Regular functions on the Shilov boundary

Olga Bershtein

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email: bershtein@ilt.kharkov.ua

Abstract

In this paper a $^*$-algebra of regular functions on the Shilov boundary $S(D)$ of bounded symmetric domain $D$ is constructed. The algebras of regular functions on $S(D)$ are described in terms of generators and relations for two particular series of bounded symmetric domains. Also, the degenerate principal series of quantum Harish-Chandra modules related to $S(D) = U_n$ is investigated.

1 Introduction

Quantum Harish-Chandra modules (see the next Sec.) form a broad class of infinite dimensional representations of quantum universal enveloping algebras. Construction and classification of simple quantum Harish-Chandra modules are important open problems (cf. [1]).

In this paper we investigate a quantum analog of degenerate principal series realized in the spaces of regular functions on the Shilov boundary of bounded symmetric domain of tube type [2].

Let $(a_{ij})_{i,j=1,...,l}$ be a Cartan matrix of positive type, $\mathfrak{g}$ the corresponding simple complex Lie algebra. So the Lie algebra can be defined by the generators $e_i, f_i, h_i, i = 1, ..., l$, and the well-known relations (see [3]). Let $\mathfrak{h}$ be the linear span of $h_i, i = 1, ..., l$. Fix simple roots $\{\alpha_i \in \mathfrak{h}^*| i = 1, ..., l\}$ via $\alpha_i(h_j) = a_{ji}$. Also, let $\{\varpi_i| i = 1, ..., l\}$ be the fundamental weights and $P = \bigoplus_{i=1}^l \mathbb{Z}\varpi_i$ the weight lattice.

Fix $l_0 \in \{1, ..., l\}$ and the Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ generated by

$$e_i, f_i, \quad i \neq l_0; \quad h_i, \quad i = 1, ..., l.$$ 

We consider Lie algebras $\mathfrak{g}$ equipped with $\mathbb{Z}$-grading as follows:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}, \quad \mathfrak{g}_j = \{\xi \in \mathfrak{g}| [h_0, \xi] = 2j\xi\},$$

(1)

where $h_0 \in \mathfrak{h}$ and

$$\alpha_i(h_0) = 0, \quad i \neq l_0; \quad \alpha_{l_0}(h_0) = 2.$$ 

Let $\delta$ be the maximal root, and $\delta = \sum_{i=1}^l c_i \alpha_i$. (1) holds if and only if $c_{l_0} = 1$. In this case $\mathfrak{g}_0 = \mathfrak{k}$ and we call the pair $(\mathfrak{g}, \mathfrak{k})$ a Hermitian symmetric pair.
Harish-Chandra established the one-to-one correspondence between Hermitian symmetric pairs and irreducible bounded symmetric domains considered up to biholomorphic isomorphisms \[2\].

Let \( W \) be the Weyl group of the root system \( R \), \( w_0 \in W \) the element of maximal length. The irreducible bounded symmetric domain \( \mathbb{D} \) related to the pair \((g, \mathfrak{k})\) is a domain of tube type if and only if \( \varpi_{w_0} = -w_0 \varpi_{l_0} \).

Fix a Hermitian symmetric pair \((g, \mathfrak{k})\). Let \( G \) be a connected simply connected complex Lie group with Lie \((G) = g\) and \( K \subset G \) a connected complex Lie subgroup with Lie \((K) = \mathfrak{k}\). Consider the Lie subalgebra \( p \subset g \) generated by \( e_i, h_i, i = 1, \ldots, l, \) and \( f_j, j \neq l_0 \). There is the corresponding connected complex Lie subgroup \( P \subset G \). The homogeneous space \( G/P \) is a simply connected projective variety.

There exists a distinguished noncompact real form \( G_0 \) of \( G \) with a unique closed \( G_0 \)-orbit in \( G/P \) \[2\]. The Shilov boundary \( S(\mathbb{D}) \) corresponds to it under the Borel embedding \( i: \mathbb{D} \hookrightarrow G/P \). These reasons allow us to obtain quantum analogs of regular functions on the Shilov boundary and series of Harish-Chandra modules related to it. This approach to the quantum Shilov boundaries belongs to L. Vaksman (private communication).

### 2 Quantum analog of algebra of regular functions on the Shilov boundary

In this section we introduce quantum analogs of regular functions on the Shilov boundaries \( S(\mathbb{D}) \) of bounded symmetric domains \( \mathbb{D} \) of tube type.

First of all, recall some notions from the quantum group theory \[3\]. In the sequel the ground field is \( \mathbb{C} \), \( q \in (0, 1) \), and all algebras are associative and unital.

Denote by \( d_i > 0, i = 1, \ldots, l \), such coprime numbers that the matrix \((d_i a_{ij})_{i,j=1,\ldots,l}\) is symmetric. Recall that the quantum universal enveloping algebra \( U_qg \) is a Hopf algebra defined by the generators \( K_i, K_i^{-1}, E_i, F_i, i = 1, 2, \ldots, l \) and the relations:

\[
\begin{align*}
K_i K_j &= K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\
K_i E_j &= q_i^{a_{ij}} E_j K_i, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i, \\
E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\
\sum_{m=0}^{1-a_{ij}} (-1)^m \left[ 1 - a_{ij} \atop m \right]_{q_i} E_i^{1-a_{ij}-m} E_j E_i^m &= 0, \quad m \leq a_{ij}, \\
\sum_{m=0}^{1-a_{ij}} (-1)^m \left[ 1 - a_{ij} \atop m \right]_{q_i} F_i^{1-a_{ij}-m} F_j F_i^m &= 0,
\end{align*}
\]

where \( q_i = q^{d_i}, 1 \leq i \leq l \) and

\[
\left[ \begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q! [m-n]_q!}, \quad [n]_q = [n]_q [2]_q [1]_q, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]
The comultiplication \( \Delta \), the counit \( \varepsilon \), and the antipod \( S \) are defined on generators by the following formulas:

\[
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i,
\]

\[
S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i, \quad S(K_i) = K_i^{-1},
\]

\[
\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.
\]

We need two important classes of \( U_q\mathfrak{g} \)-modules. A representation \( \rho: U_q\mathfrak{g} \to \text{End}V \) is called weight (and \( V \) is called a weight module, respectively), if \( V \) admits a decomposition into a sum of weight subspaces

\[
V = \bigoplus_{\lambda} V_{\lambda}, \quad V_{\lambda} = \{ v \in V | \rho(K_j^{\pm 1})v = q_j^{\pm \lambda_j}v, j = 1, \ldots, l \},
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{Z}^l \). The subspace \( V_{\lambda} \) is called a weight subspace of weight \( \lambda \).

