Electromagnetic Force in Dispersive and Transparent Media

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A hydrodynamic-type, macroscopic theory was set up recently to simultaneously account for dissipation and dispersion of electromagnetic field, in nonstationary condensed systems of nonlinear constitutive relations. Since it was published in the letter format, some algebra and the more subtle reasonings had to be left out. Two of the missing parts are presented in this paper: How algebraically the new results reduce to the known old ones; and more thoughts on the range of validity of the new theory, especially concerning the treatment of dissipation.

1. INTRODUCTION

The macroscopic Maxwell equations, given in terms of $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, and $\mathbf{B}$, need constitutive relations linking them to be closed. The form of these relations depend crucially on two physical parameters, the frequency and the field strength. Weak fields are necessary for the validity of the linear response theory, as these expressions are only known explicitly vanishing velocity field of the condensed system (in this first step) neglects magnetization.

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41.20.Bt, 47.10.+g, 52.35.Mw, 52.25.-b

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transparent interpretation for the old and classic results which, having relied heavily on algebra, are somewhat lacking in appropriate physical pictures.

The hydrodynamic theory of dispersion will be presented in section II to render the present manuscript self-contained. Because we shall only list the relevant formulas and abstain from repeating all the reasoning and arguments that lead to the new theory, the reader is advised to consult [11] at the beginning of the next section. Section IV incorporates the specified approximation and deduce four results. They are compared to the energy density by Brillouin, and to three formulas by Pitaevskii’s: the total stress tensor, the “nonmagnetic” magnetization, and the time dependent permittivity. Section V ends with a brief summary.

II. THE HYDRODYNAMIC THEORY OF DISPERSION

In this section, we shall first discuss in some details the range of validity of the new theory, then present its complete set of equations, and specify the theory by an expansion of the thermodynamic energy to third order in the field variables.

A. The Range of Validity

The proper hydrodynamic theory of electromagnetism accounts for the macroscopic dynamics of continuous media in the low frequency limit, for a system that is charged or exposed to external fields. Local thermodynamic equilibrium holds, and the set of hydrodynamic variables is identical to that of the thermodynamic variables. The equations of motion are conservation laws and the Maxwell equations, including irreversible terms accounting for dissipation.

At higher frequencies, microscopic variables deviate more and more from equilibrium, becoming independent, and finally ballistic. Denoting the time $\tau_{\text{loc}}$ needed to establish local equilibrium, this starts to happen when $\omega \tau_{\text{loc}}$ is no longer small. To account for this circumstance, we usually have to abandon the hydrodynamic theory and embrace the Boltzmann theory which, despite its undeniable usefulness, is both a far more complex and a less general theory – it considers the vast number of microscopic degrees of freedom explicitly, and it is confined to dilute systems. The question therefore is whether an appropriately generalized hydrodynamic theory can be made to account for some of the more interesting aspects at higher frequencies, and save us from the complexities of the Boltzmann theory.

Let us concentrate on one specific microscopic variable, the polarization $\mathbf{P}$. Actually, as far as its spatial extent is concerned, it is a macroscopic rather than a microscopic variable, but it is certainly dependent in the hydrodynamic limit, as long as $\omega \tau_p \ll 1$, where $\tau_p$ is $\mathbf{P}$’s longest time scale. (All this can also be said of the magnetization, which we however shall not consider here.)

In a dielectric medium, $\mathbf{P}$ has many characteristic times, given by the resonance frequencies and their widths. If they are well separated, then the equation of motion, close to one resonance and in the simplest case considered below is

$$\dot{\mathbf{P}}/\omega_p^2 - \tau \ddot{\mathbf{P}} + \mathbf{P} = \chi \mathbf{D}, \quad (1)$$

cf Eq(14) of the next section. The given resonance may be overdamped ($\tau \gg \omega_p$) or sharply resonating ($\tau \ll \omega_p$), and the characteristic time $\tau_p$, after which the polarization is no longer independent, is respectively $\tau$ and $2/\omega_p^2 \tau$, while $1/\omega_p$ is typical for the time scale of $\mathbf{P}$’s motion in the resonating case. (Note that going to a different resonance, $\tau_p$ will change, it therefore depends on the frequency of the external field.)

If the polarization $\mathbf{P}$ is a specially slow variable, $\tau_p \gg \tau_{\text{loc}}$ (where $\tau_{\text{loc}}$ is around $10^{-10}$s at usual temperatures and densities,) we may increase the range of validity of the hydrodynamic electromagnetic theory, from $\omega \ll 1/\tau_p$ to $\omega \ll 1/\tau_{\text{loc}}$. By taking the energy as a local function also of $\mathbf{P}$, $\mathbf{P}$, and derive the equation of motion for $\mathbf{P}$. There are quite a number of systems with a large $\tau_p$: All electro-rheological fluids have especially large $\tau_p$, of the order of $10^{-4}$s, but other complex fluids with large molecules and a permanent molecular dipole moment (such as nematic liquid crystals) also have a slow polarization. (Even the comparatively small water molecule with its permanent dipole moment has a $\tau_p$ around $10^{-9}$s, just slightly too fast.) The hydrodynamic theory of dispersion presented below is unqualifiedly valid for these systems, (though it should usually be enough to neglect $\mathbf{P}$ in the equation of motion, or equivalently, exclude $\mathbf{P}$ as an additional variable.) We shall refer to this scenario as the hydrodynamic dispersion.

Interestingly, essentially the same set of equations also accounts for a system in the ballistic regime, for quickly oscillating electric fields and polarizations, $1/\omega_p \ll \tau_{\text{loc}}$ — if we confine our curiosity to questions such as what is the averaged force that a high frequency external field exerts on a volume element. (Note that the low resolution is quite sufficient for resolving the hydrodynamic responses to a high frequency field.) This scenario we shall refer to as ballistic dispersion. Clearly, we need to understand why the same set of equations also works for the ballistic dispersion, and what the restrictions are.

