H ; a, b \in A \] , \quad \text{(B').15}

\[ \Delta_K(h \otimes a) = h_{(1)} \otimes a^{(1)}(1) \otimes h^{(2)}(2) a^{(2)} \otimes a^{(2)} \] , \quad \text{(B').16}

\[ \epsilon_K = \epsilon_H \otimes \epsilon_A , \quad 1_A = 1_H \otimes 1_A \] , \quad \text{(B').17}

\[ S(h \otimes a) = (1_H \otimes S_A(a^{(1)}))(S_H(h a^{(2)}) \otimes 1_A) \] . \quad \text{(B').18}

In \( K = H \otimes A \), \( h \equiv h \otimes 1_A \) and \( a \equiv 1_H \otimes a \); thus, \( ah = a^{(1)} \otimes h \otimes a^{(2)} \). There are two cases of special interest [15] (see also [11]). When \( \bar{\beta} = I \otimes 1_H \) i.e. \( \bar{\beta}(a) =

* If \( A \) is cocommutative and \( H \) commutative, condition (B').14 is automatically satisfied.
a \otimes 1_H \text{ (trivial coaction) and } A \text{ is cocommutative, } K \text{ is the semidirect product of Hopf algebras since then } \Delta_K(h \otimes a) = (a_1 \otimes h_{(1)}) \otimes (b_2 \otimes g_{(2)}). \text{ When } \bar{\alpha} \text{ is trivial, } 
\bar{\alpha} = \epsilon_A \otimes 1_H \text{ (}a \bar{\triangleright} h = h \epsilon_A(a)\text{) and } H \text{ is commutative, } K \text{ is the semidirect coproduct of Hopf algebras since } (h \otimes a)(g \otimes b) = hg \otimes ab. \text{ When } \bar{\alpha} \text{ is trivial, } \bar{\beta}(ab) = \bar{\beta}(a)\bar{\beta}(b) \text{ (algebra homomorphism)}.

As for } H \triangleright A, \text{ the above construction may be extended to accommodate cocycles } [11, 12]. \text{ Let } A \text{ is a left Hopf algebras since } \alpha = \epsilon_K \text{ Hopf algebras since then } \Delta a = \bar{\psi} \otimes 1_A \text{ (dual cocycle condition) and } (B'.4) \text{ is replaced by}

\[ \bar{\psi}(a \otimes 1_A) = 1_H \epsilon(a) = \bar{\xi}(1_A \otimes a) \quad \bar{\xi}(1_A \otimes 1_A) = 1_H \] \quad \text{(B'.19)}

\[ a_{(1)} \bar{\triangleright} \bar{\xi}(b_{(1)} \otimes c_{(1)}) \bar{\xi}(a_{(2)} \otimes b_{(2)} c_{(2)}) = \bar{\xi}(a_{(1)} \otimes b_{(1)}) \bar{\xi}(a_{(2)} b_{(2)} c_{(2)}) , \forall a, b, c \in A \] \quad \text{(B'.20)}

\underbrace{\text{cyclics [11, 12]}} \text{ Let } A \text{ is a left Hopf algebra if } (B'.3), (B'.5) \text{ are fulfilled and there is a linear (two-cocycle) map } \bar{\xi} : A \otimes A \to H \text{ such that}

\[ \bar{\xi}(a \otimes 1_A) = 1_H \epsilon(a) = \bar{\xi}(1_A \otimes a) \quad \bar{\xi}(1_A \otimes 1_A) = 1_H \] \quad \text{(B'.19)}

\[ a_{(1)} \bar{\triangleright} \bar{\xi}(b_{(1)} \otimes c_{(1)}) \bar{\xi}(a_{(2)} \otimes b_{(2)} c_{(2)}) = \bar{\xi}(a_{(1)} \otimes b_{(1)}) \bar{\xi}(a_{(2)} b_{(2)} c_{(2)}) , \forall a, b, c \in A \] \quad \text{(B'.20)}

