Holonomy groups of flat manifolds with $R_\infty$ property

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Abstract

Let $M$ be a flat manifold. We say that $M$ has $R_\infty$ property if the Reidemeister number $R(f) = \infty$ for every homeomorphism $f: M \to M$. In this paper, we investigate a relation between the holonomy representation $\rho$ of a flat manifold $M$ and the $R_\infty$ property. In case when the holonomy group of $M$ is solvable we show that, if $\rho$ has a unique $\mathbb{R}$-irreducible subrepresentation of odd degree, then $M$ has $R_\infty$ property. The result is related to conjecture 4.8 from [3].

1 Introduction

Let $M^n$ be a closed Riemannian manifold of dimension $n$. We shall call $M^n$ flat if, at any point, the sectional curvature is equal to zero. Equivalently, $M^n$ is isometric to the orbit space $\mathbb{R}^n/\Gamma$, where $\Gamma$ is a discrete, torsion-free and co-compact subgroup of $O(n) \ltimes \mathbb{R}^n = \text{Isom}(\mathbb{R}^n)$. From the Bieberbach theorem (see [1], [9], [10]) $\Gamma$ defines a short exact sequence of groups

\begin{equation}
0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{\rho} G \to 0,
\end{equation}

where $G$ is a finite group. $\Gamma$ is called a Bieberbach group and $G$ its holonomy group.

Since $\Gamma = \pi_1(M^n)$, any continuous map $f: M^n \to M^n$ induces a morphism $\rho_f : \Gamma \to \Gamma$. We say that two elements $\alpha, \beta \in \Gamma$ are $f_\sharp$-conjugated if there exists $\gamma \in \Gamma$ such that $\beta = \gamma \alpha \rho_f(\gamma)^{-1}$. The $f_\sharp$-conjugacy class $\{\gamma \alpha \rho_f(\gamma)^{-1} |$
\( \gamma \in \Gamma \) of \( \alpha \) is called a Reidemeister class of \( f \). The number of Reidemeister classes is called the Reidemeister number \( R(f) \) of \( f \). A manifold \( M^n \) has the \( R_\infty \) property if \( R(f) = \infty \) for every homeomorphism \( f : M^n \to M^n \), see [3]. It is evident that we can also define the above number \( R(f) \) for a countable discrete group \( E \) and its automorphism \( f \). We say that a group \( E \) has the \( R_\infty \) property if \( R(f) = \infty \) for any automorphism \( f \). Moreover, the following groups (see [4] for the complete bibliography) have the \( R_\infty \) property: non-elementary Gromov-hyperbolic groups, Baumslag-Solitar groups \( \text{BS}(m,n) = \langle a,b \mid ba^mb^{-1} = a^n \rangle \) except for \( \text{BS}(1,1) \).

In this paper we shall consider the case of Bieberbach groups. We can define a holonomy representation \( \rho : G \to \text{GL}(n, \mathbb{Z}) \) by the formula:

\[
(1.2) \quad \forall g \in G, \rho(g)(e_i) = \tilde{g}e_i(\tilde{g})^{-1},
\]

where \( e_i \in \Gamma \) are generators of the free abelian group \( \mathbb{Z}^n \) for \( i = 1, 2, ..., n \), and \( \tilde{g} \in \Gamma \) such that \( p(\tilde{g}) = g \). In this article we describe relations between \( R_\infty \) property on the flat manifold \( M^n \) (Bieberbach group \( \Gamma \)) and a structure of its holonomy representation. The connections between geometric properties of \( M^n \) and algebraic properties of \( \rho \) was already considered in different cases. For example, \( \text{Out}(\Gamma) \) is finite if and only if a holonomy representation is \( \mathbb{Q} \)-mutiplicity free and any \( \mathbb{Q} \)-irreducible components of a holonomy representation is \( \mathbb{R} \)-irreducible, see [3]. A similar equivalence says that an Anosov diffeomorphism \( f : M^n \to M^n \) exists if and only if any \( \mathbb{Q} \)-irreducible component of a holonomy representation that occurs with multiplicity one is reducible over \( \mathbb{R} \), see [5]. We want to define conditions of this kind for the holonomy representation of a flat manifold with \( R_\infty \) property. We already know that, in this way, the complete characteristic is not possible. There are examples [3 Th.5.9] of flat manifolds \( M_1, M_2 \) with the same holonomy representation such that \( M_1 \) has \( R_\infty \) property and \( M_2 \) has not. In [3 Corollary 4.4] it is proved that if there exists Anosov diffeomorphism \( f : M^n \to M^n \) then \( R(f) \) is finite and \( M^n \) does not have the \( R_\infty \) property. Moreover there exists \( M \), such that its holonomy representation has \( \mathbb{Q} \)-irreducible component which is irreducible over \( \mathbb{R} \) and occurs with multiplicity one, and \( M \) does not have the \( R_\infty \) property, [3 Example 4.6]. Nevertheless in [3 Th. 4.7] is proved:

**Theorem 1.1.** ([3 Th. 4.7]) Let \( M \) be a flat manifold with a holonomy representation \( \rho : G \to \text{GL}(n, \mathbb{Z}) \) and let \( \rho' : G \to \text{GL}(n', \mathbb{Z}) \) be a \( \mathbb{Q} \)-irreducible \( \mathbb{Q} \)-subrepresentation of \( \rho \) such that \( \rho'(G) \) is not \( \mathbb{Q} \)-conjugated to \( \tilde{\rho}(G) \) for any other \( \mathbb{Q} \)-subrepresentation \( \tilde{\rho} \) of \( \rho \). Suppose moreover that for every \( D' \in N_{\text{GL}(n',\mathbb{Z})}(\rho'(G)) \), there exists \( A \in G \) such that \( \rho'(A)D' \) has eigenvalue 1. Then \( M \) has the \( R_\infty \) property.
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Remark 1.1. If we assume that

$$ N_{\text{GL}(n',\mathbb{Q})}(\rho'(G))/C_{\text{GL}(n',\mathbb{Q})}(\rho'(G)) \cong \text{Aut}(G), \tag{1.3} $$

then the above requirement that $\rho'(G)$ is not $\mathbb{Q}$-conjugated to $\tilde{\rho}(G)$ is equivalent to the condition that $\rho' \subset \rho$ has multiplicity one. For example, if we take the diagonal representation $\rho: (\mathbb{Z}_2)^{2n} \to \text{SL}(2n + 1, \mathbb{Z})$ of the elementary abelian 2-group, then the above equation (1.3) is not satisfied for any $\mathbb{Q}$-irreducible subrepresentation of $\rho$.

We shall prove:

**Theorem 1.2.** Let $M$ be a flat manifold with a holonomy representation $\rho: G \to \text{GL}(n, \mathbb{Z})$ and let $G$ be a solvable group and $\rho': G \to \text{GL}(n', \mathbb{Z})$ be a $\mathbb{Q}$-irreducible $\mathbb{Q}$-subrepresentation of $\rho$ of odd dimension. If $\rho'(G)$ is not $\mathbb{Q}$-conjugated to $\tilde{\rho}(G)$, for any other $\mathbb{Q}$-subrepresentation $\tilde{\rho}$ of $\rho$ then $M$ has the $R_{\infty}$ property.

If we restrict our consideration to the class of finite groups which satisfy a condition (1.3) we have.

**Theorem 1.3.** Let $M$ be a flat manifold with a holonomy representation $\rho: G \to \text{GL}(n, \mathbb{Z})$ and let $G$ be a solvable group and $\rho': G \to \text{GL}(n', \mathbb{Z})$ be a $\mathbb{Q}$-irreducible $\mathbb{Q}$-subrepresentation of $\rho$ of multiplicity one and odd dimension which satisfies a condition (1.3), then $M$ has the $R_{\infty}$ property.