First, a coarse grained, hydrodynamic-type description is at all possible because the field variables $\mathbf{P}$, $\mathbf{D}$ and $\mathbf{B}$ vary (due to the largeness of the light velocity) on macroscopic, hydrodynamic length scales. Second, most of the general principles used as input to consider hydrodynamic dispersion are also valid here. Especially, the total energy and momentum remain conserved. The one
exception is local equilibrium, the lack of which introduces some caveats with respect to temporal resolution and to dissipation. More specifically, they are:

- In the ballistic regime, the variables of the theory divides into two types, fast and slow. The field variables are fast, the rest is slow. They are the densities of mass $\rho$, entropy $s$, total energy $u$, and total momentum $g^\text{tot}$. Therefore, the equations of motion of the field variables are highly accurate, resolving temporal increments much less than $1/\omega_p$, while the actual hydrodynamic equations are much coarser, with a resolution low compared to $\tau_\text{loc}$. And because every differential equation, consistently, has a unique resolution, all field terms appearing in the slow, hydrodynamic equations need to be appropriately averaged.

- The generalized hydrodynamic theory presented in the next section is shown below to be clearly valid for ballistic dispersion in the transparent region, where electromagnetic dissipation is negligible, and the time $\tau_p$ with which $P$ looses energy long. But as argued below, it should remain valid even if field dissipation is strong.

Taking electromagnetic dissipation into account, the total, conserved energy divides into three parts,

$$ U = U^\text{mat} + U^\text{em} + U^\text{mic}. \quad (2) $$

The first is the thermodynamic energy in the absence of an external field; the second is the additional energy in the presence of a field; and the third is the rest, the energy of all microscopic variables not given in the first two explicitly, $U^\text{mic}(x_1^1, x_2^1 \ldots)$. The variables $x_i$ are defined such that they vanish in local equilibrium, so they are irrelevant for the consideration of hydrodynamic dispersion. In the ballistic regime, $U^\text{mic}$ is finite and serves as a transit hall: External energy is being fed continually into $U^\text{em}$, the electromagnetic dissipation excite some microscopic degrees of freedom $x_i$, and convert $U^\text{em}$ into $U^\text{mic} \ldots$ which after the comparably long time of $\tau_\text{loc}$ becomes heat, $U^\text{em} \rightarrow U^\text{mic} \rightarrow \int T\,ds$. The rate at which $U^\text{em}$ is lost is approximately $\dot{U}^\text{em} \approx -U^\text{em}/\tau_p$, the average time this energy stays in the transit hall is $\tau_\text{loc}$, so $U^\text{mic} \approx (\tau_\text{loc}/\tau_p)U^\text{em}$. The right side translates into $(\tau_\text{loc}/\tau)U^\text{em}$ for the overdamped oscillation, and into $\frac{1}{2}(\omega_p^2 \tau_\text{loc})U^\text{em}$ for the resonating one. In the first case, we always have $U^\text{mic} \gg U^\text{em}$, and in the second we mostly do, rendering the transit hall usually large.

Including nonhydrodynamic variables such as $P$ leads to contributions $\sim P$ and $\partial U/\partial P$ in the energy and momentum flux, see next section. If $U^\text{mic}$ is nonzero, we would expect similar terms $\sim x_i$ and $\partial U/\partial x_i$. These we may neglect in the transparent region of vanishing dissipation, defined as the frequency regime where the stringent condition $\omega_p^2 \tau_\text{loc} \ll 1$ holds, or equivalently $U^\text{mic} \ll U^\text{em}$, so $U^\text{mic}$ and its contributions may be neglected. Outside these regimes, circumstances are not as certain and in need of a clarifying, more microscopic approach such as the Boltzmann theory, to confirm the considerations in the next paragraph.

There are reasons why we may quite generally neglect $U^\text{mic}$: While terms such as $P^2$ and $P^2$, of macroscopic extent, coherently add up over many periods to yield diverse, slowly varying contributions $\sim (P^2)$ and $\sim (P^2)$ in the momentum and energy flux, and thereby directly alter the slow, hydrodynamic variables, the quantities $x_i$ are random and of microscopic spatial scales. They further dissipate and degrade, to eventually turn into heat. So on a coarse, hydrodynamic time scale, we may simply lump $\langle U^\text{mic}\rangle$ into heat $\int T\,ds$, and $\langle \dot{U}^\text{mic} \rangle$ into the heat production $R$. Then, clearly, $U^\text{mic}$ is neglected as an independent entity.

On a more fundamental level, the very criterion by which we have singled out $P$ and $\dot{P}$ from the lot of microscopic degrees of freedom is because they are qualitatively different: Given a certain energy content in the field $D$ and $B$, there is a back and forth of energy flow between $D$, $B$, $P$ and $\dot{P}$ within each period; while the field energy that leaks into the other microscopic degrees of freedom is usually lost. In fact, for an overdamped resonance, it is (as mentioned) appropriate to exclude $\dot{P}$ as an explicit variable, and consider it as one of the many ordinary microscopic degrees of freedom, as the energy leaked into $\dot{P}$ is lost to the field. On the other hand, if a system involves more variables in the tidal-like transfer of field energy, the present theory needs to be generalized to also include them — one example comes readily to mind: an independent magnetization.

Note that being a function also of the slow variables, $U^\text{em}$ is not conserved by itself, and the permeability $\varepsilon$ will in general contain an imaginary part to account for this fact, even without any dissipation (or electric charge).

### B. The Equations of Motion

The complete hydrodynamic theory of dispersion consists of a closed set of partial differential equations that governs the dynamics of the medium and the electromagnetic field. The structure of the equations is determined by general principles: the Maxwell equations, the Lorentz-Galilean transformation, the thermodynamic theory and the relevant conservation laws.

Combining the two macroscopic energy densities

$$ U^\text{Mac} \equiv U^\text{mat} + U^\text{em}, \quad (3) $$

we take it as a function of the entropy density $s$, mass density $\rho$, the electric and magnetic field $D$ and $B$, the electric polarization $P$, its canonical conjugate $a$ (that will turn out to be essentially $\dot{P}$), and the thermodynamic momentum density $g$. 


\[dU^{Mac} = T ds + \mu d\rho + v \cdot dq + E \cdot dD + H \cdot dB + h \cdot dP + b \cdot da, \]

where the thermodynamic momentum density \(g\) is related to the total momentum density

\[g^{tot} = \rho v + (E \times H)/c \]

through

\[g = g^{tot} - D \times B/c. \]

As soon as the energy function \(U^{Mac}\) is known, the temperature \(T\), chemical potential \(\mu\), velocity \(v\), field strengths \(E\) and \(H\) are also determined. (In accordance with \(\Box\), the polarization defined here is a rest frame quantity, \(P \equiv D_0 - E_0\).

