QUASI-CARLEMAN OPERATORS AND THEIR SPECTRAL PROPERTIES

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Abstract. The Carleman operator is defined as integral operator with kernel \((t + s)^{-1}\) in the space \(L^2(\mathbb{R}_+)\). This is the simplest example of a Hankel operator which can be explicitly diagonalized. Here we study a class of self-adjoint Hankel operators (we call them quasi-Carleman operators) generalizing the Carleman operator in various directions. We find explicit formulas for the total number of negative eigenvalues of quasi-Carleman operators and, in particular, necessary and sufficient conditions for their positivity. Our approach relies on the concepts of the sigma-function and of the quasi-diagonalization of Hankel operators introduced in the preceding paper of the author.

1. Introduction

1.1. Hankel operators can be defined as integral operators

\[
(Hf)(t) = \int_0^\infty h(t + s)f(s)ds
\]

in the space \(L^2(\mathbb{R}_+)\) with kernels \(h\) that depend on the sum of variables only. We refer to the books [11, 12, 13] for basic information on Hankel operators. Of course \(H\) is symmetric if \(h(t) = h(t)\). There are very few cases when Hankel operators can be explicitly diagonalized. The most simple and important case \(h(t) = t^{-1}\) was considered by T. Carleman in [3].

Here we study a class of Hankel operators (quasi-Carleman operators) with kernels

\[
h(t) = (t + r)^{-q}e^{-\alpha t}, \quad r \geq 0,
\]

where \(\alpha\) and \(q\) are real numbers. We will see that a Hankel operator with kernel (1.2) can be correctly defined as a self-adjoint operator in the Hilbert space \(L^2(\mathbb{R}_+)\) if

\[
either \alpha > 0 \text{ or } \alpha = 0, \quad q > 0,
\]

that is, \(h(t) \to 0\) as \(t \to \infty\). The singularity of \(h(t)\) at the point \(t = 0\) may be arbitrary. There are of course no chances to explicitly find the spectrum and eigenfunctions of quasi-Carleman operators. The only exceptions are the cases \(q = 1, \alpha = 0\) and \(q = 1, \alpha > 0\).

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To obtain information about spectral properties of quasi-Carleman operators, we here use the method of quasi-diagonalization of Hankel operators suggested in [21]. Roughly speaking, this method relies on the identity
\[ H = L^* \Sigma L \] (1.4)
where \( L \) is the Laplace transform defined by the relation
\[ (Lf)(\lambda) = \int_0^\infty e^{-t\lambda} f(t) dt \] (1.5)
and \( \Sigma \) is the multiplication operator by a function \( \sigma(\lambda) \). We use the term “sigma-function” of the Hankel operator \( H \) (or of the kernel \( h(t) \)) for this function. It is formally linked to the kernel \( h(t) \) of \( H \) by the relation
\[ h(t) = \int_{-\infty}^{\infty} e^{-t\lambda} \sigma(\lambda) d\lambda, \] (1.6)
that is, \( h(t) \) is the two-sided Laplace transform of \( \sigma(\lambda) \). We consider here \( L \) as a mapping of appropriate spaces of test functions so that \( L^* \) is the corresponding mapping of dual spaces (of distributions).

It is clear from formula (1.6) that \( \sigma(\lambda) \) can be a regular function only for kernels \( h(t) \) satisfying some specific analytic conditions. Without such very restrictive assumptions, \( \sigma \) is necessarily a distribution. For example, for kernels (1.2) the sigma-function is given by the explicit formula
\[ \sigma(\lambda) = \frac{1}{\Gamma(q)} (\lambda - \alpha)^{q-1} e^{-r(\lambda-\alpha)}, \quad -q \notin \mathbb{Z}_+, \] (1.7)
where \( \Gamma(\cdot) \) is the gamma function. The function \( \mu_+^{q-1} \) is regular for \( q > 0 \) (in this case \( \mu_+^{q-1} = \mu^{q-1} \) for \( \mu > 0 \) and \( \mu_+^{q-1} = 0 \) for \( \mu \leq 0 \)), but it is a singular distribution for \( q \leq 0 \). For \(-n-1 < q < -n\) where \( n \in \mathbb{Z}_+ \) and a test function \( \varphi(\lambda) \), it is defined by the standard formula
\[ \int_{-\infty}^{\infty} (\lambda - \alpha)^{q-1} \varphi(\lambda) d\lambda = \int_{\alpha}^{\infty} (\lambda - \alpha)^{q-1} \left( \varphi(\lambda) - \sum_{p=0}^{n} \frac{1}{p!} \varphi^{(p)}(\alpha)(\lambda - \alpha)^p \right) d\lambda. \] (1.8)
If \(-q \in \mathbb{Z}_+\), then \( \sigma(\lambda) \) is a linear combination of derivatives \( \delta^{(p)}(\lambda - \alpha) \) of delta-functions for \( p = 0, 1, \ldots, -q \) (note that the corresponding Hankel operator \( H \) has finite rank if \( \alpha > 0 \)). Thus, in general, \( \sigma(\lambda) \) is a singular distribution so that \( \Sigma \) need not even be defined as an operator. Therefore, instead of operators, we work with the corresponding quadratic forms which is both more general and more convenient. So, to be precise, instead of (1.4) we consider the identity
\[ (H f, f) = (\Sigma L f, L f) \] (1.9)
on the set of elements \( f \in C_0^\infty(\mathbb{R}_+) \) and assume only that \( h \in C_-^\infty(\mathbb{R}_+) \). Note that \( L \) acts as an isomorphism of \( C_0^\infty(\mathbb{R}_+) \) onto a set (denoted \( Y \)) of analytic functions and for kernels (1.2) quadratic forms in (1.9) are well defined for all values of \( \alpha, q \in \mathbb{R} \) and \( r \geq 0 \).

It follows from (1.9) that the total numbers of positive \( N_+(H) \) and negative \( N_-(H) \) eigenvalues of the operator \( H \) equal the same quantities for \( \Sigma \):

\[
N_\pm(H) = N_\pm(\Sigma).
\]

In particular, \( \pm H \geq 0 \) if and only if \( \pm \Sigma \geq 0 \). In general, we have to speak about quadratic forms \( (Hf, f) \) for \( f \in C_0^\infty(\mathbb{R}_+) \) and \( (\Sigma w, w) \) for \( w \in \mathcal{Y} \) instead of the operators \( H \) and \( \Sigma \). Under the only assumption \( h \in C_0^\infty(\mathbb{R}_+) \)' the number \( N_+(H) \) is defined as the maximal dimension of linear sets in \( C_0^\infty(\mathbb{R}_+) \) where \( \pm(Hf, f) > 0 \) for all \( f \neq 0 \).

This general construction was applied to kernels (1.2) in [21] where the numbers \( N_\pm(H) \) were explicitly calculated (see Theorem 2.7 below) as a function of \( q \). In particular, it turns out that \( N_\pm(H) \) do not depend on \( \alpha \) and \( r \).

1.2. This paper can be considered as a continuation of [21]. It has two goals. The first is to define Hankel operators \( H \) with kernels (1.2) as self-adjoint operators in the space \( L^2(\mathbb{R}_+) \). We note that such operators are bounded in the following two cases:

1. if \( \alpha > 0 \), then either \( r > 0 \) and \( q \) is arbitrary or \( r = 0 \) and \( q \leq 1 \),
2. if \( \alpha = 0 \), then either \( r > 0 \) and \( q \geq 1 \) or \( r = 0 \) and \( q = 1 \).

It is easy to see that under assumption (1.3) all unbounded quasi-Carleman operators have kernels

\[
h(t) = t^{-q}e^{-\alpha t}, \quad \alpha > 0, \quad q > 1,
\]

or

\[
h(t) = (t + r)^{-q}, \quad q > 0,
\]

where either \( r > 0, q < 1 \) or \( r = 0, q \neq 1 \).

A study of unbounded integral operators goes back to T. Carleman [3] (see also Appendix I to the book [1]). In particular, his general results apply to Hankel operators with kernels satisfying the condition

\[
\int_t^\infty |h(s)|^2 ds < \infty, \quad \forall t > 0.
\]

This condition allows one to define \( H \) as a symmetric but not as a self-adjoint operator. We note that for kernels (1.12) condition (1.13) is not satisfied if \( q \leq 1/2 \); in this case the corresponding operator \( H \) is not defined even on the set \( C_0^\infty(\mathbb{R}_+) \).

We proceed from the identity (1.9) and define \( H \) in terms of the corresponding quadratic form. In view of formula (1.7) for kernels (1.2) the identity (1.9) reads as

\[
\int_0^\infty \int_0^\infty \overline{f(t)} f(s)(t + s + r)^{-q}e^{-\alpha(t+s)} dt ds = \frac{e^{\alpha r}}{\Gamma(q)} \int_{-\infty}^\infty (\lambda - \alpha)^{q-1}e^{-r\lambda}|(Lf)(\lambda)|^2 d\lambda
\]

(1.14)
where \( f \in C_0^\infty(\mathbb{R}+) \) is arbitrary. The following result defines \( H \) as a self-adjoint operator.

**Theorem 1.1.** Let the function \( h(t) \) be given by formula \( (1.2) \) where either \( \alpha > 0 \), \( q \geq 1 \) or \( \alpha = 0 \), \( q > 0 \) (the parameter \( r \geq 0 \) is arbitrary). Then the form \( (1.14) \) defined on the set of functions \( f \in C_0^\infty(\mathbb{R}+) \) admits the closure in the space \( L^2(\mathbb{R}+) \), and it is closed on the set of all \( f \in L^2 \) such that the integral in the right-hand side of \( (1.14) \) is finite. The corresponding Hankel operator is strictly positive.

In particular, this result applies to kernels \( (1.11) \) and \( (1.12) \).

Actually, we consider a more general problem of defining a Hankel operator by means of its sigma-function given by some measure \( dM(\lambda) \) on \([0, \infty)\). Thus we assume that

\[
h(t) = \int_0^\infty e^{-t\lambda} dM(\lambda). \tag{1.15}
\]

Recall that, as shown by H. Widom in \([16]\), the Hankel operator \( H \) with kernel \( (1.15) \) is bounded if and only if the measure in \( (1.15) \) satisfies the condition

\[
M([0, \lambda)) = O(\lambda) \text{ as } \lambda \to 0 \text{ and as } \lambda \to \infty. \tag{1.16}
\]

In particular, for the Lebesgue measure \( dM(\lambda) = d\lambda \) on \( \mathbb{R}^+ \), we have \( h(t) = t^{-1} \) and \( H \) is the Carleman operator.

Our goal is to study the singular case when condition \( (1.16) \) is not satisfied, but \( H \) can be defined as an unbounded positive operator via its quadratic form \((Hf, f)\). We find sufficient (and practically necessary) conditions on the measure \( dM(\lambda) \) guaranteeing that the form \((Hf, f)\) defined on \( C_0^\infty(\mathbb{R}+) \) admits the closure in \( L^2(\mathbb{R}+) \) and describe the domain of its closure. These results are deduced from properties of the Laplace transform \( L \) considered as a mapping of \( L^2(\mathbb{R}^+; dM(\lambda)) \) into \( L^2(\mathbb{R}^+; dM(\lambda)) \). Perhaps, the results on the Laplace transform are of independent interest.

For quasi-Carleman operators with homogeneous kernels, we prove the following spectral result.

**Theorem 1.2.** Let \( H \) be the Hankel operator with kernel \( h(t) = t^{-q} \) where \( q > 0 \) and \( q \neq 1 \). The spectrum of the operator \( H \) coincides with the positive half-line, it has a constant multiplicity and is absolutely continuous.

1.3. Another goal of the paper is to study perturbations of singular Hankel operators \( H_0 \) constructed in Theorem 1.1 by bounded quasi-Carleman operators \( V \) with kernels

\[
v(t) = v_0(t + \rho)^k e^{-\beta t}, \quad v_0 \in \mathbb{R}, \beta \geq 0, \rho \geq 0, k \in \mathbb{R}. \tag{1.17}
\]

The cases \( k < 0 \) and \( k \geq 0 \) turn to to be qualitatively different. According to formula \( (1.7) \) in the first case the sigma-function \( \sigma(\lambda) = \sigma_0(\lambda) + \sigma_v(\lambda) \) of the operator \( H = H_0 + V \) belongs to the set \( L^1_{\text{loc}}(\mathbb{R}^+) \). It implies that \( H \geq 0 \) if \( \sigma(\lambda) \geq 0 \) and \( H \) has infinite negative spectrum if \( \sigma(\lambda) < 0 \) on a set of positive measure. The precise result is stated in Theorem 4.6.

In the case \( k \geq 0 \) we have the following result.
Theorem 1.3. Let $H_0$ be the Hankel operator with kernel $h_0(t)$ given by formula (1.2) where either $q \geq 1$ for $\alpha > 0$ or $q > 0$ for $\alpha = 0$. Let $V$ be the Hankel operator with kernel (1.17) where $\beta > 0$ and $k \geq 0$. Then

$$N_-(H_0 + V) = N_-(V).$$

(1.18)

Thus we obtain the striking result: the total multiplicity of the negative spectrum of the operator $H = H_0 + V$ does not depend on the operator $H_0$. The inequality $N_-(H) \leq N_-(V)$ is of course obvious because $H_0 \geq 0$. On the contrary, the opposite inequality $N_-(H) \geq N_-(V)$ looks surprising because the operator $H_0$ may be “much stronger” than $V$; for example, the Hankel operator with kernel $h_0(t) = t^{-q}$ is never compact and is unbounded unless $q = 1$. Nevertheless its adding to $V$ does not change the total number of negative eigenvalues of the operator $V$. A heuristic explanation of this phenomenon can be given in terms of the sigma-functions. The sigma-function $\sigma_0(\lambda)$ of the operator $H_0$ is continuous and positive while the sign-function $\sigma_v(\lambda)$ of $V$ has a strong negative singularity at the point $\lambda = \beta$. Therefore the sigma-functions of $H$ and $V$ have the same negative singularity. Very loosely speaking, the supports of the functions $\sigma_0(\lambda)$ and $\sigma_v(\lambda)$ are essentially disjoint so that the operators $H_0$ and $V$ “live in orthogonal subspaces”, and hence the positive operator $H_0$ does not affect the negative spectrum of $V$.

Relation (1.18) is also true for perturbations of singular Hankel operators by finite rank Hankel operators. The kernels of these operators are linear combinations of functions (1.17) where $k \in \mathbb{Z}_+$, $r = 0$ and $\text{Re} \beta > 0$. The sigma-function of such kernel (1.17) consists of the delta-function and its derivatives supported at the point $\lambda = \beta$. If $\text{Im} \beta \neq 0$, this sigma-function is more singular than functions (1.7) which impedes the proof of relation (1.18).

Let us compare the results on the negative spectrum of Hankel and differential operators. Let $H = D^2 + V(x)$ be the Schrödinger operator in the space $L^2(\mathbb{R})$. Suppose that $V(x) \leq 0$. If $V(x)$ decays sufficiently rapidly as $|x| \to \infty$, then $N_-(H) < \infty$ and $N_-(H) = \infty$ in the opposite case. Contrary to the Schrödinger case, the negative spectrum of Hankel operators $H = H_0 + V$ is determined not by the behavior of $v(t)$ at singular points $t = 0$ and $t = \infty$ but exclusively by the corresponding sigma-functions.

1.4. Let us briefly describe the structure of the paper. We collect necessary results of [21] in Section 2. Singular Hankel operators are studied in Section 3. In particular, Theorems 1.1 and 1.2 are proven there. Perturbations of singular Hankel operators by bounded quasi-Carleman operators are considered in Section 4. Similar results for finite rank perturbations are discussed in Section 5. In particular, the results of Sections 4 and 5 imply Theorem 1.3. Finally, in Appendix we study the Fourier transform sandwiched by functions one of which is unbounded. This problem is adjacent to that considered in Section 3.
Let us introduce some standard notation: \( S = S(\mathbb{R}) \) is the Schwartz space, \( \Phi \) is the Fourier transform, 
\[
(\Phi u)(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} u(x)e^{-ix\xi}dx.
\]
For spaces of test functions, for example \( S \) and \( C_{0}^{\infty}(\mathbb{R}_{+}) \), we denote by \( S' \) and \( C_{0}^{\infty}(\mathbb{R}_{+})' \) the dual classes of distributions (continuous antilinear functionals). We use the notation \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) for the duality symbols in \( L^{2}(\mathbb{R}_{+}) \) and \( L^{2}(\mathbb{R}) \), respectively. They are linear in the first argument and antilinear in the second argument.

