Non–Contact Friction for Ion–Surface Interactions

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Non-contact friction forces are exerted on physical systems through dissipative processes, when the two systems are not in physical contact with each other, or, in quantum mechanical terms, when the overlap of their wave functions is negligible. Non-contact friction mediated by the exchange of virtual quanta, with the additional requirement that the scattering process needs to have an inelastic component. For finite-temperature ion-surface interactions, the friction is essentially caused by Ohmic resistance due to the motion of the image charge moving in a dielectric material. A conceivable experiment is difficult because the friction force needs to be isolated from the interaction with the image charge, which significantly distorts the ion’s flight path. We propose an experimental setup which is designed to minimize the influence of the image charge interaction through a compensation mechanism, and evaluate the energy loss due to non-contact friction for helium ions (He\textsuperscript{+}) interacting with gold, vanadium, titanium and graphite surfaces. Interactions with the infinite series of mirror charges in the plates are summed in terms of the logarithmic derivatives of the Gamma function, and of the Hurwitz zeta function.

PACS numbers: 31.30.jh, 12.20.Ds, 31.30.J–, 31.15.–p

I. INTRODUCTION

If an ion trajectory is parallel and close to a surface (close to a “half-space filled with a conducting material”), then it is intuitively obvious that the motion of the image charge inside the conductor will lead to some sort of Ohmic heating. Unless one can show that the motion of the image charge is frictionless, this Ohmic heating can only be compensated by a corresponding loss in the kinetic energy of the projectile ion. Here, we devote special attention to a rederivation of the non-contact friction in ion-surface interactions, with an emphasis on a proposed experimental setup which serves to eliminate the “signal” (the quantum friction) from the “noise” (the interaction with the image charge). The latter can otherwise lead to a net gain or loss of kinetic energy for the ion.

Provided one accepts the existence of non-contact friction for ion-surface interactions, the same effect (but at a smaller scale) can be expected for atom-surface interactions. Namely, any atom constitutes, due to its quantum mechanical nature, an “oscillating dipole” whose frequency-dependent response to exciting radiation is described by the (dynamic dipole) polarizability. Because the dipole moment fluctuates, there is a fluctuating image charge moving in the medium, which also leads to Ohmic heating and by consequence, non-contact friction. However, the friction force in this case will be much smaller, numerically, than in the case of ion-surface interactions.

One can draw a distant analogy to ion-atom versus atom-atom interactions. It is well known that the ion-atom interaction (or more generally, the interaction of a charged particle with an atom) has the functional form $1/R^4$ where $R$ is the ion-atom distance. For two neutral atoms, the dominant term in the non-retardation limit is the van-der-Waals interaction, which gives rise to a $1/R^6$-interaction. For the quantum friction effect, with an ion-surface distance $Z$, the damping coefficient $\eta$ in the friction force $F = -\eta v$ is proportional to $1/Z^3$ for ion-surface interactions, which is modified to a $1/Z^5$-law for atom-surface interactions [1]. The pattern is clear: There are two more powers of the distance ($R$ viz. $Z$) in the denominator for interactions with atoms, as opposed to the corresponding interaction involving an ion “at the other end”.

Alternatively, we may treat the case of finite temperature [2–6] as follows [1, 7–9]. When two physical systems are in contact, then the charge fluctuations in the one will induce mirror charge fluctuations in the other. At zero temperature, the alteration of the available modes for quantum fluctuations due to the presence of the surface (due to the “boundary conditions”) gives rise to atom-wall interactions of the Casimir–Polder type (see Chap. 8 of Ref. [10]). At finite temperature, one has to replace the integral over the virtual photon frequency by a summation over Matsubara frequencies [see Eq. (30) of Ref. [1]]. The zero–temperature limit is obtained by the replacement $\beta^{-1} \sum_{n=-\infty}^{\infty} \rightarrow (2\pi)^2 \int_\omega^{\infty} d\omega/(2\pi)$, where the $n$th Matsubara frequency is $\omega_n = 2\pi/(\hbar\beta)$, and $\beta = 1/(k_B T)$ is the Boltzmann factor. The same observation is made in Sec. 81 of Ref. [11], where it is shown that in the zero-temperature and short-distance limit, the Casimir force between two solids reduces to an expression which is in agreement with the van-der-Waals force between atoms [limit of two dilute media, see Eqs. (81.1), (81.9) and (82.3) ff. of Ref. [11]]. The calculation of static forces is most easily accomplished in the imaginary-time formalism, which leads to the Matsubara frequencies, while the calculation of friction forces is usually done using the real-time retarded fluctuation-dissipation theorem [1, 9]. Indeed, in Ref. [9], it is shown that the friction force on an atom in the vicinity of a dielectric surface can be obtained from a calculation of the track.
van-der-Waals interaction for an atom which undergoes a small periodic mechanical oscillation, after subtracting the conservative static, and the conservative oscillatory component (vibration component) of the van-der Waals force [see Eq. (24) of Ref. [9]].

In order to calculate the friction force, one has to describe the correlation of electric field fluctuations at different points, in the presence of a dielectric material filling the half-space \( z < 0 \), due to charge fluctuations in the dielectric [2–4, 11]. We here use the real-time formulation of the fluctuation-dissipation theorem and apply it from first principles. Another point is that one has to apply the Wick theorem to the thermal fluctuations, remembering the conservative static, and the conservative oscillatory component (vibration component) of the van-der-Waals interaction for an atom which undergoes a small periodic mechanical oscillation, after subtracting the conservative static, and the conservative oscillatory component (vibration component) of the van-der Waals force [see Eq. (24) of Ref. [9]].