The above result is a corollary from [7, Th. 5.4.4], the Theorem 1.1 and the following theorem:

**Theorem A** Let $G$ be a finite group with a non-trivial normal abelian subgroup $A$ and let $\rho: G \to \text{GL}(n, \mathbb{Z})$ be a faithful $\mathbb{R}$-irreducible representation. Suppose $n$ is odd. Then for every $D \in N_{\text{GL}(n,\mathbb{Z})}(\rho(G))$, there exists $g \in G$ such that $\rho(g)D$ has eigenvalue 1.

Remark 1.2. A conjecture 4.8 in [3] says that the above theorem A is true for any finite group. We do not know whether it holds in general.

We prove Theorem A in the next section.

**Acknowledge:** We would like to thanks G. Hiss for helpful conversation and particulary for putting our attention on the Clifford’s theorem.
2 Proof of Theorem A

Theorem 2.1. Let \( G \) be a finite group and \( n \) be an odd integer. Let \( \rho : G \to \text{GL}(n, \mathbb{Z}) \) be a faithful representation of \( G \), which is irreducible over \( \mathbb{R} \). Then \( \rho \) is irreducible over \( \mathbb{C} \).

Proof. Assume, that \( \rho \) is reducible over \( \mathbb{C} \) and let \( \tau \) be any \( \mathbb{C} \)-irreducible subrepresentation of \( \rho \). By [6, Theorem 2], the representation \( \rho \) is uniquely determined by \( \tau \) and, if \( \chi \) is the character of \( \tau \), then the character of \( \rho \) is given by

\[ \chi + \overline{\chi}. \]

Hence \( \rho \) is of even degree. This proves the theorem.

For the rest of this section we assume, that \( \rho : G \to \text{GL}(n, \mathbb{Z}) \) is an absolutely irreducible representation of \( G \), where \( n \) is an odd integer.

Proposition 2.2. If \( A \) is normal abelian subgroup of \( G \), then \( A \) is elementary abelian 2-group.

Proof. Let \( \tau \) be a \( \mathbb{R} \)-irreducible subrepresentation of \( \rho|_A \). By Clifford’s theorem ([2, Theorem 49.2]) all \( \mathbb{R} \)-subrepresentations of \( \rho|_A \) are conjugates of \( R \)-irreducible subrepresentation \( \tau \), i.e. there exists \( g_1 = 1, g_2, \ldots, g_l \in G \) such that

\[ \rho|_A = \tau^{(g_1)} \oplus \ldots \oplus \tau^{(g_l)}, \]

where

\[ \forall 1 \leq i \leq l \forall g \in G \quad \tau^{(g_i)}(g) = \tau(g^{-1}gg_i). \]

Let \( a \in A \) be an element of order greater than 2. Since \( \rho \) is faithful, there exists \( 1 \leq i \leq l \), such that \( \tau^{(g_i)}(a) \) is a real matrix of order at least 2. Hence \( \deg(\tau^{(g_i)}) = \deg(\tau) = 2 \) and \( n = \deg(\rho) = \deg(\rho|_A) = l \deg(\tau) = 2l \) is an even integer. This contradiction finishes the proof.

Since \( A \) is an elementary abelian 2-group, the decomposition (2.1) may be realized over the rationals. By [2, Theorem 49.7] we may assume, that

\[ \rho|_A = e\tau^{(g_1)} \oplus \ldots \oplus e\tau^{(g_k)}, \]

i.e. one-dimensional representations \( \tau^{(g_1)}, \ldots, \tau^{(g_k)} \) occur with the same multiplicity \( e = n/k \). Let \( \rho_i := e\tau^{(g_i)} \), for \( i = 1, \ldots, k \). By the suitable choice of basis of \( \mathbb{Q}^n \) we may assume, that for every \( a \in A \), \( \rho(a) \) is a diagonal matrix, such that

\[ \forall 1 \leq i \leq k \text{Img}(\rho_k) = \langle -1 \rangle, \]
where $I$ is the identity matrix of degree $e$.

Since $A \triangleleft G$ and $\rho$ is faithful, we have

$$\rho(A) \triangleleft \rho(G) \subset N_{\text{GL}(n,\mathbb{Q})}(\rho(A)) = \{ m \in \text{GL}(n,\mathbb{Q}) \mid m^{-1}\rho(A)m = \rho(A) \}.$$  

In the next two subsections we will focus on the above normalizer.