We often use the same notation for a function and for the operator of multiplication by this function. The Dirac function is standardly denoted \( \delta \) symbol, i.e., \( \delta_{n,m} \) is the Kronecker symbol, i.e., \( \delta_{n,n} = 1 \) and \( \delta_{n,m} = 0 \) if \( n \neq m \). The letter \( C \) (sometimes with indices) denotes various positive constants whose precise values are inessential.

We use the special notation \( C \) for the Hankel operator \( H \) with kernel \( h(t) = t^{-1} \), that is, for the Carleman operator. Recall that \( C \) is bounded and it has the absolutely continuous spectrum \([0, \pi]\) of multiplicity 2. Its sigma-function \( \sigma(\lambda) \) equals 1 for \( \lambda \geq 0 \) and it equals 0 for \( \lambda < 0 \).

2. The sigma-function and the main identity

Here we collect some necessary results of \[21\]. In particular, we give the precise definition of the sigma-function \( \sigma(\lambda) \) and discuss the main identity \[1.9\].

2.1. We work on test functions \( f \in C_{0}^{\infty}(\mathbb{R}_{+}) \) and require that \( h \) belong to the dual space \( C_{0}^{\infty}(\mathbb{R}_{+})' \). Let the set \( \mathcal{Y} \) consist of entire functions \( \varphi(\lambda) \) satisfying, for all \( \lambda \in \mathbb{C} \), bounds
\[
|\varphi(\lambda)| \leq C_n(1 + |\lambda|)^{-n}e^{r_{\pm}|\text{Re}\lambda|}, \quad \pm \text{Re}\lambda \geq 0,
\]
for all \( n \) and some \( r_{+} = r_{+}(\varphi) < 0 \); the number \( r_{-} = r_{-}(\varphi) \) may be arbitrary. Thus functions in \( \mathcal{Y} \) exponentially decay as \( \text{Re}\lambda \to +\infty \), and they are exponentially bounded as \( \text{Re}\lambda \to -\infty \). The space \( \mathcal{Y}' \) is of course invariant with respect to the complex conjugation \( \varphi(\lambda) \mapsto \varphi^*(\lambda) := \varphi(\overline{\lambda}) \). By definition, \( \varphi_k(\lambda) \to 0 \) as \( k \to \infty \) in \( \mathcal{Y} \) if all functions \( \varphi_k(\lambda) \) satisfy bounds \[2.1\] with the same constants \( r_{+} < 0, r_{-}, C_n \) and \( \varphi_k(\lambda) \to 0 \) as \( k \to \infty \) uniformly on compact subsets of \( \mathbb{C} \).

Let the Laplace transform \( L \) be defined by formula \[1.5\]. By one of the versions of the Paley-Wiener theorem (see, e.g., the book \[5\] for similar assertions), \( L : C_{0}^{\infty}(\mathbb{R}_{+}) \to \mathcal{Y} \) is the one-to-one continuous mapping of \( C_{0}^{\infty}(\mathbb{R}_{+}) \) onto \( \mathcal{Y} \) and the inverse mapping \( L^{-1} : \mathcal{Y} \to C_{0}^{\infty}(\mathbb{R}_{+}) \) is also continuous. In such cases we say that \( L \) is an isomorphism. Passing to the dual spaces, we see that the mapping \( L^* : \mathcal{Y}' \to C_{0}^{\infty}(\mathbb{R}_{+})' \) is also an isomorphism.

Let us construct the sigma-function.

**Definition 2.1.** Assume that \( h \in C_{0}^{\infty}(\mathbb{R}_{+})' \). The distribution \( \sigma \in \mathcal{Y}' \) defined by the formula
\[
\sigma = (L^*)^{-1}h
\]
is called the sigma-function of the kernel \( h \) or of the corresponding Hankel operator \( H \).

According to this definition for all \( F \in C_0^\infty(\mathbb{R}_+) \), we have the identity
\[
\langle \langle \langle h, F \rangle \rangle \rangle = \langle \langle \langle L^* \sigma, F \rangle \rangle \rangle = \langle \sigma, LF \rangle.
\] (2.3)

By virtue of (2.2), the kernel \( h(t) \) can be recovered from its sigma-function \( \sigma(\lambda) \) by the formula
\[
h(t) = L^* \sigma
\] which gives the precise sense to formal relation (1.6). Thus there is the one-to-one correspondence between kernels \( h \in C_0^\infty(\mathbb{R}_+) \) and their sigma-functions \( \sigma \in \mathcal{Y}' \).

Let us introduce the Laplace convolution
\[
(\bar{f}_1 \ast f_2)(t) = \int_0^t \bar{f}_1(s)f_2(t-s)ds
\]
of functions \( \bar{f}_1, f_2 \in C_0^\infty(\mathbb{R}_+) \). Then it formally follows from (1.1) that
\[
(H f_1, f_2) = \langle h, \bar{f}_1 \ast f_2 \rangle
\]
where we write \( \langle \cdot, \cdot \rangle \) instead of \( (\cdot, \cdot) \) because \( h \) may be a distribution. The following result was established in [21].

**Theorem 2.2.** Let \( h \in C_0^\infty(\mathbb{R}_+)' \), and let \( \sigma \in \mathcal{Y}' \) be defined by formula (2.2). Then the identity
\[
\langle h, \bar{f}_1 \ast f_2 \rangle = \langle \sigma, (Lf_1)^*Lf_2 \rangle
\] (2.4)
holds for arbitrary \( f_1, f_2 \in C_0^\infty(\mathbb{R}_+) \).

The identity (2.4) attributes a precise meaning to (1.4) or (1.9).

2.2. Let \( \mathbb{H}^p_r \) be the Hardy space of functions analytic in the right half-plane. By the Paley-Wiener theorem the operator \((2\pi)^{-1/2}L : L^2(\mathbb{R}_+) \rightarrow \mathbb{H}^2_r \) is unitary. If a Hankel operator \( H \) is bounded, then necessarily the sigma-function \( \sigma \) belongs to the space \((\mathbb{H}^1_r)' \) dual to \( \mathbb{H}^1_r \), and the identity (2.4) extends to all \( f_1, f_2 \in L^2(\mathbb{R}_+) \).

If \( \text{supp} \sigma \) belongs to the right half-plane, it is sometimes convenient to make the exponential change of variables and to define the function (we call it the sign-function)
\[
s(x) = \sigma(e^{-x}), \quad -\pi/2 < \text{Im} x < \pi/2.
\] (2.5)

In particular, for sigma-function (1.7) we have
\[
s(x) = \frac{e^{\alpha r}}{\Gamma(-k)}(e^{-x} - \alpha)^{-k-1}e^{-re^{-x}},
\]

obviously \( s \in \mathcal{S}' \). It follows from (2.5) that
\[
s[u,u] := \langle s, u^*u \rangle = \langle \sigma, w^*w \rangle =: \sigma[w,w] \quad (2.6)
\]
if
\[
u(x) = e^{-x/2}w(e^{-x}).
\] (2.7)
Note that \( w(\lambda) \) is analytic in the half-plane \( \Re \lambda > 0 \) if and only if the corresponding function \( u(x) \) is analytic in the strip \( -\pi/2 < \Im x < \pi/2 \). Moreover, the conditions
\[
\sup_{-\pi/2 < a < \pi/2} \int_{-\infty}^{\infty} |u(x + ia)|^2 dx < \infty
\]
are equivalent.

According to identity (2.6) we can work either with sigma-functions \( \sigma(\lambda) \) and test functions \( w(\lambda) \) or with sign-functions \( s(x) \) and test functions \( u(x) \). Both points of view are equivalent, and we frequently pass from one to another at our convenience.

To recover \( f(t) \) from \( w(\lambda) = (Lf)(\lambda) \), we have to invert the Laplace transform \( L \). To that end, we use its factorization. Let \( \Gamma(z) \) be the gamma function and
\[
(\Gamma g)(\xi) = \Gamma(1/2 + i\xi)g(\xi), \quad \xi \in \mathbb{R}. \tag{2.8}
\]
Note that
\[
|\Gamma(1/2 + i\xi)| = \sqrt{\frac{\pi}{\cosh(\pi\xi)}} =: v(\xi). \tag{2.9}
\]
Put \((Uw)(x) = e^{x/2}w(e^x)\). Obviously the operator \( U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}) \) is unitary, and hence the Mellin transform \( M = \Phi U \) is also unitary. Let \( \mathcal{J} \), \((\mathcal{J} g)(\xi) = g(-\xi)\), be the reflection. Then the Laplace transform factorizes as
\[
L = M^{-1} \mathcal{J} M \tag{2.10}
\]
so that \( L^{-1} = M^{-1} \Gamma^{-1} \mathcal{J} M \) and hence
\[
(M f)(\xi) = \Gamma(1/2 + i\xi)^{-1}(Mw)(-\xi) = \Gamma(1/2 + i\xi)^{-1}(\Phi u)(\xi). \tag{2.11}
\]
This formula allows us to recover \( f(t) \) if either \( w(\lambda) \) or \( u(x) = e^{-x/2}w(e^{-x}) \) are given.

2.3. Suppose now that \( h(t) = \overline{h(t)} \) for all \( t > 0 \), or to be more precise \( \langle h, F \rangle = \langle h, F \rangle \) for all \( F \in C_0^\infty(\mathbb{R}_+) \). Then it follows from (2.3) that the sigma-function is also real, that is, \( \langle \sigma, w \rangle = \langle \sigma, w^* \rangle \) for all \( w \in \mathcal{Y} \).

Below we use the following natural definition.

**Definition 2.3.** Let \( h[\varphi, \varphi] \) be a real quadratic form defined on a linear set \( D \). We denote by \( N_\pm(h) = N_\pm(h; D) \) the maximal dimension of linear sets \( M_\pm \subset D \) such that \( \pm h[\varphi, \varphi] > 0 \) for all \( \varphi \in M_\pm, \varphi \neq 0 \).

Definition 2.3 means that there exists a linear set \( M_\pm \subset D \), \( \dim M_\pm = N_\pm(h; D) \), such that \( \pm h[\varphi, \varphi] > 0 \) for all \( \varphi \in M_\pm, \varphi \neq 0 \), and for every linear set \( M'_\pm \subset D \) with \( \dim M'_\pm > N_\pm(h; D) \) there exists \( \varphi \in M'_\pm, \varphi \neq 0 \), such that \( \pm h[\varphi, \varphi] \leq 0 \).

Of course, if the set \( D \) is dense in a Hilbert space \( \mathcal{H} \) and \( h[\varphi, \varphi] \) is semibounded and closed on \( D \), then for the self-adjoint operator \( H \) corresponding to \( h \), we have \( N_\pm(H) = N_\pm(h; D) \). In particular, this is true for bounded operators \( H \).

We apply Definition 2.3 to the forms \( h[f, f] = \langle h, f \ast f \rangle \) on \( f \in C_0^\infty(\mathbb{R}_+) \) and \( \sigma[w, w] = \langle \sigma, w^*w \rangle \) on \( w \in \mathcal{Y} \).
Since \( L : C_0^\infty(\mathbb{R}_+) \to \mathcal{Y} \) is an isomorphism, the following assertion is a direct consequence of Theorem 2.2

**Theorem 2.4.** Let \( h \in C_0^\infty(\mathbb{R}_+)'. \) Then \( \sigma = (L^*)^{-1}h \in \mathcal{Y}' \) and

\[ N_\pm(h;C_0^\infty(\mathbb{R}_+)) = N_\pm(\sigma;\mathcal{Y}). \]

In particular, the form \( \pm\langle h, \bar{f} \ast f \rangle \geq 0 \) for all \( f \in C_0^\infty(\mathbb{R}_+) \) if and only if the form \( \pm\langle \sigma, w \ast w \rangle \geq 0 \) for all \( w \in \mathcal{Y} \).

Thus a Hankel operator \( H \) is positive (or negative) if and only if its sigma-function \( \sigma(\lambda) \) is positive (or negative).

**2.4.** The following assertion (see [18]) is very convenient for calculation of the numbers \( N_\pm(\sigma;\mathcal{Y}) \). Its proof relies on formula (2.10).

**Lemma 2.5.** Suppose that distribution \( \sigma \) belongs to the class \( S' \). Then

\[ N_\pm(\sigma;\mathcal{Y}) = N_\pm(\sigma;C_0^\infty(\mathbb{R}_+)). \]

Putting together Theorem 2.4 and Lemma 2.5 we obtain the following result.

**Theorem 2.6.** Let \( h \in C_0^\infty(\mathbb{R}_+)'. \) Suppose that the corresponding distribution \( \sigma \) belongs to the class \( S' \). Then

\[ N_\pm(h;C_0^\infty(\mathbb{R}_+)) = N_\pm(\sigma;C_0^\infty(\mathbb{R}_+)). \]

In applications to quasi-Carleman operators, the following result was obtained also in [21].

**Theorem 2.7.** Let \( h(t) \) be given by formula (1.2) where \( \alpha \in \mathbb{R} \) and \( r \geq 0 \). If \( q > 0, \) then \( \langle h, \bar{f} \ast f \rangle \geq 0 \) for all \( f \in C_0^\infty(\mathbb{R}_+) \). If \( q < 0 \) but \( |q| \not\in \mathbb{Z}_+ \), then \( N_+(h;C_0^\infty(\mathbb{R}_+)) = \lfloor |q| \rfloor/2 + 1, N_-(h;C_0^\infty(\mathbb{R}_+)) = \infty \) for even \( |q| \) and \( N_+(h;C_0^\infty(\mathbb{R}_+)) = (\lfloor |q| \rfloor + 1)/2, N_-(h;C_0^\infty(\mathbb{R}_+)) = \infty \) for odd \( |q| \).

This result is of course deduced from formula (1.7) for the sigma-function. If \( q > 0, \) then \( \sigma(\lambda) \geq 0 \). If \( q < 0, \) then \( \Gamma(q) < 0 \) for \( \lfloor |q| \rfloor \) even and \( \Gamma(q) > 0 \) for \( \lfloor |q| \rfloor \) odd. Therefore, for example, for even \( \lfloor |q| \rfloor \), the sigma-function is continuous and negative everywhere except the point \( \lambda = \alpha \) which ensures that \( N_-(h;C_0^\infty(\mathbb{R}_+)) = \infty \). The singularity at the point \( \lambda = \alpha \) produces a finite number of positive eigenvalues.

3. **Singular quasi-Carleman operators**

3.1. Let us now consider Hankel operator with kernels (1.15) where the (locally finite nonnegative) measure \( dM(\lambda) \) on \( [0, \infty) \) satisfies the condition

\[ \int_0^\infty e^{-t\lambda}dM(\lambda) < \infty, \quad \forall t > 0. \] (3.1)

Then the Hankel quadratic form admits the representation

\[ \langle h, \bar{f} \ast f \rangle = \int_0^\infty \|\lambda f(\lambda)\|^2dM(\lambda), \quad \forall f \in C_0^\infty(\mathbb{R}_+), \] (3.2)
where $L$ is the Laplace transform defined by formula (1.5). According to formula (1.7) kernels (1.11) and (1.12) satisfy assumption (3.1) with $dM(\lambda) = \sigma(\lambda) d\lambda$.

For the study of the form (3.2), we consider $L$ as the mapping of $L^2(\mathbb{R}⁺) =: L^2$ into $L^2(\mathbb{R}_+; dM) =: L^2(M)$; the scalar product in the space $L^2(M)$ will be denoted $(\cdot, \cdot)_M$. Note that for an arbitrary $f \in L^2$, the integral $(Lf)(\lambda)$ converges and the function $(Lf)(\lambda)$ is continuous for all $\lambda > 0$; moreover,

$$|(Lf)(\lambda)| \leq (2\lambda)^{-1/2} \|f\|.$$  

Let $\mathcal{E}$ (resp. $\mathcal{E}_M$) consist of functions $f \in L^2$ (resp. $g \in L^2(M)$) compactly supported in $\mathbb{R}_+$. If $f \in \mathcal{E}$, then $(Lf)(\lambda)$ is a continuous function for all $\lambda \geq 0$ and $(Lf)(\lambda) = O(e^{-c\lambda})$ with some $c = c(f) > 0$ as $\lambda \to \infty$. In particular, $Lf \in L^2(M)$ for such $f$.