The Kallen–Welton thermal factor \( \Theta(\omega, T) \) is given by

\[
\Theta(\omega, T) = \frac{1}{\exp(\beta h \omega) - 1}.
\]

The Kallen–Welton thermal factor \( \Theta(\omega, T) \) is given by the relation

\[
\Theta(\omega, T) = h \omega \left( \frac{1}{2} + n(\omega) \right) = \frac{1}{2} h \omega \coth \left( \frac{1}{2} \beta \omega \right).
\]

It has the properties,

\[
\Theta(-\omega, T) = -\Theta(\omega, T),
\]

\[
(1 + n(\omega)) (1 + n(-\omega)) = \frac{1}{\beta} \frac{\partial n(\omega)}{\partial \omega},
\]

\[
(1 + n(\omega)) n(\omega) = -\frac{1}{\beta} \frac{\partial n(\omega)}{\partial \omega}.
\]

In the high-temperature limit, we have \( \Theta(\omega, T) \to 1/\beta = k_B T \). Let now \( x \) be an observable of a dynamical system with Hamiltonian \( H_0(x) \) subject to thermal fluctuations. We assume that \( x(t) \) fluctuates around its mean value \( \langle x \rangle_0 \) with fluctuations characterized by a power spectrum

\[
S_x(\omega) = \langle x(t) x(0) \rangle_\omega = \int \frac{d\omega}{2\pi} e^{-i\omega t} \langle x(t) x(0) \rangle.
\]

Let us consider the conjugate variable of \( x \), namely, a scalar force field \( f \) which alters the Hamiltonian to

\[
H = H_0(x) + x(t) f(t).
\]

The response of an observable \( x \) to the field term is characterized (to first order) by the susceptibility of the linear response function \( \chi(t) \) of the system,

\[
\langle x(t) \rangle = \langle x \rangle_0 + \int_{-\infty}^{t} \chi(t - \tau) f(\tau) d\tau,
\]

where the mean value \( \langle x \rangle_0 \) is obtained as a thermal average taken over the distribution governed by the unperturbed Hamiltonian \( H_0(x) \), i.e., with respect to the weight function

\[
W_0(x) = \frac{\exp\left[-\beta H_0(x)\right]}{\int dx' \exp\left[-\beta H_0(x')\right]}.
\]

The field term is adiabatically switched on at \( \tau = -\infty \). Expanding the weight function of the full Hamiltonian given in Eq. (5) for small \( f(t) \), one can easily motivate [12] that the the imaginary part of the complex susceptibility

\[
\chi(\omega) = \text{Re} \left[ \chi(\omega) \right] + i \text{Im} \left[ \chi(\omega) \right],
\]

is related to the power spectrum of fluctuations, which by the fluctuation-dissipation theorem [2–6] reads as

\[
\langle x(t) x(0) \rangle_\omega = S_x(\omega) = \frac{2\Theta(\omega, T)}{\omega} \text{Im} \left[ \chi(\omega) \right].
\]

This involves the Kallen–Welton thermal factor. In the high-temperature limit, one may replace \( \Theta(\omega, T) \to 1/\beta = k_B T \). We should note that the conventions for the Fourier transform used in the current work follow those commonly used in physics,

\[
\tilde{f}(\omega) = \int dt e^{i\omega t} f(t), \quad f(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{f}(\omega),
\]

and these are different from those in electrical engineering (see Refs. [2–4]).

### B. Correlation Function for the Electric Field

Let us try to motivate a formula for the thermal correlation of the electric field in the vicinity of a dielectric material, based on the thermal charge fluctuations inside
the dielectric. We first observe that the scalar potential \( \Phi(\vec{r}) \) is given as
\[
\Phi(\vec{r}) = \int \mathrm{d}^3 r' G(\vec{r}, \vec{r}') \rho(\vec{r}'),
\]
where \( \rho \) is the charge density. In free space, the Green function is given as \( G(\vec{r}, \vec{r}') = \frac{1}{4\pi \epsilon_0 |\vec{r} - \vec{r}'|} \). In the vicinity of a dielectric wall, one has to modify the Green function as follows (non-retardation limit, see Ref. [1]),
\[
G(\vec{r}, \vec{r}') \to G(\omega, \vec{r}, \vec{r}') = \frac{1}{4\pi \epsilon_0 |\vec{r} - \vec{r}'|} - \frac{e(\omega)}{\epsilon(\omega) + 1} \frac{1}{4\pi \epsilon_0 |\vec{r} - \vec{r}' + 2n_\perp (\vec{r}' \cdot \vec{n}_\perp)|}.
\]
(12)

We here ignore a possible path difference of the emitted perturbation (at \( \vec{r} \)) and the incoming wave (at \( \vec{r}' \)). The normal vector \( \vec{n}_\perp \) is the outward normal pointing away from the half-space filled with the dielectric material. Indeed, we shall need the Green function in the limit \( \vec{r} \to \vec{r}' \). In this limit, the path difference of the reflected and the emitted wave vanishes. Corrections to this result due to the velocity of the atom in the \( x \) direction are of order \( Z \omega n_\perp^2 / \epsilon_0^2 \) and are negligible on the order of interest for the current paper. Here, \( Z \) denotes the ion-surface distance.