### 2.1 Centralizer

In the beginning we describe the centralizer

$$C_{\text{GL}(n,\mathbb{Q})}(\rho(A)) = \{ m \in \text{GL}(n,\mathbb{Q}) \mid \forall a \in A \rho(a)m = \rho(a)m \}.$$  

Let $m = (m_{ij}) \in \text{GL}(n,\mathbb{Q})$ be a block matrix, such that $m_{\rho|A} = \rho_{|Am}$. We get

$$\begin{pmatrix} m_{11} & \cdots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \cdots & m_{kk} \end{pmatrix} \begin{pmatrix} \rho_1 & 0 \\ \vdots & \ddots \\ 0 & \cdots \end{pmatrix} = \begin{pmatrix} \rho_1 & 0 \\ \vdots & \ddots \\ 0 & \cdots \end{pmatrix} \begin{pmatrix} m_{11} & \cdots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \cdots & m_{kk} \end{pmatrix},$$

and thus

$$\forall 1 \leq i,j \leq k m_{ij} = \rho_i m_{ij}.$$  

Since for $i \neq j$, $\rho_i$ and $\rho_j$ have no common subrepresentation, by Schur’s Lemma (see \cite[(27.3)]{2}) $m_{ij} = 0$ for $i \neq j$ and $m_{ii} \in \text{GL}(n/k,\mathbb{Q})$, for $i = 1, \ldots, k$. We just have proved

**Lemma 2.3.** Let $\rho: G \to \text{GL}(n,\mathbb{Q})$ be a faithful, absolutely irreducible representation of finite group $G$ of odd degree $n$. Let $A$ be normal abelian subgroup of $G$, such that conditions (2.2) and (2.3) hold. Then

$$C_{\text{GL}(n,\mathbb{Q})}(\rho(A)) = \{ \text{diag}(c_1,\ldots,c_k) \mid c_i \in \text{GL}(n/k,\mathbb{Q}), i = 1,\ldots,k \},$$

where $k$ is equal to the number of pairwise nonisomorphic irreducible subrepresentations of $\rho|_A$.

### 2.2 Normalizer

Since the group $A$ is finite, $\text{Aut}(A)$ is a finite group. Moreover, we have a monomorphism

$$N_{\text{GL}(n,\mathbb{Q})}(\rho(A))/C_{\text{GL}(n,\mathbb{Q})}(\rho(A)) \to \text{Aut}(A).$$

Hence any coset $mC_{\text{GL}(n,\mathbb{Q})}(\rho(A)), m \in N_{\text{GL}(n,\mathbb{Q})}(\rho(A))$ corresponds to some automorphism of $A$.  

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Let \( \varphi \in \text{Aut}(A) \) and \( m = (m_{ij}) \in \text{GL}(n, \mathbb{Q}) \) be a block matrix, which represents this automorphism, with blocks of degree \( n/k \), i.e.

\[
\forall c \in C_{\text{GL}(n, \mathbb{Q})(\rho(A))} \forall a \in A (mc \rho(a)(mc)^{-1} = m \rho(a) m^{-1} = \rho(\varphi(a)).
\]

We have

\[
\begin{pmatrix}
m_{11} & \ldots & m_{1k} \\
\vdots & \ddots & \vdots \\
m_{k1} & \ldots & m_{kk}
\end{pmatrix}
\begin{pmatrix}
\rho_1 & 0 \\
\vdots & \ddots \\
0 & \rho_k
\end{pmatrix}
= 
\begin{pmatrix}
\rho_1 \varphi & 0 \\
\vdots & \ddots \\
0 & \rho_k \varphi
\end{pmatrix}
\begin{pmatrix}
m_{11} & \ldots & m_{1k} \\
\vdots & \ddots & \vdots \\
m_{k1} & \ldots & m_{kk}
\end{pmatrix}.
\]