Let \( U_q\mathfrak{k} \subset U_q\mathfrak{g} \) be a Hopf subalgebra generated by \( E_i, F_i, i = 1, \ldots, l, i \neq l_0 \) and \( K_j^{\pm 1}, j = 1, \ldots, l \). A finitely generated weight \( U_q\mathfrak{g} \)-module \( V \) is called a quantum Harish-Chandra module if \( V \) is a sum of finite dimensional simple \( U_q\mathfrak{k} \)-modules and \( \dim \text{Hom}_{U_q\mathfrak{k}}(W, V) < \infty \) for every finite dimensional simple \( U_q\mathfrak{k} \)-module \( W \).

We restrict our consideration to quantum Harish-Chandra modules only.

Equip \( U_q\mathfrak{g} \) with a \(*\)-Hopf algebra structure via the antilinear involution \(*\):

\[
E_{l_0}^* = -K_{l_0}F_{l_0}, \quad F_{l_0}^* = -E_{l_0}K_{l_0}^{-1}, \quad K_{l_0}^* = K_{l_0},
\]

\[
E_j^* = K_jF_j, \quad F_j^* = E_jK_j^{-1}, \quad K_j^* = K_j, \quad j \neq l_0.
\]

Recall the notion of quantum analog of the algebra \( \mathbb{C}[G] \) of regular functions on \( G \). Denote by \( \mathbb{C}[G]_q \subset (U_q\mathfrak{g})^* \) the Hopf subalgebra of all matrix coefficients of weight finite dimensional \( U_q\mathfrak{g} \) representations. \( \mathbb{C}[G]_q \) is a \( U_q\mathfrak{g} \) module algebra:

\[
(\xi f)(\eta) = f(\eta \xi), \quad \xi, \eta \in U_q\mathfrak{g}, f \in \mathbb{C}[G]_q.
\]

The algebra \( \mathbb{C}[G]_q \) is called the algebra of regular functions on the quantum group \( G \).

Introduce special notations for the elements of \( \mathbb{C}[G]_q \). Consider the finite dimensional simple \( U_q\mathfrak{g} \)-module \( L(\varpi_{l_0}) \) with the highest weight \( \varpi_{l_0} \in P_+ \). Equip it with an invariant scalar product \( \langle \cdot, \cdot \rangle \) (as usual in the compact quantum group theory). Following N. Reshetikhin and V. Lakshmibai (see [5]), choose nonzero vectors \( \{ v_\mu \} \in L(\varpi_{l_0})_\mu \) for all weights \( \mu \in W \varpi_{l_0} \). Let

\[
c_{\lambda, \mu}(\xi) = \frac{\langle \xi v_\mu, v_\lambda \rangle}{\|v_\mu\|\|v_\lambda\|} \quad \mu, \lambda \in W \varpi_{l_0}.
\]

For brevity, put

\[
t = c_{\varpi_{l_0}, -\varpi_{l_0}}, \quad t' = c_{\varpi_{l_0}, \varpi_{l_0}}.
\]

Denote by \( \mathbb{C}[X]_q \subset \mathbb{C}[G]_q \) the minimal \( U_q\mathfrak{g} \)-module subalgebra generated by \( t \).

Equip \( \mathbb{C}[X]_q \) with an antilinear involution \(*\) compatible with the involution in \( U_q\mathfrak{g} \), i.e.

\[
(\xi f)^* = (S(\xi))^* f^*, \quad \xi \in U_q\mathfrak{g}, f \in \mathbb{C}[X]_q.
\]
Proposition 1 There exists a unique involution $*$ which equips the algebra $\mathbb{C}[X]_q$ with the $(U_q\mathfrak{g},*)$-module algebra structure, such that $t^* = q^{c_{\omega_0,\rho}} t'$, where $\rho$ is the half-sum of positive roots and the scalar product is defined by $(\alpha_i, \alpha_j) = \delta_{ij} a_{ij}$.

Let $x = tt^*$. Recall that $\mathbb{C}[X]_q$ is an integral domain.

Proposition 2  
1. $x^{\mathbb{Z}^+}$ is an Ore set.
2. There exists a unique extension of the $(U_q\mathfrak{g},*)$-module algebra structure from $\mathbb{C}[X]_q$ to the localization $\mathbb{C}[X]_{q,x}$ of $\mathbb{C}[X]_q$ with respect to the multiplicative set $x^{\mathbb{Z}^+}$. Equip $\mathbb{C}[X]_{q,x}$ with the $U_q\mathfrak{g}$-invariant $\mathbb{Z}$-grading by $\deg t = 1$. (E.g. $\deg(c_{\omega_{\mu,\nu}}) = 1$ for all weights $\mu \in W \omega_{l_0}$.) Define the subalgebra $\mathbb{C}[S(\mathbb{D})]_q = \{ f \in \mathbb{C}[X]_{q,x} | \deg f = 0 \}$. This subalgebra inherits the $(U_q\mathfrak{g},*)$-module algebra structure. This algebra is a quantum analog of the algebra of regular functions on the Shilov boundary $S(\mathbb{D})$ of a bounded symmetric domain of tube type.

3 Examples of algebras of regular functions on the Shilov boundaries

We repeat below the general construction from the previous Sec. for the special case $\mathfrak{g} = \mathfrak{sl}_n$. A Hermitian symmetric pair $(\mathfrak{g},\mathfrak{t})$ is related to the irreducible bounded symmetric domain of tube type only if $N = 2n$ and $l_0 = n$. The corresponding bounded symmetric domain is the unit ball in the standardly normed space of complex matrices $\mathbb{D} = \{ A \in \text{Mat}_{n,n} | AA^* \leq I \}$. The Shilov boundary is isomorphic to the closed $SU_{n,n}$-orbit of the Grassmanian $Gr_{n,\mathbb{C}^{2n}}$ under the Borel embedding.

Recall the well known notation $U_q\mathfrak{su}_{n,n}$ for the $*$-Hopf algebra $(U_q\mathfrak{sl}_{2n},*)$ and $C[\text{Mat}_{2n}]_q$ for the quantum $2n \times 2n$-matrix space defined by the generators $\{t_{ij}\}_{i,j=1,\ldots,2n}$ and the relations (cf. [21])

$$
t_{ik} t_{jk} = qt_{jk} t_{ik}, \quad t_{ki} t_{kj} = qt_{kj} t_{ki}, \quad i < j,
$$

$$
t_{ij} t_{kl} = t_{kl} t_{ij}, \quad i < k \& j > l, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
The compatible involution on \( \mathbb{C}[X]_q \) is given by \( t^* = q^{\frac{n(n+1)}{2}} t' \). The elements \( t, t' \in \mathbb{C}^{\wedge n}_{\{1, \ldots, n\}} \), \( t' = t_{\{n+1, \ldots, 2n\}} \{1, \ldots, n\} \), and \( x = tt^* \) quasi-commute with \( t_{ij} \) for all \( i, j = 1, \ldots, 2n \).