Our formula (12) is in agreement with Eq. (6) of Ref. [1], which gives the fluctuations of the scalar instead of the vector potential, due to charge fluctuations in the vicinity of a dielectric wall. The \( 1/(2\pi) \) prefactor in Eq. (6) of Ref. [1] is due to the different Fourier transforms conventions used therein, which follow the conventions commonly adopted in electrical engineering. The correlation function of the fluctuations of the scalar potential reads, in full agreement with the general paradigm set by the fluctuation-dissipation theorem given in Eq. (9),
\[
\langle \Phi(\vec{r}) \Phi(\vec{r}') \rangle_\omega = \frac{2 \Theta(\omega, T)}{\omega} \Im [G(\omega; \vec{r}, \vec{r}')].
\]
(13)

Here, \( \rho \) is the “fluctuating force”, whereas \( \Phi(\vec{r}) \) takes the role of the fluctuating signal. The correlation function for the electric field follows by differentiation,
\[
\langle E_i(\vec{r}) E_j(\vec{r}') \rangle_\omega = - \langle \nabla_i \Phi(\vec{r}) \nabla_j \Phi(\vec{r}') \rangle_\omega
\]
\[
= - \frac{2 \Theta(\omega, T)}{\omega} \Im \left[ \nabla_i \nabla_j G(\omega; \vec{r}, \vec{r}') \right],
\]
\[
= \frac{2 \Theta(\omega, T)}{\omega} \Im \left[ \frac{\epsilon(\omega) - 1}{\epsilon(\omega) + 1} \right]
\]
\[
\times \nabla_i \nabla_j \frac{1}{4\pi \epsilon_0 |\vec{r} - \vec{r}' + 2n_\perp (\vec{r}' \cdot \vec{n}_\perp)|}.
\]
(14)

The minus sign is explained because negatively correlated charge fluctuations at two points along a reference line generate fluctuations of the electric field which are positively correlated along the line joining the two points. Here, \( \epsilon(\omega) \) denotes the relative dielectric function (relative permittivity), which has to be multiplied by the vacuum permittivity \( \epsilon_0 \), if one would like to obtain the full electric permittivity of the medium.

There is one last subtlety to discuss. We should be careful interpreting the correlation function \( \langle E_i(\vec{r}) E_j(\vec{r}') \rangle_\omega \). Namely, according to Eqs. (32) and (49) of Ref. [1], it is more accurate to relate the following, symmetrized correlation function to the imaginary part of the susceptibility, according to the replacement
\[
\langle E_i(\vec{r}) E_j(\vec{r}') \rangle_\omega \to \int_{-\infty}^{\infty} \mathrm{d}t \cos(\omega t) \langle E_i(\vec{r}, t) E_j(\vec{r}', 0) \rangle
\]
\[
= \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}t \left( e^{i\omega t} + e^{-i\omega t} \right) \langle E_i(\vec{r}, t) E_j(\vec{r}', 0) \rangle
\]
\[
+ \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}t e^{i\omega t} \langle E_i(\vec{r}, -t) E_j(\vec{r}', 0) \rangle
\]
\[
= \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}t e^{i\omega t} \langle \{ E_i(\vec{r}, t), E_j(\vec{r}', 0) \} \rangle_\omega.
\]
(15)

The distinction between \( \langle E_i(\vec{r}) E_j(\vec{r}') \rangle_\omega \) and \( \langle \{ E_i(\vec{r}, t), E_j(\vec{r}', 0) \} \rangle_\omega \) becomes important when more than two operators are involved [1, 13]. So, we should write, more correctly,
\[
\langle \{ E_i(\vec{r}, t), E_j(\vec{r}', 0) \} \rangle_\omega = \frac{2 \Theta(\omega, T)}{\omega} \Im \left[ \frac{\epsilon(\omega) - 1}{\epsilon(\omega) + 1} \right]
\]
\[
\times \nabla_i \nabla_j \frac{1}{4\pi \epsilon_0 |\vec{r} - \vec{r}' + 2n_\perp (\vec{r}' \cdot \vec{n}_\perp)|}.
\]
(16)

The result in Eq. (16) is consistent with Eq. (15) of Ref. [14].

### III. QUANTUM FRICTION FOR IONS

#### A. Ion–Surface Interactions

Armed with the results from Sec. II, we are now in the position to evaluate the friction force due to the mirror charge running in the dielectric, which generates Ohmic heating. The force on a charged particle is given by
\[
\vec{F}(t) = Z e \vec{E}(\vec{r}, t),
\]
(17)

where we assume \( e \) to denote the electron charge, and the ion is \( Z \)-fold negatively charged. We also use the result \( [\vec{r} = (x, y, Z)] \) and \( [\vec{r}' = (x', y', Z')] \)
\[
\lim_{\vec{r} \to \vec{r}'} \nabla_x \nabla_x' \frac{1}{|\vec{r} - \vec{r}' + 2n_\perp (\vec{r}' \cdot \vec{n}_\perp)|} = \frac{1}{8 Z^3}.
\]
(18)

