Noncommutativity of Space and Rotation of Polarization of Light in a Background Magnetic Field

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Recently the PVLAS collaboration has reported the observation of rotation of polarization of light propagating in a background magnetic field. In this letter we explore the possibility that such a rotation is a result of noncommutativity in the background space-time. To explain the reported polarization rotation within noncommutative QED we need the noncommutativity parameter \( \theta \approx (30 \text{ GeV})^{-2} \).

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I. INTRODUCTION

Soon after the conception of quantum mechanics, according to which the phase space describing a quantum system is noncommutative, possible noncommutative (NC) structure in the space-time was also explored \[\text{[1]}\]. This noncommutative space-times were, however, not considered extensively until the recent re-appearance through specific string theory setups (for a review, see \[\text{2}\] and references therein). The natural question which arises is what are the phenomenological implications of such a noncommutativity, if it exists. The natural setup for asking this question is either the quantum mechanics or quantum field theory on these space-times. Let us consider the simplest noncommutative space-time, defined by:

\[ [\hat{x}_i, \hat{x}_j] = i \theta_{ij}, \quad i,j = 1,2,3. \] (I.1)

Here we have assumed noncommutativity only in space and kept time and space commuting and also taken \( \theta_{ij} \) to be an antisymmetric constant matrix. Time-space noncommutativity, \( \theta_{0i} \neq 0 \), leads to the well-known problems with unitarity and causality \[\text{[2, 3]}\]. The above commutation relation between the space coordinate operators will have physical effects which could be observable in the physical experiments. One of the most important effects is the Lorentz invariance and even the rotational invariance violations. The noncommutative quantum field theory can still be constructed starting from the representations of the Poincaré algebra, because it has twisted-Poincaré symmetry, and consequently the same representation content as a theory with usual Poincaré invariance \[\text{[3]}\]. However, the violation of usual Lorentz invariance is manifest. For example, it can change the spectrum of hydrogen atom and have an impact on the Lamb-shift \[\text{[4]}\] or affect the differences between to atomic hyperfine or Zeeman transition frequencies, in clock-comparison experiments \[\text{[5]}\]. Using the present experimental data one may then extract some bounds on the noncommutativity parameter \( \theta_{ij} \). It is convenient to introduce the noncommutativity vector \( \theta_i \) as

\[ \theta_i = \frac{1}{2} \epsilon_{ijk} \theta_{jk}. \] (I.2)

and the norm of the vector \( \theta \) as inverse of the square of the noncommutativity scale \( \Lambda_{\text{NC}} \):

\[ \Lambda_{\text{NC}}^2 = \frac{1}{|\theta|^2}. \] (I.3)

The experimental lower bounds are usually represented on the \( \Lambda_{\text{NC}} \) and they are generically of order of \( 100 - 10^4 \) GeV, depending on the experiments and their precision.

The noncommutativity of space in the quantum field theories formulated on noncommutative space (the NC-QFTs) appears through the modification of the product of the fields which appear in the action. For the noncommutativity of the form of \( \text{[6]} \), this modified product is the so-called Moyal star product which is defined as

\[ (f \star g)(x) = \exp(i \frac{\theta_{ij}}{2} \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j})f(x)g(y)|_{x=y} = f(x) + i \theta_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} + O(\theta^2), \] (I.4)

where \( x \) denotes the coordinate on the commutative (Moyal) counterpart of the noncommutative space.

In a similar manner one can construct the Yang-Mills gauge theories on a noncommutative space. In particular one may construct NC \( U(1) \) gauge theory, which upon addition of fermions (electron) becomes the NCQED \[\text{[8, 9]}\]; or one may try to construct noncommutative versions of the electro-weak Standard Model (NCSM) \[\text{[10, 11]}\].

In the present work we would like to study one of the consequences of noncommutativity when we have a sizable background electromagnetic field. Perhaps the most
extensively studied consequence of the presence of a constant background electric field is the Schwinger pair production, whose noncommutative version has been discussed in [12] where it is shown that in a noncommutative space the pair production in QED and NCQED are the same. The effects of a constant background magnetic field (at tree level) does not lead to a pair production effect.