Let us also introduce the operator $L_*$ formally adjoint to $L$ by the equality

$$(L_*g)(t) = \int_0^\infty e^{-t\lambda} g(\lambda) dM(\lambda). \quad (3.3)$$

For an arbitrary $g \in L^2(M)$, integral (3.3) converges for all $t > 0$, the function $(L_*g)(t)$ is continuous and is bounded as $t \to \infty$. If $g \in \mathcal{E}_M$, then $(L_*g)(t)$ is a continuous function for all $t \geq 0$ and $(L_*g)(t) = O(e^{-ct})$ with some $c = c(g) > 0$ as $t \to \infty$; in particular, $L_*g \in L^2$.

The following assertion is a direct consequence of the Fubini theorem.

**Lemma 3.1.** Let assumption (3.1) be satisfied. Suppose that either $f \in \mathcal{E}$ and $g \in L^2(M)$ or $f \in L^2$ and $g \in \mathcal{E}_M$. Then

$$(Lf, g)_M = (f, L_*g). \quad (3.4)$$

Define now the operator $A_0 : L^2 \to L^2(M)$ by the equality $A_0 f = Lf$ on the domain $\mathcal{D}(A_0) = \mathcal{E}$. Let us construct its adjoint $A_0^* : L^2(M) \to L^2$. Let the set $\mathcal{D}_* \subset L^2(M)$ consist of $g \in L^2(M)$ such that $L_*g \in L^2$. As we have seen, $\mathcal{E}_M \subset \mathcal{D}_*$.

**Lemma 3.2.** Under assumption (3.1) the operator $A_0^*$ is given by the equality $A_0^*g = L_*g$ on the domain $\mathcal{D}(A_0) = \mathcal{D}_*$.

**Proof.** It follows from identity (3.4) for $f \in \mathcal{D}(A_0)$ and $g \in \mathcal{D}_*$ that $\mathcal{D}_* \subset \mathcal{D}(A_0^*)$ and $A_0^*g = L_*g$ for $g \in \mathcal{D}_*$. Conversely, if $g \in \mathcal{D}(A_0^*)$, then $|(L_f g)(\lambda)| \leq C \|f\|$ for all $f \in \mathcal{D}(A_0)$. In view again of (3.4), this estimate implies that $|L_f g| \leq C \|f\|$ and hence $L_*g \in L^2$. Thus $\mathcal{D}(A_0^*) \subset \mathcal{D}_*$. \hfill $\square$

**Corollary 3.3.** The operator $A_0$ admits the closure if and only if $M(\{0\}) = 0$.

**Proof.** If $M(\{0\}) = 0$, then the set $\mathcal{E}_M$ is dense in $L^2(M)$. Since $\mathcal{E}_M \subset \mathcal{D}(A_0^*)$, it follows that the operator $A_0^*$ is densely defined, or equivalently, that the operator $A_0$ admits the closure. Conversely, suppose that $M(\{0\}) > 0$ and denote by $dM_0(\lambda)$ the restriction of $dM(\lambda)$ on $\mathbb{R}_+$. Since $\mathcal{E}_M$ is dense in $L^2(M_0)$, we see that $(L_*g)(t) \to 0$ as $t \to \infty$ for all $g \in L^2(M_0)$ and hence $(L_*g)(t) \to M(\{0\}) g(0)$ as $t \to \infty$ for all $g \in L^2(M)$. It follows that $g(0) = 0$ for all $g \in \mathcal{D}_*$ so that $\mathcal{D}_*$ is not dense in $L^2(M)$. \hfill $\square$
Remark 3.4. The choice of the set $\mathcal{E}$ in the definition of the operator $A_0$ is not essential because its restriction, for example, on the set $C_0^\infty(\mathbb{R}_+)$ has the same adjoint as $A_0$ defined on $\mathcal{E}$.

Next, we construct the second adjoint $A_0^{\ast\ast}$. Let the operator $A$ be defined by the equality $Af = Lf$ on the domain $\mathcal{D}(A)$ which consists of $f \in L^2$ such that $Lf \in L^2(M)$. Obviously, $A_0 \subset A$.

Lemma 3.5. Under the assumptions $M\{\{0\}\} = 0$ and (3.1), we have $A_0^{\ast\ast} \subset A$.

Proof. If $f \in \mathcal{D}(A_0^{\ast\ast})$, then $|(f, L_sg)| \leq C \|g\|_M$ for all $g \in \mathcal{D}_s$ and, in particular, for $g \in \mathcal{E}_M$. Using the identity (3.4) for $f \in L^2$ and $g \in \mathcal{E}_M$, we see that $|(Lf, g)_M| \leq C \|g\|_M$ and hence $Lf \in L^2(M)$. Thus $f \in \mathcal{D}(A)$ and $A_0^{\ast\ast} f = Lf$ according again to (3.4). □

3.2. The proof of the opposite inclusion $A \subset A_0^{\ast\ast}$ is essentially more difficult. Now we have to check relation (3.4) for all $f \in L^2$ such that $Lf \in L^2(M)$ and all $g \in L^2(M)$ such that $L_sg \in L^2$. To that end, we require that, for some $k > 0$, the measure $dM(\lambda)$ satisfies the condition

$$\int_0^\infty (1 + \lambda)^{-k} dM(\lambda) < \infty$$

(3.5)

which is stronger than (3.1). Let $\chi_n$ be the operator of multiplication by the function $\chi_n(\lambda) = e^{-n^2\lambda/4}$. Consider $(Lf, \chi_n g)_M$. Since $Lf \in L^2(M)$, $g \in L^2(M)$ and $\chi_n \to I$ strongly in this space, we see that

$$\lim_{n \to \infty} (Lf, \chi_n g)_M = (Lf, g)_M.$$  (3.6)

Let us now show that

$$\lim_{n \to \infty} (Lf, \chi_n g)_M = (f, L_sg).$$  (3.7)

Astonishingly, this turns out to be a substantial problem. We put

$$\hat{\chi}_n = \Phi U\chi_n U^{-1}\Phi^{-1}.$$  (3.8)

Since the operator $U\chi_n U^{-1}$ acts as the multiplication by the function $e^{-x^2/4}$, we have

$$(\hat{\chi}_n g)(\xi) = \int_{-\infty}^{\infty} \hat{\chi}_n(\xi - \eta)g(\eta) d\eta$$  (3.9)

where

$$\hat{\chi}_n(\xi) = (2\sqrt{\pi})^{-1}ne^{-\xi^2/4}.$$  (3.10)

Lemma 3.6. Let the operator $\Gamma$ be given by formula (2.8). Then the operators

$$T_n = \Gamma^* \hat{\chi}_n(\Gamma^*)^{-1}.$$  (3.11)

defined on $C_0^\infty(\mathbb{R})$ extend to bounded operators on the space $L^2(\mathbb{R})$. Moreover, their norms are bounded uniformly in $n$ and $T_n \to I$ strongly as $n \to \infty$. 

Proof. Let the function \( v(\xi) \) be defined by formula (2.9). It follows from (3.9), (3.11) that
\[
\left| (T_n g)(\xi) \right| \leq v(\xi) \int_{-\infty}^{\infty} \hat{\chi}_n(\xi - \eta) v(\eta)^{-1} |g(\eta)| d\eta
\]
and hence
\[
\left| (T_n g)(\xi) \right|^2 \leq q_n(\xi) \int_{-\infty}^{\infty} \hat{\chi}_n(\xi - \eta) |g(\eta)|^2 d\eta
\]
where
\[
q_n(\xi) = v(\xi)^2 \int_{-\infty}^{\infty} \hat{\chi}_n(\xi - \eta) v(\eta)^{-2} d\eta.
\]

It is easy to see that
\[
q_n(\xi) \leq Q
\]
where the constant \( Q \) does not depend on \( \xi \) and \( n \). Indeed, in view of formulas (3.10) and (2.9), we have to estimate the expression
\[
nv(\xi)^2 \int_{-\infty}^{\infty} e^{-n^2(\xi-\eta)^2/4} e^{\pi\eta} d\eta = nv(\xi)^2 e^{\pi\xi} e^{\pi^2/4} \int_{-\infty}^{\infty} e^{-(nx-2\pi/n)^2/4} dx
\]
where we have set \( x = \eta - \xi \). Since the last integral equals \( 2\sqrt{\pi}/n \), expression (3.14) is uniformly bounded which proves (3.13).

Integrating now estimate (3.12) over \( \xi \), we obtain that \( \|T_n\|^2 \leq Q \). Using that the operators \( \hat{\chi}_n \to I \) strongly as \( n \to \infty \), we see that \( T_n g \to g \) for all \( g \in C_0^\infty(\mathbb{R}) \) and hence for all \( g \in L^2(\mathbb{R}) \). □

Let us return to relation (3.7). Now we use the factorization (2.10) of the Laplace operator.

Lemma 3.7. Let the operator \( T_n \) be defined by formula (3.11), and let \( M = \Phi U \) be the Mellin transform. Then
\[
\chi_n L = LM^{-1}T_n M.
\]

Proof. Putting together relations (2.10) (where we use that \( \Phi J = J \Phi \), (3.8) and (3.11), we see that
\[
LM^{-1}T_n M = (U^{-1} J \Phi^{-1} \Gamma M)M^{-1}(\Gamma^{-1} \hat{\chi}_n \Gamma)M
= U^{-1} J \Phi^{-1} \hat{\chi}_n \Gamma M = U^{-1} J (U \chi_n U^{-1}) \Phi^{-1} \Gamma M.
\]
Since the operators \( J \) and \( U \chi_n U^{-1} \) commute, the right-hand side of (3.16) equals
\[
\chi_n U^{-1} J \Phi^{-1} \Gamma M = \chi_n L,
\]
which proves (3.15). □
It follows from identity (3.15) that
\[(L f, \chi_n g)_M = (L u_n, g)_M\] (3.17)
where
\[u_n = M^{-1} T_n^* M f.\] (3.18)
Let us show that, for all \(g \in L^2(M)\),
\[(L u_n, g)_M = (u_n, L^* g).\] (3.19)
To that end, we use the Fubini theorem, which requires some estimates on the functions \(u_n\). They are given in the following assertion.

**Lemma 3.8.** If \(M f \in C_0^\infty(\mathbb{R})\), then functions (3.18) satisfy estimates
\[|u_n(t)| \leq C_n(b) t^b, \quad \forall b \in \mathbb{R}.\] (3.20)

**Proof.** Put \(\varphi = \Gamma M f \in C_0^\infty(\mathbb{R})\). Then, by definitions (3.11) and (3.18),
\[u_n = M^{-1} \hat{\chi}_n \varphi.\]
According to (3.9), we have
\[u_n(t) = \int_{-\infty}^{\infty} G_n(t, \eta) \varphi(\eta) d\eta\] (3.21)
where
\[G_n(t, \eta) = (2\pi)^{-1/2} t^{-1/2} \int_{-\infty}^{\infty} t^i \Gamma(1/2 + i\xi)^{-1} \hat{\chi}_n(\xi - \eta) d\xi\] (3.22)
and \(\hat{\chi}_n\) is function (3.10). Here we have used equality (2.9) and the Fubini theorem to interchange integrations over \(\xi\) and \(\eta\) in the right-hand side of (3.21). For an arbitrary \(a \in \mathbb{R}\), the integration in (3.22) can be shifted to the line \(\mathbb{R} + ia\) whence
\[G_n(t, \eta) = t^{-a-1/2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} t^i \Gamma(1/2 - a + i\xi)^{-1} \hat{\chi}_n(\xi + ia - \eta) d\xi.\] (3.23)
The modulus of the integrand here does not depend on \(t\). In view of the Stirling formula we have
\[|
\Gamma(1/2 - a + i\xi)| = (2\pi)^{1/2} |\xi|^{a-\pi|\xi|} (1 + O(|\xi|^{-1})), \quad |\xi| \to \infty.\]
Therefore it follows from equalities (3.10) and (3.23) that
\[|G_n(t, \eta)| \leq C_n(a) t^{-a-1/2} \int_{-\infty}^{\infty} (1 + |\xi|)^{-a} e^{\pi|\xi|} e^{-n^2(\xi-\eta)^2/4} d\xi.\]
The integral here is bounded by a constant which depends on \(a\) and \(n\) but does not depend on \(\eta\) in compact intervals. According to (3.21) this yields estimate (3.20). \(\square\)

Observe now that under the assumptions of Lemma 3.8 the integral
\[(L u_n, g)_M = \int_{0}^{\infty} \left( \int_{0}^{\infty} e^{-\lambda t} u_n(t) dt \right) g(\lambda) dM(\lambda)\] (3.24)
converges absolutely because
\[ \int_0^\infty e^{-\lambda t}|u_n(t)|dt \leq C(a)(1 + \lambda)^{-k/2}, \quad \forall k > 0, \]
and, by the Schwarz inequality and condition (3.5),
\[ \int_0^\infty (1 + \lambda)^{-q}|g(\lambda)|dM(\lambda) \leq \sqrt{\int_0^\infty (1 + \lambda)^{-k}dM(\lambda)}\|g\|_M < \infty. \]

Therefore, by the Fubini theorem, we can interchange the order of integrations in (3.24) which yields equality (3.19).

It follows from relations (3.17) – (3.19) that
\[ (Lf, \chi_0 g)_M = (Mf, T_n ML_\ast g) \]
if \(Mf \in C^\infty_0(\mathbb{R}_+)\) and \(g \in L^2(M)\), \(L_\ast g \in L^2\). Since the operators \(T_n\) are bounded, this equality extends to all \(f \in L^2\). If \(Lf \in L^2(M)\), then using Lemma 3.6 we can pass here to the limit \(n \to \infty\) which yields relation (3.7).

Putting together (3.6) and (3.7), we obtain relation (3.1) for all \(f \in L^2\) such that \(Lf \in L^2(M)\) and all \(g \in L^2(M)\) such that \(L_\ast g \in L^2\). Hence \(A \subset A_0^\ast\).

Let us summarize the results obtained.

**Theorem 3.9.** Let \(dM(\lambda)\) be a measure on \([0, \infty)\) satisfying condition (3.1). Then the operator \(A_0 : L^2 \to L^2(M)\) defined on the domain \(D(A_0) = \mathcal{E}\) (or \(D(A_0) = C^\infty_0(\mathbb{R}_+)\)) by the equality \(A_0 f = Lf\) admits the closure if and only if \(M(\{0\}) = 0\). If \(M(\{0\}) = 0\) and assumption (3.3) is satisfied for some \(k > 0\), then the closure \(A_0 =: A\) of \(A_0\) is given by the same equality \(Af = Lf\) on the domain \(D(A)\) which consists of all \(f \in L^2\) such that \(Lf \in L^2(M)\).

3.3. Now we return to Hankel operators. Let us reformulate Theorem 3.9 in their terms.

**Theorem 3.10.** Let \(dM(\lambda)\) be a measure on \([0, \infty)\) satisfying condition (3.1). Then the form (3.2) defined on the set \(\mathcal{E}\) (or \(C^\infty_0(\mathbb{R}_+)\)) admits the closure in the space \(L^2(\mathbb{R}_+)\) if and only if \(M(\{0\}) = 0\). If \(M(\{0\}) = 0\) and assumption (3.3) is satisfied for some \(k > 0\), then the form (3.2) is closed on the set \(D[h]\) of all \(f \in L^2\) such that the integral in the right-hand side of (3.2) is finite.

Let us now take for \(H\) the self-adjoint nonnegative operator corresponding to the closed form \(h[f, f] = \langle h, \tilde{f} \ast f \rangle\). It means that \(D(H) \subset D[h]\) and \((Hf_1, f_2) = h[f_1, f_2]\) for all \(f_1 \in D(H)\) and all \(f_2 \in D[h]\). Note that \(D[h] = D(\sqrt{H}) = D(A)\) and \(\|\sqrt{H} f\| = \|Af\|_M\) for all \(f \in D(A)\).

**Remark 3.11.** If \(M(\{0\}) = 0\) and only condition (3.1) is satisfied, then the form (3.2) is closed on \(D[h] = D(A_0^\ast) \subset D(A)\). In this case \(H\) is still defined as a self-adjoint operator, but we do not have an explicit description of \(D(\sqrt{H})\).
We can also give a condition on $dM(\lambda)$ guaranteeing that $H > 0$.