Note, that

\[
(2.4) \quad \forall 1 \leq i \leq k \text{Img}(\rho_i) = \text{Img}(\rho_i \varphi) = (-I).
\]

Since, for \( i \neq j \), \( \rho_i \) and \( \rho_j \) does not have common subrepresentations, the same applies to \( \rho_i \varphi \) and \( \rho_j \varphi \). Hence, using Shur’s lemma again for every \( 1 \leq i \leq k \) there exists exactly one \( 1 \leq j \leq k \), such that

\[
m_{ji} \rho_i = \rho_j \varphi \rho_{ji}
\]

and \( m_{ji} \neq 0 \). Moreover, \( \det(m) \neq 0 \) and also \( \det(m_{ij}) \neq 0 \). By \((2.4)\) \( \rho_i = \rho_j \varphi \) and there exists a permutation \( \sigma \in S_k \), s.t.

\[
(2.5) \quad m \text{diag}(\rho_1, \ldots, \rho_k) m^{-1} = \text{diag}(\rho_{\sigma(1)}, \ldots, \rho_{\sigma(k)}).
\]

Let \( \tau \in S_k \) be any permutation and let \( P_\tau \in \text{GL}(n, \mathbb{Q}) \) be a block matrix, with blocks of degree \( n/k \), such that

\[
(2.6) \quad (P_\tau)_{i,j} = \begin{cases} 
1 & \text{if } \tau(i) = j, \\
0 & \text{otherwise}, 
\end{cases}
\]

where \( 1 \leq i, j \leq k \). By \((2.5)\) we may take

\[
m = P_\sigma
\]

as a representative of a coset in \( N_{\text{GL}(n, \mathbb{Q})(\rho(A))/C_{\text{GL}(n, \mathbb{Q})(\rho(A))}} \), which realizes the automorphism \( \varphi \).

Let

\[
S := \{ \tau \in S_k \mid P_\tau \in N_{\text{GL}(n, \mathbb{Q})(\rho(A))} \}. 
\]

Then \( S \) is a subgroup of \( S_k \) and

\[
P := \{ P_\tau \mid \tau \in S \}
\]

is a subgroup of the normalizer. By the above and the Lemma \((2.3)\) we get

**Proposition 2.4.** The normalizer \( N_{\text{GL}(n, \mathbb{Q})(\rho(A))} \) is a semidirect product of \( C_{\text{GL}(n, \mathbb{Q})(\rho(A))} \) and \( P \). Moreover

\[
N_{\text{GL}(n, \mathbb{Q})(\rho(A))} = C_{\text{GL}(n, \mathbb{Q})(\rho(A))} \cdot P \cong C_{\text{GL}(n, \mathbb{Q})(\rho(A))} \rtimes S \cong \text{GL}(n/k, \mathbb{Q}) \wr S,
\]

where \( \text{GL}(n/k, \mathbb{Q}) \wr S \) denotes the wreath product of \( \text{GL}(n/k, \mathbb{Q}) \) and \( S \).
2.3 Properties of a group $G$

Let

\[ C := \rho^{-1}(\rho(G) \cap C_{\text{GL}(n, \mathbb{Q})}(\rho(A))) \]

and

\[ Q := \rho^{-1}(\rho(G) \cap P). \]

Then

\[ (2.7) \quad G = C \cdot Q \cong C \rtimes Q \]

is a semidirect product of $C$ and $Q$. Without lose of generality, we can assume, that $Q \subset S_k$.

The representations $\rho_i$, $i = 1, \ldots, k$ are defined on the group $A$. Lemma 2.3 gives us a possibility to extend domain of these representations to $C$. Let $V_i$ be subspaces of $\mathbb{Q}^n$ corresponding to representations $\rho_i$, $i = 1, \ldots, k$. In fact, since $\rho|_C$ is in block diagonal form, we have

\[ \forall 1 \leq i \leq k \quad V_i = \Theta \oplus \ldots \oplus \Theta \oplus \mathbb{Q}^{n/k} \oplus \Theta \oplus \ldots \oplus \Theta \subset \mathbb{Q}^n, \]

where $\Theta$ is considered as a zero-dimensional subspace (zero vector) of $\mathbb{Q}^{n/k}$. Moreover, every element of the group $\rho(Q) = P$ permutes elements of the set

\[ \{V_1, \ldots, V_k\}. \]

We want to prove that this action is transitive.