Although the presence of a background constant magnetic field does not have an observable effect within the tree level QED setup (or even the Standard Model), it can have observable effects and implications for beyond the Standard Models or at one or higher loop levels in QED. One of the areas where the presence of the constant background magnetic field can be felt is the propagation of light (photons) in models which contain an axion field [13], where we have axion production by photons propagating in a static magnetic field (the Primakoff effect) [14]. The background magnetic field can also affect propagation of photons through a photon splitting process ($F_{\mu\nu}^4$ terms in the one-loop effective QED) [15].

There have been many experiments and proposals to test the effects of the axion, and in general the background magnetic field, most of them focusing on the solar axions. The most famous of such experiments is the PVLAS collaboration at CERN [16]. The PVLAS experiment [17, 18], however, is an experiment set up to test the polarization rotation of the equation

$$S = -\frac{1}{4\pi} \int d^4x F_{\mu\nu}^4 \star F_{\mu\nu}^4$$  \hspace{1cm} (II.1)

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i e [A_\mu, A_\nu]$$  \hspace{1cm} (II.2)

and $[A_\mu, A_\nu]_s$ is the Moyal bracket, defined as

$$[A_\mu, A_\nu]_s = A_\mu \star A_\nu - A_\nu \star A_\mu.$$  

In our case, when there is a background magnetic field turned on,

$$A_\mu = A^0_\mu + A_\mu$$  \hspace{1cm} (II.3)

where $A^0_\mu$ is the background gauge field and $A_\mu$ is the gauge field corresponding to the propagating light.

Plugging (II.3) into (II.2), we have

$$F_{\mu\nu}^0 = F_{\mu\nu}^0 + F_{\mu\nu}$$

where $F_{\mu\nu}^0$ is the background field strength:

$$F_{\mu\nu}^0 = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu + i e [A^0_\mu, A^0_\nu]_s.$$  \hspace{1cm} (II.4)

Since in the PVLAS experiment the applied background is a constant magnetic field, $\vec{B}_0$, it follows that

$$A^0_0 = 0$$

and $A^0_\mu$ is found from the relation of definition of $\vec{B}_0$, i.e.

$$F_{ij}^0 = \epsilon_{ijk} B^k_0,$$  \hspace{1cm} (II.5)

leading to the equation

$$\partial_i A^0_j - \partial_j A^0_i + i e [A^0_i, A^0_j]_s = \epsilon_{ijk} B^k_0.$$  \hspace{1cm} (II.6)

Solving (II.6) for $A^0_i$, we obtain the gauge potential of the applied background field as a power series in $\theta$ (in effect, a power series in $e \vec{\theta} \cdot \vec{B}_0$):

$$A^0_i = \frac{1}{2} \epsilon_{ijk} B^k_0 S \left( e \vec{\theta} \cdot \vec{B}_0 \right) x^k$$  \hspace{1cm} (II.7)

where the power series $S \left( e \vec{\theta} \cdot \vec{B}_0 \right)$ is obtained as solution of the equation

$$\left( \frac{e \vec{\theta} \cdot \vec{B}_0}{4} \right) S^2 - S + 1 = 0.$$  

From (II.7) we can extract an effective background magnetic field $\vec{B}$, which is related to the applied magnetic field $\vec{B}_0$ as

$$\vec{B} = \vec{B}_0 S \left( e \vec{\theta} \cdot \vec{B}_0 \right).$$  \hspace{1cm} (II.8)

The rest of the paper is organized as follows. First we present the action of the noncommutative $U(1)$ Yang-Mills gauge theory and work out the modified energy-momentum dispersion relation in this setup. We then compute the polarization rotation in an external magnetic field and show that the PVLAS experiment results can be explained within this noncommutative model with the noncommutativity scale $\Lambda_{NC}$ around 30 GeV.