We consider the formula for the friction force,
\[
F_x = -\eta v_x.
\]
(19)
According to Eq. (8.15) of Ref. [15] (for a more modern perspective Ref. [16]), the fluctuation-dissipation theorem determines the friction force via a Green–Kubo formula. In the derivation, we use the fact that for $t > 0$, we have $\langle F_x(t) F_x(0) \rangle = \langle F_x(0) F_x(-t) \rangle$ due to time-translation invariance. We can thus symmetrize the integrand as follows,

$$\eta = \beta \int_0^{\infty} dt \langle F_x(t) F_x(0) \rangle$$

$$= \frac{\beta}{2} \int_{-\infty}^{\infty} dt \left\{ \frac{1}{2} \{ F_x(t), F_x(0) \} \right\}$$

$$= \frac{\beta (Ze)^2}{2} \lim_{r \to 0^+} \int_{-\infty}^{\infty} dt \left\{ \frac{1}{2} \{ E_x(r, t), E_x(r', 0) \} \right\}$$

$$= \frac{\beta (Ze)^2}{2} \lim_{\omega \to 0} \left\{ \frac{2\Theta(\omega, T)}{\omega} \Im \left( \frac{\epsilon(\omega) - 1}{\epsilon(\omega) + 1} \right) \right\}$$

$$\times \left( \frac{1}{4\pi \epsilon_0} \right) |r - r'| + 2n_1 \cdot (r \times r_1)|$$

$$= \frac{\beta (Ze)^2}{4\pi \epsilon_0} \lim_{\omega \to 0} \left\{ \frac{1}{\beta \omega} \Im \left( \frac{\epsilon(\omega) - 1}{\epsilon(\omega) + 1} \right) \right\} \frac{1}{2\omega^3}$$

$$= \frac{(Ze)^2}{32\pi \epsilon_0} \lim_{\omega \to 0} \left\{ \frac{1}{\omega} \Im \left( \frac{\epsilon(\omega) - 1}{\epsilon(\omega) + 1} \right) \right\} \frac{1}{2\omega^3}. \quad (20)$$

The end result is

$$\eta = \frac{(Ze)^2 L}{32\pi \epsilon_0 2\omega^3}. \quad (21a)$$

$$L = \lim_{\omega \to 0} \left\{ \frac{1}{\omega} \Im \left( \frac{\epsilon(\omega) - 1}{\epsilon(\omega) + 1} \right) \right\}. \quad (21b)$$

We note that the drag force $F_x = -\eta v_x$ is independent of the mass of the projectile particle; it only depends on its charge state. The result (21) depends on the asymptotic shape of the Boltzmann factor $n(\omega) = 1/(\exp(\beta \omega) - 1)$, which goes as $1/(\beta \omega)$, for small $\omega$. The final evaluation of the limit $L$ proceeds via a consideration of the low-frequency limit of the dielectric function of the material and crucially depends on the lowest electronic resonance frequency of the material.

A final word on retardation corrections is in order. In the treatment of Casimir-Polder interactions of neutral atoms, the parameter governing the retardation effects is [10, 17]

$$\xi = \frac{\omega Z}{c}, \quad \omega \sim \frac{\alpha^2 m_e e^2}{\hbar},$$

where we indicate a typical value of an atomic transition frequency $\omega$ (the fine-structure constant is $\alpha$). Retardation sets in when the phase of the atomic oscillation changes significantly over the time it takes light to travel to the surface and back, i.e., when $\xi \sim 1$ and thus

$$Z \sim \frac{a_0}{\alpha} \approx 137 a_0, \quad a_0 = \frac{\hbar}{\alpha m_e c}.$$  

Retardation changes the leading $1/Z^3$ interaction for short distances to a $1/Z^4$ term at long range [10, 17].

For the ion-surface interaction, the leading conservative term, for any distance, is given by the $1/Z$ attractive interaction with the mirror charge, and, for the dissipative friction term, by the $1/Z^3$ form given in Eq. (21). The estimate of the retardation corrections then has to proceed differently; it is necessary to evaluate the retardation correction to the (in principle electrostatic) interaction of the ion with its mirror charge. One should compare (i) the time it takes light to travel to the surface and back to the ion moving alongside and parallel to the surface, to (ii) the time it would light to travel back to a static ion. The relative difference is a measure of the retardation correction to the electrostatic interaction. The expansion parameter in this case is easily found to be

$$\xi' = \frac{\nu}{c}, \quad (24)$$

where $\nu$ is the ion’s velocity. Because these fly-by velocities are much smaller than $c$ in typical atomic-beam or ion-beam experiments, we can safely ignore the retardation corrections. Further retardation effects due to the frequency of the exchange photon can be ignored; the result given in Eq. (21b) is formulated in terms of a limit for small frequencies $\omega \to 0$.