**Lemma 2.5.** $Q \subset S_k$ is a transitive permutation group.

**Proof.** If we assume that $Q$ is not transitive, then

\[ \exists 1 \leq j \leq k \quad \forall \tau \in Q \quad \tau(i) \neq j. \]

Let

\[ \hat{V}_j = \bigoplus_{i=1 \atop i \neq j}^k V_i \]

and $c\tau$, where $c \in C$, $\tau \in Q$, be any element of $G$. Then

\[ \rho(c\tau)(\hat{V}_j) = \rho(c)(\rho(\tau)(\hat{V}_j)) = \rho(c)\left( \bigoplus_{i=1 \atop i \neq j}^k V_{\tau(i)} \right) = \rho(c)(\hat{V}_j) = \hat{V}_j. \]

Thus $\hat{V}_j \subset \mathbb{Q}^n$ is an invariant subspace of $\rho$ and hence $\rho$ is reducible (over $\mathbb{Q}$). This contradiction proves the lemma. \qed
The following lemma helps us to understand the structure of the group $G$.

**Lemma 2.6.** Representations $\rho_1, \ldots, \rho_k : C \to \text{GL}(n, \mathbb{Q})$ are absolutely irreducible.

**Proof.** Let $\phi : C \to \text{GL}(d, \mathbb{C})$ be a $\mathbb{C}$-irreducible subrepresentation of $\rho|_{C}$. By Clifford’s theorem, for the group $C \triangleleft G$ the representation $\rho|_{C}$ is a sum of conjugates of $\phi$, i.e.

$$\rho|_{C} = \bigoplus_{s=1}^{m} \phi^{(g_s)},$$

where $g_s \in G, s = 1, \ldots, m$ and $g_1 = 1$. For every $1 \leq s \leq m$, $\phi^{(g_s)}$ is a complex subrepresentation of some $\rho_i, i = 1, \ldots, k$. Counting dimensions, we can see, that for every $1 \leq i \leq k$

$$\rho_i = \bigoplus_{j=1}^{m/k} \rho_{i,j},$$

where

$$\forall 1 \leq j \leq m/k \rho_{i,j} \in \{ \phi^{(g_s)} | 1 \leq s \leq m \}.$$

Let $V_{i,j} \subset V_i$ be an invariant space under the action of $\rho_{i,j}$, for $1 \leq i \leq k, 1 \leq j \leq m/k$. Taking a suitable basis for $V_i, 1 \leq i \leq k$, we can assume, that the decomposition

$$\rho_i = \bigoplus_{j=1}^{m/k} \rho_{i,j}$$

is given in a block diagonal form:

$$\forall 1 \leq j \leq m/k \, V_{i,j} = \Theta \oplus \ldots \oplus \Theta \oplus \mathbb{C}^{n/m} \oplus \Theta \oplus \ldots \oplus \Theta \subset V_i,$$

where $\Theta$ is a zero-dimensional subspace (zero vector) of $\mathbb{C}^{n/m}$. Note that the images of $\rho_{i\mid A}, i = 1, \ldots, k$, remain the same in this new basis. Hence the description of the representatives of the normalizer given in the subsection 2.2 remains the same for the group $\text{GL}(n, \mathbb{C})$ and we can assume, that $\rho(Q) = P$.

If the representations $\rho_i, i = 1, \ldots, k$, are $\mathbb{C}$-reducible, then $m > k$. Let

$$W = \bigoplus_{i=1}^{k} V_{i,1}$$
and $c\tau, c \in C, \tau \in Q$, be any element of $G$. We get
\[
\rho(c\tau)(W) = \rho(c)\rho(\tau)(W) = \rho(c)\left( \bigoplus_{i=1}^{k} V_{\tau(i),1} \right) = \rho(c)(W) = W.
\]
Hence $W \subseteq C^n$ is an invariant subspace of $\rho$ and thus $\rho$ cannot be absolutely irreducible. Contradiction.