Isotropy of space results in the identity

\[E \times D + H \times B + h \times P + b \times a + v \times g = 0. \]

The Maxwell equations

\[\nabla B = 0, \quad D = \rho v, \quad \nabla D = \rho v, \quad \nabla E = c \nabla \times H - \rho v \]

account for the motion of \(D\) and \(B\). Here, the dot indicates partial temporal derivative \(\partial/\partial t\) and \(\rho v\) denotes the macroscopic charge density. The variables \(\rho, U, g^{tot}\) are conserved, their equations of motion take the form

\[\dot{\rho} + \nabla \cdot (\rho v) = 0, \quad \dot{U} + \nabla \cdot Q = 0, \quad \dot{g}^{tot} + \nabla_j (\Pi_{ij} - \Pi_{ij}^{D}) = 0, \]

where \(Q\) is the total energy flux, and \((\Pi_{ij} - \Pi_{ij}^{D})\) the symmetric total momentum flux, or total stress tensor. The entropy is not conserved, and has a positive source \(R\),

\[s + \nabla \cdot (sv - f^{D}) = R/T, \quad R \geq 0, \]

The dissipative part of entropy flux \(f^{D}\) describes especially thermoconduction, while \(\Pi_{ij}^{D}\) accounts primarily for viscosity-related phenomena.

The macroscopic variables \(P\) and \(a\) are governed by equations that are essentially of the Hamiltonian type,

\[\dot{P} = \partial U/\partial a = b, \quad \dot{a} = -\partial U/\partial P = -h, \]

with some supplementary terms needed to ensure the proper transformation behavior, and to account for dissipation. First, the temporal derivative is replaced by the Galilean invariant operator that takes into account the effect of the local movement of the medium,

\[D_i = \partial_i + (v \nabla_j) - \Omega_j, \]

where \(\Omega \equiv \frac{1}{2} \nabla \times v\). Second, a dissipative force \(h^{D}\) is introduced in the equation for \(a\) to account for electromagnetic dissipation that (in the linear case) is usually taken care of by an imaginary term in the electric permittivity \(\varepsilon\). Third, the polarization is changed if the medium undergoes volume dilatation, as a term \(-P(\nabla v)\) appears in the equation of motion for \(P\),

\[D_i P = b - P(\nabla \cdot v), \quad D_i a = -h - h^{D}. \]

The dissipative terms are determined with the familiar method of irreversible thermodynamics: We first identify the entropy production as

\[R + \dot{U}^{mic} = f^{D} \cdot (\nabla T) + h^{D} \cdot b + \Pi_{ij}^{D} v_{ij}, \]

where \(v_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)\). Then take the fluxes as proportional to the thermodynamic forces,

\[(\Pi_{ij}^{D}) = \left( \begin{array}{ccc} n_{ijk} & o_{ijk} & \beta_{ijk} \\ \bar{o}_{ij} & \bar{s}_{ij} & \bar{\beta}_{ij} \\ \bar{\alpha}_{ij} & \bar{\lambda}_{ij} & \bar{\zeta}_{ij} \end{array} \right) \times \left( \begin{array}{c} v_{ij} \\ v_{il} \\ b_{j} \end{array} \right) \]

(Appropriate Onsager reciprocity relations are implied.)

The energy flux is

\[Q_i = (Ts + \mu \rho + v_k g_k) v_i - T f_{ij}^{D} - v_j \Pi_{ij}^{D} + (E \times H)_i + v_i (h \cdot P) + \frac{1}{2} (v \times (h \times P + b \times a))_i. \]

(The last two terms were erroneously omitted from \(\Box\).) This expression may be rewritten as

\[Q_i = c (E^{0} \times H^{0})_i - f_{ij}^{D} T + U^{Mac} v_i + (\Pi_{ij} - \Pi_{ij}^{D}) v_{ij} - v_k g_k^{tot} v_{ij}, \]

to see that the velocity-dependent parts do come from an Lorentz-Galilean transformation, discussed eg in \(\Box\).

\[(E^{0} \equiv E + v \times B/c \text{ and } H^{0} \equiv H - v \times D/c \text{ are the restframe fields.} \]

The stress tensor is symmetric and given as

\[\Pi_{ij} = \frac{1}{2} (v_i g_j - E_i D_j - H_i B_j + (i \leftrightarrow j)) + (Ts + \mu \rho + g \cdot v + H \cdot B + E \cdot D + h \cdot P - U^{Mac}) \delta_{ij}. \]

Frequently, there are many different resonance frequencies of the polarization, not just the single one, given here as \(\omega_p\). This fact can be accounted for by introducing as many "subpolarizations",

\[D - E = P = \sum P_{\alpha}, \]

chosen such that the two squared order terms of the energy are diagonal,

\[U_{0}^{ren} = \ldots + \sum (P_{\alpha}^{2}/\chi_{\alpha} + \chi_{\alpha} \omega_{p}^{2} a_{\alpha}^{2})/2 + \ldots \]

Close to one resonance \(\alpha\), if it is well separated, as all the other subpolarizations are not excited, we may simply substitute \(P_{\alpha}\) for \(P\).
C. Some Explicit Expressions

Now, the above equations are rendered more explicit by an expansion of the energy function in the vector-variables \( \mathbf{D}, \mathbf{B}, \mathbf{P}, \mathbf{a} \) and \( \mathbf{v} \) to third order, as this is sufficient for a comparison to the linear results by Brillouin and Pitaevskii's. For a magnetically inactive medium (ie taking the static magnetic permeability as 1), such an expansion yields

\[
U^\text{Mac} = U^\text{mat} + \frac{1}{2} B^2 + \frac{1}{2} D^2 - D \cdot P + \frac{1}{2} P^2/\chi + \frac{1}{2} \chi \omega_p^2 a^2 - \xi B \cdot (P \times a) + \frac{1}{2} \rho v^2 + O^4,
\]

where \( O^n \) denotes terms of n-th or higher order in the vector variables \( \mathbf{D}, \mathbf{B}, \mathbf{P}, \mathbf{a}, \mathbf{v} \). The energy density in the absence of electromagnetic fields is \( U^\text{mat}(s, \rho) \); the coefficient \( \chi \) is related to the conventional static dielectric susceptibility \( \chi' = P/E \) by \( \chi^{-1} = \chi^{-1} - 1; \) \( \omega_p^2 \) is the dielectric resonance frequency; \( \xi \) is connected to the magnetic cyclotron-frequency \( \omega_B \); all these parameters are in principle functions of \( \rho \) and \( s \).