B. Material and Friction Force

A rather general functional form for the dielectric function of a material is given as [17, 18]

$$\epsilon(\omega) = 1 + \sum_{n=0}^{\infty} \frac{a_n - i b_n \omega}{\omega_n^2 - \omega^2 - i \gamma_n \omega}. \quad (25)$$

The expansion coefficient $a_n$ is an amplitude which can be written as

$$a_n = \Delta \epsilon_n \omega_n^2, \quad b_n = \Delta \epsilon_n \gamma_n. \quad (26)$$

Recently, an approach with manifestly complex expansion coefficients has led to an excellent fit of the dielectric function of silicon over a wide frequency range [18]. The static limit of the dielectric function then reads as

$$\epsilon(\omega = 0) = 1 + \sum_{n=0}^{\infty} \Delta \epsilon_n. \quad (27)$$

The functional behaviour of $\epsilon(\omega)$ for small $\omega$ crucially depends on the value of the lowest resonance frequency $\omega_n = 0$, where we reserve the subscript $n = 0$ for the case of an excitation at zero resonance frequency $\omega_0 = 0$ and otherwise set its multiplying coefficient to zero in the case of the absence of such a resonance. We ascertain that if a resonance with $\omega_0 = 0$ exists, then $\epsilon(\omega) = 1 = O(\omega^{-1})$ for small angular frequencies. This is the case for metals and other conductors, where, by definition, the energy
constituting the conduction band electrons. This leads to such as gases, the evaluation of the limit friction coefficient vanishes. By contrast, if \( a_0 = 0 \) and the lowest resonance frequency of the material is \( \omega_1 \neq 0 \), then \( \epsilon(\omega) = \epsilon(0) + \mathcal{O}(\omega) \), so that \( \text{Im}[\epsilon(\omega)] = \mathcal{O}(\omega) \) for small \( \omega \). In both cases, the limit \( L \) defined in Eq. (21b) is finite.

Let us first consider the case of a conductor. Then, the dominant contribution for small \( \omega \) is given by the term with \( n = 0 \) in Eq. (25),

\[
\epsilon(\omega) \approx 1 - \frac{a_0}{\omega(\omega + \gamma_0)} = 1 - \frac{\omega^2}{\omega(\omega + \gamma_0)},
\]

where \( \omega_p \) is the plasma frequency of the free electron gas constituting the conduction band electrons. This leads to

\[
L = \lim_{\omega \to 0} \left\{ \frac{1}{\omega} \text{Im} \left( \frac{\epsilon(\omega) - 1}{\epsilon(\omega) + 1} \right) \right\} = \frac{2\gamma_0}{\omega_p} = \frac{2\epsilon_0}{\sigma_T(0)},
\]

where

\[
\sigma_T(0) = \epsilon_0 \frac{\omega_p^2}{\gamma_0}
\]

is the temperature-dependent conductivity (measured in inverse \( \Omega \) m) of the material, for a direct current (alternating current of zero frequency). We recover the result given in Eq. (35) of Ref. [1] for a conductor,

\[
\eta = \frac{(Ze)^2}{16 \pi \sigma_T(0) Z^3}.
\]

The only temperature-dependence remaining in this result is via the implicit dependence of the “static” conductivity \( \sigma_T(0) \) on the temperature. In the low-temperature limit, many materials become superconducting, which implies that their conductivity \( \sigma_T(0) \) diverges and the friction coefficient \( \eta \) vanishes.

For an undoped semiconductor and other materials such as gases, the evaluation of the limit \( L \) proceeds via a consideration of the lowest resonance. In the case \( b_1 = 0 \), we have

\[
\epsilon(\omega) \approx 1 + \frac{a_1}{\omega^2 - \omega^2 - i\gamma_1 \omega},
\]

\[
L = \frac{2a_1 \gamma_1}{(a_1 + 2\omega_1^2)^2},
\]

\[
\eta = \frac{(Ze)^2 a_1 \gamma_1}{16 \pi (a_1 + 2\omega_1^2)^2 Z^3}.
\]

Here, we assume that the lowest resonance dominates, i.e., that \( \omega_1^2 \gg \omega_0^2 \gg a_1 \). If a modified Lorentz profile with \( \gamma'_1 \) is chosen (according to Ref. [18]), then the result simplifies to

\[
\epsilon(\omega) \approx 1 + \sum_{n=1}^{\infty} \frac{\Delta\epsilon_n (\omega^2 - i\gamma'_n \omega)}{\omega_n^2 - \omega^2 - i\gamma_n \omega},
\]

\[
L = \sum_{n=1}^{\infty} \frac{2 \Delta\epsilon_n (\gamma_n - \gamma'_n)}{(2 + \sum_m \Delta\epsilon_m)^2 \omega_n^2},
\]

\[
\eta = \frac{(Ze)^2}{16 \pi Z^3} \sum_{n} \frac{\Delta\epsilon_n (\gamma_n - \gamma'_n)}{(2 + \sum_m \Delta\epsilon_m)^2 \omega_n^2},
\]

where in the last line both sums extend over \( m, n = 1, \ldots, \infty \). Let us match these formulas with the known expressions for a dilute gas (bulk material), whose dielectric function is given by

\[
\epsilon(\omega) = 1 + \sum_{n} \frac{N_V}{\epsilon_0} \alpha(\omega) = 1 + \sum_{n} \frac{N_V}{\epsilon_0} \sum_{n} E_{n0}^2 \frac{f_n}{E_{n0}^2 - i\Gamma_n (\hbar\omega) - (\hbar\omega^2)}.
\]

Here, \( N_V = N/V \) is the volume density of atoms, and the \( E_{n0} \) are the excitation energies from the ground state to the \( n \)th excited state. The (dipole) polarizability is \( \alpha(\omega) \), and the dipole oscillator strength for a transition from the