2.4 Abelian normal subgroups

Without lose of generality, we can assume, that $A$ is maximal abelian subgroup of $G$, i.e. if $A' \triangleleft G$ is abelian and $A \subset A'$, then $A = A'$. We will show, that $A$ is unique in $G$ and hence – characteristic.

**Lemma 2.7.** $A$ is unique in $C$.

**Proof.** Let $A' \triangleleft G$ be an abelian group, such that $A' \subset C$. Since all elements of $A$ commute with all elements of $C$, they commute with all elements of $A'$. Hence $AA'$ is normal abelian subgroup of $G$. Since $A$ is maximal, we have

$$AA' = A \Rightarrow A' \subset A.$$ 

If we can prove, that $A \subset C$, then $A$ is going to be unique in $G$. Recall, that by (2.7) we have a short exact sequence

$$1 \rightarrow C \rightarrow G \xrightarrow{p} Q \rightarrow 1.$$

Assuming $A \notin C$, we get

$$1 \neq p(A) \triangleleft Q.$$

We prove that it is impossible.

**Lemma 2.8.** Let $Q \subset S_k$ be a transitive permutation group and $k \in \mathbb{N}$ be an odd natural number. Then $Q$ does not contain nontrivial normal elementary abelian 2-groups.

**Proof.** Let us denote by $N(\tau), \tau \in S_k$, a set

$$N(\tau) := \{1 \leq i \leq k \mid \tau(i) \neq i\}.$$

Assume, that $H \triangleleft Q$ is a normal nontrivial elementary abelian 2-group. Let $\tau$ be any element of $Q$. Without lose of generality we may assume $1 \in N(\tau)$. Since $Q$ is transitive, we have

$$\forall 1 \leq i \leq k \exists \sigma_i \in Q \sigma_i(1) = i.$$
Moreover
\[ \forall 1 \leq i \leq k N_i := N(\sigma_i \tau \sigma_i^{-1}) = \sigma_i(N(\tau)) \]
and hence
\[ \bigcup_{i=1}^{k} N_i = \{1, \ldots, k\}. \]

Let \( \mathcal{I} \) be any element of the set
\[ \left\{ \mathcal{K} \subset \{1, \ldots, k\} \left| \left| \bigcup_{i \in \mathcal{K}} N_i \right| \text{ is odd} \right. \right\} \]
with a minimum number of elements. Since \( \tau \) and all of its conjugates has order 2, \( \mathcal{I} \) has at least two elements. Let \( s \in \mathcal{I} \). Since \( H \) is normal in \( Q \), we have
\[ \forall i \in \mathcal{I} \sigma_i \tau \sigma_i^{-1} \in H. \]

By the minimality of \( \mathcal{I} \), the set
\[ N^{(s)} := \bigcup_{i \in \mathcal{I} \setminus \{s\}} N_i \]
contains even number of elements. Moreover, the same applies to the set \( N_s = N(\sigma_s \tau \sigma_s^{-1}) \). Hence, the intersection
\[ N^{(s)} \cap N_s \]
has an odd number of elements. Recall, that \( \sigma_i \tau \sigma_i^{-1} \), for \( i \in \mathcal{I} \), as elements of order 2, are products of disjoint transpositions. By the above, there exist \( t \in \mathcal{I} \setminus \{s\} \) and \( a, b, c \in \{1, \ldots, k\} \) such that
\[ a \in N_t \setminus N_s, b \in N_t \cap N_s, c \in N_s \setminus N_t \]
and \((a, b)\) and \((b, c)\) are transpositions in \( \sigma_t \tau \sigma_t^{-1} \) and \( \sigma_s \tau \sigma_s^{-1} \), respectively. But then
\[ \sigma_t \tau \sigma_t^{-1} \cdot \sigma_s \tau \sigma_s^{-1} \]
is an element of order greater than 2 in the elementary abelian 2-group \( H \).