Obtaining the differential form from Eqs(22, 23)

\[
d(U^\text{Mac} - \mathbf{v} \cdot g) = T ds + \mu d\rho + E \cdot d\mathbf{D} + H \cdot d\mathbf{B}
\]

we can derive the thermodynamically conjugate variables,

\[
T = \frac{\partial U^\text{mat}}{\partial s} + \frac{P^2}{2} \frac{\partial \chi^{-1}}{\partial s} + \frac{q^2}{2} \frac{\partial \chi \omega_p^2}{\partial s},
\]

\[
- B \cdot (P \times a) \frac{\partial \xi}{\partial s} + O^4,
\]

\[
\mu = \frac{\partial U^\text{mat}}{\partial \rho} + \frac{P^2}{2} \frac{\partial \chi^{-1}}{\partial \rho} + \frac{a^2}{2} \frac{\partial \chi \omega_p^2}{\partial \rho} - \frac{B(P \times a)}{\partial \rho} \frac{\partial \xi}{\partial \rho} - \frac{1}{2} v^2 + O^4,
\]

\[
E = D - P - \nu \times M + O^3,
\]

\[
H = B - \xi (P \times a) + \frac{1}{c} \nu \times P + O^3,
\]

\[
h = \frac{1}{\chi} \frac{\partial H}{\partial E} - \xi a \times B + \frac{1}{c} B \times v + O^3,
\]

\[
b = \chi \omega_p^2 a - \xi B \times P + O^3.
\]

Note that the magnetization \( \mathbf{M} = \mathbf{B} - \mathbf{H} \) as given in Eq(27) is a term of order \( O^2 \). So the difference in the polarization \( \nu \times \mathbf{M} \), between the rest frame quantity \( \mathbf{D}_0 - \mathbf{E}_0 \) and the laboratory quantity \( \mathbf{D} - \mathbf{E} \), is of order \( O^3 \). Within the accuracy of the above equations, it is therefore ignored.

Inserting (29) in (14), we obtain the expression

\[
a = \frac{1}{\chi \omega_p^2} [D_t P + P(\nabla v) + \xi B \times P] + O^3
\]

that may be used to eliminate \( \mathbf{a} \) in the above formulas, and write them instead with \( \mathbf{P} \). Especially, the magnetization in the rest frame and to lowest order is then \( (\xi/\chi \omega_p^2) \mathbf{P} \times \mathbf{P} \). The nonmagnetic magnetization is now seen to result from rotations of the polarization.

For a qualitative estimate of the coefficient \( \xi \), envision electrons revolving around their ion centers \( q_i \). Assuming the rotations to occur in phase, the magnetization associated with it is \( (q_i n_i/m_e) L \), where \( q_i \) and \( m_e \) are the charge and mass of the electrons, while \( n_i \) denotes their density. The angular moment of the electrons, \( \mathbf{L} = m_e r_e \times r_e \) (with \( r_e \) the radius of the circular motion) can also be written as \( \mathbf{L} = (m_e / q_i \omega_p^2) \mathbf{P} \times \mathbf{P} \), because the polarization \( \mathbf{P} \) is \( q_i n_e r_e \). The attendant magnetization is \( \mathbf{M} = (1/2q_i n_e) \mathbf{P} \times \mathbf{P} \). So the coefficient \( \xi \) is

\[
\xi = \chi \omega_p^2/2q_i n_e.
\]

Particularly for an electron plasma, \( \chi = 1 \), and \( \omega_p \) can be considered as the plasma frequency \( (q_i^2 n_e/m_e)^{1/2} \). Eq(31) reduces to \( \xi = q_i/2m_e = -\omega_B/2B \), with \( \omega_B = -B q_i/m_e \) the plasma cyclotron frequency.

III. MONOCHROMATIC APPROXIMATIONS

With the help of a closed L-C circuit, Pitaevskii obtained a number of important results on the effects of a high-frequency field in a medium \( \mathbf{1} \). Because of the special setup, the results are subject to certain restrictions. In order to compare our theory with his work, the same limits will be taken in our theory. Therefore, we shall consider a transparent medium exposed to a strictly monochromatic electric field:

\[
\mathbf{E} = \frac{1}{2} \mathbf{E} e^{-i\omega t} + c.c., \quad \dot{\mathbf{E}} = 0,
\]

where \( \mathbf{E} \) is the constant amplitude and \( \omega \) the frequency. From now on, we shall always assume that the medium is at rest, so any velocity-dependent terms will be discarded. Because only the electric properties are of interest, we also omit the quickly oscillating part of magnetic field in the medium, as in \( \mathbf{1} \), though a strong, constant magnetic field is allowed to be present. If the material coefficients \( \chi, \omega_p^2, \xi \) are constant with respect to time, the induction \( \mathbf{D} \) and polarization \( \mathbf{P} \) will also take the monochromatic form

\[
\mathbf{D} = \frac{1}{2} \mathbf{D} e^{-i\omega t} + c.c., \quad \mathbf{P} = \frac{1}{2} \mathbf{P} e^{-i\omega t} + c.c.
\]

However, if \( \chi, \omega_p^2, \xi \) and \( \mathbf{B} \) are allowed to vary — slowly — via their dependence on the density or temperature, the fields \( \mathbf{D} \) and \( \mathbf{P} \) will become quasi-monochromatic. In this case, we may still hold \( \omega \) to be strictly constant, while allowing the amplitudes \( \mathbf{D}, \mathbf{P} \) to change slowly with time. The quasi-monochromatic situation will be studied in subsection 3.F.
In what follows, the dynamic equations given in the previous section will be investigated, under the preconditions mentioned above. We will show in detail the derivations of four formulas, all well-known in the literature \[4\].