FIG. 1: Basic proposed setup for the measurement of ionic non-contact friction. An incoming ion beam is velocity-selected and enters an interaction region with a compensating slit configuration, involving two parallel bars of (preferentially single-crystal) material, whose induced non-contact friction is to be studied in the experiment. A further free beam path of a few meters in length serves to eliminate conceivable contributions to the ionic energy loss due to imperfect alignment of the interaction region. Finally, the energy of the emerging ion can be measured after it has passed through an aperture.
reference state $k$ to the virtual state $n$ is $f_{nk} \equiv f_{nk}^{(\ell=1)}$, where

$$f_{n0} = 2 \left(\frac{4\pi e^2}{(2\ell + 1)^2} \hbar \omega_{n0}\right) \times \sum_{m_k} \sum_{m} \left| \left\langle \phi_k \mid \sum_i (r_i)_{\ell m} (r_i) \right\rangle \phi_0 \right|^2.$$

Here, $\ell = 1$ denotes the dipole contribution to the polarizability, where $2^\ell$ is the multipole order [19]. We sum over the magnetic projections $m$ of the spherical harmonic and over the magnetic projections $m_k$ of the excited state $\phi_k$. The identification of Eqs. (33) and (34) then proceeds as follows,

$$\omega_n = \frac{E_{n0}}{\hbar}, \quad \gamma_n = \frac{\Gamma_n}{\hbar}, \quad \gamma'_n = 0, \quad \Delta \epsilon_n = \frac{N_V f_{n0}}{\epsilon_0 E_{n0}^2}. \quad \text{(36)}$$

These quantities can directly be used in Eq. (33).

IV. “SNIPER” SETUP CONFIGURATION

A. Mirror Charges for Parallel Conducting Plates

We consider an experimental setup as given in Fig. 1 and strive to calculate the interaction potential of the ion with the mirror charges in both the upper, as well as the lower slab, as well as the induced friction force. For simplicity, we shall consider, in the following, two conducting parallel plates. The upper surface of the lower plate is in the $xy$ plane ($z = 0$), whereas the upper plate is at $z = a$. The ion’s $z$ coordinate is denoted as $Z$. The following analysis is in part inspired by Refs. [20, 21]. The (positively charged) ion generates two image charges in the two conductors, which, in turn, by mirroring them against the respective other conducting plate, generate an infinite series of mirror charges, which can be identified as follows. Iterative calculation of the positions of the mirror charges leads to are two “upper” series of mirror charges, negative ones at position $2na - Z$ [the distance to the reference point $Z = 2(na - Z)$], and positive ones at positions $2na + Z$ (the distance is $2na$). There are also two series of mirror charges in the lower slab, positive ones at positions $-2na + Z$ (with a distance of $2na$), and negative ones at position $-2(n - 1)a - Z$, whose distance to the reference point is $2(n - 1)a + Z$.

Let us consider the limiting case of the ion being close to the lower plate at $z = 0$, i.e., the limit $Z \to 0$. The dominant interaction potential will be due to the closest mirror charge in the lower slab, which is located at $z = -Z$. The distance of charge and mirror charge is $2Z$, and the interaction potential is $-(Ze)^2/(16\pi\epsilon_0 Z^3)$, where the prefactor takes care of the distance $2Z$, as well as the fact that the electric field is zero inside the conductor (see the Complement to Chap. 11 of Ref. [22]). Alternatively, the additional factor 1/2 can be understood as follows:

If one is to move the charge upward by a given distance, then the mirror charge moves “automatically”, and thus the work required to move the charge (and its interaction potential) is halved. The sum over all other mirror charges leads to the following total interaction potential,

$$V = -\frac{(Ze)^2}{4\pi\epsilon_0} C(Z). \quad \text{(37)}$$

A series representation of the correction factor is as follows (again in the non-retardation limit, see Ref. [21]),

$$C(Z) = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{2a(n-1) + 2Z} + \frac{1}{2an - 2Z} - \frac{1}{an} \right)$$

$$= -\frac{1}{4a} \left[ \Psi \left( \frac{Z}{a} \right) + \Psi \left( 1 - \frac{Z}{a} \right) + 2 \gamma_E \right], \quad \text{(38)}$$

where $\gamma_E = 0.57721\ldots$ is the Euler–Mascheroni constant. The logarithmic derivative of the Gamma function and its generalization $\Psi(n)(x)$ read as follows,

$$\Psi(x) = \frac{d}{dx} \ln[\Gamma(x)], \quad \Psi(n)(x) = \frac{d^n}{dx^n} \psi(x). \quad \text{(39)}$$

We anticipate that we shall need $\Psi(n=2)(x)$ in the following derivations. For small $Z$, the asymptotic expansion reads as

$$C(Z) = \frac{1}{4Z} + \frac{\zeta(3)}{2a^3} Z^2 + O(Z^4), \quad \text{(40)}$$

in agreement with the consideration sketched above.

For the “sniper” configuration to be discussed below, the setup sketched in Fig. 1 suggests to assume an (almost) symmetric configuration, with the atom in the middle between the two conducting slabs, i.e., one expands about the point $Z \approx a/2$. For perfect symmetry, we have

$$C \left( Z = \frac{a}{2} \right) = \frac{\ln(2)}{a} = 0.693147 \frac{a}{a} < \frac{1}{a}. \quad \text{(41)}$$

If we assume perfect symmetry, then the two nearest mirror charges are at a distance $a$ from the ion, which, together with the correction factor 1/2 of Ref. [22], would suggest a value of $2 \times 1/(2a) = 1/a$ for the $C$ coefficient. The contribution of the remaining mirror charges reduces this result to the value $\ln(2)/a$.