Consider the \*-algebra \( \mathbb{C}[X]_{q,x} \). Obtain a description for the subalgebra of zero-degree elements of \( \mathbb{C}[X]_{q,x} \).

Let \( \mathbb{C}[\text{Mat}_{n}] \subset \mathbb{C}[X]_{q,x} \) be the subalgebra generated by

\[
z^b_a = t_{\{1, \ldots, n\}}^{\wedge n} J_{a,b}, \text{ where } J_{a,b} = \{n + 1, n + 2, \ldots, 2n\} \setminus \{2n + 1 - b\} \cup \{a\}
\]

The defining relations between \( z^b_a \) are similar to (2) [7]:

\[
\begin{align*}
z_{a_1}^{b_1} z_{a_2}^{b_2} &= q z_{a_1}^{b_2} z_{a_2}^{b_1}, & b_1 < b_2, \\
z_{a_1}^{b_1} z_{a_2}^{b_2} &= q z_{a_2}^{b_1} z_{a_1}^{b_2}, & a_1 < a_2, \\
z_{a_1}^{b_1} z_{a_2}^{b_2} &= z_{a_2}^{b_1} z_{a_1}^{b_2}, & b_1 < b_2 & a_1 > a_2, \\
z_{a_1}^{b_1} z_{a_2}^{b_2} - z_{b_1}^{b_2} z_{a_2}^{a_1} &= (q - q^{-1}) z_{b_2}^{b_1} z_{a_2}^{a_1}, & b_1 < b_2 & a_1 < a_2.
\end{align*}
\]

Similarly to (3), put

\[
\sum \prod (-q)^{l(s)} z_{a_1}^{b_1(1)} \ldots z_{a_k}^{b_1(k)} \subset \mathbb{C}[S(\mathbb{D})]_q \]

It is easy to prove that \( \det_q z = t_{1}^{q-1} \). The algebra \( \mathbb{C}[S(\mathbb{D})]_q \) is isomorphic to the localization of the algebra \( \mathbb{C}[\text{Mat}_{n}]_q \) with respect to the Ore system \( (\det_q z)^{\overline{z}^b_a} \) and

\[
(\overline{z}^b_a)^* = (-q)^{a+b-2n} (\det_q z)^{-1} \det_q z^b_a,
\]

where \( \det_q z^b_a \) is the \( q \)-determinant of the matrix derived from \( z \) by deleting the line \( b \) and the column \( a \).

The \( U_q \mathfrak{su}_{n,n} \)-action on \( \mathbb{C}[S(\mathbb{D})]_q \) can be described explicitly:

\[
K^\pm_1 z^b_a = \begin{cases} q^{\pm 1} z^b_a, & a = n & b = n, \\ q^{\pm 1} z^b_a, & a = n & b \neq n & a \neq n & b = n, \\ z^b_a, & \text{otherwise}, \end{cases}
\]

\[
E_n z^b_a = q^{1/2} z^b_a, \quad a = n & b = n, \\
0, & \text{otherwise},
\]

and for all \( k \neq n \)

\[
K^\pm_k z^b_a = \begin{cases} q^{\pm 1} z^b_a, & k < n & a = k, & a > n & b = 2n - k, \\ q^{\pm 1} z^b_a, & k < n & a = k + 1, & > n & b = 2n - k + 1, \\ z^b_a, & \text{otherwise}, \end{cases}
\]

\[
E_k z^b_a = q^{1/2} z^b_a, \quad k < n & a = k, \\
0, & \text{otherwise},
\]

\[
E_{k-1} z^b_a = q^{-1/2} z^b_a, \quad k > n & a = k + 1, \\
0, & \text{otherwise},
\]

The \*-algebra \( \mathbb{C}[S(\mathbb{D})]_q \) is isomorphic to the algebra of regular functions on the quantum group \( U_q \). The latter algebra was equipped with the \( U_q \mathfrak{k} \)-module algebra structure.
Our general construction gives us hidden symmetry (additional structure of $U_q\mathfrak{g}$-module algebra).

Now turn to type $C_n$ Lie algebras. Fix $\mathfrak{g} = \mathfrak{sp}_{2n}$. The pair $(\mathfrak{g}, \mathfrak{t})$ is Hermitian symmetric only if $\mathfrak{t} = \mathfrak{gl}_n$. In this case the bounded symmetric domain is the unit ball in the standardly normed space of complex symmetric $n \times n$-matrices. The $*$-Hopf algebra $(U_q\mathfrak{sp}_{2n}, *)$ is a quantum analog of the universal enveloping algebra $\mathfrak{sp}_{2n}(\mathbb{R})$.

We need an algebra $\mathbb{C}[\text{Mat}^{\text{sym}}_n]_q$ defined by the generators $z_{ij}$ for $1 \leq j \leq i \leq n$ and the relations\footnote{This is a quantum analog of the polynomial algebra on complex symmetric $n \times n$-matrices.}

\[
\begin{align*}
z_{ii}z_{kk} &= q^2 z_{ki}z_{ii}, & i < k \\
z_{ki}z_{kk} &= q^2 z_{kk}z_{ki}, & i < k \\
z_{ij}z_{ik} &= q z_{ik}z_{ij}, & j < k < i \\
z_{ij}z_{kj} &= q z_{kj}z_{ij}, & j < i < k \\
z_{ij}z_{kl} &= z_{kl}z_{ij}, & j < l < k < i \\
z_{ii}z_{jj} &= z_{jj}z_{ii} + q(q^2 - q^{-2}) z_{jj}^2, & i < j \\
z_{ii}z_{jk} &= z_{jk}z_{ii} + (q^2 - q^{-2}) z_{ki}z_{ji}, & i < k < j \\
z_{ik}z_{jj} &= z_{jj}z_{ik} + (q^2 - q^{-2}) z_{kj}z_{ji}, & k < i < j \\
z_{ij}z_{kl} &= z_{kl}z_{ij} + (q - q^{-1})(q z_{li}z_{kj} + z_{ki}z_{lj}), & j < i < l < k \\
z_{ij}z_{kl} &= z_{kl}z_{ij} + (q - q^{-1}) z_{il}z_{kj}, & j < l < i < k \\
z_{ij}z_{kl} &= q z_{kl}z_{ij} + (q - q^{-1}) z_{il}z_{kj}, & j < i < l < k \\
\end{align*}
\]

Remark. The algebra generated by linear functionals on $\{z_{ij}\}$ was introduced in [9], where it was considered as a $U_q\mathfrak{t}$-module algebra.