**A. The permittivity**

The frequency-dependent permittivity \(\varepsilon_{ij}\) is calculated from the equation of motion for \(\mathbf{P}\). The expression is given by inserting \((30)\) and \((28)\) to \((14)\), taking the coefficients \(\chi, \xi\) and the magnetic field \(\mathbf{B}\) as constants. Neglecting the velocity-dependent terms, we have

\[
\mathbf{D} - \frac{\mathbf{P}}{\chi} - 2\frac{\xi \mathbf{B} \times \dot{\mathbf{P}}}{\chi \omega_p^2} - \frac{\dot{\mathbf{P}}}{\chi \omega_p^2} = 0. \tag{34}
\]

If the fields \(\mathbf{E}, \mathbf{P}\) assume the monochromatic form of Eqs\((32,33)\), the above equation becomes

\[
\left(1 - \frac{1}{\chi} + \frac{\omega^2}{\chi \omega_p^2}\right)\mathbf{P} + 2i\frac{\xi \omega}{\chi \omega_p^2} \mathbf{B} \times \mathbf{P} + \mathbf{E} = 0. \tag{35}
\]

Solving it for \(\mathbf{P}\), we obtain

\[
\mathbf{P} = (\varepsilon_1 - 1)\mathbf{E} + \varepsilon_2(\mathbf{B} \mathbf{E}) + i \varepsilon_3(\mathbf{E} \times \mathbf{B}), \tag{36}
\]

with

\[
\varepsilon_1 = 1 - \frac{\chi \omega_p^2 (\omega^2 - \omega_p^2 + \chi \omega_p^2)}{(\omega^2 - \omega_p^2 + \chi \omega_p^2)^2 - \omega^2 \omega_p^2 B^2}, \tag{37}
\]

\[
\varepsilon_2 = \frac{\chi \omega_p^2 \omega B^2}{(\omega^2 - \omega_p^2 + \chi \omega_p^2)^2 - \omega^2 \omega_p^2 B^2}, \tag{38}
\]

and

\[
\varepsilon_3 = \frac{\chi \omega_p^2 B^2 \omega B}{(\omega^2 - \omega_p^2 + \chi \omega_p^2)^2 - \omega^2 \omega_p^2 B^2}, \tag{39}
\]

where

\[
\omega_B = -2\xi B. \tag{40}
\]

Using the fact that \(\mathbf{P}_i = (\varepsilon_{ik} - \delta_{ik})\mathbf{E}_k\), we observe from \(34\) that the permittivity is

\[
\varepsilon_{ik} = \varepsilon_1 \delta_{ik} + \varepsilon_2 B_i B_k + i \varepsilon_3 \varepsilon_{ikl} B_l, \tag{41}
\]

where \(\varepsilon_{ikl}\) is the total antisymmetric tensor, \(\varepsilon_{123} = 1\). In the low frequency limit \(\omega \to 0\), we have according to \(37\)

\[
\varepsilon_1 = 1/(1 - \chi). \tag{41}
\]

\(\varepsilon_1\) is not connected to dissipation. It is a purely reactive term. This can best be seen from its invariance under the time-reversal operation: \(\omega \to -\omega, B_i \to -B_i\).

**B. The energy density**

Eliminating the quantity \(a\) in the energy function \((22)\) with the help of \((30)\), we get, for a medium at rest and including terms of third order in the field

\[
\langle U \rangle = U^{\text{mat}} + \frac{B^2}{2} + \frac{D^2}{2} - \mathbf{D} \cdot \mathbf{P} + \frac{\mathbf{P}^2}{2\chi} + \frac{\dot{\mathbf{P}}^2}{2\chi\omega_p^2}. \tag{42}
\]

Now consider the monochromatic case \((32,33)\) and apply a time-average procedure denoted as \((\cdots)\), the energy density is then given as

\[
\langle U \rangle = U^{\text{mat}} + \frac{B^2}{2} + \frac{1}{4} \varepsilon_k \varepsilon_k^* + \frac{1}{4\chi} \left(1 + \frac{\omega^2}{\omega_p^2} - \chi\right) \mathbf{P}_k \mathbf{P}_k^* \tag{43}
\]

or

\[
\langle U \rangle = U^{\text{mat}} + \frac{B^2}{2} + \frac{1}{4} |\mathbf{E}|^2
\]

\[+ \frac{1}{4} \left(1 + \frac{\omega^2}{\omega_p^2} - 1\right) (\varepsilon_{km} - \delta_{km})(\varepsilon_{kn}^* - \delta_{kn}) \varepsilon_m \varepsilon_n^*, \tag{44}\]

where \(\ast\) denotes complex conjugation. Because in our work the permittivity is given by \((41)\) and \((37-39)\), we can verify by direct computations that the equation

\[
\frac{\partial \varepsilon_{mn}}{\partial \omega} = \delta_{mn} + \frac{1}{\chi} \left(1 + \frac{\omega^2}{\omega_p^2} - \chi\right) (\varepsilon_{km} - \delta_{km}) \tag{44}
\]

holds for this form of permittivity. So the time-averaged energy density could be expressed as

\[
\langle U \rangle = U^{\text{mat}} + \frac{B^2}{2} + \frac{1}{4\omega^2} \frac{\partial \varepsilon_{mn}}{\partial \omega} \varepsilon_m \varepsilon_n^*, \tag{45}\]

which is the Brillouin’s expression for the time-averaged energy density of the electric field \(\mathbf{E}\).

**C. Pitaevskii magnetization**

The equation \((27)\) shows that a magnetization could be induced dynamically in an electrically polarizable medium, although the static magnetic permeability is 1. Inserting \((30)\) in \((27)\), we obtain, for a medium at rest, the magnetization

\[
\mathbf{M} = \frac{\varepsilon}{\varepsilon_0} \mathbf{P} \times \mathbf{P} + \mathbf{O}^3. \tag{46}\]

In the monochromatic approximation, the time-average of this magnetization is the same as that obtained by Pitaevskii \[3\]. Indeed, inserting \((22,33)\) in \((16)\) we have
\[ \langle M \rangle_i = \frac{i \omega \xi}{2 \chi \omega_p^2} (P \times P^*)_i \]

\[ = \frac{i \omega \xi}{2 \chi \omega_p^2} \varepsilon_{ijk} (\varepsilon_{jm} - \delta_{jm}) (\varepsilon_{kn} - \delta_{kn}) \mathcal{E}_m \mathcal{E}_n^*. \] 

(47)

With the help of the expression of the permittivity (41), with the help of (27), (28), (30), the stress becomes

\[ \langle \mathcal{M} \rangle = \frac{1}{4} \frac{\partial \varepsilon_{nm}}{\partial B} \varepsilon_m \varepsilon_n. \] 

(49)