**Proposition 3.12.** Let $M(\{0\}) = 0$ and let assumption (3.5) be satisfied for some $k > 0$. Suppose that every set $X \subset \mathbb{R}_+$ of full $M$-measure (that is, $M(\mathbb{R}_+ \setminus X) = 0$) contains infinite number of points $\lambda_1, \lambda_2, \ldots$ such that $\lambda_n \to \lambda_0 > 0$ as $n \to \infty$. Then 0 is not an eigenvalue of the operator $H$, that is, $H > 0$.

**Proof.** Indeed, if $Hf = 0$, then according to (3.2) we have $(Lf)(\lambda) = 0$ for almost all $\lambda \in \mathbb{R}_+$ with respect to the measure $M$. It follows that $(Lf)(\lambda_n) = 0$ for some infinite sequence $\lambda_n \to \lambda_0 > 0$. Since the function $(Lf)(\lambda)$ is analytic in the right half-plane, we see that $(Lf)(\lambda)$ for all $\lambda > 0$. This implies that $f = 0$ because the kernel of the operator $L$ considered in the space $L^2(\mathbb{R}_+)$ is trivial. □

In view of formula (1.7) under the assumptions of Theorem 1.1, the measure $dM(\lambda) = \sigma(\lambda)d\lambda$ satisfies the conditions of Proposition 3.12. Thus both Theorem 3.10 and Proposition 3.12 are applicable in this case. This yields all the results stated in Theorem 1.1. More generally, it is true for sums

$$h(t) = \sum_{n=1}^{N} \kappa_n h_n(t), \quad \kappa_n > 0,$$

where each function $h_n(t)$ satisfies the assumptions of Theorem 1.1.

Observe that kernels (1.11) and (1.12) may have arbitrary powers at the point $t = 0$, but it is always required that $h(t) \to 0$ as $t \to \infty$. Without this assumption, there is no reasonable way to define a Hankel operator. For example, for $h(t) = 1$ representation (1.15) holds with the measure such that $M(\mathbb{R}_+) = 0$ and $M(\{0\}) = 1$. Therefore the form (3.2) does not admit the closure.

**Remark 3.13.** Theorem 3.10 can be formulated in a somewhat more general form. Suppose that $h \in C_0^\infty(\mathbb{R}_+)$, and let $\sigma \in \mathcal{V}'$ be the corresponding sigma-function. Assume that for all $w \in \mathcal{V}$

$$\langle \sigma, w^*w \rangle = \int_{0}^{\infty} |w(\lambda)|^2dM(\lambda) - \sigma_0 \|w\|^2_{L^2(\mathbb{R}_+)}$$

(3.25)

where the measure $dM(\lambda)$ satisfies the conditions of Theorem 3.10 and $\sigma_0 < 0$ (if $\sigma_0 > 0$, then (3.25) implies that (3.2) holds true with the measure $dM(\lambda) + \sigma_0 d\lambda$). It follows from (3.25) that the kernel $\tilde{h}(t) = h(t) + \sigma_0 t^{-1}$ satisfies the assumptions of Theorem 3.10 which allows us to define the operator $\tilde{H}$ with kernel $\tilde{h}(t)$. Since the Carleman operator $C$ is bounded, we can now set $H = \tilde{H} - \sigma_0 C$.

**3.4.** In the case $\alpha = 0, r = 0$ one can obtain an additional spectral information about the operator $H$. Let us introduce the unitary operator

$$(D(\gamma)f)(t) = \gamma^{1/2} f(\gamma t), \quad \gamma > 0,$$

of dilations in the space $L^2(\mathbb{R}_+)$. The following assertion is intuitively obvious, but it requires a proof because we do not have an explicit description of $D(H)$. 
Lemma 3.14. If \( h(t) = t^{-q} \), then \( \mathcal{D}(H) \) is invariant with respect to \( \mathcal{D}(\gamma) \) and

\[
\mathcal{D}(\gamma)^* H \mathcal{D}(\gamma) = \gamma^{q-1} H. \tag{3.26}
\]

Proof. According to (1.7) we now have \( \sigma(\lambda) = \Gamma(q)^{-1} \lambda^{q-1} \). Since \( L \mathcal{D}(\gamma) = \mathcal{D}(\gamma^{-1}) L \), it follows from Theorem 3.10 that \( \mathcal{D}[h] = \mathcal{D}(\sqrt{H}) \) is invariant with respect to the dilations \( \mathcal{D}(\gamma) \) and

\[
(\sqrt{H} \mathcal{D}(\gamma)f_1, \sqrt{H} \mathcal{D}(\gamma)f_2) = \gamma^{q-1}(\mathcal{H}f_1, \mathcal{H}f_2) \tag{3.27}
\]

for all \( f_1, f_2 \in \mathcal{D}(\sqrt{H}) \).

Let us set \( G = \gamma^{(1-q)/2} \mathcal{D}(\gamma)^* \sqrt{H} \mathcal{D}(\gamma) \). Then equality (3.27) can be rewritten as \( (Gf_1, Gf_2) = (\sqrt{H}f_1, \sqrt{H}f_2) \). If \( f_1 \in \mathcal{D}(H) \), then \( (Gf_1, Gf_2) = (Hf_1, f_2) \) so that \( Gf_1 \in \mathcal{D}(G^*) \) and \( (G^* Gf_1, f_2) = (Hf_1, f_2) \). Since \( G = G^* \), it follows that \( G^2 = H \) which is equivalent to (3.26).

Let \( X \subset \mathbb{R}_+ \) be an arbitrary Borel set, and let \( E(X) \) be the spectral measure of the operator \( H \). Then relation (3.26) can equivalently be rewritten as

\[
\mathcal{D}(\gamma)^* E(X) \mathcal{D}(\gamma) = E(\gamma^{1-q} X). \tag{3.28}
\]

Each of relations (3.26) or (3.28) implies that if \( \lambda > 0 \) belongs to the spectrum \( \text{spect}(H) \) of the operator \( H \), then all points \( \gamma^{q-1} \lambda \) also belong to the set \( \text{spect}(H) \). It follows that \( \text{spect}(H) = [0, \infty) \). If \( \lambda > 0 \) is an eigenvalue of \( H \), then all points \( \gamma^{q-1} \lambda \) are also eigenvalues of \( H \). This is impossible so that the operator \( H \) does not have eigenvalues.

Actually, we have a more general statement[1]

Proposition 3.15. Let \( H \) be a self-adjoint positive operator such that the operators \( H \) and \( aH \) are unitarily equivalent for all \( a > 0 \). Then the spectrum of the operator \( H \) coincides with the positive half-line, it has a constant multiplicity and is absolutely continuous.

Proof. According to the spectral theorem we can realize \( H \) (see, e.g., the book [2]) as the operator of multiplication by independent variable \( \lambda \) in the space \( L^2(\mathbb{R}_+; dM(\lambda); \mathfrak{N}(\lambda)) \) where \( dM(\lambda) \) is a measure of maximal type with respect to \( H \) and \( \dim \mathfrak{N}(\lambda) \) equals the multiplicity of the spectrum of the operator \( H \) for almost all (with respect to \( dM(\lambda) \)) \( \lambda \in \mathbb{R}_+ \). Since the operators \( H \) and \( aH \) are unitarily equivalent, for an arbitrary Borelian set \( X \subset \mathbb{R}_+ \) the conditions \( M(X) = 0 \) and \( M(aX) = 0 \) are equivalent for all \( a > 0 \). This implies that the measure \( dM(\lambda) \) is equivalent to the Lebesgue measure on \( \mathbb{R}_+ \). This is proven in Problem 2.12 of Chapter X of the book [7]. To be precise, the invariance of measures with respect to translations was considered in [7], but the invariance with respect to dilations reduces to this case by a change of variables. Thus the operator \( H \) is absolutely continuous.

[1]The author thanks A. A. Lodkin and B. M. Solomyak for useful consultations on the measure theory.
It remains to check that the multiplicity of the spectrum of $H$ is constant. Let $X = X_k$ be the Borelian set where this multiplicity is $k$. Suppose that the Lebesgue measure $|X| > 0$. We have to check that $X$ has full measure in $\mathbb{R}_+$. Since the operators $H$ and $aH$ are unitarily equivalent, the sets $X$ and $aX$ coincide up to a set of the Lebesgue measure zero. Let $\lambda$ be a density point of $X$, that is
\[
\lim_{\varepsilon \to 0} (2\varepsilon)^{-1}|(\lambda - \varepsilon, \lambda + \varepsilon) \cap X| = 1.
\]
Recall (see, e.g., the book [10]) that almost all points of $X$ possess this property. Evidently, $a\lambda$ is a density point for the set $aX$. Since the sets $X$ and $aX$ coincide up to a set of the Lebesgue measure zero, $a\lambda$ is a density point for the set $X$ for all $a > 0$. Suppose that $|\mathbb{R}_+ \setminus X| > 0$ and take a density point $\mu \in \mathbb{R}_+ \setminus X$. Choosing $a = \mu/\lambda$, we see that $\mu = a\lambda$ is also a density point for the set $X$. It follows that
\[
|(\mu - \varepsilon, \mu + \varepsilon)| = |(\mu - \varepsilon, \mu + \varepsilon) \cap X| + |(\mu - \varepsilon, \mu + \varepsilon) \cap (\mathbb{R}_+ \setminus X)| = 4\varepsilon(1 + o(\varepsilon))
\]
while the left-hand side of this relation is $2\varepsilon(1 + o(\varepsilon))$. □

In view of relation (3.26) this result can be directly applied to Hankel operators which yields Theorem 1.2. As shown in the paper [8], the multiplicity of the spectrum of a positive bounded Hankel operator does not exceed 2. Most probably, the multiplicity of the spectrum of the operator $H$ considered in Theorem 1.2 is 1 because its kernel $h(t) = t^{-a}$ has only one singular point $t = \infty$ for $q < 1$ and only one singular point $t = 0$ for $q > 1$. But this question is out of the scope of the present article.

The results of Theorem 1.2 are of course true for all operators unitarily equivalent to $H$. For example, let $J$ be the involution in $L^2(\mathbb{R}_+)$ defined by the relation $(Jf)(t) = t^{-1}f(t^{-1})$. Then the operator $K = J^*HJ$ acts according to the formula
\[
(Kf)(t) = \int_0^\infty (t + s)^{-q}(ts)^{q-1}f(s)ds.
\]
Therefore all the conclusions of Theorem 1.2 are satisfied for such operators $K$ (if $q \neq 1$).

### 3.5. In the general case some spectral information is also available. Below we consider the quadratic form of $H$ on the characteristic functions $\mathbb{1}_{(a,b)}(t)$ of intervals $(a, b) \subset \mathbb{R}_+$. Obviously, $\mathbb{1}_{(a,b)} \in \mathcal{E}$ if $0 < a < b < \infty$, $\|\mathbb{1}_{(a,b)}\|^2 = b - a$ and
\[
(L\mathbb{1}_{(a,b)})(\lambda) = (e^{-a\lambda} - e^{-b\lambda})\lambda^{-1}.
\]

#### Proposition 3.16. If $M(\{0\}) = 0$ and condition (3.1) is satisfied, then the point zero belongs to the spectrum of the corresponding Hankel operator $H$.

**Proof.** Let $f_n = \mathbb{1}_{(n,n+1)}$. According to (3.29) the functions $(Lf_n)(\lambda)$ are uniformly bounded by $e^{-\lambda}$ and tend to zero as $n \to \infty$ for all $\lambda > 0$. Hence, by the dominated convergence theorem, it follows from (3.2) that $\|\sqrt{H}f_n\| \to 0$ as $n \to \infty$. Thus $0 \in \text{spec}(H)$. □
Of course this result is consistent with the general fact (see, e.g., the book [12]) that 0 ∈ spec(H) for all bounded Hankel operators.

**Proposition 3.17.** Let \( M(\{0\}) = 0 \), and let condition (3.1) be satisfied. If at least one of the conditions (1.10) is violated, then the corresponding Hankel operator \( H \) is unbounded.

**Proof.** If the first condition (1.10) is violated, then there exists a sequence \( \varepsilon_n \to 0 \) such that \( \varepsilon_n^{-1} M(0, \varepsilon_n) \to \infty \) as \( n \to \infty \). Put \( f_n = \mathbb{1}_{(1, \varepsilon_n^{-1})} \). It follows from (3.29) that \(|(L f_n)(\lambda)| \geq (e \varepsilon_n)^{-1} \) for \( \lambda \in (0, \varepsilon_n) \). Hence according to (3.2) we have \( \|\sqrt{H} f_n\|^2 \geq (e \varepsilon_n)^{-2} M(0, \varepsilon_n) \) so that \( \|f_n\|^2 \|\sqrt{H} f_n\|^2 \geq \varepsilon_n^2 M(0, \varepsilon_n) \to \infty \) as \( n \to \infty \).

Similarly, if the second condition (1.16) is violated, then there exists a sequence \( l_n \to \infty \) such that \( l_n^{-1} M(0, l_n) \to \infty \) as \( n \to \infty \). Put \( f_n = \mathbb{1}_{(l_n^{-2}, l_n)} \). It follows from (3.29) that \(|(L f_n)(\lambda)| \geq (e l_n)^{-1} \) for \( \lambda \in (0, l_n) \). Hence according to (3.2) we have \( \|\sqrt{H} f_n\|^2 \geq (e l_n)^{-2} M(0, l_n) \) so that \( \|f_n\|^2 \|\sqrt{H} f_n\|^2 \geq e^{-2} l_n^{-1} M(0, l_n) \to \infty \) as \( n \to \infty \).

In view of formula (1.7) Propositions 3.16 and 3.17 directly apply to Hankel operators \( H \) with kernels (1.11) and (1.12).

Proposition 3.17 is essentially equivalent to the result of H. Widom mentioned in Section 1, but our proof relies on the construction of trial functions and is quite different from that in [16].

### 4. Perturbation theory

Here we study perturbations of singular quasi-Carleman operators \( H_0 \) introduced in Section 3 by bounded and, in particular, compact self-adjoint Hankel operators \( V \) with kernels (1.11).

**4.1.** As far as the unperturbed operator \( H_0 \) is concerned, we accept the following

**Assumption 4.1.** The sigma-function \( \sigma_0(\lambda) \) of the operator \( H_0 \) is nonnegative, \( \text{supp} \sigma_0 \subset [0, \infty) \), \( \sigma_0 \in L^\infty_{\text{loc}}(\mathbb{R}_+) \) and

\[
\sigma_0(\lambda) = O(\lambda^{-l_+}) \text{ as } \lambda \to 0, \quad \sigma_0(\lambda) = O(\lambda^{l_-}) \text{ as } \lambda \to \infty
\]  

(4.1)

where \( l_+ < 1 \) and \( l_- \) may be arbitrary large.

This assumption can of course be equivalently reformulated in terms of the sign-function \( s_0(x) = \sigma_0(e^{-x}) \colon s_0(x) \geq 0, \text{supp} s_0 \subset \mathbb{R}, s_0 \in L^\infty_{\text{loc}}(\mathbb{R}) \) and

\[
s_0(x) = O(e^{l_+ x}) \text{ as } x \to +\infty \quad \text{and} \quad s_0(x) = O(e^{l_- |x|}) \text{ as } x \to -\infty.
\]

Of course, Assumption 4.1 does not guarantee that \( s_0 \in S' \).

Since the measure \( dM_0(\lambda) = \sigma_0(\lambda) d\lambda \) satisfies condition (3.5), according to Theorem 3.10 the form

\[
h_0[f, f] = \int_0^\infty |(L f)(\lambda)|^2 \sigma_0(\lambda) d\lambda
\]  

(4.2)
is closed on the set $\mathcal{D}[h_0]$ of all functions $f \in L^2(\mathbb{R}_+)$ such that integral (4.2) is finite. Setting $\lambda = e^{-x}$, we can equivalently rewrite definition (4.2) as
\[
 h_0[f, f] = \int_{-\infty}^{\infty} |u(x)|^2 s_0(x)dx
\] (4.3)
where $u(x)$ and $f(t)$ are linked by formula (2.11). We denote by $H_0$ the self-adjoint operator corresponding to the form (4.2) (or (4.3)). Of course, $H_0 \geq 0$. Moreover, by Propositions 3.12 and 3.16 the point $0 \in \text{spec}(H_0)$, but it is not the eigenvalue of $H_0$.