Denote a quantum analog of determinant of symmetric matrix

\[
\det_q^{\text{sym}} z \overset{\text{def}}{=} \sum_{s \in S_n} (-q)^{-l(s)} q^{n - \sum_{i=1}^n \delta_{s(i)} z_{s(n)} \cdots z_{s(1)}}
\]

with $z_{kl} = q^{-2} z_{lk}$ for $l > k$.

Similarly to the case $A_n$, the algebra $\mathbb{C}[S(\mathbb{D})]_q$ is isomorphic to the localization of the algebra $\mathbb{C}[\text{Mat}^{\text{sym}}_n]_q$ with respect to the Ore system $(\det_q^{\text{sym}} z)^\mathbb{Z}_+$ with the involution

\[
\det_q^{\text{sym}} z_{nn} = \det_q^{\text{sym}} z_{nn} (\det_q^{\text{sym}} z)^{-1},
\]

where $\det_q^{\text{sym}} z_{nn}$ is the $q$-determinant of the matrix derived from $z$ by deleting the line $n$ and the column $n$.

From the general construction, $\mathbb{C}[S(\mathbb{D})]_q$ is equipped with the $(U_q\mathfrak{sp}_{2n}, *)$-module algebra structure. The $U_q\mathfrak{sp}_{2n}$-action is described explicitly via the following:

\[
\begin{align*}
K_n z_{ij} &= \begin{cases}
q^4 z_{ij}, & i = j = n, \\
q^2 z_{ij}, & i = n > j \text{ or } j = n > i, \\
z_{ij}, & \text{otherwise}.
\end{cases} \\
F_n z_{ij} &= q \begin{cases}
z_{ij}, & i = j = n, \\
0, & \text{otherwise}.
\end{cases} \\
E_n z_{ij} &= -q \begin{cases}
z_{nn} z_{ij}, & i = n \geq j \text{ or } j = n \geq i, \\
q^{-1} z_{ni} z_{nj}, & \text{otherwise}.
\end{cases}
\end{align*}
\]
and for $k \neq n$ and $i \geq j$

$$E_k z_{ij} = q^{-1/2} \begin{cases} (q + q^{-1}) z_{i-1,j} & i = j = k + 1, \\
\frac{q}{2} & i = k + 1 \& j < k + 1, \\
\frac{q}{2} & i > k + 1 \& j = k + 1, \\
0 & \text{otherwise}. \end{cases}$$

$$F_k z_{ij} = q^{1/2} \begin{cases} (q + q^{-1}) z_{i+1,j} & i = j = k, \\
\frac{q}{2} & i = k \& j < k, \\
\frac{q}{2} & i > k \& j = k, \\
0 & \text{otherwise}. \end{cases}$$

$$K_k z_{ij} = \begin{cases} q^2 z_{ij} & i = j = k, \\
q^{-2} z_{ij} & i = j = k + 1, \\
q z_{ij} & (i = k \& j < k) \text{ or } (i > k + 1 \& j = k), \\
q^{-1} z_{ij} & (i = k + 1 \& j < k) \text{ or } (i > k + 1 \& j = k + 1), \\
z_{ij} & \text{otherwise}. \end{cases}$$

Here is another interesting fact about the Shilov boundaries.

In the cases $A_n$ and $C_n$ there are points on the Shilov boundaries, i.e. $(U_q \mathfrak{g}, *)$-morphisms $p : \mathbb{C}S(\mathbb{D})_q \to \mathbb{C}$. The respective $*$-morphisms can be rebuilt from the formulas

$$p(z_{ab}^a) = \begin{cases} q^{n-a} & a = b, \\
0 & a \neq b. \end{cases} \quad p(z_{ij}) = \begin{cases} q^{n-j} & i = j, \\
0 & i \neq j. \end{cases}$$

Using these points, we can construct in both cases a $(U_q \mathfrak{g}, *)$-morphism $i : \mathbb{C}S(\mathbb{D})_q \to \mathbb{C}K_q$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mathbb{C}S(\mathbb{D})_q & \xrightarrow{i} & \mathbb{C}K_q \\
p \downarrow & & \downarrow \varepsilon \\
\mathbb{C} & & \mathbb{C}
\end{array}$$

The subalgebra $i(\mathbb{C}S(\mathbb{D})_q)$ in these cases admits a description in the spirit of M. Noumi’s paper [10].

### 4 Degenerate principal series of quantum Harish-Chandra modules related to the Shilov boundary for the case $A_n$

In this section we investigate a quantum analog of the degenerate principal series of $U_q \mathfrak{su}_{n,n}$-modules related to the Shilov boundary of the quantum $n \times n$-matrix unit ball. We give necessary and sufficient conditions for the representations to be irreducible and unitarizable.

In this section we provide $q$-analogs of classical results obtained by K.D. Johnson, S. Sahi, G. Zhang, R.E. Howe, and E.-C. Tan [11, 12, 13, 14, 15]. Another degenerate
principal series is considered in the A. Klimyk and S. Pakuliak paper [16]. Our results are quantum analogs of results from [17]. More detailed results and proofs are given in [18].

Assume first that $\alpha, \beta \in \mathbb{Z}$. Define a representation $\pi_{\alpha, \beta} : U_q \mathfrak{s}l_{2n} \to \text{End}(\mathbb{C}[S(\mathbb{D})]|_q)$ as follows:

$$\pi_{\alpha, \beta}(\xi)f = (\xi \cdot (f(t^\alpha t^\beta)))t^{-\beta}(t)^{-\alpha} = (\xi \cdot (f(\text{det}_q z)^\alpha t^\beta))t^{-\alpha-\beta}(\text{det}_q z)^{-\alpha}$$

for every $\xi \in U_q \mathfrak{s}l_{2n}, f \in \mathbb{C}[S(\mathbb{D})]|_q$. For each $\lambda \in \mathbb{Z}$ we have

$$E_j t^\lambda = 0, \, F_j t^\lambda = 0, \, K_j t^\lambda = t^\lambda, \quad j = 1, \ldots, 2n - 1, \ j \neq n$$

$$E_n t^\lambda = q^{-3/2} \frac{1 - q^{-2\lambda}}{1 - q^{-2}} z_n t^\lambda, \quad F_n t^\lambda = 0, \quad K_{n\pm 1} t^\lambda = q^{ \mp 1} t^\lambda, \quad j = 1, \ldots, 2n - 1, \ j \neq n$$

$$K_{n\pm 1}((\text{det}_q z)^\lambda) = q^{ \pm 2\lambda}(\text{det}_q z)^\lambda, \quad E_n((\text{det}_q z)^\lambda) = -q^{1/2} \frac{1 - q^{-2\lambda}}{1 - q^{-2}} z_n(\text{det}_q z)^\lambda,$$

$$F_n((\text{det}_q z)^\lambda) = q^{1/2} \frac{1 - q^{-2\lambda}}{1 - q^{-2}} z_n^{(\lambda-n-1)|_{1,...,n-1}}(\text{det}_q z)^{\lambda-1}, \quad \lambda \neq 0.$$