D. The stress tensor

Inserting the expressions (22), (23), (25) for \( U, T, \mu \) into the stress (49) and eliminating the fields \( B, a, \mathbf{h} \) with the help of (27), (28), (34), the stress becomes

\[ \Pi_{ij} = \left[ p_0 + \frac{H^2}{2} + \frac{E^2}{2} + \frac{1}{2} \left( \frac{1}{\chi} - 1 \right) P^2 - \frac{\hat{p}^2}{2 \chi \omega_p^2} \right] - \frac{\xi}{\chi \omega_p^2} H (P \times P) + \frac{P^2}{2} \left( \frac{\partial}{\partial \rho} \frac{1}{\chi} \right) - \frac{\hat{p}^2}{2 \chi \omega_p^2} \frac{\partial \chi}{\partial \rho} \]

\[ - \frac{1}{2} \left[ \frac{\partial \mathbf{H}}{\partial \rho} \mathbf{H} \right]_{ij} - \frac{1}{2} \left[ \frac{\partial \mathbf{H}}{\partial s} \mathbf{H} \right]_{ij} + \mathcal{O}^4, \] 

(50)

where \( p_0 \) is pressure of the medium in the case without electromagnetic fields,

\[ p_0(\rho, s) = -U^\text{mat} + \rho \frac{\partial U^\text{mat}}{\partial \rho} + s \frac{\partial U^\text{mat}}{\partial s}. \] 

(51)

Here, in order to avoid unnecessarily complicated formulas, we also neglect the entropy dependence of the parameters \( \chi, \omega_p^2, \xi \) in (50). When the two electric fields take the monochromatic form (32, 33), we obtain after the time-averaging procedure,

\[ \langle \Pi_{ij} \rangle = \left\{ p_0 + \frac{H^2}{2} + \frac{1}{4} \mathcal{E} \mathcal{E}^* - \frac{\omega^2 - \omega_p^2 + \chi \omega_p^2}{4 \chi \omega_p^2} \mathcal{P} \mathcal{P}^* \right\} + \frac{i \omega \xi}{2 \chi \omega_p^2} (P \times P^*) H + \frac{1}{4 \chi \omega_p^2} H \]

\[ \times \left[ H (\omega^2 - \omega_p^2) \mathcal{P} \mathcal{P}^* - i \chi \omega (\mathcal{P} \times \mathcal{P}^*) H \right] \frac{\partial \chi}{\partial \rho} \]

\[ + \frac{1}{4 \chi \omega_p^2} H \left[ H (\omega^2 \mathcal{P} \mathcal{P}^* - i \chi \omega (\mathcal{P} \times \mathcal{P}^*) H \right] \frac{\partial \omega_p^2}{\partial \rho} \]

\[ + \frac{i \omega}{4 \chi \omega_p^2} (P \times P^*) H \frac{\partial \omega_p^2}{\partial \rho} \}

\[ \delta_{ij} - \frac{1}{8} (\mathcal{E} \mathcal{D}^*_l + \mathcal{E} \mathcal{D}_l^* + c.c.) - \frac{1}{2} \left( \langle M \rangle_i H_j + \langle M \rangle_j H_i \right). \] 

(52)

Because the difference between \( B \) and \( H \) (i.e., the Pitaevskii magnetization (49)) is of second order in \( \mathcal{E} \), we can write the formula (53) for \( \mathcal{P} \) to the same accuracy as

\[ \mathcal{P} = (\varepsilon_1 - 1) \mathcal{E} + \varepsilon_2 (H \mathcal{E}) H + i \varepsilon_3 (\mathcal{E} \times H), \] 

(53)

with

\[ \varepsilon_1 = 1 - \frac{\chi \omega_p^2 (\omega^2 - \omega_p^2 + \chi \omega_p^2)}{(\omega^2 - \omega_p^2 + \chi \omega_p^2)^2 - \omega^2 \omega_p^2}, \]

(54)

\[ \varepsilon_2 = \frac{\chi \omega_p^2 \omega H}{[(\omega^2 - \omega_p^2 + \chi \omega_p^2)^2 - \omega^2 \omega_p^2] \omega_H}, \]

(55)

and

\[ \varepsilon_3 = \frac{\chi \omega_p^2 \omega H}{[(\omega^2 - \omega_p^2 + \chi \omega_p^2)^2 - \omega^2 \omega_p^2] \omega \omega_H}. \]

(56)

where

\[ \omega_H = -2 \xi H. \]

(57)

Now inserting in the stress (52) the expression (53), it becomes

\[ \langle \Pi_{ij} \rangle = \left\{ p_0 + \frac{H^2}{2} - \frac{1}{4} \left( \frac{\partial \varepsilon_{im}}{\partial \rho} - \varepsilon_{im} \right) \mathcal{E}_i \mathcal{E}_m \right\} \delta_{ij} \]

\[ - \frac{1}{2} \left( \langle M \rangle_i H_j + \langle M \rangle_j H_i \right), \]

(58)

where we have used the following equations valid for the \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) given by (54-56).
\[ \frac{\partial \varepsilon_1}{\partial \chi} = \frac{\omega_p^2 - \omega^2}{\chi^2 \omega_p^2} [\varepsilon_1 (1 - 1)^2 + \varepsilon_3^2 H^2] - \frac{2 \omega \omega H H}{\chi^2 \omega_p^2} \varepsilon_3 (\varepsilon_1 - 1), \]  
(59)

\[ \frac{\partial \varepsilon_2}{\partial \chi} = \frac{\omega_p^2 - \omega^2}{\chi^2 \omega_p^2} [2 \varepsilon_2 (\varepsilon_1 - 1) + \varepsilon_2^2 H^2 - \varepsilon_3^2] \]

\[ + \frac{2 \omega \omega H}{\chi^2 \omega_p^2} \varepsilon_3 (\varepsilon_1 - 1), \]  
(60)

\[ \frac{\partial \varepsilon_3}{\partial \chi} = \frac{2 \omega_p^2 - \omega^2}{\chi^2 \omega_p^2} \varepsilon_3 (\varepsilon_1 - 1) - \frac{2 \omega \omega H}{\chi^2 \omega_p^2} \varepsilon_3 (\varepsilon_1 - 1)^2 + \varepsilon_3^2 H^2], \]  
(61)

\[ \frac{\partial \varepsilon_1}{\partial \omega_p} = \frac{-\omega^2}{\chi^4 \omega_p^4} [(\varepsilon_1 - 1)^2 + \varepsilon_3^2 H^2] - \frac{2 \omega \omega H}{\chi^4 \omega_p^4} \varepsilon_3 (\varepsilon_1 - 1), \]  
(62)

\[ \frac{\partial \varepsilon_2}{\partial \omega_p} = -\frac{\omega^2}{\chi^4 \omega_p^4} [2 \varepsilon_2 (\varepsilon_1 - 1) + \varepsilon_2^2 H^2 - \varepsilon_3^2] \]