It follows from formula (1.7) that Assumption 4.1 is satisfied for kernels $h_0(t)$ given by equality (1.2) where either $\alpha = 0$ and $q > 0$ or $\alpha > 0$ and $q \geq 1$. Now $l_+ = 1 - q$ for $\alpha = 0$ and $l_+$ is arbitrary for $\alpha > 0$; $l_- = q - 1$ for $r = 0$ and $l_-$ is arbitrary for $r > 0$. In the case $\alpha = 0$, $q > 0$ the operators $H_0$ are unbounded unless $q = 1$ when $H_0 = C$ is the Carleman operator. If $\alpha > 0$, but $r = 0$, then the operators $H_0$ are unbounded for $q > 1$, but $H_0$ is bounded for $q = 1$. If $\alpha > 0$ and $r > 0$, then the operators $H_0$ are compact for all values of $q$.

Another interesting example are Hankel operators $H_0$ with kernels
\[
 h_0(t) = P(\ln t)t^{-1}
\] (4.4)
where $P(x) = \sum_{k=0}^{K} p_k x^k$, $p_K > 0$, is an arbitrary real polynomial of even degree $K$. Such operators were studied in [20] where, in particular, it was shown that $H_0$ are semibounded and the essential spectrum $\text{spec}_{\text{ess}}(H_0) = [0, \infty)$ unless $K = 0$. The sign-function of kernel (4.4) is given by the polynomial $s_0(x) = \sum_{k=0}^{K} q_k x^k$ where the coefficients $q_k$ admit an explicit expression in terms of the coefficients $p_k$, $p_{k+1}, \ldots, p_K$.

Of course $s_0 \in \mathcal{S}'$ and Assumption 4.1 is satisfied for kernels (4.4) provided $s_0(x) \geq 0$. Note that the inequality $s_0(x) \geq 0$ implies that $P(x) \geq 0$ but not vice versa.

On the contrary, sigma-functions of finite rank Hankel operators are singular distributions (see Section 5) so that Assumption 4.1 is violated for such operators.

4.2. Let us start with a general statement on perturbations of operators $H_0$ satisfying Assumption 4.1 by bounded Hankel operators $V$. Recall that, for all $f \in L^2(\mathbb{R}_+)$, we have
\[
 (Vf, f) = \langle \sigma_v, w^*w \rangle =: \sigma_v[w, w]
\]
where $w = Lf \in \mathbb{H}^2_\nu$ and $\sigma_v \in (\mathbb{H}^1_\nu)'$. We put $H = H_0 + V$. This operator is defined by its quadratic form
\[
 h[f, f] = h_0[f, f] + (Vf, f)
 = \int_{0}^{\infty} \sigma_0(\lambda)|w(\lambda)|^2d\lambda + \sigma_v[w, w] =: \sigma[w, w], \quad w = Lf.
\] (4.5)

This form is closed on the set $\mathcal{D}[h]$ of all $f \in L^2(\mathbb{R}_+) \text{ (or equivalently of all } w \in \mathbb{H}^2_\nu)$ such that the integral in the right-hand side converges. Of course $\mathcal{D}[h] = \mathcal{D}[h_0]$. The following auxiliary result (cf. Theorem 2.4) is a direct consequence of equality (4.5).
Lemma 4.2. Let Assumption 4.1 be satisfied, and let a Hankel operator $V$ be bounded. Then $N(H)$ equals the maximal dimension of linear sets $K \subset \mathbb{H}_r^2$ such that $\sigma[w, w] < 0$ for all $w \in K$, $w \neq 0$.

Note that if $w \in \mathbb{H}_r^2$ and $\sigma[w, w] < 0$, then $\sigma_0[w, w] < \infty$ and hence $f = L^{-1}w \in D[h]$.

If $s_0 \not\in S'$, we cannot use Theorem 2.6 and study the form $(4.5)$ on the set $C^0(\mathbb{R}_+)$ of test functions $w(\lambda)$ (this reduction was essentially used by the proof of Theorem 2.7 in [21]). Now we are obliged to work with analytic functions $w \in \mathbb{H}_r^2$ (or the corresponding test functions $u(x) = e^{-x^2/2}w(e^{-x})$) which is technically more involved. In the case $s_0 \in S'$ (for example, for Hankel operators with kernels $(4.4)$) the proofs below can be considerably simplified. On the other hand, the advantage of the method suggested here is that it directly yields, by formula $(2.11)$, trial functions $f(t)$ for which $h[f, f] < 0$.

In many cases it suffices to consider gaussian trial functions

$$u_{\varepsilon}(x; a) = \varepsilon^{-1/2}e^{-(x-a)^2/\varepsilon^2}, \quad \varepsilon > 0, \quad a \in \mathbb{R}. \quad (4.6)$$

Obviously, $\|u_{\varepsilon}(a)\|^2 = \sqrt{\pi}/2$. Since

$$(\Phi u_{\varepsilon}(a))(\xi) = 2^{-1/2}\varepsilon^{-1/2}e^{-i\xi a}e^{-\varepsilon^2\xi^2/4}, \quad (4.7)$$

functions $f_{\varepsilon}(t; a)$ defined by $(2.11)$ belong to $L^2(\mathbb{R}_+)$ or, equivalently, the corresponding functions $w_{\varepsilon}(\lambda; a)$ belong to $\mathbb{H}_r^2$. Note however that $\|f_{\varepsilon}(a)\| \to \infty$ as $\varepsilon \to 0$.

The following assertion is almost obvious.

Lemma 4.3. If the parameters $a_1, \ldots, a_N$ are pairwise different, then the functions $u_{\varepsilon}(x; a_1), \ldots, u_{\varepsilon}(x; a_N)$ are linearly independent.

Proof. It suffices to check that the functions $(\Phi u_{\varepsilon}(a_1))(\xi), \ldots, (\Phi u_{\varepsilon}(a_N))(\xi)$ are linearly independent or according to $(4.7)$ that the functions $e^{-i\xi a_1}, \ldots, e^{-i\xi a_N}$ are linearly independent. If $\sum_{j=1}^N c_j e^{-i\xi a_j} = 0$, then differentiating this identity and putting $\xi = 0$, we obtain the system $\sum_{j=1}^N c_j a_j^n = 0$ where $n = 0, 1, \ldots, N - 1$ for $c_1, \ldots, c_N$. The determinant of this system is the Vandermonde determinant. Since it is not zero, we see that $c_1 = \cdots = c_N = 0$.

For Hankel operators $V$ with regular sign-functions, we use the following assertion.

Theorem 4.4. Let Assumption 4.1 be satisfied, and let $H_0$ be the corresponding Hankel operator. Suppose that a Hankel operator $V$ is bounded and that its sign-function $s_v \in L^1_{\text{loc}}(\mathbb{R})$. Put $H = H_0 + V$ and $s = s_0 + s_v$. Then:

1. The operator $H \geq 0$ if $s(x) \geq 0$ for almost all $x \in \mathbb{R}$.

2. The operator $H$ has infinite negative spectrum if $s(x) \leq -s_0 < 0$ for almost all $x$ in some interval $\Delta \subset \mathbb{R}$ and $s(x)$ is exponentially bounded away from $\Delta$.

Proof. The assertion 1 is obvious because (cf. formula $(4.3)$ for $h_0[f, f]$) for all $f \in D[h]$ we have

$$h[f, f] = \int_{-\infty}^{\infty} s(x) |u(x)|^2 \, dx \quad (4.8)$$
where \( u(x) \) and \( f(t) \) are related by (2.11).

Let us prove 2.0. For an arbitrary \( N \), let us choose points \( a_1, \ldots, a_N \in \Delta = (a_0, a_{N+1}) \) in such a way that \( a_{j+1} - a_j = a_j - a_{j-1} =: \delta \) for all \( j = 1, 2, \ldots, N \) and define functions \( u_\varepsilon(x; a_j) \) by formula (4.6) where \( \varepsilon \) is sufficiently small number. By Lemma 4.3 the functions \( u_\varepsilon(x; a_1), \ldots, u_\varepsilon(x; a_N) \) are linearly independent. By our condition on \( s(x) \), we have

\[
\int_\Delta s(x)u_\varepsilon^2(x; a_j)\,dx \leq -s_0\varepsilon^{-1} \int_{a_0}^{a_{N+1}} e^{-2(x-a_j)^2/\varepsilon^2} \,dx = -s_0 \int_{(a_{N+1}-a_j)/\varepsilon}^{(a_{N+1}-a_j)/\varepsilon} e^{-2y^2} \,dy = -s_0(\sqrt{\pi/2} + O(\varepsilon)) \leq -s_0
\]

because \((a_0 - a_j)/\varepsilon \to -\infty\) and \((a_{N+1} - a_j)/\varepsilon \to \infty\) as \( \varepsilon \to 0 \). Moreover, for all \( \delta > 0 \) and a sufficiently large \( l > 0 \),

\[
\int_{\mathbb{R} \setminus \Delta} s(x)u_\varepsilon^2(x; a_j)\,dx \leq C\varepsilon^{-1} \int_0^\infty e^{lx} e^{-2x^2/\varepsilon^2} \,dx = C \int_\delta/\varepsilon^2 e^{\delta y} e^{-2y^2} \,dy = O(\varepsilon). \tag{4.9}
\]

For \( j \neq k \), we have

\[
|\int_\Delta s(x)u_\varepsilon(x; a_j)u_\varepsilon(x; a_k)\,dx| \leq C\varepsilon^{-1} e^{-\delta^2/2\varepsilon^2} \int_\Delta |s(x)|\,dx = O(\varepsilon),
\]

and according to (4.9) the corresponding integral over \( \mathbb{R} \setminus \Delta \) is also \( O(\varepsilon) \).

Putting together the estimates obtained, we see that

\[
|s(\sum_{j=1}^N \nu_j u_\varepsilon(a_j), \sum_{j=1}^N \nu_j u_\varepsilon(a_j))| \leq -s_0 \sum_{j=1}^N |\nu_j|^2 (1 + O(\varepsilon)).
\]

Therefore the operator \( H \) has at least \( N \) negative eigenvalues. \( \square \)

**Remark 4.5.** Under the assumptions of Theorem 4.3 the operator \( V \) is not supposed to be compact, but if it is compact then, in the assertion 2.0, the negative spectrum of the operator \( H \) consists of infinite number of eigenvalues.

4.3. Let us now consider perturbations of singular operators \( H_0 \) defined in Theorem 3.10 by Hankel operators \( V \) with kernels (1.17). According to formula (1.7) the corresponding sigma-function \( \sigma_v(\lambda) \) is given by the equality

\[
\sigma_v(\lambda) = v_0 e^{\beta \rho} \frac{\Gamma(-k)}{\Gamma(k)} (\lambda - \beta)^{-k-1} e^{-\rho \lambda}. \tag{4.10}
\]

We impose conditions on the parameters \( \beta, \rho \) and \( k \) such that the operators \( V \) are bounded.

It turns out that the cases \( k < 0 \) and \( k \geq 0 \) are qualitatively different. In the first case the sign-function of \( V \) is regular, so that the negative spectrum of \( H = H_0 + V \) is governed by Theorem 4.3. In the second case the negative spectrum of \( H = H_0 + V \) is determined solely by the singularity of the sigma-function \( \sigma_v(\lambda) \) at the point \( \lambda = \beta \).
Consider first perturbations (1.17) for $k < 0$. If $v_0 \geq 0$, then, by Theorem 2.7, the operator $V \geq 0$ and hence $H = H_0 + V \geq 0$. So we suppose that $v_0 < 0$. If $k \in (-1, 0)$ and $\beta > 0$, then $\sigma_v(\lambda)$ is continuous for $\lambda > \beta$, $\sigma_v(\lambda) \to -\infty$ and hence $\sigma(\lambda) = \sigma_0(\lambda) + \sigma_v(\lambda) \to -\infty$ as $\lambda \to \beta + 0$. Since $\sigma_v \in L_{\text{loc}}(\mathbb{R}_+)$, it follows from Theorem 4.4 that the operator $H$ has infinite negative spectrum.

If $k = -1$ or $k < -1$ but $\rho > 0$, then $\sigma_v(\lambda)$ is a bounded negative function. By Theorem 4.4, the operator $H \geq 0$ if and only if $\sigma(\lambda) \geq 0$ (for all $\lambda \geq \beta$). In the opposite case it has infinite negative spectrum.

Let us summarize the results obtained.

**Theorem 4.6.** Let $H = H_0 + V$ where the sigma-function $\sigma_0(\lambda)$ of the operator $H_0$ satisfies Assumption 4.1 and let $V$ be the Hankel operator with kernel (1.17) where $k < 0$. Then:

1. If $k > -1$ and $\beta > 0$, then the operator $H > 0$ for $v_0 \geq 0$ and $H$ has infinite negative spectrum for all $v_0 < 0$.

2. Let $k \leq -1$. If $k < -1$ suppose that $\rho > 0$. Put

$$
\nu = \Gamma(-k)e^{-\beta\rho} \inf_{\lambda \geq \beta} (\lambda - \beta)^k e^{\rho(\lambda)} \sigma_0(\lambda).
$$

Then the operator $H \geq 0$ for $v_0 \geq -\nu$, and it has infinite negative spectrum for $v_0 < -\nu$.

For the Carleman operator $H_0 = C$, we have $\sigma_0(\lambda) = 1$ and hence $\nu = 1$ for all values of $\beta$ and $\rho$ if $k = -1$ and

$$
\nu = \Gamma(-k)e^{-k-1} \left(\frac{\rho}{-k-1}\right)^{-k-1}, \quad k < -1.
$$

In particular, the critical coupling constant $\nu$ does not depend on $\beta$ in this case.

**Remark 4.7.** In part 1 of Theorem 4.6, the operator $V$ is compact so that for $v_0 < 0$ the operator $H$ has infinite number of negative eigenvalues accumulating to the point zero. The same is true in part 2 if $v_0 < -\nu$ and either $\beta > 0$ or $\beta = 0$, $k < -1$ (in these cases $V$ is compact). If $V$ is not compact, then the operator $H = H_0 + V$ may have negative continuous spectrum. For example, if $h(t) = t^{-1}$ and $v(t) = v_0 t^{-1}$, then the spectrum of $H = (1 + v_0)C$ coincides with the interval $[0, (1 + v_0)\pi]$ for $v_0 > -1$ and with the interval $[(1 + v_0)\pi, 0]$ for $v_0 < -1$.

The case $k > 0$ when the operator $V$ is not sign-definite is essentially more difficult. Our goal is to prove the following result.

**Theorem 4.8.** Let $H = H_0 + V$ where the sigma-function $\sigma_0(\lambda)$ of the operator $H_0$ satisfies Assumption 4.1 and $V$ is the Hankel operator with kernel (1.17). Suppose that $\beta > 0$, $\rho \geq 0$ and $k > 0$ for some $k \notin \mathbb{Z}_+$. Then:

1. If $v_0 > 0$ and $[k]$ is odd, then $N_-(H) = \lfloor [k] \rfloor/2$.

2. If $v_0 < 0$ and $[k]$ is even, then $N_-(H) = \lfloor [k] \rfloor/2 + 1$.

3. If $v_0 > 0$ and $[k]$ is even or $v_0 < 0$ and $[k]$ is odd, then $N_-(H) = \infty$. 

Putting together Theorems 2.7 and 4.8 we obtain relation (1.18). This implies Theorem 1.3 for \( k \notin \mathbb{Z}_+ \). The case \( k \in \mathbb{Z}_+ \) will be considered in the next section.

**4.4.** Let us prove parts 1 and 2 of Theorem 4.8. Set \( n = \lfloor k \rfloor \) and \( \ell = \lfloor n/2 \rfloor + 1 \).

Since \( H_0 \geq 0 \), we have \( N_-(H) \leq N_-(V) \), which in view of Theorem 2.7 yields the upper estimate \( N_-(H) \geq \ell \). According to Lemma 4.2, to prove the lower estimate \( N_-(H) \geq \ell \), we have to construct a linear subspace \( K \subset H^2_\mathbb{r} \) of dimension \( \ell \) such that \( \sigma[w, w] < 0 \) for all \( w \in K \), \( w \neq 0 \). Note that our assumptions \( v_0 \Gamma(\rho-k)^{-1} > 0 \).