Contradiction. \( \square \)

We have just proved.

**Proposition 2.9.** The maximal, normal elementary abelian subgroup \( A \triangleleft G \) is unique maximal in \( G \) and hence it is a characteristic subgroup.

**Corollary 2.10.**
\[ N_{GL(n,\mathbb{Q})}(\rho(G)) \subset N_{GL(n,\mathbb{Q})}(\rho(A)). \]
2.5 The proof

Let us first restate the theorem.

**Theorem A** Let $G$ be a finite group with a non-trivial normal abelian subgroup $A$ and let $\rho: G \to \text{GL}(n, \mathbb{Z})$ be a faithful $\mathbb{R}$-irreducible representation. Suppose $n$ is odd. Then for every $D \in N_{\text{GL}(n, \mathbb{Z})}(\rho(G))$, there exists $g \in G$ such that $\rho(g)D$ has eigenvalue 1.

**Proof.** Note first, that eigenvalues of matrices and their products does not depend on their conjugacy class. Hence, we can change the basis of $\rho$, with conjugating the group $N_{\text{GL}(n, \mathbb{Z})}(\rho(G))$ by appropriate invertible rational matrix simultaneously, and prove the theorem with these new forms of $\rho$ and $N = N_{\text{GL}(n, \mathbb{Z})}(\rho(G))$. Note that, by $\mathbb{R}$-irreducibility of $\rho$, $N$ is a finite group (see [8, pages 587-588]).

From above, we can assume, that $\rho(A)$ is a group of diagonal matrices. Using Corrolary 2.10 Proposition 2.4 and a fact, that

$$N \subset N_{\text{GL}(n, \mathbb{Z})}(\rho(G)),$$

we get

$$N \subset C_{\text{GL}(n, \mathbb{Q})}(\rho(A)) \cdot P.$$  

Recall, that

$$C_{\text{GL}(n, \mathbb{Q})}(\rho(A)) = \bigoplus_{i=1}^{k} \text{GL}(n/k, \mathbb{Q})$$

and elements of $P$ are "block permutation matrices" (see Lemma 2.3 and (2.6) respectively).

Let $D \in N$, then $D$ has the form

$$D = P_{\sigma}\text{diag}(c_1, \ldots, c_k),$$

where $\sigma \in S_k$ and $c_i \in \text{GL}(n/k, \mathbb{Q})$, for $i = 1, \ldots, k$. Recall, that $G = CQ$, where $Q \subset S_k$ is a transitive permutation group (see Lemma 2.5). Hence there exists $\tau \in Q$, such that

$$\tau(1) = \sigma^{-1}(1).$$

We get

$$P_{\tau}P_{\sigma}\text{diag}(c_1, \ldots, c_k) = P_{\sigma\tau}\text{diag}(c_1, \ldots, c_k) = \text{diag}(c_1, X),$$

where $X$ is a matrix of rows of $\text{diag}(c_2, \ldots, c_k)$ permuted by $\sigma\tau$. Since $c_1 \in \text{GL}(n/k, \mathbb{Q})$ has an odd degree, it must have real eigenvalue and since $N$ is
of a finite order, this eigenvalue is \(\pm 1\). If the eigenvalue is 1, then we take \(g = \tau\) and the theorem is proved. Otherwise, by the Clifford’s theorem and the faithfulness of \(\rho\), we can take such \(a \in A\), that \(\rho_1(a) = -I\). Then \(\rho_1(a)c_1\) has an eigenvalue 1 and hence, taking \(g = a\tau\), the element

\[
\rho(g)D = \rho(a\tau)D = \rho(a)\rho(\tau)D = \rho(a)P_\tau P_\sigma \text{diag}(c_1, \ldots, c_k) = \\
= (\rho_1 \oplus \ldots \oplus \rho_k)(a) \cdot \text{diag}(c_1, X) = \\
= \text{diag}(\rho_1(a)c_1, (\rho_2 \oplus \ldots \oplus \rho_k)(a)X)
\]

has an eigenvalue equal to 1 also. This finishes the proof. \(\square\)

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