\text{(cocycle condition) and } (B'.4) \text{ is replaced by}

\[ a_{(1)} \bar{\triangleright} (b_{(1)} \bar{\triangleleft} h) \bar{\xi}(a_{(2)} \otimes b_{(2)}) = \bar{\xi}(a_{(1)} \otimes b_{(1)}) ((a_{(2)} b_{(2)} \bar{\triangleleft} h) \bar{\triangleright} h) , \forall h \in H, \forall a, b \in A \] \quad \text{(B'.21)}

\text{which for } \bar{\xi} \text{ trivial reproduces } (B'.4). \text{ Similarly, } A \text{ is a right } H\text{-comodule coalgebra cocycle if } (B'.6), (B'.8), (B'.9) \text{ are fulfilled, and there is a linear map } \bar{\psi} : A \to H \otimes H\text{,}

\[ \bar{\psi}(a) = \bar{\psi}(a)^{(1)} \otimes \bar{\psi}(a)^{(2)} \text{, such that} \]

\[ \epsilon(\bar{\psi}(a)^{(1)}) \bar{\psi}(a)^{(2)} = 1 \epsilon(a) = \bar{\psi}(a)^{(1)} \epsilon(\bar{\psi}(a)^{(2)}) , \quad [(\epsilon \otimes \text{id}) \circ \bar{\psi} = (\text{id} \otimes \epsilon) \circ \bar{\psi}] \quad \text{(B'.22)} \]

\[ \Delta \bar{\psi}(a_{(1)})^{(1)} \bar{\psi}(a_{(2)})^{(2)} = \bar{\psi}(a_{(1)})^{(1)} \otimes \Delta \bar{\psi}(a_{(2)})^{(2)} \bar{\psi}(a_{(2)}) , \forall a \in A \] \quad \text{(B'.23)}

\text{(dual cocycle condition) and } (B'.7) \text{ is replaced by}

\[ (1 \otimes \bar{\psi}(a_{(1)}))((\bar{\beta} \otimes \text{id}) \circ \bar{\beta}(a_{(2)})) = a_{(1)}^{(1)} \otimes \Delta a_{(1)}^{(2)} \bar{\psi}(a_{(2)}) \]

\[ = \left((\text{id} \otimes \Delta) \bar{\beta}(a_{(1)})(1 \otimes \bar{\psi}(a_{(2)})) \right) \quad \text{(B'.24)} \]

\text{Then, if the compatibility conditions } (B'.10), (B'.12), (B'.14) \text{ and }

\[ \Delta(a_{(1)} \bar{\triangleleft} h) \bar{\psi}(a_{(2)}) = \bar{\psi}(a_{(1)}) [a_{(2,1)} \bar{\triangleleft} h_{(1)} \otimes a_{(2,2)} (a_{(3)} \bar{\triangleleft} h_{(2)})] \] \quad \text{(B'.25)}

\[ (1 \otimes \bar{\xi}(a_{(1)} \otimes b_{(1)})) \bar{\beta}(a_{(2)} b_{(2)}) = a_{(1)}^{(1)} b_{(1)}^{(1)} \otimes a_{(1)}^{(2)} (a_{(2)} b_{(2)}^{(2)} \bar{\triangleleft} h_{(2)}) \bar{\psi}(a_{(3)} \otimes b_{(2)}) \] \quad \text{(B'.26)}
(which replace (B’.11), (B’.13)), together with

\[
\Delta \bar{\xi}(a(1) \otimes b(1)) \bar{\psi}(a(2)b(2)) = \bar{\psi}(a(1)) \left[ (a(2)\bar{\Delta} \bar{\psi}(b(1))(1)) \bar{\xi}(a(4) \otimes b(1)) \otimes (a(2)\bar{\Delta} \bar{\psi}(b(1))(2)) a(4) (a(5)\bar{\Delta} b(2)) \bar{\xi}(a(6) \otimes b(3)) \right],
\]