From these equalities we see that for each $\xi \in U_q \mathfrak{s}l_{2n}, f \in \mathbb{C}[S(\mathbb{D})]|_q$ the vector function $p_{f, \xi}(q^\alpha, q^\beta) \overset{\text{def}}{=} \pi_{\alpha, \beta}(\xi)(f)$ is a Laurent polynomial of the variables $q^\alpha, q^\beta$. These Laurent polynomials are uniquely defined by their values on the set $\{(q^\alpha, q^\beta)| \alpha, \beta \in \mathbb{Z}\}$ and deliver the canonical ”analytic continuation” for $\pi_{\alpha, \beta}(\xi)(f)$ to $\alpha, \beta \in \mathbb{C}^2$.

Let $(\alpha, \beta) \in \mathbb{C}^2$. Define a representation $\pi_{\alpha, \beta}(\xi)(f) \overset{\text{def}}{=} p_{f, \xi}(\alpha, \beta)$. To prove that the representation $\pi_{\alpha, \beta}$ is well defined for $(q^\alpha, q^\beta) \in \mathbb{C}^2$, it is sufficient to verify some identities for Laurent polynomials. These identities hold for $\alpha, \beta \in \mathbb{Z}$.

Introduce a ”deformation parameter” $h$ by the equality $q = e^{-h/2}$. Clearly, if $\alpha_1 = \alpha_2 + \frac{2\pi}{h}$ and $\beta_1 = \beta_2 + \frac{2\pi}{h}$, then $\pi_{\alpha_1, \beta_1} = \pi_{\alpha_2, \beta_2}$. Then it suffices to consider $(\alpha, \beta) \in D$, where

$$D = \{(\alpha, \beta) \in \mathbb{C}^2 | 0 \leq \text{Im} \alpha < \frac{2\pi}{h}, \ 0 \leq \text{Im} \beta < \frac{2\pi}{h}\}.$$  

It is clear that $\pi_{\alpha, \beta}$ defines a Harish-Chandra module if and only if $q^{\alpha-\beta} \in q^Z$. Note that for any complex $\alpha, \beta$ such that $0 \leq \text{Im} \alpha < \frac{2\pi}{h}, \ 0 \leq \text{Im} \beta < \frac{2\pi}{h}$, the statements $q^{\alpha-\beta} \in q^Z$ and $\alpha - \beta \in \mathbb{Z}$ are equivalent.

**4.1 Equivalence of the representations**

In this subsection we obtain a parameter set $D$ and prove that each representation of the degenerate principal series is equivalent to a representation $\pi_{\alpha, \beta}$ for some $(\alpha, \beta) \in D$.

The representations $\pi_{\alpha, \beta}$ and $\pi_{\alpha-1, \beta+1}$ are equivalent for all $\alpha, \beta$. The corresponding intertwining operator $T : \mathbb{C}[S(\mathbb{D})]|_q \to \mathbb{C}[S(\mathbb{D})]|_q$ is defined as follows: $T(f) = f \cdot (\text{det}_q z)^{-1}$ for every $f \in \mathbb{C}[S(\mathbb{D})]|_q$. Indeed, since for each $f \in \mathbb{C}[S(\mathbb{D})]|_q, \xi \in U_q \mathfrak{s}l_{2n}$

$$\pi_{\alpha-1, \beta+1}(\xi)(f) = (\xi \cdot (f(\text{det}_q z)^{\alpha-1} t^{\beta+\alpha}))t^{-\alpha-\beta}(\text{det}_q z)^{1-\alpha} =$$

$$(\xi \cdot (f(\text{det}_q z)^{-1}(\text{det}_q z)^{\alpha} t^{\beta+\alpha}))t^{-\alpha-\beta}(\text{det}_q z)^{-\alpha}(\text{det}_q z) = \pi_{\alpha, \beta}(\xi)(f(\text{det}_q z)^{-1})\text{det}_q z,$$
we see that $T$ intertwines the representations $\pi_{\alpha,\beta}$ and $\pi_{\alpha-1,\beta+1}$. Therefore without loss of generality we can assume that $\alpha, \beta \in D$, where

$$D = \{(\alpha, \beta) \in \mathbb{C}^2 \mid \alpha - \beta \in \{0, 1\}, 0 \leq \text{Im} \alpha < \frac{2\pi}{h}, 0 \leq \text{Im} \beta < \frac{2\pi}{h}\}.$$ 

**Proposition 3** If $\alpha, \beta \notin \mathbb{Z}$, then the representations $\pi_{\alpha,\beta}$ and $\pi_{-n-\beta,-n-\alpha}$ are equivalent.

The proof follows from the explicit formulas for the intertwining operators given in Section 4.3.

If $\alpha, \beta \in \mathbb{Z}$, then the representations $\pi_{\alpha,\beta}$ and $\pi_{-n-\beta,-n-\alpha}$ are not equivalent. This fact follows from the statement that only one of the representations $\pi_{\alpha,\beta}$ and $\pi_{-n-\beta,-n-\alpha}$ has a finite dimensional subrepresentation for integral $\alpha, \beta$. An explanation of this fact is given in the end of Section 4.2.

Introduce an equivalence relation on $D$. The equivalence class of $(\alpha, \beta)$ consists of one point for $\alpha, \beta \in \mathbb{Z}$ and two points for $\alpha, \beta \notin \mathbb{Z}$:

$$(\alpha_1, \beta_1) \sim (\alpha_2, \beta_2), \quad \text{iff} \quad \begin{cases} \alpha_1 = -n - \beta_2, \beta_1 = -n - \alpha_2 \text{ for } \text{Im} \alpha_1 = \text{Im} \alpha_2 = 0, \\ \alpha_1 = \frac{2\pi}{h} - n - \beta_2, \beta_1 = \frac{2\pi}{h} - n - \alpha_2, \text{ otherwise.} \end{cases}$$

**Proposition 4** The set of equivalence classes $D/\sim$ is in the one-to-one correspondence $(\alpha, \beta) \mapsto \pi_{\alpha,\beta}$ with the set of equivalence classes of representations of the degenerate principal series.