We need an elementary assertion about distributions \( \mu - k - 1 + \).

**Lemma 4.9.** Suppose that \( k \in \mathbb{R}_+ \setminus \mathbb{Z}_+ \). Let a bounded \( C^\infty \) function \( \varphi(\mu) \) of \( \mu \in \mathbb{R} \) satisfy the conditions

\[
\varphi(0) = 1 \quad \text{and} \quad \varphi'(0) = \cdots = \varphi^{(n)}(0) = 0 \quad \text{if} \quad n \geq 1, \tag{4.11}
\]

and let \( Q(\mu) \) be a polynomial of \( \deg Q \leq n \). Then

\[
\int_{-\infty}^{\infty} \mu^{\ell-1}Q(\mu)\varphi^2(\mu)d\mu = \int_{0}^{\infty} \mu^{\ell-1}Q(\mu)(\varphi^2(\mu) - 1)d\mu. \tag{4.12}
\]

**Proof.** According to (4.11) for the function \( \psi(\mu) = Q(\mu)\varphi^2(\mu) \), we have \( \psi^{(\mu)}(0) = Q^{(\mu)}(0) \) for all \( p = 0, \ldots, n \), whence

\[
\sum_{p=0}^{n} \frac{1}{p!}\psi^{(\mu)}(0)\mu^p = \sum_{p=0}^{n} \frac{1}{p!}Q^{(\mu)}(0)\mu^p = Q(\mu)
\]

if \( n \geq \deg Q \). Therefore relation (4.12) is a direct consequence of definition (1.8). \( \square \)

Put

\[
w_\varepsilon(\lambda) = P(\lambda - \beta)R(\lambda - \beta)\exp \left( -\varepsilon^{-2m}\ln^{2m}(\lambda/\beta) \right) \tag{4.13}
\]

where \( P(\mu) = \sum_{j=0}^{\ell-1}p_j\mu^j \) is an arbitrary polynomial of \( \deg P \leq \ell - 1 \) and \( R(\mu) = \sum_{j=0}^{n}r_j\mu^j \) is a special polynomial of \( \deg R \leq n \), \( \varepsilon \) is a small parameter and \( m \) is a sufficiently large number. It is easy to see that functions (4.13) belong to the Hardy space \( \mathbb{H}_r^{2} \) for all \( m = 1, 2, \ldots \).

First we construct the polynomial \( R(\mu) \). We require that the function

\[
\theta(\mu) = R(\mu)e^{-\rho\mu/2} \tag{4.14}
\]

satisfy the equations

\[
\theta(0) = 1 \quad \text{and} \quad \theta'(0) = \cdots = \theta^{(n)}(0) = 0 \quad \text{if} \quad n \geq 1. \tag{4.15}
\]

Solving these equations for \( r_0, \ldots, r_n \), we find successively all the coefficients \( r_0 = 1, r_1 = \rho/2, r_2, \ldots, r_n \). Note that in the case \( \rho = 0 \), we have \( R(\mu) = 1 \) so that this construction is not necessary.

Let us estimate the sigma-form of the operator \( V \).
Lemma 4.10. Let $\sigma_v(\lambda)$ be function (4.10), and let the functions $w_\varepsilon$ be defined by equality (4.13) where $2m > n$. Suppose that function (4.14) satisfies conditions (4.15). Then there exist $\varepsilon > 0$ and $c > 0$ such that

$$v_0^{-1}\Gamma(-k)\sigma_v[w_\varepsilon, w_\varepsilon] \leq -c\|P\|^2, \quad \|P\|^2 := \sum_{j=0}^{\ell-1} |p_j|^2, \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (4.16)$$

Proof. Put

$$\varphi_\varepsilon(\mu) = \theta(\mu) \exp \left(-\varepsilon^{-2m} \ln^{2m}(1 + \mu/\beta)\right)$$

where $\theta(\mu)$ is function (4.14). According to (4.10) and (4.13) we have the expression

$$\sigma_v[w_\varepsilon, w_\varepsilon] = \frac{v_0}{\Gamma(-k)} \int_{-\infty}^{\infty} \mu^{-k-1} |P(\mu)|^2 \varphi_\varepsilon^2(\mu)d\mu. \quad (4.17)$$

Since $2m > n$, equations (4.15) for $\theta(\mu)$ imply equations (4.11) for $\varphi_\varepsilon(\mu)$. Therefore Lemma 4.9 applied to $Q(\mu) = |P(\mu)|^2$ yields the representation

$$v_0^{-1}\Gamma(-k)\sigma_v[w_\varepsilon, w_\varepsilon] = \int_{0}^{\infty} |P(\mu)|^2 (\theta^2(\mu)e^{-2\varepsilon^{-2m}\ln^{2m}(1+\mu/\beta)} - 1) \mu^{-k-1}d\mu. \quad (4.18)$$

Note that

$$\int_{0}^{\infty} |P(\mu)|^2 (e^{-2\varepsilon^{-2m}\ln^{2m}(1+\mu/\beta)} - 1) \mu^{-k-1}d\mu \leq -c\|P\|^2 \quad (4.19)$$

for all $\varepsilon \leq 1$. Indeed, the left-hand here is maximal for $\varepsilon = 1$ and hence it suffices to use that

$$\min_{\|P\|=1} \int_{0}^{\infty} |P(\mu)|^2 (1 - e^{-2\ln^{2m}(1+\mu/\beta)}) \mu^{-k-1}d\mu \geq c > 0.$$

If $\rho = 0$, then $\theta(\mu) = 1$, and hence estimates (4.19) and (4.16) coincide. If $\rho > 0$, we have to get rid of $\theta(\mu)$ in the right-hand side of (4.18). It follows from conditions (4.15) that $|\theta^2(\mu) - 1|\mu^{-k-1} \leq C\mu^{n-k}$. Using also an obvious estimate

$$|P(\mu)|^2 \leq C\|P\|^2(1 + \mu^{2\ell-2}), \quad (4.20)$$

we see that

$$\int_{0}^{\infty} |P(\mu)|^2 |\theta^2(\mu) - 1| e^{-2\varepsilon^{-2m}\ln^{2m}(1+\mu/\beta)} \mu^{-k-1}d\mu \leq C\|P\|^2 \int_{0}^{\infty} e^{-2\varepsilon^{-2m}\ln^{2m}(1+\mu/\beta)} (\mu^{n-k} + \mu^{n-k+2\ell-2})d\mu. \quad (4.21)$$

Since $n - k > -1$ and $\ell \geq 1$, the integral in the right-hand side of (4.21) tends to zero as $\varepsilon \to 0$, and hence the right-hand side of (4.21) is bounded by $c\|P\|^2/2$ for sufficiently small $\varepsilon > 0$. Putting together this result with (4.19), we obtain estimate (4.16). □
Let us now consider the sigma-form \( \sigma_0[w_\varepsilon, w_\varepsilon] \) on functions \( w_\varepsilon \) defined by formula (4.13). Using again estimate (4.20), conditions (4.1) where \( l_+ < 1 \) and making the change of variables \( \lambda = \beta e^{\varepsilon x} \), we see that
\[
\sigma_0[w_\varepsilon, w_\varepsilon] \leq C \|P\|^2 \int_0^\infty (\lambda^{-l_+} + \lambda^{2(\ell+n-1)+l_+}) \exp \left(-2\varepsilon^{-2m} \ln^2(\lambda/\beta)\right) d\lambda \leq C_1 \varepsilon \|P\|^2.
\]

Putting together this estimates with (4.16), we see that \( \sigma[w_\varepsilon, w_\varepsilon] < 0 \) if \( w_\varepsilon \) is defined by formula (4.13) where \( \varepsilon \) is sufficiently small and \( P(\mu) \) is an arbitrary nontrivial polynomial of \( \deg P \leq \ell + 1 \). Since the dimension of such polynomials equals \( \ell \), this yields us the linear subspace \( \mathcal{K} \subset \mathbb{H}_r^2 \) of dimension \( \ell \) where the form \( \sigma \) is negative. This shows that \( N_-(H) \geq \ell \) and hence concludes the proof of parts 1 and 2 of Theorem 4.4.

**4.5.** It remains to prove part 3 of Theorem 4.4. We use essentially the same construction of trial functions as in part 2 of Theorem 4.4. Actually, it is slightly more convenient to work with trial functions
\[
w_\varepsilon(\lambda; A) = (\varepsilon \lambda)^{-1/2} e^{-\varepsilon^{-2} \ln^2(\lambda/A)}.
\] (4.22)

If \( A = e^{-a} \), they are linked by relation (2.47) to functions \( u_\varepsilon(x; a) \) defined by formula (4.6) and hence belong to \( \mathbb{H}_r^2 \). The proof below is significantly more complicated than that of part 2 of Theorem 4.4 because the parameter \( A \) in definition (4.22) will be chosen in rather a special way. We need the following auxiliary result.

**Proposition 4.11.** Let the sigma-function \( \sigma_v \) and trial functions \( w_\varepsilon(\lambda; A) \) be defined by formulas (4.10) and (4.22), respectively. Then, for any \( A > \beta \), we have the relation
\[
\lim_{\varepsilon \to 0} \sigma_v[w_\varepsilon(A), w_\varepsilon(A)] = v_0 \Gamma(-k)^{-1} \sqrt{\pi/2} (A - \beta)^{-k-1} e^{-\rho(A - \beta)}. \tag{4.23}
\]

Moreover, if \( B > \beta, B \neq A \), then
\[
\lim_{\varepsilon \to 0} \sigma_v[w_\varepsilon(A), w_\varepsilon(B)] = 0. \tag{4.24}
\]

We emphasize that limits (4.23) and (4.24) as well as all limits below are uniform with respect to \( A \) and \( B \) in compact subintervals of \((\beta, \infty)\). The proof of Proposition 4.11 will be split in several simple lemmas. In view of formula (4.10) and definition (4.8) we have
\[
\sigma_v[w_\varepsilon(A), w_\varepsilon(B)] = \frac{v_0 e^{\beta \rho}}{\Gamma(-k)} \int_\beta^\infty (\lambda - \beta)^{-k-1} \left( \psi_\varepsilon(\lambda; A, B) - \Psi_\varepsilon^{(n)}(\lambda; A, B) \right) d\lambda \tag{4.25}
\]
where
\[
\psi_\varepsilon(\lambda; A, B) = e^{-1} e^{-\rho \lambda} \lambda^{1-1} \exp \left(-e^{-2}(\ln^2(\lambda/A) + \ln^2(\lambda/B))\right), \tag{4.26}
\]
\[
\Psi_\varepsilon^{(n)}(\lambda; A, B) = \sum_{p=0}^n \frac{1}{p!} \psi_\varepsilon^{(p)}(\beta; A, B)(\lambda - \beta)^p, \quad n = [k]. \tag{4.27}
\]

Our study of integral (4.25) relies on the following arguments. First, term (4.27) is important in a neighborhood of the point \( \lambda = \beta \) only, and it can be neglected away from this point. Second, if \( A = B \), then the asymptotics of integral (4.25) as \( \varepsilon \to 0 \)
is determined by a neighborhood of the point \( \lambda_0 = A = B \) (the exponential term in (4.26) equals 1 at \( \lambda_0 \)), but there is no such point if \( A \neq B \).

Differentiating definition (4.26), we obtain bounds on derivatives of function (4.26).

**Lemma 4.12.** For all \( p \in \mathbb{Z}_+ \) and \( \lambda \geq \beta > 0 \),
\[
|\psi^{(p)}_\varepsilon(\lambda; A, B)| \leq C_p(A, B)\varepsilon^{-1} \lambda^{-1} \exp \left( -\varepsilon^{-2} (\ln^2 \lambda/A + \ln^2 \lambda/B) \right). \tag{4.28}
\]

We suppose for definiteness that \( B \leq A \) and put \( \lambda_0 = (A + B)/2 \). First, we consider a neighborhood of the point \( \lambda = \beta \).

**Lemma 4.13.** Let \( \beta < \lambda_1 < \lambda_0 \). Then
\[
\lim_{\varepsilon \to 0} \int_{-\beta}^{\lambda_1} (\lambda - \beta)^{-k-1} |\psi_\varepsilon(\lambda; A, B) - \Psi^{(n)}_\varepsilon(\lambda; A, B)| d\lambda = 0. \tag{4.29}
\]

**Proof.** Since
\[
|\psi_\varepsilon(\lambda; A, B) - \Psi^{(n)}_\varepsilon(\lambda; A, B)| \leq (n + 1)^{-1} (\lambda - \beta)^{n+1} \max_{\lambda \in [\beta, \lambda_1]} |\psi^{(n+1)}_\varepsilon(\lambda; A, B)|
\]
and \( \ln^2(\lambda_1/\lambda_0) \leq \ln^2(\lambda/A) \), it follows from (4.28) that
\[
|\psi_\varepsilon(\lambda; A, B) - \Psi^{(n)}_\varepsilon(\lambda; A, B)| \leq C_n(A, B)(\lambda - \beta)^{-3} \exp^{-2} \ln^2(\lambda_1/\lambda_0).
\]
This implies (4.29) because \( n > k - 1 \). \( \square \)

Away from the point \( \lambda = \beta \) term (4.27) is negligible.

**Lemma 4.14.** If \( \lambda_1 > \beta \), then
\[
\lim_{\varepsilon \to 0} \int_{-\beta}^{\lambda_1} (\lambda - \beta)^{-k-1} |\Psi^{(n)}_\varepsilon(\lambda; A, B)| d\lambda = 0. \tag{4.30}
\]

**Proof.** In view of (4.28) where \( \lambda = \beta \) the integral here is bounded by
\[
C_n(A, B) \exp \left( -\varepsilon^{-2} \ln^2(\beta/A) \right) \sum_{p=0}^{n} \varepsilon^{-2p} \int_{-\beta}^{\lambda_1} (\lambda - \beta)^{-j-k-1} d\lambda.
\]
The integrals here are convergent because \( p - k \leq n - k < 0 \), and hence this expression tends to zero as \( \varepsilon \to 0 \) because \( \beta < A \). \( \square \)

In the next result, we have to distinguish the cases \( A \neq B \) and \( A = B \).

**Lemma 4.15.** If \( A \neq B \), then for any \( \lambda_1 > \beta \)
\[
\lim_{\varepsilon \to 0} \int_{-\beta}^{\lambda_1} (\lambda - \beta)^{-k-1} \psi_\varepsilon(\lambda; A, B) d\lambda = 0. \tag{4.31}
\]

If \( A = B \), then for all \( \lambda_1 \in (\beta, \lambda_0) \)
\[
\lim_{\varepsilon \to 0} \int_{-\beta}^{\lambda_1} (\lambda - \beta)^{-k-1} \psi_\varepsilon(\lambda; A, A) d\lambda = \sqrt{\pi/2} (A - \beta)^{k-1} e^{-\rho A}. \tag{4.32}
\]
we have seen,

where

\[ A \epsilon \]

the estimate

we see that this expression is bounded by a constant

\[ w \]

if \( \lambda < \lambda_0 \), then \( \lambda < A \) and \(-\ln^2(\lambda/A) \leq -\ln^2(\lambda_0/A)\). Therefore the first term in (4.33) is estimated by \( C \epsilon^{-1} \exp(-\epsilon^{-2} \ln^2(\lambda_0/A)) \) which tends to zero as \( \epsilon \to 0 \) because \( \lambda_0 < A \).

Making the change of variables \( \lambda = Be^{\epsilon x} \), we see that the second term in (4.33) is estimated by the integral of \( e^{-x^2} \) over the interval \((x_0(\epsilon), \infty)\) where \( x_0(\epsilon) = \epsilon^{-1} \ln(\lambda_0/B) \to +\infty \) as \( \epsilon \to 0 \) because \( \lambda_0 > B \).

Making the change of variables \( \lambda = Ae^{\epsilon x} \), we see that integral (4.32) equals

\[
\epsilon^{-1} \int_{\lambda_1}^{\infty} (\lambda - \beta)^{-k-1} e^{-\rho A \epsilon x - 2\epsilon^2 \ln^2(\lambda/A)} \lambda^{-1} d\lambda = \int_{x_1(\epsilon)}^{\infty} (Ae^{\epsilon x} - \beta)^{-k-1} e^{-\rho Ae^{\epsilon x} x - 2\epsilon^2 x^2} dx
\]

where \( x_1(\epsilon) = \epsilon^{-1} \ln(\lambda_1/A) \to -\infty \) as \( \epsilon \to 0 \) because \( \lambda_1 < A \). Therefore the right-hand side here converges as \( \epsilon \to 0 \) to the corresponding integral over \( \mathbb{R} \). It equals the right-hand side of (4.32).