(B’.27)

\[
\epsilon(\bar{\xi}(a \otimes b)) = \epsilon(a)\epsilon(b), \quad \bar{\psi}(1A) = 1_H \otimes 1_H
\]

(B’.28)

hold, \((A, H, \bar{\alpha}, \bar{\beta}, \bar{\xi}, \bar{\psi})\) determine a cocycle left-right bicrossproduct bialgebra \(H_{\bar{\xi}} \ltimes \bar{\psi} A\). In it, the counit and unit are defined by (B’.17) and the product and coproduct (B’.15), (B’.16) are replaced by

\[
(h \otimes a)(g \otimes b) = h(a(1)\bar{\Delta} g) \bar{\xi}(a(2) \otimes b(1)) \otimes a(3)b(2),
\]

(B’.29)

\[
\Delta(h \otimes a) = h(1)\bar{\psi}(a(1))(1) \otimes a(2)(1) \otimes h(2)\bar{\psi}(a(1))(2) a(2)(2) \otimes a(3).
\]

(B’.30)

It is convenient to have the explicit expression of (B’.27) in the more simple cases. For \(\bar{\psi}\) trivial it reads

\[
\Delta \bar{\xi}(a \otimes b) = \bar{\xi}(a(1) \otimes b(1)) \otimes a(2)(1) (a(2)\bar{\Delta} b(1)) \bar{\xi}(a(3) \otimes b(2)).
\]

(B’.31)

For \(\bar{\psi}\) trivial, it gives

\[
\Delta \bar{\xi}(a(1) \otimes b(1)) \bar{\psi}(a(2)b(2)) = \\
\bar{\psi}(a(1))[\bar{\psi}(b(1))(1) \bar{\xi}(a(2) \otimes b(2)) \otimes \bar{\psi}(b(1))(2) a(2) b(2) \bar{\xi}(a(3) \otimes b(3))].
\]

(B’.32)

For \(\bar{\xi}\) and \(\bar{\psi}\) trivial, it reduces to

\[
\Delta \bar{\xi}(a \otimes b) = \bar{\xi}(a(1) \otimes b(1)) \otimes a(2)(1) \bar{\xi}(a(2) \otimes b(2)).
\]

(B’.33)

For \(\bar{\xi}\) trivial \([\bar{\xi}(a \otimes b) = \epsilon(a)\epsilon(b)1_H]\) (B’.21) reduces to (B’.4), (B’.26) to (B’.13) and (B’.29) to (B’.15). For \(\bar{\psi}\) trivial \([\bar{\psi}(a) = 1_H \otimes 1_H\epsilon(a)]\), (B’.24) reduces to (B’.7), (B’.25) to (B’.11) and (B’.30) to (B’.16). For \(\beta(a) = a \otimes 1\) trivial, (B’.30) gives for the elements of \(A\) with original primitive coproduct the cocycle extension expression

\[
\Delta(1 \otimes a) = 1 \otimes a \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 + a + \bar{\psi}(a)(1) \otimes 1 \otimes \bar{\psi}(a)(2) \otimes 1,
\]

(B’.34)

which in \(K\) simply reads \(\Delta(a) = 1 \otimes a + a \otimes 1 + \bar{\psi}(a)\). This was used for (5.7) [(5.6)], (5.8) [(5.10)] and (5.11) [(5.12)].

\(\dagger\) With \(\bar{\xi}(a(1) \otimes b(1))\bar{\xi}^{-1}(a(2) \otimes b(2)) = \epsilon(a)\epsilon(b)\) (convolution invertible [11]), eq. (B.26) gives \(\beta(ab) = a(2)(1) \otimes \bar{\xi}^{-1}(a(1) \otimes b(1)) a(2)(2) a(3) b(2) \bar{\xi}(a(4) \otimes b(3))\). If \(A\) is Abelian, as is always the case in the cocycle bicrossproduct structures in the main text, this formula reduces to (B’.13).
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