**Proof.** By the above, each representation of the degenerate principal series is equivalent to the representation $\pi_{\alpha,\beta}$ for some $(\alpha, \beta) \in D$.

Prove that the representations $\pi_{\alpha_1,\beta_1}$ and $\pi_{\alpha_2,\beta_2}$, with $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in D$, are equivalent if and only if $(\alpha_1, \beta_1) \sim (\alpha_2, \beta_2)$. For that, we calculate the action of a central element $C \in U_q\mathfrak{sl}_{2n}^{\text{ext}}$ (see [19] for the definition). It can be proved that $\pi_{\alpha,\beta}(C)$ is a scalar operator for all $\alpha, \beta \in D$.

From [20] it follows that there exists a unique central element $C$ which acts on the $U_q\mathfrak{sl}_{2n}$-highest vector $v^{\text{high}}$ with weight $\lambda$ as follows:

$$C(v^{\text{high}}) = \sum_{j=0}^{2n-1} q^{-2(\mu_j, \lambda + \rho)} v^{\text{high}},$$

where $\mu_0 = -1$, $\mu_j = -\varpi_j + \varpi_{j+1}$ for $j = 1, \ldots, 2n - 2$, $\mu_{2n-1} = -\varpi_{2n-1}$.

First let $\alpha, \beta$ be integers. It can be proved that

$$\pi_{\alpha,\beta}(C)(\det_q \mathbf{Z})^\beta = 4 \text{ch} \frac{h}{2}(\alpha + \beta + n)(\sum_{j=0}^{n-1} \text{ch} \frac{h}{2} j)(\det_q \mathbf{Z})^\beta.$$

Hence $\pi_{\alpha,\beta}(C) = 4 \text{ch} \frac{h}{2}(\alpha + \beta + n)(\sum_{j=0}^{n-1} \text{ch} \frac{h}{2} j) \cdot \text{Id}$ for all $(\alpha, \beta) \in D$.

Suppose that $\pi_{\alpha_1,\beta_1}$ and $\pi_{\alpha_2,\beta_2}$ are equivalent. Equivalent representations have the same weight lattice. Therefore $(\alpha_1 - \beta_1) - (\alpha_2 - \beta_2) \in 2\mathbb{Z}$. Since $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in D$, we see that $(\alpha_1 - \beta_1) - (\alpha_2 - \beta_2) = 0$. 

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Then the equivalent representations $\pi_{a_1, b_1}$ and $\pi_{a_2, b_2}$ have the same values of central characters, which means that
\[
(\text{ch} \frac{h}{2}(a_1 + b_1 + n) - \text{ch} \frac{h}{2}(a_2 + b_2 + n)) \sum_{j=0}^{n-1} \text{ch} \frac{h}{2} j = 0
\]
Since $0 \leq \text{Im} \alpha_1 < \frac{2\pi}{h}$, $0 \leq \text{Im} \beta_1 < \frac{2\pi}{h}$, $0 \leq \text{Im} \alpha_2 < \frac{2\pi}{h}$, and $0 \leq \text{Im} \beta_2 < \frac{2\pi}{h}$, we have that $a_1 + b_1 = a_2 + b_2$, or $a_1 + b_1 = -a_2 - b_2 - 2n$, or $a_1 + b_1 = -a_2 - b_2 - 2n - \frac{4\pi i}{h}$. If $a_1 + b_1 = a_2 + b_2$, then $a_1 = a_2$ and $b_1 = b_2$. For any fixed non-integral $a_1, b_1$ there is a unique pair $(a_2, b_2) \in D$ such that either $a_1 + b_1 = -a_2 - b_2 - 2n$ or $a_1 + b_1 = -a_2 - b_2 - 2n - \frac{4\pi i}{h}$, and $(a_1, b_1) \sim (a_2, b_2)$. Although for integral parameters $\pi_{a_1, b_1}$ and $\pi_{a_2, b_2}$ are not equivalent, because the only one of them has a finite-dimensional subrepresentation. This can be deduced from Corollary \[\square\] Thus each equivalence class in $D$ is assigned to a unique equivalence class of representations of the degenerate principal series $\pi_{a, b}$.

4.2 Reducibility of $\pi_{a, b}$

Let $U_q \mathfrak{k}_{ss} \subset U_q \mathfrak{sl}_{2n}$ be the Hopf subalgebra generated by $E_j, F_j, K_j^{\pm 1}$, $j = 1, \ldots, 2n - 1$, $j \neq 0$ and $U_q \mathfrak{k} \subset U_q \mathfrak{sl}_{2n}$ be the Hopf subalgebra generated by $K_n^{\pm 1}$ and $U_q \mathfrak{k}_{ss}$.

Note that $\pi_{a, b}|_{U_q \mathfrak{k}_{ss}}$ does not depend on $a, b$. The following preliminary result on reducibility of $\pi_{a, b}$ is well known in the classical case. For brevity, set\[^2\]
\[z^k = z^{k(1, \ldots, k)}_{\{1, \ldots, k\}}.\]

Introduce the following notation: $\hat{K} = \{k = (k_1, \ldots, k_n) \in \mathbb{Z}^n| k_1 \geq k_2 \geq \ldots \geq k_n\}$, $e_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^n$.

**Proposition 5** The representation space $\mathbb{C}[S(D)]_q$ for $\pi_{a, b}$ splits into a sum of simple pairwise non-isomorphic $U_q \mathfrak{k}$-modules as follows.\[^3\]
\[\mathbb{C}[S(D)]_q = \bigoplus_{\hat{K} \in \hat{K}} V_{\hat{K}}, \text{ with } V_{\hat{K}} = \pi_{a, b}(U_q \mathfrak{k}) \cdot \psi_{\hat{K}} \text{ and } \psi_{\hat{K}} = (z^{\hat{1}})^{k_1} \cdots (z^{\hat{n}})^{k_n}(z^{\hat{1}})^{k_n}.
\]

**Remark.** It can be easily verified that $\psi_{\hat{K}}$ is a $U_q \mathfrak{k}$-highest vector with weight $(k_1 - k_2, \ldots, k_{n-1} - k_n, 2k_n + \alpha - \beta, k_{n-1} - k_n, \ldots, k_1 - k_2)$. Then the highest weight of the simple $U_q \mathfrak{k}$-module $V_{\hat{K}}$ is equal to $(k_1 - k_2, \ldots, k_{n-1} - k_n, 2k_n + \alpha - \beta, k_{n-1} - k_n, \ldots, k_1 - k_2)$.