Let us return to representation (4.25). In view of Lemma 4.13 the integral over \((\beta, \lambda_1)\) can be neglected for any \( \lambda_1 < (A + B)/2 \). If \( A \neq B \), then the integral (4.25) over \((\lambda_1, \infty)\) also tends to zero according to relations (4.30) and (4.31). This yields (4.24). In the case \( A = B \) the integral (4.25) over \((\lambda_1, \infty)\) tends to the right-hand side of (4.32) according to relations (4.30) and (4.32). This yields (4.23) and concludes the proof of Proposition 4.11.

4.6. Now we are in a position to prove part 3 of Theorem 4.8. Let the functions \( w_\epsilon(\lambda; A) \) be defined by formula (4.22). Observe that under Assumption (4.1) we have the estimate

\[
\sigma_0[w_\epsilon(A), w_\epsilon(A)] = \epsilon^{-1} \int_{-\infty}^{\infty} s_0(x) e^{-2\epsilon x^2} dx \leq C \epsilon^{-1} \int_{-\infty}^{\infty} e^{(l+|a|)|x|} e^{-2\epsilon x^2} dx
\]

where \( a = -\ln A \) and \( l = \max\{l_-, l_+\} \). Making here the change of variables \( x = \epsilon y \), we see that this expression is bounded by a constant \( c_0 > 0 \) which does not depend on \( \epsilon \in (0, 1) \) and on the parameter \( a \) in a compact interval of \( \mathbb{R} \).

Let \( N \) be given. We look for trial functions in the form

\[
w_\epsilon(\lambda) = \sum_{j=1}^{N} \nu_j w_\epsilon(\lambda; A_j)
\]

where \( A_j > \beta \) for all \( j = 1, \ldots, N \) and \( \nu_1, \ldots, \nu_N \) are arbitrary complex numbers. As we have seen,

\[
\sigma_0[w_\epsilon, w_\epsilon] \leq N \sum_{j=1}^{N} |\nu_j|^2 \sigma_0[w_\epsilon(A_j), w_\epsilon(A_j)] \leq c_0 N \sum_{j=1}^{N} |\nu_j|^2.
\]

(4.34)
Next, we consider the form $\sigma_v[w_\varepsilon, w_\varepsilon]$. Observe that under our assumptions $v_0 \Gamma(-k) < 0$. We choose the points $A_1, \ldots, A_N$ so close to $\beta$ that

$$v_0 \Gamma(-k)^{-1} \sqrt{\pi/2} (A_j - \beta)^{-k-1} e^{-\rho(A_j - \beta)} \leq -3c_0, \quad j = 1, \ldots, N.$$ 

Then it follows from Proposition 4.11 that for a sufficiently small $\varepsilon > 0$

$$\sigma_v[w_\varepsilon, w_\varepsilon] \leq -2c_0 \sum_{j=1}^{N} |\nu_j|^2. \quad (4.35)$$

Comparing estimates (4.34) and (4.35), we see that $\sigma[w_\varepsilon, w_\varepsilon] \leq -c_0 \sum_{j=1}^{N} |\nu_j|^2.$

for arbitrary numbers $\nu_1, \ldots, \nu_N \in \mathbb{C}$. In view of formula (4.5) where $w_\varepsilon$ and $f_\varepsilon$ are related by equation (2.11), this yields us the subspace in $D[h]$ of dimension $N$ where the form $h[f_\varepsilon, f_\varepsilon]$ is negative. This concludes the proof of Theorem 4.8.

5. Quasi-Carleman and finite rank Hankel operators

Here we consider perturbations of singular quasi-Carleman operators $H_0$ by finite rank self-adjoint Hankel operators $V$. We shall prove that $N_-(H_0 + V) = N_-(V)$, that is, adding $H_0$ to $V$ does not change the total number of negative eigenvalues of a finite rank Hankel operator $V$. Our proof here is relatively similar to that of Theorem 4.8 but new difficulties arise because the singularities of the sign-function $s_v(x)$ may lie in the complex plane; in this case $s_v(x)$ is even more singular than function (4.10).

5.1. The unperturbed operator $H_0$ is the same as in Section 4. Thus we accept Assumption 4.1 and define $H_0$ by its quadratic form (4.3).

Recall that integral kernels of finite rank Hankel operators $V$ are given (this is the classical Kronecker theorem — see, e.g., Sections 1.3 and 1.8 of the book [12]) by the formula

$$v(t) = \sum_{m=1}^{M} P_m(t) e^{-\beta_m t} \quad (5.1)$$

where $\Re \beta_m > 0$ and $P_m(t)$ are polynomials. We consider self-adjoint $V$ when $v(t) = v(t)$. If $\Im \beta_m \neq 0$, then necessarily the sum in (5.1) contains both terms $P_m(t) e^{-\beta_m t}$ and $\overline{P_m(t)} e^{-\bar{\beta}_m t}$. Let $\Im \beta_m = 0$ for $m = 1, \ldots, M_0$ and $\Im \beta_m > 0$, $\beta_{M_1+m} = \bar{\beta}_m$ for $m = M_0+1, \ldots, M_0+M_1$. Thus $M = M_0 + 2M_1$; of course the cases $M_0 = 0$ or $M_1 = 0$ are not excluded. We have $P_m(t) = \overline{P_m(t)}$ for $m = 1, \ldots, M_0$ and $P_{M_1+m}(t) = \overline{P_m(t)}$ for $m = M_0+1, \ldots, M_0+M_1$. Let $K_m = \deg P_m$. Then rank $V = \sum_{m=1}^{M} K_m + M$.

For $m = 1, \ldots, M_0$, we set

$$p_m = P_m(K_m),$$
that is, $p_m/K_m$ is the coefficient at $t^{K_m}$ in the polynomial $P_m(t)$, and
\[
\begin{aligned}
N_m &= (K_m + 1)/2 \quad \text{if } K_m \text{ is odd} \\
N_m &= K_m/2 \quad \text{if } K_m \text{ is even and } p_m > 0 \\
N_m &= K_m/2 + 1 \quad \text{if } K_m \text{ is even and } p_m < 0.
\end{aligned}
\]

Our main result is formulated as follows.

**Theorem 5.1.** Let a function $s_0(x)$ satisfy Assumption 4.1 and let $H_0$ be the self-adjoint positive operator defined by the quadratic form (4.3). Let $V$ be the self-adjoint Hankel operator of finite rank with kernel $v(t)$ given by formula (5.1), and let the numbers $N_m$ be defined by formula (5.2). Then the total number $N_-(H)$ of (strictly) negative eigenvalues of the operator $H = H_0 + V$ is given by the formula
\[
N_-(H) = \sum_{m=1}^{M_0} N_m + \sum_{m=M_0+1}^{M_0+M_1} K_m + M_1 =: \mathcal{N}.
\]

Theorem 5.1 generalizes the corresponding result of [19] where the sign-function $s_0(x)$ was supposed to be bounded; in this case the operator $H_0$ is also bounded. In particular, formula (5.3) was established in [19] in the case $H_0 = 0$, that is, for finite rank Hankel operators $H$. We emphasize that the right-hand side of (5.3) does not depend on the operator $H_0$. Therefore using (5.3) for $H_0 = 0$, we obtain equality (1.18).

Since $H_0 \geq 0$, we have $N_-(H) \leq N_-(V) = \mathcal{N}$. Thus we only have to prove that
\[
N_-(H) \geq \mathcal{N}. \tag{5.4}
\]

To that end, we construct trial functions $f$ such that $h[f,f] < 0$. We emphasize that the constructions for the terms in (5.1) corresponding to $\text{Im} \beta_m = 0$ and to $\text{Im} \beta_m \neq 0$ are essentially different.

**5.2.** In this subsection we collect some results of [19] which we use below. First we recall the explicit expression for the sign-function $s_v(x)$ of the Hankel operator with kernel $v(t) = t^j e^{-\beta t}$.

**Lemma 5.2.** Let $v(t) = t^j e^{-\beta t}$ where $\text{Re} \beta > 0$ and $j \in \mathbb{Z}_+$. If $j = 0$, then
\[
s_v(x) = \beta^{-1} \delta(x - \kappa), \quad \kappa = -\ln \beta, \quad -\pi/2 < \text{Im} \kappa < \pi/2. \tag{5.5}
\]

If $j \geq 1$, then
\[
s_v(x) = \beta^{-1-j} (1 - \partial) \cdots (j - \partial) \delta(x - \kappa). \tag{5.6}
\]

Clearly, $s_v \notin S'$ unless $\text{Im} \beta = 0$, but the corresponding sigma-function $\sigma_v \in \mathcal{Y}$. Actually, distributions (5.5) and (5.6) are well defined as antilinear functionals on test functions $u(z)$ analytic in the strip $-\pi/2 < \text{Im} z < \pi/2$. We put $u^*(z) = \overline{u(\bar{z})}$. It follows from Lemma 5.2 that $s_v[u,u] = \langle s_v, u^* u \rangle$ is determined by values of $u(z)$ and its derivatives at the points $z = \kappa$ and $z = \bar{\kappa}$. To be more precise, Lemma 5.2 implies the following result.
Lemma 5.3. Let $v(t) = P(t)e^{-\beta t}$ where $\Re \beta > 0$ and $P(t) = p_K t^K + \cdots, p_K \neq 0$, is a polynomial of degree $K$. Put

$$J_K(\kappa)u = (u(\kappa), u'(\kappa), \ldots, u^{(K)}(\kappa))^\top \in \mathbb{C}^{K+1}. \quad (5.7)$$

Then

$$s_v[u, u] = (S(P, \kappa) J_K(\kappa)u, J_K(\bar{\kappa})u)_{K+1}, \quad \kappa = -\ln \beta,$$

where $(\cdot, \cdot)_{K+1}$ is the scalar product in $\mathbb{C}^{K+1}$ and the matrix $S(P, \beta)$ is skew triangular, that is, its entries $s_{j,\ell} = 0$ for $j + \ell > K$. Moreover, we have

$$s_{j,\ell} = C^j_K \beta^{-1-K} p_K \quad \text{for} \quad j + \ell = K \quad (5.8)$$

and $S(\bar{P}, \bar{\beta}) = S(P, \beta)^*$.\footnote{The upper index "\top" means that a vector is regarded as a column.}

According to (5.8) we have $\text{Det} S \neq 0$. Actually, all entries $s_{j,\ell}$ of the matrix $S(P, \beta)$ (we call it the sign-matrix of the kernel $v(t)$) admit simple expressions in terms of the coefficients of the polynomial $P(t)$, but below we need only the information collected in Lemma 5.3.

In the symmetric case, we use the following result on spectra of the sign-matrices.

Lemma 5.4. Let $\beta = \bar{\beta}$ and $P(t) = \overline{P(t)}$. If $K$ is odd, then $S(P, \beta)$ has $(K + 1)/2$ positive and $(K + 1)/2$ negative eigenvalues. If $K$ is even, then $S(P, \beta)$ has $K/2 + 1$ positive and $K/2$ negative eigenvalues for $p_K > 0$ and it has $K/2$ positive and $K/2 + 1$ negative eigenvalues for $p_K < 0$.

In the complex case, Lemma 5.3 implies the following assertion.

Proposition 5.5. Let

$$v(t) = P(t)e^{-\beta t} + \overline{P(t)}e^{-\bar{\beta} t}, \quad \Re \beta > 0, \quad \Im \beta > 0.$$ 

Put

$$\tilde{J}_K(\kappa)u = (J_K(\kappa)u, J_K(\bar{\kappa})u)^\top. \quad (5.9)$$

Then

$$s_v[u, u] = (\tilde{S}(P, \beta)(\tilde{J}_K(\kappa)u, (\tilde{J}_K(\kappa)u)_{2K+2}, \quad \kappa = -\ln \beta,$$

where the sign-matrix

$$\tilde{S}(P, \beta) = \begin{pmatrix} 0 & S(P, \beta)^* \\ S(P, \beta) & 0 \end{pmatrix}. \quad (5.10)$$

Obviously, the spectrum of matrix (5.10) is symmetric so that it consists of $K + 1$ positive and $K + 1$ negative eigenvalues.

Let us collect the results obtained together.
Theorem 5.6. Let \( v(t) \) be kernel (5.1), let the operators \( J_{K_m}(\kappa_m) \) and \( \tilde{J}_{K_m}(\kappa_m) \) be defined by formulas (5.7) and (5.9), respectively, and let \( S(P_m, \beta_m) \) and \( \tilde{S}(P_m, \beta_m) \) be the corresponding sign-matrices. Then for all functions \( u(z) \) analytic in the strip \(-\pi/2 < \text{Im} \, z < \pi/2\), we have

\[
s_v[u, u] = \sum_{m=1}^{M_0} (S(P_m, \beta_m)J_{K_m}(\kappa_m)u, J_{K_m}(\kappa_m)u)_{K_m+1} + \sum_{m=M_0+1}^{M_0+M_1} (\tilde{S}(P_m, \beta_m)\tilde{J}_{K_m}(\kappa_m)u, \tilde{J}_{K_m}(\kappa_m))_{2K_m+2}. \tag{5.11}
\]

5.3. For the construction of trial functions \( u(x) \) where the form \( s[u, u] < 0 \), we need the following assertion. We emphasize that the considerations of real and complex \( \kappa_m \) in (5.11) are essentially different.

Lemma 5.7. Let \( \kappa \in \mathbb{C}, \varepsilon > 0 \) (\( \varepsilon \) is a small parameter) and let \( \omega(z) \) be a polynomial such that \( \omega(\kappa) \neq 0 \). If \( \kappa = \bar{\kappa} \), we set

\[
\varphi(z; \varepsilon) = \omega(z)e^{-(z-\kappa)^2/\varepsilon^2}. \tag{5.12}
\]

If \( \kappa = \kappa' + i\kappa'' \) where \( \kappa'' \neq 0 \), we set

\[
\varphi(z; \varepsilon) = \omega(z)e^{-i\text{sgn} \kappa''(z-\kappa)/\varepsilon}e^{-(z-\kappa')^2}. \tag{5.13}
\]

Let \( a_0, a_1, \ldots, a_K \) be any given numbers. Then there exists a polynomial

\[
Q(z; \varepsilon) = \sum_{p=0}^{K} q_p(\varepsilon)(z - \kappa)^p \tag{5.14}
\]

such that the function

\[
\psi(z; \varepsilon) = Q(z; \varepsilon)\varphi(z; \varepsilon) \tag{5.15}
\]

satisfies the conditions

\[
\psi^{(j)}(\kappa; \varepsilon) = a_j, \quad j = 0, 1, \ldots, K. \tag{5.16}
\]

Moreover, the coefficients of polynomial (5.14) satisfy estimates

\[
|q_p(\varepsilon)| \leq C\varepsilon^{-p}, \quad p = 0, 1, \ldots, K. \tag{5.17}
\]

Proof. In view of (5.14), (5.15) conditions (5.16) yield the equations

\[
j! \sum_{p=0}^{j} (p-j)!^{-1} q_p(\varepsilon)\varphi^{(j-p)}(\kappa; \varepsilon) = a_j, \quad j = 0, 1, \ldots, K, \tag{5.18}
\]

for the coefficients \( q_p(\varepsilon) \). Let us consider these equations successively starting from \( j = 0 \). Observe that \( \varphi(\kappa; \varepsilon) = \varphi(\kappa; 0) = \omega(\kappa)e^{\kappa''^2} \neq 0 \). Therefore

\[
q_0 = \varphi(\kappa; 0)^{-1}a_0.
\]
Then equation (5.18) determines \( q_j(\varepsilon) \) if \( q_0, q_1(\varepsilon), \ldots, q_{j-1}(\varepsilon) \) are already found:

\[
\varphi(\kappa; 0)q_j(\varepsilon) = j!^{-1}a_j - \sum_{p=0}^{j-1}(p-j)!^{-1}q_p(\varepsilon)\varphi^{(j-p)}(\kappa; \varepsilon).
\]

Since for both functions (5.12) and (5.13) estimates (5.17) on \( q_0(\varepsilon), \ldots, q_{j-1}(\varepsilon) \) imply the same estimate on \( q_j(\varepsilon) \).