We now consider the contribution of the remaining mirror charges to the friction. To this end, we first recall the result from Eq. (21), which is initially valid for a single surface,

$$\eta = \frac{(Ze)^2 L}{32\pi\epsilon_0 Z^3} \to \eta(Z) = \frac{(Ze)^2 L}{32\pi\epsilon_0} D(Z). \quad \text{(42)}$$

Here, the variable $Z$ in the expression for $\eta$ denotes the distance of the ion and the conducting wall in the case of the presence of a single wall. This result will need to be matched against a function $D(Z)$, which measures the
The friction force in the interaction region is given as
\[ F_\parallel = -\eta(z) v_x, \]
where \( \eta(z) \) is the damping coefficient for a configuration with two parallel plates, which we consider according to Eq. (42), within the approximation (45) for the ion in the middle of the parallel plates. The conductivities of metals strongly depend on the concentration of dopants. We use the following values, which represent estimates of the room-temperature direct-current conductivity \( \sigma = \sigma(0) \) of a number of metals which can easily be formed into almost perfect slabs, either as a polycrystalline material with a well-polished surface, or even as single crystals (as in the case of vanadium),
\[
\begin{align*}
\sigma_{\text{Al}} &\approx 2.30 \times 10^7 (\Omega \text{m})^{-1}, & \sigma_{\text{V}} &\approx 5.08 \times 10^6 (\Omega \text{m})^{-1}, \\
\sigma_{\text{Ti}} &\approx 1.27 \times 10^6 (\Omega \text{m})^{-1}, & \sigma_{C} &\approx 1.28 \times 10^5 (\Omega \text{m})^{-1}.
\end{align*}
\]
(47a)
(47b)
Here, \( \sigma_C \) refers to graphite, where we assume that the basal plane of the hexagonal crystal lattice is aligned with the surface plane of the slabs. [The data in Eq. (47) has been compiled as the average of data given by manufacturers for several commercially available metals with different polycrystalline structure and different dopants; in a precision experiment, it would seem indicated to measure the low-frequency (direct-current) limit of the conductivity \( \sigma \) of a specific sample independently.] For reference purposes, we here indicate that an evaluation of \( L \) according to Eqs. (29) and (47) for graphite leads to a value of \( L = 1.38 \times 10^{-16} \text{ (rad/s)}^{-1} \). For helium ions at a distance \( z \) from the plates, with
\[
\begin{align*}
m &\approx 6.646 \times 10^{-27} \text{kg}, & Z &\approx 0.5 \mu \text{m},
\end{align*}
\]
(48)
one has \( m \, dv_x/dt = -\eta(z) v_x \) or
\[
\begin{align*}
\frac{dv_x}{dt} = -\frac{2 \eta(z)}{m} v_x = -\Gamma v_x.
\end{align*}
\]
(49)

B. "Sniper" Setup Configuration

The basic idea is obvious from Fig. 1. An ion enters the beamline, pre-accelerated. Energy selection (velocity selection) with a control measurement of the ion’s energy proceeds before entering the interaction region between the two aligned, parallel plates. For an accelerating voltage of order 20 V, and single-charged helium ions (He\(^+\)), the de Broglie wavelength is of the order of \( 5.1 \times 10^{-13} \text{ m} \). Thus, we can safely ignore diffraction effects which could otherwise occur when entering the interaction region in between the plates. Interactions with the image charges on both sides of the beam track can be ignored under perfectly aligned ("sniper") conditions. The idea is that the atom loses energy on its trajectory due to non-contact friction, with a corresponding measurable energy loss after its has left the interaction region. An aperture before the ion’s energy measurement area, about a meter away, ensures that no significant distortion of the ion’s path due to interaction with the image charges has occurred. This is necessary because the Coulomb interaction with the image charges does work on the projectile ion, potentially altering its kinetic energy.
leaving the interaction region. Let us assume that roughly $D = 1$ m further down the beam line, we restrict the available angular region for the arriving ions to a circular aperture of diameter (see also Fig. 1)

$$a_s = 100 \mathcal{Z} = 50 \mu m.$$  \hfill (52)

For non-perfect alignment, the image charge interaction may exert a force in the interaction region, in the positive or negative $z$ direction, without altering the $x$ component $v_x$. The calculation of the interaction energy with the mirror charge(s) has been discussed in Sec. IV A. However, in order to obtain a figure-of-merit for our proposed experimental setup, it is not sufficient to consider the geometric requirement of passage through the geometric aperture indicated in Fig. 1; it corresponds to a restriction of the modulus of the velocity change according to

$$v_x \sqrt{1 + \left(\frac{a_s}{D}\right)^2} - v_x \approx \frac{1}{2} \left(\frac{a_s}{D}\right)^2 v_x = \xi v_x,$$  \hfill (53)

where we redefined $\xi$ as compared to Eq. (22). The figure-of-merit of the measurement is obtained as the ratio of the relative change in the velocity due to non-contact friction, divided by the geometrically restricted “uncertainty” in the velocity measurement,

$$f = \frac{r}{\xi} = \frac{2 (1 - \exp(-\Gamma \Delta t))}{(a_s/D)^2}.$$  \hfill (54)