**Proposition 6** The representation $\pi_{a, b}$ is irreducible if and only if $a, b$ satisfy the following equivalent conditions.\[^4\]
1. $a \notin \mathbb{Z}$;
2. $b \notin \mathbb{Z}$.

---

\[^2\]Note that, obviously, $z^{\hat{n}} = \det_q z$.

\[^3\]These isotypic components are $U_q \mathfrak{k}_{ss}$-isomorphic. However, they are not $U_q \mathfrak{k}$-isomorphic, since the action of $\pi_{a, b}(K_n)$ depends on $a, b$.

\[^4\]Since $\alpha - \beta \in \mathbb{Z}$, these conditions are equivalent.
Now suppose $\alpha, \beta \in \mathbb{Z}$. We investigate reducibility and proper subrepresentations of $\pi_{\alpha, \beta}$. We describe results with figures as in [11, 17].

Each $U_q$-isotypic component $V_k$ is assigned to the point $(k_1, \ldots, k_n) \in \mathbb{R}^n$. Thus $\hat{K}$ is assigned to the set $K^+ = \{(k_1, \ldots, k_n) \mid k_1 \geq \ldots \geq k_n\} \subset \mathbb{R}^n$. Consider $2n$ hyperplanes:

$$L_j^+: k_j = \beta + j - 1; \quad L_j^-: k_j = -\alpha - n + j.$$  

These hyperplanes are parallel to the coordinate axis and pass through points with integral coordinates. The distance between $L_j^+$ and $L_j^-$ is equal to $\alpha + \beta + n - 1$.

Investigate the example $n = 2$. In this case $L_j^+$, $j = 1, 2$, are just lines on the plane $\mathbb{R}^2$, parallel to coordinate axis. Consider different values of $\alpha + \beta$.

Case 1. $\alpha + \beta \geq 0$. In this case $L_1^+$ lies to the right of $L_1^-$, $L_2^+$ lies higher than $L_2^-$ (see Fig.1). The intersection point of $L_1^+$ and $L_2^-$ has the coordinates $(\beta, -\alpha)$ and belongs to $K^+$. Arrows attached to $L_j^\pm$ show the direction where $\pi_{\alpha, \beta}$ "moves" the isotypic components. There exists a unique simple submodule $V^s = \bigoplus_{\{k \in \hat{K} \mid k_1 \leq \beta, k_2 \geq -\alpha\}} V_k$ in $\mathbb{C}[S(D)]_q$.

Case 2. $\alpha + \beta = -1$. In this case $L_1^+$ and $L_1^-$, $L_2^+$ and $L_2^-$ coincide. The intersection point of $L_1^+$ and $L_2^+$ does not belong to $K^+$ (Fig.2). There are two simple submodules in $\mathbb{C}[S(D)]_q$: $V^1_s = \bigoplus_{\{k \in \hat{K} \mid k_1 = -1 - \alpha\}} V_k$ and $V^2_s = \bigoplus_{\{k \in \hat{K} \mid k_2 = -\alpha\}} V_k$.

Fig.1. The structure of $\pi_{\alpha, \beta}$ with $\alpha + \beta \geq 0$.

Fig.2. The structure of $\pi_{\alpha, \beta}$ with $\alpha + \beta = -1$. 

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Fig. 2. The structure of $\pi_{\alpha,\beta}$ with $\alpha + \beta = -1$.

Case 3. $\alpha + \beta = -2$. In this case $\mathcal{L}^+_1$ lies to the left of $\mathcal{L}^-_1$, $\mathcal{L}^+_2$ lies lower than $\mathcal{L}^-_2$. However, $\mathcal{L}^-_1$ and $\mathcal{L}^+_2$ meet at the point with coordinates $(-\alpha - 1, \beta + 1)$ (see Fig.3). Besides, the distance between $\mathcal{L}^+_j$ and $\mathcal{L}^-_j$ is 1. This shows that $\mathbb{C}[S(D)]_q$ is a direct sum of three submodules:

$$
V_1^* = \bigoplus_{\{\mathbf{k} \in \hat{K}|k_1 \leq \beta\}} V_{\mathbf{k}}, \quad V_2^* = \bigoplus_{\{\mathbf{k} \in \hat{K}|k_2 \geq -\alpha\}} V_{\mathbf{k}}, \quad V_3^* = \bigoplus_{\{\mathbf{k} \in \hat{K}|k_1 \geq -\alpha - 1, k_2 \leq \beta + 1\}} V_{\mathbf{k}}.
$$
Fig.3. The structure of $\pi_{\alpha,\beta}$ with $\alpha + \beta = -2$.

Case 4. $\alpha + \beta \leq -3$. In this case the intersection point of $L^{-}_1$ and $L^{+}_2$ belongs to $K^+$ (see Fig.4). Also, there are simple submodules $V_1^\ast$, $V_2^\ast$, $V_3^\ast$ in $\mathbb{C}[S(D)]_q$, but $\mathbb{C}[S(D)]_q$ does not decompose into their direct sum.

Fig.4. The structure of $\pi_{\alpha,\beta}$ with $\alpha + \beta \leq -3$. 
Turn now to the general case. Consider all possible values of $\alpha + \beta + n - 1$.

**Case 1.** $\alpha + \beta + n - 1 \geq 1$. In this case the hyperplanes $\mathcal{L}_j^\pm$, $j = 1, \ldots, n$, bound in $K^+$ a subset that corresponds to a unique simple finite dimensional submodule

$$V^s = \bigoplus_{\{k \in \mathcal{K} | -\alpha - n + j \leq \beta + j - 1 \text{ for all } j = 1, \ldots, n\}} V_k.$$  

**Case 2.** $\alpha + \beta + n - 1 = 0$. In this case the hyperplanes $\mathcal{L}_j^+$ and $\mathcal{L}_j^-$ coincide. There are $n$ simple submodules in $\mathbb{C}[S(\mathbb{D})]_q$:

$$V_j^s = \bigoplus_{\{k \in \mathcal{K} | k_j = \beta + j - 1\}} V_k, \quad j = 1, \ldots, n. \quad (4)$$

**Case 3.** $\alpha + \beta = -n$. Here the distance between $\mathcal{L}_j^+$ and $\mathcal{L}_j^-$ is 1. This allows one to decompose the set $\mathcal{K}$ into a direct sum of $n + 1$ subsets $\mathcal{K}_i$, $i = 1, \ldots, n + 1$, those correspond to the simple submodules: $V_i^s = \bigoplus_{k \in \mathcal{K}_i} V_k \subset \mathbb{C}[S(\mathbb{D})]_q$. The subsets $\mathcal{K}_i$ are defined as follows:

$$\mathcal{K}_i = \{k \in \mathcal{K} | k_{i-1} \geq -\alpha - n + i - 1, \beta + i - 1 \geq k_i\}$$

(for $i = 1$ and $i = n + 1$ we put respectively $\mathcal{K}_1 = \{k \in \mathcal{K} | k_1 \leq \beta\}$ and $\mathcal{K}_{n+1} = \{k \in \mathcal{K} | k_n \geq -\alpha\}).$

**Case 4.** $\alpha + \beta + n - 1 \leq -2$. Also, there are simple submodules corresponded to $\mathcal{K}_i$. However, $\mathbb{C}[S(\mathbb{D})]_q$ is not their direct sum.