II. PROPAGATION OF A PHOTON IN A NONCOMMUTATIVE SPACE IN BACKGROUND MAGNETIC FIELD

We start with the action of a NC $U(1)$ gauge theory:

$$S = -\frac{1}{4\pi} \int d^4x F_{\mu\nu}^4 \star F_{\mu\nu}^4$$  \hspace{1cm} (II.1)

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i e [A_\mu, A_\nu]$$  \hspace{1cm} (II.2)

and $[A_\mu, A_\nu]_s$ is the Moyal bracket, defined as

$$[A_\mu, A_\nu]_s = A_\mu \star A_\nu - A_\nu \star A_\mu.$$  

In our case, when there is a background magnetic field turned on,

$$A_\mu = A^0_\mu + A_\mu$$  \hspace{1cm} (II.3)

where $A^0_\mu$ is the background gauge field and $A_\mu$ is the gauge field corresponding to the propagating light.

Plugging (II.3) into (II.2), we have

$$F_{\mu\nu} = F_{\mu\nu}^0 + F_{\mu\nu}$$

where $F_{\mu\nu}^0$ is the background field strength:

$$F_{\mu\nu}^0 = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu + i e [A^0_\mu, A^0_\nu]_s.$$  \hspace{1cm} (II.4)

Since in the PVLAS experiment the applied background is a constant magnetic field, $\vec{B}_0$, it follows that

$$A^0_0 = 0$$

and $A^0_\mu$ is found from the relation of definition of $\vec{B}_0$, i.e.

$$F_{ij}^0 = \epsilon_{ijk} B^k_0,$$  \hspace{1cm} (II.5)

leading to the equation

$$\partial_i A^0_j - \partial_j A^0_i + i e [A^0_i, A^0_j]_s = \epsilon_{ijk} B^k_0.$$  \hspace{1cm} (II.6)

Solving (II.6) for $A^0_i$, we obtain the gauge potential of the applied background field as a power series in $\theta$ (in effect, a power series in $e \vec{\theta} \cdot \vec{B}_0$):

$$A^0_i = \frac{1}{2} \epsilon_{ijk} B^k_0 S \left( e \vec{\theta} \cdot \vec{B}_0 \right) x^k$$  \hspace{1cm} (II.7)

where the power series $S \left( e \vec{\theta} \cdot \vec{B}_0 \right)$ is obtained as solution of the equation

$$\left( \frac{e \vec{\theta} \cdot \vec{B}_0}{4} \right) S^2 - S + 1 = 0.$$  

From (II.7) we can extract an effective background magnetic field $\vec{B}$, which is related to the applied magnetic field $\vec{B}_0$ as

$$\vec{B} = \vec{B}_0 S \left( e \vec{\theta} \cdot \vec{B}_0 \right).$$  \hspace{1cm} (II.8)
In the $A_0 = 0$ gauge, we have:

$$\mathcal{F}_{0i} = \partial_0 A_i ,$$

$$\mathcal{F}_{ij} = \partial_i A_j + i e ( [A_0^i, A_j] + [A_i, A_j^0] ) + \mathcal{O}(A^2)$$

$$(1 - \frac{e}{2} \vec{\theta} \cdot \vec{B}) \partial_i A_j + \frac{e}{2} (\theta_i \vec{B} \cdot \theta_j A_j - i \leftrightarrow j) + \mathcal{O}(A^2)$$

Inserting the above expressions for the components of $\mathcal{F}_{\mu \nu}$ into the action (II.1), and dropping the parts only involving the background $B_0$ field and the total derivative terms we obtain

$$S = \frac{1}{4 \pi} \int d^4x \left( 2 (\partial_0 A_i)^2 - \mathcal{F}_{ij}^2 \right)$$

and hence the equations of motion are

$$\left[ \delta_{ij} \left( \partial_i^2 - (1 - \frac{1}{2} e \vec{\theta} \cdot \vec{B})^2 \nabla^2 \right) - e (1 - \frac{1}{2} e \vec{\theta} \cdot \vec{B}) \delta_{ij} (\vec{\theta} \cdot \nabla) - \frac{1}{2} \theta_i \partial_j \right] \vec{B} \cdot \nabla \nabla = 0$$