\[ + \frac{2 \omega \omega H}{\chi^4 \omega_p^4} \varepsilon_3 (\varepsilon_1 - 1), \]  
(63)

\[ \frac{\partial \varepsilon_3}{\partial \omega_p} = \frac{-2 \omega^2}{\chi^4 \omega_p^4} \varepsilon_3 (\varepsilon_1 - 1) - \frac{2 \omega \omega H}{\chi^4 \omega_p^4} [(\varepsilon_1 - 1)^2 + \varepsilon_3^2 H^2], \]  
(64)

\[ \frac{\partial \varepsilon_1}{\partial \omega_H} = \frac{2 \omega H}{\chi^2 \omega_p^2} \varepsilon_3 (\varepsilon_1 - 1), \]  
(65)

\[ \frac{\partial \varepsilon_2}{\partial \omega_H} = -\frac{2 \omega}{\chi^2 \omega_p^2} \varepsilon_3 (\varepsilon_1 - 1), \]  
(66)

\[ \frac{\partial \varepsilon_3}{\partial \omega_H} = \frac{\omega}{\chi^2 \omega_p^2} [(\varepsilon_1 - 1)^2 + \varepsilon_3^2 H^2]. \]  
(67)

Using (49) and the fact that \( D_i = \varepsilon_{im} \varepsilon_m \), we can also write the tensor \( \delta_{ij} \) into the form

\[ \langle \Pi_{ij} \rangle = \left\{ p_0 + \frac{H^2}{2} - \frac{\varepsilon^* \varepsilon_m}{4} \left( \frac{\partial \varepsilon_{lm}}{\partial \rho} - \varepsilon_{lm} \right) \right\} \delta_{ij} - H_i H_j \]

\[ - \frac{1}{4} \left\{ \varepsilon_1 \varepsilon^*_i H_j + \varepsilon_2 (\varepsilon^*_j H_i + \varepsilon_2 (\varepsilon^*_j H_i) \right\} \]

\[ + H_i H_j \left( \frac{\partial \varepsilon_1}{\partial H^2} \varepsilon^* + \frac{\partial \varepsilon_2}{\partial H^2} (\varepsilon^*_j H_i) \varepsilon^*_m \right) \]

\[ + i \frac{\partial \varepsilon_3}{\partial H^2} (\varepsilon^*_j \varepsilon^*_m) \right\} \]  
(68)

If the following identity is noted

\[ \varepsilon_i^* (\varepsilon \times H)_j + \frac{\partial \varepsilon_i}{\partial t} (\varepsilon \times H)_j \]

\[ + \left( i \leftrightarrow j \right) \right\}. \]  
(69)

we finally obtain

\[ \langle \Pi_{ij} \rangle = \left\{ p_0 + \frac{H^2}{2} - \frac{\varepsilon^* \varepsilon_m}{4} \left( \frac{\partial \varepsilon_{lm}}{\partial \rho} - \varepsilon_{lm} + \varepsilon^*_{lm} \right) \right\} \delta_{ij} \]

\[ - \frac{1}{4} \left\{ \varepsilon_1 \varepsilon^*_i H_j + \varepsilon_2 (\varepsilon^*_j H_i + \varepsilon_2 (\varepsilon^*_j H_i) \right\} \]

\[ - H_i H_j \left( \frac{\partial \varepsilon_1}{\partial H^2} \varepsilon^* + \frac{\partial \varepsilon_2}{\partial H^2} (\varepsilon^*_j H_i) \varepsilon^*_m \right) \]

\[ + i \frac{\partial \varepsilon_3}{\partial H^2} (\varepsilon^*_j \varepsilon^*_m) \right\}. \]  
(70)

This agrees with the Pitaevskii’s stress tensor of a variable electric field in a liquid located in a strong magnetic field \( \mathbf{B} \).

**E. Time-dependent permittivity**

In the previous subsections, we assumed that the parameters \( \chi, \omega_p^2, \xi \) and the magnetic field \( \mathbf{B} \) are time-independent. Consequently, the permittivity discussed in the subsection III.A is static. Now, we will abandon the restriction and allow \( \chi, \omega_p^2, \xi, \mathbf{B} \) to change slowly with time. This case is naturally accounted for by (13), (28) and (30), the equation of motion for \( \mathbf{P} \),

\[ \frac{\partial}{\partial t} \left( \frac{P}{\chi \omega_p^2} + \frac{2 \varepsilon B}{\chi \omega_p^2} \times \mathbf{P} \right) - \left( \frac{\partial \varepsilon \mathbf{B}}{\partial t} \right) \times \mathbf{P} \]

\[ + \left( \frac{1}{\chi} - 1 \right) \mathbf{P} - \mathbf{E} = 0. \]  
(71)

Here, we again neglected the dissipation \( h^D \) and considered a stationary medium: \( \mathbf{v}(r, t) \equiv 0 \). Comparing the equation (7) with (14), we can see that the temporal variations of \( \chi, \omega_p^2, \xi, \mathbf{B} \) give rise to additional terms, which result in a dynamic correction \( \varepsilon^{\text{dyn}}_{ij} \) to the static dielectric permittivity obtained in subsection III.A. In other words, the relationship between the amplitude of polarization \( \mathbf{P} \) and that of electric field \( \mathbf{E} \) is no longer given by (30), but by
\[ \mathcal{P}_i = (\varepsilon_{ij} - \delta_{ij}) \mathcal{E}_j + \varepsilon_{ij}^{\text{dyn}} \mathcal{E}_j, \tag{72} \]

where \( \varepsilon_{ij} \) is given by [11] and [37][39]. \( \varepsilon_{ij}^{\text{dyn}} \) may be calculated by retaining a monochromatic electric field in (71): \( \dot{\mathcal{E}} = 0 \). Yet, because \( \varepsilon \) is now time-dependent, the amplitude of polarization \( \mathcal{P} \) will change with time, \( \dot{\mathcal{P}} = \dot{\varepsilon}_{ij} \mathcal{E}_j \).