Thus we have proved the following

**Corollary 1** For $\alpha, \beta \in \mathbb{Z}$ only one of the representations $\pi_{\alpha, \beta}$ and $\pi_{-n-\beta, -n-\alpha}$ has an irreducible finite dimensional subrepresentation.

### 4.3 Intertwining operators

In this section we construct the intertwining operators between the representations $\pi_{\alpha, \beta}$ and $\pi_{-n-\beta, -n-\alpha}$ for non-integral $\alpha, \beta$. This allows one to prove Proposition 3.

Let $A : \mathbb{C}[S(\mathbb{D})]_q \rightarrow \mathbb{C}[S(\mathbb{D})]_q$ be an intertwining operator, i.e., for all $\xi \in U_q\mathfrak{sl}_2$, $v \in \mathbb{C}[S(\mathbb{D})]_q$, we have $A \pi_{\alpha, \beta}(\xi)(v) = \pi_{-n-\beta, -n-\alpha}(\xi)(Av)$. The operators $\pi_{\alpha, \beta}(U_q^\mathfrak{g})$ are independent of $\alpha, \beta$ and $\pi_{\alpha, \beta}(K_n) = \pi_{-n-\beta, -n-\alpha}(K_n)$. Also, $V_{\mathcal{K}}$ and $V_{\mathcal{M}}$ are non-isomorphic $U_q^\mathfrak{g}$-modules for $\mathcal{K} \neq \mathcal{M}$. Then $A(\alpha, \beta)|_{V_\mathcal{K}} = a_{\mathcal{K}}(\alpha, \beta)$, $a_{\mathcal{K}}(\alpha, \beta) \in \mathbb{C}$. By the additional assumption $a_{\mathcal{K}}(\alpha, \beta) = 1$, we have the explicit formulas for the coefficients $a_{\mathcal{K}}(\alpha, \beta) = A(\alpha, \beta)|_{V_\mathcal{K}}$ of the intertwining operator $A$

$$a_{\mathcal{K}}(\alpha, \beta) = \prod_{j=1}^n P_j(\alpha, \beta), \quad (5)$$

where

$$P_j(\alpha, \beta) = \begin{cases} \prod_{i=0}^{k_j-1} \frac{1-q^{2(\alpha+i+j+1)}}{1-q^{2(-\beta+i+j+1)}}, & \text{for } k_j > 0, \\ 1, & \text{for } k_j = 0, \\ \prod_{i=1+k_j}^0 \frac{1-q^{2(-\beta+i+j)}}{1-q^{2(\alpha+i+j+1)}}, & \text{for } k_j < 0. \end{cases}$$
For fixed $\alpha - \beta \in \mathbb{Z}$, $A$ is a meromorphic operator-valued function with simple poles at integral points. Note that $A$ coincides up to a multiplicative constant with the so-called standard intertwining operator, and the constant can be expressed from a $q$-analog of the Harish-Chandra $c$-function.

4.4 Unitarizable representations of the degenerate principal series

In this section we list necessary and sufficient conditions for modules of the degenerate principal series and their simple submodules to be unitarizable.

Recall the definition of unitarizable module. Let $A$ be a $*$-Hopf algebra, $W$ an $A$-module. An $A$-module $W$ is unitarizable if there exists a scalar product $5$ $(\cdot, \cdot)$, which is $A$-invariant, i.e.,

$$(au, v) = (u, a^*v) \quad \text{for any } u, v \in W, \ a \in A.$$ 

We can present the following series of simple unitary representations of the degenerate principal series related to the Shilov boundary.

The principal unitary series: $\Re(\alpha + \beta) = -n, \alpha, \beta \notin \mathbb{Z}$. In this case all representations are unitarizable. The invariant scalar product is obtained from the results of [22].

The complementary series: $\Im(\alpha + \beta) = 0, \ [-\alpha - n] = [\beta], \alpha, \beta \notin \mathbb{Z}$. In this case the representations $\pi_{\alpha,\beta}$ are unitarizable too.

The strange series: $\Im \alpha = \pi \overline{\frac{\pi}{n}}$. For such values of the parameters the respective representations $\pi_{\alpha,\beta}$ are irreducible and unitarizable. This series of representations has no classical analog (cf. [16]).

Now let $\alpha, \beta \in \mathbb{Z}$. (Recall that in this case $\pi_{\alpha,\beta}$ is reducible.) For such $\alpha, \beta$ there might exist unitarizable simple submodules in the respective module (we will mention them below), although the module is not unitarizable. Consider all possible cases:

**Case 1.** $\alpha + \beta \geq 2 - n$. In this case the representation is not unitarizable and its unique irreducible subrepresentation is not unitarizable too.

**Case 2.** $\alpha + \beta = 1 - n$. In this case there exist $n$ irreducible unitarizable subrepresentations of the representation $\pi_{\alpha,1-n-\alpha}$. Precisely, $V^s_j$ (see (4)) is a simple submodule in $\mathbb{C}[S(D)]_q$ for any $j = 1, \ldots, n$. Notice that each $V^s_j$ can be equipped with a $U_qsu_{n,n}$-invariant scalar product $(\cdot, \cdot)$. Such modules are called small representations because they have ”poor” decompositions into isotypic components.

**Case 3.** $\alpha + \beta = -n$. In this case the representations are completely reducible, their irreducible subrepresentations $V^s_i, \ i = 1, \ldots, n + 1$, (see Section 4.2) are unitarizable (actually, the required invariant scalar product is the same as that for the principal unitary series).

**Case 4.** $\alpha + \beta \leq -1 - n$. In this case the submodules $V^s_i, \ i = 1, \ldots, n + 1$, are unitarizable, although there exist non-unitarizable quotient modules in $\mathbb{C}[S(D)]_q$.

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5I.e., sesquilinear Hermitian-symmetric positive form.
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