Expanding

$$A_i(x) = \int d^3k \; e_i(k) e^{i (\omega t - \vec{k} \cdot \vec{x})}$$

we obtain

$$\left[ \delta_{ij} \left( \omega^2 - (1 - \frac{1}{2} e \vec{\theta} \cdot \vec{B})^2 k^2 \right) - e (1 - \frac{1}{2} e \vec{\theta} \cdot \vec{B}) \vec{k} \cdot (\vec{k} \cdot \vec{B} \delta_{ij} - \delta_{ij} k_i) \right] e_j(k) = 0 .$$

Next we note that in the PVLAS setup the external magnetic field is transverse to the direction of the light beam propagation, i.e. $\vec{B}_0 \cdot \vec{k} = 0$. For this case, after imposing the transversality condition which is necessary to fix the remainder of the NC $U(1)$ gauge freedom, the equation of motion simplifies considerably and leads to the following modified dispersion relation

$$\omega^2 = (1 - \frac{1}{2} e \vec{\theta} \cdot \vec{B})^2 k^2$$

To relate the above dispersion relation to the rotation of the polarization vector we note that the rotation of the polarization resulting from the change in the dispersion relation of the form

$$\omega - k = k \Delta$$

is

$$\delta \phi = k L \Delta ,$$

where $L$ is the length the photon has passed through the external magnetic field (or the length of the Fabry-Perot cavity in the PVLAS setup). In our model

$$\Delta = \frac{1}{2} e \vec{\theta} \cdot \vec{B}$$

Putting (II.13) and (II.14) together we obtain

$$\delta \phi = \frac{\pi L}{\lambda} e \vec{\theta} \cdot \vec{B} .$$

Recall that $\vec{B}$ is not exactly the applied background magnetic field. The effective background magnetic field $\vec{B}$, in our notations, is given in (II.8) and slightly differs from $B_0$, by a power series in $(e \vec{\theta} \cdot \vec{B}_0)$. For $(e \vec{\theta} \cdot \vec{B}_0)$ small, however, we can discard the higher-order terms in the power series and practically treat the background field as a commutative object.

### III. COMPARISON TO THE PVLAS RESULTS

Now that we have shown and computed the polarization rotation in the background magnetic field in the noncommutative setup we are ready to compare our result with the PVLAS experiment. In the PVLAS experiment the background $B_0$ field is on a turntable which rotates with a frequency of $\nu_B = 0.33$ Hz and the magnitude of $B_0 = 5.5$ T. In our computations we have ignored the time dependence of the external magnetic field and that the magnet is placed on a turn-table. This will, however, not alter our result, if used for the PVLAS experiment, as the frequency of the light beam $\omega \sim 10^9$ GHz is much larger than that of the magnetic field $\nu_B \sim 0.1$ Hz and hence in our revolution of the photon wave the magnetic field is essentially a constant. Moreover, as is seen from (II.13) the rotation in the polarization is in phase with the background magnetic field.

In the PVLAS experiment [18]:

$$B_0 = 5.5 \text{ T} , \quad \delta \phi = (2.2 \pm 0.3) \times 10^{-7} \text{ rad}$$

$$\lambda = 1064 \text{ nm} , \quad L = N \cdot l_0 ,$$

where $N$ is the number of passes through the magnetic field region which has length $l_0$. In the PVLAS experiment, $N = \frac{2\pi}{\lambda}$ with the finesse parameter $\mathcal{F} = 8.2 \times 10^5$, leading to $N = 5.22 \times 10^5$, and $l_0 = 1.333$ m.

Noting that $e B_0 = 3.25 \times 10^{-10} \text{ MeV}^2$ for a magnetic field of 5.5 T, defining the noncommutativity scale $\Lambda_{NC}$ as in (I.3) and assuming that $\vec{\theta}$ is parallel to $\vec{B}_0$ we obtain

$$\Lambda_{NC} \simeq 30 \text{ GeV} .$$

(III.2) is our main result.
IV. DISCUSSIONS AND CONCLUSIONS

The value obtained for the noncommutativity energy scale from the PVLAS data, \( \mu \), is by two orders of magnitude lower than the one obtained from other considerations, such as NC Lamb shift, clock comparison experiments, or precision data of Standard Model. As such, noncommutativity cannot explain the amount of the polarization rotation in the PVLAS experiment. Of course, it is also possible that the noncommutativity scale discussed in the literature are all based so far on the “non-observation” of the noncommutative effects. In this sense, PVLAS might serve as the first experiment in which the noncommutative effects are observed. Thus space-time noncommutativity could be a plausible candidate, would the improved PVLAS data change to sufficiently lower values. As such, an improvement in the PVLAS results, which we are awaiting, would shed light on the noncommutativity of the space-time.