And Eq. (71) becomes,

\[
\left( 1 - \frac{1}{\chi} + \frac{\omega^2}{\chi \omega_p^2} \right) \mathcal{P} + 2i \frac{\xi \omega}{\chi \omega_p^2} \mathbf{B} \times \mathcal{P} + \mathcal{E} + \frac{\partial}{\partial t} \left( \frac{2i \omega}{\chi \omega_p^2} \dot{\mathcal{E}} - \frac{2 \xi \mathbf{B}}{\chi \omega_p^2} \mathbf{B} \times \mathcal{E} \right) - i \omega \left( \frac{\partial}{\partial t} \frac{1}{\chi \omega_p^2} \right) (\dot{\varepsilon} + 1) \mathcal{E} + \left( \frac{\partial}{\partial t} \frac{\xi \mathbf{B}}{\chi \omega_p^2} \right) \times (\dot{\varepsilon} + 1) \mathcal{E} = 0, \tag{73} \]

where \( \dot{\varepsilon} \) is matrix notation of the permittivity [41]. Solving for \( \mathcal{P} \), we obtain

\[
\varepsilon_{ij}^{\text{dyn}} = (\varepsilon_{im} - \delta_{im}) \left[ \frac{2i \omega}{\chi \omega_p^2} \varepsilon_{mj} + i \omega \left( \frac{\partial}{\partial t} \frac{1}{\chi \omega_p^2} \right) (\varepsilon_{mj} - \delta_{mj}) \right] - \frac{2 \xi}{\chi \omega_p^2} \varepsilon_{mn} B_n \varepsilon_{ij} - \varepsilon_{mn} \left( \frac{\partial}{\partial t} \frac{\xi \mathbf{B}}{\chi \omega_p^2} \right) (\varepsilon_{ij} - \delta_{ij}) \tag{74}. \]

Together with the static permittivity derived before, this dynamic correction provides the expression for the full time-dependent permittivity.

To compare with [3], we decompose the dynamic correction (74) into a hermitian and an antihermitian part:

\[
\varepsilon_{ij}^{\text{dyn}} = \varepsilon_{ij}^{\text{ih}} + \varepsilon_{ij}^{\text{ah}}. \tag{75} \]

Both matrices \( \varepsilon_{ij}^{\text{ih}} \) and \( \varepsilon_{ij}^{\text{ah}} \) may have complex elements, but must be hermitian. In accordance to (75), we may also call them the real and imaginary parts of \( \varepsilon_{ij}^{\text{dyn}} \).

Particularly, the imaginary part is

\[
\varepsilon_{ij}^{\text{ih}} = \frac{i}{2} (\varepsilon_{ij}^{\text{ih}} - \dot{\varepsilon}_{ij}^{\text{ih}}), \tag{76} \]

which can be also written as

\[
\frac{\partial}{\partial t} \left[ \varepsilon_{im} - \delta_{im} \right] \left( \frac{\omega \delta_{mn} + i \xi \varepsilon_{mkn} B_k}{\chi \omega_p^2} \right) (\varepsilon_{nj} - \delta_{nj}), \tag{77} \]

here the fact \( \varepsilon_{im}^{\text{ih}} = \varepsilon_{ji} \) is used. From the expression [41] and [37][39] for \( \varepsilon \), one can show that the equation

\[
(\varepsilon_{im} - \delta_{im}) \left( \frac{\omega \delta_{mn} + i \xi \varepsilon_{mkn} B_k}{\chi \omega_p^2} \right) (\varepsilon_{nj} - \delta_{nj}) = \frac{1}{2} \frac{\partial \varepsilon_{ij}}{\partial \omega}, \tag{78} \]

is valid. So the imaginary part [70] of the dynamic contribution to the permittivity is

\[
\varepsilon_{ij}^{\text{ih}} = \frac{1}{2} \frac{\partial^2 \varepsilon_{ij}}{\partial \omega \partial t}. \tag{78} \]

This formula was first obtained in [3].

**IV. DISCUSSIONS**

Because both the dispersion and nonlinearity are accounted for, the hydrodynamic theory of dispersion sketched in section II is a fairly complete theory for the dynamics of a fluid interacting with varying fields. The theory is derived by generalizing the hydrodynamic approach, but the result is consistent with the work of Pitaevskii, who starts from rather different physics.

Though not shown here, the present theory reduces to the hydrodynamic one [3], in the low frequency limit \( \omega \to 0 \). It is also in agreement with the Barash-Karpman’s stress tensor derived for quasi-monochromatic field (ie including the lowest-order effects of temporal variation of the field amplitude \( \mathcal{E} \)) [4]. All these features support the statement that the basic equations shown in section II are correctly formulated, particularly the fundamental differential relation [4].

In our theory for dispersive media, the Pitaevskii’s magnetization appears as a consequence of circular motions of the polarization. In contrast to the conventional magnetization of atomic origin, the Pitaevskii’s magnetization is macroscopic. Because the circular motion of \( \mathbf{P} \) is usually accompanied by a rotating electric field, the Pitaevskii’s magnetization is less suitably generated by linearly polarized electromagnetic fields, as suggested in [3]. It is remarkable that the explanation to this phenomenon was given by the first nonlinear term in the expansion of the energy, Eq.(22). We expect it to be an important nonlinear effect, and to play a significant role in the nonlinear interaction between matter and intense laser lights.

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Because the imaginary parts of the permeabilities, $\varepsilon''$ and $\mu''$, are odd functions of the frequency $\omega$, and the real parts, $\varepsilon'$ and $\mu'$ even ones, and because the first frequency dependent effects when leaving the equilibrium are linear in $\omega$, they belong to $\varepsilon''$, $\mu''$ and are dissipative. Lack of dissipation therefore characterizes a frequency region disjunct from $\omega = 0$.

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The Heaviside-Lorentz units, or the so-called rational units, employed in this papers renders the formulas and their manipulations much simpler: In this system, all four fields have the same dimension, square root of the energy density, (ie $\sqrt{\text{J/m}^3}$ in MKS and $\sqrt{\text{erg/cm}^3}$ in cgs,) sensibly with $H = B$ and $D = E$ in vacuum, and of similar magnitudes in ponderable media. As a result, the ubiquitous factors of $4\pi$, $\epsilon_0$ and $\mu_0$ — the actual reason why we have to continually look up the formulas — simply vanish. To revert to MKSA ($\hat{E}, \hat{H}$...), employ

\[
\hat{H} = H/\sqrt{\mu_0}, \quad \hat{B} = B/\sqrt{\mu_0}, \quad \hat{E} = E/\sqrt{\epsilon_0}, \quad \hat{D} = D/\sqrt{\epsilon_0},
\]

\[
\hat{\rho}_e = \rho_e/\sqrt{\epsilon_0}, \quad \hat{\jmath}_e = \jmath_e/\sqrt{\epsilon_0}.
\]