We obtain

$$f_{\text{Au}} = 1.24 \times 10^2, \quad f_{\text{V}} = 5.62 \times 10^2, \quad f_{\text{Ti}} = 2.25 \times 10^3, \quad f_{\text{C}} = 2.23 \times 10^4.$$  \hfill (55a)

These results show that a measurement should be feasible. We also have done an evaluation based on the extensive reference volume [23], analyzing available data for the dielectric function of $\alpha$-quartz, along the ordinary (o) and extraordinary (e) axis. We obtain the following values for the $L$ coefficient,

$$L_o = 1.40 \times 10^{-17} \text{ (rad/s)}^{-1}, \quad L_e = 2.14 \times 10^{-17} \text{ (rad/s)}^{-1},$$  \hfill (56a)

which translates into the following loss coefficients and figures-of-merit,

$$r_o = 2.82 \times 10^{-6}, \quad r_e = 4.30 \times 10^{-6}, \quad f_o = 9.04 \times 10^3, \quad f_e = 1.37 \times 10^4.$$  \hfill (57a)

A measurement using single-crystalline quartz surfaces also seems possible. The “sniper” configuration is “auto-correcting” in the sense that the ions can only pass through the detector aperture under well-aligned conditions.

V. CONCLUSIONS

We investigate the energy loss of an ion in the vicinity of two conducting surfaces, in a configuration where the electrostatic interaction with the two image charges compensate each other. Based on a rederivation of the frictional force, we relate the general expression for the friction force to the functional form of the permittivity at low frequencies, which we parameterize in terms of a sum of generalized Drude and Lorentz profiles. The definition of a conductor implies the existence of a zero-resonance-frequency term in the permittivity and we show that it is this term which dominates in the evaluation of the frictional force.

Specifically, the result for the non-contact friction coefficient, given in Eq. (21) for a single wall, is generalized to the interaction with two parallel walls in Eq. (42). The $L$ coefficient can be written in terms of a parameterization of the dielectric response functions of the material, according to Eq. (33). For a conductor, the limit $L$ is exclusively determined by the conduction band [see Eq. (29)].

We identify the main problem in a conceivable experiment as the isolation of the frictional force from the strong electrostatic interaction with the image charge. In general, the functional form of the interaction potentials of ions with other electromagnetically interacting objects is different from the corresponding expressions for atoms. In Sec. I, we recall that the ion-atom interaction $(1/R^3)$ is stronger than the atom-atom interaction $(1/R^6)$, and that the friction force in the vicinity of a surface $(1/Z^3)$ is larger than that for an atom $(1/Z^3)$. In a typical case, there is a difference in the power law involving two more inverse powers of the distance for “atom-something” interactions as compared to the corresponding “ion-something” interaction.

In principle, the enhanced functional form of the non-contact friction coefficient for ions as compared to atoms would recommend a measurement of the quantum friction effect using ions. However, the strong interaction with the image charge implies that a careful compensation of the interactions with the images becomes necessary in the vicinity of a conducting surface, in order to minimize a deflection of the ion’s trajectory. We aim to identify a collection of suitable parameters, in terms of a pre-accelerating voltage, to find a compromise between a slow flight velocity (which enhances the “risk of deflection”) and a too short interaction time (which would make the quantum friction effect undetectable). A proposed set-up with certain auto-compensating features, to isolate the friction effect, is described in Sec. IV. The most suitable materials for the experiment are be those which are readily available as large-scale single crystals, and constitute conductors, but with a low value of the conductivity, in order to increase the Ohmic heating due to the flow of the image charge. We find that the best figure-of-merit is obtained for graphite, while a readily available metal like vanadium, which has excellent sur-
face properties, also would be available for a precision experiment.

Acknowledgments

The authors acknowledge helpful conversations with Professor K. Pachucki. This research has been supported by the National Science Foundation (Grants PHY–1068547 and PHY–1403973) and by the Polish Ministry of Science (MNiSW, Grant No. 0307/IP3/2011/71). Early stages of this research have also been supported by the Deutsche Forschungsmeinschaft (DFG, contract Je285/5–1).

Appendix A: Fourier Transform in Signal Theory

In signal theory, different conventions are used for the Fourier transform. Normally, in physics, one integrates the Fourier frequency transform with the “integration measure” $d\omega/(2\pi)$. However, according to the Eq. (3.118) of Ref. [3], and Eqs. (1.90) and (1.91) of Ref. [4], we have

\begin{align}
    f(t) &= \int d\omega \, e^{-i\omega t} \tilde{f}(\omega), \\
    \tilde{f}(\omega) &= \int \frac{dt}{2\pi} e^{i\omega t} f(t),
\end{align}

and the same conventions are employed in signal theory for the corresponding transformations from coordinate space to wave number space, namely, one integrates with the integration measures $d^3r/(2\pi)^3$ and $d^3k$ instead of $d^3r$ and $d^3k/(2\pi)^3$. The difference in the integration measures also explains the occurrence of multiplicative factors of $2\pi$ in other formulations of the fluctuation-dissipation theorem, such as Eq. (3.118) of Ref. [3].

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