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[1] H. S. Snyder, “Quantized Space-Time,” Phys. Rev. 71 (1947) 38.
[2] R. J. Szabo, “Quantum field theory on noncommutative spaces,” Phys. Rept. 378, 207 (2003) arXiv:hep-th/0109162.
[3] J. Gomis and T. Mehen, “Space-time noncommutative field theories and unitarity,” Nucl. Phys. B 591, 265 (2000) arXiv:hep-th/0005129.
[4] N. Seiberg, L. Susskind and N. Toumbas, “Space/time non-commutativity and causality,” JHEP 0006, 044 (2000) arXiv:hep-th/0005015.
[5] M. Chaichian, K. Nishijima and A. Tureanu, “Spin-statistics and CPT theorems in noncommutative field theory,” Phys. Lett. B 568, 146 (2003) arXiv:hep-th/0209008.
[6] M. Chaichian, P. P. Kulish, K. Nishijima and A. Tureanu, “On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT,” Phys. Lett. B 604, 98 (2004) arXiv:hep-th/0408069.
[7] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane and T. Okamoto, “Noncommutative field theory and Lorentz violation,” Phys. Rev. Lett. 87, 141601 (2001) arXiv:hep-th/0101075.
[8] S. K. K. Hayakawa, “Perturbative analysis on infrared aspects of noncommutative QED on \( R^4 \),” Phys. Rev. B 478, 394 (2000) arXiv:hep-th/9903001.
[9] I. F. Riad and M. M. Sheikh-Jabbari, “Noncommutative QED and anomalous dipole moments,” JHEP 0008, 045 (2000) arXiv:hep-th/0008132.
[10] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari and A. Tureanu, “Noncommutative standard model: Model building,” Eur. Phys. J. C 29, 413 (2003) arXiv:hep-th/0107055.
[11] X. Calmet, B. Jurco, P. Schupp, J. Wess and M. Wohlgenannt, “The standard model on noncommutative space-time,” Eur. Phys. J. C 23, 363 (2002) arXiv:hep-ph/0111115.
[12] L. Alvarez-Gaumé and J. L. F. Barbon, “Non-linear vacuum phenomena in non-commutative QED,” Int. J. Mod. Phys. A 16, 1123 (2001) arXiv:hep-th/0006290.
[13] R. D. Peccei and H. R. Quinn, “CP conservation in the presence of instantons,” Phys. Rev. Lett. 38, 1440 (1977).
[14] R. Cameron et al., “Search for nearly massless, weakly coupled particles by optical techniques,” Phys. Rev. D 47 (1993) 3707.
[15] S. L. Adler, “Photon splitting and photon dispersion in a strong magnetic field,” Annals Phys. 67, 599 (1971).
[16] S. Andrianome et al., “First results from the CERN axion solar telescope (CAST),” Phys. Rev. Lett. 94, 121301 (2005) arXiv:hep-ex/0411033. For more information about the CERN Axion Solar Telescope (CAST) collaboration visit the website: http://cast.web.cern.ch/CAST/.
[17] E. Zavattini et al. [PVLAS Collaboration], “Experimental observation of optical rotation generated in vacuum by a magnetic field,” arXiv:hep-ex/0507107.
[18] U. Gastaldi, “PVLAS results”, Presented at 40th Rencontres de Moriond on Electroweak Interactions and Unified Theories, March 2005, arXiv:hep-ex/0507061.
[19] K. Van Bibber, N. R. Dadgeviren, S. E. Koonin, A. Kerman and H. N. Nelson, “An experiment to produce and detect light pseudoscalars,” Phys. Rev. Lett. 59, 759 (1987).