If \( \omega = x \in \mathbb{R} \), then functions (5.12) and (5.13) satisfy estimates

\[
|\varphi(x; \varepsilon)| \leq C(1 + |x - \kappa|^{\deg \omega})e^{-(x-\kappa)^2/\varepsilon^2}, \quad \kappa = \bar{\kappa},
\]

and

\[
|\varphi(x; \varepsilon)| \leq Ce^{-|\kappa''|/\varepsilon}(1 + |x - \kappa'|^{\deg \omega})e^{-(x-\kappa')^2}, \quad \kappa'' \neq 0,
\]

respectively. Here we have taken into account that

\[
e^{-isgn \kappa''(z-\kappa)/\varepsilon} = e^{-|\kappa''|/\varepsilon}.
\]

Note also that in view of (5.17) polynomial (5.14) satisfies the estimate

\[
|Q(x; \varepsilon)| \leq C(1 + |x - \kappa'|^{K}e^{-K}).
\]

This leads to the following assertion.

**Corollary 5.8.** Under the assumptions of Lemma 5.7, for any \( l \in \mathbb{R} \),

\[
\int_{-\infty}^{\infty} e^{l|x|}|\psi(x; \varepsilon)|^2dx = O(\varepsilon), \quad \varepsilon \to 0.
\]

Proof. If \( \kappa'' = 0 \), we use estimates (5.19) and (5.21). Making the change of variables \( x - \kappa = \varepsilon y \), we see that integral (5.22) is bounded by \( C\varepsilon \). If \( \kappa'' \neq 0 \), then this integral tends to zero exponentially due to the factor \( e^{-|\kappa''|/\varepsilon} \) in (5.20).

**5.4.** Let us return to Hankel operators \( H = H_0 + V \). For kernels (5.1), we put \( \kappa_m = -\ln \beta_n \). Then \( \kappa_m = \bar{\kappa}_m \) for \( m = 1, \ldots, M_0 \) and \( \kappa_m = \bar{\kappa}_{m+M_1} \) for \( m = M_0 + 1, \ldots, M_0 + M_1 \). For all \( k = 0, \ldots, K_m, m = 1, \ldots, M \), we construct the functions \( \psi_{k,m}(z, \varepsilon) \) by formulas of Lemma 5.7 where \( \kappa = \kappa_m \). Of course we use formula (5.12) if \( \text{Im} \kappa_m = 0 \) and (5.13) if \( \text{Im} \kappa_m \neq 0 \). We require that \( \psi^{(l)}_{k,m}(\kappa_m; \varepsilon) = \delta_{k,l} \) for all \( k, l = 0, \ldots, K_m \). Let us set \( \omega_1(z) = 1 \) if \( M = 1 \) and

\[
\omega_m(z) = \prod_{n=1; n \neq m}^{M} (z - \kappa_n)^{K_n+1} \quad \text{if} \quad M \geq 2.
\]

Then \( \omega_m(\kappa_m) \neq 0 \) and due to this factor in (5.12) and (5.13) the function \( \psi_{k,m}(z; \varepsilon) \) satisfies the conditions \( \psi^{(l)}_{k,m}(\kappa_m; \varepsilon) = 0 \) for all \( n \neq m \) and \( l = 0, \ldots, K_m \).
of the conditions at the points \( \kappa_m \), \( m = 1, \ldots, M \), for every fixed \( \varepsilon > 0 \) all functions \( \psi_{k,m}(z; \varepsilon) \) are linearly independent.

For arbitrary complex numbers \( \nu_{k,m} \), we put
\[
u_m(z; \varepsilon) = \sum_{k=0}^{K_m} \nu_{k,m} \psi_{k,m}(z; \varepsilon) \tag{5.23}\]
for \( m = 1, \ldots, M_0 \) and
\[
u_m(z; \varepsilon) = \sum_{k=0}^{K_m} \left( \nu_{k,m} \psi_{k,m}(z; \varepsilon) + \nu_{k,m+M_1} \psi_{k,m+M_1}(z; \varepsilon) \right) \tag{5.24}\]
for \( m = M_0 + 1, \ldots, M_0 + M_1 \). The functions \( u_1(z; \varepsilon), \ldots, u_{M_0+M_1}(z; \varepsilon) \) are of course linearly independent.

Let \( a_m = (\nu_{0,m}, \nu_{1,m}, \ldots, \nu_{K_m,m})^\top \in \mathbb{C}^{K_m+1} \) for \( m = 1, \ldots, M \). We put \( a_m = a_m \), \( J_mu = J_m(k_m)u, S_m = S(P_m, \beta_m) \) for \( m = 1, \ldots, M_0 \) and \( a_m = (a_m, a_{m+M_1})^\top \), \( J_mu = J_{m}(k_m)u, S_m = \bar{S}(P_m, \beta_m) \) for \( m = M_0 + 1, \ldots, M_0 + M_1 \). Note that each matrix \( S_m \) has \( N_m \) negative eigenvalues with \( N_m \) defined by formula (5.2). For \( m = 1, \ldots, M_0 \), this result follows from Lemma 5.4. For \( m = M_0 + 1, \ldots, M_0 + M_1 \), this is a direct consequence of representation (5.10).

For functions (5.23) and (5.24), we have \( J_mu_m(\varepsilon) = a_m \) and \( J_nu_m(\varepsilon) = 0 \) if \( n \neq m \). Therefore it follows from formula (5.11) that
\[s_0[u_m(\varepsilon), u_n(\varepsilon)] = (S ma_m, a_m), \quad s_0[u_m(\varepsilon), u_n(\varepsilon)] = 0, \quad \forall \varepsilon > 0,\]
for all \( m, n = 1, \ldots, M_0 + M_1, n \neq m \). According to Corollary 5.8 under Assumption 4.1 we have
\[s_0[u_m(\varepsilon), u_n(\varepsilon)] \leq C\varepsilon ||a_m||||a_n||, \quad \forall m, n = 1, \ldots, M_0 + M_1.\]
Thus if \( a_m \) is an eigenvector of the matrix \( S_m \) corresponding to its negative eigenvalue \( -\mu_m \), then
\[s[u_m(\varepsilon), u_n(\varepsilon)] \leq -(\mu_m - C\varepsilon) ||a_m||^2.\]
This expression is negative if \( C\varepsilon < \mu_m \). This yields the estimate
\[s[u(\varepsilon), u(\varepsilon)] \leq -c||u||^2, \quad c > 0, \tag{5.25}\]
for all linear combinations \( u(z; \varepsilon) \) of functions \( u_m(z; \varepsilon) \). Since we have \( N_m \) linearly independent functions \( u_m(z; \varepsilon) \), the dimension of their linear combinations equals \( N = N_1 + \cdots + N_{M_0+M_1} \).

Let \( f(t; \varepsilon) \) be the functions linked to \( u(x; \varepsilon) \) by formula (2.11). Since the functions \( \psi_{k,m}(x; \varepsilon) \) defined by equalities (5.12), (5.13) and (5.15) decay super-exponentially as \( |x| \to \infty \), the functions \( f(\varepsilon) \in L^2(\mathbb{R}_+) \). These functions belong to \( D[h] \) because \( s[u(\varepsilon), u(\varepsilon)] < \infty \). It now follows from identity (4.8) and estimate (5.25) that
\[h[f, f] = ||\sqrt{H}f(\varepsilon)||^2 + (V f(\varepsilon), f_m(\varepsilon)) = s[u(\varepsilon), u(\varepsilon)] < 0\]
for all \( f(\varepsilon) \neq 0 \) in the linear space of dimension \( \mathcal{N} \). This yields estimate (5.4) and thus concludes the proof of Theorem 5.1.

**Remark 5.9.** The condition \( s_0 \in L^\infty_{\text{loc}}(\mathbb{R}) \) of regularity of the sign-function \( s_0(x) \) in Assumption 4.1 cannot be omitted. Indeed, consider, for example, the kernels \( h_0(t) = 2e^{-t}, v(t) = -e^{-t} \). Then \( N_-(V) = 1 \) while \( N_-(H) = 0 \).

**APPENDIX A. Sandwitched Fourier transforms**

For absolutely continuous measures \( dM(\lambda) = \sigma(\lambda)d\lambda \), the results of Section 3 on the operators \( L : L^2 \to L^2(M) \) can be reformulated in terms of the operators

\[
A = s(x)\Phi^*v(\xi)
\]

where \( v(\xi) = \Gamma(1/2 - i\xi) \) and \( s(x) = \sigma(e^{-x}) \). Now we study operators \( A \) in the space \( L^2 := L^2(\mathbb{R}) \) for sufficiently arbitrary functions \( v(\xi) \) and \( s(x) \). Properties of operators \( (A.1) \) for \( v \) and \( \sigma \) belonging to some spaces \( L^p \) were extensively discussed in the literature (see, e.g., the book [14]). Here we consider operators \( (A.1) \) with functions \( s(x) \) growing at infinity. Our goal is to define operators \( (A.1) \) as closed (unbounded) operators in \( L^2 \).

In contrast to Section 3, our assumptions on \( s(x) \) exclude its exponential growth at infinity. We now suppose that

\[
|s(x)| \leq C(1 + |x|)^K
\]

for some \( K \in \mathbb{R} \). With respect to \( v(\xi) \), we assume that \( v \in C^\infty \) and

\[
|v^{(n)}(\xi)| \leq C_n(1 + |\xi|)^{k_n}
\]

for all \( n = 0, 1, \ldots \) and some numbers \( k_n \in \mathbb{R} \). Obviously, for \( f \in \mathcal{S} \), we have \( vf \in \mathcal{S} \), \( \Phi^*(vf) \in \mathcal{S} \) and hence \( s\Phi^*(vf) \in L^2 \). It means that \( \mathcal{A} : \mathcal{S} \to L^2 \). If \( g \in L^2 \), then \( \bar{sg} \in \mathcal{S}' \), \( \Phi(\bar{sg}) \in \mathcal{S}' \) and

\[
\mathcal{A}_s g := \bar{v}\Phi\bar{sg} \in \mathcal{S}'.
\]

It means that \( \mathcal{A}_s : L^2 \to \mathcal{S}' \). Moreover, for all \( f \in \mathcal{S} \) and all \( g \in L^2 \), we have the identity

\[
\langle \mathcal{A}_s f, g \rangle = \langle f, \mathcal{A}_s g \rangle.
\]

Define now the operator \( A_0 \) in \( L^2 \) on the domain \( \mathcal{D}(A_0) = \mathcal{S} \) by the equality \( A_0 f = \mathcal{A} f \). Let us construct its adjoint operator \( A_0^* \). Let \( \mathcal{D}_s \subset L^2 \) consist of \( g \in L^2 \) such that \( \mathcal{A}_s g \in L^2 \).

**Lemma A.1.** Under assumptions \( (A.2) \) and \( (A.3) \) the operator \( A_0^* \) is given by the equality \( A_0^* g = \mathcal{A}_s g \) on the domain \( \mathcal{D}(A_0^*) = \mathcal{D}_s \).

**Proof.** If \( f \in \mathcal{D}(A_0) \) and \( g \in \mathcal{D}_s \), then it follows from identity \( (A.5) \) that

\[
(A_0 f, g) = (f, \mathcal{A}_s g).
\]

Hence \( g \in \mathcal{D}(A_0^*) \) and \( A_0^* g = \mathcal{A}_s g \).
Conversely, if \( g \in D(A_n^*) \), then \( |(Af, g)| \leq C\|f\| \) for all \( f \in S \). In view of (A.5), this estimate implies that \( \mathcal{A}_*g \in L^2 \) and hence \( D(A_0^*) \subset D_* \).

**Corollary A.2.** Suppose additionally that \( k_0 = 0 \) in condition (A.3). Then the operator \( A_0 \) admits the closure.

**Proof.** Indeed, for \( g \in S \), we have \( \check{s}g \in L^2 \) so that \( \mathcal{A}_*g \in L^2 \) if the function \( v \) is bounded. It follows that \( S \subset D_* = D(A_0^*) \), and hence the operator \( A_0^* \) is densely defined.

Next, we construct the second adjoint \( A_0^{**} \). Observe that \( \mathcal{A} : L^2 \to S' \). Let the operator \( A \) be defined by the equality \( Af = \mathcal{A}f \) on the domain \( D(A) \) which consists of all \( f \in L^2 \) such that \( Af \in L^2 \).

**Lemma A.3.** Under assumptions (A.2) and (A.3) where \( k_0 < -1/2 \), the inclusion \( A_0^{**} \subset A \) holds.

**Proof.** For \( f \in L^2 \) we have \( vf \in L^1 \), and for \( g \in S \) we have \( \check{s}g \in L^1 \). Therefore according to the Fubini theorem, the identity (A.5) is now true for all \( f \in L^2 \) and all \( g \in S \). If \( f \in D(A_0^{**}) \), then \( |(f, A_0g)| \leq C\|g\| \) for all \( g \in D_* \) and, in particular, for \( g \in S \). Thus it follows from (A.5) that \( |(A\bar{f}, g)| \leq C\|g\| \) and hence \( A\bar{f} \in L^2 \). Moreover, \( A_0^{**}f = A\bar{f} \) according again to (A.5).

Let the assumptions of Lemma A.3 hold. For the proof of the opposite inclusion \( A \subset A_0^{**} \), we have to check relation (A.5) for all \( f \in L^2 \) and \( g \in L^2 \) such that \( Af \in L^2 \) and \( A_*g \in L^2 \). Suppose now that \( v(\xi) \neq 0 \) for all \( \xi \in \mathbb{R} \). Let \( \chi_n \) be the operator of multiplication by the function \( \chi_n(x) = \chi(x/n) \) where \( \chi = \check{\chi} \), \( \chi(0) = 1 \) and the Fourier transform \( \check{\chi} = \Phi \chi \in C_0^{\infty}(\mathbb{R}) \). Since \( vf \in L^1 \) and \( \chi_n \check{s}g \in L^1 \), it follows from the Fubini theorem that

\[
(Af, \chi_n g) = (f, A_* \chi_n g), \quad \forall f, g \in L^2.
\]

We have to pass here to the limit \( n \to \infty \). Since \( Af \in L^2 \), \( g \in L^2 \) and \( \chi_n \to I \) strongly in this space, the left-hand side of (A.6) converges to \( (Af, g) \).

Let us now consider the right-hand side of (A.6). According to (A.4) we have

\[
A_*\chi_n g = T_n A_* g
\]

where

\[
T_n = \check{v}\Phi \chi_n \Phi^* v^{-1}.
\]

Quite similarly to Lemma 3.6 we obtain the following assertion.

**Lemma A.4.** Suppose that \( v(\xi) \neq 0 \) for all \( \xi \in \mathbb{R} \) and that

\[
\max_{|\xi - \eta| \leq 1} |v(\eta)||v(\xi)|^{-1} < \infty.
\]

Then the operators \( T_n \) defined by formula (A.8) are bounded in the space \( L^2 \), and their norms are bounded uniformly in \( n \). Moreover, \( T_n \to I \) strongly as \( n \to \infty \).
Note that condition (A.9) admits an exponential decay of the function $v(\xi)$ as $|\xi| \to \infty$, but not a more rapid one. In particular, function (2.9) is allowed.

It follows from equality (A.7) and Lemma A.4 that if $A_*g \in L^2$, then $A_\ast \chi_n g \to A_*g$ as $n \to \infty$. This allows us to pass to the limit $n \to \infty$ in the right-hand side of (A.6) which yields relation (A.5) for all $f \in L^2$ and $g \in L^2$ such that $Af \in L^2$ and $A_*g \in L^2$. This proves that $A \subset A_0^{**}$.

Let us summarize the results obtained.

**Theorem A.5.** Let the operator $A$ be defined by formula (A.1) on the set $S$. Let the functions $s(x)$ and $v(\xi)$ satisfy assumptions (A.2) and (A.3) where $k_0 = 0$, respectively. Then the operator $A_0$ in $L^2$ defined on the domain $D(A_0) = S$ by the equality $A_0 f = Af$ admits the closure. Suppose additionally that $k_0 < -1/2$ in (A.3) and that the assumptions of Lemma A.4 are satisfied. Then the closure $A_0 := A$ is given by the same equality $Af = A_f$ on the domain $D(A)$ which consists of all $f \in L^2$ such that $Af \in L^2$.

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